HEEKAARD SPLITTINGS OF SUFFICIENTLY COMPLICATED 3-MANIFOLDS II: AMALGAMATION

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Abstract. We show that when two unstabilized, ∂-unstabilized Heegaard splittings are amalgamated by a "sufficiently complicated" map, the resulting splitting is unstabilized. We also show that in a 3-manifold formed by a sufficiently complicated gluing, a low genus, unstabilized Heegaard splitting can be expressed in a unique way as an amalgamation over the gluing surface.

1. Introduction.

Given an analogous result about connected sums [Bac08], [SQ], it is natural to conjecture that amalgamations of unstabilized Heegaard splittings are unstabilized. That is, if \(M_1\) and \(M_2\) are glued along a surface \(F\) of non-zero genus, and \(H_i\) is an unstabilized Heegaard surface in \(M_i\), then \(H_1\) and \(H_2\) can be amalgamated in \(M_1 \cup M_2\) to an unstabilized Heegaard surface. Unfortunately, even if the genus of \(F\) is one, then Schultens and Weidmann have shown this to be false [SW].

Here we show that if we assume the gluing map between \(M_1\) and \(M_2\) is "sufficiently complicated," then the desired result holds. Our main result is the following:

**Theorem 8.2.** Let \(M_1\) and \(M_2\) be compact, orientable, irreducible 3-manifolds with incompressible boundary, neither of which is an I-bundle. Let \(M\) denote the manifold obtained by gluing some component \(F\) of \(\partial M_1\) to some component of \(\partial M_2\) by some homeomorphism \(\phi\). Let \(H_i\) be an unstabilized, boundary-unstabilized Heegaard splitting of \(M_i\). If \(\phi\) is "sufficiently complicated" then the amalgamation of \(H_1\) and \(H_2\) in \(M\) is unstabilized.

As a corollary to this result we construct an example whose existence has been conjectured by Yoav Moriah: a non-minimal genus Heegaard splitting which has Hempel distance [Hem01] exactly one. Moriah has called the search for such examples the “nemesis of Heegaard splittings” [Mor]. In fact, in Corollary 8.3 we go further and produce manifolds that have an arbitrarily large number of such splittings.

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Finally, we prove the following theorem, which is also analogous to a result about connected sums established by the author in [Bac08].

**Theorem 9.1.** Let \( M_1 \) and \( M_2 \) be compact, orientable, irreducible 3-manifolds with incompressible boundary, neither of which is an \( I \)-bundle. Let \( M \) denote the manifold obtained by gluing some component \( F \) of \( \partial M_1 \) to some component of \( \partial M_2 \) by some homeomorphism \( \phi \). Let \( H_i \) and \( G_i \) be unstabilized, boundary-unstabilized Heegaard splittings of \( M_i \). If \( \phi \) is "sufficiently complicated" and the amalgamation of \( H_1 \) and \( H_2 \) in \( M \) is isotopic to the amalgamation of \( G_1 \) and \( G_2 \), then \( H_i \) is isotopic to \( G_i \) for \( i = 1, 2 \).

Theorems 8.2 and 9.1 together give a complete picture of the “low genus” Heegaard splittings of a 3-manifold constructed by gluing two manifolds \( M_1 \) and \( M_2 \) together by a “sufficiently complicated” map, \( \phi \). Every unstabilized, low genus Heegard splitting of \( M_1 \cup_\phi M_2 \) is an amalgamation of “component” unstabilized splittings of \( M_1 \) and \( M_2 \). Two such splittings are isotopic if and only if their component splittings are isotopic. Hence, the unstabilized splittings of \( M_1 \cup_\phi M_2 \) are completely determined by the unstabilized splittings of \( M_1 \) and \( M_2 \).

This paper is organized as follows. In Sections 2 through 6 we review the definitions and results given in [Bac08]. These include critical surfaces, Generalized Heegaard splittings (GHSs), and Sequences of GHSs (SOGs). In Section 7 we review the main result from [Baca], which says that complicated amalgamations act as barriers to low genus incompressible, strongly irreducible, and critical surfaces. Everything in Sections 2 through 7 is contained the same numbered sections of [Bacb]. Those who have read those sections in [Bacb] can skip them here. In Sections 8 and 9 we use all of this machinery to give the proofs of Theorems 8.2 and 9.1.

## 2. Incompressible, Strongly Irreducible, and Critical Surfaces

**Definition 2.1.** Let \( F \) be a properly embedded surface in \( M \). Let \( \gamma \) be a loop in \( F \). \( \gamma \) is *essential* on \( F \) if it is a loop that does not bound a disk in \( F \). A *compression* for \( F \) is a disk, \( D \), such that \( D \cap F = \partial D \) is essential on \( F \).

**Definition 2.2.** Let \( F \) be a properly embedded surface in \( M \). The surface \( F \) is *compressible* if there is a compression for it. Otherwise it is *incompressible*. 
Definition 2.3. Let $H$ be a separating, properly embedded surface in $M$. Let $V$ and $W$ be compressions on opposite sides of $H$. Then we say $(V, W)$ is a weak reducing pair for $H$ if $V \cap W = \emptyset$.

Definition 2.4. Let $H$ be a separating, properly embedded surface in $M$ which is not a torus. Then we say $H$ is strongly irreducible if there are compressions on opposite sides of $H$, but no weak reducing pairs.

Definition 2.5. Let $H$ be a properly embedded, separating surface in $M$. The surface $H$ is critical if the compressions for $H$ can be partitioned into sets $C_0$ and $C_1$ such that:

1. For each $i = 0, 1$ there is at least one pair of disks $V_i, W_i \in C_i$ such that $(V_i, W_i)$ is a weak reducing pair.
2. If $V \in C_0$ and $W \in C_1$ then $(V, W)$ is not a weak reducing pair.

3. Generalized Heegaard Splittings

Definition 3.1. A compression body $C$ is a manifold formed in one of the following two ways:

1. Starting with a 0-handle, attach some number of 1-handles. In this case we say $\partial_- C = \emptyset$ and $\partial_+ C = \partial C$.
2. Start with some (possibly disconnected) surface $F$ such that each component has positive genus. Form the product $F \times I$. Then attach some number of 1-handles to $F \times \{1\}$. We say $\partial_- C = F \times \{0\}$ and $\partial_+ C$ is the rest of $\partial C$.

Definition 3.2. Let $H$ be a properly embedded, transversally oriented surface in a 3-manifold $M$, and suppose $H$ separates $M$ into $V$ and $W$. If $V$ and $W$ are compression bodies and $V \cap W = \partial_+ V = \partial_+ W = H$, then we say $H$ is a Heegaard surface in $M$.

Definition 3.3. The transverse orientation on the Heegaard surface $H$ in the previous definition is given by a choice of normal vector. If this vector points into $V$, then we say any subset of $V$ is above $H$ and any subset of $W$ is below $H$.

Definition 3.4. Suppose $H$ is a Heegaard splitting of a manifold $M$ with non-empty boundary. Let $F$ denote a component of $\partial M$. Then the surface $H'$ obtained from $H$ by attaching a copy of $F$ to it by an unknotted tube is also a Heegaard surface in $M$. We say $H'$ was obtained from $H$ by a boundary-stabilization along $F$. The reverse operation is called a boundary-destabilization along $F$. 
Definition 3.5. A generalized Heegaard splitting (GHS) $H$ of a 3-manifold $M$ is a pair of sets of transversally oriented, connected, properly embedded surfaces, $\text{Thick}(H)$ and $\text{Thin}(H)$ (called the thick levels and thin levels, respectively), which satisfy the following conditions.

1. Each component $M'$ of $M \setminus \text{Thin}(H)$ meets a unique element $H_+ \in \text{Thick}(H)$. The surface $H_+$ is a Heegaard surface in $M'$ dividing $M'$ into compression bodies $\mathcal{V}$ and $\mathcal{W}$. Each component of $\partial_- \mathcal{V}$ and $\partial_- \mathcal{W}$ is an element of $\text{Thin}(H)$. Henceforth we will denote the closure of the component of $M \setminus \text{Thin}(H)$ that contains an element $H_+ \in \text{Thick}(H)$ as $M(H_+)$. 

2. Suppose $H_- \in \text{Thin}(H)$. Let $M(H_+)$ and $M(H_+')$ be the submanifolds on each side of $H_-$. Then $H_-$ is below $H_+$ in $M(H_+)$ if and only if it is above $H_+'$ in $M(H_+').$

3. The term “above” extends to a partial ordering on the elements of $\text{Thin}(H)$ defined as follows. If $H_-$ and $H_-'$ are subsets of $\partial M(H_+)$, where $H_-$ is above $H_+$ in $M(H_+)$ and $H_-'$ is below $H_+$ in $M(H_+)$, then $H_-$ is above $H_-'$ in $M$.

Definition 3.6. Suppose $H$ is a GHS of an irreducible 3-manifold $M$. Then $H$ is strongly irreducible if each element $H_+ \in \text{Thick}(H)$ is strongly irreducible in $M(H_+)$. The GHS $H$ is critical if each element $H_+ \in \text{Thick}(H)$ but exactly one is strongly irreducible in $M(H_+)$, and the remaining element is critical in $M(H_+)$. 

The strongly irreducible case of the following result is due to Scharlemann and Thompson [ST94]. The proof in the critical case is similar.

Theorem 3.7. ([Bac08], Lemma 4.6) Suppose $H$ is a strongly irreducible or critical GHS of an irreducible 3-manifold $M$. Then each thin level of $H$ is incompressible.

4. Reducing GHSs

Definition 4.1. Let $H$ be an embedded surface in $M$. Let $D$ be a compression for $H$. Let $\mathcal{V}$ denote the closure of the component of $M \setminus H$ that contains $D$. (If $H$ is non-separating then $\mathcal{V}$ is the manifold obtained from $M$ by cutting open along $H$.) Let $N$ denote a regular neighborhood of $D$ in $\mathcal{V}$. To surger or compress $H$ along $D$ is to remove $N \cap H$ from $H$ and replace it with the frontier of $N$ in $\mathcal{V}$. We denote the resulting surface by $H/D$.

It is not difficult to find a complexity for surfaces which decreases under compression. Incompressible surfaces then represent “local minima” with respect to this complexity. We now present an operation
that one can perform on GHSs that also reduces some complexity (See Lemma 5.14 of [Bac08]). Strongly irreducible GHSs will then represent “local minima” with respect to such a complexity. This operation is called weak reduction.

**Definition 4.2.** Let $H$ be a properly embedded surface in $M$. If $(D, E)$ is a weak reducing pair for $H$, then we let $H/DE$ denote the result of simultaneous surgery along $D$ and $E$.

**Definition 4.3.** Let $M$ be a compact, connected, orientable 3-manifold. Let $G$ be a GHS. Let $(D, E)$ be a weak reducing pair for some $G_+ \in \text{Thick}(G)$. Define

$$T(H) = \text{Thick}(G) - \{G_+\} \cup \{G_+/D, G_+/E\},$$

$$t(H) = \text{Thin}(G) \cup \{G_+/DE\}.$$ 

A new GHS $H = \{\text{Thick}(H), \text{Thin}(H)\}$ is then obtained from $\{T(H), t(H)\}$ by successively removing the following:

1. Any sphere element $S$ of $T(H)$ or $t(H)$ that is inessential, along with any elements of $t(H)$ and $T(H)$ that lie in the ball that it (co)bounds.
2. Any element $S$ of $T(H)$ or $t(H)$ that is $\partial$-parallel, along with any elements of $t(H)$ and $T(H)$ that lie between $S$ and $\partial M$.
3. Any elements $H_+ \in T(H)$ and $H_- \in t(H)$, where $H_+$ and $H_-$ cobound a submanifold $P$ of $M$, such that $P$ is a product, $P \cap T(H) = H_+$, and $P \cap t(H) = H_-$. 

We say the GHS $H$ is obtained from $G$ by weak reduction along $(D, E)$.

The first step in weak reduction is illustrated in Figure 1.

![Figure 1. The first step in weak reduction.](image)

**Definition 4.4.** The weak reduction of a GHS given by the weak reducing pair $(D, E)$ for the thick level $G_+$ is called a destabilization if $G_+/DE$ contains a sphere.
We will see later that GHSs have a coarse measure of complexity called genus. Destabilizations are precisely those weak reductions that reduce genus.

5. Amalgamations

Let $H$ be a GHS of a connected 3-manifold $M$. In this section we use $H$ to produce a complex that is the spine of a Heegaard splitting of $M$. We call this splitting the amalgamation of $H$. Most of this material is reproduced from [Bac08]. First, we must introduce some new notation.

**Definition 5.1.** Let $H$ be a Heegaard surface in $M$. Let $Σ$ denote a properly embedded graph in $M$. Let $\partial M'$ denote the union of the boundary components of $M$ that meet $Σ$. Then we say $\partial M' ∪ Σ$ is a spine of $H$ if the frontier of a neighborhood of $\partial M' ∪ Σ$ is isotopic to $H$.

Suppose $H$ is a GHS of $M$ and $H_+ ∈ \text{Thick}(H)$. Recall that $H_+$ is transversely oriented, so that we may consistently talk about those points of $M(H_+)$ that are “above” $H_+$ and those points that are “below.” The surface $H_+$ divides $M(H_+)$ into two compression bodies. Henceforth we will denote these compression bodies as $V(H_+)$ and $W(H_+)$, where $V(H_+)$ is below $H_+$ and $W(H_+)$ is above. When we wish to make reference to an arbitrary compression body which lies above or below some thick level we will use the notation $V$ and $W$. Define $\partial_- M(H_+)$ to be $\partial_- V(H_+)$ and $\partial_+ M(H_+)$ to be $\partial_- W(H_+)$. That is, $\partial_- M(H_+)$ and $\partial_+ M(H_+)$ are the boundary components of $M(H_+)$ that are below and above $H_+$, respectively. If $N$ is a union of manifolds of the form $M(H_i)$ for some set of thick levels $\{H_i\} ⊂ \text{Thick}(H)$ then we let $\partial_± N$ denote the union of those boundary components of $N$ that are components of $\partial_± M(H_i)$, for some $i$.

We now define a sequence of manifolds $\{M_i\}$ where

$$M_0 \subset M_1 \subset ... \subset M_n = M.$$ 

The submanifold $M_0$ is defined to be the disjoint union of all manifolds of the form $M(H_+)$, such that $\partial_- M(H_+) ⊂ \partial M$. The fact that the thin levels of $H$ are partially ordered guarantees $M_0 \neq ∅$. Now, for each $i$ we define $M_i$ to be the union of $M_{i-1}$ and all manifolds $M(H_+)$ such that $\partial_- M(H_+) ⊂ \partial M_{i-1} ∪ \partial M$. Again, it follows from the partial ordering of thin levels that for some $i$ the manifold $M_i = M$.

We now define a sequence of complexes $Σ_i$ in $M$. The final element of this sequence will be a complex $Σ$. This complex will be a spine of the desired Heegaard surface. The intersection of $Σ$ with some $M(H_+)$ is depicted in Figure 2.
Each \( \mathcal{V} \subset M_0 \) is a compression-body. Choose a spine of each, and let \( \Sigma_0' \) denote the union of these spines. The complement of \( \Sigma_0' \) in \( M_0 \) is a (disconnected) compression body, homeomorphic to the union of the compression bodies \( \mathcal{W} \subset M_0 \). Now let \( \Sigma_0 \) be the union of \( \Sigma_0' \) and one vertical arc for each component \( H_- \) of \( \partial_+ M_0 \), connecting \( H_- \) to \( \Sigma_0' \).

We now assume \( \Sigma_{i-1} \) has been constructed and we construct \( \Sigma_i \). Let \( M'_i = M_i - M_{i-1} \). For each compression body \( \mathcal{V} \subset M'_i \) choose a set of arcs \( \Gamma \subset \mathcal{V} \) such that \( \partial \Gamma \subset \Sigma_{i-1} \cap \partial M_{i-1} \), and such that the complement of \( \Gamma \) in \( \mathcal{V} \) is a product. Let \( \Sigma'_i \) be the union of \( \Sigma_{i-1} \) with all such arcs \( \Gamma \), and all components of \( \partial_- \mathcal{V} \) that are contained in \( \partial M \). Now let \( \Sigma_i \) be the union of \( \Sigma'_i \) and one vertical arc for each component \( H_- \) of \( \partial_+ M_i \), connecting \( H_- \) to \( \Sigma'_i \).

**Figure 2.** The intersection of \( \Sigma \) with \( \mathcal{V}(H_+) \) and \( \mathcal{W}(H_+) \).

**Lemma 5.2.** ([Bac08], Lemma 7.2) If \( H \) is a GHS of \( M \) then the complex \( \Sigma \) defined above is the spine of a Heegaard splitting of \( M \).

**Definition 5.3.** Let \( H \) be a GHS and \( \Sigma \) be the complex in \( M \) defined above. The Heegaard splitting that \( \Sigma \) is a spine of is called the amalgamation of \( H \) and will be denoted \( \mathcal{A}(H) \).

Note that although the construction of the complex \( \Sigma \) involved some choices, its neighborhood is uniquely defined up to isotopy at each stage. Hence, the amalgamation of a GHS is well defined, up to isotopy.

For the next lemma, recall the definition of destabilization, given in Definition 4.4.

**Lemma 5.4.** ([Bac08], Corollary 7.5) Suppose \( M \) is irreducible, \( H \) is a GHS of \( M \) and \( G \) is obtained from \( H \) by a weak reduction which is not a destabilization. Then \( \mathcal{A}(H) \) is isotopic to \( \mathcal{A}(G) \).
It follows that if a GHS $G$ is obtained from a GHS $H$ by a weak reduction or a destabilization then the genus of $A(G)$ is at most the genus of $A(H)$.

**Definition 5.5.** The *genus* of a GHS is the genus of its amalgamation.

**Definition 5.6.** Suppose $H$ is a GHS of $M$. Let $N$ denote a submanifold of $M$ bounded by elements of $\text{Thin}(H)$. Then we may define a GHS $H(N)$ of $N$. The thick and thin levels of $H(N)$ are the thick and thin levels of $H$ that lie in $N$.

### 6. Sequences of GHSs

**Definition 6.1.** A *Sequence Of GHSs* (SOG), $\{H^i\}$ of $M$ is a finite sequence such that for each $i$ either $H^i$ or $H^{i+1}$ is obtained from the other by a weak reduction.

**Definition 6.2.** If $H$ is a SOG and $k$ is such that $H^{k-1}$ and $H^{k+1}$ are obtained from $H^k$ by a weak reduction then we say the GHS $H^k$ is *maximal* in $H$.

It follows that maximal GHSs are larger than their immediate predecessor and immediate successor.

Just as there are ways to make a GHS “smaller”, there are also ways to make a SOG “smaller”. These are called *SOG reductions*, and are explicitly defined in Section 8 of [Bac08]. If the first and last GHS of a SOG are strongly irreducible and there are no SOG reductions then the SOG is said to be *irreducible*. For our purposes, all we need to know about SOG reduction is that the maximal GHSs of the new SOG are obtained from the maximal GHSs of the old one by weak reduction, and the following lemma holds:

**Lemma 6.3.** ([Bac08], Lemma 8.9) Every maximal GHS of an irreducible SOG is critical.

**Definition 6.4.** The *genus* of a SOG is the maximum among the genera of its GHSs.

**Lemma 6.5.** If a SOG $\Lambda$ is obtained from an SOG $\Gamma$ by a reduction then the genus of $\Gamma$ is at least the genus of $\Lambda$.

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7. Barrier surfaces

We begin this section with a brief description of the complexity of a gluing map, as defined in [Baca]. Let $M$ be a compact, irreducible, (possibly disconnected) 3-manifold with incompressible boundary, such that no component of $M$ is an $I$-bundle. Suppose boundary components $F_1$ and $F_2$ of $M$ are homeomorphic. Let $M_\phi$ be the manifold obtained from $M$ by gluing these boundary components together by the map $\phi : F_1 \to F_2$.

Let $Q$ denote a properly embedded (possibly disconnected) surface in $M$ of maximal Euler characteristic, which is both incompressible and $\partial$-incompressible, and is incident to both $F_1$ and $F_2$. Then we define the distance of $\phi$ to be the distance between the loops of $\phi(F_1 \cap Q)$ and $F_2 \cap Q$. When the genus of $F_2$ is at least two, then this distance is measured in the curve complex of $F_2$. If $F_2 \cong T^2$, then this distance is measured in the Farey graph.

We are now prepared to state the main result of [Baca].

**Theorem 7.1.** [Baca] Let $F$ denote the image of $F_1$ in $M_\phi$. There is a constant $K$, depending linearly on $\chi(Q)$, such that if the distance of $\phi \geq Kg$, then any incompressible, strongly irreducible, or critical surface $H$ in $M_\phi$ of genus at most $g$ can be isotoped to be disjoint from $F$.

This theorem motivates us to make the following definition:

**Definition 7.2.** An incompressible surface $F$ in a 3-manifold $M$ is a $g$-barrier surface if any incompressible, strongly irreducible, or critical surface in $M$ whose genus is at most $g$ can be isotoped to be disjoint from $F$.

By employing Theorem 7.1 we may construct 3-manifolds with any number of $g$-barrier surfaces. Simply begin with a collection of 3-manifolds and successively glue boundary components together by “sufficiently complicated” maps.

**Lemma 7.3.** Let $M$ be a (possibly disconnected) 3-manifold which has a $g$-barrier surface $F$. Let $H$ be a genus $g$ strongly irreducible or critical GHS of $M$. Then $F$ is isotopic to a thin level of $H$.

**Proof.** Since the genus of $H$ is $g$, it follows that the genus of every thick and thin level of $H$ is at most $g$. By Theorem 3.7 we know that each thin level of $H$ is incompressible. Since $F$ is a $g$-barrier surface, it can be isotoped to be disjoint from every thin level. But then $F$ is contained in $M(H_+)$, for some thick level, $H_+$. The surface $H_+$ is either strongly irreducible or critical, so again since $F$ is a $g$-barrier surface
it may be isotoped to be disjoint from $H_+$. The surface $F$ can thus be isotoped into a compression body, $C$. But every incompressible surface in $C$ is parallel to some component of $\partial_-C$. Each such component is a thin level of $H$. □

**Lemma 7.4.** Let $M$ be a (possibly disconnected) 3-manifold which has a $g$-barrier surface $F$. Let $H$ be a genus $g$ irreducible SOG of $M$. Then $F$ is isotopic to a thin level of every element of $H$.

**Proof.** By Lemma 6.3 each maximal GHS of $H$ is critical. Hence, by Lemma 7.3 $F$ is isotopic to a thin level of every maximal GHS of $H$. But every other GHS of $H$ is obtained from a maximal GHS by a sequence of weak reductions and destabilizations. Such moves may create new thin levels, but will never destroy an incompressible thin level. Hence, $F$ is isotopic to a thin level of every element of $H$. □

8. **Irreducible Heegaard splittings of amalgamated 3-manifolds.**

We now establish a result analogous to a conjecture of Cameron Gordon. This conjecture asserts that the connected sum of unstabilized Heegaard splittings is unstabilized. This was proved by the author in [Bac08], and by Scharlemann in Qiu in [SQ]. The higher genus analogue would be:

**Conjecture 8.1.** Let $M_1$ and $M_2$ denote irreducible, orientable 3-manifolds with homeomorphic, incompressible boundary. Let $M$ be the 3-manifold obtained from $M_1$ and $M_2$ by gluing their boundaries by some homeomorphism. If $H_i$ is an unstabilized Heegaard splitting of $M_i$ then the amalgamation of $H_1$ and $H_2$ in $M$ is unstabilized.

Unfortunately, without any assumptions on the gluing map, Schultens and Weidmann have shown this conjecture to be false [SW]. Hence, to prove the conjecture we will at least need to add the hypothesis that the gluing map is “sufficiently complicated”. But here is another thing that can go wrong.

Let $M_1$ be a compression body with genus $g$ and genus 1 boundary components. Let $H_1$ be the splitting you get by tubing copies of these boundary components together. Then $H_1$ is a genus $g + 1$ Heegaard splitting of $M_1$ that has both a genus $g$ and a genus 1 boundary component on the same side. It must therefore be irreducible.

Let $M_2$ be a manifold with a genus 1 boundary component, and a $\gamma$-primitive Heegaard splitting (see [MS04]). Such a Heegaard splitting
is unstabilized, but has the property that boundary-stabilizing it produces a stabilized splitting. Then no matter how we glue $M_1$ to $M_2$, the amalgamation of $H_1$ and $H_2$ will be stabilized.

Given this example, and those of Schultens and Weidmann, we deduce the following: In order for the conclusion of Conjecture 8.1 to be true, we would at least have to know that $H_1$ and $H_2$ are not stabilized, not boundary-stabilized, and that the gluing map is “sufficiently complicated.” In the next theorem we show that these hypotheses are enough to obtain the desired result.

**Theorem 8.2.** Let $M_1$ and $M_2$ be compact, orientable, irreducible 3-manifolds with incompressible boundary, neither of which is an $I$-bundle. Let $M$ denote the manifold obtained by gluing some component $F$ of $\partial M_1$ to some component of $\partial M_2$ by some homeomorphism $\phi$. Let $H_i$ be an unstabilized, boundary-unstabilized Heegaard splitting of $M_i$. If $\phi$ is “sufficiently complicated” then the amalgamation of $H_1$ and $H_2$ in $M$ is unstabilized.

Here the term “sufficiently complicated” means that the distance of $\phi$ is high enough so that by Theorem 7.1 $F$ becomes a $g$-barrier surface, where $g = \text{genus}(H_1) + \text{genus}(H_2) - \text{genus}(F)$.

**Proof.** Let $\Gamma$ be the SOG depicted in Figure 3. The second GHS pictured is the GHS whose thick levels are $H_1$ and $H_2$. The first GHS in the figure is obtained from this one by a maximal sequence of weak reductions, and is hence strongly irreducible. The third GHS is the amalgamation $H$ of $H_1$ and $H_2$. The next GHS pictured is obtained from $H$ by some number of destabilizations. Finally, the last GHS is obtained from the second to last by a maximal sequence of weak reductions. Note that by construction, $\text{genus}(\Gamma) = \text{genus}(H)$. By Lemma 5.4 this is the same as the genus of the second GHS in Figure 3. It follows that $\text{genus}(H) = g$ (see, for example, Lemma 5.7 of [Bac8]).

Now let $\Lambda = \{\Lambda^i\}_{i=1}^n$ be the SOG obtained from $\Gamma$ by a maximal sequence of SOG reductions. When the first and last GHS of a SOG are strongly irreducible, then they remain unaffected by SOG reduction. Hence, $\Lambda^1$ is the first element of $\Gamma$ and $\Lambda^n$ is the last element of $\Gamma$.

By Lemma 7.4 the surface $F$ is isotopic to a thin level of every GHS of $\Lambda$. Let $m$ denote the largest number such that $F$ is isotopic to a unique thin level $F_i$ of $\Lambda^i$, for all $i \leq m$. The surface $F_i$ then divides $M$ into manifolds $M_1^i$ and $M_2^i$, homeomorphic to $M_1$ and $M_2$, for each $i \leq m$.

Now note that there are no stabilizations in the original SOG $\Gamma$. It thus follows from Lemma 8.12 of [Bac08] that the first destabilization
in \( \Lambda \) happens before the first stabilization. Furthermore, as the genus of \( \Lambda^n \) is less than the genus of \( \Lambda^1 \), there is at least one destabilization in \( \Lambda \). Let \( p \) denote the smallest value for which \( \Lambda^{p+1} \) is obtained from \( \Lambda^p \) by a destabilization. Then for all \( i \leq p \), either \( \Lambda^i \) or \( \Lambda^{i-1} \) is obtained from the other by a weak reduction that is not a destabilization.

If \( p \leq m \), then for all \( i \leq p \), either \( \Lambda^i(M^i) = \Lambda^{i-1}(M^{i-1}) \) or one of \( \Lambda^i(M^i) \) and \( \Lambda^{i-1}(M^{i-1}) \) is obtained from the other by a weak reduction that is not a destabilization. It follows from Lemma 5.4 that \( \widehat{M^i_1} = \mathcal{A}(\Lambda^i(M^i)) \) is the same for all \( i \leq p \). But \( \widehat{M^1_1} = H_1 \), so \( \widehat{M^p_1} = H_1 \).

By identical reasoning \( \widehat{M^p_2} = \mathcal{A}(\Lambda^p(M^p)) = H_2 \). But \( H_1 \) and \( H_2 \) are unstabilized, so neither \( \widehat{M^p_1} \) nor \( \widehat{M^p_2} \) can be obtained from \( \widehat{M^p_1} \) or \( \widehat{M^p_2} \) by destabilization, a contradiction.

We thus conclude \( p > m \), and thus \( \widehat{M^m_1} = H_1 \) and \( \widehat{M^m_2} = H_2 \). In particular, it follows that \( m \) is strictly less than \( n \). That is, there exists a GHS \( \Lambda^{m+1} \) which has two thin levels isotopic to \( F \).

Since \( \Lambda^{m+1} \) has a thin level that is not a thin level of \( \Lambda^m \), it must be obtained from \( \Lambda^m \) by a weak reduction. It follows that there is some thin level \( F_{m+1} \) of \( \Lambda^{m+1} \) that is identical to \( F_m \). The other thin level of \( \Lambda^{m+1} \) that is isotopic to \( F \) we call \( F'_m \). The surface \( F'_{m+1} \) either lies in \( M^m \) or \( M^n \). Assume the former. Let \( M^{m+1}_1 \) denote the side of \( F_{m+1} \) homeomorphic to \( M_1 \). It follows that \( \Lambda^{m+1}(M^{m+1}_1) \) is obtained from \( \Lambda^m(M^m) \) by a weak reduction. Thus, by Lemma 5.4,

\[
\widehat{M^{m+1}_1} = \mathcal{A}(\Lambda^{m+1}(M^{m+1}_1)) = \mathcal{A}(\Lambda^m(M^m)) = \widehat{M^m_1} = H_1.
\]

The surfaces \( F_{m+1} \) and \( F'_{m+1} \) cobound a product region \( P \) of \( M \). A GHS of \( P \) is given by \( \Lambda^{m+1}(P) \), and thus \( \widehat{P} = \mathcal{A}(\Lambda^{m+1}(P)) \) is a

**Figure 3.** The initial SOG, \( \Gamma \).
Heegaard splitting of a product. This splitting is obtained from some strongly irreducible splitting of $P$ by some number of stabilizations. If this number of stabilization is non-zero, then $\hat{M}_1^{m+1}$ would be stabilized. But since $\hat{M}_1^{m+1} = H_1$, and $H_1$ is unstabilized, this is not the case.

We conclude $\hat{P}$ is a strongly irreducible Heegaard splitting of $P$, and hence $\hat{P}$ is the unique thick level of $\Lambda^{m+1}(P)$. From [ST93] this splitting is either a copy of $F$, or two copies of $F$ connected by a single unknotted tube. In the former case we have a contradiction, as the thick level of $\Lambda^{m+1}(P)$ would be parallel to the two thin levels $F_{m+1}$ and $F'_{m+1}$, and would thus have been removed during weak reduction. In the latter case $\hat{M}_1^{m+1}$ is boundary-stabilized. As this Heegaard splitting is $H_1$, which is not boundary-stabilized, we again have a contradiction.

**Corollary 8.3.** There exists manifolds that contain arbitrarily many non-minimal genus, unstabilized Heegaard splittings which are not strongly irreducible.

**Proof.** Let $M$ denote a 3-manifold with torus boundary, and strongly irreducible Heegaard splittings of arbitrarily high genus. (Such an example has been constructed by Casson and Gordon. See [Sed97]. The manifold they construct is closed, but there is a solid torus that is a core of one of the handlebodies bounded by each Heegaard surface. Thus, removing this solid torus produces a manifold with torus boundary that has arbitrarily high genus strongly irreducible Heegaard splittings.)

Now let $M_1$ and $M_2$ be two copies of $M$, and let $H^i_g$ denote a genus $g$ strongly irreducible splitting in $M_i$. As $H^i_g$ is strongly irreducible, it is neither stabilized nor boundary-stabilized. Hence, if $M_1$ is glued to $M_2$ by a sufficiently complicated homeomorphism, it follows from Theorem 8.2 that the amalgamation of $H^1_g$ and $H^2_g$ is unstabilized, for all $g \leq G$. (One can make $G$ as high as desired, by gluing $M_1$ to $M_2$ by more and more complicated maps.)

Finally, note that every amalgamation is weakly reducible. $\Box$

9. ISOTOPIC HEegaRd splittings in amalgamated 3-manifolds.

In [Bac08] we prove that the connected sum of non-isotopic Heegaard splittings is non-isotopic. Here we prove the analogous, higher genus result.

**Theorem 9.1.** Let $M_1$ and $M_2$ be compact, orientable, irreducible 3-manifolds with incompressible boundary, neither of which is an I-bundle. Let $M$ denote the manifold obtained by gluing some component
$F$ of $\partial M_1$ to some component of $\partial M_2$ by some homeomorphism $\phi$. Let $H_i$ and $G_i$ be unstabilized, boundary-unstabilized Heegaard splittings of $M_i$. If $\phi$ is “sufficiently complicated” and the amalgamation of $H_1$ and $H_2$ in $M$ is isotopic to the amalgamation of $G_1$ and $G_2$, then $H_i$ is isotopic to $G_i$ for $i = 1, 2$.

As in Theorem 8.2, the term “sufficiently complicated” means that the distance of $\phi$ is high enough so that by Theorem 7.1 $F$ becomes a $g$-barrier surface, where $g = \text{genus}(H_1) + \text{genus}(H_2) - \text{genus}(F)$.

**Proof.** Let $\Gamma$ be the SOG depicted in Figure 4. The second GHS pictured is the GHS whose thick levels are $H_1$ and $H_2$. The first GHS in the figure is obtained from this one by a maximal sequence of weak reductions, and is hence strongly irreducible. The third GHS is the amalgamation of $H_1$ and $H_2$. The fourth GHS is the one whose thick levels are $G_1$ and $G_2$. By assumption, the third GHS is also the amalgamation of the fourth GHS. Finally, the last GHS is obtained from the fourth one by a maximal sequence weak reductions.

Figure 4. The initial SOG, $\Gamma$.

Now let $\Lambda = \{\Lambda^i\}_{i=1}^n$ be the SOG obtained from $\Gamma$ by a maximal sequence of SOG reductions. Note that $\Lambda^1$ is the first element of $\Gamma$ and $\Lambda^n$ is the last element of $\Gamma$.

By Lemma 7.4 the surface $F$ is isotopic to a thin level of every GHS of $\Lambda$. If we assume there is a destabilization in $\Lambda$, then the argument given in the proof of Theorem 8.2 provides a contradiction. Similarly, if, for some $i$, we assume the surface $F$ is isotopic to two elements of Thin($\Lambda^i$), then the argument given in the proof of Theorem 8.2 provides a contradiction.
We conclude, then, that for each $i$ either $\Lambda^i$ or $\Lambda^{i+1}$ is obtained from the other by a weak reduction that is not a destabilization. Furthermore, since for all $i$ the surface $F$ is isotopic to a unique thin level of $\Lambda^i$, it follows that for each $i$, $M_1(\Lambda^i) = M_1(\Lambda^{i+1})$, or either $M_1(\Lambda^i)$ or $M_1(\Lambda^{i+1})$ is obtained from the other by a weak reduction that is not a destabilization. It thus follows from Lemma 5.4 that for each $i$ the surface $A(M_1(\Lambda^i))$ is the same (up to isotopy). But $A(M_1(\Lambda^1)) = H_1$ and $A(M_1(\Lambda^n)) = G_1$. Hence, $H_1$ is isotopic to $G_1$. A symmetric argument shows $H_2$ must be isotopic to $G_2$, completing the proof. □

Our picture of a 3-manifold constructed by gluing two component manifolds $M_1$ and $M_2$ together by a “sufficiently complicated” map $\phi$ is now complete. Every unstabilized “low genus” Heegaard splitting of $M \cup_\phi M_2$ is an amalgamation of “component” unstabilized, boundary-unstabilized splittings of $M_1$ and $M_2$. Two such splittings are isotopic if and only if their component splittings are isotopic. In other words, there is a one-to-one correspondence between the set of pairs of low genus unstabilized, boundary-unstabilized Heegaard splittings $(H_1, H_2)$, where $H_i \subset M_i$, and the set of unstabilized Heegaard splittings of $M_1 \cup_\phi M_2$. Precisely how low “low genus” is depends on how complicated the map $\phi$ is.

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