Coarse Network Coding: A Simple Relay Strategy to Resolve Interference

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Abstract—Reminiscent of the parity function in network coding for the butterfly network, it is shown that forwarding an even/odd indicator bit for a scalar quantization of a relay observation recovers 1 bit of information at the two destinations in a noiseless interference channel where interference is treated as noise. Based on this observation, a coding strategy is proposed to improve the rate of both users at the same time using a relay node in a noisy interference channel. In this strategy, the relay observes a linear combination of signals sent by the two sources, and broadcasts a common message to the two destinations over a shared out-of-band link of constant rate \( R_0 \) bits per channel use. The relay message consists of the bin index of a structured binning scheme obtained from a \( 2^{R_0} \)-way partition of the squared lattice in the complex plane. We show that such scalar quantization-binning relay strategy asymptotically achieves the cut-set bound in an interference channel with a common out-of-band relay link of limited rate, improving the sum rate by two bits for every bit relayed, asymptotically at high signal to noise ratios (SNR) and when interference is treated as noise. We then use low-density parity-check (LDPC) codes along with bit-interleaved coded-modulation (BICM) as a practical coding scheme for the proposed strategy. We consider matched and mismatched scenarios, depending on whether the input alphabet of the interference signal is known or unknown to the decoder, respectively. For the matched scenario, we show the proposed strategy results in significant gains in SNR. For the mismatched scenario, we show that the proposed strategy results in rate improvements that, without the relay, cannot be achieved by merely increasing transmit powers. Finally, we use generalized mutual information analysis to characterize the theoretical performance of the mismatched scenario and validate our simulation results.

I. INTRODUCTION

Wireless communication is changing gears. The most promising way to meet higher and higher demands is to move towards a much higher spatial reuse, with very small cells where the distance between transmitters and receivers is dramatically reduced. On the other hand, this approach calls for a more and more unregulated user-based deployment of wireless networks, in contrast to the carefully and centrally planned conventional macro-cellular layout. In this context, femtocells (i.e., small home-based base stations) is rapidly emerging. Soon, people could be operating their own base stations (BS) in home or office, selectively controlling user access to their privately-operated BS.

Yet, interference remains as a major obstacle to widely deploy ad-hoc user-managed cellular systems, requiring further attention and new solutions. In a dense area, like a high-rise residential building, inter-cell interference is inevitable among user-controlled BSs. Because of sizable number of interfering cells and small number of users in each cell, traditional frequency reuse strategies are ineffective and spectrally inefficient. Hence, new strategies to handle interference are needed.

This paper introduces a simple such strategy based on relays. Relays are traditionally used as middle terminals to retransmit the message. In an interference scenario, however, we envision an alternative application of relays as “helpers” to resolve interference. Fig. 1 shows an out-of-band relay placed between two neighboring interfering cells. The relay has a noiseless link of a limited rate to the two destinations, and can broadcast a common signal to the two destinations. The relay could be connected over an orthogonal wireless connection to the two users in the case of downlink (e.g., WiFi), or even wired to the two BSs for the uplink communication (e.g., Ethernet).

We focus on relay strategies that involve no decoding of the user message. This is because the relay also experiences interference, rendering the decoding difficult. In our strategy, the relay operation involves a symbol-level quantization of the relay observation using a planar lattice. The relay sends a bin index for its quantized observation, obtained through a suitable partition of the square lattice \([1]\). See Fig. 4 and Fig. 5 for examples. The relay message is incorporated in the user
decoder to enhance the channel (initial) log-likelihood ratios (LLR) values. In other words, only initial LLR computations in the receiver decoder has to be modified to integrate the relay message in decoder.

We simulate the performance of the proposed strategy using bit-interleaved coded modulation (BICM) and low-density parity-check codes (LDPC), assuming perfect channel state information at the decoder. In practice, channel parameters can be estimated using pilot subchannels with relatively small overhead, specially in femtocell scenarios with slow channel variations. We study matched and mismatched decoding metrics, where the decoder is aware of the interference signal alphabet (constellation), or treats interference as Gaussian, respectively. In the matched scenario, where the decoder searches over the product of user constellations to compute LLRs, our results indicate that an SNR gain of approximately 1 dB can be achieved for every bit relayed. The mismatched scenario is more interesting from a practical perspective, since it leads to a significantly lower decoding complexity, and our results indicate that a significant gain is obtained using the relay. See Fig. [12]

The proposed relay strategy can be viewed as an analog version of digital network coding for wireless channels [2], [3]. We explain this connection to network coding and the relation of the proposed scheme with respect to the compress-and-forward (CF) relaying scheme [4]–[6] in Section II. We show that for an interference channel where interference is treated as noise, a scalar quantization strategy is asymptotically optimal. That is, we show that asymptotically in the low noise regime, every bit relayed improves the achievable sum rate by two bits, using a scalar quantize-and-forward strategy. This two-for-one improvement in sum rate is analogous to the rate improvement due to parity forwarding in the celebrated butterfly network example in network coding [7].

The interference channel with an out-of-band relay has been the subject of a number of recent studies, where the fundamental information theoretic aspects of the interference relay channel are investigated [8]–[11]. However, practical coding schemes to utilize relays in interference scenarios are not as widely studied. Moreover, not many practical compress-and-forward relay coding schemes are available, since the decode-and-forward scheme is often favored upon the compress-and-forward relay strategy for multihop relaying. In this work, we make further progress in bringing some of the theoretical achievements on the application of relays in interference scenarios to practice by devising a simple, yet effective, coding strategy based on the compress-and-forward scheme, tailored specifically for the interference channel.

The rest of this paper is organized as follows: Section III describes some of the connections between the proposed strategy, digital network coding, and compress-and-forward relay strategy. Section IV illustrates the system model and encoding and decoding procedures, and Section VI provides some simulation results. Finally, a few remarks conclude the paper in Section VII.

![Fig. 2](image-url) A two-user interference channel with a common out-of-band relay link of rate $R_0$.

![Fig. 3](image-url) Coarse parity: the relay forwards a parity bit for a quantized value of $Y_r$.

## II. BACKGROUND

### A. Network Coding

A simple illustration of coarse network coding can be given using a toy example. Consider a noiseless version of the interference channel shown in Fig. 2 with $N_1 = N_2 = N_r = 0$. In this model, the source signal $X_1$ is corrupted with interference $X_2$ and is received as $h_{11}X_1 + h_{21}X_2$ at the destination, with $X_2$ treated as noise. Assume that all channel gains are available at destination nodes and $X_1$ and $X_2$ are real independent Gaussian random variables. Consider now a relay that observes $Y_r = g_1X_1 + g_2X_2$ and wishes to assist the destination by forwarding 1 bit of information over a shared noiseless link. A simple relay strategy to improve the achievable rate at the destination by close to one bit is to send a 0 if $|Y_r/d|$ is even, or a 1 otherwise, for a very small value of $d > 0$; see Fig. 3. To see this, let $X_r = |Y_r/d| \mod 2$ denote the relay message, and then the rate improvement is given by:

$$I(X_1; Y_1, X_r) = I(X_1; Y_1) + I(X_1; X_r | Y_1)$$

$$= I(X_1; Y_1) + I(X_1; X_r | h_{11}X_1 + h_{21}X_2)$$

$$= I(X_1; Y_1) + H(X_r | Y_1) - H(X_r | X_1, Y_1)$$

$$(a) = I(X_1; Y_1) + H(X_r | h_{11}X_1 + h_{21}X_2) - 0$$

$$(b) = I(X_1; Y_1) + 1$$

In the above derivation, (a) follows since $Y_r$ is known given $X_1$ and $Y_1$, and $H(X_r | X_1, Y_1) = 0$, and (b) follows since for very small values of $d$, $|Y_r/d| \mod 2$ is a Bernoulli 1/2 random variable given $Y_1$, provided that the relay observation is not statistically identical with the receiver observations, i.e.,

$$|Y_r/d| \mod 2.$$
the matrix

\[
\begin{bmatrix}
  h_{11} & g_1 \\
  h_{21} & g_2
\end{bmatrix}
\]

is full rank. (See Appendix.)

Now, switch the roles of \(X_1\) and \(X_2\), and consider the second destination who is interested in decoding \(Y_2\) is full rank. (See Appendix.) This is an example of coarse network coding, where an even/odd indicator for a quantized version of relay observation recovers two bits of information, and is reminiscent of the celebrated butterfly network example in network coding [7]. A simple generalization of the above coarse parity function can be used to allow for higher relay data rates, and more importantly, to account for background noise.

Although focusing on different problems, there are interesting connections between the proposed coarse network coding and the analog network coding strategy devised in [12]. Analog network coding proposed in [12] is a symbol-wise nonlinear amplify-and-forward, where a scalar relay function is optimized (assuming differentiability) such that the end-to-end mutual information between the source and destination is maximized. Interestingly, the optimized nonlinear relay functions have a semi-periodic form, resembling a smooth version of the scalar quantization and binning of coarse network coding (see Section III-A). The analog network coding strategy, however, is devised only to serve a single destination since the relay operation depends on the destination channel, and the system optimization searches only for differentiable relay functions, which may not be necessarily optimal.

The problem considered in this paper differs from the wireless network coding approach of [2], where it is assumed that the relay observes clean versions of the source data. Coarse network coding also differs from the compute-and-forward scheme of [13] in that structured codes at the sources are not required in coarse network coding as no decoding, even of a function of the two source messages, is performed at the relay. Further, coarse network coding is a symbol-wise strategy in the same spirit of the XOR strategy of digital network coding.

B. Compress-and-Forward Relaying

Coarse network coding is essentially a scalar compress-and-forward strategy ([4, Theorem 6]) where quantization and binning are performed at the symbol level. Though a scalar quantization and binning are not generally the strategies of choice for a single-relay channel, this paper shows that there are substantial gains to be obtained from a simple scalar compress-and-forward strategy in an interference channel.

The asymptotic incremental optimality of scalar compress-and-forward shown in the previous section (i.e., one bit rate improvement per one bit relayed as noise tends to zero) is a consequence of cross-determinism in a noiseless interference channel. A cross deterministic relay channel was first introduced in [14], and consists of a three-terminal network of a source \(X\), a relay \(Y_r\), and a destination \(Y\), with a deterministic relationship between the relay observation \(Y_r\), and the sent and received signals \(X\) and \(Y\), in way that \(Y_r\) is a deterministic function of \(X\) and \(Y\). The noiseless interference channel in Section III is an instance of two coupled cross-deterministic relay channels: here, from \(X_1\) and \(Y_1\) or from \(X_2\) and \(Y_2\), we can compute the relay observation \(Y_r\); for both users, the relay channel is cross-deterministic.

To see the role of cross-determinism in the compress-and-forward strategy consider the achievable rate of compress-and-forward with list decoding for the interference channel with interference treated as noise [6]:

\[
\begin{align*}
R_1 &\leq I(X_1;Y_1) + \min\{R_0, I(Y_1;\hat{Y}_r|Y_1)\} - I(Y_r;\hat{Y}_r|X_1,Y_1) \\
R_2 &\leq I(X_2;Y_2) + \min\{R_0, I(Y_2;\hat{Y}_r|Y_2)\} - I(Y_r;\hat{Y}_r|X_2,Y_2),
\end{align*}
\]

where \(\hat{Y}_r\) is an auxiliary random variable that plays the role of quantization of the relay observation \(Y_r\). When \(Y_r\) is a deterministic function of \(X_1\) and \(Y_1\), and also \(X_2\) and \(Y_2\), the penalty terms in (1) vanish, since

\[
\begin{align*}
I(\hat{Y}_r;Y_r|X_1,Y_1) &= 0 \\
I(Y_r;\hat{Y}_r|X_2,Y_2) &= 0.
\end{align*}
\]

Consequently, the rate of each user is improved by \(R_0\) bits, as long as the minimums in (1) occur at \(R_0\) for the quantization scheme used. In other words, performance is not very sensitive with respect to the quantization scheme and in particular, the block length over which quantization is performed.

Unfortunately, optimal vector quantization with side information and list decoding as developed in [5] or [6] is not quite amenable to practical code construction. The list decoding scheme of [6], or the more recent noisy network coding approach of [5], search over all possible quantized relay observations \(\hat{Y}_r\) and transmitted source codewords over a large block. In other words, an exponential number of codewords have to be examined in order to decode the source message.

However, this paper shows that much of the promised gain can be achieved at high SNRs by simply using scalar quantization and binning. This is because the requirements on \(\hat{Y}_r\) to achieve \(R_0\) bits of improvement are quite loose at high SNRs, in the sense that \(\hat{Y}_r\) is admissible as long as it is sufficiently close to \(Y_r\) (measured in terms of \(I(Y_r;\hat{Y}_r;Y_1)\) and \(I(Y_r;\hat{Y}_r;Y_2)\)). Driven by the above intuition, we can replace the high-dimensional vector quantizer of the general CF scheme with a simple scalar quantizer. Most remarkably, when this strategy is combined with conventional (single-user) powerful coded-modulation based on binary LDPC with BICM and iterative decoding, and the iterative decoder inputs are enhanced by incorporating the relayed bits, it is shown that a significant gain is achieved even at practical (non-asymptotically high) SNR.
III. SYSTEM DESIGN

Fig. 2 shows the baseband-equivalent model of a two-user interference channel with a relay defined as:

\[
Y_1 = h_{11}X_1 + h_{21}X_2 + N_1 \\
Y_2 = h_{22}X_2 + h_{12}X_1 + N_2,
\]

(2a)

(2b)

where \(X_1\) and \(X_2\) are source signals of maximum power \(P_1\) and \(P_2\), and \(Y_1\), \(Y_2\), and \(Y_r\) denote the channel observation at the two destinations and the relay, respectively. The relay assists both destinations simultaneously by broadcasting \(X_r\) over a common digital link of rate \(R_0\). Here, \(N_1\), \(N_2\), and \(N_r\) are circularly-symmetric complex Gaussian noise of zero mean and variance \(\frac{N_0}{2}\) per real dimension, and \(h_{ij}, g_i, i, j = 1, 2\) represent channel gains. We assume that the channel undergoes fast fading with independent channel gains across different time slots, and \(h_{ij}, g_i, i, j = 1, 2\) are modeled as independent circularly-symmetric complex Gaussian random variables of zero mean and variance 0.5 per real dimension. Also, relevant channel state information (CSI) is known to each user, specifically, \(g_1\) and \(g_2\) are known to the destinations and the relay, and \(h_{22}, h_{12}\) are known to destination 1, and \(h_{22}, h_{12}\) are known to destination 2. In practice, the i.i.d. fast fading model with CSI known to all can be approached by using interleaving and pilot-assisted channel estimation, for a time- and frequency-selective fading channel with OFDM\(^1\), where sufficiently long codewords are interleaved in the time-frequency domain, as currently done in today’s OFDM systems.

A. Encoding:

The source message is encoded using a conventional LDPC code, then the binary codeword is interleaved and mapped to a sequence of M-ary constellation symbols using Gray labeling. The sequence of constellation symbols is sent over the channel using quadrature amplitude modulation (QAM). In (2), \(X_1\) and \(X_2\) represent the constellation symbol sent by the two users in each time slot, and \(X_r\) represent the relay message of rate \(R_0\), i.e., \(H(X_r) = R_0\).

In each slot, \(X_r\) is computed according to a scalar quantization scheme that we refer to as chessboard quantization, since the relay quantization Voronoi pattern resembles a chessboard. Chessboard quantization can be described as cost partitioning of the square lattice into \(2^{R_0}\) sublattices \([1]\). Denote the the square integer lattice in the complex plane by \(\mathbb{Z}^2\), and let \(\Lambda\) denote a nested sublattice of \(\mathbb{Z}^2\) with generator matrix \(G\) and nesting ratio of \(2^{R_0}\), i.e.,

\[
\Lambda = \{G \cdot i : i \in \mathbb{Z}^2\},
\]

(3)

where \(G\) is a matrix of integers with

\[
det(G) = 2^{R_0}.
\]

(4)

\(^1\)Orthogonal Frequency-Division Multiplexing.

We use \(\Lambda\) to partition \(\mathbb{Z}^2\). Chessboard quantization can then be formally defined as:

\[
X_r = B \lfloor Q(Y_r/d) \rfloor,
\]

(5)

where \(d\) is a complex scaling factor, \(Q(\cdot)\) represents the nearest neighbor quantizer with respect to \(\mathbb{Z}^2\), and \(B[\cdot i ]\) denotes the coset leader for \(i \in \mathbb{Z}^2\) with respect to the coarse lattice \(\Lambda\).

For given \(R_0\), \(g_1, g_2, N_0\), and input alphabets, we need to optimize the quantization, i.e, \(d\), and \(\Lambda\). For practical feasibility, we restricted the quantization lattice to be \(\mathbb{Z}^2\) and we have optimized \(\Lambda\) independently of the channel realization, i.e., \(\Lambda\) is chosen depending on the statistics of the channel coefficients and kept fixed for all channel uses. A more complex alternative would be to choose \(\Lambda\) among all sublattices with nesting ratio \(2^{R_0}\) based on the value of the channel coefficients \(g_1, g_2\). However, this more complex solution was not pursued here. In summary, only the scaling coefficient \(d\) is optimized on a symbol-by-symbol basis, based on the realization of \(g_1, g_2\).

In principle, the best partition for given source constellations and channel gains may be very complicated if we remove the lattice structure. Fixing the sublattice partition pattern first significantly reduces the optimization complexity.

To find a suitable lattice partition, we can optimize \(\Lambda\) such
that the normalized second moment of $\Lambda$ is maximized $\Pi$. For special cases $R_0 = 1, 2$, we choose $\Lambda$ as the lattice generated by
\[ G = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \] (6)
for $R_0 = 1$, and
\[ G = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \] (7)
for $R_0 = 2$. See Fig. 4 and Fig. 5.

The parameter $d$ is found such that $H(X_r|X_1, X_2)$ for given $g_1, g_2, N_0$ and input alphabets is minimized. An explanation for choosing $H(X_r|X_1, X_2)$ as the optimization metric is given later based on log-likelihood ratios. As an alternative motivation, minimizing $H(X_r|X_1, X_2)$ is related to maximizing the achievable rates. For a given channel realization, the rate improvement due to the relay for user one and user two is given by:
\[ \Delta R_1 = I(X_1; X_r|Y_1) = H(X_r|Y_1) - H(X_r|X_1, Y_1), \]
\[ \Delta R_2 = I(X_2; X_r|Y_2) = H(X_r|Y_2) - H(X_r|X_2, Y_2). \]

Since the relay sends a common signal to both destinations and direct channel gains $h_{ij}, i, j = 1, 2$ are not known at the relay, $\Delta R_1$ and $\Delta R_2$ cannot be jointly maximized at the relay for each time slot. An alternative would be to minimize $H(X_r|X_1, X_2)$ for given $g_1$ and $g_2$ (which are available at the relay), since for a given channel realization:
\[
\begin{align*}
H(X_r|X_1, Y_1) &> H(X_r|X_1, X_2, Y_1) = H(X_r|X_1, X_2), \\
H(X_r|X_2, Y_2) &> H(X_r|X_1, X_2, Y_2) = H(X_r|X_1, X_2)
\end{align*}
\]
where (a) follows from conditional entropy inequality, and (b) follows since for given channel gains, $X_r - (X_1, X_2) - Y_1$ forms a Markov chain. Thus, $H(X_r|X_1, X_2)$ is a lower bound on both $H(X_r|X_1, Y_1)$ and $H(X_r|X_2, Y_2)$, which becomes tight at high SNR where background noise is negligible. Consequently, minimizing $H(X_r|X_1, X_2)$ asymptotically results in maximized $\Delta R_1$ and $\Delta R_2$.

IV. Decoding

This section describes the decoding scheme for user one. Decoding at user two follows similar steps. Given $Y_1$ and $X_r$, destination one computes LLRs for corresponding bit positions of the underlying LDPC code, and then conventional sum-product algorithm is used to iteratively decode the source binary codeword. In other words, $X_r$ only affects the initial computation of LLRs at the destination.

We consider two forms of decoding: A matched scheme where the actual statistics for the interference signal is used for decoding, and a mismatched scheme where the distribution of the interference signal is approximated to be Gaussian with the same mean and variance. For QAM modulation, the matched decoder searches over the Cartesian product of the source and interference constellations to compute LLRs. Such an exhaustive search over the product constellation generally achieves a better performance at the cost of higher complexity.

On the other hand, the mismatched decoder searches only over the source constellation and approximates the LLRs by treating the interference signal (of discrete-alphabet) to be Gaussian.

In practical settings, the interference signal constellation is often unknown to the decoder, since the control channel of the interfering user usually needs to be decoded to obtain the encoding parameters like the constellation size. In such scenarios, the mismatch decoding scheme is an attractive strategy with a lower decoding complexity and robust performance.

A. Matched Decoder

Let $b_1 \cdots b_k$ denote the binary Gray label for constellation symbol $s \in \mathcal{M}$, where $\mathcal{M}$ is a constellation of size $M = 2^k$. Define also the reverse mapping $b_i = A_i(s), i = 1, \ldots, k$, where $b_i$ denotes the $i$’th bit of the binary label of $s$.

Given $y_1$ and $x_r$, the destination computes $\lambda_i$, the LLR corresponding to the $i$’th bit position, for $i = 1, \ldots, k$ as follows:
\[
\lambda_i = \log \sum_{x_1 \in \mathcal{M}_1, x_2 \in \mathcal{M}_2} \frac{p(y_1, x_r|x_1, x_2)}{\sum_{x'_1 \in \mathcal{M}_1, x'_2 \in \mathcal{M}_2} p(y_1, x_r|x'_1, x'_2)},
\]
where $\mathcal{M}_1, \mathcal{M}_2$ are the constellations for user one and two. Now, to compute $p(y_1, x_r|x_1, x_2)$, we have:
\[
p(y_1, x_r|x_1, x_2) = p(y_1|x_1, x_2)p(x_r|x_1, x_2),
\]
where
\[
p(y_1|x_1, x_2) = \frac{1}{\pi N_0} \exp \left\{ -\frac{\|y_1 - h_{11}x_1 - h_{21}x_2\|^2}{N_0} \right\},
\]
and
\[
p(x_r|x_1, x_2) = \frac{1}{\pi N_0} \int_{B(x_r)} \exp \left\{ -\frac{\|y_r - g_1x_1 - g_2x_2\|^2}{N_0} \right\} dy_r
\]
where $B(x_r)$ encompasses all quantization regions with $x_r = x_r$; for example, $B(0)$ represents all gray-colored squares corresponding to $x_r = 0$ in Fig. 4.

The LLR functions in (8) and (9) also suggest that minimizing $H(X_r|X_1, X_2)$ amounts to better initial LLRs with larger magnitude. This can be illustrated by (9), where the effect of the relay signal $X_r$ on the destination LLRs is controlled by $p(x_r|x_1, x_2)$. Hence, the more deterministic $p(x_r|x_1, x_2)$ is, the larger the magnitude of initial LLRs are, which results in a lower decoding error rate.

B. Mismatched Decoder

Computing the LLR values in (8) requires summing over all pairs of symbols in $\mathcal{M}_1$ and $\mathcal{M}_2$, which can be computationally intensive depending on the size of $\mathcal{M}_1$ and $\mathcal{M}_2$. This section describes a low-complexity mismatched decoding scheme that does not require the knowledge of $\mathcal{M}_2$.

The mismatched decoder treats the interference signal $X_2$ as a Gaussian random variable with the same mean and variance. Yet, the mismatched scheme becomes matched when $X_2$ is indeed Gaussian; for example, asynchronous cross sub-channel interference in an OFDM system due to misaligned timing can be well approximated to be Gaussian.
Decoding is again performed by computing the initial mismatched log-likelihood ratios (mLLR). Given \( y_1 \) and \( x_r \), user one computes \( \gamma_i \), the mLLR corresponding to the \( i \)’th bit position, for \( i = 1, \ldots, k \) as follows:

\[
\gamma_i = \log \frac{\sum_{x_1 \in \mathcal{M}_1} q(y_1, x_r | x_1) \mathcal{A}(x_1=1)}{\sum_{x_1 \in \mathcal{M}_1} q(y_1, x_r | x_1) \mathcal{A}(x_1=0)},
\]

where \( \mathcal{M}_1 \) is again the constellations for user one. The metric \( q(y_1, x_r | x_1) \) is given by:

\[
q(y_1, x_r | x_1) = q_1(y_1 | x_1) q_r(x_r | x_1, y_1),
\]

where

\[
q_1(y_1 | x_1) = \frac{1}{\pi \|h_{21}\|^2 P_2 + N_0} \exp \left\{ - \frac{\|y_1 - h_{11} x_1\|^2}{\|h_{21}\|^2 P_2 + N_0} \right\},
\]

and

\[
q_r(x_r | x_1, y_1) = \int_{y_r \in \mathcal{B}(x_r)} \frac{1}{\pi \sigma} \exp \left\{ - \frac{\|y_r - \mu\|^2}{\sigma^2} \right\} dy_r,
\]

with

\[
\sigma^2 = N_0 + \frac{\|g_2\|^2 P_2 N_0}{\|h_{21}\|^2 P_2 + N_0} \mu = g_1 x_1 + \frac{g_2 h_{21} P_2}{\|h_{21}\|^2 P_2 + N_0} (y_1 - h_{11} x_1).
\]

Note that \( q(y_1, x_r | x_1) = p(y_1, x_r | x_1) \) when \( X_2 \) is Gaussian. The mLLRs computed for every bit position are used as initial LLRs fed to the receiver’s iterative decoder.

V. ACHIEVABLE RATES

Under the matched decoding scenario, the achievable rates of the above strategy can be computed using standard mutual information analysis. However, characterizing the capacity limits under the mismatched decoding scenario requires a different set of techniques. Based on the analysis in this section, numeric results for the achievable rates under matched and mismatched scenarios are presented in Section [VI].

A. Matched Metric

In the matched scenario, the decoder uses \( p(y_1, x_r | x_i) \) as the likelihood metric, \( i = 1, 2 \) for user one and two. For this decoding metric, the achievable rate is given by the mutual information between channel input \( Y_i \) and the channel outputs \( Y_1, Y_r \) for \( i = 1, 2 \). Thus, we have

\[
R_1 = I(X_1; Y_1, X_r) = \mathbb{E} \left[ \log \frac{p(X_1, Y_1, X_r)}{p(X_1)p(Y_1, X_r)} \right],
\]

\[
R_2 = I(X_2; Y_2, X_r) = \mathbb{E} \left[ \log \frac{p(X_2, Y_2, X_r)}{p(X_2)p(Y_2, X_r)} \right],
\]

where the expectation \( \mathbb{E} \) is also with respect to channel coefficients\(^2\). Using (13), \( R_1 \) can be computed for discrete-alphabet \( X_1 \) and \( X_2 \) inputs using

\[
p(y_1, x_1, x_r) = \sum_{x_2 \in \mathcal{M}_2} p(y_1, x_1, x_2)p(x_1)p(x_2),
\]

along with (9), where \( p(x_1) \) and \( p(x_2) \) are uniform over the source constellations. When \( X_2 \) is continuous, an equivalent expression can be found for \( p(y_1, x_1, x_r) \) by using integral in place of sum in (15). The rate \( R_2 \) can be computed similarly.

B. Mismatched Metric

The mismatched decoder uses \( q(y_1, x_r | x_1) \) as the likelihood metric. The supremum of achievable rates of a mismatched decoder is unknown in general. However, error exponent analysis with random coding can be used to establish an achievable rate for a mismatched decoder [15].

Theorem 1 (Generalized Mutual Information [15]): Consider a point-to-point channel defined by \( p(y|x) \) for \( X \in \mathcal{X} \) and \( Y \in \mathcal{Y} \). Using a mismatched decoder with mismatched likelihood metric \( q(x, y) \), all positive rates \( R \) are achievable for this channel with a vanishing error probability for large block lengths if

\[
R < I^{\text{gmi}}(X; Y) \triangleq \max_{s > 0} I_s^{\text{gmi}}(X; Y),
\]

where

\[
I_s^{\text{gmi}} \triangleq \mathbb{E} \left[ \log \frac{q(X, Y)^s}{\sum_{x' \in \mathcal{X}} p(x') q(x', Y)^s} \right].
\]

Using Theorem 1 computing lower bounds for the achievable capacity with the mismatched decoding metric in (11) is quite straightforward. For user one, we have

\[
R_1(s) = \mathbb{E} \left[ \log \frac{q(Y_1, X_r | X_1)^s}{\sum_{x_1 \in \mathcal{M}_1} q(Y_1, X_r | x_1)^s} \right],
\]

where \( M_1 \) is the size of \( M_1 \). The expectation \( \mathbb{E} \) is with respect to probability distribution \( p(y_1, x_1, x_r) \) given in (15), and also the channel coefficients. To compute \( R_1(s) \) numerically with channel coefficients, the average \( I_s^{\text{gmi}} \) is computed for different realizations of the channel coefficients, with \( s \) optimized for each realization. A similar expression can be found for user two.

VI. SIMULATION RESULTS

This section first investigates the theoretical achievable rate for the matched and mismatched scenarios. Then, the performance of iterative decoding for LDPC codes with BICM modulation is studied using simulations.

A. Achievable Rates

For the matched decoding scenario, Table IV lists the improvement in minimum SNR required to achieve a source rate of 1 bit/symbol for different constellation sizes as channel inputs. As it is shown, when both the desired signal and interference are taken from 4-QAM constellations, an SNR gain of 0.86 dB is obtained with 1 bit relayed. With 2 bits relayed, the SNR gain is more substantial: 2.75 dB improvement to achieve a data rate of 1 bit/symbol. The gain due to coarse
network coding improves at higher bit-per-symbol rates. The second and third row of Table I list the gain in minimum SNR required to achieve $R_1 = 1$, when interference $X_2$ is taken from a 16-QAM constellation. In this case, 1 bit of relaying results in about 2 dB gain in SNR, and the SNR gain is almost doubled in dB with 2 bits relayed.

Fig. 6 shows the achievable rate $R_1$ when both $X_1$ and $X_2$ are taken from a 4-QAM (QPSK) constellation of power $P$, and SNR is defined as $10 \log_{10}(P/N_0)$. With discrete input alphabets, the destination can uniquely identify the source symbol sent by $X_1$ at high SNRs, and thus asymptotically, the achievable rate $R_1$ tends to $\log_2(4) = 2$, as shown in Fig. 6.

Fig. 7 shows the achievable rate curves when interference comes from a 16-QAM constellation. In this figure, $X_1$ and $X_2$ are of the same power $P$, and SNR is $10 \log_{10}(P/N_0)$. Comparing the first and second row in Table I it is revealed that coarse network coding is more helpful for a QPSK source when interference $X_2$ comes from a 16-QAM constellation as opposed to 4-QAM. A possible conclusion is that the more damaging the interfering signal is, the more the relay can help.

Fig. 8 shows the achievable rate improvement obtained by coarse network coding when the source signals $X_1$ is taken from a 16-QAM and 64-QAM constellations and interference is a circularly-symmetric complex Gaussian random variable. Both $X_1$ and $X_2$ are of the same power $P$ and SNR is again $10 \log_{10}(P/N_0)$. In this case, coarse network coding almost linearly improves the achievable rate. Highest gains are achieved when $X_1$ is taken from a 64-QAM constellation, where asymptotically at high SNRs every relayed bit improves the achievable rate by slightly less than 1 bit.

From Fig. 6, Fig. 7 and Table I it is expected that coarse network coding improves the achievable rates by larger extents as the size of the source alphabets increase. Fig. 9 shows the achievable rates when both $X_1$ and $X_2$ are circularly-symmetric Gaussian random variables. It is shown in Fig. 9 that for Gaussian inputs, the achievable rate of both users is improved by close to 1 bits at $R_0 = 1$, and 1.5 bits at $R_0 = 2$, at high SNRs.

Fig. 10 shows the achievable rate obtained by the generalized mutual information analysis for the mismatched decoding

### Table I

| $\mathcal{M}_1$ | $\mathcal{M}_2$ | $R_0 = 1$ gain | $R_0 = 2$ gain |
|-----------------|-----------------|----------------|----------------|
| 4-QAM           | 4-QAM           | 0.86 dB        | 2.75 dB        |
| 4-QAM           | 16-QAM          | 2.03 dB        | 4.74 dB        |
| 16-QAM          | 16-QAM          | 2.18 dB        | 4.78 dB        |

Fig. 6. Achievable rate with matched decoding and QPSK $X_1$ and $X_2$.

Fig. 7. Achievable rate with matched decoding and 16-QAM $X_2$.

Fig. 8. Achievable rate with matched decoding for discrete-alphabet $X_1$ and Gaussian interference $X_2$.

Fig. 9. Achievable rate with mismatched decoding for discrete-alphabet $X_1$ and Gaussian interference $X_2$. The destination can uniquely identify the source symbol sent by $X_1$ at high SNRs, and thus asymptotically, the achievable rate $R_1$ tends to $\log_2(4) = 2$, as shown in Fig. 6. Fig. 9 also indicates that a substantial gain in minimum SNR required to achieve a certain rate is achieved. Highest SNR gains are obtained for larger values of $R_1$. At SNR=25 dB, the SNR gain is close to 4 dB for one bit relayed, and 8 dB for 2 bits relayed.

Fig. 10 shows the achievable rate improvement obtained by coarse network coding when the source signals $X_1$ is taken from a 16-QAM and 64-QAM constellations and interference is a circularly-symmetric complex Gaussian random variable.
metric and QPSK inputs. Although both the inputs $X_1$ and $X_2$ are discrete, the achievable rate with mismatched decoding is saturated even at high SNR values due to interference. Yet, adding the relay alleviates the rate saturation at high SNRs by improving the asymptotic achievable rate in proportion to the number of bits relayed. The improvement in achievable rates for QPSK inputs and mismatched decoding is less than a bit for every bit relayed. However, this rate improvement tends to one as the size of signal constellations increases. At the extreme case where both $X_1$ and $X_2$ are continuous Gaussian random variables, the mismatched decoding reduces to the matched strategy, and every bit relayed results in two bits improvement in sum rate for large SNR.

### B. Performance with LDPC-Encoded BICM

For the matched scenario, Fig. 11 shows the performance of BICM-LDPC coding along with coarse network coding and iterative decoding at source rate $R_{1} = 1$. The BICM system is comprised of an LDPC code of rate 0.5 and block length 20,000, and gray-labeled 4-QAM modulation for both $X_1$, and interference $X_2$. The underlying LDPC code is the Gallager regular (3,6) LDPC code [16], with a random graph construction. Decoding is performed as described in Section IV by enhancing the initial LLRs using the extra bit received from the relay.

An SNR gain of approximately 1 dB is achieved using the (3,6) LDPC code of rate $1/2$ along with coarse network coding at $R_0 = 1$. At $R_0 = 2$, the gain in SNR is approximately close to 2.5 dB. Improved performance could perhaps be achieved by optimizing LDPC codes for the specific enhanced LLR density distribution. Because of the decoding approach used, the achieved gain is controlled by the shape of enhanced LLR density distribution. The more the relay message skews the initial LLR density distribution, the higher the achieved SNR gain would be, which also requires tuning the LDPC code for the specific input LLR density distribution to fully exploit potential gains. In this paper, standard LDPC codes are used to simulate the performance, as our focus is more on the multiuser aspects, rather than the impact of LDPC graph itself. Also, since the relay message is only used for initial LLR computations, the iterative decoding procedure at each destination is equivalent to that of a single-user point-to-point channel, which can be optimized by standard tools such as density evolution, EXIT charts, and protograph-based structures [17]–[19].

Fig. 12 shows the performance of the (3,6) LDPC code of rate 0.5 with mismatched decoding and gray-labeled QPSK BICM signaling. Using the relay results in substantial improvements when the decoder is unaware of the interfering signal constellation. At rate $R_1 = 1$, receiver is unable to decode the source codeword with mismatched LLR values, even at high SNRs, without relay assistance. Yet, using a relay link of rate $R_0 = 1$ enhances the LLR values in a way that decoding is possible at SNR values around 10 dB,
which is consistent with GMI achievable rates in Fig. 11. A relay link of rate $R_0 = 2$ results in an additional 5 dB SNR gain as compared to $R_0 = 1$, again matching the result from GMI simulation in Fig. 10. The SNR gains are quite substantial, especially if we consider the very simple relay strategy, requiring no decoding and only scalar quantization.

C. An intuitive interpretation

It is helpful to intuitively understand how (8) and (11) enhance the LLR values at the decoder. We can gain some insights by looking at likelihood computation for a particular constellation symbol $s_1$ sent by user one. Assuming that $s_1$ is sent by user one, the decoder can subtract $s_1$ from its observation $y_1$ to find an estimate for $s_2$, the symbol sent by user two. Now, using the channel state information, decoder can find an estimate for the relay observation and the corresponding quantization index. If the decoders estimate of the relay message matches the message received from the relay, the likelihood of $s_1$ being sent is improved, otherwise, it is less likely that $s_1$ is sent by user one. Such small adjustments to the LLR values at symbol level can be combined over a larger block length using, for example, an LDPC code with iterative decoding. Small enhancements in initial LLR values translate to significant improvement in overall bit error rate (BER) performance, because of the waterfall characteristic of BER curves for LDPC codes.

VII. Conclusion

Interference is one of the main obstacles to deploy user-based uncoordinated wireless cellular networks. We propose a simple relay strategy to improve the achievable rates in a two-user interference channel, where interference is treated as noise. This relay strategy is called coarse network coding, since the relay operation involves a scalar quantization followed by a scalar binning, reminiscent of the parity generation in digital network coding. In addition, asymptotically at high SNRs, a single bit sent by the relay in coarse network coding improves the rate of both users by one bit, resembling the rate improvement obtained from a single parity bit at two different destinations in digital network coding. In this relay scheme, the relay quantizes its observation using a scalar quantization obtained by suitable partitioning of the square lattice. We consider decoding scenarios using matched and mismatched metrics, depending on whether where the alphabet of interfering signal is known or unknown to the decoder, respectively. Using LDPC-encoded BICM modulation, we show that the expected theoretical SNR gains due to coarse network coding are mostly achieved using a generic regular (3,6) LDPC code; higher SNR gains are expected by designing the LDPC graph for the specific density distribution of the relay-enhanced LLRs. This is left for future work.

The proposed relay strategy can also be used to allow receiver cooperation in an interference channel. This is illustrated in Fig. 13 where each receiver is connected via a digital link of limited rate to the other. Each receiver plays the role of a relay for the other user, quantizes its observation prior to decoding using the same chessboard-like quantization strategy, and forwards the index corresponding to its quantized observation to the other user. Decoding is performed again by computing enhanced LLR values and iterative decoding.

Note that the analysis for the shared relay scenario shows that the same relay message can be used by different receivers; thus, generating relay messages are independent of the end receiver and the relay messages could be consolidated in a server to be used by any relevant interfered receiver. As an application, Fig. 14 shows a possible uplink scenario in a residential cellular wireless communication network where the two base stations use their Internet connections to communicate with each other as limited-rate relay nodes via a server. In the uplink, the base station playing the role of a relay shares its channel state information and bin indices for its quantized observation with neighboring base stations using, for example, an interference resolution server. A base station experiencing interference queries the server for relayed information from neighboring interfering base stations. The receiver then incorporates the retrieved information to enhance the initial LLR calculations, which are then used by the iterative decoder.
APPENDIX

In this appendix, we prove the asymptotic incremental optimality of scalar quantization/binning for a Gaussian interference channel for $R_0 = 1$, and with Gaussian inputs, where interference is treated as noise. That is, we prove that one bit relayed improves the rate of each user by one bit asymptotically at high SNRs.

The incremental rate improvement for user one can be found as follows:

$$
I(X_1; Y_1, X_r) = I(X_1; Y_1) + I(X_1; X_r|Y_1)
$$

$$
= I(X_1; Y_1) + H(X_r|X_1, Y_1) - H(X_r|X_1, Y_1)
$$

$$
= I(X_1; Y_1) + H(X_r|h_{11}X_1 + h_{21}X_2 + N_1) - H(X_r|h_{11}X_1 + h_{21}X_2 + N_1)
$$

We prove that as $N_0 \to 0$,

$$
H(X_r|h_{11}X_1 + h_{21}X_2 + N_1) \to 1 \quad (19)
$$

and

$$
H(X_r|h_{11}X_1 + h_{21}X_2 + N_1, X_1) \to 0. \quad (20)
$$

Since $X_r = \lfloor Y_r/d \rfloor \mod 2$, we first need to find the conditional distributions of $Y_r$ given $Y_1$ and $(Y_1, X_1)$. We have:

$$
p(y_r|x_1, y_1) = \frac{1}{\pi \sigma} \exp \left\{-\frac{\|y_r - \mu_1\|^2}{\sigma_1^2} \right\} dy_r, \quad (21)
$$

where

$$
\sigma_1^2 = N_0 + \frac{\|g_2\|^2 P_2 N_0}{\|h_{21}\|^2 P_2 + N_0}
$$

$$
\mu_1 = g_1 x_1 + \frac{g_2 h_{21} P_2}{\|h_{21}\|^2 P_2 + N_0} y_1 - h_{11} x_1
$$

and

$$
p(y_r|y_1) = \frac{1}{\pi \sigma_2} \exp \left\{-\frac{\|y_r - \mu_2\|^2}{\sigma_2^2} \right\} dy_r, \quad (22)
$$

where

$$
\sigma_2^2 = N_0 + \frac{\|g_1 h_{21} - g_2 h_{11}\|^2 P_1 P_2 + N_0 (\|g_1\|^2 P_1 + \|g_2\|^2 P_2)}{\|h_{11}\|^2 P_1 + \|h_{21}\|^2 P_2 + N_0}
$$

$$
\mu_2 = g_1 h_{21} P_1 + g_2 h_{21} P_2
$$

$$
= \frac{g_2 h_{21} P_2}{\|h_{21}\|^2 P_2 + N_0} y_1.
$$

Let

$$
d = N_0^\alpha,
$$

with $0 < \alpha < 0.5$. Then, we have:

$$
\text{var}(Y_r/d|X_1, Y_1) = N_0^{1-2\alpha} + \frac{\|g_2\|^2 P_2}{\|h_{21}\|^2 P_2 + N_0} \cdot N_0^{-2\alpha} \to 0, \quad (23)
$$

as $N_0 \to 0$ for $\alpha < 1/2$. Thus, given $X_1$ and $Y_1$, $Y_r/d$ is asymptotically deterministic, and consequently, $[Y_r/d]$ and also $[Y_r/d] \mod 2$ are deterministic. This proves (20).

To prove (19), notice that:

$$
\text{var}(Y_r/d|X_1, Y_1) = \left(1 + \frac{\|g_1\|^2 P_1 + \|g_2\|^2 P_2}{\|h_{11}\|^2 P_1 + \|h_{21}\|^2 P_2 + N_0} \right) \cdot N_0^{-2\alpha} \to \infty, \quad (24)
$$

as $N_0 \to 0$ for $\alpha > 0$, provided that $g_1 h_{21} - g_2 h_{11} \neq 0$. Thus, given $Y_1$, the variance of $Y_r/d$ asymptotically is unbounded, and hence, $[Y_r/d] \mod 2$ tends to a Bernoulli 1/2 random variable. This proves (19). A similar analysis proves that the rate of the second user also improves by one bit.

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