SOME CONVERGENCE AND OPTIMALITY RESULTS OF ADAPTIVE MIXED METHODS IN FINITE ELEMENT EXTERIOR CALCULUS

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Abstract. In this paper, we present several new a posteriori error estimators for decoupled errors and two adaptive mixed finite element methods AMFEM1 and AMFEM2 for the Hodge Laplacian problem in finite element exterior calculus. We prove that AMFEM1 and AMFEM2 are both convergent starting from any initial coarse mesh. A suitably defined quasi error is crucial to the convergence analysis. In addition, we prove the optimality of AMFEM2. The main technical contribution is a localized discrete upper bound. Comparing to existing literature, our results work on Lipschitz domains with nontrivial cohomology and provide the first norm convergence and optimality results.

Key words. a posteriori error estimate, adaptive mixed finite element method, finite element exterior calculus, Hodge Laplacian, convergence, optimality

AMS subject classifications. 65N30, 65N15

1. Introduction. In this paper, we present some a posteriori error estimators, convergence and optimality results of adaptive finite element method (FEM) for the Hodge Laplacian problem on Lipschitz domains in finite element exterior calculus (FEEC). Arnold, Falk, and Winther [3] proposed the framework of FEEC for solving the Hodge Laplacian problem on the de Rham complex, which is a wild generalization of the traditional mixed formulations for scalar and vector Laplacian problems in $\mathbb{R}^2$ and $\mathbb{R}^3$. In [4], they extended FEEC to closed Hilbert complex. A priori error estimates were given under suitable regularity assumption on the exact solution and the domain. Such assumptions fail at the presence of singularity, e.g., jump coefficients, nonsmooth sources, domains with nonsmooth boundary or reentrant corners. In these cases, standard FEMs on quasi-uniform meshes suffer from a slow rate of convergence. To overcome the singularity barrier, adaptive FEMs (AFEM) were proposed, cf.[6] and references therein. Convergence and optimality results of AFEMs for the primal formulation of scalar elliptic equations are extensive in the literature, cf.[18, 26, 9, 30, 12, 20] and references therein. See also [34, 11, 13, 7, 22] for related results in Maxwell equations and mixed methods for scalar 2nd and 4th order elliptic equations. However, there are only a few papers on convergence and optimality of AFEMs in the FEEC framework, see, e.g., [21, 15, 16]. Instead of solving the Hodge Laplacian problem, [16] is devoted to computing the harmonic space by AFEM.

The first step of designing AFEMs is developing a posteriori error estimates. A posteriori error estimates in mixed finite element methods (MFEM) are technical and relying on delicate decomposition results, see, e.g., [1, 10, 28, 22, 21]. Until recently, Demlow and Hirani [17] constructed the first residual-type a posteriori error estimator for the coupled error $\|\sigma - \sigma_h\|_{H^k} + \|u - u_h\|_{H^k} + \|p - p_h\|$ in MFEM for the Hodge Laplacian. Important tools in [17] include a stable regular decomposition and a commuting quasi-interpolation, see Theorem 2.1. Relevant a posteriori error estimates and quasi-interpolants for $H(\text{div})$ and $H(\text{curl})$ in $\mathbb{R}^3$ can be found in [28]. In contrast to the estimator for the coupled error in [17], we will derive several separate estimators for the decoupled errors $\|\sigma - \sigma_h\|_{H^{k-1}}, \|p - p_h\|$, and $\|d(u - u_h)\|$. All of

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these estimators are locally efficient while the error estimator in [17] have not been shown to be efficient because of the intractable term $\|P_S u_h\|$. Our separate estimator $\eta_\sigma(K)$ (see Theorem 3.3) is favorable when $\sigma$ is a practically relevant quantity. For example, the traditional MFEM for Poisson’s equation is designed to obtain a better finite element solution $\sigma_h$ approximating $\nabla u$. In addition, when using the pair $P_{r+1}\Lambda^{k-1}(\mathcal{T}_h) \times P_r\Lambda^{k}(\mathcal{T}_h)$ to discretize $H\Lambda^{k-1}(\Omega) \times H\Lambda^{k}(\Omega)$ (cf.[4]), we have the a priori estimate for the coupled error

$$\|\sigma - \sigma_h\|_{H\Lambda^{k-1}} + \|u - u_h\|_{H\Lambda^{k}} + \|p - p_h\| = O(h^r),$$

while the improved a priori estimates hold

$$\|\sigma - \sigma_h\|_{H\Lambda^{k-1}} = O(h^{r+1}), \quad \|\sigma - \sigma_h\| = O(h^{r+2}).$$

In this case, $\|d(u - u_h)\| = O(h^r)$ dominates the coupled error, which makes a coupled error estimator inefficient for the purpose of estimating $\sigma - \sigma_h$.

In [21], Holst, Mihalik, and Szypowski developed an optimally convergent adaptive MFEM (AMFEM) for the boundary Hodge Laplacian ($k = n$), which is in fact the traditional MFEM for Poisson’s equation on domains with general topology in any dimension. [15] seems to be the only convergence result of AMFEM for the Hodge Laplacian with general index $1 \leq k \leq n - 1$. In particular, Chen and Wu designed a convergent AMFEM on domains with vanishing de Rham cohomology, e.g., a contractible domain. Their algorithm can reduce $\|d(\sigma - \sigma_h)\|^{2} + \|d(u - u_h)\|^{2}$ below any given error tolerance in finite steps. By assuming fineness of the initial mesh, they also claimed optimality of their algorithm. Since $d$ has a large kernel, [15] is a semi-norm convergence, i.e., $\|d(\sigma - \sigma_h)\|^{2} + \|d(u - u_h)\|^{2} \rightarrow 0$ does not imply $\sigma_h \rightarrow \sigma$ nor $u_h \rightarrow u$ in the $L^2(\Omega)$, $H\Lambda$, or any other Sobolev norm. In view of literature on traditional mixed methods and MFEMs in FEEC, researchers are more concerned with errors measured in the $L^2$-norm $\|\cdot\|$ or the $V$-norm $\|\cdot\|_{H\Lambda}$. For example, the authors in [4, 5] made a great effort to obtain improved decoupled a priori error estimates for $\|\sigma - \sigma_h\|$ and $\|u - u_h\|$.

In summary, our main contributions are listed as follows. All of our results are in full generality, i.e., they work for the Hodge Laplacian with any index $1 \leq k \leq n$ in any dimension $n \geq 2$ and on domains with general topology ($\delta^k \neq \{0\}$).

1. We derive reliable and efficient a posteriori error estimators $\eta_\sigma$, $\eta_p$, and $\eta_{du}$ for $\|\sigma - \sigma_h\|_{H\Lambda^{k-1}}$, $\|p - p_h\|$, and $\|d(u - u_h)\|$, respectively. Although $\eta_{du}$ appears in [15] in the case $\delta^k = \{0\}$, $\eta_\sigma$ and $\eta_p$ are firstly proposed in this manuscript and building blocks of AMFEM1 and AMFEM2.

2. We develop a convergent adaptive algorithm AMFEM1 w.r.t. $\|\sigma - \sigma_h\|_{H\Lambda^{k-1}}^{2} + \|p - p_h\|^{2} + \|d(u - u_h)\|^{2}$. The convergence analysis works on any initial coarse mesh. Prior to this manuscript, we are not aware of any norm convergence result (w.r.t. $\|\sigma - \sigma_h\|_{H\Lambda^{k-1}}$, $\|\sigma - \sigma_h\|$, $\|u - u_h\|_{H\Lambda^{k}}$, $\|u - u_h\|$ or any combination of them) of AMFEM on the de Rham complex with index $1 \leq k \leq n - 1$. In addition, AMFEM1 gives new result on AMFEM for Poisson’s equation when $k = n$, see Remark 4.1.

3. By dropping several marking steps in AMFEM1, we obtain an convergent algorithm AMFEM2 w.r.t. $\|\sigma - \sigma_h\|_{H\Lambda^{k-1}}$, with optimal complexity. The new ingredient of the optimality proof is a localized discrete upper bound. As far as we know, there is no convergence with optimal complexity of AMFEM for the Hodge Laplacian with index $1 \leq k \leq n - 1$ in the literature.
Ingredients of convergence analysis of AMFEM1 include global reliability of $\eta_\sigma$, $\eta_p$, $\eta_{du}$, the error reduction Lemma 4.1, and the estimator reduction Lemma 4.2. Proofs of a posteriori upper bounds rely on the Hodge decomposition, the regular decomposition, and a gap estimate Lemma 3.2. The role of Lemma 4.1 is similar to the orthogonality or quasi-orthogonality in standard convergence analysis of AFEMs. However, Lemma 4.1 is elementary comparing to many technical quasi-orthogonality results in the AFEM literature. In fact, convergence of $\|\sigma - \sigma\|_{H^{k+1}}$ in AMFEM1 is realized by discovering the suitably weighted quasi error $E_h = \|\sigma - \sigma_h\|^2 + \zeta\|\sigma - \sigma_h\|^2 + \rho_\sigma \eta^2_\sigma(T_h)$, see the proof of Theorem 4.3. Convergence of $\|p - p_h\|$ and $\|u - u_h\|$ is guaranteed by separate Dörfler markings (4.2) and (4.3). The estimator reduction Lemma 4.2 results from dependence of our estimators on $\sigma_h, p_h$ and $du_h$. In contrast, the coupled error estimator in [17] depends on the triple ($\sigma_h, u_h, p_h$), in particular on $\mu\|u_h\|$ (see (3.10)), which is an obstacle against proving estimator reduction.

Separate marking may cause problems in proving optimality, see Section 6 in [12] for details. Hence we consider the standard adaptive method AMFEM2 by dropping markings (4.2) and (4.3). AMFEM2 seems to be the first convergent AMFEM with optimal complexity for the Hodge Laplacian in FEEC w.r.t. $\|\sigma - \sigma_h\|_{H^{k+1}}$. Comparing to the pioneering AFEM literature (cf.[26, 23]), the mesh refinement in AMFEM2 does not require an interior node property. Unlike AMFEMs for Poisson’s equation [7, 13] using separate marking for data oscillation, the marking step in AMFEM2 is solely based on the estimator $\eta_\sigma$. In addition, there is no fineness assumption on the initial mesh size in the contraction proof of AMFEM2, in contrast to some AFEMs for indefinite problems (e.g., Maxwell equations in $\mathbb{R}^3$ [34]). In fact, convergence of AMFEM2 is guaranteed by discovering the aforementioned quasi error $E_h$.

The new ingredient in the optimality proof is Theorem 5.2, a localized discrete upper bound, which is a discrete and local version of the global reliability Theorem 3.3. The proof is similar to Theorem 3.3 in the sense that the discrete Hodge decomposition instead of the Hodge decomposition is applied. To localize the upper bound, we combine the Falk–Winther locally bounded cochain projection [19] and the standard Scott–Zhang interpolation [29]. Demlow (see Lemma 3 in [16]) originally used this technique to prove a localized discrete a posteriori estimate for the gap between discrete harmonic spaces $\delta_h$ and $\delta_H$ on two nested meshes.

The rest of this paper is organized as follows. In Section 2, we introduce the background of FEEC. In Section 3, we derive several a posteriori error estimates. Section 4 is devoted to convergence analysis of AMFEM1 and AMFEM2. In Section 5, we construct a localized discrete upper bound, and prove the optimality of AMFEM2.

2. Preliminaries. In this section, we follow the convention of [4] and [3] to introduce necessary background of FEEC in our analysis.

2.1. Closed Hilbert complex. Consider the closed Hilbert complex $(W, d)$:

$$\cdots \to W^{k-1} \xrightarrow{d^{k-1}} W^k \xrightarrow{d^k} W^{k+1} \xrightarrow{d^{k+1}} \cdots,$$

i.e., for each index $k$, $W^k$ is a Hilbert space, $d^k : W^k \to W^{k+1}$ is a densely defined closed operator, the range of $d^k$ is closed in $W^{k+1}$ and contained in the domain of $d^{k+1}$ and $d^{k+1} \circ d^k = 0$. Let $Z^k = N(d^k)$ denote the null space of $d^k$, $\Omega^k = R(d^{k-1})$ the range of $d^{k-1}$, $\delta^k = Z^k \cap \Omega^{k\perp}$ the space of harmonic forms, where $\perp$ is the notation of orthogonal complement w.r.t. the inner product $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{W^k}$ in $W^k$. For a closed Hilbert complex $(W, d)$, we have the Hodge decomposition

\begin{equation}
W^k = \Omega^k \oplus \delta^k \oplus Z^k \perp, \end{equation}
where ⊕ denotes the direct sum w.r.t. ⟨·,·⟩. Correspondingly, let $P_{\mathfrak B}$, $P_{\mathfrak A}$, and $P_{\mathfrak A^*}$ be projections onto $\mathfrak B^k$, $\mathfrak A^k$, and $\mathfrak A^k$ w.r.t. ⟨·,·⟩, respectively. The complex $(W,d)$ is closely related to the dual complex $(W,d^*)$:

$$\cdots \rightarrow W^{k+1} \xrightarrow{d_{k+1}^*} W^k \xrightarrow{d_k^*} W^{k-1} \xrightarrow{d_{k-1}^*} \cdots$$

where $d_k^*$ is the adjoint of $d^{k-1}$. It is straightforward to show that $(W,d^*)$ is also a closed Hilbert complex. Let $\mathfrak B^*_k = N(d_k^*)$ and $\mathfrak A^*_k = \mathfrak B^*_k \oplus B^*_k$. By the closed range theorem, $\mathfrak B^{k+1} = \mathfrak B_k^k$ and the Hodge decomposition (2.1) can be written as

$$V_k = \mathfrak B^k \oplus \mathfrak A^k \oplus B^*_k.$$

Associated with any Hilbert complex $(W,d)$ is the domain complex $(V,d)$:

$$\cdots \rightarrow V^{k-1} \xrightarrow{d_{k-1}} V^k \xrightarrow{d_k} V^{k+1} \xrightarrow{d_{k+1}} \cdots,$$

where $V^k$ is the domain of $d^k$ in $W^k$, equipped with the inner product $\langle u,v \rangle_V = \langle u,v \rangle_{W^k} + \langle d^k u, d^k v \rangle_{W^{k+1}}$. The domain complex of $(W,d^*)$ is $(V^*,d^*)$, where $V_k^*$ is the domain of $d_k^*$. We use $\| \cdot \|$ to denote the norm of $W$ and $\| \cdot \|_V$ the norm of $V$. By inverse mapping theorem, the Poincaré inequality $\|v\| \leq c_p \|d^k v\|$ holds, provided $v \in 3^{k+1} \cap V^k$. For the dual complex $(V^*,d^*)$, a Poincaré inequality still holds with the same constant, i.e., $\|v\| \leq c_p \|d_{k+1}^* v\|$ provided $v \in \mathfrak B^{k+1} \cap V^*_k$. In fact, let $v = d^k w \in \mathfrak B^{k+1} = \mathfrak B_k^k$ with $w \in \mathfrak B^{k+1}$,

$$\|v\|^2 = \langle d_{k+1}^* v, w \rangle \leq \|d_{k+1}^* v\| \|w\| \leq c_p \|d_{k+1}^* v\| \|d^k w\| = c_p \|d_{k+1}^* v\| \|v\|. $$

The proof is complete.

Given $v \in W^k$, by the Hodge decomposition (2.2), there exist $v_1 \in V^{k-1}$, $v_2 \in V^*_k$, and $q \in \mathfrak A^k$, such that $v = d^{k-1} v_1 + d_{k+1} v_2 + q$. Clearly, we can assume $v_1 \in (3^{k-1} \cap V^k)$, $v_2 \in 3^{k+1} \cap V^*_k$, and $\|v_1\| \leq c_p \|d^{k-1} v_1\|$, $\|v_2\| \leq c_p \|d_{k+1}^* v_2\|$. This decomposition will be applied in proofs of many theorems in this paper.

Given a Hilbert complex $(W,d)$, the unbounded operator $L = dd^* + d^* d : W^k \rightarrow W^k$ is called the abstract Hodge Laplacian. The abstract Hodge Laplacian problem is to solve $Lu = f$ mod $\mathfrak A^k$. The variational formulation is to find $(u,p) \in (V^k \cap V^*_k) \times \mathfrak A^k$, such that

$$\langle du, dv \rangle + \langle d^* u, d^* v \rangle + \langle v, p \rangle = \langle f, v \rangle, \quad v \in V^k \cap V^*_k,$$

$$\langle u, q \rangle = 0, \quad q \in \mathfrak A^k.$$

However, it is generally difficult to construct a finite element subspace of $V^k \cap V^*_k$ and develop a convergent FEM for (2.3) in practice. Alternatively, Arnold, Falk, and Winther [4] considered the equivalent and well-posed mixed formulation of the abstract Hodge Laplacian: find $(\sigma, u, p) \in V^{k-1} \times V^k \times \mathfrak A^k$, such that

$$\langle \sigma, \tau \rangle - \langle \tau, u \rangle = 0, \quad \tau \in V^{k-1},$$

$$\langle d\sigma, v \rangle + \langle du, dv \rangle + \langle v, p \rangle = \langle f, v \rangle, \quad v \in V^k,$$

$$\langle u, q \rangle = 0, \quad q \in \mathfrak A^k.$$
Following [4], we use the adjoint subcomplex of $(V, d)$, the completion of $\Lambda^k V$, the discrete Hodge decomposition $\langle \cdot, \cdot \rangle_{W^k}$. For the restricted exterior derivative $d : V^k_h \to V^{k+1}_h$, the adjoint $d^*_h : V^{k+1}_h \to V^k_h$ is defined as

$$\langle d^*_h u, v \rangle = \langle u, dv \rangle, \quad u \in V^{k+1}_h, \ v \in V^k_h.$$ 

We keep in mind that $d^*_h \oplus \perp$ and $\perp$ are all based on the $W$-inner product $\langle \cdot, \cdot \rangle$. In order to understand how well the approximation of $V$ by $V^k_h$ is, we need a bounded cochain projection $\pi_h$ from $(V, d)$ to $(V_h, d)$. To be precise, for each index $k$, $\pi^h_k$ maps $V^k$ onto $V^k_h$, $\pi^h_k : V^k \to V^k_h$, $\pi^h_k = \pi^h_{k+1} d^k$, and $\|\pi^h_k\|_V = \|\pi^h_k\|_{V^k \to V^k_h} < \infty$ uniformly with respect to the discretization parameter $h$. Theorem 3.6 in [4] gives the discrete Poincaré inequality $\|v\| \leq c_h \|\pi^h_k\|_V \|dv\|$ provided $v \in \mathfrak{B}^k_h$. Similar to the continuous case, another discrete Poincaré inequality $\|v\| \leq c_h \|\pi^h_k\|_V \|d^*_h v\|$ holds provided $v \in \mathfrak{B}^k_h \perp \mathfrak{B}^k_h$.

Using the discrete complex $(V_h, d)$, the discrete mixed formulation corresponding to (2.4) is to find $(\sigma_h, u_h, p_h) \in V^{k-1}_h \times V^k_h \times \mathfrak{B}^k_h$, such that

$$\langle \sigma_h, \tau \rangle - \langle d\tau, u_h \rangle = 0, \quad \tau \in V^{k-1}_h,$$

$$\langle d\sigma_h, v \rangle + \langle dv_h, dv \rangle + \langle v, p_h \rangle = \langle f, v \rangle, \quad v \in V^k_h,$$

$$\langle u_h, q \rangle = 0, \quad q \in \mathfrak{B}^k_h.$$ 

Since $\mathfrak{B}^k_h \perp \mathfrak{B}^k$ is not contained in $\mathfrak{B}^k_h$, (2.6) is not a standard Galerkin method. The authors in [4] proved a discrete inf-sup condition and a priori error estimates of (2.6).

Combining (2.4) and (2.6), we obtain the error equation

$$\langle \sigma - \sigma_h, \tau \rangle - \langle d\tau, u - u_h \rangle = 0,$$

$$\langle d(\sigma - \sigma_h), v \rangle + \langle d(u - u_h), dv \rangle + \langle v, p - p_h \rangle = 0,$$

for all $(\tau, v) \in V^{k-1}_h \times V^k_h$.

**2.3. De Rham complex and approximation.** The de Rham complex is a canonical example of the closed Hilbert complex $(W, d)$. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Given $0 \leq k \leq n$, let $\Lambda^k(\Omega)$ denote the space of smooth $k$-forms $\omega$ which can be written as

$$\omega = \sum_{1 \leq \sigma_1 < \cdots < \sigma_k \leq n} a_\sigma dx_{\sigma_1} \wedge \cdots \wedge dx_{\sigma_k},$$

where $a_\sigma \in C^\infty(\Omega)$ and $\wedge$ is the wedge product. The space $\Lambda^k(\Omega)$ is naturally endowed with the $L^2$-inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $d : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$ denote the exterior derivative for differential forms. The Sobolev version of $\Lambda^k(\Omega)$ is $L^2\Lambda^k(\Omega)$, the completion of $\Lambda^k(\Omega)$ under $\| \cdot \|$. Corresponding to the closed Hilbert complex $(W, d)$ is the $L^2$ de Rham complex

$$L^2\Lambda^0(\Omega) \xrightarrow{d} L^2\Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} L^2\Lambda^{n-1}(\Omega) \xrightarrow{d} L^2\Lambda^n(\Omega),$$

where $d : \Lambda^{k-1}(\Omega) \to \Lambda^k(\Omega)$ is the boundary operator.
where $d$ is the weak exterior derivative. Define $H\Lambda^k(\Omega) := \{ \omega \in L^2\Lambda^k(\Omega) : d\omega \in L^2\Lambda^{k+1}(\Omega) \}$. The domain complex [corresponds to $(V, d)$] is

$$H\Lambda^0(\Omega) \xrightarrow{d} H\Lambda^1(\Omega) \xrightarrow{d} \cdots \xrightarrow{d} H\Lambda^{n-1}(\Omega) \xrightarrow{d} H\Lambda^n(\Omega).$$

In order to define the dual complex, we need the Hodge star operator $\star : \Lambda^k(\Omega) \rightarrow \Lambda^{n-k}(\Omega)$,

$$\int_{\Omega} \omega \wedge \mu = \langle \star \omega, \mu \rangle, \quad \mu \in \Lambda^{n-k}(\Omega).$$

The coderivative $\delta : \Lambda^k(\Omega) \rightarrow \Lambda^{k-1}(\Omega)$ is then defined as $\star d \omega = (-1)^k d \star \omega$. $d$ and $\delta$ are related by the integrating by parts formula

$$(2.8) \quad \langle d\omega, \mu \rangle = \langle \omega, \delta \mu \rangle + \int_{\partial \Omega} \omega \wedge \mu, \quad \omega, \mu \in \Lambda^k(\Omega),$$

where the trace operator $\text{tr} = i^* \mid \partial \Omega$ is the pullback of the inclusion $i : \partial \Omega \rightarrow \overline{\Omega}$. 

$(2.8)$ holds on any Lipschitz subdomain $\Omega_0 \subset \Omega$ and in this case $\text{tr}$ denotes the trace on $\partial \Omega_0$ by abuse of notation. The dual complex [corresponds to $(W, d^*)$] is

$$L^2\Lambda^n(\Omega) \xrightarrow{\delta} L^2\Lambda^{n-1}(\Omega) \xrightarrow{\delta} \cdots \xrightarrow{\delta} L^2\Lambda^1(\Omega) \xrightarrow{\delta} L^2\Lambda^0(\Omega).$$

For the above $L^2$ complex, $\delta$ is understood in the weak sense. In order to have $\delta = d^*$, we define the domain of $\delta$ to be $\delta^* \Lambda^k(\Omega) := \{ \omega \in L^2\Lambda^k(\Omega) : \delta \omega \in L^2\Lambda^{k-1}(\Omega) \}$, $\text{tr} \circ \delta = 0$ on $\partial \Omega$. Then

$$\delta^* \Lambda^k(\Omega) \xrightarrow{\delta} \delta^* \Lambda^{k-1}(\Omega) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \delta^* \Lambda^1(\Omega) \xrightarrow{\delta} \delta^* \Lambda^0(\Omega)$$

is the domain complex [corresponds to $(V^*, d^*)$]. The Hodge Laplacian problem $(d\delta + \delta d)u = f \mod S^k$ is a concrete example of the abstract Hodge Laplacian. The corresponding mixed formulation is $(2.4)$ with $\delta$ replacing the abstract adjoint $d^*$. A more compact form is

$$(2.9) \quad \sigma = \delta u, \quad d\sigma + \delta d\sigma + p = f, \quad u \perp S^k.$$

Since $u \in \text{Dom}(\delta)$ and $d\delta u \in \text{Dom}(\delta)$, we have $\text{tr} \circ \delta \circ u = 0$, $\text{tr} \circ d \circ u = 0$ on $\partial \Omega$.

Let $T_h$ be a simplicial triangulation of $\Omega$ with the mesh size $h := \max_{K \in T_h} h_K$, $h_K := |K|^{\frac{1}{n}}$, where $|K|$ is the volume of $K$. We still use $V_h^k$ to denote a suitable finite element space of $k$-forms with piecewise polynomial coefficients on $T_h$. Arnold, Falk, and Winther [3] gave all possible choices of $V_h^{k-1} \times V_h^k$ for discretizing the pair $H\Lambda^{k-1}(\Omega) \times H\Lambda^k(\Omega)$. They also constructed a uniformly bounded cochain projection from $(H\Lambda, \delta)$ to $(V_h, d)$ on quasi-uniform $T_h$. Quasi-uniform meshes are not suitable for AFEMs. Instead, we only assume that $T_h$ is shape regular, i.e., $\max_{K \in T_h} \frac{r_K}{p_K} \leq C_{T_h} < \infty$, where $r_K$ and $p_K$ are radii of circumscribed and inscribed spheres of the simplex $K$, respectively. The constant $C_{T_h}$ quantifies the shape regularity of $T_h$. For a uniformly bounded cochain projection under shape regular but non quasi-uniform meshes, readers are referred to [15].

Let $H^s \Lambda^k(\Omega)$ be the space of $k$-forms whose coefficients are in $H^s(\Omega)$ and be endowed with the $H^s$ Sobolev inner product and norm. Formula $(2.8)$ still holds for $\omega \in H^1 \Lambda^k$ and $\mu \in H^1 \Lambda^{k+1}$. Consider $H^1 \Lambda^k(T_h) := \{ \omega \in L^2 \Lambda^k(\Omega) : \omega|_K \in H^1 \Lambda^k(K) \}$ for all $K \in T_h$, the space of piecewise $H^1$ $k$-forms. Let $E_h$ be the set of $(n-1)$-faces in $T_h$. For each interior face $e \in E_h$ and $\omega \in H^1 \Lambda(T_h)$, let $[\omega] :=$
\( \omega |_{K_1} - \omega |_{K_2} \) denote the jump of \( \omega \) on \( e \), where the two adjacent simplices \( K_1 \) and \( K_2 \) share \( e \) as an \((n-1)\)-face. For each boundary face \( e \), \( [\omega] := \omega |_{K} \), where \( K \) is the simplex having \( e \) as a face. We use the notation \( A \subseteq B \) provided \( A \subseteq C \cdot B \) and \( C \) is a generic constant depending only on \( \Omega \) and shape regularity of the underlying mesh. We use \( \| \cdot \|_{H^1} \) and \( \| \cdot \|_{H^k_{A}} \) to denote the \( H^1 \Lambda^l(\Omega) \) norm with some \( l \) and \( H^k_{A}(\Omega) \) norm, respectively. \( \langle \cdot, \cdot \rangle_K \) denotes the \( L^2 \) inner product restricted to \( K \). \( \| \cdot \|_K \) and \( \| \cdot \|_{\partial K} \) denote the \( L^2 \) norm restricted to \( K \) and \( \partial K \), respectively.

To derive our error estimators and prove global reliability as well as localized discrete upper bound, we need the regular decomposition and commuting quasi-interpolation developed by Demlow and Hirani, see Lemma 5 and Lemma 6 in [17].

**Theorem 2.1** (regular decomposition and quasi-interpolation). Assume that \( \Omega \) is a bounded Lipschitz domain in \( \mathbb{R}^n \) and \( 0 \leq k \leq n - 1 \). Given \( v \in H^k(\Omega) \), there exist \( \varphi \in H^1(\Omega) \) and \( z \in H^1(\Omega) \), such that \( v = d\varphi + z \) and

\[
\| \varphi \|_{H^1} + \| z \|_{H^1} \lesssim \| v \|_{H^k_{A}}.
\]

Let \( \Pi_h^k : L^2(\Omega) \to V_h^k \) be the commuting quasi-interpolation in [17], then \( \Pi_{h+1} \circ d^k = d^k \circ \Pi_h^k \) and the following approximation property holds. If \( k = 0 \), we have

\[
\sum_{K \in T_h} (h_K^{-2} \| v - \Pi_h^k v \|_K^2 + h_K^{-1} \| v - \Pi_h^k v \|_{\partial K}^2 + |v - \Pi_h^k v|_{H^1(K)}^2) \lesssim \| v \|_{H^k_{A}}^2.
\]

If \( 1 \leq k \leq n - 1 \), we have

\[
\sum_{K \in T_h} (h_K^{-2} \| \varphi - \Pi_h^k \varphi \|_K^2 + h_K^{-1} \| z - \Pi_h^k z \|_K^2 + h_K^{-1} \| \text{tr}(\varphi - \Pi_h^k \varphi) \|_{\partial K}^2 \lesssim \| v \|_{H^k_{A}}^2.
\]

Similar decomposition holds for the space \( \tilde{H}^k(\Omega) \) with essential boundary condition.

The discrete Hodge Laplacian (2.6) with \( k = 0 \) reduces to the standard FEM for solving Poisson’s equation with pure Neumann boundary condition. On the other hand, [21] gave a convergent AMFEM with optimal complexity for the Hodge Laplacian when \( k = n \). Hence we focus on the index \( 1 \leq k \leq n - 1 \) although our results can be directly extended to \( k = n \). To avoid cumbersome notations, we may drop the index \( k \) appearing in \( Z_h^k, B_h^k, S_h^k, \Pi_h^k \) etc. provided no confusion arises.

In the end of this section, we recall the four Poincaré inequalities

\[
\| v_1 \| \leq c_p \| dv_1 \|, \quad v_1 \in Z^{1,1}, \quad \| v_2 \| \leq c_p \| \delta v_2 \|, \quad v_2 \in Z^{2,1},
\]

\[
\| v_3 \| \leq c_p \| \pi_h v \| \| dv_1 \|, \quad v_1 \in Z_h^1, \quad \| v_4 \| \leq c_p \| \pi_h v \| \| \delta_h v_2 \|, \quad v_2 \in Z_h^{1,1},
\]

which are universal in the rest of this paper.

**3. A posteriori error estimates.** In this section, we derive a posteriori estimates for \( \| \sigma - \sigma_h \|_{H^k_{A+1}} \), \( \| p - p_h \| \), and \( \| d(u - u_h) \| \) in order and separately. An error estimator for \( \| d(\sigma - \sigma_h) \| \) was developed in [15].

**Theorem 3.1** (Chen and Wu’s a posteriori estimate for \( \| d(\sigma - \sigma_h) \| \)). For \( 1 \leq k \leq n - 1 \) and \( f \in H^k(\Omega) \), we have the following a posteriori error estimate

\[
\| d(\sigma - \sigma_h) \|^2 \lesssim \eta_{d\sigma}^2(T_h) = \sum_{K \in T_h} \eta_{d\sigma}^2(K),
\]
where

\[ \eta_{\delta}^2(K) = h_K^2 \| \delta(f - d\sigma_h) \|_K^2 + h_K \| [\text{tr}*(f - d\sigma_h)] \|_K^2. \]

Theorem 3.1 holds on any Lipschitz domain \( \Omega \) with nontrivial cohomology although [15] is concerned with domains without harmonic forms.

To derive upper bounds for other error quantities in the mixed method (2.6), the Hodge decomposition (2.2) is frequently applied. In order to control the harmonic component, we need to estimate the gap between \( \mathcal{H} \) and \( \mathcal{H}_h \)

\[
\delta(\mathcal{H}, \mathcal{H}_h) := \sup_{\|q\|=1, q \in \mathcal{H}} \| q - P_{\mathcal{H}_h} q \|,
\]

\[
\delta(\mathcal{H}_h, \mathcal{H}) := \sup_{\|q\|=1, q \in \mathcal{H}_h} \| q - P_{\mathcal{H}} q \|,
\]

\[
\text{gap}(\mathcal{H}, \mathcal{H}_h) := \max \{ \delta(\mathcal{H}, \mathcal{H}_h), \delta(\mathcal{H}_h, \mathcal{H}) \}.
\]

The next lemma is essentially a combination of results in [21, 17, 4]. To be complete and precise, we give an explicit upper bound for the gap.

**Lemma 3.2.** Let \( \pi_h \) be a cochain projection from \((V, d)\) to \((V_h, d)\). Then

\[
\text{gap}(\mathcal{H}, \mathcal{H}_h) = \delta(\mathcal{H}, \mathcal{H}_h) = \delta(\mathcal{H}_h, \mathcal{H}) \leq \left( 1 - \frac{1}{\|\pi_h\|_V + 1} \right)^{\frac{1}{2}}.
\]

Let \((V_H, d)\) be a subcomplex of \((V, d)\) and \( \pi_H \) be a cochain projection from \((V, d)\) to \((V_H, d)\). Then

\[
\text{gap}(\mathcal{H}_h, \mathcal{H}_H) = \delta(\mathcal{H}_h, \mathcal{H}_H) = \delta(\mathcal{H}_H, \mathcal{H}_h) \leq \left( 1 - \frac{1}{\|\pi_H\|_V + 1} \right)^{\frac{1}{2}}.
\]

**Proof.** Lemma 2 in [17] gives \( \delta(\mathcal{H}, \mathcal{H}_h) = \delta(\mathcal{H}_h, \mathcal{H}) \). For any \( q \in \mathcal{H}_h \), we have the estimate (see Theorem 3.5 in [4])

\[
\| q - P_{\mathcal{H}} q \| \leq \| (I - \pi_h) P_{\mathcal{H}} q \|.
\]

Note that \( \| q \| = \| q \|_V \) for any \( q \in \mathcal{H} \) or \( \mathcal{H}_h \). Combining (3.3) with a triangle inequality gives \( \| q \| \leq \| q - P_{\mathcal{H}} q \| + \| P_{\mathcal{H}} q \| \leq \| (I - \pi_h) \|_V + 1 \| P_{\mathcal{H}} q \| \). Since \( \pi_h \) is a projection, we have \( \| (I - \pi_h) \|_V = \| \pi_h \|_V \) (see [33]) and thus

\[
\| q \| \leq \| \pi_h \|_V + 1 \| P_{\mathcal{H}} q \|.
\]

It then follows from the above inequality and \( \| q - P_{\mathcal{H}} q \|^2 = \| q \|^2 - \| P_{\mathcal{H}} q \|^2 \) that

\[
\delta(\mathcal{H}_h, \mathcal{H}) = \sup_{\|q\|=1, q \in \mathcal{H}_h} \sqrt{1 - \| P_{\mathcal{H}} q \|^2} \leq \left( 1 - \frac{1}{\|\pi_h\|_V + 1} \right)^{\frac{1}{2}}.
\]

For any \( q \in \mathcal{H}_H \subset \mathcal{H}_h = \mathcal{B}_h \oplus \mathcal{H}_h, q - P_{\mathcal{H}_h} q \in \mathcal{B}_h \) and thus \( \pi_H(q - P_{\mathcal{H}_h} q) \in \mathcal{B}_H \). Hence \( \pi_H(q - P_{\mathcal{H}_h} q) \perp q - P_{\mathcal{H}_h} q \), and

\[
\| q - P_{\mathcal{H}_h} q \| \leq \| q - P_{\mathcal{H}_h} q - \pi_H(q - P_{\mathcal{H}_h} q) \| = \| (I - \pi_H) P_{\mathcal{H}_h} q \|.
\]

Replacing (3.3) by (3.4) and following the same argument in the proof of (3.1), we obtain (3.2). \( \square \)
Christiansen and Winther [14] gave a $\pi_h$ that is uniformly bounded in the $L^2$-norm (thus in the $V$-norm) on shape regular meshes. Let $V_H^k \subset V_h^k$ be two nested finite element spaces generated by AMFEM. Then $\text{gap}(\mathcal{S}_h^k, \mathcal{S}_H^k)$ and $\text{gap}(\mathcal{S}_h^k, \mathcal{S}_H^k)$ are uniformly bounded below from 1 provided the mesh refinement algorithm preserves shape regularity.

The major component of this section is a separate a posteriori error estimator for $\|\sigma - \sigma_h\|_{H^{k-1}}$, which is crucial for proving convergence and optimality of our adaptive algorithms w.r.t. the error $\|\sigma - \sigma_h\|_{H^{k-1}}$.

**Theorem 3.3** (separate error estimator for $\|\sigma - \sigma_h\|_{H^{k-1}}$). For $f \in H^1(\mathcal{T}_h)$ with $1 \leq k \leq n - 1$ or $f \in L^2(\Omega)$, there exists a constant $C_{up}$ depending solely on $\Omega$ and the shape regularity of $\mathcal{T}_h$, such that

$$
\|\sigma - \sigma_h\|_{H^{k-1}} \leq C_{up} \eta^2(\mathcal{T}_h) = C_{up} \sum_{K \in \mathcal{T}_h} \eta^2(K),
$$

where

$$
\eta^2(K) = \begin{cases}
  h^2_k \|\delta(f - d\sigma_h)\|_{H^1}^2 + h_K \|\text{tr}(f - d\sigma_h)\|_{\partial K}^2, & k = 1, \\
  h^2_K \|\delta(f - d\sigma_h)\|_{H^1}^2 + h^2_K \|\delta\sigma_h\|_{H^1}^2 \\
  + h_K \|\text{tr}(f - d\sigma_h)\|_{\partial K}^2 + h_K \|\text{tr}\sigma_h\|_{\partial K}^2, & 2 \leq k \leq n - 1, \\
  h^2_K \|\delta\sigma_h\|_{H^1}^2 + h_K \|\text{tr}\sigma_h\|_{\partial K}^2 + \|f - f_{\mathcal{T}_h}\|_{L^2}^2, & k = n,
\end{cases}
$$

$f_{\mathcal{T}_h}$ is the $L^2$ projection of $f$ onto $V_h^n$.

**Proof.** Assume $2 \leq k \leq n - 1$. To estimate $\|d(\sigma - \sigma_h)\|$, we apply Theorem 3.1. We then need to estimate $\|\sigma - \sigma_h\|$. Let $\sigma - \sigma_h = dv_1 + \delta v_2 + q$ be the Hodge decomposition of $\sigma - \sigma_h$, where $v_1 \in \mathcal{F}^{1,V}$, $v_2 \in \mathcal{F}^{1,V^*}$, and $q \in \mathcal{S}$. Our strategy is to estimate each component in the decomposition separately. In doing so, let $v_1 = d\varphi_1 + z_1$ be a regular decomposition of $v_1$ and recall that $\Pi_h$ is the commuting quasi-interpolation in Theorem 2.1. Then $dv_1 = dz_1$ and $d(v_1 - \Pi_h v_1) = dz_1 - \Pi_h z_1)$. By Theorem 2.1 and a Poincaré inequality,

$$
\|z\|_{H^1} \lesssim \|v_1\|_{H^{k-2}} \lesssim \|dv_1\|.
$$

By $\sigma \in \mathcal{F}$ and $\sigma_h \in \mathcal{S}_h$, and element-wise integration by parts, we have

$$
\|dv_1\|^2 = \langle \sigma - \sigma_h, dv_1 \rangle = \langle -\sigma_h, d(v_1 - \Pi_h v_1) \rangle = \langle -\sigma_h, d(z_1 - \Pi_h z_1) \rangle = \sum_{K \in \mathcal{T}_h} \langle \delta \sigma_h, z_1 - \Pi_h z_1 \rangle_K - \int_{\partial K} \text{tr}\sigma_h \wedge \text{tr}(z_1 - \Pi_h z_1).
$$

Since $z_1 - \Pi_h z_1 \in H^{k-2}(\Omega)$, the trace $\text{tr}(z_1 - \Pi_h z_1) \in H^{-1/2}(\Lambda^{k-2}(e))$ is well-defined and thus has no jump across the boundary $e \subset \partial K$. It then follows by regrouping the
edge sum and the Cauchy–Schwarz inequality that

\[ \|dv_1\|^2 = \sum_{K \in T_h} -\langle \delta \sigma_h, z_1 - \Pi_h z_1 \rangle_K - \sum_{e \in E_h} \int_e [\text{tr} \star \sigma_h] \wedge \text{tr}(z_1 - \Pi_h z_1) \]

\[ \lesssim \left( \sum_{K \in T_h} h_K^2 \| \delta \sigma_h \|^2_K + h_K \| \text{tr} \star \sigma_h \|^2_{\partial K} \right)^{\frac{1}{2}} \]

\[ \times \left( \sum_{K \in T_h} h_K^{-2} \| z - \Pi_h z \|^2_K + h_K^{-1} \| \text{tr}(z - \Pi_h z) \|^2_{\partial K} \right)^{\frac{1}{2}}. \]

Using the approximation property of \( \Pi_h \) in Theorem 2.1 and the bounds (3.5), we obtain

\[ \|dv_1\| \lesssim \left( \sum_{K \in T_h} h_K^2 \| \delta \sigma_h \|^2_K + h_K \| \text{tr} \star \sigma_h \|^2_{\partial K} \right)^{\frac{1}{2}}. \]

For the component \( \delta v_2 \) with \( v_2 \in \mathcal{Z}^{+1} \subset \mathcal{Z}^* = \mathcal{B} \), let \( v_2 = dw \) where \( w \in \mathcal{Z}^{1V} \). Let \( w = d\varphi_2 + z_2 \) be a regular decomposition of \( w \). By the error equation (2.7b), we have

\[ \|\delta v_2\|^2 = \langle d(\sigma - \sigma_h), dw \rangle \\
= \langle d(\sigma - \sigma_h), d(w - \Pi_h w) \rangle - \langle d\Pi_h w, p - p_h \rangle. \]

It then follows from \( p \perp \mathcal{B} \supset \mathcal{B}_h \), \( p_h \perp \mathcal{B}_h \), \( d(w - \Pi_h w) = d(z_2 - \Pi_h z_2) \) and \( d\sigma = f - \delta du - p \) that

\[ \|\delta v_2\|^2 = \langle f - p - \delta du - d\sigma_h, d(z_2 - \Pi_h z_2) \rangle \\
= \langle f - d\sigma_h, d(z_2 - \Pi_h z_2) \rangle. \]

Similarly using (3.7), element-wise integration by parts, Theorem 2.1, and bounds

\[ \|z_2\|_{H^1} \lesssim \|w\|_{H^k} \lesssim \|dv_1\| \lesssim \|\delta v_2\|, \]

we obtain

\[ \|\delta v_2\|^2 = \sum_{K \in T_h} \langle \delta(f - d\sigma_h), z_2 - \Pi_h z_2 \rangle_K \\
+ \sum_{e \in E_h} \int_e [\text{tr} \star (f - d\sigma_h)] \wedge \text{tr}(z_2 - \Pi_h z_2) \]

\[ \lesssim \left( \sum_{K \in T_h} h_K^2 \| \delta(f - d\sigma_h) \|^2_K + h_K \| \text{tr} \star (f - d\sigma_h) \|^2_{\partial K} \right)^{\frac{1}{2}} \]

\[ \times \left( \sum_{K \in T_h} h_K^{-2} \| z_2 - \Pi_h z_2 \|^2_K + h_K^{-1} \| \text{tr}(z_2 - \Pi_h z_2) \|^2_{\partial K} \right)^{\frac{1}{2}} \]

\[ \lesssim \left( \sum_{K \in T_h} \eta_{\delta \sigma}^2(K) \right)^{\frac{1}{2}} \|w\|_{H^k} \lesssim \sum_{K \in T_h} \eta_{\delta \sigma}^2(K). \]
In the end, we estimate the harmonic component.

\[
\|q\| = \langle \sigma - \sigma_h, \frac{q}{\|q\|} \rangle \\
= \langle \sigma - \sigma_h, \frac{q}{\|q\|} - P_{D_h} \frac{q}{\|q\|} \rangle \\
\leq \delta(\delta_h, \delta_h) \|\sigma - \sigma_h\|.
\]

By (3.1) in Lemma 3.2, \(\delta(\delta_h, \delta_h) \leq C(T_h, \Omega) < 1\), where \(C(T_h, \Omega)\) is a constant depending on \(\Omega\) and the shape regularity of \(T_h\). Combining the above three bounds (3.6), (3.8), and (3.9), we have

\[
\|\sigma - \sigma_h\|^2 = \|dv_1\|^2 + \|\delta v_2\|^2 + \|q\|^2 \\
\leq \frac{1}{1 - C(T_h, \Omega)^2} \left( \|dv_1\|^2 + \|\delta v_2\|^2 \right) \\
\lesssim \sum_{K \in \mathcal{T}_h} h_K^2 \|\delta\sigma_h\|^2_K + h_K \|\text{tr} \star \sigma_h\|^2_{\partial K} + \eta_{\delta\sigma}(K).
\]

When \(k = 1\), \(\sigma - \sigma_h \in H^0(\Omega)\) and then there is no boundary component \(dv_1\) in the Hodge decomposition. In this case, \(\|\sigma - \sigma_h\|^2 \lesssim \sum_{K \in \mathcal{T}_h} \eta_{\delta\sigma}(K)\). When \(k = n\), \(d\sigma = f\), \(d\sigma_h = f_{T_h}\), and we replace \(\eta_{\delta\sigma}(T_h)\) by \(\|f - f_{T_h}\|\). The proof is complete. \(\Box\)

We compare \(\eta_{\sigma}(T_h)\) with the a posteriori error estimator

\[
\eta_{DH}(T_h) = \left( \sum_{K \in \mathcal{T}_h} \eta_{\sigma K}^2(K) + \eta_{\delta\sigma}^2(K, p_h) \right) \frac{1}{2} + \mu \|u_h\|,
\]

in [17], where \(\mu\) is some assumed a posteriori upper bound on \(\text{gap}(\delta_h, \delta_h)\). Practical a posteriori estimates for \(\text{gap}(\delta_h, \delta_h)\) can be found in [16]. However, there is no efficiency result on the term \(\mu \|u_h\|\), i.e., \(\mu \|u_h\| \lesssim \|\sigma - \sigma_h\|_{H^{k-1}(\Omega)} + \|u - u_h\|_{H^{k} + \|p - p_h\|}\).

In contrast, a local efficiency result of \(\eta_{\sigma}(K)\) can be directly established by using the Verfürth bubble function technique (see [17] and [32]). In efficiency results, we define \(\Omega_K\) to be the local patch surrounding \(K \in \mathcal{T}_h\) which is the union of \(K\) and neighboring simplices sharing an \((n-1)\)-face with \(K\).

**Theorem 3.4 (efficiency).** For \(f \in H^1(\Omega)\) with \(1 \leq k \leq n - 1\) or \(f \in L^2(\Omega)\), there exists a constant \(C_{\text{low}}\) depending solely on \(\Omega\) and the shape regularity of \(T_h\), such that

\[
C_{\text{low}} \eta_{Dh}^2(K) \leq \|\sigma - \sigma_h\|^2_{H^{k-1}(\omega_K)} + \text{osc}_{T_h}^2(\sigma_h, K),
\]

when \(k = n\), \(\text{osc}_{T_h}(\sigma_h, K) = 0\); when \(1 \leq k \leq n - 1\),

\[
\text{osc}_{T_h}^2(\sigma_h, K) = h_K^2 \|(I - Q_K)\delta(f - d\sigma_h)\|^2_K \\
+ h_K \|(I - Q_{\partial K})\text{tr} \star (f - d\sigma_h)\|^2_{\partial K},
\]

where \(Q_K\) is the \(L^2\)-projection onto the space of polynomial \((k-1)\)-forms of degree \(r\) on \(K\), \(Q_{\partial K}\) is the \(L^2(\partial K)\)-projection onto the space of discontinuous piecewise \((n-k)\)-forms of degree \(r'\) on \(\partial K\), \(r\) and \(r'\) are arbitrary but fixed nonnegative integers.

Note that \(\text{osc}_{T_h}(\sigma_h, K)\) here is different from the data oscillation defined by (5.13)-(5.15) in [17]. But using our notation, we have the dominance in Theorem 5.3, which is helpful for proving optimality.

We then give an a posteriori estimate for the harmonic error.
The error equation (2.7b) implies
\[
\|p - p_h\|^2 \lesssim \eta_p^2(T_h) = \sum_{K \in T_h} \eta_p^2(K),
\]
where
\[
\eta_p^2(K) = h_K^2 \|\delta p_h\|^2_K + h_K \|\text{tr} \ast p_h\|^2_{\partial K} + \eta_{\text{ab}}^2(K).
\]

**Proof.** Let \( p - p_h = (p - P_\mathcal{B}p_h) + (P_\mathcal{B}p_h - p_h). \) Since \( p_h \in \mathcal{F}_h \subset \mathcal{F} = \mathcal{B} \oplus \mathcal{F}, \)
\( p_h - P_\mathcal{B}p_h \in \mathcal{B} \). Therefore \( p_h - P_\mathcal{B}p_h \perp q \) and
\[
\|p - P_\mathcal{B}p_h\| \leq \sup_{q \in \mathcal{F}, \|q\|=1} \langle p - P_\mathcal{B}p_h, q \rangle = \sup_{q \in \mathcal{F}, \|q\|=1} \langle p - p_h, q \rangle \tag{3.11}
\]
\[
= \sup_{q \in \mathcal{F}, \|q\|=1} (\langle p - p_h, q - P_\mathcal{B}p_h, q \rangle + \langle p - p_h, P_\mathcal{B}p_h, q \rangle) \leq \delta(\mathcal{F}, \mathcal{F}_h)\|p - p_h\| + \sup_{q \in \mathcal{F}, \|q\|=1} \langle p - p_h, P_\mathcal{B}p_h, q \rangle.
\]
The error equation (2.7b) implies
\[
\sup_{q \in \mathcal{F}, \|q\|=1} \langle p - p_h, P_\mathcal{B}p_h, q \rangle = \sup_{q \in \mathcal{F}, \|q\|=1} \langle d(\sigma - \sigma_h), -P_\mathcal{B}p_h, q \rangle \tag{3.12}
\]
\[
= \sup_{q \in \mathcal{F}, \|q\|=1} \langle d(\sigma - \sigma_h), q - P_\mathcal{B}p_h, q \rangle \leq \delta(\mathcal{F}, \mathcal{F}_h)\|d(\sigma - \sigma_h)\|.
\]

By the equation (2.12) and Lemma 9 in [17],
\[
\|P_\mathcal{B}p_h - p_h\| \lesssim \sup_{\|\phi\|_{H^k} = 1} \langle p_h, d(\phi - \Pi_h \phi) \rangle \tag{3.13}
\]
\[
\lesssim \left( \sum_{K \in T_h} h_K^2 \|\delta p_h\|^2_K + h_K \|\text{tr} \ast p_h\|^2_{\partial K} \right)^{\frac{1}{2}}.
\]
Combining (3.11)-(3.13) and using Theorem 3.1, we obtain
\[
\|p - p_h\| \leq \|p - P_\mathcal{B}p_h\| + \|P_\mathcal{B}p_h - p_h\|
\]
\[
\leq \frac{1}{1 - \delta(\mathcal{F}, \mathcal{F}_h)} (\delta(\mathcal{F}, \mathcal{F}_h)\|d(\sigma - \sigma_h)\| + \|P_\mathcal{B}p_h - p_h\|)
\]
\[
\lesssim \left( \sum_{K \in T_h} h_K^2 \|\delta p_h\|^2_K + h_K \|\text{tr} \ast p_h\|^2_{\partial K} \right)^{\frac{1}{2}} + \eta_d(\mathcal{T}_h).
\]
The proof is complete. \( \square \)

When \( k = n, \mathcal{F}_h^k = \mathcal{F}_h^n = \{0\} \), and \( \eta_p \) is useless. The efficiency of \( \eta_p \) follows from the efficiency of \( \eta_{-1}(K), \eta_0(K), \) and \( \eta_{\mathcal{F}}(p_h, K) \) in [17].
Theorem 3.6 (efficiency of $\eta_p(K)$). For $1 \leq k \leq n - 1$ and $f \in H^1 \Lambda^k(T_h)$, the local efficiency holds:

$$\eta_p(K) \lesssim \|p - pH\|_{\omega_K} + \|d(\sigma - \sigma_h)\|_{\omega_K} + h_K^2 \|\delta(f - pHf)\|_{\omega_K} + h_K^2 \|\|tr \ast(f - pHf)\|\|_{\partial\omega_K},$$

where $P_Hf$ is the $L^2$-projection of $f$ onto the space of $k$-forms with discontinuous piecewise polynomial coefficients of arbitrary but fixed degree.

The next theorem gives a posteriori estimate on $\|d(u - uh)\|$, which is similar to the error estimator in [15] but involving harmonic terms here.

Theorem 3.7 (separate estimator for $\|d(u - uh)\|$). For $1 \leq k \leq n - 1$ and $f \in H^1 \Lambda^k(T_h)$, we have the a posteriori estimate

$$\|d(u - uh)\|^2 \lesssim \eta_{du}^2(T_h) = \sum_{K \in T_h} \eta_{du}^2(K),$$

where

$$\eta_{du}^2(K) = h_K^2 \|f - d\sigma_h - \delta du_h - pHf\|^2_{\omega_K} + h_K^2 \|\delta(f - d\sigma_h - pHf)\|^2_{\omega_K} + h_K \|\|tr \ast(f - d\sigma_h - pHf)\|\|_{\partial\omega_K} + h_K \|\|tr \ast du_h\|\|_{\partial\omega_K}.$$

Proof. Let $v \in \mathbf{3}^{\perp V}$ such that $dv = d(u - uh)$. Let $v = d\varphi + z$ be the regular decomposition of $v$. (2.7b) implies

$$\|d(u - uh)\|^2 = \langle d(u - uh), dv \rangle = \langle d(u - uh), d(v - \Pi_h v) \rangle = \langle d(\sigma - \sigma_h), \Pi_h v \rangle - \langle \Pi_h v, p - pH \rangle.$$ 

Then by $v \perp \mathbf{3}^k$, $f = d\sigma + \delta du + p$, and $v - \Pi_h v = d(\varphi - \Pi_h \varphi) + (z - \Pi_h z)$, we have

$$\|d(u - uh)\|^2 = \langle d(u - uh), d(v - \Pi_h v) \rangle + \langle d(\sigma - \sigma_h), v - \Pi_h v \rangle + \langle v - \Pi_h v, p - pH \rangle = \langle f - d\sigma_h - pHf, v - \Pi_h v \rangle - \langle du_h, d(v - \Pi_h v) \rangle = \langle f - d\sigma_h - pHf, d(\varphi - \Pi_h \varphi) + (z - \Pi_h z) \rangle - \langle du_h, d(z - \Pi_h z) \rangle.$$ 

The theorem then follows from the standard element-wise integration by parts, Theorem 2.1, and the Poincaré inequality $\|v\|_{H^1} \lesssim \|dv\|$. \hfill \Box

In fact, $\eta_{du}(K)$ in Theorem 3.7 is just $\eta_h(K)$ in [17]. Local efficiency of $\eta_{du}$ can be found from Lemma 12 in [17].

In the end, we can proceed to derive the a posteriori error estimate for $\|u - uh\|$ by following the proof of Theorem 3.3. However, the corresponding error bound would be similar to the coupled error bound $\eta_{DH}$ (3.10). In particular, it involves the term $\|PHH_u\| \leq \mu \|u_h\|$. Therefore the separate error estimator for $\|u - uh\|$ has no advantage.

4. Convergence. Given a subset $M \subset T_h$, define

$$\eta_p^2(M) := \sum_{K \in M} \eta_p^2(K),$$

and similar for $\eta_p(M)$ and $\eta_{du}(M)$. We now present the adaptive algorithm AMFEM1 for solving the Hodge-Laplacian problem (2.4) based on the standard adaptive
feedback loop

\[ \text{SOLVE} \quad \rightarrow \quad \text{ESTIMATE} \quad \rightarrow \quad \text{MARK} \quad \rightarrow \quad \text{REFINE}. \]

AMFEM1 is designed to reduce the error \( \| \sigma - \sigma_h \|_{H^1}^2 + \| p - p_h \|^2 + \| d(u - u_h) \|^2 \). It is unconditionally convergent starting from any initial coarse mesh.

**AMFEM1.** Given an initial mesh \( T_0 \), marking parameters \( 0 < \theta_\sigma, \theta_p, \theta_{du} < 1 \), and an error tolerance \( \text{tol} > 0 \). Let \( \ell = -1 \) and \( \eta_{-1} = \text{tol} + 1 \). While \( \eta_\ell > \text{tol} \), do

**Step 1** Solve (2.6) on \( T_\ell \) to obtain the finite element solution \((\sigma_\ell, u_\ell, p_\ell)\).

**Step 2** Compute the error estimators \( \eta_\sigma(\ell), \eta_p(\ell) \) and \( \eta_{du}(\ell) \) on each element \( K \in T_\ell \) and \( \eta_\ell = (\eta_\sigma^2(\ell) + \eta_p^2(\ell) + \eta_{du}^2(\ell))^{\frac{1}{2}} \).

**Step 3** Select a subset \( M_\ell \) of \( T_\ell \) such that

\[
\begin{align*}
(4.1) & \ \ \ \ \eta_\sigma(M_\ell) \geq \theta_\sigma \eta_\sigma(T_\ell), \\
(4.2) & \ \ \ \ \eta_p(M_\ell) \geq \theta_p \eta_p(T_\ell), \\
(4.3) & \ \ \ \ \eta_{du}(M_\ell) \geq \theta_{du} \eta_{du}(T_\ell).
\end{align*}
\]

**Step 4** Refine each element in \( M_\ell \) and necessary neighboring elements by a mesh refinement algorithm preserving shape regularity to get a conforming mesh \( T_{\ell+1} \). Set \( \ell \leftarrow \ell + 1 \).

**Step 3** is often called Dörfler marking in the literature. Marking properties (4.1)-(4.3) can be achieved by first selecting \( M_\ell \) such that (4.1) holds, then successively enlarging \( M_\ell \) to make (4.2) and (4.3) satisfied. The marking step can be flexible, see Remark 4.1 for details. Candidates for mesh refinement in **Step 4** include bisection [24] or quad-refinement with bisection closure [8, 32].

Let \{\( (\sigma_\ell, p_\ell, u_\ell), T_\ell \}_{\ell \geq 0} \) be a sequence of finite element solutions and meshes produced by AMFEM1. Let

\[
\begin{align*}
\sigma_{\ell+1} = \sigma_\ell - \Delta \sigma_\ell, \\
\eta_{\sigma,\ell+1} = \eta_{\sigma,\ell} + \varepsilon_{\ell-1} \Delta \sigma_\ell, \\
\eta_{p,\ell+1} = \eta_{p,\ell} - (1 - \varepsilon_2) \Delta p_\ell + \varepsilon_1^{-1} \eta_{\sigma,\ell+1}, \\
\eta_{du,\ell+1} = \eta_{du,\ell} - (1 - \varepsilon_3) \Delta du_\ell + 2 \varepsilon_3^{-1} (\eta_{\sigma,\ell+1} + \eta_{p,\ell+1}).
\end{align*}
\]

**Lemma 4.1.** For \( \ell \geq 0 \),

\[
\eta_{\sigma,\ell+1} = \eta_{\sigma,\ell} - \Delta \sigma_\ell.
\]

In addition, for arbitrary \( \{\varepsilon_i\}_{i=1}^3 \subset (0, 1) \),

\[
\begin{align*}
(4.5a) & \ \quad \varepsilon_{\ell+1} \leq \frac{1}{1 - \varepsilon_1} \varepsilon_{\ell} - \Delta \sigma_\ell + \frac{\varepsilon_1^{-1}}{1 - \varepsilon_1} \Delta du_\ell, \\
(4.5b) & \ \quad \varepsilon_{p,\ell+1} \leq \varepsilon_{p,\ell} - (1 - \varepsilon_2) \Delta p_\ell + \varepsilon_1^{-1} \varepsilon_{\ell+1}, \\
(4.5c) & \ \quad \varepsilon_{du,\ell+1} \leq \varepsilon_{du,\ell} - (1 - \varepsilon_3) \Delta du_\ell + 2 \varepsilon_3^{-1} (\varepsilon_{\ell+1} + \varepsilon_{p,\ell+1}).
\end{align*}
\]

**Proof.** The orthogonality (4.4) directly follows from (2.7b). Let \( Y_{\ell+1} \) be \( Y_h \) based on \( T_{\ell+1} \). By \( \sigma - \sigma_{\ell+1} \perp Y_{\ell+1} \) and the Poincaré inequality \( \| P_{Y_{\ell+1}} (\sigma_\ell - \sigma_{\ell+1}) \| \lesssim \)
\[ \|dP_{\overline{3}}(\sigma_{\ell} - \sigma_{\ell+1})\| = \sqrt{\Delta_{\sigma,\ell}}, \]

we obtain

\[ e_{\sigma,\ell+1} = e_{\sigma,\ell} - \Delta_{\sigma,\ell} + 2(\sigma - \sigma_{\ell+1}, P_{\overline{3}}(\sigma_{\ell} - \sigma_{\ell+1})) \]
\[ \leq e_{\sigma,\ell} - \Delta_{\sigma,\ell} + 2\varepsilon_{\sigma,\ell+1}^2 \Delta_{\sigma,\ell}^2, \]
\[ \leq e_{\sigma,\ell} - \Delta_{\sigma,\ell} + \varepsilon_1 v_{\sigma,\ell+1} + \varepsilon_1^{-1} \Delta_{\sigma,\ell}. \]

The proof of (4.5a) is complete. Similarly (2.7b) implies

\[ e_{p,\ell+1} = e_{p,\ell} - \Delta_{p,\ell} + 2(p - p_{\ell+1}, p - p_{\ell+1}) \]
\[ = e_{p,\ell} - \Delta_{p,\ell} - 2(d(\sigma - \sigma_{\ell+1}), p - p_{\ell+1}) \]
\[ \leq e_{p,\ell} - \Delta_{p,\ell} + 2\varepsilon_{p,\ell,\ell+1} \Delta_{p,\ell}^2, \]
\[ \leq e_{p,\ell} - (1 - \varepsilon_2) \Delta_{p,\ell} + \varepsilon_2^{-1} e_{p,\ell+1}. \]

In the end, let \( v_{\ell+1} \in \overline{3}_{\ell+1} \) such that \( dv_{\ell+1} = d(u_{\ell} - u_{\ell+1}) \). Using (2.7b) and

\[ \|v_{\ell+1}\| \leq \|dv_{\ell+1}\| = \Delta_{\overline{3},\ell+1}^2, \]
(4.5b) can be proved by

\[ e_{d/\ell,\ell+1} = e_{d/\ell,\ell} - \Delta_{d/\ell,\ell} + 2(d(u - u_{\ell+1}), d(u_{\ell} - u_{\ell+1})) \]
\[ = e_{d/\ell,\ell} - \Delta_{d/\ell,\ell} + 2(d(\sigma - \sigma_{\ell+1}), v_{\ell+1}) - 2(d_{\ell+1}, p - p_{\ell+1}) \]
\[ \leq e_{d/\ell,\ell} - \Delta_{d/\ell,\ell} + 2(e_{d/\ell,\ell+1} + e_{\overline{3},\ell+1}) \Delta_{d/\ell,\ell}^2, \]
\[ \leq e_{d/\ell,\ell} - \Delta_{d/\ell,\ell} + \varepsilon_1 \Delta_{d/\ell,\ell} - 2\varepsilon_1^{-1} e_{d/\ell,\ell+1}. \]

The proof is complete. \( \Box \)

Lemma 4.1 deals with error reduction on two consecutive meshes. It is quite elementary comparing to many technical quasi-orthogonality results in the literature. Another ingredient of convergence analysis is the estimator reduction.

**Lemma 4.2.**

\[
\begin{align}
\eta_{\sigma,\ell+1} &\leq \beta_{\sigma} \eta_{\sigma,\ell} + C_{\sigma}(\Delta_{d/\ell,\ell} + \Delta_{\sigma,\ell}), \\
\eta_{p,\ell+1} &\leq \beta_{p} \eta_{p,\ell} + C_{p}(\Delta_{d/\ell,\ell} + \Delta_{p,\ell}), \\
\eta_{d/\ell,\ell+1} &\leq \beta_{d} \eta_{d/\ell,\ell} + C_{d}(\Delta_{d/\ell,\ell} + \Delta_{d/\ell,\ell}).
\end{align}
\]

where \( 0 < \beta_{\sigma}, \beta_{p}, \beta_{d} < 1 \) and \( C_{\sigma}, C_{p}, C_{d/\ell} > 0 \) depend only on marking parameters \( \theta_{\sigma}, \theta_{p}, \theta_{d} \) and \( T_{0} \).

**Proof.** We only prove (4.6a) since proofs of other inequalities are the same. Recall the definition of \( \eta_{\sigma,\ell} = \eta_{\sigma}^2(T_{\ell}) \) in Theorem 3.3. Using the same argument in the proof of Corollary 3.4 in [12], for arbitrary \( \delta_{\ast} > 0 \), we have

\[
\eta_{\sigma,\ell+1} \leq (1 + \delta_{\ast})(\eta_{\sigma,\ell} - \lambda \eta_{n/2}(\mathcal{M}_{\ell})) + (1 + \delta_{\ast}^{-1}) C_{\sigma}(\Delta_{d/\ell,\ell} + \Delta_{\sigma,\ell}),
\]

where \( \lambda = 1 - 2^{-\frac{\mu}{2}} < 1, b > 0 \) is an integer depending on mesh refinement strategy. Then (4.6a) follows from (4.7) and the marking property (4.2) \( \eta_{\sigma,\ell}^2(\mathcal{M}_{\ell}) \geq \theta_{\sigma}^2 \eta_{\sigma,\ell} \) with \( \beta_{\sigma} = (1 + \delta_{\ast})(1 - \lambda \theta_{\sigma}). \) \( \beta_{\sigma} < 1 \) holds provided \( \delta_{\ast} < \frac{\lambda \theta_{\sigma}^2}{1 - \lambda \theta_{\sigma}^2}. \) \( \Box \)

Now we are in the position to prove the convergence of AMFEM.

**Theorem 4.3** (convergence of AMFEM). For \( 1 \leq k \leq n-1 \) and \( f \in H^{k+1}(\Omega) \), let \( \{ (\sigma_{\ell}, u_{\ell}, p_{\ell}), T_{\ell} \}_{\ell \geq 0} \) be a sequence of finite element solutions and meshes generated
by Algorithm AMFEM1. Then there exist $\zeta, \rho_p, \rho_{du}, C_0 > 0$, and $\gamma \in (0, 1)$, depending only on $\Omega$, $T_0$ and $\theta_\sigma, \theta_p, \theta_{du}$, such that
\[
\|\sigma - \sigma_\ell\|^2 + \zeta \|d(\sigma - \sigma_h)\|^2 + \|p - p_\ell\|^2 + \|d(u - u_\ell)\|^2 + \rho_\sigma \eta_\sigma^2(T_\ell) + \rho_p \eta_p^2(T_\ell) + \rho_{du} \eta_{du}^2(T_\ell) \leq C_0 \gamma^\ell.
\]

Proof. For convenience, we may use $C$ as a generic constant, depending only on $\Omega$, $T_0$ and possibly $\theta_\sigma, \theta_p, \theta_{du}$. Let $\rho_\sigma = \frac{1}{C_\sigma}$, and $E_\ell = e_{d\sigma, \ell} + e_{\sigma, \ell} + \rho_\sigma \eta_{\sigma, \ell}$. By (4.5a) and (4.6a),
\[
E_{\ell+1} \leq e_{d\sigma, \ell} - \Delta_{d\sigma, \ell} + \frac{1}{1 - \varepsilon_1} e_{\sigma, \ell} + \frac{1}{1 - \varepsilon_1} e_{\sigma, \ell} + \frac{1}{1 - \varepsilon_1} \Delta_{d\sigma, \ell}
\]
where $C_{\varepsilon_1} = \frac{\varepsilon_1}{1 - \varepsilon_1} > 1$. Let $\alpha_1 \in (0, 1)$ be an undetermined constant. Then by Theorem 3.3,
\[
E_{\ell+1} \leq \alpha_1 (e_{d\sigma, \ell} + e_{\sigma, \ell}) + \left\{ \left( \frac{1}{1 - \varepsilon_1} - \alpha_1 \right) C_{up} + \frac{C_{up} \Delta_{d\sigma, \ell}}{\alpha_1 C_{up} + \rho_\sigma} \right\} \Delta_{d\sigma, \ell}.
\]
Solving the equation
\[
\left( \frac{1}{1 - \varepsilon_1} - \alpha_1 \right) C_{up} + \frac{C_{up} \Delta_{d\sigma, \ell}}{\alpha_1 C_{up} + \rho_\sigma} = \alpha_1 \rho_\sigma
\]
for $\alpha_1$, we obtain
\[
\alpha_1 = \frac{\frac{C_{up}}{\varepsilon_1} + \frac{\rho_\sigma \beta_\sigma}{C_{up} + \rho_\sigma}}{C_{up} + \rho_\sigma}.
\]
$\alpha_1 < 1$ provided
\[
0 < \varepsilon_1 < \frac{1 - \beta_\sigma}{C_\sigma C_{up} + 1 - \beta_\sigma}
\]
holds. In this case, (4.8) reduces to
\[
E_{\ell+1} \leq \alpha_1 E_\ell + C_{\varepsilon_1} \Delta_{d\sigma, \ell}.
\]
Let $\alpha_2 \in (0, 1)$ be a constant to be determined. Combining (4.9), (4.4), and $e_{d\sigma, \ell} \leq E_\ell$, we have
\[
E_{\ell+1} + C_{\varepsilon_1} e_{d\sigma, \ell+1} \leq \alpha_1 E_\ell + C_{\varepsilon_1} e_{d\sigma, \ell}
\]
where $\gamma_1 = \alpha_1 + C_{\varepsilon_1} \alpha_2$. We require that
\[
\gamma_1 < 1, \quad \frac{1 - \alpha_2}{\alpha_1 + C_{\varepsilon_1} \alpha_2} \leq 1,
\]
which is equivalent to
\[ 0 < \frac{1 - \alpha_1}{1 + C_{\varepsilon_1}} \leq \alpha_2 < \frac{1 - \alpha_1}{C_{\varepsilon_1}} < 1. \]

By taking any \( \alpha_2 \) satisfying the above criterion, we proved
\[ (4.10) \quad E_{\ell+1} + C_{\varepsilon_1} e_{d_\sigma,\ell+1} \leq \gamma_1(E_{\ell} + C_{\varepsilon_1} e_{d_\sigma,\ell}) \leq C_{\gamma_1}^{\ell+1}, \]
which gives contraction on \( e_{\sigma,\ell}, e_{d_\sigma,\ell} \), and \( \eta_{\sigma,\ell} \).

Similarly, using (4.5b), (4.6b), and Theorems 3.5, we have
\[ (4.11) \quad e_{p,\ell+1} + \rho_{p} \eta_{p,\ell+1} \leq \alpha_3(e_{p,\ell} + \rho_{p} \eta_{p,\ell}) + \rho_{p} C_{p} \Delta_{d_\sigma,\ell} + \varepsilon_2^{-1} e_{d_\sigma,\ell+1} \]
\[ \leq \alpha_3(e_{p,\ell} + \rho_{p} \eta_{p,\ell}) + C(\eta_{\sigma,\ell} + \eta_{\sigma,\ell+1}), \]
where \( \rho_{p} = (1 - \varepsilon_2)/C_{p} \), \( \alpha_3 = C_{p,\uparrow} + \rho_{p} \beta_{p} \frac{C_{p,\uparrow}}{C_{p}_{\downarrow}} < 1 \), and \( C_{p,\uparrow} \) is the multiplicative constant in Theorem 3.5. Let \( \tilde{\gamma}_2 = \max(\alpha_3, \gamma_1) \). It then follows from (4.11) and (4.10) that
\[ e_{p,\ell+1} + \rho_{p} \eta_{p,\ell+1} \leq \tilde{\gamma}_2(e_{p,\ell} + \rho_{p} \eta_{p,\ell}) + C_{\gamma_2}^{\ell}. \]

By induction and taking \( \gamma_2 = \sqrt{\tilde{\gamma}_2} \), we obtain
\[ (4.12) \quad e_{p,\ell} + \rho_{p} \eta_{p,\ell} \leq \gamma_2^{\ell}(e_{p,0} + \rho_{p} \eta_{p,0}) + C_{\gamma_2}^{\ell} \leq C_{\gamma_2}^{\ell}. \]

In the end, it follows from (4.5c), (4.6c), Theorems 3.3 and 3.5 that
\[ (4.13) \quad e_{du,\ell+1} + \rho_{du} \eta_{du,\ell+1} \leq \alpha_4(e_{du,\ell} + \rho_{du} \eta_{du,\ell}) + \rho_{du} C_{du}(\Delta_{d_\sigma,\ell} + \Delta_{p,\ell}) \]
\[ + 2\varepsilon_3^{-1}(e_{d_\sigma,\ell+1} + e_{p,\ell+1}), \]
\[ \leq \alpha_4(e_{du,\ell} + \rho_{du} \eta_{du,\ell}) + C(\eta_{\sigma,\ell} + \eta_{p,\ell} + \eta_{\sigma,\ell+1} + \eta_{p,\ell+1}), \]
where \( \rho_{du} = (1 - \varepsilon_3)/C_{du} \), \( \alpha_4 = \frac{C_{du,\uparrow} + \rho_{du} \beta_{du}}{C_{du,\uparrow} + \rho_{du}} < 1 \), and \( C_{du,\uparrow} \) is the multiplicative constant in Theorem 3.7. Then using (4.13), (4.10), (4.12), and induction, we have
\[ (4.14) \quad e_{du,\ell} + \rho_{du} \eta_{du,\ell} \leq C_{\gamma_3}^{\ell}, \]
for \( \gamma_3 := \sqrt{\max(\alpha_4, \gamma_1, \gamma_2)} \). Combining (4.10), (4.12), and (4.14), the proof is complete by taking \( \gamma = \max(\gamma_1, \gamma_2, \gamma_3) \) and \( \zeta = 1 + C_{\varepsilon_1}. \)

Unlike common convergence analysis of AMFEM in the literature (cf.[7, 13]), the proof of Theorem 4.3 does not really rely on sharp quasi-orthogonality results. In fact, convergence of \( \|\sigma - \sigma_\ell\|_{H^{k-1}} \) is realized by the Dörfler marking (4.1) and discovering the quasi error \( \|\sigma - \sigma_h\|^2 + \rho_{\sigma} \eta_{\sigma}^2(T_h) + (1 + C_{\varepsilon_1})\|d(\sigma - \sigma_h)\|^2 \) in (4.10). Convergence of \( \|p - p_h\| \) is guaranteed by markings (4.1) and (4.2); convergence of \( \|d(u - u_\ell)\| \) is enforced by all markings (4.1)-(4.3). Hence the marking procedure at Step 3 can be easily adjusted according to practical purpose.

**Remark 4.1.** If \( \delta^k = \{0\} \), marking (4.2) disappears. In this case, the contraction in Theorem 4.3 reduces to
\[ \|\sigma - \sigma_\ell\|^2 + \zeta\|d(\sigma - \sigma_\ell)\|^2 + \|d(u - u_\ell)\|^2 \]
\[ + \rho_{\sigma} \eta_{\sigma}^2(T_h) + \rho_{du} \eta_{du}^2(T_h) \leq C_0 \gamma^k, \]
which is different from the contraction given in Theorem 20 in [15]. In particular, it guarantees \( \|\sigma - \sigma_\ell\|_{H^{k-1}} \to 0 \) while the AMFEM in [15] only provided \( \|d(\sigma - \sigma_\ell)\| \to 0 \).
Remark 4.2. Consider the case \( k = n \), (4.2) is just the traditional mixed method for solving the mixed formulation of Poisson’s equation in \( \mathbb{R}^n \). In this case, markings (4.2) and (4.3) disappear. To prove convergence of their AMFEM on a simply connected polygon in \( \mathbb{R}^2 \) (denoted by AMFEM-CHX), Chen, Holst, and Xu [13] constructed a technical quasi-orthogonality result

\[
(1 - \varepsilon)\varepsilon_{\sigma,\ell+1} \leq \varepsilon_{\sigma,\ell} - \Delta_{\sigma,\ell} + \frac{C_{T_0}}{\varepsilon} \text{osc}_{\ell},
\]

for any \( 0 < \varepsilon < 1 \), \( \text{osc}_{\ell} := ||h_\ell(f - f_{\ell})||^2 \), where \( h_\ell \) is the meshsize function with \( h_\ell|_K = |K|^{1/2} \) for \( K \in T_\ell \). Comparing to (4.5a), (4.15) is sharper because \( \text{osc}_{\ell} \ll \Delta_{\sigma,\ell} = ||f_{\ell+1} - f_{\ell+2}||^2 \). However for convergence analysis of AMFEM-1, the elementary result (4.5a) is enough. In addition, AMFEM-CHX involves an extra marking for data oscillation while AMFEM-2 is a really standard adaptive method using a single marking step. To prove optimality, [13] assumed a preadaptation step enforcing \( \text{osc}_0 = 0 \):

\[
\hat{T} = \text{APPROX}(f, T_0, \varepsilon); \quad f \leftarrow f_{\hat{T}}, \quad T_0 \leftarrow \hat{T}.
\]

In this case, \( \Delta_{\sigma,\ell} = 0 \) for all \( \ell \geq 0 \) and (4.5a) is actually an orthogonality result. Then optimality of AMFEM-1 follows from the same optimality proof of AMFEM-CHX in [13]. For the Hodge Laplacian problem with \( k \leq n - 1 \), such preadaptation is not reasonable because \( d(\sigma_{\ell} - \sigma_{\ell+1}) \) cannot be represented by the data \( f \).

**AMFEM2.** Given an initial mesh \( T_0 \), marking parameters \( 0 < \theta < 1 \), and an error tolerance \( \text{tol} > 0 \). Let \( \ell = -1 \) and \( \eta_{-1} = \text{tol} + 1 \). While \( \eta_k > \text{tol} \), do

**Step 1** Solve (2.6) on \( T_\ell \) to obtain the finite element solution \( (\sigma_\ell, u_\ell, p_\ell) \).

**Step 2** Compute \( \eta_\sigma(K) \) on each element \( K \in T_\ell \) and \( \eta_\ell = \eta_\sigma(T_\ell) \).

**Step 3** Select a subset \( M_\ell \) of \( T_\ell \) such that

\[
\eta_\sigma(M_\ell) \geq \theta \eta_\sigma(T_\ell).
\]

**Step 4** Refine each element in \( M_\ell \) and necessary neighboring elements by a mesh refinement algorithm preserving shape regularity to get a conforming mesh \( T_{\ell+1} \). Set \( \ell \leftarrow \ell + 1 \).

**5. Optimality.** For the purpose of proving optimality, it seems questionable to impose separate markings (4.2) and (4.3) in AMFEM1. See [12] for the drawback of separate marking based on data oscillation. Hence we propose AMFEM2 with a single marking step. As a result, we are not able to control \( ||p - p_\ell|| \) and \( ||u - u_\ell|| \). However, optimality of AMFEM2 will follow as a compensation. AMFEM2 is a contraction by (4.10) in the proof of Theorem 4.3.

**Theorem 5.1** (contraction of AMFEM2). For \( f \in H^{k-1}(\Omega) \) with \( 1 \leq k \leq n-1 \) or \( f \in L^2(\Omega, \Omega) \), let \( \{(\sigma_\ell, u_\ell, p_\ell), T_\ell\}_{\ell \geq 0} \) be a sequence of finite element solutions and meshes generated by Algorithm AMFEM2. Then there exist \( \zeta, \rho > 0 \) and \( \alpha \in (0, 1) \), depending only on \( \Omega, \text{tol} \) and \( \theta \), such that

\[
\|\sigma - \sigma_{\ell+1}\|^2 H_k^{-1} + \zeta \|d(\sigma - \sigma_{\ell+1})\|^2 + \rho \eta_\sigma^2(T_{\ell+1}) \leq \alpha \left\{ \|\sigma - \sigma_\ell\|^2 H_{k-1}^{-1} + \zeta \|d(\sigma - \sigma_\ell)\|^2 + \rho \eta_\sigma^2(T_\ell) \right\}.
\]

To prove the optimality of AMFEM2, we need a localized discrete upper bound, which can be viewed as a discrete and local version of the global upper bound \( \eta_\sigma(T_\ell) \).
for $\|\sigma - \sigma_h\|_{H^{k-1}}$. Following [16], we repeatedly apply the locally defined bounded cochain projection $\tilde{\pi}_h$ given by Falk and Winther [19]. Let $T_h$ be a refinement of $T_h$ and $R_h = R_{T_h} - T_h$ be the set of refined elements in $T_h$. Given $\tilde{\pi}_h$ on $T_h$ and $K \in T_h$, there is a subdomain $\hat{\omega}_K$ which contains $K$ and aligns with $T_h$, such that $\# \{ K' \in T_h : K' \subset \hat{\omega}_K \} \lesssim 1$. In addition, $\tilde{\pi}_h$ is a local projection (see also (2.9) in [16]) in the sense that

$$v|_{\hat{\omega}_K} \in V_K$$

To specify the dependence of $\eta_{\sigma}(K)$ on the mesh and finite element solution, we use the notation $\eta_{\sigma}(h, K) = \eta_{\sigma}(K)$ and $\eta_{\sigma, T_h}(h, K) = \sum_{T \in \mathcal{M}} \eta_{\sigma, T_h}(h, K)$ for $\mathcal{M} \subset T_h$ in the next theorem.

**Theorem 5.2 (discrete localized upper bound).** Let $T_h$ be a conforming refinement of $T_h$, $\sigma_h$ and $\sigma_H$ be the finite element solution approximating $\sigma$ on $T_h$ and $T_h$, respectively. For $f \in H^{k-1}(\Omega) \quad 1 \leq k \leq n - 1$ or $f \in L^2(\Omega)$, there exists $R_h \subset T_h$, which is the union of $R_h$ and a collection of some neighboring elements of $R_h$, $\# R_h \lesssim \# R_h$, such that

$$\|\sigma_h - \sigma_H\|^2_{H^{k-1}} \leq C_{\text{loc}} \eta_{\sigma, T_h}(\sigma_h, R_h),$$

where $C_{\text{loc}}$ depends solely on $\Omega$ and the shape regularity of $T_h, T_h$.

**Proof.** Consider $2 \leq k \leq n$. We first focus on the proof of $\|\sigma_h - \sigma_H\|^2 \lesssim \eta_{\sigma, T_h}(\sigma_h, R_h)$. Throughout the proof, $R_h$ always denotes the union of $R_h$ and a collection of some neighboring elements of $R_h$, $\# R_h \lesssim \# R_h$. However, $R_h$ can vary from step to step and $\# R_h$ depends on the local property of $\tilde{\pi}_h$ and the coefficient-wise Scott–Zhang interpolation $I_h$ [29]. Similar to the proof of Theorem 3.3, we consider the discrete Hodge decomposition $\sigma_h - \sigma_H = dv_h + \delta \pi_h + q_h$, where $v_h \in T_h, v_h = d\omega_h \in T_h = T_h$ and $q_h \in T_h$.

Let $v_h = d\varphi + z$ be a regular decomposition of $v_h$. In view of (5.1),

$$\text{supp}(\tilde{\pi}_h z - \tilde{\pi}_h \pi_h z) \subseteq \bigcup_{\mathcal{K} \in R_h} K.$$  

Since $\tilde{\pi}_h$ is a cochain projection, $dv_h = dz = d\tilde{\pi}_h z$. Then by $\sigma_h \in T_h, \sigma_H \in T_h$, element-wise integration by parts, and (5.2), we have

$$\|dv_h\|^2 = \langle \sigma_h - \sigma_H, dv_h \rangle = \langle -\sigma_H, d(\tilde{\pi}_h z - \tilde{\pi}_h \pi_h z) \rangle = \sum_{\mathcal{K} \in R_h} -\langle \delta \sigma_H, \tilde{\pi}_h z - \tilde{\pi}_h \pi_h z \rangle_K - \int_{\partial K} \text{tr} \sigma_H \wedge \text{tr}(\tilde{\pi}_h z - \tilde{\pi}_h \pi_h z) \lesssim \sum_{\mathcal{K} \in R_h} \left( h_{\mathcal{K}} \|\delta \sigma_H\|^2_K + h_{\mathcal{K}} \|\text{tr} \sigma_H\|^2_{\partial \mathcal{K}} \right)^{\frac{1}{2}} (I_K + J_K),$$

where

$$I_K = h_{\mathcal{K}}^{-1} \|\tilde{\pi}_h z - \tilde{\pi}_h \pi_h z\|_K, \quad J_K = h_{\mathcal{K}}^{-\frac{1}{2}} \|\text{tr}(\tilde{\pi}_h z - \tilde{\pi}_h \pi_h z)\|_{\partial \mathcal{K}}.$$  

Splitting $\tilde{\pi}_h z - \tilde{\pi}_h \pi_h z = (\tilde{\pi}_h z - I_h z) + (I_h z - \tilde{\pi}_h \pi_h z) + \tilde{\pi}_h (I_h z - \tilde{\pi}_h z)$, Demlow proved that (see the proof Lemma 3 in [16])

$$I_K + J_K \lesssim |z|_{H^1(\omega_K)}.$$
Combining (5.3), (5.4) with the Cauchy–Schwarz inequality and \( |z|_{H^1} \lesssim \|v_h^1\|_{H^{\alpha - 2}} \lesssim \|dv_h^1\| \), we have

\[
\|dv_h^1\|^2 \lesssim \sum_{K \in \mathcal{K}_H} h_K^2 \|\delta \sigma_H\|_K^2 + h_K \|\nabla (f - d\sigma_H)\|_{\partial K}^2.
\] (5.5)

Let \( w_h = d\varphi_2 + z_2 \) be a regular decomposition of \( w_h \). It follows from (2.6) and \( v_h^2 = dw_h = dz_2 = d\bar{\pi}_h z_2 \) that

\[
\|\delta_h v_h^2\|^2 = \langle \sigma_h - \sigma_H, \delta_h v_h^2 \rangle
= \langle \sigma_h - \sigma_H, \bar{\pi}_h z_2 - \bar{\pi}_H \bar{\pi}_h z_2 \rangle
= \langle f - d\sigma_H, \bar{\pi}_h z_2 - \bar{\pi}_H \bar{\pi}_h z_2 \rangle.
\]

Then by similar argument in proving (5.5) and a series of bounds

\[
\|z_2\|_{H^1} \lesssim \|w_h\|_{H^{\alpha - 1}} \lesssim \|dv_h\| = \|v_h^2\| \lesssim \|\delta_h v_h^2\|,
\]

we have

\[
\|\delta_h v_h^2\|^2 \lesssim \sum_{K \in \mathcal{K}_H} (h_K^2 \|\delta (f - d\sigma_H)\|_K^2 + h_K \|\nabla (f - d\sigma_H)\|_{\partial K}^2)^{\frac{1}{2}}
\times (h_K^{-1} \|\bar{\pi}_h z_2 - \bar{\pi}_H \bar{\pi}_h z_2\|_K^2 + h_K^{-2} \|\nabla (\bar{\pi}_h z_2 - \bar{\pi}_H \bar{\pi}_h z_2)\|_{\partial K}^2)
\lesssim \sum_{K \in \mathcal{K}_H} h_K^2 \|\delta (f - d\sigma_H)\|_K^2 + h_K \|\nabla (f - d\sigma_H)\|_{\partial K}^2.
\] (5.6)

In the end, the harmonic component is controlled by

\[
\|q_h\| = \langle \sigma_h - \sigma_H, q_h \rangle_{\|\cdot\|_{q_h}}
= \langle \sigma_h - \sigma_H, q_h \rangle_{\|\cdot\|_{q_h}} - P_{\mathcal{H}_H} q_h
\leq \delta(\mathcal{H}_h, \mathcal{H}_H) \|\sigma_h - \sigma_H\|,
\] (5.7)

where \( \delta(\mathcal{H}_h, \mathcal{H}_H) \leq C(\mathcal{T}_0, \Omega) < 1 \) by Lemma 3.2.

Combining the above three bounds (5.5), (5.6), and (5.7), we prove that

\[
\|\sigma_h - \sigma_H\|^2 = \|dv_h^1\|^2 + \|\delta_h v_h^2\|^2 + \|q_h\|^2
\leq \frac{1}{1 - C(\mathcal{T}_0, \Omega)^2} \left( \|dv_h^1\|^2 + \|\delta_h v_h^2\|^2 \right)
\lesssim \sum_{K \in \mathcal{K}_H} \eta_{\mathcal{H}_h, \mathcal{H}_H}^2(\sigma_h, K).
\] (5.8)

When \( k = 1 \), the proof is the same but without the boundary component \( dv_h^1 \).

When \( 1 \leq k \leq n - 1 \), let \( \sigma_h - \sigma_H = d\varphi_3 + z_3 \) be a regular decomposition of \( \sigma_h - \sigma_H \). Then \( d(\sigma_h - \sigma_H) = dz_3 = d\bar{\pi}_h z_3 \) and

\[
\|d(\sigma_h - \sigma_H)\|^2 = \langle d(\sigma_h - \sigma_H), d(\bar{\pi}_h z_3 - \bar{\pi}_H \bar{\pi}_h z_3) \rangle
= \langle f - d\sigma_H, d(\bar{\pi}_h z_3 - \bar{\pi}_H \bar{\pi}_h z_3) \rangle.
\]
Following the proof of (5.6), we obtain
\[
\|d(\sigma_h - \sigma_H)\|^2 \lesssim \sum_{K \in \mathcal{R}_H} h_K^2 \|\delta(f - d\sigma_H)\|_{K}^2 + h_K \|\text{tr}*(f - d\sigma_H)\|_{dK}^2.
\]
When \(k = n\),
\[
\|d(\sigma_h - \sigma_H)\|^2 = \|f_{T_h} - f_{T_H}\|^2 = \sum_{K \in \mathcal{R}_H} \|f_{T_h} - f_{T_H}\|_{K}^2.
\]
Combining (5.8), (5.9), and (5.10), the proof is complete. 

In addition, we need several ingredients as in [12]. For a subset \(\mathcal{M} \subset \mathcal{T}_h\), define \(\text{osc}_h^2(\tau_h, \mathcal{M}) := \sum_{K \in \mathcal{M}} \text{osc}_h^2(\tau_h, K)\). Keep in mind that \(V^k_h\) and \(V^{k-1}_H\) are finite element subspaces of \(HA^{k-1}(\Omega)\) based on \(T_h\) and \(T_H\), respectively. The first part of the following lemma follows from the same proof of Corollary 3.5 in [12]. The second part is straightforward by the definition of \(\text{osc}_{T_h}\).

**Lemma 5.3 (properties of oscillation).** Let \(T_h\) be a conforming refinement of \(T_H\). For any \(\tau_h \in V^{k-1}_h\) and \(\tau_H \in V^{k-1}_H\), we have
\[
\text{osc}_h^2(\tau_h, T_H \cap T_h) \leq 2 \text{osc}_h^2(\tau_h, T_H \cap T_h) + C_{\text{osc}}\|d(\tau_h - \tau_H)\|^2.
\]
where \(C_{\text{osc}}\) depends only on \(T_0\). In addition, the dominance holds
\[
\text{osc}_{T_h}(\tau_h, K) \leq \eta_{\sigma, T_h}(\tau_h, K), \quad K \in T_h.
\]
Before presenting the optimality result, we make several assumptions.

**Assumption 5.1.** We assume the following properties of Algorithm AMFEM2.
(a) The marking parameter \(\theta\) satisfies \(0 < \theta < 1\), where
\[
\theta^2_1 = \frac{C_{\text{low}}}{1 + (C_{\text{osc}} + 1)C_{\text{loc}}}.
\]
(b) Step 3 selects a set \(\mathcal{M}_i\) with minimal cardinality.
(c) Step 4 generates a sequence of meshes \(\{\mathcal{M}_i\}_{i \geq 0}\) satisfying the cardinality estimate
\[
\# T_{\tilde{h}} - \# T_0 \lesssim \sum_{i=0}^{\tilde{t}-1} \# \mathcal{M}_i.
\]
Assumption (b) can be guaranteed by sorting \(\{\eta_{\sigma, T_h}(\sigma, K)\}_{K \in T_i}\). Assumption (c) holds provided the newest vertex bisection is applied and a suitable labelling for newest vertices on the initial mesh is chosen, see [31, 9].

Let \(T_N = \{T_h\} \) be a conforming refinement of \(T_0\) : \(\#T_h - \#T_0 \leq N\). For \(s > 0\), we define the approximation class
\[
\mathcal{A}_s = \{((\tau, f) \in HA^{k-1}(\Omega) \times L^2 H^k(\Omega) : ||(\tau, f)||_s = \sup_{N > 0} \{E(N; \tau, f)\} < \infty\},
\]
where
\[
E(N; \tau, f) := \inf_{T_h \in T_N} \inf_{\tau_h \in V^{k-1}_h} \left\{\|\tau - \tau_h\|^2_{HA^{k-1}} + \text{osc}_h^2(\tau_h, T_h)\right\}^{1/2}.
\]
Combining the upper bound Theorem 3.3, the lower bound Theorem 3.4, the contraction Theorem 5.1, and the localized discrete upper bound Theorem 5.2, the proof of the optimality of AMFEM2 is almost the same as the optimality proof in [12], see Lemmas 5.9, 5.10 and Theorem 5.11 in [12] for details.
Theorem 5.4 (optimality of AMFEM2). Let Assumption 5.1 be satisfied. For $f \in H^{k}(\Omega)$ with $1 \leq k \leq n-1$ or $f \in L^2(\Omega)$, let $\{(\sigma_\ell, u_\ell, p_\ell), T_\ell\}_{\ell \geq 0}$ be a sequence of finite element solutions and meshes generated by Algorithm AMFEM2. Then there exists a constant $C_{\text{opt}}$ depending only on $\Omega, T_0, \theta$ and $s$, such that

$$\|\sigma - \sigma_\ell\|_{H^{k-1}}^2 + \frac{\text{osc}_T^2(\sigma_\ell, T_\ell)}{\|T_\ell\|} \leq C_{\text{opt}} \eta_s((\sigma, f), (#T_\ell - #T_0)^{-s}$$

6. Concluding remarks. In this paper, we have developed two adaptive mixed finite element methods AMFEM1 and AMFEM2 for solving the Hodge Laplacian problem on bounded Lipschitz domains. AMFEM1 is convergent w.r.t. $\|\sigma - \sigma_h\|_{H^{k-1}} + \|p - p_h\| + \|d(u - u_h)\|_2$ while AMFEM2 is convergent with optimal complexity w.r.t. $\|\sigma - \sigma_h\|_{H^{k-1}}$. Our results are presented in the language of FEEC. For translation of results from FEEC into $H(\text{curl})$ and $H(\text{div})$ in $\mathbb{R}^3$, readers are referred to [3, 4, 17, 15].

Although the a posteriori upper bound $\eta_{DH}(T_h)$ for $NE_h = (\|\sigma - \sigma_h\|_{H^{k-1}} + \|u - u_h\|_{H^{k}} + \|p - p_h\|_2)^{\frac{1}{2}}$ is available, we are not able to develop a convergent AMFEM w.r.t. the error $NE_h$ in the natural norm. The main difficulty comes from the term $\mu\|u_h\|_{\text{DH}}$, which is an obstacle against proving efficiency and estimator reduction in Lemma 4.2. However, in the absence of harmonic forms, the Arnold–Falk–Winther method (2.6) for the Hodge Laplacian reduces to a conforming method and $\eta_{DH}|_K = (\eta_0^2(K) + \eta_{-1}^2(K) + \eta_{-2}^2(K, p_h))^\frac{1}{2}$ is locally efficient up to data oscillation. In this case, the convergence of AMFEM based on $\eta_{DH}$ follows from the plain convergence result in [27].

References

[1] A. Alonso. Error estimators for a mixed method. Numer. Math., 74 (1996) pp. 385-395.
[2] C. Amrouche, C. Bernardi, M. Dauge and V. Girault, Vector potentials in three-dimensional non-smooth domains. Math. Methods Appl. Sci., 21 (1998) pp. 823-864.
[3] D. N. Arnold, R. S. Falk, and R. Winther. Finite element exterior calculus, homological techniques, and applications. Acta Numer. 15 (2006) 1-155.
[4] D. N. Arnold, R. S. Falk, and R. Winther. Finite element exterior calculus: from Hodge theory to numerical stability. Bull. Amer. Math. Soc. 47 (2010) no. 2, 281-354.
[5] D. N. Arnold and L. Li. Finite element exterior calculus with lower order terms. Math. Comp. (2016)
[6] I. Babuška and T. Strouboulis. The finite element method and its reliability. Numerical Mathematics and Scientific Computation. The Clarendon Press, Oxford University Press, New York, 2001.
[7] R. Becker and S. Mao. An optimally convergent adaptive mixed finite element method. Numer. Math., 111 (2008) pp. 35-54.
[8] J. Bey. Simplicial grid refinement: on Freudenthal’s algorithm and the optimal number of congruence classes. Numer. Math., 85 (2000) No. 1, pp. 1-29.
[9] P. Binev, W. Dahmen, and R. DeVore. Adaptive finite element methods with convergence rates. Numer. Math., 97 (2004), No.2, pp. 219-268.
[10] C. Carstensen. A posteriori error estimate for the mixed finite element method. Math. Comp., 111 (2008) pp. 35-54.
[11] C. Carstensen and R. H. W. Hoppe. Error reduction and convergence for an adaptive mixed finite element method. Math. Comp., 75 (2006) No. 255, pp. 1033-1042.
[12] J. M. Cascon, C. Kreuzer, R. H. Nochetto, and K. G. Siebert. Quasi-optimal convergence rate for an adaptive finite element method. SIAM J. Numer. Anal. 46 (2008) No. 5, 2524-2550.
[13] L. Chen, M. Holst, and J. Xu. Convergence and optimality of adaptive mixed finite element methods. Math. Comp., 78 (2009), pp. 35-53.
[14] S. H. Christiansen and R. Winther. Smoothed projections in finite element exterior calculus, Math. Comp. 77 (2008) no. 262, 813-829.
[15] L. Chen and Y. Wu. Convergence of adaptive mixed finite element methods for the Hodge Laplacian equation: without harmonic forms. SIAM J. Numer. Anal. 15 (2017) No.6, pp. 2905-2929.
[16] A. Demlow. Convergence and quasi-optimality of adaptive finite element methods for harmonic forms. Numer. Math. (2017)
[17] A. Demlow and A. N. Hirani. A posteriori error estimates for finite element exterior calculus: The de Rham complex. Found. Comput. Math. (2014)
[18] W. Dörfler. A convergent adaptive algorithm for Poisson’s equation. SIAM J. Numer. Anal., 33 (1996) pp. 1106-1124.
[19] R. S. Falk and R. Winther. Local bounded cochain projections. Math. Comp., 83 (2014) pp. 2631-2656.
[20] M. Feischl, T. Führer, and D. Praetorius. Adaptive FEM with optimal convergence rates for a certain class of nonsymmetric and possibly nonlinear problems. SIAM J. Numer. Anal., 52 (2014) No.2, pp. 601-625.
[21] M. Holst, A. Mihalik, and R. Szypowski. Convergence and optimality of adaptive methods in the finite element exterior calculus framework. https://arxiv.org/abs/1306.1886, 2013.
[22] J. Huang, X. Huang, and Y. Xu. Convergence of an adaptive mixed finite element method for Kirchhoff plate bending problems. SIAM J. Numer. Anal., 49 (2011) No.2, pp. 1803-1827.
[23] K. Mekchay and R. H. Nochetto. Convergence of adaptive finite element methods for general second order linear PDEs. SIAM J. Numer. Anal., 43 (2005) No.5, pp. 1803-1827.
[24] W. F. Mitchell. A comparison of adaptive refinement techniques for elliptic problems. ACM Transaction on Math. Software, 15 (1989) pp. 326-347
[25] D. Mitrea, M. Mitrea, and M. Taylor. Layer potentials, Hodge-Laplacian and global boundary problems in nonsmooth Riemannian manifolds. Mem. Amer. Math. Soc., 150 (2001).
[26] P. Morin, R. H. Nochetto, and K. G. Siebert. Data oscillation and convergence of adaptive FEM. SIAM J. Numer. Anal., 38 (2000), No. 2, pp. 466-488.
[27] P. Morin, K. G. Siebert, and A. Veeser. A basic convergence result for conforming adaptive finite elements. Math. Models Methods Appl. Sci. 18 (2008), no. 5, pp. 707-737.
[28] J. Schöberl. A posteriori error estimates for Maxwell equations. Math. Comp., 77 (2008), pp. 633-649.
[29] L. R. Scott and S. Zhang. Finite element interpolation of nonsmooth functions satisfying boundary conditions. Math. Comp., 54 (1990) 483-493.
[30] R. Stevenson. Optimality of a standard adaptive finite element method. Found. Comput. Math., 7 (2007), pp. 245-269.
[31] R. Stevenson. The completion of locally refined simplicial partitions created by bisection. Math. Comp., 77 (2008), pp. 227-241.
[32] R. Verfürth. A Review of a Posteriori Error Estimation and Adaptive Mesh Refinement Techniques. Wiley, Chichester, 1996.
[33] J. Xu and L. Zikatanov. Some observations on Babuska and Brezzi theories. Numer. Math., 94 (2003) pp. 195-202.
[34] L. Zhong, L. Chen, S. Shu, G. Wittum, and J. Xu. Convergence and Optimality of adaptive edge finite element methods for time-harmonic Maxwell equations. Math. Comp., 81 (2012) pp. 623-642.