The center of the twisted Heisenberg category, factorial Schur $Q$-functions, and transition functions on the Schur graph

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Abstract
We establish an isomorphism between the center of the twisted Heisenberg category and the subalgebra $\Gamma_1$ of the symmetric functions generated by odd power sums. We give a graphical description of the factorial Schur $Q$-functions and inhomogeneous power sums as closed diagrams in the twisted Heisenberg category and show that the bubble generators of the center correspond to two sets of generators of $\Gamma_1$ which encode data related to up/down transition functions on the Schur graph. Finally, we describe an action of the trace of the twisted Heisenberg category, the $W$-algebra $W^- \subset W_{1+\infty}$, on $\Gamma$.

Keywords
Hecke algebras · Spin representation theory · Schur $Q$-functions · Schur graph

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1 Introduction

In [14], Khovanov describes a linear monoidal category $\mathcal{H}$ which conjecturally categorifies the Heisenberg algebra. The morphisms of $\mathcal{H}$ are governed by a graphical calculus of planar diagrams. This category has connections to many interesting areas of representation theory and combinatorics. The trace of $\mathcal{H}$, which can be defined diagrammatically as the algebra of diagrams on the annulus, is shown in [5] to be isomorphic to the $W$-algebra $W_{1+\infty}$ at level one. The center of $\mathcal{H}$, which is the algebra $\text{End}_{\mathcal{H}}(1)$ of endomorphisms of the monoidal identity, is shown in [16] to be isomorphic to the algebra of shifted symmetric functions $\Lambda^*$ of Okounkov and Olshanski [20].

A twisted version of Khovanov’s Heisenberg category was introduced by Cautis and Sussan [6]. The twisted Heisenberg category $\mathcal{H}_{tw}$ is a $\mathbb{C}$-linear additive monoidal category, with an additional $\mathbb{Z}/2\mathbb{Z}$-grading. It conjecturally categorifies the twisted Heisenberg algebra. The center of $\mathcal{H}_{tw}$, $\text{End}_{\mathcal{H}_{tw}}(1)$, was studied in [21] where it was shown that as a commutative $\mathbb{C}$-algebra, $\text{End}_{\mathcal{H}_{tw}}(1) \cong \mathbb{C}[d_2, d_4, d_6, \ldots] \cong \mathbb{C}[ar{d}_2, \bar{d}_4, \bar{d}_6, \ldots]$, where $d_{2k}$ and $\bar{d}_{2k}$ are certain clockwise and counterclockwise bubble generators, respectively. While symmetric groups play a central role for $\mathcal{H}$ in [14], finite Sergeev superalgebras $\{S_n\}_{n \geq 0}$ (also known as finite Hecke–Clifford algebras of type $A$) play the central role for $\mathcal{H}_{tw}$. In particular, Cautis and Sussan construct a family of functors $\{F_{n,\mathcal{H}_{tw}}\}_{n \geq 0}$ from $\mathcal{H}_{tw}$ to bimodule categories of Sergeev algebras in order to categorify the Fock space representation of the twisted Heisenberg algebra. When restricted to $\text{End}_{\mathcal{H}_{tw}}(1)$, each functor gives a surjective algebra homomorphism $F_{n,\mathcal{H}_{tw}} : \text{End}_{\mathcal{H}_{tw}}(1) \rightarrow \mathbb{Z}(S_n)$, where $\mathbb{Z}(S_n)$ is the even center of $S_n$.

In this paper, we study the combinatorial and representation theoretic properties of $\text{End}_{\mathcal{H}_{tw}}(1)$. Our main result, Theorem 5.2, establishes an isomorphism $\phi : \text{End}_{\mathcal{H}_{tw}}(1) \sim \Gamma$, where $\Gamma$ is a subalgebra of the algebra of symmetric functions $\Gamma = \mathbb{C}[p_1, p_3, p_5, \ldots]$. ($\Gamma$ is sometimes known as the algebra of supersymmetric [8] or doubly symmetric [22] functions.) The construction of $\phi$ relies on the fact that there are embeddings of both $\text{End}_{\mathcal{H}_{tw}}(1)$ and $\Gamma$ into the algebra of functions on strict partitions, $\text{Fun}(\mathcal{SP}, \mathbb{C})$. In our proof of Theorem 5.2, we identify the images of certain algebraically independent generators of these algebras in $\text{Fun}(\mathcal{SP}, \mathbb{C})$: the closures of $n$-cycles from $\text{End}_{\mathcal{H}_{tw}}(1)$ and inhomogeneous analogues of odd power sums $p_n$ in $\Gamma$. The latter were first investigated by Ivanov in his study of the asymptotic behavior of characters of projective representations of symmetric groups [8]. We go on to identify the closure of idempotents of $S_n$ with scalar multiples of Ivanov’s factorial Schur $Q$-functions. Intriguingly, the coefficients that appear on the image of idempotent closures when written in terms of factorial Schur $Q$-functions count the number of paths between specific vertices in the graph of all strict partitions (also known as the Schur graph). A similar phenomenon was observed in [16]. A dictionary between $\Gamma$ and $\text{End}_{\mathcal{H}_{tw}}(1)$ is found in Table 1.
In parallel to the surjective homomorphisms $\{F_n^{\mathcal{H}_{tw}}\}_{n \geq 0}$ from $\text{End}_{\mathcal{H}_{tw}}(\mathbb{1})$ to $\{Z(\mathbb{S}_n)^{\mathbb{1}}\}_{n \geq 0}$, for all $n \geq 0$ one can also construct surjective homomorphisms $F_n^\Gamma : \Gamma \rightarrow Z(\mathbb{S}_n)^{\mathbb{1}}[8]$. Our isomorphism $\varphi$ is canonical in the sense that it intertwines the pair $F_n^{\mathcal{H}_{tw}}$ and $F_n^\Gamma$ for each $n \geq 0$.

One interesting feature of the center of the non-twisted Heisenberg category $\mathcal{H}$ is that, as shifted symmetric functions, the curl generators are best understood in terms of moments of Kerov’s transition and cotransition measures on Young diagrams; fundamental tools used to answer probabilistic questions related to the asymptotic representation theory of symmetric groups [12]. In this paper, we show that this connection to asymptotic representation theory extends to the twisted Heisenberg category. Specifically, we identify the clockwise bubble generators $\{d_{2k}\}_{k \geq 0}$ and counterclockwise bubble generators $\{\bar{d}_{2k}\}_{k \geq 1}$ with two sets of algebraically independent generators for $\Gamma$ discovered by Petrov [22], $\{g^\downarrow_k\}_{k \geq 0}$ and $\{g^\uparrow_k\}_{k \geq 0}$, respectively. The functions $\{g^\downarrow_k\}_{k \geq 0}$ (respectively, $\{g^\uparrow_k\}_{k \geq 0}$) encode down (resp. up) Markov transition kernels on the Schur graph. In particular, the difference between up and down transition functions manifests itself graphically in $\text{End}_{\mathcal{H}_{tw}}(\mathbb{1})$ as a difference in orientation of diagrams. This seems to be yet another indication of the “planar nature” of structures arising from noncommutative probability theory.

There is a natural action of the trace of a category on its center, which can be diagrammatically defined as gluing annular diagrams (elements of the trace) around planar ones (elements of the center). In the case of Khovanov’s Heisenberg category, the results of [5,16] give rise to an action of $W_{1+\infty}$ on the algebra of shifted symmetric functions. This representation of $W_{1+\infty}$ was described in terms of symmetric group representation theory by Lascoux and Thibon [17]. The trace of $\mathcal{H}_{tw}$ is shown in [21] to be isomorphic to a classical-type subalgebra of $W_{1+\infty}$ discovered by Kac, Wang, and Yan called $W^-$ [11]. This result along with Theorem 5.2 gives a representation of $W^-$ on $\Gamma$. In this paper, we describe this representation which is a twisted version of the representation described in [17].

The paper is structured as follows. In Sect. 2, we describe necessary background material on Schur’s graph and the representation theory of Sergeev algebras. In Sect. 3, we describe the subalgebra $\Gamma$ of the symmetric functions and several of its bases. In Sect. 4, we recall the definition of the twisted Heisenberg category $\mathcal{H}_{tw}$ and review the functors $\{F_n^{\mathcal{H}_{tw}}\}_{n \geq 0}$. In Sect. 5, we first establish the isomorphism between $\text{End}_{\mathcal{H}_{tw}}(\mathbb{1})$ and $\Gamma$ and then describe the $W$-algebra $W^-$ and its induced action on $\Gamma$. 

\[ \text{End}_{\mathcal{H}_{tw}}(\mathbb{1}) \xleftrightarrow{\varphi} \Gamma \]

\[ F_n^{\mathcal{H}_{tw}} \downarrow \quad Z(\mathbb{S}_n)^{\mathbb{1}} \uparrow \quad F_n^\Gamma \]

\[ \downarrow \quad \Gamma \quad \uparrow \]

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2 The Schur graph and Sergeev algebras

2.1 Transition functions on the Schur graph

Let $\mathcal{P}_n$ be the set of all partitions of $n$ and set

$$\mathcal{P} := \bigcup_{n \geq 0} \mathcal{P}_n.$$ 

We freely identify a partition $\rho$ with its corresponding Young diagram. If $\rho \in \mathcal{P}_n$, then we write $|\rho| = n$. If $\rho = (\rho_1, \rho_2, \ldots, \rho_r)$ and $\eta = (\eta_1, \eta_2, \ldots, \eta_t) \in \mathcal{P}$, then we write $\eta \subset \rho$ when $\eta_i \leq \rho_i$ for all $i \geq 1$. We denote the number of parts (or length) of a partition $\rho$ by $\ell(\rho)$. A partition $\mu = (\mu_1, \ldots, \mu_r) \in \mathcal{P}_n$ is called an odd partition if $\mu_i$ is odd for all $1 \leq i \leq r$. We denote the collection of odd partitions of $n$ by $\mathcal{OP}_n$ and set $\mathcal{OP} := \bigcup_{n \geq 0} \mathcal{OP}_n$.

We call a partition $\lambda \in \mathcal{P}_n$ strict if all its nonzero parts are distinct. Let $\mathcal{SP}_n$ be the set of all strict partitions of $n$ and set $\mathcal{SP} := \bigcup_{n \geq 0} \mathcal{SP}_n$. To a strict partition $\lambda$, we can associate its shifted Young diagram $S(\lambda)$ which is obtained from the Young diagram (using English notation) by shifting all rows so that the $i$th row is shifted rightward by $(i - 1)$ cells.

**Example** Let $\lambda = (6, 5, 2, 1) \in \mathcal{SP}_{14}$, then

$$\lambda = \begin{array}{ccccccc}
\square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square \\
\end{array} , \\
S(\lambda) = \begin{array}{ccccccc}
\square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & \square & \square \\
\end{array} .$$

Henceforth, we reserve the variables $\lambda$ and $\nu$ for strict partitions and the variables $\mu$ and $\gamma$ for odd partitions.

For $\nu, \lambda \in \mathcal{SP}$, we write $\nu \nearrow \lambda$ (respectively, $\nu \searrow \lambda$) when we can obtain $\lambda$ from $\nu$ by adding (resp. removing) a single cell $\square$. Set

$$\kappa(\nu, \lambda) := \begin{cases} 
2 & \text{if } \nu \nearrow \lambda, \ell(\lambda) = \ell(\nu), \\
1 & \text{if } \nu \nearrow \lambda \text{ and } \ell(\lambda) = \ell(\nu) + 1, \\
0 & \text{otherwise}. 
\end{cases}$$

Note that when $\nu \nearrow \lambda, \kappa(\nu, \lambda) = 1$ exactly when we obtain $S(\lambda)$ from $S(\nu)$ by adding a cell to the central diagonal of the diagram $S(\nu)$.

**Definition** The Schur graph $G$ is the graded graph such that:

- the vertex set of $G$ corresponds to $\mathcal{SP}$, and the $n$th graded component is $\mathcal{SP}_n$,
- the number of edges from $\nu$ to $\lambda$ is given by $\kappa(\nu, \lambda)$.

The version of $G$ that we consider here is the same as that studied in [22]. Another version of the Schur graph without edge multiplicity was investigated in [4].
graphs have the same down transition functions [see (1) below], so in principle, we could have chosen to use either.

A standard shifted Young tableau of shape $\lambda \in \mathcal{SP}_n$ is a bijective labeling of the cells of $S(\lambda)$ by the integers $\{1, \ldots, n\}$ such that entries increase from left to right across rows and down columns. Let $g_\lambda$ be the number of standard shifted Young tableaux of shape $\lambda$. We can compute $g_\lambda$ explicitly [18, Section III.8, Example 12] as:

$$g_\lambda = \frac{n!}{\lambda_1!\lambda_2! \cdots \lambda_r!} \prod_{1 \leq i < j \leq \ell(\lambda)} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}.$$ 

Following [22], we denote the number of paths from $\emptyset$ to $\lambda$ in $\mathcal{G}$ by $h(\lambda)$. Then,

$$h(\lambda) = 2^{|\lambda| - \ell(\lambda)} g_\lambda.$$

In [3], Borodin and Olshanski used coherent families of measures on partitions to construct infinite-dimensional diffusion processes. Petrov studied analogous processes on the Schur graph [22]. We review some basic definitions related to the latter of these works below.

The down transition function $p^\downarrow : \mathcal{G} \times \mathcal{G} \to \mathbb{Q}$ on $\mathcal{G}$ is defined so that for $\nu, \lambda \in \mathcal{SP}$,

$$p^\downarrow(\lambda, \nu) := \frac{h(\nu)}{h(\lambda)} \kappa(\nu, \lambda).$$

(1)

In particular, when the first argument of $p^\downarrow$ is restricted to $\mathcal{SP}_n$ and the second to $\mathcal{SP}_{n-1}$, the function $p^\downarrow$ gives a Markov transition kernel from $\mathcal{G}_n$ to $\mathcal{G}_{n-1}$.

A coherent system on $\mathcal{G}$ with respect to down transition function $p^\downarrow$ is a collection of probability measures $\{M_n\}_{n \geq 0}$, with $M_n$ a probability measure on $\mathcal{G}_n$, such that if $\nu \in \mathcal{SP}_{n-1}$, then

$$M_{n-1}(\nu) = \sum_{\lambda \searrow \nu} p^\downarrow(\lambda, \nu) M_n(\lambda).$$

By abuse of notation, we write $M_n(\lambda)$ for $M_n(\{\lambda\})$. One choice of coherent system with respect to the $p^\downarrow$ defined by (1) is the collection of Plancherel measures $\{Pl_n\}_{n \geq 0}$ where for $\lambda \in \mathcal{SP}_n$,

$$Pl_n(\lambda) := \frac{2^{\ell(\lambda) - n} h(\lambda)^2}{n!}.$$ 

Given the coherent system $\{Pl_n\}_{n \geq 0}$ and the down transition function $p^\downarrow$, the corresponding up transition function $p^\uparrow : \mathcal{G} \times \mathcal{G} \to \mathbb{Q}$ is defined as:

$$p^\uparrow(\nu, \lambda) := \frac{Pl_{n+1}(\lambda)}{Pl_n(\nu)} p^\downarrow(\lambda, \nu) = \frac{h(\lambda)}{h(\nu)(|\nu| + 1)}.$$

(In [22], this is denoted by $p^\uparrow_\infty$.)

In the next section, we will make a connection between induction and restriction of simple Sergeev supermodules and $p^\uparrow$, $p^\downarrow$. 

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2.2 The Sergeev algebra and the twisted hyperoctahedral group

Let $S_n$ be the symmetric group on $n$ elements with $s_1, s_2, \ldots, s_{n-1}$, the Coxeter generators of $S_n$. We will work with the Clifford algebra $C\uparrow_n$ which is the unital associative algebra with $n$ generators:

$$C\uparrow_n := \mathbb{C}(c_1, \ldots, c_n \mid c_i^2 = -1, c_ic_j = -c_jc_i \text{ for } i \neq j).$$  (2)

**Remark 2.1** There is another common presentation of $C\uparrow_n$ in which $C\uparrow_n$ is generated by $c'_1, \ldots, c'_n$ subject to the relations $c'_i^2 = 1$ and $c'_ic'_j = -c'_jc'_i$ for $i \neq j$ (see [15,28] for example). An equivalence between this presentation and (2) can be obtained by setting $c_i = \sqrt{-1}c'_i$.

**Definition** The finite Sergeev algebra (also known as the finite Hecke–Clifford algebra of type A) is

$$S_n \simeq C\uparrow_n \rtimes \mathbb{C}[S_n]$$  (3)

where the action of $S_n$ on the Clifford generators is by permuting indices, i.e.,

$$s_ic_i = c_{i+1}s_i, \quad s_is_{i+1} = c_is_i, \quad \text{and} \quad s_ic_j = c_js_i \quad \text{for } j \neq i, \ i + 1.$$

There is a $\mathbb{Z}/2\mathbb{Z}$-grading making $S_n$ into a superalgebra in which the $S_n$ generators $\{s_i \mid 1 \leq i \leq n - 1\}$ are even and the $C\uparrow_n$ generators $\{c_j \mid 1 \leq j \leq n\}$ are odd. For homogeneous element $x \in S_n$, we write $|x|$ for the degree of $x$. The Sergeev algebras form a tower of superalgebras via the embedding $S_{n-1} \hookrightarrow S_n$ which sends $s_i \mapsto s_i$ for $1 \leq i \leq n - 2$ and $c_i \mapsto c_i$ for $1 \leq i \leq n - 1$. We call this the standard embedding and use it implicitly throughout this paper. We set $S_0 = \mathbb{C}$.

It will be convenient to realize $S_n$ as the quotient of a group algebra. Let $C_2$ denote the cyclic group of order two. Define the group

$$\Pi_n := \langle z, a_1, \ldots, a_n \mid a_i^2 = z, a_ia_j = za_ja_i, za_i = a_iz, z^2 = 1 \rangle.$$

The group $\Pi_n$ is a double cover of $C^2_2$ via the short exact sequence:

$$1 \longrightarrow C_2 \longrightarrow \Pi_n \longrightarrow C^2_2 \longrightarrow 1$$  (4)

in which $C_2$ is mapped to the subgroup $\{1, z\} \subset \Pi_n$ and $z \in \Pi_n$ is mapped to 1.

**Definition** The twisted hyperoctahedral group is defined as $\hat{B}_n := \Pi_n \rtimes S_n$ where $S_n$ acts on $\{a_i\}$ by permuting their indices, and acts trivially on $z$.

The algebra $\mathbb{C}[\hat{B}_n]$ is also a superalgebra via the $\mathbb{Z}/2\mathbb{Z}$-grading which sets $\deg(a_j) = 1$ for $1 \leq j \leq n$ and $\deg(z) = \deg(s_i) = 0$ for $1 \leq i \leq n - 1$. Using (4), one can show that $\hat{B}_n$ is a double cover of the hyperoctahedral group $B_n = C^2_2 \times S_n$ (i.e., the type $B$ Weyl group) via the short exact sequence:

$$1 \longrightarrow C_2 \longrightarrow \hat{B}_n \overset{f}{\longrightarrow} B_n \longrightarrow 1$$
where $f$ sends $z$ to 1. On the other hand, from a comparison of generators and relations it is clear that
\[ S_n \simeq \mathbb{C}[\widehat{B}_n]/\langle z + 1 \rangle. \tag{5} \]

We denote the corresponding projection by $\pi_n : \mathbb{C}[\widehat{B}_n] \to S_n$.

Since $z$ is central and $z^2 = 1$, for any $\mathbb{C}[\widehat{B}_n]$-supermodule $L$, we have that $z$ must act by multiplication by either 1 or $-1$. Hence, studying $S_n$-supermodules is equivalent to studying $\mathbb{C}[\widehat{B}_n]$-supermodules in which $z$ acts as multiplication by $-1$. (These are commonly referred to as spin representations of $\widehat{B}_n$.) Furthermore, via the super Wedderburn theorem it follows that
\[ \mathbb{C}[\widehat{B}_n] \cong \mathbb{C}[\widehat{B}_n]/(z - 1) \oplus \mathbb{C}[\widehat{B}_n]/(z + 1) \cong \mathbb{C}[B_n] \oplus S_n. \tag{6} \]

The group algebras $\mathbb{C}[\widehat{B}_n]$ also form a tower of algebras with the embedding $\mathbb{C}[\widehat{B}_{n-1}] \hookrightarrow \mathbb{C}[\widehat{B}_n]$ which sends $s_i \mapsto s_i$, $a_i \mapsto a_i$, and $z \mapsto z$. Note that this maps the subalgebra $\mathbb{C}[S_{n-1}]$ into the subalgebra $\mathbb{C}[S_n]$ in the usual way, and projected down to $S_{n-1}$ and $S_n$, this becomes the standard embedding described above. We set $\mathbb{C}[\widehat{B}_0]$ to be the subalgebra generated by $z$.

**Lemma 2.2** For $n \geq 2$,
\[ \{ s_1 \ldots s_{n-1}a_n^\epsilon \mid 1 \leq i \leq n, \epsilon \in \{0, 1\} \} \tag{7} \]
is a collection of left coset representatives of $\widehat{B}_{n-1}$ in $\widehat{B}_n$.

Note that we follow the convention that the elements corresponding to $i = n$ are $a_n^\epsilon$ for $\epsilon \in \{0, 1\}$ in Lemma 2.2.

**Proof** The set $\{ s_1 \ldots s_{n-1} \mid 1 \leq i \leq n \}$ forms a collection of minimal length left coset representatives of $S_{n-1}$ in $S_n$. It follows from this and the fact that $\widehat{B}_n := \Pi_n \rtimes S_n$ that any element $g \in \widehat{B}_n$ can be written as $g = s_i \ldots s_{n-1} \omega a_n^\epsilon a_J z^\beta$ where $1 \leq i \leq n$, $\omega \in S_{n-1}$, $a_J = a_{j_1} \ldots a_{j_l}$ for some $J = \{j_1, \ldots, j_l\} \subseteq \{1, 2, \ldots, n-1\}$, and $\epsilon, \beta \in \{0, 1\}$. Since $a_n$ commutes with $S_{n-1}$, we have $g = s_i \ldots s_{n-1} a_n^\epsilon \omega a_J z^\beta$. Since $\omega a_J z^\beta \in \widehat{B}_{n-1}$, the set (7) contains a set of left coset representatives. The result then follows from the observation that the size of (7) is $2n$ while $|\widehat{B}_n| = 2^{n+1}n!$ and $|\widehat{B}_{n-1}| = 2^n(n-1)!$.

**Remark 2.3** If $g = s_i \ldots s_{n-1} a_n^\epsilon$ for $\epsilon \in \{0, 1\}$, then $g^{-1} = a_n^{-\epsilon} s_{n-1} \ldots s_i$, and consequently, while $\pi_n(g) = s_i \ldots s_{n-1} c_n^\epsilon$, we have
\[ \pi_n(g^{-1}) = (-1)^{1-\epsilon} c_n^\epsilon s_{n-1} \ldots s_i = (-1)^{|g|} c_n^\epsilon s_{n-1} \ldots s_i. \]

We use the inclusions $\widehat{B}_1 \subset \widehat{B}_2 \subset \cdots \subset \widehat{B}_{n-1} \subset \widehat{B}_n \subset \cdots$ to iterate Lemma 2.2 to get that for all $1 \leq k < n$,
\[ \widehat{L}c_n^\epsilon := \{ (s_{i_1} \ldots s_{n-1} a_n^\epsilon)(s_{i_1} \ldots s_{n-2} a_n^{\epsilon-1}) \ldots (s_{i_k+1} \ldots s_k a_{k+1}^{\epsilon_k}) \mid 1 \leq i_j \leq j, \epsilon_j \in \{0, 1\} \} \]

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is a collection of left coset representatives of $\hat{B}_k$ in $\hat{B}_n$. Note in particular that

$$|\hat{\mathcal{LC}}^n_k| = \frac{|\hat{B}_n|}{|B_k|} = n^\downarrow k 2^{n-k} \quad (8)$$

where $n^\downarrow k$ is the falling factorial

$$n^\downarrow k := \frac{n!}{(n-k)!} = n(n-1)\ldots(n-k+1)$$

for $1 \leq k < n$. The projection $\pi_n : \mathbb{C}[\hat{B}_n] \to S_n$ sends the elements of $\hat{\mathcal{LC}}^n_k$ to distinct nonzero elements of $S_n$, and we set

$$\mathcal{LC}^n_k := \pi_n(\hat{\mathcal{LC}}^n_k).$$

The set of conjugacy classes of $\hat{B}_n$ is indexed by pairs of partitions $(\rho_+, \rho_-)$ such that $|\rho_+| + |\rho_-| = n$, plus an additional parameter $\epsilon \in \{0, 1\}$ when either $(\rho_+, \rho_-) = (\mu, \emptyset)$ with $\mu \in \mathcal{OP}_n$ or $(\rho_+, \rho_-) = (\emptyset, \lambda)$ with $\lambda \in \mathcal{SP}_n$. We denote this indexing set by Conj. A detailed description of the conjugacy class structure of $\hat{B}_n$ can be obtained by analyzing the conjugacy class structure of $B_n$ (which follows from the basic theory for the conjugacy class structure of wreath products [18, Appendix B]) and investigating how the inverse image of these sets under the map $\gamma : \hat{B}_n \to B_n$ splits into new conjugacy classes [23]. The additional parameter $\epsilon \in \{0, 1\}$ appears precisely when a conjugacy class in $B_n$ splits into two conjugacy classes in $\hat{B}_n$. Since we are ultimately interested in the center of $S_n$, which can be described using information about the conjugacy classes indexed by $(\mu, \emptyset, \epsilon)$ for $\mu \in \mathcal{OP}_n$ and $\epsilon \in \{0, 1\}$, we limit ourselves to considering these classes. We call this set of conjugacy classes $\text{Conj}^\text{odd}_1 \subset \text{Conj}$. For $\beta \in \text{Conj}$, we write $\text{Conj}(\beta)$ for the corresponding conjugacy class.

We introduce a family of elements of $S_n$ which will be useful for constructing representatives for the conjugacy classes from $\text{Conj}^\text{odd}_1$. For $\mu = (\mu_1, \ldots, \mu_r) \in \mathcal{OP}_k$, set $\pi_{\mu} = 1$ if $\mu = (1^k)$ and otherwise

$$\pi_{\mu} := (s_{k-1} \ldots s_{k-\mu_r+1}) \ldots (s_{\mu_1+\mu_2-1} \ldots s_{\mu_1+1})(s_{\mu_1-1} \ldots s_{\mu_1}) = (k, k-1, \ldots, k-\mu_r+1) \ldots (\mu_1+\mu_2, \ldots, \mu_1+1)(\mu_1, \ldots, 2, 1) \in S_k. \quad (9)$$

For $n \geq k$, we define $\sigma_{\mu,n} := \tau_0 \pi_{\mu} \tau_0^{-1}$, where $\tau_0$ is the longest element of $S_n$ by Coxeter length. Notice that $\sigma_{\mu,n}$ has cycle type $(\mu, 1^{n-k}) \in \mathcal{OP}_n$ and fixes $1, 2, \ldots, n-k$ pointwise.

**Proposition 2.4** The elements $\{ \sigma_{\mu,n}, \ z\sigma_{\mu,n} \ | \ \mu \in \mathcal{OP}_n \}$ form a complete set of conjugacy class representatives for the conjugacy classes $\text{Conj}^\text{odd}_1$ in $\hat{B}_n$ with $\sigma_{\mu,n}$ corresponding to $(\mu, \emptyset, 0) \in \text{Conj}^\text{odd}$ and $z\sigma_{\mu,n}$ corresponding to $(\mu, \emptyset, 1) \in \text{Conj}^\text{odd}$.

**Proof** This follows from the description of the conjugacy classes of $B_n$ and results on conjugacy class splitting in $\hat{B}_n$ [23] (see [28, Section 2.5] for an overview). \qed

\(
\end{verbatim}

\(\hat{\mathcal{LC}}_{\downarrow}^n \)
Note that under the projection map \( \pi_n : \mathbb{C}[\hat{\mathcal{B}}_n] \to \mathbb{S}_n \), the two sets of conjugacy classes \( \{ \sigma_{\mu;n} \}_{\mu \in \mathcal{P}_n} \) and \( \{ z\sigma_{\mu;n} \}_{\mu \in \mathcal{P}_n} \) are identified since \( \pi_n(z) = -1 \).

The size of the conjugacy classes \( \text{Conj}(\mu, \emptyset, \epsilon) \) will be important to us later. For \( \rho \in \mathcal{P}_n \), we denote by \( z_\rho \) the size of the stabilizer of an element of \( \mathbb{S}_n \) of cycle type \( \rho \) under the conjugation action. Recall that

\[
 z_\rho = \prod_{i \in \mathbb{Z}_{\geq 0}} i^{m_i(\rho)} m_i(\rho)!
\]

where \( m_i(\rho) \) is the number of parts of size \( i \) in \( \rho \).

**Lemma 2.5 [8]** For \( \mu \in \mathcal{O}_n, \epsilon \in \{0, 1\}, \)

\[
 |\text{Conj}(\mu, \emptyset, \epsilon)| = \frac{n!}{z_\mu} 2^{n-\ell(\mu)}. 
\]

\( \mathbb{S}_n \) has analogues to the classical Jucys–Murphy elements of \( \mathbb{C}[\mathbb{S}_n] \). These elements \( \{ J_i \}_{i=1}^n \), which we also call Jucys–Murphy elements, are defined by:

\[
 J_1 := 0, \quad J_k := \sum_{j=1}^{k-1} (1 + c_j c_k)(j,k).
\]

They generate a commutative subalgebra of \( \mathbb{S}_n \), and their spectra have a combinatorial interpretation analogous to that of the classical Jucys–Murphy elements [7,19,26,27].

### 2.3 The super representation theory of \( \mathbb{S}_n \) and \( \hat{\mathcal{B}}_n \)

In this subsection, we will review basic facts about the super representation theory of \( \mathbb{S}_n \). Recall that any \( \mathbb{S}_n \)-supermodule is by definition a spin representation of \( \hat{\mathcal{B}}_n \), so all statements about \( \mathbb{S}_n \)-supermodules also hold for \( \hat{\mathcal{B}}_n \) spin representations. We refer the reader to [15,28] for thorough accounts of these topics as well as a review of super representation theory.

Let \( \delta : \mathcal{SP} \to \{0, 1\} \) be defined by:

\[
 \delta(\lambda) := \begin{cases} 
 0 & \ell(\lambda) \text{ is even} \\
 1 & \ell(\lambda) \text{ is odd}. 
\end{cases}
\]

The function \( \delta \) will be useful for describing quantities related to the representation theory of \( \mathbb{S}_n \).

**Theorem 2.6 [25]** The set of simple \( \mathbb{S}_n \)-supermodules is indexed by \( \mathcal{SP}_n \), and for any \( \lambda \in \mathcal{SP}_n \), the simple \( \mathbb{S}_n \)-supermodule indexed by \( \lambda \) has dimension

\[
 \dim(L^\lambda) = 2^{n-\frac{(\ell(\lambda)+\delta(\lambda))}{2}} g_\lambda. 
\]
The algebras $\{S_n\}_{n \geq 0}$ are semisimple. When $N$ and $M$ are $S_n$-supermodules, we write

$$[M : N] := \dim(\text{Hom}_{S_n}(M, N)).$$

The following theorem describes the branching for $\{S_n\}_{n \geq 0}$.

**Theorem 2.7** [15] Let $\lambda \in \mathcal{SP}_n$ and $\nu \in \mathcal{SP}_{n-1}$, then:

1. $[L^\nu : \text{Res}_{S_{n-1}}^{S_n} L^\lambda] = \begin{cases} \frac{2^{2+\ell(\nu) - \delta(\nu) - \ell(\lambda)} - \delta(\lambda)}{2} & \lambda \downarrow \nu \\ 0 & \text{otherwise.} \end{cases}$
2. $[\text{Ind}_{S_{n-1}}^{S_n} L^\nu : L^\lambda] = \begin{cases} \frac{2^{2+\ell(\nu) - \delta(\nu) + \delta(\lambda)}}{2} & \nu \nearrow \lambda, \\ 0 & \text{otherwise.} \end{cases}$

The next theorem describes how $S_n$ as a left $S_n$-supermodule decomposes into a direct sum of simple $S_n$-supermodules.

**Proposition 2.8** $S_n$ as a left $S_n$-supermodule decomposes as

$$S_n \cong \bigoplus_{\lambda \in \mathcal{SP}_n} (L^\lambda)^{d(\lambda)/2}.$$

**Proof** For $\lambda \in \mathcal{SP}_n$, the multiplicity of $L^\lambda$ in $S_n$ is equal to the multiplicity of the corresponding spin representation $\hat{L}^\lambda$ of $\hat{B}_n$ in the left regular super representation. If $\hat{L}^\lambda$ is of type M (i.e., $\delta(\lambda) = 0$), then $L^\lambda$ is simple as an ungraded $\hat{B}_n$-module [15, Section 12.2]. When $\hat{L}^\lambda$ is of type Q (i.e., $\delta(\lambda) = 1$) then considered as an ungraded $\mathbb{C}[\hat{B}_n]$-module, $\hat{L}^\lambda$ splits into a direct sum of two simple $\mathbb{C}[\hat{B}_n]$-modules, each with dimension $\dim(\hat{L}^\lambda)/2$. The result then follows from the ungraded representation theory of $\hat{B}_n$. \qed

We can now relate $p^\downarrow(\cdot, \cdot)$ and $p^\uparrow(\cdot, \cdot)$ to the representation theory of the algebras $\{S_n\}_{n \geq 0}$.

**Proposition 2.9** Let $\lambda \in \mathcal{SP}_n$ and $\nu \in \mathcal{SP}_{n-1}$. Then,

1. $p^\downarrow(\lambda, \nu) = \frac{[L^\nu : \text{Res}_{S_{n-1}}^{S_n} L^\lambda] \dim(L^\nu)}{\dim(L^\lambda)},$
2. $2^{\delta(\lambda) - \delta(\nu)} p^\uparrow(\nu, \lambda) = \frac{[\text{Ind}_{S_{n-1}}^{S_n} L^\nu : L^\lambda] \dim(L^\lambda)}{\dim(\text{Ind}_{S_{n-1}}^{S_n} L^\nu)}.$

**Proof** Both 1 and 2 follow from the branching rules for Sergeev algebras in Theorem 2.7 and the dimension formula in Theorem 2.6. \qed

For $\lambda \in \mathcal{SP}_n$, we denote by $\hat{\chi}^\lambda$ the character corresponding to simple $\mathbb{C}[\hat{B}_n]$-supermodule $\hat{L}^\lambda$. This descends to a character $\chi^\lambda$ for simple $S_n$-supermodule $L^\lambda$ with

$$\chi^\lambda(\pi_n(g)) = \hat{\chi}^\lambda(g).$$
The normalized character \( \tilde{\chi}^\lambda \) is defined such that for \( x \in S_n \),

\[
\tilde{\chi}^\lambda(x) := \frac{\chi^\lambda(x)}{\dim(L^\lambda)} = \frac{\chi^\lambda(x)}{\chi^\lambda(1)}.
\]

**Proposition 2.10**  For \( \lambda \in \mathcal{SP}_n \), the character \( \chi^\lambda \) is uniquely determined by its value on the elements \( \{ \sigma_{\mu;n} \mid \mu \in \mathcal{OP}_n \} \).

**Proof**  This follows from a similar statement [8, Proposition 1.9] where each element \( \sigma_{\mu;n} \) is replaced by an element of \( S_n \) that is conjugate to it. Since characters are constant across conjugacy classes, the result follows. \( \square \)

Given Proposition 2.10, for \( \mu \in \mathcal{OP}_k \) with \( k \leq n \), it is convenient to write \( \chi^\lambda(\mu \cup 1^{n-k}) := \chi^\lambda(\sigma_{\mu;n}) \).

### 2.4 The centers of \( S_n \) and \( \mathbb{C}[\hat{B}_n] \)

As a superalgebra the center of \( S_n \) breaks up into even and odd components of supercommutative elements. In this paper, we will focus on \( Z(\mathbb{S}_n)_{\overline{\mathbb{1}}} \), which corresponds to the ungraded center of \( \mathbb{S}_n \). Note that \( Z(\mathbb{S}_n)_{\overline{\mathbb{1}}} \) is exactly those elements that act on all simple \( \mathbb{S}_n \)-modules as multiplication by a scalar. Following [8], we will construct a basis for \( Z(\mathbb{S}_n)_{\overline{\mathbb{1}}} \) via the surjection \( \pi : \mathbb{C}[\hat{B}_n] \rightarrow \mathbb{S}_n \).

Recall that the set Conj indexes the conjugacy classes of \( \hat{B}_n \). For \( \beta \in \text{Conj} \), set

\[
\hat{C}_\beta := \sum_{g \in \text{Conj}(\beta)} g.
\]

It is clear that \( \{ \hat{C}_\beta \}_{\beta \in \text{Conj}} \) is a basis for the ungraded center of \( \mathbb{C}[\hat{B}_n] \). In [8], Ivanov uses the subset of this basis corresponding to elements of \( \text{Conj}_{\text{odd}} \) of the form \((\mu, \emptyset, 0)\) to construct a basis for \( Z(\mathbb{S}_n)_{\overline{\mathbb{1}}} \). For \( \mu \in \mathcal{OP}_n \), let

\[
C_\mu := \pi_n(\hat{C}_{(\mu, \emptyset, 0)}).
\]

**Proposition 2.11** [8]  The set \( \{ C_\mu \mid \mu \in \mathcal{OP}_n \} \) is a linear basis for \( Z(\mathbb{S}_n)_{\overline{\mathbb{1}}} \).

We now define a scaled version of Ivanov’s basis of \( Z(\mathbb{S}_n)_{\overline{\mathbb{1}}} \) which naturally appears from the center of the twisted Heisenberg category.

**Definition**  For \( k \leq n \) and \( \mu \in \mathcal{OP}_k \), define

\[
\hat{A}_{\mu;n} := \sum_{g \in \hat{\text{Conj}}_{n-k}} g \sigma_{\mu;n} g^{-1}
\]

and

\[
A_{\mu;n} := \pi(\hat{A}_{\mu;n}).
\]
Proposition 2.12 Let $k \leq n$ and $\mu \in \mathcal{OP}_k$, then:

1. $\hat{A}_{\mu;n} \in Z(\mathbb{C}[\hat{B}_n])$ and $A_{\mu;n} \in Z(\mathbb{S}_n)$.  
2. $\hat{A}_{\mu;n} = 2^{k-n+\ell(\mu)}z_{\mu \cup 1^{n-k}}\hat{C}(\mu \cup 1^{n-k}, \emptyset, 0)$.  
3. Let $h \in \hat{B}_n$ be an element not belonging to the same conjugacy class as $\sigma_{\mu;n}$ or $z\sigma_{\mu;n}$ for some $\mu \in \mathcal{OP}_n$ (i.e., $h$ does not belong to a conjugacy class indexed by $(\mu, \emptyset, \epsilon)$, $\epsilon \in \{0, 1\}$). Then,

$$
\pi_n\left( \sum_{g \in \hat{B}_n} ghg^{-1} \right) = 0.
$$

Proof. 1. Recall that we defined $\sigma_{\mu;n}$ as a distinguished element from the conjugacy class of $\hat{B}_n$ indexed by $(\mu \cup 1^{n-k}, \emptyset, 0)$. Since $\sigma_{\mu;n}$ is by definition a product of $s_{n-1}, \ldots, s_{n-k+1}$, it commutes with $\hat{B}_{n-k}$. Since $\mathcal{L}_n^{\hat{n}}$ is a collection of left coset representatives of $\hat{B}_{n-k}$ in $\hat{B}_n$, any element $g \in \hat{B}_n$ can be written uniquely as $g = \sigma h$ for $\sigma \in \mathcal{L}_n^{\hat{n}}$ and $h \in \hat{B}_{n-k}$. Thus, $g\sigma_{\mu;n}g^{-1} = \sigma h\sigma_{\mu;n}h^{-1}\sigma^{-1} = \sigma\sigma_{\mu;n}\sigma^{-1}$, and hence, $g\sigma_{\mu;n}g^{-1}$ is completely determined by the left coset to which $g$ belongs. It follows that

$$
\sum_{g \in \hat{B}_n} g\sigma_{\mu;n}g^{-1} = |\hat{B}_{n-k}| \sum_{g \in \mathcal{L}_n^{\hat{n}}} g\sigma_{\mu;n}g^{-1} = |\hat{B}_{n-k}|\hat{A}_{\mu;n}
$$

and $\hat{A}_{\mu;n} \in Z(\mathbb{C}[\hat{B}_n])$ since $\hat{A}_{\mu;n}$ is a scalar multiple of a central element.

Finally, note that $\pi_n$ is a degree-preserving homomorphism and $\hat{A}_{\mu;n}$ is even, so $\pi(\hat{A}_{\mu;n}) = A_{\mu;n} \in Z(\mathbb{S}_n)$.

2. It follows from Lemma 2.5 and the orbit stabilizer theorem that

$$
\sum_{g \in \hat{B}_n} g\sigma_{\mu;n}g^{-1} = 2^{\ell(\mu)+1}z_{\mu \cup 1^{n-k}}\hat{C}(\mu \cup 1^{n-k}, \emptyset, 0).
$$

Then, (10) implies that

$$
|\hat{B}_{n-k}|\hat{A}_{\mu;n} = 2^{\ell(\mu)+1}z_{\mu \cup 1^{n-k}}\hat{C}(\mu \cup 1^{n-k}, \emptyset, 0).
$$

The result follows.

3. It follows from the proofs of [24, Proposition 3.4] and [24, Proposition 3.9] that the image in $\mathbb{S}_n$, $\pi_n(h)$, of any such element, is conjugate to its negative. Hence, each term in the image of the conjugacy class sum of any such $h$ appears in a pair with its negative. The result follows.

It follows from Proposition 2.12.2. that $\{A_{\mu;n} \mid \mu \in \mathcal{OP}_n\}$ is also a linear basis of $Z(\mathbb{S}_n)$.

For a spin representation $\hat{L}^\lambda$ of $\hat{B}_n$, the corresponding character $\hat{\chi}^\lambda$ is a homomorphism when restricted to $Z(\mathbb{C}[\hat{B}_n])_0$ and $\chi^\lambda$ and $\hat{\chi}^\lambda$ are homomorphisms on $Z(\mathbb{S}_n)$.
Proposition 2.13 Let $\lambda \in \mathcal{SP}_n$ and $\mu \in \mathcal{OP}_k$. Then,

$$\widetilde{\chi}^\lambda(A_{\mu,n}) = 2^k n^\frac{k}{2} \chi^\lambda(\sigma_\mu; n).$$

Proof This follows from the fact that characters are invariant under conjugation and (8).

Another basis for $Z(\mathcal{S}_n\hat{G})$ is given by the set of central idempotents of $\mathcal{S}_n$ corresponding to the simple $\mathcal{S}_n$-supermodules. We denote these central idempotents by \{e_\lambda | \lambda \in \mathcal{SP}_n\}.

Lemma 2.14 For $\lambda \in \mathcal{SP}_n$, the central idempotent $e_\lambda \in \mathcal{S}_n$ corresponding to the simple $\mathcal{S}_n$-supermodule $L^\lambda$ can be written as:

$$e_\lambda = 2^{-\frac{\ell(\lambda)}{2} - \frac{\delta(\lambda)}{2}} g_\lambda \sum_{\mu \in \mathcal{OP}_n} \chi^\lambda(\mu) C_\mu.$$

Proof The definition of $\mathcal{S}_n$ implies that $e_\lambda$ is the image of the corresponding central idempotent $\hat{e}_\lambda$ in $\hat{B}_n$ under the projection map $\pi_n$. If $L^\lambda$ is of type M, then $\hat{L}^\lambda$ is simple as an ungraded $\mathbb{C}[\hat{B}_n]$-module; if $L^\lambda$ is of type Q, then $\hat{L}^\lambda$ viewed as an ungraded $\mathbb{C}[\hat{B}_n]$-module breaks into the direct sum of two simples of equal dimension, $\hat{L}^\lambda = \hat{L}^{\lambda_0} \oplus \hat{L}^{\lambda_1}$. Thus, the central idempotent corresponding to $\hat{L}^\lambda$ is given by:

$$\hat{e}_\lambda = \frac{\dim(\hat{L}^\lambda)}{2^\delta(\lambda) |\hat{B}_n|} \sum_{\beta \in \text{Conj}} \hat{\chi}^\lambda(g) \hat{C}_\beta,$$

where if $L^\lambda$ is of type Q, then $\hat{\chi}^\lambda(g) = \hat{\chi}^{\lambda_0}(g) + \hat{\chi}^{\lambda_1}(g)$.

Applying $\pi_n$ to $\hat{e}_\lambda$, Proposition 2.12.3. implies that most terms go to zero and we are left with

$$e_\lambda = 2^{-\frac{\ell(\lambda)}{2} - \frac{\delta(\lambda)}{2}} - 1 g_\lambda \sum_{\beta \in \text{Conj}_{\text{odd}}} \chi^\lambda(\beta) \pi_n(\hat{C}_\beta).$$

Recall that $\beta \in \text{Conj}_{\text{odd}}$ contains pairs $\beta = (\mu, \emptyset, 0)$ and $\hat{\beta} = (\mu, \emptyset, 1)$ for $\mu \in \mathcal{OP}_n$ such that if $x \in \text{Conj}(\beta)$, then $zx \in \text{Conj}(\hat{\beta})$. It follows that $\pi_n(\hat{C}_\beta) = \pi_n(\hat{C}_\beta)$. At the same time, since $z$ acts as multiplication by $-1$ on $\hat{L}^\lambda$, then $\chi^\lambda(z\sigma_\mu; n) = -\chi^\lambda(\sigma_\mu; n)$. It follows that

$$2^{-\frac{\ell(\lambda)}{2} - \frac{\delta(\lambda)}{2}} - 1 g_\lambda \sum_{\beta \in \text{Conj}_{\text{odd}}} \chi^\lambda(\beta) \pi_n(\hat{C}_\beta) = 2^{-\frac{\ell(\lambda)}{2} - \frac{\delta(\lambda)}{2}} g_\lambda \sum_{\mu \in \mathcal{OP}_n} \chi^\lambda(\mu) C_\mu.$$
2.5 Interlacing coordinates for strict partitions

In [13], Kerov developed a useful way to parametrize Young diagrams via their interlacing coordinates. Petrov showed that shifted strict Young diagrams can be similarly parametrized with only slight modification [22]. For \( \lambda \in \mathcal{SP} \), and \( \Box \in S(\lambda) \) with coordinates \((i, j)\), the \textit{content} of \( \Box \) is defined to be

\[
\text{cont}(\Box) := i - j.
\]

Note that when \( \Box \) comes from a shifted diagram, \( \text{cont}(\Box) \) is always nonnegative.

Let

1. \( X(\lambda) \) be the set of contents for cells that we can add to \( S(\lambda) \) to get another shifted strict partition.
2. \( Y(\lambda) \) be the set of contents for cells that we can remove from \( S(\lambda) \) to get another shifted strict partition.

The set \((X(\lambda), Y(\lambda))\) uniquely characterizes \( S(\lambda) \) and is called the \textit{Kerov coordinates} of \( S(\lambda) \). We follow [22] and denote the shifted diagram obtained by adding a cell \( \Box \) with content \( x \in X(\lambda) \) to \( S(\lambda) \) by \( S(\lambda) + \Box(x) \) and the shifted diagram obtained by removing a cell \( \Box \) from \( S(\lambda) \) with content \( y \in Y(\lambda) \) by \( S(\lambda) - \Box(y) \).

\textbf{Example} In the case of \( \lambda = (6, 5, 2, 1) \in \mathcal{SP}_{14}, Y(\lambda) = \{0, 4\}, X(\lambda) = \{2, 6\}, \) and

\[
S(\lambda) = \begin{array}{cccccc}
\textbf{6} & & & & \\
\textbf{4} & & & & \\
\textbf{2} & & & & \\
\textbf{0} & & & & \\
\end{array}
\]

We set

\[
s(i) := i(i + 1).
\]

Petrov defined two families of functions \( g_k^\uparrow: \mathcal{SP} \rightarrow \mathbb{Q} \) and \( g_k^\downarrow: \mathcal{SP} \rightarrow \mathbb{Q} \), such that

\[
g_k^\uparrow(\lambda) := \sum_{x \in X(\lambda)} p^\uparrow(\lambda, \lambda + \Box(x)) s(x)^k
\]

and

\[
g_{k+1}^\downarrow(\lambda) := 2|\lambda| \sum_{y \in Y(\lambda)} p^\downarrow(\lambda, \lambda - \Box(y)) s(y)^k
\]

and investigated their properties in [22]. Note that \( g_0^\uparrow = 1 \).

\textbf{Remark 2.15} \( g_k^\uparrow(\lambda) \) and \( g_k^\downarrow(\lambda) \) are the strict partition analogue to moments of Kerov’s transition and cotransition measure [13]. They will play a similar role to the one they played in [16].
Proposition 2.16 [22, Proposition 5.4] For $\lambda \in \mathcal{SP}_n$,

$$g_k^\uparrow = g_k^\downarrow + \sum_{i,j>0, \ i+j=k} g_i^\uparrow g_j^\downarrow.$$ 

We now give algebraic interpretations of $g_k^\uparrow(\lambda)$ and $g_k^\downarrow(\lambda)$ analogous to those found by Biane for Kerov’s transition and cotransition measure on Young diagrams [2]. Let $\text{pr}_{n-1} : \mathbb{S}_n \to \mathbb{S}_{n-1}$ be the linear map defined such that for $x \in \mathbb{S}_n$,

$$\text{pr}_{n-1}(x) := \begin{cases} x & \text{if } x \in \mathbb{S}_{n-1} \\ 0 & \text{otherwise}. \end{cases}$$

Proposition 2.17 Let $\lambda \in \mathcal{SP}_n$ for $n \geq 1$ and $k \geq 0$, then

1. $\tilde{\chi}^\lambda(\text{pr}_n(J_{2k}^{n+1})) = g_k^\uparrow(\lambda)$.

2. $\tilde{\chi}^\lambda \left( \sum_{x \in \mathcal{LC}_{n-1}} x J_{n+1}^r \right) = \begin{cases} g_k^\downarrow(\lambda) & \text{if } r = 2k \text{ is even} \\ 0 & \text{otherwise}. \end{cases}$

Proof

1. Consider the character $\tau_n : \mathbb{S}_n \to \mathbb{C}$ corresponding to $\mathbb{S}_n$ acting on itself by left multiplication. For $x \in \mathbb{S}_n$,

$$\tau_n(x) := \begin{cases} 2^n n! & \text{if } x = 1 \\ 0 & \text{otherwise}. \end{cases} \quad (11)$$

It follows from (11) that

$$2(n+1) \tau_n(\text{pr}_n(x)) = \tau_{n+1}(x).$$

Also note that if $y \in \mathbb{S}_n$ and $x \in \mathbb{S}_{n+1}$, then

$$\text{pr}_n(yx) = y \text{pr}_n(x).$$

Recall that $e_\lambda$ is the central idempotent of $\mathbb{S}_n$ corresponding to simple $\mathbb{S}_n$-supermodule $L^\lambda$. Then, by Lemma 2.8 there are $2^{-\delta(\lambda)} \dim(L^\lambda)$ copies of $L^\lambda$ in the $\mathbb{S}_n$-supermodule $\mathbb{S}_n$ so that

$$2(n+1) \tau_n(\text{pr}_n(e_\lambda J_{2k}^{n+1})) = 2(n+1) \tau_n(e_\lambda \text{pr}_n(J_{2k}^{n+1}))$$

$$= 2^{1-\delta(\lambda)}(n+1) \dim(L^\lambda) \tilde{\chi}^\lambda(\text{pr}_n(J_{2k}^{n+1})). \quad (13)$$
On the other hand, the weight space decomposition for the Jucys–Murphy operators on $\mathbb{S}_n$-supermodules implies that

$$2(n + 1)\tau_n(\text{pr}_n(e_\lambda J_{n+1}^{2k})) = \tau_{n+1}(e_\lambda J_n^{2k})$$

$$= \sum_{x \in X(\lambda)} \left[ L^\lambda : \text{Res}_{\mathbb{S}_n}^{\mathbb{S}_{n+1}} L^\lambda + \square(x) \right] \frac{\dim(L^\lambda) \dim(L^\lambda + \square(x))}{2^\delta(\lambda + \square(x))} s(x)^k.$$  

Thus, taking the normalized character gives

$$\tilde{\chi}_\lambda(\text{pr}_{n-1}(J_n^{2k})) = \sum_{x \in X(\lambda)} 2^{\delta(\lambda + \square(x)) - \delta(\lambda)} \left[ L^\lambda : \text{Res}_{\mathbb{S}_n}^{\mathbb{S}_{n+1}} L^\lambda + \square(x) \right] \frac{\dim(L^\lambda + \square(x))}{2(n+1) \dim(L^\lambda)} s(x)^k.$$  

As $\dim(\text{Ind}_{\mathbb{S}_n}^{\mathbb{S}_{n+1}} L^\lambda) = 2(n + 1) \dim(L^\lambda)$ and by Frobenius reciprocity

$$\left[ L^\lambda : \text{Res}_{\mathbb{S}_n}^{\mathbb{S}_{n+1}} L^\lambda + \square(x) \right] = \left[ \text{Ind}_{\mathbb{S}_n}^{\mathbb{S}_{n+1}} L^\lambda : L^\lambda + \square(x) \right],$$  

applying Lemma 2.9.2, gives the desired result.

2. The elements $c_i$ and the Jucys–Murphy elements $J_i$ satisfy $J_i c_i = -c_i J_i$, and for $x = s_i \ldots s_{n-1} c_n$, we have $x^{-1} = (-1)^\epsilon c_n s_{n-1} \ldots s_i$. Therefore,

$$\sum_{x \in \mathcal{L}^n_{c_n}} xJ^n_r x^{-1} = \sum_{i=1}^{n} s_i \ldots s_{n-1} J^n_r s_{n-1} \ldots s_i - s_i \ldots s_{n-1} c_n J^n_r c_n s_{n-1} \ldots s_i$$

$$= \sum_{i=1}^{n} s_i \ldots s_{n-1} J^n_r s_{n-1} \ldots s_i - (-1)^{r+1} s_i \ldots s_{n-1} J^n_r s_{n-1} \ldots s_i.$$

(14)

When $r$ is odd, this is then equal to zero. When $r = 2k$, (14) is equal to

$$2 \sum_{i=1}^{n} s_i \ldots s_{n-1} J_n^{2k} s_{n-1} \ldots s_i.$$  

Since characters are invariant under conjugation, we have

$$\tilde{\chi}_\lambda\left(2 \sum_{i=1}^{n} s_i \ldots s_{n-1} J_n^{2k} s_{n-1} \ldots s_i\right) = 2n \tilde{\chi}_\lambda(J_n^{2k}).$$
Decomposing $J_n$ into its weight spaces then gives

$$2n\tilde{\chi}_{\lambda}(J_n^{2k}) = 2n \sum_{y \in Y(\lambda)} \frac{\left[ L^\lambda - \square(y) : \text{Res}_{S_{n-1}}^S L^\lambda \right] \dim(L^\lambda - \square(y)) s(y)^k}{\dim(L^\lambda)}$$

$$= 2n \sum_{y \in Y(\lambda)} p^\downarrow(\lambda, \lambda - \square(y)) s(y)^k$$

where the last equality uses Lemma 2.9.1.

\[\square\]

3 The subalgebra $\Gamma$

We recall relevant facts about the algebra $\Gamma$ following [18]. Let $p_k$ be the $k$th power sum symmetric function, and recall that, for $\rho \in \mathcal{P}$,

$$p_\rho := \prod_{k=1}^{\ell(\rho)} p_{\rho_k}.$$

$\Gamma$ can be described as the subalgebra of the symmetric functions generated by the odd power sums

$$\Gamma = \mathbb{C}[p_1, p_3, p_5, \ldots].$$

Elements of $\Gamma$ can be evaluated on partitions in the following way. Let $f \in \Gamma$ and $\rho \in \mathcal{P}$, and define

$$f(\rho) := f(\rho_1, \rho_2, \ldots, \rho_{\ell(\rho)}, 0, \ldots). \quad (15)$$

Let $\text{Fun}(\mathcal{S}P, \mathbb{C})$ denote the algebra of functions from $\mathcal{S}P$ to $\mathbb{C}$ with pointwise multiplication.

**Proposition 3.1** [10, Proposition 6.2] The algebra $\Gamma$ embeds into $\text{Fun}(\mathcal{S}P, \mathbb{C})$ via the evaluation map (15).

An important linear basis of $\Gamma$ is the Schur $Q$-functions $\{Q_\lambda\}$, indexed by $\mathcal{S}P$ (cf. [18, Section III.8]).

Define numbers $X^\lambda_\mu$ for $\lambda \in \mathcal{S}P_n, \mu \in \mathcal{O}P_n$, via

$$p_\mu = \sum_{\lambda \in \mathcal{S}P_n} 2^{-\ell(\lambda)} X^\lambda_\mu Q_\lambda. \quad (16)$$
There is a “factorial” version of the Schur $Q$-functions, defined in [9]. For $\lambda \in SP$, the factorial Schur $Q$-polynomial corresponding to $\lambda$ is defined as:

$$Q_{\lambda|N}(x_1, \ldots, x_N) := \frac{2^{\ell(\lambda)}}{(N - 1)!} \sum_{\omega \in S_N} \omega \left( x_1^{\downarrow \lambda_1} x_2^{\downarrow \lambda_2} \cdots x_l^{\downarrow \lambda_l} \prod_{\substack{1 \leq i \leq l \leq N \atop i < j}} \frac{x_i + x_j}{x_i - x_j} \right). \quad (17)$$

If $\ell(\lambda) > N$, then $Q_{\lambda|N}$ is defined to be 0. The collection $(Q_{\lambda|N})_{N=1,2,\ldots}$ defines an element of $\Gamma$, the factorial Schur $Q$-function $Q^*_\lambda$. Factorial Schur $Q$-functions have the following useful properties.

**Proposition 3.2** [8] Let $\lambda, \nu \in SP$.

1. There exists $g \in \Gamma$ of degree less than $|\lambda|$ such that

$$Q^*_\lambda = Q_{\lambda} + g.$$ 

2. The collection $\{Q^*_\lambda\}_{\lambda \in SP}$ is a linear basis of $\Gamma$.

3. If $\nu \in SP_{k}$, $\lambda \in SP_{n}$ for $k \leq n$ and $\nu \not\subseteq \lambda$, $Q^*_\lambda(\nu) = 0$.

Let $\psi : \Gamma \to \Gamma$ be the linear map that sends $Q_\lambda \mapsto Q^*_\lambda$. For any $\mu \in OP$, define the inhomogeneous analogue of the power sum $p_\mu := \psi(p_\mu) \in \Gamma$. Applying $\psi$ to both sides of (16) gives

$$p_\mu = \sum_{\lambda \in SP_k} 2^{-\ell(\lambda)} X_\mu^\lambda Q^*_\lambda.$$ 

It also follows from the fact that $X_\mu^\lambda = 2^{-\ell(\mu) + \frac{(\ell(\lambda) - \ell(\mu))}{2}} X_{\mu}^{\lambda}(\mu)$ [8, Proposition 3.3] and

$$Q_\lambda = \sum_{\mu \in OP_n} 2^{\ell(\mu)} z_\mu X_\mu^\lambda p_\mu$$

that

$$Q^*_\lambda = 2 \frac{\ell(\lambda) - \ell(\mu)}{2} \sum_{\mu \in OP_n} X_\mu^\lambda(\mu) p_\mu.$$ 

(18)

The elements $\{p_\mu\}_{\mu \in SP}$ were first studied in [8], where Ivanov proves that they satisfy the following properties.

**Proposition 3.3** [8] Let $\mu \in OP_k$ and $\lambda \in SP_n$.

1. There exists $g \in \Gamma$ of degree less than $|\mu|$ such that

$$p_\mu = p_{\mu} + g.$$ 

2. The family $(p_\mu)_{\mu \in OP}$ is a linear basis of $\Gamma$. 

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3. $p_\mu(\lambda) = \begin{cases} n^{\downarrow k} \cdot \frac{X_\lambda^\mu \cup (1^n - k)}{g_\lambda} & \text{if } |\lambda| \geq |\mu|, \\ 0 & \text{otherwise} \end{cases}$

where in particular $g_\lambda = X_\lambda^{\downarrow |\lambda|}$.

4. Let $\gamma \in \mathcal{OP}$. Define $\mu \cup \gamma$ to be the partition formed by taking the disjoint union of parts of $\mu$ and $\gamma$ and rearranging them in decreasing order. Then, there exists $g \in \Gamma$ of degree less than $|\mu \cup \gamma|$ such that

$$p_\mu \cdot p_\gamma = p_{\mu \cup \gamma} + g.$$

As a Corollary to part 3. of the above Proposition, we have another formula for the value of $p_\rho$.

**Corollary 3.4** [8] Let $\mu \in \mathcal{OP}_k$ and $\lambda \in \mathcal{SP}_n$. We have

$$p_\mu(\lambda) = 2^{k - \ell(\mu)} n^{\downarrow k} \frac{X_\lambda^\mu \cup (1^n - k)}{X_\lambda(1^n)}.$$

**Corollary 3.5** The elements $\{p_{2k+1}\}_{k \geq 0}$ are algebraically independent and generate $\Gamma$.

It is shown in [22] that viewed as elements of $\text{Fun}(\mathcal{SP}, \mathbb{C})$, $\{g^\uparrow_k\}_{k \geq 1}$ and $\{g^\downarrow_k\}_{k \geq 1}$ belong to $\Gamma$.

**Proposition 3.6** [22, Corollary 4.7] The elements $\{g^\uparrow_k\}_{k \geq 1}$ and $\{g^\downarrow_k\}_{k \geq 1}$ are each sets of algebraically independent generators of $\Gamma$, and

$$\deg(g^\uparrow_k) = \deg(g^\downarrow_k) = 2k - 1.$$

4 The twisted Heisenberg category

4.1 The definition of $\mathcal{H}_{tw}$

The twisted Heisenberg category $\mathcal{H}_{tw}$ was introduced by Cautis and Sussan in [6]. It is a $\mathbb{Z}/2\mathbb{Z}$-graded additive monoidal category whose morphisms are described diagrammatically as oriented compact 1-manifold immersed in $\mathbb{R} \times [0, 1]$. There is an injective algebra homomorphism from the twisted Heisenberg algebra into the split Grothendieck group of $K_0(\text{Kar}(\mathcal{H}_{tw}))$, where Kar denotes the Karoubi envelope (idempotent completion). As in the untwisted case, this map is conjecturally surjective.

**Remark 4.1** Cautis and Sussan define their version of $\mathcal{H}_{tw}$ to be idempotent complete; since the center of a category remains invariant under passage to the idempotent completion, we use the non-idempotent complete version here. All results that hold for the center of $\mathcal{H}_{tw}$ also hold for the center of its idempotent completion.

The objects of $\mathcal{H}_{tw}$ are monoidally generated by $P$ and $Q$, so that a generic object in $\mathcal{H}_{tw}$ is a direct sum of sequences of $P$’s and $Q$’s. We denote the empty sequence,
which is the unit object of $\mathcal{H}_{tw}$, by $\mathbb{1}$. The morphisms of $\mathcal{H}_{tw}$ are generated by oriented planar diagrams up to boundary fixing isotopies, with generators
\[
\begin{align*}
\uparrow, \quad \downarrow, \quad \begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram1.png}
\end{array}, \quad \begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram2.png}
\end{array}, \quad \begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram3.png}
\end{array}, \quad \begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram4.png}
\end{array}, \quad \begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram5.png}
\end{array}, \quad \begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram6.png}
\end{array}
\end{align*}
\]
(19)

where the first diagram corresponds to a map $P \rightarrow P\{1\}$ and the second diagram corresponds to a map $Q \rightarrow Q\{1\}$, where $\{1\}$ denotes the $\mathbb{Z}/2\mathbb{Z}$-grading shift. These generators satisfy the following relations:
\[
\begin{align*}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram7.png}
\end{array} &= \begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram8.png}
\end{array},
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram9.png}
\end{array} &= \begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram10.png}
\end{array},
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram11.png}
\end{array} &= \begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram12.png}
\end{array},
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram13.png}
\end{array} &= \begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram14.png}
\end{array},
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram15.png}
\end{array} &= \begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram16.png}
\end{array},
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram17.png}
\end{array} &= \begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram18.png}
\end{array},
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram19.png}
\end{array} &= \begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram20.png}
\end{array},
\end{align*}
\]
(20)

Generators commute in all other situations. (For instance, hollow dots commute with crossings.)

If we denote a right twist curl by a dot $\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram21.png}
\end{array} := \begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram22.png}
\end{array}$ then we have the following relations:
\[
\begin{align*}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram23.png}
\end{array} &= - \begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram24.png}
\end{array},
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram25.png}
\end{array} &= \begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram26.png}
\end{array},
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram27.png}
\end{array} &= \begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram28.png}
\end{array},
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram29.png}
\end{array} &= \begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram30.png}
\end{array},
\end{align*}
\]
(21)

From [21], we have the following “dot sliding” relations:
\[
\begin{align*}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram31.png}
\end{array} &= \begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram32.png}
\end{array} + \begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram33.png}
\end{array} + \begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram34.png}
\end{array},
\end{align*}
\]
(26)

\[
\begin{align*}
\begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram35.png}
\end{array} &= \begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram36.png}
\end{array} + \begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram37.png}
\end{array} - \begin{array}{c}
\includegraphics[width=0.1\textwidth]{diagram38.png}
\end{array}.
\end{align*}
\]
(27)

We can also move clockwise “bubbles” with dots on them through strands.
Lemma 4.2 Let \( n \geq 0 \), then
\[
2n = 2n + (4n+2) - 2 \sum_{a+b=2n-1}^{b} \sum_{k=1}^{h} a+k b-k.
\]

Proof This follows from the proof of [21, Lemma 4.7] along with the dot sliding relation (27).

As a consequence of relations (20) and (21), there are homomorphisms \( T_n : S_n^{opp} \to \text{Hom}_{H_{tw}}(P^n) \) which send
\[
\begin{align*}
S_k & \xrightarrow{T_n} \cdots \cdots \\
& \text{k-1 strands} \quad \text{n-k-1 strands}
\end{align*}
\]
\[
\begin{align*}
c_k & \xrightarrow{T_n} \cdots \cdots \\
& \text{k-1 strands} \quad \text{n-k strands}
\end{align*}
\]

In order to simplify our diagrams, we write the image of \( x \in S_n \) under \( T_n \) as:
\[
T_n(x) =:
\begin{array}{c}
\cdots \\
\text{n strands}
\end{array}
\]

(28)

4.2 The center of \( H_{tw} \)

The center of a \( k \)-linear monoidal category \( C \) is defined to be the endomorphism algebra of the monoidal unit \( \mathbb{1} \) of \( C \), that is, \( \text{End}_C(\mathbb{1}) \). In a diagrammatic category such as \( H_{tw} \), \( \text{End}_{H_{tw}}(\mathbb{1}) \) is then by definition the commutative algebra of closed diagrams where multiplication of two closed diagrams corresponds to placing them next to each other.

We define the following elements of \( \text{End}_{H_{tw}}(\mathbb{1}) \):
\[
d_{2n} := 2n \quad \text{and} \quad \bar{d}_{2n} := 2n.
\]

It follows from the defining relations of \( H_{tw} \) that bubbles with an even number of hollow dots are equivalent to bubbles with no hollow dots, and that bubbles with an odd number of hollow dots are zero. Additionally, bubbles with odd numbers of solid dots are zero. Hence, it suffices to consider the bubbles with an even number of solid dots and no hollow dots.

Proposition 4.3 [21, Proposition 4.4] The elements \( \{d_{2n}\}_{n \geq 0} \) are algebraically independent generators of \( \text{End}_{H_{tw}}(\mathbb{1}) \), i.e., there is an isomorphism
\[
\text{End}_{H_{tw}}(\mathbb{1}) \cong \mathbb{C}[d_0, d_2, d_4, \ldots].
\]
The elements \( \{d_{2n}\}_{n \geq 0} \) and \( \{\bar{d}_{2n}\}_{n \geq 0} \) are related via a recursive relation.

**Proposition 4.4** [21, Lemma 4.3] For \( n \geq 1 \),

\[
\bar{d}_{2n} = \sum_{2a + 2b = 2n - 2} \bar{d}_{2a} \bar{d}_{2b}.
\]

**Corollary 4.5** The elements \( \{\bar{d}_{2n}\}_{n \geq 1} \) are another algebraically independent generating set of \( \text{End}_{\mathcal{H}_{tw}(\mathbb{1})} \).

Another natural set of diagrams in \( \text{End}_{\mathcal{H}_{tw}(\mathbb{1})} \) comes from the closure of permutations. We define

\[
\vdots
\]

For \( \nu = (\nu_1, \ldots, \nu_r) \in \mathcal{P}_k \), let

\[
\nu := \nu_1 \cdots \nu_r
\]

then we define

\[
\alpha_{\nu} := \nu
\]

We set \( \alpha_k := \alpha_{(k)} \).

**Remark 4.6** By an argument given in [16], this notation is consistent with (28) in the sense that the closures of the diagrams \( \mathcal{T}_n(\omega_1) \) and \( \mathcal{T}_n(\omega_2) \), for \( \omega_1, \omega_2 \in S_n \) and \( \omega_1 \) and \( \omega_2 \) having the same cycle type \( \nu \), are both equal to \( \alpha_{\nu} \).

One can impose a grading on \( \text{End}_{\mathcal{H}_{tw}(\mathbb{1})} \) by setting:

\[
\text{deg}(d_0) = 0 \quad \text{and} \quad \text{deg}(d_{2k}) = 2k + 1.
\]

**Lemma 4.7** In terms of the grading defined by (31),

\[
\alpha_{2k+1} = d_{2k} + \text{l.o.t.}
\]
Proof We can reduce the diagram $\alpha_{2k+1}$ to a polynomial in $d_0, d_2, d_4, \ldots$ via repeated application of the dot sliding moves (26) and (27) and clockwise bubble sliding move from Lemma 4.2. Specifically, the innermost crossing in $\alpha_{2k+1}$ forms part of a right twist curl, i.e., a solid dot. The case $k = 1$ is an illuminating example; hence, we present it here:

$$\alpha_3 = \bigcirc \bigcirc = \bigcirc \bigcirc = \bigcirc \bigcirc - \bigcirc = 2 \bigcirc - \bigcirc + 2 \bigcirc = d_2 - d_0^2 + 2d_0$$

Modulo terms with at least two fewer crossings (all of which also have exactly two connected components), this dot can be moved to the outside of the diagram via repeated applications of relations (26) and (27). The resulting diagram once again has a right twist curl in its center. Repeating this process $2k$ times yields $d_{2k}$, plus terms which have two fewer crossings (including those in right twist curls written as solid dots) and exactly two connected components.

It remains to reduce the lower-order terms to polynomials in $d_0, d_2, d_4, \ldots$. Note that the dot sliding moves (26) and (27) and bubble sliding relation in Lemma 4.2 never increase the number of crossings and can only increase the number of connected components by at most 1 with a corresponding decrease in crossings by 2 (see the second and third term on the right side of (26)-(27)). It follows that while reducing each of these additional terms to monomials in bubbles, in each diagram

$$\text{#crossings} + \text{#connected components} \leq (2k - 2j) + (j + 1) = 2k - j + 1$$

where $1 \leq j \leq k$ is half the number of crossings that have been resolved via a dot or bubble slide. Hence, the degree of each resulting monomial in bubbles is strictly less than $2k + 1$. Since $d_{2k}$ has degree $2k + 1$, the result follows.

Corollary 4.8 End$_{\mathcal{H}_{tw}}(1)$ is generated by $\{\alpha_{2k+1}\}_{k \geq 0}$, and these elements are algebraically independent.

4.3 Diagrams as bimodule homomorphisms

An action of $\mathcal{H}_{tw}$ on the category $\mathcal{S}$ whose objects are compositions of induction and restriction functors between $\mathbb{Z}/2\mathbb{Z}$-graded finite dimensional $S_n$-supermodules, for all $n \geq 0$, is described in [6, Section 6.3]. Because induction and restriction functors for the algebras $S_n$ can be written as:

$$\text{Ind}_{S_n}^{S_{n+1}}(-) = S_{n+1} \otimes S_n - \text{ and } \text{Res}_{S_n}^{S_{n+1}}(-) = S_n \otimes S_{n+1} -$$

(where above $S_{n+1} \otimes S_n$ is a $(S_{n+1}, S_n)$-bimodule and $S_n \otimes S_{n+1}$ is a $(S_n, S_{n+1})$-bimodule), the objects of $\mathcal{S}$ can alternatively be described as tensor products of certain $(S_{k_1}, S_{k_2})$-bimodules for all $k_1, k_2 \geq 0$. We will use this interpretation extensively below. Let $k_1, k_2 \leq n$, then we write
• \( (n) \) for \( S_n \) considered as a \((S_n, S_n)\)-bimodule,
• \( (n)_{k_2} \) for \( S_n \) considered as a \((S_n, S_{k_2})\)-bimodule,
• \( k_1(n) \) for \( S_n \) considered as a \((S_{k_1}, S_n)\)-bimodule,
• \( k_1(n)_{k_2} \) for \( S_n \) considered as a \((S_{k_1}, S_{k_2})\)-bimodule.

The morphisms in \( \mathcal{G} \) are certain natural transformations of these compositions of induction/restriction functors (or, equivalently, certain bimodule homomorphisms). To describe the morphisms of \( \mathcal{G} \), we use the diagrams of \( \mathcal{H}_{tw} \), but this time we also label the rightmost region with a nonnegative integer \( n \). Since an upward strand denotes the identity endomorphism of induction, it increases the label by one as one reads from right to left. Similarly, a downward strand decreases the label by one.

We also set any diagram with a negative label to be zero. Descriptions of other morphisms in \( \mathcal{G} \) are most easily given in terms of bimodules, so we henceforth use this language exclusively.

Below are the generating morphisms and the corresponding bimodule maps:

\[
(n + 1)_n \to (n + 1)_n \quad (n + 1) \to n \quad n \to n + 1
\]

\[
g \mapsto (-1)^{|g|} gc_{n+1}
\]

\[
n - 1(n) \to n - 1(n) \quad n - 1 \to n
\]

\[
g \mapsto c_{n+1}g
\]

\[
q_n : (n) \to (n)_{n-1}(n)
\]

\[
g \mapsto c_{n+1}g
\]

\[
pr_n : n(n + 1)_n \to (n)
\]

\[
g \mapsto g
\]

\[
(n + 1)(n + 1) \to (n + 1) \quad (n + 1) \to n + 1
\]

\[
g_1 \otimes g_2 \mapsto g_1 g_2
\]

\[
i_n : (n) \to (n)(n + 1)_n
\]

\[
g \mapsto g
\]

\[
(n + 2)_n \to (n + 2)_n \quad n + 2 \to n
\]

\[
g \mapsto gs_{n+1}
\]

where \( pr_n : n(n + 1)_n \to (n) \) is the projection map given by \( pr_n(g) = g \) if \( g \in S_n \), \( pr_n(g) = 0 \) if \( g \not\in S_n \), and \( q_n : (n) \to (n)_{n-1}(n) \) is the bimodule map determined by \( q_n(1) = \sum_{x \in LC_{n-1}^n} x \otimes x^{-1} \).
**Remark 4.9** The action of $H_{tw}$ on $S$ can be lifted to the idempotent closures of these categories. This then becomes a categorification of the Fock space representation [6].

Following Khovanov’s approach from [14], let $S_n$ be the full subcategory of $S$ whose objects are $(S_k, S_n)$-bimodules, for all $k \in \mathbb{Z}_{\geq 0}$. For every $n \in \mathbb{Z}_{\geq 0}$, there is a functor $F_n^H_{tw} : H_{tw} \rightarrow S_n$ which is defined on objects of $H_{tw}$ such that it sends the rightmost $P$ (respectively, $Q$) from a $P, Q$ sequence to $(n+1)_n$ (resp. $(n-1)_n$) and all other $P$ and $Q$ in the sequence are determined by this choice. Hence, $F_n^H_{tw}$ maps all objects of $H_{tw}$ into $S_n$. Under $F_n^H_{tw}$, a morphism (or diagram) is mapped to a morphism in $S_n$ by labeling the rightmost region by $n$ which determines the labelings of all other regions. An upward strand increases the label by 1, and a downward strand decreases the label by 1 as we read from right to left.

Note that the image of a closed diagram $D$ under $F_n^H_{tw}$ will be an $(S_n, S_n)$-bimodule endomorphism of $S_n$ which we denote as $f : S_n \rightarrow S_n$. $f$ is fully determined by the value $f(1)$ since for any $x \in S_n$, $f(x) = xf(1)$. Furthermore, $f(1)$ is an element of $Z(S_n)_n$ because $xf(1) = f(x) = f(1)x$. In this way, we can identify the image of $\text{End}_{H_{tw}}(1)$ under $F_n^H_{tw}$ with elements of $Z(S_n)_n$.

We next study the image of some of the elements of $\text{End}_{H_{tw}}(1)$ from Sect. 4.2 under the functor $F_n^H_{tw}$.

**Lemma 4.10**

1. The diagram

$$
\begin{array}{c}
n \uparrow \quad n-1 \quad \cdots \quad n-k \\
\end{array}
$$

corresponds to the $(S_n, S_n)$-bimodule homomorphism $(n) \rightarrow (n)_{n-k}(n)$ which sends

$$1 \mapsto \sum_{x \in LC^n_{n-k}} x \otimes x^{-1}.$$  

2. For $\mu \in \mathcal{OP}_k$ with $k \leq n$, the diagram

$$
\begin{array}{ccc}
\mu & \rightarrow & \vdots \\
n & \downarrow & n-k \\
\end{array}
$$

corresponds to the $(S_n, S_n)$-bimodule homomorphism $(n)_{n-k}(n) \rightarrow (n)_{n-k}(n)$ which for $x, y \in S_n$ sends

$$x \otimes y \mapsto x\sigma_{\mu,n} \otimes y.$$  

**Proof** Both 1 and 2 follow from calculations using the definitions of cup (34) and crossing (38) maps.
**Proposition 4.11** For $\mu \in OP_k$,

$$F_{n}^{\mathcal{H}_{tw}}(\alpha_{\mu}) = \begin{cases} A_{\mu;n} & \text{if } k \leq n \\ 0 & \text{otherwise}. \end{cases}$$

**Proof** The diagram for $\alpha_{\mu}$ can be broken into three components

![Diagram](image)

Reading from bottom to top, the first component corresponds to Lemma 4.10.1., and the second corresponds to Lemma 4.10.2. The composition of these two maps sends

$$1 \mapsto \sum_{x \in \mathcal{LC}_{n-k}} x \sigma_{\mu;n} \otimes x^{-1}.$$ 

The top component of $k$ nested caps is the multiplication map which sends

$$\sum_{x \in \mathcal{LC}_{n-k}} x \sigma_{\mu;n} \otimes x^{-1} \mapsto \sum_{x \in \mathcal{LC}_{n-k}} x \sigma_{\mu;n} x^{-1} = A_{\mu;n}.$$

**Lemma 4.12** [21] For $n - 1 \geq 0$, the right twist curl

![Diagram](image)

corresponds to the $(\mathbb{S}_n, \mathbb{S}_{n-1})$-bimodule homomorphism, $(n)_{n-1} \mapsto (n)_{n-1}$ which multiplies $x \in \mathbb{S}_n$ on the right by the Jucys–Murphy element $J_n$

$$x \mapsto x J_n.$$

**Proposition 4.13** Let $k \geq 0$ and $n \geq 1$, then

1. $F_{n}^{\mathcal{H}_{tw}}(\delta_{2k}) = pr_n(J_{n+1}^{2k})$,
2. $F_{n}^{\mathcal{H}_{tw}}(d_{2k}) = \sum_{x \in \mathcal{LC}_{n-1}} x J_n^{2k} x^{-1}$.

**Proof** These follow from direct calculation using the definitions of the cup (34) and cap (36) maps and Lemma 4.12.
5 Main results

5.1 An isomorphism between $\text{End}_{\mathcal{H}_{tw}}(\mathbb{1})$ and $\Gamma$

In this subsection, we establish an isomorphism between $\text{End}_{\mathcal{H}_{tw}}(\mathbb{1})$ and $\Gamma$. The key step in the construction of this map will be identifying the elements of $\text{End}_{\mathcal{H}_{tw}}(\mathbb{1})$ with functions on $\mathcal{S}P$, i.e., as elements of $\text{Fun}(\mathcal{S}P, \mathbb{C})$. To do this, let $\lambda \in \mathcal{S}P_n$ and $x \in \text{End}_{\mathcal{H}_{tw}}(\mathbb{1})$. Then, we evaluate $x$ on $\lambda$ by:

$$x(\lambda) := \tilde{\chi}^\lambda(F_n^\mathcal{H}_{tw}(x)).$$

Because $F_n^\mathcal{H}_{tw}$ is a homomorphism on $\text{End}_{\mathcal{H}_{tw}}(\mathbb{1})$ which maps into $Z(\mathbb{S}_n)^0$ and $\tilde{\chi}^\lambda$ is a homomorphism when restricted to $Z(\mathbb{S}_n)^0$, this defines a homomorphism into $\text{Fun}(\mathcal{S}P, \mathbb{C})$.

Proposition 5.1 For $\mu \in \mathcal{O}P_k$ and $\lambda \in \mathcal{S}P_n$, we have

$$\alpha_\mu(\lambda) = \begin{cases} 2^k n \binom{n}{k} \frac{\chi^\lambda(\mu \cup [n-k])}{\chi^\lambda([n])} & \text{if } k \leq n \\ 0 & \text{otherwise} \end{cases}$$

Proof This follows from Propositions 2.13 and 4.11.

Theorem 5.2 There is an algebra isomorphism $\varphi : \text{End}_{\mathcal{H}_{tw}}(\mathbb{1}) \rightarrow \Gamma$ which for any $\mu \in \mathcal{O}P$ sends

$$\alpha_\mu \mapsto 2^{\ell(\mu)}p_\mu.$$

Proof It is clear from Proposition 5.1 and Corollary 3.4 that $2^{-\ell(\mu)}\alpha_\mu$ and $p_\mu$ map to the same function in $\text{Fun}(\mathcal{S}P, \mathbb{C})$. Furthermore, the collection of functions which are the image of $\{p_{2k+1}\}_{k \geq 0}$ are algebraically independent by Proposition 3.1 and Corollary 3.5. By Proposition 4.8, $\text{End}_{\mathcal{H}_{tw}}(\mathbb{1})$ is generated by the algebraically independent elements $\{\alpha_{2k+1}\}_{k \geq 0}$. It then follows that the map that sends $\alpha_\mu \mapsto 2^{\ell(\mu)}p_\mu$ is an isomorphism.

Let $\mu \in \mathcal{O}P_n$. It follows from Lemma 2.5, Remark 4.6, and Theorem 5.2 that

$$C_\mu = \frac{n!}{z_\mu} 2^{n-\ell(\mu)} \mathcal{S}P_n \mapsto \varphi \frac{n!}{z_\mu} 2^np_\mu.$$  

(39)

Theorem 5.3 Let $\lambda \in \mathcal{S}P_n$. Under the isomorphism $\varphi : \text{End}_{\mathcal{H}_{tw}}(\mathbb{1}) \rightarrow \Gamma$, the closure of the central idempotent $e_\lambda$ of $\mathbb{S}_n$ maps to $h(\lambda)Q^*_\lambda$. 

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Table 1 A dictionary between $\Gamma$ and diagrams in $\text{End}_{\mathcal{H}_{tw}}(1)$

| $\Gamma$ | $p_\mu$ | $Q^*_\lambda$ | $g^\uparrow_k$ | $g^\uparrow_{k+1}$ |
|----------|---------|---------------|----------------|-------------------|
| Diagram in $\text{End}_{\mathcal{H}_{tw}}(1)$ | $\frac{1}{2^{\ell(\mu)}}$ | $\frac{1}{h(\lambda)}$ | $2k$ | $2k$ |

**Proof** Recall from Lemma 2.14 that

$$e_\lambda = 2^{\frac{-\delta(\lambda) - \delta(\lambda)}{2}} g_\lambda \sum_{\mu \in OP_n} \chi^\lambda(\mu) C_\mu$$

while by (18)

$$Q^*_\lambda = 2^{\frac{\ell(\lambda) - \delta(\lambda)}{2}} \sum_{\mu \in OP_n} \frac{\chi^\lambda(\mu)}{z_\mu} p_\mu.$$

Combining these facts with Theorem 5.2 and (39), it follows that the closure of $e_\lambda$ is equal to $2^{n-\ell(\lambda)} g_\lambda Q^*_\lambda = h(\lambda) Q^*_\lambda$. $\square$

**Remark 5.4** Recall that the Schur $Q$-functions are related to the Schur $P$-functions by $P_\lambda = 2^{-\ell(\lambda)} Q_\lambda$. Ivanov also studied factorial Schur $P$-functions $\{P^*_\lambda\}_{\lambda \in \mathcal{SP}_n}$ where $P^*_\lambda = 2^{-\ell(\lambda)} Q^*_\lambda$ [8]. Then, one alternative description of the closure of $e_\lambda$ in $\Gamma$ is as $2^n g_\lambda P^*_\lambda$.

Moving in the opposite direction, we can also identify the elements of $\Gamma$ corresponding to the generators $\{d_{2k}\}_{k \geq 0}$ and $\{\bar{d}_{2k}\}_{k \geq 0}$.

**Theorem 5.5** For $k \geq 0$,

1. $\psi(\bar{d}_{2k}) = g^\uparrow_k(\cdot)$,
2. $\psi(d_{2k}) = g^\uparrow_{k+1}(\cdot)$.

**Proof** This follows from Propositions 2.17 and 4.13. $\square$

**Remark 5.6** In light of Theorem 5.5, Proposition 4.4 can be seen as a diagrammatic manifestation of Proposition 2.16.

### 5.2 An action of $\text{Tr}(\mathcal{H}_{tw})_0$ on $\Gamma$

Aside from taking the Grothendieck group or center, another method for decategorifying a category $\mathcal{C}$ is taking the categorical trace of $\mathcal{C}$, $\text{Tr}(\mathcal{C})$ (also known as the zeroth Hochschild homology of $\mathcal{C}$). See [1] for a discussion of this method of decategorification. In [21], it is shown that the even part of the trace of $\mathcal{H}_{tw}$, $\text{Tr}(\mathcal{H}_{tw})_0$, is isomorphic to $\mathcal{C}$. Springer
to the vertex algebra $W^-$ at level one, a subalgebra of $W_{1+\infty}$ defined by Kac, Wang, and Yan [11].

In a diagrammatic setting such as this, the trace can be realized as the algebra of closed diagrams on an annulus. There is a natural action of $\text{Tr}(C)$ on the center of the category $\mathcal{C}$, $\text{End}_C(1)$, where diagrammatically a closed diagram on an annulus acts on a closed diagram in a disk by plugging the annulus with the disk, resulting in a new diagram in the disk. The results of [21] along with Theorem 5.2 imply that $W^-$ acts on $\Gamma$. This action is similar to the action of $W_{1+\infty}$ on the centers of symmetric group algebras described in [17]. In this section, we will first review $W^-$ and then describe the action of the generators of $W^-$ on basis elements of $\Gamma$.

We first review the vertex algebra $\hat{W}$, which appears in the trace of $\mathcal{H}_{tw}$. Below, $t$ refers to a multiplication operation by the variable $t$, $\frac{d}{dt}$ is differentiation with respect to $t$, and $D = t \frac{d}{dt}$. We will also use the elements $\omega_{(k, \ell)} := t^k D^\ell$.

Let $\hat{D}^{-}$ be the Lie algebra over the vector space spanned by $\{C\} \cup \{t^{2k-1}g(D + (2k - 1)/2); \ g \text{ even}\} \cup \{t^{2k}f(D + k); \ f \text{ odd}\}$ where $k \in \mathbb{Z}$ and even and odd refer to even and odd polynomial functions. The Lie bracket in $\hat{D}^{-}$ is defined by:

$$[t^r f(D), t^s g(D)] = t^{r+s} (f(D + s)g(D) - f(D)g(D + r)) + \psi(t^r f(D), t^s g(D))C,$$

where

$$\psi(t^r f(D), t^s g(D)) = \begin{cases} \sum_{-r \leq j \leq -1} f(j)g(j + r) & r = -s \geq 0 \\ 0 & r + s \neq 0 \end{cases},$$

(41)

and $C$ is a central element. Denote by $W^-$, the universal enveloping algebra of $\hat{D}^{-}$. The trace of $\mathcal{H}_{tw}$ is shown in [21] to be isomorphic to the quotient $W^-/\langle \omega_{0,0}, C - 1 \rangle$.

A generating set for $W^-/\langle \omega_{0,0}, C - 1 \rangle$ is given by $\omega_{1,0}$, $\omega_{-1,0}$, and $\omega_{\pm 2,1} \pm \omega_{\pm 2,0}$ [21, Lemma 2.2]. In order to explicitly write down an action of the algebra $W^-$ on $\Gamma$, we will work with different generating sets.

**Proposition 5.7** The algebra $W^-/\langle \omega_{0,0}, C - 1 \rangle$ is also generated by $\omega_{1,0}$, $\omega_{-1,0}$ and $\omega_{0,3}$.

**Proof** We will show that we can obtain the aforementioned generators $\omega_{\pm 2,1} \pm \omega_{\pm 2,0}$ via the elements $\omega_{1,0}$, $\omega_{-1,0}$, and $\omega_{0,3}$. It is a straightforward computation that

$$\omega_{0,1} = -\frac{1}{20} [\langle \omega_{0,3}, \omega_{-1,0} \rangle, \omega_{1,0}] + \frac{1}{5} \omega_{-1,0} \omega_{1,0},$$

and using $\omega_{0,1}$, we can obtain $\omega_{-1,2} - \omega_{-1,1}$ as follows:

$$\omega_{-1,2} - \omega_{-1,1} = \frac{1}{6} [\omega_{0,3}, \omega_{-1,0}] + \frac{1}{3} \omega_{-1,0} \omega_{0,1}.$$
Then, one of the elements we are looking for is given by:
\[ \omega_{-2,1} - \omega_{-2,0} = \frac{1}{2}[\omega_{-1,2} - \omega_{-1,1}, \omega_{-1,0}] \].

To obtain \( \omega_{2,1} + \omega_{2,0} \), we follow a very similar computation:
\[ \omega_{1,2} + \omega_{1,1} = -\frac{1}{6}[\omega_{0,3}, \omega_{1,0}] + \frac{1}{3} \omega_{0,1} \omega_{1,0}, \]
and finally
\[ \omega_{2,1} + \omega_{2,0} = -\frac{1}{2}[\omega_{1,2} + \omega_{1,1}, \omega_{1,0}] \].

The images of these generators under the isomorphism \( W^-/\langle \omega_{0,0}, C - 1 \rangle \to \text{Tr}(\mathcal{H}_w) \) from [21] are given below. In the following diagrams, we draw an asterisk inside the diagrams to emphasize that they live on an annulus, not on the plane:

\[ \sqrt{2} \omega_{-1,0} \mapsto \star ; \]
\[ \sqrt{2} \omega_{1,0} \mapsto \star ; \]
\[ -2 \omega_{0,3} \mapsto 2 \star = \star + \star + \star \]. \hspace{1cm} (42)

We will also use the elements \( \omega_{-(2n+1),0} \) and their images in \( \text{Tr}(\mathcal{H}_w) \):

\[ \sqrt{2} \omega_{-(2n+1),0} \mapsto \tau \star \]. \hspace{1cm} (43)

where \( \tau \) is a \((2n+1)\)-cycle.

We now describe the action of the generating set \( \{\omega_{1,0}, \omega_{-1,0}, \omega_{0,3}\} \) of \( W^- \) on the vector space basis \( \{p_\mu\}_{\mu \in \mathcal{O}_P} \) of \( \Gamma \). We achieve this by describing the action of the corresponding generators of \( \text{Tr}(\mathcal{H}_w) \) on the basis \( \{\alpha_\mu\}_{\mu \in \mathcal{O}_P} \) of \( \text{End}_{\mathcal{H}'}(1) \).
Lemma 5.8 We have

\[ \alpha(\mu, 1) = \alpha_{\mu} \alpha_1 - 2|\mu| \alpha_{\mu} . \]

**Proof** This simply follows from the local bubble sliding relation

applied \(|\mu|\) times to the diagram \(\alpha(\mu, 1)\), as we pull the clockwise bubble \(\alpha_1\) from within \(\alpha_{\mu}\).

\[ \square \]

Lemma 5.9 We have

\[ \omega_{1,0} \cdot \alpha_{\mu} \alpha_1 = (\alpha_1 + 2) \omega_{1,0} \cdot \alpha_{\mu} . \]

**Proof** We compute:

\[ \omega_{1,0} \cdot \alpha_{\mu} \alpha_1 = \sqrt{2} \alpha(\mu, 1) + (\alpha_1 + 2 \sqrt{2}) \hat{\mu} \]

as desired.

\[ \square \]

Theorem 5.10 The generators \(\text{Tr}(H_{tw})\) act on the basis elements \(\{p_\mu\}_{\mu \in OP}\) of \(\Gamma\) as follows:

1. \(\omega_{-1,0} \cdot p_\mu = \sqrt{2} p(\mu, 1)\)
2. \(\omega_{1,0} \cdot p_\mu = \frac{1}{\sqrt{2}} p_\mu + \frac{k}{\sqrt{2}} p_{\hat{\mu}}\)
3. \(\omega_{0,3} \cdot p_\mu = -p_3 p_\mu - 2 p_{(1,1)} p_\mu\)

where \(k\) is the number of parts of size 1 of \(\mu\) and \(\hat{\mu}\) stands for the partition obtained by removing one part of size 1 from \(\mu\) if this is possible. When \(\mu = (1)\), then \(p_{(1)} = 1\).
**Proof** For the action of $\omega_{-1,0}$, note that the action of $\ast$ on $\alpha_\mu$ is diagrammatically just enclosing the diagram of $\alpha_\mu$ by a clockwise oriented strand:

$$\ast \cdot \alpha_\mu = \alpha_\mu$$  \hspace{1cm} (44)

and the resulting diagram is the diagram of $\alpha_{(\mu,1)}$. Replacing $\alpha_\mu$ by $2^{\ell(\mu)} p_\mu$ and the clockwise bubble by $\sqrt{2} \omega_{-1,0}$, we get $\omega_{-1,0} \cdot p_\mu = \sqrt{2} p_{(\mu,1)}$.

More generally, it is easy to see that $\omega_{-(2n+1),0} \cdot p_\mu = \sqrt{2} p_{(\mu,2n+1)}$ from (43). Note that we have $[\omega_{-1,0}, \omega_{1,0}] = -1$ and, for $n \geq 0$, we have $[\omega_{-(2n+1),0}, \omega_{1,0}] = 0$.

To simplify the notation in the following computations, denote $\omega_+ := \sqrt{2} \omega_{1,0}$.

We start by showing that if the partition $\mu$ does not contain any parts of size one, then $\omega_+ \cdot \alpha_\mu = \alpha_\mu$ by induction on $\ell(\mu)$. We provide a diagrammatic proof for the base case $\ell(\mu) = 1$ (i.e., $\alpha_\mu = \alpha_k$ for $k \neq 1$ odd).

In the diagram $\sqrt{\alpha_k}$, we claim that we can pass $\alpha_k$ through the outer strand for free, meaning that all the resolution terms that appear as a result of relation 21 are zero.

We provide the computation for the case of $\alpha_k = \alpha_5$ and explain how the arguments generalize to any $\alpha_k$. We have

and the two hollow dots appearing in the last term cancel with each other if we slide them along the outermost strand. This observation will hold for the rest of the computation, so we will omit drawing the second resolution term and instead write the
first resolution term with coefficient 2. We will show that all resolution terms coming from crossings on the outermost strand, innermost strand, and intermediate strands are zero.

For the resolution term coming from the crossing of outermost strands, we have

\[ \text{Diagram} \quad = \quad 0 \]

where the last equality follows from (21).

For the resolution term coming from the crossing of intermediate strands, consider a generic intermediate strand. We have

\[ \text{Diagram} \quad = \quad 0 \]

where the second and third equalities follow from a Reidemeister III move, and the fourth is a result of relation (20). Hence, these resolution terms are zero as well. In general, for a resolution term coming from a crossing of intermediate strands, we can first pull the red string above the permutation using Reidemeister III moves and then pull the red string into the permutation using relation (20) to get a left twist curl.
Finally, for the resolution term coming from the crossing of intermediate strands the situation is simpler:

\[
\begin{align*}
\includegraphics{diagram1} &= \includegraphics{diagram2} = \includegraphics{diagram3} = 0.
\end{align*}
\]

Hence, all the resolution terms are zero. This leaves us with

\[
\begin{align*}
\includegraphics{diagram4} &= \includegraphics{diagram5} = \includegraphics{diagram6} = \includegraphics{diagram7}.
\end{align*}
\]

and a counterclockwise-oriented bubble is equal to 1 by the defining relation (21). These diagrammatic arguments clearly hold for arbitrary \( k > 1 \). Hence, the action of \( \omega_{(1,0)} \) on \( \alpha_k \) for \( k \neq 1 \) is trivial.

This concludes the proof of the base case \( \ell(\mu) = 1 \). Now, suppose \( \omega_+ \cdot \alpha_\mu = \alpha_\mu \) for some \( \mu \in \mathcal{OP} \) such that \( \ell(\mu) = m - 1 \), and let \( n \) be a positive integer. Then,

\[
0 = [\sqrt{2}\omega_-(2n+1),0,\omega_+] \cdot \alpha_\mu = \sqrt{2}\omega_-(2n+1),0 \cdot (\omega_+ \cdot \alpha_\mu) - \omega_+ \cdot (\sqrt{2}\omega_-(2n+1),0 \cdot \alpha_\mu)
\]

\[
= \sqrt{2}\omega_-(2n+1),0 \cdot \alpha_\mu - \omega_+ \cdot (\sqrt{2}\omega_-(2n+1),0 \cdot \alpha_\mu)
\]

\[
= \alpha(\mu,2n+1) - \omega_+ \cdot \alpha(\mu,2n+1),
\]

and the result follows by induction.

Hence, if \( \mu \) does not contain any parts of size 1, then \( \omega_+ \cdot \alpha_\mu = \alpha_\mu \).

Now, suppose \( \gamma \) is an odd partition without parts of size 1. We will prove that

\[
\omega_+ \cdot \alpha(\gamma,1^k) = \alpha(\gamma,1^k) + 2k\alpha(\gamma,1^{k-1})
\]
by induction on $k$. The base case $k = 0$ was proved above. Suppose the formula holds for $\alpha(\gamma, 1^k)$.

\[
\omega_+ \cdot \alpha(\gamma, 1^{k+1}) = \omega_+ \cdot (\alpha(\gamma, 1^k) \alpha_1 - 2(\gamma, 1^k) \alpha(\gamma, 1^t)) \quad \text{by Lemma 5.8}
\]
\[
= \alpha_1 \omega_+ \cdot \alpha(\gamma, 1^k) + 2\omega_+ \cdot \alpha(\gamma, 1^t) - 2(\gamma, 1^k) \omega_+ \cdot \alpha(\gamma, 1^t) \quad \text{by Lemma 5.9}
\]
\[
= (\alpha_1 + 2 - 2(\gamma, 1^k)) \omega_+ \cdot \alpha(\gamma, 1^t)
\]
\[
= (\alpha_1 + 2 - 2(\gamma, 1^k))(\alpha(\gamma, 1^t) + 2k\alpha(\gamma, 1^{t-1})) \quad \text{by the inductive hypothesis}
\]
\[
= (\alpha_1 + 2 - 2(\gamma, 1^k))\alpha(\gamma, 1^t) + 2k(\alpha_1 + 2 - 2(\gamma, 1^k))\alpha(\gamma, 1^{t-1})
\]
\[
= (\alpha_1 - 2(\gamma, 1^k))\alpha(\gamma, 1^t) + 2\alpha(\gamma, 1^t) + 2k(\alpha_1 - 2(\gamma, 1^{k-1}))\alpha(\gamma, 1^{t-1})
\]
\[
= \alpha(\gamma, 1^{k+1}) + 2\alpha(\gamma, 1^t) + 2k\alpha(\gamma, 1^t) \quad \text{by Lemma 5.8}
\]
\[
= \alpha(\gamma, 1^{k+1}) + 2(k + 1)\alpha(\gamma, 1^t),
\]

and the result follows after the identification $\alpha_\mu \rightarrow 2^\ell(\mu)p_\mu$.

For the action of $\omega_{0,3}$, note that this element acts on the center as multiplication by itself. Therefore,

\[
-2\omega_{0,3} \cdot \alpha_\mu = \alpha_3 \alpha_\mu + \alpha(1, 1) \alpha_\mu \quad \text{by the Eq. 42},
\]
\[
-2\omega_{0,3} \cdot 2^\ell(\mu)p_\mu = 2^\ell(\mu)+1p_3p_\mu + 2^\ell(\mu)+2p(1, 1)p_\mu.
\]

\[
\square
\]

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