GAMMA-CONVERGENCE OF CHEEGER ENERGIES WITH RESPECT TO INCREASING DistANCES

DANKA LUČIĆ AND ENRICO PASQUALETTO

Abstract. We prove a Γ-convergence result for Cheeger energies along sequences of metric measure spaces, where the measure space is kept fixed, while distances are monotonically converging from below to the limit one. As a consequence, we show that the infinitesimal Hilbertianity condition is stable under this kind of convergence of metric measure spaces.

1. Introduction

In the successful theory of weakly differentiable functions over metric measure spaces, a leading role is played by the so-called Cheeger \( p \)-energy, which was introduced in [2] and generalises the classical Dirichlet \( p \)-energy functional. The purpose of this paper is to study the convergence of Cheeger \( p \)-energies along a sequence of metric measure spaces, where the underlying set and the measure are fixed, while distances monotonically converge from below.

More precisely, given a metric measure space \((X, d, m)\) and a sequence \((d_i)_{i \in \mathbb{N}}\) of distances on \(X\) inducing the same topology as \(d\) and satisfying \(d_i \nearrow d\), we prove (in Theorem 4.1) that for any \(p \in (1, \infty)\) the Cheeger \( p \)-energies \(E_{d_i}^{\text{Ch}, p} : L^p(m) \to [0, +\infty]\) associated with \((X, d_i, m)\) converge to \(E_d^{\text{Ch}, p}\) in the sense of Mosco. As shown in Example 4.4, this kind of statement might totally fail in the case where \(d_i \not\searrow d\). Since the family of quadratic forms is closed under Mosco-convergence, an interesting consequence of Theorem 4.1 is the stability of the infinitesimal Hilbertianity condition (that was introduced in [5] and states the quadraticity of the Cheeger \( 2 \)-energy functional) with respect to increasing limits of the involved distances.

Sub-Riemannian manifolds constitute a significant example of metric structures where the above results apply, as the induced length distances can be monotonically approximated from below by Riemannian ones; cf. the discussion in Remark 4.3.

A previous result on the Mosco-convergence of Cheeger energies was obtained in [6, Theorem 6.8] for sequences of \(\text{CD}(K, \infty)\) spaces that converge with respect to (a variant of) the pointed measured Gromov–Hausdorff topology. However, since measured Gromov–Hausdorff convergence is a zeroth-order concept, while the Cheeger energy is a first-order one, we cannot expect such Mosco-convergence result to hold on arbitrary metric measure spaces. Indeed, given an arbitrary metric measure space \((X, d, m)\), one can easily construct a sequence of discrete measures \((m_i)_{i \in \mathbb{N}}\) that weakly converge to \(m\); consequently, since the Cheeger energies...
associated with the spaces $(X, d, m)$ are identically zero, the Mosco-convergence result will generally fail. In the case of CD$(K, \infty)$ spaces, the convergence of the Cheeger energies is boosted by the uniform lower bound on the Ricci curvature (encoded in the CD condition), which is a second-order notion. Conversely, in our main Theorem 4.1 we do not require any regularity at the level of the involved metric measure spaces, but instead we consider a notion of convergence that is much stronger than the pointed measured Gromov–Hausdorff one.

We conclude the introduction by briefly describing an approximation result for Lipschitz functions (Proposition 3.3) that will have an essential role in the proof of Theorem 4.1. Under the same assumptions as in the Mosco-convergence result for Cheeger energies, we prove that every $d$-Lipschitz function $f$ can be approximated (in the integral sense) by a $d_i$-Lipschitz function $g$, for some index $i \in \mathbb{N}$ sufficiently large, such that the integral of the $p$-power of the asymptotic slope of $g$ is close to that of $f$. This goal is achieved by appealing to the asymptotic-slope-preserving extension result for Lipschitz functions obtained in [4].

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2. Preliminaries

Let $(X, d)$ be a given metric space. We denote by $\tau(d)$ the topology on $X$ induced by the distance $d$. The open ball and the closed ball of center $x \in X$ and radius $r > 0$ are given by

$$B^d_r(x) := \{ y \in X \mid d(x, y) < r \}, \quad \bar{B}^d_r(x) := \{ y \in X \mid d(x, y) \leq r \},$$

respectively. The space of $d$-Lipschitz functions $f : X \to \mathbb{R}$ will be denoted by $\mathrm{LIP}_d(X)$. Given any $f \in \mathrm{LIP}_d(X)$ and $E \subseteq X$, we denote by $\text{Lip}_d(f; E) \in [0, \infty)$ and $\text{lip}_d^d(f) : X \to [0, +\infty)$ the Lipschitz constant of $f|_E$ and the asymptotic slope of $f$, respectively. Videlicet, we set

$$\text{Lip}_d(f; E) := \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} \mid x, y \in E, x \neq y \right\},$$

$$\text{lip}_d^d(f)(x) := \inf_{r > 0} \text{Lip}_d(f; B^d_r(x)), \quad \text{for every } x \in X,$$

where we adopt the convention that $\text{Lip}_d(f; \emptyset) = \text{Lip}_d(f; \{x\}) := 0$. For the sake of brevity, we will use the shorthand notation $\text{Lip}_d(f) := \text{Lip}_d(f; X)$. Observe that $\text{lip}_d^d(f) \leq \text{Lip}_d(f)$.

Remark 2.1. Let $X$ be a non-empty set. Let $d$ and $d'$ be distances on $X$ such that $d \leq d'$. Then for any $x, x', y, y' \in X$ it holds that

$$|d(x, y) - d(x', y')| \leq d(x, x') + d(y, y') \leq d'(x, x') + d'(y, y') \leq \sqrt{2} (d' \times d')( (x, y), (x', y')), $$

thus $d : X \times X \to [0, +\infty)$ is $(d' \times d')$-continuous, where $(d' \times d')$ stands for the product distance

$$(d' \times d')( (x, y), (x', y')) := \sqrt{d'(x, x')^2 + d'(y, y')^2}, \quad \text{for every } (x, y), (x', y') \in X \times X.$$
Moreover, given \( f \in \text{LIP}_d(X) \) and \( E \subseteq X \), we can estimate \( |f(x) - f(y)| \leq \text{Lip}_d(f; E) d(x, y) \) for every \( x, y \in E \). This shows that \( \text{LIP}_d(X) \subseteq \text{LIP}_{d'}(X) \) and that

\[
\text{Lip}_{d'}(f; E) \leq \text{Lip}_d(f; E), \quad \text{for every } f \in \text{LIP}_d(X) \text{ and } E \subseteq X.
\]

In particular, we obtain that \( \text{lip}_{d'}(f) \leq \text{lip}_{d}(f) \) for every \( f \in \text{LIP}_d(X) \).

By a metric measure space \((X, d, m)\) we mean a complete and separable metric space \((X, d)\), which is endowed with a boundedly-finite Borel measure \( m \geq 0 \). One of the possible ways to introduce Sobolev spaces on \((X, d, m)\) is via relaxation of upper gradients. Instead of the original approach that was introduced by Cheeger [2], we present its equivalent reformulation (via relaxation of the asymptotic slope) that was studied by Ambrosio–Gigli–Savaré in [1].

Given a metric measure space \((X, d, m)\) and an exponent \( p \in (1, \infty) \), let us define the asymptotic \( p \)-energy functional \( \mathcal{E}_{a,p}^d : L^p(m) \to [0, +\infty] \) as

\[
\mathcal{E}_{a,p}^d(f) := \begin{cases} \frac{1}{p} \int \text{lip}_d(f)^p \, dm, & \text{if } f \in \text{LIP}_d(X) \text{ is boundedly-supported,} \\ +\infty, & \text{otherwise.} \end{cases}
\]

Then the Cheeger \( p \)-energy functional \( \mathcal{E}_{\text{Ch},p}^d : L^p(m) \to [0, +\infty] \) is defined as the \( L^p(m) \)-lower semicontinuous envelope of \( \mathcal{E}_{a,p}^d \). Videlicet, for any function \( f \in L^p(m) \) we define

\[
\mathcal{E}_{\text{Ch},p}^d(f) := \inf \left\{ \lim_{n \to \infty} \mathcal{E}_{a,p}^d(f_n) \mid (f_n)_n \subseteq L^p(m), f_n \to f \text{ strongly in } L^p(m) \right\}.
\]

It turns out that \( \mathcal{E}_{\text{Ch},p}^d \) is weakly lower semicontinuous, meaning that \( \mathcal{E}_{\text{Ch},p}^d(f) \leq \liminf_{n \to \infty} \mathcal{E}_{\text{Ch},p}^d(f_n) \) whenever \( f \in L^p(m) \) and \((f_n)_n \subseteq L^p(m)\) satisfy \( f_n \to f \) weakly in \( L^p(m) \). The \( p \)-Sobolev space on \((X, d, m)\) is then defined as the finiteness domain of \( \mathcal{E}_{\text{Ch},p}^d \), videlicet

\[
W^{1,p}(X) := \{ f \in L^p(m) \mid \mathcal{E}_{\text{Ch},p}^d(f) < +\infty \}.
\]

It holds that \( W^{1,p}(X) \) is a Banach space if endowed with the following norm:

\[
\|f\|_{W^{1,p}(X)} := \left( \|f\|_{L^p(m)}^p + p \mathcal{E}_{\text{Ch},p}^d(f) \right)^{1/p}, \quad \text{for every } f \in W^{1,p}(X).
\]

In general, the \( 2 \)-Sobolev space is not Hilbert. A metric measure space \((X, d, m)\) is said to be infinitesimally Hilbertian [5] provided the associated \( 2 \)-Sobolev space \( W^{1,2}(X) \) is Hilbert, or equivalently provided \( \mathcal{E}_{\text{Ch},2}^d \) is a quadratic form.

**Remark 2.2.** Let \((X, d, m)\) be a metric measure space. Let \( d' \) be a distance on \( X \) with \( d \leq d' \) and \( \tau(d) = \tau(d') \), thus \((X, d', m)\) is a metric measure space as well. Then Remark 2.1 yields

\[
\mathcal{E}_{a,p}^{d'} \leq \mathcal{E}_{a,p}^d, \quad \mathcal{E}_{\text{Ch},p}^{d'} \leq \mathcal{E}_{\text{Ch},p}^d,
\]

for any given exponent \( p \in (1, \infty) \).■
3. An approximation result

Aim of this section is to achieve an approximation result for Lipschitz functions (i.e., Proposition 3.3), which will be a key tool in order to prove our main Theorem 4.1.

Remark 3.1. Let $X$ be a non-empty set and $(d_i)_{i \in \mathbb{N}}$ a sequence of distances on $X$ satisfying $d_i(x, y) \not\succ d_\infty(x, y)$, for every $x, y \in X$.

Then $d_i \to d_\infty$ uniformly on each subset of $X \times X$ that is compact with respect to $\tau(d_\infty \times d_\infty)$. Indeed, Remark 2.1 grants that $d_i : X \times X \to \mathbb{R}$ is $(d_\infty \times d_\infty)$-continuous for all $i \in \mathbb{N}$. \hfill \blacksquare

We begin with a preliminary approximation result, where the given Lipschitz function is uniformly approximated on a compact set and just the global Lipschitz constant is controlled.

Lemma 3.2. Let $(X, d)$ be a metric space. Suppose there exists a sequence $(d_i)_{i \in \mathbb{N}}$ of distances on $X$ such that $d_i(x, y) \not\succ d(x, y)$ as $i \to \infty$ for every $x, y \in X$. Let $f \in \text{LIP}_d(X)$ be given. Then for any $K \subseteq X$ compact and $\varepsilon > 0$ there exist $i \in \mathbb{N}$ and $g \in \text{LIP}_{d_i}(X)$ such that

$$\max_{K} |g - f| \leq \varepsilon,$$

$$\text{Lip}_{d_i}(g) \leq \text{Lip}_d(f). \hspace{1cm} (3.1b)$$

Proof. Call $L := \text{Lip}_d(f)$ and fix a dense sequence $(x_j)_{j \in \mathbb{N}}$ in $K$. Given any $n \in \mathbb{N}$, we define

$$\tilde{g}_n(x) := \left( -L d(x, x_1) + f(x_1) \right) \lor \cdots \lor \left( -L d(x, x_n) + f(x_n) \right) - \frac{1}{n}, \quad \text{for every } x \in X.$$

Note that $(\tilde{g}_n)_{n \in \mathbb{N}} \subseteq \text{LIP}_{d_i}(X)$ and $\tilde{g}_n \leq \tilde{g}_{n+1} \leq f$ for all $n \in \mathbb{N}$. We claim that $\tilde{g}_n(x) \to f(x)$ as $n \to \infty$ for every $x \in K$. To prove it, fix $x \in K$ and $\delta > 0$. Pick $\tilde{n} \in \mathbb{N}$ such that $1/\tilde{n} \leq \delta$ and $d(x, x_{\tilde{n}}) \leq \delta$. Then for any $n \geq \tilde{n}$ it holds that

$$\tilde{g}_n(x) \geq -L d(x, x_{\tilde{n}}) + f(x_{\tilde{n}}) - \frac{1}{n} \geq f(x) - 2L d(x, x_{\tilde{n}}) - \frac{1}{n} \geq f(x) - (2L + 1)\delta,$$

which grants that $\tilde{g}_n(x) \succ f(x)$ by arbitrariness of $\delta$. Therefore, we have that $\tilde{g}_n \to f$ uniformly on $K$, so that there exists $n \in \mathbb{N}$ for which the function $\tilde{g} := \tilde{g}_n$ satisfies $|\tilde{g} - f| \leq \varepsilon/2$ on $K$. Given any $i \in \mathbb{N}$, let us define the function $g_i \in \text{LIP}_{d_i}(X)$ as

$$g_i(x) := \left( -L d_i(x, x_1) + f(x_1) \right) \lor \cdots \lor \left( -L d_i(x, x_n) + f(x_n) \right) - \frac{1}{n}, \quad \text{for every } x \in X.$$

Note that $g_i \not\succ \tilde{g}$ pointwise on $K$, as a consequence of the assumption $d_i \not\succ d$. Since each $g_i$ is continuous with respect to $d_i$, we deduce that $g_i \to \tilde{g}$ uniformly on $K$, thus for some $i \in \mathbb{N}$ the function $g := g_i$ satisfies $|g - \tilde{g}| \leq \varepsilon/2$ on $K$. Hence, it holds that $|g - f| \leq \varepsilon$ on $K$, yielding (3.1a). Finally, we have that $\text{Lip}_{d_i}(g) \leq L = \text{Lip}_d(f)$ by construction, whence (3.1b) and accordingly the statement follow. \hfill \square

By combining Lemma 3.2 with a partition of unity argument and the extension result in [4], we show that also the asymptotic slope can be kept under control (in an integral sense).
Proposition 3.3. Let \((X, d, m)\) be a metric measure space. Suppose to have a sequence \((d_i)_{i \in \mathbb{N}}\) of distances on \(X\) such that \(d_i(x, y) /\!\!/ d(x, y)\) as \(i \to \infty\) for every \(x, y \in X\) and \(\tau (d_i) = \tau (d)\) for every \(i \in \mathbb{N}\). Fix an exponent \(p \in (1, \infty)\) and a boundedly-supported function \(f \in \text{LIP}_d(X)\). Then for any \(\varepsilon > 0\) there exist \(i \in \mathbb{N}\) and \(g \in \text{LIP}_{d_i}(X)\) boundedly-supported such that

\[
\int |g - f|^p \, dm \leq \varepsilon, \tag{3.2a}
\]

\[
\int \text{lip}_{d_i}(g)^p \, dm \leq \int \text{lip}_d(f)^p \, dm + \varepsilon. \tag{3.2b}
\]

Proof. First of all, fix a point \(\bar{x} \in X\) and a radius \(R > 0\) such that \(\text{spt}(f) \subseteq B^d_{R^n}(\bar{x})\). Denote by \(B\) the ball \(B^d_{R^n}(\bar{x})\). Moreover, fix any \(\varepsilon' \in (0, 1/4)\) such that

\[
\left[ \left( 3p \text{ Lip}_d(f)^{p-1} + 1 \right) m(B) + \left( 15 \text{ Lip}_d(f) + \sup_x |f| + 7 \right)^p \right] \varepsilon' \leq \varepsilon. \tag{3.3}
\]

**Step 1: Construction of the auxiliary function \(\tilde{h}\).** Since \(X \ni x \mapsto \text{LIP}(f; B^d_{1/n}(x))\) is a Borel function for any \(n \in \mathbb{N}\) and \(\text{lip}_d(f)(x) = \lim_{n \to \infty} \text{LIP}(f; B^d_{1/n}(x))\) for every \(x \in X\), by virtue of Egorov’s theorem there exist \(K \subseteq B\) compact and \(r > 0\) with \(m(B \setminus K) \leq \varepsilon'\) and

\[
\text{LIP}(f; B^d_{4r}(x)) \leq \text{lip}_d(f)(x) + \varepsilon', \quad \text{for every } x \in K. \tag{3.4}
\]

Choose some points \(x_1, \ldots, x_k \in K\) for which \(K \subseteq \bigcup_{j=1}^k B^d_r(x_j)\). Fix a \(d_1\)-Lipschitz partition of unity \(\{\psi_1, \ldots, \psi_k\}\) of \(K\) subordinated to \(\{K \cap B^d_r(x_1), \ldots, K \cap B^d_r(x_k)\}\). Videlicet, each function \(\psi_j : K \to [0, 1]\) is \(d_1\)-Lipschitz, satisfies \(\text{spt}(\psi_j) \subseteq K \cap B^d_r(x_j)\), and \(\sum_{j=1}^k \psi_j(x) = 1\) for every \(x \in K\). Since \(d_i \to d\) uniformly on \(K \times K\) (by Remark 3.1), there exists \(i_0 \in \mathbb{N}\) such that \(d(x, y) \leq d_i(x, y) + \varepsilon'r\) for every \(x, y \in K\) and \(i \geq i_0\). Given any \(j = 1, \ldots, k\), pick some function \(f_j \in \text{LIP}_d(X)\) such that \(f_j|_{B^d_{3r}(x_j)} = f|_{B^d_{3r}(x_j)}\) and \(\text{LIP}_d(f_j) = \text{LIP}_d(f; B^d_{2r}(x_j))\), thus we can find (by Lemma 3.2) an index \(i(j) \geq i_0\) and a function \(h_j \in \text{LIP}_{d_i}(X)\) such that

\[
|h_j(x) - f_j(x)| \leq \frac{\varepsilon'}{k \text{ Lip}_d(\psi_j)} \vee 1, \quad \text{for every } x \in K, \tag{3.5a}
\]

\[
\text{LIP}_{d_i}(h_j) \leq \text{LIP}_d(f_j) = \text{LIP}_d(f; B^d_{2r}(x_j)). \tag{3.5b}
\]

Let us denote \(i := \max \{i(1), \ldots, i(k)\}\) and \(d := d_i|_{K \times K}\). Moreover, we define \(\tilde{h} : K \to \mathbb{R}\) as

\[
\tilde{h}(x) := \sum_{j=1}^k \psi_j(x) h_j(x), \quad \text{for every } x \in K.
\]

**Step 2: Estimates for the Lipschitz constant of \(\tilde{h}\).** We claim that \(\tilde{h} \in \text{LIP}_d(K)\) and that \(\text{LIP}_d(\tilde{h}) \leq \varepsilon' + 5 \text{ Lip}_d(f)\). In order to prove it, fix any \(y, z \in K\). Then we have that

\[
|\tilde{h}(y) - \tilde{h}(z)| \leq \left| \sum_{j=1}^k \psi_j(y) (h_j(y) - h_j(z)) \right| + \left| \sum_{j=1}^k (\psi_j(y) - \psi_j(z)) (h_j(z) - f(z)) \right| \tag{3.6}
\]

\[
\leq \sum_{j=1}^k \psi_j(y) |h_j(y) - h_j(z)| + \sum_{j=1}^k |\psi_j(y) - \psi_j(z)| |h_j(z) - f(z)|.
\]
Observe that the first term in the second line of the above formula can be estimated as

$$\sum_{j=1}^{k} \psi_j(y) |h_j(y) - h_j(z)| \leq \sum_{j=1}^{k} \psi_j(y) \text{Lip}_d(h_j) d_i(y, z) \leq \text{Lip}_d(f) d_i(y, z). \quad (3.7)$$

In order to estimate the second term in (3.6), fix $j = 1, \ldots, k$. We consider three cases:

i) If $z \in B_{2r}(x_j)$, then $f(z) = f_j(z)$ and accordingly

$$\left| \psi_j(y) - \psi_j(z) \right| |h_j(z) - f(z)| \leq \text{Lip}_d(\psi_j) d_i(y, z) \frac{\varepsilon'}{k \text{Lip}_d(\psi_j)} \leq \frac{\varepsilon'}{k} d_i(y, z). \quad (3.5a)$$

ii) If $z \notin B_{2r}(x_j)$ and $y \in B_{r}(x_j)$, then $f_j(y) = f(y)$ and $d(y, z) > r$. In particular,

$$\frac{d(y, z)}{d_i(y, z)} \leq \frac{d_i(y, z) + \varepsilon' r}{d_i(y, z)} \leq 1 + \frac{\varepsilon' r}{d(y, z) - \varepsilon' r} < 1 + \frac{\varepsilon'}{1 - \varepsilon'} < 2, \quad (3.8)$$

whence it follows that

$$\left| \psi_j(y) - \psi_j(z) \right| |h_j(z) - f(z)| \leq \left| \psi_j(y) - \psi_j(z) \right| \left[ |h_j(z) - f_j(z)| + |f_j(z) - f_j(y)| + |f_j(y) - f(z)| \right] \leq \text{Lip}_d(\psi_j) d_i(y, z) |h_j(z) - f_j(z)| + \psi_j(y) |f_j(z) - f_j(y)| + \psi_j(y) |f(y) - f(z)| \leq \text{Lip}_d(\psi_j) d_i(y, z) \frac{\varepsilon'}{k \text{Lip}_d(\psi_j)} + \psi_j(y) \text{Lip}_d(f) d(y, z) + \psi_j(y) \text{Lip}_d(f) d_i(y, z) \leq \frac{\varepsilon'}{k} d_i(y, z) + 2 \psi_j(y) \text{Lip}_d(f) d(y, z) \leq \left( \frac{\varepsilon'}{k} + 4 \psi_j(y) \text{Lip}_d(f) \right) d_i(y, z). \quad (3.5b)$$

iii) If $z \notin B_{2r}(x_j)$ and $y \notin B_{r}(x_j)$, then trivially $|\psi_j(y) - \psi_j(z)| |h_j(z) - f(z)| = 0$.

By combining the estimates we obtained in i), ii), iii) with (3.7) and (3.6), we deduce that

$$|\tilde{h}(y) - \tilde{h}(z)| \leq (\varepsilon' + 5 \text{Lip}_d(f)) d_i(y, z), \quad \text{for every } y, z \in K.$$

This proves that $\tilde{h} \in \text{Lip}_d(K)$ and $\text{Lip}_d(\tilde{h}) \leq \varepsilon' + 5 \text{Lip}_d(f)$, yielding the sought conclusion.

**Step 3: Estimates for the asymptotic slope of $\tilde{h}$**. Next we claim that

$$\text{lip}_d(\tilde{h})(x) \leq \text{lip}_d(f)(x) + 2\varepsilon', \quad \text{for every } x \in K. \quad (3.9)$$

To prove it, fix any $\delta < \varepsilon' r$ and $y, z \in B_{\delta}(x)$. Define $F := \{ j = 1, \ldots, k : d(x, x_j) < 3r/2 \}$.

If $j \notin F$, then $y, z \notin B_{\delta}(x_j)$ and thus $\psi_j(y) = \psi_j(z) = 0$, as it is granted by the estimates

$$d(y, x_j) \geq d(x, x_j) - d(x, y) \geq \frac{3r}{2} - d_i(x, y) - \varepsilon' r > \left( \frac{3}{2} - \varepsilon' \right) r - \delta > \left( \frac{3}{2} - 2\varepsilon' \right) r > r,$$

and similarly for $d(z, x_j)$. If $j \in F$, then $B_{\delta}(x_j) \subseteq B_{3r/2}(x)$ and $f_j(z) = f(z)$. The latter claim follows from the fact that $z \in B_{2r}(x_j)$, which is granted by the estimates

$$d(z, x_j) \leq d(z, x) + d(x, x_j) < d_i(z, x) + \varepsilon' r + \frac{3r}{2} < \delta + \left( \varepsilon' + \frac{3}{2} \right) r < \left( 2\varepsilon' + \frac{3}{2} \right) r < 2r.$$
Therefore, by using (3.6) and the above considerations, we obtain that
\[
\begin{align*}
|\tilde{h}(y) - \tilde{h}(z)| & \leq \sum_{j \in F} \psi_j(y) \text{Lip}_d(h_j; d_{i(j)}(y, z)) + \sum_{j \in F} \text{Lip}_d(\psi_j) \text{d}_i(y, z) \frac{\varepsilon'}{k \text{Lip}_d(\psi_j)} \\
& \leq \left[ \sum_{j \in F} \psi_j(y) \text{Lip}_d(f; B_d^2(x_j)) + \varepsilon' \right] \text{d}_i(y, z) \\
& \leq \left[ \text{Lip}_d(f; B_d^2(x)) + \varepsilon' \right] \text{d}_i(y, z) \leq \frac{\text{lip}_d^d(f) + 2\varepsilon'}{\text{Lip}_d(\tilde{h}; B_d^2(x))} \text{d}_i(y, z).
\end{align*}
\]

Thanks to the arbitrariness of \(y, z \in B_d^2(x)\), we deduce that \(\text{Lip}_d(h; B_d^2(x)) \leq \text{lip}_d^d(f) + 2\varepsilon'\), whence by letting \(\delta \searrow 0\) we can finally conclude that the inequality in (3.9) is verified.

**Step 4: Construction of the function \(g\).** Given any point \(x \in K\), it holds that
\[
|\tilde{h}(x) - f(x)| \leq \sum_{j=1}^{k} \psi_j(x) |h_j(x) - f(x)| = \sum_{j=1}^{k} \psi_j(x) |h_j(x) - f_j(x)| \leq \varepsilon' \tag{3.10}
\]
In particular, we have that \(\sup_K |\tilde{h}| \leq \sup_X |f| + 1\). Recall also that \(\text{Lip}_d^d(\tilde{h}; B_d^2(x)) \leq \text{lip}_d^d(f) + \varepsilon'\), as proven in Step 2. Therefore, by applying [4, Theorem 1.1] we can find a function \(h \in \text{LIP}_d(X)\) with \(h|_K = \tilde{h}\) such that \(\text{lip}_d^d(h)(x) = \text{lip}_d^d(\tilde{h})(x)\) for every \(x \in K\) and
\[
\text{Lip}_d^d(h) \leq \text{Lip}_d^d(\tilde{h}) + \varepsilon' \leq 5 \text{Lip}_d(f) + 2\varepsilon' =: C. \tag{3.11}
\]
Define \(G := \{x \in X : d_i(x, \text{spt}(f) \cap K) \leq 2\}\) and observe that \(\sup_{G} |h| \leq 2C + \sup_X |f| + 1\). Indeed, given any point \(x \in G\), one has that
\[
|h(x)| \leq \inf_{y \in \text{spt}(f) \cap K} \left[ |h(x) - h(y)| + |h(y)| \right] \leq \text{Lip}_d^d(h) \inf_{y \in \text{spt}(f) \cap K} d_i(x, y) + \sup_K |h| \leq C d_i(x, \text{spt}(f) \cap K) + \sup_K |\tilde{h}| \leq 2C + \sup_X |f| + 1. \tag{3.11}
\]
Moreover, we have that \(G \subseteq B = B_{d_i}^{d_i}(\bar{x})\). Indeed, by using that \(\text{spt}(f) \subseteq B_{d_i}^{d_i}(\bar{x})\), we get
\[
d_i(x, \bar{x}) \leq \inf_{y \in \text{spt}(f) \cap K} [d_i(x, y) + d_i(y, \bar{x})] \leq \inf_{y \in \text{spt}(f) \cap K} d_i(x, y) + R \leq R + 2,
\]
for every \(x \in G\). Let us now define the \(d_i\)-Lipschitz cut-off function \(\eta: X \to [0, 1]\) as
\[
\eta(x) := \left( (2 - d_i(x, \text{spt}(f) \cap K)) \wedge 1 \right) \vee 0, \quad \text{for every } x \in X.
\]
It holds that \(\eta = 1\) on a neighbourhood of \(\text{spt}(f) \cap K\) and that \(\text{Lip}_d^d(\eta) \leq 1\). Given that \(\eta = 0\) in \(X \setminus G\), it also holds that \(\text{spt}(\eta) \subseteq G\). We then define the function \(g: X \to \mathbb{R}\) as \(g := \eta h\).

**Step 5: Conclusion.** Note that \(g \in \text{LIP}_d(X), \text{spt}(g) \subseteq G, \text{ and } \sup_X |g| \leq 2C + \sup_X |f| + 1\). Let us estimate \(\text{Lip}_d^d(g)\). Since \(|g(x) - g(y)| \leq |\eta(x)| |h(x) - h(y)| + |\eta(x) - \eta(y)||h(y)|\) holds for every \(x, y \in X\), we obtain that \(|g(x) - g(y)| \leq (C + \sup_G |h|) d_i(x, y)\) whenever \(y \in G\), whence it follows that \(\text{Lip}_d^d(g) \leq 3C + \sup_X |f| + 1\). The same computations give
\[
\text{Lip}_d^d(g; E) \leq \text{Lip}_d^d(h; E) + \sup_E |h|, \quad \text{for every } E \subseteq X. \tag{3.12}
\]
On the one hand, since $g$ and $h$ agree on a neighbourhood of $\text{spt}(f) \cap K$, for any $x \in \text{spt}(f) \cap K$ we have that $|g(x) - f(x)| \leq \varepsilon'$ by (3.10) and $\operatorname{lip}_a^d(g)(x) \leq \operatorname{lip}_a^d(f)(x) + 2\varepsilon'$ by (3.9). On the other hand, if $x \in K \setminus \text{spt}(f)$, then $f(x) = \operatorname{lip}_a^d(f)(x) = 0$, thus accordingly we can deduce from (3.10) that $|g(x) - f(x)| = \eta(x)|h(x)| \leq \varepsilon'$, while (3.9), (3.10), and (3.12) ensure that

$$\operatorname{lip}_a^d(g)(x) = \lim_{\delta \searrow 0} \operatorname{Lip}_a^d(g; B^d_\delta(x)) \leq \lim_{\delta \searrow 0} \operatorname{Lip}_a^d(h; B^d_\delta(x)) + \lim_{\delta \searrow 0} \sup_{B^d_\delta(x)} |h|
= \operatorname{lip}_a^d(h)(x) + |h(x)| \leq 3\varepsilon'.$$

All in all, we have shown that

$$|g(x) - f(x)| \leq \begin{cases} \varepsilon', & \text{if } x \in K, \\ 2C + \sup_X |f| + 1, & \text{if } x \in X \setminus K, \end{cases} \tag{3.13a}$$

$$\operatorname{lip}_a^d(g)(x) \leq \begin{cases} \operatorname{lip}_a^d(f)(x) + 3\varepsilon', & \text{if } x \in K, \\ 3C + \sup_X |f| + 1, & \text{if } x \in X \setminus K. \tag{3.13b} \end{cases}$$

It remains to check that $g$ satisfies (3.2a) and (3.2b). Recall that $\text{spt}(f), \text{spt}(g) \subseteq B$. Then

$$\int |g - f|^p \, dm = \int_K |g - f|^p \, dm + \int_{B \setminus K} |g - f|^p \, dm \leq m(K) (\varepsilon')^p + m(B \setminus K) \left(2C + \sup_X |f| + 1\right)^p \leq \left[m(B) + \left(2C + \sup_X |f| + 1\right)^p\right] \varepsilon'. $$

Moreover, it holds that

$$\int \operatorname{lip}_a^d(g)^p \, dm = \int_K \operatorname{lip}_a^d(g)^p \, dm + \int_{B \setminus K} \operatorname{lip}_a^d(g)^p \, dm \leq \int_K \left(\operatorname{lip}_a^d(f)^p + 3\varepsilon'\right)^p \, dm + m(B \setminus K) \left(3C + \sup_X |f| + 1\right)^p \leq \int \operatorname{lip}_a^d(f)^p \, dm + 3p\varepsilon' \int_B \operatorname{lip}_a^d(f)^{p-1} \, dm + \left(3C + \sup_X |f| + 1\right)^p \varepsilon' \leq \int \operatorname{lip}_a^d(f)^p \, dm + \left[3p \operatorname{Lip}_a^d(f)^{p-1} m(B) + \left(3C + \sup_X |f| + 1\right)^p\right] \varepsilon'. $$

By taking (3.3) into account, we can finally conclude that (3.2a) and (3.2b) are verified. \hfill \Box

4. Mosco-convergence of Cheeger energies

By applying Proposition 3.3, we can easily obtain our main $\Gamma$-convergence result.

**Theorem 4.1.** Let $(X, \mathbf{d}, \mathbf{m})$ be a metric measure space. Let $(\mathbf{d}_i)_{i \in \mathbb{N}}$ be a sequence of complete distances on $X$ such that $\mathbf{d}_i \nearrow \mathbf{d}$ as $i \to \infty$. Suppose $\tau(\mathbf{d}_i) = \tau(\mathbf{d})$ for all $i \in \mathbb{N}$. Fix $p \in (1, \infty)$. Then $\mathcal{E}^{\mathbf{d}_i}_{\text{Ch},p}$ Mosco-converges to $\mathcal{E}^{\mathbf{d}}_{\text{Ch},p}$ as $i \to \infty$. Videlicet, the following properties hold:
i) **Weak $\Gamma$-lim inf.** If $(f_i)_{i \in \mathbb{N}} \subseteq L^p(m)$ weakly converges to $f \in L^p(m)$, then it holds

$$
\mathcal{E}_{Ch,p}^d(f) \leq \lim_{i \to \infty} \mathcal{E}_{Ch,p}^d(f_i).
$$

ii) **Strong $\Gamma$-lim sup.** Given any $f \in L^p(m)$, there exists a sequence $(f_i)_{i \in \mathbb{N}} \subseteq L^p(m)$ that strongly converges to $f$ and satisfies

$$
\mathcal{E}_{Ch,p}^d(f) \geq \lim_{i \to \infty} \mathcal{E}_{Ch,p}^d(f_i).
$$

**Proof.** Item i) can be easily proven: given any $f \in L^p(m)$ and $(f_i)_{i \in \mathbb{N}} \subseteq L^p(m)$ with $f_i \to f$ weakly in $L^p(m)$, the weak lower semicontinuity of $\mathcal{E}_{Ch,p}^d: L^p(m) \to [0, +\infty]$ grants that

$$
\mathcal{E}_{Ch,p}^d(f) \leq \lim_{i \to \infty} \mathcal{E}_{Ch,p}^d(f_i) \quad \text{(2.1)}
$$

Let us then pass to the verification of item ii). Let $f \in L^p(m)$ be given. If $f \notin W^{1,p}(X)$, then $\mathcal{E}_{Ch,p}^d(f) = +\infty$ and accordingly the $\Gamma$-lim sup inequality is trivially verified (by taking, for instance, $f_i := f$ for every $i \in \mathbb{N}$). Now suppose $f \in W^{1,p}(X)$. By definition of $\mathcal{E}_{Ch,p}^d$, we can find a sequence $(\tilde{f}_n)_n \subseteq \operatorname{LIP}_p(X)$ of boundedly-supported functions such that $\tilde{f}_n \to f$ strongly in $L^p(m)$ and $\mathcal{E}_{Ch,p}^d(f) = \lim_n \mathcal{E}_{a,p}^d(\tilde{f}_n)$. By Proposition 3.3, we can find $\iota: \mathbb{N} \to \mathbb{N}$ increasing and a sequence $(g_n)_n$ of boundedly-supported functions $g_n \in \operatorname{LIP}_{p,1}(X)$ such that

$$
\int |g_n - \tilde{f}_n|^p \, dm \leq \frac{1}{n},
$$

$$
\mathcal{E}_{a,p}^d(g_n) \leq \mathcal{E}_{a,p}^d(\tilde{f}_n) + \frac{1}{n}.
$$

In particular, $g_n \to f$ strongly in $L^p(m)$ and $\mathcal{E}_{Ch,p}^d(f) \geq \lim_n \mathcal{E}_{Ch,p}^d(g_n) \geq \lim_n \mathcal{E}_{Ch,p}^d(\tilde{f}_n)$. Finally, we define the recovery sequence $(f_i)_i \subseteq L^p(m)$ in the following way:

$$
(f_i) := g_{\iota(n)}, \quad \text{for every } n \in \mathbb{N} \text{ and } i \in \{\iota(n), \ldots, \iota(n+1) - 1\}.
$$

Notice that $f_i \to f$ strongly in $L^p(m)$. Moreover, Remark 2.2 grants that $\mathcal{E}_{Ch,p}^d(f_i) \leq \mathcal{E}_{Ch,p}^d(g_n)$ whenever $\iota(n) \leq i < \iota(n+1)$, which implies that $\lim_i \mathcal{E}_{Ch,p}^d(f_i) = \lim_n \mathcal{E}_{Ch,p}^d(g_n) \leq \mathcal{E}_{Ch,p}^d(f)$. This gives the $\Gamma$-lim sup inequality, thus accordingly the statement is achieved. 

It readily follows from Theorem 4.1 that the infinitesimal Hilbertianity condition is stable under taking increasing limits of the distances (while keeping the measure fixed). Videlicet:

**Corollary 4.2.** Let $(X, d, m)$ be a metric measure space. Let $(d_i)_{i \in \mathbb{N}}$ be a sequence of complete distances on $X$ such that $d_i \rightarrow d$ as $i \to \infty$ and $\tau(d_i) = \tau(d)$ for every $i \in \mathbb{N}$. Suppose $(X, d_i, m)$ is infinitesimally Hilbertian for every $i \in \mathbb{N}$. Then $(X, d, m)$ is infinitesimally Hilbertian.

**Proof.** Theorem 4.1 implies that $\mathcal{E}_{Ch,2}^d \Gamma \mathcal{E}_{Ch,2}^d$ with respect to the strong topology of $L^2(m)$, thus $[3, \text{Theorem 11.10}]$ grants that $\mathcal{E}_{Ch,2}^d$ is a quadratic form, which gives the statement.

**Remark 4.3.** Let $(M, d)$ be the metric space associated with a generalised sub-Riemannian manifold, in the sense of [7, Definition 4.1]. Then there exists a sequence $(d_i)_{i \in \mathbb{N}}$ of distances on $M$, induced by Riemannian metrics, such that $d_i \rightarrow d$; cf. [7, Corollary 5.2]. Suppose $d$ and each $d_i$ are complete distances. Fix a Radon measure $m$ on $M$. Then [8, Theorem 4.11]
ensures that each \((M, d_i, m)\) is infinitesimally Hilbertian. Therefore, by applying Corollary 4.2 we can conclude that \((M, d, m)\) is infinitesimally Hilbertian as well. This argument provides an alternative proof of [7, Corollary 5.6].

We conclude the paper by illustrating an example which shows that the results of this section cannot hold if the assumption of monotone convergence from below of the distances is replaced by a monotone convergence from above.

**Example 4.4.** Let \((X, d, m)\) be any metric measure space such that \(d \leq 1\). Given any \(i \in \mathbb{N}\), we define the ‘snowflake’ distance \(d_i\) on \(X\) as \(d_i(x, y) := d(x, y)^{1 - \frac{1}{i}}\) for every \(x, y \in X\). Then we have \(d_i(x, y) \searrow d(x, y)\) as \(i \to \infty\) for all \(x, y \in X\) and \(\tau(d_i) = \tau(d)\) for all \(i \in \mathbb{N}\). Since absolutely continuous curves in \((X, d_i)\) are constant, it follows from the results in [1] that

\[
\mathcal{E}^i_{Ch,p}(f) = 0, \quad \text{for every } p \in (1, \infty) \text{ and } f \in L^p(m).
\]

In particular, each space \((X, d_i, m)\) is infinitesimally Hilbertian. This shows that Theorem 4.1 and Corollary 4.2 might fail if we replace the assumption \(d_i \nearrow d\) with \(d_i \searrow d\). ■

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