Abstract

We present a general procedure for calculating the partition function of an Ising Model on a one dimensional Fibonacci lattice in presence of magnetic field. This partition function can be written as a sum of partition functions of usual open Ising chains in presence of magnetic field with coefficients having Fibonacci symmetries. An exact expression for the partition function of the usual open open Ising Model is found. We observe that 'Mirror Symmetry' is a characteristic property of all Fibonacci generations. Further $n$th and $(n + 6)$th generations have the same topology. We have also established a recurrence relation among partition functions of different Fibonacci generations from $n$th to $(n+6)$th.

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Introduction

Studying Ising model on a Fibonacci chain in presence of magnetic field is interesting in its own right since no proper analytical procedure exists for evaluating the partition function. However scaling forms of thermodynamic functions for such system have been studied using renormalization group technique [1] through one step decimation. Here we reformulate the expression for the partition function by breaking the transfer matrix into two particular noncommuting matrices. This formulation enables us to calculate the partition function for the usual open Ising chain. The result is quite nontrivial in contrast to the expression for the partition function of the closed one[2]. The same formulation helps us to express the partition function of Ising model on Fibonacci chain in presence of magnetic field as a sum of partition functions of usual Ising open chains with coefficients containing Fibonacci symmetry. We have also studied some symmetry properties of the Fibonacci chain. Using a special symmetry property ('Mirror Symmetry') and the usual trace map relation we have established a recurrence relation among the partition functions of different Fibonacci generations. This includes all the partition functions starting from $n$th up to $(n + 6)$th generations. We observe that mirror symmetry is a characteristic property of each Fibonacci generation with $n$th and $(n + 6)$th generations having same topology.

Exact partition function for open Ising chain with magnetic field

The one dimensional Ising model consists of a chain of $N$ spins $S_i = \pm 1$
\( i = 1, 2, \ldots, N \) with nearest neighbour interactions \( \epsilon_{i,i+1} \). The Hamiltonian is given by:

\[
\mathcal{H} = - \sum_{i=1}^{N-1} \epsilon_{i,i+1} S_i S_{i+1} - H \sum_{i=1}^{N} S_i
\]

For a uniform lattice \( \epsilon_{i,i+1} = \epsilon \), the partition function is given by:

\[
Z_o^\epsilon(T, H) = \sum_{S_1, S_2, \ldots, S_N = -1} f(S_1, S_2) f(S_2, S_3) \cdots f(S_{N-1}, S_N) f_0(S_N, S_1)
\]

with \( f(S_i, S_{i+1}) = \exp[\beta \epsilon S_i S_{i+1} + \frac{1}{2} \beta H (S_i + S_{i+1})] \); \( f_0(S_N, S_1) = [f(S_N, S_1)]_{\epsilon=0} \). Here the superscript \( o \) stands for the chain with open boundary condition. Therefore the partition function (2) can be written in terms of transfer matrix as:

\[
Z_o^\epsilon(T, H) = \text{Tr} P \prod_{i=1}^{N-1} P
\]

where

\[
P = \sqrt{r} (1 + \frac{\lambda}{r}) \sigma_1 = \sqrt{r} \sigma_1 (1 + \frac{\lambda^T}{r})
\]

\[
P_0 = [P]_{\epsilon=0} = (1 + \lambda) \sigma_1 = \sigma_1 (1 + \lambda^T)
\]

with \( r = \exp(-2\beta \epsilon) \), \( \lambda = \begin{pmatrix} 0 & e^{\beta H} \\ e^{-\beta H} & 0 \end{pmatrix} \), \( \lambda^T = \begin{pmatrix} 0 & e^{\beta H} \\ e^{-\beta H} & 0 \end{pmatrix} \) and \( \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

The general formula of the partition function for even and odd number of spins (i.e., odd and even number of bonds) can be derived by using equations (3) and (4) as:

\[
Z_o^\epsilon(2N, T, H) = \prod_{i=1}^{N-1} \text{Tr} (1 + x_1) (1 + x_2) \cdots (1 + x_{2N-1}) (1 + \lambda^T) \quad (6)
\]

and

\[
Z_o^\epsilon(2N+1, T, H) = \prod_{i=1}^{N} \text{Tr} (1 + x_1) (1 + x_2) \cdots (1 + x_{2N}) (1 + \lambda) \sigma_1 \quad (7)
\]

where

\[
x_{2i+1} = \frac{\lambda}{r}, x_{2i} = \frac{\lambda^T}{r}; i = \text{integer}
\]

The above equations show that \( \lambda, \lambda^T \) are the signatures for the transfer matrices corresponding to bonds in odd and even positions. In the case of a chain with closed boundary condition the last factor in eqn. (2) is \( f(S_N, S_1) \) and consequently the partition function takes the form:

\[
Z_c^\epsilon(2N, T, H) = \prod_{i=1}^{N-1} \text{Tr} (1 + x_1) (1 + x_2) \cdots (1 + x_{2N-1}) (1 + \lambda^T) \quad (9)
\]

and

\[
Z_c^\epsilon(2N+1, T, H) = \prod_{i=1}^{N} \text{Tr} (1 + x_1) (1 + x_2) \cdots (1 + x_{2N}) (1 + \lambda) \sigma_1 \quad (10)
\]

where

\[
Z_N^c(T, H) = \lambda_+^N + \lambda_-^N
\]

The superscript \( c \) indicates closed chain. One can show by elementary calculation that eqns. (9) and (10) reduce to the well known form [3]
\[
\lambda_{\pm} = r^{-\frac{1}{2}}[\cosh(\beta H) \pm \sqrt{\sinh^2(\beta H) + r^2}]
\]  \hspace{1cm} \text{(12)}

are the eigenvalues of the transfer matrix \( P \). The expression for the partition function in the case of an open chain with even number of spins can be derived from eqn.(6) as follows:

\[
Z_{2N}^o(T, H) = r^{N-\frac{1}{2}}Tr(1 + x_1)(1 + x_2).....(1 + x_{2N-1})(1 + \lambda^T)
\]
\[
= \sqrt{r}Z_{2N}^c(T, H) + r^{N-\frac{1}{2}}(1 - r)Tr(1 + x_1)(1 + x_2).....(1 + x_{2N-1})
\]
\[
= \sqrt{r}Z_{2N}^c(T, H) + \sqrt{r}(1 - r)Z_{2(N-1)}^c(T, H) + r^{N-\frac{1}{2}}(1 - r)
\]
\[
\times Tr(1 + x_1)(1 + x_2).....(1 + x_{2N-2})\lambda_{\pm} + 2(1 - r)
\]  \hspace{1cm} \text{(13)}

The last term in the above expression can be written in terms of the eigenvalues of the transfer matrix \( P \) viz. \( \lambda_{\pm} \). By following the method of induction:

\[
r^{N-\frac{1}{2}}(1 - r)Tr(1 + x_1)(1 + x_2).....(1 + x_{2N-2})x_{2N-1}
\]
\[
= (1 - r)r^{N-\frac{1}{2}}\frac{4}{r^2}\cosh^2(\beta H) \sum_{i=0}^{N-2} \left( \frac{\lambda_+}{r} \right)^{N-2-i} \left( \frac{\lambda_-}{r} \right)^i
\]
\[
= 4(1 - r)r^{N-\frac{3}{2}}\cosh^2(\beta H) \frac{\lambda_+^{2(N-1)} - \lambda_-^{2(N-1)}}{\lambda_+^2 - \lambda_-^2}
\]  \hspace{1cm} \text{(14)}

So eqn.(13) becomes:

\[
Z_{2N}^o(T, H) = \sqrt{r}Z_{2N}^c(T, H) + \sqrt{r}(1 - r)Z_{2(N-1)}^c(T, H)
\]
\[
+ 4(1 - r)r^{N-\frac{3}{2}}\cosh^2(\beta H) \times \frac{\lambda_+^{2(N-1)} - \lambda_-^{2(N-1)}}{\lambda_+^2 - \lambda_-^2}
\]  \hspace{1cm} \text{(15)}

Similarly the expression (7) for the open chain partition function with odd number of spins takes the form:

\[
Z_{2N+1}^o(T, H) = \sqrt{r}Z_{2N+1}^c(T, H) + 2(1 - r)\cosh(\beta H)
\]
\[
\times \frac{\lambda_+^{2N} - \lambda_-^{2N}}{\lambda_+^2 - \lambda_-^2}
\]  \hspace{1cm} \text{(16)}

Exact expressions for the thermodynamic functions can be calculated by well known methods [3].

**Ising model on Fibonacci chain**

A Fibonacci chain can be inflated by two bonds \( L \text{(large)} \) and \( S \text{(small)} \) by the inflation rule \( L \rightarrow LS, S \rightarrow L \). The chain can be represented by the sequence:

\[
L \rightarrow LS \rightarrow LSL \rightarrow LSLLS \rightarrow LSLLSLSL \rightarrow ....
\]  \hspace{1cm} \text{(17)}

In this case the interaction strengths in the Hamiltonian (1) \( \epsilon_{i,i+1} = \epsilon \) for long bonds and \( \epsilon_{i,i+1} = \bar{\epsilon} \) for the short ones where the bonds are arranged according to the Fibonacci
The corresponding partition function of the \( n \)th generation Fibonacci chain having \( N \) spins with \( N - 1 \) bonds is given by:

\[
Z_0^N(F) = \text{Tr} P \tilde{P} PP \tilde{P} .... P_0
\] (18)

where for long bonds the transfer matrix \( P \) is given by eqn.(4) and for short bonds the transfer matrix \( \tilde{P} \) is given by eqn.(4) with \( r \) replaced by \( \tilde{r} = r_{|\epsilon=\tilde{\epsilon}} \). Henceforth \( Z_0^N(F) \) and \( Z_0^N(I) \) will represent partition functions for Ising models on an open Fibonacci chain and on an open regular lattice respectively. The expressions for the partition functions with odd and even number of bonds take the same forms as shown in eqns. (6) and (7) with \( x_i \)'s given in eqn.(8) for long bonds whereas for short bonds we replace \( r \) by \( \tilde{r} \) in eqn.(8). The explicit expressions for the partition functions for open and closed chains are:

\[
Z_0^{2N}(F) = r^{N_L} \tilde{r}^{N_S} \text{Tr} (1 + x_1)(1 + x_2).....(1 + x_{2N-1})(1 + \lambda^T)
\] (19)

\[
Z_0^{2N+1}(F) = r^{N_L} \tilde{r}^{N_S} \text{Tr} (1 + x_1)(1 + x_2).....(1 + x_{2N})(1 + \lambda)\sigma_1
\] (20)

Similarly for closed chain eqs. (18),(19) and (20) take the following forms:

\[
Z_c^N(F) = \text{Tr} P \tilde{P} PP \tilde{P} .... P
\] (21)

\[
Z_c^{2N}(F) = r^{N_L} \tilde{r}^{N_S} \text{Tr} (1 + x_1)(1 + x_2).....(1 + x_{2N-1})(1 + x_{2N})
\] (22)

\[
Z_c^{2N+1}(F) = r^{N_L} \tilde{r}^{N_S} \text{Tr} (1 + x_1)(1 + x_2).....(1 + x_{2N})(1 + x_{2N+1})\sigma_1
\] (23)

where \( N_L, N_S \) are number of long and short bonds in a particular sequence.

Now \( \tilde{P} \) is related to \( P \) through the following equation:

\[
\tilde{P} = \sqrt{\tilde{r}}(1 - \frac{r}{\tilde{r}})\sigma_1 + \sqrt{\frac{r}{\tilde{r}}}P
\] (24)

Using eqn.(24) in eqns.(19) and (20) the Fibonacci partition function for any generation can be written in terms of open Ising partition functions as follows:

\[
Z_0^{2N}(F) = h_0(\epsilon, \tilde{\epsilon}) + \sum_{i=1}^{N} h_{2i}(\epsilon, \tilde{\epsilon})Z_0^{2i}(I)
\] (25)

\[
Z_0^{2N-1}(F) = l_0(\epsilon, \tilde{\epsilon}) + \sum_{i=1}^{N} l_{2i-1}(\epsilon, \tilde{\epsilon})Z_0^{2i-1}(I)
\] (26)

where \( Z_0^{2i}(I) \) and \( Z_0^{2i-1}(I) \) are given by eqns.(15) and (16) respectively. We observe that the quasiperiodic nature of the Fibonacci chain is encoded in the functions \( h(\epsilon, \tilde{\epsilon}) \) and \( l(\epsilon, \tilde{\epsilon}) \). Though for small generations these functions can be derived exactly still we could not find out their general forms.

To circumvent this difficulty we study the recurrence relations among the partition functions of different Fibonacci generations. A survey of different Fibonacci generations depicted by eqn.(17) shows a symmetric pattern in terms of the number of bonds ,viz.,
Recurrence relation among partition functions

Let $P_{n-2}$ be the $(n-2)th$ Fibonacci generation with even number of bonds. This automatically ensures that the previous as well as the next two consecutive generations will have odd number of bonds. The recurrence relation for Fibonacci generations is given by:

$$P_n = P_{n-1}P_{n-2}$$

Now adding a term $D_{n-2}P_{n-3}$ in the above equation gives

$$P_n + D_{n-2}P_{n-3} = P_{n-1}P_{n-2} + D_{n-2}P_{n-3}$$

where $D_{n-2} = Det(P_{n-2})$. The following operations are applied sequentially on eqn.(29):

Substitute $P_{n-2}^{-1}P_{n-1}$ in place of $P_{n-3}$ on the right hand side and finally use Cayley-Hamilton theorem to get the usual trace map relation on the Fibonacci lattice:

$$TrP_n = TrP_{n-1}TrP_{n-2} - D_{n-2}TrP_{n-3}$$

The above equation will be necessary for calculating recurrence relation among different Fibonacci generations. For this purpose we must understand symmetry properties of Fibonacci chain. Inspecting different generations of the Fibonacci chain it reveals that if the total number of bonds $N$ of a particular generation is odd then there is a mirror reflection symmetry around the $\left(\frac{N-1}{2}\right)_{th}$ bond; except the last two bonds. If the special bond around which mirror symmetry occurs is a short (long) one the Fibonacci generation will have equal number of odd and even short (long) bonds. However if the total number of bonds $N$ is even, the mirror reflection symmetry is around a cluster of two successive long bonds at the $\left(\frac{N}{2}\right)_{th}$ and $\left(\frac{N-2}{2}\right)_{th}$ positions of the chain. So “Mirror reflection symmetry” is a characteristic property of a Fibonacci chain.

The $nth$ and $(n \pm 3)th$ generations have the mirror reflection symmetry property around the same kind of bond with last two bonds interchanged, while the $(n \pm 6)th$ generations are topologically same as the $nth$ one.

Using recurrence relation (28) we can write

$$D_{n-2}P_{n-3} = D_{n-2}P_{n-2}^{-1}P_{n-1}$$

Using Cayley-Hamilton theorem on the right hand side of eq.(31) we get:

$$D_{n-2}P_{n-3} = (TrP_{n-2})P_{n-1} - P_{n-2}P_{n-1}$$

Multiplying eq.(32) by P from the right and taking trace we obtain:
\[ D_{n-2} Z_{n-3}^c = (\text{Tr} P_{n-2}) Z_{n-1}^c - \text{Tr}(P_{n-2} P_{n-1} P) \]  
(33)

In a similar fashion we obtain:

\[ D_{n-2} Z_{n-3}^a = (\text{Tr} P_{n-2}) Z_{n-1}^a - \text{Tr}(P_{n-2} P_{n-1} P_0) \]  
(34)

The expression \( P_{n-2} P_{n-1} \) in eqns.(33) and (34) is similar to \( P_n = P_{n-1} P_{n-2} \) with last two bonds interchanged, i.e., both of them have the same mirror symmetric part \( \Omega_n \). Therefore eqns.(33) and (34) can be written as:

\[ D_{n-2} Z_{n-3}^c = Z_{n-1}^c (\text{Tr} P_{n-2}) - \text{Tr}(\Omega_n P \bar{P} P) \]  
(35)

\[ D_{n-2} Z_{n-3}^a = Z_{n-1}^o (\text{Tr} P_{n-2}) - \text{Tr}(\Omega_n P \bar{P} P_0) \]  
(36)

The transfer matrix has the property that \( P = P^T \) and \( \bar{P} = \bar{P}^T \). If such transfer matrices are arranged in a mirror symmetric fashion then the resulting matrix \( (\Omega_n) \) will have the following properties:

i) Off diagonal elements are same i.e., \( (\omega_n)_{12} = (\omega_n)_{21} \)

ii) Diagonal elements are not same but satisfy the condition:

\( (\omega_n)_{11}(p, q) = (\omega_n)_{22}(q, p) \); where \( p = e^{\beta H}, q = e^{-\beta H} \).

Thus the matrix \( \Omega_n \) in eqns. (34) and (35) is of the form:

\[ \Omega_n = \begin{pmatrix} (\omega_n)_{11} & (\omega_n)_{12} \\ (\omega_n)_{21} & (\omega_n)_{22} \end{pmatrix} \]

Eqns.(35) and (36) can be written explicitly in the following way:

\[
D_{n-2} Z_{n-3}^c = Z_{n-1}^c (\text{Tr} P_{n-2}) - r \sqrt{r} [(\omega_n)_{11}(y + \frac{pu}{r}) + (\omega_n)_{12}(u + v + \frac{px + qy}{r})] + (\omega_n)_{22}(x + \frac{qv}{r}) \]
(37)

\[
D_{n-2} Z_{n-3}^a = Z_{n-1}^o (\text{Tr} P_{n-2}) - \sqrt{r} \bar{r} [(\omega_n)_{11}(y + pu) + (\omega_n)_{12}(u + v + px + qy)] + (\omega_n)_{22}(x + qv) \]
(38)

where we have used

\[ P \bar{P} = \sqrt{r} \bar{r} \begin{pmatrix} u & y \\ x & v \end{pmatrix} \]

with

\[ x = \frac{p}{r} + \frac{q}{\bar{r}}, y = \frac{p}{r} + \frac{q}{\bar{r}}, u = 1 + \frac{p^2}{r \bar{r}} \text{ and } v = 1 + \frac{q^2}{r \bar{r}}. \]

Eliminating \( \text{Tr} P_{n-2} \) from eqns.(37) and (38) we have:

\[
\frac{y V_{n-1} + pu V'_{n-1}}{(u + v) V_{n-1} + (px + qy) V'_{n-1}} (\omega_n)_{11} + \frac{x V_{n-1} + qv V'_{n-1}}{(u + v) V_{n-1} + (px + qy) V'_{n-1}} (\omega_n)_{22} = \frac{D_{n-2}}{r \sqrt{r}} \times \frac{Z_{n-3}^c Z_{n-3}^c - Z_{n-3}^o Z_{n-3}^o}{(u + v) V_{n-1} + (px + qy) V'_{n-1}} \]
(39)
where 
\[ V_{n-1} = \frac{Z_n}{\sqrt{r}} - Z_{n-1}^o \] and \[ V'_{n-1} = \frac{Z_n}{\sqrt{r}} - \frac{Z_{n-1}}{r}. \]

To solve for \((\omega_n)_{11}, (\omega_n)_{12}\) and \((\omega_n)_{22}\) we need another two equations. These equations are obtained from the usual formulae:

\[ Z_n^c = Tr(\Omega_n \tilde{P} \tilde{P} P), \quad Z_n^o = Tr(\Omega_n \tilde{P} \tilde{P} P_0). \]

The explicit forms of these two relations are:

\[
\begin{align*}
\frac{x + pu}{u + v + qx + py} (\omega_n)_{11} + \frac{y + qv}{u + v + qx + py} (\omega_n)_{12} + (\omega_n)_{22} &= \frac{Z_n^c}{r} \frac{1}{u + v + qx + py} \\
(\omega_n)_{11} &= \frac{\Gamma_{n-1} - \Gamma_{n-1}'}{r} \frac{1}{\sqrt{rr}} \times \Delta_{n-1} \\
-\frac{1}{\sqrt{rr}} \frac{1}{\Gamma_{n-1} - \Gamma_{n-1}'} \times (\Gamma_{n-1} Z_n^o K' - \Gamma' \frac{Z_n^c}{\sqrt{r}} K) \quad (42)
\end{align*}
\]

\[
\begin{align*}
(\omega_n)_{22} &= -\frac{\Lambda_{n-1} - \Lambda_{n-1}'}{r} \frac{1}{\sqrt{rr}} \times \Delta_{n-1} \\
+\frac{1}{\sqrt{rr}} \frac{1}{\Gamma_{n-1} - \Gamma_{n-1}'} \times (\Gamma_{n-1} Z_n^o K' - \Gamma' \frac{Z_n^c}{\sqrt{r}} K) \quad (43)
\end{align*}
\]

and

\[
(\omega_n)_{12} = \frac{1}{(qx + py)(1 - \frac{1}{r})} \left[ \frac{1}{\sqrt{rr}} \times V_n - (1 - \frac{1}{r}) \left( pu(\omega_n)_{11} + qv(\omega_n)_{22} \right) \right] \quad (44)
\]

where

\[
\begin{align*}
\Lambda_{n-1}(x, y) &= \frac{yV_{n-1} + puV'_{n-1}}{(u + v)V_{n-1} + (px + qy)V'_{n-1}} - \frac{x + pu}{u + v + qx + py} \\
\Lambda'_{n-1}(x, y) &= \frac{yV_{n-1} + puV'_{n-1}}{(u + v)V_{n-1} + (px + qy)V'_{n-1}} - \frac{x + pu}{u + v + qx + py} \\
\Gamma_{n-1}(x, y) &= \frac{xV_{n-1} + qV'_{n-1}}{(u + v)V_{n-1} + (px + qy)V'_{n-1}} - \frac{y + vw}{u + v + qx + py}
\end{align*}
\]
\[
\Gamma'_{n-1}(x, y) = \frac{xV_{n-1} + qyV'_{n-1}}{(u + v)V_{n-1} + (px + qy)V'_{n-1}} - \frac{y + qy}{u + v + qx + py}
\]

(48)

\[
\Delta_{n-1}(x, y) = \frac{Z^o_{n-1}Z^c_{n-3} - Z^c_{n-1}Z^o_{n-3}}{(u + v)V_{n-1} + (px + qy)V'_{n-1}}
\]

(49)

\[
K(x, y) = \frac{1}{u + v + \frac{qx + py}{r}}
\]

(50)

\[
K'(x, y) = \frac{1}{u + v + qx + py}
\]

(51)

Eliminating \(D_{n-2}\) from eqns. (37) and (38) we obtain:

\[
TrP_{n-2} = \frac{r\sqrt{r} \times V_{n-3}}{Z^o_{n-1}Z^c_{n-3} - Z^c_{n-1}Z^o_{n-3}} \left[ \left( \alpha_{xy} + \beta_{xy} V'_{n-3} \right) (\omega_n)_{11}(x, y) + \left( \alpha'_{xy} + \beta'_{xy} V'_{n-3} \right) (\omega_n)_{22}(x, y) \right] + \frac{1}{\sqrt{r\sqrt{r}(1 - \frac{1}{r})(qx + py)}} \left( (u + v) + (px + qy) \frac{V'_{n-3}}{V_{n-3}} \right) V_n
\]

(52)

where

\[
\alpha_{xy} = y - \frac{u + v}{py + qx} \times pu, \quad \beta_{xy} = \frac{(x - y)(q - p)}{py + qx} \times pu
\]

\[
\alpha'_{xy} = x - \frac{u + v}{py + qx} \times qy, \quad \beta'_{xy} = \frac{(x - y)(q - p)}{py + qx} \times qy
\]

In general \(n\)th and \((n + 2)\)th generations have same arrangement of the last two bonds apart from their respective mirror symmetric parts. That is why \(TrP_{n}\) and \(TrP_{n+2}\) will have similar expressions. Since we have assumed \(P_n = \Omega_{n}PP\) it follows from the expression (27) that \(P_{n-3} = \Omega_{n-3}PP\). Therefore proceeding in a similar way as above we get:

\[
TrP_{n-3} = \frac{r\sqrt{r} \times V_{n-4}}{Z^o_{n-2}Z^c_{n-4} - Z^c_{n-2}Z^o_{n-4}} \left[ \left( \alpha_{yx} + \beta_{yx} V'_{n-4} \right) (\omega_{n-1})_{11}(y, x) + \left( \alpha'_{yx} + \beta'_{yx} V'_{n-4} \right) (\omega_{n-1})_{11}(y, x) \right] + \frac{1}{\sqrt{r\sqrt{r}(1 - \frac{1}{r})(px + qy)}} \left( (u + v) + (qy + px) \frac{V'_{n-4}}{V_{n-4}} \right) V_{n-1}
\]

(53)

Using eqns. (52) and (53) and similar expressions for \(TrP_{n}\), \(TrP_{n-1}\) in the trace map relation (30) we obtain the following recurrence relation among partition functions of different Fibonacci generations as:

\[
\frac{V_{n-1}}{Z^o_{n+1}Z^c_{n-1} - Z^c_{n+1}Z^o_{n-1}} \left[ \left( \alpha_{xy} + \beta_{xy} \frac{V'_{n-1}}{V_{n-1}} \right) (\omega_{n+2})_{11}(x, y) + \left( \alpha'_{xy} + \beta'_{xy} \frac{V'_{n-1}}{V_{n-1}} \right) (\omega_{n+2})_{22}(x, y) \right] - \frac{1}{\sqrt{r\sqrt{r}(1 - \frac{1}{r})(qx + py)}} \left( u + v + (px + qy) \frac{V'_{n-1}}{V_{n-1}} \right) V_{n+2} = \frac{r\sqrt{r} \times V_{n-2}}{Z^o_{n+2}Z^c_{n-2} - Z^c_{n+2}Z^o_{n-2}}
\]
\[
\begin{align*}
&\times \frac{V_{n-3}}{Z_{n-1}^o Z_{n-3}^c - Z_{n-1}^c Z_{n-3}^o} \left[ \left( \alpha_{yx} + \beta_{yx} \frac{V'_{n-2}}{V_{n-2}} \right) (\omega_{n+1})_{11}(y,x) + \left( \alpha'_{yx} + \beta'_{yx} \frac{V'_{n-2}}{V_{n-2}} \right) (\omega_{n+1})_{22}(y,x) \\
&\quad - \frac{1}{\sqrt{r(1 - \frac{1}{r})}} \left( u + v + (qx + py) \frac{V'_{n-2}}{V_{n-2}} \right) V_{n+1} \right] \times \left[ \left( \alpha_{xy} + \beta_{xy} \frac{V'_{n-3}}{V_{n-3}} \right) (\omega_n)_{11}(x,y) + \left( \alpha'_{xy} + \beta'_{xy} \frac{V'_{n-3}}{V_{n-3}} \right) (\omega_n)_{22}(x,y) \\
&\quad - \frac{1}{\sqrt{r(1 - \frac{1}{r})}} \left( u + v + (qx + py) \frac{V'_{n-3}}{V_{n-3}} \right) V_n \right] \\
&\quad + \left( \alpha'_{xy} + \beta'_{xy} \frac{V'_{n-3}}{V_{n-3}} \right) (\omega_n)_{22}(x,y) - \frac{1}{\sqrt{r(1 - \frac{1}{r})}} \left( u + v + (qx + py) V'_{n-3} \right) V_n \\
&\quad - \frac{1}{\sqrt{r(1 - \frac{1}{r})}} \left( u + v + (qx + py) V'_{n-4} \right) V_{n-1} \right] \times \left[ \left( \alpha_{xy} + \beta_{xy} \frac{V'_{n-4}}{V_{n-4}} \right) (\omega_{n-1})_{11}(y,x) + \left( \alpha'_{xy} + \beta'_{xy} \frac{V'_{n-4}}{V_{n-4}} \right) (\omega_{n-1})_{22}(y,x) \\
&\quad - \frac{1}{\sqrt{r(1 - \frac{1}{r})}} \left( u + v + (qx + py) \frac{V'_{n-4}}{V_{n-4}} \right) V_{n-1} \right] \times \left[ \left( \alpha_{yx} + \beta_{yx} \frac{V'_{n-4}}{V_{n-4}} \right) (\omega_{n-1})_{11}(x,y) + \left( \alpha'_{yx} + \beta'_{yx} \frac{V'_{n-4}}{V_{n-4}} \right) (\omega_{n-1})_{22}(x,y) \\
&\quad - \frac{1}{\sqrt{r(1 - \frac{1}{r})}} \left( u + v + (qx + py) \frac{V'_{n-4}}{V_{n-4}} \right) V_{n-1} \right]
\end{align*}
\]

The above equation reveals the recurrence relation among the partition functions of different Fibonacci generations from \((n - 4)th\) to \((n + 2)th\). The partition functions have entered in the above equation through the quantities given by equations from (42) to (51). This is in conformity with the symmetry properties of the Fibonacci chain.

**Conclusion**

We have found an exact expression for the partition function of an Ising model on a regular lattice in presence of magnetic field with open boundary conditions. This always includes closed partition functions because of the fact that out of four different spin configurations at the end points of the chain two configurations \(\uparrow\uparrow\) and \(\downarrow\downarrow\) satisfy closed boundary conditions. We have also shown that ”Mirror Symmetry” is a characteristic property of all Fibonacci generations. This is the property which leads to a recurrence relation among the partition functions of different Fibonacci generations from \(nth\) to \((n \pm 6)th\). Since \(nth\) and \((n \pm 6)th\) generations are topologically same, one must go through six times decimation renormalization group procedure to find scaling forms of thermodynamic functions.

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