VERY BADLY APPROXIMABLE MATRIX FUNCTIONS

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Abstract. We study in this paper very badly approximable matrix functions on the unit circle $\mathbb{T}$, i.e., matrix functions $\Phi$ such that the zero function is a superoptimal approximation of $\Phi$. The purpose of this paper is to obtain a characterization of the continuous very badly approximable functions.

Our characterization is more geometric than algebraic characterizations earlier obtained in [PY1] and [AP]. It involves analyticity of certain families of subspaces defined in terms of Schmidt vectors of the matrices $\Phi(\zeta)$, $\zeta \in \mathbb{T}$. This characterization can be extended to the wider class of admissible functions, i.e., the class of matrix functions $\Phi$ such that the essential norm $\|H_\Phi\|_e$ of the Hankel operator $H_\Phi$ is less than the smallest nonzero superoptimal singular value of $\Phi$.

In the final section we obtain a similar characterization of badly approximable matrix functions.

1. Introduction

A well-known classical result in complex analysis says that any bounded measurable function $\varphi$ on the unit circle $\mathbb{T}$ has a best uniform approximation by bounded analytic functions, i.e., there exists a function $f \in H^\infty$ such that

$$\|\varphi - f\|_\infty = \text{dist}_{L^\infty}(\varphi, H^\infty) = \inf_{h \in H^\infty} \|\varphi - h\|_\infty.$$ 

It is even more remarkable, that in many cases the best approximation $f$ is unique. For example, this is true if $\varphi$ is continuous on $\mathbb{T}$; this was first proved for the first time in [Kh] and was rediscovered later by several other mathematicians.

A function $\varphi \in L^\infty$ is called badly approximable if

$$\|\varphi\|_\infty = \text{dist}_{L^\infty}(\varphi, H^\infty),$$

i.e., if its norm cannot be reduced by subtracting an $H^\infty$ function. Another way to describe badly approximable functions is to say that any such function is the difference between a function and its best approximation in $H^\infty$.

There is an elegant characterization of the set of continuous badly approximable functions: a nonzero continuous function $\varphi \neq 0$ on the unit circle $\mathbb{T}$ is badly approximable if and only if it has constant modulus and its winding number $\text{wind} \varphi$.

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is negative (see [AAK], [Po]). Recall that the winding number of a continuous function \( \varphi : \mathbb{T} \to \mathbb{C} \setminus \{0\} \), is the number of turns of the point \( \varphi(e^{it}) \) around the origin when \( t \) runs from 0 to \( 2\pi \) (see, e.g., [Pe], Ch. 3, §3).

This characterization can be extended to broader classes of functions, for which the winding number is not defined. For such functions the result can be stated in terms of Hankel and Toeplitz operators.

It is well known (see e.g., [D]) that if \( \varphi \in C(\mathbb{T}) \) and \( \varphi \) does not vanish on \( \mathbb{T} \), then the Toeplitz operator \( T_\varphi \) on the Hardy class \( H^2 \) is Fredholm and \( \text{ind} \, T_\varphi = -\text{wind} \, \varphi \) (recall that for a Fredholm operator \( A \), its index is defined as \( \dim \text{Ker} \, A - \dim \text{Ker} \, A^* \)). The above characterization of badly approximable functions can be easily generalized in the following way: if \( \varphi \in L^\infty \) such that the essential norm \( \|H_\varphi\|_e \) of the Hankel operator \( H_\varphi \) is less than its norm, then \( \varphi \) is badly approximable if and only if \( \varphi \) has constant modulus almost everywhere on \( \mathbb{T} \), \( T_\varphi \) is Fredholm, and \( \text{ind} \, T_\varphi > 0 \) (see e.g., [Pe], Ch. 7, §5). Recall that the Toeplitz operator \( T_\varphi : H^2 \to H^2 \) and the Hankel operator \( H_\varphi : H^2 \to H^2 \) are defined by

\[
T_\varphi f = \mathbb{P}_+ \varphi f, \quad H_\varphi f = \mathbb{P}_- \varphi f,
\]

where \( \mathbb{P}_- \) and \( \mathbb{P}_+ \) are the orthogonal projections onto the subspaces \( H^2 \) and \( H^2_\varphi \).

Recall also that

\[
\|H_\varphi\| = \text{dist}_{L^\infty}(\varphi, H^\infty) \quad \text{and} \quad \|H_\varphi\|_e = \text{dist}_{L^\infty}(\varphi, H^\infty + C)
\]

(see, e.g. [Pe]).

1.1. Badly approximable matrix functions. In this paper we deal with matrix-valued functions. The notion of a badly approximable matrix function can be defined in a similar way. A matrix function \( \Phi \) with values in the space \( \mathbb{M}_{m,n} \) of \( m \times n \) matrices is called badly approximable if

\[
\|\Phi\|_{L^\infty} = \inf \{\|\Phi - F\|_{L^\infty} : F \in H^\infty(\mathbb{M}_{m,n})\}.
\]

Here

\[
\|\Phi\|_{L^\infty} \overset{\text{def}}{=} \text{ess sup}_{\zeta \in \mathbb{T}} \|\Phi(\zeta)\|_{\mathbb{M}_{m,n}},
\]

\( \mathbb{M}_{m,n} \) is equipped with the standard operator norm, and \( H^\infty(\mathbb{M}_{m,n}) \) is the space of bounded analytic functions with values in \( \mathbb{M}_{m,n} \).

While it is possible (and it is done in this paper) to describe badly approximable matrix-functions, the problem does not look very natural. The main reason is, that even for continuous matrix-valued functions a best \( L^\infty \) approximation by analytic matrix functions is almost never unique. For example, suppose that \( m = n = 2 \) and suppose that \( u \) is a scalar badly approximable unimodular function (i.e., \( |u(\zeta)| = 1 \) almost everywhere on \( \mathbb{T} \)). Consider now the matrix function \( \Phi = \begin{pmatrix} u & 0 \\ 0 & 0 \end{pmatrix} \). It is easy to see that for any scalar function \( f \) in the unit ball of \( H^\infty \), the matrix
function \( \begin{pmatrix} 0 & 0 \\ 0 & f \end{pmatrix} \) is a best approximation of \( \Phi \). Clearly, if \( \psi \) is an arbitrary scalar function in the unit ball of \( L^\infty \), then the matrix function \( \begin{pmatrix} u & 0 \\ 0 & \psi \end{pmatrix} \) is badly approximable. However, \( \psi \) can as “bad” as possible.

The problem of describing all badly approximable functions such that \( 0 \) is the unique best approximation looks slightly more natural. This problem is also solved in this paper, see Theorem 6.2 below. But the most natural problem appears when one considers the approximation method that gives a unique “very best” approximation (for continuous matrix-valued functions).

Thus in our opinion, in the case of matrix functions it is most natural to consider the notion of very badly approximable matrix functions, which was introduced in [PY1]. To define a very badly approximable matrix function, we need the notion of superoptimal approximation (see [PY1]).

1.2. Superoptimal approximations and very badly approximable matrix functions. Recall that for a matrix \( A \) the singular value \( s_j(A) \), \( j \geq 0 \), is, by definition, the distance from \( A \) to the set of matrices of rank at most \( j \). Clearly, \( s_0(A) = \|A\| \).

**Definition.** Given a matrix function \( \Phi \in L^\infty(\mathbb{M}_{m,n}) \) we define inductively the sets \( \mathcal{O}_j \), \( 0 \leq j \leq \min\{m,n\} - 1 \), by

\[
\mathcal{O}_0 = \{ F \in H^\infty(\mathbb{M}_{m,n}) : F \text{ minimizes } t_0 \text{ def } = \sup_{\zeta \in T} \|\Phi(\zeta) - F(\zeta)\| \};
\]

\[
\mathcal{O}_j = \{ F \in \mathcal{O}_{j-1} : F \text{ minimizes } t_j \text{ def } = \sup_{\zeta \in T} s_j(\Phi(\zeta) - F(\zeta)) \}, \quad j > 0.
\]

Functions in \( \bigcap_{k \geq 0} \mathcal{O}_k = \mathcal{O}_{\min\{m,n\} - 1} \) are called superoptimal approximations of \( \Phi \) by bounded analytic matrix functions. The numbers \( t_j = t_j(\Phi) \) are called the superoptimal singular values of \( \Phi \). Note that the functions in \( \mathcal{O}_0 \) are just the best approximations by analytic matrix functions.

A matrix function \( \Phi \) is called very badly approximable if the zero function is a superoptimal approximation of \( \Phi \). Again, a very badly approximable function can be interpreted as the difference between a function and its superoptimal approximation.

1.3. Some known results. The notion of superoptimal approximation seems very natural for the approximation theory of matrix-valued functions, for the superoptimal approximation is unique for continuous functions: it was designed to have uniqueness! Namely, it was shown in [PY1] that if \( \Phi \in (H^\infty + C)(\mathbb{M}_{m,n}) \) (i.e., all entries of \( \Phi \) belong to \( H^\infty + C \)), then \( \Phi \) has a unique superoptimal approximation \( Q \) by bounded analytic matrix functions. Moreover, it was shown in
that
\[ s_j(\Phi(\zeta) - Q(\zeta)) = t_j(\Phi) \quad \text{for almost all} \quad \zeta \in \mathbb{T}. \]  

(1.2)

The problem to describe the very badly approximable functions was posed in [PY1]. It follows from (1.2) that if \( \Phi \) is a very badly approximable function in \((H^\infty + C')(\mathbb{M}_{m,n})\), then the singular values \( s_j(\Phi(\zeta)) \) are constant for almost all \( \zeta \in \mathbb{T} \). Moreover, it was shown in [PY1] that if in addition to this \( m \leq n \) and \( s_{m-1}(\Phi(\zeta)) \neq 0 \) almost everywhere, then the Toeplitz operator \( T_{z\Phi} : H^2(\mathbb{C}^n) \to H^2(\mathbb{C}^m) \) has dense range (if \( \Phi \) is a scalar function, the last condition is equivalent to the fact that \( \text{ind} \ T_{\Phi} > 0 \)). Note that the Toeplitz and the Hankel operators whose symbols are matrix functions can be defined in the same way as in the scalar case (see [PY1]). Obviously, this necessary condition is equivalent to the condition \( \text{Ker} \ T_{z\Phi^*} = \{ 0 \} \). In fact, the proof of necessity given in [PY1] allows one to obtain a more general result: if \( \Phi \) is an arbitrary very badly approximable function in \((H^\infty + C')(\mathbb{M}_{m,n})\) and \( f \in \text{Ker} \ T_{z\Phi^*} \), then \( \Phi^* f = 0 \).

On the other hand, in [PY1] an example of a continuous \( 2 \times 2 \) function \( \Phi \) was given such that \( s_0(\Phi(\zeta)) = 1 \), \( s_1(\Phi(\zeta)) = \alpha < 1 \), \( \zeta \in \mathbb{T} \), \( T_{z\Phi} \) is invertible but \( \Phi \) is not even badly approximable.

The very badly approximable matrix functions of class \((H^\infty + C')(\mathbb{M}_{m,n})\) were characterized in [PY1] algebraically, in terms of so-called thematic factorizations. Later in [PT] the above results of [PY1] were generalized to the broader context of matrix functions \( \Phi \) such that the essential norm \( \| H_\Phi \|_e \) of the Hankel operator \( H_\Phi \) is less than the smallest nonzero superoptimal singular value of \( \Phi \). We call such matrix functions \( \Phi \) admissible. In particular, if \( \Phi \) is an admissible very badly approximable \( m \times n \) matrix function, then the functions \( s_j(\Phi(z)) \) are constant almost everywhere on \( \mathbb{T} \) and

\[ \text{Ker} \ T_{z\Phi^*} = \{ f \in H^2(\mathbb{C}^n) : \Phi^* f = 0 \}. \]

In [AP] another algebraic characterization of the set of very badly approximable admissible matrix functions was given in terms of canonical factorizations (see §2 for the definition).

We refer the reader to the book [Pe], which contains all the above information and results on superoptimal approximation and very badly approximable functions.

1.4. What is done in the paper. Although a complete description (necessary and sufficient condition) of very badly approximable matrix functions was obtained in [PY1] and [AP], this description is rather complicated: it says that a function is very badly approximable if and only if it admits some special factorization. While such characterizations are very helpful for constructing very badly approximable functions, it is not easy to check, using such characterizations, that a function is very badly approximable.
The main result of the paper is Theorem 4.1 in §4, which gives another description of admissible very badly approximable matrix-functions. In particular, it gives a complete description of the very badly approximable matrix-functions with entries in \( H^\infty + C \). This description is more geometric and closer in spirit to the scalar result stated at the beginning of this paper than the algebraic characterizations obtained in [PY1] or [AP].

Note, that the result is new and highly nontrivial even for continuous functions. The main difficulty is to understand the structure of very badly approximable functions, not to extend the results to a wider class of functions.

The paper is organized as follows: In §3 we find a new necessary condition for an admissible matrix functions to be very badly approximable. It involves analyticity of certain families of subspaces. However, we will see in §3 that if we add this analyticity condition to the above two necessary conditions, the three conditions will still remain insufficient.

In §4 we slightly modify this necessary conditions to obtain a description of the very badly approximable admissible matrix functions. In §5 we give a new approach to this problem that is based on the notion of a superoptimal weight.

Finally, in §6 we obtain a characterization of the set badly approximable matrix functions \( \Phi \) satisfying the condition \( \|H\Phi\|_e < \|\Phi\|_{L^\infty} \) and we obtain a characterization of badly approximable matrix functions, for which \( 0 \) is the unique best approximation.

In §2 we define canonical factorizations and state several results we are going to use in §3 and later to establish the main result of the paper.

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1.6. Notation. Throughout this paper we use the following notation:

- \( I_n \) is the identity matrix of size \( n \times n \);
- \( I_n \) is the matrix function on \( \mathbb{T} \) equal to \( I_n \) almost everywhere;
- \( 0 \) denotes a scalar or matrix function on \( \mathbb{T} \) that is equal to zero almost everywhere;
- \( 1 \) is the scalar function identically equal to 1.
- \( z \) denotes the identical function: \( z(\zeta) = \zeta, \zeta \in \mathbb{T} \).

2. Preliminaries

To define canonical factorizations, we need the notion of balanced unitary-valued functions. Recall that a matrix function \( G \in H^\infty(\mathbb{M}_{m,n}) \) is called \textit{inner} if \( G^*G = I_n \). A matrix function \( G \in H^\infty(\mathbb{M}_{m,n}) \) is called \textit{outer} if \( GH^2(\mathbb{C}^n) \) is dense in \( H^2(\mathbb{C}^m) \). Finally, \( G \in H^\infty(\mathbb{M}_{m,n}) \) is called \textit{co-outer} if the transposed function \( G^t \in H^\infty(\mathbb{M}_{n,m}) \) is outer.
It is easy to deduce from the definition of co-outer functions that if \( G \) is a co-outer function in \( H^\infty(M_{m,n}) \) and \( f \) is a function in \( L^2(\mathbb{C}^n) \) such that \( Gf \in H^2(\mathbb{C}^m) \), then \( f \in H^2(\mathbb{C}^n) \) (see e.g., [2], Ch. 14, §1).

By the Beurling–Lax–Halmos theorem (see e.g., [N]), a nonzero subspace \( L \) of \( H^2(\mathbb{C}^n) \) is invariant under multiplication by \( z \) if and only if \( L = \Upsilon H^2(\mathbb{C}^r) \), where \( 1 \leq r \leq n \) and \( \Upsilon \) is an inner \( n \times r \) matrix function. It is easy to see that

\[
    r = \dim \{ f(\zeta) : f \in L \} \quad \text{for almost all} \quad \zeta \quad \text{in the unit disk} \quad \mathbb{D}. \tag{2.1}
\]

**Definition.** Let \( n \) be a positive integer and let \( r \) be an integer such that \( 0 < r < n \). Suppose that \( \Upsilon \) is an \( n \times r \) inner and co-outer matrix function and \( \Theta \) is an \( n \times (n - r) \) inner and co-outer matrix function. If the matrix function

\[
    \mathcal{V} = (\begin{pmatrix} \Upsilon & \Theta \end{pmatrix}^{\dagger})
\]

is unitary-valued, it is called an \( r \)-**balanced matrix function**. If \( r = 0 \) or \( r = n \), it is natural to say that an \( r \)-balanced matrix is a constant unitary matrix. An \( n \times n \) matrix function \( V \) is called **balanced** if it is \( r \)-balanced for some \( r \), \( 0 \leq r \leq n \). 1-balanced matrix functions are also called **thematic**.

It is well known (see [V]) that each inner and co-outer matrix function \( \Upsilon \) has a balanced completion \( (\begin{pmatrix} \Upsilon & \Theta \end{pmatrix}^{\dagger}) \).

The following result was obtained in [AP].

**Theorem A.** Let \( \mathcal{V} \) be a balanced matrix function. Then the Toeplitz operators \( T_\mathcal{V} \) and \( T_{\mathcal{V}^t} \) have trivial kernel and dense range.

We also need the following fact from [AP].

**Theorem B.** Suppose that \( \Phi \in L^\infty(M_{m,n}) \) and \( \| H_\Phi \|_e < \| H_\Phi \| \). Let \( \mathcal{L} \) be the minimal invariant subspace of multiplication by \( z \) on \( H^2(\mathbb{C}^n) \) that contains all maximizing vectors of \( H_\Phi \). Then

\[
    \mathcal{L} = \Upsilon H^2(\mathbb{C}^r),
\]

where \( r \) is the number of superoptimal singular values of \( \Phi \) equal to \( \| H_\Phi \| \) and \( \Upsilon \) is an inner and co-outer \( n \times r \) matrix function.

If we apply Theorem A to the transposed function \( \Phi^t \), we find an \( m \times r \) inner and co-outer matrix function \( \Theta \) such that the invariant subspace of multiplication by \( z \) on \( H^2(\mathbb{C}^m) \) spanned by all maximizing vectors of \( H_{\Phi^t} \) coincides with \( \Theta H^2(\mathbb{C}^r) \).

Consider now balanced completions \( (\begin{pmatrix} \Upsilon & \Theta \end{pmatrix}^{\dagger}) \) and \( (\begin{pmatrix} \Theta & \Xi \end{pmatrix}^{\dagger}) \) of \( \Upsilon \) and \( \Theta \) and define the unitary-valued functions \( \mathcal{V} \) and \( \mathcal{W} \) by

\[
    \mathcal{V} = (\begin{pmatrix} \Upsilon & \Theta \end{pmatrix}^{\dagger}), \quad \mathcal{W} = (\begin{pmatrix} \Theta & \Xi \end{pmatrix}^{\dagger}).
\]

**Theorem C.** Under the hypotheses of Theorem A the matrix functions \( \Upsilon, \Theta, \Theta, \Xi \) are left invertible in \( H^\infty \).
Recall that a matrix function $\Phi \in H^\infty(M_{m,n})$ is said to be left invertible in $H^\infty$ if there exists $\Psi \in H^\infty(M_{n,m})$ such that $\Psi \Phi = I_n$. Theorem C was established in [AP], see also [PT] where it was proved in the case when $V$ and $W^*$ are 1-balanced.

The following result can also be found in [AP].

**Theorem D.** Suppose that $\Phi \in L^\infty(M_{m,n})$ and $\|H\Phi\|_e < \|H\Phi\|$. Then $\Phi$ is badly approximable if and only if it admits a factorization

$$\Phi = W^*(\sigma U \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Psi) V^*$$

where $V$ and $W^*$ are $r$-balanced matrix functions, $r$ is the number of superoptimal singular values of $\Phi$ equal to $\|\Phi\|_{L^\infty}$, $\sigma = \|\Phi\|_{L^\infty}$, $U$ is an $r \times r$ very badly approximable unitary-valued function such that $\|H_U\|_e < 1$, and $\Psi$ is an $(m-r) \times (n-r)$ matrix function such that $\|\Psi\|_{L^\infty} \leq \sigma$, $\|H_\Psi\| < \sigma$, and $\|H_\Psi\|_e \leq \|H_\Phi\|_e$.

Moreover, $\Phi$ is very badly approximable if and only if $\Psi$ is very badly approximable.

**Remark 1.** If $m = r$ or $n = r$, by $\begin{pmatrix} \sigma U & 0 \\ 0 & \Psi \end{pmatrix}$ we mean $\begin{pmatrix} \sigma U & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} \sigma U \\ 0 \end{pmatrix}$ respectively, in which case $\Phi$ is very badly approximable if and only if $\Phi$ is badly approximable.

Such factorizations are a special case of partial canonical factorizations. Partial canonical factorizations in the general case are defined in [AP].

**Remark 2.** Actually, if $\Phi$ admits a factorization as above, then $\Phi$ must be badly approximable even without the assumption $\|H_\Phi\|_e < \|H_\Phi\|$.

**Remark 3.** Note that if $U$ is a very badly approximable unitary-valued function such that $\|H_U\|_e < 1$, then the Toeplitz operator $T_U$ is Fredholm, see [AP].

Let us now define a canonical factorization. Let $\sigma_0, \ldots, \sigma_{i-1}$ be all distinct nonzero superoptimal singular values of $\Phi$. Suppose that $d_j$ is the multiplicity of the superoptimal singular value $\sigma_j$ of $\Phi$. A canonical factorization of $\Phi$ is a representation of $\Phi$ of the form

$$\Phi = \mathcal{W}_0^* \cdots \mathcal{W}_{i-1}^* \begin{pmatrix} \sigma_0 U_0 & 0 & \cdots & 0 & 0 \\ 0 & \sigma_1 U_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \sigma_{i-1} U_{i-1} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \mathcal{V}_{i-1}^* \cdots \mathcal{V}_0^*, \quad (2.2)$$

where the $U_j$ are $d_j \times d_j$ unitary-valued very badly approximable matrix functions such that $\|H_{U_j}\|_e < 1$, the matrix functions $\mathcal{V}_j$ and $\mathcal{W}_j$, $1 \leq j \leq i-1$, have the
form
\[ V_j = \begin{pmatrix} I_{d_0 + \cdots + d_{r-1}} & 0 \\ 0 & \tilde{V}_j \end{pmatrix} \quad \text{and} \quad W_j = \begin{pmatrix} I_{d_0 + \cdots + d_{r-1}} & 0 \\ 0 & \tilde{W}_j \end{pmatrix}, \quad 1 \leq j \leq \iota - 1, \]

\( V_0 \) and \( W_0 \) are \( d_0 \)-balanced matrix functions and \( \tilde{V}_j \) and \( \tilde{W}_j \) are \( d_j \)-balanced matrix functions. Note that the last zero row has size \((m - (d_0 + \cdots + d_{\iota-1})) \times n\). If \( m = d_0 + \cdots + d_{\iota-1} \), this means that there is no zero row in (2.2). A similar remark can be made about the last zero column in (2.2).

It was shown in [AP] that an admissible matrix function \( \Phi \) is very badly approximable if and only if it admits a canonical factorization. Again, if \( \Phi \) is an arbitrary bounded matrix function (not necessarily admissible) that admits a canonical factorization, then \( \Phi \) must be very badly approximable.

Finally, we need the following result from [AP].

**Theorem E.** Let \( U \) be a unitary-valued matrix function such that \( \| H_U \|_e < 1 \). Then \( U \) is very badly approximable if and only if the Toeplitz operator \( T_{zU^*} \) has trivial kernel.

Note that all the above results can be found in Chapter 14 of the book [Pe].

### 3. Analytic Families of Subspaces

In this section we are going to state one more necessary condition for an admissible matrix function to be very badly approximable. This condition involves analyticity of certain families of subspaces.

Let \( \Phi \) be a matrix function in \( L^\infty(M_{m,n}) \) and let \( \sigma > 0 \). For \( \zeta \in \mathbb{T} \) we denote by \( \mathcal{S}_\Phi^{(\sigma)}(\zeta) \) the linear span of all Schmidt vectors \(^1\) of \( \Phi(\zeta) \) that correspond to the singular values of \( \Phi(\zeta) \) that are greater than or equal to \( \sigma \). The subspaces \( \mathcal{S}_\Phi^{(\sigma)}(\zeta) \) are defined for almost all \( \zeta \in \mathbb{T} \).

As we have mentioned in the introduction, in [PY2] an example of a continuous \( 2 \times 2 \) matrix function \( \Phi \) was given such that \( T_{z\Phi} \) is invertible, \( s_0(\Phi(\zeta)) = 1 \), \( s_1(\Phi(\zeta)) = \alpha \), \( \alpha \in (0, 1) \), but \( \Phi \) is not badly approximable. If we look at the subspace of maximizing vectors of \( \Phi(\zeta) \), \( \zeta \in \mathbb{T} \), in that example, we can easily observe that the family of subspaces \( \mathcal{S}_\Phi^{(1)}(\zeta) \), \( \zeta \in \mathbb{T} \), is not analytic in the following sense.

**Definition.** Let \( \mathcal{L}_n \) be the set of all subspaces of \( \mathbb{C}^n \). A family of subspaces \( \mathcal{L}(\zeta) \), \( \zeta \in \mathbb{T} \), defined for almost all \( \zeta \in \mathbb{T} \) is called **analytic** if there exist functions

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\(^1\)Recall that if \( A \) is an \( m \times n \) matrix and \( s \) is a singular value of \( A \), a nonzero vector \( x \in \mathbb{C}^n \) is called a **Schmidt vector** corresponding to \( s \) if \( A^*Ax = s^2x \).
ξ_1, \cdots, ξ_k in H^2(\mathbb{C}^n) such that L(ζ) = \text{span}\{ξ_j(ζ) : 1 \leq j \leq k\} for almost all ζ ∈ T.

**Remark 1.** It is easy to see that if L(ζ), ζ ∈ T, is an analytic family of subspaces, then there exists r ∈ \mathbb{Z}_+ such that dim L(ζ) = r everywhere on T and there exist ξ_1, \cdots, ξ_r in H^2(\mathbb{C}^n) such that L(ζ) = \text{span}\{ξ_j(ζ) : 1 \leq j \leq r\} for almost all ζ ∈ T.

In the Introduction we have mentioned the following necessary conditions for an admissible matrix function Φ to be very badly approximable:

(C1) the functions ζ ↦ s_j(Φ(ζ)), 0 \leq j \leq \min\{m, n\} − 1, are constant almost everywhere on T;
(C2) Ker T_{2 Φ^*} = \{f ∈ H^2(\mathbb{C}^n) : Φ^* f = 0\} and Ker T_{\overline{Φ}} = \{f ∈ H^2(\mathbb{C}^n) : \overline{Φ} f = 0\}.

In this section we consider the following important condition:

(C3) if σ > 0, then \mathcal{S}_{Φ}^{(σ)}(ζ), ζ ∈ T, and \mathcal{S}_{Φ^*}^{(σ)}(ζ), ζ ∈ T, are analytic families of subspaces.

**Theorem 3.1.** Let Φ be an admissible very badly approximable matrix function in L^\infty(\mathcal{M}_{m,n}). Then Φ satisfies (C3).

We will see later that Theorem 3.1 is an immediate consequence of Theorem 4.1.

**Remark 2.** Note that it follows easily from the above Remark 1 that the analyticity of the families \mathcal{S}_{Φ}^{(σ)}(ζ), ζ ∈ T, for σ > 0 implies condition (C1). A fortiori (C3) implies (C1).

Indeed, for any σ > 0 the analytic family of subspaces \mathcal{S}_{Φ}^{(σ)} has constant dimension a.e. on T, and as one can easily see, this is possible only if the functions ζ ↦ s_j(Φ(ζ)) are constant almost everywhere on T.

We show in this section that if Φ is an admissible matrix function satisfying (C3), then Φ admits a factorization of the form (2.2) with \mathcal{V}_j and \mathcal{W}_j as in (2.2) and unitary-valued functions U_j such that \|H_{U_j}\|_e < 1. We call such factorizations quasicanonical. (A quasicanonical factorization is canonical if the unitary-valued functions U_j are very badly approximable).

Then we show that conditions (C1)–(C3) are not sufficient for an admissible function Φ to be very badly approximable.

Note here that the condition that the families \mathcal{S}_{Φ}^{(σ)}(ζ), ζ ∈ T, are analytic for σ > 0 does not imply that the families \mathcal{S}_{Φ^*}^{(σ)}(ζ), ζ ∈ T are analytic for σ > 0 (even under condition (C2)) as the following example shows.

**Example 1.** Let

\[
W = \begin{pmatrix}
w_1 & -\overline{w_2} \\
w_2 & \overline{w_1}
\end{pmatrix}
\]
be a thematic (1-balanced) matrix function, i.e., \( w_1, w_2 \in H^\infty, |w_1|^2 + |w_2|^2 = 1 \), and \( w_1 \) and \( w_2 \) are coprime. Consider the function
\[
\Phi = W^* \begin{pmatrix} \bar{z} & 0 \\ 0 & \frac{\bar{w}}{2} \end{pmatrix}
\]
Clearly, \( \mathcal{S}_\Phi^{(\sigma)} \) is a constant function for each \( \sigma > 0 \), and so the family \( \mathcal{S}_\Phi^{(\sigma)}, \zeta \in \mathbb{T} \), is analytic.

Let us verify that \( \Phi \) satisfies (C2). Suppose that \( g \in \text{Ker} T_{\bar{z}\Phi} \). Clearly, this means that \( Wg \in H^2(\mathbb{C}^2) \), i.e., \( g \in \text{Ker} T_W \). By Theorem A in [2] \( g = 0 \). Similarly, it is easy to see that \( \text{Ker} T_{\bar{z}\Phi} = \{0\} \) if and only is \( \text{Ker} T_{W^*} = \{0\} \). The last equality also follows from Theorem A.

Let us show that the family \( \mathcal{S}_\Phi^{(\sigma)}(\zeta), \zeta \in \mathbb{T} \), does not have to be analytic. Suppose that \( f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in H^2(\mathbb{C}^2) \) and \( f(\zeta) \) is a maximizing vector of \( \Phi^t(\zeta) \) for almost all \( \zeta \in \mathbb{D} \). Clearly, \( \overline{W^t}f \) must be of the form
\[
\overline{W^t}f = \begin{pmatrix} \varphi \\ 0 \end{pmatrix}, \quad \varphi \in L^2.
\]
Since \( W \) is a unitary-valued matrix function, it follows that
\[
f = W^t \overline{W^t}f = W^t \begin{pmatrix} \varphi \\ 0 \end{pmatrix} = \begin{pmatrix} \varphi w_1 \\ -\bar{\varphi} \bar{w}_2 \end{pmatrix}.
\]
Thus the function \( \mathcal{S}_\Phi^{(1)} \) is analytic if and only if there exists a function \( \varphi \in L^2 \) such that \( \varphi w_1 \in H^2 \) and \( \varphi \bar{w}_2 \in H^2 \). Suppose now that \( w_1 \) is invertible in \( H^\infty \). Then \( \varphi \) must be in \( H^2 \). Then the function \( w_2 \) must have a meromorphic pseudocontinuation (see [N], Lect. II, Sect. 1). Hence, if \( w_1 \) is invertible in \( H^\infty \) and \( w_2 \) does not have a pseudocontinuation, the function \( \mathcal{S}_\Phi^{(1)} \) is not analytic. \( \blacksquare \)

The following example shows that none of the two conditions in (C2) implies the other one (even under conditions (C1) and (C3)).

**Example 2.** Let \( V = \begin{pmatrix} v_1 & -\bar{v}_2 \\ v_2 & \bar{v}_1 \end{pmatrix} \) be a continuous thematic (1-balanced) matrix function. Consider the matrix function \( \Phi \) defined by
\[
\Phi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} V^* = \begin{pmatrix} \bar{v}_1 & \bar{v}_2 \\ 0 & 0 \end{pmatrix}.
\]
Obviously, \( \Phi \) satisfies (C1) and (C3). Let us show that \( \Phi \) satisfies the first condition in (C2). Suppose that \( \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \text{Ker} T_{\bar{z}\Phi} \). Then
\[
\bar{z} \Phi^* \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} \bar{z}v_1 g_1 \\ \bar{z}v_2 g_1 \\ 10 \end{pmatrix} \in H^2(\mathbb{C}^2).
\]
It follows that both $v_1g_1$ and $v_2g_1$ are constant functions. Suppose now that both $v_1$ and $v_2$ are nonzero functions such that the function $v_1v_2^{-1}$ is nonconstant. It is easy to see that in this case $g_1 = 0$. Thus
\[
\Phi^* \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \Phi^* \begin{pmatrix} 0 \\ g_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

However, $\text{Ker} T_{\Phi} \neq \{ f \in H^2(\mathbb{C}^n) : \Phi f = 0 \}$. Indeed,
\[
\bar{z} \Phi^* \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \bar{z} \begin{pmatrix} v_1g_1 + v_2g_2 \\ 0 \end{pmatrix}.
\]

Clearly, we can choose nonzero functions $g_1$ and $g_2$ in $H^2$ such that $v_1g_1 + v_2g_2 = 1$.

\[\blacksquare\]

**Theorem 3.2.** Let $\Phi$ be a matrix function in $L^\infty(\mathbb{M}_{m,n})$ that satisfies (C3). Then $\Phi$ admits a quasicanonical factorization.

**Proof.** As we have already observed, (C3) implies (C1). Let $\sigma_0 > \cdots > \sigma_{i-1}$ be positive numbers such that for almost all $\zeta \in T$ the distinct nonzero singular values of $\Phi(\zeta)$ are precisely, $\sigma_0, \cdots, \sigma_{i-1}$. We argue by induction on $i$. If $i = 0$, then $\Phi = 0$.

Let now $\iota > 0$. Suppose that $\dim \mathcal{S}_\Phi^{(\sigma_0)}(\zeta) = r$ for almost all $\zeta \in T$. Obviously, $\dim \mathcal{S}_\Phi^{(\sigma_0)}(\zeta) = r$ for almost all $\zeta \in T$. Let us first show that $\Phi$ admits a factorization of the form
\[
\Phi = \mathcal{W}^* \begin{pmatrix} \sigma_0 U & 0 \\ 0 & \Psi \end{pmatrix} \mathcal{V}^*,
\]

in which $\mathcal{V}$ and $\mathcal{W}$ are $r$-balanced unitary-valued matrix functions, $U$ is an $r \times r$ unitary-valued matrix function such that $\|H_U\|_e < 1$. The proof is similar to the proof of Theorem 4.3 of [AP].

Let $\xi_1, \cdots, \xi_r$ and $\eta_1, \cdots, \eta_r$ are functions in $H^2(\mathbb{C}^r)$ such that
\[
\mathcal{S}_\Phi^{(\sigma_0)}(\zeta) = \text{span}\{\xi_1(\zeta), \cdots, \xi_r(\zeta)\} \quad \text{and} \quad \mathcal{S}_\Phi^{(\sigma_0)}(\zeta) = \text{span}\{\eta_1(\zeta), \cdots, \eta_r(\zeta)\}
\]

almost everywhere on $T$. Let $\mathcal{L}$ be the minimal invariant subspace of multiplication by $z$ on $H^2(\mathbb{C}^n)$ that contains $\xi_1, \cdots, \xi_r$ and let $\mathcal{M}$ be the minimal invariant subspace of multiplication by $z$ on $H^2(\mathbb{C}^n)$ that contains $\eta_1, \cdots, \eta_r$.

It is easy to see from (2.1) that there exist $n \times r$ inner functions $\Upsilon$ and $\Theta$ such that $\mathcal{L} = \Upsilon H^2(\mathbb{C}^r)$ and $\mathcal{M} = \Theta H^2(\mathbb{C}^r)$. Let us show that $\Upsilon$ and $\Theta$ are co-outer.

Indeed, suppose that $\Upsilon^t = LF$, where $L$ is an inner matrix function and $F$ is an outer matrix function. Since $\dim \mathcal{S}_\Phi^{(\sigma_0)}(\zeta) = r$ for almost all $\zeta \in T$, it follows that $\text{rank} L(\zeta) = r$ almost everywhere on $T$, and so $L$ is an $r \times r$ inner function, and so $F^t$ is inner. Since $\Upsilon = F^t L^t$, it follows that $\mathcal{L} = \Upsilon H^2(\mathbb{C}^r) \subset F^tH^2(\mathbb{C}^r)$.

Clearly, for every $d \in \mathbb{C}^r$, the vector $\Upsilon(\zeta)d$ belongs to $\mathcal{S}_\Phi^{(\sigma_0)}(\zeta)$ for almost all $\zeta \in T$. It follows that $F^t(\zeta)d = \Upsilon(\zeta)L(\zeta)d \in \mathcal{S}_\Phi^{(\sigma_0)}(\zeta)$ for almost all $\zeta \in T$, and
so $F^t H^2(\mathbb{C}^r) = \mathcal{L} = \Upsilon H^2(\mathbb{C}^r)$. Hence, $\mathcal{L}$ is a constant matrix (see [N]) and $\Upsilon$ is co-outer.

Let now $\Theta$ and $\Xi$ be inner and co-outer matrix functions such that the matrix functions
\[
\mathcal{V} = \begin{pmatrix} \Upsilon & \Theta \end{pmatrix} \quad \text{and} \quad \mathcal{W}^t = \begin{pmatrix} \emptyset & \Xi \end{pmatrix}
\]
are $r$-balanced.

It is easy to see that if $q_1, \cdots, q_r$ are scalar polynomials and $\xi = q_1 \xi_1 + \cdots + q_r \xi_r$, then $\xi(\zeta)$ is a maximizing vector of $\Phi(\zeta)$ for almost all $\zeta \in \mathbb{T}$. It follows that for any function $f \in \mathcal{L}$ the vector $f(\zeta)$ is a maximizing vector of $\Phi(\zeta)$ for almost every $\zeta \in \mathbb{T}$. In particular, the columns of $\Upsilon(\zeta)$ are maximizing vectors of $\Phi(\zeta)$ almost everywhere on $\mathbb{T}$. For the same reason, the columns of $\emptyset(\zeta)$ are maximizing vectors of $\Phi^t(\zeta)$ for almost every $\zeta \in \mathbb{T}$.

We need two obvious and well known lemmas.

**Lemma 3.3.** Let $A \in \mathbb{M}_{m,n}$ and $\|A\| = 1$. Suppose that $v_1, \cdots, v_r$ is an orthonormal family of maximizing vectors of $A$ and $w_1, \cdots, w_r$ is an orthonormal family of maximizing vectors of $A^t$. Then
\[
\begin{pmatrix} w_1 & \cdots & w_r \end{pmatrix}^t A \begin{pmatrix} v_1 & \cdots & v_r \end{pmatrix}
\]
is a unitary matrix.

**Lemma 3.4.** Let $A$ be a matrix in $\mathbb{M}_{m,n}$ such that $\|A\| = 1$ and $B$ has the form
\[
A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},
\]
where $A_{11}$ is a unitary matrix. Then $A_{12}$ and $A_{21}$ are the zero matrices.

Consider the matrix function
\[
\begin{pmatrix} U & X \\ Y & \Psi \end{pmatrix} \overset{\text{def}}{=} \sigma_0^{-1} \mathcal{W} \Phi \mathcal{V}.
\]
It follows easily from Lemmas [3.3] and [3.4] that $U = \sigma_0^{-1} \emptyset^t \Phi \Upsilon$ is a unitary-valued matrix function while $X$ and $Y$ are the zero matrix functions. Thus [3.1] holds.

Let us show that $\|H_U\|_e < 1$. We have
\[
\begin{align*}
\|H_U\|_e &= \text{dist}_{L^\infty} \left( U, (H^\infty + C)(\mathbb{M}_{r,r}) \right) \\
&= \sigma_0^{-1} \text{dist}_{L^\infty} \left( \emptyset^t \Phi \Upsilon, (H^\infty + C)(\mathbb{M}_{r,r}) \right) \\
&\leq \sigma_0^{-1} \text{dist}_{L^\infty} \left( \Phi, (H^\infty + C)(\mathbb{M}_{m,n}) \right) = \sigma_0^{-1} \|H_\Phi\|_e < 1.
\end{align*}
\]

It is sufficient to show that $\Psi$ has a quasicanonical factorization. Clearly, for almost all $\zeta \in \mathbb{T}$, $\sigma_1, \cdots, \sigma_{r-1}$ are all distinct nonzero singular values of $\Psi(\zeta)$. If $r = 1$, then $\Psi = 0$, and everything is trivial. Let us show that the families $\mathcal{S}_\Psi^{(\sigma_j)}(\zeta)$, $\zeta \in \mathbb{T}$, are analytic for $j \geq 1$. 

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Consider the family $\mathcal{S}_\psi^{(\sigma)}(\zeta), \zeta \in \mathbb{T}$. Let $\xi_1, \ldots, \xi_\kappa$ be functions in $H^2(\mathbb{C}^n)$ such that

$$\mathcal{S}_\psi^{(\sigma)}(\zeta) = \text{span}\{\xi_1(\zeta), \ldots, \xi_\kappa(\zeta)\}$$

for almost all $\zeta \in \mathbb{T}$.

Since $V$ is unitary-valued, we have

$$\mathcal{S}_\psi^{(\sigma)}(\zeta) = \text{span}\{(V^*\xi_1)(\zeta), \ldots, (V^*\xi_\kappa)(\zeta)\}$$

for almost all $\zeta \in \mathbb{T}$.

We have $V^*\xi_r = (\Upsilon^*\xi_r)(\zeta)$, and so it belongs to $\mathcal{S}_\psi^{(\sigma)}(\zeta)$. It is easy to see that $(\Theta^*\xi_r)(\zeta) \in \mathcal{S}_\psi^{(\sigma)}(\zeta)$. Moreover, it is evident that

$$\mathcal{S}_\psi^{(\sigma)}(\zeta) = \text{span}\{(\Theta^*\xi_r)(\zeta) : 1 \leq r \leq \kappa\}$$

for almost all $\zeta \in \mathbb{T}$, which proves that $\mathcal{S}_\psi^{(\sigma)}(\zeta), \zeta \in \mathbb{T}$, is an analytic family of subspaces. The same reasoning shows that the functions $\mathcal{S}_\psi^{(\sigma)}(\zeta) \subset \mathbb{T}$ is an analytic family of subspaces. for $\sigma > 0$.

By the inductive hypothesis, $\Psi$ admits a quasicanonical factorization. ■

It turns out, however, that conditions (C1)–(C3) do not imply that the matrix function $\Phi$ is very badly approximable.

**Example 3.** Consider the function

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2}z^2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \bar{z} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \bar{z} \end{pmatrix},$$

Let us show that $\Phi$ satisfies (C1)–(C3), but $\Phi$ is not even badly approximable. Note that

$$\bar{z}\Phi = \begin{pmatrix} \bar{z} & 0 \\ 0 & \frac{1}{2}\bar{z}^2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \bar{z} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \bar{z} \end{pmatrix}$$

is a canonical factorization of $\bar{z}\Phi$, and so $\bar{z}\Phi$ is very badly approximable (see §2). Hence, it satisfies (C1)–(C3). Clearly, conditions (C1) and (C3) are invariant under multiplication by $z$. Thus $\Phi$ satisfies (C1) and (C3).

Let us show that $\text{Ker} T_{\bar{z}\Phi^*} = \{0\}$. Suppose that $\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \text{Ker} T_{\bar{z}\Phi^*}$. We have

$$\bar{z}\Phi^* \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 2g_1 - zg_2 \\ 2\bar{z}g_1 + g_2 \end{pmatrix}.$$
Let us prove now that $\ker T_{\bar{z}\Phi} = \{0\}$. Suppose that \( \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \ker T_{\bar{z}\Phi} \). We have
\[
\bar{z}\Phi \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 2g_1 + 2\bar{z}g_2 \\ -\bar{z}g_1 + g_2 \end{pmatrix}.
\]
It follows that $g_1 + \bar{z}g_2 \in H_2$ and $-\bar{z}g_1 + g_2 \in H_2$. Again, multiplying the second inclusion by $\bar{z}$, we obtain $g_1 - \bar{z}g_2 \in H_2$, and so both $g_1$ and $\bar{z}g_2$ belong to $H_2$. Thus $g_1 = \mathbf{0}$, and it follows from the second inclusion that $g_2 = \mathbf{0}$.

We can show now that $\Phi$ is not even badly approximable. Clearly, $\|\Phi\|_{L_\infty} = 1$. If $\Phi$ is badly approximable, then $\|H_\Phi\| = 1$. Since $\Phi$ is continuous, $H_\Phi$ is compact and so $H_\Phi$ has a maximizing vector $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$. Put
\[
V = \begin{pmatrix} \frac{1}{\sqrt{2}} \bar{z} \\ \frac{1}{\sqrt{2}} \end{pmatrix}^* \begin{pmatrix} v_1 \\ -\bar{v}_2 \end{pmatrix}.
\]
Clearly, the second component of the vector function $V^*f$ must be zero and $\Phi f$ must belong to $H^2(\mathbb{C}^2)$. Let $V^*f = \begin{pmatrix} h \\ 0 \end{pmatrix}$, where $h \in L^2$. We have
\[
f = VV^*f = V \begin{pmatrix} h \\ 0 \end{pmatrix} = \begin{pmatrix} v_1h \\ v_2h \end{pmatrix} \in H^2(\mathbb{C}^2).
\]
Since the matrix function $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is co-outer, it is easy to see that $h \in H^2$. We have
\[
\Phi f = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2}\bar{z}^2 \end{pmatrix} V^*f = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2}\bar{z}^2 \end{pmatrix} \begin{pmatrix} h \\ 0 \end{pmatrix} = \begin{pmatrix} h \\ 0 \end{pmatrix} \in H^2_\bar{z}(\mathbb{C}^2),
\]
and so $h = 0$. Hence, $H_\Phi$ has no maximizing vector and we get a contradiction. ■

4. Very Badly Approximable Matrix Functions

We obtain in this section a necessary and sufficient condition for an admissible matrix function to be very badly approximable. To do this, we slightly modify the necessary conditions stated in the previous section.

**Definition.** Let $\mathfrak{L}_n$ be the set of all subspaces of $\mathbb{C}^n$. Suppose that $L : \mathbb{T} \to \mathfrak{L}_n$ is an $\mathfrak{L}_n$-valued function defined almost everywhere. We say that functions $\xi_1, \ldots, \xi_l$ in $H^2(\mathbb{C}^n)$ span the function $L$ if $L(\zeta) = \text{span}\{\xi_j(\zeta) : 1 \leq j \leq l\}$ for almost all $\zeta \in \mathbb{T}$. 

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It is easy to see that if functions $\xi_1, \ldots, \xi_l$ in $H^2(\mathbb{C}^n)$ span an $\mathfrak{L}_n$-valued function $L$, then $\dim L(\zeta)$ is constant for almost all $\zeta \in \mathbb{T}$ and there exist functions $\eta_1, \ldots, \eta_k$ in $\text{span}\{\xi_j : 1 \leq j \leq l\}$ such that $k = \dim L(\zeta)$ and $L(\zeta) = \text{span}\{\eta_j(\zeta) : 1 \leq j \leq k\}$ almost everywhere on $\mathbb{T}$.

As in §3, we consider a matrix function $\Phi$ in $L^\infty(\mathbb{M}_{m,n})$ and for $\sigma > 0$ we associate with $\Phi$ the linear span $\mathfrak{S}_\Phi^{(\sigma)}(\zeta)$ of all Schmidt vectors of $\Phi(\zeta)$ that correspond to the singular values greater than or equal to $\sigma$.

We consider in this section the following condition:

(C4) for each $\sigma > 0$, the analytic family of subspaces $\mathfrak{S}_\Phi^{(\sigma)}$ is spanned by finitely many functions in $\text{Ker} T_\Phi$.

Remark 1. Clearly, condition (C4) implies that $\mathfrak{S}_\Phi^{(\sigma)}(\zeta), \zeta \in \mathbb{T}$, is an analytic family of subspaces, and it is easy to see that Theorem 4.1 implies Theorem 3.1.

Remark 2. As we have already observed (see Remark 2 after Theorem 3.1), condition (C4) implies that the functions $\zeta \mapsto s_j(\Phi(\zeta))$ are constant almost everywhere on $\mathbb{T}$.

Remark 3. It is interesting to observe that to prove that (C4) implies that $\Phi$ is very badly approximable, we do not need the fact that $\Phi^t$ satisfies (C4).

Proof of Theorem 4.1. Suppose first that $\Phi$ is admissible and very badly approximable. Then $s_j(\Phi(\zeta)) = t_j(\Phi), 0 \leq j \leq \min\{m, n\} - 1$, almost everywhere on $\mathbb{T}$ (see (1.2)). Let us prove by induction on $\kappa$ that if $\Phi$ is an admissible very badly approximable matrix function and for almost all $\zeta \in \mathbb{T}$,

$$
\sigma_0 = \sigma_0(\Phi) > \sigma_1 = \sigma_1(\Phi) > \sigma_2 = \sigma_2(\Phi) > \cdots
$$

are all distinct nonzero singular values of $\Phi(\zeta)$, then $\mathfrak{S}_\Phi^{(\sigma_\kappa)} = \mathfrak{S}_\Phi^{(\sigma_\kappa(\Phi))}$ is spanned by finitely many functions in $\text{Ker} T_\Phi$.

By Theorem D stated in §2, $\Phi$ admits a factorization

$$
\Phi = W^* \begin{pmatrix} \sigma_0 U & 0 \\ 0 & \Psi \end{pmatrix} V^*,
$$

(4.1)
where \( \sigma_0 = \| \Phi \|_{L^\infty} \), \( \mathcal{V} \) and \( \mathcal{W}_r \) are \( r \)-balanced unitary-valued functions, \( 1 \leq r \leq \min\{m, n\} \), \( U \) is an \( r \times r \) very badly approximable unitary-valued function such that \( T_U \) is Fredholm and \( \| H_U \|_\infty < 1 \), and \( \Psi \) is an admissible very badly approximable matrix function with \( \| \Psi \|_{L^\infty} = \sigma_1 < \sigma_0 \).

Let us prove first that \( \mathcal{S}_{\Phi}(\sigma_0) \) is spanned by finitely many functions in Ker \( T_\Phi \). Since \( T_U \) is Fredholm (see Remark 3 after Theorem D in §2), it admits a Wiener–Hopf factorization
\[
U = G^* D F,
\]
where
\[
D = \begin{pmatrix}
d_1 & 0 & \cdots & 0 \\
0 & d_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & d_r
\end{pmatrix},
\]
\( F \) and \( G \) are \( r \times r \) matrix functions such that \( F^{\pm 1} \in H^2(\mathbb{M}_{r,r}) \) and \( G^{\pm 1} \in H^2(\mathbb{M}_{r,r}) \), and \( d_1, \ldots, d_r \in \mathbb{Z} \) (Simonenko’s theorem; see e.g., [3], Ch. 3, §5). By Theorem E, Ker \( T_{z^U} \) = \{0\}, which implies easily that the indices \( d_1, \ldots, d_r \) are negative. Let \( c_1, \ldots, c_r \) be a basis in \( \mathbb{C}^r \). Consider the functions
\[
\mathcal{V}^* F^{-1} c_j, \ldots, \mathcal{V}^* F^{-1} c_r,
\]
where \( c_j \) denotes the constant function identically equal to \( c_j \). Since \( \mathcal{V} \) is a unitary-valued function, it is easy to see that
\[
\mathcal{V}^* \mathcal{V}^* F^{-1} c_j = \begin{pmatrix} F^{-1} c_j \\
0
\end{pmatrix},
\]
and so
\[
\Phi \mathcal{V}^* F^{-1} c_j = \mathcal{W}_{\sigma} \begin{pmatrix} G^* D c_j \\
0
\end{pmatrix} = \mathcal{W}_{\sigma} G^* D c_j \in H^2(\mathbb{C}^m),
\]
since the Wiener–Hopf indices \( d_j \) are negative. It is easy to see now that the functions in \( \mathcal{S}_{\Phi}(\sigma_0) \) belong to Ker \( T_\Phi \) and span \( \mathcal{S}_{\Phi}(\sigma_0) \).

Let now that \( \sigma > 0 \). Clearly, for almost all \( \zeta \in \mathbb{T} \), \( \sigma_1 > \sigma_2 \cdots > \sigma_l \) are all nonzero singular values of \( \Psi(\zeta) \) and \( \sigma_{\sigma} = \sigma_{\sigma}(\Phi) = \sigma_{\sigma}(\Psi) \). By the inductive hypothesis, there exist functions \( \xi_1, \ldots, \xi_l \) in Ker \( H_\Phi \) that span \( \mathcal{S}_{\Psi}(\sigma_\sigma) = \mathcal{S}_{\Psi}(\sigma_{\sigma}(\Psi)) \). By Theorem C, the functions \( \Theta \) and \( \Xi \) are left invertible in \( H^\infty \). Let \( Q \in H^\infty(\mathbb{M}_{n-r,n}) \) and \( R \in H^\infty(\mathbb{M}_{m-r,m}) \) such that \( Q \Theta = I_{n-r} \) and \( R \Xi = I_{m-r} \). Put
\[
\eta_j = Q^t \xi_j + \mathcal{V} q_j, \quad 1 \leq j \leq l,
\]
where the functions \( q_j \in H^2(\mathbb{C}^r) \) will be chosen later. We have
\[
\mathcal{V}^* \eta_j = \begin{pmatrix} \mathcal{V}^* \Theta^t \\
\mathcal{V}^* \Xi^t
\end{pmatrix} (Q^t \xi_j + \mathcal{V} q_j) = \begin{pmatrix} \mathcal{V}^* Q^t \xi_j + q_j \\
\xi_j
\end{pmatrix},
\]
(4.3)
since \( \mathcal{V} \) is unitary-valued and \( \Theta^t Q^t = I_{n-r} \).
Since $\mathcal{W}$ is a unitary-valued function, we obtain

$$I_m = \mathcal{W}^* \mathcal{W} = \overline{\mathcal{O}} \mathcal{O}^* + \Xi \Xi^*,$$

and so

$$\Xi = \Xi(R \Xi)^* = \Xi \Xi^* R^* = (I_m - \overline{\mathcal{O}} \mathcal{O}^*) R^*. \tag{4.4}$$

We have now from (4.3) and (4.4)

$$\Phi(\eta_j) = \mathcal{W}^* \left( \begin{array}{c} \sigma U \\ 0 \end{array} \Psi \right) \left( \begin{array}{c} \Upsilon^* Q^j \xi_j + q_j \\ \xi_j \end{array} \right)
= \mathcal{W}^* \left( \begin{array}{c} \sigma U(\Upsilon^* Q^j \xi_j + q_j) \\ \Psi \xi_j \end{array} \right)
= \sigma \overline{\mathcal{O}} U(\Upsilon^* Q^j \xi_j + q_j) + (I_m - \overline{\mathcal{O}} \mathcal{O}^*) R^* \Psi \xi_j
= R^* \Psi \xi_j + \overline{\mathcal{O}}(\sigma U(\Upsilon^* Q^j \xi_j + q_j) - \mathcal{O}^* R^* \Psi \xi_j).$$

In order that $\Phi(\eta_j) \in H^2(\mathbb{C}^m)$, it is sufficient that

$$P_+(\sigma U(\Upsilon^* Q^j \xi_j + q_j) - \mathcal{O}^* R^* \Psi \xi_j) = 0,$$

which means that

$$\sigma T_U q_j = P_+(\mathcal{O}^* R^* \Psi \xi_j - \sigma U \Upsilon^* Q^j \xi_j).$$

Since $\text{Range} T_U = H^2(\mathbb{C}^r)$, we can find a solution $q_j \in H^2(\mathbb{C}^r)$. This proves that $\eta_j \in \text{Ker} T_U$, $1 \leq j \leq l$. It is also easy to see that the functions

$$\Upsilon^* F^{-1} c_1, \ldots, \Upsilon^* F^{-1} c_r, \eta_1, \ldots, \eta_l$$

span $\mathcal{S}_{\delta}^{(\sigma_\nu)}$.

Note that the above reasoning is similar to the proof of Lemma 1.2 in [PY 2].

Suppose now that $\Phi$ satisfies (C4). Let us show that it is very badly approximable. As we have already observed (see Remark 2 after the statement of Theorem 4.1), the singular values $s_j(\Phi(\zeta))$ are constant almost everywhere on $\mathbb{T}$. Let $\sigma_0 > \cdots > \sigma_{i-1}$ be positive numbers such that for almost all $\zeta \in \mathbb{T}$ the distinct nonzero singular values of $\Phi(\zeta)$ are precisely $\sigma_0, \ldots, \sigma_{i-1}$. We argue by induction on $i$. If $i = 0$, the situation is trivial. Suppose that $i > 0$. Suppose that $\xi \in \text{Ker} T_{\Phi}$ and $\xi(\zeta)$ is a maximizing vector of $\Phi(\zeta)$ for almost all $\zeta \in \mathbb{T}$. Clearly, $H_{\Phi} \xi = \Phi(\xi)$ and $\|H_{\Phi} \xi\| = \sigma_0 \|\xi\|$. It follows that $\xi$ is a maximizing vector of $H_{\Phi}$ and $\Phi$ is badly approximable. Conversely, if $\xi$ is a maximizing vector of $H_{\Phi}$, then $\xi \in \text{Ker} T_{\Phi}$ and $\xi(\zeta)$ is a maximizing vector of $\Phi(\zeta)$ for almost all $\zeta \in \mathbb{T}$ (see §2).

Suppose that $\dim \mathcal{S}_{\Phi}^{(\sigma_\nu)}(\zeta) = r$ for almost all $\zeta \in \mathbb{T}$. Obviously, $\dim \mathcal{S}_{\Phi^*}^{(\sigma_\nu)}(\zeta) = r$ for almost all $\zeta \in \mathbb{T}$. Let us first show that $\Phi$ admits a partial canonical factorization (1.1) in which $\mathcal{V}$ and $\mathcal{W}^*$ are $r$-balanced unitary-valued matrix functions, $U$ is an $r \times r$ very badly approximable unitary-valued matrix function such that $\|H_U\|_e < 1$. 17
It is well known and it is easy to verify that if \( \xi \) is a maximizing vector of \( H_{\Phi} \) and \( \eta = \overline{z H_{\Phi} \xi} \), then \( \eta \) is a maximizing vector of \( H_{\Phi^t} \) and vice versa.

Let \( \mathcal{L} \) be the minimal invariant subspace of multiplication by \( z \) on \( H^2(\mathbb{C}^n) \) that contains all maximizing vectors of \( H_{\Phi} \) and let \( \mathcal{M} \) be the minimal invariant subspace of multiplication by \( z \) on \( H^2(\mathbb{C}^m) \) that contains all maximizing vectors of \( H_{\Phi^t} \).

By Theorem B, there exist \( n \times r \) inner and co-outer matrix functions \( \Upsilon \) and \( \Theta \) such that \( \mathcal{L} = \Upsilon H^2(\mathbb{C}^r) \) and \( \mathcal{M} = \Theta H^2(\mathbb{C}^r) \). Let \( \Theta \) and \( \Xi \) are inner and co-outer matrix functions such that the matrix functions

\[
V = \begin{pmatrix} \Upsilon & \Theta \end{pmatrix} \quad \text{and} \quad W^t = \begin{pmatrix} \Theta & \Xi \end{pmatrix}
\]

are \( r \)-balanced. Then \( \Phi \) admits a factorization (see Remark 3 after Theorem D in \( \S 2 \)). Moreover, to show that \( \Phi \) is very badly approximable, it suffices to verify that \( \Psi \) is very badly approximable. Clearly, \( \Psi \) satisfies (C1). Let us verify that \( \Psi \) satisfies (C4).

Clearly, for almost all \( \zeta \in \mathbb{T} \), \( \sigma_1, \ldots, \sigma_{i-1} \) are all distinct nonzero singular values of \( \Psi(\zeta) \). If \( i = 1 \), then \( \Psi = 0 \), and so \( \Phi \) is very badly approximable (see Theorem D). Suppose now that \( i > 1 \). Consider the function \( S(\sigma_d) \Psi, 1 \leq d \leq i - 1 \).

Let \( \xi_1, \ldots, \xi_\kappa \) be functions in \( \text{Ker} T_{\Phi} \) such that

\[
S(\sigma_d) \Phi(\zeta) = \text{span} \{ \xi_1(\zeta), \ldots, \xi_\kappa(\zeta) \}
\]

for almost all \( \zeta \in \mathbb{T} \).

Since \( \mathcal{V} \) is unitary-valued, we have

\[
S_{\Phi_V}^{(\sigma_d)}(\zeta) = \text{span} \{ (\mathcal{V}^* \xi_1)(\zeta), \ldots, (\mathcal{V}^* \xi_\kappa)(\zeta) \}
\]

for almost all \( \zeta \in \mathbb{T} \). We have \( \mathcal{V}^* \xi_j = \begin{pmatrix} \Upsilon^* \xi_j \\ \Theta^t \xi_j \end{pmatrix} \). Obviously, \( \begin{pmatrix} (\mathcal{V}^* \xi_j)(\zeta) \\ 0 \end{pmatrix} \) is a maximizing vector of \((\Phi \mathcal{V})(\zeta)\), and so it belongs to \( S_{\Phi_V}^{(\sigma_d)}(\zeta) \). Thus \( \begin{pmatrix} 0 \\ (\Theta^t \xi_j)(\zeta) \end{pmatrix} \in S_{\Phi_V}^{(\sigma_d)}(\zeta) \). It is easy to see that \( (\Theta^t \xi_j)(\zeta) \in S_{\Phi_V}^{(\sigma_d)}(\zeta) \). Moreover, it is evident that

\[
S_{\Psi}^{(\sigma_d)}(\zeta) = \text{span} \{ (\Theta^t \xi_j)(\zeta) : 1 \leq j \leq \kappa \}
\]

for almost all \( \zeta \in \mathbb{T} \).

Let us show that \( \Theta^t \xi_j \in \text{Ker} T_{\Psi} \).

It follows from (4.1) that

\[
\Psi = \Xi S_{\Psi} \Phi T_{\Psi}.
\]

Hence,

\[
\Psi \Theta^t \xi_j = \Xi S_{\Psi} \Phi T_{\Psi} \Theta^t \xi_j = \Xi S_{\Psi} \Phi \xi_j \in H^2(\mathbb{C}^m),
\]

since \( \xi_j \in \text{Ker} T_{\Phi} \). By the inductive hypothesis, \( \Psi \) is very badly approximable, and so \( \Phi \) is very badly approximable. ■
5. An alternative approach: weighted estimates and superoptimal weights

In this section we present an alternative, more geometric proof of the main result (Theorem 4.1). Main ideas of this proof go back to [T], where the so-called superoptimal weights were used to prove the uniqueness of superoptimal approximation.

Although we do not use superoptimal weight per se in this proof, the main ideas from [T] (weighted estimates, optimal vectors, “pinching” the weights, etc) are present here, so we wanted to mention the origin of the ideas.

5.1. Matrix weights and weighted Nehari Problem. Let $W$ be an $n \times n$ matrix weight, i.e., a bounded matrix-valued function on $T$, whose values are nonnegative $n \times n$ matrices. Given a matrix weight, one can introduce the weighted norm $\| \cdot \|_W$ on $L^2(C^n)$:

$$\|f\|_W^2 := \langle Wf, f \rangle_{L^2(C^n)} = \int_T (W(\xi)f(\xi), f(\xi))dm(\xi), \quad f \in L^2(C^n),$$

with the corresponding weighted inner product $\langle \cdot, \cdot \rangle_W$, $\langle f, g \rangle_W = \langle Wf, g \rangle_{L^2(C^n)}$.

Given a Hankel operator $H_\Phi : H^2(C^n) \to H^2(C^m)$, we call the weight $W$ admissible (for the Hankel operator $H_\Phi$) if the following inequality

$$\|H_\Phi f\|_2^2 \leq \|f\|_W^2 = \langle Wf, f \rangle_{L^2(C^n)} := \int_T (W(\xi)f(\xi), f(\xi))dm(\xi), \quad f \in H^2(C^n),$$

holds.

We need the following weighted analogue of the classical Nehari Theorem.

**Weighted Nehari Theorem.** Let $\Phi \in L^\infty(M_{m,n})$ and let $W \in L^\infty(M_{n,n})$ be an admissible weight for $\Phi$. Then there exists $F \in H^\infty(M_{m,n})$ such that the function $\Psi = \Phi - F$ satisfies the inequality $\Psi(\xi)^*\Psi(\xi) \leq W(\xi)$ a.e. on $T$.

This theorem (and even its operator-valued version) easily follows from the classical operator Nehari Theorem. We refer the reader to [T] for the proof.

5.2. The necessity of condition (C4). Suppose that $\Psi$ is a very badly approximable function. By (1.2), $s_k(\Phi(\zeta)) = t_k(\Phi)$ for almost all $\zeta \in T$, where the $t_k(\Phi)$ are the superoptimal singular values of $\Phi$. Let $\sigma_k$, $k = 0, 1, 2, \cdots$, be the sequence of distinct nonzero superoptimal singular values of $\Phi$ arranged in the decreasing order. In other words, for almost all $\zeta \in T$, the sequence $\sigma_k$, $k = 0, 1, 2, \cdots$, is the sequence of distinct singular values of $\Phi(\zeta)$ arranged in the decreasing order.

Define the functions $\varphi_k$ by $\varphi_k(x) = \max\{x, \sigma_k^2\}$, $x \geq 0$, $k = 0, 1, 2, \cdots$, and define the weights $W_k$ by $W_k(\zeta) = \varphi_k(\Phi(\zeta)^*\Phi(\zeta))$, $\zeta \in T$.

Since $\Phi^*\Phi \leq W_k$, the weights $W_k$ are admissible for the Hankel operator $H_\Phi$. 
For \( k = 0, 1, \ldots \), we denote by \( \mathcal{E}_k \) the set of all extremal functions for the weighted estimate \( \|H_\Phi f\|^2 \leq (W_k f, f) \), i.e., the set of all functions \( f \in H^2(\mathbb{C}^n) \) satisfying
\[
\|H_\Phi f\|^2 = (W_k f, f)_{L^2(\mathbb{C}^n)}.
\]

Since \( \|H_\Phi\| = t_0(\Phi) > \|H_\Phi\|_e \), the norm of \( H_\Phi \) is attained, and \( \mathcal{E}_0 \) is a nontrivial finite-dimensional subspace of \( H^2(\mathbb{C}^n) \). Since by the assumption of the theorem \( \sigma_k > \|H_\Phi\|_e \), the subspaces \( \mathcal{E}_k \) are finite-dimensional, and since the sequence \( \mathcal{E}_k \) is clearly increasing, all \( \mathcal{E}_k \) are nontrivial subspaces.

Denote by \( E_k(\zeta) \defeq \text{span}\{f(\zeta) : f \in \mathcal{E}_k\} \), \( \zeta \in \mathbb{T} \). More precisely, take some basis in \( \mathcal{E}_k \), select a function \( f_j \) from each equivalence class, and define \( E_k(\zeta) = \text{span}\{f_j(\zeta) : j = 1, 2, \ldots\} \). Note that different choices of bases and representatives give us different functions \( E_k \), but any two such functions coincide almost everywhere. Thus the corresponding equivalence class of subspace-valued functions is well defined.

It is easy to show that the function \( \dim E_k(\zeta) \) is constant almost everywhere on \( \mathbb{T} \) and that the projection-valued functions \( \zeta \mapsto P_{E_k(\zeta)} \) are measurable, cf [T].

Our goal is to show, that \( E_k(\zeta) = \mathcal{G}_\Phi^{(\sigma_k)}(\zeta) \) for almost all \( \zeta \in \mathbb{T} \). Then we are done, because any \( \mathcal{E}_k \subset \text{Ker}\ T_\Phi \). Indeed, for \( f \in \mathcal{E}_k \),
\[
(\Phi^* \Phi f, f) \leq (W_k f, f) = \|H_\Phi f\|^2 \leq (\Phi^* \Phi f, f),
\]
whence \( \|\Phi f\| = \|H_\Phi f\| \). Keeping in mind that \( \|\Phi f\|^2 = \|H_\Phi f\|^2 + \|T_\Phi f\|^2 \), we get \( \|T_\Phi f\| = 0 \).

Let us show first that \( E_k(\zeta) \subset \mathcal{G}_\Phi^{(\sigma_k)}(\zeta) \) for almost all \( \zeta \in \mathbb{T} \). Assume the contrary. Then there exists a function \( f \in \mathcal{E}_k \) such that \( f(\zeta) \notin \mathcal{G}_\Phi^{(\sigma_k)}(\zeta) \) on a set of positive measure. Since for any finite collection of functions \( f_1, f_2, \ldots, f_N \in H^2(\mathbb{C}^n) \), the dimension \( \dim \text{span}\{f_1, f_2, \ldots, f_N\} \) is constant almost everywhere on \( \mathbb{T} \) (the minors belong to the Nevanlinna class), it follows that \( f(\zeta) \notin \mathcal{G}_\Phi^{(\sigma_k)}(\zeta) \) for almost all \( \zeta \in \mathbb{T} \). By the definition of \( \mathcal{G}_\Phi^{(\sigma_k)} \), we have \( \|\Phi(\zeta) f(\zeta)\|_{\mathbb{C}^n} < \sigma_k \|f(\zeta)\|_{\mathbb{C}^n} \), \( \zeta \in \mathbb{T} \), and so
\[
\|H_\Phi f\|_2 \leq \|\Phi f\|_2 < \|\Phi f\|_2.
\]
However, this contradicts the definition of \( \mathcal{E}_k \). Hence, \( E_k(\zeta) \subset \mathcal{G}_\Phi^{(\sigma_k)}(\zeta) \) for almost all \( \zeta \in \mathbb{T} \).

Let us now prove that \( E_k(\zeta) = \mathcal{G}_\Phi^{(\sigma_k)}(\zeta) \). Suppose that \( E_k(\zeta) \) is a proper subspace of \( \mathcal{G}_\Phi^{(\sigma_k)}(\zeta) \) for almost all \( \zeta \in \mathbb{T} \). Let us show that in this case \( \Phi \) is not a very badly approximable function.

Let \( N \) be the largest integer such that \( s_N(\Phi(\zeta)) = \sigma_k \) for almost all \( \zeta \in \mathbb{T} \) (recall that the functions \( \zeta \mapsto s_j(\Phi(\zeta)) \) are constant almost everywhere). This means that for almost all \( \zeta \) there are exactly \( N + 1 \) singular values of \( \Phi(\zeta) \) (counting multiplicities) that are greater than or equal to \( \sigma_k \).
We want to construct a function $\Psi$ such that $\Phi - \Psi \in H^\infty(\mathbb{M}_{m,n})$,

$$s_j(\Psi(\zeta)) \leq s_j(\Phi(\zeta)), \quad \zeta \in \mathbb{T} \text{ for } j < N,$$

but

$$s_N(\Psi(\zeta)) < s_N(\Phi(\zeta)), \quad \zeta \in \mathbb{T},$$

which would imply that $\Phi$ is not very badly approximable.

To do that we “pinch” the weight $W_k(\xi)$ in the directions orthogonal to $E_k(\xi)$ to make it smaller (but still admissible) and then solve the Weighted Nehari Problem.

Namely, consider the one-parametric family of weights $W_k^a$, $a > 0$, defined by

$$W_k^a(\xi) = P_{E_k(\xi)}W_kP_{E_k(\xi)} \oplus a^2P_{E_k(\xi)\perp};$$

here we use the symbol $\oplus$ to emphasize that both operators on the right-hand side act on orthogonal subspaces of $\mathbb{C}^n$, i.e., the operators $W_k^a(\xi)$ have block-diagonal form with respect to the orthogonal decomposition $\mathbb{C}^n = E_k(\xi) \oplus E_k(\xi)\perp$.

If we can show that for some $a < \sigma_k$ the weight $W_k^a$ is still admissible, the necessity is proved. Indeed, let $\Psi$ be a solution of the Weighted Nehari Problem, i.e., a function such that $\Phi - \Psi \in H^\infty$ and $\Psi^*(\zeta)\Psi(\zeta) \leq W_k^a(\zeta), \zeta \in \mathbb{T}$. Then the minimax property of the singular values implies that for $j < N$

$$s_j(\Psi(\zeta)) \leq s_j(W_k^a(\zeta))^{1/2} \leq s_j(W_k(\zeta))^{1/2} = s_j(\Phi(\zeta)), \quad \zeta \in \mathbb{T},$$

but

$$s_N(\Psi(\zeta)) \leq a < s_N(\Phi(\zeta)), \quad \zeta \in \mathbb{T}.$$

We will need the following simple fact, whose proof is left as an exercise.

**Lemma 5.1.** Let $T$ be an operator (acting from one Hilbert space to another one), and let $f$ be a maximizing vector of $T$. Then for any vector $g$, the condition $g \perp f$ implies $Tg \perp Tf$.

Let us now apply this lemma. We treat the Hankel operator $H_\Phi$ as a operator, acting from $H^2(\mathbb{C}^n)$ endowed with the weighted norm $\|\cdot\|_{W_k}$ to the space $H^2(\mathbb{C}^m)$. The nonzero vectors in $E_k$ are exactly the maximizing vectors for this operator. Therefore by Lemma 5.1 for any function $g \in H^2(\mathbb{C}^n)$ orthogonal to $E_k$ with respect to the weighted inner product $(\cdot, \cdot)_W$, we have

$$H_\Phi g \perp H_\Phi E_k$$

(with respect to the usual, unweighted scalar product).

Put

$$q = \sup \{\|H_\Phi f\| : \|f\|_{W_k} = 1, \text{ and } f \text{ is } W_k\text{-orthogonal to } E_k\}.$$ 

(5.1)

Since $W_k$ is an admissible weight, $q \leq 1$. Moreover, the following lemma says that actually $q < 1$. 

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Lemma 5.2. Let $W$ be an invertible admissible weight for a Hankel operator $H_\Phi$ such that $W(\xi) \geq a^2I$, $a > \|H_\Phi\|_e$, and let $K$ be a closed subspace of $H^2(\mathbb{C}^n)$. If

$$q = \sup\{\|H_\Phi f\| : f \in K, \|f\|_W \leq 1\} = 1,^2$$

then there exists a vector $f_0 \in K$ such that $\|H_\Phi f_0\| = \|f_0\|_W$.

Proof. Putting $g = W^{1/2}f$, we can rewrite the condition $q = 1$ in the following way:

$$\sup\{\|H_\Phi W^{-1/2}g\| : g \in W^{1/2}K, \|g\| = 1\} = 1,$$

which simply means that the norm of the operator $(H_\Phi W^{-1/2})|W^{1/2}K$ is 1. Since the norm of multiplication by $W^{-1/2}$ is at most $a^{-1}$, the essential norm of the operator $(H_\Phi W^{-1/2})|W^{1/2}K$ can be estimated as

$$\|(H_\Phi W^{-1/2})|W^{1/2}K\|_e \leq \|H_\Phi\|_e W^{-1/2}\|_\infty \leq \|H_\Phi\|_e a^{-1} < 1.$$

Therefore the norm of this operator is attained on some vector $g_0 \in W^{1/2}K$, and so $f_0 = W^{-1/2}g_0$ is a maximizing vector in $K$. ■

Let us apply Lemma 5.2 to the weight $W_k$ and the subspace $K$ of $H^2(\mathbb{C}^n)$ of vectors that are $W_k$-orthogonal to $E_k$. If $q = 1$ in (5.1), the lemma asserts that there is a maximizing vector in $K$, which is impossible, since $E_k$ contains all maximizing vectors.

To complete the proof of necessity, we have to show that for $a = qa_k$, the weight $W_k$ is still admissible. First of all, note that $E_k(\zeta)$ is an invariant subspace of all $W_k^{|a|}(\zeta)$ (including $W_k(\zeta)$) for almost all $\zeta \in \mathbb{T}$. Since for any $f \in E_k$, we have $f(\zeta) \in E_k(\zeta), \zeta \in \mathbb{T}$, and so for $f \in E_k$ and $g \in H^2(\mathbb{C}^n)$, we obtain

$$\langle W_k^{|a|}(\zeta)f(\zeta), g(\zeta) \rangle = \langle PE_k(\zeta)W_k^{|a|}(\zeta)f(\zeta), g(\zeta) \rangle = \langle PE_k(\zeta)W_k(\zeta)f(\zeta), g(\zeta) \rangle = \langle W_k(\zeta)f(\zeta), g(\zeta) \rangle, \zeta \in \mathbb{T}. \quad (5.2)$$

In particular, it follows that $K$, being the $W_k$-orthogonal complement of $E_k$, is also the $W_k^{|a|}$-orthogonal complement of $E_k$ for all $a > 0$.

Let $f \in E_k$ and let $g$ be $W_k$ orthogonal to $E_k$. Then $\|f\|_{W_k^{|a|}}$ does not depend on $a$, and for $a = qa_k$ we have $q\|g\|_{W_k} \leq \|g\|_{W_k^{|a|}}$. (If $g(\zeta)$ were pointwise orthogonal to $E_k(\zeta)$, then equality would hold. But $g(\zeta)$ is not necessarily pointwise orthogonal to $E_k(\zeta)$, so we can guarantee only inequality.) By Lemma 5.1, $H_\Phi f \perp H_\Phi g$, and so

$$\|H_\Phi(f + g)\|^2 = \|H_\Phi f\|^2 + \|H_\Phi g\|^2 \leq \|f\|_{W_k}^2 + q\|g\|_{W_k}^2 \leq \|f\|_{W_k^{|a|}}^2 + \|g\|_{W_k^{|a|}}^2 = \|f + g\|_{W_k^{|a|}}^2,$$

whence the weight $W_k^{|a|}$ is admissible. This completes the proof of necessity. ■

^2Note that the supremum is always at most 1.
5.2. Sufficiency. Suppose that a function $\Phi$ satisfies condition (C4). Let us show that $\Phi$ is very badly approximable.

As we already discussed above, (C4) implies that singular values of $\Phi(z)$ (i.e. the functions $\zeta \mapsto s_k(\Phi(\zeta)))$ are constant almost everywhere on $\mathbb{T}$. Let $s_0, s_1, s_2, \cdots$ denote these singular values arranged in the nonincreasing order (counting multiplicity), and let $\sigma_0, \sigma_1, \sigma_2, \cdots$ be all distinct singular values arranged in the decreasing order (i.e., $\sigma_0, \sigma_1, \sigma_2, \cdots$ be the singular values of $\Phi(\zeta)$ not counting multiplicity).

Let $F$ be a superoptimal approximation of $\Phi$, $\Psi = \Phi - F$, and let $t_0, t_1, t_2, \cdots$ be the superoptimal singular values of $\Phi$ (equivalently, of $\Psi$). Let $N_k$ be the largest integer such that $s_{N_k} = \sigma_k$, which means that there are exactly $N_k + 1$ singular values that are greater than or equal to $\sigma_k$.

As in the proof of necessity, let us introduce the weight $W = \Phi^* \Phi$, and let $W_k(\xi) = \varphi_k(W)$, where $\varphi_k(x) = \max\{x, \sigma_k\}$ for $x \geq 0$.

We are going to prove using induction on $k$ that for all $k$ the following conditions are satisfied:

(i) $\Psi^* (\zeta) \Psi(\zeta) \leq W_k(\xi)$ for almost all $\zeta \in \mathbb{T}$;
(ii) $\Psi_k(\zeta)|\mathcal{G}_k^{(\sigma_k)}(\zeta) = \Phi_k(\zeta)|\mathcal{G}_k^{(\sigma_k)}(\zeta)$ for almost all $\zeta \in \mathbb{T}$;
(iii) $t_j = s_j$ for $0 \leq j \leq N_k$.

This will immediately prove that $\Psi \equiv \Phi$, and so $0$ is the unique superoptimal approximation of $\Phi$.

Consider first the case $k = 0$. By the definition of superoptimal approximation $t_0 \leq s_0 = \sigma_0$, and hence,

$$\Psi^*(\zeta)\Psi(\zeta) \leq s_0^2 I = W_0(\zeta), \quad \zeta \in \mathbb{T},$$

i.e., condition (i) is satisfied.

Suppose that $f_0, f_1, \cdots, f_{N_0} \in \text{Ker} T_\Phi$ are functions that span $\mathcal{G}_\Phi^{(\sigma_0)}$. Since

$$\Phi f = T_\Phi f + H_\Phi f,$$

and $T_\Phi f_j = 0$, we have $H_\Phi f_j = \Phi f_j$ for $0 \leq j \leq N_0$. Hence,

$$\sigma_0 \|f_j\| = \|\Phi f_j\| = \|H_\Phi f_j\| = \|H_\Psi f_j\| \leq \|\Psi f_j\| \leq t_0 \|f_j\|.$$

Since $t_0 \leq \sigma_0$, the above inequalities are actually equalities and (5.3) implies that

$$\Psi f_j = H_\Psi f_j = H_\Phi f_j = \Phi f_j.$$

Since span$\{f_j(\zeta) : 0 \leq j \leq N_0\} = \mathcal{G}_\Phi^{(\sigma_0)}(\zeta)$ for almost all $\zeta \in \mathbb{T}$, condition (ii) is satisfied. Condition (iii) is an immediate consequence of (i) and (ii).

Let us assume now that the inductive hypotheses (i)–(iii) are proved for $k$, and we want to prove them for $k+1$. It follows from (iii) and the definition of superoptimal approximation that $t_{N_k+1} \leq s_{N_k+1} = \sigma_{k+1}$, and so $\Psi^* \Psi \leq W_{k+1}$. This proves (i).

The proof of the other two condition is very similar to that of in the case $k = 0$.

First of all note that the case $\sigma_{k+1} = 0$ is trivial, since in this case $W_{k+1}(\zeta), \Phi(\zeta)$ and $\Psi(\zeta)$ must be zero on $\mathcal{G}_\Phi^{(\sigma_k)}(\zeta)$.
Let us assume that $\sigma_{k+1} > 0$ and let $f_j$, $0 \leq j \leq N_{k+1}$ be functions in $\text{Ker } T_\Phi$ that span $\mathcal{S}_{\Phi}^{(\sigma_{k+1})}$. The condition $T_\Phi f_j = 0$ and (5.3) implies that $H_\Phi f_j = \Phi f_j$ and using the fact that $f_j(\zeta) \in \mathcal{S}_{\Phi}^{(\sigma_{k+1})}(\zeta)$ almost everywhere on $\mathbb{T}$, we can write

$$(W_{k+1} f_j, f_j) = \|\Phi f_j\|^2 = \|H_\Phi f_j\|^2 = \|H_\Psi f_j\|^2 \leq \|\Psi f_j\|^2 = (\Psi^* \Psi f_j, f_j).$$

We have already proved that $\Psi^* \Psi \leq W_{k+1}$, and so the inequality in the above chain turns into equality. Thus (5.3) implies that

$$\Psi f_j = H_\Phi f_j = H_\Psi f_j = \Phi f_j,$$

which in turn implies condition (ii) follows, since

$$\text{span}\{f_j(\zeta) : 0 \leq j \leq N_{k+1}\} = \mathcal{S}_{\Phi}^{(\sigma_{k+1})}(\zeta), \quad \zeta \in \mathbb{T}.$$ 

Condition (iii) is again an immediate consequence of (i) and (ii).

6. Badly Approximable Matrix Functions

In this section we obtain a characterization of the badly approximable matrix functions $\Phi$ satisfying the condition $\|H_\Phi\|_e < \|\Phi\|_{L^\infty}$. Finally, under the same assumption we characterize matrix functions $\Phi$, for which $0$ is the only best approximation.

**Theorem 6.1.** Let $\Phi$ be a matrix function in $L^\infty(M_{m,n})$ such that $\|H_\Phi\|_e < \|\Phi\|_{L^\infty}$. Then $\Phi$ is badly approximable if and only if the following conditions are satisfied:

(i) $\|\Phi(\zeta)\|_{M_{m,n}}$ is constant for almost all $\zeta \in \mathbb{T}$; 
(ii) there exists a function $f$ in $\text{Ker } T_\Phi$ such that $f(\zeta)$ is a maximizing vector of $\Phi(\zeta)$ for almost all $\zeta \in \mathbb{T}$.

**Remark 1.** It will be clear from the proof that if $\Phi$ is an arbitrary matrix function satisfying (i) and (ii), then it is badly approximable. In other words, to prove that (i) and (ii) imply that $\Phi$ is badly approximable, we do not need the condition $\|H_\Phi\|_e < \|\Phi\|_{L^\infty}$.

**Proof.** Suppose that $\Phi$ is badly approximable. Then it admits a factorization

$$\Phi = W^* \begin{pmatrix} \sigma u & 0 \\ 0 & \Psi \end{pmatrix} V^*$$

where $V$ and $W^t$ are thematic (1-balanced) matrix functions, $\sigma = \|\Phi\|_{L^\infty}$, $u$ is a scalar unimodular badly approximable function such that $\|H_u\|_e < 1$, and $\Psi$ is an $(m-1) \times (n-1)$ matrix function such that $\|\Psi\|_{L^\infty} \leq \sigma$ (see [AP] or [Pe], Ch. 14, §4). Let

$$V = \begin{pmatrix} v & \overline{\Theta} \end{pmatrix}, \quad W = \begin{pmatrix} w & \overline{\Xi} \end{pmatrix}^t,$$
where \( \mathbf{v} \) and \( \mathbf{w} \) are inner and co-outer column functions while \( \Theta \) and \( \Xi \) are inner and co-outer matrix functions.

It follows from the characterization of badly approximable scalar functions mentioned in the introduction that \( T_u \) is Fredholm and \( \text{ind} \, T_u > 0 \). Therefore \( \text{Ker} \, T_u \neq \{0\} \). Let \( h \) be a nonzero function in \( \text{Ker} \, T_u \). Put \( f = h \mathbf{v} \). We have

\[
\Phi f = W^* \begin{pmatrix} \sigma u & 0 \\ 0 & \Psi \end{pmatrix} \begin{pmatrix} \mathbf{v}^* \\ \Theta^t \end{pmatrix} h \mathbf{v} = W^* \begin{pmatrix} \sigma u & 0 \\ 0 & \Psi \end{pmatrix} \begin{pmatrix} h \\ 0 \end{pmatrix} = (\mathbf{w} \Xi) \begin{pmatrix} \sigma u h \\ 0 \end{pmatrix} = \sigma u h \mathbf{w} \in H^2(\mathbb{C}^m),
\]

since \( h \in \text{Ker} \, T_u \). Thus \( f \in \text{Ker} \, \Phi \).

On the other hand,

\[
\|H_\Phi f\| = \|\Phi f\| = \|\Phi\|_{L^\infty} \|f\|,
\]

i.e., \( \|H_\Phi\| = \|\Phi\|_{L^\infty} \), and so \( \Phi \) is badly approximable. ■

The following theorem describes badly approximable functions, for which \( 0 \) is the only best approximation. If \( \Phi \) is a nonzero matrix function in \( L^\infty(\mathbb{M}_{m,n}) \), we can normalize it by the condition \( \|\Phi\|_{L^\infty} = 1 \).

**Theorem 6.2.** Let \( \Phi \) be a matrix function in \( L^\infty(\mathbb{M}_{m,n}) \) such that

\[ \|H_\Phi\|_e < \|\Phi\|_{L^\infty}. \]

Then \( 0 \) is the only best approximation of \( \Phi \) if and only if the following conditions are satisfied:

(i) \( \Phi \) takes isometric values if \( n \leq m \) and \( \Phi^t \) takes isometric values if \( n > m \) almost everywhere on \( \mathbb{T} \);

(ii) the function \( \zeta \mapsto (\text{Ker} \, \Phi(\zeta))^\perp, \zeta \in \mathbb{T} \), is spanned by finitely many functions in \( \text{Ker} \, T_\Phi \).

**Remark 2.** If \( n \leq m \) and \( \Phi \) satisfies (i), then \( \text{Ker} \, \Phi(\zeta) = \{0\} \) for almost all \( \zeta \in \mathbb{T} \), and so (ii) means that there are finitely many functions \( f_j \in \text{Ker} \, T_\Phi \) such that \( \text{span} \{f_j(\zeta) : j = 1, 2, \ldots\} = \mathbb{C}^n \) for almost all \( \zeta \in \mathbb{T} \). Note that if \( n > m \), then instead of \( \Phi \) we can consider the transposed function \( \Phi^t \).

**Remark 3.** As in the case of Theorem to prove that (i) and (ii) imply that \( 0 \) is the only best approximation of \( \Phi \), we do not need the condition \( \|H_\Phi\|_e < \|\Phi\|_{L^\infty} \).

**Proof.** Let \( \Phi \) be a badly approximable matrix function in \( L^\infty(\mathbb{M}_{m,n}) \) such that \( \|H_\Phi\|_e < \|\Phi\|_{L^\infty} = 1 \). Let \( r \) be the number of superoptimal singular values of \( \Phi \).
equal to $\|\Phi\|_{L^\infty} = 1$. Suppose that $r < \min\{m, n\}$. By Theorem D in [2] $\Phi$ admits a factorization

$$\Phi = W^* \begin{pmatrix} U & 0 \\ 0 & \Psi \end{pmatrix} V^*,$$

where $V$ and $W^a$ are $r$-balanced matrix functions, $U$ is an $r \times r$ very badly approximable unitary-valued function such that $\|H_U\|_e < 1$, and $\Psi$ is an $(m-r) \times (n-r)$ matrix function such that $\|\Psi\|_{L^\infty} \leq 1$, $\|H_\Psi\|_e \leq \|H_\Phi\|_e$. Since $\|H_\Psi\| < 1$, there exist infinitely many matrix functions $F \in H^\infty(\mathbb{M}_{m-r,n-r})$ such that $\|\Psi - F\|_{\infty} \leq 1$. Note, that if $F \neq 0$, then the function $W^* \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix} V^*$ is a nonzero function in $H^\infty(\mathbb{M}_{m,n})$. Hence, $\Phi$ has infinitely many best approximations. Thus $r = \min\{m, n\}$, which means that (i) holds.

Note that any superoptimal approximation is a best approximation. Thus if $0$ is the only best approximation, it is also the only superoptimal approximation. So $\Phi$ is a very badly approximable function, and condition (ii) follows from Theorem 4.1.

Suppose now that a function $\Phi$ satisfies (i) and (iii). Let $F$ be a best approximation of $\Phi$, and let $\Psi = \Phi - F$. Let $f_j$ be functions in Ker $T_\Phi$ that span the function $(\text{Ker } \Phi(\zeta))^\perp$. The condition $T_\Phi f_j = 0$ implies that $H_\Phi f_j = \Phi f_j$ (see [5.3]), and so

$$\|f_j\|_2 = \|\Phi f_j\|_2 = \|H_\Phi f_j\|_2 = \|H_\Psi f_j\|_2 \leq \|\Psi f_j\|_2 \leq \|f_j\|_2.$$

Therefore all inequalities in the above chain must be equalities, and it follows from [5.3] that

$$\Psi f_j = H_\Psi f_j = H_\Phi f_j = \Phi f_j.$$

Hence,

$$\Psi(\zeta)|\langle \text{Ker } \Phi(\zeta) \rangle^\perp = \Psi(\zeta)|\langle \text{Ker } \Phi(\zeta) \rangle^\perp, \quad \zeta \in \mathbb{T}.$$

If $n \leq m$, then $(\text{Ker } \Phi(\zeta))^\perp = \mathbb{C}^n$, and therefore $\Phi \equiv \Psi$.

To show that $\Phi \equiv \Psi$ for $m < n$ one more step is needed. Namely, let us observe that $\Psi(\zeta)$ are contractions and that $\Phi(\zeta)$ are co-isometries (i.e., $\Phi(\zeta)^*$ are isometries) for almost all $\zeta \in \mathbb{T}$. It follows from Lemma 5.1 that if a contraction $T$ and a co-isometry $U$ coincide on $(\text{Ker } U)^\perp$, then $T|\text{Ker } U = 0$, and so $T = U$.

Thus we have proved that $\Psi = \Phi$, i.e., $F = 0$, and so $0$ is the only best approximation of $\Phi$. ■

References

[AAK] V.M. Adamyan, D.Z. Arov, and M.G. Krein, On infinite Hankel matrices and generalized problems of Carathéodory-Fejér and F. Riesz, Funktsional. Anal. i Prilozhen. 2:1 (1968), 1-19 (In Russian).
[AP] R.B. Alexeev and V.V. Peller, Badly approximable matrix functions and canonical factorizations, Indiana Univ. Math. J. 49 (2000), 1247-1285.

[D] R.G. Douglas, Banach algebra techniques in operator theory, Academic Press, New York–London 1972.

[Kh] S. Khavinson, On some extremal problems of the theory of analytic functions, Uchen. Zapiski Mosk. Universiteta, Matem. 144:4 (1951), 133-143. English Translation: Amer. Math. Soc. Translations (2) 32 (1963), 139-154.

[N] N.K. Nikol’skii, Treatise on the shift operator. Spectral function theory, Springer-Verlag, Berlin–Heidelberg–New York–Tokyo, 1986.

[Pe] V.V. Peller, Hankel operators and their applications, Springer-Verlag, New York, 2003.

[PT] V.V. Peller and S.R. Treil, Approximation by analytic matrix functions. The four block problem, J. Funct. Anal. 148 (1997), 191-228.

[PY1] V.V. Peller and N.J. Young, Superoptimal analytic approximations of matrix functions, J. Funct. Anal. 120 (1994), 300-343.

[PY2] V.V. Peller and N.J. Young, Superoptimal singular values and indices of matrix functions, Int. Eq. Op. Theory 20 (1994), 35-363.

[Po] S. J. Poreda, A characterization of badly approximable functions, Trans. Amer. Math. Soc., 169 (1972), 249-256.

[T] S.R. Treil, On superoptimal approximation by analytic and meromorphic matrix-valued functions, J. Funct. Anal. 131 (1995), 386-414.

[V] V.I. Vasyunin, Formula for multiplicity of contractions with finite defect indices, Oper. Theory: Adv. Appl., Birkhäuser 4 (1989), 281-304.

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