Solitons and entanglement in the double sine-Gordon model

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Abstract

The bipartite ground state entanglement in a finite linear harmonic chain of particles is numerically investigated. The particles are subjected to an external on-site periodic potential belonging to a family parametrized by the unit interval encompassing the sine-Gordon potential at both ends of the interval. Strong correspondences between the soliton entanglement entropy and the kink energy distribution profile as functions of the subchain length are found.

Keywords: soliton entanglement, double sine-Gordon potential, entanglement entropy

(Some figures may appear in colour only in the online journal)

1. Introduction

Entanglement in systems displaying solitonic solutions has been studied for some specific models [1–5]. Recently, solitons in discrete chains presenting bounded internal modes have received increasing attention from both experimental and theoretical points of view [6–13]. Since solitonic solutions of nonlinear models are characterized by being localized and topologically protected, investigation focusing on their use for quantum information tasks is in progress. As for example, in [14] the authors studied the behavior of entanglement in a chain of particles using the Frenkel–Kontorova model [15–17], and they have investigated the possibility to use solitons as the carrier of quantum information. In this model, the particles of the 1D chain interact by means of an elastic force with the nearest neighbors and are subjected to an on-site (substrate) sine-Gordon potential $V$. The strength of the elastic couplings $g$ between adjacent particles becomes a parameter in the model that rules the width (proportional to $\sqrt{g}$) of the unique soliton energy density lump. The influence of the sG soliton on the bipartite entanglement was analyzed by means of the von Neumann entropy, where one of two sub-blocks that conform to the chain is always centered at the kink energy density maximum (kink center). The authors study the dependence of the von Neumann entropy on the size of the sub-blocks for several values of $g$, i.e., for several values of the soliton width. The result is that the presence of the soliton enhances the entanglement, reaching a maximum when the size of the sub-block coincides approximately with the soliton width. The investigation of this issue in interacting multisoliton solutions is achieved in [18].

Solitons and/or kinks are extended states whose energy densities are spatially localized in bounded regions, in contrast with vacuum energy densities, which are spatially homogeneous. In scalar field theory on the real line, the existence of kinks comes from quantization of solitary wave solutions of the classical field equations. These traveling waves have energy densities showing one or several lumps of energy moving together with their center of mass. Thus, the kink center and the centers and widths of these lumps characterize the extended state; see e.g., [19]. We shall address a system in this class determined by the family of double sine-Gordon potential energy densities:

$$V(\phi; a) = 1 - (1 - a) \cos(2\pi\phi) - a \cos(4\pi\phi), \quad a \in [0, 1].$$

The family parameter $a$ is chosen to be in the unit interval in order to have a function $V$ semidefinite positive. At $a = 0$, the standard sine-Gordon (sG) model is included, whereas at $a = 1$ the sG model reappears, although the argument is twice the standard sine-Gordon angle. Crucial to us in this work is the existence of kink-traveling waves in any member of this family of models. The structure of the kink profiles, however, differs with varying values of $a$. In particular, in the range...
0 ≤ a ≤ \frac{1}{2} the kink energy density shows a single maximum at the kink center, i.e., it is formed by only one lump of energy density. In the range \frac{1}{2} < a < 1, however, the kink energy density presents two separated maxima symmetrically distributed with respect to the kink center. The kink profile is formed by two separated energy density lumps if a is greater than \frac{1}{2} and less than 1. The distance between the lump centers grows with a and becomes infinity at a = 1, where the one-lump sine-Gordon kink of double angle emerges; see [20, 21].

We shall work in the Frenkel–Kontorova scheme choosing as an on-site substrate the potential of the double sine-Gordon model, i.e., we shall focus on the double sine-Gordon model defined on a finite chain of points instead of on the whole real line. Following the ideas disclosed in [14], we shall study soliton entanglement entropy, replacing the standard sine-Gordon on-site potential by one potential belonging to the double sine-Gordon (dSG) family. Proceeding in this way we enlarge the number of physical parameters. Besides the spring constant \( g \), the family parameter \( a \in [0, 1] \) plays a significant rôle. The length where the kink energy density departs from the vacuum energy, the kink width, is proportional to \( \sqrt{g} \), whereas \( a \) determines the kink profile shape. These facts together make the dSG model specially suitable to investigate entanglement entropy when the kink energy distributions are more or less extended over the chain and/or their shape exhibits one or two energy lumps with different separations. With this in mind we divide the chain following [14] into two complementary sub-blocks or subchains with no common points. Unlike Marcovitz et al., we select the two subchains starting from the two chain endpoints. We expect that the entanglement between fluctuations in different subchains will be influenced by how many of the kink energy distribution is contained in each subchain. The \( g \) parameter leads us to distinguish three characteristic regimes: (1) First, we address the situation where the kink width is much shorter than the total chain length. For instance, for the choice of \( g = 10^5 \) on a chain of length 501 points (used in this paper), the kink energy density occupies one fifth of the chain. The kink is clearly localized. (2) Second, in a chain of the same length, a coupling of \( g = 10^5 \) concentrates the kink in three-fifths of the chain. The kink is spread more throughout the chain in this regime than in the previous one. (3) Third, choosing again the same chain but having \( g = 10^6 \), we encounter the last regime: the kink energy distribution encompasses the whole chain. Clearly, similar regimes for longer chains are found for higher values of \( g \). In all these considerations it is assumed that the kink center sits at the chain midpoint.

We shall perform numerical calculations of the correlation functions and the von Neumann entropy in these three regimes. Both the quantum vacuum and kink sectors will be investigated, whereas three values of the \( a \) parameter will be considered distinguishing kink profiles with a single lump, two close lumps, and two widely separated lumps. Our results offer a refinement in several directions on the structure of the soliton entanglement entropy discussed in [14]. First, the spatial correlation functions computed before the partition into two sub-chains show shapes in strong correspondence with the kink energy distribution profile. These correspondences are unveiled in the three regimes previously described, in each of them for the three characteristic values of \( a \). Comparison with spatial correlation between vacuum fluctuations is also offered. In a second step, we use the knowledge of the correlation functions to compute, also numerically, the entanglement entropy arising in a partition of the system into two sub-chains between fluctuations living in different parts of the chain. Our \( N \)-particle chain will thus be divided into one \( \ell \)-particle and another \( N-\ell \) particle subblock with all the \( \ell \) points in the first subchain located at the left of the \( N-\ell \) points of the second subchain. We shall compute the entanglement for different choices of the partition length \( \ell \). The main conclusion is that entanglement between fluctuations in the two different subchains is maximum when \( \ell \) coincides with one of the kink lump centers. If the kink energy density shows two energy lumps the entanglement entropy as a function of \( \ell \) reaches two maxima too precisely when the length of the sub-block \( \ell \) coincides with the kink lump centers. In more general terms, we claim that the entanglement is enhanced when the kink energy density at the sub-block endpoint increases, and it is maximum when the kink energy density is maximum.

2. Quantization

We shall address a 1D linear harmonic chain of \( N \) particles interacting with the nearest neighbors and subjected to an on-site (substrate) potential \( V \). We assume that the equilibrium position of the \( n \)th particle in the chain is \( x_n^0 = n, n = 0, ..., N \). The classical Hamiltonian governing the dynamics of this model is

\[
H = \sum_{n=0}^{N-1} H_{n,n+1} = \sum_{n=0}^{N-1} \left[ \pi_n^2 + 2V(\phi_n) + g(\phi_{n+1} - \phi_n)^2 \right],
\]

where \( \phi_n \) represents the displacement of the \( n \)th particle with respect to its equilibrium position, \( X_n = n + \phi_n, \pi_n = \dot{\phi}_n \) is the momentum of the \( n \)th particle, \( g \) is the non-dimensional strength of the elastic couplings between adjacent neighbor particles, and \( V \) represents the external on-site potential; see [16]. We assume that \( V(\phi) \) is a non-negative function that vanishes when the displacements reach an equilibrium point. In formula (1), all the magnitudes involved, particle positions, momenta, and potential, are rescaled to non-dimensional quantities. The system of coupled ODE’s

\[
\frac{d^2 \phi_n}{dt^2} = -\frac{\partial V(\phi_n)}{\partial \phi_n} + g(\phi_{n+1} - 2\phi_n + \phi_{n-1}),
\]

\[
n = 0, 1, ..., N
\]

are the equations of motion describing the classical dynamics of the model, whereas time-independent solutions are
determined from the difference equations system:

\[
\frac{dV(\phi_n)}{d\phi_n} + g(\phi_{n+1} - 2\phi_n + \phi_{n-1}) = 0. \tag{3}
\]

We shall distinguish two types of time-independent finite energy solutions:

1. Homogeneous solutions of (3), denoted as \(V^V(n) = \phi^n_V\):

\[
\phi^V(n) = \phi^n_V = 0 \quad \text{with} \quad n = 0, 1, \ldots, N.
\]

All the particles sit at their equilibrium points \(X_n = n\) in such a way that their classical energy is zero: \(H(\phi^n_V) = 0\). We shall refer to them as vacuum solutions foreseeing future treatment in a quantum setting.

2. Discrete soliton/kink solutions, which we shall denote as \(\phi^S(n) = \phi^n_S\). Soliton and kink are frequently used as synonyms in the literature. The strict concept of soliton, however, is more restrictive. Both kinks and solitons are solitary nonlinear waves, but they differ in the fact that solitons preserve shapes after collisions. We shall denote as kinks the non-homogeneous solutions from now on throughout this paper. Kinks are thus solutions of (3), obeying the following boundary conditions:

\[
\phi^S(0) = \phi^S_0 = 0, \quad \phi^S(N+1) = \phi^S_N = 1, \tag{4}
\]

characterized accordingly by the non-null ‘topological charge’: \(T = |\phi^S_N - \phi^S_0| = 1\). The boundary conditions (4), tantamount to \(X^0 = 0\) and \(X^N = N + 1\), demand that the kink profile will depart from the particle equilibrium positions somewhere in the middle of the chain. The kink energy distribution (density in the continuum limit) is \(E(\phi^S_k) = V(\phi^S_k) + \frac{1}{2}g(\phi^S_{k+1} - \phi^S_k)^2\).

Small fluctuations \(\eta(n, t)\) around static solutions \(\phi^{sta}(n) = \phi^n_{sta}\), either vacuum or kink configurations,

\[
\phi(n, t) = \phi^{sta}(n) + \eta(n, t)
\]

are still solutions of the ODE’s system (2) if and only if the following ODE’s linear system

\[
\left(\frac{d^2}{dt^2} + \Omega(n) + 2g \right)\eta(n, t) - g \eta(n - 1, t)
- g \eta(n + 1, t) = \Omega'\eta
\]

is satisfied; see [20]. In formula (5), \(\Omega\) is the second derivative of the potential evaluated at the static solution:

\[
\Omega(n) = \frac{d^2V}{d\phi^2} \bigg|_{\phi^{sta}(n)}.
\]

If \(\phi^{sta}(n) = \phi^n_S\) then \(\Omega(n)\) is the curvature of the potential at the equilibrium points, where all the particles of the chain sit. If \(\phi^{sta}(n) = \phi^n_V\), then \(\Omega(n)\) is evaluated at kink solutions and because kinks are not homogeneous, i.e., they vary with \(n\), and it takes different values at different points of the chain. From the spectral problem of the \((N + 1) \times (N + 1)\) Hessian matrix \(B(n, m)\)

\[
B(n, m) = (\Omega + 2g)\delta_{nm} - g \delta_{n,m-1} - g \delta_{n,m+1},
\]

and it takes different values at different points of the chain. From the spectral problem of the \((N + 1) \times (N + 1)\) Hessian matrix \(B(n, m)\)

\[
\sum_{n=0}^{N} B(n, m)\psi_k(n) = \omega_k^2\psi_k(n), \tag{6}
\]

where \(\Omega\) obeys either to vacuum or kink static solutions, the general solution of the ODE’S system (5)

\[
\eta(n, t) = \sum_{k=0}^{N} \frac{1}{2\sqrt{2\omega_k}} \left(\delta_k e^{i\omega_k t} + \delta_k^* e^{-i\omega_k t}\right)\psi_k(n), \tag{7}
\]

is obtained in terms of the \((N + 1)\) normal modes of fluctuation \(\psi_k(n)\), whose eigenvalues have been denoted as \(\omega^2_k\); see (6). Dirichlet boundary conditions ensure that the eigenmodes are real.

Quantization of this prequantum picture is achieved by promoting the \(a_k\) and \(a_k^\dagger\) coefficients to annihilation and the creation of bosonic operators: \([\hat{a}^\dagger, \hat{a}] = \delta_{kj}\). The bosonic quantum field and its conjugate momenta operator read:

\[
\hat{\eta}(n, t) = \sum_{k=0}^{N} \frac{1}{2\sqrt{2\omega_k}} \left(\hat{a}_k e^{i\omega_k t} + \hat{a}^\dagger_k e^{-i\omega_k t}\right)\psi_k(n),
\]

and

\[
\hat{\kappa}(n, t) = i \sum_{k=0}^{N} \frac{\omega_k}{2} \left(\hat{a}_k e^{i\omega_k t} - \hat{a}^\dagger_k e^{-i\omega_k t}\right)\psi_k(n). \tag{8}
\]

States characterized by a set of occupation numbers \(n_k\) of each \(k\)th normal mode are eigenfunctions of the linearized Hamiltonian \(H_0 = \sum_{k=0}^{N} \left(\hat{a}^\dagger_k (\hat{a}^\dagger k) (\hat{a}_k + \frac{i}{\sqrt{2}})\right)\omega_k\) of energy:

\[
\langle n_k \mid \hat{H}_0 \mid \{ n_k \} \rangle = E_{\{n_k\}} \langle \{ n_k \} \mid \{ n_k \} \rangle,
\]

\[
E_{\{n_k\}} = \sum_{k=0}^{N} \left(n_k + \frac{1}{2}\right)\omega_k. \tag{9}
\]

These Fock space states correspond to the fundamental mesons of the system, both in the vacuum sector and in the kink sector, as distinct alternatives distinguished by the appropriate choice of the \(B\) matrix.

3. The double sine-Gordon model

We choose the substrate potential between the members of the one-parametric family of double sine-Gordon (dSG) potentials

\[
V(\phi; a) = 1 - (1 - a) \cos(2\phi) - a \cos(4\phi), \quad a \in [0, 1],
\]

plotted in figure 1 as a function of \(\phi\) for three characteristic values: \(a = 0.0, 0.6, \) and 0.99; see [22].

The set of absolute minima or zeroes of \(V(\phi; a)\) in the interval \(a \in [0, 1)\) is independent of \(a\): \(M = \{\phi^{(j)} = \frac{j\pi}{2}\} \in \mathbb{Z}\); see figure 2. As a function of \(\phi\) (10), however, it exhibits more critical points, \(\frac{dV}{d\phi}(\phi; a) = 0\) if
We could choose another absolute vacuum as the vacuum state of the quantum system. Quasi-particles tells us that the three types of critical points Evaluation of the second derivative of the potential at the second type contains maxima if \( f_i \) complies with the contour conditions (4) at the endpoints dependent solutions in these models. These kinks must reach finite chain; see [14]. A qualitative analysis of (3) conforms this claim.

Fig. 1. Double sine-Gordon potential for the parameters \( a = 0 \) (blue dashed line), 0.6 (pink thick solid line) and 0.99 (red thin line).

and only if:

\[
\phi^{(0)} = j, \quad \phi^{(\pm 2)} = j + \frac{1}{2},
\]

\[
\phi^{(\mp j)} = \pm \frac{1}{\pi} \arctan \sqrt{\frac{3a + 1}{5a - 1}} + j.
\]

(11)

Evaluation of the second derivative of the potential at the three types of critical points

\[
\frac{d^2V}{d\phi^2}(0; a) = 4(1 + 3a)\pi^2,
\]

\[
\frac{d^2V}{d\phi^2}(\frac{1}{2}; a) = 4(5a - 1)\pi^2,
\]

\[
\frac{d^2V}{d\phi^2}(\phi^{(\pm 0)}; a) = \frac{1 - a(2 + 15a)}{a} \pi^2
\]

(12)

tells us that the first type is always formed by minima of \( V \), the second type contains maxima if \( a < \frac{1}{5} \) that become relative minima when \( a \) reaches \( \frac{1}{5} \) and the last type formed only by real critical points for \( a > \frac{1}{5} \) that are always maxima; see figures 1 and 2. Only, quantization over absolute minima is safe, relative minima give rise to false vacua through tunnel effects, and maxima produce tachyons. We thus choose the state with no fluctuations over the classical configuration \( \phi^{(0)} \) as the vacuum state of the quantum system. Quasi-particles of mass \( 4(1 + 3a)\pi^2 \) arise when the creation operator \( \hat{a}^\dagger \) acts on the vacuum state.

Besides the homogeneous solutions of the field equations (3) listed in (11), there are other static but \( n \)-dependent solutions in these models. These kinks must comply with the contour conditions (4) at the endpoints because the system is defined on a finite chain; see figure 3.

The system of finite difference equations (3) shows that these solutions depend both on \( a \) and the strength of the elastic coupling \( g \). As stated before, the parameter \( g \) rules the extension of this topological solution along the chain; see [14]. A qualitative analysis of (3) confirms this claim. If \( g \) is strong enough, the particle distribution solving (3) is such that \( \phi_{n+1} - \phi_n \approx \frac{1}{a} \) for almost all \( n \). \( \phi(n) \) is almost a straight line with a slope as small as that allowed by the boundary conditions. The influence of the external potential is very small, and the kink profiles depart from the homogeneous vacuum practically along the whole chain. At weak coupling \( g \to 0 \), however, the distribution of particles favoured by kink solutions of (3) presents abrupt changes from one to the next particle somewhere in the middle of the chain. The potential provides strong forces away the thin kink region restoring the particles to lie on consecutive absolute minima of \( V \) at two sides of the kink. In this case, the effect of the kink is confined within a small part of the chain. In figures 4–6, we show graphics of numerically generated kink solutions with unit topological charge for several values of the elastic constant \( g \) in a chain of \( N + 1 = 501 \) particles together with their energy distributions where the above-mentioned pattern is observed. Having fixed this length for the chain, we have selected two values of \( g \) in order to characterize the strong and weak regimes mentioned above, plus a third value in between to analyze the intermediate regime. It is clear that for chains of different length, one must adopt other values of \( g \) to deal with kinks that cover \( \frac{1}{3} \) and/or the whole chain. We thus distinguish three different regimes:

1. **Substrate potential dominant regime** (SubsReg). We select the value \( g = 10^5 \) as the regime representative in a \( N = 500 \) particle chain. The dynamics are fundamentally determined by the substrate potential to the detriment of the elastic interparticle forces. The kink energy distribution is localized in an interval of length \( \frac{N}{2} = 100 \).

2. **Balanced Substrate potential/Elastic force regime** (BalReg). \( g = 10^5 \) is the representative value of the elastic parameter in a chain of the same length. Elastic forces and those induced by the substrate potential act on each particle in a balanced manner. Under these circumstances, the kink energy distribution spreads on an interval of length \( \frac{N}{2} = 300 \).

3. **Elastic force dominant regime** (ElasReg). On the same chain \( g = 10^5 \) is a representative value of this third regime. Forces induced by the substrate potential are very weak compared to the elastic interparticle forces. The kink energy distribution is smoothly distributed over the whole chain.

In sum, the set of graphics displayed in figures 4–6 reveals the rôle of the parameter \( a \). For \( a \) less than but close to the kink profile consists of two-step climbing, with a long plateau in between, from one vacuum solution up to the next one. The energy distribution is concentrated around the midpoints of each jump, and thus a two-energy lump configuration arises. The reason is the existence of a relative minima of \( V \) between each two absolute minima when \( a > \frac{1}{5} \); recall formulas (11) and (12). Moreover, because \( V(\phi^{(i+2)}) = 2(1 - a) \), the value of the potential at these
The climb from relative minima is close to zero when $a$ is slightly less than 1. The climb from $\phi^{(0)}$ to $\phi^{(1)}$, ‘resting’ at $\phi^{(1/2)}$ as much as possible. Thus, the distance between the two jumps is very large when, e.g., $a = 0.99$. When $a$ decreases but is still greater than $\frac{1}{2}$ the situation is the same, but the value of the potential at the relative minimum is higher, and the distance between the jumps is shorter, i.e., the two lumps in the energy distribution are closer. Finally, if $a < \frac{1}{2}$ the relative minima become maxima and the climbing takes place in one step because there is not a relative minima in between to rest. The $a = 1$ limit is special. The relative minima become absolute, zeroes of the substrate potential $V(\phi; 1)$, that form the set $\mathcal{M}(1) = \{\phi^{(j)} = \frac{1}{2}, j \in \mathbb{Z}\}$. For this value the dSG potential (10) becomes a pure sG potential again. Certainly, the model in the limit $a = 1$ involves a two-kink solution of a single-well sine-Gordon potential with argument twice the usual angle. In the field theory model on the continuous real line, this limit implies that one of the kinks goes to infinity because the relative minimum becomes absolute. In a finite discrete chain, we selected the value $a = 0.99$ to offer a description of kink profiles close to this limit: the two lumps of energy move towards the endpoints of the chain.

At this point, it is important to emphasize that the surge of two lumps on kink profiles when $a > \frac{1}{3}$ in the unit topological charge sector of the double sine-Gordon does not mean that these soliton structures are multisoliton or breather solutions akin to those arising in pure sine-Gordon models, i.e., the models found in the boundary points $a = 0$ and $a = 1$ of the family. Sine-Gordon multisolitons live in sectors with topological charges different from one another and are time-dependent even in their center of mass. Unlike in the standard sine-Gordon model, there exist forces between kinks, either formed by one or two lumps in the double sine-Gordon models, specifically between kinks and antikinks; see [21]. Isolated kinks, however, are static in their center of mass, and are formed either by one or two lumps. The only time-dependence arises from the motion of the center of mass obeying Lorentz transformations.

Next, we study how the rich structure of these dSG solitons influences quantum entanglement between vacuum and kink fluctuations. In particular, we begin by analyzing the spatial correlation functions between two particles in the chain in the kink and vacuum sectors.

4. Spatial Correlation Functions

To compute spatial correlation functions in this $N$-particle system, we follow the procedure established in [14, 23]. The canonical variables are assembled into a vector $Y = (\eta, \pi)$, which is a point in phase space. The $N$-component vectors $\eta = (\eta_1, \eta_2, \ldots, \eta_N)$ and $\pi = (\pi_1, \pi_2, \ldots, \pi_N)$ encompass, respectively, the particle positions and momenta. Quantization forces the following commutation rules:

$$[\hat{Y}_\alpha, \hat{Y}_\beta] = i\lambda_{\alpha \beta}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (13)$$

where the $2N \times 2N$ symplectic matrix $J$ is written in this formula as $N \times N$ blocks of the null and identity matrices. Expectation values of two canonical variables in a quantum state are the matrix elements of the $2N \times 2N$ covariance matrix $M = \text{Re} \{\langle \hat{Y}^T \hat{Y} \rangle\}$, which may be diagonalized in the Williamson form $W = S_W M S_W^{-1} = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N, \lambda_1, \lambda_2, \ldots, \lambda_N)$, where the eigenvalues $\lambda_i \geq \frac{1}{2}$ are the moduli of the eigenvalues of the matrix $JM$. Gaussian states are completely characterized by the covariance matrix $M$ and by their first momenta. Partitioning the $N$-mode system into two sets $A$ and $B$, the reduced

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Figure 2. Pictures of vacuum solutions for a chain of 11 particles subjected to a double sine-Gordon substrate potential, corresponding to the $a$ parameter values: (a) $a = 0.0$, (b) $a = 0.6$, and (c) $a = 0.99$. 

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5
covariance matrices $M_k = \text{Re} \left\{ \langle \hat{Y}^T \rangle \right\}$, $k = A, B$, must be put in the Williamson form. Matrix elements of the covariance matrix $M$ are thus of the form 

$$\langle \hat{Y}^{(n)} \hat{Y}^{(m)} \rangle = \text{tr} \hat{Y}^{(n)} \hat{Y}^{(m)}, \langle \hat{X}^{(m)} \hat{Y}^{(n)} \rangle = \text{tr} \hat{X}^{(m)} \hat{Y}^{(n)}$$

and 

$$\langle \hat{X}^{(n)} \hat{X}^{(m)} \rangle = \text{tr} \hat{X}^{(n)} \hat{X}^{(m)} \rangle = \text{tr} \hat{X}^{(m)} \hat{X}^{(n)} \rangle$$

Note that the particle on the right boundary is displaced in one site with respect to the same particle in the vacuum. The particles not in the boundary are displaced away from their equilibrium positions.

Figure 3. Graphics of the kink solution numerically obtained for a chain of 11 particles under the dSG substrate potential and the parameter value $a = 0.6$. Note that the particle on the right boundary is displaced in one site with respect to the same particle in the vacuum. The particles not in the boundary are displaced away from their equilibrium positions.

Figure 4. Kink profiles (solid line) and their energy distributions (dashed line) for $a = 0$, $a = 0.6$, and $a = 0.99$ in the substrate potential dominant regime of the dSG model on an $N = 500$ chain. The jump from a homogeneous solution to the next one in an abrupt manner. Accordingly, the kink energy distributions neatly depart from zero only on a short interval and have one maximum if $a < \frac{1}{5}$, but two maxima appear if $a > \frac{1}{5}$.

Figure 5. Kink profiles (solid line) and their energy distributions (dashed line) for $a = 0$, $a = 0.6$ and $a = 0.99$ in the balanced substrate potential/elastic force regime of the dSG model on an $N = 500$ chain. The jump from a vacuum solution to the next one is less steep than in the regime discussed in figure 4. The kink energy distributions are also less localized and spread over longer intervals. The distinct number of lumps, one if $a < \frac{1}{5}$, two for $a > \frac{1}{5}$, is also observed.

Figure 6. Kink profiles (solid line) and their energy distributions (dashed line) for $a = 0$, $a = 0.6$, and $a = 0.99$ in the elastic force dominant regime of the dSG model on an $N = 500$ chain. The kink profiles climb almost uniformly, starting from one vacuum solution at the chain left endpoint and ending at the next vacuum solution on the chain right endpoint. The kink energy distributions are even less localized than those in the previously described regime. The existence of energy distributions formed by one or two lumps when $a$ is below or above $\frac{1}{5}$ is also appreciated.
functions between particle positions and momenta

\[
\text{Re} \left\{ \langle \tilde{\eta}(m) \tilde{\eta}(n) \rangle \right\} = \frac{1}{2} \sum_{l} \psi_{l}(m) \psi_{l}(n) \alpha_{l},
\]

\[
\text{Re} \left\{ \langle \tilde{\pi}(m) \tilde{\pi}(n) \rangle \right\} = \frac{1}{2} \sum_{l} \psi_{l}(m) \psi_{l}(n). \tag{14}
\]

whereas positions and momenta verify

\[
\text{Re} \left\{ \langle \tilde{\pi}(m) \tilde{\eta}(n) \rangle \right\} = \text{Re} \left\{ \langle \tilde{\eta}(m) \tilde{\pi}(n) \rangle \right\} = 0.
\]

Correlations between two particle positions are obtained from the normal mode eigenvalues and eigenfunctions:

\[
\xi_{mn} = \frac{1}{2} \sum_{k=0}^{\infty} \psi_{k}(m) \psi_{k}(n) \alpha_{k}. \tag{15}
\]

In figures 7(a)–(c), the dependence on \( n \) of the spatial correlation function when one of the particles sits at the left endpoint of the chain \( \xi_{0n} = \xi_{n} \) is displayed. Numerical computations relying on formula (15) have been performed for our chain of 501 particles, accounting for vacuum and kink normal modes, in the nine cases where the kink profiles were depicted: three characteristic values of \( a \) and \( g \). The properties of correlation functions as functions of \( n \) are clearly observed in the \( g = 10^{4} \) regime. Vacuum and kink correlations \( \xi_{v} \) do not differ from each other for small \( n \) where the kink profile does not depart too much from the vacuum solution. At larger \( n \), kink fluctuation correlations remarkably grow bigger than vacuum correlations until reaching one or two maximum values, precisely at the points where the kink energy distribution also reaches its maximum values. If \( n_{x} \) denotes the position of the maximum farther away from the origin, kink correlations start to decrease for \( n > n_{x} \). The difference between kink correlations for \( a < \frac{1}{2} \) and \( a > \frac{1}{2} \) is that in the first case there is only a single maximum \( n_{0} \) whereas there are two \( n_{x} \) if \( a > \frac{1}{2} \). Kink correlations in the last case decrease after \( n_{-} \), pass a relative minimum at \( n_{0} \), and increase again up to \( n_{+} \); see figure 7(a). We conclude that two-point particle spatial correlations in the kink sector recognize somehow the kink profile shape. This behavior, albeit less pronounced, is reproduced in the BalReg regime, figure 7(b), whereas in the ElasReg regime [see figure 7(c)], the curvatures at the maxima are very mild and the correlation functions \( \xi_{v} \) for vacuum and kink fluctuations look similar. We thus conclude that the logarithmic correlations in the kink sector behave in a remarkable different manner than the vacuum correlations of standard logarithmic class: \( \xi_{v} = \text{log} \frac{1}{|m-n|} \).

The correlation function shapes as functions of the interparticle distance just described follow a similar pattern independently of the fixed particle position. In order to illustrate this fact, correlation functions between the \( m = 100 \)th particle and a second particle located in an arbitrary
5. Entanglement

Entanglement between fluctuations supported by Gaussian states is measured by taking advantage of the correspondence between the diagonal form of the covariance matrix 

\[ M = \text{Re} \left( Y Y^T \right) \]

for the Gaussian state with the covariance matrix of a thermal state, also diagonal in Fock space, having an average phonon number \( \bar{n} = \lambda - \frac{1}{2} \) [23]. Here, the \( \lambda_i \) are the Williamson eigenvalues of the covariance matrix and entanglement entropy is accordingly defined as:

\[
E_S = \sum_j S(\lambda_j) \quad \text{with} \\
S(\lambda) = \left( \lambda + \frac{1}{2} \right) \ln \left( \lambda + \frac{1}{2} \right) - \left( \lambda - \frac{1}{2} \right) \ln \left( \lambda - \frac{1}{2} \right).
\]

(16)

After choosing a sub-block of \( \ell \) particles forming a subchain \( A \) of the whole chain of 501 particles, the eigenvalues of the reduced \( 2 \ell \times 2 \ell \) covariance matrix to this sub-block are numerically computed. Formula (16) is then used to estimate also numerically the entanglement entropy. In contrast with the usual choice predominant in the literature where the subchain is centered at the chain midpoint (see e.g., [14]), we take the sub-block starting from the left endpoint of the chain; see figure 8. The reason is that performing these calculations on subchains of fixed but arbitrary length \( \ell \leq 501 \) a striking similarity between the entanglement entropy as a function of \( \ell \), and the kink energy distribution along the chain clearly emerges. More precisely, entanglement between fluctuations in the subchains \( A \) and \( B \) is maximum when \( \ell \) is chosen at a maximum of the kink energy distribution.

In figures 9–11, the entanglement entropies numerically obtained from (16) are displayed corresponding to the three regimes where the kink profiles were shown before: SubsReg (figure 9(a)–(c)), BalReg (figure 10(a)–(c)), and ElasReg (figure 11(a)–(c)). In each regime, the graphics for the three values of \( a, a = 0, a = 0.6, \) and \( a = 0.99 \), are depicted.

Entanglement entropies between vacuum fluctuations depend only slightly on the subchain location. For fixed \( g \) and \( a, \) they vary with \( \ell \) in the same qualitative way as those entropies found for centered subchains. The differences are only quantitative: the heights of the plateau of these functions are smaller in this new location of the sub-block. Kink fluctuation entanglement entropies require a much more detailed analysis. We shall describe in turn the results in the nine different cases:

5.1. A. Substrate potential dominant regime

The numerically computed entanglement entropy is displayed in figures 9(a)–(c) for the three habitual values of the \( a \) parameter. The kink energy distributions are also shown as a shadowed zone in the same figures. Note the correspondence between the maxima of the kink energy distributions and the entanglement entropy as functions of \( \ell \), in concordance also with the spatial correlation function maxima.

We remark the following facts: (1) For sine-Gordon kinks, \( a = 0 \), the entanglement entropy between kink–kink and vacuum–vacuum fluctuations are identical when the sub-block length \( \ell \) is small. The kink-supported entropy \( E_S^{kink} \) begins to increase and starts to differ from \( E_S^{vacuum} \) when \( \ell \) approaches 150. The kink entanglement entropy reaches its maximum value at a sub-block length of \( \ell = 250 \). From this point onwards, \( E_S^{kink} \) decreases until a sub-block length of approximately \( \ell = 350 \), a point where again the kink-supported entropy equals the vacuum entropy; see figure 9(a). The maximum of the entanglement entropy and the center of the kink energy distribution are identical and lie at the midpoint of the chain, a fact shown by the vertical dashed line. The kink entanglement entropy is maximum for a subchain whose endpoint coincides with the kink energy distribution maximum point. Again this is in concordance with the spatial correlation function, which reaches its maximum just at the center of the single kink; see figure 7.

(2) When \( a = 0.6 \), the entanglement entropy between kink fluctuations behaves in a slightly different way. As in the previous case, \( E_S^{kink} \) fits in the function \( E_S^{vacuum} \) approximately until \( \ell = 150 \). For sub-blocks of length 150 < \( \ell \leq 236 \) the kink fluctuation entropy increases and reaches a maximum at \( \ell = 236 \), precisely at the center of the first lump of the kink energy distribution. The entropy of subchains of lengths in the interval \( \ell \in (236, 250) \) decreases until hitting a relative minimum \( \ell = 250 \), half the whole chain length. Beyond this length the entanglement entropy between kink fluctuations starts to increase again reaching a second maximum at the center of the second lump of the kink energy distribution \( \ell = 264 \). For sub-block lengths in the interval 264 < \( \ell < 350 \), \( E_S^{kink} \) decreases until approximately \( \ell = 350 \). Beyond this sub-block length, the kink fluctuation entropy coincides again with \( E_S^{vacuum} \); see figure 9(b).

(3) If \( a = 0.99 \), the entanglement entropy \( E_S^{kink} \) follows the same pattern explained in the previous case; see figure 9(c). Here, the distance between the two maxima of the entanglement entropy is greater than the distance between the maxima of the \( a = 0.6 \) kink. The positions of the two maxima occur in this case for sub-block lengths: \( \ell = 221 \) and \( \ell = 279 \), the maxima of the kink energy distribution. The kink
fluctuation entropy exhibits a relative minimum between the two maxima, but the value of the kink entropy at this minimum is much greater than $E_{S_{\text{vacuum}}}$ in the same range.

In the three cases the kink entanglement entropies and energy distributions are curves as functions of $\ell$ with similar shapes and only the scales are different. One also checks that the correlation functions in figure 7(a)–(d) have the same critical points than the kink energy distributions.

5.2. B. Balanced Substrate potential-Elastic force Regime

In figures 10(a)–(c) graphics of both the entanglement entropy between vacuum–vacuum and kink–kink fluctuations in an $N = 500$ particle chain are shown within the BalReg Regime $g = 10^4$. The kink energy distributions are plotted as curves enclosing a shadowed area in the scale set by $E_S$. Dashed vertical lines have been included to point out the relative positions of the kink energy distribution and entanglement entropy maxima.

There are only quantitative discrepancies with respect to the pattern just described in the previous $g = 10^4$ regimes. The curves in this $g = 10^5$ regime are smoother, and consequently, the differences between the $E_{S_{\text{kink}}}$ and $E_{S_{\text{vacuum}}}$ entropies are less pronounced than those occurring for $g = 10^4$. The centers of the kink energy distributions, however, slightly differ from the maxima of $E_{S_{\text{kink}}}$ in both the cases when the kink is formed by one or two lumps. More precisely: (1) For $a = 0$ the entropy $E_{S_{\text{kink}}}$ shows a maximum at the midpoint of the chain where the kink center is localized. (2) If $a = 0.6$, the function $E_{S_{\text{kink}}}$ exhibits two maxima situated at $\ell = 207$ and $\ell = 291$, and despite that, the respective maxima do not exactly coincide the kink entanglement entropy and the energy distribution has similar shapes as functions of $\ell$. (3) For $a = 0.9$, the lengths of the sub-blocks
for which the entanglement entropy is maximum are $\ell = 157$
and $\ell = 342$. This means that two sub-blocks that remarkably differ
in length support maximum entropy. The lump centers
nearly coincide with the entanglement entropy maxima. The little
discrepancies between entropy and kink energy maxima are due to
discreteness effects: the kink energy distribution is more extended
in the chain than in the previous SubsReg regime, and the location of the energy distribution maximum
on a discrete chain is less accurate. In figure 7(b), one observes that the spatial correlation functions give a clue of the
entanglement entropy shapes also in this regime as one can check in figures 10(a)–(c).

5.3. C. Elastic force dominant regime

In figures 11(a)–(c), the graphics of the entanglement entro-
pies between kink–kink and vacuum–vacuum fluctuations are plotted in the $g = 10^6$ ElasReg Regime for the three usual
representative values of the parameter $a$.

Differences between kink–kink and vacuum–vacuum
fluctuation entanglement entropies become small in this
regime. In general, these differences get smaller when the
elastic constant $g$ increases and the kinks become less con-
centrated. Nevertheless, the shape of the kink entropy and
energy distribution remains correlated, despite the fact that
the center of the kink energy distribution differs from the
maxima of $E_{\text{kink}}^a$ for $a = 0.6$, and $a = 0.9$ as in the BallReg
regime. These discrepancies are not only due to discreteness
effects. Contour effects also affect them because the kink
profile differs from a vacuum solution practically along the
whole chain, and the boundaries have a stronger influence on the
kink energy distributions. In fact, notice that in this
regime, the kink loses its characteristic smooth staircase
shape, adopting an almost linear profile; see figure 6. We note
three points: (1) The function $E_{\text{kink}}^a$ exhibits a single max-
imum at the middle of the chain if $a = 0$ as in the other two
regimes previously discussed. (2) For $a = 0.6$, the kink fluctu-
tion entropy exhibits two low curvature maxima at the
values $\ell = 155$ and $\ell = 343$. (3) The two maxima in the
$a = 0.99$ case are placed at $\ell = 126$ and $\ell = 374$. In each case, the
maxima coincide with that of the spatial correlation function, figure 7(c), whereas the entanglement maxima only
nearly coincide with the centers of the lumps of the kink
energy distribution because of the reasons mentioned above.

6. Conclusions

We have studied entanglement in a chain of 501 particles
subjected to a substrate potential belonging to the one-par-
metric family of double sine-Gordon potentials. We
considered the entanglement arising when a portion of the chain of
length $\ell$ is taken from the left of the chain, instead of from the
center [14]. This different approach allowed us to see in a
clearer way that, when focusing on kink solutions carrying an
energy distribution either formed by one or two lumps, the
maximum entanglement exhibits a strong correspondence
with both the spatial correlation function and the centers of
the lumps in the kink energy distribution. Regarding the
spatial correlation function, its maxima occur at the same
value of the length of the sub-block that makes the entan-
glement maxima, whereas the maximum entanglement nearly
coincide with the maximum of the lumps in the kink energy
distribution. We conclude that whenever the spatial correla-
tion function between kink fluctuations presents maxima, also
the kink entanglement entropy and the kink energy distribu-
tions exhibit maxima. In figure 7, together with figures 9–11,
the pattern previously mentioned is observed. The entangle-
ment grows when the kink energy distribution evaluated at the
sub-block endpoint increases. In general, the maximum
for the entanglement is reached when the sub-block endpoint
coincides with the kink lump centers. This pattern is clearly
observed if the kink is confined in a small region; see
figures 9 and 10, where the small discrepancies between kink
entanglement entropy and energy distribution maxima is
understood as being due to discreteness effects. In figure 11, a
greater discrepancy between the maxima of the entanglement
and the kink energy distribution is displayed. In this case, the
entanglement entropy maximum shift with respect to the
maximum of the energy distribution is mainly associated with
boundary effects. Recall that in this regime the kink occupy
essentially all the chain such that the boundaries influence the
kink solution. Indeed, the profile loses its kinky shape; see
figure 6, in favor of an almost equidistant particle
configuration.

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