Heegner Points and Exceptional Zeros of Garrett $p$-Adic $L$-Functions

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Abstract. This article proves a case of the $p$-adic Birch and Swinnerton–Dyer conjecture for Garrett $p$-adic $L$-functions of (Bertolini et al. in On $p$-adic analogues of the Birch and Swinnerton–Dyer conjecture for Garrett $L$-functions, 2021), in the imaginary dihedral exceptional zero setting of extended analytic rank 4.

1. Statement of the Main Result

Let $A$ be an elliptic curve defined over the field $\mathbb{Q}$ of rational numbers, having multiplicative reduction at a rational prime $p > 3$. Let $K$ be a quadratic imaginary field of discriminant $d_K$ coprime to the conductor $N_A$ of $A$, and let

$$\nu_g : G_K \longrightarrow \mathbb{Q}^* \quad \text{and} \quad \nu_h : G_K \longrightarrow \mathbb{Q}^*$$

be finite order characters of the absolute Galois group $G_K = \text{Gal}(\bar{\mathbb{Q}}/K)$ of $K$, where $\mathbb{Q}$ is the field of algebraic complex numbers. Write $N_A = N_A^+ \cdot N_A^-$, where each prime divisor of $N_A^+$ (resp., $N_A^-$) splits (resp., is inert) in $K$. We make the following

Assumption 1.1. 1. (Heegner assumption) The prime $p$ is inert in $K$ (id est divides $N_A^-$) and $N_A^-$ is a square-free product of an even number of primes.
2. (Self-duality) The central characters of $\nu_g$ and $\nu_h$ are inverse to each other.
3. (Cuspidality) The characters $\nu_g$ and $\nu_h$ are not induced by Dirichlet characters.
4. (Local signs) The conductors of $\nu_g$ and $\nu_h$ are coprime to $d_K \cdot N_A$.

Let $f = \sum_{n \geq 1} a_n(f) \cdot q^n$ in $S_2(\Gamma_0(N_f))$ be the newform of conductor $N_f = N_A$ attached to $A$ by the modularity theorem. For $\nu_\xi = \nu_g, \nu_h$, let $g_\xi : G_Q \longrightarrow \text{GL}_2(\mathbb{C})$ be the odd irreducible (cf. Assumption 1.1.(3)) Artin representation of $G_Q$ induced by $\nu_\xi$, corresponding by modularity to the cuspidal weight one theta series

$$\xi = \sum_{(a, f_\xi) = 1} \nu_\xi(a) \cdot q^{N_a} \in S_1(N_\xi, \chi_\xi).$$

Here $a$ runs the set of non-zero ideals of $O_K$ coprime to the conductor $f_\xi$ of $\nu_\xi$, $N_a = |O_K/a|$, $N_\xi = d_K \cdot N_\xi$ and $\chi_\xi = \varepsilon_K \cdot \nu_\xi^{\text{cen}}$, where $\varepsilon_K : (\mathbb{Z}/d_K \mathbb{Z})^* \longrightarrow \mu_2$ is
the quadratic character of $K$ and $\nu_\xi^\text{cen} : G_Q \rightarrow \hat{Q}^*$ is the central character of $\nu_\xi$. Since $p$ is inert in $K$ by Assumption 1.1.(1), the $p$-th Hecke polynomial of $\xi$ equals $X^2 + \chi_\xi(p)$ (i.e., the $p$-th Fourier coefficient of $\xi$ is equal to zero). In addition $\chi_\xi(p)$ is non-zero by Assumption 1.1.(4), hence $X^2 + \chi_\xi(p) = (X - \alpha_\xi) \cdot (X - \beta_\xi)$ has distinct roots $\alpha_\xi$ and $\beta_\xi = -\alpha_\xi$. According to Assumption 1.1.(2) one has $\alpha_g \cdot \alpha_h = \beta_g \cdot \beta_h = \pm 1$ and $\alpha_g \cdot \beta_h = \beta_g \cdot \alpha_h = -\alpha_g \cdot \alpha_h$, hence we can, and will, assume

$$\alpha_f = \alpha_g \cdot \alpha_h = \beta_g \cdot \beta_h$$

by reordering the roots $(\alpha_\xi, \beta_\xi)$ of $X^2 + \chi_\xi(p)$ if necessary, where $\alpha_f = a_p(f) = \pm 1$.

Fix an algebraic closure $\bar{Q}_p$ of $Q_p$, an embedding $i_p : Q \hookrightarrow \bar{Q}_p$, and a finite extension $L$ of $Q_p$ containing (the images under $i_p$ of) the values of $\nu_\xi$ and $\alpha_\xi$, for $\xi = g, h$. Denote by $\xi_\alpha$ in $S_1(pN_\xi, \chi_\xi)$ the $p$-stabilisation of $\xi$ with $U_p$-eigenvalue $\alpha_\xi$. According to [1, 7], there exist unique Hida families

$$f = \sum_{n \geq 1} a_n(f) \cdot q^n \in \mathcal{O}_f[q] \quad \text{and} \quad \xi_\alpha = \sum_{n \geq 1} a_n(\xi_\alpha) \cdot q^n \in \mathcal{O}_{\xi_\alpha}$$

specialising to $f = f_2$ and $\xi_\alpha = \xi_{\alpha,1}$ in weights two and one respectively. Here $\mathcal{O}_f$ is the ring of bounded analytic functions on a (small) connected open disc $U_f$ centred at 2 in the weight space $W = \text{Hom}_{\text{cont}}(Z_p, C_p)$ over $Q_p$. For each $k$ in $U_f \cap \mathbb{Z}_{\geq 4}$, the weight-$k$ specialisation $f_k$ of $f$ is the ordinary $p$-stabilisation of a $p$-ordinary newform $f_k$ of weight $k$ and level $\Gamma_0(N_f/p)$. Similarly $\mathcal{O}_{\xi_\alpha}$ is the ring of bounded analytic functions on a connected open disc $U_{\xi_\alpha}$ centred at 1 in $W_L = W \otimes Q_p$, and $\xi_{\alpha, u}$ is the $p$-stabilisation of a newform $\xi_u$ of weight $u$ and level $\Gamma_1(N_{\xi})$ for each $l$ in $U_{\xi_\alpha} \cap \mathbb{Z}_{\geq 1}$, with $\xi_1 = \xi$. In order to lighten the notation, we write $U_{\xi} = U_{\xi_\alpha}$ and $\mathcal{O}_{\xi} = \mathcal{O}_{\xi_\alpha}$.

Set $\varrho = \varrho_g \otimes \varrho_h$ and $\mathcal{O}_{fgh} = \mathcal{O}_f \otimes \mathcal{O}_g \otimes L \mathcal{O}_h$. Under Assumption 1.1, Theorem A of [8] associates with $(f, g_\alpha, h_\alpha)$ a Garrett–Hida square root $p$-adic $L$-function

$$L_p^{\alpha}(A, \varrho) = L_p(f, g_\alpha, h_\alpha) \in \mathcal{O}_{fgh}$$

(denoted $L^f_p$ in loc. cit., where $F = (f, g_\alpha, h_\alpha)$), whose square

$$L^{\alpha}(A, \varrho) = L_p(f, g_\alpha, h_\alpha) = L_p(f, g_\alpha, h_\alpha)^2$$

interpolates the central critical values $L(f_k \otimes g_l \otimes h_m, (k + l + m - 2)/2)$ of the Garrett $L$-functions attached to $(f_k, g_l, h_m)$ for classical triples $(k, l, m)$ in the $f$-unbalanced region, viz. triples $(k, l, m)$ in $U_f \times U_g \times U_h \cap \mathbb{Z}_{\geq 1}$ satisfying $k \geq l + m$. The first equality in (1) implies that $L^{\alpha}(A, \varrho)$ has an exceptional zero in the sense of [9] at the “Birch and Swinnerton–Dyer point” $w_\varrho = (2, 1, 1)$ (cf. [5, Section 1.2]).

Fix a number field $Q(\varrho)$ containing the values of $\nu_g$ and $\nu_h$, and for $\xi = g, h$ fix a $Q(\varrho)[G_Q]$-module $V_\xi$, two-dimensional over $Q(\varrho)$, affording the Artin representation $\varrho_\xi$. Define $A(K_\varrho)^{\varrho} = H^0(\text{Gal}(K_\varrho/Q), A(K_\varrho)^{\varrho} \otimes \mathbb{Z} V_{gh})$, where $V_{gh} = V_g \otimes Q(\varrho) V_h$ and $K_\varrho$ is the number field cut-out by $\varrho = \varrho_g \otimes \varrho_h$. Following [9] one exploits Tate’s $p$-adic uniformisation to define an extended Mordell–Weil group

$$A^\dagger(K_\varrho)^{\varrho} = A(K_\varrho)^{\varrho} \oplus \mathcal{Q}_p(A, \varrho),$$
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where $Q_p(A, g)$ is a two-dimensional $Q(g)$-vector space depending only on the base change of $A$ to $Q_p$ and on the restriction of $V_{gh}$ to $G_{Q_p}$ (cf. Sect. 2.1.3 below). Moreover, Section 2 of [4] constructs a Garrett–Nekovář height-pairing

$$\langle \cdot, \cdot \rangle_{f g, h, A} : A^\dagger(K^{\circ}) \otimes Q(g) A^\dagger(K^{\circ}) \to \mathcal{I} / \mathcal{J}^2,$$

where $\mathcal{J}$ is the kernel of evaluation at $w_0$ on $\mathcal{O}_{f\mathcal{g}h}$. It is a skew-symmetric bilinear form, arising from an application of Nekovář’s theory of Selmer complexes to the big self-dual Galois representation associated with $(f, g, h)$. After setting

$$r^\dagger = \dim_{Q(g)} A(K^{\circ}),$$

Conjecture 1.1 of [4] predicts that $L_p^{\alpha}(A, g)$ belongs to $\mathcal{J} - \mathcal{J}^5$, and that its image in $(\mathcal{J}^r / \mathcal{J}^{r+1})/Q(g)^*2$ is equal to the discriminant

$$R_p^{\alpha}(A, g) = \det \langle (P_i, P_j)_{f g, h, A} \rangle_{1 \leq i, j \leq r^\dagger}$$

of the $p$-adic height $\langle \cdot, \cdot \rangle_{f g, h, A}$, where $P_1, \ldots, P_{r^\dagger}$ is any $Q(g)$-basis of $A^\dagger(K^{\circ})$.

The following theorem is the main result of this note.

**Theorem.** Assume that Assumptions 1.1 and 1.2 (stated below) are satisfied. If the complex $L$-function $L(f \otimes g \otimes h, s)$ has order of vanishing $2$ at $s = 1$, then

$$\dim_{Q(g)} A^\dagger(K^{\circ}) = 4, \quad L_p^{\alpha}(A, g) \in \mathcal{J}^4 - \mathcal{J}^5$$

and the equality

$$L_p^{\alpha}(A, g) \pmod{\mathcal{J}^5} = R_p^{\alpha}(A, g)$$

holds in the quotient of $\mathcal{J}^4 / \mathcal{J}^5$ by the multiplicative action of $Q(g)^*2$.

In the present setting, the Garrett $L$-function $L(f \otimes g \otimes h, s)$ factors as the product of the Rankin–Selberg $L$-functions $L(A/K, \varphi, s)$ and $L(A/K, \psi, s)$, where $\varphi = \nu_g \cdot \nu_h$ and $\psi = \nu_g \cdot \nu_h^\circ$, and $\nu_h^\circ$ is the conjugate of $\nu_h$ by the nontrivial element of $\text{Gal}(K/Q)$. Note that $\varphi$ and $\psi$ are dihedral by Assumption 1.1.(2), and that both $L(A/K, \varphi, s)$ and $L(A/K, \psi, s)$ have sign $-1$ in their functional equation by Assumption 1.1.(1). In particular the assumptions of the Theorem imply that $L(A/K, \chi, s)$ has a simple zero at $s = 1$ for $\chi = \varphi$ and $\chi = \psi$, hence $A(K^{\circ})^\dagger$ is two-dimensional over $Q(g)$ and generated by Heegner points by the Kolyvagin–Gross–Zagier–Zhang theorem.

If $\chi = \varphi, \psi$ is quadratic, $Q^{\ker(\chi)} = Q(\sqrt{cd_1}, \sqrt{cd_2})$, where $c, d_1$ and $d_2$ are fundamental discriminants such that $d_K = d_1 \cdot d_2$. (We consider $1$ as a fundamental discriminant). In this case $L(A/K, \chi, s)$ further factors as the product of the Hasse-Weil $L$-functions $L(A/Q, \chi_1, s)$ and $L(A/Q, \chi_2, s)$ of the twists of $A$ by the quadratic characters $\chi_i$ of $Q(\sqrt{cd_i})$. By Assumptions 1.1.(1) and 1.1.(4), we can order $\chi_1$ and $\chi_2$ in such a way that $\text{sign}(A, \chi_1) = -1$ and $\text{sign}(A, \chi_2) = +1$, where $\text{sign}(A, \chi)$ is the sign in the functional equation satisfied by $L(A/Q, \chi, s)$.

**Assumption 1.2.** If $\chi = \varphi$ or $\chi = \psi$ is quadratic, then $\chi_1(p) = \alpha_f$.

Under the assumptions of the Theorem, the results of [2, 5] imply that $L_p^{\alpha}(A, g)$ belongs to $\mathcal{J}^4 - \mathcal{J}^5$. The actual contribution of this note is the proof of the identity

$$L_p^{\alpha}(A, g) \pmod{\mathcal{J}^5} = R_p^{\alpha}(A, g),$$

which grounds on the results of loc. cit. and an extension of the techniques of [10–12].
2. Proof of the Main Result

2.1. Preliminaries

2.1.1. Galois Representations. To lighten the notation, set \((g, h) = (g_\alpha, h_\alpha)\). For \(\xi = f, g, h\) let \(V(\xi)\) be the big Galois representation attached to \(\xi\) (cf. Section 5 of [5]). Under the current assumptions, it is a free \(\mathcal{O}_\xi\)-module of rank two, equipped with a continuous \(\mathcal{O}_\xi\)-linear action of \(G_\mathbb{Q}\). For each \(u\) in \(U_\xi \cap \mathbb{Z}_{\geq 2}\), evaluation at \(u\) on \(U_\xi\) induces a natural specialisation isomorphism

\[\rho_u : V(\xi) \otimes_u E \cong V(\xi_u),\]

where \(E = \mathbb{Q}_p\) if \(\xi = f\) and \(E = L\) if \(\xi = g, h\), where \(\cdot \otimes_u E\) denotes the base change along evaluation at \(u\) on \(\mathcal{O}_\xi\), and where \(V(\xi_u)\) is the homological \(p\)-adic Deligne representation of \(\xi_u\) with coefficients in \(E\) (cf. Section 2.4 of [5]).

When \(\xi = f\) and \(u = 2\), the representation \(V(f) = V(f_2)\) is equal to the \(f\)-isotypic component of the cohomology group \(H^1_{\text{ét}}(X_1(N_f)\bar{\mathbb{Q}}, \mathbb{Q}_p(1))\), where \(X_1(N_f)\bar{\mathbb{Q}}\) is the base change to \(\bar{\mathbb{Q}}\) of the compact modular curve \(X_1(N_f)\) of level \(\Gamma_1(N_f)\) defined over \(\mathbb{Q}\). Fix a modular parametrisation (viz. a non-constant morphism of \(\mathbb{Q}\)-curves)

\[\varphi_\infty : X_1(N_f) \to A,\]

which induces an isomorphism of \(\mathbb{Q}_p[G_\mathbb{Q}]\)-modules between \(V(f)\) and the \(p\)-adic Tate module \(V_p(A) = H^1_{\text{ét}}(A_{\bar{\mathbb{Q}}}, \mathbb{Q}_p(1))\) of \(A\) with \(\mathbb{Q}_p\)-coefficients.

When \(\xi = g, h\) and \(u = 1\), the \(L[G_\mathbb{Q}]\)-module

\[V(\xi) = V(\xi) \otimes_1 L\]

affords the dual of the Deligne–Serre representation of \(\xi\), id est the induced from \(G_K\) to \(G_\mathbb{Q}\) of the character \(\nu_\xi\) with coefficients in \(L\). (Recall that \(\xi_1 = \xi_\alpha\). Here we favour the lighter notation \(V(\xi)\) for \(V(\xi) \otimes_1 L\) over the more consistent one \(V(\xi_\alpha)\).)

There exists a perfect \(G_\mathbb{Q}\)-equivariant and skew-symmetric pairing

\[\pi_\xi : V(\xi) \otimes_{\mathcal{O}_\xi} V(\xi) \to \mathcal{O}_\xi(\chi_{\xi} : \chi_{\text{cyc}}^{u-1}),\]

where \(\chi_{\text{cyc}} : G_\mathbb{Q} \to \mathbb{Z}_p^*\) is the \(p\)-adic cyclotomic character and \(\chi_{\text{cyc}}^{u-1} : G_\mathbb{Q} \to \mathcal{O}_\xi^*\) satisfies \(\chi_{\text{cyc}}^{u-1}(\sigma)(u) = \chi_{\text{cyc}}(\sigma)^{u-1}\) for each \(\sigma\) in \(G_\mathbb{Q}\) and each \(u\) in \(U_\xi \cap \mathbb{Z}\). (With the notations of [5, Section 5], the pairing \(\pi_\xi\) is the composition of the twist by \(\chi_{\xi} : \chi_{\text{cyc}}^{u-1}\) of the \(\mathcal{O}_\xi\)-adic Poincaré duality \(\langle \cdot, \cdot \rangle_f : V(\xi) \otimes_{\mathcal{O}_\xi} V^*(\xi) \to \mathcal{O}_\xi\) defined in [5, Equation (103)] with \(\text{id}_{V(\xi)} \otimes w_{N_p}^{-1}\), where \(w_{N_p} : V^*(\xi) (\chi_{\xi} : \chi_{\text{cyc}}^{u-1}) \simeq V(\xi)\) is the \(\mathcal{O}_\xi\)-adic Atkin–Lehner isomorphism defined in [5, Equation (114)].) Up to sign, the pairing \(\pi_f : V(f) \otimes_{\mathbb{Q}_p} V(f) \to \mathbb{Q}_p(1)\) arising from the base change of \(\pi_f\) along evaluation at \(k = 2\) on \(\mathcal{O}_f\) and the specialisation isomorphism \(\rho_2\) is the one induced on the \(f\)-isotypic components by the Poincaré duality on \(H^1_{\text{ét}}(X_1(N_f)\bar{\mathbb{Q}}, \mathbb{Q}_p(1))\). If \(\xi = g, h\), the weight-one specialisation of \(\pi_\xi\) yields a perfect skew-symmetric duality

\[\pi_\xi : V(\xi) \otimes_L V(\xi) \to L(\chi_{\xi}).\]

Identify \(G_{\mathbb{Q}_p}\) with a subgroup of \(G_\mathbb{Q}\) via the embedding \(i_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p\) fixed at the outset, and let \(\tilde{a}_p(\xi) : G_{\mathbb{Q}_p} \to \mathcal{O}_\xi^*\) be the unramified character sending an
arithmetic Frobenius to the $p$-th Fourier coefficient $a_p(\xi)$ of $\xi$. In the present setting there is a natural short exact sequence of $\mathcal{O}_\xi[\mathbb{Q}_p]$-modules

$$V(\xi)^+ \hookrightarrow V(\xi) \twoheadrightarrow V(\xi)^-,$$

where $V(\xi)^+$ and $V(\xi)^-$ are free $\mathcal{O}_\xi$-modules of rank one and $G_{\mathbb{Q}_p}$ acts on them via the characters $\chi_{\xi}\cdot\chi_{\text{cyc}}^{-1}\cdot\hat{\alpha}_p(\xi)^{-1}$ and $\hat{\alpha}_p(\xi)$ respectively (cf. Section 5 of [5]). If $\xi = f$, the specialisation isomorphism $\rho_2 : V(f) \otimes_2 \mathbb{Q}_p \simeq V(f)$ identifies $V(f)^- \otimes_2 \mathbb{Q}_p$ with the maximal $p$-unramified quotient of $V(f)$ and $V(\xi)^+ \otimes_2 \mathbb{Q}_p$ with the kernel $V(f)^+$ of the projection $V(f) \twoheadrightarrow V(f)^-$. If $\xi = g, h$ define

$$V(\xi)_\alpha = V(\xi)^- \otimes_1 L \quad \text{and} \quad V(\xi)_\beta = V(\xi)^+ \otimes_1 L,$$

so that $V(\xi)_\gamma$ (for $\gamma = \alpha, \beta$) is the submodule of $V(\xi)$ on which an arithmetic Frobenius in $G_{\mathbb{Q}_p}$ acts as multiplication by $\gamma$, and (as $L[\mathbb{Q}_p]$-modules)

$$V(\xi) = V(\xi)_\alpha \oplus V(\xi)_\beta.$$

Define

$$V = V(f, g, h) = V(f)\hat{\otimes}_{\mathbb{Q}_p} V(g)\hat{\otimes}_L V(h)(\Xi_{fg})$$

where $\Xi_{fg} = (4k^2 - 1 - m)/2 : G_{\mathbb{Q}} \to \mathcal{O}_{fg}^*$ satisfies $\Xi_{fg}(\sigma)(w) = \chi_{\text{cyc}}(\sigma)\chi_{\text{cyc}}^{-1}\cdot\hat{\alpha}_p(\xi)^{-1}$ for each $\sigma$ in $G_{\mathbb{Q}}$ and each $w = (k, l, m)$ in $U_f \times U_g \times U_h \cap \mathbb{Z}^3$, and

$$V = V(f, g, h) = V(f) \otimes_{\mathbb{Q}_p} V(g) \otimes_L V(h).$$

Evaluation at $w_0 = (2, 1, 1)$ on $\mathcal{O}_{fg}$ induces a specialisation isomorphism

$$\rho_{w_0} : V \otimes_{w_0} L \simeq V.$$

The product of the pairing $\pi_{\xi}$ for $\xi = f, g, h$ yields a perfect, $G_{\mathbb{Q}}$-equivariant and skew-symmetric duality (cf. Assumption 1.1.(2))

$$\pi_{fg} : V \otimes_{\mathcal{O}_{fg}} V \to \mathcal{O}_{fg}(1),$$

whose base change along evaluation at $w_0$ on $\mathcal{O}_{fg}$ recasts (via $\rho_{w_0}$) the perfect duality

$$\pi_{fg} : V \otimes_L V \to L(1)$$

defined by the product of the perfect pairings $\pi_{\xi}$ for $\xi = f, g, h$.

For $\xi = f, g, h$ let $\mathcal{F}^* V(\xi)$ be the decreasing filtration on the $\mathcal{O}_{fg}[G_{\mathbb{Q}_p}]$-module $V(\xi)$ defined by $\mathcal{F}^1 V(\xi) = V(\xi)^+$, $\mathcal{F}^2 V(\xi) = V(\xi)^-$ for each $i \leq 0$ and $\mathcal{F}^i V(\xi) = 0$ for each $i \geq 2$. Define the balanced submodule $\mathcal{F}^2 V$ of $V$ by

$$\mathcal{F}^2 V = \left[ \sum_{a+b+c=2} \mathcal{F}^a V(f)\hat{\otimes}_{\mathbb{Q}_p} \mathcal{F}^b V(g)\hat{\otimes}_L \mathcal{F}^c V(h) \right] \otimes_{\mathcal{O}_{fg}} \Xi_{fg},$$

and the $f$-unbalanced submodule $V^+$ of $V$ by

$$V^+ = V(f)^{+}\hat{\otimes}_{\mathbb{Q}_p} V(g)\hat{\otimes}_L V(h) \otimes_{\mathcal{O}_{fg}} \Xi_{fg}.$$
These are $G_{Q_p}$-invariant free $\mathcal{O}_{fgh}$-submodules of $V$ of rank $4 = 1/2 \text{rank} \mathcal{O}_{fgh} V$, which are maximal isotropic with respect to the skew-symmetric duality $\pi_{fgh}$. After setting
\[ V^- = V/V^+ \quad \text{and} \quad V_f = V(f)^- \cdot \bigotimes Q_p V(g)^+ \cdot L V(h)^+ \cdot \mathcal{O}_{fgh} \Xi_{fgh}, \]
one has a commutative diagram of $\mathcal{O}_{fgh}[G_{Q_p}]$-modules
\[
\begin{array}{ccc}
\mathcal{F}^2 V & \xleftarrow{i_f} & V \\
p_f \downarrow & & \downarrow p^- \\
V_f & \xleftarrow{i_f} & V^-
\end{array}
\] (2)
with $i_\mathcal{F}$ and $i_f$ the natural inclusions and $p^-$ the natural projection. Note that $p^- \circ i_\mathcal{F}$ and $i_f$ have the same image, hence the morphism $p_f$ is defined by the commutativity of the diagram. One defines the balanced local subspace $H^1_{\text{bal}}(Q_p, V)$ of $H^1(Q_p, V)$ to be the image of the morphism induced in cohomology by $i_\mathcal{F}$. This morphism is injective (cf. Section 7.2 of [5]), hence gives a natural identification
\[ H^1_{\text{bal}}(Q_p, V) = H^1(Q_p, \mathcal{F}^2 V) \] (3)

Set $V^\pm = V(f)^\pm \otimes Q_p V(g) \otimes L V(h)$. For each pair $(i, j)$ of elements of $\{\alpha, \beta\}$ define $V_{ij} = V(f)^i \otimes Q_p V(g)_j \otimes L V(h)_j$, where $\cdot$ is one of symbols $\emptyset$, $+$ and $-$. Then
\[ V^\pm = V_{\alpha \alpha} \oplus V_{\alpha \beta} \oplus V_{\beta \alpha} \oplus V_{\beta \beta} \]
as $L[G_{Q_p}]$-modules, and Eq. (1) implies
\[ H^0(Q_p, V^-) = V_{\alpha \alpha} \oplus V_{\beta \beta} \quad \text{and} \quad H^0(Q_p, V^+(-1)) = V_{\alpha \alpha}(-1) \oplus V_{\beta \beta}^+(-1). \] (4)
The specialisation isomorphism $\rho_{w_o}$ identifies $V^\pm \otimes w_o L$, $\mathcal{F}^2 V \otimes w_o L$ and $V_f \otimes w_o L$ with $V^\pm$, $\mathcal{F}^2 V = V_{\beta \beta} + V_{\alpha \beta}^+ + V_{\beta \alpha}^+$ and $V_{\beta \beta}$ respectively. In particular the base change of the commutative diagram (2) along evaluation at $w_o$ on $\mathcal{O}_{fgh}$ is equal to
\[
\begin{array}{ccc}
\mathcal{F}^2 V & \xleftarrow{i_f} & V \\
p_f \downarrow & & \downarrow p^- \\
V_{\beta \beta} & \xleftarrow{i_f} & V^-
\end{array}
\] (5)
with $i_\mathcal{F}$ and $i_f$ the natural inclusions and $p^-$ the natural projection.

The Bloch–Kato finite subspace of $H^1(Q_p, V)$ is equal to the kernel of the map $p^- : H^1(Q_p, V) \to H^1(Q_p, V^-)$, cf. Section 9.1 of [5]. (With a slight abuse of notation, we denote by the same symbol $a$ morphism of $G_{Q_p}$-modules and the maps it induces in cohomology.) By construction (cf. Eqs. (2) and (5)), the specialisation $\kappa = \rho_{w_o}(\kappa)$ in $H^1(Q_p, V)$ at $w_o$ of a local balanced class $\kappa$ in $H^1_{\text{bal}}(Q_p, V)$ belongs to the kernel of the map $H^1(Q_p, V) \to H^1(Q_p, V_{ij})$ for $ij = \alpha \alpha, \alpha \beta, \beta \alpha$. Then $\kappa$ is crystalline precisely if it belongs to the kernel of $H^1(Q_p, V) \to H^1(Q_p, V_{\beta \beta})$, if and only if $p_f(\kappa)$ in $H^1(Q_p, V_f)$ (cf. Eq. (3)) belongs to the kernel of the specialisation map $\rho_{w_o} : H^1(Q_p, V_f) \to H^1(Q_p, V_{\beta \beta})$. Since the ideal $\mathcal{I}$ of $\mathcal{O}_{fgh}$ is generated
by a regular sequence and \( H^2(\mathbb{Q}_p, V_{\beta \beta}) = 0 \), the specialisation map \( \rho_{w_o} \) induces an isomorphism \( H^1(\mathbb{Q}_p, V_f) \otimes_{w_o} L \simeq H^1(\mathbb{Q}_p, V_{\beta \beta}) \). We have proved the following

**Lemma 2.1.** Let \( \kappa \) be a local balanced class in \( H^1_{\text{bal}}(\mathbb{Q}_p, V) \) and set \( \kappa = \rho_{w_o}(\kappa) \) in \( H^1(\mathbb{Q}_p, V) \). Then \( \kappa \) is crystalline if and only if \( p_f(\kappa) \) belongs to \( \mathfrak{S} \cdot H^1(\mathbb{Q}_p, V_f) \).

**2.1.2. p-Adic Periods.** Let \( \hat{Q}^\text{nr}_p \) be the \( p \)-adic completion of the maximal unramified extension of \( \mathbb{Q}_p \), let \( c = c(\chi_g) \) be the conductor of \( \chi_g \), and for \( \xi = g, h \) define

\[
G(\chi_\xi) = (-c)^i \xi \cdot \sum_{a \in (\mathbb{Z}/c\mathbb{Z})^*} \chi_\xi(a)^{-1} \otimes e^{2\pi i a/c} \in D_{\text{cris}}(\chi_\xi),
\]

where \( i_g = 0, i_h = -1 \) and \( D_{\text{cris}}(\chi_\xi) \) is a shorthand for \( H^0(\mathbb{Q}_p, L(\chi_\xi) \otimes Q_p \hat{\Omega}^\text{nr}_p) \).

As explained in Section 3.1 of [4], for \( \xi = f, g, h \) the module \( D(\xi)^- \) of \( G\hat{Q}_p \)-invariants of \( V(\xi)^- \otimes Q_p \hat{\Omega}^\text{nr}_p \) is free of rank one over \( \mathcal{O}_{\xi} \), and its base change

\[
D(\xi)^- = D(\xi)^- \otimes_u L
\]

along evaluation at a classical weight \( u \) in \( U_{\xi} \cap \mathbb{Z}_{\geq 2} \) on \( \mathcal{O}_{\xi} \) is canonically isomorphic to the \( \xi_u \)-isotypic component \( L \cdot \xi_u \) of \( S_u(pN_\xi, \chi_\xi)_L \). Moreover there exists an \( \mathcal{O}_{\xi} \)-basis

\[
\omega_\xi \in D(\xi)^-
\]

whose image \( \omega_{\xi_u} \) in \( D(\xi)^- \) corresponds to \( \xi_u \) under the aforementioned isomorphism for each \( u \) in \( U_{\xi} \cap \mathbb{Z}_{\geq 2} \). (We refer to loc. cit. and the references therein for the details.) The weight-two specialisation of \( \omega_f \) equals the de Rham class

\[
\omega_f \in D_{\text{cris}}(V(f)^-) \simeq \text{Fil}^0 D_{\text{dR}}(V(f))
\]

associated with \( f \) under the Faltings–Tsuji comparison isomorphism between the étale and de Rham cohomology of \( X_1(N_f)_{\text{Q}_p} \). (The isomorphism in the previous equation arises from the projection \( V(f) \rightarrow V(f)^- \).) Denote by

\[
\langle \cdot, \cdot \rangle_f : D_{\text{dR}}(V(f)) \otimes L D_{\text{dR}}(V(f)) \rightarrow L
\]

the perfect duality induced by \( \pi_f \), and define \( \eta_f \) in \( D_{\text{dR}}(V(f)) / \text{Fil}^0 \) by the identity

\[
\langle \eta_f, \omega_f \rangle_f = 1.
\]

For \( \xi = g, h \), the weight-one specialisation of \( \omega_\xi \) yields a class

\[
\omega_{\xi_a} \in D_{\text{cris}}(V(\xi)_a) = D_{\text{cris}}(V(\xi))^{\varphi = \alpha^{-1}}
\]

(with \( \varphi \) the crystalline Frobenius). The pairing \( \pi_\xi = \pi_\xi \otimes_1 L \) induces a perfect duality

\[
\langle \cdot, \cdot \rangle_\xi : D_{\text{cris}}(V(\xi)) \otimes L D_{\text{cris}}(V(\xi)) \rightarrow D_{\text{cris}}(\chi_\xi)
\]

and one defines \( \eta_{\xi_a} \) in \( D_{\text{cris}}(V(\xi)_\beta) = D_{\text{cris}}(V(\xi))^{\varphi = \beta^{-1}} \) by the identity

\[
\langle \eta_{\xi_a}, \omega_{\xi_a} \rangle_\xi = G(\chi_\xi).
\]

Along with \( \omega_f \), it is important to consider another \( p \)-adic period

\[
q(f) \in D_{\text{cris}}(V(f)^-) = \text{Fil}^0 D_{\text{dR}}(V(f))
\]
arising from the Tate uniformisation of $A_{Q_p}$, cf. Section 2 of [3]. Let $K_p$ be the completion of $K$ at $p$ (namely the quadratic unramified extension of $Q_p$). Tate’s theory gives a rigid analytic uniformisation $\varphi_{\text{Tate}} : G_{m,K_p}^{\text{rig}} \rightarrow A_{K_p}$, unique up to sign, with kernel the lattice generated by the Tate period $q_A$ in $pZ_p$ of $A_{Q_p}$. One sets

$$q(A) = p^{-1}(\varphi_{\text{Tate}}(n\sqrt{q_A})) \in V_p(A)^{-}, \quad q(f) = \sqrt{m_p} \cdot \varphi^{-1}_\infty(q(A)),$$

(6)

where $n\sqrt{q_A}$ is any compatible system of $p^n$th roots of $q_A$. $\varphi_{\infty} : V(f)^{-} \simeq V_p(A)^{-}$ is the isomorphism arising from the fixed modular parametrisation $\varphi_{\infty}$, $m_p = 1$ if $\alpha_f = 1$ and $m_p = d_K$ if $\alpha_f = -1$. As in loc. cit., define the generators

$$q_{\alpha\alpha} = q(f) \otimes \omega_g \otimes \omega_h, \quad q_{\beta\beta} = q(f) \otimes \eta_g \otimes \eta_h,$$

of the subspaces $V_{\alpha\alpha}^{-}$ and $V_{\beta\beta}^{-}$ respectively of $H^0(Q_p, V^{-}) = D_{\text{cris}}(V^{-})^{\varphi=1}$.

**2.1.3. The Garrett–Nekovář $p$-Adic Height Pairing.** Section 2 of [4] constructs a canonical skew-symmetric $p$-adic height pairing

$$\langle \cdot, \cdot \rangle_{fg,h} : \tilde{H}^1_f(Q, V) \otimes_L \tilde{H}^1_f(Q, V) \rightarrow \mathcal{J}/\mathcal{J}^2$$

on the extended Selmer group $\tilde{H}^1_f(Q, V)$ associated with the Greenberg local condition at $p$ arising from the inclusion $\iota^+ : V^+ \hookrightarrow V$. Let $\text{Sel}(Q, V)$ denote the Bloch–Kato Selmer group of $V$, which is equal to the kernel of $H^1(Q, V) \rightarrow H^1(Q_p, V^{-})$ in the present setting (cf. [5, Section 9.1]). One has a commutative exact diagram

$$0 \longrightarrow H^0(Q_p, V^{-}) \overset{j}{\longrightarrow} \tilde{H}^1_f(Q, V) \overset{\pi}{\longrightarrow} \text{Sel}(Q, V) \longrightarrow 0 \quad (7)$$

and there exists a unique section $\iota_{ur} : \text{Sel}(Q, V) \hookrightarrow \tilde{H}^1_f(Q, V)$ of $\pi$ such that the composition $\iota_{ur} \circ \gamma$ takes values in the finite subspace $H^1_{\text{fin}}(Q_p, V^+) \subset H^1(Q_p, V^+)$ (cf. Section 2.3 of [4]). As in loc. cit. we use the maps $j$ and $\iota_{ur}$ to identify Nekovář’s extended Selmer group $\tilde{H}^1_f(Q, V)$ with the naive extended Selmer group

$$\text{Sel}^1(Q, V) = H^0(Q_p, V^-) \oplus \text{Sel}(Q, V).$$

Enlarging $L$ if necessary, for $\xi = g, h$ fix an isomorphism of $L[G_Q]$-modules

$$\gamma_\xi : V_\xi \otimes Q(\xi) \overset{L}{\approx} V(\xi) \quad \text{and} \quad \pi_\xi(\gamma_\xi(x) \otimes \gamma_\xi(y)) \in Q(\xi)(\chi_\xi) \quad (8)$$

for each $x$ and $y$ in $V_\xi$ (cf. Eq. (4) of [4]). Set (cf. Eq. (6))

$$Q_p(A, g) = H^0(Q_p, Q(g) \cdot q(A) \otimes Q(g), V_{gh}).$$

(9)

The modular parametrisation $\varphi_{\infty} : X_1(N_f) \rightarrow A$ fixed in Sect. 2.1.1, the global Kummer map on $A(K_\ell) \otimes Q_p$ and the isomorphisms $\gamma_g$ and $\gamma_h$ induce an embedding

$$\gamma_{gh} : A^1(K_\ell)^{\theta} \hookrightarrow \text{Sel}^1(Q, V) = \tilde{H}^1_f(Q, V),$$

(10)

and one defines the Garrett–Nekovář $p$-adic pairing (cf. Sect. 1)

$$\langle \cdot, \cdot \rangle_{fg,h} : A^1(K_\ell)^{\theta} \otimes Q(\xi) A^1(K_\ell)^{\theta} \rightarrow \mathcal{J}/\mathcal{J}^2.$$
to be the restriction of the canonical height $\langle \cdot, \cdot \rangle_{fgh}$ on $\tilde{H}^1_f(Q, V)$ along $\gamma_{gh}$. Note that the discriminant $R^\alpha_p(A, \varrho)$ of $\langle \cdot, \cdot \rangle_{fgh}$ on $A^\iota(K_\varrho)^\circ$ (cf. Sect. 1) is independent of the choice of the modular parametrisation $\varphi_\infty$ and the isomorphisms $\gamma_g$ and $\gamma_h$.

### 2.1.4. Logarithms

Let $V_{dR} = D_{dR}(V)$ be the de Rham module of $V = V(f, g, h)$. The duality $\pi_{fgh} : V \otimes_L V \to L(1)$ induces a perfect pairing

$$\langle \cdot, \cdot \rangle_{fgh} : V_{dR} \otimes_L V_{dR} \to L.$$ 

After identifying $V_{dR}$ with $D_{dR}(V(f)) \otimes Q_p \cdot D_{\text{cris}}(V(g)) \otimes L \cdot D_{\text{cris}}(V(h))$ and $L$ with $D_{\text{cris}}(\chi_g) \otimes L \cdot D_{\text{cris}}(\chi_h)$ under the natural isomorphisms (cf. Assumption 1.1.(2)), the pairing $\langle \cdot, \cdot \rangle_{fgh}$ agrees with the product of the pairings $\langle \cdot, \cdot \rangle_\xi$ for $\xi = f, g, h$.

The Bloch–Kato exponential map $\exp_p$ gives an isomorphism between the tangent space $V_{dR}/\Fil^0$ of $V$ and the finite (viz. crystalline) subspace $H^1_{\text{fin}}(Q_p, V)$ of $H^1(Q_p, V)$. Denote by $\log_p$ the inverse of $\exp_p$ and define the $\alpha\alpha$-logarithm

$$\log_{\alpha\alpha} = \langle \log_p, \omega_f \otimes \eta_{g\alpha} \otimes \eta_{h\alpha} \rangle_{fgh} : H^1_{\text{fin}}(Q_p, V) \to L$$

to be the composition of $\log_p$ with evaluation at $\omega_f \otimes \eta_{g\alpha} \otimes \eta_{h\alpha}$ in $\Fil^0 V_{dR}$ under the perfect duality $\langle \cdot, \cdot \rangle_{fgh}$. Similarly define the $\beta\beta$-logarithm

$$\log_{\beta\beta} = \langle \log_p, \omega_f \otimes \omega_{g\alpha} \otimes \omega_{h\alpha} \rangle : H^1_{\text{fin}}(Q_p, V) \to L.$$ 

(Note that $\log_{ii}$ factors through the projection $H^1_{\text{fin}}(Q_p, V) \to H^1(Q_p, V_{ii})$.)

Set $\tg_{dR, K_p}(f) = H^0(K_p, V(f) \otimes Q_p \cdot B_{dR})/\Fil^0$ and consider the composition

$$\log_{A,p} : A(K_p) \cdot Q_p \simeq H^1_{\text{fin}}(K_p, V_p(A)) \simeq H^1_{\text{fin}}(K_p, V(f)) \simeq \tg_{dR, K_p}(f),$$

where the first isomorphism is the local Kummer map, the second is induced by the fixed modular parametrisation $\varphi_\infty : X_1(N_f) \to A$ (cf. Sect. 2.1.1), and the third is the inverse of the Bloch–Kato exponential map. For $\chi = \varphi, \psi$ (cf. Sect. 1) define

$$\log_{\omega_f} = \langle \log_{A,p}, \omega_f \rangle : A(K_\chi) \to K_p,$$

where $K_\chi$ is the ring class field of $K$ cut-out by $\chi$ and $A(K_\chi)$ is viewed as a subgroup of $A(K_p)$ via the embedding $i_p : Q \hookrightarrow Q_p$ fixed at the outset. (Recall that $p$ is inert in $K$ and that $\chi$ is dihedral, hence $p\mathcal{O}_K$ splits completely in $K_\chi$.)

### 2.2. Big Logarithms and Diagonal Classes

Let

$$\mathcal{L}_f : H^1(Q_p, V_f) \to \mathcal{I}$$

be the big logarithm map constructed in Proposition 7.3 of [5] using the work of Coleman, Perrin-Riou et alii. (Note that the tame character $\chi_f$ of $f$ is trivial in the present setting and that the logarithm $\mathcal{L}_f$ takes values in $\mathcal{I}$ by the exceptional zero condition $\alpha_f = \alpha_g \cdot \alpha_h$.) With a slight abuse of notation denote by

$$\mathcal{L}_f : H^1_{\text{bal}}(Q_p, V) \to \mathcal{I}$$

also the composition $\mathcal{L}_f \circ p_f$ (cf. Eq. (3)).
Let \( H^1_{\text{bal}}(Q, V) \) be the group of global classes in \( H^1(Q, V) \) whose restriction at \( p \) belongs to the balanced local condition \( H^1_{\text{bal}}(Q_p, V) \). According to Theorem A of [5] (cf. [2, Section 2.1]) there exists a canonical big diagonal class

\[
\kappa(f, g, h) = \kappa(f, g_\alpha, h_\alpha) \in H^1_{\text{bal}}(Q, V)
\]
such that

\[
\mathcal{L}_f(\text{res}_p(\kappa(f, g, h))) = \mathcal{L}_p^{\alpha\alpha}(A, g).
\]

Define the diagonal class

\[
\kappa(f, g_\alpha, h_\alpha) = \rho_{w_\alpha}(\kappa(f, g_\alpha, h_\alpha))
\]
to be the image in \( H^1(Q, V) \) of \( \kappa(f, g, h) \) under the map induced in cohomology by the specialisation isomorphism \( \rho_{w_\alpha} : V \otimes_{w_\alpha} L \simeq V \). Since by assumption the complex Garrett \( L \)-function \( L(A, g, s) = L(f \otimes g \otimes h, s) \) vanishes at \( s = 1 \), Theorem B of [5] implies that \( \kappa(f, g_\alpha, h_\alpha) \) is crystalline at \( p \), hence a Selmer class:

\[
\kappa(f, g_\alpha, h_\alpha) \in \text{Sel}(Q, V).
\]

Identify \( \rho_{fgh} \) with a subring of the power series ring \( L[[k - 2, l - 1, m - 1]] \), where \( k - 2 \in \mathcal{O}_f \) is a uniformiser at the centre 2 of \( U_f \), and \( l - 1 \) and \( m - 1 \) are defined similarly. In light of Eq. (12) and Lemma 2.1 there exist local classes \( \mathfrak{Y}_k, \mathfrak{Y}_l \) and \( \mathfrak{Y}_m \) in \( H^1(Q_p, V_f) \) satisfying the identity

\[
p_f(\text{res}_p(\kappa(f, g, h))) = \sum_u \mathfrak{Y}_u \cdot (u - u_0).
\]

Equation (11) gives

\[
\mathcal{L}_p^{\alpha\alpha}(A, g) = \sum_u \mathcal{L}_f(\mathfrak{Y}_u) \cdot (u - u_0) \in \mathcal{I}^2.
\]

The following key lemma, proved in Part 1 of Proposition 9.3 of [5], gives an explicit description of the linear term of \( \mathcal{L}_f(\mathfrak{Y}_u) \) at \( w_\alpha \). Identify the \( p \)-adic completion of the maximal abelian extension of \( Q_p \) with that of \( Q^*_p \) via the local Artin map, normalised in such a way that \( p^{-1} \) corresponds to the arithmetic Frobenius. This identifies \( H^1(Q_p, Q^*_p) \) with \( \text{Hom}_{\text{cont}}(Q^*_p, Q_p) \), hence (recalling that \( G_{Q_p} \) acts trivially on \( V_{\beta\beta} \), cf. Eq. (4))

\[
H^1(Q_p, V^-_{\beta\beta}) = \text{Hom}_{\text{cont}}(Q^*_p, Q_p) \otimes_{Q_p} V^-_{\beta\beta},
\]

and the Bloch–Kato dual exponential \( \exp^*_p \) on \( H^1(Q_p, V^-_{\beta\beta}) \) satisfies

\[
\exp^*_p(\varphi \otimes v) = \varphi(e(1)) \cdot v
\]
in \( \text{D}_{\text{cris}}(V^-_{\beta\beta}) = V^-_{\beta\beta} \) for each \( \varphi \) in \( \text{Hom}_{\text{cont}}(Q^*_p, Q_p) \) and \( v \) in \( V^-_{\beta\beta} \), where

\[
e(1) = (1 + p) \otimes \log_p(1 + p)^{-1} \in Z_p^* \otimes_{Q_p} Q_p.
\]

For \( x = \varphi \otimes v \) in \( H^1(Q_p, V^-_{\beta\beta}) \) (with \( \varphi \) and \( v \) as above) and \( q \) in \( Q^*_p \otimes Q_p \), set

\[
x(q) = \varphi(q) \cdot v \quad \text{and} \quad x(q)_f = \langle x(q), \eta_f \otimes \omega_{g_\alpha} \otimes \omega_{h_\alpha} \rangle_{fgh}.
\]

If \( (\xi, u) \) denotes one of the pairs \( (f, k), (g, l) \) and \( (h, m) \), define

\[
\tilde{D}_u : H^1(Q_p, V_f) \rightarrow L
\]
to be the linear map which on $\mathfrak{Y}$ in $H^1(Q_p, V_f)$ takes the value
\[
\tilde{D}_u(\mathfrak{Y}) = \frac{(-1)^{u_o}}{2(1-p^{-1})} \cdot (\eta(p^{-1})f - \mathcal{L}^\text{an}_\xi \cdot \eta(e(1)))f.
\] (16)

Here $\eta = \rho_{w_o}(\mathfrak{Y})$ in $H^1(Q_p, V_{f,\beta})$ is the $w_o$-specialisation of $\mathfrak{Y}$, $u_o = 2$ if $u = k$ and $u_o = 1$ if $u = l, m$, and $\mathcal{L}^\text{an}_\xi$ in $L$ is the analytic $\mathcal{L}$-invariant of $\xi$, defined by
\[
\mathcal{L}^\text{an}_\xi = -2 \cdot d\log a_p(\xi)(u_o)
\]
(where $d\log a = a'/a$ for $a$ in $\mathcal{O}^*_\xi$). We can finally state the aforementioned key lemma.

**Lemma 2.2.** For each local class $\mathfrak{Y}$ in $H^1(Q_p, V_f)$ one has
\[
\mathcal{L}_f(\mathfrak{Y}) (\text{mod } \mathcal{I}^2) = \sum_u \tilde{D}_u(\mathfrak{Y}) \cdot (u - u_o).
\]

For each pair $(u, v)$ of distinct elements of $\{k, l, m\}$, define (cf. Eq. (13))
\[
\tilde{D}_{u,v}(\kappa(f, g, h)) = \tilde{D}_u(\mathfrak{Y}_u) \quad \text{and} \quad \tilde{D}_{u,v}(\kappa(f, g, h)) = \tilde{D}_u(\mathfrak{Y}_v) + \tilde{D}_v(\mathfrak{Y}_v).
\]

Equation (14) and Lemma 2.2 give the following lemma (which implies that the derivatives $\tilde{D}_\cdot(\kappa(f, g, h))$ are independent of the choice of the classes $\mathfrak{Y}_u$ satisfying (13)).

**Lemma 2.3.** One has the following equality in $\mathcal{I}^2/\mathcal{I}^3$.
\[
\mathcal{L}^\text{an}_p(A, g) (\text{mod } \mathcal{I}^3) = \sum_{u,v} \tilde{D}_{u,v}(\kappa(f, g, h)) \cdot (u - u_o)(v - v_o)
\]

### 2.3. An Exceptional Zero Formula à la Rubin–Perrin–Riou

For a positive integer $n$ and each $2n$-tuple $y = (y_1, \ldots, y_{2n})$ of elements of $\tilde{H}^1_f(Q, V)$ denote by
\[
\mathcal{R}^\alpha_p(y) = \text{Pf}(\langle y_i, y_j \rangle_{fgh})_{1 \leq i, j \leq 2n} \in \mathcal{I}^n/\mathcal{I}^{n+1}
\]
the Pfaffian of the skew-symmetric $2n \times 2n$ matrix whose $ij$-entry is $\langle y_i, y_j \rangle_{fgh}$, and define the extended Garrett–Nekovář $p$-adic height pairing
\[
\tilde{h}^\alpha_p : \text{Sel}(Q, V) \otimes_L \text{Sel}(Q, V) \rightarrow \mathcal{I}^2/\mathcal{I}^3
\]
to be the bilinear form which on $y \otimes y'$ in $\text{Sel}(Q, V)^{\otimes 2}$ takes the value
\[
\tilde{h}_p^\alpha(y \otimes y') = \mathcal{R}_p^\alpha(q_{\alpha\alpha}, q_{\beta\beta}, y, y').
\]

The aim of this section is to prove the following proposition.

**Proposition 2.4.** Up to sign, one has the equality
\[
\tilde{h}^\alpha_p(\kappa(f, g_\alpha, h_\alpha) \otimes \cdot) = c_A \cdot \log_{\alpha\alpha}(\text{res}_p(\cdot)) \cdot \mathcal{L}^\alpha_p(A, g) (\text{mod } \mathcal{I}^3)
\]
of $\mathcal{I}^2/\mathcal{I}^3$-valued $L$-linear forms on $\text{Sel}(Q, V)$, where $c_A = \frac{m_p^{-1} - p^{-1} \cdot \text{ord}_p(q_A)}{\deg(\varphi_\infty)}$. 

We divide the proof of Proposition 2.4 in a series of lemmas. Define
\[ c_p(f) = \langle q(f), \eta_f \rangle_{fgh} \]
in \( L^* \) (cf. Sect. 2.1.2). As in Sect. 2.2, identify \( H^1(\mathbb{Q}_p, \mathbb{Q}_p) \) with \( \text{Hom}_{\text{cont}}(\mathbb{Q}_p^*, \mathbb{Q}_p) \) via the local Artin map (sending \( p^{-1} \) to an arithmetic Frobenius), and set
\[ \log_\xi = \log_p - \xi_{\text{an}}^* \cdot \text{ord}_p \in H^1(\mathbb{Q}_p, \mathbb{Q}_p) \otimes \mathbb{Q}_p L, \]
where \( \log_p : \mathbb{Q}_p^* \to \mathbb{Q}_p \) is the (branch of the) \( p \)-adic logarithm (vanishing at \( p \)) and \( \text{ord}_p : \mathbb{Q}_p^* \to \mathbb{Z} \) is the \( p \)-adic valuation normalised by \( \text{ord}_p(p) = 1 \).

**Lemma 2.5.** For each Selmer class \( y \) in \( \text{Sel}(\mathbb{Q}, V) \) one has
\[ -2 \cdot \langle q_{\beta \gamma}, y \rangle_{fgh} = c_p(f) \cdot \log_{\alpha \alpha}(\text{res}_p(y)) \cdot (k - l - m) \]
and
\[ 2 \cdot \deg(\varphi_\infty) \cdot (m_p \cdot \text{ord}_p(q_A)) \cdot \langle q_{\beta \gamma}, q_{\alpha \alpha} \rangle_{fgh} = \langle \xi^* f, \xi^* g \rangle (l - 1) + \langle \xi^* f, \xi^* h \rangle (m - 1) \]

**Proof.** See Equations (17) and (27) of [3]. (Note that the \( p \)-adic logarithm denoted by \( \log_\alpha \) in [3] is equal to \( \langle \log_p, q_{\beta \gamma} \rangle_{fgh} = -c_p(f) \cdot \log_{\alpha \alpha} \).)

Let \( C^\bullet_{\text{cont}}(\mathbb{Q}_p, V^-) \) be the complex of (inhomogeneous) continuous cochains of \( G_{\mathbb{Q}_p} \) with values in the quotient \( p^\cdot : V \to V^- \) of \( V \) (cf. Sect. 2.1.1), and let
\[ \langle \cdot, \cdot \rangle_{\text{Tate}} : H^1(\mathbb{Q}_p, V^-) \otimes_L H^1(\mathbb{Q}_p, V^+) \to L \]
the local Tate pairing arising from the perfect duality \( \pi_{fgh} : V \otimes_L V \to L(1) \). Recall the morphism \( \cdot^+ : \tilde{H}^1_f(\mathbb{Q}, V) \to H^1(\mathbb{Q}_p, V^+) \) introduced in Diagram (7).

**Lemma 2.6.** There exist 1-cochains \( X_k, X_l \) and \( X_m \) in \( C^1_{\text{cont}}(\mathbb{Q}_p, V^-) \) such that
\[ p^\cdot(\text{res}_p(\kappa(f, g, h))) = \text{cl} \left( \sum_u X_u \cdot (u - u_0) \right), \]
\[ \text{id est} \sum_u X_u \cdot (u - u_0) \text{ is a 1-cocycle representing } p^\cdot(\text{res}_p(\kappa(f, g, h))), \]
and
\[ \langle \kappa(f, g, h_\alpha, \eta), y \rangle_{fgh} = \sum_u \langle \tau_u, y^+ \rangle_{\text{Tate}} \cdot (u - u_0) \]
for each extended Selmer class \( y \) in \( \tilde{H}^1_f(\mathbb{Q}, V) \), where
\[ \tau_u = \text{cl}(\rho_{w_0}(X_u)) \]
is the local class in \( H^1(\mathbb{Q}_p, V^-) \) represented by the 1-cocycle \( \rho_{w_0}(X_u) \).

**Proof.** This follows from Equations (30)–(37) in Section 3.4 of [4]. (The paragraphs containing the aforementioned equations do not use the non-exceptionality assumption [4, Equation (26)] imposed in [4, Section 3].)

Fix in what follows 1-cochains \( X_k, X_l \) and \( X_m \) satisfying the conclusions of Lemma 2.6. For \( i = \alpha \alpha, \beta \beta \) let \( \text{pr}_i : H^1(\mathbb{Q}_p, V^-) \to H^1(\mathbb{Q}_p, V^-) \) be the natural projection.
Lemma 2.7. For \( u \) equal to one of \( k, l \) and \( m \), one has
\[
pr_{\alpha\alpha}(r_u) = \mu_u \cdot \log_f \otimes q_{\alpha\alpha}
\]
in \( H^1(Q_p, V_{\alpha\alpha}) = H^1(Q_p, Q_p) \otimes Q_p V_{\alpha\alpha} \) for some \( \mu_u \) in \( L \).

Proof. Set \( \kappa_{\alpha\alpha} = \kappa(f, g_\alpha, h_\alpha) \). As explained in Section 3.3 of [3] (cf. Equation (15) of loc. cit.) one has (cf. Diagram (7))
\[
q_{\beta\beta}^+ = \frac{m_p}{\deg(\varphi_\infty)} \cdot (q_A \otimes 1) \otimes q_{\alpha\alpha}^\ast
\]
in the direct summand
\[
H^1(Q_p, V_{\beta\beta}^+) = H^1(Q_p, Q_p(1)) \otimes Q_p V_{\beta\beta}^+(-1)
\]
of \( H^1(Q_p, V^+) \), where \( q_{\alpha\alpha}^\ast \) in \( V_{\beta\beta}^+(-1) \) is the dual basis of \( q_{\alpha\alpha} \) under the pairing \( \pi_{fgh}(-1) \). It then follows from Lemma 2.6 and local class field theory that
\[
\langle \kappa_{\alpha\alpha}, q_{\beta\beta} \rangle_{fgh} = \sum_u r_u^{\alpha\alpha}(q_A) \cdot (u - u_o),
\]
where the class \( r_u^{\alpha\alpha} \) in \( H^1(Q_p, Q_p) \) is defined by the identity
\[
pr_{\alpha\alpha}(r_u) = r_u^{\alpha\alpha} \otimes q_{\alpha\alpha}.
\]
On the other hand, since \( \log_{\alpha\alpha}(\text{res}_p(\kappa_{\alpha\alpha})) = 0 \) (because \( \kappa_{\alpha\alpha} \) is a balanced class, cf. Section 6.1 of [5]), Lemma 2.5 and the skew-symmetry of \( \langle \cdot, \cdot \rangle_{fgh} \) yield
\[
\langle \kappa_{\alpha\alpha}, q_{\beta\beta} \rangle_{fgh} = - \langle q_{\beta\beta}, \kappa_{\alpha\alpha} \rangle_{fgh} = 0,
\]
hence \( r_u^{\alpha\alpha}(q_A) = 0 \), id est \( r_u^{\alpha\alpha} \) is a multiple of \( \log_{q_A} \). The lemma follows from this and Theorem 3.18 of [6], according to which \( \log_{q_A} \) equals \( \log_f \). \( \square \)

Lemma 2.8. Assume that either \( \Sigma_f^\an \neq \Sigma_g^\an \) or \( \Sigma_f^\an \neq \Sigma_h^\an \). Then the local classes \( r_k \), \( r_l \) and \( r_m \) belong to the direct summand \( H^1(Q_p, V_{\beta\beta}^-) \) of \( H^1(Q_p, V^-) \).

Proof. The proof uses the main properties of the Bockstein map
\[
\beta_{fgh} : H^0(Q_p, V^-) \longrightarrow H^1(Q_p, V^-) \otimes_L \mathcal{I} / \mathcal{I}^2
\]
introduced in [3, Section 3.1.1]. As \( \kappa(f, g_\alpha, h_\alpha) = \rho_{w_o}(\kappa(f, g, h)) \) is crystalline at \( p \), Lemma 2.1 shows that there exist \( 3_k \), \( 3_l \) and \( 3_m \) in \( H^1(Q_p, V_f) \) such that
\[
p_f(\text{res}_p(\kappa(f, g, h))) = \sum_u 3_u \cdot (u - u_o). \tag{18}
\]
Recall the specialisation isomorphism \( \rho_{w_o} : V_f \otimes_{w_o} L \simeq V_{\beta\beta}^- \) arising from evaluation at \( w_o \) on \( \mathcal{O}_{fgh} \) (cf. Sect. 2.1.1), set \( 3_u = \rho_{w_o}(3_u) \) in \( H^1(Q_p, V_{\beta\beta}^-) \) and define
\[
\nabla_f = \sum_u 3_u \cdot (u - u_o)
\]
in \( H^1(Q_p, V_{\beta\beta}^-) \otimes \mathcal{I} / \mathcal{I}^2 \). It follows from Eqs. (17) and (18) and Lemma 3.2 of [3] that the difference \( \sum_u r_u \cdot (u - u_o) - \nabla_f \) belongs to the image of the Bockstein map \( \beta_{fgh}^- \). There exist then \( \mu \) and \( \nu \) in \( L \) such that
\[
\sum_u r_u \cdot (u - u_o) - \nabla_f - \nu \cdot \beta_{fgh}^- (q_{\beta\beta}) = \mu \cdot \beta_{fgh}^- (q_{\alpha\alpha}). \tag{19}
\]
Equation (8) of [3] shows that \( \beta_{fgh}^- (q_{\beta \beta}) \) belongs to \( H^1(Q_p, V_{\beta \beta}^-) \otimes_L \mathcal{I} / \mathcal{I}^2 \), hence Lemma 2.7 and the previous equation give

\[
\sum_u \mu_u \cdot \log f \otimes q_{\alpha \alpha} \cdot (u - u_o) = \sum_u \text{pr}_{\alpha \alpha}(r_u) \cdot (u - u_o) = \mu \cdot \text{pr}_{\alpha \alpha}(\beta_{fgh}^-(q_{\alpha \alpha})) \tag{20}
\]

(where in the right-most term we write again \( \text{pr}_{\alpha \alpha} \) to denote the \( \mathcal{I} / \mathcal{I}^2 \)-base change of the projection \( \text{pr}_{\alpha \alpha} : H^1(Q_p, V^-) \to H^1(Q_p, V_{\alpha \alpha}^-) \)). The computations carried out in Sections 3.3 and 3.4 of [3] (see in particular Equation (30) of loc. cit. and the discussion preceding it) give the following equality in \( H^1(Q_p, V_{\alpha \alpha}^-) \otimes_L \mathcal{I} / \mathcal{I}^2 \):

\[
2 \cdot \text{pr}_{\alpha \alpha}(\beta_{fgh}^-(q_{\alpha \alpha})) = \sum_u \log f \otimes q_{\alpha \alpha} \cdot (u - u_o),
\]

where \( (\xi, u) = (f, k), (g, l), (h, m) \). Together with Eq. (20) this implies

\[
2 \mu_k = \mu, \quad 2 \mu_l \cdot \log f = \mu \cdot \log g \quad \text{and} \quad 2 \mu_m \cdot \log f = \mu \cdot \log h,
\]

thus \( \mu = \mu_k = \mu_l = \mu_m = 0 \) by the assumption on the analytic \( \mathcal{L} \)-invariants made in the statement. The lemma follows from this and Eq. (19).

Let \( (u, \xi) \) denote one of \( (k, f), (l, g) \) and \( (m, h) \). For each local class \( x \) in \( H^1(Q_p, V^-) \), denote by \( x_{\beta \beta} = \text{pr}_{\beta \beta}(x) \) in \( H^1(Q_p, V_{\beta \beta}^-) \) its \( \beta \beta \)-component and (with the notations introduced in Sect. 2.2) set

\[
\ell_u(x) = (-1)^{u_o} \cdot (x_{\beta \beta}(p^{-1})_f - \alpha_{\xi} \cdot x_{\beta \beta}(e(1)))_f.
\]

For each pair \( (u, v) \) of distinct elements of \( \{k, l, m\} \) define

\[
\tilde{D}_{u,v} = \ell_u(x_u) \quad \text{and} \quad \tilde{D}_{u,v} = \ell_u(x_v) + \ell_v(x_u).
\]

**Lemma 2.9.** For each pair \( (u, v) \) of elements of \( \{k, l, m\} \) one has

\[
2 \left( 1 - p^{-1} \right) \cdot \tilde{D}_{u,v}(\kappa(f, g, h)) = \tilde{D}_{u,v}.
\]

**Proof.** We give the proof for \( (u, v) = (k, l) \) and \( (u, v) = (k, k) \), the other cases being similar. We use the notations introduced in the proof of Lemma 2.8. Section 3 of [3] (see in particular Equations (8) and (30) of loc. cit.) gives the identities

\[
2 \cdot \beta_{fgh}^- (q_{\beta \beta}) = \sum_u (-1)^{u_o} \cdot \log f \otimes q_{\beta \beta} \cdot (u - u_o) \quad \text{and} \quad \text{pr}_{\beta \beta}(\beta_{fgh}^-(q_{\alpha \alpha})) = 0.
\]

Equation (19) (and the definition of derivatives \( \tilde{D}_{u,v} \)) then yields

\[
2 \left( 1 - p^{-1} \right) \cdot \tilde{D}_{k,l}(\kappa(f, g, h)) - \ell_k(x_l) = \ell_l(x_k) = \frac{\nu}{2} (\ell_k(\log g \otimes q_{\beta \beta}) - \ell_l(\log f \otimes q_{\beta \beta})) = 0
\]

and

\[
2 \left( 1 - p^{-1} \right) \cdot \tilde{D}_{k,k}(\kappa(f, g, h)) - \ell_k(x_k) = -\frac{\nu}{2} \cdot \ell_k(\log f \otimes q_{\beta \beta}) = 0,
\]

quod erat demonstrandum.

**Lemma 2.10.** Assume that either \( \mathcal{L}_f^\text{an} \neq \mathcal{L}_g^\text{an} \) or \( \mathcal{L}_f^\text{an} \neq \mathcal{L}_h^\text{an} \). Then one has

\[
c_p(f) \cdot \langle q_{\alpha \alpha}, \kappa(f, g_{\alpha}, h_{\alpha}) \rangle_{fgh} = -\frac{m_p \cdot \text{ord}_p(q_{\alpha})}{\deg(q_{\alpha \infty})} \cdot \sum_u \ell_k(x_u) \cdot (u - u_o).
\]
Proof. Under the assumption in the statement \( r_u \) belongs to \( H^1(Q_p, V_{\beta\bar{\beta}}) \) by Lemma 2.8. Together with the equality \( \mathcal{L}^\alpha_f = \frac{\log_p(q_\alpha)}{\text{ord}_p(q_\alpha)} \) (cf. [6]), this gives

\[
\ell_k(r_u) = r_u(p^{-1}f) - \mathcal{L}^\alpha_f \cdot r_u(e(1)f) = -\frac{1}{\text{ord}_p(q_\alpha)} \cdot r_u(q_\alpha)f.
\]

(21)

According to Equation (15) of [3], one has

\[
q_{\alpha\alpha}^+ = \frac{m_p}{\deg(\rho_{\infty})} \cdot (q_\alpha \hat{\otimes} 1) \otimes q_{\beta\beta}^*,
\]

where \( q_{\beta\beta}^* \) in \( V_{\alpha\alpha}^+ \) is the dual basis of \( q_{\beta\beta} \) under the perfect pairing \( \pi_{fg\beta}(\cdot, \cdot) \). Lemma 2.6, the skew-symmetry of \( \langle \cdot, \cdot \rangle_{fg\beta} \) and local class field theory then give

\[
\langle q_{\alpha\alpha}, \kappa(f, g_\alpha, h_\alpha) \rangle_{fg\beta} = -\langle \kappa(f, g_\alpha, h_\alpha), q_{\alpha\alpha} \rangle_{fg\beta} = \frac{m_p}{\deg(\rho_{\infty})} \cdot \sum_u r_{u\beta}^{\beta\beta}(q_\alpha) \cdot (u - u_0),
\]

where \( r_{u\beta}^{\beta\beta} \) in \( H^1(Q_p, Q_p) \) is defined by \( r_u = r_{u\beta}^{\beta\beta} \otimes q_{\beta\beta} \). The lemma follows from the previous equation, Eq. (21) and the identity \( r_u(q_\alpha)f = r_{u\beta}^{\beta\beta}(q_\alpha) \cdot \varepsilon_p(f) \).

\[\square\]

**Lemma 2.11.** Assume that either \( \mathcal{L}^\alpha_f \neq \mathcal{L}^\alpha_g \) or \( \mathcal{L}^\alpha_f \neq \mathcal{L}^\alpha_h \), so that \( r_u \) belongs to \( H^1(Q_p, V_{\beta\bar{\beta}}) \) for \( u = k, l, m \) by Lemma 2.8. Then

\[
\langle \kappa(f, g_\alpha, h_\alpha) \rangle_{fg\beta} = \log_{\alpha\alpha}(\varepsilon_p(\cdot)) \cdot \sum_u r_u(e(1)f) \cdot (u - u_0)
\]

as \( I / I^2 \)-valued \( L \)-linear forms on the Bloch–Kato Selmer group \( \text{Sel}(Q, V) \).

Proof. Let \( y \) be a Selmer class in \( \text{Sel}(Q, V) \), and let \( \tilde{y} = r_{ur}(y) \) in \( \tilde{H}^1_f(Q, V) \) be the corresponding class in the extended Selmer group (cf. Section 2.3 of [4]). By construction \( \tilde{y}^+ \) belongs to the Bloch–Kato finite subspace of \( H^1(Q, V^+) \), and \( \text{res}_p(y) = i^+(\tilde{y}^+) \) is its image under the map \( i^+ \) induced in cohomology by the inclusion \( V^+ \hookrightarrow V \). Define \( \tilde{g}^+_{\alpha\alpha} \) in \( Z_p^* \otimes_{Z_p} L \) by the identity

\[
\text{pr}_{\alpha\alpha}(\tilde{g}^+) = \tilde{g}^+_{\alpha\alpha} \otimes (\eta_f \otimes \omega_{\alpha\alpha} \otimes \omega_{h_\alpha}),
\]

in \( H^1(\text{fin}(Q_p, V_{\alpha\alpha}^+)) = H^1_{\text{fin}}(Q_p, L(1)) \otimes_L V_{\alpha\alpha}^+(-1) \) (where as usual \( H^1_{\text{fin}}(Q_p, L(1)) \) is identified with \( Z_p^* \otimes_{Z_p} L \) via the local Kummer map). Then one has

\[
\log_{\alpha\alpha}(\varepsilon_p(y)) = \log_p(\tilde{g}^+_{\alpha\alpha})
\]

where \( \log_p \) is the \( L \)-linear extension of the \( p \)-adic logarithm on \( Z^*_p \). Write similarly

\[
r_u = \text{pr}_{\beta\beta}(r_u) = r_{u\beta}^{\beta\beta} \otimes (\omega_f \otimes \eta_{\alpha\alpha} \otimes \eta_{h_\alpha})
\]

in \( H^1(Q_p, V_{\beta\bar{\beta}}) = H^1(Q_p, L) \otimes_L V_{\beta\bar{\beta}}^\beta \) for some \( r_{u\beta}^{\beta\beta} \) in \( H^1(Q_p, L) \), so that

\[
\langle r_u, \tilde{y}^+ \rangle_{\text{Tate}} = -r_{u\beta}^{\beta\beta}(\tilde{g}^+_{\alpha\alpha}) = -\log_p(\tilde{g}^+_{\alpha\alpha}) \cdot r_{u\beta}^{\beta\beta}(e(1)) = \log_{\alpha\alpha}(\varepsilon_p(y)) \cdot r_u(e(1)f)
\]

by local class field theory. The statement then follows from Lemma 2.6.

\[\square\]

We can finally conclude the proof of Proposition 2.4.

**Proof of Proposition 2.4.** To lighten the notation set \( \kappa_{\alpha\alpha} = \kappa(f, g_\alpha, h_\alpha) \). By definition the extended height \( h_{\alpha\alpha}^\alpha(k_{\alpha\alpha} \otimes y) \) is equal (up to sign) to

\[
\langle q_{\alpha\alpha}, q_{\beta\beta} \rangle_{fg\beta} \cdot \langle \kappa_{\alpha\alpha}, y \rangle_{fg\beta} - \langle q_{\alpha\alpha}, \kappa_{\alpha\alpha} \rangle_{fg\beta} \cdot \langle q_{\beta\beta}, y \rangle_{fg\beta}
\]

\[\Box\]
for each Selmer class $y$ in $\text{Sel}(\mathbb{Q}, V)$. Since $\kappa_{\alpha\alpha}$ is (the specialisation of) a balanced class, one has $\log_{\alpha\alpha}(\text{res}_p(\kappa_{\alpha\alpha})) = 0$ (cf. Section 9.1 of [5]), hence $\langle q_{\beta\beta}, \kappa_{\alpha\alpha} \rangle_{fg}^{gh}$ is equal to zero by Lemma 2.5. As a consequence

$$\tilde{h}^\alpha_p(\kappa_{\alpha\alpha} \otimes y) = \text{det}\left(\frac{\langle q_{\alpha\alpha}, q_{\beta\beta} \rangle_{fg}^{gh} \cdot \langle q_{\alpha\alpha}, \kappa_{\alpha\alpha} \rangle_{fg}^{gh}}{\langle q_{\beta\beta}, y \rangle_{fg}^{gh} \cdot \langle \kappa_{\alpha\alpha}, y \rangle_{fg}^{gh}}\right). \quad (22)$$

Assume first $\mathcal{L}_f^{an} = \mathcal{L}_g^{an} = \mathcal{L}_h^{an}$. Then $\langle q_{\alpha\alpha}, q_{\beta\beta} \rangle_{fg}^{gh}$ is equal to zero by Lemma 2.5, so that Eq. (22) and Lemmas 2.5 and 2.10 yield the equality (up to sign)

$$\tilde{h}^\alpha_p(\kappa_{\alpha\alpha} \otimes y) = \frac{m_p \cdot \text{ord}_p(q_A)}{2 \cdot \text{deg}(\varphi_\infty)} \cdot \log_{\alpha\alpha}(\text{res}_p(y)) \cdot (k - l - m) \cdot \sum_u \ell_k(r_u) \cdot (u - u_o).$$

Moreover one has (by definition) $\ell_k = -\ell_l = -\ell_m$, hence

$$(k - l - m) \cdot \sum_u \ell_k(r_u) \cdot (u - u_o) = \sum_{u,v} \tilde{\delta}_{u,v} \cdot (u - u_o)(v - v_o).$$

Proposition 2.4 follows from the previous two equations and Lemmas 2.3 and 2.9.

Assume from now on that the analytic $\mathcal{L}$-invariants $\mathcal{L}_f^{an}, \mathcal{L}_g^{an}$ and $\mathcal{L}_h^{an}$ are not all equal. Then Eq. (22), Lemmas 2.5, 2.10 and 2.11 yield

$$\tilde{h}^\alpha_p(\kappa_{\alpha\alpha} \otimes y) = \frac{m_p \cdot \text{ord}_p(q_A)}{2 \cdot \text{deg}(\varphi_\infty)} \cdot \log_{\alpha\alpha}(\text{res}_p(y)) \cdot \text{det}(H) \quad (23)$$

in $\mathcal{I}^2 / \mathcal{I}^3$ for each Selmer class $y$ in $\text{Sel}(\mathbb{Q}, V)$, where

$$H = \left(\begin{array}{cc}
\left(\mathcal{L}_f^{an} - \mathcal{L}_g^{an}\right) \cdot (l - 1) + (\mathcal{L}_f^{an} - \mathcal{L}_h^{an}) \cdot (m - 1) & -\sum_u \ell_k(r_u) \cdot (u - u_o) \\
l + m - k & \sum_u r_u(e(1)f) \cdot (u - u_o)
\end{array}\right).$$

A direct computation gives

$$\text{det}(H) = -\sum_{u,v} \tilde{\delta}_{u,v} \cdot (u - u_o)(v - v_o). \quad (24)$$

Proposition 2.4 follows from Eqs. (23) and (24) and Lemmas 2.3 and 2.9. \hfill \square

### 2.4. Heegner Points and Diagonal Classes

Assume from now on

$$\text{ord}_{s = 1} L(f \otimes g \otimes h, s) = 2 \quad (25)$$

and that Assumption 1.2 (stated in Sect. 1) is satisfied.

For each finite order character $\mu : G_K \rightarrow \mathbb{Q}(\varphi)^*$, let $\text{Ind}_K^Q \mu$ be the $\mathbb{Q}(\varphi)$-module of functions $c : G_Q \rightarrow \mathbb{Q}(\varphi)$ satisfying $c(\tau \sigma) = \mu(\tau) \cdot c(\sigma)$ for each $\tau$ in $G_K$ and $\sigma$ in $G_Q$, equipped with the action of $G_Q$ defined by $(\sigma' \cdot c)(\sigma) = c(\sigma \sigma')$
for each $\sigma$ and $\sigma'$ in $G_\mathbb{Q}$. For $\xi = g, h$, the $\mathbb{Q}(g)[G_\mathbb{Q}]$-module $\text{Ind}_K^G \nu_\xi$ affords the representation $\rho_\xi$. With the notations of Sect. 1 we can then take

$$V_\xi = \text{Ind}_K^G \nu_\xi.$$ 

One has an isomorphism of $\mathbb{Q}(g)[G_\mathbb{Q}]$-modules

$$V_{gh} = V_g \otimes_{\mathbb{Q}(g)} V_h \simeq \text{Ind}_K^G \varphi \oplus \text{Ind}_K^G \psi,$$

where $\varphi = \nu_g \cdot \nu_h$ and $\psi = \nu_g \cdot \nu_h^c$ are dihedral characters of $K$ (cf. Sect. 1). The Artin formalism then yields the factorisation

$$L(f \otimes g \otimes h, s) = L(A/K, \varphi, s) \cdot L(A/K, \psi, s),$$

where $L(A/K, \chi, s) = L(f \otimes \vartheta_\chi, s)$ is the Hasse–Weil $L$-function of the base change of $A$ to $K$ twisted by $\chi = \varphi, \psi$ (viz. the Rankin–Selberg convolution of $f$ and the weight-one theta series $\vartheta_\chi$ associated with $\chi$).

Let $\chi$ denote either $\varphi$ or $\psi$, let $K_\chi$ be the ring class field of $K$ cut out by $\chi$, and let $A(K_\chi)^{\chi}$ be the submodule of $A(K_\chi) \otimes \mathbb{Z} \mathbb{Q}(g)$ on which $\text{Gal}(K_\chi/K)$ acts via $\chi$. Fix a primitive Heegner point $P$ in $A(K_\chi)$ and set

$$P_\chi = \sum_{\sigma \in \text{Gal}(K_\chi/K)} \chi(\sigma)^{-1} \cdot \sigma(P) \in A(K_\chi)^{\chi}.$$ 

Equations (25) and (27) and Assumption 1.1.(1) imply that $L(A/K, \chi, s)$ has a simple zero at $s = 1$, hence the Gross–Zagier–Kolyvagin–Zhang theorem yields

$$P_\chi \neq 0 \quad \text{and} \quad A(K_\chi)^{\chi} \otimes \mathbb{Q}(g) L = L \cdot P_\chi = \text{Sel}(K_\chi, V_p(A))^\chi,$$

where $\text{Sel}(K_\chi, V_p(A))$ is the Bloch–Kato Selmer group of the restriction of $V_p(A)$ to $G_{K_\chi}$, one denotes by $\text{Sel}(K_\chi, V_p(A))^\chi$ the submodule of $\text{Sel}(K_\chi, V_p(A)) \otimes \mathbb{Q}_p L$ on which the Galois group of $K_\chi/K$ acts via the character $\chi$, and one considers $A(K_\chi)^{\chi}$ as a submodule of $\text{Sel}(K_\chi, V_p(A))^\chi$ via the $K_\chi$-rational Kummer map.

Let $\sigma_p$ in $G_{\mathbb{Q}} - G_K$ be an arithmetic Frobenius at $p$. For $\xi = g, h$ and each pair $(a, b)$ of elements of $\mathbb{Q}(g)$, denote by $[a, b]_\xi$ in $V_\xi$ the $\mathbb{Q}(g)$-valued function on $G_{\mathbb{Q}}$ sending the identity to $a$ and $\sigma_p$ to $b$. Then $G_K$ acts on the line $L \cdot [1, 0]_\xi$ via $\nu_\xi$, and on the line $L \cdot [0, 1]_\xi$ via the conjugate $\nu_\xi^c$ of $\nu_\xi$ by the nontrivial element $c = \sigma_p|_K$ of $\text{Gal}(K/\mathbb{Q})$. Moreover, since $\nu_\xi(\sigma_p^2) = \nu_\xi(p) = \varepsilon_K(p) \cdot \chi_\xi(p) = -\chi_\xi(p) = \alpha_\xi^2$ (cf. Sect. 1), one has $\sigma_p \cdot [a, b]_\xi = [b, \alpha_\xi^2 \cdot a]_\xi$ for each $a$ and $b$ in $\mathbb{Q}(g)$. Set

$$v_{\xi, \alpha} = [1, \alpha_\xi]_\xi \in V_\xi^{\sigma_p=\alpha_\xi} \quad \text{and} \quad v_{\xi, \beta} = [1, -\alpha_\xi]_\xi \in V_\xi^{\sigma_p=\beta_\xi}.$$ 

(recall that $\beta_\xi = -\alpha_\xi$), and for each pair $(i, j)$ of elements of $\{\alpha, \beta\}$ set

$$v_{ij} = v_{g, i} \otimes v_{h, j} \in V_g^{\sigma_p=i_g} \otimes \mathbb{Q}(g) V_h^{\sigma_p=j_h} \rightarrow V_{gh}^{\sigma_p=i_{gh}+j_{gh}}.$$ 

A direct computation shows that the vectors

$$v_\varphi = v_{\alpha\alpha} + v_{\alpha\beta} + v_{\beta\alpha} + v_{\beta\beta} \quad \text{and} \quad v_\psi = v_{\alpha\alpha} - v_{\alpha\beta} + v_{\beta\alpha} - v_{\beta\beta}$$

of $V_{gh}$ are equal to $4 \cdot [1, 0]_g \otimes [1, 0]_h$ and $4\alpha_\xi \cdot [1, 0]_g \otimes [0, 1]_h$ respectively, hence $G_K$ acts on them via $\varphi = \nu_g \cdot \nu_h$ and $\psi = \nu_g \cdot \nu_h^c$ respectively. For $\chi = \varphi, \psi$ define

$$P(\chi) = \gamma_{gh}(P_\chi \otimes \sigma_p(v_\chi) + \sigma_p(P_\chi) \otimes v_\chi).$$
in $\text{Sel}(Q,V)$ to be image of $P_{\chi} \otimes \sigma_p(v_{\chi}) + \sigma_p(P_{\chi}) \otimes v_{\chi}$ in $A(K_p)^{g}$ under the embedding $\gamma_{gh}$ introduced in Eq. (10), so that (cf. Eqs. (26) and (28))

$$\text{Sel}(Q,V) = L \cdot P(\varphi) \oplus L \cdot P(\psi).$$

(29)

Write $\varepsilon = \alpha_f$ and for $\chi$ equal to $\varphi$ or $\psi$ define

$$P_{\chi}^\varepsilon = P_{\chi} + \varepsilon \cdot \sigma_p(P_{\chi}).$$

The point $P_{\chi}^\varepsilon$ is non-zero. This follows from Eq. (28) if $\chi$ is not quadratic. When $\chi$ is quadratic, one has $\sigma_p(P_{\chi}) = \chi_1(p) \cdot P_{\chi}$, hence $P_{\chi}^\varepsilon$ is non-zero by Eq. (28) and Assumption 1.2. In order to lighten the notation, set $\kappa_{\alpha\alpha} = \kappa(f,g,\alpha)$. The main result Theorem A of [2] proves the identity

$$\log_{\beta\beta}(\text{res}_p(\kappa(f,g,\alpha))) = \log_{\omega_f}(P_{\varphi}^\varepsilon) \cdot \log_{\omega_f}(P_{\psi}^\varepsilon) \in L^*/Q(g)^*.$$

(30)

Here $\log_{\omega_f} : A(K_\chi) \otimes_{\mathbb{Z}} L \longrightarrow L \otimes_{\mathbb{Q}_p} K_p$ denotes the $L$-linear extension of the logarithm $\log_{\omega_f}$ on $A(K_\chi)$ introduced in Sect. 2.1.4. (Note that the right hand side of the previous identity is an element of $L \otimes_{\mathbb{Q}_p} K_p$ fixed by the action of $\sigma_p$, ide est of $L$.)

Recall that the roots $\alpha_\xi$ and $\beta_\xi = -\alpha_\xi$ of the $p$th Hecke polynomial of $\xi = g,h$ are distinct, and that $\alpha_g \cdot \alpha_h = \alpha_f = \beta_g \cdot \beta_h$ (cf. Eq. (1)). We can then replace in the above constructions the Hida family $\xi = \xi_\alpha$ with the one $\xi_\beta$ specialising to the $p$-stabilisation $\xi_\beta(q) = \xi(q) - \alpha_\xi \cdot \xi(q^p)$ at weight one, for $\xi = g,h$. This produces a diagonal class $\kappa(f,g_\beta,h_\beta)$ in the Selmer group $\text{Sel}(Q,W)$ of the $p$-adic representation $W = V(f,g_\beta,h_\beta) \otimes_{\text{G}_{\text{Q}}}{L}$. Fix an isomorphism of $L[\text{G}_{\text{Q}}]$-modules $\mu : W \simeq V$, and let

$$\kappa_{\beta\beta} = \mu(\kappa(f,g_\beta,h_\beta)) \in \text{Sel}(Q,V)$$

be the image of $\kappa(f,g_\beta,h_\beta)$ under the isomorphism it induces in cohomology. The analogue of Eq. (30) proves that the $\alpha\alpha$-logarithm of $\kappa_{\beta\beta}$ is non-zero:

$$\log_{\alpha\alpha}(\text{res}_p(\kappa_{\beta\beta})) \in L^*.$$ 

(31)

Since by the definition of the balanced local condition (cf. Sect. 2.1.1) one has

$$\log_{\alpha\alpha}(\text{res}_p(\kappa_{\alpha\alpha})) = \log_{\beta\beta}(\text{res}_p(\kappa_{\beta\beta})) = 0,$$

(32)

it follows that the diagonal classes $\kappa_{\alpha\alpha}$ and $\kappa_{\beta\beta}$ are linearly independent, hence

$$\text{Sel}(Q,V) = L \cdot \kappa_{\alpha\alpha} \oplus L \cdot \kappa_{\beta\beta}.$$ 

(33)

2.4.1. Conclusion of the Proof. Consider the $L$-basis (cf. Eqs. (6) and (8))

$$q_g = \frac{1}{\sqrt{m_p}} \cdot q(f) \otimes v_g^\alpha \otimes v_h^\alpha$$

and

$$q_h = \frac{1}{\sqrt{m_p}} \cdot q(f) \otimes v_g^\beta \otimes v_h^\beta$$

of $H^0(Q_p,V^-)$, where $v_\xi = \gamma_\xi(v_\xi)$ for $\xi = g,h$ and $\cdot = \alpha,\beta$. It is the image of the $Q(g)$-basis $\{q(A) \otimes v_{g,\alpha} \otimes v_{h,\alpha}, q(A) \otimes v_{g,\beta} \otimes v_{h,\beta}\}$ of $Q_p(A,g)$ (cf. Eq. (9)) under the isomorphism $Q_p(A,g) \otimes_{\text{G}_{\text{Q}}}{L} \simeq H^0(Q_p,V)$ arising from the modular parametrisation $\varphi_\infty$ fixed in Sect. 2.1.1 and the embeddings $\gamma_g$ and $\gamma_h$ fixed in Eq. (8). Define $\mathbb{M}$ and $\mathbb{N}$ in $\text{GL}_2(L)$ by the identities (cf. Eqs. (29) and (33))

$$\begin{pmatrix} \kappa_{\alpha\alpha} \\ \kappa_{\beta\beta} \end{pmatrix} = \mathbb{M} \begin{pmatrix} P(\chi) \\ P(\psi) \end{pmatrix}$$

and

$$\begin{pmatrix} q_{\alpha\alpha} \\ q_{\beta\beta} \end{pmatrix} = \mathbb{N} \begin{pmatrix} q_g \\ q_h \end{pmatrix}.$$
By the definition of the $p$-adic regulator $R_p^{\alpha\alpha}(A, \varrho)$ and Proposition 2.4 one has
\[ R_p^{\alpha\alpha}(A, \varrho) = \frac{\log_2 (\res_p (\kappa_{\beta\beta}))}{\det(M)^2 \cdot \det(N)^2} \cdot L_p^{\alpha\alpha}(A, \varrho) \pmod{\mathcal{I}^5} \tag{34} \]
in the quotient of $\mathcal{I}^4 / \mathcal{I}^5$ by the multiplicative action of $Q(\varrho)^*$. Set $\hat{L} = L \otimes_{q_p} \hat{Q}_p^{nr}$ and for $\xi = g, h$ denote by
\[ \hat{\pi}_\xi : V(\xi) \otimes_L V(\xi) \otimes_{Q_p} \hat{Q}_p^{nr} \longrightarrow \hat{L} \]
the $\hat{Q}_p^{nr}$-base change of the perfect pairing $\pi_\xi$ introduced in Sect. 2.1.1. Since
\[ \hat{\pi}_g(\eta_{g_\alpha} \otimes \omega_{g_\alpha}) \cdot \hat{\pi}_h(\eta_{h_\alpha} \otimes \omega_{h_\alpha}) = G(\chi_g) \cdot G(\chi_h) = 1 \]
(cf. Assumption 1.1.(2) and the definitions introduced in Sect. 2.1.2), one has
\[ N = \frac{1}{\sqrt{m_p}} \begin{pmatrix} \hat{\pi}_g(v^\alpha_g \otimes \eta_{g_\alpha}) \cdot \hat{\pi}_h(v^\alpha_h \otimes \eta_{h_\alpha}) & 0 \\ 0 & \hat{\pi}_g(v^\beta_g \otimes \omega_{g_\alpha}) \cdot \hat{\pi}_h(v^\beta_h \otimes \omega_{h_\alpha}) \end{pmatrix} \]
(in $H^0(\sigma_p, \GL_2(\hat{L})) = \GL_2(L)$), hence
\[ \det(N) = m_p^{-1} \cdot \hat{\pi}_g(v^\alpha_g \otimes v^\beta_g) \cdot \hat{\pi}_h(v^\alpha_h \otimes v^\beta_h) \in Q(\varrho)^* \tag{35} \]
by the normalisation imposed on the embeddings $\gamma_g$ and $\gamma_h$ (cf. Eq. (8)).

According to Eqs. (30), (31) and (32) one has
\[ M^{-1} = \begin{pmatrix} \log_{\beta\beta}(P(\varphi)) & \log_{\alpha\alpha}(P(\varphi)) \\ \log_{\beta\beta}(P(\psi)) & \log_{\alpha\alpha}(P(\psi)) \end{pmatrix} \]
where $\log_{i\alpha} : \Sel(Q, V) \longrightarrow L$, for $i = \alpha, \beta$, is a shorthand for $\log_{i\alpha} \circ \res_p$. After retracing the definitions given in Sect. 2.4, a direct computation yields
\[ \log_{\alpha\alpha}(P(\chi)) = \varepsilon \cdot \log_\omega f(P^e_\chi) \cdot \hat{\pi}_g(v^\alpha_g \otimes \eta_{g_\alpha}) \cdot \hat{\pi}_h(v^\beta_h \otimes \omega_{h_\alpha}) \]
(in $H^0(\sigma_p, \hat{L}) = L$, where as usual $\chi$ denotes either $\varphi$ or $\psi$) and
\[ \log_{\beta\beta}(P(\chi)) = \varepsilon_\varphi \cdot \varepsilon \cdot \log_\omega f(P^e_\chi) \cdot \hat{\pi}_g(v^\beta_g \otimes \omega_{g_\alpha}) \cdot \hat{\pi}_h(v^\beta_h \otimes \omega_{h_\alpha}) \]
where $\varepsilon_\varphi = 1$ and $\varepsilon_\psi = -1$. As a consequence
\[ \frac{\log_{\alpha\alpha}(\kappa_{\beta\beta})}{\det(M)} = 2 \cdot \frac{\log_\omega f(P^e_\varphi) \cdot \log_\omega f(P^e_\psi)}{\log_{\beta\beta}(\kappa_{\alpha\alpha})} \cdot \hat{\pi}_g(v^\alpha_g \otimes v^\beta_g) \cdot \hat{\pi}_h(v^\alpha_h \otimes v^\beta_h) \in Q(\varrho)^* \tag{36} \]
by Eqs. (30) and (8). Equations (34), (35) and (36) give the identity
\[ L_p^{\alpha\alpha}(A, \varrho) \pmod{\mathcal{I}^5} = R_p^{\alpha\alpha}(A, \varrho) \]
in the quotient of $\mathcal{I}^4 / \mathcal{I}^5$ by the multiplicative action of $Q(\varrho)^*$. To conclude the proof of the Theorem stated in Sect. 1, it remains to prove that both sides of the previous identity are non-zero. This follows by combining Eq. (30) with [5, Theorem A] and [2, Proposition 2.2], which prove the equality
\[ \frac{\partial^2 \mathcal{L}_p^{\alpha\alpha}(A, \varrho)}{\partial k^2}(w_o) = c_p(f) \cdot \frac{\deg(\varphi_\infty)}{2m_p \ord_p(q_A)} \left( 1 - \frac{1}{p} \right)^{-1} \cdot \log_{\beta\beta}(\kappa_{\alpha\alpha}) \].
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