ENTROPY OF LYAPUNOV MAXIMIZING MEASURES OF
$GL(2, \mathbb{R})$ TYPICAL COCYCLES

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ABSTRACT. In this paper we study ergodic optimization problems for one-step cocycles. We consider one-step $GL(2, \mathbb{R})$-cocycles that satisfy pinching and twisting conditions (in the sense of [10]). We prove that the Lyapunov maximizing measures have zero entropy under the additional assumption of nonoverlapping between the cones on the Mather set, thus extending a result by Bochi and Rams [7].

1. Introduction

Our main subject is ergodic optimization. The large-scale image of ergodic optimization is that one is interested in optimizing potential functions over the (typically externally complex) class of invariant measures for a dynamical system; see [19, 3]. The field also has both local aspects in which the optimization is studied on individual functions and has general aspects in which the optimization is considered in the large on whole Banach spaces. There has been input from physicists with numerical simulations suggesting that the optimizing measures are typically supported on periodic orbits, which is the main question of this subject. That is converted to the dynamical system language as follows, for hyperbolic base dynamics and for typical functions, optimizing orbits should have low complexity. This is proved by Contreras [12], who showed that the optimizing orbits with respect to generic Hölder/Lipschitz potentials over an expanding base are periodic.

In this paper, we would like to study ergodic optimization in a non-commutative setting where matrix cocycles are a well-known example of a noncommutative system. In the ergodic optimization of Lyapunov exponents, the quantities we want to maximize are the associated Lyapunov exponents of matrix cocycles. We would like to investigate the low complexity phenomena mentioned above in the matrix cocycle case. For 2-dimensional one-step cocycles, Bochi and Rams [7] showed that Lyapunov-maximizing measures have zero entropy. Their sufficient conditions for zero entropy are domination and existence of strictly invariant families of cones satisfying a non-overlapping condition, which is open, but not a typical assumption. Jenkinson and Pollicott [18] also investigated the zero entropy of Lyapunov maximizing measures under the almost same assumption, but their techniques are

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different. Bousch and Mairesse [11] showed that the maximizing products are not always periodic, thus disproving the so-called Finiteness Conjecture. Their examples are difficult to construct, and are broadly believed to be rare. The complexity of the matter already appears in the simple setting one-step cocycles. Indeed, such sets appear in the literature both as finiteness counterexamples [9, 15, 24].

In this paper, we deal with \( GL(2, \mathbb{R}) \) one-step cocycles. The smaller size of this body of research on the entropy of Lyapunov optimization measures can perhaps be explained by the fact that the behavior of the Lyapunov exponents of matrix cocycles is much more complicated than the behavior of Birkhoff averages. The goal of the paper is to show that the entropy of Lyapunov maximizing measures is zero under pinching, twisting and existence of strictly invariant families of cones satisfying a non-overlapping conditions on the Mather set.

This should be also contrasted with the situation typically found in the thermodynamic formalism of ergodic theory, in which the measures picked out by variational principles tend to have wide support. Ergodic optimization may be seen as the zero-temperature limit of thermodynamic formalism; see [21].

1.1. Ergodic optimization of Birkhoff averages. We call \((X, T)\) a topological dynamical system (TDS) if \(T : X \to X\) is a continuous map on the compact metric space \(X\). We denote by \(\mathcal{M}(X, T)\) the space of all \(T\)-invariant Borel probability measures on \(X\), which the space is a nonempty convex set and is compact with respect to the weak* topology. Let \(\mathcal{E}(X, T)\) be the subset formed by ergodic measures, which are exactly the extremal points of \(\mathcal{M}(X, T)\).

Let \(f : X \to \mathbb{R}\) be a continuous function. We denote by \(S_n f(x) := \sum_{k=0}^{n-1} f(T^k(x))\) the Birkhoff sum, and we call \(\lim_{n \to \infty} \frac{1}{n} S_n f(x)\), and \(\mu\)-almost every \(x \in X\), the Birkhoff average is well-defined. We denote by \(\beta(f)\) and \(\alpha(f)\) the supremum and infimum of the Birkhoff average over \(x \in X\), respectively; we call these numbers the maximal and minimal ergodic averages of \(f\). Since \(\alpha(f) = -\beta(-f)\), let us focus the discussion on the quantity \(\beta\) that can also be characterized as

\[
\beta(f) = \sup_{\mu \in \mathcal{M}(X, T)} \int f d\mu.
\]

By compactness of \(\mathcal{M}(X, T)\), the supremum is attained; such measures will be called Birkhoff maximizing measures, we denote them by \(\mathcal{M}_{\text{max}}(f)\).

1.2. Ergodic optimization of Lyapunov exponents. We assume that \(X\) is a compact metric space and \(T : X \to X\) is a homeomorphism. Let \(A : X \to GL(d, \mathbb{R})\) be a continuous map. We define a linear cocycle \(F : X \times \mathbb{R}^d \to X \times \mathbb{R}^d\) as

\[
F(x, v) = (T(x), A(x)v).
\]

We say that \(F\) is generated by \(T\) and \(A\); we will also denote it by \((A, T)\). Observe \(F^n(x, v) = (T^n(x), A^n(x)v)\) for each \(n \geq 1\), where

\[
A^n(x) = A(T^{n-1}(x))A(T^{n-2}(x)) \cdots A(x).
\]
By Kingman’s subadditive ergodic theorem, for any \( \mu \in \mathcal{M}(X, T) \) and \( \mu \) almost every \( x \in X \) such that \( \log^+ \|A\| \in L^1(\mu) \), the following limit, called the top Lyapunov exponent at \( x \), exists:

\[
\chi(x, A) := \lim_{n \to \infty} \frac{1}{n} \log \|A^n(x)\|. \tag{1.2}
\]

Let us denote \( \chi(\mu, A) = \int \chi(., A) d\mu \). If the measure \( \mu \) is ergodic then \( \chi(x, A) = \chi(\mu, A) \) for \( \mu \)-almost every \( x \in X \).

Similarly to what we did for Birkhoff averages, we can either maximize or minimize the top Lyapunov exponent \( \chi(., A) \); we define the maximal and minimal Lyapunov exponent as follows

\[
\beta(A) := \limsup_{n \to \infty} \frac{1}{n} \log \sup_{x \in X} \|A^n(x)\| \tag{1.3}
\]

\[
\alpha(A) := \liminf_{n \to \infty} \frac{1}{n} \log \inf_{x \in X} \|A^n(x)\| \tag{1.4}
\]

Feng and Huang [14, Lemma A.3] showed that the limit exists in (1.3). Moreover, the maximal and minimal Lyapunov exponent can be characterized as the supremum and infimum of the Lyapunov exponents of measures over invariant measures, respectively, i.e.

\[
\beta(A) = \sup_{\mu \in \mathcal{M}(X, T)} \chi(\mu, A), \tag{1.5}
\]

\[
\alpha(A) = \inf_{\mu \in \mathcal{M}(X, T)} \chi(\mu, A) \tag{1.6}
\]

In (1.5), the supremum is always attained by an ergodic measure- this follows from the fact that \( \mathcal{M}(X, T) \) is a compact convex set whose extreme points are exactly the ergodic measure, and upper semi continuity of \( \chi(., A) \) with respect to the weak* topology. On the other hand, the infimum in (1.6) does not necessarily attend; see [22] for more information.

Let us define the set of Lyapunov maximizing measures of the cocycle \( A \) to be the set of invariant measures on \( X \) given by

\[
\mathcal{M}_{max}(A) := \{ \mu \in \mathcal{M}(X, T) : \beta(A) = \chi(\mu, A) \}. \]

The \( \mathcal{M}_{max} \) is non-empty, compact and convex. Another significant strand of research in ergodic optimization of Lyapunov exponents, again already present in early works, was its interpretation (see [21]) as a limiting zero temperature version of the subadditive thermodynamic formalism, with maximizing measures (referred to as ground states by physicists) arising as zero temperature accumulation points of equilibrium measures of the subadditive potentials; working this area has primarily focused on understanding convergence and non-convergence in the zero temperature limit.
In the ergodic optimization of Birkhoff averages, a maximizing set is a closed subset such that an invariant probability measure is a maximizing measure if and only if its support lies on this subset. If Mañé Lemma holds, then the existence of such sets is guaranteed in any setting. The smallest maximizing set is the Mather set that is defined as the union of the supports of all maximizing measures, which is borrowed from Lagrangian dynamics; see more information [17].

Following Morris [23], we define the Mather set $\text{Mather}(\Phi_A)$ for the subadditive potential $\Phi_A := \{\log \|A^n\|\}_{n=1}^\infty$ as the union of the supports of all Lyapunov maximizing measures, i.e.,

$$\text{Mather}(\Phi_A) = \bigcup_{\mu \in \mathcal{M}_{\max}(A)} \text{supp} \mu.$$  

We denote $\mathcal{K} = \text{Mather}(\Phi_A)$ to simplify the notations in the proofs. The Mather set is a nonempty, compact, and $T$-invariant set; see [5, Proposition 6.1].

1.3. Typical cocycles. Let $Q = (q_{ij})$ be a $k \times k$ matrix with $q_{ij} \in \{0, 1\}$. The one side subshift of finite type associated to the matrix $Q$ is a left shift map $T : \Sigma^+_Q \to \Sigma^+_Q$, that is, $T(x_n)_{n \in \mathbb{N}_0} = (x_{n+1})_{n \in \mathbb{N}_0}$, where $\Sigma^+_Q$ is the set of sequences

$$\Sigma^+_Q := \{x = (x_i)_{i \in \mathbb{N}_0} : x_i \in \{1, \ldots, k\} \text{ and } Q_{x_i,x_{i+1}} = 1 \text{ for all } i \in \mathbb{N}_0\}.$$  

Similarly, one defines two-sided subshift of finite type $T : \Sigma_Q \to \Sigma_Q$, where

$$\Sigma_Q := \{x = (x_i)_{i \in \mathbb{Z}} : x_i \in \{1, \ldots, k\} \text{ and } Q_{x_i,x_{i+1}} = 1 \text{ for all } i \in \mathbb{Z}\}.$$  

When the matrix $Q$ has entries all equal to 1 we say this $\Sigma^+_Q, \Sigma_Q$ is the full shift. For simplicity, we denote that $\Sigma^+_Q = \Sigma^+$ and $\Sigma_Q = \Sigma$. We denote by $\Sigma_* = \cup_{n \geq 0} \{1, \ldots, k\}^n$ the set of words of the alphabet $\{1, \ldots, k\}$.

We say that $i_1 \ldots i_k$ is an admissible word if $Q_{i_n,i_{n+1}} = 1$ for all $1 \leq n \leq k - 1$.

Observe that the partition of $\Sigma_Q$ (or $\Sigma^+_Q$) into first level cylinders is generating, for this reason the partition into first level cylinders is the partition canonically used in symbolic dynamics to calculate the metric entropy.

A well known example of linear cocycles is one-step cocycles, which is defined as follows. Let $X = \{1, \ldots, k\}^\mathbb{Z}$ be a symbolic space. Let $T : X \to X$ be a full shift map, i.e., $T(x_i)_i = (x_{i+1})_i$. Given a finite set of matrices $A = (A_1, \ldots, A_k) \subset GL(d, \mathbb{R})$, we define the function $A : X \to GL(d, \mathbb{R})$ by $A(x_i)_i = A_{x_i}$. Let $(A, T)$ be a one-step cocycle. For $i = (i_1, \ldots, i_n)$, we define

$$A_i = A_{i_1} \ldots A_{i_n}.$$  

For one-step cocycles, the value of the maximal Lyapunov exponent can be alternatively defined as follows

$$\beta(A) := \lim_{n \to \infty} \frac{1}{n} \log \sup_{i_1, \ldots, i_n} \|A_{i_n} \ldots A_{i_1}\|.$$  

We denote by $\text{Grass}(k, d)$ the $k$-th Grassmannian, i.e., the set of all $k$-dimensional subspaces of $\mathbb{R}^d$. 
Definition 1.1. We say that a one-step cocycle $A$

- is $i$–pinching if there is $\ell \in \Sigma_*$ such that the matrix $A_\ell$ is $i$–simple, where the logarithms $\theta_i = \log \sigma_i(A_\ell)$ of the singular values $\sigma_i(A)$ satisfy the following inequality
  \[ \theta_i > \theta_{i+1} \]

- $i$–twisting if, given $1 \leq i \leq d - 1$, any $F \in \text{Grass}(i, d)$, and any finite family $G_1, \ldots, G_n \in \text{Grass}(d - i, d)$, there exists $t \in \Sigma_*$ such that $A_t(F) \cap G_j = \{0\}$ for every $1 \leq j \leq n$.

We say that the cocycle $A$ is $1$–typical if it is $1$–pinching and $1$–twisting. We also say that the cocycle $A$ is typical if it is $i$–pinching and $i$–twisting for all $1 \leq i \leq d - 1$.

In this paper, we are interested in $1$–typical cocycles. Bonatti and Viana [10] showed that

\[ \mathcal{W}_1^d := \{ A \in GL(d, \mathbb{R}) : A \text{ is } 1\text{-pinching and } 1\text{-twisting} \} \]

is open and dense.

1.4. Multicone. Let $(A, T)$ be a one-step cocycle where $A : \Sigma \to GL(2, \mathbb{R})$. Let $\mathbb{R}^2_+ := \mathbb{R}^2 \setminus \{0\}$. The standard symmetric cone in $\mathbb{R}^2_+$ is

\[ C_+ := \{(x, y) \in \mathbb{R}^2_+ : xy \geq 0\} \]

An image of $C_+$ by a linear isomorphism is a cone in $\mathbb{R}^2_+$ and a multicone in $\mathbb{R}^2_+$ is a disjoint union of finitely many cones.

We say that a multicone $M \subset \mathbb{R}^2_+$ is strictly forward-invariant with respect to $A = (A_1, \ldots, A_k)$ if the image multicone $\bigcup_i A_i(M)$ is contained in the interior of $M$ (see figure 1).

If $M$ is a multicone, its complementary multicone $M_{co}$ is defined as the closure (relative to $\mathbb{R}^2_+$) of $\mathbb{R}^2_+ \setminus M$. If $M$ is strictly forward-invariant with respect to $(A_1, \ldots, A_k)$ then $M_{co}$ is strictly backwards-invariant, i.e., strictly forward-invariant with respect to $(A_k^{-1}, \ldots, A_1^{-1})$.

We say that the cocycle $A = (A_1, \ldots, A_k)$ satisfies the forward nonoverlapping condition (NOC) if the cocycle has a strictly forward-invariant multicone $M \subset \mathbb{R}^2_+$ such that

\[ A_i(M) \cap A_j(M) = \emptyset \text{ whenever } i \neq j. \]

We say that the cocycle $A = (A_1, \ldots, A_k)$ satisfies the backwards NOC if $(A_1^{-1}, \ldots, A_k^{-1})$ satisfies the forward NOC. We say that the cocycle $A$ satisfies the NOC if the cocycle satisfies both the forward and the backwards NOC (see figure 1).
Figure 1. This is an example of a uniformly hyperbolic $A = (A_1, A_2)$ and a multicone with with 5 components that satisfies the NOC condition. Inner arrows indicate stable and unstable directions of $A_1$ and $A_2$. Blue and red outer arrows indicate the action of $A_1$ and $A_2$ in the components of the multicone, respectively.

1.5. Precise setting and statements. Let $(A, T)$ be a one-step cocycle where $A : \Sigma \to GL(2, \mathbb{R})$. We assume that the cocycle $(A, T)$ satisfying in 1-pinning and 1-twisting conditions.

The main result of this paper is the following:

Theorem 1.2. Let $(A, T)$ be a two dimensional one-step cocycle satisfying 1-pinning and 1-twisting conditions. Assume that the cocycle $A$ satisfies the NOC condition on the Mather set $K$. Then the entropy of any Lyapunov maximizing measure is zero.

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\footnote{See section \ref{section3} for the precise definition.}
2. Domination

Let $X$ be a compact metric space. Let $A : X \to GL(d, \mathbb{R})$ be a matrix cocycle function over a homeomorphism map $(X, T)$.

We say that the cocycle $(A, T)$ is $i-$dominated on $X$ if there are constants $C > 0$ and $0 < \tau < 1$ such that

$$\frac{\sigma_{i+1}(A^n(x))}{\sigma_i(A^n(x))} \leq C\tau^n \ \forall x \in X, \forall n \in \mathbb{N}.$$  

We say that the cocycle $(A, T)$ is dominated on $X$ if and only if the cocycle is $i-$dominated for all $1 \leq i \leq d - 1$.

Assume that $2 \times 2$ matrix cocycle $(A, T)$ is dominated or uniformly hyperbolic. Important properties of domination are that there are two continuous maps $e_1$ and $e_2$ from $X$ to $\mathbb{P}\mathbb{R}^2$, which are a splitting of $\mathbb{R}^2$ as the sum of two one-dimensional subspaces $e_1(\omega), e_2(\omega)$ for each $\omega \in X$, and the following properties hold:

- equivariance: $A(\omega)(e_i(\omega)) = e_i(T(\omega))$ for all $\omega \in X$ and $i \in \{1, 2\}$;
- dominance: there are constants $c > 0$ and $0 < \delta < 1$ such that

$$\frac{|A^{(n)}(\omega) e_1(\omega)|}{|A^{(n)}(\omega) e_2(\omega)|} \geq ce^{\delta n} \ \text{for all } \omega \in X \text{ and } n \geq 1;$$

see [29, 6].

It was proved in [1, 6] that the one step cocycle generated by $A$ is dominated if and only if $A$ has a strictly forward-invariant multicone.

**Theorem 2.1.** Let $A \in W^1_d$ be a 1-typical cocycle over a full shift $(X, T)$. Then, the Mather set $K$ is 1-dominated.

**Proof.** It follows from [5, Theorem 3.7] and [5, Remark 3.12]. $\square$

2.1. Domination for Subshift cocycles. We are going to introduce a special class of linear cocycles called subshift cocycles.

Let $\Sigma$ be any subshift of finite type. If $\alpha$ and $\beta$ are symbols in the alphabet $\{1, \ldots, k\}$, we write $\alpha \to \beta$ to indicate that the symbol $\alpha$ can be followed by the symbol $\beta$. Let $T : \Sigma \to \Sigma$ be a subshift of finite type and we fix a cocycle $A : \Sigma \to GL(d, \mathbb{R})$ over $(T, \Sigma)$ which is locally constant i.e., there exists $\varphi : \{1, \ldots, k\} \to GL(d, \mathbb{R})$ such that $A(x) = \varphi(x_0)$ for all $x = (x_k)_{k \in \mathbb{Z}} \in \Sigma$. The pair $(A, T)$ is called the subshift cocycle.

**Theorem 2.2.** Consider a subshift linear cocycle $(A, T)$ where $A : \Sigma \to GL(2, \mathbb{R})$. The following statements are equivalent:

(a) the cocycle $(A, T)$ is dominated (or uniformly hyperbolic) w.r.t. $\Sigma$;
(b) there are non-empty open sets $M_\alpha \subset \mathbb{P}\mathbb{R}^2$, one for each symbol $\alpha$ with $M_\alpha \neq \mathbb{P}\mathbb{R}^2$, and such that

$$\alpha \to \beta \ \text{implies} \ A_\beta(M_\alpha) \subset M_\beta,$$
We can take each $M_\alpha$ with finitely many connected components, and those components with disjoint closures.

Moreover, in (b) we can always choose a family composed of multicones.

Proof. It follows from \cite[Theorem 5.1]{4}.

For any subshift of finite type $\Sigma$, one can define the dual subshift $\Sigma^*$ as follows: if $\alpha \to \beta$ are the allowed transitions for $\Sigma$, so the allowed transitions for $\Sigma^*$ are $\beta \to \alpha$. If $A$ is a dominated (or uniformly hyperbolic) $N$-tuple w.r.t. $\Sigma$, with a family of multicones $(M_\alpha)$, then the $N$-tuple $(A_\alpha^{-1})$ is dominated w.r.t. $\Sigma^*$, with family of multicones $(M_\alpha^{-1}) = (\mathbb{P}R^2 \setminus M_\alpha)$.

Remark 1. Let $T : \Sigma \to \Sigma$ be a subshift of finite type and let the cocycle associated to $(A_1, \ldots, A_k)$ be uniformly hyperbolic with respect to $\Sigma$. For $x = (x_i)_{i \in \mathbb{Z}}$, $e_1(x)$ depends only on $(\ldots, x_{-2}, x_{-1})$ while $e_2(x)$ depends only on $(x_0, x_1, \ldots)$. (That is, $e_1$, resp. $e_2$, is constant on local unstable, resp. stable, manifolds.)

3. Proof of Theorem 1.2

Assume that $(A, T)$ is a typical cocycle where $T : \Sigma \to \Sigma$ is a full shift and $A : \Sigma \to GL(2, \mathbb{R})$ is given by $A(x) = A_{x_0}$. We also assume that the NOC condition holds on its Mather set.

By the theorem 2.1 the Mather set is dominated (or we say that the cocycle $A$ is uniformly hyperbolic with respect to the Mather set $\mathcal{K}$). Therefore, it can be approximated by a symbolic space $\Sigma_Q$ (see \cite[Section 4.5]{25} and \cite{8}). We consider a subshift of finite type $T' : \Sigma_Q \to \Sigma_Q$ and $B : \Sigma_Q \to GL(2, \mathbb{R})$ where $B(x) := A(x)$. The subshift cocycle $(B, T')$ is dominated and typical.

$\Sigma_Q$ will impose some allowed transitions. We denote by $S$ the set of all allowed transitions that imposed by $\Sigma_Q$. The cocycle $(B, T')$ is dominated, so there are non-empty open sets $M_\alpha \subset \mathbb{P}R^2$ with finitely many connected components for each symbol $\alpha$ such that

$$\alpha \to \beta \implies A_\beta(M_\alpha) \subset M_\beta,$$

by Theorem 2.2. We denote by $\mathcal{M}$ the family composed of multicones.

Since the cocycle $(A, T)$ satisfies the NOC condition on the Mather set $\mathcal{K}$, the cocycle $(B, T')$ satisfies the NOC condition. In other words, the cocycle $B$ satisfies the forward nonoverlapping condition (NOC) i.e., there is a strictly invariant family composed of multicones $\mathcal{M} \subset \mathbb{R}^2$ such that

$$A_i(\mathcal{M}) \cap A_j(\mathcal{M}) = \emptyset$$

for any symbol $\alpha$ and $\beta$ such that $\alpha \to i$ and $\beta \to j \in S$. Moreover, the cocycle $B$ satisfies the backward nonoverlapping condition i.e., $B^{-1}$ satisfies the forward NOC.

Bochi and Rams considered an $GL(2, \mathbb{R})$—one-step cocycle and they showed that the entropy of any Lyapunov maximizing measure is zero under domination\footnote{$\alpha$ can be equal to $\beta$.}.
property and the NOC condition, which is equivalent to the restriction of $T$ to the compact invariant set $K$ has zero topological entropy by the entropy variational principle. To finish the proof of Theorem 1.2 we need to show that the Bochi and Rams’s result is true for subshift cocycles.

**Theorem 3.1.** Let $(A, T)$ be a subshift cocycle satisfying domination property as well as the NOC condition, then the entropy of any Lyapunov maximizing measure is zero.

Before proving Theorem 3.1 let us discuss Barabanov norms.

### 3.1. Barabanov functions.

Let $(A, T)$ be a one-step cocycle. Barabanov [2] proved that there exist extremal norms $\|\cdot\|$ on $\mathbb{R}^d$ with the following stronger property:

$$\forall u \in \mathbb{R}^d, \quad \max_{i \in \{1, \ldots, N\}} \| A_i u \| = e^{\beta(A)}\| u \|,$$

whenever $A$ is irreducible (i.e., there is no common invariant non-trivial subspace). For more information on the Barabanov norms, see [19] and [27].

Bochi and Garibaldi [5, Theorem 5.7] established the existence of extremal norms in a far more general setting. In particular, if a subshift cocycle $(A, T)$ satisfies 1-pinching and twisting conditions, then

$$\forall u \in \mathbb{R}^d, \quad \max_{i \in \{1, \ldots, 1\}} \| A_i u \| = e^{\beta(A)}\| u \|.$$

Now, we are going to prove Theorem 3.1. We will explain how one can change the proof of [7, Theorem 2] such that it works for subshift cocycles Theorem 3.1.

They consider a set of optimal future trajectories on $\Sigma^+ \times M$ by using Barabanov functions [7, Theorem 3.1], where $M$ is a multicone. We can define the set of optimal future trajectories by using (3.1) that is

$$J = \{(\omega, v) \in \Sigma^+_Q \times M : \log \| B^{(n)}(\omega, v) \| = n\beta(B) + \log \| v \|\}.$$

Now, we need to prove [7, Lemma 4.1] for subshift cocycles.

**Lemma 3.2.** If $(\omega, v) \in J$ and $u \in M$ are such that $v - u \in e_2(\omega, v)$ then

$$\| v \| \leq \| u \|.$$

**Proof.** Suppose that $\omega \in \Sigma^+_Q$ and $v, u \in M$ be such that $v - u \in e_2(\omega, v)$. Assume that $v_n := B^{(n)}(\omega, v)$ and $u_n := B^{(n)}(\omega, u)$, for $n \geq 0$. Since $v, u \notin e_2(\omega, v)$, it follows from [7, Lemma 2.1] that the quantities

$$\frac{\| v_n - u_n \|}{\| v_n \|} \quad \text{and} \quad \frac{\| v_n - u_n \|}{\| u_n \|} \quad \text{tend to 0 as } n \to \infty.$$

We are going to show that

$$\lim_{n \to \infty} \left| \log \| u_n \| - \log \| v_n \| \right| = 0.$$
By [5] Remark 3.12 and [5] Theorems 5.2 and 5.7, there is $C > 0$ such that for every $w \in \mathbb{R}^2$,

\[(3.4) \quad C^{-1}||w|| \leq ||w|| \leq C||w||.\]

Now, \[(3.3)\] can be estimated

\[
|\log ||u_n|| - \log ||v_n||| \leq \max\left(\frac{||u_n||}{||v_n||} - 1, \frac{||v_n||}{||u_n||} - 1\right)
\]

\[
\leq \frac{||v_n - u_n||}{\min(||u_n||, ||v_n||)} \leq 2C \frac{||v_n - u_n||}{\min(||u_n||, ||v_n||)},
\]

which by \[(3.2)\] goes to zero as well. This proves \[(3.3)\].

Since $\omega + v \in J$, for all $n \geq 0$,

\[
\log ||v_n|| = n\beta(B) + \log ||v||.
\]

By \[(3.1)\],

\[
\log ||u_n|| \leq n\beta(B) + \log ||u||.
\]

In particular,

\[
|\log ||u_n|| - \log ||v_n||| \leq \log ||u|| - \log ||v||.
\]

Taking limits as $n \to \infty$. \hfill \Box

If $A, B, C, D$ are four points in $\mathbb{R}^2$, no three of them equal, then we define their cross ratio to be

\[
[A, B; C, D] := \frac{A \times C}{A \times D} \frac{B \times D}{B \times C},
\]

where $\times$ denotes cross-product in $\mathbb{R}^2$, i.e., determinant.

Now, we use Lemma 3.2 to prove the following Lemma.

**Lemma 3.3.** If $(x, v_1), (y, w_1) \in J$ and non-zero vectors $v_2 \in e_2(x), w_2 \in e_2(y)$, then

\[
||[v_1, w_1; v_2, w_2]|| \geq 1.
\]

**Proof.** Since $e_1$ direction is different from any $e_2$ direction, $v_1$ or $w_1$ can not be collinear to $v_2$ or $w_2$, so the cross-ratio is well defined. Furthermore, one can write

\[
v_1 = c_1 v_2 + c_2 w_1 \quad \text{and} \quad w_1 = \beta_1 w_2 + \beta_2 v_1.
\]

By Lemma 3.2,

\[
||v_1|| \leq ||c_1 v_2|| \leq \frac{||c_2 \beta_2 v_1||}{||v_1||} = |c_2 \beta_2||v_1||.
\]

Hence, $|c_2 \beta_2| \geq 1$. We substitute

\[
c_2 = \frac{v_1 \times v_2}{w_1 \times v_2} \quad \text{and} \quad \beta_2 = \frac{w_1 \times w_2}{v_1 \times w_2}.
\]

The assertion is obtained. \hfill \Box
The rest of the proof of their paper will work in our case up to some minor modifications.

4. Ergodic optimization of Lyapunov exponents of typical cocycles and Birkhoff averages are equivalent

In this section, we are going to show that there is a continuous function \( f \in C(\Sigma) \) such that one can look at the ergodic optimization of the Birkhoff sum of \( f \) instead of optimizing Lyapunov exponents of typical cocycles. The related result was previously obtained by Jenkinson [16]: if \( \mu \) is any ergodic optimization, then there exists a continuous function \( \varphi \) whose unique maximizing measure is \( \mu \), but there is no relation between \( \varphi \) and the cocycle \( A \). On the other hand, our result provides a relation between the Birkhoff sum \( S_n f \) and the norm of the matrix product \( A^n \).

Let \((X, T)\) be a topological dynamical systems. We say that \( A : X \to GL(d, \mathbb{R}) \) is almost multiplicative if there is a constant \( C > 0 \) such that

\[
\|A^{m+n}(x)\| \geq C \|A^m(x)\| \|A^n(T^m(x))\| \quad \forall x \in X, m, n \in \mathbb{N}.
\]

We note that since clearly \( \|A^{m+n}(x)\| \leq \|A^m(x)\| \|A^n(T^m(x))\| \) for all \( x \in X, m, n \in \mathbb{N} \), the condition of almost multiplicativity of \( A \) is equivalent to the statement that \( \Phi_A := \{\log \|A^n\|\}_{n=1}^\infty \) is almost additive.

The author [22] showed that if a cocycle is 1–domained, which can be characterized in terms of existence of invariant cone fields (or multicones) [1, 6, 4], then the potential \( \Phi_A := \{\log \|A^n\|\}_{n=1}^\infty \) is almost additive.

Lemma 4.1 ([22 Theorem 7.5]). Let \( X \) be a compact metric space, and let \( A : X \to GL(d, \mathbb{R}) \) be a matrix cocycle over a TDS \((T, X)\). Let \( (C_r)_{r \in \mathbb{R}} \) be an invariant cone field (or multicones) on \( \mathbb{R}^d \). Then, there exists \( \kappa > 0 \) such that for every \( m, n > 0 \) and for every \( x \in X \) we have

\[
\|A^{m+n}(x)\| \geq \kappa \|A^m(x)\| \cdot \|A^n(T^m(x))\|.
\]

Cuneo [13] showed that every almost additive potential sequence is actually equivalent to an additive potential in the sense that there exists a continuous potential with the same equilibrium states, topological pressure, weak Gibbs measures, variational principle, level sets (and irregular set) for the Lyapunov exponent.

Theorem 4.2 ([13 Theorem 1.2]). Let \( \Phi = \{\log \phi_n\}_{n=1}^\infty \) be an almost additive sequence of potentials over the topological dynamical systems \((X, T)\). Then, there exists \( f \in C(X) \) such that

\[
\lim_{n \to \infty} \frac{1}{n} \|\log \phi_n - S_n f\| = 0
\]

Theorem 4.3. For any \( A \in W^1_d \), there is a continuous function \( f \in C(\Sigma) \) such that

\[
\mathcal{M}_{\max}(A) = \mathcal{M}_{\max}(f).
\]
Proof. As we explained in the proof of Theorem 1.2, since the cocycle $A$ is $1$-dominant with respect to the Mather set $K$ by Theorem 2.1, one can find dominated subsystem $(B, T')$ on the Mather set that is a subshift cocycle. By Lemma 4.1, $\Phi_B = \{\log \|B^n\|\}_{n=1}^\infty$ is almost additive. Therefore, there is $f \in C(\Sigma_Q)$ such that

$$
\lim_{n \to \infty} \frac{1}{n} \|\log \|B^n\| - S_n f\| = 0,
$$

(4.1)

By Theorem 4.2. Thus,

$$
\chi(\mu, B) = \int f d\mu,
$$

for any maximizing measure $\mu$. \qed

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