HOMOLOGICAL PROPERTIES OF PINCHED VERONESE RINGS

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Abstract. Pinched Veronese rings are formed by removing an algebra generator from a Veronese subring of a polynomial ring. We study the homological properties of such rings, including the Cohen-Macaulay, Gorenstein, and complete intersection properties. Greco and Martino classified Cohen-Macaulayness of pinched Veronese rings by the maximum entry of the exponent vector of the pinched monomial; we re-prove their results with semigroup methods and correct an omission of a small class of examples of Cohen-Macaulay pinched Veronese rings. When the underlying field is of prime characteristic, we show that pinched Veronese rings exhibit a variety of F-singularities, including F-regular, F-injective, and F-nilpotent. We also compute upper bounds on the Frobenius test exponents of pinched Veronese rings, a computational invariant which controls the Frobenius closure of all parameter ideals simultaneously.

1. Introduction

A pinched Veronese ring is formed by removing any one of the algebra generators of a Veronese subring of a polynomial ring over a field. Let \( k \) be a field and \( \mathbf{m} = (m_1, \ldots, m_n) \) be an exponent vector in \( \mathbb{N}^n \) with \( m_1 + \cdots + m_n = d \), we denote by \( \mathcal{P}_{n,d,m} \) the pinched Veronese ring in \( n \) variables formed by removing the monomial generator \( x_1^{m_1} \cdots x_n^{m_n} \) from the degree \( d \) Veronese subring of \( k[x_1, \ldots, x_n] \). For a vector \( \mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n \), we let \( \max(\mathbf{m}) = m_j \) if \( m_j \geq m_i \) for all \( 1 \leq i \leq n \).

These affine semigroup rings are rarely normal and provide examples of rings for which certain homological properties are difficult to prove. For instance, consider the pinched Veronese ring:

\[
R = k[x^3, y^3, z^3, x^2y, xy^2, x^2z, xz^2, y^2z, yz^2]
\]

formed by removing the generator \( xyz \) of the third Veronese subring of \( k[x, y, z] \), i.e. \( R = \mathcal{P}_{3,3,(1,1,1)} \). It was not known whether \( R \) was Koszul; the question was raised by Sturmfels in 1993 and it was settled in the affirmative in 2009 by Caviglia in [Cav09], who used the techniques of Gröbner bases and Koszul filtrations (See also [CC13]).

The pinched Veronese ring \( \mathcal{P}_{2,4,(2,2)} = k[x^4, x^3y, xy^2, y^4] \) is well known to not be Cohen-Macaulay (see [Mac94]). Both of the examples above raise two important questions which we attempt to address in this paper – first, which pinched Veronese rings are Cohen-Macaulay, if any? Furthermore, can we understand the singularity types of these rings?

In Section 2 we discuss the normalizations of pinched Veronese rings and develop combinatorial tools which compare the affine semigroups defining pinched Veronese rings to those defining the corresponding Veronese rings. These combinatorial results serve as the technical heart of the paper and help to illuminate the homological properties of pinched Veronese rings in the later sections. Section 2 also includes a brief discussion of the prime characteristic singularity types which arise for pinched Veronese rings. In particular, Lemma 2.14 outlines when certain subrings of \( F \)-rational rings are \( F \)-nilpotent.
In Section 3, we use local cohomology and the combinatorial tools from Section 2 to study the Cohen-Macaulay and Gorenstein properties of pinched Veronese rings. Indeed, we show that the above homological properties of these rings are sensitive to the algebra generator that is pinched out.

**Theorem A.** The pinched Veronese ring $\mathcal{P}_{n,d,m}$ is Cohen-Macaulay if and only if one of the following three conditions hold.

- $\max(m) = d$.
- $n = 2$ and $\max(m) = d - 1$.
- $n = 3$, $d = 2$, and $\max(m) = 1$.

Further, when $\max(m) = d - 1$, $\mathcal{P}_{2,d,m}$ is a Gorenstein ring with a-invariant zero, and when $\max(m) = 1$, $\mathcal{P}_{3,2,m}$ is a complete intersection ring.

This theorem re-proves the result [GM17, Theorem B] of Greco and Martino by using purely semigroup techniques and corrects an omission of a class of Cohen-Macaulay rings. In their original proof, Greco-Martino calculated the Betti numbers of pinched Veronese rings by means of the reduced homology of squarefree divisor complexes.

The Cohen-Macaulay property of affine semigroup rings has been an area of active research since the 1970s. In [Hoc72], Hochster proved the following famous result.

**Theorem** (Hochster, [Hoc72]). If $Q$ is a normal semigroup, then the affine semigroup ring $k[Q]$ is Cohen-Macaulay.

In [TH86], Trung and Hoa describe the Cohen-Macaulay property of an affine semigroup ring in terms of combinatorial and topological properties of the convex rational polyhedral cone spanned by the affine semigroup.

In Section 4, we show that when the characteristic of the underlying field is positive, pinched Veronese rings exhibit a variety of $F$-singularities. We show that a large class of pinched Veronese rings are $F$-nilpotent. Interestingly, a certain class of these rings are either $F$-nilpotent or $F$-injective depending on the characteristic of the field.

**Theorem B.** Let $k$ be a field of characteristic $p > 0$. The $F$-singularity type of the pinched Veronese ring $\mathcal{P}_{n,d,m}$ is as follows.

- $\mathcal{P}_{n,d,m}$ is $F$-regular for $\max(m) = d$.
- When $d > 2$, $\mathcal{P}_{n,d,m}$ is $F$-nilpotent for $\max(m) < d$.
- $\mathcal{P}_{n,2,m}$ is $F$-nilpotent if $p = 2$ and $F$-injective if $p > 2$. Further, $\mathcal{P}_{3,2,m}$ is $F$-pure if $\max(m) = 1$ and $p > 2$.

We also compute upper bounds on the Hartshorne-Speiser-Lyubeznik numbers and the Frobenius test exponents of pinched Veronese rings.

In Section 5, we prove similar results for affine semigroup rings obtained by removing larger subsets of the Veronese generators. In particular, we consider removing any subset of algebra generators $x_1^{m_1} \cdots x_n^{m_n}$ with $\max(m) < d - 1$ from the Veronese subring and show that this situation is remarkably similar to the $\max(m) < d - 1$ case for single pinches. Finally, we conclude the paper with several open questions.

### 2. Preliminaries

Throughout, fix a field $k$ and let $n > 1$ and $d > 1$ be natural numbers. We let $S$ denote the standard graded polynomial ring $k[x_1, \ldots, x_n]$. For a vector $m = (m_1, \ldots, m_n) \in \mathbb{N}^n$, we...
we let \( |m| = \sum_{i=1}^{n} m_i \) and \( \max(m) = m_j \) if \( m_j \geq m_i \) for all \( 1 \leq i \leq n \). We say \( m \) is in **descending order** if \( m_1 \geq m_2 \geq \ldots \geq m_n \).

For a semigroup \( A \subset \mathbb{N}^n \), we have the **affine semigroup ring** \( k[A] = k[x_1^{a_1}, \ldots, x_n^{a_n} | (a_1, \ldots, a_n) \in A] \). One well-understood example is given by the semigroup \( A_{n,d} \subset \mathbb{N}^n \) generated by the set \( \{ a \in \mathbb{N}^n | |a| = d \} \) and the corresponding affine semigroup ring \( V_{n,d} = k[A_{n,d}] \), also known as the \( d \)th Veronese subring of the polynomial ring \( S \). Since \( V_{n,d} \) is a direct summand of \( S \), many important ring theoretic properties descend from \( k[x_1, \ldots, x_n] \) to \( V_{n,d} \). These include the Cohen-Macaulay property, unique factorization property, and \( F \)-regularity in positive characteristic among others.

Now fix any \( m \in A_{n,d} \) with \( |m| = d \). Our primary object of interest throughout this paper is the semigroup \( A_{n,d,m} \) generated by \( \{ a \in \mathbb{N}^n | |a| = d \) and \( a \neq m \} \) and the corresponding affine semigroup ring \( P_{n,d,m} = k[A_{n,d,m}] \), also known as the **pinched Veronese ring**.

**Lemma 2.1.** The field of fractions of \( P_{2,2,(1,1)} \) is \( k(x^2, y^2) \). Otherwise, the field of fractions of \( P_{n,d,m} \) is the same as that of \( V_{n,d} \):

\[
\text{Frac}(P_{n,d,m}) = \text{Frac}(V_{n,d}) = k(x_2/x_n, x_3/x_n, \ldots, x_{n-1}/x_n, x_n^d).
\]

**Proof.** The first statement is clear since \( P_{2,2,(1,1)} \) is a regular ring. Now assume without loss of generality that \( m \) is arranged in descending order, and we observe that \( x_i/x_1 \in \text{Frac}(P_{n,d,m}) \) for each \( 1 \leq i \leq n-1 \):

\[
\frac{x_i}{x_n} = \frac{x_i x_n^{d-1}}{x_n^d}
\]

where both the numerator and denominator are monomials in \( P_{n,d,m} \). Since \( P_{n,d,m} \) is an \( n \)-dimensional domain, and the claimed \( k \)-algebra generators of \( \text{Frac}(P_{n,d,m}) \) are clearly algebraically independent, the result follows. \( \square \)

**Remark 2.2.** It follows immediately from the above lemma that the extension of fields \( k \subset \text{Frac}(P_{n,d,m}) \) is **rational** for all choices of \( n, d, \) and \( m \).

**Theorem 2.3.** The pinched Veronese ring \( P_{2,2,(1,1)} \) is normal. The normalizations of the other pinched Veronese rings \( P_{n,d,m} \) are as follows.

\[
\overline{P_{n,d,m}} = \begin{cases} 
V_{n,d} & \text{if } \max(m) < d \\
\overline{P_{n,d,m}} & \text{if } \max(m) = d
\end{cases}
\]

**Proof.** The first statement is clear since \( P_{2,2,(1,1)} \) is a regular ring. We may assume, without loss of generality, that \( m \) is in descending order. Denote \( P = P_{n,d,m} \) and \( V = V_{n,d} \). Begin with the case that \( \max(m) < d \).

Note that if \( f = x_1^{m_1} \cdots x_n^{m_n} \in V \) then \( f^d = (x_1^d)^{m_1} \cdots (x_n^d)^{m_n} \in P \) (here we have used \( \max(m) < d \)). Thus \( f \) satisfies the monic polynomial \( T^d - f^d \in P[T] \). So, \( P \subseteq V \) is an integral extension.

Now we show that \( f \in \text{Frac}(P) \). By Lemma 2.1 we may write:

\[
f = \left( \frac{x_1}{x_n} \right)^{m_1} \left( \frac{x_2}{x_n} \right)^{m_2} \cdots \left( \frac{x_{n-1}}{x_n} \right)^{m_{n-1}} x_n^d \in \text{Frac}(P).
\]

This shows that \( V \) is contained in the normalization of \( P \). Since \( V \) is normal (as it is a direct summand of a polynomial ring), it follows that the normalization of \( P \) is \( V \).

To complete the proof, it suffices to show that \( P_{n,d,m} \) is a normal semigroup ring for \( m = (d, 0, \ldots, 0) \). Since the affine semigroup ring corresponding to a normal affine semigroup
is normal, we show that the semigroup $A_{n,d,m}$ is normal, i.e., we show that for a positive integer $m$, if $ma \in A_{n,d,m}$ then $a \in A_{n,d,m}$ by a contrapositive argument. So, assume that $a \notin A_{n,d,m}$. This happens precisely when $a = (td - q, q)$ where $q$ is a vector in $\mathbb{N}^{n-1}$ with $|q| = q \leq t - 1$. Then, $ma = (mtd - mq, mq)$. Notice that $ma \notin A_{n,d,m}$ since:

$$|mq| = mq \leq mt - m \leq mt - 1.$$

This finishes the proof (cf. [GM17, Theorem 3.1]).

**Remark 2.4.** It follows immediately from Hochster’s theorem and Theorem 2.3 that for $\max(m) = d$, $P_{n,d,m}$ is a Cohen-Macaulay ring.

### 2.1. An overview of prime characteristic singularities.

We are interested in exploring the singularities that pinched Veronese rings may have, and in the prime characteristic case we will see in Section 4 that nearly all pinched Veronese rings are $F$-nilpotent. First, we need to establish some prime characteristic background. We will assume that all prime characteristic rings $R$ mentioned in this section are $F$-finite, that is, the Frobenius endomorphism $F: R \to R$ by $r \mapsto r^p$ is finite as a map of rings.

Fix $(R, m)$ a local ring of prime characteristic $p > 0$. The map $F : R \to R$ has been a central object of study in the singularity theory of prime characteristic rings and beyond since Ernst Kunz’s famous theorem that $R$ is regular if and only if $F$ is a flat map ([Kim09, Theorem 2.1]). One of many consequences of flatness is that all ideals of a regular ring are Frobenius closed, as outlined below.

**Definition 2.5.** Let $R$ be a ring of prime characteristic $p > 0$ and dimension $d$, and let $I \subset R$. The $e$th Frobenius bracket power $I^{[p^e]}$ of $I$ is the ideal generated by the set of elements $\{x^{p^e} \mid x \in I\}$. The Frobenius closure $I^F$ of $I$ is the ideal:

$$I^F = \{x \in R \mid x^{p^e} \in I^{[p^e]} \text{ for some } e \in \mathbb{N}\}.$$  

The ideal $I$ is **Frobenius closed** if $I^F = I$.

Another important ideal closure operation in prime characteristic defined similarly is tight closure $I^*$ of an ideal $I$, but we will not have need for it in this paper, and the definition is similar to that of the tight closure of the zero submodule given below. We refer the reader to [HH93] for the details of tight closure.

Two classical $F$-singularities are defined in terms of these ideal closure operations.

**Definition 2.6.** Let $R$ be a local ring of prime characteristic $p > 0$. Then, if $R$ is a $\mathbb{N}$-graded local ring, $R$ is **$F$-regular** if all ideals of $R$ are tightly closed. If $R$ is reduced and excellent, $R$ is **$F$-pure** if all ideals of $R$ are Frobenius closed.

The notions of $F$-regularity and $F$-purity are defined outside of the cases given above, but many technicalities ensue. Since all of our prime characteristic rings will be standard graded $F$-finite domains, the definitions above suffice for us. For different notions of $F$-regularity and their equivalence in the graded case, we refer the reader to Lyubeznik-Smith [LS99]. For different notions of $F$-purity, we refer the reader to Hochster [Hoc77].

Other commonly studied $F$-singularity types are defined in terms of the Frobenius action on local cohomology. In particular, the Frobenius map $F : R \to R$ induces a map $F : H^j_I(R) \to H^j_I(R)$ for any $j \in \mathbb{N}$, and $I \subset R$. It is important to note that this map is not $R$-linear, but $p$-linear, in that $F(x \xi) = x^p F(\xi)$ for any $x \in R$, $\xi \in H^j_I(R)$. We will focus on the case $I = m$ when $(R, m)$ is local.
**Definition 2.7.** Let \((R, \mathfrak{m})\) be a local domain of prime characteristic \(p > 0\) and dimension \(d\). The **tight closure** of 0 in \(H^d_m(R)\) is the \(R\)-submodule:

\[
0^*_{H^d_m(R)} = \{ \xi \in H^d_m(R) \mid \text{there is a } 0 \neq c \in R \text{ with } cF^e(\xi) = 0 \text{ for all } e \gg 0 \}.
\]

A Cohen-Macaulay ring \(R\) is **\(F\)-rational** if \(0^*_{H^d_m(R)} = 0\).

Tight closure in this sense can be defined outside of the domain case but we will avoid the technicalities here. There are also other equivalent notions of \(F\)-rationality which make use of tight closure of parameter ideals. In particular, the two papers of Smith ([Smi94] and [Smi97]) help outline the equivalence of these definitions.

**Remark 2.8.** We note that \(F\)-rational rings are normal Cohen-Macaulay domains. Furthermore, if \(R\) is a direct summand of a polynomial ring, then \(R\) is \(F\)-regular, which implies it is \(F\)-rational. We refer the reader to Hochster and Huneke’s seminal paper on tight closure ([HH94]) for an overview of these singularity types.

Weaker than \(F\)-rationality is \(F\)-injectivity. In particular a local ring \((R, \mathfrak{m})\) of prime characteristic \(p > 0\) is **\(F\)-injective** if the Frobenius action \(F : H^d_m(R) \to H^d_m(R)\) is injective for all \(j\). Of recent interest have been singularity types opposite to \(F\)-rationality. Weaker than \(F\)-nilpotence was first given in Srinivas-Takagi ([ST17]) and an interesting ideal-theoretic characterization was given in Polstra-Quy ([PQ18]), which we will explore in Section 4.

**Definition 2.9.** Let \((R, \mathfrak{m})\) be a local ring of prime characteristic \(p > 0\) and dimension \(d\). If for each \(0 \leq j < d\) we have that \(F^e : H^j_m(R) \to H^j_m(R)\) is the zero map some \(e \gg 0\), then \(R\) is **weakly \(F\)-nilpotent**. If \(R\) is weakly \(F\)-nilpotent and additionally \(F^e(0^*_{H^d_m(R)}) = 0\) for some \(e \gg 0\), then \(R\) is **\(F\)-nilpotent**.

**Remark 2.10.** Notice that a local ring \((R, \mathfrak{m})\) is \(F\)-rational if and only if it is both \(F\)-nilpotent and \(F\)-injective.

The definition of \(F\)-nilpotence was first given in Srinivas-Takagi ([ST17]) and an interesting ideal-theoretic characterization was given in Polstra-Quy ([PQ18]), which we will explore in Section 4.

**Remark 2.11.** If \(R\) is a graded ring, then \(F\) is not a homogeneous map, but does multiply the degree of a homogeneous element by \(p\). Consequently, if \(\mathfrak{m}\) is the homogeneous maximal ideal of \(R\), the induced map \(F : H^j_m(R) \to H^j_m(R)\) also multiplies the degree of a homogeneous class by \(p\).

**Definition 2.12.** Given a graded \(R\)-module \(M\), a **graded Frobenius action** on \(M\) is a \(p\)-linear map \(f : M \to M\) so that \(f([M]^n) \subset [M]_{np}\). We say \(M\) is **nilpotent** if \(f^e = 0\) for some \(e \in \mathbb{N}\).

A graded Frobenius action \(f : M \to M\) on an \(R\)-module \(M\) induces another graded Frobenius action, which we also call \(f\), on the graded local cohomology \(H^j_I(M)\) for any homogeneous ideal \(I \subset R\), and if \(f : M \to M\) is nilpotent, so is \(f : H^j_I(M) \to H^j_I(M)\) for all \(j\).

**Remark 2.13.** Let \(R\) be a graded ring of prime characteristic \(p > 0\) and suppose \(A, B, C\) are graded \(R\)-modules which fit into a commutative diagram as below:

\[
\begin{array}{ccc}
0 & \longrightarrow & A \overset{\alpha}{\longrightarrow} B \overset{\beta}{\longrightarrow} C \longrightarrow 0 \\
\downarrow f_A & & \downarrow f_B \\
0 & \longrightarrow & A \overset{\alpha}{\longrightarrow} B \overset{\beta}{\longrightarrow} C \longrightarrow 0
\end{array}
\]

\[
\begin{array}{ccc}
0 & \longrightarrow & A \overset{\alpha}{\longrightarrow} B \overset{\beta}{\longrightarrow} C \longrightarrow 0
\end{array}
\]
such that the rows are exact and the vertical maps are graded Frobenius actions. Then, for any homogeneous ideal \( I \subset R \), we also get a commutative diagram of local cohomology modules:

\[
\begin{array}{cccccc}
\cdots & \rightarrow & H^j_1(A) & \rightarrow & H^j_1(B) & \rightarrow & H^j_1(C) & \rightarrow & \cdots \\
& \downarrow f_A & & \downarrow f_B & & \downarrow f_C & & \\
\cdots & \rightarrow & H^j_1(A) & \rightarrow & H^j_1(B) & \rightarrow & H^j_1(C) & \rightarrow & \cdots
\end{array}
\]

where the rows are the induced long exact sequences of each row and the vertical maps are the induced Frobenius actions in each place. We refer the reader to [MM21, Remark 2.2] for further details.

We now conclude this subsection with a lemma of independent interest which we will apply to the normalization map \( P_{n,d,m} \rightarrow V_{n,d} \).

**Lemma 2.14.** Let \( (R, m) \rightarrow (S, n) \) be an inclusion of local rings (or graded rings with homogeneous maximal ideals \( m \) and \( n \) respectively) of prime characteristic \( p > 0 \), and let \( C \) be the cokernel of the map. Suppose that \( S \) is \( F \)-rational and the induced Frobenius action \( F : C \rightarrow C \) is nilpotent. Then \( R \) is \( F \)-nilpotent.

**Proof.** Since \( S \) is \( F \)-rational, \( S \) is a domain, which implies \( R \) is a domain. Further, notice that for some \( e \in \mathbb{N} \) we have \( F^e : C \rightarrow C \) is the zero map implies for all \( s \in S, s^{p^e} \in R \). We now also note that \( \sqrt{mS} = n \), as if \( s \in n, s^{p^e} \in R \), and if \( s^{p^e} \notin m \subset mS \), then \( s^{p^e} \) is a unit of \( R \), and consequently a unit of \( S \), contradicting that \( s \in n \). This also implies that \( R \) and \( S \) are both of the same dimension \( n \).

As in Remark 2.13, associated to the short exact sequence:

\[
0 \rightarrow R \rightarrow S \rightarrow C \rightarrow 0
\]

we obtain the long exact sequence:

\[
\cdots \rightarrow H^{j-1}_m(S) \rightarrow H^j_m(R) \rightarrow H^j_m(S) \rightarrow H^j_m(C) \rightarrow \cdots
\]

in which we know \( H^j_m(S) = 0 \) for \( j < n \) as \( S \) is Cohen-Macaulay. Consequently, \( H^{j-1}_m(C) \cong H^j_m(R) \) for all \( j < n \), and as \( F^e : H^{j-1}_m(C) \rightarrow H^{j-1}_m(C) \) is the zero map, we also have \( F^e : H^j_m(R) \rightarrow H^j_m(R) \) is the zero map for \( j < n \). Thus, \( R \) is weakly \( F \)-nilpotent.

To show that \( R \) is \( F \)-nilpotent, let \( \xi \in 0^*_{H^n_m(R)} \) so that there is a nonzero \( c \in R \) with \( cF^e(\xi) = 0 \) for all \( e \gg 0 \). When \( j = n \) in the long exact sequence above, we get an exact sequence:

\[
H^{n-1}_m(C) \xrightarrow{\delta} H^n_m(R) \xrightarrow{\alpha} H^n_m(S)
\]

and for all \( e \gg 0 \) we have \( \alpha(cF^e(\xi)) = cF^e(\alpha(\xi)) = 0 \) by Remark 2.13. We also have \( c \) is nonzero in \( S \), so \( \alpha(\xi) \in 0^*_{H^n_m(S)} \), which implies \( \alpha(\xi) = 0 \) as \( S \) is \( F \)-rational.

Thus \( \xi \in \text{im}(\delta) \) by exactness, and we have \( \xi = \delta(\xi') \) for some \( \xi' \in H^{n-1}_m(C) \). Since \( C \) is nilpotent, we have \( F^e(\xi') = F^e(\delta(\xi')) = \delta(F^e(\xi')) = 0 \), and thus, \( 0^*_{H^n_m(R)} \) is nilpotent. So, \( R \) is \( F \)-nilpotent. \( \square \)
2.2. Key Combinatorial Results. Our primary tool in the later sections is to control the cokernel $C$ of the natural inclusion of $P_{n,d,m}$ into its normalization $V_{n,d}$. Since this map corresponds to the natural ring homomorphism induced by the semigroup inclusion $A_{n,d,m} \subset A_{n,d}$, we can understand the cokernel by computing the set difference $A_{n,d} \setminus A_{n,d,m}$.

Throughout this subsection, we let $T = T_{n,d} = \{ e \in \mathbb{N}^n \mid \|e\| = d \}$ be the generating set for $A_{n,d}$ and for any fixed $m \in T$, we let $T_m = T \setminus \{ m \}$ be the generating set for $A_{n,d,m}$. We will often re-write elements of $A_{n,d}$ in terms of elements of $A_{n,d,m}$ and for brevity the following definition will be useful.

**Definition 2.15.** Let $a \in \mathbb{Z}^n$. The $(i,j)$-perturbation of $a = (a_1, \ldots, a_i, \ldots, a_j, \ldots, a_n)$ is the vector:

$$a' = (a_1, \ldots, a_i + 1, \ldots, a_j - 1, \ldots, a_n).$$

We will refer to $b$ as a perturbation of $a$ if $b$ is the $(i,j)$-perturbation of $a$ for some $1 \leq i \leq n$ and $1 \leq j \leq n$.

**Remark 2.16.** Notably, if $b$ is a perturbation of $a$, $|b| = |a|$, and in particular, $(i,j)$-perturbations of vectors in $A_{n,d}$ are still in $A_{n,d}$ as long as $j \neq 0$. Furthermore, if $\text{max}(m) < d$, then there is always a pair $(i,j)$ so that the $(i,j)$-perturbation and the $(j,i)$-perturbation of $m$ is in $A_{n,d,m}$.

We also remark that when trying to re-write an element $e$ of $A_{n,d}$ as a sum of vectors in $A_{n,d,m}$, if $e \neq m + f$ for any $f \in A_{n,d}$ then $e$ can be written purely as a sum of vectors in $T_m$ and is thus in $A_{n,d,m}$. We will implicitly use this fact in the proofs later in this subsection.

First, we consider the case $\text{max}(m) < d - 1$ (for which we require $d > 2$), where show that only point of $A_{n,d}$ missing from $A_{n,d,m}$ is $m$.

**Lemma 2.17.** Suppose $m \in T_{n,d}$ has $\text{max}(m) < d - 1$. Then $A_{n,d} \setminus A_{n,d,m} = \{ m \}$.

**Proof.** Without loss of generality, we assume $m$ is in descending order. Let $f \in A_{n,d}$ with $f = m + e$ for some $e \in A_{n,d}$. First, we show if $\|m + e\| = 2d$, then $m + e \in A = A_{n,d,m}$. We have that either $\text{max}(e) = d$ or $e$ has at least two nonzero entries.

In the former case, assume $e_1 = d$. Then, letting $m' = (1,2)$-perturbation of $m$ and $e'$ the $(2,1)$-perturbation of $e$, we get $m + e = m' + e'$. If $e_i = d$ for some $i > 1$, we can re-write $m + e = m' + e'$ similarly using the $(i,1)$-perturbation $m'$ of $m$ and the $(1,i)$-perturbation $e'$ of $e$. Since $\text{max}(m) < d - 1$ and $m$ is in descending order, all of the perturbations above are in $A$, we have shown $m + e \in A$ in any of the cases.

If $e$ has at least two nonzero entries, say $e_i \neq 0$ and $e_j \neq 0$ with $i < j$, then we can re-write similarly. First, let $m'$ be the $(i,1)$-perturbation of $m$ and $e'$ be the $(1,i)$-perturbation of $e$, and then $m + e = m' + e'$ and $e'$ is in $A$ unless $e' = m$. In this case, $e_i = m_i + 1$ and $e_j = m_j$, and then instead of using $e'$, we can re-write $m + e = m' + e''$ where $m''$ is the $(j,1)$-perturbation of $m$ and $e''$ is the $(1,j)$-perturbation of $e$. This shows that $A$ has every vector in the hyperplane $x_1 + x_2 + \cdots + x_n = 2d$ of $\mathbb{N}^n$.

Finally, if $e \in A_{n,d}$ with $|e| = td$ for some $t > 2$ and $t$ is even, we can write $e$ as a sum of vectors in the $\sum x_i = 2d$ hyperplane, which shows $e \in A$. If $t$ is odd, $e = f + e'$, where
|e'| = (t - 1)d and |f| = d. Since t - 1 is even, we can write e' = a + b, where a, b ∈ A and |b| = (t - 2)d. Then e = f + a + b, and f + a ∈ A since |f + a| = 2d, thus e ∈ A also. 

Now we consider the max(m) = d - 1 case, in which we show that the behavior depends on whether d = 2 or d > 2.

Lemma 2.18. Suppose m ∈ Tm,n has max(m) = d - 1, and without loss of generality we may assume m = (d - 1, 1, 0, . . . , 0). Then, An,d \ A_{n,d,m} depends on d.

\[ A_{n,d} \setminus A_{n,d,m} = \begin{cases} \{(2s + 1, 2t + 1, 0, . . . , 0) | s, t \geq 0\} & \text{if } d = 2 \\ \{(ds - 1, 1, 0, . . . , 0) | s \geq 1\} & \text{if } d > 2 \end{cases} \]

Proof. First, suppose n = 2. If d = 2, then Tm = \{(2, 0), (0, 2)\} thus \(A_{2,2,(1,1)}\) consists of exactly the ordered pairs (i, j) so that i and j are even, showing the result in this case.

If d > 2, then no vector (d - j, j) of Tm has j = 1. Consequently, all the vectors (i, j) in \(A_{2,d,m}\) have j ≠ 1, and so \(\{(ds - 1, 1) | s \geq 1\} \subset A_{2,d} \setminus A_{2,d,m}\). We now wish to show that if j ≠ 1, then e = (i, j) is in \(A_{2,d,m}\). To see this, let |e| = td with t > 1, and reduce (i, j) mod d; if i = ad + u and j = bd + v with 0 ≤ u, v ≤ d and v ≠ 1, we can re-write e = a(d, 0) + b(0, d) + (u, v), so e ∈ \(A_{2,d,m}\). If (u, v) = (d - 1, 1), notice we must have b > 0. Then, write e = (ad + 1, bd - 1) + (d - 2, 2), and the previous case applies to (ad + 1, bd - 1) since bd - 1 ≠ 1 mod d when d > 2.

This demonstrates all possible behavior in the plane. If n > 2, the same arguments above work in the x1x2-plane to show that \(A_{n,d} \setminus A_{n,d,m}\) contains the given sets, and we will show that no other vectors of \(A_{n,d}\) are missing from \(A_{n,d,m}\).

If d = 2, let e = (e1, . . . , en) ∈ \(A_{n,2}\) with e ≠ (2s + 1, 2t + 1, 0, . . . , 0) for any s, t ∈ N. We may assume e has e1, e2, and ei nonzero for some i > 2. If both of e1 and e2 are even, the result is trivial. If e1 is odd but e2 is even, then we can re-write e using the (1, i)-perturbation to apply the previous case, and similarly if e2 is odd but e1 is even. So now we suppose e1 and e2 are odd, and if ei > 1 then we can use both the (1, i)- and (2, i)-perturbations to re-write e as a sum of vectors in \(A_{n,2,m}\). If e1 = 1, then since e1, e2, and ei are all odd we must have a fourth nonzero entry ej, and we can use both (1, i)- and (2, j)-perturbations to end up in previous cases, completing the proof when d = 2.

If d > 2, let e = (e1, . . . , en) ∈ \(A_{n,d}\) with e ≠ (ds - 1, 1, 0, . . . , 0) for any s ∈ N. Suppose |e| = td, and we will induce on t. When t = 1, e ∈ Tm and there is nothing to show. If t = 2 and ei = 0 for i > 2, e lies on the x1x2-plane of \(\mathbb{N}^n\) and we can rely on the n = 2 case to show e ∈ \(A_{n,d,m}\). If ei ≠ 0 for some i > 2, consider the possible values of e1. If e1 = d - 1, then we can re-write e as:

\[(d - 1, e2, . . . , ei, . . . , en) = (d - 1, 0, . . . , 1, . . . , 0) + (0, e2, . . . , ei - 1, . . . , en)\]

and if e1 ≥ d then instead we can use:

\[(e1, e2, . . . , ei, . . . , en) = (d, 0, . . . , 0) + (e1 - d, e2, . . . , ei, . . . , en),\]

and in either case all of the vectors on the right hand side of the equations above are in Tm.

If t > 2, then write e = m + f, where f = (f1, . . . , fn) must have fi > 0 for some i > 1. If i = 2, then we can re-write e = m' + f', where m' is the (2, 1)-perturbation of m and f' is the (1, 2)-perturbation of f, unless f' = ((t - 1)d - 1, 1, 0, . . . , 0). In this case, we have f1 = (t - 1)d - 2 and f2 = 2, so instead we re-write e = m'' + f'' where m'' is the (1, 2)-perturbation of m and f'' is the (2, 1)-perturbation of f''. In either case, by
induction $f'$ (or $f''$, as necessary) is in $A_{n,d,m}$, and $m'$ (or $m''$) is in $T_m$. If $i > 2$, then we can similarly re-write using the $(i,2)$-perturbation of $m$ and the $(2,i)$-perturbation $f'$ of $f$, unless $f' = ((t-1)d - 1, 1, \ldots, 0)$ and instead we can use the $(1,2)$-perturbation of $m$ and the $(2,1)$-perturbation of $f$. In either of these cases the result lives in $A_{n,d,m}$. By induction, the proof is complete. \[\square\]

These combinatorial results allow us to explicitly describe the cokernel of $P_{n,d,m} \to V_{n,d}$ whenever $\max(m) < d$.

**Corollary 2.19.** If $\max(m) < d$, the cokernel $C$ of the inclusion of the pinched Veronese ring $P = P_{n,d,m}$ in its normalization $V = V_{n,d}$ is principally generated as a $P$-module by $x_1^{m_1} \cdots x_n^{m_n} + P$.

**Proof.** If $\max(m) < d - 1$, certainly $y = x_1^{m_1} \cdots x_n^{m_n}$ is in $V \setminus P$ so forms a nonzero element in $C$. However, by Lemma 2.17 we evidently have any nonzero element of $C$ is simply $\lambda y + P$ for any $\lambda \in k$, which shows $C$ is the $P$-module generated by $y + P$.

If $\max(m) = d - 1$, assume without loss of generality that $m = (d - 1, 1, 0, \ldots, 0)$. If $d > 2$, by Lemma 2.18 any nonzero element of $C$ is a $k$-linear sum of monomials of the form $x_1^{d-1} x_2 + P$ for some $s \geq 1$. However, $x_1^{d-1} x_2 + P = (x_1^{d-1})^s (x_1^{d-1} x_2 + P)$, and so any element of $C$ can be written as $r(x_1^{d-1} x_2 + P)$ for some $r \in P$.

Similarly, if $\max(m) = 1$ and $d = 2$, then any element of $C$ is a $k$-linear sum of monomials of the form $x_1^{2s+1} x_2^{t+1} + P$ for some $s \in N$ and $t \in N$. However, $x_1^{2s+1} x_2^{t+1} + P = (x_1^{2s} x_2^t)(x_1 x_2 + P)$, and so any element of $C$ can be written as $r(x_1 x_2 + P)$ for some $r \in P$. \[\square\]

3. The Cohen-Macaulay and Gorenstein properties

Our aim in this section is to prove Theorem $A$.

**Theorem A.** The pinched Veronese ring $P_{n,d,m}$ is Cohen-Macaulay if and only if one of the following three conditions hold.

- $\max(m) = d$.
- $n = 2$ and $\max(m) = d - 1$.
- $n = 3$, $d = 2$, and $\max(m) = 1$.

Further, when $\max(m) = d - 1$, $P_{2,d,m}$ is a Gorenstein ring with $a$-invariant zero, and when $\max(m) = 1$, $P_{3,2,m}$ is a complete intersection ring.

We recover the result of Greco and Martino in [GM17, Theorem B] by using purely semigroup techniques and correct an omission of a class of Cohen-Macaulay rings. We prove Theorem $A$ by studying the possible depths of the pinched Veronese rings using the combinatorial results from Section 2.

**Lemma 3.1.** The depth of the pinched Veronese ring $P_{n,d,m}$ is as follows.

1. If $\max(m) = d$ then $\text{depth}(P_{n,d,m}) = n$.
2. If $n > 2$, $d = 2$, and $\max(m) = 1$, then $\text{depth}(P_{n,d,m}) = 3$.
3. If $\max(m) = d - 1$, then $\text{depth}(P_{n,d,m}) = 2$. Also, $\text{depth}(P_{2,2,m}) = 2$.
4. If $\max(m) < d - 1$ then $\text{depth}(P_{n,d,m}) = 1$.

**Proof.** By Theorem 2.3 when $\max(m) = d$, $P = P_{n,d,m}$ is normal. By Hochster’s theorem, normal affine semigroups are Cohen-Macaulay. Since $P$ is $n$ dimensional, it follows that the depth of $P$ is also $n$. This proves case 1.
Let $V = V_{n,d}$ denote the Veronese ring, which is the normalization of $P = \mathcal{P}_{n,d,m}$ by Theorem 2.3, and let $m$ denote the homogeneous maximal ideal of $P$. We observe that the inclusion of affine semigroups $\mathcal{A}_{n,d,m} \subset \mathcal{A}_{n,d}$ induces the short exact sequence of $P$-modules below.

$$
0 \longrightarrow P \longrightarrow V \longrightarrow C \longrightarrow 0
$$

Let $d$ denote the depth of $P$. Since $V$ is a Cohen-Macaulay ring, applying the local cohomology functor supported at $m$ to the above short exact sequence we get the inclusion $H_{m}^{d-1}(C) \subset H_{m}^{d}(P)$.

In case 2, the cokernel $C$ has depth 2 since $x_1^2$ and $x_2^2$ form a maximal regular sequence on $C$ by Lemma 2.18. Since the first nonvanishing of the local cohomology module $H_{m}^{i}(C)$ occurs at the depth of $C$ as a $P$-module, we get $H_{m}^{2}(C)$ is nonzero. Consequently, $H_{m}^{3}(P)$ is nonzero and the depth of $P$ is 3.

In case 3, for $n = 2$, both $x_1^2$ and $x_2^2$ are regular on $C$ by the first part of Lemma 2.18 so that $P$ has depth 2. For $n > 2$, since $x_1^d$ is a maximal regular sequence on $C$ by the second part of Lemma 2.18 so the depth of $P$ is again 2. Further, $\mathcal{P}_{2,2,(1,1)} \cong k[x^2, y^2]$ is regular and hence depth 2 as well.

In case 4, the cokernel $C$ is an Artinian $P$-module by Lemma 2.17 so $P$ has depth 1. □

The lemma above, together with the fact that $\dim(\mathcal{P}_{n,d,m}) = n$, settles the first part of Theorem A.

**Remark 3.2.** The above argument shows that when $\max(m) < d - 1$, all the lower local cohomology modules of $\mathcal{P}_{n,d,m}$ are finitely generated, and thus of finite length. Therefore $\mathcal{P}_{n,d,m}$ is a generalized Cohen-Macaulay ring when $\max(m) < d - 1$.

When $\max(m) = d$, $\mathcal{P}_{n,d,m}$ is Cohen-Macaulay. In dimension 2 we can be more explicit.

**Remark 3.3.** If $\max(m) = d$, then $\mathcal{P}_{2,d,m}$ is isomorphic to the Veronese ring $V_{2,d-1}$. We assume $m = (d,0)$ and let $P$ be the sub-semigroup of $\mathbb{Z}^2$ generated by $\{(d-1,1),(d-2,2),\ldots,(1,d-1),(0,d)\}$. If $\langle P \rangle$ is the subgroup of generated by $P$ in $\mathbb{Z}^2$, we see $\langle P \rangle = \langle (0,d),(1,-1) \rangle$, and these generators are $\mathbb{Z}$-linearly independent. Consequently $\langle P \rangle$ is a free abelian group of rank 2.

We define a lattice isomorphism $\langle P \rangle \cong \mathbb{Z}^2$ given by $(0,d) \mapsto (1,0)$ and $(1,-1) \mapsto (0,1)$. Under this transformation, the points $(d-1,1),(d-2,2),\ldots,(0,d)$ map to $(1,d-1),(1,d-2),\ldots,(1,0)$ respectively. We let $Q$ be the sub-semigroup of $\mathbb{Z}^2$ generated by these points.

We see that $Q$ is isomorphic as a semigroup to the normal semigroup $\mathcal{A}_{2,d-1}$ and the semigroup isomorphism $\mathcal{A}_{2,d,m} \cong \mathcal{A}_{2,d-1}$ induces a ring isomorphism $P \cong k[\mathcal{A}_{2,d-1}] = V_{2,d-1}$.

We now study the Gorenstein property of pinched Veronese rings. Recall for a graded ring $R$ of dimension $d$ with homogeneous maximal ideal $m$, Goto and Watanabe in [GW78] define the $a$-invariant of $R$ to be the highest integer $a(R) = a$ such that the grade piece $[H_{m}^{d}(R)]_a$ is nonzero.

**Remark 3.4.** If $\max(m) = d$, then $P = \mathcal{P}_{2,d,m}$ is Gorenstein if and only if $d = 2, 3$. This is because when $\max(m) = d$, then $P \cong V_{2,d-1}$ which is a Veronese subring of $S = k[x,y]$. $S$ is a regular (hence Gorenstein) ring with $a(S) = -2$, so by [GW78], Corollary 3.1.5 and Theorem 3.2.1, $V_{2,d-1} = S^{(d-1)}$ is Gorenstein if and only if $d - 1 \mid 2$. Thus $d = 2, 3$.\(^{10}\)
Theorem 3.5. Let $\max(\mathfrak{m}) = d - 1$. Then $\mathcal{P}_{2,d,\mathfrak{m}}$ is a Gorenstein ring of $a$-invariant zero. When $\max(\mathfrak{m}) = 1$, $\mathcal{P}_{3,2,\mathfrak{m}}$ is a complete intersection ring.

Proof. Without loss of generality, we may assume that $\mathfrak{m} = (d - 1, 1)$. By Lemma 3.1, $P = \mathcal{P}_{2,d,\mathfrak{m}}$ is Cohen-Macaulay. Thus, the system of parameters $x^d, y^d$ is a regular sequence in $P$. Consequently, it is enough to show that the zero-dimensional ring $R = P/(x^d, y^d)$ is Gorenstein. We claim that:

$$R = k \oplus kx^{d-2}y^2 \oplus \cdots \oplus kx^2y^{d-2} \oplus kxy^{d-1} \oplus kx^{d-1}y^{d+1}.$$ 

This is because for $i + j \neq d - 1$, $(x^iy^{d-i})(x^jy^{d-j})$ lies in the ideal $(x^d, y^d)$, as shown by the computation below.

$$(x^iy^{d-i})(x^jy^{d-j}) = \begin{cases} x^d(x^i+jy^{2d-(i+j)}) & \text{if } i + j \geq d \\ y^d(x^{i+j}y^{d-(i+j)}) & \text{if } i + j < d \end{cases}$$

Thus the socle of $R$ is singly generated by $x^{d-1}y^{d+1}$. Therefore $P$ is Gorenstein.

Further, notice that the (nonzero) element of maximal degree in the top local cohomology module of $P$ supported at its homogeneous maximal ideal is:

$$\eta = \left[ \frac{x^{d-1}y^{d+1}}{x^dy^d} \right].$$

Under the standard grading, the degree of $\eta$ is $(d - 1) + (d + 1) - 2d = 0$. So, the $a$-invariant of $P$ is zero.

Next, it suffices to observe that $P = \mathcal{P}_{3,2,(1,1,0)}$ is a complete intersection ring. Consider the surjection:

$$\varphi : R = k[a, b, c, d, e] \rightarrow P = k[x^2, xz, y^2, yz, z^2]$$

given by $a \mapsto x^2$, $b \mapsto xz$, $e \mapsto z^2$.

The ideal $I = (ae - b^2, ce - d^2)$ is clearly contained in the kernel of $\varphi$ and defines a prime ideal generated by a regular sequence of length two in the ring $R$. Since $R/I$ is a three dimensional domain, it follows that $\varphi$ is an isomorphism of rings so that $P$ is a complete intersection. \hfill \Box

4. Pinched Veronese Rings in Prime Characteristic

In this section, we will explore the $F$-singularities of pinched Veronese rings which are of prime characteristic $p > 0$. As an application, we will show that all pinched Veronese rings have finite Frobenius test exponents, which is an invariant that controls the Frobenius closure of all parameter ideals simultaneously. We again assume that the underlying field $k$ is $F$-finite, which by a theorem of Kunz [Kun76, Theorem 2.5], implies any finite type $k$-algebra is excellent.

4.1. F-nilpotence of Pinched Veroneses. Our aim in this subsection is to prove Theorem B by cases on $\max(\mathfrak{m})$.

Theorem B. Let $k$ be a field of characteristic $p > 0$. The $F$-singularity type of the pinched Veronese ring $\mathcal{P}_{n,d,\mathfrak{m}}$ is as follows.

- $\mathcal{P}_{n,d,\mathfrak{m}}$ is $F$-regular for $\max(\mathfrak{m}) = d$.
- $\mathcal{P}_{n,d,\mathfrak{m}}$ is $F$-nilpotent for $d > 2$ and $\max(\mathfrak{m}) < d$. 

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• \( P_{n,2,m} \) is \( F \)-nilpotent if \( p = 2 \) and \( F \)-injective if \( p > 2 \). Further, \( P_{3,2,m} \) is \( F \)-pure if \( \text{max}(m) = 1 \) and \( p > 2 \).

**Remark 4.1.** If \( \text{max}(m) = d \), then \( P_{n,d,m} \) is a direct summand of a polynomial ring, and is hence \( F \)-regular, as noted before.

When \( \text{max}(m) < d \), we will use Lemma 2.14 to conclude that \( P_{n,d,m} \) is \( F \)-nilpotent.

**Theorem 4.2.** If \( d > 2 \) and \( \text{max}(m) < d \), then the pinched Veronese ring \( P_{n,d,m} \) is \( F \)-nilpotent.

**Proof.** Write \( P = P_{n,d,m} \). First, suppose \( \text{max}(m) = d - 1 \) and assume without loss of generality that \( m = (d - 1, 1, 0, \ldots, 0) \). If \( d = 2 \), then \( P \simeq k[x^2, y^2] \) is regular and is thus \( F \)-nilpotent. If \( d > 2 \), By Lemma 2.18, the cokernel \( C \) of \( P \to V_{n,d} \) is principally generated as a \( P \)-module by \( x_1^{d-1}x_2 + P \). Then, \( \overline{F}(x_1^{d-1}x_2 + P) = x_1^{d-p}x_2^p + P = P \) since the exponent on \( x_2 \) is not 1. Consequently, \( \overline{F} : C \to C \) is the zero map.

Now assume \( \text{max}(m) < d - 1 \). Then, by Lemma 2.17 we know the cokernel \( C \) of \( P \to V_{n,d} \) is concentrated in a single positive degree, and so \( \overline{F}(C) = 0 \) as well, since \( \overline{F} \) multiplies degrees by \( p \).

In either case, we can apply Lemma 2.14 to conclude the proof. \( \square \)

To handle the \( d = 2 \) case, we will need to utilize the Nagel-Schenzel isomorphism given in [NS94] for local cohomology modules. We re-state the isomorphism below in the context that we need – the original statement utilizes filter-regular sequences, which are a generalization of regular sequences.

**Theorem** (Nagel-Schenzel). Let \((R, m)\) be a local ring and let \(x_1, \ldots, x_t \in m\) be a regular sequence on a finitely-generated \(R\)-module \(M\), with \(I = (x_1, \ldots, x_t)\). Then, \(H^t_m(M) \simeq H^0_m(H^t_I(M))\).

We now show the \( F \)-singularity type in the \( d = 2 \) case depends on the parity of \( p \).

**Theorem 4.3.** Let \( k \) be a field of characteristic \( p > 0 \). The pinched Veronese ring \( P_{n,2,m} \) with \( \text{max}(m) = 1 \) is \( F \)-nilpotent for \( p = 2 \) and is \( F \)-injective for \( p > 2 \).

**Proof.** Assume without loss of generality that \( m = (1, 1, 0, \ldots, 0) \). We saw in Lemma 2.18 that \( C \) is principally generated as a \( P \)-module by \( x_1x_2 + P \). If \( p = 2 \), \( \overline{F}(x_1x_2 + P) = x_1^2x_2^2 + P = P \) since \( x_1^2 \) and \( x_2^2 \) are in \( P \). Hence, if \( p = 2 \), \( \overline{F} : C \to C \) is the zero map. We can now apply Lemma 2.14 to see that \( P \) is \( F \)-nilpotent.

If \( p \) is odd, however, then for any nonzero \( r \cdot x_1x_2 + P \in C \), we have \( \overline{F}(r \cdot x_1x_2 + P) = r^p \cdot x_1^px_2^p + P \), and since \( p \) is odd, \( x_1^px_2^p \not\in P \). Furthermore, we can see that \( x_1^2 \) and \( x_2^2 \) are regular elements on \( C \) but all other generators of the homogeneous maximal ideal \( m \) of \( P \) annihilate \( C \), so that if \( r \cdot x_1x_2 + P \neq P \) then \( r^{p^j} x_1^{2^j}x_2^p + P \neq P \) as well, as \( r \) can only be of the form \( \lambda x_1^{2^j}x_2^{2^j} \) for some \( \lambda \in k \) and \( i, j \in \mathbb{N} \). Consequently, \( \overline{F} : C \to C \) is injective as well.

Now we will now show that the induced Frobenius map \( \overline{F} \) on \( H^2_m(C) \) is also injective. Letting \( I = (x_1^2, x_2^2) \), by the Nagel-Schenzel isomorphism we have \( H^2_m(C) \simeq H^0_m(H^2_I(C)) \). We consider \( H^2_I(C) \) as the direct limit:

\[
\lim_{i \to \infty} \left( C/(x_1^{2i}, x_2^{2i})C \xrightarrow{x_1x_2^i} C/(x_1^{2i+2}, x_2^{2i+2})C \right).
\]
If \([z + (x_1^{2i}x_2^{2i})C]| is a class in the direct limit, then \(\overline{F}([z + (x_1^{2i}, x_2^{2i})C]) = [\overline{F}(z) + (x_1^{2ip}, x_2^{2ip})C]. \) As \(x_1^2\) and \(x_2^2\) form a regular sequence on \(C\), the direct limit system is injective, and consequently \(\overline{F}(z) + (x_1^{2ip}, x_2^{2ip})C = 0\) if and only if \(\overline{F}(z) \in (x_1^{2ip}, x_2^{2ip})C\). However, if \(z \not\in (x_1^{2i}, x_2^{2i})C\), then \(z\) is a \(k\)-linear sum of monomials of the form \(x_1^{2s+1}x_2^{2i+1} + P\) with \(s < i\) and \(t < i\). Then, \(\overline{F}(z)\) is a \(k\)-linear sum of monomials of the form \(x_1^{2ip+p}x_2^{2ip+p} + P\). We can see that this is not in \((x_1^{2ip}, x_2^{2ip})C\) as \(s < i\) implies \(2s + 1 < 2i\), so \(2sp + p < 2ip\), and similarly, \(2tp + p < 2ip\). Thus, \(\overline{F}\) is injective on \(\lim\) \(C/(x_1^{2i}, x_2^{2i})C = H^3_i(C)\), and so \(\overline{F}\) is injective on the submodule \((0 : H^3_3(C)m^\infty) \simeq H^3_m(C)\).

If \(n = 3\), to see that \(P\) is \(F\)-injective we need only see that the Frobenius action on \(H^3_m(P)\) is injective as \(P\) is Cohen-Macaulay. We have the short exact sequence:

\[
0 \longrightarrow H^3_m(C) \longrightarrow H^3_m(P) \longrightarrow H^3_m(V) \longrightarrow 0
\]

and the Frobenius actions on the outer two modules are injective, which implies the Frobenius action on \(H^3_m(P)\) is also. If \(n > 3\), we get that the local cohomology of \(P\) is completely described by the isomorphisms \(H^3_m(C) \simeq H^3_m(P)\) and \(H^n_m(P) \simeq H^n_m(V)\), and since \(\overline{F}\) on \(H^3_m(C)\) and \(F\) on \(H^3_m(V)\) are injective, \(P\) is \(F\)-injective. \(\square\)

**Corollary 4.4.** By Theorem [A] we have \(\mathcal{P}_{3,2,m}\) with \(\text{max}(m) = 1\) is Gorenstein, and the previous theorem shows that it is also \(F\)-injective in odd characteristic. Hence, by a result of Fedder ([Fed83 Lemma 3.3]), \(\mathcal{P}_{3,2,m}\) with \(\text{max}(m) = 1\) is \(F\)-pure in odd characteristic.

**Remark 4.5.** As noted in Section 2, \(F\)-rational rings are both \(F\)-nilpotent and \(F\)-injective. Interestingly, the class of pinched Veronese rings in the above theorem are never \(F\)-rational (since they are not normal), but may have either \(F\)-nilpotent or \(F\)-injective singularities depending on the characteristic of the field.

In [Pan21 Example 2.5], the second author calculates the cohomological dimension of the ideal defining \(\mathcal{P}_{(2,4,2,2)}\) over the integers by using the normalization map. We have now finished the proof of Theorem [B].

### 4.2. Frobenius Test Exponents for Pinched Veroneses

In this subsection, we explore the Frobenius test exponent for parameter ideals of pinched Veronese rings. This numerical invariant measures how far any parameter ideal is from being Frobenius closed in the following sense.

For any ideal \(I\), \(I^F\) is finitely generated, so there is an \(e \in \mathbb{N}\) so that \((I^F)^{[p^e]} = I^{[p^e]}\). If we restrict to the class of parameter ideals, there may possibly be a uniform exponent which trivializes the Frobenius closure of all parameter ideals simultaneously, which motivates the following definition.

**Definition 4.6.** Let \((R, \mathfrak{m})\) be a (graded) local ring of prime characteristic \(p > 0\) and let \(\mathfrak{q}\) be an ideal generated by a full (homogeneous) system of parameters of \(R\). Then, the **Frobenius test exponent** of \(\mathfrak{q}\) is the smallest \(e \in \mathbb{N}\) so that \((\mathfrak{q}^F)^{[p^e]} = \mathfrak{q}^{[p^e]}\). The **Frobenius test exponent** for \(R\) is:

\[
\text{Fte}(R) = \sup\{\text{Fte}(\mathfrak{q}) \mid \mathfrak{q} \text{ is a (homogeneous) parameter ideal of } R\}.
\]

In this sense, \(\text{Fte}(R)\) uniformly annihilates Frobenius closure relations for all parameter ideals. Frobenius test exponents have been shown to be finite in several important cases, typically related to nilpotence properties. See the introduction of [Mad19] for a historical survey.
Recently, Quy showed in [Quy19] that weakly $F$-nilpotent rings have finite Frobenius test exponent as well. We will utilize Quy's upper bound to bound Frobenius test exponents for pinched Veronese rings.

When $R$ is $F$-nilpotent, Polstra-Quy show in [PQ18, Theorem 5.11] that $q^e = q^{[e]}$ for all parameter ideals $q$ of $R$. Since nearly all pinched Veronese rings are $F$-nilpotent, we can also treat the Frobenius test exponent as a measure of how far parameter ideals in these rings are from being tightly closed, as letting $e = Fte(R)$, then $(q^e)^{[e]} = q^{[e]}$.

Upper bounds for Frobenius test exponents are typically given in terms of the Hartshorne-Speiser-Lyubeznik numbers of $R$.

**Definition 4.7.** Let $(R, m)$ be a graded local ring of prime characteristic $p > 0$ and let $0 \leq j \leq d = \dim(R)$. Write $0^F_{H^j_m(R)}$ for the set of elements of the graded local cohomology module $H^j_m(R)$ which are in the kernel of $F^e$ for some $e$. Then the **Hartshorne-Speiser-Lyubeznik number of** $H^j_m(R)$ is defined as:

$$HSL(H^j_m(R)) = \inf \left\{ e \in \mathbb{N} \mid F^e \left( 0^F_{H^j_m(R)} \right) = 0 \right\}.$$  

Furthermore, the **Hartshorne-Speiser-Lyubeznik number of** $R$ is defined as:

$$HSL(R) = \sup \{ HSL(H^j_m(R)) \mid 0 \leq j \leq d \}.$$  

Amazingly, the Hartshorne-Speiser-Lyubeznik numbers of $R$ must be finite since $H^j_m(R)$ is Artinian. Initial results about finiteness for these numbers is due to Hartshorne and Speiser, and hypotheses were removed by Lyubeznik and later Sharp. Their results are collected below.

**Theorem** (Hartshorne-Speiser [HS77], Lyubzenik [Lyu97], Sharp [Sha97]). Let $(R, m)$ be a (graded) local ring of prime characteristic $p > 0$. Then, $HSL(R) < \infty$.

Notice that $R$ is $F$-injective if and only if $HSL(R) = 0$. Since $V = V_{n,d}$ is $F$-regular, it is in particular $F$-injective, and so $HSL(V) = 0$. We can then find upper bounds on the Hartshorne-Speiser-Lyubeznik numbers of $P = P_{n,d,m}$ using the cokernel of $P \to V$.

**Theorem 4.8.** The pinched Veronese ring $P_{n,d,m}$ has $HSL(P) \leq 1$.

**Proof.** Let $P = P_{n,d,m}$, $V = V_{n,d}$, $m$ be the homogeneous maximal ideal of $P$, and let $C$ be the cokernel of $P \to V$. We now analyze the Hartshorne-Speiser-Lyubeznik numbers of $P$.

Three simple cases are when $\max(m) = d$, when $n = d = 2$ and $m = (1,1)$, and finally when $d = 2$, $\max(m) = 1$ and $p > 2$, since in these cases, $P$ is $F$-injective, so $HSL(P) = 0$.

In the remaining cases, we have seen that the Frobenius action $\overline{F} : C \to C$ is the zero map, which implies that the induced Frobenius action $\overline{F} : H^j_m(C) \to H^j_m(C)$ is the zero map as well. Notably, as observed in the proof of Lemma 3.1, in all cases $C$ is a Cohen-Macaulay $P$-module.

In the cases where $P$ itself is not Cohen-Macaulay, write $\text{depth}(P) = j$. We then get $H^{j-1}_m(C) \simeq H^j_m(P)$, and in particular, we also have $F : H^j_m(P) \to H^j_m(P)$ is also the zero map. This implies $HSL(H^j_m(P)) = 1$. The only other local cohomology module for $P$ is $H^n_m(P) \simeq H^n_m(V)$, and since $F : H^n_m(V) \to H^n_m(V)$ is injective, we have $HSL(H^n_m(P)) = 0$. This shows $HSL(P) = 1$ when $P$ is not Cohen-Macaulay.
When $P$ is Cohen-Macaulay but not $F$-injective, then we get the short exact sequence:

$$0 \longrightarrow H^{n-1}_m(C) \xrightarrow{\delta} H^n_m(P) \xrightarrow{\alpha} H^n_m(V) \longrightarrow 0$$

and for any $\xi \in H^n_m(P)$ such that $F^e(\xi) = 0$, we have $\alpha(F^e(\xi)) = F^e(\alpha(\xi)) = 0$. Since $F^e$ is injective on $H^n_m(V)$, we know $\alpha(\xi) = 0$ and $\xi = \delta(\xi')$ for some $\xi' \in H^{n-1}_m(C)$. Then, $F(\xi) = F(\delta(\xi')) = F(\xi') = 0$, which shows $\text{HSL}(H^n_m(P)) = 1$ and $\text{HSL}(P) = 1$.

Now we will utilize the primary result of [Quy19], where Quy shows that the Hartshorne-Speiser-Lyubeznik numbers bound the Frobenius test exponent.

**Theorem (Quy).** Let $(R, \mathfrak{m})$ be a local ring of prime characteristic $p > 0$ and dimension $n$. If $R$ is weakly $F$-nilpotent, then:

$$\text{Fte}(R) \leq \sum_{j=0}^{n} \binom{n}{j} \text{HSL}(H^i_m(R)).$$

Quy’s result translates to the graded local setting with minimal adjustment, so we can apply it here to $\mathcal{P}_{n,d,m}$.

**Corollary 4.9.** The pinched Veronese ring $P = \mathcal{P}_{n,d,m}$ has an upper bound on its Frobenius test exponents given below.

- In the cases that $\max(\mathfrak{m}) = d$; $n = d = 2$ and $\max(\mathfrak{m}) = 1$; or $n = 3$, $d = 2$, $\max(\mathfrak{m}) = 1$ and $p > 2$, then $\text{Fte}(P) = 0$, i.e., every parameter ideal of $P$ is Frobenius closed.
- If $d > 2$ and $\max(\mathfrak{m}) = d - 1$, then $\text{Fte}(P) \leq \binom{n}{2}$. In particular, when $n = 2$, $\text{Fte}(P) = 1$.
- If $d = 2$, $n \geq 3$, $\max(\mathfrak{m}) = 1$, and $p = 2$, then $\text{Fte}(P) \leq \binom{n}{3}$. In particular, when $n = 3$, $\text{Fte}(P) = 1$.
- If $\max(\mathfrak{m}) < d - 1$, then $\text{Fte}(P) \leq n$.

**Proof.** We handle each case in turn. In the first case, no matter the conditions on $n,d$, and $\mathfrak{m}$, we have that $P$ is Cohen-Macaulay and $F$-injective, so by Quy-Shimomoto [QS17 Corollary 3.9], $\text{Fte}(P) = 0$.

Now we let $\mathfrak{m}$ be the homogeneous maximal ideal of $P$. If $d > 2$ and $\max(\mathfrak{m}) = d - 1$, we can apply Quy’s upper bound to get:

$$\text{Fte}(P) \leq \sum_{j=0}^{n} \binom{n}{j} \text{HSL}(H^1_m(P)) = \binom{n}{2} \text{HSL}(H^2_m(P)) = \binom{n}{2}$$

and if $n = 2$, then $\text{Fte}(P) \leq 1$. But in this case, $P$ is Cohen-Macaulay, and Katzman-Sharp show that $\text{Fte}(P) = \text{HSL}(P)$ when $P$ is Cohen-Macaulay as mentioned above.

If $d = 2$, $\max(\mathfrak{m}) = 1$, and $p = 2$, the situation is very similar to the $\max(\mathfrak{m}) = d - 1$ case. When $n = 3$, $P$ is Cohen-Macaulay and $\text{Fte}(P) = \text{HSL}(P) = 1$, and when $n > 3$ then $\text{Fte}(P) \leq \binom{n}{3} \text{HSL}(H^3_m(P)) + \binom{n}{2} \text{HSL}(H^2_m(P)) = \binom{n}{3}$.

Finally, if $\max(\mathfrak{m}) < d - 1$, then Quy’s upper bound gives:

$$\text{Fte}(P) \leq \sum_{j=0}^{n} \binom{n}{j} \text{HSL}(H^j_m(P)) = \binom{n}{1} \text{HSL}(H^1_m(P)) = n,$$

as required. □
Remark 4.10. Notably missing from the list above is the case that \( n > 3, d = 2, \) \( \max(m) = 1, \) and \( p > 2. \) In this case, \( P = \mathcal{P}_{n,2,m} \) is \( F \)-injective but not \( F \)-nilpotent or Cohen-Macaulay, so none of the techniques used in Corollary 4.9 apply. The authors currently do not know if these examples have finite Frobenius test exponents.

5. Multi-pinched Veronese rings

Throughout this section, let \( T = T_{n,d} \) be as before with \( d > 2 \) and \( n \geq 2. \) Most of the results in this paper so far have focused semigroups generated by removing only a single vector of \( T. \) However, we show that even if we remove a larger subset of \( T, \) we can still control the set difference between \( A_{n,d} \) and the corresponding semigroup as long as we do not remove any vector \( e \) of \( T \) with \( \max(e) \geq d - 1. \) The authors are grateful to Mark Denker for his combinatorial insight and inspiration for the lemma and proof below.

Lemma 5.1. We let \( S \subset T \) be the set of vectors in \( \mathbb{N}^n \) below.

\[
S = \{ m \in T \mid \max(m) \geq d - 1 \}
\]

Let \( A \) be the semigroup generated by \( S. \) Then, \( A_{n,d} \setminus A \) is a finite set. In particular, if \( e \in A_{n,d} \) and \( \max(e) \geq (n - 1)(d^2 - d), \) then \( e \in A. \)

Proof. We will show that if \( t \geq (n - 1)(d^2 - d), \) then \( e = (t, e_2, \ldots, e_n) \) is in \( A, \) which will imply the result since the argument is symmetric in each coordinate. Using the division algorithm, write \( e_i = dp_i + q_i. \) We can re-write \( e = d(0, p_2, \ldots, p_n) + (t, q_2, \ldots, q_n) \) with \( d(0, p_2, \ldots, p_n) \in A. \) Then, for \( 2 \leq i \leq n, \) we let \( a_i = (a_1, \ldots, a_n) \) where \( a_1 = d - 1 \) and \( a_j = 1 \) if \( j = i \) or \( a_j = 0 \) otherwise, notably each \( a_i \in A. \) We can then then re-write \( (t, q_2, \ldots, q_n): \)

\[
(t, q_2, \ldots, q_n) = \left( t - \sum_{i=2}^{n} (d-1)q_i + \sum_{i=2}^{n} (d-1)q_i, q_2, \ldots, q_n \right)
\]

\[
= \left( t - \sum_{i=2}^{n} (d-1)q_i, 0, \ldots, 0 \right) + q_2a_2 + \cdots + q_n a_n
\]

Furthermore, \( t - \sum_{i=2}^{n} (d-1)q_i > 0 \) since \( t \) was assumed to be larger than \( (n - 1)(d^2 - d), \) and we may replace each \( q_i \) with the upper bound \( d. \)

Finally, a direct computation shows that \( t - \sum_{i=2}^{n} (d-1)q_i \) is a multiple of \( d. \) Consequently, \( (t - \sum_{i=2}^{n} (d-1)q_i, 0, \ldots, 0) \in A, \) and thus \( e \in A. \)

Remark 5.2. In fact, the proof above shows that if \( B \) is a semigroup generated by any set \( U \) with \( S \subset U \subset T, \) then \( A_{n,d} \setminus B \) is also a finite set.

We now fix \( d > 2, n \geq 2, \) and a semigroup \( B \) generated as in the remark above. The homological properties of the affine semigroup ring \( k[B] \) are similar to the \( \max(m) < d - 1 \) case for single pinches, as the cokernel \( k[B] \to V_{n,d} \) is still finite-dimensional over \( k \) and hence finite length as a \( k[B] \)-module.

Theorem 5.3. Let \( B \) be as in the paragraph above. Then, \( \text{depth}(k[B]) = 1 \) and so \( k[B] \) is not Cohen-Macaulay.
Proof. As in the $d > 2$, $\max(m) < d - 1$ case of Lemma 3.1, $A_{n,d} \setminus B$ is finite so the cokernel $C$ of $k[B] \to V_{n,d}$ is dimension zero as a $k[B]$-module. So, letting $m$ be the homogeneous maximal ideal of $k[B]$, we get $0 \neq H^0_m(C) = C \simeq H^1_m(k[B])$, which shows that $\text{depth}(k[B]) = 1$. Since $k[B]$ still contains the system of parameters $x_1^d, \ldots, x_n^d$ of $V_{n,d}$, we also know $\dim(k[B]) = n$ which shows that $k[B]$ is not Cohen-Macaulay. □

We now show these rings $k[B]$ are $F$-nilpotent.

**Theorem 5.4.** Let $B$ be as established above. Then, the affine semigroup ring $k[B]$ is $F$-nilpotent.

Proof. Write $R = k[B]$, $V = V_{n,d} = k[A_{n,d}]$, and $m$ and $n$ for the homogeneous maximal ideals of $R$ and $V$ respectively. Then, $\sqrt{mV} = n$ since $x_i^d \in R$ for each $i$. Furthermore, the cokernel of $R \to V$ is concentrated in finitely many strictly positive degrees since $A_{n,d} \setminus B \subset A_{n,d} \setminus A$, and is hence nilpotent under the induced Frobenius action. Thus we can apply Lemma 2.14 to conclude that $R$ is $F$-nilpotent, as required. □

We are now ready to compute an upper bound on the Frobenius test exponents for these multi-pinched Veronese rings.

**Theorem 5.5.** Let $B$ be a semigroup as defined in Theorem 5.4 with $B \subset A_{n,d}$ for $n > 2$. Then, $\text{Fte}(k[B]) \leq n\lceil \log_p((n - 1)(d^2 - d)) \rceil$.

Proof. Since $A_{n,d} \setminus B \subset A_{n,d} \setminus A$ is finite, we have the cokernel $C$ of $k[B] \to V_{n,d}$ is a finite-dimensional $k$-vector space and is thus dimension 0 as a $k[B]$-module. We can then completely analyze the local cohomology of $k[B]$; for simplicity we let $R = k[B]$, $V = V_{n,d}$, and $m$ and $n$ be the homogeneous maximal ideals of $R$ and $V$ respectively. Then, $\sqrt{mV} = n$, and we get that $H^0_m(C) = C \simeq H^1_m(R)$, $H^2_m(R) = H^j_m(V) = 0$ for $1 < j < n$, and $H^n_m(R) \simeq H^n_m(V)$.

Then, similarly to the proof of Theorem 4.8, $HSL(H^j_m(R)) = 0$ for $j \neq 1$ and $HSL(H^1_m(R))$ is the minimum $e \in \mathbb{N}$ so that $F^e(C) = 0$, which we can see is bounded above by $\lceil \log_p((n - 1)(d^2 - d)) \rceil$ by Lemma 5.1. Thus, in Quy’s upper bound for Frobenius test exponents, we get $\text{Fte}(R) \leq {n \choose 1} HSL(H^1_m(R)) \leq n \lceil \log_p((n - 1)(d^2 - d)) \rceil$. □

This provides a coarse upper bound independent of which particular semigroup $B$ is chosen. If $|m| = d$ and $\max(m) < d - 1$, then $A_{2,d,m}$ is a semigroup for which Theorem 5.3 applies, but we have shown that $\text{Fte}(\mathcal{P}_{2,d,m}) \leq 2$, which is much sharper than the bound of $2 \lceil \log_p(d^2 - d) \rceil$ provided above.

We conclude the paper with some further questions.

**Question 1.** We have shown in Theorem 5.3 that pinching any number of generators $x_1^{m_1} \cdots x_n^{m_n}$ of $V_{n,d}$ where $\max(m_1, \ldots, m_n) < d - 1$ provides a non Cohen-Macaulay affine semigroup ring. Can we determine all subsets $S$ of $T_{n,d}$ which generate semigroups $A$ such that $k[A]$ is Cohen-Macaulay?

**Question 2.** We have shown in Corollary 4.4 that the ring $\mathcal{P}_{3,2,(1,1,0)}$ is $F$-pure when $p > 2$, but it seems likely that $\mathcal{P}_{n,2,m}$ for $\max(m) = 1$ is also $F$-pure when $p > 2$. However, since $\mathcal{P}_{n,2,m}$ is not even Cohen-Macaulay for $n > 3$, the same technique we used to show $\mathcal{P}_{3,2,m}$ is $F$-pure will not apply. Can we show that $\mathcal{P}_{n,2,m}$ is always $F$-pure when $p > 2$ and $\max(m) = 1$?
**Question 3.** Hochster’s theorem establishes normality as a sufficient intrinsic condition on an affine semigroup $A$ so that $k[A]$ is Cohen-Macaulay, and in prime characteristic $p > 0$, normality further implies that $k[A]$ is $F$-regular, a restrictive $F$-singularity type.

In the $\max(m) < d$ case, the proofs we provide of $F$-nilpotence and $F$-injectivity depend on extrinsic qualities, i.e. the embedding $A_{n,d,m} \subset A_{n,d}$. Can we determine intrinsic conditions on an affine semigroup so its corresponding affine semigroup ring has a certain classes of $F$-singularity?

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