Core structure and exactly solvable models in dilaton gravity coupled to Maxwell and antisymmetric tensor fields

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Abstract

We consider the D-dimensional massive dilaton gravity coupled to Maxwell and antisymmetric tensor fields (EMATD). We derive the full separability of this theory in static case. This discloses the core structure of the theory and yields the simple procedure of how to generate integrability classes. As an example we take a certain new class, obtain the two-parametric families of dyonic solutions. It turns out that at some conditions they tend to the D-dimensional dyonic Reissner-Nordström-deSitter solutions but with “renormalized” dyonic charge plus a small logarithmic correction. The latter has the significant influence on the global structure of the non-perturbed solution - it may shift and split horizons, break down extremality, and dress the naked singularity. We speculate on physical importance of the deduced integrability classes, in particular on their possible role in understanding of the problem of unknown dilaton potential in modern cosmological and low-energy string models.

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I. INTRODUCTION

We start with the following action

$$S = \int d^D x \sqrt{-g} \left[ R - \frac{\beta}{2} (\partial \phi)^2 + \Xi F^2 + \Psi F_{(p)}^2 + \Lambda \right],$$

(1)

where $R$ is the Ricci scalar, $p = D - 2$, $F$ and $F_{(p)}$ are two- and $p$-forms respectively, $\Xi$, $\Psi$ and $\Lambda$ are functions of dilaton $\phi$, and $\beta$ is some unspecified constant (not necessary positive). The models of such a kind appear in modern cosmological and low-energy string and supergravity theories. We will be interested in static solutions of this system hence further we will work with the metric ansatz

$$ds^2 = -e^{U(r)} dt^2 + e^{-U(r)} dr^2 + e^{A(r)} ds^2_{(p,k)},$$

(2)

where $ds^2_{(p,k)}$ is a $p$-dimensional maximally symmetrical space with $k$ being $-1$, $0$, $+1$ depending on geometry ($\mathcal{H}^P$, $\mathcal{E}^P$, $\mathcal{S}^p$) - we are going to handle them simultaneously and uniformly. The Maxwell and $p$-form fields are assumed being in the form

$$F = Q e^{-pA/2} \Xi dt \wedge dr, \quad F_{(p)M...N} = P e^{-pA/2} e_{M...N}$$

(3)

in an orthonormal frame, where $Q$ and $P$ are electric and magnetic charges. Then the field equations become

$$A'' + A'(U' + pA'/2) - \frac{2}{p} \hat{\Xi} e^{-pA - U} - \frac{2}{p} \Lambda e^{-U} - 2k(p - 1)e^{-A - U} = 0,$$

(4)

$$\beta \phi'' + \beta \phi'(U' + pA'/2) + \hat{\Xi} e^{-pA - U} + \Lambda \phi e^{-U} = 0,$$

(5)

$$A'' + A'^2/2 + \beta \phi'^2/p = 0,$$

(6)

where $' \equiv \partial_r$ and $\hat{\Xi} \equiv 2Q^2 \Xi^{-1} + p! P^2 \Psi$.

The paper is arranged as follows. In next section we prove the full separability of static EMATD theory and reveal its core structure. The latter is formulated in terms of the classes of integrability (more correctly, solvability). Sec. II deals with a particular example of the so-called linear class. We rule out the relevant models and their solutions’ families, and study the physical relevance of latter. Sec. III is devoted to how the discovered core structure of EMATD gravity might help theorists in the situation when neither couplings nor potential are known precisely. Conclusions are made in Sec. IV.

II. SEPARABILITY AND CORE STRUCTURE OF EMATD GRAVITY

Now we will rule out the full separability of the static dyonic EMATD theory (the case without antisymmetric tensor has been considered in ref. [3]). Due to that separability, we will be equipped with the straightforward procedure of generating of the numerous classes of integrability that are dyonic besides $\Lambda$ is non-zero in general case.

Applying the approach of ref. [3] one can obtain the following system (which is similar to that from the EMD case)

$$2k(p - 1) + \frac{2e^\lambda}{p} \left( \Lambda + e^{-p\hat{A}\phi} \right) + e^{U + 2Y} \times \left( \frac{\beta}{pA^2_{\phi}} - \frac{U_{\phi}}{A_{\phi}} - \frac{p - 1}{2} \right) = 0,$$

(7)

$$2k(p - 1) + \frac{2e^\lambda}{p} \left( \Lambda + \frac{p}{2\beta} \Lambda_{\phi} \hat{A}_{\phi} \right) + e^{U + 2Y} \left( \frac{1}{A_{\phi}} \right),$$

(8)
where \( U(r) \equiv \tilde{U} (\phi (r)) \) and \( A(r) \equiv \tilde{A}(\phi (r)) \), the subscript \( \tilde{} \) stands for the derivative with respect to dilaton. This system is equivalent to the initial one but has much more capabilities. First of all, it is explicitly separable: \( \tilde{U} \) is algebraically given by Eq. (8) so one can easily exclude it from Eq. (9) to receive the so-called class equation

\[
\frac{H}{A} + \left( \frac{\beta}{pA} + \frac{p-1}{2} \right) H + k(p-1) + e^{\tilde{A}} \left( \Lambda + \frac{p}{2\beta} \tilde{\Lambda} \tilde{A} \right) = 0,
\]

(10)

where \( H \equiv \frac{1}{p(1/A)_{,\phi} \phi} \left( kp(p-1) + e^{\tilde{A}} \left( \Lambda + \frac{p}{2\beta} \tilde{\Lambda} \tilde{A} \right) + e^{-(p-1)\tilde{A}} \left( \tilde{\Xi} + \frac{p}{2\beta} \tilde{\Xi} \tilde{A} \right) \right) \), and thus to come to the system of autonomous equations yielding \( A, \phi, U \) consecutively. Eq. (10) is a non-linear third-order ODE with respect to \( \tilde{A}(\phi) \) so the direct task (finding of \( \tilde{A} \) at given \( \tilde{\Xi}, \Psi, \Lambda \)) is still hard to solve without supplementary symmetries or assumptions. However, using this equation one may study the inverse problem, i.e., the obtaining of the \( \tilde{\Xi}\Psi\Lambda \) triplets corresponding to a concrete fixed \( \tilde{A} \). Thus, with every \( \tilde{A} \) it is associated the appropriate class of integrability determined by the equation above. It will help that the equation is a linear (at most) second-order ODE with respect to \( \tilde{\Xi} \) and \( \Lambda \), besides, having only one equation it is much easier to study the integrability classes numerically, e.g., to clarify whether they always have stable solutions, see ref. 8 and references on appropriate methods therein.

Moreover, there is an exceptional class in this construction. If \( \tilde{A} \sim \phi \) then \( \tilde{U} \) immediately disappears in Eq. (9), so the latter becomes a linear first-order ODE with respect to \( \Lambda \) and \( \tilde{\Xi} \). The linear class is of interest both by itself and in connection with supergravity models, so below we will study it in more details. Then, as an example, we will pick some concrete concrete EMATD model to obtain its general-in-class solutions.

### III. AN EXAMPLE: LINEAR CLASS

Let us impose

\[
\tilde{A} = \frac{4d_1}{p} \phi - \ln d_2,
\]

(11)

with \( d_i \) being arbitrary constants. Then Eq. (8) yields

\[
\phi = \left\{ \frac{4pd_1}{\pi_{p+1}} \ln \left[ \frac{\sqrt{2\pi}(4d_1^2+4d_2^2)(r-r_0)}{4d_1^2} \right] - i\sqrt{\frac{\pi}{2}} \phi \right\} (r-r_0),
\]

\[
d_i = \frac{i}{2} \sqrt{\frac{\pi}{2}},
\]

(12)

and Eq. (8) becomes the equation of integrability class

\[
\frac{e^{4d_1\phi}}{d_2} \left( \Lambda + \frac{2d_1}{\beta} \Lambda \phi \right) + \frac{4d_1}{d_2} (d_1 + 1) \phi + kp(p-1) = 0,
\]

(13)

where \( \tilde{U} \) can be easily found from Eq. (9) which is the linear first-order ODE with respect to \( e^{\tilde{U}} \). It should be noted that the extended Lambda-Maxwell duality (discussed in ref. 2 at \( \Psi = 0 \)) appears to be broken at \( D \neq 4 \); it is curious that electric-magnetic duality is also broken if \( D \neq 4 \), therefore, \( D = 4 \) turns out to be a magic number again. Now it is time to take some concrete narrow physical system and obtain its solutions within the frameworks of the linear class.

**String-inspired model: solutions**

We choose the physically important model which was first integrated (at \( \Lambda = 0 \)) by Gibbons and Maeda 4:

\[
\Xi = -e^{-\frac{4d_1}{\pi_{p+1}} \phi}, \quad \Psi = -\frac{2}{p!} e^{-\frac{4d_1}{\pi_{p+1}} \phi}, \quad \beta = 8/p,
\]

(14)

where \( g \)'s are coupling constants. When \( \Psi \) vanishes then \( g_2 = 1 \) corresponds to field theory limit of superstring model, \( g_2 = \sqrt{1+2n/p} \) corresponds to the toroidal \( T^n \) reduction of \( (D + n) \)-spacetime to \( D \)-spacetime, \( g_2 = 0 \) is a usual Einstein-Maxwell system); \( \Lambda \) is precisely unknown in string theory 9. The models of such a type have been intensively studied in the case \( \Lambda = 0 \) but the progress in the models which contain both an antisymmetric tensor and a massive dilaton is still rather slow despite their obvious importance. With the settings (14) Eq. (12) becomes

\[
\phi = \left\{ \frac{4pd_1}{\pi_{p+1}} \ln \left[ \frac{\sqrt{2\pi}(4d_1^2+4d_2^2)(r-r_0)}{4d_1^2} \right] - i\sqrt{\frac{\pi}{2}} \phi \right\} (r-r_0),
\]

\[
d_1 = i \frac{1}{2} \sqrt{\frac{\pi}{2}},
\]

(15)

For further it is convenient to define the three polynomials:

\[
\pi_1 = d_1^2 - 1, \quad \pi_2 = pd_1^2 - g_2d_1 - 1, \quad \pi_3 = g_1d_2 + d_2^2 - 1.
\]

The integration of the class equation above reveals the following cases (note that additionally each case may contain the multiple subcases determined by the combinations of parameters apart from \( \pi_1 = 0 \) ones at which an initial \( \tilde{U} \), but not \( \Lambda \), becomes singular):

(i) \( \pi_1 = 0, \pi_2 \neq 0, \pi_3 \neq 0 \). Integrating Eq. (13) with (14) we see that \( \Lambda \) must be

\[
\Lambda = a_0 e^{\pi_{d_1} \phi} + \frac{k d_1}{\pi_1} p(p-1) e^{\frac{4d_1}{\pi_{p+1}} \phi} - 2d_1 g_1 e^{-4d_1 \phi}
\]

\[
\times \left[ \frac{Q^2}{\pi_2} (1 + g_2d_1) e^{\frac{4d_1}{\pi_{p+1}} \phi} + \frac{P^2}{\pi_3} (1 - g_2d_1) e^{-\frac{4d_1}{\pi_{p+1}} \phi} \right],
\]

(16)
where \(a_0\) is integration constant, corresponding \(A, \phi\) are given by Eqs. (11), (15), and \(U\) is given by Eq. (7) provided (11) and (14):

\[
e^U = ec^{2\frac{(p-1)d_1^2}{2}} + 4k(p-1)de^{\frac{4\phi}{\pi_1}} + 4a_0d_1^2e^{\frac{4\phi}{\pi_1}} - \frac{1}{d_2p(d_2^2 + \pi_1)} + 8Q^2d_1^2d_2^2e^{-\frac{4(\pi_2 - d_2^2)}{\pi_1}} + \frac{8P^2d_1^2d_2^2e^{-\frac{4(\pi_3 - d_2^2)}{\pi_1}}}{\pi_3(\pi_3 - d_1(d_1 - d_2^2))} \tag{17}
\]

where \(c\) is another integration constant related to mass. As was alerted above, this case contains cases \(p_1^2 \pm \pi_1 = 0\), \(p_2 - d_2(d_2 + g_2) = 0\), \(\pi_1 - d_1(d_1 - g_1) = 0\) making the last equation, but not Eq. (5), singular. For the sake of brevity, we do not present them here.

(ii) \(\pi_1 = 0, \pi_2 \neq 0, \pi_3 \neq 0\). We choose the positive root \(d_1 = 1\) then in the same way as above can one show that \(\Lambda\) must be

\[
\Lambda = [a_0 - 4kd_2(p-1)\phi] e^{-\frac{4\phi}{\pi_1}} - 2d_2^2e^{-4\phi} \times \left[\frac{Q^2(1 + g_2)e^{\frac{2\phi}{\pi_2}}}{p_2} + \frac{P^2(1 - g_1)e^{\frac{4\phi}{\pi_3}}}{p_2}\right] \tag{18}
\]

corresponding \(A, \phi\) are given by Eqs. (11), (13) with \(d_1\) being as above, whereas \(U\) turns out to be

\[
e^U = e^{\frac{4\phi}{p_2}} \left[ce^{-2\phi} + 4a_0d_2^2e^{-4\phi} \times \left[\frac{Q^2(1 + g_2)e^{\frac{2\phi}{\pi_2}}}{p_2} + \frac{P^2(1 - g_1)e^{\frac{4\phi}{\pi_3}}}{p_2}\right]
\]

\[
\\frac{8Q^2d_1^2d_2^2e^{-\frac{4(\pi_2 - d_2^2)}{\pi_1}}}{(p_1 - 2g_2)(p_1 - 1 - g_2)} + \frac{8P^2d_1^2d_2^2e^{-\frac{4(\pi_3 - d_2^2)}{\pi_1}}}{(p_1 - 2g_2)(p_1 - 1 - g_2)} \cdot \tag{19}
\]

(iii) \(\pi_1 \neq 0, \pi_2 = 0, \pi_3 \neq 0\). To avoid root branches we will work in terms of \(d_1\) assuming that it is related to \(g_2\) via the relation \(g_2 = pd_1 - 1/d_1\), then \(\Lambda\) is

\[
\Lambda = (a_0 + 8Q^2d_1^2\phi) e^{-\frac{4\phi}{\pi_1}} + \frac{kd_2p}{\pi_1}(p_1 - 1) e^{-\frac{4\phi}{\pi_1}} - \frac{2P^2d_2^2e^{-\frac{4(\pi_2 - d_2^2)}{\pi_1}}}{\pi_3(\pi_3 - d_1(d_1 - g_2))} \tag{20}
\]

corresponding \(A, \phi\) are given by Eqs. (11), (13) with \(d_1\) being as above, whereas \(U\) turns out to be

\[
e^U = ec^{2\frac{(p-1)d_1^2}{2}} + 4kd_1^2(p_1 - 1)e^{\frac{4\phi}{\pi_1}} + 8Q^2d_1^2d_2^2e^{-\frac{4(\pi_2 - d_2^2)}{\pi_1}} \frac{p_2d_1^2 + \pi_1}{\pi_1} \times \frac{4d_1\phi}{p} \left[4d_1\phi + a_0 + 8Q^2d_1^2e^{-\frac{4\phi}{\pi_1}} - 3pd_1^2 + \pi_1\right] + \frac{8P^2d_1^2d_2^2e^{-\frac{4(\pi_3 - d_2^2)}{\pi_1}}}{\pi_3(\pi_3 - d_1(d_1 - g_2))} \cdot \tag{21}
\]

(iv) \(\pi_1 \neq 0, \pi_2 \neq 0, \pi_3 = 0\). This case is identical to the previous one if one replaces everywhere \(g_2\) with \(-g_p\) and interchanges \(Q\) and \(P\).

(v) \(\pi_1 = 0, \pi_2 = 0, \pi_3 \neq 0\). Therefore, we have the following two sets \(\{g_2, d_1\} = \pm\{p + 1, 1\}\). We choose the plus branch then \(\Lambda\) is

\[
\Lambda = \left[a_0 + 4d_2(2Q^2d_1^2e^{-\frac{4(\pi_2 - d_2^2)}{\pi_1}} - k(p-1)\phi) e^{-\frac{4\phi}{\pi_1}} - 2P^2d_2^2(1 - g_p)e^{-\frac{4(\pi_2 - d_2^2)}{\pi_1}} + \frac{8P^2d_1^2d_2^2e^{-\frac{4(\pi_3 - d_2^2)}{\pi_1}}}{\pi_3(\pi_3 - d_1(d_1 - g_2))}\right] \tag{22}
\]

corresponding \(A, \phi\) are given by Eqs. (11), (13) with \(d_1\) being as above, whereas \(U\) turns out to be

\[
e^U = e^{\frac{4\phi}{p^2}} \left[ce^{-2\phi} + 4a_0d_2^2k(p^2 + p - 2) - 6Q^2 \right] + 16\phi \left(2Q^2d_2^2 - k(p - 1)\right) + \frac{8P^2d_1^2d_2^2e^{-\frac{4(\pi_2 - d_2^2)}{\pi_1}}}{(p - 1 + g_p)(p - 2 + 2g_p)}. \tag{23}
\]

(vi) \(\pi_1 = 0, \pi_2 \neq 0, \pi_3 = 0\). Similarly, we have the following two sets \(\{g_2, d_1\} = \pm\{1 - p, 1\}\). One can use the expressions from the previous case but has to replace in them \(g_p\) with \(-g_2\) and interchange \(Q\) and \(P\).

(vii) \(\pi_1 \neq 0, \pi_2 = 0, \pi_3 = 0\). It contains the condition \(g_2 + g_p = 0\), so we can exclude \(g_p\). Besides, to avoid root branches we will work again in terms of \(d_1\) assuming that it is related to \(g_2\) via the relation \(g_2 = pd_1 - 1/d_1\). We have

\[
\Lambda = \left[a_0 + 8d_1d_2(2Q^2 + P^2)\phi\right] e^{-\frac{4\phi}{\pi_1}} + \frac{kd_2p}{\pi_1} \tag{24}
\]

corresponding \(A, \phi\) are given by Eqs. (11), (13) with \(d_1\) being as above, whereas \(U\) one can formally use Eq. (7) without the last term \((\sim P^2)\) and with \(Q^2\) being replaced with \(Q^2 + P^2\).

(viii) \(\pi_1 = 0, \pi_2 = 0, \pi_3 = 0\). Therefore, \(g_2 + g_p = 0\) and we have the following two sets \(\{g_2, g_p, d_1\} = \pm\{1 - p, 1 \pm p\}\). We choose the plus branch then \(\Lambda\) is

\[
\Lambda = \left[a_0 + 4d_2\left(2d_2^2(1 - Q^2 - P^2) - k(p - 1)\phi\right) e^{-\frac{4\phi}{\pi_1}} + \frac{kd_2p}{\pi_1} \right]. \tag{25}
\]

corresponding \(A, \phi\) are given by Eqs. (11), (13) with \(d_1\) being as above, whereas for \(U\) one can formally use the corresponding expression from (v) but without the last term \((\sim P^2)\) and with \(Q^2\) being replaced with \(Q^2 + P^2\).

Thus, we have enumerated the basic solutions, which correspond to the model (14) within frameworks of the linear class. Of course, we have mentioned just a few examples.

**String-inspired model: discussion of solutions**

Analyzing the solutions above, one can see that the set (i) is the largest set of solutions due to the parameter \(d_1\) being non-fixed there. In this section we will study the
solutions (i) in details. In view of future considerations, let us first redefine the constants
\[ c = -8 \mu d_2 \frac{p+1}{d_1^2}, \quad a_0 = \Lambda_0 d_2^{-2}, \]  
(24)
where \( p_1 \equiv p - 1 \). The next step is to switch coordinates to the infinite-observer frame of reference
\[ e^{A(r)} = \left[ \frac{1 + d_2^2}{2d_2 \frac{p+1}{d_1^2} r} \right]^{\frac{2d_2^2}{1-p+1}} \rightarrow r^2, 2d_2^2 t \rightarrow t, \]  
(25)
then in new coordinates we obtain
\[ ds^2 = -dt^2 \left[ \frac{k_1 d_1^4 t^{2d_2^2}}{\pi_1(p_1 d_1^2 + 1)} - \frac{2 \mu}{r^{p+1}} + \frac{\Lambda_0 d_2^{-2}}{p(p+1)d_1^2 - 1} + \frac{2d_2^2}{\pi_2(p_1 d_1^2 - 2g_2d_1 - 1)} \right] + dr^2 \left[ \frac{2d_2^2}{\pi_1(p_1 d_1^2 + 1)} - \frac{2 \mu}{r^{p+1}d_1^2 - 1} + \frac{\Lambda_0 d_2^{-2}}{p(p+1)d_1^2 - 1} + \frac{2d_2^2}{\pi_2(p_1 d_1^2 - 2g_2d_1 - 1)} \right] + r^2 d_s^2(p,k), \]
\[ e^\phi = \left( \sqrt{d_2} \right)^{\frac{2d_2^2}{1-p+1}}, \]
\[ F = \frac{d_2 d_1^{-1} Q}{r^{p-2g_2d_1^2 - 1}} dr \wedge dt, \quad F(p)_{M \ldots N} = P \frac{\varepsilon_{M \ldots N}}{g_2^2} \]  
(26)
The remainder of this section will be devoted to the studies of this solution at non-fixed large values of \(|d_1|\). Assuming \(|d_1| \gg \text{max} \{ \text{Reissner-Nordström} \} \), we obtain that up to the order \( O(1/d_1^2) \) (here and below it is supposed to be the default precision of calculations) the metric above takes the habitual form
\[ ds^2 = -e^{U(r)} dt^2 + e^{-U(r)} dr^2 + r^2 d_s^2(p,k), \]
with
\[ e^U(r) = k - \frac{2 \mu}{r^{p+1}} + \frac{\Lambda_0 r^2}{p(p+1)} + \frac{\Delta - \Theta \ln (r^{p+1}/\eta)}{r^{2p+1}}, \]  
(27)
where we have defined the following constants
\[ \Delta = \frac{2}{pp_1} \left( Z^2 + \frac{3p - 1}{pp_1} W \right), \quad \Theta = -\frac{4W}{p^2 p_1 d_1^2}, \]
\[ \eta = d_2^\frac{1}{p_1}, \quad W = g_2 Q^2 - g_p P^2, \quad Z^2 = Q^2 + P^2, \]  
(28)
and it is implied that \( p > 1 \) (lower-dimensional cases will be separately considered after). Also, the \( O(d_1^{-2}) \)-asymptotical form of the dilaton potential \( \Lambda_0 \) is
\[ \Lambda = \Lambda_0 - 2 \frac{d_2^2}{p_1} e^{-4d_1 \phi}. \]  
(29)

The first, second and third terms in the metric above is the D-dimensional Schwarzschild-deSitter. The term proportional to \( \Delta \) is nothing but the D-dimensional Reissner-Nordström with the only difference that the effective dyonic charge is the standard one plus a small correction of order \( d_1^{-1} \). The last term, proportional \( \Theta \), is definitely something new, and below we will study its influence in details.

From now we will work with the spherical case \( k = 1 \), besides we will neglect the cosmological constant for simplicity. Then the information about the global structure of the metric can be read off from the intersection of two curves described by the following algebraic equation
\[ x^2 - 2\mu x + \Delta = (x - \delta_+)(x - \delta_-) = \Theta \ln(x/\eta), \]  
(30)
where \( x = r^{p_1} \) and \( \delta_\pm = \mu(1 \pm \sqrt{1 - \Delta/\mu^2}) \). It is useful to keep in mind that \( \Theta \) is small \( (\sim d_1^{-1}) \) that simplifies subject matter. This smallness in fact means that for the whole region except perhaps \( x \rightarrow 0 \) and \( x \rightarrow +\infty \) the value of the logarithm in the equation above should be assumed small in comparison with the parameters \( \mu \), \( \Delta \) and \( \eta \).

Case \( \mu^2 > \Delta \). If \( \Theta \equiv 0 \) (that may happen not only when \( d_1 = \infty \) but also when \( g_2 Q^2 - g_p P^2 \)) this case corresponds to the D-dimensional Reissner-Nordström black hole. Otherwise we have to solve the transcendental equation \( (\text{b}) \) with real \( \delta \)’s. Fortunately, it can be done analytically with the use of \( \Theta \)’s smallness. Solving it, we obtain that we still have two horizons but their radii acquire a correction:
\[ r_{H \pm} = \frac{\delta_\pm + \Theta \ln(\delta_\pm/\eta)}{2(\delta_\pm - \mu)} \pm \frac{\pi_1}{2}, \]  
(31)
and the corresponding Hawking temperatures are calculated to be
\[ T_{H \pm} = \frac{p_1 \delta_\pm}{2\pi} \left[ \frac{\delta_\pm - \mu - \Theta}{2\delta_\pm} \right] \times \left( 1 + \frac{(\delta_\pm - \mu)p_1 \ln(\delta_\pm/\eta)}{p_1(\delta_\pm - \mu)} \right), \]  
(32)
an absolute value is implied.

Case \( \mu^2 > \Delta \). Without the \( \Theta \)-perturbation this case corresponds to the D-dimensional extremal Reissner-Nordström black hole. It turns out that the series expansion used in the previous case fails (diverges) so we have to invent another one. The non-perturbed horizon appears at \( x = \mu \). We are interested in small deviations from the non-perturbed case so it is natural to expand \( \text{Eq. (31)} \) with respect to \( x \) up to the third order near this point. We obtain that \( \text{Eq. (31)} \) becomes the quadratic equation,
\[ \left( 1 + \frac{\Theta}{2\mu^2} \right)x^2 - 2\mu \left( 1 + \frac{\Theta}{\mu^2} \right)x + \mu^2 = \Theta \left[ \ln(\mu/\eta) - \frac{3}{2} \right], \]
from which one concludes that extremality is broken and the extreme horizon is shifted and split into two ones, with the radii
Here, the term proportional to $\Theta$ shifts the horizon outward or inward (depending on the sign of $W/d_1$) whereas the term proportional to $\sqrt{\Theta}$ describes the split. It is curious that in the particular case $\eta = \mu$ the extremality is again restored up to $O[d_1^{-1}]$. The corresponding Hawking temperatures are

$$T_{H^z} = \left[ \mu + \frac{\Theta}{2\mu} \pm \sqrt{\Theta} \ln \left( \frac{\mu}{\eta} \right) \right]^{1/\pi}.$$  \hspace{1cm} (33)

and they do vanish not only when $d_1 = \infty$ but also at $\eta = \mu$.

Case $\mu^2 < \Delta$. If $\Theta \equiv 0$ then the solution describes the naked Reissner-Nordström singularity. There is a strong hope that the $\Theta$-perturbation “dresses” the singularity, i.e., of the form (35), provided $\mu$ does not cross an $x$-axis. The intuitive solution for this is to require the minimum point of the parabola to be as close as possible to the $x$-axis, hence, to the logarithmic curve, because the latter is small. The distance from the minimum point of the parabola to the $x$-axis equals to $\Delta - \mu^2$, so $\Delta$ must be equal to $\mu^2$ plus a small positive correction, say

$$\Delta = \mu^2 + |\text{const} d_1^{-1}|.$$ \hspace{1cm} (35)

Again, we expand Eq. (33) near the minimum point of parabola and obtain the quadratic equation

$$\left( 1 + \frac{\Theta}{2\mu^2} \right) x^2 - 2\mu \left( 1 + \frac{\Theta}{\mu^2} \right) x + \Delta = \Theta \left[ \ln (\mu/\eta) - \frac{3}{2} \right].$$

If it has complex roots then the singularity is naked otherwise it is hidden under at least one horizon. One can check that this equation in general case does not have real roots but if $\Delta$ is

$$\Delta = \mu^2 - \frac{2 \Theta \mu^2 \ln (\mu/\eta)}{\Theta - 4 \mu^2} = \mu^2 + \frac{\Theta}{2} \ln (\mu/\eta),$$

i.e., of the form (35), provided $d_1^{-1} W \ln (\mu/\eta)$ is non-positive, then the imaginary part vanishes, so one does have the purely real double root. It means that we have found an example when a singularity is dressed by the single horizon. Its radius is

$$r_{H^z} = \left[ \mu + \frac{\Theta}{2\mu} \right]^{1/\pi},$$ \hspace{1cm} (36)

but with the Hawking temperature,

$$T_{H} = \frac{p_1 \Theta \ln (\mu/\eta)}{4 \pi \mu \frac{\Theta}{\pi^2}}.$$ \hspace{1cm} (37)

being of order $O[d_1^{-1}]$, rather than $O[d_1^{-1/2}]$ as in previous case.

As a final part of this section, we have to study the low-dimensional case. Indeed, the majority of Eqs. (27)-(37) are not applicable when $D = 3$ or 2, i.e., when the number of spatial dimensions is, respectively, two and one. The two-dimensional case is of no interest here because all the solutions were derived assuming $p \neq 0$ for obvious reasons. In the 3D case when $|d_1|$ is large, instead of Eq. (27) we obtain

$$e^{U(r)} = \zeta - 2 Z^2 \ln (r \sqrt{d_2}) + \frac{\Lambda_0 r^2}{2},$$ \hspace{1cm} (38)

where $Z^2$ is as above and it is denoted

$$\zeta = \frac{g_p^2 Q^2 + g_z^2 P^2}{2 g_z^2 g_p} - \frac{d_1 (g_p Q^2 - g_z P^2)}{g_z g_p} - Z^2 - 2\mu,$$

d_2 is assumed positive for definiteness. The scalar, Maxwell and $p$-form fields (24) do not undergo principal changes in the sense that they are not singular when $p \to 1$. However, it is worth to note that $p$-form becomes the plan vertex-type vector field with the only non-zero component $F_{(1)} = f(r) \, d\phi$ where $\phi$ is an angular coordinate. It cannot be represented as an external derivative of some potential, therefore, it is not possible to derive it from the 3D variational principle - in the action (1) it may appear only as a non-dynamical source term

$$\Psi F_{(1)} F_{(1)}^\alpha = \Psi f(r)^2 = \hat{\Lambda}(\phi),$$ \hspace{1cm} (39)

because dilaton is an invertible function of $r$. Nevertheless, one can consider the 1-form contribution formally, so below we will not impose $P \equiv 0$.

The solution is essentially cosmological - the metric (38) tends to de Sitter one (provided $\zeta$ is positive), and has the only singularity at $r = 0$ provided $Z \neq 0$. Its global structure is, however, non-trivial and crucially depends on values of the parameters. Simple analysis shows that: (a) when $\Lambda_0$ is negative, we have a single horizon regardless of what other parameters are; (b) when $\Lambda_0$ is positive, we have the naked singularity, one extreme horizon, two horizons, depending on whether the value

$$\zeta + Z^2 - Z^2 \ln (2d_2 Z^2 / \Lambda_0)$$

is positive, zero or negative, respectively; (c) when $\Lambda_0$ vanishes, we have single horizon with

$$r_H = \sqrt{d_2} e^{\zeta/Z^2},$$ \hspace{1cm} (40)

but the solution is not asymptotically flat (despite the curvature invariants do vanish asymptotically) so this is still cosmological, rather than black hole, horizon.

To summarize this section: we have demonstrated that the two-parametric family of exact solutions (i) at large values of one of the parameters describes the solution which is the D-dimensional Reissner-Nordström-deSitter
solution but with “renormalized” dyonic charge plus a perturbative non-constant (logarithmic) correction \(22\). It is also shown that this correction despite its smallness has significant influence on the global structure of the non-perturbed solution - it may shift and split horizons, break down extremality, and dress the naked singularity.

IV. INTEGRABILITY CLASSES AND DILATON POTENTIAL PROBLEM

In the previous section we studied some particular class as the fruitful example of the proposed approach’s power. Other integrability classes (not talking about models) are so diverse and numerous that in principle never can be covered all. Other \(\Xi\), \(\Psi\), \(\Lambda\) that may appear from a concrete problem can be paired up within our class in a similar manner. Despite this pairing is an artificial procedure the generated exact solutions are better than numerical studies from scratch, besides ones can verify or falsify qualitative approaches and results. Be that as it may, the good news is that now one has a powerful tool to study numerous models in a straightforward and uniform way, even having no background in the theory of differential equations.

Now, it is a good time to coin the advantages that come after the proven separability of static EMATD theory. This section will be devoted to the generic physical significance of the integrability classes given by relations between \(A\) and dilaton alike \(10\). Here we are going to justify the point that the integrability classes of such a kind is not only a mathematical object but also can play key role in some fundamental aspects of field theory and theoretical cosmology.

For instance, the integrability classes may help with the problem of unknown dilaton potential \(\Lambda\) (but, of course, by themselves they cannot provide a complete answer). When supersymmetry is unbroken then the dilaton is in the same supermultiplet as the graviton and hence cannot acquire a mass. However, in low-energy region the supersymmetry is broken so the assumption \(\Lambda \equiv 0\) is inconsistent with observations \(7\). Nowadays it is the strong problem that the dependence of \(\Lambda\) on scalar field is precisely known neither from string theory nor from cosmology-related experiments. The one of the ideas of getting \(\Lambda(\phi)\) comes from supersymmetry and supergravity - despite the theories with the dilaton potential of general type are non-supersymmetric as a rule, some non-trivial potentials can be justified by supergravity models. The flaw, however, is that supergravity cannot predict dilaton potential uniquely - it has been already proposed an enormous amount of them \(8\).

The integrability classes, which are incidentally based on dependence \(A(\phi)\), bring the view on the \(A\)-problem from the viewpoint different from the above-mentioned ones. First, one should make the important observation that the dependence \(A(\phi)\) is more universal than, e.g., \(U(\phi)\) or \(A(U)\). Indeed, if the metric is in the gauge \(12\) then \(A \sim \ln \det q\) is related to the radius of (compact) factor space and thus determines the geometrical scale of extra \(p\) dimensions. On the other hand, the dilaton field was introduced historically to consider the gravitational constant as a variable, and it describes thus the rigidity of spacetime. Therefore, \(A(\phi)\) symbolizes the dependence

\[ A(\phi) \sim \text{graviton-scale(dilaton-scale)}, \tag{41} \]

or, the “size(rigidity)” one. In view of this, the existence of above-mentioned integrability classes claims that with each such a dependence it is associated a unique \(\Xi\)-\(\Psi\)-\(\Lambda\) triplet. Let us for clarity disregard the \(p\)-form, as in ref. \(2\). Then, if one knows both the dependence \(A(\phi)\) and the explicit form of the dilaton-Maxwell coupling \(\Xi\) then \(\Lambda\) is uniquely determined by the class equation. Occasionally, in our case \(\Xi\) is known from (perturbative) string theory so the problem now is what is the explicit relation of \(A\) to dilaton.

So far, we do not have any clear idea on the latter. But we sure that the above-studied linear class (and, therefore, models therein) is at least a first-order approximation if one expands the yet unknown for sure True Function \(A(\phi)\) in Taylor series with respect to dilaton. It should be also noted that the linear class is distinct from others not only because it is given by a first- rather than second-order ODE with respect to \(\Xi\) and \(\Lambda\) but also because it possesses a certain discrete symmetry that alike the electric-magnetic duality in pure EMD is broken at \(D \neq 4\) and thus it forces \(D = 4\) to be a magic number again, see the paragraph after Eq. \(13\) and ref. \(3\).

V. CONCLUSION

We have deduced the full separability of the static D-dimensional massive dilaton gravity coupled to Maxwell and antisymmetric tensor fields for arbitrary dilaton potential and dilaton-Maxwell and dilaton-tensor coupling. This fact allowed us to achieve the two following aims.

First, in Sec. \(\text{II}\) the core structure of the theory has been revealed. It turned out that it is the universal relation between characteristic scales of gravity and scalar field that lies in the very heart of EMATD gravity and, probably, of any other Einstein gravity that contains scalar field. In Sec. \(\text{IV}\) we have demonstrated how the knowledge of this structure can help us with the situation when neither coupling nor potential are known precisely. Then the core structure of EMATD gravity suggests the self-consistency requirement: with each above-mentioned relation it is associated a unique dilaton coupling-coupling-potential “triplet”. Therefore, if one knows couplings then one can determine potentials and vice versa.

Second, separability has also led us to the practical concept of integrability classes that appeared to be powerful tool for getting of exactly solvable EMATD mod-
els and their solutions. As an example, we have studied the new class of solutions in Sec. III. It has been observed that in 4D this class obeys a certain duality between dilaton potential and dilaton-Maxwell coupling. There we have studied the physical properties of the two-parametric family of dyonic solutions for the case of the exponential Gibbons-Maeda couplings related to higher-dimensional gravities including superstrings. It turned out that these solutions at some conditions resemble the D-dimensional dyonic Reissner-Nordström-deSitter solutions but with "renormalized" dyonic charge plus a small logarithmic correction. The latter has the significant influence on the global structure of the non-perturbed solution - it may shift and split horizons, break down extremality, and dress the naked singularity.

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