The Affine Quantum Gravity Program

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Abstract

The central principle of affine quantum gravity is securing and maintaining the strict positivity of the matrix \( \{\hat{g}_{ab}(x)\} \) composed of the spatial components of the local metric operator. On spectral grounds, canonical commutation relations are incompatible with this principle, and they must be replaced by noncanonical, affine commutation relations. Due to the partial second-class nature of the quantum gravitational constraints, it is advantageous to use the recently developed projection operator method, which treats all quantum constraints on an equal footing. Using this method, enforcement of regularized versions of the gravitational operator constraints is formulated quite naturally by means of a novel and relatively well-defined functional integral involving only the same set of variables that appears in the usual classical formulation. It is anticipated that skills and insight to study this formulation can be developed by studying special, reduced-variable models that still retain some basic characteristics of gravity, specifically a partial second-class constraint operator structure. Although perturbatively nonrenormalizable, gravity may possibly be understood nonperturbatively from a hard-core perspective that has proved valuable for specialized models. Finally, developing a procedure to pass to the genuine physical Hilbert space involves several interconnected steps that require careful coordination.

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INTRODUCTION

Despite being a very difficult problem, quantization of the gravitational field has attracted considerable attention because of its fundamental importance. Among the currently favored approaches to quantize gravity is work associated with string theory, or work that is part of the canonical program, both important schemes; see, e.g., \[1, 2\].

A relatively new effort to study quantum gravity is the affine quantum gravity program the highlights of which are reviewed in this article and which is also intended to introduce this program to those who are not familiar with it. Although the principles involved are quite conservative and fairly natural, this program nevertheless involves a somewhat unconventional approach when compared with more traditional techniques. (Several precursors to the present program are briefly discussed in \[3\], while details are available in \[3, 4, 5\].)

Basic principles of affine quantum gravity

The program of affine quantum gravity is founded on \textit{four basic principles} which we briefly review here. \textit{First}, like the corresponding classical variables, the 6 components of the spatial metric field operators $\hat{g}_{ab}(x) = \hat{g}_{ba}(x)$, $a, b = 1, 2, 3$, form a \textit{positive-definite} $3 \times 3$ matrix for all $x$. \textit{Second}, to ensure self-adjoint kinematical variables when smeared, it is necessary to adopt the \textit{affine commutation relations} (with $\hbar = 1$)

\begin{align}
[\hat{\pi}^a_b(x), \hat{\pi}^c_d(y)] &= i\frac{1}{2} [\delta^c_b \hat{\pi}^a_d(x) - \delta^a_d \hat{\pi}^c_b(x)] \delta(x, y), \\
[\hat{g}_{ab}(x), \hat{\pi}^c_d(y)] &= i\frac{1}{2} [\delta^c_a \hat{g}_{bd}(x) + \delta^c_b \hat{g}_{ad}(x)] \delta(x, y), \\
[\hat{g}_{ab}(x), \hat{g}_{cd}(y)] &= 0
\end{align}

between the metric and the 9 components of the “scale” field operator $\hat{\pi}^a_b(x)$. \textit{Third}, the principle of \textit{quantization before any constraints are introduced}, due to Dirac, strongly suggests that the basic fields $\hat{g}_{ab}$ and $\hat{\pi}^a_b$ are initially realized by \textit{ultralocal representations}, which is explained below. \textit{Fourth}, and last, introduction and enforcement of the gravitational constraints not only leads to the physical Hilbert space but it has the added virtue that all vestiges of the temporary ultralocal representation vanish and are replaced by physically acceptable alternatives. In attacking these basic issues full use of \textit{coherent...
state methods and the projection operator method for constrained system quantization is made.

The affine coherent states are defined (for $\hbar = 1$) by

$$|\pi, \gamma\rangle \equiv e^{i \int \pi^{ab} g_{ab} \, dx} e^{-i \int \gamma^a_\pi g_{ab} \, dx} |\eta\rangle \quad (2)$$

for general, smooth, $c$-number fields $\pi^{ab} = \pi^{ba}$ and $\gamma^a_b$ of compact support, and are chosen so that the coherent state overlap function becomes

$$\langle \pi'', \gamma'' | \pi', \gamma' \rangle = \exp\left(-2 \int b(x) \, dx \right) \times \ln \left\{ \frac{\det\left\{ \frac{1}{2} \left[ g''^{kl}(x) + g'^{kl}(x) \right] + i \frac{1}{2} b(x)^{-1} \left[ \pi''^{kl}(x) - \pi'^{kl}(x) \right] \right\}}{\left( \det[g''^{kl}(x)] \right)^{1/2} \left( \det[g'^{kl}(x)] \right)^{1/2}} \right\} \equiv \langle \pi''', \gamma''' | \pi', g' \rangle \quad (3)$$

First, observe that the matrices $\gamma''$ and $\gamma'$ do not explicitly appear in (3); they have each been replaced by

$$g(x) \equiv e^{\gamma(x)/2} \bar{g}(x) e^{-\gamma(x)/2} \equiv \{g_{ab}(x)\} \quad (4)$$

where $\langle \eta|g_{ab}(x)|\eta\rangle \equiv \bar{g}_{ab}(x) = \{ \bar{g}_{ba}(x) \}$, a fixed reference metric that only serves to define the underlying topology of the space being quantized. Since only $g$ (and not $\gamma$) appears in the chosen functional form, we have renamed the overlap function $\langle \pi'', g'' | \pi', g' \rangle$ without loss of generality. This fact implies that the coherent states themselves are equally well denoted by $|\pi, g\rangle$.

Second, note that the representation (3) is ultralocal, i.e., specifically of the form

$$\exp\left\{-\int b(x) \, dx \left[ L[\pi''(x), g''(x); \pi'(x), g'(x)] \right] \right\} \quad (5)$$

and thus, by design, there are no correlations between or among spatially separated field values, a neutral position towards correlations before any constraints are introduced. On invariance grounds, (3) necessarily involves a scalar density $b(x)$, $0 < b(x) < \infty$, for all $x$; this arbitrary and nondynamical auxiliary function $b(x)$ is only temporary and it will disappear when the gravitational constraints are fully enforced, at which point proper field correlations will arise. In addition, note well that the coherent-state overlap functional is invariant under general spatial coordinate transformations.
Third, and last, we emphasize that the expression $\langle \pi'', g'' | \pi', g' \rangle$ is a continuous, positive-definite functional and thus may be used as a reproducing kernel to define a reproducing kernel Hilbert space $\mathcal{C}$ composed of continuous phase-space functionals $\psi(\pi, g)$ on which the initial, ultralocal representation of the affine field operators acts in a natural fashion. Some further explanation at this point may be helpful.

Although not commonly used, reproducing kernel Hilbert spaces are very natural and readily understood. By definition, the vectors $\{|\pi, g\rangle\}$ span the Hilbert space, and therefore elements of a dense set of vectors have the form

$$|\psi\rangle = \sum_{k=1}^{K} \alpha_k |\pi(k), g(k)\rangle ,$$  \hspace{1cm} (6)

for general sets $\{\alpha_k\}_{k=1}^{K}$ and $\{\pi(k), g(k)\}_{k=1}^{K}$, and some $K < \infty$. The inner product of two such vectors is clearly given by

$$\langle \phi | \psi \rangle = \sum_{j,k=1}^{J,K} \beta_{\ast j}^k \alpha_k \langle \pi(j), g(j) | \pi(k), g(k) \rangle .$$  \hspace{1cm} (7)

To represent the abstract vectors themselves as functionals, we adopt the natural coherent-state representation, namely

$$\psi(\pi, g) \equiv \langle \pi, g | \psi \rangle = \sum_{k=1}^{K} \alpha_k \langle \pi, g | \pi(k), g(k) \rangle .$$  \hspace{1cm} (8)

Thus, we have a dense set of continuous functions, $\{\psi(\pi, g)\}$, and a definition of an inner product between pairs of such functions, $\langle \phi, \psi \rangle \equiv \langle \phi | \psi \rangle$, as defined in (7). It only remains to complete the space to a (separable) Hilbert space $\mathcal{C}$ by adding the limit points of all Cauchy sequences in the norm $\|\psi\| \equiv (\psi, \psi)^{1/2}$. Observe that these definitions imply that $\langle \cdot, | \pi', g' \rangle, \psi \rangle = \psi(\pi', g')$ and so the kernel $\langle \pi, g | \pi', g' \rangle$ reproduces the original vector $\psi(\pi, g)$. Note well that all properties of the reproducing kernel Hilbert space $\mathcal{C}$ follow as direct consequences from just the reproducing kernel $\langle \pi'', g'' | \pi', g' \rangle$ itself; for details see, e.g., [1]. (Of course, $f(y) = \int \delta(x-y) f(x) \, dx$ as well, but the difference is that the kernel $\delta(x-y)$ is neither continuous nor is an element of the Hilbert space of square integrable functions. Therefore, $L^2(\mathbb{R})$ is not a reproducing kernel Hilbert space.)
During the past several years, a functional integral formulation has been developed \[4\] that, in effect, within a single formula captures the essence of all four of the basic principles described above. This “Master Formula” takes the form

\[
\langle \pi'', g'' | E \pi', g' \rangle = \lim_{\nu \to \infty} \mathcal{N}_\nu \int e^{-i \int [g_{ab} \dot{\pi}_{ab} + N^a H_a + NH] d^3x dt} \times \exp\left\{-\frac{1}{2\nu} \int [b(x)^{-1} g_{ab} g_{cd} \pi_{bc} \pi_{da} + b(x) g_{ab} g_{cd} \dot{g}_{bc} \dot{g}_{da}] d^3x \right\} \times [\Pi_{x,t} \Pi_{a \leq b} \pi_{ab}(x,t) d\pi_{ab}(x,t)] DR(N^a, N) .
\]

(9)

Let us explain the meaning of (9).

As an initial remark, let us artificially set $H_a = H = 0$, and use the fact that $\int DR(N^a, N) = 1$. Then the result is that $E = 1$, and the remaining functional integral yields the coherent state overlap $\langle \pi'', g'' | \pi', g' \rangle$ as given in (3). This is the state of affairs before the constraints are imposed, and remarks below regarding the properties of the functional integral on the right-hand side of (9) apply in this case as well. We next turn to the full content of (9).

The expression $\langle \pi'', g'' | E | \pi', g' \rangle$ denotes the coherent state matrix elements of the projection operator $E$ which projects onto a subspace of the original Hilbert space on which the quantum constraints are fulfilled in a regularized fashion. Furthermore, the expression $\langle \pi'', g'' | E | \pi', g' \rangle$ is another manifestly positive-definite functional that can be used as a reproducing kernel and thus used directly to generate the reproducing kernel physical Hilbert space on which the quantum constraints are fulfilled in a regularized manner. The right-hand side of equation (9) denotes a reasonably well-defined functional integral over fields $\pi_{ab}(x,t)$ and $g_{ab}(x,t)$, $0 < t < T$, designed to calculate this important reproducing kernel for the regularized physical Hilbert space and which entails functional arguments defined by their smooth initial values $\pi_{ab}(x,0) = \pi''_{ab}(x)$ and $g_{ab}(x,0) = g''_{ab}(x)$ as well as their smooth final values $\pi_{ab}(x,T) = \pi''_{ab}(x)$ and $g_{ab}(x,T) = g''_{ab}(x)$, for all $x$ and all $a,b$. Up to a surface term, the phase factor in the functional integral represents the canonical action for general relativity, and specifically $N^a$ and $N$ denote Lagrange multiplier fields (classically interpreted as the shift and lapse), while $H_a$ and $H$ denote phase-space symbols (since $\hbar \neq 0$) associated with the quantum diffeomorphism and Hamiltonian constraint field operators, respectively. The $\nu$-dependent factor in the integrand formally tends to unity in
the limit $\nu \to \infty$; but prior to that limit, the given expression regularizes and essentially gives genuine meaning to the heuristic, formal functional integral that would otherwise arise if such a factor were missing altogether \cite{4}. The functional form of the given regularizing factor ensures that the metric variables of integration strictly fulfill the positive-definite domain requirement. The given form and in particular the need for the nondynamical, nonvanishing, arbitrarily chosen scalar density $b(x)$, is very welcome since this form—and quite possibly only this form—leads to a reproducing kernel Hilbert space for gravity having the needed infinite dimensionality; a seemingly natural alternative \cite{7} using $\sqrt{\det g_{ab}(x)}$ in place of $b(x)$ fails to lead to a reproducing kernel Hilbert space with the required dimensionality \cite{8}. The choice of $b(x)$ determines a specific ultralocal representation for the basic affine field variables, but this unphysical and temporary representation disappears entirely after the gravitational constraints are fully enforced (as soluble examples explicitly demonstrate \cite{5}). The integration over the Lagrange multiplier fields ($N^a$ and $N$) involves a rather specific measure $R(N^a, N)$ (described in \cite{9}), which is normalized such that $\int \mathcal{D} R(N^a, N) = 1$. This measure is designed to enforce (a regularized version of) the quantum constraints; it is manifestly not chosen to enforce the classical constraints, even in a regularized form. The consequences of this choice are profound in that no (dynamical) gauge fixing is needed, no ghosts are required, no Dirac brackets are necessary, etc. In short, no auxiliary structure of any kind is introduced. (These facts are general properties of the projection operator method of dealing with constraints \cite{9, 10} and are not limited to gravity.)

How one uses (9) to proceed further is detailed below, but the general idea, roughly speaking, is as follows. A major goal in the general analysis of (9) involves reducing the regularization imposed on the quantum constraints to its appropriate minimum value, and, in particular, for constraint operators that are partially second class, such as those of gravity, the proper minimum of the regularization parameter is nonzero. Achieving this minimization involves fundamental changes of the representation of the basic kinematical operators, which, as models show \cite{5}, are so significant that any unphysical aspect of the original, ultralocal representation disappears completely. When the appropriate minimum regularization is achieved, then the quantum constraints are properly satisfied. The result is the reproducing kernel for the physical Hilbert space which then permits a variety of physical questions to be studied.
One may wonder why we have stressed the reproducing kernel definition of the inner product in the resultant Hilbert space and have not mentioned any of the standard integral expressions for forming inner products usually associated with coherent states. The reason for this is that the functional representation of Hilbert space involved does not possess a local integral expression to define the inner product of vectors despite the fact that it is a coherent state representation. Such representations are said to involve weak coherent states [11, 12] and they are based on so-called nonsquare integrable representations of the associated group. Despite the lack of a usual local integral representation for the inner product, it is especially interesting that there is, nevertheless, a nearly unchanged phase-space path integral representation. This fascinating and indeed somewhat surprising story arises for elementary systems as well and is described in [12], a paper that was motivated by the present study of quantum gravity.

Quantum constraints and their treatment

The quantum gravitational constraints, $\mathcal{H}_a(x)$, $a = 1, 2, 3$, and $\mathcal{H}(x)$, formally satisfy the commutation relations

$$
\begin{align*}
[\mathcal{H}_a(x), \mathcal{H}_b(y)] &= i\frac{1}{2}[\delta_{a}(x, y)\mathcal{H}_b(y) + \delta_{b}(x, y)\mathcal{H}_a(x)] , \\
[\mathcal{H}_a(x), \mathcal{H}(y)] &= i\delta_{a}(x, y)\mathcal{H}(y) , \\
[\mathcal{H}(x), \mathcal{H}(y)] &= i\frac{1}{2}\delta_{a}(x, y)[g^{ab}(x)\mathcal{H}_b(x) + \mathcal{H}_b(x)g^{ab}(x) \\
&\quad + g^{ab}(y)\mathcal{H}_b(y) + \mathcal{H}_b(y)g^{ab}(y)] .
\end{align*}
$$

(10)

Following Dirac, we first suppose that $\mathcal{H}_a(x)|\psi\rangle_{\text{phys}} = 0$ and $\mathcal{H}(x)|\psi\rangle_{\text{phys}} = 0$ for all $x$ and $a$, where $|\psi\rangle_{\text{phys}}$ denotes a vector in the physical Hilbert space $\mathcal{H}_{\text{phys}}$. However, these conditions are incompatible since $[\mathcal{H}_b(x), g^{ab}(x)] \neq 0$ and almost surely $g^{ab}(x)|\psi\rangle_{\text{phys}} \not\in \mathcal{H}_{\text{phys}}$, even when smeared. This means that the quantum gravity constraints are partially second class. While others may resist this conclusion, we accept it for what it is. One advantage of the projection operator method is that it treats all quantum constraints, e.g., first- and second-class constraints, on an equal footing; see [9, 10]. In brief, if $\{\Phi_a\}$ denotes a set of self-adjoint quantum constraint operators, then

$$
E = E(\sum \Phi_a^2 \leq \delta(h)^2) = \int Te^{-i\int L^{\alpha}(t)\Phi_a dt} DR(\lambda)
$$

(11)
denotes a projection operator onto a regularized physical Hilbert space. Sometimes, just by reducing the regularization parameter $\delta(h)^2$ to its appropriate size, the proper physical Hilbert space arises. Thus, e.g., if $\Sigma\Phi^2 = J_1^2 + J_2^2 + J_3^2$, the Casimir operator of $su(2)$, then $0 \leq \delta(h)^2 < 3h^2/4$ works for this first class example. If $\Sigma\Phi^2 = P^2 + Q^2$, where $[Q, P] = i\hbar I$, then $\hbar \leq \delta(h)^2 < 3\hbar$ covers this second class example. Other cases may be more involved but the principles are similar. The time-ordered integral representation for $E$ given in (11) is useful in path integral representations and this expression is entirely analogous to the origin of $R(N^a, N)$ in (9).

It is fundamentally important to make clear how Eq. (9) was derived and how it is to be used [4]. The left-hand side of (9) is an abstract operator construct in its entirety that came first and corresponds to one of the basic expressions one would like to calculate. The functional integral on the right-hand side of (9) came second and is a valid representation of the desired expression. However, the final goal is to turn that order around and to use the functional integral to define and evaluate (at least partially) the desired operator-defined expression on the left-hand side. In no way should it be thought that the functional integral was “simply postulated” as a “guess as how one might represent the proper expression”, however suggestive that guess may be.

**DIRECTIONS FOR FUTURE RESEARCH**

There are several directions for further research that are worthwhile, and let us elaborate on some of these. In a certain sense, they all take the Master Formula, Eq. (9), as their starting point.

**Reduced variable models**

To gain insight into (9) it is useful to study models with a finite number of degrees of freedom. This reduction, however, should be done so as to retain some of the second class nature of the quantum constraints characteristic of gravity. Most workers want to avoid quantum second class constraints at any cost assuming they must be “wrong” (i.e., unphysical). In fact, they arise as a direct consequence of the fundamentally different invariance groups of classical and quantum mechanics embodied in *fundamentally different* bracket
relations, e.g., as given between generator elements (for $\hbar = 1$) by

$$\{e^{aq+bp}, e^{cq+dp}\} = (ad - bc) e^{(a+c)q+(b+d)p},$$

$$[e^{aQ+bP}, e^{cQ+dP}] = 2i \sin \left( \frac{1}{2}(ad - bc) \right) e^{(a+c)Q+(b+d)P},$$

where $a, b, c,$ and $d$ represent free parameters. So disturbing has the loss of full classical invariance been to some workers, it has prompted the program of “Geometric quantization” which changed the rules of operator representation so that the new quantum commutator bracket agreed in structure with the classical Poisson bracket. While such a modification is mathematically possible, it has essentially nothing to do with physics.

We accept the implication that when a classical invariance is lost it is nevertheless replaced with a quantum invariance that reduces to the classical one as $\hbar \to 0$. Therefore, in the present philosophy, reduced variable models should be studied that retain the important feature of gravitational constraints that are first class classically and (partially) second class quantum mechanically. (Such features are avoided in traditional minisuperspace models.)

**Metrical quantization**

We have emphasized that (9) was first obtained by starting with the putative operator approach and deriving from it a valid functional integral representation. In so doing we have automatically been led to the all-important $\nu$-dependent continuous-time regularization factor that renders the functional integral representation so nearly well defined (see [4] for details). In particular, observe that there is a phase space metric that appears in the regularization factor in (9).

Alternative to how (9) was originally derived, one could accept the need for some such regularization and adopt the viewpoint that it is the choice of a metric on gravitational phase space that is the first and key ingredient in deciding how to initiate quantization by functional integration. This is the viewpoint of Metrical quantization [13]. A preliminary study of alternative metrics has begun [8], and a more thorough examination is well warranted in order to determine whether the metric given in (9) is indeed optimal or perhaps other metrics are also worthy of study and would have the potential of leading to qualitatively different quantizations (as other examples demonstrate [14]).
Nonrenormalizability and symbols

Viewed perturbatively, gravity is nonrenormalizable. However, the (nonperturbative) hard-core picture of nonrenormalizability [15, 16] holds that the nonlinearities in such theories are so strong that, from a functional integration point of view, a nonzero set of functional histories that were allowed in the support of the linear theory is now forbidden by the nonlinear interaction. This picture is qualitatively like that of a hard-core potential in nuclear physics which forbids particles from coming closer than a specified distance. In each case, expansion of the hard-core interaction in a perturbation series is surely inappropriate and alternative analyses are called for.

Various highly specialized field theory models exhibit analogous hard-core behavior and nevertheless possess suitable nonperturbative solutions [16]. It is believed that gravity and also $\phi^4$ field theories in high enough spacetime dimensions can be understood in similar terms. A computer study to analyze the $\phi^4$ theory has begun, and there is hope to clarify that particular theory. Any progress in the scalar field case could strengthen the gravitational field case as well.

Evidence from soluble examples points to the appearance of a nonclassical (i.e., $\propto \hbar$) and nontraditional counterterm in the functional integral representing the irremovable effects of the hard core. These counterterms have an important role to play as part of the symbols representing the diffeomorphism and Hamiltonian constraints in the functional integral since for them $\hbar \neq 0$ as well; indeed, in the chosen units $\hbar = 1$. In brief, the form taken by the symbols $H_\alpha$ and $H$ in (9) is intimately related to the proper understanding of how to handle the perturbative nonrenormalizability and the concomitant hard-core nature of the overall theory. These are clearly difficult issues, but it is equally clear that they may be illuminated by studies of other nonrenormalizable models such as $\phi^4$ in sufficiently many dimensions.

Classical limit

Suppose one starts with a classical theory, quantizes it, and then takes the classical limit. It seems obvious that the classical theory obtained at the end should coincide with the classical theory one started with. However, there are counterexamples to this simple wisdom! For example, the $\phi^4$ theory in five spacetime dimensions has a nontrivial classical behavior. But,
if one quantizes it by the lattice limit of a natural lattice formulation, the result is a free (or generalized free) quantum theory whose classical limit is also free and thus differs from the original theory [17]. This unsatisfactory behavior is yet another side of the nonrenormalizability puzzle. However, those nonrenormalizable models for which the quantum hard-core behavior has been accounted for do have satisfactory classical limits [16]. The conjectured hard-core nature of $\phi^4$ models is under present investigation, and it is anticipated that a proper classical limit should arise. It is further conjectured that a favorable consequence of clarifying and including the hard-core behavior in gravity will ensure that the resultant quantum theory enjoys the correct classical limit.

A few general remarks may be useful. It is a frequent misconception that passage to the classical limit requires that the parameter $\hbar \to 0$. To argue against this view, just note that the macroscopic world we know and describe so well by classical mechanics is the same real world in which $\hbar \neq 0$. In point of fact, classical and quantum formalisms must coexist, and this coexistence is very well expressed with the help of coherent states. It is characteristic of coherent state formalisms that classical and quantum “generators”, loosely speaking, are simply related to each other through the Weak Correspondence Principle [18]. In the case of the gravitational field, prior to the introduction of constraints, this connection takes the general form

$$\langle \pi, g | W | \pi, g \rangle = W(\pi, g).$$

(14)

Here $W$ denotes a quantum generator and $W(\pi, g)$ the corresponding classical generator (which is generally a “symbol” still since $\hbar \neq 0$). The simplest example of this kind is given by $\langle \pi, g | \hat{g}_{ab}(x) | \pi, g \rangle = g_{ab}(x)$.

In soluble models where the appropriate classical limit has been obtained [16], coherent state methods were heavily used. It is expected that they will prove equally useful in the case of gravity.

**Passage to the physical Hilbert space**

The reproducing kernel for the regularized physical Hilbert space given in (9) contains the seeds needed to define the reproducing kernel for the genuine physical Hilbert space. Rather than using (9) directly, however, we need to recognize that the Hamiltonian constraint, in particular, needs to be regularized since in its unregularized form it is incompatible with the original ul-
tralocal representation of the basic kinematical operators. The regularization may be removed when the basic kinematical operators are in the proper representation, and the proper representation is determined by a proper choice of fiducial vector. In practice, we can say that to obtain a proper reproducing kernel for the physical Hilbert space one must reduce the regularization parameter combined with any necessary rescaling of the reproducing kernel as well as recenter the reproducing kernel, which means to suitably change the fiducial vector, all the while that one removes the regularization introduced into the Hamiltonian constraint.

One possible procedure to accomplish these goals would be as follows. Let us first introduce the notation

\[ U[(\pi, \gamma)] \equiv e^{i \int \pi^a \partial_a \, d^3x} e^{-i \int \gamma_a \partial_a \, d^3x} \quad (15) \]

for elements of the affine group. Next let us define the parameter product \((\pi, \gamma) \cdot (\pi', \gamma')\) as the group multiplication law so that

\[ U[(\pi, \gamma) \cdot (\pi', \gamma')] \equiv U[(\pi, \gamma)] U[(\pi', \gamma')] . \quad (16) \]

As a third step, let us form the states

\[ |\pi, \gamma; \{a\} \rangle \equiv \sum_{k=1}^{K} a_k \left( U[(\pi, \gamma)] U[(\pi(k), \gamma(k))] |\eta\rangle \right) \]

\[ = \sum_{k=1}^{K} a_k U[(\pi, \gamma) \cdot (\pi(k), \gamma(k))] |\eta\rangle \]

\[ = \sum_{k=1}^{K} a_k \left( (\pi, \gamma) \cdot (\pi(k), \gamma(k)) \right) \]

\[ \equiv \sum_{k=1}^{K} a_k |\pi((k)), \gamma((k))\rangle \]

\[ \equiv \sum_{k=1}^{K} a_k |\pi((k)), g((k))\rangle , \quad (17) \]

for suitable sets \(\{a_k\}_{k=1}^{K}\) and \(\{\pi((k)), g((k))\}_{k=1}^{K}\), and some \(K < \infty\). Observe, by this procedure, that we have been able to change the fiducial vector in
the original reproducing kernel, since, according to the first line of (17), we see that

\[ | \pi, \gamma; \{ a \} \rangle = U[(\pi, \gamma)] \left\{ \sum_{k=1}^{K} a_k U[(\pi(k), \gamma(k))] \right\} | \eta \rangle \equiv U[(\pi, \gamma)] \{ \{ a \} \} \]. \quad (18)

Let us denote by \( E_{\Lambda} \) the projection operator appearing in (9) where \( \Lambda \) denotes the cutoff introduced into the Hamiltonian constraint. The \( \Lambda \), based on the earlier discussion, we can form a set of quotients determined by suitable linear sums of terms given by (9). Specifically, let us consider the set given by

\[ \left\{ \frac{\langle \pi'', \gamma''; \{ a \} | E_{\Lambda} | \pi', \gamma'; \{ a \} \rangle}{\langle 0, \tilde{\gamma}; \{ a \} | E_{\Lambda} | 0, \tilde{\gamma}; \{ a \} \rangle} : \text{for all nonzero } | \{ a \} \rangle \text{ as defined in (18)} \right\} \quad (19) \]

(It is assumed, of course, that no terms with a vanishing denominator are included.) Among all elements of this set, one seeks to find those vectors \( | \{ a \} \rangle \) (i.e., those sets \( \{ a_k \} \) and \( \{ \pi((k)), g((k)) \} \)) that maintain joint continuity in the arguments \( (\pi'', \gamma'') \) and \( (\pi', \gamma') \). As the cutoff is removed, i.e., as \( \Lambda \to \infty \), any given quotient will signal incompatibility with the current representation of the basic kinematical operators by losing joint continuity in the arguments \( (\pi'', \gamma'') \) and \( (\pi', \gamma') \). Ideally, to regain continuity one must reject the current operator representation and search for the proper representation, or provisionally at least, a more favorable representation. This search may be effected by examining various limits involving different fiducial vectors \( | \{ a \} \rangle \). In particular, exploring inequivalent operator representations involves changing Hilbert spaces, so to speak, and this entails taking suitable limits as \( K \to \infty \) that force the fiducial vector to leave its original Hilbert space. However, prior to taking all the required limits, indications of the proper direction to proceed should already be seen in the pattern of behavior within the original Hilbert space. This fact makes the study of the insipient discontinuity of various elements of (13) a vital clue to decide which direction to take when “leaving the original Hilbert space”.

This picturesque view can be made more precise and done so to an extent, hopefully, that some specific algorithm can be drafted which provides a practical way to find and study properties of changing the fiducial vector in an optimal fashion. Achieving the goal of finding the appropriate fiducial
vector will lead to a new and proper representation of the basic kinematical operators that at the same time supersedes the original and provisional ultralocal representation.

It is notable that the relevant variables of the resultant functional representation arise as a direct consequence of the very reduction procedure outlined above, an especially welcome consequence of using reproducing kernels. In particular, the physical reproducing kernel Hilbert space, generally speaking, has one or more “spectator” variables that are not necessary to span the Hilbert space, and even more importantly, they are not necessary to form the inner product in the associated reproducing kernel Hilbert space. It is among such variables that a suitable parameter to serve as “time” may be found for theories that originally were reparametrization invariant. Several soluble examples have confirmed this possibility; see, e.g., [10].

**SUMMARY**

The approach to quantum gravity described in this article may seem unusual to the general reader, but it should be appreciated that its basic underlying principles are in fact quite conservative. A brief summary of the main points may be helpful.

Central to the present procedure is an insistence on the strict positive-definite character of the matrix of operators for the spatial metric. (A similar insistence does not seem to appear in either the canonical or string programs.) Compatibility with the metric spectrum requires using the affine commutation relations which are decidedly noncanonical and, instead, more like current algebras. Specific affine coherent states, projection operators to enforce constraints, and continuous-time regularized functional integral representations complete the formalism as presently constituted. Suitable limits to change initial operator representations have the potential of determining the proper physical Hilbert space for quantum gravity, and achieving that would enable many physical questions to be studied.

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