Logical systems II: Free semantics

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Abstract—This paper is a sequel to [1]. It provides a general
2-categorical setting for extensional calculi and shows how
intensional and extensional calculi can be related in logical
systems. We focus on transporting the notion of Day convolution
to a 2-categorical framework, and as a complementary result
we prove the convolution theorem for internal categories. We
define the concept of Yoneda triangle, and show how objects in
a Yoneda bitriangle get extensional semantics “for free”. This
includes the usual semantics for propositional calculi, Kripke
semantics for intuitionistic calculi and ternary frame semantics
for substructural calculi including Lambek’s lambda calculi,
relevance and linear logics. We show how in this setting, one
may use a model-theoretic logic to induce a structure of a proof-
theoretic logic.

I. INTRODUCTION

A well-known Mac Lane’s slogan says “adjunctions arise
everywhere”. One may find adjunctions in variety of con-
cepts from theoretical computer science: definitions of Galois
correspondence between syntax and semantics together with
Dedekind-MacNeille completion as the fixed-point of the ad-
junction, power objects, function spaces (in the form of lambda
abstraction), logical connectives and quantifications, to classi-
cal mathematics: free structures, definitions of tensor products,
distributions, and many more. In some cases, however, the
definition of an adjunction is too restrictive.

Example 1.1 (Topological spaces): Although category of
topological spaces is not cartesian closed, very many interest-
ing topological spaces are exponentiable. In fact for a topolog-
ical category 

A 0 ⊆ A 1 ⊆ A 2 ⊆ · · · that is collectively universal, which
means that for every set A there exists U k and exactly one
X ∈ U k with A ≈ X.

We should think of universes as of 2-dimensional analogue of
the internal truth-values object ∊ in a topos — just like

Π 0 ≈ Σ 0 ≈ Ω internal logic of a category, a universal object
tries to classify the external logic. The attempt to classify the
full external logic is, however, futile, as stated in the above
example. Therefore, we have to focus on a classification of
some parts of the external logic.

In this paper we set forth categorical foundations for “2-
powers”, which shall generalise partial classifiers of external
logics. We show how internal logical systems in any 2-
category with 2-powers carry free semantics on their objects.
We propose a notion of a Yoneda (bi)triangles as relativisations
of internal adjunctions, and use them to characterise universes
that admit a notion of convolution. As a complementary result
we prove the convolution theorem for internal categories.

II. CATEGORICAL 2-POWERS

To better understand our definition of “2-powers”, let
us first recall how one may define ordinary powers. With
every (locally small) regular category [8] there is associated the (poset-enriched) category of its internal re-

In other words — if the coding is effective then it has to be ambiguous.

The result generalises to any higher-order type theory [11].

Otherwise, by the axiom of union we could form A 0 = U k ∪ U k, and
A = P(A 0) would not be classified by any U k.

Poset-enriched category, also called a 2-poset, is a category enriched in
the category of posets.
relations \( \text{Rel}(\mathcal{C}) \) together with a bijective-on-objects functor \( J: \mathcal{C} \to \text{Rel}(\mathcal{C}) \). Furthermore, the right adjoint of \( J \), if it exists, \( P: \mathcal{C} \to \text{Rel}(\mathcal{C}) \) induces the natural isomorphism:

\[
\frac{\text{hom}_{\text{Rel}(\mathcal{C})}(A, B)}{\text{hom}_{\mathcal{C}}(A, P(B))}
\]

If additionally \( \mathcal{C} \) has a terminal object \( 1 \) then recalling the definition of an internal relation gives:

\[
\frac{\text{sub}(\mathcal{C})(A)}{\text{hom}_{\text{Rel}(\mathcal{C})}(A, 1)}
\]

making \( \mathcal{C} \) a topos with power functor \( P \) and the subobject classifier \( \Omega = P(1) \). All of the above may be abstractly characterised by starting with a regular fibration \( p: E \to \mathcal{C} \) on a finitely complete category \( \mathcal{C} \), then constructing the category of \( p \)-internal relations and a bijective-on-objects functor \( J: \mathcal{C} \to \text{Rel}(p) \). We shall recover the classical situation by taking for \( p \) the usual subobject fibration. Now we would like to argue that the right notion of the category of relations over \( \mathcal{C} \) is encapsulated by any bijective-on-objects functor \( J: \mathcal{C} \to \mathbb{D} \), where \( \mathbb{D} \) is a 2-post. First, let us recall that any such bijective-on-objects functor corresponds to a poset-enriched module monad:

\[
\text{hom}(J(-), J(=)): \mathcal{C}^{op} \times \mathcal{C} \to \text{Pos}
\]

with the unit \( \eta: \text{hom}(-, =) \to \text{hom}(J(-), J(=)) \) given by the action of \( J \) and multiplication \( \mu: \int^{B \in \mathcal{C}} \text{hom}(J(-), J(B)) \times \text{hom}(J(B), J(=)) \to \text{hom}(J(-), J(=)) \) induced by the composition from \( \mathbb{D} \). Therefore bijective-on-objects functors are “essentially the same” as fibrwise preorder monoidal fibrations. If we assume that \( \mathcal{C} \) has a terminal object \( 1 \), then by Grothendieck construction:

\[
\text{hom}(J(1), J(-)): \mathcal{C} \to \text{Pos}
\]

corresponds to an opfibration:

\[
\pi_{\text{hom}(J(1), J(-))}: \int \text{hom}(J(1), J(-)) \to \mathcal{C}
\]

and:

\[
\text{hom}(J(-), J(1)): \mathcal{C}^{op} \to \text{Pos}
\]

corresponds to a fibration:

\[
\pi_{\text{hom}(J(-), J(1))}: \int \text{hom}(J(-), J(1)) \to \mathcal{C}
\]

In case \( \mathbb{D} = \text{Rel}(p) \) these two functors are equivalent and encode the fibration \( p: E \to \mathcal{C} \); one may check that our fibred span arises by pulling back \( p: E \to \mathcal{C} \) along the Cartesian product functor \( \times: \mathcal{C} \times \mathcal{C} \to \mathcal{C} \) to obtain a bifibration \( \text{rel}(p): \text{Rel}(p) \to \mathcal{C} \times \mathcal{C} \) and postcomposing it with two projections. For this reason the functor \( J: \mathcal{C} \to \mathbb{D} \) does not lose any information about the regular logic associated to \( \mathcal{C} \). Second, we do believe that a more natural setting for relations is a fibred span than a bifibration — this allows us to distinguish between relations \( A \to B \) from relations \( B \to A \) and generalise the construction to higher categories.

For example, as suggested by Benabou, the role of relations between categories should be played by distributors. For any complete and cocomplete symmetric monoidal category \( \mathcal{C} \), we may define a 2-fibred span:

\[
\text{hom}_{\text{Dist}(\mathcal{C})}(\mathcal{C})^{op} \times \text{Cat}(\mathcal{C}) \to \text{Cat}
\]

sending \( \mathcal{C} \)-enriched categories \( A, B \) to the category of \( \mathcal{C} \)-enriched distributors \( A \to B \) and \( \mathcal{C} \)-enriched natural transformations. Because:

\[
\text{hom}_{\text{Dist}(\mathcal{C})}(A, B) \neq \text{hom}_{\text{Dist}(\mathcal{C})}(B, A)
\]

the fibred span \( \text{hom}_{\text{Dist}(\mathcal{C})} \) is not induced by any 2-(op)fibration.

**Example 2.1 (Allegory):** Another way to look at these concepts is through the notion of an allegory \cite{benabou}. An allegory is a pair \( (\mathcal{A}, (-)^*: \mathcal{A} \to \mathcal{A}^{op}) \), where \( \mathcal{A} \) is a poset-enriched category (2-post), \( (\mathcal{A}^*)^*: \mathcal{A} \to \mathcal{A}^{op} \) is an identity-on-objects duality involution, and:

- for each \( A, B \in \mathcal{A} \) the poset \( \text{hom}(A, B) \) has binary conjunctions
- for each triple of morphisms \( A \overset{h}{\to} B \overset{g}{\to} C \) the following holds:

\[
g \circ f \land h \leq g \circ (g^* \circ h \land f)
\]

Every allegory \( \mathcal{A} \) induces a bijective on objects embedding \( J: \mathcal{C} \to \mathcal{A} \) by forming a subcategory \( \mathcal{C} \) consisting of morphisms that has right adjoints. Moreover, if \( \mathcal{A} \) is a tabular allegory\footnote{An allegory is tabular if for every morphism \( h \) admits a decomposition \( h = g^* \circ f \) such that \( f^* \circ f \land g^* \circ g = \text{id} \) and.} then \( \text{Rel}(\mathcal{C}) \approx \mathcal{A} \) and \( \mathcal{C} \) is (locally) regular \cite{benabou}.

As mentioned earlier, a (locally small) category with finite limits has power objects iff it is regular and the induced functor \( J: \mathcal{C} \to \text{Rel}(\mathcal{C}) \) has right adjoint. It is natural then to provide the following generalisation of a power functor. If \( J: \mathcal{C} \to \mathbb{D} \) is a bijective on objects functor, then we say that \( P(B) \in \mathbb{D} \) is a J-power of \( B \in \mathbb{D} \) if there is a representation:

\[
\text{hom}(J(-), B) \approx \text{hom}(-, P(B))
\]

If a representation \( P(B) \) exists for every \( B \in \mathbb{D} \), i.e. \( J \) has the right adjoint \( P: \mathbb{D} \to \mathcal{C} \), we say that \( \mathcal{C} \) has \( J \)-powers.

**Example 2.2 (Topos):** Let \( \mathcal{C} \) be a finitely complete (locally small) regular category, and \( J: \mathcal{C} \to \text{Rel}(\mathcal{C}) \) its inclusion functor into the category of relations. \( \mathcal{C} \) is a topos iff it has \( J \)-powers.

**Example 2.3 (Quasitopos):** Let \( \mathcal{C} \) be a finitely complete and cocomplete locally cartesian closed category, such that its fibration of regular subobjects\footnote{A regular subobject of \( A \) is a (equivalence class of) regular monomorphism with codomain \( A \). A regular monomorphism is a (necessary) monomorphism that arises as an equaliser.} is regular, and \( J: \mathcal{C} \to \text{RegRel}(\mathcal{C}) \) its inclusion functor into the category of regular relations. \( \mathcal{C} \) is a quasitopos iff it has \( J \)-powers.

**Example 2.4 (Regular fibration):** More generally, let \( p: E \to \mathcal{C} \) be a regular fibration on a finitely complete category with initial object \( 0 \).

- for each \( \mathcal{A}, \mathcal{B}, \mathcal{C} \in \text{Cat}(\mathcal{A}) \) let \( \mathcal{C} \) be a topos with power functor \( P \) and \( \text{hom}_{\text{Dist}(\mathcal{C})}(\mathcal{C})^{op} \times \text{Cat}(\mathcal{C}) \to \text{Cat} \) sending \( \mathcal{C} \)-enriched categories \( A, B \) to the category of \( \mathcal{C} \)-enriched distributors \( A \to B \) and \( \mathcal{C} \)-enriched natural transformations. Because:

\[
\text{hom}_{\text{Dist}(\mathcal{C})}(A, B) \neq \text{hom}_{\text{Dist}(\mathcal{C})}(B, A)
\]

the fibred span \( \text{hom}_{\text{Dist}(\mathcal{C})} \) is not induced by any 2-(op)fibration.
category $\mathbb{C}$. If $J: \mathbb{C} \to \text{Rel}(p)$ has a right adjoint, then $p: \mathbb{E} \to \mathbb{C}$ has a generic object. The converse is true provided that $\mathbb{C}$ is cartesian closed.

Still, as exposed in the introduction, such definition is too strong to embrace many interesting examples. Here is another one. Let $\text{Cat}$ be the 2-category of small categories, and $\text{Dist}$ the 2-category of distributors, with the usual bijective on object embedding $J: \text{Cat} \to \text{Dist}$ defined on functors $J(F) = \text{hom}(\bullet, F(=))$. Then $\text{Cat}$ does not have $\mathcal{J}$-powers due to the size issues — distributors $\mathbb{A} \to \mathbb{B}$ correspond to functors $\mathbb{A} \to \text{Set}^{\mathbb{B}^{\text{op}}}$, but the category $\text{Set}^{\mathbb{B}^{\text{op}}}$ usually is not small, nor even equivalent to a small one. Unfortunately, these size issues are fundamental — there is no sensible restriction on the sizes of objects and morphism to make $\text{Cat}$ admit $\mathcal{J}$-powers. However, some of the distributors are classified in such a way. These observations lead to the concept of a Yoneda triangle.

**Definition 2.1 (Yoneda triangle):** Let $\mathcal{W}$ be a 2-category. A Yoneda triangle in $\mathcal{W}$, written $\eta : y \triangleright (f, g)$, consists of three morphisms $y : A \to \mathcal{X}$, $f : A \to B$ and $g : B \to \mathcal{X}$ together with a 2-morphism $\eta : y \Rightarrow g \circ f$ which exhibits $g$ as a pointwise left Kan extension of $y$ along $f$, and exhibits $f$ as an absolute left Kan lifting$^8$ of $y$ along $g$:

$$
\begin{array}{ccc}
A & \xrightarrow{y} & \mathcal{X} \\
\downarrow f = \text{Lift}_g(y) & & \downarrow g = \text{Lan}_f(y) \\
B & & \\
\end{array}
$$

The absoluteness of a Kan lifting means that the lifting is preserved by any morphism $k : K \to A$ — i.e. the 2-morphism $\eta \circ k$ exhibits $\text{Lift}_g(y) \circ k$ as $\text{Lift}_g(y \circ k)$.

The idea of a Yoneda triangle is that we have a morphism $y : A \to \mathcal{X}$ which plays the role of a “defective identity” and for a given morphism $f : A \to B$ we try to characterise its right adjoint up to the “defective identity” $y$.

**Example 2.5 (Adjunction as Yoneda triangle):** A 1-morphism $f : A \to B$ in a 2-category $\mathcal{W}$ has a right adjoint $g : B \to A$ with unit $\eta : id \Rightarrow g \circ f$ precisely when $\eta : id \triangleright (f, g)$ is a Yoneda triangle:

$$
\begin{array}{ccc}
B & \xrightarrow{g} & A \\
\downarrow id & & \downarrow f = \text{Lift}_g(id) \\
\downarrow g = \text{Lan}_f(id) & & \end{array}
$$

Since $f = \text{Lift}_g(id)$ is an absolute lifting, $f \circ g$ is a lifting of $g$ through $g$ with $\eta \circ g : g \Rightarrow g \circ f \circ g$. By the universal property of the lifting, there is a unique 2-morphism $\epsilon : f \circ g \Rightarrow id$ such that $g \circ \epsilon \cdot \eta \circ g = id$, which may be defined as the counit of the adjunction. We have to show that also the other triangle equality holds. Let us first postcompose the equation $g \circ \epsilon \cdot \eta \circ g = id$ with $f$ to obtain $g \circ \epsilon \circ f \cdot \eta \circ g \circ f = id$. Then postcompose it with $\eta$ to get $g \circ \epsilon \circ f \cdot \eta \circ g \circ f \cdot \eta = \eta$. But this equation under bijection provided by $\text{Lift}_g(\eta)$ corresponds to the equation $\epsilon \circ f \cdot f \circ \eta = id$, which is the required triangle equality.

On the other hand, let us assume that $f$ is left adjoint to $g$ with unit $\eta : id \Rightarrow g \circ f$ and counit $\epsilon : f \circ g \Rightarrow id$. We shall see that for every $k : C \to A$ the composite $\eta \circ k$ exhibits $f \circ k$ as the left Kan lifting of $k$ along $g$:

$$
\begin{array}{ccc}
C & \xrightarrow{k} & A \\
\downarrow h & & \downarrow g \circ h \\
B & & \\
\end{array}
$$

We have to show that the assignments:

$$
\begin{array}{ccc}
f \circ k & \xrightarrow{\alpha} & h \circ k \circ \text{Lift}_g(\eta) \circ k \\
\downarrow \epsilon \circ h \cdot f \circ \beta & & \downarrow \epsilon \circ \text{Lift}_g(\eta) \circ \beta \\
\end{array}
$$

are inverse of each other. Let us check the first composition:

$$
\epsilon \circ h \cdot f \circ (g \circ \alpha \cdot \eta \circ k) = \epsilon \circ h \cdot f \circ g \circ \alpha \cdot f \circ \eta \circ k
$$

$$
= \alpha \cdot \epsilon \circ f \circ k \cdot f \circ \eta \circ k
$$

$$
= \alpha
$$

where the first and second equality follows from the interchange law of a 2-category, and the last one is the triangle equation. Similarly we may check the second composition:

$$
\epsilon \circ (\epsilon \circ h \cdot f \circ \beta) \cdot \eta \circ k = \epsilon \circ \epsilon \circ h \cdot g \circ \beta \cdot \eta \circ k
$$

$$
= \epsilon \circ \epsilon \circ h \cdot \eta \circ g \circ h \cdot \beta
$$

$$
= \beta
$$

The fact that $g$ is a pointwise left extension of $id$ along $f$ follows from a more general observation that a left Kan extension along a left adjoint always exists and is pointwise$^{14}$. However, it is illustrative to see how the bijections defining Kan extensions are constructed in our particular case. Let us extend the diagram of adjunction $f \dashv g$ by generalised elements $a \in_X A$, $b \in_X B$:

$$
\begin{array}{ccc}
X & \xrightarrow{f} & A \\
\downarrow \pi_1 & & \downarrow f \circ \pi_1 \\
\downarrow \pi_2 & & \downarrow g \circ \pi_2 \\
B & \xleftarrow{a \circ b} & A \\
\end{array}
$$

$^8$The concept of a Kan lifting is the opposite of the concept of Kan extension — i.e. a Kan lifting in $\mathcal{W}$ is a Kan extension in $\mathcal{W}^{\text{op}}$. 

where \( h: X \to f \downarrow b \) is the unique morphism to the comma object induced by the counit \( \epsilon_6: f(g(b)) \to b \). Then, one part of the bijective correspondence is given by assigning to \( \beta: g(b) \to a \) an arrow \( \beta \circ \pi_2 \cdot g \circ \pi \cdot \eta \circ \pi_1: \pi_1 \to a \circ \pi_2 \), and the other is given by composition with \( h \).

Generally, a Yoneda-like triangle \( \eta: y \Rightarrow (f,g) \) where \( g \) is not assumed to be the left Kan extension of \( y \) along \( f \) is called an adjunction relative to \( y \). Note however, that in such a case \( g \) need not be uniquely determined by \( f \).

Just like in \([1]\) we provided an elementary description of pointwise Kan extensions, we shall now give a similar characterisation of absolute Kan liftings. Let us extend the diagram of \( \hom(\_ , \_ ) \)

in case the category is not tensored over \( \mathbf{Cat} \)

and natural transformations, then the condition that \( g \) is a pointwise left Kan extension of \( y \) along \( f \) is called a pointwise Kan lifting of \( y \) along \( f \).

Furthermore, if \( Y \) is dense, then \( G \) is automatically a pointwise Kan extension in a canonical way — from density we have:

\[
G(-) \approx \int_{A \in \mathcal{A}} \hom(Y(A), G(-)) \times Y(A)
\]

and using the formula for an absolute lifting:

\[
G(-) \approx \int_{A \in \mathcal{A}} \hom(F(A), -) \times Y(A)
\]

This example needs more elaboration. In the literature, there exist two essentially different notions of pointwise Kan extensions. The older, provided by Eduardo Dubuc \([16]\) for enriched categories, defines pointwise Kan extensions as appropriate enriched (co)ends:

\[
\text{Ran}_F(Y) = \int_A Y(A) \hom(-, F(C))
\]

\[
\text{Lan}_F(Y) = \int_A \hom(F(C), -) \otimes Y(A)
\]

The newer, provided by Ross Street \([14]\), works in the general context of (sufficiently complete) 2-categories, and uses this in this dissertation. As pointed in \([14, 6]\) these definitions agree for categories enriched in \( \mathbf{Set} \), and in categories enriched in the 2-valued Boolean algebra \( \mathbb{2} \), but Street’s definition is stronger than Dubuc’s one for general enriched categories (it is strictly stronger for categories enriched in abelian groups \( \mathbf{Ab} \), and for categories enriched in \( \mathbf{Cat} \)). Steve Lack \([17]\) blamed for this mismatch the definition of a category of \( \mathcal{C} \)-enriched categories, which “can’t see” the extra structure of a \( \mathcal{C} \)-enriched category on functor categories \( \hom(A, B) \).

Whilst it is certainly true that the category \( \mathbf{Cat}(\mathcal{C}) \) of \( \mathcal{C} \)-enriched categories is more than a 2-category — after all, it is a \( \mathbf{Cat}(\mathcal{C}) \)-enriched category with an underlying 2-category — the reasoning is not correct. Technically, the reasoning cannot be right, because treating a 2-category as a \( \mathbf{Cat}(\mathcal{C}) \)-enriched category and carrying to this setting Street’s definition of pointwise Kan extension may only strengthen the concept of a Kan extension, which is, actually, in its ordinary 2-categorical form, stronger than Dubuc’s one. More importantly, also philosophically the reasoning cannot be right — the enrichment of \( \mathbf{Cat}(\mathcal{C}) \) in \( \mathbf{Cat}(\mathcal{C}) \) is a self-enrichment, which means that it is completely recoverable from its underlying 2-category; the idea behind Street’s pointwise Kan extensions\([4]\) was to define the Kan extension at “every generalised 2-point” just to evade defining it on “enriched objects” — the sufficiency of such definitions may be explained by the usual Yoneda yoga.

**Example 2.7 (Yoneda triangle along Yoneda embedding):** For any functor \( F: \mathcal{A} \to \mathcal{B} \) between locally small categories, there is a Yoneda triangle:

\[
\begin{array}{c}
\mathcal{A} \\
F \\
\mathcal{B}
\end{array}
\]

which reassembles the fact that every functor always has a “distributional” right adjoint\([10]\). The same is true for inter-

\[\text{As explained in } [1].\]

\[\text{Every functor has a right adjoint in the weak 2-category of distributors.}\]
nal categories and for categories enriched in a cocomplete symmetric monoidal closed category, and generally (almost by definition) for any 2-category equipped with a Yoneda structure in the sense of [4].

The essence of the example is that because the Yoneda functor \( y_A : B \to \text{Set}^{\text{op}} \) is a full and faithful embedding, functors \( F : A \to B \) may be thought of as distributors

\[
y_A \circ F = \text{hom}(-, F(-))
\]

Every distributor arisen in this way has a right adjoint distributor \( \text{hom}(F(-), -) \) in the bicategory of distributors. The distributor \( \text{hom}(F(-), -) \) has actually type \( B \to \text{Set}^{\text{op}} \), which is the only think that may prevent \( F \) of having the ordinary (functorial) right adjoint \( G : B \to A \). Formally, we say that \( F \) has a right adjoint, if there exists \( G \) such that:

\[
y_A \circ G \approx \text{hom}(F(-), -)
\]

which means:

\[
\text{hom}(-, G(-)) \approx \text{hom}(F(-), -)
\]

Of course, a Yoneda 2-triangle is a Yoneda triangle in the (2-)category of 2-categories, 2-functors, and 2-natural transformations. However, in light of our elaboration on “pointwiseness”, we shall weaken the definition of pointwise Kan extension to the one suitable for enriched categories — as it is much easier and convenient to work with.

**Definition 2.2 (Yoneda bitriangle):** \( A \) Yoneda bitriangle \( \eta : Y \triangleright (F,G) \) consists of pseudo-functors \( Y : A \to \tilde{A} \), \( F : A \to B \), \( G : B \to A \) between (weak) 2-categories \( A, \tilde{A}, B \) and a 2-natural transformation \( \eta : Y \to G \circ F \) that induces natural equivalences between functors:

\[
\text{hom}(Y(-), G(-)) \approx \text{hom}(F(-), -)
\]

and:

\[
G(-) \approx \int^{A \in A} \text{hom}(F(A), -) \times Y(A)
\]

We shall be mostly interested in Yoneda triangles arisen from proarrow equipment \([2][3]\). Let \( J : A \to B \) be a (weak) 2-functor from a (strict) 2-category \( A \) to a (weak) 2-category \( B \). We say that \( J \) equips \( A \) with proarrows if the following holds:

- \( J \) is bijective on objects
- \( J \) is locally fully faithful, which means that for every pair of objects \( X, Y \in A \) the induced functor \( \text{hom}_A(X,Y) \to \text{hom}_B(J(X), J(Y)) \) is fully faithful
- for every 1-morphism \( f : X \to Y \) in \( A \) the corresponding morphism \( J(f) : J(X) \to J(Y) \) in \( B \) has a right adjoint

A proarrow equipment reassembles the concept of an allegory in a 2-dimensional context.

**Definition 2.3 (2-power):** Let \( J : A \to B \) be an equipment of \( A \) with proarrows, and \( Y : A \to \tilde{A} \) a 2-functor making \( A \) a full 2-subcategory of \( \tilde{A} \). Then \( A \) has \( J \)-relative 2-powers if \( J \) and \( Y \) can be completed to a 2-Yoneda triangle \( \eta : Y \triangleright (J, P) \) with \( P : B \to \tilde{A} \) and \( \eta : Y \to P \circ J \).

**Example 2.8 (Categorical 2-powers):** The archetypical situation is when we take \( \eta : Y : \text{cat} \to \text{Cat} \triangleright (J : \text{cat} \to \text{Dist}, P : \text{Dist} \to \text{Cat}) \), where \( \text{cat} \) is the 2-category of small categories, \( \text{Cat} \) is the 2-category of locally small categories, and \( \text{Dist} \) is the bicategory of distributors between small categories. Then \( J : \text{cat} \to \text{Dist}, Y : \text{cat} \to \text{Cat} \) are the usual embeddings, \( P : \text{Dist} \to \text{Cat} \) is the covariant 2-power pseudofunctor \( \text{Set}^{\text{op}} \to \text{Dist} \) defined on distributors via left Kan extensions, and \( \eta_A : A \to \text{Set}^{\text{op}} \) is the Yoneda embedding of a small category \( A \). There are isomorphisms of categories:

\[
\text{hom}_{\text{Dist}}(A, B) \approx \text{hom}_{\text{Cat}}(A, \text{Set}^{\text{op}})
\]

where \( A \) and \( B \) are small. Therefore, to show that \( P \) is a (weak) pointwise left Kan extension it suffices to show that \( Y \) is 2-dense. However, \( Y \) is obviously 2-dense, because the terminal category is a 2-dense subcategory of \( \text{Cat} \) and \( Y \) is fully faithful.

Here is a similar result for internal categories.

**Theorem 2.1 (C-internal 2-powers):** Let \( C \) be a finitely complete locally cartesian closed category. There is a Yoneda bitriangle:

\[
\eta : \text{fam} \triangleright (J : \text{cat}(C) \to \text{Dist}(C), P : \text{Dist}(C) \to \text{Cat}^{C\text{op}})
\]

where \( \text{cat}(C) \) is the 2-category of \( C \)-internal categories, \( \text{Dist}(C) \) is the (weak) 2-category of \( C \)-internal distributors with \( J \) the usual embedding, and:

\[
fam : \text{cat}(C) \to \text{Cat}^{C\text{op}}
\]

is the canonical family functor (the externalisation functor). Pseudofunctor:

\[
P : \text{Dist}(C) \to \text{Cat}^{C\text{op}}
\]

given by:

\[
P(A) = \text{fam}(C) \text{fam}(A)^{\text{op}}
\]

\[
P(A \to B) = \text{Lan}_{y_A}(F)
\]

where \( \text{fam}(C) \) is a split indexed category corresponding to the fundamental (i.e. codomain) fibration \( \text{cod}(C) \), and:

\[
y_A : \text{fam}(A) \to \text{fam}(C) \text{fam}(A)^{\text{op}}
\]

is the usual internal Yoneda embedding defined as the cartesian transposition of:

\[
\text{hom}(-, -) : \text{fam}(A) \times \text{fam}(A)^{\text{op}} \to \text{fam}(C)
\]

**Proof:** Since Kan extensions are (pseudo)functorial, \( P \) is a pseudofunctor \( \text{Dist}(C) \to \text{Cat}^{C\text{op}} \). There is an equivalence of categories\([4][23]\):

\[
\text{hom}_{\text{Dist}(C)}(A, B) \approx \text{hom}_{\text{Cat}^{C\text{op}}}(\text{fam}(A), \text{cod}(C) \text{fam}(B)^{\text{op}})
\]

To show that \( P \) is a (pointwise) left Kan extension it suffices to show that \( \text{fam} \) is 2-dense. However, \( \text{fam} \) on discrete internal categories is clearly 2-dense by (weak) 2-Yoneda lemma, and

\[\text{In fact this equivalence is almost a definition of the category } \text{Dist}(C)\text{.}\]
discrete internal categories form a full 2-subcategory of all categories. Therefore \( \text{fan} \) is 2-dense. It requires much more work to obtain analogous result for enriched categories. The issue is of the same kind as we encountered earlier — discrete objects in the category of enriched categories are generally not dense (more — they hardly constitute a generating family) and there is no canonical candidate for any subcategory giving a dense notion of discreteness. First, let us observe that every enriched category is a canonical limit over its full subcategories consisting of at most three objects.

**Lemma 2.2 (On a 2-dense subcategory of \( \mathbf{Cat}(C) \)):** Let \( (I, \otimes, C) \) be a complete and cocomplete symmetric monoidal closed category. The category of small \( C \)-enriched categories is a 2-dense subcategory of all \( C \)-enriched categories.

**Proof:** We have to show that the following categories of natural transformations are isomorphic in a canonical way for all \( C \)-enriched categories \( A, B \in \mathbf{Cat}(C) \):

\[
\text{hom}(\text{hom}(\cdot, A), \text{hom}(\cdot, B)) \cong \text{hom}(\text{hom}(Y(\cdot), A), \text{hom}(Y(\cdot), B))
\]

where \( Y : \mathbf{Cat}_S(C) \to \mathbf{Cat}(C) \) is the embedding of small \( C \)-enriched categories \( \mathbf{Cat}_S(C) \) into all categories \( \mathbf{Cat}(C) \). To simplify the proof, let us observe that it suffices to show that the underlying sets of the above natural transformation objects are bijective (i.e. that \( \mathbf{Cat}_S(C) \) is 1-dense in \( \mathbf{Cat}(C) \)). Since \( \mathbf{Cat}_S(C) \) is cotensored we have natural bijections:

\[
\text{hom}(\text{hom}(\cdot, A), \text{hom}(Y(\cdot), X \triangleleft B)) \cong \text{hom}(\text{hom}(\cdot, A), \text{hom}(Y(\cdot), B)X) \cong \text{hom}(X, \text{hom}(\text{hom}(\cdot, A), \text{hom}(\cdot, B)))
\]

and similarly:

\[
\text{hom}(\text{hom}(Y(\cdot), A), \text{hom}(Y(\cdot), X \triangleleft B)) \cong \text{hom}(\text{hom}(Y(\cdot), A), \text{hom}(Y(\cdot), B)X) \cong \text{hom}(X, \text{hom}(\text{hom}(Y(\cdot), A), \text{hom}(Y(\cdot), B)))
\]

By the usual Yoneda argument categories:

\[
\text{hom}(\text{hom}(\cdot, A), \text{hom}(\cdot, B))
\]

and:

\[
\text{hom}(\text{hom}(Y(\cdot), A), \text{hom}(Y(\cdot), B))
\]

are isomorphic iff the sets:

\[
\text{hom}(X, \text{hom}(\text{hom}(Y(\cdot), A), \text{hom}(Y(\cdot), B)))
\]

and:

\[
\text{hom}(X, \text{hom}(\text{hom}(\cdot, A), \text{hom}(\cdot, B)))
\]

are naturally bijective. Therefore, if the canonical function \( \text{hom}(\text{hom}(\cdot, A), \text{hom}(\cdot, X \triangleleft B)) \to \text{hom}(\text{hom}(\cdot, A), \text{hom}(\cdot, X \triangleleft B)) \) is a bijection, then the canonical functor \( \text{hom}(\text{hom}(\cdot, A), \text{hom}(\cdot, B)) \to \text{hom}(\text{hom}(Y(\cdot), A), \text{hom}(Y(\cdot), B)) \) is an isomorphism.

Denote by \( \mathbf{Cat}_3(C) \) the full 1-subcategory of \( \mathbf{Cat}(C) \) consisting of categories with at most three objects, and by \( \mathbf{K} : \mathbf{Cat}_3(C) \to \mathbf{Cat}(C) \) its embedding. We show that \( \mathbf{Cat}_3(C) \) is a 1-dense subcategory of \( \mathbf{Cat}(C) \), which by fullness-faithfulness of \( Y \) implies that \( \mathbf{Cat}_S(C) \) is 1-dense subcategory of \( \mathbf{Cat}(C) \), and by the above that it is 2-dense.

One direction is easy — if \( \alpha : \text{hom}(\cdot, A) \to \text{hom}(\cdot, B) \) is a natural transformation, then its restriction \( \tilde{\alpha} : \text{hom}(K(\cdot), A) \to \text{hom}(K(\cdot), B) \) to a subcategory is natural as well, and since \( \mathbf{Cat}_3(C) \) is clearly a generating subcategory, then this assignment is injective. So let us focus on the other direction.

For the other direction, observe that every \( C \)-enriched category \( A \) may be canonically represented as a colimit over at most three-object categories:

- for every triple of objects \( X, Y, Z \in A \), let \( A_{X,Y,Z} \) be the full subcategory of \( A \) on this triple with projection \( j_{X,Y,Z}^A : A_{X,Y,Z} \to A \); similarly define \( B_{X,Y,Z} : A_{X,Y} \to A_{X,Y,Z} \) for the full subcategory \( A_{X,Y} \) of \( A_{X,Y,Z} \) on every pair \( X, Y \in A_{X,Y,Z} \), and \( j_{X,Y}^A : A_X \to A_{X,Y} \) for the full one-object subcategory on every object \( X \in A_{X,Y} \).
- for diagram \( D_A \) consisting of all such defined injections \( j_{X,Y,Z}^A : A_{X,Y} \to A_{X,Y,Z} \), \( j_{X,Y}^A : A_X \to A_{X,Y} \), the category \( A \) together with \( \alpha_A : A_{X,Y,Z} \to A \) is the colimit of \( D_A \) — if \( B \) is another category with cocone \( \tau_{X,Y,Z}^A : A_{X,Y,Z} \to B \) then the unique functor \( H : A \to B \) is given on objects by \( H(X) = (\tau_{X,Y.Z}^A \circ j_{X,Y,Z}^A)(X) \), and similarly on morphisms; the compositions are preserved by \( H \), because they are preserved pairwise by each \( \tau_{X,Y,Z}^A \), and preservation of identities is obvious.

Let \( \tilde{\alpha} : \text{hom}(K(\cdot), A) \to \text{hom}(K(\cdot), B) \) be a natural transformation. By naturality the diagram \( D_A \) is mapped to a cocone under \( B \). By universal property of colimits, this cocone induces a morphism \( c : A \to B \), which by Yoneda lemma is tantamount to a natural transformation:

\[
\text{hom}(\cdot, c) : \text{hom}(\cdot, A) \to \text{hom}(\cdot, B)
\]

We have to show, that \( \text{hom}(\cdot, c) \) on \( \mathbf{Cat}_3(C) \) is equal to \( \tilde{\alpha} \), that is: for any at most three-element category \( M \) and a functor \( f : M \to A \) the composite \( c \circ f \) is equal to \( \tilde{\alpha}(f) \). But that is easy. Let us assume that \( M \) has exactly three objects \( X, Y, Z \) then \( f \) : \( M \to A \) as factors as \( g : M \to A_{f(X),f(Y),f(Z)} \) through injection \( j_{f(X),f(Y),f(Z)} : A_{f(X),f(Y),f(Z)} \to A \). By naturality of \( \tilde{\alpha} \) we have: \( \tilde{\alpha}(j_{f(X),f(Y),f(Z)}) \circ g = \tilde{\alpha}(j_{f(X),f(Y),f(Z)}) \circ g = \tilde{\alpha}(f) \) and by the definition: \( c \circ f = \tilde{\alpha}(f) \). A similar argument exhibits equality between components of natural transformations on less than three object categories. ■

**Theorem 2.3 (\( C \)-enriched 2-powers):** Let \( (I, \otimes, C) \) be a complete and cocomplete symmetric monoidal closed category. There is a Yoneda bitriangle:

\[
\eta : Y \dashv (J : \mathbf{Cat}_S(C) \to \mathbf{Dist}(C), P : \mathbf{Dist}(C) \to \mathbf{Cat}(C))
\]
where $\mathbf{Cat}_2(\mathcal{C})$ is the 2-category of small $\mathcal{C}$-enriched categories, $\mathbf{Cat}(\mathcal{C})$ is the 2-category of all (i.e. locally small) $\mathcal{C}$-enriched categories, $\mathbf{Dist}(\mathcal{C})$ is the (weak) 2-category of $\mathcal{C}$-internal distributors between small categories, and $J, Y$ are the canonical embeddings. Pseudofunctor:

$$P: \mathbf{Dist}(\mathcal{C}) \rightarrow \mathbf{Cat}(\mathcal{C})$$

is given by:

$$P(A) = \mathcal{C}^{A_{op}}$$

$$P(A \overset{e}{\to} B) = \text{Lan}_{yA}(F)$$

where $yA: A \rightarrow \mathcal{C}^{A_{op}}$ is the enriched Yoneda functor.

**Proof:** Since Kan extensions are pseudofunctorial, $P$ is a pseudofunctor $\mathbf{Dist}(\mathcal{C}) \longrightarrow \mathbf{Cat}^{\mathcal{C}_{op}}$. By definition of $\mathbf{Dist}(\mathcal{C})$ there is an equivalence of categories:

$$\text{hom}_{\mathbf{Dist}(\mathcal{C})}(A, B) \cong \text{hom}_{\mathbf{Cat}(\mathcal{C})}(A, \mathcal{C}^{B_{op}})$$

By Lemma 2.2 category $\mathbf{Cat}_2(\mathcal{C})$ is a 2-dense subcategory of $\mathbf{Cat}(\mathcal{C})$; therefore $P$ is a pointwise left Kan extension of $Y$ along $G$.

It should be noted that proarrow equipments in the above examples are canonically determined by the 2-categories of internal and enriched categories respectively — in fact the categories of distributors are equivalent to the (weak) 2-categories of codiscrete cofibrant spans $\mathbf{Span}_{\mathbf{Cat}}$ in these categories. One can seek for a characterisation of a 2-topos along this line, but we leave it for a careful reader, as it is mostly irrelevant for our considerations.

### III. POWER SEMANTICS

If $\models S \rightarrow M \subseteq M \times S$ is a binary relation between two sets: $M$, which is thought of as a set of models, and $S$, which is thought of as a set of syntactic elements (sentences), then we have for free a Boolean semantics for propositional connectives formed over set $S$:

- $M \models \top$ iff $\top$  
- $M \models \bot$ iff $\bot$  
- $M \models x \land y$ iff $M \models x \land M \models y$  
- $M \models x \lor y$ iff $M \models x \lor M \models y$  
- $M \models x \rightarrow y$ iff $M \models x \rightarrow M \models y$

More generally, in any topos with a subobject classifier $\Omega$, a relation $\models: S \rightarrow M$ corresponds to a morphism $\nu: S \rightarrow \Omega^M$. Since for every object $M$ the power object $\Omega^M$ inherits an internal Heyting algebra structure from $\Omega$, we may give the valuation semantics for propositional connectives in $S$ via the composition:

$$\nu(x \land y) \models \nu(x) \land \nu(y)$$

$$\nu(x \lor y) \models \nu(x) \lor \nu(y)$$

$$\nu(x \rightarrow y) \models \nu(x) \rightarrow \nu(y)$$

where $x, y \in S$ are generalised elements. The above should be read as follows — given any generalised elements $X \rightarrow_{x,y} S$ there is a diagram:

$$\begin{array}{ccc}
X & \xrightarrow{x} & S \\
\downarrow & & \downarrow \nu \\
\uparrow & & \uparrow \leftarrow \Omega^M
\end{array}$$

then the semantics of meta-formula “$x \land y$” is:

$$\land \circ (\nu \circ x \land \nu \circ y)$$

where $\Omega^M \times \Omega^M \xrightarrow{\land} \Omega^M$ is the internal conjunction morphism in internal Heyting algebra $\Omega^M$; similarly for the other connectives.

**Example 3.1 (Free propositional semantics):** Let us start with a set $\mathbf{Var}$ and the equality relation $\models: \mathbf{Var} \rightarrow \mathbf{Var}$. Since every set is isomorphic to a coproduct on singletons, all generalised elements are recoverable from global elements. Therefore, we may restrict our semantics to global elements only. For every pair of elements $x, y \in \mathbf{Var}$ the free semantics for the meta-conjunction $x \land y$ is $\nu(x) \land \nu(y) = \nu(x \land (y = v))$, and similarly for other connectives. Observe, that this gives semantics for a pair $x, y \in \mathbf{Var}$ interpreted as conjunction $x \land y$, without saying what exactly $x \land y$ is. If one is not comfortable with such semantics, then one may “materialize” elements by forming an initial algebra. Formally, for a given set $\mathbf{Var}$ let us define an endofunctor on Set:

$$F(X) = (X \times X) \sqcup (X \times X) \sqcup (X \times \bot) \sqcup \bot \sqcup \bot$$

and $\text{Prop}_{\mathbf{Var}}$ as the initial algebra for $F(X) \sqcup \mathbf{Var}$. Now, the free semantics of $\models: \mathbf{Var} \rightarrow \mathbf{Var}$ may be extended to the semantics for $\text{Prop}_{\mathbf{Var}}$ via the unique morphism from the initial algebra to the algebra $s = [\land, \lor, \rightarrow, \bot, \top, \models]$: $\nu: \text{Prop}_{\mathbf{Var}} \rightarrow \mathbf{Var}$.

Much more is true. Not only does the power object $\Omega^M$ have all propositional connectives, in a sense, which we make precise in this section, $\Omega^M$ has all possible connectives.

**Example 3.2 (Relational semantics in Set):** Let $r \subseteq \mathbf{M} \times M \times M$ be a ternary relation on a set $M$. Then there is a corresponding binary operation $\otimes_r$ on $\Omega^M$ defined as follows:

$$f \otimes_r g = \lambda x \mapsto \exists a, b \in \mathbf{M} (f(a) \land g(b) \land r(x, a, b))$$

Moreover, $r$ has “exponentiations” on each of its coordinates. They are given by the following formulae:

$$f \circ_L g = \lambda a \mapsto \forall x, b \in \mathbf{M} (f(b) \land r(x, a, b) \rightarrow g(x))$$

$$f \circ_R g = \lambda b \mapsto \forall x, a \in \mathbf{M} (f(a) \land r(x, a, b) \rightarrow g(x))$$

We get the usual propositional connectives by considering relations associated to the unique comonoid structure $(\top: \mathbf{M} \rightarrow 1, \Delta: \mathbf{M} \rightarrow \mathbf{M} \times \mathbf{M})$ in a cartesian closed category $\mathbf{Set}$.

One may recognise in the above example the concept of ternary frame semantics for substructural logics [19]. The
crucial point however, is that such defined semantics have 2-dimensional analogues. The next example was the subject of Brain Day’s thesis [21].

**Example 3.3 (Day convolution):** Let \( \langle C, \otimes, I \rangle \) be a complete and cocomplete monoidal closed category. Suppose \( M: A \otimes A \to A \) is a \( C \)-enriched distributor. The convolution of \( M \) is a functor \( \otimes_M: C^{op} \otimes C^{op} \to C^{op} \) defined via coend:

\[
(F \otimes_M G)(-) = \int_{B,C \in A} F(B) \otimes G(C) \otimes M(-, B, C)
\]

If \( \langle A, M: A \otimes A \to A, J: A^{op} \to C \rangle \) is a \( C \)-monoidal category (i.e. a weak monoid in a (weak) 2-category of \( C \)-enriched distributors). The induced operation on \( C^{op} \) via the convolution yields a monoidal structure \( \langle C^{op}, \otimes_M, J \rangle \). First observe that \( J \) is the right unit of \( \otimes_M \):

\[
(F \otimes_M J)(-) = \int_{B,C \in A} F(B) \otimes J(C) \otimes M(-, B, C) \\
\approx \int_{B \in A} F(B) \otimes \text{hom}(-, B) \\
\approx F(-)
\]

where \( \int_{C \in A} J(C) \otimes M(-, B, C) \approx \text{hom}(-, B) \) because \( J \) is the promonoidal right unit of \( M \). Similarly, \( J \) is the left unit of \( \otimes_M \). If the promonoidal structure on \( A \) is induced by a monoidal structure — i.e. if:

\[
M(-, B, C) = \text{hom}(-, B \otimes_M C)
\]

then this structure is preserved by the Yoneda embedding — there is a natural isomorphism:

\[
\text{hom}(-, X) \otimes_M \text{hom}(-, Y) \approx M(-, X, Y)
\]

By definition \( \text{hom}(-, X) \otimes_M \text{hom}(-, Y) = \int_{B,C \in C} \text{hom}(B, X) \otimes \text{hom}(C, Y) \otimes M(-, B, C) \) which via Yoneda reduction is isomorphic to \( M(-, X, Y) \).

Brain Day showed more — every monoidal structure induced via convolution is a (biclosed) monoidal structure. The left linear exponent is defined by:

\[
(F \to^L_M G)(-) = \int_{A,C \in C} G(A)^{F(C) \otimes M(A, -, C)}
\]

and the right linear exponent by:

\[
(F \to^R_M G)(-) = \int_{A,B \in C} G(A)^{F(B) \otimes M(A, B, -)}
\]

Indeed, we have to show that:

\[
\text{hom}(H, F \to^L_M G) \approx \text{hom}(H \otimes_M F, G)
\]

Unwinding the right hand side, we get:

\[
\text{hom}(H \otimes_M F, G) = \\
= \text{hom}(\int_{B,C \in C} H(B) \otimes F(C) \otimes M(-, B, C), G) \approx \\
\approx \int_{B,C \in C} \text{hom}(H(B) \otimes F(C) \otimes M(-, B, C), G) \approx \\
\approx \int_{A,B,C \in C} G(A)^{H(B) \otimes F(C) \otimes M(A, B, C)} \approx \\
\approx \int_{A,B,C \in C} \text{hom}(H, G(A)^{F(C) \otimes M(A, B, C)}) \approx \\
\approx \text{hom}(H, \int_{A,C \in C} G(A)^{F(C) \otimes M(A, -, C)}) \approx \\
\approx \text{hom}(H, F \to^L_M G)
\]

and similarly for the other variable.

We show that a similar phenomenon occurs for internal categories. In [24] Brain Day and Ross Street defined a notion of convolution within a monoidal (weak) 2-category. For a reason that shall become clear in a moment, we are willing to call it “virtual convolution”. Here is the definition. Let:

\( \langle A, \delta: A \to A \otimes A, \epsilon: A \to I \rangle \)

be a weak comonoid, and:

\( \langle B, \mu: B \otimes B \to B, \eta: I \to B \rangle \)

be a weak monoid in a monoidal (weak) 2-category with tensor \( \otimes \) and unit \( I \), then:

\( \langle \text{hom}(A, B), \star, i \rangle \)

is a monoidal category by:

\[
f \star g = \mu \circ (f \otimes g) \circ \delta
\]

\[
i = \eta \circ \epsilon
\]

So the “virtual convolution” structure exists “virtually” — on hom-categories. If a monoidal 2-category admits all right Kan liftings, then the induced monoidal category \( \langle \text{hom}(I, B), \star, i \rangle \) for trivial comonoid on \( I \) is monoidal (bi)closed by:

\[
f \overset{L}{\to} h = \text{Rift}_{\mu \circ (f \otimes i)}(h)
\]

\[
f \overset{R}{\to} h = \text{Rift}_{\mu \circ (id \otimes f)}(h)
\]

Taking for the monoidal 2-category the category of distributors, we obtain the well-known formula for convolution. However, in the general setting, such induced structure is far weaker than one would wish to have — for example in the category of distributors enriched over a monoidal category \( C \)
the induced convolution instead of giving a monoidal structure on the category of enriched presheaves:

\[ \mathcal{C}^{B^{op}} \]

merely gives a monoidal structure on the underlying (Set-enriched) category.\(^{13}\)

\[ \hom(I, \mathcal{C}^{B^{op}}) \]

The solution is to find a way to “materialize” the “virtual convolution”. Here is a materialisation for internal categories.

**Theorem 3.1 (Internal convolution):** Let \( \mathcal{C} \) be a locally cartesian closed category with finite coproducts. For every \( \mathcal{C} \)-internal distributor \( \mu : A \times A \to A \) there exists a (bi)closed magma on \( \text{fam}(\mathcal{C})^{\text{fam}(A)^{op}} \). Furthermore, if \( \mu : A \times A \to A \) together with \( \eta : 1 \to A \) is a weak (symmetric) monoid, then the induced magma is weak (symmetric) monoidal.

**Proof:** Since \( \mathcal{C} \) is locally cartesian closed, every existing colimit in \( \mathcal{C} \) is stable under pullbacks. In particular, coequalisers are stable under pullbacks, and we may form the (weak) 2-category of \( \mathcal{C} \)-internal distributors in the usual tensor-like manner.\(^{18}\) Moreover, local cartesian closedness ensures that the category of distributors admits all right Kan liftings.\(^{20}\) We have to show that given a monoidal structure

\[ \langle A, \mu : A \times A \to A, \eta : 1 \to A \rangle \]

there is a corresponding monoidal (bi)closed structure on:

\[ \text{fam}(\mathcal{C})^{\text{fam}(A)^{op}} \]

which just means,\(^{22}\) that each fibre of \( \text{fam}(\mathcal{C})^{\text{fam}(A)^{op}} \) is a monoidal closed category and reindexing functors preserve these monoidal structures. For \( K \in \mathcal{C} \) interpreted as a discrete \( \mathcal{C} \)-internal category, there are isomorphisms:

\[ \begin{align*}
\text{fam}(\mathcal{C})^{\text{fam}(A)^{op}}(K) & \approx \hom(\hom(-, K), \text{fam}(\mathcal{C})^{\text{fam}(A)^{op}}) \\
& \approx \hom(1, \text{fam}(\mathcal{C})^{\text{hom}(-, K) \times \text{fam}(A)^{op}}) \\
& = \hom_{\text{fam}(\mathcal{C})}(1, K \times A)
\end{align*} \]

where the first isomorphism is the fibre Yoneda lemma, and the second is induced by cartesian closedness of \( \text{Cat}^{\mathcal{C}^{op}} \) and the fact that \( K = K^{op} \) for discrete internal category \( K \).

Since \( K \) has a trivial promonoidal structure:

\[ \langle K, K \times K \xrightarrow{\Delta^*} K, 1 \xrightarrow{\eta^*} K \rangle \]

we obtain a “product” promonoidal structure on \( K \times A \):

\[ \begin{array}{c}
K \times A \xrightarrow{\Delta^* \times \eta^*} K \times A \\
1 \xrightarrow{(\eta^*)^*} K \times A
\end{array} \]

In more details, because \( \mathcal{C} \) is cartesian, every object \( K \in \mathcal{C} \) carries a unique comonoid structure:

\[ \begin{align*}
K & \xrightarrow{\Delta} K \times K \\
K & \xrightarrow{1} 1
\end{align*} \]

\(^{12}\)There is a work-around for this issue in the context of enriched categories, as suggested in the paper, but the general weakness of “virtual convolution” is obvious.

which has a promonoidal right adjoint structure \( \langle \Delta^*, \eta^* \rangle \) in the category of internal distributors. The product of the above two promonoidal structures is given by the usual cartesian product of internal categories (note, it is not a product in the category of internal distributors) followed by the internal product functor \( \text{fam}(\mathcal{C}) \times \text{fam}(\mathcal{C})^{\text{prod}} \to \text{fam}(\mathcal{C}) \).

Then, by “virtual convolution” there is a monoidal (bi)closed structure on \( \hom_{\text{fam}(\mathcal{C})}(1, K \times A) \). Therefore each fibre \( \text{fam}(\mathcal{C})^{\text{fam}(A)^{op}}(K) \) is a monoidal (bi)closed category. In a cartesian category \( \text{cat}(\mathcal{C}) \) every morphism is a homomorphism of comonoids, and so it is also a homomorphism of monoids obtained by taking right adjoints to the comonoid structures in \( \text{Dist}(\mathcal{C}) \). Therefore, every reindexing morphism, being the product of a homomorphism and and identity, preserves the convolution structure.

Let us work out the concept of internal Day convolution in case \( \mathcal{C} = \text{Set} \), and see that it agrees with the usual formula for convolution.

**Example 3.4 (Set-internal convolution):** The split family (or more accurately, the indexed functor corresponding to the family fibration) for a locally small category \( A \):

\[ \text{fam}(A) : \text{Set}^{op} \to \text{Cat} \]

is defined as follows:

\[ \text{fam}(A)(K \in \text{Set}) = A^K \]

\[ \text{fam}(A)(K \xrightarrow{f} L) = A^L \xrightarrow{(-)^{op} f} A^K \]

where \( K, L \) are sets and \( K \xrightarrow{f} L \) is a function between sets. One may think of category \( A^K \) as of the category of \( K \)-indexed tuples of objects and morphisms from \( A \). Given any monoidal structure on a small category

\[ \langle A, \otimes : A \times A \to A, I : 1 \to A \rangle \]

the usual notion of convolution induces a monoidal structure on \( \text{Set}^{A^{op}} \):

\[ \langle F, G \rangle \mapsto F \otimes G = \int_{B,C \in A} F(B) \times G(C) \times \hom(-, B \otimes C) \]

The split fibration:

\[ \text{fam}(\text{Set})^{\text{fam}(A)^{op}} : \text{Set}^{op} \to \text{Cat} \]

may be characterised as follows:

\[ \begin{align*}
\text{fam}(\text{Set})^{\text{fam}(A)^{op}}(K \in \text{Set}) & = \text{Set}^{A^{op} \times K} \\
\text{fam}(\text{Set})^{\text{fam}(A)^{op}}(K \xrightarrow{f} L) & = \text{Set}^{A^{op} \times L \xrightarrow{(-)^{op} \circ (id \times f)} \text{Set}^{A^{op} \times K}}
\end{align*} \]

Since \( \text{Set}^{A^{op} \times K} \approx \left( \text{Set}^{A^{op}} \right)^K \) we may think of \( \text{Set}^{A^{op} \times K} \) as of \( K \)-indexed tuples of functors \( A^{op} \to \text{Set} \). In fact:

\[ \text{fam}(\text{Set})^{\text{fam}(A)^{op}} \approx \text{fam}(\text{Set}^{A^{op}}) \]

It is natural then to extend the monoidal structure induced on \( \text{Set}^{A^{op}} \) pointwise to \( (\text{Set}^{A^{op}})^{K} \):

\[ (F \otimes G)(k) = \int_{B,C \in A} F(k)(B) \times G(k)(C) \times \hom(-, B \otimes C) \]
where $k \in K$. On the other hand, using the internal formula for convolution, we get (up to a permutation of arguments):

$$
\int_{B \otimes C} \frac{F(B, k) \times G(C, k) \times \text{hom}(\Delta(k), (\beta, \gamma)) \times \text{hom}(-, B \otimes C) - \text{hom}(-, B \otimes C)}{B \otimes C, \beta, \gamma}.
$$

where the first equivalence is the definition of a diagonal $\Delta$, and the second one is by “Yoneda reduction” applied twice.

Note that the local cartesian closedness of the ambient category $C$ was crucial for the proof. There is always the trivial (cartesian) monoidal structure on the terminal category $1$ internal to $C$, but if $C$ is not locally cartesian closed than its fundamental fibration $\text{fan}(C) \approx \text{fan}(C)^1$ is not a cartesian closed fibration.

There are various possibilities to define universes that induce free semantics. Here is the weakest one.

**Definition 3.1 (Power semantics universe):** Let $\eta: \mathbf{Y} \triangleright (F, G)$ be a Yoneda bitriality $\mathbf{Y}: \mathbb{A} \to \mathbb{A}$, $F: \mathbb{A} \to \mathbb{B}$, $G: \mathbb{B} \to \mathbb{A}$, where $\mathbb{B}$ is a finitely cocomplete monoidal 2-category, and $\mathbb{A}$ has finite limits and admits a notion of discreteness, and suppose that $G$ maps magmas from $\mathbb{B}$ to internally (bi)closed magmas in $\mathbb{A}$. We shall call the triangle $\eta: \mathbf{Y} \triangleright (F, G)$ a power semantics universe if for any $V \in \mathbb{A}$ the 2-functor:

$$
F_V(X) \to Y(V) \sqcup (X \times X) \sqcup (|X| \times X) \sqcup (|X| \times X)
$$

has a (pseudo)-initial 2-algebra $\text{Lambek}_V$.

If $\eta: \mathbf{Y} \triangleright (F, G)$ is a power semantics universe, then for every magma $R: M \otimes M \to M$ and every $\models: V \to M$ in $\mathbb{B}$, the free semantics of $V$ by $R$ is defined to be the the unique morphism $\text{Lambek}_V \to P(M)$ from the (pseudo)-initial algebra $\text{Lambek}_V$ to the algebra $Y(V) \sqcup (P(M) \times P(M)) \sqcup ([P(M) \times P(M)] \sqcup [P(M) \times P(M)])$.

$$
\models = \begin{cases} 0 & \text{if } \phi \land \psi \models \phi \land \psi \\ 1 & \text{if } \phi \lor \psi \models \phi \lor \psi
\end{cases}
$$

The compatibility condition on variables implies compatibility condition on all formulae, so every Kripke structure gives rise to a logical system $\vdash: \langle S, \models \rangle^{op} \to \text{Prop}_V$ where $\langle S, \models \rangle^{op} = \{S, \subseteq\}$ is a degenerated category, and $\text{Prop}_V$ is the category induced by the logical consequence of $\vdash$.

Kripke structures may be rediscovered as power semantics for trivial comonoidal structure in the power semantics universe of 2-enriched categories. A poset $\leq: S \times S$ is exactly a 2-enriched category $S$. Moreover, $S$ has the trivial comonoidal structure $\Delta: S \to S \times S$, which induces a promonoidal structure $\Delta^*: S \times S \to S$.

Given a “forcing” relation on variables $\models V \subseteq S \times V$ that satisfies compatibility condition (i.e. is a 2-enriched distributor $\models V: V \to S^{op}$), there is a unique semantic homomorphism $\text{Lambek}_V \to 2^S$ induced by initiality of $\text{Lambek}_V$ and algebraic structure $(\models V, \times, \Rightarrow, \models)$. Given a “forcing” relation on variables $\models V \subseteq S \times V$ that satisfies compatibility condition (i.e. is a 2-enriched distributor $\models V: V \to S^{op}$), there is a unique semantic homomorphism $\text{Lambek}_V \to 2^S$ induced by initiality of $\text{Lambek}_V$ and algebraic structure $(\models V, \times, \Rightarrow, \models)$.

$$
\models \subseteq \text{Mod} \times \text{Sen}
$$

thought of as a satisfaction relation between a set of models $\text{Mod}$ and a set of sentences $\text{Sen}$. By transposition, relation $\models$ yields the “theory” function $\text{th}: \text{Mod} \to 2^{\text{Sen}}$, where $2^{\text{Sen}}$ is the poset of function $\text{Sen} \to 2$, or equivalently the poset of subsets of $\text{Sen}$.

Since “power” posets $2^{\text{Sen}}$ are internally complete in the 2-category $\text{Cat}(2)$, the stable density product of $\text{th}: \text{Mod} \to 2^{\text{Sen}}$ exists:

$$
\text{Mod} \xrightarrow{\text{th}} 2^{\text{Sen}}
$$

and is given by the 2-enriched end [14]:

$$
\text{Th}((\Gamma)) = \int_{M \in \text{Mod}} \text{th}(M)(\psi)^{\text{hom}(\Gamma, \text{th}(M)(\psi))}
$$
We are interested in values of $T_{th}$ on representables $\text{hom}(\cdot, \phi)$:

$$T_{th}(\text{hom}(\cdot, \phi))(\psi) \approx \int_{M \in \text{Mod}} \text{th}(M)(\psi)^{\text{th}(M)(\phi)}$$

where the isomorphism follows from the Yoneda reduction. Observe that the exponent $\text{th}(M)(\psi)^{\text{th}(M)(\phi)}$ in a 2-enriched world may be expressed by the implication “$\text{th}(M)(\phi) \Rightarrow \text{th}(M)(\psi)$”, or just “$M \models \phi \Rightarrow M \models \psi$”, where every component of the implication is interpreted as a logical value in the 2-valued Boolean algebra. Furthermore, ends turn into universal quantifiers, when we move to 2-enriched world. So, the end $\int_{M \in \text{Mod}} \text{th}(M)(\psi)^{\text{th}(M)(\phi)}$ is equivalent to the meta formula $\forall_{M \in \text{Mod}}M \models \phi \Rightarrow M \models \psi$, which is just the definition of logical consequence:

$$\phi \models_{\text{Sen}} \psi \iff \forall_{M \in \text{Mod}}M \models \phi \Rightarrow M \models \psi$$

The general case, where $\Gamma$ is not necessary representable, is similar:

$$T_{th}(\Gamma)(\psi) \iff \forall_{M \in \text{Mod}}(\forall_{\phi \in \Gamma}M \models \phi) \Rightarrow M \models \psi$$

Therefore, the density product of a satisfaction relation re-assembles the semantic consequence relation. In this example the satisfaction relation $\models \subseteq \text{Mod} \times \text{Sen}$ induces semantic consequence relation $\models_{\text{Sen}} \subseteq \text{Sen} \times \text{Sen}$ via the density product. We have also seen in [1], that density products are always equipped with a monad structure. In fact $\models_{\text{Sen}} \subseteq \text{Sen} \times \text{Sen}$ thought of as a 2-enriched distributor acquires the monad structure from the density product. Because the 2-category of 2-enriched distributors is cocomplete, this monad has a resolution as a Kleisli object $\text{Sen}_K$. In more details, a Kleisli object in the 2-category of 2-enriched distributors may be described by generalised Grothendieck construction [15] — objects in $\text{Sen}_K$ are the same as in $\text{Sen}$, whereas morphisms in $\text{Sen}_K$ are defined by:

$$\text{hom}_{\text{Sen}_K}(\phi, \psi) = \phi \models_{\text{Sen}} \psi$$

Identities and compositions are induced by monad’s unit and multiplication respectively. Then by definition of density product the relation $\models \subseteq \text{Mod} \times \text{Sen}$ extends to the relation:

$$\models \subseteq \text{Mod} \times \text{Sen}_K$$

In an essentially the same manner one may extend the forcing relation $\models \subseteq S \times \text{Prop}_V$ of the above Example [3,5] to the relation:

$$\models \subseteq S \times \text{Prop}_V$$

where $\text{Prop}_V$ is the Kleisli resolution for the density product on the forcing relation.

The next example generalises semantics in Kripke structures.

**Example 3.7 (Ternary frame):** A ternary frame [19] is a pair $\langle X, R \rangle$, where $X$ is a set, and $R: X \times X \times X \to 2$ is a ternary relation on $X$. Ternary frames were proposed as generalisations of Kripke structures to model substructural logics. Let $\Sigma_{\text{Lambek}}$ be the signature consisting of three binary symbols $\otimes$, $\Rightarrow$, and $\to$. The semantics for Lambek syntax in ternary frame $(X, \mathcal{R})$ is a relation $\models \subseteq X \times \text{Lambek}_V$ satisfying:

- $x \models \phi \otimes \psi$ iff $\exists y, z \in X \, y \models \phi \land z \models \psi \land \mathcal{R}(x, y, z)$
- $y \models \phi \to \psi$ iff $\forall z \in X \, z \models \phi \land \mathcal{R}(x, y, z) \Rightarrow x \models \psi$
- $z \models \phi \Rightarrow \psi$ iff $\forall x \in X \, y \models \phi \land \mathcal{R}(x, y, z) \Rightarrow x \models \psi$

Connectives are defined according to the nonassociative Lambek calculus defined on $2^X$ via the convolution of $\mathcal{R}$.

**IV. CONCLUSIONS**

In the paper we defined a general 2-categorical setting for extensional calculi and shows how intensional and extensional calculi can be combined to form logical systems. We provided a notion of a generalised adjunction, which we call a Yoneda triangle, and showed that many concepts in category theory may be characterised as (higher) Yoneda triangles. We showed that the natural setting for convolution is a Yoneda bitriple, and prove Day convolution theorem for internal categories. Such Yoneda bitriples admitting convolutions provide a semantic universe, where objects get their semantics (almost) for free — this includes the usual semantics for propositional calculus, Kripke semantics for intuitionistic calculi and ternary frame semantics for substructural calculi including Lambek’s lambda calculi, relevance and linear logics.

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