Old and new results on density of stable mappings

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The analysis of the conditions for a map-germ to be finitely determined and of the degree of determinacy involves the most important of the local aspects of singularity theory.

– C.T.C. Wall[108]

Abstract Density of stable maps is the common thread of this paper. We review Whitney’s contribution to singularities of differentiable mappings and Thom-Mather theories on $C^\infty$ and $C^0$-stability. Infinitesimal and algebraic methods are presented in order to prove Theorem A and Theorem B on density of proper stable and topologically stable mappings $f : N^n \to P^p$. Theorem A states that the set of proper stable maps is dense in the set of all proper maps from $N$ to $P$, if and only if the pair $(n, p)$ is in nice dimensions, while Theorem B shows that density of topologically stable maps holds for any pair $(n, p)$. A short review of results by du Plessis and Wall on the range in which proper smooth mappings are $C^1$- stable is given. A Thom-Mather map is a topologically stable map $f : N \to P$ whose associated $k$-jet map $j^k f : N \to P$ is transverse to the Thom-Mather stratification in $J^k(N, P)$. We give a detailed description of Thom-Mather maps for pairs $(n, p)$ in the boundary of the nice dimensions. The main open question on density of stable mappings is to determine the pairs $(n, p)$ for which Lipschitz stable mappings are dense. We discuss recent results by Nguyen, Ruas and Trivedi on this subject, formulating conjectures for the density of Lipschitz stable mappings in the boundary of the nice dimensions. At the final section, Damon’s results relating $A$-classification of map-germs and $\mathcal{K}_V$ classification of sections of the discriminant $V = \Delta(F)$ of a stable unfolding of $f$ are reviewed and open problems are discussed.

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1 Introduction

Although Riemann, Klein, Poincaré and other great mathematicians of the nineteenth century already used deep topological concepts in their work, the birth of algebraic and differential topology as formal sub-areas of Mathematics occurred in the first half of the twentieth century.

After previous works of Whitehead, Veblen and others, the American mathematician Hassler Whitney introduced fundamental concepts and proved strong results in differential topology such as the well known strong Whitney embedding theorem and weak Whitney embedding theorem. The first one states that any smooth real \( m \)-dimensional manifold can be smoothly embedded in \( \mathbb{R}^{2m} \), while the latter says that any continuous mapping of an \( n \)-dimensional manifold to an \( m \)-dimensional manifold may be approximated by a smooth embedding provided that \( m > 2n \). Furthermore, replacing embedding by immersion in this last statement the result holds for all \( m \geq 2n \). His survey paper Topological properties of differentiable manifolds published in 1937 [111] contains many contributions he made in those early years of differential topology.

In 1944, Whitney [113] studied the first pair of dimensions not covered by his immersion theorem. For mappings \( f \) from \( \mathbb{R}^n \) to \( \mathbb{R}^{2n-1} \) Whitney proved that singularities cannot be avoided in general. He introduced the semi regular mappings as proper mappings \( f : \mathbb{R}^n \to \mathbb{R}^{2n-1} \) whose only singularities are the generalized cross-caps (Whitney umbrellas) points. Away from singular points, \( f \) is an immersion with transverse double points, and when \( n = 2 \), a finite number of triple points may also appear in the image of \( f \). These are the only stable singularities in these dimensions. However, only later, Whitney introduced the notion of stable mappings.

Abstract spaces and their topological properties were known by then, so that the notion of stability of systems and mappings appeared naturally. It appeared first in dynamical systems, introduced by A. Andronov and L. Pontryagin [1] for a class of autonomous differential systems on the plane, under the name of “systèmes grossiers”. The term “structural stability” appears in the english language edition of
the book by Andronov and Chaikin, edited under the direction of Solomon Lefschetz in 1949 [2] (see also [91]). It also appears in other pioneering papers on the subject, among them the paper On structural stability by Mauricio Peixoto [78], published in 1959.

The notion of stable mappings was formulated by Whitney in [115] around the middle of last century. He characterized stable mappings from \( \mathbb{R}^n \) to \( \mathbb{R}^p \) with \( p \geq 2n - 1 \) in [112] and stable mappings from the plane into the plane in [114], showing in these cases that stable mappings form a dense set in the space of smooth proper mappings.

The article Whitney [114] published in 1955 is a landmark, considered by many to be the cornerstone of the theory of singularities. The stable singularities of mappings of the plane into the plane are folds and cusps and any proper smooth mapping \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) can be approximated by a stable mapping. Whitney conjectured that density of stable mappings would hold for any pair \( (n, p) \). However René Thom showed, in his 1959 lecture at Bonn, that this is not the case by given an example of a map \( f : \mathbb{R}^9 \to \mathbb{R}^9 \) that appears generically in a 1-parameter family of maps.

Thom conjectured that the topologically stable maps are always dense and gave an outline of the proof. The complete proof was given by John Mather, who from 1965 to 1975, solved almost completely the program drawn by René Thom for the problem of stability.

Mather found several characterizations of stability and proved that the set \( S^\infty(N, P) \) of stable mappings is dense in the set \( C^\infty_{pr}(N, P) \) of smooth proper mappings, from the \( n \)-dimensional manifold \( N \) to the \( p \)-dimensional manifold \( P \), if and only if \( (n, p) \) is in the nice dimensions, which he completely characterized in [63]. Based on Thom’s ideas, he also proved in [65, 66] that the set of topologically stable mappings \( S^0(N, P) \) in \( C^0_{pr}(N, P) \) is residual for all pairs \( (n, p) \).

The 70’s was blooming period for singularity theory. Along with Mather’s work, René Thom’s book on catastrophe theory [94] and Arnold’s seminal classification of simple singularities of functions [3] also had a great impact. These works paved the intense development of the theory of the following decades. The deep understanding of stable mappings, versal unfoldings and finite determinacy transformed singularity theory into an organizing center for several areas of mathematics and sciences.

The common thread of these notes is the question of density of stable mappings in \( C^\infty_{pr}(N, P) \). We outline the solutions of the various formulations of this problem: \( C^\infty, C^0 \) and \( C^l, 1 \leq l < \infty \) stability. The remaining open problem in this setting is density of Lipschitz stable mappings. Recent progress in the solution of this problem appear in [88, 75].

We give an account of tools for the proofs of the main theorems including the notion of infinitesimal stability, the generalized Malgrange’s theorem, Thom’s transversality theorem, mappings of finite singularity type and finite determinacy of Mather’s groups. Whitney and Thom’s results on stratified sets and maps are fundamental pieces of the theory. For an account of these topics we refer to David Trotman’s article in Volume 1 of this Handbook.

In these notes we concentrate on the discussion of real singularities. The infinitesimal methods discussed here also hold true for holomorphic mappings. For an
account on Mather’s theory of \(\mathcal{A}\)-equivalence and the description of the topology of stable perturbations of \(\mathcal{A}\)-finitely determined holomorphic germs the reader may consult the notes by David Mond and Juan José Nuño-Ballesteros in this Handbook [70].

Related topics to those discussed in these notes, as well as new developments of the theory, are given in the subsections Notes at the end of each section. The final section includes a discussion of open problems in the theory of singularities of smooth mappings.

## 2 Setting the problem

Let \(C^\infty(N, P) = \{f : N \to P, f \in C^\infty\}\) be the set of smooth mappings from \(N\) to \(P\), where \(N\) and \(P\) are smooth manifolds of dimension \(n\) and \(p\) respectively. The topology on \(C^\infty(N, P)\) is the \(C^\infty\)-Whitney topology.

We review here the contributions of singularity theory to solve the following problem.

**Problem 2.1** Find an open and dense set \(S\) in \(C^\infty(N, P)\) and describe all singularities of mappings \(f \in S\).

The relevant equivalence is \(\mathcal{A}\)-equivalence.

**Definition 2.2** Two smooth maps \(f, g : N \to P\) are \(\mathcal{A}\)-equivalent if there exist \(C^\infty\) diffeomorphisms \(h : N \to N\) and \(k : P \to P\) such that the following diagram commutes

\[
\begin{array}{ccc}
N & \xrightarrow{f} & P \\
\downarrow{h} & \circ & \downarrow{k} \\
N & \xrightarrow{g} & P
\end{array}
\]

**Definition 2.3** The map \(f : N \to P\) is stable (\(\mathcal{A}\)-stable) if there exists a neighborhood \(W\) of \(f\) in \(C^\infty(N, P)\), such that \(g \sim f\) for every \(g \in W\).

Replacing \(C^\infty\)-diffeomorphisms by homeomorphisms, \(C^l\)-diffeomorphisms, \(l \geq 0\) or bi-Lipschitz homeomorphisms in definitions 2.2 and 2.3 we get respectively the definitions of \(C^0\)-\(\mathcal{A}\), \(C^l\)-\(\mathcal{A}\) \((l \geq 0)\), bi-Lipschitz-\(\mathcal{A}\) equivalences and of topological stability, \(C^l\)-stability, or Lipschitz stability of maps in \(C^\infty(N, P)\).

Before starting the discussion of Problem 2.1, we review some notation and definitions.

The Whitney \(C^\infty\)-topology in \(C^\infty(N, P)\) was defined by John Mather in [57]. We review it here (more details can be found in the book of Golubitsky and Guillemin [40]).

For \(x \in N, y \in P\) and for a non-negative integer \(k\), we denote by \(J^k(N, P)_{x, y}\) the set of \(k\)-jets of map-germs \((N, x) \to (P, y)\). When \(N = \mathbb{R}^n, P = \mathbb{R}^p\), we denote
The set $J^k(N, P)$ is the set of polynomial mappings $f : \mathbb{R}^n \to \mathbb{R}^p$ of degree $\leq k$, such that $f(0) = 0$.

The set $J^k(N, P) = \bigcup_{x \in N, y \in P} J^k(N, P)_{x,y}$ is the $k$-jet space of mappings from $N$ to $P$. The set $J^k(N, P)$ is a smooth manifold (Theorem 2.7 in [40]). Moreover, it has the structure of a fibre bundle with basis $N \times P$.

Let $U$ be an open set in $J^k(N, P)$ and

$$M(U) = \{f \in C^{\infty}(N, P) \mid j^k f(N) \subset U\}.$$ 

The family of sets $\{M(U)\}$ where $U$ is an open set of $J^k(N, P)$ is a basis for a topology in $C^{\infty}(N, P)$ (note that $M(U) \cap M(V) = M(U \cap V)$). This topology is called the Whitney $C^k$-topology.

Denote by $W_k$ the set of open subsets of $C^{\infty}(N, P)$ in the Whitney $C^k$-topology. The Whitney $C^\infty$-topology is the topology whose basis is $W = \bigcup_{k=0}^\infty W_k$.

Given a metric $d$ on $J^k(N, P)$, compatible with its topology and a nonnegative continuous function $\delta : N \to \mathbb{R}$, we can define a basic neighborhood of $f \in C^{\infty}(N, P)$ as follows

$$B_\delta(f) = \{g \in C^{\infty}(N, P) \mid d(j^k f(x), j^k g(x)) < \delta(x), \forall x \in N\}.$$ 

When $N$ is compact, $f_n$ converges to $f$ in the Whitney $C^k$-topology if and only if $j^k f_n$ converges uniformly to $j^k f$. On noncompact manifolds $f_n$ converges to $f$ in the Whitney $C^k$-topology if and only there exists a compact $K \subset N$, such $j^k f_n$ converges to $j^k f$ uniformly in $K$, and there exists $n_0$ such that $f_n \equiv f$ in $N \setminus K$ for any $n \geq n_0$ (for details see the book by Golubitsky and Guillemin [40]).

Thus we can see that there is a great difference in the Whitney topology depending on whether or not the domain $N$ is a compact manifold.

When $N$ is not compact, the Whitney $C^k$-topology is a very fine topology, with many open sets. As a consequence, dense sets in $C^{\infty}(N, P)$ are very large sets, and theorems characterizing these sets in $C^{\infty}(N, P)$ are strong results.

### 2.1 The work of Hassler Whitney: from 1944 to 1958

The foundations of the theory were Whitney’s work, in which he formulated the problem of classifying singularities that can not be eliminated by small perturbations, and completely succeeded in solving it for maps from $\mathbb{R}^n$ to $\mathbb{R}^p$ with $p \geq 2n - 1$ in Whitney [112] and from $\mathbb{R}^2$ to $\mathbb{R}^2$ in Whitney [114].

The article [114] published in 1955 is a magnificent work dedicated to maps from the plane into the plane. In the introduction to the article, Whitney presents a complete review of the existing results and future perspectives of the theory. We reproduce it here: “Let $f_0$ be a mapping of an open set $R$ in $n$-space $E^m$ into $m$-space $E^n$. Let us consider, along with $f_0$, all the mappings $f$ which are sufficiently good approximations to $f_0$. By the Weierstrass Approximation Theorem, there are
such mappings $f$ which are analytic; in fact, (see [110, Lemma1]) we may make $f$ approximate to $f_0$ throughout $R$ arbitrarily well, and if $f_0$ is $r$-smooth, (i.e., has continuous partial derivatives of order $\leq r$), we may make corresponding derivatives of $f$ approximate those of $f_0$.

Supposing $f$ is smooth, (i.e., $1$-smooth), the Jacobian matrix $J(f)$ of $f$ is defined (using fixed coordinate systems); we say the point $p \in R$ is a regular point or singular point of $f$, according as $J(f)$ is of maximal rank (i.e., of rank $\min(n,m)$) or lesser rank. In general we cannot expect $f$ to be free of singular points. A fundamental problem is to determine what sort of singularities any good approximations $f$ to $f_0$ must have; what sort of sets they occupy, what $f$ is like near such points, what topological properties hold with references to them, etc.

Some special cases of this problem have been studied as follows:

a) For $m = 1$, we have a real valued function in $R$. It was shown by M. Morse in Theorem 1.6 of [73], that $f$ may be chosen so that the singular points (called critical points here) are isolated, the “Hessian” being non-zero at each.” “Moreover, each critical point may be assigned a “type number”; topological relations among these were given by Morse [72].

b) If $m \geq 2n$, we may find an $f$ with no singular points; see (a) and (b) of Theorem 2 in [110].

c) If $m = 2n - 1$, we may obtain an $f$ with singular points: see [112]. For each such point $p \in R$, coordinate systems $(x_1, x_2, \ldots, x_n)$ in $E^n$ and $(u_1, u_2, \ldots, u_m)$ in $E^m$ may be chosen, in which $f$, near $p$, has the form

$$u_1 = x_1^2, \quad u_i = x_i, \quad u_{n+i-1} = x_1 x_i, \quad (i = 2, \ldots, n).$$

The singularities are studied from a topological point of view in [113].

d) Some beginnings have been made for the other pairs of values $(n,m)$ by N. Wolfsohn, [120], but no complete classification of the singularities exist in these cases. Thus the smallest pair of values for which the problem is open is the pair $(2,2)$, i.e for mappings of the plane into the plane; it is this case that we treat here. In this case, there can be “folds” lying along curves and isolated “cusps” on the folds (Figure 1).

We review Whitney’s results in this section.

Let $f : U \to \mathbb{R}^2$ be a smooth mapping defined on the open set $U \subset \mathbb{R}^2$. With coordinates systems $(x, y)$ in $U$ and $(u, v)$ in the target, the Jacobian of $f$ is given by

$$J(f) = u_x v_y - u_y v_x.$$
(ii) $(f \circ \phi)'(0) = 0$ and $(f \circ \phi)''(0) \neq 0$, we say $p$ is a cusp point of $f$. These definitions are independent of the parametrization chosen for $S(f)$ in a neighborhood of $p$.

One can easily see that at a fold point, the restriction of $f$ to its singular set is non singular, while a cusp point is a singular point of this restriction.

It follows from the definition that cusp points are isolated.

**Definition 2.4 (Whitney [114], p. 379)** Let $f$ be a good map. We say that $p$ is an excellent point of $f$ if it is a regular, fold or cusp point of $f$. If each point $p \in U$ is excellent we say $f$ is excellent.

Any smooth map can be approximated in the $C^r$-Whitney topology, $r \geq 3$, by an excellent map.

**Theorem 2.5 (Whitney [114], Theorem 13A)** Let $f_0$ be a mapping from $U \subset \mathbb{R}^2$ to $\mathbb{R}^2$, where $U$ is an open set in $\mathbb{R}^2$. Then arbitrarily near $f_0$ there is an excellent mapping $f$. If $f_0$ is $r$-smooth and $\varepsilon$ is a positive continuous function in $U$, we make $f$ an $(r, \varepsilon)$-approximation of $f$.

Prior to Thom’s transversality theorem ([92]), Whitney introduced the method of characterizing in the jet space the set of jets with degenerate singularities, the so called “bad set”.

In addition, methods of producing generic $C^r$-perturbations of any given map were also introduced by him. The goal was to find sufficiently close perturbations that would avoid the bad set.

For polynomial maps from the plane into plane, the bad set are the polynomial maps admitting singularities more degenerate than folds and cusps.

Folds and cusps have simple normal forms.

**Theorem 2.6 (Whitney [114], Theorems 15A and 15B)**

1. Let $p$ be a fold point of the $r$-smooth mapping $f$ of $\mathbb{R}^2$ into $\mathbb{R}^2$, with $r \geq 3$. Then $(r - 3)$-smooth coordinate systems $(x, y)$, $(u, v)$ may be introduced about $p$ and $f(p)$ respectively, in terms of which $f$ takes the form

$$u = x^2, \quad v = y$$

(1)

2. Let $p$ be a cusp point of the $r$-smooth mapping $f$ of $\mathbb{R}^2$ into $\mathbb{R}^2$, with $r \geq 12$. Then $(\frac{r}{2} - 5)$-smooth coordinate systems $(x, y)$, $(u, v)$ may be introduced about $p$ and $f(p)$ respectively, in terms of which $f$ takes the form

$$u = xy - x^3, \quad v = y$$

(2)

While the proof of (1) is not hard, Whitney’s proof of the normal form in a neighborhood of a cusp point $p$ follows by an ingenious sequence of changes of coordinates in the source and target. The tool is essentially the implicit function theorem.
Today, there are simpler proofs of this result, based on current tools of singularity theory: see for instance, Theorem 2.4, Chapter VI in Golubitsky and Guillemin’s book [40] or Example 3.6 in Mond and Ballesteros [69].

The notion of stable mappings is due to Whitney. In order to characterize them, in addition to the local behavior of stable singularities, it is necessary to explain the behavior of multiple points. For maps from the plane into the plane the following holds.

**Theorem 2.7** Let \( f : N^2 \rightarrow P^2 \) be a smooth map, \( N \) and \( P \) 2-dimensional manifolds, \( N \) compact. Then \( f \) is \( C^\infty \)-stable if and only if the following conditions hold.

1. \( f \) is excellent and hence \( S(f) \) is a regular curve, with at most a finite number of cusp points.
2. If \( p_1 \) and \( p_2 \) are singular points of \( f \), \( f(p_1) = f(p_2) \), then \( p_1 \) and \( p_2 \) are not cusp points. Moreover the fold lines intersect transversally at \( f(p_1) = f(p_2) \).
3. The restriction of \( f \) to \( S(f) \) has no triple points.

Whitney formulated in [115] a general approach to defining a stratification in jet space and to define locally generic mappings as those whose \( r \)-jets were transversal to the strata of the stratification, for every \( r \in \mathbb{N}^* \). The article contains an explicit description of generic singularities for pairs \((n, p)\) such that \( n, p \leq 5 \).

He asked the question whether for any pair of dimensions \((n, p)\), the stable maps could be characterized by transversality to a finite collection of submanifolds in jet space, so that one could apply Thom’s transversality theorem to prove that a smooth map could be always approximated by stable maps.

However, in a course taught at the University of Bonn in 1959, René Thom showed with an example that it is not always possible to approximate a given map by \( C^\infty \)
stable mappings (See section 6, on Thom’s example). In fact, in the notes Singularity of differentiable mappings I, written by Harold Levine [96], Thom sketched the proof that $C^2$-stable mappings do not form an open set in $C^\infty(N, P)$, when $n = p = 9$ and he formulated conjectures that promoted a great development in the theory in the following decades. In particular, Thom conjectured the density of topologically stable mappings, proved by John Mather in 1971. We discuss René Thom and John Mather’s contributions in the next section.

2.2 René Thom and John Mather: from 1958 to 1970

We start by reviewing the subjects covered by R. Thom in his course at the University of Bonn. H. Levine’s notes are divided into three chapters.

Chapter I, named “Jets” introduces the notion of jet spaces, the action of the group $A$ in jet space and $A^r$-invariant manifolds, denominated, in the notes, critical varieties in $J^r(n, p)$. The set $S_k$ of 1–jets of corank $k$ and its topological closure $S_k$ in $J^1(n, p)$ were defined.

In Chapter II, entitled “Singularities of mappings”, Thom’s transversality theorem was stated and proved. We remark however that the topology in the space of mappings in Thom’s proof was the weakest topology making the mapping

$$j^r : C^\infty(N, P) \to C^\infty(N, J^r(N, P))$$

$$f \mapsto j^r f$$

continuous. The topology in the second space was the compact open topology. The transversality theorem in [96] was stated as follows: For $s > r \geq 0$, let $W$ be a codimension $q$, $C^{s-r}$ submanifold of $J^r(N, P)$, $s - r > \dim N - q$. Then the set of mappings $f \in C^\infty(N, P)$, such that $j^r f \pitchfork W$ is dense in $C^\infty(N, P)$. The notion of second order singularities $S_{h,k}$ in $J^2(n, p)$ was introduced. These sets are connected to the singular points $S_h \subset J^1(n, p)$ by the relation: if $j^1 f \pitchfork S_h$, then $(j^2 f)^{-1}(S_{h,k}) = S_h(S_k(f))$. The general definition of the singular varieties $S_{h_1,\ldots,h_k} \subset J^r(N, P)$, introduced in [96] was better formulated by J.M. Boardman, in 1967, in [11]. Mather’s account in [64] is the clearest.

**Remark 2.8** In the following sections the sets $S_k$ and $S_{k,h}$ will be denoted by $\Sigma^k$ and $\Sigma^{k,h}$, respectively.

In Chapter III, “Equivalence and stability”, Thom formulated the problem of characterizing singularities determined by their jet of some order. The name finitely determined germs, was later given by John Mather [58], who also gave necessary and sufficient conditions for finite determinacy. The notion of $C^s$-stable mappings and the example illustrating that $C^2$ stable mappings are not dense when $n = p = 9$ were discussed in that chapter.

The notion of homotopic stability was also introduced. A mapping $f : N \to P$ is homotopically stable if for every homotopy $F : N \times I \to P$ of $f$, there exist $t_0$ and
homotopies of diffeomorphisms \( \phi_t : N \to N \), 0 \( \leq t \leq t_0 \), \( \psi_t : P \to P \), of \( 1_N \) and \( 1_P \) such that \( F_t = \psi_t \circ f \circ \phi_t \), \( t < t_0 \).

The program for the theory of stable mappings originated from the contributions of Whitney and Thom consisted of finding pairs of dimensions \( (n, p) \), for which there exists a set of mappings \( S \subset C^\infty(N^n, P^p) \), with the following properties:

1. \( S \) is a residual set in \( C^\infty(N^n, P^p) \),
2. The maps \( f \in S \) are \( C^\infty \)-stable,
3. There exists a finite number of polynomial normal forms such that every singular point of \( f \in S \) is equivalent to a normal form in this list.

In a memorable series of six articles from 1968 to 1971, John Mather found several characterizations of stability and provided theorems answering almost completely the question of density of stable maps.

The main results on density of stable mappings are stated below. The proofs are based on ideas of René Thom developed by Mather in the sequence of papers, on Stability of \( C^\infty \)-mappings, I to VI, [57, 60, 58, 61, 62, 63] and [56, 65, 66]. In these notes we review the main steps leading to the proofs of Theorems A and B.

Let \( C^\infty_{pr}(N, P) \) be the set of proper smooth mappings \( f : N \to P \).

**Theorem A (Density of stable mappings in the nice dimensions, Mather [63, 61])**

The set \( S^{\infty}(N, P) \) of proper stable mappings \( f : N \to P \) is dense in \( C^\infty_{pr}(N, P) \) if and only if \( (n, p) \) is in the nice dimensions.

See section 5 for the definition of the nice dimensions.

**Theorem B (Density of topologically stable mappings, Mather [65, 66])**

The set \( S^{0}(N, P) \) of proper topologically stable mappings is dense in \( C^\infty_{pr}(N, P) \).

The main tools in the proofs of theorems A and B are the notion of infinitesimal stability, Thom’s transversality theorem, the generalized Malgrange theorem, the notions of mappings of finite singularity type and contact equivalence, finite determinacy and unfoldings of Mather’s groups, properties of Whitney stratified sets and Thom’s isotopy theorems. Such notions and results form the framework of the theory of singularities of differentiable mappings.

We organize the contents of the next sections as follows.

In section 3 we introduce infinitesimally stable and transverse stable mappings. The main goal of the section is to discuss theorem 3.11 which establishes the equivalence between these notions and stable mappings.

Section 4 gives a short presentation of the infinitesimal machinery of singularity theory. We introduce the contact group \( \mathcal{K} \) defined by Mather as a tool to classify stable singularities. For Mather’s groups \( \mathcal{G} = \mathcal{R}, \mathcal{L}, \mathcal{A}, \mathcal{C} \) and \( \mathcal{K} \) we define \( \mathcal{G} \)-finitely determined germs and prove the Infinitesimal Criterion for \( \mathcal{G} \)-determinacy. We finish the section with a discussion of maps of finite singularity type (FST), a global version of \( \mathcal{K} \)-finitely determined germs, which plays a central role in the proof of theorem B.

In section 5 we define the nice dimensions and give an outline of the proof of theorem A.
Section 6 gives a detailed presentation of Thom’s example, illustrating that the set of stable maps in $C^\infty_p(\mathbb{R}^9, \mathbb{R}^9)$ is not dense.

Section 7 is dedicated to the proof of density of topologically stable mappings $f : N \to P$, when $N$ is compact manifold. The general lines of the proof are discussed, although the details are omitted.

Section 8 gives a systematic presentation of the topologically stable singularities in the boundary of the nice dimensions. Much of the section is well known to experts, however the organized presentation of the Thom-Mather stratification in jet space and the discussion of properties of topologically stable mappings in these dimensions do not appear in the literature.

The question of the density of Lipschitz stable mappings is still open. We report on section 9 some recent results of Ruas and Trivedi [88] and Nguyen, Ruas and Trivedi [75] on this subject.

In section 10, Damon’s results relating $\mathcal{A}$-classification of map-germs and $\mathcal{K}_V$ classification of sections of the discriminant $V = \Delta(F)$ of a stable unfolding of $f$ are reviewed and open problems are discussed.

3 Equivalent notions of stability

Mather defined infinitesimally stable mappings in [57], in order to introduce infinitesimal deformations of a map as a tool to study stability. The main goal in this section is to review Mather’s result that, for proper mappings, stability and infinitesimal stability are equivalent notions.

First, we introduce some notation. Let $C^\infty(N) = \{\lambda : N \to \mathbb{R}\}$ be the ring of smooth functions defined on the smooth manifold $N$.

We denote by $\Theta_f$ the $C^\infty(N)$-module of vector fields along $f$, defined as follows

$$\Theta_f = \{\sigma : N \to TP \mid \pi_2 \circ \sigma = f\}$$

where $\pi_2 : TP \to P$ is the projection of the tangent bundle $TP$ into $P$.

Let $f^*(TP)$ denote the pull-back bundle over $N$ via $f$. Then the module $\Theta_f$ is the set of sections of this bundle.

Similarly,

$$\Theta_N = \{\xi : N \to TN \mid \pi_1 \circ \xi = I_N\}$$

is the set of sections of the tangent bundle of $N$, and

$$\Theta_P = \{\eta : P \to TP \mid \pi_2 \circ \eta = I_P\},$$

the set of sections of the tangent bundle of $P$, where $I_N$ and $I_P$ are the identities.

The set $\Theta_N$ is a $C^\infty(N)$-module, while $\Theta_P$ is a module over the ring $C^\infty(P)$.

We have the following diagram and homomorphisms
The map $t_f$ is a homomorphism of $C_1^1$-modules. The map $f : N \to P$ induces a ring homomorphism $f : C_1^1 P \to C_1^1 N$.

**Definition 3.1** The map $f : N \to P$ is infinitesimally stable if for any $\sigma \in \Theta_f$, there are sections $\xi \in \Theta_N$ and $\eta \in \Theta_P$ such that $\sigma = t_f(\xi) + \eta$. Equivalently, we can say that $\Theta_f = t_f(\Theta_N) + \omega f(\Theta_P)$.

**Example 3.2** If $N$ is compact, $1 - 1$ immersions and submersions $f : N \to P$ are infinitesimally stable.

Infinitesimal stability has a local counterpart that we define now. Recall that two maps $f, g : N^n \to P^p$ define the same germ at $x = a$ if they agree in some neighborhood of $a$. The point $x = a$ is the source of the germ and $b = f(a)$ is its target. The analogues of the above notations for a germ $f : (N, a) \to (P, b)$ can be obtained replacing $N$ by $(N, a)$ and $P$ by $(P, b)$ in the previous notation. However to simplify notation, we take local coordinates such that $a = 0 \in \mathbb{R}^n$ and $f(a) = 0 \in \mathbb{R}^p$, denoting the germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$. In this case, we use the usual notation:

$E_n = \{ \lambda : (\mathbb{R}^n, 0) \to \mathbb{R} \}$ is the local ring of $C_\infty$ function germs at the origin. Its unique maximal ideal is $M_n = \{ \lambda \in E_n | \lambda(0) = 0 \}$.

$E_n^p = \{ f : (\mathbb{R}^n, 0) \to \mathbb{R}^p \}$ is a free $E_n$-module of rank $p$, also denoted by $E_{n,p}$.

The local version of the previous diagram is

$$
\begin{array}{ccc}
TN & \xrightarrow{df} & TP \\
\pi_1 & \sigma & \pi_2 \\
N & f & P
\end{array}
$$

$t_f : \Theta_N \to \Theta_f$

$\xi \mapsto t_f(\xi)$

where $t_f(\xi)(x) = df_x(\xi(x))$.

$\omega f : \Theta_P \to \Theta_f$

$\eta \mapsto \omega f(\eta) = \eta \circ f$

The map $t_f$ is a homomorphism of $C_\infty(N)$-modules. The map $f : N \to P$ induces a ring homomorphism $f^* : C_\infty(P) \to C_\infty(N)$.

$\phi \mapsto f^*(\phi) = \phi \circ f$.
The set
\[ \Theta_f = \{ \sigma : (\mathbb{R}^n, 0) \to (T\mathbb{R}^p, 0) | \, \pi_2 \circ \sigma = f \} \]
is the $E_n$-module of rank $p$ consisting of germs of vector fields along $f$. When $f$ is
the identity in $\mathbb{R}^n$, respectively in $\mathbb{R}^p$, we obtain
\[ \Theta_n = \{ \xi : (\mathbb{R}^n, 0) \to (T\mathbb{R}^n, 0) | \, \pi_1 \circ \xi = \text{id}_{\mathbb{R}^n} \} \]
and
\[ \Theta_p = \{ \eta : (\mathbb{R}^p, 0) \to (T\mathbb{R}^p, 0) | \, \pi_2 \circ \eta = \text{id}_{\mathbb{R}^p} \} \]

We now define the groups acting on $E_n^p$.

**Definition 3.3** Let
\[ \mathcal{R} = \{ h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0), \text{ germs of } C^\omega \text{ diffeomorphisms in } (\mathbb{R}^n, 0) \}, \]
\[ \mathcal{L} = \{ k : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0), \text{ germs of } C^\omega \text{ diffeomorphisms in } (\mathbb{R}^p, 0) \}, \]
and \[ \mathcal{A} = \mathcal{R} \times \mathcal{L}. \]

The actions of the groups $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{A}$ are as follows

\[ \mathcal{R} \times E_n^p \to E_n^p, \quad \mathcal{L} \times E_n^p \to E_n^p, \quad \mathcal{A} \times E_n^p \to E_n^p \]
\[ (h, f) \mapsto f \circ h^{-1}, \quad (k, f) \mapsto k \circ f, \quad ((h, k), f) \mapsto k \circ f \circ h^{-1}. \]

These notions extend to multigerms. Let $S = \{x_1, x_2, \ldots, x_s\}$ be a finite subset of \[ \mathbb{R}^n. \]

**Definition 3.4** A multigerm at $S = \{x_1, \ldots, x_s\}$ is the germ of a smooth map
\[ f = \{f_1, f_2, \ldots, f_s\} : (\mathbb{R}^n, S) \to (\mathbb{R}^p, y), \quad f_i(x_i) = y, \, i = 1, \ldots, s. \]

By a local change of coordinates at each $x_i \in S$, we can take $f_i : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$
and we let $M_S E_n^p$ be the vector space of these map-germs, and call $f_i, i = 1, \ldots, s$
a branch of $f$.

The previous notations for monogerms extend naturally to multigerms. As before
$\Theta_f$ and $\Theta_{n,S}$ are $E_{n,S}$-modules. The map $tf : \Theta_{n,S} \to \Theta_f$ is an $E_{n,S}$-module
homomorphism defined by $tf(\xi)(x) = df_x(\xi(x))$.

The map-germ $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ induces the ring homomorphism
$f^* : \mathcal{E}_p \to \mathcal{E}_{n,S}$

$\gamma \mapsto f^*(\gamma) = \gamma \circ f,$

and we say that the map

$\omega f : \Theta_p \to \Theta_f.$

$\eta \mapsto \omega f(\eta) = \eta \circ f$

is a homomorphism over $f^*(\mathcal{E}_p)$ (or alternatively, an $\mathcal{E}_p$-module homomorphism via $f$).

**Definition 3.5** Two germs $f, g : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ are $\mathcal{A}$-equivalent ($f \sim g$) if there exist $h : (\mathbb{R}^n, S) \to (\mathbb{R}^n, S)$ and $k : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$ such that $g = k \circ f \circ h^{-1}.$

**Definition 3.6** The germ $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ is infinitesimally stable if

$t f(\Theta_{n,S}) + \omega f(\Theta_p) = \Theta_f$

**Remark 3.7** When we refer to an infinitesimally stable multigerm $f : (N, S) \to (P, y),$ we use the notation

$t f(\Theta_{(N,S)}) + \omega f(\Theta_{(P,y)}) = \Theta_f.$

**Definition 3.8** For the groups $\mathcal{G} = \mathcal{R}, \mathcal{L}, \mathcal{A},$ and any multigerm $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0),$ we define the tangent space $T\mathcal{G}_f$ and the extended tangent space $T\mathcal{G}_e f$ as follows:

$TR f = t f(M_n \Theta_{n,S})$  
$TR_e f = t f(\Theta_{n,S})$  
$TL f = \omega f(M_p \Theta_p)$  
$TL_e f = \omega f(\Theta_p)$  
$TA f = t f(M_S \Theta_{n,S}) + \omega f(M_p \Theta_p)$  
$TA_e f = t f(\Theta_{n,S}) + \omega f(\Theta_p)$

One can give a heuristic justification for the definition of the tangent space for the groups $\mathcal{G}$ in the above definition. They can be seen as the set of “tangent vectors” at the origin, to “paths” $f_i,$ such that $f_0 = f,$ and $f_i$ is contained in the $\mathcal{G}$-orbit of $f.$ A careful calculation in the case $\mathcal{G} = \mathcal{A},$ beginning with $f_i = \psi_i \circ f_j \circ \phi_i,$ and differentiating with respect to $t,$ is done on pages 60-61 of the book of Mond and Nuño-Ballesteros [69].

For any group $\mathcal{G}$ acting on $\mathcal{E}_{n,S}$ the $\mathcal{G}$-codimension and the $\mathcal{G}_e$-codimension to the $\mathcal{G}$-orbit of $f,$ are given by

$\mathcal{G}$-cod $f = \dim \mathbb{R} \frac{M_S \Theta_f}{T\mathcal{G} f}$  
$\mathcal{G}_e$-cod $f = \dim \mathbb{R} \frac{\Theta_f}{T\mathcal{G}_e f}.$

Note that a map-germ $f \in \mathcal{E}_{n,S}$ is infinitesimally stable if and only if $\mathcal{A}_e$-cod $f = 0.$

**Definition 3.9** A mapping $f : N \to P$ is locally infinitesimally stable at $S = \{x_1, \ldots, x_s\} \subset N$ if the germ of $f$ at $S$ is infinitesimally stable.
The next theorem shows that for proper mappings infinitesimal stability is locally a condition of finite order. That is, if the equations can be solved locally to order $p = \dim P$, then they can be solved globally.

**Theorem 3.10 (Theorem 1.6, Chapter 5, [40])** Let $f : N \to P$ be a smooth and proper $C^\infty$ mapping. Then $f$ is infinitesimally stable if and only if for every $y \in P$ and every finite set $S \subset f^{-1}(y)$, with no more than $(p + 1)$ points, we have

$$
\Theta_f = tf(\Theta_{(N,S)}) + \omega f(\Theta_{(P,y)}) + \mathcal{M}_S^{p+1}\Theta_f .
$$

The proof of the necessity in theorem 3.10 is obvious. To prove the sufficiency, the main tool is the generalized Malgrange Preparation Theorem proved by Mather in [57]. See Proposition 4.21 and Corollary 4.23. A complete proof of this theorem is given in Chapter 5, section 1 of [40].

Our main goal in this section is to discuss the following theorem.

**Theorem 3.11 (Mather [62], Theorem 4.1)** The following conditions are equivalent in $C^m_{pr}(N, P)$ for a proper mapping $f : N \to P$.

1. $f$ is stable,
2. $f$ is infinitesimally stable.
3. $f$ is transverse stable.

We present the main steps of the proof of Theorem 3.11. Initially we discuss the notion of transverse stability.

### 3.1 Transverse stability and the proof of 2. $\Leftrightarrow$ 3.

The idea of transverse stability consists in defining a stratification in jet space, such that the strata of this stratification are invariant by the action of the group $\mathcal{A}$ in jet space. A map is transverse stable if its $k$-jet is transversal to this stratification. To make this notion more precise, we introduce the $r$-fold $k$-jet bundle, following Mather [62].

Let $N$ and $P$ be manifolds. Let $N^{(r)} = \{(x_1, x_2, \ldots, x_r) \in N^r \mid x_i \neq x_j$ if $i \neq j\}$. Let $\pi_N : J^k(N, P) \to N$ denote the projection where $J^k(N, P)$ is the bundle of $k$-jets. We define $rJ^k(N, P) = (\pi_N^r)^{-1}(N^{(r)})$ where $\pi_N^r : J^k(N, P)^r \to N^r$ is the projection.

It follows that

$$rJ^k(N, P) = \{(z_1, \ldots, z_r) \in J^k(N, P)^r, \text{ such that } \pi_N(z_i) \neq \pi_N(z_j), \text{ if } i \neq j\}.$$

The set $rJ^k(N, P)$ is a fibre bundle over $N^{(r)} \times P^r$, and we call it the $r$ fold $k$-jet bundle of mappings of $N$ into $P$.

If $f : N \to P$ is a $C^\infty$ mapping, we define
Theorem 3.15 (Mather [60], Theorem 2) then there exists a neighborhood
then it is stable, that is
now, transversality is preserved by
An important remark is that in order to understand the local structure of the orbits in 
ín the proof of theorem 3.11. The next proposition gives a characterization of transversality of 
transverse to every submanifold.
Proposition 3.12 (Mather [62], Proposition 1.4) An orbit W in 
is a submanifold.
\[ \text{Definition 3.13 } f : N \to P \text{ is transverse stable if } r^k f : N^{(r)} \to r^k (N, P) \text{ is transverse to every } \mathcal{A}^k \text{ orbit } W \text{ in } r^k (N, P). \]
An important remark is that in order to understand the local structure of the orbits in 
ín it is sufficient to understand the structure of the orbits in \( \pi'_p (\Lambda_r) \), where \( \Lambda_r \subset P^r \) is the diagonal (see Mather [62] for details). In other words, it suffices to take jets with sources \( S = \{x_1, \ldots, x_r\} \) for which \( f(x_1) = \cdots = f(x_r) \).
\[ \text{The next proposition gives a characterization of transversality of } r^k f \text{ to } W; \text{ it is an important step in the proof of theorem 3.11.} \]
\[ \text{Proposition 3.14 (Mather [62], Proposition 2.6) } r^k f \text{ is transverse to } W \text{ at } x \text{ if and only if, } \]
\[ tf(\Theta_{(N,S)}) + \omega f(\Theta_{(P,y)}) + M_{S}^{k+1} \Theta_{f} = \Theta_{f}, \]
where \( y = f(x), S = f^{-1}(y) = \{x_1, \ldots, x_r\} \).
\[ \text{From proposition 3.14 and theorem 3.10 we obtain the proof of } 2. \iff 3. \text{ in theorem 3.11.} \]
That 1. implies 3. in Theorem 3.11 follows from a general fact, and it is not hard to show.
In fact, let \( f : N \to P \) be a stable mapping. It follows from the transversality theorem that \( f \) can be well approximated by a mapping \( g : N \to P \), such that \( g \) is transverse stable as \( g \sim f \). That is, there is \( (h, k) \in \mathcal{A} \) such that \( g = k \circ f \circ h^{-1} \).
Now, transversality is preserved by \( \mathcal{A} \)-equivalence, hence \( f \) is transverse stable as well, as we wanted to show.
We have proved 1. \( \Rightarrow 2. \iff 3. \).
Mather proved in [60], Theorem 1 that if \( f \) is proper and infinitesimally stable then it is stable, that is 2. \( \Rightarrow 1. \).
His proof follows from the following result.
\[ \text{Theorem 3.15 (Mather [60], Theorem 2) If } f \text{ is proper and infinitesimally stable, then there exists a neighborhood } U \text{ of } f \text{ in } C^\infty (N, P) \text{ and continuous mappings } H_1 : U \to \text{Diff}^\infty (N) \text{ and } H_2 : U \to \text{Diff}^\infty (P) \text{ such that } H_1(f) = 1_N, H_2(f) = 1_P \text{ and } g = H_2(g) \circ f \circ H_1(g), \text{ for } g \in U. \]
Du Plessis and Wall [82] introduced the notion of $W$-strongly stable mappings as stable mappings $F: N \to P$ admitting a neighborhood $U$ in $C^\infty(N, P)$ satisfying the conditions stated in theorem 3.15.

The main difficult to prove that stable mappings are $W$-strongly stable is that in the Whitney $C^1$ topology, the composition of mappings is not continuous. However continuity holds when one restricts to proper mappings. The strong stability of non proper functions was recently discussed by Kenta Hayano in [42].

It follows that the result $2. \Rightarrow 1.$ is an easy consequence of theorem 3.15.

The hypothesis that $f$ is proper cannot be omitted, as we see in the following example.

**Example 3.16 ([60], pp. 267)** Let $N = (-1, 1) \cup (1, 2), P = (-1, 1)$, and

$$ f|_{(-1,1)} : (-1, 1) \to (-1, 1), \quad f|_{(1,2)} : (1, 2) \to (-1, 1) $$

$$ x \mapsto x^2, \quad x \mapsto 2 - x $$

We can verify that $f$ is infinitesimally stable, as the restrictions to $(-1, 1)$ and $(1, 2)$ are.

However, $f$ is not stable since it has the following non-stable property: for any $a \in P, f^{-1}(a)$ contains either 0, 1 or 3 points.

The reader can find in [62] the discussion of which implications in theorem 3.11 depend on the hypothesis that $f$ is proper.

In the next example we illustrate the role of the Whitney $C^\infty$-topology in the characterization of stable mappings.

**Example 3.17** The cusp map

$$ F : \mathbb{R}^2 \to \mathbb{R}^2 $$

$$ (x, y) \mapsto F(x, y) = (x, y^3 + xy) $$

is a stable mapping when the topology in $C^\infty_{pr}(\mathbb{R}^2, \mathbb{R}^2)$ is the Whitney topology. This follows from Whitney’s theorem as we discussed in section 2.1. We can also apply Mather’s result: the map $F$ is proper and infinitesimally stable, hence it is stable.

Let $F_n(x, y) = (x, y^3 + xy + \frac{x^5}{n}y).$ The singular set of $F_n$ is the set $\Sigma_n$ defined by $3y^2 + x + \frac{x^3}{n} = 0.$ For each $n, F_n$ has two cusp points: $(0, 0)$ and $(-n, 0)$.

We can easily see that $F_n \to F$ in $C^\infty_{pr}(\mathbb{R}^2, \mathbb{R}^2)$ with the topology of uniform convergence on compact sets. Hence $F$ is not stable when one considers this topology in $C^\infty_{pr}(\mathbb{R}^2, \mathbb{R}^2)$.

### 3.2 Notes

The definitions and properties of infinitesimally stable mappings also hold for real and complex analytic germs. However, care is necessary to characterize stable maps.
when $f$ is a holomorphic map between complex manifolds $N$ and $P$. In fact, Thom’s transversality theorem does not hold in general in this case. See discussion by F. Forstnerič, [34] and examples given by S. Kaliman and M. Zaïdenberg in [46]. In a recent paper, S. Trivedi [99] proves that the set of maps between Stein manifolds and Oka manifolds, transverse to a countable collection of submanifolds in the target is dense in the space of holomorphic maps with the weak topology. The results hold, in particular, for holomorphic maps $f : \mathbb{C}^n \to \mathbb{C}^p$, as the complex spaces satisfy the hypothesis of the theorem.

A related problem is the characterization of topologically stable polynomial mappings $f : \mathbb{C}^n \to \mathbb{C}^p$. M. Farnick, Z. Jelonek and M.A. S. Ruas [32], characterize topologically stable polynomial mappings $F : \mathbb{C}^2 \to \mathbb{C}^2$ in the space $\Omega_{C^2}(d_1, d_2)$ of polynomial mappings of degree bounded by $(d_1, d_2)$. Locally stable singularities are folds and cusps, but the behavior of generic polynomial mappings at infinity imposes new restrictions. The number of cusps of a topologically stable $F \in \Omega_{C^2}(d_1, d_2)$ is given by $c(F) = d_1^2 + d_2^2 + 3d_1d_2 - 6d_1 - 6d_2 + 7$. In particular, when $d_1 = 1$ and $d_2 = 3$, $c(F) = 2$.

### 4 Finite determinacy of Mather’s groups

Mather’s groups are the groups $G = \mathcal{R}, \mathcal{L}, \mathcal{A}, \mathcal{K}$ and $C$.

The contact group $\mathcal{K}$, defined by Mather in [58] plays a fundamental role in the classification of stable singularities. In subsections 4.1 and 4.3 we define the group $\mathcal{K}$, discuss properties of $\mathcal{K}$-equivalence and their role in the study of stable mappings.

The problem of classification of stable singularities motivated the introduction of the notion of $G$-finitely determined germs [58]. For the groups $G = \mathcal{R}$ or $\mathcal{K}$, finite determinacy was studied by J. Tougeron in [97] and chapter II of [98]. When $G = \mathcal{A}$ or $\mathcal{L}$, the first results are due to Mather’s in [58]. Infinitesimal criteria of finite determinacy for $G = \mathcal{A}$ and $\mathcal{L}$ depend on the Preparation Theorem. We discuss the infinitesimal criterion for Mather’s group in section 4.2. In section 4.4 we introduce the basic properties of maps of finite singularity type.

#### 4.1 The contact group

**Definition 4.1** The contact group $\mathcal{K}$ is the set of pairs of germs of diffeomorphisms $(h, H)$, where $h : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$, $H : (\mathbb{R}^n \times \mathbb{R}^p, 0) \to (\mathbb{R}^n \times \mathbb{R}^p, 0)$ such that $\pi_1 \circ H = h$, $\pi_2 \circ H)(x, 0) = 0$ where $\pi_1$ and $\pi_2$ are the projections into $\mathbb{R}^n$ and $\mathbb{R}^p$, respectively.

Notice that $H(x, y) = (h(x), H_2(x, y))$, $H_2(x, 0) = 0$.

The set of pairs $(h, H) \in \mathcal{K}$, such that $h$ is the identity $I_{\mathbb{R}^n}$, form a subgroup of $\mathcal{K}$, usually denoted by $C$. 
**Definition 4.2** Let \( f, g \in \mathcal{E}^p_n \). We say that \( f \) and \( g \) are contact equivalent, \( f \sim_K g \), if there is a pair \((h, H) \in \mathcal{K}\) such that \( H(x, f(x)) = (h(x), g(h(x))) \).

**Remark 4.3** Notice that if \( f \sim_K g \), then the diffeomorphism \( H : (\mathbb{R}^n \times \mathbb{R}^p, 0) \to (\mathbb{R}^n \times \mathbb{R}^p, 0) \) sends \( \text{graph}(f) \) into \( \text{graph}(g) \), leaving \( \mathbb{R}^n \times \{0\} \) invariant (see Figure 2). This geometric viewpoint of contact equivalence was extended by Montaldi [71] as follows: two pairs of germs of submanifolds of \( \mathbb{R}^m \) have the same contact type if there is a germ of diffeomorphism of \( \mathbb{R}^m \) taking one pair to the other. Moreover, he proved in [71], that the contact type of a pair of germs of manifolds is completely characterized by the \( \mathcal{K} \)-equivalence class of a convenient map. This result is one the fundamental pieces of the applications of singularity theory to differential geometry (see Bruce and Giblin [13] and Izumiya, Romero-Fuster, Ruas and Tari, [45]).

The tangent space and the extended tangent space of \( \mathcal{K} \)-equivalence are, respectively

\[
T\mathcal{K}f = tf(M_n \Theta_n) + f^*(M_p) \Theta f \\
T\mathcal{K}_e f = tf(\Theta_n) + f^*(M_p) \Theta f
\]

We also define \( \mathcal{K} \)-cod \( f = \dim_{\mathbb{R}} \frac{M_n \Theta f}{T\mathcal{K}f} \) and \( \mathcal{K}_e \)-cod \( f = \dim_{\mathbb{R}} \frac{\Theta f}{T\mathcal{K}_e f} \).

![Fig. 2 Contact equivalence](image-url)

The following result was first proved by Mather in [61].

**Proposition 4.4** (Gibson [38], Proposition 2.2, Mond and Nuño-Ballesteros [69], Section 4.4)

The following statements are equivalent.

1. Two map-germs \( f, g \in \mathcal{E}^p_n \) are \( \mathcal{K} \)-equivalent.
(2) There exists a germ of diffeomorphism $h : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^n, 0)$ such that

$$h^* f(M_p)E_n = g^*(M_p)E_n.$$ 

The local algebra we introduce now is an useful invariant of $\mathcal{K}$-equivalence. For a given map-germ $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ we define the local algebra of $f$ as

$$Q(f) = \frac{E_n}{f^*(M_p)E_n}.$$ 

It follows from the previous proposition that the isomorphism class of $Q(f)$ is a $\mathcal{K}$-invariant. Furthermore, it is a complete invariant of $\mathcal{K}$-equivalence for germs $f$ with finite $\mathcal{K}$-codimension. More precisely, we have

**Theorem 4.5** If $f$ and $g$ are map-germs with finite $\mathcal{K}$-codimension it follows that

$f \sim \mathcal{K} g$ if and only if the local algebras $Q(f)$ and $Q(g)$ are isomorphic.

**Remark 4.6** For complex analytic germs the hypothesis of $\mathcal{K}$-determinacy in Theorem 4.5 is not needed.

**Example 4.7** Let $F : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ be a germ of rank $r$. Then, up to $\mathcal{A}$-equivalence, we can take $F$ in the normal form $F(x, y) = (x, f(x, y)), x \in \mathbb{R}^r, y \in \mathbb{R}^{n-r}$, with $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^{p-r}, 0)$ and $j^1 f(0, 0) \equiv 0$. Let $f_0 : (\mathbb{R}^{n-r}, 0) \rightarrow (\mathbb{R}^{p-r}, 0)$ be the rank zero germ $f_0(y) = f(0, y)$. Then $Q(F) = Q(f_0)$.

If $\mathcal{K}$-cod $f_0 < \infty$ and $Q(F) \cong Q(f_0)$ it follows that $F$ is $\mathcal{K}$-equivalent to the suspension $F_0(x, y) = (x, f_0(y))$ of $f_0$.

As we shall see in the next section, germs $f \in E_n^p$ of finite $\mathcal{K}$-codimension are finitely $\mathcal{K}$-determined, and in this case $\mathcal{K}(f) = \mathcal{K}(z)$, where $z = j^k f(0)$ for some $k$.

Now, for each positive integer $k$, we set

$$Q_k(f) = \frac{E_n}{f^*(M_p)E_n + M^n_{k+1}}.$$ 

$Q_k(f)$ is the local algebra of $z = j^k f(0)$. We can also write $Q_k(f) = Q(z)$.

It is not hard to show that $z \sim \mathcal{K} z'$ if and only if $Q_k(z)$ and $Q_k(z')$ are isomorphic.

This definition can be extended to $k$-jets of a multigerm $f : (\mathbb{R}^n, S) \rightarrow (\mathbb{R}^p, 0)$ $S = \{x_1, x_2, \ldots, x_s\}$. By a contact class in $J^k(N, P)$ we mean an equivalence class of $J^k(N, P)$ under the relation of $\mathcal{K}^k$-equivalence.

### 4.2 Finitely determined germs

Let $G$ be a group acting in the space of germs $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$. We say that $f$ is finitely $G$-determined if there exists a positive integer $k$ such that for all
Results on density of stable mappings

$g : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ with $j^k g(0) = j^k f(0)$, it follows that $f \sim g$. We say that $f$ is $\mathcal{G}$-\textit{finitely determined} if $f$ is $k$-determined for some $k$. The denomination $\mathcal{G}$-\textit{finite germs} is also widely used.

Finite determinacy has been an important subject in singularity theory for many decades and the bibliography in this topic is extensive.

With regard to results on necessary and sufficient conditions of finite determinacy and estimates of the order of determinacy we refer to Mather [58], Gaffney [36, 37], du Plessis [79], Damon [24] and Du Plessis, Bruce and Wall [14]. The survey article by Terry Wall [108] is a complete account of the theory of finite determinacy for Mather’s groups $\mathcal{G} = \mathcal{A}, \mathcal{R}, \mathcal{L}, \mathcal{K}$ and $\mathcal{C}$ until 1981. See also the clear presentation (with examples) in Chapter 6 of the book of Mond and Nuño-Ballesteros [69].

An important advance appeared in [24] in which J. Damon defined the geometric subgroups of $\mathcal{K}$, a large class of subgroups for which the theory of finite determinacy can be formulated as for Mather’s group.

The following theorem, known as \textit{infinitesimal criterion} gives necessary and sufficient conditions for finite determinacy. The original result is due to Mather [58]. We give here an improved version due to Gaffney [37] and du Plessis [79]. The statement and proof of Theorem 4.8 are slight modifications of T. Wall [108, Theorem 1.2]. The reader can also compare the statement for the group $\mathcal{A}$ in section 1.2.3 (Theorem 1.2.12) of the article of Mond and Nuño-Ballesteros in this Handbook [70].

**Theorem 4.8** For each $f \in \mathcal{E}^n_\mathcal{G}$, $\mathcal{G} = \mathcal{R}, \mathcal{L}, \mathcal{A}, \mathcal{C}, \mathcal{K}$ the following conditions are equivalent

1. $f$ is finitely $\mathcal{G}$-determined,
2. for some $r$, $T\mathcal{G} f \supset M_n^r \Theta_f$,
3. $\mathcal{G}$-cod $f < \infty$,
4. $\mathcal{G}_e$-cod $f < \infty$.

More precisely, if we set $\epsilon = 1$ for $\mathcal{G} = \mathcal{R}, \mathcal{C}$ or $\mathcal{K}$ and $\epsilon = 2$ for $\mathcal{G} = \mathcal{L}, \mathcal{A}$,

(i) If $f$ is $k$-\textit{\mathcal{G}}-determined then $T\mathcal{G} f \supset M_n^{k+1} \Theta_f$,
(ii) If $T\mathcal{G} f \supset M_n^{k+1} \Theta_f$, then $f$ is $(ek + 1)$-\textit{\mathcal{G}}-determined.
(iii) If $T\mathcal{G} f + M_n^{ek+2} \Theta_f \supset M_n^{k+1} \Theta_f$, then $T\mathcal{G} f \supset M_n^{k+1} \Theta_f$.

This section is mainly devoted to describe this result. Although the theory applies to multigerms, for simplicity we restrict our discussion to monogerms $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$.

The successful approach to finite determinacy was inspired by the action of a Lie group on finite dimensional manifolds. The following lemma is due to Mather.

**Lemma 4.9 (Mather [61], Lemma 3.1)** Let $G$ be a Lie group, $M$ a $C^\infty$ manifold and $\alpha : G \times M \to M$ a $C^\infty$ action. Let $V$ be a connected $C^\infty$-submanifold of $M$. Then $V$ is contained in an orbit of $\alpha$ if and only if

(a) For all $v \in V$, $T_v G \cdot v \supset T_v V$, and
(b) $\dim T_v (G \cdot v)$ is the same for all $v \in V$.  

Our groups are not Lie groups, and our function spaces are not Banach manifolds. But, the solution to the problem of finding sufficient conditions for a germ \( f \in E^p_n \) to be finitely determined, consists in reducing our infinitesimal approach to jet spaces.

Suppose \( f \) is \( k\)-\( G \)-determined. Then, given \( g \in E^p_n \), \( j^k g(0) = j^k f(0) \), the one-parameter family

\[
\tilde{f} : (\mathbb{R}^n \times \mathbb{R}, 0 \times \mathbb{R}) \to (\mathbb{R}^p \times \mathbb{R}, 0) \\
(x, t) \mapsto \tilde{f}(x, t) = (1 - t)f(x) + tg(x)
\]

has a constant \( k \)-jet \( j^k \tilde{f}(0) = j^k f(0) + tj^k (g - f)(0) = j^k f(0) \).

We will identify \( \tilde{f} \) with a “line” \( L_t \) in \( E^p_n \). Our problem is to show that \( L_t \) is contained in a unique orbit.

A sufficient condition is to find a 1-parameter family \( h_t \) of elements in \( \mathcal{G} \) such that \( h_0 = 1 \in \mathcal{G} \), \( h_t(0) = 0 \), \( h_t \cdot f_t = f \), for any \( t \in \mathbb{R} \). These conditions say that the family \( \tilde{f} \) is \( \mathcal{G} \)-trivial. As in the case of stable singularities, the next step is to search for an infinitesimal condition, giving an equivalent characterization of triviality in terms of vector fields.

This step, in principle, is not hard: the equation \( h_t \cdot f_t = f \) implies that \( \frac{\partial}{\partial t}(h_t \cdot f_t) = 0 \) leading to the desired infinitesimal condition. The converse follows from integration of vector fields.

For any group \( \mathcal{G} \) acting on \( E^p_n \), we call this result “the Thom-Levine lemma.”

We now specialize to \( \mathcal{G} = \mathcal{A} \), as this case includes all difficulties of the proof of the infinitesimal criterion.

**Definition 4.10** A 1-parameter family \( \tilde{f} : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^p, 0), \tilde{f}(x, 0) = f(x) \) is \( \mathcal{A} \)-trivial if there is a pair \((h, k)\) of 1-parameter families of germs of diffeomorphisms

\[
h : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n, 0) \quad k : (\mathbb{R}^p \times \mathbb{R}, 0) \to (\mathbb{R}^p, 0) \\
(x, t) \mapsto h(x, t) \quad (y, t) \mapsto k(y, t)
\]

such that \( h(x, 0) = x, k(y, 0) = y, h_t(0) = 0, k_t(0) = 0 \) and

\[k_t \circ f_t \circ h_t = f.\]

**Remark 4.11** We also use the notation \( F(x, t) = (\tilde{f}(x, t), t), H(x, t) = (h(x, t), t) \) and \( K(y, t) = (k(y, t), t) \) for the corresponding 1-parameter unfoldings. In this notation \( F \) is \( \mathcal{A} \)-trivial if \( K \circ F \circ H = f \times \text{Id}_\mathbb{R} \). We denote by \( \partial \cdot F \) the vector field in \((\mathbb{R}^n \times \mathbb{R}, 0)\) with zero component in the \( \frac{\partial}{\partial t} \) direction, that is \( dF(\frac{\partial}{\partial t}) = (\partial \cdot F, 1) \).

The next result is known as the Thom-Levine lemma (see [58, 79, 69]).

**Proposition 4.12** Let \( f \in E^p_n \) and \( F : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^p \times \mathbb{R}, 0), F(x, t) = (\tilde{f}(x, t), t), \tilde{f}(0, t) = 0, \tilde{f}(x, 0) = f(x), \) the germ at 0 of a 1-parameter unfolding of \( F \). Then \( F \) is \( \mathcal{A} \)-trivial if and only there exist vector fields \( V : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^n \times \mathbb{R}, 0) \) with \( V(x, t) = v(x, t) + \frac{\partial}{\partial t}, v(x, t) = \sum_{i=1}^n v_i(x, t) \frac{\partial}{\partial x_i}, v_i(0, t) = 0 \) for
\[ i = 1, \ldots, n \quad \text{and} \quad W : (\mathbb{R}^p \times \mathbb{R}, 0) \to (\mathbb{R}^p \times \mathbb{R}, 0) \quad \text{with} \quad W(y, t) = w(y, t) + \frac{\partial}{\partial t}. \]
\[ w(y, t) = \sum_{j=1}^{p} w_j(y, t) \frac{\partial}{\partial y_j}, \quad w_j(0, t) = 0 \quad \text{for} \quad j = 1, \ldots, p. \quad \text{such that} \]
\[ \partial \cdot F(x, t) = \sum_{i=1}^{n} \frac{\partial \tilde{f}}{\partial x_i}(x, t) \cdot v_i(x, t) + w \circ F(x, t). \quad (3) \]

**Proof** We give here an idea of the proof. The reader may consult, for instance, Mather [58, p. 144], du Plessis [79, p. 174], or Mond and Nuño-Ballesteros [69, p. 37] for a complete proof.

If \( F \) is a trivial unfolding of \( f, K \circ F \circ H = f \times 1_{\mathbb{R}} \) and then \( \partial \cdot (K \circ F \circ H) = 0 \) and we apply the chain rule to get (3).

Conversely, if condition (3) holds, we consider the systems of differential equations in \((\mathbb{R}^n \times \mathbb{R}, 0)\) and \((\mathbb{R}^p \times \mathbb{R}, 0)\), respectively:

\[
\begin{align*}
\dot{x} &= v(x, t) \\
v(0, t) &= 0
\end{align*} \quad \begin{align*}
\dot{y} &= w(y, t) \\
w(0, t) &= 0 \quad (4)
\end{align*}
\]

We can integrate these vector fields to obtain 1-parameter families \( h_t \) and \( k_t \) of diffeomorphisms of \((\mathbb{R}^n \times \mathbb{R}, 0)\) and \((\mathbb{R}^p \times \mathbb{R}, 0)\), respectively, such that \( h_0(x) = x, h_t(0) = 0; k_0(y) = y, k_t(0) = 0 \) and \( k_t \circ \tilde{f}_t \circ h_t = f \).

Condition (3) in Proposition 4.12 admits an useful algebraic formulation. First, we introduce some notation.

Given the 1-parameter unfolding \( F : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^p \times \mathbb{R}, 0), F(x, t) = (\tilde{f}(x, t), t) \) with \( \tilde{f}(x, 0) = f(x) \), as before, \( \Theta_F \) denotes the \( \mathcal{E}_{n+1} \) module of vector fields along \( F \). However, here it will be more convenient to consider the submodule of \( \Theta_F \) defined as:

\[ \Psi_F = \{ \sigma \in \Theta_F \mid \text{the } \mathbb{R} \text{-component of } \sigma \text{ is zero} \}. \]

Similarly, \( \Psi_{n+1} \) and \( \Psi_{p+1} \) denote vector fields in \((\mathbb{R}^n \times \mathbb{R}, 0)\) and \((\mathbb{R}^p \times \mathbb{R}, 0)\) respectively, with zero \( \mathbb{R} \)-components.

The restrictions of the homomorphisms \( tF \) and \( \omega F \) give respectively the \( \mathcal{E}_{n+1} \)-homomorphism \( tF : \Psi_{n+1} \to \Psi_F \) and the \( \mathcal{E}_{p+1} \)-homomorphism via \( F^* \), \( \omega F : \Psi_{p+1} \to \Psi_F \).

With this notation, we can see that (3) holds if and only if

\[ \partial \cdot F \in tF(\mathcal{M}_n \Psi_{n+1}) + \omega F(\mathcal{M}_p \Psi_{p+1}) \quad (5) \]

holds.

We call \( T\mathcal{A}_{un}(F) = tF(\mathcal{M}_n \Psi_{n+1}) + \omega F(\mathcal{M}_p \Psi_{p+1}) \), the \( \mathcal{A} \)-tangent space of the unfolding \( F \). Similarly \( T\mathcal{K}_{un}(F) = tF(\mathcal{M}_n \Psi_{n+1}) + F^*(\mathcal{M}_{p+1})\Psi_{p+1} \) is the \( \mathcal{K} \)-tangent space of \( F \).
We now turn to the algebraic tools we need in the proof of theorem 4.8.

In the cases $\mathcal{G} = \mathcal{R}, \mathcal{C}$ or $\mathcal{K}$ the proof of the infinitesimal criterion of $\mathcal{G}$-determinacy will follow from the following elementary result.

**Lemma 4.13 (Nakayama’s Lemma)** Let $\mathcal{R}$ be a commutative ring, $M$ an ideal such that for $x \in M$, $(1 + x)$ is invertible. Let $C$ be a finitely generated $\mathcal{R}$-module, $A$ a submodule, then

(i) if $A + M \cdot C = C$, then $A = C$,

(ii) if $\mathcal{R}$ is a $k$-algebra, and $\dim_k \left( \frac{C}{A + M \cdot C} \right) \leq d$ then $M^d \cdot C \subseteq A$.

An equivalent formulation of condition (i) in Lemma 4.13 is the following

(i') If $M \cdot C = C$, then $C = 0$.

When $\mathcal{G} = \mathcal{L}$ or $\mathcal{A}$, we need a fairly deep result, the generalized Malgrange preparation theorem (see Golubitsky and Guillemin [40], Martinet [54, 55], Wall [108]).

**Theorem 4.14 (Preparation Theorem)** Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a $C^\infty$ map-germ, $E$ a finitely generated $E_n$-module. If $\dim_k \left( \frac{E}{f^*(M_p) + E} \right) < \infty$, then $E$ is finitely generated as $E_p$-module (via $f$).

The next proposition is a consequence of the Preparation theorem. It is an useful tool to study $\mathcal{A}$-finite determinacy.

**Proposition 4.15 (Bruce, du Plessis and Wall [14], Lemma 2.6)** Let $C$ be a finitely generated $E_n$-module, $B \subset C$ a finitely generated $E_n$-submodule, $A \subset f^*(M_p)C$ a finitely generated $E_p$-submodule (via $f$), and $M$ a proper, finitely generated ideal in $E_n$. If

$$MC \subset A + B + M(f^*(M_p) + M)C$$

then $MC \subset A + B$.

We are now ready to prove Theorem 4.8.

**Proof (of Theorem 4.8)** First we notice that (i) and (ii) give respectively the implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1). The implication (2) $\Rightarrow$ (3) is trivial since $M_p^k \Theta_f$ has finite codimension.

It is easy to prove the equivalence between (3) and (4). The implication (3) $\Rightarrow$ (2) will follow from (iii), as we now explain.

For any $\mathcal{G} = \mathcal{R}, \mathcal{C}, \mathcal{K}, \mathcal{L}, \mathcal{A}$ let

$$c_k = \dim_k \left( \frac{M_p \Theta_f}{T \mathcal{G} f + M_p^k \Theta_f} \right), \quad k \geq 1.$$

Since $\mathcal{G}$-cod $f < \infty$, the sequence

$$0 = c_1 \leq c_2 \leq \cdots \leq \mathcal{G}$-cod $f$$

converges to a finite codimension.
is finite.

Then, there exists \( s \) such that \( c_k = c_s \) for all \( k \geq s + 1 \). It follows that \( TG f + M^s f \mathcal{O}_f = TG f + M^s f \mathcal{O}_f \) for all \( k \geq s + 1 \). In particular \( M^s f \mathcal{O}_f \subseteq TG f + M^s f \mathcal{O}_f \) for all \( k \geq s + 1 \). Taking \( k = s + 1 \), when \( \mathcal{G} = \mathcal{A}, \mathcal{C}, \mathcal{K} \) and \( k = 2s \), when \( \mathcal{G} = \mathcal{A}, \mathcal{L} \), we obtain the statement in (iii) from which the result follows.

It suffices to prove (i), (ii) and (iii). For a clearer presentation, we first prove (iii).

Let \( f \in \mathcal{G} \) and \( \mathcal{G} f = f \mathcal{O}_f \). We leave the details as an exercise to the reader. (i) **Necessary condition for finite determinacy.**

This is not hard. A map-germ \( f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0) \) is \( k \)-determined if \( \mathcal{G} f \) contains all germs \( g \in \mathcal{E}_n^p \), such that \( j^k g(0) = j^k f(0) \). Let us denote this set by \( \mathcal{W} \).

Let

\[
\pi^l : \mathcal{E}_n^p \to J'(n,p)
\]

As \( \mathcal{G} f \supseteq \mathcal{W} \), then \( \pi^l (\mathcal{G} f) \supseteq \pi^l (\mathcal{W}) \). Thus we also get that

the tangent space of \( \pi^l (\mathcal{G} f) \supseteq \pi^l (\mathcal{W}) \). (6)

Notice that for all \( l \geq k \), the set \( \pi^l (\mathcal{W}) \) is the affine subspace of \( J'(n,p) \) consisting of all \( l \)-jets whose \( k \)-jet is \( j^k f(0) \). Hence we can rewrite (6) as

\[
TG f + M^{k+1} f \mathcal{O}_f \supseteq M^{k+1} f \mathcal{O}_f , l \geq k.
\]

The result now follows from (iii) taking \( l = k + 1 \) for \( \mathcal{G} = \mathcal{A} \) or \( \mathcal{K} \) and \( l = 2k + 1 \) when \( \mathcal{G} = \mathcal{A} \) or \( \mathcal{L} \).

(ii) **Sufficient condition for finite determinacy.**

Let \( f, g \in \mathcal{E}_n^p, j^k f(0) = j^k g(0), \epsilon = 1 \) or \( 2 \), \( F(x,t) = (\tilde{f}(x,t),t) \), where \( \tilde{f}(x,t) = (1-t)f(x) + tg(x), t \in [0,1] \).

(1) \( \mathcal{G} = \mathcal{A}, \mathcal{C} \) or \( \mathcal{K} \).

In these cases the hypothesis

\[
TG f \supseteq M^{k+1} f \mathcal{O}_f
\]

implies

\[
TG_{un}(F) + M^{k+2} \Psi_F \supseteq M^{k+1} \Psi_F.
\]

The proof that (7) implies (8) is not hard, but we omit it (the reader may consult Wall [108] or du Plessis [79]).

The tangent spaces \( TG_{un}(F) \), \( \mathcal{G} = \mathcal{A}, \mathcal{C} \) or \( \mathcal{K} \), are finitely generated \( \mathcal{E}_{n+1} \)-modules, so we can apply Nakayama’s lemma to (8) with \( C = TG_{un}(F) + M^{k+1} \Psi_{n+1}, A = TG_{un}(F) \) and \( M = M_{n+1} \) to get \( TG_{un}(F) \supseteq M^{k+1} \Psi_F \).
Now, $\partial \cdot F = g - f \in M^{k+2}_n \Psi_F$, and we can apply the Thom-Levine lemma to prove that $F$ is $\mathcal{G}$-trivial in some neighborhood of $t = 0$. For a proof of the Thom-Levine lemma for $\mathcal{G} = \mathcal{K}$ see du Plessis et al. [39]. Notice that $j^{k+1} \overline{f}_l(0) = j^{k+1} f(0)$, and the hypothesis (ii) holds for $\overline{f}_a$, for any $a \in [0, 1]$, so the arguments of the proof also hold to prove that $F$ is $\mathcal{G}$-trivial in a small neighborhood of $t = a$ for any $a \in [0, 1]$. Hence $\overline{f}$ is $(k + 1)$-$\mathcal{G}$-determined, $\mathcal{G} = \mathcal{R}$, $\mathcal{C}$ or $\mathcal{K}$.

(II) $\mathcal{G} = \mathcal{L}$ or $\mathcal{A}$.

In these cases, $TG_\mathcal{G}(F)$ is not an $E_{n+1}$-module in general. Let $\mathcal{G} = \mathcal{A}$ (the case $\mathcal{G} = \mathcal{L}$ follows as a particular case).

\[
TA_{\mathcal{G}}(F) = tF(M_n \Psi_{n+1}) + \omega F(M_p \Psi_{p+1}),
\]

and $j^{2k+1} f(0) = j^{2k+1} g(0)$.

First notice that if $F_0(x, t) = (f(x), t)$ is the suspension of $f$, the hypothesis $M^{k+1}_n \Theta_f \subseteq t f(M_n \Theta_n) + \omega f(M_p \Theta_p)$ implies that

\[
M^{k+1}_n \Theta_{F_0} \subseteq t F_0(M_n \Psi_{n+1}) + \omega F_0(M_p \Psi_{p+1}) + (tM^{k+1}_n + M^{2k+2}_n) \Psi_{F_0}.
\]

Notice that $M^{k+1}_n \Psi_{F_0} \subseteq M^{k+1}_n \Theta_f + tM^{k+1}_n \Psi_{F_0}$.

The next step is to verify that similar inclusion holds replacing $F_0$ by $F$, $j^{2k+1} \overline{f}_l(0) = j^{2k+1} f(0)$, that is

\[
M^{k+1}_n \Psi_F \subset tF(M_n \Psi_{n+1}) + \omega F(M_p \Psi_{p+1}) + (tM^{k+1}_n + M^{2k+2}_n) \Psi_F \quad (9)
\]

(see sublemma 2.2 in du Plessis [79]).

If we can show that the term $(tM^{k+1}_n + M^{2k+2}_n) \Psi_F$ can be eliminated in (9) then the Thom-Levine lemma can be applied to prove that $F$ is $\mathcal{A}$-trivial.

To achieve this goal Malgrange’s preparation theorem will be the fundamental tool.

Multiplying (9) by $M^{k+1}_n$ and since $M^{k+1}_n \omega F(M_p \Psi_{p+1}) \subset F^\ast(M_p) M^{k+1}_n \Psi_F$, we get

\[
M^{2k+2}_n \Psi_F \subset tF(M^{k+2}_n \Psi_{n+1}) + F^\ast(M_p) M^{k+1}_n \Psi_F + (t + M^{k+1}_n M^{2k+2}_n \Psi_F. \quad (10)
\]

The $E_{n+1}$-module

\[
E = \frac{tF(M^{k+2}_n \Psi_{n+1}) + F^\ast(M_p) M^{k+1}_n \Psi_F + M^{2k+2}_n \Psi_F}{tF(M^{k+2}_n \Psi_{n+1}) + F^\ast(M_p) M^{k+1}_n \Psi_F}
\]

is finitely generated, since it is a quotient of finitely generated modules. Moreover, from (10) we get that $E = (t + M^{k+1}_n) E$, and by Nakayama’s lemma it follows that $E = 0$. Then, we get

\[
M^{2k+2}_n \Psi_F \subset tF(M^{k+2}_n \Psi_{n+1}) + F^\ast(M_p) M^{k+1}_n \Psi_F. \quad (11)
\]
Using (11) to replace part of the remainder term in (9), we get
\[ M_n^{k+1}Ψ_F \subseteq tF(M_nΨ_{n+1}) + \omega F(M_pΨ_{p+1}) + (t + F^*(M_p))M_n^{k+1}Ψ_F. \] (12)

Let \( E' \) be the \( F^*(E_{p+1}) \)-module
\[ E' = \frac{tF(M_nΨ_{n+1}) + \omega F(M_pΨ_{p+1}) + M_n^{k+1}Ψ_F}{tF(M_nΨ_{n+1}) + \omega F(M_pΨ_{p+1})}. \]

Using (12), it follows that \( E' = (t + F^*(M_p))E' \). Notice that the ideal \( (t) + F^*(M_p) \) is contained in \( F^*(M_{p+1}) \), so it follows that \( E' = F^*(M_{p+1})E' \).

To apply Nakayama’s lemma, one has to show that \( E' \) is a \( F^*(E_{p+1}) \)-module finitely generated. For this, let the finitely generated \( E' \)-module
\[ E'' = \frac{tF(M_nΨ_{n+1}) + M_n^{k+1}Ψ_F}{tF(M_nΨ_{n+1})} \]

Notice that the inclusion
\[ tF(M_nΨ_{n+1}) + M_n^{k+1}Ψ_F \subseteq tF(M_nΨ_{n+1}) + \omega F(M_pΨ_{p+1}) + M_n^{k+1}Ψ_F \]
induces an epimorphism of \( F^*(E_{n+1}) \)-modules \( E'' \to E' \) so that if \( E'' \) is a finitely generated \( F^*(E_{p+1}) \)-module, then \( E' \) also is.

From Malgrange preparation theorem, \( E'' \) is a finitely generated \( F^*(E_{p+1}) \)-module if and only if
\[ \dim_{\mathbb{R}} E'' = \frac{E''}{F^*(M_{p+1})E''} < \infty. \] (13)

Now
\[ \frac{E''}{F^*(M_{p+1})E''} \cong \frac{tF(M_nΨ_{n+1}) + M_n^{k+1}Ψ_F}{tF(M_nΨ_{n+1}) + F^*(M_{p+1})M_n^{k+1}Ψ_F} \]

It follows from (11) that
\[ tF(M_n^{k+2}Ψ_{n+1}) + F^*(M_{p+1})M_n^{k+1}Ψ_F \supset M_n^{2k+2}Ψ_F. \]

As \( t \in F^*(M_{p+1}) \), we also get that
\[ tF(M_n^{k+2}Ψ_{n+1}) + F^*(M_{p+1})M_n^{k+1}Ψ_F \supset M_n^{k+1}M_n^{k+1}Ψ_F, \]
so that
\[ \dim_{\mathbb{R}} \frac{E''}{F^*(M_{p+1})E''} \leq \dim_{\mathbb{R}} \frac{M_n^{k+1}Ψ_F}{M_n^{k+1}M_n^{k+1}Ψ_F} < \infty \]

Then we can apply Nakayama’s lemma to (12) to get that \( E' = 0 \), so that \( M_n^{k+1}Ψ_F \subseteq tF(M_nΨ_{n+1}) + \omega F(M_pΨ_{p+1}) \).

To conclude we proceed as in part (I).
The following result follows from Theorem 4.8 and Mather’s lemma.

**Proposition 4.16** Let \( f \in E^n_p \), \( \epsilon = 1 \) when \( G = R, C \) or \( K \) and \( \epsilon = 2 \) when \( G = L, A \). Then \( f \) is \( k \)-\( G \)-determined if and only if \( M^{k+1}_n \Theta_g \subset TGF + M^{(k+1)}_n \Theta_g \) for all \( g \in E^n_p \) such that \( j^k g(0) = j^k f(0) \).

We see in the next example that the converse of condition (i) in Theorem 4.8 does not hold, that is, the condition \( TGF \supseteq M^{k+1}_n \Theta_f \) does not imply in general that \( f \) is \( k \)-\( G \)-determined.

**Example 4.17** Let \( f : (\mathbb{R}^2, 0) \to (\mathbb{R}, 0) \), \( f(x, y) = x^3 + y^3 \), and \( G = R \). Then

\[
TRf = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) M_2 = M_3^3
\]

but \( f \) is not 2-\( G \)-determined as \( j^2 f(0) \equiv 0 \).

A successful approach to a necessary and sufficient condition for finite determinacy appears in [14] where J. Bruce, A. du Plessis and C.T.C. Wall prove this condition for unipotent subgroups of \( G = R, C, K, L \) or \( A \).

Let \( G_s = \{ h \in G \mid j^s h(0) = j^s 1_G \} \) where \( 1_G \) is the identity of \( G \), and \( J^s G \) the Lie group of \( s \)-jets of elements of \( G \). The sets \( G_s \), \( s \geq 1 \) are unipotent subgroups of \( G \). A special case of the main result in [14] is the following:

**Theorem 4.18** (Bruce, du Plessis, Wall [14]) A \( C^m \) map-germ \( f : (\mathbb{R}^n, 0) \to (\mathbb{R}, 0) \) is \( r-G_s \)-determined \( (s \geq 1) \) if and only if \( M^{r+1}_n \Theta_f \subset TGF_s(f) \).

### 4.3 Classification of stable singularities

We consider here the problem of classification of stable germs with respect to \( \mathcal{A} \)-equivalence. The main result is the following

**Theorem 4.19** (Mather [61]) If \( f, g \) are stable germs then \( f \sim \mathcal{A} g \) if and only if the algebras \( Q(f) \) and \( Q(g) \) are isomorphic.

The proof of this theorem follows from the following property holding for infinitesimally stable germs: \( \mathcal{A}^{p+1} z = K^{p+1} z \cap St^{p+1} \), where \( z = j^{p+1} f(0) \), and \( St^{p+1} \) is the set of all stable jets in \( J^{p+1}(n, p) \). We omit the complete proof, however the main steps leading to the proof are given.

**Example 4.20** The hypothesis that \( f \) and \( g \) are stable is essential. For instance, let \( f(x, y) = (x, y^3 + xy) \) and \( g(x, y) = (x, y^3) \). Both algebras \( Q(f) \) and \( Q(g) \) are isomorphic to \( \frac{E^3_1}{(y^3)} \), but \( f \) and \( g \) are not \( \mathcal{A} \)-equivalent. In fact, \( f \) is stable and \( g \) is not.

The condition that \( f \in E^n_p \) is infinitesimally stable is determined by its \( p + 1 \)-jet. In fact the following holds:
Proposition 4.21 (Mather [61], Proposition I.1) \textit{The map-germ } $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0)$ \textit{is stable if and only if}

$$tf(\Theta(n,S)) + \omega f(\Theta_p) + (f^*(M_p) + M^{p+1})\Theta_f = \Theta_f. \tag{14}$$

\textbf{Proof} We need to show that (14) implies

$$tf(\Theta(n,S)) + \omega f(\Theta_p) = \Theta_f.$$

The proof is similar to the proof of Proposition 4.15 but simpler. Let $D = tf(\Theta(n,S)) + f^*(M_p)\Theta_f$. Note that

$$\omega f(M_p\Theta_f) \subset f^*(M_p)\Theta_f \subset D.$$

Then

$$\dim \frac{\Theta_f}{M^{p+1}\Theta_f + D} \leq \dim \frac{\omega f(\Theta_p)}{\omega f(M_p\Theta_p)} \leq p.$$

The result then follows by Lemma 4.13 (ii).

\textbf{Remark} Mather gives in [61], Proposition (I.6), a simple geometric characterization of a stable multigerm $f : (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0), S = \{x_1, x_2, \ldots, x_r\}$. Recall that if $V$ is a vector space and $H_1, \ldots, H_r$ are subspaces of $V$, then $H_1, \ldots, H_r$ are in general position if for every sequence of integers $i_1, \ldots, i_l$ with $1 \leq i_1 \leq \cdots \leq i_l \leq r$, we have $\text{cod}(H_{i_1} \cap \cdots \cap H_{i_l}) = \text{cod}(H_{i_1}) + \cdots + \text{cod}(H_{i_l})$.

Let $f_i : U_i \to \mathbb{R}^p, i = 1, \ldots, r$ be a representative of the germ $f_i : (\mathbb{R}^n, x_i) \to (\mathbb{R}^p, 0)$. Denote by $X_i = \{x \in U_i | (f_i, x) \sim (f_i, x_i)\}$ where $(f_i, x)$ denotes the germ $f_i : (\mathbb{R}^n, x) \to (\mathbb{R}^p, 0), i = 1, \ldots, r$. Since $f$ is infinitesimally stable, the sets $X_i$ are submanifolds. Mather’s result states that the multigerm $f$ is stable if and only if each branch $f_i : (\mathbb{R}^n, x_i) \to (\mathbb{R}^p, 0), i = 1 \ldots r$ is infinitesimally stable and the images $f_i(X_i), i = 1, \ldots, r$ are in general position.

\textbf{Corollary 4.23} An infinitesimally stable germ $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ is $(p+1)$-$\mathcal{A}$-determined.

\textbf{Proof} Notice that Proposition 4.21 implies that if $j^{p+1}g(0) = j^{p+1}f(0)$, then $g$ is also infinitesimally stable.

It is also clear that every such $g$ is $\mathcal{A}$-finitely determined, say $l$-$\mathcal{A}$-determined. Then, we can apply Proposition 4.16 to get the result. \hfill \Box

As the local algebra is a complete invariant for the classification of stable germs, we can ask:

– Can we provide a normal form of a stable germ whose local algebra is a given algebra $Q$?

The answer was given by Mather [61] and we review it here (see also section 1.2.5 of the Mond and Nuñó-Ballesteros in this Handbook [70]).

We start with a rank zero $\mathcal{K}$-finitely determined $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0), f = (f_1, f_2, \ldots, f_p)$. Let
\[ Q(f) = \frac{\mathcal{E}_n}{f^*(M_p)\mathcal{E}_n} = \frac{\mathcal{E}_n}{(f_1, \ldots, f_p)\mathcal{E}_n}. \]

Since \( f \) is \( \mathcal{K} \)-finitely determined, the quotient
\[ N_f = \Theta_f \]
\[ = \frac{\Theta_f}{tf(\Theta_n) + f^*(M_p)\Theta_f + \omega f(\Theta_p)} \quad (15) \]
is a finite dimensional \( \mathbb{R} \)-vector space of dimension \( r \) and we can choose \( \sigma_i \in \mathcal{E}_n^{M_p}, i = 1, \ldots, r \) such that
\[ N_f = \mathbb{R}\{\sigma_1, \ldots, \sigma_r\}, \quad (16) \]

For practical purposes, note that the vector space \( N_f \) admits the following simpler characterization:
\[ N_f \approx \frac{M_n\Theta_f}{tf(\Theta_n) + f^*M_p\Theta_f} \]

Let \( F : (\mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}^p \times \mathbb{R}^r, 0) \) be the linear \( r \)-parameter unfolding of \( f \) defined by
\[ F(x, u) = (f(x) + \sum_{i=1}^n u_i \sigma_i(x), u). \quad (17) \]

Then \( F \) is infinitesimally stable. In fact from (16) we get
\[ \Theta_f = tf(\Theta_n) + \omega f(\Theta_p) + f^*(M_p)\Theta_f + \mathbb{R}\{\sigma_1, \ldots, \sigma_r\}, \]

which implies that
\[ \Psi_F = tf(\Psi_{n+r}) + \omega F(\Psi_{p+r}) + F^*(M_{p+r})\Psi_F + \mathcal{E}_r\{\sigma_1, \ldots, \sigma_r\}, \quad (18) \]

where \( \mathcal{E}_r\{\sigma_1, \ldots, \sigma_r\} \) denotes the \( \mathcal{E}_r \)-module generated by \( \{\sigma_1, \ldots, \sigma_r\} \). Notice that \( F^*(M_p)\mathcal{E}_{n+r} \supset u_1, \ldots, u_r \mathcal{E}_{n+r} \). Then, it follows from that
\[ \Theta_F = tf(\Theta_{n+r}) + \omega F(\Theta_{p+r}) + F^*(M_{p+r})\Theta_F, \]

and it follows from Proposition 4.21 that \( F \) is infinitesimally stable.

**Example 4.24 (a) \( \mathcal{A}_k \) singularities**

Let \( f : (\mathbb{R}, 0) \to (\mathbb{R}, 0), f(x) = x^{k+1} \). Then \( N_f = \mathbb{R}\{1, x, \ldots, x^{k-1}\} \). From the above construction, we obtain that
\[ F : \mathbb{R} \times \mathbb{R}^{k-1} \to \mathbb{R} \times \mathbb{R}^{k-1} \]
\[ (x, u) \mapsto F(x, u) = (x^{k+1} + \sum_{i=1}^{k-1} u_i x^i, u), \]
is infinitesimally stable.

**(b) \( \Sigma^{2,0} \) singularities \( B^2_{2,2} = (x^2 \pm y^2, xy) \)
Results on density of stable mappings

(We use here du Plessis and Wall notation [82]. They are denoted \( I_{2,2} = (x^2 + y^2, xy) \) and \( II_{2,2} = (x^2 - y^2, xy) \) by Mather [61].)

Normal forms for infinitesimally stable singularities whose local algebra are \( B^+_{2,2} \) are

\[
F: (\mathbb{R}^2 \times \mathbb{R}^2, 0) \to (\mathbb{R}^2 \times \mathbb{R}^2, 0)
\]

\[(x, y, u, v) \mapsto F(x, y, u, v) = (x^2 \pm y^2 + ux + vy, xy, u, v).\]

As a consequence of the results of this section we can state the following addendum to Theorem 3.11.

**Theorem 4.25 (Mather [62], Addendum to Theorem 4.1)** Let \( r \leq p + 1 \) and \( k \geq p \). Let \( f: N \to P \) be a proper \( C^\infty \) mapping. Then the following conditions are equivalent

\( (a) \) \( f \) is stable.

\( (b) \) \( r^k \) is transversal to every contact class in \( j^k(N, P) \).

\( (c) \) For every subset \( S \) of \( N \) having \( r \) or fewer points, such that \( f(S) \) is a single point \( y \in P \), we have

\[ tf(\Theta(\Lambda, S)) + \omega f(\Theta(\Lambda, y)) + M^{k+1}S = \Theta f.\]

4.4 Maps of finite singularity type

Another fundamental notion introduced by Mather in [65] was the notion of mappings of finite singularity type, denoted by FST. Properties of such mappings are also discussed in [39].

A mapping \( f: N \to P \) will be said of finite singularity type if \( E = \frac{\Theta f}{tf(\Theta N)} \) is a finite module over \( C^\infty(P) \) via \( f \).

We can also define similarly the notion of FST for multigerms \( f: (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0) \).

Local properties of mappings of finite singularity type follow from our previous discussion. The critical set of \( f \) is the set \( \Sigma(f) \) of non-submersive points of \( f \).

Let \( F: (\mathbb{R}^n \times \mathbb{R}^r, 0) \to (\mathbb{R}^p \times \mathbb{R}^r, 0) \) with \( F(x, u) = (f(x, u), u) \) and \( \tilde{f}(x, 0) = f(x) \). If \( F \) is a stable germ, we say that \( F \) is a parametrized stable unfolding of \( f \).

**Theorem 4.26** Let \( f: (\mathbb{R}^n, S) \to (\mathbb{R}^p, 0) \). The following are equivalent.

\( (1) \) \( f \) is of FST.

\( (2) \) \( f \) is \( \mathcal{K} \)-finitely determined.

\( (3) \) \( f \) admits a stable parametrized unfolding.

Moreover, these conditions imply

\( (4) \) for every sufficiently small representative \( f: U \to V, f|\Sigma(f): \Sigma(f) \to V \) is proper and has finite fibers.
Remark 4.27 We say that \( f : X \to Y \) has finite fibers (or, is finite-to-one) if for every \( y \in Y \), \( f^{-1}(y) \) has a finite number of points.

**Proof** The equivalence (1) \( \iff \) (2) follows from the Preparation Theorem. In fact 
\[
E = \frac{\Theta_f}{\Theta_f(\Theta_{(n,x)})}
\]
is a finitely generated \( f^*(E_P) \)-module if and only if \( \mathcal{K}_e \)-cod \( f = \dim_{\mathbb{R}} \frac{\Theta_f}{\Theta_f(\Theta_{(n,x)}) + f^*(M_P)} < \infty \).

We saw in section 4.3 that a \( \mathcal{K} \)-finitely determined germ has a stable unfolding; so that (2) \( \implies \) (3). We saw in Example 4.7 that \( Q(f) = Q(f_0) \), so that (3) \( \implies \) (2).

It is sufficient to prove (4) for infinitesimally stable germs. In this case, the general position condition implies that for any \( y \in V \), \( f^{-1}(y) \cap \Sigma(f) \) has at most \( p \) points (see Remark 4.22).

We shall need some extra conditions to formulate the theory of FST mappings \( f : N \to P \). The condition that \( f \) has a parametrized stable unfolding is fairly easily computable, but it does not always have a global version (see Mather [65] for counter examples).

**Definition 4.28** Let \( f : N \to P \) be smooth. We say that \( \{ F, N', P', i, j \} \) is an unfolding of \( f \) if we have a commutative diagram

\[
\begin{array}{ccc}
N' & \xrightarrow{F} & P' \\
\downarrow{i} & & \downarrow{j} \\
N & \xrightarrow{f} & P
\end{array}
\]

where \( N', P' \) are smooth manifolds, \( F \) is a smooth mapping, \( i, j \) are closed smooth embeddings, \( i(N) = F^{-1}(j(P)) \) and \( F \) is transverse to \( j \).

**Theorem 4.29 (Mather [66], Proposition 7.2)**

Let \( f : N \to P \) be smooth and \( N \) compact. Then \( f \) is of finite singularity type if and only if there exists an unfolding \( \{ F, N', P', i, j \} \) of \( f \) such that \( F \) is proper and infinitesimally stable.

### 4.5 Notes

All the results in this section remain true if we replace smooth germs by real analytic or complex analytic germs. In particular, the notion of \( \mathcal{G} \)-finite determinacy for \( \mathcal{G} = \mathcal{R}, \mathcal{L}, \mathcal{A}, \mathcal{C} \) and \( \mathcal{K} \) is independent of whether we consider \( f \) as a real analytic, \( C^\infty \) or complex analytic map-germ. The Infinitesimal Criterion of \( \mathcal{G} \)-finite determinacy holds with essentially the same proof replacing Malgrange Preparation Theorem by Weirstrass Preparation Theorem. We use the same notation \( O_n \) for the local rings of real analytic or complex analytic map-germs at the origin. The maximal ideal in both cases is also denoted by \( M_n \). The set \( O_n^\mathbb{K} \) denotes the \( O_n \)-module of real or complex analytic map-germs from \( (\mathbb{K}^n, 0) \to (\mathbb{K}^n, 0) \), \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \). The following result explains the relation among finite determined germs in these different modules.
Proposition 4.30 Let $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ be a real analytic map-germ. The following are equivalent

(i) $f$ is $k\cdot \mathcal{G}$-determined in the space of real analytic map-germs $\mathcal{O}_n^p$.
(ii) $f$ is $k\cdot \mathcal{G}$-determined in $E_n^p$.
(iii) The complexification of $f$, $f_C : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, is $k\cdot \mathcal{G}$-determined in the space $\mathcal{O}_n^p$ of holomorphic map-germs.

In the complex case there are useful geometric characterization of $\mathcal{G}$-finite determinacy. The main result characterizes $\mathcal{G}$-finite determined germs as map-germs with isolated instability. The case $\mathcal{G} = \mathcal{A}$ was stated by Mather and proved by Gaffney. For a complete account we refer to Wall [108] or Mond and Nuño-Ballesteros [69]. See also Mond and Nuño-Ballesteros article in this Handbook [70].

Theorem 4.31 (Geometric criterion of finite determinacy) A holomorphic map-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, is $\mathcal{A}$-finite if and only if there is a neighborhood $U$ of $0$ in $\mathbb{C}^n$ such that for every finite subset $S \subset U \setminus \{0\}$, the multigerm of $f$ at $S$ is $\mathcal{A}$-stable.

The geometric condition of this theorem (isolated instability) holds for any real $\mathcal{A}$-finite map-germ. However, the converse statement does not hold. For a simple example, let $f(x, y) = (x^2 + y^2)^2$. As $\Sigma(f) = \{0\}$, the origin is an isolated instability, but $f$ is not $\mathcal{A}$-finitely determined.

5 The nice dimensions

We discuss in this section the main steps in the proof of theorem A. Mather proved in [61] that for a pair of positive integers $(n, p)$, there exists a smallest Zariski closed $\mathcal{R}^k$-invariant set $\Pi^k(n, p)$ in the set $J^k(n, p)$ such that $J^k(n, p) \setminus \Pi^k(n, p)$ is the union of finitely many $\mathcal{R}^k$-orbits. The set $\Pi^k(n, p)$ is the “bad set.” It is in fact the set of $k$-jets in $J^k(n, p)$ of “modality” ($\mathcal{K}$-modality) greater than or equal to 1 (see Section 8.1 for the definition of modality).

We review Mather’s construction of $\Pi^k(n, p)$. For each $r, k \in \mathbb{N}$ we define $W_r^k(n, p)$ as the set of $z \in J^k(n, p)$ such that $\mathcal{R}^k$-cod $z \geq r$. This set is a closed algebraic subset of $J^k(n, p)$. Let $W_r^k(n, p)^*$ denote the union of all irreducible components of $W_r^k(n, p)$ whose codimension is less than $r$. We let $\Pi^k(n, p) = \bigcup_{r \geq 0} W_r^k(n, p)^*$. The following properties hold:

- $\Pi^k(n, p)$ is a closed algebraic subset of $J^k(n, p)$.
- Let $\pi_k : J^{k+1}(n, p) \to J^k(n, p)$ be the projection. It follows that $\pi_k^{-1}(\Pi^k(n, p)) \subset \Pi^{k+1}(n, p)$, hence cod $\Pi^{k+1}(n, p) \leq \text{cod} \Pi^k(n, p)$.
- There exists a $k$ big enough for which the codimension of $\Pi^k(n, p)$ attains its minimum. For this $k$, $\text{cod} \Pi^k(n, p)$ is denoted $\sigma(n, p)$.

Mather calculated $\sigma(n, p)$ in [63] and the result is as follows:
Case 1: $n \leq p$
\[ \sigma(n, p) = \begin{cases} 
6(p - n) + 8 & \text{if } p - n \geq 4 \text{ and } n \geq 4 \\
6(p - n) + 9 & \text{if } 3 \geq p - n \geq 0 \text{ and } n \geq 4 \text{ or if } n = 3 \\
7(p - n) + 10 & \text{if } n = 2 \\
\infty & \text{if } n = 1
\end{cases} \]

Case 2: \( n > p \)

\[ \sigma(n, p) = \begin{cases} 
9 & \text{if } n = p + 1 \\
8 & \text{if } n = p + 2 \\
n - p + 7 & \text{if } n \geq p + 3
\end{cases} \]

**Definition 5.1** A pair \((n, p)\) is in the *nice dimensions* if \(n < \sigma(n, p)\).

Suppose \(k\) has the property that \(\text{cod } \Pi^k(n, p) = \sigma(n, p)\). If \((n, p)\) is in the nice dimensions, then there exists an analytically trivial stratification \(S^k(n, p)\) of \(J^k(n, p) \setminus \Pi^k(n, p)\) such that the strata are a finite number of \(K\)-orbits. To get a stratification of the whole jet space \(J^k(N, P)\), we add to \(S^k(n, p)\) a Whitney regular stratification of \(\Pi^k(n, p)\) (it exists since \(\Pi^k(n, p)\) is an algebraic closed set of \(J^k(n, p)\)).

This stratification of \(J^k(n, p)\) induces a partition of \(J^k(N, P)\) by \(K\)-orbit bundles whose restriction to \(J^k(N, P) \setminus \Pi^k(N, P)\) is denoted by \(S^k(N, P)\).
As we saw in Theorem 4.25, stable mappings can be characterized by transversality of the $k$-jet extension $j^k f : N \to J^k(N, P)$ to the $\mathcal{K}^k$-orbits.

When $\sigma(n, p) > n$, transversality to the strata of the stratification $J^k(N, P)$, implies that $j^k f (N) \cap \Pi^k(N, P) = \emptyset$. Hence Theorem A follows from Thom’s transversality theorem.

**Example 5.2 (Stable singularities when $n = p \leq 8$)** We refer to [69] for the list of stable singularities in the nice dimensions.

When $n = p$, $\sigma(n, p) = 9$, then $(n, n)$ is a nice pair of dimensions if and only if $n \leq 8$. The set $\Pi^k(n, n) \subset J^k(n, n)$, $k \geq n + 1$, $n \leq 8$ is the closure of all $\mathcal{K}^k$-orbits of $\mathcal{K}^k$-codimension greater than or equal to $n + 1$. In particular, $\Sigma^3(n, n) \subset \Pi^2(n, n)$, where $n \leq 8$ since cod $\Sigma^3 = 9$. The strata of the stratification $S^k(n, n)$, $k \geq n + 1$, $n \leq 8$ are presented in Table 1:

| Type $\Sigma^k$ | Name $\Sigma^k$ | Conditions | $\mathcal{K}$-cod $\leq n$ |
|-----------------|-----------------|------------|-----------------------------|
| $\Sigma^1$      | $A_j(x^j)$      | $1 \leq j \leq n$ | $j$                       |
| $\Sigma^{2,0}$  | $B_{q,p}^{p,q}(xy + x^p y^q)$ | $2 \leq p, q \leq n - 2$ | $p + q$              |
| $\Sigma^{2,1}$  | $B_{q,p}^{p,q}(x^2 + y^p, x^p)$ | $3 \leq p \leq 4$ | $2p$                      |
| $\Sigma^{2,1}$  | $C_{q,p}^{p,q}(x^2 + y^q, y^p)$ | $7$ |                           |
| $\Sigma^{2,1}$  | $C_{q,p}^{p,q}(x^2 + y^q, y^p)$ | $8$ |                           |

**Remark 5.3 Classification of stable singularities in the nice dimensions.** Mather classified the stable germs in the nice dimensions as an application of results and arguments in [63]. He gave complete proofs of the classification of the local algebras of singularities of type $\Sigma^1$ and $\Sigma^{2,0}$ and outlined the classification of $\Sigma^{2,1}$ and $\Sigma^{n-p+1}$ singularities. Further classification of simple and unimodular algebras were performed by Arnold [4], Wall [109], Dimca and Gibson [27, 28, 29] and Damon [18, 20, 19].

A remarkable property of stable map-germs in the nice dimensions is that, with respect to suitable coordinates, all singularities are weighted homogeneous. For many years, this property was considered to be true but there was no reference of a written proof.

This result was recently proved by Mond and Nuño-Ballesteros [69] theorem 7.6. Their proof is based on Mather’s classification of local algebras of stable germs in the nice dimensions and on the direct construction of the normal forms of their minimal stable unfoldings. This property of the nice dimensions plays a crucial role in the proof of Damon and Mond [26] that the $\mathcal{A}_c$-codimension is less than or equal to the rank of the vanishing homology of the discriminant (the discriminant Milnor number) for map germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ with $n \geq p$ and $(n, p)$ nice dimensions.
5.1 Notes

Non proper stable mappings. $C^\infty$ non-proper stable mappings were discussed by du Plessis and Vosegaard [81] and more recently by Kenta Hayano [42].

For proper maps $f : N \to P$, Mather proves that stability, strong stability, infinitesimal stability and local infinitesimal stability are equivalent notions. In [81], du Plessis and Vosegaard prove that these notions are equivalent when $f$ is a quasi-proper map with closed discriminant.

The purpose of Hayano’s paper, [42], is to give a sufficient condition for strong stability of non-proper smooth functions $f : N \to \mathbb{R}$. He introduces the notion of end-triviality of smooth mappings, which controls the behavior of $f$ around the ends of the source manifold $N$. He shows that a Morse function is stable if it is end-trivial at any point in its discriminant. The extra-nice dimensions. When the pair $(n, p)$ is in the nice dimensions and the source $N$ is compact, an important problem in the applications of singularity theory to topology of manifolds is the characterization of generic singularities of 1-parameter paths between two stable maps; they are also known as pseudo-isotopies. A 1-parameter family $F : N \times [0, 1] \to P$ connecting two non equivalent stable maps always intersects the set of non stable maps at a finite number of values of the parameter, the bifurcation points. The classification of singularities of bifurcation points in generic families of maps is an important step in results on elimination of singularities (see for instance [50, 7]) and on results about the topology of the space of smooth maps such as [16, 44, 104].

We say that a family $F : N \times [0, 1] \to P$ is a locally stable family if $F_t : N \to P$ is stable for all $t \in [0, 1]$ except possibly a finite number of values $\{t_1, \ldots, t_k\}$ and the non stable singularities of $F_t$ are a finite number of points $x_j$ at which $\mathcal{A}_e\text{-cod}(F_t) = 1$.

In [6] Sinha, Ruas and Atique obtain a result parallel to Mather’s characterization of the nice dimensions. They define the extra-nice dimensions and (see Figure 4) prove that the subset of stable 1-parameter families in $C^\infty(N \times [0, 1], P)$ is dense if and only $(n, p)$ is in the extra-nice dimensions.

In section 10 we relate the condition that $(n, p)$ is in the extra-nice dimensions to the geometry of sections of the discriminant of stable maps in dimensions $(n + 1, p + 1)$.

6 Thom’s example

If a pair of dimensions $(n, p)$ is not in the nice range of dimensions, then there exists an open non void subset $U$ of $C^\infty(N, P)$, such that $U$ is the union of an uncountable number of $\mathcal{A}_e$-orbits. This property was first proved by René Thom when $n = p = 9$. We review Thom’s example ([96]) here. The pair $n = p = 9$ is in the boundary of the nice dimensions, which consists of pairs $(n, p)$ such that $\sigma(n, p) = n$.

The construction of Thom’s example was based on the following
1. The set of mappings $F : N \to P$, $\dim N = \dim P = n$, such that $j^k F \pitchfork \Sigma^r (N, P)$, where $\Sigma^r (N, P) = \{ \sigma \in J^k (N, P) | \operatorname{corank} \sigma = r \}$, $0 \leq r \leq n$ is a residual set of $C^\infty (N, P)$.

2. $\operatorname{cod} j^k (N, P) \Sigma^r (N, P) = r^2$.

3. When $r = 3$, $n = 9$, there exists a 1-parameter family of non $\mathcal{K}$-equivalent mappings $F_A : \mathbb{R}^9 \to \mathbb{R}^9$, such that $j^k_1 F : \mathbb{R} \times \mathbb{R}^9 \to J^k (\mathbb{R}^9, \mathbb{R}^9)$ is transversal to $\Sigma^3 (\mathbb{R}^9, \mathbb{R}^9)$, where $j^k_1 F$ denotes the $k$-jet with respect to the variable $x$.

The sets $\Sigma^r$ are the first order Boardman symbols and it is an easy exercise to prove that they are codimension $r^2$ submanifolds of $J^k (N, P)$ when $\dim (N) = \dim (P)$. Hence (1) follows from Thom’s transversality theorem.

It is sufficient to verify (3) for map-germs $F : (\mathbb{R}^3, 0) \to (\mathbb{R}^9, 0)$, such that $\operatorname{corank} F (0) = 3$. By changing coordinates in source and target, it follows that $F$ can be written in the form $F (x, u) = (f (x, u), u)$, $x = (x_1, x_2, x_3)$, $u = (u_1, \ldots, u_6)$, $f_0 (x) = f (x, 0)$, where $f_0 : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ has zero rank.
The local algebras $Q(F)$ and $Q(f_0)$ are isomorphic. As we saw in Example 4.7, $F$ is $\mathcal{K}$-equivalent to a suspension of $f_0$. The 2-jet $j^2f_0$ is a quadratic polynomial mapping $q : \mathbb{R}^3 \to \mathbb{R}^3$, which determines a net of real quadrics. Non degenerate nets of quadrics over the complex numbers were classified by C. T. C. Wall in [107]. Over the reals, the classification was given by Wall and Edwards in [30]. The complete classification of real nets of quadrics can be found in [82] chapter 8, table 8.21.

For our purpose here, it suffices to remark that the set $\Sigma^{3,3}$ has a Zariski open set, denoted by $W_2$, defined by the union of the $J^2\mathcal{K}$-orbits of the unimodular family:

$$(f_0)_4 : (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$$
$$(x_1, x_2, x_3) \to (x_1^2 + \lambda x_2 x_3, x_2^2 + \lambda x_1 x_3, x_3^2 + \lambda x_2 x_3)$$

with $\lambda(\lambda^3 + 8)(\lambda^3 - 1) \neq 0$.

For each $\lambda$, $(f_0)_4$ is a homogeneous polynomial map of degree 2, hence the $J^2\mathcal{K}$-action in $W_2$ coincides with the action of the linear group $G = GL(3) \times GL(3)$ on $W_2$. Notice that the dimension of the linear group $G$ is 18, as well as the dimension of $W_2$.

However $G$ contains a one dimensional subgroup which acts trivially on $W_2$, namely $\{ (ct_{\mathbb{R}^3}, \frac{1}{c} t_{\mathbb{R}^3}) \}$, $c$ a non zero number. Hence the orbits have codimension at least 1 in $W_2$.

We can prove that the family (19) is 2-determined with respect to $\mathcal{K}$-equivalence. It follows that $W_2$ determines the $\mathcal{K}$-invariant sets $W_2^k = (\pi_2^k)^{-1}(W_2)$, where $\pi_2^k : J^k(9, 9) \to J^2(9, 9)$. Moreover, $\text{cod}_{J^k(9, 9)} W_2 = 9$, and $\mathcal{K}\text{-cod}(f_0)_4 = 10$.

In other words, $\sigma(9, 9) = \text{cod} W_2 = 9$, so that the unimodular stratum $W_2$ cannot be avoided by a generic set of proper mappings $F : \mathbb{R}^9 \to \mathbb{R}^9$. As a consequence, stable mappings are not dense when $n = p = 9$.

For each $\lambda \notin \{0, -2, 1\}$, $(f_0)_4$ admits the topologically stable unfolding

$$F_{\lambda} : (\mathbb{R}^9, 0) \to (\mathbb{R}^9, 0)$$
$$(x, u) \to (f_{\lambda}(x, u), u)$$

where $f_{\lambda}(x, u) = (x_1^2 + \lambda x_2 x_3 + u_1 x_2 + u_2 x_3, x_2^2 + \lambda x_1 x_3 + u_3 x_1 + u_4 x_2, x_3^2 + \lambda x_1 x_2 + u_5 x_1 + u_6 x_2)$.

We will discuss the topological stability of $F_{\lambda}$ in section 8.

7 Density of topologically stable mappings

From the previous example, it becomes clear that outside the nice dimensions, one has to loosen the formulation of Problem 2.1 to obtain a solution. Mather considered in [64] two possible ways.
One might hope that the space of mappings $f$ whose germ $f_x$ at each point $x \in N$ is $\mathcal{A}$-finitely determined is an open and dense subset in $C^m_p(N, P)$. However, Mather gave in [59] an example which shows that this set is not always dense. In [80] du Plessis defined the semi-nice dimensions as the pairs $(n, p)$ for which finite determinacy holds in general (see Definition 7.6). The complement of the semi-nice dimensions is essentially made of pairs $(n, p)$ where singularities of $\mathcal{K}$-modality greater than or equal to 2 occur generically (see [80], [109]).

The second way to try to solve the problem is based on ideas due to Thom, and led to Theorem B on density of $C^0$ stable mappings in $C^m_p(N, P)$.

In his article "Local topological properties of differentiable mappings" [93], Thom describes the topological structure of differentiable mappings, outlining the proof of the topological stability theorem.

**Theorem 7.1 (Theorem 4, [93])** Let $z$ be any jet in $J^r(n, p)$. Then, there exists a positive integer $s$ depending only on $r, n$ and $p$, and a proper algebraic variety $\Sigma$ in $\pi^{-1}_r(z) \subset J^s(n, p)$ such that any jet in $\pi^{-1}_r(z)$ outside $\Sigma$ is $C^0, \mathcal{A}$-finitely determined. Moreover, any two mappings realizing such jet are locally weakly stratified and isotopic.

A complete proof of this theorem follows from the proof of the Main Theorem in A. Varchenko’s article with the same title, Local topological properties of differentiable mappings [103] (see also [102, 101]). He also proves in [103] a stratification theorem, although he states in the paper he does not know whether Mather’s density theorem follows from his stratification theorem, or whether the stratification theorem can be proved by Mather’s methods.

Mather gave in 1970, an outline of a complete proof of Theorem B. His proof was published in the Proceedings of the Symposium of Dynamical Systems, held in Salvador, Bahia [64]. As remarked by him, he expected to publish a book in which the details of the proof would appear. In the Spring 1970, he gave a series of lectures and the notes appeared as a booklet published in the same year by the Harvard Printing Office. The notes also discuss the Thom-Whitney theory of stratified sets and stratified mappings. They were recently republished in the Bulletin of the American Mathematical Society [56].

Complete proofs of Theorem B were given in 1976, independently, by Gibson, Wirthmüller, du Plessis and Looijenga in [39] and by Mather in [66]. Both proofs are based on Thom’s ideas and Mather’s outline [64]. In what follows we refer to Theorem B as the Thom-Mather theorem.

The book [39] comprises the notes of a seminar on Topological Stability of Smooth Mappings held at the Department of Pure Mathematics in the University of Liverpool, during the academic year 1974-75. The main objective was to organize a complete proof of the Topological Stability Theorem, for which no published complete account existed. The book has become a fundamental reference on the subject.

The proof in [39] and [66] are similar and they rely on the following ingredients:

1. Properties of Whitney regular stratifications
Łojasiewicz theorem, giving the existence of Whitney regular stratification of semialgebraic sets.

Properties of stable mappings and mappings of finite singularity type (FST). A fundamental property of mappings of FST is the existence of a stable unfolding.

Thom’s second isotopy theorem, applied to show that families of mappings transverse to the Thom-Mather stratification are topologically trivial.

For a review of stratification theory and Thom’s isotopy theorems in the differentiable category, we also refer to the paper by David Trotman, in Volume I of this Handbook. We only make a brief presentation of basic concepts and results.

Let \( V \) be a subset of a smooth manifold \( N \) of class \( C^k \). A \( C^k \)-stratification of \( V \) is a filtration by closed subsets

\[
V = V_d \supset V_{d-1} \supset \cdots \supset V_1 \supset V_0
\]

such that each difference \( V_i \setminus V_{i-1} \) is \( C^k \)-manifold of dimension \( i \), or is empty. Each connected component of \( V_i \setminus V_{i-1} \) is a stratum of dimension \( i \). It follows that \( V \) is disjoint union of strata \( \{X_\alpha\}_{\alpha \in A} \), and we say that \( V \) is a stratified set.

For the purposes of these notes we assume that the stratified sets \( V = \bigcup_{\alpha \in A} X_\alpha \) are locally finite and satisfy the frontier condition (see Gibson et al. book \[39\] or Trotman \[100\] for the definition).

Let \( V \) be a subset of \( \mathbb{R}^n \) and \( \{X_\alpha\}_{\alpha \in A} \) a stratification of \( V \). Whitney defined regularity conditions (a) and (b), seeking for stratifications topologically trivial along strata.

**Definition 7.2 (Whitney’s conditions (a) and (b))** Let \( X \) and \( Y \) be strata of \( \{X_\alpha\}_{\alpha \in A} \), such that \( Y \subset X \setminus X_\alpha \).

(a) The pair \((X, Y)\) satisfies Whitney’s condition (a) at \( y \in Y \) if: for all sequences \((x_m) \in X \) with \( x_m \to y \), such that \( T_{x_m}X \) converges to a subspace \( T \subset \mathbb{R}^n \) (in Grassmannian of dim \( X \)-planes in \( \mathbb{R}^n \)), then \( T \supset T_{x_m}Y \).

(b) The pair \((X, Y)\) satisfies Whitney’s condition (b) at \( y \in Y \) if: for all sequences \((x_m) \in X \) and \((y_m) \in Y \), with \( x_m \to y \), \( y_m \to y \), such that \( \{T_{x_m}X\} \) converges to \( T \) and the lines \( \overline{x_my_m} \) converges to a line \( \ell \) one has \( \ell \in T \).

It was pointed out by Mather in his notes on topological stability that Whitney (b) implies Whitney (a). The reader may verify this as an exercise. We say that the stratification is **Whitney regular** if every pair of strata \((X_\alpha, X_\beta)\) satisfies (b) (hence also satisfies (a)) at every point in \( X_\beta \).

These regularity conditions are local and can be easily extended to stratified sets of a manifold \( N \).

Whitney \[117, 116\] proved in 1965 that any analytic variety in \( \mathbb{R}^n \) or \( \mathbb{C}^n \) admits a regular stratification whose strata are analytic. This result was extended to semi-analytic sets by Łojasiewicz \[49\], also in 1965. For the purposes of this section, the relevant result is the existence theorem for semialgebraic sets. We refer to Thom \[95\] and Wall \[105\] for accessible proofs.
Definition 7.3 Let \( f : N \to P \) be a smooth mapping and \( A \subseteq N, B \subseteq P \) sets with \( f(A) \subset B \). A stratification of \( f : A \to B \) is a pair \((X, X')\), such that \( X \) is a Whitney stratification of \( A \), \( X' \) is a Whitney stratification of \( B \), and the following conditions hold:

- \( f \) maps strata to strata.
- If \( X \in X, X' \in X' \), \( f(X) \subset X' \) then \( f : X \to X' \) is a submersion.

Definition 7.4 Let \( f : N \to P \) and \( X \) and \( X' \) as in definition 7.3. Given \( X_{\alpha}, X_{\beta} \) strata of \( X \), \( x \in X_{\beta} \) we say that \( X_{\alpha} \) is Thom regular over \( X_{\beta} \) at \( x \) relative to \( f \) when the following holds: for every sequence \((x_i) \in X_{\alpha}, x_i \to x \) such that \( \ker (d_{x_i}(f|_{x_i})) \) converges to \( T \) in the appropriate Grassmannian, then \( d_{x}(f|_{x_i}) \subseteq T \). We say that \( X_{\alpha} \) is Thom regular over \( X_{\beta} \) relative to \( f \) when this condition hold for all \( x \in X_{\beta} \). The triple \((X, X')\) is a Thom stratification for \( f \) when \( X \)’s regularity condition holds for all pair of strata \((X_{\alpha}, X_{\beta})\) with \( X_{\beta} \subset X_{\alpha} \). The triple \((f, X, X')\) with \( f \) a smooth mapping and \((X, X')\) a Thom stratification for \( f \) is called a Thom stratified mapping.

7.1 How to stratify mappings and jet spaces

We first discuss the Thom-Mather stratification in jet space and how to stratify stable mappings and mappings of finite singularity type. Then, we discuss why mappings transverse to the Thom-Mather stratification are topologically stable.

The idea of the proof is to construct a stratification \( \mathcal{A}^l(N, P) \), of a big open subset of \( J^l(N, P) \), with the following property: if \( l \) is sufficiently large, then for any mapping \( f : N \to P \) which is multitransverse to \( \mathcal{A}^l(N, P) \), then the locally finite manifold partition \( \mathcal{B} = ((j^l f)^{-1} \mathcal{A}^l(N, P)) \) is a Whitney stratification which extends to a Thom stratification \((\mathcal{B}, \mathcal{B}')\) of \( f \).

Let \( z \in j^l(n, p) \) and let \( f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0) \) such that \( j^l f(0) = z \).

Following Gibson et al. [39], we let

\[
\chi_z = \dim_{\mathbb{R}} \Theta_f - \dim_{\mathbb{R}} (j^l f)(\Theta_n) + (j^l f)(M_n) + M_n^l \Theta_f
\]

We define \( W^l(n, p) = \{ z \in j^l(n, p) \mid \chi_z \leq l \} \). \( W^l(n, p) \) is the bad set, and the following hold:

(a) If \( z \in j^l(n, p) \setminus W^l(n, p) \), then any \( f \in E^l_n \) such that \( j^l f(0) = z \) is \( l \)-\( \mathcal{K} \)-determined.
(b) \( W^l(n, p) \) is \( \mathcal{K} \)-invariant.
(c) \( W^l(n, p) \) is a real algebraic variety in \( J^l(n, p) \).

To verify (a) notice that, if \( \chi_z \leq l - 1 \), then

\[
t_f(\Theta_n) + (j^l(f)(M_n) + M_n^l) \Theta_f \supset M_n^{l-1} \Theta_f.
\]

Then we can multiply (21) by \( M_n \) and the result follows from Theorem 4.8.
It follows from (a) that map-germs $f \in \mathcal{E}_n^p$ such that $z = j^l f(0)$ satisfy $\chi_z \leq l - 1$ are of finite singularity type. In the following proposition we prove that the property of FST holds in general.

**Proposition 7.5** (Gibson et all [39], Theorem 7.2) The following conditions hold:

(i) $\text{cod } \mathcal{W}^{l+1}(n, p) \geq \text{cod } \mathcal{W}^l(n, p)$.

(ii) $\lim_{p \to \infty} \text{cod } \mathcal{W}^l(n, p) = \infty$.

(iii) There is a subbundle $\mathcal{W}^l(N, P) \subset \mathcal{J}^l(N, P)$ naturally associated to $\mathcal{W}^l(n, p)$.

Moreover, when $N$ is compact, mappings $f : N \to P$ such that $j^l f(N) \cap \mathcal{W}^l(N, P) = \emptyset$ are of finite singularity type.

**Definition 7.6** We say that a property $\mathcal{P}$ of map-germs holds in general if the sets $\mathcal{W}^l_p(n, p) = \{ z \in \mathcal{J}^l(n, p) | z \text{ does not satisfy } \mathcal{P} \}$, satisfy (i) and (ii) (see [108]).

While condition (i) in Proposition 7.5 can be easily verified, we can prove (ii) as follows.

Given $z \in \mathcal{W}^l(n, p)$, find $z' \in \mathcal{W}^{l+q}(n, p)$, $\pi_l(z') = z$, where $\pi_l : \mathcal{W}^{l+q}(n, p) \to \mathcal{W}^l(n, p)$ is the projection, such that $z' \notin \mathcal{W}^{l+q}(n, p)$ (see Bruce, Ruas and Saia [15], for a simpler proof of this result).

As $\mathcal{W}^l(n, p)$ is a real algebraic variety, it follows from Łojasiewicz’s result [49] that it has a Whitney stratification with semialgebraic strata. Condition (iii) is immediate. Notice that conditions (i) and (ii) imply that we can choose sufficiently high $l$ for which $\text{cod } \mathcal{W}^l(n, p) > n$. Then, the mappings $f : N \to P$ which are multitransverse to $\mathcal{A}^l(N, P)$ satisfy the condition $j^l f(N) \cap \mathcal{W}^l(N, P) = \emptyset$.

Our problem now is to construct a stratification $\mathcal{A}^l(n, p)$ of $\mathcal{J}^l(n, p) \setminus \mathcal{W}^l(n, p)$ whose members are $\mathcal{K}$-invariant sets $S_j = \{ z \in \mathcal{J}^l(n, p) \setminus \mathcal{W}^l(n, p) | \text{cod } z = j \}$, for $j = 0, 1, 2, \ldots$. The definition of $\text{cod } z$ will be given in the sequel.

We shall see that $\mathcal{K}^l$-equivalent jets $z$ and $z'$ have the same codimension, i.e., $\text{cod } z = \text{cod } z'$. This number does not coincide with the $\mathcal{K}^l$-codimension.

Although we know that contact classes are smooth submanifolds of the jet spaces, it is not clear at this point that the collection $S_j$ defines a stratification of $\mathcal{J}^l(n, p) \setminus \mathcal{W}^l(n, p)$. To define $\text{cod } z$ and to understand the structure of the strata $S_j$ in $\mathcal{A}^l(n, p)$, we first discuss shortly how to stratify infinitesimally stable mappings and mappings of FST. Recall that for any smooth map $f : N \to P$, the critical set of $f$ is $\Sigma(f) = \{ x \in N | df_x : T_x N \to T_{f(x)} f(N) \text{ is not surjective} \}$ and the discriminant of $f$ is $\Delta(f) = f(\Delta(f))$.

We saw in section 4 that if $f : N \to P$ is infinitesimally stable, the restriction $f|_{\Sigma(f)} : \Sigma(f) \to P$ is proper and uniformly finite-to-one. In fact for any $y \in P$, $\#(f^{-1}(y) \cap \Sigma(f)) \leq p$. Moreover, if $f^{-1}(y) \cap \Sigma(f) = \{ x_1, x_2, \ldots, x_s \}$ the multigerm $f : (N, S) \to (P, y)$ has a representative equivalent to a polynomial mapping $f : U \subset \mathbb{R}^n \to V \subset \mathbb{R}^p$, where $U$ and $V$ are open sets in $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively. In other words $f$ is a semialgebraic map defined on semialgebraic subsets. Then we can apply the basic theorems of Whitney and Łojasiewicz to construct Whitney stratifications $\mathcal{S}$ of $N$ and $\mathcal{S}'$ of $P$ with the following properties

1. For each stratum $X$ of $\mathcal{S}$, there is a stratum $Y$ of $\mathcal{S}'$ such that $f(X) \subset Y$. 

2. For each stratum \( Y \) of \( \mathcal{S} \), it follows that \( f^{-1}(Y) \setminus \Sigma(f) \) is a stratum of \( \mathcal{S} \).

3. For each stratum \( X \) of \( \mathcal{S} \), such that \( X \subset \Sigma(f) \), we have that \( \dim X = \dim Y \) and \( f : X \to Y \) is an immersion, where \( Y \) is the stratum of \( \mathcal{S}' \) which contains \( f(X) \).

Notice that from 2. it follows that \( N \setminus \Sigma(f) \) is a union of strata. Hence, \( \Sigma(f) \) is also a union of strata.

Now, if \( f : (N, x_0) \to (P, y_0) \) is a stable germ, for any small representative that we also denote by \( f \), the stratum \( X \in \mathcal{S} \) which contains \( x_0 \) is connected and its codimension is strictly greater than the codimension of any other stratum of \( \mathcal{S} \). This number depends only of \( f \). We call it the codimension of \( f \), and we write \( \text{cod } f \). A germ \( f \) has codimension zero if and only if it is of maximal rank.

This notion generalizes to map-germs of finite singularity type.

**Definition 7.7** Let \( f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0) \) be a map of finite singularity type. We define \( \text{cod } f \) at \( x = 0 \) as the codimension of a stable unfolding of \( f \).

Notice that this number is well defined. In fact, if \( F : (\mathbb{R}^n \times \mathbb{R}^s, 0) \to (\mathbb{R}^p \times \mathbb{R}^r, 0) \) and \( F' : (\mathbb{R}^n \times \mathbb{R}^s, 0) \to (\mathbb{R}^p \times \mathbb{R}^r, 0) \) are stable unfoldings of \( f \) and if, say, \( r = s + k \), then it follows that \( F \times Id \) is equivalent to \( F' \), where \( Id \) is the identity map in \( \mathbb{R}^k \). Then \( \text{cod } (F \times Id) = \text{cod } F' \), and it is easy to see that \( \text{cod } F = \text{cod } (F \times Id) \). Now the following result follows easily.

**Proposition 7.8** If \( f \sim f' \) then \( \text{cod } f = \text{cod } f' \).

The properties of the stratification \( \mathcal{A}^l(N, P) \) can be summarized in the following results.

**Proposition 7.9** Let \( f : (N, x_0) \to (P, y_0) \) be a smooth map-germ with an unfolding \( F : (N', x'_0) \to (P', y'_0) \), as in the diagram

\[
\begin{array}{ccc}
(N', x'_0) & \xrightarrow{F} & (P', y'_0) \\
\downarrow i & & \downarrow j \\
(N, x_0) & \xrightarrow{f} & (P, y_0).
\end{array}
\]

Then the following conditions are equivalent

(i) \( j^l f \notin W^l(N, P) \) and \( j^l f \) is transverse to \( \mathcal{A}^l(N, P) \).

(ii) \( j^l F \notin W^l(N', P') \) and \( j^l F \) is transverse to \( \mathcal{A}^l(N', P') \), and in addition if \( X \in (j^l F)^{-1} \mathcal{A}^l(N', P') \) contains \( x'_0 \), then \( i \) is transverse to \( N' \).

**Proposition 7.10** (Gibson et al., [39], Proposition 3.3, Chapter 4) Let \( f : N \to P \) be a proper smooth mapping multi-transverse to \( \mathcal{A}^l(N, P) \) and such that \( j^l f(N) \cap W^l(N, P) = 0 \). Let \( \mathcal{S} = (j^l f)^{-1} \mathcal{A}^l(N, P) \) and \( \mathcal{S}' = \{ f(X) \mid X \in \mathcal{S} \} \cup \{ P \setminus f(N) \} \). Then \( (\mathcal{S}, \mathcal{S}') \) is a Thom stratification of \( f \).

**Remark 7.11** The pair \( (\mathcal{S}, \mathcal{S}') \) in Proposition 7.10 has a minimality property which uniquely characterizes it among all possible pairs. We refer to Gibson et al., [39] or Mather [66] for details.
7.2 Proof that topologically stable mappings are dense (Mather, [66], §8)

Initially, we state the Thom-Mather topological stability theorem, whose proof we outline in this section. Theorem B will follow from this result and Thom’s transversality theorem.

**Theorem 7.12** If \( f : N \to P \) is proper and for some (and hence for all) \( k \geq p + 1 \), \( j^k f \) is multitransverse to the Thom-Mather stratification of \( J^k(N,P) \), then \( f \) is strongly \( C^\infty \)-stable.

Given \( f : N \to P \), we will show that we can approximate it by a topologically stable mapping. First, we approximate \( f \) by a mapping \( f_1 : N \to P \) of finite singularity type (Proposition 7.5). Then, we can choose an unfolding \( (F, N', P', i, j) \) of \( f_1 \) such that \( F \) is proper and infinitesimally stable. Let \( S'_N \) and \( S'_P \) be stratifications of \( N' \) and \( P' \), respectively satisfying conditions (1)-(3) in Section 7.1.

By Thom’s transversality theorem, we can approximate \( j \) by \( j_2 : P \to P' \) such that \( j_2 \) is transverse to the strata of \( S'_P \). Moreover we may suppose \( j_2 = j \) outside a compact neighborhood of \( f(N) \).

Since \( F \) is transverse to \( j \), it follows that \( F \) is transverse to \( j_2 \) for \( j_2 \) sufficiently close to \( j \).

The set \( N_2 = F^{-1}(j_2(P)) \) is a smooth manifold. One can show that there is a diffeomorphism \( i_2 : N \to N_2 \) close to \( i : N \to N' \).

We let \( f_2 : j_2^{-1} \circ F \circ i_2 : N \to P \). It follows from construction that \( f_2 \) is close to \( f \) in the \( C^\infty \) topology. We claim that \( f_2 \) is topologically stable.

The proof is based in the following facts from the construction we have made:

(i) \( (F, N', P', i_2, j_2) \) is an unfolding of \( f_2 \);
(ii) \( F \) is proper and infinitesimally stable;
(iii) \( j_2 \) is transverse to the stratification \( S'_P \) of \( P' \).

Let \( g \) be a small perturbation of \( f_2 \), so that we can suppose \( f_2 \) and \( g \) are connected by a small arc \( g_t \) in \( C^\infty(N,P) \), \( t \in [0,1] \), \( g_0 = f_2, g_1 = g \). We can lift \( g_t \) to an arc \( G_t \) in \( C^\infty(N',P') \) such that \( G_0 = F \) and \( (G_t, N', P', i, j) \) is an unfolding of \( g_t \). Moreover, we may suppose that \( G_t = F \) outside of a sufficiently small compact neighborhood of \( i(N) \).

From Theorem 3.15, it follows that there exist one parameter families of diffeomorphisms \((H_t, K_t) \in \mathcal{A}, H_0 = Id_{N'}, K_0 = Id_{P'} \), such that \( F = K_t \circ G_t \circ H_t^{-1} \), for all \( t \in [0,1] \).

Now consider the commutative diagram

\[
\begin{array}{ccc}
N & \xrightarrow{i} & N' \\
\downarrow{g_t} & & \downarrow{F} \\
N & \xrightarrow{H_t} & N' \\
\downarrow{G_t} & & \downarrow{F} \\
P & \xrightarrow{j} & P' \\
\downarrow{K_t} & & \downarrow{P'}
\end{array}
\]
Since \((G_t, N', P', i, j)\) is an unfolding of \(g_t\), it follows that \((F, N', P', H_t \circ i, K_t \circ j)\) is also an unfolding of \(g_t\). Let \(G(x, t) = (g_t(x), t), \tilde{H}(x, t) = H_t(x)\) and \(\tilde{K}(y, t) = K_t(y)\). Then we have the following commutative diagram

\[
\begin{array}{ccc}
N \times I & \xrightarrow{\tilde{H}} & N' \\
\downarrow{G} & & \downarrow{F} \\
P \times I & \xrightarrow{\tilde{K}} & P'
\end{array}
\]

So, we have that the triple \((F, S_{N'}, S_{P'})\) is a Thom stratified map, and \(i\) and \(j\) are transverse respectively to \(S_{N'}\) and \(S_{P'}\). Then taking \(g\) sufficiently close to \(f_2\), \(H_t \circ i\) and \(K_t \circ j\) are also transverse to \(S_{N'}\) and \(S_{P'}\), respectively.

It follows that these stratifications pull back to the Whitney’s stratifications \(\tilde{H}^*(S_{N'})\) and \(\tilde{K}^*(S_{P'})\) in \(N \times I\) and \(P \times I\), respectively.

Moreover, each \(N \times \{t\}, P \times \{t\}\) is transverse to \(\tilde{H}^*(S_{N'})\) and \(\tilde{K}^*(S_{P'})\), and conditions (1)-(3) are satisfied.

Then, we may apply the Thom’s second isotopy lemma (Gibson et al., [39], theorem 5.8, Chapter II) and conclude that \(f_2 = g_0\) is topologically equivalent to \(g = g_1\).

### 7.3 The geometry of topological stability

Whether \(C^0\)-stability and \(C^\infty\)-stability are equivalent notions in the nice dimensions is a question not answered by the Thom-Mather theory. The first steps towards such result appear in Robert May’s thesis [67, 68]. Mays’s results were followed by a series of papers by Damon [20, 21, 19], who proved in [21] that \(C^\infty\)-stability is equivalent to a stronger notion of \(C^0\)-stability.

Some of the ideas introduced in these papers form part of the basis for Andrew du Plessis and Terry Wall’s book on topological stability. The book, *The geometry of topological stability*, [82] published in 1995, is a deep contribution to the subject of topological stability of smooth mappings. They are motivated by the problems left unanswered in the Thom-Mather theory. One such problem is that it is very difficult to determine explicitly the Thom-Mather stratification \(\mathcal{A}^k(n, p)\) in the complement of the nice dimensions and its boundary. Another problem is that the transversality to the Thom-Mather stratification is not a necessary condition for topological stability. In fact, this follows from a combination of results of Looijenga [51] and Bruce[12] as we see in examples 7.15 and 7.16 below. du Plessis and Wall give partial answers to the following two conjectures:

**Conjecture (i)** (Conjecture 1.3 in [82]) The smooth map \(f : N \to P\) is \(W\)-strongly \(C^0\)-stable if and only if it is quasi-proper and locally \(C^0\)-stable.

Following [82], we say that a map \(f\) is *quasi-proper* if there is a neighborhood \(V\) of the discriminant \(\Delta(f)\) in \(P\) such that the restriction of \(f\) to \(f^{-1}(V)\), \(f : f^{-1}(V) \to V\), is a proper map.


Conjecture (ii) If \( N \) is compact, \( f : N \to P \) is \( C^0 \)-stable if and only if it is locally \( C^0 \)-stable.

Conjecture (iii) (Conjecture 1.4 in [82]) There exist a \( \mathcal{K} \)-invariant semi-algebraic stratification \( \mathcal{B}^k(n, p) \) of \( J^k(n, p) \setminus W^k(n, p) \) such that a smooth map \( f : N \to P \) is locally \( C^0 \)-stable if and only if, for \( k \) such that \( \text{cod} W^k(n, p) > n \), \( j^k f \) avoids \( W^k(n, p) \) and is multitransverse to \( \mathcal{B}^k(n, p) \).

We summarize now the main results of [82].

**Theorem 7.13 (Theorem 1.5, [82])**

(i) If \( f : N \to P \) is \( W \)-strongly \( C^0 \)-stable, then it is quasi-proper and locally \( C^0 \)-stable.

(ii) If \( f : N \to P \) is quasi-proper, of a finite singularity type over a neighborhood of its discriminant, and locally tamely \( P \)-\( C^0 \)-stable, then it is \( W \)-strongly \( C^0 \)-stable.

The local \( P \)-\( C^0 \)-stability is a very strong form of local \( C^0 \)-stability. We refer to [82, p. 113], for the definition of tame \( P \)-\( C^0 \)-stability.

**Theorem 7.14 (Theorem 1.6, [82])** There exist \( \mathcal{K} \)-invariant algebraic subsets \( Y^k(n, k) \) in \( J^k(n, k) \) with \( W^k(n, k) \subseteq Y^k(n, k) \), and a \( \mathcal{K} \)-invariant stratification \( \mathcal{B}^k(n, p) \) of \( J^k(n, k) \setminus Y^k(n, k) \) with the following properties:

(a) If \( f : N \to P \) is locally \( C^0 \)-stable, or if \( N \) is compact and \( f \) is \( C^0 \)-stable, then \( j^k f \) is multitransverse to \( \mathcal{B}^k(N, P) \); moreover, if \( \text{codim} Y^k(n, P) \geq n \), then \( j^k f \) avoids \( Y^k(N, P) \).

(b) If \( f : N \to P \) is such that \( j^k f \) avoids \( Y^k(N, P) \) and is multitransverse to \( \mathcal{B}^k(N, P) \), then \( f \) is locally tamely \( C^0 \)-stable.

As remarked by the authors, in the range of dimensions \( n < \text{codim} Y^k(n, P) \), the results imply that Conjectures 1.3 and 1.4, with \( W^k \) replaced by \( Y^k \), hold.

We finish this section with two examples illustrating two rather delicate questions in the theory of \( C^0 \)-stability.

**Example 7.15 (The simple elliptic singularity \( \tilde{E}_8 \))** The simple elliptic singularities \( \tilde{E}_k \) in \( \mathbb{K}^3 \), \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \), is the \( \mathcal{K} \)-unimodular family of hypersurfaces with isolated singularities defined by

\[
\tilde{E}_8: \quad f_{\lambda}(x_0, x_1, x_2) = x_0^3 + x_1^3 + x_2^3 + \lambda x_0 x_1 x_2.
\]

The family \( f_{\lambda} \) is weighted homogeneous of type \( (3, 2, 1; 6) \), then the Milnor number \( \mu(f_{\lambda}) \) is constant and equal to 10. When \( \mathbb{K} = \mathbb{C} \), it was shown by Looijenga [51] that the stable unfolding of \( f_{\lambda} \) is topologically trivial along the moduli parameter \( \lambda \).

From section 4.3, (17), it follows that the stable unfolding of \( f_{\lambda} \) can be given as

\[
F : (\mathbb{C}^3 \times \mathbb{C}^8 \times \mathbb{C}, 0) \to (\mathbb{C} \times \mathbb{C}^8 \times \mathbb{C}, 0) \quad (x, u, \lambda) \mapsto (\tilde{f}(x, u, \lambda), u, \lambda)
\]
with \( x = (x_0, x_1, x_2) \), \( u = (u_1, \ldots, u_8) \), \( \tilde{f}_1(x, u) = \tilde{f}(x, u, \lambda) \), \( \tilde{f}_1(x, 0) = f_1(x) \), and

\[
\tilde{f}(x, u, \lambda) = x_0^2 + x_1^3 + x_2^6 + \lambda x_0 x_1 x_2 + u_1 x_1 + u_2 x_2 + u_3 x_1 x_2
+ u_4 x_1^2 + u_5 x_1 x_2^2 + u_6 x_2^3 + u_7 x_1 x_2^3 + u_8 x_2^4.
\]

For all \( \lambda \) sufficiently small, including \( \lambda = 0 \),\( F_1 : (\mathbb{C}^{11}, 0) \to (\mathbb{C}^9, 0) \) is topologically stable. See Looijenga \([51]\) and Bruce \([12]\).

On the other hand, the construction of the Thom-Mather stratification \( \mathcal{A}^k(n, p) \) in \( J^k(n, p) \setminus W^k(n, p) \) as discussed in sections 7.2 and 7.3 reduces to the problem of finding a minimal Whitney stratification of jets of finite singularity type. However, Bruce proved that at \( \lambda = 0 \) the Whitney condition \((b)\) fails (see \([12]\), Proposition 2 and Example 3(a)). The failure of condition \((b)\) can be geometrically detected as follows: the number of cusps (\( A_2 \)-singularities) of the intersection of the discriminant \( \Delta(F) \) with a family of 2-planes transversal to \( \Delta(F) \) jumps from 12 to 13 at \( \lambda = 0 \). This number is an invariant of the Thom-Mather stratification \((12)\), Proposition 2).

If follows that the germ \( F_0 : (\mathbb{C}^{11}, 0) \to (\mathbb{C}^9, 0) \) is topologically stable, but \( j^kF_0 \) is not transverse to the Thom-Mather stratification.

**Example 7.16 (May \([67]\) and du Plessis and Wall \([82]\), Section 4.1)**

Let \( f : \mathbb{R} \to \mathbb{R} \) be the proper map whose graph is illustrated in Figure 5. Its singular set \( \Sigma(f) \) is \( \mathbb{Z} \subset \mathbb{R} \), and the critical values are \( F(0) = 0 \), \( f(n) = n + 1 \), for \( n > 0 \) and odd, and \( f(n) = n - 1 \) for \( n > 0 \) and even; while \( f(-x) = -f(x) \). For example, we may define, as in du Plessis and Wall \([82]\),

\[
f(x) = \begin{cases} 
  x^3 & x \in [-\frac{1}{2}, \frac{1}{2}], \\
  n + 1 - (x - n)^2 & x \in [n - \frac{1}{2}, n + \frac{1}{2}], n \in \mathbb{N}, n \text{ odd} \\
  n - 1 + (x - n)^2 & x \in [n - \frac{1}{2}, n + \frac{1}{2}], n \in \mathbb{N}, n \text{ even}
\end{cases}
\]

with \( f \) defined on the remaining intervals so that it is monotone (with \( f' \neq 0 \)) on each interval and \( C^\infty \) everywhere.

Fig. 5 \( C^0 \)-stable non transversal map.
One can see that $f$ is $C^0$-stable. However it is not transverse to the Boardman manifold $\Sigma^1$ at the origin. In fact, $f$ cannot be transverse to any invariant stratification of jet space. Thus $C^0$-stability of proper maps $f : \mathbb{R} \to \mathbb{R}$ cannot be characterized by multitransversality to any stratification.

Notice that $f$ is not locally $C^0$-stable, then it follows from Theorem 7.13(i) that $f$ is not strongly stable.

### 7.4 Notes

In the recent paper *On the smooth Whitney fibering conjecture* [74] Murolo, du Plessis and Trotman give a remarkable improvement of the first Thom-Mather isotopy theorem for Whitney stratified sets. The result follows from their proof, in the same paper, of the smooth version of the Whitney fibering conjecture for Bekka (c)-regular stratifications. The original conjecture made by Whitney in [116] in the real and complex, local analytic and global algebraic cases, was proved by Parusinski and Paunescu [77] in 2014.

As an application of the results, in section 9 of the paper, the authors give a sufficient condition for a smooth map between two smooth manifolds to be strongly topologically stable ([74, Theorem 13]).

This result in turn, implies the long-awaited improvements of Mather’s topological stability theorem, which we state below.

**Corollary 7.17 (Corollary 11, [74])** Let $f : N \to P$ be a quasi-proper smooth map of finite singularity type whose $l$-jet avoids $W^l(N, P)$ and is multi-transverse to $A^l(N, P)$. Then $f$ is strongly topologically stable.

Corollary 7.17 has the following immediate consequence.

**Corollary 7.18 (Corollary 12, [74])** The space of strong topologically stable maps is dense in the space of quasi-proper maps between two smooth manifolds.

### 8 The boundary of the nice dimensions

In this section we give a systematic presentation of the Thom-Mather singularities in the boundary of the nice dimensions (BND). Much of the material presented here is well known to experts. However, it seems that the organized presentation of the construction of the Thom-Mather stratification of $J^k(n, p)$ when $(n, p)$ is a pair in BND combined with the discussion of the properties of topologically stable mappings in these dimensions do not appear in the literature. The results come from Mather [61, 63], Damon [22, 23], du Plessis and Wall [82] and Ruas [90] and recent results by Ruas and Trivedi [88].

We only give an outline of most of the proofs but we present the full details in the case $n = p = 9$. 
We also review du Plessis and Wall main result in [83] that $C^1$-stable mappings are dense if and only if $(n, p)$ is in the nice dimensions.

### 8.1 A candidate for the Thom-Mather stratification in BND

The main reference for this section is Ruas and Trivedi [88]. We saw that a pair $(n, p)$ is in the boundary of nice dimensions if $\sigma(n, p) = n$, where $\sigma(n, p) = \text{cod } \pi^k(n, p)$, $k \geq p + 1$, and $\pi^k(n, p)$ is the smallest Zariski closed $\mathcal{K}^k$-invariant set in $J^k(n, p)$ such that its complement in $J^k(n, p)$ is the union of finitely many $\mathcal{K}^k$-orbits.

In the nice dimensions $\sigma(n, p) > n$, so it follows that the strata of the stratification of $J^k(n, p) \setminus \pi^k(n, p)$ are the simple $\mathcal{K}^k$-orbits of $\mathcal{K}$-codimension $\leq n$. However, at the BND, there are strata of codimension $n$ in $\pi^k(n, p)$; these strata cannot be avoided by transversal maps. We shall see that for all pairs $(n, p)$ in BND with the exception of the pair $(10, 7)$ these strata are unimodular strata consisting of the union of a one-parameter family of $\mathcal{K}$-orbits. When $(n, p) = (10, 7)$, surprisingly, the Thom-Mather stratification also has a bimodal strata which is the union of a two parameter family of $\mathcal{K}$-orbits. We call the pair $(10, 7)$ the exceptional pair in BND.

We recall here the notion of modality (or modularity). This notion can be defined for any geometric subgroup of $\mathcal{K}$, but here we refer to modularity for group $\mathcal{K}$.

Let $z \in J^k(n, p)$ and denote by $K^*(z)$ the union of all $\mathcal{K}^k$-orbits of codimension equal to the codimension of $\mathcal{K}^k(z)$ in $J^k(n, p)$. Suppose $K_*(z)$ is the connected component of $K^*(z)$ in which $z$ lies. Then we say that $z \in J^k(n, p)$ is $r$-modular if

$$\text{cod } K_*(z) = \text{cod } K^k \cdot z - r.$$  

We say that 1-modular jets are unimodular, 2-modular jets are bimodular and so on. Also, if the union of unimodular jets is a submanifold of $J^k(n, p)$, as it happens in our case, we call this union a unimodular stratum.

The bad set $\tilde{\Pi}^k(n, p)$ in this case is a proper Zariski closed subset of $\Pi^k(n, p)$ such that $\text{cod } \tilde{\Pi}^k(n, p) \geq n + 1$ and $\Pi^k(n, p) \setminus \tilde{\Pi}^k(n, p)$ is the union of the connected components of a unique unimodular family, while for the pair $(10, 7)$ this set is the union of the unimodular and the bimodular families.

We stratify $J^k(n, p) \setminus \tilde{\Pi}^k(n, p)$ by taking as strata the $\mathcal{K}$-orbits of the stable maps and the modular strata. We call this stratification $\Sigma^k_{\text{bnd}}(n, p)$ (see [88]).

In the global setting we have the following situation. Let $N, P$ and $J^k(N, P)$ as before. Denote by $\tilde{\Pi}(N, P)$ the subbundle of $J^k(N, P)$ with fibers $\tilde{\Pi}(n, p)$. Then the codimension of $J^k(N, P) \setminus \tilde{\Pi}(N, P)$ is equal to the codimension of $\tilde{\Pi}(n, p)$ in $J^k(n, p)$. Moreover, the stratification $\Sigma^k_{\text{bnd}}(n, p)$ induces a stratification on $J^k(N, P) \setminus \tilde{\Pi}(N, P)$, denoted by $\Sigma^k_{\text{bnd}}(N, P)$.

The following result appears in [88].

**Theorem 8.1 (Ruas and Trivedi, [88], Theorem 3.1)** The set of maps $f : N \to P$ such that $j^k f(N) \cap \tilde{\Pi}(N, P) = \emptyset$ and $j^k f$ is transverse to the strata of $\Sigma^k_{\text{bnd}}(N, P)$ is open in $C^\infty(N, P)$ with the Whitney topology.
The (a) regularity of $\Sigma^k_{bnd}(N, P)$ follows from the above result and the Main Theorem in Trotman [100].

**Corollary 8.2** The stratification $\Sigma^k_{bnd}(n, p)$ is (a)-regular.

We prove in Theorem 8.4 that maps transverse to $\Sigma^k_{bnd}(N, P)$ are Thom-Mather maps for any pair $(n, p)$ in BND.

### 8.2 The unimodular strata in BND

The results in this section are local and hold for map-germs $f : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $f \in \mathcal{E}^p_0$ or $f \in \mathcal{O}^p_0$. From Mather’s calculations in [63], it follows that the following pairs lie in the boundary of the nice dimensions:

(i) $n \leq p$:

1. The case $\sigma(n, p) = 6(p - n) + 9$ for $3 \geq p - n \geq 0$ and $n \geq 4$ or $n = 3$, gives $(n, p) \in \{(9, 9), (15, 16), (21, 23), (27, 30)\}$.
2. The case $\sigma(n, p) = 6(p - n) + 8$ for $p - n \geq 4$ and $n \geq 4$, gives $(n, p) \in \{(6t + 2, 7t + 1) ; t \geq 5\}$.

(ii) $n > p$:

1. The case $\sigma(n, p) = 9$ for $n = p + 1$, gives $(n, p) = (9, 8)$.
2. The case $\sigma(n, p) = 8$ for $n = p + 2$, gives $(n, p) = (8, 6)$.
3. The case $\sigma(n, p) = n - p + 7$ for $n \geq p + 3$ gives $(n, p) \in \{(10 + k, 7) ; k \geq 0\}$.

The strategy to find the strata of $\Sigma_{bnd}(n, p)$ has the following steps:

1. Inspecting the classification of the local algebras $Q(z)$, $z \in J^k(n, p)$, such that $K$-cod$(z) \leq n$. By Mather’s results these algebras are simple and for each such algebra $Q(z)$ there exists a stable germ $f : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$, such that $Q(f) = Q(z)$;
2. Listing the unimodular algebras of $K$-codimension $n + 1$, whose union makes the unimodular strata of the stratification;
3. Excluding the existence of bimodular strata of codimension $n$ for pairs $(n, p)$ in BND except $(10, 7)$. For $(n, p) = (10, 7)$ we include the classification of the bimodular strata.

A detailed discussion of simple and unimodular algebras appears in Chapter 8 of the book of du Plessis and Wall [82]. For the convenience of the reader we give the precise references of the classifications. First a word about the notation. We use mainly Thom’s notation, and the relevant here are the first and second order the Thom-Boardman symbols $\Sigma^r$ and $\Sigma^{r,s}$, respectively, $r = 1, 2, 3, 4$. Mather’s adaptation $\Sigma^{r(s)}$ also appears, as it is useful for 2-jet classification. A germ $f$ in $\Sigma^r$ may be regarded as an unfolding of a germ $f_0$ with rank zero and source dimension $r$. When we look at the second degree terms, the notation $s$ in $\Sigma^{r(s)}$ indicates how many independent components the 2-jet of $f_0$ has.

We first describe the unimodular strata in the boundary of the nice dimensions, based on the presentation in Ruas and Trivedi [88].
8.2.1 Case 1: $n \leq p$

(1) $(n, p) = (9, 9)$

The first unimodular family in this case is the one parameter family of type $\Sigma^{3,0}$ ($\Sigma^{3(3)}$ in Mather’s notation) introduced in section 6:

$$f_1: (\mathbb{K}^3, 0) \rightarrow (\mathbb{K}^3, 0)$$

$$(x, y, z) \mapsto (x^2 + \lambda yz, y^2 + \lambda xz, z^2 + \lambda xy)$$

with $\lambda \neq 0, -2, 1$.

Calculating the $K$-tangent space of $f_1$ we find that $\mathcal{K}\text{-cod}(f_1) = 10$, for $\lambda \neq 0, -2, 1$. The sets $(-\infty, 0), (0, -2), (-2, 1), (1, \infty)$ parametrize orbits in the connected components of the unimodular strata of codimension 9.

(2) $(n, p) = (15, 16)$

The unimodular stratum in these dimensions is related to the moduli stratum in dimensions $(9, 9)$ in the following way. From a result of Serre and Berger (see Eisenbud [31], Proposition 2) it follows that for analytic map-germs $f : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^n, 0)$ the class of the Jacobian $J(f)$ is a non-zero element in the local algebra $Q(f)$. Moreover, the ideal generated by $J(f)$ in this algebra is the unique minimal non-zero ideal in $Q(f)$. It also follows that the residue class of $J^2(f)$ in $Q(f)$ is zero.

The unimodular family here is

$$f_{1,1}: (\mathbb{K}^3, 0) \rightarrow (\mathbb{K}^4, 0), \quad f_{1,1}(x, y, z) = (f_1(x, y, z), J(f_1)(x, y, z)),$$  \hspace{1cm} (23)

where $f_1$ is the map given in (22) and $J(f_1)(x, y, z) = xyz$. The following holds

$$\mathcal{K}\text{-cod}(f_{1,1}) = \mathcal{K}\text{-cod}(f_1) + (\delta(f_1) - 2) = 16$$

where $\delta(f_1) = \dim_{\mathbb{R}} Q(f_1) = 8$. The unimodular stratum in $J^k(15, 16)$, $k \geq 3$ is the union of all corank 3 $k$-jets $z \in J^k(15, 16)$, $\mathcal{K}$-equivalent to a suspension of $f_{1,1}$.

(3) $(n, p) = (21, 23)$

In this case the unimodular family is

$$f_{2,1}: (\mathbb{K}^3, 0) \rightarrow (\mathbb{K}^5, 0), \quad f_{2,1}(x, y, z) = (f_{1,1}(x, y, z), 0).$$  \hspace{1cm} (24)

(4) $(n, p) = (27, 30)$

The unimodular family here is

$$f_{3,1}: (\mathbb{K}^3, 0) \rightarrow (\mathbb{K}^6, 0), \quad f_{3,1}(x, y, z) = (f_{2,1}(x, y, z), 0).$$  \hspace{1cm} (25)

Remark 8.3 The following formula holds (du Plessis and Wall [82], Chapter 8)

$$\mathcal{K}\text{-cod}(f_{i,1}) = \mathcal{K}\text{-cod}(f_1) + (p - n)(\dim_{\mathbb{R}} Q(f_1) - 2),$$

for $i = 1, 2, 3, p = n + i$. 

(5) \((n, p) = (6t + 2, 7t + 1)\) for \(t \geq 5\)

When \(t = 5\) the unimodular stratum is defined by

\[
f_A : (\mathbb{P}^4, 0) \to (\mathbb{P}^8, 0), f_A(x, y, z, w) = (u_1, u_2, \ldots, u_8)
\]

where

\[
egin{align*}
  u_1 &= x^2 + y^2 + z^2 \\
  u_2 &= y^2 + \lambda z^2 + w^2 \\
  u_3 &= xy \\
  u_4 &= xz \\
  u_5 &= xw \\
  u_6 &= yz \\
  u_7 &= yw \\
  u_8 &= zw
\end{align*}
\]

8.2.2 Case 2: \(n > p\)

(6) \((n, p) = (8, 6)\)

The smallest pair \((n, p)\) with \(n > p\) in the boundary of the nice dimensions is \((8, 6)\). The unimodular stratum is given by the following one-parameter family of maps

\[
f_A : (\mathbb{P}^4, 0) \to (\mathbb{P}^2, 0),
f_A(x, y, z, w) = (x^2 + y^2 + z^2, y^2 + \lambda z^2 + w^2), \lambda \neq 0, 1.
\]

(7) \((n, p) = (9, 8)\)

The unimodular family here is

\[
f_A : (\mathbb{P}^2, 0) \to (\mathbb{P}, 0),
f_A(x, y) = x^4 + y^4 + \lambda x^2 y^2, \lambda \neq \pm 2.
\]

(8) \((n, p) = (10 + k, 7)\) for \(k \geq 0\)

In this case, the unimodular family is

\[
f_A : (\mathbb{P}^{4+k}, 0) \to (\mathbb{P}, 0),
f_A(x, y, z, w_0, \ldots, w_k) = x^3 + y^3 + z^3 + \lambda xyz + \sum_{i=0}^{k} \delta_i w_i^2,
\]

for \(\delta = \pm 1, i = 0, \ldots, k, \lambda^3 \neq -1\).

The pair \((n, p) = (10, 7)\) is the exceptional pair in BND. It follows from Wall [109] that the following two parameter moduli family of \(\Sigma^5\) singularities has codimension \(n = 10\), providing for this pair of dimensions a new relevant strata.

\[
f_A : (\mathbb{P}^3, 0) \to (\mathbb{P}^2, 0),
f_A(x) = \left( \sum_{i=1}^{5} a_i x_i^2, \sum_{i=1}^{5} b_i x_i^2 \right), a_i b_j - a_j b_i \neq 0, i \neq j.
\]
Theorem 8.4 For each pair \((n, p)\) in the boundary of the nice dimensions the following hold:

(a) If \((n, p) \neq (10, 7)\) the strata of \(\Sigma_{\text{bnd}}^k(n, p)\) are are the \(\mathcal{K}^k\)-orbits of the stable germs of \(\mathcal{K}\)-codimension \(\leq n\) and the unimodular strata of codimension \(n\) defined by the connected components of the unimodular families described in 8.2.1 and 8.2.2. If \((n, p) = (10, 7)\), besides the unimodular strata defined in 8.2.2(8), there is an exceptional bimodular strata as defined in (26).

(b) Maps \(f : N \to P\) such that \(j^kf\) is transverse to the strata of \(\Sigma_{\text{bnd}}^k(n, p)\) are Thom-Mather maps for any pair \((n, p)\) in BND.

Proof The proof consists on a careful inspection of the tables of simple and unimodular singularities in order to list the relevant strata and to verify that the codimension of the set \(\tilde{\Pi}^k(n, p)\), \(k \geq p + 1\) is greater than or equal to \(n + 1\). We give an outline of the proof.

I. \(n \leq p\)

For \((n, p) \in \{(9, 9), (15, 16), (21, 23), (27, 30)\}\) the relevant Boardman types are \(\Sigma^1, \Sigma^{2,0}, \Sigma^{2,1}\) and \(\Sigma^3\). We first analyze the pair \((9, 9)\).

Case (1) \((n, p) = (9, 9)\)

All singularities of type \(\Sigma^1\) and \(\Sigma^{2,0}\) are simple. A complete list of strata of type \(\Sigma^{2,1}\) has been given by Dimca and Gibson [28]. See also Table 8.4 in du Plessis and Wall [82].

The first unimodular family of type \(\Sigma^{2,1}\) is

\[
I_{2,3} : (x^2 - \eta y^4, xy^3 + cy^5), \quad c^2 \neq 0, \eta.
\]

(27)

It follows that the \(\mathcal{K}\)-codimension of each orbit is 12, the unimodular stratum has codimension 11, so that this family does not appear generically when \(n = p = 9\). As a consequence, the relevant \(\Sigma^{2,1}\) strata in this case are simple \(\mathcal{K}\)-orbits. Notice that \(\text{cod} \Sigma^{2,2}(9, 9) \geq 10\) and then the \(\Sigma^{2,2}\) singularities do not appear generically in \(J^k(9, 9)\).

The next Boardman symbol is \(\Sigma^3\), and as we saw in 8.2.1, the relevant strata are the connected components of the unimodular family (1).

We list all the strata in Table 2.

The set \(\tilde{\Pi}^k(9, 9)\) is the finite union of the following Zariski closed sets of codimension \(\geq 10\) in \(J^k(9, 9)\), \(k \geq 10\):

\[
\tilde{\Pi}^k(9, 9) = \tilde{\Pi}^k_1 \cup \tilde{\Pi}^k_2 \cup \tilde{\Pi}^k_{j \geq 3}
\]

where

\[
\tilde{\Pi}^k_1 = \{\sigma \in J^k(9, 9), \sigma \in \Sigma^1, \mathcal{K}^k\text{-cod}(\sigma) \geq 10\}
\]

\[
\tilde{\Pi}^k_2 = \{\sigma \in J^k(9, 9), \sigma \in \Sigma^2, \mathcal{K}^k\text{-cod}(\sigma) \geq 10\}
\]

\[
\tilde{\Pi}^k_{j \geq 3} = \{\sigma \in J^k(9, 9), \sigma \in \Sigma^j, j \geq 3, \mathcal{K}^k\text{-cod}(\sigma) \geq 11\}
\]

Cases (2) \((15, 16)\); (3) \((21, 23)\); (4) \((27, 30)\)
The singularities of type \( \Sigma^1 \) and \( \Sigma^{2,0} \) are simple. The classification of the singularities of type \( \Sigma^{2.1} \) and their invariants in these cases can be found in Tables 8.7, 8.8 and 8.9 of \[82\]. The first unimodular family of type \( \Sigma^{2.1} \), when \( n < p \), is \( \overline{D}_{3,5} \) (also denoted by \( \overline{J}_{2,3,5,5} \) in \[82\]).

The normal forms are

\[
\begin{align*}
  f_{14}(x, y) &= (x^2 \pm y^4, xy^3 + cy^5, y^6) \\
  f_{24}(x, y) &= (x^2 \pm y^4, xy^3 + cy^5, y^6, 0) \\
  f_{34}(x, y) &= (x^2 \pm y^4, xy^3 + cy^5, y^6, 0, 0)
\end{align*}
\]

From (27), we get

\[
\mathcal{K}\text{-cod}(f_{i4}) = \mathcal{K}\text{-cod}(f_i) + \dim \mathbb{R}Q(f_i) - 2,
\]

for \( i = 1, 2, 3 \) where

\[
f_i(x, y) = (x^2 \pm y^4, xy^3 + cy^5).
\]

Then \( \mathcal{K}\text{-cod}(f_{i4}) = 12 + i(10 - 2), i = 1, 2, 3 \) and these singularities do not appear generically in BND. As in Case (1), for \( n = 9 + 6i \), \( i = 1, 2, 3 \) with the help of Tables 8.7, 8.8, 8.9 and 8.11 in \[82\] we can verify that the strata of type \( \Sigma^1, \Sigma^{2,0}, \Sigma^{2.1} \) and \( \Sigma^{2,2} \) are \( \mathcal{K}\)-orbits of \( \mathcal{K}\)-codimension \( \leq 9 + 6i \), \( i = 1, 2, 3 \) and the unimodular strata defined in (23), (24) and (25). Moreover, \( \cod \Pi^k(n, p) \geq n + 1 \).

**Cases (5) \((6t + 2, 7t + 1), t \geq 5\)**

The relevant Boardman types here are \( \Sigma^1, \Sigma^{2,0}, \Sigma^{2.1}, \Sigma^{2.2}, \Sigma^3 \) and \( \Sigma^4 \). As before, \( \Sigma^1, \Sigma^{2,0} \) are simple, and the moduli strata of type \( \Sigma^{2.1} \) has normal form \( f_{4}: (\mathbb{K}^2, 0) \rightarrow (\mathbb{K}^{t+1}, 0), t \geq 5, \)

\[
f_{34}(x, y) = (x^2 \pm y^4, xy^3 + cy^5, y^6, 0, \ldots, 0),
\]

where \( f_{4}(x, y) = (x^2 \pm y^4, xy^3 + cy^5) \). Since \( \mathcal{K}\text{-cod}(f_{4}) = 12 \), then \( \mathcal{K}\text{-cod}(f_{34}) \geq 12 + (t - 1)(10 - 2) = 4 + 8t > 6t + 2 \), and it follows that this family is not generic when \( (n, p) = (6t + 2, 7t + 1), t \geq 5 \).

The \( \Sigma^{2.2} \) germs of order 3 appear in du Plessis and Wall \[82\], Section 8.5, Tables 8.10 and 8.11. The type \( \Sigma^{2.2} \) is subdivided (see \[82\]) into types \( \Sigma^{2.2(j)} \),
where \( j \) is the rank of the kernel of the third intrinsic derivative. It follows that \( \text{codim} \Sigma_{2,2}(j) = 6e + 10 + j(e + j - 2) \), where \( e = p - n \). With a simple calculation we get that the relevant are \( j = 0, 1 \). Based on Table 8.10 of [82] we can verify that \( \bar{\Pi}(6t + 2, 7t + 1) \) contains the closure of the \( \mathcal{K} \)-orbit \((x^3 \pm xy^2, x^2y, y^3, 0, 0, 0)\) (type \( E-\mathcal{Q}^1_1 \)).

Germs of type \( \Sigma^n \), \( n = 3, 4 \) are classified in [82], Section 8.6.

For \( n = 3 \), the more delicate analysis is that of singularities of type \( \Sigma^{3(2)} \). Based on Tables 8.15, 8.17 and 8.20 in [82], it follows that the moduli does not occur in strata of codimension \( \leq 6t - 2, t \geq 5 \). It follows then that \( \bar{\Pi}(6t + 2, 7t + 1) \cap \Sigma^{3(2)} \) is the closure of \( \mathcal{K} \)-orbits of codimensions \( > 6t + 2 \).

For the singularities of type \( \Sigma^{3(3)} \), the best algebra of this type is the unimodular family whose normal form is \( f_{\delta,4} = (f_{3,4}, 0) \), where \( f_{3,4} \) is as in 8.2.1 (4).

We know that \( \mathcal{K} \)-cod\( (f_{3,4}) = 28 \) and \( \delta(f_{3,4}) = 7 \), so that \( \mathcal{K} \)-cod\( (f_{3,4}) = 28 + 6 = 34 > 32 \). As the family is 1-modal it follows that the codimension of the stratum is 33, then this singularity does not occur generically in (32, 26). It is easy to extend this argument to all pairs \((6t + 2, 7t + 1), t > 5 \).

The first singularity of type \( \Sigma^4 \) in (32, 36) is the unimodular family 8.2.1 (5). The \( \mathcal{K} \)-cod \( (f_{1}) = 33 \) and the codimension of the stratum is 32.

It follows from our description that \( \text{codim} \bar{\Pi}(6t + 2, 7t + 1) \geq 6t + 2 \).

**Cases** (6) (8, 6); (7) (10 + \( k \), 7) \( k > 0 \)

These cases are simpler, since the deformations of the algebras have to be a simple function singularity, i.e., a singularity from Arnold’s list of simple singularities of functions[3]. We can obtain the complete list from the adjacencies of simple and unimodular singularities from Arnold’s [5].

The exceptional pair (10, 7) has two modular strata

(i) The unimodular family \( f_{11}(x, y, z, w) = x^3 + y^3 + z^3 + \lambda xyz + w^2 \) with \( \mathcal{K} \)-cod\( (f_{11}) = 11 \) and codimension of the stratum equal to 10.

(ii) The bimodular family \( f_{3,4}(x) = (\sum_{i=1}^{5} a_{ij}x_i^2, \sum_{j=1}^{5} a_{ij}x_j^2), a_{ij}b_j - a_{ji}b_i \neq 0, 1 \leq i, j \leq 5, i \neq j \).

### 8.3 Topological triviality of unimodular families

Results on \( C^0,\mathcal{A} \)-triviality of the unimodular families of mappings appeared few years after Mather’s theorem, due mainly to Eduard Looijenga [51, 52] and Jim Damon [22, 23].

In the 1977 paper Looijenga obtained explicit examples of topologically stable map-germs which are not analytically stable. He studied the simple elliptic singularities:

- \( \tilde{E}_6 : f(z_0, \ldots, z_n) = z_1(z_1 - z_0)(z_1 - \lambda z_0) + z_0z_2^2 + Q(z_3, \ldots, z_n), (n \geq 2) \);
- \( \tilde{E}_7 : f(z_0, \ldots, z_n) = z_1z_0(z_1 - z_0)(z_1 - \lambda z_0) + Q(z_2, \ldots, z_n), (n \geq 1) \);
- \( \tilde{E}_8 : f(z_0, \ldots, z_n) = z_1(z_1 - z_0^2)(z_1 - \lambda z_0^2) + Q(z_2, \ldots, z_n), (n \geq 1) \).
where $Q$ is any nondegenerate quadratic form. He proved that two simple-elliptic singularities in the same family have topologically equivalent semi-universal deformations. As a consequence he obtained the $C^0$-$\mathcal{A}$-triviality of the stable unfolding of these singularities along the moduli parameter.

**Remark 8.5** The family $\tilde{E}_6$ is analytically equivalent to the family 8.2.2 (8) and $\tilde{E}_7$ is analytically equivalent to the family 8.2.2 (7). The family $\tilde{E}_8$ does not occur generically in BND.

Looijenga’s approach to this problem is based on the weighted homogeneity of the germs together with algebraic calculations to solve a localized form of equation for infinitesimal $C^1$-or analytic triviality.

Wirthmüller [119] extended Looijenga’s results proving the topological triviality of the versal unfolding of non-simple hypersurfaces germs along the Hessian deformation parameter. These results were further extended by J.Damon [22, 23] for unfoldings $F$ of “non-negative weight” of a weighted homogeneous polynomial germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$. His main result applies to a large class of unimodular families, which includes all unimodular families in the boundary of the nice dimensions.

**Theorem 8.6 (Damon, [22])** If $f$ is a weighted homogeneous $\mathcal{A}$-finitely determined germ, then any polynomial unfolding of $f$ of non-negative weight is topologically trivial.

Damon’s result apply to weighted homogeneous $\mathcal{A}$-finitely determined germs $f$ of type $(w_1, \ldots, w_n; d_1, d_2, \ldots, d_p)$ and their unfoldings of weighted degree equal to or higher than the weighted degree of $f$.

The unimodular families in the boundary of the nice dimensions satisfy an even stronger condition: up to the addition of a quadratic form, the $\mathcal{K}$-orbits $\mathcal{K}(f_\lambda)$ in 8.2.1 and 8.2.2 have a homogeneous normal form; in other words we can take weights $w_1 = w_2 = \cdots = w_n = 1$, and if we write $f_\lambda : (\mathbb{K}^t, 0) \to (\mathbb{K}^t, 0)$, $f_\lambda = (f_{1,\lambda}, f_{2,\lambda}, \ldots, f_{t,\lambda})$, then $f_{i,\lambda}$ is homogeneous of degree $d_i$, $i = 1, \ldots, t$. As in section 4.3 let

$$N(f_\lambda) = \frac{\Theta(f_\lambda)}{TK_\mathcal{E}(f_\lambda) + \omega f_\lambda(\Theta_\mathcal{E})},$$

Notice that since $f_\lambda$ has rank 0, it follows that $N(f_\lambda) = \frac{\Theta(f_\lambda)}{TK_\mathcal{E}(f_\lambda)}$.

Let $J(f_\lambda)$ be the ideal generated by the $t \times t$ minors of $f_\lambda$ and let $I(f_\lambda) = J(f_\lambda) + f_\lambda^* (\mathcal{M}_p)$. Notice that when $s < t$, $I(f_{i,\lambda}) = f_{i,\lambda}^* (\mathcal{M}_p)$.

**Lemma 8.7(a)** If

$$I_1^1 = \langle x^2 + \lambda yz, y^2 + \lambda xz, z^2 + \lambda xy, xyz \rangle, \quad \lambda \neq -2, 0, 1$$

and

$$I_1^2 = \langle x^2 + y^2 + z^2, y^2 + \lambda z^2 + w^2, xy, xz, xw, yz, yw \rangle, \quad \lambda \neq 0, 1$$
then \( I_{\lambda}^i \supseteq M^3, \ i = 1, 2 \).

(b) For each normal form (1) to (5) in 8.2.1 and (6) in 8.2.2, \( TK_\epsilon(f_A) \supseteq M^1 \Theta(f_A) \).

(c) For the normal form (8) in 8.2.2, \( J(f_A) \supseteq M^4 \).

(d) For the normal form (7) in 8.2.2, \( J(f_A) \supseteq M^5 \).

**Proof** (a), (c) and (d) follows from easy calculations, using the corresponding normal forms.

To prove (b) notice that if \( I(f_A) = J(f_A) + f_A^*(M_t) \), it follows that \( I(f_A) \Theta(f_A) \subset TK_\epsilon(f_A) \), and the result follows from (a).

With the help of the above Lemma it is an easy task to find, for each normal form, (1) to (5) in 8.2.1 and (6) to (8) in 8.2.2, a monomial basis for the normal space \( N(f_A) \), so that we can write

\[
N(f_A) \cong \mathbb{K}\{\sigma_1, \sigma_2, \ldots, \sigma_r, \sigma_m\}
\]

where the \( r \) generators \( \sigma_j = (\sigma_{1j}, \sigma_{2j}, \ldots, \sigma_{tj}) \in \Theta(f_A), \ j = 1, \ldots, r \) have the following property: each coordinate \( \sigma_{ij}, i = 1, \ldots, t \) of \( \sigma_j \) satisfies the following condition

\[
\text{degree} \sigma_{ij} < \text{degree} f_i, \ i = 1, \ldots, t, \ j = 1, \ldots, r.
\]

The generator \( \sigma_m = (\sigma_{1m}, \sigma_{2m}, \ldots, \sigma_{tm}) \) is the direction of the modulus and the degree \( \sigma_{im} = \text{degree} f_i \) for \( i = 1, \ldots, t \).

For each \( \lambda = \lambda_0 \), the stable unfolding of \( f_{\lambda_0} \) is the map-germ

\[
F : (\mathbb{K}^s \times \mathbb{K}^r \times \mathbb{K}, 0) \to (\mathbb{K}^s \times \mathbb{K}^r \times \mathbb{K}, 0)
\]

\[
(x, u, \lambda) \mapsto (\tilde{f}(x, u, \lambda), u, \lambda),
\]

\( x = (x_1, \ldots, x_s), u = (u_1, \ldots, u_r) \), and

\[
\tilde{f}(x, u, \lambda) = f(x, \lambda_0) + \sum_{j=1}^r u_j \sigma_j(x) + \lambda \sigma_m(x).
\]

For each \( \lambda_0 \), with the exception of a finite number of exceptional values, we obtain the normal form of the unimodular topologically stable singularity:

\[
F_{\lambda_0} : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0),
\]

with

\[
F_{\lambda_0}(x, u) = (\tilde{f}_{\lambda_0}(x, u), u),
\]

(30)

where

\[
\tilde{f}_{\lambda_0}(x, u) = f(x, \lambda_0) + \sum_{j=1}^r u_j \sigma_j(x). 
\]

(31)

and \( n = s + r, \ p = t + r \).
Remark 8.8 Notice that $F_{b_0}$ is unfolding of $f_{b_0}(x)$ by terms $\sigma_j$ of smaller degree. Damon’s in [22] refers to $F_{b_0}$ as unfolding of negative weight of $f_{b_0}$ (see section 2 in Damon [23]).

A similar construction can be made for the exceptional pair $(n, p) = (10, 7)$. The bimodal family $f_t = (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}^2, 0)$, $t = \lambda_1, \lambda_2$ has a normal space

$$N(f_t) \cong \mathfrak{R}\{\sigma_1, \ldots, \sigma_r, \sigma_m^1, \sigma_m^2\},$$

where $\{\sigma_m^1, \sigma_m^2\}$ generates the bimodal plane and degree $\sigma_m^i = \text{degree } f_t = 2$, $i = 1, 2$. The normal form of the topologically stable singularity is given by (30).

We display these normal forms in tables below. To simplify notation we denote $A$ as unfolding of negative weight of $x$. Notice that, with convenient choices of weights for the variables $u_1, \ldots, u_r$, each normal form $F_{b_0}$ is a weighted homogeneous germ. To apply Damon’s result (Theorem 8.6) we need to show that $F_{b_0}$ is $\mathcal{A}$-finitely determined. The relevant property of $F_{b_0}$ is that the $\mathcal{A}$-orbit is open in the $\mathcal{K}$-orbit, as we now explain.

**Definition 8.9** Let $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ be a $\mathcal{A}$-finitely determined map-germ. The $\mathcal{A}$-orbit of $f$ is open in the $\mathcal{K}$-orbit of $f$ if $T\mathcal{A}(f) = T\mathcal{K}(f)$.  

---

### Table 3

| $(n, p)$ | $f = (f_i)$ | Unfolding monomials $< m$ | $r$ | $\sigma_m^i$ |
|----------|-------------|---------------------------|-----|-----------|
| $(9, 9)$ | $f_t = (x^2 + y^2 + z^2 + w^2, xz, xy, yz, zw, 0, \ldots, 0)$ | $\{x, y\}e_1, \{z, x\}e_2$ | $6$ | $yze_1 + xze_1 + yxe_1$ |
| $(15, 16)$ | $f_t = (f_t, 0)$ | $\{y, z\}e_1, \{x, z\}e_2, \{x, y\}e_3, \{x, y, z\}e_4, \{y, x, z, y\}e_1, \{y, x, z, x, y\}e_3$ | $12$ | $yze_1 + xze_1 + yxe_1$ |
| $(21, 23)$ | $f_t = (f_t, 0)$ | $\{y, z\}e_1, \{x, z\}e_2, \{x, y\}e_3, \{x, y, z\}e_4, \{y, x, z, y\}e_1, \{y, x, z, x, y\}e_3 \{x, y, z\}e_4$ | $18$ | $yze_1 + xze_1 + yxe_1$ |
| $(27, 30)$ | $f_t = (f_t, 0)$ | $\{y, z\}e_1, \{x, z\}e_2, \{x, y\}e_3, \{x, y, z\}e_4, \{y, x, z, y\}e_1, \{y, x, z, x, y\}e_3 \{x, y, z\}e_4$ | $24$ | $yze_1 + xze_1 + yxe_1$ |

Table 3 $6(p - n) + 9 = n, 3 \leq p - n \leq 0$

### Table 4

| $(n, p)$ | $f = (f_i)$ | Unfolding monomials $< m$ | $r$ | $\sigma_m^i$ |
|----------|-------------|---------------------------|-----|-----------|
| $(6s + 2, 7s + 1)$ | $f_t := (x^2 + y^2 + z^2, x^2 + z^2 + w^2, x, y, x, x, w, y, y, w, z, w, 0, \ldots, 0)$ | $\{x, y\}e_1, \{z, x\}e_2, \{x, y, z, w\}e_{3i}$ | $6s - 2$ | $z^2e_2$ |
| $s \geq 5$ | $t = s + 3, s \geq 5$ | $s - 5$ | $0 \leq i \leq s, s \geq 5$ | |

Table 4 $6(p - n) + 8, p - n \geq 4, n \geq 4$
Table 5 $n > p$

| exceptional pair | complex normal form | Unfolding monomials $< m$, $m = 2$ | $p$ | $\sigma_m$ |
|-------------------|---------------------|-----------------------------------|-----|---------|
| $(10, 7)$         | $f_{1,4} = (\rho(x), q(x))$ | $\rho(x) = \sum_{i=1}^{4} x_i^4$ | $x_2^2 + A_1 x_2^3 + A_2 x_2^4 + x_2^5$ | $(x_2, x_1, x_4, x_3) e_1$ | $(1, 2)$ |
|                   |                     | $A_i \neq 0, 1 \leq i \leq 2$ | | $(x_2, x_4) e_2$ | |

Table 6 Bimodular strata

Given a pair $(n, p)$ and a $\mathcal{K}^k$-orbit in $J^k(n, p)$, if this $\mathcal{K}^k$-orbit does not contain an infinitesimally stable map-germ $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$, we can ask whether there exist $f$ such that $\mathcal{A}^k(f)$ is open in $\mathcal{K}^k(f)$. This was introduced by Rua [90] as an approach to the $\mathcal{A}$-classification problem. The non existence of $f$ with such property implies that all map-germs $f \in \mathcal{K}^k$ are non-simple. The following necessary and sufficient condition for the existence of an open orbit in $\mathcal{K}(f)$ was given in [90] (see also Rieger and Rua [85]).

**Proposition 8.10 (Rua, [90],Theorem 5.1, Rieger and Rua, [85], Prop.4.6)***

Let $f : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ be a $\mathcal{K}$-finitely determined germ and denote by $\{v_1, v_2, \ldots, v_r\}$ a basis for $N = \frac{T\mathcal{A} f + f^* M_p}{\theta f + f^* M_p \theta f}$. The $\mathcal{A}$-orbit of $f$ is open in the $\mathcal{K}$-orbit of $f$ if $f_i v_j \in T\mathcal{A} f$, mod($f^* M_p \theta f$) for $i = 1, \ldots, p$, $j = 1, \ldots, r$.

To apply proposition 8.10 to the unimodular singularities at BND we introduce the following notation, where $F_A$ is as in equation (30).

Let

$T_{F_A} = F_A^*(M_p)\{\sigma_1, \sigma_2, \ldots, \sigma_r\} + t F_A(M_{p+r} \Psi_{\sigma r}) + \omega F_A(M_{p+r} \Psi_{\sigma r})$.

This is a $F_A^*(E_{\sigma r})$-submodule of $\Psi_{F_A}$ consisting of elements of $T\mathcal{A} F_A$ with zero components in the $R'$ direction (see section 4.2).

**Corollary 8.11** Let $F_A$ as in (30). Then $\mathcal{A}(F_A)$ is open in $\mathcal{K}(F_A)$ is and only if

(i) $(F_A) \cdot \sigma_m \subset T_{F_A} + F_A^*(M_p^2 \Psi_{F_A})$, $i = 1, \ldots, t$.

(ii) $u_j \cdot \sigma_m \subset T_{F_A} + F_A^*(M_p^2 \Psi_{F_A})$, $i = 1, \ldots, r$.

**Remark 8.12** Taking the quotient $\frac{T_{F_A}}{M_r T_{F_A}}$ in condition (i) of Corollary 8.11, we get

$(0) \quad (F_A) \cdot \sigma_m \subset \frac{T_{F_A}}{M_r T_{F_A}} \simeq f^*(m_1)\{\sigma_1, \ldots, \sigma_r\} + t f_A(m_\Theta) + \omega f_A(M_\Theta)$. (32)
The $f^*(\theta_i)$-module $\frac{T_{\mathcal{A}}}{\mathcal{M}_{\mathcal{A}}F_{\mathcal{A}}}$ is $im(z_0)$ in Damon’s notation (see definition of $z_0$ in section 1 of Damon [23]).

Condition (ii) is a necessary condition for the property $T\mathcal{A}(F_{\lambda}) = T\mathcal{K}(F_{\lambda})$ to hold.

We collect in the following proposition the relevant properties of $F_{\lambda_0}$.

**Theorem 8.13** Let $(n, p)$ be a pair in BND and $F_{\lambda_0} : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$ the unimodular map-germ as in (30). Then for all $\lambda_0 \in \mathbb{R}$, except a finite number of exceptional values the following hold:

(a) $F_{\lambda_0}$ is $\mathcal{A}$-finitely determined.
(b) $\mathcal{A}c$-cod $F_{\lambda_0} = 1$.
(c) The $\mathcal{A}$-orbit of $F_{\lambda_0}$ is open in $\mathcal{K}(F_{\lambda_0})$.

**Proof** First notice that (c) $\Leftrightarrow$ (b) $\Rightarrow$ (a). In fact if (c) holds, $T\mathcal{A}(F_{\lambda_0}) = T\mathcal{K}(F_{\lambda_0})$.

We saw that $\mathcal{K}$-cod$(F_{\lambda_0}) = n + 1$. Now, for any $\mathcal{A}$-finitely determined $f : (\mathbb{R}^n, S) \rightarrow (\mathbb{R}^p, 0)$, $S = \{x_1, \ldots, x_s\}$, the following formula due to L. Wilson [118] holds (see Rieger [84] for a proof):

$$\mathcal{A}c\text{-cod}(f) = \mathcal{A}\text{-cod}(f) + s(p - n) - p.$$  

Applying this formula with $s = 1$, it follows that $\mathcal{A}c\text{-cod}(F_{\lambda_0}) = 1 \Leftrightarrow \mathcal{A}\text{-cod}(f) = n + 1$ and the equivalence (c) $\Leftrightarrow$ (b) follows from this. It is also clear that (b) $\Rightarrow$ (a).

We now want to verify (c) (or equivalently (b)). For each normal form $F_A : (\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^p, 0)$, with $F_A(x, u) = (\tilde{f}_A(x, u), u)$, $\tilde{f}_A(x, u) = f_A(x) + \sum_{j=1}^{r} u_j \cdot \sigma_j(x)$, degree$(\sigma_j) < \text{degree}(f_A)$, $j = 1, \ldots, r$.

To verify (c), we verify condition (i) and (ii) in Corollary 8.11 to $F_A$. We do it case by case, collecting calculations that appeared previously in the literature.

(1) Cases $(n, p) = \{(9, 9), (15, 16), (21, 23), (27, 30)\}$.

These were solved by Damon in Example 2 and Proposition 8.2, §8 in [23].

Notice that Damon uses Wall’s normal form for the $\Sigma^{3,0}$ unimodular family

$$f_A = (2xz + y^2, 2yz, x^2 + 3gy^2 - cz^2), c \neq 0, \quad c + 9g^2 \neq 0.$$  

Here $c$ is fixed and $g$ is the modulus.  

(2) Cases $(n, p) = (8, 6)$ and $(n, p) = (32, 36)$.

We first consider $(n, p) = (8, 6)$.

$F_A : (\mathbb{R}^8, 0) \rightarrow (\mathbb{R}^6, 0)$, $F_A = (\tilde{f}_A, u)$, where

$$\tilde{f}_A(x, y, z, w, u) = (x^2 + y^2 + z^2 + u_1x + u_2y + y^2 + \lambda z^2 + w^2 + u_3x + u_4z).$$  

It follows from Lemma 8.7 that $F_A$ is 2-determined with respect to the group $\mathcal{K}$, if $\lambda \neq 0, 1$. The following follow from simple calculations

(i) $J(f_A) + f_A'(M_2)$ contains the mixed monomials $xy, xz, xw, yz, yw, zw$.

(ii) If $a = x^3, y^3, z^3, w^3$, then $a \epsilon_1 \in T\mathcal{A}f_A, i = 1, 2(\mod J(f_A) \Theta(f_A))$. 




Using (i) and (ii) it follows that the conditions of Corollary 8.11 hold, and $\mathcal{A}(F_i)$ is open in $\mathcal{K}(F_i)$.

We leave the calculations of the pair $(n, p) = (32, 36)$ as an exercise for the reader.

(3) Cases $(n, p) = (9, 8)$ and $(n, p) = (10 + k, 7)$, $k \geq 0$.

These cases follows from Looijenga [52], Lemma 2.2.

Remark 8.14 A similar result holds for the bimodular strata in the pair $(10, 7)$ replacing $\mathcal{A}_c$-cod$(F_i) = 1$ by $\mathcal{A}_c$-cod$(F_i) = 2$.

We summarize the discussion of this section stating the following results.

**Corollary 8.15** Let $(n, p)$ be a pair in BND and $F_{\lambda_0} : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0)$ the unimodular map-germ as in (30). Then for all $\lambda_0 \in \mathbb{K}$, except for a finite number of exceptional values, the one parameter unfolding $F : (\mathbb{K}^n \times \mathbb{K}, 0) \rightarrow (\mathbb{K}^p \times \mathbb{K}, 0)$ of $F_{\lambda_0}$, as in (29), is $\mathcal{A}$-topologically trivial.

**Proof** The proof follows from Theorem 8.13 and Damon’s result (Theorem 8.6). □

**Corollary 8.16** Let $(n, p)$ be a pair in BND. Then a Thom-Mather map $f : N^n \rightarrow P^p$ has at most a finite set of points $S = \{x_1, \ldots, x_r\}$ such that for all $x_i \in S$, $j^k f(x_i) \in \mathcal{A}_M$, $j^k f \cap \mathcal{A}_M$, where $\mathcal{A}_M$ is any of the modal stratum of $\mathcal{A}^k(N, P)$.

Moreover, if $f(x_i) = y_i$, $i = 1, \ldots, r$ then $f^{-1}(y_i) \cap \Sigma(f) = \{x_i\}$, $i = 1, \ldots, r$. The restriction of $f$ to $N \setminus S$ is an infinitesimally stable map.

**8.4 Notes**

_Density of $C^1$ stable mappings._ In [83], du Plessis and Wall determine the precise range of dimensions where $C^1$-stable maps are dense. This property holds if and only if the pair $(n, p)$ is in the nice dimensions.

A parallel result is also obtained when $C^1$-stability is replaced by $\infty$-$C^1$ determinacy. We say that a map-germ $f \in \mathcal{E}^p_n$ is $\infty$-determined with respect to $\mathcal{A}$-equivalence if the $C^1$-$\mathcal{A}$-orbit of $f$ contains all $g \in \mathcal{E}^p_n$ such that $j^\infty g(0) = j^\infty f(0)$.

The paper [83] appeared in 1989. In contrast with the $C^0$ and $C^\infty$ cases much less was known in the $C^1$ case. Wall [106] sketched in 1980 the proof that $C^1$-stable maps are not dense when $n = 8$ and $p = 6$ and Mather [59] proved that finite $\mathcal{A}^{(1)}$ determinacy does not hold in general for map-germs $(\mathbb{R}^n, 0) \rightarrow (\mathbb{R}^{n+1}, 0)$, with $n \geq 15$.

The main result of [83] is the following theorem: (A) if $(n, p)$ is in complement of the nice dimensions, then for any smooth manifolds $N, P$ there is a nonempty open subset $U \subset C^\infty$ containing no $C^1$-stable mapping. (B) If $(n, p)$ is in the complement of semi-nice dimensions (see [80, 109] for details) with the exception of the pairs $(14, 14)$, $(15, 15)$, $(16, 16)$, $(12, 11)$, $(14, 12)$ and $(15, 13)$, then for any pair of
smooth manifolds $N, P$ there is a nonempty open subset $U \subset C^\infty$ containing no map all of whose point-germs are $\infty$-$\mathcal{A}^1$-determined.

The proof of this theorem follows the line of the proof of the corresponding $C^\infty$ result. It is shown that $C^1$ stability implies transversality and $\infty$-$\mathcal{A}^{(1)}$-determinacy implies transversality off the base-point to the fibres of a $\mathcal{K}$-invariant fibred submanifold of $J'(n, p)$ in the complement of the set $W^r(n, p)$ of $r$-jets with $\mathcal{K}^r$-modality $\geq 1$. This follows from the property that stability and determinacy conditions imply a weak form of transversality (the preimage is a $C^1$-submanifold). To strengthen this to actual transversality the use of unfolding theory and a perturbation lemma of R.D. May [67] were the important tools.

Several notions of $C^1$-invariance of submanifolds of jet space are discussed in [82]. In particular, the $C^1$-invariance of the Thom-Boardman varieties and, in some cases, of $\mathcal{K}^r$-orbits within them are obtained.

### 9 Density of Lipschitz stable mappings

We discuss here the problem of density of Lipschitz stable mappings, which is still widely open.

In [76] Nguyen, Ruas and Trivedi introduced the Lipschitz nice dimensions (LND) as the pairs $(n, p)$ for which the set $S^{Lip}(N, P)$ of Lipschitz stable mappings is dense in $C^\infty(N^n, P^p)$.

When $N$ is compact, it is clear that the LND contains Mather’s nice dimensions, since every $C^\infty$ stable mapping is Lipschitz stable. The main purpose in Nguyen, Ruas and Trivedi [76] is to give an answer for the following conjectures.

**Conjecture 9.1** The Lipschitz nice dimensions contains Mather’s nice dimensions and its boundary.

**Conjecture 9.2** The result in Conjecture 9.1 is sharp, that is, if $(n, p)$ is in the complement of the nice dimensions or its boundary then $S^{Lip}(N, P)$ is not a dense set in $C^\infty(N, P)$.

The following result is proved by Ruas and Trivedi [88].

**Theorem 9.3** (Section 6, [88]) The unimodular strata in the boundary of the nice dimensions are bi-Lipschitz $\mathcal{K}$-trivial.

**Remark 9.4** The exceptional unimodular strata when $(n, p) = (10, 7)$ also satisfies bi-Lipschitz $\mathcal{K}$-triviality condition.

We first review the notions of $\mathcal{K}$-equivalence and $\mathcal{K}$-triviality of $r$-parameter deformations.

**Definition 9.5** A bi-Lipschitz $\mathcal{K}$-equivalence of $r$-parameter deformations is a pair $(H, K)$ of bi-Lipschitz germs $H : (\mathbb{R}^r \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R}^r \times \mathbb{R}^n, 0)$ and $K : (\mathbb{R}^r \times \mathbb{R}^n \times \mathbb{R}^n, 0) \rightarrow (\mathbb{R}^r \times \mathbb{R}^n \times \mathbb{R}^n, 0)$.
Results on density of stable mappings 63

\(\mathbb{R}^r, 0 \to (\mathbb{R}^r \times \mathbb{R}^n \times \mathbb{R}^r, 0)\) with \(H\) an \(r\)-parameter unfolding at 0 of the germ of the identity map of \(\mathbb{R}^n\), and \(K\) an \(r\)-parameter unfolding at 0 of the germ of the identity in \(\mathbb{R}^n \times \mathbb{R}^r\) such that the following diagram commutes

\[
\begin{array}{ccc}
(\mathbb{R}^r \times \mathbb{R}^n, 0) & \xrightarrow{i} & (\mathbb{R}^r \times \mathbb{R}^n \times \mathbb{R}^r, 0) \\
\downarrow H & & \downarrow \pi \\
(\mathbb{R}^r \times \mathbb{R}^n, 0) & \xrightarrow{j} & (\mathbb{R}^r \times \mathbb{R}^n \times \mathbb{R}^r, 0)
\end{array}
\]

Here \(i\) is the canonical inclusion and \(\pi\) is the canonical projection. Two \(r\)-parameter deformations \(\Phi\) and \(\Psi\) of \(f\) are bi-Lipschitz \(K\)-equivalent if there exist a bi-Lipschitz \(K\)-equivalence \((H, K)\) as above such that

\[K \circ (id, \phi) = (id, \Psi) \circ H.\]

If \((H, K)\) has the special property that \(H\) is the germ of the identity on \(\mathbb{R}^n\), then \((H, K)\) is said to be a \(C\)-equivalence and \(\phi\) and \(\Psi\) are said to be \(C\)-equivalent deformations.

**Definition 9.6** An \(r\)-parameter deformation \(\Phi\) of a germ \(f : (\mathbb{R}^n, 0) \to (\mathbb{R}^r, 0)\) is bi-Lipschitz \(K\)-trivial (resp. bi-Lipschitz \(C\)-trivial) if it is bi-Lipschitz \(K\)-equivale (resp. bi-Lipschitz \(C\)-equivale) to the deformation \(\Psi : (\mathbb{R}^r \times \mathbb{R}^n, 0) \to (\mathbb{R}^r, 0)\), given by \(\Psi(u, x) = f(x)\).

A sufficient condition for bi-Lipschitz \(K\)-triviality is the following Thom-Levine type lemma.

**Lemma 9.7** Let \(F : (\mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}^p, 0)\) be a one parameter deformation of \(f : (\mathbb{R}^r, 0) \to (\mathbb{R}^p, 0)\). If there exist a \(p \times p\) matrix \((a_{ij})\) (not necessarily invertible) with entries germs of Lipschitz functions \((\mathbb{R}^n \times \mathbb{R}, 0)\) and a germ of a Lipschitz vector field \(X\) of the form

\[
X = \frac{\partial}{\partial t} + \sum_{i=1}^{n} X_i(x, t) \frac{\partial}{\partial x_i}
\]

with \(X_i(0, t) = 0\) such that

\[
X \cdot \begin{bmatrix}
F_1 \\
\vdots \\
F_p
\end{bmatrix} = \begin{bmatrix}
a_{11} & \cdots & a_{1p} \\
\vdots & \ddots & \vdots \\
a_{p1} & \cdots & a_{pp}
\end{bmatrix} \begin{bmatrix}
F_1 \\
\vdots \\
F_p
\end{bmatrix}
\]

Then, \(F\) is a bi-Lipschitz \(K\)-trivial deformation.

The proof follows from the fact the integration of a Lipschitz vector field gives a bi-Lipschitz flow. In fact, the bi-Lipschitz trivialization in source is given by integrating the vector field \(X\) and that in the product is given by integration of the vector field \(W\), where
\[
W(x, y, t) = \frac{\partial}{\partial t} + \sum_{i=1}^{p} W_i(x, y, t) \frac{\partial}{\partial y_i}
\]

where \( W_i(x, y, t) = \sum_{j=1}^{p} a_{ij} y_j \).

The converse of the above lemma is not known and so we say that a one parameter deformation is \textit{strongly bi-Lipschitz} \( K \)-trivial if the conditions of the above lemma hold.

If \( X_t(x, t) \equiv 0, i = 1, \ldots, n \), condition (33) implies that \( F \) is \( C \)-trivial.

A case by case proof of the bi-Lipschitz \( K \)-triviality of the unimodular strata 8.2.1 and 8.2.2 is given in Ruas and Trivedi [88]. The cases \( n \leq p \) and \( n > p \) are treated separately.

When \( n \leq p \), the modal families are families of finite maps. For them, \( K \)-determinacy holds if and only if \( C \)-determinacy holds (see Wall [108], Prop. 2.4). In this case, we can apply the Lipschitz Thom-Levine lemma to prove the bi-Lipschitz \( C \)-triviality of these families.

We discuss here the case \( n = p = 9 \).

\textbf{Lemma 9.8 (Ruas and Trivedi, [88], Lemma 6.1)} The unimodular family 8.2.1 (1)

\[
F(x, y, z, \lambda) = (x^2 + \lambda yz, y^2 + \lambda xz, z^2 + \lambda xy),
\]

\( \lambda \neq -2, 0, 1 \), is \textit{strongly bi-Lipschitz} \( C \)-trivial.

\textbf{Proof} Let \( I \) be the \( E_4 \)-ideal generated by the components of \( F \), i.e.,

\[
I = (x^2 + \lambda yz, y^2 + \lambda xz, z^2 + \lambda xy).
\]

We can prove that \( I \supset M_4^1 E_4 \), where \( M_3 \) is the ideal generated by \( x, y, z \). More precisely

\[
I \cdot M_4^1 E_4 = M_4^3 E_4 \tag{34}
\]

Consider the following control function \( \rho(x, y, z, \lambda) = \sqrt{F_1^2 + F_2^2 + F_3^2} \). Since \( F_1 \)

is \( C \)-finitely determined and homogeneous of degree 2 for all \( \lambda \neq -2, 0, 1 \), there exist constants \( c \) and \( c' \), (see Ruas [86]), such that

\[
c'||(x, y, z)||^2 \leq \rho(x, y, z, \lambda) \leq c||(x, y, z)||^2
\]

From (34) it follows that there exists a \( 3 \times 3 \) matrix \( (a_{ij}) \) with entries in \( M_4^3 E_4 \) such that

\[
\rho^2(x, y, z, \lambda) \begin{bmatrix}
\frac{\partial F_1}{\partial \lambda} \\
\frac{\partial F_2}{\partial \lambda} \\
\frac{\partial F_3}{\partial \lambda}
\end{bmatrix} = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix} \begin{bmatrix}
F_1 \\
F_2 \\
F_3
\end{bmatrix}
\]

Now consider the germ of the vector field \( V \) on \((\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R})\) defined by

\[
V = \frac{\partial}{\partial \lambda} + \frac{1}{\rho^2} \left( \sum_{j=1}^{3} a_{1j} Y_j \frac{\partial}{\partial Y_1} + \sum_{j=1}^{3} a_{2j} Y_j \frac{\partial}{\partial Y_2} + \sum_{j=1}^{3} a_{3j} Y_j \frac{\partial}{\partial Y_3} \right)
\]
where \((Y_1, Y_2, Y_3) = Y\) are the target coordinates. Notice that \(\frac{a_i Y_j}{p^r}\) are continuous in a neighborhood of the origin in \((\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}, 0)\), but the derivative with respect to \(x, y, z\) are not bounded, so that \(V\) is not Lipschitz. However we can modify \(V\) to get a Lipschitz vector field \(V' = pV\) where \(p : (\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}, 0) \rightarrow (\mathbb{R}, 0)\) is defined as follows.

Let \(D_1 = \{|Y| \leq c_1||(x, y, z, \lambda)||\}\) and \(D_2 = \{|Y| \geq c_2||(x, y, z, \lambda)||\}\) be cones in \((\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R})\) with \(c_1 < c_2\) and let \(p\) be defined by

\[
p(x, y, z, \lambda, Y) = \begin{cases} 
1 & \text{if } (x, y, z, \lambda, Y) \in D_1 \\
0 & \text{if } (x, y, z, \lambda, Y) \in D_2
\end{cases}
\]

and \(0 < p(x, y, z, \lambda, Y) < 1\) if \(c_1||(x, y, z, \lambda)|| < |Y| < c_2||(x, y, z, \lambda)||\), such that the derivative of \(p(x, y, z, \lambda, Y)\) with respect to any coordinate is bounded by a real number \(K\) (see Ruas [86] for details).

The integration of \(V'\) will give a bi-Lipschitz \(C\)-trivialization of \(F\) by the Thom-Levine criterion. This completes the proof.

**Remark 9.9** For any fixed \(\lambda = \lambda_0 \neq -2, 0, 1\), the deformation \(F(x, y, z, \lambda)\) in Lemma 9.8 is semialgebraic and satisfies the condition \(\frac{\|F_j(x, y, z)\|}{\|F_0(x, y, z)\|}\) is bounded for any \((x, y, z, \lambda)\) in \((\mathbb{R}^3 \times \mathbb{R}, 0)\). Then we can also apply Theorem 3.1 of Ruas and Valette [89] to prove that \(F_j\) is semialgebraically bi-Lipschitz \(K\)-trivial. Notice however that the conclusion in Lemma 9.8 is stronger, as we prove that the family \(F_j\) is strongly bi-Lipschitz \(K\)-trivial.

The bi-Lipschitz \(K\)-triviality of the Thom-Mather stratification along the unimodular strata in the boundary of the nice dimensions suggest that mappings transverse to this stratification are bi-Lipschitz stable.

A natural approach to prove Conjecture 9.1 is to follow the proof of Theorem 8.6, taking into account that the pair \((n, p)\) is in the boundary of the nice dimensions.

We saw in Corollary 8.16 that a Thom-Mather map \(f : N^p \rightarrow P^p\), \((n, p)\) in the boundary of the nice dimensions has at most a finite set of points \(S = \{x_1, \ldots, x_\ell\}\) such that for all \(x_i \in S\), \(j^k f(x_i) \in \mathcal{A}_M\), \(j^k f \pitchfork \mathcal{A}_M\), where \(\mathcal{A}_M\) is the modal stratum. Moreover by multi-transversality, if \(f(x_i) = y_i, i = 1, \ldots, \ell\) then

\[
f^{-1}(y_i) \cap \Sigma(f) = \{x_i\}, i = 1, \ldots, \ell.
\]

Clearly, \(f\) is an infinitesimally stable mapping in the complement of \(S\).

To prove that \(f\) is Lipschitz stable it would be sufficient to prove that each unimodular family \(F_j\) (see Section 8.3), and also the bimodular family when \((n, p) = (10, 7)\), is bi-Lipschitz \(A\)-trivial.

Let

\[
F(x, u, \lambda) = (\bar{f}(x, u, \lambda), u, \lambda)
\]

be the (weighted homogeneous) normal form of a unimodular family in BND as in (29), where \(x = (x_1, \ldots, x_s), u = (u_1, \ldots, u_r), s + r = n\) and \(\bar{f} = (\bar{f}_1, \ldots, \bar{f}_i)\).
Following the proof of Theorem 8.6, we can find weighted homogeneous vector fields \( V \) and \( W \) in source and target respectively, given by:

\[
V(x, u, \lambda) = \sum_{j=1}^{x} v_j(x, u, \lambda) \frac{\partial}{\partial x_j} + \sum_{i=1}^{r} \tilde{v}_i(f, u, \lambda) \frac{\partial}{\partial u_i} + \frac{\partial}{\partial \lambda}
\]

where \( x = (x_1, \ldots, x_s) \), \( u = (u_1, u_2, \ldots, u_r) \) and \( \tilde{v}_i(0, 0, \lambda) = v_j(0, 0, \lambda) = 0 \),

\[
W(X, U, \lambda) = \sum_{j=1}^{t} w_j(X, U, \lambda) \frac{\partial}{\partial X_j} + \sum_{i=1}^{r} \tilde{w}_i(X, U, \lambda) \frac{\partial}{\partial U_i} + \frac{\partial}{\partial \lambda}
\]

where \( X = (X_1, \ldots, X_t) \), \( U = (U_1, \ldots, U_r) \), and \( \tilde{w}_i(0, 0, \lambda) = w_j(0, 0, \lambda) = 0 \),

(capital letters denote the coordinates in the target), and a weighted homogeneous control function \( \rho(X, U, \lambda) \) such that

\[
(\rho \circ F)(x, u, \lambda) \frac{\partial \tilde{f}}{\partial \lambda} = -3 \sum_{j=1}^{s} \frac{\partial \tilde{f}}{\partial x_j} v_j(x, u, \lambda) - 6 \sum_{i=1}^{r} \frac{\partial \tilde{f}}{\partial u_i} \tilde{v}_i(f, u, \lambda) + \tilde{W}(\tilde{f}, u, \lambda) \tag{35}
\]

where \( \tilde{W} = (w_1, \ldots, w_t) \).

It follows from (35) that the vector fields \( X(x, u, \lambda) = \frac{1}{(\rho \circ F)(x, u, \lambda)} V(x, u, \lambda) \) and \( Y(X, U, \lambda) = \frac{1}{\rho(X, U, \lambda)} W(X, U, \lambda) \) satisfy the equation \( D F(X) = Y \circ F \). Moreover, they are continuous and can be integrated to give the topological \( \mathcal{A} \)-triviality of \( F \) along the moduli space.

If we can prove that \( X \) and \( Y \) are Lipschitz vector fields in the source and target, respectively, the bi-Lipschitz \( \mathcal{A} \)-triviality of \( F \) would follow from the Lipschitz version of the Thom-Levine lemma.

### 10 Sections of discriminant of stable germs. Open Problems

An important consequence of Theorems A and B is that we can approximate any map \( f : U \subset \mathbb{K}^n \rightarrow \mathbb{K}^p \), \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \), by a stable mapping if \((n, p)\) is in the nice dimensions or else by a topologically stable map if \((n, p)\) is not in the nice dimensions.

For a map-germ of finite singularity type \( f : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^p, 0) \), a stable perturbation can be realized as the generic member of a 1-parameter unfolding \( \tilde{f}(x, t) = (f_t(x), t) \) of \( f \). More precisely, \( \tilde{f} \) is a stabilization of \( f \) if there exists a representative \( \tilde{f} : U \rightarrow V \times T \) such that \( f_t : U \cap (\mathbb{K}^n \times \{t\}) \rightarrow V \) is stable for all \( t \neq 0 \).

When \( \mathbb{K} = \mathbb{C} \), the stable perturbation of \( f \) is uniquely determined up to \( \mathcal{A} \)-equivalence when \((n, p)\) is in the nice dimensions and up to \( C^0, \mathcal{A} \)-equivalence otherwise. When \( \mathbb{K} = \mathbb{R} \), there may exists a finite number of nonequivalent stabili-
tions of \( f \). On the reals, in general \( t > 0 \) and \( t < 0 \) give non-equivalent perturbations of \( f \) (see Mond and Nuño-Ballesteros in this Handbook or \[69\] for details).

The geometry of the stable perturbations \( \tilde{f} \) are associated to invariants of the germ \( f \).

We discuss here this important tool in singularity theory.

Let \( f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0), \mathbb{K} = \mathbb{R} \) or \( \mathbb{C} \) be a germ of a finite singularity type and \( F \) its stable unfolding:

\[
\begin{align*}
(\mathbb{K}^n, 0) & \xrightarrow{f} (\mathbb{K}^p, 0) \\
(\mathbb{K}^n, 0) & \xrightarrow{g} (\mathbb{K}^p, 0)
\end{align*}
\]

where \( g \) is the germ of an immersion transverse to \( F \).

Let \( \mathcal{K} = \Delta(F) \) be the discriminant of \( F \) (recall that when \( n < p \) the discriminant is the image \( F(\mathbb{K}^n) \).) Damon in \[25\] described a relation between \( \mathcal{A} \)-equivalence and properties of the discriminant \( \mathcal{A} \). This relation is valid for all pairs \( n - p \) and directly relates \( \mathcal{A} \)-codimension of \( f \) with a codimension of the germ at 0 of \( g(\mathbb{K}^p) \) as a section of the discriminant. The idea of using sections of the discriminant to determine \( \mathcal{A} \)-determinacy properties of \( f \) was derived in \[25\].

Given the germ of a variety \( V = \Delta(\mathcal{K}) \) of contact equivalences preserving \( V \) which acts on the set of germs \( g : (\mathbb{K}^p, 0) \to (\mathbb{K}^p, 0) \) (the map-germs \( g \) are in \( \mathcal{E}_p \) when \( \mathbb{K} = \mathbb{R} \) or in \( \mathcal{O}_p \) when \( \mathbb{K} = \mathbb{C} \).

The contact group \( \mathcal{K}_V \) is defined as follows:

\[
\mathcal{K}_V = \{(h, H) \in \mathcal{K} | H(\mathbb{K}^p \times V) \subseteq \mathbb{K}^p \times V\}
\]

(see definition \[4.1\]).

The action of \( \mathcal{K}_V \) on \( \mathcal{E}_p \) or \( \mathcal{O}_p \) is defined as in definition \[4.1\]. We can also define the similar notions for unfoldings. The group \( \mathcal{K}_V \) is a geometric subgroup of the contact group, so that the machinery of singularity theory applies to \( \mathcal{K}_V \)-equivalence. In particular the infinitesimal and the geometric criteria for \( \mathcal{K}_V \)-determinacy.

We can define

\[
\begin{align*}
\mathcal{T} \mathcal{K}_V \cdot g &= \mathcal{T} g(\mathcal{M}_p \Theta_p) + \mathcal{E}_p \{ \eta_i \circ g, i = 1, \ldots, m \} \\
\mathcal{T} \mathcal{K}_V \cdot g &= \mathcal{T} g(\Theta_p) + \mathcal{E}_p \{ \eta_i \circ g, i = 1, \ldots, m \}
\end{align*}
\]

where \( \eta_i, i = 1, \ldots, m \) are the generators of \( \Theta_V \), the \( \mathcal{E}_p \)-module of vector fields in \( \mathbb{K}^p \) tangent to the variety \( V \) at its smooth points. Equivalently, \( \Theta_V \) is the \( \mathcal{E}_p \)-module of derivations of \( \Theta_p \) which preserve the ideal defining \( V \). The notation \( \text{Der}(-\log V) \) proposed by Saito for the module of these vector fields as well the notation \( \gamma^*(\text{Der}(-\log V)) \) for the \( \mathcal{E}_p \)-module \( \mathcal{E}_p \{ \eta_i \circ g, i = 1, \ldots, m \} \) are also widely.
used. See section 2.9 of the article of Nuño-Ballesteros and David Mond in this Handbook [70].

$T\mathcal{K}Vg$ and $T\mathcal{K}V_e g$ are $O_p$-modules when $\mathbb{K} = \mathbb{C}$ and $E_p$-modules when $\mathbb{K} = \mathbb{R}$.

With the notations as in (36) we can state the main results in [25] as follows.

1. $g$ has finite $\mathcal{K}V$-codimension if and only if $f$ has finite $\mathcal{A}$-codimension.
2. If $N\mathcal{A}e f$ and $N\mathcal{K}V_e g$ denote the normal spaces to $\mathcal{A}e f$ and $\mathcal{K}V_e g$–respectively,
   then
   $$N\mathcal{A}e f \cong N\mathcal{K}V_e g.$$
3. $\mathcal{A}e$-codimension$(f) = \mathcal{K}V_e$-codimension$(g)$.
4. Conditions 1. to 3. hold for multigerms $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$.

The geometric characterization of $\mathcal{K}V$-equivalence holds only for holomorphic map-germs $f \in O_p^n$, namely: $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is $\mathcal{A}$-finitely determined if and only if $g$ is transverse to the strata of $V$ away from the origin. For real germs, the geometric condition is a necessary condition for $\mathcal{K}V$ finite determinacy, but the converse does not hold.

Damon’s theory builds a solid bridge between singularity theory of mappings and topology of singular varieties. This connection has been used successfully for the past three decades. We follow this approach to formulate some open problems in singularity theory, related to the subject discussed in this paper.

10.1 Geometry of sections of discriminant of stable mappings in the nice dimensions

Let $(n + 1, p + 1)$ be a nice pair of dimensions and $F : (\mathbb{K}^{n+1}, 0) \to (\mathbb{K}^{p+1}, 0)$ a minimal stable map-germ. Minimal here means that $\{0\} \in \mathbb{K}^{n+1}$ is a stratum on the stratification of $F$ by stable types. A hyperplane section $H = g(\mathbb{K}^p)$ transversal to the discriminant $V = \Delta(F) \subset \mathbb{K}^{p+1}$ away from the origin pulls back by $F$ to an $\mathcal{A}$-finite map-germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$.

From Damon’s result 3. above, it follows that if $(n, p)$ is in the semi-nice dimensions (see section 7.1) there exists an open and dense set $\mathcal{I}$ of immersions $g : (\mathbb{K}^p, 0) \to (\mathbb{K}^{p+1}, 0)$ such that the pull back of $g$ by $F$ is an $\mathcal{A}$-finite map germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ whose $\mathcal{A}e$-codimension is minimal, that is,

$$\mathcal{A}e\text{-cod} f \leq \mathcal{A}e\text{-cod} f', \text{ for all } f' \sim f.$$

As $F$ is a minimal stable unfolding of $f$ we may ask: is there a map-germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$, $Q(f) \equiv Q(F)$ such that $\mathcal{A}e\text{-cod} f = 1$, which in this case implies that $\mathcal{A}$-orbit of $f$ is open in its $K$-orbit?

It follows from Proposition 8.10 that this condition holds if and only if it holds for a general linear hyperplane section (see [41] for the case $(n, n + 1)$). Notice however that sections of $\Delta(F)$ minimizing $\mathcal{A}e$-codimension are not necessarily linear (see section 3.1 in [6]). The complete answer to the question above appears in [6].
Theorem 10.1 ([6], Theorem 4.6) If the pair \((n, p)\) is in the extra-nice dimensions, then every stable germ \(F : (\mathbb{K}^{n+1}, 0) \to (\mathbb{K}^{p+1}, 0)\) admits a section of \(\mathcal{A}_e\)-codimension 1 \(f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)\). The converse is true if \((n+1, p+1)\) is in the nice dimensions.

Corollary 10.2 If \(\mathbb{K} = \mathbb{C}\) and \((n, p)\) is in the extra-nice dimensions any two generic hyperplane sections \(g, g'\) of the discriminant \(\Delta(F)\) of a stable germ \(F : (\mathbb{K}^{n+1}, 0) \to (\mathbb{K}^{p+1}, 0)\) pull back by \(F\) to \(\mathcal{A}\)-equivalent germs \(f, f' : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)\). Moreover \(\mathcal{A}_e\)-cod \(f = \mathcal{A}_e\)-cod \(f'\).

Remark 10.3 When \(\mathbb{K} = \mathbb{C}\), \(p \leq n+1\) and \((n, p)\) is in the nice dimensions, the topology of the stabilization of holomorphic \(\mathcal{A}_e\)-codimension 1, corank 1 germs and multigerms is well understood. See [17] where T. Cooper, D. Mond and Wik-Atique classify these singularities and study the topology of their stabilizations.

Problem 1. To study the geometry of generic hyperplane sections of the discriminant of stable mappings in \((n+1, p+1)\) when \((n, p)\) is in extra-nice dimensions and its boundary.

Problem 2. To study equisingularity of families of generic hyperplane sections \(g, g' : (\mathbb{K}^p, 0) \to (\mathbb{K}^p, 0)\) of the discriminant \(\Delta(F)\) of stable map-germs \(F : (\mathbb{K}^{n+1}, 0) \to (\mathbb{K}^{p+1}, 0)\) where \(g, g' : (\mathbb{K}^p, 0) \to (\mathbb{K}^p, 0)\) are germs of immersions, when \((n, p)\) is in the boundary of extra-nice dimensions. These pair of extra-nice dimensions have been calculated in [6].

(i) \(n \leq p\), \(4p = 5n - 5\), \(p \geq 5\).
(ii) \(n > p\), \((n, p) = \{(5, 4), (7, 5), (9 + k, 6), k \geq 0\}\).

Observe that these families are always topologically trivial. However the Whitney equisingularity and the bi-Lipschitz triviality of these families are open questions.

Conjecture 10.4 At the boundary of the extra-nice dimensions any two generic immersions \(g, g' : (\mathbb{K}^p, 0) \to (\mathbb{K}^p, 0)\) are bi-Lipschitz \(\mathcal{K}_V\)-equivalent and they define bi-Lipschitz \(\mathcal{A}\)-equivalent germs \(f, f' : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)\).

Problem 3. Apply the geometric approach discussed in this section to study the bi-Lipschitz \(\mathcal{G}\)-classification of analytic map-germs \(f \in O^\mathcal{G}_n\) where \(\mathcal{G} = \mathcal{R}, \mathcal{C}, \mathcal{K}, \mathcal{L}, \mathcal{A}\) or more generally, any geometric subgroup of \(\mathcal{K}\). The Lipschitz theory of singularity is an almost completely open problem. See [87] for an account on bi-Lipschitz \(\mathcal{G}\)-classification of function germs \(\mathcal{G} = \mathcal{R}, \mathcal{C}, \mathcal{K}\) and references therein [8, 89, 43, 75, 10, 9, 33, 47, 48, 35].

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