Caustics and Intermittency in Turbulent Suspensions of Heavy Particles

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The statistics of velocity differences between very heavy inertial particles suspended in an incompressible turbulent flow is found to be extremely intermittent. When particles are separated by distances within the viscous subrange, the competition between quiet regular regions and multi-valued caustics leads to a quasi bi-fractal behavior of the particle velocity structure functions, with high-order moments bringing the statistical signature of caustics. Contrastingly, for particles separated by inertial-range distances, the velocity-difference statistics is characterized in terms of a local Hölder exponent, which is a function of the scale-dependent particle Stokes number only. Results are supported by high-resolution direct numerical simulations. It is argued that these findings might have implications in the early stage of rain droplets formation in warm clouds.

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Two effects have recently been singled out to explain the speed-up of collisions between small finite-size particles suspended in turbulent flows 1, 2, 3: preferential concentration, that is the development of strong inhomogeneities in their spatial distribution (Fig. 1a) 4, 5, 6, and the formation of fold caustics (also called the sling effect), which results in large probabilities that close particles have important velocity differences (Fig. 1b) 7, 8, 9, improving the collision kernels used in kinetic models for atmospheric physics, astrophysics, and engineering requiring quantifying precisely the individual contribution of these two effects and, in particular, to what extent turbulence might affect them, i.e. how they depend on the Taylor-scale Reynolds number $R_L$ of the flow 10, 11.

In this Letter, to straighten out these questions, we consider suspensions of small, heavy, and dilute particles, which is a setting relevant to the early stage of rain droplets formation in clouds 2. In these conditions, particles are simply dragged by viscous forces and each individual trajectory $X(t)$ solves the equation

$$\tau \ddot{X} = -\dot{X} + u(X, t),$$

where dots denote time derivatives, $u$ the fluid velocity field, solution of the incompressible Navier–Stokes equation, and $\tau = 2a^2 \alpha/(9\nu)$ the Stokes time, depending on particle radius $a$, density contrast $\alpha$ with the fluid, and fluid kinematic viscosity $\nu$, see 12 for a recent review. The importance of inertia in the particle dynamics is quantified by the *Stokes number* $St = \tau / \tau_\eta$, where $\tau_\eta$ is the fluid eddy turnover time associated to the Kolmogorov dissipative scale $\eta$. The collision rate between particles is evaluated using the ghost-particle approach 4, which assumes that particles can occupy any point of space independently of the positions of others. This approximation is valid in the asymptotics of very diluted suspensions, and has the advantage of relying on stationary dynamical statistics: the geometrical collision rate is then determined by the value at $r = 2a$ of the approaching rate 4

$$\kappa(r; St) = -\left( \langle \frac{d}{dt} \frac{R}{\hat{R}} \right)_{R = r \text{ and } \hat{R} \leq 0} p_2(r). \quad (2)$$

Here $R$ denotes the distance between two particles with Stokes number $St$, and $p_2$ its probability density. The average is performed over time and particle pairs, with the condition to be separated by a distance $r$ and to approach each other. Clearly the behavior of $\kappa(r; St)$ prescribes the dependence of the collision rate upon the particle radius $a$.

**FIG. 1:** (a) Snapshot of the position of particles for $St = 2$ in a slice of size $5\eta \times 100\eta \times 100\eta$ for $R_L \approx 400$. (b) Particle velocity field in the same slice for a larger Stokes, $St = 20$, showing the existence of regions where particles have different velocities (highlighted by gray and black arrows).
particle attributes (size and mass density contrast). Caustics and preferential concentration (Fig. 1) intricately appear in [2] affecting the conditional average of the velocity difference $\langle \delta u \rangle$ and the $r$-dependence of $p_2$, respectively. In particular, as shown in [13], $p_2(r)$, which is straightforwardly related to the radial distribution function of [3], behaves as a power law in the dissipative range, namely $p_2(r) \propto r^{D_2-1}$ for $r \ll \eta$, where $D_2 \in [0:3]$ is the correlation dimension of the particle distribution and non-trivially depends on the Stokes number.

We focus here on the velocity contribution by studying the behavior as a function of the separation $r$ of the longitudinal particle velocity structure functions

$$S_p(r; St) = \left\langle |\dot{R}|^p \mid R=r \right\rangle. \quad (3)$$

The choice of defining structure functions with absolute values is motivated by the definition of the collision kernel (via the approaching rate), since we do not expect important differences between negative and positive velocity fluctuations. One can therefore estimate: $\kappa(r) \propto r^{D_2-1}S_2(r; St)$ (see [1]). Evaluating $S_p(r; St)$ for values of $p$ different from 1, besides providing a more complete characterization of the velocity statistics, allows one to account also for fluctuations of the local approaching rate, which can vary significantly from place to place. In the limit of small inertia, the particle dynamics approaches that of tracers and consequently the velocity difference $\dot{R}$ is essentially coincident with the fluid longitudinal increment over a separation $R$. Conversely, when $St \to \infty$, particles move ballistically in the flow with uncorrelated velocities and the structure functions $S_p(r; St)$ become independent of $r$. For intermediate values of the Stokes number, one expects a non-trivial behavior of $S_p$ as a function of $r$ and $St$. Data analyzed in this study are from a direct numerical simulation at $R_\lambda \approx 400$ described in [14][15].

Figure 2 represents the behavior of the second-order structure function $S_2(r; St)$, measured in direct numerical simulations, for two different values of the carrier flow Reynolds number (see [14] for details on the simulations). One distinguishes different regimes, depending whether $r$ is within the dissipative or inertial range of the turbulent carrier flow. While the dissipative-range behavior directly relates to inter-particle collisions, the inertial-range behavior has important implications on the relative dispersion of particles in turbulent flow [14] in general and for pollution models in particular. In the sequel we investigate these two regimes separately.

In the dissipative range, the structure functions display a power-law behavior $S_p(r; St) \propto r^{\xi_p}$. The two asymptotics of weak and strong inertia imply that $\xi_p \approx p$ for $St \ll 1$ and $\xi_p \to 0$ for $St \to \infty$. For intermediate values of the Stokes number, $\xi_p \equiv \xi_p(St)$ is a convex function of the order $p$ with values in $[0 : p]$. Figure 3 shows the first-order exponent $\xi_1$ as a function of the Stokes number. One can clearly observe that for $St = O(1)$, the exponent $\xi_1$ takes non-trivial values spanning the whole interval $[0:1]$. The present accuracy of data does not allow for settling either the issue of a possible saturation of the exponent to the limiting values at the two extrema, nor a possible dependence of the exponent upon $R_\lambda$. Despite a factor two in $R_\lambda$, data differ by less than the errors made in the determination of the exponents or in the value of $\tau_\eta$ that enters the definition of $St$.

At first glance the continuous variation of the exponent $\xi_1$ from 1 to 0 at increasing $St$ seems inconsistent with a naive picture of the role of caustics in velocity statistics. Fold caustics are a part of catastrophe theory [16]: they occur when fast particles catch up with slower ones to create regions where several velocities can be found at the same location as in Fig. 1b. If particles conserve their velocity and move ballistically, such caustics will extend over the whole domain (whence the analogy with caustics formed by light rays [8]). The typical velocity difference between two particles becomes in that case independent of their distance, meaning that structure functions tend to a constant as $r \to 0$, and thus $\xi_p = 0$. However, there are two clear reasons why this continuous-field picture may fail. First, because of their dissipative dynamics, particles concentrate on dynamical attractors in the position-velocity phase space [7]. Such sets are fractal.
the caustics are randomly distributed with a typical size and dominate the velocity statistics at large moments only, while small orders are controlled by the smooth regions of the particle velocity. In that case the structure function would display a bi-fractal behavior similar to that present in random solutions to the Burgers equation (see, e.g., [18]), namely $\xi_p = p$ for $p \leq \xi_\infty$ and $\xi_p = \xi_\infty$ for $p \geq \xi_\infty$. Current numerical results do not permit to distinguish between these two possibilities. As seen from Fig. [4] the measured exponents show some deviations from the bi-fractal behavior. However as already observed in other settings [10], this apparent multiscaling could be an artifact due to the presence of sub-leading terms or logarithmic corrections.

To further disentangle the question of the contribution of caustics to velocity scaling, we investigate the statistical properties of $\sigma = \dot{R}/R$, which can interpreted as a longitudinal velocity gradient of an effective particle velocity field. This quantity is at the center of much work devoted to the relative motion of a pair of particles in time-uncorrelated flows [3, 20, 21, 22]. There, the dynamics of $\sigma$ becomes independent of $R$ at very small scales, a far from obvious feature for particles transported by real flows, where time correlations and structures play important roles. Further, results in random flows suggest that the conditional probability density $p(\sigma | R = r)$ is independent of $\sigma$ at small scales and has power-law tails. As seen from Fig. [5] numerical measurements in turbulent flows suggest features similar to those of structure-less random flows. The core of the distributions associated to different scales $r$ collapse for $|\sigma| \lesssim \sigma^* (r)$ on a distribution with a fat, almost algebraic behavior. Interestingly, the associated power-law exponent is close to $-(1 + \xi_\infty)$, suggesting that $\langle \sigma^p \rangle$ diverges for $p > \xi_\infty$, a behavior favoring the bi-fractal scenario. Indeed $\langle \sigma^p \rangle (r; St) = r^{p \xi_p} (\sigma^p | R = r) \propto r^{p} (\sigma^p)$ for $r \rightarrow 0$ and $p$ such that $\langle \sigma^p \rangle < \infty$. However, for $|\sigma| \gtrsim \sigma^* (r)$, the distributions display stretched-exponential tails, whose contribution to the structure function is for the moment unsettled. They are connected to the caustics size prob-
ability distribution and could lead to multiscaling. A related open question is the non-trivial entanglement between clusters and large velocity differences, as already stressed in random flows \[23\]. This latter feature might imply an intricate dependence on $r$ of velocity difference statistics, that might lead to multiscaling. Settling numerically the issue of bi- versus multiscaling would require to explore systems were statistics can be handled in a more systematic way, as for instance in random correlated flows or real flows at smaller Reynolds numbers.

We finally turn to the behavior of the velocity structure function for separations within the inertial range of turbulence, i.e. for $\eta \ll r \ll L$. As seen from Fig. 2 particle velocity structure functions recover the fluid ones when $r$ becomes very large. Indeed as $r$ increases the associated eddy turnover time grows as $r^{2/3}$ (where we used the Kolmogorov 1941 scaling) so that the effective strength of inertia reduces. Similarly to random self-similar carrier flows \[22\], this effect can be put on a quantitative ground in terms of a scale-dependent Stokes number $St(r) = \varepsilon^{1/3}r^{2/3}$ defined as the ratio between the particle response time and the turnover time associated to the scale $r$, where $\varepsilon$ denotes the mean dissipation rate of kinetic energy. We check whether the local scaling exponent $\xi_1(r, \tau) \equiv (\ln S_\eta(r; St))/ (\ln r)$ does depend on $St(r)$ only, as observed in random self-similar flows \[22\]. Figure 5 shows a good collapse of the values of $\xi_1(r, \tau)$ associated to various $\tau$ and of $r$, once represented as a function of $St(r)$. Moreover, the curve $\xi_1(St)$ has a shape qualitatively very similar to that of $\xi_1(St)$ observed in the dissipative range and shown in Fig. 3 this fact is relevant to heavy particle dispersion in turbulent flows \[14\]. Let us stress that data corresponding to small $St(r)$ in Fig. 5 show deviations from the K41 scaling that are similar to those expected for tracers-like statistics.

To conclude we briefly discuss the applicability of the present results to atmospheric physics. The main shortcoming of the proposed approach is that the gravity force is neglected. As observed in \[24\] for dynamics of water droplets in warm clouds, gravitational settling is found to dominate the statistics of velocity differences between particles. However, this effect acts mainly between particles of different sizes that fall at different speeds. Present results should apply to earlier stage of rain formation during which the droplets are almost mono-disperse and might play an important role in explaining the observed fast spectral broadening observed in clouds.

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