Investigation of flexural vibrations in poroelastic solid sphere in the presence of static stresses

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Abstract. In this paper, the three dimensional wave propagation in a homogeneous isotropic poroelastic solid sphere is investigated in the presence of static stresses. The frequency equation is obtained in the framework of Biot’s theory. Non-dimensional phase velocity is computed as a function of uniaxial static stress for fixed wavenumber. For the numerical results, three types of poroelastic spherical solids are employed, and the results are presented graphically.

1. Introduction
The analysis for transient problems of spherical structures is important and interesting research fields for engineers and scientists. The applications for homogeneous solid sphere have continuously increased in some engineering areas, including aerospace, frequency control filters, chemical vessels, information storage devices, and signal processing devices. Even in the human body most of the bones are in spherical shape. Mott investigated elastic motion of an isotropic medium in the presence of body forces and static stresses [1]. In the paper [1], Mott derived equations of elastic motion which include the effect of static surface forces and body forces. When the body forces imposed on a solid which is at rest gives static stresses and strains. Effect of static axial stress upon the velocity of the lowest-order flexural mode is investigated in [2]. Employing Biot’s theory of poroelasticity [3], detailed investigation of wave propagation is given in [4]. From the mathematics perspective of poroelastic materials, several authors are investigated wave propagation phenomena which characterize Biot’s theory [5-9]. Flexural vibrations in poroelastic elliptic cone against the angle made by the major axis of the cone in spheroconal coordinate system are given in [10]. Shah and Tajuddin [11] discussed torsional vibrations of poroelastic spheroidal shells. In this paper, authors derived frequency equations for poroelastic thin spherical shell, thick spherical shell, poroelastic solid sphere, and concluded that the frequency is same for all the three cases. Axially symmetric vibrations of fluid filled poroelastic spherical shell are studied by Shah and Tajuddin [12]. In the said paper, frequency equations are derived for radial and rotatory vibrations of fluid-filled and empty poroelastic spherical shells. A comparative study is made between the modes of composite spherical shell and its ring modes [13]. Shanker et. al. [14] investigated vibration analysis of a poroelastic composite hollow sphere. In this paper, authors derived frequency equations for radial and rotatory vibrations of fluid filled and empty poroelastic shells with rigid core. Torsional vibrations of thick-walled hollow poroelastic spheres are studied by Ahmed Shah and Tajuddin [15]. In this paper, authors derived complex frequencies of torsional vibrations of thick-walled hollow poroelastic spheres for different dissipations and it is concluded that as the dissipation increases,
the propagation increases while the attenuation remains almost same. Flexural vibrations of poroelastic solids in presence of static stress are studied by Rajitha et al. [16]. In the paper, governing equations are derived in the presence of static stress, which were not available in the earlier literature. However, in all the above papers, static stress is not employed in the case of flexural vibrations of poroelastic solid sphere. Therefore, in this paper, the same is investigated in the framework of Biot’s theory.

This paper is organized as follows. In section 2, governing equations, and solution of the problem are given. In section 3, the case of static uniaxial stress is considered. Numerical results are described in section 4. Finally, conclusion is given in section 5.

2. Governing equations and solution of the problem
Consider an isotropic poroelastic solid sphere in spherical coordinate system \((r, \theta, \phi)\). Let \(\vec{u}(u, v, w)\) and \(\vec{U}(U, V, W)\) be the solid and fluid displacements. When the body forces are large, displacements will be large. Consequently second order coupling between stresses and strains cannot be ignored. Effective stresses must be inserted in the place of usual ones. The effective stress-strain relations are as follows [1]:

\[
\begin{align*}
\sigma'_{rr} &= (\sigma_{rr} + s) - \sigma_{r\theta} \frac{\partial u}{\partial r} - \sigma_{r\phi} \frac{\partial u}{\partial \phi}, \\
\sigma'_{r\theta} &= \sigma_{r\theta} - (\sigma_{rr} + s) \frac{\partial v}{\partial r} - \sigma_{r\phi} \frac{\partial v}{\partial \phi}, \\
\sigma'_{r\phi} &= \sigma_{r\phi} - (\sigma_{rr} + s) \frac{\partial w}{\partial r} - \sigma_{r\theta} \frac{\partial w}{\partial \phi}, \\
\sigma'_{\theta\theta} &= (\sigma_{\theta\theta} + s) - \sigma_{\theta r} \frac{\partial u}{\partial \theta} - \sigma_{\theta\phi} \frac{\partial u}{\partial \phi}, \\
\sigma'_{\theta r} &= \sigma_{\theta r} - (\sigma_{\theta\theta} + s) \frac{\partial u}{\partial \theta} - \sigma_{\theta\phi} \frac{\partial u}{\partial \phi}, \\
\sigma'_{\theta\phi} &= \sigma_{\theta\phi} - (\sigma_{\theta r} \frac{\partial v}{\partial \theta} \frac{\partial v}{\partial \phi} - \sigma_{\theta\phi} \frac{\partial v}{\partial \phi}) \\
\sigma'_{\phi\phi} &= (\sigma_{\phi\phi} + s) - \sigma_{\phi r} \frac{\partial w}{\partial \phi} - \sigma_{\phi\theta} \frac{\partial w}{\partial \theta}.
\end{align*}
\]

(1)

The expressions of effective stresses involve only shear components. As there are no shear components for the case of fluid pressure, the expression for the fluid pressure remains the same. In equation (1), usual stresses \(\sigma_{ij}\) and fluid pressure \(s\) are given under [3].

\[
\begin{align*}
\sigma_{ij} &= 2Ne_{ij} + (Ae + Q\varepsilon) \delta_{ij}, \quad (i, j = r, \theta, \phi), \\
s &= Qe + Re\varepsilon.
\end{align*}
\]

(2)

In equation (2), \(e_{ij}\)’s strain displacements, \(A, N, Q, R\) are poroelastic constants, \(e\) and \(\varepsilon\) are the dilatations of solid and fluid, respectively, and \(\delta_{ij}\) is the well-known Kronecker delta function. Substitution of effective stresses for usual stresses, the equations of motion obtained in the following manner.
\[ \frac{\partial}{\partial r} (\sigma_{rr}'(1 + \frac{\partial u}{\partial r} + \frac{\partial w}{\partial r})) + \frac{1}{r} \frac{\partial}{\partial \theta} (\sigma_{\theta\theta}'(1 + \frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial \theta})) + \frac{1}{r \sin \phi} \frac{\partial}{\partial \phi} (\sigma_{\phi\phi}'(1 + \frac{\partial u}{\partial \phi} + \frac{\partial w}{\partial \phi})) + \frac{2}{r} \sigma_{rr}'(1 + \frac{\partial u}{\partial r} + \frac{\partial w}{\partial r}) - \frac{1}{r} \sigma_{\theta\theta}'(1 + \frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial \theta}) - \frac{1}{r} \sigma_{\phi\phi}'(1 + \frac{\partial u}{\partial \phi} + \frac{\partial w}{\partial \phi}) \cot \theta + F_r = \frac{\partial^2}{\partial r^2} (\rho_{11} u + \rho_{12} u), \]

\[ \frac{\partial}{\partial r} (\sigma_{r\theta}'(1 + \frac{\partial u}{\partial r} + \frac{\partial w}{\partial r})) + \frac{1}{r} \frac{\partial}{\partial \theta} (\sigma_{\theta\theta}'(1 + \frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial \theta})) + \frac{1}{r \sin \phi} \frac{\partial}{\partial \phi} (\sigma_{\phi\phi}'(1 + \frac{\partial u}{\partial \phi} + \frac{\partial w}{\partial \phi})) + \frac{1}{r} \sigma_{r\theta}'(1 + \frac{\partial u}{\partial r} + \frac{\partial w}{\partial r}) - \frac{1}{r \sin \phi} \sigma_{\theta\theta}'(1 + \frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial \theta}) + 3 \sigma_{r\theta}'(1 + \frac{\partial u}{\partial r} + \frac{\partial w}{\partial r}) + F_\theta = \frac{\partial^2}{\partial r^2} \left( \rho_{11} v + \rho_{12} v \right), \]

\[ \frac{\partial}{\partial r} (\sigma_{r\phi}'(1 + \frac{\partial u}{\partial r} + \frac{\partial w}{\partial r})) + \frac{1}{r} \frac{\partial}{\partial \theta} (\sigma_{\theta\phi}'(1 + \frac{\partial u}{\partial \theta} + \frac{\partial w}{\partial \theta})) + \frac{1}{r \sin \phi} \frac{\partial}{\partial \phi} (\sigma_{\phi\phi}'(1 + \frac{\partial u}{\partial \phi} + \frac{\partial w}{\partial \phi})) + \frac{2}{r} \sigma_{r\phi}'(1 + \frac{\partial u}{\partial r} + \frac{\partial w}{\partial r}) + 3 \sigma_{r\phi}'(1 + \frac{\partial u}{\partial r} + \frac{\partial w}{\partial r}) + F_\phi = \frac{\partial^2}{\partial r^2} \left( \rho_{11} w + \rho_{12} w \right). \]

In equation (3), \( \rho_{ij} \) are mass coefficients, \( F_r, F_\theta, \) and \( F_\phi \) are the components of the body force vector \( \vec{F} \). Equations of motion and the boundary conditions together form the basis for the solution of wave propagation problems in poroelastic solids. The equations (2) and (3) have to be satisfied at every point of the poroelastic solid, and on surface of that solid. Thus the solution of the problem can be completely determined. To originate connection between the time variant and the static poroelastic quantities, the displacements and the stresses are written in the following form [1, 16]:

\[ u = \sum u_n(r, \theta, \phi, \omega_n t) = u_0 + u_1 + u_2 + \ldots, \]

\[ v = \sum v_n(r, \theta, \phi, \omega_n t) = v_0 + v_1 + v_2 + \ldots, \]

\[ w = \sum w_n(r, \theta, \phi, \omega_n t) = w_0 + w_1 + w_2 + \ldots, \]

\[ U = \sum U_n(r, \theta, \phi, \omega_n t) = U_0 + U_1 + U_2 + \ldots, \]

\[ V = \sum V_n(r, \theta, \phi, \omega_n t) = V_0 + V_1 + V_2 + \ldots, \]

\[ W = \sum W_n(r, \theta, \phi, \omega_n t) = W_0 + W_1 + W_2 + \ldots, \]

\[ \sigma_{ij} = \sum \sigma_{ij_n}(r, \theta, \phi, \omega_n t) = \sigma_{i,j_0} + \sigma_{i,j_1} + \sigma_{i,j_2} + \ldots, \]

where \( \omega_n \) is the \( n^{th} \) angular frequency. Using equations (1), (2) and (4) in the equation (3), we get the following static equations, \( n = 0 \):
Similarly, the harmonic equations

\[
\frac{\partial}{\partial r} \left( \sigma_{r\theta} - (\sigma_{r\theta} + s) \rho_{\theta \theta} - \sigma_{r\phi} \frac{\partial \rho_{\theta \phi}}{\partial \theta} \right) + \frac{1}{r} \sigma_{r\phi} \frac{\partial \rho_{\phi \theta}}{\partial \phi} \frac{\partial \rho_{\phi \theta}}{\partial \theta} - \frac{1}{r} \sigma_{\rho \phi} \frac{\partial \rho_{\rho \phi}}{\partial \phi} \frac{\partial \rho_{\rho \phi}}{\partial \theta} + \frac{1}{r} \sigma_{\rho \phi} \frac{\partial \rho_{\rho \phi}}{\partial \phi} \frac{\partial \rho_{\rho \phi}}{\partial \theta} = 0,
\]

\[
Q_{\frac{d\varepsilon}{d\theta}} + R_{\frac{d\varepsilon}{d\theta}} = 0,
\]

\[
Q_{\frac{d\varepsilon}{d\phi}} + R_{\frac{d\varepsilon}{d\phi}} = 0,
\]

\[
Q_{\frac{d\varepsilon}{d\rho}} + R_{\frac{d\varepsilon}{d\rho}} = 0.
\]

In the above,

\[
\varepsilon = \frac{\partial^2 U_0}{\partial r^2} + \frac{2}{r} \frac{\partial U_0}{\partial r} + \frac{\cot \theta}{\partial \theta} \frac{\partial U_0}{\partial \phi} + \frac{1}{r^2} \frac{\partial^2 U_0}{\partial \phi^2},
\]

and

\[
\varepsilon = \frac{\partial^2 U_0}{\partial r^2} + \frac{2}{r} \frac{\partial U_0}{\partial r} + \frac{\cot \theta}{\partial \theta} \frac{\partial U_0}{\partial \phi} + \frac{1}{r^2} \frac{\partial^2 U_0}{\partial \phi^2}.
\]

Similarly, the harmonic equations \(n = 1\) are obtained, which are:

\[
\frac{\partial}{\partial r} \left( \sigma_{r\theta} - (\sigma_{r\theta} + s) \rho_{\theta \theta} - \sigma_{r\phi} \frac{\partial \rho_{\theta \phi}}{\partial \theta} \right) + \frac{1}{r} \sigma_{r\phi} \frac{\partial \rho_{\phi \theta}}{\partial \phi} \frac{\partial \rho_{\phi \theta}}{\partial \theta} - \frac{1}{r} \sigma_{\rho \phi} \frac{\partial \rho_{\rho \phi}}{\partial \phi} \frac{\partial \rho_{\rho \phi}}{\partial \theta} + \frac{1}{r} \sigma_{\rho \phi} \frac{\partial \rho_{\rho \phi}}{\partial \phi} \frac{\partial \rho_{\rho \phi}}{\partial \theta} = 0,
\]

\[
Q_{\frac{d\varepsilon}{d\theta}} + R_{\frac{d\varepsilon}{d\theta}} = 0,
\]

\[
Q_{\frac{d\varepsilon}{d\phi}} + R_{\frac{d\varepsilon}{d\phi}} = 0,
\]

\[
Q_{\frac{d\varepsilon}{d\rho}} + R_{\frac{d\varepsilon}{d\rho}} = 0.
\]
\[
\frac{\partial}{\partial r} \left( (\sigma_{r\phi} - (\sigma_{rr} + s)\frac{\partial w_0}{\partial r} - (\sigma_{rr0} + s)\frac{\partial w_0}{\partial \theta} - \sigma_{\phi\theta} \frac{\partial w_0}{\partial \phi} + (\sigma_{r\phi0} - (\sigma_{rr} + s)\frac{\partial w_0}{\partial r} + \frac{\partial w_0}{\partial \phi}) + (\sigma_{\phi\theta} - (\sigma_{rr} + s)\frac{\partial w_0}{\partial r} + \frac{\partial w_0}{\partial \phi}) \right) \frac{1}{1 + \frac{\partial w_0}{\partial r} + \frac{\partial w_0}{\partial \phi}} + (\sigma_{r\phi0} - (\sigma_{rr} + s)\frac{\partial w_0}{\partial r} + \frac{\partial w_0}{\partial \phi}) \right) + \frac{1}{\rho} \frac{\partial}{\partial r} \left( \left( (\sigma_{\phi\theta} - (\sigma_{rr} + s)\frac{\partial w_0}{\partial r} + \frac{\partial w_0}{\partial \phi}) + (\sigma_{\phi\theta} - (\sigma_{rr} + s)\frac{\partial w_0}{\partial r} + \frac{\partial w_0}{\partial \phi}) \right) \frac{1}{1 + \frac{\partial w_0}{\partial r} + \frac{\partial w_0}{\partial \phi}} + (\sigma_{\phi\theta} - (\sigma_{rr} + s)\frac{\partial w_0}{\partial r} + \frac{\partial w_0}{\partial \phi}) \right) + \frac{1}{\rho} \frac{\partial}{\partial r} \left( \left( (\sigma_{\phi\theta} - (\sigma_{rr} + s)\frac{\partial w_0}{\partial r} + \frac{\partial w_0}{\partial \phi}) + (\sigma_{\phi\theta} - (\sigma_{rr} + s)\frac{\partial w_0}{\partial r} + \frac{\partial w_0}{\partial \phi}) \right) \frac{1}{1 + \frac{\partial w_0}{\partial r} + \frac{\partial w_0}{\partial \phi}} + (\sigma_{\phi\theta} - (\sigma_{rr} + s)\frac{\partial w_0}{\partial r} + \frac{\partial w_0}{\partial \phi}) \right) + F_{\phi_1} = \frac{\partial^2}{\partial r^2} (p_{11} w_1 + \rho_1 W_1),
\]

\[
Q_{w r} + R \frac{\partial^2}{\partial r} = \frac{\rho^2}{\rho^2_r} (\rho_1 u_1 + \rho_2 U_r),
\]

\[
Q_{\phi r} + R \frac{\partial^2}{\partial r} = \frac{\rho^2}{\rho^2_r} (\rho_1 u_1 + \rho_2 U_r),
\]

\[
Q_{w \phi} + R \frac{\partial^2}{\partial r} = \frac{\rho^2}{\rho^2_r} (\rho_1 u_1 + \rho_2 U_r).
\]

In the all the above, \(F_{r_1}, F_{\theta_1}, F_{\phi_1}\) are the body forces which have the frequency \(\omega\). The equations (5) and (6) are static equations, and first harmonic equations respectively.

3. The case of uniaxial static stress

When the large static uniaxial stress is applied on a solid medium one can ignore the effect of body forces [1]. If we assume that applied static uniaxial stress is acting in the direction of \(\phi\)–axis, then we have

\[
\sigma_{r\phi} = e_{r\phi} = 0, (i \neq j),
\]

\[
\sigma_{r\phi} = \sigma_{\phi\theta} = 0,
\]

\[
F_{r} = F_{\theta} = F_{\phi} = 0,
\]

where \(\sigma_{\phi\theta}\) is the applied uniform static uniaxial stress. Substituting the equation(7) in the equation(5), it can be seen that equation(5) are automatically satisfied and the static strains are obtained as follows:

\[
e_{r\phi} = -\frac{\partial e_{\phi\theta}}{\partial r}.
\]

Substituting these equations in the first equation of equation(2), the following equation is obtained:

\[
e_{\phi\theta} = \frac{\sigma_{\phi\theta} - Q_{\varepsilon}}{Y}.
\]

In the above, \(\vartheta = \frac{A + 2A}{2(2A + N)}\) is the Poisson ratio, and \(Y = \frac{N(F + 2A)}{A + N}\) is the Young’s modulus. Substituting the equations(7) and (8) in equation(6), and due to the assumption of infinitesimal deformation in the linear theory of elasticity, we can neglect the product terms consequently equations of motion in this case are obtained as follows:

\[
N((\nabla^2 - \frac{\partial^2}{\partial r^2}) u + \frac{1 + \frac{\partial^2 u}{\partial r^2}}{1 + \frac{\partial^2 u}{\partial r^2}} + \frac{\partial^2 u}{\partial r^2} - \frac{3}{4}) (\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \theta^2} + \frac{N c_{\phi\theta}}{c_{\phi\theta}} + \frac{\partial^2 u}{\partial \phi^2} - \frac{2 N c_{\phi\theta}}{c_{\phi\theta}} - 2 N c_{\phi\theta} + 3 N c_{\phi\theta} + \frac{\partial^2 u}{\partial r^2} - \frac{\partial^2 u}{\partial \phi^2}) + \frac{A}{A + N} (\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial \phi^2} - \frac{2 N c_{\phi\theta}}{c_{\phi\theta}} - 2 N c_{\phi\theta} + 3 N c_{\phi\theta} + \frac{\partial^2 u}{\partial r^2} - \frac{\partial^2 u}{\partial \phi^2}) = \frac{\partial^2}{\partial r^2} (p_{11} u + \rho_{12} U).
\]

5
\[ (N(\nabla^2 - \frac{2}{r} \frac{\partial}{\partial r})) v + \frac{N}{r} \frac{\partial^2 u}{\partial \theta^2} + \frac{N}{r} \frac{\partial^2 u}{\partial \phi^2} + \frac{3 + \cot \theta}{2} \frac{N}{r} u - \frac{N}{r} v + \frac{N}{r} \frac{\partial^2 u}{\partial \theta \partial \theta} + \frac{N \cot \theta}{r} \frac{\partial u}{\partial \phi} + \frac{N}{r^2} \frac{\partial w}{\partial \phi} + \left( \frac{\partial u}{\partial \theta} + \cot \theta \right) + \frac{1}{r} \left( (A + Q) e + 4 \frac{Q}{r} e \right) \frac{\partial u}{\partial \theta} - \cot \theta - \frac{2N \cot \theta}{r} \frac{\partial w}{\partial \theta} + usin \theta + vcos \theta + \frac{N}{r^2} \frac{\partial^2 u}{\partial \theta \partial \theta} - \frac{N}{r^2} \frac{\partial^2 v}{\partial \phi^2} + \frac{N}{r^2} \frac{\partial w}{\partial r} + \frac{1}{r^2} \frac{\partial w}{\partial \phi} + (A + Q) e + (Q + R) e + (3 + \cot \theta) \frac{2N}{r} \frac{\partial w}{\partial \phi} - \frac{N}{r} \frac{\partial w}{\partial \theta} + (N \nabla^2) \left( \epsilon \frac{\partial u}{\partial \theta} - \cot \theta \right) - \frac{2N \cot \theta}{r} \frac{\partial w}{\partial \phi} + usin \theta + vcos \theta \right) + \frac{N}{r^2} \frac{\partial^2 u}{\partial \theta \partial \phi}
\]
4. Numerical results

For the numerical results, we consider the wave propagation in the the φ– direction and at a point \( r = a \). In this case, \( k_1 = k_2 = 0 \), and equation (11) reduces to the following matrix form:

\[
[A_{lm}][C] = 0, \quad (l, m = 1, 2, 3, 4, 5).
\]

Equation(12) results in a system of five homogeneous equations in five arbitrary constants \( C_1, C_2, C_3, C_4, C_5 \). For a non trivial solution, determinant of coefficients is zero. Accordingly we obtain the following frequency equation.

\[
| A_{lm} | = 0, \quad (l, m = 1, 2, 3, 4, 5).
\]

Equation (13) degenerated as follows:

\[
A_{33} = 0.
\]

\[
\begin{bmatrix}
A_{11} & A_{12} & A_{14} & 0 \\
A_{21} & A_{22} & 0 & A_{25} \\
A_{41} & 0 & A_{44} & 0 \\
A_{51} & 0 & A_{55} & 0 \\
\end{bmatrix} = 0.
\]
here

\[ A_{33} = \frac{-2k_3^2}{\sin^2 \theta} (N + A\sigma_{\phi\theta} + (\csc^2 \theta - 4)(\frac{N}{a^2} + \frac{A}{a^2(P+2A)}\sigma_{\phi\theta}) + \omega^2 NV_s^{-2}) , \]

\[ A_{11} = \frac{-Nk_3^2}{a^2\sin^2 \theta} + \frac{Ak_3^2}{a^2\sin^2 \theta}\sigma_{\phi\theta} + \rho_{11}\omega^2, \]

\[ A_{12} = \frac{-3N\cot \theta}{a^2} - \frac{P\cot \theta}{(P+2A)2a^2}\sigma_{\phi\theta} + \frac{2A}{(P+2A)}\cot \theta\sigma_{\phi\theta}, \]

\[ A_{21} = \frac{6N}{a^2} + \frac{2A}{(P+2A)}\csc \theta \sigma_{\phi\theta} + \frac{P(\cot \theta+3)}{a^2(P+2A)}\sigma_{\phi\theta}, \]

\[ A_{22} = \frac{k_3^2}{\sin^2 \theta} (N + \frac{A}{(P+2A)}\sigma_{\phi\theta}) + \frac{2A}{(P+2A)}\csc \theta \sigma_{\phi\theta} + \rho_{12}\omega^2, \]

\[ A_{14} = A_{25} = A_{41} = A_{51} = \rho_{12}\omega^2, \]

\[ A_{44} = A_{55} = \rho_{22}\omega^2, \]

where \( V_s \) is the shear wave velocity [3]. Equations (14) and (15) are investigated by introducing the non-dimensionalised quantities given below:

\[ a_1 = \frac{P}{H}, \quad a_2 = \frac{Q}{H}, \quad a_3 = \frac{R}{H}, \quad a_4 = \frac{N}{H}, \]

\[ d_1 = \frac{a_{11}}{\rho}, \quad d_2 = \frac{a_{12}}{\rho}, \quad d_3 = \frac{a_{22}}{\rho}, \quad \tilde{z} = \left( \frac{\tilde{V}_0}{a^2} \right)^2, \]

\[ \rho = \rho_{11} + 2\rho_{12} + \rho_{22}, \quad H = P + 2Q + R, \quad V_0^2 = \frac{H}{\rho}, \]

\[ m = \frac{c}{c_0}, \quad c = \frac{v}{k}, \quad \frac{a}{a_0} = \frac{N}{\rho}. \]

In equation (17), \( c \) is the phase velocity, \( c_0 \) and \( V_0 \) are the reference velocities, \( k \) is wavenumber and \( m \) is non-dimensional phase velocity. Using the equation (17) in equation (14), we obtain implicit relation between non-dimensional phase velocity \( m \), non-dimensional static uniaxial stress and non-dimensional wavenumber \( k_3a \) as follows.

\[ m = \left[ \frac{1}{(\csc^2 \theta - 4)\sigma_{\phi\theta}(k_3a)^2 + (a_4(k_3a))^2 + \frac{(a_1 - 2a_4)\sigma_{\phi\theta}}{3a_1 - 2a_4 H}}{(a_4 + \frac{(a_1 - 2a_4)\sigma_{\phi\theta}}{3a_1 - 2a_4 H})} \right] \frac{1}{2}. \]

Employing the non-dimensional quantities in the frequency equation (15), we obtain implicit relation between non-dimensional phase velocity \( m \), non-dimensional static uniaxial stress and non-dimensional wavenumber \( k_3a \). For numerical process, three poroelastic solids are considered and then discussed. Of these three poroelastic solids, two are sandstone saturated with water and kerosene, respectively [17,18], and the third one is bony element. The physical parameter values of first two materials pertaining to the equation (17) are given in the Table 1. Further, the values of bone poroelastic parameters \( A, N, Q, R \) and its mass coefficients are computed following the paper [19]. The values of Young’s modulus and Poisson ratio are taken to be \( 3 \times 10^6 \) lb/inch\(^2 \), and 0.28, respectively as suggested in the paper [19]. The value of \( \theta \) is taken to be \( \frac{\pi}{2} \) respectively. Phase velocity is computed using the bisection method implemented in MATLAB, and the results are depicted in the figure 1. Figure 1 shows the plots of non-dimensional phase velocity against the static uniaxial stress for fixed wavenumber. From this figure it is clear that material-1 values are much greater than that of material-2. This discrepancy is due to fluid present in the materials. The values of bone are much greater than that of material-1 and material-2.
Table 1. Material parameters

| Material parameter | Material-1 | Material-2 |
|--------------------|------------|------------|
| $a_1$              | 0.843      | 0.96       |
| $a_2$              | 0.65       | 0.006      |
| $a_3$              | 0.28       | 0.028      |
| $a_4$              | 0.234      | 0.412      |
| $d_1$              | 0.901      | 0.887      |
| $d_2$              | -0.001     | 0          |
| $d_3$              | 0.101      | 0.123      |
| $\tilde{z}$        | 3.851      | 2.129      |

Figure 1. Variation of non-dimensional phase velocity with the non-dimensional static uniaxial stress.

5. Conclusion

Employing Biot’s theory, flexural vibrations of poroelastic solid sphere in presence of static stress are investigated. Non-dimensional phase velocity against the non-dimensional static uniaxial stress for fixed wavenumber is computed for three types of poroelastic solids. Material-1 value is much greater than that of material-2 for the same value of applied uniaxial static stress. Both the materials are sandstone related and differ in only the fluid part. Hence, it can be inferred that the fluid part is causing the above discrepancy. This kind of analysis can be made for any poroelastic solid sphere if the values of parameters are available.

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