On a singular limit for the compressible rotating Euler system

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Abstract

The work addresses a singular limit for a rotating compressible Euler system in the low Mach number and low Rossby number regime. Based on the concept of dissipative measure-valued solution, the quasi-geostrophic system is identified as the limit problem in the case of ill-prepared initial data. The ill-prepared initial data will cause rapidly oscillating acoustic waves. Using dispersive estimates of Strichartz type, the effect of the acoustic waves in the asymptotic limit is eliminated.

Key words: compressible Euler equations, singular limit, low Mach number, low Rossby number, dissipative measure-valued solutions.

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1 Introduction

Earth’s graceful rotation is an unignorable factor at geophysical fluids models. These models play an important role in the analysis of complex Earth phenomena in meteorology, geophysical and astrophysics. In order to describe the effect of rotation, people introduce two factors: Coriolis acceleration and centrifugal acceleration. In many real world applications, the action of centrifugal force is neglected, as it is in equilibrium with stratification caused by the gravity of the Earth. Under the above assumptions, we consider the following scaled Euler equations in an infinite slap Ω = \(\mathbb{R}^2 \times (0, 1)\):

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \frac{1}{Ma^2} \nabla_x p(\rho) + \frac{1}{Ro} \rho \omega \times u &= 0,
\end{align*}
\]

where the unknown fields \(\rho = \rho(t, x)\) and \(u = u(t, x)\) represent the density and the velocity of an inviscid compressible fluid, \(\omega = (0, 0, 1)\) is the rotation axis. The Mach number \(Ma\), proportional

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to the characteristic velocity field divided by the sound speed, and the Rossby number $Ro$, defined as the ratio of the displacement due to Coriolis forces, play the role of singular (small) parameters. The symbol $p = p(\rho)$ denotes the barotropic pressure (assumptions on the pressure see (3.1)).

The system is supplemented by the far field conditions

$$u \to 0, \quad \rho \to \bar{\rho}, \quad \text{as } |x| \to \infty, \quad \text{where } \bar{\rho} > 0,$$

and boundary condition

$$u \cdot n|_{\partial \Omega} = 0,$$

where $n$ is outer normal vector to $\partial \Omega$.

From modeling of geophysical fluids, the value of Mach number and Rossby number can be considered very small. It is well known that the compressible fluid flow becomes incompressible in the low Mach number limit, as the density distribution is constant and the velocity field becomes solenoidal. On the other hand, low Rossby number corresponds to fast rotation and the fast rotating fluids will lead to the so-called Taylor-Proudman columns phenomena. Therefore, it is interesting to observe the phenomenon if the two effects take place simultaneously. In this paper, we address the problem of the double limit for $Ma = Ro = \epsilon$. Let $\rho = \rho_\epsilon$, $u = u_\epsilon$, the system (1.1) takes the form

$$\begin{cases}
\partial_t \rho_\epsilon + \text{div} (\rho_\epsilon u_\epsilon) = 0, \\
\partial_t (\rho_\epsilon u_\epsilon) + \text{div} (\rho_\epsilon u_\epsilon \otimes u_\epsilon) + \frac{1}{\epsilon} \nabla_x p(\rho_\epsilon) + \frac{1}{\epsilon} \rho_\epsilon (\omega \times u_\epsilon) = 0.
\end{cases}$$

Our goal is to study the singular limit $\epsilon \to 0$ at the case of the ill–prepared initial data for the scaled system (1.4). The definition of ill–prepared initial data will be introduced in Section 3.3. Supposing we know that in the corresponding spaces,

$$\rho_\epsilon^{(1)} = \frac{\rho_\epsilon - \bar{\rho}}{\epsilon} \to q, \quad u_\epsilon \to v,$$

we can find that $q$ and $v$ satisfy the following equations:

$$\omega \times v + \frac{p'(\bar{\rho})}{\bar{\rho}} \nabla_x q = 0,$$

$$\partial_t (\Delta_h q - \frac{1}{p'(\bar{\rho})} q) + \nabla_h q \cdot \nabla_h (\Delta_h q) = 0.$$

Equations (1.5) (1.6) can be interpreted as a kind of stream function, according to physicists, named as quasi-physical flows [32]. For non-rotating compressible Euler fluids, a great number of well-posedness results have been obtained. However, some classical literatures show that smooth solutions of the Euler system will exhibit blow-up phenomena in a finite time no matter how smooth or small the initial data are. Therefore, it seems more appropriate to consider a suitable class of admissible weak solutions to (1.4). By admissible we mean that solutions will satisfy some form of the energy balance. The need for global admissible solutions of the Euler system leads to the concept of more general dissipative measure–valued (DMV) solutions introduced in the context of the full Euler system in [3, 4].

The measure-valued solutions to hyperbolic conservation laws were introduced by DiPerna [10]. He used Young measures to pass to the artificial viscosity limit. In the case of the incompressible Euler equations, DiPerna and Majda [11] proved the global existence of measure-valued
solutions for any initial data with finite energy. They introduced generalized Young measures to take into account oscillations and concentrations. Further, the existences of measure-valued solutions were shown for further models of fluids, e.g. compressible Euler and Navier-Stokes equations [26, 23]. The measure-valued solution to the non-Newtonian case was proved by Novotný and Nečasová [25]. The generalization was given by Alibert and Bouchitté [1]. The weak-strong uniqueness for generalized measure-valued solutions of isentropic Newtonian Euler equations were proved in [21]. Inspired by previous results, the concept of dissipative measure-valued solution was finally applied to the barotropic compressible Navier-Stokes system [19].

The reader may consult [17, 18, 23, 24, 26] for applications of the theory of (DMV) solutions in fluid mechanics or their counterparts [8, 9] in other areas of mathematical physics.

Let us discuss the main differences between weak solutions and (DMV) solutions. First important advantage of (DMV) solution is that DMV solutions to the compressible Euler system exist globally in time. Secondly (DMV) solutions convergence to the limit system holds for any ill-prepared initial data, which in both case are not valid for weak solutions.

Due to the above fascinating advantage, there are some new results concerning singular limits in the context of measure-valued solutions. The low Mach number limit was studied in [18], where it is shown that (DMV) solutions approach the smooth solutions of incompressible Euler system both for well-prepared and ill-prepared data. Moreover, the singular limit of compressible Euler system in the low Mach number and strong stratification regime for the ill-prepared data was identified, see [20]. However, to the best of our knowledge, compared with non-rotating case, there is a few results concerning on the singular limit of rotating compressible Euler system no matter weak solutions or strong solutions. Nilasis [28] proved the singular limit of a rotating compressible Euler system with stratification at the case of well-prepared initial data. Our goal is to consider the asymptotic limit of (DMV) solutions to the compressible Euler equations with ill-prepared initial data. We prove it converges to the strong solutions of quasi-physical flows. Moreover, we should emphasize that boundary conditions in this paper can be replaced by the periodical conditions in $x_3$ direction with a few changes. All the choices of boundary conditions prevent the flow from creating a viscous boundary layer. The periodic domain with well-prepared initial data was considered by Feireisl et al., [18]. If the whole domain is torus $T^3$, it is difficult to obtain the analysis of acoustic waves at the case of ill-prepared initial data. It seems interesting to compare the results of the present paper with those obtained in [18]. The analysis in [18] leans that the (DMV) solutions of Euler system will converge to incompressible Euler system. Moreover, there is obvious difference about acoustic wave analysis between rotating and non-rotating case. The extension of the results of [18] to the rotating Euler system is therefore not straightforward. Last but not least, we should emphasize that there are huge results about rotating Navier-Stokes system such as [5, 6, 14, 15, 16].

The paper is organized as follows. In Section 2, we introduce the dissipative measure solutions, relative energy and the other necessary material. In Section 3, we state our main theorem. Section 4 is devoted to deriving uniform bounds of the Euler system independent of $\epsilon$. In Section 5, we perform the necessary analysis of the acoustic waves. The proof of the main theorem is completed in Section 6.
2 Preliminaries

First let us observe that it is more convenient to rewrite the Euler system in terms of the conservative variables \( \rho, \ m = \rho u \). Let \( Q = \{ (\rho, m) | \rho \in [0, \infty), m \in \mathbb{R}^3 \} \) be the natural phase space associated to solutions \([\rho, m] = [\rho, \rho u] \).

2.1 Dissipative measure–valued solutions

A dissipative measure-valued (DMV) solution of the Euler system (1.1) is a parameterized family of probability measures
\[
\{ Y_{t,x} \}_{t \in [0, T], x \in \Omega} \mapsto Y_{t,x} \in L^\infty_{\text{weak} – (\ast)}((0, T) \times \Omega; \mathcal{P}(Q)),
\]
satisfying
- the continuity equation
  \[
  \int_0^T \int_\Omega [(Y_{t,x}; \rho) \partial_t \varphi + (Y_{t,x}; m) \nabla_x \varphi] dx dt = - \int_\Omega (Y_{0,x}; \rho) \varphi(0) dx,
  \]
  for all \( \varphi \in C_\infty_c([0, T] \times \Omega) \);
- the momentum equation
  \[
  \int_0^T \int_\Omega [(Y_{t,x}; m) \partial_t \varphi + (Y_{t,x}; m \otimes m \rho) \nabla_x \varphi] dx dt + \int_0^T \int_\Omega (Y_{t,x}; p(\rho)) \text{div} \varphi dx dt + \int_0^T \int_\Omega (Y_{t,x}; \omega \times m) \varphi dx dt = - \int_\Omega (Y_{0,x}; m) \varphi(0) dx - \int_0^T \int_\Omega \nabla_x \varphi : d\mu_c,
  \]
  for all \( \varphi \in C_\infty_c([0, T] \times \Omega) \), where \( \mu_c \in M([0, T] \times \Omega) \) is the so–called momentum concentration measure;
- the energy inequality
  \[
  \int_\Omega [(Y_{t,x}; \frac{1}{2} |m|^2 \rho) + (P(\rho) - P'(\rho)(\rho - \overline{\rho}) - P(\overline{\rho}))] dx + D(\tau) \\
  \leq \int_\Omega (Y_{0,x}; \frac{1}{2} |m|^2 \rho) + (P(\rho) - P'(\rho)(\rho - \overline{\rho}) - P(\overline{\rho})) dx,
  \]
  for a.a \( \tau \in (0, T) \), where
  \[
  P(\rho) = \rho \int_\mathcal{P} \frac{p(z)}{z^2} dz,
  \]
  and \( D \) is a non-negative function \( D \in L^\infty(0, T) \), satisfying the compatibility condition
  \[
  \int_0^\tau \int_\Omega |\mu_c| dx dt \leq C \int_0^\tau \xi(t) D(t) dt, \quad \text{for some} \ \xi \in L^1(0, T).
  \]

Remark 2.1. The notion of (DMV) solutions can be founded in many works as it was already mentioned in the Introduction, see e.g. [1, 2, 3, 4, 11]. For convenience of readers, we give more details.

Let \( L^\infty_{\text{weak} – (\ast)}((0, T) \times \Omega; \mathcal{P}(Q)) \) be the space of essentially bounded weakly*– measure maps \( Y : (0, T) \times \Omega \to \mathcal{P}(Q), (t,x) \mapsto Y_{t,x} \). By virtue of fundamental theorem on Young measures (see
there exists a subsequence of \( \{ \rho_\epsilon, m_\epsilon \}_{\epsilon > 0} \) and parameterized family of probability measures 
\( \{ Y_{t,x} \}_{(t,x) \in (0,T) \times \Omega} \)

\[ [(t,x) \mapsto Y_{t,x}] \in L^\infty_{\text{weak}*}((0,T) \times \Omega; P(Q)), \]

such that a.a. \((t,x) \in (0,T) \times \Omega\)

\[ \langle Y_{t,x}; G(\rho, m) \rangle = G(\rho, m)(t,x) \text{ for any } G \in C_c(Q), \]

whenever

\[ G(\rho_\epsilon, m_\epsilon) \rightharpoonup G(\rho, m)(t,x) \text{ weakly } \star \text{ in } L^\infty((0,T) \times \Omega). \]

Moreover, if \( G \in C(Q) \) is such that

\[ \int_0^T \int_\Omega |G(\rho_\epsilon, m_\epsilon)| dx \leq C, \]

then \( G \) is \( Y_{t,x} \) integrable for almost all \((t,x) \in (0,T) \times \Omega\) and

\[ [(t,x) \mapsto \langle Y_{t,x}; G(\rho, m) \rangle] \in L^1((0,T) \times \Omega), \]

and

\[ G(\rho_\epsilon, m_\epsilon) \rightharpoonup G(\rho, m)(t,x) \text{ weakly } \star \text{ in } \mathcal{M}((0,T) \times \Omega). \]

The difference

\[ \mu_G \equiv G(\rho, m) - [(t,x) \mapsto \langle Y_{t,x}; G(\rho, m) \rangle] \in \mathcal{M}((0,T) \times \Omega), \]

is called concentration defect measure.

For more details, please see [7].

**Remark 2.2.**

- The measure \( Y_{0,x} \) plays the role of initial conditions.
- The proof of an existence of (DMV) solutions of Euler system was done in the pioneer work by Neustupa, [26]. Recently, see [27], the authors proved the local strong solutions of rotating compressible Euler system in \( \mathbb{R}^3 \). Feireisl et al. [3, 4] proved the existence of (DMV) solutions to the non-rotating full Euler system. As the rotating term does not bring any trouble in the proof of existence, the existence of (DMV) solutions to (1.4) can be obtained by analogous methods as in [4].

**Remark 2.3.** We need to define the function

\[ [\rho, m] \mapsto \frac{|m|^2}{\rho} \]

on the vacuum set as

\[ [\rho, m] \mapsto \frac{|m|^2}{\rho} = \begin{cases} \infty, & \text{if } \rho = 0 \text{ and } m \neq 0, \\ \frac{|m|^2}{\rho}, & \text{if } \rho > 0, \\ 0, & \text{otherwise}. \end{cases} \] (2.7)

Accordingly, it follows from the energy inequality (2.4) that

\[ \text{Supp}[Y_{t,x}] \cap \{ [\rho, m] \in Q | \rho = 0, m \neq 0 \} = \emptyset \text{ for a.a. } (t,x). \] (2.8)
2.2 Relative entropy inequality

Motivated by [12][13][4], we introduce the relative energy functional

\[ \mathcal{E}(\rho, m|r, U) = \int_{\Omega} (Y_{t,x}; \frac{1}{2} \rho \frac{m}{\rho} - U(t, x))^2 + (P(\rho) - P'(r)(\rho - r) - P(r))dx, \]

where \( r > 0, U \) are smooth "test" functions, \( r - \mathcal{J}, U \) compactly supported in \( \Omega \).

As shown in [3], any (DMV) solution of (1.1) satisfies the relative entropy inequality

\[
\begin{align*}
\mathcal{E}(\rho, m|r, U)_{|t=\tau}^{t=T} + \mathcal{D}(\tau) &\leq \int_0^\tau \int_{\Omega} (Y_{t,x}; (\partial_t U + \frac{m}{\rho} \nabla_x U)(\rho U - m))dxdt \\
&+ \int_0^\tau \int_{\Omega} (Y_{t,x}; (r - \rho) \partial_t P'(r) + (r U - m) \nabla_x P'(r))dxdt + \int_0^\tau \int_{\Omega} (Y_{t,x}; \omega \times \frac{m}{\rho})(\rho U - m)dxdt \\
&- \int_0^\tau \int_{\Omega} (Y_{t,x}; p(\rho) - p(r)) \text{div} U dxdt + \int_0^\tau \int_{\Omega} \nabla_x U : d\mu_c.
\end{align*}
\]

for a.a. \( \tau \in [0, T] \), and any \( r, U \in C^1([0, T] \times \Omega), r - \mathcal{J}, U \) compactly supported in \( \Omega \).

3 Main result

Before stating our main result, we introduce some notations and collect several mostly technical hypotheses and known facts concerning the limit system. \( x = (x_h, x_3) \) with \( x_h \in \mathbb{R}^2 \) denoting its horizontal component. For a vector field \( b = [b_1, b_2, b_3] \), we introduce the horizontal component \( b_h = [b_1, b_2] \) writing \( b = [b_h, b_3] \). Similarly, we use the symbols \( \nabla_h, \text{div}_h \) to denote the differential operators acting on the horizontal variables. The following assumptions and results will be used in the proof.

3.1 Pressure

We suppose the pressure \( p \) is a continuously differentiable function of the density such that for some \( \gamma > 1 \),

\[ p \in C^1[0, \infty) \cap C^\infty(0, \infty), \quad p(0) = 0, \quad p'(\rho) > 0 \text{ for all } \rho > 0, \quad \lim_{\rho \to \infty} \frac{p'(\rho)}{\rho^{\gamma-1}} = p_\infty > 0. \]

Remark 3.1. Similarly to [18], we deduce that

\begin{align*}
p(\rho) - p'(r)(\rho - r) - p(r) \text{ is dominated by } &P(\rho) - P'(r)(\rho - r) - P(r), \text{ specifically,} \\
|\rho - r|^2 &\leq c(\delta)(P(\rho) - P'(r)(\rho - r) - P(r)) \quad \text{when } 0 < \delta \leq \rho, \quad r \leq \frac{1}{\delta}, \quad \delta > 0, \\
1 + |\rho - r| + P(\rho) &\leq c(\delta)(P(\rho) - P'(r)(\rho - r) - P(r)) \quad \text{if } 0 < 2\delta < r < \frac{1}{2\delta}, \\
&\quad \rho \in [0, \delta) \cup \left(\frac{1}{\delta}, \infty\right), \quad \delta > 0.
\end{align*}

3.2 Quasi-geophysical equation

The expected limit problem reads

\[ \omega \times \nu + \frac{p'(\tilde{\rho})}{\tilde{p}} \nabla_x q = 0, \quad \nu = [\nu_h(x_h), 0], \quad q = q(x_h), \]

(3.2)
\[ \partial_t (\Delta h q - \frac{1}{p'(\rho)} q) + v_h \cdot \nabla h (\Delta h q) = 0. \] (3.3)

supplement with the initial condition
\[ q|_{t=0} = q_0. \]

As shown by Oliver \[29\], the problem (3.2) – (3.3) possesses a unique classical solution
\[ q \in C([0, T]; W^{m,2}(\mathbb{R}^2)) \cap C^1([0, T]; W^{m-1,2}(\mathbb{R}^2)), \quad m \geq 4, \] (3.4)
for any initial solution
\[ q_0 \in W^{m,2}(\mathbb{R}^2). \] (3.5)

### 3.3 Ill prepared initial–data

The ill-prepared initial data for the scaled system (1.4) take the form
\[ \rho_\epsilon(0, \cdot) = \rho_{0,\epsilon} = \overline{\rho} + \epsilon s_{0,\epsilon}, \quad u_\epsilon(0, \cdot) = u_{0,\epsilon}, \] (3.6)
where
\[ s_{0,\epsilon} \rightarrow s_0 \text{ in } W^{k,2}(\Omega) \cap W^{k,1}(\Omega), \quad u_{0,\epsilon} \rightarrow u_0 \text{ in } W^{k,2}(\Omega) \cap W^{k,1}(\Omega), \quad (k > 3), \]
\[ u_0 = v_0 + \nabla x \Phi_0. \] (3.7)

### 3.4 Singular limit – main result

For simplicity, we assume \( \overline{p} = p'(\overline{\rho}) = P''(\overline{\rho}) = 1 \). Now, we are ready to state our main result.

**Theorem 3.1.** Let \( \{Y_{\epsilon,x}\}_{(t,x) \in [0,T] \times \Omega} \) be a family of (DMV) solutions to the scaled Euler system (1.4) satisfying the compatibility condition (2.6) with a function \( \xi \) independent of \( \epsilon \). Let the initial data \( \{X_{\epsilon,x}\}_{x \in \Omega} \) be ill-prepared, namely
\[ \int_{\Omega} (Y_{0,x}^\epsilon \cdot \frac{1}{2} \frac{m}{p} \rho - u_{0,\epsilon}(x))^2 + \frac{1}{\epsilon^2} (P(\rho) - P'(\rho_{0,\epsilon})(\rho - \rho_{0,\epsilon}) - P(\rho_{0,\epsilon}))) dx \rightarrow 0, \]
where \( \rho_{0,\epsilon}, u_{0,\epsilon} \) are ill prepared data introduced in Section 3.3.

Then
\[ D^\epsilon \rightarrow 0 \text{ in } L^\infty(0,T), \]
\[ Y_{\epsilon,x}^\epsilon \rightarrow \delta_{\|x\|} \text{ in } L^p(0,T; L^1(\Omega; M^+([Q]_{\text{weak}}(\cdot,\cdot)) \text{ for any finite } p \geq 1, \]
where \( q \) and \( v \) is the unique solution of problem (3.2)-(3.3) starting from the initial data \( q_0 \) and where \( q_0 \in W^{k+1,2}(\mathbb{R}^2) \cap W^{k+1,1}(\mathbb{R}^2) \) is the unique solution of the elliptic problem
\[ -\Delta_h q_0 + q_0 = \int_0^1 \text{curl}_h |u_0|_h dx_3 + \int_0^1 s_0 dx_3. \] (3.9)

The rest of the paper is devoted to the proof of Theorem 3.1.
4 Energy bounds

We start by deriving uniform bounds on solutions to (1.4) independent of $\epsilon$. Similarly to [18], we introduce the decomposition

$$h(\rho, m) = [h]_{ess}(\rho, m) + [h]_{res}(\rho, m), \quad [h]_{ess} = \psi(\rho)h(\rho, m), \quad [h]_{res} = (1 - \psi(\rho))h(\rho, m),$$

where

$$\psi \in C^\infty_c(0, \infty), \quad 0 \leq \psi(\rho) \leq 1, \quad \psi(\rho) = 1 \text{ on an open interval containing } \rho = 1.$$ 

As the initial data are ill-prepared, the expression on the right-hand side of the energy inequality (2.4) remains bounded uniformly for $\epsilon \to 0$. Consequently, we deduce the following bound:

$$\text{ess sup}_{t \in (0, T)} \int_{\Omega} \langle Y^{\epsilon}_{t,x}; \frac{1}{2} |m_{t,x}|^2 \rho + \frac{1}{\epsilon^2}(P(\rho_{t,x}) - P'(1)(\rho_{t,x} - 1) - P(1)) \rangle dx \leq C. \quad (4.1)$$

Thus, exactly as in [18], we use the structural properties of the function $p$ to deduce

$$\text{ess sup}_{t \in (0, T)} \int_{\Omega} \langle Y^{\epsilon}_{t,x}; \rho_{t,x} - 1 \epsilon \rangle_{ess} \rangle_{ess} dx \leq C;$$

$$(t, x) \mapsto (Y^{\epsilon}_{t,x}; m_{t,x}) \text{ bounded in } L^\infty(0, T; L^2(\Omega)) + L^{\frac{2}{\gamma+1}}(\Omega);$$

$$(t, x) \mapsto (Y^{\epsilon}_{t,x}; \frac{\rho_{t,x} - 1}{\epsilon}) \text{ bounded in } L^\infty(0, T; L^2(\Omega));$$

$$\epsilon \rightarrow \epsilon - \epsilon^2 \langle Y^{\epsilon}_{t,x}; \rho_{t,x} \rangle \text{ bounded in } L^\infty(0, T; L^\gamma(\Omega)). \quad (4.2)$$

Using the same argument in [17], there exist functions $\rho^{(1)} \in L^\infty(0, T; L^2(\Omega))$ and $m \in L^\infty(0, T; L^q(\Omega))$ for some $q > 1$ and a subsequence such that

$$(Y^{\epsilon}_{t,x}; m_{t,x}) \rightarrow m \text{ weakly in } L^\infty(0, T; L^6(\Omega));$$

$$(Y^{\epsilon}_{t,x}; \frac{\rho_{t,x} - 1}{\epsilon}) \rightarrow \rho^{(1)} \text{ weakly in } L^\infty(0, T; L^2(\Omega)).$$

Recalling (2.2) and (2.3), we deduce

$$\int_0^T \int_{\Omega} m \cdot \nabla_x \varphi dx dt = 0, \quad \int_0^T \int_{\Omega} [\omega \times m] \cdot \varphi + \rho^{(1)} \text{div} \varphi |dx dt = 0,$$

for $\varphi \in C^1([0, T] \times \Omega)$. In other words,

$$\text{div}_x m = 0, \quad \omega \times m + \nabla_x \rho^{(1)} = 0, \quad (4.3)$$

in the sense of distribution.

It is easy to check that

$$\rho^{(1)} = \rho^{(1)}(x_h), \quad m = (m_h, 0), \quad \text{div}_x m = \text{div}_h m_h = 0.$$

Moreover, the detail of derivation of (3.9) can be seen in [14, 15, 16].
5 Acoustic waves

It is well-known that ill-prepared data give rise to rapidly oscillating acoustic waves. Similarly to [15], the relevant acoustic equation reads

\[
\begin{align*}
\epsilon \partial_t s_\epsilon + \text{div}(\nabla_x \Phi_\epsilon) &= 0, \\
\epsilon \partial_t \nabla_x \Phi_\epsilon + \omega \times \nabla_x \Phi_\epsilon + \nabla_x s_\epsilon &= 0,
\end{align*}
\]

supplemented with the initial data

\[ s_\epsilon(0, \cdot) = s_0, \quad \nabla_x \Phi_\epsilon(0, \cdot) = \nabla_x \Phi_0, \]

where \( s_0, \nabla_x \Phi_0 \) have been introduced in Section 3.3.

As a matter of fact, the initial data must be smoothed and cut-off via suitable regularization operators, namely

\[ s_\epsilon(0, \cdot) = s_0,\delta = [s_0]_\delta; \quad \nabla_x \Phi_\epsilon(0, \cdot) = \nabla_x \Phi_0,\delta = [\Phi_0]_\delta, \]

where \([\cdot]_\delta\) denotes the regularization introduced in [15].

Denoting the corresponding solutions \( s_\epsilon,\delta, \Phi_\epsilon,\delta \) we report the following energy and dispersive estimates proved in [15, Section 6]:

\[ \sup_{t \in [0, T]} ||\Phi_{\epsilon,\delta}(t, \cdot)||_{W^{m, 2}} + ||s_{\epsilon,\delta}(t, \cdot)||_{W^{m, 2}} = ||\nabla_x \Phi_{0,\delta}||_{L^2} + ||s_{0,\delta}||_{L^2}, \]

and

\[ \int_0^T ||\Phi_{\epsilon,\delta}(t, \cdot)||_{W^{\infty, \infty}} + ||s_{\epsilon,\delta}(t, \cdot)||_{W^{\infty, \infty}} \leq \omega(\epsilon, m, \delta)||\nabla_x \Phi_{0,\delta}||_{L^2} + ||r_{0,\delta}||_{L^2}, \]

where \( \omega(\epsilon, m, \delta) \to 0 \) as \( \epsilon \to 0 \) for any fixed \( m \geq 0 \) and \( \delta > 0 \). More details about Strichartz estimates and acoustic waves, readers can refer to [30, 31].

6 Convergence

The proof of convergence is based on the ansatz

\[ r_\epsilon = 1 + \epsilon(q + s_\epsilon,\delta), \quad U_\epsilon = v + \nabla_x \Phi_{\epsilon,\delta}, \]

in the relative energy inequality (2.10). The \([s_\epsilon, \delta, \nabla_x \Phi_{\epsilon, \delta}]\) are solutions of the acoustic system (5.1), and \([q, v]\) is solution of the target problem

\[ \omega \times v + \nabla_x q = 0, \]

\[ \partial_t(\Delta_h q - q) + \nabla_h^\perp q \cdot \nabla_h(\Delta_h q) = 0. \]

In addition, to avoid technicalities, we shall assume that \( s_0 \) and \( \Phi_0 \) are sufficiently regular so that the \( \delta \)-regularization is not needed in (5.1–5.3). Accordingly, we have \( s_{\epsilon,\delta} = s_\epsilon, \Phi_{\epsilon,\delta} = \Phi_\epsilon \). The general case may be handled as in [15].

First note that the relative energy for the scaled system reads

\[ E_\epsilon(\rho_\epsilon, m_\epsilon|r_\epsilon, U_\epsilon) = \int_\Omega (Y_{\epsilon,x}; \frac{1}{2\rho_\epsilon} \frac{m_\epsilon}{\rho_\epsilon} - U_\epsilon)^2 + \frac{1}{\epsilon^2}(P(\rho_\epsilon) - P'(\rho_\epsilon)(\rho_\epsilon - r_\epsilon) - P(r_\epsilon))dx, \]

where \( \epsilon \) is small and \( m_\epsilon \) is close to 1. The energy estimates (5.2) and (5.3) for \( \epsilon \) small imply that

\[ E_\epsilon(\rho_\epsilon, m_\epsilon|r_\epsilon, U_\epsilon) \to E(\rho_\epsilon, m_\epsilon|q, v), \]

and hence

\[ \epsilon \to 0, \quad \rho_\epsilon \to \rho, \quad m_\epsilon \to m. \]

Therefore, the \( \Phi_{\epsilon,\delta} \) converge strongly in the \( H^m \) norm to \( \Phi_{\delta} \), which is a solution of the target problem.

Finally, we show that \( r_\epsilon \) converges strongly in the energy space to \( q \). The energy estimates (5.2) and (5.3) imply that

\[ \epsilon \to 0, \quad \rho_\epsilon \to \rho, \quad m_\epsilon \to m, \quad r_\epsilon \to q. \]

Therefore, the \( \Phi_{\epsilon,\delta} \) converge strongly in the \( H^m \) norm to \( \Phi_{\delta} \), which is a solution of the target problem.
with the corresponding relative energy inequality:

\[
\mathcal{E}_c(\rho_\epsilon, m_\epsilon | r_\epsilon, U_\epsilon) \big|_{\epsilon = 0}^{\epsilon} + D'(\tau) \leq \int_0^\tau \int_\Omega (Y_{t,x}^\epsilon; r_\epsilon - \rho_\epsilon) \partial_t P'(r_\epsilon) + (Y_{t,x}^\epsilon; r_\epsilon U_\epsilon - m_\epsilon) \nabla_x P'(r_\epsilon) dx dt + \frac{1}{\epsilon^2} \int_0^\tau \int_\Omega \nabla_x \cdot (\rho_\epsilon U_\epsilon - m_\epsilon) dx dt - \frac{1}{\epsilon} \int_0^\tau \int_\Omega (\nabla_x p(\rho_\epsilon) - p(r_\epsilon)) div U_\epsilon dx dt + \int_\Omega \nabla_x U_\epsilon : d\mu_\epsilon. \tag{6.4}
\]

Our goal is to show that, with the ansatz (6.1), the relative energy \(\mathcal{E}_c(\rho_\epsilon, m_\epsilon | r_\epsilon, U_\epsilon)\) tends to zero for \(\epsilon \to 0\) uniformly in \(t \in [0, T]\). In view of the dispersive estimates (5.2) – (5.3), this will yield the conclusion claimed in Theorem 3.1. To this end, we use a Gronwall type argument showing that all integrals in the right-hand side of (6.4) are either small or can be absorbed by the left-hand side as \(\epsilon \to 0\). This programme will be carried over by means of several steps.

### 6.1 Step 1

First, we compute

\[
\int_0^\tau \int_\Omega \frac{1}{\epsilon} [(Y_{t,x}^\epsilon; r_\epsilon - \rho_\epsilon) \partial_t P'(r_\epsilon) + (Y_{t,x}^\epsilon; r_\epsilon U_\epsilon - m_\epsilon) \nabla_x P'(r_\epsilon)] + \langle Y_{t,x}^\epsilon; p(\rho_\epsilon) - p(r_\epsilon) \rangle div U_\epsilon dx dt - \frac{1}{\epsilon^2} \int_0^\tau \int_\Omega \nabla_x U_\epsilon : d\mu_\epsilon.
\]

Note that, in view of (6.2),

\[
\partial_t r_\epsilon + \nabla_x (r_\epsilon U_\epsilon) = \epsilon \partial_t q + \partial_t s_\epsilon + div(r_\epsilon (v + \nabla_x \Phi_\epsilon)) = \epsilon \partial_t q + div((q + s_\epsilon) U_\epsilon).
\]

Next, by virtue of (5.1) and (6.1),

\[
\nabla_x P'(r_\epsilon)(\rho_\epsilon U_\epsilon - m_\epsilon) = \nabla_x (P'(r_\epsilon) - P''(1)(r_\epsilon - 1) - P'(1))(\rho_\epsilon U_\epsilon - m_\epsilon) + \epsilon \nabla_x s_\epsilon \cdot (\rho_\epsilon U_\epsilon - m_\epsilon) + \epsilon \nabla_x q \cdot (\rho_\epsilon U_\epsilon - m_\epsilon)
\]

\[
= \nabla_x (P'(r_\epsilon) - P''(1)(r_\epsilon - 1) - P'(1))(\rho_\epsilon U_\epsilon - m_\epsilon) + \epsilon \nabla_x q \cdot (\rho_\epsilon U_\epsilon - m_\epsilon)
\]

\[
- \epsilon^2 (\rho_\epsilon U_\epsilon - m_\epsilon) \cdot \partial_t \nabla \Phi_\epsilon - \epsilon (\rho_\epsilon U_\epsilon - m_\epsilon) (\omega \times \nabla \Phi_\epsilon).
\]

Furthermore, by virtue of the compatibility condition (2.6), we can control the concentration measure,

\[
\int_0^\tau \int_\Omega \nabla_x U : d\mu_\epsilon \leq \|\nabla_x U\|_{L^\infty} \int_0^\tau \xi(t) d\xi'(t) dt.
\]
Finally, as the hypotheses about the ill-prepared initial data, we have

\[ \mathcal{E}_\epsilon(\rho_\epsilon, \mathbf{m}_\epsilon | r_\epsilon, U_\epsilon)(0) \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \]

Thus we may conclude that

\[
\mathcal{E}_\epsilon(\rho_\epsilon, \mathbf{m}_\epsilon | r_\epsilon, U_\epsilon)(\tau) + D'(\tau) \leq \int_0^\tau \int_\Omega \left< \frac{\mathbf{m}_\epsilon}{\rho_\epsilon} \nabla x U_\epsilon \right>(\rho_\epsilon U_\epsilon - \mathbf{m}_\epsilon) dxdt \\
+ \frac{1}{\epsilon} \int_0^\tau \int_\Omega \left< \frac{\mathbf{m}_\epsilon}{\rho_\epsilon} \nabla x U_\epsilon \right>(\rho_\epsilon U_\epsilon - \mathbf{m}_\epsilon) dxdt \\
+ \frac{1}{\epsilon} \int_0^\tau \int_\Omega \left< \frac{\mathbf{m}_\epsilon}{\rho_\epsilon} \nabla x U_\epsilon \right>(\rho_\epsilon U_\epsilon - \mathbf{m}_\epsilon) dxdt \\
- \frac{1}{\epsilon} \int_0^\tau \int_\Omega \left< \frac{\mathbf{m}_\epsilon}{\rho_\epsilon} \nabla x U_\epsilon \right>(\rho_\epsilon U_\epsilon - \mathbf{m}_\epsilon) dxdt \\
+ \frac{1}{\epsilon} \int_0^\tau \int_\Omega \left< \frac{\mathbf{m}_\epsilon}{\rho_\epsilon} \nabla x U_\epsilon \right>(\rho_\epsilon U_\epsilon - \mathbf{m}_\epsilon) dxdt + c \int_0^\tau \xi(t) D'(t) dt + \omega(\epsilon),
\]

where \( \omega(\epsilon) \) denotes a generic quantity satisfying

\[ \omega(\epsilon) \rightarrow 0 \text{ in } L^1(0,T) \text{ as } \epsilon \rightarrow 0. \]

Using (6.2), we get the following conclusion:

\[
\mathcal{E}_\epsilon(\rho_\epsilon, \mathbf{m}_\epsilon | r_\epsilon, U_\epsilon)(\tau) + D'(\tau) \leq \int_0^\tau \int_\Omega \left< \frac{\mathbf{m}_\epsilon}{\rho_\epsilon} \nabla x U_\epsilon \right>(\rho_\epsilon U_\epsilon - \mathbf{m}_\epsilon) dxdt \\
+ \frac{1}{\epsilon} \int_0^\tau \int_\Omega \left< \frac{\mathbf{m}_\epsilon}{\rho_\epsilon} \nabla x U_\epsilon \right>(\rho_\epsilon U_\epsilon - \mathbf{m}_\epsilon) dxdt \\
+ \frac{1}{\epsilon} \int_0^\tau \int_\Omega \left< \frac{\mathbf{m}_\epsilon}{\rho_\epsilon} \nabla x U_\epsilon \right>(\rho_\epsilon U_\epsilon - \mathbf{m}_\epsilon) dxdt \\
- \frac{1}{\epsilon} \int_0^\tau \int_\Omega \left< \frac{\mathbf{m}_\epsilon}{\rho_\epsilon} \nabla x U_\epsilon \right>(\rho_\epsilon U_\epsilon - \mathbf{m}_\epsilon) dxdt \\
+ \frac{1}{\epsilon} \int_0^\tau \int_\Omega \left< \frac{\mathbf{m}_\epsilon}{\rho_\epsilon} \nabla x U_\epsilon \right>(\rho_\epsilon U_\epsilon - \mathbf{m}_\epsilon) dxdt + c \int_0^\tau \xi(t) D'(t) dt + \omega(\epsilon),
\]

6.2 Step 2

We write

\[
\int_0^\tau \int_\Omega \left< \frac{\mathbf{m}_\epsilon}{\rho_\epsilon} \nabla x U_\epsilon \right>(\rho_\epsilon U_\epsilon - \mathbf{m}_\epsilon) dxdt \\
= \int_0^\tau \int_\Omega \left< \frac{\mathbf{m}_\epsilon}{\rho_\epsilon} \nabla x U_\epsilon \right>(\rho_\epsilon U_\epsilon - \mathbf{m}_\epsilon) dxdt \\
+ \int_0^\tau \int_\Omega \left< \frac{\mathbf{m}_\epsilon}{\rho_\epsilon} \nabla x U_\epsilon \right>(\rho_\epsilon U_\epsilon - \mathbf{m}_\epsilon) dxdt \\
+ \int_0^\tau \int_\Omega \left< \frac{\mathbf{m}_\epsilon}{\rho_\epsilon} \nabla x U_\epsilon \right>(\rho_\epsilon U_\epsilon - \mathbf{m}_\epsilon) dxdt \\
= I_1 + I_2 + I_3.
\]
Using the uniform bounds (4.2), we can split the functions in $I_2$ into their essential and residual parts obtaining

\[
\left| \int_{\mathbb{R}^3} (Y^\epsilon_{t,x}; \rho_\epsilon U_\epsilon - m_\epsilon) (v \cdot \nabla_x \Phi_\epsilon + \nabla_x \Phi_\epsilon \nabla_x U_\epsilon) \, dx \right| \\
\leq \|\nabla_x \Phi_\epsilon\|_{L^1_{T}} (\|v\|_{L^3} + \|\nabla_x U_\epsilon\|_{L^3})^2 + c\mathcal{E}_\epsilon(\rho_\epsilon, m_\epsilon, r_\epsilon, U_\epsilon),
\]

where the first term on the right–hand side can be controlled by means of the dispersive estimate (5.2) and (5.3).

Summing up the previous observations, we may infer that the relative energy inequality with the ansatz (6.1) reduces to

\[
\mathcal{E}_\epsilon(\rho_\epsilon, m_\epsilon, r_\epsilon, U_\epsilon)(\tau) + \mathcal{D}'(\tau) \leq \int_0^\tau \int_\Omega (Y^\epsilon_{t,x}; \rho_\epsilon U_\epsilon - m_\epsilon) (\partial_t v + v \cdot \nabla_x v) \, dx \, dt \\
+ \frac{1}{\epsilon^2} \int_0^\tau \int_\Omega (Y^\epsilon_{t,x}; \nabla_x (P'(r_\epsilon) - P''(1)(r_\epsilon - 1) - P'(1)) \rho_\epsilon U_\epsilon - m_\epsilon) \, dx \, dt \\
- \frac{1}{\epsilon^2} \int_0^\tau \int_\Omega (Y^\epsilon_{t,x}; \rho_\epsilon - p(r_\epsilon) - p'(r_\epsilon)(\rho_\epsilon - r_\epsilon)) \, div U_\epsilon \, dx \, dt \\
- \frac{1}{\epsilon} \int_0^\tau \int_\Omega (Y^\epsilon_{t,x}; \rho_\epsilon - r_\epsilon) P''(r_\epsilon) (\partial_t q + div ((q + s_\epsilon) U_\epsilon)) \, dx \, dt \\
+ C \int_0^\tau \mathcal{E}_\epsilon(\rho_\epsilon, m_\epsilon, r_\epsilon, U_\epsilon) \, dt + C \int_0^\tau \xi(t) \mathcal{D}'(t) \, dt + \omega(\epsilon).
\]

### 6.3 Step 3

Now, we will deal with pressure term and corresponding term. First, using direct calculation, the Taylor formula and dispersive estimates (5.2-5.3), we deduce that

\[
\frac{1}{\epsilon^2} |\nabla_x (P'(r_\epsilon) - P'(1) - P''(1)(r_\epsilon - 1))| \\
= \frac{1}{\epsilon} |(P''(r_\epsilon) - P'(1)) \nabla_x (q + s_\epsilon)| \\
\rightarrow P''(1) q \nabla_x q \quad \text{as} \quad \epsilon \rightarrow 0.
\]

Therefore, combining the previous energy bounds and convergence, we get

\[
\frac{1}{\epsilon^2} \int_0^\tau \int_\Omega (Y^\epsilon_{t,x}; \nabla_x (P'(r_\epsilon) - P'(1) - P''(1)(r_\epsilon - 1))) (\rho_\epsilon U_\epsilon - m_\epsilon) \, dt \rightarrow 0 \quad \text{as} \quad \epsilon \rightarrow 0.
\]

The remaining pressure term is

\[
\frac{1}{\epsilon^2} \int_0^\tau \int_\Omega (Y^\epsilon_{t,x}; \rho_\epsilon - p(r_\epsilon) - p'(r_\epsilon)(\rho_\epsilon - r_\epsilon)) \, div U_\epsilon \, dt \\
= \frac{1}{\epsilon^2} \int_0^\tau \int_\Omega (Y^\epsilon_{t,x}; \rho_\epsilon - p(r_\epsilon) - p'(r_\epsilon)(\rho_\epsilon - r_\epsilon)) (div v + \Delta \Phi_\epsilon) \, dt \\
\leq c \frac{1}{\epsilon^2} \int_0^\tau \int_\Omega (Y^\epsilon_{t,x}; \rho_\epsilon - P(r_\epsilon) - P'(r_\epsilon)(\rho_\epsilon - r_\epsilon)) (div v + \Delta \Phi_\epsilon) \, dt \\
\leq C \int_0^\tau \mathcal{E}_\epsilon(\rho_\epsilon, m_\epsilon, r_\epsilon, U_\epsilon) \, dt,
\]
where we have used the previous dispersive estimates (5.2) and (5.3). Thus, we can conclude that
\[
\mathcal{E}_c(\rho_c, m_c, U_c)(\tau) + D'(\tau) \leq \omega(\epsilon) + C \int_0^\tau \mathcal{E}(\rho_c, m_c, U_c) dt + C \int_0^\tau \xi(t) D'(t) dt \\
+ \int_0^\tau \int_\Omega (Y_{t,x}^\epsilon; \mathbf{v})^2 (\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v}) dx dt \\
- \frac{1}{\epsilon} \int_0^\tau \int_\Omega (Y_{t,x}^\epsilon; (\rho_c - \rho_c) P''(\rho_c)) (\partial_t q + \text{div}(q + s_c U_c)) dx dt.
\]

### 6.4 Step 4

Finally, we deal with the remaining pressure terms. Similar to the previous analysis, we obtain
\[
\mathcal{E}_c(\rho_c, m_c, U_c)(\tau) + D'(\tau) \leq \omega(\epsilon) + C \int_0^\tau \mathcal{E}(\rho_c, m_c, U_c) dt + C \int_0^\tau \xi(t) D'(t) dt \\
+ \int_0^\tau \int_\Omega (Y_{t,x}^\epsilon; \mathbf{v} - \mathbf{m})(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v}) dx dt + \int_0^\tau \int_\Omega (Y_{t,x}^\epsilon; q - \rho(1)(\partial_t q + \text{div}(q \mathbf{v}))) dx dt,
\]
where
\[
\int_0^\tau \int_\Omega (Y_{t,x}^\epsilon; \mathbf{v} - \mathbf{m})(\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v}) dx dt + \int_0^\tau \int_\Omega (Y_{t,x}^\epsilon; q - \rho(1)(\partial_t q + \text{div}(q \mathbf{v}))) dx dt
\]
\[
= \frac{1}{2} \int_0^\tau \int_\Omega (Y_{t,x}^\epsilon; \partial_t |\mathbf{v}|^2 + |\partial_t q|^2) dx dt - \int_0^\tau \int_\Omega (Y_{t,x}^\epsilon; \partial_t \mathbf{v} \cdot \mathbf{m} + \partial_t q \cdot \rho(1)) dx dt
\]
\[
- \int_0^\tau \int_\Omega (Y_{t,x}^\epsilon; \mathbf{v} \cdot \nabla_x \mathbf{v} \cdot \mathbf{m} + \rho(1) \text{div}(q \mathbf{v})) dx dt
\]

By virtue of (4.3) and (6.2), we have
\[
\text{div}_x(q \mathbf{v}) = \nabla_x q \cdot \mathbf{v} = \nabla_x q \cdot \nabla_x^\perp q = 0
\]
and
\[
- \int_0^\tau \int_\Omega (Y_{t,x}^\epsilon; \partial_t \mathbf{v} \cdot \mathbf{m} + \partial_t q \rho(1)) dx dt = - \int_0^\tau \int_\Omega (Y_{t,x}^\epsilon; \omega \times \mathbf{m}) \mathbf{v} \Delta_h q dx dt.
\]

Moreover, it is easy to check that
\[
\mathbf{v} \cdot \nabla_x \mathbf{v} \cdot \mathbf{m} + (\omega \times \mathbf{m}) \cdot \mathbf{v} \Delta_h q = \mathbf{m} \cdot \nabla_h \frac{|\mathbf{v}|^2}{2}.
\]
So we deduce that
\[
\frac{1}{2} \int_0^\tau \int_\Omega (Y_{t,x}^\epsilon; \partial_t |\mathbf{v}|^2 + |\partial_t q|^2) dx dt = \frac{1}{2} \int_0^\tau \int_\Omega (Y_{t,x}^\epsilon; \partial_t |\nabla_h^\perp q|^2 + |\partial_t q|^2) dx dt
\]
\[
= \int_0^\tau \int_\Omega (Y_{t,x}^\epsilon; \mathbf{v} \cdot \nabla_h (\Delta_h q)) dx dt
\]
\[
= - \int_0^\tau \int_\Omega (Y_{t,x}^\epsilon; \mathbf{v} \cdot \nabla_h q \Delta_h q) dx dt = 0,
\]
where the last equality due to \( \mathbf{v} = \nabla^\perp q \).
Putting together Step 1 to Step 4, we conclude that
\[ E(\rho_\epsilon, m_\epsilon, r_\epsilon, U_\epsilon) + D'(\tau) \leq \omega(\epsilon) + \int_0^T (1 + \xi(t))(E(\rho_\epsilon, m_\epsilon, r_\epsilon, U_\epsilon) + D'(t))dt, \]
where \( r_\epsilon, U_\epsilon \) are given by (6.1). Letting \( \epsilon \to 0 \) and applying the Gronwall's lemma, we complete the proof of Theorem 3.1.

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