A BREZIS-NIRENBERG TYPE RESULT FOR MIXED LOCAL AND NONLOCAL OPERATORS

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Abstract. We study a critical problem for an operator of mixed order obtained by the superposition of a Laplacian with a fractional Laplacian.

In particular, we investigate the corresponding Sobolev inequality, detecting the optimal constant, which we show that is never achieved.

Moreover, we present an existence (and nonexistence) theory for the corresponding subcritical perturbation problem.

1. Introduction

In this paper we are concerned with elliptic operators of mixed local and nonlocal type, in relation to the possible existence of positive solutions for critical problems and in connection with possible optimizers of suitable Sobolev inequalities.

The investigation of operators of mixed order is a very topical subject of investigation, arising naturally in several fields, for instance as the superposition of different types of stochastic processes such as a classical random walk and a Lévy flight, which has also interesting application in the study of optimal animal foraging strategies, see [19, 20, 25, 26].

From the technical point of view, these operators offer quite relevant challenges caused by the combination of nonlocal difficulties with the lack of invariance under scaling. The contemporary investigation has specifically focused on several problems in the existence and regularity theory (see [14, 15, 16, 17, 21, 22, 24, 25, 31]) symmetry and classification results (see [7, 9, 10]), etc.

The stirring motivation for the problems presented in this paper comes from the study of nonlinear problems with critical exponents, as the ones suggested by the optimizers of the Sobolev inequality. Roughly speaking (see Section 2 for a formal definition of the functional setting) the strategy adopted here is to consider a fractional exponent \( s \in (0, 1) \), an open set \( \Omega \subseteq \mathbb{R}^n \), not necessarily bounded or connected, and all functions \( u : \mathbb{R}^n \rightarrow \mathbb{R} \) which vanish outside \( \Omega \), accounting for a mixed type Sobolev inequality of the sort

\[
S_{n,s}(\Omega) \| u \|_{L^{2^*(\infty)}(\mathbb{R}^n)}^2 \leq \| \nabla u \|_{L^2(\mathbb{R}^n)}^2 + \int_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy.
\]

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Here above, the constant $S_{n,s}(\Omega)$ is taken to be the largest possible one for which such an inequality holds true and, as usual, $n \geq 3$ and $2^* := \frac{2n}{n-2}$.

We observe below that indeed (1.1) is satisfied by choosing the constant on the left hand side to be (less than or) equal to the classical Sobolev constant

\[(1.2) \quad S_n := \frac{1}{n(n-2)\pi} \left( \frac{\Gamma(n)}{\Gamma(n/2)} \right)^{2/n},\]

since by the standard Sobolev inequality (in $\Omega$, or even in $\mathbb{R}^n$),

\[S_n \|u\|_{L^{2^*}(\mathbb{R}^n)}^2 \leq \|\nabla u\|_{L^2(\mathbb{R}^n)}^2 \leq \|\nabla u\|_{L^2(\mathbb{R}^n)}^2 + \iint_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} \, dx \, dy.\]

As a result, the largest possible constant in (1.1) certainly satisfies $S_{n,s}(\Omega) \geq S_n$.

In principle, one may suspect that in fact a strict inequality occurs (because, for instance, $S_n$ is independent on $\Omega$, as well as on $s$), but this is not the case, according to the following result:

**Theorem 1.1.** Let $s \in (0,1)$ and $\Omega \subseteq \mathbb{R}^n$ be an arbitrary open set. Then, we have $S_{n,s}(\Omega) = S_n$.

A natural question related to this problem is whether or not the optimal constant in (1.1) is achieved, i.e. whether or not a minimizer exists. The next result answers this question.

**Theorem 1.2.** Let $\Omega \subseteq \mathbb{R}^n$ be an arbitrary open set. Then, the optimal constant $S_{n,s}(\Omega)$ in (1.1) is never achieved.

As customary, the study of possible optimizers for inequalities of Sobolev type is intimately connected with the possible existence of nontrivial solutions for critical problems like

\[
\begin{cases}
L u = u^{2^*-1} & \text{in } \Omega, \\
u \geq 0 & \text{in } \Omega, \\
u \equiv 0 & \text{in } \mathbb{R}^n \setminus \Omega.
\end{cases}
\]

As for the classical case (i.e. $L = -\Delta$), when $\Omega$ is star-shaped, the above problem does not admit positive solutions, see [27, Theorem 1.3] thanks to some proper variants of the Pohozaev identity. This fact naturally drive us to study critical problem of the form

\[(1.3) \quad \begin{cases}
L u = u^{2^*-1} + \lambda u^p & \text{in } \Omega, \\
u \geq 0 & \text{in } \Omega, \\
u \equiv 0 & \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}\]

where $p \in [1, 2^*-1)$ and $\lambda \in \mathbb{R}$. We remind that when $L = -\Delta$, (1.3) is the famous Brezis-Nirenberg problem [14], and it has already been studied for the fractional Laplacian as well (see [30]).

Our first result is that there do not exist solutions to this problem when $\lambda \leq 0$, at least on bounded and star-shaped domains.

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1Strictly speaking, to formally address the problem of the existence of this minimizer, one should carefully define the functional space to which this object may in principle belong. This will be done in details in Section 2. Suffices here to say that one can, roughly speaking, just consider the natural closure of the space of smooth functions compactly supported in $\Omega$.

Further clarifications about this point will be highlighted in Remark 3.2.
Theorem 1.3. Let \( \lambda \leq 0 \). Assume that \( \Omega \subseteq \mathbb{R}^n \) is bounded and star-shaped. Then, there do not exist solutions to problem (1.3), whatever the exponent \( p \in [1, 2^* - 1) \).

To further analyze the existence theory for problem (1.3) in dependence of the parameter \( \lambda \), let us briefly recall the main strategy used in [14] in the case \( p = 1 \).

Due to the lack of compactness caused by the critical exponent, an idea borrowed from [33] consists in proving that

\[
S_{\lambda} := \inf \left\{ \| \nabla u \|_{L^2(\Omega)}^2 - \lambda \| u \|_{L^2(\Omega)}^2 : u \in H_0^1(\Omega) \text{ and } \| u \|_{L^{2^*_n}(\Omega)} = 1 \right\}
\]

is achieved under some restrictions on \( \lambda \); in order to do this, the key step is to show that \( S_{\lambda} < S_n \), where \( S_n \) is as in (1.2).

Now, the strict inequality \( S_{\lambda} < S_n \) is obtained in [14] via the following approach: first of all, taking into account that the minimizers in the Sobolev inequality are given by Aubin-Talenti functions, one consider the function

\[
u_{\varepsilon} = \frac{\phi}{(\varepsilon^2 + |x|^2)^{(n-2)/2}} \quad (\varepsilon > 0).
\]

Then, using \( \nu_{\varepsilon} \) as a competitor function, one gets (at least for \( n \geq 5 \)) that

\[
\frac{\| \nabla \nu_{\varepsilon} \|_{L^2(\Omega)}^2 - \lambda \| \nu_{\varepsilon} \|_{L^2(\Omega)}^2}{\| \nu_{\varepsilon} \|_{L^{2^*_n}(\Omega)}^2} = S_n - c\lambda \varepsilon^2 + O(\varepsilon^{n-2}) \quad \text{as } \varepsilon \to 0^+,
\]

where \( c > 0 \) is a suitable constant. From this, choosing \( \varepsilon \) sufficiently small, one immediately conclude that \( S_{\lambda} < S_n \). A similar approach works for \( 1 < p < 2^* - 1 \) as well, and it has also been used in the nonlocal framework [30].

Differently from what one can expect, in our mixed setting the situation changes. The main reason is that, in trying to repeat the above argument, one is led to consider the following minimization problem

\[
S_{\lambda, s} := \inf \left\{ \| \nabla u \|_{L^2(\Omega)}^2 + [u]_s^2 - \lambda \| u \|_{L^2(\Omega)}^2 : u \in H_0^1(\Omega) \text{ and } \| u \|_{L^{2^*_n}(\Omega)} = 1 \right\}
\]

and to prove that

\[
S_{\lambda, s} < S_{n, s},
\]

where \([u]_s\) denotes the usual Gagliardo seminorm of \( u \) (here, we agree to identify \( u \) with its zero-extension out of \( \Omega \)) and \( S_{n, s}(\Omega) \) is as in (1.1). Now, since we know from Theorem (1.3) that \( S_{n, s}(\Omega) = S_n \), in order to prove (1.4) it is natural to consider as a competitor function the same function \( \nu_{\varepsilon} \) defined above; however, the presence of the nonlocal part \([u_{\varepsilon}]_s^2\) gives

\[
\frac{\| \nabla \nu_{\varepsilon} \|_{L^2(\Omega)}^2 + [u_{\varepsilon}]_s^2 - \lambda \| \nu_{\varepsilon} \|_{L^2(\Omega)}^2}{\| u_{\varepsilon} \|_{L^{2^*_n}(\Omega)}^2} = S_n + O(\varepsilon^{2-2s}) - c\lambda \varepsilon^2 + O(\varepsilon^{n-2}) \quad \text{as } \varepsilon \to 0^+
\]

(see, precisely, identity (4.14) in the proof of Lemma 4.11), and the term \( O(\varepsilon^{2-2s}) \) is not negligible when \( \varepsilon \to 0^+ \).

All that being said, in the linear case \( p = 1 \) we obtain that problem (1.3) does not admit any solution both in the range of “small” and “large” values of \( \lambda \), but it does possess solutions for an “intermediate” regime of values of \( \lambda \); more precisely, denoting by \( \lambda_{1, s} \) the first Dirichlet eigenvalue of \((-\Delta)^s\) in a bounded open set \( \Omega \),
and by $\lambda_1$ the first Dirichlet eigenvalue of $L$ in $\Omega$ (a precise summary of the related spectral property being recalled on page 12), we have the following result.

**Theorem 1.4.** Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set and $p = 1$. There exists some $\lambda^* \in [\lambda_{1,s}, \lambda_1)$ such that problem (1.3) possesses at least one solution if

$$\lambda^* < \lambda < \lambda_1.$$ 

Moreover, the following facts hold:

1. there do not exist solutions to problem (1.3) if $\lambda \geq \lambda_1$;
2. for every $0 < \lambda \leq \lambda_{1,s}$ there do not exist solutions to problem (1.3) belonging to the closed ball $B \subset L^2(\mathbb{R}^n)$ defined as

$$B := \{u \in L^2(\mathbb{R}^n) : \|u\| \leq S_{n-n/4}(n-2)\}.$$ 

We believe that a similar result should hold for sublinear perturbations (i.e. $0 < p < 1$), but we will come back to this in a future work.

As regards the case of the superlinear perturbation, instead, the situation is quite different: as a matter of fact, we can adapt the variational argument in [6] (based on the Mountain Pass Theorem,) to prove the following result.

**Theorem 1.5.** Let $n \geq 3$ and $p \in (1, 2^*-1)$. Set

$$(1.5) \quad \kappa_{s,n} := \min\{2 - 2s, n - 2\}, \quad \beta_{p,n} := n - \frac{(p+1)(n-2)}{2}.$$ 

Then, the following assertions hold.

1. If $\kappa_{s,n} > \beta_{p,n}$, then problem (1.3) admits a solution for every $\lambda > 0$.
2. If $\kappa_{s,n} \leq \beta_{p,n}$, then problem (1.3) admits a solution for $\lambda$ large enough.

A ‘dichotomy’ similar to that in Theorem 1.5 appears also in the purely local setting, see [14, Section 2], where we have the same value of $\beta_{p,n}$ and

$$(1.6) \quad \kappa_{n} = \kappa_{n} = n - 2.$$ 

The main difference between our mixed setting and the purely local one is that, in our context, when $p \sim 1$ we can prove the existence of solutions to problem (1.3) only for large values of $\lambda$, independently of how large the dimension is.

To clarify this phenomenon, let us assume that $n \geq 4$, so that $\kappa_{s,n} = 2 - 2s$. Under this assumption, the condition $\kappa_{s,n} > \beta_{s,p}$ boils down to

$$n > 2 + \frac{4s}{p-1} =: \theta_{s,p}. $$

Since $\theta_{s,p} \to \infty$ as $p \to 1^+$, when $p \sim 1$ we have $\kappa_{s,n} \leq \beta_{s,p}$, and thus, by Theorem 1.5 we deduce that problem (1.3) possesses solutions for $\lambda$ sufficiently large.

In the purely local setting, instead, the situation is quite different: indeed, by (1.6), since $\kappa_{n} = n - 2$, we have

$$\kappa_{n} > \beta_{p,s} \iff n > 2 + \frac{4}{p+1} =: \theta_{p}. $$

Thus, since $\theta_{p} < 4$ for every $p > 1$, when $n \geq 4$ we obtain in the classical case the existence of solutions for every $\lambda > 0$, independently of the exponent $p \in (1, 2^*-1)$.

In any case, the restriction to large values of $\lambda$ for finding solutions of this type of problems is a common occurrence also in the local scenario, see in particular the case $n = 3$ in [14, Corollary 2.4] (see also the nonlocal counterpart in [6]).
We also remark that the case \( p = 1 \) in Theorem 1.4 is structurally different than the case \( p \in (1, 2^* - 1) \) in Theorem 1.5. Indeed, on the one hand, in both cases we cannot establish the existence of solutions for \( \lambda \) close to zero; on the other hand, while Theorem 1.5 guarantees the existence of solutions for all \( \lambda \) large enough, Theorem 1.4 only detects solutions for \( \lambda \) in a certain interval, showing also that no solutions exist when \( \lambda \) is too large (therefore, the case \( p = 1 \) cannot be seen as a limit case of the setting \( p \in (1, 2^* - 1) \)).

1.1. Plan of the paper. The rest of this paper is organized as follows. Section 2 contains the preliminary material needed to set up the appropriate functional spaces and to formalize the problems that we treat.

Then, in Section 3, we focus on the mixed order Sobolev-type inequality and prove Theorems 1.1 and 1.2.

Finally, the analysis of the critical problem (1.3) occupies Section 4, where we prove Theorems 1.3, 1.4 and 1.5.

2. The functional setting

In this section we collect the notation and some preliminary results which will be used in the rest of the paper. More precisely, we introduce the adequate function spaces to study problem (1.3), and we investigate the existence of extremals for some mixed Sobolev-type inequalities.

Let \( s \in (0, 1) \). If \( u : \mathbb{R}^n \to \mathbb{R} \) is a measurable function, we set

\[
[u]_s := \left( \int\int_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy \right)^{1/2},
\]

and we refer to \([u]_s\) as the Gagliardo seminorm of \( u \) (of order \( s \)).

Let \( \emptyset \neq \Omega \subseteq \mathbb{R}^n \) (with \( n \geq 3 \)) be an arbitrary open set, not necessarily bounded. We define the function space \( \mathcal{X}^{1,2}(\Omega) \) as the completion of \( C_0^\infty(\Omega) \) with respect to the global norm

\[
\rho(u) := (\|\nabla u\|_{L^2(\mathbb{R}^n)}^2 + [u]_s^2)^{1/2}, \quad u \in C_0^\infty(\Omega).
\]

Remark 2.1. A couple of observations concerning the space \( \mathcal{X}^{1,2}(\Omega) \) are in order.

1. The norm \( \rho(\cdot) \) is induced by the scalar product

\[
\langle u, v \rangle := \int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dx + \int\int_{\mathbb{R}^{2n}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dx \, dy,
\]

where \( \cdot \) denotes the usual scalar product in the Euclidean space \( \mathbb{R}^n \), and \( \mathcal{X}^{1,2}(\Omega) \) is a Hilbert space.

2. Even if the function \( u \in C_0^\infty(\Omega) \) identically vanishes outside \( \Omega \), it is often still convenient to consider in the definition of \( \rho(\cdot) \) the \( L^2 \)-norm of \( \nabla u \) on the whole of \( \mathbb{R}^n \), rather than restricted to \( \Omega \) (though of course the result would be the same): this is to stress that the elements in \( \mathcal{X}^{1,2}(\Omega) \) are functions defined on the entire space \( \mathbb{R}^n \) and not only on \( \Omega \) (and this is consistent with the nonlocal nature of the operator \( \mathcal{L} \)). The benefit of having this global functional setting is that these functions can be globally approximated on \( \mathbb{R}^n \) (with respect to the norm \( \rho(\cdot) \)) by smooth functions with compact support in \( \Omega \).
In particular, when \( \Omega \neq \mathbb{R}^n \), we will see that this global definition of \( \rho(\cdot) \) implies that the functions in \( X^{1,2}(\Omega) \) naturally satisfy the nonlocal Dirichlet condition prescribed in problem (1.3), that is,
\[
\tag{2.2}
\quad u \equiv 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \text{ for every } u \in X^{1,2}(\Omega).
\]

In order to better understand the nature of the space \( X^{1,2}(\Omega) \) (and to recognize the validity of (2.2)), we distinguish two cases.

(i) If \( \Omega \) is bounded. In this case we first recall the following inequality, which expresses the continuous embedding of \( H^1(\mathbb{R}^n) \) into \( H^s(\mathbb{R}^n) \) (see, e.g., [13, Proposition 2.2]): there exists a constant \( c = c(s) > 0 \) such that, for every \( u \in C_0^\infty(\Omega) \), one has
\[
(2.3) \quad [u]_s^2 \leq c(s)\|u\|_{H^1(\mathbb{R}^n)}^2 = c(s)(\|u\|_{L^2(\mathbb{R}^n)}^2 + \|\nabla u\|_{L^2(\mathbb{R}^n)}^2).
\]
This, together with the classical Poincaré inequality, implies that \( \rho(\cdot) \) and the full \( H^1 \)-norm in \( \mathbb{R}^n \) are actually equivalent on the space \( C_0^\infty(\Omega) \), and hence
\[
X^{1,2}(\Omega) = \overline{C_0^\infty(\Omega)}_{\|\cdot\|_{H^1(\mathbb{R}^n)}} = \{ u \in H^1(\mathbb{R}^n) : u|_{\Omega} \in H^1_0(\Omega) \text{ and } u \equiv 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \}.
\]

(ii) If \( \Omega \) is unbounded. In this case, even if the embedding inequality (2.3) is still satisfied, the Poincaré inequality does not hold; hence, the norm \( \rho(\cdot) \) is no more equivalent to the full \( H^1 \)-norm in \( \mathbb{R}^n \), and \( X^{1,2}(\Omega) \) is not a subspace of \( H^1(\mathbb{R}^n) \).

On the other hand, by the classical Sobolev inequality we infer the existence of a constant \( S = S_n > 0 \), independent of the open set \( \Omega \), such that
\[
(2.4) \quad S_n\|u\|_{L^2^*(\mathbb{R}^n)}^2 \leq \|\nabla u\|_{L^2(\mathbb{R}^n)}^2 \leq \rho(u)^2 \quad \text{for every } u \in C_0^\infty(\Omega).
\]
From this, we deduce that every Cauchy sequence in \( C_0^\infty(\Omega) \) (with respect to the norm \( \rho(\cdot) \)) is also a Cauchy sequence in the space \( L^2^*(\mathbb{R}^n) \); as a consequence, since the functions in \( C_0^\infty(\Omega) \) identically vanish out of \( \Omega \), we obtain
\[
X^{1,2}(\Omega) = \{ u \in L^2^*(\mathbb{R}^n) : u \equiv 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega, \nabla u \in L^2(\mathbb{R}^n) \text{ and } |u|_s < \infty \}.
\]
In particular, when \( \Omega = \mathbb{R}^n \) we have
\[
X^{1,2}(\mathbb{R}^n) = \{ u \in L^2^*(\mathbb{R}^n) : \nabla u \in L^2(\mathbb{R}^n) \text{ and } |u|_s < \infty \}.
\]

Remark 2.2. We stress that the mixed Sobolev-type inequality (2.4) holds for every open set \( \Omega \subset \mathbb{R}^n \) (bounded or not); thus, we always have
\[
(2.5) \quad X^{1,2}(\Omega) \hookrightarrow L^2^*(\Omega).
\]
Furthermore, by exploiting the density of \( C_0^\infty(\Omega) \) in \( X^{1,2}(\Omega) \), we can extend inequality (2.4) to every function \( u \in X^{1,2}(\Omega) \), thereby obtaining
\[
S_n\|u\|_{L^2^*(\mathbb{R}^n)}^2 \leq \rho(u)^2 = \|\nabla u\|_{L^2(\mathbb{R}^n)}^2 + |u|_s^2 \quad \text{for every } u \in X^{1,2}(\Omega).
\]

3. The mixed Sobolev-type inequality

Now we further develop our analysis on the mixed Sobolev inequality (2.4), that is,
\[
S_n\|u\|_{L^2^*(\mathbb{R}^n)}^2 \leq \rho(u)^2 \quad \text{for every } u \in C_0^\infty(\Omega),
\]
with the aim of proving Theorems 1.1 and 1.2.
To this end, our first goal is to find the sharp constant in (2.4), namely,
\begin{equation}
S_{n,s}(\Omega) := \inf \{\rho(u)^2 : u \in C_0^\infty(\Omega) \cap M(\Omega)\},
\end{equation}
where $M(\Omega)$ is the unit sphere in $L^2^*(\mathbb{R}^n)$, that is,
\[M(\Omega) := \{u \in L^2^*(\mathbb{R}^n) : \|u\|_{L^2^*(\mathbb{R}^n)} = 1\}.
\]
We stress that, since $C_0^\infty(\Omega)$ is dense in the space $\mathcal{X}^{1,2}(\Omega)$ (with respect to the norm $\rho(\cdot)$), by the continuous embedding (2.5) we have
\[S_{n,s}(\Omega) = \inf \{\rho(u)^2 : u \in \mathcal{X}^{1,2}(\Omega) \cap M(\Omega)\}.
\]
Moreover, $S_{n,s}(\Omega)$ is translation-invariant with respect to $\Omega$, that is,
\[S_{n,s}(x_0 + \Omega) = S_{n,s}(\Omega) \quad \text{for every } x_0 \in \mathbb{R}^n.
\]

\textbf{Remark 3.1.} Before starting with the study of the minimization problem (3.1), we review here below some well-known properties of the best constant $S_n$ in the classical Sobolev inequality, which is defined as follows:
\[S_n := \inf \{\|\nabla u\|_{L^2(\Omega)} : u \in C_0^\infty(\Omega) \cap M(\mathbb{R}^n)\}.
\]
For a proof of these facts we refer to [3, 32] (see also [14, Section 1]).

1. The constant $S_n$ is independent of the open set $\Omega$, and depends only on the dimension $n$. More precisely, we have the explicit expression
\begin{equation}
S_n = \frac{1}{n(n-2)\pi} \left(\frac{\Gamma(n)}{\Gamma(n/2)}\right)^{2/n},
\end{equation}
where $\Gamma(\cdot)$ denotes the usual Euler Gamma function.
2. Given any open set $\Omega \subseteq \mathbb{R}^n$ (bounded or not), we have
\[S_n = \inf \{\|\nabla u\|_{L^2(\mathbb{R}^n)}^2 : u \in D_{0}^{1,2}(\Omega) \cap M(\Omega)\},
\]
where $D_{0}^{1,2}(\Omega)$ is the space defined as the completion of $C_0^\infty(\Omega)$ with respect to the gradient norm $u \mapsto \mathcal{G}(u) := \|\nabla u\|_{L^2(\mathbb{R}^n)}$.
3. If $\Omega \subseteq \mathbb{R}^n$ is a bounded open set, then $S_n$ is never achieved.
4. If $\Omega = \mathbb{R}^n$, then $S_n$ is not achieved in $C_0^\infty(\mathbb{R}^n)$ but it is achieved in the bigger space $D_{0}^{1,2}(\mathbb{R}^n)$ by the family of functions
\[\mathcal{F} = \{U_{t,x_0}(x) = t^{\frac{2-n}{2}}U((x-x_0)/t) : t > 0, x_0 \in \mathbb{R}^n\},
\]
where
\[U(z) := c(1+|z|^2)^{\frac{2-n}{2}}
\]
and $c > 0$ is such that $\|U\|_{L^2(\mathbb{R}^n)} = 1$.

We are thus ready to prove Theorem 1.1.

\textbf{Proof of Theorem 1.1.} Since $\rho(u) \geq \|\nabla u\|_{L^2(\Omega)}$ for every $u \in C_0^\infty(\Omega)$, we have
\[S_{n,s}(\Omega) \geq \inf \{\|\nabla u\|_{L^2(\Omega)}^2 : u \in C_0^\infty(\Omega) \cap M(\Omega)\} = S_n.
\]
To prove the reverse inequality, by the translation-invariance of $S_{n,s}(\Omega)$ we assume that $x_0 = 0 \in \Omega$, and we let $r > 0$ be such that $B_r(0) \subseteq \Omega$. We now observe that, given any $u \in C_0^\infty(\mathbb{R}^n) \cap M(\mathbb{R}^n)$, there exists $k_0 = k_0(u) \in \mathbb{N}$ such that
\[\text{supp}(u) \subseteq B_{kr}(0) \quad \text{for every } k \geq k_0,
\]
as a consequence, setting \(u_k := k^{\frac{2-s}{2}} u(kx)\) (for \(k \geq k_0\)), we readily see that

\[
\text{supp}(u_k) \subseteq B_r(0) \subseteq \Omega \quad \text{and} \quad \|u_k\|_{L^{2^*}(\mathbb{R}^n)} = 1.
\]

Taking into account the definition of \(S_{n,s}(\Omega)\), we find that, for every \(k \geq k_0\),

\[
S_{n,s}(\Omega) \leq \rho(u_k)^2 = \|\nabla u_k\|^2_{L^2(\mathbb{R}^n)} + [u_k]^2_s = \|\nabla u\|^2_{L^2(\mathbb{R}^n)} + k^{2s-2}[u^s].
\]

From this, letting \(k \to \infty\) (and recalling that \(s < 1\)), we obtain

\[
S_{n,s}(\Omega) \leq \|\nabla u\|^2_{L^2(\mathbb{R}^n)}.
\]

By the arbitrariness of \(u \in C^\infty_0(\Omega) \cap M(\mathbb{R}^n)\) and the fact that \(S_n\) is independent of the open set \(\Omega\) (see Remark 3.1-(1)), we finally infer that

\[
S_{n,s}(\Omega) \leq \inf \{ \|\nabla u\|^2_{L^2(\mathbb{R}^n)} : C^\infty_0(\Omega) \cap M(\mathbb{R}^n) \} = S_n,
\]

and hence \(S_{n,s}(\Omega) = S_n\). \(\square\)

We are now in the position of proving Theorem 1.2.

**Proof of Theorem 1.2.** Arguing by contradiction, let us suppose that there exists a nonzero function \(u_0 \in X^{1,2}(\Omega)\) such that \(\|u_0\|_{L^{2^*}(\mathbb{R}^n)} = 1\) and

\[
\rho(u_0)^2 = \|\nabla u_0\|^2_{L^2(\mathbb{R}^n)} + [u_0]^2_s = S_n.
\]

Taking into account that \(X^{1,2}(\Omega) \subseteq D_0^{1,2}(\Omega)\) (this inclusion being a straightforward of the fact that \(\rho(\cdot) \geq G(\cdot)\), see Remark 3.1-(2)), we infer that

\[
S_n \leq \|\nabla u_0\|^2_{L^2(\Omega)} \leq \|\nabla u_0\|^2_{L^2(\mathbb{R}^n)} + [u_0]^2_s = \rho(u_0)^2 = S_n,
\]

from which we derive that \([u_0]^s = 0\). As a consequence, the function \(u_0\) must be constant in \(\mathbb{R}^n\), but this is contradiction with the fact that \(\|u_0\|_{L^{2^*}(\mathbb{R}^n)} = 1\). \(\square\)

**Remark 3.2.** Even if Theorem 1.2 show that the constant \(S_{n,s}(\Omega) = S_n\) is never achieved in the space \(X^{1,2}(\Omega)\) (independently of the open set \(\Omega\)), in the particular case \(\Omega = \mathbb{R}^n\) we can prove that \(S_n\) is achieved ‘in the limit’: more precisely, if \(F = \{U_{t,x_0}\}\) is as in Remark 3.1-(4), we have

\[
\rho(U_{t,x_0})^2 \to S_n \quad \text{as} \quad t \to \infty.
\]

Indeed,

\[
|U(z)| \leq C \min\{1, |z|^{2-n}\} \quad \text{and} \quad |\nabla U(z)| \leq C \min\{|z|, |z|^{1-n}\},
\]
for some $C > 0$, therefore (up to renaming $C$ line after line)

$$
[U(z)]^2_s = \int_{\mathbb{R}^n} \frac{|U(x + y) - U(x)|^2}{|y|^{n+2s}} \, dx \, dy
\leq \int_{\mathbb{R}^n \times B_1} \left( \left| \nabla U(x + ty) \cdot y \right|^2 \frac{dx}{|y|^{n+2s}} + 2 \int_{|x + ty|^2 + |U(x)|^2} \frac{dx}{|y|^{n+2s}} \right) \\
\leq \int_{\mathbb{R}^n \times B_1 \times (0,1)} \min\{|x + ty|^2, |x + ty|^{2(1-s)}\} \frac{dx \, dy \, dt}{|y|^{n+2s-2}} \\
+ C \int_{\mathbb{R}^n \times \mathbb{R}^n \setminus B_1} \min\{1, |z|^{2(2-s)}\} \frac{dz \, dy}{|y|^{n+2s}},
$$

which is finite.

Hence, $U \in \mathcal{X}^{1,2}(\mathbb{R}^n)$ and consequently $U_{t,x_0} \in \mathcal{X}^{1,2}(\mathbb{R}^n)$ for every $t > 0$ and $x_0 \in \mathbb{R}^n$.

Moreover, recalling that

$$
U_{t,x_0}(x) = t^{\frac{2-n}{s}} U \left( \frac{x - x_0}{t} \right) \quad \text{and} \quad \|U_{t,x_0}\|_{L^2(\mathbb{R}^n)} = \|U\|_{L^2(\mathbb{R}^n)} = 1,
$$

by arguing as in the proof of Theorem [1.1] we have

$$
\rho(U_{t,x_0})^2 = \rho(U_{t,0})^2 = \|\nabla U\|_{L^2(\mathbb{R}^n)}^2 + t^{2s-2}[U]_s^2.
$$

From this, since $U = U_{1,0}$ is an optimal function in the classical Sobolev inequality (and since $s < 1$), by letting $t \to \infty$ we obtain

$$
\rho(U_{t,x_0})^2 \to \|\nabla U\|_{L^2(\mathbb{R}^n)}^2 = S_n.
$$

4. Critical problems driven by $\mathcal{L}$

We now develop our study concerning the solvability of problem [1.3]. Throughout this section, we tacitly understand that

(i) $\Omega \subseteq \mathbb{R}^n$ is a bounded open set;
(ii) $\lambda \in \mathbb{R}$;
(iii) $1 \leq p < 2^* - 1$.

Moreover, we inherit all the definitions and notation of Section [2] in particular, due to their relevance in what follows, we recall that

(a) $s \in (0, 1)$ and $[u]_s$ is the Gagliardo seminorm of $u$, see (2.1);
(b) $\mathcal{X}^{1,2}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$
\rho(u) = (\|\nabla u\|_{L^2(\mathbb{R}^n)}^2 + [u]_s^2)^{1/2};
$$

(c) $\mathcal{M}(\Omega)$ is the unit sphere in $L^{2^*}(\mathbb{R}^n)$, that is,

$$
\mathcal{M}(\Omega) = \{u \in L^{2^*}(\mathbb{R}^n) : \|u\|_{L^{2^*}(\mathbb{R}^n)} = 1\};
$$
We begin by giving the precise definition of solution $F$ where

$$\mathcal{S}_n = \inf \left\{ \| \nabla u \|_{L^2(\mathbb{R}^n)}^2 : u \in D_0^{1,2}(\Omega) \cap \mathcal{M}(\Omega) \right\}$$

(\text{4.1})

and for a future reference, we have

$$\mathcal{X}_+^{1,2}(\Omega) := \{ u \in \mathcal{X}_+^{1,2}(\Omega) : u \geq 0 \text{ a.e. in } \Omega \}.$$

Finally, we introduce the convex cone

$$\mathcal{X}_+^{1,2}(\Omega) := \{ u \in \mathcal{X}_+^{1,2}(\Omega) : u \geq 0 \text{ a.e. in } \Omega \}.$$

We begin by giving the precise definition of solution to (1.3).

**Definition 4.1.** We say that a function $u \in \mathcal{X}_+^{1,2}(\Omega)$ is a solution to problem (1.3) if it satisfies the following properties:

1. $|\{ x \in \Omega : u(x) > 0 \}| > 0$;
2. for every test function $v \in \mathcal{X}_+^{1,2}(\Omega)$ we have

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dx \, dy \leq \int_{\Omega} (u^{2^*-1} + \lambda u^p) v \, dx.$$

(\text{4.2})

**Remark 4.2.** Some observations concerning Definition 4.1 are in order.

1. First of all we recall that, since $\Omega \subset \mathbb{R}^n$ is bounded, we have

$$\mathcal{X}_+^{1,2}(\Omega) = \{ u \in H^1(\mathbb{R}^n) : u|_{\Omega} \in H^1_0(\Omega) \text{ and } u \equiv 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \}.$$

As a consequence, the assumption $u \in \mathcal{X}_+^{1,2}(\Omega)$ contains both the Dirichlet boundary condition $u \equiv 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega$ and the sign condition $u \geq 0 \text{ a.e. in } \Omega$.

2. We observe that Definition 4.1 is well-posed, in the sense that all the integrals in (4.2) are finite. Indeed, if $u, v \in \mathcal{X}_+^{1,2}(\Omega)$, we have

$$\int_{\Omega} |\nabla u| \, dx + \int_{\mathbb{R}^n} \frac{|u(x) - u(y)| |v(x) - v(y)|}{|x - y|^{n+2s}} \, dx \, dy \leq \| \nabla u \|_{L^2(\mathbb{R}^n)} \cdot \| \nabla v \|_{L^2(\mathbb{R}^n)} + [u]_s \cdot [v]_s \leq 2\rho(u)\rho(v) < \infty.$$

Moreover, since $\mathcal{X}_+^{1,2}(\Omega) \hookrightarrow L^{2^*}(\mathbb{R}^n)$ and $p < 2^* - 1$, using Hölder’s inequality (and taking into account that $u, v \equiv 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega$) we also have

$$\int_{\Omega} (u^{2^*-1} + \lambda u^p) v \, dx \leq \| u \|_{L^{2^*}(\mathbb{R}^n)} \cdot \| v \|_{L^{2^*}(\mathbb{R}^n)} + |\lambda| \| u \|_{L^{2^*}(\mathbb{R}^n)} \cdot \| v \|_{L^{2^*-p}(\mathbb{R}^n)} < \infty.$$

Before proceeding we highlight that, if $u \in \mathcal{X}_+^{1,2}(\Omega)$ is a given function, the validity of identity (4.2) (which expresses the fact that $u$ is a solution to problem (1.3)) is actually equivalent to the following condition

$$d\mathcal{F}_{\lambda,p}(u) \equiv 0 \text{ on } \mathcal{X}_+^{1,2}(\Omega),$$

where $\mathcal{F}_{\lambda,p}$ is the $C^1$-functional defined on the Hilbert space $\mathcal{X}_+^{1,2}(\Omega)$ by

$$\mathcal{F}_{\lambda,p}(u) := \frac{1}{2} \rho(u)^2 - \frac{1}{2s} \int_{\Omega} |u|^{2s} \, dx - \frac{\lambda}{p + 1} \int_{\Omega} |u|^{p+1} \, dx.$$

(\text{4.3})
Thus, we derive that the solutions to problem (1.3) are precisely the nonnegative critical points of the functional $\mathcal{F}_{\lambda,p}$. This characterization will be used, as a crucial tools, in order to study the existence of solutions to problem (1.3).

With Definition 4.1 at hand, we establish the following boundedness and positivity result.

**Theorem 4.3.** Assume that there exists a solution $u_0 \in \mathcal{X}_{1,2}^1(\Omega)$ to problem (1.3) for some $\lambda \in \mathbb{R}$ and $p \in [1,2^*-1)$. Then, the following facts hold:

1. $u_0 \in L^\infty(\mathbb{R}^n)$;
2. if, in addition, $\lambda \geq 0$, then $u_0 > 0$ a.e. in $\Omega$.

**Proof.** The boundedness of $u_0$ is a direct consequence of [11, Theorem 4.1], applied here with the choice $f(x,t) := t^{2^*-1} + \lambda|t|^p$: indeed, $u_0$ is a solution to

$$\begin{cases}
Lu = f(x,u) & \text{in } \Omega, \\
u \geq 0 & \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}$$

and the function $f$ satisfies the following properties:

(i) $f : \Omega \times [0,\infty) \to \mathbb{R}$ is a Carathéodory function;
(ii) $f(\cdot,t) \in L^\infty(\Omega)$ for every $t \geq 0$;
(iii) $|f(x,t)| \leq (1 + |\lambda|)(1 + t^{2^*-1})$ for every $t \geq 0$.

In view of these properties of $f$, we are then entitled to invoke [11, Theorem 4.1] (see also [11, Remark 4.2]), ensuring that $u_0 \in L^\infty(\mathbb{R}^n)$. This, jointly with the fact that $u_0 \equiv 0$ a.e. in $\mathbb{R}^n \setminus \Omega$, implies that $u_0 \in L^\infty(\mathbb{R}^n)$, as desired.

Assume now that $\lambda \geq 0$. Recalling that $u_0$ is a solution to (1.3) (in the sense of Definition 4.1), and since $u_0 \geq 0$ a.e. in $\Omega$, for every $v \in \mathcal{X}_{1,2}^1(\Omega)$ we have

$$\begin{align*}
\int_{\Omega} \nabla u \cdot \nabla v
dx + \int_{\mathbb{R}^{2n}} \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+2s}}
dx dy \\
= \int_{\Omega} (u^{2^*-1} + \lambda|t|^p)v
dx \geq 0.
\end{align*}$$

As a consequence, we can apply the Strong Maximum Principle in [11, Theorem 3.1] (this time with the choice $f(x,t) \equiv 0$), ensuring that either $u_0 > 0$ a.e. in $\Omega$ or $u_0 \equiv 0$ a.e. in $\Omega$.

On the other hand, since $|\{u_0 > 0\}| > 0$, we have that $u_0$ is not identically vanishing in $\Omega$; thus, $u_0 > 0$ a.e. in $\Omega$, and the proof is complete. □

Thanks to Theorem 4.3, we can now prove Theorem 1.3.

**Proof of Theorem 1.3.** Arguing by contradiction, let us suppose that there exists a nonzero solution $u_0$ to problem (1.3). Owing to Theorem 4.3, we have $u_0 \in L^\infty(\mathbb{R}^n)$; moreover, defining $f(x,t) := |t|^{2^*-2}t + \lambda|t|^{p-1}t$ (with $t \in \mathbb{R}$), we see that $u_0$ solves

$$\begin{cases}
Lu = f(x,u) & \text{in } \Omega, \\
u \equiv 0 & \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}$$

and the nonlinearity $f$ satisfies the following properties:
(i) $f \in C^{0,1}_{\text{loc}}(\overline{\Omega} \times \mathbb{R})$;

(ii) for every $x \in \Omega$ and every $t \in \mathbb{R}$, we have
\[
\frac{n-2}{2} tf(x,t) = \frac{n-2}{2} \left( |t|^{2^*} + \lambda |t|^{p+1} \right) = n \left( \frac{|t|^{2^*}}{2^*} + \frac{\lambda |t|^{p+1}}{2} \right) \tag{since $\lambda \leq 0$ and $p \leq 2^* - 1$}
\]
\[
\geq n \left( \frac{|t|^{2^*}}{2^*} + \frac{\lambda |t|^{p+1}}{p+1} \right) = n \int_0^t f(x,\tau) \, d\tau.
\]

As a consequence, since $u_0$ is a bounded solution of (4.5), $f$ satisfies (i)-(ii) and $\Omega$ is star-shaped, we are entitled to apply [27, Theorem 1.3], ensuring that $u_0 \equiv 0$ a.e. in $\Omega$.

This is clearly a contradiction with the fact that $u_0 \geq 0$ in $\Omega$ (recall that $u_0$ is a solution to (1.3), see Definition 4.1), and the proof is complete. \qed

From now on we assume that $\lambda > 0$.

Under this assumption (which is somewhat in agreement with Theorem 1.3) and taking into account all the discussion carried out so far, we can start our study of the solvability of problem (1.3). To this end, since the linear case $p = 1$ and the superlinear case $p > 1$ present some subtle differences, it is worth treating these cases separately.

4.1. The linear case $p = 1$. We begin by studying the solvability of (1.3) in the linear case $p = 1$, that is, we begin by considering the problem

\[
\begin{aligned}
L u &= u^{2^*-1} + \lambda u \quad \text{in } \Omega, \\
\{ u \} &= 0 \quad \text{in } \Omega, \\
\{ u \} &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,
\end{aligned}
\]

(4.6)

In this linear context, the existence of solutions to problem (4.6) for a given $\lambda > 0$ is deeply influenced by the first Dirichlet eigenvalues of $(-\Delta)^s$ and of $L$.

We recall that, given the bounded open set $\Omega$, one defines

(i) the first Dirichlet eigenvalue of $(-\Delta)^s$ in $\Omega$ as
\[
\lambda_{1,s} := \inf \{ [u]_s^2 : u \in C_0^\infty(\Omega) \text{ and } \|u\|_{L^2(\mathbb{R}^n)} = 1 \};
\]

(ii) the first Dirichlet eigenvalue of $L$ in $\Omega$ as
\[
\lambda_1 := \inf \{ \rho(u)^2 : u \in C_0^\infty(\Omega) \text{ and } \|u\|_{L^2(\mathbb{R}^n)} = 1 \}.
\]

Clearly, both $\lambda_{1,s}$ and $\lambda_1$ depend on the open set $\Omega$; however, in order to simplify the notation, we avoid to make explicit this dependence in what follows.

For the sake of completeness (and for a future reference), we then present in the next remark a short overview of the main properties of $\lambda_{1,s}$ and $\lambda_1$.

Remark 4.4 (Properties of $\lambda_{1,s}$ and $\lambda_1$). As regards $\lambda_{1,s}$ we first observe that, since the map $u \mapsto [u]_s := \mathcal{N}(u)$ is a norm on $C_0^\infty(\Omega)$ which is equivalent to the full $H^s$-norm (see, e.g., [18, Theorem 6.5] and recall that $\Omega$ is bounded), one has
\[
\lambda_{1,s} = \inf \{ [u]_s^2 : u \in \mathcal{D}_0^{s,2}(\Omega) \text{ and } \|u\|_{L^2(\mathbb{R}^n)} = 1 \},
\]
where $\mathcal{D}^{s,2}_0(\Omega) \subseteq L^2(\mathbb{R}^n)$ is the completion of $C_0^\infty(\Omega)$ with respect to $\mathcal{N}$. Moreover, since the embedding $\mathcal{D}^{s,2}_0(\Omega) \hookrightarrow L^2(\mathbb{R}^n)$ is compact, it is not difficult to recognize that $\lambda_{1,s}$ is actually achieved in this larger space $\mathcal{D}^{s,2}_0(\Omega)$, that is,

$$\exists \phi_0 \in \mathcal{D}^{s,2}_0(\Omega) : \|\phi_0\|_{L^2(\mathbb{R}^n)} = 1 \text{ and } |\phi_0|^2 = \lambda_{1,s} > 0.$$ 

As a matter of fact, the above function $\phi_0$ can be chosen to be strictly positive a.e. in $\Omega$ (see, e.g., [29 Proposition 9]); in addition, since $\phi_0$ is a constrained minimizer of the functional $u \mapsto |u|^2$, by the Lagrange Multiplier Rule we easily see that

$$\int\int_{\mathbb{R}^{2n}} \frac{(\phi_0(x) - \phi_0(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dx \, dy = \lambda_{1,s} \int \phi_0 v \, dx \quad \forall v \in \mathcal{D}^{s,2}_0(\Omega),$$ 

and this shows that $\phi_0$ is weak solution to the eigenvalue problem

$$\begin{cases}
(-\Delta)^su = \lambda_{1,s}u & \text{in } \Omega, \\
u \neq 0 & \text{in } \Omega, \\
u \equiv 0 & \text{in } \mathbb{R}^n \setminus \Omega.
\end{cases}$$

As regards $\lambda_1$, we have completely analogous results (with the obvious modifications): first of all, since $\rho(\cdot)$ defines a norm on $C_0^\infty(\Omega)$ which is equivalent to the full $H^1$-norm in $\mathbb{R}^n$ (as $\Omega$ is bounded), we have

$$\lambda_1 = \inf \{\rho(u)^2 : u \in \mathcal{X}^{1,2}(\Omega) \text{ and } \|u\|_{L^2(\mathbb{R}^n)} = 1\},$$

Moreover, since the embedding $\mathcal{X}^{1,2}(\Omega) \hookrightarrow L^2(\mathbb{R}^n)$ is compact, it is not difficult to see that $\lambda_1(\Omega)$ is actually achieved in this larger space $\mathcal{X}^{1,2}(\Omega)$, that is,

$$\exists \psi_0 \in \mathcal{X}^{1,2}(\Omega) : \|\psi_0\|_{L^2(\mathbb{R}^n)} = 1 \text{ and } \rho(\psi_0)^2 = \lambda_1 > 0.$$ 

As a matter of fact, the above function $\psi_0$ can be chosen to be strictly positive a.e. in $\Omega$ (see, e.g., [11, Proposition 5.1]); in addition, since $\psi_0$ is a constrained minimizer of the functional $\rho(\cdot)^2$, by the Lagrange Multiplier Rule we easily see that

$$\int \nabla \psi_0 \cdot \nabla v \, dx + \int\int_{\mathbb{R}^{2n}} \frac{(\psi_0(x) - \psi_0(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dx \, dy = \lambda_1 \int \psi_0 v \, dx \quad \forall v \in \mathcal{X}^{1,2}(\Omega),$$

and this shows that $\psi_0$ is weak solution to the eigenvalue problem

$$\begin{cases}
\mathcal{L}u = \lambda_1 u & \text{in } \Omega, \\
u \neq 0 & \text{in } \Omega, \\
u \equiv 0 & \text{in } \mathbb{R}^n \setminus \Omega.
\end{cases}$$

We are now approaching the proof of Theorem 1.4. We will achieve this through several independent results. To begin with, we prove a lemma linking the existence of solutions to (4.6) with the existence of constrained minimizers for a suitable functional.

As a matter of facts, we have already recognized at the beginning of this section that the solutions to problem (4.3) are precisely the unconstrained nonnegative critical points of the functional $\mathcal{F}_{\lambda,p}$ defined in (4.3) (for all $p \geq 1$); we now aim at
proving that, in this linear context, there is a link between existence of solutions to (4.6) and existence of minimizers for the functional

\[ Q_\lambda(u) := \rho(u)^2 - \lambda \|u\|_{L^2(\mathbb{R}^n)}^2 \quad (u \in \mathcal{X}^{1,2}(\Omega)), \]

constrained to the manifold \( \mathcal{V}(\Omega) := \mathcal{X}^{1,2}(\Omega) \cap \mathcal{M}(\Omega) \).

**Lemma 4.5.** For every fixed \( \lambda > 0 \), we define

\[ S_n(\lambda) := \inf_{u \in \mathcal{V}(\Omega)} Q_\lambda(u). \]

We assume that \( S_n(\lambda) > 0 \) and that \( S_n(\lambda) \) is achieved, that is, there exists some function \( w \in \mathcal{V}(\Omega) \) such that \( Q_\lambda(w) = S_n(\lambda) \).

Then, there exists a solution to (4.6).

Before giving the proof of Lemma 4.5, we list in the next remark some properties of the number \( S_n(\lambda) \) which will be used in the sequel.

**Remark 4.6 (Properties of \( S_n(\lambda) \)).** We begin by observing that, on account of (4.1) and using Hölder’s inequality, for every \( u \in \mathcal{V}(\Omega) \) we have

\[ Q_\lambda(u) \geq \inf_{u \in \mathcal{V}(\Omega)} \rho^2(u) - \lambda \|u\|_{L^2(\mathbb{R}^n)}^2 \geq S_n - \lambda \|u\|_{L^2(\mathbb{R}^n)}^2 \]

(recall that \( u \equiv 0 \) a.e. in \( \mathbb{R}^n \setminus \Omega \))

\[ \geq S_n - \lambda |\Omega|^{4/n}. \]

As a consequence, we obtain

\[ S_n(\lambda) \geq S_n - \lambda |\Omega|^{4/n} > -\infty \quad \text{for every} \ \lambda > 0. \]

In addition, from the definition of \( S_n(\lambda) \) we easily infer that

(i) \( S_n(\lambda) \leq S_n \) for every \( \lambda > 0 \);

(ii) \( S_n(\lambda) \leq S_n(\mu) \) for every \( 0 < \mu < \lambda \).

Finally, owing to the definition of \( \lambda_1 \) in (4.8) (and recalling that \( \lambda_1 \) is achieved in the space \( \mathcal{X}^{1,2}(\Omega) \), see Remark 4.4), it is easy to see that

\[ S_n(\lambda) \geq 0 \iff 0 < \lambda \leq \lambda_1. \]

We are now ready to provide the proof of Lemma 4.5.

**Proof of Lemma 4.5.** Let \( w \in \mathcal{V}(\Omega) \) be a constrained minimizer for the functional \( Q_\lambda \) (whose existence is guaranteed by the assumptions), that is,

\[ Q_\lambda(w) = S_n > 0. \]

Since \( Q_\lambda(|w|) \leq Q_\lambda(w) \), by possibly replacing \( w \) with \( |w| \) we may and do assume that \( w \geq 0 \) a.e. in \( \Omega \); moreover, by the Lagrange Multiplier Rule we have

\[ dQ_\lambda(w) \equiv \mu \ d\left(u \mapsto \|u\|_{L^2(\mathbb{R}^n)}^2 \right)(w) \quad \text{on} \ \mathcal{X}^{1,2}(\Omega), \]

for some \( \mu \in \mathbb{R} \). This means that

\[ \int_{\Omega} \nabla w \cdot \nabla \phi + \int_{\mathbb{R}^{2n}} \frac{(w(x) - w(y))(\phi(x) - \phi(y))}{|x - y|^{n+2s}} = \int_{\Omega} (\mu w^{2s-1} + \lambda w) \phi \ dx \quad \text{for every} \ \phi \in \mathcal{X}^{1,2}(\Omega). \]
Now, if we choose \( \phi = w \) in this last identity, we get
\[
\mu = \mu \| w \|_{L^2_2(\mathbb{R}^n)}^2 = \rho(w)^2 - \lambda \| w \|_{L^2(\mathbb{R}^n)}^2 = Q_\lambda(w) = S_n(\lambda) > 0.
\]
As a consequence, setting \( u := S_n(\lambda)^{(n-2)/4}w \), we see that

(i) \( u \in \mathcal{X}^{1,2}_+(\Omega) \setminus \{0\} \), since \( w \geq 0 \) in \( \Omega \) and \( S_n(\lambda) > 0 \);

(ii) for every \( \phi \in \mathcal{X}^{1,2}(\Omega) \), from (4.10) we get
\[
\int_\Omega \nabla u \cdot \nabla \phi + \int_{\mathbb{R}^n} \frac{(u(x) - u(y)) \phi(x) - \phi(y))}{|x-y|^{n+2s}} = \int_\Omega (u^{2^*-1} + \lambda w) \phi \, dx.
\]
Gathering (i)-(ii), we then conclude that \( u \) is a solution to (4.6), as desired. \( \square \)

On account of Lemma 4.9 an important information to study the solvability of problem (4.6) is the sign of the real number \( S_n(\lambda) \). In this perspective, we have already recognized in Remark 4.6 that
\[
S_n(\lambda) \geq 0 \iff 0 < \lambda \leq \lambda_1.
\]
Thus, we prove the following result, which precis\( e \)s this information.

**Lemma 4.7.** For every \( 0 < \lambda \leq \lambda_{1,s} \), we have
\[
S_n(\lambda) = S_n > 0.
\]

**Proof.** Let \( \lambda \in (0, \lambda_{1,s}) \) be given. On the one hand, we have already recognized in Remark 4.6 that \( S_n(\lambda) \leq S_n \); on the other hand, taking into account the definition of \( \lambda_{1,s} \) in (4.7), for every \( u \in C_0^\infty(\Omega) \cap \mathcal{M}(\Omega) \) we have
\[
Q_\lambda(u) = \| \nabla u \|_{L^2(\mathbb{R}^n)}^2 + (|u|^2 - \lambda \| u \|_{L^2(\mathbb{R}^n)}^2)
\geq \| \nabla u \|_{L^2(\mathbb{R}^n)}^2 + (\lambda_{1,s} - \lambda) \| u \|_{L^2(\mathbb{R}^n)}^2
\geq \| \nabla u \|_{L^2(\mathbb{R}^n)}^2.
\]
As a consequence, since \( C_0^\infty(\Omega) \) is dense in \( \mathcal{X}^{1,2}(\Omega) \), from (4.11) we obtain
\[
S_n(\lambda) = \inf \{ Q_\lambda(u) : u \in C_0^\infty(\Omega) \cap \mathcal{M}(\Omega) \}
\geq \inf \{ \| \nabla u \|_{L^2(\mathbb{R}^n)}^2 : u \in C_0^\infty(\Omega) \cap \mathcal{M}(\Omega) \} = S_n,
\]
from which we deduce that \( S_n(\lambda) = S_n \), as desired. \( \square \)

By combining Remark 4.6 with Lemma 4.7 we conclude that

(i) \( S_n(\lambda) = S_n \) for every \( 0 < \lambda \leq \lambda_{1,s} \);

(ii) \( S_n(\lambda) \geq 0 \) for every \( 0 < \lambda \leq \lambda_1 \);

(iii) \( S_n(\lambda) < 0 \) for every \( \lambda > \lambda_1 \).

We now turn to prove that \( S_n(\cdot) \) is continuous function of \( \lambda \in (0, \infty) \); this, together with the above (i)-to-(iii), ensures that \( S_n(\cdot) \) assumes all the values between 0 and the constant \( S_n \) when \( \lambda \) ranges in the interval \( (0, \lambda_1] \).

**Lemma 4.8.** The function \( \lambda \mapsto S_n(\lambda) \) is continuous on \( (0, \infty) \).

**Proof.** We have already recognized in Remark 4.6 that \( S_n(\cdot) \) is nonincreasing on the interval \( (0, \infty) \).

As a consequence, for every \( \lambda > 0 \) we have
\[
\exists S_n(\lambda-) := \lim_{\lambda \to 0^-} S_n(\lambda) \in \mathbb{R} \quad \text{and} \quad \exists S_n(\lambda+) := \lim_{\lambda \to 0^+} S_n(\lambda) \in \mathbb{R}.
\]
To prove the continuity of $S_n(\cdot)$ on $(0, \infty)$, we then pick any $\lambda_0 > 0$ and we show that $S_n(\cdot)$ is continuous both from the left and from the right at $\lambda_0$, that is,

$$S_n(\lambda_0-) = S_n(\lambda_0+) = S_n(\lambda_0).$$

As regards the continuity from the left, we proceed as follows: first of all, given any $\varepsilon > 0$, by definition of $S_n(\lambda_0)$ there exists $u = u_{\varepsilon, \lambda_0} \in V(\Omega)$ such that

$$0 < S_n(\lambda) - S_n(\lambda_0) \leq Q_\lambda(u) - S_n(\lambda_0) = (Q_{\lambda_0}(u) - S_n(\lambda_0)) + (\lambda_0 - \lambda)\|u\|^2_{L^2(\mathbb{R}^n)} < \frac{\varepsilon}{2} + (\lambda_0 - \lambda)\|u\|^2_{L^2(\mathbb{R}^n)} \leq \frac{\varepsilon}{2} + (\lambda_0 - \lambda)|\Omega|^{4/n}$$

As a consequence, setting $\delta_\varepsilon := \varepsilon/(2|\Omega|^{4/n})$, we conclude that

$$0 < S_n(\lambda) - S_n(\lambda_0) < \varepsilon \quad \text{for every } \lambda_0 - \delta_\varepsilon < \lambda \leq \lambda_0,$$

and this proves that $S_n(\cdot)$ is continuous from the left at $\lambda_0$.

As regards the continuity from the right, we proceed essentially as above: first of all, given any $\varepsilon > 0$ and any $\lambda > \lambda_0$, we can find $u = u_{\varepsilon, \lambda} \in V(\Omega)$ such that

$$S_n(\lambda) \leq Q_\lambda(u) < S_n(\lambda) + \frac{\varepsilon}{2}.$$

From this, by the monotonicity of $S_n(\cdot)$ and Hölder’s inequality, we obtain

$$0 < S_n(\lambda_0) - S_n(\lambda) \leq Q_{\lambda_0}(u) - S_n(\lambda) = (Q_{\lambda}(u) - S_n(\lambda)) + (\lambda_0 - \lambda)\|u\|^2_{L^2(\mathbb{R}^n)} \leq \frac{\varepsilon}{2} + (\lambda_0 - \lambda)|\Omega|^{4/n} < \varepsilon,$$

provided that $\lambda_0 \leq \lambda < \lambda_0 + \delta_\varepsilon$ (where $\delta_\varepsilon > 0$ is as above).

We then conclude that $S_n(\cdot)$ is also continuous from the right at $\lambda_0$, and the proof is complete. \hfill \square

In the light of the results established so far, we can finally provide the

**Proof of Theorem 1.4** To begin with, we define

$$\lambda^* := \sup \{\lambda > 0 : S_n(\mu) = S_n \text{ for all } 0 < \mu \leq \lambda\}.$$  

On account of Lemma 4.7, we see that $\lambda_{1,s} \leq \lambda^* < \infty$; moreover, since we know from Lemma 4.8 that $S_n(\cdot)$ is continuous, we have $S_n(\lambda^*) = S_n$.

We also notice that, since $S_n(\cdot) \geq 0$ on $(0, \lambda_1]$ and $S_n(\cdot) < 0$ on $(\lambda_1, \infty)$, again by the continuity of $S_n(\cdot)$ we infer that $S_n(\lambda_1) = 0$; as a consequence, recalling that $S_n(\cdot)$ is nonincreasing on $(0, \infty)$, we conclude that

$$\lambda^* \in [\lambda_{1,s}, \lambda_1).$$

We now turn to prove that the assertion of Theorem 1.4 holds with this choice of $\lambda^*$, that is, we show that the following assertions hold:
and this shows that the best Sobolev constant $S_n$ never achieved in a solution we have the following computation, based on the properties of $\lambda$. This is the proof of [14, Lem. 1.2], we recognize that $S_n$ and this proves that (4.11)

As a consequence, owing to Lemma 4.7, we obtain that

Now we have proved that

To this end, we treat the following three cases separately.

Case I: $0 < \lambda \leq \lambda_{1,s}$. Arguing by contradiction, let us suppose that there exists a solution $u$ to problem (4.6) such that $u \in \mathcal{B}$. Then, setting $v := u / \|u\|_{L^2(\Omega)}$,

we have the following computation, based on the properties of $u$:

As a consequence, owing to Lemma 4.7, we obtain

(4.11) $Q_\lambda(w) = S_n.$

and this proves that $S_n(\lambda) = S_n$ is achieved by $v$.

Now, recalling (4.1), and since $v \in \mathcal{X}^{1,2}(\Omega) \subseteq D_0^{1,2}(\Omega)$, from (4.11) we get

$S_n \leq \|\nabla v\|^2_{L^2(\Omega)} = Q_\lambda(v) - (\|v\|^2 - \lambda \|v\|_{L^2(\Omega)})$

(since $\lambda \leq \lambda_{1,s}$ and $v \in \mathcal{X}^{1,2}(\Omega) \subseteq D_0^{1,2}(\Omega))

$\leq Q_\lambda(v) = S_n,$

and this shows that the best Sobolev constant $S_n$ is achieved by $w \in D_0^{1,2}(\Omega)$.

On the other hand, since $\Omega$ is bounded, we know from Remark 3.1 that $S_n$ is never achieved in $\mathcal{X}^{1,2}(\Omega)$, and so we have a contradiction.

Case II: $\lambda^* < \lambda < \lambda_1$. In this case, owing to the definition of $\lambda^*$ (and recalling that $S_n(\cdot)$ is continuous), we have $0 \leq S_n(\lambda) < S_n$; from this, by arguing exactly as in the proof of [14] Lem. 1.2], we recognize that $S_n(\lambda)$ is achieved, that is, there exists a nonzero function $w \in \mathcal{V}(\Omega)$ such that

$Q_\lambda(w) = S_n(\lambda).$

In particular, since $\lambda < \lambda_1$ (and $w \not= 0$), we obtain

$S_n(\lambda) = \|w\|^2_{L^2(\mathbb{R}^n)} \left[ \frac{\rho(w)^2}{\|w\|_{L^2(\mathbb{R}^n)}} - \lambda \right] \geq \|w\|^2_{L^2(\mathbb{R}^n)} (\lambda_1 - \lambda) > 0.$

Now we have proved that $S_n(\lambda)$ is strictly positive and is achieved, we are then entitled to apply Lemma 4.7 ensuring that there exists a solution to (4.6).
Case III: $\lambda \geq \lambda_1$. In this last case, since $S_n(\cdot)$ is nonincreasing (and since we have already recognized that $S_n(\lambda_1) = 0$), we infer that

$$S_n(\lambda) \leq 0.$$ 

Owing to this fact, we can argue once again as in [14, Lem. 1.2] to prove that $S_n(\lambda)$ is achieved; however, since $S_n(\lambda) \leq 0$, we are not entitled to apply Lemma 4.7 to derive the existence of a solution to problem (4.6). Rather, we show that there cannot exist solutions to (4.6) by proceeding as in [14, Remark 1.1].

Arguing by contradiction, let us assume that there exists (at least) one solution $u$ to problem (4.6) in this case. On account of Remark 4.4, we know that exists a nonzero function $\psi_0 \in X^{1,2}(\Omega)$ such that $\psi_0 > 0$ a.e. in $\Omega$ and

$$\int_{\Omega} \nabla \psi_0 \cdot \nabla v \, dx + \iint_{\mathbb{R}^n} \frac{(\psi_0(x) - \psi_0(y))(v(x) - v(y))}{|x - y|^{n+2s}} \, dx \, dy = \lambda_1 \int_{\Omega} \psi_0 v \, dx \quad \forall \ v \in X^{1,2}(\Omega),$$

that is, $\psi_0$ is an eigenfunction for $L$ associated with $\lambda_1$.

In particular, choosing $v = u \in X^{1,2}(\Omega)$ in the above identity and recalling that $u$ is a solution to problem (1.3) (according to Definition 4.1), we obtain

$$\lambda_1 \int_{\Omega} \psi_0 u \, dx = \int_{\Omega} \nabla \psi_0 \cdot \nabla u \, dx + \iint_{\mathbb{R}^n} \frac{(\psi_0(x) - \psi_0(y))(u(x) - u(y))}{|x - y|^{n+2s}} \, dx \, dy$$

$$= \int_{\Omega} (u^{2^* - 1} + \lambda u) \psi_0 \, dx$$

(since $u, \psi_0 > 0$ a.e. in $\Omega$, see Theorem 1.3)

$$> \lambda \int_{\Omega} u \psi_0 \, dx,$$

but this is clearly in contradiction with the fact that $\lambda \geq \lambda_1$. \hfill $\square$

4.2. The superlinear case $1 < p < 2^* - 1$. Now we continue the study of the solvability of (1.3) in the case $1 < p < 2^* - 1$, and we address the proof of Theorem 1.5. Compared to the linear case previously treated, the main difference is that we cannot link the solvability of problem (1.3) to the existence of constrained minimizers for the quadratic form $Q_\lambda$ in (4.9) (since the rescaling argument in the proof of Lemma 4.5 cannot be applied when $p > 1$); as a consequence, we may try to prove the existence of solutions to (1.3) by showing the existence of unconstrained and nonnegative critical points for the functional $F_{\lambda,p}$ in (4.3).

However, this approach presents a drawback: even if we are able to prove the existence of a nonzero critical point $v$ for $F_{\lambda,p}$, there is no reason for $u := |v|$ to be a critical point, and so we cannot be sure that there exist nonnegative critical points. For this reason, and following the approach in [14], we consider a slightly modified functional with respect to (1.3), namely,

$$J_{\lambda,p}(u) := \frac{1}{2} \rho(u)^2 - \frac{1}{2} \int_{\Omega} (u_+)^{2^*} \, dx - \frac{\lambda}{p+1} \int_{\Omega} (u_+)^{p+1} \, dx,$$

where $u_+ := \max\{u, 0\}$ denotes the positive part of the function $u$.

Now, it is easy to recognize that any (nonzero) critical point of $J_{\lambda,p}$ is a solution to (1.3): indeed, if $u \neq 0$ is a critical point of this functional, by writing down the
associated Euler-Lagrange equation we see that $u$ is a weak solution of

\[
\begin{cases}
Lu = (u^+) ^{2^* - 1} + \lambda (u^+) ^{p-1} & \text{in } \Omega, \\
u \equiv 0 & \text{in } \mathbb{R}^n \setminus \Omega.
\end{cases}
\]

In particular, since we are assuming $\lambda > 0$, we deduce that $Lu \geq 0$ in the weak sense in $\Omega$ (see, precisely, (4.4) in the proof of Theorem 4.3). This, together with the fact that $u \equiv 0$ a.e. in $\mathbb{R}^n \setminus \Omega$, allows us to apply the Weak Maximum Principle in [5, Theorem 1.2], ensuring that $u \geq 0$ a.e. in $\mathbb{R}^n$. We then conclude that $u^+ \equiv u$,

and thus $u$ is a solution to problem (1.3) (according to Definition 4.1).

In view of this fact, to prove Theorem 1.5 it then suffices to prove the existence of a nonzero critical point for $J_{\lambda,p}$.

As in the purely local (see [14]) or purely nonlocal case (see [6]), the main difficulty in the application of the Mountain Pass Theorem consists in proving the validity of a Palais–Smale condition at level $c \in \mathbb{R}$, usually denoted by $(PS)_c$. In particular, we have to prove that the Mountain Pass critical level of $J_{\lambda,p}$ lies below the threshold of application of the $(PS)_c$ condition. In doing this, we will understand better the strange behaviour in the linear case and we will notice how the local part of $L$ affects the problem.

To this end, the first step is to recognize that the functional $J_{\lambda,p}$ has a nice Mountain Pass geometry, as described by the following lemma.

**Lemma 4.9.** There exist two positive constants $\alpha, \beta > 0$ such that

i) for any $u \in X^{1,2}(\Omega)$ with $\rho(u) = \alpha$, it holds that $J_{\lambda,p}(u) \geq \beta$;

ii) there exists a positive function $e \in X^{1,2}(\Omega)$ such that $\rho(e) > \alpha$ and $J_{\lambda,p}(e) < \beta$.

Moreover, for every positive function $u \in X^{1,2}(\Omega)$, it holds that

\[
\lim_{t \to 0^+} J_{\lambda,p}(tu) = 0.
\]

**Proof.** The proof is quite standard and exploits the validity of a Sobolev embedding for the space $X^{1,2}(\Omega)$. See e.g. [6, Proposition 3.1].

We now move to show that $J_{\lambda,p}$ satisfies a local Palais-Smale condition at a level $c \in \mathbb{R}$ related to the best Sobolev constant $S_n$.

**Lemma 4.10.** The functional $J_{\lambda,p}$ satisfies the $(PS)_c$ for every $c < \frac{1}{n}(S_n)^{n/2}$.

**Proof.** Let $\{u_m\}$ be a $(PS)_c$ sequence for the functional $J_{\lambda,p}$ in $X^{1,2}(\Omega)$, i.e.

i) $J_{\lambda,p}(u_m) \to c$, as $m \to +\infty$;

ii) $J_{\lambda,p}'(u_m) \to 0$, as $m \to +\infty$.

By standard arguments, this ensures that $\{u_m\}$ is bounded in the Hilbert space $X^{1,2}(\Omega)$. In particular, this implies the existence of a function $u_\infty \in X^{1,2}(\Omega)$ such that, up to subsequences,

\[
\int_{\Omega} \langle \nabla u_m, \nabla \varphi \rangle \, dx + \iint_{\mathbb{R}^{2n}} \frac{(u_m(x) - u_m(y))(\varphi(x) - \varphi(y))}{|x - y|^n + 2s} \, dxdy \\
\to \int_{\Omega} \langle \nabla u_\infty, \nabla \varphi \rangle \, dx + \iint_{\mathbb{R}^{2n}} \frac{(u_\infty(x) - u_\infty(y))(\varphi(x) - \varphi(y))}{|x - y|^n + 2s} \, dxdy,
\]

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for every $\varphi \in X^{1,2}(\Omega)$. Moreover, $u_m$ converges to $u_\infty$ (as $m \to +\infty$) weakly in $L^2(\Omega)$, almost everywhere and strongly in $L^r(\Omega)$ for every $r \in [1,2^*)$ (and hence in norm). Putting everything together, we get that $u_\infty$ is a weak solution to (1.3) (with $p > 1$).

Now, by exploiting [13] Theorem 1, we find that
\[
\mathcal{J}_{\lambda,p}(u_m) = \frac{1}{2} \int_\Omega |\nabla u_m|^2 + \frac{1}{2} \int_{\mathbb{R}^2} \frac{|u_m(x) - u_m(y)|^2}{|x-y|^{n+2s}} \, dxdy
- \frac{1}{2^*} \int_\Omega (|u_m|_+^{2^*} - \lambda \int_\Omega (|u_m|_+^{p+1})^\frac{p}{p+1} \, dx
= \mathcal{J}_{\lambda,p}(u_\infty) + \frac{1}{2} \rho(u_m - u_\infty)^2 - \frac{1}{2^*} \int_\Omega |(u_m|_+ - (u_\infty|_+|)^2^\frac{1}{2} \, dx + o(1).
\]

Arguing as in the proof of Claim 3 of [6, Proposition 3.2], we can further prove that
\[
\rho(u_m - u_\infty)^2 = \int_\Omega |(u_m|_+ - (u_\infty|_+|)^2^\frac{1}{2} \, dx + o(1)
\leq \int_\Omega |u_m - u_\infty|^{2^*} \, dx + o(1).
\]

We are now ready to finish the proof of Lemma 4.10. By (4.12), we have that
\[
\frac{1}{2} \rho(u_m - u_\infty)^2 - \frac{1}{2} \int_\Omega |(u_m|_+ - (u_\infty|_+|)^2^\frac{1}{2} \, dx = \frac{1}{n} \rho(u_m - u_\infty)^2 + o(1),
\]
and therefore
\[
\mathcal{J}_{\lambda,p}(u_\infty) + \frac{1}{n} \rho(u_m - u_\infty)^2 = \mathcal{J}_{\lambda,p}(u_m) + o(1) = c_2 + o(1),
\]
as $m \to +\infty$. On the other hand, thanks to the boundedness of $\{u_m\}$, possibly passing to a subsequence, we have that there exists $L > 0$ such that
\[
\rho(u_m - u_\infty)^2 \to L,
\]
as $m \to +\infty$,
and that there exists $\tilde{L} \geq L$ such that
\[
\int_\Omega |u_m - u_\infty|^{2^*} \, dx \to \tilde{L}.
\]

Recalling that
\[
S_{n,s}(\Omega) = \inf \frac{\rho(u)^2}{\|u\|^{2^*}_{L^{2^*}(\Omega)}},
\]
we find that
\[
L \geq S_{n,s}(\Omega) \tilde{L}^{2/2^*} = S_n \tilde{L}^{2/2^*},
\]
which implies that either $L = 0$ or $L \geq S_n \tilde{L}^{n/2}$. Arguing as in [6] Proposition 3.2 we can show that the only possible case is $L = 0$, and this shows that the sequence $\{u_m\}$ strongly converges in $X^{1,2}(\Omega)$.

The last ingredient needed to establish Theorem 1.5 consists in showing the existence of a path whose energy is below the critical threshold $S_n/n$. To this aim, we first introduce an auxiliary function reminiscent of the one used in [13]: let $\phi_0 \in C^\infty(\mathbb{R})$ be a nonincreasing cut–off function given by
\[
\phi_0(t) := \begin{cases} 1, & \text{if } t \in [0, \frac{1}{2}], \\ 0, & \text{if } t \geq 1. \end{cases}
\]
Then, we define the function $\phi_r(x) : \mathbb{R}^n \to \mathbb{R}$ as

$$
\phi_r(x) := \phi_0 \left( \frac{|x|}{r} \right),
$$

for a given $r > 0$ such that $B_r(0) \subset \Omega$. Finally, for all $\varepsilon > 0$, let

$$
U_\varepsilon(x) := \frac{\varepsilon^{(n-2)/2}}{(|x|^2 + \varepsilon^2)^{(n-2)/2}} \quad \text{and} \quad \eta_\varepsilon(x) := \frac{\phi_r(x) U_\varepsilon(x)}{\|\phi_r U_\varepsilon\|_{L^{2^*}(\Omega)}}.
$$

In this framework, we have:

**Lemma 4.11.** Let $n \geq 3$ and $p \in (1, 2^* - 1)$. Moreover, let $\kappa_{s,n}, \beta_{p,n}$ be as in (1.5). Then, the following assertions hold.

(i) If $\kappa_{s,n} > \beta_{p,n}$, then there exists $\varepsilon > 0$ small enough such that

$$
\sup_{t \geq 0} J_{\lambda,p}(t \eta_\varepsilon) < \frac{1}{n} (S_n)^{n/2} \quad \text{for all} \ \lambda > 0.
$$

(ii) If, instead, $\kappa_{s,n} \leq \beta_{p,n}$, then there exist $\varepsilon > 0$ and $\lambda_0 > 0$ such that

$$
\sup_{t \geq 0} J_{\lambda,p}(t \eta_\varepsilon) < \frac{1}{n} (S_n)^{n/2} \quad \text{for all} \ \lambda \geq \lambda_0.
$$

**Proof.** We closely follow the argument in [6, Lemma 3.4], which in turn is a modification of the original approach by Brezis and Nirenberg [14].

To begin with we observe that, by definition, we have

$$
J_{\lambda,p}(t \eta_\varepsilon) = \frac{t^2}{2} \int_\Omega |\nabla \eta_\varepsilon|^2 \, dx + \frac{t^2}{2} \iint_{\mathbb{R}^{2n}} \frac{|\eta_\varepsilon(x) - \eta_\varepsilon(y)|^2}{|x-y|^{n+2s}} \, dxdy - \frac{t^{2^*}}{2^*} - \frac{\lambda^{p+1}}{p+1} \int_\Omega \eta_\varepsilon(x)^{p+1} \, dx; \tag{4.13}
$$

we then turn to estimate the three integrals in the right-hand side of (4.13). As regards the $L^{p+1}$-norm of $\eta_\varepsilon$, setting $N := n - (n-2)(p+1)$, one has

$$
\int_{\mathbb{R}^n} \eta_\varepsilon(x)^{p+1} \, dx = C \int_{B_r(0)} U_\varepsilon(x)^{p+1} \, dx
$$

$$
= C \varepsilon^{-(p+1) \frac{n-2}{2}} \int_0^r \frac{r^{n-1}}{(\varepsilon^2 + 1)(p+1)^{\frac{n-2}{2}}} \, dr
$$

$$
= C \varepsilon^{-(p+1) \frac{n-2}{2}} \int_0^{r/\varepsilon} \frac{r^{n-1}}{(t^2 + 1)(p+1)^{\frac{n-2}{2}}} \, dt
$$

$$
\geq C \varepsilon^{-(p+1) \frac{n-2}{2}} \int_1^{r/\varepsilon} \frac{r^{n-1}}{(t^2 + 1)(p+1)^{\frac{n-2}{2}}} \, dt
$$

$$
\geq C \varepsilon^{-(p+1) \frac{n-2}{2}} \int_1^{r/\varepsilon} \frac{r^{n-1-(n-2)(p+1)}}{1} \, dt
$$

$$
= C \varepsilon^{-(p+1) \frac{n-2}{2}} \left( \left( \frac{r}{\varepsilon} \right)^N + 1 \right),
$$
where the constant $C > 0$ is adjusted step by step, but it depends only on $n$. As regards the $L^2$-norm of $\nabla \eta_\varepsilon$, instead, we know from [14] Lemma 1.1 that
\[
\int_\Omega |\nabla \eta_\varepsilon|^2 \, dx = S_n + O(\varepsilon^{n-2}) \quad \text{as } \varepsilon \to 0^+.
\]

We are then left to estimate the Gagliardo seminorm of $\eta_\varepsilon$. First of all, arguing as in [30, Proposition 21], we fix $\delta > 0$ such that $B_{\delta}(0) \subset \Omega$ and we define
\[
\mathcal{D} := \{(x, y) \in \mathbb{R}^{2n} : x \in B_\delta(0), y \in B_\delta(0) \text{ and } |x - y| > \delta/2\},
\]
\[
\mathcal{E} := \{(x, y) \in \mathbb{R}^{2n} : x \in B_\delta(0), y \in B_\delta(0) \text{ and } |x - y| \leq \delta/2\}.
\]

With this notation at hand, we get that
\[
\int_{\mathbb{R}^{2n}} \frac{|\eta_\varepsilon(x) - \eta_\varepsilon(y)|^2}{|x - y|^{n+2s}} \, dx dy = \int_{B_\delta(0) \times B_\delta(0)} \frac{|U_\varepsilon(x) - U_\varepsilon(y)|^2}{|x - y|^{n+2s}} \, dx dy
\]
\[
+ 2 \int_{\mathcal{D}} \frac{|\eta_\varepsilon(x) - \eta_\varepsilon(y)|^2}{|x - y|^{n+2s}} \, dx dy
\]
\[
+ 2 \int_{\mathcal{E}} \frac{|\eta_\varepsilon(x) - \eta_\varepsilon(y)|^2}{|x - y|^{n+2s}} \, dx dy
\]
\[
+ \int_{B_\delta(0) \times B_\delta(0)} \frac{|\eta_\varepsilon(x) - \eta_\varepsilon(y)|^2}{|x - y|^{n+2s}} \, dx dy.
\]

Repeating the proof of [30, Proposition 21], one realizes that the last three integrals in the above identity behave like $O(\varepsilon^{n-2})$ as $\varepsilon \to 0^+$. On the other hand, after the change of variables $x = \varepsilon \xi$ and $y = \varepsilon \zeta$, we get
\[
\int_{B_\delta(0) \times B_\delta(0)} \frac{|U_\varepsilon(x) - U_\varepsilon(y)|^2}{|x - y|^{n+2s}} \, dx dy = O(\varepsilon^{2-2s}).
\]

Gathering all these information, from (1.13) we then obtain
\[
\mathcal{J}_{\lambda, p}(t \eta_\varepsilon) \leq \frac{t^2}{2} (S_n + O(\varepsilon^{n-2}) + O(\varepsilon^{2-2s})) - \frac{t^2}{2} - C \lambda \frac{\varepsilon^n p - 1}{p + 1} \varepsilon^{n - (p + 1)\frac{2-2}{2}}
\]
\[
= \frac{t^2}{2} (S_n + O(\varepsilon^{n,s,n})) - \frac{t^2}{2} - C \lambda \frac{\varepsilon^n p - 1}{p + 1} \varepsilon^{\beta_{p,n}}
\]
\[
\leq \frac{t^2}{2} (S_n + C \varepsilon^{n,s,n}) - \frac{t^2}{2} - C \lambda \frac{\varepsilon^n p - 1}{p + 1} \varepsilon^{\beta_{p,n}} =: g(t),
\]

provided that $\varepsilon > 0$ is sufficiently small and for a suitable constant $C > 0$.

To proceed further, following the proof of [6] Lemma 3.4, we turn to study the maximum/maximum points of the function $g$. To this end we first observe that, since $g(t) \to -\infty$ as $t \to \infty$, there exists some point $t_{\varepsilon, \lambda} > 0$ such that
\[
\sup_{t \geq 0} g(t) = g(t_{\varepsilon, \lambda}).
\]

If $t_{\varepsilon, \lambda} = 0$, we have $g(t) \leq 0$ for all $t \geq 0$, and the lemma is trivially established as a consequence of (4.14). If, instead, $t_{\varepsilon, \lambda} > 0$, since $g \in C^1((0, \infty))$ we get
\[
0 = g'(t_{\varepsilon, \lambda}) = t_{\varepsilon, \lambda} (S_n + C \varepsilon^{n,s,n}) - \frac{t^2}{2} - C \lambda \frac{\varepsilon^n p - 1}{p + 1} \varepsilon^{\beta_{p,n}}.
\]

In particular, recalling that $t_{\varepsilon, \lambda} > 0$, we can rewrite the above identity as
\[
S_n + C \varepsilon^{n,s,n} = \frac{t^2}{2} + C \lambda \frac{\varepsilon^n p - 1}{p + 1} \varepsilon^{\beta_{p,n}}
\]
from which we easily derive that
\[ t_{\varepsilon,\lambda} < (S_n + C\varepsilon^{\kappa_{s,n}})^{1/(2^* - 2)}. \]

We now distinguish two cases, according to the assumptions.

**Case (i):** \( \kappa_{s,n} > \beta_{p,n} \). In this case we first observe that, since \( t_{\varepsilon,\lambda} > 0 \), from identity (4.15) we easily infer the existence of some \( \mu_\lambda > 0 \) such that
\[ t_{\varepsilon,\lambda} \geq \mu_\lambda > 0 \quad \text{provided that } \varepsilon \text{ is small enough}. \]

This, together with the fact that since the map
\[ t \mapsto \frac{t^2}{2} (S_n + C\varepsilon^{\kappa_{s,n}}) - \frac{t^{2^*}}{2^*} \]
is increasing in the closed interval \([0, (S_n + C\varepsilon^{\kappa_{s,n}})^{1/(2^* - 2)}]\), implies
\[ \sup_{t \geq 0} g(t) = g(t_{\varepsilon,\lambda}) < \frac{1}{n}(S_n)^{n/2} + C\varepsilon^{\kappa_{s,n}} < \frac{1}{n}(S_n)^{n/2}, \]
provided that \( \varepsilon > 0 \) is sufficiently small. We explicitly stress that, in the last estimate, we have exploited in a crucial way the assumption \( \kappa_{s,n} > \beta_{p,n} \).

**Case (ii):** \( \kappa_{s,n} \leq \beta_{p,n} \). In this second case, we begin by claiming that
\[ (4.16) \quad \lim_{\lambda \to \infty} t_{\varepsilon,\lambda} = 0. \]

Indeed, suppose by contradiction that \( \ell := \limsup_{\lambda \to \infty} t_{\varepsilon,\lambda} > 0 \): then, by possibly choosing a sequence \( \{\lambda_k\}_k \) diverging to \( \infty \), from (4.15) we get
\[ S_n + C\varepsilon^{\kappa_{s,n}} = t_{\varepsilon,\lambda_k}^{2^* - 2} + C\lambda_k \varepsilon^{p - 1} \to \infty \]
buts this is clearly absurd. Now we have established (4.16), we can easily complete the proof of the lemma: indeed, by combining (4.14) with (4.16), we have
\[ 0 \leq \sup_{t \geq 0} J_{\lambda,p}(t\eta_\varepsilon) \leq g(t_{\varepsilon,\lambda}) < \frac{t_{\varepsilon,\lambda}^2}{2} (S_n + C\varepsilon^{\kappa_{s,n}}) - \frac{t_{\varepsilon,\lambda}^{2^*}}{2^*} \to 0 \quad \text{as } \lambda \to \infty, \]
and this readily implies the existence of \( \lambda_0 = \lambda_0(p, s, n, \varepsilon) > 0 \) such that
\[ \sup_{t \geq 0} J_{\lambda,p}(t\eta_\varepsilon) < \frac{1}{n}(S_n)^{2/n} \quad \text{for all } \lambda \geq \lambda_0, \]
provided that \( \varepsilon > 0 \) is small enough but fixed. This ends the proof.

**Proof of Theorem 1.3** The desired result in Theorem 1.5 now follows from the Mountain Pass Theorem, thanks to Lemmata 4.9, 4.10 and 4.11.
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