Hypersurfaces of a Projective Randers conformal change

V. K. Chaubey and *Pradeep Kumar

Department of Applied Sciences, Buddha Institute of Technology
Sector-7, Gida, Gorakhpur (U.P.)-273209, INDIA, E-Mail: vkchaubey@outlook.com

*Department of Mathematics and Statistics, DDU Gorakhpur University
Gorakhpur (U.P.)-273209, INDIA

Abstract

In the year 1984 Shibata investigated the theory of a change which is called a $\beta$-change of a Finsler metric. On the other hand in 1985 a systematic study of geometry of hypersurfaces in Finsler spaces was given by Matsumoto. In the present paper is devoted to the study of a condition for a Randers conformal change to be projective and find out when a totally geodesic hypersurface $F^{n-1}$ remains to be a totally geodesic hypersurface $F^{n-1}$ under the projective Randers conformal change. Further obtained the condition under which a Finslerian hypersurfaces given by the projective Randers conformal change are projectively flat.

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1 Introduction

Let \((M^n, L)\) be an \(n\)-dimensional Finsler space on a differentiable manifold \(M^n\), equipped with the fundamental function \(L(x, y)\). In 1984, Shibata [13] introduced the transformation of Finsler metric:

\[ L'(x, y) = f(L, \beta) \]

where \(\beta = b_i(x)y^i\), \(b_i(x)\) are components of a covariant vector in \((M^n, L)\) and \(f\) is positively homogeneous function of degree one in \(L\) and \(\beta\). This change of metric is called a \(\beta\)-change.

The conformal theory of Finsler spaces has been initiated by M.S. Knebelman [8] in 1929 and has been investigated in detail by many authors [2, 3, 4, 8] etc. The conformal change is defined as

\[ L(x, y) \rightarrow e^{\sigma(x)}L(x, y), \]

where \(\sigma(x)\) is a function of position only and known as conformal factor.

On the other hand in 1985 M. Matsumoto investigated the theory of Finslerian hypersurface [10]. He has defined three types of hypersurfaces that were called a hyperplane of the first, second and third kinds.

Finslerian Hypersurfaces for a change in a Finsler metric was studied by several authors [5, 6, 7, 12, 16] and obtained so many important results in the standpoint of Finsler geometry.

In the year 2012 Shukla and Mishra we studied Randers conformal change by defining as

\[ L(x, y) \rightarrow L^*(x, y) = e^{\sigma(x)}L(x, y) + \beta(x, y), \]

where \(\sigma(x)\) is a function of \(x\) and \(\beta(x, y) = b_i(x)y^i\) is a 1-form on \(M^n\). This change generalizes various types of changes. When \(\beta = 0\), it reduces to a conformal change. When \(\sigma = 0\), it reduces to a Randers change. When \(\beta = 0\) and \(\sigma\) is a non-zero constant then it reduces to homothetic change.

In the present paper we have obtained the condition for a Randers conformal change to be projective and find out when a totally geodesic hypersurface \(F^{n-1}\) remains to be a totally geodesic hypersurface \(F^{n-1}\) under the projective Randers conformal change. Further we obtained the condition under which a Finslerian hypersurfaces given by the projective Randers conformal change are projectively flat.
2 Preliminaries

Let \((M^n, L)\) be a Finsler space \(F^n\), where \(M^n\) is an \(n\)-dimensional differentiable manifold equipped with a fundamental function \(L\). A change in fundamental metric \(L\), defined by equation (1.2), is called Randers conformal change, where \(\sigma(x)\) is conformal factor and function of position only and \(\beta(x, y) = b_i(x)y^i\) is a 1-form on \(M^n\). A space equipped with fundamental metric \(L^*(x, y)\) is called Randers conformally changed space \(F^*n\).

Differentiating equation (1.2) with respect to \(y^i\), the normalized supporting element \(l_i^* = \partial_i L^*\) is given by
\[
(2.1) \quad l_i^*(x, y) = e^{\sigma(x)}l_i(x, y) + b_i(x),
\]
where \(l_i = \partial_i L\) is the normalized supporting element in the Finsler space \(F^n\).

Differentiating (2.1) with respect to \(y^j\), the angular metric tensor \(h_{ij}^* = L^*\partial_i \partial_j L^*\) is given by
\[
(2.2) \quad h_{ij}^* = e^{\sigma(x)}\frac{L^*}{L}h_{ij},
\]
where \(h_{ij} = L\partial_i \partial_j L\) is the angular metric tensor in the Finsler space \(F^n\).

Again the fundamental tensor \(g_{ij}^* = \partial_i \partial_j L^*\) is given by
\[
(2.3) \quad g_{ij}^* = \tau g_{ij} + b_i b_j + e^{\sigma(x)}L^{-1}(b_i y_j + b_j y_i) - \beta e^{\sigma(x)}L^{-3} y_i y_j,
\]
where we put \(y_i = g_{ij}(x, y)y^j\), \(\tau = e^{\sigma(x)}\frac{L^*}{L}\) and \(g_{ij}\) is the fundamental tensor of the Finsler space \(F^n\). It is easy to see that the \(\det(g_{ij}^*)\) does not vanish, and the reciprocal tensor with components \(g^{*ij}\) is given by
\[
(2.4) \quad g^{*ij} = \tau^{-1}g^{ij} + \phi y^i y^j - L^{-1}\tau^{-2}y^i b^j + y^j b^i,
\]
where \(\phi = e^{-2\sigma(x)}(L e^{\sigma(x)} b^2 + \beta)L^{-3}\), \(b^2 = b_i b^i\), \(y^i = g^{ij} b_j\) and \(g^{ij}\) is the reciprocal tensor of \(g_{ij}\).

Here it will be more convenient to use the tensors
\[
(2.5) \quad h_{ij} = g_{ij} - L^{-2}y_i y_j, \quad a_i = \beta L^{-2} y_i - b_i
\]
both of which have the following interesting property:
\[
(2.6) \quad h_{ij} y^i = 0, \quad a_i y^i = 0
\]
Now differentiating equation (2.3) with respect to $y^k$ and using relation (2.5),
the Cartan covariant tensor $C^\ast$ with the components $C^\ast_{ijk} = \partial_k(\frac{g^T_{ij}}{2})$ is given as:

$$
(2.7) \quad C^\ast_{ijk} = \tau[C_{ijk} - \frac{1}{2L^*}(h_{ij}a_k + h_{jk}a_i + h_{ki}a_j)]
$$

where $C_{ijk}$ is (h)hv-torsion tensor of Cartan’s connection $C\Gamma$ of Finsler space $F^n$.

In order to obtain the tensor with the components $C^\ast_{ikj}$, paying attention to (2.6), we obtain from (2.4) and (2.7),

$$
(2.8) \quad C^\ast_{ikj} = C_{ikj} - \frac{1}{2L^*}(h_i^i a_k + h_j^j a_i + h_k^k a_j) - (\tau L)^{-1}C_{ikr}y^i b^r - \frac{\tau^{-1}}{2LL^*}(2a_i a_k + a^2 h_{ik})y^j
$$

where $a_i a^i = a^2$. We denote by the symbol $(|)$ the h-covariant differentiation with respect to the Cartan connection $C\Gamma = (F^i_{jk}, N^i_{j}, C^i_{jk})$ and put

$$
(2.9) \quad 2E_{ij} = b_{ij} + b_{ji} \quad 2F_{ij} = b_{ij} - b_{ji}
$$

Now we deal with well-known functions $G^i(x, y)$ which are (2)p-homogeneous in $y^i$ and are written as $2G^i = \gamma^i_{jk}y^j y^k$ by putting $\gamma^i_{jk} = g^r_{ir}(\partial_k g_{jr} + \partial_j g_{kr} - \partial_r g_{jk})$.

Owing to (2.3) and (2.4), a straightforward calculation leads to

$$
(2.10) \quad G^\ast^i(x, y) = \frac{\gamma^i_{jk}y^j y^k}{2} = G^i + D^i
$$

where the vector $D^i$ is given by

$$
D^i = \frac{1}{2}\{\tau^{-1}g^{ir} + 2\phi g^y y^r - 2L(-1)\tau^{-1}(y^b b^r + y^r b^i)}\{\tau_0 2b_r E_{00} + 4\beta F_{r0} - 2l_r - \tau_{ir} + \epsilon^r L(-1)\{2\sigma_0 (b_r + \beta y_r) - 2\beta \sigma_{ir} 4F_{r0} + 2E_{00} y_{r}\} - \epsilon^r L(-3)\{2\beta_0 y_{r} - \beta_{ir}\} - \beta \epsilon^r L(-3)\{2\sigma_0 y_{r} - \sigma_{ir}\}\}.
$$

$F^i_j = g^b F_{bj}$ and the subscript ’0’ means the contraction by $y^i$.

3 Relation between projective change and Randers conformal change

For two Finsler spaces $F^n = (M^n, L)$ and $F^{*n} = (M^n, L^*)$, if any geodesic on $F^n$ is also a geodesic on $F^{*n}$ and the inverse is true, the change $L \rightarrow L^*$
of the metric is called \textit{projective}. A geodesic on $F^n$ is given by a system of differential equations

\begin{equation}
\frac{d^2y^i}{dt^2} + 2G^i(x, y) = y^i, \quad y^i = \frac{dx^i}{dt}
\end{equation}

where $G^i(x, y)$ are \(2\) \(p\) -homogeneous functions in \(y^i\). We are now in a position to find a condition for a Randers conformal change to be projective. For this purpose we deal with Euler-Lagrange equations $B_i = 0$, where $B_i$ is defined by

$$B_i = \partial_i L - \frac{d}{dt}(\partial_i L).$$

Therefore from the Euler-Lagrange differential equations $B_i^* = 0$ for $F^*$ are given by

$$B_i^* = e^\sigma B_i + e^\sigma L \partial_i \sigma + \partial_i \beta - \frac{db}{dt} = 0$$

Thus the above equation can be written as

\begin{equation}
B_i^* = e^\sigma B_i + A_i
\end{equation}

where $A_i$ is a covariant vector and defined as $A_i = e^\sigma L \partial_i \sigma + \partial_i \beta - \frac{db}{dt}$. Thus we have

\textbf{Proposition 3.1.} Let $F \ast n = (M^n, L^*)$ be an \(n\)-dimensional Finsler space obtained from the Randers Conformal change of the Finsler space $F^n = (M^n, L)$, and 1-form metric, then the Finsler metric $L^*$ is projective if the covariant vector $A_i$ of the equation (3.2) vanishes identically.

\section{Hypersurface given by projective Randers conformal change}

Hereafter, we assume that metrics $L^2$ and $L^{*2}$ are positive-definite respectively and we consider hypersurfaces. According to [9], a hypersurface $M^{n-1}$ of the underlying smooth manifold $M^n$ may be parametrically represented by the equation $x^i = x^i(u^\alpha)$, where $u^\alpha$ are Gaussian coordinates on $M^{n-1}$ and Greek indices vary from 1 to \(n-1\). Here we shall assume that the matrix consisting of the projection factors $B^i_\alpha = \frac{\partial^i}{\partial u^\alpha}$ is of rank \(n-1\). The following notations are also employed:

$$B^{i}_{\alpha \beta} = \frac{\partial^2 x^i}{\partial u^\alpha \partial u^\beta}, \quad B^{i}_{\alpha 0} = v^\alpha B^{i}_{\alpha \beta}$$

If the supporting element $y^i$ at a point $(u^\alpha)$ of $M^{n-1}$ is assumed to be tangential to $M^{n-1}$, we may then write $y^i = B^i_\alpha (u)v^\alpha$ i.e. \(v^\alpha\) is thought of as the supporting element of $M^{n-1}$ at the point $(u^\alpha)$. Since the function
\( \bar{L}(u, v) = L\{x(u), y(u, v)\} \) gives rise to a Finsler metric of \( M^{n-1} \), we get a \((n - 1)\)-dimensional Finsler space \( F_{n-1} = \{M^{n-1}, \bar{L}(u, v)\} \).

At each point \((u^a)\) of \( F_{n-1} \), the unit normal vector \( N^i(u, v) \) is defined by

\[
(4.1) \quad g_{ij}B^i_aN^j = 0, \quad g_{ij}N^iN^j = 1
\]

If \( B^a_i, N_i \) is the inverse matrix of \((B^i_a, N^i)\), we have

\[
B^a_iB^i_\beta = \delta^\beta_\alpha, \quad B^a_iN_i = 0, \quad N^iN_i = 1 \quad \text{and} \quad B^i_aB^a_j + N^iN_j = \delta^i_j.
\]

Making use of the inverse matrix \((g^{\alpha\beta})\) of \((g_{\alpha\beta})\), we get

\[
(4.2) \quad B^a_i = g^{\alpha\beta}g_{ij}B^j_\beta, \quad N_i = g_{ij}N^j
\]

For the induced Cartan’s connection \( IC = (F^a_\beta, N^i, C^a_\beta_\gamma) \) on \( F_{n-1} \), the normal curvature vector \( H^i_\alpha \) is given by

\[
(4.3) \quad H^i_\alpha = N_i(B^i_a + N^iB^a_j)\]

Consider a Finslerian hypersurface \( F^{n-1} = \{M^{n-1}, \bar{L}(u, v)\} \) of the \( F^n \) and another Finslerian hypersurface \( F^{*n-1} = \{M^{n-1}, \bar{L}^*(u, v)\} \) of the \( F^{*n} \) given by the Randers conformal change. Let \( N^i \) be the unit vector at each point of \( F^{n-1} \) and \((B^a_i, N_i)\) be the inverse matrix of \((B^i_a, N^i)\). The function \( B^a_i \) may be considered as components of \((n - 1)\) linearly independent tangent vectors of \( F^{n-1} \) and they are invariant under Randers conformal change. Thus we shall show that a unit normal vector \( N^{*i}(u, v) \) of \( F^{*n-1} \) is uniquely determined by

\[
(4.4) \quad g^{*i}_jB^{*i}_aN^{*j} = 0, \quad g^{*i}_jN^{*i}N^{*j} = 1
\]

Contracting (2.3) by \( N^iN^j \) and paying attention to (4.1) and the fact that \( l_iN^i = 0 \), we have

\[
(4.5) \quad g^{*i}_jN^iN^j = \tau + (b_iN^i)^2
\]

Therefore we obtain

\[
 g^{*i}_j\left\{ \pm \frac{N^i}{\sqrt{\tau + (b_iN^i)^2}} \right\}\left\{ \pm \frac{N^j}{\sqrt{\tau + (b_iN^i)^2}} \right\} = 1
\]

Hence we can put

\[
(4.6) \quad N^{*i} = \frac{N^i}{\sqrt{\tau + (b_iN^i)^2}}
\]

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where we have chosen the positive sign in order to fix an orientation.

Using equation (2.3), (4.6) and from first condition of (4.4) we have

\[(4.7) \quad (b_i B^i_\alpha + e^{\sigma(x)} l_i B^i_\alpha) \frac{b_j N^j}{\sqrt{\tau + (b_i N^i)^2}} = 0 \]

If \(b_i B^i_\alpha + e^{\sigma(x)} l_i B^i_\alpha = 0\), then contracting it by \(v^\alpha\) and using \(y^i = B^i_\alpha v^\alpha\) we get \(\beta + e^{\sigma(x)} L = L^* = 0\) which is contradiction to the assumption that \(L^* > 0\). Hence \(b_i N^i = 0\). Therefore equation (4.6) can be written as

\[(4.8) \quad N^{*i} = \frac{1}{\sqrt{\tau}} N^i \]

Summarizing the above Shukla, Chaubey and Mishra [16] obtained the following result

**Proposition 4.1.** If \(\{(B^i_\alpha, N^i), \alpha = 1, 2, \ldots, (n-1)\}\) be the filed of linear frame of the Finsler space \(F^n\), there exist a field of linear frame \(\{(B^i_\alpha, N^{*i} = \frac{1}{\sqrt{\tau}} N^i), \alpha = 1, 2, \ldots, (n-1)\}\) of the Finsler space \(F^m\) such that (15) is satisfied along \(F^{*n-1}\) and then \(b_i\) is tangential to both the hypersurfaces \(F^{n-1}\) and \(F^{*n-1}\).

The quantities \(B^{*i}_{\alpha}\) are uniquely defined along \(F^{*n-1}\) by

\[B^{*i}_{\alpha} = g^{*\alpha\beta} g^i_{\beta j} B^j_\beta \]

where \(g^{*\alpha\beta}\) is the inverse matrix of \(g^i_{\alpha j}\). Let \(\{B^{*i}_{\alpha}, N^{*i}\}\) be the inverse matrix of \(\{B^i_{\alpha}, N^i\}\), then we have

\[B^i_{\alpha} B^{*i}_{\beta} = \delta^\beta_{\alpha}, \quad B^i_{\alpha} N^*_i = 0, \quad N^{*i} N^*_i = 1 \]

Furthermore \(B^i_{\alpha} B^{*i}_{\alpha} + N^{*i} N^*_j = \delta^i_j\). We also get \(N^*_i = g^i_{\beta j} N^{*j}\) which in view of (2.1), (2.3) and (4.8) gives

\[(4.9) \quad N^*_i = \sqrt{\tau} N^i \]

Now we assume that a Randers conformal change of the metric is projective. Using (2.10) and Proposition 3.1, we have

\[(4.10) \quad D^i = G^{*i} - G^i \]

Since \(D^i_j = \dot{\delta}_j D^i \) and \(N^i_j = \dot{\delta}_j G^i\) the above gives

\[(4.11) \quad D^i_j = N^{*i}_j - N^i_j \]

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Further contracting the above equation by $N_iB^j_\alpha$ we have

\[(4.12)\]

\[N_iD^j_iB^j = 0\]

If each geodesic of $F^{(n_1)}$ with respect to the induced metric is also a geodesic of $F^{(n)}$, then $F^{(n_1)}$ is called totally geodesic. A totally geodesic hypersurface $F^{(n_1)}$ is characterized by $H_\alpha = 0$.

From equation (4.3), (4.9) and (4.11) we have

\[(4.13)\]

\[H^*_\alpha = \sqrt{\tau}H_\alpha + N_iD^j_iB^j\]

Thus using the equation (4.12) in the above equation we have

\[(4.14)\]

\[H^*_\alpha = \sqrt{\tau}H_\alpha\]

Thus we have

**Theorem 4.1.** A hypersurface $F^{n-1}$ of a Finsler space $F^n(n > 3)$ is totally geodesic, if and only if the hypersurface $F^*(n-1)$ of the space $F^*$ obtained from $F^n$ by a projective Randers conformal change, is totally geodesic.

### 5 Hypersurfaces of Projectively Flat Finsler spaces

In this section, we shall consider a projective Randers conformal change and we are concerned with the Berwald connection $B\Gamma$ on $F^n = (M^n, L)$ and $B\bar{\Gamma}$ on $F^*n = (M^m, L)$. In the theory of projective changes in Finsler spaces, we have two essential projective invariants, one is the Weyl torsion tensor $W^h_{ij}$ and the other is the Douglas tensor $D^h_{ijk}$, so that under the projective Randers conformal change, we get $W^*_i{}^h_{ij} = W^h_{ij}$ and $D^*_i{}^h_{ijk} = D^h_{ijk}$.

Now we are concerned with a projectively flat Finsler space defined as follows: If there exists a projective change $L \to L^*$ of a Finsler space $F^n = (M^n, L)$ such that the Finsler space $F^*n = (M^m, L)$ is a locally Minkowski space then $F^n$ is called projectively flat Finsler space. We have already known the following theorems:

**Theorem 5.1.** [6] A Finsler space $F^n(n > 2)$ is projectively flat, if and only if $W^h_{ij} = 0$ and $D^h_{ijk} = 0$.

**Theorem 5.2.** [14] A Finsler space $F^n(n > 3)$ is projectively flat then the totally geodesic hypersurface $F^{n-1}$ is also projectively flat.
Thus from theorem (4.1), theorem (5.1) and theorem (5.2) we have

**Theorem 5.3.** Let $F^n(n > 3)$ be a projectively flat Finsler space. If the hypersurface $F^{n-1}$ is totally geodesic, then the hypersurface $F^{*(n-1)}$ of the space $F^*$ obtained from $F^n$ by a projective Randers conformal change, is projectively flat.

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