Research Article

Blow-Up Solution of a Porous Medium Equation with Nonlocal Boundary Conditions

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In this paper, we devote to studying the blow-up phenomena for a porous medium equation under nonlocal boundary conditions. Based on auxiliary functions and differential inequality technique, we derive the sufficient conditions to guarantee the existence of blow-up solutions under different measures and obtain an upper bound for blow-up time. Moreover, we demonstrate the lower bounds for blow-up time under some appropriate measures in $\mathbb{R}^3$ and in the higher-dimensional space $\mathbb{R}^n$ ($n \geq 3$), respectively. At last, two examples are given to illustrate the applications of main results.

1. Introduction

As is known to all, porous medium equations usually describe processes involving fluid flow, heat transfer, or diffusion. There are a lot of applications in physical phenomena, mathematical biology, lubrication, boundary layer theory, and other fields [1, 2].

The paper is focused on the following porous medium equation with time-dependent coefficient:

$$u_t = \Delta u^m + a(x)k_1(t)f(u), \quad (x,t) \in \Omega \times (0,t^*),$$

subject to nonlocal boundary and initial conditions:

$$\frac{\partial u}{\partial v} = k_2(t)\int_{\Omega} g(u)dx, \quad (x,t) \in \partial\Omega \times (0,t^*),$$

$$u(x,0) = u_0(x) \geq 0, \quad x \in \overline{\Omega},$$

where $m > 1$ and $\Omega \subset \mathbb{R}^n (n \geq 3)$ is a bounded convex domain with smooth boundary $\partial\Omega$. $\partial u/\partial v$ is the outward normal derivative on $\partial \Omega$, and $t^*$ is the maximal existence time of $u$. Set $\mathbb{R}_+ = (0, +\infty)$. Throughout the paper, we assume that $f, g \in C^1(\mathbb{R}_+)$ are nonnegative, the weight function $a \in C(\overline{\Omega})$ is positive, and $k_1, k_2 \in C^1(\mathbb{R}_+)$ are positive functions. $u_0 \in C(\overline{\Omega})$ is the initial value which satisfies compatibility conditions. By the degenerate parabolic theory in [3], one can deduce that the local classical solution of (1) exists uniquely and is nonnegative.

During the past decades, there have been many works to study the blow-up phenomena of parabolic equations and systems. The related results include the existence of blow-up and global solutions, the bounds of the blow-up time, and blow-up rate. We refer to [4–12] and references therein. Porous medium equations as representative examples of parabolic equations have been widely investigated by many scholars [13–17]. In order to research problem (1), the recent papers have aroused our interest. Xiao and Fang [14] considered the following porous medium equation under nonlinear boundary conditions:

$$u_t = \Delta u^m - k(t)f(u), \quad (x,t) \in \Omega \times (0,t^*),$$

$$\frac{\partial u}{\partial v} = g(u), \quad (x,t) \in \partial\Omega \times (0,t^*),$$

$$u(x,0) = u_0(x) \geq 0, \quad x \in \overline{\Omega},$$
where \( m \geq 1 \) and \( \Omega \subset \mathbb{R}^n (n \geq 2) \) is a bounded star-shaped region with smooth boundary \( \partial \Omega \). Using the auxiliary function method and modified differential inequality technique, they established some conditions on time-dependent coefficient and nonlinear functions to guarantee that the solution \( u(x, t) \) exists globally or blows up at some finite time. Moreover, the upper and lower bounds for the blow-up time were derived in the higher-dimensional space.

In [17], the authors dealt with the blow-up problem of the following porous medium equation subject to nonlinear boundary and initial conditions:

\[
\begin{aligned}
  u_t &= \Delta u^m + k_1(t)(u) \quad (x, t) \in \Omega \times (0, t^*), \\
  \frac{\partial u}{\partial y} &= g(u), \quad (x, t) \in \partial \Omega \times (0, t^*), \\
  u(x, 0) &= u_0(x) \geq 0, \quad x \in \overline{\Omega},
\end{aligned}
\]

where \( m > 1 \) and \( \Omega \subset \mathbb{R}^n (n \geq 2) \) is a bounded convex region with smooth boundary \( \partial \Omega \). Under appropriate assumptions on the data, a criterion was given to guarantee that solution \( u \) blows up at finite time, and an upper bound and a lower bound for blow-up time were derived.

Recently, there are some papers on the issue of studying reaction-diffusion equations under nonlocal boundary conditions. Marras and Vernier Piro [18] studied the following reaction-diffusion equation under nonlocal boundary conditions:

\[
\begin{aligned}
  u_t &= \Delta u + k_1(t)(u) \quad (x, t) \in \Omega \times (0, t^*), \\
  \frac{\partial u}{\partial y} &= k_2(t)\int_{\Omega} g(u)dx, \quad (x, t) \in \partial \Omega \times (0, t^*), \\
  u(x, 0) &= u_0(x) \geq 0, \quad x \in \overline{\Omega},
\end{aligned}
\]

where \( \Omega \) is a bounded convex region in \( \mathbb{R}^n (n \geq 2) \) with smooth boundary \( \partial \Omega \). Under some conditions on data, the authors showed the solution must blow up at finite time \( t^* \). At the same time, they obtained upper bounds of \( t^* \). When \( \Omega \subset \mathbb{R}^3 \), lower bounds of \( t^* \) were derived.

In [19], the authors were concerned about the following equations:

\[
\begin{aligned}
  u_t &= \nabla \cdot (\rho(u) \nabla u) + k_1(t)f(u) \quad (x, t) \in \Omega \times (0, t^*), \\
  \frac{\partial u}{\partial y} &= k_2(t)\int_{\Omega} g(u)dx, \quad (x, t) \in \partial \Omega \times (0, t^*), \\
  u(x, 0) &= u_0(x) \geq 0, \quad x \in \overline{\Omega},
\end{aligned}
\]

where \( \Omega \) is a bounded convex region in \( \mathbb{R}^n (n \geq 2) \) and the boundary \( \partial \Omega \) is smooth. By constructing some auxiliary functions and using differential inequality technique, they derived that the solution blows up at some finite time. Moreover, upper and lower bounds of the blow-up time were obtained.

In this paper, our main interest lies in blow-up phenomena of problem (1). Obviously, boundary conditions (2) and (3) are different from the one in [14, 17], and the main part of equation (1) is different from the problem in [18, 19]. Therefore, we need to establish the new auxiliary functions to prove our main results. By means of new auxiliary functions and differential inequality technique, we prove the existence of blow-up solutions under different measures and obtain an upper bound for blow-up time. Moreover, we demonstrate the lower bounds for blow-up time under some appropriate measures in \( \mathbb{R}^3 \) and in the higher-dimensional space \( \mathbb{R}^n (n \geq 3) \), respectively.

The rest of this paper is arranged as follows. In Section 2, a criterion for the blow-up solutions of problems (1)–(3) is established, and we obtain an upper bound for blow-up time. Section 3 shows that a lower bound is presented under two different measures when blow-up does occur. In Section 4, two examples are given to illustrate our main results.

### 2. Blow-Up Solution and Upper Bound for \( t^* \)

The section shows the blow-up solution for problem (1) in two different measures. And we get the sufficient conditions to ensure an upper bound for the blow-up time under two different measures. Suppose that the functions \( k_1, a, f \) satisfy

\[
\begin{align}
  f(s) &\geq s^{(1/a)+1}, \quad s > 0, \\
  k_1(t) &\geq \gamma, \quad t \geq 0, \\
  a(x) &\geq \beta, \quad x \in \mathbb{R}^n,
\end{align}
\]

where \( \alpha, \beta, \gamma \) are positive constants, and they satisfy

\[
\alpha m < \alpha + 1.
\]

To obtain our first result, we define the following auxiliary function:

\[
\Phi(t) = k_1(t) \int_{\Omega} \omega_1^2 dx.
\]

Let \( \lambda_1 \) be the first positive eigenvalue and \( \omega_1 \) be the corresponding eigenfunction of the following problem:

\[
\begin{aligned}
  \Delta \omega + \lambda \omega &= 0, \quad x \in \Omega, \\
  \omega &= 0, \quad x \in \partial \Omega,
\end{aligned}
\]

with

\[
\int_{\Omega} \omega^2 dx = 1.
\]

Now, we show our theorem for the blow-up solution in the measure \( \Phi(t) \).

**Theorem 1.** Let \( u \) be a nonnegative classical solution of problem (1). Assume that (8)–(10) and the following assumptions hold:
\[ \frac{k_1(t)}{k_1(t)} \geq \eta, \quad t \geq 0, \quad (14) \]
\[ \beta k_1^{a-n}(0) \Phi(0)^{a+1-an/a} - 2\lambda_1 > 0, \quad (15) \]
\[ \beta \Phi(0)^{a+1-an/a} - 2\lambda_1 \gamma^{a-an} > 0, \quad (16) \]

where \( \eta \) is a positive constant. Then, \( u \) blows up at some finite time \( t' \) in the measure \( \Phi(t) \), and
\[ t' \leq \int_{\Phi(0)}^{\Phi(t')} \frac{ds}{\alpha \eta s + \beta s^{a+1/n} - 2\lambda_1 \gamma^{a-an} m}. \quad (17) \]

**Proof.** Applying the divergence theorem and conditions (8)–(14), we have
\[ \Phi'(t) = ak_1^{a-n}(t)k_1(t) \int_{\Omega} \omega_1^2udx + k_1(t) \int_{\Omega} \omega_1^2u dx \]
\[ = \frac{k_1'(t)}{k_1(t)} k_1(t) \int_{\Omega} \omega_1^2udx + k_1(t) \int_{\Omega} \omega_1^2u dx \]
\[ \geq \alpha \eta \Phi(t) + k_1(t) \int_{\Omega} \omega_1^2(\Delta u^m + a(x)k_1(t)f(u)) dx \]
\[ \geq \alpha \eta \Phi(t) + k_1(t) \int_{\Omega} \Delta \omega_1^2u^m dx + \beta k_1^{a+1}(t) \int_{\Omega} \omega_1^2u^{(1/a)+1} dx \]
\[ \geq \alpha \eta \Phi(t) + 2k_1(t) \int_{\Omega} u^m[\omega_1^2]^2 dx + 2k_1(t) \int_{\Omega} \omega_1^2u^{(1/a)+1} dx \]
\[ \geq \alpha \eta \Phi(t) + 2k_1(t) \int_{\Omega} u^m[\omega_1^2]^2 dx + 2k_1(t) \int_{\Omega} \omega_1^2u^{(1/a)+1} dx \]
\[ \cdot \int_{\Omega} \omega_1^2u^m \Delta \omega_1 dx + \beta k_1^{a+1}(t) \int_{\Omega} \omega_1^2u^{(1/a)+1} dx. \quad (18) \]

From Hölder inequality, we get
\[ \int_{\Omega} \omega_1^2u^m dx \leq \left( \int_{\Omega} \omega_1^2u^{(1/a)+1} dx \right)^{a/(a+1)} \left( \int_{\Omega} \omega_1^2u dx \right)^{1/(a+1)} \]
\[ = \left( \int_{\Omega} \omega_1^2u^{(1/a)+1} dx \right)^{a/(a+1)} \cdot \int_{\Omega} \omega_1^2u^{(1/a)+1} dx. \quad (19) \]

Then, we rewrite (18) to gain
\[ \Phi'(t) \geq \alpha \eta \Phi(t) - 2\lambda_1k_1(t) \left( \int_{\Omega} \omega_1^2u^{(1/a)+1} dx \right)^{a/(a+1)} \]
\[ + \beta k_1^{a+1}(t) \int_{\Omega} \omega_1^2u^{(1/a)+1} dx \]
\[ \geq \alpha \eta \Phi(t) + k_1(t) \left( \int_{\Omega} \omega_1^2u^{(1/a)+1} dx \right)^{a/(a+1)} \]
\[ + \beta k_1(t) \left( \int_{\Omega} \omega_1^2u^{(1/a)+1} dx \right)^{1/(a+1)} - 2\lambda_1. \quad (20) \]

Using Hölder inequality again, we have
\[ \int_{\Omega} \omega_1^2u^m dx \leq \left( \int_{\Omega} \omega_1^2u^{(1/a)+1} dx \right)^{a/(a+1)} \left( \int_{\Omega} \omega_1^2u dx \right)^{1/(a+1)} \]
\[ = \left( \int_{\Omega} \omega_1^2u^{(1/a)+1} dx \right)^{a/(a+1)}. \quad (21) \]

We substitute (21) into (20) to get
\[ \Phi'(t) \geq \alpha \eta \Phi(t) + k_1(t) \left( \int_{\Omega} \omega_1^2u^{(1/a)+1} dx \right)^{a/(1+a)} \]
\[ \beta k_1(t) \left( \int_{\Omega} \omega_1^2u^{(1/a)+1} dx \right)^{a/(a+1)} - 2\lambda_1 \]
\[ \cdot \left\{ \beta k_1^{a+1}(t) \left( \int_{\Omega} \omega_1^2u^{(1/a)+1} dx \right)^{a/(a+1)} \right\} \]
\[ \cdot \left\{ \beta k_1^{a-n}(t) \Phi(t)^{a+1-an/a} - 2\lambda_1 \right\}. \quad (22) \]

Now, we prove that
\[ \beta k_1^{a-n}(t) \Phi(t)^{a+1-an/a} - 2\lambda_1 > 0, \quad t > 0. \quad (23) \]

Suppose that (23) does not hold. Set \( t_1 = \min \{ t > 0 \mid \beta k_1^{a-n}(t) \Phi(t)^{a+1-an/a} - 2\lambda_1 \leq 0 \} \). Then, for \( 0 \leq t < t_1 \), we have
\[ \beta k_1^{a-n}(t) \Phi(t)^{a+1-an/a} - 2\lambda_1 > 0. \quad (24) \]

According to (22), for \( 0 \leq t < t_1 \), we have \( \Phi'(t) > 0 \). Hence, (15) yields
\[ \Phi(t_1) > \Phi(0) > \frac{2\lambda_1}{\beta k_1^{a-n}(0)} \]
\[ \phi^{a+1-an/m}, \quad (25) \]
that is,
\[ \beta k_1^{a-n}(0) \Phi(t_1)^{a+1-an/a} - 2\lambda_1 > 0. \quad (26) \]

Since \( k_1(t) \) is increasing, we get
\[ \beta k_1^{a-n}(t_1) \Phi(t_1)^{a+1-an/a} - 2\lambda_1 > 0. \quad (27) \]

There is a contradiction. Therefore, (23) holds. Similarly, from (16), we deduce
\[ \beta \Phi(t)^{(a+1-an)/m} - 2\lambda_1 \gamma^{a-an} > 0. \quad (28) \]

Next, applying (21) to (22), we obtain
\[ \Phi'(t) \geq \alpha \eta \Phi(t) + k_1(t) \Phi(t)^{a+1-an/a} \]
\[ \beta k_1^{a-n}(t) \Phi(t)^{a+1-an/a} - 2\lambda_1 \]
\[ = \alpha \eta \Phi(t) + \beta \Phi(t)^{a+1/an} - 2\lambda_1k_1^{a-n}(t) \Phi(t)^{a+1-an/a} \]
\[ \geq \alpha \eta \Phi(t) + \beta \Phi(t)^{a+1/an} - 2\lambda_1 \gamma^{a-an} \Phi(t)^{a+1-an/a} \]
\[ = \alpha \eta \Phi(t) + \Phi(t)^{a+1-an/a} - 2\lambda_1 \gamma^{a-an} \]
\[ > 0, \quad t > 0, \quad (29) \]

where (23) and (28) are used. Integrating (29) from 0 to \( t \) yields
\[ t \leq \int_{\Phi(0)}^{\Phi(t)} \frac{ds}{\alpha \eta s + \beta s^{a+1/an} - 2\lambda_1 \gamma^{a-an} m}. \quad (30) \]
From (30), we can conclude that the solution $u$ blows up at some finite time $t^*$ in the measure $\Phi(t)$. In fact, assume that $u$ remains global in the measure $\Phi(t)$; then, $\Phi(t) < +\infty$, $t > 0$, and
\[
t \leq \int_{\phi(0)}^t \frac{ds}{a \eta s + \beta s^{1+1/\alpha} - 2 \lambda_1 y^{\alpha - am} g^m}.
\] (31)

When $t \rightarrow +\infty$, we have
\[
\int_{\phi(0)}^\infty \frac{ds}{a \eta s + \beta s^{1+1/\alpha} - 2 \lambda_1 y^{\alpha - am} g^m} = +\infty.
\] (32)

However, since $\Phi(0) > 0$, $(\alpha + 1/\alpha) > m > 1$, we have
\[
\int_{\phi(0)}^\infty \frac{ds}{a \eta s + \beta s^{1+1/\alpha} - 2 \lambda_1 y^{\alpha - am} g^m} < +\infty,
\] (33)

that is, it is a contradiction. Then, the solution $u$ blows up at some finite time $t^*$ in the measure $\Phi(t)$. Taking the limit as $t \rightarrow t^*$ in (30), we obtain
\[
t^* \leq \int_{\phi(0)}^\infty \frac{ds}{a \eta s + \beta s^{1+1/\alpha} - 2 \lambda_1 y^{\alpha - am} g^m}.
\] (34)

In the following corollary, we will present the blow-up solution for (1) in the measure $\Psi(t)$, where
\[
\Psi(t) = \int_\Omega \omega_1^2 x dx.
\] (35)

From the proof similar to Theorem 1, we get the following conclusion:

**Corollary 1.** Let $u$ be a nonnegative classical solution of problem (1). Assume that (8)–(10) and the following assumptions hold:
\[
\beta \Psi(0) y^{1-\alpha/m} - 2 \lambda_1 > 0.
\] (36)

Then, $u$ blows up at some finite time $T$ in the measure $\Psi(t)$, and
\[
T \leq \int_{\Psi(0)}^{\infty} \frac{ds}{\beta \Psi s^{1+1/\alpha} - 2 \lambda_1 s g^m}.
\] (37)

**3. Lower Bound for $t^*$**

The section presents lower bounds for blow-up time $t^*$ under two different measures when the solution of (1)–(3) blows up at some finite time. Firstly, we state two lemmas from [20] and some inequalities, which play a basic role in the process of our proof.

**Lemma 1.** Let $\Omega$ be a bounded star-shaped domain in $\mathbb{R}^N$, $(N \geq 2)$. Then, we have
\[
\int_{\Omega} u^n ds \leq \frac{N}{\rho_0} \int_{\Omega} u^n dx + \frac{n d}{\rho_0} \int_{\Omega} \frac{u^{m-1} |Vu|^2 dx}{x},
\] (38)

where $u$ is a nonnegative $C^1$-function, $\rho_0 = \min_{\Omega} x \cdot \nu$, $d = \max_{\overline{\Omega}} |x|$, and $\nu$ is the unit outward normal vector on $\partial \Omega$.

**Lemma 2.** Let $\Omega$ be a bounded domain in $\mathbb{R}^3$ assumed to be star-shaped and convex in two orthogonal directions. Then, we have
\[
\int_{\Omega} u^{(3/2)m} dx \leq \left[ \frac{3}{2 \rho_0} \int_{\Omega} u^n dx + \frac{n}{2} \left( 1 + \frac{d}{\rho_0} \right) \int_{\Omega} \frac{u^{m-1} |Vu|^2 dx}{x} \right]^{3/2},
\] (39)

where $u$ is the nonnegative $C^1$-function, $n \geq 1$, $\rho_0$ and $d$ have the same meaning as in Lemma 1.

In the proof process of the main results, we need to use the following inequality:
\[
(a + b)^l \leq a^l + b^l, \quad a, b > 0, \quad 0 < l < 1,
\] (40)

and the following Sobolev inequality $(n \geq 3)$ given in [21]:
\[
\|u^{n/2}\|_{L(2n/(n-2); \Omega)} \leq C \|u^{n/2}\|_{W^{1/2}(\Omega)},
\] (41)

where $C$ is a constant depending on $\Omega$ and $n$, that is,
\[
\left( \int_{\Omega} u^{n/2} dx \right)^{2/2n} \leq C \left( \int_{\Omega} u^n dx + \int_{\Omega} |\nabla u|^{2n} dx \right)^{1/2},
\] (42)

and Young’s inequality yields that
\[
a^{\tau_1} b^{\tau_2} \leq \left( 1 - \tau_2 \right) e^{\tau_1/\tau_2} a^{\tau_1/\tau_2 - 1} + \tau_2 b^{\tau_2},
\] (43)

\[
a, b > 0, \quad 0 < \tau_2 < 1, \quad \epsilon > 0.
\] (44)

For convenience, we suppose that the following assumptions on functions $a, f, g$ are satisfied:
\[
a(x) \leq \rho_0, \quad x \in \Omega.
\] (44)

\[
0 \leq g(s) \leq s^{\epsilon 1},
\] (45)

\[
0 \leq f(s) \leq s^{\epsilon 2},
\] (46)

\[
s > 0.
\] (47)

In the sequel, we restrict that $\Omega \subset \mathbb{R}^3$, and we define the new auxiliary function:
\[
A(t) = k_1(t) \int_{\Omega} u^\theta dx, \quad t \geq 0,
\] (48)

where $\mu, \theta$ are positive constants, and
\[
\theta > \max \left\{ \frac{3}{2} (m - 3), 2 (m - 2), m - 1 \right\}, \quad m > 3.
\] (49)

Now, we show our main results.

**Theorem 2.** Let $u$ be a nonnegative classical solution of problem (1). Suppose that (44)–(45) hold, and the following conditions are satisfied:
Assume that $u$ becomes unbounded in the measure $A(t)$ at some finite time $t^*$. Then, $t^*$ is bounded below by

$$
t^* \geq \left( \int_0^t F_2(s) \left( \int_0^s F_1(\tau) d\tau \right)^{-1} \left( \frac{1}{2A^2(0)} \right) \right),
$$

where

$$
F_1(t) = \mu \eta_0 + \frac{3m(2\theta - m - r_2 + 2)}{2\rho_0} \left( \frac{3 \sqrt{3}}{2} \right)^{2(\theta - m)/3} |\Omega|^{1-(r_1/\rho)} \left( \frac{1}{\rho_0} \right)^{3(2\theta - m)/3} k_2(t) k_1(\mu^{(m+2r_1-2)\theta})(t) + \frac{4\theta^2 - 2\theta r_2 - (8\theta - 3m + 3)(m - 2)}{4\theta^2 - 6\theta (m - 2)} \left( 1 + \frac{d\theta}{\rho_0} \right)^{3(m-1)/2} |\Omega|^{3(3)(m-1) + 3\theta^2 - 4\theta r_1 + (8\theta - 3m + 3)(m - 2)} (t) + \frac{dm(\theta + m - 2)(2\theta - m - 2r_1 + 3)}{4\rho_0}.
$$

$$
F_2(t) = \frac{3m(m + r_1 - 2)}{2\rho_0} \left( \frac{3 \sqrt{3}}{2} \right)^{2(\theta - m)/3} |\Omega|^{1-(r_1/\rho)} \left( \frac{1}{\rho_0} \right)^{3(2\theta - m)/3} k_2(t) k_1(\mu^{(m+2r_1-2)\theta})(t) + \frac{2\theta r_1 + (2\theta - 3m + 3)(m - 2)}{2\rho_0} \cdot \epsilon_1^{-(3(m-1) - 2\theta - 3)(m-2)} \left( \frac{1}{\rho_0} \right)^{2(\theta - m)/3} |\Omega|^{3(m-1) + 3\theta^2 - 4\theta r_1 + (8\theta - 3m + 3)(m - 2)} (t) + \frac{dm(\theta + m - 2)(2\theta - m + 3)}{4\rho_0}.
$$
where $\epsilon_i (i = 1, 2, 3, 4)$ are defined by the following equalities:

\[
\begin{align*}
\epsilon_1 &= \frac{1}{6} \left\{ 9(m - 2) \left( \frac{1}{2} \right)^{2(m - 2)/\theta} \left( \theta + \frac{d\theta}{\rho_0} \right)^{2(m - 2)/\theta} |\Omega|^{3(m - 1)(m - 2) + 2\theta(\theta - r_1)/2\theta} k_2(t) \right\}^{-1} \theta(\theta - 1), \\
\epsilon_2 &= \frac{1}{6} \left\{ d(\theta + m - 2) \left( \frac{3\sqrt{3}}{2} \right)^{m - 3/\theta} \left( \frac{1}{\rho_0} \right)^{3(m - 2)/\theta} |\Omega|^{1 - (1/\theta)} k_2(t) \right\}^{-1} (\theta - 1), \\
\epsilon_3 &= \frac{1}{6} \left\{ d(\theta + m - 2)(2\theta + m - 9) \left( \frac{1}{2} \right)^{m - 3/\theta} \left( \theta + \frac{d\theta}{\rho_0} \right)^{3(m - 2)/2\theta} |\Omega|^{1 - (m - 1)(m - 3) + 4\theta - 4r_1/2\theta} k_2(t) \right\}^{-1} \cdot \theta(\theta - 1), \\
\epsilon_4 &= \frac{1}{2} \left\{ \frac{3b_0(r_2 - 1)}{2} \left( \frac{r_1 - 1}{2} \right)^{\theta(1 - 1/\theta)} \left( \theta + \frac{d\theta}{\rho_0} \right)^{3(r_1 - 1)/\theta} |\Omega|^{1 - (r_1 - 1)/2\theta} k_1(t) \right\}^{-1} \theta(\theta - 1)m,
\end{align*}
\]

$|\Omega|$ is the volume of $\Omega$, $\rho_0 = \min_d x \cdot v$, and $d = \max_{\Omega} |x|$. Proof. With the help of (44), (45), (48) and the divergence theorem, we have

\[
A'(t) = \mu \kappa_1(t) k_\mu(t) \int_{\Omega} u^\theta dx + \theta k_\mu(t) \int_{\Omega} u^{\theta - 1} (\Delta u + a(x)k_1(t)f(u)) dx 
\]

\[
\leq \mu \eta_0 A(t) + \theta m k_2(t) \kappa_\mu(t) \int_{\Omega} u^{\theta + m - 2} \int_{\Omega} g(u) dx - \theta(\theta - 1) mk_\mu(t) \int_{\Omega} u^{\theta + m - 3} \cdot |\nabla u|^2 dx 
\]

\[
+ \theta b_0 k_\mu(t) \int_{\Omega} u^{\theta - 1} f(u) dx 
\]

\[
\leq \mu \eta_0 A(t) + \theta m k_2(t) \kappa_\mu(t) \int_{\Omega} u^{\theta + m - 2} \int_{\Omega} u^\gamma dx - \theta(\theta - 1) mk_\mu(t) \int_{\Omega} u^{\theta + m - 3} \cdot |\nabla u|^2 dx 
\]

\[
+ \theta b_0 k_\mu(t) \int_{\Omega} u^{\theta + r_1 - 1} dx.
\]

In the following, we deal with the second term of the right-hand side of (58). According to Lemma 1, we have

\[
\int_{\Omega} u^{\theta + m - 2} dx \leq \frac{3}{\rho_0} \int_{\Omega} u^{\theta + m - 2} dx + \frac{d(\theta + m - 2)}{\rho_0} \int_{\Omega} u^{\theta + m - 3} |\nabla u| dx. 
\]

From Hölder's inequality, we have

\[
\int_{\Omega} u^{\theta + m - 2} dx \leq \left( \int_{\Omega} u^{(3/2)m dx} \right)^{2m - 4/\theta} \left( \int_{\Omega} u^\theta dx \right)^{1 - (2m - 4/\theta)},
\]

where $0 < (2m - 4/\theta) < 1$. According to Lemma 2, we have

\[
\int_{\Omega} u^{(3/2)m dx} \leq \frac{3}{2\rho_0} \int_{\Omega} u^\theta dx + \frac{d}{\rho_0} \int_{\Omega} u^{\theta - 1} |\nabla u| dx \right]^{3/2}.
\]

The inequality $(a + b)^{3/2} \leq \sqrt{2a^{3/2}} + \sqrt{2b^{3/2}}$ implies
\[
\int_\Omega u^{(3/2)\theta} \, dx \leq \sqrt{2} \left( \frac{3}{2\rho_0} \right)^{3/2} \left( \int_\Omega u^\theta \, dx \right)^{3/2} + \sqrt{2} \left( \frac{\theta}{2} \left( 1 + \frac{d}{\rho_0} \right) \right)^{3/2} \left( \int_\Omega u^{\theta-1} |\nabla u| \, dx \right)^{3/2} \\
\leq \sqrt{2} \left( \frac{3}{2\rho_0} \right)^{3/2} \left( \int_\Omega u^\theta \, dx \right)^{3/2} + \sqrt{2} \left( \frac{\theta}{2} \left( 1 + \frac{d}{\rho_0} \right) \right)^{3/2} \left( \int_\Omega u^{\theta-m+1} \, dx \right)^{3/4} \left( \int_\Omega u^{\theta-m+3} |\nabla u|^2 \, dx \right)^{3/4} \\
\leq \sqrt{2} \left( \frac{3}{2\rho_0} \right)^{3/2} \left( \int_\Omega u^\theta \, dx \right)^{3/2} + \sqrt{2} \left( \frac{\theta}{2} \left( 1 + \frac{d}{\rho_0} \right) \right)^{3/2} |\Omega|^{3(m-1)/4 \theta} \left( \int_\Omega u^\theta \, dx \right)^{3(\theta-m+1)/4 \theta} \\
\cdot \left( \int_\Omega u^{\theta-m+3} |\nabla u|^2 \, dx \right)^{3/2},
\]

where Hölder inequality is used. Then, after inserting (59) into (57), inequality (41) yields

\[
\int_\Omega u^{\theta-m+1} \, dx \leq \frac{3 \sqrt{3}}{2} \left( \frac{1}{\rho_0} \right)^{3(\theta-1)/2} \left( \int_\Omega u^\theta \, dx \right)^{1+(\theta-1)/2} + \frac{1}{2} \left( \theta + \frac{d}{\rho_0} \right)^{3(\theta-1)/2} \left( \int_\Omega u^{\theta-m+3} |\nabla u|^2 \, dx \right)^{3(\theta-1)/2}
\]

Similarly, from (57)–(60), we can get

\[
\int_\Omega u^{\theta-m-3} |\nabla u| \, dx \leq \left( \int_\Omega u^{\theta-m-3} \, dx \right)^{1/2} \left( \int_\Omega u^{\theta-m-3} |\nabla u|^2 \, dx \right)^{1/2} \\
\int_\Omega u^{\theta-m-3} \, dx \leq \frac{3 \sqrt{3}}{2} \left( \frac{1}{\rho_0} \right) \left( \int_\Omega u^\theta \, dx \right)^{1+(\theta-m-3)/2} + \frac{1}{2} \left( \theta + \frac{d}{\rho_0} \right) \left( \int_\Omega u^{\theta-m+3} |\nabla u|^2 \, dx \right)^{3(\theta-m-3)/2}
\]

Then, combining (61) and (62), we have

\[
\int_\Omega u^{\theta-m-3} |\nabla u| \, dx \leq \left( \frac{3 \sqrt{3}}{2} \right)^{2(\theta-1)/2} \left( \frac{1}{\rho_0} \right)^{3(\theta-1)/2} \left( \int_\Omega u^\theta \, dx \right)^{1+(\theta-1)/2} + \frac{1}{2} \left( \theta + \frac{d}{\rho_0} \right)^{3(\theta-1)/2} \left( \int_\Omega u^{\theta-m+3} |\nabla u|^2 \, dx \right)^{3(\theta-1)/2}
\]

\[
\cdot |\Omega|^{3(m-1)(m-3)/2 \theta^2} \left( \int_\Omega u^\theta \, dx \right)^{1-(\theta+3m-3)(m-3)/2 \theta^2} \left( \int_\Omega u^{\theta-m+3} |\nabla u|^2 \, dx \right)^{3(m-3)/2 \theta} \left( \int_\Omega u^{\theta-m-3} |\nabla u|^2 \, dx \right)^{1/2}
\]

\[
\cdot \left( \int_\Omega u^{\theta-m-3} |\nabla u|^2 \, dx \right)^{1/2}
\]
Similarly, from (57)–(60), we get

\[
\begin{align*}
\int_{\Omega} u^{\theta + m - 3} |\nabla u|^2 \, dx 
&\leq \left( \frac{3\sqrt{3}}{2} \right)^{m - 3/\theta} \left( \frac{1}{\rho_0} \right)^{3(m-1)/2\theta} \left( \int_{\Omega} u^\theta \, dx \right)^{1/2} (\int_{\Omega} u^{\theta + m - 3} |\nabla u|^2 \, dx)^{1/2} \\
&\quad + \left( \frac{1}{2} \right)^{m - 3/\theta} \left( \theta + d \theta \rho_0 \right) \left( \frac{3(m-3)/2\theta}{\rho_0} \right)^{3(m-1)/2\theta} \left( \int_{\Omega} u^\theta \, dx \right)^{1/2 - \left( (\theta + 3m - 3)(m - 3)/4\theta \right)}
\end{align*}
\]

(63)

where inequality (41) is applied again. According to (60), (63), and Hölder inequality, the second term of the right-hand side of (55) is changed into

\[
\begin{align*}
\int_{\partial \Omega} u^{\theta + m - 2} \, dS &\leq \left( \frac{3}{\rho_0} \right)^{2(m-2)/\theta} \left\{ \left( \frac{1}{\rho_0} \right)^{3(m-2)/\theta} \left( \int_{\Omega} u^\theta \, dx \right)^{1/\theta} (\int_{\Omega} u^{\theta + m - 3} |\nabla u|^2 \, dx)^{1/\theta} \\
&\quad + \left( \frac{1}{2} \right)^{3(m-2)/\theta} \left( \theta + d \theta \rho_0 \right) \left( \frac{3(m-3)/2\theta}{\rho_0} \right)^{3(m-1)/2\theta} \left( \int_{\Omega} u^\theta \, dx \right)^{1/\theta - \left( (2\theta r_1 - (\theta + 3m - 3)(m - 2)/2\theta \right)}
\end{align*}
\]

(64)

Now, we estimate the last term of the right-hand side of (55). Similarly, from (57)–(60), we get

\[
\begin{align*}
\int_{\Omega} u^{\theta + r - 1} \, dx &\leq \left( \frac{3\sqrt{3}}{2} \right)^{2(r_{-1}/\theta)} \left( \frac{1}{\rho_0} \right)^{3(r_{-1}/\theta)} \left( \int_{\Omega} u^\theta \, dx \right)^{1 + (r_{-1}/\theta)} (\int_{\Omega} u^{\theta + m - 3} |\nabla u|^2 \, dx)^{2(r_{-1}/\theta)} \left( \theta \left( 1 + d \theta \rho_0 \right) \right)^{3(r_{-1}/\theta)}
\end{align*}
\]

(65)
Complexity

Then, we deal with the terms in (64) and (65) including
\[ \int_{\Omega} u^{(\theta, \rho, m)} \|\nabla u\|^2 \, dx. \]
It follows from (43) that

\[
\begin{align*}
\left( \int_{\Omega} u^\theta \, dx \right)^{1 + \frac{(2\theta m - (\theta + 3m - 3)(m - 2)/2\theta)}{2\theta}} & + \frac{2\theta - 3(m - 2)}{2\theta} \left( \int_{\Omega} u^{(\theta, \rho, m)} \|\nabla u\|^2 \, dx \right) \\
\leq & \frac{2\theta - 3(m - 2)}{4\theta} \left( \int_{\Omega} u^\theta \, dx \right)^{1 + \frac{2\theta m - (\theta + 3m - 3)(m - 2)/2\theta}{2\theta}} \left( \int_{\Omega} u^{(\theta, \rho, m)} \|\nabla u\|^2 \, dx \right) \\
& \leq \frac{2\theta - 3(m - 2)}{2\theta} \epsilon_4 \left( \int_{\Omega} u^{(\theta, \rho, m)} \|\nabla u\|^2 \, dx \right)^{3(r - 1)/2\theta}
\end{align*}
\]

where \( \epsilon_i \) (i = 1, 2, \ldots, 4) are defined by (56). Therefore, we substitute (64)–(69) into (55) and get

\[
\begin{align*}
A'(t) \leq & \mu_1 A(t) + \frac{3\rho m}{\rho_0} \left( \frac{3\sqrt{3}}{2} \right)^{(\alpha-1)/\alpha} \left( \frac{1}{\rho_0} \right)^{3(m-2)/2\theta} k_2(t) k_1(\rho_0) A^{*+(\alpha-1)/\alpha}(t) \\
& + \frac{3m(2\theta - 3m + 6)}{2\theta} \left( \frac{3\sqrt{3}}{2} \right)^{3(m-2)/2\theta} \epsilon_1 \left( \int_{\Omega} u^{(\theta, \rho, m)} \|\nabla u\|^2 \, dx \right) \\
& + \frac{3\rho m(2\theta - 3m + 6)}{4\theta} \left( \frac{3\sqrt{3}}{2} \right)^{3(m-2)/2\theta} \epsilon_3 \left( \int_{\Omega} u^{(\theta, \rho, m)} \|\nabla u\|^2 \, dx \right) \\
& + \frac{3\rho m(2\theta - 3m + 6)}{4\theta} \left( \frac{3\sqrt{3}}{2} \right)^{3(m-2)/2\theta} \epsilon_4 \left( \int_{\Omega} u^{(\theta, \rho, m)} \|\nabla u\|^2 \, dx \right)
\end{align*}
\]

From (47)–(50), note that
Taking the limit as $t \mapsto t^*$ in (75), we get
\[
\int_0^{t^*} F_2(s) e^{2 \int_0^s F_1(r) \, dr} \, ds \geq \frac{1}{2A^2(0)}.
\] (76)

Note that $\int_0^{t^*} F_2(s) e^{2 \int_0^s F_1(r) \, dr} \, ds$ is a strictly increasing function, and we can conclude that
\[
t^* \geq \left( \int_0^{t^*} F_2(s) e^{2 \int_0^s F_1(r) \, dr} \, ds \right)^{-1} \left( \frac{1}{2A^2(0)} \right).
\] (77)

Then, we finish the proof.

And we assume that $\Omega \subset R^n (n \geq 3)$ is a smooth bounded domain. The functions $k_1$ and $k_2$ satisfy
\[
k_1(t) \leq \eta_1, \\
k_2(t) \leq \eta_2,
\] (78)
t > 0,

where $\eta_1$ and $\eta_2$ are some positive constants. Now, we define the new auxiliary function:
\[
B(t) = \int_{\Omega} u^\delta \, dx, \quad t \geq 0,
\] (79)

where $\delta$ is a positive constant, and it satisfies
\[
\delta > \max \left\{ 3 - m, r_1, \frac{n(r_2 - m)}{2} \right\}, \quad r_2 > 1.
\] (80)

**Theorem 3.** Let $u$ be a nonnegative classical solution of problem (1). Suppose that (44), (45), and (78) hold. Assume that $u$ becomes unbounded in the measure $B(t)$ at some finite time $t^*$. Then, $t^*$ is bounded below by
\[
t^* \geq \int_{B(\Omega)} \frac{dr}{\varphi(t)}.
\] (81)

where
\[
\varphi(t) = Q_1 t^{m + 8 + \alpha_2 - 2/\delta} + Q_2 t^{\delta + \alpha_2 - 1/\delta} + Q_3 t^{[2\delta(\delta + m - 2) + r_1(m + 2\delta - n)](\delta + m - 1)/2\delta^2 (\delta + m - 1) + \delta n(m - 1)}
\]
\[
+ Q_4 t^{(m + 8 + \alpha_2 - 2/\delta)(\delta + m - 1) + \delta (\delta + m - 1) + \delta n(m - 1)}
\]
\[
+ Q_5 t^{[2\delta (\delta + m - 2) + r_1(m + 2\delta - n)](\delta + m - 1)/2\delta^2 (\delta + m - 1) + \delta n(m - 1) + \delta^2 (m + 1)}
\]
\[
+ Q_6 t^{(m + 2\delta - n) \alpha_2 \alpha_2 (r_2 - 1)(4\delta + n(m - 1)(4 - \delta))/[m + 2\delta - n] \alpha_2 \alpha_2 (m - 1)(\delta - 1) + \delta n(m - 1) + \delta^2 (m + 1)}
\]
\[
+ Q_7 t^{4\delta - (\delta - 1)(\delta - 2) + \delta n(m - 1)}
\] (82)

where $Q_i (i = 1, 2, \ldots, 7)$ are defined by the following equalities:
\[ Q_1 = \frac{\delta m \eta_2 n}{\rho_0} C^\alpha (m-1)(\delta + m - 2)/(\delta + m - 1) |\Omega|^{(1/\delta + m - 1) + (\delta - r_1/\delta)}, \]

\[ Q_2 = \delta b_0 \eta_1 C^{Q_1(r_2 - 1)/\delta}, \]

\[ Q_3 = 2\delta (\delta + m - 1) + n(m - 1) e^{-(n(m - 1)(\delta + m - 2)/(\delta + m - 1) + n(m - 1)) C_1}, \]

\[ Q_4 = \frac{1}{2} e^{-1} C_2, \]

\[ Q_5 = \frac{mn - n + \delta (\delta + m - 1)}{(mn + 2\delta - n)(\delta + m - 1)} e^{-(n(m - 1)(\delta + m - 2)/(\delta + m - 1)/mn + \delta (\delta + m - 1)) C_3}, \]

\[ Q_6 = \left(1 - \frac{n^2 (m - 1)(r_2 - 1)}{(mn + 2\delta - n)^2}\right) e^{-(n^2 (m - 1)(r_2 - 1)/(mn + 2\delta - n)^2 - n^2 (m - 1)/r_2 - 1)) C_4}, \]

\[ Q_7 = \frac{mn + 2\delta - n r_2}{mn + 2\delta - n} e^{-(n(r_2 - 1)(mn + 2\delta - n)) C_7 C_8 Q_1(2n(r_2 - 1)(mn + 2\delta - n))}. \]

Here,

\[ C_1 = \frac{\delta m \eta_2 n}{\rho_0} \left(\frac{2\delta + n(m - 1)}{2\delta}\right)^{\delta + m - 2/\delta + m - 1} C^{2n(m - 1)(\delta + m - 2)/(mn + 2\delta - n)(\delta + m - 1)} |\Omega|^{(1/\delta + m - 1) + (\delta - r_1/\delta)}, \]

\[ C_2 = \frac{2 \delta m \eta_1 (\delta + m - 2)}{2\delta} C^{2n(m - 1)(\delta + m - 3/2)/(\delta + m - 1)} |\Omega|^{(1/\delta + m - 1) + (\delta - r_1/\delta)}, \]

\[ C_3 = \frac{2 \delta m \eta_1 (\delta + m - 2)}{\rho_0 (\delta + m - 1)} \left(\frac{2\delta + n(m - 1)}{2\delta}\right)^{\delta + m - 3/2(\delta + m - 1)} C^{2n(m - 1)(\delta + m - 3)/(mn + 2\delta - n)(\delta + m - 1)} |\Omega|^{(1/\delta + m - 1) + (\delta - r_1/\delta)}, \]

\[ C_4 = \delta b_0 \eta_1 \frac{2\delta + n(m - 1)}{2\delta} C^{4n(r_2 - 1)(mn + 2\delta - n)(mn + 2\delta - n)^2}, \]

\[ \varepsilon = \left\{ \begin{array}{l} n(m - 1)(\delta + m - 2)/(mn + 2\delta - n)(\delta + m - 1) C_1 + \frac{1}{2} C_2 + \left(\frac{n(m - 1)(\delta + m - 3)}{2(mn + 2\delta - n)(\delta + m - 1)} + \frac{1}{2}\right) C_3 + \left(\frac{n^2 (m - 1)(r_2 - 1)}{mn + 2\delta - n} C_4 + \frac{n(r_2 - 1)}{mn + 2\delta - n}\right)^{-1} C_7 C_8 Q_1(2n(r_2 - 1)(mn + 2\delta - n)) \end{array} \right. \]
that is the volume of $\Omega$, $\rho_0 = \min_{\Omega} x \cdot v$, and $d = \max_{\Omega} |x|$.

**Proof.** By assumptions (44), (45), and (78) and the divergence theorem, we obtain

$$B'(t) = \delta \int_{\Omega} u^{\delta-1} u dx$$

$$= \delta \int_{\Omega} u^{\delta-1} (\Delta u^m + a(x) k_1(t) f(u)) dx$$

$$= \delta \int_{\Omega} u^{\delta-1} \Delta u^m dx + \delta k_1(t) \int_{\Omega} a(x) u^{\delta-1} f(u) dx$$

$$= \delta \int_{\Omega} \nabla \cdot (u^{\delta-1} \nabla u^m) dx - \delta \int_{\Omega} \nabla u^{\delta-1} \cdot \nabla u^m dx$$

$$+ \delta k_1(t) \int_{\Omega} a(x) u^{\delta-1} f(u) dx$$

$$= \delta m \int_{\partial \Omega} u^{\delta m-2} \frac{\partial u}{\partial v} dS - \delta(\delta - 1) n \int_{\Omega} u^{\delta m-3} |\nabla u|^2 dx$$

$$+ \delta k_1(t) \int_{\Omega} a(x) u^{\delta-1} f(u) dx$$

Now, we estimate the first term of the right-hand side of (101). Note that $0 < (r_1/d) < 1$, and Hölder inequality implies that

$$\int_{\Omega} u^r dx \leq \left( \int_{\Omega} u^\delta dx \right)^{r/\delta} |\Omega|^{1-(r/\delta)}. \quad (86)$$

Owing to Lemma 1, we have

$$\int_{\Omega} u^{\delta m-2} dx \leq \left( \int_{\Omega} u^{\delta m-1} dx \right)^{\delta m-2/\delta m-1} |\Omega|^{1/\delta m-1}, \quad (88)$$

$$\int_{\Omega} u^{\delta m-3} |\nabla u| dx \leq \left( \int_{\Omega} u^{\delta m-3} dx \right)^{1/2} \left( \int_{\Omega} u^{\delta m-3} |\nabla u|^2 dx \right)^{1/2} \quad (89)$$

$$\int_{\Omega} u^{\delta m-1} dx \leq \left( \int_{\Omega} u^{\delta m-1} dx \right)^{1/\delta m-1} \left( \int_{\Omega} u^{\delta m-3} |\nabla u|^2 dx \right)^{1/2}, \quad (90)$$

From Hölder inequality, we have

$$\int_{\Omega} u^{\delta m-2} ds \leq \frac{n}{\rho_0} \int_{\Omega} u^{\delta m-2} dx + \frac{d(\delta + m - 2)}{\rho_0} \int_{\Omega} u^{\delta m-3} |\nabla u| dx. \quad (87)$$
where $0 < ((m - 1)(n - 2)/mn + 2\delta - n) < 1$. And it follows from Sobolev inequality (42) and inequality (41) that

$$
\int_{\Omega} u^{\delta + m - 2} \, dx \leq \left( \int_{\Omega} u^{\delta} \, dx \right)^{n(m - 1)/\delta} + \frac{2\delta + n(m - 1)}{2\delta} C^{2n(m - 1)/mn + 2\delta - n} \left( \int_{\Omega} u^{\delta + m - 1} \, dx \right)
$$

Inserting (92) into (88) and (89), respectively, we have

$$
\int_{\Omega} u^{\delta + m - 2} \, dx \leq \left( \int_{\Omega} u^{\delta} \, dx \right)^{n(m - 1)/\delta} + \frac{2\delta + n(m - 1)}{2\delta} C^{2n(m - 1)/mn + 2\delta - n} \left( \int_{\Omega} u^{\delta + m - 1} \, dx \right)
$$
\[
\int_{\Omega} u^{\delta + m - 3} |\nabla u| \, dx \leq \left( C^{n(m-1)/\delta} \left( \int_{\Omega} u^{\delta} \, dx \right)^{m+\delta - 1/\delta} + \frac{2\delta + n(m - 1)}{2\delta} C^{2n(m-1)(m+\delta-n)} \left( \int_{\Omega} u^{\delta} \, dx \right)^{2(m+\delta-1)/(m+\delta-n)} \right) \\
\cdot \left( \int_{\Omega} |\nabla u|^{\delta + m - 1/2} \, dx \right)^{\frac{n(m-1)/\delta}{\delta + m - 1/2(\delta + m - 1)}} \left( \int_{\Omega} u^{\delta} \, dx \right)^{1/2} \\
\leq \frac{2}{\delta + m - 1} C^{n(m-1)/(\delta+m-3)/2\delta(\delta+m-1)} |\Omega|^{1/\delta+m-1} \left( \int_{\Omega} u^{\delta} \, dx \right)^{m+\delta - 3/2\delta} \\
\cdot \left( \int_{\Omega} |\nabla u|^{\delta + m - 1/2} \, dx \right)^{1/2} \\
\cdot \left( \int_{\Omega} u^{\delta} \, dx \right)^{n(m-1)/(\delta+m-3)/(m+\delta-n)(\delta+m-1)} \delta^{1/2},
\]

where inequality (41) is used again. Combining (86) and (87) and (93) and (94), we deduce

\[
\int_{\partial \Omega} u^{\delta + m - 2} \, d\Omega \leq \left( \int_{\Omega} u^{\delta} \, dx \right)^{1/\delta} |\Omega|^{-1/\delta} \left( \frac{n}{\rho_0} \int_{\Omega} u^{\delta+m - 2} \, dx + \frac{d(\delta + m - 2)}{\rho_0} \int_{\Omega} u^{\delta+3m-3}|\nabla u| \, dx \right) \\
\leq \frac{n}{\rho_0} C^{n(m-1)/(\delta+m-2)/\delta(\delta+m-1)} |\Omega|^{(1/\delta+m-1)\delta - r_1/\delta} \left( \int_{\Omega} u^{\delta} \, dx \right)^{m+\delta r_1 - 3/2\delta} \\
+ \frac{n}{\rho_0} \left( \frac{2\delta + n(m - 1)}{2\delta} \right)^{\delta + m - 2/\delta + m - 1} C^{2n(m-1)/(\delta+m-3)/2\delta(\delta+m-1)} |\Omega|^{(1/\delta+m-1)\delta - r_1/\delta} \left( \int_{\Omega} u^{\delta} \, dx \right)^{m+\delta r_1 - 3/2\delta} \\
\cdot \left( \int_{\Omega} |\nabla u|^{\delta + m - 1/2} \, dx \right)^{1/2} + \frac{2d(\delta + m - 2)}{\rho_0} \left( \frac{2\delta + n(m - 1)}{2\delta} \right)^{\delta + m - 2/\delta + m - 1} \\
\cdot \left( \int_{\Omega} u^{\delta} \, dx \right)^{m+\delta - 3/2m+\delta-n+1/\delta} \\
\cdot \left( \int_{\Omega} |\nabla u|^{\delta + m - 1/2} \, dx \right)^{1/2} \cdot \left( \int_{\Omega} u^{\delta} \, dx \right)^{n(m-1)/(\delta+m-3)/(m+\delta-n)(\delta+m-1)\delta^{1/2}}.
\]
Next, we deal with the last term of the right-hand side of (101). From Hölderö inequality, Sobolev inequality (42), and inequality (41), we obtain

\[
\int_{\Omega} u^{\delta r_1 - 1} v \leq \left( \int_{\Omega} u^{\delta} \right)^{1 - \left( (r_1 - 1) (n - 2) (\frac{m}{m + 2} - 1) - n \right) (r_1 - 1) (m + 2)} \cdot \left( \int_{\Omega} \left| \nabla u^{\delta m - 1} \right|^2 \cdot \left( \int_{\Omega} \left| \nabla v^{\delta m - 1} \right|^2 \right) \right)^{n (r_1 - 1) (m + 2)}
\]

\[
\leq \left( \int_{\Omega} u^{\delta} \right)^{1 - \left( (r_1 - 1) (n - 2) (\frac{m}{m + 2} - 1) - n \right) (r_1 - 1) (m + 2)} \cdot C^2n (r_1 - 1) (m + 2)^n \cdot \left( \int_{\Omega} \left| \nabla u^{\delta m - 1} \right|^2 \cdot \left( \int_{\Omega} \left| \nabla v^{\delta m - 1} \right|^2 \right) \right)^{n (r_1 - 1) (m + 2)}
\]

\[
\leq \left( \int_{\Omega} u^{\delta} \right)^{1 - \left( (r_1 - 1) (n - 2) (\frac{m}{m + 2} - 1) - n \right) (r_1 - 1) (m + 2)} \cdot C^2n (r_1 - 1) (m + 2)^n \cdot \left( \int_{\Omega} \left| \nabla u^{\delta m - 1} \right|^2 \cdot \left( \int_{\Omega} \left| \nabla v^{\delta m - 1} \right|^2 \right) \right)^{n (r_1 - 1) (m + 2)}
\]

where (92) is substituted. We insert (95) and (96) into (85), and we get

\[
B' (t) \leq Q_1 \left( \int_{\Omega} u^{\delta} \right)^{m + \delta a - 1 - \frac{2\delta}{2}} + Q_2 \left( \int_{\Omega} u^{\delta} \right)^{1 + \left( r_1 - 1 \right) \delta} + C^2n (r_1 - 1) (m + 2)^n \cdot \left( \int_{\Omega} \left| \nabla u^{\delta m - 1} \right|^2 \cdot \left( \int_{\Omega} \left| \nabla v^{\delta m - 1} \right|^2 \right) \right)^{n (r_1 - 1) (m + 2)}
\]

\[
\cdot \left( \int_{\Omega} \left| \nabla u^{\delta m - 1} \right|^2 \cdot \left( \int_{\Omega} \left| \nabla v^{\delta m - 1} \right|^2 \right) \right)^{n (r_1 - 1) (m + 2)} \cdot \left( \int_{\Omega} \left| \nabla u^{\delta m - 1} \right|^2 \cdot \left( \int_{\Omega} \left| \nabla v^{\delta m - 1} \right|^2 \right) \right)^{n (r_1 - 1) (m + 2)}
\]

\[
\cdot \left( \int_{\Omega} \left| \nabla u^{\delta m - 1} \right|^2 \cdot \left( \int_{\Omega} \left| \nabla v^{\delta m - 1} \right|^2 \right) \right)^{n (r_1 - 1) (m + 2)} \cdot \left( \int_{\Omega} \left| \nabla u^{\delta m - 1} \right|^2 \cdot \left( \int_{\Omega} \left| \nabla v^{\delta m - 1} \right|^2 \right) \right)^{n (r_1 - 1) (m + 2)}
\]

where (92) is substituted. We insert (95) and (96) into (85), and we get
where \(Q_1, Q_2, C_i (i = 1, 2, 3, 4)\) are defined by (83) and (84). It follows from (43) that

\[
\left( \int_\Omega u^\delta dx \right)^{2(m+\delta-2)/(m+2\delta-n)} \cdot \left( \int_\Omega |\nabla u^{\delta_{m-1/2}}|^2 dx \right)^{n(m-1)} \frac{\delta + m - 1}{(m+2\delta-n)(\delta + m - 1)^2 - n(m-1)}
\]

\[
\leq 2^\delta (\delta + m - 1) + n(m-1)
\]

\[
\left( \int_\Omega u^\delta dx \right)^{2(\delta + m - 2)/(m+2\delta-n)(\delta + m - 1)} \left( \int_\Omega |\nabla u^{\delta_{m-1/2}}|^2 dx \right)^{n(m-1)} \frac{\delta + m - 1}{(m+2\delta-n)(\delta + m - 1)^2 - n(m-1)}
\]

\[
+ \frac{n(m-1)}{(m+2\delta-n)(\delta + m - 1)} \cdot \int_\Omega |\nabla u^{\delta_{m-1/2}}|^2 dx,
\]

\[
\left( \int_\Omega u^\delta dx \right)^{m + \delta - 3/2} \cdot \left( \int_\Omega |\nabla u^{\delta_{m-1/2}}|^2 dx \right)^{1/2} \leq \frac{1}{2} \left( \int_\Omega u^\delta dx \right)^{m + \delta - 3/2} + \frac{1}{2} \epsilon \int_\Omega |\nabla u^{\delta_{m-1/2}}|^2 dx,
\]

\[
\left( \int_\Omega u^\delta dx \right)^{((m+\delta-3)/(m+2\delta-n)+r_1/\delta)} \cdot \left( \int_\Omega |\nabla u^{\delta_{m-1/2}}|^2 dx \right)^{n(m-1)} \frac{(m+\delta-3)/(m+2\delta-n)(\delta + m - 1)}{r_1/\delta} + \frac{1}{2} \epsilon \int_\Omega |\nabla u^{\delta_{m-1/2}}|^2 dx,
\]

\[
\left( \int_\Omega u^\delta dx \right)^{1 - \left( (r_2-1)/(4\delta + n(m-1)(4-n)/(m+2\delta-n)^2) \right)} \cdot \left( \int_\Omega |\nabla u^{\delta_{m-1/2}}|^2 dx \right)^{n(r_2-1)/(m+2\delta-n)^2}
\]

\[
\leq 1 - \frac{n^2(m-1)(r_2-1)}{(m+2\delta-n)^2} \epsilon \left( (r_2-1)/(4\delta + n(m-1)(4-n)/(m+2\delta-n)^2) - n^2(m-1)(r_2-1) \right)
\]

\[
\cdot \left( \int_\Omega u^\delta dx \right)^{n(r_2-1)/(m+2\delta-n)^2} \cdot \frac{n^2(m-1)(r_2-1)}{(m+2\delta-n)^2} \epsilon \int_\Omega |\nabla u^{\delta_{m-1/2}}|^2 dx,
\]

\[
\left( \int_\Omega u^\delta dx \right)^{1 - \left( (r_2-1)/(4\delta + n(m-1)(4-n)/(m+2\delta-n)^2) \right)} \cdot \left( \int_\Omega |\nabla u^{\delta_{m-1/2}}|^2 dx \right)^{n(r_2-1)/(m+2\delta-n)}
\]

\[
\leq \frac{mn + 2\delta - n}{mn + 2\delta - n} \epsilon \left( (r_2-1)/(4\delta + n(m-1)(4-n)/(m+2\delta-n)^2) - n^2(m-1)(r_2-1) \right)
\]

\[
\cdot \left( \int_\Omega u^\delta dx \right)^{n(r_2-1)/(m+2\delta-n)^2} \cdot \frac{n^2(m-1)(r_2-1)}{(m+2\delta-n)^2} \epsilon \int_\Omega |\nabla u^{\delta_{m-1/2}}|^2 dx.
\]

Therefore, by using (98)–(101), we rewrite (97) to get

\[
B'(t) \leq Q_1 \left( \int_\Omega u^\delta dx \right)^{m + \delta - r_1 - 2/\delta} + Q_2 \left( \int_\Omega u^\delta dx \right)^{\delta r_1 - 1/\delta} + \frac{2^\delta \delta + m - 1 + n(m-1)}{(2\delta + m - 2)/(\delta + m - 1) + n(m-1)}
\]

\[
\cdot \left( \int_\Omega u^\delta dx \right)^{m + \delta - r_1 - 2/\delta}
\]

\[
\cdot \left( \int_\Omega u^\delta dx \right)^{\delta r_1 - 1/\delta} + \frac{2^\delta \delta + m - 1 + n(m-1)}{(2\delta + m - 2)/(\delta + m - 1) + n(m-1)}
\]

\[
\cdot \left( \int_\Omega u^\delta dx \right)^{m + \delta - r_1 - 2/\delta}
\]
Because of the definition of $\epsilon$, we get

$$B'(t) \leq Q_1(B(t))^{m+1+\tau_2-2/\delta} + Q_2(B(t))^{\delta \tau_2-1/\delta} + Q_3(B(t))^{2\delta(\delta+1)+\tau_1, (mv+2\delta-n)}(\delta+1)+\delta(m-1)$$

$$+ Q_4(B(t))^{mv+\tau_2-3/\delta} + Q_5(B(t))^{(m+\delta-1)+\tau_1, (mv+2\delta-n)}(\delta+1)+\delta(m-1)$$

$$+ Q_6(B(t))^{mv+2\delta-n}t(r_1-1)(4\delta+2(m-1)+\delta(m-1)}{(m+2\delta-n)}(\delta+1)+\delta(m-1)$$

$$+ Q_7(B(t))^{mv+2\delta-n}t(r_1-1)(4\delta+2(m-1)+\delta(m-1)\tau_2) + Q_8(B(t))^{mv+2\delta-n}t(r_1-1)(n+2\delta-n),$$

where $Q_i (i = 1, 2, \ldots, 6)$ are given by (83). Integrating (104) from 0 to $t$, we deduce

$$\int_0^{B(t)} \frac{d\tau}{B(0) \varphi(\tau)} \leq t.$$  

Hence, if $u$ blows up in the measure $B(t)$ at some finite time $t^*$, we pass to the limit as $t \to t^*$ to obtain

$$t^* \geq \int_0^\infty \frac{d\tau}{B(0) \varphi(\tau)}.$$  

(107)

The proof is complete.  

4. Applications

In this section, some examples are given to demonstrate the applications of the main results.
where $\Omega = \{x = (x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 < 1\}$. Then,

$$m = 4,$$

$$a(x) = 27 - |x|^2, f(u) = u^5,$$

$$g(u) = u^3,$$

$$k_1(t) = e^{t/2},$$

$$k_2(t) = \frac{1260}{1303\pi} e^{t/2},$$

$$u_0(x) = \frac{1}{2} + |x|^2.$$

From (12) and (13), we get $\lambda_1 = \pi^2$ and $\omega_1 = \sin \pi|x|/\sqrt{2\pi|x|}$. Set $\alpha = 1/4$, $\beta = 26$, $\eta = 1/2$, and $\gamma = 1$. Therefore,

$$\Phi(t) = e^{(1/8)t} \int_0^t \left(\frac{\sin \pi r}{\sqrt{2\pi r}}\right)^2 \mathrm{d}x.$$

We can compute

$$F_1(t) = \frac{1}{2} + 110.622 e^{(1/4)t} + 463.634 e^{(3/10)t} + 384.366 e^{(59/140)t} + 531.608 e^{(13/20)t} + 1702.72 e^{(269/340)t} + 166040.1221 e^{(39/40)t},$$

$$F_2(t) = 36.874 e^{(1/4)t} + 115.908 e^{(3/10)t} + 92.2609 e^{(59/140)t} + 286.25 e^{(13/20)t} + 1062.25 e^{(269/340)t} + 62980.7 e^{(39/40)t}.$$

Then,

$$t^* \geq \left(\int_0^t F_2(s)^2 e^{\int_0^t F_2(r) \mathrm{d}r} \right)^{-1} \left(\frac{1}{2A^2(0)}\right) = 3.641 \times 10^{-9}.$$

It follows from (112) and (115) that

$$3.641 \times 10^{-9} \leq t^* \leq 0.161936.$$
Next, we will show a lower bound for the blow-up time. Set $b_0 = 7.8$, $r_1 = 2$, $r_2 = 4$, $\eta_1 = 2.01$, $\eta_2 = 725/726\pi$, $\rho_0 = d = 1$, and $|\Omega| = 4/3\pi$. We define

$$B(t) = \int_\Omega u^5dx, \quad t \geq 0,$$

$$B(0) = \int_\Omega u_0^5dx = \int_\Omega \left(1 + |x|^2\right)^5dx = 54.3484. \quad (122)$$

Obviously, all conditions of Theorem 3 are valid. And since $u$ blows up at a finite time $t^*$ in the measure $\Phi(t)$, $u$ can blow up in the measure $B(t)$. It follows from [22] that the Sobolev embedding constant $C = 5.6958$. According to (98) and (100), we can compute $C_1 = 135.5476$, $C_2 = 45.7226$, $C_3 = 31.8926$, $C_4 = 1505.5976$, and $\varepsilon = 0.012$. And we have

$$\varphi(s) = 545.771s^{8/5} + 1795.24s^{8/5} + 1623.15s^{161/95},$$

$$+ 1905.11s^{9/5} + 4530.8s^{399/205} + 3905.83s^{167/101},$$

$$+ 476413.607s^{13/6}. \quad (123)$$

Then, by Theorem 3, we get a lower bound of $t^*$ as follows:

$$t^* \geq \int_{18.5105}^{\infty} \frac{ds}{\varphi(s)} = 1.06861 \times 10^{-8}. \quad (124)$$

It follows from (121) and (124) that

$$1.06861 \times 10^{-8} \leq t^* \leq 0.107917. \quad (125)$$

### Data Availability

No data were used to support this study.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

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