AUTOMATIC BOUNDEDNESS OF ADJOINTABLE OPERATORS ON BARRELED VH-SPACES

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Abstract. We consider the space of adjointable operators on barreled VH (Vector Hilbert) spaces and show that such operators are automatically bounded. This generalizes the well known corresponding result for locally Hilbert $C^*$-modules. We pick a consequence of this result in the dilation theory of VH-spaces and show that, when barreled VH-spaces are considered, a certain boundedness condition for the existence of VH-space linearisations, equivalently, of reproducing kernel VH-spaces, is automatically satisfied.

Introduction

VE (Vector Euclidean) and VH (Vector Hilbert) spaces were introduced by Loynes in 1965, in [22] and [23]. These are generalisations of the notion of an inner product space, where, the "inner product" has its values in a certain ordered vector space, called "ordered $*$-space", and otherwise has the similar properties with the usual inner product, see Section II for precise definitions. He was motivated by stochastic processes [24] and obtained a generalisation of B. Sz. Nagy Dilation Theorem [38] for operator valued positive semidefinite maps on $*$-semigroups along with other results about the spectral theory of order bounded linear operators of VH-spaces, see [22] and [23]. The notion of VH-spaces were used in prediction theory [40], [41], as well as in dilation theory, [10], [11], [12], [13], [4], [5], [6] and some others.

VH-spaces can be considered as a rather general type of spaces and indeed, it is possible to consider VH-spaces over many different ordered $*$-spaces. This is the case of Hilbert modules over $C^*$-algebras, introduced in 1973 by Paschke in [30], following [19], and independently by Rieffel in [33]; more generally, Hilbert modules over locally $C^*$-algebras, introduced in 1985 by Mallios in [25] and later by Phillips in [32], also VH-spaces over $H^*$-algebras, considered by Saworotnow in [34], are all special cases of a VH-space.

Operator theory of VH-spaces has been considered by various authors. Loynes singled out and studied the $C^*$-algebra $B^*(\mathcal{H})$ of all order bounded adjointable linear operators on a VH-space $\mathcal{H}$ in [23]. In [13], a Stinespring type theorem was obtained for positive semidefinite kernels with values in $B^*(\mathcal{H})$, generalizing the Stinespring Theorem, [37] in the VH-space direction. Given the generality of VH-spaces, it is clear that more general classes of linear operators must also be given consideration, and in [9], [29], [5], [6] and some others, more general classes of continuous linear operators on VH-spaces were considered. Such operators found applications, in particular, in the dilation theory of invariant positive semidefinite

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kernels on a set under an action of a \(*\)-semigroup and valued in a topologically ordered \(*\)-space, see [6].

On the other hand, operator theory on locally Hilbert \(C^*\)-modules was considered in [32], [33], [18], more recently in [14] and probably many others. The techniques to study these operators usually come from the corresponding techniques of Hilbert \(C^*\)-modules, but the arguments are not always straightforward.

One of the well known results of locally Hilbert \(C^*\)-modules states that the class of adjointable operators \(L^*(\mathcal{H})\) on a locally Hilbert \(C^*\)-module \(\mathcal{H}\) consists of bounded operators and is a locally \(C^*\)-algebra with the locally convex Hausdorff topology given by the operator seminorms, see Subsection 1.4 for a review. In [1] it was shown that this result is responsible for controlling one of the boundedness conditions of a dilation theorem for the existence of linearisations, or equivalently, reproducing kernel spaces of operator valued positive semidefinite kernels invariant under action of \(*\)-semigroups. This dilation theorem leads to a Kasparov type theorem in the context of locally Hilbert \(C^*\)-modules, generalizing the Hilbert \(C^*\)-module theorem in [20], first obtained in [18], and proven in a different way in [5], using VH-space dilations and following the ideas in [27]. An impetus to write this article was to understand whether a similar result holds for adjointable operators of VH-spaces, or more generally, of topological VE-spaces, and if so, what conditions must be imposed on them. We show that barreledness is a sufficient condition. A related notion of \(m\)-barreledness on certain \(*\)-algebras was considered in [15].

From that point of view, Theorem 3.7 is the main result of this article. Main idea of the proof consists of passing to the quotient spaces by kernels of increasing \(*\)-seminorms and then employ the Uniform Boundedness Theorem on a certain family of vector valued functions on the quotient spaces. In addition, VH-space tools are used. The classical result above on locally Hilbert \(C^*\)-modules can be recovered by a variation of Theorem 3.7, see Corollary 3.8 and the discussion in the end of Section 3.

We now briefly describe the contents of this article. In Section 1 we recall definitions and some properties of topologically ordered \(*\)-spaces, VE-spaces, VH-spaces and their operators. We prove Lemma 1.3 which shows that we can always assume that the cone of a topologically ordered \(*\)-space is closed. We recall two Schwarz type inequalities in Lemmas 1.4 and 1.5 that will be used several times in the article. Subsection 1.4 briefly reviews locally \(C^*\)-algebras and locally Hilbert \(C^*\)-modules, mainly for convenience.

In Section 2 given two topological VE-spaces \(\mathcal{E}\) and \(\mathcal{F}\) over the same topologically ordered \(*\)-space \(Z\), we study the class \(L^*_c(\mathcal{E}, \mathcal{F})\) of continuous and continuously adjointable operators and \(L^*_b(\mathcal{E}, \mathcal{F})\) of bounded and adjointable operators. In particular, if \(\mathcal{E}\) is a VH-space, then \(L^*_b(\mathcal{E})\) is a locally \(C^*\)-algebra, see Corollary 2.5.

The main result of this article, Theorem 3.7, states that, an adjointable operator \(T: \mathcal{E} \to \mathcal{F}\) from a barreled topological VE-space \(\mathcal{E}\) to a topological VE-space \(\mathcal{F}\) over the same topologically ordered \(*\)-space \(Z\) is automatically bounded, hence continuous. Here boundedness is in a rather strong sense that becomes the usual boundedness of linear operators when normed VE-spaces are considered, see the definitions in Section 2. A series of corollaries of the main result are proven, showing that the behaviour of adjointable operators of barreled topological VE-spaces are in a certain sense close to the adjointable operators of locally Hilbert \(C^*\)-modules, and in the normed case, to the adjointable operators of Hilbert \(C^*\)-modules.

In the last section we turn to the dilation theory of VH-spaces and prove Theorem 4.2 as an application of Theorem 3.7, which asserts that, when a barreled VH-space \(\mathcal{H}\) is considered,
a boundedness condition (b2) from a characterization of the existence of invariant VH-space linearizations (equivalently, existence of reproducing kernel VH-spaces with \(*\)-representations) of positive semidefinite kernels valued in adjointable operators of \(\mathcal{H}\), see Theorem 4.1, is satisfied automatically. This provides another partial answer to a question raised in [5], asking when the condition (b2) in Theorem 4.1 is satisfied automatically.

1. Notation and Preliminary Results

1.1. Ordered \(*\)-Spaces. A complex vector space \(Z\) is called an ordered \(*\)-space, see e.g. [1] or [31] if:

(a1) \(Z\) has an involution \(*\), that is, a map \(Z \ni z \mapsto z^* \in Z\) that is conjugate linear \(((sx + ty)^* = \overline{sx}^* + t\overline{y}^*\) for all \(s, t \in \mathbb{C}\) and all \(x, y \in Z\) and involutive \(((z^*)^* = z\) for all \(z \in Z\).

(a2) In \(Z\) there is a convex cone \(Z_+ (sx + ty \in Z_+\) for all numbers \(s, t \geq 0\) and all \(x, y \in Z_+)\), that is strict \((Z_+ \cap -Z_+ = \{0\}\) and consisting of selfadjoint elements only \((z^* = z\) for all \(z \in Z_+)\). This cone is used to define a partial order in selfadjoint elements \(Z^h\) by: \(z_1 \geq z_2\) if \(z_1 - z_2 \in Z_+\).

Note that, the set of all selfadjoint elements \(Z^h\) is a real vector space.

The complex vector space \(Z\) is called a topologically ordered \(*\)-space if it is an ordered \(*\)-space and:

(a3) \(Z\) is a Hausdorff separated locally convex space.

(a4) The involution \(*\) is continuous with respect to this topology.

Condition (a4) is new compared to the definition in [5] and [6].

(a5) The cone \(Z_+\) is closed with respect to this topology.

(a6) The topology of \(Z\) is compatible with the partial ordering in the sense that there exists a base of the topology, linearly generated by a family of neighbourhoods \(\{N_j\}_{j \in J}\) of the origin, such that all of them are absolutely convex and solid, that is, whenever \(x \in N_j\) and \(0 \leq y \leq x\) then \(y \in N_j\).

It can be proven that axiom (a6) is equivalent with the following one, see [5]:

(a6') There exists a collection of seminorms \(\{p_j\}_{j \in J}\) defining the topology of \(Z\) that are increasing, that is, \(0 \leq x \leq y\) implies \(p_j(x) \leq p_j(y)\).

We denote the collection of all increasing continuous seminorms on \(Z\) by \(S(Z)\). \(Z\) is called an admissible space if, in addition to the axioms (a1)–(a6),

(a7) The topology on \(Z\) is complete.

If the topology of a topologically ordered \(*\)-space \(Z\) can be given by a single increasing norm, then we call \(Z\) a normed ordered \(*\)-space. If the topology on a normed ordered \(*\)-space \(Z\) is complete, then \(Z\) is called a Banach ordered \(*\)-space.

Remark 1.1. Notice that given two increasing seminorms \(p\) and \(q\) on a topologically ordered \(*\)-space \(Z\), the seminorms \(p + q\) and \(\alpha p\) for any constant \(\alpha \geq 0\) are also increasing. In addition, the maximum seminorm \(r(z) := \max\{p(z), q(z)\}\), \(z \in Z\) is increasing. In particular, the collection of all continuous and increasing seminorms \(S(Z)\) is closed under addition, multiplication with a positive scalar and the maximum.
Remark 1.2. Let us call a seminorm $p$ on a topologically ordered $*$-space a $*$-seminorm if $p(z^*) = p(z)$ for all $z \in Z$. It turns out that, the topology of a topologically ordered $*$-space can always be given by a family of continuous increasing $*$-seminorms. To see this, given a continuous increasing seminorm $q$, define $p(z) := \frac{1}{2}(q(z) + q(z^*))$ for any $z \in Z$. It is then easy to see that $p$ is a continuous $*$-seminorm. To show that it is increasing as well, let $0 \leq z_1 \leq z_2$; in particular, $z_1 = z_1^*$ and $z_2 = z_2^*$. Then

$$p(z_1) = \frac{1}{2}(q(z_1) + q(z_1^*)) = q(z_1) \leq q(z_2) = \frac{1}{2}(q(z_1) + q(z_2^*)) = p(z_2)$$

and $p$ is increasing.

Now given a collection of continuous increasing seminorms $\{q_j\}_{j \in J}$ defining the topology of $Z$ as in [(a5')], let $\{p_j\}_{j \in J}$ be the corresponding collection of continuous increasing $*$-seminorms obtained by the above formula. We show that they give the same locally convex topology: Since the involution is continuous, for any $q_j$, there exists $\{q_{ji}\}_{i=1}^n$ such that $q_j(z^*) \leq \sum_{i=1}^n q_{ji}(z)$ for all $z$. Therefore $p_j(z) = \frac{1}{2}(q_j(z) + q_j(z^*)) \leq q_j(z) + \sum_{i=1}^n q_{ji}(z)$. We also have $q_j(z) \leq 2p_j(z)$ clearly and the equivalence of the topologies is shown.

We denote the set of all continuous increasing $*$-seminorms by $S_*(Z)$. Similar to $S(Z)$, $S_*(Z)$ is closed under addition, multiplication with a positive scalar and the maximum.

The following lemma is basically Corollary 2.24 in [1]. It shows that, closedness of the cone can always be assumed, by passing to the closure if necessary.

Lemma 1.3. Assume that an ordered $*$-space $Z$ satisfies all axioms (a3)–(a6) except for (a5), that is, the cone $Z^+$ is not necessarily closed with respect to the specified topology. Then by passing to the closure $\overline{Z^+}$ of the cone $Z^+$ we obtain a topologically ordered $*$-space.

Proof. By passing to the closure $\overline{Z^+}$, we consider the space $Z$ with positive elements $\overline{Z^+}$. First we show that $(Z, \overline{Z^+})$ is an ordered $*$-space: It is clear that $\overline{Z^+}$ is closed under addition and multiplication with positive scalars.

In order to see that $\overline{Z^+}$ consists of selfadjoint elements, let $z \in \overline{Z^+}$. Then there exists a net $(z_i) \subseteq Z^+$, $i \in I$ such that $z_i \to z$. By continuity of $*$, $z_i^* \to z^*$. As $z_i^* = z_i$ for any $i \in I$, it follows that $z_i \to z^*$ and since the topology is Hausdorff, $z = z^*$.

We now show that $\overline{Z^+}$ is strict. Assume that there is $z \in \overline{Z^+}$ such that $-z \in \overline{Z^+}$ as well. Then there exist nets $(z_i) \subseteq Z^+$, $i \in I$ and $(w_j) \subseteq Z^+$, $j \in J$ such that $z_i \to z$ and $w_j \to -z$. Let $p \in S_*(Z)$ be arbitrary and let $\epsilon > 0$. Pick $i_0 \in I$ such that $p(z - z_i) < \epsilon/2$ for $i \geq i_0$ and also pick $j_0 \in J$ such that $p(w_j + z) < \epsilon/2$ for $j \geq j_0$. Observe that, by the increasing property of $p$,

$$0 \leq p(z_i) \leq p(z_i + w_j) = p(z_i - z + z + w_j) \leq p(z - z_i) + p(z + w_j) < \epsilon$$

from which it follows that $p(z_i) \to 0$. But we also have $p(z_i) \to p(z)$, so $p(z) = 0$. By the Hausdorff property and since $p$ was arbitrary, $z = 0$ and therefore $\overline{Z^+}$ is strict. This completes the proof that $(Z, \overline{Z^+})$ is an ordered $*$-space.

Finally we show that any $p \in S_*(Z)$ is increasing when $\overline{Z^+}$ is considered instead of $Z^+$ as well: Let $y, z \in \overline{Z^+}$ be arbitrary and pick nets $z_i \to z$ with $(z_i) \subseteq Z^+$, $i \in I$ and $y_j \to y$ with $(y_j) \subseteq Z^+$, $j \in J$. By increasing property of $p$, we have $p(z_i) \leq p(z_i + y_j)$ for all $i \in I$ and $j \in J$. By passing to limits, it follows that $p(z) \leq p(z + y)$ and the increasing property is shown. In particular, any $p \in S_*(Z)$ is an increasing $*$-seminorm and the proof is complete. \[\square\]
1.2. Vector Euclidean Spaces and Their Linear Operators. Given a complex linear space \( \mathcal{E} \) and an ordered \(*\)-space \( Z \), a \( Z\)-valued inner product or \( Z\)-gramian is, by definition, a mapping \( \mathcal{E} \times \mathcal{E} \ni (x, y) \mapsto [x, y] \in Z \) subject to the following properties:

\[
\begin{align*}
\text{(ve1)} & \quad [x, x] \geq 0 \text{ for all } x \in \mathcal{E}, \text{ and } [x, x] = 0 \text{ if and only if } x = 0. \\
\text{(ve2)} & \quad [x, y] = [y, x]^* \text{ for all } x, y \in \mathcal{E}. \\
\text{(ve3)} & \quad [x, ay + by] = a[x, y_1] + b[x, y_2] \text{ for all } a, b \in \mathbb{C} \text{ and all } x_1, x_2 \in \mathcal{E}.
\end{align*}
\]

A complex linear space \( \mathcal{E} \) onto which a \( Z\)-valued inner product \([\cdot, \cdot]\) is specified, for a certain ordered \(*\)-space \( Z \), is called a VE-space (Vector Euclidean space) over \( Z \).

In any VE-space \( \mathcal{E} \) over an ordered \(*\)-space \( Z \) the familiar polarisation formula

\[
(1.1) \quad 4[x, y] = \sum_{k=0}^{3} i^k[(x + i^k y, x + i^k y)], \quad x, y \in \mathcal{E},
\]

holds, which shows that the \( Z\)-valued inner product is perfectly defined by the \( Z\)-valued quadratic map \( \mathcal{E} \ni x \mapsto [x, x] \in Z \).

The concept of VE-spaces isomorphism is also naturally defined: this is just a linear bijection \( U : \mathcal{E} \to \mathcal{F} \), for two VE-spaces over the same ordered \(*\)-space \( Z \), which is isometric, that is, \([Ux, Uy]_\mathcal{F} = [x, y]_\mathcal{E} \) for all \( x, y \in \mathcal{E} \).

Given two VE-spaces \( \mathcal{E} \) and \( \mathcal{F} \), over the same ordered \(*\)-space \( Z \), one can consider the vector space \( \mathcal{L}(\mathcal{E}, \mathcal{F}) \) of all linear operators \( T : \mathcal{E} \to \mathcal{F} \). The operator \( T \) is called order bounded if there exists \( C \geq 0 \) such that

\[
(1.2) \quad \|Te, Te\|_\mathcal{F} \leq C^2[e, e]_\mathcal{E}, \quad e \in \mathcal{E}.
\]

Note that the inequality (1.2) is in the sense of the order of \( Z \) uniquely determined by the cone \( Z_+ \), see the axiom (a2). The infimum of these scalars is denoted by \( \|T\| \) and it is called the order operator norm or order norm of \( T \), more precisely,

\[
(1.3) \quad \|T\| = \inf\{C > 0 \mid \|Te, Te\|_\mathcal{F} \leq C^2[e, e]_\mathcal{E}, \text{ for all } e \in \mathcal{E}\}.
\]

Let \( \mathcal{B}(\mathcal{E}, \mathcal{F}) \) denote the collection of all order bounded linear operators \( T : \mathcal{E} \to \mathcal{F} \). Then \( \mathcal{B}(\mathcal{E}, \mathcal{F}) \) is a linear space and \( \|\cdot\| \) is a norm on it, cf. Theorem 1 in [23]. In addition, if \( T \) and \( S \) are order bounded linear operators acting between appropriate VE-spaces over the same ordered \(*\)-space \( Z \), then \( \|TS\| \leq \|T\|\|S\| \), in particular \( TS \) is bounded. If \( \mathcal{E} = \mathcal{F} \) then \( \mathcal{B}(\mathcal{E}) = \mathcal{B}(\mathcal{E}, \mathcal{E}) \) is a normed algebra, more precisely, the operator norm is submultiplicative.

A linear operator \( T \in \mathcal{L}(\mathcal{E}, \mathcal{F}) \) is called adjointable if there exists \( T^* \in \mathcal{L}(\mathcal{F}, \mathcal{E}) \) such that

\[
(1.4) \quad [Te, f]_\mathcal{F} = [e, T^* f]_\mathcal{E}, \quad e \in \mathcal{E}, \ f \in \mathcal{F}.
\]

The operator \( T^* \), if it exists, is uniquely determined by \( T \) and called its adjoint. Since an analog of the Riesz Representation Theorem for VE-spaces does not exist, in general, there may be not so many adjointable operators. We denote by \( \mathcal{L}^*(\mathcal{E}, \mathcal{F}) \) the vector space of all adjointable operators from \( \mathcal{L}(\mathcal{E}, \mathcal{F}) \). Note that \( \mathcal{L}^*(\mathcal{E}) = \mathcal{L}^*(\mathcal{E}, \mathcal{E}) \) is a \(*\)-algebra with respect to the involution \(* \) determined by the operation of taking the adjoint.

An operator \( A \in \mathcal{L}(\mathcal{E}) \) is called selfadjoint if

\[
(1.5) \quad [Ae, f] = [e, Af], \quad e, f \in \mathcal{E}.
\]

Clearly, any selfadjoint operator \( A \) is adjointable and \( A = A^* \). By the polarisation formula (1.11), \( A \) is selfadjoint if and only if

\[
(1.6) \quad [Ae, e] = [e, Ae], \quad e \in \mathcal{E}.
\]
An operator \( A \in \mathcal{L}(\mathcal{E}) \) is positive if
\[
(1.7) \quad [Ae, e] \geq 0, \quad e \in \mathcal{E}.
\]
Since the cone \( \mathcal{Z}_+ \) consists of selfadjoint elements only, any positive operator is selfadjoint and hence adjointable.

Let \( \mathcal{B}^*(\mathcal{E}) \) denote the collection of all adjointable order bounded linear operators \( T: \mathcal{E} \to \mathcal{E} \). Then \( \mathcal{B}^*(\mathcal{E}) \) is a pre-\( C^* \)-algebra, that is, it is a normed \( * \)-algebra with the property
\[
(1.8) \quad \|A^*A\| = \|A\|^2, \quad A \in \mathcal{B}^*(\mathcal{E}),
\]
and hence adjointable. For a proof, see Lemma 2.2 of [6]. In the special case when \( \mathcal{E} \) is a VH-space, see subsection 1.3, by Lemma 1.5 of [23].

1.3. Vector Hilbert Spaces and Their Linear Operators. If \( \mathcal{Z} \) is a topologically ordered \( * \)-space, any VE-space \( \mathcal{E} \) can be made in a natural way into a Hausdorff separated locally convex space by considering the weakest locally convex topology on \( \mathcal{E} \) that makes the mapping \( \mathcal{E} \ni h \mapsto [h, h] \in \mathcal{Z} \) continuous, more precisely, letting \( \{N_j\}_{j \in \mathcal{J}} \) be the collection of convex and solid neighbourhoods of the origin in \( \mathcal{Z} \) as in axiom (a5'), the collection of sets
\[
(1.9) \quad U_j = \{x \in \mathcal{E} \mid [x, x] \in N_j\}, \quad j \in \mathcal{J},
\]
is a topological base of neighbourhoods of the origin of \( \mathcal{E} \) that linearly generates the weakest locally convex topology on \( \mathcal{E} \) that makes all mappings \( \mathcal{E} \ni h \mapsto [h, h] \in \mathcal{Z} \) continuous, cf. Theorem 1 in [22]. In terms of seminorms, this topology can be defined in the following way: let \( \{p_j\}_{j \in \mathcal{J}} \) be a family of increasing seminorms defining the topology of \( \mathcal{Z} \) as in axiom (a5') and let
\[
(1.10) \quad \tilde{p}_j(h) = p_j([h, h])^{1/2}, \quad h \in \mathcal{E}, \quad j \in \mathcal{J}.
\]
Then each \( \tilde{p}_j \) is a seminorm on \( \mathcal{E} \) and its topology is fully determined by the family \( \{\tilde{p}_j\}_{j \in \mathcal{J}} \), see Lemma 1.3 in [5]. With respect to this topology, we call \( \mathcal{E} \) a topological VE-space over \( \mathcal{Z} \). In the special case when \( \mathcal{Z} \) is a normed ordered \( * \)-space, we call \( \mathcal{E} \) a normed VE-space over \( \mathcal{Z} \).

The following is a Schwarz type inequality that we will use in the rest of the article. For a proof, see Lemma 2.2 of [6].

**Lemma 1.4.** Let \( \mathcal{E} \) be a topological VE-space over the topologically ordered \( * \)-space \( \mathcal{Z} \) and \( p \in S(\mathcal{Z}) \). Then
\[
(1.11) \quad p([e, f]) \leq 4p([e, e])^{1/2}p([f, f])^{1/2} = 4\tilde{p}(e)\tilde{p}(f), \quad e, f \in \mathcal{E}.
\]

It turns out that if positive operators are considered, a stronger Schwarz type inequality holds. For a proof of the following lemma, see Lemma 2.13 of [5].

**Lemma 1.5.** Let \( T \in \mathcal{L}^*(\mathcal{E}) \) be a positive operator on a topological VE-space \( \mathcal{E} \) over the topologically ordered \( * \)-space \( \mathcal{Z} \). Let \( p \in S(\mathcal{Z}) \). Then
\[
p([Th, h]) \leq p([Th, Th])^{1/2}p([h, h])^{1/2}, \quad h \in \mathcal{H}.
\]
If both of $Z$ and $E$ are complete with respect to their specified locally convex topologies, then $E$ is called a \textit{VH-space} (Vector Hilbert space), sometimes also called a \textit{pseudo Hilbert space}. In the case when $Z$ is normed, then $E$ is called a \textit{normed VH-space}. Any topological VE-space $E$ on a topologically ordered $*$-space can be embedded as a dense subspace of a VH-space $H$, uniquely determined up to an isomorphism, cf. Theorem 2 in \cite{22}. Note that, given two VH-spaces $H$ and $K$, over the same admissible space $Z$, any isomorphism $U : H \rightarrow K$ in the sense of VE-spaces, is automatically bounded and adjointable, hence $U \in B(H,K)$, and it is natural to call this operator \textit{unitary}.

Now we provide some examples of admissible spaces, topological VE-spaces and VH-spaces. For a more comprehensive list of examples, we refer to \cite{3}.

\textbf{Examples 1.6.} (1) Any $C^*$-algebra $A$ is an admissible space, as well as any closed $*$-subspace $S$ of a $C^*$-algebra $A$, with the positive cone $S^+ = A^+ \cap S$ and all other operations (addition, multiplication with scalars, and involution) inherited from $A$.

(2) Any pre-$C^*$-algebra is a topologically ordered $*$-space. Any $*$-subspace $S$ of a pre-$C^*$-algebra $A$ is a topologically ordered $*$-space, with the positive cone $S^+ = A^+ \cap S$ and all other operations inherited from $A$. Note that, by Lemma \textbf{1.3} we can pass to the closure of $S^+$ if needed.

(3) Any locally $C^*$-algebra, cf. \cite{10}, \cite{32}, see subsection \textbf{1.4} is an admissible space. In particular, any closed $*$-subspace $S$ of a locally $C^*$-algebra $A$, with the cone $S_+ = A^+ \cap S$ and all other operations inherited from $A$, is an admissible space.

(4) Any locally pre-$C^*$-algebra, see subsection \textbf{1.4} is a topologically ordered $*$-space. Any $*$-subspace $S$ of a locally pre-$C^*$-algebra is a topologically ordered $*$-space, with $S^+ = A^+ \cap S$ and all other operations inherited from $A$. Again, by Lemma \textbf{1.3} we can pass to the closure of $S^+$ if needed.

(5) Any Hilbert $C^*$-module $H$ over a $C^*$-algebra $A$, see e.g. \cite{21} and \cite{26}, as well as any closed linear subspace $S$ of a Hilbert $C^*$-module $H$ over the same $C^*$-algebra $A$ is an example of a normed VH-space. A pre-Hilbert $C^*$-module over a pre-$C^*$-algebra, as well as any linear subspace of a pre-Hilbert $C^*$-module, is an example of a normed VE-space.

(6) More generally, any locally Hilbert $C^*$-module $H$ over a locally $C^*$-algebra $A$, see subsection \textbf{1.4} as well as any closed linear subspace $S$ of a locally Hilbert $C^*$-module over the same locally $C^*$-algebra $A$ is an example of a VH-space. A pre-locally Hilbert $C^*$-module over a pre-locally $C^*$-algebra, as well as any linear subspace of a pre-locally Hilbert $C^*$-module, is an example of a topological VE-space.

(7) Let $H$ be an infinite dimensional separable Hilbert space and let $C_1$ be the trace-class ideal, that is, the collection of all linear bounded operators $A$ on $H$ such that $\text{tr}(|A|) < \infty$. $C_1$ is a $*$-ideal of $B(H)$ and complete under the norm $||A||_1 = \text{tr}(|A|)$. Positive elements in $C_1$ are defined in the sense of positivity in $B(H)$. In addition, the norm $|| \cdot ||_1$ is increasing, since $0 \leq A \leq B$ implies $\text{tr}(A) \leq \text{tr}(B)$, hence $C_1$ is a normed admissible space.

(8) With notation as in (7), consider $C_2$ the ideal of Hilbert-Schmidt operators on $H$. Then $[A, B] = A^*B$, for all $A, B \in C_2$, is a gramian with values in the admissible space $C_1$ with respect to which $C_2$ becomes a VH-space. Since $C_1$ is a normed admissible space, $C_2$ is a normed VH-space, with norm $||A||_2 = \text{tr}(|A|^2)^{1/2}$, for all $A \in C_2$, cf. \textbf{1.10}. More abstract versions of this example have been considered by Saworotnow in \cite{34}.
(9) Let $\mathcal{H}$ be a Hilbert space and $\mathcal{E}$ a VH-space over the admissible space $Z$. On the algebraic tensor product $\mathcal{H} \otimes \mathcal{E}$ define a gramian by
\[ [h \otimes e, l \otimes f]_{\mathcal{H} \otimes \mathcal{E}} = \langle h, l \rangle_{\mathcal{H}} [e, f]_{\mathcal{E}} \in Z, \quad h, l \in \mathcal{H}, \ e, f \in \mathcal{E}, \]
and then extend it to $\mathcal{H} \otimes \mathcal{E}$ by linearity. It can be proven that, in this way, $\mathcal{H} \otimes \mathcal{E}$ is a VE-space over $Z$. Since $Z$ is an admissible space, $\mathcal{H} \otimes \mathcal{E}$ can be topologised as in (1.10) and then completed to a VH-space $\tilde{\mathcal{H}} \otimes \mathcal{E}$ over $Z$.

1.4. **Locally $C^*$-Algebras and Locally Hilbert $C^*$-Modules.** In this subsection we recall the definitions of locally $C^*$-algebras and locally Hilbert $C^*$-modules, and review some known facts.

A $*$-algebra $\mathcal{A}$ that has a complete Hausdorff topology induced by a family of $C^*$-seminorms, that is, seminorms $p$ on $\mathcal{A}$ that satisfy the $C^*$-condition $p(a^* a) = p(a)^2$ for all $a \in \mathcal{A}$, is called a locally $C^*$-algebra [16] (equivalent names are (Locally Multiplicatively Convex) LMC*-algebras [35], [25], or $b^*$-algebra [2], [3], or pro $C^*$-algebra [39], [32]). Note that, any $C^*$-seminorm is submultiplicative, $p(ab) \leq p(a) p(b)$ for all $a, b \in \mathcal{A}$, cf. [36], and is a $*$-seminorm, $p(a^*) = p(a)$ for all $a \in \mathcal{A}$. Denote the collection of all continuous $C^*$-seminorms by $S^*(\mathcal{A})$. Then $S^*(\mathcal{A})$ is a directed set under pointwise maximum seminorm, namely, given $p, q \in S^*(\mathcal{A})$, letting $r(a) : = \max\{p(a), q(a)\}$ for all $a \in \mathcal{A}$, then $r$ is a continuous $C^*$-seminorm and $p, q \leq r$. Locally $C^*$-algebras were studied in [2], [3], [16], [35], [32], and [43], to cite a few.

It follows from Corollary 2.8 in [16] that any locally $C^*$-algebra is, in particular, an admissible space, more precisely, a directed family of increasing seminorms generating the topology in axiom (a5)’ in Subsection 1.1 is $S^*(\mathcal{A})$. Note that $S^*(\mathcal{A}) \subset S_*\mathcal{A}$ and, although they generate the same topology on $\mathcal{H}$, these two sets are quite different. For instance, while $S_*\mathcal{A}$ is a cone, $S^*(\mathcal{A})$ is not even stable under positive scalar multiplication.

A pre-Hilbert module over a locally $C^*$-algebra $\mathcal{A}$, or a pre-Hilbert $\mathcal{A}$-module is a topological VE-space $\mathcal{H}$ over $\mathcal{A}$ where, in addition we have a right module action of $\mathcal{A}$ on $\mathcal{H}$ that respects the gramian, namely, for every $a \in \mathcal{A}$ and $h, g \in \mathcal{H}$ we have $[g, ha] = [g, h] a$. Note that the topology on the pre-Hilbert $\mathcal{A}$-module $\mathcal{H}$ is given by the family of seminorms $\{\tilde{p}\}_{p \in S^*(\mathcal{A})}$, where $\tilde{p}(h) = p([h, h])^{1/2}$ for all $p \in S^*(\mathcal{A})$ and all $h \in \mathcal{H}$. A pre-Hilbert $\mathcal{A}$-module $\mathcal{H}$ is called a Hilbert $\mathcal{A}$-module if it is complete, e.g. see [32] as well as [14].

Let $\mathcal{H}$ be a pre-Hilbert $\mathcal{A}$-module, let $p \in S^*(\mathcal{A})$ and let $x, y \in \mathcal{H}$. Then a Schwarz type inequality holds, e.g. see [43], as follows
\begin{equation}
(1.12) \quad p([h, k]_\mathcal{H}) \leq p([h, h]_\mathcal{H})^{1/2} p([k, k]_\mathcal{H})^{1/2}, \quad h, k \in \mathcal{H}.
\end{equation}
It is interesting to compare this inequality to the Schwarz type inequality of Lemma [1.4] which is weaker but works for the larger class of seminorms $S(\mathcal{A})$.

2. **Classes of Continuous Linear Operators on Topological VE-Spaces**

In this section, we study the properties of some classes of linear continuous operators acting between topological VE-spaces over the same topologically ordered $*$-space.

For topological VE-spaces $\mathcal{E}$ and $\mathcal{F}$ over the same topologically ordered $*$-space $Z$, we denote the space of all linear continuous operators $T : \mathcal{E} \to \mathcal{F}$ by $\mathcal{L}_c(\mathcal{E}, \mathcal{F})$, and in particular, $\mathcal{L}_c(\mathcal{E}, \mathcal{E})$ by $\mathcal{L}_c(\mathcal{E})$. By Remarks [1.1] and [1.2] a characterization of elements of $\mathcal{L}_c(\mathcal{E}, \mathcal{F})$ is as follows: Fix an arbitrary family of generating seminorms $S_0(\mathcal{Z}) \subseteq S_*\mathcal{Z}$. Then a linear operator $T : \mathcal{E} \to \mathcal{F}$ is in $\mathcal{L}_c(\mathcal{E}, \mathcal{F})$ if given any $p \in S_0(\mathcal{Z})$, there exists $q \in S_*\mathcal{Z}$ with $\tilde{q}(Tx) \leq \tilde{q}(x)$ for all $x \in \mathcal{H}$. It is easy to see that the choice of $S_0(\mathcal{Z})$ is immaterial; in particular, we can
always choose $S_0(Z)$ to be $S_+(Z)$. The space of all continuous and continuously adjointable linear operators $T: \mathcal{E} \rightarrow \mathcal{F}$ are denoted by $\mathcal{L}_c^*(\mathcal{E}, \mathcal{F})$, and $\mathcal{L}_c^*(\mathcal{E}) = \mathcal{L}_c^*(\mathcal{E}, \mathcal{E})$, where the latter is a $*$-algebra, and in fact even an ordered $*$-algebra, see [3] for the definition and a more detailed account.

For $\mathcal{E}$ and $\mathcal{F}$ as above, we denote by $\mathcal{L}_b(\mathcal{E}, \mathcal{F}, S_0(Z))$ the set of linear and $S_0(Z)$-bounded operators; this is the subset of $\mathcal{L}_b(\mathcal{E}, \mathcal{F})$ consisting of operators for which the seminorm $\tilde{q}$ can be chosen to be $C\tilde{p}$, where $S_0(Z) \subseteq S_+(Z)$ is a fixed generating family of seminorms, $C \geq 0$ is a constant dependent only on $T$ and $p$; i.e. we have, given $T \in \mathcal{L}_b(\mathcal{E}, \mathcal{F}, S_0(Z))$ that, given any $p \in S_0(Z)$, there exists a constant $C \geq 0$ such that $\tilde{p}(Tx) \leq C\tilde{p}(x)$ for all $x \in \mathcal{E}$. Let $\overline{\mathcal{P}}(T)$ be the infimum of all such $C \geq 0$. Then it is routine to see that $\overline{\mathcal{P}}$ is a seminorm on $\mathcal{L}_b(\mathcal{E}, \mathcal{F}, S_0(Z))$ and that $\mathcal{L}_b(\mathcal{E}, \mathcal{F}, S_0(Z))$ with the family of seminorms $\{\overline{\mathcal{P}} \mid p \in S_0(Z)\}$ is turned into a Hausdorff locally convex space. In particular, we denote by $\mathcal{L}_b(\mathcal{E}, \mathcal{F})$ the space $\mathcal{L}_b(\mathcal{E}, \mathcal{F}, S_+(Z))$, this is the space of all bounded linear operators.

Finally, we define $\mathcal{L}_b^*(\mathcal{E}, \mathcal{F}, S_0(Z)) := \mathcal{L}_b(\mathcal{E}, \mathcal{F}, S_0(Z)) \cap \mathcal{L}_b^*(\mathcal{E}, \mathcal{F})$, the space of all $S_0(Z)$-bounded and adjointable operators. We use $\mathcal{L}_b^*(\mathcal{E}, S_0(Z))$ for $\mathcal{L}_b^*(\mathcal{E}, \mathcal{E}, S_0(Z))$. In particular, we denote by $\mathcal{L}_b^*(\mathcal{E}, \mathcal{F}) := \mathcal{L}_b(\mathcal{E}, \mathcal{F}) \cap \mathcal{L}_b^*(\mathcal{E}, \mathcal{F})$, and call it the space of all bounded and adjointable operators.

From now on, throughout the article, by $S_0(Z)$ we denote any fixed generating family of seminorms of a given topologically ordered $*$-space $Z$ with $S_0(Z) \subseteq S_+(Z)$.

**Lemma 2.1.** Let $\mathcal{E}$ and $\mathcal{F}$ be VE-spaces over the topologically ordered $*$-space $Z$. Let $T \in \mathcal{L}_b^*(\mathcal{E}, \mathcal{F}, S_0(Z))$. Then $T^* \in \mathcal{L}_b^*(\mathcal{F}, \mathcal{E}, S_0(Z))$, and for any $p \in S_0(Z)$ we have $\overline{\mathcal{P}}(T) = \overline{\mathcal{P}}(T^*)$.

**Proof.** Let $T \in \mathcal{L}_b^*(\mathcal{E}, \mathcal{F}), p \in S_0(Z)$ and $y \in \mathcal{F}$. We have

\[
p([T^*y, T^*y]) = p(TT^*y, y) \leq \overline{\mathcal{P}}(T)p(T^*y)p(y)
\]

where for the first inequality Lemma 1.3 is used. A standard argument now implies that $\overline{\mathcal{P}}(T) = \overline{\mathcal{P}}(T^*)$. \qed

It follows from the preceding lemma that, $\mathcal{L}_b^*(\mathcal{E}, \mathcal{F}, S_0(Z))$ is a Hausdorff locally convex $*$-space; i.e. it is a $*$-space whose locally convex topology is given by $*$-seminorms and is Hausdorff.

**Proposition 2.2.** Let $\mathcal{F}$ be a VH-space and $\mathcal{E}$ be a topological VE-space over the admissible space $Z$. Then $\mathcal{L}_b(\mathcal{E}, \mathcal{F}, S_0(Z))$ is a complete Hausdorff locally convex $*$-space.

**Proof.** We only have to prove completeness and the proof goes using the same steps as in the corresponding proof for Banach spaces. For completeness, we briefly discuss the details. Let $(T_i)_{i \in I} \subseteq \mathcal{L}_b(\mathcal{E}, \mathcal{F}, S_0(Z))$ be a Cauchy net. Then for any $p \in S_0(Z)$ and $\epsilon > 0$, there exists $i_0 \in I$ such that $\overline{\mathcal{P}}(T_i - T_j) < \epsilon$ for all $i, j > i_0$, therefore $\tilde{p}(T_i x - T_j x) \leq \epsilon \tilde{p}(x)$ for any $x \in \mathcal{E}$. It follows that $(T_i x)_{i \in I}$ is a Cauchy net in $\mathcal{F}$ for each $x \in \mathcal{E}$, so $T_i x \to y$ for some $y \in \mathcal{F}$ by completeness of $\mathcal{F}$. Let $T: \mathcal{E} \to \mathcal{F}$ be the linear operator $Tx := y$. For any $x \in \mathcal{E}$, taking the limit over $j$ in the second inequality above, by continuity of $\tilde{p}$ we obtain $\tilde{p}(T_i x - T x) \leq \epsilon \tilde{p}(x)$ and hence $T_i - T \in \mathcal{L}_b(\mathcal{E}, \mathcal{F}, S_0(Z))$ for $i > i_0$. Consequently $\overline{\mathcal{P}}(T_i - T) = \overline{\mathcal{P}}(T_i - T_i) \leq \overline{\mathcal{P}}(T_i)$ and clearly $T_i \to T$ in $\mathcal{L}_b(\mathcal{E}, \mathcal{F}, S_0(Z))$, which finishes the proof. \qed

**Proposition 2.3.** Let $\mathcal{F}$ be a VH-space and $\mathcal{E}$ be a topological VE-space over the admissible space $Z$. Then $\mathcal{L}_b^*(\mathcal{E}, \mathcal{F}, S_0(Z))$ is a complete Hausdorff locally convex $*$-space.
Proof. We only have to show completeness. For this, we show that, $L^*_b(E, F, S_0(Z))$ is closed in $L_b(E, F, S_0(Z))$. Let $(T_i)_{i \in I} \in L^*_b(E, F, S_0(Z))$ be a convergent net with $T_i \to T$, $T \in L_b(E, F, S_0(Z))$. For any $p \in S_0(Z)$ since $\bar{p}(T^*_i) = \bar{p}(T_i)$ for all $i \in I$ by Lemma 2.1, $(T^*_i)_{i \in I}$ is a Cauchy net in $L_b(E, F, S_0(Z))$. By completeness of $L_b(E, F, S_0(Z))$ from Proposition 2.2 there exists $W \in L_b(E, F, S_0(Z))$ such that $T^*_i \to W$. Then we have, for any $x \in E$ and $y \in F$

$$[T x, y] = \lim_i [T_i x, y] = \lim_i [x, T^*_i y] = [x, W y]$$

as $T_i x \to T x$ for each $x$, $T^*_i y \to W y$ for each $y$, $S_0(Z)$ is generating, and the gramian is continuous. Consequently, $[T x, y] = [x, W y]$ for all $x \in E$ and $y \in F$. Therefore $T$ is adjointable with $T^* = W$, completing the proof.

Proposition 2.4. Let $E$ be a VE-space over the topologically ordered $*$-space $Z$. Then $L^*_b(E, S_0(Z))$ is a pre-locally $C^*$-algebra.

Proof. $L^*_b(E, S_0(Z))$ is a Hausdorff locally convex $*$-space by Lemma 2.1. For any $p \in S_0(Z)$ and $T_1, T_2 \in L^*_b(E, S_0(Z))$ it is easy to see that we have $\bar{p}(T_1 T_2) \leq \bar{p}(T_1) \bar{p}(T_2)$, and it follows that $L^*_b(E, S_0(Z))$ is, in particular, a $*$-algebra. In addition, for any $p \in S_0(Z)$, $T \in L^*_b(E, S_0(Z))$ and $x \in E$ we have

$$\bar{p}(T^* T x) \leq \bar{p}(T^*) \bar{p}(T) \bar{p}(x) = \bar{p}(T)^2 \bar{p}(x)$$

, where we use Lemma 2.1 for the equality, and we obtain $\bar{p}(T^* T) \leq p(T)^2$. Finally, for any $p \in S_0(Z)$, $T \in L^*_b(E)$ and $x \in E$ we have

$$\bar{p}(T x)^2 = p([T x, T x]) = p([T^* T x, x]) \leq \bar{p}(T^* T) \bar{p}(x) \leq \bar{p}(T^* T) \bar{p}(x)^2$$

where for the first inequality we used Lemma 1.5. Therefore $\bar{p}(T)^2 \leq \bar{p}(T^* T)$ and combining the two inequalities we obtain $\bar{p}(T)^2 = \bar{p}(T^* T)$ and the $C^*$ identity is shown for the seminorms $\bar{p}$, completing the proof.

In the end of this section, we provide some immediate corollaries of the facts proven so far.

Corollary 2.5. Let $E$ be a VH-space over the admissible space $Z$. Then $L^*_b(E, S_0(Z))$ and in particular $L^*_b(E)$ is a locally $C^*$-algebra.

Proof. Combine the statements of Propositions 2.3 and 2.4.

Corollary 2.6. Let $E$ and $F$ be two VH-spaces over the admissible space $Z$. Then $L^*_b(E, F, S_0(Z))$ is a locally Hilbert $C^*$-module over the locally $C^*$-algebra $L^*_b(E, S_0(Z))$. In particular, $L^*_b(E, F)$ is a locally Hilbert $C^*$-module over the locally $C^*$-algebra $L^*_b(E)$.

Proof. Given $T \in L^*_b(E, F, S_0(Z))$ and $A \in L^*_b(E, S_0(Z))$, define a right module action of $L^*_b(E, S_0(Z))$ on $L^*_b(E, F, S_0(Z))$ by $T A$. Define a pairing $[\cdot, \cdot]: L^*_b(E, F, S_0(Z)) \times L^*_b(E, F, S_0(Z)) \to L^*_b(E, S_0(Z))$ by $[T_1, T_2] := T_1^* T_2$. Then routine checking shows that $[\cdot, \cdot]$ is a gramian respecting the right module action. Together with Proposition 2.3 this proves the corollary.
3. Adjointable Operators of Barreled Topological VE-Spaces

3.1. Definition and Examples of Barreled Topological VE-Spaces. Barreled spaces were introduced by Bourbaki in [8]. Recall that, a set in a topological vector space is called a barrel if it is closed, absorbent, balanced and convex. A locally convex space is called barreled if each barrel is a neighborhood of 0. In this article, we will use standard results on barreled locally convex spaces, see e.g. Chapter 11 of [17] or [28]. We will refer to these results as needed.

Let $E$ be a topological VE-space over a topologically ordered $*$-space $Z$. If the underlying locally convex topology of $E$ is barreled, then $E$ is called a barreled topological VE-space. If $Z$ is a normed ordered $*$-space and the underlying normed topology of $E$ is barreled, then we call $E$ a barreled normed VE-space.

Note that any normed VH-space over a Banach ordered $*$-space, as well as any subspace of it is automatically barreled as it is Banach. In particular, any Hilbert $C^*$-module and its closed linear submanifolds are examples. Given a countable family of VH-spaces, by forming their countable direct product VH-space, see Examples 1.4. (4) of [9] for the details, we obtain examples of barreled VH-spaces, as in particular the underlying locally convex space is a Fréchet space, which is well known to be barreled. Also, any barreled subspace of the previous examples are barreled topological VE-spaces. In the following, we provide some non-examples.

**Example 3.1.** It is a natural question that, given a barreled topologically ordered $*$-space $Z$, if we automatically have that any topological VE-space over $Z$ is also barreled or not. It is not difficult to see that there is no such permanence property, simply by considering any non-barreled subspace of an inner product space. In the following, we provide such an example.

Consider the complex sequence space of all finite sequences $c_{00}$ with the $l^2$ inner product, so $c_{00}$ is an inner product space and hence a topological VE-space over $\mathbb{C}$. By considering the sequence of linear bounded functionals $f_n(x) := \sum_{i=1}^{n} x_i$, $n = 1, 2, \ldots$, we see that the set $(f_n)_{n=1}^\infty$ is pointwise bounded, but since $\|f_n\| = \sqrt{n}$, it is not uniformly bounded. Since the Principle of Uniform Boundedness fails, $c_{00}$ is not barreled.

Notice that, this example in particular shows that, a topological VE-space need not be barreled.

Another natural question is, in the case when we have a VH-space $H$ if we automatically have barreledness of $H$ or not. Notice that, a counter example must be out of the category of normed spaces, since a normed VH-space, being a Banach space in particular, is automatically barreled. In the following, we provide an example of a non-barreled locally $C^*$-module, hence an example of a non-barreled locally Hilbert $C^*$-module, and hence an example of a non-barreled VH-space, see Examples [17].

**Example 3.2.** We adopt Example 11.12.6 in [28], and observe that it is a locally $C^*$-algebra. For the convenience of the reader, we provide all the details here. Let $T$ be a set of uncountable cardinality and let $X$ be the complex vector space $X := \{x \in \mathbb{C}^T \mid x(t) = 0 \text{ except for finitely many } t\}$. Define product of two elements in $X$ pointwise; i.e. $(xy)(t) := x(t)y(t)$, clearly the product is commutative. Similarly, the involution $*$ is defined pointwise; $x^*(t) := \overline{x(t)}$. It is easy to check that $X$ is a $*$-algebra. For each $t \in T$, consider the seminorm $p_t(x) := |x(t)|$ and notice that it is a $C^*$-seminorm. The family of $C^*$-seminorms...
\{p_t\}_{t \in T} \text{ turn } X \text{ into a pre-locally } C^*-\text{algebra. However, by direct checking, it is not difficult to see that } X \text{ is a complete locally convex space, therefore } X \text{ is a locally } C^*-\text{algebra.}

Now consider the set \( B := \{ x \in X \mid \sum_{t \in T} |x(t)| \leq 1 \} \). It is straightforward to check that \( B \) is a barrel in \( X \). We show that \( B \) cannot contain a basic neighborhood \( U := \{ x \in X \mid p_t(x) \leq r_t \text{ for all } t \} \) of 0, where \( r_t > 0 \), hence it cannot be a neighborhood of 0: Assume \( B \supseteq U \) for some \( U \). Consider the sets \( A_n := \{ t \in T \mid |r_t| > 1/n \} \) for \( n \geq 2 \). Since \( T \) is uncountable it is clear that there exists \( n_0 \) such that \( A_{n_0} \) contains infinitely many elements. Hence there exists a finite subset \( H \subset T \) such that \( \sum_{t \in H} r_t > 1 \). Now form the element \( x(t) = \begin{cases} r_t, & t \in H \\ 0, & t \notin H \end{cases} \) which is in \( U \), but not in \( B \). Therefore \( X \) is not barreled.

### 3.2. Automatic Boundedness of Adjointable Operators of Barreled Topological VE-Spaces

In this section, we prove the main result of this article, Theorem 3.7, which states that, an adjointable operator \( T : E \to F \) from a barreled topological VE-space \( E \) to a topological VE-space \( F \) over the same topologically ordered *-space \( Z \) is automatically bounded, hence continuous.

We now do some preparation in order to prove Theorem 3.7. For a topologically ordered *-space \( Z \), let \( p \in S_*(Z) \) and consider the kernel of \( p \), \( I_p := \{ z \in Z \mid p(z) = 0 \} \). It is easy to see that \( I_p \) is a closed ordered *-subspace of \( Z \). In addition, since \( p \) is an increasing *-seminorm it follows that \( I_p \) is an order ideal, that is, \( I_p \) is selfadjoint and if \( z_0 \in I_p \) with \( z_0 \geq 0 \), for any \( z \in Z \) with \( 0 \geq z \geq z_0 \) we have \( z \in I_p \).

It follows from Example 1.2(9) of [4] that, the quotient space \( Z_p := Z/I_p \) is an ordered *-space with the cone \( Z^+_p := Z^+/I_p \), and the involution of \( Z_p \) is defined by \((z + I_p)^* := z^* + I_p \).

It is standard that, \( \|z + I_p\|_p := p(z) \) defines a norm on \( Z_p \) for any \( p \in S_*(Z) \). It is easy to see that \( \|\cdot\|_p \) is a *-norm, as \( p \) is a *-seminorm, and to see that it is increasing, we just observe that for \( y, z \in Z^+ \) we have \( \|z + I_p\|_p = p(z) \leq p(z + y) = \|(z + y) + I_p\|_p = \|(z + I_p) + (y + I_p)\|_p \).

Therefore the ordered *-space \( (Z_p, Z^+_p) \) with the norm \( \|\cdot\|_p \) satisfies all axioms (a3) - (a6) of a topologically ordered *-space except for (a5). By Lemma 1.3 it follows that for any \( p \in S_*(Z) \) the space \( (Z_p, \overline{Z^+_p}) \) where \( \overline{Z^+_p} \) is the closure of \( Z^+_p \) is a normed ordered *-space with the norm \( \|\cdot\|_p \) an increasing *-norm.

We record the discussions above as a lemma:

**Lemma 3.3.** Let \( Z \) be a topologically ordered *-space and \( p \in S_*(Z) \). Then the quotient space \( (Z_p, Z^+_p) \) is an ordered *-space. Moreover, the space \( (Z_p, \overline{Z^+_p}, \|\cdot\|_p) \) satisfies all axioms of a normed ordered *-space except for closedness of its cone and by passing to the closure \( \overline{Z^+_p} \) of the cone, the space \( (Z_p, \overline{Z^+_p}, \|\cdot\|_p) \) is a normed ordered *-space.

**Proof.** See the preceding discussion. \( \square \)

From now on we will use \( Z_p \) for the space \( (Z_p, \overline{Z^+_p}, \|\cdot\|_p) \) as in Lemma 3.3 where \( p \in S_*(Z) \).

Now let \( E \) be a topological VE-space over the topologically ordered *-space \( Z \) and \( p \in S_*(Z) \). Following the notation in [5], define \( \tilde{I}_p^E := \{ x \in E \mid [x, x] \in I_p \} \). Equivalently, \( \tilde{I}_p^E = \{ x \in E \mid \tilde{p}(x) = 0 \} \), where \( \tilde{p} \) is as in (1.10). We will use the notation \( \tilde{I}_p \) as well if the space \( E \) is clear from the context.

**Lemma 3.4.** Let \( E \) be a topological VE-space over the topologically ordered *-space \( Z \) and let \( p \in S_*(Z) \). Then \( \tilde{I}_p \) is a closed vector subspace of \( E \) and the quotient \( E_p := E/\tilde{I}_p \) is a normed
VE-space over the normed ordered $*$-space $Z_p$, with norm given by $\|x + \tilde{I}_p\| = \tilde{p}(x)$ for any $x \in \mathcal{E}$.

**Proof.** Let $p \in S_\ast(Z)$. For any $x \in \tilde{I}_p$ and $\alpha \in \mathbb{C}$, it is immediate to see that $\alpha x \in \tilde{I}_p$. Now letting $x_1, x_2 \in \tilde{I}_p$ we have

$$\tilde{p}^2(x_1 + x_2) = p([x_1 + x_2, x_1 + x_2]) \leq p([x_1, x_1]) + 2p([x_1, x_2]) + p([x_2, x_2]),$$

$$\leq \tilde{p}^2(x_1) + 8\tilde{p}(x_1)\tilde{p}(x_2) + \tilde{p}^2(x_2) = 0$$

where for the second inequality we used the Schwarz type inequality in Lemma 1.4. This completes the proof that $\tilde{I}_p$ is a vector subspace of $\mathcal{E}$. It is a routine check to see that $\tilde{I}_p$ is closed.

We form the quotient normed space $\mathcal{E}_p := \mathcal{E}/\tilde{I}_p$ with quotient norm $\|x + \tilde{I}_p\|_p = \tilde{p}(x)$ for any $x \in \mathcal{E}$ and show that it is a normed VE-space over $Z_p$. Define the pairing $[\cdot, \cdot]_p : \mathcal{E}_p \times \mathcal{E}_p \to Z_p$ by

$$[x + \tilde{I}_p, y + \tilde{I}_p]_p := [x, y] + I_p$$

for any $x, y \in \mathcal{E}$. To see that this is well defined, let $x_1, x_2, y_1, y_2 \in \mathcal{E}$ such that $x_1 + \tilde{I}_p = x_2 + \tilde{I}_p$ and $y_1 + \tilde{I}_p = y_2 + \tilde{I}_p$, equivalently, $\tilde{p}(x_1 - x_2) = 0$ and $\tilde{p}(y_1 - y_2) = 0$. Observe that we have

$$p([x_1, y_1] - [x_2, y_2]) = p([x_1, y_1] + [x_1, y_2] - [x_1, y_2] - [x_2, y_2])$$

$$\leq p([x_1, y_1 - y_2]) + p([x_1 - x_2, y_2])$$

$$\leq 4\tilde{p}(x_1)\tilde{p}(y_1 - y_2) + 4\tilde{p}(x_1 - x_2)\tilde{p}(y_2) = 0$$

where for the second inequality we used Lemma 1.4 again. Then we have

$$[x_1 + \tilde{I}_p, y_1 + \tilde{I}_p]_p = [x_1 + y_1] + I_p$$

$$= [x_2, y_2] + I_p$$

$$= [x_2 + \tilde{I}_p, y_2 + \tilde{I}_p]_p$$

so the pairing $[\cdot, \cdot]_p$ is well defined. It is straightforward to check that $[\cdot, \cdot]_p$ defines a gramian on $\mathcal{E}_p$. Finally, the natural norm topology of $\mathcal{E}_p$ as in Subsection 1.3 given by

$$\|x + \tilde{I}_p\| := \|[x + \tilde{I}_p, x + \tilde{I}_p]_p\|_{\mathcal{E}_p}^{1/2}$$

$$= \|[x, x] + I_p\|_{\mathcal{E}_p}^{1/2} = p([x, x])^{1/2} = \tilde{p}(x)$$

for any $x \in \mathcal{E}$, agrees with the quotient norm $\|x + \tilde{I}_p\|_p = \tilde{p}(x)$ and therefore the quotient norm turns $\mathcal{E}_p$ into a normed VE-space. That completes the proof. □

**Corollary 3.5.** Let $\mathcal{E}$ be a barreled topological VE-space over the topologically ordered $*$-space $Z$ and let $p \in S_\ast(Z)$. Then $\mathcal{E}_p$ is a barreled normed VE-space over $Z_p$.

**Proof.** Since $\tilde{I}_p$ is a closed subspace of $\mathcal{E}$ and $\mathcal{E}_p$ has the quotient topology by Lemma 3.4, by e.g. Proposition 1(a) in 11.3 of [17], the normed topology of $\mathcal{E}_p$ is barreled as well. Therefore $\mathcal{E}_p$ is a barreled normed VE-space over $Z_p$. □

**Corollary 3.6.** Let $\mathcal{E}$ and $\mathcal{F}$ be two topological VE-spaces over the same topologically ordered $*$-space $Z$. Let $p \in S_\ast(Z)$ and $T \in \mathcal{L}^\ast(\mathcal{E}, \mathcal{F})$. Then the naturally defined quotient operator $T_p : \mathcal{E}_p \to \mathcal{F}_p$ is a well defined linear operator and $T_p \in \mathcal{L}^\ast(\mathcal{E}_p, \mathcal{F}_p)$. 

Proof. Define \( T_p : \mathcal{E}_p \to \mathcal{F}_p \) by \( T_p(x + \tilde{I}_p) := Tx + \tilde{I}_p \). In order to see that \( T_p \) is well defined, it is enough to show that, if \( x + \tilde{I}_p = 0 \), then \( Tx + \tilde{I}_p = 0 \); equivalently, \( \tilde{p}_e(x) = 0 \) implies \( \tilde{p}_e(Tx) = 0 \). We have, for such \( x \in \mathcal{E} \),

\[
\tilde{p}_e(Tx) = p([Tx, Tx])\frac{1}{2} = p([T^*Tx, x])\frac{1}{2} \leq \tilde{p}_e(T^*Tx) \tilde{p}_e(x) = 0
\]

where for the inequality we used Lemma 1.3.

It is easy to see that \( T_p \) is linear. To see that it is adjointable with adjoint \( (T^*)_p \), let \( x \in \mathcal{E} \) and \( y \in \mathcal{F} \) be arbitrary. Then

\[
[T_p(x + \tilde{I}_p), y + \tilde{I}_p]_{\mathcal{F}_p} = [Tx + \tilde{I}_p, y + \tilde{I}_p]_{\mathcal{F}_p} = [x, T^*y] + I_p = [x + \tilde{I}_p, T^*y + \tilde{I}_p]_{\mathcal{E}_p} = [x + \tilde{I}_p, (T^*)_p(y + \tilde{I}_p)]_{\mathcal{E}_p}
\]

and it is shown that \( T_p \) is adjointable with \( T^*_p = (T^*)_p \).

The following theorem establishes boundedness of an adjointable operator on a barreled topological VE-space. To some extent, we use an idea going back to [30], see the argument in page 8 of [21] as well.

**Theorem 3.7.** Let \( \mathcal{E} \) be a barreled topological VE-space and \( \mathcal{F} \) be a topological VE-space over the same topologically ordered \(*\)-space \( Z \). Then \( \mathcal{L}^*(\mathcal{E}, \mathcal{F}) = \mathcal{L}^*_\ast(\mathcal{E}, \mathcal{F}) \).

**Proof.** Let \( T \in \mathcal{L}^*(\mathcal{E}, \mathcal{F}) \) be an operator, and \( p \in S_+(Z) \). Using Lemma 3.4 and Corollaries 3.5 and 3.6 we obtain the barreled normed VE-space \( \mathcal{E}_p \), the normed VE-space \( \mathcal{F}_p \), both over \( Z_p \) and the quotient operator \( T_p : \mathcal{E}_p \to \mathcal{F}_p \). Now for any \( y \in \mathcal{F} \) with \( \tilde{p}(y) \leq 1 \) define the vector valued linear functional \( f_y : \mathcal{E}_p \to \mathcal{F}_p \) by \( f_y(x + \tilde{I}_p) = [T^*_p(y + \tilde{I}_p), x + \tilde{I}_p]_{\mathcal{E}_p} \).

We show that for \( y \) fixed, \( f_y \) is bounded. We have

\[
\|f_y(x + \tilde{I}_p)\|_{Z_p} = \|([T^*_p(y + \tilde{I}_p), x + \tilde{I}_p]_{\mathcal{E}_p})\|_{Z_p} = \|[T^*y, x]_{\mathcal{E}} + I_p\|_{Z_p} = p([T^*y, x]) \leq 4\tilde{p}(T^*y)\tilde{p}(x) = 4\tilde{p}(T^*y)\|((x + \tilde{I}_p))\|_{\mathcal{E}_p}
\]

for all \( x \in \mathcal{E} \), where for the inequality we used Lemma 1.4 and it follows that \( f_y \) is bounded with \( \|f_y\| \leq 4\tilde{p}(T^*y) \). In addition we show that the family \( \{f_y\} \) is pointwise bounded. For any fixed \( x \in \mathcal{E} \) we have

\[
\|f_y(x + \tilde{I}_p)\|_{Z_p} = \|([T^*_p(y + \tilde{I}_p), x + \tilde{I}_p]_{\mathcal{E}_p})\|_{Z_p} = \|[T^*y, x]_{\mathcal{E}} + I_p\|_{Z_p} = p([T^*y, x]) = p([y, Tx]) \leq 4\tilde{p}(y)\tilde{p}(Tx) \leq 4\tilde{p}(Tx)
\]

where we used Lemma 1.4 again for the first inequality. Hence the family \( \{f_y\} \) is pointwise bounded with constant \( C_x = 4\tilde{p}(Tx) \) for any fixed \( x \in \mathcal{E} \).

It follows by the Principle of Uniform Boundedness applied to the family \( \{f_y\} \) that, there exists a constant \( C > 0 \) such that \( \|f_y(x + \tilde{I}_p)\|_{Z_p} \leq C\|x + \tilde{I}_p\|_{\mathcal{E}_p} \) for all \( x \in \mathcal{E} \) and \( y \in \mathcal{F} \) with
\( \tilde{p}(y) \leq 1 \). Equivalently, we obtain \( p([y,Tx]) \leq C\tilde{p}(x) = Cp([x,x])^{1/2} \) for all \( x \in \mathcal{E} \) and \( y \in \mathcal{F} \) with \( \tilde{p}(y) \leq 1 \).

Now let \( y \in \mathcal{F} \) be any element, and let \( \epsilon > 0 \). By applying the inequality above with the element \((\tilde{p}(y) + \epsilon)^{-1}y\) in place of \( y \) we obtain

\[
p([y,Tx]) \leq C(\tilde{p}(y) + \epsilon)\tilde{p}(x)
\]

for all \( x \in \mathcal{E} \) and \( y \in \mathcal{F} \). Letting \( \epsilon \to 0 \), it follows that

\[
p([y,Tx]) \leq C\tilde{p}(y)\tilde{p}(x)
\]

for all \( x \in \mathcal{E} \) and \( y \in \mathcal{F} \). Finally, choosing \( y := Tx \) in the last inequality and a standard argument implies that \( \tilde{p}(Tx) \leq C\tilde{p}(x) \) for all \( x \in \mathcal{E} \). It follows that \( T \) is bounded and the proof is finished. \( \Box \)

Notice that, in the preceding proof, if the topological VE-space \( \mathcal{E} \) is such that the quotient normed spaces \( \mathcal{E}_p \) are barreled, then the proof still works even without the barreled assumption on \( \mathcal{E} \). Consequently, we obtain the following corollary:

**Corollary 3.8.** Let \( \mathcal{E} \) be a topological VE-space over the topologically ordered \(*\)-space \( Z \). Assume that the quotient normed VE-spaces \( \mathcal{E}_p \) are barreled for all \( p \in S_0(Z) \), where \( S_0(Z) \subseteq S_*(Z) \) is a family defining the topology of \( Z \). Let \( F \) be a topological VE-space over the same topologically ordered \(*\)-space \( Z \). Then \( L^*_p(\mathcal{E},F) = L^*_p(\mathcal{E},F,S_0(Z)) \).

**Remark 3.9.** As a consequence of Theorem 3.7, with \( \mathcal{E} \) and \( F \) as in the theorem, we have \( L^*_p(\mathcal{E},F) = L^*_p(\mathcal{E},F) = L^*(\mathcal{E},F) \). In particular, any adjointable operator is bounded and hence continuous, with a bounded and hence continuous adjoint.

Let \( \mathcal{H} \) be a VH-space over a normed admissible space \( Z \). Then \( \mathcal{H} \) is in particular a Banach space, see subsection 1.3, consequently it is a barreled VH-space. Hence the following corollary of Theorem 3.7 follows:

**Corollary 3.10.** Let \( \mathcal{H} \) be a normed VH-space and \( \mathcal{K} \) be a normed VE-space over the same normed admissible space \( Z \). Then \( L^*_p(\mathcal{H},\mathcal{K}) = L^*(\mathcal{H},\mathcal{K}) \).

We obtain the following structural theorem for \( L^*(\mathcal{E}) \) when \( \mathcal{E} \) is a barreled topological VE-space.

**Theorem 3.11.** Let \( \mathcal{E} \) be a barreled topological VE-space over a topologically ordered \(*\)-space \( Z \). Then \( L^*(\mathcal{E}) \) is a pre-locally \( C^*\)-algebra. In addition, if \( \mathcal{E} \) is a barreled VH-space over an admissible space \( Z \), then \( L^*(\mathcal{E}) \) is a locally \( C^*\)-algebra.

**Proof.** For the first statement, combine the statements of Proposition 2.4 and Theorem 3.7. For the second, combine the statements of Corollary 2.5 and Theorem 3.7. \( \Box \)

**Corollary 3.12.** Let \( \mathcal{H} \) be a barreled VH-space over the admissible space \( Z \) and \( \mathcal{K} \) be a VH-space over \( Z \). Then the space \( L^*(\mathcal{H},\mathcal{K}) \) is a locally Hilbert \( C^*\)-module over the locally \( C^*\)-algebra \( L^*(\mathcal{H}) \).

**Proof.** Combine the statements of Corollary 2.6, Theorem 3.7 and Corollary 3.11. \( \Box \)

We obtain the following corollary from Theorem 3.11:

**Corollary 3.13.** Let \( \mathcal{H} \) be a normed VH-space over the normed admissible space \( Z \). Then \( L^*(\mathcal{H}) \) is a \( C^*\)-algebra.
The hypothesis of Corollary 3.8 holds when $\mathcal{E}$ is a locally Hilbert $C^*$-module over a locally $C^*$-algebra $\mathcal{A}$ and the family $S_0(Z)$ is taken as a generating family of $C^*$-seminorms $S(\mathcal{A})$ of $\mathcal{A}$. It is known in that case, see [3], that the quotient spaces $\mathcal{A}_p$, $p \in S(\mathcal{A})$ are $C^*$-algebras and corresponding quotient spaces $\mathcal{E}_p$ are Hilbert $C^*$-modules. In particular $\mathcal{E}_p$ are complete and hence barreled, e.g. see [32] and [43].

As a result, we recover the following well known result, see [32] and [43], as a special case of Corollary 3.8.

**Corollary 3.14.** Let $\mathcal{E}$ be a locally Hilbert $C^*$-module over the locally $C^*$-algebra $\mathcal{A}$ and $\mathcal{F}$ be a pre-locally Hilbert $C^*$-module over $\mathcal{A}$. Then $\mathcal{L}^*(\mathcal{E}, \mathcal{F}) = \mathcal{L}^*_b(\mathcal{E}, \mathcal{F}, S^*(\mathcal{A}))$. Moreover, $\mathcal{L}^*(\mathcal{E}) = \mathcal{L}^*_b(\mathcal{E}, S^*(\mathcal{A}))$ is a locally $C^*$-algebra. If, in addition $\mathcal{F}$ is also a locally Hilbert $C^*$-module, then $\mathcal{L}^*(\mathcal{E}, \mathcal{F}) = \mathcal{L}^*_b(\mathcal{E}, \mathcal{F}, S^*(\mathcal{A}))$ is a locally Hilbert $C^*$-module over the locally $C^*$-algebra $\mathcal{L}^*(\mathcal{E}, S^*(\mathcal{A}))$.

**Proof.** In order to prove the first statement, note that $\mathcal{E}$ is a VH-space in particular and $\mathcal{F}$ is a topologically VE-space over the admissible space $\mathcal{A}$, see Examples 1.6, then use Corollary 3.8 with $S_0(\mathcal{A}) = S^*(\mathcal{A})$. For the second statement, combine Corollary 2.5 and the first statement. Finally, for the third statement, combine Corollary 2.6 and first statement. \qed

4. **Dilation Theory of Barreled VH-Spaces**

In this section we pick an implication of Theorem 3.7 on dilations of barreled VH-spaces, as an application of Theorem 3.7. More precisely, we will show that when barreled VH-spaces are considered, an operator boundedness condition from a characterization of the existence of invariant VH-space linearizations (equivalently, existence of reproducing kernel VH-spaces with *-representations) of kernels valued in operators of VH-spaces, see Theorem 4.1, is satisfied automatically.

First we recall some definitions from [5] that will be necessary in the remainder of the paper, mainly for the convenience of the reader. For detailed discussions of these definitions and related results we refer to [5].

Let $X$ be a nonempty set and let $\mathcal{E}$ be a VE-space over the ordered $*$-space $Z$. A map $k: X \times X \to \mathcal{L}(\mathcal{E})$ is called a kernel on $X$ and valued in $\mathcal{L}(\mathcal{E})$. In case the kernel $k$ has all its values in $\mathcal{L}^*(\mathcal{E})$, an adjoint kernel $k^*: X \times X \to \mathcal{L}^*(\mathcal{E})$ can be associated by $k^*(x, y) = k(y, x)^*$ for all $x, y \in X$. The kernel $k$ is called Hermitian if $k^* = k$. In what follows we will always consider kernels valued in $\mathcal{L}^*(\mathcal{E})$.

Given $n \in \mathbb{N}$, the kernel $k$ is called $n$-positive if for any $x_1, x_2, \ldots, x_n \in X$ and any $h_1, h_2, \ldots, h_n \in \mathcal{H}$ we have

$$\sum_{i,j=1}^{n} [k(x_i, x_j)h_j, h_i]_{\mathcal{E}} \geq 0. \quad (4.1)$$

The kernel $k$ is called positive semidefinite (or of positive type) if it is $n$-positive for all natural numbers $n$. A 2-positive kernel is Hermitian, see [13].

Given an $\mathcal{L}^*(\mathcal{E})$-valued kernel $k$ on a nonempty set $X$, for some VE-space $\mathcal{E}$ on an ordered $*$-space $Z$, a VE-space linearisation or, equivalently, a VE-space Kolmogorov decomposition of $k$ is, by definition, a pair $(\mathcal{K}; V)$, subject to the following conditions:

(vel1) $\mathcal{K}$ is a VE-space over the same ordered $*$-space $Z$.

(vel2) $V: X \to \mathcal{L}^*(\mathcal{E}, \mathcal{K})$ satisfies $k(x, y) = V(x)^*V(y)$ for all $x, y \in X$. 


The VE-space linearisation \((\mathcal{K}; V)\) is called \textit{minimal} if
\begin{equation}
(\text{vel3}) \text{ Lin} V(X) \mathcal{E} = \mathcal{K}.
\end{equation}

Two VE-space linearisations \((V; \mathcal{K})\) and \((V'; \mathcal{K}')\) of the same kernel \(k\) are called \textit{unitary equivalent} if there exists a VE-space isomorphism \(U: \mathcal{K} \rightarrow \mathcal{K}'\) such that \(UV(x) = V'(x)\) for all \(x \in X\).

A minimal VE-space linearisation \((\mathcal{K}; V)\) of a positive semidefinite kernel \(k\), is unique modulo unitary equivalence, see [4].

Let \(\mathcal{E}\) be a VE-space over the ordered \(*\)-space \(Z\), and let \(X\) be a nonempty set. Let \(\mathcal{F} = \mathcal{F}(X; \mathcal{E})\) denote the complex vector space of all functions \(f: X \rightarrow \mathcal{E}\). A VE-space \(\mathcal{R}\), over the same ordered \(*\)-space \(Z\), is called an \(\mathcal{H}\)-\textit{reproducing kernel VE-space on} \(X\) if there exists a Hermitian kernel \(k: X \times X \rightarrow \mathcal{L}^*(\mathcal{E})\) such that the following axioms are satisfied:
\begin{itemize}
  \item[(rk1)] \(\mathcal{R}\) is a subspace of \(\mathcal{F}(X; \mathcal{E})\), with all algebraic operations.
  \item[(rk2)] For all \(x \in X\) and all \(h \in \mathcal{E}\), the \(\mathcal{H}\)-valued function \(k_x h = k(\cdot, x)h \in \mathcal{R}\).
  \item[(rk3)] For all \(f \in \mathcal{R}\) we have \([f(x), h]\mathcal{E} = [f, k_x h]\mathcal{R}\), for all \(x \in X\) and \(h \in \mathcal{E}\).
\end{itemize}

As a consequence of (rk2), \(\text{Lin}\{k_x h \mid x \in X, \ h \in \mathcal{E}\} \subseteq \mathcal{R}\). The reproducing kernel VE-space \(\mathcal{R}\) is called \textit{minimal} if the following property holds as well:
\begin{itemize}
  \item[(rk4)] \(\text{Lin}\{k_x h \mid x \in X, \ h \in \mathcal{E}\} = \mathcal{R}\).
\end{itemize}

Let \(\Gamma\) be a (multiplicative) \(*\)-semigroup, that is, there is an \textit{involution} \(*\) on \(\Gamma\): \((\xi \eta)^* = \eta^* \xi^*\) and \((\xi^*)^* = \xi\) for all \(\xi, \eta \in \Gamma\). Note that, in case \(\Gamma\) has a unit \(e\) then \(e^* = e\). Let \(\Gamma\) act on a nonempty set \(X\), denoted by \(\xi \cdot x\), for all \(\xi \in \Gamma\) and all \(x \in X\). By definition, we have
\begin{equation}
(4.2) \quad \alpha \cdot (\beta \cdot x) = (\alpha \beta) \cdot x \text{ for all } \alpha, \beta \in \Gamma \text{ and all } x \in X.
\end{equation}

Given a VE-space \(\mathcal{E}\) we consider those Hermitian kernels \(k: X \times X \rightarrow \mathcal{L}^*(\mathcal{E})\) that are \textit{invariant} under the action of \(\Gamma\) on \(X\), that is,
\begin{equation}
(4.3) \quad k(y, \xi \cdot x) = k(\xi^* \cdot y, x) \text{ for all } x, y \in X \text{ and all } \xi \in \Gamma.
\end{equation}

A triple \((\mathcal{K}; \pi; V)\) is called an \textit{invariant VE-space linearisation} of the kernel \(k\) and the action of \(\Gamma\) on \(X\), shortly a \(\Gamma\)-\textit{invariant VE-space linearisation} of \(k\), if:
\begin{itemize}
  \item[(ikd1)] \((\mathcal{K}; V)\) is a VE-space linearisation of the kernel \(k\).
  \item[(ikd2)] \(\pi: \Gamma \rightarrow \mathcal{L}^*(\mathcal{K})\) is a \(*\)-representation, that is, a multiplicative \(*\)-morphism.
  \item[(ikd3)] \(V\) and \(\pi\) are related by the formula: \(V(\xi \cdot x) = \pi(\xi) V(x)\), for all \(x \in X, \xi \in \Gamma\).
\end{itemize}

If \((\mathcal{K}; \pi; V)\) is a \(\Gamma\)-invariant VE-space linearisation of the kernel \(k\) then \(k\) is invariant under the action of \(\Gamma\) on \(X\).

If, in addition to the axioms (ikd1)–(ikd3), the triple \((\mathcal{K}; \pi; V)\) has the property
\begin{itemize}
  \item[(ikd4)] \(\text{Lin} V(X) \mathcal{E} = \mathcal{K}\),
\end{itemize}
that is, the VE-space linearisation \((\mathcal{K}; V)\) is minimal, then \((\mathcal{K}; \pi; V)\) is called a \textit{minimal} \(\Gamma\)-\textit{invariant VE-space linearisation} of \(k\) and the action of \(\Gamma\) on \(X\).

Now let \(\mathcal{H}\) be a VH-space over the admissible space \(Z\), and consider a kernel \(k: X \times X \rightarrow \mathcal{L}_c^*(\mathcal{H})\). A \(\text{VH-space linearisation of} \ k\), or \(\text{VH-space Kolmogorov decomposition of} \ k\), is a pair \((\mathcal{K}; V)\), subject to the following conditions:
\begin{itemize}
  \item[(vh1)] \(\mathcal{K}\) is a VH-space over the same ordered \(*\)-space \(Z\).
  \item[(vh2)] \(V: X \rightarrow \mathcal{L}_c^*(\mathcal{H}, \mathcal{K})\) satisfies \(k(x, y) = V(x)^* V(y)\) for all \(x, y \in X\).
\end{itemize}
The VH-space linearisation \((\mathcal{K}; V)\) is called \textit{minimal} if
(vhl3) \( \text{Lin} V(X)\mathcal{H} \) is dense in \( \mathcal{K} \).

Two VH-space linearisations \( (V; \mathcal{K}) \) and \( (V'; \mathcal{K}') \) of the same kernel \( k \) are called \textit{unitary equivalent} if there exists a unitary operator \( U: \mathcal{K} \rightarrow \mathcal{K}' \) such that \( UV(x) = V'(x) \) for all \( x \in X \).

As in the case for VE-space linearisations, a minimal VH-space linearisation \( (\mathcal{K}; V) \) of a positive semidefinite kernel \( k \), is unique modulo unitary equivalence, see [13].

A VH-space \( \mathcal{R} \) over the ordered \( * \)-space \( Z \) is called an \( \mathcal{H} \)-reproducing \textit{kernel VH-space on} \( X \) if there exists a Hermitian kernel \( k: X \times X \rightarrow \mathcal{L}_c^*(\mathcal{H}) \) such that the following axioms are satisfied:

\begin{align*}
(\text{rk1}) & \mathcal{R} \text{ is a subspace of } \mathcal{F}(X; \mathcal{H}), \text{ with all algebraic operations.} \\
(\text{rk2}) & \text{For all } x \in X \text{ and all } h \in \mathcal{H}, \text{ the } \mathcal{H}-\text{valued function } k_x h = k(\cdot, x)h \in \mathcal{R}. \\
(\text{rk3}) & \text{For all } f \in \mathcal{R} \text{ we have } \[ f(x), h \]_\mathcal{H} = [f, k_x h]_\mathcal{R}, \text{ for all } x \in X \text{ and } h \in \mathcal{H}. \\
(\text{rk4}) & \text{For all } x \in X \text{ the evaluation operator } \mathcal{R} \ni f \mapsto f(x) \in \mathcal{H} \text{ is continuous.}
\end{align*}

We now describe invariant linearisations for VH-spaces. Let \( \mathcal{H} \) be a VH-space over an admissible space \( Z \), let \( k: X \times X \rightarrow \mathcal{L}_c^*(\mathcal{H}) \) be a kernel on some nonempty set \( X \), and let \( \Gamma \) be a \( * \)-semigroup that acts on the nonempty set \( X \). As in the case of VE-space operator valued kernels, we call \( k \) \( \Gamma \)-\textit{invariant} if

\begin{equation}
(4.4) \quad k(\xi \cdot x, y) = k(x, \xi^* \cdot y), \quad \xi \in \Gamma, \ x, y \in X.
\end{equation}

A triple \( (\mathcal{K}; \pi; V) \) is called a \( \Gamma \)-\textit{invariant VH-space linearisation} for \( k \) if

\begin{align*}
(\text{ihl1}) & (\mathcal{K}; V) \text{ is a VH-space linearisation of } k. \\
(\text{ihl2}) & \pi: \Gamma \rightarrow \mathcal{L}_c^*(\mathcal{K}) \text{ is a } * \text{-representation.} \\
(\text{ihl3}) & V(\xi \cdot x) = \pi(\xi)V(x) \text{ for all } \xi \in \Gamma \text{ and all } x \in X.
\end{align*}

Also, \( (\mathcal{K}; \pi; V) \) is \textit{minimal} if the VH-space linearisation \( (\mathcal{K}; V) \) is minimal, that is, \( \mathcal{K} \) is the closure of the linear span of \( V(X)\mathcal{H} \).

The following theorem is Theorem 2.10 from [5], see also the Remark 2.11 following it. It characterizes the existence of invariant VH-space linearisations, equivalently, existence of reproducing kernel VH-spaces (with \( * \)-representations of the underlying \( * \)-semigroup) of kernels valued in the class \( \mathcal{L}_c^*(\mathcal{H}) \) of a VH-space \( \mathcal{H} \).

\textbf{Theorem 4.1.} Let \( \Gamma \) be a \( * \)-semigroup that acts on the nonempty set \( X \) and let \( k: X \times X \rightarrow \mathcal{L}_c^*(\mathcal{H}) \) be a kernel, for some VH-space \( \mathcal{H} \) over an admissible space \( Z \). Let \( S_0(Z) \subseteq S(Z) \) be a family of seminorms generating the topology of \( Z \). Then the following assertions are equivalent:

1. \( k \) is positive semidefinite, in the sense of (4.1), and invariant under the action of \( \Gamma \) on \( X \), that is, (4.3) holds, and, in addition, the following conditions hold:
   a1. For any \( \xi \in \Gamma \) and any seminorm \( p \in S_0(Z) \), there exists a seminorm \( q \in S(Z) \) and a constant \( c_p(\xi) \geq 0 \) such that for all \( n \in \mathbb{N} \), \( \{h_i\}_{i=1}^n \in \mathcal{H} \), \( \{x_i\}_{i=1}^n \in X \) we have
      \[
      p(\sum_{i,j=1}^n [k(\xi \cdot x_i, \xi \cdot x_j)h_j, h_i]_\mathcal{H}) \leq c_p(\xi) q(\sum_{i,j=1}^n [k(x_i, x_j)h_j, h_i]_\mathcal{H}).
      \]
   b2. For any \( x \in X \) and any seminorm \( p \in S_0(Z) \), there exists a seminorm \( q \in S(Z) \) and a constant \( c_p(x) \geq 0 \) such that for all \( n \in \mathbb{N} \), \( \{y_i\}_{i=1}^n \in X \), \( \{h_i\}_{i=1}^n \in \mathcal{H} \) we
have
\[ p\left(\sum_{i,j=1}^{n}[k(x, y_i)h_i, k(x, y_j)h_j]_H\right) \leq c_p(x) q\left(\sum_{i,j=1}^{n}[k(y_j, y_i)h_i, h_j]_H\right). \]

(2) \(k\) has a \(\Gamma\)-invariant VH-space linearisation \((K; \pi; V)\).

(3) \(k\) admits an \(H\)-reproducing kernel VH-space \(R\) and there exists a \(*\)-representation \(\rho: \Gamma \to \mathcal{L}^*_\pi(\mathcal{R})\) such that \(\rho(\xi)k_\xi h = k_{\xi \cdot h}\) for all \(\xi \in \Gamma\), \(x \in X\), \(h \in H\).

In addition, in case any of the assertions (1), (2), or (3) holds, then a minimal \(\Gamma\)-invariant VH-space linearisation of \(k\) can be constructed, any minimal \(\Gamma\)-invariant VH-space linearisation of \(k\) is unique up to unitary equivalence, and the pair \((R; \rho)\) as in assertion (3) is uniquely determined by \(k\) as well.

In the final part of the paper, we show that, when barreled VH-spaces are considered, condition (b2) of Theorem 4.1 is automatically satisfied. Precisely, we have the following theorem:

**Theorem 4.2.** Let \(\Gamma\) be a \(*\)-semigroup that acts on the nonempty set \(X\) and let \(k: X \times X \to \mathcal{L}^*(H)\) be a kernel, for a barreled VH-space \(H\) over an admissible space \(Z\). Let \(S_0(Z) \subseteq S_\pi(Z)\) be a family of seminorms generating the topology of \(Z\). Then the following assertions are equivalent:

(1) \(k\) is positive semidefinite, in the sense of [4.1], and invariant under the action of \(\Gamma\) on \(X\), that is, [4.3] holds, and, in addition, the following condition hold:

(b1) For any \(\xi \in \Gamma\) and any seminorm \(p \in S_0(Z)\), there exists a seminorm \(q \in S(Z)\) and a constant \(c_p(\xi) \geq 0\) such that for all \(n \in \mathbb{N}\), \(\{h_i\}_{i=1}^n \in H\), \(\{x_i\}_{i=1}^n \in X\) we have
\[ p\left(\sum_{i,j=1}^{n}[k(\xi \cdot x_i, \xi \cdot x_j)h_j, h_i]_H\right) \leq c_p(\xi) q\left(\sum_{i,j=1}^{n}[k(x_i, x_j)h_j, h_i]_H\right). \]

(2) \(k\) has a \(\Gamma\)-invariant VH-space linearisation \((K; \pi; V)\).

(3) \(k\) admits an \(H\)-reproducing kernel VH-space \(R\) and there exists a \(*\)-representation \(\rho: \Gamma \to \mathcal{L}^*_\pi(\mathcal{R})\) such that \(\rho(\xi)k_\xi h = k_{\xi \cdot h}\) for all \(\xi \in \Gamma\), \(x \in X\), \(h \in H\).

In addition, in case any of the assertions (1), (2), or (3) holds, then a minimal \(\Gamma\)-invariant VH-space linearisation of \(k\) can be constructed, any minimal \(\Gamma\)-invariant VH-space linearisation of \(k\) is unique up to unitary equivalence, and the pair \((R; \rho)\) as in assertion (3) is uniquely determined by \(k\) as well.

Before the proof of this theorem, let us recall the definition of an \(m\)-topologisable operator: For a topological VE-space \(\mathcal{E}\) over the topologically ordered \(*\)-space \(Z\), following [32], also see [7], an operator \(T \in \mathcal{L}(\mathcal{E})\) is called \(m\)-topologisable if for every \(p \in S_0(Z)\), where \(S_0(Z) \subseteq S(Z)\) is an arbitrary generating family of seminorms, there exists a constant \(D_p \geq 0\) and a continuous seminorm \(r\) on \(\mathcal{E}\) such that, for every \(n \in \mathbb{N}\) and every \(h \in \mathcal{E}\),
\[ \hat{p}(T^n h) = p([T^n h, T^n h])^{1/2} \leq D_p^n r(h). \]

It is easy to see that the definition does not depend on the particular family of seminorms \(S_0(Z)\) chosen.
Let us observe that, if $T \in \mathcal{L}_b(E)$, then for every $n \in \mathbb{N}$, $p \in S_+(Z)$ and $h \in \mathcal{E}$ we have

$$\bar{p}(T^nh) = p((T^nh, T^nh))^{\frac{1}{2}} \leq C_p \, p([T^{n-1}h, T^{n-1}h])^{\frac{1}{2}}$$

$$\leq C_p \, p([T^{n-2}h, T^{n-2}h])^{\frac{1}{2}} \leq \cdots \leq C_p \, p(h, h)]^{\frac{1}{2}} = C_p \, \bar{p}(h)$$

for some constant $C_p \geq 0$, hence $T$ is m-topologisable.

**Proof.** Using barreledness of $\mathcal{H}$, by Remark 3.3 we have $\mathcal{L}^*_b(\mathcal{H}) = \mathcal{L}_b^*(\mathcal{H}) = \mathcal{L}^*(\mathcal{H})$. Therefore, in particular, $k(x, x) \in \mathcal{L}^*(\mathcal{H})$ is m-topologizable for any $x \in X$. By Proposition 2.17 in [5], in this case, condition (b2) is automatically satisfied and the theorem follows. \hfill \Box

**Remark 4.3.** By Corollary 3.8 the assumption of barreledness of the VH-space $\mathcal{H}$ in Theorem 4.2 can be replaced by the assumption that, all the quotient normed VE-spaces $\mathcal{H}_p$ are barreled, for $p \in S_0(Z)$ with $S_0(Z) \subseteq S_+(Z)$ any generating family of seminorms.

**Remark 4.4.** Theorem 4.2 provides a partial answer to a question raised in [5]: When the condition (b2) in Theorem 4.1 is satisfied automatically? Thus, one answer is when the VH-space $\mathcal{H}$ is a barreled VH-space, or when the quotient normed VE-spaces $\mathcal{H}_p$, $p \in S_0(Z)$ are barreled. A full characterization of the condition (b2) remains an open problem.

**Remark 4.5.** It would be interesting to know if the linearisation spaces $\mathcal{K}$ and/or $\mathcal{R}$ in Theorem 4.2 are barreled VH-spaces in general.

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