ON THE NONPARAMETRIC MAXIMUM LIKELIHOOD ESTIMATOR FOR GAUSSIAN LOCATION MIXTURE DENSITIES WITH APPLICATION TO GAUSSIAN DENOISING

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We study the Nonparametric Maximum Likelihood Estimator (NPMLE) for estimating Gaussian location mixture densities in $d$-dimensions from independent observations. Unlike usual likelihood-based methods for fitting mixtures, NPMLEs are based on convex optimization. We prove finite sample results on the Hellinger accuracy of every NPMLE. Our results imply, in particular, that every NPMLE achieves near parametric risk (up to logarithmic multiplicative factors) when the true density is a discrete Gaussian mixture without any prior information on the number of mixture components. NPMLEs can naturally be used to yield empirical Bayes estimates of the Oracle Bayes estimator in the Gaussian denoising problem. We prove bounds for the accuracy of the empirical Bayes estimate as an approximation to the Oracle Bayes estimator. Here our results imply that the empirical Bayes estimator performs at nearly the optimal level (up to logarithmic multiplicative factors) for denoising in clustering situations without any prior knowledge of the number of clusters.

1. Introduction. In this paper, we study the performance of the Nonparametric Maximum Likelihood Estimator (NPMLE) for estimating a Gaussian location mixture density in multiple dimensions. We also study the performance of the empirical Bayes estimator based on the NPMLE for estimating the Oracle Bayes estimator in the problem of Gaussian denoising.

By a Gaussian location mixture density in $\mathbb{R}^d$, $d \geq 1$, we refer to a density of the form

$$f_G(x) := \int \phi_d(x - \theta)dG(\theta)$$

for some probability measure $G$ on $\mathbb{R}^d$ where $\phi_d(z) := (2\pi)^{-d/2} \exp\left(-\|z\|^2/2\right)$ is the standard $d$-dimensional normal density ($\|z\|$ is the usual Euclidean norm of $z$). Note that $f_G$ is the density of the random vector $X = \theta + Z$ where $\theta$ and $Z$ are independent $d$-dimensional random vectors with $\theta$ having distribution $G$ (i.e., $\theta \sim G$) and $Z$ having the Gaussian distribution with zero mean and identity covariance matrix (i.e., $Z \sim N(0,I_d)$). We let $\mathcal{M}$ to be the class of all Gaussian location mixture densities i.e., densities of the form $f_G$ as $G$ varies over all probability measures on $\mathbb{R}^d$.

Given $n$ independent $d$-dimensional data vectors $X_1, \ldots, X_n$ (throughout the paper, we assume that $n \geq 2$) generated from an unknown Gaussian location mixture density $f^* \in \mathcal{M}$, we study the problem of estimating $f^*$ from $X_1, \ldots, X_n$. This problem is fundamental to the area of estimation in mixture models to which a number of books (see, for example, Everitt and Hand [23], Titterington et al. [59], Lindsay [37], Böning [8], McLachlan and Peel [45], Schlattmann [53]) and papers have been devoted. We focus on the situation where $d$ is small or moderate, $n$ is large and where no

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specific prior information is available about the mixing measure corresponding to \( f^* \). Consistent estimation in the case where \( d \) is comparable in size to \( n \) needs simplifying assumptions on \( f^* \) (such as that the mixing measure is discrete with a small number of atoms and that it is concentrated on a set of sparse vectors in \( \mathbb{R}^d \)) which we do not make in this paper. Let us also note here that we focus on the problem of estimating \( f^* \) and not on estimating the mixing measure corresponding to \( f^* \).

There are two well-known likelihood-based approaches to the estimation of Gaussian location mixture densities: (a) the first approach involves fixing an integer \( k \) and performing maximum likelihood estimation over the class \( \mathcal{M}_k \) which is the collection of all densities \( f_G \in \mathcal{M} \) where \( G \) is discrete and has at most \( k \) atoms, and (b) the second approach involves performing maximum likelihood estimation over the entire class \( \mathcal{M} \). This results in the Nonparametric Maximum Likelihood Estimator (NPMLE) for \( f^* \) and is the focus of this paper.

The first approach (maximum likelihood estimation over \( \mathcal{M}_k \) for a fixed \( k \)) is quite popular. However, it suffers from the two well-known issues: choosing \( k \) is non-trivial and, moreover, maximizing likelihood over \( \mathcal{M}_k \) results in a non-convex optimization problem. This non-convex algorithm is usually approximately solved by the EM algorithm (see, for example, Dempster et al. [16], McLachlan and Krishnan [44], Watanabe and Yamaguchi [63]). Recent progress on obtaining a theoretical understanding of the behaviour of the non-convex EM algorithm has been made by Balakrishnan et al. [3]. For the issue of choosing \( k \), one can adapt standard model selection methodology such as those based on the AIC [2] or BIC [54]. However theoretical properties of the resulting estimator are not well understood because the usual regularity conditions that are required for AIC or BIC to work do not hold in this mixture model setting. More recently, Maugis and Michel [42] (see also Maugis-Rabusseau and Michel [43]) proposed a penalization likelihood criterion to choose \( k \) by suitably employing the general theory of non-asymptotic model selection via penalization due to Birgé and Massart [6], Barron et al. [4] and Massart [40]. Maugis and Michel [42] also established nonasymptotic risk properties of the resulting estimator. The computational aspects of their estimator are quite involved however (see Maugis and Michel [41]) as their estimators are based on solving multiple non-convex optimization problems.

The present paper concentrates on second likelihood-based approach involving nonparametric maximum likelihood estimation of \( f^* \). This method is not affected by the issues of non-convexity and the need for choosing \( k \). Formally, by an NPMLE, we refer to any maximizer \( \hat{f}_n \) of \( \sum_{i=1}^n \log f(X_i) \) as \( f \) varies over \( \mathcal{M} \) i.e.,

\[
\hat{f}_n \in \arg\max_{f \in \mathcal{M}} \frac{1}{n} \sum_{i=1}^n \log f(X_i). \tag{1.2}
\]

Note that because the maximization is done over the entire class \( \mathcal{M} \) of all Gaussian location mixture densities (and not on any non-convex subset such as \( \mathcal{M}_k \)), the optimization in (1.2) is a convex optimization problem. Indeed, the objective function in (1.2) is concave in \( f \) and the constraint set \( \mathcal{M} \) is a convex class of densities.

The idea of using NPMLEs for estimating mixture densities has a long history (see, for example, the classical references Kiefer and Wolfowitz [30], Lindsay [35, 36, 37], Böning [8]). The optimization problem (1.2) and its solutions have been studied by many authors. It is known that maximizers of \( f \mapsto \sum_{i=1}^n \log f(X_i) \) exist over \( \mathcal{M} \) which implies that NPMLEs exist. Maximizers are non-unique however so there exist multiple NPMLEs. Nevertheless, for every NPMLE \( \hat{f}_n \), the values \( f(X_i) \) for \( i = 1, \ldots, n \) are unique (this is essentially because the objective function in the optimization (1.2) only depends on \( f \) through the values \( f(X_1), \ldots, f(X_n) \)). Proofs of these basic facts can be found, for example, in Böning [8, Chapter 2].
There exist many algorithms in the literature for approximately solving the optimization (1.2) (note that though (1.2) is a convex optimization problem, it is infinite-dimensional which is probably why exact algorithms seem to be unavailable). These algorithms range from: (a) vertex direction methods and vertex exchange methods (see the review papers: Böhnig [7], Lindsay and Lesperance [38] and the references therein), (b) EM algorithms (see Laird [32] and Jiang and Zhang [28]), and (c) modern large-scale interior point methods (see Koenker and Mizera [31] and Feng and Dicker [24]). Most of these methods focus on the case \( d = 1 \) and involve maximizing the likelihood over mixture densities where the mixing measure is supported on a fixed fine grid in the range of the data. The algorithm of Koenker and Mizera [31] is highly scalable (relying on the commercial convex optimization library Mosek [46]) and can obtain an approximate NPMLE efficiently even for large sample sizes (\( n \) of the order 100,000). See Section 4 for more algorithmic and implementation details as well as some simulation results.

Let us now describe the main objectives and contributions of the current paper. Our first goal is to investigate the theoretical properties of NPMLEs. In particular, we study the accuracy of \( \hat{f}_n \) as an estimator of the density \( f^* \) from which the data \( X_1, \ldots, X_n \) are generated. We shall use, as our loss function, the squared Hellinger distance:

\[
\mathcal{H}_2(f, g) := \int (\sqrt{f(x)} - \sqrt{g(x)})^2 \, dx,
\]

which is one of the most commonly used loss functions for density estimation problems. We present a detailed analysis of the risk, \( \mathbb{E}\mathcal{H}_2^2(\hat{f}_n, f^*) \), of every NPMLE (the expectation here is taken with respect to \( X_1, \ldots, X_n \) distributed independently according to \( f^* \)). The other common loss function used in density estimation is the total variation distance. The total variation distance is bounded from above by a constant multiple of \( \mathcal{H}_2 \) so that upper bounds for risk under the squared Hellinger distance automatically imply upper bounds for risk in squared total variation distance.

Our results imply that, for a large class of true densities \( f^* \in \mathcal{M} \), the risk of every NPMLE \( \hat{f}_n \) is parametric (i.e., \( n^{-1} \)) up to multiplicative factors that are logarithmic in \( n \). In particular, our results imply that when the true \( f^* \in \mathcal{M}_k \) for some \( 1 \leq k \leq n \), then every NPMLE has risk \( k/n \) up to a logarithmic multiplicative factor in \( n \). It is not hard to see that the minimax risk over \( \mathcal{M}_k \) is bounded from below by \( k/n \) which implies therefore that every NPMLE is nearly minimax over \( \mathcal{M}_k \) (ignoring logarithmic factors in \( n \)) for every \( k \geq 1 \). This is interesting because NPMLEs do not use any a priori knowledge of \( k \). The price in squared Hellinger risk that is paid for not knowing \( k \) in advance is only logarithmic in \( n \). Our results are non-asymptotic and the bounds for risk over \( \mathcal{M}_k \) hold even when \( k \) grows with \( n \). Our results also imply that NPMLEs have parametric risk (again up to multiplicative logarithmic factors) when the mixing measure of \( f^* \) is supported on a fixed compact subset of \( \mathbb{R}^d \). Note that we have assumed that the covariance matrix of every Gaussian component of mixture densities in the class \( \mathcal{M} \) is the identity matrix. Our results can be extended to the case of arbitrary covariance matrices provided a lower bound on the eigenvalues is available (see Proposition 2.5) (on the other hand, when no a priori information on the covariance matrices is available, it is well-known that likelihood based approaches are infeasible). These results are described in detail in Section 2.

Previous results on the Hellinger accuracy of NPMLEs were due to Ghosal and van der Vaart [26] and Zhang [66] who dealt with the univariate (\( d = 1 \)) case. They studied the Hellinger accuracy under conditions on the moments of the mixing measure corresponding to \( f^* \). The accuracy of NPMLEs in the interesting case when \( f^* \in \mathcal{M}_k \) does not appear to have been studied previously even in \( d = 1 \). We study the Hellinger risk of NPMLEs for all \( d \geq 1 \) and also under a much broader set of assumptions on \( f^* \) compared to existing papers.
We would like to mention here that numerous papers have appeared in the theoretical computer science community establishing rigorous theoretical results for estimating densities in $M_k$. For example, the papers Daskalakis and Kamath [15], Suresh et al. [56], Bhaskara et al. [5], Chan et al. [13, 12], Acharya et al. [1], Li and Schmidt [34] have results on estimating densities in $M_k$ with rigorous bounds on the error in estimation. The estimation error is mostly measured in terms of the total variation distance which is smaller (up to constant multiplicative factors) compared to the Hellinger distance used in the present paper. Their sample complexity results imply rates of estimation of $k/n$ up to logarithmic factors in $n$ for densities in $M_k$ in terms of the squared total variation distance and hence these results are comparable to our results for the NPMLE. The estimation procedures used in these papers range from (a) hypothesis selection over a set of candidate estimators via an improved version of the Scheffé estimate ([15, 56]; see Devroye and Lugosi [17, Chapter 6] for background on the Scheffé estimate), (b) reduction to finding sparse solutions to a non-negative linear systems ([5]), and (c) fitting piecewise polynomial densities ([13, 12, 1, 34]; these papers have the sharpest results). These methods are very interesting and, remarkably, come with precise time complexity guarantees. They are not based on likelihood maximization however and, in our opinion, conceptually more involved compared to the NPMLE studied in this paper. An additional minor difference between our work and this literature is that $k$ is taken to be a constant (and sometimes even known) in these papers while we allow $k$ to grow with $n$ and the NPMLE does not need prior knowledge of $k$.

Let us now describe briefly the proof techniques underlying our risk results for the NPMLEs. Our technical arguments are based on standard ideas from the literature on empirical processes for assessing the performance of maximum likelihood estimators (see Van der Vaart and Wellner [61], Wong and Shen [64], Zhang [66]). These techniques involve bounding the covering numbers of the space of Gaussian location mixture densities. For each compact subset $S \subseteq \mathbb{R}^d$, we prove covering number bounds for $M$ under the supremum distance ($L_\infty$) on $S$. Our bounds can be seen as extensions of the one-dimensional covering number results of Zhang [66] (which are themselves enhancements of corresponding results in Ghosal and van der Vaart [26]). The covering number results of Zhang [66] can be viewed as special instances of our bounds for the case when $S = [-M, M]$.

The extension to arbitrary compact sets $S$ is crucial for dealing with rates for densities in $M_k$. For proving the final Hellinger risk bounds of $\hat{f}_n$ from these $L_\infty$ covering numbers, we use appropriate modifications of tail arguments from Zhang [66].

The second goal of the present paper is to use NPMLEs to yield empirical Bayes estimates in the Gaussian denoising problem. By Gaussian denoising, we refer to the problem of estimating vectors $\theta_1, \ldots, \theta_n \in \mathbb{R}^d$ from independent $d$-dimensional observations $X_1, \ldots, X_n$ generated as

$$X_i \sim N(\theta_i, I_d) \quad \text{for } i = 1, \ldots, n. \quad (1.4)$$

The naive estimator in this denoising problem simply estimates each $\theta_i$ by $X_i$. It is well-known that, depending on the structure of the unknown $\theta_1, \ldots, \theta_n$, it is possible to achieve significant improvement over the naive estimator by using information from $X_j, j \neq i$ in addition to $X_i$ for estimating $\theta_i$. An ideal prototype for such information sharing across observations is given by the Oracle Bayes estimator which will be denoted by $\hat{\theta}^*_1, \ldots, \hat{\theta}^*_n$ and is defined in the following way:

$$\hat{\theta}^*_i := \mathbb{E}(\theta|X = X_i) \quad \text{where } \theta \sim G_n \text{ and } X|\theta \sim N(\theta, I_d)$$

and $G_n$ is the empirical measure corresponding to the true set of parameters $\theta_1, \ldots, \theta_n$. In other words, $\hat{\theta}^*_i$ is the posterior mean of $\theta$ given $X = X_i$ under the model $X|\theta \sim N(\theta, I_d)$ and the prior $\theta \sim G_n$. This is an Oracle estimator that is infeasible in practice as it uses information on the unknown parameters $\theta_1, \ldots, \theta_n$ via their empirical measure $G_n$. It is well-known (see, for example,
Jiang and Zhang [28] argued that $\hat{\theta}_i^*$ has the following alternative expression as a consequence of Tweedie’s formula:

$$\hat{\theta}_i^* = X_i + \frac{\nabla f_{\bar{G}_n}(X_i)}{f_{\bar{G}_n}(X_i)}$$

(1.5)

where $f_{\bar{G}_n}$ is the Gaussian location mixture density with mixing measure $\bar{G}_n$ (defined as in (1.1)). From the above expression, it is clear that the Oracle Bayes estimator can be estimated from the data $X_1, \ldots, X_n$ provided one can estimate the Gaussian location mixture density, $f_{\bar{G}_n}$, from the data $X_1, \ldots, X_n$. For this purpose, as insightfully observed in Jiang and Zhang [28], any NPMLE, $\hat{f}_n$, as in (1.2) can be used. Indeed, if $\hat{f}_n$ denotes any NPMLE based on the data $X_1, \ldots, X_n$, then Jiang and Zhang [28] argued that $\hat{f}_n$ is a good estimator for $f_{\bar{G}_n}$ under (1.4) so that $\hat{\theta}_i^*$ is estimable by

$$\hat{\theta}_i := X_i + \frac{\nabla \hat{f}_n(X_i)}{\hat{f}_n(X_i)}.$$ 

(1.6)

This yields a completely tuning-free solution to the Gaussian denoising problem (note however that the noise distribution is assumed to be completely known as $N(0, I_d)$). This is the General Maximum Likelihood empirical Bayes estimator of Jiang and Zhang [28] who proposed it and studied its theoretical properties in detail for estimating sparse univariate normal means. To the best of our knowledge, the properties of the estimator (1.6) for multidimensional denoising problems have not been previously explored. More generally, the empirical Bayes approach to the Gaussian denoising problem goes back to Robbins [49, 50, 52]. The effectiveness of nonparametric empirical Bayes estimators for estimating sparse normal means has been explored by many authors including Johnstone and Silverman [29], Brown and Greenshtein [11], Jiang and Zhang [28], Donoho and Reeves [19], Koenker and Mizera [31] but most work seems restricted to the univariate setting. On the other hand, there exists prior work on parametric empirical Bayes methods in the multivariate Gaussian denoising problem (see, for example, [21, 22]) but the role of nonparametric empirical Bayes methods in multivariate Gaussian denoising does not seem to have been explored previously.

We perform a detailed study of the accuracy of $\hat{\theta}_i$ in (1.6) as an estimator of the oracle Bayes estimator $\theta_i^*$ for $i = 1, \ldots, n$ in terms of the following squared error risk measure:

$$\mathcal{R}_n(\hat{\theta}, \theta^*) := \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} ||\hat{\theta}_i - \theta_i^*||^2 \right]$$

(1.7)

where the expectation is taken with respect to $X_1, \ldots, X_n$ generated independently according to (1.4). The risk $\mathcal{R}_n(\hat{\theta}, \theta^*)$ depends on the configuration of the unknown parameters $\theta_1, \ldots, \theta_n$ and we perform a detailed study of the risk for natural configurations of the points $\theta_1, \ldots, \theta_n \in \mathbb{R}^d$. Our results imply that, under natural assumptions on $\theta_1, \ldots, \theta_n$, the risk $\mathcal{R}_n(\hat{\theta}, \theta^*)$ is bounded by the parametric rate $1/n$ up to logarithmic multiplicative factors. For example, when the number of distinct vectors among $\theta_1, \ldots, \theta_n$ equals $k$ for some $k \leq n$ (an assumption which makes sense in clustering situations), we prove that the risk $\mathcal{R}_n(\hat{\theta}, \theta^*)$ is bounded from above by the parametric rate $k/n$ up to logarithmic multiplicative factors in $n$. This result is especially remarkable because the estimator (1.6) is tuning free and does not have knowledge of $k$. We also prove that the analogous minimax risk over this class is bounded from below by $k/n$ implying that the empirical Bayes estimate is minimax up to logarithmic multiplicative factors. Our result also implies that when $\theta_1, \ldots, \theta_n$ take values in a bounded region on $\mathbb{R}^d$, then also the risk $\mathcal{R}_n(\hat{\theta}, \theta^*)$ is nearly parametric.

Summarizing, our results imply that, under a wide range of assumptions on $\theta_1, \ldots, \theta_n$, the empirical
Bayes estimator \( \hat{\theta}_i \) performs comparably to the Oracle Bayes estimator \( \hat{\theta}^*_i \) for denoising. These results are in Section 3. The results and the proof techniques are inspired by the arguments of Jiang and Zhang [28] who studied the univariate denoising problem under sparsity assumptions. We generalize their arguments to multidimensions.

In addition to theoretical results, we also present simulation evidence for the effectiveness of \( \hat{\theta}_i \) in the Gaussian denoising problem in Section 4 (where we also present some implementation and algorithmic details for computing approximate NPMLEs). Here, we illustrate the performance of (1.6) for denoising when the true parameter vectors \( \theta_1, \ldots, \theta_n \) take values in certain natural regions in \( \mathbb{R}^2 \). We also numerically analyze the performance of (1.6) in clustering situations when \( \theta_1, \ldots, \theta_n \) take \( k \) distinct values for some small \( k \). Here we compare the performance of (1.6) to other natural procedures such as \( k \)-means with \( k \) selected via the gap statistic (see Tibshirani et al. [58]). We argue that (1.6) performs very efficiently in terms of the risk measure \( \mathcal{R}_n(\hat{\theta}, \hat{\theta}^*) \). In terms of a purely clustering based comparison index (such as the Adjusted Rand Index), we argue that the performance of (1.6) is still reasonable.

The rest of the paper is organized in the following manner. In Section 2, we state our results on the Hellinger accuracy of NPMLEs for estimating Gaussian location mixture densities. Section 3 has statements of our results on the risk \( \mathcal{R}_n(\hat{\theta}, \theta^*) \) in the denoising problem. Section 4 has algorithmic details and simulation evidence for the effectiveness of (1.6) for denoising. Proofs for results in Section 2 are given in Section 5 while proof for Section 3 are in Section 6. Metric entropy results for multivariate Gaussian location mixture densities play a crucial role in the proofs of the main results; these results are stated and proved in Section 7. Section 8 contains the statement and proof for a crucial ingredient for the proof of the main denoising theorem. Finally, additional technical results needed in the proofs of the main results are collected in Section A together with their proofs.

### 2. Hellinger Accuracy of NPMLE

Given data \( X_1, \ldots, X_n \), let \( \hat{f}_n \) be any NPMLE defined as in (1.2). In this section, we shall study the accuracy of \( \hat{f}_n \) in terms of the squared Hellinger distance (defined in (1.3)). All the results in this section are proved in Section 5.

For investigations into the performance of \( \hat{f}_n \), it is most natural to assume that the data \( X_1, \ldots, X_n \) are independent observations having common density \( f^* \in \mathcal{M} \) in which case we seek bounds on \( \mathcal{L}^2(\hat{f}_n, f^*) \). However, following Zhang [66], we work under the more general assumption that \( X_1, \ldots, X_n \) are independent but not identically distributed and that each \( X_i \) has a density that belongs to the class \( \mathcal{M} \). This additional generality will be used in Section 3 for proving results on the Empirical Bayes estimator (1.6) for the Gaussian denoising problem.

Specifically, we assume that \( X_1, \ldots, X_n \) are independent and that each \( X_i \) has density \( f_{G_i} \) for some probability measures \( G_1, \ldots, G_n \) on \( \mathbb{R}^d \). This distributional assumption on the data \( X_1, \ldots, X_n \) includes the following two important special cases: (a) \( G_1, \ldots, G_n \) are all identically equal to \( G \) (say): in this case, the observations \( X_1, \ldots, X_n \) are identically distributed with common density \( f^* = f_G \in \mathcal{M} \), and (b) Each \( G_i \) is degenerate at some \( \theta_i \in \mathbb{R}^d \); here each data point \( X_i \) is normal with \( X_i \sim N_d(\theta_i, I_d) \).

We let \( \bar{G}_n := (G_1 + \cdots + G_n)/n \) to be the average of the probability measures \( G_1, \ldots, G_n \). In the case when \( G_1, \ldots, G_n \) are all identically equal to \( G \), then clearly \( \bar{G}_n = G \). On the other hand, when each \( G_i \) is degenerate at some \( \theta_i \in \mathbb{R}^d \), then \( \bar{G}_n \) equals the empirical measure corresponding to \( \hat{\theta}_1, \ldots, \hat{\theta}_n \).

Under the above independent but not identically distributed assumption on \( X_1, \ldots, X_n \), it has been insightfully pointed out by Zhang [66] that every NPMLE \( \hat{f}_n \) based on \( X_1, \ldots, X_n \) (defined as in (1.2)) is really estimating \( f_{\bar{G}_n} \). Note that \( f_{\bar{G}_n} \) denotes the average of the densities of \( X_1, \ldots, X_n \).

In this section, we shall prove bounds for the accuracy of any NPMLE \( \hat{f}_n \) as an estimator for \( f_{\bar{G}_n} \) under the Hellinger distance i.e., for \( \mathcal{L}(\hat{f}_n, f_{\bar{G}_n}) \). For every compact set \( S \subseteq \mathbb{R}^d \) and \( M \geq \sqrt{10 \log n} \),
we shall prove an upper bound for $\mathcal{H}(\hat{f}_n, f_{G_n})$ in terms of $S$ and $M$. As will be seen later in this section, under some simplifying assumptions on $G_n$, our bound for $\mathcal{H}(\hat{f}_n, f_{G_n})$ can be optimized over $S$ and $M$ to produce an explicit bound.

In order to state our main theorem, we need to introduce the following notation. For nonempty sets $S \subseteq \mathbb{R}^d$, we define the function $\mathcal{A}_S : \mathbb{R}^d \to [0, \infty)$ by

$$
(2.1) \quad \mathcal{A}_S(x) := \inf_{u \in S} \|x - u\| \quad \text{for } x \in \mathbb{R}^d
$$

where $\|\cdot\|$ is the usual Euclidean norm on $\mathbb{R}^d$. Also for $S \subseteq \mathbb{R}^d$, we let

$$
(2.2) \quad S^1 := \{x : \mathcal{A}_S(x) \leq 1\}.
$$

Our bound on $\mathcal{H}(\hat{f}_n, f_{G_n})$ will be controlled by the following quantity. For every non-empty compact set $S \subseteq \mathbb{R}^d$ and $M \geq \sqrt{10 \log n}$, let $\epsilon_n(M, S)$ be defined via

$$
(2.3) \quad \epsilon_n^2(M, S) := \text{Vol}(S^1) \frac{M^d}{n} \left(\sqrt{\log n}\right)^{(4-d)_+} + \left(\log n\right) \inf_{p \geq \frac{d + 1}{2 \log n}} \left(\frac{\mu_p(\mathcal{A}_S)}{M}\right)^p
$$

where $S^1$ is defined in (2.2), $(4 - d)_+$ is defined by way of $x_+ := \max(x, 0)$ and $\mu_p(\mathcal{A}_S)$ is defined as the moment

$$
\mu_p(\mathcal{A}_S) := \left(\int_{\mathbb{R}^d} (\mathcal{A}_S(\theta))^p d\tilde{G}_n(\theta)\right)^{1/p} \quad \text{for } p > 0.
$$

Note that the moments $\mu_p(\mathcal{A}_S)$ quantify how the probability (under $\tilde{G}_n$) decays as one moves away from the set $S$.

The next theorem proves that $\mathcal{H}^2(\hat{f}_n, f_{G_n})$ is bounded (with high probability and in expectation) by a constant (depending on $d$) multiple of $\epsilon_n^2(M, S)$ for every estimator $\hat{f}_n$ having the property that the likelihood of the data at $\hat{f}_n$ is not too small compared to the likelihood at $f_{G_n}$ (made precise in inequality (2.4)). Every NPMLE trivially satisfies this condition (as it maximizes likelihood) but the theorem also applies to certain approximate likelihood maximizers.

**Theorem 2.1.** Let $X_1, \ldots, X_n$ be independent random vectors with $X_i \sim f_{G_i}$ and let $G_n := (G_1 + \cdots + G_n)/n$. Fix $M \geq \sqrt{10 \log n}$ and a non-empty compact set $S \subseteq \mathbb{R}^d$ and let $\epsilon_n(M, S)$ be as defined in (2.3). Then there exists a positive constant $C_d$ (depending only on $d$) such that for every estimator $\hat{f}_n$ based on the data $X_1, \ldots, X_n$ satisfying

$$
(2.4) \quad \prod_{i=1}^n \frac{\hat{f}_n(X_i)}{f_{G_n}(X_i)} \geq \exp \left[\frac{C_d(\beta - \alpha)}{\min(1 - \alpha, \beta)} n\epsilon_n^2(M, S)\right] \quad \text{for some } 0 < \beta \leq \alpha < 1,
$$

we have

$$
(2.5) \quad \mathbb{P}\left\{\mathcal{H}(\hat{f}_n, f_{G_n}) \geq \frac{t\epsilon_n(M, S)\sqrt{C_d}}{\sqrt{\min(1 - \alpha, \beta)}}\right\} \leq 2n^{-t^2} \quad \text{for every } t \geq 1.
$$

and

$$
(2.6) \quad \mathbb{E}\mathcal{H}^2(\hat{f}_n, f_{G_n}) \leq \frac{4C_d}{\min(1 - \alpha, \beta)} \epsilon_n^2(M, S).
$$
Theorem 2.1 asserts that the risk $\mathbb{E}\delta^2(\hat{f}_n, \bar{f}_{G_n})$ is bounded from above by a constant (depending on $d$, $\alpha$ and $\beta$) multiple of $\epsilon_n^2(M,S)$ for every $M \geq \sqrt{d \log n}$ and compact subset $S \subseteq \mathbb{R}^d$. This is true for every estimator $\hat{f}_n$ satisfying (2.4). Every NPMLE satisfies (2.4) with $\alpha = \beta = 0.5$ (note that the right hand side of (2.4) is always less than or equal to one because $\beta \leq \alpha$).

Theorem 2.1 is novel to the best of our knowledge. When $d = 1$ and $S$ is taken to be $[-R,R]$ for some $R \geq 0$, then the conclusion given by Theorem 2.1 appears implicitly in Zhang [66, Proof of Theorem 1]. The advantages of allowing $S$ to be an arbitrary compact set will be clear from the special cases of Theorem 2.1 that are given below. Our proof of Theorem 2.1 (given in Section 5) is greatly inspired by Zhang [66, Proof of Theorem 1].

To get the best rate for $\delta(\hat{f}_n, f_{\bar{G}_n})$ from Theorem 2.1, we need to choose $M$ and $S$ so that $\epsilon_n(M,S)$ is small. These choices depend on $\bar{G}_n$, and in the next result, we describe how to choose $M$ and $S$ based on reasonable assumptions on $\bar{G}_n$. This leads to explicit rates for $\delta(\hat{f}_n, f_{\bar{G}_n})$. For simplicity, we shall assume, for the next result, that $\hat{f}_n$ is an NPMLE so that (2.4) is satisfied with $\alpha = \beta = 0.5$. We shall also only state the results on the risk $\mathbb{E}\delta^2(\hat{f}_n, f_{\bar{G}_n})$.

**Corollary 2.2.** Let $X_1, \ldots, X_n$ be independent random vectors with $X_i \sim f_{G_i}$ and let $\bar{G}_n := (G_1 + \cdots + G_n)/n$. Let $\hat{f}_n$ be an NPMLE based on $X_1, \ldots, X_n$ defined as in (1.2). Below $C_d$ denotes a positive constant depending on $d$ alone.

1. Suppose that $\bar{G}_n$ is supported on a compact subset $S$ of $\mathbb{R}^d$. Then

   \begin{equation}
   \mathbb{E}\delta^2(\hat{f}_n, f_{\bar{G}_n}) \leq C_d \frac{\text{Vol}(S^1)}{n} \left(\sqrt{\log n}\right)^{d+(4-d)_+}.
   \end{equation}

2. Suppose there exist a compact subset $S \subseteq \mathbb{R}^d$ and real numbers $0 < \alpha \leq 2$ and $K \geq 1$ such that

   \begin{equation}
   \mu_p(\partial S) \leq K p^{1/\alpha} \quad \text{for all } p \geq 1.
   \end{equation}

   Then

   \begin{equation}
   \mathbb{E}\delta^2(\hat{f}_n, f_{\bar{G}_n}) \leq C_d \frac{\text{Vol}(S^1)(Ke^{1/\alpha})^d}{\sqrt{\log n}} \left(\sqrt{\log n}\right)^{(2d/\alpha)+(4-d)_+}.
   \end{equation}

3. Suppose there exists a compact set $S \subseteq \mathbb{R}^d$ and real numbers $\mu > 0$ and $p > 0$ such that

   \begin{equation}
   \mu_p(\partial S) \leq \mu. \quad \text{Then there exists a positive constant } C_{d,\mu,p} \text{ (depending only on } d, \mu \text{ and } p) \text{ such that}
   \end{equation}

   \begin{equation}
   \mathbb{E}\delta^2(\hat{f}_n, f_{\bar{G}_n}) \leq C_{d,\mu,p} \left(\frac{\text{Vol}(S^1)}{n}\right)^{p/(p+d)} \left(\sqrt{\log n}\right)^{(2d+p(4-d_+))/(p+d)}.
   \end{equation}

Corollary 2.2 is a generalization of Zhang [66, Theorem 1] as the latter result can be seen as a special case of Corollary 2.2 for $d = 1$ and $S = [-R,R]$ for some $R \geq 0$. The fact that $S$ can be arbitrary in Corollary 2.2 allows us to deduce the following important adaptation results of NPMLEs for estimating Gaussian mixtures whose mixing measures are discrete. These results are, to the best of our knowledge, novel.

**Proposition 2.3 (Near parametric risk for discrete Gaussian mixtures).** Let $X_1, \ldots, X_n$ be independent random vectors with $X_i \sim f_{G_i}$ and let $\bar{G}_n := (G_1 + \cdots + G_n)/n$. Let $\hat{f}_n$ be an NPMLE based on $X_1, \ldots, X_n$ defined as in (1.2). Then there exists a positive constant $C_d$ depending only on
such that whenever $\tilde{G}_n$ is a discrete probability measure that is supported on a set of cardinality $k$, we have

$$
\mathbb{E} d^2(\hat{f}_n, f_{\tilde{G}_n}) \leq C_d \left( \frac{k}{n} \right) \left( \sqrt{\log n} \right)^{d+(4-d)}.
$$

The significance of Proposition 2.3 is the following. Note that the right hand side of (2.11) is the parametric risk $k/n$ up to an additional multiplicative factor that is logarithmic in $n$. This inequality shows important adaptation properties of NPMLEs. When the true unknown Gaussian mixture $f_{\tilde{G}_n}$ is a discrete mixture having $k$ Gaussian components, then every NPMLE nearly (up to logarithmic factors) achieves the parametric squared Hellinger risk $k/n$. For a fixed $k$, it is well-known that fitting a $k$-component Gaussian mixture via maximum likelihood is a non-convex problem that is usually solved by the EM algorithm. On the other hand, NPMLE is given by a convex optimization algorithm, does not require any prior specification of $k$ and still achieves the $k/n$ rate (up to logarithmic factors) when the truth is a $k$-component Gaussian mixture.

Note that Proposition 2.3 applies to the case of independent but not identically distributed $X_1, \ldots, X_n$ which is more general compared to the i.i.d assumption. This implies, in particular, that (2.11) also applies to the case when $X_1, \ldots, X_n$ are i.i.d having density $f^* \in \mathcal{M}$. In this case, we have

$$
\sup_{f^* \in \mathcal{M}_k} \mathbb{E} d^2(\hat{f}_n, f^*) \leq C_d \left( \frac{k}{n} \right) \left( \sqrt{\log n} \right)^{d+(4-d)}.
$$

The interesting aspect of this inequality is that it holds for every $k \geq 1$ and that the estimator $\hat{f}_n$ does not know or use any information about $k$.

It is straightforward to prove a minimax lower bound over $\mathcal{M}_k$ that complements Proposition 2.3. The following result proves that the minimax risk over $\mathcal{M}_k$ is bounded from below by a constant multiple of $k/n$. This implies that the NPMLE is minimax optimal over $\mathcal{M}_k$ ignoring logarithmic factors of $n$. Moreover, this optimality is adaptive since MLE does not require knowledge of $k$. This minimax lower bound is stated for the i.i.d case which implies that it holds for the more general independent but not identically distributed case as well.

**Lemma 2.4.** For $k \geq 1$, let

$$
\mathcal{R}(\mathcal{M}_k) := \inf_{\tilde{f}} \sup_{f \in \mathcal{M}_k} \mathbb{E}_f d^2(\tilde{f}, f)
$$

where $\mathbb{E}_f$ denotes expectation when the data $X_1, \ldots, X_n$ are independent observations drawn from the density $f$. Then there exists a universal positive constant $C$ such that

$$
\mathcal{R}(\mathcal{M}_k) \geq C \frac{k}{n} \quad \text{for every } 1 \leq k \leq n.
$$

Inequality (2.12) and Lemma 2.4 together imply that every NPMLE $\hat{f}_n$ is minimax optimal up to logarithmic factors in $n$ over the class $\mathcal{M}_k$ for every $k \geq 1$. This optimality is adaptive since the NPMLE requires no information on $k$. The logarithmic terms in (2.12) are likely suboptimal but we are unable to determine the exact power of $\log n$ in (2.12).

So far we have studied estimation of Gaussian location mixture densities where the covariance matrix of each Gaussian component is fixed to be the identity matrix. We next show that the same estimator (NPMLE defined as in (1.2)) can be modified to estimate arbitrary Gaussian mixtures
(where the covariance matrices can be different from identity) provided a lower bound on the eigenvalues of the covariance matrices is available. Suppose that \( h^* \) is the Gaussian mixture density

\[
h^*(x) := \sum_{j=1}^{k} w_j \phi_d(x; \mu_j, \Sigma_j) \quad \text{for } x \in \mathbb{R}^d
\]

where \( k \geq 1, \mu_1, \ldots, \mu_k \in \mathbb{R}^d \) and \( \Sigma_1, \ldots, \Sigma_k \) are \( d \times d \) positive definite matrices. Here \( \phi_d(\cdot; \mu, \Sigma) \) denotes the \( d \)-variate normal density with mean \( \mu \) and covariance matrix \( \Sigma \). Suppose \( \sigma_2^\text{min} \) and \( \sigma_2^\text{max} \) are two positive numbers that are, respectively, smaller and larger than all the eigenvalues of \( \Sigma_1, \ldots, \Sigma_k \), i.e.,

\[
\sigma_2^\text{min} \leq \min_{1 \leq j \leq k} \lambda_{\text{min}}(\Sigma_j) \leq \max_{1 \leq j \leq k} \lambda_{\text{max}}(\Sigma_j) \leq \sigma_2^\text{max} \tag{2.15}
\]

Consider the problem estimating \( h^* \) from i.i.d observations \( Y_1, \ldots, Y_n \). It turns out that for every NPMLE \( \hat{f}_n \) computed as in (1.2) based on the data \( X_1 := Y_1/\sigma_{\text{min}}, \ldots, X_n := Y_n/\sigma_{\text{min}} \) can be coverted to a very good estimator for \( h^* \) via

\[
\hat{h}_n(x) := \sigma_{\text{min}}^{-d} \hat{f}_n(\sigma_{\text{min}}^{-1} x) \quad \text{for } x \in \mathbb{R}^d. \tag{2.16}
\]

Our next result shows that the squared Hellinger risk of \( \hat{h}_n \) is bounded from above by \( (k/n) \) up to a logarithmic factor in \( n \) provided that \( \sigma_{\text{max}}/\sigma_{\text{min}} \) is bounded by a constant.

**Proposition 2.5.** Let \( Y_1, \ldots, Y_n \) be independent and identically distributed observations having density \( h^* \) defined in (2.14). Consider the estimator \( \hat{h}_n \) for \( h^* \) defined in (2.16). Then

\[
\mathbb{E} \Delta^2(\hat{h}_n, h^*) \leq C_d \left( \frac{k}{n} \right) (\max(1, \tau))^d \left( \sqrt{\log n} \right)^{d+(4-d)_+}, \quad \text{where } \tau := \sqrt{\frac{\sigma_{\text{max}}^2}{\sigma_{\text{min}}^2} - 1}. \tag{2.17}
\]

Proposition 2.5 shows that the estimator \( \hat{h}_n \) achieves near parametric risk \( k/n \) (up to logarithmic factors in \( n \)) provided \( \tau \) is bounded from above by a constant. Note that this estimator \( \hat{h}_n \) uses knowledge of \( \sigma_{\text{min}}^2 \) but does not use knowledge of any other feature of \( h^* \) including the number of components \( k \). In particular, this is an estimation procedure which (without knowing the value of \( k \)) achieves nearly the \( k/n \) rate for \( k \)-component well-conditioned Gaussian mixtures provided a lower bound \( \sigma_{\text{min}}^2 \) on eigenvalues is known a priori.

It is natural to compare Proposition 2.5 to the main results in Maugis and Michel [42] where an adaptive procedure is developed for estimating \( k \)-component Gaussian mixtures at the rate \( k/n \) (up to a logarithmic factor) without prior knowledge of \( k \). The estimator of Maugis and Michel [42] is very different from ours. They first fit \( m \)-component Gaussian mixtures for different values of \( m \) and then select one of these estimators by optimizing a penalized model-selection criterion. Thus, their procedure is based on solving multiple non-convex optimization problems. Also, Maugis and Michel [42] impose upper and lower bounds on the means and the eigenvalues of the covariance matrices of the components of the mixture densities. On the contrary, our method is based on convex optimization and we only need a lower bound on the eigenvalues of the covariance matrices (no bounds on the means are necessary). On the flip side, the result of Maugis and Michel [42] has much better logarithmic factors compared to Proposition 2.5 and it is also stated in the form of an Oracle inequality.
3. Application to Gaussian Denoising. In this section, we explore the role of the NPMLE for estimating the Oracle Bayes estimator in the Gaussian denoising problem. The goal is to estimate unknown vectors \( \theta_1, \ldots, \theta_n \) in \( \mathbb{R}^d \) from independent random vectors \( X_1, \ldots, X_n \) such that \( X_i \sim N(\theta_i, I_d) \) for \( i = 1, \ldots, n \). The Oracle estimator is \( \hat{\theta}_i^* \), \( i = 1, \ldots, n \) which is given by (1.5) where \( G_n \) is the empirical measure corresponding to \( \theta_1, \ldots, \theta_n \).

It is natural to estimate the Oracle Bayes estimator by the Empirical Bayes estimator \( \hat{\theta}_i \) which is defined as in (1.6) for \( i = 1, \ldots, n \). Here \( \hat{f}_n \) is any NPMLE based on \( X_1, \ldots, X_n \) (defined as in (1.2)). We will gauge the performance of \( \hat{\theta}_i \) as an estimator for \( \theta^* \) in terms of the squared error risk measure \( \mathcal{R}_n(\theta, \theta^*) \) defined in (1.7). The main theorem of this section is given below. This is stated in a form that is similar to the statement of Theorem 2.1. It proves that, for every compact set \( S \subseteq \mathbb{R}^d \) and \( M \geq \sqrt{10 \log n} \), the risk \( \mathcal{R}_n(\theta, \theta^*) \) is bounded from above by \( \epsilon_n^2(M, S) \) up to an additional logarithmic multiplicative factor in \( n \).

**Theorem 3.1.** Let \( X_1, \ldots, X_n \) with independent random vectors with \( X_i \sim N(\theta_i, I_d) \) for \( i = 1, \ldots, n \). Let \( G_n \) denote the empirical measure corresponding to \( \theta_1, \ldots, \theta_n \). Let \( \hat{f}_n \) denote an NPMLE based on \( X_1, \ldots, X_n \) defined as in (1.2). Let \( \hat{\theta}_1, \ldots, \hat{\theta}_n \) be as defined in (1.6) and let \( \hat{\theta}_i^* \) be as in (1.5). Also, let \( \mathcal{R}(\hat{\theta}, \theta^*) \) be as in (1.7). Fix a non-empty compact set \( S \subseteq \mathbb{R}^d \) and \( M \geq \sqrt{10 \log n} \) and let \( \epsilon_n(M, S) \) be defined as in (2.3). Then there exists a positive constant \( C_d \) (depending only on \( d \)) such that

\[
\mathcal{R}_n(\theta, \theta^*) \leq C_d \epsilon_n^2(M, S) \left( \log n \right)^{\max(d,3)}.
\]

**Remark 3.1.** For the case of \( d = 1 \), Jiang and Zhang [28, Theorem 5] established a related result on the risk of \( \hat{\theta}_i \) in comparison to \( \hat{\theta}_i^* \). They proved that, for every compact set \( S \subseteq \mathbb{R}^d \) and \( M \geq \sqrt{10 \log n} \), the risk \( \mathcal{R}_n(\theta, \theta^*) \) is bounded from above by \( \epsilon_n^2(M, S) \) up to an additional logarithmic multiplicative factor in \( n \). The statement of Theorem 3.1 and its proof as well as the following corollary are inspired by Jiang and Zhang [28, Proof of Theorem 5].

Under specific reasonable assumptions on \( G_n \), it is possible to choose \( M \) and \( S \) explicitly which leads to the following result that is analogous to Corollary 2.2.

**Corollary 3.2.** Consider the same setting and notation as in Theorem 3.1. Below \( C_d \) denotes a positive constant depending on \( d \) alone.

1. Suppose that \( G_n \) is supported on a compact subset \( S \) of \( \mathbb{R}^d \). Then

\[
\mathcal{R}_n(\hat{\theta}, \theta^*) \leq C_d \frac{\text{Vol}(S^1)}{n} \left( \sqrt{\log n} \right)^{d+(4-d)+} (\log n)^{\max(d,3)}.
\]

2. Suppose there exist a compact subset \( S \subseteq \mathbb{R}^d \) and real numbers \( 0 < \alpha \leq 2 \) and \( K \geq 1 \) such that (2.8) holds. Then

\[
\mathcal{R}_n(\hat{\theta}, \theta^*) \leq C_d \frac{\text{Vol}(S^1)(Ke^{1/\alpha})^d}{n} \left( \sqrt{\log n} \right)^{(2d/\alpha)+(4-d)+} (\log n)^{\max(d,3)}.
\]
3. Suppose there exists a compact set $S \subseteq \mathbb{R}^d$ and real numbers $\mu > 0$ and $p > 0$ such that $\mu_p(\partial S) \leq \mu$. Then there exists a positive constant $C_{d,\mu,p}$ (depending only on $d, \mu$ and $p$) such that

$$
R_n(\hat{\theta}, \hat{\theta}^*) \leq C_{d,\mu,p} \left( \frac{\text{Vol}(S^1)}{n} \right)^{(p-d)/2} \left( \sqrt{\log n} \right)^{(2d+p(4-d)_+)/p+d} \left( \log n \right)^{\max(d,3)}.
$$

Corollary 3.2 has interesting consequences. Inequality (3.1) states that when $G_n$ is supported on a fixed compact set $S$, then the risk $R_n(\hat{\theta}, \hat{\theta}^*)$ is parametric up to logarithmic multiplicative factors in $n$. This is especially interesting because $\hat{\theta}_1, \ldots, \hat{\theta}_n$ do not use any knowledge of $S$.

Corollary 3.2 also leads to the following result with gives an upper bound for $R_n(\hat{\theta}, \hat{\theta}^*)$ when $\theta_1, \ldots, \theta_n$ are clustered into $k$ groups.

**Proposition 3.3.** Consider the same setting and notation as in Theorem 3.1. Suppose that $\theta_1, \ldots, \theta_n$ satisfy

$$
\max_{1 \leq i \leq n} \min_{1 \leq j \leq k} \| \theta_i - a_j \| \leq R
$$

for some $a_1, \ldots, a_k \in \mathbb{R}^d$ and $R \geq 0$. Then

$$
R_n(\hat{\theta}, \hat{\theta}^*) \leq C_d (1 + R)^d \left( \frac{k}{n} \right) \left( \sqrt{\log n} \right)^{d+(4-d)_+} \left( \log n \right)^{\max(d,3)}.
$$

The assumption (3.4) means that $\theta_1, \ldots, \theta_n$ can be grouped into the $k$ balls each of radius $R$ centered at the points $a_1, \ldots, a_k$. When $R$ is not large, this implies $\theta_1, \ldots, \theta_n$ can be clustered into $k$ groups. In particular, when $R = 0$, the assumption (3.4) implies that $\theta_1, \ldots, \theta_n$ take only $k$ distinct values. In words, Proposition 3.3 states that when $\theta_1, \ldots, \theta_n$ are clustered into $k$ groups, then $\hat{\theta}_1, \ldots, \hat{\theta}_n$ estimate $\hat{\theta}^*_1, \ldots, \hat{\theta}^*_n$ in squared error loss with accuracy $k/n$ up to logarithmic multiplicative factors in $n$. The notable aspect about this result is that the estimator does not use any knowledge of $k$ and is tuning-free. It is well-known in the clustering literature that choosing the optimal number of clusters is a challenging task (see, for example, Tibshirani et al. [58]). It is therefore helpful that the estimator $\hat{\theta}_1, \ldots, \hat{\theta}_n$ achieves nearly the $k/n$ rate in (3.4) without explicitly getting into the pesky problem of estimating $k$. Moreover, $\hat{\theta}_1, \ldots, \hat{\theta}_n$ is given by convex optimization (on the other hand, one usually needs to deal with non-convex optimization problems for solving clustering-type problems even if the number of clusters $k$ is known).

There exist techniques for estimating the number of clusters and subsequently employing algorithms for minimizing the $k$-means objective (notably, the “gap statistic” of Tibshirani et al. [58]). However, we are not aware of any result analogous to Proposition 3.4 for such techniques. There also exist other techniques for clustering based on convex optimization such as the method of convex clustering (see, for example, Lindsten et al. [39], Hocking et al. [27], Chen et al. [14]) which is based on a fused lasso-type penalized optimization. This method requires specification of tuning parameters. While interesting theoretical development exists for convex clustering (see, for example, Radchenko and Mukherjee [47], Zhu et al. [67], Tan and Witten [57], Wu et al. [65], Wang et al. [62]), to the best of our knowledge, a result similar to Proposition 3.4 is unavailable.

It is straightforward to see that it is impossible to devise estimators that achieve a rate that is faster than $k/n$ for the risk measure $R_n$. We provide a proof of this via a minimax lower bound in the following lemma. The logarithmic factors can probably be improved in Proposition 3.4 but we are unable to do so at the present moment. For the lower bound, let $\Theta_{n,d,k}$ denote the class of all $n$-tuples $(\theta_1, \ldots, \theta_n)$ with each $\theta_i \in \mathbb{R}^d$ and such that the number of distinct vectors among
\[ \theta_1, \ldots, \theta_n \text{ is equal to } k. \] Equivalently, \( \Theta_{n,d,k} \) consists of all \( n \)-tuples \( (\theta_1, \ldots, \theta_n) \) whose empirical measure is supported on a set of cardinality \( k \). The minimax risk for estimating \( \hat{\theta}_1, \ldots, \hat{\theta}_n \) with \( (\theta_1, \ldots, \theta_n) \in \Theta_{n,d,k} \) in squared error loss from the observations \( X_1, \ldots, X_n \) can be defined as

\[ R^*(\Theta_{n,d,k}) := \inf_{\hat{\theta}_1, \ldots, \hat{\theta}_n} \sup_{(\theta_1, \ldots, \theta_n) \in \Theta_{n,d,k}} \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^{n} \left\| \hat{\theta}_i - \hat{\theta}_i \right\|^2 \right] \]

The following result proves that \( R^*(\Theta_{n,d,k}) \) is at least \( Ck/n \) for a universal positive constant \( C \).

**Lemma 3.4.** Let \( \Theta_{n,d,k} \) and \( R^*(\Theta_{n,d,k}) \) be defined as above. There exists a universal positive constant \( C \) such that

\[ R^*(\Theta_{n,d,k}) \geq C \frac{k}{n} \quad \text{for every } 1 \leq k \leq n. \]

Lemma 3.4, together with Proposition 3.3, implies that \( \hat{\theta}_1, \ldots, \hat{\theta}_n \) is nearly minimax optimal (up to logarithmic multiplicative factors) for estimating \( \hat{\theta}_1, \ldots, \hat{\theta}_n \) over the class \( \Theta_{n,d,k} \). Moreover, this optimality is adaptive over \( k \) because the estimator does not use any knowledge of \( k \).

**4. Implementation Details and Some Simulation Results.** In this section, we shall discuss some computational details concerning the NPMLE and also provide numerical evidence for the effectiveness of the estimator (1.6) based on the NPMLE for denoising.

For the optimization problem (1.2), it can be shown that \( \hat{f}_n \) exists and is non-unique. However \( \hat{f}_n(X_1), \ldots, \hat{f}_n(X_n) \) are unique and they solve the finite dimensional optimization problem:

\[ \arg\max \sum_{i=1}^{n} \log f_i \]

subject to \( (f_1, \ldots, f_n) \in \text{ConvexHull} \left\{ (\phi(X_1 - \theta), \ldots, \phi(X_n - \theta)) : \theta \in \mathbb{R}^d \right\} \).

The constraint set in the above problem however involves every \( \theta \in \mathbb{R}^d \). A natural way of computing an approximate solution is to fix a finite data-driven set \( \{a_1, \ldots, a_m\} \subseteq \mathbb{R}^d \) and restrict the infinite convex hull to the convex hull over \( \theta \) belonging to this set. This leads to the problem:

\[ \arg\max \sum_{i=1}^{n} \log f_i \]

subject to \( (f_1, \ldots, f_n) \in \text{ConvexHull} \left\{ (\phi_d(X_1 - a_j), \ldots, \phi_d(X_n - a_j)) : j = 1, \ldots, m \right\} \).

This can also be seen as an approximation to (1.2) where the densities \( f \in \mathcal{M} \) are restricted to have atoms in \( \{a_1, \ldots, a_m\} \subseteq \mathbb{R}^d \). (4.2) is a convex optimization problem over the probability simplex in \( m \) dimensions and can be solved using many algorithms (for example, standard interior point methods as implemented in the software, Mosek, can be used here).

The effectiveness of (4.2) as an approximation to (1.2) depends crucially on the choice of \( \{a_1, \ldots, a_m\} \). For \( d = 1 \), Koenker and Mizera [31] propose the use of a uniform grid within the range of the observations [\( \min_{1 \leq i \leq n} X_i, \max_{1 \leq i \leq n} X_i \)]. Dicker and Zhao [18] discuss this approach in more detail and recommend the choice \( m := \lceil \sqrt{n} \rceil \). They also prove (see Dicker and Zhao [18, Theorem 2]) that the resulting approximate MLE, \( \hat{f}_n \), has a squared Hellinger accuracy, \( 5^2/2n \), of \( O_p((\log n)^2/n) \) when the mixing measure corresponding to \( f_0 \) has bounded support. For \( d \geq 1 \), Feng and Dicker [24] recommend taking a regular grid in a compact region containing the data. They also mention that empirical results seem "fairly insensitive" to the choice of \( m \).
A proposal for selecting \( \{a_1, \ldots, a_m\} \) that is different from gridding is the so called “exemplar” choice where one takes \( m = n \) and \( a_i = X_i \) for \( i = 1, \ldots, n \). This choice is proposed in Böning [8] for \( d = 1 \) and in Lashkari and Golland [33] for \( d \geq 1 \). This avoids gridding which can be problematic in multiple dimensions. Also, this method is computationally feasible as long as \( n \) is moderate (up to a few thousands) but becomes expensive for larger \( n \). In such instances, a reasonable strategy is to take \( a_1, \ldots, a_m \) as a random subsample of the data \( X_1, \ldots, X_n \). For fast implementations, one can also extend the idea of Koenker and Mizera [31] by binning the observations and weighting the likelihood terms in (1.2) by relative multinomial bin counts.

We shall now provide some graphical evidence of the effectiveness of the NPMLE for denoising. In all these plots, the NPMLE is approximately computed via the algorithm (4.2) where the \( a_1, \ldots, a_m \) are chosen to be the data points \( X_1, \ldots, X_n \) with \( m = n \) (i.e., we follow the exemplar recommendation of Böning [8] and Lashkari and Golland [33]). We use the software, Mosek, to solve the problem (4.2). The theorems of this paper do not apply directly to these approximate NPMLEs and extending them is the subject of future work. However, we shall argue via simulations that these approximate NPMLEs work well for denoising.

In Figure 1, we illustrate the performance of \( \hat{\theta}_1, \ldots, \hat{\theta}_n \) (defined as in (1.6)) for denoising when the true vectors \( \theta_1, \ldots, \theta_n \) take values in a bounded region of \( \mathbb{R}^2 \). The plots refer to these estimates as the Empirical Bayes estimates and the quantities (1.5) as the Oracle Bayes estimates. In each of the four subfigures in Figure 1, we generate \( n \) vectors \( \theta_1, \ldots, \theta_n \) from a bounded region in \( \mathbb{R}^d \) for \( d = 2 \): they are generated from two concentric circles in the first subfigure, a triangle in the second subfigure, the digit 8 in the third subfigure and the uppercase letter A in the last subfigure. Note that, in each of these cases, the empirical measure \( \hat{G}_n \) is supported on a bounded region so that Corollary 3.2 yields the near parametric rate \( 1/n \) up to logarithmic multiplicative factors in \( n \) for every NPMLE. In each of the subfigures in Figure 1, we plot the true parameter values \( \theta_1, \ldots, \theta_n \) in black, the data \( X_1, \ldots, X_n \) (generated independently according to \( X_i \sim N(\theta_i, I_d) \)) are plotted in gray, the Oracle Bayes estimates \( \hat{\theta}_1, \ldots, \hat{\theta}_n \) are plotted in blue while the estimates \( \hat{\theta}_1, \ldots, \hat{\theta}_n \) are plotted in red. The mean squared discrepancies:

\[
\frac{1}{n} \sum_{i=1}^{n} \| \hat{\theta}_i^* - \theta_i \|^2, \quad \frac{1}{n} \sum_{i=1}^{n} \| \hat{\theta}_i - \theta_i \|^2 \quad \text{and} \quad \frac{1}{n} \sum_{i=1}^{n} \| \hat{\theta}_i^* - \hat{\theta}_i \|^2
\]

are given in each figure in the legend at the upper-right corner. Note that the third MSE is much smaller than the other two in each subfigure.

As can be observed from Figure 1, the Empirical Bayes estimates (1.6) approximate their targets (1.5) quite well. The most noteworthy fact is that the estimates (1.6) do not require any knowledge of the underlying structure in \( \hat{G}_n \), for instance, concentric circles, or triangle or a letter of the alphabet etc. We should also note here that the noise distribution here is completely known to be \( N(0, I_d) \) which implies, in particular, that there is no unknown scale parameter representing the noise variance.

We shall now illustrate the denoising performance when the true vectors \( \theta_1, \ldots, \theta_n \) have a clustering structure. Here we take \( d = 2 \) and consider the following four simulation settings:

1. Setting One: We generate \( \theta_1, \ldots, \theta_n \) as i.i.d from the distribution which puts equal probability (0.5) at (0,0) and (2,2).
2. Setting Two: We generate \( \theta_1, \ldots, \theta_n \) as i.i.d from the distribution which puts 1/4 probability at (0,0) and 3/4 probability at (2,2).
3. Setting Three: We generate \( \theta_1, \ldots, \theta_n \) as i.i.d from the distribution which puts 1/4 probability each at (0,0) and (0,2) and 1/2 probability at (2,−2).
(a) **Two circles:** \( n = 1000. \) Half of \( \{ \theta_i \}_{i=1}^n \) are drawn uniformly at random from each of the concentric circles of radii 2 and 6 respectively.

(b) **Triangle:** \( n = 999. \) A third of \( \{ \theta_i \}_{i=1}^n \) are drawn uniformly at random from each edge of the triangle with vertices \((-3, 0), (0, 6)\) and \((3, 0)\)

(c) **Digit 8:** \( n = 1000. \) Half of \( \{ \theta_i \}_{i=1}^n \) are drawn uniformly at random from each of the circles of radii 3 centered at \((0, 0)\) and \((0, 6)\) respectively.

(d) **Letter A:** \( n = 1000. \) A fifth of \( \{ \theta_i \}_{i=1}^n \) are drawn uniformly at random from each of the line segments joining the points \((-4, -6), (-2, 0), (0, 6), (2, 0)\) and \((4, 6)\) so as to form the letter A.

Fig 1: Illustrations of denoising using the Empirical Bayes estimates (1.6)
4. Setting Four: We generate a random probability vector \((\alpha_1, \alpha_2, \alpha_3, \alpha_4)\) from the Dirichlet distribution with parameters \((1, 1, 1, 1)\) and then generate \(\theta_1, \ldots, \theta_n\) as i.i.d from the probability distribution with puts probabilities \(\alpha_1, \alpha_2, \alpha_3\) and \(\alpha_4\) at the four points \((0, 0), (0, 3), (3, 0)\) and \((3, 3)\).

The observed data \(X_1, \ldots, X_n\) are, as usual, generated independently as \(X_i \sim N(\theta_i, I_d)\). We allow the sample size \(n\) to range in the set \(\{300, 600, 900, 1200, 1500, 1800, 2100\}\). For each \(n\), we perform 1000 replicates to get accurate estimates of mean squared error. For each dataset, we compute the Empirical Bayes estimates \((1.6)\). For comparison, we also computed \(k\)-means estimates based on the true (Oracle) value of \(k\) and those based on the gap statistic (from Tibshirani et al. \[58\]). These estimates will be referred to, in the sequel, as \(k\text{-means-Oracle}\) and \(k\text{-means-gap}\) respectively. For \(k\)-means, we used the standard Lloyd’s algorithm based on 10 random starts and the best solution is considered of the random starts. Note that because of non-convexity, no implementation of \(k\)-means can provably reach global optimum.

For each of the these three estimates, we plotted the mean squared errors in Figure 2 (see the first plot in each pair of plots for the different settings). From these MSE plots, it is clear that the Empirical Bayes estimates based on the NPMLE are more accurate than \(k\text{-means-gap}\). In fact, with the exception of the first setting, the Empirical Bayes estimates are even more accurate than \(k\text{-means-Oracle}\). This is probably because of the non-convexity of \(k\)-means.

In addition to the MSE, we also compared the clusterings given by the different methods based on the Adjusted Rand Index (ARI) \[48\]. The Empirical Bayes is designed to work well for the squared error objective and not quite for the ARI. We plotted the average ARI of each of the three methods as well as the average ARI of the Oracle Bayes estimate. Higher ARIs are preferred to lower values. Here the Oracle Bayes estimate is the best; the \(k\text{-means-Oracle}\) method is superior to the Empirical Bayes estimate as well as \(k\text{-means-gap}\). The comparison between the Empirical Bayes and the \(k\text{-means-gap}\) estimates in terms of ARI can be summarized as follows. In the first setting, the performance of \(k\text{-means-gap}\) is very good and is indistinguishable from \(k\text{-means-Oracle}\). In more complicated settings with more than two clusters and/or with imbalanced cluster proportions, a distinction between the two methods becomes apparent. In the second and fourth settings, the Empirical Bayes method outperforms \(k\text{-means-gap}\). In the third setting, the performances of the two methods start to coincide for larger sample sizes.

5. Proofs of results in Section 2. The following notation will be used in the proofs in the sequel.

1. For \(x \in \mathbb{R}^d\) and \(a > 0\), let

\[ B(x, a) := \{u \in \mathbb{R}^d : \|u - x\| \leq a\} \]

denote the closed ball of radius \(a\) centered at \(x\).

2. For a subset \(S \subseteq \mathbb{R}^d\) and \(a > 0\), we denote the set \(S^a\) by

\[ S^a := \cup_{x \in S} B(x, a) = \{y : d_S(y) \leq a\} \]

where \(d_S(\cdot)\) is defined as in \((2.1)\).

3. For a compact subset \(S\) of \(\mathbb{R}^d\) and \(\epsilon > 0\), we denote the \(\epsilon\)-covering number of \(S\) in the usual Euclidean distance by \(N(\epsilon, S)\) i.e., \(N(\epsilon, S)\) stands for the smallest number of closed balls of radius \(\epsilon\) whose union contains \(S\).

4. Given a pseudometric \(\varrho\) on \(\mathcal{M}\), let \(N(\epsilon, \mathcal{M}, \varrho)\) denote the \(\epsilon\)-covering number of \(\mathcal{M}\) under the pseudometric \(\varrho\) by \(N(\epsilon, \mathcal{M}, \varrho)\) i.e., \(N(\epsilon, \mathcal{M}, \varrho)\) denotes the smallest positive integer \(N\) for
(a) Setting 1. Two equally sized clusters centered at $(0,0)$ and $(2,2)$. For clarification, in the ARI plot the red and green curves coincide.

(b) Setting 2. Two clusters centered at $(0,0)$ and $(2,2)$ with cluster proportions $1/4$ and $3/4$. For clarification, in the ARI plot the red and green curves coincide.

(c) Setting 3. Three clusters centered at $(0,0), (0,2), (2,-2)$ with cluster proportions $1/4, 1/4, 1/2$ respectively.

(d) Setting 4. Four cluster centers centered at $(0,0), (0,3), (3,0), (3,3)$ with cluster proportions drawn from Dirichlet distribution with parameters $(1,1,1,1)$.

Fig 2: Empirical performance of methods in the denoising problem in four different clustering settings. A method with lower MSE is preferred over one with higher MSE. In contrast, a method with higher ARI is preferred over one with lower ARI. The lines show mean of the metric in question over 1000 replicates.
which there exist densities \( f_1, \ldots, f_N \in \mathcal{M} \) satisfying
\[
\sup_{f \in \mathcal{M}} \inf_{1 \leq i \leq N} \varrho(f, f_i) \leq \epsilon.
\]

In the proof below, we will be concerned with \( N(\epsilon, \mathcal{M}, \varrho) \) for the following choice of \( \varrho \). For a compact set \( S \), let \( \|\cdot\|_{\infty, S} \) denote the pseudonorm on \( \mathcal{M} \) defined by
\[
\|f\|_{\infty, S} := \sup_{x \in S} |f(x)|.
\]

This pseudonorm naturally induces a pseudometric on \( \mathcal{M} \) given by \( \varrho(f, g) := \|f - g\|_{\infty, S} \). The covering number for this pseudometric will be denoted by \( N(\epsilon, \mathcal{M}, \|\cdot\|_{\infty, S}) \). In the proofs for the results in Section 3, we will need to deal with covering numbers for other pseudometrics \( \varrho \) on \( \mathcal{M} \) as well. These pseudometrics will be introduced in Section 6.

With the above notation in place, we are now ready to give the proof of Theorem 2.1. This proof uses additional ingredients which are proved in later sections. Arguably the most important ingredient for the proof of this theorem is a bound on the covering numbers \( N(\epsilon, \mathcal{M}, \|\cdot\|_{\infty, S}) \). These bounds are given in Section 7 (specifically inequality (7.1) in Theorem 7.1 will be used). Other ingredients include inequality (A.13) (which is a consequence of Lemma A.4) and a standard fact (Lemma A.8) giving a volumetric upper bound for Euclidean covering numbers.

5.1. Proof of Theorem 2.1.

Proof of Theorem 2.1. We shall prove inequalities (2.5) and (2.6) under the assumption that the sample size \( n \) satisfies
\[
n \geq \max \left( \exp \left( \frac{d + 1}{2} \right), \frac{1}{2} (2\pi)^{(d-1)/2} \right).
\]

If (5.3) is not satisfied, then \( \epsilon_n(M, S) \) and also the larger quantity \( \epsilon_n(M, S)/\min(1 - \alpha, \beta) \) will be bounded from below by a positive constant \( \kappa_d \). We can then therefore choose \( C_d \) in (2.5) and (2.6) large enough so that \( \epsilon_n(M, S) \sqrt{C_d} > \sqrt{2} \min(1 - \alpha, \beta) \). Because the Hellinger distance \( H(\hat{f}_n, \bar{G}_n) \) is always bounded from above by \( \sqrt{2} \), the probability on the left hand side of (2.5) will then equal zero so that (2.5) holds trivially. Inequality (2.6) will also be trivial because its right hand side will then be larger than 2.

Let us therefore fix \( n \) satisfying (5.3). Fix a positive sequence \( \{\gamma_n\} \) and assume that \( \hat{f}_n \) satisfies
\[
\prod_{i=1}^{n} \frac{\hat{f}_n(X_i)}{f_{\bar{G}_n}(X_i)} \geq \exp \left( (\beta - \alpha) n \gamma_n^2 \right) \quad \text{for some } 0 < \beta \leq \alpha < 1.
\]

We shall then bound the probability
\[
\mathbb{P}\{H(\hat{f}_n, \bar{G}_n) \geq t\gamma_n\}
\]
for \( t \geq 1 \).

Fix a non-empty compact set \( S \subseteq \mathbb{R}^d \) and \( M \geq \sqrt{10 \log n} \). We shall work with the set \( S^M \) (defined as in (5.1)) and the pseudometric given by the pseudonorm \( \|\cdot\|_{\infty, S^M} \) (defined as in (5.2)).

Let \( \eta := 1/n^2 \) and let \( \{h_1, \ldots, h_N\} \subseteq \mathcal{M} \) denote an \( \eta \)-covering set of \( \mathcal{M} \) in the pseudometric given by \( \|\cdot\|_{\infty, S^M} \) where \( N = N(\eta, \mathcal{M}, \|\cdot\|_{\infty, S^M}) \) i.e.,
\[
\sup_{h \in \mathcal{M}} \inf_{1 \leq j \leq N} \|h - h_j\|_{\infty, S^M} \leq \eta.
\]
Inequality (7.1) in Theorem 7.1 gives an upper bound for $N$ that will be crucially used in this proof.

Let $J$ denote the set of all $j \in \{1, \ldots, N\}$ for which there exists a density $h_{0j} \in \mathcal{M}$ satisfying

$$\|h_{0j} - h_j\|_{\infty, SM} \leq \eta \text{ and } \delta_{\beta}(h_{0j}, f_{\bar{G}_n}) \geq t_{\gamma_n}.$$  

Because $h_1, \ldots, h_N$ cover $\mathcal{M}$, there will exist $1 \leq j \leq N$ such that $\|h_j - \hat{f}_n\|_{\infty, SM} \leq \eta$. If $\delta_{\beta}(\hat{f}_n, f_{\bar{G}_n}) \geq t_{\gamma_n}$, then $j \in J$ and consequently

$$\|\hat{f}_n - h_{0j}\|_{\infty, SM} \leq 2\eta.$$  

We now define a function $v := v_{S,M} : \mathbb{R}^d \to (0, \infty)$ via

$$v(x) := \begin{cases} \eta & \text{if } x \in S^M \\ \eta \left(\frac{M}{\delta_S(x)}\right)^{d+1} & \text{otherwise} \end{cases}$$

where $\delta_S : \mathbb{R}^d \to [0, \infty)$ is defined as in (2.1).

Inequality (5.5) clearly implies that $\hat{f}_n(X_i) \leq h_{0j}(X_i) + 2\eta = h_{0j}(X_i) + 2v(X_i)$ whenever $X_i \in S^M$ which allows us to write

$$\prod_{i=1}^{n} \hat{f}_n(X_i) \leq \prod_{i:X_i \in S^M} \{h_{0j}(X_i) + 2v(X_i)\} \prod_{i:X_i \notin S^M} (2\pi)^{-d/2}$$

where we used the bound $\hat{f}_n(X_i) \leq \sup_x \hat{f}_n(x) \leq (2\pi)^{-d/2}$ for $X_i \notin S^M$ (the bound $\sup_x f(x) \leq (2\pi)^{-d/2}$ holds for every $f \in \mathcal{M}$ as can easily be seen). From here, we deduce

$$\prod_{i=1}^{n} \hat{f}_n(X_i) \leq \prod_{i=1}^{n} \{h_{0j}(X_i) + 2v(X_i)\} \prod_{i:X_i \notin S^M} \frac{(2\pi)^{-d/2}}{h_{0j}(X_i) + 2v(X_i)}$$

$$\leq \prod_{i=1}^{n} \{h_{0j}(X_i) + 2v(X_i)\} \prod_{i:X_i \notin S^M} \frac{(2\pi)^{-d/2}}{2v(X_i)}$$

We have therefore proved that the inequality

$$\prod_{i=1}^{n} \frac{\hat{f}_n(X_i)}{f_{\bar{G}_n}(X_i)} \leq \max_{j \in J} \prod_{i=1}^{n} \frac{h_{0j}(X_i) + 2v(X_i)}{f_{\bar{G}_n}(X_i)} \prod_{i:X_i \notin S^M} \frac{(2\pi)^{-d/2}}{2v(X_i)}$$

holds on the event $\delta_{\beta}(\hat{f}_n, f_{\bar{G}_n}) \geq t_{\gamma_n}$.

Because $\hat{f}_n$ satisfies (5.4), we obtain

$$\mathbb{P}\left(\delta_{\beta}(\hat{f}_n, f_{\bar{G}_n}) \geq t_{\gamma_n}\right) \leq \mathbb{P}\left\{ \max_{j \in J} \prod_{i=1}^{n} \frac{h_{0j}(X_i) + 2v(X_i)}{f_{\bar{G}_n}(X_i)} \prod_{i:X_i \notin S^M} \frac{(2\pi)^{-d/2}}{2v(X_i)} \geq \exp\left(\frac{t_{\gamma_n}}{\sqrt{\alpha_n t_n^2 \gamma_n^2}}\right) \right\}$$

$$\leq \mathbb{P}\left\{ \max_{j \in J} \prod_{i=1}^{n} \frac{h_{0j}(X_i) + 2v(X_i)}{f_{\bar{G}_n}(X_i)} \geq e^{-\alpha_n t_n^2 \gamma_n^2} \right\}$$

$$+ \mathbb{P}\left\{ \prod_{i:X_i \notin S^M} \frac{(2\pi)^{-d/2}}{2v(X_i)} \geq e^{\beta n t_n^2 \gamma_n^2} \right\}.$$  

(5.7)
We shall bound the two probabilities above separately. For the first probability:

\[
\mathbb{P} \left\{ \max_{j \in J} \prod_{i=1}^{n} \frac{h_{0j}(X_i) + 2v(X_i)}{f_{\tilde{G}_n}(X_i)} \geq e^{-\alpha n^2 \gamma_n^2} \right\} \leq \sum_{j \in J} \mathbb{P} \left\{ \prod_{i=1}^{n} \frac{h_{0j}(X_i) + 2v(X_i)}{f_{\tilde{G}_n}(X_i)} \geq e^{-\alpha n^2 \gamma_n^2} \right\}
\]

\[
\leq e^{\alpha n^2 \gamma_n^2 / 2} \sum_{j \in J} \mathbb{E} \left\{ \prod_{i=1}^{n} \frac{h_{0j}(X_i) + 2v(X_i)}{f_{\tilde{G}_n}(X_i)} \right\} = e^{\alpha n^2 \gamma_n^2 / 2} \sum_{j \in J} \mathbb{E} \left\{ \prod_{i=1}^{n} \frac{h_{0j}(X_i) + 2v(X_i)}{f_{\tilde{G}_n}(X_i)} \right\}.
\]

Now for each fixed \( j \in J \), we have

\[
\prod_{i=1}^{n} \mathbb{E} \left\{ \frac{h_{0j}(X_i) + 2v(X_i)}{f_{\tilde{G}_n}(X_i)} \right\} = \exp \left( \sum_{i=1}^{n} \log \mathbb{E} \left\{ \frac{h_{0j}(X_i) + 2v(X_i)}{f_{\tilde{G}_n}(X_i)} \right\} \right)
\]

\[
\leq \exp \left( \sum_{i=1}^{n} \mathbb{E} \left\{ \frac{h_{0j}(X_i) + 2v(X_i)}{f_{\tilde{G}_n}(X_i)} - n \right\} \right)
\]

\[
\leq \exp \left( \sum_{i=1}^{n} \int \sqrt{\frac{h_{0j} + 2v}{f_{\tilde{G}_n}}} - n \right) = \exp \left( n \int \sqrt{\frac{h_{0j} + 2v}{f_{\tilde{G}_n}}} - n \right).
\]

Because of \( \sqrt{\alpha + \beta} \leq \sqrt{\alpha} + \sqrt{\beta} \) and the Cauchy-Schwartz inequality (along with \( \int f_{\tilde{G}_n} = 1 \)), we obtain

\[
\int \sqrt{(h_{0j} + 2v)f_{\tilde{G}_n}} \leq \int \sqrt{h_{0j}f_{\tilde{G}_n}} + \sqrt{2} \int \sqrt{vf_{\tilde{G}_n}}
\]

\[
\leq \int \sqrt{h_{0j}f_{\tilde{G}_n}} + \sqrt{2} \sqrt{\int v = 1 - \frac{1}{2} S_j^2(h_{0j}, f_{\tilde{G}_n}) + \sqrt{2} \sqrt{\int v}}.
\]

We now use Lemma A.7 which gives an upper bound on \( \int v \). This (along with the fact that \( S_j(h_{0j}, f_{\tilde{G}_n}) \geq t \gamma_n \)) allows us to deduce:

\[
\int \sqrt{(h_{0j} + 2v(X_i))f_{\tilde{G}_n}} \leq 1 - \frac{t^2}{2} \gamma_n^2 + C_d \sqrt{2 \eta \text{Vol}(S^M)}.
\]

We have therefore proved that

\[
\mathbb{P} \left\{ \max_{j \in J} \prod_{i=1}^{n} \frac{h_{0j}(X_i) + 2v(X_i)}{f_{\tilde{G}_n}(X_i)} \geq e^{-\alpha n^2 \gamma_n^2} \right\} \leq \exp \left( \frac{\alpha}{2} n t^2 \gamma_n^2 + \log |J| - \frac{1}{2} n t^2 \gamma_n^2 + n C_d \sqrt{\eta \text{Vol}(S^M)} \right)
\]

\[
\leq \exp \left( \frac{\alpha - 1}{2} n t^2 \gamma_n^2 + \log N + C_d \sqrt{\text{Vol}(S^M)} \right)
\]

(5.8)

because \( \eta := n^{-2} \) and \( |J| \leq N \) (as \( J \subseteq \{1, \ldots, N\} \)).

We now use the upper bound on \( N \) from inequality (7.1) in Theorem 7.1. Because \( \eta = 1/n^2 \) and \( n \geq 2 \), the quantity \( a \) appearing in Theorem 7.1 satisfies

\[
a = \sqrt{2 \log \frac{2\sqrt{2\pi}}{(2\pi)^{d/2} n}} = \sqrt{2 \log \frac{2\sqrt{2\pi}}{(2\pi)^{d/2}} + 4 \log n} \leq \sqrt{6 \log n}.
\]
Also because of (5.3), we have $2n \geq (2\pi)^{(d-1)/2}$ so that

$$a = \sqrt{2 \log \frac{2\sqrt{2\pi}}{(2\pi)^{d/2} \eta}} = \sqrt{2 \log \frac{2\sqrt{2\pi}}{(2\pi)^{d/2}}} + 4 \log n \geq \sqrt{2 \log(1/n)} + 4 \log n = \sqrt{2 \log n}.$$  

Thus Theorem 7.1 gives

$$\log N \leq C_d N(a, (S^M)^a)(\log n)^2 \leq C_d N(\sqrt{2 \log n}, (S^M + \sqrt{10 \log n})(\log n)^2).$$

Using Lemma A.8 to bound the Euclidean covering number appearing in the right hand side above, we deduce that

$$N(\sqrt{2 \log n}, (S^M + \sqrt{10 \log n})) \leq C_d(\sqrt{2 \log n})^{-d} Vol(S^M + \sqrt{10 \log n} + \sqrt{2 \log n} / 2) \leq C_d(\log n)^{-d/2} Vol(S^M + \sqrt{10 \log n}) \leq C_d(\log n)^{-d/2} Vol(S^M)$$

as $M \geq \sqrt{10 \log n}$. Thus

$$\log N \leq C_d(\log n)^2 - (d/2) Vol(S^M).$$

Using the above in (5.8), we obtain

$$\mathbb{P} \left\{ \max_{j \in J} \prod_{i=1}^n \frac{h_{ij}(X_i) + 2v(X_i)}{f_{G_a}(X_i)} \geq e^{-\alpha nt^2 \gamma_n^2} \right\} \leq \exp \left( \frac{\alpha - 1}{2} nt^2 \gamma_n^2 \right) + C_d(\log n)^2 - (d/2) Vol(S^M) + C_d \sqrt{Vol(S^M)}.$$

We shall now bound the second probability in (5.7). First observe, by Markov’s inequality, that

$$\mathbb{P} \left\{ \prod_{i : X_i \notin S^M} \frac{(2\pi)^{-d/2}}{2v(X_i)} \geq e^{\beta nt^2 \gamma_n^2} \right\} \leq \exp \left( -\frac{\beta nt^2 \gamma_n^2}{2 \log n} \right) \mathbb{E} \left( \prod_{i : X_i \notin S^M} \frac{(2\pi)^{-d/2}}{2v(X_i)} \right)^{1/(2 \log n)}$$

The expectation above can be bounded as (recall the formula for $v(\cdot)$ from (5.6))

$$\mathbb{E} \left( \prod_{i : X_i \notin S^M} \frac{(2\pi)^{-d/2}}{2v(X_i)} \right)^{1/(2 \log n)} \leq \mathbb{E} \left( \prod_{i : X_i \notin S^M} \frac{1}{v(X_i)} \right)^{1/(2 \log n)} = \mathbb{E} \left( \prod_{i : X_i \notin S^M} \frac{\delta_S(X_i)}{M \eta^{1/(d+1)}} \right)^{(d+1)/(2 \log n)} = \mathbb{E} \left[ \prod_{i=1}^n \left( \frac{\delta_S(X_i)}{M \eta^{1/(d+1)}} \right)^{I\{\delta_S(X_i) \geq M\}} \right]^{(d+1)/(2 \log n)}$$

The above term will be controlled below by using inequality (A.13) (which is a consequence of Lemma A.4) with

$$a := \frac{1}{M \eta^{1/(d+1)}} \quad \text{and} \quad \lambda := \frac{d + 1}{2 \log n}$$
As a result, gives which follows from inequality (A.27) in Lemma A.8. This, along with the definition of $\epsilon$

We now note that

We have proved therefore that for every $p$

Therefore the second probability in (5.7) satisfies the inequality:

Thus (5.11) holds for $p \geq (d + 1)/(2 \log n)$ so we can also write

We have proved therefore that for every $t > 0$

We now note that

which follows from inequality (A.27) in Lemma A.8. This, along with the definition of $\epsilon_n^2(M, S)$ in (2.3), gives

As a result,
Now suppose that
\begin{equation}
\gamma_n^2 = C'_d \frac{\epsilon_n^2(M, S)}{\min(1 - \alpha, \beta)}
\end{equation}
for some $C'_d \geq 4C_d$. We deduce then that, for every $t \geq 1$,
\begin{align}
\mathbb{P}\left\{ \hat{f}(\hat{f}_n, f_{\bar{G}_n}) \geq t\gamma_n \right\} & \leq \exp\left(-\frac{1 - \alpha}{2} nt^2\gamma_n^2 + \frac{1 - \alpha}{4} n\gamma_n^2\right) + \exp\left(-\frac{\beta}{2 \log n} nt^2\gamma_n^2 + \frac{\beta}{4 \log n} n\gamma_n^2\right) \\
& \leq 2 \exp\left(-\frac{\min((1 - \alpha), \beta)}{4 \log n} nt^2\gamma_n^2\right)
\end{align}
(5.13)
Observe now that (because $M \geq \sqrt{10 \log n}$)
\begin{align*}
\epsilon_n^2(M, S) & = \text{Vol}(S^1) \frac{M^d}{n} \left(\sqrt{\log n}\right)^{(4-d)_+} \geq \text{Vol}(B(0, 1)) \frac{(\log n)^2}{n}
\end{align*}
so that we can choose the constant $C'_d$ such that
\begin{align*}
n \min(1 - \alpha, \beta) \gamma_n^2 & \geq C'_d n \epsilon_n^2(M, S) \geq 4(\log n)^2.
\end{align*}
This gives, via (5.13),
\begin{align*}
\mathbb{P}\left\{ \hat{f}(\hat{f}_n, f_{\bar{G}_n}) \geq t\gamma_n \right\} & \leq 2 n^{-t^2}.
\end{align*}
We have therefore proved the above inequality for $\gamma_n$ as chosen in (5.12) (provided $C'_d$ is chosen sufficiently large) for every estimator $\hat{f}_n$ satisfying (5.4). This completes the proof of (2.5).

For (2.6), we multiply both sides of (2.5) by $t$ and then integrate from $t = 1$ to $t = \infty$ to obtain
\begin{align*}
\mathbb{E}\left( \hat{f}(\hat{f}_n, f_{\bar{G}_n}) \min(1 - \alpha, \beta) \right) & \leq 1 + 4 \int_1^\infty tn^{-t^2} dt \leq 1 + \frac{2}{n \log n} \leq 4
\end{align*}
which proves (2.6) and completes the proof of Theorem 2.1.

5.2. Proof of Corollary 2.2.

To prove (2.7), assume that $\bar{G}_n$ is supported on a compact set $S$. We then apply Theorem 2.1 to this $S$ and $M = \sqrt{10 \log n}$. Because $\bar{G}_n$ is supported on $S$, we have $\mu_p(\partial S) = 0$ for every $p > 0$ so that $\epsilon_n^2(M, S)$ (defined in (2.3)) becomes
\begin{align*}
\epsilon_n^2(M, S) & = \text{Vol}(S^1) \frac{M^d}{n} \left(\sqrt{\log n}\right)^{(4-d)_+} = \frac{\text{Vol}(S^1)}{n} \left(\sqrt{\log n}\right)^{d+(4-d)_+}.
\end{align*}
Inequality (2.7) then immediately follows from Theorem 2.1.

We next prove (2.9) assuming the condition (2.8). Let
\begin{equation}
M := 4K(e \log n)^{1/\alpha}.
\end{equation}
This quantity $M \geq \sqrt{10 \log n}$ because $K \geq 1$ and $\alpha \leq 2$. We shall apply (2.6) with this $M$. Let
\begin{align*}
T_2(M, S) := (\log n) \inf_{p \geq \frac{d+1}{2 \log n}} \left(\frac{2 \mu_p(\partial S)}{M}\right)^p
\end{align*}
be the second term on the right hand side of (2.3) in the definition of $\epsilon_n^2(M, S)$. The infimum over $p$ above is easily seen to be achieved at $p = (M/(2K))^{\alpha}(1/e)$. By the expression (5.14) for $M$, it is easy to see that $p \geq (d + 1)/(2 \log n)$ provided

$$n \geq \exp\left(\sqrt{(d + 1)/2}\right).$$

We then deduce that

$$T_2(M, S) \leq (\log n)^{\alpha} \left(\frac{2K}{M}\right)^{\alpha}.$$  

It follows from here that $T_2(M, S) \leq (\log n)/n$ because $M \geq (4K)(e \log n)^{1/\alpha} \geq (2K)(ae \log n)^{1/\alpha}$. Thus

$$\epsilon_n^2(M, S) = \text{Vol}(S^1) \left(\frac{M^{d/\alpha}}{n} \left(\sqrt{\log n}\right)^{d/\alpha} + T_2(M, S)\right)$$

and hence (2.9) readily follows as a consequence of Theorem 2.1. When the assumption (5.15) does not hold, inequality (2.9) becomes trivially true when $C_d$ is chosen sufficiently large.

We now turn to (2.10). Assume that $S$ is such that $\mu_p(\partial S) \leq \mu$ for fixed $\mu > 0$ and $p > 0$. Then Theorem 2.1 gives

$$\mathbb{E}S^2(f_n, f_{G_n}) \leq C_d \inf_{M \geq \sqrt{10 \log n}} \epsilon_n^2(M, S)$$

where we assumed that $n$ is large enough so that $p \geq (d + 1)/(2 \log n)$. Taking

$$M = \left(\sqrt{\log n}\right)^{(2-(4-d))/(p+d)} \left(\frac{n\mu p}{\text{Vol}(S^1)}\right)^{1/(p+d)}$$

results in (2.10). When $n$ is large enough, $M$ chosen as above exceeds $\sqrt{10 \log n}$. For smaller $n$, the inequality (2.10) trivially holds provided $C_{d, \mu, p}$ is chosen large enough.

5.3. Proof of Proposition 2.3. This directly follows from inequality (2.7). Suppose that $\bar{G}_n$ is supported on a finite set $S$ of cardinality $k$. We then apply inequality (2.7) to this $S$. It is easy to see then that $\text{Vol}(S^1) \leq C_d k$ which proves (2.11).

5.4. Proof of Lemma 2.4. The following uses standard ideas involving Assouad’s lemma (see, for example, Tsybakov [60, Chapter 2].

**Proof of Lemma 2.4.** Fix $\delta > 0$ and $M > 0$. Let $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ be points in $\mathbb{R}^d$ such that

$$\min_{i \neq j} \|a_i - a_j\|, \min_{i \neq j} \|b_i - b_j\|, \min_{i \neq j} \|a_i - b_j\| \geq M$$

and such that

$$\|a_i - b_i\| = \delta$$

for every $1 \leq i \leq k$. 
Now for every $\tau \in \{0, 1\}^k$, let

$$f_\tau(x) = \frac{1}{k} \sum_{i=1}^k \phi_d(x - a_i(1 - \tau_i) - b_i\tau_i)$$

where $\phi_d(\cdot)$ is the standard normal density on $\mathbb{R}^d$. Clearly $f_\tau \in \mathcal{M}_k$ for every $\tau \in \{0, 1\}^k$. We shall now employ Assouad’s lemma which gives

$$\mathcal{R}(\mathcal{M}_k) \geq \frac{k}{8} \min_{\tau \neq \tau'} \frac{\delta^2(f_\tau, f_{\tau'})}{\Upsilon(\tau, \tau')} \min_{\tau(\tau, \tau') = 1} \left(1 - \|P_{f_\tau} - P_{f_{\tau'}}\|_{TV}\right)$$

where $\Upsilon(\tau, \tau') := \sum_{i=1}^k I\{\tau_j \neq \tau_j'\}$ denotes Hamming distance and $P_f$ (for $f \in \mathcal{M}$) denotes the joint distribution of $X_1, \ldots, X_n$, which are independently distributed according to $f$. We now fix $\tau \neq \tau' \in \{0, 1\}^k$ and bound $\delta^2(f_\tau, f_{\tau'})$ from below. For simplicity, let $f = f_\tau$ and $g = f_{\tau'}$. Also, for $i = 1, \ldots, k$, let

$$f_i(x) := \phi_d(x - a_i(1 - \tau_i) - b_i\tau_i) \quad \text{and} \quad g_i(x) := \phi_d(x - a_i(1 - \tau_i') - b_i\tau_i')$$

so that $f = \sum_{i=1}^k f_i/k$ and $g = \sum_{i=1}^k g_i/k$. This gives

$$\frac{1}{2}\delta^2(f, g) = 1 - \int \sqrt{f(x)g(x)}\,dx = 1 - \frac{1}{k^2} \sum_{i,j} f_i(x)g_j(x)dx \geq 1 - \frac{1}{k} \sum_{i,j} \int \sqrt{f_i(x)}g_j(x)dx$$

Because $f_i$ and $g_j$ are normal densities, by a straightforward computation, we obtain

$$\int \sqrt{f_i(x)}g_j(x)dx = \exp \left(-\|a_i(1 - \tau_i) + b_i\tau_i - a_j(1 - \tau_j') - b_j\tau_j'\|^2/8\right)$$

so that by (5.16) and (5.17), we obtain that

$$\int \sqrt{f_i(x)}g_j(x)dx = I\{\tau_i = \tau_j'\} + I\{\tau_i \neq \tau_j'\}e^{-\delta^2/8} \quad \text{for } i = j$$

and

$$\int \sqrt{f_i(x)}g_j(x)dx \leq e^{-M^2/8} \quad \text{for } i \neq j.$$ 

As a result, we obtain

$$\frac{1}{2}\delta^2(f_\tau, f_{\tau'}) = 1 - \frac{1}{k} \sum_{i=1}^k \int \sqrt{f_i(x)}g_i(x)dx - \frac{1}{k^2} \sum_{i \neq j} \int \sqrt{f_i(x)}g_j(x)dx$$

$$\geq 1 - \frac{1}{k} \sum_{i=1}^k I\{\tau_i = \tau_j'\} - \frac{e^{-\delta^2/8}}{k} \Upsilon(\tau, \tau') - \frac{k^2 - k}{k} e^{-M^2/8}$$

$$= \frac{1}{k} \Upsilon(\tau, \tau') \left(1 - e^{-\delta^2/8}\right) - (k - 1) e^{-M^2/8}$$

(5.18)

for every $\tau \neq \tau' \in \{0, 1\}^k$. Now let us fix $\tau, \tau'$ with $\Upsilon(\tau, \tau') = 1$ and bound from above the total variation distance between $P_{f_\tau}$ and $P_{f_{\tau'}}$. Without loss of generality, we can assume that $\tau_1 \neq \tau_1'$ and that $\tau_i = \tau_i'$ for $i \geq 2$. Below $D(P_{f_\tau}||P_{f_{\tau'}})$ denotes the Kullback-Leibler divergence between $P_{f_\tau}$ and $P_{f_{\tau'}}$. Also $D(f_\tau||f_{\tau'})$ and $\chi^2(f_\tau, f_{\tau'})$ denote the Kullback-Leibler divergence and chi-squared
divergence between the densities $f_r$ and $f_{r'}$ respectively. By Pinsker’s inequality and the fact that $D(f_r\|f_{r'}) \leq \chi^2(f_r,f_{r'})$, we obtain

$$\|P_{f_r} - P_{f_{r'}}\|_{TV} \leq \sqrt{\frac{1}{2} D(P_{f_r}||P_{f_{r'}})} = \sqrt{\frac{n}{2} D(f_r||f_{r'})} \leq \sqrt{\frac{n}{2} \chi^2(f_r||f_{r'})}.$$  

Further

$$\chi^2(f_r||f_{r'}) = \int \frac{(f_r(x) - f_{r'}(x))^2}{f_{r'}(x)} dx$$

$$= \frac{1}{k^2} \int \frac{(\phi_d(x - a_1(1 - \tau_1) - b_1\tau_1) - \phi_d(x - a_1(1 - \tau'_1) - b_1\tau'_1))^2}{\phi_d(x - a_1(1 - \tau'_1) - b_1\tau'_1)} dx$$

$$\leq \frac{1}{k} \int \frac{(\phi_d(x - a_1(1 - \tau_1) - b_1\tau_1) - \phi_d(x - a_1(1 - \tau'_1) - b_1\tau'_1))^2}{\phi_d(x - a_1(1 - \tau'_1) - b_1\tau'_1)} dx.$$  

By a routine calculation, it now follows that

$$\chi^2(f_r||f_{r'}) \leq \frac{1}{k} \left\{ \exp \left( ||a_1(1 - \tau_1) + b_1\tau_1 - a_1(1 - \tau'_1) - b_1\tau'_1||^2 \right) - 1 \right\}$$

$$= \frac{1}{k} \left\{ \exp \left( ||a_1 - b_1||^2 \right) - 1 \right\} = \frac{1}{k} \left( e^{\delta^2} - 1 \right).$$

We have therefore proved that

$$\|P_{f_r} - P_{f_{r'}}\|_{TV} \leq \sqrt{\frac{n}{2k} \left( e^{\delta^2} - 1 \right)} \quad \text{for every } \tau, \tau' \in \{0, 1\}^k \text{ with } \Upsilon(\tau, \tau') = 1.$$  

Combining (5.18) and (5.19), we obtain

$$\mathcal{R}(\mathcal{M}_k) \geq \frac{k}{4} \left( 1 - e^{-\delta^2/8} \right) - \frac{(k - 1)}{\Upsilon(\tau, \tau')} \left( e^{-M^2/8} \right) \left( 1 - \sqrt{\frac{n}{2k} \left( e^{\delta^2} - 1 \right)} \right).$$

This inequality holds for every $\delta > 0$ and $M > 0$. So we can let $M$ tend to $\infty$ to deduce

$$\mathcal{R}(\mathcal{M}_k) \geq \frac{1}{4} \left( 1 - e^{-\delta^2/8} \right) \left( 1 - \sqrt{\frac{n}{2k} \left( e^{\delta^2} - 1 \right)} \right)$$

for every $\delta > 0$. The inequalities $1 - e^{-t} \geq t/2$ and $e^t - 1 \leq 2t$ for $0 \leq t \leq 1$ imply that

$$\mathcal{R}(\mathcal{M}_k) \geq \frac{\delta^2}{64} \left( 1 - \sqrt{\frac{n}{k} \delta} \right) \quad \text{for every } 0 \leq \delta \leq 1.$$  

The choice $\delta = \sqrt{k/4n}$ now proves (2.13). \hfill \Box

5.5. Proof of Proposition 2.5.

Proof of Proposition 2.5. Note that

$$h^*(x) = \sum_{j=1}^k w_j \phi_d(x; \mu_j, \Sigma_j) = \sum_{j=1}^k w_j \det(\Sigma_j^{-1/2}) \phi_d \left( \Sigma_j^{-1/2} (x - \mu_j) \right)$$
where \( \phi_d(z) := (2\pi)^{-d/2} \exp\left(-\|z\|^2/2\right) \) denotes the standard \( d \)-dimensional normal density. It is then easy to see that \( X_1, \ldots, X_n \) (where \( X_i = Y_i/\sigma_{\min} \)) are independent observations having the density \( f^* \) where

\[
f^*(x) = \sigma_{\min}^d h^*(\sigma_{\min}x) = \sum_{j=1}^k w_j \left[ \det \left( \sigma_{\min}^{-2} \Sigma_j \right)^{-1/2} \right] \phi_d \left( \left\{ \sigma_{\min}^{-2} \Sigma_j \right\}^{-1} (x - \sigma_{\min}^{-1} \mu_j) \right).
\]

This means that \( f^* \) is the density of the normal mixture:

\[
\sum_{j=1}^k w_j N\left( \sigma_{\min}^{-1} \mu_j, \sigma_{\min}^{-2} \Sigma_j \right)
\]

where \( N(\mu, \Sigma) \) denotes the multivariate normal distribution with mean vector \( \mu \) and covariance matrix \( \Sigma \). It follows from here that \( f^* \) equals \( f_{G^*} \) (in the notation (1.1)) where \( G^* \) is the distribution of the normal mixture

\[
\sum_{j=1}^k w_j N\left( \sigma_{\min}^{-1} \mu_j, \sigma_{\min}^{-2} \Sigma_j - I_d \right)
\]

where \( I_d \) is the \( d \times d \) identity matrix.

We can now use Corollary 2.2 to bound \( H_2(\hat{f}_n, f^*) \) (note that \( \hat{f}_n \) is an NPMLE based on \( X_1, \ldots, X_n \)). Specifically we shall use inequality (2.9) with \( S := \left\{ \sigma_{\min}^{-1} \mu_1, \ldots, \sigma_{\min}^{-1} \mu_k \right\} \).

In order to verify (2.8), observe first that \( \bar{G}_n \) in Corollary 2.2 is \( G^* \) since \( X_1, \ldots, X_n \) are i.i.d \( f_{G^*} \) and that

\[
d_S(\theta) = \min_{1 \leq i \leq k} \| \sigma_{\min}^{-1} \mu_i - \theta \|
\]

As a result, for every \( p \geq 1 \) and \( Z \sim N(0, I_d) \), we have

\[
\mu_p(d_S) \leq \left( \mathbb{E} \max_{1 \leq j \leq k} \left\| (\sigma_{\min}^{-2} \Sigma_j - I_d)^{1/2} Z \right\|^p \right)^{1/p} \leq \sqrt{\frac{\sigma_{\max}^2}{\sigma_{\min}^2} - 1} \left( \mathbb{E} \|Z\|^p \right)^{1/p} \leq C_d \tau \sqrt{p}.
\]

Thus (2.8) holds with \( K := C_d \max(1, \tau) \) and \( \alpha = 2 \) and inequality (2.9) then gives

\[
\mathbb{E} \delta^2(\hat{f}_n, f^*) \leq C_d \frac{\text{Vol}(S^1)}{n} \left( \max(1, \tau) \right)^d \left( \sqrt{\log n} \right)^{d+(4-d)_+}.
\]

As \( S \) is a finite set of cardinality \( k \), we have \( \text{Vol}(S^1) \leq k C_d \) so that

\[
\mathbb{E} \delta^2(\hat{f}_n, f^*) \leq C_d \left( \frac{k}{n} \right) \left( \max(1, \tau) \right)^d \left( \sqrt{\log n} \right)^{d+(4-d)_+}.
\]

We now use the fact that the Hellinger distance is invariant under scale transformations which implies that \( \delta(\hat{f}_n, f^*) = \delta(h_n, h^*) \). This proves inequality (2.17).

\[
\square
\]

6. Proofs of Results in Section 3.
6.1. Proof of Theorem 3.1. The proof of Theorem 3.1 is similar to Jiang and Zhang [28, Proof of Theorem 5]. It uses ingredients that are proved in Section 7, Section 8 and Section A. More precisely, crucial roles are played by the metric entropy results of Section 7 (specifically Corollary 7.2) and Theorem 8.1 in Section 8 which relates the denoising error to Hellinger distance (thereby allowing the application of Theorem 2.1). Additionally, Lemma A.2, Lemma A.4, Lemma A.5, Lemma A.8 and Lemma A.9 from Section A will also be used.

The notation described at the beginning of Section 5 will be followed in this section as well.

Proof of Theorem 3.1. The goal is to bound

$$\mathcal{R}_n(\hat{\theta}, \hat{\theta}^*) = \mathbb{E} \left( \frac{1}{n} \sum_{i=1}^{n} ||\hat{\theta}_i - \hat{\theta}_i^*||^2 \right) = \mathbb{E} \left( \frac{1}{n} \sum_{i=1}^{n} \left\| X_i + \frac{\nabla \hat{f}_n(X_i)}{\hat{f}_n(X_i)} - X_i - \frac{\nabla f_{\hat{G}_n}(X_i)}{f_{\hat{G}_n}(X_i)} \right\|^2 \right)$$

It is convenient to introduce some notation here. Let $X$ denote the $d \times n$ matrix whose columns are the observed data vectors $X_1, \ldots, X_n$. For a density $f \in \mathcal{M}$, let $T_f(X)$ denote the $d \times n$ matrix whose $i^{th}$ column is given by the $d \times 1$ vector:

$$X_i + \frac{\nabla f(X_i)}{f(X_i)} \quad \text{for } i = 1, \ldots, n.$$

With this notation, we can clearly rewrite $\mathcal{R}_n(\hat{\theta}, \hat{\theta}^*)$ as

$$\mathcal{R}_n(\hat{\theta}, \hat{\theta}^*) = \mathbb{E} \left( \frac{1}{n} \left\| T_{\hat{f}_n}(X) - T_{f_{\hat{G}_n}}(X) \right\|_F^2 \right)$$

where $\| \cdot \|_F$ denotes the usual Frobenius norm for matrices.

To bound the above, we first observe that since $\hat{f}_n$ is an NPMLE defined as in (1.2), it follows from the general maximum likelihood theorem (see, for example, Böning [8, Theorem 2.1]) that

$$\frac{1}{n} \sum_{i=1}^{n} \frac{\phi_d(X_i - \theta)}{\hat{f}_n(X_i)} \leq 1$$

for every $\theta \in \mathbb{R}^d$. Taking $\theta = X_i$ in the above inequality, we deduce that

$$1 \geq \frac{\phi_d(X_i)}{n\hat{f}_n(X_i)} = \frac{\phi_d(0)}{n\hat{f}_n(X_i)}$$

so that $\hat{f}_n(X_i) \geq \phi_d(0)/n = (2\pi)^{-d/2}n^{-1}$. Since this is true for each $i = 1, \ldots, n$, this means that

$$\min_{1 \leq i \leq n} \hat{f}_n(X_i) \geq \rho_n := \frac{(2\pi)^{-d/2}}{n}.$$ 

As a result, $\hat{f}_n(X_i) = \max(\hat{f}_n(X_i), \rho_n)$ for each $i$ so that $T_{\hat{f}_n}(X) = T_{\hat{f}_n}(X, \rho_n)$ where for $f \in \mathcal{M}$ and $\rho > 0$, we define $T_f(X, \rho)$ to be the $d \times n$ matrix whose $i^{th}$ column is given by the $d \times 1$ vector:

$$X_i + \frac{\nabla f(X_i)}{\max(f(X_i), \rho)} \quad \text{for } i = 1, \ldots, n.$$

This gives

$$\mathcal{R}_n(\hat{\theta}, \hat{\theta}^*) = \mathbb{E} \left( \frac{1}{n} \left\| T_{\hat{f}_n}(X, \rho_n) - T_{f_{\hat{G}_n}}(X) \right\|_F^2 \right).$$
A difficulty in dealing with the expectation on the right hand side above comes from the fact that \( \hat{f}_n \) is random. This is handled by covering the random \( \hat{f}_n \) by an \( \epsilon \)-net for a specific \( \epsilon \) in the following way. First fix a compact set \( S \subseteq \mathbb{R}^d \) and \( M \geq \sqrt{10 \log n} \). Note that by Theorem 2.1 (specifically inequality (2.5) applied to \( \alpha = \beta = 0.5 \) and \( t = 1 \)), we deduce that the following inequality holds with probability at least \( 1 - (2/n) \):

\[
\mathcal{E}(\hat{f}_n, \hat{f}_{G_n}) \leq \tilde{C}_d \epsilon_n(M, S). \tag{6.3}
\]

Here \( \tilde{C}_d \) is a positive constant depending on \( d \) alone and \( \epsilon_n(M, S) \) is defined as in (2.3). Let \( E_n \) denote the event that (6.3) holds. We now obtain a covering of

\[
\{ f \in M : \mathcal{E}(f, \hat{f}_{G_n}) \leq \tilde{C}_d \epsilon_n(M, S) \}
\]

under the pseudometric given by

\[
\|f - g\|_{SM, \nabla}^{\rho_n} := \sup_{x \in SM} \left\| \frac{\nabla f(x)}{\max(f(x), \rho_n)} - \frac{\nabla g(x)}{\max(g(x), \rho_n)} \right\|
\]

where \( SM := \{ x \in \mathbb{R}^d : d_S(x) \leq M \} \). We have proved covering number bounds under this pseudo-metric in Corollary 7.2 which will be used in this proof. Let \( f_{G_1}, \ldots, f_{G_N} \) denote a maximal subset of (6.4) such that for every \( i \neq j \), we have

\[
\|f_{G_i} - f_{G_j}\|_{SM, \nabla}^{\rho_n} \geq 2 \eta^*
\]

where \( \eta^* \) is defined in terms of

\[
\eta^* := \left( \frac{1}{\rho_n} + \sqrt{\frac{1}{\rho_n^2 \log \frac{1}{(2\pi)^d \rho_n^2}}} \right) \eta \quad \text{and} \quad \eta := \frac{\rho_n}{n}. \tag{6.7}
\]

By the usual relation between packing and covering numbers, the integer \( N \) is then bounded from above by \( N(\eta^*, M, \| \cdot \|_{SM, \nabla}) \) which is bounded in Corollary 7.2. Specifically, Corollary 7.2 (applied to \( SM \)) gives

\[
\log N \leq C_d N(a, (SM)^a) \| \log (\eta)^2 \leq C_d N(a, SM^a) (\log n)^2
\]

where

\[
a := \sqrt{2 \log(2\sqrt{2\pi n^2})}. \tag{6.8}
\]

This further implies (via the use of inequality (A.26) in Lemma A.8 to bound \( N(a, SM^a) \))

\[
\log N \leq C_d (\log n)^2 a^{-d} \text{Vol}(SM^{(3a/2)}) \leq C_d (\log n)^{2-(d/2)} \text{Vol}(SM^{(3a/2)}).
\]

Using (A.27) in Lemma A.8 to bound \( \text{Vol}(SM^{(3a/2)}) \) in terms of \( \text{Vol}(S^1) \) (and the fact that \( a \leq C \sqrt{10 \log n} \leq CM \)), we obtain

\[
\log N \leq C_d \text{Vol}(S^1)M^d(\log n)^{2-(d/2)} .\tag{6.9}
\]

Also because \( f_{G_1}, \ldots, f_{G_N} \) is a maximal subset of (6.4) satisfying (6.6), we have

\[
\max_{1 \leq j \leq N} \mathcal{E}(f_{G_j}, \hat{f}_{G_n}) \leq \tilde{C}_d \epsilon_n(M, S) \tag{6.10}
\]
and, on the event $E_n$,

$$\min_{1 \leq j \leq N} \left\| \hat{f}_n - f_{G_j} \right\|_{S_M, \mathcal{V}}^\rho_n \leq 2\eta^*.$$  

We are now ready to bound the risk $\mathcal{R}_n(\hat{\theta}, \theta^*)$. The strategy is to break down the risk into various terms involving the densities $f_{G_1}, \ldots, f_{G_N}$.

**Breakdown of the risk:** The risk

$$\mathcal{R}_n(\hat{\theta}, \theta^*) = \mathbb{E} \left( \frac{1}{n} \left\| T_{f_n}(X, \rho_n) - T_{f_{G_n}}(X) \right\|_F^2 \right)$$

will be broken down via the inequality:

$$\left\| T_{f_n}(X, \rho_n) - T_{f_{G_n}}(X) \right\|_F \leq \left\| T_{f_n}(X, \rho_n) - T_{f_{G_n}}(X, \rho_n) \right\|_F + \left\| T_{f_{G_n}}(X, \rho_n) - T_{f_{G_n}}(X) \right\|_F$$

(6.12)

$$\leq (\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 + \zeta_5)$$

where

$$\zeta_1 := \left\| T_{f_n}(X, \rho_n) - T_{f_{G_n}}(X, \rho_n) \right\|_F I(E_n^c)$$

$$\zeta_2 := \left( \left\| T_{f_n}(X, \rho_n) - T_{f_{G_n}}(X, \rho_n) \right\|_F - \max_{1 \leq j \leq N} \left\| T_{f_{G_j}}(X, \rho_n) - T_{f_{G_n}}(X, \rho_n) \right\|_F \right) + I(E_n)$$

$$\zeta_3 := \max_{1 \leq j \leq N} \left( \left\| T_{f_{G_j}}(X, \rho_n) - T_{f_{G_n}}(X, \rho_n) \right\|_F - \mathbb{E}\left[I_{f_{G_j}}(X, \rho_n) - T_{f_{G_n}}(X, \rho_n) \right] \right) + I(E_n)$$

$$\zeta_4 := \max_{1 \leq j \leq N} \mathbb{E}\left[I_{f_{G_j}}(X, \rho_n) - T_{f_{G_n}}(X, \rho_n) \right]$$

$$\zeta_5 := \left\| T_{f_{G_n}}(X) - T_{f_{G_n}}(X, \rho_n) \right\|_F$$

In conjunction with the elementary inequality $(a_1 + \cdots + a_5)^2 \leq 5(a_1^2 + \cdots + a_5^2)$, inequality (6.12) gives

$$\mathcal{R}_n(\hat{\theta}, \theta^*) \leq 5 \sum_{i=1}^5 \frac{\mathbb{E}\zeta_{i,n}^2}{n}.$$ 

The proof of Theorem 3.1 will be completed below by showing the existence of a positive constant $C_d$ such that, for every $i = 1, \ldots, 5$,

$$\mathbb{E}\zeta_{i,n}^2 \leq C_d n \epsilon_n^2 (M, S) (\log n)^{\max(d,3)}$$

(6.13)

$$= C_d \left( \text{Vol}(S^1) M^d \left( \sqrt{\log n} \right)^{(4-d)_+} + n (\log n) \inf_{p \geq \frac{1}{d+1}} \left( \frac{2M}{p} \frac{\delta(S)}{M} \right)^p \right) (\log n)^{\max(d,3)}.$$ 

It may be noted that $\zeta_{4,n}$ is non-random so that the expectation above can be removed for $i = 4$. Every other $\zeta_{i,n}$ is random.

**Bounding $\mathbb{E}\zeta_{1,n}^2$:** We write

$$\mathbb{E}\zeta_{1,n}^2 = \mathbb{E} \left( \left\| T_{f_n}(X, \rho_n) - T_{f_{G_n}}(X, \rho_n) \right\|_F^2 I(E_n^c) \right)$$

$$= \sum_{i=1}^n \mathbb{E} \left( \left\| \nabla f_n(X_i) \max(f_n(X_i), \rho_n) - \nabla f_{G_n}(X_i) \max(f_{G_n}(X_i), \rho_n) \right\|^2 I(E_n^c) \right).$$
Inequality (A.2) in Lemma A.2 now gives
\[
\left\| \nabla \hat{f}_n(X_i) - \frac{\nabla f_{G_n}(X_i)}{\max(f_n(X_i), \rho_n)} \right\|^2 \leq 4 \log \left( \frac{(2\pi)^d}{\rho_n^2} \right)
\]
provided \( \rho_n \leq (2\pi)^{-d/2} e^{-1/2} \) which is equivalent to \( n \geq \sqrt{e} \) and hence holds for all \( n \geq 2 \). This gives (note that \( \mathbb{P}(E_n^c) \leq 2/n \))
\[
\mathbb{E} \xi_{1n}^2 \leq 4n \left( \log \left( \frac{(2\pi)^d}{\rho_n^2} \right) \right) \mathbb{P}(E_n^c) \leq 4 \cdot \left( \log \left( \frac{(2\pi)^d}{\rho_n^2} \right) \right) \leq C_d \log n \leq C_d \text{Vol}(S^d) M^d \left( \sqrt{\log n} \right)^{(4-d)_+}
\]
which proves (6.13) for \( i = 1 \).

**Bounding \( \mathbb{E} \xi_{2n}^2 \):** For this, we write
\[
\xi_{2n}^2 \leq \min_{1 \leq j \leq N} \left\| T_{f_n}(X, \rho_n) - T_{f_{G_j}}(X, \rho_n) \right\|_F^2 I(E_n)
\]
\[
= \min_{1 \leq j \leq N} \sum_{i=1}^n \left\| \frac{\nabla \hat{f}_n(X_i)}{\max(f_n(X_i), \rho_n)} - \frac{\nabla f_{G_j}(X_i)}{\max(f_{G_j}(X_i), \rho_n)} \right\|^2 I(E_n)
\]
\[
\leq \min_{1 \leq j \leq N} \left( \left\| \frac{\hat{f}_n - f_{G_j}}{\rho_n} \right\|_{S^d, \nabla}^2 \right) \left( \sum_{i=1}^n I\{X_i \in S^d\} \right) I(E_n) + \left( 4 \log \left( \frac{(2\pi)^d}{\rho_n^2} \right) \right) \left( \sum_{i=1}^n I\{X_i \notin S^d\} \right) I(E_n).
\]
where we have used the notation (6.5) in the first term above and the inequality (6.14) in the second term. We can simplify the above bound as
\[
\xi_{2n}^2 \leq n \left( \min_{1 \leq j \leq N} \left\| \frac{\hat{f}_n - f_{G_j}}{\rho_n} \right\|_{S^d, \nabla}^2 \right) \left( \sum_{i=1}^n I\{X_i \notin S^d\} \right) + \left( 4 \log \left( \frac{(2\pi)^d}{\rho_n^2} \right) \right) \left( \sum_{i=1}^n I\{X_i \notin S^d\} \right).
\]
Inequality (6.11) and the expression (6.7) for \( \eta^* \) now give
\[
\mathbb{E} \xi_{2n}^2 \leq \frac{4}{n} \left( 1 + \left( \log \left( \frac{1}{(2\pi)^d \rho_n^2} \right) \right)^2 \right) + \left( 4 \log \left( \frac{(2\pi)^d}{\rho_n^2} \right) \right) \left( \sum_{i=1}^n \mathbb{P}\{X_i \notin S^d\} \right)
\]
\[
\leq C_d \log n + C_d (\log n) \left( \sum_{i=1}^n \mathbb{P}\{X_i \notin S^d\} \right).
\]
To control the second term above, we use inequality (A.14) (which is a consequence of Lemma A.4). Note that \( \mathbb{P}\{X_i \notin S^d\} \leq \mathbb{P}\{S(X_i) \geq M\} \). Inequality (A.14) therefore gives
\[
\mathbb{E} \xi_{2n}^2 \leq \left( \log n \right) M^d - 2 + C_d (\log n) \inf_{p \geq \frac{d+1}{2 \log n}} \left( \frac{2 \mu_p(\theta_S)}{M} \right)^p.
\]
This proves (6.13) for \( i = 2 \) (note that \( \left( \log n \right) M^d - 2 \leq M^d \) as \( M \geq \sqrt{10 \log n} \)).

**Bounding \( \xi_{3n}^2 \):** Here Lemma A.5 and the bound (6.9) will be crucially used. Let us first write \( \xi_{3n} := \max_{1 \leq j \leq N} \xi_{3n,j} \) where
\[
\xi_{3n,j} := \left( \left\| T_{f_{G_j}}(X, \rho_n) - T_{f_{G_n}}(X, \rho_n) \right\|_F - \mathbb{E} \left\| T_{f_{G_j}}(X, \rho_n) - T_{f_{G_n}}(X, \rho_n) \right\|_F \right)_{+}.
\]
Lemma A.5 then gives
\[ \mathbb{P}\{\zeta_{3n,j} \geq x\} \leq \exp\left(\frac{-x^2}{8L^4(\rho_n)}\right) \quad \text{for every } 1 \leq j \leq N \text{ and } x > 0. \]

where
\[ (6.15) \quad L(\rho_n) = \sqrt{\log \frac{1}{(2\pi)^d \rho_n^2}} = \sqrt{\log n}. \]

By the union bound, we have
\[ \mathbb{P}\{\zeta_{3n} \geq x\} \leq N \exp\left(\frac{-x^2}{8L^4(\rho_n)}\right) \quad \text{for every } x > 0 \]
so that, for every \( x_0 > 0 \),
\[ \mathbb{E}\zeta_{3n}^2 \leq \int_0^{\infty} \mathbb{P}\{\zeta_{3n} \geq \sqrt{x}\} dx \]
\[ \leq x_0 + \int_{x_0}^{\infty} N \exp\left(\frac{-x}{8L^4(\rho_n)}\right) dx = x_0 + 8NL^4(\rho_n) \exp\left(\frac{-x_0}{8L^4(\rho_n)}\right). \]

Minimizing the above bound over \( x_0 > 0 \), we deduce that
\[ \mathbb{E}\zeta_{3n}^2 \leq 8L^4(\rho_n) \log (eN). \]

The bound (6.9) (along with (6.15)) then gives
\[ \mathbb{E}\zeta_{3n}^2 \leq C_d \text{Vol}(S^1)^d \left(\sqrt{\log n}\right)^{8-d} \leq C_d \text{Vol}(S^1)^d \left(\sqrt{\log n}\right)^{(4-d)+ (\log n)^3} \]
which proves (6.13) for \( i = 3 \).

**Bounding \( \zeta_{4n}^2 \)**: To bound the non-random quantity \( \zeta_{4n}^2 \), we only need to bound
\[ \Gamma_j^2 := \mathbb{E}\left\| T_{f_{G_j}}(X, \rho_n) - T_{f_{G_n}}(X, \rho_n) \right\|^2_F \]
for each \( 1 \leq j \leq N \). We can clearly write
\[ \Gamma_j^2 = \sum_{i=1}^{n} \mathbb{E}\left\| \nabla f_{G_j}(X_i) - \nabla f_{G_n}(X_i) \right\|^2 \]
\[ = n \int \left\| \frac{\nabla f_{G_j}(x)}{\max(f_{G_j}(x), \rho_n)} - \frac{\nabla f_{G_n}(x)}{\max(f_{G_n}(x), \rho_n)} \right\|^2 f_{G_n}(x) dx. \]

The above term can be bounded by a direct application of Theorem 8.1 which furnishes a bound in terms of \( \mathfrak{J}(f_{G_j}, f_{G_n}) \). Indeed, because \( n \geq 2 \), we have \( \rho_n \leq (2\pi)^{d/2} e^{-1/2} \) so that Theorem 8.1 applies (with \( G = G_j \) and \( G_0 = G_n \)) and we obtain
\[ \frac{1}{n} \Gamma_j^2 \leq C_d \max \left\{ \left( \frac{\log (2\pi)^{-d/2}}{\rho_n} \right)^3, \left| \log \mathfrak{J}(f_{G_j}, f_{G_n}) \right| \right\} \mathfrak{J}^2(f_{G_j}, f_{G_n}) \]
\[ = C_d \max \left\{ (\log n)^3, \left| \log \mathfrak{J}(f_{G_j}, f_{G_n}) \right| \right\} \mathfrak{J}^2(f_{G_j}, f_{G_n}). \]
We now use the fact that \( \mathcal{S}(f_{G_1}, f_{G_n}) \) is bounded from above by \( \tilde{C}_d \epsilon_n(M, S) \) (see (6.10)). We can then work with two cases. If \( \tilde{C}_d \epsilon_n(M, S) \leq e^{-1/2} \), then using the fact that \( h \mapsto h^2|\log h| \) is increasing on \((0, e^{-1/2}]\), we have

\[
\frac{1}{n} \Gamma_j^2 \leq C_d \tilde{C}_d^2 \max \left\{ (\log n)^2, \log(\tilde{C}_d \epsilon_n(M, S)) \right\} \epsilon_n^2(M, S).
\]

The trivial observation \( \epsilon_n(M, S) \geq K_d/n \) for a constant \( K_d \) now gives

\[
(6.16) \quad \Gamma_j^2 \leq nC_d(\log n)^3 \epsilon_n^2(M, S).
\]

On the other hand when \( \tilde{C}_d \epsilon_n(M, S) > e^{-1/2} \), then we can simply bound \( |\log \mathcal{S}(f_{G_1}, f_{G_n})| \) \( |\mathcal{S}(f_{G_1}, f_{G_n})| \) by a constant (the function \( h \mapsto h^2|\log h| \) is bounded on \( h \in (0, 2] \)) so that the inequality (6.16) still holds. The bound in the right hand side of (6.16) does not depend on \( j \) so that it is an upper bound for \( \epsilon_n^2 \) as well. This proves (6.13) for \( i = 4 \).

**Bounding \( \mathbb{E} \xi_{G_n}^2 \):** We write

\[
\mathbb{E} \xi_{G_n}^2 = \mathbb{E} \left\| T_{f_{G_n}}(X) - T_{f_{G_n}}(X, \rho_n) \right\|^2_F
\]

\[
= \sum_{i=1}^n \mathbb{E} \left\| \nabla f_{G_n}(X_i) f_{G_n}(X_i) - \nabla f_{G_n}(X_i) \max(f_{G_n}(X_i), \rho_n) \right\|^2
\]

\[
= n \int \left\| \nabla f_{G_n}(x) f_{G_n}(x) - \nabla f_{G_n}(x) \max(f_{G_n}(x), \rho_n) \right\|^2 f_{G_n}(x)dx
\]

\[
= n \int \left( 1 - \frac{f_{G_n}}{\max(f_{G_n}, \rho)} \right)^2 \left\| \nabla f_{G_n} \right\|^2 f_{G_n} = n \Delta(G_n, \rho_n)
\]

where we define

\[
\Delta(G, \rho) := \int \left( 1 - \frac{f_G}{\max(f_G, \rho)} \right)^2 \left\| \nabla f_G \right\|^2 f_G
\]

for probability measures \( G \) on \( \mathbb{R}^d \) and \( \rho > 0 \). We now use Lemma A.9 to bound \( \Delta(G_n, \rho_n) \). Specifically, inequality (A.29) in Lemma A.9 applied to the compact set \( S^M \) gives

\[
(6.17) \quad \Delta(G_n, \rho_n) \leq C_d N \left( \frac{4}{L(\rho_n)}, S^M \right) L^d(\rho_n) \rho_n + d \cdot G_n((S^M)^c).
\]

The first term above is bounded using Lemma A.8 as follows (note that \( \rho_n = (2\pi)^{-d/2}/n \) and \( L(\rho_n) = \sqrt{\log n} \) as shown in (6.15)):

\[
N \left( \frac{4}{L(\rho_n)}, S^M \right) L^d(\rho_n) = N \left( \frac{4}{\sqrt{\log n}}, S^M \right) (\log n)^{d/2}(2\pi)^{-d/2} \frac{n}{\log n}
\]

\[
\leq C_d (4/\sqrt{\log n})^{-d} \text{Vol}(S^M)^{d/2} \left( \frac{\log n}{n} \right)^{d/2} \text{(using inequality (A.26))}
\]

\[
\leq \frac{C_d}{n} (\log n)^d \text{Vol}(S^M)^{d} \frac{n}{\sqrt{\log n}}
\]

\[
\leq \frac{C_d}{n} (\log n)^d \text{Vol}(S^1)^{d} \left( 1 + \frac{M}{4} + \frac{1}{2\sqrt{\log n}} \right)^d \text{(using inequality (A.27))}
\]

\[
\leq \frac{C_d}{n} (\log n)^d M^d \text{Vol}(S^1).
\]
For the second term in (6.17), note that
\[
\bar{G}_n((S^M)^c) \leq \int I \{ \Theta_S(\theta) \geq M \} d\bar{G}_n(\theta) \leq \inf_{p \geq \frac{d+1}{2\log n}} \left( \frac{2\mu_p(\Theta_S)}{M} \right)^p.
\]
We have therefore proved that
\[
\mathbb{E} \zeta_2^2 \leq n \Delta(\bar{G}_n, \rho_n) \leq C_d \left\{ (\log n)^d M^d \text{Vol}(S^1) + n \inf_{p \geq \frac{d+1}{2\log n}} \left( \frac{2\mu_p(\Theta_S)}{M} \right)^p \right\}
\]
which evidently implies (6.13). The proof of Theorem 3.1 is now complete.

6.2. Proof of Corollary 3.2. The idea is to choose \( M \) and \( S \) appropriately under each of the assumptions on \( \bar{G}_n \) and then to appropriately bound \( \epsilon_n(M, S) \). The necessary work for this is already done in Corollary 2.2 from which Corollary 3.2 immediately follows.

6.3. Proof of Proposition 3.3. The assumption (3.4) implies that the empirical measure \( \bar{G}_n \) of \( \theta_1, \ldots, \theta_n \) is supported on \( S := \bigcup_{j=1}^k B(a_j, R) \) where \( B(a_j, R) := \{ x \in \mathbb{R}^d : \| x - a_j \| \leq R \} \).

We can therefore apply inequality (3.1) in Corollary 3.2 to bound \( R_n(\hat{\theta}, \hat{\theta}^*) \). The conclusion (3.5) then immediately follows from (3.1) because
\[
\text{Vol}(S^1) \leq \sum_{j=1}^k \text{Vol}(B(a_j, 1 + R)) \leq C_d k (1 + R)^d.
\]

6.4. Proof of Lemma 3.4. The proof of Lemma 3.4 uses Assouad’s lemma (see, for example, Tsybakov [60, Chapter 2] as well as Lemma A.11 (stated and proved in Section A).

**Proof of Lemma 3.4.** Fix \( k \) and \( n \) with \( 1 \leq k \leq n \). Also fix \( \delta > 0 \) and \( M \geq 2 \). Let \( a_1, \ldots, a_k \) and \( b_1, \ldots, b_k \) be points in \( \mathbb{R}^d \) such that
\[
\min_{i \neq j} \{ \min_{i \neq j} \| a_i - a_j \|, \min_{i \neq j} \| b_i - b_j \|, \min_{i \neq j} \| a_i - b_j \| \} \geq M
\]
and such that
\[
\| a_i - b_i \| = \delta \quad \text{for every } 1 \leq i \leq k.
\]
We now define a partition \( S_1, \ldots, S_k, S_{k+1} \) of \( \{1, \ldots, n\} \) via
\[
S_i := \{(i-1)m + 1, \ldots, im\} \quad \text{for } i = 1, \ldots, k
\]
and \( S_{k+1} := \{km + 1, \ldots, n\} \) where \( m := [n/k] \) (for \( x > 0 \), we define \( [x] \) as usual to be the largest integer that is smaller than or equal to \( x \)). Note that the cardinality of \( S_j \) equals \( m \) for \( i = 1, \ldots, k \) and that \( S_{k+1} \) will be empty if \( n \) is a multiple of \( k \).

Now for every \( \tau \in \{0, 1\}^k \), we define \( n \) vectors \( \theta_1(\tau), \ldots, \theta_n(\tau) \) in \( \mathbb{R}^d \) via
\[
\theta_i(\tau) := (1 - \tau_j) a_j + \tau_j b_j \quad \text{provided } i \in S_j \text{ for some } 1 \leq j \leq k
and for \( i \in S_{k+1} \), we take \( \theta_i(\tau) := a_1 \).

Let \( \Theta(\tau) \) denote the collection of all \( n \)-tuples \( (\theta_1(\tau), \ldots, \theta_n(\tau)) \) as \( \tau \) ranges over \( \{0, 1\}^k \). It is easy to see that \( \Theta(\tau) \subseteq \Theta_{n, d, k} \) so that

\[
\mathcal{R}^*(\Theta_{n, d, k}) \geq \mathcal{R}^*(\Theta(\tau)) := \inf_{\hat{\theta}_1, \ldots, \hat{\theta}_n (\theta_1, \ldots, \theta_n) \in \Theta(\tau)} \sup_{\theta_1, \ldots, \theta_n, \tau} E \left[ \frac{1}{n} \sum_{i=1}^{n} \left\| \hat{\theta}_i - \theta_i^* \right\|^2 \right].
\]

The elementary inequality \( \|a - b\|^2 \geq \|a\|^2/2 \) for vectors \( a, b \in \mathbb{R}^d \) gives

\[
\frac{1}{n} \sum_{i=1}^{n} \left\| \hat{\theta}_i - \theta_i^* \right\|^2 \geq \frac{1}{2n} \sum_{i=1}^{n} \left\| \hat{\theta}_i - \theta_i \right\|^2 - \frac{1}{n} \sum_{i=1}^{n} \left\| \theta_i^* - \theta_i \right\|^2
\]

for every \( \theta_1, \ldots, \theta_n \) and estimators \( \hat{\theta}_1, \ldots, \hat{\theta}_n \). As a result, we deduce that

\[
(6.20) \quad \mathcal{R}^*(\Theta(\tau)) \geq \mathcal{R}(\Theta(\tau)) - \sup_{(\theta_1, \ldots, \theta_n) \in \Theta(\tau)} E \left[ \frac{1}{n} \sum_{i=1}^{n} \left\| \hat{\theta}_i - \theta_i \right\|^2 \right]
\]

where

\[
\mathcal{R}(\Theta(\tau)) := \inf_{\hat{\theta}_1, \ldots, \hat{\theta}_n (\theta_1, \ldots, \theta_n) \in \Theta(\tau)} \sup_{\theta_1, \ldots, \theta_n, \tau} E \left[ \frac{1}{n} \sum_{i=1}^{n} \left\| \hat{\theta}_i - \theta_i \right\|^2 \right].
\]

We first bound \( \mathcal{R}(\Theta(\tau)) \) from below via Assouad’s lemma. For \( \tau, \tau' \in \{0, 1\}^k \), let

\[
\Sigma(\tau, \tau') := \frac{1}{n} \sum_{i=1}^{n} \left\| \theta_i(\tau) - \theta_i(\tau') \right\|^2.
\]

Also let \( P_\tau \) denote the joint distribution of the independent random variables \( X_1, \ldots, X_n \) with \( X_i \sim N(\theta_i(\tau), I_d) \) for \( i = 1, \ldots, n \). Assouad’s lemma then gives

\[
(6.21) \quad \mathcal{R}(\Theta(\tau)) \geq k \min_{8/\tau \neq \tau'} \frac{\Sigma(\tau, \tau')}{\Upsilon(\tau, \tau')} \min_{\tau, \tau'} (1 - \|P_\tau - P_{\tau'}\|_{TV})
\]

where \( \Upsilon(\tau, \tau') := \sum_{j=1}^{k} I\{\tau_j \neq \tau'_j\} \) is the Hamming distance and \( \|P_\tau - P_{\tau'}\|_{TV} \) denotes the variation distance between \( P_\tau \) and \( P_{\tau'} \). We now bound the terms appearing in the right hand side of (6.21). For \( \tau, \tau' \in \{0, 1\}^k \), observe that

\[
(6.22) \quad \Sigma(\tau, \tau') = \frac{1}{n} \sum_{j=1}^{k} \sum_{i : i \in S_j} \|a_j - b_j\| I\{\tau_j \neq \tau'_j\} = \frac{1}{n} \sum_{j=1}^{k} |S_j| \|a_j - b_j\|^2 I\{\tau_j \neq \tau'_j\} = \frac{m \delta^2}{n} \Upsilon(\tau, \tau')
\]

where \( |S_j| \) denotes the cardinality of \( S_j \). We have used above the fact that \( |S_j| = m \) for \( 1 \leq j \leq k \) and (6.19).

To bound the last term in (6.21), we use Pinsker’s inequality (below \( D \) stands for Kullback-Leibler divergence) to obtain

\[
\|P_\tau - P_{\tau'}\|_{TV} \leq \sqrt{\frac{1}{2} D(P_\tau \| P_{\tau'})} = \frac{1}{2} \sqrt{\sum_{i=1}^{n} \left\| \theta_i(\tau) - \theta_i(\tau') \right\|^2} = \frac{1}{2} \sqrt{n \Sigma(\tau, \tau')}.
\]
Thus, from (6.22), we deduce that for $\Upsilon(\tau, \tau') = 1$,
\[
\| P_\tau - P_{\tau'} \|_{TV} \leq \frac{1}{2} \sqrt{m \delta^2}.
\]
Inequality (6.21) thus gives
\[
(6.23) \quad \bar{R}(\Theta(\tau)) \geq \frac{km\delta^2}{8n} \left( 1 - \frac{\sqrt{m\delta^2}}{2} \right).
\]
To bound the second term in (6.20), we use Lemma A.11 which gives that for every $\theta_1, \ldots, \theta_n \in \Theta(\tau)$, we have
\[
E \left[ \frac{1}{n} \sum_{i=1}^{n} \left\| \hat{\theta}_i^\tau - \theta_i \right\|^2 \right] \leq \frac{k}{2 \sqrt{2\pi}} \sum_{j,l,j \neq l} (p_j + p_l) \| c_j - c_l \| \exp \left( -\frac{1}{8} \| c_j - c_l \|^2 \right)
\]
where $c_1, \ldots, c_{k+1}$ denote the distinct elements from $\theta_1, \ldots, \theta_n$ and $p_j, j = 1, \ldots, k + 1$ are nonnegative real numbers summing to one. Now each $c_j$ equals either $a_j$ or $b_j$ and hence, by (6.18), we have $\| c_j - c_l \| \geq M$ for every $j \neq l$. As $x \mapsto xe^{-x^2/8}$ is decreasing for $x > 2$ and $M > 2$, we deduce that
\[
(6.24) \quad E \left[ \frac{1}{n} \sum_{i=1}^{n} \left\| \hat{\theta}_i^\tau - \theta_i \right\|^2 \right] \leq \frac{k}{\sqrt{2\pi} M e^{-M^2/8}} \sum_{j,l,j \neq l} (p_j + p_l) \leq \frac{k}{\sqrt{2\pi} M e^{-M^2/8}}.
\]
We obtain therefore from (6.20), (6.23) and (6.24), that
\[
R^\ast(\Theta_{n,d,k}) \geq \frac{km\delta^2}{8n} \left( 1 - \frac{\sqrt{m\delta^2}}{2} \right) - \frac{k}{\sqrt{2\pi} M e^{-M^2/8}}.
\]

The left hand side above does not depend on $M$ so we can let $M \to \infty$ to obtain
\[
R^\ast(\Theta_{n,d,k}) \geq \frac{km\delta^2}{8n} \left( 1 - \frac{\sqrt{m\delta^2}}{2} \right).
\]

We now make the choice $\delta := 1/\sqrt{m}$ to obtain $R^\ast(\Theta_{n,d,k}) \geq k/(16n)$ which proves Lemma 3.4. \qed

7. Main Metric Entropy Results and Proofs. For a compact set $S \subseteq \mathbb{R}^d$, let $\| \cdot \|_S$ and $\| \cdot \|_{S,\nabla}$ denote two pseudonorms given by
\[
\| f \|_{S} := \sup_{x \in S} |f(x)| \quad \text{and} \quad \| f \|_{S,\nabla} := \sup_{x \in S} \| \nabla f(x) \|
\]
for densities $f \in \mathcal{M}$. These naturally lead to two pseudometrics on $\mathcal{M}$ and we shall denote the $\eta$-covering numbers of $\mathcal{M}$ under these pseudometrics by $N(\eta, \mathcal{M}, \| \cdot \|_S)$ and $N(\eta, \mathcal{M}, \| \cdot \|_{S,\nabla})$ respectively. The notion of covering numbers is defined at the beginning of Section 5. The following theorem gives upper bounds for $N(\eta, \mathcal{M}, \| \cdot \|_S)$ and $N(\eta, \mathcal{M}, \| \cdot \|_{S,\nabla})$. Recall the notation introduced at the beginning of Section 5.

Theorem 7.1. There exists a positive constant $C_d$ depending on $d$ alone such that for every compact set $S \subseteq \mathbb{R}^d$ and $0 < \eta \leq \frac{2\sqrt{2\pi}}{\sqrt{m \delta \| S \|^2}}$, we have
\[
(7.1) \quad \log N(\eta, \mathcal{M}, \| \cdot \|_S) \leq C_d N(a, S^a) \log \eta^2
\]
and

\[(7.2) \quad \log N(\eta, \mathcal{M}, \|\cdot\|_{S,\nabla}) \leq C_d N(a, S^a) |\log \eta|^2 \]

where \(a\) is defined as

\[(7.3) \quad a := \sqrt{2 \log \frac{2\sqrt{2\pi}}{(2\pi)^{d/2}}}.\]

Theorem 7.1 immediately implies a covering number result for \(\mathcal{M}\) in terms of another pseudo-metric that is defined in terms of both \(f(x)\) and \(\nabla f(x)\). This is given in the next corollary which was used in the proof of Theorem 3.1.

**Corollary 7.2.** For a compact set \(S \subseteq \mathbb{R}^d\) and \(\rho > 0\), define the pseudometric:

\[(7.4) \quad \|f - g\|_{S,\nabla}^{\rho} := \sup_{x \in S} \left\| \frac{\nabla f(x)}{\max(f(x), \rho)} - \frac{\nabla g(x)}{\max(g(x), \rho)} \right\| \]

for functions \(f : \mathbb{R}^d \to \mathbb{R}\) which are bounded on \(S\) and whose derivatives are bounded on \(S\). Let the \(\epsilon\)-covering number of \(\mathcal{M}\) in the pseudometric given by (7.4) be denoted by \(N(\epsilon, \mathcal{M}, \|\cdot\|_{S,\nabla}^{\rho})\). Then there exists a positive constant \(C_d\) depending on \(d\) alone such that for every \(\rho > 0, 0 < \eta \leq \frac{2\sqrt{\pi}}{(2\pi)^{d/2}\sqrt{\rho}}\) and compact subset \(S \subseteq \mathbb{R}^d\), we have

\[(7.5) \quad \log N(\eta^*, \mathcal{M}, \|\cdot\|_{S,\nabla}^{\rho}) \leq C_d N(a, S^a) |\log \eta|^2 \]

where \(a\) is defined as in (7.3) and

\[(7.6) \quad \eta^* := \left(1 + \frac{1}{\rho^2} \log \frac{1}{(2\pi)^{d/2}\rho^2} \right) \eta.\]

**Remark 7.1.** When \(d = 1\) and \(S = [-M, M]\), we have

\[N(a, S^a) \leq C \max \left\{ \frac{M}{\sqrt{|\log \eta|}}, 1 \right\} \]

so that inequalities (7.1), (7.2) and (7.5) become

\[(7.7) \quad \log N(\eta, \mathcal{M}, \|\cdot\|_{[-M,M]}) \leq C |\log \eta|^2 \max \left\{ \frac{M}{\sqrt{|\log \eta|}}, 1 \right\}, \]

\[(7.8) \quad \log N(\eta, \mathcal{M}, \|\cdot\|_{[-M,M],\nabla}) \leq C |\log \eta|^2 \max \left\{ \frac{M}{\sqrt{|\log \eta|}}, 1 \right\}, \]

and

\[(7.9) \quad \log N(\eta^*, \mathcal{M}, \|\cdot\|_{[-M,M],\nabla}) \leq C |\log \eta|^2 \max \left\{ \frac{M}{\sqrt{|\log \eta|}}, 1 \right\}\]

respectively. Inequality (7.7) has previously appeared in Zhang [66, Lemma 2] (improving an earlier result of Ghosal and van der Vaart [25]). Inequality (7.8) does not seem to have been stated explicitly previously but is implicit in Jiang and Zhang [28, Proof of Proposition 3]. Inequality (7.9) has previously appeared as Jiang and Zhang [28, Proposition 3]. Our contribution therefore lies in generalizing these results to multiple dimensions and further in allowing $S$ to take the form of any compact subset of $\mathbb{R}^d$.

The rest of this section is devoted to the proofs of Theorem 7.1 and Corollary 7.2.

7.1. Proof of Theorem 7.1.

7.1.1. Moment Matching Lemma. Recall that for $x \in \mathbb{R}^d$ and $a > 0$, we denote the closed Euclidean ball of radius $a$ centered at $x$ by $B(x, a)$. We also let

$$B(x, a) := \{ u \in \mathbb{R}^d : \| u - x \| < a \}$$

denote the open ball of radius $a$ centered at $x$.

**Lemma 7.3.** Let $G$ and $G'$ be two arbitrary probability measures on $\mathbb{R}^d$. Fix $a \geq 1$ and $x \in \mathbb{R}^d$. Let $A$ be a subset of $\mathbb{R}^d$ such that

$$\tilde{B}(x, a) \subseteq A \subseteq B(x, ca)$$

for some $c \geq 1$. Suppose that, for some $m \geq 1$, we have

$$\int_A \theta_j^k dG(\theta) = \int_A \theta_j^k dG'(\theta) \quad \text{for every } 1 \leq j \leq d \text{ and } 0 \leq k \leq 2m + 1. $$

Then

$$|f_G(x) - f_{G'}(x)| \leq \frac{1}{(2\pi)^{(d+1)/2}} \left( \frac{c^2 a^2 e}{2(m+1)} \right)^{m+1} + \frac{e^{-a^2/2}}{(2\pi)^{d/2}},$$

and

$$\| \nabla f_G(x) - \nabla f_{G'}(x) \| \leq \frac{ca}{(2\pi)^{(d+1)/2}} \left( \frac{c^2 a^2 e}{2(m+1)} \right)^{m+1} + \frac{ae^{-a^2/2}}{(2\pi)^{d/2}}.$$

**Proof of Lemma 7.3.** First write

$$f_G(x) - f_{G'}(x) = \int \phi_d(x - \theta) \left( G(d\theta) - G'(d\theta) \right)$$

and

$$\nabla f_G(x) - \nabla f_{G'}(x) = \int (\theta - x) \phi_d(x - \theta) \left( G(d\theta) - G'(d\theta) \right).$$

We split each integral above into two terms by restricting their range first over $A$ and then over $A^c$, the complement set of $A$:

$$f_G(x) - f_{G'}(x) = \int_A \phi_d(x - \theta) \left( dG(\theta) - dG'(\theta) \right) + \int_{A^c} \phi_d(x - \theta) \left( dG(\theta) - dG'(\theta) \right)$$

and

$$\nabla f_G(x) - \nabla f_{G'}(x) = \int_A (\theta - x) \phi_d(x - \theta) \left( dG(\theta) - dG'(\theta) \right) + \int_{A^c} (\theta - x) \phi_d(x - \theta) \left( dG(\theta) - dG'(\theta) \right).$$
Because $A \supseteq \hat{B}(x,a)$, it is clear that

$$
\sup_{\theta \in A^c} \phi_d(x - \theta) \leq \sup_{\theta : \|x - \theta\| \geq a} \phi_d(x - \theta) \leq (2\pi)^{-d/2} \exp(-a^2/2)
$$

$$
\sup_{\theta \in A^c} \|\theta - x\| \phi_d(x - \theta) \leq \sup_{\theta : \|x - \theta\| \geq a} \|x - \theta\| \phi_d(x - \theta) \leq (2\pi)^{-d/2} \sup_{u \geq a} ue^{-u^2/2} = (2\pi)^{-d/2} ea^{-a^2/2}
$$

because $a \geq 1$. Therefore the second terms on the right hand side on (7.13) and (7.14) are respectively bounded in absolute value by the final terms in (7.11) and (7.12). It only remains to prove the following pair of inequalities

$$
\left| \int_{\mathbb{A}} \phi_d(x - \theta) (dG(\theta) - dG'(\theta)) \right| \leq \frac{1}{(2\pi)^{(d+1)/2}} \left( \frac{c^2a^2e}{2(m+1)} \right)^m
$$

$$
\left| \int_{\mathbb{A}} (\theta - x)\phi_d(x - \theta) (dG(\theta) - dG'(\theta)) \right| \leq \frac{ca}{(2\pi)^{(d+1)/2}} \left( \frac{c^2a^2e}{2(m+1)} \right)^m
$$

For this, we use Taylor expansion and the moment matching condition (7.10). Taylor’s formula for $e^u$ is

$$
e^u = \sum_{i=0}^{m} \frac{u^i}{i!} + \frac{u^{m+1}}{(m+1)!} e^v
$$

for every $u$ where $v$ is some real number lying between 0 and $u$. Using this for $u = -t^2/2$, we obtain

$$
\exp(-t^2/2) = \sum_{i=0}^{m} \frac{(-t^2/2)^i}{i!} + (-1)^{m+1} \frac{(t^2/2)^{m+1}}{(m+1)!} e^v
$$

where $v$ lies between 0 and $-t^2/2$. Because $e^v \leq 1$, this gives

$$
\left| \exp(-t^2/2) - \sum_{i=0}^{m} \frac{(-t^2/2)^i}{i!} \right| \leq \frac{(t^2/2)^{m+1}}{(m+1)!}.
$$

We can therefore write $\phi_d(z) = P_d(z) + R_d(z)$ for every $z \in \mathbb{R}^d$ where $P_d(z)$ is a polynomial of degree $2m$ in $z$ and $R_d(z)$ is a remainder term which satisfies

$$
|R_d(z)| \leq \frac{(||z||^2/2)^{m+1}}{(2\pi)^{d/2}(m+1)!}.
$$

Using this for $z = x - \theta$, we can write

$$
\left| \int_{\mathbb{A}} \phi_d(x - \theta) (dG(\theta) - dG'(\theta)) \right| \leq \left| \int_{\mathbb{A}} P_d(x - \theta) (dG(\theta) - dG'(\theta)) \right| + \left| \int_{\mathbb{A}} R_d(x - \theta) (dG(\theta) - dG'(\theta)) \right|
$$

and similarly,

$$
\left| \int_{\mathbb{A}} (\theta - x)\phi_d(x - \theta) (dG(\theta) - dG'(\theta)) \right| \leq \left| \int_{\mathbb{A}} (\theta - x) P_d(x - \theta) (dG(\theta) - dG'(\theta)) \right| + \left| \int_{\mathbb{A}} (\theta - x) R_d(x - \theta) (dG(\theta) - dG'(\theta)) \right|
$$
The first terms in the above two equations are zero because of condition (7.10) and the fact that $P_d(x - \theta)$ is a polynomial in $\theta$ with degree $2m$ (implying that for every $j$, $(\theta_j - x_j)P_d(x - \theta)$ is a polynomial of degree $2m + 1$). Because $A \subseteq B(x, ca)$, we have $\|x - \theta\| \leq ca$ for every $\theta \in A$ so that

$$|R_d(x - \theta)| \leq \frac{(2\pi)^{-d/2}}{(m + 1)!} \left( \frac{\|x - \theta\|^2}{2} \right)^{m+1} \leq \frac{(2\pi)^{-d/2}}{(m + 1)!} \left( \frac{c^2a^2e}{2} \right)^{m+1}.$$

Stirling’s formula $n! \geq \sqrt{2\pi n}(n/e)^n \geq \sqrt{2\pi(n/e)^n}$ applied to $n = m + 1$ yields

$$|R_d(x - \theta)| \leq \frac{1}{(2\pi)^{(d+1)/2}} \left( \frac{c^2a^2e}{2(m + 1)} \right)^{m+1}$$

for every $\theta \in A$

and

$$\|\theta - x\| |R_d(x - \theta)| \leq \frac{ca}{(2\pi)^{(d+1)/2}} \left( \frac{c^2a^2e}{2(m + 1)} \right)^{m+1}$$

for every $\theta \in A$

which completes the proof.

7.1.2. Approximation by mixtures with discrete mixing measures. Given any distribution $f_G$, what is a bound on $\ell$ such that we can approximate $f_G$ by another gaussian mixture $f_{G'}$ where $G'$ is a discrete measure with at most $\ell$ atoms. The following lemma addresses this question where approximation is in terms of the pseudometrics $\sup_{x \in S} |f_G(x) - f_{G'}(x)|$ as well as $\sup_{x \in S} \|\nabla f_G(x) - \nabla f_{G'}(x)\|$.

Recall that for a subset $S$ of $\mathbb{R}^d$, we write $N(\eta, S)$ to mean its $\eta$ covering number (defined as the smallest number of closed balls of radius $\eta$ whose union contains $S$).

**Lemma 7.4.** Let $G$ be an arbitrary probability measure on $\mathbb{R}^d$ and let $S$ denote an arbitrary compact subset of $\mathbb{R}^d$. Also let $a \geq 1$. Then there exists a discrete probability measure $G'$ that is supported on $S^a := \cup_{x \in S} B(x, a)$ and having at most

$$\ell := d \left(2\left[(13.5)a^2 \right] + 2 \right) N(a, S^a) + 1$$

atoms such that

$$\sup_{x \in S} |f_G(x) - f_{G'}(x)| \leq \left(1 + \frac{1}{\sqrt{2\pi}}\right)(2\pi)^{-d/2}e^{-a^2/2}$$

and

$$\sup_{x \in S} \|\nabla f_G(x) - \nabla f_{G'}(x)\| \leq \left(a + \frac{3a}{\sqrt{2\pi}}\right)(2\pi)^{-d/2}e^{-a^2/2}.$$
We shall then prove below that inequalities (7.18) and (7.19) are satisfied. Fix \( x \in S \). Because \( \hat{B}(x,a) \) is contained in \( S^a \), the sets \( E_1, \ldots, E_L \) cover \( \hat{B}(x,a) \) i.e.,
\[
\hat{B}(x,a) \subseteq \bigcup_{i \in F} E_i
\]
where \( F := \{ 1 \leq i \leq L : E_i \cap \hat{B}(x,a) \neq \emptyset \} \). Also because the diameter of \( E_i \subseteq B_i \) is at most \( 2a \), we deduce that
\[
\hat{B}(x,a) \subseteq \bigcup_{i \in F} E_i \subseteq B(x,3a).
\]
We now use Lemma 7.3 with \( A = \bigcup_{i \in F} E_i \) and \( c = 3 \) to deduce that
\[
|f_G(x) - f_{G'}(x)| \leq \frac{1}{\sqrt{2\pi}} \left( \frac{9a^2e}{2(m+1)} \right)^{m+1} + \frac{e^{-a^2/2}}{(2\pi)^{d/2}}
\]
\[
\|\nabla f_G(x) - \nabla f_{G'}(x)\| \leq \frac{3a}{\sqrt{2\pi}} \left( \frac{9a^2e}{2(m+1)} \right)^{m+1} + \frac{ae^{-a^2/2}}{(2\pi)^{d/2}}
\]
Because \( m := \lfloor 13.5a^2 \rfloor \), we have \( m + 1 \geq 13.5a^2 \) and consequently,
\[
\left( \frac{9a^2e}{2(m+1)} \right)^{m+1} \leq \left( \frac{e}{3} \right)^{m+1} \leq \exp \left( -\frac{m + 1}{12} \right) \leq \exp \left( \frac{-27a^2}{24} \right) \leq e^{-a^2/2}
\]
where we have also used that \( (e/3)^6 \leq e^{-1/2} \). This proves both inequalities (7.18) and (7.19).

It therefore remains to prove that a discrete probability \( G' \) satisfying (7.20) can be chosen with at most \( \ell \) atoms where \( \ell \) is given by (7.17). This is guaranteed by Caratheodory’s theorem as argued below. Let \( \mathcal{P}(\mathbb{R}^d) \) denote the collection of all probability measures on \( \mathbb{R}^d \) and let
\[
T := \left\{ \left( \int \theta_j^k \{ \theta \in E_i \} dG(\theta), 1 \leq j \leq d, 0 \leq k \leq 2m + 1, 1 \leq i \leq L \right) : G \in \mathcal{P}(\mathbb{R}^d) \right\}.
\]
This set \( T \) is clearly a convex subset of \( \mathbb{R}^p \) for \( p := d(2m + 2)L \). Moreover, it is easy to see that \( T \) is simply the convex hull of
\[
C := \left\{ \left( \theta_j^k \{ \theta \in E_i \}, 1 \leq j \leq d, 0 \leq k \leq 2m + 1, 1 \leq i \leq L \right) : \theta \in S^a \right\}.
\]
Therefore, by Caratheodory’s theorem, every element of \( T \) can be written as a convex combination of at most \( p + 1 \) elements of \( C \). We therefore take \( G' \) to be the discrete probability measure supported upon these elements with probabilities given by the weights of this convex combination. Note that the number of atoms of \( G' \) is bounded from above by \( \ell \) given in (7.17). It is also easy to see that \( G' \) is supported on \( S^a \). This completes the proof.

7.1.3. Proof of Theorem 7.1.

**Proof.** Fix \( 0 < \eta \leq \frac{2\sqrt{2\pi}}{2\pi^{d/2}\sqrt{e}} \) and define \( a \) as in (7.3). Note that \( a \geq 1 \). Fix \( G \in \mathcal{G} \). According to Lemma 7.4, there exists a discrete probability measure \( G' \) supported on \( S^a \) and having \( \ell \) atoms (with \( \ell \) as in (7.17)) such that:
\[
\sup_{x \in S} |f_G(x) - f_{G'}(x)| \leq \left( 1 + \frac{1}{\sqrt{2\pi}} \right) (2\pi)^{-d/2} e^{-a^2/2}
\]
Similarly, and consequently
\[ \| \nabla f_G(x) - \nabla f_{G'}(x) \| \leq \left( a + \frac{3\alpha}{\sqrt{2\pi}} \right) (2\pi)^{-d/2} e^{-\alpha^2/2}. \]

(7.22)

Now let \( \alpha > 0 \) and let \( s_1, \ldots, s_D \) be an \( \alpha \)-cover of \( S^a \) (i.e., \( \sup_{s \in S^a} \inf_{i \leq D} \| s - s_i \| \leq \alpha \)) with \( D = N(\alpha, S^a) \). Now if \( G' = \sum_{i=1}^\ell w_i \delta_{a_i} \) (for some probability vector \( (w_1, \ldots, w_\ell) \) and atoms \( a_1, \ldots, a_\ell \in S^a \)), then let \( G'' := \sum_{i=1}^\ell w_i \delta_{b_i} \) where \( b_i \in \{ s_1, \ldots, s_D \} \) and \( \| a_i - b_i \| \leq \alpha \). Then, for every \( x \in S \),

\[
|f_{G'}(x) - f_{G''}(x)| = \left| \sum_{i=1}^\ell w_i \phi_d(x - a_i) - \sum_{i=1}^\ell w_i \phi_d(x - b_i) \right|
\leq \sum_{i=1}^\ell w_i |\phi_d(x - a_i) - \phi_d(x - b_i)|
\leq \sum_{i=1}^\ell w_i \sup_t \| \nabla \phi_d(t) \| \alpha
\leq \alpha \sup_t \| \nabla \phi_d(t) \| = \alpha (2\pi)^{-d/2} \sup_t \| t \| e^{-\| t \|^2/2} = \alpha (2\pi)^{-d/2} e^{-1/2}.
\]

We shall now bound \( \| \nabla f_{G'}(x) - \nabla f_{G''}(x) \| \) using similar arguments. By the mean value theorem, there exists \( u_i \) on the line segment joining \( x - a_i \) and \( x - b_i \) such that,

\[
\phi_d(x - b_i) = \phi_d(x - a_i) + (a_i - b_i) \nabla \phi_d(u_i)
\]

and consequently
\[
x - b_i = u_i + \zeta_i \quad \text{for some } \zeta_i \text{ satisfying } \| \zeta_i \| \leq \alpha.
\]

Similarly,
\[
\| \nabla f_{G'}(x) - \nabla f_{G''}(x) \| = \sum_{i=1}^\ell w_i \| \nabla \phi_d(x - a_i) - \nabla \phi_d(x - b_i) \|
= \sum_{i=1}^\ell w_i \| (a_i - x) \phi_d(x - a_i) - (b_i - x) \phi_d(x - b_i) \|
= \sum_{i=1}^\ell w_i \| (b_i - a_i) \phi_d(x - a_i) + (u_i + \zeta_i) (a_i - b_i) \nabla \phi_d(u_i) \|
\leq \alpha \sup_t \phi_d(t) + \alpha \sup_t \| t \| + \alpha \| \nabla \phi_d(t) \|
\leq \alpha \sup_t \phi_d(t) + \alpha \sup_t \| t \|^2 \phi_d(t) + \alpha^2 \sup_t \| t \| \phi_d(t)
= \frac{\alpha}{(2\pi)^{d/2}} \left[ 1 + \frac{2}{e} + \alpha \frac{1}{\sqrt{e}} \right]
\]
Now if $G'' := \sum_{i=1}^{\ell} w'_i \delta_{b_i}$ for some other probability vector $w' := (w'_1, \ldots, w'_\ell)$, then clearly

$$|f_{G'}(x) - f_{G''}(x)| = \left| \sum_{i=1}^{\ell} (w_i - w'_i) \phi(x - b_i) \right| \leq (2\pi)^{-d/2} \sum_{i=1}^{\ell} |w_i - w'_i|$$

$$\|\nabla f_{G'}(x) - \nabla f_{G''}(x)\| = \left\| \sum_{i=1}^{\ell} (w_i - w'_i) \nabla \phi(x - b_i) \right\| \leq \sum_{i=1}^{\ell} |w_i - w'_i| \left[ \sup_t \|t\phi_t(t)\| \right] = (2\pi)^{-d/2} e^{-1/2} \sum_{i=1}^{\ell} |w_i - w'_i|$$

Therefore if $\sum_{i=1}^{\ell} |w_i - w'_i| \leq v$, then

$$\sup_{x \in S} |f_{G'}(x) - f_{G''}(x)| \leq \left( 1 + \frac{1}{\sqrt{2\pi}} \right) (2\pi)^{-d/2} e^{-a/2} + \alpha (2\pi)^{-d/2} e^{-1/2} + (2\pi)^{-d/2} v$$

and

$$\sup_{x \in S} \|\nabla f_{G'}(x) - \nabla f_{G''}(x)\| \leq \left( a + \frac{3a}{\sqrt{2\pi}} \right) (2\pi)^{-d/2} e^{-a/2} + \alpha (2\pi)^{-d/2} \left[ 1 + \frac{2}{e} + \alpha \frac{1}{\sqrt{e}} \right]$$

By choosing

$$v = \alpha = \frac{(2\pi)^{d/2}}{2\sqrt{2\pi}} \eta \quad \text{and} \quad a = \sqrt{2 \log \frac{2\sqrt{2\pi}}{(2\pi)^{d/2} \eta}} = \sqrt{2 \log \frac{1}{\alpha}},$$

we obtain

$$\sup_{x \in S} |f_{G'}(x) - f_{G''}(x)| \leq \frac{a}{(2\pi)^{d/2}} \left[ 2 + \frac{1}{\sqrt{2\pi}} + \frac{1}{\sqrt{e}} \right] < \eta$$

$$\sup_{x \in S} \|\nabla f_{G'}(x) - \nabla f_{G''}(x)\| \leq \frac{a \alpha}{(2\pi)^{d/2}} \left[ 2 + \frac{3}{\sqrt{2\pi}} + \frac{3}{e} + \frac{1}{\sqrt{e}} \right] < a \eta$$

where we have noted that $a \geq 1$ and $\alpha \leq e^{-1/2}$.

It only remains to count the number of ways of choosing the discrete probability measure $G''$. The number of ways of choosing the atoms of $G''$ is clearly

$$\binom{D}{\ell} \leq D^\ell \leq \left( \frac{De}{\ell} \right)^\ell$$

where we used that $\ell! \geq (\ell/e)^\ell$, a fact that follows from Stirling’s formula.

The probability vector $w' = (w'_1, \ldots, w'_\ell)$ can be chosen to belong to a $v$-covering set for all $\ell$-dimensional probability vectors under the $L^1$ norm. This covering number is well known to be at most: $(1 + (2/v)^\ell$. Therefore $N(\eta, M, d_S, a_1, a_2)$ is bounded from above by:

$$\left[ D e \left( 1 + \frac{2}{v} \right) \right]^\ell = A^\ell \quad \text{where} \quad A := \frac{De}{\ell} \left( 1 + \frac{2}{v} \right).$$

We shall bound $A$ below. Below $C_d$ will denote a constant that depends on $d$ alone. Because $v \leq e^{-1/2}$,

$$1 + \frac{2}{v} \leq \left( \frac{1}{\sqrt{e}} + 2 \right) \frac{1}{v} = \frac{C_d}{\eta}.$$
Also note that from the expression for \( \ell \) given in (7.17), we have \( \ell \geq N(a, S^a) \) and hence
\[
\frac{D}{\ell} \leq \frac{N(\alpha, S^a)}{N(a, S^a)} \leq N(\alpha, B(0, a)) \leq \left( 1 + \frac{a}{\alpha} \right)^d \leq C_d \left( \frac{1}{\eta} \right)^{3d/2}.
\]
where we have used the trivial fact that
\[
(7.23) \quad a = \sqrt{2 \log \frac{1}{\alpha}} \leq \sqrt{\frac{4}{\alpha}} = C_d \frac{1}{\sqrt{\eta}}.
\]
We thus have
\[
A \leq C_d \eta^{-1 - 3d/2}
\]
so that,
\[
\log N(\eta, \mathcal{M}, \|\cdot\|_S) \leq \ell \log A \leq C_d \ell \log \frac{1}{\eta}
\]
which along with the expression (7.17) for \( \ell \) proves (7.13). Similarly,
\[
\log N(a\eta, \mathcal{M}, \|\cdot\|_{S^\alpha}) \leq C_d \ell \log \frac{1}{\eta} \leq C_d N(a, S^a) \log \eta^2.
\]
This implies that
\[
\log N(\eta, \mathcal{M}, \|\cdot\|_{S^\alpha}) \leq C_d N(a, S^a) \left| \log \frac{\eta}{a} \right| \leq C_d N(a, S^a) \log \eta^2
\]
where the last inequality follows from (7.23). This completes the proof of (7.14) and consequently Theorem 7.1.

\[\square\]

7.2. Proof of Corollary 7.2.

**Proof of Corollary 7.2.** Fix \( \rho > 0, 0 < \eta \leq \frac{2\sqrt{2\pi}}{(2\pi)^d/2\sqrt{e}} \) and compact subset \( S \subseteq \mathbb{R}^d \). For \( a, b \in \mathbb{R} \), we shall denote the maximum of \( a \) and \( b \) by \( a \vee b \). Note first that for every pair of densities \( f_G, f_H \in \mathcal{M} \) and \( x \in S \), we have
\[
\left\| \frac{\nabla f_G(x)}{\rho \vee f_G(x)} - \frac{\nabla f_H(x)}{\rho \vee f_H(x)} \right\| \leq \left\| \frac{\nabla f_G(x)}{\rho \vee f_G(x)} - \frac{\nabla f_G(x)}{\rho \vee f_H(x)} + \frac{\nabla f_G(x)}{\rho \vee f_H(x)} - \frac{\nabla f_H(x)}{\rho \vee f_H(x)} \right\|
\]
\[
\leq \frac{\|\nabla f_G(x)\|}{\rho \vee f_G(x)} \left\| \frac{\rho \vee f_G(x) - \rho \vee f_H(x)}{\rho \vee f_H(x)} \right\| + \frac{1}{\rho} \|\nabla f_G(x) - \nabla f_H(x)\|.
\]
Using inequality (A.2) (in Lemma A.2) and the fact that \( t \mapsto \rho \vee t \) is 1-Lipschitz, we deduce from the above that
\[
\left\| \frac{\nabla f_G(x)}{\rho \vee f_G(x)} - \frac{\nabla f_H(x)}{\rho \vee f_H(x)} \right\| \leq \sqrt{\frac{1}{\rho^2} \log \frac{1}{(2\pi)^d \rho^2} \|f_G(x) - f_H(x)\|} + \frac{1}{\rho} \|f_G(x) - f_H(x)\|.
\]
Because this is true for every \( x \in S \), we have
\[
\|f_G - f_H\|_{S^\alpha} \leq \sqrt{\frac{1}{\rho^2} \log \frac{1}{(2\pi)^d \rho^2} \|f_G - f_H\|_S} + \frac{1}{\rho} \|f_G - f_H\|_{S^\alpha}.
\]
We thus have
\[
N(\eta^*, T_{\rho}, \|\cdot\|_S) \leq N(\eta, \mathcal{M}, \|\cdot\|_S) + N(\eta, \mathcal{M}, \|\cdot\|_{S^\alpha})
\]
from which (7.5) follows. \[\square\]
8. Bounding Bayes Discrepancy via Hellinger Distance. The purpose of this section is to state and prove the following theorem relating the quantity:

$$\Gamma(G_0, G, \rho) := \left( \int \left\| \nabla f_G(x) \right\|_{\max(f_G(x), \rho)}^2 f_G(x) dx \right)^{1/2}$$

for \( \rho > 0 \) and two probability measures \( G_0 \) and \( G \) on \( \mathbb{R}^d \) in terms of the squared Hellinger distance between \( f_G \) and \( f_{G_0} \). This result is crucial for the proof of Theorem 3.1.

**Theorem 8.1.** There exists a universal positive constant \( C \) such that for every pair of probability measures \( G \) and \( G_0 \) on \( \mathbb{R}^d \) and \( 0 < \rho \leq (2\pi)^{-d/2}e^{-1/2} \), we have

$$\Gamma^2(G_0, G, \rho) \leq C d \max \left\{ \left( \log \left( \frac{2\pi}{\rho} \right)^{-d/2} \right)^3, |\log H|, |\log \delta_H| \right\} \delta^2(f_G, f_{G_0})$$

where \( \Gamma(G_0, G, \rho) \) is defined as in (8.1).

The above theorem is a generalization of Jiang and Zhang [28, Theorem 3] to the case when \( d \geq 1 \). Its proof given below follows Jiang and Zhang [28, Proof of Theorem 3] with appropriate changes to deal with the \( d \geq 1 \) case. Lemma A.2 and Lemma A.3 from Section A will be used in this proof.

**Proof of Theorem 8.1.** For real numbers \( a \) and \( b \), we denote \( \max(a, b) \) by \( a \vee b \). For functions \( u: \mathbb{R}^d \rightarrow \mathbb{R}^d \), we let

$$\|u\|_0 := \left( \int \|u(x)\|^2 f_{G_0}(x) dx \right)^{1/2}$$

so that

$$\Gamma(G_0, G, \rho) = \left\| \frac{\nabla f_G}{f_G \vee \rho} - \frac{\nabla f_{G_0}}{f_{G_0} \vee \rho} \right\|_0$$

$$= \left\| \frac{\nabla f_G}{f_G \vee \rho} - \frac{2\nabla f_G}{f_G \vee \rho + f_{G_0} \vee \rho} + \frac{\nabla f_G - \nabla f_{G_0}}{f_G \vee \rho + f_{G_0} \vee \rho} + \frac{2\nabla f_{G_0}}{f_G \vee \rho + f_{G_0} \vee \rho} - \frac{\nabla f_{G_0}}{f_{G_0} \vee \rho} \right\|_0$$

$$\leq 2 \max_{H \in \{G, G_0\}} \frac{\| (\nabla f_H) |f_G \vee \rho - f_{G_0} \vee \rho| \|_0}{(f_G \vee \rho)(f_G \vee \rho + f_{G_0} \vee \rho)} + 2 \frac{\| \nabla f_G - \nabla f_{G_0} \|_0}{f_G \vee \rho + f_{G_0} \vee \rho},$$

where we have used the triangle inequality for \( ||||_0 \) in the last step. Let us represent the two terms on the right hand side above by \( T_1 \) and \( T_2 \) respectively so that \( \Gamma(G_0, G, \rho) \leq T_1 + T_2 \). We shall now bound \( T_1 \) and \( T_2 \) separately. For \( T_1 \), we use inequality (A.2) in Lemma A.2 (note that we have...
assumed $0 < \rho \leq (2\pi)^{-d/2}e^{-1/2})$. This inequality allows us to bound $T_1$ as follows:

\[
\frac{1}{4} T_1^2 = \max_{H \in \{G,G_0\}} \int \frac{\|\nabla f_H\|^2 (f_G \lor \rho - f_G \lor \rho)^2}{(f_H \lor \rho)^2 (f_G \lor \rho + f_G \lor \rho)^2} f_G
\leq \left[ \log \left( \frac{(2\pi)^{-d}}{\rho^2} \right) \right] \int \frac{(f_G \lor \rho - f_G \lor \rho)^2}{(f_G \lor \rho + f_G \lor \rho)^2} f_G
\leq \left[ \log \left( \frac{(2\pi)^{-d}}{\rho^2} \right) \right] \int \frac{(f_G - f_G_0)^2}{(f_G \lor \rho + f_G \lor \rho)^2} f_G
\leq 2 \left[ \log \left( \frac{(2\pi)^{-d}}{\rho^2} \right) \right] \int \frac{(\sqrt{f_G} - \sqrt{f_G_0})^2}{(f_G \lor \rho + f_G \lor \rho)^2} f_G
\leq 2 \left[ \log \left( \frac{(2\pi)^{-d}}{\rho^2} \right) \right] \int (\sqrt{f_G} - \sqrt{f_G_0})^2 f_G
\leq 2 \left[ \log \left( \frac{(2\pi)^{-d}}{\rho^2} \right) \right] \int (\sqrt{f_G} - \sqrt{f_G_0})^2 = 2 \left[ \log \left( \frac{(2\pi)^{-d}}{\rho^2} \right) \right] \delta^2(f_G, f_G_0)
\]

which gives

(8.3) \[ T_1 \leq 2\sqrt{2} \delta(f_G, f_G_0) \sqrt{\log \left( \frac{(2\pi)^{-d}}{\rho^2} \right)}. \]

We shall now deal with $T_2$. This requires an elaborate argument. Start by writing

\[
\frac{1}{4} T_2^2 = \int \frac{\|\nabla f_G - \nabla f_G_0\|^2}{f_G \lor \rho + f_G \lor \rho} f_G
\leq \int \frac{\|\nabla f_G - \nabla f_G_0\|^2}{f_G \lor \rho + f_G \lor \rho} \left( \frac{f_G_0}{f_G \lor \rho + f_G \lor \rho} \right) \leq \int \frac{\|\nabla f_G - \nabla f_G_0\|^2}{f_G \lor \rho + f_G \lor \rho} = \sum_{i=1}^d \Delta_{i,1}^2
\]

where, for $1 \leq i \leq d$ and $k \geq 0$,

\[
\Delta_{i,k}^2 := \int \frac{(\partial^k_i (f_G - f_G_0))^2}{f_G \lor \rho + f_G \lor \rho} \quad \text{with} \quad \partial^k_i f := \partial^k_i \frac{\partial}{\partial \rho} f.
\]

The next task therefore is to bound $\Delta_{i,1}^2$ from above. Before dealing with $\Delta_{i,1}^2$, let us first note that it is easy to bound $\Delta_{i,0}$ by the Hellinger distance between $f_G$ and $f_G_0$. Indeed, we can write

\[
\Delta_{i,0}^2 = \int \frac{(f_G - f_G_0)^2}{f_G \lor \rho + f_G \lor \rho} = \int \frac{(\sqrt{f_G} - \sqrt{f_G_0})^2}{f_G \lor \rho + f_G \lor \rho} \left( \frac{f_G + f_G_0}{f_G \lor \rho + f_G \lor \rho} \right) \leq 2 \int \frac{(\sqrt{f_G} - \sqrt{f_G_0})^2}{f_G \lor \rho + f_G \lor \rho} \leq 2 \delta^2(f_G, f_G_0).
\]

A simple upper bound for $\Delta_{i,k}^2$ for general $k \geq 1$ can be obtained via Lemma A.3. Indeed, noting that (via $f_G \lor \rho + f_G \lor \rho \geq 2\rho$)

\[
\Delta_{i,k}^2 \leq \frac{1}{2\rho} \int \left( \partial^k_i (f_G - f_G_0) \right)^2
\]
we can apply Lemma A.3 to deduce that

\begin{equation}
\Delta_{i,k}^2 \leq \frac{2(2\pi)^{-d/2}}{\rho} \left\{ a^{2k} \mathbb{E}^2(f_G, f_{G_0}) + \sqrt{\frac{2}{\pi} a^{2k-1} e^{-a^2}} \right\} \quad \text{for every } a \geq \sqrt{2k - 1}.
\end{equation}

The problem with this bound is the presence of $\rho$ in the denominator. This $\rho$ will be, in applications of Theorem 8.1, of the order $n^{-1}$ which makes the above bound quite large. The more refined argument below will get rid of the $\rho$ factor in the denominator. This argument involves integration by parts for bounding $\Delta_{i,1}^2$. It will be clear that the use of integration by parts will result in expressions involving $\Delta_{i,k}^2$ for $k \geq 2$. It will then become necessary to deal with $\Delta_{i,k}^2$ for $k \geq 2$ even though we are only interested in $\Delta_{i,1}^2$. Indeed, integration by parts gives, for $k \geq 1$,

\begin{equation}
\Delta_{i,k}^2 = -\int \left[ \partial_i f_G - f_G \right] \left[ \partial_i f_{G_0} \right] \partial_i \left( \frac{1}{f_G \lor \rho + f_{G_0} \lor \rho} \right)
\end{equation}

Note now that, almost surely

\begin{equation}
\left| \partial_i \left( \frac{1}{f_G \lor \rho + f_{G_0} \lor \rho} \right) \right| \leq \frac{|\partial_i f_G| + |\partial_i f_{G_0}|}{(f_G \lor \rho + f_{G_0} \lor \rho)^2}
\end{equation}

\begin{equation}
\leq \frac{|\partial_i f_G|/(f_G \lor \rho) + |\partial_i f_{G_0}|/(f_{G_0} \lor \rho)}{f_G \lor \rho + f_{G_0} \lor \rho}
\end{equation}

\begin{equation}
\leq \frac{\|\nabla f_G\|/(f_G \lor \rho) + \|\nabla f_{G_0}\|/(f_{G_0} \lor \rho)}{f_G \lor \rho + f_{G_0} \lor \rho}
\end{equation}

\begin{equation}
\leq \frac{2}{f_G \lor \rho + f_{G_0} \lor \rho} \sqrt{\log \left( \frac{(2\pi)^{-d}}{\rho^2} \right)}
\end{equation}

where, in the last inequality, we used (A.2) in Lemma A.2. Imputing the above inequality into (8.7), we obtain

\begin{equation}
\Delta_{i,k}^2 \leq 2 \sqrt{\log \left( \frac{(2\pi)^{-d}}{\rho^2} \right)} \int \left[ \frac{\partial_i f_G - f_G}{f_G \lor \rho + f_{G_0} \lor \rho} \right] \left[ \frac{\partial_i f_{G_0} - f_{G_0}}{f_{G_0} \lor \rho + f_G \lor \rho} \right]
\end{equation}

Applying the Cauchy-Schwarz inequality to each of the two terms on the right hand side above, we obtain

\begin{equation}
\Delta_{i,k}^2 \leq 2 \sqrt{\log \left( \frac{(2\pi)^{-d}}{\rho^2} \right)} \sqrt{\int \left( \frac{\partial_i f_G - f_G}{f_G \lor \rho + f_{G_0} \lor \rho} \right)^2 \int \left( \frac{\partial_i f_{G_0} - f_{G_0}}{f_{G_0} \lor \rho + f_G \lor \rho} \right)^2}
\end{equation}

\begin{equation}
\text{which can be rewritten as}
\end{equation}

\begin{equation}
\Delta_{i,k}^2 \leq \Upsilon \Delta_{i,k-1} \Delta_{i,k} + \Delta_{i,k-1} \Delta_{i,k+1} \quad \text{where } \Upsilon := 2 \sqrt{\log \left( \frac{(2\pi)^{-d}}{\rho^2} \right)}.
\end{equation}
The strategy to bound $\Delta_{i,1}$ is now as follows. Divide both sides of (8.8) by $\Delta_{i,k-1}\Delta_{i,k}$ to get

$$\frac{\Delta_{i,k}}{\Delta_{i,k-1}} \leq \Upsilon + \frac{\Delta_{i,k+1}}{\Delta_{i,k}} \quad \text{for every } k \geq 1. \tag{8.9}$$

Fix an integer $k_0 \geq 1$ and a real number $\beta > 0$. Our bound on $\Delta_{i,1}$ will depend on $k_0$ and $\beta$ and the bound will be optimized for $k_0$ and $\beta$ at the end.

Suppose first that there exists an integer $1 \leq k \leq k_0$ such that $\Delta_{i,k+1} = \beta \Delta_{i,k}$. Then applying (8.9) recursively for $1, \ldots, k$, we obtain

$$\frac{\Delta_{i,1}}{\Delta_{i,0}} \leq k \Upsilon + \beta$$

so that, by (8.5),

$$\Delta_{i,1} \leq (k \Upsilon + \beta) \Delta_{i,0} \leq \sqrt{2} (k \Upsilon + \beta) \mathcal{E}(f_G, f_{G_0}) \leq \sqrt{2} (k_0 \Upsilon + \beta) \mathcal{E}(f_G, f_{G_0}). \tag{8.10}$$

Now suppose that $\Delta_{i,k+1} > \beta \Delta_{i,k}$ for every integer $1 \leq k \leq k_0$. In this case, we deduce from (8.9) that

$$\frac{\Delta_{i,k+1}}{\Delta_{i,k}} \leq \left(1 + \frac{\Upsilon}{\beta}\right) \frac{\Delta_{i,k+1}}{\Delta_{i,k}} \leq \left(1 + \frac{\Upsilon}{\beta}\right) \frac{\Delta_{i,k+1}}{\Delta_{i,k}} \quad \text{for every } k = 0, \ldots, k_0.$$

A recursive application of this inequality implies that

$$\frac{\Delta_{i,1}}{\Delta_{i,0}} \leq \left(1 + \frac{\Upsilon}{\beta}\right)^{k} \frac{\Delta_{i,k+1}}{\Delta_{i,k}} \quad \text{for every } k = 0, \ldots, k_0.$$

To obtain a bound for $\Delta_{i,1}/\Delta_{i,0}$ that depends only on $\Delta_{i,k_0+1}$ and $\Delta_{i,0}$, one can take the geometric mean of the above inequality for $k = 0, 1, \ldots, k_0$. This gives

$$\frac{\Delta_{i,1}}{\Delta_{i,0}} \leq \left(\prod_{k=0}^{k_0} \left(1 + \frac{\Upsilon}{\beta}\right)^{k} \frac{\Delta_{i,k+1}}{\Delta_{i,k}}\right)^{1/(k_0+1)} = \left(1 + \frac{\Upsilon}{\beta}\right)^{k_0/2} \Delta_{i,k_0+1}^{1/(k_0+1)} \Delta_{i,0}^{-1/(k_0+1)}$$

which is same as

$$\Delta_{i,1} \leq \left(1 + \frac{\Upsilon}{\beta}\right)^{k_0/2} \Delta_{i,k_0+1}^{1/(k_0+1)} \Delta_{i,0}^{-1/(k_0+1)}$$

Now using (8.5) and the bound (8.6) (with $k = k_0 + 1$), we obtain

$$\Delta_{i,1} \leq \left(1 + \frac{\Upsilon}{\beta}\right)^{k_0/2} \left(\frac{2(2\pi)^{-d/2}}{\rho} \left[a^{2k_0+2} \mathcal{E}^2(f_G, f_{G_0}) + \sqrt{\frac{2}{\pi}} a^{2k_0+1} e^{-a^2}\right]\right)^{1/(2k_0+2)} \frac{\gamma}{2k_0+2} \Delta_{i,k_0+1}^{1/(k_0+1)} \Delta_{i,0}^{-1/(k_0+1)}$$

for every $a \geq \sqrt{2k_0+1}$. The final bound obtained for $\Delta_{i,1}$ is the maximum of the right hand side above and the right hand side of (8.10). This bound will need to be optimized by choosing $k_0$, $\beta$ and $a \geq \sqrt{2k_0+1}$ appropriately.

$\beta$ will be chosen as $\beta = k_0 \Upsilon$ so that the bound (8.10) becomes $2\sqrt{2k_0} \Upsilon \mathcal{E}(f_G, f_{G_0})$ and the term $(1 + \Upsilon/\beta)^{k_0/2}$ appearing in (8.11) is bounded by $e$. To select $k_0$, the key is to focus on the term involving $\rho$ in (8.11) which is

$$\left(\frac{2\pi}{\rho}\right)^{-d/2} \exp\left(\frac{\Upsilon^2}{16(k_0+1)}\right).$$
This suggests taking \( k_0 \) to be the smallest integer \( \geq 1 \) such that \( k_0 + 1 \geq \Upsilon^2/8 \) so that the above term is at most \( \sqrt{e} \). Finally \( a \) will be taken to be
\[
\begin{align*}
a := \max \left( \sqrt{2k_0 + 1}, \sqrt{2} \log \frac{\mathcal{S}(f_G, f_{G_0})}{\mathcal{S}(f_G, f_{G_0})} \right)
\end{align*}
\]
which will ensure that \( e^{-a^2} \leq \mathcal{S}^2(f_G, f_{G_0}) \) and the term involving \( a \) in (8.11) can then be bounded by
\[
\begin{align*}
\left( a^{2k_0+2}\mathcal{S}^2(f_G, f_{G_0}) + \sqrt{\frac{2}{\pi}}a^{2k_0+1}e^{-a^2} \right)^{2k_0+2} \leq a \left( 1 + \sqrt{\frac{2}{\pi}} \right)^{2k_0+2} \left( \mathcal{S}(f_G, f_{G_0}) \right)^{\frac{1}{2k_0+2}} \\
\leq \left( 1 + \sqrt{\frac{2}{\pi}} \right) a \left( \mathcal{S}(f_G, f_{G_0}) \right)^{\frac{1}{2k_0+2}} \\
\leq 2a \left( \mathcal{S}(f_G, f_{G_0}) \right)^{\frac{1}{2k_0+2}}.
\end{align*}
\]

We have therefore proved that the right hand side in (8.11) is bounded from above by \( 2\sqrt{2}ae\mathcal{S}(f_G, f_{G_0}) \). Because \( \Delta_{i,1} \) is bounded by the maximum of the bounds given by (8.10) and (8.11), we obtain:
\[
\begin{align*}
\Delta_{i,1} \leq 2\sqrt{2} \max \left\{ k_0\Upsilon, ea \right\} \mathcal{S}(f_G, f_{G_0}) \leq 2\sqrt{2} \max \left\{ k_0\Upsilon, e\sqrt{2k_0+1}, e\sqrt{2} \log \frac{\mathcal{S}(f_G, f_{G_0})}{\mathcal{S}(f_G, f_{G_0})} \right\} \mathcal{S}(f_G, f_{G_0}).
\end{align*}
\]

Now because \( k_0 \) is chosen to be the smallest integer \( \geq 1 \) such that \( k_0 + 1 \geq \Upsilon^2/8 \), we have
\[
\begin{align*}
k_0 \leq 1 + \frac{\Upsilon^2}{8} = \log \frac{e(2\pi)^{-d/2}}{\rho} \leq \frac{3}{2} \log \frac{(2\pi)^{-d/2}}{\rho}
\end{align*}
\]
because \( \rho \leq (2\pi)^{-d/2}e^{-1/2} \). This, along with the expression for \( \Upsilon \), gives
\[
\begin{align*}
\Delta_{i,1} \leq C \max \left\{ \left( \log \frac{(2\pi)^{-d/2}}{\rho} \right)^{3/2}, \sqrt{\log \frac{\mathcal{S}(f_G, f_{G_0})}{\mathcal{S}(f_G, f_{G_0})}} \right\} \mathcal{S}(f_G, f_{G_0})
\end{align*}
\]
where \( C \) is a universal positive constant. Combining with (8.4), we deduce that
\[
\begin{align*}
\mathcal{T}_2^2 \leq C d \max \left\{ \left( \log \frac{(2\pi)^{-d/2}}{\rho} \right)^{3}, \left| \log \frac{\mathcal{S}(f_G, f_{G_0})}{\mathcal{S}(f_G, f_{G_0})} \right| \right\} \mathcal{S}^2(f_G, f_{G_0}).
\end{align*}
\]
The proof of Theorem 8.1 is now completed by combining the above inequality with the bound (8.3) and the fact that \( \Gamma(G_0, G, \rho) \leq \mathcal{T}_1 + \mathcal{T}_2 \) (which implies that \( \Gamma^2(G_0, G, \rho) \leq 2\mathcal{T}_1^2 + 2\mathcal{T}_2^2 \)).

**APPENDIX A: AUXILIARY RESULTS**

This section collects various results which were used in the proofs of the main results of the paper.

We start with the following lemma which is standard and is stated without proof.

**Lemma A.1.** Suppose \( X|\theta \sim N(\theta, \sigma^2 I_d) \) and \( \theta \sim G \). Then
\[
\mathbb{E}(\theta|X) = X + \sigma^2 \frac{\nabla f_G(X)}{f_G(X)}
\]
and
\[
\mathbb{E} \left\| X + \sigma^2 \frac{\nabla f_G(X)}{f_G(X)} - \theta \right\|^2 = d\sigma^2 - \sigma^4 \int \left\| \frac{\nabla f_G}{f_G} \right\|^2 f_G.
\]
The following lemma generalizes Jiang and Zhang [28, Lemma A.1] to the case $d \geq 1$.

**Lemma A.2.** Fix a probability measure $G$ on $\mathbb{R}^d$. For every $x \in \mathbb{R}^d$, we have ($\|\cdot\|$ denotes the usual Euclidean norm on $\mathbb{R}^d$)

\[(A.1) \quad \left( \frac{\|\nabla f_G(x)\|}{f_G(x)} \right)^2 \leq \text{tr} \left( I_d + \frac{H f_G(x)}{f_G(x)} \right) \leq \log \left( \frac{2\pi}{{\rho}^2} \right) \rho^2 \leq (2\pi)^{-d/4} e^{-1/4}
\]

where $\nabla$ and $H$ stand for gradient and Hessian respectively and $\text{tr}$ denotes trace.

Also for every $x \in \mathbb{R}^d$, we have

\[(A.2) \quad \frac{\|\nabla f_G(x)\|}{\max(f_G(x), \rho)} \leq \sqrt{\log \left( \frac{2\pi}{{\rho}^2} \right) \rho^2} \quad 0 < \rho \leq (2\pi)^{-d/2} e^{-1/2}
\]

and

\[(A.3) \quad \left( \frac{\|\nabla f_G(x)\|}{f_G(x)} \right)^2 \frac{f_G(x)}{f_G(x) \vee \rho} \leq \log \left( \frac{2\pi}{{\rho}^2} \right) \rho^2 \quad \text{for } 0 < \rho \leq (2\pi)^{-d/2} e^{-1}.
\]

**Proof of Lemma A.2.** If $\theta \sim G$ and $X|\theta \sim N(\theta, I_d)$, then it is easy to verify that, for every $x \in \mathbb{R}^d$,

\[\nabla f_G(x) = \mathbb{E}(\theta - X | X = x)\]

and

\[H f_G(x) = -I_d + \mathbb{E}((\theta - X)(\theta - X)^T | X = x).
\]

From here, we can deduce that

\[I_d + \frac{H f_G(x)}{f_G(x)} = \mathbb{E}((\theta - X)(\theta - X)^T | X = x)
\]

\[= (\mathbb{E}(\theta - X | X = x))(\mathbb{E}(\theta - X | X = x))^T + \mathbb{E}((\theta - \mathbb{E}(\theta | X = x))(\theta - \mathbb{E}(\theta | X = x))^T | X = x)
\]

\[= \frac{\nabla f_G(x)}{f_G(x)} (\nabla f_G(x))^T + \mathbb{E}((\theta - \mathbb{E}(\theta | X = x))(\theta - \mathbb{E}(\theta | X = x))^T | X = x)
\]

and hence

\[(A.5) \quad I_d + \frac{H f_G(x)}{f_G(x)} \succeq \frac{\nabla f_G(x)}{f_G(x)} (\nabla f_G(x))^T f_G(x)
\]

where $A \succeq B$ means that $A - B$ is non-negative definite.

Also from (A.4) and the convexity of $A \mapsto \exp(tr(A)/2)$ ($tr(A)$ denotes the trace of the $d \times d$ matrix $A$), we have

\[\exp \left( \frac{1}{2} tr \left( I_d + \frac{H f_G(x)}{f_G(x)} \right) \right) = \exp \left( \frac{1}{2} tr \left( \exp \left( \frac{1}{2} tr((\theta - X)(\theta - X)^T) \right) | X = x \right) \right)
\]

\[\leq \mathbb{E} \left( \exp \left( \frac{1}{2} tr((\theta - X)(\theta - X)^T) \right) | X = x \right)
\]

\[= \mathbb{E} \left( \exp \left( \frac{1}{2} \|X - \theta\|^2 \right) | X = x \right) = \left( \frac{2\pi}{f_G(x)} \right)^{-d/2}
\]
so that we have
\[ tr \left( I_d + \frac{H f_G(x)}{f_G(x)} \right) \leq \log \left( \frac{(2\pi)^{-d}}{f_G^2(x)} \right). \]

Combining with (A.5), we obtain (A.1).

To prove (A.2), note first from (A.1) that
\[ \frac{\|\nabla f_G(x)\|}{\max(f_G(x), \rho)} \leq \sqrt{\frac{(2\pi)^{-d}}{f_G^2(x)}} \frac{f_G(x)}{\max(f_G(x), \rho)} = \begin{cases} \sqrt{\frac{(2\pi)^{-d}}{f_G^2(x)}} \leq \sqrt{\frac{(2\pi)^{-d}}{\rho^2}} & \text{if } f_G(x) > \rho \\ \sqrt{\frac{(2\pi)^{-d}}{f_G^2(x)}} \frac{f_G(x)}{\rho} & \text{if } f_G(x) \leq \rho \end{cases} \]

The function \( v \mapsto v \log \left( \frac{(2\pi)^{-d}}{v} \right) \) is non-decreasing on \((0, (2\pi)^{-d}/e]\) and hence when \( f_G^2(x) \leq \rho^2 \leq (2\pi)^{-d}/e\), the inequality
\[ \sqrt{\frac{(2\pi)^{-d}}{f_G^2(x)}} \frac{f_G(x)}{\rho} \leq \sqrt{\frac{(2\pi)^{-d}}{\rho^2}} \]
holds and this proves (A.2).

We now turn to (A.3). Whenever \( f_G(x) \geq \rho \), note that (A.3) follows directly from (A.2). Thus, (A.3) only needs to be established when \( f_G(x) < \rho \). In this case using (A.1),
\[ \left( \frac{\|\nabla f_G(x)\|}{f_G(x)} \right)^2 \frac{f_G(x)}{\max\{f_G(x), \rho\}} \leq \left( \frac{f_G(x)}{\rho} \right) \log \left( \frac{(2\pi)^{-d}}{f_G^2(x)} \right) = 2 \log \left( \frac{(2\pi)^{-d/2}}{f_G(x)} \right) \frac{f_G(x)}{\rho} \]
From here we note that \( v \mapsto v \log \left( \frac{(2\pi)^{-d}}{v^2} \right) \) is non-decreasing on \((0, (2\pi)^{-d/2}/e]\). This, along with \( f_G(x) < \rho \), immediately implies (A.3). \( \square \)

For an infinitely differentiable function \( u : \mathbb{R}^d \to \mathbb{R} \), \( 1 \leq i \leq d \) and \( k \geq 1 \), let \( \partial_i^k u : \mathbb{R}^d \to \mathbb{R} \) denote the function
\[ (\partial_i^k u)(x) := \frac{\partial^k}{\partial x_i^k} u(x). \]

**Lemma A.3.** For every pair of probability measures \( G \) and \( G_0 \) on \( \mathbb{R}^d \), \( 1 \leq i \leq d \) and \( k \geq 1 \), we have
\[ \left( \int \partial_i^k (f_G(x) - f_{G_0}(x)) \right)^2 dx \leq \frac{4(2\pi)^{-d/2}}{a \geq \sqrt{2k-1}} \inf_{a \geq \sqrt{2k-1}} \left\{ a^{2k} \mathbb{F}^2(f_G, f_{G_0}) + \sqrt{\frac{2}{\pi}} a^{2k-1} e^{-a^2} \right\}. \]

**Proof of Lemma A.3.** Fix \( a \geq \sqrt{2k-1} \) and assume, without loss of generality, that \( i = 1 \). Let
\[ f_{G,1}^*(u, x_2, \ldots, x_d) := \int e^{iu_1} f_G(x) dx_1 \]
denote the Fourier transform of \( f_G \) treated as a function of \( x_1 \). The function \( f_{G,1}^* \) is defined analogously. For ease of notation, we shall suppress the dependence of \( f_{G,1}^* \) on \( u, x_2, \ldots, x_d \) below and write it simply as \( f_{G,1}^*(u) \) (resp. \( f_{G,0,1}^* \)).

For every \( x_2, \ldots, x_d \), we then have (by Plancherel’s identity)
\begin{align*}
2\pi \int \left\{ \partial_1^k (f_G(x) - f_{G_0}(x)) \right\}^2 dx_1 &= \int u^{2k} \left| f_{G,1}^*(u) - f_{G,0,1}^*(u) \right|^2 du \\
&\leq a^{2k} \int \left| f_{G,1}^*(u) - f_{G,0,1}^*(u) \right|^2 du + \int_{|u| > a} u^{2k} \left| f_{G,1}^*(u) - f_{G,0,1}^*(u) \right|^2 du \\
&= (2\pi)a^{2k} \int (f_G(x) - f_{G,0}(x))^2 dx_1 + \int_{|u| > a} u^{2k} \left| f_{G,1}^*(u) - f_{G,0,1}^*(u) \right|^2 du \\
&\leq (2\pi)a^{2k} \int (f_G(x) - f_{G,0}(x))^2 dx_1 + \frac{\sqrt{\frac{2}{\pi}} a^{2k-1} e^{-a^2}}{a \geq \sqrt{2k-1}} \left( a^{2k} \mathbb{F}^2(f_G, f_{G_0}) + \sqrt{\frac{2}{\pi}} a^{2k-1} e^{-a^2} \right) \int_{|u| > a} u^{2k} \left| f_{G,1}^*(u) - f_{G,0,1}^*(u) \right|^2 du.
\end{align*}
for every $a > 0$. Also note that for every $u, x_2, \ldots, x_d \in \mathbb{R},$

$$f_{G,1}^*(u) = \int e^{iu_1} \left( \int \phi_d(x - \theta) dG(\theta) \right) dx_1$$

$$= \int \left( \int e^{iu_1} \phi_d(x - \theta) dx_1 \right) dG(\theta)$$

$$= \int (2\pi)^{-d/2} \left[ \int e^{iu_1} e^{-(x_1 - \theta_1)^2/2} dx_1 \right] \exp \left( - \sum_{j \neq 1} (x_j - \theta_j)^2 / 2 \right) dG(\theta)$$

$$= (2\pi)^{-(d-1)/2} \int e^{iu_1} e^{-u^2/2} \exp \left( - \sum_{j \neq 1} (x_j - \theta_j)^2 / 2 \right) dG(\theta)$$

so that

$$|f_{G,1}^*(u)| \leq (2\pi)^{-(d-1)/2} 2^{-u^2/2} \int \exp \left( - \sum_{j \neq 1} (x_j - \theta_j)^2 / 2 \right) dG(\theta).$$

An analogous bound also holds for $|f_{G_0,1}^*(u)|$. Using these bounds for $f_{G,1}^*(u)$ and $f_{G_0,1}^*(u)$, the second term in (A.7) can be bounded from above as

$$\int_{|u| > a} u^{2k} |f_{G,1}^*(u) - f_{G_0,1}^*(u)|^2 du \leq 2(2\pi)^{-(d-1)} \int \exp \left( - \sum_{j \neq 1} (x_j - \theta_j)^2 \right) \{dG(\theta) + dG_0(\theta)\} \int_{|u| > a} u^{2k} e^{-u^2} du$$

Thus integrating both sides of (A.7) with respect to $x_2, \ldots, x_d$, we deduce that

$$2\pi \int \left\{ \partial^k (f_G(x) - f_{G_0}(x)) \right\}^2 dx \leq (2\pi)a^{2k} \int (f_G - f_{G_0})^2 + 4(2\pi)^{-(d-1)/2} \int_{|u| > a} u^{2k} e^{-u^2} du.$$

which implies that

$$\int \left\{ \partial^k (f_G(x) - f_{G_0}(x)) \right\}^2 dx \leq a^{2k} \int (f_G - f_{G_0})^2 + 8(2\pi)^{-(d+1)/2} \int_{u > a} u^{2k} e^{-u^2} du.$$

We now use the integration by parts argument in Jiang and Zhang [28, Page 1675] which gives

$$\int_{u > a} u^{2k} e^{-u^2} du \leq a^{2k-1} e^{-a^2} \quad \text{provided} \quad a \geq \sqrt{2k-1}.$$

The proof of Lemma A.3 is now completed by noting that

$$\int (f_G - f_{G_0})^2 \leq \int \left( \sqrt{f_G} - \sqrt{f_{G_0}} \right)^2 \left( \sqrt{f_G} + \sqrt{f_{G_0}} \right)^2 \leq 4(2\pi)^{-d/2} \mathcal{E}^2(f_G, f_{G_0})$$

where we have used that every Gaussian mixture density $f_G$ is bounded from above by $(2\pi)^{-d/2}$. \hfill \Box

**Lemma A.4.** Let $X_1, \ldots, X_n$ be independent random variables with $X_i \sim f_{G_i}$ and $G_n := (G_1 + \cdots + G_n)/n$. Let $g : \mathbb{R}^d \to [0, \infty)$ be a $1$-Lipschitz function i.e.,

$$g(x) - g(y) \leq ||x - y|| \quad \text{for all} \quad x, y \in \mathbb{R}^d.$$
Also let \( \mu_p(g) \) denote the \( p \)th moment of \( g \) under the measure \( \tilde{G}_n \), i.e.,

\[
\mu_p(g) := \left( \int_{\mathbb{R}^d} g(\theta)^p d\tilde{G}_n(\theta) \right)^{1/p}.
\]

There then exists a positive constant \( C_d \) depending only on \( d \) such that

\[
\mathbb{E} \left\{ \prod_{i=1}^{n} |ag(X_i)|^{I\{g(X_i)\geq M\}} \right\}^\lambda \leq \exp \left\{ C_d a^\lambda M^{\lambda+d-2} + (aM)^\lambda n \left( \frac{2\mu_p(g)}{M} \right)^p \right\}
\]

for every \( a > 0, M \geq \sqrt{8 \log n} \) and \( 0 < \lambda \leq \min(1, p) \).

Further, there exists a positive constant \( C_d \) depending only on \( d \) such that

\[
\frac{1}{n} \sum_{i=1}^{n} \mathbb{P} \left( g(X_i) \geq M \right) \leq C_d \frac{M^{d-2}}{n} + \inf_{p \geq 2} \left( \frac{2\mu_p(g)}{M} \right)^p
\]

for any \( M \geq \sqrt{8 \log n} \).

**Proof of Lemma A.4.** We write

\[
\mathbb{E} \left\{ \prod_{i=1}^{n} |ag(X_i)|^{I\{g(X_i)\geq M\}} \right\}^\lambda = \prod_{i=1}^{n} \mathbb{E} |ag(X_i)|^{\lambda M \{g(X_i)\geq M\}}
\]

\[
\leq \prod_{i=1}^{n} \left\{ 1 + a^\lambda \mathbb{E} \left[ (g(X_i))^\lambda I\{g(X_i) \geq M\} \right] \right\}
\]

\[
\leq \prod_{i=1}^{n} \exp \left( a^\lambda \mathbb{E} (g(X_i))^\lambda I\{g(X_i) \geq M\} \right)
\]

\[
= \exp \left( a^\lambda \sum_{i=1}^{n} \mathbb{E} [g(X_i))^\lambda I\{g(X_i) \geq M\}] \right)
\]

\[
= \exp \left( na^\lambda \int (g(x))^\lambda I\{g(x) \geq M\} f_{\tilde{G}_n}(x) dx \right) = \exp \left( na^\lambda U \right)
\]

where

\[
U := \int (g(x))^\lambda I\{g(x) \geq M\} f_{\tilde{G}_n}(x) dx = \mathbb{E} \left[ (g(\theta + Z))^\lambda I\{g(\theta + Z) \geq M\} \right]
\]

with independent random variables \( Z \sim N(0, I_d) \) and \( \theta \sim G_n \). Because of the 1-Lipschitz property of \( g \), we have \( g(\theta + z) \leq g(\theta) + \|z\| \) so that

\[
U \leq \mathbb{E} (2\|Z\|)^\lambda I\{2\|Z\| \geq M\} + \mathbb{E} (2g(\theta))^\lambda I\{2g(\theta) \geq M\}.
\]

The first term above will be bounded as

\[
\mathbb{E} \left[ (2\|Z\|)^\lambda I\{2\|Z\| \geq M\} \right] = M^{\lambda E} \left[ \left( \frac{\|Z\|}{M/2} \right)^\lambda I\{\|Z\| \geq M/2\} \right]
\]

\[
\leq M^{\lambda E} \left[ \left( \frac{\|Z\|}{M/2} \right)^\lambda I\{\|Z\| \geq M/2\} \right] \quad \text{since } \lambda \leq 1
\]

\[
= 2M^{\lambda - 1} \frac{1}{(2\pi)^{d/2}} \int_{\|x\| \geq M/2} \|x\| e^{-\|x\|^2/2} dx
\]

\[
\leq C_d M^{\lambda - 1} \int_{r \geq M/2} r e^{-r^2/2} r^{d-1} dr \leq C_d M^{\lambda+d-2} e^{-M^2/8}
\]
Lemma A.4 then gives the inequality

$$E\left[(2\|Z\|)\lambda I\{2\|Z\| \geq M\}\right] \leq \frac{C_d}{n} M^{\lambda+d-2}.$$  

(A.11)

For the second term in (A.10), note that (because $\lambda \leq p$)

$$E\left[(2g(\theta))^\lambda I\{2g(\theta) \geq M\}\right] = M^{\lambda} \int_{g(\theta) \geq M/2} \left(\frac{g(\theta)}{M/2}\right)^\lambda G_n(d\theta)$$

$$\leq M^{\lambda} \int \left(\frac{g(\theta)}{M/2}\right)^p G_n(d\theta) = M^{\lambda} \left(\frac{2\mu_p(g)}{M}\right)^p.$$  

(A.12)

The proof of (A.8) is now completed by putting together inequalities (A.10), (A.11) and (A.12).

For (A.9), we first use an argument similar to the above to write

$$\frac{1}{n} \sum_{i=1}^n \mathbb{P}[g(X_i) \geq M] = \mathbb{P}[g(\theta + Z) \geq M]$$

where $\theta \sim G_n$ and $Z \sim N(0, I_d)$ are independent. Since $g$ is 1-Lipschitz, $g(\theta + z) \leq g(\theta) + \|z\|$. Consequently,

$$\mathbb{P}[g(\theta + Z) \geq M] \leq \mathbb{P}[2g(\theta) \geq M] + \mathbb{P}[2\|Z\| \geq M]$$

Applying (A.11) and (A.12) with $\lambda = 0$ then concludes the proof of (A.9). \hfill \square

**Remark A.1.** We shall apply Lemma A.4 to the function

$$d_S(x) := \inf_{u \in S} \|x - u\|$$

for a fixed subset $S$ of $\mathbb{R}^d$. This function is clearly nonnegative and 1-Lipschitz. Inequality (A.8) in Lemma A.4 then gives the inequality

$$E\left(\prod_{i=1}^n |a d_S(X_i)|^{I\{d_S(X_i) \geq M\}}\right)^\lambda \leq \exp\left\{C_d a^\lambda M^{\lambda+d-2} + (aM)^\lambda n \left(\frac{2\mu_p(d_S)}{M}\right)^p\right\}$$

(A.13)

for all $a > 0$, $M \geq \sqrt{\log n}$ and $0 < \lambda \leq \min(1, p)$.

Further, inequality (A.9) for $g = d_S$ gives

$$\frac{1}{n} \sum_{i=1}^n \mathbb{P}[d_S(X_i) \geq M] \leq C_d \frac{M^{d-2}}{n} + \inf_{p \geq \frac{1}{\sqrt{\log n}}} \left(\frac{2\mu_p(d_S)}{M}\right)^p$$

(A.14)

for all $M \geq \sqrt{\log n}$.

These two inequalities (A.13) and (A.14) hold under the same assumptions on $X_1, \ldots, X_n$ as in Lemma A.4.

**Lemma A.5.** Fix $\theta_1, \ldots, \theta_n \in \mathbb{R}^d$. Suppose $X_1, \ldots, X_n$ are independent random vectors with $X_i \sim N(\theta_i, I_d)$ for $i = 1, \ldots, n$. Let $X$ denote the $d \times n$ matrix whose columns are $X_1, \ldots, X_n$. For $f \in \mathcal{M}$ and $\rho$, let $T_f(X, \rho)$ be defined as in the proof of Theorem 3.1 as the $d \times n$ matrix whose $i^{th}$ column is given by the $d \times 1$ vector:

$$X_i + \frac{\nabla f(X_i)}{\max(f(X_i), \rho)} \quad \text{for } i = 1, \ldots, n.$$
Then for every \( f \in \mathcal{M}, 0 < \rho \leq (2\pi)^{-d/2}e^{-3/2} \) and \( x > 0 \), we have

\[
\text{Pr} \left\{ \left\| T_f(X, \rho) - T_{f\hat{G}_n}(X, \rho) \right\|_F \geq \mathbb{E} \left\| T_f(X, \rho) - T_{f\hat{G}_n}(X, \rho) \right\|_F + x \right\} \leq \exp \left( -\frac{x^2}{8L^4(\rho)} \right)
\]

where

\[
L(\rho) := \sqrt{\log \frac{1}{(2\pi)^d \rho^2}}
\]

and \( \hat{G}_n \) denotes the empirical measure corresponding to \( \theta_1, \ldots, \theta_n \).

**Proof of Lemma A.5.** Let

\[
F(X) := \left\| T_f(X, \rho) - T_{f\hat{G}_n}(X, \rho) \right\|_F.
\]

We shall prove that \( F(X) \), as a function of \( X \), is Lipschitz with constant \( 2L^2(\rho) \) under the Frobenius matrix on \( X \) i.e.,

\[
|F(X) - F(Y)| \leq 2L^2(\rho) \|X - Y\|_F.
\]

Inequality (A.5) would then directly follow from the standard concentration inequality for Lipschitz functions of Gaussian random vectors (see, for example, Boucheron et al. [9, Theorem 5.6]). To prove (A.16), note first that

\[
|F(X) - F(Y)| = \left| \left\| T_f(X, \rho) - T_{f\hat{G}_n}(X, \rho) \right\|_F - \left\| T_f(Y, \rho) - T_{f\hat{G}_n}(Y, \rho) \right\|_F \right|
\leq \left\| T_f(X, \rho) - T_f(Y, \rho) \right\|_F + \left\| T_{f\hat{G}_n}(X, \rho) - T_{f\hat{G}_n}(Y, \rho) \right\|_F.
\]

Note now that

\[
\left\| T_f(X, \rho) - T_f(Y, \rho) \right\|_F^2 = \sum_{i=1}^n \left\| t_f(X_i, \rho) - t_f(Y_i, \rho) \right\|^2
\]

where

\[
t_f(x, \rho) := x + \frac{\nabla f(x)}{\max(f(x), \rho)}.
\]

To bound \( \|t_f(X_i, \rho) - t_f(Y_i, \rho)\| \), we compute the Jacobian of the map \( x \mapsto t_f(x, \rho) \) as

\[
Jt_f(x, \rho) = \begin{cases} 
I_d + \frac{Hf(x)}{\rho} & \text{if } f(x) < \rho \\
I_d + \frac{Hf(x)}{f(x)} - \left( \frac{\nabla f(x)}{f(x)} \right)^T & \text{if } f(x) > \rho
\end{cases}
\]

where \( \nabla \) and \( H \) denote gradient and Hessian respectively. We shall now argue that

\[
0 \leq Jt_f(x, \rho) \leq L^2(\rho)I_d
\]

where \( A \preceq B \) means that \( B - A \) is a nonnegative definite matrix. Before proving (A.18), let us first note that (A.18) implies

\[
\|t_f(x, \rho) - t_f(y, \rho)\| \leq L^2(\rho) \|x - y\|
\]

which further implies, via (A.17), that

\[
\left\| T_f(X, \rho) - T_f(Y, \rho) \right\|_F^2 \leq L^2(\rho) \|X - Y\|_F^2.
\]
Since this inequality holds for every \( f \in \mathcal{M} \), it also holds for \( f_G \) which gives (A.16) and completes the proof of Lemma A.5.

It remains to prove (A.18). For this, we shall use the above expression for \( Jt_f(x, \rho) \) as well as inequality (A.1) from Lemma A.2 and inequality (A.5) from the proof of Lemma A.2. First when \( f(x) > \rho \), note that

\[
Jt_f(x, \rho) = I_d + \frac{Hf(x)}{f(x)} \left( \nabla f(x) \right)^T f(x)
\]

which is \( \geq 0 \) from (A.5) and, by (A.1), we get

\[
0 \leq Jt_f(x, \rho) \leq I_d + \frac{Hf(x)}{f(x)} \leq \text{tr} \left( I + \frac{Hf(x)}{f(x)} \right) I_d < L^2(f(x)) I_d \leq L^2(\rho) I_d
\]

where, in the last inequality, we have used that \( L(\cdot) \) is a decreasing function. Here \( \text{tr} \) denotes trace. This proves (A.18) when \( f(x) > \rho \). Now let \( f(x) < \rho \). Then

\[
Jt_f(x, \rho) = I_d + \frac{Hf(x)}{\rho} = \left( 1 - \frac{f(x)}{\rho} \right) I_d + \frac{f(x)}{\rho} \left( I_d + \frac{Hf(x)}{f(x)} \right)
\]

which is \( \geq 0 \) because \( f(x) < \rho \) and because of (A.5). Also, by (A.1),

\[
Jt_f(x, \rho) = \left( 1 - \frac{f(x)}{\rho} \right) I_d + \frac{f(x)}{\rho} \left( I_d + \frac{Hf(x)}{f(x)} \right)
\]

\[
\leq \left( 1 - \frac{f(x)}{\rho} \right) I_d + \frac{f(x)}{\rho} \text{tr} I_d \left( I_d + \frac{Hf(x)}{f(x)} \right)
\]

\[
\leq \left( 1 + \frac{f(x)}{\rho} \left( \log \left( \frac{2\pi}{\rho^d} \right) - 1 \right) \right) I_d = \left( 1 + \frac{f(x)}{\rho} \left( L^2(f(x)) - 1 \right) \right) I_d
\]

The right hand side above is \( \leq L^2(\rho) I_d \) because \( t \mapsto t(L^2(t) - 1) \) is non-decreasing on \( t \in (0, (2\pi)^{-d/2} e^{-3/2}] \) so that when \( f(x) < \rho \), we have

\[
1 + \frac{f(x)}{\rho} \left( L^2(f(x)) - 1 \right) \leq L^2(\rho)
\]

This proves (A.18) which completes the proof of Lemma A.5. \( \square \)

**Lemma A.6.** There exists a positive constant \( A_d \) depending only on \( d \) such that for every \( M \geq 1 \) and \( d \in \{0, 1, 2, \ldots \} \), we have

\[
I(d) := \int_{r \geq M} r^d e^{-r^2/2} dr \leq A_d M^{d-1} e^{-M^2/2}.
\]

**Proof of Lemma A.6.** Let \( A_0 := 1 \), \( A_1 := 1 \) and define \( A_d \) for \( d \geq 2 \) via the recursion \( A_d := 1 + (d-1)A_{d-2} \). Clearly

\[
I(0) = \int_{r \geq M} e^{-r^2/2} dr \leq \int_{r \geq M} \frac{r}{M} e^{-r^2/2} = M^{-1} e^{-M^2/2}
\]

and

\[
I(1) = \int_{r \geq M} r e^{-r^2/2} dr = e^{-M^2/2}
\]

and thus inequality (A.19) holds for \( d = 0 \) and \( d = 1 \). For \( d \geq 2 \), integration by parts gives

\[
I(d) = M^{d-1} e^{-M^2/2} + (d-1)I(d-2).
\]

Inequality (A.19) for \( d \geq 2 \) now easily follows by induction on \( d \). \( \square \)
Lemma A.7. Let $S$ be a compact subset of $\mathbb{R}^d$. For $\eta, M > 0$, define

\[(A.20)\quad v(x) := \begin{cases} \eta & \text{if } x \in S^M \\ \eta \left( \frac{M}{\delta_S(x)} \right)^{d+1} & \text{otherwise} \end{cases} \]

Then, for some constant $C_d$ depending only on $d$,

\[(A.21)\quad \int v(x) dx \leq C_d \eta \text{Vol}(S^M) \]

Proof of Lemma A.7. We first write

\[(A.22)\quad \int v(x) dx = \eta \text{Vol}(S^M) + \eta M^{d+1} \int_{x \notin S^M} \frac{1}{\delta_S(x)^{d+1}} dx \]

Let $N$ be the maximal integer such that there exist $u_1, \ldots, u_N \in S$ with

\[(A.23)\quad \min_{i \neq j} \|u_i - u_j\| \geq M/2. \]

The maximality of $N$ implies that $\sup_{u \in S} \min_{1 \leq i \leq N} \|u - u_i\| \leq M/2$. As a result, for every $x \in \mathbb{R}^d$, by triangle inequality, we have

$$\delta_S(x) = \min_{u \in S} \|x - u\| \geq \min_{1 \leq i \leq N} \|x - u_i\| - \frac{M}{2}$$

so that

\[(A.24)\quad \int_{x \notin S^M} \frac{dx}{\delta_S(x)^{d+1}} \leq \int_{x \notin S^M} \left( \frac{1}{\min_{1 \leq i \leq N} \|x - u_i\| - M/2} \right)^{d+1} dx \]

\[\leq \sum_{i=1}^{N} \int_{x \notin S^M} \left( \frac{1}{\|x - u_i\| - M/2} \right)^{d+1} dx \]

\[\leq \sum_{i=1}^{N} \int_{\|x - u_i\| \geq M} \left( \frac{1}{\|x - u_i\| - M/2} \right)^{d+1} dx \]

\[= N \int_{\|x\| \geq M} \left( \frac{1}{\|x\| - M/2} \right)^{d+1} dx \]

\[= NC_d \int_{M/2}^{\infty} \left( \frac{1}{r - M/2} \right)^{d+1} r^{d-1} dr \]

Note now that because of (A.23), the balls $B(u_i, M/4), i = 1, \ldots, N$ have disjoint interiors and are all contained in $S^{M/4}$. As a result

\[(A.25)\quad N \leq \frac{\text{Vol}(S^{M/4})}{\text{Vol}(B(0, M/4))} \leq C_d \frac{\text{Vol}(S^M)}{M^d}. \]

The proof of Lemma A.7 is completed by putting together inequalities (A.22), (A.24) and (A.25).
**Lemma A.8.** There exists a positive constant $C_d$ such that for every compact set $K \subseteq \mathbb{R}^d$ and real numbers $\epsilon > 0$ and $M > 0$, we have

\begin{equation}
N(\epsilon, K) \leq C_d e^{-d \text{Vol}(K^{\epsilon/2})}
\end{equation}

and

\begin{equation}
\text{Vol}(K^{2M}) \leq C_d \text{Vol}(K^{\epsilon/2}) \left(1 + \frac{M}{\epsilon}\right)^d
\end{equation}

**Proof of Lemma A.8.** Let us first prove (A.26). Let $a_1, \ldots, a_N \in K$ be a maximal set of points such that $\min_{i \neq j} \|a_i - a_j\| \geq \epsilon$. Then clearly $N(\epsilon, K) \leq N$. The balls $B(a_i, \epsilon/2)$ for $i = 1, \ldots, N$ have disjoint interiors and are all contained in $K^{\epsilon/2}$. As a result

\begin{equation}
N(\epsilon, K) \leq \frac{\text{Vol}(K^{\epsilon/2})}{\text{Vol}(B(0, \epsilon/2))}
\end{equation}

from which (A.26) follows.

To prove (A.27), note that the $K$ is contained in the union of the balls $B(a_i, \epsilon)$ for $i = 1, \ldots, N$. This implies that

\[ K^{2M} \subseteq \bigcup_{i=1}^N B(a_i, \epsilon + 2M) \]

so that

\[ \text{Vol}(K^{2M}) \leq N \text{Vol}(B(0, \epsilon + 2M)). \]

Inequality (A.28) then gives

\[ \text{Vol}(K^{2M}) \leq \frac{\text{Vol}(K^{\epsilon/2})}{\text{Vol}(B(0, \epsilon/2))} \text{Vol}(B(0, \epsilon + 2M)) \leq C_d \text{Vol}(K^{\epsilon/2}) \left(1 + \frac{M}{\epsilon}\right)^d. \]

\[ \square \]

**Lemma A.9.** Fix a probability measure $G$ on $\mathbb{R}^d$ and let $0 < \rho \leq (2\pi)^{-d/2}/\sqrt{\epsilon}$. Let

\[ L(\rho) := \sqrt{\log \frac{1}{(2\pi)^d \rho^2}}. \]

Then there exists a positive constant $C_d$ such that for every compact set $S \subseteq \mathbb{R}^d$, we have

\begin{equation}
\Delta(G, \rho) := \int \left(1 - \frac{f_G}{\max(f_G, \rho)}\right)^2 \frac{\|\nabla f_G\|^2}{f_G} \leq C_d N \left(\frac{4}{L(\rho)}, S\right) L^d(\rho) \rho + d G(S^c).
\end{equation}

**Proof of Lemma A.9.** The proof uses Lemma A.10.

Fix a compact set $S$. Suppose first that $G$ is supported on $S$ so that the second term in (A.29) equals 0.

We consider two further special cases. First assume that $S$ is contained in a ball of radius $a := 4/L(\rho)$. Without loss of generality, we may assume that the ball is centered at the origin. Because $G$ is assumed to be supported on $S$, we have $\|\theta\| \leq a$ almost surely under $G$.

For $\theta \sim G$ and $X|\theta \sim N(\theta, I_d)$, we can write

\[ \frac{\nabla f_G(x)}{f_G(x)} = \mathbb{E}(\theta - X|X = x) \]
so that
\begin{equation}
\frac{\|\nabla f_G(x)\|}{f_G(x)} = \|E(\theta - X|X = x)\| \leq \|E(\|\theta - X\| |X = x)\| \leq \|x\| + a.
\end{equation}

Note also that
\begin{equation}
(2\pi)^{-d/2} \exp\left(-\frac{1}{2} (\|x\| + a)^2\right) \leq f_G(x) \leq (2\pi)^{-d/2} \exp\left(-\frac{1}{2} (\|x\| - a)^2\right)
\end{equation}
because \( (\|x\| - a, + \leq \|x - \theta\| \leq \|x\| + a \) whenever \( \|\theta\| \leq a \). This also implies that whenever \( f_G(x) \leq \rho \), we have
\[ \rho \geq (2\pi)^{-d/2} \exp\left(-\frac{1}{2} (\|x\| + a)^2\right) \]
which gives
\begin{equation}
\|x\| + a \geq L(\rho) := \sqrt{\log \frac{1}{(2\pi)^{d}\rho^2}}.
\end{equation}

Putting together (A.30), (A.31) and (A.32), we deduce that
\[ \Delta(G, \rho) \leq \int \{f_G \leq \rho\} \left(\frac{\|\nabla f_G\|}{f_G}\right)^2 f_G \]
\[ \leq \int_{\{\|x\| \geq L(\rho)\}} (\|x\| + a)^2 (2\pi)^{-d/2} \exp\left(-\frac{1}{2} (\|x\| - a)^2\right) dx. \]

Moving to polar coordinates, we deduce
\[ \Delta(G, \rho) \leq C_d \int_{(L(\rho) - a)^+}^{\infty} (r + a)^2 \exp\left(-(r - a)^2/2\right) r^{d-1} dr. \]

Note now that with \( a := 4/L(\rho) \) and \( \rho \leq (2\pi)^{-d/2}/\sqrt{e} \), we have \( 4a \leq L(\rho) \) so that
\[ \Delta(G, \rho) \leq C_d \int_{L(\rho) - a}^{\infty} (r + a)^2 \exp\left(-(r - a)^2/2\right) r^{d-1} dr. \]

By a change of variable \( r - a \mapsto r \), we obtain
\[ \Delta(G, \rho) \leq C_d \int_{L(\rho) - 2a}^{\infty} (s + 2a)^2 \exp\left(-s^2/2\right) (s + a)^{d-1} ds. \]

Because \( 4a \leq L(\rho) \), we have
\[ s + a \leq s + 2a \leq s + L(\rho) - 2a \leq 2s \]
whenever \( s \geq L(\rho) - 2a \). Thus
\[ \Delta(G, \rho) \leq C_d \int_{L(\rho) - 2a}^{\infty} s^{d+1} e^{-s^2/2} ds. \]

By Lemma A.6, we deduce that
\[ \Delta(G, \rho) \leq C_d (L(\rho))^d \exp\left(-\frac{1}{2} (L(\rho) - 2a)^2\right) \leq C_d (L(\rho))^d e^{2aL(\rho)} e^{-L^2(\rho)/2} = C_d \rho (L(\rho))^d e^{2aL(\rho)}. \]
We now take
\[ a := \frac{4}{L(\rho)} \]
which gives
\[ \Delta(G, \rho) \leq C_d \rho (L(\rho))^d \]  
whenever \( G \) is supported on a set that is contained in a ball of radius \( a = 4/L(\rho) \).

For the rest of the proof, we shall use Lemma A.10. Now suppose that \( G \) is supported on a general compact set \( S \). Then, for \( N := N(a,S) \) (where \( a := 4/L(\rho) \)), let \( E_1, \ldots, E_N \) denote a disjoint covering of \( S \) such that each \( E_i \) is contained in a ball of radius \( a \). We can then write
\[ G := \sum_{j=1}^{N} w_j H_j \]
where \( w_j := G(E_j) \) and \( H_j \) is the probability measure \( G \) conditioned on \( H_j \). The bound (A.35) in Lemma A.10 then gives
\[ \Delta(G, \rho) \leq \sum_{j=1}^{N} w_j \Delta(H_j, \rho/w_j). \]

Because \( H_j \) is supported on a ball of radius at most \( a \), we can use (A.33) on each \( H_j \) to deduce that
\[ \Delta(G, \rho) \leq C_d \sum_{j=1}^{N} w_j \rho \frac{L(\rho/w_j)}{L(\rho)}. \]  

To bound \( \Delta(G, \rho) \) for an arbitrary probability measure \( G \), we write
\[ G = w_1 H_1 + w_2 H_2 \]
where \( w_1 = G(S) = 1 - w_2 \) and \( H_1 \) and \( H_2 \) are the probability measures obtained by conditioning \( G \) on \( S \) and \( S^c \) respectively. Then clearly \( H_1 \) is supported on a compact set \( S \) so that the bound (A.34) can be used for \( \Delta(H_2, \rho/w_2) \). For \( \Delta(H_1, \rho/w_1) \), we use the trivial bound \( d \) (see the first part of Lemma A.10). This gives (via (A.35))
\[ \Delta(G, \rho) \leq C_d G(S) N(a,S) L^d(\rho) \rho + d G(S^c) \leq C_d N(a,S) L^d(\rho) \rho + d G(S^c) \]
which completes the proof of Lemma A.9.

**Lemma A.10.** For a probability measure \( G \) on \( \mathbb{R}^d \) and \( \rho > 0 \), let
\[ \Delta(G, \rho) := \int \left( 1 - \frac{f_G}{\max (f_G, \rho)} \right)^2 \frac{||\nabla f_G||^2}{f_G} \]
The following pair of statements are then true.

1. For every \( G \) and \( \rho > 0 \), we have \( \Delta(G, \rho) \leq d \).
2. Suppose \( G = \sum_{j=1}^{m} w_j H_j \) for some probability measures \( H_1, \ldots, H_m \) and weights \( w_1, \ldots, w_m \). Then
\[ \Delta(G, \rho) \leq \sum_{j=1}^{m} w_j \Delta(H_j, \rho/w_j). \]
The above expression and the fact that

\[ X \sim N(\theta, I_d), \]

As a result

\[ \Delta(G, \rho) \leq \int \frac{||\nabla f_G||^2}{f_G} = \mathbb{E} \| \mathbb{E}(\theta - X | X = x) \| \leq \mathbb{E} \| \theta - X \|^2 = d. \]

For proving (A.35), note first that by the convexity of \( x \mapsto \|x\|^2 \), we have

\[
\frac{||\nabla f_G||^2}{f_G} = \frac{\left( \sum_j w_j \nabla f_{H_j} \right)^2}{\sum_j w_j f_{H_j}} \\
= \left\{ \sum_j \left( \frac{w_j f_{H_j}}{\sum_j w_j f_{H_j}} \right) \nabla f_{H_j} \right\}^2 \left( \sum_j w_j f_{H_j} \right) \\
\leq \left\{ \sum_j \left( \frac{w_j f_{H_j}}{\sum_j w_j f_{H_j}} \| \nabla f_{H_j} \|^2 \right) \left( \sum_j w_j f_{H_j} \right) \right\} = \sum_j w_j \| \nabla f_{H_j} \|^2.
\]

This, along with the trivial inequality (here \( a \lor b \) stands for \( \max(a, b) \))

\[
\left( 1 - \frac{f_G}{f_G \lor \rho} \right)^2 \leq \left( 1 - \frac{f_{H_j}}{f_{H_j} \lor (\rho/w_j)} \right)^2 \quad \text{for every } 1 \leq j \leq m
\]
yields (A.35).

\[
\text{Proof of Lemma A.11.} \quad \text{Suppose } X_1, \ldots, X_n \text{ are independent observations with } X_i \sim N(\theta_i, I_d) \text{ for some } \theta_1, \ldots, \theta_n \in \mathbb{R}^d. \text{ Let the Oracle Bayes estimators } \hat{\theta}_1^*, \ldots, \hat{\theta}_n^* \text{ be defined as in } (1.5) \text{ where } \hat{G}_n \text{ is the empirical measure of } \theta_1, \ldots, \theta_n. \text{ Suppose that } G_n \text{ is supported on a set } \{a_1, \ldots, a_k\} \text{ of cardinality } k \text{ with } \hat{G}_n \{a_i\} = p_i \text{ for } i = 1, \ldots, k \text{ with } p_i \geq 0 \text{ and } \sum_{i=1}^k p_i = 1. \text{ Then}
\]

\[
(A.36) \quad \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n \left\| \hat{\theta}_i^* - \theta_i \right\|^2 \right] \leq \frac{k-1}{2\sqrt{2\pi}} \sum_{j,l:j \neq l} (p_j + p_l) \left\| a_j - a_l \right\| \exp \left( -\frac{1}{8} \left\| a_j - a_l \right\|^2 \right).
\]

\[
\text{Proof of Lemma A.11.} \quad \text{Note first that } \hat{\theta}_i^* \text{ has the following expression}
\]

\[
\hat{\theta}_i^* = \frac{\sum_{j=1}^k a_j p_j \phi_d(X_i - a_j)}{\sum_{j=1}^k p_j \phi_d(X_i - a_j)} \quad \text{for } i = 1, \ldots, n.
\]

The above expression and the fact that \( X_i - \theta_i \sim N(0, I_d) \) lets us write

\[
R := \mathbb{E} \left[ \frac{1}{n} \sum_{i=1}^n \left\| \hat{\theta}_i^* - \theta_i \right\|^2 \right] = \sum_{i=1}^k p_i \mathbb{E} \left[ \frac{\sum_{j=1}^k a_j p_j \phi_d(a_l + Z - a_j)}{\sum_{j=1}^k p_j \phi_d(a_l + Z - a_j)} - a_l \right]^2
\]

\[
= \sum_{i=1}^k p_i \mathbb{E} \left[ \frac{\sum_{j=1}^k (a_j - a_l) p_j \phi_d(a_l + Z - a_j)}{\sum_{j=1}^k p_j \phi_d(a_l + Z - a_j)} \right]^2
\]

\[
= \sum_{i=1}^k \mathbb{E} \left[ \sum_{j: j \neq l} (a_j - a_l) w_{jl}(Z) \right]^2
\]
where $Z \sim N(0, I_d)$ and
\[
w_{jl}(Z) := \frac{p_j \phi_d(a_l + Z - a_j)}{\sum_{u=1}^k p_u \phi_d(a_l + Z - a_u)} \quad \text{for } 1 \leq j, l \leq k.
\]
The elementary inequality $\|\sum_{i=1}^m \alpha_i\|^2 \leq m \sum_{i=1}^m \|\alpha_i\|^2$ for vectors $\alpha_1, \ldots, \alpha_m \in \mathbb{R}^d$ now lets us write
\[
R \leq (k - 1) \sum_{l=1}^k p_l \sum_{j: j \neq l} \|a_j - a_l\|^2 \mathbb{E} w_{jl}^2(Z).
\]
We now bound $\mathbb{E} w_{jl}^2(Z)$ in the following way. Let
\[
U := \left\{ z \in \mathbb{R}^d : \|a_j - a_l\|^2 \geq 2 \langle Z, a_j - a_l \rangle \right\}.
\]
When $Z \notin U$, we shall use the trivial upper bound $w_{jl}^2(Z) \leq 1$. When $Z \in U$, we shall use the bound
\[
w_{jl}^2(Z) \leq w_{jl}(Z) \leq \frac{p_j \phi_d(a_l + Z - a_j)}{p_l \phi_d(a_l + Z - a_l)} = \frac{p_j \phi_d(a_l + Z - a_j)}{p_l \phi_d(Z)}.
\]
This gives
\[
\mathbb{E} w_{jl}^2(Z) \leq \mathbb{P} \{ Z \notin U \} + \int \frac{p_j \phi_d(a_l + z - a_j)}{p_l \phi_d(z)} I \{ \|a_j - a_l\|^2 \geq 2 \langle z, a_j - a_l \rangle \} \phi_d(z) dz.
\]
The change of variable $x = a_l + z - a_j$ in the integral above allows us to write
\[
\mathbb{E} w_{jl}^2(Z) \leq \mathbb{P} \left\{ \langle Z, a_j - a_l \rangle > \frac{1}{2} \|a_j - a_l\|^2 \right\} + \frac{p_j}{p_l} \mathbb{P} \left\{ \langle Z, a_j - a_l \rangle \leq -\frac{1}{2} \|a_j - a_l\|^2 \right\}
\leq \left( 1 + \frac{p_j}{p_l} \right) \left( 1 - \Phi \left( \frac{1}{2} \|a_j - a_l\|^2 \right) \right)
\]
where $\Phi$ is the standard univariate Gaussian cumulative distribution function. The bound $1 - \Phi(t) \leq \phi(t)/t$ for $t > 0$ now gives
\[
\mathbb{E} w_{jl}^2(Z) \leq \frac{1}{2\sqrt{2\pi}} \left( 1 + \frac{p_j}{p_l} \right) \frac{1}{\|a_j - a_l\|} \exp \left( -\frac{1}{8} \|a_j - a_l\|^2 \right).
\]
This bound, when combined with (A.37), yields (A.36) and hence completes the proof of Lemma A.11. □

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