Comparison principle for non-cooperative elliptic systems

G. Boyadzhiev

28.02.2007

1 Introduction

In this paper are considered weakly coupled linear elliptic systems of the form

\[ L_M u = 0 \text{ in a bounded domain } \Omega \in \mathbb{R}^n \text{ with smooth boundary} \]

and boundary data \( u(x) = g(x) \) on \( \partial \Omega \), where \( L_M = L + M \), \( L \) is a matrix operator with null off-diagonal elements \( L = \text{diag}(L_1, L_2, \ldots, L_N) \), and matrix \( M = \{ m_{ik}(x) \}_{i,k=1}^N \). Scalar operators

\[ L_k u_k = -\sum_{i,j=1}^n D_j \left( a_{kk}^{ij}(x) D_i u_k \right) + \sum_{i=1}^n b_{ik}^i(x) D_i u_k + c_k u_k \text{ in } \Omega \]

are uniformly elliptic ones for \( k = 1, 2, \ldots, N \), i.e. there are constants \( \lambda, \Lambda > 0 \) such that

\[ \lambda |\xi|^2 \leq \sum_{i,j=1}^n a_{kk}^{ij}(x) \xi_i \xi_j \leq \Lambda |\xi|^2 \]

for every \( k \) and any \( \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \).

Coefficients \( c_k \) and \( m_{ik} \) in (1) are supposed continuous in \( \overline{\Omega} \), and \( a_{kk}^{ij}(x), b_{ik}^i(x) \in W^{1,\infty}(\Omega) \cap C(\overline{\Omega}) \).

Quasi-linear weakly coupled elliptic systems

\[ Q^l(u) = -\text{div} a^l(x, u^l, Du^l) + F^l(x, u^1, \ldots, u^N, Du^l) = f^l(x) \text{ in } \Omega \]
(4) \[ u^l(x) = g^l(x) \text{ on } \partial \Omega \]

\[ l = 1, \ldots, N \] are considered as well.

System (3) is supposed uniformly elliptic one, i.e. there are continuous and positive functions \( \lambda(|u|), \Lambda(|u|), |u| = \left( (u^1)^2 + \ldots + (u^N)^2 \right)^{1/2} \), such that \( \lambda(s) \) is monotone-decreasing one, \( \Lambda(s) \) is monotone increasing one and

\[ \lambda(|u|) \left| \xi^l \right|^2 \leq \sum_{i,j=1}^N \frac{\partial a^l}{\partial p_j} (x, t, u^1, \ldots, u^N, p^l) \xi^l_i \xi^l_j \leq \Lambda(|u|) \left| \xi^l \right|^2 \]

for every \( u^l \) and \( \xi^l = (\xi^l_1, \ldots, \xi^l_n) \in \mathbb{R}^n, l = 1, 2, \ldots, N. \)

The coefficients \( a^l(x, u, p), F^l(x, u, p), f^l(x), g^l(x) \) are supposed to be at least measurable functions with respect to the \( x \) variable and locally Lipschitz continuous on \( u^l, u \) and \( p \), i.e.

\[ \left| F^l(x, u, p) - F^l(x, v, q) \right| \leq C(K) \left( |u - v| + |p - q| \right), \]

\[ \left| a^l(x, u^l, p) - a^l(x, v^l, q) \right| \leq C(K) \left( |u^l - v^l| + |p - q| \right) \]

for every \( (x) \in \Omega, |u| + |v| + |p| + |q| \leq K, l = 1, \ldots, N. \)

Hereafter by \( f^-(x) = \min(f(x), 0) \) and \( f^+(x) = \max(f(x), 0) \) are denoted the non-negative and, respectively, the non-positive part of the function \( f \). The same convention is valid for matrices as well. For instance, we denote by \( M^+ \) the non-negative part of \( M \), i.e.\( M^+ = \{m^+_{ij}(x)\}_{i,j=1}^N \).

This paper concerns the validity of the comparison principle for weakly-coupled elliptic systems. Let us briefly recall the definition of the comparison principle in a weak sense for linear systems.

The comparison principle holds in a weak sense for the operator \( L_M \) if \( (L_M u, v) \leq 0 \) and \( u|_{\partial \Omega} \leq 0 \) imply \( (u, v) \leq 0 \) in \( \Omega \) for every \( v > 0, v \in \left( W^{1,\infty}(\Omega) \cap C_0(\Omega) \right)^N \) and \( u \in \left( W^{1,\infty}(\Omega) \cap C(\Omega) \right)^N \).

As it is well-known, there is no comparison principle for an arbitrary elliptic system /see Theorem 6 below/. On the other hand, there are broad classes of elliptic systems, such that the comparison principle holds for their members.
According to Theorem 1 below, one of these classes can be constructed using the following condition:

\[(6) \quad \text{There is real-valued principal eigenvalue } \lambda_{\Omega_0} \text{ of } L_M \text{ and its adjoint operator } L^*_M \text{ for every } \Omega_0 \subseteq \Omega, \text{ such that the corresponding eigenfunctions } w_{\Omega_0}, w_{\Omega_0} \in \left(W^{2, \infty}_{\text{loc}}(\Omega_0) \cap C_0(\overline{\Omega_0})\right)^N \text{ are positive ones.} \]

\[\blacksquare\]

**Remark 1**: By adjoint operator we mean \(L^*_M = L^* + M^t, \) \(L^* = \text{diag}(L^*_1, L^*_2, ..., L^*_N),\) and \(L^*_k\) are \(L^2\)-adjoint operators to \(L_k.\) The principal eigenvalue is the first one, or the smallest eigenvalue.

More precisely, the class is \(C^6 = \{L_M \text{ satisfies (6) and } \lambda_{\Omega_0} > 0 \text{ for every } \Omega_0 \subseteq \Omega\} \) i.e. \(C^6\) contains the elliptic systems possessing a positive principal eigenvalue with positive corresponding eigenfunction in \(\Omega_0.\) In this case the necessary and sufficient condition for the validity of the comparison principle for systems (Theorem 1) is the same as the one for a single equation (See [2]).

**Theorem 1**: Assume that (2) and (6) are satisfied. The comparison principle holds for system (1) if the principal eigenvalue \(\lambda_{\Omega_0} > 0, \) where \(\lambda_{\Omega_0}\) is the principal eigenvalue of the operator \(L_M\) on \(\Omega_0 \subseteq \Omega.\) If the principal eigenvalue \(\lambda = \lambda_\Omega \leq 0,\) then the comparison principle does not hold.

If we consider classical solutions, then comparison principle holds if and only if \(\lambda = \lambda_\Omega \leq 0.\)

**Proof**: 1. Assume that the comparison principle does not hold for \(L_M.\) Let \(\underline{u}, \underline{v} \in \left(W^{1, \infty}(\Omega) \cap C(\overline{\Omega})\right)^N\) be an arbitrary weak sub- and super-solution of \(L_M.\) Then \(u = \underline{u} - \underline{v} \in \left(W^{1, \infty}(\Omega) \cap C(\overline{\Omega})\right)^N\) is a weak sub-solution of \(L_M,\) i.e. \((L_M(u), v) \leq 0\) in \(\Omega\) for any \(v \in \left(W^{1, \infty}(\Omega) \cap C_0(\overline{\Omega})\right)^N, v > 0\) and \(u^+ \equiv 0\) on \(\partial\Omega.\) Suppose \(u^+ \neq 0.\) Then

\[0 \geq (L_M u^+, w_{\Omega_0}) = (u^+, L^*_M w_{\Omega_0}) = \lambda (u^+, w_{\Omega_0}) > 0\]

for \(\lambda_{\Omega_0}, w_{\Omega_0}\) defined in (6).

Therefore \(u^+ \equiv 0,\) i.e for any sub- and super-solution of \(L_M\) we obtain \(\underline{u} \leq \underline{v}.\)

2. Suppose \(\lambda \leq 0\) and \(\tilde{w}\) is the corresponding positive eigenfunction of \(L_M.\) Then \(\tilde{w} > 0\) but \(L_M(\tilde{w}) = \lambda \tilde{w} \leq 0.\) Therefore the comparison principle
does not hold for (1). □

Unfortunately, there are some odds in the application of this general theorem since the condition (6) is uneasy to check. First of all, the system (1) may have no principal eigenvalue at all (See [10]). Another obstacle is the computation of $\lambda$ even when it exists.

Comparison principle holds for members of another broad class, so-called cooperative elliptic systems, i.e. the systems with $m_{ij}(x) \leq 0$ for $i \neq j$ (See [9]). Most results on the positivity of the classical solutions of linear elliptic systems with non-negative boundary data are obtained for the cooperative systems (See [6,7,13,15,16,18,19,21]). As it is well known, the positiveness and the comparison principle are equivalent for linear systems. As for the non-linear ones, the positiveness of the solutions is a weaker statement than the comparison principle; positiveness can hold without ordering of sub-and super-solutions or uniqueness of the solutions at all.

Comparison principle for the diffraction problem for weakly coupled quasi-linear elliptic systems is proved in [3].

The spectrum properties of the cooperative $L_M$ are studied as well. A powerful tool in the cooperative case is the theory of the positive operators (See [17]) since the inverse operator of the cooperative $L_{M^-}$ is positive in the weak sense. Unfortunately, this approach cannot be applied to the general case $M \neq M^-$ since $(L_M)^{-1}$ is not a positive operator at all. Nevertheless in [20] is proved the validity of the comparison principle for non-cooperative systems obtained by small perturbations of cooperative ones.

Using unconventional approach, an interesting result is obtained in [14] for two-dimensional system (1) with $m_{11} = m_{22} = 0$ and $m_{ij} = p_i(x) > 0$ for $i \neq j$, $i = 1, 2$. Theorem 6.5 [14] states the existence of a principal eigenvalue with positive principal eigenfunction in the cone $C_U = P_U \times (-P_U)$, where $P_U$ is the cone of the positive functions in $W_1^\infty(\Omega)$. In the same paper, Theorem 6.3, are provided sharp conditions for the validity of the comparison principle with respect to the order in $C_U = P_U \times (-P_U)$, i.e. $(u_1, u_2) \leq (v_1, v_2)$ if and only if $u_1 \leq v_1$ and $u_2 \geq v_2$.

In [12] are studied existence and local stability of positive solutions of systems with $L_k = -d_k \Delta$, linear cooperative and non-linear competitive part, and Neumann boundary conditions. Theorem 2.4 in [12] is similar to Theorem 2 in the present article for $L_k = -d_k \Delta$.

Let us recall that the comparison principle was proved in [11] for the viscosity sub-and super-solutions of general fully non-linear elliptic systems $G^l(x, u^l, ...u^N, Du^l, D^2u^l) = 0, l = 1, ...N$ /See also the references there/. The systems considered in [11] are degenerate elliptic ones and satisfy the
same structure-smoothness condition as the one for a single equation. The first main assumption in [11] guarantees the quasi-monotonicity of the system. Quasi-monotonicity in the non-linear case is an equivalent condition to the cooperativeness in the linear one.

The second main assumption in [11] comes from the method of doubling of the variables in the proof.

This work extends the results obtained for cooperative systems to the non-cooperative ones. The general idea is the separation of the cooperative and competitive part of system (1). Then using the appropriate spectral properties of the cooperative part, in Theorems 3 and 4 are derived conditions for the validity of the comparison principle for the initial system. In particular in Theorem 3 is employed the fact that irreducible cooperative system possesses a principal eigenvalue and the corresponding eigenfunction is a positive one, i.e. condition (6) holds. This way are obtained some sufficient conditions for validity of the comparison principle for the non-cooperative system as well. Analogously, in Theorem 4 are derived the corresponding conditions for the validity of comparison principle for competitive systems. The conditions derived in Theorems 3 and 4 are not sharp.

Since predator-prey systems are basic model example for non-cooperative systems, in Theorem 5 is adapted the main idea of Theorem 4 to systems which cooperative part is a triangular matrix. Sufficient condition for the validity of comparison principle for predator-prey systems is derived in Theorem 5.

In Theorems 6 and 7 are given conditions for failure of the comparison principle.

The results of Theorems 3 and 4 are adapted to quasi-linear systems in Theorem 8.

2 Comparison principle for linear elliptic systems

As a preliminary statement we need the following well known fact

**Theorem 2**: Every irreducible cooperative system $L_{M-}$ has unique principal eigenvalue and the corresponding eigenfunction is positive.

The principal eigenfunction for linear operators is unique up to positive multiplicative constants, but for our purpose the positiveness is of importance.

In fact, Theorem 2 is in the scope of Theorems 11 and 12 in [1]. Theorems
11 and 12 in [1] concern second order cooperative linear elliptic systems with cooperative boundary conditions and are more general than Theorem 2. In sake of completeness, a sketch of the proof of Theorem 2 follows. It is based on the idea of adding a big positive constant to the operator. The same idea appears for instance in [16] and many other works.

Sketch of the proof: Let us consider the operator $L_c = L_{M^-} + cI$ where $c \in \mathbb{R}$ is a constant and $I$ is the identity matrix in $\mathbb{R}^n$. Then $L_c$ satisfies the conditions of Theorem 1.1.1 [16] if $c$ is large enough, namely

1. $L_c$ is a cooperative one;
2. $L_c$ is a fully coupled;
3. There is a super-solution $\varphi$ of $L_c \varphi = 0$.

Conditions 1 and 2 above are obviously fulfilled by $L_c$, since $L_{M^-}$ is a cooperative and a fully coupled one, and $L_c$ inherits these properties from $L_{M^-}$.

As for the condition 3, we construct the super-solution $\varphi$ using the principal eigenfunctions of the operators $L_k - c_k$. More precisely, $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_N)$, where $(L_k - c_k) \varphi_k = \lambda_k \varphi_k$, and $\lambda_k, \varphi_k > 0$ in $\Omega$. The existence of $\varphi_k$ is a well-known fact.

We claim that if $c$ is large enough then $\varphi$ is a super-solution of $L_c \varphi = 0$, i.e. $\varphi \in \left(W^{2,n}_{loc}(\Omega) \cap C(\overline{\Omega})\right)^N$ and $\varphi \geq 0$, $L_c \varphi \geq 0$ and $\varphi$ is not identical to null in $\Omega$.

Since we have chosen $\varphi_k$ being the principal eigenfunctions of $L_k - c_k$, we have $\varphi_k \in \left(C^2(\Omega) \cap C(\overline{\Omega})\right)$ and $\varphi_k > 0$. It remains to prove that $L_c \varphi \geq 0$.

Let

$$A_k = (L_c \varphi)_k = - \sum_{i,j=1}^{n} D_j \left(a_{ij}^k(x) D_i \varphi_k\right) + \sum_{i=1}^{n} b_{ik}^j(x) D_i \varphi_k + \sum_{i=1}^{n} m_{ki}(x) \varphi_i + (c_k + c) \varphi_k =$$

$$= (\lambda_k + c_k + c) \varphi_k + \sum_{i=1}^{n} m_{ki}(x) \varphi_i.$$

Then $A_k \geq 0$ for every $i$.

First of all, if we denote by $n$ the outer unitary normal vector to $\partial \Omega$, then

$$\frac{dA_k}{dn}|_{\partial \Omega} = (\lambda_k + c_k + c) \frac{d\varphi_k}{dn} + \sum_{i=1}^{n} m_{ki}(x) \frac{d\varphi_i}{dn}$$

since $\varphi_i|_{\partial \Omega} = 0$. Therefore there is a constant $c'$, such that $\frac{dA_k}{dn}|_{\partial \Omega} < 0$ for $c > c'$ since $\frac{d\varphi_i}{dn} < 0$ on $\partial \Omega$ (See [14], Theorem 7, p.65) and $\lambda_i$ is independent on $c$. 
Hence there is a neighbourhood \( \Omega_{\varepsilon} = \{ x \in \Omega : \text{dist}(x, \partial \Omega) < \varepsilon \} \) for some \( \varepsilon > 0 \), such that
\[
\frac{dA_k}{dn}|_{\Omega_{\varepsilon}} < 0
\]

Since \( A_k = 0 \) on \( \partial \Omega \), then \( A_k > 0 \) in \( \Omega_{\varepsilon} \)

The set \( \Omega \setminus \Omega_{\varepsilon} \) is compact, therefore there is \( c'' > 0 \) such that \( A_k > 0 \) in the compact set \( \Omega \setminus \Omega_{\varepsilon} \) for \( c > c'' \), since \( \varphi_k > 0 \) in \( \Omega \setminus \Omega_{\varepsilon} \).

Considering \( c > \max(c', c'') \) we obtain \( A_k > 0 \) in \( \Omega \), therefore \( \varphi \) is indeed a super-solution of \( L_c \).

The rest of the proof follows the proof of Theorem 1.1.1 [16]. \( \square \)

A reasonable question is: could the non-cooperative part of the system "improve" the spectral facilities of the cooperative system? In other words, if the cooperative part of the system has non-positive principal eigenvalue, what are conditions on the competitive part, such that the comparison principle holds for the system? An answer of this question is given in the following

**Theorem 3:** Let (1) be a weakly coupled system with irreducible cooperative part of \( L^*_M \) such that (2) is satisfied. Then the comparison principle holds for system (1) if there is \( x_0 \in \Omega \) such that
\[
(7) \quad \left( \lambda + \sum_{k=1}^{N} m_{kj}^+(x_0) \right) > 0 \text{ for } j = 1...N
\]
and
\[
(8) \quad \lambda + m_{jj}^+(x) \geq 0 \text{ for every } x \in \Omega \text{ and } j = 1...N
\]

where \( \lambda = \inf_{\Omega_{\lambda}} \{ \lambda \Omega_0 : \lambda \Omega_0 \text{ is the principal eigenvalue of the operator } L_M \text{ on } \Omega_0 \} \).

It is obvious, that if \( \lambda > 0 \), then the comparison principle holds. More interesting case is \( \lambda < 0 \). Then \( m_{kj}^+ \) can "improve" the properties of \( L_M \) with respect to the validity of the comparison principle. Furthermore, if \( \lambda + m_{jj}^+(x) > 0 \), then (7) is consequence of (8). Condition (7) is important when \( \lambda + m_{jj}^+(x) \equiv 0 \).

**Remark 2:** If \( L^*_M \) is irreducible, then \( L_M \) is irreducible as well. In fact \( L^*_M = L^* + M^{-t} \) and if \( M^{-t} \) is irreducible, then such is \( M^* \).

**Proof:** Suppose all conditions of Theorem 3 are satisfied by \( L_M \) but the comparison principle does not hold for \( L_M \). Let \( u, \pi \in \left( W^{1,\infty}(\Omega) \cap C(\overline{\Omega}) \right)^N \).
be an arbitrary weak sub- and super-solution of $L_M$. Then $u = u - \bar{u} \in W^{1,\infty}(\Omega) \cap \mathcal{C}(\Omega)^N$ is a weak sub-solution of $L_M$ as well, i.e. $(L_M(u), v) \leq 0$ in $\Omega$ for any $v \in W^{1,\infty}(\Omega) \cap C_0(\overline{\Omega})^N$, $v > 0$ and $u^+ \equiv 0$ on $\partial \Omega$.

Assume $u^+ \neq 0$. Let $\Omega_{\text{supp}(u^+)} \subseteq \text{supp}(u^+)$ has smooth boundary. Then for any $v > 0$, $v \in W^{1,\infty}(\Omega_{\text{supp}(u^+)} \cap C(\Omega_{\text{supp}(u^+)}))^N$ is satisfied since $L_M(u^+) \leq 0$.

Since $L_{M^-}$ is a cooperative operator, such is $(L_{M^-})^* = L^* + (M^-)^t$ as well. According to Theorem 2 above, there is a unique positive eigenfunction $w \in W^{2,n}_{\text{loc}}(\Omega_{\text{supp}(u^+)}) \cap C(\Omega_{\text{supp}(u^+)})^N$ such that $w > 0$ and $L_{M^-} w = \lambda w$ for some $\lambda > 0$.

Then $w$ is a suitable test-function for (9). Rewriting the inequality (9) for $v = w$ we obtain

$$0 \geq (L_M u^+, v) = (u^+, u^*_M - v) + (M^+ u^+, w)$$

or componentwise

$$(10) \quad 0 \geq (u^+, \lambda w) + \left( \sum_{j=1}^N m_{kj}^+ u_j^+, w_k \right)$$

for $k = 1, \ldots, n$.

The sum of inequalities (10) is

$$0 \geq \sum_{k=1}^N \left( (u_k^+, \tilde{L}_k^* w_k) + \left( \sum_{j=1}^N m_{kj}^+ u_j^+, w_k \right) \right) = \sum_{k=1}^N \left( u_k^+, \lambda w_k \right) + \sum_{k,j=1}^N \left( u_j^+, m_{kj}^+ w_k \right) = \sum_{j=1}^N \left( u_j^+, \sum_{k=1}^N \left( \delta_{jk}^\lambda + m_{kj}^+ \right) w_k \right) > 0$$

since $u^+ > 0$, $w_k > 0$, (7) and (8). Condition (8) is used in $\left( u_k^+, (\lambda + m_{kk}^+) w_k \right) \geq 0$.

The above contradiction proves that $u^+ \equiv 0$ and therefore the comparison principle holds for operator $L_M$. $\square$
Since in [1] and [18] are considered only systems with irreducible cooperative part, the ones with reducible \( L_{M^-} \) are excluded of the range of Theorem 3. Nevertheless the same idea is applicable to some systems with reducible cooperative part as well, as it is given it Theorem 4.

**Theorem 4:** Assume \( m_{ij} \equiv 0 \) for \( i \neq j \) and (2) is satisfied. Then the comparison principle holds for system (1) if there is \( x_0 \in \Omega \) such that

\[
\lambda_j + \sum_{k=1}^{N} m_{kj}^+(x_0) > 0 \text{ for } j = 1 \ldots N
\]

and

\[
\lambda_j + m_{jj}^+(x) \geq 0 \text{ for every } x \in \Omega, j = 1 \ldots N,
\]

where \( \lambda_j = \inf_{\lambda_{j}\Omega_0} \{ \lambda_{j}\Omega_0 : \lambda_{j}\Omega_0 \text{ is the principal eigenvalue of the operator } L_j + m_{jj}^- \text{ on } \Omega_0 \} \).

Theorem 4 is formulated for diagonal matrix \( M^- \). The statement is valid with obvious modification if \( M^- \) has block structure, i.e.

\[
M^- = \begin{pmatrix}
M_1^- & 0 & \cdots & 0 \\
0 & M_2^- & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & M_r^-
\end{pmatrix}
\]

where \( M_k^- \) are \( d_k \)-dimensional square matrixes, \( \sum d_k \leq N \).

Proof: Let all conditions of Theorem 4 be satisfied by \( L_M \) but the comparison principle does not hold for \( \tilde{L}_{M^+} \). Let \( \underline{u}, \overline{u} \in \left( W^{1,\infty}(\Omega) \cap C(\overline{\Omega}) \right)^N \) be an arbitrary weak sub- and super-solution of \( \tilde{L}_{M^+} \). Then \( u = \overline{u} - \underline{u} \in \left( W^{1,\infty}(\Omega) \cap C(\overline{\Omega}) \right)^N \) is a weak sub-solution of \( \tilde{L}_{M^+} \) as well, i.e. \( (\tilde{L}_{M^+}(u), v) \leq 0 \) in \( \Omega \) for any \( v \in \left( W^{1,\infty}(\Omega) \cap C(\overline{\Omega}) \right)^N \), \( v > 0 \) and \( u^+ \equiv 0 \) on \( \partial \Omega \).

Suppose that \( u^+ \neq 0 \). Let \( \Omega_{\text{supp}(u^+)} \subseteq \text{supp}(u^+) \) has smooth boundary. Then for any \( v > 0 \), \( v \in W^{1,\infty}(\Omega_{\text{supp}(u^+)} \cap C(\overline{\Omega}_{\text{supp}(u^+)}) \)

\[
0 \geq (\tilde{L}_{M^+}u^+, v) = \left( u^+, \tilde{L}^* v \right) + \left( M^+u^+, v \right)
\]
is satisfied since \( \tilde{L}_M^+ u^+ \leq 0 \).

According to Theorem 2.1 in [2], there is a positive principal eigenfunction for the operator \( \tilde{L}_k^* \) in \( \Omega_{\text{supp}(u^+)} \), i.e. \( \exists w_k(x) \in C^2(\Omega_{\text{supp}(u^+)} \cap R^1) \) such that \( \tilde{L}_k^* w_k(x) = \lambda_k w_k(x) \) and \( w_k(x) > 0 \). Note that \( w_k \) are classical solutions.

Then the vector-function \( w(x) = (w_1(x), ..., w_n(x)) \), composed of the principal eigenfunctions \( w_k(x) \), is suitable as a test-function in (13).

Writing componentwise inequality (13) for \( v = w \) we obtain

\[
0 \geq (u_k^+, \tilde{L}_k^* w_k) + \left( \sum_{j=1}^N m_{kj}^+ u_j^+, w_k \right)
\]

for \( k = 1, ..., N \).

The sum of inequalities (14) is

\[
0 \geq \sum_{k=1}^N \left( (u_k^+, \tilde{L}_k^* w_k) + \left( \sum_{j=1}^N m_{kj}^+ u_j^+, w_k \right) \right) = \sum_{k=1}^N \left( u_k^+, \lambda_k w_k \right) + \sum_{k,j=1}^N \left( u_j^+, m_{kj}^+ w_k \right) = \sum_{j=1}^N \left( u_j^+, \sum_{k=1}^N \left( \delta_{jk} \lambda_j + m_{kj}^+ \right) w_k \right) > 0
\]

since \( u^+ > 0, w_k > 0, (11) \) and (12).

The above contradiction proves that \( u^+ \equiv 0 \) and therefore the comparison principle holds for operator \( L_M^+ \).\[\square\]

Remark 3: It is obvious that conditions (7), (8), and respectively, (11), (12), can be substituted by the sharper condition \( \sum_{k=1}^N \left( \delta_{jk} \lambda_k + m_{kj}^+ \right) w_k > 0 \) for every \( x \in \Omega \) and every \( j = 1, ..., N \), which is useful only if the exact values of the eigenfunctions \( w_k \) can be computed.

The main idea in Theorem 4 could be modified for systems with triangular cooperative part, for instance with null elements above the main diagonal. For instance predator-prey systems have triangular cooperative part. Of course, if \( m_{ij}^- (x) > 0 \) for every \( x \in \Omega \) and \( i = 1, ..., N, j < i \), then the system is in the scope of Theorem 3. In Theorem 5 this condition is not necessary, i.e. some of the species can extinguish in some subarea of \( \Omega \).

**Theorem 5:** Assume (2) is satisfied and the cooperative part \( M^- \) is triangular for the system (1), i.e. \( m_{ij}^- = 0 \) for \( i = 1, ..., N, j > i \). Then the comparison principle holds for system (1), if there is \( \varepsilon > 0 \) such that
(15) \( (\lambda_j - (1 - \delta_{ij})\varepsilon + \sum_{k=1}^{N} m_{kj}^+(x_0)) > 0 \) for \( j = 1...N \) for some \( x_0 \in \Omega \)

and

(16) \( \lambda_j - (1 - \delta_{ij})\varepsilon + m_{jj}^+(x) \geq 0 \) for every \( x \in \Omega \) and \( j = 1...N \),

where \( \lambda_j = \inf_{\Omega_0 \subseteq \Omega} \{ \lambda_{j\Omega_0} : \lambda_{j\Omega_0} \text{ is the principal eigenvalue of the operator } L_j + m_{jj}^- \text{ on } \Omega_0 \} \).

Note that the condition for triangular cooperative part does not exclude \( m_{ij}^-(x_0) = 0 \) for some \( x_0 \in \Omega, i,j = 1,...N \).

Proof: 1. The first equation in \( L_{M-} \) is not coupled, and there are principal eigenvalue \( \lambda_1 \) and principal eigenfunction \( w_1 > 0 \) of \( L_1 + m_{11}^- \) (See Theorem 2.1 in [2]). We put \( \tilde{w}_1 = w_1 \).

2. The equation \( (L_2 + m_{22}^-)\tilde{w}_2 - \lambda \tilde{w}_2 = m_{21}\tilde{w}_1 \) with null boundary conditions has unique solution for \( \lambda < \lambda_2 \), where \( \lambda_2 \) is the principal eigenvalue of \( L_2 + m_{22}^- \). We put \( \lambda = \lambda_2 - \varepsilon \). Since the right-hand side \( m_{21}\tilde{w}_1 \) is positive, the solution \( \tilde{w}_2 \) is positive as well.

3. By induction we construct positive functions \( \tilde{w}_j, j = 3,...N \) as solutions of \( (L_j + m_{jj}^-)\tilde{w}_j - (\lambda_j - \varepsilon)\tilde{w}_j = \sum_{i=1}^{j-1} m_{ji}\tilde{w}_i \) with null boundary conditions. As usual \( \lambda_j \) are the principal eigenfunctions of \( L_j + m_{jj}^- \).

4. The rest of the proof follows the proof of Theorem 4 where \( \lambda_j \) is substituted with \( \lambda_j - \varepsilon \) and \( w_j \) is substituted with \( \tilde{w}_j \).

For the simplest predator-prey system, \( N = 2 \), \( m_{11} = m_{22} = 0 \), \( m_{12} > 0 \) and \( m_{21} < 0 \), conditions (15) and (16) are \( \lambda_1 \geq 0 \), \( \lambda_2 > 0 \), where \( \lambda_j \) is the principal eigenvalue of the operator \( L_j, j = 1,2 \).

Condition (12) in Theorem 2 is useful for construction of counter-example for the non-validity of comparison principle in general.

**Theorem 6**: Let (1) be a weakly coupled system with reducible cooperative part \( L_{M-} \) and (2) be satisfied. Suppose that (12) is not true, i.e there is some \( j \in \{1...N\} \) such that \( (\lambda_j + m_{jj}^+(x)) < 0 \) for any \( x \in \Omega \), and \( m_{jj}^+ = 0 \) for \( l \neq j \), \( l = 1,...N \). Then comparison principle does not hold for system (1).

Proof: Let us suppose for simplicity that \( j = 1 \) and \( m_{1j}^- = 0 \) for \( j = 2,...N \). We consider vector-function \( w(x) = w_1(x),0,...,0 \), where \( w_1(x) \) is the principal eigenfunction of \( L_1 + m_{11}^- \).
Then for the first component \((L_M)_1\) of \(L_M\) is valid \((L_M w)_1 = \lambda w_1(x) + m_{11}^+ w_1(x) < 0\) in \(\Omega\), where \(\lambda_j\) is the principal eigenvalue of \(L_1\), and \((L_M w)_k = 0\) for \(k = 1, \ldots, N\). Therefore, \(L_M w \leq 0\) but \(w(x) \geq 0\) and comparison principle fails. \(\square\)

The simplest case to illustrate Theorems 4 and 6 is \(N = 2\). Let us consider irreducible competitive system

\[(17)\quad L_j u_j + \sum_{j,k=1}^{2} m_{jk} u_k = f_j, \quad j = 1, 2,\]

where \(m_{11} = m_{22} = 0, \; m_{12} > 0, \; m_{21} > 0\).

Suppose \(\lambda_j\) is the principal eigenvalue of \(L_j^*\), \(j = 1, 2\). If \(\lambda_j \geq 0\) and there is \(x_0 \in \Omega\) such that \(\lambda_1 + m_{21}(x_0) > 0\) and \(\lambda_2 + m_{12}(x_0) > 0\), then according to Theorem 4 the comparison principle holds for system (1), i.e. if \(f_1 > 0, \; f_2 > 0\), then \(u_1 > 0\) and \(u_2 > 0\), where \(u = u - \pi\) is defined in the proof of Theorem 3.

If \(\lambda_2 + m_{12}(x) < 0\) for every \(x \in \Omega\), then according to Theorem 6 there is no comparison principle for system (1) in the lexicographic order, used in this paper.

More detailed analysis of the validity of the comparison principle for system (1) could be done if we consider order in the cone \(C_U = P_U \times (-P_U)\), i.e. \((u_1, u_2) \leq (v_1, v_2)\) if and only if \(u_1 \leq v_1\) and \(u_2 \geq v_2\). Then Theorem 6.5 [14] states the existence of a principal eigenvalue \(\lambda\) of \(L^*\) with positive in \(C_U\) principal eigenfunction \(w_1(x) > 0, \; w_2(x) < 0\).

If \(\lambda > 0\), then according to Theorem 6.3 [14] the comparison principle holds in the order in \(C_U\), i.e. if \(f_1 > 0, \; f_2 < 0\), then \(u_1 > 0\) and \(u_2 < 0\).

If \(\lambda < 0\), then \((L_1(-u_1) + m_{12} u_1, w_1) + (L_2 u_2 + m_{21}(-u_1), w_2) = (-u_1, \lambda w_1 + m_{21} w_2) + (u_2, m_{21} w_1 + \lambda w_2) > 0\). Hence \(u_1 < 0\) and \(u_2 > 0\) for \(f_1 > 0, \; f_2 > 0\).

A statement analogous to Theorem 6 is valid for irreducible systems as well.

**Theorem 7:** Let (1) be a weakly coupled system with irreducible cooperative part \(L_M^\|\) and (2) be satisfied. Suppose that (7) is not true, i.e there is some \(j \in \{1 \ldots N\}\) such that \((\lambda + m_{jj}^-(x)) < 0\) for any \(x \in \Omega\), and \(m_{jj}^- = 0\) for \(l \neq j, \; l = 1, \ldots, N\). Then comparison principle does not hold for system (1).

Note that in Theorem 6 and Theorem 7 we need the violation of condition (12) and, respectively, condition (7) in all \(\Omega\). The proof of Theorem 7 follows the proof of Theorem 6 with obvious adaptation.
3 Comparison principle for quasi-linear elliptic systems

Considering quasi-linear system (3), (4), we use the results of the previous section to derive conditions for the validity of comparison principle.

Let $u(x) \in \left(W^{1,\infty}(\Omega) \cap C(\overline{\Omega})\right)^N$ be a sub-solution and $v(x) \in \left(W^{1,\infty}(\Omega) \cap C(\overline{\Omega})\right)^N$ be a super-solution of (3), (4). Comparison principle holds for (3), (4), if $Q(u) \leq Q(v)$ in $\Omega$, $u \leq v$ on $\partial \Omega$ imply $u \leq v$ in $\Omega$. Last three inequalities are considered in the weak sense.

Recall that the vector-function $u(x)$ is a weak sub-solution of (3), (4) if

$$\int_{\Omega} (a^{li}(x, u^l, Du^l)\eta_{x_i} + F^l(x, u^1, \ldots u^N, Du^l)\eta_l - f^l(x)\eta_l) \, dx \leq 0$$

for $l = 1, \ldots N$ and for every nonnegative vector function $\eta \in \left(W^{1,\infty}(\Omega) \cap C(\overline{\Omega})\right)^N$ (i.e. $\eta = (\eta^1, \ldots, \eta^N)$, $\eta_l \geq 0$, $\eta_l \in \left(W^{1,\infty}(\Omega) \cap C(\overline{\Omega})\right)$ and $\eta_l = 0$ on $\partial \Omega$.

Analogously, $v(x) \in \left(W^{1,\infty}(\Omega) \cap C(\overline{\Omega})\right)^N$ is a super-solution of (3), (4), if

$$\int_{\Omega} (a^{li}(x, v^l, Dv^l)\eta_{x_i} + F^l(x, v^1, \ldots v^N, Dv^l)\eta_l - f^l(x)\eta_l) \, dx \geq 0$$

for $l = 1, \ldots N$ and for every nonnegative vector function $\eta \in \left(W^{1,\infty}(\Omega) \cap C(\overline{\Omega})\right)^N$.

Since $u(x)$ and $v(x)$ are sub-and super-solution respectively, then $\tilde{w}(x) = u(x) - v(x)$ is a weak sub-solution of the following problem

$$- \sum_{i,j=1}^n D_i \left( B^{li}_j D_j \tilde{w}^l + B^{li}_0 \tilde{w}^l \right) + \sum_{k=1}^N E^l_k \tilde{w}^k + \sum_{i=1}^n H^l_i D_i \tilde{w}^l = 0 \text{ in } \Omega$$

with non-positive boundary data on $\partial \Omega$. Here

$$B^{li}_j = \int_0^1 \frac{\partial a^l}{\partial p_j}(x, P^l) ds, \quad B^{li}_0 = \int_0^1 \frac{\partial a^l}{\partial u^l}(x, P^l) ds, \quad E^l_k = \int_0^1 \frac{\partial F^l}{\partial u^k}(x, S^l) ds,$$

$$H^l_i = \int_0^1 \frac{\partial F^l}{\partial v^i}(x, S^l) ds, \quad P^l = \left(v^l + s(u^l - v^l), Dv^l + sD(u^l - v^l)\right),$$

$$S^l = \left(v + s(u - v), Dv + sD(u - v)\right).$$
Therefore, \( \tilde{w}_+(x) = \max(\tilde{w}(x), 0) \) is a sub-solution of

\[
\sum_{i,j=1}^n D_i \left( B_j^i D_j \tilde{w}_+^i + B_0^i \tilde{w}_+^i \right) + \sum_{k=1}^N E_k^i \tilde{w}_+^k + \sum_{i=1}^n H_i^i D_i \tilde{w}_+^i = 0
\]
in \( \Omega \)

with zero boundary data on \( \partial \Omega \).

Equation (18) is equivalent in terms of matrix to

\[
B E \tilde{w}_+ = (B + E) \tilde{w}_+ = 0 \text{ in } \Omega,
\]

where \( B = \text{diag}(B_1, B_2, \ldots B_N) \), \( B_i = \sum_{j=1}^n D_i \left( B_j^i D_j \tilde{w}_+^i + B_0^i \tilde{w}_+^i \right) + \sum_{i=1}^n H_i^i D_i \tilde{w}_+^i \)
and \( E = \{ E_k^i \}_{i,k=1}^N \).

If we denote \( D^k_{ij} \) by \( a_{ij}^k \), \( B_0^k + H^k \) by \( b^k \), \( \sum_{i=1}^n D_i B_{ij}^i + E_k^i \) by \( m_{ik}(x) \) for \( i, j = 1, \ldots n, \ k = 1, \ldots N \) and \( E_k^i \) by \( m_{ik}(x) \) for \( k, l = 1, \ldots N, \ k \neq l \), system (18) looks like system (1). Hereafter we follow the notations for system (1).

Suppose now that \( \tilde{w}_+(x) \) is not identical equal to zero in \( \Omega \), i.e. comparison principle fails for (3), (4). Suppose \( L_M^- \) is irreducible. Then

\[
0 \geq (L_M \tilde{w}_+, w) = (\tilde{w}_+, L_{M^-} w) + (M^+ \tilde{w}_+, w) = (\tilde{w}_+, \lambda w) + (M^+ \tilde{w}_+, w)
\]

where \( \lambda \) is the principal eigenvalue of \( L_{M^-} \) and \( w \) is the corresponding eigenfunction.

Suppose \( a_{ij}^k \) and \( m_{ik}(x) \) satisfy the conditions (2), (7) and (8) in Theorem 3. Following the proof of Theorem 3, we obtain that \( \tilde{w}_+ \equiv 0 \) in \( \Omega \), i.e. comparison principle holds for the system (3), (4).

If \( L_M^- \) is reducible, then

\[
0 \geq (L_M \tilde{w}_+, w) = (\tilde{w}_+, L_{M} w) + (M^+ \tilde{w}_+, w) = (\tilde{w}_+, \lambda w) + (M^+ \tilde{w}_+, w)
\]

where \( \lambda w = (\lambda_1 w_1, \lambda_2 w_2, \ldots \lambda_N w_N) \), \( \lambda_k \) is the principal eigenvalue of \( L_k^+ \) and \( w_k \) is the corresponding eigenfunction for \( k = 1, \ldots N \).

Suppose \( a_{ij}^k \) and \( m_{ik}(x) \) satisfy the conditions (2), (11) and (12) in Theorem 4. Following the proof of Theorem 4, we obtain that \( \tilde{w}_+ \equiv 0 \) in \( \Omega \), i.e. comparison principle holds for the system (3), (4).

We have sketched the proof the following
Theorem 8: Suppose (3), (4) is a quasi-linear system and the corresponding system $B_{E^-}$ in (19) is elliptic. Then the comparison principle holds for system (3), (4) if

\( (i) \quad B_{E^-} \text{ in (19) is irreducible and for every } j = 1...n \)

\( (ii) \quad \lambda + \left( \sum_{k=1}^N \frac{\partial F_k}{\partial p^j}(x, p, Dp^j) + \sum_{i=1}^N D_i \frac{\partial a^{ji}}{\partial p^j}(x, p^j, Dp^j) \right)^+ > 0, \)

\( (iii) \quad \lambda + \left( \sum_{i=1}^n D_i \frac{\partial a^{ji}}{\partial p^j}(x, p^j, Dp^j) + \frac{\partial F_j}{\partial p^j}(x, p, Dp^j) \right)^+ \geq 0 \)

where $x \in \Omega$, $p \in \mathbb{R}^n$ and $\lambda = \inf_{\Omega_0 \subseteq \Omega} \{ \lambda_{\Omega_0} : \lambda_{\Omega_0} \text{ is the principal eigenvalue of the operator } B_{E^-} \text{ on } \Omega_0 \};$

or

\( (i') \quad B_{E^-} \text{ in (19) is reducible and for every } j = 1...n \)

\( (ii') \quad \lambda_j + \left( \sum_{k=1}^n \frac{\partial F_k}{\partial p^j}(x, p^j, Dp^j) + \sum_{i=1}^n D_i \frac{\partial a^{ji}}{\partial p^j}(x, p^j, Dp^j) \right)^+ > 0, \)

\( (iii') \quad \lambda_j + \left( \sum_{i=1}^n D_i \frac{\partial a^{ji}}{\partial p^j}(x, p^j, Dp^j) + \frac{\partial F_j}{\partial p^j}(x, p, Dp^j) \right)^+ \geq 0 \)

where $x \in \Omega$, $p \in \mathbb{R}^n$ and $\lambda_l = \inf_{\Omega_0 \subseteq \Omega} \{ \lambda_{\Omega_0} : \lambda_{\Omega_0} \text{ is the principal eigenvalue of the operator } B_l \text{ on } \Omega_0 \}.$

4 Final remarks

The sufficient conditions in Theorems 3 and 4 are derived from the spectral properties of the cooperative part of (1) - the operator $L_{M^-}$, or, in other words, comparing the principal eigenvalue of $L_{M^+}$ with the quantities in $M^+$. In fact the positive matrix $M^+$ causes a migration of the principal eigenvalue of $L_{M^-}$ to the left.

Theorems 3 and 4 provide a huge class of non-cooperative systems such that the comparison principle is valid for. The idea of migrating the spectrum of a positive operator on the right works in this case, though the spectrum itself is not studied in this article. The results for non-cooperative systems in this paper are not sharp and the validity of the comparison principle is to be determined more precisely in the future.
5 Acknowledgment

The author would like to acknowledge Professor Alexander Sobolev for the very useful talks on the theory of positive operators, during the author’s stay at University of Sussex as Maria Curie fellow.

6 REFERENCES

[1] H.Amann, Maximum Principles and Principal Eigenvalues, 10 Mathematical Essays on Approximation in Analysis and Topology (J.Ferrera, J.Lopez-Gomez and F.R.Ruiz del Portal Eds.), Elsevier, Amsterdam (2005), 1-60.

[2] H.Berestycki, L.Nirenberg, S.R.S. Varadhan : The principal eigenvalue and maximum principle for second-order elliptic operators in general domains, Commun. Pure Appl. Math. 47, No.1, 47-92 (1994).

[3] G.Boyadzhiev, N.Kutev : Diffraction problems for quasilinear reaction-diffusion systems, Nonlinear Analysis 55 (2003), 905-926.

[4] G.Caristi, E. Mitidieri : Further results on maximum principle for non-cooperative elliptic systems. Nonl.Anal.T.M.A., 17 (1991), 547-228.

[5] C.Coosner, P.Schaefer : Sign-definite solutions in some linear elliptic systems. Proc.Roy.Soc.Edinb., Sect.A 111, (1989), 347-358.

[6] D.di Figueredo, E.Mitidieri : Maximum principles for cooperative elliptic systems. C.R.Acad.Sci. Paris, Ser. I, 310 (1990), 49-52.

[7] D.di Figueredo, E.Mitidieri : A maximum principle for an elliptic system and applications to semi-linear problems, SIAM J.Math.Anal. 17 (1986), 836-849.

[8] Gilbarg, D and Trudinger, N. Elliptic partial differential equations of second order. 2nd ed., Springer - Verlag, New York.

[9] M.Hirsch : Systems of differential equations which are competitive or cooperative I. Limit sets, SIAM J. Math. Anal. 13 (1982), 167-179.

[10] P.Hess : On the Eigenvalue Problem for Weakly Coupled Elliptic Systems, Arch. Ration. Mech. Anal. 81 (1983), 151-159.

[11] Ishii, Sh. Kofke : Viscosity solutions for monotone systems of second order elliptic PDEs. Commun. Part.Diff.Eq. 16 (1991), 1095 - 1128.

[12] Li Jun Hei, Juan Hua Wu : Existence and Stability of Positive Solutions for an Elliptic Cooperative System. Acta Math. Sinica Oct.2005, Vol.21, No 5, pp 1113-1130.
[13] J.Lopez-Gomez, M. Molina-Meyer: The maximum principle for cooperative weakly coupled elliptic systems and some applications. Diff.Int.Eq. 7 (1994), 383-398.

[14] J.Lopez-Gomez, J.C.Sabina de Lis, Coexistence states and global attractivity for some convective diffusive competing species models, Trans.Amer.Math.So. 347, 10 (1995), 3797-3833.

[15] E.Mitidieri, G.Sweers: Weakly coupled elliptic systems and positivity. Math.Nachr. 173 (1995), 259-286.

[16] M. Protter, H.Weinberger: Maximum Principle in Differential Equations, Prentice Hall, 1976.

[17] M.Reed, B.Simon: Methods of modern mathematical Physics, v.IV: Analysis of operators, Academic Press, New York, (1978).

[18] G.Sweers: Strong positivity in $C(\overline{\Omega})$ for elliptic systems. Math.Z. 209 (1992), 251-271.

[19] G.Sweers: Positivity for a strongly coupled elliptic systems by Green function estimates. J Geometric Analysis, 4, (1994), 121-142.

[20] G.Sweers: A strong maximum principle for a noncooperative elliptic systems. SIAM J. Math. Anal., 20 (1989), 367-371.

[21] W.Walter: The minimum principle for elliptic systems. Appl.Anal.47 (1992), 1-6.

Author’s address:
Institute of Mathematics and Informatics,
Bulgarian Academy of Sciences,
Acad.G.Bonchev st., bl.8,
Sofia, Bulgaria