Theoretical and Numerical Analysis of One Euler Two-Layer Completely Conservative Difference Scheme of Gas Dynamics with Adaptive Viscosity

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Abstract. For the equations of gas dynamics in Euler variables, using the operator approach, a family of two-layer in time completely conservative difference schemes (CCDS) with space-profiled weight factors used to approximate the momentum and energy fluxes in time is constructed, theoretically and numerically studied. The schemes have second-order accuracy and are implemented using simple iterative processes. A class of divergent adaptive viscosities for these CCDS is developed and a theoretical analysis of their stability is carried out. The regularization of flux terms of the gas dynamics equations with the help of adaptive artificial viscosity is proposed and, using the example of well-known Einfeld problem, is numerically investigated. This regularization effectively eliminates nonphysical oscillations of the solution, entropy peaks and preserves the property of complete conservatism of schemes of this class.

1. Introduction
The present work is devoted to the theoretical study of stability and the numerical analysis of a family of two-layer in time completely conservative difference schemes (CCDS) with space-profiled time weights for the system of gas dynamics equations in Euler variables using adaptive artificial viscosity [1, 2]. The purpose of the work is to study and improve numerical approaches that correctly model the entropic evolution of the system and determine the quantitative characteristics of unstable flows in the form of spatially distributed viscous accumulations in a discrete medium.

Regularization of divergent fluxes of momentum and internal energy masses of equations of gas dynamics is proposed using adaptive artificial viscosity that does not violate the properties of complete conservatism of schemes of this class. The analysis of the amplitude of this regularization and the possibility of its use on non-uniform grids are considered. Regularized fluxes make the scheme quasi-monotonic and ensure the coordination of momentum, kinetic and internal energy (with correct entropic evolution) balances while maintaining the property of complete conservatism. At the same time, in the constructed class of divergent difference schemes, in accordance with the laws of thermodynamics, there are no approximating permanent sources (or sinks) of internal energy. Schemes are second-order accurate. The paper describes the approximation, the introduction of an artificial viscosity of the CCDS and the study of the stability of the obtained difference schemes. The structure of iterative CCDS algorithms with dynamically generated viscous accumulations in a discrete medium.
is described. The developed algorithm was tested on the basis of the well-known Einfeld problem [3, 4] about the propagation of two symmetric rarefaction waves in opposite directions. Numerical calculations using the class of divergent adaptive viscosities developed for CCDS show a significant improvement in the quality of the obtained approximate solutions in terms of their high-frequency monotonization and conservation of entropy properties. The entropy peaks in the temperature profiles disappear with the refinement of the spatial grid.

2. Formulation

The flow of a compressible medium is considered in Eulerian variables in a Cartesian coordinate system. Let \( \mathbf{u} \) be the velocity of the flow. The mass flux density is denoted by \( \rho \mathbf{u} \) (\( \rho \) is the medium density). Then the system of Euler equations [1, 2] for the medium flow will take the form:

\[
\frac{D(dM)}{Dt} = -dV \text{div} \, \mathbf{u},
\]

\[
\frac{D(\rho dM)}{Dt} = -dV \text{grad} \, P - dV \text{div} \, (\rho \mathbf{u} \cdot \mathbf{u}),
\]

\[
\frac{D(\rho u^2/2dM)}{Dt} = -\rho dV \text{grad} \, P - dV \text{div} \, (\rho \mathbf{u} \cdot \mathbf{u}^2/2),
\]

\[
\frac{D[(\epsilon + u^2/2)dM]}{Dt} = -dV \text{div}(P \mathbf{u}) - dV \text{div} \, [\rho (\mathbf{u} + \mathbf{u}^2/2)] + dQ,
\]

with the laws of conservation of mass, momentum, as well as the internal, kinetic and total energies of the system. When writing the balance equations, we used the identity:

\[
\rho D(\rho dM)/Dt = D(\rho u^2/2dM)/Dt + \rho u^2/2D(dM)/Dt.
\]

Here \( P \) is the pressure, \( \epsilon \) is the specific internal energy. The traditional model of the medium and the corresponding designations are adopted [5]: a particle of mass \( dM \) is enclosed in a volume \( dV \), through the boundaries of which a flux of mass (related to its density \( \rho \)) flows, carrying momentum (determined by the density \( \rho \mathbf{u} \)) and internal energy (with a density \( \rho \mathbf{u} \cdot \mathbf{u} \)). \( dQ \) is the external energy released in the volume \( dV \). For practical calculations, the system must be supplemented with an equation of state for the gas that relates pressure, density and temperature.

3. Time two-layer completely conservative difference scheme in Eulerian variables

Figure 1 shows the difference grid, where \( \omega \) are the nodes of the difference grid, \( \Omega \) are its cells. Thermodynamic quantities \( \rho, P, \epsilon \), as well as the volume of the cell \( V \) and its mass \( M = \rho V \) will be referred to the cells \( \Omega \). Velocity \( \mathbf{u} \), near-node mass \( m \) and volume \( v \) will be referred to nodes \( \omega \).

Figure 1. Difference grid.

Obviously:

\[
m_\omega = \frac{1}{2} \sum_\Omega(\omega) M_\Omega, \quad V_\omega = h_\omega, \quad v_\omega = \frac{1}{2} \sum_\Omega(\omega) V_\Omega = h_{\omega+0.5}, \quad \mu_\omega = \frac{1}{2} \sum_\Omega(\omega) \mu_\omega, \quad \rho_\omega = \frac{m_\omega}{v_\omega} = \rho_{\omega+0.5},
\]

where \( \mu_\omega \) and \( \mu_P \) are respectively introduced nodal and cell mass fluxes (\( \mathbf{u} = \rho \mathbf{u} \)). Also, by the momentum referred to the node we mean the quantity \( \hat{I}_\omega = \rho_{\omega+0.5} \mathbf{u}_\omega \).

Further, for the continual operations of vector analysis \( \text{div} \, \mathbf{u}, \text{grad} \, P, \text{div} \, (\mathbf{u} \cdot \mathbf{u}) \) we introduce their difference analogs \( \text{DIV} : (\omega) \rightarrow (\Omega), \text{GRAD} : (\Omega) \rightarrow (\omega) \), as well as dyad divergence \( \text{DIT}_D : (\Omega) \rightarrow (\omega) \) to approximate the transfer processes. So, we obtain:

\[
\text{DIV} \, \mathbf{u} = \frac{1}{V} \sum_\Omega(\omega) S_\omega(\Omega) \mathbf{u}(\omega), \quad \text{GRAD} \, P = \frac{\Delta P}{v}.
\]
where \( \Delta P = -\sum_{\omega} S_{\omega}(\Omega) P_{\Omega} + S_{\partial \omega} P_{\partial \omega} \), \( \text{DIV}_D \vec{\mu}_D = -\frac{1}{\nu} \sum_{\omega} S_{\omega}(\Omega) \mu_D(\Omega), \text{DIV}_D(\vec{\mu}_D \cdot \vec{u}_D) = -\frac{1}{\nu} \sum_{\omega} S_{\omega}(\Omega) \mu_D(\Omega) \vec{u}_D(\Omega). \) In the expression for \( \Delta P \), if the node \( \omega = \partial \omega \) is a boundary, a term with the value \( P_{\partial \omega} \) at the boundary and the sign function \( S_{\partial \omega} = \pm 1 \) depending on the direction of the boundary normal are added.

Let us write down a completely conservative difference scheme [2]:

\[
(\mu u)_t = -v \text{GRAD} P^\circ - v \text{DIV}_D(\vec{\mu}_D \cdot \vec{u}_D),
\]

\[
(\mu u)_t = -P^\circ \text{DIV} \vec{u} - P \text{DIV}[(\rho \vec{u})^\circ],
\]

\[
(m \vec{u}^2/2)_t = -v(u^\circ, \text{GRAD} P^\circ) - v \text{DIV}_D(\vec{\mu}_D \vec{u}^2 - /2).
\]

Here \( \vec{\mu}_D \) is understood to be some approximation of the internal energy flux at the node \( \omega \) and \( u^\circ_D = 0.5(\vec{u}^{(\omega)}_\omega + \vec{u}^{(\partial \omega)}_\omega) \), \( \vec{u}^2 _D = 0.5(\vec{u}^{(\omega)}_\omega \cdot \vec{u}^{(\partial \omega)}_\omega) \). In the difference form, we used the identity (the theorem of living forces):

Further, by \( u^\circ \) we will mean \( \vec{u}^\circ = \vec{u}^{(\omega)} \). On time layers \( t \) and \( \bar{t} = t + \tau (\tau > 0 \text{ is time step}) \), we introduce difference time derivatives and space-point (i.e., at grid nodes \( \omega \)) time interpolations:

\[
a_t = (a - a)/\tau, a^{(\delta)} = \delta a + (1 - \delta)a. \]

Here the interpolation weight \( \delta \) can depend on the node of the spatial grid according to the law \( \delta = \sqrt{m}/(\sqrt{m} + \sqrt{m}) \). Also, by arbitrary interpolation in time of grid functions \( a, \hat{a} \) between layers \( t \) and \( \bar{t} \) we mean some grid value \( a^\circ \).

4. Artificial viscosity for completely conservative difference schemes in Euler variables

For simplicity, the spatial grid in this section will be considered uniform with a step \( h_i = h_{i+0.5} = h \).

4.1. Regularization of the continuity equation

Let’s write the equation of continuity in a cell:

\[
M_t + V \text{DIV} \vec{\mu} = 0,
\]

or in index form:

\[
\frac{\rho_i - \rho_i}{\tau} + \frac{\mu_i \rho_i - \mu_{i+0.5} - \mu_{i-0.5}}{h_i} = 0,
\]

from which (3.1) follows and \( \vec{\mu}_D = 0.5 \sum_{\omega(\Omega)} \vec{\mu}^\circ(\omega) \). \( \vec{\mu}^\circ(\omega) = 0.5 \left( \vec{\mu}(\omega) + \vec{\mu}(\omega) \right) \), \( \vec{\mu} = 0.5 \sum_{\omega(\Omega)} \vec{\mu}(\omega) \). The adaptive viscosity with the coefficient \( v \) of the continuity equation (4.1.1) is presented on the explicit and implicit layers in time in the nodal mass fluxes:

\[
\mu^\circ = \rho^\circ u^\circ = (v \text{GRAD} \rho)^\circ.
\]

Here \( \rho^\circ \) is the nodal approximation of the density.

According to [1, 6] on the explicit and implicit time layers in equation (4.1.1) in each cell \( i \): a) with local “freezing” of velocity \( u_{i-0.5} = u_{i+0.5} = u = \text{const} \); b) when choosing viscosity with a lower estimate \( |u|/2 < (v/h)^{0.5} \); c) only on explicit time layer in choosing viscosity with an upper estimate \( [(v/h)^{0.5} + (v/h)^{0.5}]/2 < h_i/\tau \). In the norm \( C \), there is the stability estimate of the difference solution \( \| \vec{\mu} \|_C \leq \| \rho \|_C \). This result is used when choosing adaptive artificial viscosity in continuity equation (4.1.1), (4.1.2). Conditions b) and c) lead to the necessity of fulfilling Courant criterion in form of \( |u|/2 < h_i/\tau \).

4.2. Regularization of the momentum

We rewrite the momentum equation (3.2) in the form:

\[
l_t = -\text{GRAD} P^\circ - \text{DIV}_D(\vec{\mu}_D \cdot \vec{u}_D).
\]

\[
l_t = -\text{GRAD} P^\circ - \
\]

\[
l_t = -\text{GRAD} P^\circ - \text{DIV}_D(\vec{\mu}_D \cdot \vec{u}_D).
\]
To study stability, as an additive to the pressure $P_\Omega$ specified in the cells $\Omega$, we use the linear 
artificial von Neumann-Richtmyer viscosity $(v_D DIV \vec{I}_\Omega)$ in the adaptive form [1, 7], so that by the 
variable $P^-$ in (5.2.1) we mean

$$
P^-_\Omega = \left[ P_\Omega - (v_D DIV \vec{I}_\Omega) \right]^{(0.5)} = \left[ P_i - \left( \frac{v}{u} \right)_i (I_{i+0.5} - I_{i-0.5}) \right]^{(0.5)}. \tag{4.2.2}$$

When viscosity coefficient $v_D$ is selected, the mass flux density in the cells is locally "frozen"
$\mu_{D,i+1} = \mu_D = \text{const}$, as well as the near-node masses $m_{i-0.5} = m_{i+0.5} = m_{i+1.5} = m = \text{const}$ 
not included in the near-node momentum $I^m = \mu u = \nu I$ on explicit and implicit layers in time. Then 
from (4.2.1) we obtain

$$
\left( I_{i+0.5}^m \right)_t = \frac{1}{m} \left( \frac{v}{u} \right)_i \left( I_{i+1.5}^m - I_{i+0.5}^m \right) - \left( \frac{v}{u} \right)_i \left( I_{i+0.5}^m - I_{i-0.5}^m \right) \right)^{(0.5)} - \mu_D \left[ u_{D,i+1} - u_{D,i} \right] - 
\left( \Delta P \right)^{(0.5)}_{i+0.5},
$$

here in the cells it is supposed that $u = \rho \nu$. The value $-(\Delta P)_{i+0.5}^{(0.5)}$ on right side of the equation is not 
used in calculating the viscosity $v_D$. Also, for $m = \text{const}$, the relations $\delta = \sqrt{\mu}/(\sqrt{\nu} + \sqrt{m}) = 0.5$, 
$u^* = u^{(0.5)}$, $u^*_{D,i} = 0.5(u_{i+0.5} + u_{i-0.5})^{(0.5)}$ hold.

Hence to calculate the viscosity we finally get the equation:

$$
\frac{I_{i+0.5}^m - I_{i-0.5}^m}{\tau} - \frac{1}{m} \left( \frac{v}{u} \right)_i \left( I_{i+1.5}^m - I_{i+0.5}^m \right) - \left( \frac{v}{u} \right)_i \left( I_{i+0.5}^m - I_{i-0.5}^m \right)^{(0.5)}
+ \frac{\mu_D}{2m} \left( I_{i+1.5}^m - I_{i-0.5}^m \right)^{(0.5)} = 0. \tag{4.2.3}
$$

According to [1, 6, 7] in equation (4.2.1) at each node $i + 0.5$: a) with local "freezing" of the mass 
flux density in the cells $\mu_{D,i} - \mu_{D,i+1} = \mu_D = \text{const}$, as well as not included in the near-node momentum $(\nu u/h)$ the masses $m_{i-0.5} = m_{i+0.5} = m_{i+1.5} = m = \text{const}$ on explicit and implicit layers 
in time obtaining the resulting equation (4.2.3); b) on explicit and implicit time layers when choosing 
viscosity with a lower estimate $|\mu_D|/2 < \left( \frac{v}{u} \right)_i^{(1)}$; c) on an explicit time layer when choosing 
viscosity with an upper estimate $\left( \frac{v}{u} \right)_i^{(1)} + \left( \frac{v}{u} \right)_i^{(1)}/2 < m/\tau$; in the norm $C$, there is the 
stability estimate of the difference solution $\|I^m\|_C < \|I^m\|_C$. It will be used in future when choosing 
the adaptive artificial viscosity in the momentum balance equation (4.2.1), (4.2.2). Conditions b) and 
c) lead to the necessity of fulfilling the Courante criterion in the form of $\tau |\mu_D|/(2m) < 1$.

4.3. Regularization of internal energy

We rewrite the internal energy balance equation (3.3) in the cells $\Omega$ in the form:

$$
E_t = -P^- DIV \vec{u}^- - DIV \mu_E^-, \tag{4.3.1}
$$

where $E_\Omega = \rho_\Omega e_\Omega$ is the internal energy per volume unit in a cell $\Omega$, $\mu_E^- = \mu_E^{(0.5)}$. The adaptive 
viscosity with the coefficient $v_E$ for the internal energy balance equation (4.3.1) is presented on the 
explicit and implicit layers in time in the nodal internal energy fluxes similarly to section 4.1:

$$
\mu_E^-(\omega) = E_\omega u_\omega - (v_E GRAD E)_\omega = \mu_E^{(i+0.5)}. \tag{4.3.2}
$$

Here $E_\omega$ is the nodal approximation of internal energy ($E = \rho e$).

The value $-P^- DIV \vec{u}^-$ on the right-hand side of (4.3.1) is not used when calculating the viscosity 
v_E. Hence, to calculate the viscosity, we obtain the equation:

$$
(E_t - E)_{/\tau} + \left( \frac{\mu_E^{(i+0.5)} - \mu_E^{(i-0.5)}}{h_i} \right) = 0. \tag{4.3.3}
$$

According to [1, 6, 7], on the explicit and implicit layers in time in equation (4.3.3) in each cell $i$: 
a) with local "freezing" of velocity $u_{i-0.5} = u_{i+0.5} = u = \text{const}$; b) when choosing viscosity with a
lower estimate $|u|/2 < (v_E/h)_{i \pm 0.5}$; c) only on an explicit time layer in choosing viscosity with an upper estimate $[(v_E/h)_{i-0.5} + (v_E/h)_{i+0.5}]/2 < h_i/\tau$; - in the norm $C$, there is the stability estimate of the difference solution $\|\tilde{E}\|_C < \|E\|_C$. It will be used in future to select the adaptive artificial viscosity in the internal energy balance equation (4.3.1), (4.3.2). Conditions b) and c) lead to the necessity of fulfilling the Courante criterion $|u|/2 < h_i/\tau$.

We also note that for two-layer in time CCDS (3.1)-(3.4) in Euler variables, the regularization of the fluxes of mass, momentum, and internal energy of adaptive viscosity in the forms of (4.1.2) for the continuity equation, (4.2.2) for the momentum equation and (4.3.2) for the internal energy equation does not violate the properties of complete conservatism of this family of difference schemes.

5. Testing the algorithm
To test the algorithm, the Einfeld problem [3, 4, 8] was chosen. This problem is about the propagation of two symmetric rarefaction waves in opposite directions. Here the feature is associated with the behavior of the internal energy on the numerical solution. The problem is solved as a special case of the problem of discontinuity decay. The segment [-1, 1] is used as the computational domain. The decay is located in the center of this segment, at the point $x = 0$. The initial conditions are presented in table 1. SI is taken as the system of units of measurement in the calculations.

| Table 1. Initial conditions. |
|-----------------------------|
|                           | Left area ($x<0$) | Right area ($x>0$) |
| $\rho$                    | $u$ | $p$ | $\rho$ | $u$ | $p$ |
| 1                         | -2  | 0.4 | 1       | 2   | 0.4 |

Over time, an expanding stationary section (plateau) with constant values of density and pressure of gas, which are rather small, forms in the center of the region. Since the equation of state of an ideal gas is satisfied, the specific internal energy remains constant in this section, and the entropy also remains constant in the entire computational domain at $t > 0$ (isentropic process). Numerical solutions of this problem on the basis of many known methods unsatisfactorily convey the behavior of the specific internal energy. We will show that the algorithm proposed above significantly improves the approximation of these thermodynamic variables in comparison with the most well known methods.

The numerical solution of the problem under consideration in Euler variables is presented in figures 2, 3, 4, 5 for $N = 2000$ computational cells and 6 for $N = 500$. Blue and orange curves correspond to analytical and numerical solutions of the corresponding variables.

The calculations were carried out for $N = 500, 1000, 2000$ points. In the first three figures, there is more than good agreement between the analytical and numerical solutions, not only for 2000, but also for $N = 500$ and 1000. Temperature graphs (figures 5 and 6) are of particular interest. In figure 6, there is a noticeable deviation of the numerical solution in the locality of the plateau, but it dissolves quite well when the mesh is compacted (figure 5). In most of the currently used numerical algorithms, conservative variables are traditionally used, the amplitude of the entropy peak is about 70% of the solution. One of the most famous works on the optimization of the entropy trace was studied in [4] based on the discrete Galerkin method, where a noticeable improvement in the entropy peak is observed, but the smoothing effect is not observed when the mesh is refined.

6. Conclusions
A numerical experiment was carried out with a class of divergent adaptive viscosities developed in relation to completely conservative difference schemes with temporal weights associated with variable masses of moving nodal particles of the medium. The proposed algorithm showed a significant improvement in the quality of the numerical solution of the Einfeld problem. Compared to other methods, no entropy peak is observed. Effective preservation of the balance of internal energy in this
type of divergent difference schemes is ensured by the absence of permanent sources of difference origin, which produce computational entropy (including the singularities of the solution).

7. References
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