Right Amenable Left Group Sets and the Tarski-Følner Theorem

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Abstract We introduce right amenability, right Følner nets, and right paradoxical decompositions for left homogeneous spaces and prove the Tarski-Følner theorem for left homogeneous spaces with finite stabilisers. It states that right amenability, the existence of right Følner nets, and the non-existence of right paradoxical decompositions are equivalent.

Keywords: amenability, group actions, Tarski-Følner theorem

The notion of amenability for groups was introduced by John von Neumann in 1929, see the paper ‘Zur allgemeinen Theorie des Maßes’[4]. It generalises the notion of finiteness. A group $G$ is left or right amenable if there is a finitely additive probability measure on $\mathcal{P}(G)$ that is invariant under left and right multiplication respectively. Groups are left amenable if and only if they are right amenable. A group is amenable if it is left or right amenable.

The definitions of left and right amenability generalise to left and right group sets respectively. A left group set $(M,G,\trianglelefteq)$ is left amenable if there is a finitely additive probability measure on $\mathcal{P}(M)$ that is invariant under $\trianglelefteq$. There is in general no natural action on the right that is to a left group action what right multiplication is to left group multiplication. Therefore, for a left group set there is no natural notion of right amenability.

A transitive left group action $\triangleright$ of $G$ on $M$ induces, for each element $m_0 \in M$ and each family $\{g_{m_0,m} \}_{m \in M}$ of elements in $G$ such that, for each point $m \in M$, we have $g_{m_0,m} \triangleright m_0 = m$, a right quotient set semi-action $\trianglelefteq$ of $G/G_0$ on $M$ with defect $G_0$ given by $m \trianglelefteq gG_0 = g_{m_0,m}^{-1} g_{m_0,m} \triangleright m$, where $G_0$ is the stabiliser of $m_0$ under $\triangleright$. Each of these right semi-actions is to the left group action what right multiplication is to left group multiplication. They occur in the definition of global transition functions of cellular automata over left homogeneous spaces as defined in [5]. A coordinate system is a choice of $m_0$ and $\{g_{m_0,m} \}_{m \in M}$.

A left homogeneous space is right amenable if there is a coordinate system such that there is a finitely additive probability measure on $\mathcal{P}(M)$ that is semi-invariant under $\trianglelefteq$. For example finite left homogeneous spaces, abelian groups, and finitely right generated left homogeneous spaces of sub-exponential growth are right amenable, in particular, quotients of finitely generated groups of sub-exponential growth by finite subgroups acted on by left multiplication.
A net of non-empty and finite subsets of $M$ is a right Følner net if, broadly speaking, these subsets are asymptotically invariant under $\ltimes$. A finite subset $E$ of $G/G_0$ and two partitions $\{A_e\}_{e \in E}$ and $\{B_e\}_{e \in E}$ of $M$ constitute a right paradoxical decomposition if the map $\_ \ltimes e$ is injective on $A_e$ and $B_e$, and the family $\{(A_e \ltimes e) \cup (B_e \ltimes e)\}_{e \in E}$ is a partition of $M$. The Tarski-Følner theorem states that right amenability, the existence of right Følner nets, and the non-existence of right paradoxical decompositions are equivalent.

The Tarski alternative theorem and the theorem of Følner, which constitute the Tarski-Følner theorem, are famous theorems by Alfred Tarski and Erling Følner from 1938 and 1955, see the papers ‘Algebraische Fassung des Maßproblems’ [3] and ‘On groups with full Banach mean value’ [2]. This paper is greatly inspired by the monograph ‘Cellular Automata and Groups’ [1] by Tullio Ceccherini-Silberstein and Michel Coornaert.

For a right amenable left homogeneous space with finite stabilisers we may choose a right Følner net. Using this net we show in [6] that the Garden of Eden theorem holds for such spaces. It states that a cellular automaton with finite set of states and finite neighbourhood over such a space is surjective if and only if it is pre-injective.

In Sect. 1 we introduce finitely additive probability measures and means, and kind of right semi-actions on them. In Sect. 2 we introduce right amenability. In Sect. 3 we introduce right Følner nets. In Sect. 4 we introduce right paradoxical decompositions. In Sect. 5 we prove the Tarski alternative theorem and the theorem of Følner, and in Sect. 6 we show under which assumptions left implies right amenability and give two examples of right amenable left homogeneous spaces.

**Preliminary Notions.** A left group set is a triple $(M, G, \triangleright)$, where $M$ is a set, $G$ is a group, and $\triangleright$ is a map from $G \times M$ to $M$, called left group action of $G$ on $M$, such that $G \rightarrow \text{Sym}(M), \ g \mapsto [g \triangleright \_], $ is a group homomorphism. The action $\triangleright$ is transitive if $M$ is non-empty and for each $m \in M$ the map $\_ \triangleright m$ is surjective; and free if for each $m \in M$ the map $\_ \triangleright m$ is injective. For each $m \in M$, the set $G \triangleright m$ is the orbit of $m$, the set $G_m = (\_ \triangleright m)^{-1}(m)$ is the stabiliser of $m$, and, for each $m' \in M$, the set $G_{m,m'} = (\_ \triangleright m)^{-1}(m')$ is the transporter of $m$ to $m'$.

A left homogeneous space is a left group set $(M, G, \triangleright)$ such that $\triangleright$ is transitive. A coordinate system for $M$ is a tuple $K = (m_0, \{g_{m_0,m}\} \in \mathcal{G})$, where $m_0 \in M$ and, for each $m \in M$, we have $g_{m_0,m} \triangleright m_0 = m$. The stabiliser $G_{m_0}$ is denoted by $G_0$. The tuple $R = (M, K)$ is a cell space. The set $\{gG_0 \mid g \in G\}$ of left cosets of $G_0$ in $G$ is denoted by $G/G_0$. The map $\equiv : M \times G/G_0 \rightarrow M$, $(m, gG_0) \mapsto g_{m_0,m} \triangleright m_0$ is a right semi-action of $G/G_0$ on $M$ with defect $G_0$, which means that

$$\forall m \in M : m \equiv G_0 = m,$$

$$\forall m \in M \forall g \in G \exists g_0 \in G_0 : \forall g' \in G/G_0 : m \equiv g \cdot g' = (m \equiv gG_0) \equiv g_0 \cdot g'. $$
It is transitive, which means that the set $M$ is non-empty and for each $m \in M$ the map $m \trianglelefteq -$ is surjective; and free, which means that for each $m \in M$ the map $m \trianglelefteq -$ is injective; and semi-commutes with $\triangleright$, which means that

$$\forall m \in M \forall g \in G \exists g_0 \in G_0 : g \triangleright m \trianglelefteq g_0 \triangleright (m \trianglelefteq g_0 \cdot g').$$

For each $A \subseteq M$, let $\mathbb{1}_A : M \to \{0, 1\}$ be the indicator function of $A$.

\section{Finitely Additive Probability Measures and Means}

In this section, let $\mathcal{R} = ((M, G, \triangleright), (m_0, \{g_{m_0, m}\}_{m \in M})$ be a cell space.

**Definition 1.** Let $\mu : \mathcal{P}(M) \to [0, 1]$ be a map. It is called

1. normalised if and only if $\mu(M) = 1$;
2. finitely additive if and only if, for each $A \subseteq M$ and $B \subseteq M$ such that $A \cap B = \emptyset$, $\mu(A \cup B) = \mu(A) + \mu(B)$;
3. finitely additive probability measure on $M$ if and only if it is normalised and finitely additive.

The set of all finitely additive probability measures on $M$ is denoted by $\mathcal{P}M(M)$.

**Definition 2.** The group $G$ acts on $[0, 1]^{\mathcal{P}(M)}$ on the left by

$$\triangleright : G \times [0, 1]^{\mathcal{P}(M)} \to [0, 1]^{\mathcal{P}(M)},$$

$$(g, \varphi) \mapsto [A \mapsto \varphi(g^{-1} \triangleright A)],$$

such that $G \triangleright \mathcal{P}M(M) \subseteq \mathcal{P}M(M)$.

**Definition 3.** The quotient set $G/G_0$ kind of semi-acts on $[0, 1]^{\mathcal{P}(M)}$ on the right by

$$\trianglelefteq : [0, 1]^{\mathcal{P}(M)} \times G/G_0 \to [0, 1]^{\mathcal{P}(M)},$$

$$(\varphi, g) \mapsto [A \mapsto \varphi(A \trianglelefteq g)].$$

**Definition 4.** Let $\varphi$ be an element of $[0, 1]^{\mathcal{P}(M)}$. It is called $\trianglelefteq$-semi-invariant if and only if, for each element $g \in G/G_0$ and each subset $A$ of $M$ such that the map $A \trianglelefteq g$ is injective on $A$, we have $\varphi(A) = \varphi(A \trianglelefteq g)$.

**Remark 1.** Let $\mathcal{R}$ be the cell space $((G, G, \cdot), (e_G, \{g\}_{g \in G}))$. Then, $G_0 = \{e_G\}$ and $\trianglelefteq = \cdot$. Hence, $\triangleright : (\varphi, g) \mapsto [A \mapsto \varphi(A \cdot g)]$. Except for $g$ not being inverted, this is the right group action of $G$ on $\mathcal{P}M(M)$ as defined in [1, Sect. 4.3, Paragraph 4]. Moreover, for each element $g \in G$, the map $A \trianglelefteq g$ is injective. Hence, being $\trianglelefteq$-semi-invariant is the same as being right-invariant as defined in [1] Sect. 4.4, Paragraph 2.
Definition 5. The vector space of bounded real-valued functions on \( M \) with pointwise addition and scalar multiplication is denoted by \( \ell^\infty(M) \), the supremum norm on \( \ell^\infty(M) \) is denoted by \( \| \cdot \|_\infty \); the topological dual space of \( \ell^\infty(M) \) is denoted by \( \ell^\infty(M)^* \); the pointwise partial order on \( \ell^\infty(M) \) is denoted by \( \leq \), and the constant function \( \{ m \mapsto 0 \} \) is denoted by \( 0 \).

Definition 6. Let \( \nu : \ell^\infty(M) \to \mathbb{R} \) be a map. It is called

1. normalised if and only if \( \nu(1_M) = 1 \);
2. non-negativity preserving if and only if
   \[
   \forall f \in \ell^\infty(M) : (f \geq 0 \implies \nu(f) \geq 0);
   \]
3. mean on \( M \) if and only if it is linear, normalised, and non-negativity preserving.

The set of all means on \( M \) is denoted by \( M(M) \).

Definition 7. Let \( \Psi \) be a map from \( \ell^\infty(M) \) to \( \ell^\infty(M) \). It is called non-negativity preserving if and only if

\[
\forall f \in \ell^\infty(M) : (f \geq 0 \implies \Psi(f) \geq 0).
\]

Lemma 1. Let \( G_0 \) be finite, let \( A \) be a finite subset of \( M \), and let \( g \) be an element of \( G/G_0 \). Then, \(| (\_ \trianglelefteq g)^{-1}(A) | \leq |G_0| \cdot |A| \).

Proof. Let \( a \in A \) such that \( (\_ \trianglelefteq g)^{-1}(a) \neq \emptyset \). There are \( m \) and \( m' \in M \) such that \( G_{m,m'} = g \) and \( m' \trianglelefteq g = a \). For each \( m'' \in M \), we have \( m'' \trianglelefteq g = g_{m_0,m''} \triangleright m \) and hence

\[
m'' \trianglelefteq g = a \iff m'' \trianglelefteq g = m' \trianglelefteq g \\
\iff g_{m_0,m''}^{-1} m = m \\
\iff g_{m_0,m''} G_m = g_{m_0,m''} G_m.
\]

Moreover, for each \( m'' \) and each \( m''' \in M \) with \( m'' \neq m''' \), we have \( g_{m_0,m''} \neq g_{m_0,m'''} \). Thus,

\[
|(\_ \trianglelefteq g)^{-1}(a)| = |\{ m'' \in M \mid m'' \trianglelefteq g = a \}| \\
= |\{ m'' \in M \mid g_{m_0,m''} G_m \}| \\
\leq |g_{m_0,m''} G_m| \\
= |G_m| \\
= |G_0|.
\]

Therefore, because \( (\_ \trianglelefteq g)^{-1}(A) = \bigcup_{a \in A} (\_ \trianglelefteq g)^{-1}(a) \), we have \( |(\_ \trianglelefteq g)^{-1}(A)| \leq |G_0| \cdot |A| \). \( \square \)
Definition 8. The group $G$ acts on $\ell^\infty(M)$ on the left by
\[ g \cdot f : G \times \ell^\infty(M) \to \ell^\infty(M), \]
\[ (g, f) \mapsto [m \mapsto f(g^{-1} \triangleright m)]. \]

Lemma 2. Let $G_0$ be finite. The quotient set $G/G_0$ kind of semi-acts on $\ell^\infty(M)$ on the right by
\[ g \cdot f : \ell^\infty(M) \times G/G_0 \to \ell^\infty(M), \]
\[ (f, g) \mapsto [m \mapsto \sum_{m' \in (\_ \trianglelefteq g)^{-1}(m)} f(m')], \]
such that, for each tuple $(f, g) \in \ell^\infty(M) \times G/G_0$, we have $\|f \triangleright g\|_\infty \leq |G_0| \cdot \|f\|_\infty$.

Proof. Let $g \in G/G_0$. Furthermore, let $f \in \ell^\infty(M)$. Moreover, let $m \in M$. Because $G_0$ is finite, according to Lemma 1, we have $|(\_ \trianglelefteq g)^{-1}(m)| \leq |G_0| < \infty$. Hence, the sum in the definition of $\triangleright$ is finite. Furthermore,
\[ |(f \triangleright g)(m)| \leq \sum_{m' \in (\_ \trianglelefteq g)^{-1}(m)} |f(m')| \]
\[ \leq \left( \sum_{m' \in (\_ \trianglelefteq g)^{-1}(m)} 1 \right) \cdot \|f\|_\infty \]
\[ = |(\_ \trianglelefteq g)^{-1}(m)| \cdot \|f\|_\infty \]
\[ \leq |G_0| \cdot \|f\|_\infty. \]

Therefore, $f \triangleright g \in \ell^\infty(M)$, $\|f \triangleright g\|_\infty \leq |G_0| \cdot \|f\|_\infty$, and $\triangleright$ is well-defined. □

Remark 2. In the situation of Remark 1, we have $\triangleright : (f, g) \mapsto [m \mapsto f(m \cdot g^{-1})]$. Hence, $\triangleright$ is the right group action of $G$ on $\mathbb{R}^G$ as defined in [1, Sect. 4.3, Paragraph 5].

Lemma 3. Let $G_0$ be finite and let $g$ be an element of $G/G_0$. The map $\_ \trianglelefteq g$ is linear, continuous, and non-negativity preserving.

Proof. Linearity follows from linearity of summation, continuity follows from linearity and $\|\_ \trianglelefteq g\|_\infty \leq |G_0| \cdot \|\_\|_\infty$, and non-negativity preservation follows from non-negativity preservation of summation. □

Lemma 4 ([1, Proposition 4.1.7]). Let $\nu$ be a mean on $M$. Then, $\nu \in \ell^\infty(M)^*$ and $\|\nu\|_{\ell^\infty(M)^*} = 1$. In particular, $\nu$ is continuous. □

Definition 9. The group $G$ acts on $\ell^\infty(M)^*$ on the left by
\[ g \cdot \psi : G \times \ell^\infty(M)^* \to \ell^\infty(M)^*, \]
\[ (g, \psi) \mapsto [f \mapsto \psi(g^{-1} \triangleright f)], \]
such that $G \triangleright \mathcal{M}(M) \subseteq \mathcal{M}(M)$. 
Definition 10. Let $G_0$ be finite. The quotient set $G/G_0$ kind of semi-acts on $\ell^\infty(M)^*$ on the right by

$$\equiv : \ell^\infty(M)^* \times G/G_0 \to \ell^\infty(M)^*,$$
$$\equiv (\psi, g) \mapsto [f \mapsto \psi(f \equiv g)].$$

Proof. Let $\psi \in \ell^\infty(M)^*$ and let $g \in G/G_0$. Then, $\psi \equiv g = \psi \circ (\equiv \equiv g)$. Because $\psi$ and $\equiv \equiv g$ are linear and continuous, so is $\psi \equiv g$. $\square$

Definition 11. Let $G_0$ be finite and let $\psi$ be an element of $\ell^\infty(M)^*$. It is called $\equiv$-invariant if and only if, for each element $g \in G/G_0$ and each function $f \in \ell^\infty(M)$, we have $(\psi \equiv g)(f) = \psi(f)$.

Remark 3. In the situation of Remark 2 we have $\equiv : (\psi, g) \mapsto [f \mapsto \psi(f \equiv g)]$. Except for $g$ not being inverted, this is the right group action of $G$ on $\ell^\infty(G)^*$ as defined in [1, Sect. 4.3, Paragraph 6]. Hence, being $\equiv$-invariant is the same as being right-invariant as defined in [1, Sect. 4.4, Paragraph 3].

Theorem 1 ([1, Theorem 4.1.8]). The map

$$\Phi : \mathcal{M}(M) \to \mathcal{P}\mathcal{M}(M),$$
$$\nu \mapsto [A \mapsto \nu(\mathcal{L}_A)],$$

is bijective. $\square$

Theorem 2 ([1, Theorem 4.2.1]). The set $\mathcal{M}(M)$ is a convex and compact subset of $\ell^\infty(M)^*$ equipped with the weak-$*$ topology. $\square$

2 Right Amenability

In Definition 13 we introduce the notion of right amenability using finitely additive probability measures. And in Theorem 3 we characterise right amenability of cell spaces with finite stabilisers using means.

Definition 12. Let $(M, G, \circlearrowright)$ be a left group set. It is called left amenable if and only if there is an $\equiv$-invariant finitely additive probability measure on $M$.

Definition 13. Let $\mathcal{M} = (M, G, \circlearrowright)$ be a left homogeneous space. It is called right amenable if and only if there is a coordinate system $\mathcal{K} = (m_0, \{g_{m_0,m}\}_{m \in M})$ for $\mathcal{M}$ such that there is a $\equiv$-semi-invariant finitely additive probability measure on $\mathcal{M}$, in which case the cell space $\mathcal{R} = (\mathcal{M}, \mathcal{K})$ is called right amenable.

Remark 4. In the situation of Remark 3, being right amenable is the same as being amenable as defined in [1, Definition 4.4.5].

In the remainder of this section, let $\mathcal{R} = (\mathcal{M}, \circlearrowright, (m_0, \{g_{m_0,m}\}_{m \in M}))$ be a cell space such that the stabiliser $G_0$ of $m_0$ under $\circlearrowright$ is finite.
Lemma 5. Let $g$ be an element of $G/G_0$ and let $A$ be a subset of $M$ such that the map $\_ \trianglelefteq g$ is injective on $A$. Then, $1_{A \trianglelefteq g} = 1_A \trianglelefteq g$.

Proof. For each $m \in M$, because $\_ \trianglelefteq g$ is injective on $A$, 

$$1_{A \trianglelefteq g}(m) = \begin{cases} 1, & \text{if } m \in A \trianglelefteq g, \\ 0, & \text{otherwise}, \end{cases}$$

$$= |\{m' \in A \mid m' \trianglelefteq g = m\}|$$

$$= \sum_{m' \in (\_ \trianglelefteq g)^{-1}(m)} 1_A(m')$$

$$= (1_A \trianglelefteq g)(m).$$

In conclusion, $1_{A \trianglelefteq g} = 1_A \trianglelefteq g$. \hfill $\square$

Lemma 6 (Proposition 4.1.9]). The vector space

$$E(M) = \{f : M \to \mathbb{R} \mid f(M) \text{ is finite} \} (= \text{span}\{1_A \mid A \subseteq M\})$$

is dense in the Banach space $(\ell^\infty(M), \|\_\|_\infty)$. \hfill $\square$

Lemma 7. Let $\psi$ be an element of $\ell^\infty(M)^*$ such that, for each element $g \in G/G_0$ and each subset $A$ of $M$ such that the map $\_ \trianglelefteq g$ is injective on $A$, we have $(\psi \trianglelefteq g)(1_A) = \psi(1_A)$. The map $\psi$ is $\trianglelefteq$-invariant.

Proof. Let $g \in G/G_0$.

First, let $A \subseteq M$. Moreover, let $m \in M$. According to Lemma 1, we have $k_m = \|(\_ \trianglelefteq g)^{-1}(m)\| \leq |G_0|$. Hence, there are pairwise distinct $m_{m,1}, m_{m,2}, \ldots, m_{m,k_m} \in M$ such that $(\_ \trianglelefteq g)^{-1}(m) = \{m_{m,1}, m_{m,2}, \ldots, m_{m,k_m}\}$. For each $i \in \{1, 2, \ldots, |G_0|\}$, put

$$A_i = \{m_{m,i} \mid m \in M, k_m \geq i\} \cap A.$$

Because, for each $m \in M$ and each $m' \in M$ such that $m \neq m'$, we have $(\_ \trianglelefteq g)^{-1}(m) \cap (\_ \trianglelefteq g)^{-1}(m') = \emptyset$, the sets $A_1, A_2, \ldots, A_{|G_0|}$ are pairwise disjoint and the map $\_ \trianglelefteq g$ is injective on each of these sets. Moreover, because

$$\bigcup_{m \in M}(\_ \trianglelefteq g)^{-1}(m) = M,$$

we have $\bigcup_{i=1}^{\lfloor G_0 \rfloor} A_i = A$. Therefore, $1_A = \sum_{i=1}^{\lfloor G_0 \rfloor} 1_{A_i}$. Thus, because $\psi \trianglelefteq g$ and $\psi$ are linear,

$$(\psi \trianglelefteq g)(1_A) = (\psi \trianglelefteq g) \left( \sum_{i=1}^{\lfloor G_0 \rfloor} 1_{A_i} \right) = \sum_{i=1}^{\lfloor G_0 \rfloor} (\psi \trianglelefteq g)(1_{A_i}) = \sum_{i=1}^{\lfloor G_0 \rfloor} \psi(1_{A_i}) = \psi(1_A).$$

Therefore, $\psi \trianglelefteq g = \psi$ on the set of indicator functions. Thus, because the indicator functions span $E(M)$, and $\psi \trianglelefteq g$ and $\psi$ are linear, $\psi \trianglelefteq g = \psi$ on $E(M)$. Hence, because $E(M)$ is dense in $\ell^\infty(M)$, and $\psi \trianglelefteq g$ and $\psi$ are continuous, $\psi \trianglelefteq g = \psi$ on $\ell^\infty(M)$. In conclusion, $\psi$ is $\trianglelefteq$-invariant. \hfill $\square$
Theorem 3. The cell space $\mathcal{R}$ is right amenable if and only if there is a $\equiv$-invariant mean on $M$.

Proof. Let $\Phi$ be the map in Theorem \[1\]
First, let $\mathcal{R}$ be right amenable. Then, there is $\equiv$-semi-invariant finitely additive probability measure $\mu$ on $M$. Put $\nu = \Phi^{-1}(\mu)$. Then, for each $g \in G/G_0$ and each $A \subseteq M$ such that $\_ \equiv g$ is injective on $A$, according to Lemma \[4\]

$$\nu(\_ \equiv g)(A) = \nu(\_ \equiv g(A) = \nu(A \equiv g) = \mu(A \equiv g) = \mu(A) = \nu(A).$$

Thus, according to Lemma \[5\] the mean $\nu$ is $\equiv$-invariant.

Secondly, let there be a $\equiv$-invariant mean $\nu$ on $M$. Put $\mu = \Phi(\nu)$. Then, for each $g \in G/G_0$ and each $A \subseteq M$ such that $\_ \equiv g$ is injective on $A$, according to Lemma \[5\]

$$\mu(\_ \equiv g)(A) = \mu(A \equiv g) = \nu(\_ \equiv g(A) = \nu(\_ \equiv g(A) = \nu(A) = \mu(A).$$

Hence, $\mu$ is $\equiv$-semi-invariant. \[\square\]

3 Right Følner Nets

In this section, let $\mathcal{R} = ((M,G,\triangleright), (m_0, \{g_{m_0,m}\}_{m \in M}))$ be a cell space.

Definition 14. Let $\{F_i\}_{i \in I}$ be a net in $\{F \subseteq M \mid F \neq \emptyset, F$ finite\} indexed by $(I, \leq)$. It is called right Følner net in $\mathcal{R}$ indexed by $(I, \leq)$ if and only if

$$\forall g \in G/G_0 : \lim_{i \in I} \frac{|F_i \setminus (\_ \equiv g)^{-1}(F_i)|}{|F_i|} = 0.$$

Remark 5. In the situation of Remark \[1\] for each element $g \in G$ and each index $i \in I$, we have $(\_ \equiv g)^{-1}(F_i) = F_i \cdot g^{-1}$. Hence, right Følner nets in $\mathcal{R}$ are exactly right Følner nets for $G$ as defined in \[1\] First paragraph after Definition 4.7.2.

Lemma 8. Let $V$ be a set, let $W$ be a set, and let $\Psi$ be a map from $V \times W$ to $\mathbb{R}$. There is a net $\{v_i\}_{i \in I}$ in $V$ indexed by $(I, \leq)$ such that

$$\forall w \in W : \lim_{i \in I} \Psi(v_i, w) = 0, \quad (1)$$

if and only if, for each finite subset $Q$ of $W$ and each positive real number $\varepsilon \in \mathbb{R}_{>0}$, there is an element $v \in V$ such that

$$\forall q \in Q : \Psi(v, q) < \varepsilon. \quad (2)$$

Proof. First, let there be a net $\{v_i\}_{i \in I}$ in $V$ indexed by $(I, \leq)$ such that \[1\] holds. Furthermore, let $Q \subseteq W$ be finite and let $\varepsilon \in \mathbb{R}_{>0}$. Because \[1\] holds, for each $q \in Q$, there is an $i_q \in I$ such that,

$$\forall i \in I : (i \geq i_q \implies \Psi(v_i, q) < \varepsilon).$$
Because \((I, \leq)\) is a directed set and \(Q\) is finite, there is an \(i \in I\) such that, for each \(q \in Q\), we have \(i \geq i_q\). Put \(v = v_i\). Then, (2) holds.

Secondly, for each finite \(Q \subseteq W\) and each \(\varepsilon \in \mathbb{R}_{>0}\), let there be a \(v \in V\) such that \((2)\) holds. Furthermore, let

\[
I = \{Q \subseteq W \mid Q \text{ is finite}\} \times \mathbb{R}_{>0}
\]

and let \(\leq\) be the preorder on \(I\) given by

\[
\forall (Q, \varepsilon) \in I \forall (Q', \varepsilon') \in I : (Q, \varepsilon) \leq (Q', \varepsilon') \iff Q \subseteq Q' \land \varepsilon \geq \varepsilon'.
\]

For each \((Q, \varepsilon) \in I\) and each \((Q', \varepsilon') \in I\), the element \((Q \cup Q', \min(\varepsilon, \varepsilon'))\) of \(I\) is an upper bound of \((Q, \varepsilon)\) and of \((Q', \varepsilon')\). Hence, \((I, \leq)\) is a directed set.

By precondition, for each \(i = (Q, \varepsilon) \in I\), there is a \(v_i \in V\) such that

\[
\forall q \in Q : \psi(v_i, q) < \varepsilon.
\]

Let \(w \in W\) and let \(\varepsilon_0 \in \mathbb{R}_{>0}\). Put \(i_0 = (\{w\}, \varepsilon_0)\). For each \(i = (Q, \varepsilon) \in I\) with \(i \geq i_0\), we have \(w \in Q\) and \(\varepsilon \leq \varepsilon_0\). Hence,

\[
\forall i \in I : (i \geq i_0 \implies \psi(v_i, w) < \varepsilon_0).
\]

Therefore, \(\{v_i\}_{i \in I}\) is a net in \(V\) indexed by \((I, \leq)\) such that (1) holds. \(\square\)

**Lemma 9.** There is a right Følner net in \(R\) if and only if, for each finite subset \(E\) of \(G/G_0\) and each positive real number \(\varepsilon \in \mathbb{R}_{>0}\), there is a non-empty and finite subset \(F\) of \(M\) such that

\[
\forall e \in E : \frac{|F \setminus (\_ \trianglelefteq e)^{-1}(F)|}{|F|} < \varepsilon.
\]

**Proof.** This is a direct consequence of Lemma 8 with

\[
\Psi : \{F \subseteq M \mid F \neq \emptyset, F \text{ finite}\} \times G/G_0 \to \mathbb{R},
\]

\[
(F, g) \mapsto \frac{|F \setminus (\_ \trianglelefteq g)^{-1}(F)|}{|F|}.
\]

\(\square\)

**Lemma 10.** Let \(m\) be an element of \(M\), and let \(g\) be an element of \(G/G_0\). There is an element \(g' \in g\) such that

\[
\forall g' \in G/G_0 : (m \trianglelefteq g) \trianglelefteq g' = m \trianglelefteq g \cdot g'.
\]

**Proof.** There is a \(g \in G\) such that \(gG_0 = g\). Moreover, because \(\trianglelefteq\) is a semi-action with defect \(G_0\), there is a \(g_0 \in G_0\) such that

\[
\forall g' \in G/G_0 : (m \trianglelefteq gG_0) \trianglelefteq g' = m \trianglelefteq g \cdot (g_0^{-1} \cdot g').
\]

Because \(g \cdot (g_0^{-1} \cdot g') = gg_0^{-1} \cdot g'\) and \(gg_0^{-1} \in \mathcal{g}\), the statement holds. \(\square\)
Lemma 11. Let $A$ and $A'$ be two subsets of $M$, and let $g$ and $g'$ be two elements of $G/G_0$. Then, for each element $m \in (\_ \triangleleft g)^{-1}(A) \setminus (\_ \triangleleft g')^{-1}(A')$,

$$m \triangleleft g \in \bigcup_{g' \in g} A \setminus (\_ \triangleleft g^{-1} \cdot g')^{-1}(A'),$$
$$m \triangleleft g' \in \bigcup_{g \in g'} (\_ \triangleleft g^{-1} \cdot g)^{-1}(A) \setminus A'.$$

Proof. (Lemma 12). Let $m \in (\_ \triangleleft g)^{-1}(A) \setminus (\_ \triangleleft g')^{-1}(A')$. Then, $m \triangleleft g \in A$ and $m \triangleleft g' \notin A'$. According to Lemma 11, there is a $g \in g$ and $g' \in g'$ such that $(m \triangleleft g) \in g^{-1} \cdot g' = m \triangleleft g' \notin A'$ and $(m \triangleleft g') \in (g')^{-1} \cdot g = m \triangleleft g \in A$.

Hence, $m \triangleleft g \notin (\_ \triangleleft g^{-1} \cdot g')^{-1}(A')$ and $m \triangleleft g' \in (\_ \triangleleft (g')^{-1} \cdot g^{-1})^{-1}(A)$. Therefore, $m \triangleleft g \in A \setminus (\_ \triangleleft g^{-1} \cdot g')^{-1}(A')$ and $m \triangleleft g' \in (\_ \triangleleft (g')^{-1} \cdot g^{-1})^{-1}(A) \setminus A'$. In conclusion, $m \triangleleft g \in \bigcup_{g \in g} A \setminus (\_ \triangleleft g^{-1} \cdot g')^{-1}(A')$ and $m \triangleleft g' \in \bigcup_{g' \in g'} (\_ \triangleleft (g')^{-1} \cdot g^{-1})^{-1}(A) \setminus A'$.

Lemma 12. Let $G_0$ be finite, let $F$ and $F'$ be two finite subsets of $M$, and let $g$ and $g'$ be two elements of $G/G_0$. Then,

$$|(\_ \triangleleft g)^{-1}(F) \setminus (\_ \triangleleft g')^{-1}(F')| \leq \begin{cases} |G_0|^2 \cdot \max_{g \in g} |F \setminus (\_ \triangleleft g^{-1} \cdot g')^{-1}(F')|, \\
|G_0|^2 \cdot \max_{g' \in g'} |(\_ \triangleleft g^{-1} \cdot g)^{-1}(F) \setminus F'|. \end{cases}$$

Proof. Put $A = (\_ \triangleleft g)^{-1}(F) \setminus (\_ \triangleleft g')^{-1}(F')$. For each $g \in g$, put $B_g = F \setminus (\_ \triangleleft g^{-1} \cdot g')^{-1}(F')$. For each $g' \in g'$, put $B'_{g'} = (\_ \triangleleft (g')^{-1} \cdot g)^{-1}(F) \setminus F'$. According to Lemma 11, the restrictions $(\_ \triangleleft g)_{A \to \bigcup_{g \in g} B_g}$ and $(\_ \triangleleft g')_{A \to \bigcup_{g' \in g'} B'_{g'}}$ are well-defined. Moreover, for each $m \in M$, according to Lemma 11, we have $|(\_ \triangleleft g)^{-1}(m)| \leq |G_0|$ and $|(\_ \triangleleft g')^{-1}(m)| \leq |G_0|$. Therefore, because $|g| = |G_0|$, $|A| \leq |G_0| \cdot \bigcup_{g \in g} |B_g| \leq |G_0| \cdot \sum_{g \in g} |B_g| \leq |G_0|^2 \cdot \max_{g \in g} |B_g|$ and analogously $|A| \leq |G_0|^2 \cdot \max_{g' \in g'} |B'_{g'}|$.

Lemma 13. Let $G_0$ be finite and let $\{F_i\}_{i \in I}$ be a net in $\{F \subseteq M \mid F \neq \emptyset, F \text{ finite}\}$ indexed by $(I, \leq)$. The net $\{F_i\}_{i \in I}$ is a right Følner net in $\mathcal{R}$ if and only if

$$\forall g \in G/G_0 : \lim_{i \to I} \frac{|(\_ \triangleleft g)^{-1}(F_i) \setminus F_i|}{|F_i|} = 0. \tag{3}$$

Proof. Let $g \in G/G_0$. Furthermore, let $i \in I$. Because $F_i = (\_ \triangleleft g)^{-1}(F_i)$, according to Lemma 12,

$$|(\_ \triangleleft g)^{-1}(F_i) \setminus F_i| \leq |G_0|^2 \cdot \max_{g \in g} |F_i \setminus (\_ \triangleleft g^{-1}G_0)^{-1}(F_i)|$$
and
\[|F_i \setminus (_\trianglelefteq g)^{-1}(F_i)| \leq |G_0|^2 \cdot \max_{g \in G_0}(|(_\trianglelefteq g^{-1}G_0)^{-1}(F_i) \setminus F_i|).
\]
Moreover, \(|g| = |G_0| < \infty\). Therefore, if \(\{F_i\}_{i \in I}\) is a right Følner net in \(\mathcal{R}\), then
\[
\lim_{i \in I} \frac{|(_\trianglelefteq g)^{-1}(F_i) \setminus F_i|}{|F_i|} = 0;
\]
and, if (3) holds, then
\[
\lim_{i \in I} \frac{|F_i \setminus (_\trianglelefteq g)^{-1}(F_i)|}{|F_i|} = 0.
\]
In conclusion, \(\{F_i\}_{i \in I}\) is a right Følner net in \(\mathcal{R}\) if and only if (3) holds.

**Lemma 14.** Let \(G_0\) be finite. There is a right Følner net in \(\mathcal{R}\) if and only if, for each finite subset \(E\) of \(G/G_0\) and each positive real number \(\varepsilon \in \mathbb{R}_{>0}\), there is a non-empty and finite subset \(F\) of \(M\) such that
\[
\forall e \in E : \frac{|(_\trianglelefteq e)^{-1}(F) \setminus F|}{|F|} < \varepsilon.
\]

**Proof.** This is a direct consequence of Lemma 13 and Lemma 8 with
\[
\Psi : \{F \subseteq M \mid F \neq \emptyset, F \text{ finite}\} \times G/G_0 \to \mathbb{R},
\]
\[
(F, g) \mapsto \frac{|(_\trianglelefteq g)^{-1}(F) \setminus F|}{|F|}.
\]

\(\square\)

### 4 Right Paradoxical Decompositions

In this section, let \(\mathcal{R} = ((M, G, \triangleright), (m_0, \{g_{rn,i}\}_{m \in M}))\) be a cell space.

**Definition 15.** Let \(A\) and \(A'\) be two sets. The set \(A \cup A'\) is denoted by \(A \cup \cdot A'\) if and only if the sets \(A\) and \(A'\) are disjoint.

**Definition 16.** Let \(E\) be a finite subset of \(G/G_0\), and let \(\{A_e\}_{e \in E}\) and \(\{B_e\}_{e \in E}\) be two families of subsets of \(M\) indexed by \(E\) such that, for each index \(e \in E\), the map \(\_ \trianglelefteq e\) is injective on \(A_e\) and on \(B_e\), and
\[
M = \bigcup_{e \in E} A_e = \bigcup_{e \in E} B_e = \left(\bigcup_{e \in E} A_e \trianglelefteq e\right) \cup \left(\bigcup_{e \in E} B_e \trianglelefteq e\right).
\]
The triple \((N, \{A_e\}_{e \in E}, \{B_e\}_{e \in E})\) is called right paradoxical decomposition of \(\mathcal{R}\).

**Remark 6.** In the situation of Remark 1, for each element \(g \in G\), the map \(\_ \trianglelefteq g\) is injective. Hence, right paradoxical decompositions of \(\mathcal{R}\) are the same as right paradoxical decompositions of \(G\) as defined in [1] Definition 4.8.1.
Lemma 15. Let $G_0$ be finite and let $(N, \{A_e\}_{e \in E}, \{B_e\}_{e \in E})$ be a right paradoxical decomposition of $\mathcal{R}$. Then,

$$\mathbf{1}_M = \sum_{e \in E} \mathbf{1}_{A_e} = \sum_{e \in E} \mathbf{1}_{B_e} = \sum_{e \in E} (\mathbf{1}_{A_e} \trianglelefteq e) + \sum_{e \in E} (\mathbf{1}_{B_e} \triangleleft e).$$

Proof. This is a direct consequence of Definition 15 and Lemma 5.

5 Tarski’s and Følner’s Theorem

In this section, let $\mathcal{R} = ((M, G, \triangleright), (m_0, \{g_{m_0,m}\}_{m \in M}))$ be a cell space such that the stabiliser $G_0$ of $m_0$ under $\triangleright$ is finite.

Lemma 16. Let $g$ be an element of $G/G_0$. The map $\_ \trianglelefteq g$ is continuous, where $\ell^\infty(M)^*$ is equipped with the weak-* topology.

Proof. For each $f \in \ell^\infty(M)$, let $ev_f : \ell^\infty(M)^* \to \mathbb{R}$, $\psi \mapsto \psi(f)$. Furthermore, let $f \in \ell^\infty(M)$. Then, for each $\psi \in \ell^\infty(M)^*$,

$$(ev_f \circ (\_ \trianglelefteq g))(\psi) = ev_f(\psi \trianglelefteq g) = (\psi \trianglelefteq g)(f) = \psi(f \trianglelefteq g) = ev_{f \trianglelefteq g}(\psi).$$

Thus, $ev_f \circ (\_ \trianglelefteq g) = ev_{f \trianglelefteq g}$. Hence, because $ev_{f \trianglelefteq g}$ is continuous, so is $ev_f \circ (\_ \trianglelefteq g)$. Therefore, the map $\_ \trianglelefteq g$ is continuous.

Lemma 17. Let $\{\nu_i\}_{i \in I}$ be a net in $\mathcal{M}(M)$ such that, for each element $g \in G/G_0$, the net $\{\nu_i \trianglelefteq g - \nu_i\}_{i \in I}$ converges to $0$ in $\ell^\infty(M)^*$ equipped with the weak-* topology. The cell space $\mathcal{R}$ is right amenable.

Proof. Let $g \in G/G_0$. According to Theorem 2, the set $\mathcal{M}(M)$ is compact in $\ell^\infty(M)^*$ equipped with the weak-* topology. Hence, there is a subnet $\{\nu_{i_j}\}_{j \in J}$ of $\{\nu_i\}_{i \in I}$ that converges to a $\nu \in \mathcal{M}(M)$. Because, according to Lemma 10, the map $\_ \trianglelefteq g$ is continuous, the net $\{\nu_i \trianglelefteq g - \nu_i\}_{i \in I}$ converges to $(\nu \trianglelefteq g) - \nu$ in $\ell^\infty(M)^*$. Because it is a subnet of $\{\nu_i \trianglelefteq g - \nu_i\}_{i \in I}$, it also converges to $0$ in $\ell^\infty(M)^*$. Because the space $\ell^\infty(M)^*$ is Hausdorff, we have $(\nu \trianglelefteq g) - \nu = 0$ and hence $\nu \trianglelefteq g = \nu$. Altogether, $\nu$ is a $\trianglelefteq$-invariant mean. In conclusion, according to Theorem 3 the cell space $\mathcal{R}$ is right amenable.

Lemma 18. Let $m$ be an element of $M$, and let $E$ and $E'$ be two subsets of $G/G_0$. There is a subset $E''$ of $G/G_0$ such that $(m \triangleleft E) \triangleleft E' = m \triangleleft E''$; if $G_0 \subset E \cap E'$, then $G_0 \subset E''$; if $E$ and $E'$ are finite, then $|E''| \leq |E| \cdot |E'|$; and if $G_0 \cdot E' \subset E''$, then $E'' = \{g \cdot e' \mid e \in E, e' \in E', g \in e\}$. 

Proof. For each $e \in E$, according to Lemma 10 there is a $g_e \in e$ such that

$$\forall g \in G/G_0 : (m \triangleleft e) \trianglelefteq g = m \triangleleft g_e \cdot g.$$ 

Put $E'' = \{g_e \cdot e' \mid e \in E, e' \in E'\}$. Then, $(m \triangleleft E) \triangleleft E' = m \triangleleft E''$. Moreover, if $G_0 \subset E \cap E'$, then $G_0 = gG_0 \cdot G_0 \subset E''$; if $E$ and $E'$ are finite, then $|E''| \leq |E| \cdot |E'|$; and if $G_0 \cdot E' \subset E''$, then $E''$ is as stated. □
Main Theorem 4. Let $\mathcal{R} = ((M, G, \vartriangleright), (m_0, \{g_{m_0, m}\}_{m \in M}))$ be a cell space such that the stabiliser $G_0$ of $m_0$ under $\vartriangleright$ is finite. The following statements are equivalent:

1. The cell space $\mathcal{R}$ is not right amenable;
2. There is no right Følner net in $\mathcal{R}$;
3. There is a finite subset $E$ of $G/G_0$ such that $G_0 \in E$ and, for each finite subset $F$ of $M$, we have $|F \cdot E| \geq 2|F|$;
4. There is a 2-to-1 surjective map $\phi : M \to M$ and there is a finite subset $E$ of $G/G_0$ such that
   $$\forall m \in M \exists e \in E : \phi(m) \trianglelefteq e = m;$$
5. There is a right paradoxical decomposition of $\mathcal{R}$.

Proof. $[\text{1}]$ implies $[\text{2}]$. Let there be a right Følner net $\{F_i\}_{i \in I}$ in $\mathcal{R}$. Furthermore, let $i \in I$. Put

$$\nu_i : \ell^\infty(M) \to \mathbb{R},
\quad
f \mapsto \frac{1}{|F_i|} \sum_{m \in F_i} f(m).$$

Then, $\nu_i \in \mathcal{M}(M)$. Moreover, let $g \in G/G_0$ and let $f \in \ell^\infty(M)$. Then,

$$\begin{align*}
(\nu_i \equiv g)(f) &= \nu_i(f \equiv g) \\
&= \frac{1}{|F_i|} \sum_{m \in F_i} (f \equiv g)(m) \\
&= \frac{1}{|F_i|} \sum_{m \in F_i} \sum_{m' \in (\_ \trianglelefteq g)^{-1}(m)} f(m') \\
&= \frac{1}{|F_i|} \sum_{m \in (\_ \trianglelefteq g)^{-1}(F_i)} f(m).
\end{align*}$$

Hence,

$$\begin{align*}
(\nu_i \equiv g - \nu_i)(f) &= \frac{1}{|F_i|} \left( \sum_{m \in (\_ \trianglelefteq g)^{-1}(F_i) \setminus F_i} f(m) - \sum_{m \in F_i \setminus (\_ \trianglelefteq g)^{-1}(F_i)} f(m) \right).
\end{align*}$$

Therefore,

$$\begin{align*}
|\nu_i \equiv g - \nu_i(f)| &\leq \frac{1}{|F_i|} \left( \sum_{m \in (\_ \trianglelefteq g)^{-1}(F_i) \setminus F_i} |f(m)| + \sum_{m \in F_i \setminus (\_ \trianglelefteq g)^{-1}(F_i)} |f(m)| \right) \\
&\leq \left( \frac{|(\_ \trianglelefteq g)^{-1}(F_i) \setminus F_i|}{|F_i|} + \frac{|F_i \setminus (\_ \trianglelefteq g)^{-1}(F_i)|}{|F_i|} \right) \|f\|_{\infty}.
\end{align*}$$
According to Definition 14 and Lemma 13, the nets \[ \{\underline{\alpha}g^{-1}(F_i) \setminus F_i \}/|F_i| \}_{i \in I} \] and \[ \{F_i \setminus (\underline{\alpha}g^{-1}(F_i)) \}/|F_i| \}_{i \in I} \] converge to 0. Hence, so does \[ \{\|\nu_i g - \nu_i(f)\| \}_{i \in I} \] converges to 0 in \( \ell^\infty(M)^* \) equipped with the weak* topology. Hence, according to Lemma 17, the cell space \( \mathcal{R} \) is right amenable. In conclusion, by contraposition, if \( \mathcal{R} \) is not right amenable, then there is no right Følner net in \( \mathcal{R} \).

**2 implies 3.** Let there be no right Følner net in \( \mathcal{R} \). According to Lemma 9, there is a finite \( E_1 \subseteq G/G_0 \) and an \( \varepsilon \in \mathbb{R}_{>0} \) such that, for each non-empty and finite \( F \subseteq M \), there is an \( e_F \in E_1 \) such that

\[
\frac{|F \setminus (\underline{\alpha} e_F)^{-1}(F)|}{|F|} \geq \varepsilon.
\]

Put \( E_2 = \{G_0\} \cup E_1 \).

Let \( F \subseteq M \) be non-empty and finite. Then, \( F \subseteq F \cup (F \cap E_1) = F \cup E_2 \).

Thus,

\[
|F \cup E_2| - |F| = |(F \cup E_2) \setminus F| = |(F \cap E_1) \setminus F| \geq |(F \cap e_F) \setminus F|.
\]

Moreover, according to Lemma 1, we have \( |(\underline{\alpha} e_F)^{-1}((F \cap e_F) \setminus F)| \leq |G_0| \cdot |(F \cap e_F) \setminus F| \).

Hence,

\[
|F \cup E_2| - |F| \geq \frac{|(\underline{\alpha} e_F)^{-1}((F \cap e_F) \setminus F)|}{|G_0|}.
\]

Therefore, because \( F \setminus (\underline{\alpha} e_F)^{-1}(F) \subseteq (\underline{\alpha} e_F)^{-1}((F \cap e_F) \setminus F) \),

\[
|F \cup E_2| - |F| \geq \frac{|F \setminus (\underline{\alpha} e_F)^{-1}(F)|}{|G_0|} \geq \frac{\varepsilon}{|G_0|}|F|.
\]

Put \( \xi = 1 + \varepsilon/|G_0| \). Then, \( |F \cup E_2| \geq \xi|F| \). Because \( \varepsilon \) does not depend on \( F \), neither does \( \xi \). Therefore, for each non-empty and finite \( F \subseteq M \), we have \( |F \cup E_2| \geq \xi|F| \).

Let \( F \subseteq M \) be non-empty and finite. Because \( \xi > 1 \), there is an \( n \in \mathbb{N} \) such that \( \xi^n \geq 2 \). Hence,

\[
|((F \cup E_2) \cup \cdots) \cup E_2| \geq \xi^{|(F \cup E_2) \cup \cdots) \cup E_2|} \geq \cdots \geq \xi^n|F| \geq 2|F|.
\]

Moreover, according to Lemma 13, there is an \( E \subseteq G/G_0 \) such that \( E \) is finite, \( G_0 \in E \), and \( F \subseteq E = (((F \cup E_2) \cup \cdots) \cup E_2) \cup E_2 \). In conclusion, \( |F \cup E| \geq 2|F| \).
Theorem 15. \( M \) \( \subseteq \) \( N_l(F) = \{ m \in M \mid (m \trianglelefteq E) \cap F \neq \emptyset \} \)

\[ \{ m \in M \mid (m \trianglelefteq E) \cap F \neq \emptyset \} = N_l(F) = F \trianglelefteq E \]

Figure 1: Schematic representation of the set-up of the proof of Theorem 15. Each region enclosed by one of the two rectangles is \( M \); the regions enclosed by the smaller circles with solid borders are subsets \( F \) and \( F' \) of \( M \) respectively; the regions enclosed by the circles with dashed borders are \( m \trianglelefteq E \) and \( m' \trianglelefteq E \) respectively; the region enclosed by the circle with dotted border is \( \bigcup_{e \in E} (\_\trianglelefteq e')^{-1}(m') \); the regions enclosed by the larger circles are \( N_r(F) \) and \( N_l(F') \) respectively.

**3 implies 4** (see Fig. 1). Let there be a finite \( E \subseteq G/G_0 \) such that, for each finite \( F \subseteq M \), we have \( |F \trianglelefteq E| \geq 2|F| \). Furthermore, let \( \mathcal{G} \) be the bipartite graph

\[ (M, M, \{(m, m') \in M \times M \mid \exists e \in E : m \trianglelefteq e = m'\}). \]

Moreover, let \( F \subseteq M \) be finite. The right neighbourhood of \( F \) in \( \mathcal{G} \) is

\[ N_r(F) = \{ m' \in M \mid \exists e \in E : F \trianglelefteq e \supseteq m' \} = F \trianglelefteq E \]

and the left neighbourhood of \( F \) in \( \mathcal{G} \) is

\[ N_l(F) = \{ m \in M \mid \exists e \in E : m \trianglelefteq e \in F \} = \bigcup_{e \in E} (\_\trianglelefteq e)^{-1}(F). \]

By precondition \( |N_r(F)| = |F \trianglelefteq E| \geq 2|F| \). Moreover, because \( G_0 \in E \), we have \( F = (\_\trianglelefteq G_0)^{-1}(F) \subseteq N_l(F) \) and hence \( |N_l(F)| \geq |F| \geq 2^{-1}|F| \). Therefore, according to the Hall harem theorem, there is a perfect \((1,2)\)-matching for \( \mathcal{G} \). In conclusion, there is a 2-to-1 surjective map \( \phi : M \to M \).
such that, for each \( m \in M \), we have \((\phi(m), m) \in E\), that is, there is an \( e \in E \) such that \( \phi(m) \trianglelefteq e = m \).

**II** implies **III** (see Fig. 2). Let there be a 2-to-1 surjective map \( \phi : M \rightarrow M \) and a finite subset \( E \) of \( G/G_0 \) such that

\[
\forall m \in M \exists e \in E : \phi(m) \trianglelefteq e = m.
\]

By the axiom of choice, there are two injective maps \( \psi \) and \( \psi' \) : \( M \rightarrow M \) such that, for each \( m \in M \), we have \( \phi^{-1}(m) = \{\psi(m), \psi'(m)\} \). For each \( e \in E \), let

\[
A_e = \{ m \in M \mid m \trianglelefteq e = \psi(m) \} \quad \text{and} \quad B_e = \{ m \in M \mid m \trianglelefteq e = \psi'(m) \}.
\]
Let \( m \in M \). There is an \( e \in E \) such that \( \phi(\psi(m)) \circ e = \psi(m) \). Because \( \phi(\psi(m)) = m \), we have \( m \in A_e \). And, because \( \circ \) is free, for each \( e' \in E \setminus \{e\} \), we have \( m \circ e' \neq m \circ e = \psi(m) \) and thus \( m \notin A_{e'} \). Therefore,

\[
M = \bigcup_{e \in E} A_e \quad \text{and analogously} \quad M = \bigcup_{e \in E} B_e.
\]

Moreover, \( \psi(A_e) = A_e \odot e \) and \( \psi'(B_e) = B_e \odot e \). Hence, because \( M = \psi(M) \cup \psi'(M) \), and \( \psi \) and \( \psi' \) are injective,

\[
M = \left( \bigcup_{e \in E} \psi(A_e) \right) \cup \left( \bigcup_{e \in E} \psi'(B_e) \right) = \left( \bigcup_{e \in E} A_e \odot e \right) \cup \left( \bigcup_{e \in E} B_e \odot e \right).
\]

Furthermore, because \( \psi \) and \( \psi' \) are injective, for each \( e \in E \), the maps \( (\_ \odot e)|_{A_e} = \psi|_{A_e} \) and \( (\_ \odot e)|_{B_e} = \psi'|_{B_e} \) are injective. In conclusion, \( (N, \{A_e\}_{e \in E}, \{B_e\}_{e \in E}) \) is a right paradoxical decomposition of \( \mathcal{R} \).

[5] implies [1]. Let there be a right paradoxical decomposition \( (N, \{A_e\}_{e \in E}, \{B_e\}_{e \in E}) \) of \( \mathcal{R} \). According to Lemma [15]

\[
1 = \nu(\mathbb{1}_M) = \sum_{e \in E} \nu(\mathbb{1}_{A_e}) = \sum_{e \in E} \nu(\mathbb{1}_{B_e}) = \sum_{e \in E} (\nu \odot e)(\mathbb{1}_{A_e}) + \sum_{e \in E} (\nu \odot e)(\mathbb{1}_{B_e}) = \sum_{e \in E} \nu(\mathbb{1}_{A_e}) + \sum_{e \in E} \nu(\mathbb{1}_{B_e}) = \nu(\mathbb{1}_M) + \nu(\mathbb{1}_M) = 1 + 1 = 2,
\]

which contradicts that \( 1 \neq 2 \). In conclusion, \( \mathcal{R} \) is not right amenable. \( \square \)

**Corollary 1** (Tarski alternative theorem; Alfred Tarski, 1938). Let \( M \) be a left homogeneous space with finite stabilisers. It is right amenable if and only if there is a coordinate system \( K \) for \( M \) such that there is no right paradoxical decomposition of \((M, K)\).

**Corollary 2** (Theorem of Fölner; Erling Fölner, 1955). Let \( M \) be a left homogeneous space with finite stabilisers. It is right amenable if and only if there is a coordinate system \( K \) for \( M \) such that there is a right Fölner net in \((M, K)\).

**Remark 7.** In the situation of Remark [1] Corollaries [1] and [2] constitute [1] Theorem 4.9.1].
6 From Left to Right Amenability

Lemma 19. Let \( \mathcal{R} = ((M,G,\triangleright),(m_0,\{g_{m_0,m}\}_{m \in M})) \) be a cell space and let \( H \) be a subgroup of \( G \) such that, for each element \( g \in G/G_0 \), there is an element \( h \in H \) such that the maps \( \_ \triangleright g \) and \( h \triangleright \_ \) are inverse to each other. If \( (M,H,\triangleright|_{H \times M}) \) is left amenable, then \( \mathcal{R} \) is right amenable.

Proof. Let \( \mu \in PM(M) \). Furthermore, let \( g \in G/G_0 \). There is an \( h \in H \) such that \( \_ \triangleright g \) and \( h \triangleright \_ \) are inverse to each other. Moreover, let \( A \subseteq M \). Because \( \_ \triangleright g = (h \triangleright \_)^{-1} = h^{-1} \triangleright \_ \), we have \( A \triangleright g = h^{-1} \triangleright A \). Therefore,

\[
(\mu \triangleright g)(A) = \mu(A \triangleright g) = \mu(h^{-1} \triangleright A) = (h^{-1} \triangleright \mu)(A).
\]

Thus, \( \mu \triangleright g = h^{-1} \triangleright \mu \). Hence, if \( \mu \) is \( \triangleright|_{H \times \{0,1\}}\)-invariant, then \( \mu \) is \( \triangleright \)-semi-invariant. In conclusion, if \( (M,H,\triangleright|_{H \times M}) \) is left amenable, then \( \mathcal{R} \) is right amenable. \( \square \)

Lemma 20. Let \( \mathcal{R} = ((M,G,\triangleright),(m_0,\{g_{m_0,m}\}_{m \in M})) \) be a cell space and let \( H \) be a subgroup of \( G \) such that \( G = G_0H \), for each element \( g \in G/G_0 \), the map \( \_ \triangleright g \) is injective,

\[
\forall h \in H : \_ \triangleright hG_0 = h \triangleright \_,
\]

and

\[
\forall h \in H \forall g \in G/G_0 : (_ \triangleright hG_0) \triangleright g = _ \triangleright h \cdot g.
\]

If \( (M,H,\triangleright|_{H \times M}) \) is left amenable, then \( \mathcal{R} \) is right amenable.

Proof. Let \( gG_0 \in G/G_0 \). Because \( g^{-1} \in G = G_0H \), there is a \( g_0 \in G_0 \) and there is an \( h \in H \) such that \( g^{-1} = g_0h \). Thus, \( h = g_0^{-1}g^{-1} \in H \). Hence, for each \( m \in M \),

\[
((\_ \triangleright gG_0) \circ (h \triangleright \_))(m) = (h \triangleright m) \triangleright gG_0 = (m \triangleright hG_0) \triangleright gG_0 = m \triangleright hG_0 = m \triangleright g_0^{-1}g^{-1}gG_0 = m \triangleright gG_0 = m.
\]

Therefore, \( h \triangleright \_ \) is right inverse to \( _ \triangleright gG_0 \). Hence, \( _ \triangleright gG_0 \) is surjective and thus, because it is injective by precondition, bijective. Therefore, \( _ \triangleright gG_0 \) and \( h \triangleright \_ \) are inverse to each other. In conclusion, according to Lemma 19 if \( (M,H,\triangleright|_{H \times M}) \) is left amenable, then \( \mathcal{R} \) is right amenable. \( \square \)

Definition 17. Let \( G \) be a group. The set

\[
Z(G) = \{ z \in G \mid \forall g \in G : zg = gz \}
\]

is called centre of \( G \).
Lemma 21. Let $G$ be a group. The centre of $G$ is a subgroup of $G$. □

Lemma 22. Let $\mathcal{R} = ((M, G, \triangleright),(m_0, \{g_{m_0,m}\}_{m \in M}))$ be a cell space and let $H$ be a subgroup of $G$ such that $G$ is equal to $G_0H$, $\triangleright|_{H \times M}$ is free, and $\{g_{m_0,m}\}_{m \in M}$ is included in $Z(H)$. If $(M, H, \triangleright|_{H \times M})$ is left amenable, then $\mathcal{R}$ is right amenable.

Proof. Let $g \in G$. For each $m \in M$,

$$m \trianglelefteq gG_0 = g_{m_0,m} \triangleright m_0 = g_{m_0,m} \triangleright (g \triangleright m_0).$$

Let $m \in M$. For each $m' \in M$, because $\triangleright|_{Z(H) \times M}$ is free and $g_{m_0,m}, g_{m_0,m'} \in Z(H)$,

$$m' \trianglelefteq gG_0 = m \trianglelefteq gG_0 \iff g_{m_0,m'} = g_{m_0,m} \iff m' = m.$$

Therefore, $\_ \trianglelefteq gG_0$ is injective.

Let $m \in M$ and let $h \in H$. Because $g_{m_0,m} \in Z(G)$,

$$m \trianglelefteq hG_0 = g_{m_0,m} h \triangleright m_0 = h g_{m_0,m} \triangleright m_0 = h \triangleright m.$$

Put $m' = m \trianglelefteq hG_0$. Then,

$$g_{m_0,m} h \triangleright m_0 = h g_{m_0,m} \triangleright m_0 = h \triangleright m = m'.$$

Hence, because $g_{m_0,m'} \triangleright m_0 = m'$ also and $\triangleright|_{H \times M}$ is free, $g_{m_0,m'} = g_{m_0,m} h$.

Therefore,

$$(m \trianglelefteq hG_0) \trianglelefteq gG_0 = m' \trianglelefteq gG_0 = g_{m_0,m} \trianglelefteq gG_0 = g_{m_0,m} g \triangleright m_0 = g_{m_0,m} h \triangleright m_0 = m \trianglelefteq h_gG_0.$$

In conclusion, according to Lemma 22, if $(M, H, \triangleright|_{H \times M})$ is left amenable, then $\mathcal{R}$ is right amenable. □

Example 1. Let $M = \mathbb{K}$ be a field, let

$$G = \{f: M \to M, x \mapsto ax + b | a, b \in M, a \neq 0\}$$

be the group of affine functions with composition as group multiplication, and let

$$H = \{f: M \to M, x \mapsto x + b | b \in M\}$$
be the group of translations also with composition as group multiplication. The

group $H$ is an abelian subgroup of $G$, which in turn is a non-abelian subgroup

del symmetry group of $M$. Moreover, according to [1, Example 4.6.2 and

Theorem 4.6.3], the group $G$ is left amenable and hence, according to [1, Pro-

duction 4.5.1], so is its subgroup $H$. Furthermore, the group $G$ acts transitively

on $M$ by function application by $\circ$ and so does $H$ by $\circ_{H \times M}$, even freely so. 

Because the groups $G$ and $H$ are left amenable, so are the left group sets $(M,G,\circ)$

and $(M,H,\circ_{H \times M})$. The stabiliser of $m_0 = 0$ is the group of dilations

$$G_0 = \{f : M \to M, x \mapsto a x \mid a \in M - \{0\}\}.$$ 

We have $G = G_0H$. For each $m \in M$, let

$$g_{m_0,m} : M \to M, \ x \mapsto x + m,$$

be the translation by $m$. Then, $\{g_{m_0,m}\}_{m \in M}$ is included in $Z(H) = H$. Hence,

according to Lemma 22 the cell space $R = ((M,G,\circ),(m_0,\{g_{m_0,m}\}_{m \in M}))$ is

right amenable.

Lemma 23. Let $H$ and $N$ be two groups, let $\phi : H \to \text{Aut}(N)$ be a group ho-

momorphism, let $G$ be the Cartesian product $H \times N$, and let

$$\cdot : G \times G \to G, \ \ ((h,n),(h',n')) \mapsto (hh',n\phi(h)(n')).$$

The tuple $(G,\cdot)$ is a group, called semi-direct product of $H$ and $N$ with respect
to $\phi$, and denoted by $H \ltimes_\phi N$. □

Lemma 24. Let $G$ be a semi-direct product of $H$ and $N$ with respect to $\phi$. The
neutral element of $G$ is $(e_H, e_N)$ and, for each element $(h,n) \in G$, the inverse
of $(h,n)$ is $(h^{-1}, \phi(h^{-1}))(n))$. □

Definition 18. Let $(M,G,\circ)$ be a left homogeneous space. It is called principal
if and only if the action $\circ$ is free.

Lemma 25. Let $(M,H,\circ_H)$ be a principal left homogeneous space. Furthermore, let $G_0$
be a group, let $\phi : G_0 \to \text{Aut}(H)$ be a group homomorphism, let $m_0$
an element of $M$, for each element $m \in M$, let $h_{m_0,m}$ be the unique
element of $H$ such that $h_{m_0,m} \circ m_0 = m$, and let

$$\circ_{G_0} : G_0 \times M \to M, \ \ (g_0,m) \mapsto \phi(g_0)(h_{m_0,m}) \circ_H m_0.$$ 

Moreover, let $G$ be the semi-direct product of $G_0$ and $H$ with respect to $\phi$, and let

$$\circ : G \times M \to M,$$
\[(g_0, m) \mapsto h \triangleright_H (g_0 \triangleright G_0 m).\]

The triple \((M, G_0, \triangleright G_0)\) is a left group set and the group \(G_0\) is the stabiliser of \(m_0\) under \(\triangleright G_0\). Furthermore, the tuple \(R = ((M, G, \triangleright), (m_0, \{(e_{G_0}, h_{m_0, m})\}_{m \in M}))\) is a cell space and the group \(G_0 \times \{e_H\}\) is the stabiliser of \(m_0\) under \(\triangleright\). Moreover, under the identification of \(G_0\) with \(G_0 \times \{e_H\}\) and of \(H\) with \(\{e_{G_0}\} \times H\), the left group sets \((M, G_0, \triangleright G_0)\) and \((M, H, \triangleright H)\) are left group subsets of \((M, G, \triangleright)\).

**Proof.** Because \(\phi(e_{G_0}) = \text{id}_{\text{Aut}(H)}\), for each \(m \in M\),

\[e_{G_0} \triangleright G_0 m = \phi(e_{G_0})(h_{m_0, m}) \triangleright_H m_0 = h_{m_0, m} \triangleright_H m_0 = m.\]

Let \(g_0, g'_0 \in G_0\), and let \(m \in M\). Because \(\triangleright_H\) is free and \(h_{m_0, \rho(g'_0)(h_{m_0, m}) \triangleright_H m_0} = \phi(g'_0)(h_{m_0, m}) \triangleright_H m_0\), we have \(h_{m_0, \rho(g_0)(h_{m_0, m}) \triangleright_H m_0} = \phi(g'_0)(h_{m_0, m})\). Therefore,

\[g_0 g'_0 \triangleright G_0 m = \phi(g_0 g'_0)(h_{m_0, m}) \triangleright_H m_0 = (\phi(g_0) \circ \phi(g'_0))(h_{m_0, m}) \triangleright_H m_0 = \phi(g_0)(\phi(g'_0)(h_{m_0, m})) \triangleright_H m_0 = \phi(g_0)(h_{m_0, \rho(g'_0)(h_{m_0, m}) \triangleright_H m_0}) \triangleright_H m_0 = g_0 \triangleright G_0 (\phi(g'_0)(h_{m_0, m}) \triangleright_H m_0) = g_0 \triangleright G_0 (g'_0 \triangleright G_0 m).\]

In conclusion, \((M, G_0, \triangleright G_0)\) is a left group set. Because \(h_{m_0, m_0} = e_H\), for each \(g_0 \in G_0\),

\[g_0 \triangleright G_0 m_0 = \phi(g_0)(e_H) \triangleright_H m_0 = e_H \triangleright_H m_0 = m_0.\]

In conclusion, \(G_0\) is the stabiliser of \(m_0\) under \(\triangleright G_0\).

For each \(m \in M\),

\[(e_{G_0}, e_H) \triangleright m = e_H \triangleright_H (e_{G_0} \triangleright G_0 m) = m.\]

Let \(g_0 \in G_0\), let \(h \in H\), and let \(m \in M\). Because \(h h_{m_0, m} \triangleright_H m_0 = h \triangleright_H m\), we have \(h h_{m_0, m} = h_{m_0, h \triangleright_H m}\). Hence,

\[\phi(g_0)(h) \triangleright_H (g_0 \triangleright G_0 m) = \phi(g_0)(h) \triangleright_H (\phi(g_0)(h_{m_0, m}) \triangleright_H m_0) = \phi(g_0)(h) \phi(g_0)(h_{m_0, m}) \triangleright_H m_0 = \phi(g_0)(h h_{m_0, m}) \triangleright_H m_0 = \phi(g_0)(h_{m_0, h \triangleright_H m}) \triangleright_H m_0 = g_0 \triangleright G_0 (h \triangleright_H m).\]
Therefore, for each \( g_0 \in G_0 \), each \( g'_0 \in G_0 \), each \( h \in H \), each \( h' \in H \), and each \( m \in M \),
\[
(g_0, h)(g'_0, h') \triangleright m = (g_0g'_0, h\phi(g_0)(h')) \triangleright m
\]
\[
= h\phi(g_0)(h') \triangleright_H (g_0g'_0 \triangleright_G_0 m)
\]
\[
= h \triangleright_H \left( \phi(g_0)(h') \triangleright_H \left( g_0 \triangleright_G_0 \left( g'_0 \triangleright_G_0 m \right) \right) \right)
\]
\[
= h \triangleright_H \left( g_0 \triangleright_G_0 \left( h' \triangleright_H \left( g'_0 \triangleright_G_0 m \right) \right) \right)
\]
\[
= (g_0, h) \triangleright \left( (g'_0, h') \triangleright m \right).
\]
In conclusion, \((M, G, \triangleright)\) is a left group action.

Because \( \triangleright_H \) is transitive and, for each \( h \in H \) and each \( m \in M \), we have \((e_{G_0}, h) \triangleright m = h \triangleright m\), the left group action \( \triangleright \) is transitive and hence \( M = (M, G, \triangleright) \) is a left homogeneous space. Moreover, because, for each \( m \in M \),
\[
(e_{G_0}, h_{m_0}, m) \triangleright m_0 = h_{m_0, m} \triangleright_H (e_{G_0} \triangleright_G_0 m_0)
\]
\[
= h_{m_0, m} \triangleright_H m_0
\]
\[
= m,
\]
the tuple \( \mathcal{K} = \langle m_0, \{ (e_{G_0}, h_{m_0, m}) \}_{m \in M} \rangle \) is a coordinate system for \( M \). Therefore, \( \mathcal{R} = \langle M, \mathcal{K} \rangle \) is a cell space.

Because \( G_0 \) is the stabiliser of \( m_0 \) under \( \triangleright_{G_0} \), for each \((g_0, h) \in G\), we have \((g_0, h) \triangleright m_0 = h \triangleright (g_0 \triangleright m_0) = h \triangleright m_0\). Because \( \triangleright_H \) is free, \( G_0 \times \{ e_H \} \) is the stabiliser of \( m_0 \) under \( \triangleright \).

Under the identification of \( G_0 \) with \( G_0 \times \{ e_H \} \) and of \( H \) with \( \{ e_{G_0} \} \times H \), we have \( \triangleright_{| G_0 \times M } = \triangleright_{G_0} \) and \( \triangleright_H_{| H \times M } = \triangleright_H \).

\textbf{Corollary 3.} In the situation of Lemma 22 let \( H \) be abelian. The cell space \( \mathcal{R} \) is right amenable.

\textbf{Proof.} According to [1, Theorem 4.6.1], because \( H \) is abelian, it is left amenable. Therefore, \( (M, H, \triangleright_H) \) is left amenable. Identify \( G_0 \) with \( G_0 \times \{ e_H \} \) and identify \( H \) with \( \{ e_{G_0} \} \times H \). Then, \( H \) is a subgroup of \( G \), and \( G = G_0 H \), and \( \triangleright_H_{| H \times M } = \triangleright_H \) is free, and, for each \( m \in M \), we have \((e_{G_0}, h_{m_0, m}) \in H = Z(H)\). Hence, according to Lemma 22 the cell space \( \mathcal{R} \) is right amenable. \( \square \)

\textbf{Example 2.} Let \( d \) be a positive integer; let \( E \) be the \( d \)-dimensional Euclidean space, that is, the symmetry group of the \( d \)-dimensional Euclidean space, in other words, the isometries of \( \mathbb{R}^d \) with respect to the Euclidean metric with function composition; let \( T \) be the \( d \)-dimensional translation group; and let \( O \) be the \( d \)-dimensional orthogonal group. The group \( T \) is abelian, a normal subgroup of \( E \), and isomorphic to \( \mathbb{R}^d \) with addition; the group \( O \) is isomorphic to the quotient \( E/T \) and to the \( (d \times d) \)-dimensional orthogonal matrices with matrix multiplication; the group \( E \) is isomorphic to the semi-direct product \( O \ltimes T \).
where \( \iota: O \to \text{Aut}(\mathbb{R}^d) \) is the inclusion map. The groups \( T, O, \) and \( E \) act on \( \mathbb{R}^d \) on the left by function application, denoted by \( \triangleright_T, \triangleright_O, \) and \( \triangleright, \) respectively; under the identification of \( T \) with \( \mathbb{R}^d \) by \( t \mapsto [v \mapsto v + t], \) of \( O \) with the orthogonal matrices of \( \mathbb{R}^{d \times d} \) by \( A \mapsto [v \mapsto Av], \) and of \( E \) with \( O \rtimes T \) by \( (A, t) \mapsto [v \mapsto Av + t], \) we have

\[
\triangleright_T: T \times \mathbb{R}^d \to \mathbb{R}^d,
(t, v) \mapsto v + t,
\]

and

\[
\triangleright_O: O \times \mathbb{R}^d \to \mathbb{R}^d,
(A, v) \mapsto Av,
\]

and

\[
\triangleright: E \times \mathbb{R}^d \to \mathbb{R}^d,
((A, t), v) \mapsto Av + t,
\]

and

\[
\iota: O \to \text{Aut}(\mathbb{R}^d),
A \mapsto [v \mapsto Av].
\]

Hence, for each vector \( v \in \mathbb{R}^d, \) we have \( v \triangleright_T 0 = v, \) therefore, \( \triangleright_O = [(A, v) \mapsto \iota(A)(v) \triangleright_T 0], \) and thus \( \triangleright = [((A, t), v) \mapsto t \triangleright_T (A \triangleright_O v)]. \) Moreover, because the group \((T, \circ) \cong (\mathbb{R}^d, +)\) is abelian, according to [II Theorem 4.6.1], it is left amenable and so is \((\mathbb{R}^d, \mathbb{R}^d, +) \cong (\mathbb{R}^d, T, \triangleright). \) In conclusion, according to Corollary 3, the cell space \(((\mathbb{R}^d, E, \triangleright), (0, \{-v \mid v \in \mathbb{R}^d\}))\) is right amenable.

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6. Simon Wacker. The Garden of Eden Theorem for Cellular Automata on Group Sets. [arXiv:1603.07272] [math.GR].
In Appendix A we present the basic theory of topologies. In Appendix B we present the basic theory of nets. In Appendix C we introduce initial and product topologies. In Appendix D we introduce the notion of compactness for topological spaces. In Appendix E we introduce topological dual spaces of topological spaces. In Appendix F we state Hall’s marriage and harem theorems. And in Appendix G we state Zorn’s lemma.

A Topologies

The theory of topologies and nets as presented here may be found in more detail in Appendix A in the monograph ‘Cellular Automata and Groups’ [1].

Definition 19. Let $X$ be a set and let $T$ be a set of subsets of $X$. The set $T$ is called topology on $X$ if, and only if

1. $\{\emptyset, X\}$ is a subset of $T$,
2. for each family $\{O_1\}_{i \in I}$ of elements in $T$, the union $\bigcup_{i \in I} O_i$ is an element of $T$,
3. for each finite family $\{O_i\}_{i \in I}$ of elements in $T$, the intersection $\bigcap_{i \in I} O_i$ is an element of $T$.

Definition 20. Let $X$ be a set, and let $T$ and $T'$ be two topologies on $X$. The topology $T$ is called

1. coarser than $T'$ if, and only if $T \subseteq T'$;
2. finer than $T'$ if, and only if $T \supseteq T'$.

Definition 21. Let $X$ be a set and let $T$ be a topology on $X$. The tuple $(X, T)$ is called topological space if, and only if

1. each subset $O$ of $X$ with $O \in T$ is called open in $X$,
2. each subset $A$ of $X$ with $X \setminus A \in T$ is called closed in $X$,
3. each subset $U$ of $X$ that is both open and closed is called clopen in $X$.

The set $X$ is said to be equipped with $T$ if, and only if it shall be implicitly clear that $T$ is the topology on $X$ being considered. The set $X$ is called topological space if, and only if it is implicitly clear what topology on $X$ is being considered.

Example 3. Let $X$ be a set. The set $\mathcal{P}(X)$ is the finest topology on $X$. Itself as well as the topological space $(X, \mathcal{P}(X))$ are called discrete.

Definition 22. Let $(X, T)$ be a topological space, let $x$ be a point of $X$, and let $N$ be a subset of $X$. The set $N$ is called neighbourhood of $x$ if, and only if there is an open subset $O$ of $X$ such that $x \in O$ and $O \subseteq N$.

Definition 23. Let $(X, T)$ be a topological space and let $x$ be a point of $X$. The set of all open neighbourhoods of $x$ is denoted by $T_x$. 

A $\emptyset$
B Nets

**Definition 24.** Let $I$ be a set and let $\leq$ be a binary relation on $I$. The relation $\leq$ is called preorder on $I$ and the tuple $(I, \leq)$ is called preordered set if, and only if the relation $\leq$ is reflexive and transitive.

**Definition 25.** Let $\leq$ be a preorder on $I$. It is called directed and the preordered set $(I, \leq)$ is called directed set if, and only if

$$\forall i \in I \forall i' \in I \exists i'' \in I : i \leq i'' \land i' \leq i''.$$  

**Definition 26.** Let $\leq$ be a preorder on $I$, let $J$ be a subset of $I$, and let $i$ be an element of $I$. The element $i$ is called upper bound of $J$ in $(I, \leq)$ if, and only if

$$\forall i' \in J : i' \leq i.$$  

**Definition 27.** Let $M$ be a set, let $I$ be a set, and let $f : I \rightarrow M$ be a map. The map $f$ is called family of elements in $M$ indexed by $I$ and denoted by $\{m_i\}_{i \in I}$, where, for each index $i \in I$, $m_i = f(i)$.

**Definition 28.** Let $I$ be a set, let $\leq$ be a binary relation on $I$, and let $\{m_i\}_{i \in I}$ be a family of elements in $M$ indexed by $I$. The family $\{m_i\}_{i \in I}$ is called net in $M$ indexed by $(I, \leq)$ if, and only if the tuple $(I, \leq)$ is a directed set.

**Definition 29.** Let $\{m_i\}_{i \in I}$ and $\{m'_j\}_{j \in J}$ be two nets in $M$. The net $\{m'_j\}_{j \in J}$ is called subnet of $\{m_i\}_{i \in I}$ if, and only if there is a map $f : J \rightarrow I$ such that $\{m'_j\}_{j \in J} = \{m_{f(j)}\}_{j \in J}$ and

$$\forall i \in I \exists j \in J : \forall j' \in J : (j' \geq j \implies f(j') \geq i).$$

**Definition 30.** Let $(X, T)$ be a topological space, let $\{x_i\}_{i \in I}$ be a net in $X$ indexed by $(I, \leq)$, and let $x$ be a point of $X$. The net $\{x_i\}_{i \in I}$ is said to converge to $x$ and $x$ is called limit point of $\{x_i\}_{i \in I}$ if, and only if

$$\forall O \in T_x \exists i_0 \in I : \forall i \in I : (i \geq i_0 \implies x_i \in O).$$

**Definition 31.** Let $(X, T)$ be a topological space and let $\{x_i\}_{i \in I}$ be a net in $X$ indexed by $(I, \leq)$. The net $\{x_i\}_{i \in I}$ is called convergent if, and only if there is a point $x \in X$ such that it converges to $x$.

**Remark 8.** Let $\{m_i\}_{i \in I}$ be a net that converges to $x$. Each subnet $\{m'_j\}_{j \in J}$ of $\{m_i\}_{i \in I}$ converges to $x$.

**Lemma 26.** Let $(X, T)$ be a topological space, let $Y$ be a subset of $X$, and let $x$ be an element of $X$. Then, $x \in \overline{Y}$ if, and only if there is a net $\{y_i\}_{i \in I}$ in $Y$ that converges to $x$.

**Proof.** See Proposition A.2.1 in ‘Cellular Automata and Groups’ [1].

\[\square\]
Lemma 27. Let \((X, \mathcal{T})\) be a topological space. It is Hausdorff if, and only if each convergent net in \(X\) has exactly one limit point.

Proof. See Proposition A.2.2 in ‘Cellular Automata and Groups’ \([1]\). \(\square\)

Definition 32. Let \((X, \mathcal{T})\) be a Hausdorff topological space, let \(\{x_i\}_{i \in I}\) be a convergent net in \(X\) indexed by \((I, \leq)\), and let \(x\) be the limit point of \(\{x_i\}_{i \in I}\). The point \(x\) is denoted by \(\lim_{i \in I} x_i\), and we write \(x_i \to x\).

Definition 33. Let \((X, \mathcal{T})\) be a topological space, let \(\{x_i\}_{i \in I}\) be a net in \(X\) indexed by \((I, \leq)\), and let \(x\) be an element of \(X\). The point \(x\) is called cluster point of \(\{x_i\}_{i \in I}\) if, and only if

\[ \forall O \in \mathcal{T}_x \forall i \in I \exists i' \in I : (i' \geq i \land x_{i'} \in O). \]

Lemma 28. Let \((X, \mathcal{T})\) be a topological space, let \(\{x_i\}_{i \in I}\) be a net in \(X\) indexed by \((I, \leq)\), and let \(x\) be an element of \(X\). The point \(x\) is a cluster point of \(\{x_i\}_{i \in I}\) if, and only if there is a subnet of \(\{x_i\}_{i \in I}\) that converges to \(x\).

Proof. See Proposition A.2.3 in ‘Cellular Automata and Groups’ \([1]\). \(\square\)

Lemma 29. Let \((X, \mathcal{T})\) and \((X', \mathcal{T}')\) be two topological spaces, let \(f\) be a continuous map from \(X\) to \(X'\), let \(\{x_i\}_{i \in I}\) be a net in \(X\), and let \(x\) be an element of \(X\).

1. If \(x\) is a limit point of \(\{x_i\}_{i \in I}\), then \(f(x)\) is a limit point of \(\{f(x_i)\}_{i \in I}\).
2. If \(x\) is a cluster point of \(\{x_i\}_{i \in I}\), then \(f(x)\) is a cluster point of \(\{f(x_i)\}_{i \in I}\).

Proof. Confer the last paragraph of Sect. A.2 in ‘Cellular Automata and Groups’ \([1]\). \(\square\)

Definition 34. Let \(\mathbb{R} = \mathbb{R} \cup \{\pm \infty\}\) be the affinely extended real numbers and let \(\{r_i\}_{i \in I}\) be a net in \(\mathbb{R}\) indexed by \((I, \leq)\).

1. The limit of the net \(\{\inf_{i' \geq i} r_{i'}\}_{i \in I}\) is called limit inferior of \(\{r_i\}_{i \in I}\) and denoted by \(\liminf_{i \in I} r_i\).
2. The limit of the net \(\{\sup_{i' \geq i} r_{i'}\}_{i \in I}\) is called limit superior of \(\{r_i\}_{i \in I}\) and denoted by \(\limsup_{i \in I} r_i\).

C Initial and Product Topologies

Definition 35. Let \(X\) be a set, let \(I\) be a set, and, for each index \(i \in I\), let \((Y_i, \mathcal{T}_i)\) be a topological space and let \(f_i\) be a map from \(X\) to \(Y_i\). The coarsest topology on \(X\) such that, for each index \(i \in I\), the map \(f_i\) is continuous, is called initial with respect to \(\{f_i\}_{i \in I}\).

Lemma 30. Let \((X, \mathcal{T})\) be a topological space, where \(\mathcal{T}\) is the initial topology with respect to \(\{f_i : X \to Y_i\}_{i \in I}\), let \((Z, \mathcal{S})\) be a topological space, and let \(g\) be a map from \(Z\) to \(X\). The map \(g\) is continuous if, and only if, for each index \(i \in I\), the map \(f_i \circ g\) is continuous. \(\square\)
Lemma 31. Let \((X, \mathcal{T})\) be a topological space, where \(\mathcal{T}\) is the initial topology with respect to \(\{f_i : X \to Y_i\}_{i \in I}\), let \(\{x^i\}_{i \in I'}\) be a net in \(X\), and let \(x\) be a point in \(X\). The point \(x\) is a limit point or cluster point of \(\{x^i\}_{i \in I'}\) if, and only if, for each non-empty and connected subset \(\{y_j\}_{j \in J}\) of \(X\), there is a finite subset \(J'\subseteq J\) such that \(\bigcap_{j \in J'} (y_j - x) \neq \emptyset\).

Proof. Confer the last paragraph of Sect. A.3 in ‘Cellular Automata and Groups’ [1].

Definition 36. Let \((X, \mathcal{T}_i)\) be a family of topological spaces, let \(X\) be the set \(\prod_{i \in I} X_i\), and, for each index \(i \in I\), let \(\pi_i\) be the projection of \(X\) onto \(X_i\). The initial topology on \(X\) with respect to \(\\{\pi_i\}_{i \in I}\) is called product.

Remark 9. The product topology on \(X\) has for a base the sets \(\prod_{i \in I} O_i\), where, for each index \(i \in I\), the set \(O_i\) is an open subset of \(X_i\), and the set \(\{i \in I \mid O_i \neq X_i\}\) is finite.

Definition 37. Let \((X, \mathcal{T}_i)\) be a family of discrete topological spaces and let \(X\) be the set \(\prod_{i \in I} X_i\). The product topology on \(X\) is called prodiscrete.

Lemma 32. Let \((X, \mathcal{T}_i)\) be a family of Hausdorff topological spaces. The set \(\prod_{i \in I} X_i\), equipped with the product topology, is Hausdorff.

Proof. See Proposition A.4.1 in ‘Cellular Automata and Groups’ [1].

Definition 38. Let \(X\) be a topological space. It is called totally disconnected if, and only if, for each non-empty and connected subset \(A\) of \(X\), we have \(|A| = 1\).

Lemma 33. Let \((X, \mathcal{T}_i)\) be a family of totally disconnected topological spaces. The set \(\prod_{i \in I} X_i\), equipped with the product topology, is totally disconnected.

Proof. See Proposition A.4.2 in ‘Cellular Automata and Groups’ [1].

Lemma 34. Let \((X, \mathcal{T}_i)\) be a family of topological spaces and, for each index \(i \in I\), let \(A_i\) be a closed subset of \(X_i\). The set \(\prod_{i \in I} A_i\) is a closed subset of \(\prod_{i \in I} X_i\), equipped with the product topology.

Proof. See Proposition A.4.3 in ‘Cellular Automata and Groups’ [1].

D Compactness

Definition 39. Let \((X, \mathcal{T})\) be a topological space and let \(\{O_i\}_{i \in I}\) be a family of elements of \(\mathcal{T}\). The family \(\{O_i\}_{i \in I}\) is called open cover of \(X\) if, and only if, \(\bigcup_{i \in I} O_i = X\).

Definition 40. Let \((X, \mathcal{T})\) be a topological space. It is called compact if, and only if, for each open cover \(\{O_i\}_{i \in I}\) of \(X\), there is a finite subset \(J\) of \(I\) such that \(\{O_j\}_{j \in J}\) is an open cover of \(X\).
Lemma 35. Let $(X, \mathcal{T})$ be a topological space. It is compact if, and only if, for each family $\{A_i\}_{i \in I}$ of closed subsets of $X$ such that, for each finite subset $J$ of $I$, $\bigcap_{j \in J} A_j \neq \emptyset$, we have $\bigcup_{i \in I} A_i \neq \emptyset$.

Proof. Confer first paragraph of Sect. A.5 in ‘Cellular Automata and Groups’[1].

Theorem 5. Let $(X, \mathcal{T})$ be a topological space. The following statements are equivalent:

1. The space $(X, \mathcal{T})$ is compact;
2. Each net in $X$ has a cluster point with respect to $\mathcal{T}$;
3. Each net in $X$ has a convergent subnet with respect to $\mathcal{T}$.

Proof. See Theorem A.5.1 in ‘Cellular Automata and Groups’[1].

Theorem 6 (Andrey Nikolayevich Tikhonov, 1935). Let $\{(X_i, \mathcal{T}_i)\}_{i \in I}$ be a family of compact topological spaces. The set $\prod_{i \in I} X_i$, equipped with the product topology, is compact.

Proof. See Theorem A.5.2 in ‘Cellular Automata and Groups’[1].

Corollary 4. Let $\{(X_i, \mathcal{T}_i)\}_{i \in I}$ be a family of finite topological spaces. The set $\prod_{i \in I} X_i$, equipped with the product topology, is compact.

Proof. See paragraph before Corollary A.5.3 in ‘Cellular Automata and Groups’[1].

E Dual Spaces

The theory of dual spaces as presented here may be found in more detail in Appendix F in ‘Cellular Automata and Groups’[1].

In this section, let $(X, \| \|)$ be a normed $\mathbb{R}$-vector space.

Definition 41. The vector space

$$X^* = \{ \psi : X \to \mathbb{R} \text{ linear} \mid \psi \text{ is continuous} \}$$

is called topological dual space of $X$.

Definition 42. 1. The norm

$$\| \|_{X^*} : X^* \to \mathbb{R},$$

$$\psi \mapsto \sup_{x \in X \setminus \{0\}} \frac{|\psi(x)|}{\|x\|},$$

is called operator norm on $X^*$.

2. The topology on $X^*$ induced by $\| \|_{X^*}$ is called strong topology on $X^*$. 
Definition 43. Let $x$ be an element of $X$. The map
\[ ev_x : X^* \to \mathbb{R}, \quad \psi \mapsto \psi(x), \]
is called evaluation map at $x$.

Definition 44. The initial topology on $X^*$ with respect to \( \{ ev_x \}_{x \in X} \) is called weak-$^*$ topology on $X^*$.

Lemma 36. Let $\{ \psi_i \}_{i \in I}$ be a net in $X^*$, let $\psi$ be an element of $X^*$, and let $X^*$ be equipped with the weak-$^*$ topology. The net $\{ \psi_i \}_{i \in I}$ converges to $\psi$ if, and only if, for each element $x$ of $X$, the net $\{ \psi_i(x) \}_{i \in I}$ converges to $\psi(x)$.

Lemma 37. The weak-$^*$ topology on $X^*$ is coarser than the strong topology on $X^*$.

Corollary 5. Let $\{ \psi_i \}_{i \in I}$ be a net in $X^*$ that converges to $\psi$ with respect to the strong topology on $X^*$. The net $\{ \psi_i \}_{i \in I}$ converges to $\psi$ with respect to the weak-$^*$ topology on $X^*$.

Definition 45. Let $Y$ be a subset of $X$. The set $Y$ is called convex if, and only if,
\[ \forall (y, y') \in Y \times Y \forall t \in [0, 1] : ty + (1 - t)y' \in Y. \]

Definition 46. The topological vector space $X$ is called locally convex if, and only if, the origin has a neighbourhood base of convex sets.

Lemma 38. Let $X^*$ be equipped with the weak-$^*$ topology and let $\psi$ be an element of $X^*$. A neighbourhood base of $\psi$ is given by the sets
\[ B(\psi, F, \varepsilon) = \{ \psi' \in X^* \mid \forall x \in F : |\psi(x) - \psi'(x)| < \varepsilon \}, \]
for $F \subseteq X$ finite and $\varepsilon \in \mathbb{R}_{>0}$.

Corollary 6. Let $X^*$ be equipped with the weak-$^*$ topology. The space $X^*$ is locally convex.

Lemma 39. The space $X^*$, equipped with the weak-$^*$ topology, is Hausdorff.

Proof. Confer last paragraph of Sect. F.2 in ‘Cellular Automata and Groups’[1].

Theorem 7 (Stefan Banach, 1932; Leonidas Alaoglu, 1940). Let $X^*$ be equipped with the weak-$^*$ topology. The unit ball $\{ \psi \in X^* \mid \| \psi \|_{X^*} \leq 1 \}$, equipped with the subspace topology, is compact.

Proof. See Theorem F.3.1 in ‘Cellular Automata and Groups’[1].
F  Hall’s Theorems

The theory concerning Hall’s theorems as presented here may be found in more detail in Appendix H in the monograph ‘Cellular Automata and Groups’[1].

**Definition 47.** Let $X$ and $Y$ be two sets, and let $E$ be a subset of $X \times Y$. The triple $(X,Y,E)$ is called bipartite graph, each element $x$ of $X$ is called left vertex, each element $y$ of $Y$ is called right vertex, and each element $e$ of $E$ is called edge.

**Definition 48.** Let $(X,Y,E)$ and $(X',Y',E')$ be two bipartite graphs. The graph $(X,Y,E)$ is called bipartite subgraph of $(X',Y',E')$ if, and only if, $X \subseteq X'$, $Y \subseteq Y'$, and $E \subseteq E'$.

**Definition 49.** Let $(X,Y,E)$ be a bipartite graph, and let $(x,y)$ and $(x',y')$ be two elements of $E$. The edges $(x,y)$ and $(x',y')$ are called adjacent if, and only if, $x = x'$ or $y = y'$.

**Definition 50.** Let $(X,Y,E)$ be a bipartite graph.

1. Let $x$ be an element of $X$. The set
   \[ N_r(x) = \{ y \in Y \mid (x,y) \in E \} \]
   is called right neighbourhood of $x$.
2. Let $A$ be a subset of $X$. The set $\bigcup_{a \in A} N_r(a)$ is called right neighbourhood of $A$ and is denoted by $N_r(A)$.
3. Let $y$ be an element of $Y$. The set
   \[ N_l(y) = \{ x \in X \mid (x,y) \in E \} \]
   is called left neighbourhood of $y$.
4. Let $B$ be a subset of $Y$. The set $\bigcup_{b \in B} N_l(b)$ is called left neighbourhood of $B$ and is denoted by $N_l(B)$.

**Definition 51.** Let $(X,Y,E)$ be a bipartite graph. It is called

1. finite if, and only if, the sets $X$ and $Y$ are finite.
2. locally finite if, and only if, for each element $x$ of $X$, the set $N_r(x)$ is finite, and for each element $y$ of $Y$, the set $N_l(y)$ is finite.

**Remark 10.** Let $(X,Y,E)$ be a locally finite bipartite graph. Then, for each finite subset $A$ of $X$, the set $N_r(A)$ is finite; and, for each finite subset $B$ of $Y$, the set $N_l(B)$ is finite.

**Definition 52.** Let $(X,Y,E)$ be a bipartite graph and let $M$ be a subset of $E$. The set $M$ is called matching if, and only if, for each element $(e,e')$ of $M \times M$ with $e \neq e'$, the edges $e$ and $e'$ are non-adjacent.
Definition 53. Let \((X, Y, E)\) be a bipartite graph and let \(M\) be a matching. The matching \(M\) is called
1. left-perfect if, and only if,
\[ \forall x \in X \exists y \in Y : (x, y) \in M; \]
2. right-perfect if, and only if,
\[ \forall y \in Y \exists x \in X : (x, y) \in M; \]
3. perfect if, and only if, it is left-perfect and right-perfect.

Definition 54. Let \((X, Y, E)\) be a locally finite bipartite graph. It is said to satisfy the
1. left Hall condition if, and only if,
\[ \forall A \subseteq X \text{ finite} : |N_r(A)| \geq |A|; \]
2. right Hall condition if, and only if,
\[ \forall B \subseteq Y \text{ finite} : |N_l(B)| \geq |B|; \]
3. Hall marriage conditions if, and only if, it satisfies the left and right Hall conditions.

Theorem 8. Let \((X, Y, E)\) be a locally finite bipartite graph. It satisfies the left or right Hall condition if, and only if, there is a left- or right-perfect matching, respectively.

Proof. See Theorem H.3.2 in ‘Cellular Automata and Groups’[1]. □

Theorem 9. Let \((X, Y, E)\) be a bipartite graph such that there is a left-perfect matching and there is a right-perfect matching. There is a perfect matching.

Proof. See Theorem H.3.4 in ‘Cellular Automata and Groups’[1]. □

Corollary 7 (Georg Ferdinand Ludwig Philipp Cantor, Friedrich Wilhelm Karl Ernst Schröder, Felix Bernstein). Let \(X\) and \(Y\) be two sets such, that there is an injective map \(f\) from \(X\) to \(Y\) and there is an injective map \(g\) from \(Y\) to \(X\). There is a bijective map from \(X\) to \(Y\).

Proof. See Theorem H.3.5 in ‘Cellular Automata and Groups’[1]. □

Theorem 10 (Hall’s marriage theorem, Philip Hall, 1935). Let \((X, Y, E)\) be a locally finite bipartite graph. It satisfies the Hall marriage conditions if, and only if, there is a perfect matching.

Proof. See Theorem H.3.6 in ‘Cellular Automata and Groups’[1]. □
Definition 55. Let $X$ and $Y$ be two sets, and let $f$ be a surjective map from $X$ to $Y$. The map $f$ is called $k$-to-1 if, and only if,
\[ \forall y \in Y : |f^{-1}(y)| = k. \]

Definition 56. Let $(X,Y,E)$ be a bipartite graph, let $k$ be a positive integer, and let $M$ be a subset of $E$. The set $M$ is called perfect $(1,k)$-matching if, and only if,
\[ \forall x \in X : |\{y \in Y | (x,y) \in E\}| = k \]
and
\[ \forall y \in Y : |\{x \in X | (x,y) \in E\}| = 1. \]

Remark 11. The set $M$ is a perfect $(1,k)$-matching if, and only if, there is a $k$-to-1 surjective map $\psi : Y \to X$ such that $\{(\psi(y),y) \mid y \in Y\} = M$. 
Remark 12. The set $M$ is a perfect $(1,1)$-matching if, and only if, it is a perfect matching.

Definition 57. Let $(X,Y,E)$ be a locally finite bipartite graph and let $k$ be a positive integer. The graph $(X,Y,E)$ is said to satisfy the Hall $k$-harem conditions if, and only if, for each finite subset $A$ of $X$, we have $|N_r(A)| \geq k|A|$, and for each finite subset $B$ of $Y$, we have $|N_l(B)| \geq k^{-1}|B|$.

Theorem 11 (Hall’s harem theorem, Philip Hall). Let $(X,Y,E)$ be a locally finite bipartite graph and let $k$ be a positive integer. The graph $(X,Y,E)$ satisfies the Hall $k$-harem conditions if, and only if, there is a perfect $(1,k)$-matching.

Proof. See Theorem H.4.2 in ‘Cellular Automata and Groups’[1]. \qed

G  Zorn’s Lemma

Definition 58. Let $\leq$ be a preorder on $I$. It is called partial order on $I$ and the preordered set $(I,\leq)$ is called partially ordered set if, and only if the relation $\leq$ is antisymmetric.

Definition 59. Let $\leq$ be a partial order on $I$. It is called total order on $I$ and the partially ordered set $(I,\leq)$ is called totally ordered set if, and only if the relation $\leq$ is total.

Definition 60. Let $\leq$ be a preorder on $I$ and let $i$ be an element of $I$. The element $i$ is called maximal in $(I,\leq)$ if, and only if
\[ \forall i' \in I : (i' \geq i \implies i' \leq i). \]

Definition 61. Let $\leq$ be a preorder on $I$ and let $J$ be a subset of $I$. The set $J$ is called chain in $(I,\leq)$ if, and only if the restriction of $\leq$ to $J$ is a total order on $J$.

Lemma 40 (Zorn’s Lemma). Let $(I,\leq)$ be a preordered set such that each chain in $I$ has an upper bound. Then, $I$ has a maximal element. \qed