Investigating quark confinement from the viewpoint of lattice gauge-scalar models

Ryu Ikeda\textsuperscript{a,\dagger} and Kei-Ichi Kondo\textsuperscript{a,b}

\textsuperscript{a}Department of Physics, Graduate School of Science and Engineering, Chiba University, Chiba 263-8522, Japan
\textsuperscript{b}Department of Physics, Graduate School of Science, Chiba University, Chiba 263-8522, Japan

E-mail: cdna0955@chiba-u.jp, kondok@faculty.chiba-u.jp

In this talk, first, we show that the color $N$-dependent area law falloffs of the double-winding Wilson loop averages for the $SU(N)$ lattice gauge model are reproduced from the $Z_N$ lattice Abelian gauge model due to the center group dominance in quark confinement. Next, we discuss lattice gauge-scalar models which allow analytic continuation for gauge invariant operators between confinement region and Higgs region. Applying the cluster expansion, we try to understand non-trivial contribution from scalar field in quark confinement mechanism. In order to understand quark confinement further, moreover, we study double-winding Wilson loop averages in the analytical region on the phase diagram.
1. Introduction

In the lattice gauge theory, a double-winding Wilson loop operator \( W(C_1 \cup C_2) \) has been introduced in [1] to examine the possible mechanisms for quark confinement. The double-winding Wilson loop operator is defined as a trace of the path-ordered product of gauge link variables \( U_\ell \) along a closed loop \( C \) composed of two loops \( C_1 \) and \( C_2 \):

\[
W(C_1 \cup C_2) \equiv \text{tr} \left[ \prod_{\ell \in C_1 \cup C_2} U_\ell \right].
\]  

(1)

The double-winding Wilson loop is called *coplanar* if the two loops \( C_1 \) and \( C_2 \) lie in the same plane, while it is called *shifted* if the two loops \( C_1 \) and \( C_2 \) lie in planes parallel to the \( x-y \) plane, but are displaced from one another in the transverse \( z \)-direction by distance \( R \), and are connected by lines running parallel to the \( z \)-axis to keep the gauge invariance. See Fig. 1. Note that the double-winding Wilson loop operators are defined as a gauge invariant operator.

![Figure 1: (a) a “coplanar” double-winding Wilson loop, (b) a “shifted” double-winding Wilson loop.](image)

The area dependence of the expectation value \( \langle W(C_1 \cup C_2) \rangle \) has been first investigated in [1] to show that the coplanar double-winding Wilson loop average obeys the “difference-of-areas law” in the lattice \( SU(2) \) Yang-Mills model by using the strong coupling expansion and the numerical simulations:

\[
\langle W(C_1 \cup C_2) \rangle_{R=0} \approx \exp[-\sigma (|S_1| - |S_2|)].
\]  

(2)

where \( S_1 \) and \( S_2 \) are respectively the minimal areas bounded by loops \( C_1 \) and \( C_2 \).

In the continuum \( SU(N) \) Yang-Mills model, general multiple-winding Wilson loops have been investigated in [2] to show that there is a novel “max-of-areas law” which is neither difference-of-areas law nor sum-of-areas law for multiple-winding Wilson loop average, provided that the string tension obeys the Casimir scaling for quarks in the higher representations.

In the lattice \( SU(N) \) Yang-Mills model, it has been shown in [3] that the coplanar double-winding Wilson loop average has the \( N \)-dependent area law falloff in the strong coupling region: “difference-of-areas law” for \( N = 2 \), “max-of-areas law” for \( N = 3 \) and “sum-of-areas law” for
$N \geq 4$:

$$
\langle W(C_1 \cup C_2) \rangle_{R=0} = \begin{cases} 
\exp[-\sigma(|S_1| - |S_2|)] & (N = 2) \\
\exp[-\sigma \max(|S_1|, |S_2|)] & (N = 3) \\
\exp[-\sigma(|S_1| + |S_2|)] & (N \geq 4)
\end{cases}.
$$

Moreover, a shifted double-winding Wilson loop average as a function of the distance $R$ in a transverse direction has the long distance behavior which does not depend on $N$, while the short distance behavior depends on $N$.

In our investigation in [4], we examine the center group dominance for a double winding Wilson loop average. It has been shown in [5] that the ordinary single-winding Wilson loop average in the non-Abelian lattice gauge theory with the gauge group $G$ is bounded from above by the same Wilson loop average in the Abelian lattice gauge theory with the center gauge group $Z(G)$:

$$
|\langle W_{R(G)}(C) \rangle_{G} | \leq 2 \text{tr}(1) \langle W_{R(Z(G))}(C) \rangle_{Z(G)} (2 \text{dim}(G) \beta).
$$

We have extended the above statement to the double winding Wilson loop average, beyond the case of the ordinary single-winding Wilson loop average:

$$
|\langle W_{R(G)}(C_1 \cup C_2) \rangle_{G} | \leq 2 \text{tr}(1) \langle W_{R(Z(G))}(C_1 \cup C_2) \rangle_{Z(G)} (2 \text{dim}(G) \beta).
$$

From this point of view, we introduce the character expansion to the weight $\exp^{S_G[U]}$ coming from the action and perform the group integration, in order to estimate the expectation value in the $Z_N$ lattice gauge model. We evaluate the double-winding Wilson loop average up to the leading contribution to show that the $N$-dependent area law falloff in the $SU(N)$ lattice gauge model can be reproduced by using the (Abelian) $Z_N$ lattice gauge model. By taking the limit $N \to \infty$, we show the center group dominance for a double-winding Wilson loop average in the $U(N)$ lattice gauge model through the $U(1)$ lattice gauge model.

Finally, we extend the above arguments for the lattice gauge-scalar model on the “analytic region”. For this purpose, we estimate the area law falloff, the string tension, and the mass gap by using the cluster expansion.

### 2. Lattice $Z_N$ gauge model

First, we consider the lattice $Z_N$ gauge model with the coupling constant defined by $\beta := 1/g^2$ on a $D$-dimensional lattice $\Lambda$ with unit lattice spacing, which is specified by the action

$$
S_G[U] = \beta \sum_{p \in \Lambda} \text{Re} \ U_p , \quad U_p := \prod_{\ell \in \partial p} U_{\ell} ,
$$

where $\ell$ labels a link, $p$ labels an elementary plaquette. To examine this $Z_N$ gauge model analytically, we introduce the character expansion to the weight $\exp^{S_G[U]}$ to obtain the expanded form of the expectation value of an operator $\mathcal{F}$:

$$
\langle \mathcal{F} \rangle_\Lambda := Z_\Lambda^{-1} \int \prod_{\ell \in \Lambda} dU_\ell \, \exp^{S_G[U]} \mathcal{F} = Z_\Lambda^{-1} \int \prod_{\ell \in \Lambda} dU_\ell \, \prod_{p \in \Lambda} \prod_{n=0}^{N-1} b_n(\beta) U_p^n \mathcal{F} ,
$$

$$
Z_\Lambda := \int \prod_{\ell \in \Lambda} dU_\ell \, \exp^{S_G[U]} ,
$$

where $b_n(\beta)$ are the coefficients of the character expansion, and $\mathcal{F}$ is an operator with $\mathcal{F} = 1$.
where the coefficients $b_n(\beta)$ is defined by

$$b_n(\beta) := \frac{1}{N} \sum_{\xi \in \mathbb{Z}_N} \xi^{-n} e^{\beta \text{Re} \, \xi}.$$  \hfill (9)

We define $c_n(\beta) := b_n(\beta)/b_0(\beta)$. For $N = 2, 3, 4$ and $\infty$, $c_1(\beta)$ and $c_2(\beta)$ are written in the form

$$c_1(\beta) = \frac{e^\beta - e^{-\beta}}{e^\beta + e^{-\beta}} \quad (N = 2), \quad c_1(\beta) = \frac{e^\beta - e^{-\beta/2}}{e^\beta + 2e^{-\beta/2}} = c_2(\beta) \quad (N = 3),$$

$$c_1(\beta) = \frac{e^\beta - e^{-\beta}}{e^\beta + 2 + e^{-\beta}}, \quad c_2(\beta) = \frac{e^\beta - 2 + e^{-\beta}}{e^\beta + 2 + e^{-\beta}} \quad (N = 4),$$

$$c_1(\beta) = \frac{I_1(\beta)}{I_0(\beta)}, \quad c_2(\beta) = \frac{I_2(\beta)}{I_0(\beta)} \quad (N = \infty).$$  \hfill (10)

Note that $b_{N-n}(\beta) = b_n(\beta)$ and $0 \leq c_n(\beta) < 1$ for $0 \leq \beta < \infty$. For $N = 2, 3, 4$ and $\infty$, the behavior of $c_1(\beta)$ and $c_2(\beta)$ as functions of $\beta$ are indicated in Fig.2. We find that $c_1(\beta) \sim O(\beta)$ ($N \geq 2$) and $c_2(\beta) \sim O(\beta^2)$ ($N \geq 4$) for $\beta \ll 1$.

![Figure 2: The character expansion coefficient as a function of $\beta$, (a) $c_1(\beta)$, (b) $c_2(\beta)$](image)

Next, we evaluate the expectation value of a coplanar double-winding Wilson loop in the lattice $\mathbb{Z}_N$ pure gauge model. The leading contribution to a coplanar double-winding Wilson loop average is given by the tiling of a planar set of plaquettes, as shown in the Fig.3. (These result are exact for all $\beta$ when $D = 2$, while valid for $\beta \ll 1$ when $D > 2$.)

The result of the coplanar double-winding Wilson loop average up to the leading contribution is given by

$$\langle W(C_1 \cup C_2) \rangle_{K=0} \approx \begin{cases} 
  c_1(\beta)|S_1|^{-|S_2|} & (N = 2) \\
  c_1(\beta)|S_1| & (N = 3) \\
  c_2(\beta)|S_2|c_1(\beta)|S_1|^{-|S_2|} & (N \geq 4)
\end{cases}.$$  \hfill (11)

Then we obtain the (non-zero) string tension from this result:

$$\sigma(\beta) \approx \ln \frac{1}{c_1(\beta)} > 0.$$  \hfill (12)
Investigating quark confinement from the viewpoint of lattice gauge-scalar models

Ryu Ikeda

Figure 3: A coplanar double-winding Wilson loop, (a) $N = 2$ , (b) $N \geq 3$

In the strong coupling region, this result reproduces the area law falloff in the $SU(N)$ lattice gauge model obtained in [3]. Moreover, by taking the continuous group limit $N \rightarrow \infty$, we find that the area law for $N \geq 4$ persists in the $U(1)$ lattice gauge model.

Furthermore, we also evaluate the expectation value of a shifted double-winding Wilson loop in the lattice $Z_N$ pure gauge model. The leading contribution to a shifted double-winding Wilson loop average can be given by the 2 types of tiling by a set of plaquettes, as shown in the Fig. 4.

Figure 4: A shifted double-winding Wilson loop, (a) R-independent contribution, (b) R-dependent contribution

The result of the shifted double-winding Wilson loop average up to the leading contribution is given by

$$
\langle W(C_1 \cup C_2) \rangle_{R \neq 0} \approx \begin{cases} 
    c_1(\beta)^{|S_1|+|S_2|} + c_1(\beta)^2 R(L_2+T) \cdot c_1(\beta)^{|S_1|-|S_2|} & (N = 2) \\
    c_1(\beta)^{|S_1|+|S_2|} + c_1(\beta)^2 R(L_2+T) \cdot c_1(\beta)^{|S_1|} & (N = 3) \\
    c_1(\beta)^{|S_1|+|S_2|} + c_1(\beta)^2 R(L_2+T) \cdot c_2(\beta)^{|S_2|} c_1(\beta)^{|S_1|-|S_2|} & (N \geq 4)
\end{cases}
$$

(13)

This result reproduces the $R$-dependent behavior of the shifted double-winding Wilson loop average in [3]. In particular, we obtain the (non-zero) mass gap from the case of $S_1 = S_2 = 1$ and $R \gg 1$ in the above result:

$$
\Delta(\beta) = 4 \ln \left( \frac{1}{c_1(\beta)} \right) > 0 .
$$

(14)
3. Lattice $Z_N$ gauge-scalar theory

Next, we consider the lattice $Z_N$ gauge-scalar model with the frozen scalar field norm $R$ for simplicity. The action of this model with the coupling constants defined by $\beta := 1/g^2$ and $K := R^2$ on a $D$-dimensional lattice $\Lambda$ with unit lattice spacing is given by

$$S[U, \varphi] = \beta \sum_{p \in \Lambda} \text{Re} U_p + K \sum_{\ell \in \Lambda} \text{Re} (\varphi_x U_{\ell} \varphi^*_{x+\ell}) ,$$

(15)

where $\ell$ labels a link, and $p$ labels an elementary plaquette. $U_\ell$ is a $Z_N$ link variable on link $\ell$ and $\varphi_x$ is a $Z_N$ scalar field at site $x$ which transforms according to the fundamental representation of the gauge group $Z_N$.

In this model, the expectation value of an operator $\mathcal{F}$ has the form

$$\langle \mathcal{F} \rangle_{\Lambda} := Z_{\Lambda}^{-1} \int \prod_{\ell \in \Lambda} dU_\ell \prod_{x \in \Lambda} d\varphi_x \ e^{S[U, \varphi]} \mathcal{F} = Z_{\Lambda}^{-1} \int \prod_{\ell \in \Lambda} dU_\ell \ h[U] \ e^{\beta \sum_{p \in \Lambda} \text{Re} U_p} \mathcal{F} ,$$

$$Z_{\Lambda} := \int \prod_{\ell \in \Lambda} dU_\ell \prod_{x \in \Lambda} d\varphi_x \ e^{S[U, \varphi]} , \quad h[U] := \int \prod_{x \in \Lambda} d\varphi_x \ e^{K \sum_{\ell \in \Lambda} \text{Re} (\varphi_x U_\ell \varphi^*_{x+\ell})} .$$

(16)

According to [6], we can perform the cluster expansion by introducing the new variable $\rho_p$ and the new measure $d\mu_{\Lambda}$ which absorbs the scalar part $h[U]$: 

$$\langle \mathcal{F} \rangle_{\Lambda} = \frac{\int d\mu_{\Lambda} \prod_{p \in \Lambda} (1 + \rho_p) \mathcal{F}}{\int d\mu_{\Lambda} \prod_{p \in \Lambda} (1 + \rho_p)} = \sum_{Q, Q^c} \int d\mu_{\Lambda} \mathcal{F} \prod_{p \in Q(Q^c)} \rho_p \frac{Z_{Q(Q^c) \cup Q^c}}{Z_{\Lambda}} ,$$

(17)

$$d\mu_{\Lambda} := \frac{\prod_{\ell \in \Lambda} dU_\ell \ h[U]}{\prod_{\ell \in \Lambda} dU_\ell \ h[U]} , \quad \rho_p := e^{\beta \text{Re} U_p} - 1 ,$$

(18)

where $Q_0$ is the set of plaquettes which is the support of $\mathcal{F}$ and $Q(Q_0)$ is the set of plaquettes which is connected to $Q_0$. For the general set of plaquettes $Q$, $Q^c$ represents the complement of $Q$. Here, $Z_Q$ is defined by

$$Z_Q := \sum_{Q^c} \int d\mu_{\Lambda} \prod_{p \in Q^c} \rho_p .$$

(19)

Note that $\rho_p \sim O(\beta)$ for $\beta \ll 1$. It has been showed in [7] that the confinement region ($0 \leq \beta \ll 1, K \ll 1$) and the Higgs region ($\beta \gg 1, K_c \ll K < \infty$) are analytically continued in a single "analytic region", where the cluster expansion converges uniformly. See Fig.5.

To evaluate $h[U]$, we apply the character expansion and perform the group integration. Ignoring the contributions from multiple plaquettes, then we obtain the expression which is valid up to the lowest plaquettes order:

$$h[U] = \int \prod_{x \in \Lambda} d\varphi_x \prod_{\ell \in \Lambda} \left[ b_0(K) + b_1(K) \varphi_x U_{\ell} \varphi^*_{x+\ell} + \cdots + b_{N-1}(K) (\varphi_x U_{\ell} \varphi^*_{x+\ell})^{N-1} \right]$$

$$= N^{|\Lambda|} b_0(K)^D |\Lambda| \prod_{p \in \Lambda} \sum_{n=0}^{N-1} c_n(K)^4 t_p^n + \cdots .$$

(20)

We estimate the leading contribution to the double-winding Wilson loop average with the above $h[U]$, we also apply the character expansion for $\rho_p$ and evaluate the upper bound of the cluster.
expansion by using the binominal expansion. We find that there is an correspondence between the
evaluation for the $Z_N$ lattice gauge model and for the estimated upper bound for the $Z_N$ lattice
gauge-scalar model:

$$
c_n(\beta) = \frac{\left[b_0(\beta) - e^\beta\right] c_n(K)^4 + b_1(\beta)c_{n+1}(K)^4 + \cdots + b_{N-1}(\beta)c_{N+n-1}(K)^4}{b_0(\beta) + b_1(\beta)c_1(K)^4 + \cdots + b_{N-1}(\beta)c_{N-1}(K)^4 + c_n(K)^4} \pmod{N, n = 1, \cdots, N-1}
$$

(21)

Note that $a_n(\beta, 0) = c_n(\beta)$ and $a_n(\beta, \infty) = 1$. The above estimation is valid only for the values of parameter $\beta$ and $K$ on the analytic region in the range where the string breaking does not occur.

By applying the same method as the above, we obtain the estimation for the coplanar double-winding Wilson loop average:

$$
\langle W(C_1 \cup C_2) \rangle_{R=0} \leq \begin{cases} 
    a_1(\beta, K)^{|S_1|-|S_2|} & (N = 2) \\
    a_1(\beta, K)^{|S_1|} & (N = 3) \\
    a_2(\beta, K)^{|S_2|}a_1(\beta, K)^{|S_1|-|S_2|} & (N \geq 4)
\end{cases}
$$

(22)

and we obtain the (non-zero) string tension from the above result:

$$
\sigma(\beta, K) \geq \ln \frac{1}{a_1(\beta, K)} > 0.
$$

(23)

This estimation suggests that the area law falloff in the $Z_N$ lattice gauge model persists in the
$Z_N$ lattice gauge-scalar model and the $K \to 0$ limit agrees with the pure gauge case. Moreover,
for $\sigma(\beta, K)$, the $K \to 0$ limit agrees with $\sigma(\beta)$ in the $Z_N$ lattice gauge model, and $K \to \infty$ limit converges to 0 uniformly in $\beta$. In other words, the string tension is non-zero on the analytic region.

Additionally, we also estimate the shifted double-winding Wilson loop average:

$$
\langle W(C_1 \cup C_2) \rangle_{R=0} \leq \begin{cases} 
    a_1(\beta, K)^{|S_1|+|S_2|} + a_1(\beta, K)^{2R(L_2+T)} \cdot a_1(\beta, K)^{|S_1|-|S_2|} & (N = 2) \\
    a_1(\beta, K)^{|S_1|+|S_2|} + a_1(\beta, K)^{2R(L_2+T)} \cdot a_1(\beta, K)^{|S_1|} & (N = 3) \\
    a_1(\beta, K)^{|S_1|+|S_2|} + a_1(\beta, K)^{2R(L_2+T)} \cdot a_2(\beta, K)^{|S_2|}a_1(\beta, K)^{|S_1|-|S_2|} & (N \geq 4)
\end{cases}
$$

(24)
and we obtain the (non-zero) mass gap from the case of $S_1 = S_2 = 1$ and $R \gg 1$ in the above result:

$$\Delta(\beta, K) \gtrsim 4 \ln \frac{1}{a_1(\beta, K)} > 0.$$  \hspace{1cm} (25)

For $\Delta(\beta, K)$, the $K \to 0$ limit agrees with $\Delta(\beta)$ in the $Z_N$ lattice gauge model, and $K \to \infty$ limit converges to 0 uniformly in $\beta$. In other words, the mass gap is non-zero on the analytic region.

4. Conclusion

We investigated the area law falloff of the double-winding Wilson loops in the $Z_N$ lattice gauge model and $Z_N$ lattice gauge-scalar model, where the gauge group is the center group of the original $SU(N)$. First, we evaluated the $N$-dependent area law falloff for the coplanar double-winding Wilson loop average up to the leading contribution. We found the $N$-dependence of the area law falloff in the $Z_N$ lattice gauge model, which reproduces the area law falloff in the $SU(N)$ lattice gauge model obtained in [3]. Secondly, we also checked the limit $N \to \infty$, the area law falloff for $N \geq 4$ persists in the $U(1)$ lattice gauge model. This result implies that the coplanar double-winding Wilson loop average in the $U(N)$ lattice gauge model and the $SU(N)(N \geq 4)$ lattice gauge model obeys the same area law up to the leading contribution. Furthermore, we also considered the shifted double-winding Wilson loop average up to the leading contributions. This result reproduces the $R$-dependent behavior in the $SU(N)$ lattice gauge model obtained in [3]. We obtained the (non-zero) mass gap $\Delta(\beta)$ from this result. Finally, we extended the above study for the $Z_N$ lattice gauge-scalar model on the analytic region. We found that the area law falloff in the $Z_N$ lattice gauge model persists in the $Z_N$ lattice gauge-scalar model. We discovered that the string tension and the mass gap are non-zero on the analytic region from this estimation.

References

[1] J. Greensite and R. Höllwieser, Phys. Rev. D\textbf{91}, 054509 (2015) [arXiv:1411.5091 [hep-lat]]

[2] R. Matsudo and K.-I. Kondo, Phys. Rev. D\textbf{96}, 105011 (2017) [arXiv:1706.05665 [hep-th]]

[3] S. Kato, A. Shibata and K.-I. Kondo, Phys.Rev. D\textbf{102}, 094521 (2020) [arXiv:2008.03684 [hep-lat]]

[4] R. Ikeda and K.-I. Kondo, Prog. Theor. Exp. Phys. \textbf{2021}, ptab114 (2021) [arXiv:2106.14416 [hep-lat]]

[5] J. Fröhlich, Phys. Lett. \textbf{83B}, 195 (1979).

[6] K. Osterwalder and E. Seiler, Annals Phys. \textbf{110}, 440 (1978).

[7] E.H. Fradkin and S.H. Shenker, Phys.Rev. D\textbf{19}, 3682–3697 (1979).