We study the statistics of single particle Lagrangian velocity in a shell model of turbulence. We show that the small scale velocity fluctuations are intermittent, with scaling exponents connected to the Eulerian structure function scaling exponents. The observed reduced scaling range is interpreted as a manifestation of the intermediate dissipative range, as it disappears in a Gaussian model of turbulence.

In the recent years there has been a great improvement in the experimental investigation of transport and mixing by particles initially at position $x_0$. This is natural description for studying transport and mixing of neutrally advected substances in turbulent flows. The remarkable coincidence that the fractal dimension $D(h)$ recovers in a more compact form the prediction $\xi(p) = p/2$ but one expects corrections to dimensional scaling in presence of intermittency.

A generalization of (1) which takes into account intermittency corrections can be easily developed within the multifractal model of turbulence (4). The dimensional argument is repeated for the local scaling exponent $h$, giving $\delta v(t) \sim V(t/t_0)^{h/(1-h)}$. Integrating over the $h$ distribution one ends with the prediction

$$\langle \delta v(t)^p \rangle \sim V^p \int dh \left( \frac{1}{\tau_0} \right)^{p h - D(h) + 3}$$

(3)

where, in the limit $t/\tau_0 \rightarrow 0$, the integral can be estimated by a steepest descent argument giving finally

$$\xi(p) = \min_h \left[ \frac{p h - D(h) + 3}{1 - h} \right]$$

(4)

The fractal dimension $D(h)$ is related to the Eulerian structure function scaling exponents $\xi(p)$ by the Legendre transform $\xi(p) = \min_h [ph - D(h) + 3]$. The standard inequality in the multifractal model (following from the exact result $\xi(3) = 1$, $D(h) \leq 3h + 2$ implies for (4) that even in presence of intermittency $\xi(2) = 1$. This is a direct consequence of the fact that energy dissipation enters in (4) at the first power. Our expression for scaling exponents (4) recovers in a more compact form the prediction obtained on the basis of an “ergodic hypothesis” of the statistics of energy dissipation (3).

Recent experimental results [3] have shown that indeed Lagrangian velocity fluctuations are intermittent and display anomalous scaling exponents, as predicted by the above argument. Despite the relative high Reynolds number of the experiments, a true temporal scaling range is not observed. Thus the estimation of the scaling exponents in (4) can be done only relatively to a reference moment (the so-called ESS procedure [4]).

In this work we use a dynamical toy model of turbulence for investigating scaling (4) and prediction (4) at very high Reynolds numbers. The model is a simple shell model of turbulence (5) in which the velocity fluctuations of the eddies at the scale $\ell_n = L 2^{-n}$ are represented by a single variable $u_n$ ($n = 1, 2, ..., N$). Only local interaction among shells are represented and therefore no sweeping effects are present. In this sense shell
models are dynamical models of velocity fluctuations in a Lagrangian framework, and have been already used for the study of turbulent dispersion [1]. The equation of motion for the complex shell variable \( u_n \) is [2,3]

\[
\frac{du_n}{dt} = ik_n \left( u_{n+2}u^*_{n+1} - \frac{\delta}{2} u_{n+1}u^*_{n-1} + \frac{1-\delta}{4} u_{n-1}u_{n-2} \right) - \nu k^2 u_n + f_n
\]

(5)

where \( k_n = \ell_n^{-1} \) and \( f_n \) is a deterministic forcing acting on the first two shells only. Shell model [3] is characterized by a chaotic dynamics with a statistically steady state with a constant flux of energy from large to small scales. The fluctuations generate by chaoticity induce a breaking of the global scaling invariance and corrections to the Kolmogorov exponents for the structure functions close to the experimental values [4].

Lagrangian velocity in the shell model framework can be represented as the superposition of the contributions of all the different eddies. Let us thus define

\[
v(t) \equiv \sum_{n=1}^{N} Re(u_n)
\]

(6)

where we have taken, rather arbitrarily, only the real part of the shell variables with unit coefficient. From the definition of the shell model, there is not a precise recipe for reconstructing the Lagrangian velocity. More in general, one could think of a representation in which shell variables are multiplied by an appropriate wavelet functions. Of course numerical prefactor, such as \( C_0 \) in [1] will depend on the wavelet basis. Nevertheless one expects that different choices should not affect Lagrangian scaling exponents \( \xi(p) \) which are determined by the dynamical properties of the model.

Previous studies of multi-time correlations in shell models of turbulence have demonstrated the existence of a set of correlation times compatible with the multifractal picture of the turbulent cascade [5]. This is an indication that, as we will see, Lagrangian velocity defined as [1] will be affected by intermittency.

Very long and accurate numerical simulations of the shell model [3] with \( N = 24 \) shells and \( \delta = 1/2 \) have been performed. The energy is injected at a constant flux \( \varepsilon = 0.01 \) is the first 2 shells and is removed at the smallest shells by viscosity \( \nu = 10^{-7} \). With these parameters, our simulations correspond to a Reynolds number \( Re \approx 10^8 \). For each realization, Lagrangian structure functions are computed from the Lagrangian velocity [1] up to the large-scale time \( \tau_0 \). Average is then taken over \( 10^5 \) independent realizations.

In Fig. 1 we plot the set of numerically determined Eulerian structure function scaling exponent \( \zeta(q) \) together with the fractal dimension \( D(\ell) \) reconstructed by means of the Legendre transform. We observe strong intermittency in velocity statistics with scaling exponent which deviates from Kolmogorov prediction. The scaling exponents are not universal with respect to the particular shell model. Model [4] gives a set of exponents which are a little more intermittent than, but not far from, the experimentally observed exponents [5]: \( \zeta(2) \approx 0.72, \zeta(4) \approx 1.25, \zeta(6) \approx 1.71 \). We thus expect that the values of \( \xi(p) \) obtained from [4] using the \( D(\ell) \) of Fig. 1 will be directly comparable with real experimental data.

Figure 2 shows the second-order Lagrangian structure function [4] as a function of time. The linear behavior is evident (see the inset) even if a long crossover from the ballistic scaling at short time \( \langle \delta v(t)^2 \rangle \sim t^2 \) is present. In spite of the very high Reynolds numbers achievable in the shell model, the extension of the temporal scaling [4] is still moderate. For a comparison with the available experimental data, in the inset we also plot the result obtained from two simulations at lower resolution, with \( Re \approx 2 \times 10^6 \) and \( Re \approx 10^5 \). In the latter case almost no scaling range is observable. Despite these limitations, we will see that high \( Re \) simulations allow the determination of the Lagrangian scaling exponents with good accuracy.

The long crossover in Fig. 2 can be understood in terms of intermediate dissipative range as a consequence of the fluctuating dissipative scale [6,7,8]. The smallest time at which one can expect scaling [4] is the Kolmogorov time \( \tau_\eta \approx \tau_0 Re^{-(1-h)/(1+h)} \) which fluctuates with \( h \). A demonstration of the effects induced by intermittency is given by considering a non-intermittent Gaussian model.

Setting \( f_n = \nu = 0 \), [4] becomes a conservative system with two conserved quantities which depends on the value of the \( \delta \) [9]. In statistical stationary condition, the model shows equipartition of the conserved quantities among the shell, in agreement with statistical mechanics prediction [6]. For \( \delta = 1 + 2^{-2/3} \) the equipartition state leads at small scales to Kolmogorov scaling \( \langle |u_n|^2 \rangle \sim \ell_n^{2/3} \) with Gaussian statistics. In Fig. 2 we plot the second-order Lagrangian structure function [4] for the Gaussian model. Both the ballistic and the diffusive scaling is clearly observable and the crossover is strongly reduced with respect to Fig. 1.

In Fig. 4 we plot the probability density functions of \( \delta v(t) \) computed at different \( t \) in the linear scaling range of Fig. 2 rescaled with their variances. The form of the pdf varies continuously from almost Gaussian at large time \( (t \sim \tau_0) \) to the development of stretched exponential tails at short times, similar to what observed in laboratory experiments [1]. Flatness \( F \) grows from Gaussian value \( F = 3 \) up to \( F \approx 20 \) at smallest times. This is an indication of Lagrangian intermittency, in the sense that the Lagrangian statistics cannot be described in term of a single scaling exponent.

In Fig. 5 we plot the set of Lagrangian scaling exponents \( \xi(p) \) obtained from a direct fit of temporal structure functions. The nonlinear behavior in \( p \) confirms the presence of Lagrangian intermittency already observed from the pdf. We present the result for moments up to \( p = 6 \) which approximatively corresponds, from [4], to Eulerian structure function of order \( q = 8 \). In this sense temporal structure functions are more intermittent.
Figure 5 shows that the agreement with the multifractal prediction (4) is very good up to the moment achievable with our statistics. What is even more remarkable is that our prediction is very close to experimentally determined exponents. For example we find $\xi(3) \approx 1.31$, $\xi(4) \approx 1.58$, $\xi(5) \approx 1.85$, while the experimental data give $\xi_{\text{exp}}(3) = 1.34 \pm 0.02$, $\xi_{\text{exp}}(4) = 1.56 \pm 0.06$ and $\xi_{\text{exp}}(5) = 1.8 \pm 0.2$.

In this work we have investigated the statistical properties of single particle Lagrangian velocity in fully developed turbulence. A prediction for intermittent scaling exponents of Lagrangian structure function is given within the multifractal framework. Very high Reynolds number simulations in shell model confirm the multifractal prediction, even if rather small scaling ranges are observed. In particular, our simulations show that at experimental Reynolds number presently available almost no scaling is observable. The reduction of the scaling range in Lagrangian statistics is interpreted as an effect of the intermediate dissipative range. A Gaussian, non-intermittent version of the shell model confirms this interpretation. An important consequence of our findings is that usual models of particle dispersion, based on stochastic model (16), are incorrect in a fundamental sense, and one should take into account the modifications due to intermittency.

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FIG. 1. Shell model Eulerian structure function scaling exponents $\zeta(q)$ determined from over the statistics of $10^5$ independent configurations. In the inset we plot the codimension $3 - D(h)$ as determined from numerically solving the Legendre transform.

FIG. 2. Second-order Lagrangian structure function $\langle \delta v(t)^2 \rangle$ as a function of time delay $t$ for the simulation at $Re = 10^8$. Continuous line is the ballistic behavior $t^2$ at short time. Dashed line represents the linear growth (1). Inset: $\langle \delta v(t)^2 \rangle$ compensated with the dimensional prediction $\varepsilon t$ at $Re = 10^6$ (continuous line), $Re = 2 \times 10^6$ (dashed line) and $Re = 10^5$ (dotted line).

FIG. 3. Second-order Lagrangian structure function $\langle \delta v(t)^2 \rangle$ as a function of time delay $t$ for the equilibrium Gaussian model. Continuous line is the ballistic behavior $t^2$ at short time. Dashed line represents the linear growth (1).

FIG. 4. Probability density functions of velocity differences $\delta v(t)$ normalized with the variance at time lags $t/\tau_0 = 0.002(\square), 0.01(\ast), 0.06(\times), 0.35(+)$. Continuous line represents a Gaussian. Inset: flatness $F = \langle \delta v(t)^4 \rangle / \langle \delta v(t)^2 \rangle^2$ as function of time and Gaussian value $F = 3$ (dashed line).

FIG. 5. Lagrangian structure function scaling exponents $\zeta(p)$ numerically determined by a best fit on (2). The line represents the multifractal prediction (4) with $D(h)$ obtained from Fig. 1.