Bounds on Unique-Neighbor Codes

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Abstract

Let $A$ be an $m \times n$ parity-check matrix of a linear binary code of length $n$, rate $R$, and distance $\delta n$. This means that for every $0 < k < \delta n$, every $m \times k$ submatrix of $A$ has a row of odd weight. Message-passing decoding algorithms require the stronger unique-neighbor property. Namely, that every such submatrix has a row of weight 1. This concept arises naturally in the context of efficient decoding of LDPC expander codes as well as in the study of codes for the binary erasure channel, where $A$ is said to have stopping distance $\delta n$. It is well known that if $\frac{1}{2} \leq \delta$, then $R = o(1)$ whereas for every $\delta < \frac{1}{2}$ there exist linear codes of length $n \to \infty$ and distance $\geq \delta n$ with $R$ bounded away from zero. We believe that the unique-neighbor property entails sharper upper bounds on the rate.

Concretely, we conjecture that for a proper choice of $f(m) = o(m)$ and some positive constant $\epsilon_0$, every $m \times (m + f(m))$ binary matrix has an $m \times m'$ submatrix with $0 < m' \leq (\frac{1}{2} - \epsilon_0)m$ where no row has weight 1. In other words, that every linear code of non-vanishing rate has a normalized stopping distance of at most $\frac{1}{2} - \epsilon_0$. We make several contributions to the study of this conjecture. Concretely, we (1) prove the conjecture for sufficiently dense matrices (2) find tight upper bounds for the stopping distance of matrices in standard form, and prove the conjecture in this special case (3) show that the conjecture can hold only if $f(t) \geq \log_2(t)$ (4) find tight upper bounds for both distance and stopping distance of matrices where $n - m$ is small, and provide minimal $(m, n)$ where the upper bound on the stopping distance is strictly smaller than that of the distance of such matrices.

1 Introduction and Main Problems

We consider here only binary codes $C \subseteq \{0,1\}^n$ of length $n$. As usual, we denote the rate of $C$ by $R = R(C) = \frac{1}{n} \log_2 |C|$ and its distance by $\text{dist}(C) = \min_{x \neq y, x, y \in C} d_H(x, y)$, where $d_H$ stands for the Hamming distance. A fundamental open problem in coding theory seeks the best possible tradeoff between $R$ and $0 \leq \delta \leq 1$. We refer to this as

**Problem 1.1.** Determine, or estimate the real function

$$R(\delta) = \limsup_{n \to \infty} \left\{ R(C) \mid C \subseteq \{0,1\}^n, \text{dist}(C) \geq \delta n \right\}$$

A linear code is a linear subspace of the vector space $\mathbb{F}_2^n$, which we identify with $\{0,1\}^n$. Such a code can be defined in terms of a parity-check matrix $A$ which is a $[(1 - R)n] \times n$ binary matrix. Namely, $C = \{x \in \mathbb{F}_2^n \mid Ax = 0\}$.

Let $S$ be a nonempty set of columns in a binary matrix and let $z$ be the sum over the integers of the columns in $S$. We say that the set $S$ is 1-free if no entry of $z$ equals 1, and we call $S$ even, if all entries of $z$ are even integers. We refer to the number of 1-entries in a binary vector as its weight or sum.

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Definition 1.2. Let $A$ be a binary matrix.

- The distance of the binary code with parity-check $A$, which we denote by $\varepsilon(A)$, is the smallest cardinality of a nonempty even set of columns in $A$.
- The stopping distance of $A$, which we denote by $u(A)$ is, the smallest cardinality of a nonempty 1-free set of columns in $A$.
- The maximum value of $\varepsilon(A)$ over all binary $m \times n$ matrices is denoted $\varepsilon(m,n)$.
- The maximum value of $u(A)$ over all binary $m \times n$ matrices is denoted $u(m,n)$.

With this, the question analogous to Problem 1.1 for linear codes suggests itself:

Problem 1.3. Determine, or estimate the real function

$$R_L(\delta) := \limsup_{n \to \infty} \{ R \mid \text{there exists a } [(1-R)n] \times n \text{ binary matrix } A \text{ with } \varepsilon(A) \geq \lfloor \delta n \rfloor \}$$

in other words, $R_L(\delta)$ is the smallest real $R$ such that

For any $\rho > R$ and large enough $n$, every $[(1-\rho)n] \times n$ binary matrix has an even set of $\leq \delta n$ columns.

And for stopping distance:

Problem 1.4. Determine, or estimate the real function

$$R_U(\delta) := \limsup_{n \to \infty} \{ R \mid \text{there exists a } [(1-R)n] \times n \text{ binary matrix with } u(A) \geq \lfloor \delta n \rfloor \}$$

In other words, $R_U(\delta)$ is the smallest real $R$ such that

For any $\rho > R$ and large enough $n$, every $[(1-\rho)n] \times n$ binary matrix has a 1-free set of $\leq \delta n$ columns.

Equivalently

For any $\rho > R$ and large enough $n$, every $[(1-\rho)n] \times n$ binary matrix has a stopping distance $\leq \delta n$.

Clearly,

$$R_U(\delta) \leq R_L(\delta) \leq R(\delta) \quad \text{for all } 0 \leq \delta$$

At present, we cannot even rule out the possibility that all these three functions are, in fact, identical.

It is easily verified that (i) All three are decreasing functions of $\delta$, and (ii) $R(0) = R_L(0) = R_U(0) = 1$.

It is also well known that $R(\delta), R_L(\delta)$ are positive for $\delta < \frac{1}{2}$ and $R(\delta), R_L(\delta) = 0$ for $\frac{1}{2} \leq \delta$.

We believe that the strict inequality $R_U(\delta) < R_L(\delta)$ holds for at least some of the range $0 < \delta < \frac{1}{2}$.

More specifically that $R_U$ vanishes already at some $\delta_0 < \frac{1}{2}$. Concretely, we state

Conjecture 1.5. There is some positive function $f = f(m) = o(m)$ and some positive $\epsilon_0$ such that every $m \times (m + f(m))$ binary matrix has a 1-free set of at most $(\frac{1}{2} - \epsilon_0)n$ columns.

Let $A$ be a parity-check matrix of a linear code $C \subseteq \{0,1\}^n$. Of course $C$ remains invariant under elementary row operations on $A$. Also distances among vectors in $C$ remain unchanged as $A$’s columns get permuted. Consequently, in the study of $R_L(\delta)$ as in Problem 1.3, there is no loss of generality in assuming that $A$ is in standard form, i.e., its first $n$ columns form an order-$n$ identity matrix. We pose:

Problem 1.6. Let $n, k$ be positive integers, and let $A$ be a binary $n \times (n + k)$ matrix $A$ whose first $n$ columns form the order-$n$ identity matrix. How large can $u(A)$ be?
We often use the fact that $u$ and $\varepsilon$ are invariant under row and column permutations. The matrices that we consider have size $m \times n$ or $n \times (n + k)$ in different parts of the paper.

### 1.1 General Context and the Meaning of Conjecture 1.5

Note that whereas the distance is an inherent parameter of a code, the stopping distance is defined for binary matrices and varies with the choice of the code’s parity-check matrix. We briefly review below some of the prior work on the stopping distance of parity-check matrices. We further elaborate on this in Section 1.2.

**Message-passing algorithms** offer a powerful approach to the decoding problem of linear codes $C = \{x \in F_2^n \mid Ax = 0\}$. In the analysis of such algorithms, $A$ is viewed as the bipartite adjacency matrix of the code’s *factor graph* (also known in the literature as a Tanner graph [25]). This is a bipartite graph $(U, V; E)$, where $U$ is the set of $A$‘s rows and $V$ is its set of $A$‘s columns, and edges correspond to 1-entries in $A$. Message-passing algorithms such as belief propagation [6] work by iteratively passing messages between vertices in $U$ with those in $V$.

#### 1.1.1 In LDPC Codes

When the factor graph is a bounded-degree expander graph, we say that $C$ is an *expander code*. Such codes belong to the class of low density parity-check (LDPC) codes introduced by Gallager [6]. A major reason for the great interest in such codes [6, 20, 27] is that message-passing algorithms can decode them efficiently. Specifically, in a highly influential paper [23], Sipser and Spielman showed that message-passing algorithms can efficiently decode expander codes even when linearly many (in $n$) errors occur. The performance of the algorithm depends on the unique-neighbor expansion of the graph’s bipartite adjacency matrix. We briefly describe the algorithm and refer the reader to the original paper and to survey articles [9, 19, 22] for a full account:

- We associate with an $m \times n$ binary matrix $A$ a bipartite graph $(U, V; E)$ with $|U| = m, |V| = n$. The $i$-th vertex in $U$ is adjacent to the $j$-vertex in $V$ iff $a_{ij} = 1$. Let $y$ be (an $n$-bit) received word. We think of $y$ as the indicator vector of some subset $Y \subseteq V$. A vertex $u \in U$ is considered *satisfied* if it has an even number of neighbors in $Y$. Notice that $y \in C$ iff all vertices in $U$ are satisfied.

- Every vertex $v_j \in V$ counts how many of its neighbors in $U$ are satisfied resp. unsatisfied. If the number of unsatisfied neighbors exceeds that of the satisfied ones, we flip the $j$-th bit of $y$.

- Recalculate and repeat, until a word in $C$ is reached.

The assumptions made in [23] are expressed in terms of $G$’s *expansion*, a notion and concept of immense importance in computer science mathematics and more (see [10] for a comprehensive survey). We say that $G = (U, V; E)$ has *$(\alpha, \beta)$-expansion from $V$ to $U$* if every subset $Z \subseteq V$ of cardinality $|Z| \leq \alpha|V|$ has at least $\beta|Z|$ distinct neighbors in $U$. In defining $G$’s *unique-neighbor expansion* we consider for $Z$ as above, only those vertices in $U$ that have *exactly one neighbor* in $Z$. In contrast with the extensive literature on expansion, unique-neighbor expansion is still not very well understood.

#### 1.1.2 In BEC decoding

In decoding for *binary erasure channels*, the quantity $u(A)$ is called $A$’s *stopping distance*, e.g., [21]. The setup is this: Again we start with a linear code $C = \{x \in F_2^n \mid Ax = 0\}$. Upon receiving a word $y \in \{0, 1, ?\}^n$, we seek to replace the question marks by bits in such a way that the resulting word belongs to $C$. Again we construct the factor graph $G = (U, V; E)$ as before. We let $Q \subseteq V$ correspond to all coordinates $i \in [n]$, where $y_i = ?$. Suppose that $v_i \in V$ belongs to $Q$. If in $G$ this vertex $v_i \in Q$, has
a unique neighbor in $U$, then we know whether to make $y_i$ equal either zero or one. We do that and proceed. Consequently, if $|Q| \leq u(A)$, the algorithm terminates with the (unique) vector in $C$ that gave rise to $y$. On the other hand, if $Q$ constitutes a 1-free set of columns in $A$, namely a stopping set, then the algorithm terminates prematurely.

The above discussion explains the interest in binary matrices with large stopping distance, and suggests the following problem: Given a parity-check matrix $A$ of a linear code $C$, can we add some rows to $A$ so that the resulting $A'$ defines the same code $C$ and yet has the largest possible stopping distance? This question is solved in the affirmative in [21]. By adding redundant (i.e. linearly dependent) rows to $A$, the resulting matrix $A'$ clearly defines the same linear code as $A$, and as they show, these redundant rows can be chosen such that $u(A') = \varepsilon(A') = \varepsilon(A)$. Unfortunately, the least number of such additional redundant rows needed (called $A$'s stopping redundancy) may be exponential.

1.1.3 What Conjecture 1.5 Says

As is well known, linear codes with positive rate have normalized distance less than a half. We believe that the analogous parameter for stopping distance is strictly less than a half.

1.2 Some Further Background and References to the Literature

1.2.1 Unique-Neighbor Expanders

The literature on unique-neighbor codes is fascinating and yet we still know very little. In particular, quantitative results in this area are few and far between. It is this lacuna that motivates our work. Alon and Capalbo [1] found explicit constructions of bipartite graphs which are $(\alpha, \beta)$-unique-neighbor-expander graphs, where $\alpha, \beta$ are some positive absolute constants and $|V|/|U|$ is bounded away from 1. More recently Becker [2] showed how to construct such graphs that are also Cayley graphs.

The work of Sipser and Spielman has been subsequently improved several times. Viderman [26] proved that the same conclusions hold as well for graphs with lower expansion rate, resp. unique-neighbor expansion rate. We recall that a Tanner code $T(G, C)$ [23] is defined by its inner graph $G$ and its inner code $C$. Here $G = (U, V; E)$ is a bipartite $(c, d)$-regular graph with $U = [n]$, and $C$ is a linear code of block length $d$. The corresponding Tanner code $T(G, C)$ is the set $\{x \in \mathbb{F}_2^n \mid \forall i \in V, x|_{N(i)} \in C\}$. Here $N(i)$ is the set of $i$'s neighbors in $G$, and $x|_{N(i)} \in \{0, 1\}^d$ is the restriction of the codeword $x$ to the coordinates in $N(i)$. Dowling and Gao [5] determined a range of parameters for which linear-time decoding of Tanner codes is possible. Namely, the unique-expansion rate of the inner graph, the distance of the inner code and the error rate.

Ben-Sasson and Viderman [3] used unique-neighbor expanders to construct robustly-testable codes by taking their tensor products with another code with good distance and rate. They stress that an expander code of good distance is necessarily also a unique-neighbor expander. A linear code with parity-check matrix $A$ is called smooth if its distance remains large also after a few rows and columns are removed from $A$. It is called weakly-smooth if the above holds provided the removed rows have a small total weight. Ben-Sasson and Viderman showed that unique-neighbor expander codes are weakly-smooth, and can therefore be used to form robustly-testable codes. It remains open whether or not they are smooth.

1.2.2 Stopping Distance and Stopping Redundancy

The distance of a linear code, namely the function $\varepsilon$ receives, of course, a lot of attention in coding theory. The function $u$ was previously (e.g., [4, 13, 21, 16]) defined and called the stopping distance of a matrix. A stopping set in a parity-check matrix is what we called above a 1-free set. This name originates in the area of iterative decoding algorithms for BEC channels (see, e.g., [9]). Such an algorithm terminates
prematurely upon encountering a stopping set, and might fail decoding the received input. For further algorithmic aspects of stopping sets see [11, 12, 18, 15, 17].

2 Our New Results

Our work addresses Problems 1.4 and 1.6. Problems 1.1 and 1.3 are mentioned here for context only.

1. We prove Conjecture 1.5 under the assumption of a lower bound on the weight of each row in $A$ (Theorem 3.1). In that case the statement holds in fact with $f(m) = 0$.

2. We show (Theorem 6.1) that Conjecture 1.5 cannot hold unless $f(m)$ exceeds $\log_2(m)$.

3. We answer Problem 1.6 in full (Theorem 4.1).

4. Clearly $u(m, n) \leq \varepsilon(m, n)$ for all $m$ and $n$. We find the smallest pair $m < n$ for which the inequality is strict (Theorem 7.1, Item 1).

3 The Effect of Large Row Weights

As we show next, matrices of sufficiently large row weights satisfy Conjecture 1.5. In contrast, finding small 1-free sets in sparse matrices seems harder, and in Theorem 6.1 we use matrices with row sums 3 to derive a lower bound on $f = f(n)$ without which the conclusion of Conjecture 1.5 fails to hold.

**Theorem 3.1.**  
1. If $A$ is a binary $n \times n$ matrix in which every row has weight at least 9, then it has an $n \times n'$ ($n' \leq n$) submatrix with $n' \leq 0.49n$ with all row sums at least 2.

2. On the other hand, there exist binary $n \times n$ matrices where every row has weight 4, such that every $n \times n'$ ($n' \leq n$) submatrix with no row of weight 1 must satisfy $n' \geq n/2$.

**Proof.** Let $c$ be the smallest Hamming weight of the rows in such a matrix. Let us sample a random set of columns by picking every column independently with probability $\rho$. We denote by $X_0, X_1$ the (random) sets of rows in the resulting submatrix of weight zero, resp. one. Next we iteratively correct every row of weight 1 by adding one column to make its weight at least 2. It follows from our assumptions that this procedure terminates and requires at most $|X_0| + |X_1|$ additional columns. This yields a set of columns as stated in the theorem with cardinality at most $(\rho + \mathbb{E}(|X_0|) + \mathbb{E}(|X_1|))n$. Note that

$$
\mathbb{E}(|X_0|) \leq n(1 - \rho^c) + o(n) \quad \mathbb{E}(|X_1|) \leq n_c(1 - \rho)^{c-1} + o(n)
$$

The first part now follows by observing that $(\rho + \mathbb{E}(|X_0|) + \mathbb{E}(|X_1|))n < 0.481n$ for $c = 9, \rho = 0.4$.

For the second part, let $A$ be the $n \times n$ binary matrix whose rows are comprised of all $n$ cyclic rotations of the vector $1^40^{n-4}$. Given a vector $x \in \{0, 1\}^n \setminus \{0\}$ (here $0$ is the all-zero vector of length $n$), let the vector $Ax$ be defined by real arithmetic. Clearly, $Ax \in \{0, 1, 2, 3, 4\}^n$. Multiply on the left by the all-1 vector to conclude that $\|Ax\|_1 = 4\|x\|_1$. Note next that if $(Ax)_i = 0$, then $(Ax)_{i+1 \mod n}$ is either 0 or 1. Therefore, if $Ax$ has no 1 coordinates, then all its coordinates are at least 2, so that $4\|x\|_1 = \|Ax\|_1 \geq 2n$ and $\|x\|_1 \geq n/2$, as claimed.

Item 2 of Theorem 3.1 reflects on the validity of Conjecture 1.5. It shows that to guarantee the existence of small 1-free sets of columns, we must consider matrices with more columns than rows. This statement is made quantitative in Theorem 6.1.

We suspect that Item 1 of Theorem 3.1 remains valid even when all row weights are at least 5. However, this seems to require a substantial new idea.
4 Matrices in Standard Form

We denote by \( u_I(m, n) \) the maximum of \( u(A) \) for a binary \( m \times n \) matrix in standard form \( A = [I_m|B] \), where \( I_m \) stands for the identity \( m \times m \) matrix. Answering Problem 1.6, we give an upper bound on \( u_I(m, n) \) that is tight in infinitely many cases.

**Theorem 4.1.** For every positive integer \( k \) and \( n \to \infty \), every binary \( n \times (n + k) \) matrix of the form \( A = [I_n|B] \) has a 1-free set of at most \( \frac{H_k}{k} + O(k) \) columns where \( H_k = \sum_{\ell=1}^{k} \frac{1}{\ell} \) is the \( k \)-th harmonic sum. The bound is tight, that is \( u_I(n, n + k) = \frac{H_k}{k} + O(k) \).

**Proof.** We denote by \( \langle u, v \rangle \) the inner product of the two real vectors \( u \) and \( v \). It is easy to describe all 1-free sets of columns in \( A \): Start with a submatrix of \( B \) with column set \( S \subset [k] \) and observe the weight-1 rows in this submatrix. Then add all the corresponding columns in \( I_n \) to make the set 1-free. So, given a matrix \( B \), we can express the least size of a 1-free set of columns in \( A \) as the optimum of an integer linear program. For a binary vector \( u \in \{0, 1\}^k \), let \( c_u \) be the number of rows in \( B \) that equal \( u \). Clearly, \( c_u \) is a nonnegative integer, and \( \sum_{u \in \{0, 1\}^k} c_u = n \).

If \( s \in \{0, 1\}^k \) is the indicator vector of \( S \), then we must add at least \( \sum_{u} \{ c_u | \langle u, s \rangle = 1 \} \) columns from \( I_n \) to reach a 1-free set of columns. So, let \( M \) be the \( 2^k \times 2^k \) binary matrix that is indexed by \( \{0, 1\}^k \). The \( (u, v) \) entry of \( M \) equals 1 iff \( \langle u, v \rangle = 1 \) (integer arithmetic), and conclude that

\[
m \leq u_I(n, n + k) \leq m + k
\]

where

\[
m = \max_{y} \quad \text{subject to} \quad Mz \geq 1 \cdot y, \\
\langle z, 1 \rangle = n \text{ and } z \geq 0 \text{ is a vector of integers}
\]

where \( 1 = 1_{2^k \times 1} \) is the all-1 vector of length \( 2^k \) here.

The \( \pm k \) uncertainty in our bound on \( u_I(n, n + k) \) has to do with the size of \( S \subset [k] \) mentioned above.

We turn to solve the rational relaxation of the above ILP.

\[
\max_{y} \quad \text{subject to} \quad Mz \geq 1 \cdot y, \\
\langle z, 1 \rangle = n \text{ and } z \geq 0
\]

We consider the dual and find the \( 2^k \)-dimensional vector \( w \) with \( w_0 = 0 \) and \( w_u = \frac{1}{(k-1)} \) for every \( u \neq 0 \) in \( \{0, 1\}^k \)

It follows that \( (w^T M)_0 = 0 \), and if \( u \in \{0, 1\}^k \) with \( |v| = j \) for some \( 1 \leq j \leq k \) then

\[
(w^T M)_u = \sum_{i=1}^{k} \frac{1}{(i-1)!} j! \binom{k-j}{i-1} = \frac{j!(k-j)!}{(k-1)!} \sum_{i=1}^{k} \frac{k-i}{j-1} = \frac{j!(k-j)!}{(k-1)!} \frac{k}{j} = k.
\]

The first equality follows from the definition. The second only involves reorganizing terms. The third
one uses the standard and easy fact that for all positive integers $s \leq N$ it holds that

$$\sum_{s \leq r \leq N} \binom{r}{s} = \binom{N+1}{s+1}$$

In other words, $w^T M = k(1 - e_0)$, and hence $\langle w, 1 \rangle \cdot y = w^T \cdot 1 y \leq w^T M z \leq nk$.

Also

$$\langle w, 1 \rangle = \sum_{i=1}^{k} \binom{k}{i-1} = kH_k.$$ 

It follows that $m \leq \frac{n}{H_k}$ and hence

$$u_I(n, n+k) \leq \frac{n}{H_k} + O(k)$$

since an upper bound on the LP applies as well to the corresponding ILP, both of which seek to maximize the same objective function.

The reverse inequality follows by letting

$$z := n\frac{w}{kH_k}$$

and observing that with similar calculations we get $\langle z, 1 \rangle = n$, and $1_{\frac{n}{H_k}} = Mz$, hence the stopping distance of an $n \times (n+k)$ matrix $A = [I_n|B]$ where the rows of $B$ correspond to such $z$ is $\frac{n}{H_k} + O(k)$ and $m \geq \frac{n}{H_k}$.

To get the lower bound on the ILP, let $H_k := \frac{a_k}{b_k}$ written as a reduced rational. If $n$ is divisible by $a_k$, say $n = pa_k$, then $u_I(n, n+k) = pb_k + O(k)$, because in this case the optimal solutions to our LP and the ILP coincide. More generally, if $n = pa_k + q$ then $\frac{n}{H_k} + O(k) - 1 \leq \lfloor \frac{n}{a_k} \rfloor b_k + O(k) \leq u_I(n, n+k)$. 

5 Some Useful Constructions

In this section we introduce a $(2^k - 1 - k) \times (2^k - 1)$ binary matrix $U_k$ for $k = 2, 3, \ldots$ to be used below. We define $U_k$ both recursively and directly. It is easy to verify by a simple inductive argument that the two definitions coincide. Here is the recursive one:

$$U_2 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

(1)

$$U_{k+1} = \begin{pmatrix} I_{2^k-1} & 1_{2^k-1} & I_{2^k-1} \\ 0 & 1_{2^k-1} & 0 \\ 0 & 0 & U_k \end{pmatrix}$$

(2)

where $1_{m,n}$ is the all-1 matrix of dimensions $m \times n$.

In the direct definition of $U_k$ we index its columns by all integers $2^k - 1, 2^k - 2, \ldots, 1$, in this order. The rows are indexed by the subsequence of the above excluding the powers of 2. Each row of $U_k$ has weight 3. If the integer $m \in \{1, \ldots, 2^k - 1\}$ is not a power of 2, such that $2^t < m < 2^{t+1}$ for an integer $t$, then the three 1 entries in row $m$ appear in columns $m, m - 2^t$ and $2^t$. 

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For example,

\[
U_3 = \begin{pmatrix}
1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}
\]

with rows called 7, 6, 5, 3 in this order and columns called 7, . . . , 1.

We note that \(U_k\) is a generator matrix of the \([2^k - 1, 2^k - 1 - k, 2^k - 1 - k, 2^k - 1 - k, 2^k - 1 - k, 2^k - 1 - k, 2^k - 1 - k, 2^k - 1 - k, 2^k - 1 - k]\) Hamming code. Clearly every such triplet belongs to the relevant generalized Hamming code. Also, \(U_k\) has a full row rank since it contains an upper-triangular square submatrix called \(T_k\) which is attained by erasing those columns of \(U_k\) that correspond to a power of two. The remaining submatrix is called \(R_k\), namely the columns of \(U_k\) whose index is a power of 2. For example,

\[
T_3 = \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
R_3 = \begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 1
\end{pmatrix}
\]

An interesting aspect of this construction is that it exploits the duality between Hamming codes and simplex codes. Since the former are generated by a matrix with row weights 3, it is relatively easy to derive a lower bound on \(u(U_k)\). On the other hand, due to the fact that all nonzero codewords in an order-\(N\) simplex code have weight \(N/2\), for a proper choice of \(N\) it yields an upper bound on \(\varepsilon(U_k)\), which is an upper bound on \(u(U_k)\). Since these bounds coincide they are both tight. Moreover, in the next section we show that these matrices attain the bounds \(u(n, n + k), \varepsilon(n, n + k)\) (for the corresponding \(n, k\)) and these quantities are, in fact, equal.

We next construct for every \(k \geq 2, m \geq 1\) a \(((2^k - 1) - m - k) \times ((2^k - 1) - m)\) binary matrix as follows.

\[
U_{k,m} = \begin{pmatrix}
T_k & \cdots & T_k \\
& & \\
& & R_k \\
& & \vdots \\
& & R_k \\
& & \vdots \\
& & I_k \\
& & \vdots \\
& & I_k
\end{pmatrix}
\]

(3)

where the empty blocks are all-zero.

6 Conjecture 1.5 Can Hold Only If \(f\) is at Least Logarithmic

We prove next that \(u(n - \log_2 n - 1, n - 1) = \varepsilon(n - \log_2 n - 1, n - 1) = \frac{n}{2}\) for infinitely many integers \(n\). Concretely,

**Theorem 6.1.** For every integer \(k \geq 2\) it holds that

\[
u(2^k - 1 - k, 2^k - 1) = \varepsilon(2^k - 1 - k, 2^k - 1) = 2^{k-1}.
\]
Proof. The proof proceeds by showing that

\[ \varepsilon(2^k - 1 - k, 2^k - 1) \leq 2^{k-1} \] \quad \text{and} \quad u(U_k) \geq 2^{k-1}.

We recall the following well-known fact:

**Proposition 6.2.** Every \( n \times (n+k) \) binary matrix \( A \) has an even set of at most \( \left(1 + \frac{1}{|\mathcal{C}|} \right) \frac{n+k}{2} \) columns.

This inequality is a direct consequence of Griesmer’s bound \[8\]: The linear code \( \mathcal{C} := \ker A \) has block length \( n+k \) and dimension \( \log |\mathcal{C}| \leq k \). Then from Griesmer’s bound, denoting \( d = \varepsilon(A) \) the minimum distance of \( \mathcal{C} \), \( n+k \geq \sum_{i=0}^{\log |\mathcal{C}| - 1} \frac{d}{2^i} = d \cdot \frac{1}{1-\frac{1}{2^k}} \), so \( d \leq \frac{1}{1-\frac{1}{2^k}} \frac{n+k}{2} = \left(1 + \frac{1}{|\mathcal{C}|-1} \right) \frac{n+k}{2} \leq \left(1 + \frac{1}{2^{k-1}} \right) \frac{n+k}{2} \).

We note that Proposition 6.2 can be alternatively proven without Griesmer’s bound, using linear algebra, and the resulting proof is simpler than that of Griesmer’s bound.

This yields \( \varepsilon(U_k) \leq \varepsilon(2^k - 1 - k, 2^k - 1) \leq 2^{k-1} \) as \( U_k \) is a \( 2^k - 1 - k \times 2^k - 1 \) binary matrix.

We turn to prove that \( u(U_k) \geq 2^{k-1} \) by induction on \( k \geq 2 \). For \( k = 2 \) the claim clearly holds. For the induction step we use the recursive description of \( U_{k+1} \) in Section 5. Consider a 1-free column set of \( U_{k+1} \). If it contains column \( 2^k \), then it must include at least \( 2^k - 1 \) additional columns (from either side of the column), for a total of at least \( 2^k \) columns, as claimed.

Thus it suffices to consider a 1-free set of columns of the form \( L \sqcup R \), where \( R, L \) are the subsets of columns from the \( 2^k - 1 \) rightmost, leftmost ones (respectively). By the induction hypothesis \( |R| \geq 2^{k-1} \).

For every \( r \in R \), one of its first (upper) \( 2^k - 1 \) coordinates contains a 1. Since column \( 2^k \) is absent from \( L \sqcup R \), the (unique) matching column from the \( 2^k - 1 \) leftmost columns should be included in \( L \), in order for \( L \sqcup R \) to be 1-free. It follows that \( |L \sqcup R| \geq 2^k \), completing the proof. \( \square \)

Theorem 6.1 and the weak monotonicity of \( u, \varepsilon \) (see Proposition 7.2 below) yield

**Corollary 6.3.** For every \( k, n \) it holds that \( u(n, n+k) \geq \varepsilon(n, n+k) - 2^{k-1} - 1 \).

Further cases where \( \varepsilon \) and \( u \) coincide are provided by the following extension of Theorem 6.1:

**Theorem 6.4.** For every integers \( k \geq 2 \) and \( m \geq 1 \) it holds that \( u((2^k-1)m - k, (2^k-1)m) = \varepsilon((2^k-1)m - k, (2^k-1)m) = 2^{k-1}m \).

Proof. Again we bound \( u \) from below and \( \varepsilon \) from above. The bound on \( u \) uses the matrices \( U_{k,m} \) from Section 5 and the bound on \( \varepsilon \) follows from Proposition 6.2. To show that \( u(U_{k,m}) \geq 2^{k-1}m \), we consider 1-free sets of columns in \( U_{k,m} \). The matrix \( U_{k,m} \) with its last \( k \) columns removed has no nonempty 1-free sets, since it is an upper-triangular, full-rank matrix. So consider a 1-free set that includes \( t > 0 \) columns among the last \( k \) columns of \( U_{k,m} \). By Theorem 6.1 at least \( (2^{k-1} - t)m \) additional columns are needed, namely at least \( 2^{k-1} - t \) from every \( T_k \) in the direct sum. The lower part of \( U_{k,m} \) necessitates exactly \( (m-1)t \) columns from the columns that contain the block \( I_{(m-1)k} \). In total the cardinality of the 1-free set at hand is at least \( t + (2^{k-1} - t)m + (m-1)t = 2^{k-1}m \), as claimed. \( \square \)

7 Between \( u \) and \( \varepsilon \) When \( n - m \) is Bounded

In this section we compare between \( u(m,n), \varepsilon(m,n) \) when \( n-m \geq 1 \) is bounded. Here is our main result:

**Theorem 7.1.**

1. \( u(4,8) = 3 \) whereas \( \varepsilon(4,8) = 4 \). This is the first case where \( u < \varepsilon \).

2. \( u(n,n+1) = \varepsilon(n, n+1) = n+1 \). The case of equality is fully characterized.

3. \( u(n,n+2) = \varepsilon(n, n+2) = \lfloor \frac{2n+4}{3} \rfloor \).

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4. If \( n \neq -1 \mod 7 \), then \( u(n, n+3) = \varepsilon(n, n+3) = \lfloor \frac{4n+12}{7} \rfloor \). Also, \( u(7m-1, 7m+2) = 4m \) for every positive integer \( m \).

**Proof.** We start with several simple observations:

**Proposition 7.2.**

1. \( u(m, n) \leq \varepsilon(m, n) \leq m \).

2. Both \( u(m, n) \) and \( \varepsilon(m, n) \) weakly increase with \( m \) and decrease with \( n \).

3. \( u(m, n) \leq u(m+1, n+1) \), \( \varepsilon(m, n) \leq \varepsilon(m+1, n+1) \).

4. If a binary matrix \( A \) has a row of weight 1, then \( u(B) = u(A) \), \( \varepsilon(B) = \varepsilon(A) \) where \( B \) is attained from \( A \) by an elementary collapse, i.e., by deleting the corresponding row and column of \( A \).

**7.1 Proof of Item 1:** \( u(4, 8) = 3 < 4 = \varepsilon(4, 8) \).

**Proof.** Proposition 6.2 implies that \( \varepsilon(4, 8) \leq 4 \). On the other hand \( \varepsilon(A) = 4 \) for

\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 1
\end{pmatrix}
\]

We now show that \( u(4, 8) = 3 \). The following matrix yields \( u(4, 8) \geq 3 \)

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 1
\end{pmatrix}
\]

Next we show that \( u(A) \leq 3 \) for every binary \( 4 \times 8 \) matrix \( A \). We reduce to the case that every column of \( A \) has weight at least 2. If \( A \) has a zero column, then clearly \( u(A) = 1 \). If some column of \( A \) has weight 1, say \( a_{1,1} = 1 \) and \( a_{i,1} = 0 \) for \( i = 2, 3, 4 \), consider the submatrix \( B \) of \( A \) that is obtained by erasing its first row and column. If \( B \) has an all-zero column, then \( u(A) \leq 2 \), and if \( B \) has two equal columns, then \( u(A) \leq 3 \). The only remaining case is when

\[
B = \begin{pmatrix}
0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1
\end{pmatrix}
\]

up to permutations of the rows and columns. Consider the weight \( w = \sum_j a_{1,j} \) of row 1 in \( A \). If \( w = 1 \), then \( a_{1,j} = 0 \) for all \( j \geq 2 \). Consequently, \( u(A) = u(B) = 3 \), since every set of columns that is 1-free in \( B \) is also 1-free in \( A \). If \( w \geq 3 \) there are at least two indices \( 1 < \beta < \alpha \) such that \( a_{1,\alpha} = a_{1,\beta} = 1 \). If columns \( \alpha, \beta, \gamma \) of \( B \) are the vectors \( u \) resp. \( v \), then some column \( \gamma \) corresponds to \( u \oplus v \) (mod 2 sum). Columns \( \alpha, \beta, \gamma \) form a 1-free set in \( A \). Finally, if \( w = 2 \), there is exactly one index \( \delta > 1 \) such that \( a_{1,\delta} = 1 \). But then we can find a triplet of columns of the form \( u, v, u \oplus v \) in \( B \) none of which is column \( \delta \).

We can now assume that every column of \( A \) has weight 2, 3 or 4. \( A \) has no repeated columns, or else \( u(A) = 2 \). Also \( A \) can have at most two columns of weight at least 3, for any three such distinct vectors form a 1-free set. Consequently \( A \) has exactly two columns of weight at least 3 and each of the six columns of weight 2. But the latter 6-tuple contains a 1-free set of three columns. \( \Box \)

Note that 4, 8 are the **minimal** \( m, n \) for which \( u(m, n) < \varepsilon(m, n) \): The other parts of the present theorem show that equality holds when \( n - m \leq 3 \). For \((m, n) = (1, 5), (2, 6)\) equality trivially holds
also the upper bound on $u$ with no isolated vertices. Consequently, at most $k$ above
$\sum A a$ 1-free set of a single new vertex $v$. This contradicts our assumption that $u$ $L$
On the other hand, if $K$ In particular $n$ $V$ $T$
By induction it is the edge-vertex matrix of $G$ $u$ $N$
that the edge $L$ 1-free set, contrary to our assumption. If $u$ $G$
The proof for $u$, $A$
Let $B$ be the connected components of $(V, L)$. By the above $\sum |V_i| = n + 1$, $|L_i| \geq |V_i| - 1$, so that $|L| = \sum |L_i| \geq n + 1 - k$, with equality iff $(V, L)$ is a forest with no isolated vertices. Consequently, at most $k - 1$ edges in $E$ are heavy.
Let $B$ be the edges vs. vertices matrix of the hypergraph that results from $G$ by shrinking each $V_i$ to a single new vertex $v_i$. Since $L \neq \emptyset$ this actually reduces the size of the matrix and we can use induction to prove the proposition. Every 1-free set $S$ of $B$ yields a 1-free set in $A$ by inflating each $v_i \in S$ to $V_i$. In particular $u(B) < k$ would imply $u(A) \leq n$. Consequently, $B$ is a $(k - 1) \times k$ matrix with $u(B) = k$. By induction it is the edge-vertex matrix of $K$, a tree with vertex set $\{v_1, \ldots, v_k\}$. Say that $v_1$ is a leaf of $K$, and let $e$ be the single edge of $K$ that is incident with $v_1$. We claim that either $V_1$ or $V \setminus V_1$ comprise a 1-free set of $A$. Indeed, only the row corresponding to $e$ may have weight 1 in the submatrix of $A$ corresponding to either $V_1$ or $V \setminus V_1$. But it is impossible that both cases occur, for that would mean that the edge $e$ has size 2 contrary to the fact that $e$ is a heavy edge. Since both $V_1, V \setminus V_1$ are nonempty, this contradicts our assumption that $u(A) = n + 1$. This establishes Item 2 of Theorem 7.1.

7.3 Proof of Items 3, 4
The proof for $k = 2$ splits to cases according to the value of $n \mod 3$. When $n \equiv 1 \mod 3$ we have $u(3m - 2, 3m) = \varepsilon(3m - 2, 3m) = 2m$ by Theorem 6.4. By Proposition 6.2, $u, \varepsilon$ do not change as we move to $n = 3m - 1$. Finally, for $n = 3m$ we introduce the matrix

$$A := \begin{pmatrix} U_{2,m} & 0 \\ 0 & I_2 \\ I_2 & 0 \end{pmatrix}$$

with $U_{2,m}$ as defined in Section 5. It is easy to see that $u(A) = \varepsilon(A) = 2m + 1$. By Proposition 6.2 this is also the upper bound on $u(3m, 3m + 2), \varepsilon(3m, 3m + 2)$. We conclude that $u(n, n + 2) = \varepsilon(n, n + 2) = \lfloor \frac{2n + 4}{3} \rfloor$ as claimed.
The analysis when $k = 3$ is somewhat more involved and proceeds according to the value of $n \bmod 7$. We start with the upper bound: By Proposition 6.2, $\varepsilon(n, n + 3) \leq \left\lceil \frac{4m + 1}{7} \right\rceil$. This bound is tight, except if $n \equiv -1 \bmod 7$, when it can be reduced by 1 due to Griesmer's bound [8]:

**Proposition 7.4.** Every $k$-dimensional binary linear code of distance $d$ has length at least $\sum_{i=0}^{k-1} \left\lfloor \frac{d}{7i} \right\rfloor$.

Indeed, our general upper bound is $\varepsilon(7m - 1, 7m + 2) \leq \left\lfloor \frac{28m + 8}{7} \right\rfloor = 4m + 1$, but by Griesmer’s bound if the code’s distance is $4m + 1$, then its length, is at least $4m + 1 + \left\lfloor \frac{4m + 1}{2} \right\rceil + \left\lceil \frac{4m + 1}{4} \right\rceil = 7m + 3$.

We proceed to deal with the lower bounds. The case $k = 3$ of Theorem 6.4 gives $u(7m - 3, 7m) = \varepsilon(7m - 3, 7m) = 4m$. Namely, $u = \varepsilon$ when $n \equiv 4 \bmod 7$.

Item 3 of Proposition 7.2 and Proposition 6.2 yield

$$u(n - 1, n + 2) \leq u(n, n + 3) \leq \varepsilon(n, n + 3) \leq \left\lceil \frac{4}{7} (n + 3) \right\rceil.$$  

When $u(n - 1, n + 2) = \left\lfloor \frac{3}{7} (n + 3) \right\rfloor$, this trivially allows to derive the case $n \equiv r + 1 \bmod 7$ from the case $n \equiv r \bmod 7$. This works verbatim for $r \equiv \pm 2 \bmod 7$. When $n \equiv 0, 1, 3 \bmod 7$, an additional argument is needed. To this end, we extend $U_{3,m}$ from Section 5 to an $n \times (n + 3)$ matrix for the appropriate $n$. This resembles the construction of $U_{k,m}$ from $U_k$, and the case $k = 2$. In all three cases, these matrices show that $u(n, n + 3)$ attains the upper bound on $\varepsilon(n, n + 3)$, namely $\left\lceil \frac{3}{7} (n + 3) \right\rfloor$. Hence we get in each case a matrix $U$ such that $\varepsilon(n, n + 3) = u(U) \leq u(n, n + 3)$. For illustration, when $n = 7m$, we use the matrix $U_{3,m}$ to construct

$$U := \begin{pmatrix} U_{3,m} & 0 \\ 0 & I_3 & I_3 \end{pmatrix}$$

Note that $4m + 1 = u(U) \leq u(7m, 7m + 3)$ and $\varepsilon(7m, 7m + 3) \leq 4m + 1$ from Proposition 6.2, so in total $u(7m, 7m + 3) = \varepsilon(7m, 7m + 3) = 4m + 1$.

We note that Item 4 holds as well when $n = 1, 2, 3$, but we skip this verification.

This concludes the proof of Theorem 7.1.

---

### 8 Open Problems

**Problem 8.1.** The most obvious question is Conjecture 1.5 which remains open.

**Problem 8.2.** What is the smallest $c$ for which the conclusion of Theorem 3.1 holds? Is it 5?

**Problem 8.3.** Let $u_3(m, n)$ denote $\max u(A)$ of an $m \times n$ binary matrix $A$ where every row has weight 3. Proposition 7.3 implies that $u_3(n, n + 1) < u(n, n + 1)$, but perhaps $u_3(m, n) = u(m, n)$ when $m + 1 < n$.

Some supportive evidence for this is that $u_3(4,8) = u(4,8)$, $u_3(2^k - 1 - k; 2k - 1) = u(2^k - 1 - k; 2k - 1)$. We note that more generally, $u_3((2^k - 1)m - 1, (2^k - 1)m) = u((2^k - 1)m - 1, (2^k - 1)m)$ holds, because the matrices $U_{k,m}$ can be modified so all rows have weight 3 without changing $u, \varepsilon$.

**Problem 8.4.** The proof of Theorem 3.1 suggests a more general setup. We seek a 1-free set of columns in a binary matrix $A$. Having committed to some subset of columns, the rows of $A$ are split into: $I_0 \sqcup I_1 \sqcup I_\ast$, those of weight 0, 1 and $\geq 2$, respectively. To extend our initially chosen set into a 1-free set, we need an additional set of columns $J$, the weight of whose $I_0$ and $I_1$ rows differ from 1,0 respectively. Under what conditions is it possible to pre-specify which row sums we wish to be $\neq 0$ and which $\neq 1$?
Remark 8.5. Assuming that Conjecture 1.5 is valid, it is not clear how it can be established. As Theorem 6.1 shows, methods that work for square matrices and matrices with only a few more columns than rows as in Theorem 3.1 are not likely to deliver a full answer.

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