Curvature estimates for properly immersed $\phi_h$-bounded submanifolds

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Abstract

Jorge-Koutrofiotis [14] & Pigola-Rigoli-Setti [23] proved sharp sectional curvature estimates for extrinsically bounded submanifolds. Alias, Bessa and Montenegro in [27], showed that these estimates hold on properly immersed cylindrically bounded submanifolds. On the other hand, in [11], Alias, Bessa and Dajczer proved sharp mean curvature estimates for properly immersed cylindrically bounded submanifolds. In this paper we prove these sectional and mean curvature estimates for a larger class of submanifolds, the properly immersed $\phi$-bounded submanifolds, Thms. [2.3 & 2.5] These ideas, in fact, we prove stronger forms of these estimates, see the results in section 4.

keywords: Curvature estimates, $\phi$-bounded submanifolds, Omori-Yau pairs, Omori-Yau maximum principle.

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1 Introduction

The classical isometric immersion problem asks whether there exists an isometric immersion $\phi: M \to N$ for given Riemannian manifolds $M$ and $N$ of dimension $m$ and $n$ respectively, with $m < n$. The model result for this type of problem is the celebrated Efimov-Hilbert Theorem [11], [13] that says that there is no isometric immersion of a geodesically complete surface $M$ with sectional curvature $K_M \leq -\delta^2 < 0$ into $\mathbb{R}^3$, $\delta \in \mathbb{R}$. On the other hand, the Nash Embedding Theorem shows that there is always an isometric embedding into the Euclidean $n$-space $\mathbb{R}^n$ provided the codimension $n - m$ is sufficiently large, see [17]. For small codimension, meaning in this paper that $n - m \leq m - 1$, the answer in general depends on the geometries of $M$ and $N$. For instance, a classical result of C. Tompkins [27] states that a compact, flat, $m$-dimensional Riemannian manifold can not be isometrically immersed into $\mathbb{R}^{2m-1}$. C. Tompkin’s result was extended in a series of papers, by

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Chern and Kuiper [9], Moore [16], O’Neill [19], Otsuki [20] and Stiel [25], whose results can be summarized in the following theorem.

**Theorem 1.1 (C. Tompkins et al.)** Let \( \varphi : M \to N \) be an isometric immersion of compact Riemannian \( m \)-manifold \( M \) into a Cartan-Hadamard \( n \)-manifold \( N \) with small codimension \( n - m \leq m - 1 \). Then the sectional curvatures of \( M \) and \( N \) satisfy

\[
\sup_M K_M > \inf_N K_N. \tag{1}
\]

L. Jorge and D. Koutrofiotis [14], considered complete extrinsically bounded submanifolds with scalar curvature bounded from below and proved the curvature estimates (3). Pigola, Rigoli and Setti [23] proved an all general and abstract version of the Omori-Yau maximum principle [8], [28] and in consequence they were able to extend Jorge-Koutrofiotis’ Theorem to complete \( m \)-submanifolds \( M \) immersed into regular balls of any Riemannian \( n \)-manifold \( N \) with scalar curvature bounded below as \( s_M \geq -c \cdot \rho_M^2 \cdot \prod_{j=1}^{k} \left( \log^{(j)}(\rho_M) \right)^2 \), \( \rho_M \gg 1 \).

Their version of Jorge-Koutrofiotis Theorem is the following.

**Theorem 1.2 (Jorge-Koutrofiotis & Pigola-Rigoli-Setti)** Let \( \varphi : M \to N \) be an isometric immersion of a compact Riemannian \( m \)-manifold \( M \) into a \( n \)-manifold \( N \), with \( n - m \leq m - 1 \), with \( \varphi(M) \subset B_N(r) \), where \( B_N(r) \) is a regular geodesic ball of \( N \). If the scalar curvature of \( M \) satisfies

\[
s_M \geq -c \cdot \rho_M^2 \cdot \prod_{j=1}^{k} \left( \log^{(j)}(\rho_M) \right)^2, \quad \rho_M \gg 1, \tag{2}
\]

for some constant \( c > 0 \) and some integer \( k \geq 1 \), where \( \rho_M \) is the distance function on \( M \) to a fixed point and \( \log^{(j)} \) is the \( j \)-th iterate of the logarithm. Then

\[
\sup_M K_M \geq C^2_b(r) + \inf_{B_N(r)} K_N, \tag{3}
\]

where \( b = \sup_{B_N(r)} K_N^{\text{rad}} \leq b \)

\[
C_b(t) = \begin{cases} 
\sqrt{b} \cot(\sqrt{b} t) & \text{if } b > 0 \text{ and } 0 < t < \pi/2\sqrt{b} \\
1/t & \text{if } b = 0 \text{ and } t > 0 \\
\sqrt{-b} \coth(\sqrt{-b} t) & \text{if } b < 0 \text{ and } t > 0.
\end{cases} \tag{4}
\]

\(^1\)Meaning: immersed into regular geodesic balls of a Riemannian manifold.
Remark 1.3 If $B(r) \subset \mathbb{N}^n(b)$ is a geodesic ball of radius $r$ in the simply connected space form of sectional curvature $b$, $\partial B(r)$ its boundary and $\varphi : \partial B(r - \varepsilon) \rightarrow B(r)$ is the canonical immersion, where $\varepsilon > 0$ is small, then we have

$$
\sup_M K_{M} = K_{\partial B(r - \varepsilon)} = \begin{cases} 
\frac{b}{\sin^2(\sqrt{b}(r - \varepsilon))} & \text{if } b > 0 \\
\frac{1}{(r - \varepsilon)^2} & \text{if } b = 0 \\
\frac{-b}{\sinh^2(\sqrt{-b}(r - \varepsilon))} & \text{if } b < 0.
\end{cases}
$$

Therefore, $\sup_M K_M - [C_b^2(r) + \inf_{B_N(r)} K_{rad}] = [C_b^2(r - \varepsilon) - C_b^2(r)] \rightarrow 0$ as $\varepsilon \rightarrow 0$, showing that the inequality (3) is sharp.

Remark 1.4 One may assume without loss of generality that $\sup_M K_M < \infty$. This together with the scalar curvature bounds (2) implies that

$$
K_M \geq -c^2 \cdot \rho_M^2 \cdot \prod_{j=1}^{k} \left( \log (j)(\rho_M) \right)^2 \cdot \rho_M \gg 1
$$

for some positive constant $c > 0$. This curvature lower bound implies that $M$ is stochastically complete, which is equivalent to the fact that $M$ hold the weak maximum principle, (a weaker form of Omori-Yau maximum principle, see details in [22]), and that is enough to reproduce Jorge-Koutrofitis original proof of the curvature estimate (3).

Recently, Alias, Bessa and Montenegro [2] extended Theorem 1.2 to the class of cylindrically bounded, properly immersed submanifolds, where an isometric immersion $\varphi : M \hookrightarrow N \times \mathbb{R}^\ell$ is cylindrically bounded if $\varphi(M) \subset B_N(r) \times \mathbb{R}^\ell$. Here $B_N(r)$ is a regular geodesic ball in $N$ of radius $r > 0$. They proved the following theorem.

Theorem 1.5 (Alias-Bessa-Montenegro) Let $\varphi : M \rightarrow N \times \mathbb{R}^\ell$ be a cylindrically bounded isometric immersion, $\varphi(M) \subset B_N(r) \times \mathbb{R}^\ell$, where $B_N(r)$ is a regular geodesic ball of $N$ and $b = \sup K_{B_N(r)}^{rad}$. Let $\dim(M) = m$, $\dim(N) = n - \ell$ and assume that $n - m \leq m - \ell - 1$. If either

i. the scalar curvature of $M$ is bounded below as (2), or

ii. the immersion $\varphi$ is proper and

$$
\sup_{\varphi^{-1}(B_N(r) \times \partial B_M(t))} \|\alpha\| \leq \sigma(t),
$$

where $\alpha$ is the second fundamental form of $\varphi$ and $\sigma : [0, +\infty) \rightarrow \mathbb{R}$ is a positive function satisfying $\int_0^{+\infty} dt / \sigma(t) = +\infty$, then

$$
\sup_M K_M \geq C_b^2(r) + \inf_{B_N(r)} K_N.
$$
Remark 1.6 The idea is to show that the hypotheses, in both items i. & ii. implies that \( M \) is stochastically complete, then Remark 1.4 applies.

In the same spirit, Alias, Bessa and Dajczer [1], had proved the following mean curvature estimates for cylindrically bounded submanifolds properly immersed into \( N \times \mathbb{R}^\ell \) immersed submanifolds.

**Theorem 1.7 (Alias-Bessa-Dajczer)** Let \( \varphi : M \to N \times \mathbb{R}^\ell \) be a cylindrically bounded isometric immersion, \( \varphi(M) \subset B_N(r) \times \mathbb{R}^\ell \), where \( B_N(r) \) is a regular geodesic ball of \( N \) and \( b = \sup K_{B_N(r)}^{\text{rad}} \). Here \( M \) and \( N \) are complete Riemannian manifolds of dimension \( m \) and \( n - \ell \) respectively, satisfying \( m \geq \ell + 1 \). If the immersion \( \varphi \) is proper, then

\[
\sup_M |H| \geq (m - \ell) \cdot C_b(r). \tag{7}
\]

2 Main results

The purpose of this paper is to extend these curvature estimates to a larger class of submanifolds, precisely, the properly immersed \( \phi \)-bounded submanifolds. To describe this class we need to introduce few preliminaries.

2.1 \( \phi \)-bounded submanifolds

Consider \( G \in C^\infty([0, \infty)) \) satisfying

\[
G_- \in L^1(\mathbb{R}^+), \quad t \int_0^{+\infty} G_-(s)ds \leq \frac{1}{4} \quad \text{on} \quad \mathbb{R}^+, \tag{8}
\]

and \( h \) the solution of the following differential equation

\[
\begin{cases}
    h''(t) - G(t)h(t) = 0, \\
    h(0) = 0, \quad h'(0) = 1.
\end{cases} \tag{9}
\]

In [6, Prop. 1.21], it is proved that the solution \( h \) and its derivative \( h' \) are positive in \( \mathbb{R}^+ = (0, \infty) \), provided \( G \) satisfies (8) and furthermore \( h \to +\infty \) whenever the stronger condition

\[
G(s) \geq -\frac{1}{4s^2} \quad \text{on} \quad \mathbb{R}^+ \tag{10}
\]

holds. Define \( \phi_h \in C^\infty([0, \infty)) \) by

\[
\phi_h(t) = \int_0^t h(s)ds. \tag{11}
\]
Since $h$ is positive and increasing in $\mathbb{R}^+$, we have that $\lim_{t \to \infty} \phi_h(t) = +\infty$. Moreover, $\phi_h$ satisfies the differential equation

$$\phi_h''(t) - \frac{h'}{h}(t)\phi_h'(t) = 0$$

for all $t \in [0, \infty)$.

**Notation.** In this paper, $N$ will always be a complete Riemannian manifold with a distinguished point $z_0$ and radial sectional curvatures along the minimal geodesic issuing from $z_0$ bounded above by $K_{rad}^N(z) \leq -G(\rho_N(z))$, where $G$ satisfies the conditions (8). Let $h$ be the solution of (9) associated to $G$ and $\phi_h = \int h(s)ds$. Finally, $\rho_N(z) = \text{dist}_N(z_0, z)$ will be the distance function on $N$. For any given complete Riemannian manifold $(L, y_0)$ with a distinguished point $y_0$ and radial sectional curvature bounded below ($K_{rad}^L \geq -\Lambda^2$) and $\varepsilon \in (0, 1)$ consider the subset $\Omega_{\phi_h}(\varepsilon) \subset N \times L$ given by

$$\Omega_{\phi_h}(\varepsilon) = \{(x, y) \in N \times L : \phi_h(\rho_N(x)) \leq \log(\rho_L(y) + 1)^{1-\varepsilon}\}.$$  

Here $\rho_L(y) = \text{dist}_L(y_0, y)$, $y_0 \in L$.

**Definition 2.1** An isometric immersion $\varphi : M \to N \times L$ of a Riemannian manifold $M$ into the product $N \times L$ is said to be $\phi_h$-bounded if there exists a compact $K \subset M$ and $\varepsilon \in (0, 1)$ such that $\varphi(M \setminus K) \subset \Omega_{\phi_h}(\varepsilon)$.

**Remark 2.2** The class of $\phi$-bounded submanifolds contains the class of cylindrically bounded submanifolds.

### 2.2 Curvature estimates for $\phi$-bounded submanifolds

In this section, we extend the cylindrically bounded version of Jorge-Koutrofiotis’s Theorem, Thm. 1.5-ii. and the mean curvature estimates of Thm. 1.7 to the class of $\phi_h$-bounded properly immersed submanifolds. These extensions are done in two ways. First: the class we consider is larger than the class of cylindrically bounded submanifolds. Second: there are no requirements on the growth on the second fundamental form as in Thm. 1.5. We also should observe that although $\phi$-bounded properly immersed submanifolds, ($\varphi : M \to N \times L$) are stochastically complete, provided $L$ has an Omori-Yau pair, see Section 4 we do not need that to prove the following result.

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2 Along the geodesics issuing from $y_0$.  

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Theorem 2.3 Let $\varphi: M \to N^{n-\ell} \times L^\ell$ be a $\phi_b$-bounded isometric immersion of a complete Riemannian $m$-manifold $M$ with $n - m \leq m - \ell - 1$. If $\varphi$ is proper and $-G \leq b \leq 0$ then

$$\sup_M K_M \geq |b| + \inf_N K_N.$$  \hspace{1cm} (12)

With strict inequality $\sup_M K_M > \inf_N K_N$ if $b = 0$.

Corollary 2.4 Let $\varphi: M \to N^{n-\ell} \times L^\ell$ be a properly immersed, cylindrically bounded submanifold, $\varphi(M) \subset B_N(r) \times L^\ell$, where $B_N(r)$ is a regular geodesic ball of $N$. Suppose that $n - m \leq m - \ell - 1$. Then the sectional curvature of $M$ satisfies the following inequality

$$\sup_M K_M \geq C^2_b(r) + \inf_N K_N,$$  \hspace{1cm} (13)

where $b = \sup_{B_N(r)} K^\text{rad}_N$ and $C_b$ is defined in (4).

Our next main result extends the mean curvature estimates (7) to $\phi$-bounded submanifolds.

Theorem 2.5 Let $\varphi: M \to N^{n-\ell} \times L^\ell$ be a $\phi_b$-bounded isometric immersion of a complete Riemannian $m$-manifold $M$ with $m \geq \ell + 1$. If $\varphi$ is proper then the mean curvature vector $H = \text{tr}\alpha$ of $\varphi$ satisfies

$$\sup_M |H| \geq (m - \ell) \cdot \inf_{r \in [0,\infty)} \frac{h'(r)}{h(r)}.\hspace{1cm} (14)$$

If $-G \leq b \leq 0$ then

$$\sup_M |H| \geq (m - \ell) \cdot \sqrt{|b|}.\hspace{1cm} (15)$$

With strict inequality $\sup_M |H| > 0$ if $b = 0$.

3 Proof of the main results

3.1 Basic results

Let $M$ and $W$ be Riemannian manifolds of dimension $m$ and $n$ respectively and let $\varphi: M \to W$ be an isometric immersion. For a given function $g \in C^\infty(W)$ set $f = g \circ \varphi \in C^\infty(M)$. Since

$$\langle \text{grad}_w f, X \rangle = \langle \text{grad}_w g, X \rangle$$

for every vector field $X \in TM$, we obtain

$$\text{grad}_w g = \text{grad}_w f + (\text{grad}_w g \perp)$$
according to the decomposition $TW = TM \oplus T^\perp M$. An easy computation using the Gauss formula gives the well-known relation (see e.g. [14])

$$\text{Hess}_w f(X, Y) = \text{Hess}_w g(X, Y) + \langle \text{grad}_w g, \alpha(X, Y) \rangle$$

(16)

for all vector fields $X, Y \in TM$, where $\alpha$ stands for the second fundamental form of $\varphi$. In particular, taking traces with respect to an orthonormal frame $\{e_1, \ldots, e_m\}$ in $TM$ yields

$$\triangle_w f = \sum_{i=1}^m \text{Hess}_w g(e_i, e_i) + \langle \text{grad}_w g, H \rangle.$$  

(17)

where $H = \sum_{i=1}^m \alpha(e_i, e_i)$.

In the sequel, we will need the following well known results, see the classical Greene-Wu [12] for the Hessian Comparison Theorem and Pigola-Rigoli-Setti’s “must looking at” book [24, Lemma 2.13], see also [26, 6, Thm.1.9] for the Sturm Comparison Theorem.

**Theorem 3.1 (Hessian Comparison Thm.)** Let $W$ be a complete n-manifold and $\rho_w(x) = \text{dist}_w(x_0, x)$, $x_0 \in W$ fixed. Let $D_{x_0} = W \setminus \{x_0\} \cup \text{cut}(x_0)$ be the domain of normal geodesic coordinates at $x_0$. Let $G \in C^0([0, \infty))$ and let $h$ be the solution of (9). Let $[0, R)$ be the largest interval where $h > 0$. Then

i. If the radial sectional curvatures along the geodesics issuing from $x_0$ satisfies

$$K_w^{\text{rad}} \geq -G(\rho_w), \text{ in } B_w(R)$$

then

$$\text{Hess}_w \rho \leq \frac{h'}{h}(\rho_w) [\langle \cdot, \cdot \rangle - d\rho \otimes d\rho] \text{ on } D_{x_0} \cap B_w(R)$$

ii. If the radial sectional curvatures along the geodesics issuing from $x_0$ satisfy

$$K_w^{\text{rad}} \leq -G(\rho_w), \text{ in } B_w(R)$$

then

$$\text{Hess}_w \rho \geq \frac{h'}{h}(\rho) [\langle \cdot, \cdot \rangle - d\rho \otimes d\rho] \text{ on } D_{x_0} \cap B_w(R)$$

**Lemma 3.2 (Sturm Comparison Thm.)** Let $G_1, G_2 \in L^\infty_{\text{loc}}(\mathbb{R})$, $G_1 \leq G_2$ and $h_1$ and $h_2$ solutions of the following problems:

a.) $\begin{cases} h_1''(t) - G_1(t)h_1(t) & \leq 0 \\ h_1(0) = 0, \quad h_1'(0) > 0 \end{cases}$  

b.) $\begin{cases} h_2''(t) - G_2(t)h_2(t) & \geq 0 \\ h_2(0) = 0, \quad h_2'(0) > h_1'(0) \end{cases}$

(18)

and let $I_1 = (0, S_1)$ and $I_2 = (0, S_2)$ be the largest connected intervals where $h_1 > 0$ and $h_2 > 0$ respectively. Then

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1. $S_1 \leq S_2$. And on $I_1$, $\frac{h_1'}{h_1} \leq \frac{h_2'}{h_2}$ and $h_1 \leq h_2$.

2. If $h_1(t_o) = h_2(t_o)$, $t_o \in I_1$ then $h_1 \equiv h_2$ on $(0,t_o)$.

For a more detailed Sturm Comparison Theorem one should consult the beautiful book [24, Chapter 2]. If $-G = b \in \mathbb{R}$ then the solution of $h''_0(t) - G \cdot h_0(t) = 0$ with $h_0(0) = 0$ and $h'_0(0) = 1$ is given by

$$h_b(t) = \begin{cases} \frac{1}{\sqrt{-b}} \cdot \sinh(\sqrt{-b}t) & \text{if } b < 0 \\ t & \text{if } b = 0 \\ \frac{1}{\sqrt{b}} \cdot \sin(\sqrt{b}t) & \text{if } b > 0. \end{cases}$$

In particular, if the radial sectional curvatures along the geodesics issuing from $x_0$ satisfy $K_w^\text{rad}(x) \leq -G(\rho_w(x)) \leq b$, $x \in B_w(R) = \{x, \text{dist}_w(x_0,x) = \rho_w(x) < R\}$, then the solution $h$ of (9), satisfies $(h'/h)(t) \geq (h''_0/h_b)(t) = C_b(t)$, $t \in (0,R)$, $R < \pi/2\sqrt{b}$, if $b > 0$. Therefore, $\text{Hess}_w \rho_w \geq C_b(\rho_w)(\langle \cdot \rangle - d\rho_w + d\rho_w)$. Likewise, if $K_w^\text{rad}(x) \geq -G(\rho_w(x)) \geq b$, $x \in B_w(R)$ then $(h'/h)(t) \leq C_b(t)$, $t \in (0,R)$ and $\text{Hess}_w \rho_w \leq C_b(\rho_w)(\langle \cdot \rangle - d\rho_w + d\rho_w)$.

3.2 Proof of Theorem 2.3

Assume without loss that there exists a $x_0 \in M$ such that $\varphi(x_0) = (z_0,y_0) \in N \times L$, $z_0$, $y_0$ the distinguished points of $N$ and $L$. For each $x \in M$, let $\varphi(x) = (z(x),y(x))$. Define $g: N \times L \to \mathbb{R}$ by $g(z,y) = \phi_h(\rho_n(z)) + 1$, recalling that $\phi_h(t) = \int_0^t h(s)ds$, and define $f = g \circ \varphi: M \to \mathbb{R}$ by $f(x) = g(\varphi(x)) = \phi_h(\rho_n(z(x)))+1$. For each $k \in \mathbb{N}$, set $g_k(x) = f(x) - \frac{1}{k} \log(\rho_n(y(x)) + 1)$. Observe that $g_k(x_0) = 1$ for all $k$, since $\rho_n(z_0) = \rho_n(y_0) = 0$. First, let us prove the item i.

If $x \to \infty$ in $M$ then $\varphi(x) \to \infty$ in $N \times L$ since $\varphi$ is proper. On the other hand, $\varphi(M \setminus K) \subset \Omega_h(\varepsilon)$ for some compact $K \subset M$ and $\varepsilon \in (0,1)$. This implies that $y(x) \to \infty$ in $L$ and

$$\frac{g_k(x)}{\log(\rho_n(y(x)) + 1)} = \frac{f(x)}{\log(\rho_n(y(x)) + 1)} - \frac{1}{k} \log(\rho_n(y(x)) + 1) \leq \frac{1}{k} < 0$$

for $\rho_n(x) \gg 1$. This implies that $g_k(x) < 0$ for $\rho_n(x) \gg 1$. Therefore each $g_k$ reach a maximum at a point $x_k \in M$. This yields a sequence $\{x_k\} \subset M$ so that $\text{Hess}_n g_k(x_k)(X,X) \leq 0$ for all $X \in T_{x_k}M$, this is, $\forall X \in T_{x_k}M$

$$\text{Hess}_n f(x_k)(X,X) \leq \frac{1}{k} \cdot \text{Hess}_n \log(\rho_n(y(x_k)) + 1)(X,X).$$

(19)
Observe that \( \log(\rho_L(y(x_k))) + 1 = \log(\rho_L \circ \pi_L + 1) \circ \varphi(x_k), \pi_L: N \times L \rightarrow L \) the projection on the second factor, thus the right hand side of (19), using the formula (16), is given by

\[
\text{Hess}_M \log(\rho_L(y(x_k)) + 1)(X,X) = \text{Hess}_{N \times L} \log(\rho_L \circ \pi_L + 1)(\varphi(x_k))(X,X) \\
+ \langle \text{grad}_{N \times L} \log(\rho_L \circ \pi_L + 1), \alpha(X,X) \rangle 
\]

(20)

Where \( \alpha \) is the second fundamental form of \( \varphi \). For simplicity, set \( \psi(t) = \log(t + 1) \), \( z_k = z(x_k) \), \( y_k = y(x_k) \), \( s_k = \rho_x(z_k) \) and \( t_k = \rho_L(y_k) \). Decomposing \( X \in TM \) as \( X = X^N + X^L \in TN \oplus TL \), we see that the first term of the right hand side of (20) is

\[
\text{Hess}_{N \times L} \psi \circ \rho_L \circ y(x_k)(X,X) = \psi''(t_k)|X^L|^2 + \psi'(t_k)\text{Hess}_L \rho_L(\varphi(x_k))(X,X) \\
\leq \psi''(t_k)|X^L|^2 + C_{-\Lambda^2}(t_k)\frac{|X^N|^2}{(t_k + 1)} \\
\leq C_{-\Lambda^2}(t_k)\frac{|X^N|^2}{(t_k + 1)},
\]

since \( \text{Hess}_L \rho_L(\varphi(x_k))(X,X) \leq C_{-\Lambda^2}(t_k)|X^N|^2 \) (by Theorem 3.1) and \( \psi'' \leq 0 \).

The second term of the right hand side of (20) is

\[
\langle \text{grad}_{N \times L} \psi \circ \rho_L \circ y(x_k), \alpha(X,X) \rangle = \psi'(t_k)\langle \text{grad}_L \rho_L(\varphi(x_k), \alpha(X,X) \rangle \\
\leq \frac{1}{(t_k + 1)}\|\alpha\| \cdot |X|^2 
\]

(22)

From (21) and (22) we have the following

\[
\text{Hess}_M \psi \circ \rho_L \circ y(x_k)(X,X) \leq \frac{C_{-\Lambda^2}(t_k) + \|\alpha\|}{(t_k + 1)} \cdot |X|^2 
\]

(23)

And from (19) and (23) we have that

\[
\text{Hess}_M f(x_k)(X,X) \leq \frac{1}{k} \frac{(C_{-\Lambda^2}(t_k) + \|\alpha\|)}{(t_k + 1)}|X|^2 
\]

(24)

We will compute the left hand side of (19). Using the formula (16) again we have

\[
\text{Hess}_M f(x_k) = \text{Hess}_{N \times L} g(\varphi(x_k)) + \langle \text{grad}_{N \times L} g, \alpha \rangle 
\]

(25)
Recalling that $f = g \circ \varphi$ and $g$ is given by $g(z,y) = \phi_h(\rho_N(z))$, where $\phi_h$ is defined in (11) and $\rho_N(z) = dist_N(z_0, z)$. Let us consider an orthonormal basis \((26)\)

\[
\{\text{grad} \rho_N, \partial / \partial \theta_1, \ldots, \partial / \partial \theta_{n-1}, \partial / \partial \gamma_1, \ldots, \partial / \partial \gamma_l\}
\]

for $T_{\varphi(x)}(N \times L)$. Thus if $X \in T_x M$, $|X| = 1$, we can decompose

\[
X = a \cdot \text{grad} \rho_N + \sum_{j=1}^{n-l-1} b_j \cdot \partial / \partial \theta_j + \sum_{i=1}^l c_i \cdot \partial / \partial \gamma_i
\]

with $a^2 + \sum_{j=1}^{n-l-1} b_j^2 + \sum_{i=1}^l c_i^2 = 1$. Recalling that $s_k = \rho_N(z(x_k))$, we can see that the first term of the right hand side of (25)

\[
\text{Hess}_{N \times L}g(\varphi(x))(X,X) = \phi''_h(s_k) \cdot a^2 + \phi'_h(s_k) \sum_{j=1}^{n-l-1} b_j \cdot \text{Hess} \rho_N(z_k)(\partial / \partial \theta_j, \partial / \partial \theta_j)
\]

\[
\geq \phi''_h(s_k) \cdot a^2 + \phi'_h(s_k) \sum_{j=1}^{n-l-1} b_j \cdot \frac{h'}{h}(s_k)
\]

\[
= \phi''_h(s_k) \cdot a^2 + (1 - a^2 - \sum_{i=1}^l c_i^2) \cdot \phi'_h(s_k) \cdot \frac{h'}{h}(s_k)
\]

\[
= \left(1 - \sum_{i=1}^l c_i^2\right) \cdot \phi'_h(s_k) \cdot \frac{h'}{h}(s_k)
\]

Thus

\[
\text{Hess}_{N \times L}g(\varphi(x))(X,X) \geq (1 - \sum_{i=1}^l c_i^2) \cdot \phi'_h(s_k) \cdot \frac{h'}{h}(s_k). \tag{27}
\]

The second term of the right hand side of (25) is the following

\[
\langle \text{grad}_{N \times L}g, \alpha(X,X) \rangle = \phi'_h(s_k) \langle \text{grad} \rho_N(z_k), \alpha(X,X) \rangle
\]

\[
\geq -\phi'_h(s_k) |\alpha(X,X)|\tag{28}
\]

From (25), (27), (28) we have that,
Lemma 3.3 (Otsuki)

We will need the following lemma known as Otsuki’s Lemma [15, p. 28].

\[
\text{Hess}_m f(x_k)(X,X) \geq \left(1 - \sum_{i=1}^{\ell} c_i^2 \right) \cdot \frac{h'(s_k)}{h(s_k)} - |\alpha(X,X)| \phi'_b(s_k) \tag{29}
\]

Recall that \(n + \ell \leq 2m - 1\). This dimensional restriction implies that \(m \geq \ell + 2\), since \(n \geq m + 1\). Therefore, for every \(x \in M\) there exists a sub-space \(V_x \subset T_x M\) with \(\dim(V_x) \geq (m - \ell) \geq 2\) such that \(V \perp TL\), this is equivalent to \(c_i = 0\). If we take any \(X \in V_{x_k} \subset T_{x_k} M, |X| = 1\) we have by (29) that

\[
\left( \frac{C_{-\Lambda^2}(t_k) + |\alpha(X,X)|}{k(t_k + 1)} \right) \geq \text{Hess}_m f(x_k)(X,X) \geq \left[ \frac{h'(s_k)}{h(s_k)} - |\alpha(X,X)| \right] \phi'_b(s_k)
\]

Thus, reminding that \(\phi'_b = h\),

\[
\left[ \frac{1}{k(t_k + 1) h(s_k)} + h(s_k) \right] |\alpha(X,X)| \geq \frac{h'(s_k)}{h(s_k)} - \frac{C_{-\Lambda^2}(t_k)}{k(t_k + 1)}
\]

(30)

Since \(-G \leq b \leq 0\), we have by Lemma 3.2 (Sturm’s argument) that the solution \(h\) of (9) satisfies \(\left( h'/h \right)(t) \geq C_b(t) > \sqrt{|b|}\) and that \(h(t) \to +\infty\) as \(t \to +\infty\), where \(C_b\) is defined in (31). Let us assume that \(x_k \to \infty\) in \(M\), (the case \(\rho_M(x_k) \leq C^2 < \infty\) will be considered later), then \(s_k \to \infty\) as well as \(t_k \to \infty\). Thus from (30), for sufficiently large \(k\), we have at \(\phi(x_k)\) that

\[
\left[ \frac{1}{k(t_k + 1) h(s_k)} + 1 \right] |\alpha(X,X)| \geq \frac{h'(s_k)}{h(s_k)} - \frac{C_{-\Lambda^2}(t_k)}{k(t_k + 1) h(s_k)} \geq C_b(s_k) - \frac{C_{-\Lambda^2}(t_k)}{k(t_k + 1) h(s_k)} > 0
\]

(31)

Thus, at \(x_k\) and \(X \in T_{x_k} M\) with \(|X| = 1\) we have

\[
|\alpha(X,X)| \geq \left[ C_b(s_k) - \frac{C_{-\Lambda^2}(t_k)}{k(t_k + 1) h(s_k)} \right] \left[ \frac{1}{k(t_k + 1) h(s_k)} + 1 \right]^{-1} > 0.
\]

(32)

We will need the following lemma known as Otsuki’s Lemma [15, p. 28].

**Lemma 3.3 (Otsuki)** Let \(\beta : \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{R}^d, d \leq q - 1\), be a symmetric bilinear form satisfying \(\beta(X,X) \neq 0\) for \(X \neq 0\). Then there exists linearly independent vectors \(X, Y\) such that \(\beta(X,X) = \beta(Y,Y)\) and \(\beta(X,Y) = 0\).

The horizontal subspace \(V_{x_k}\) has dimension \(\dim(V_{x_k}) \geq m - \ell \geq 2\). Thus, by the inequality (32) and \(n - m \leq m - \ell - 1 \leq \dim(V_{x_k}) - 1\), we may apply Otsuki’s Lemma to \(\alpha(x_k) : V_{x_k} \times V_{x_k} \to T_{x_k} M^\perp \simeq \mathbb{R}^{n-m}\) to obtain \(X, Y \in V_{x_k}, |X| \geq |Y| \geq 1\) such that \(\alpha(x_k)(X,X) = \alpha(x_k)(Y,Y)\) and \(\alpha(x_k)(X,Y) = 0\).
By the Gauss equation we have that
\[ K_M(x_k)(X,Y) - K_N(\varphi(x_k))(X,Y) = \frac{\langle \alpha(x_k)(X,X), \alpha(x_k)(Y,Y) \rangle}{|X|^2Y|^2 - (X,Y)^2} = \frac{|\alpha(x_k)(X,X)|^2}{|X|^2Y|^2} \geq \left( \frac{|\alpha(x_k)(X,X)|}{|X|^2} \right)^2 \geq \left( \frac{|\alpha(x_k)(X,X)|}{|X|^2} \right)^2 \]

This implies by (32) that
\[ \sup K_M - \inf K_N > \left( \left[ \frac{h'(s_k)}{h(s_k)} - \frac{C^{-A^2}(t_k)}{k(t_k + 1)h(s_k)} \right] \frac{1}{k(t_k + 1)h(s_k)} + 1 \right)^{-1} > 0. \]

Therefore, \( \sup K_M - \inf K_N > 0 \) regardless \( b = 0 \) or \( b < 0 \). If \( b < 0 \) we let \( k \to +\infty \) and then we have
\[ \sup K_M - \inf K_N \geq \lim_{s_k \to \infty} \left[ \frac{h'(s_k)}{h(s_k)} \right]^2 = |b| \]
(33)

The case where the sequence \( \{x_k\} \subset M \) remains in a compact set, we proceed as follows. Passing to a subsequence we have that \( x_k \to x_\infty \in M \). Thus \( t_k \to t_\infty < \infty \) and \( s_k \to s_\infty < \infty \). By (24)
\[ \text{Hess}_M f(x_\infty)(X,X) \leq \lim_{k \to \infty} \left( \frac{C^{-A^2}(t_\infty) + |\alpha(x_\infty)(X,X)|}{k(t_\infty + 1)} \right) = 0, \]
(34)
for all \( X \in T_{x_\infty} M \). Using the expression on the right hand side of (29) we obtain for every \( X \in V_{x_\infty} \)
\[ 0 \geq \text{Hess} f(x_\infty)(X,X) \geq \left[ 1 - \sum_{i=1}^{\ell} c_i \cdot \frac{h'(s_\infty)}{h(s_\infty)} - |\alpha(X,X)| \right] \phi_{\ell}(s_\infty). \]

There exists a sub-space \( V \subset T_{x_\infty} M \) with \( \dim(V) \geq (m - \ell) \geq 2 \) such that \( V \perp T \mathbb{R}^\ell \), this is equivalent to \( c_i = 0 \). If we take any \( X \in V_{x_\infty} \subset T_{x_\infty} M, |X| = 1 \) we have hence
\[ |\alpha_{x_\infty}(X,X)| \geq \frac{h'(s_\infty)}{h(s_\infty)} |X|^2. \]

Again, using Otsuki’s Lemma and Gauss equation, we conclude that
\[ \sup M K_M - \inf_{B_{s_\infty}(r)} K_N \geq \frac{h'(s_\infty)}{h(s_\infty)} > |b|. \]
(35)
3.3 Proof of Theorem 2.5

We will follow the proof of Theorem 2.3 closely. Recall that $g_k$ reaches a maximum at $x_k \in M, k = 1, 2, \ldots$, thus so that $\Delta_M g_k(x_k) \leq 0$. Thus

$$\Delta_M f(x_k) \leq \frac{1}{k} \cdot \Delta_M (\log(\rho \circ \pi_L + 1) \circ \varphi(x_k)).$$

(36)

Using the formula (17)

$$\Delta_M(\log(\rho \circ \pi_L + 1) \circ \varphi(x_k)) = \sum_{i=1}^m \text{Hess}_{N \times L}(\log(\rho \pi_L' + 1)\varphi(x_k))(X_i, X_i)$$

$$+ \langle \text{grad}_{N \times L}(\log(\rho \circ \pi_L + 1)), H \rangle$$

(37)

where $H = \sum_{i=1}^m \alpha(X_i, X_i)$ is the mean curvature vector while $\alpha$ is the second fundamental form of the immersion $\varphi$ and $\{X_i\}$ is an orthonormal basis of $T_{x_k}M$.

As before, decomposing $X \in TM$ as $X = X^N + X^L \in TN \oplus TL$ and setting $\psi(t) = \log(t + 1), y_k = y(x_k)$ and $t_k = \rho_L(y_k)$ we have that the right hand side of (37)

$$\sum_{i=1}^m \text{Hess}_{N \times L}(\log(\rho \circ \pi) \circ y(x_k))(X_i, X_i) = \psi''(t_k) \sum_{i=1}^m |X_i^L|^2$$

$$+ \psi'(t_k) \sum_{i=1}^m \text{Hess}_L \rho_L(y_k)(X_i, X_i)$$

$$\leq \frac{C_{-\Delta^2}(t_k)}{(t_k + 1)} \sum_{i=1}^m |X_i^N|^2,$$

$$\leq \frac{m \cdot C_{-\Delta^2}(t_k)}{(t_k + 1)}$$

(38)

since $\psi'' \leq 0$ and

$$\langle \text{grad}_{N \times L}(\log(\rho \circ \pi) \circ y(x_k)), H \rangle = \psi'(t_k) \langle \text{grad} \rho_L(y_k), H \rangle$$

$$\leq \frac{1}{(t_k + 1)} |H|$$

(39)

From (37), (38) and (39) we have

$$\Delta_M \log(\rho_L(y(x_k)) + 1) \leq \frac{m \cdot C_{-\Delta^2}(t_k) + |H|}{(t_k + 1)}$$

(40)
And from (36) and (40) we have that
\[
\Delta_M f(x_k) \leq \frac{m \cdot C_{-\Lambda^2}(t_k) + |H|}{k(t_k + 1)} \tag{41}
\]

We will compute the left hand side of (36). Recall that \( f = g \circ \varphi \) and \( g \) is given by \( g(z, y) = \phi_h(\rho_N(z)) \), where \( \phi \) is defined in (11). Using the formula (17) again we have
\[
\Delta_M f(x_k) = \sum_{i=1}^{m} \text{Hess}_{N \times L}g(\varphi(x_k))(X_i, X_i) + \langle \text{grad} g, H \rangle \tag{42}
\]
Consider the orthonormal basis (26) for \( T_{\varphi(x_k)}(N \times L) \). Thus if \( X_i \in T_{x_k}M, |X_i| = 1 \), we can decompose
\[
X_i = a_i \cdot \text{grad} \rho_N + \sum_{j=1}^{n-\ell-1} b_{ij} \cdot \partial / \partial \theta_j + \sum_{l=1}^{\ell} c_{il} \cdot \partial / \partial \gamma_l
\]
with \( a_i^2 + \sum_{j=1}^{n-\ell-1} b_{ij}^2 + \sum_{l=1}^{\ell} c_{il}^2 = 1 \). Set \( z_k = z(x_k) \) and \( s_k = \rho_N(z_k) \). We have as in (27)
\[
\text{Hess}_{N \times L}g(\varphi(x))(X_i, X_i) \geq (1 - \sum_{l=1}^{\ell} c_{il}^2) \cdot \phi_h'(s_k) \cdot \frac{h'}{h}(s_k) \tag{43}
\]
The second term of the right hand side of (42) is the following, if \( |X| = 1 \),
\[
\langle \text{grad} g, H \rangle = \phi_h'(s_k) \langle \text{grad} \rho_N(z_k), H \rangle \geq -\phi_h'(s_k)|H| \tag{44}
\]
Therefore from (42), (43), (44) we have that,
\[
\Delta_M f(x_k) \geq \left[ (m - \sum_{i=1}^{m} \sum_{l=1}^{\ell} c_{il}^2) \cdot \frac{h'}{h}(s_k) - |H| \right] \phi_h'(s_k) \tag{45}
\]
From (41) and (45) we have
\[
\frac{m \cdot C_{-\Lambda^2}(t_k) + |H|}{k(t_k + 1)} \geq \Delta_M f(x_k) \geq \left[ (m - \ell) \cdot \frac{h'}{h}(s_k) - |H| \right] \phi_h'(s_k) \tag{46}
\]
Therefore
\[
\sup_M |H| \left[ \frac{1}{h(s_k) \cdot k \cdot (t_k + 1)} + 1 \right] \geq (m - \ell) \cdot \frac{h'}{h}(s_k) - \frac{m \cdot C_{-\Lambda^2}(t_k)}{h(s_k) \cdot k \cdot (t_k + 1)}
\]

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Letting $k \to \infty$ we have

$$\sup_M |H| \geq (m - \ell) \cdot \lim_{k \to \infty} \frac{h'}{h}(s_k).$$

If in addition, we have that $-G \leq b \leq 0$ then $(h'/h)(s_k) \geq C_b(s_k)$. The case that $b = 0$ we have $(h'/h)(s_k) \geq 1/s_k$ and $h(s_k) \geq s_k$. Since the immersion is $\phi$-bounded we have $s_k^2 \leq 2 \log(t_k + 1)^{(1-\varepsilon)}$. Thus for sufficient large $k$

$$\sup_M |H| \geq \frac{1}{s_k \cdot k \cdot (t_k + 1)} + 1 \geq \frac{m - \ell}{s_k} - \frac{m \cdot C_{-\Lambda^2}(t_k)}{s_k \cdot k \cdot (t_k + 1)} > 0.$$

This shows that $\sup_M |H| > 0$.

In the case $b < 0$, we have $(h'/h)(s_k) \geq C_b(s_k) \geq \sqrt{|b|}$ and

$$\sup_M |H| \geq (m - \ell) \cdot \lim_{k \to \infty} \frac{h'}{h}(s_k) \geq \sqrt{|b|}.$$ 

**Remark 3.4** The statements of Theorems 2.3 and 2.5 are also true in a slightly more general situation. This is, if, instead a proper $\phi$-bounded immersion, one asks a proper immersion $\varphi: M \to N \times L$ with the property

$$\lim_{x \to \infty_M} \frac{\phi_h(\rho_N(z(x)))}{\log(\rho_L(y(x))) + 1} = 0,$$

where $\varphi(x) = (z(x), y(x)) \in N \times L$.

### 4 Omori-Yau pairs

Omori, in [18], discovered an important global maximum principle for complete Riemannian manifolds with sectional curvature bounded below. Omori’s maximum principle was refined and extended by Cheng and Yau, [8], [28], [29], to Riemannian manifolds with Ricci curvature bounded below and applied to find elegant solutions to various analytic-geometric problems on Riemannian manifolds. There were others generalizations of the Omori-Yau maximum principle under more relaxed curvature requirements in [7], [10] and an extension to an all general setting by S. Pigola, M. Rigoli and A. Setti in their beautiful book [23]. There, they introduced the following terminology.

**Definition 4.1 (Pigola-Rigoli-Setti)** The Omori-Yau maximum principle holds on a Riemannian manifolds $W$ if for any $u \in C^2(W)$ with $u^* := \sup_W u < \infty$, there exists a sequence of points $x_k \in W$, depending on $u$ and on $W$, such that

$$\lim_{k \to \infty} u(x_k) = u^*, \quad \|\nabla u\| (x_k) < \frac{1}{k}, \quad \Delta u(x_k) < \frac{1}{k^2}.$$  

(47)
Likewise, the Omori-Yau maximum principle for the Hessian holds on $W$ if
\[
\lim_{k \to \infty} u(x_k) = u^*, \quad \text{grad} u(x_k) < \frac{1}{k}, \quad \text{Hess}_u u(x_k)(X,X) < \frac{1}{k} \cdot |X|^2, \quad (48)
\]
for every $X \in T_{x_k}W$.

A natural and important question is, what are the Riemannian geometries the Omori-Yau maximum principle holds on? It does hold on complete Riemannian manifolds with sectional curvature bounded below holds [18], it holds on complete Riemannian manifolds with Ricci curvature bounded below [8], [28], [29]. Follows from the work of Pigola-Rigoli-Setti [23] that the Omori-Yau maximum principle holds on complete Riemannian manifolds $W$ with Ricci curvature with strong quadratic decay,
\[
\text{Ric}_W \geq -c^2 \cdot \rho^2 \cdot \Pi_{i=1}^{k} (\log(i) (\rho_W + 1), \rho_W \gg 1).
\]
The notion of Omori-Yau pair was formalized in [3], after the work of Pigola-Rigoli-Setti. The Omori-Yau pair is, here, described for the Laplacian and for the Hessian however, it certainly can be extended to other operators or bilinear forms.

**Definition 4.2** Let $W$ be a Riemannian manifold. A pair $(G, \gamma)$ of smooth functions $G: [0, +\infty) \to (0, +\infty)$, $\gamma: W \to [0, +\infty)$, $G \in C^1([0, \infty))$, $\gamma \in C^2([0, \infty))$, forms an Omori-Yau pair for the Laplacian in $W$, if they satisfy the following conditions:

h.1) $\gamma(x) \to +\infty$ as $x \to \infty$ in $W$.

h.2) $G(0) > 0$, $G'(t) \geq 0$ and $\int_0^{+\infty} \frac{ds}{\sqrt{G(s)}} = +\infty$.

h.3) $\exists A > 0$ constant such that $|\text{grad}_u \gamma| \leq A \sqrt{G(\gamma)} \left( \int_0^{\gamma} \frac{ds}{\sqrt{G(s)}} + 1 \right)$ off a compact set.

h.4) $\exists B > 0$ constant such that $\triangle_u \gamma \leq B \sqrt{G(\gamma)} \left( \int_0^{\gamma} \frac{ds}{\sqrt{G(s)}} + 1 \right)$ off a compact set.

The pair $(G, \gamma)$ forms an Omori-Yau pair for the Hessian if instead h.4) one has

h.5) $\exists C > 0$ constant such that $\text{Hess} \gamma \leq C \sqrt{G(\gamma)} \left( \int_0^{\gamma} \frac{ds}{\sqrt{G(s)}} + 1 \right)$ off a compact set, in the sense of quadratic forms.
The result \[23\] Thm.1.9 captured the essence of the Omori-Yau maximum principle and it can be stated as follows.

**Theorem 4.3** If a Riemannian manifold \( M \) has an Omori-Yau pair \( (G, \gamma) \) then the Omori-Yau maximum principle on it.

The main step in the proof of Alias-Bessa-Montenegro’s Theorem (Thm 1.5) and Alias-Bessa-Dajczer’s Theorem (Thm 1.7) is to show that a cylindrically bounded submanifold, properly immersed into \( N \times L \), with *controlled* second fundamental form or *controlled* mean curvature vector, has an Omori-Yau pair, provided \( L \) has an Omori-Yau pair. Thus, the Omori-Yau maximum principle holds on those submanifolds and their proof follows the steps of Jorge-Koutrofiotis’s Theorem. On the other hand, the idea behind the proof of Theorems 2.3 & 2.5 is that: the factor \( L \) has bounded sectional curvature it has a natural Omori-Yau pair \((G, \gamma)\). This Omori-Yau pair together with the geometry of the factor \( N \) allows us to consider an unbounded region \( \Omega_\phi \) such that if \( \varphi : M \to \Omega_\phi \subset N \times L \) is an isometric immersion then there exists a function \( f \in C^2(M) \), not necessarily bounded, and a sequence \( x_k \in M \) satisfying \( \Delta f(x_k) \leq 1/k \). We show that a properly immersed \( \phi \)-bounded submanifold has an Omori-Yau pair for the Laplacian, provided the fiber \( L \) has an Omori-Yau pair for the Hessian. We show in Theorem 4.5 that an Omori-Yau pair for the Hessian guarantee the Omori-Yau sequence for certain unbounded functions, as this unbounded function \( f \) we are working. This leads to stronger forms of Theorem 2.3 & Theorem 2.5.

Let \( M, N, L \) be complete Riemannian manifolds of dimension \( m, n - \ell \) and \( \ell \), with distinguished points \( x_0, z_0 \) and \( y_0 \) respectively. Suppose that \( K_{N}^{\text{rad}} \leq -G(\rho_\gamma) \), \( G \) satisfying (8). Let \( h \) solution of (9) and \( \varphi_h \) as in (11). Suppose in addition that \( L \) has an Omori-Yau pair for the Hessian. We show in Theorem 4.5 that an Omori-Yau pair for the Hessian guarantee the Omori-Yau sequence for certain unbounded functions, as this unbounded function \( f \) we are working. This leads to stronger forms of Theorem 2.3 & Theorem 2.5.

**Theorem 4.4** Let \( \varphi : M \to N \times L \) be a properly immersed submanifold such that \( \varphi(M \setminus K) \subset \Omega_{h,T,G}(\epsilon) \) for some compact \( K \subset M \) and positive \( \epsilon \in (0,1) \).

1. If \( K_{N}^{\text{rad}} \leq -G \leq b \leq 0 \) and the codimension satisfies \( n - m \leq m - \ell - 1 \) then

\[
\sup_M K_M \geq |b| + \inf_N K_N.
\] (49)

With strict inequality \( \sup_M K_M > \inf_N K_N \) if \( b = 0 \).
2. If \( m \geq \ell + 1 \) then

\[
\sup_M |H| \geq (m - \ell) \cdot \inf_{r \in [0, \infty)} \frac{h'}{h}(r).
\]  

(50)

If \(-G \leq b \leq 0\) then

\[
\sup_M |H| \geq (m - \ell) \cdot \sqrt{|b|}.
\]  

(51)

With strict inequality \( \sup_M |H| > 0 \) if \( b = 0 \).

Assume without loss of generality that there exists \( x_0 \in M \) such that \( \varphi(x_0) = (z_0, y_0) \in N \times L \). As before, \( \varphi(x) = (z(x), y(x)) \) and \( g, p : N \times L \to \mathbb{R} \) given by

\( g(z, y) = \phi_b(\rho_N(z)) + \psi(\gamma(y)), p(z, y) = \psi(\gamma(y)) \).

For each \( k \in \mathbb{N} \), let \( g_k : M \to \mathbb{R} \) given by \( g_k(x) = g \circ \varphi(x) - p \circ \varphi(x)/k \). Observe that \( g_k(x_0) = 1 \) and for \( \rho_M(x) \gg 1 \), we have that \( g_k(x) < 0 \). This implies that \( g_k \) has a maximum at a point \( x_k \), yielding in this way a sequence \( \{x_k\} \subset M \) such that \( \text{Hess}_M g_k(x_k) \leq 0 \) in the sense of quadratic forms. Proceeding as in the proof of Theorem 2.3 we have that for \( X \in T_{x_k} M \),

\[
\text{Hess}_M \circ \varphi(x_k)(X, X) \leq \frac{1}{k} \text{Hess}_N g(\varphi(x_k))(X, X).
\]  

(52)

We have to compute both terms of this inequality. Considering once more the orthonormal basis \( (26) \) for \( T_{\varphi(z_k)}(N \times L) \) we can decompose, \( X \in T_{x_k} M, |X| = 1 \), (after identifying \( X \) with \( d\varphi X \)), as

\[
X = a \cdot \text{grad} \rho_N + \sum_{j=1}^{n-\ell-1} b_j \cdot \partial / \partial \theta_j + \sum_{i=1}^{\ell} c_i \cdot \partial / \partial \gamma_i
\]

with \( a^2 + \sum_{j=1}^{n-\ell-1} b_j^2 + \sum_{i=1}^{\ell} c_i^2 = 1 \). Setting \( s_k = \rho_N(z(x_k)), t_k = \gamma(y(x_k)) \), we have as in (29),

\[
\text{Hess}_M \circ \varphi(x_k)(X, X) = \text{Hess}_{N \times L} g(\varphi(x_k))(X, X) + \langle \text{grad}_{N \times L} g, \alpha(X, X) \rangle
\]

\[
\geq \left[ (1 - \sum_{i=1}^{\ell} c_i^2) \cdot \frac{h'}{h}(s_k) - |\alpha(X, X)| \right] \phi_b(s_k)
\]  

(53)
\[ \text{Hess}_M p \circ \varphi(x_k)(X, X) = \text{Hess}_{N \times L} p(\varphi(x_k))(X, X) + \langle \text{grad}_{N \times L} p, \alpha(X, X) \rangle \]

\[ = \psi''(t_k) \langle X, \text{grad}_L \gamma \rangle^2 + \psi'(t_k) \text{Hess}_L \gamma(X, X) \]

\[ + \psi'(t_k) \langle \text{grad}_L \gamma, \alpha(X, X) \rangle \]

\[ \leq \psi'(t_k) \left( \text{Hess}_L \gamma(X, X) + |\text{grad}_L \gamma| \cdot |\alpha(X, X)| \right) \quad (54) \]

\[ \leq \left[ \sqrt{G(\gamma(t_k))} \left( \int_0^{t_k} \frac{ds}{\sqrt{G(\gamma(s))}} + 1 \right) \right] \left[ C + A \cdot |\alpha(X, X)| \right] \]

\[ \leq C + A \cdot |\alpha(X, X)|, \]

since \( \psi'' \leq 0 \). Taking into consideration the bounds (53) & (54), the inequality (52) yields, \( (\phi'(s) = h(s)) \),

\[ \left[ \frac{A}{k \cdot h(s_k)} + 1 \right] |\alpha(X, X)| \geq (1 - \sum_{i=1}^\ell c_i^2) \frac{h'}{h}(s_k) - \frac{C}{k \cdot h(s_k)}. \quad (55) \]

Under the hypotheses of item 1. we have that \( (h'/h)(s) \geq C_b(s) > \sqrt{|b|} \) and \( h(s) \to \infty \) as \( s \to \infty \). Moreover, there exists a subspace \( V_{s_k} \subset T_{s_k} M \), \( \dim V_{s_k} \geq 2 \), such that if \( X \in V_{s_k} \), then \( X = a \cdot \text{grad} \rho_N + \sum_{j=1}^{n-\ell-1} b_j \cdot \partial/\partial \theta_j \). Therefore, for \( X \in V_{s_k}, |X| = 1 \), we have for \( k \gg 1 \).

\[ \left[ \frac{A}{k \cdot h(s_k)} + 1 \right] |\alpha(X, X)| \geq \frac{h'}{h}(s_k) - \frac{C}{k \cdot h(s_k)} \]

\[ > |b| - \frac{C}{k \cdot h(s_k)} \]

\[ > 0. \quad (56) \]

The proof follows exactly the steps of the proof of Theorem 2.3 and we obtain that \( \sup_M K_M \geq |b| + \inf_N K_N \) if \( b < 0 \) and \( \sup_M K_M > \inf_N K_N \) if \( b = 0 \).

To prove item 2., take an onthormal basis \( X_1, \ldots, X_q, \ldots, X_m \in T_{s_k} M \),

\[ X_q = a_q \cdot \text{grad} \rho_N + \sum_{j=1}^{n-\ell-1} b_{jq} \cdot \partial/\partial \theta_j + \sum_{i=1}^\ell c_{iq} \cdot \partial/\partial \gamma_i \]
with \( a^2_q + \sum_{j=1}^{n-\ell-1} b^2_{jq} + \sum_{i=1}^{\ell} c^2_{iq} = 1 \). Tracing the inequality (55) to obtain

\[
\left[ \frac{A}{k \cdot h(s_k)} + 1 \right] |H| \geq (m - \sum_{q=1}^{m} \sum_{i=1}^{\ell} c^2_{iq} \frac{h'(s_k)}{h(s_k)} - \frac{C}{k \cdot h(s_k)} \\
\geq (m - \ell) C_b(s_k) - \frac{C}{k \cdot h(s_k)}
\]

(57)

for \( k \gg 1 \). If \( b = 0 \) then \( C_b(s) = 1/s \) then, coupled with the estimate \( h(s) \geq s \sqrt{s} \), see \[6\], we deduce that \( \sup_M |H| \geq 0 \). And if \( b < 0 \) then \( C_b(s) \geq \sqrt{|b|} > 0 \), then letting \( k \to \infty \) we have \( \sup_M |H| \geq (m - \ell) \sqrt{|b|} > 0 \) if \( b < 0 \). We can see these curvature estimates as geometric applications of the following extension of the Pigola, Rigoli, Setti [23, Thm.1.9].

**Theorem 4.5** Let \( W \) be a complete Riemannian manifold with an Omori-Yau pair \((G, \gamma)\) for the Hessian (Laplacian). If \( u \in C^2(W) \) satisfies \( \lim_{x \to \infty} \frac{u(x)}{\psi(\gamma(x))} = 0 \), where \( \psi(t) = \log \left( \int_0^t \frac{ds}{\sqrt{G(s)}} + 1 \right) \), then there exist a sequence \( x_k \in M, k \in \mathbb{N} \) such that

\[
|\nabla u|(x_k) \leq \frac{A}{k}, \quad \text{Hess}_w u(x_k) \leq \frac{C}{k} \quad \text{and} \quad \left( \Delta_w u(x_k) \right) \leq \frac{B}{k}
\]

(58)

If \( u^* = \sup_M u < \infty \) then \( u(x_k) \to u^* \). The constants \( A, B \) and \( C \) come from the Omori-Yau pair \((G, \gamma)\), see Definition [4,2].

This result above should be compared with [21, Cor. A1.], due to Pigola, Rigoli, and Setti where they proved an Omori-Yau for quite general operators, applicable to certain unbounded functions with growth to infinity faster than ours. However, we could replace the distance function of their result by an Omori-Yau pair. It would be interesting to understand these facts.

Assume that the Omori-Yau pair \((G, \gamma)\) is for the Hessian. The case of the Laplacian is similar. Fix a point \( x_0 \in M \) such that \( \gamma(x_0) > 0 \) and define for each \( k \in \mathbb{N} \), \( g_k : M \to \mathbb{R} \) by \( g_k(x) = u(x) - \frac{1}{k} \psi(\gamma(x)) + 1 - u(x_0) - \frac{1}{k} \psi(\gamma(x_0)) \). We have that \( g_k(x_0) = 1 \) and \( g_k(x) \leq 0 \) for \( \rho_w(x) = \text{dist}_w(x_0, x) \gg 1 \). Thus there is a point \( x_k \) such that \( g_k \) reaches a maximum. This way we find a sequence \( x_k \in M \) such that
for all $X \in T_x W$

\[
\text{Hess}_w u(X,X) \leq \frac{1}{k} \text{Hess}_w \psi(\gamma)(X,X) \\
= \frac{1}{k} \left[ \psi''(\gamma) \langle \text{grad}_w \psi, X \rangle^2 + \psi'(\gamma) \text{Hess}_w \psi(X,X) \right] \\
\leq \frac{1}{k} \left[ \frac{1}{\sqrt{G(\gamma)}} \left( \int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1 \right) C \sqrt{G(\gamma)} \left( \int_0^\gamma \frac{ds}{G(s)} + 1 \right) \right] |X|^2 \\
= \frac{C}{k} |X|^2.
\]

We used that $\psi'' \leq 0$ and $\text{Hess}_w \psi(X,X) \leq \frac{C}{k} \sqrt{G(\gamma)} \left( \int_0^\gamma \frac{ds}{G(s)} + 1 \right)$.

\[
|\text{grad}_w u| = \frac{1}{k} |\text{grad}_w \psi(\gamma)| \\
\leq \frac{1}{k} \left[ \frac{1}{\sqrt{G(\gamma)}} \left( \int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1 \right) A \sqrt{G(\gamma)} \left( \int_0^\gamma \frac{ds}{G(s)} + 1 \right) \right] \\
\leq \frac{A}{k}.
\]

### 4.1 Omori-Yau pairs and warped products

Let $(N, g_n)$ and $(L, g_L)$ be complete Riemannian manifolds of dimension $n - \ell$ and $\ell$ respectively and $\xi : L \to \mathbb{R}^+$ be a smooth function. Let $\varphi : M \to L \times \xi N$ be an isometric immersion into the warped product $L \times \xi N = (L \times N, ds^2 = g_L + \xi^2 g_n)$.

The immersed submanifold $\varphi(M)$ is cylindrically bounded if $\pi_N(\varphi(M)) \subset B_N(r)$, where $\pi_N : L \times N \to N$ is the canonical projection in the $N$-factor and $B_N(r)$ is a regular geodesic ball of radius $r$ of $N$. Aliás and Dajczer in the proof of [4, Thm.1], showed that if $\varphi$ is proper in $L \times N$ then the existence of an Omori-Yau pair for the Hessian in $L$ induces an Omori-Yau pair for the Laplacian on $M$ provided the mean curvature $|H|$ is bounded. We can prove a slight extension of this result.

**Lemma 4.6** Let $\varphi : M \to L \times \xi N$ be an isometric immersion, proper in the first entry, where $L$ carries an Omori-Yau pair $(G, \gamma)$ for the Hessian, $\xi \in C^\infty(L)$ is a positive function satisfying

\[
|\text{grad} \log \xi(y)| \leq \ln \left( \int_0^{\gamma(y)} \frac{ds}{\sqrt{G(s)}} + 1 \right),
\]

(59)
Letting $\varphi(x) = (y(x), z(x))$ and if
\[ |H(\varphi(x))| \leq \ln \left( \int_0^{\gamma(y(x))} \frac{ds}{\sqrt{G(s)}} + 1 \right), \tag{60} \]
then $M$ has an Omori-Yau pair for the Laplacian. In particular, $M$ holds the Omori-Yau maximum principle for the Laplacian.

The idea of the proof is presented in [4] and therefore will try to follow the same notation to simplify the demonstration. Let $(\mathcal{G}, \gamma)$ the Omori-Yau pair for the Hessian of $L$. Assume w.l.o.g. that $M$ is non-compact and denote $\varphi(x) = (y(x), z(x))$.

Define $\varphi = \gamma(y)$ and define $\varphi(x) = \Gamma \circ \varphi = \gamma(y(x))$. We will show that $(\mathcal{G}, \varphi)$ is an Omori-Yau pair for the Laplacian in $M$. Indeed, let $q_k \in M$ a sequence such that $q_k \to \infty$ in $M$ as $k \to +\infty$. Since $\varphi$ is proper in the first entry, we have that $y(q_k) \to \infty$ in $L$. Since $\varphi(q_k) = \gamma(y(q_k))$ we have $\varphi(q_k) \to \infty$ as $q_k \to \infty$ in $M$.

We have that
\[ \nabla L \times \xi^N \Gamma(z, y) = \nabla L \gamma(y). \tag{61} \]
Since $\xi = \Gamma \circ \varphi$, we obtain at $\varphi(q)$
\[ \nabla_{\xi^N} \Gamma = (\nabla_{\xi^N} \Gamma)^T + (\nabla_{\xi^N} \Gamma)^\perp \]
\[ = \nabla_{\xi^N} \xi + (\nabla_{\xi^N} \Gamma)^\perp. \]
By hypothesis we have
\[ |\nabla_{\xi} \xi(q)| \leq |\nabla_{\xi^N} \Gamma|(\varphi(q)) = |\nabla_{\xi} \gamma(y)| \]
\[ \leq \sqrt{G(\gamma(y(q)))} \left( \int_0^{\gamma(y(q))} \frac{ds}{\sqrt{G(s)}} + 1 \right) \]
out of a compact subset of $M$.

Let $T, S \in TL, X, Y \in TN$ and $\nabla^{L \times \xi^N}, \nabla^L$ and $\nabla^N$ be the Levi-Civita connections of the metrics $ds^2 = g_L + \xi^2 g_N$, $g_L$ and $g_N$ respectively. It is easy to show that $\nabla^{L \times \xi^N} T = \nabla^T S$ and $\nabla^{L \times \xi^N} X = \nabla^T \xi^N X = T(\eta)X$ where $\eta = \log \xi$. Therefore,
\[ \nabla^T \nabla^{L \times \xi^N} \Gamma = \nabla^T \nabla_{\xi^N} \gamma \]
\[ \nabla_X \nabla^{L \times \xi^N} \Gamma = \nabla_{\xi^N} \gamma(\eta) X. \]
Hence,
\[
\text{Hess}_{L \times N} \Gamma(T, S) = \text{Hess}_L \gamma(T, S), \quad \text{Hess}_{L \times N} \Gamma(T, X) = 0
\]
\[
\text{Hess}_{L \times N} \Gamma(X, Y) = \langle \text{grad}_L \eta, \text{grad}_L \gamma \rangle \langle X, Y \rangle.
\]

For any unit vector \( e \in T_q M \), decompose \( e = e^L + e^N \), where \( e^L \in T_{y(q)} L \) and \( e^N \in T_{z(q)} N \). Then we have at \( \phi(q) \)
\[
\text{Hess}_L \times N \Gamma(e, e) = \text{Hess}_L \gamma(y(q))(e^L, e^L) + \langle \text{grad}_L \gamma, \text{grad}_L \eta \rangle \langle y(q) \rangle |e^N|^2.
\]
On the other hand, \( \text{Hess}_{L \times N} \xi(q)(e, e) = \text{Hess}_{L \times N} \Gamma(e, e) + \langle \text{grad}_{L \times N} \Gamma, \alpha(e, e) \rangle \). Therefore,
\[
\text{Hess}_{L \times N} \xi(q)(e, e) = \text{Hess}_L \gamma(e^L, e^L) + \langle \text{grad}_L \gamma, \text{grad}_L \eta \rangle \langle z(q) \rangle |e^P|^2
\]
\[+ \langle \text{grad}_L \gamma, \alpha(e, e) \rangle. \tag{62} \]

However,
\[
\text{Hess}_L \gamma \leq \sqrt{G(\gamma)} \left( \int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1 \right), \tag{63} \]
out of a compact subset of \( L \). By hypothesis, see (59),
\[
\langle \text{grad}_L \gamma, \text{grad}_L \eta \rangle \langle y(q) \rangle \leq |\text{grad}_L \gamma| \cdot |\text{grad}_L \eta|
\]
\[\leq \sqrt{G(\gamma)} \left( \int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1 \right) \ln \left( \int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1 \right). \tag{64} \]

Considering (63), (64) and (62) we have that (off a compact set)
\[
\text{Hess}_{L \times N} \xi(q)(e, e) \leq C \cdot \sqrt{G(\gamma)} \left( \int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1 \right) \ln \left( \int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1 \right)
\[+ \langle \text{grad}_L \gamma, \alpha(e, e) \rangle,
\]
for some positive constant \( C \). Thus, by (60) it follows that
\[
\triangle \gamma \leq B \sqrt{G(\gamma)} \left( \int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1 \right) \ln \left( \int_0^\gamma \frac{ds}{\sqrt{G(s)}} + 1 \right)
\]
for some positive constant \( B \). Concluding that \((G, \xi)\) is an Omori-Yau pair for the Laplacian in \( M \). The proof of [4, Thm.1] coupled with Lemma 4.6 allows us to state the following technical extension of Alias-Dajczer’s Theorem [4, Thm.1].
Theorem 4.7 (Alias-Dajczer) Let \( \varphi : M \to L \times \xi N \) be an isometric immersion, proper in the first entry, where \( L \) carries an Omori-Yau pair \( (G, \gamma) \) for the Hessian, \( \xi \in C^\infty(L) \) is a positive function satisfying
\[
|\text{grad} \log \xi(y)| \leq \ln \left( \int_0^{\gamma(y)} \frac{ds}{\sqrt{G(s)}} + 1 \right).
\] (65)

Letting \( \varphi(x) = (y(x), z(x)) \) and if
\[
|H(\varphi(x))| \leq \ln \left( \int_0^{\gamma(y(x))} \frac{ds}{\sqrt{G(s)}} + 1 \right).
\] (66)

Suppose that \( \varphi(M) \subset \{(y, z) : y \in L, z \in B_N(r)\} \) then
\[
\sup_M \xi |H| \geq (m - \ell)C_b(r),
\]
where \( b = \sup_{B_N(r)} K_{\text{rad}} \).

Remark 4.8 The Theorems 2.3 & 2.5 should have versions for \( \phi \)-bounded submanifold of warped product \( L \times \xi N \). Specially interesting should be the Jorge-Koutrofios Theorem in this setting. We leave to the interested reader to pursue it.

As a last application of Theorem 4.5, let \( N^{n+1} = I \times \xi P^n \) the product manifold endowed with the warped product metric, \( I \subset \mathbb{R} \) is an open interval, \( P^n \) is a compact Riemannian manifold and \( \xi : I \to \mathbb{R}_+ \) is a smooth function. Given an isometrically immersed hypersurface \( \varphi : M^n \to N^{n+1} \), define \( h : M^n \to I \) the \( C^\infty(M^n) \) height function by setting \( h = \pi_I \circ \varphi \), where \( \pi_I : I \times P \to I \) is a projection. This result below is a technical extension of [5, Thm.7] its proof is exactly as there, we just relaxed the hypothesis guaranteeing an Omori-Yau sequence.

Theorem 4.9 Let \( \varphi : M^n \to N^{n+1} \) be an isometrically immersed hypersurface. If \( M^n \) has an Omori-Yau pair \( (G, \gamma) \) for the Laplacian and the height function \( h \) satisfies
\[
\lim_{x \to \infty} \frac{h(x)}{\gamma'(x)} = 0
\]
then
\[
\sup_{M^n} |H| \geq \inf_{M^n} \mathcal{H}(h),
\]
(67)
with \( H \) being the mean curvature and \( \mathcal{H}(t) = \frac{\rho'(t)}{\rho(t)} \).

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