Theoretic–model Properties of Regular Polygons

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Abstract
This work is dedicated to the results were got in the model theory of the regular polygons. We give the characterization of the monoids with axiomatizable and model complete class of regular polygons. We describe the monoids with complete class of regular polygons which satisfy the additional conditions. We study the monoids, whose regular core is presented as a union of the finite number of principal right ideals, all regular polygons over which have the stable and superstable theory. We prove the stability of the axiomatizable model complete class of regular polygons and we also describe the monoids with the superstable and \( \omega \)–stable class of regular polygons when this class is axiomatizable and model complete.

This work is dedicated to the results were got in the model theory of the regular polygons.

Under the left polygon \( sA \) over monoid \( S \) or simply polygon we understand a set \( A \) upon which \( S \) acts on left with the identity of \( S \) acting as the identity map on \( A \). As it can be seen from this definition we can look at the polygon over monoid as the generalization of the notion of the module over ring. So many notions and problems came to the model theory of the polygons from the model theory of the modules. In particular, the notion of the regular polygon did. In the module theory there are several different definitions of the regular polygon. In the polygon theory we are using the analogue of the regular of Zelmanowitz module [Zel], introducing by Tran [Tra].

One of the standard problems in the model theory of the polygons is a problem of monoids description, over which some class of polygons possesses property \( P \), where \( P \) can be the axiomatizability, completeness, model completeness and others. In the given work these questions are considered for regular polygon class.

We tried to make the interpretation as closed as possible, giving all necessary definitions and statements, which became classical already.

In the first two paragraphs we give information from the polygon theory which will be necessary in future.

In the third paragraph we state the information from the model theory.

In the fourth paragraph the characterization of the monoids with axiomatizable class of regular polygons is given (Theorem 4.1). As a consequence the axiomatizability of the class of regular polygons over group is received (Corollary 5.1).

In the fifth paragraph the monoids with axiomatizable model complete class of regular polygons are described (Theorem 5.1). In this paragraph the model completeness of the class of regular polygons over infinite group is proved (Corollary 5.2).

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In the sixth paragraph the questions of the completeness of the class of regular polygons is regarded. Here we give the characterization of monoids with complete class of regular polygons which satisfy the additional conditions, namely, the monoids over which the class of regular polygons satisfies the condition of formula definability of isomorphic orbits (Theorem 6.1) and the linearly ordered monoids of depth 2 (Theorem 6.2). Note that the question of the description of the monoids with complete class of regular polygons is still open.

In the seventh and eighth paragraphs we study the monoids, whose regular core is presented as a union of the finite number of principal right ideals, all regular polygons over which have the stable (Theorem 7.1) and superstable (Theorem 8.1) theory.

In the ninth paragraph we prove the stability of the axiomatizable model complete class of regular polygons and we also describe the monoids with the superstable and $\omega$–stable class of regular polygons when this class is axiomatizable and model complete (Theorem 9.1).

§ 1. Polygons

Throughout, by $S$ we denote a monoid, and by 1 unity in $S$. An algebraic system $\langle A; s \rangle_{s \in S}$ of the language $L_S = \{ s \mid s \in S \}$ is a (left) $S$–polygon (or polygon over $S$, or polygon) if $s_1(s_2a) = (s_1s_2)a$ and $1a = a$ for all $s_1, s_2 \in S$ and $a \in A$. A polygon $\langle A; s \rangle_{s \in S}$ is denoted by $S_A$. All the polygons treated in the article, unless specified otherwise, are left $S$–polygons. Similarly the notion of the right $S$-polygon is defined. We denote the class of all polygons by $S \mathcal{A}c_t$. The subsystem $sB$ of the polygon $sA$ is called the subpolygon of the polygon $sA$. The polygon $sA$ is called finite generated if there exists $a_1, \ldots, a_n \in A$ such that $sA = \bigcup_{i=1}^n Ssa_i$. The polygon $sA$ is called cyclic if $sA$ is one-generated polygon that is $sA = Ssa$ for some $a \in S$. The coproduct of the polygons $sA_i$ is a disjunctive union of this polygons. The coproduct of the polygons $sA_i$ we denote by $\biguplus_{i \in I} S A_i$.

The following proposition we will often use without references.

**Proposition 1.1.** For any $a, e, f \in S$, $e^2 = e$, $f^2 = f$ we have:
1) $aS \subseteq eS \iff ea = a$;
2) $Sa \subseteq Se \iff ae = a$. $\square$

**Proposition 1.2.** Let $e, f \in S$, $e^2 = e$, $f^2 = f$, $Se \subseteq Sf$, $fS \subseteq eS$. Then $e = f$.

**Proof.** Suppose $Se \subseteq Sf$, $fS \subseteq eS$, where $e, f \in S$ are the idempotents. Then from the first including by Proposition 1.1(2) it follows $e = ef$, from second one by Proposition 1.1(1) it follows $f = ef$. Consequently, $e = f$. $\square$

**Proposition 1.3.** Let $T$ be a semigroup, $e \in T$ is idempotent. Then the left ideal $Te$ is minimum by inclusion among the left ideals of the semigroup $T$, generated by the idempotens if and only if the right ideal $eT$ is minimum by inclusion among the right ideals of the semigroup $T$, generated by the idempotens.

**Proof. Necessity.** Suppose the left ideal $Te$ is minimum among the left ideals of the semigroup $T$, generated by the idempotens, $gT \subseteq eT$, $g^2 = g$. Then $g = eg$. In view of minimality of the ideal $Te$ the equality $Tge = Te$ is hold. Consequently, $e = kge$ for some $k \in T$ and $e = kge = kgge = kgege = ege = ge$. Thus $e \in gT$ and $gT = eT$. 


Sufficiency is proved similarly.

A semigroup $T$ is said to be a rectangular band of the groups if $T = \bigcup \{T_{ij} \mid i \in I(T), j \in J(T)\}$ is a decomposition of the semigroup $T$ on groups $T_{ij}$ and herewith $T_{ij} \cdot T_{kl} \subseteq T_{il}$.

Remark 1.1. If a semigroup $(T, \ast)$ is a rectangular band of groups $T_{ij}$ with units $e_{ij}$, $i \in I(T), j \in J(T), \ast$, then for any $i, k \in I(T), j, l \in J(T)$ the following conditions are true:

1) $e_{ij} \cdot e_{kj} = e_{ij}; e_{ij} \cdot e_{il} = e_{il};$
2) $e_{ij} \cdot T_{kl} = T_{ij} \cdot e_{kl} = T_{ij} \cdot T_{kl} = T_{il};$
3) $\langle T_{ij}, \ast \rangle \equiv \langle T_{kl}, \ast \rangle;$
4) $Te_{ij} = \bigcup_{p \in I(T)} T_{pi}; e_{ij}T = \bigcup_{p \in J(T)} T_{ip};$
5) for any $a \in T$ the condition $Ta = Te_{ij}$ ($aT = e_{ij}T$) is equivalent to $a \in T_{pj} (a \in T_{ip})$
6) for any $a \in T$ the set $Ta (aT)$ forms a minimal left (right) ideal.

The union of all minimal left ideal of a semigroup $T$ is called its kernel and will be denoted by $K(T)$.

Proposition 1.4 [Su]. If the kernel $K(S)$ contains an idempotent, then $K(S)$ is a rectangular band of groups.

The set of all idempotents of a semigroup $T$ will be denoted by $E(T)$.

Proposition 1.5. If $Te$ is a minimal left ideal of a semigroup $T$ and $e \in E(T)$, then the set $G_e \doteq \{a \mid ea = ae = a\}$ forms a subgroup of the semigroup $T$.

Proof. If $a, b \in G_e$, then $e(ab) = (ea)b = ab$ and $(ab)e = a(be) = ab$. Consequently, the set $G_e$ is closed under a semigroup operation. Obviously, the idempotent $e$ is an unit in $G_e$.

Let $a$ be an arbitrary element of $G_e$. By $ae = a$ and the minimality of left ideal $Se$ we get $Sa = Se$. Then there exists an element $b \in S$ such that $ba = e$. Since $ebe \in G_e$ and $(ebe)a = ebe(a) = eba = e$, then it is not difficult check that element $a \cdot ebe = g$ is an idempotent. Since $g \in Se$, then $eg = ge = g$, that is $g \in G_e$. In view of minimality $Se$ we have $Se = Se$, consequently, $eg = e$. Since $ea = a$ and $g = a \cdot ebe$, then $eg = g$. Thereby, $g = e, a^{-1} = b$ and $G_e$ forms a subgroup in the semigroup $S$.

§ 2. Regular Polygons

Let $S A$ be a polygon. We call $a \in A$ an act–regular element if there exists a homomorphism $\varphi : S A \longrightarrow S S$ such that $\varphi(a)a = a$, and $S A$ is called a regular polygon if every $a \in A$ is an act–regular element.

Proposition 2.1 [KKM]. Let $S A$ be a polygon. The following conditions for $a \in A$ are equivalent:

1) the element $a$ is an act–regular element;
2) there exists an idempotent $e \in S$ and an isomorphism $\psi : S S a \longrightarrow S S e$ such that $\psi(a) = e$;
3) there exists an idempotent $e \in S$ such that $S S a \cong S S e$, i.e. the polygon $S S a$ is projective.

Proof. 1) $\Rightarrow$ 2). Suppose $a \in A$ is act–regular element, $\varphi : S S a \longrightarrow S S$ is a homomorphism such that $\varphi(a)a = a$. Let $e = \varphi(a)$. Then $e = \varphi(a) = \varphi(\varphi(a)a) = \varphi(a)\varphi(a) = e^2$
and $ea = \varphi(a)a = a$. Furthermore an equality $sa = ta$ implies $se = s\varphi(a) = \varphi(sa) = \varphi(a)sa = sa$.
\( \varphi(ta) = t\varphi(a) = te \) for any \( s,t \in S \). Then the mapping \( \psi : sSa \rightarrow SSe \) such that \( \psi(sa) = s \psi(a) \) for any \( s \in S \), is an polygon isomorphism.

The implication 2)\( \Rightarrow \)3) is obviously.

3)\( \Rightarrow \)1). Let \( e \) be an idempotent of monoid \( S \), \( \psi : sSa \rightarrow SSe \) be an isomorphism, \( \psi(a) = ve, \psi(va) = e \). Then \( eva = va, e = \psi(va) = v\psi(a) = ve \), \( \psi(veva) = \psi(veva) = 2v \psi(a) = ve \), i.e. \( e = vue \). Suppose \( f = vue \in S \), \( \varphi : S \rightarrow S \) is the mapping such that \( \varphi(sa) = sf \) for any \( s \in S \). The equality \( f^2 = uev = vea = f \) implies that \( f \) is an idempotent. If \( sa = ta \) for some \( s,t \) then \( vueva = tue \) and \( sf = tf \), consequently, \( \varphi \) is the homomorphism. Since \( \varphi(a)a = vueva = a \), then \( a \) is an act–regular element. \( \square \)

**Corollary 2.1.** The following conditions for a polygon \( sA \) are equivalent:

1) \( sA \) is a regular polygon;

2) for any \( a \in A \) there exist an idempotent \( e \in S \) and an isomorphism \( \psi : sSa \rightarrow SSe \) such that \( \psi(a) = e \);

3) for any \( a \in A \) there exists an idempotent \( e \in S \) such that \( sSa \cong SSe \). \( \square \)

Recall that an element \( a \) of semigroup \( T \) is (von Neumann) regular if \( a = aba \) for some \( b \in T \). The semigroup is called (von Neumann) regular if all its elements are regular.

**Proposition 2.2** [KKM]. If \( S \) is a (von Neumann) regular monoid then \( sS \) is a regular polygon but the converse is not true.

**Proof.** Let \( S \) be a (von Neumann) regular monoid, \( a \in S \). Then \( a = aba \) for some \( b \in S \). Denote the element \( ba \) of monoid \( S \) by \( b \). Clear that \( e = s \) is an idempotent. Furthermore, \( Sa = Saba = Sae \subseteq Se \), \( Se = Sba \subseteq Sa \), that is \( Sa = Se \). Consequently, \( sS \) is a regular polygon.

On the other hand, let \( S \) be a right cancellative monoid which is not a group. Then for any \( a \in S \) a polygon \( sSa \) is isomorphic to a polygon \( sS \). Consequently, \( sS \) is a regular polygon but \( S \) is not a (von Neumann) regular monoid. \( \square \)

Let \( sA \) be a polygon which has a regular subpolygon. Note that the union of all regular subpolygons of the polygon \( sA \) is also a regular subpolygon. This subpolygon is called the regular core of the polygon \( sA \) and denote by \( R(sA) \). Instead \( R(sS) \) we will write \( sR \). A subsemigroup \( R \) of monoid \( S \) is called the the regular core of the monoid \( S \). Hereinafter, we assume that \( R \neq \emptyset \). Denote the class all regular polygons by \( \mathcal{R} \).

The elements \( x, y \) of the polygon \( sA \) are called connected (denoted by \( x \sim y \)) if there exist \( n \in \omega, a_0, \ldots, a_n \in A, s_1, \ldots, s_n \in S \) such that \( x = a_0, y = a_n \), and \( a_i = s_ia_{i-1} \) or \( a_{i-1} = s_ia_i \). The polygon \( sA \) is called connected if we have \( x \sim y \) for any \( x, y \in sA \). It is easy to check that \( \sim \) is a congruence relation on the polygon \( sA \). Let \( B \subseteq A \). The elements \( x, y \in A \setminus B \) are called connected out of \( B \), if there exist \( n \in \omega, a_0, \ldots, a_n \in A \setminus B, s_1, \ldots, s_n \in S \) such that \( x = a_0, y = a_n \), and \( a_i = s_ia_{i-1} \) or \( a_{i-1} = s_ia_i \)

Let \( sA \) be a polygon. We will denote by \( \text{Con}(sA) \) the lattice of congruences of the polygon \( sA \), by \( 1_{sA} \) and \( 0_{sA} \) unit and zero in the lattice \( \text{Con}(sA) \) accordingly. The congruence \( \theta \in \text{Con}(sA) \) is called an amalgam congruence if \( \theta \cap \sim = 0_{sA} \). Notice that amalgam congruences identify elements, not connected with each other.

For an arbitrary class of polygons \( K \) we will denote by

- \( \mathcal{H}_A(K) \) the class of all polygons, isomorphic to factor-polygons of polygons from \( K \) by amalgam congruences;
- \( \mathcal{S}(K) \) the class of all polygons, isomorphic to subpolygons from \( K \);
- \( \mathcal{D}(K) \) the class of all polygons, isomorphic to coproducts of polygons from \( K \).
Proposition 2.3. [Ov1] For any monoid $S$ the class of all regular $S$-polygons $sR$ coincides with the following classes: $H_1 DS(sR)$, $SH_1 D(sR)$, $H_1 SD(sR)$. □

A monoid $S$ is called regularly linearly ordered if for any $a \in R$ the set $\{Sb \mid Sb \subseteq Sa\}$ is linearly ordered by inclusion.

Proposition 2.4. 1) If $r \in R$, $e \in S$, $e^2 = e$ and $rS = eS$ then $e \in R$ and $rR = eR$.

2) If $e, f \in R$, $e^2 = e$ and $f^2 = f$ then the equality $eS = fS$ is equivalent to the equality $eR = fR$.

Proof. Suppose $r \in R$, $e \in S$, $e^2 = e$ and $rS = eS$. From the last equality we get $Sr \cong Se$. Since $r \in R$ then $e \in R$. Since $e \in rS$ then $e = rt$, where $t \in S$. Since $e \in R$ then $te \in R$. Consequently, $e = rt = rte \in rR$, i.e. $eR \subseteq rR$. In view of the inclusion $rS \subseteq eS$ we have $r = er \in eR$ that is $rR \subseteq eR$. Thus, $rR = eR$ and 1) is proved.

Suppose $e, f \in R$, $e^2 = e$ and $f^2 = f$. If $eR = fR$ then $e = ee \in eR = fR \subseteq fS$ and $eS \subseteq fS$; similarly, $fS \subseteq eS$ that is $eS = fS$. If $eS = fS$ then $e = fe \in fR$ and $eR \subseteq fR$; similarly, $fR \subseteq eR$ that is $eR = fR$ and 2) is proved. □

Proposition 2.5. If the monoid $S$ is regularly linear ordered, $sA \in R$ and $a \in A$ then the set $\{Sb \mid Sb \subseteq Sa\}$ is linear ordered by inclusion.

Proof. Suppose $sA \in R$, $a \in A$ and $b_1, b_2 \in Sa$. Since $sSa \in R$ then there exists the isomorphism $\varphi : sSa \rightarrow sSe$, where $e^2 = e \in R$. Hence in view of regular linear ordered of the monoid $S$ either $S\varphi(b_1) \subseteq S\varphi(b_2)$ or $S\varphi(b_2) \subseteq S\varphi(b_1)$. Consequently, either $Sb_1 \subseteq Sb_2$ or $Sb_1 \subseteq Sb_2$.

By the left depth (or simply depth) of the semigroup $T$ we call the greatest length of chain of principal left ideals of this semigroup if it exists and finite, and the symbol $\infty$ otherwise. We will denote the left depth of the semigroup $T$ by $ld(T)$.

Proposition 2.6. If the depth $ld(R)$ of core of monoid $S$ is finite, then the kernel $K(R)$ is a rectangular band of groups.

Proof. The finiteness of $ld(R)$ implies the existence of minimal left ideal $Sa$. In view of the regularity of the polygon $sR$ on Corollary 2.1 there exist an idempotent $e \in R$ and an isomorphism $\psi : sSa \rightarrow sSe$ such that $\psi(a) = e$. Then the ideal $Se$ is also minimal and the element $e$ belongs to $K(R)$. On Proposition 1.4 we get that $K(R)$ is a rectangular band of groups. □

§ 3. Information from Model Theory

The initial information from the model theory, used in this article, may be found in [EP], [ChK] and [Sac]. We will remind some of it.

Let us fix some complete theory $T$ of language $L$ and a rather large and saturate model $C$ of the theory $T$, which we call it as a monster-model, because we suppose that all considered models of the theory $T$ are its elementary submodels. All elements and sets will also be taken from the monster-model. All formulae, considered in this paragraph, will have a language $L$.

The finite sequences are called the corteges. A set of the corteges of a set $A$ are denoted by $A^{<\omega}$. A length of a cortege $\bar{a}$ is denoted by $l(\bar{a})$. The corteges of the length $n$ are called $n$-corteges. For the simplicity instead of a denotement $\bar{a} \in A^{<\omega}$ we will often use a denotement $\bar{a} \in A$. If $\Phi(\bar{x}, \bar{y})$ is a formula of a language $L$, $\bar{a}$ is a cortege of the elements and $l(\bar{a}) = l(\bar{y})$, then by $\Phi(C, \bar{a})$ we will denote a set $\{\bar{b} \mid C \models \Phi(\bar{b}, \bar{a})\}$.
Class $K$ of the structures is called axiomatizable if there exist a language $L$ and a set of the sentences $Z$ of the language $L$ such that for any structure $\mathcal{A}$

$$\mathcal{A} \in K \iff (\text{the language of } \mathcal{A} \text{ is } L \text{ and } \mathcal{A} \models \Phi \text{ for all } \Phi \in Z).$$

If the condition (3.1) holds for the class $K$, then $L$ is called the language of $K$, and the set $Z$ is called the set of the axioms for $K$ (we denote it as $K = K_L(Z)$). If all structures of the class $K$ have a language $L$, then the set of the sentences of the language $L$, which are true in all structures from $K$, is called an elementary theory of the class $K$ and denoted by $Th(K)$. If $K = \{A\}$ then we will write $Th(A)$ instead $Th(K)$.

We will say that the class $K$ of structures is closed under the elementary equivalence (the isomorphism, the subsystems, the ultraproduct and others) if with structures $A_i, \ i \in I$, it contains all structures which are elementary equivalent to them (isomorphic to them, subsystems of them, ultraproduct of the structures $A_i$ and others.)

When we will study the axiomatizability of the classes we will use the following criterion.

**Theorem 3.1 [EP].** Class $K$ of structures of the language $L$ is axiomatizable if and only if $K$ is closed under elementary equivalence and ultraproducts.

The set of the sentences of the language $L$, closed under deducibility, is called the elementary theory or simply theory of the language $L$. The structure, in which all sentences of the theory $T$ are true, is called a model of the theory $T$. The consistent theory $T$ of the language $L$ is called complete if $\Phi \in T$ or $\neg \Phi \in T$ for any sentence $\Phi$ of the language $L$. A substructure $\mathcal{A}$ of a structure $\mathcal{B}$ is called elementary (it denoted by $\mathcal{A} \prec \mathcal{B}$), if for any formula $\varphi(\bar{x})$ of the language $L$ and any $\bar{b} \in \mathcal{B}$

$$\mathcal{A} \models \varphi(\bar{b}) \iff \mathcal{B} \models \Phi(\bar{b}).$$

Note that in this definition the condition ”$\iff$” we can exchange to ”$\Leftarrow$” (it is necessary to go to the negation of the formula).

The consistent theory $T$ of the language $L$ is called model complete if

$$\mathcal{A} \subseteq \mathcal{B} \implies \mathcal{A} \prec \mathcal{B}$$

for any models $\mathcal{A}, \mathcal{B}$ of theory $T$ of the language $L$. Theory $T$ of the language $L$ is called the theory with elimination of quantifiers if any formula $\Phi$ of the language $L$ is equivalent in $T$ some quantifier–free formula $\Psi$. Obviously, the consistent theory with elimination of quantifiers is model complete.

The formula of a form $\exists \bar{x}\Psi(\bar{x}; \bar{y}) \ (\forall \bar{x}\Psi(\bar{x}; \bar{y}))$ for a quantifier-free formula $\Psi(\bar{x}; \bar{y})$ is called existential (universal).

Let $\mathcal{A}$ be a structure of the language $L$, $X \subseteq A$, the language $L_X$ is obtained from $L$ by adding a new constant symbol to $L$ for each element of the set $X$, $\mathcal{A}_x$ be enrichment of the structure $\mathcal{A}$ up to the language $L_X$ with the natural interpretation of the new constant symbols. A set $D(\mathcal{A})$ of the atomic sentences of the language $L_A$, which are true in a structure $\mathcal{A}_A$, is called a diagram of the structure $\mathcal{A}$.

**Theorem 3.2 [ChK].** If $T$ is a theory of the language $L$, then the following conditions are equivalent:

1) the theory $T$ is model complete;

2) if $\mathcal{A}$ and $\mathcal{B}$ are the models of the theory $T$ and $\mathcal{A} \subseteq \mathcal{B}$, then for any existential formula $\psi(\bar{y})$ of the language $L$ and any $\bar{b} \in A$

$$\mathcal{B} \models \psi(\bar{b}) \implies \mathcal{A} \models \psi(\bar{b});$$
3) for any formula $\varphi(\bar{x})$ of the language $L$ there exists an existential formula $\psi(\bar{x})$, which is equivalent in the theory $T$ to a formula $\varphi(\bar{x})$.

**Proof.** The statement $1 \Rightarrow 2$ is trivial. Denote by $3'$ the condition which obtains from the condition 3 by exchange ”existential” on ”universal”. It is clear that the conditions 3 and $3'$ are equivalent (it is enough to go to the negation of the formula). Since the existential formulae are preserved under the extending of structures and the universal formulae are preserved under substructures, then the statement $3 \Rightarrow 1$ holds.

Let us proof the statement $2 \Rightarrow 3$. Suppose the condition 2 holds and $\varphi(\bar{x})$ is some formula of the language $L$. We proof the property 3 by induction on the number of the quantifiers in the formula $\varphi(\bar{x})$. Since the existential quantifier is expressing through the universal quantifier and negation, we will consider that the formula $\varphi(\bar{x})$ does not contain the existential quantifier. Assume $\varphi(\bar{x})$ has a form $\forall y \psi(y, \bar{x})$. By the induction supposition and the equivalence of the conditions 3 and $3'$, the formula $\psi(y, \bar{x})$ is equivalent in the theory $T$ to some universal formula. Hence, the formula $\varphi(\bar{x})$ is also equivalent in the theory $T$ to some universal formula. Let a set $Q$ consists of all existential formulae $\xi(\bar{x})$ of the language $L$ with the condition $(T \cup \{\xi(\bar{x})\}) \vdash \varphi(\bar{x})$ and a set $Q^*$ consists of the negations of the formulae from $Q$. If the set $(Q^* \cup \{\varphi(\bar{x})\})$ is not consistent with the theory $T$, then the formula $\varphi(\bar{x})$ is equivalent in $T$ to the disjunction of some formulae from $Q$, so to some existential formula, and the property 3 is proved.

Suppose the set $(Q^* \cup \{\varphi(\bar{x})\} \cup T)$ is consistent. Assume for some model $A$ of the theory $T$ and $\bar{b} \in A$ we have $A \models \chi(\bar{b})$ for all formulae $\chi(\bar{x}) \in (Q^* \cup \{\varphi(\bar{x})\})$. We claim that

$$(D(A) \cup T) \vdash \varphi(\bar{c}_b).$$

If it is wrong then there exists a model $B$ of a set $(T \cup D(A) \cup \{\neg \varphi(\bar{c}_b)\})$. Since $B$ is a model of a set $D(A)$, then we can suppose that $A \subseteq B$. Since $\neg \varphi$ is equivalent in $T$ to the existential formula and $A \models \varphi(\bar{b})$, then it contradicts to the condition 2. Since $D(A)$ consists of the quantifier–free formulae then we have $(\{\Phi(\bar{c}_b; \bar{c}_b)\} \cup T) \vdash \varphi(\bar{c}_b)$ for some quantifier–free formula $\Phi$ and cortege $\bar{a} \in A$. Consequently, we have $A \models \exists z \Phi(z; \bar{b})$ and $\exists z \Phi(z; \bar{x}) \in Q$, but all formulae from the set $Q^*$ are true in $A$ on the cortege $\bar{b}$, contradiction.

Consistent theory $T$ of the language $L$ is called submodel complete if the theory $T \cup D(B)$ of the language $L_B$ is complete for any substructure $B$ of any model $A$ of the theory $T$. Note that any submodel complete theory is model complete.

**Theorem 3.3** [Sac]. A theory $T$ is submodel complete if and only if $T$ is the theory with elimination of quantifiers.

**Proof.** **Necessity.** Let $T$ be a submodel complete theory of the language $L$ and $\varphi(\bar{x})$ be an arbitrary formula of the language $L$. Let $Q$ be a set of all quantifier–free formulae $\xi(\bar{x})$ of the language $L$ with the property $(T \cup \{\xi(\bar{x})\}) \vdash \varphi(\bar{x})$ and $Q^*$ be a set of all negations of formulae from $Q$. If a set $(Q^* \cup \{\varphi(\bar{x})\})$ is not consists with the theory $T$, then the formula $\varphi(\bar{x})$ is equivalents in $T$ to disjunction of some formulae from the set $Q$, consequence, to some quantifier–free formula, that the necessity is proved.

Suppose the set $(Q^* \cup \{\varphi(\bar{x})\} \cup T)$ is consistent. Assume for some model $A$ of the theory $T$ and $\bar{b} \in A$ we have $A \models \chi(\bar{b})$ for all formulae $\chi(\bar{x}) \in (Q^* \cup \{\varphi(\bar{x})\})$. Let $B$ be a substructure of a structure $A$, generated in $A$ by elements of the cortege $\bar{b}$. Since the theory $T$ is a submodel complete, then one of two cases are hold: 1) $(D(B) \cup T) \vdash \varphi(\bar{c}_b)$, 2) $(D(B) \cup T) \vdash \neg \varphi(\bar{c}_b)$. 

Suppose the first case holds. Since $D(B)$ is consists of quantifier–free formulae and a structure $B$ is generated by a cortege $b$, then we have $\{\Phi(\bar{c}_b)\} \cup T \vdash \varphi(\bar{c}_b)$ for some quantifier–free formula $\Phi$. Therefore, $\mathcal{A} \models \Phi(b)$ and $\Phi(\bar{x}) \in Q$, but in $\mathcal{A}$ all formulae from the set $Q^*$ are true on the cortege $b$, contradiction.

The second case is also impossible, because a structure $\mathcal{A}$ is a model of the set $(D(B)\cup T)$ and we have $\mathcal{A} \models \varphi(\bar{b})$.

**Sufficiency.** Since any quantifier–free formula is equivalent to the disjunction of formulae which are the conjunctions of the atomic formulae and their negations, and a set $D(\mathcal{A})$ contains $\varphi$ or its negation for any atomic sentence $\varphi$ of the language $L_A$, then for any quantifier–free sentence $\Phi$ of the language $L_A$ we have $D(\mathcal{A}) \vdash \Phi$ or $D(\mathcal{A}) \vdash \neg\Phi$. Thus, if $T$ is the theory with elimination of quantifiers, then $T$ is submodel complete. □

If $K$ is a class of the structures of the language $L$ then by $K^\infty$ we denote the class of infinite structures from $K$. Class $K$ is called complete (model complete) if the theory $Th(K^\infty)$ of the class $K^\infty$ is complete (model complete).

Let $T$ be a consistent theory of the language $L$, $X = \{x_i \mid 1 \leq i \leq n\}$, $L_n = L_X$. A set of all formulae of language $L$ with free variables from $X$ and parameters from $A \subseteq \mathcal{C}$ is denote by $F_X(A)$. Any set of the sentences $p$ of the language $L_n$ is called $n$–type of the language $L$. If the theory $p \cup T$ is consistent, then $p$ is called $n$–type over $T$. If $p$ is a complete theory, then $p$ is called complete $n$–type of the language $L$. If additionally $T \subseteq p$, then $p$ is called a complete $n$–type over $T$. A set of all complete $n$–types over $T$ is denote by $S_n(T)$.

Let $\mathcal{A}$ be a structure of the language $L$, $X \subseteq A$, $a \in A$. A set $tp(a, X) = \{\Phi(x) \mid \mathcal{A}_X \models \Phi(a)\}$ is called a type of an element $a$ over a set $X$. It is not difficult to understand that $tp(a, X)$ is a complete 1–type over $Th(\mathcal{A}_X)$. By $S_n(X)$ we denote $S_n(Th(\mathcal{A}_X))$. Often we will write $S(X)$ instead $S_1(X)$.

The theory $T$ is called stable in a cardinal $\kappa$ or $\kappa$–stable if $|S(X)| \leq \kappa$ for any model $\mathcal{A}$ of the theory $T$ and any $X \subseteq A$ of cardinal $\kappa$. If the theory $T$ is $\kappa$–stable for some infinite $\kappa$, then $T$ is called stable. If the theory $T$ $\kappa$–stable for all $\kappa \geq 2^{|T|}$, then $T$ is called superstable. If the theory $T$ is not stable, then $T$ is called unstable.

**Theorem 3.4** [She]. A complete theory is unstable if and only if there exists a formula $\Phi(\bar{x}, \bar{y})$ on $2n$ variables, a model $\mathcal{A}$ of the theory $T$ and $\bar{a}_i \in A^n$, $i \in \omega$, such that for any $i, j$, $i \neq j$,

\[ i < j \iff \mathcal{A} \models \Phi(\bar{a}_i, \bar{a}_j). \]

□

**Theorem 3.5.** If the theory $T$ is stable in a countable cardinality ($\omega$–stable), then it stable in all infinite cardinality.

**Proof.** Let the theory $T$ is $\omega$–stable. Assume there exists a subset $A$ of monster-model $\mathcal{C}$ of the cardinality $\lambda \geq \omega$, such that the cardinality of the set $S(A)$ strictly more then a cardinal $\lambda$. From $\omega$–stability it follows that there exists the countable language $L' \subseteq L$, such that for any predicate or function of the language $L$ there exists equivalent in theory $T$ predicate or function accordingly of the language $L'$. So we can consider that the language $L$ is countable. Then the set $F_X(A)$, where $X = \{x\}$, has a cardinality $\lambda$. Let $S(A)$ be the Stone space of $A$. Remark that this space is defined by the basis of open sets $\{U_\Phi \mid \Phi \in F_X(A)\} \cup \{t \mid \Phi \in t \in S(A)\}$. For the topological space $X$ we denote by $X'$ the derived space, that is the space, which is obtained from the space $X$ removing the isolated points. By induction on ordinal $\alpha$ we define the subspaces $S^{(\alpha)}$...
The definition of the stationary theory implies that for any formula $\Phi \in F_X(A)$ there exists not more than one ordinal $\beta < \gamma$ such that the set $(U_\Phi \cap S^{(\beta)})$ consists of one point. So the cardinality of the ordinal $\gamma$ is not more than the cardinality of the set $F_X(A)$, which is equal to $\lambda$. Since the number of isolated points of the space, whose basis of open sets has the cardinality $\lambda$, is also not more than $\lambda$, and the cardinality of $S(A)$ is strictly more than $\lambda$, it follows that the cardinality of $S^{(\gamma)}$ is also more than $\lambda$. Since the space $S^{(\gamma)}$ has not isolated points and is Hausdorff, then for any cortego $\varepsilon \in 2^{<\omega}$ of elements of the set $\{0, 1\}$ there exist nonempty sets $X(\varepsilon)$ of the form $\Phi_\varepsilon(\mathfrak{C}; a_\varepsilon)$ for the formula $\Phi(x; a) \in F_X(A)$ with the following properties:

1) $X(\emptyset) = \emptyset$;
2) $X(\varepsilon \cup L) \subseteq X(\varepsilon)$, $L \in \{0, 1\}$;
3) $(X(\varepsilon \cup 0) \cap X(\varepsilon \cup 1)) = \emptyset$.

Write $A_0$ for the countable set $= \bigcup\{a_\varepsilon \mid \varepsilon \in 2^{<\omega}\}$. By the properties 1–3 the cardinality of the set $S(A_0)$ is equal to $2^\omega$ which contradicts with the $\omega$-stability of the theory $T$. \(\square\)

In [Mus] T.G.Mustafin gives the notion of stationary theory of polygons, which we will use hereinafter. A complete theory $T$ of polygons is called stationary if for any $sM \models T$ and $a, b \in \mathfrak{C} \setminus M$,

$$a \in Sb \iff M \cap Sa = M \cap Sb.$$  

Let $sA$ be a polygon and $a \in \mathfrak{C} \setminus A$. An element $c \in A$ is called input element from $a$ in $sA$ if $c \in Sa$ and $Sb \subseteq Sc$ for all $b \in A \cap Sa$.

**Theorem 3.6** [Mus]. Let $T$ be a stationary theory, $sM \models T$, $a, b \in \mathfrak{C} \setminus M$ and $c$ be an input element from $a$ in $sM$. The following conditions are equivalent:

1) $tp(a, M) = tp(b, M)$;
2) $c$ is an input element from $b$ in $sM$ and $tp(a, \{c\}) = tp(b, \{c\})$;
3) $M \cap Sa = M \cap Sb \iff tp(a, (M \cap Sa)) = tp(b, M \cap Sa)$.

**Proof.** 1 $\Rightarrow$ 2. Assume the condition 1 is satisfied. The equality $tp(a, \{c\}) = tp(b, \{c\})$ is obviously. Let us prove $M \cap Sa = M \cap Sb$. Suppose $m \in M \cap Sa$. Then $m = sa$ for some $s \in S$. Hence, $m = sx \in tp(a, M)$. Therefore, $m = sx \in tp(b, M)$, that is $m = sb \in M \cap Sb$. Thus, $M \cap Sa \subseteq M \cap Sb$. Similarly $M \cap Sb \subseteq M \cap Sa$. Consequently, $M \cap Sa = M \cap Sb$. Since $c$ is an input element from $a$ in $sM$ then $c$ is an input element from $b$ in $sM$.

2 $\Rightarrow$ 3. Assume the condition 2 is satisfied. The equalities $M \cap Sa = Sc$ and $M \cap Sb = Sc$ follow from the definition of the input element. Hence, $M \cap Sa = M \cap Sb$. The equality $tp(a, \{c\}) = tp(b, \{c\})$ implies the existence of an identity on $\{c\}$ automorphism $\varphi$ of the polygon $\mathfrak{C}$ such that $\varphi(a) = b$. If $d \in Sc$ and $d = sc$, where $s \in S$, then $\varphi(d) = \varphi(sc) = s\varphi(c) = sc = d$. Therefore, $\varphi$ is identity on $Sc$. Thus, $tp(a, (M \cap Sa)) = tp(b, M \cap Sa)$.

3 $\Rightarrow$ 1. Assume the condition 3 is satisfied. The equality $tp(a, M \cap Sa) = tp(b, M \cap Sa)$ implies the existence of an identity on $M \cap Sa$ automorphism $\varphi$ of the polygon $\mathfrak{C}$ such that $\varphi(a) = b$. Denote the set $\{d \in \mathfrak{C} \mid d \text{ connected with } u \text{ out of } M\}$ by $C_M(u)$, where $u \in \mathfrak{C}$.

We claim that $\varphi(C_M(a)) \subseteq C_M(b)$. Let $a_0, \ldots, a_n \in \mathfrak{C} \setminus M$, $a = a_0$ and $a_i \in Sa_{i+1}$ or $a_i \in Sb_i$ for all $i$, $0 \leq i \leq n - 1$. By the induction on $n$ let us prove that $\varphi(a_n) \in C_M(b)$. For $n = 0$ the statement follows from the condition $b \in \mathfrak{C} \setminus M$. Suppose $\varphi(a_i) \notin M$. If $a_i \in Sa_{i+1}$, then $\varphi(a_i) \in S\varphi(a_{i+1})$ and $\varphi(a_{i+1}) \notin M$. Assume $a_{i+1} \in Sa_i$ and $\varphi(a_{i+1}) \in M$. The definition of the stationary theory implies $M \cap S\varphi(a_{i+1}) = M \cap Sb_i$. Therefore, $\varphi(a_{i+1}) \in$
Thus, $\varphi(M) \subseteq C_M(b)$. Similarly, $\varphi^{-1}(C_M(b)) \subseteq C_M(a)$, that is $\varphi(C_M(a)) = C_M(b)$. The equality $\varphi(C_M(b)) \subseteq C_M(a)$ is proved the same way.

We construct the automorphism $\psi$ of polygon $\mathcal{C}$ as follows: $\psi(C_M(a) \cap C_M(b)) = \varphi(C_M(a) \cap C_M(b))$ and $\psi$ is an identity mapping on the set $\mathcal{C} \setminus (C_M(a) \cap C_M(b))$. Clearly, that $\psi(a) = b$. Hence, the condition 1 of lemma holds. 

3.7 [Mus]. Each stationary theory is stable.

Proof. Let $sM$ be a model of the theory $T$, $|M| = \kappa$, $\kappa = \kappa^{|T|}$. If $a \in \mathcal{C}$, then $|sA| \leq |S| \leq |T|$ and $|M \cap sA| \leq |T|$. So on Lemma 3.1 $|S_1(M)| \leq |M|^{|T|} \cdot 2^{|T|} = |M|^{|T|} = \kappa^{|T|} = \kappa$. Hence, $T$ is $\kappa$-stable theory.

Let $K$ be a class of polygons. Monoid $S$ is called $K$-stabilizer ($K$-superstabilizer, $K$-\(\omega\)-stabilizer) if $T(h_{sA})$ is stable (superstable, \(\omega\)-stable accordingly) for any polygon $sA \in K$. If $K = S - Act$, then $K$-stabilizer ($K$-superstabilizer, $K$-\(\omega\)-stabilizer) are called stabilizer (superstabilizer, \(\omega\)-stabilizer accordingly).

Let us give the characterization of stabilizer and superstabilizer, which was got by T.G. Mustafin. A monoid $S$ is called linearly ordered if the set $\{S a \mid a \in S\}$ is linearly ordered by inclusion.

Theorem 3.8 [Mus]. Monoid $S$ is stabilizer if and only if $S$ is linear ordered monoid.

A linearly ordered monoid $S$ is called well-ordered if it satisfies the ascending chain condition for principal left ideals.

Theorem 3.9 [Mus]. Let $S$ be a countable monoid. Monoid $S$ is superstabilizer if and only if $S$ well-ordered monoid.

§ 4. Axiomatizability of Class for Regular Polygons

The main result in this paragraph is Theorem 4 which give the characterization for the monoids with the axiomatizable class of the regular polygons. In particular we get the axiomatizability of the class of regular polygons over the group (Corollary 4.3).

Theorem 4.1 [Ste1]. Class $\mathfrak{R}$ for the regular polygons axiomatizable is if and only if

1) the semigroup $R$ is satisfied the descending chain condition for principal right ideals which are generated by the idempotens;

2) for any $n \geq 1$, $s_i, t_i \in S$ ($1 \leq i \leq n$) the set $\{x \in R \mid \bigwedge_{i=1}^{n} s_i x = t_i x\}$ is empty or finite generated as a right ideal of the semigroup $R$.

Proof. Necessity. Let $\mathfrak{R}$ be an axiomatizable class. Assume condition 1 is not hold. This means that there exists a decreasing sequence of principal right ideals:

$$f_1 S \supset f_2 S \supset \ldots \supset f_n S \supset \ldots,$$

where $f_n \in R$, $f_n^2 = f \ (n \geq 1)$. For any $n, m, 1 \leq n \leq m$, the inclusion $f_n S \supset f_m S$ implies the equality $f_n f_m = f_m$. Suppose $\tilde{f} = (f_n)_{n \in \omega} \in R^\omega$ and $D$ is an arbitrary non-principal ultrafilter on $\omega$. Then the equality $f_n \cdot \tilde{f} / D = \tilde{f} / D$ is true in $sR^\omega / D$ for any $n \geq 1$. In view of the axiomatizability of the class $\mathfrak{R}$ by Theorem 3.1 we have $sR^\omega / D \in \mathfrak{R}$. On Corollary 2.1 there exist an idempotent $e \in R$ and an isomorphism $\varphi : s(S \cdot \tilde{f} / D) \rightarrow sSe$ such that $\varphi(\tilde{f} / D) = e$. Then $e \cdot \tilde{f} / D = \tilde{f} / D$. The equality $f_n e = e$ implies the equality $f_n \cdot \tilde{f} / D = \tilde{f} / D$ for any $n \geq 1$. Consequently, there exists
Assume condition 2 is not hold. Then there exist $n \geq 1, s_i, t_i \in S$ ($1 \leq i \leq n$) such that $X = \{x \in R \mid \bigwedge_{i=1}^{n} s_i x = t_i x\}$ is the non–empty set and is not finite generated as a right ideal of $R$. So there are the infinite ordinal $\gamma$ and $x_\tau \in X$ ($\tau < \gamma$) such that $X = \bigcup \{x_\tau R \mid \tau < \gamma\}$ and $x_\beta R \not\subseteq \bigcup \{x_\tau R \mid \tau < \beta\}$ for all $\beta < \gamma$. Suppose $\bar{x} = (x_\tau)_{\tau < \gamma} \in R^\gamma$ and $D$ is the ultrafilter on $\gamma$ such that $|Y| = \gamma$ for $Y \in D$. In vire of the axiomatizability of the class $\mathcal{R}$ by Theorem 3.1 we get $s\bar{R}/D \in \mathcal{R}$. On Corollary 2.1 there exist an idempotent $e \in R$ and an isomorphism $\varphi: s\bar{S}/D \longrightarrow sSe$ such that $\varphi(\bar{x}/D) = e$. Since $x_\tau \in X$ ($\tau < \gamma$) we have $\bigwedge_{i=1}^{n} s_i x/D = t_i x/D$ and $e \in X$. Consequently, $eR \subseteq \bigcup \{x_\tau R \mid \tau < \gamma\}$ that is $eR \subseteq x_{\tau_0} R$ for some $\tau_0 < \gamma$. Since $e = ee$ it follows that $\bar{x}/D = e \cdot \bar{x}/D$. In particular, $x_\tau \in eR$ for some $\tau > \tau_0$ and $x_\tau R \subseteq x_{\tau_0} R$. We get the contradiction. Thus condition 2 is proved.

**Sufficiency.** Assume conditions 1,2 of this theorem hold. Suppose $n \geq 1$, $\bar{s} = \langle s_1, \ldots, s_n \rangle$, $\bar{t} = \langle t_1, \ldots, t_n \rangle \in S^n$, $X_{\bar{s} \bar{t}} = \{x \in R \mid \bigwedge_{i=1}^{n} s_i x = t_i x\}$. Let us show that either $X_{\bar{s} \bar{t}} = \emptyset$ or $X_{\bar{s} \bar{t}} = \bigcup \{e_i R \mid 1 \leq i \leq k\}$ for some $k \geq 1$ and idempotents $e_i \in X_{\bar{s} \bar{t}}$ ($1 \leq i \leq k$). Suppose $X_{\bar{s} \bar{t}} \neq \emptyset$. Under condition of the theorem there exist $k \geq 1$, $r_i \in X_{\bar{s} \bar{t}}$ ($1 \leq i \leq k$) such that $X_{\bar{s} \bar{t}} = \bigcup \{r_i R \mid 1 \leq i \leq k\}$. We can consider that $r_i R \not\subseteq r_j R$ ($i \neq j$). Fix $i$, $1 \leq i \leq k$. Since $r_i \in R$, on Corollary 2.1 there exist an idempotent $e_i \in R$ and an isomorphism $\varphi: sS r_i \longrightarrow sS e_i$ such that $\varphi(r_i) = e_i$. Then $e_i r_i = r_i$. Since $r_i \in X_{\bar{s} \bar{t}}$ we have $e_i \in X_{\bar{s} \bar{t}} = \bigcup \{r_i R \mid 1 \leq i \leq k\}$, that is, $e_i = r_j s$ for some $j, 1 \leq j \leq k$, and $s \in R$. Consequently, $r_i = e_i r_i = r_j s r_i \subseteq r_j R$. In vire of the choice of the element $r_j$ ($1 \leq j \leq k$) this means that $r_i = r_j$. Since $r_i = e_i r_i$ it follows that $r_i \in e_i R$. In view of $e_i = r_j s$ we have $e_i \in r_j S$. Consequently, $r_i S = e_i S$. On Proposition 2.4 (2), $r_i R = e_i R$. Thereby, $X_{\bar{s} \bar{t}} = \bigcup \{e_i R \mid 1 \leq i \leq k\}$, where $e_i \in X_{\bar{s} \bar{t}}$.

Define a set of formulae $\Gamma$ as follows: for all $n \geq 1$, $\bar{s} = \langle s_1, \ldots, s_n \rangle$, $\bar{t} = \langle t_1, \ldots, t_n \rangle \in S^n$

\[
\neg \exists x \bigwedge_{i=2}^{n} s_i x = t_i x \in \Gamma, \text{ if } sR \models \neg \exists x (x \in X_{\bar{s} \bar{t}}); \]

\[
\forall x \bigwedge_{i=2}^{n} s_i x = t_i x \longrightarrow \bigwedge_{j=1}^{k} x = e_j x \in \Gamma, \text{ if } sR \models \exists x (x \in X_{\bar{s} \bar{t}}), \]

where $X_{\bar{s} \bar{t}} = \{x \in R \mid \bigwedge_{i=1}^{n} s_i x = t_i x\} = \bigcup \{e_j R \mid 1 \leq j \leq k\}$, $e_j^2 = e_j \in X_{\bar{s} \bar{t}}$. Let us show that

\[
sA \in \mathcal{R} \iff sA \models \Gamma.\]

Let $sA \in \mathcal{R}$. Suppose $\bigwedge_{i=1}^{n} s_i a = t_i a$ for some $a \in A$. On Corollary 2.1 there exist an idempotent $f \in R$ and an isomorphism $\varphi: sS a \longrightarrow sS f$ such that $\varphi(a) = f$. Then $f \in X_{\bar{s} \bar{t}}$. Consequently, $X_{\bar{s} \bar{t}} \neq \emptyset$. Assume $X_{\bar{s} \bar{t}} = \bigcup \{e_j R \mid 1 \leq j \leq k\}$ and $e_j^2 = e_f \in X_{\bar{s} \bar{t}}$. Then $sR \models \bigwedge_{j=1}^{k} e_j f = f$ and $sA \models \bigwedge_{j=1}^{k} e_j a = a$. 

$m \geq 1$ such that $f_m = e f_m \in eS \subseteq f_n S$ for any $n \geq 1$ that contradict to the condition $f_{m+1} S \subseteq f_m S$. Thus condition 1 is proved.
Suppose $sA \models \Gamma$, $a \in A$. Let us prove that $sSa \cong sSe$, where $e$ is some idempotent from $R$. Suppose $sa = ta$, $s, t \in S$. Since $sA \models \Gamma$ we have $sR \models \exists x(sx = tx)$ and $a = fa$ for some idempotent $f \in R$. Suppose $\{f_x \mid \tau < \gamma\} = \{f \mid f^2 = f, fa = a, f \in R\}$. By induction on $\gamma$ we will show that there exists $\gamma_0 < \gamma$ such that

$$f_{\gamma_0}S = \cap \{f_xS \mid \tau < \gamma\}.$$

Let $\gamma$ be a limit ordinal, $\tau_0 < \gamma$. Under suggestion of the induction there exists $\beta_0 < \tau_0$ such that $f_{\beta_0}S = \cap \{f_xS \mid \tau < \tau_0\}$. If $f_{\beta_0}S \not\subseteq \cap \{f_xS \mid \tau < \gamma\}$ then there are $\beta_1, \gamma_1 < \tau < \gamma$ such that

$$f_{\beta_0}S \supset f_{\beta_1}S \subseteq \cap \{f_xS \mid \tau < \gamma\}$$

and etc. Since $f_{\beta_n}S \supset f_{\beta_{n+1}}S$ we have $f_{\beta_n}f_{\beta_{n+1}} = f_{\beta_n}f_{\beta_{n+1}}$. Consequently, $f_{\beta_n}R \supset f_{\beta_{n+1}}R$ and on Proposition 2.4 (2) $f_{\beta_n}R \supset f_{\beta_{n+1}}R, n \geq 0$. Under condition of the theorem the decreasing chain of ideals $f_{\beta_n}R \supset f_{\beta_{n+1}}R \supset \ldots \supset f_{\beta_n}R \supset \ldots$ is become stabilize. On Proposition 2.4 (2) the decreasing chain of ideals $f_{\beta_0}S \supset f_{\beta_1}S \supset \ldots \supset f_{\beta_n}S \supset \ldots$ is become stabilize also that is $f_{\beta_k}S = \cap \{f_xS \mid \tau < \gamma\}$ for some $k \geq 0$.

Let $\gamma$ be a non–limit ordinal. Assume there exists $\beta_0 < \gamma - 1$ such that $f_{\beta_0}S = \cap \{f_xS \mid \tau < \gamma - 1\}$. Then $sA \models a = f_{\beta_0}a \land a = f_{\gamma_1}a$. Since $sA \models \Gamma$ we have $sR \models \exists x(x = f_{\beta_0}x \land x = f_{\gamma_1}x)$ and there exists $f \in R$ such that $a = fa$, $f = f_{\beta_0}f = f_{\gamma_1}f$.

Consequently,

$$f = f_{\gamma_0}, \quad \gamma_0 < \gamma, \quad f_{\gamma_0}S \subseteq f_{\beta_0}S \cap f_{\gamma_1}S, \quad f_{\gamma_0}S = \cap \{f_xS \mid \tau < \gamma\}.$$ 

We put $e = f_{\gamma_0}$. Then $ea = a$ and the equality $ga = a$ implies $eS \subseteq gS$ for any idempotent $g \in R$ that is $e = ge$. Let us show that the mapping $\varphi : Sa \rightarrow Se$ such that $\varphi(sa) = se$ for any $s \in S$, is a polygon isomorphism. Suppose $ra = ka$, $r, k \in S$. Since $sA \models \Gamma$ then there exists an idempotent $g \in R$ such that $rg = kg$ and $ga = a$. So $ge = e$ and $re = ke$. Suppose $re = ke, r, k \in S$. Since $ea = a$ we get $ra = ka$. Thus, in view of arbitrary of the choice of the element $a$, $sSa \cong sSe$ and $sA \in \mathfrak{R}$. □

From proof of sufficiency it follows

**Corollary 4.1.** Let $\mathfrak{R}$ be an axiomatizable class and $X_{\vec{s}} = \{x \in R \mid \bigwedge_{i=1}^{n} s_i x = t_i x\}$ be a non–empty set, where $n \geq 1$, $\vec{s} = (s_1, \ldots, s_n)$, $\vec{t} = (t_1, \ldots, t_n) \in S^n$. Then the set $X_{\vec{s}}$ is finite generated as a right ideal of the semigroup $R$ if and only if $X_{\vec{s}} = \bigcup\{e_i R \mid 1 \leq i \leq k\}$ for some $k \geq 1$ and some idempotants $e_i \in X_{\vec{s}}$ ($1 \leq i \leq k$). □

**Corollary 4.2.** If the class $\mathfrak{R}$ of the regular polygons is axiomatizable then $R = \bigcup\{e_i R \mid 1 \leq i \leq n\}$ for some $n \geq 1$, $e_i \in R$, $e_i^2 = e_i$ ($1 \leq i \leq n$).

**Proof** follows from Theorem 4.1, Corollary 4.1 and the equality $R = \{x \in R \mid x = x\}$. □

The following statement is obviously.

**Corollary 4.3.** The class $\mathfrak{R}$ for regular polygons over group is axiomatizable. □

§ 5. Model Completeness of Class for Regular Polygons

In this paragraph it is formulated and proved the criterion of the model completeness for an axiomatizable class of regular polygons (Theorem 5.2). As a consequence we derive a model completeness of the class of regular polygons over infinite group (Corollary 5.1).
Lemma 5.1. Let a monoid $S$ be regularly linearly ordered, $sB \subseteq sA \in \mathcal{R}$, $a_i \in A$ $(1 \leq i \leq k)$. Then for any $i$, $1 \leq i \leq k$, the following conditions hold:
1) $\cap \{Sa_j \mid Sa_j \cap Sa_i \neq \emptyset\} \neq \emptyset$;
2) if $Sa_i \cap B \neq \emptyset$ then $B \cap \{Sa_j \mid Sa_j \cap Sa_i \neq \emptyset\} \neq \emptyset$.

Proof. Suppose that the conditions of the lemma hold. Let us prove the statement 2 (the statement 1 is proved similarly). Assume that $\{j_0, \ldots, j_s\} = \{j \mid Sa_j \cap Sa_i \neq \emptyset\}$, where $j_0 = i$;

$$B_n = B \cap \{Sa_j \mid j \in \{j_0, \ldots, j_n\}\} \text{ (} n \leq s \text{).}$$

It is sufficiently to show that for any $n$, $n < s$, the inequality $B_n \neq \emptyset$ implies the inequality $Sa_{j_{n+1}} \cap B_n \neq \emptyset$. Let $c \in B_n \subseteq Sa_i$, $b \in Sa_{j_{n+1}} \cap Sa_i$. Since $b \in Sa_i$, on Proposition 2.5 either $Sc \subseteq Sb$ or $Sb \subseteq Sc$, that is either $c \in Sa_{j_{n+1}} \cap B_n$ or $b \in Sa_{j_{n+1}} \cap B_n$. □

Lemma 5.2. Let $\mathcal{R}$ be an axiomatizable class and for any $a \in R$ and an idempotent $e \in R$ from the inclusion $Sa \subseteq Se$ follows the existence an idempotent $f \in R$ such that $Sa = Sf$ and $fS \subseteq eS$. Then for any idempotent $g \in R$ there exists an idempotent $h \in R$ such that $Sh \subseteq Sg$, the polygon $sSh$ is minimum by inclusion and the right ideal $hS$ is minimum for the principal right ideals of the monoid $S$, generated by idempotents.

Proof. Suppose the conditions of the lemma hold and $g \in R$, $g = e^2$. Assume there exist the infinitely decreasing chain of polygons

$$sSg \supset sSk_1 \supset \ldots \supset sSk_n \supset \ldots,$$

where $k_i \in R$ $(i \geq 1)$. On the condition there exist the idempotents $h_i \in R$ $(i \geq 0)$ such that $h_0 = g$, $Sk_i = Sh_i$ $(i \geq 1)$ and

$$gS = h_0S \supset h_1S \supset \ldots \supset h_nS \supset \ldots.$$ 

In view of the axiomatizability of the class $\mathcal{R}$ and Theorem 4.1 there exists $n \in \omega$ such that $h_iS = h_jS$ for all $i \geq n$, $j \geq n$. On Proposition 1.2 the inclusion $Sh_{n+1} \subseteq Sh_n$ and the equality $h_{n+1}S = h_nS$ imply $h_{n+1} = h_n$ which contradicts with the suggestion $Sh_{n+1} \neq Sh_n$. Consequently, there exists a minimum by inclusion polygon $sSh$, such that $Sh \subseteq Sg$. On Proposition 1.3 the right ideal $hS$ is minimum for the principal right ideals of the monoid $S$, generated by idempotents. □

Theorem 5.1 [Ste1]. Let the class $\mathcal{R}$ for regular polygons be axiomatizable. The class $\mathcal{R}$ is model complete if and only if the following conditions hold:

1) $S$ is a regularly linearly ordered monoid;
2) for any idempotent $e \in R$ and for $a, a_i \in S$, $1 \leq i \leq m$ if $Sa \subseteq Se$ and $e \notin \cup \{a_iS \mid 1 \leq i \leq m\}$ then there exist the idempotents $e_j \in R$ $(j \in \omega)$ such that

$$e_j \neq e_k, \quad Sa = Se_j, \quad e_j \in eS \cup \cup \{a_iS \mid 1 \leq i \leq m\}$$

for any $j, k \in \omega$, $j \neq k$;
3) $|eSf| \geq \omega$ for any idempotents $e, f \in R$.

Proof. Necessity. Suppose that the class $\mathcal{R}$ is model complete.

Let us prove condition 1. Assume $e^2 = e \in R$. Since any cyclic regular polygon is isomorphic to a subpolygon of the polygon $sS$, generated by idempotent, it follows that it is enough to show that the set $\{Sa \mid Sa \subseteq Se\}$ is linearly ordered by inclusion. Let $Sa_1 \subseteq Se$ and $Sa_2 \subseteq Se$. Then $a_1e = a_1$, $a_2e = a_2$ and $sSe \models \exists x(a_1x = a_1 \land a_2x = a_2)$. In view of model completeness of the class $\mathcal{R}$ we have $s(Sa_1 \cup Sa_2) \lesssim sSe$, that is

$$s(Sa_1 \cup Sa_2) \models \exists x(a_1x = a_1 \land a_2x = a_2).$$
Suppose $a_1c = a_1$, $a_2c = a_2$, where $c \in Sa_1 \cup Sa_2$. If for instance $c \in Sa_1$ then $a_2 = a_2c \in Sa_1$. Thus, $Sa_2 \subseteq Sa_1$.

Let us prove condition 2. Suppose $Sa \subset Se$, $e \not\in \cup\{a_iS \mid 1 \leq i \leq m\}$, $e^2 = e \in R$. Assume

$$\Phi(y, a) \iff a = ay \wedge \bigwedge_{i=1}^m \neg \exists x(y = ax) \wedge y = ey.$$ 

Then $Se \models \Phi(e, a)$, that is $Se \models \exists y \Phi(y, a)$. Since the class $\mathfrak{R}$ is model complete, we have $sSa \rightarrow sSe$ and $sSa \models \exists y \Phi(y, a)$. Consequently, $sSa \models \Phi(e_1, a)$ for some $e_1 \in Sa \subseteq R$. Hence $a = ae_1$ and $Sa = Se_1$. Let $e_1 = ka$. Since $e_1 = ee_1$ it follows that $e_1 \in eS$. We claim that $e_1$ is an idempotent. Let $e_1e_1 = kae_1 = ka = e_1$. Furthermore, $sSa \models \bigwedge_{i=1}^m \neg \exists x(e_1 = ai)$, that is $e_1 \not\in \cup\{a_iS \mid 1 \leq i \leq m\}$.

Suppose there exist the idempotents $e_1, \ldots, e_k \in R$ satisfying the conditions:

$$Sa = Se_i, \ e_i \in eS \setminus \cup\{a_rS \mid 1 \leq r \leq m\}, \ e_i \neq e_j,$$

for any $i, j$, $i \neq j$, $1 \leq i, j \leq k$. We claim that there exists an idempotent $e_{k+1} \in R$ satisfying the same conditions with substitution $i$ on $k + 1$, that is the idempotent $e_{k+1} \in R$ such that $sSe \models \Phi(e_{k+1})$, and $e_{k+1}$ is not equal to $e_1, \ldots, e_k$. Under condition $Se_j = Sa \subset Se$, so $e \neq e_j (1 \leq j \leq k)$. So $sSe \models \exists y \Psi(y)$, where

$$\Psi(y) \iff \bigwedge_{j=1}^k \neg y = e_j \wedge \Phi(y).$$

Consequently, $sSa \models \Psi(e_{k+1})$ for some $e_{k+1} \in Sa$. Thus, $sSe \models \Phi(e_{k+1})$ and $e_{k+1}$ is not equal to $e_1, \ldots, e_k$. As for $e_1$ it is proved that $e_{k+1}$ is an idempotent.

Let us prove condition 3. Suppose $e^2 = e \in R$, $f^2 = f \in R$, $sSf_i$ ($i \in \omega$) are mutually disjoint copies of the polygon $sSf$. Since $sSf \models e(ef) = ef$ it follows that $sSf \models \exists x(ex = x)$ and $sSf \cup \bigcup_{i \in \omega} sSf_i \models \exists x_1, \ldots, x_n (\bigwedge_{i \neq j} x_i \neq x_j \wedge \bigwedge_{i \in n} ex_i = x_i)$ for any $n \geq 1$. In view of the model completeness of the class $\mathfrak{R}$ and inclusion $Sf \subseteq Sf \cup \bigcup_{i \in \omega} Sf_i$, we have $sSf \models \exists x_1, \ldots, x_n (\bigwedge_{i \neq j} x_i \neq x_j \wedge \bigwedge_{i \in n} ex_i = x_i)$ for any $n \geq 1$. Consequently, $|Sf| \geq \omega$.

**Sufficiency.** Let conditions 1–3 of the theorem hold. Suppose $sA, sB \in \mathfrak{R}$, $sB \subseteq sA$, $\bar{d} = \langle d_1, \ldots, d_r \rangle \in B^r$,

$$sA \models \exists \bar{x} \bigwedge_{j=1}^4 \Phi_j(\bar{x}, \bar{d}),$$

where $\bar{x} = \langle x_1, \ldots, x_k \rangle$,

$$\Phi_1(\bar{x}, \bar{d}) \iff \bigwedge\{nx_i = mx_j \mid \langle i, j, n, m \rangle \in L_1\},$$

$$\Phi_2(\bar{x}, \bar{d}) \iff \bigwedge\{nx_i = md_j \mid \langle i, j, n, m \rangle \in L_2\},$$

$$\Phi_3(\bar{x}, \bar{d}) \iff \bigwedge\{-nx_i = mx_j \mid \langle i, j, n, m \rangle \in L_3\},$$

$$\Phi_4(\bar{x}, \bar{d}) \iff \bigwedge\{-nx_i = md_j \mid \langle i, j, n, m \rangle \in L_4\},$$

$L_t \subseteq \tilde{k} \times \tilde{k} \times S \times S$ ($t \in \{1, 3\}$), $L_t \subseteq \tilde{k} \times \tilde{r} \times S \times S$ ($t \in \{2, 4\}$), $\tilde{k} = \{1, \ldots, k\}$, $\tilde{r} = \{1, \ldots, r\}$. 

moreover, if \((i, j, n, m) \in L_r\) then \((j, i, m, n) \in L_r\) \((r \in \{1, 3\})\). On the definition of the model completeness of the theory and on Theorem 3.2 to proving \(S B \prec S A\) it is enough to show

\[ S B \models \exists \bar{x} \bigwedge_{j=1}^4 \Phi_j (\bar{x}, \bar{d}). \]

Suppose \(S A \models \bigwedge_{j=1}^4 \Phi_j (\bar{a}, \bar{d})\), \(\bar{a} = \langle a_1, \ldots, a_k \rangle \in A^k\). We can consider that for any \(i, j\), which are satisfying condition \(S a_i \cap S a_j \neq \emptyset\), there exist \(n, m \in S\) such that \((i, j, n, m) \in L_1\). Since \(a_i \in A\), \(S A \in \mathcal{R}\) and on Proposition 2.1 there exist an idempotent \(f_i \in R\) and an isomorphism \(\varphi_i : S S A_i \rightarrow S S f_i\) \((1 \leq i \leq k)\). If \(S a_i \subset S a_j\) then on condition 2 we can consider \(S f_i = S \varphi_i (a_i) \subseteq S f_j\) and \(\varphi_i = \varphi_j |_{S a_i}\). If \(S a_i = S a_j\) then we suppose \(f_i = f_j\) and \(\varphi_i = \varphi_j\).

For \(i\) \((1 \leq i \leq k)\) such that \(S a_i \cap B \neq \emptyset\) suppose

\[ S c_i = \max \{ S m d_j | m d_j \in S a_i, \langle i', j', m', n \rangle \in L_2 \text{ for some } i', 1 \leq i' \leq k, m' \in S\} \cup \]

\[ \cup \{ S m a_j \mid m a_j \in B \cap S a_i, \langle i', j', m', n \rangle \in L_1 \text{ for some } i', 1 \leq i' \leq k, m' \in S\} \cup \{ S b \}, \]

where \(b \in B \cap \bigcap \{ S a_j \mid S a_j \cap S a_i \neq \emptyset\}\). The correctness of the definition of the set \(S c_i\) follows from Lemma 5.1 and Proposition 2.5.

Let us fix \(i, j\), \(1 \leq i, j \leq k\). Assume \(B \cap S a_i \cap S a_j \neq \emptyset\),

\[ S d_{ij} = \max \{ S m d_j | m d_j \in S a_i \cap S a_j, \langle i', j', n, m \rangle \in L_2 \text{ for some } i', 1 \leq i' \leq k, n \in S\} \cup \]

\[ \cup \{ S m a_j \mid m a_j \in S a_i \cap S a_j, \langle i', j', m, n \rangle \in L_1 \text{ for some } i', 1 \leq i' \leq k, n \in S\}. \]

In view of Proposition 2.5 this denotement is correct. Clearly, \(S d_{ij} = S d_{ji}\). Suppose \(d_{ij} = d_{ji}\).

Reenumber (if it is necessary) the elements of the set \(\{ a_1, \ldots, a_k\}\) so as for any \(i\), \(1 \leq i < k\), we have:

\[ S d_{i+1} = \max \{ S d_{ji} \mid i < j \leq k\}. \]

The proof of the sufficiency will contain of some lemmas.

**Lemma 5.3.** For any \(i, j\), \(1 \leq i < k\), \(0 \leq j < k - i\), we have

1) \(S d_{i+j+1} \subseteq S d_{i+j+1}\);
2) \(S d_{i+j+1} \subseteq \cap \{ S d_{i+j+1} | 0 \leq s \leq j\}\).

**Proof.** Let \(i, j\) be any numbers such that \(1 \leq i < k, 0 \leq j < k - i\).

1) We will prove by the induction on \(j\). If \(j = 0\) then the inclusion \(S d_{i+1} \subseteq S d_{ii}\) follows from the definition of \(S d_{ii}\). Assume the statement 1 of this lemma is proved for all \(j' < j\), that is

\[ S d_{i+j} \subseteq S d_{i+j-1} \subseteq \ldots \subseteq S d_{i+1} \subseteq S d_{ii}. \]

We claim that \(S d_{i+j+1} \subseteq S d_{i+j}\).

Suppose the contraries. In view of the regular linear ordering of monoid \(S\) and on Proposition 2.5 we have \(S d_{i+j+1} \subseteq S d_{i+j+1}\). Let \(a\) be an element of the form \(m d_{i'}\) or \(m a_{i'}\) from the definition \(S d_{i+j+1}\) such that \(S d_{i+j+1} = S a\) and \(a \in S a_{i+j+1} \cap S a_i\). Then \(a \notin S a_{i+j}\). By the induction on \(r\), \(0 \leq r \leq j - 1\), we will prove that \(a \notin S a_{i+j-r}\). Suppose \(a \notin S a_{i+j-r+1}\), \(a \in S a_{i+j-r}\). Since \(a \in S a_{i+j+1}\), on (5.1) and the definition of \(S d_{i+j-r+1}\) we derive \(a \in S d_{i+j-r+1}\). Consequently, \(a \in S a_{i+j-r+1}\), which contradicts with the induction suggestion. Thus, \(a \notin S a_{i+j-r}\) for any \(r\), \(0 \leq r \leq j - 1\).
In particular, \( a \not\in S_{a_{i+1}} \). Since \( a \in S_{i} \cap S_{i+j+1} \) and on \((5.1)\) we have \( i = i + j + 1 \) that impossible.

2) On the statement 1 of this lemma \( S_{d_{i+j+1}} \subseteq S_{d_{i+j}} \subseteq \ldots \subseteq S_{d_{i+j+1}} \subseteq S_{d_{i}} \). Suppose the inclusion \( S_{d_{i+j+1}} \subseteq S_{d_{i+j}} \cap \ldots \cap S_{d_{i+j+r-1}} \) is proved for \( 0 \leq r \leq j - 1, 1 \leq j < k \). We claim that \( S_{d_{i+j+1}} \subseteq S_{d_{i+r+1}} \). Let \( b \in S_{d_{i+j+1}} \), where \( b \) is an element of the form \( md_{r} \) or \( ma_{r} \) from the definition of \( S_{d_{i+j+1}} \), and \( b \in S_{i+j+1} \cap S_{a_{i}} \). Since \( S_{d_{i+j+1}} \subseteq S_{d_{i+r+1}} \), it follows that \( b \in S_{d_{i+r+1}} \). The equality \( i + j + 1 > i + r + 1 \) and definition of \( S_{d_{i+r+1}} \) imply that \( S_{d_{i+j+1}} \subseteq S_{d_{i+r+1}} \).

**Lemma 5.4.** For any \( i, j, l \), \( 1 \leq i, j \leq k \), there exists an idempotent \( e_{ij} \) such that \( Se_{ij} = S\varphi_{1}(d_{ij}) \), moreover, if \( Se_{ij} \subseteq Se_{ij} \) then \( e_{ij}S \subseteq e_{ij}S \subseteq f_{i}S \).

**Proof.** On condition 2 of this theorem there exist the idempotents \( e'_{ij} \) such that \( Se'_{ij} = S\varphi_{1}(d_{ij}) \). Let \( 1 \leq i < k \). In view of the regular linear ordering of the monoid \( S \) the set \( \{Se'_{ij} | 1 \leq j \leq k \} \) is linear ordering by inclusion. Assume \( Se'_{ij} \subseteq Se'_{ij} \subseteq \ldots \subseteq Se'_{i,j_{k}} \subseteq Sf_{ij} \), where \( \{j_{1}, \ldots, j_{k}\} = \{1, \ldots, k\} \). Suppose there exist the idempotents \( e_{ij}, e_{ij+1}, \ldots, e_{ij} \in R \) such that \( Se_{ij} = Se_{ij}, Se_{ij+1} = Se_{ij+1}, \ldots, Se_{ij} = Se_{ij}, Se_{ij} \subseteq \ldots \subseteq Se_{ij} \subseteq Sf_{ij}, \) where \( 1 < t < k \). We claim that there exists a idempotent \( e_{ij-1} \) such that \( Se_{ij-1} = Se_{ij-1} \), \( e_{ij-1} \subseteq Se_{ij} \). If \( Se'_{ij-1} = Se_{ij} \), we suppose \( e_{ij-1} = e_{ij} \). If \( Se'_{ij-1} \subseteq Se_{ij} \), then \( e_{ij}S \subseteq e_{ij} \subseteq e_{ij}S \) (otherwise, on Proposition 1.2 \( e_{ij-1} = e_{ij} \)), and on condition 2 of this theorem there exists an idempotent \( e_{ij-1} \) such that \( Se_{ij-1} = Se_{ij-1} \), \( e_{ij-1}S \subseteq e_{ij}S \).

Since \( \varphi_{1} \) is an isomorphism and \( \varphi_{1}(d_{ij}) = \varphi_{1}(d_{ij})e_{ij} \in \varphi_{1}(d_{ij})e_{ij}Se_{ij} \) we derive \( d_{ij} \in \varphi_{1}(d_{ij})e_{ij}Sd_{ij} \). Since \( \varphi_{1} \) is the isomorphism we have \( \varphi_{1}(d_{ij})e_{ij}S\varphi_{1}(d_{ij}) = \varphi_{1}(d_{ij})e_{ij}Se_{ij} \). Choose \( t_{ij} \in e_{ij}Se_{ij} \) such that \( \varphi_{1}(d_{ij}) = \varphi_{1}(d_{ij})t_{ij} \). If \( \varphi_{1}d_{ij} = \varphi_{1}d_{ij} \), then we put \( t_{ij} = e_{ij} \), \( t_{ij} = e_{ij} \), in particular, \( t_{ij} = e_{ij} \).

**Lemma 5.5.** For any \( i, j, 1 \leq i, j \leq k \), the follows are hold:

1) \( \varphi_{j}(x) = \varphi_{i}(x)t_{ij} \) for any \( x \in Sd_{ij} \);
2) \( t_{ij} \cdot t_{ij} = e_{ij} \);
3) \( \varphi_{1}^{-1}(y) = \varphi_{1}^{-1}(yt_{ij}) \) for any \( y \in Se_{ij} \).

**Proof.** 1) Let \( x \in Sd_{ij} \). Then \( x = sd_{ij} = sd_{ij} \) for some \( s \in S \). Consequently, \( \varphi_{j}(x) = \varphi_{j}(sd_{ij}) = s\varphi_{j}(d_{ij})t_{ij} = \varphi_{i}(d_{ij})t_{ij} = \varphi_{i}(x)t_{ij} \).

2) Since \( t_{ij} \in e_{ij}S \) we have \( e_{ij}t_{ij} = t_{ij} \). Since \( Se_{ij} = S\varphi_{1}(d_{ij}) \) we derive \( e_{ij} = \varphi_{i}(x) \) for some \( x \in Sd_{ij} \). On the statement 1 of this lemma \( \varphi_{j}(x) = \varphi_{i}(x)t_{ij} \) and \( \varphi_{i}(x) = \varphi_{j}(x)t_{ij} \).

Consequently, \( t_{ij} \cdot t_{ij} = e_{ij}t_{ij}t_{ij} = \varphi_{i}(x)t_{ij} \cdot t_{ij} = \varphi_{i}(x)t_{ij} = \varphi_{i}(x) = e_{ij} \).

3) Let \( y \in Se_{ij} = S\varphi_{1}(d_{ij}) \). Then \( \varphi_{1}^{-1}(y) = \varphi_{1}^{-1}(yt_{ij}) \) is \( \varphi_{1}^{-1}(y) = \varphi_{1}^{-1}(yt_{ij}) \).

**Lemma 5.6.** Let \( S_{a_{i}} \cap B \neq \emptyset \) and \( c_{i} \in S_{a_{j}} \) for all \( i, j, 1 \leq i, j \leq k \). Then there exist the idempotents \( g_{1}, \ldots, g_{k} \in R \) such that for any \( i, j, 1 \leq i, j \leq k \), the following conditions hold:

1) \( g_{i} \in S_{a_{i}} \); 2) \( g_{i} \in S_{a_{i}} \);
3) \( g_{i} \in e_{i-1}S \);
4) \( g_{i} = t_{i-1}g_{i-1}t_{i-1} \);
5) \( e_{i-1}g_{i-1} = t_{i-1}g_{i}t_{i-1} \).
6) \(e_{i_1}g_i = t_{i_1}g_1t_{i_1};\)
7) \(g_i = t_{i_1}g_1t_{i_1}.\)

**Proof.** Suppose conditions of this lemma hold. Then \(Sc_i = Sc_j\) \((1 \leq i, j \leq k)\). Let \(Sc = Sc_1\). So \(Sc \subseteq Sd_{i_j}\) for all \(i, j, 1 \leq i, j \leq k\). Choose an idempotent \(g_1\) so that \(S\varphi_1(c) = S\varphi_1\) and \(g_1 \in e_{i_1}S\) \((1 \leq i \leq k)\). It is possible to do it because condition 2 of this theorem hold and by Lemma 5.4 the set \(\{e_{i_j}S \mid 1 \leq j \leq k\}\) is linear ordering by inclusion. Then

\[g_i \in S\varphi_1(c) \subseteq S\varphi_1(d_{i_1}) = Se_{i_1} \quad (1 \leq i \leq k)\]

and for \(i = 1\) conditions 1, 2 of this lemma hold. For \(i = 1\) conditions 6, 7 of this lemma hold trivially, since \(t_{i_1} = e_{i_1}\).

Suppose \(i \geq 2\) and for the idempotents \(g_1, \ldots, g_{i-1}\) conditions 1-7 hold. We put \(g_i = t_{i_1}g_{i_1}t_{i_1}^{-1}\) and prove that \(g_i\) is an idempotent. On Lemma 5.5 (2) \(t_{i_1}^{-1} \cdot t_{i_1} = e_{i_1}\). Since \(g_{i_1} \in Se_{i_1}\) we have \(g_{i_1}e_{i_1} = g_{i_1}\). Consequently,

\[g_i = t_{i_1}g_{i_1}t_{i_1}^{-1} \cdot t_{i_1}^{-1} = t_{i_1}g_{i_1}e_{i_1} = t_{i_1}g_{i_1}t_{i_1}^{-1} = g_i\]

We claim condition 1 of this lemma, that is \(S\varphi_i = S\varphi_i(c)\). Since \(S\varphi_{i-1} = S\varphi_{i-1}(c)\) it follows that \(g_{i-1} = \varphi_{i-1}(x), \varphi_{i-1}(c) = r\varphi_{i-1}\), where \(x \in Sc \subseteq Sd_{i_1}, r \in S\). On Lemma 5.5 (1) we have

\[g_i = t_{i_1}g_{i_1}t_{i_1}^{-1} = t_{i_1}^{-1}\varphi_{i-1}(x) = t_{i_1}^{-1}\varphi_{i-1}(x) = S\varphi_i(c);\]

\[\varphi_i(c) = \varphi_{i-1}(c) = r\varphi_{i-1}(c) = r\varphi_{i-1}(c) = S\varphi_i(c),\]

Since \(g_i \in S\varphi_i(c) \subseteq S\varphi_i(d_{i_1}) = Se_{i_1}\) for any \(j, 1 \leq j \leq k\) condition 2 of this lemma hold.

On the definition of \(g_i\) and the building of \(t_{i_1}^{-1}\) (on the building we have \(t_{i_1}^{-1} \in e_{i_1}S\)) condition 3 hold.

We claim condition 5 of this lemma. On Lemma 5.5 (2) and the equality \(g_{i_1}e_{i_1} = g_{i_1}\), which is true on the suggestion of the induction, we have

\[t_{i_1}g_i = t_{i_1}t_{i_1}^{-1}g_{i_1}t_{i_1}^{-1} = t_{i_1}^{-1}g_{i_1}t_{i_1}^{-1} = e_{i_1}g_{i_1} = e_{i_1}g_{i} = e_{i_1}g_{i_1} = e_{i_1}g_{i_1}.\]

We claim condition 6 of this lemma. Since \(e_{i_1} \in S\varphi_i(d_{i_1})\) and \(g_1 \in S\varphi_1(c)\) it follows that \(e_{i_1} = \varphi_i(y), g_1 = \varphi_i(z), y \in Sd_{i_1}, z \in Sc\). On Lemma 5.3 \(Sd_{i_1} \subseteq \cap\{Sd_{2+r+1+r} \mid 0 \leq r \leq i-2\}\); on the definition of \(Sc\) we have \(Sc \subseteq \cap\{Sd_{j_r} \mid 1 \leq j, r \leq k\}\). Consequently, on Lemma 5.5 (1)

\[e_{i_1}t_{i_1} = \varphi_i(y)t_{i_1} = \varphi_i(y) = \varphi_2(y)t_{i_1} = \varphi_3(y)t_{i_1} = \cdots = \varphi_i(y)t_{i_1} = t_{i_1}t_{i_1}^{-1} \cdot \cdots \cdot t_{i_1} = e_{i_1}t_{i_1}^{-1} \cdot \cdots \cdot t_{i_1};\]

\[g_1t_{i_1} = \varphi_i(z)t_{i_1} = \varphi_i(z) = \varphi_{i-1}(z)t_{i-1} = \varphi_{i-1}(z)t_{i-1} = \varphi_{i-1}(z)t_{i-1} = \cdots \cdot t_{i-1} = g_1t_{i_1} = g_1t_{i_1}.\]

Since condition 4 of this lemma is true for all index, not greater than \(i\) we have

\[t_{i_1}g_1t_{i_1} = e_{i_1}t_{i_1}g_1t_{i_1} = e_{i_1}t_{i_1} = t_{i_1}g_1t_{i_1} = e_{i_1}g_1t_{i_1}.\]

The follow equalities, which are true on Lemma 5.5 (2) and condition 6 of this lemma, imply condition 7:

\[g_1 = e_{i_1}g_1e_{i_1} = e_{i_1}t_{i_1}t_{i_1}g_1t_{i_1}t_{i_1} = e_{i_1}t_{i_1}e_{i_1}g_1t_{i_1}.\]

Thus, we build the idempotents \(g_1, \ldots, g_k \in R\) for which conditions 1-7 of this lemma hold. \(\Box\)
Remark 5.1. The idempotent $g_1$ is chose arbitrarily with the regard for the conditions
\begin{equation}
S\varphi_1(c) = Sg_1 \quad g_1 \in e_i S \text{ for all } i, \ 1 \leq i \leq k.
\end{equation}

In lemmas 5.7–5.9 we suggest that the elements $a_1, \ldots, a_k$ satisfy the conditions: $S_{a_i} \cap B \neq \emptyset$ and $S_{a_i} \cap S_{a_j} \neq \emptyset$ for all $i, j \leq 1 \leq i, j \leq k$. On the set $\{a_1, \ldots, a_k\}$ we define the following binary relation:

\[ a_i \sim a_j \iff c_i, c_j \in S_{a_i} \cap S_{a_j}. \]

On the definition of the sets $S_{c_i}$ ($1 \leq i \leq k$) this relation is an equivalence. For each class of this equivalence we build the idempotent $g_i$ satisfying Lemma 5.6.

Lemma 5.7. $sB \models \Phi_1(b, d)$, where $b = (b_1, \ldots, b_k) \in B^k$, $b_i = \varphi_i^{-1}(\varphi_i(a_i)g_i)$ ($1 \leq i \leq k$).

\textbf{Proof.} Let $\langle i, j, n, m \rangle \in L_3$. Suppose $a_i \sim a_j$. We claim that $sB \models nb_i = mb_j$ for $b_r = \varphi_r^{-1}(\varphi_r(a_r)g_r)$, $r \in \{i, j\}$. Let $i = j + l$, where $l \geq 0$. Since $na_i = ma_j \in S_{a_i} \cap S_{a_j}$ we have $na_i \in S_{d_{ij}} = S_{d_{i+l}}$. On Lemma 5.3

\[ S_{d_{j+l}} \subseteq \bigcap \{S_{d_{j+r+1+j+r}} \mid 0 \leq r \leq l - 1\} \]

for $l \geq 1$. Consequently, on Lemma 5.5 (1,3) and Lemma 5.6 (4) we have

\[ mb_j = m\varphi_j^{-1}(\varphi_j(a_j)g_j) = \varphi_j^{-1}(\varphi_j(ma_j)g_j) = \varphi_j^{-1}(\varphi_j(ma_j)g_j + 1) = \]

\[ = \varphi_j^{-1}(\varphi_j(ma_j)g_j + 1) = \ldots = \varphi_j^{-1}(\varphi_j(ma_j)g_j) = \varphi_j^{-1}(\varphi_j(na_i)g_i) = n\varphi_j^{-1}(\varphi_j(a_i)g_i) = nb_i, \]

that is $mb_j = nb_i$. If $l = 0$ then the equality $mb_j = nb_i$ is obviously.

Suppose $a_i \sim a_j$ is false. Let, for instance, $c_i \notin S_{a_j}$. We claim that $S_{a_i} \cap S_{a_j} \subseteq B$. Let $a \in S_{a_i} \cap S_{a_j}$. On the definition of $S_{c_i}$ we have $S_{c_i} \subseteq S_{a_i} \cap S_{a_j}$. On Proposition 2.5

\[ \text{either } S_a \subseteq S_{c_i}, \text{ or } S_{c_i} \subseteq S_a \]

If $S_{c_i} \subseteq S_a$, then $c_i \in S_{a_j}$ which is contradicts with the suggestion. Consequently, $S_{a_i} \cap S_{a_j} \subseteq S_{c_i} \subseteq B$.

Since $na_i = ma_j$ we have $na_i, ma_j \in S_{a_i} \cap S_{a_j} \subseteq S_{c_i}$. Consequently, $ma_j \in B$, $ma_j \subseteq S_{a_j}$ and $na_i \subseteq S_{c_i}$. Thus,

\[ \varphi_j(na_i) = S\varphi_j(c_i) = Sg_i, \quad \varphi_j(ma_j) = S\varphi_j(c_j) = Sg_j, \]

\[ nb_i = n\varphi_j^{-1}(\varphi_j(a_i)g_i) = \varphi_j^{-1}(\varphi_j(na_i)g_i) = \varphi_j^{-1}(\varphi_j(a_i)g_i) = na_i. \]

Similarly, $mb_j = ma_j$. Hence $nb_i = na_i = ma_j = mb_j$, and lemma is proved.

Lemma 5.8. $sB \models \Phi_2(b, d)$, where $b = (b_1, \ldots, b_k) \in B^k$, $b_i = \varphi_i^{-1}(\varphi_i(a_i)g_i)$ ($1 \leq i \leq k$).

\textbf{Proof.} Let $\langle i, j, n, m \rangle \in L_3$. We claim that $sB \models nb_i = md_j$ for $b_i = \varphi_i^{-1}(\varphi_i(a_i)g_i)$. On the definition of $S_{c_i}$ we have $md_j \in S_{c_i}$. Since $na_i = md_j$ it follows that $na_i \in S_{c_i}$. On Lemma 5.6 (1) $S\varphi_i(c_i) = Sg_i$. Consequently, $\varphi_i(na_i)g_i = \varphi_i(na_i)$ and

\[ nb_i = n\varphi_i^{-1}(\varphi_i(a_i)g_i) = \varphi_i^{-1}(\varphi_i(na_i)g_i) = \varphi_i^{-1}(\varphi_i(na_i)) = na_i = md_j, \]

that is $nb_i = md_j$.

On Remark 5.1 for each class of the equivalence by the relation ~ there exists some freedom of the choice of one of the idempotents $g_i$, which were build for this class. Let $g_{t_1}, \ldots, g_{t_s}$ be all such idempotents.

Lemma 5.9. The idempotents $g_{t_1}, \ldots, g_{t_s}$ can be choose such that for them it is held condition (5.2) with the change $g_1$ on $g_{t_t}$ ($1 \leq t \leq s$) and $sB \models \bigwedge_{i=1}^s \Phi_i(b, d)$, where $b = (b_1, \ldots, b_k) \in B^k$, $b_i = \varphi_i^{-1}(\varphi_i(a_i)g_i)$ ($1 \leq i \leq k$).
Proof. Let \( \langle i, j, n, m \rangle \in L_3 \). Note that \( \varphi_i(b_i) = \varphi_i(\varphi_i^{-1}(\varphi_i(a_i)g_i)) = \varphi_i(a_i)g_i \in Sg_i \), that is \( b_i \in Sc_i \subseteq B \). Furthermore, \( b_i \in Sa_i \). Add an element \( b_i \) to the set \( \{ d_1, \ldots, d_r \} \). We can consider that \( b_i = d'_i \in \{ d_1, \ldots, d_r \} \). Add an element \( b_i \) to the set \( \{ a_1, \ldots, a_r \} \). We can consider that \( b_i = a'_i \in \{ a_1, \ldots, a_k \} \). Similarly, we can consider that \( b_i = d'_i \in \{ d_1, \ldots, d_r \} \) and \( b_j = a'_j \in \{ a_1, \ldots, a_k \} \). Denote the formula \( \Phi_i(\bar{x}, \bar{d}) \land mx_i = nd'_i \land mx_j = md'_j \) by \( \Phi_1(\bar{x}) \) once again. Since \( b_i \in Sc_i \), the sets \( Sc_i \) and \( Sd_{ij} \) are not changed, where \( a_i \sim a_j \). On the same reason the sets \( Sc_j \) and \( Sd_{ji} \) are not changed too, where \( a_i \sim a_j \). Since \( a'_i \in Sa_i \), it follows that \( \varphi_i' = \varphi_i | Sa_i' \), \( \varphi_i'(a'_i) \in Sg_i' \) and \( b'_i = \varphi_i^{-1}(\varphi_i'(a'_i)g'_i) = a'_i = a_i \) under any choice of idempotents \( g'_i \) satisfying Lemma 5.6. Similarly, \( b'_j = b_j \). If \( na_i = md'_i \) and \( ma_j = nd'_i \) then \( nb'_i = n\varphi_i^{-1}(\varphi_i(a_i)g'_i) = a'_i = na_i \) under any choice of idempotents \( g'_i \) satisfying Lemma 5.6; similarly, \( mb'_j = ma_j \), that is \( nb'_i \neq mb'_j \). So we will consider only the case: \( na_i \neq md'_i \) or \( ma_j \neq nd'_i \). Without the restriction of generality we can consider \( ma_j \neq nd'_i \). Denote the formula \( \Phi_4(\bar{x}, \bar{d}) \land \neg mx_j = nd'_i \) by \( \Phi_4(\bar{x}, \bar{d}) \) once again. If \( na_i \neq md'_i \) or \( ma_j \neq nd'_i \) denote the formula, getting from the formula \( \Phi_3(\bar{x}, \bar{d}) \) by cancellation of subformulae \( \neg mx_j = nx_i \) and \( \neg nx_i = mx_j \) by \( \Phi_3(\bar{x}, \bar{d}) \). The same way we do with all \( \langle i, j, n, m \rangle \in L_3 \). As a result, in the set \( L_3 \) it is remained only such collections of \( \langle i, j, n, m \rangle \), for which \( nb'_i \neq mb'_j \) under any choice of the idempotents \( g'_i \) satisfying Lemma 5.6. If we will choose the idempotents \( g_{i1}, \ldots, g_in \) (under which it is built all another \( g_i \), \( 1 \leq i \leq k \)) satisfying (5.2) with substitution \( g_i \) on \( g_{it} \) (\( 1 \leq t \leq s \)) such that \( sB = \Phi_4(\bar{x}, \bar{d}) \) it follows that as it was note above and on Lemmas 5.7 and 5.8, \( sB = \bigcup_{i=1}^{4} \Phi_i(\bar{b}, \bar{d}) \), where \( \bar{b} = \langle b_1, \ldots, b_k \rangle \in B^k \).

Without loss of generality we can consider that \( g_{i1} = g_1 \). Show, how can be choose the element \( g_1 \) (another idempotents \( g_{i2}, \ldots, g_in \) are chosen similarly). Let \( K_j = \{ 1, 2, \ldots, k \} = \{ i \mid a_i \sim a_1 \} \) and Lemma 5.3 holds with substitution \( k \) on \( k_i \). Note that \( \langle i, j, n, m \rangle \in L_4 \), \( i \in K_j \) and \( md_j \in Sa_i \) imply \( nb_i \neq md_j \) under any choice \( g_i \), satisfying conditions (5.2). For all \( i \in K_j \) and \( j \in K_1 \) we will build \( e_{ij} \) and \( t_{ij} \) satisfying Lemma 5.4 with substitution \( c_{ij} \) on \( e_{ij} \), Lemma 5.5 with substitution \( t_{ij} \) on \( t_{ij} \) and the condition

\[
\varphi_i(na_i)t'_{i-1,1} \cdot t'_{i-1,1-2} \cdot \ldots \cdot t'_{i,1+i,1}e_{ij} \neq \varphi_i(md_i)t'_{i-1,1} \cdot t'_{i-1,1-2} \cdot \ldots \cdot t'_{i,1+i,1}e_{ij},
\]

where \( md_i \in Sa_i \), \( l \geq i \), \( \langle l, t, n, m \rangle \in L_4 \). Suppose the elements \( e_{k1}, e_{k1-1,1}, \ldots, e_{i+1,1} \) and \( t'_{k1}, t'_{k1-1,1}, \ldots, t'_{i,1+i} \) are built for all \( j \in K_1 \). Let

\[Y_i = \{ x \in R \mid \varphi_i(na_i)t'_{i-1,1} \cdot t'_{i-1,1-2} \cdot t'_{i,1+i,1}x = \varphi_i(md_i)t'_{i-1,1} \cdot t'_{i-1,1-2} \cdot t'_{i,1+i,1}x, \; md_i \in Sa_i, \; \langle l, t, n, m \rangle \in L_4 \}
\]

\[Y_i = \bigcup \{ Y_i^l \mid l \geq i \}.
\]

Let use prove that \( f_i \not\subseteq Y_i \). Let \( f_i \in Y_i \). If \( l = i \) then \( \varphi_i(na_i) = \varphi_i(na_i)f_i = \varphi_i(ma_j) = \varphi_i(ma_j) \), that is \( na_i = ma_j \), a contradiction. On the definition, \( t'_{i,1+i} \in Sc_{i+1} \subseteq Sf_i \), that is \( t'_{i,1+i}f_i = t'_{i,1+i} \). Consequently, \( \varphi_i(na_i)t'_{i-1,1} \cdot t'_{i-1,1-2} \cdot \ldots \cdot t'_{i,1+i,1} = \varphi_i(md_i)t'_{i-1,1} \cdot t'_{i-1,1-2} \cdot \ldots \cdot t'_{i,1+i,1} \). Then \( \varphi_i(na_i)t'_{i-1,1} \cdot t'_{i-1,1-2} \cdot \ldots \cdot t'_{i,1+i,1} = \varphi_i(md_i)t'_{i-1,1} \cdot t'_{i-1,1-2} \cdot \ldots \cdot t'_{i,1+i,1} \). On Lemma 5.5 (2) \( \varphi_i(na_i)t'_{i-1,1} \cdot t'_{i-1,1-2} \cdot \ldots \cdot t'_{i,1+i,1}e_{ij} = \varphi_i(md_i)t'_{i-1,1} \cdot t'_{i-1,1-2} \cdot \ldots \cdot t'_{i,1+i,1}e_{ij} \) which contradicts with the choice of the element \( e_{i+1,1} \). Let \( Y_i \neq \emptyset \). In view of the axiomatizability of the class \( \mathcal{R} \) we have \( Y_i = \bigcup \{ k_sR \mid 1 \leq s \leq t \} \) for some \( t > 0, \; k_s \in S \) \( 1 \leq s \leq t \). We put \( X_i = \bigcup \{ k_sS \mid 1 \leq s \leq t \} \). Clearly,

\[h \in Y_i \iff h \in X_i\]
for any idempotent \( h \in R \). Consequently, \( f_i \not\in X_i \). On Lemmas 5.3 and 5.4 the set \( \{ Se_{ij} \mid j \in K_1 \} \) is linearly ordered by inclusion. Let

\[
Se_{ij_1} \subseteq Se_{ij_2} \subseteq \ldots \subseteq Se_{ij_r} \subseteq Sf_i,
\]

where \( \{ j_1, \ldots, j_r \} = \{ j \mid j \in K_1 \} \). Let \( e_{ij_1}' = f_i \). Suppose the idempotents \( e_{ij_1}', e_{ij_2}', \ldots, e_{ij_r}' \) are already built, moreover \( Se_{ij_i}' = Se_{ij_i} \). Since \( e_{ij_i}' \not\in X_i \), by condition 2 of this theorem there exists an idempotent \( e_{ij_{i-1}}' \) such that \( Se_{ij_{i-1}}' = Se_{ij_{i-1}}, e_{ij_{i-1}}' S \subseteq e_{ij_i}' S \) and \( e_{ij_{i-1}}' \not\in X_i \). By the idempotents \( e_{ij_i}' \) we build \( t_{ij}' \in S \), satisfying conditions of Lemma 5.5. On the building, \( g_1 \in Se_{ij_1} = Se_{ij_1}' \), where \( j \in K_1 \). Then on condition 2 of this theorem there exists an idempotent \( g_1' \) such that \( Sg_1' = Sg_1, g_1'S \subseteq e_{ij_i}' S \) for all \( j \in K_1 \) and \( g_1' \not\in X_1 \). Similarly we build the idempotents \( g_{i_1}, \ldots, g_{i_s} \). Another idempotents \( g_i \) we build such that they satisfy Lemma 5.6. The elements \( b_i \) are defined as previously:

\[
b_i = \varphi_i^{-1}(\varphi_i(a_i)g_i'), \quad \text{where} \ i \in K_1. \]

Then, as it is noticed above, \( sB = \bigvee_{i=1}^k \Phi_i(b_1, \ldots, b_k) \).

Let us prove that \( nb_i \neq md_j \), where \( \langle i, j, n, m \rangle \in L_2 \), \( i \in K_1 \). Let \( nb_i = md_j \). Then \( \varphi_i(na_i)g_i' = \varphi_i(md_j) \). On Lemma 5.6 (2), \( \varphi_i(na_i)t_{ii_1} \cdot g_i' \cdot t_{ii_1}^{-1} = \varphi_i(md_j) \cdot t_{ii_1}^{-1} \). On Lemma 5.6 (2), \( \varphi_i(na_i)t_{ii_1} \cdot g_i' \cdot t_{ii_1}^{-1} = \varphi_i(md_j) \cdot t_{ii_1}^{-1} \). Continuing this process, we have \( \varphi_i(na_i)t_{ii_1} \cdot t_{ii_1}^{-1} \cdot \cdots \cdot t_{jj_1}^{-1} \cdot g_i' = \varphi_i(md_j) \cdot t_{ii_1}^{-1} \cdot \cdots \cdot t_{jj_1}^{-1} \cdot g_1' \). Consequently, \( g_i' \in X_1 \) which contradicts with the building of the element \( g_1' \). Thus, \( sB = \bigvee_{i=1}^k \Phi_i(b_i, d_i) \).

Let \( \{ a_1, \ldots, a_k \} \) be an arbitrary maximum subset of the set \( \{ a_1, \ldots, a_k \} \) such that

\[
(5.3) \quad Sa_i \cap B = \emptyset, \quad Sa_i \cap Sa_j \neq \emptyset \quad \text{for any} \ i, j \in \{ i_1, \ldots, i_s \},
\]

\( b^1 \) be an arbitrary element of \( B \). On the definition of the regular polygon there exist an idempotent \( f^1 \in R \) and an isomorphism \( \psi_1 : sB^1 \to sSf^1 \). On Lemma 5.2 there is an idempotent \( f \in R \) such that \( Sf \subseteq Sf^1 \) and \( sSf \) is a minimum by inclusion polygon. We put: \( Sb = S\psi_1^{-1}(f_1), \psi = \psi_1|_{sb} \). Since \( Sa_i \cap Sa_j \neq \emptyset \) for any \( i, j \in \{ i_1, \ldots, i_s \} \), on Lemma 5.1 (1) there exists \( a^1 \in S \) such that \( a^1 \in \bigcap\{ Sa_i \mid i \in \{ i_1, \ldots, i_s \} \} \). In view of the regularity of the polygon \( S \) there exist an idempotent \( e^1 \in R \) and an isomorphism \( \varphi : sSa^1 \to sSe^1 \). On Lemma 5.2 there exists an idempotent \( e \in R \) such that \( Se \subseteq Se^1 \), the polygon \( sSe \) is minimum and the right ideal \( eS \) is minimum for the principal right ideals of the monoid \( S \), generated by idempotents. Suppose \( B' = Sa = S\varphi^{-1}(e) \), where \( a \in A \). Then \( Sa_i \cap B' \neq \emptyset \) for any \( i \in \{ i_1, \ldots, i_s \} \). Replacing \( B' \) by \( B' \) in all previous arguments, which concern to the proof of sufficiency, we have in particular \( b_i = \varphi_i^{-1}(\varphi_i(a_i)g_i') \in B', \) where \( i \in \{ i_1, \ldots, i_s \} \). Let \( \alpha \) be an arbitrary isomorphism from \( sSa \) to \( sSe \) (for instance, \( \alpha = \varphi|_{sa} \)). Denote the element \( \psi^{-1}(\alpha(b_i)f) \) by \( b_i \). Clearly, \( b_i \in SB \subseteq B (i \in \{ i_1, \ldots, i_s \} \). For the element \( a_i \in \{ a_1, \ldots, a_k \} \) such that \( Sa_i \cap B \neq \emptyset \), we build the element \( b_i \in B \) as in Lemma 5.9.

Let us prove that \( sB = \bigvee_{i=1}^k \Phi_i(b, d), \) where \( b = \langle b_1, \ldots, b_k \rangle \in B^k \).

If \( \langle i, j, n, m \rangle \in L_2 \) then \( Sa_i \cap B \neq \emptyset \) and on Lemma 5.9 \( sB = nb_i = md_j \).

If \( \langle i, j, n, m \rangle \in L_1 \) and \( Sa_i \cap B \neq \emptyset \) then \( Sa_i \cap Sa_j \neq \emptyset \). On condition 1 of this theorem \( Sa_i \cap B \neq \emptyset \); on Lemma 5.9 \( sB = nb_i = md_j \).
Suppose $\langle i, j, n, m \rangle \in L_1$, $Sa_i \cap B = \emptyset$, $Sa_j \cap B = \emptyset$. Then on Lemma 5.9 and choice $b^i, b^j \in B'$ we have $sSa_i \models nb^i = mb^j$. Consequently,

$$nb_i = n\psi^{-1}(\alpha(b^i)f) = \psi^{-1}(\alpha(nb^i)f) = \psi^{-1}(\alpha(mb^j)f) = m\psi^{-1}(\alpha(b^j)f) = mb_j.$$

If $\langle i, j, n, m \rangle \in L_3$ or $\langle i, j, n, m \rangle \in L_4$ and $Sa_i \cap Sa_j \neq \emptyset$, $Sa_i \cap B \neq \emptyset$, $Sa_j \cap B \neq \emptyset$ then on Lemma 5.9 $nb_i \neq mb_j$ or $nb_i \neq md_j$, accordingly.

Let $i$, $1 \leq i \leq k$, be such that $Sa_i \cap B = \emptyset$; the set $\{a_1, \ldots, a_i\}$ be a maximum subset of the set $\{a_1, \ldots, a_k\}$ with properties (5.3), where $i_1 = i$. Above for the set $\{a_1, \ldots, a_i\}$ it is defined the elements $a \in \cap \{Sa_j \mid j \in \{i_1, \ldots, i_s\}\}$, $e \in S$ and it is chose an arbitrary isomorphism $\alpha : sSa \rightarrow sSe$, moreover the polygon $sSe$ is minimum and the right ideal $eS$ is minimum for the principal right ideals of the monoid $S$ generated by idempotents.

Suppose $\langle i, j, n, m \rangle \in L_3$, $j \in \{i_1, \ldots, i_s\}$. On Lemma 5.9 $nb_i \neq mb^j$. Since $\alpha(nb^i), \alpha(mb^j) \in Se$ we have $e \not\in X = \{x \mid \alpha(nb^i)x = \alpha(mb^j)x\}$. Clearly, $eS \subseteq eS$. Since $Sef \subseteq Se$, on condition 2 of this theorem, $efS = gS$ for some idempotent $g \in S$. In view of the minimality of the right ideal $eS$ and the inclusion $gS \subseteq eS$ we have the equality $eS = gS = efS$, that is $e = efr$ for some $r \in S$. If $f \in X$ then $ef \in X$ and $e \in X$. Consequently, $f \not\in X$ and

$$nb_i = n\psi^{-1}(\alpha(b^i)f) = \psi^{-1}(\alpha(nb^i)f) \neq \psi^{-1}(\alpha(mb^j)f) = m\psi^{-1}(\alpha(b^j)f) = mb_j.$$ 

Suppose $\langle i, j, n, m \rangle \in L_3$ and $j \not\in \{i_1, \ldots, i_s\}$ or $\langle i, j, n, m \rangle \in L_4$. Define a homomorphism $\alpha_t : sSe \rightarrow sSe$ as follows: $\alpha_t(se) = sete$ for any $s \in S$. In view of the minimality of the polygon $sSe$ the equality $Se = Se$ hold, that is $e = lete$ for some $l \in S$ and $\alpha_t$ is an epimorphism. The minimality of the right ideal $eS$ implies the equality $eS = eteS$, that is $e = etel'$ for some $l' \in S$ and $\alpha_t$ is an isomorphism. Note that $\alpha_t \neq \alpha_s$ implies $\alpha_t(c) \neq \alpha_s(c)$ for any $c \in Se$. Really, let $sete = \alpha(c) = \alpha_at = cese$. Since $Se = Se$, that is $e = cles$ for some $l \in S$, we have $ete = clecete = cese$ as it contradicts with suggestion. On condition 3 of this theorem, $|eSe| \geq \omega$. Consequently, there exists $t_i$ ($i \in \omega$) such that $\alpha_{t_i} \neq \alpha_{t_j} (i \neq j)$. In view of the minimality of the polygon $Sf$ the equality $Sef = Sf$ hold, that is

(5.4) 

$$f = uef$$

for some $u \in S$. For $i \in \{i_1, \ldots, i_s\}$ and $n \in S$ we put:

$$X^*_{i_n} = \{x \in Se \mid x = \psi(mb_j)ue, \text{ where } mb_j \in Sb, \langle i, j, n, m \rangle \in L_3\},$$

$$Y^*_{i_n} = \{x \in Se \mid x = \psi(md_j)ue, \text{ where } md_j \in Sb, \langle i, j, n, m \rangle \in L_4\}.$$

If $i \in \{i_1, \ldots, i_s\}$ and $n \in S$ such that either the set $X^*_{i_n}$ or the set $Y^*_{i_n}$ is not determined, then we assume either $X^*_{i_n} = \emptyset$ or $Y^*_{i_n} = \emptyset$ accordingly. Since the sets $X^*_{i_n}$ and $Y^*_{i_n}$ are finite for all $i \in \{i_1, \ldots, i_s\}$ and $n \in S$, but the set of the different isomorphisms $\alpha_i$ is infinite, then there exists $t \in S$ such that $\alpha_i \alpha_i \alpha_i \alpha_i \not\in X^*_{i_n}$ for any $i \in \{i_1, \ldots, i_s\}$, $n \in S$. For the isomorphism $\alpha$ we will take the isomorphism $\alpha : sSa \rightarrow sSe$, which we denote by $\alpha$. Then $\alpha(nb^i) \not\in X^*_{i_n}$ for any $i \in \{i_1, \ldots, i_s\}$, $n \in S$.

Suppose $\langle i, j, n, m \rangle \in L_3$ and $j \not\in \{i_1, \ldots, i_s\}$. If $nb_i = mb_j$ then $\psi^{-1}(\alpha(nb^i)f) = mb_j$ by the definitions of $b_i$ and $mb_j \in Sb$, that is

$$\alpha(nb^i)e = \alpha(nb^i)f = \psi(mb_j) = \psi(mb_j)^{-1}.$$ 

The equality (5.4) implies $\alpha(nb^i)e = \psi(mb_j)ue$. In view of the minimality of the right ideal $eS$ and the equality $efS = eS$ we have $\alpha(nb^i)e = \alpha(nb^i) = \psi(mb_j)ue$, that is
\[ \alpha(nb^i) \in X^n_i. \] We have a contradict with the choice of the isomorphism \( \alpha. \) Consequently, \( nb^i \neq mb^j. \)

If \( \langle i, j, n, m \rangle \in L_4 \) then the inequality \( nb^i \neq md^j \) is proved similarly. Theorem is proved. \( \square \)

**Corollary 5.1.** The class \( \mathcal{R} \) for the regular polygons over the infinite group is axiomatizable and model complete.

**Proof** follows from Theorem 5.1 and Corollary 4.3. \( \square \)

### § 6. Completeness of Class for Regular Polygons

In this paragraph we investigate the monoids with the complete class of regular polygons. It is known that the model completeness of the class \( \mathcal{R} \) for regular polygons implies the completeness of this class (Lemma 6.1). Theorems 6.1 and 6.2 assert that the completeness of the class \( \mathcal{R} \) implies the model completeness of this class if we put some conditions on the monoid: the class \( \mathcal{R} \) satisfies the condition of formula definability of isomorphic orbits or the monoid is linearly ordered and has depth 2. Theorems 6.3 and 6.4 assert that this conditions in Theorems 6.1 and 6.2 are essential.

If for element \( a \in S_A \) and \( b \in S_B \) there exists an isomorphism \( f: S_A \rightarrow S_B \) such that \( f(a) = b \), then this fact will be denoted by \( S_A \sim S_B. \)

We will say that a class \( \mathcal{R} \) satisfies the condition of formula definability of isomorphic orbits if for each idempotent \( e \in R \) there exists a formula \( \Phi_e(x) \) such that for any regular polygon \( S_A \) and any \( a \in S_A \) the following is true:

\[ S_A \models \Phi_e(a) \iff S_A \sim S_e. \]

**Theorem 6.1 [Ov2].** The following conditions are equivalent:

1) the axiomatizable class \( \mathcal{R} \) is model complete and satisfies the condition of formula definability of isomorphic orbits;

2) the axiomatizable class \( \mathcal{R} \) is complete and satisfies the condition of formula definability of isomorphic orbits;

3) the semigroup \( R \) is a rectangular band of infinite groups, and the set \( I(R) \) is finite certainly.

**Proof** contains some lemmas.

**Lemma 6.1.** If the class of regular polygons \( \mathcal{R} \) is model complete, then \( \mathcal{R} \) is complete.

**Proof.** If \( S_A, S_B \in \mathcal{R} \), then the polygon \( S_C \equiv S_A \cup S_B \) is also regular. In view of model completeness of the class \( \mathcal{R} \) the polygons \( S_A \) and \( S_B \) are elementary submodels of the polygon \( S_C \). Then \( \text{Th}(S_A) = \text{Th}(S_C) = \text{Th}(S_B) \), that is \( S_A \equiv S_B \). \( \square \)

**Lemma 6.2.** If the class \( \mathcal{R} \) is complete and satisfies the condition of formula definability of isomorphic orbits then for any idempotent \( e \in R \) and any element \( a \in R \) the condition \( Sa \subseteq Se \) implies the equality \( Sa = Se \).

**Proof.** Let \( e \) be an idempotent belonging to \( R \), and formula \( \Phi_e(x) \) be such that for any \( S_A \in \mathcal{R} \) and any \( a \in S_A \) the following is true:

\[ S_A \models \Phi_e(a) \iff S_A \sim S_e. \]

Assume we have \( Sa \subseteq Se \) for some element \( a \in R \). Since \( a \in T \), the polygon \( S_A \) is regular. In view of completeness of the class \( \mathcal{R} \) we get \( S_A \equiv S_e \). Since \( S_e \models \Phi_e(e) \) then \( S_e \models \exists x \Phi_e(x) \) and, consequently, \( S_e \models \exists x \Phi_e(x) \). Let \( b \) be a realization of the
formula $\Phi_e(x)$ in the polygon $sSa$. Since $Sa \subset Se$, then $b \neq e$ and $Sb \subset Se$. Then $be = b$ and

$$sSe \models \Phi_e(b) \land be = b \land \Phi_e(e).$$

Consequently,

$$sSe \models \Phi_e(b) \land \exists y \ (by = b \land \Phi_e(y)).$$

Denote by $\Psi(x)$ the formula $\Phi_e(x) \to \exists y \ (by = x \land \Phi_e(y))$. Obviously, $sSe \models \Psi(b)$.

We will show that $sSe \not\models \Psi(e)$. Since $sSe \models \Phi_e(e)$, it is enough to check that

$$sSe \not\models \exists y \ (by = e \land \Phi_e(y))$$
or

$$sSe \models \forall y \ (\Phi_e(y) \to by \neq e).$$

Assume there exists an element $s \in Se$ such that $sSe \models \Phi_e(c)$ and $b \cdot c = e$. Since $sSe \models \Phi_e(c)$, then $sSe \cong sSe$. Let $f : sSe \to sSe$ be an isomorphism for which $f(c) = e$. Since $c \in S$ then $ce = c$. From the equality $b \cdot c = e$ we get $cbe = c$. Then $cbe = e$, but $be = b$. Consequently, $cb = e$. So $e \in Sb$ which contradicts with the condition $Sb \subset Se$. Thus $sSe \not\models \Psi(e)$ and consequently, $sSe \not\models \forall x \Psi(x)$.

On induction we will build now a regular polygon $sA$, on which the formula $\forall x \Psi(x)$ is true.

1. We set $sA_0 = sSe$. Obviously, the polygon $sA_0$ is regular and connected.

2. If a regular connected polygon $sA_k$ is built, then we denote by $X_k$ the set

$$\{x \mid x \in A_k, \ sSx \cong sSe, \ sA_k \models \forall y \ (\Phi_e(y) \to by \neq x)\}.$$

For each $x \in X_k$ we consider the regular polygon $sSa_x$ such that $sSa_x \cong sSe$. Herewith without loss of generality we can assume that $Sa_x \cap Sa_y = \emptyset$ for $x \neq y$ and $Sa_x \cap A_k = \emptyset$ for any $x \in X_k$. Let $f_x : sSa_x \to sSe$ be an isomorphism for which $f(a_x) = e$. We set

$$s\tilde{A}_{k+1} = \left( sA_k \bigcup \left( \bigcup_{x \in X_k} sSa_x \right) \right) / \theta,$$

where $\theta$ is a congruence generated by the set $\{(x, ba_x) \mid x \in X_k\}$. Since $Sb = sS(be) \cong sSe$ and $sSa_x \cong sSe$, then

$$sS(ba_x) \cong sS(be) \cong sSe.$$

By the definition of the set $X_k$ for any $x \in X_k$ it is executed $sSx \cong sSe$. Since there exists an isomorphism $f_x : sSx \to sSba_x$ such that $f_x(x) = b \cdot a_x$, then

$$\langle x_0, y_0 \rangle \in \theta \iff x_0 = y_0, \mathrm{or} \ x_0 = sx \mathrm{\ and} \ y_0 = sba_x \mathrm{\ for\ some} \ s \in S, \ x \in X_k.$$

Since for any $x \in X_k$ we have $x \in A_k$ and $ba_x \in Sa_x$, then $x \nmid ba_x$. Consequently, $sx \nmid sba_x$ and $\theta$ is an amalgam congruence. Since $sA_k, sSa_x$ is a regular polygon, then on Proposition 2.21 the polygon $s\tilde{A}_{k+1}$ is regular.

We will show that $c \sim d$ for any $c, d \in sA_{k+1}$. Let $c_1, d_1$ be some elements, for which $c = c_1/\theta, \ d = d_1/\theta$. If $c_1, d_1 \in A_k$, then $c \sim d$ in view of connectivity of the polygon $sA_k$. If $c_1 \in A_k$ and $d_1 \in Sa_x$, then $d_1 \sim a_x \sim ba_x$ and in view of connectivity of the polygon $sA_k, c_1 \sim x$. Then $c_1/\theta \sim x/\theta$ and $d_1/\theta \sim ba_x/\theta$. Since $x/\theta = ba_x/\theta$, then $c_1/\theta \sim d_1/\theta$, that is $c \sim d$. But if $c_1 \in Sa_x$ and $d_1 \in Sa_y$ for some $x, y \in X_k$ then for any $b_1 \in A_k$ it is executed $c_1/\theta \sim b_1/\theta$ and $d_1/\theta \sim b_1/\theta$. By the transitivity of relation $\sim$ we get that $c_1/\theta \sim d_1/\theta$, that is i.e. $c \sim d$. Thereby, $s\tilde{A}_{k+1}$ is a connected polygon.
We will show that for any \( x_0 \in X_k \) it is executed \( sA_{k+1} \models \Psi (x_0/\theta) \). Since the polygons \( sA_k \) and \( sSa_{x_0} \) are connected and \( \theta \) is an amalgam congruence, then the restrictions on \( sA_k \) and \( sSa_{x_0} \) of a natural homomorphism corresponding to \( \theta \) are isomorphic embeddings. Since \( sS(a_{x_0}) \cong sSe \), then

\[
S (x_0/\theta) \cong sSe, \quad sS \cong s\bar{A}_{k+1} = \Phi (x_0/\theta), \quad s\bar{A}_{k+1} = \Phi (x_0/\theta).
\]

Since \( b(a_{x_0}/\theta) = (ba_{x_0})/\theta = x_0/\theta \), then \( s\bar{A}_{k+1} \models b(a_{x_0}/\theta) = x_0/\theta \) and \( s\bar{A}_{k+1} \models \Phi (x_0/\theta) \).

Since \( sA_k \) is isomorphically embeddable in \( s\bar{A}_{k+1} \), then there exists a polygon \( sA_{k+1} \) such that \( sA_k \subseteq sA_{k+1} \) and \( sA_{k+1} \cong s\bar{A}_{k+1} \). Then for any \( x_0 \in X_k \) it is executed \( sA_{k+1} \models \Psi (x_0/\theta) \).

As a result of the induction process we get a chain of regular polygons

\[
S A_0 \subseteq sA_1 \subseteq \ldots \subseteq sA_k \subseteq sA_{k+1} \subseteq \ldots
\]

We set \( sA = \bigcup_{i \in \omega} sA_i \). As the class \( \mathcal{R} \) is closed relative to unions, the polygon \( sA \) is regular.

We will show that \( sA \models \forall x \Psi (x) \). Notice that since for any regular polygon \( s\bar{A} \) and any \( a \in \bar{A} \) it is executed

\[
\bar{A} \models \Phi (a) \iff sSa \cong sSe,
\]

then for any element \( a \in A \) and for any subpolygon \( sB \), containing the element \( a \) and any extended polygon \( sC \) we have

\[
sB \models \Phi (a), \quad sC \models \Phi (a).
\]

Since

\[
\Psi (x) = \Phi (x) \rightarrow \exists y (by = x \land \Phi (y)),
\]

then from \( sA \models \Psi (a) \) we get \( sC \models \Psi (a) \) for any extended regular polygon \( sC \).

Assume there exists an element \( d \in A \) such that \( sA \not\models \Psi (d) \). Choose the number \( k \in \omega \), for which \( d \in A_k \). Since \( sA \not\models \Psi (d) \) and \( sA_k \) is a subpolygon of the polygon \( sA \), then \( sA_k \not\models \Psi (d) \), that is \( sA_k \models \Phi (d) \) and \( sA_k \models \forall y (\Phi (y) \rightarrow by \neq d) \). Then \( d \in X_k \), but by the construction of the polygon \( sA_{k+1} \) we get \( sA_{k+1} \models \Psi (d) \) and, consequently, \( sA \models \Psi (d) \). Thereby, \( sA \models \forall x \Psi (x) \).

Since \( sS \not\models \forall x \Psi (x) \), then one get a contradiction to completeness of the class \( \mathcal{R} \). \( \square \)

**Lemma 6.3.** If \( \text{ld}(R) = 1 \), then the class \( \mathcal{R} \) axiomatizable if and only if the set \( I(R) \) is finite.

**Proof.** Assume the class \( \mathcal{R} \) is axiomatizable. Then on Theorem 4.1 the set \( \{ x \in R \mid 1 \cdot x = x \} \) is equal to \( R \) and finitely generated as a right ideal of the semigroup \( R \). Since by Proposition 2.6 the semigroup \( R \) is a rectangular band of groups, then principal right ideals in \( R \) are pairwise disjoint and \( R \) is their union. Since the set of all principal right ideals in \( R \) has the same cardinality as the set \( I(R) \), we conclude that the set \( I(R) \) is finite.

For a proof of sufficiency we notice that the number of principal right ideal coincides with the number of element of the set \( I(R) \) and, consequently, is finite. Then, obviously, the conditions of Theorem 4.1 are satisfied and the class \( \mathcal{R} \) axiomatizable. \( \square \)

**Lemma 6.4.** If the class \( \mathcal{R} \) is complete, \( sA \in \mathcal{R} \) and \( sA \models \exists x \Phi (x) \), then \( sA \models \exists^{\omega^\omega} x \Phi (x) \).
Proof. Let \( sA_i, i \in \omega \), be disjoint isomorphic copies of the polygon \( sA \). Then \( \bigcup_{i \in \omega} sA_i \models \exists \omega x \Phi(x) \) and in view of \( \bigcup_{i \in \omega} sA_i \equiv sA \) we get \( sA \models \exists \omega x \Phi(x) \). \( \square \)

Lemma 6.5. If the class \( \mathcal{R} \) is complete then \( |eSg| \geq \omega \) for any idempotents \( e \in S \) and \( g \in R \).

Proof. Denote by \( a \) the element \( eg \). Obviously, \( a \in eSg \) and \( Sg \models ea = a \). Consequently, \( Sg \models \exists x (ex = x) \) and, on Lemma 6.4, \( Sg \models \exists \omega x (ex = x) \). Since the condition \( x \in eSg \) is equivalent to \( ex = x \) and \( x \in Sg \), then \( |eSg| \geq \omega \). \( \square \)

Lemma 6.6. If the kernel \( K(R) \) is not empty and the class \( \mathcal{R} \) is complete, then \( K(R) \) is a rectangular band of infinite groups.

Proof. Proposition 1.4 and the regularity of polygon \( sR \) imply that \( K(R) \) is a rectangular band of groups. On Remark 1.1 it is enough to prove that the group \( G_e \) is infinite for any idempotent \( e \in K(R) \).

Let \( e \) be an arbitrary idempotent from \( K(R) \). Then \( sSe \models (ee = e) \) and, consequently, \( sSe \models \exists x (ex = x) \). On Lemma 6.4 we get \( sSe \models \exists \omega x (ex = x) \). Since \( K(R) \) is a rectangular band of groups, then the ideal \( S \) is minimal, and, consequently, for any \( a \in Se \) we have \( Se \models (ea = a) \) if and only if \( a \in G_e \). Thereby, for any \( e \in K(R) \) the group \( G_e \) is infinite and the kernel \( K(R) \) is a rectangular band of infinite groups. \( \square \)

Lemma 6.7. If \( H \) is a subgroup of the monoid \( S \), \( e \) is a unit in \( H \), then \( sSa \rightarrow sSe \) for any \( a \in H \).

Proof. Define the map \( \varphi : sSa \rightarrow sSe \) by the rule \( \varphi(x) = xa^{-1} \), where \( a^{-1} \) is inverse for \( a \) in the group \( H \). Then \( \varphi(a) = a \cdot a^{-1} = e \) and

\[ \varphi(sx) = (sx) \cdot a^{-1} = s(x \cdot a^{-1}) = s\varphi(x) \]

for any \( s \in S \) and \( x \in Sa \). Consequently, \( \varphi \) is a homomorphism.

If \( x, y \in Sa \) and \( \varphi(x) = \varphi(y) \), then \( x \cdot a^{-1} = y \cdot a^{-1} \). So \( xa^{-1}a = ya^{-1}a, xe = ye, x = y \).

Thus \( \varphi \) is an one-to-one correspondence. If \( x \in Se \), then \( x = xe = xa \cdot a^{-1} = \varphi(xa) \).

Thereby, \( \varphi \) is an isomorphism. \( \square \)

Lemma 6.8. If \( R \) is a rectangular band of groups then for any element \( a \in R \) and for any idempotent \( e \in R \) the condition \( sSa \rightarrow sSe \) is true if and only if \( e \cdot a = a \).

Proof. If \( sSa \rightarrow sSe \), then the equality \( e \cdot a = a \) is obvious.

Let the equality \( e \cdot a = a \) be true and \( R \) be a rectangular band of groups \( S \) with units \( e_{ij}, i \in I, j \in J \). Then \( e = e_{ij} \) for some \( i \in I \) and \( j \in J \). Since \( e_{ij} \cdot a = a, e_{ij}T = aT \) and on Remark 1.1 we get \( a \in S_{ik} \) for some \( k \in J \).

We will prove now that \( sSe_{ik} \rightarrow sSe_{ij} \). Define the map \( \varphi \) from \( Se_{ik} \) to \( Se_{ij} \) by the rule \( \varphi(x) = xe_{ij} \). Assume \( s \in S \), \( x \in Se_{ik} \). Then \( sx \in Se_{ik} \) and

\[ \varphi(sx) = (sx) \cdot e_{ij} = s(x \cdot e_{ij}) = s\varphi(x), \]

that is \( \varphi \) is a homomorphism. If \( a, b \in Se_{ik} \) and \( \varphi(a) = \varphi(b) \), then \( ae_{ij} = be_{ij} \). Consequently, \( ae_{ik} = be_{ik}, a = b \). Thus the map \( \varphi \) is injective. If \( c \in Se_{ij} \), then

\[ c = c \cdot e_{ij} = c \cdot e_{ik}e_{ij} = \varphi(ce_{ik}), \]

that is \( \varphi \) is surjective. Thereby, \( \varphi \) is an isomorphism and \( \varphi(e_{ik}) = e_{ik} \cdot e_{ij} = e_{ij} \). Consequently, \( sSe_{ik} \rightarrow sSe_{ij} \). Since \( sSa \rightarrow sSe_{ik} \) on Lemma 6.7 and \( e_{ij} = e \), then \( sSa \rightarrow sSe \). \( \square \)

The implication \( 1 \Rightarrow 2 \) results from Lemma 6.1.
2 $\Rightarrow$ 3. Assume the class $\mathcal{R}$ is complete. Then on Lemma 6.2 we have $\text{Id}(R) = 1$ and on Lemma 6.3 $R$ is a rectangular band of groups.

On Lemma 6.3 the set $I(R)$ is finite, and on Lemma 6.8 groups forming $R$ are infinite.

3 $\Rightarrow$ 1. Since in rectangular band of groups any principal right ideal in $R$ is minimal, then $S$ satisfies condition 1 of Theorem 4.1. Since the set $I(R)$ is finite, the set of principal right ideal is also finite. Consequently, the monoid $S$ satisfies condition 2 of Theorem 4.1, and the class $\mathcal{R}$ is axiomatizable.

Since in rectangular band of groups any left ideal in $R$ is minimal, then $S$ satisfies conditions 1 and 2 of Theorem 5.1. If idempotents $e$ and $g$ belong to $R$, then $e = e_{ij}$ and $g = e_{kl}$ for some $i, k \in I(R), j, l \in J(R)$. Then using properties of the rectangular bands of the groups we have

$$eSg = e_{ij}Te_{kl} = \left( \bigcup_{p \in J(T)} S_{ip} \right) e_{kl} = \bigcup_{p \in J(T)} (S_{ip}e_{kl}) = S_{il}.$$  

In view of infinity of the group $S_{il}$ we get condition 3 of Theorem 5.1. So the class $\mathcal{R}$ is model complete.

The condition of formula definability of isomorphic orbits follows from Lemma 6.8.  

**Corollary 6.1** [Ov2]. If a semigroup $R$ contains a finite number of idempotents, then the following conditions are equivalent:

1) the class $\mathcal{R}$ is model complete;
2) the class $\mathcal{R}$ is complete;
3) the semigroup $T$ is a rectangular band of a finite number of infinite groups.

**Proof.** Since in any regular polygon $sA$ for any element $a \in A$ there exists an idempotent $e \in R$ such that $sAs \sim sSe$, and the semigroup $R$ contains only finite number of idempotents then for each idempotent $e \in R$ there exists a formula $\Phi_e(x)$ such that

$$sA \models \Phi_e(x) \iff sAs \sim sSe,$$

that is the class $\mathcal{R}$ satisfies the condition of formula definability of isomorphic orbits. Thus, on Theorem 6.1 the corollary holds. 

**Corollary 6.2.** If a semigroup $R$ contains single idempotent, then the following conditions are equivalent:

1) the class $\mathcal{R}$ is model complete;
2) the class $\mathcal{R}$ is complete;
3) $R$ is an infinite group.

**Corollary 6.3.** If $S$ is a commutative monoid, then the following conditions are equivalent:

1) the class $\mathcal{R}$ is model complete;
2) the class $\mathcal{R}$ is complete;
3) $R$ is an infinite Abelian group.

**Proof.** Considering Corollary 6.2 it is enough to show that if $\mathcal{R}$ is a complete class of regular polygons over commutative monoid, then $R$ contains single idempotent. Let $e_1, e_2$ be some idempotents from $R$. Then for arbitrary $x \in Se_1$ we have $xe_1 = x$. In view of commutativity of monoid $S$ we get $e_1x = x$. Consequently, $sSe_1 \models \forall x (e_1x = x)$. Since class $\mathcal{R}$ is complete, then $sSe_2 \models \forall x (e_1x = x)$ and so $e_1e_2 = e_2$. Similarly we get $e_2e_1 = e_1$. Thereby, the commutativity of monoid $S$ implies $e_1 = e_2$. 

\[\square\]
Theorem 6.2 [Ov3]. If $S$ is a linearly ordered monoid of the depth 2 and $\mathcal{R}$ is an axiomatizable class, then the following conditions equivalent:

1) the class $\mathcal{R}$ is model complete;
2) the class $\mathcal{R}$ is complete;
3) the semigroup $K(S)$ is a rectangular band of infinite number of infinite groups.

Proof contains some lemmas.

Lemma 6.9. If a monoid $S$ has a finite depth, then $K(S) = K(R)$.

Proof. The condition $K(R) \subseteq K(S)$ is obvious. Since the depth $\text{ld}(S)$ is finite, then the depth $\text{ld}(R)$ is finite too. So on Proposition 2.6 the kernel $K(R)$ is a rectangular band of groups. Let $e$ be an element from $E(S) \cap K(R)$. Then $e$ belongs to $K(S)$ and, consequently, on Proposition 1.4 the kernel $K(S)$ is a rectangular band of groups. Thence we get $K(S) \subseteq R$ and, obviously, $K(S) \subseteq K(R)$. ☐

Lemma 6.10. If a monoid $S$ has a depth 2, then $R = S$.

Proof. As the monoid $S$ has the depth 2 then for any element $a \in S$ the condition $Sa \neq S1$ implies $Sa \subseteq S1$ and $Sa \subseteq K(S)$. On Lemma 6.9 and Proposition 2.6 the kernel $K(S)$ is a rectangular band of groups. Consequently, $K(S) \subseteq R$. If $a \in S \setminus K(S)$, then $Sa = S = S1$. So the polygon $Sa$ belongs to the class $\mathcal{R}$, that is $a \in R$. Thereby, $R = S$.

Lemma 6.11. If $S$ is a rectangular band of groups and $|J(S)| = 1$ then for some group $G$ the semigroup $S$ is isomorphic to a semigroup with the universe $S^0 = \{\langle b, i \rangle \mid b \in G, i \in I(S)\}$ and the operation $\cdot$, defined by equalities

$$\langle c, j \rangle \cdot \langle d, k \rangle = \langle c \cdot d, j \rangle$$

for any elements $\langle c, j \rangle, \langle c, j \rangle \in S^0$.

Proof. Let $S$ be a rectangular band of groups $G_i$ with idempotents $e_i$ for $i \in I(S)$. We fix an element $i_0 \in I(S)$ and we set $G = G_{i_0}, S' = G \times I(S)$. Define a map $\varphi : S' \rightarrow S$, acting by the rule $\varphi(\langle b, i \rangle) = e_i \cdot b, b \in G, i \in I(S)$. On Remark 1.1 (3) for any $i \in I(S)$ the group $G_i$ is isomorphic to the group $G$, as well as to the group $G_i'$ with the universe $\{\langle b, i \rangle \mid b \in G\}$ and the operation, defined by the rule $\langle b_1, i \rangle \cdot \langle b_2, i \rangle = \langle b_1 \cdot b_2, i \rangle$. Herewith, obviously, the restriction of the map $\varphi$ on the set $G_i'$ realizes an isomorphism between groups $G_i'$ and $G_i$. Besides it is directly checked that the map $\varphi$ itself is a bijection.

Define the operation $\cdot$ on elements of the set $S'$ by the following rule: $\langle c, j \rangle \cdot \langle d, k \rangle = \langle c \cdot d, j \rangle$, $\langle c, j \rangle, \langle c, j \rangle \in S'$. We will show that the map $\varphi$ conserves the operation $\cdot$. Really, using properties of the rectangular band of groups, for any $j, k \in I(S)$ and $c, d \in G$ we get

$$\varphi(\langle c, j \rangle) \cdot \varphi(\langle d, k \rangle) = e_j c \cdot e_k d = e_j (c e_{i_0}) \cdot e_k d =$$

$$= e_j c (e_{i_0} e_k) d = e_j c e_{i_0} d = e_j (c e_{i_0}) d = e_j (c d) = \varphi(\langle c \cdot d, j \rangle).$$

Thereby, the map $\varphi$ realizes an isomorphism between $S'$ and $S$, and the map $\varphi^{-1} : S \rightarrow S'$ is a required isomorphism.

Hereinafter under $J(S) = 1$ the rectangular band of the groups $S$ will be identified with the semigroup $S'$.

Lemma 6.12. If a monoid $S$ has a finite depth and $R = S$ then $1 \in aS$ for any element $a \in S$ such that $Sa = S$.

Proof. Denote by $G_1$ the set $\{x \in S \mid Sx = S\}$. We will show that $G_1 \cap E(S) = \{1\}$. Really if $e \in G_1 \cap E(S)$ then $1 \cdot e = 1$. Since $1 \cdot e = e$ it follows that $e = 1$. On the other hand, obviously, the unit belongs to $G_1 \cap E(S)$. Thereby, $G_1 \cap E(S) = \{1\}$.
Denote by $n$ the depth $\text{ld}(S)$. Since for any $a \in S$ it is executed $Sa \subseteq S \cdot 1$, then any chain of the length $n$ of principal left ideals contains the set $S \cdot 1$ as maximal element, that is $S \cdot 1$ is single principal left ideal of the depth $n$. Since $R = S$ then for any element $a \in G_1$ there exists an idempotent $g \in S$ such that $sSa \sim sSg$. Since $Sa = S$ and the depth of $Sa$ is equal to $n$, we get $sSa \sim sS1$. Let $\varphi$ be an isomorphism from $sSa$ to $sS1$ such that $\varphi(a) = 1$. Since $1 \in Sa$ then the element $\varphi(1)$ is defined. So $a \cdot \varphi(1) = \varphi(a \cdot 1) = \varphi(a) = 1$, that is $1 \in aS$.

**Lemma 6.13.** If $2 \leq \text{ld}(R) < \infty$ and the class $\mathfrak{R}$ is complete, then $|I(T)| \geq \omega$.

**Proof.** Since $\text{ld}(R) \geq 2$, then there exist an idempotent $g \in R$ and an element $a \in R$ such that $Sa \subseteq Sg$. In view of finiteness of $\text{ld}(R)$ one can suppose that $Sa$ is a minimal left ideal. Then the element $a$ belongs to $K(R)$. On Proposition 2.6 the kernel $K(R)$ is a rectangular band of groups, and in view of properties of the rectangular band of groups we have $Sa = \bigcup_{i \in I(R)} S_{ij}$ for some $j_0 \in J(R)$.

Assume the set $I(R)$ is finite. Then the set $\{e_{ij_0} \mid i \in I(R)\}$ is finite and

$$sSa |= \forall x \left( \bigvee_{i \in I(R)} e_{ij_0}x = x \right).$$

But since $Se_{ij_0} \subseteq Sg$, then $e_{ij_0}g = e_{ij_0} \neq g$ for all $i \in I(R)$ and $sSg \not\subseteq e_{ij_0}g = g$. Consequently, $sSg \not\subseteq sSa$ that contradicts to the completeness of the class $\mathfrak{R}$. \hfill $\square$

The implication $1 \Rightarrow 2$ results from Lemma 6.1.

$2 \Rightarrow 3$. On Lemmas 6.9 and 2.6 $K(S)$ is a rectangular band of groups. On Lemma 6.13 the set $I(K(S))$ is infinite, that is there are infinitely many groups forming $K(S)$. The infinity of the groups follows from Lemma 6.6.

$3 \Rightarrow 1$. Since the class $\mathfrak{R}$ is axiomatizable, we can use Theorem 5.1.

On Lemma 6.10 we have $R = S$. Since $S$ is a linearly ordered monoid of depth 2, there exists single chain of principal left ideals $Sa \subseteq S \cdot 1$. Consequently, condition 1 of Theorem 5.1 holds.

Since $\text{ld}(S) = 2$, the condition $Sa \subseteq Se$ implies $e = 1$, where $a \in S$, $e \in E(S)$. Then $Sa \subseteq K(S)$. In view of linear ordering of $S$ the equality $Sa = K(S)$ holds, that is on Lemmas 6.9 and 2.6 semigroup $Sa$ is a rectangular band of the groups. Thereby, kernel $K(S)$ consists of one principal left ideal and $J(S) = 1$. Let us fix the group $\langle G; \cdot \rangle$ with unit $e'$ such that all groups from $Sa$ are isomorphous to it. Then on Lemma 6.11 elements from $K(S)$ can be present as $\langle b, i \rangle$, where $b \in G$, $i \in I(K(S))$. Consequently,

$$S = \{x \mid Sx = S \cdot 1\} \cup \{\langle b, i \rangle \mid b \in G, i \in I(K(S))\}.$$

Assume $1 \not\in \bigcup_{i=1}^{m} a_iS$ for some $a_i \in S$, $1 \leq i \leq m$. Then we have $1 \not\in a_iS$ for each $i \in \{1, \ldots, m\}$, and in view of Lemma 6.12 for any $i \in \{1, \ldots, m\}$ there exist $b_i \in G$, $k_i \in I(K(S))$ such that $a_i = \langle b_i, k_i \rangle$. In view of properties of the rectangular band of groups the equality $|J(K(S))| = 1$ implies $\langle b_i, k_i \rangle S = S_{k_i}$ for all $i \in \{1, \ldots, m\}$, where $S_{k_i} = \{\langle b, k_i \rangle \mid b \in G\}$. Since for any $j \in I(K(S)) \setminus \{k_i\}$ it is satisfied $\langle e', j \rangle \not\in S_{k_i}$, $S(e', j) = S\{b_i, k_i\}$ and $\langle e', j \rangle = 1 \cdot \langle e', j \rangle \in 1 \cdot S$, then the set $E_{k_i} = E(S) \cap \{x \mid x \in (1 \cdot S) \setminus a_iS\}$ coincides with the set $\{\langle e', j \rangle \mid j \in I(K(S)) \setminus \{k_i\}\}$. On Lemma 6.13 the set $I(K(S))$ is infinite, and, consequently, the set $E_{k_i}$ is also infinite for any $i \in I(K(S))$. 
Since

\[(1 \cdot S) \setminus \bigcup_{i=1}^{m} a_i S = \bigcap_{i=1}^{m} ((1 \cdot S) \setminus a_i S),\]

then

\[E(S) \cap \{ x \mid x \in (1 \cdot S) \setminus \bigcup_{i=1}^{m} a_i S \} = \{ \langle e', j \rangle \mid j \in I(S) \setminus \{ k_i \mid 1 \leq i \leq m \} \}\]

and in view of infinity of the set \(I(K(S))\) condition 2 of Theorem 5.1 holds.

For the checking of condition 3 of Theorem 5.1 we will consider all possible variants for values of idempotents \(e, g\).

If \(e = 1\) and \(g = 1\), then \(|eSg| = |1S \cdot 1| = |S| \geq \omega\).

If \(e = 1\) and \(g = \langle e', i \rangle\) for some \(i \in I(K(S))\), then

\[|eSg| = |1S \langle e', i \rangle| = |S \langle e', i \rangle| = |K(S)| \geq \omega\].

If \(e = \langle e', i \rangle\) for some \(i \in I(K(S))\) and \(g = 1\) then on Lemma 6.6 we have

\[|eSg| = |\langle e', i \rangle S \cdot 1| = |\langle e', i \rangle S| = |S_i| \geq \omega\].

If \(e = \langle e', i \rangle\) and \(g = \langle e', j \rangle\) for some \(i, j \in I(K(S))\) then by properties of the rectangular band of groups it is satisfied

\[|eSg| = |\langle e', i \rangle S \langle e', j \rangle| = |S_i \langle e', j \rangle| = |S_i| \geq \omega\].

Thus, all conditions of Theorem 5.1 hold and the class \(\mathfrak{R}\) is model complete.

The theorem is proved. \(\square\)

The following two theorems show that there exist monoids over which classes for all regular polygons are complete, but not model complete. Here we will only build such monoids but detailed proofs of their properties can be found in [Ov4, Ov5]

**Theorem 6.3.** There exists a not linearly ordered monoid of depth 2, over which the class of all regular polygons is complete, but not model complete. Here we will only build such monoids but detailed proofs of their properties can be found in [Ov4, Ov5]

**Proof.** For an Abelian group \(G = (G, +)\), nonempty sets \(I, J\) and a function \(\varphi : I \times J \rightarrow G\) we call by \(\langle G, I, J, \varphi \rangle\)-band the semigroup \(\langle G \times I \times J, \ast \rangle\), in which the operation \(\ast\) is defined by the following way:

\[\langle a, i, j \rangle \ast \langle b, k, l \rangle = \langle a + b + \varphi(k, j), i, l \rangle.\]

For arbitrary elements \(i \in I\) and \(j \in J\) we denote by \(S_{ij}\) the set \(\{ \langle a, i, j \rangle \mid a \in G \}\). Then the algebra \(S_{ij} = \langle S_{ij}, \ast \rangle\) is a subgroup of \(\langle G, I, J, \varphi \rangle\)-band with the idempotent \(\langle -\varphi(i, j), i, j \rangle\) as the unit element. Since \(S_{ij} = S_{kl}\) then \(\langle G, I, J, \varphi \rangle\) is a rectangular band of groups, which will be denoted by RB\((G, I, J, \varphi)\).

Remind [KM] that for any group \(G\) with its unit \(e\) and any ordinal \(\alpha\) by \(G^\alpha\) one denote the direct degree of group \(G\), that is the set of all function \(f : \alpha \rightarrow G\) such that the set \(\{ x \mid f(x) \neq e \}\) is finite.

Let us consider the group \(Z_{2}^\omega\), where \(Z_{2} = \{0, 1\}, +\). Elements of the group \(Z_{2}^\omega\) will be denoted by \(\tilde{a}, \tilde{b}, \ldots\), where \(\tilde{a} = (a_0, a_1, \ldots), \tilde{b} = (b_0, b_1, \ldots)\). We will denote zero of the group \(Z_{2}^\omega\) by \(\tilde{0}\). For arbitrary element \(\tilde{a} \in Z_{2}^\omega\) we introduce the following notations:

\[h(\tilde{a}) = \begin{cases} 
(0), & \tilde{a} = \tilde{0}, \\
(a_0, \ldots, a_{n-1}), & a_{n-1} = 1 \text{ and } a_k = 0 \text{ for all } k \geq n,
\end{cases}\]
$l(\bar{a}) \rightarrow \begin{cases} 
0, & \bar{a} = \bar{0}, \\
n, & h(\bar{a}) = (a_0, \ldots, a_{n-1}), 
\end{cases}$

$r(\bar{a}) \equiv \sum_{i=0}^{l(\bar{a})} a_i \cdot 2^i.$

If $a = \langle \bar{a}, i, j \rangle \in S$, then we set $l(a) \equiv l(\bar{a}), r(a) \equiv r(\bar{a})$.

Define the function $\psi$ from $\omega$ to $\mathbb{Z}_2^\omega$ by the following way:

$$\psi(n) = r^{-1}(\lfloor \sqrt{n} \rfloor),$$

where $[x]$ means the integer part of real number $x$.

Define the multiplying operation of elements of group $\mathbb{Z}_2^\omega$ on 0 and 1 as follows:

$$\bar{a} \cdot 0 \equiv 0,$$

$$\bar{a} \cdot 1 \equiv \bar{a}.$$

We set $\mathcal{G} = \mathbb{Z}_2^\omega$, $I \equiv \omega$, $J \equiv \{0, 1\}$, $\varphi(i, j) \equiv \psi(i) \cdot j$.

Define a monoid $S$ by equality

$$S = \langle \text{RB}(\mathcal{G}, I, \varphi) \cup \mathbb{Z}_2^\omega, \ast \rangle,$$

where action of operation $\ast$ between elements of the same nature is defined by natural way, but between element from $\text{RB}(\mathcal{G}, I, \varphi)$ and $\mathbb{Z}_2^\omega$ by the following rule:

$$\langle \bar{a}, i, j \rangle \ast \bar{b} = \bar{b} \ast \langle \bar{a}, i, j \rangle = \langle \bar{a} + \bar{b}, i, j \rangle.$$}

The unit of the monoid $S$ is the element $\bar{0}$. All idempotents of monoid $S$ form the set

$$E(S) = \{\langle \varphi(i, j), i, j \rangle \mid i \in I, j \in J\} \cup \{\bar{0}\}.$$

The monoid $S$ is a union of the groups, isomorphic $\mathbb{Z}_2^\omega$. Consequently, $R = S.$ \hfill $\Box$

**Theorem 6.4.** There exists a linearly ordered monoid of depth 3, over which the class of all regular polygons is complete, but not model complete.

**Proof.** Let $G = \langle G; \ast, 0 \rangle$ be a countable group.

Define a monoid $S = \langle S; \cdot \rangle$ as follows: $S = (\cup_{i \in \omega} G_{1i}) \cup (\cup_{i \in \omega} G_{2i}) \cup (\cup_{i \in \omega} G_{3i}) \cup G_4 \cup G_5$, where

$$G_{1i} = \{[1, i, \bar{j}] \mid \bar{j} \in G^\omega\}, \quad G_{2i} = \{[2, i, \bar{j}] \mid \bar{j} \in G^\omega\},$$

$$G_{3i} = \{[3, i, \bar{j}] \mid \bar{j} \in G^\omega\}, \quad G_4 = \{[4, \bar{i}] \mid \bar{i} \in G^\omega\},$$

$$G_5 = \{[5, \bar{i}] \mid \bar{i} \in G^\omega\}.$$

An action of operation $\cdot$ will be defined by the following table:

|     | $[1, m, \bar{k}]$ | $[2, m, \bar{k}]$ | $[3, m, \bar{k}]$ | $[4, \bar{k}]$ | $[5, \bar{k}]$ |
|-----|------------------|------------------|------------------|---------------|---------------|
| $[1, i, \bar{j}]$ | $[1, i, \bar{j} * \bar{k}]$ | $[1, i, \bar{j} * \bar{k}]$ | $[1, i, \bar{j} * \bar{k}]$ | $[1, i, \bar{j} * \bar{k}]$ | $[1, i, \bar{j} * \bar{k}]$ |
| $[2, i, \bar{j}]$ | $[2, i, \bar{j} * \bar{k}]$ | $[2, i, \bar{j} * \bar{k}]$ | $[2, i, \bar{j} * \bar{k}]$ | $[2, i, \bar{j} * \bar{k}]$ | $[2, i, \bar{j} * \bar{k}]$ |
| $[3, i, \bar{j}]$ | $[3, i, \bar{j} * \bar{k}]$ | $[3, i, \bar{j} * \bar{k}]$ | $[3, i, \bar{j} * \bar{k}]$ | $[3, i, \bar{j} * \bar{k}]$ | $[3, i, \bar{j} * \bar{k}]$ |
| $[4, \bar{j}]$ | $[1, m, \bar{j} * \bar{k}]$ | $[2, m, \bar{j} * \bar{k}]$ | $[2, m, \bar{j} * \bar{k}]$ | $[4, \bar{j} * \bar{k}]$ | $[4, \bar{j} * \bar{k}]$ |
| $[5, \bar{j}]$ | $[1, m, \bar{j} * \bar{k}]$ | $[2, m, \bar{j} * \bar{k}]$ | $[3, m, \bar{j} * \bar{k}]$ | $[4, \bar{j} * \bar{k}]$ | $[5, \bar{j} * \bar{k}]$ |
Note that the monoid $S$ is a union of the groups. The groups, forming $S$, are isomorphic to the group $G^\omega$. The unit of the monoid is an element $[5,0]$. The polygon $sS$ has three different inclusions $S[1,0,0] \subset S[4,0] \subset S[5,0]$, each of which is generated by an idempotent. Consequently, $R = S$ and $\text{Id}(S) = 3$. □

§ 7. Stability of Class for Regular Polygons

In this paragraph we give the characterization of $\mathcal{R}$–stabilizer whose regular core is presented as a union of the finite number of principal right ideals (Theorem 7.1). As the consequence corollary it is proved that for the axiomatizable class of all regular polygons the stability of this class is equivalent to the regularly linearly ordering of monoid $S$. The regular linearly ordering of monoid is the necessary (Proposition 7.1) but not sufficient (Example 7.1) condition for the stability of any regular polygon. The Example 7.2 shows in particular that there exists a regularly linearly ordering monoid with non–axiomatizable class of all regular polygons over which all regular polygons are stable.

**Proposition 7.1** If $S$ is a $\mathcal{R}$–stabilizer then $S$ is a regularly linearly ordered monoid.

**Proof.** Assume $S$ is a $\mathcal{R}$–stabilizer which is not a regularly linearly ordered monoid, that is, there exist $a, b, c \in R$ such that $Sb \not\subseteq Sc \subseteq Sa$ and $Sc \not\subseteq Sb \subseteq Sa$. Then $b = ta$ and $c = sa$ for some $t, s \in S$. Let $K = \{\langle i, j \rangle \mid j \leq i < \omega\}$; $sA_{ij}$ is a copy of the polygon $sSa ((\langle i, j \rangle) \in K)$, and $A_{ij} \cap A_{kl} = \emptyset$ if $\langle i, j \rangle \neq \langle k, l \rangle$; $d_{ij}$ is a copy of the element $d \in Sa$ in $A_{ij}$. Write $sA$ for a polygon $\bigcup_{\langle i, j \rangle \in K} sA_{ij}/\theta$, where $\theta$ is a congruence on $\bigcup_{\langle i, j \rangle \in K} sA_{ij}$ generated by the set $\{\langle b_{ij}, b_{kl} \rangle \mid \langle i, j \rangle \in K, \langle i, l \rangle \in K\} \cup \{\langle c_{ij}, c_{kl} \rangle \mid \langle i, j \rangle \in K, \langle j, k \rangle \in K\}$. Denote by $b_i$ an equivalence class of $\theta$ with a representative $b_{ij} ((\langle i, j \rangle) \in K)$, and by $c_j$ an equivalence class of $\theta$ with a representative $c_{ij} ((\langle i, j \rangle) \in K)$. Write $\varphi(x, y)$ to abbreviate the formula $\exists z(x = tz \land y = sz)$. Since a restriction of $\theta$ to the polygon $sA_{ij} ((\langle i, j \rangle) \in K)$ is zero congruence, $sA \in \mathcal{R}$. Note that $b_i = ta_{ij}/\theta$ and $c_j = sa_{ij}/\theta$. Moreover,

$$sB \models \varphi(b_i, c_j) \iff i \geq j,$$

which contradicts the stability of $\text{Th}(sA)$ on Theorem 3.5. □

**Theorem 7.1** [Ste2]. Let

$$R = \bigcup_{i=0}^n a_i R$$

for some $n \geq 0$ and $a_i \in R$ ($0 \leq i \leq n$). The monoid $S$ is an $\mathcal{R}$–stabilizer if and only if $S$ is a regularly linearly ordered monoid.

**Proof.** Assume the theorem condition is satisfied.

**Necessity** follows from Proposition 7.1.

**Sufficiency.** Let $S$ be a regularly linearly ordered monoid and $sA \in \mathcal{R}$. We claim that $\text{Th}(sA)$ is stationary. Suppose $sM \equiv sA$, $a, b \in \mathcal{C} \setminus M$, $a = sb$, and $c = rb, c \in M$. Suppose $d \in A$. On Corollary 2.1, there exist an idempotent $e \in R$ and an isomorphism $\varphi : Sd \to Se$ such that $\varphi(d) = e$. On (7.1), there exist $i$, $0 \leq i \leq n$, and $u \in R$ such that $e = a_i u$. Therefore, $e = a_i u e$. Consequently, $sSd \models \exists y(d = a_i y)$, that is, $sA \models \forall x \exists y \bigvee_{i=0}^n (x = a_i y)$. Since $sM \equiv sA$, $sM \sim s\mathcal{C}$, we have $b = a_i v$ for some $i$, $0 \leq i \leq n$, and $v \in \mathcal{C}$. Since the set $\{Sa \mid Sa \subseteq Sa_i\}$ is linearly ordered, one of the inclusions $Ssa_i \subseteq Sra_i$ or $Sra_i \subseteq Ssa_i$ holds. If $Ssa_i \subseteq Sra_i$ then $a = sb = sa_i v \in$
Sra_{i}v = Srb = Sc \subseteq M$, which contradicts the choice of a. Consequently, $Sra_{i} \subseteq Ssa_{i}$. In this case $c = rb = ra_{i}v \in Ssa_{i}v = Ssb = Sa$, that is, the theory $\text{Th}(sA)$ is stationary. On Theorem 3.7, $\text{Th}(sA)$ is stable, that is, $S$ is an $\mathcal{R}$–stabilizer.

**Corollary 7.1.** Let class $\mathcal{R}$ for regular polygons be axiomatizable. Class $\mathcal{R}$ is stable if and only if $S$ is a linearly ordered monoid.

The proof follows from Theorem 7.1 and Corollary 4.2. □

From the proof of Theorem 7.1 it follows

**Corollary 7.2.** If the condition of Theorem 7.1 holds and $S$ is a linearly ordered monoid then the theory of any regular polygons is stationary. □

**Example 7.1.** We construct a non $\mathcal{R}$–stabilizer which is a regularly linearly ordered monoid.

Let $S$ be a union of five disjoint sets:

$$S = A \cup B \cup C \langle \alpha, \beta \rangle \cup \{1\},$$

where $A = \{a_{i} | i \in \omega\}$, $B = \{b_{i} | i \in \omega\}$, $C = \{c_{ij} | i \geq j, i, j \in \omega\}$ and $\langle \alpha, \beta \rangle$ is a free two-generated commutative semigroup with $\alpha$ and $\beta$ generators. Equip $S$ with a binary operation defined as follows: $1$ is a identity element of $S$; $a_{i}x = a_{i}$, $b_{i}x = b_{i}$, $c_{ij}x = c_{ij}$, $(\gamma\alpha)c_{ij} = a_{i}$, $(\gamma\beta)c_{ij} = b_{j}$, $(\gamma\alpha)y = y$, and $(\gamma\beta)y = y$ for any $x \in S$, $y \in S \setminus (C \langle \alpha, \beta \rangle)$, $\gamma \in \langle \alpha, \beta \rangle \cup \{1\}$, $a_{i} \in A$, $b_{i} \in B$, $c_{ij} \in C$. It is easy to verify that $S$ is a monoid under the operation given. Furthermore, $R = A \cup B \cup C$ and $S_{a_{i}} = S_{b_{i}} = S_{c_{ij}} = R$ for any $a_{i} \in A$, $b_{i} \in B$, $c_{ij} \in C$. Hence $S$ is a regularly linearly ordered monoid. Let $\Phi(x, y) = \exists z (x = \alpha z \land y = \beta z)$. Then

$$S_{a}S \models \exists z (a_{i} = \alpha z \land b_{j} = \beta z) \iff i \geq j,$$

and so on Theorem 3.5 $\text{Th}(sR)$ is not stable, that is, $S$ is not $\mathcal{R}$–stabilizer. □

**Example 7.2.** We construct a regularly linearly ordered $\mathcal{R}$–stabilizer which does not satisfy the condition of Theorem 7.1.

Let $S = \bigcup_{i \in \omega} Z_{i} \cup \langle \alpha, \beta \rangle \cup \{1\}$, where $Z_{i} = \{n_{i} | n \in Z\}$ is a copy of the set $Z$ of integers on which addition is defined naturally, $Z_{i} \cap Z_{j} = \emptyset (i \neq j)$, and $\langle \alpha, \beta \rangle$ is a free two-generated commutative semigroup with $\alpha$ and $\beta$ generators. Equip $S$ with a binary operation defined as follows: $1$ is an identity element of $S$, $n_{i} \cdot m_{j} = (n + m)_{\min(i,j)}$, $\alpha m_{i} = n_{i}\alpha = (n + 3)_{i}$, $\beta n_{i} = n_{i}\beta = (n + 2)_{i}$, and $(\gamma_{1}\gamma_{2})n_{i} = \gamma_{1}(\gamma_{2}n_{i})$, where $\gamma_{1}, \gamma_{2} \in \langle \alpha, \beta \rangle$, $i, j \in \omega$. It is easy to verify that $S$ is a monoid under the operation given, $\{1, 0_{0}, 0_{1}, \ldots, 0_{i}, \ldots\}$ is the set of all idempotents of $S$ and $S \cdot n_{i} = n_{i} \cdot S = \bigcup_{j=0}^{i} Z_{j}$ for all $i \in \omega$, $n \in Z$.

Consequently, $R = \bigcup_{i \in \omega} Z_{i}$ and $S$ is a regularly linearly ordered monoid which does not satisfy the condition of Theorem 7.1.

We claim that $S$ is an $\mathcal{R}$–stabilizer. It suffices to prove that $\text{Th}(sA)$ is stationary for any $sA \in \mathcal{R}$. Let $sA \in \mathcal{R}$, $sM \equiv sA$, $c = sb$, $a = tb$, $a, b \in C \setminus M$, and $c \in M$. If $s, t \in R$, that is, $s = n_{i}$ and $t = m_{j}$, then the equalities $n_{i} = (n - m)_{i} m_{j}$ and $c = (n - m)_{i} m_{j} b = (n - m)_{i} a$ hold for all $i \leq j$, and $c \in Sa$; for $i > j$, likewise we derive $a \in Sc$, which is impossible. Let $s \in \langle \alpha, \beta \rangle$. Note that

$$S \cdot 0_{i} \models \forall x \exists y (x = sy) \quad (7.2)$$

for each $i \in \omega$. Let us prove that $sA \models \forall x \exists y (x = sy)$. In fact, on Corollary 2.1 for any $d \in A$ there exist $i \in \omega$ such that $sSd \cong sS0_{i}$. Since the formula $\forall x \exists y (x = sy)$ is true in
$S_0 \), then this formula is true in $S_d$. Hence it is true in $S_A$. Let $d = s d_1$ and $d = s d_2$ for some $d_1, d_2 \in A$. On Corollary 2.1 there exist $i \in \omega$ and isomorphism $\varphi : S d_1 \to S \cdot 0_i$. Then $\varphi d_1, \varphi d \in Z_i$ and $S \varphi d_1 = S \varphi d$. Consequently, $S d_1 = S d$. Similarly $S d_2 = S d$, that is, $S d_1 = S d_2 \cong S \cdot 0_i$. In view of (7.2), $d_1 = d_2$. Since $s M \equiv s A$, it follows that $s M \models \exists y (c = s y)$ and $s \mathcal{C} \models \exists y (c = s y)$; on the other hand, $c = s b, b \notin M$, which is impossible. Thus $s \in R$ and $t \in \langle \alpha, \beta \rangle$. Let $s = n_i$. There exists a $k \in \omega$, $k \geq 2$, such that $s R \models \forall x(n_i x = (n - k)_i t x)$. In view of regularity of polygon $S A$ on Corollary 2.1 $s A \models \forall x(n_i x = (n - k)_i t x)$. Since $S A \equiv s \mathcal{C}$, it follows that $s \mathcal{C} \models \forall x(n_i x = (n - k)_i t x)$; in particular, $c = s b = n_i b = (n - k)_i t b = (n - k)_a$, that is, $c \in S a$. Consequently, $Th(s A)$ is stationary, and $S$ is an $\mathfrak{R}$–stabilizer in view of Theorem 3.7.

\section{8. Superstability of Class for Regular Polygons}

In this paragraph we give the characterization of an $\mathfrak{R}$–superstabilizer whose regular core is presented as the union of finite number of principal right ideals (Theorem 8.1). As in stable case it may be weakened to the condition when the regular core is presented as a union of finite number of principal right ideals replacing it by the axiomatizing of the class $\mathfrak{R}$ (Corollary 8.1). Since the class regular polygons over group is axiomatizable, it is superstable (Corollary 8.2). Proposition 8.1 gives necessary condition for the superstability of all regular polygons. But as Example 8.1 shows, this condition is not sufficient. In this paragraph we also construct the example of an $\mathfrak{R}$–superstabilizer, but not stabilizer (Example 8.3) and the example of $\mathfrak{R}$–stabilizer, but not an $\mathfrak{R}$–superstabilizer (Example 8.4).

**Proposition 8.1.** If $S$ is an $\mathfrak{R}$–superstabilizer then $S$ is a regularly linearly ordered monoid and the semigroup $S a$ satisfies the ascending chain condition for left ideals, where $a \in R$.

**Proof.** Let $S$ be an $\mathfrak{R}$–superstabilizer. Then $S$ is an $\mathfrak{R}$–stabilizer, which is regularly linearly ordered by Proposition 7.1. Assume $a, a_i \in R$ are such that $S a_i \subseteq S a_{i+1} \subseteq S a$, $a_i = s_i a$ ($s_i \in S$, $i \in \omega$). Let $T$ be the theory of $\mathfrak{R}$; $\kappa$ be an arbitrary cardinal, $\kappa > 2^{\aleph_0}$; $Q = \{\eta \in \kappa^\omega \mid \exists n < \omega \forall m \geq n(\eta(m) = 0)\}; 0 \in \kappa^\omega$ is such that $0(m) = 0$ for each $m \in \omega$, and $r(\eta) = \min\{n \in \omega \mid \forall m \geq n(\eta(m) = 0)\}$. On $Q$, we define the following relation:

$$\eta < \varepsilon \Leftrightarrow r(\eta) < r(\varepsilon);$$

moreover, if $r(\eta) = r(\varepsilon)$, then there exists a $k \in \omega$ such that $\eta|_k = \varepsilon|_k$, but $\eta(k) < \varepsilon(k)$. The set $Q$ equipped with a relation $\leq$ conforming to the relation $<$ above is well ordered. Denote the element from $\kappa^{k-1}$ by $\eta_k$ if $\eta|_k = \eta|_k$, where $\eta \in Q$, $k \in \omega$. We put $\eta_0 = \emptyset$. For any $\eta \in Q$ and $k \in \omega$, we construct an $S$-polygons $S N_\eta$, $S N'_\eta$ and elements $b_\eta, b_\eta k \in N_\eta$. Let $N_0 = S a$, $b_0 = a$, and $b_0 k = a k$, $k \in \omega$. Assume the polygons $S \tilde{N}_\xi$, $S \tilde{N}'_\xi$ and the elements $b_\xi, b_\xi k$ are constructed for all $\xi \in Q$, $\xi < \eta$, $k \in \omega$. Note that element $b_{\eta k}$ has constructed on the previous steps. If it exists the largest element $\xi \in Q$ such that $\varepsilon < \eta$, then $S N'_\eta = s N_\varepsilon$; in the other ways, $S N'_\eta = \bigcup_{\xi < \eta} s N_\xi$. We put $s N_\eta = s(N'_\eta \cup S a)/\Theta_\eta$, where $\Theta_\eta$ is a congruence on the polygon $(S(N'_\eta \cup S a))$ generated by the pair $\langle b_{\eta k}, a_{\eta k} \rangle$; in this case elements of $N'_\eta$ are identified with congruence classes $\eta$ whose representatives they are of. Redenote the element $a/\theta_\eta$ by $b_\eta$, and elements $a_i/\theta_\eta$ by $b_\eta i$, where $i > r - 1$. 

\[\]
Write $sN$ for $\bigcup_{\eta \in Q} sN_\eta$. Clearly, $sN \in \mathfrak{R}$. Let $A = \{b_\eta : \eta \in \kappa^\omega, k \in \omega\}$. Since

$$b_\eta = s_kx \in \text{tp}(b_\varepsilon, A) \iff \eta(\xi) = \varepsilon(\xi) \text{ for all } \xi \leq k,$$

where $k \geq 0$, we have $\text{tp}(b_\eta, A) \neq \text{tp}(b_\varepsilon, A)$, $\eta \not= \varepsilon, \eta, \varepsilon \in Q$. Since $|A| = \sum k^k$, we obtain $|S(A)| \geq |\{b_\eta : \eta \in \kappa^\omega\}| = \kappa^\omega > \kappa$. Thus $\text{Th}(sN)$ is not superstable, and consequently $S$ fails as an $\mathfrak{R}$-superstabilizer. \hfill \Box

**Theorem 8.1** [Ste2] Let

$$R = \bigcup_{i=0}^n a_iR$$

for some $n \geq 0$ and $a_i \in R$ ($0 \leq i \leq n$). The monoid $S$ is an $\mathfrak{R}$-superstabilizer if and only if $S$ is a regularly linearly ordered monoid and the semigroup $Sa$ satisfies the ascending chain condition for left ideals, where $a \in R$.

**Proof.** Assume the theorem condition is satisfied.

**Necessity** follows from Proposition 7.1.

**Sufficiency.** Let $S$ be a regularly linearly ordered monoid, the semigroup $Sa$ satisfies the ascending chain condition for left ideals, where $a \in R$; $sA \in \mathfrak{R}$; $sM \equiv sA; a \in \mathfrak{C} \setminus M$; $Sa \cap M = \emptyset$.

We claim that there exists an entering element from $a$ into $M$, that is, there exists $c \in M$ for which $c \in Sa$ and $Sb \subseteq Sc$ with all $b \in M \cap Sa$. Let $d$ be an arbitrary element of $A$ and $\Phi = \forall x \bigvee_{i \leq n} \exists y(x = a_iy)$. Since $sA$ is regular on Corollary 2.1 it exists an idempotent $e \in R$ such that $Sd \cong Se$. Under conditions theorem $sSe \models \Phi$. Consequently, $sSd \models \Phi$. In view of the arbitrary of element $d$ we have $sA \models \Phi$. Since the polygons $sA$, $sM$ and $s\mathfrak{C}$ are elementary equivalents it follows that $sM \equiv \Phi$ and $s\mathfrak{C} \models \Phi$. Consequently, $a = a_i a'$ for some $i \leq n$ and $a' \in \mathfrak{C}$. Assume $m, m' \in Sa \cap M$. Then $m = sa = sa_a'$ and $m' = ta = ta_a'$ for some $s, t \in S$. Since $S$ is a regularly linearly ordered monoid, either $rsa_i \subseteq sta_t$ or $sta_t \subseteq rsa_i$. Hence either $sa_i = rsa_t$ or $ta_t = rsa_t$, $r \in S$, that is, either $m = rm'$ or $m' = rm$. Consequently, either $Sm \subseteq Sm'$ or $Sm' \subseteq Sm$. Suppose that there exists no entering element from $a$ into $M$, which means that there exist $m_i \in Sa \cap M, i \in \omega$, such that $Sm_i \subsetneq Sm_{i+1}$ and any polygon $sSm_m, m \in Sa \cap M$, coincides with some polygon $sSm_i$. For any $j \in \omega$, there then exists an $s_j \in S$ for which $m_j = s_j a = s_j a_a'$. Let $s_{j+1} a_i \subseteq s_{j+1} a_i$, that is $s_{j+1} a_i = ks_j a_i$ for some $k \in S$. Hence $s_{j+1} a_a' = ks_j a_a'$. Then $s_{j+1} a_a' = ks_j a_a'$, that is $m_{j+1} = km_j$ and $Sm_{j+1} \subseteq Sm_j$, which contradicts the assumption. In view of the regularity linear ordering $S$ we claim that $sS_{j+1} a_i \subseteq sS_{j+1} a_i$, which contradicts the ascending left chain condition for ideals of semigroup $Sa_i$. We have thus proved that the desired entering element from $a$ into $M$ exists.

Let $m$ be an entering element from $a$ into $M$, $a, b \in \mathfrak{C}$. On Corollary 7.4, the theory $\text{Th}(sM)$ is stationary. On Theorem 3.6,

$$\text{tp}(a, M) = \text{tp}(b, M) \iff \text{tp}(a, \{m\}) = \text{tp}(b, \{m\})$$

and $m$ is an entering element from $b$ into $M$. Let $T = \text{Th}(sM), |M| = \kappa \geq 2^{[T]}$, $Q_0 = \{\text{tp}(a, M) : a \in M\}$, $Q_1 = \{\text{tp}(a, M) : a \in \mathfrak{C} \setminus M, M \cap Sa = \emptyset\}$, and $Q_2 = \{\text{tp}(a, M) : a \in \mathfrak{C} \setminus M, M \cap Sa \neq \emptyset\}$. Then $|Q_0| = \kappa$; $|Q_1| \leq |S(\emptyset)| \leq 2^{[T]} \leq \kappa$;
$|Q_2| \leq |M| \cdot 2^{|T|} = \kappa \cdot 2^{|T|} = \kappa$, that is, $|S(M)| = |Q_0 \cup Q_1 \cup Q_2| = \kappa$. Consequently, $T$ is superstable and $S$ is an $\mathcal{R}$–superstabilizer.

Note that the ascending chain condition for left ideals of $Sa$ $(a \in R)$ in Theorem 8.1 may be replaced by the ascending chain condition for left ideals of $Sa_i$ $(1 \leq i \leq n)$. It follows from the proof of Theorem 8.1.

**Corollary 8.1.** Let the class $\mathcal{R}$ for regular polygons is axiomatizable. The class $\mathcal{R}$ is superstable if and only if $S$ is a regularly linearly ordered monoid and the semigroup $Sa$ satisfies the ascending chain condition for left ideals, where $a \in R$.

The proof follows from Theorem 8.1 and Corollary 4.2.

**Corollary 8.2.** The class $\mathcal{R}$ for regular polygons over group is superstable.

The proof follows from Corollaries 4.3 and 8.1.

**Example 8.1.** Example 7.1 exemplifies a regularly linearly ordered, non–$\mathcal{R}$–superstabilizer $S$ such that the semigroup $Sa$ satisfies the ascending chain condition for left ideals, where $a \in R$.

**Example 8.2.** We construct a regularly linearly ordered superstabilizer $S$, such that the semigroup $Sa$ satisfies the ascending chain condition for left ideals, where $(a \in R)$, for which the assumption of Theorem 8.1 fails.

Let $S = \omega \cup \langle \alpha \rangle \cup \{1\}$, where $\langle \alpha \rangle$ is a one-generated free semigroup with generator $\alpha$. Equip $S$ with a binary operation defined as follows: 1 is an identity element of $S$; $n \cdot m = n$ and $\alpha^i \cdot n = n \cdot \alpha^i = n$ for any $n, m \in \omega$, $i \geq 1$. It is easy to verify that $S$ is a monoid under the operation given, $Sn = \omega = R$ and $nS = \{n\}$, where $n \in \omega$. Then $S$ is a regularly linearly ordered monoid for which the assumption of Theorem 8.1 fails and the semigroup $Sa$ satisfies the ascending chain condition for left ideals, where $a \in R$. Since $Sm \subset \alpha^{i+1}S \subset \alpha^iS$ for any $m \in \omega$, $i \geq 1$ then $S$ is a superstabilizer by Theorem 3.9.

**Example 8.3.** We construct an $\mathcal{R}$–stabilizer $S$ which is not a stabilizer.

The monoid $S$ is obtained from the monoid specified in Example 7.2 by replacing the ordinal $\omega$ by $\{0\}$ and preserving the operation. It follows that $R = Z_0 = 0_0 \cdot R$, and by Theorem 8.1, $S$ is a regular superstabilizer. Since $S$ is not linearly ordered, $S$ is an $\mathcal{R}$–superstabilizer. Since $S$ is not linearly ordered then by Theorem 3.8 $S$ is not a stabilizer.

**Example 8.4.** We construct an $\mathcal{R}$–stabilizer $S$ which is not an $\mathcal{R}$–superstabilizer.

The monoid $S$ is obtained via the monoid in Example 7.2 by replacing the ordinal $\omega$ by $\omega \cup \{\omega\}$ and preserving the operation. It follows that $R = \bigcup_{i \in \omega \cup \{\omega\}} Z_i = Z_\omega$ and $S0_i \subset S0_{i+1} \subset S0_\omega$ for any $i \in \omega$. That is, $S$ fails as an $\mathcal{R}$–superstabilizer by Proposition 8.1. That $S$ is an $\mathcal{R}$–stabilizer is explicated as in Example 7.2.

**§ 9. $\omega$–stability of Class for Regular Polygons**

In [FG] there were considered the questions of the stabilities of the theories, which are the model companion of the theory of all polygons. Such theory exists if a monoid $S$ is left coherent. In this work it is proved that each complete type of this theory is characterized by triple consisting of a left ideal of the monoid $S$, a left congruence on the monoid $S$ and a polygon homomorphism. In this paragraph in the case of an axiomatizability and a model
completeness of the class $\mathfrak{R}$ each complete type of the theory of $\mathfrak{R}$ is characterized by the triple consisting of a left ideal of the semigroup $Se_i$, left congruency of the semigroup $Se_i$ and a polygon homomorphism, where the idempotents $e_i$ are taken from Corollary 4.2 (Lemma 9.1). It is this result which allows us to translate model theoretic properties of the class $\mathfrak{R}$ into algebraic properties of the monoid $S$. The consequence is that we can easily find upper bounds for the number of the full types of the class $\mathfrak{R}$. Theorem 9.1 gives us a criterion of superstability and $\omega$–stability of axiomatizable and a model complete class $\mathfrak{R}$ of regular polygons. In this theorem the stability of the class $\mathfrak{R}$ it proved. At the end of the paragraph there is an example of $\mathfrak{R}$–superstabilizer, but not $\omega$–$\mathfrak{R}$–stabilizer and not superstabilizer.

Assume $\mathfrak{s}A \in \mathfrak{R}$, $R = \bigcup\{e_iR \mid 1 \leq i \leq n\}$, where $n \in \omega$, $e_i^2 = e_i \in R$ $(1 \leq i \leq n)$. Write $Tr^i(A)$ $(1 \leq i \leq n)$ for the set of the triples $\langle \theta, I, \alpha \rangle$, satisfying the following conditions:

1. $\theta$ is a congruency of a polygon $\mathfrak{s}Se_i$;
2. $\mathfrak{s}I$ is a subpolygon of a polygon $\mathfrak{s}Se_i$; $Se_i$;
3. $\alpha : \mathfrak{s}I \to \mathfrak{s}A$, $\alpha$ is a polygon homomorphism;
4. $\mathfrak{s}I$ is a $\theta$–saturated polygon that is $\langle a, b \rangle \in \theta$ and $a \in I$ imply $b \in I$;
5. $Ker\alpha = \theta \cap (I \times I)$;
6. $\mathfrak{s}(Se_i/\theta) \in \mathfrak{R}$.

Suppose $Tr(A) = \bigcup\{Tr^i(A) \mid 1 \leq i \leq n\}$.

**Lemma 9.1.** Let $\mathfrak{R}$ be an axiomatizable and model complete class, $R = \bigcup\{e_iR \mid 1 \leq i \leq n\}$, where $n \in \omega$, $e_i^2 = e_i \in R$ $(1 \leq i \leq n)$, $\mathfrak{s}A \in \mathfrak{R}$. There exists a surjective mapping $\varphi : Tr(A) \to S_1(A)$ such that $\varphi|_{Tr^i(A)}$ is an injective mapping for any $i$, $1 \leq i \leq n$, where $S_1(A)$ is the set of 1–types over $A$.

**Proof.** Assume $\mathfrak{s}A \in \mathfrak{R}$ and $\langle \theta, I, \alpha \rangle \in Tr^i(A)$ for some $i$, $1 \leq i \leq n$. Write $p'(x)$ for the following type over $A$:

$$\{sx = a \mid \alpha(se_i) = a, se_i \in I\} \cup \{sx \neq a \mid se_i \notin I, a \in A\} \cup$$

$$\{sx = tx \mid \langle se_i, te_i \rangle \in \theta\} \cup \{sx \neq tx \mid \langle se_i, te_i \rangle \notin \theta\}.$$ We claim that the type $p'(x)$ is consist with the theory $T$ of the class $\mathfrak{R}$. We can assume that $A \cap Se_i = \emptyset$ (otherwise we replace $A$ by the isomorphic copy $A'$, $A' \cap Se_i = \emptyset$, a homomorphism $\alpha$ by $\alpha' = \beta\alpha$, where $\beta : A \to A'$ is an isomorphism). On the set $A \cup Se_i$ we define a relation $\sigma$:

$$\sigma = \{\langle \alpha(t), t \rangle \mid t \in I\}.$$ Suppose $\tau = \varepsilon \cup \sigma \cup \sigma^{-1} \cup \theta$, where $\varepsilon$ is a zero congruence of the polygon $\mathfrak{s}(A \cup Se_i)$. We claim that $\tau$ is a congruence of the polygon $\mathfrak{s}(A \cup Se_i)$. For this it is enough to prove that $\tau$ is a transitive relation. Assume $x, y, z \in A \cup Se_i$, $x\tau y \tau z$, $x \neq y$, $y \neq z$.

If $x\sigma y$ then $x = \alpha(y)$, $y \in I$. So $z = \alpha(y)$ or $y\theta z$. In the first case we have $x = y$, in the second case in view of $\theta$–saturation of the polygon $\mathfrak{s}I$ we have $z \in I$, and the equality $Ker\alpha = \theta \cap (I \times I)$ implies $\alpha(y) = \alpha(z)$, that is, $x\sigma z$.

If $x\sigma^{-1} y$ then $y = \alpha(x)$, $y \in A$, consequently, $y = \alpha(z)$, $x, y \in I$ and $x\theta z$. Thus, $\tau$ is a congruence of the polygon $\mathfrak{s}(A \cup Se_i)$. Let $\mathfrak{s}C = \mathfrak{s}(A \cup Se_i)/\tau$, $c = e_i/\tau$. Then an element $c$ realizes the type $p'(x)$ in $\mathfrak{s}C$. Furthermore, since $\mathfrak{s}A \in \mathfrak{R}$ and $\mathfrak{s}(Se_i/\theta) \in \mathfrak{R}$ it follows that $\mathfrak{s}C \in \mathfrak{R}$. Consequently, the type $\Sigma(x) = p'(x) \cup T \cup D(\mathfrak{s}A)$ is realized by element $c \in C$, where $D(\mathfrak{s}A)$ is a diagram of the polygon $\mathfrak{s}A$. Hence this type is consistent.
We claim that the type $p(x) = tp(c, A)$ of an element $c$ over a set $A$ is a unique complete type, containing the type $\Sigma(x)$. It is clear that $\Sigma(x) \subseteq p(x)$. Let $\Phi(x)$ be an atomic formula of the language $L_S(A)$, which is an enriching the language $L_S$ by adding in constant symbols for all elements $A$. If $\Phi(x)$ is a sentence then $\Phi(x) \in D(SA)$ or $\neg \Phi(x) \in D(SA)$. If $x$ is a free variable of the formula $\Phi(x)$, $\Phi(x) \not\in \Sigma$ and $\neg \Phi(x) \not\in \Sigma$ then the definition of $p'(x)$ implies that $\Phi(x)$ has the form of $sx = a$, where $se_i \in I$, $\alpha(se_i) \neq a$, $a \in A$. Since $se_i \in I$ it follows that $\alpha(se_i) = b \in A$, $b \neq a$, consequently, $(sx = b) \in \Sigma(x)$ and $\Sigma(x) \models \neg \Phi(x)$. Thus for any quantifier–free formula $\psi(x)$ either $\Sigma(x) \models \psi(x)$ or $\Sigma(x) \models \neg \psi(x)$. Since $T$ is a submodel complete theory, $T$ has elimination of quantifiers by Theorem 3.3. Consequently, $\Sigma(x) \models p(x)$ for any type $p(x) \in S_1(A)$. We correspond the type $p$ to the triple $(\theta, I, \alpha) \in Tr^i(A)$. The mapping $\varphi$ is built. Obviously, $\varphi|_{Tr^i(A)}$ is an injective mapping for any $i$, $1 \leq i \leq n$.

We claim that $\varphi$ is a surjective mapping. Let $p(x) \in S_1(A)$; $SB$ be a saturated model of $T$, realizing the type $p(x)$ and being elementary extension of $SA$; element $b_0 \in B$ realizes

the type $p(x)$. Since $\mathfrak{R}$ is axiomatizable class and $SA \in \mathfrak{R}$, $SB \in \mathfrak{R}$ and $SB_0 \in \mathfrak{R}$. On Corollary 2.1 there exist an idempotent $f \in R$ and an isomorphism $\eta : SB_0 \rightarrow SSf$ such that $\eta(b_0) = f$. Since $f \in R = \bigcup\{e_iR \mid 1 \leq i \leq n\}$ it follows that $f \in e_iS$, that is, $f = ef$ for a certain $e \in \{e_1, \ldots, e_n\}$. It is easy to understand that for any $s, r \in S$

\begin{equation}
se = re \implies sef = ref \iff sf = rf \Longleftrightarrow sx = rx \in p(x).
\end{equation}

Let

\begin{align*}
\theta &= \{(re, se) \in (Se)^2 \mid (rx = sx) \in p(x), r, s \in S\}, \\
I &= \{re \in Se \mid (rx = a) \in p(x), r \in S, a \in A\},
\end{align*}

$\alpha : I \rightarrow A$,

where $\alpha(re) = a$ for all $re \in I$ such that $(rx = a) \in p(x)$, $a \in A$.

In view of the completeness of type $p(x)$ and (9.1) it is easy to prove conditions (1)-(5) for the triple $(\theta, I, \alpha)$. Condition (6) follows from the correlation $Se/\theta \cong SB_0 \in \mathfrak{R}$. Clearly, $\varphi((\theta, I, \alpha)) = p(x)$. Lemma is proved. □

\textbf{Theorem 9.1} [Ste3]. Let the class $\mathfrak{R}$ for all regular polygons be axiomatizable and model complete. Then $|R| \geq \omega$, $R = \bigcup\{e_iR \mid 1 \leq i \leq n\}$ for some $n \in \omega$, $e_1, \ldots, e_n \in R$, $e_i^2 = e_i \ (1 \leq i \leq n)$, and

1) the class $\mathfrak{R}$ is stable;

2) the class $\mathfrak{R}$ is superstable if and only if for any $i$, $1 \leq i \leq n$, a semigroup $Se_i$ satisfies the ascending chain condition for left ideals;

3) if a semigroup $R$ is countable, then the class $\mathfrak{R}$ is $\omega$–stable if and only if for any $i$, $1 \leq i \leq n$, a semigroup $Se_i$ satisfies the the ascending chain condition for left ideals and has no more than countable number of the left congruence $\theta$, such that $S(Se_i/\theta) \in \mathfrak{R}.$

\textbf{Proof}. Let the class $\mathfrak{R}$ for all regular polygons be axiomatizable and model complete. Using Theorem 5.1 (3) we have $|R| \geq \omega$. The presentation of the semigroup $R$ as the union of the finite number of right ideals is established in Corollary 4.2.

Let $SA \in \mathfrak{R}$. Note that the number of all subpolygons $SI$ of polygon $SSe_i \ (1 \leq i \leq n)$ is not greater than $2^{|R|}$, the number of all left congruences $\theta$ of semigroup $Se_i \ (1 \leq i \leq n)$ is not greater than $2^{|R|^2}$, the number of all homomorphisms from the subpolygon $SI$ of the polygon $SSe_i \ (1 \leq i \leq n)$ to the polygon $SA$ is not greater than $2 |A|^{|R|}$. Thus, $|Tr(A)| \leq 2^{|R|} \cdot 2^{|R|^2} \cdot |A|^{|R|}$. 


If $|A| \leq 2^{|R|}$ then $|Tr(A)| \leq 2^{|R|}$ and, on Lemma 9.1, $|S_1(A)| \leq 2^{|R|}$. Consequently, the class $\mathcal{R}$ is stable and 1) is proved.

Suppose for any $i$, $1 \leq i \leq n$, the semigroup $S_{e_i}$ satisfies the ascending chain condition for left ideals. Then each subpolygon $sI$ of the polygon $sS_{e_i}$ $(1 \leq i \leq n)$ is finite-generated. Consequently, the number of such subpolygons is not greater than $|R|$ and the number of all homomorphisms from the subpolygon $sI$ of the polygon $sS_{e_i}$ $(1 \leq i \leq n)$ to the polygon $sA$ is not greater than $\omega \cdot |A| \cdot |R|$. So $|Tr(A)| \leq |A| \cdot |R| \cdot \omega \cdot |A|^{|R|}$. If $|A| \geq 2^{|R|}$ then $|Tr(A)| \leq |A|$ and on Lemma 9.1 $|S_1(A)| \leq |A|$. Consequently, the class $\mathcal{R}$ is superstable.

Assume $|R| = \omega$ and for any $i$, $1 \leq i \leq n$, the semigroup $S_{e_i}$ satisfies the ascending chain condition for left ideals and has no more than countable number of the left congruence $\theta$ such that $s(S_{e_i}/\theta) \in \mathcal{R}$. Then, on proved above, $|Tr(A)| \leq |A| \cdot |R| \cdot \omega$. If $|A| = \omega$ then a countability of the semigroup $R$ implies $|Tr(A)| \leq \omega$ and on Lemma 9.1 $|S_1(A)| \leq \omega$. Consequently, the class $\mathcal{R}$ is $\omega$-stable.

Let the class $\mathcal{R}$ be superstable. Since the class $\mathcal{R}$ is axiomatizable, the monoid $S$ is $\mathcal{R}$-superstable and, on Proposition 8.1, for any $i$, $1 \leq i \leq n$, the semigroup $S_{e_i}$ satisfies the ascending chain condition for left ideals. Thus 2) is proved.

Let the class $\mathcal{R}$ be $\omega$-stable. Then on Theorem 3.4 the class $\mathcal{R}$ is superstable. Let us show that for any $i$, $1 \leq i \leq n$, the semigroup $S_{e_i}$ has no more than countable number of the left congruences $\theta$ such that $s(S_{e_i}/\theta) \in \mathcal{R}$. Since $sR \in \mathcal{R}$ and $|R| = \omega$, it follows that $|S_1(R)| \leq \omega$. On Lemma 9.1, $|Tr^i(R)| \leq |S_1(R)|$ for any $i$, $1 \leq i \leq n$, that is $|Tr^i(R)| \leq \omega$. Let $\theta$ be a left congruence of the semigroup $sS_{e_i}$ such that $s(S_{e_i}/\theta) \in \mathcal{R}$. Suppose $I = S_{e_i}$. Since $s(S_{e_i}/\theta) = sS(e_i/\theta) \in \mathcal{R}$ it follows that on Corollary 2.1 there exist an idempotent $f \in R$ and an isomorphism $\alpha' : sS(e_i/\theta) \rightarrow sSf$. We construct a homomorphism $\alpha : sI \rightarrow sR$ as follows: $\alpha(a) = \alpha'(a/\theta)$ for any $a \in S_{e_i}$. Then $\langle \theta, I, \alpha \rangle \in Tr^i(R)$ and the number of the congruences of polyg $sS_{e_i}$ is not greater than $Tr^i(R)$. Since $|Tr^i(R)| \leq \omega$, the semigroup $S_{e_i}$ has no more than countable number of the left congruences $\theta$ such that $s(S_{e_i}/\theta) \in \mathcal{R}$ for any $i$, $1 \leq i \leq n$. Thus, 3) is proved. □

Note that the statements 1 and 2 of Theorem 9.1 we can derive as a consequence of Theorems 7.1 and 8.1 accordingly.

**Corollary 9.1.** The class of regular polygons over a countable semigroup is $\omega$-stable.

**Proof.** On Corollary 5.1, the class of regular polygons over countable group is axiomatizable and model complete. In a group $S$ a unique idempotent is unit, a unique left ideal is a group $S$ itself. Let $\theta$ be a left congruence of the group $S$ such that $s(S/\theta) \in \mathcal{R}$. Then $S/\theta = S \cdot 1/\theta$. On Corollary 2.1 there exists an isomorphism $\varphi : S \cdot 1/\theta \rightarrow S$ such that $\varphi(1/\theta) = 1$. If $t \in 1/\theta$ then $t \cdot 1/\theta = 1/\theta$, consequently, $t = 1$ and $|1/\theta| = 1$. Thus, on Theorem 9.1, the class of regular polygons over a countable semigroup is $\omega$-stable. □

We will use a formula $\exists^n x \Psi(x,x_1,\ldots,x_n)$ as an abbreviation of a formula "there exist exactly $n$ elements $x$ such that $\Psi(x,x_1,\ldots,x_n)$".

**Lemma 9.1.** Let $a,b,c \in R$ be such that $Sc \subseteq Sb \subseteq Sa$, $b = aa$, $c = b\beta$ and there exist a formula $\Phi(x,y,z)$ and $n \in \omega$, $n > 0$, such that $\Phi(S,S,c) = \Phi(Sa \setminus Sb,Sb \setminus Sc,c)$ and $sSa = \Phi(a,b,c) \land \forall y(\beta y = c \land \exists x(\Phi(x,y,c) \land ax = y) \rightarrow \exists^n x(\Phi(x,y,c) \land ax = y))$. Then $S$ is not an $\mathcal{R}$-$\omega$-stabilizer.
Proof. Assume the conditions of lemma are satisfied. Let $K \subseteq \omega$; $sM_K = \bigcup_{i \in K} sS(a, j^i, K)/\theta_K$; $sM = \bigcup_{K \subseteq \omega} sM_K$, where $S(a, j^i, K) = \{\langle x, j^i, K \rangle | x \in Sa\}$, $s \cdot \langle x, j^i, K \rangle = \langle sx, j^i, K \rangle$ for all $x \in Sa$, $i \in K$, $j \leq i$; $\theta_K$ be a congruence of the polygon $\bigcup_{i \in K} sS(a, j^i, K)$, which is generated by a set $\{\langle \langle b_1, j_1^i, K \rangle, \langle b_2, j_2^i, K \rangle \rangle \mid j_1, j_2 \leq i, i_1 \leq i, i_2 \leq i_1, i_2, i \in K\}$. White $\Gamma_K(x)$ for a set of the formulae
\[
\{\exists y \exists^{n(i+1)} z(x = \beta y \land y = \alpha z \land \Phi(z, y, x) \mid i \in K\} \cup \\
\{\exists y \exists^{n(i+1)} z(x = \beta y \land y = \alpha z \land \Phi(z, y, x) \mid i \notin K\}
\]
It is not difficult to understand that the set of the formulae $\Gamma_K(x)$ is realized by an element $\langle c, j^i, K \rangle /\theta_K$ and is not realized by element $\langle c, j^i, K' \rangle /\theta_{K'}$ for any $K' \subseteq \omega$, $K \neq K'$. Consequently, $|S(\emptyset)| = 2^n$, that is, $S$ is not $\mathcal{R}$--$\omega$--stabilizer.

Example 9.1. We construct a non $\mathcal{R}$--$\omega$--stabilizer and non stabilizer which is $\mathcal{R}$--superstabilizer.

Let $S = \{a, b, c \cup \{\alpha, \beta\} \cup \{1\}$, where $\langle \alpha, \beta \rangle$ is a free two--generated commutative semigroup with $\alpha$ and $\beta$ generators. On $S$ we define the binary operation as follows: 1 unit in $S$, $a \cdot x = a$, $b \cdot x = x$ for any $x \in \{a, b\}$, $c \cdot y = y \cdot c = y$ for any $y \in \{a, b, c\}$, $\gamma \cdot z = z \cdot \gamma = z$ for any $z \in \{a, b, c\}$, $\gamma \in \{\alpha, \beta\}$. It is easy to check that $S$ equipped with the operation is a monoid, $\{a, b, c\}$ is a set of all idempotents of monoid $S$, $Sa \subseteq Sb \subseteq Sc = S$ and $R = Sc = cS$. On Theorem 8.1 $S$ is $\mathcal{R}$--superstabilizer. For the elements $a, b, c$, a formula $\Phi(x, y, z) = bx = y \land x \neq y$, and an $n = 1$ the conditions of Lemma 9.4 are hold. Consequently, $S$ is non $\mathcal{R}$--$\omega$--stabilizer. Since a monoid $S$ is not linear ordered, $S$ is non stabilizer.

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