THE SOLVABILITY OF GROUPS
WITH NILPOTENT MINIMAL COVERINGS

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In memory of László Kovács

Abstract. A covering of a group is a finite set of proper subgroups whose union is the whole group. A covering is minimal if there is no covering of smaller cardinality, and it is nilpotent if all its members are nilpotent subgroups. We complete a proof that every group that has a nilpotent minimal covering is solvable, starting from the previously known result that a minimal counterexample is an almost simple finite group.

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1. Introduction

A covering (or cover) for a group $G$ is a finite collection of proper subgroups whose union is all of $G$. A minimal covering for $G$ is a covering which has minimal cardinality among all the coverings of $G$. The size of a minimal covering of a group $G$ is denoted $\sigma(G)$ and is called the covering number of $G$. Since the first half of the last century a lot of attention has been given to determining which numbers can occur as covering numbers for groups, and, when possible, to characterize groups having the same value of $\sigma(G)$. The earlier works date back to G. Scorza ([24]) and D. Greco ([13], [14], [15]). The terminology “minimal covering” appears in the celebrated paper [26] of M. J. Tomkinson. Also worth mentioning are [16], [3] and [9]. More recent works determine bounds, and also exact values of $\sigma(G)$, for various classes of finite groups (see for instance [4], [21], [23], [17] and [18]).

Here we are interested in minimal coverings of groups by proper subgroups with restricted properties. For example, in [5, Theorem 2] R. Bryce and L. Serena show that a group that has a minimal covering consisting of abelian subgroups is solvable of very restricted structure. In [6] the same authors treat the case of groups that admit a minimal covering with all members nilpotent, that is, a nilpotent minimal covering. They state the following:

Conjecture. Only solvable groups can admit a nilpotent minimal covering.

Their main result ([6, Proposition 2.1]) is a reduction to the almost simple case, namely if there is an insolvable group with a nilpotent minimal covering then there is a finite almost simple such group. Bryce and Serena also show that several classes of finite almost simple groups (among them the alternating and symmetric groups, the projective special/general linear groups, the Suzuki groups, and the 26 sporadic groups) do not have nilpotent minimal coverings.
Our main result is the following.

**Theorem 1.** No finite almost simple group has a nilpotent minimal covering.

As an immediate corollary we complete the proof of the aforementioned conjecture.

**Theorem 2.** Every group that has a nilpotent minimal covering is solvable.

The structure of solvable groups with such a minimal covering is well understood and can be found in [5, Theorem 11].

A reasonable indication of the truth of Theorem 1 is suggested by the fact that in a finite non abelian simple group the order of the largest nilpotent subgroups is always much smaller than the order of the group (see [27]).

Our proof of Theorem 1 makes use of the classification of finite non abelian simple groups, and it can be outlined as follows. We start by taking a minimal order counterexample $G$, which is therefore an almost simple group, say $S \leq G \leq \text{Aut}(S)$, where $S$ is a non abelian simple group. If $S$ is a group of Lie type we reduce to the cases when $S$ has Lie rank one or twisted Lie rank one, or $S$ has Lie rank two and $G$ contains a graph, or a graph-field, automorphism of $S$. Then we reduce to $G/S$ cyclic and we use a technical lemma (Lemma 5) to eliminate the possibility that $G$ is itself not simple (Proposition 6). Finally we prove that no finite simple group can be a counterexample (Proposition 7).

Recently nilpotent coverings and their connections with maximal non-nilpotent subsets in finite simple groups of Lie type have been studied in [2].

The notation of this paper is standard and mostly follows the book [12]. We remark that for the classical groups we have preferred to use the ‘classical’ notation rather than Artin’s single letter notation. Therefore we use $\text{PSL}(n,q)$ instead of $A_{n-1}(q)$ or $L_n(q)$, and similarly $\text{PSp}(2n,q)$ for $B_{2n}(q)$ and $\text{PSU}(n,q)$ for $^2A_{n-1}(q)$ or $U_n(q)$. Note also that whenever we write $\text{PSU}(n,q)$ we mean that this group is defined over the field of order $q^2 = p^f$ ($p$ a prime). Differently from [12], we denote the Suzuki and the Ree groups over the fields $F_{2^f}$ and $F_{3^f}$ ($f$ odd), by $^2B_2(2^f)$, $^2G_2(3^f)$, instead of $^2B_2(2^f)$ and $^2G_2(3^f)$.

2. Proofs of Theorems 1 and 2

We start with a simple but important observation. Assume that $A = \{A_1, \ldots, A_\sigma\}$ is a minimal covering of a group $G$, that is,

$$G = A_1 \cup A_2 \cup \ldots \cup A_\sigma$$

and $G$ is not the setwise union of fewer than $\sigma$ proper subgroups. Then for every $1 \leq i < j \leq \sigma$, $(A_i, A_j) = G$, since otherwise we could replace the subgroups $A_i$ and $A_j$ in $A$ with $(A_i, A_j)$, obtaining a covering of $G$ with fewer than $\sigma$ members. We will use this simple fact often.

The proof of Theorem 1 depends on understanding the structure of the finite simple groups of Lie type and the corresponding simple linear algebraic groups. Lemma 2 is a key step in our proof. We first recall some important facts regarding algebraic groups.
A regular unipotent element of an algebraic group $G$ is an element $g$ of $G$ such that $\dim(G_{\geq}(g)) = \text{rk}(G)$. The following result can be found in [8], Propositions 5.1.2 and 5.1.3.

**Proposition 1.** Let $G$ be a connected reductive group. Then $G$ admits regular unipotent elements and these elements form a unique conjugacy class in $G$. Moreover, if $u$ is a unipotent element of $G$, then the following conditions on $u$ are equivalent:

(a) $u$ is regular,

(b) $u$ lies in a unique Borel subgroup of $G$, and

(c) $u$ is conjugate to an element of the form $\prod_{\alpha \in \Phi^+} x_\alpha(\lambda_\alpha)$ with $\lambda_\alpha \neq 0$ for all $\alpha \in \Delta$ (where $\Phi^+$ and $\Delta$ denote, respectively, a positive system of roots and its fundamental system).

In particular, if $G$ is a simple linear algebraic group over an algebraically closed field in characteristic $p$ and $F$ is a Frobenius endomorphism of $G$, then the finite group of Lie type $G = G^F$ contains $p$-elements (that we still call regular unipotent) which have the property that each lies in a unique Sylow $p$-subgroup of $G$ (see [8, Proposition 5.1.7]).

**Lemma 2.** Let $G$ be an almost simple group whose socle $S$ is a group of Lie type in characteristic $p$. Suppose that $\mathcal{A} = \{A_i\}_{i=1}^\sigma$ is a nilpotent minimal covering of $G$. Then the following hold:

(a) $\sigma$ is greater than the number $n_p(S)$ of Sylow $p$-subgroups of $S$, and

(b) if $U$ is a Sylow $p$-subgroup of $S$, then $N_G(U)$ is a maximal subgroup of $G$.

**Proof.** Let $u$ be a regular unipotent element of $S$ and let $U$ be the unique Sylow $p$-subgroup of $S$ containing $u$. Assume that $u \in A_i$. Since $A_i$ is nilpotent, $O_p'(A_i) \leq C_G(u)$; in particular $O_p'(A_i) \leq C_G(u)$. We now prove that $O_p'(A_i) \leq N_G(U)$. Let $y \in O_p'(A_i)$. Then $y$ normalizes $O_p'(A_i) \cap S$, and since $u \in O_p'(A_i) \cap S = (O_p'(A_i) \cap S)^y \subseteq U^y$ and $U$ is the unique Sylow $p$-subgroup of $S$ containing $u$, we have $U^y = U$, as we wanted. It follows that $A_i \leq N_G(U)$. As $\langle A_i, A_j \rangle = G$ for $i \neq j$, two different members of $\mathcal{A}$ cannot normalize the same Sylow $p$-subgroup of $S$. This shows that $\sigma \geq |\text{Syl}_p(S)|$. Moreover, since a finite group is never the union of conjugates of a unique proper subgroup ([11, Theorem 1]), in fact $\sigma > |\text{Syl}_p(S)|$.

Assume now that $N_G(U) < K \leq G$. Then for every $k \in K \setminus N_G(U)$, the element $u^k$ is still regular unipotent in $S$ and lies in $U^k \neq U$. If $u \in A_i$ and $u^k \in A_j$, we have that for $i \neq j$, the subgroup $A_j$ is contained in $N_G(U)^k$ and

$$G = \langle A_i, A_j \rangle \leq \langle N_G(U), N_G(U)^k \rangle \leq K,$$

which is a contradiction. Thus $N_G(U)$ is maximal in $G$. \hfill \Box

We next determine in which of these groups $G$ the normalizer of a Sylow $p$-group of $S$ is a maximal subgroup of $G$.

**Proposition 3.** Let $G$ be a finite almost simple group whose socle $S$ is a group of Lie type in characteristic $p$. Let $U$ be a Sylow $p$-subgroup of $S$. Then $N_G(U)$ is maximal in $G$ if and only if one of the following holds:

(a) $S \in \{\text{PSL}(2, q), \text{PSU}(3, q), 2B_2(q), 2G_2(q)\}$, or
(b) $S \in \{ \text{PSL}(3,q), \text{PSp}(4,2^f), G_2(3^f) \}$ and $G$ contains a graph or graph-field automorphism of $S$.

Proof. Assume first that $G = S$ is simple. Then $B = N_S(U)$ is a Borel subgroup of $S$ and, by general BN-pair theory ([7, Proposition 8.2.1 and Theorem 13.5.4]), the lattice of overgroups of $B$ in $S$ consists of $B$, the parabolic subgroups of $S$, and $S$. In particular, $N_S(U)$ is maximal in $S$ if and only if it is the unique parabolic subgroup of $S$, which is the case exactly when $S$ is of Lie rank one or, respectively, of twisted Lie rank one. Only the finite simple groups listed in (a) have this property.

Assume now that $G > S$ and let $S^*$ be the extension of $S$ by the diagonal and field automorphisms of $S$. The group $S^*$ has a BN-pair whose Borel subgroup is $B^* = N_{S^*}(U)$, since to construct $S^*$ from $S$ we can choose diagonal and field automorphisms that normalize every root subgroup of $U$. Of course, the BN-pair restricts to $G \cap S^*$. Therefore, if $G \leq S^*$, we have immediately that $N_G(U)$ is maximal in $G$ precisely when $G$ is an extension of some simple group that appears in (a).

Suppose then that $G \not\leq S^*$, that is, that $G$ contains a graph or graph-field automorphism of $S$. Note that this happens exactly when $S$ is one of the following (see [7] or [10]):

$$\text{PSL}(n,q), n \geq 3, \text{PSp}(4,2^f), D_n(q), G_2(3^f), F_4(2^f), E_6(q).$$

Moreover, non-trivial graph automorphisms, modulo the field automorphisms, always have order 2 or 3 (order 3 occurs only in the case $S = D_4(q)$), and such automorphisms interchange the fundamental root subgroups. By looking at the action of such graph automorphisms on the Dynkin diagrams, only when the Lie rank of $S$ is two can it be the case that $N_G(U)$ is maximal. This condition excludes all the possible groups except when $S$ is one of the following: $\text{PSL}(3,q)$, $\text{PSp}(4,2^f)$ or $G_2(3^f)$. Finally we claim that in these groups $N_G(U)$ is indeed a maximal subgroup of $G$. By our earlier argument, the group $G^* = G \cap S^*$ has a BN-pair with Borel subgroup $B^* = N_{G^*}(U)$, whose overgroups are $B^*$, $P_1^*$, $P_2^*$ and $G^*$, where $P_1^*$ and $P_2^*$ are the meets of $G$ with the extensions, by diagonal and field automorphisms, of the two parabolic subgroups of $S$ that contain $B$. Now $|G : G^*| = 2$ and any element of $N_G(U) \setminus N_{G^*}(U)$ interchanges $P_1^*$ and $P_2^*$, since it interchanges the two fundamental root subgroups. Suppose that $M$ is a maximal subgroup of $G$ containing $N_G(U)$. Then $M \leq S^*$ contains $B^* = N_{G^*}(U)$, and so $M \cap S^* \in \{ B^*, P_1^*, P_2^*, G^* \}$. By the Frattini argument, $G = SN_G(U)$. Thus there is an element $g$ in $N_G(U) \setminus N_{G^*}(U)$, such that, by the above, $g$ interchanges $P_1^*$ and $P_2^*$. But $g$ normalizes $M \cap S^* =$; thus $M \cap S^* = B^*$ or $S^*$. If $M \cap S^* = S^*$, then $M = S^*N_G(U) = G$, a contradiction. Hence $M \cap S^* = B^*$ and so $M = N_G(U)$. We conclude that $N_G(U)$ is maximal in $G$ in all these cases.

We make a further reduction that applies to any minimal counterexample to Theorem 1.

Lemma 4. If $G$ is a minimal counterexample to Theorem 1, where $S \leq G \leq \text{Aut} (S)$ and $S$ is a finite non abelian simple group, then $G/S$ is a cyclic group.

Proof. Assume by contradiction that $G$ has a nilpotent minimal covering with $\sigma = \sigma(G)$ subgroups and that $G/S$ is not cyclic. Note that this assumption automatically excludes the cases when $S$ is an alternating group $A_n$ with $n \neq 6$ or...
a sporadic group, since in those cases $|G/S| = 2$. Thus $S$ is a simple group of Lie type, and, by Lemma 2 and Proposition 3, the pair $(G,S)$ is one that appears in the statement of Proposition 3. We may also assume $S$ is not one of $2B_2(2^f), 2G_2(3^f), PSp(4, 2^f)$, or $G_2(q)$, since for these groups Out $(S)$ is cyclic of order $f$ or $2f$. Trivially, we may cover $G/S$ using all its non-trivial cyclic subgroups, so in particular $σ(G/S) < |G/S|$. Since $σ ≤ σ(G/S)$, we deduce that $σ < |\text{Out}(S)|$. By Lemma 2, then, we have that $n_p(S) < |\text{Out}(S)|$. Thus, as before, $n_p(G)$ denotes the number of Sylow $p$-subgroups of $S$, that is, the index of a Borel subgroup of $S$ in $S)$. But for the remaining possible groups listed in Proposition 3 we have

(a) $n_p(PSL(2, q)) = q + 1$ and $|\text{Out}(PSL(2, q))| = df$, where $q = p^f$ and $d = (2, q - 1)$,

(b) $n_p(PSL(3, q)) = (q + 1)(q^2 + q + 1)$ and $|\text{Out}(PSL(3, q))| = 2df$, where $q = p^f$ and $d = (3, q - 1)$, and

(c) $n_p(PSU(3, q)) = q^3 + 1$ and $|\text{Out}(PSU(3, q))| = df$, where $q^2 = p^f$ and $d = (3, q + 1)$,

and it is straightforward to show in each case that $n_p(S) > |\text{Out}(S)|$. \□

The following technical lemma is the key ingredient to reduce to the case that a minimal counterexample to Theorem 1 is necessarily a finite simple group.

**Lemma 5.** Let $G$ be an almost simple group with socle $S$ such that $G/S$ is a cyclic group. Assume also that if $S$ is of Lie type, then the pair $(G, S)$ appears in the statement of Proposition 3. Then there exist some element $s ∈ S$ and some maximal subgroup $K$ of $G$ containing $S$ such that $\text{g.c.d.}(|s|, |G/K|) = 1$ and $G ≠ KC_G(s)$.

**Proof.** Let $S = A_n$ be an alternating group, with $n ≥ 5$. If $n ≠ 6$, or $n = 6$ and $G = S_6$, take $s$ to be an $n$-cycle if $n$ is odd, or an $(n - 1)$-cycle if $n$ is even, and take $K = S$. In both cases $|s|$ is odd and $C_{S_n}(s) ≤ A_n$. If $n = 6$ and $G$ is a cyclic extension of $A_6$ distinct from $S_6$, we may always take $s$ to be a 3-cycle (see [10]).

If $S$ is a sporadic group, then Out $(S)$ is always cyclic of order at most two. The following table lists possible choices for the order of $s$, depending on the pair $(G, S)$ when $G ≠ S$ (our reference is [10]). Then $s$ can be chosen to be any element of the given order.

| $S$ | $M_{12}$ | $M_{23}$ | $J_2$ | $HS$ | $J_3$ | $McL$ | $He$ | $Suz$ | $O'N$ | $Fi_{22}$ | $HN$ | $Fi_{24}$ |
|-----|----------|----------|-------|------|-------|-------|------|-------|-------|----------|------|-------|
| $|s|$ | 11 | 11 | 5 | 11 | 19 | 7 | 17 | 31 | 13 | 19 | 29 |

We assume now that $G$ is a cyclic extension of a finite simple group $S$ of Lie type in characteristic $p$ and that the pair $(G, S)$ satisfies the conclusions of Proposition 3. Let $δ$ be a diagonal automorphism of $S$ of maximal order $d$, modulo $S$, and set $\hat{S} = S(δ)$. Let $φ$ be a field automorphism of $\hat{S}$ of order $f$, where $q = p^f$ except when $S$ is unitary, when $q^2 = p^f$. Set $S^* = \hat{S}(φ)$. Then

$$S ≤ \hat{S} ≤ S^* ≤ \text{Aut}(S),$$

where the indices are respectively $d$, $f$ and $g$, where $g ∈ \{1, 2\}$, for the groups under consideration. We treat separately the following three cases: a) $G ≤ \hat{S}$, b) $G ≤ S^* \setminus \hat{S}$ and c) $G ≤ S^*$.

a) Assume $G ≤ \hat{S}$.

According to Proposition 3, $S ∈ \{PSL(2, q), PSU(3, q)\}$ and $G = \hat{S}$ with the index
of $S$ in $G$ being respectively 2 or 3. In particular, $p$ is coprime with $|G/S|$. Let $s$ be a regular unipotent element of $S$. Since by [29, Lemma 3.1] (respectively by [25, Table 2]) we have that $C_G(s) < S$, taking $K = S$ we have that $G \neq KC_G(s)$, as we wanted.

b) Assume $G \leq S^* \setminus \hat{S}$.

According to Proposition 3, $S$ is one of the following groups:

$$PSL(2, q), PSU(3, q), 2B_2(q), 2G_2(q).$$

Note that in the last two cases $q$ is respectively $2^f$ or $3^f$, with $f$ odd and $f \geq 3$ (since $2B_2(2)$ and $2G_2(3)$ are not simple groups). Moreover, as $G \not\leq \hat{S}$, we always have $f > 1$ in this case.

Let $\mathbb{F}_p$ be the algebraic closure of the field $\mathbb{F}_p$ of order $p$. We first claim that for any of the aforementioned simple groups $S$ there is a least integer $m$ whose values are displayed in Table A and an embedding

$$\iota: \hat{S} \rightarrow PGL(m, \mathbb{F}_p),$$

such that $\varphi$ is the restriction to $S$ of the standard Frobenius automorphism of $PGL(m, \mathbb{F}_p)$, which later we will still call $\varphi$. This claim is trivial when $S = PSL(2, q)$ or $S = PSU(3, q)$, respectively, when $m = 2$ or 3 and $\iota$ is the natural inclusion. For the case $S = 2B_2(2^f)$, note that $S$ is the centralizer in $S_0 = PSp(4, 2^f)$ of a graph involution $x$ ([20, Proposition 2.4.4]) and that

$$\text{Aut} (S_0) = \text{Inn} (S_0) : (\langle \varphi \rangle \times \langle x \rangle) \simeq S_0 : (C_f \times C_2).$$

Since the existence of an embedding $\iota$ with the aforementioned property, of $S_0$ into $PGL(4, \mathbb{F}_2)$ is guaranteed, the same is true for $2B_2(2^f)$. Similarly, $S = 2G_2(3^f)$ is the centralizer in $S_0 = P\Omega^+(8, 3^f)$ of the full group of graph automorphisms of $S_0$ (see [19]). As

$$\text{Aut} (S_0) = \text{Inn} (S_0) : (\langle \varphi \rangle \times \langle x, y \rangle) \simeq S_0 : (C_f \times C_3),$$

for suitable graph automorphisms $x$ and $y$, and such an embedding $\iota$ exists for $P\Omega^+(8, 3^f)$ into $PGL(8, \mathbb{F}_3)$, the same is true for $2G_2(3^f)$ and our claim is proved.

Now we assume that there exists a primitive prime divisor of $p^f z - 1$, with $z$ as in Table A, and let $r$ be such a prime divisor. Note that $r$ divides the order of $S$. Also, trivially, $r \neq d$, and, if $r$ divides $f$, then, writing $f = rf'$, we have $0 \equiv p^f z - 1 \equiv p^{f'} z - 1$ (mod $r$), which contradicts the fact that $r$ is a primitive prime divisor of $p^f z - 1$. Therefore $r$ is coprime with $|G/S|$. Let $t_1$ be an element of $S$ of order $r$. Note that $t_1$ is a power of a generator of a cyclic maximal torus $T$ of $S$, whose order is displayed in Table A. Suppose that $C_G(t_1)$ contains an element of the form $g \delta^h \varphi^k$, with $g \in S$, and $0 \leq h \leq d - 1$, $0 < k \leq f - 1$. Then

$$t_1^{\delta^h} = t_1^{\varphi^k}.$$

This, of course, implies that

$$(\iota(t_1^{\delta^h}))^L = (\iota(t_1)^{\varphi^k})^L,$$

where $L = PGL(m, \mathbb{F}_p)$ and $(g)^L$ denotes the $L$-conjugacy class of $g \in L$. Now $\iota(t_1)$ is $L$-conjugate to the projection $\tilde{\alpha}$ of a diagonal $m \times m$ matrix $\alpha$, and $\varphi$ sends $\tilde{\alpha}$ to its $p$-th power $\tilde{\alpha}^p$. As $\iota(g \delta^h) \in L$, it follows that

$$(\iota(t_1^{\delta^h}))^L = (\iota(t_1))L = (\tilde{\alpha})^L = (\tilde{\alpha}^{p^{-k}})^L.$$
We want to prove that if $0 < k < f$ the two $L$-classes $(\bar{a})^L$ and $(\bar{a}^p)^L$ are different. Note that as $t_1$ has order $r$ and the matrix $\alpha \in SL(m, \mathbb{F}_p)$ is determined modulo the scalars, we can choose $\alpha$ in such a way that its eigenvalues are either 1 or have order $r$ in the multiplicative group of $\mathbb{F}_p$. Moreover, $\alpha$ has at least one eigenvalue $\mu$ of order $r$. Note that $\mu$ belongs to the field $\mathbb{F}_{p^f}$, but to no smaller field. Since $\nu(t_1) \in PSL(m, p^f)$, the characteristic polynomial $\chi$ of $\alpha$ has coefficients in $\mathbb{F}_{p^f}$, so $\mu, \mu^p, \ldots, \mu^{p^{f-1}}$ are all distinct roots of $\chi$. If $S \neq 2G_2(q)$ then $\chi$ has degree $m = z$ and the eigenvalues of $\alpha$ are precisely $\mu, \mu^p, \ldots, \mu^{p^{f-1}}$. If $S = 2G_2(q)$, then $m = 6$, $z = 8$, and $\chi$ factors as $\chi = \chi_1\chi_2$, where $\chi_1$ is the minimum polynomial of $\mu$ and has degree 6, and $\chi_2$ has degree 2. Now the roots of $\chi_2$, which are eigenvalues of $\alpha$, cannot have order $r$, because $r \mid q^2 - 1$, so they must be 1, and the eigenvalues of $\alpha$ are: $\mu, \mu^p, \ldots, \mu^{p^2}$, 1, 1. The non-zero entries of the matrix $\alpha^{p^{-1}}$ are the $p^k$-th powers of the eigenvalues of $\alpha$ and it is straightforward to see that if $0 < k < f$ it cannot happen that $\alpha = \lambda^0\alpha^{p^{-k}}$ for some $\lambda \in \mathbb{F}_p$. This proves that $C_G(t_1) \leq G \cap \hat{S}$. Thus, by taking $s = t_1$ and $K$ any maximal subgroup containing $\hat{S} \cap G$, we have that $G \not= KC_G(s)$, as we wanted.

We consider now the cases in which no primitive prime divisor of $p^{f^2} - 1$ exists. Then by Zsigmondy’s Theorem (see [30]), either $(p, zf) = (2, 6)$ or $p$ is a Mersenne prime and $zf = 2$. The last condition cannot happen, since in this case both $z$ and $f$ are greater than one. The first condition reduces to considering the cases when $S$ is either $PSL(2, 8)$ or $PSU(3, 2)$. But $PSU(3, 2)$ is not simple, while if $S = PSL(2, 8)$, then $G = \text{Aut}(S) = S^*$, and we can take $s$ to be an element of $S$ of order 7 and $K = S$ (see [10]).

| $S$     | $|T|$, $T$ a max. torus of $S$ | $d$   | $|\text{Out}(S)|$ | $z$ | $m$ |
|---------|-----------------|------|------------------|----|-----|
| $PSL(2, q)$ | $(q + 1)/d$ | $(q + 1)/d$ | $(q - 1, 2)$ | df | 2   | 2   |
| $PSU(3, q)$ | $(q^2 - q + 1)/d$ | $(q - 1, 2)$ | $(q + 1, 3)$ | df | 3   | 3   |
| $2B_2(q)$    | $\{q + \sqrt{2q} + 1$ or $q - \sqrt{2q} + 1$ | 1 | $f$ | 4   | 4   |
| $2G_2(q)$    | $\{q + \sqrt{3q} + 1$ or $q - \sqrt{3q} + 1$ | 1 | $f$ | 6   | 8   |

c) Assume $G \not\leq S^*$. Then, according to Proposition 3, $S$ is one of the following groups:

$$PSL(3, q), PSp(4, 2^f), G_2(3^f),$$

with $f$ an integer greater than 1, and $|G : G^*| = 2$, where $G^* = G \cap S^*$. We choose $s$ to be a generator of a cyclic maximal torus $T$ of $S$, whose order is respectively $(q^2 + q + 1)/d$, $q^2 + 1$, or $q^2 - q + 1$, according to whether $S$ is $PSL(3, q)$, $PSp(4, 2^f)$ or $G_2(3^f)$, and $K = G^*$. Note that $|s| = |T|$ is odd, and thus coprime with $|G/K|$. We first claim that $|C_{G^*}(T)|$ is odd. If not, let $y$ be an involution in $C_{G^*}(T)$. Since $|C_{G^*}(T)| = d|T|$ is odd, $y \not\in \hat{S}$. By Proposition 4.9.1 in [12], we have that $f$ is even and $y$ is $\hat{S}$-conjugate to a field automorphism of order two. In particular, $C_S(y)$ is isomorphic respectively to $PSL(3, p^{f/2})$, $PSp(4, 2^{f/2})$ or $G_2(3^{f/2})$. In each of these cases, by order reasons, $C_S(y)$ cannot contain $T$. Thus $|C_G(T)|$ is odd, and if we argue by contradiction assuming $G = G^*C_G(s)$, there exists some involution $x$ in $C_G(s) \setminus G^*$. Again by Proposition 4.9.1 in [12], we have that $C_G(x)$ is isomorphic
respectively to $\text{PSU}(3,q)$, $2B_2(q)$ or $^2G_2(q)$. But none of these groups contains a cyclic maximal torus $T$ of $S$, a contradiction. \hfill \Box

The following proposition eliminates the possibility that a minimal counterexample to Theorem 1 can be an almost simple group but not simple.

**Proposition 6.** Let $G$ be an almost simple group which is not simple. Then $G$ does not admit a nilpotent minimal covering.

**Proof.** Suppose that $G$ has a nilpotent minimal covering

$$G = A_1 \cup \ldots \cup A_n,$$

with $\sigma = \sigma(G)$, and all $A_i$ nilpotent. Suppose further that $S < G < \text{Aut}(S)$, where $S$ is a finite non-abelian simple group. If $S$ is of Lie type, then by Lemma 2 the pair $(G, S)$ is one that appears in Proposition 3. Also by Lemma 4, we can assume that $G/S$ is cyclic. According to Lemma 5, we may choose an element $s$ in $S$ and a maximal subgroup $K$ of $G$ containing $S$ such that $|s|$ is coprime with the prime $r = |G/K|$ and $G \neq KCG(s)$. Note that $r$ is prime since $G/S$ is cyclic. Let $s \in A_i$ for some $i \in \{1, \ldots, \sigma\}$. We claim that $A_i$ lies in $K$. For if not let $\alpha \in A_i \setminus K$, so that $G = K(\alpha)$ by the maximality of $K$. If $|\alpha| = r^av$, with $r$ not dividing $v$, we also have that $G = K(\alpha^v)$, and, moreover, that $\alpha^v$ is an element of $A_i$ of order $r^a$, which is coprime with $|s|$. Thus, since $A_i$ is nilpotent, $\alpha^v \in CG(s)$, forcing $G$ to be equal to $KCG(s)$, contradicting the choices of $s$ and $K$. Thus $A_i \leq K$. We may choose some $g \in S$ such that $s^g \notin A_i$. Such a $g \in S$ exists, for otherwise by the simplicity of $S$ we would have that $\langle s^g | g \in S \rangle = S \leq A_i$, contradicting the nilpotence of $A_i$. Suppose that $s^g \in A_j$. Then, arguing as before, $A_j \leq K$, and so we conclude that $G = \langle A_i, A_j \rangle \leq K$, a contradiction. \hfill \Box

We are now in a position to complete the proof of Theorem 1.

**Proposition 7.** No finite simple group $S$ admits a nilpotent minimal covering.

**Proof.** In [6] the cases $S$ alternating and sporadic are completely settled. We can assume therefore that $S$ is a finite simple group of Lie type in characteristic $p$. By Lemma 2 and Proposition 3, $S$ lies in one of the following families:

$$\text{PSL}(2,q), \text{PSL}(3,q), 2B_2(q), ^2G_2(q).$$

The two families of projective special linear groups $\text{PSL}(2,q)$ and the Suzuki groups $^2B_2(q)$ have also been settled in [6] (respectively Lemma 4.2 and Theorem 4.3). We need only to analyze the two remaining families.

Let therefore $S = \text{PSU}(3,q)$, with $q > 2$, or $S = ^2G_2(q)$, with $q = 3^f$, where $f$ is odd and $f \geq 3$. Note that $|\text{PSU}(3,q)| = \frac{1}{2}q^3(q^2-1)(q^3+1)$, where $d = (3, q+1)$, and $|^2G_2(q)| = q^3(q-1)(q^3+1)$. Assume first that there is an odd prime $r$ dividing $q-1$ (observe that this is always the case when $S = ^2G_2(q)$). Since $q \equiv 1 \pmod{r}$, we have that $r$ is coprime with $|S|/(q-1)$. Thus if $R$ is a Sylow $r$-subgroup of $S$, $R$ lies in a Levi complement $H$ of a Borel subgroup $B = UH$. In particular, $R$ is cyclic, $R = \langle x \rangle$, and $H = CS(R)$, by [25, Table 2] for $\text{PSU}(3,q)$, and [28], or [22, Lemma 2.2] for $^2G_2(q)$. If $x \in A_i$, then, since $A_i$ is nilpotent and the Sylow $r$-subgroups are cyclic, $A_i \leq CS(x) = H$. Now, let $u \in U$ be a regular unipotent element of $S$. Then $CS(u)$ is a $p$-subgroup (again by [25, Table 2] and [28]). In particular if $u \in A_j$, then $A_j \leq U$. But then we get a contradiction, since $\langle A_i, A_j \rangle \leq B$. 
It remains to consider the case when $S = PSU(3,q)$ and $q - 1$ is a power of 2. Note that this happens if and only if $q = 9$ or $q$ is a Fermat prime, say $q = 2^m + 1$. Let $\mathcal{A} = \{A_i\}_{i=1}^\sigma$ be a nilpotent minimal covering of $S = PSU(3,q)$. The centralizer of any regular unipotent element $u$ of $S$ is a $p$-subgroup ([25, Table 2]), and therefore there exists a unique maximal nilpotent subgroup of $S$ containing $u$, and this subgroup is a Sylow $p$-subgroup of $S$. We may therefore assume that all the Sylow $p$-subgroups of $S$ appear as members of the nilpotent covering $\mathcal{A}$. Now let $U$ be a Sylow $p$-subgroup and let $H$ be a Levi complement of it in a Borel subgroup $N_S(U) = UH$. In particular, $H$ is cyclic of order $(q^2 - 1)/d$. Let $h$ be a generating element of $H$ and assume that $h \in A_i$ for some $i \in \{1, \ldots, \sigma\}$. If $A_i = H$ then we can replace the subgroups $U$ and $A_i$ of $\mathcal{A}$ with the subgroup $N_G(U)$, obtaining a covering of $S$ with fewer than $\sigma$ members, which contradicts the minimality of $\sigma$. Therefore $A_i$ must be a nilpotent subgroup of $S$ that strictly contains $H$. Now the Sylow 2-subgroup of $H$ is cyclic (of order 16 if $q = 9$ and $2^{m+1}$ if $q = 2^m + 1$) and so it is normal of index 2 in a Sylow 2-subgroup of $S$. In particular, $w$ is a central element of $A_i$, that is, $A_i \leq C_S(w)$. By Proposition 4 (iii) in [1, Chapter II, Section 2], $C_S(w)$ is a central extension of a cyclic group of order $\frac{q^2 - 1}{d}$ by a group isomorphic to $PGL(2,q)$. Note that $H^{q-1}$ is the central subgroup of $C_S(w)$ of order $(q + 1)/d$. Now the only nilpotent subgroups of $C_S(w)$ that strictly contain a cyclic subgroup of order $(q^2 - 1)/d$ are central extensions of $C_{\frac{q^2 - 1}{d}}$ by a Sylow 2-subgroup, say $P$, of $PGL(2,q)$ (and so also of $S$). Therefore $A_i$ is a group isomorphic to $\frac{C_{\frac{q^2 - 1}{d}} \times P}$, and it contains $H$ as a subgroup of index two. Since $\hat{H}$ is not normal in $C_S(w)$ we can find an element $g \in C_S(w) \setminus N_S(H)$ and consider the element $h^g$. Assume that $h^g \not\in A_j$. Arguing as before, we have that either $A_j = \langle h^g \rangle = H^g$, or $A_j$ is a subgroup of $C_S(w)^g = C_S(w)$ isomorphic to $A_i$. In the first case we obtain a contradiction by replacing the subgroups $U^g$ and $A_j$ in $\mathcal{A}$ with $(UH)^g$. In the latter case we have that $A_j \neq A_i$, since $H \neq H^g$ and a group isomorphic to $A_i$ has a unique cyclic maximal subgroup of index two. But then $G = \langle A_i, A_j \rangle \leq C_S(w)$, a contradiction. 

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