Abstract. We construct new proper biharmonic functions defined on open and dense subsets of the special unitary group SU(2). Then we employ a duality principle to obtain new proper biharmonic functions from the non-compact 3-dimensional hyperbolic space $H^3$.

1. Introduction

The literature on biharmonic functions is vast, but usually the domains are either surfaces or open subsets of flat Euclidean space. The first proper biharmonic functions, from open subsets of the classical compact simple Lie groups SU($n$), SO($n$) and Sp($n$), were recently constructed in [7]. They are all quotients of linear combinations of the matrix coefficients for the standard irreducible representation $\pi_1$ of the corresponding group. The authors make use of the well-known fact that these matrix coefficients are all eigenfunctions of the corresponding Laplace-Beltrami operator. For this see [10] or Proposition 5.28 of [11].

In this paper we continue the study for the special unitary group SU(2) to its higher finite-dimensional irreducible representations $\pi_n$. In our Theorems 5.1, 6.1 and 7.1 we construct new proper biharmonic functions. They are all rational functions in the matrix coefficients of the irreducible representations $\pi_2$, $\pi_3$ and $\pi_4$ of SU(2), respectively. When studying these cases an interesting pattern comes to light. This makes us believe that the same method will work for the general finite dimensional irreducible representation $\pi_n$ of SU(2). These thoughts are formulated in Conjecture 8.1. As an introduction to the representations $\pi_n$ of SU(2) we recommend Fegan’s book [2].

The special unitary group SU(2) is, up to a constant multiple of the metric, isometric to the 3-dimensional unit sphere $S^3$ in the Euclidean $\mathbb{R}^4$. This means that our solutions can be seen as functions from open and dense subsets of $S^3$. In Theorem 9.1 we describe a general duality principle and employ this to yield new proper biharmonic functions on the non-compact 3-dimensional hyperbolic space $H^3$.
We conclude the paper with a short appendix providing a formula that hopefully will be useful to implement our calculations by means of some suitable software.

2. Proper $r$-harmonic functions

Let $(M,g)$ be a smooth manifold equipped with a semi-Riemannian metric $g$. We complexify the tangent bundle $TM$ of $M$ to $T^CM$ and extend the metric $g$ to a complex-bilinear form on $T^CM$. Then the gradient $\nabla f$ of a complex-valued function $f : (M,g) \to \mathbb{C}$ is a section of $T^CM$. In this situation, the well-known linear Laplace-Beltrami operator (alt. tension field) $\tau$ on $(M,g)$ acts on $f$ as follows

$$\tau(f) = \text{div}(\nabla f) = \frac{1}{\sqrt{|g|}} \partial_i \left( g^{ij} \sqrt{|g|} \frac{\partial f}{\partial x_i} \right).$$

For complex-valued functions $f_1, f_2, h_1, h_2 : (M,g) \to \mathbb{C}$ we have the following well-known relations

\begin{align*}
\tau(f_1 h_1) &= \tau(f_1) h_1 + 2 \kappa(f_1, h_1) + f_1 \tau(h_1), \\
\kappa(f_1 h_1, f_2 h_2) &= h_1 h_2 \kappa(f_1, f_2) + f_2 h_1 \kappa(f_1, h_2) + f_1 h_2 \kappa(h_1, f_2) + f_1 f_2 \kappa(h_1, h_2),
\end{align*}

where the symmetric conformality operator $\kappa$ is given by

$$\kappa(f, h) = g(\nabla f, \nabla h).$$

For a positive integer $r$, the iterated Laplace-Beltrami operator $\tau^r$ is defined by

$$\tau^0(f) = f, \quad \tau^r(f) = \tau(\tau^{(r-1)}(f)).$$

**Definition 2.1.** For a positive integer $r$, we say that a complex-valued function $f : (M,g) \to \mathbb{C}$ is

(a) $r$-harmonic if $\tau^r(f) = 0$, and

(b) proper $r$-harmonic if $\tau^r(f) = 0$ and $\tau^{(r-1)}(f)$ does not vanish identically.

It should be noted that the harmonic functions are exactly the 1-harmonic and the biharmonic functions are the 2-harmonic ones. In some texts, the $r$-harmonic functions are also called polyharmonic of order $r$.

We recall that a map $\pi : (\hat{M},\hat{g}) \to (M,g)$ between two semi-Riemannian manifolds is a harmonic morphism if it pulls back germs of harmonic functions to germs of harmonic functions. The standard reference on this topic is the book [1] of Baird and Wood. We also recommend the updated online bibliography [4].

It was recently shown in [7] that there is an interesting connections between the theory of $r$-harmonic functions and the notion of harmonic morphisms. In the sequel, we shall often employ the two following results.
Proposition 2.2. [7] Let \( \pi : (\hat{M}, \hat{g}) \to (M, g) \) be a submersive harmonic morphism from a semi-Riemannian manifold \( (\hat{M}, \hat{g}) \) to a Riemannian manifold \( (M, g) \). Further let \( f : (M, g) \to \mathbb{C} \) be a smooth function and \( \hat{f} : (\hat{M}, \hat{g}) \to \mathbb{C} \) be the composition \( \hat{f} = f \circ \pi \). If \( \lambda : \hat{M} \to \mathbb{R}^+ \) is the dilation of \( \pi \) then the tension field satisfies
\[
\tau(f) \circ \pi = \lambda^{-2} \tau(\hat{f}) \quad \text{and} \quad \tau^r(f) \circ \pi = \lambda^{-2} \tau(\lambda^{-2}(r-1)(\hat{f})),
\]
for all positive integers \( r \geq 2 \).

Proposition 2.3. [7] Let \( \pi : (\hat{M}, \hat{g}) \to (M, g) \) be a submersive harmonic morphism, from a semi-Riemannian manifold \( (\hat{M}, \hat{g}) \) to a Riemannian manifold \( (M, g) \), with constant dilation. Further let \( f : (M, g) \to \mathbb{C} \) be a smooth function and \( \hat{f} : (\hat{M}, \hat{g}) \to \mathbb{C} \) be the composition \( \hat{f} = f \circ \pi \). Then the following statements are equivalent
(i) \( \hat{f} : (\hat{M}, \hat{g}) \to \mathbb{C} \) is proper \( r \)-harmonic,
(ii) \( f : (M, g) \to \mathbb{C} \) is proper \( r \)-harmonic.

3. The Riemannian Lie group \( \text{GL}_n(\mathbb{C}) \)

Let \( G \) be a Lie group with Lie algebra \( \mathfrak{g} \) of left-invariant vector fields on \( G \). Then a Euclidean scalar product \( g \) on \( \mathfrak{g} \) induces a left-invariant Riemannian metric on the group \( G \) and turns it into a homogeneous Riemannian manifold. If \( Z \) is a left-invariant vector field on \( G \) and \( f : U \to \mathbb{C} \) is a complex-valued functions defined locally on \( G \) then the first and second order derivatives satisfy
\[
Z(f)(p) = \frac{d}{ds} [f(p \cdot \exp(sZ))] |_{s=0}, \quad (3.1)
\]
\[
Z^2(f)(p) = \frac{d^2}{ds^2} [f(p \cdot \exp(sZ))] |_{s=0}. \quad (3.2)
\]

Further, assume that \( G \) is a subgroup of the complex general linear group \( \text{GL}_n(\mathbb{C}) \) equipped with its standard Riemannian metric. This is induced by the Euclidean scalar product on the Lie algebra \( \mathfrak{gl}_n(\mathbb{C}) \) given by
\[
g(Z, W) = \Re \text{trace } ZW^*.
\]

Employing the Koszul formula for the Levi-Civita connection \( \nabla \) on \( \text{GL}_n(\mathbb{C}) \), we see that
\[
g(\nabla_Z W, Z) = g([W, Z], Z)
\]
\[
= \Re \text{trace} (WZ - ZW) Z^t
\]
\[
= \Re \text{trace} W(ZZ^t - Z^t Z)^t
\]
\[
= g([Z, Z^t], W).
\]

Let \( [Z, Z^t]_\mathfrak{g} \) be the orthogonal projection of the bracket \( [Z, Z^t] \) onto the subalgebra \( \mathfrak{g} \) of \( \mathfrak{gl}_n(\mathbb{C}) \). Then the above calculations shows that
\[
\nabla_Z Z = [Z, Z^t]_\mathfrak{g}.
\]
This implies that the tension field $\tau(f)$ and the conformality operator $\kappa(f, h)$ are given by

$$
\tau(f) = \sum_{Z \in \mathcal{B}} Z^2(f) - [Z, Z^t]_g(f) \quad \text{and} \quad \kappa(f, h) = \sum_{Z \in \mathcal{B}} Z(f) Z(h),
$$

(3.3)

where $\mathcal{B}$ is any orthonormal basis for the Lie algebra $\mathfrak{g}$.

**Remark 3.1.** For $1 \leq i, j \leq n$ we shall by $E_{ij}$ denote the element of $\mathfrak{gl}_n(\mathbb{R})$ satisfying

$$(E_{ij})_{kl} = \delta_{ik} \delta_{jl}$$

and by $D_t$ the diagonal matrices

$$D_t = E_{tt}.$$

For $1 \leq r < s \leq n$ let $X_{rs}$ and $Y_{rs}$ be the matrices satisfying

$$X_{rs} = \frac{1}{\sqrt{2}}(E_{rs} + E_{sr}), \quad Y_{rs} = \frac{1}{\sqrt{2}}(E_{rs} - E_{sr}).$$

### 4. The standard irreducible representation $\pi_1$ of $\text{SU}(n)$

In this section we describe known proper biharmonic functions on the special unitary group $\text{SU}(n)$ constructed in [7]. They are quotients of first order homogeneous polynomials in the matrix coefficients of the standard irreducible representation $\pi_1$ of $\text{SU}(2)$.

The unitary group $\text{U}(n)$ is the compact subgroup of $\text{GL}_n(\mathbb{C})$ given by

$$\text{U}(n) = \{ z \in \text{GL}_n(\mathbb{C}) \mid z \cdot z^* = I_n \}$$

with its standard matrix representation

$$\pi_1 = \begin{bmatrix}
z_{11} & z_{12} & \cdots & z_{1n} \\
z_{21} & z_{22} & \cdots & z_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
z_{n1} & z_{n1} & \cdots & z_{nn}
\end{bmatrix}.$$ 

The circle group $\mathbb{S}^1 = \{ e^{i\theta} \in \mathbb{C} \mid \theta \in \mathbb{R} \}$ acts on the unitary group $\text{U}(n)$ by multiplication

$$(e^{i\theta}, z) \mapsto e^{i\theta} z$$

and the orbit space of this action is the special unitary group

$$\text{SU}(n) = \{ z \in \text{U}(n) \mid \det z = 1 \}.$$ 

The natural projection $\pi : \text{U}(n) \to \text{SU}(n)$ is a harmonic morphism with constant dilation $\lambda \equiv 1$.

The Lie algebra $\mathfrak{u}(n)$ of the unitary group $\text{U}(n)$ satisfies

$$\mathfrak{u}(n) = \{ Z \in \mathbb{C}^{n \times n} \mid Z + Z^* = 0 \}$$

and for this we have the canonical orthonormal basis

$$\{ Y_{rs}, iX_{rs} \mid 1 \leq r < s \leq n \} \cup \{ iD_t \mid t = 1, \ldots, n \}.$$
Now, by means of a direct computation based on (3.1), (3.2), and (3.3), we have the following basic result, see [8].

Lemma 4.1. For \(1 \leq j, \alpha \leq n\), let \(z_{j\alpha} : U(n) \to \mathbb{C}\) be the complex-valued matrix coefficients of the standard representation of \(U(n)\) given by
\[
z_{j\alpha} : z \mapsto e_j \cdot z \cdot e_{\alpha}^*,
\]
where \(\{e_1, \ldots, e_n\}\) is the canonical basis for \(\mathbb{C}^n\). Then the following relations hold
\[
\tau(z_{j\alpha}) = -n \cdot z_{j\alpha} \quad \text{and} \quad \kappa(z_{j\alpha}, z_{k\beta}) = -z_{k\alpha}z_{j\beta}.
\]
(4.1)
The next result describes the first known proper biharmonic functions from the unitary group \(SU(n)\), see [7].

Proposition 4.2. Let \(p, q \in \mathbb{C}^n\) be linearly independent and \(P, Q : U(n) \to \mathbb{C}\) be the complex-valued functions on the unitary group given by
\[
P(z) = \sum_{j=1}^{n} p_j z_{j\alpha} \quad \text{and} \quad Q(z) = \sum_{k=1}^{n} q_k z_{k\beta}.
\]
Further, let the rational function \(f(z) = P(z)/Q(z)\) be defined on the open and dense subset \(W_Q = \{z \in U(n) \mid Q(z) \neq 0\}\) of the unitary group. Then the following is true.

(a) The function \(f\) is harmonic if and only if \(\alpha = \beta\).

(b) The function \(f\) is proper biharmonic if and only if \(\alpha \neq \beta\).

The corresponding statements hold for the function induced on \(SU(n)\).

5. The irreducible representation \(\pi_2\) of \(SU(2)\)

In this section we extend the construction of Proposition 4.2 for \(SU(2)\) to its 3-dimensional irreducible representation \(\pi_2\) given by the following matrix
\[
\pi^2 = \begin{bmatrix}
z_{11}^2 & z_{12}^2 & z_{12}^2 \\
z_{21}^2 & z_{11}^2 + z_{12}^2 & 2z_{12}z_{21} \\
z_{21}^2 & z_{21}z_{22} & z_{22}^2
\end{bmatrix}.
\]

Theorem 5.1. Let \(p, q \in \mathbb{C}^3\) and \(P, Q : U(2) \to \mathbb{C}\) be the complex-valued functions on the unitary group given by
\[
P(z) = \sum_{j=1}^{3} p_j \pi_{j\alpha}^2 \quad \text{and} \quad Q_\beta(z) = \sum_{k=1}^{3} q_k \pi_{k\beta}^2.
\]
Let the rational functions \(f(z) = P(z)/Q(z)\) be defined on the open and dense subset \(W_Q = \{z \in U(2) \mid Q(z) \neq 0\}\) of the unitary group. If \(\alpha \neq \beta\) then the function \(f\) is proper biharmonic if \(p_2q_3 - p_3q_2 \neq 0\) and
\[
p_4q_3^2 = q_2(2p_2q_3 - p_3q_2), \quad q_1q_3 = q_2^2.
\]
The corresponding statement holds for the function induced on \(SU(2)\).
Proof. It is easily seen that for a general quotient $f = P/Q$ we have
\[ Q^3 \tau(f) = Q^2 \tau(P) - 2Q\kappa(P, Q) + 2P\kappa(Q, Q) - PQ\tau(Q). \] (5.1)

The matrix coefficients for the irreducible representation $\pi_2$ of $\text{SU}(2)$ are all eigenfunctions of the tension field $\tau$ with the same eigenvalue. This is useful since then the above formula (5.1) simplifies to
\[ Q^3 \tau(f) = -2Q\kappa(P, Q) + 2P\kappa(Q, Q). \]

Let us first consider the case when $(\alpha, \beta) = (3, 1)$. Then an elementary but lengthy calculation gives the following formula for the tension fields.
\[
\tau(f) = 8 \frac{(z_{11}z_{22} - z_{12}z_{21})}{(q_1z_{11}^2 + 2q_2z_{11}z_{21} + q_3z_{21}^2)}[(p_1q_2 - p_2q_1)z_{11}z_{12} + (p_2q_2 - p_3q_1)z_{11}z_{22} + (p_1q_3 - p_2q_2)z_{12}z_{21} + (p_2q_3 - p_3q_2)z_{12}z_{22}]
\]
and
\[
\tau^2(f)(z) = -16 \frac{(z_{11}z_{22} - z_{12}z_{21})^2(N_1z_{11}^2 - N_2z_{11}z_{21} - N_3z_{21}^2)}{(q_1z_{11}^2 + 2q_2z_{11}z_{21} + q_3z_{21}^2)^3},
\]
where
\[
N_1 = (p_1q_1q_3 - 4p_1q_2^2 + 6p_2q_1q_2 - 3p_3q_1^2),
\]
\[
N_2 = (6p_1q_2q_3 - 8p_2q_1q_3 - 4p_2q_2^2 + 6p_3q_1q_2),
\]
\[
N_3 = (3p_1q_2^2 - 6p_2q_2q_3 - p_3q_1q_3 + 4p_3q_2^2).
\]
The bitension field $\tau^2(f)$ vanishes if and only if $N_1 = N_2 = N_3 = 0$. This system of equations is equivalent to
\[ p_1q_3^2 = q_2(2p_2q_3 - p_3q_2) \quad \text{and} \quad q_1q_3 = q_2^2.
\]
By substituting these two relations into the above formula for the tension field $\tau(f)$ we finally obtain
\[
\tau(f) = 8 \frac{(z_{11}z_{22} - z_{12}z_{21})(q_2z_{12} + q_3z_{22})(p_2q_3 - p_3q_2)}{(q_2z_{11} + q_3z_{21})^3}.
\]
This shows that if $(\alpha, \beta) = (3, 1)$ then the function $f$ is proper biharmonic if and only if
\[ p_1q_3^2 = q_2(2p_2q_3 - p_3q_2), \quad q_1q_3 = q_2^2 \quad \text{and} \quad p_2q_3 - p_3q_2 \neq 0.
\]
It is easily checked that the above mentioned particular substitutions give the same stated result in all the cases when $\alpha \neq \beta$. \hfill \Box

Example 5.2. Let $p, q \in \mathbb{C}^3$ be such that
\[ p_1q_3^2 = q_2(2p_2q_3 - p_3q_2), \quad q_1q_3 = q_2^2 \quad \text{and} \quad p_2q_3 - p_3q_2 \neq 0.
\]
Then Theorem 5.1 tells us that the local function
\[
f(z, w) = \frac{p_1\pi_{11}^2(z, w) + p_2\pi_{21}^2(z, w) + p_3\pi_{31}^2(z, w)}{q_1\pi_{13}^2(z, w) + q_2\pi_{23}^2(z, w) + q_3\pi_{33}^2(z, w)}
\]
is proper biharmonic on SU(2). It is a well-known fact that SU(2) is, up to a constant conformal factor, isometric to the Lie group S^3 of unit quaternions in the standard Euclidean C^2 \cong \mathbb{R}^4 via

\[(z, w) \in S^3 \mapsto \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} = \begin{bmatrix} x_1 + ix_2 & x_3 + ix_4 \\ -x_3 + ix_4 & x_1 - ix_2 \end{bmatrix} \in SU(2).\]

If we identify C in the standard Euclidean defined locally on S argument.

\[\pi \text{ the composition } \pi \circ f \text{ then we can easily calculate the tension fields } \tau(f) \text{ and } \tau^2(f), \text{ using Proposition 2.2, as follows.}\]

\[\tau(f) = |x|^2 \Delta \hat{f} = -16|x|^2(p_2q_3 - p_3q_2)(q_2(x_1 + ix_2) - q_3(x_3 - ix_4))\]

\[= \frac{(q_2(x_1 + ix_2) - q_3(x_3 - ix_4))}{(q_3(x_1 - ix_2) + q_2(x_3 + ix_4))^3} \neq 0\]

and

\[\tau^2(f) = |x|^2\Delta(|x|^2\Delta(\hat{f})) = 0.\]

Here \(\Delta\) is the tension field on \(\mathbb{R}^4\) i.e. the classical Laplace operator given by

\[\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} + \frac{\partial^2}{\partial x_4^2}.\]

We will see later on that this induces proper biharmonic functions on the 3-dimensional hyperbolic space \(\mathbb{H}^3\).

6. The irreducible representation \(\pi_3\) of SU(2)

In this section we extend our construction of Theorem 5.1 to the 4-dimensional irreducible representation \(\pi_3\) of SU(2) given by the matrix

\[\pi^3 = \begin{bmatrix} z_{11}^3 & z_{12}^3 & z_{12}^2 & z_{12}^1 \\ z_{11}^2 & z_{22}^2 & z_{22}^1 & z_{22}^0 \\ z_{11}^1 & z_{22}^1 & z_{22}^0 & z_{22}^{-1} \\ z_{11}^0 & z_{22}^0 & z_{22}^{-1} & z_{22}^{-2} \end{bmatrix}.\]

**Theorem 6.1.** Let \(p, q \in \mathbb{C}^4\) and \(P, Q : U(2) \to \mathbb{C}\) be the complex-valued functions on the unitary group given by

\[P(z) = \sum_{j=1}^4 p_j \pi^3_{j\alpha} \text{ and } Q(z) = \sum_{k=1}^4 q_k \pi^3_{k\beta}.\]
Let the rational function \( f(z) = P(z)/Q(z) \) be defined on the open and dense subset \( W_Q = \{ z \in U(2) \mid Q(z) \neq 0 \} \) of the unitary group. If \( \alpha \neq \beta \) then the function \( f \) is proper biharmonic if \( p_5q_4 - p_4q_5 \neq 0 \) and

\[
p_1q_4^3 = q_3^2(3p_5q_4 - 2p_4q_3), \quad q_1q_4^2 = q_3^3,
\]

\[
p_2q_4^2 = q_4(2p_5q_4 - p_4q_3), \quad q_2q_4 = q_3^2.
\]

The corresponding statement holds for the function induced on \( SU(2) \).

**Proof.** The result can be proven with exactly the same method as we used for Theorem 5.1. But the elementary calculations are more involved in this case. \( \Box \)

### 7. The irreducible representation \( \pi_4 \) of \( SU(2) \)

In this section we extend the construction of Theorem 5.1 for \( SU(2) \) to its 5-dimensional irreducible representation \( \pi_4 \). The corresponding matrix \( \pi_4 \) is given by

\[
\begin{bmatrix}
  z_{11}^2 & z_{11}z_{12} & z_{12}^2 & z_{11}z_{12}^3 & z_{12}^4 \\
  z_{11}^3z_{21} & z_{11}^2(z_{11}z_{22} + 3z_{12}z_{21}) & 2z_{11}z_{12}(z_{11}z_{22} + z_{12}z_{21}) & z_{11}^2z_{12}^2(3z_{11}z_{22} + z_{12}z_{21}) & 4z_{11}z_{22}^2 \\
  2z_{11}z_{21} & 3z_{11}z_{21}(z_{11}z_{22} + z_{12}z_{21}) & z_{11}^2z_{12}^2 + 4z_{11}z_{12}z_{21}z_{22} + z_{12}z_{21}^2 & 3z_{12}z_{22}(z_{11}z_{22} + z_{12}z_{21}) & 6z_{12}z_{22}^2 \\
  5z_{12}^3 & 3z_{12}^2(z_{11}z_{22} + z_{12}z_{21}) & 2z_{21}z_{12}(z_{11}z_{22} + z_{12}z_{21}) & z_{22}^2z_{11}z_{22} + 3z_{12}z_{21} & 4z_{12}z_{22}^3 \\
  z_{21}^4 & z_{21}^3z_{22} & z_{21}^2z_{22} & z_{21}z_{22}^3 & z_{22}^4
\end{bmatrix}
\]

**Theorem 7.1.** Let \( p, q \in \mathbb{C}^5 \) and \( P, Q : U(2) \to \mathbb{C} \) be the complex-valued functions on the unitary group given by

\[
P(z) = \sum_{j=1}^{5} p_j \pi^4_{j\alpha} \quad \text{and} \quad Q(z) = \sum_{k=1}^{5} q_k \pi^4_{k\beta}.
\]

Let the rational function \( f(z) = P(z)/Q(z) \) be defined on the open and dense subset \( W_Q = \{ z \in U(2) \mid Q(z) \neq 0 \} \) of the unitary group. If \( \alpha \neq \beta \) then the function is proper biharmonic if \( p_4q_5 \neq p_5q_4 \) and

\[
p_1q_5^4 = q_4^3(4p_4q_5 - 3p_5q_4), \quad q_1q_5^3 = q_4^4,
\]

\[
p_2q_5^3 = q_4^2(3p_4q_5 - 2p_5q_4), \quad q_2q_5^2 = q_4^3,
\]

\[
p_3q_5^2 = q_4(2p_4q_5 - p_5q_4), \quad q_3q_5 = q_4^2.
\]

The corresponding statement holds for the function induced on \( SU(2) \).

**Proof.** The result can be proven with exactly the same method as we used for Theorem 5.1. But the elementary calculations are much more involved in this case. \( \Box \)
8. The irreducible representation $\pi_n$ of $\text{SU}(2)$

In this section we discuss the general $(n + 1)$-dimensional irreducible representation $\pi_n$ of $\text{SU}(2)$. After investigating $\pi_2, \pi_3$ and $\pi_4$ a clear pattern has appeared. For this reason we formulate the following conjecture, for which it seems a non-trivial exercise to find a general proof.

**Conjecture 8.1.** For $n > 1$, let $p, q \in \mathbb{C}^{n+1}$ and $P, Q : \text{U}(2) \to \mathbb{C}$ be the complex-valued functions on the unitary group given by

$$P(z) = \sum_{j=1}^{n+1} p_j \pi^n_{j\alpha} \quad \text{and} \quad Q(z) = \sum_{k=1}^{n+1} q_k \pi^n_{k\beta}.$$

Let the rational function $f(z) = P(z)/Q(z)$ be defined on the open and dense subset $W_Q = \{ z \in \text{U}(2) | Q(z) \neq 0 \}$ of the unitary group. If $\alpha \neq \beta$ then the function $f$ is proper biharmonic if $p_n q_{n+1} - p_{n+1} q_n \neq 0$ and

$$p_1 q_{n+1}^n = q_n^{n-1}(n \cdot p_n q_{n+1} - (n - 1) \cdot p_{n+1} q_n), \quad q_1 q_{n+1}^{n-1} = q_n^n,$$

$$p_2 q_{n+1}^{n-1} = q_n^{n-2}((n - 1) \cdot p_n q_{n+1} - (n - 2) \cdot p_{n+1} q_n), \quad q_2 q_{n+1}^{n-2} = q_n^{n-1},$$

$$\vdots$$

$$p_{n-1} q_{n+1}^2 = q_n (2 \cdot p_n q_{n+1} - 1 \cdot p_{n+1} q_n), \quad p_{n-1} q_{n+1} = q_n^2.$$

The corresponding statement holds for the function induced on $\text{SU}(2)$.

9. The Duality

The approach and the methods of this section were introduced in [9]. Let $G$ be a non-compact semisimple Lie group with the Cartan decomposition $g = \mathfrak{k} + \mathfrak{p}$ of the Lie algebra of $G$ where $\mathfrak{k}$ is the Lie algebra of a maximal compact subgroup $K$. Let $G^\mathbb{C}$ denote the complexification of $G$ and $U$ be the compact subgroup of $G^\mathbb{C}$ with Lie algebra $u = \mathfrak{k} + i \mathfrak{p}$. Let $G^\mathbb{C}$ and its subgroups be equipped with a left-invariant semi-Riemannian metric which is a multiple of the Killing form by a negative constant. Then the subgroup $U$ of $G^\mathbb{C}$ is Riemannian and $G$ is semi-Riemannian.

Let $f : W \to \mathbb{C}$ be a real analytic function from an open subset $W$ of $G$. Then $f$ extends uniquely to a holomorphic function $f^\mathbb{C} : W^\mathbb{C} \to \mathbb{C}$ from some open subset $W^\mathbb{C}$ of $G^\mathbb{C}$. By restricting this to $U \cap W^\mathbb{C}$ we obtain a real analytic function $f^* : W^* \to \mathbb{C}$ on some open subset $W^*$ of $U$. The function $f^*$ is called the dual function of $f$.

**Theorem 9.1.** [7] For the above situation we have the following duality. A complex-valued function $f : W \to \mathbb{C}$ is proper $r$-harmonic if and only if its dual $f^* : W^* \to \mathbb{C}$ is proper $r$-harmonic.

**Proof.** A proof of this statement can be found in [7]. \qed
Remark 9.2. We point out that a function \( f : W \to \mathbb{C} \) is \( K \)-invariant if and only if its dual \( f^* : W^* \to \mathbb{C} \) is \( K \)-invariant. In particular, the duality principle of Theorem 9.1 is valid for the corresponding functions on the quotient spaces.

We can now apply the duality principle to the local solutions \( f \), on the compact Riemannian symmetric space \( S^3 = \text{SO}(4)/\text{SO}(3) \), given in Example 5.2. This yields local proper biharmonic functions on its non-compact dual i.e. the 3-dimensional hyperbolic space \( H^3 = \text{SO}(1,3)/\text{SO}(3) \).

Example 9.3. Let \( \mathbb{R}^4_1 \) be the standard 4-dimensional Minkowski space equipped with its Lorentzian metric

\[
(x, y)_L = -x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3.
\]

Bounded by the light cone, the open set

\[
U = \{ x \in \mathbb{R}^4_1 \mid (x, x)_L < 0 \text{ and } 0 < x_0 \}
\]

contains the 3-dimensional hyperbolic space

\[
H^3 = \{ (x_0, x_1, x_2, x_3) \in \mathbb{R}^4_1 \mid (x, x)_L = -1 \text{ and } 0 < x_0 \}.
\]

Let \( \pi^* : U \to H^3 \) be the radial projection given by

\[
\pi^* : x \mapsto \frac{x}{\sqrt{-(x, x)_L}}.
\]

This is a harmonic morphism and its dilation satisfies \( \lambda^{-2}(x) = -|x|_L^2 \), see [6]. Let \( p, q \in \mathbb{C}^3 \) and \( f \) be the local proper biharmonic function on \( S^3 \) defined in Example 5.2. Then its dual function \( f^* : W \to \mathbb{C} \) is defined locally on \( H^3 \) with

\[
f^*(x_0, x_1, x_2, x_3) = f(-ix_0, x_1, x_2, x_3).
\]

Then

\[
f(x) = \frac{p_1(-ix_0 + ix_1)^2 + 2p_2(-ix_0 + ix_1)(-x_2 + ix_3) + p_3(x_2 - ix_3)^2}{q_1(x_2 + ix_3)^2 + 2q_2(-ix_0 - ix_1)(x_2 + ix_3) + q_3(-ix_0 - ix_1)^2}.
\]

Let \( \hat{f}^* \) be the composition \( f^* \circ \pi^* \) from the Minkowski space \( \mathbb{R}^4_1 \). Then

\[
\tau(f^*) = -|x|_L^2 \Box \hat{f}^* = 16|x|_L^2 (p_2q_3 - p_3q_2) \frac{(q_2(x_0 - x_1) - iq_3(x_2 - ix_3))(q_3(x_0 + x_1) + iq_2(x_2 + ix_3))^5}{(q_3(x_0 + x_1) + iq_2(x_2 + ix_3))^5} \neq 0
\]

and

\[
\tau^2(f^*) = -|x|_L^2 \Box (-|x|_L^2 \Delta(\hat{f})) = 0.
\]

Here \( \Box \) is the tension field on \( \mathbb{R}_1^4 \) i.e. the wave operator of d’Alembert given by

\[
\Box = -\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.
\]
From this we see that $f^*$ is proper biharmonic. It should be noted that $q \in \mathbb{C}^3$ can easily be chosen such that $f^* : \mathbb{H}^3 \to \mathbb{C}$ is globally defined.

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Appendix A.

The complex general linear group $\text{GL}_n(\mathbb{C})$ is an open subset of $\mathbb{C}^{n \times n} \cong \mathbb{C}^{n^2}$ and inherits its standard complex structure. As before, let $z_{ij}$ with $1 \leq i, j \leq n$ denote the matrix coefficients of a generic element $z \in \text{GL}_n(\mathbb{C})$. Let $G$ be a Lie subgroup of $\text{GL}_n(\mathbb{C})$, $F : U \to \mathbb{C}$ be a holomorphic function defined on an open subset of $\text{GL}_n(\mathbb{C})$ and $f : U \cap G \to \mathbb{C}$ be the restriction of $F$ to $G$.

Following Lemma 3.2 of [5], the equations (3.1), (3.2) and (3.3) can be utilized to compute the tension field of $f$, see also [7].

$$\tau(f) = \sum_{1 \leq i,j,k,\ell \leq n} \frac{\partial^2 f}{\partial z_{ij} \partial z_{k\ell}} \kappa(z_{ij}, z_{k\ell}) + \sum_{1 \leq i,j \leq n} \frac{\partial f}{\partial z_{ij}} \tau(z_{ij}).$$

For the unitary group $U(n)$ we can apply Lemma 4.1 to make the above formula more explicit

$$\tau(f) = -\sum_{1 \leq i,j,k,\ell \leq n} \frac{\partial^2 f}{\partial z_{ij} \partial z_{k\ell}} z_{kj} z_{i\ell} - n \sum_{1 \leq i,j \leq n} \frac{\partial f}{\partial z_{ij}} z_{ij}.$$

This formula is our main tool for carrying out computer calculations for checking the results of this paper.

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