A continuous generalization of domination-like invariants

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Abstract

In this paper, we define a new domination-like invariant of graphs. Let \( \mathbb{R}^+ \) be the set of non-negative numbers. Let \( c \in \mathbb{R}^+ - \{0\} \) be a number, and let \( G \) be a graph. A function \( f : V(G) \to \mathbb{R}^+ \) is a \( c \)-self-dominating function of \( G \) if for every \( u \in V(G) \), \( f(u) \geq c \) or \( \max\{f(v) : v \in N_G(u)\} \geq 1 \). The \( c \)-self-domination number \( \gamma^c(G) \) of \( G \) is defined as \( \gamma^c(G) := \min\{\sum_{u \in V(G)} f(u) : f \) is a \( c \)-self-dominating function of \( G\} \). Then \( \gamma^1(G) \), \( \gamma^\infty(G) \) and \( \gamma^{\frac{1}{2}}(G) \) are equal to the domination number, the total domination number and the half of the Roman domination number of \( G \), respectively.

Our main aim is to continuously fill in the gaps among such three invariants. In this paper, we give a sharp upper bound of the \( c \)-self-domination number for all \( c \geq \frac{1}{2} \).

Keywords Self-domination · Domination · Total domination · Roman domination

Mathematics Subject Classification 05C69

1 Introduction

1.1 Definitions and notations

All graphs considered in this paper are finite, simple, and undirected. Let \( G \) be a graph. We let \( V(G) \) and \( E(G) \) denote the vertex set and the edge set of \( G \), respectively. For a vertex \( u \in V(G) \), we let \( N_G(u) \) and \( d_G(u) \) denote the neighborhood and the degree of \( u \), respectively; thus \( N_G(u) = \{v \in V(G) : uv \in E(G)\} \) and \( d_G(u) = |N_G(u)| \). For a subset \( U \) of \( V(G) \), we let \( G[U] \) denote the subgraph of \( G \) induced by \( U \). We let \( P_n \) denote the path of order \( n \). For terms and symbols not defined in this paper, we refer the reader to Diestel (2016).

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We let $\mathbb{R}^+$ denote the set of non-negative numbers. Here we regard $\infty$ as a non-negative number (i.e., $\infty \in \mathbb{R}^+$). For a graph $G$ and a function $f : V(G) \to \mathbb{R}^+$, the weight $w(f)$ of $f$ is defined by $w(f) = \sum_{u \in V(G)} f(u)$.

Let $G$ be a graph. A set $S \subseteq V(G)$ is a dominating set of $G$ if each vertex in $V(G) - S$ is adjacent to a vertex in $S$. The minimum cardinality of a dominating set of $G$, denoted by $\gamma(G)$, is called the domination number of $G$. The domination number is one of the most important invariants in Graph Theory, and it can be widely applied to real problems, for example, school bus routing problem, social network theory and location of radio stations (see Haynes et al. 1998a,b). To meet various additional requirements for above problems, many domination-like invariants were defined and studied.

A set $S \subseteq V(G)$ is a total dominating set of $G$ if each vertex of $G$ is adjacent to a vertex in $S$. Note that if $G$ has no isolated vertices, then there exists a total dominating set of $G$. For a graph $G$ without isolated vertices, the minimum cardinality of a total dominating set of $G$, denoted by $\gamma_t(G)$, is called the total domination number of $G$. The total domination number is typically defined for only graphs without isolated vertices. However, in this paper, we define $\gamma_t(G)$ as $\gamma_t(G) = \infty$ if $G$ has an isolated vertex for convenience. The concept of total domination was introduced in Cockayne et al. (1980), and has been actively studied (see a book Henning and Yeo 2013).

A function $f : V(G) \to \{0, 1, 2\}$ is a Roman dominating function of $G$ if each vertex $u \in V(G)$ with $f(u) = 0$ is adjacent to a vertex $v \in V(G)$ with $f(v) = 2$. The minimum weight of a Roman dominating function of $G$, denoted by $\gamma_R(G)$, is called the Roman domination number of $G$. The Roman domination number was introduced by Stewart (1999), and was studied by Cockayne et al. (2004) in earnest. Roman domination derives from the strategy to defend the Roman Empire against the enemies. Recently, various properties on the Roman domination number have been explored, for example, Fu et al. (2009), Liedloff et al. (2008), Liu and Chang (2012).

From a mathematical point of view, Roman domination concept seems to be more artificial than original domination and total domination. However, by the following reasons, we can interpret Roman domination as a natural extension of domination and total domination.

We define a new domination-like invariant. Let $c \in \mathbb{R}^+ - \{0\}$ be a number, and let $G$ be a graph. A function $f : V(G) \to \mathbb{R}^+$ is a c-self-dominating function (or c-SDF) of $G$ if for each $u \in V(G)$, $f(u) \geq c$ or $\max\{f(v) : v \in N_G(u)\} \geq 1$.

**Remark 1** We choose a c-SDF $f$ of $G$ so that

(C1) $w(f)$ is as small as possible and
(C2) $\{u \in V(G) : f(u) \notin \{0, 1, c\}\}$ is as small as possible subject to (C1).

Suppose that there exists a vertex $x \in V(G)$ with $f(x) \notin \{0, 1, c\}$. Now we construct a function $f' : V(G) \to \mathbb{R}^+$ as follows: For $u \in V(G) - \{x\}$, let $f'(u) = f(u)$. If $0 < f(x) < \min\{1, c\}$, let $f'(x) = 0$; if $\min\{1, c\} < f(x) < \max\{1, c\}$, let $f'(x) = \min\{1, c\}$; if $f(x) > \max\{1, c\}$, let $f'(x) = \max\{1, c\}$ (for example, if $c = \infty$, then $f'(x) = 0$ or $f'(x) = 1$ according as $0 < f(x) < 1$ or $1 < f(x) < \infty$). Then $f'$ is a c-SDF of $G$. We can easily verify that if $w(f') \neq \infty$, then $w(f') < w(f)$; otherwise, $w(f) = w(f') = \infty$ and $\{u \in V(G) : f'(u) \notin \{0, 1, c\}\} < \{u \in V(G) : f(u) \notin \{0, 1, c\}\}$.
Proposition 1.1 Let \( G \) be a graph. Then the following hold.

(i) \( \gamma^1(G) = \gamma(G) \),

(ii) \( \gamma^\infty(G) = \gamma_t(G) \), and

(iii) \( \gamma^{\frac{1}{2}}(G) = \frac{1}{2} \gamma_R(G) \).

Proof (i) For a dominating set \( S \) of \( G \) with \( |S| = \gamma(G) \), the function \( f_1 : V(G) \rightarrow \mathbb{R}^+ \) with

\[
    f_1(u) = \begin{cases} 
        1, & u \in S; \\
        0, & u \notin S 
    \end{cases}
\]

is a 1-SDF of \( G \) with \( w(f_1) = |S| \), and hence \( \gamma^1(G) \leq w(f_1) = |S| = \gamma(G) \).

Let \( f \) be a \( \gamma^1 \)-function of \( G \). Then by (1.1), \( \{ f(u) : u \in V(G) \} \subseteq \{0, 1\} \). Hence the set \( S_1 := \{ u \in V(G) : f(u) = 1 \} \) is a dominating set of \( G \) with \( |S_1| = w(f) \).

Thus \( \gamma(G) \leq |S_1| = w(f) = \gamma^1(G) \).

Consequently, \( \gamma^1(G) = \gamma(G) \).

(ii) If \( G \) has an isolated vertex, then it is clear that \( \gamma^\infty(G) = \infty = \gamma_t(G) \). Thus we may assume that \( G \) has no isolated vertices. For a total dominating set \( S \) of \( G \) with \( |S| = \gamma_t(G) \), the function \( f_2 : V(G) \rightarrow \mathbb{R}^+ \) with

\[
    f_2(u) = \begin{cases} 
        1, & u \in S; \\
        0, & u \notin S 
    \end{cases}
\]

is an \( \infty \)-SDF of \( G \) with \( w(f_2) = |S| \), and hence \( \gamma^\infty(G) \leq w(f_2) = |S| = \gamma_t(G) \).

Let \( f \) be a \( \gamma^\infty \)-function of \( G \). Since the function assigning 1 to all vertices of \( G \) is an \( \infty \)-SDF of \( G \), we have \( \gamma^\infty(G) < \infty \), and hence \( f \) does not use \( \infty \).

This together with (1.1) implies that \( \{ f(u) : u \in V(G) \} \subseteq \{0, 1\} \). Hence the set \( S_2 := \{ u \in V(G) : f(u) = 1 \} \) is a total dominating set of \( G \) with \( |S_2| = w(f) \).

Thus \( \gamma_t(G) \leq |S_2| = w(f) = \gamma^\infty(G) \).

Consequently, \( \gamma^\infty(G) = \gamma_t(G) \).

(iii) For a Roman dominating function \( f \) of \( G \) with \( w(f) = \gamma_R(G) \), the function \( f_3 : V(G) \rightarrow \mathbb{R}^+ \) with \( f_3(u) = \frac{1}{2} f(u) \) \((u \in V(G))\) is a \( \frac{1}{2} \)-SDF of \( G \) with...
Theorem 1.2 Let $c$ be a $\gamma^2$-function of $G$. Then by (1.1), $[f'(u) : u \in V(G)] \subseteq \{0, 1, \frac{1}{2}\}$. Hence the function $f'_3 : V(G) \to \{0, 1, 2\}$ with $f'_3(u) = 2f'(u)$ ($u \in V(G)$) is a Roman dominating function of $G$ with $w(f'_3) = 2w(f')$. Thus $\gamma_R(G) \leq w(f'_3) = 2w(f') = 2\gamma^2(G)$. Consequently, $\gamma^{2c}(G) = \frac{1}{2} \gamma_R(G)$. \hfill \Box

1.2 Main results

By Proposition 1.1, $c$-self-domination can continuously fill in the gaps among three invariants: domination, total domination and Roman domination. On the other hand, some results concerning such invariants have been proved via different techniques. Thus the study of $c$-self-domination for $c \geq \frac{1}{2}$ may give essential boundaries of them. As the initial research for the goal, we focus on the following known upper bounds.

Theorem A (Ore (1962)) Let $G$ be a connected graph of order $n \geq 2$. Then $\gamma(G) \leq \frac{1}{2}n$.

Theorem B (Cockayne et al. (1980)) Let $G$ be a connected graph of order $n \geq 3$. Then $\gamma_t(G) \leq \frac{2}{3}n$.

Theorem C (Chambers et al. (2009)) Let $G$ be a connected graph of order $n \geq 3$. Then $\gamma_R(G) \leq \frac{4}{5}n$.

In this paper, we generalize Theorems A–C as follows.

Theorem 1.2 Let $c \in \mathbb{R}^+$ be a number with $c \geq \frac{1}{2}$. Let $G$ be a connected graph of order $n \geq 3$. Then

$$\gamma^c(G) \leq \begin{cases} \frac{m+1}{2m+3}n, & \frac{m}{m+1} \leq c \leq \frac{2m+1}{2m+3}, \ m \in \mathbb{N}; \\ \frac{cm+2c+1}{2m+5}n, & \frac{2m+1}{2m+3} \leq c \leq \frac{m+1}{m+2}, \ m \in \mathbb{N}; \\ \frac{1}{2}n, & c = 1; \\ \frac{m+2}{2m+3}n, & \frac{m+2}{m+1} < c \leq \frac{(2m+1)(m+2)}{(2m+3)m}, \ m \in \mathbb{N}; \\ \frac{cm}{2m+1}n, & \frac{(2m+1)(m+2)}{(2m+3)m} < c \leq \frac{m+1}{m}, \ m \in \mathbb{N}; \\ \frac{2}{3}n, & c > 2. \end{cases}$$

Remark 2 (i) Since the function $h(m) := \frac{m}{m+1}$ is a monotonically increasing function and $\lim_{m \to \infty} h(m) = 1$, the interval $[\frac{1}{2}, 1]$ can be partitioned by intervals $[\frac{m}{m+1}, \frac{m+1}{m+2})$ ($m \in \mathbb{N}$). In particular, for a number $c (\frac{1}{2} \leq c < 1)$, there is only one positive integer $m$ such that $\frac{m+1}{m+2} \leq c < \frac{m+1}{m+2}$. Furthermore, since $\frac{m}{m+1} < \frac{2m+1}{2m+3} < \frac{m+1}{m+2}$, the interval $[\frac{m}{m+1}, \frac{m+1}{m+2})$ can be partitioned by two intervals $[\frac{m}{m+1}, \frac{2m+1}{2m+3})$ and $[\frac{2m+1}{2m+3}, \frac{m+1}{m+2})$.

(ii) Since the function $h'(m) := \frac{m+1}{m}$ is a monotonically decreasing function and $\lim_{m \to \infty} h'(m) = 1$, the interval $(1, 2]$ can be partitioned by intervals
Thus it suffices to focus on Theorem 1.2 for the case where 1 < \( p \leq 2 \). We consider the case where 1 < \( p \leq 2 \).

Let \( T \) be a tree. For an edge \( x \), let \( T_x \) be the component of \( T - x \) containing \( x \). An edge \( x \) of \( T \) is good if \( |V(T_x)| \geq 3 \) for each \( i \in \{1, 2\} \).

For non-negative integers \( p \) and \( q \), we let \( T_{p,q} \) denote the tree with \( V(T_{p,q}) = \{x\} \cup \{y_{i,j} : 1 \leq i \leq p, j \in \{1, 2\}\} \cup \{z_i : 1 \leq i \leq q\} \)

and \( E(T_{p,q}) = \{xy_{i,1}, y_{i,1}y_{i,2} : 1 \leq i \leq p\} \cup \{xz_i : 1 \leq i \leq q\} \)

(see Fig. 1).

The following lemma might be known. However, to keep the paper self-contained, we give its proof.

**Lemma 2.1** Let \( T \) be a tree of order at least 3. Then \( T \) has no good edge if and only if \( T \) is isomorphic to \( T_{p,q} \) for some \( p \geq 0 \) and \( q \geq 0 \) with \( 2p + q \geq 2 \).

**Proof** For integers \( p \geq 0 \) and \( q \geq 0 \) with \( 2p + q \geq 2 \), it is clear that \( T_{p,q} \) has no good edge. Thus it suffices to show that the “only if” part of the lemma.

Let \( P = x_0x_1 \cdots x_d \) be a longest path of \( T \). Then \( d \) is equal to the diameter of \( T \). In particular, \( d_T(x_1) \geq 2 \) because \( d \geq 2 \). By the maximality of \( P \), every vertex in \( N_T(x_1) - \{x_0, x_2\} \) is a leaf of \( T \).
Suppose that \( d_T(x_1) \geq 3 \). Since \( x_1x_2 \) is not a good edge of \( T \), \( |V(T_{x_1x_2}^{x_2})| \leq 2 \). In particular, either \( d = 2 \) and \( V(T_{x_1x_2}^{x_2}) = \{x_2\} \) or \( d = 3 \) and \( V(T_{x_1x_2}^{x_2}) = \{x_2, x_3\} \). Let \( k = d_T(x_1) \). If \( d = 2 \) and \( V(T_{x_1x_2}^{x_2}) = \{x_2\} \), then \( T \) is isomorphic to \( T_{0,k} \); if \( d = 3 \) and \( V(T_{x_1x_2}^{x_2}) = \{x_2, x_3\} \), then \( T \) is isomorphic to \( T_{1,k-1} \). In either case, we obtain the desired conclusion. Thus we may assume that \( d_T(x_1) = 2 \) (i.e., \( N_T(x_1) = \{x_0, x_2\} \)).

For each vertex \( u \in N_T(x_2) - \{x_1\} \), since \( x_2u \) is not a good edge and \( |V(T_{x_2u}^{x_2})| \geq 3 \), \( |V(T_{x_2u}^{x_2})| \leq 2 \). Let \( p = |\{u \in N_T(x_2) - \{x_1\} : |V(T_{x_2u}^{x_2})| = 2\}| \) and \( q = |\{u \in N_T(x_2) - \{x_1\} : |V(T_{x_2u}^{x_2})| = 1\}| \). Then \( T \) is isomorphic to \( T_{p+1,q} \), as desired. \( \square \)

### 3 Upper bound on \( \gamma^c \) for \( \frac{1}{2} \leq c < 1 \)

In this section, we prove the following theorem.

**Theorem 3.1** Let \( c \in \mathbb{R}^+ \) be a number with \( \frac{1}{2} \leq c < 1 \), and let \( m \geq 1 \) be the integer such that \( \frac{m}{m+1} \leq c < \frac{m+1}{m+2} \). Let \( T \) be a tree of order \( n \geq 3 \). Then the conclusion of Theorem 1.2 holds for \( G = T \).

**Remark 3** Since the deletion of edges cannot decrease the \( c \)-self-domination number of a graph, we obtain Theorem 1.2 for the case where \( \frac{1}{2} \leq c < 1 \) as a corollary of Theorem 3.1.

We first show the following lemma.

**Lemma 3.2** Let \( c \) and \( m \) be as in Theorem 3.1. If \( T = T_{p,q} \) for integers \( p \geq 0 \) and \( q \geq 0 \) with \( 2p+q \geq 2 \), then the conclusion of Theorem 3.1 holds.

**Proof** The function \( f_1 : V(T_{p,q}) \rightarrow \mathbb{R}^+ \) with

\[
 f_1(u) = \begin{cases} 
 1, & u = x; \\
 c, & u \in \{y_i, 2 \leq i \leq p\}; \\
 0, & \text{otherwise}
\end{cases}
\]

is a \( c \)-SDF of \( T_{p,q} \) with \( w(f_1) = cp + 1 \). Hence

\[
 \gamma^c(T_{p,q}) \leq w(f_1) = cp + 1 = \frac{cp + 1}{2p + q + 1} |V(T_{p,q})|. \tag{3.1}
\]

If \( p \geq 1 \), then the function \( f_2 : V(T_{p,q}) \rightarrow \mathbb{R}^+ \) with

\[
 f_2(u) = \begin{cases} 
 1, & u \in \{y_i, 1 \leq i \leq p\}; \\
 c, & u \in \{z_i, 1 \leq i \leq q\}; \\
 0, & \text{otherwise}
\end{cases}
\]

is a \( c \)-SDF of \( T_{p,q} \) with \( w(f_2) = p + cq \). Hence

\[
 \gamma^c(T_{p,q}) \leq w(f_2) = p + cq = \frac{p + cq}{2p + q + 1} |V(T_{p,q})| \quad \text{if } p \geq 1. \tag{3.2}
\]

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Furthermore, we have
\[ m + 1 = \frac{(2m + 1)(m + 2)}{2m + 3} + 1 \leq \frac{c(m + 2) + 1}{2m + 5} \quad \text{if} \quad c \geq \frac{2m + 1}{2m + 3}. \quad (3.3) \]

We divide the proof into four cases.

**Case 1:** Either \( p = 0 \) or \((p, q) = (1, 0)\).

Note that if \( p = 0 \), then \( q = 2p + q \geq 2 \). Hence it follows from (3.1) and (3.2) that
\[ \gamma^c(T_{p, q}) \leq \frac{1}{3}|V(T_{p, q})| < \frac{m + 1}{2m + 3}|V(T_{p, q})|. \quad (3.4) \]

In particular, we obtain the desired conclusion for the case where \( \frac{m}{m + 1} \leq c < \frac{2m + 1}{2m + 3} \).

If \( \frac{2m + 1}{2m + 3} \leq c < \frac{m + 1}{m + 2} \), then it follows from (3.3) and (3.4) that
\[ \gamma^c(T_{p, q}) < \frac{m + 1}{2m + 3}|V(T_{p, q})| \leq \frac{c(m + 2) + 1}{2m + 5}|V(T_{p, q})|, \]
as desired.

**Case 2:** \( p = 1 \) and \( q \geq 1 \).

If \( \frac{m}{m + 1} \leq c < \frac{2m + 1}{2m + 3} \), then by (3.1) and the assumption that \( q \geq 1 \),
\[ \gamma^c(T_{p, q}) \leq \frac{cp + 1}{2p + q + 1}|V(T_{p, q})| < \frac{2m + 3}{4}|V(T_{p, q})| = \frac{m + 1}{2m + 3}|V(T_{p, q})|, \]
as desired. Thus we may assume that \( \frac{2m + 1}{2m + 3} \leq c < \frac{m + 1}{m + 2} \). Since
\[ 4(cm + 2c + 1) - (c + 1)(2m + 5) = c(2m + 3) - (2m + 1) \geq \frac{(2m + 1)(2m + 3)}{2m + 3} - (2m + 1) = 0, \]
we have
\[ \frac{cp + 1}{2p + q + 1} \leq \frac{c + 1}{4} \leq \frac{cm + 2c + 1}{2m + 5}. \]

This together with (3.1) leads to the desired conclusion.

**Case 3:** \( p \geq 2 \) and \( q \geq 1 \).

Since \( p \geq 2 \),
\[ 2(m + 1)(m + 2)(p + 1) - ((m + 1)p + m + 2)(2m + 3) = m(p - 1) + p - 2 > 0, \]

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and hence

\[
\frac{(m + 1)p + m + 2}{2(m + 2)(p + 1)} \leq \frac{m + 1}{2m + 3}.
\]

This together with (3.1) implies

\[
\gamma^c(T_{p,q}) \leq \frac{cp + 1}{2p + q + 1} |V(T_{p,q})| < \frac{(m + 1)p + 1}{2p + 2} |V(T_{p,q})| = \frac{(m + 1)p + m + 2}{2(m + 2)(p + 1)} |V(T_{p,q})| \leq \frac{m + 1}{2m + 3} |V(T_{p,q})|.
\]

(3.5)

In particular, we obtain the desired conclusion for the case where \( \frac{m}{m+1} \leq c < \frac{2m+1}{2m+3} \).

If \( \frac{2m+1}{2m+3} \leq c < \frac{m+1}{m+2} \), then it follows from (3.3) and (3.5) that

\[
\gamma^c(T_{p,q}) \leq \frac{m + 1}{2m + 3} |V(T_{p,q})| \leq \frac{c(m + 2) + 1}{2m + 5} |V(T_{p,q})|,
\]

as desired.

**Case 4:** \( p \geq 2 \) and \( q = 0 \).

Suppose that \( p \leq m + 1 \). Then it follows from (3.2) that

\[
\gamma^c(T_{p,q}) \leq \frac{p}{2p + 1} |V(T_{p,q})| \leq \frac{m + 1}{2m + 3} |V(T_{p,q})|.
\]

(3.6)

In particular, we obtain the desired conclusion for the case where \( \frac{m}{m+1} \leq c < \frac{2m+1}{2m+3} \).

If \( \frac{2m+1}{2m+3} \leq c < \frac{m+1}{m+2} \), then it follows from (3.3) and (3.6) that

\[
\gamma^c(T_{p,q}) \leq \frac{m + 1}{2m + 3} |V(T_{p,q})| \leq \frac{c(m + 2) + 1}{2m + 5} |V(T_{p,q})|,
\]

as desired. Thus we may assume that \( p \geq m + 2 \).

We first suppose that \( \frac{m}{m+1} \leq c < \frac{2m+1}{2m+3} \). Then

\[
(m + 1)(2p + 1) - ((2m + 1)p + 2m + 3) = p - m - 2 \geq 0,
\]

and hence

\[
\frac{(2m + 1)p + 2m + 3}{2p + 1} \leq m + 1.
\]
This together with (3.1) leads to
\[
\gamma^c(T_{p,q}) \leq \frac{cp + 1}{2p + 1} |V(T_{p,q})| \\
< \frac{(2m+1)p + 1}{2p + 1} |V(T_{p,q})| \\
= \frac{(2m + 1)p + 2m + 3}{(2p + 1)(2m + 3)} |V(T_{p,q})| \\
\leq \frac{m + 1}{2m + 3} |V(T_{p,q})|,
\]
which leads to the desired conclusion.

Next we suppose that \(\frac{2m + 1}{2m + 3} \leq c < \frac{m + 1}{m + 2}\). Then
\[
(c(m + 2) + 1)(2p + 1) - (cp + 1)(2m + 5) = (p - m - 2)(2 - c) \geq 0,
\]
and hence
\[
\frac{cp + 1}{2p + 1} \leq \frac{c(m + 2) + 1}{2m + 5}.
\]
This together with (3.1) leads to
\[
\gamma^c(T_{p,q}) \leq \frac{cp + 1}{2p + 1} |V(T_{p,q})| \leq \frac{c(m + 2) + 1}{2m + 5} |V(T_{p,q})|.
\]
This completes the proof of Lemma 3.2. □

**Proof of Theorem 3.1** We proceed by induction on \(n\). If \(T\) has no good edge, then by Lemma 2.1 and Lemma 3.2, the desired conclusion holds. Thus we may assume that \(T\) has a good edge \(x_1x_2\). Then by the induction hypothesis, for each \(i \in \{1, 2\}\),
\[
\gamma^c(T_{x_1x_2}^i) \leq \begin{cases} 
\frac{m + 1}{2m + 3} |V(T_{x_1x_2}^i)|, & \frac{m + 1}{m + 2} \leq c < \frac{2m + 1}{2m + 3} \\
\frac{cm + 2c + 1}{2m + 5} |V(T_{x_1x_2}^i)|, & \frac{2m + 1}{2m + 3} \leq c < \frac{m + 1}{m + 2}.
\end{cases}
\]
Since \(\gamma^c(T) \leq \gamma^c(T_{x_1x_2}^{x_1}) + \gamma^c(T_{x_1x_2}^{x_2})\) and \(|V(T_{x_1x_2}^{x_1})| + |V(T_{x_1x_2}^{x_2})| = n\), this leads to the desired conclusion. □

**4 Upper bound on \(\gamma^c\) for \(1 < c \leq 2\)**

In this section, we prove the following theorem.

**Theorem 4.1** Let \(c \in \mathbb{R}^+\) be a number with \(1 < c \leq 2\), and let \(m \geq 1\) be the integer such that \(\frac{m + 2}{m + 1} < c \leq \frac{m + 1}{m}\). Let \(T\) be a tree of order \(n \geq 3\). Then the conclusion of Theorem 1.2 holds for \(G = T\).
**Remark 4** Since the deletion of edges cannot decrease the \( c \)-self-domination number of a graph, we obtain Theorem 1.2 for the case where \( 1 < c \leq 2 \) as a corollary of Theorem 4.1.

We first show the following lemma.

**Lemma 4.2** Let \( c \) and \( m \) be as in Theorem 4.1. If \( T = T_{p,q} \) for integers \( p \geq 0 \) and \( q \geq 0 \) with \( 2p + q \geq 2 \), then the conclusion of Theorem 4.1 holds.

**Proof** If \( p = 0 \), then the function \( f_1 : V(T_{p,q}) \to \mathbb{R}^+ \) with

\[
  f_1(u) = \begin{cases} 
    c, & u = x; \\ 
    0, & \text{otherwise} 
  \end{cases}
\]

is a \( c \)-SDF of \( T_{p,q} \) with \( w(f_1) = c \). Hence

\[
  \gamma_c^c(T_{p,q}) \leq w(f_1) = c = \frac{c}{q+1}|V(T_{p,q})| \quad \text{if } p = 0. \tag{4.1}
\]

If \( p \geq 1 \), then the function \( f_2 : V(T_{p,q}) \to \mathbb{R}^+ \) with

\[
  f_2(u) = \begin{cases} 
    1, & u \in \{x, y_i, 1 \leq i \leq p\}; \\ 
    0, & \text{otherwise} 
  \end{cases}
\]

is a \( c \)-SDF of \( T_{p,q} \) with \( w(f_2) = p + 1 \). Hence

\[
  \gamma_c^c(T_{p,q}) \leq w(f_2) = p + 1 = \frac{p + 1}{2p + q + 1}|V(T_{p,q})| \quad \text{if } p \geq 1. \tag{4.2}
\]

If \( q = 0 \) (i.e., \( p \geq 1 \)), then the function \( f_3 : V(T_{p,q}) \to \mathbb{R}^+ \) with

\[
  f_3(u) = \begin{cases} 
    c, & u \in \{y_i, 1 \leq i \leq p\}; \\ 
    0, & \text{otherwise} 
  \end{cases}
\]

is a \( c \)-SDF of \( T_{p,q} \) with \( w(f_3) = cp \). Hence

\[
  \gamma_c^c(T_{p,q}) \leq w(f_3) = cp = \frac{cp}{2p + 1}|V(T_{p,q})| \quad \text{if } q = 0. \tag{4.3}
\]

Furthermore, we have

\[
  \frac{cm}{2m+1} \leq \frac{(2m+1)(m+2)}{2m+3} = \frac{m+2}{2m+3} \quad \text{if } c \leq \frac{(2m+1)(m+2)}{(2m+3)m}. \tag{4.4}
\]

We divide the proof into three cases.
Case 1: Either $p = 0$ or $(p, q) = (1, 0)$.

Note that if $p = 0$, then $q = 2p + q \geq 2$. Hence it follows from (4.1) and (4.3) that

$$\gamma^c(T_{p,q}) \leq \frac{c}{3} |V(T_{p,q})| \leq \frac{cm}{2m + 1} |V(T_{p,q})|. \quad (4.5)$$

In particular, we obtain the desired conclusion for the case where $\frac{(2m+1)(m+2)}{2m+3} < c \leq \frac{m+1}{m}$. If $\frac{m+2}{m+1} < c \leq \frac{(2m+1)(m+2)}{(2m+3)m}$, then it follows from (4.4) and (4.5) that

$$\gamma^c(T_{p,q}) \leq \frac{cm}{2m + 1} |V(T_{p,q})| \leq \frac{m + 2}{2m + 3} |V(T_{p,q})|,$$

as desired.

Case 2: $p \geq 1$ and $q \geq 1$.

By (4.2),

$$\gamma^c(T_{p,q}) \leq \frac{p + 1}{2p + q + 1} |V(T_{p,q})| \leq \frac{p + 1}{2p + 2} |V(T_{p,q})| = \frac{1}{2} |V(T_{p,q})| \leq \frac{(m + 2)m}{(m + 1)(2m + 1)} |V(T_{p,q})| \leq \frac{cm}{2m + 1} |V(T_{p,q})|. \quad (4.6)$$

In particular, we obtain the desired conclusion for the case where $\frac{(2m+1)(m+2)}{2m+3} < c \leq \frac{m+1}{m}$. If $\frac{m+2}{m+1} < c \leq \frac{(2m+1)(m+2)}{(2m+3)m}$, then it follows from (4.4) and (4.6) that

$$\gamma^c(T_{p,q}) < \frac{cm}{2m + 1} |V(T_{p,q})| \leq \frac{m + 2}{2m + 3} |V(T_{p,q})|,$$

as desired.

Case 3: $p \geq 2$ and $q = 0$.

Suppose that $p \leq m$. Then it follows from (4.3) that

$$\gamma^c(T_{p,q}) \leq \frac{cp}{2p + 1} |V(T_{p,q})| \leq \frac{cm}{2m + 1} |V(T_{p,q})|. \quad (4.7)$$
In particular, we obtain the desired conclusion for the case where \( \frac{(m+1)(m+2)}{(m+3)m} < c \leq \frac{m+2}{m+1} \). If \( \frac{m+2}{m+1} < c \leq \frac{(2m+1)(m+2)}{(2m+3)m} \), then it follows from (4.4) and (4.7) that

\[
\gamma^c(T_{p,q}) \leq \frac{cm}{2m+1} |V(T_{p,q})| \leq \frac{m+2}{2m+3} |V(T_{p,q})|,
\]

as desired. Thus we may assume that \( p \geq m+1 \).

We first suppose that \( \frac{m+2}{m+1} < c \leq \frac{(2m+1)(m+2)}{(2m+3)m} \). Then \( \frac{p+1}{2p+1} \leq \frac{m+2}{2m+3} \). This together with (4.2) leads to

\[
\gamma^c(T_{p,q}) \leq \frac{p+1}{2p+1} |V(T_{p,q})| \leq \frac{m+2}{2m+3} |V(T_{p,q})|,
\]

which leads to the desired conclusion.

Next we suppose that \( \frac{(2m+1)(m+2)}{(2m+3)m} < c \leq \frac{m+1}{m} \). Then

\[
\frac{p+1}{2p+1} \leq \frac{m+2}{2m+3} = \frac{(2m+1)(m+2)m}{2m+1} < \frac{cm}{2m+1}.
\]

This together with (4.2) leads to

\[
\gamma^c(T_{p,q}) \leq \frac{p+1}{2p+1} |V(T_{p,q})| < \frac{cm}{2m+1} |V(T_{p,q})|.
\]

This completes the proof of Lemma 4.2. \( \square \)

**Proof of Theorem 4.1** We proceed by induction on \( n \). If \( T \) has no good edge, then by Lemma 2.1 and Lemma 4.2, the desired conclusion holds. Thus we may assume that \( T \) has a good edge \( x_1x_2 \). Then by the induction hypothesis, for each \( i \in \{1, 2\} \),

\[
\gamma^c(T_{x_1x_2}) \leq \begin{cases} \frac{m+2}{2m+1} |V(T_{x_1x_2})|, & \frac{m+2}{m+1} < c \leq \frac{(2m+1)(m+2)}{(2m+3)m}; \\ \frac{cm}{2m+1} |V(T_{x_1x_2})|, & \frac{(2m+1)(m+2)}{(2m+3)m} < c \leq \frac{m+1}{m}. \end{cases}
\]

Since \( \gamma^c(T) \leq \gamma^c(T_{x_1x_2}^{x_1}) + \gamma^c(T_{x_1x_2}^{x_2}) \) and \( |V(T_{x_1x_2}^{x_1})| + |V(T_{x_1x_2}^{x_2})| = n \), this leads to the desired conclusion. \( \square \)

**5 Upper bound on \( \gamma^c \) for \( c > 2 \)**

**Lemma 5.1** Let \( c \in \mathbb{R}^+ \) be a number with \( c \geq 2 \). Let \( G \) be a connected graph of order at least 2. Then \( \gamma^c(G) = \gamma_t(G) \).

**Proof** Let \( f \) be a \( \gamma^c \)-function of \( G \). By (1.1), we have \( \{ f(u) : u \in V(G) \} \subseteq \{0, 1, c\} \).

Choose \( f \) so that \( \{|u \in V(G) : f(u) = 1\} \) is as large as possible.

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Suppose that $f(x) = c$ for some $x \in V(G)$, and let $y \in N_G(x)$. Then the function $f' : V(G) \to \mathbb{R}^+$ with

$$f'(u) = \begin{cases} 1, & u \in \{x, y\}; \\ f(u), & \text{otherwise} \end{cases}$$

is a $c$-SDF of $G$ with $w(f') = w(f) - c - f(y) + 2 \leq w(f)$ and $|[u \in V(G) : f'(u) = 1]| = |[u \in V(G) : f(u) = 1]|$, which contradicts the choice of $f$. Thus $\{f(u) : u \in V(G)\} \subseteq \{0, 1\}$.

Since the set $S := \{u \in V(G) : f(u) = 1\}$ is a total dominating set of $G$ with $|S| = w(f)$, $\gamma_t(G) \leq |S| = w(f) = \gamma^c(G)$. Since every $\infty$-SDF of $G$ is also a $c$-SDF of $G$, we have $\gamma^c(G) \leq \gamma^\infty(G)$. This together with Proposition 1.1 leads to $\gamma^c(G) \leq \gamma^\infty(G) = \gamma_t(G)$, we obtain the desired conclusion.

By Lemma 5.1 and Theorem 4.1 for the case where $c \geq 2$, we obtain the following proposition.

**Proposition 5.2** Let $c \in \mathbb{R}^+$ be a number with $c \geq 2$. Let $G$ be a connected graph of order $n \geq 3$. Then $\gamma^c(G) \leq \frac{2}{3}n$.

**Remark 5** We obtain Theorem 1.2 for the case where $c > 2$ as a corollary of Proposition 5.2.

### 6 Examples

In this section, we show that Theorem 1.2 is best possible.

Let $p \geq 1$ and $s \geq 1$ be integers, and let $T_{p, 0}$ be the tree defined in Sect. 2. Let $L_p^1, \ldots, L_p^s$ be vertex-disjoint copies of $T_{p, 0}$. For $l$ ($1 \leq l \leq s$), let $x^l$ and $y^l_{i, j}$ ($1 \leq i \leq p, j \in \{1, 2\}$) be the vertices of $L_p^l$ corresponding to $x$ and $y_{i, j}$, respectively. Let $T_p^s$ be the tree obtained from $L_p^1, \ldots, L_p^s$ by adding edges $x^l x^{l+1}$ ($1 \leq l \leq s - 1$).

#### 6.1 The case $\frac{1}{2} \leq c < 1$

Throughout this subsection, fix a number $c$ with $\frac{1}{2} \leq c < 1$, and let $m \geq 1$ be the integer such that $\frac{m}{m + 1} \leq c < \frac{m + 1}{m + 2}$.

**Lemma 6.1** Let $p \geq 1$ and $s \geq 1$ be integers. Let $f : V(T_p^s) \to \mathbb{R}^+$ be a $\gamma^c$-function of $T_p^s$. Then for $l$ ($1 \leq l \leq s$), $\sum_{u \in V(L_p^l)} f(u) \geq \min\{cp + 1, p\}$.

**Proof** If $f(x^l) \geq 1$, then $f(y^l_{i, 2}) \geq c$ or $f(y^l_{i, 1}) \geq 1$ for each $i$ ($1 \leq i \leq p$), and hence

$$\sum_{u \in V(L_p^l)} f(u) = f(x^l) + \sum_{1 \leq i \leq p} (f(y^l_{i, 1}) + f(y^l_{i, 2})) \geq 1 + \sum_{1 \leq i \leq p} c = 1 + cp.$$
as desired. Thus we may assume that $f(x^i) < 1$. For each $i$ ($1 \leq i \leq p$), since the restriction of $f$ on $\{y_{i,1}, y_{i,2}\}$ is a $c$-SDF of $G[\{y_{i,1}, y_{i,2}\}] (\simeq P_2)$, we have $f(y_{i,1}) + f(y_{i,2}) \geq \gamma^c(P_2) = 1$. It follows that

$$\sum_{u \in V(L_p)} f(u) = f(x^i) + \sum_{1 \leq i \leq p} (f(y^i_{1,1}) + f(y^i_{1,2})) \geq \sum_{1 \leq i \leq p} 1 = p,$$

as desired. \hfill \Box

Now we show that Theorem 1.2 for the case where $\frac{1}{2} \leq c < 1$ is best possible. We assume that $\frac{m}{m+1} \leq c < \frac{2m+1}{2m+3}$. Let $f$ be a $\gamma^c$-function of $T_{m+1}'$. Since $c(m+1)+1 \geq \frac{m(m+1)}{m+1} + 1 = m + 1$ and $|V(T_{m+1}')| = s(2m + 3)$, it follows from Lemma 6.1 that

$$\gamma^c(T_{m+1}') = w(f) = \sum_{1 \leq l \leq s} \left( \sum_{u \in V(L_{m+1})} f(u) \right) \geq \sum_{1 \leq l \leq s} \min\{c(m+1) + 1, m+1\} = s(m + 1) = \frac{m + 1}{2m + 3} |V(T_{m+1}')|.$$ 

This together with Theorem 1.2 implies that $\gamma^c(T_{m+1}') = \frac{m+1}{2m+3} |V(T_{m+1}')|$. Since $s \geq 1$ is arbitrary, there exist infinitely many connected graphs $G$ with $\gamma^c(G) = \frac{m+1}{2m+3} |V(G)|$.

Next we assume that $\frac{2m+1}{2m+3} \leq c < \frac{m+1}{m+2}$. Let $f'$ be a $\gamma^c$-function of $T_{m+2}'$. Since $c(m+2) + 1 < \frac{(m+1)(m+2)}{m+2} + 1 = m + 2$ and $|V(T_{m+2}')| = s(2m + 5)$, it follows from Lemma 6.1 that

$$\gamma^c(T_{m+2}') = w(f') = \sum_{1 \leq l \leq s} \left( \sum_{u \in V(L_{m+2})} f'(u) \right) \geq \sum_{1 \leq l \leq s} \min\{c(m+2) + 1, m+2\} = s(c(m+2) + 1) = \frac{cm + 2c + 1}{2m + 5} |V(T_{m+2}')|.$$ 

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This together with Theorem 1.2 implies that $\gamma^c(T_{m+2}^s) = \frac{cm+2c+1}{2m+5}|V(T_{m+2}^s)|$. Since $s \geq 1$ is arbitrary, there exist infinitely many connected graphs $G$ with $\gamma^c(G) =\frac{cm+2c+1}{2m+5}|V(G)|$.

Therefore, Theorem 1.2 for the case where $\frac{1}{2} \leq c < 1$ is best possible.

### 6.2 The case $c = 1$

Fink et al. (1985) and Payan and Xuong (1982) proved that a connected graph $G$ satisfies $\gamma(G) = \frac{1}{2}|V(G)|$ if and only if $G$ is isomorphic to either a cycle of order 4 or the graph obtained from a connected graph $H$ by adding a pendant edge to each vertex of $H$. This together with Proposition 1.1(i) implies that there exist infinitely many connected graphs $G$ with $\gamma^1(G) = \frac{1}{2}|V(G)|$. Consequently Theorem 1.2 for the case where $c = 1$ is best possible.

### 6.3 The case $1 < c \leq 2$

Throughout this subsection, fix a number $c$ with $1 < c \leq 2$, and let $m \geq 1$ be the integer such that $\frac{m+2}{m+1} < c \leq \frac{m+1}{m}$.

**Lemma 6.2** Let $p \geq 1$ and $s \geq 1$ be integers. Let $f : V(T_p^s) \to \mathbb{R}^+$ be a $\gamma^c$-function of $T_p^s$. Then for $l$ $(1 \leq l \leq s)$, $\sum_{u \in V(L_p^l)} f(u) \geq \min\{p+1, cp\}$.

**Proof** If $f(x^l) \geq 1$, then $f(y^l_{i,2}) \geq c$ or $f(y^l_{i,1}) \geq 1$ for each $i$ $(1 \leq i \leq p)$, and hence

$$\sum_{u \in V(L_p^l)} f(u) = f(x^l) + \sum_{1 \leq i \leq p} (f(y^l_{i,1}) + f(y^l_{i,2})) \geq 1 + \sum_{1 \leq i \leq p} 1 = 1 + p,$$

as desired. Thus we may assume that $f(x^l) < 1$. For each $i$ $(1 \leq i \leq p)$, since the restriction of $f$ on \{y_{i,1}, y_{i,2}\} is a $c$-SDF of $G[\{y_{i,1}, y_{i,2}\}] \simeq P_2$, we have $f(y_{i,1}) + f(y_{i,2}) \geq \gamma^c(P_2) = c$. It follows that

$$\sum_{u \in V(L_p^l)} f(u) = f(x^l) + \sum_{1 \leq i \leq p} (f(y^l_{i,1}) + f(y^l_{i,2})) \geq \sum_{1 \leq i \leq p} c = cp,$$

as desired. \(\square\)

Now we show that Theorem 1.2 for the case where $1 < c \leq 2$ is best possible. We assume that $\frac{m+2}{m+1} < c \leq \frac{(2m+1)(m+2)}{(2m+3)m}$. Let $f$ be a $\gamma^c$-function of $T_{m+2}^s$. Since $c(m+1) \geq \frac{(m+2)(m+1)}{m+1} = m + 2$ and $|V(T_{m+2}^s)| = s(m + 3)$, it follows from
Lemma 6.2 that

\[
\gamma^c(T_{s+1}^m) = w(f) = \sum_{1 \leq l \leq s} \left( \sum_{u \in V(L_{l+1}^m)} f(u) \right) \geq \sum_{1 \leq l \leq s} \min\{m+1, c(m+1)\} = s(m+2) = \frac{m+2}{2m+3} |V(T_{m+1}^s)|.
\]

This together with Theorem 1.2 implies that \(\gamma^c(T_{m+1}^s) = \frac{m+2}{2m+3} |V(T_{m+1}^s)|\). Since \(s \geq 1\) is arbitrary, there exist infinitely many connected graphs \(G\) with \(\gamma^c(G) = \frac{m+2}{2m+3} |V(G)|\).

Next we assume that \(\frac{(m+1)(m+2)}{2m+3} < c \leq \frac{m+1}{m}\). Let \(f'\) be a \(\gamma^c\)-function of \(T_m^s\). Since \(cm \leq \frac{(m+1)m}{m} = m+1\) and \(|V(T_m^s)| = s(2m+1)\), it follows from Lemma 6.2 that

\[
\gamma^c(T_m^s) = w(f') = \sum_{1 \leq l \leq s} \left( \sum_{u \in V(L_{l+1}^m)} f'(u) \right) \geq \sum_{1 \leq l \leq s} \min\{m+1, cm\} = scm = \frac{cm}{2m+1} |V(T_m^s)|.
\]

This together with Theorem 1.2 implies that \(\gamma^c(T_m^s) = \frac{cm}{2m+1} |V(T_m^s)|\). Since \(s \geq 1\) is arbitrary, there exist infinitely many connected graphs \(G\) with \(\gamma^c(G) = \frac{cm}{2m+1} |V(G)|\).

Therefore, Theorem 1.2 for the case where \(1 < c \leq 2\) is best possible.

### 6.4 The case \(c > 2\)

In Subsection 6.3, we show that there exist infinitely many connected graphs \(G\) with \(\gamma^2(G) = \frac{2}{3} |V(G)|\). On the other hand, it follows from Lemma 5.1 that for a number \(c > 2\) and a connected graph \(G\) of order at least 2, \(\gamma^c(G) = \gamma^2(G)\). Hence for \(c > 2\), there exist infinitely many connected graphs \(G\) with \(\gamma^c(G) = \frac{2}{3} |V(G)|\). Consequently Theorem 1.2 for the case where \(c > 2\) is best possible.
7 Concluding remarks

In this paper, our aim is to find essential boundaries among domination, total domination and Roman domination. Thus we focused on $c$-self-domination for $c \geq \frac{1}{2}$. As far as we check Theorem 1.2 and its proof, the essential parts are roughly divided into $\frac{1}{2} \leq c < 1$, $c = 1$ and $c > 1$. Hence, for example, we expect that some results on Roman domination can be extended to $c$-self-domination for $\frac{1}{2} < c < 1$.

Here one might be interested in upper bounds on $c$-self-domination for $c < \frac{1}{2}$. As a general upper bound on $c$-self-domination, we obtain the following proposition.

**Proposition 7.1** Let $0 < c \leq 1$ be a number, and let $G$ be a graph of order $n$. Then $\gamma^c(G) \leq cn$.

**Proof** Since the function $f : V(G) \to \mathbb{R}^+$ with $f(u) = c$ ($u \in V(G)$) is a $c$-SDF of $G$ with $w(f) = cn$, we get the desired conclusion. \qed

Proposition 7.1 is best possible for the case where $0 < c \leq \frac{1}{3}$. It suffices to show that $\gamma^c(P_n) \geq cn$ for all $n \geq 1$. Let $f$ be a $\gamma^c$-function of $P_n$. By (1.1), we have $\{f(u) : u \in V(P_n)\} \subseteq \{0, 1, c\}$. Choose $f$ so that $|\{u \in V(P_n) : f(u) = c\}|$ is as large as possible. Suppose that $\{f(u) : u \in V(P_n)\} \cap \{0, 1\} \neq \emptyset$. Then there exists a vertex $x \in V(P_n)$ with $f(x) = 1$. Let $f' : V(P_n) \to \mathbb{R}^+$ be the function with

$$f'(u) = \begin{cases} c, & u = x; \\ \max\{f(u), c\}, & u \in NP_n(x); \\ f(u), & \text{otherwise}. \end{cases}$$

Then $f'$ is a $c$-SDF of $P_n$ with $w(f') \leq w(f) - 1 + 3c \leq w(f)$ and $|\{u \in V(P_n) : f'(u) = c\}| > |\{u \in V(P_n) : f(u) = c\}|$, which contradicts the choice of $f$. Thus $f(u) = c$ for all $u \in V(P_n)$. Consequently, $\gamma^c(P_n) = cn$, and so Proposition 7.1 is best possible for the case where $0 < c \leq \frac{1}{3}$.

The author recently settled the remaining case in the following paper Furuya (2020), i.e., for a number $c \in \mathbb{R}^+$ with $\frac{1}{3} < c < \frac{1}{2}$, a sharp upper bound on $\gamma^c(G)$ for a connected graph $G$ is given.

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