REGULARIZED LIMIT OF DETERMINANTS FOR DISCRETE TORI

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Abstract. We consider a combinatorial Laplace operator on a sequence of discrete graphs which approximates the \(m\)-dimensional torus when the discretization parameter tends to infinity. We establish a polyhomogeneous expansion of the resolvent trace for the family of discrete graphs, jointly in the resolvent and the discretization parameter. Based on a result about interchanging regularized limits and regularized integrals, we compare the regularized limit of the log-determinants of the combinatorial Laplacian on the sequence of discrete graphs with the logarithm of the zeta determinant for the Laplace Beltrami operator on the \(m\)-dimensional torus. In a similar manner we may apply our method to compare the product of the first \(N \in \mathbb{N}\) non-zero eigenvalues of the Laplacian on a torus (or any other smooth manifold with an explicitly known spectrum) with the zeta-regularized determinant of the Laplacian in the regularized limit as \(N \to \infty\).

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1. Introduction and formulation of the results

Introduction of a zeta-regularized determinant for the Hodge Laplacian on compact Riemannian manifolds by Ray and Singer [RaSi71] provided a counterpart to the determinant of discrete Laplacian on a simplicial complex associated to a triangulation of the manifold. Relation between the zeta-regularized determinant of
the Hodge Laplacian as an analytic spectral invariant, and the determinant of the
discrete Laplacian as a combinatorial quantity, has been shown to go beyond be-
ing just formal counterparts by the proof of the Ray-Singer conjecture [RAS71] by
Cheeger [CHE79] and Müller [MÜL78].

In fact, Müller [MÜL78] proved that a specific combination of determinants for
discrete Laplacians in various degrees, defined on simplicial complexes associated
to a triangulation of a compact Riemannian manifold, converges to the correspond-
ing combination of zeta-regularized determinants of Hodge Laplacians when the
mesh of the triangulation goes to zero. Another instance of a link between determi-
nants of discrete Laplacians and zeta-regularized determinants of the correspond-
ing Hodge Laplacians is a recent joint work with Reshetikhin [REV13] which in part
motivated the Burghelea Friedlander Kappeler gluing formula for determinants by
studying the corresponding combinatorial problem.

In both instances the behavior of the individual determinants remained open,
since the discussion is rather based on existence of a well-defined limit for combina-
tions of determinants for discrete Laplacians under finer discretizations. This leads
to the general problem if the zeta-regularized determinant of a Hodge Laplacian
may indeed be recovered from its discretization. This translates into a question on
existence of an asymptotic expansion for determinants of discrete Laplacians under
refinement of discretization.

Interest in the asymptotic behaviour for determinants of discrete Laplacians arises
in various mathematical settings, even without the conjectured relationship with
its zeta-regularized counterpart. In statistical mechanics the interest stems from
identification of the determinant in terms of the number of spanning trees on a
graph by Kirchhoff [KIR47]. Moreover, in the setting of two-dimensional lattices,
determinants of certain $\mathbb{Z}^2$ subgraphs were expressed in terms of the number of
dimer coverings of related $\mathbb{Z}^2$ subgraphs by Temperley [TEM74].

In mathematical physics, existence of an asymptotic expansion for determinants
of discrete Laplacians may provide a way to construct quantum field theory of a free
scalar Bose field as a scaling limit of a Gaussian quantum field theory on a discrete
simplicial complex associated to a triangulation of the manifold, as the mesh of the
triangulation goes to zero. In fact, this intuition also lies behind Hawking [HAW77].

In fact, asymptotics for determinants of discrete Laplacians has been studied in
several instances with partial results. In the setup of rectilinear polygonal domains,
Kenyon [KEN00] derived a partial asymptotic expansion for the determinant of the
corresponding discrete Laplacian. However existence of a constant term in that par-
tial expansion, let alone its identification with the zeta-regularized determinant, re-
mained an open problem. Other related results include Burton-Pemantle [BUP93]
and Sridhar [SRI13].

In the setting of tori, the spectrum of the discrete and Hodge Laplacians is under-
stood explicitly, which naturally allows for finer asymptotic results. This setting has
been studied by Chinta, Jorgenson and Karlsson [CJL10], who equated the constant
term in the asymptotics for determinant of discrete Laplacians with the logarithm of the zeta-determinant, cf. also the preceding results by Kasteleyn \cite{Kas61} and Duplantier-David \cite{DuDa88}. Their analysis is based on a discussion of the discrete heat operator in terms of Bessel functions, and is strongly rooted in the explicit structure of the spectrum. In the setting of two-dimensional tori, Chamaud \cite{Cha06} has relaxed geometric assumptions by variational methods.

Closely related to the question, if zeta-regularized determinant of a Hodge Laplacian may be recovered from asymptotics of its discrete counterpart, is a problem of relating the zeta-regularized determinant to finite eigenvalue products. Asymptotic behavior of eigenvalue products has been studied by Szegö \cite{Sze15} for certain Toeplitz matrices, and in fact Friedlander-Guillemin \cite{FrGu08} have compared the Szegö and zeta-regularized determinants for zero-th order pseudo-differential operators. In a related work Friedlander \cite{Fri89} obtained the zeta-determinant of a higher order elliptic pseudo-differential operator \( A \) by considering the asymptotics of a determinant for some determinant class operator associated to \( A^{-1} \).

The presented references share the common idea of replacing the meromorphic continuation technique used in the definition of the zeta-determinant, by analysis of asymptotic expansions of classical determinants. Our paper studies this question in the explicit setting of \( m \)-dimensional tori and reproves the result Chinta, Jorgenson and Karlsson proposing an alternative independent ansatz. Our method is based on a polyhomogeneous asymptotic expansion of the combinatorial resolvent trace jointly in their resolvent and the discretization parameters. In particular this approach may be be viewed as part of a program initiated jointly with Lesch \cite{LeVe13}, cf. also Vertman \cite{Ver13} and Sauer \cite{Sau13}. The main technical tool is a careful analysis of the terms in the Euler Maclaurin formula, as well as a result on interchangeability of regularized limits and integrals. We focus on identification of the regularized limit of discrete determinants in terms of the zeta-regularized determinant, rather than studying other terms in the asymptotics as in \cite{CJK10}.

Remark 1.1. Our discussion is applicable beyond the setting of discrete tori. A brief look into our argument makes apparent that our results depend on existence of a polyhomogeneous expansion of the combinatorial resolvent trace, which needs not be an exclusive feature of \( m \)-dimensional tori. We therefore expect our results to have applications to explicitly computable quotients of \( \mathbb{R}^m \) under action of \( \mathbb{Z}^m \) lattice subgroups.

Remark 1.2. We point out that the same principle may be applied to identify the product of the first \( N \in \mathbb{N} \) non-zero eigenvalues of the Laplacian on a torus (or any other smooth manifold with an explicitly known spectrum) with its zeta-regularized determinant in the regularized limit as \( N \to \infty \).
1.1. Hadamard partie finie regularization. Consider $f \in C^\infty(\mathbb{R}_+, \mathbb{C})$, $\mathbb{R}_+ := (0, \infty)$ such that for $x \to \infty$ in the Landau notation

$$f(x) = \sum_{j=1}^{N-1} \sum_{k=0}^{M_j} a_{jk} x^{\alpha_j} \log^k(x) + \sum_{k=0}^{M_0} a_{0k} \log^k(x) + o(x^{\alpha_N} \log^{M_N}(x))$$

for some $N \in \mathbb{N}$ and $(\alpha_i) \subset \mathbb{C}$, such that $(\text{Re} \alpha_i)$ is a monotonously decreasing sequence with $\text{Re} \alpha_N < 0$. Then we define the regularized limit of $f(x)$ as $x \to \infty$ by

$$\text{LIM}_{x \to \infty} f(x) := a_{00}.$$

If $\text{Re} \alpha_N < -1$, the integral of $f$ over $[1, R]$ admits an asymptotic expansion of the form (1.1) as $R \to \infty$, and we set

$$\int_1^\infty f(x) \, dx := \text{LIM}_{R \to \infty} \int_1^R f(x) \, dx.$$

Similar definition holds for the regularized limit at $x = 0$ assuming an appropriate asymptotic expansion of $f(x)$ as $x \to 0$, of the form (1.1), where $(\text{Re} \alpha_i)$ monotonously increasing with $\text{Re} \alpha_N > 0$.

One crucial analytic property of regularized limits and integrals is the following interchangeability result, which has been presented in [LeVe13, Lemma 3.3] under mildly stronger assumptions on the asymptotics. We provide a proof for general asymptotic expansions of the form (1.1).

**Proposition 1.3.** Let $f \in C^\infty(\mathbb{R}_+^2, \mathbb{C})$, $\mathbb{R}_+ = (0, \infty)$, be homogeneous of order $d \in \mathbb{C}$ jointly in both variables and asymptotic expansions of the form (1.1) in each of the variables $(z, n) \to \infty$ individually (with the other variable fixed)

$$f(z, 1) = \sum_{j=1}^{N-1} \sum_{k=0}^{M_j} a_{jk} z^{\alpha_j} \log^k(z) + \sum_{k=0}^{M_0} a_{0k} \log^k(z) + o(z^{\alpha_N} \log^{M_N}(z)),$$

$$f(1, n) = \sum_{j=1}^{N'-1} \sum_{k=0}^{M'_j} b_{jk} n^{\beta_j} \log^k(n) + \sum_{k=0}^{M'_0} b_{0k} \log^k(n) + o(n^{\beta_{N'}} \log^{M'_N}(n)),$$

for some $N, N' \in \mathbb{N}$ and $(\alpha_i), (\beta_i) \subset \mathbb{C}$, such that $(\text{Re} \alpha_i), (\text{Re} \beta_i)$ are monotonously decreasing sequences with $\text{Re} \alpha_N < -1, \text{Re} \beta_{N'} < \min(0, d + 1)$. Then

$$\text{LIM}_{n \to \infty} \int_1^\infty f(z, n) \, dz = \int_1^\infty \text{LIM}_{n \to \infty} f(z, n) \, dz + \text{Corr},$$

where $\text{Corr} = \int_0^\infty f(z, 1) \, dz$ if $d = -1$ and zero otherwise.

It is an inherent part of the statement, that the regularized limits and integrals in the equality (2.2) exist.

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1In contrast to [LeVe13, Lemma 3.3] we do not assume that $f(z, 1)$ and $f(1, n)$ are smooth at $z, n = 0$, and require partial asymptotics as $z, n \to \infty$ instead. In fact the latter result is applied to some homogeneous $f(z, n)$, where smoothness of $f(z, 1)$ and $f(1, n)$ at $z, n = 0$ indeed fails.
1.2. Polyhomogeneous expansion of combinatorial resolvent traces. For any integer \( n \in \mathbb{N} \) we consider the quotient space \( \mathbb{Z}/n\mathbb{Z} \) which we refer to as a discrete circle. A finite product of \( m \) copies of \( \mathbb{Z}/n\mathbb{Z} \) defines a discrete torus \( \mathbb{T}_n^m \), which may be viewed as a discretization of the \( m \)-dimensional torus manifold \( \mathbb{T}^m \), given by a product of \( m \) copies of \( S^1 \). Here, \( S^1 \subset \mathbb{R}^2 \) is a circle with radius 1.

The combinatorial Laplacian \( \Delta_n \) on the discrete torus \( \mathbb{T}_n^m \) is the sum of the Laplacians \( L_n \) on each discrete circle \( \mathbb{Z}/n\mathbb{Z} \) component, defined for any \( f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{R} \) by the following difference operator

\[
(L_n f)([k]) = \frac{n^2}{4\pi^2} (f([k]) - f([k - 1])) + (f([k]) - f([k + 1])) .
\]

Then the spectra of \( L_n \) and \( \Delta_n \) amount to

\[
\sigma(L_n) = \left\{ \frac{n^2}{\pi^2} \sin^2 \left( \frac{\pi k}{n} \right) \right\}_{k \in \mathbb{N}_0, k < n},
\]

\[
\sigma(\Delta_n) = \left\{ \frac{n^2}{\pi^2} \sum_{i=1}^{m} \sin^2 \left( \frac{\pi k_i}{n} \right) \right\}_{k_i \in \mathbb{N}_0, k_i < n \text{ for } i = 1, \ldots, m}.
\]

In this multiset notation, eigenvalues \( \lambda \) appear multiple times according to their multiplicity \( m(\lambda) \). By the particular choice of the rescaling factor \( \frac{n^2}{\pi^2} \), eigenvalues of \( L_n \) and \( \Delta_n \) approximate the eigenvalues of the Laplace Beltrami operators on \( S^1 \) and \( \mathbb{T}^m \) respectively, as \( n \to \infty \).

Consider the combinatorial Laplacian \( \Delta_n \) on the discrete torus \( \mathbb{T}_n^m \) and traces of the corresponding resolvent powers \( \text{Tr}(\Delta_n + z^2)^{-\alpha} \) for any \( \alpha \in \mathbb{N} \) and \( z \in \mathbb{R}_+ \). We refer to the quantities \( \text{Tr}(\Delta_n + z^2)^{-m} \) and \( \text{Tr}(\Delta + z^2)^{-m} \) as resolvent traces. The resolvent trace of the Laplace Beltrami operator \( \Delta \) on the smooth torus \( \mathbb{T}^m \) can be obtained as the limit of the combinatorial trace. In fact we have the following result.

**Proposition 1.4.** The resolvent traces are smooth in \( z \in \mathbb{R}_+ \) and for \( k \in \mathbb{N} \)

\[
\lim_{n \to \infty} \partial_z^{(k)} \text{Tr}(\Delta_n + z^2)^{-m} = \partial_z^{(k)} \text{Tr}(\Delta + z^2)^{-m}.
\]

Our first central result is a polyhomogeneous expansion of the resolvent trace for the combinatorial Laplacian \( \Delta_n \), jointly in the resolvent parameter \( z \in \mathbb{R}_+ \) and the discretization parameter \( n \in \mathbb{N} \).

**Theorem 1.5.** The resolvent trace for \( \Delta_n \) admits a partial polyhomogeneous expansion

\[
\text{Tr}(\Delta_n + z^2)^{-m} = \sum_{j=0}^{m} h_{-m-j}(z, n) + H(z, n),
\]

where each \( h_{-m-j} \in C^\infty(\mathbb{R}_+^2) \) is homogeneous of order \((-m-j)\) jointly in \((z, n)\), \( h_{-m-j}(z, 1) \) and \( h_{-m-j}(1, n) \) admit an asymptotic expansion of the form (1.1) as \( z, n \to \infty \), respectively. The remainder term satisfies \( H_N(z, n) = O(z^{-2m-2}) \), as \( z \to \infty \), uniformly in \( n > 0. \)
1.3. Zeta-functions and zeta-regularized determinants. In the next step we introduce (zeta-regularized) determinants of $\Delta_n$ and $\Delta$. In the discrete case, the determinant of $\Delta_n$ is defined here as a product of its non-zero eigenvalues, counted with their multiplicities, and in fact satisfies the following integral representation

$$\log \det \Delta_n = -2 \int_0^\infty z \text{Tr}(\Delta_n + z^2)^{-1} dz,$$

which is an immediate consequence of the following computation

$$-2 \int_0^\infty \frac{z dz}{(\lambda + z^2)} = - \lim_{R \to \infty} \lim_{\varepsilon \to 0} \left[ \log(\lambda + z^2) \right]_{z=\varepsilon}^{z=R} = \left\{ \begin{array}{ll} \log(\lambda), & \text{if } \lambda \neq 0, \\ 0, & \text{if } \lambda = 0. \end{array} \right.$$

Integrating (1.4) by parts $(m-1)$ times (more precisely we perform integration by parts for the integral on $[\varepsilon, R]$ and take the regularized limit of the total expression as $\varepsilon \to 0$ and $R \to \infty$) yields

$$\log \det \Delta_n = -2 \int_0^\infty z^{2m-1} \text{Tr}(\Delta_n + z^2)^{-m} dz,$$

with the boundary terms vanishing in the regularized limit.

The zeta-regularized determinant of the Laplace Beltrami operator $\Delta$ is obtained by the following procedure. The zeta-function $\zeta$ of $\Delta$ is defined for $\text{Re}(s) > m/2$ by

$$\zeta(s, \Delta) := \sum_{\lambda \in \text{Spec} \Delta \setminus \{0\}} m(\lambda) \lambda^{-s},$$

where $m(\lambda)$ denotes the multiplicity of the eigenvalue $\lambda$. The integral expression (cf. [LeVe13, Section 1.3]) amounts after iterative integration by parts to

$$\zeta(s, \Delta) = 2 \frac{\sin \pi s}{\pi} \frac{\Gamma(1-s)\Gamma(m)}{\Gamma(m-s)} \int_0^\infty z^{2m-2s-1} \text{Tr}(\Delta + z^2)^{-m} dz,$$

with the standard asymptotic expansion of the resolvent trace $\text{Tr}(\Delta + z^2)^{-m}$ yields a meromorphic extension of $\zeta(s, \Delta)$ to the whole complex plane $\mathbb{C}$ with $s = 0$ being a regular point. We define the zeta-regularized determinant by

$$\log \det_\zeta \Delta := - \left. \frac{d}{ds} \right|_{s=0} \zeta(s, \Delta) = -2 \int_0^\infty z^{2m-1} \text{Tr}(\Delta + z^2)^{-m} dz.$$

1.4. Main result: Approximation by combinatorial determinants.

Our main result now reads as follows.

**Theorem 1.6.** The logarithmic determinant $\log \det \Delta_n$ admits a regularized limit as $n \to \infty$, which equals the logarithm of the zeta-determinant of the Laplace Beltrami operator, i.e.

$$\log \det_\zeta \Delta = \lim_{n \to \infty} \log \det \Delta_n.$$
A remark on the relation to the discussion in [LeVe13] is in order. By Proposition 1.4 we may write

\[ \log \text{det}_c \Delta = -2 \int_0^\infty z^{2m-1} \text{Tr}(\Delta_n + z^2)^{-m} \, dz \]

\[ = -2 \int_0^\infty z^{2m-1} \lim_{n \to \infty} \text{Tr}(\Delta_n + z^2)^{-m} \, dz \]

\[ = -2 \int_0^\infty z^{2m-1} \lim_{n \to \infty} \sum_{k_1=0}^{n-1} \cdots \sum_{k_m=0}^{n-1} (\omega(n, k_1, \ldots, k_m) + z^2)^{-m} \, dz, \]

where we have introduced

\[(1.7) \quad \omega(n, k_1, \ldots, k_m) := \frac{n^2}{\pi^2} \sum_{i=1}^m \sin^2 \left( \frac{\pi k_i}{n} \right), \]

and hence our main Theorem 1.6 looks as an application of the result on interchangeability of regularized sums and integrals in [LeVe13]. However in contrast to the setting considered in [LeVe13], the individual summands \((\omega(n, k_1, \ldots, k_m) + z^2)^{-m}\) here depend on the limiting parameter \(n \in \mathbb{N}\) and do not individually admit a polyhomogeneous expansion jointly in the summation parameters \((k_1, \ldots, k_m)\) and the resolvent parameter \(z\). Only the full sum in the expression of \(\text{Tr}(\Delta_n + z^2)^{-m}\) is polyhomogeneous, while in the setup of [LeVe13] polyhomogeneity is rather lost after summation.

1.5. Comparison with a theorem by Chinta, Jorgenson and Karlsson. Chinta, Jorgenson and Karlsson [CJK10] use a different method to establish Theorem 1.6. In fact they consider a slightly more general setting where they allow the individual cyclic factors \(\mathbb{Z}/n\mathbb{Z}\) to converge to \(S^1\) at different rates.

If for \(\mathbf{n} := (n_1, \ldots, n_m) \in \mathbb{N}^m\) we write \(T^m_{\mathbf{n}} = (\mathbb{Z}/n_1\mathbb{Z}) \times \cdots \times (\mathbb{Z}/n_m\mathbb{Z})\), then in our picture the resolvent trace of the corresponding combinatorial Laplacian \(\Delta_n\) admits a partial polyhomogeneous expansion as \((\mathbf{n}, z) \to \infty\). We consider here the special case of \(n_j = n\) for all \(j = 1, \ldots, m\), with the argument for the general setup going along the same lines. Then [CJK10] asserts the following.

**Theorem 1.7 ([CJK10]).** Consider the rescaled combinatorial Laplacian \(\Delta'_n\) with \(\sigma(\Delta_n) = \frac{n^2}{4\pi^2} \sigma(\Delta'_n)\), which approximates the Laplace Beltrami operator \(\Delta'\) on a rescaled torus given by \(m\) copies of \((2\pi)^{-1}S^1\). Then the logarithmic determinant \(\log \text{det} \Delta'_n\) admits a regularized limit as \(n \to \infty\), which equals the logarithm of the zeta-determinant of the \(\Delta'\), i.e.

\[ \log \text{det}_c \Delta' = \text{LIM}_{n \to \infty} \log \text{det} \Delta'_n. \]

This statement can be easily seen to correspond to Theorem 1.6, which is basically an issue of conventions. Since the number of non-zero eigenvalues of \(\Delta_n\) is \((n^m - 1)\),
and by the identity \( \zeta(s, \Delta') = (2\pi)^{-2s} \zeta(s, \Delta) \), we obtain the following relations

\[
\log \det \Delta_n' = \log \det \Delta_n - (n^m - 1) \log \frac{n^2}{4\pi^2}.
\]

\[
\log \det_\zeta \Delta' = \log \det_\zeta \Delta + 2\zeta(0, \Delta) \log 2\pi.
\]

Since on closed compact manifolds \( \zeta(0, \Delta) = -\dim \ker \Delta \), which equals \((-1)\) in the setting of tori, we conclude that Theorems 1.6 and 1.7 are equivalent.

### 2. Interchanging regularized limits and integrals

The main result of this section is presented in [LeVe13, Lemma 3.3], albeit under mildly stronger assumptions on the asymptotics.

**Proposition 2.1.** Let \( f \in C^\infty(\mathbb{R}_+^2, \mathbb{C}), \mathbb{R}_+ = (0, \infty) \), be homogeneous of order \( d \in \mathbb{C} \) jointly in both variables and asymptotic expansions of the form (1.1) in each of the variables \( z, n \to \infty \) individually (with the other variable fixed)

\[
f(z, 1) = \sum_{j=1}^{M_1} \sum_{k=0}^{M_j} a_{jk} z^{\alpha_j} \log^k(z) + \sum_{k=0}^{M_0} a_{0k} \log^k(z) + o(z^{\alpha_N} \log^{M_N}(z)),
\]

\[
f(1, n) = \sum_{j=1}^{N'} \sum_{k=0}^{M_j'} b_{jk} n^{\beta_j} \log^k(n) + \sum_{k=0}^{M_0'} b_{0k} \log^k(n) + o(n^{\beta_{N'}} \log^{M_N}(n)),
\]

for some \( N, N' \in \mathbb{N} \) and \( (\alpha_j), (\beta_j) \subset \mathbb{C} \), such that \( \text{Re}(\alpha_j), \text{Re}(\beta_j) \) are monotonously decreasing sequences with \( \text{Re}(\alpha_N) < -1, \text{Re}(\beta_{N'}) < \min(0, d + 1) \). Then

\[
\lim_{n \to \infty} \int_0^\infty f(z, n) \, dz = \int_1^\infty \lim_{n \to \infty} f(z, n) \, dz + \text{Corr},
\]

where \( \text{Corr} = \int_0^\infty f(z, 1) \, dz \) if \( d = -1 \) and zero otherwise.

**Proof.** Using (2.1) and \( \text{Re}(\beta_{N'}) < 0 \) we find for \( f(z, n) = z^d f(1, n/z) \)

\[
\lim_{n \to \infty} f(z, n) = \sum_{k=0}^{M_0'} (-1)^k b_{0k} z^d \log^k(z).
\]

For any \( \alpha \in \mathbb{C} \) and \( k \geq 0 \) we compute iteratively

\[
\int z^\alpha \log^k(z) \, dz = \begin{cases} 
\sum_{j=0}^{k} (-1)^j k! \frac{\log^{k-j}(z)}{(k-j)! (\alpha+1)^{j+1}} z^{\alpha+1}, & \text{if } \alpha \neq -1, \\
\log^{k+1}(z) \cdot \frac{1}{(k+1)}, & \text{if } \alpha = -1.
\end{cases}
\]

Consequently we find

\[
\lim_{n \to \infty} \int_1^\infty f(z, n) \, dz = \begin{cases} 
\sum_{k=0}^{M_0'} (-1)^{k} b_{0k} (d+1)^{-k-1}, & \text{if } d \neq -1, \\
0, & \text{if } d = -1.
\end{cases}
\]
For the computation of the left hand side in (2.2), we employ the coordinate change rule for regularized integrals, cf. [Les97, Lemma 2.1.4], and obtain

\[ \int_{-1/n}^{1/n} f(x, 1) \text{d}x = n^{d+1} \int_{1/n}^{\infty} f(x, 1) \text{d}x - a_{j_0} n^{d+1} \log(n) \]

(2.6)

\[ = n^{d+1} \int_{-1}^\infty f(x, 1) \text{d}x - n^{d+1} \int_{0}^{1/n} f(x, 1) \text{d}x - a_{j_0} n^{d+1} \log(n), \]

where \( \alpha_{j_0} = -1 \). Using (2.1) and (2.3) we arrive for \( d \neq -1 \) at the following expansion.

\[
\int_{-1/n}^{1/n} f(x, 1) \text{d}x \sim_{n \to \infty} \sum_{j=1, \beta_j \neq d+1}^{N'-1} \sum_{k=0}^{M_{j}'} b_{jk} n^{\beta_j - (d+1)} \sum_{j=0}^{k} \frac{k! \log^{k-j}(1/n)}{(k-j)! (d+1 - \beta_j)^{j+1}} \\
+ \sum_{j=1, \beta_j = d+1}^{N'-1} \sum_{k=0}^{M_{j}'} b_{jk} \frac{\log^{k+1}(1/n)}{k+1} \\
+ \sum_{k=0}^{M_{0}'} b_{0k} n^{-d-1} \sum_{j=0}^{k} \frac{k! \log^{k-j}(1/n)}{(k-j)! (d+1)^{j+1}} \\
+ o(n^{\beta_{N'}-(d+1)} \log^{M_{N}'}(n)).
\]

If \( d = -1 \) the asymptotic expansion changes slightly to

\[
\int_{-1/n}^{1/n} f(x, 1) \text{d}x \sim_{n \to \infty} \sum_{j=1, \beta_j \neq d+1}^{N'-1} \sum_{k=0}^{M_{j}'} b_{jk} n^{\beta_j - (d+1)} \sum_{j=0}^{k} \frac{k! \log^{k-j}(1/n)}{(k-j)! (d+1 - \beta_j)^{j+1}} \\
+ \sum_{j=1, \beta_j = d+1}^{N'-1} \sum_{k=0}^{M_{j}'} b_{jk} \frac{\log^{k+1}(1/n)}{k+1} \\
+ \sum_{k=0}^{M_{0}'} (-1)^k b_{0k} \frac{\log^{k+1}(1/n)}{k+1} \\
+ o(n^{\beta_{N'}-(d+1)} \log^{M_{N}'}(n)).
\]

We may now take regularized limit of (2.5) as \( n \to \infty \) and find using \( \text{Re}(\beta_{N'}) < d+1 \)

\[ \text{LIM}_{n \to \infty} \int_{-1}^{1} f(z, n) \text{d}z = \begin{cases} 
\sum_{k=0}^{M_{0}'} (-1)^k b_{0k} (d+1)^{k-1}, & \text{if } d \neq -1, \\
\int_{0}^{\infty} f(x, 1) \text{d}x, & \text{if } d = -1.
\end{cases} \]

Statement follows from comparison of (2.7) and (2.4). \( \Box \)
3. Polyhomogeneous expansion of the combinatorial resolvent trace

This section is devoted to a proof of Theorem 1.5, using the Euler Maclaurin summation formula in \( n \) parameters. Write for \( n \in \mathbb{N}_0 \) and \( x = (x_1, \ldots, x_m) \in \mathbb{R}^m \)

\[
\omega(n, x) := \frac{n^2}{\pi^2} \sum_{i=1}^{m} \sin^2 \left( \frac{\pi x_i}{n} \right).
\]

Consider any tuple \( J = (j_1, \ldots, j_k) \subseteq \{1, \ldots, m\} \) of pairwise distinct integers with \( k \leq m \). We write \( |J| := k \) and put

\[
\{x_j = 0\} := \{x \in \mathbb{N}_0^m \cap [0, n]^m | x_{j_1} = \ldots = x_{j_k} = 0\}.
\]

With respect to this notation we obtain for \( \alpha \in \mathbb{N} \) and \( z \in \mathbb{R}^+ \) the following representation of the resolvent trace

\[
\text{Tr}(\Delta_n + z^2)^{-m} = \sum_{k=0}^{m} (-1)^k \sum_{|J|=k, \{x_j = 0\}} (\omega(n, x) + z^2)^{-m}.
\]

The summand corresponding to \( k = 0 \) is given explicitly by

\[
S(z, n) = \sum_{x_1=0}^{n} \cdots \sum_{x_m=0}^{n} (\omega(n, x_1, \ldots, x_m) + z^2)^{-m}.
\]

We employ the Euler Maclaurin formula for summation in \( m \) parameters\(^2\) to derive a polyhomogeneous expansion of \( S(z, n) \), the other terms in the resolvent trace formula (3.1) are treated ad verbatim. For any \( M \in \mathbb{N} \) we obtain

\[
S(z, n) = \sum_{\beta \in \{1,2,3,4\}^m} P_{\beta_1,1} \circ \cdots \circ P_{\beta_m,m} \left( (\omega(n, x_1, \ldots, x_m) + z^2)^{-m} \right),
\]

where each \( P_{\beta_j,i} \) acts in the \( x_j \)-variable on \( u \in C^\infty[0, \infty) \) by

\[
P_{\beta_1,1} u := \begin{cases} 
\int_0^{n} u(x_j) \, dx_j, & \text{if } \beta_j = 1, \\
\sum_{k=1}^{M} \frac{B_{2k}}{(2k)!} \left( \partial_{x_j}^{(2k-1)}|_{x_j=n} - \partial_{x_j}^{(2k-1)}|_{x_j=0} \right) u, & \text{if } \beta_j = 2, \\
\frac{1}{(2M + 1)!} \int_0^{n} B_{2M+1}(x_j - [x_j]) \partial_{x_j}^{(2M+1)} u(x_j) \, dx_j, & \text{if } \beta_j = 3, \\
\frac{1}{2} \left( u(x_j = n) + u(x_j = 0) \right), & \text{if } \beta_j = 4.
\end{cases}
\]

Here, \( B_i(x) \) denotes the \( i \)-th Bernoulli polynomial and \( B_i \) the \( i \)-th Bernoulli number. We study the \((z, n)\)-behaviour of the various summands. Note that for finite \( n \in \mathbb{N} \) all individual operators in the composition \( P_{\beta_1,1} \circ \cdots \circ P_{\beta_m,m} \) commute.

---

\(^2\)obtained by iterating the standard Euler Maclaurin formula in one single summation parameter.
Lemma 3.1. Put \( f(n, x, z) := \omega(n, x) + z^2 \). For each \( k \in \mathbb{N} \) we may write

\[
\partial_{x_j}^{[k]} f^{-\alpha} = \sum_{\ell=0}^{k-1} H_{k,\ell}(\partial_{x_j} f) f^{-\alpha-\ell-1},
\]

where \( H_{1,0}(\partial_{x_j} f) = -\alpha \partial_{x_j} f \) and higher functionals \( H_{k,\ell} \) are defined recursively by

\[
H_{k,\ell}(\partial_{x_j} f) = \partial_{x_j} H_{k-1,\ell}(\partial_{x_j} f) + H_{k-1,\ell-1}(\partial_{x_j} f) \cdot \partial_{x_j} f,
\]

where we set \( H_{k,k}, H_{k,-1} = 0 \). For \( k \) odd we have the estimate

\[
|H_{k,\ell}(\partial_{x_j} f)| \leq C_{k\ell} \begin{cases} n^{-k+2(\ell+1)}, & \text{if } k \geq 2(\ell+1), \\ x^{2(\ell+1)-k}, & \text{if } k \leq 2(\ell+1). \end{cases}
\]

Proof. The recursive structure of \( \partial_{x_j}^{[k]} f^{-\alpha} \) follows by induction and the only intricate statement is the estimate of \( H_{k,\ell}(g), g = \partial_{x_j} f \). Here we introduce a notion of homogeneity order for an expression \( H_{k,\ell}(g) \) by counting for every individual summand in \( H_{k,\ell}(g) \) each additional \( \partial_{x_j} \) differentiation as lowering homogeneity order by \((-1)\), and each factor of \( g \) as increasing homogeneity order by \((+1)\).

With this system, \( H_{1,0}(g) = -\alpha g \) is of homogeneity order 1, and by induction \( H_{k,\ell}(g) \) is of homogeneity order \((2(\ell+1)-k)\), where in total we count \((\ell+1)\) factors of \( g \), and \((k-(\ell+1))\) derivatives. Note the series expansion

\[
g = \frac{n}{\pi} \sin \left( \frac{\pi x_j}{n} \right) \cos \left( \frac{\pi x_j}{n} \right) = n \sum_{i=0}^{\infty} a_i \left( \frac{\pi x_j}{n} \right)^{2i+1}, \quad \partial_{x_j} g = \sum_{i=0}^{\infty} a_i (2i+1) \left( \frac{\pi x_j}{n} \right)^{2i}.
\]

Consequently, each \( g \) in \( H_{k,\ell}(g) \) carries a factor of \( x_j \), a single additional \( \partial_{x_j} \) differentiation applied to \( g \), annihilates that \( x_j \) factor, and each further derivative adds a factor of \( n^{-1} \). Consequently, homogeneity order simply counts the powers of \( x_j \) and \( n \) in the expression for \( H_{k,\ell}(g) \), leading to the statement. \( \square \)

First, we derive a polyhomogeneous expansion for the summand \( S(z, n) \).

Proposition 3.2. The function \( S(n, z) \) admits a partial polyhomogeneous expansion

\[
S(n, z) = \sum_{j=0}^{m} h'_{-m-j}(z, n) + H'(z, n),
\]

where each \( h'_{-m-j} \in \mathbb{C}^\infty(\mathbb{R}_+^m) \) is homogeneous of order \((-m-j)\) jointly in \((z, n)\), \( h'_{-m-j}(z, 1) \) and \( h'_{-m-j}(1, n) \) admit an asymptotic expansion of the form (3.1) as \( z, n \to \infty \), respectively. The remainder term satisfies \( H'_N(z, n) = O(z^{-2m-2}) \), as \( z \to \infty \), uniformly in \( n > 0 \). Moreover, \( h'_{-2m}(z, n) = z^{-2m} \).

Proof. Since \( \partial_{x_j} \) differentiation of odd order applied to \((\omega(n, x) + z^2)^{-m}\), always leads to factors \( \sin(\pi x/n) \), summands in (3.2) with \( \beta_j = 2 \) for some \( j = 1, \ldots, m \) vanish. We are left to consider summands in (3.2) with \( \beta_j \in \{1, 3, 4\} \). Consider first summands
in (3.2) with \( \beta_j = 3 \) for some \( j = 1, \ldots, m \). We will use the following basic estimates

\[
(3.4) \quad \frac{x_j}{(\omega(n, x) + z^2)} \leq C \frac{x_j}{x_j^2 + z^2} \leq C(2z)^{-1},
\]

\[
(3.5) \quad \int_{[0,n]^m} (\omega(n, x) + z^2)^{-m} d^m x \leq \left( \int_0^n \left( \frac{n^2 \sin^2 (\pi x)}{\pi} + z^2 \right)^{-1} dx \right)^m \leq \left( \frac{2n}{\sqrt{z^2 + n^2/\pi^2}} \right)^m \leq (2\pi)^m,
\]

where we denote all universal constants in the estimates by the same symbol \( C > 0 \). Let \( k = 2M + 1 \geq 3m + 2 \) be some sufficiently large integer. By Lemma 3.1 (assume \( \alpha \geq m \)) we obtain in case \( k \leq 2(\ell + 1) \)

\[
\int_{[0,n]^m} |H_{k,\ell}(\partial_{x_j} f) f^{-\alpha-\ell-1}| \leq \int_{[0,n]^m} \frac{C x_j^{2(\ell+1)-k}}{(\omega(n, x) + z^2)^{2(\ell+1)-k}} (\omega(n, x) + z^2)^{-(\alpha+k-(\ell+1))} d^m x
\]

\[
\leq C z^{-2(\ell+1)+k} \int_{[0,n]^m} (\omega(n, x) + z^2)^{-(\alpha+k-(\ell+1))} d^m x
\]

\[
\leq C z^{-k} \int_{[0,n]^m} (\omega(n, x) + z^2)^{-\alpha} d^m x
\]

\[
\leq C z^{-k-2(\alpha-m)} \int_{[0,n]^m} (\omega(n, x) + z^2)^{-m} d^m x \leq C(2\pi)^m z^{-2\alpha-2},
\]

where we used (3.4) in the second inequality, and employed (3.5) together with \( k \geq 2m + 2 \) in the last inequality. In the case \( k > 2(\ell + 1) \), we compute similarly

\[
\int_{[0,n]^m} |H_{k,\ell}(\partial_{x_j} f) f^{-\alpha-\ell-1}| \leq C n^{-k+2(\ell+1)} \int_{[0,n]^m} (\omega(n, x) + z^2)^{-\alpha-(\ell+1)} d^m x
\]

\[
\leq \begin{cases} 
  z^{-2\alpha-2} \int_{[0,n]^m} (\omega(n, x) + z^2)^{-m} d^m x, & \text{if } \ell \geq m, \\
  n^{-k+2(\ell+1)} z^{-2\alpha-2} \int_{[0,n]^m} 1 d^m x, & \text{if } \ell < m,
\end{cases}
\]

\[
\leq C' z^{-2\alpha-2},
\]

where we used \( k \geq 3m \) and (3.5) in the final estimate. Consequently, if \( \beta_j = 3 \) for some single \( j = 1, \ldots, m \), we may estimate for \( M \geq (3m + 1)/2 \) uniformly in \( n > 0 \)

\[
\left| P_{\beta_1,1} \circ \cdots \circ P_{\beta_m,m} (\omega(n, x_1, \ldots, x_m) + z^2)^{-\alpha} \right| \leq C z^{-2\alpha-2}.
\]

If \( \# \{ \beta_j = 3 \} > 1 \), the estimates proceed along the same lines. Consider next the case with \( \beta_j = 1 \) for all \( j = 1, \ldots, m \). The corresponding summand is given by the
Moreover, we need to extend the computations in Proposition 3.1. The remainder term satisfies
\[ h_{-2\alpha + m}(z, 1) + h_{-2\alpha + m}(1, \cdot) \] follows e.g. from a careful application of Melrose’s push forward theorem. The expression is homogeneous of order \((-2\alpha + m)\) jointly in \((n, z)\). Existence of an asymptotic expansion for the homogeneous terms \(h'_{-2\alpha + m}(\cdot, 1), h'_{-2\alpha + m}(1, \cdot)\) follows from the explicit structure of the expansion is irrelevant in our discussion.

It remains to discuss terms with \(\beta_j = 4\) for some \(j \in \{1, \ldots, m\}\). Each \(P_{4j}\) is an evaluation operator and in fact for any family of pairwise distinct integers \(J = \{j_1, \ldots, j_k\} \subset \{1, \ldots, m\}\) we have
\[ P_{4j_1} \circ \cdots \circ P_{4j_k}(\omega(n, x_1, \ldots, x_m) + z^2)^{-\alpha} = (\omega(n, y) + z^2)^{-\alpha}, \]
where we have introduced \(y = (y_1, \ldots, y_m) := (x_{j_{k+1}}, \ldots, x_{j_m})\) and write
\[ \omega_j(n, y) := \frac{n^2}{\pi^2} \sum_{i=1}^{m-k} \sin^2 \left( \frac{n \pi y_i}{n} \right). \]
Consequently, analysis of the terms with \(\beta_j = 4\) for \(j \in J\) proceeds along the lines above, with \(m\) simply replaced by \((m - k)\). This leads to homogeneous terms of homogeneity order \((-2\alpha + m - k)\). Setting \(\alpha = m\) proves the statement, once we observe that the homogeneous term of order \((-2m)\) is given explicitly by
\[ P_{4,1} \circ \cdots \circ P_{4,m}(\omega(n, x_1, \ldots, x_m) + z^2)^{-m} = z^{-2m}. \]

We may now prove our first main result.

**Theorem 3.3.** The resolvent trace admits a partial polyhomogeneous expansion
\[ \text{Tr}(\Delta_n + z^2)^{-\alpha} = \sum_{j=0}^{2m} h_{-m-j}(z, n) + H(z, n), \]
where each \(h_{-m-j} \in C^\infty(\mathbb{R}^2_+)\) is homogeneous of order \((-m-j)\) jointly in \((z, n)\), \(h_{-m-j}(z, 1)\) and \(h_{-m-j}(1, n)\) admit an asymptotic expansion of the form (1.1) as \(z, n \to \infty\), respectively. The remainder term satisfies \(H_N(z, n) = O(z^{-2m-2})\), as \(z \to \infty\), uniformly in \(n > 0\). Moreover, \(h_{-2m}(z, n) = 0\).

**Proof.** We need to extend the computations in Proposition 3.2 to general terms in the expression (3.1). Consider any tuple \(J = \{j_1, \ldots, j_k\} \subset \{1, \ldots, m\}\) of pairwise distinct integers with \(k \leq m\) and the corresponding term
\[ \sum_{|J| = 0} (\omega(n, x) + z^2)^{-m}. \]

Note that \((\omega(1, y) + (z/n)^2)^{-m}\) lifts to a polyhomogeneous function on the blowup space \(\{(y, t) \mid y_i \in [0, 1), t \in [0, \infty]\}\). Pushforward theorem of Melrose [Mel92, Mel93] then yields an asymptotic expansion of the integral as \(t \to 0\) or \(t \to \infty\). The explicit structure of the expansion is irrelevant in our discussion.
Put \( y = (y_1, \ldots, y_{m-k}) := (x_{j+k+1}, \ldots, x_{jm}) \) and note
\[
\sum_{\{x_j=0\}} (\omega(n, x) + z^2)^{-m} = \sum_{y \in [0,n]^{m-k}} (\omega(n, y) + z^2)^{-(m-k)-k},
\]
where we have introduced
\[
\omega_j(n, y) := \frac{n_j^2}{\pi^2} \sum_{i=1}^{m-k} \sin^2 \left( \frac{\pi y_i}{n} \right).
\]
The computations follow for these terms along the lines of Proposition 3.2 with \( m \) replaced by \((m - k)\) and \( \alpha = m \). It remains to discuss the homogeneous term of order \((-2m)\). Note by the statement of Proposition 3.2 on the \((-2m)\)-homogeneity terms
\[
h_{-2m}(z, n) = \sum_{k=0}^{m} (-1)^k \sum_{|J|=k} z^{-2m} = \sum_{k=0}^{m} (-1)^k \binom{m}{k} z^{-2m} = 0.
\]
This proves the statement.

4. CONVERGENCE OF THE COMBINATORIAL RESOLVENT TRACES

In this section we prove Proposition 1.4, using an argument of Dodziuk [Dod76] on convergence of zeta functions.

Proposition 4.1. For any integer \( \alpha \geq m \)
\[
\lim_{n \to \infty} \partial_z^{(k)} \text{Tr}(\Delta_n + z^2)^{-\alpha} = \partial_z^{(k)} \text{Tr}(\Delta + z^2)^{-\alpha}.
\]

Proof. The argument does not depend on the explicit structure of the spectra for \( \Delta_n \) and \( \Delta \). Hence we write
\[
\sigma(\Delta_n) = \{\lambda_{k,n} \mid k = 0, \ldots, N(n)\},
\]
\[
\sigma(\Delta) = \{\lambda_k \mid k \in \mathbb{N}_0\},
\]
where both sets are ordered in an ascending order, \( N(n) \) is the total number of \( \Delta_n \)-eigenvalues counted with their multiplicities, \( N(n) \to \infty \) monotonously increasing and \( \lambda_{k,n} \to \lambda_k \) as \( n \to \infty \), for each fixed \( k \in \mathbb{N}_0 \).

Fix any \( \varepsilon > 0 \). Since the resolvent trace
\[
\text{Tr}(\Delta + z^2)^{-\alpha} = \sum_{k=0}^{\infty} (\lambda_k + z^2)^{-\alpha},
\]
is a convergent series, there exists \( K \in \mathbb{N} \) sufficiently large, such that
\[
\sum_{k=K}^{\infty} (\lambda_k + z^2)^{-\alpha} < \varepsilon/3.
\]
Consequently, since \((\lambda_{k,n})_{n \in \mathbb{N}}\) is an ascending sequence, by the minimax principle (cf. [Dod76]), we may estimate

\[
0 \leq \sum_{k=K}^{\infty} (\lambda_k + z^2)^{-\alpha} - \sum_{k=K}^{N(n)} (\lambda_{k,n} + z^2)^{-\alpha} < 2\varepsilon/3.
\]

Finally, given convergence of the spectrum, choose \(n \geq n_0\), such that

\[
0 \leq \sum_{k=0}^{K} ((\lambda_k + z^2)^{-\alpha} - (\lambda_{k,n} + z^2)^{-\alpha}) < \varepsilon/3.
\]

Thus, for any given \(\varepsilon > 0\), there exists \(n \in \mathbb{N}\) sufficiently large, such that

\[
0 \leq \text{Tr}(\Delta + z^2)^{-\alpha} - \text{Tr}(\Delta_n + z^2)^{-\alpha} = \sum_{k=0}^{\infty} (\lambda_k + z^2)^{-\alpha} - \sum_{k=0}^{N(n)} (\lambda_{k,n} + z^2)^{-\alpha} < \varepsilon.
\]

Hence the combinatorial resolvent trace \(\text{Tr}(\Delta_n + z^2)^{-\alpha}\) converges to \(\text{Tr}(\Delta + z^2)^{-\alpha}\) as \(n \to \infty\). The general statement follows from the observation

\[
\partial_z \text{Tr}(\Delta_n + z^2)^{-\alpha} = 2z \text{Tr}(\Delta_n + z^2)^{-\alpha - 1},
\]

\[
\partial_z \text{Tr}(\Delta + z^2)^{-\alpha} = 2z \text{Tr}(\Delta + z^2)^{-\alpha - 1}.
\]

\[\square\]

5. Proof of the main result

We may now prove our main theorem.

**Theorem 5.1.** The logarithmic determinant \(\log \det \Delta_n\) admits a regularized limit as \(n \to \infty\) and

\[
\log \det \Delta = \lim_{n \to \infty} \log \det \Delta_n.
\]

**Proof.** By (1.6) and Proposition 1.4

\[
\log \det \Delta = -2 \int_0^{\infty} z^{2m-1} \text{Tr}(\Delta + z^2)^{-m} dz
\]

\[
= -2 \int_0^{\infty} z^{2m-1} \lim_{n \to \infty} \text{Tr}(\Delta_n + z^2)^{-m} dz
\]

\[
= -2 \int_0^{\infty} z^{2m-1} \lim_{n \to \infty} \text{LIM} \text{Tr}(\Delta_n + z^2)^{-m} dz.
\]

By Theorem 1.5, \(\text{Tr}(\Delta_n + z^2)^{-m}\) admits a polyhomogeneous expansion

\[
\text{Tr}(\Delta_n + z^2)^{-m} = \sum_{j=0}^{m} h_{-m-j}(z, n) + H(z, n),
\]
where each \( h_{-m-j} \in C^\infty(\mathbb{R}^2_+) \) is homogeneous of order \((-m-j)\) jointly in \((z, n)\), and the remainder term satisfies \( H(z, n) = O(z^{-2m-2}) \) as \( z \to \infty \), uniformly in \( n > 0 \). By Proposition 1.3

\[
-2 \int_1^\infty z^{2m-1} \operatorname{LIM}_{n \to \infty} \sum_{j=0}^m h_{-m-j}(z, n) \, dz = -2 \operatorname{LIM}_{n \to \infty} \int_1^\infty z^{2m-1} \sum_{j=0}^m h_{-m-j}(z, n) \, dz \\
-2 \int_0^\infty z^{2m-1} h_{-2m}(z, 1) \, dz.
\]

Each homogeneous term \( h_{-m-j} \) admits a regularized limit as \( n \to \infty \) and hence, by Proposition 1.4, the remainder term \( H(z, n) \) admits a regularized limit as \( n \to \infty \) as well. Since the estimate \( H(z, n) = O(z^{-2m-2}) \) for \( z \to \infty \) is uniform in \( n > 0 \), the regularized limit \( \operatorname{LIM}_{n \to \infty} H(z, n) \) is in fact a true limit \( \lim_{n \to \infty} H(z, n) \). We find by dominated convergence

\[
-2 \int_1^\infty z^{2m-1} \lim_{n \to \infty} H(z, n) \, dz = -2 \lim_{n \to \infty} \int_1^\infty z^{2m-1} H(z, n) \, dz.
\]

Consequently, we arrive at the following intermediate result

\[
-2 \int_1^\infty z^{2m-1} \operatorname{Tr}(\Delta + z^2)^{-m} \, dz = -2 \int_1^\infty z^{2m-1} \lim_{n \to \infty} \operatorname{Tr}(\Delta_n + z^2)^{-m} \, dz \\
= -2 \operatorname{LIM}_{n \to \infty} \int_0^\infty z^{2m-1} \operatorname{Tr}(\Delta_n + z^2)^{-m} \, dz \\
-2 \int_0^\infty z^{2m-1} h_{-2m}(z, 1) \, dz.
\] (5.1)

Note that \( \operatorname{Tr}(\Delta + z^2)^{-m} \sim z^{-2m} \) as \( z \to 0 \) and hence \( z^{2m-1} \operatorname{Tr}(\Delta + z^2)^{-m} \) is not integrable at \( z = 0 \) due to non-trivial kernel of \( \Delta \). Similar statement holds for \( \operatorname{Tr}(\Delta_n + z^2)^{-m} \). Subtracting the contribution from harmonic functions\(^4\), we find

\[
-2 \int_0^1 z^{2m-1} \operatorname{Tr}(\Delta + z^2)^{-m} \, dz = -2 \int_0^1 z^{2m-1} (\operatorname{Tr}(\Delta + z^2)^{-m} - z^{-2m}) \, dz, \\
-2 \int_0^1 z^{2m-1} \operatorname{Tr}(\Delta_n + z^2)^{-m} \, dz = -2 \int_0^1 z^{2m-1} (\operatorname{Tr}(\Delta_n + z^2)^{-m} - z^{-2m}) \, dz,
\]

where the integrals on the right hand side exist in the usual sense. Consequently, replacing regularized integrals with the usual integrals, we may interchange integrals and limits. This yields

\[
-2 \int_0^1 z^{2m-1} \operatorname{Tr}(\Delta + z^2)^{-m} \, dz = -2 \int_0^1 z^{2m-1} \lim_{n \to \infty} \operatorname{Tr}(\Delta_n + z^2)^{-m} \, dz \\
= -2 \lim_{n \to \infty} \int_0^1 z^{2m-1} \operatorname{Tr}(\Delta_n + z^2)^{-m} \, dz.
\] (5.2)

\(^4\)Note from (1.3) that \( \ker \Delta_n \) and \( \ker \Delta \) are both one-dimensional.
By (5.1) and (5.2), we find in view of the formula (1.6)

\[(5.3) \quad \log \det_\zeta \Delta = \text{LIM}_{n \to \infty} \log \det \Delta_n - 2 \int_1^\infty z^{2m-1} h_{-2m}(z, 1) dz.\]

The statement now follows from the fact that \(h_{-2m}(z, n) = 0\) by Theorem 3.3. \(\square\)

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References

[BuPe93] R. Burton and R. Pemantle, Local characteristics, entropy and limit theorems for spanning trees and domino tilings via transfer-impedances, Ann. Probab. 21 (1993), no. 3, 1329–1371. MR 1235419 (94m:60019)

[Cha06] L. Chaumard, Discretisation de zeta-determinants d’oprateurs de Schrödinger sur le tore, Bull. Soc. Math. France 134 (2006), no. 3, 327–355.

[Che79] J. Cheeger, Analytic Torsion and the Heat Equation, Ann. of Math. (2) 109 (1979) no. 2, 259–322.

[CJK10] G. Chinta, J. Jorgenson, and A. Karlsson, Zeta functions, heat kernels, and spectral asymptotics on degenerating families of discrete tori, Nagoya Math. J. 198 (2010), 121–172. MR 2666579 (2011i:58052)

[Dod76] J. Dodziuk, Finite-difference approach to the Hodge theory of harmonic forms, Amer. J. Math. 98 (1976), no. 1, 79–104. MR 0407872 (53 #11642)

[DuDa88] B. Duplantier and F. David, Exact partition functions and correlation functions of multiple Hamiltonian walks on the Manhattan lattice, J. Statist. Phys. 51 (1988), no. 3-4, 327–434. MR 952941 (89m:82005)

[FrGu08] L. Friedlander, V. Guillemin, Determinants of zeroth order operators, J. Diff. Geom. 78, 1, (2008), 1-12.

[Fri89] L. Friedlander, The asymptotics of the determinant function for a class of operators, Proc. Amer. Math. Soc., 107, 1989, 169–178

[Haw77] S. W. Hawking Zeta function regularization of path integrals in curved spacetime, Comm. Math. Phys. 55, 2 (1977), 133-148.

[Kas61] P. W. Kasteleyn, The statistics of dimers on a lattice, i. the number of dimer arrangements on a quadratic lattice, Physica 27 (1961), 1209–12225.

[Ken00] R. Kenyon, The asymptotic determinant of the discrete Laplacian, Acta Math. 185 (2000), no. 2, 239–286. MR 1819995 (2002g:82019)

[Kir47] G. Kirchhoff, über die auflösung der gleichungen, auf welche man bei der untersuchung der linearen verteilung galvanischer sterne geführt wird, Ann. Phys. Chem. 72 (1847), 497–508.

[Les97] M. Lesch, Operators of Fuchs type, conical singularities, and asymptotic methods, Teubner-Texte zur Mathematik [Teubner Texts in Mathematics], vol. 136, B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1997. arXiv:dg-ga/9607005v1, MR 1449639 (98d:58174)

[LeVe13] M. Lesch and B. Vertman, Regularizing infinite sums of zeta-determinants, arXiv:1206.0780 [math.SP].

[Mel92] R. Melrose, Calculus of conormal distributions on manifolds with corners, Intl. Math. Research Notices 3 (1992), 51-61.

[Mel93] ______, The Atiyah-Patodi-Singer index theorem Research Notes in Math., 4, A K Peters, Massachusetts (1993)
[MÜl78] W. Müller, Analytic torsion and R-torsion of Riemannian manifolds, Adv. in Math. 28 (1978), no. 3, 233–305. MR 498252 (80j:58065b)

[RaSi71] D.B. Ray and I.M. Singer R-torsion and the Laplacian on Riemannian manifolds, Adv. Math. 7 (1971), 145-210.

[ReVe13] N. Reshetikhin and B. Vertman, Combinatorial quantum field theory and gluing formula for determinants.

[Sau13] B. Sauer, On the resolvent trace of multi-parametric Sturm-Liouville operators, Master thesis, Bonn (2013).

[SrI13] A. Sridhar, Asymptotic determinant of discrete Laplace-Beltrami operators, preprint arXiv:1501.02057 (2015)

[Sze15] G. Szegö, Ein Grenzwertsatz über die Toeplitzschen Determinanten einer reellen positiven Funktion, Math. Ann. 76: 490-503, (1915)

[Tem74] H. N. V. Temperley, Enumeration of graphs on a large periodic lattice, Combinatorics (Proc. British Combinatorial Conf., Univ. Coll. Wales, Aberystwyth, 1973), Cambridge Univ. Press, London, 1974, pp. 155-159. London Math. Soc. Lecture Note Ser., No. 13. MR 0347616 (50 #119)

[Ver13] B. Vertman, Multiparameter resolvent trace expansion for elliptic boundary problems, arXiv:1301.7293 [math.SP].

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