AN EFFICIENT ALGEBRAIC CRITERION FOR SHELLABILITY

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Abstract. In this paper, we give a new and efficient algebraic criterion for the pure as well as non-pure shellability of simplicial complex $\Delta$ over $[n]$. We also give an algebraic characterization of a leaf in a simplicial complex (defined in [8]). Moreover, we introduce the concept of Gallai-simplicial complex $\Delta_{\Gamma}(G)$ of a finite simple graph $G$. As an application, we show that the face ring of the Gallai simplicial complex associated to tree is Cohen-Macaulay.

Key words : shellable simplicial complex, face ring of a simplicial complex, facet ideal, Cohen-Macaulay ring.

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1. Introduction

The shellability of a simplicial complex is a well known combinatorial property that carries strong algebraic interpretations for instance see [9] and [13]. Algebraic criterion for the shellability of a simplicial complex has also been a reasonably important subject, firstly introduced by A. Dress in [4]. Dress [4] showed that $\Delta$ is (non-pure) shellable in the sense of Björner and Wachs [5], if and only if the face ring $K[\Delta]$ is clean. Later on Herzog and Popescu [10] extended the concept for determining the shellability of multicomplexes. The shellability criterion for multicomplexes was further refined by Popescu [12]. Cleanness is well known to be the algebraic counterpart of shellability for simplicial complexes. Eagon and Reiner [6] gave a translation of the pure shellability of a dual simplicial complex $\hat{\Delta}$ on the monomial generators of the Stanley-Reisner ideal $I_{\mathcal{V}}(\Delta)$. Their algebraic translation gave birth to an important class of ideals known as ideals with linear quotients (Eagon-Reiner [6] called them as Dually shellable ideals). A relatively new algebraic criterion for the shellability was given in [2], but it was surprisingly found defective, see [3].

The aim of this paper is to give an efficient algebraic criterion of shellability and draw attention towards finding more algebraic properties of shellable complexes in the facet ideal theory. In this paper, we give a new and the most efficient algebraic criterion for the shellability of pure as well as non-pure simplicial complex $\Delta$

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in Theorem 3.3 in terms of the monomial generators of its facet ideal $I_F(\Delta)$. We also give an algebraic characterization of a leaf in a simplicial complex in Theorem 3.6. In the last section, we use the concept of Gallai graph $\Gamma(G)$ of a planar graph $G$ to introduce Gallai simplicial complex $\Delta_{\Gamma}(G)$. The buildup of Gallai simplicial complexes from a planar graph is an abstract idea, somehow, similar to building an origami shape from a plane sheet of paper by defining a crease pattern. We use a planar graph to build a 2-dimensional simplicial complex. We discuss the connectedness of the Gallai simplicial complexes and give a characterization of its facets. As an application, we show that the Gallai simplicial complexes associated to trees are shellable.

2. Basic Setup

A simplicial complex $\Delta$ on the vertex set $[n]$ is a subset of $2^{[n]}$ with the property that if $F \in \Delta$ and $G \subset F$, then $G \in \Delta$. The members of $\Delta$ are called faces and the maximal faces under inclusion are called facets. If $\mathcal{F}(\Delta) = \{F_1, F_2, \ldots, F_s\}$ is the set of facets of $\Delta$, we write $\Delta$ as $\Delta = \langle F_1, F_2, \ldots, F_s \rangle$.

By a subcomplex of $\Delta$, we mean a simplicial complex whose facet set is a subset of $\mathcal{F}(\Delta)$. We denote the dimension of a face $F \in \Delta$ by $\dim(F)$ and it is defined as $\dim(F) = |F| - 1$. By the dimension of a simplicial complex $\Delta$, we mean that $\dim(\Delta) = \max\{\dim(F) \mid F \text{ is a facet in } \Delta\}$. We say that $\Delta$ is a pure simplicial complex of dimension $d$, if all the facets of $\Delta$ are of dimension $d$.

Definition 2.1. A simplicial complex $\Delta$ over $[n]$ is shellable if its facets can be ordered $F_1, F_2, \ldots, F_s$ such that, for all $2 \leq j \leq s$ the subcomplex

$$\hat{\Delta}_{<F_j>} = \langle F_1, F_2, \ldots, F_{j-1} \rangle \cap \langle F_j \rangle$$

is a pure of dimension $\dim(F_j) - 1$.

Shellability in the case of non-pure simplicial complexes was firstly defined by Björner and Wachs [5].

Here, we recall the definition of connected simplicial complex from [7].

Definition 2.2. A simplicial complex $\Delta$ is said to be connected if for any two facets $F$ and $G$ of $\Delta$, there exists a sequence of facets $F = F_0, F_1, \ldots, F_t = G$ such that $F_i \cap F_{i+1} \neq \emptyset$, for any $i \in \{0, 1, 2, \ldots, t - 1\}$. A disconnected simplicial complex is that which is not connected or equivalently if the vertex set $V$ of $\Delta$ can be written as disjoint union of $V_1$ and $V_2$ such that no face of $\Delta$ has vertices in both $V_1$ and $V_2$.

The following definitions serve as the bridge between algebra and simplicial complexes.

Definition 2.3. Let $\Delta$ be a simplicial complex over $[n]$ and $S = k[x_1, \ldots, x_n]$ be the polynomial ring over an infinite field $k$. Let $I_{\mathcal{N}}(\Delta)$ be the ideal of $S$ minimally generated by square-free monomials $x_{j_1}x_{j_2} \ldots x_{j_s}$, where $\{j_1, j_2, \ldots, j_s\} \subset [n]$ is not
a face of $\Delta$. $I_N(\Delta)$ is known as non-face ideal or the Stanley-Reisner ideal of $\Delta$. The quotient ring $S/I_N(\Delta)$ is called the face ring of $\Delta$ denoted by $k[\Delta]$.

**Definition 2.4.** (Faridi [4]). Let $\Delta$ be a simplicial complex over $[n]$ and $S = k[x_1, \ldots, x_n]$ be the polynomial ring over an infinite field $k$. Let $I_F(\Delta) \subset S$ be the monomial ideal minimally generated by square-free monomials $m_{F_1}, \ldots, m_{F_r}$ such that $m_{F_i} = x_{i_1}x_{i_2} \cdots x_{i_r}$, where $F_i = \{i_1, \ldots, i_r\} \subset [n]$ is a facet of $\Delta$ for all $i \in \{1, \ldots, r\}$. $I_F(\Delta)$ is known as the facet ideal of $\Delta$.

Here, we recall the definition of pure square-free monomial ideal from [1].

**Definition 2.5.** Let $I \subset S$ be a square-free monomial ideal with a minimal generating system $\{g_1, \ldots, g_m\}$. We say that $I$ is a pure square-free monomial ideal of degree $d$ if and only if $\text{supp}(I) = \{x_1, \ldots, x_n\}$ and $\beta_{0j}(I) = 0$ for all $j \neq d$.

We conclude this section with recalling following definitions from [2].

**Definition 2.6.** Let $I$ be a monomial ideal in $S$. We define the indeg($I$) as follows

$$\text{indeg}(I) = \min\{j : \beta_{0j}(I) \neq 0\}.$$

**Definition 2.7.** Let $I \subset S = k[x_1, \ldots, x_n]$ be a monomial ideal, we say that $I$ has quasi-linear quotients, if there exists an ordered minimal monomial system of generators $m_1, m_2, \ldots, m_r$ of $I$ such that $\text{indeg}(\hat{I}_{m_i}) = 1$ for all $1 < i \leq r$, where

$$\hat{I}_{m_i} = (m_1, m_2, \ldots, m_{i-1}) : (m_i).$$

### 3. Linear residuals and shellability

In this section, we describe some new algebraic notion for explaining algebraic criterion of pure as well as non-pure shellability of $\Delta$ in the sense of Björner and Wachs [3].

**Remark 3.1.** In [2] Theorem 3.4, it had been shown that $\Delta$ will be a pure shellable simplicial complex if and only if $I_F(\Delta)$ has quasi-linear quotients. But, in [3], it was mentioned that the facet ideal $I_F(\Delta) = (x_1x_2x_3, x_2x_3x_4, x_3x_4x_5, x_4x_5x_1)$ of the pure simplicial complex $\Delta = \langle\{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}, \{4, 5, 1\}\rangle$ has quasi-linear quotients but $\Delta$ is not shellable. Therefore, if a simplicial complex $\Delta$ is pure shellable then $I_F(\Delta)$ has quasi-linear quotients but not vice versa.

The following definition is essential in describing our algebraic criterion for the shellability.

**Definition 3.2.** Let $I \subset S = k[x_1, x_2, \ldots, x_n]$ be a monomial ideal. We say that $I$ has linear residuals if there exists an ordered minimal monomial system of generators $\{m_1, m_2, \ldots, m_r\}$ of $I$ such that $\text{Res}(I_i)$ is minimally generated by linear monomials for all $1 < i \leq r$, where $\text{Res}(I_i) = \{u_1, u_2, \ldots, u_{i-1}\}$ such that $u_k = \frac{m_i}{\text{gcd}(m_k, m_i)}$ for all $1 \leq k \leq i - 1$.

Here is our main result of this section.

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$^1$supp$(I) = \{x_j \mid x_j \text{ divides } u, \text{ with } u \in G(I)\}$

$^2$graded betti number of the ideal $I$
Theorem 3.3. Let $\Delta$ be a simplicial complex of dimension $d$ over $[n]$. Then $\Delta$ will be shellable if and only if $I_{\mathcal{F}}(\Delta)$ has linear residuals.

Proof. Let $\Delta = \langle F_1, F_2, \ldots, F_s \rangle$ be a simplicial complex over $[n]$ of dimension $d$. Firstly, we show that

$$\dim(\hat{\Delta}_{<F_i>}) = \dim(F_i) - \text{indeg}(\text{Res}(I_{F_i})) \text{ for all } 2 \leq i \leq s,$$

where $m_{F_1}, m_{F_2}, \ldots, m_{F_s}$ is the minimal monomial generating system of $I_{\mathcal{F}}(\Delta)$.

By [3,2] a monomial generating system of $\text{Res}(I_{F_i})$ is given as:

$$\text{Res}(I_{F_i}) = \{ u_1, u_2, \ldots, u_{i-1} \}$$

with $u_k = \frac{m_{F_i}}{\gcd(m_{F_i}, m_{F_k})}$ for $1 \leq k \leq i-1$. Then $x_j | u_k$ for some $j \in [n]$ if and only if $\{j\} \in F_i \setminus F_k$. Therefore,

$$\deg(u_k) = |F_i \setminus F_k| = \dim(F_i) - \dim(F_i \cap F_k) \text{ for all } k < i.$$

It implies that $\text{indeg}(\text{Res}(I_{F_i})) = \min\{\dim(F_i) - \dim(F_i \cap F_k) \text{ for all } k < i\}$. Hence, we have $\text{indeg}(\text{Res}(I_{F_i})) = \dim(F_i) - \dim(<F_1, F_2, \ldots, F_{i-1} > \cap <F_i>)$.

Let us consider $\Delta$ be a shellable simplicial complex of dimension $d$ over $[n]$. Then for all $2 \leq j \leq s$ the subcomplex

$$\hat{\Delta}_{<F_j>} = \langle F_1, F_2, \ldots, F_{j-1} > \cap <F_j >$$

is pure of dimension $\dim(F_j) - 1$. From above, it implies that $\text{indeg}(\text{Res}(I_{F_j})) = 1$.

Moreover, from the purity of $\hat{\Delta}_{<F_j>}$, we have $(\text{Res}(I_{F_j}))$ is minimally generated by linear monomials for all $2 \leq j \leq s$. Because, if there exists a term $u_k$ with $\deg(u_k) > 1$ in the minimal generators of $\text{Res}(I_{<F_j>})$, then it implies that $|F_j \setminus \{F_k \cap F_j\}| > 1$ and $F_k \cap F_j$ is a facet, causing $\hat{\Delta}_{<F_j>}$ non-pure.

Conversely, let $I_{\mathcal{F}}(\Delta)$ has linear residuals, then $(\text{Res}(I_{F_j}))$ is minimally generated by linear monomials for all $2 \leq j \leq s$. It implies from above that $\dim(\hat{\Delta}_{<F_j>}) = \dim(F_j) - 1$. If $(\text{Res}(I_{F_j})) = (x_{j_1}, \ldots, x_{j_t})$ then the subcomplex $\hat{\Delta}_{<F_j>} = \langle F_j \setminus \{j_1\}, \ldots, F_j \setminus \{j_t\} \rangle$ will be pure for all $1 < j \leq s$. Hence proved. \qed

The following corollary gives an equivalence of the two algebraic criterions of shellability or one can say that it is relating two different algebraic properties.

Corollary 3.4. The face ring of a simplicial complex $\Delta$ over $[n]$ is clean if and only if $I_{\mathcal{F}}(\Delta)$ has linear residuals.

Proof. We know from [1] Theorem §4, the face ring $k[\Delta]$ is clean if and only if $\Delta$ is shellable. Therefore, result follows from Theorem 3.3. \qed

Theorem 3.3 can be useful in proving the Cohen-Macaulayness of the face ring of a pure simplicial complex as follows.

Corollary 3.5. If the facet ideal $I_{\mathcal{F}}(\Delta)$ of a pure simplicial complex $\Delta$ over $[n]$ has linear residuals, then the face ring $k[\Delta]$ is Cohen-Macaulay.
A leaf of a simplicial complex (introduced by Faridi [8]) is a facet $F$ of $\Delta$ such that either $F$ is the only facet of $\Delta$, or there exists a facet $G$ in $\Delta$, $G \neq F$, such that $F \cap \hat{F} \subsetneq F \cap G$ for every facet $\hat{F} \in \Delta$, $\hat{F} \neq F$. A simplicial complex $\Delta$ is a simplicial tree if $\Delta$ is connected and every subcomplex $\hat{\Delta}$ contains a leaf. By a subcomplex, we mean any simplicial complex of the form $\hat{\Delta} = \langle F_1, \ldots, F_r \rangle$, where $\{F_1, \ldots, F_r\}$ is a subset of the set of facets of $\Delta$.

**Theorem 3.6.** Let $I_{\mathcal{F}}(\Delta) = (m_{F_1}, \ldots, m_{F_r})$ with $r > 1$ be the facet ideal of a simplicial complex $\Delta$. A facet $F_i$ of $\Delta$ will be a leaf if and only if $(\text{Res}(\hat{I}_{F_i}))$ is a principal ideal, where,

$$\text{Res}(\hat{I}_{F_i}) = \{u_j = \frac{m_{F_i}}{\gcd(m_{F_i}, m_{F_j})} \mid \text{ for all } i \neq j\}.$$ 

**Proof.** Suppose $F_i$ is a leaf in $\Delta$, then there exists some facet $F_k$ for $k \neq i$ in $\Delta$ such that $F_i \cap F_j \subseteq F_i \cap F_k$ for all $i \neq j$. It implies that $\gcd(m_{F_i}, m_{F_k})$ divides $\frac{m_{F_i}}{\gcd(m_{F_i}, m_{F_j})}$, yielding $\frac{m_{F_i}}{\gcd(m_{F_i}, m_{F_j})}$ divisible by $\frac{m_{F_i}}{\gcd(m_{F_i}, m_{F_k})}$ for all $j \neq i$, as required.

Conversely, suppose that for a facet $F_i \in \Delta$, $(\text{Res}(\hat{I}_{F_i}))$ is a principal ideal generated by a monomial $u$. Then, $u = \frac{m_{F_i}}{\gcd(m_{F_i}, m_{F_j})}$ for some $p \neq i$, divides $\frac{m_{F_i}}{\gcd(m_{F_i}, m_{F_q})}$ for all $i \neq q \neq p$. Therefore, $\gcd(m_{F_i}, m_{F_q})$ divides $\gcd(m_{F_i}, m_{F_p})$, hence we have $F_i \cap F_q \subseteq F_i \cap F_p$ for all $q \neq i$, implies $F_i$ is a leaf. \hfill $\Box$

### 4. Gallai Simplicial Complexes

From here on, $G$ denotes a finite simple graph on the vertex set $V(G) = [n]$ and edge-set $E(G)$. The Gallai graph $\Gamma(G)$ of $G$ is a graph whose vertex set is the edge set $E(G)$; two distinct edges of $G$ are adjacent in $\Gamma(G)$ if they are incident in $G$ but do not span a triangle in $G$. In [11], authors discussed various combinatorial properties of Gallai and anti-Gallai graph for various classes of graphs.

The following definition is a nice combinatorial buildup.

**Definition 4.1.** The **Gallai graph** $\Gamma(G)$ of a graph $G$ is the graph whose vertex set is the edge set of $G$; two distinct edges of $G$ are adjacent in $\Gamma(G)$ if they are incident in $G$ but do not span a triangle in $G$.

**Example 4.2.** Given below is a graph $G$ and its Gallai graph $\Gamma(G)$.
The following definition is essence in the structural study of Gallai graph $\Gamma(G)$.

**Definition 4.3.** Let $G$ be a finite simple graph with vertex set $V(G) = [n]$ and edge set $E(G) = \{e_{i,j} = \{i, j\}| i, j \in V(G)\}$. We define the set of Gallai-indices $\Omega(G)$ of the graph $G$ as the collection of subsets of $V(G)$ such that if $e_{i,j}$ and $e_{j,k}$ are adjacent in $\Gamma(G)$, then $F_{i,j,k} = \{i, j, k\} \in \Omega(G)$ or if $e_{i,j}$ is an isolated vertex in $\Gamma(G)$ then $F_{i,j} = \{i, j\} \in \Omega(G)$.

**Definition 4.4.** A Gallai simplicial complex $\Delta_{\Gamma}(G)$ of $G$ is a simplicial complex defined over $V(G)$ such that

$$\Delta_{\Gamma}(G) = \langle F | F \in \Omega(G) \rangle,$$

where $\Omega(G)$ is the set of Gallai-indices of $G$.

**Example 4.5.** Let $G$ be a given graph as below then its Gallai simplicial complex is as follow:

$$\Delta_{\Gamma}(G) = \langle \{1, 2\}, \{1, 3, 4\}, \{1, 3, 5\}, \{2, 3, 5\}, \{3, 5, 6\}, \{4, 5, 6\}, \{2, 3, 4\} \rangle$$

![Diagram of the example graph]

**Proposition 4.6.** Let $G$ be a finite simple connected graph. Then the Gallai simplicial complex $\Delta_{\Gamma}(G)$ of $G$ is one dimensional if and only if $\Delta_{\Gamma}(G) = G$.

**Proof.** The dimension of Gallai simplicial complex $\Delta_{\Gamma}(G)$ of $G$ is one if and only if $\Delta_{\Gamma}(G) = \langle F | F \in \Omega(G) \rangle$ and $|F| = 2 = \langle E(G) \rangle$ follows from 4.3. □

**Lemma 4.7.** Let $G$ be a simple connected graph with vertex set $V(G)$. Let $\Omega(G)$ be the set of Gallai-indices of the graph $G$, then for every $F = \{v_1, v_2, v_3\} \in \Omega(G)$ there exists $H \in \Omega(G)$ such that $|F \cap H| = |H| - 1$.

**Proof.** We know from 4.3 that $F = E_i \cup E_j$ for some $E_i, E_j \in E(G)$ and say $E_i \cap E_j = \{v_1\}$ and $\{v_2, v_3\} \notin E(G)$. If $\deg(v_2) = 1 = \deg(v_3)$, then for any edge $E_k = \{v_1, v_k\}$, we have $H = \{v_1, v_2, v_k\} \in \Omega(G)$ proving the result.

If $\deg(v_2) \geq 2$, then for any edge $E_m = \{v_2, v_m\}$, we have $H_1 = \{v_1, v_2, v_m\} \in \Omega(G)$ provided $\{v_1, v_m\} \notin E(G)$ proving the result, or $H_2 = \{v_3, v_1, v_m\} \in \Omega(G)$ provided $\{v_1, v_m\} \in E(G)$ but $\{v_3, v_m\} \notin E(G)$ proving the result, or $H_3 = \{v_3, v_m, v_2\} \in \Omega(G)$ and $H_4 = \{v_1, v_m\}$ provided $\{v_1, v_m\} \in E(G)$ and $\{v_3, v_m\} \in E(G)$ proving the result. □

Here, we give a small but important result about the connectedness of Gallai simplicial complexes.

**Lemma 4.8.** Let $G$ be a simple connected graph, then its Gallai simplicial complex $\Delta_{\Gamma}(G)$ will be connected.
Proof. Let $G$ be a simple connected graph then it is well known that for any two vertices $v_j$ and $v_k$ there exists a sequence of edge-set of $G$ as $\{E_0, E_1, \ldots, E_r\}$ with $x_j \in E_0$ and $x_k \in E_r$ such that $E_i \cap E_{i+1} \neq \emptyset$. If the Gallai simplicial complex $\Delta_T(G)$ of a simple graph $G$ is of dimension one then the result follows from 4.6. Now suppose $\dim(\Delta_T(G)) = 2$ for a simple connected graph $G$. Let $F$ and $H$ be any two facets, with $|F \cap H| = 0$. Let us consider two vertices $v_r \in F$ and $v_s \in H$ of the connected graph $G$. Therefore, there exists a sequence of connected edges $E_j, E_{j+1}, \ldots, E_k$. Then by 4.3, either $E_{j+i-1}$ and $E_{j+i}$ yields a facet of $\Delta_T(G)$ as $F_i = E_{j+i-1} \cup E_{j+i} \in \Omega(G)$ or giving two connected facets $F_{i-1}$ and $F_i$ containing $E_{j+i-1}$ and $E_{j+i}$ respectively, proving the fact. \hfill \Box

Here we give a general shelling for Gallai simplicial complexes associated to trees.

Theorem 4.9. The face ring of Gallai simplicial complex $\Delta_T(T)$ associated to a tree $T$ is Cohen-Macaulay.

Proof. From 3.6, it is sufficient to show that the facet ideal $I_\mathcal{F}(\Delta_T(T))$ is pure and have linear residuals. As tree $T$ does not contain any cycle, therefore $I_\mathcal{F}(\Delta_T(T))$ is pure of the form.

$$\Omega(T) = \{\{i_1, i_2, i_3\} : \text{for any two adjacent edges } \{i_1, i_2\} \text{ and } \{i_2, i_3\}\}.$$ It is well known that any two vertices in a tree $T$ are connected by exactly one path. Without the loss of generality, let us assume a path

$$P_1 = \{v_1, v_2, \ldots, v_{m-1}, v_m\}$$ such that $\deg(v_1) = 1 = \deg(v_m)$. It gives rise to a subcomplex

$$\langle F_{1,2,3}, F_{2,3,4}, \ldots, F_{m-2,m-1,m} \rangle$$
of $\Delta_T(T)$, where $F_{i,i+1,i+2} = \{v_i, v_{i+1}, v_{i+2}\}$. It is easy to see that the facet ideal of the subcomplex $m_{\mathcal{F}(P_1)} = \{m_{F_{1,2,3}}, m_{F_{2,3,4}}, \ldots, m_{F_{m-2,m-1,m}}\}$ has linear residuals, therefore, the subcomplex is pure shellable followed from the Theorem 3.3. If $\deg(v_j) = 2$, for all $2 \leq j \leq m - 1$, then we are done. Otherwise, for any $v_j$ with $\deg(v_j) > 2$, we have a path $P_2$ starting from $v_j$ ending at some vertex with degree 1. Therefore, we have

$$m_{\mathcal{F}(P_2)} = \{m_{F_{j-1,j,k}}, m_{F_{j,j+1,k}}, m_{F_{j,k,k+1}}, \ldots, m_{F_{k_1-2,k_1-1,k_1}}\},$$ such that $(m_{\mathcal{F}(P_1)}, m_{\mathcal{F}(P_2)})$ has linear residuals due to the fact that for any $m \in m_{\mathcal{F}(P_1)}$, we have $\gcd(m_{F_{j-1,j,k}}, m) \mid \gcd(m_{F_{j-1,j,k}}, m_{F_{j-1,j,j+1}})$; therefore $\text{Res}(I_{m_{F_{j-1,j,k}}})$ is a principal ideal generated by $\frac{\gcd(m_{F_{j-1,j,k}}, m_{F_{j-1,j,j+1}})}{\gcd(m_{F_{j-1,j,k}}, m_{F_{j-1,j,j+1}})} = x_k$. Similarly, we apply the same order to all possible paths starting from a vertex in $P_1$ and ending at some vertex of degree 1. Hence the facet ideal will have the linear residuals under the following ordering of generators

$$I_\mathcal{F}(\Delta_T(T)) = (m_{\mathcal{F}(P_1)}, m_{\mathcal{F}(P_2)}, \ldots, m_{\mathcal{F}(P_i)}).$$
This ordering of generating set will yield linear residual regardless to the ordering and labeling of $P_i$’s for all $i \geq 2$. Hence, the result follows from 3.3.
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