MULTIVIEW CHIRALITY

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ABSTRACT. Given an arrangement of cameras $\mathcal{A} = \{A_1, \ldots , A_m\}$, the chiral domain of $\mathcal{A}$ is the subset of $\mathbb{P}^3$ that lies in front it. It is a generalization of the classical definition of chirality. We give an algebraic description of this set and use it to generalize Hartley’s theory of chiral reconstruction [4] to $m \geq 2$ views and derive a chiral version of Triggs’ Joint Image [12, 13].

1. INTRODUCTION

In computer vision, chirality refers to the constraint that for a scene point to be visible in a camera, it must lie in front of it [4]. There is now a mature theory of multiview geometry that ignores this constraint [3]; it is not an exaggeration to say that the theory of chiral multiview geometry is still in its infancy and most of the basic questions remain unanswered. We will discuss three:

1. When can a nonchiral reconstruction be made chiral?
2. Given a set of cameras, what is the set of images of world points that lie in front of them?
3. Given image matches, when does there exist a chiral reconstruction corresponding to them?

In his seminal paper, Hartley not only introduced the term chirality, but also gave a complete answer to the first question for two views in the projective case [4]. His results are constructive and efficient, i.e. they require solving up to two linear programs that are linear in the size of the reconstruction. They do not generalize to more than two views. Concurrently, Werner et al. also discovered some of the same results [18].

If we ignore chirality, the answer to question 2 is known as the joint image [12]. Starting with the seminal work of Longuet-Higgins [6], there is now a complete algebraic and set theoretic characterization of the joint image [1,2,8,5,7,8,11]. In the chiral case, Werner et al. provide a number of necessary conditions, but a complete characterization is not available [14, 17].

Hartley raises the third question and answers it for two views in the form of a sign condition on a projective reconstruction. Werner et. al. also consider the third question and answer it for two views in image space, considering both minimal and nonminimal configurations [15, 16]. Nistér & Schaffnitzy consider the minimal problem in the Euclidean case [10]. There is no existence theory of chiral reconstruction for $m > 2$ views.

Our paper makes three contributions.

1. We introduce the chiral domain of an arrangement of cameras – a multiview generalization of the classical definition of chirality – that covers all of $\mathbb{P}^3$ (not just finite points), and give an algebraic description of this set (Section 3).
(2) We give a complete answer to question 1 for projective ((Section 4.1) and Euclidean (Section 4.2) reconstructions in an arbitrary number of views. Like Hartley’s solution for two views, our solution is also constructive and efficient. Indeed we recover Hartley’s results as special cases of ours (Section 4.3).

(3) We give a complete answer to question 2, i.e. we algebraically describe the Euclidean closure of the chiral joint image for an arbitrary number of views.

Our results are complete in the sense that, except for the assumption of distinct camera centers, we do not make any other genericity assumptions. We will not address question 3 in this paper. We begin by describing the notation used in this paper and some necessary background.

2. BACKGROUND AND NOTATION

The sets of nonnegative integers, nonnegative real numbers, and positive real numbers are \( \mathbb{N}, \mathbb{R}_{+}, \) and \( \mathbb{R}_{+} \), respectively. \( \mathbb{P}^{n} \) denotes \( n \)-dimensional projective space over the reals, which is \( \mathbb{R}^{n+1} \setminus \{0\} \) modulo the equivalence relation ~ where \( x \sim y \) if \( x \) is a scalar multiple of \( y \). If \( x \sim y \), then we say that \( x \) and \( y \) are equal in \( \mathbb{P}^{n} \), or \( x \) is identified with \( y \). We use \( = = \) to denote coordinate wise equality in \( \mathbb{R}^{n} \). The projectivization of a set \( S \subseteq \mathbb{R}^{n+1} \) is the set \( \mathbb{P}(S) = \{ x \in \mathbb{P}^{n} \mid \exists \lambda \in \mathbb{R} \setminus \{0\} \text{ s.t. } \lambda x \in S \} \).

In multiview geometry, we focus on \( \mathbb{P}^{3}, \mathbb{P}^{2}, \) and \( \mathbb{R}^{3}, \mathbb{R}^{2} \), where \( \mathbb{P}^{n} \) is a compactification of \( \mathbb{R}^{n} \) with respect to the embedding \( \mathbb{R}^{n} \to \mathbb{P}^{n} \), \( x \mapsto \tilde{x} = (x,1) \). So points whose last coordinate is nonzero are said to be finite, whereas points whose last coordinate is 0 form the hyperplane at infinity. We write the plane at infinity as \( L_{\infty} := \{ q \in \mathbb{P}^{3} : n_{\infty}^{\top}q = 0 \} \), where we fix the normal \( n_{\infty} = (0,0,0,1)^{\top} \).

We denote points in \( \mathbb{P}^{3} \) and \( \mathbb{R}^{3} \) by \( q \) allowing the context to decide where \( q \) lies. Similarly we denote points in \( \mathbb{P}^{2} \) and \( \mathbb{R}^{2} \) by \( p \). The dehomogenization of a finite point \( q \in \mathbb{P}^{3} \) is denoted as \( \hat{q} := (q_{1}/q_{4}, q_{2}/q_{4}, q_{3}/q_{4})^{\top} \).

A projective camera is a matrix \( A = \begin{bmatrix} G & t \end{bmatrix} \in \mathbb{R}^{3 \times 4} \) of rank 3. The camera \( A \) is finite if \( \det(G) \neq 0 \). The center of the camera \( \hat{A} \) is the unique point \( c \in \mathbb{P}^{3} \) such that \( \hat{A}c = 0 \). The camera \( A \) is finite if and only if its center \( c \in \mathbb{P}^{3} \) is finite. All cameras considered in this paper are finite. For consistency, we will choose the \( \mathbb{R}^{4} \) representative \( c_{A} = \left[ -G^{-1}t \right] \left[ \begin{array}{c} \lambda \end{array} \right] \) for the center of camera \( A \).

The world \( \mathbb{R}^{3} \), which is to be imaged by \( A \), is modeled as the affine patch in \( \mathbb{P}^{3} \) with \( q_{4} = 1 \). This allows the identification of a finite point \( q \in \mathbb{P}^{3} \) with the world point \( \hat{q} \in \mathbb{R}^{3} \), and a world point \( q \in \mathbb{R}^{3} \) with the finite point \( \hat{q} \in \mathbb{P}^{3} \). The image of \( q \in \mathbb{P}^{3} \), in the camera \( A \) is \( Aq \in \mathbb{P}^{2} \). The rational map \( A : \mathbb{P}^{3} \to \mathbb{P}^{2}, q \mapsto Aq, \) is defined for all \( q \in \mathbb{P}^{3} \) except the center \( c \) of \( A \).

The principal plane of a finite camera \( A = \begin{bmatrix} G & t \end{bmatrix} \) is the hyperplane \( L_{A} := \{ q \in \mathbb{P}^{3} : A_{3,\bullet}q = 0 \} \), where \( A_{3,\bullet} \) is the third row of \( A \), i.e. it is the set of points in \( \mathbb{P}^{3} \) that image to infinite points in \( \mathbb{P}^{2} \). Note that the camera center \( c \) lies on \( L_{A} \). We regard \( L_{A} \) as an oriented hyperplane in \( \mathbb{R}^{4} \) with normal vector \( n_{A} := \det(G)A_{3,\bullet}^{\top} \), which we call the principal ray of \( A \). The \( \det(G) \) factor makes sure that if we pass from \( A \) to \( \lambda A \) for some nonzero scalar \( \lambda \in \mathbb{R} \), the normal vector of the principal plane does not change sign.

\footnote{The broken arrow (\( \rightsquigarrow \)) and the phrase "rational map" mean here that the domain of the map \( A \) is not actually \( \mathbb{P}^{3} \) but rather \( \mathbb{P}^{3} \setminus \{c\} \).}
The depth of a finite point \( q \) in a finite camera \( A \) is essentially the projection of \( \tilde{q} - \tilde{c} \) along the principal ray, see [3]. Formally, it is defined as

\[
\text{depth}(q; A) := \left( \frac{1}{\|\det(G)\|} \right) \frac{n_\infty^\top(q)}{(n_\infty^\top q)}.
\]

Notice that depth \( (q; A) \) is unaffected by scaling of \( q \) and of \( A \).

Let \( A = (A_1, \ldots, A_m) \) denote an arrangement of \( m \) cameras. We use the shorthand \( n_i \) for the principal ray of camera \( A_i \). Given a pair of cameras \( A_i, A_j \) with centers \( c_i \) and \( c_j \), let \( e_{ij} \) denote the image of \( c_j \) in \( A_i \). The points \( e_{ij} \) are called epipoles. The line through \( c_i, c_j \) is called the base line of the pair of cameras \( (A_i, A_j) \). All points on the base line (except for the centers themselves) will image in the two cameras at their respective epipoles \( (e_{ij}, e_{ji}) \).

The (polyhedral) cone \( K_U \) spanned by a set of vectors \( U = \{u_1, \ldots, u_r\} \subseteq \mathbb{R}^n \) is the set of all nonnegative linear combinations of the vectors in \( U \), so

\[
K_U := \text{cone}(u_1, \ldots, u_r) = \{u_1x_1 + \cdots + u_rx_r \mid x_1, \ldots, x_r \geq 0\}.
\]

The interior of \( K_U \) is denoted by \( \text{int} K_U \). The dual cone \( K_U^\ast \) to \( K_U \) is the set of all vectors that make nonnegative inner product with every vector of \( K_U \), so

\[
K_U^\ast := \{y \mid \forall x \in K_U : y^\top x \geq 0\}.
\]

The interior of \( K_U^\ast \) is the set \( \text{int} K_U^\ast = \{y \mid \forall x \in K_U : y^\top x > 0\} \). Checking membership in the dual cone \( K_U^\ast \) is equivalent to checking the feasibility of the inequality system \( \{h^\top u \geq 0\} \) which in turn amounts to solving a linear program.

3. The Chiral Domain of an Arrangement of Cameras

In this section we define chirality for all points in \( \mathbb{P}^3 \) with respect to one or more finite cameras. Our definition encompasses Hartley’s definition of chirality [3], [3, Chapter 21], which is restricted to finite points in \( \mathbb{P}^3 \); it is a compactification of this classical definition.

The depth of a finite point \( q \in \mathbb{P}^3 \) in a finite camera \( A \) defined in Equation (1) is 0 if and only if \( n_\infty^\top q = 0 \). This happens if and only if \( q \) lies on the principal plane \( L_A \). Otherwise, \( n_\infty^\top q \neq 0 \) and the sign of depth \( (q; A) \) is the same as the sign of the product \( (n_\infty^\top A_q)(n_\infty^\top q) \) which is either positive or negative. It is then natural to say that a finite point \( q \) is in front of the camera \( A \) if depth \( (q; A) > 0 \), see [4]. Since only the sign of depth \( (q; A) \) matters, it is usual to refer to this sign as the chirality of \( q \) in \( A \), denoted as \( \chi(q; A) \), which is either 1 or \(-1\).

To extend the definition of chirality to all points in \( \mathbb{P}^3 \), not just finite points, we rely on the natural topology in \( \mathbb{P}^3 \) induced by the quotient map \( \pi : \mathbb{R}^4 \setminus \{0\} \to \mathbb{P}^3 \) in which \( \pi(v) = \pi(w) \) if and only if \( v \sim w \). In this quotient topology, a set \( U \subseteq \mathbb{P}^3 \) is open if and only if its preimage \( \pi^{-1}(U) \) is open in \( \mathbb{R}^4 \setminus \{0\} \) in the Euclidean topology. Thus \( q \in \mathbb{P}^3 \) is a limit point of a sequence \( \{q_i\} \subseteq \mathbb{P}^3 \) if and only if all open sets \( \pi^{-1}(U) \) containing the line \( \pi^{-1}(q) \) contain some line \( \pi^{-1}(q_i) \). The closure of a set \( S \subseteq \mathbb{P}^3 \) is the set of all limit points of sequences in \( S \).

**Definition 1** (Chiral Domain of \( A \) and Chirality). Let \( A \) be an arrangement of finite cameras. Then the chiral domain of \( A \), denoted as \( \mathbb{P}^3_A \), is the closure of the set

\[
\{q \in \mathbb{P}^3 \mid q \text{ finite, } \forall A \in A, \text{ depth}(q; A) > 0\}.
\]

Moreover, a point \( q \in \mathbb{P}^3 \) is said to have chirality 1 with respect to \( A \), denoted as \( \chi(q; A) = 1 \), if and only if \( q \in \mathbb{P}^3_A \).
Theorem 1. Let \( P \) Thus \( \emptyset \) if and only if the row space of the \( 4 \times (m + 1) \) matrix \( N \) with columns \( n_1, \ldots, n_m, n_\infty \) intersects the positive orthant \( \mathbb{R}^m_+ \).

In particular, for all arrangements of \( m \leq 3 \) cameras such that \( n_\infty \) and the principal rays \( n_1, \ldots, n_m \) are linearly independent, \( \mathbb{P}^3_A \neq \emptyset \).

Proof. The set \( \mathbb{P}^3_A \neq \emptyset \) if and only if there is a finite point with positive depth in all cameras, or equivalently a \( q \in \mathbb{R}^4 \) such that \( q^\top N \) lies in the positive or negative orthant. Thus \( \mathbb{P}^3_A \neq \emptyset \) if and only if the row space of \( N \) has an intersection with \( \mathbb{R}^m_+ \).

If \( m \leq 3 \) and the columns of \( N \) are linearly independent then \( N \) has row rank equal to \( m + 1 \), and the rows of \( N \) span \( \mathbb{R}^m_+ \). So the rowspace of \( N \) intersects \( \mathbb{R}^m_+ \).

\( \square \)

Remark 1. Note that \( \mathbb{P}^3_A \) is nonempty if and only if it has nonempty interior. Indeed, if it is nonempty, there is a finite point \( q \in \mathbb{P}^3 \) that has positive depth in all cameras in the arrangement. Since the depth depends continuously on the finite point, there is a neighborhood \( U \subseteq \mathbb{P}^3 \) of finite points with positive depth in all cameras. This neighborhood is in the interior of \( \mathbb{P}^3_A \).

Theorem 2. Let \( A = \{A_1, A_2, \ldots, A_m\} \) be an arrangement of finite cameras. Then \( \mathbb{P}^3_A \neq \emptyset \) if and only if the row space of the \( 4 \times (m + 1) \) matrix \( N \) with columns \( n_1, \ldots, n_m, n_\infty \)

intersects the positive orthant \( \mathbb{R}^m_+ \).

Remark 2. \( \mathbb{P}^3_A \) is defined by strict linear inequalities, it is the interior of the polyhedral cone \( Q \subseteq \mathbb{R}^4 \) defined by the inequalities \( n_\infty^\top x \geq 0 \) and \( n_i^\top x \geq 0 \). We can define the semialgebraic set \( Q \cup (-Q) \subseteq \mathbb{R}^4 \) by the quadratic inequalities \( (n_i^\top x)(n_j^\top x) \geq 0 \), where \( \{i, j\} \) ranges over all 2-element subsets of \( \{1, 2, \ldots, m, \infty\} \). The projectivization of \( Q \cup (-Q) \) is the Euclidean closure of \( \mathbb{P}(S) \subseteq \mathbb{P}^{m-1} \), given that \( S \) is nonempty.

Remark 3. The equality in (4) is only valid when \( \mathbb{P}^3_A \) is nonempty. The right hand side can be nonempty even if \( \mathbb{P}^3_A \) is empty. This is because the nonstrict inequalities admit all points that are on the principal planes of some cameras in \( A \) and have nonnegative depth in the others.

Remark 4. In fact, \( \chi(q; A) = 1 \) is not equivalent to \( \chi(q; \{A\}) = 1 \) for all \( A \in \mathcal{A} \), at least when \( q \in \mathbb{P}^3 \) is an infinite point. Indeed, specializing Theorem 2 to one camera, we get \( \mathbb{P}^3_{\{A\}} = \{q \in \mathbb{P}^3 : (n_i^\top q)(n_\infty^\top q) \geq 0\} \), which implies that \( \chi(q; \{A\}) = 1 \) if \( q \in L_\infty \cup L_A \) for one camera \( A \). So an infinite point always has chirality one with respect to a single camera. However, there are arrangements \( A \) where not every infinite point has chirality 1 with respect to \( A \), e.g. two co-incident cameras facing in opposite directions.
4. Chiral Reconstructability

A reconstruction of a collection of image correspondences

\[ \mathcal{P} := \{(p_{1k}, \ldots , p_{mk}) \in (\mathbb{R}^2)^m, \ k = 1, \ldots , n\} \]

is a set of world points \( q_k \) and cameras \( A_i \) such that \( A_i q_k \sim \hat{p}_{ik} \). Reconstructions can be transformed by homographies of \( \mathbb{P}^3 \) to other reconstructions of \( \mathcal{P} \). Since chirality is not a projective invariant, the new reconstruction may become chiral or lose chirality. This naturally leads to the question: When can a reconstruction of a given collection of image correspondences \( \mathcal{P} \) be turned into a chiral reconstruction of \( \mathcal{P} \)? We will treat the projective and Euclidean cases separately.

4.1. Projective Reconstructions. A projective reconstruction of \( \mathcal{P} \) is a pair \((A, Q)\) consisting of an arrangement of \( m \) finite cameras \( A := \{A_1, \ldots , A_m\} \) and a set of \( n \) points \( Q := \{q_1, \ldots , q_n\} \subseteq \mathbb{R}^4 \setminus \{0\} \) such that \( A_i q_k = w_{ik} \hat{p}_{ik} \) for some scalars \( w_{ik} \). Recall that \( n_i = q_k = \text{det}(G_i)w_{ik} \) for all \( i, k \). Since all of the image correspondences \( \hat{p}_{ik} \) have last coordinate 1, \( w_{ik} \neq 0 \), and since all cameras are finite, \( n_i^\top q_k = 0 \) for any \( i, k \). This implies that no point \( q_k \in Q \) can lie on the principal plane of any camera \( A_i \).

Definition 2. A chiral reconstruction of \( \mathcal{P} \) is a projective reconstruction \((A, Q)\) of \( \mathcal{P} \) such that each \( A_i \) is a finite camera and \( \chi(q_k; A) = 1 \) for all \( k \).

In the context of two cameras, [4] and [16] call a projective reconstruction of \( \mathcal{P} \) a weak realization, and a chiral reconstruction a strong realization. In fact, while our definition of chiral reconstruction requires finite cameras, by allowing world points to be infinite, we extend the notion of a strong realization.

We first state a lemma (proof in the Appendix) that describes the effect of a homography \( H \) on a reconstruction. Recall that, for the center \( c_A \) of a finite camera \( A \), we choose the representative in \( \mathbb{R}^4 \)

\[ c_A = \begin{bmatrix} -G^{-1}t \\ 1 \end{bmatrix}. \]

Lemma 1. Let \( A = [G \ t] \) be a finite camera with center \( c_A \). Let \( H \in \text{GL}_4 \) with last row \( h^\top \) and \( \delta := \text{det}(H^{-1}) \). Then

1. After the homography, the plane at infinity is \( h^\top q = 0 \).
2. The camera \( AH^{-1} \) is finite if and only if \( h^\top c_A \neq 0 \). Its center then is \( c_{AH^{-1}} = \frac{1}{\delta} Hc_A \).
3. The principal ray of \( AH^{-1} \) is \( n_{AH^{-1}} = \delta(h^\top c_A)H^{-\top}n_A \).
4. For all \( q \in \mathbb{R}^4 \), we have \( n_{AH^{-1}}(Hq) = \delta(h^\top c_A)(n_A^\top q) \).

We now address the question of when a given projective reconstruction \((A, Q)\) of \( \mathcal{P} \) can be transformed to a projectively equivalent reconstruction

\[ (AH^{-1} := \{A_1H^{-1}, \ldots , A_mH^{-1}\}, HQ := \{Hq_1, \ldots , Hq_n\}) \]

that is chiral, by a homography \( H \in \text{GL}_4 \) of \( \mathbb{P}^3 \).

Theorem 3. Given a projective reconstruction \((A, Q)\) of \( \mathcal{P} \), set \( \sigma_{ik} = \text{sign}(n_A^\top q_k) \). Then there is a \( H \in \text{GL}_4 \) with last row \( h^\top \) such that \( (AH^{-1}, HQ) \) is a chiral reconstruction if and only if one of the following sets \( S_1 \) or \( S_2 \) is nonempty:

5. \( S_1 = \{h \mid \forall i, j, k, (h^\top q_k)(h^\top c_i)(h^\top c_j)\sigma_{ik} \geq 0, (h^\top c_i)(h^\top c_j)\sigma_{ik}\sigma_{jk} > 0\} \)
6. \( S_2 = \{h \mid \forall i, j, k, (h^\top q_k)(h^\top c_i)(h^\top c_j)\sigma_{ik} \leq 0, (h^\top c_i)(h^\top c_j)\sigma_{ik}\sigma_{jk} > 0\} \)
Theorem 4. Let \( A \) be a projective reconstruction of \( \mathcal{P} \). For each pair \( i, j \), \( H^T q_i \) lies in the chiral domain \( \mathbb{P}^3_{AH^{-1}} \) of the camera arrangement \( AH^{-1} \). Therefore, from Theorem 2 and Lemma 1 and the requirement that cameras in the chiral reconstruction need to be finite, i.e., \( h^T c_i \neq 0 \) for all \( i \), \( (AH^{-1}, H \mathcal{Q}) \) is chiral if and only if there exist \( h, \delta \) such that for all \( i, j, k \),

\[
(7) \quad (n_i^T H q_k)(n_{A_i H^{-1}} H q_k) = \delta (h^T q_k)(h^T c_i)(n_i^T q_k) \geq 0,
\]

\[
(8) \quad (n_{A_i H^{-1}} H q_k)(n_{A_j H^{-1}} H q_k) = (h^T c_i)(h^T c_j)(n_i^T q_k)(n_j^T q_k) > 0.
\]

Recall that we write \( n_i \) as shorthand for \( n_{A_i} \). Substituting \( \sigma_{ik} \in \{-1, 1\} \) for \( \text{sign}(n_i^T q_k) \), we get the two sets \( S_1 \) and \( S_2 \) to account for the sign of \( \delta \). This shows that feasibility of (7) and (8) is equivalent to one of \( S_1 \) or \( S_2 \) being nonempty. Any tuple \( (h, \delta) \) can be completed to a \( H \in \text{GL}_4 \) where \( h^T \) is the last row of \( H \) and \( \det(H^{-1}) = \delta \).

We now introduce the notion of a signed reconstruction.

Definition 3. A signed reconstruction \( (A, Q^s) \) of \( \mathcal{P} \) is a projective reconstruction of \( \mathcal{P} \) in which for each camera \( i \), there exist constants \( \sigma_i^s \in \{-1, 1\} \) such that \( \text{sign}(n_i^T q_k^s) = \sigma_i^s \) for all \( k \). We say that a projective reconstruction \( (A, Q) \) can be signed if there exist \( Q_k^s \in \mathbb{R}^4 \) such that \( q_k^s \sim q_k \) in \( \mathbb{P}^3 \) and \( (A, Q^s) \) is a signed reconstruction.

Lemma 2. Suppose that a projective reconstruction \( (A, Q) \) of \( \mathcal{P} \) is projectively equivalent to a chiral reconstruction of \( \mathcal{P} \). Then for each pair \( i, j \), the product \( \sigma_{ik}\sigma_{jk} \) is constant for all \( k \), and \( (A, Q) \) can be signed.

Proof. Let \( (A, Q) \) be a projective reconstruction of \( \mathcal{P} \). For either \( S_1 \) or \( S_2 \) to be nonempty it is necessary that for each pair \( i, j \), the product \( \sigma_{ik}\sigma_{jk} \) is constant for every \( k \). In this case, we show that \( (A, Q) \) can be signed. For each \( k \), define \( q_k^s := q_k \) if \( \sigma_{ik} = 1 \) or \( q_k^s := -q_k \) if \( \sigma_{ik} = -1 \). By construction, \( \sigma_{ik}^s \sigma_{jk}^s = \text{sign}(n_i^T q_k^s) = 1 \) for all \( k \). After this change, we still have \( (\sigma_{ik}^s\sigma_{jk}^s) \) is constant for all \( k \). Then it follows that for each \( i \), \( \sigma_{ik}^s \) is constant for all \( k \), and \( (A, Q^s) \) is a signed reconstruction of \( \mathcal{P} \).

Note that signing a projective reconstruction \( (A, Q) \) only amounts to changing the sign of some world points. It does not affect the cameras or chirality of the world points in these cameras. We are now ready to state our main result.

Theorem 4. Given a signed reconstruction \( (A, Q) \) of \( \mathcal{P} \), there exists a chiral reconstruction \( (AH^{-1}, H \mathcal{Q}) \) if and only if

\[
(9) \quad K_Q^* \cap (\text{int} K_{\sigma_C}^* \cup \text{int} K_{-\sigma_C}^*) \neq \{0\}
\]

where \( K_{\sigma_C} = \text{cone}\{c_1, \ldots, c_m\} \), and \( \text{int} K_{\sigma_C}^* \) is the interior of its dual cone, and \( K_Q = \text{cone}\{q_1, \ldots, q_n\} \), and \( K_Q^* \) is its dual cone.

Proof. Since \( (A, Q) \) is a signed reconstruction, we may substitute the constants \( \sigma_i \) for \( \sigma_{ik} \), and rewrite \( S_1 \) and \( S_2 \) as

\[
(10) \quad S_1 = \{ h \mid \forall i, j, k, (h^T q_k)(h^T \sigma_i c_i) \geq 0, (h^T \sigma_i c_i)(h^T \sigma_j c_j) > 0 \}
\]

\[
(11) \quad S_2 = \{ h \mid \forall i, j, k, (h^T q_k)(h^T \sigma_i c_i) \leq 0, (h^T \sigma_i c_i)(h^T \sigma_j c_j) > 0 \}
\]

The set \( S_1 \) is the union of the cones \( (K_Q^* \cap \text{int} K_{\sigma_C}^*) \) and \( (K_Q^* \cap \text{int} K_{-\sigma_C}^*) \). Similarly, \( S_2 \) is the union of \( (K_{-\sigma_Q}^* \cap \text{int} K_{\sigma_C}^*) \) and \( (K_{-\sigma_Q}^* \cap \text{int} K_{-\sigma_C}^*) \). Since \( K_Q^* \cap \text{int} K_{\sigma_C} \neq \{0\} \) if and only if \( K_{-\sigma_Q}^* \cap \text{int} K_{\sigma_C}^* \neq \{0\} \), and \( K_{-\sigma_Q}^* \cap \text{int} K_{\sigma_C} \neq \{0\} \) if and only if
Let \( K^*_Q \cap \text{int} \, K^*-_{\sigma C} \neq \{0\} \), finding a chiral reconstruction reduces to checking whether \( K^*_Q \) intersects one of the cones \( \text{int} \, K^*_C \) or \( \text{int} \, K^*-_{\sigma C} \).

\[ \Box \]

**Remark 5.** Checking (9) amounts to checking whether one of two linear programs is feasible. This is the higher dimensional analog of checking the feasibility of Hartley’s chiral inequalities for two views [4]. The cones in Theorem 4 are polyhedral and the number of their generators scales linearly with the size of the scene. As a result, Theorem 4 provides a polynomial time method for constructively checking when a multiview projective reconstruction can be transformed into a chiral reconstruction.

**Remark 6.** Theorem 4 also implies that if a chiral reconstruction exists, there is one in which all world points are finite and do not lie on principal planes. The argument is as follows: Since the \( h \) produced in Theorem 2 lies in \( \text{int} \, K^*_C \cup \text{int} \, K^*-_{\sigma C} \), and \( (n^i_1 q_k) \neq 0 \), it follows that \( n^i_{A_0 H^{-1}} H q_k = \langle h^*, c_i \rangle (n^i_1 q_k) \neq 0 \) for all \( i, k \), i.e., transformed world points do not lie on principal planes of transformed cameras. Observe that \( \text{sign}(\sigma_1 n^i_1 q_k) = \sigma_1^2 > 0 \) for all \( k \), meaning \( \sigma_1 n_1 \in \text{int} \, K^*_Q \), which means that for any signed reconstruction \((A, Q)\), the cone \( K^*_Q \) is full dimensional. Consequently, any \( h \) in \( S_1 \) or \( S_2 \) can be perturbed to a vector in \( \text{int} \, K^*_Q \) and remain in \( S_1 \) or \( S_2 \).

4.2. Euclidean Reconstructions. In the previous section, we asked when a projective reconstruction can be transformed to a chiral reconstruction. We now ask the same question for a Euclidean reconstruction of \( \mathcal{P} \), by which we mean a reconstruction \((A, Q)\) in which each camera has the form \([R \ t]\) where \( R \in SO(3) \).

Proposition 10 in [4] shows that we can assume \( A_1 = [I \ 0] \) by applying an appropriate similarity, without affecting chirality. Under this assumption, the following two theorems (whose proofs appear in the Appendix) answer the above question for \( m = 2 \) and \( m > 2 \) views respectively.

**Theorem 5.** Let \( \{A_1 = [I \ 0], A_2 = [R \ t]\}, Q \) be a signed Euclidean reconstruction of \( \mathcal{P} \) with distinct centers. There exists a chiral Euclidean reconstruction of \( \mathcal{P} \) if and only if \( n_\infty \in K^*_Q \cup K^*_{-Q} \) or \([-\frac{2}{\|R^t t\|^2} R^t t, 1] \in K^*_Q \cup K^*_{-Q} \).

**Theorem 6.** Let \((A, Q)\) be a signed Euclidean reconstruction of \( \mathcal{P} \) with \( m > 2 \) cameras, distinct centers, and \( A_1 = [I \ 0] \). There exists a chiral Euclidean reconstruction of \( \mathcal{P} \) if and only if \( n_\infty \in K^*_Q \cup K^*_{-Q} \).

These theorems are specializations of Theorem 4. Their proofs are based on the observation that restricting the cameras to be Euclidean restricts the class of homographies in Theorem 4 to four \((m = 2)\) and two \((m > 2)\) discrete choices respectively. The four choices for \( m = 2 \) correspond to the well known twisted pair transformations and the two choices for \( m > 2 \) correspond to reflection.

4.3. Connections to Hartley’s Work on Two-View Chirality. Our development of multiview chirality was inspired by the seminal paper of Hartley on chirality for two views [4]. In this section we show that the main theorems on two view chirality from [3] and [4] follows from our present work.

We first show the connection between our work and Hartley’s notion of quasi-affine transformations that appears in both [4] and [3, Chapter 21]. In [3, Definition 21.3], a homography \( H \) is said to be quasi-affine with respect to a set \( \mathcal{X} \subseteq \mathbb{R}^4 \), with elements having last coordinate 1, if no point in the convex hull of \( \mathcal{X} \) is sent to infinity by \( H \). We observe...
that this is equivalent to saying that h, the last row of H, lies in int $K^*_X$ or int $K^*_{X'}$. To accommodate infinite points, we make a more general definition of a quasi-affine transformation.

**Definition 4.** A linear map $H \in \text{GL}_4$ is quasi-affine with respect to $X \subseteq \mathbb{R}^4$ if the last row $h$ of $H$ is in $K^*_X \cup K^*_{X'}$. Further, $H$ is strictly quasi-affine with respect to $X$ if $h \in \text{int } K^*_X \cup \text{int } K^*_{X'}$.

Geometrically, $H$ is quasi-affine with respect to $X$ if $HX$ lies in one of the closed halfspaces defined by the hyperplane $h^\top = \{x \in \mathbb{R}^4 : h^\top x = 0\}$, which is the plane sent to infinity by the homography $H$. If $HX$ lies in an open halfspace of $h^\top$ (as in Hartley’s setup) then $H$ is strictly quasi-affine with respect to $X$.

Recall that in a signed reconstruction $(A, Q)$ we have fixed the sign of the last coordinates of all $q_k \in Q \subseteq \mathbb{R}^4$ and of all $\sigma_c$, and all points in $Q$ and $\sigma C$ are considered to be in $\mathbb{R}^4$. We first show that Theorem 4 can be interpreted in terms of quasi-affine transformations.

**Theorem 7.** Suppose $(A, Q)$ is a signed reconstruction of $P$. Then there exists a chiral reconstruction of $P$ if and only if there is a homography $H$ that is quasi-affine with respect to $Q$ and strictly quasi-affine with respect to $\sigma C$.

**Proof.** The intersection $(K^*_Q \cup K^*-Q) \cap (\text{int}(K^*_C) \cup \text{int}(K^*_{-C}))$ is nonempty if and only if $K^*_Q \cap (\text{int}(K^*_C) \cup \text{int}(K^*_{-C}))$ is nonempty because the negative of a vector in $K^*_Q \cap (\text{int}(K^*_C) \cup \text{int}(K^*_{-C}))$ will lie in $K^*_Q \cap (\text{int}(K^*_C) \cup \text{int}(K^*_{-C}))$. The statement now follows from Theorem 4.

In the rest of this section we focus on two view reconstructions and derive several results from Hartley’s work. Recall that if we have a two view reconstruction $(\{A_1, A_2\}, Q)$ of $P$ such that $A_1 q_k = w_{1k} \hat{p}_{1k}$ and $A_2 q_k = w_{2k} \hat{p}_{2k}$, then $n_1\top q_k = \det(G_1) w_{1k}$ and $n_2\top q_k = \det(G_2) w_{2k}$. Recall also that $\sigma_{ik} = \text{sign}(n_i\top q_k)$. The products $w_{1k}w_{2k}$ have the same sign for all $k$ if and only if $(n_1\top q_k)(n_2\top q_k)$ have the same sign for all $k$, i.e., $\sigma_{1k}\sigma_{2k}$ is constant for all $k$. We will assume that the centers of $A_1$ and $A_2$ are distinct. In this language, Theorem 17 in [4] (also 16, Theorem 1) says the following:

**Theorem 8.** Theorem 17: A projective reconstruction $(\{A_1, A_2\}, Q)$ of $P$ can be transformed by a homography $H$ to a chiral reconstruction if and only if $(n_1\top q_k)(n_2\top q_k)$ have the same sign for all $k$.

**Proof.** The only if direction was proved in Lemma 2.

For the “if” direction, suppose $(n_1\top q_k)(n_2\top q_k)$ have the same sign for all k. We first note that $\sigma_1 n_1$ is a nonzero element of either $K^*_C$ or $K^*_{-C}$. Indeed, $(\sigma_1 n_1)^\top (\sigma_1 c_1) = 0$ and if $\text{sign}(\sigma_1 n_1)^\top (\sigma_2 c_2) = 1$ or $\text{sign}(\sigma_1 n_1)^\top (\sigma_2 c_2) = 0$, then $\sigma_1 n_1 \in K^*_C$. Otherwise if $\text{sign}(\sigma_1 n_1)^\top (\sigma_2 c_2) = -1$, then $\sigma_1 n_1 \in (K^*_C)$.

Also, since the centers $c_1$ and $c_2$ are distinct, $\sigma_1 c_1$ is not a scalar multiple of $\sigma_2 c_2$, hence $K^*_{-C}$ is a pointed cone, i.e., does not contain a line. This implies that $K^*_C$ is full-dimensional and hence has an interior. The same is true for $K^*_{-C}$.

Without loss of generality suppose $\sigma_1 n_1 \in K^*_C$. Since $\text{sign}(n_1\top q_k) = \sigma_1$, we have that $\sigma_1 n_1 \top q_k > 0$ for all $k$, and so $\sigma_1 n_1 \in \text{int } K^*_Q$. Let $U$ be a neighborhood of $\sigma_1 n_1$ contained in $\text{int } K^*_Q$. Since $\sigma_1 n_1$ is also in $K^*_{-C}$, there is some $h \in U$ that lies in the int $K^*_Q \cap \text{int } K^*_{-C}$. This $h$ is in $S_1$, so by Theorem 4 $P$ has a chiral reconstruction.
Since the \( h \) constructed at the end of the proof of Theorem 8 lies in \( \text{int} \, K_\Sigma \cap \text{int} \, K_{\sigma C}^* \), we can strengthen Theorem 7 in the case of two views as follows.

**Corollary 1.** Suppose \( \{A_1, A_2\}, Q \) is a signed reconstruction of \( P \). Then there exists a chiral reconstruction of \( P \) if and only if there is a homography \( H \) that is strictly quasi-affine with respect to both \( Q \) and \( \sigma C \).

Part (ii) of Theorem 21.7 in [3] says the following. Suppose \( \{A_1, A_2\}, Q \) is a projective reconstruction of \( P \) for which there is a projectively equivalent chiral reconstruction, and \( n_1^\top q_k \) have the same sign for all \( k \). Then the homography \( H \) that yields a chiral reconstruction \( \{A_1H^{-1}, A_2H^{-1}\}, HQ \) can be chosen to be strictly quasi-affine with respect to \( Q \). Indeed, by Theorem 8, since \( (A, Q) \) can be transformed to a chiral reconstruction by a homography, we must have that \( (n_1^\top q_k)(n_2^\top q_k) \) have the same sign for all \( k \). If now we also have that \( n_1^\top q_k \) have the same sign for all \( k \), then it follows that \( n_2^\top q_k \) have the same sign for all \( k \). Therefore, \( (A, Q) \) is already signed. The result now follows from Corollary 1.

Hartley’s work was done with the aim of upgrading a two view projective reconstruction to a metric reconstruction. In follow up work, Nistér addresses this question for multiple views [9]. He does this by transforming the projective reconstruction into one which is quasi-affine with respect to the camera centers. As can be seen from Theorem 7 above, quasi-affineness with respect to the camera centers is a necessary condition for chirality. He does not enforce quasi-affineness with respect to the scene points, because they are often noisy and their chirality may change as part of the metric upgrade. Nistér argues shows that enforcing the quasi-affineness on camera centers makes the iterative algorithm used to perform the subsequent metric upgrade easier and more reliable.

5. THE CHIRAL JOINT IMAGE

In this final section, we address question 2 from the Introduction, and algebraically describe the set of images of world points that have chirality one with respect to an arrangement of cameras. The algebraic study of this set in the nonchiral case leads to the multiview constraints. Our study will lead to the chiral multiview constraints, the semi-algebraic analog of multiview constraints.

World points are imaged in an arrangement of finite cameras \( A = \{A_1, \ldots, A_m\} \) via the rational map

\[
\varphi_A : \mathbb{P}^3 \rightarrow (\mathbb{P}^2)^m, \quad q \mapsto (A_1q, A_2q, \ldots, A_mq)
\]

Triggs calls \( \varphi_A(\mathbb{P}^3) \) the joint image [12, 13] and Heyden-Åström call it the natural descriptor [5]. The Zariski closure of \( \varphi_A(\mathbb{P}^3) \) in \((\mathbb{P}^2)^m\) by \( \varphi_A(\mathbb{P}^3)_{\text{Zar}} \). Trager et al. refer to it as the joint image variety of \( A \) [11] and characterize it as follows:

**Theorem 9** (11, Proposition 1). Given an arrangement of cameras \( A = \{A_1, \ldots, A_m\} \), with distinct camera centers, let \( E_j = e_{1j} \times \ldots, e_{m_j} \times e_{mj} \), and \( E_A = \bigcup_{j=1}^m E_j \). Then

\[
\varphi_A(\mathbb{P}^3)_{\text{Zar}} = \varphi_A(\mathbb{P}^3) \cup E_A.
\]

Recall that the epipolar and trifocal constraints cut out the joint image variety [11].

\[\text{footnote}^{2}\text{Again, the broken arrow (\(\rightarrow\)) and the words “rational map” refer to the fact that the domain of the map } \varphi_A \text{ is not } \mathbb{P}^3 \text{ but rather } \mathbb{P}^3 \setminus \{c_1, \ldots, c_m\}. \]
The problem comes from the image of points that lie on the baseline connecting pairs of cameras. For any arrangement of cameras in $\mathbb{P}^3$, we will be working with the Euclidean topology on $(\mathbb{P}^2)^m$, and write $\varphi_A(\mathbb{P}^3)$ for the Euclidean closure of the joint image. Luckily, the Euclidean and Zariski closure of the joint image are the same. The proof of the following theorem can be found in the Appendix.

**Theorem 10.** \(\varphi_A(\mathbb{P}^3)^{\text{Zar}} = \varphi_A(\mathbb{P}^3)\).

As a result, we only consider closure in the Euclidean topology going forward. Our interest in this section is in the following set.

**Definition 5 (Chiral Joint Image).** The chiral joint image of a camera arrangement $A$ is $\varphi_A(\mathbb{P}^3_A)$, the image of the chiral domain of $A$ under $\varphi_A$.

The rest of this section is devoted to the algebraic description of $\varphi_A(\mathbb{P}^3_A)$, the Euclidean closure of the chiral joint image. We begin by defining two sets.

**Definition 6.** Given an arrangement of finite cameras $A_i = [G_i \ t_i]$, define
\begin{equation}
C_A := \left\{ \mathbf{p} \in (\mathbb{P}^2)^m \mid \det(G_i) p_{ij} (a_i \times a_j) \mathbf{b}_{ij} \geq 0, \quad \det(G_i) \det(G_j) p_{ij} p_{ji} \left( (b_{ij} \times a_i) \mathbf{b}_{ij} \right) \geq 0 \right\},
\end{equation}
where $\mathbf{p} = (p_1, p_2, \ldots, p_m)$, $b_{ij} = G_i^{-1} t_i - G_j^{-1} t_j$ is a direction of the baseline connecting the centers of cameras $A_i$ and $A_j$, and $a_i = G_i^{-1} t_i$.

Equation (13) is well-defined on $(\mathbb{P}^2)^m$ because every inequality defining $C_A$ has even degree in the coordinates on the $\mathbb{P}^2$-factors. In fact, the three inequalities are all biquadratic, i.e. of degree $(2, 2)$. Moreover, the sign does not depend on the choice of the order of the cameras in the arrangement because this choice is implicit in $b_{ij}$ and explicit in the terms $(a_i \times a_j)$ in the inequalities. So a relabeling of the cameras will not change the signs involved.

We will see that $C_A$ is obtained from the description of $\mathbb{P}^3_A$ by eliminating the world points. A first guess might be that the $\varphi_A(\mathbb{P}^3_A) = \varphi_A(\mathbb{P}^3) \cap C_A$. Indeed this is roughly true, but as we will see in Theorem 11 the precise statement requires a bit more care. The problem comes from the image of points that lie on the baselines connecting pairs of cameras. This leads us to the following definition.

**Definition 7.** Given a camera arrangement $A$ define $B_{ij} := \{ \mathbf{p} \in (\mathbb{P}^2)^m \mid \exists \lambda \neq 0, \mu \neq 0 \text{ s.t. } \mathbf{p} = \varphi_A(\lambda e_i + \mu e_j) \}$ and $B_A := \bigcup_{i,j} B_{ij}$.

The set $B_A$ consists of images of points in $\mathbb{P}^3$ that lie on the baselines of pairs of cameras in $A$. We cannot hope to determine the chirality of points in $B_A$ (with respect to $A$) from only their image coordinates. This is because, given a pair of cameras $\{A_i, A_j\}$, every point $\mathbf{q}$ on the baseline gets imaged to the pair of epipoles $(e_{ij}, e_{ji})$, but different world points $\mathbf{q}$ on the baseline can have different chiralities with respect to the camera pair depending on their individual orientations.

Our main result is the following theorem, which relates the intersection $\varphi_A(\mathbb{P}^3) \cap C_A$ to the chiral joint image $\varphi_A(\mathbb{P}^3_A)$ minus the set $B_A$.

---

3Recall that the topology we use on $\mathbb{P}^n$ is induced by the Euclidean topology on $\mathbb{R}^{n+1} \setminus \{0\}$, which is a topology on the product of real projective spaces $(\mathbb{P}^2)^m$. Explicitly, a set $U_1 \times U_2 \times \ldots \times U_m \subseteq (\mathbb{P}^2)^m$ is open if and only if the sets $U_i \subseteq \mathbb{P}^2$ are all open sets.
Theorem 11. Let $\mathcal{A}$ be an arrangement of finite cameras with distinct centers and assume that the chiral domain $\mathbb{P}_\mathcal{A}^3$ is nonempty. Let $E_\mathcal{A}^0$ be the union of all $E_j$ such that $c_j$ lies in $\mathbb{P}_\mathcal{A}^3$ and on the principal plane of a camera $A_i$ with $i \neq j$. If the camera centers are not collinear, then

$$
\phi(\mathbb{P}_\mathcal{A}^3) \cap C_\mathcal{A} \setminus B_\mathcal{A} = \phi(\mathbb{P}_\mathcal{A}^3) \cup E_\mathcal{A}^0 \setminus B_\mathcal{A}.
$$

If furthermore no center $c_j$ lies on the principal plane of $A_i$ ($i \neq j$), then

$$
\phi(\mathbb{P}_\mathcal{A}^3) \cap C_\mathcal{A} \setminus B_\mathcal{A} = \phi(\mathbb{P}_\mathcal{A}^3) \setminus B_\mathcal{A}.
$$

If the camera centers are collinear, there is only one epipole $e_i$ in every camera $A_i$ and the following holds

$$
\phi(\mathbb{P}_\mathcal{A}^3) \cap C_\mathcal{A} = \phi(\mathbb{P}_\mathcal{A}^3) \cup E_\mathcal{A}^0 \cup \{(e_1, e_2, \ldots, e_m)\}.
$$

While the statement of the above theorem in its full generality may seem daunting, we wish to highlight the generic case that is captured by Equation (15). It says that except for the set $B_\mathcal{A}$, the closure of the chiral joint image is the joint image variety intersected with the set $C_\mathcal{A}$. Stated algebraically, the epipolar and trifocal constraints together with the inequalities in (13) are the chiral multiview constraints.

Specializations of Theorem 11 to Euclidean cameras and finite images are straightforward and are omitted due to lack of space.

The proof of Theorem 11 relies on characterizing a number of set intersections carried out in the following lemmas whose proofs can be found in the Appendix. We will need to understand $(\phi(\mathbb{P}_\mathcal{A}^3) \cap C_\mathcal{A}) \setminus B_\mathcal{A}$. Since $\phi(\mathbb{P}_\mathcal{A}^3) = \phi(\mathbb{P}_\mathcal{A}^3) \cup E_\mathcal{A}$, we consider each piece separately. As we will see, the set $E_\mathcal{A}$ has to be parsed carefully.

Lemma 3. Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be an arrangement of finite cameras. If the centers are not collinear, we have $E_\mathcal{A} \cap B_\mathcal{A} = \emptyset$. If the centers are collinear, we have $E_\mathcal{A} \cap B_\mathcal{A} = B_\mathcal{A} = \{(e_1, e_2, \ldots, e_m)\}$.

Lemma 4. Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be an arrangement of finite cameras and let

$$
E_\mathcal{A}^+ := \bigcup_{j \mid c_j \in \mathbb{P}_A^3} E_j.
$$

If the centers $c_i$ are not collinear then $E_\mathcal{A} \cap C_\mathcal{A} = E_\mathcal{A}^+$. Otherwise, $E_\mathcal{A} \cap C_\mathcal{A} = E_\mathcal{A}^+ \cup \{(e_1, e_2, \ldots, e_m)\}$.

Lemma 5. Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be an arrangement of finite cameras with distinct centers. Let $E_\mathcal{A}^{++}$ be the union of the sets $E_j$ such that $c_j$ has positive depth in every camera $A_i \in \mathcal{A} \setminus \{A_j\}$, then $E_\mathcal{A}^{++} \subseteq \phi(\mathbb{P}_\mathcal{A}^3)$.

Note that $E_\mathcal{A}^0, E_\mathcal{A}^+ \subseteq E_\mathcal{A}^+ \setminus E_\mathcal{A}^{++}$.

Lemma 6. Let $\mathcal{A} = \{A_1, \ldots, A_m\}$ be an arrangement of finite cameras such that $\mathbb{P}_\mathcal{A}^3$ is nonempty. Then,

$$
(\phi(\mathbb{P}_\mathcal{A}^3) \setminus B_\mathcal{A}) \cap C_\mathcal{A} = \phi(\mathbb{P}_\mathcal{A}^3) \setminus B_\mathcal{A}.
$$

In particular, $\phi(\mathbb{P}_\mathcal{A}^3) \subseteq C_\mathcal{A}$.

With these lemmas in hand, we are now ready to prove Theorem 11.
Theorem 4.1. We will first prove Equation (14) which is the case of noncollinear centers. By Lemma 6, we get \( \varphi_A(\mathbb{P}^3) \subseteq C_A \). Therefore, \( \varphi_A(\mathbb{P}^3) \subseteq \varphi_A(\mathbb{P}^3) \cap C_A \). Furthermore, Lemma 3 shows that \( E_A^+ \) (so in particular \( E_A^0 \)) is contained in \( C_A \), but since all of \( E_A \) is inside \( \varphi_A(\mathbb{P}^3) \), we also have \( E_A^0 \subseteq \varphi_A(\mathbb{P}^3) \cap C_A \). Therefore, \( \varphi_A(\mathbb{P}^3) \cup E_A^0 \subseteq \varphi_A(\mathbb{P}^3) \cap C_A \).

For the other inclusion, we consider two cases according to Theorem 9 which says that \( \varphi_A(\mathbb{P}^3) = \varphi_A(\mathbb{P}^3) \cup E_A \).

Observe that, \( (\varphi_A(\mathbb{P}^3) \cap C_A) \setminus B_A = (\varphi_A(\mathbb{P}^3) \setminus B_A) \cap C_A = \varphi_A(\mathbb{P}^3) \setminus B_A \), where the last equality follows from Lemma 6. From Lemma 4 in the noncollinear case we have that \( (E_A \cap C_A) \setminus B_A = E_A^+ \setminus B_A \). Then putting these two together we get

\[
\begin{align*}
(\varphi_A(\mathbb{P}^3) \cap C_A) \setminus B_A &\subseteq (\varphi_A(\mathbb{P}^3) \setminus B_A) \cap C_A = \varphi_A(\mathbb{P}^3) \setminus B_A \quad (18) \\
((\varphi_A(\mathbb{P}^3) \cup E_A) \cap C_A) \setminus B_A &\subseteq (\varphi_A(\mathbb{P}^3) \cup E_A^+) \setminus B_A = (\varphi_A(\mathbb{P}^3) \cup E_A^+) \setminus B_A \quad (19) \\
(\varphi_A(\mathbb{P}^3) \cup E_A^+) \cap C_A &\setminus B_A = (\varphi_A(\mathbb{P}^3) \cup E_A^+) \setminus B_A \quad (20)
\end{align*}
\]

The last equality follows from Theorem 9 and Lemma 4. Passing to the closure of \( \varphi_A(\mathbb{P}^3) \) gives us:

\[
\begin{align*}
(\varphi_A(\mathbb{P}^3) \cap C_A) \setminus B_A &\subseteq (\varphi_A(\mathbb{P}^3) \setminus B_A) \cap C_A = \varphi_A(\mathbb{P}^3) \setminus B_A \quad (21) \\
(\varphi_A(\mathbb{P}^3) \cup E_A^+) \cap C_A &\setminus B_A = (\varphi_A(\mathbb{P}^3) \cup E_A^+) \setminus B_A
\end{align*}
\]

The last equality follows from Lemma 5.

If no center of a camera lies on the principal plane of any other, then \( E_A^+ = E_A^+ \), i.e. \( E_A^0 = \emptyset \), and Equation (15) is a special case of Equation (14).

The collinear case follows the same proof mechanics as above, utilizing the fact that \( B_A = \{ (e_1, \ldots, e_m) \} \) and that \( B_A \subseteq \varphi_A(\mathbb{P}^3) \cap C_A \).

\[\square\]

**Technical Proofs**

This appendix contains the proofs of statements not proved in the main paper due to reasons of space or narrative clarity. The numbering of theorems and lemmas matches those in the main paper and they are presented here in the order in which they appear in the main paper. In some cases, these proofs rely on additional lemmas (lemmas 7 to 10) which are only present in this appendix. As a result, some lemmas appear out of order because they are presented in the order they are needed.

**Proofs from Section 4.1**

**Lemma 4.1.** Let \( A = [G \quad t] \) be a finite camera with center \( c_A \). Let \( H \in \text{GL}_4 \) with fourth row \( h^\top \) and \( \delta = \det(H^{-1}) \). Then

(1) After the homography, the plane at infinity is \( h^\top q = 0 \).

(2) The camera \( AH^{-1} \) is finite if and only if \( h^\top c_A \neq 0 \). Its center then is \( c_{AH^{-1}} = \frac{1}{h^\top c_A} H c_A \).

(3) The principal ray of \( AH^{-1} \) is \( n_{AH^{-1}} = \delta(h^\top c_A)H^{-\top} n_A \).

(4) \( \forall q \in \mathbb{R}^4, n_{AH^{-1}}^\top (Hq) = \delta(h^\top c_A)(n_A^\top q) \)

**Proof.**

(1) A point \( Hq \) lies on the plane at infinity if and only if \( h^\top q = 0 \).

(2) The equivalence in the claim follows from the previous part. Further, a representative \( c \in \mathbb{P}^3 \) for the center of \( A \) can be computed using Cramer’s rule so that its
last coordinate is \( \det(G) \), see [4]. Therefore, \( c_A = \frac{1}{\det(G)}c \). Using Cramer’s rule again shows that \( \delta Hc \) is a representative for the center of \( AH^{-1} \), which shows

\[
c_{AH^{-1}} = \frac{1}{h^\top c_A} Hc_A.
\]

(3) The last coordinate of \( \delta Hc \) is the determinant of the first \( 3 \times 3 \) block of \( AH^{-1} \).

The principal ray of \( AH^{-1} \) is therefore,

\[
n_{AH^{-1}} = (\delta Hc)_4 (A_{3\bullet} H^{-1})^\top
\]

\[
= \delta \det(G)(h^\top c_A) H^{-\top} A_{3\bullet}
\]

\[
= \delta (h^\top c_A) H^{-\top} n_A
\]

(4) Plugging in the principal ray from the previous part, we compute

\[
(\mathbf{n}_{AH^{-1}}(H q)) = \delta (h^\top c_A)(\mathbf{n}_A H^{-1} H q)
\]

\[
= \delta (h^\top c_A)(\mathbf{n}_A q)
\]

for all \( q \in \mathbb{R}^4 \). \( \square \)

**Proofs from Section 4.2**

Applying techniques from Section 4.1, we show when a Euclidean reconstruction can be made chiral using a homography. As we argue in Section 4.2, we may assume that our starting and target reconstructions have \( A_1 = [I \quad 0] \). This choice of the first cameras restricts the homographies we need to consider to \( H \) such that \( H^{-1} = \begin{bmatrix} I & 0 \\ v^\top & \delta \end{bmatrix} \) for some \( v \in \mathbb{R}^3 \) and nonzero \( \delta \in \mathbb{R} \). Note that \( \delta = \det H^{-1} \).

We now introduce the notion of a quasi-Euclidean camera.

**Definition 8.** A camera \( A = [U \quad t] \) is quasi-Euclidean if \( U U^\top = I \).

While we are interested in transforming a Euclidean reconstruction into a chiral Euclidean reconstruction, a homography may only be able to yield a reconstruction where the transformed cameras are quasi-Euclidean. However, since scaling a camera does not change chirality, a chiral quasi-Euclidean reconstruction can be turned into a chiral Euclidean reconstruction by multiplying \( A_1 \) by \( \text{sign}(\det(G_i)) \). As a result, we only need to search for a homography \( H \) that sends our starting Euclidean reconstruction to one where every camera is quasi-Euclidean, which bring us to the following lemma.

**Lemma 7.** Given a Euclidean camera \( A = [R \quad t] \) such that \( t \neq 0 \) and a homography \( H \) such that \( H^{-1} = \begin{bmatrix} I & 0 \\ v^\top & \delta \end{bmatrix} \) for some vector \( v \in \mathbb{R}^3 \) and \( \delta \neq 0 \), the camera \( AH^{-1} \) is quasi-Euclidean if and only if \( v = 0 \) or \( v = -\frac{2}{\|t\|^2} R^\top t \).

**Proof.** The requirement that \( AH^{-1} \) be quasi-Euclidean translates to

\[
I = (R + tv^\top)^\top (R + tv^\top) - R^\top R + vt^\top R + R^\top tv^\top + vt^\top tv^\top
\]

\[
= I + vt^\top R + R^\top tv^\top + \|t\|^2 vv^\top
\]

For the fixed vector \( \mathbf{c} := -R^\top t \neq 0 \), this system is equivalent to finding \( v \) such that \( M := -v\mathbf{c}^\top - \mathbf{c}v^\top + (v\mathbf{c}^\top)(\mathbf{c}v^\top) = 0 \). Certainly \( v = 0 \) is one solution. Otherwise,
applying $M$ to $v$, we get that
\begin{align}
0 &= Mv = -(c^Tv)v - (v^Tv)c + (\tilde{c}^Tv)v \\
&= ((\tilde{c}^Tv)(v^Tv) - (\tilde{c}^Tv)v - (v^Tv)\tilde{c}).
\end{align}

If $(\tilde{c}^Tv)(v^Tv) - (\tilde{c}^Tv)v = 0$ for some $v \neq 0$, then $Mv = (v^Tv)\tilde{c} \neq 0$. Therefore, Equation (28) implies that $v = \lambda \tilde{c}$ for some $\lambda \neq 0$. Solving for $\lambda$, we get $\lambda = \frac{2}{2||\tilde{c}||^2}$. Which gives us the only additional solution $v = \frac{2}{||\tilde{c}||^2}\tilde{c} = -\frac{2}{||\tilde{c}||^2}R^Tt$.

Without loss of generality, we may assume the homographies in Lemma 7 have $|\delta| = 1$, leaving us with the following four possibilities for two view Euclidean reconstructions:
\begin{align}
H_1^{-1} &= \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix}, \\
H_2^{-1} &= \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix}, \\
H_3^{-1} &= \begin{bmatrix} I & 0 \\ v^T & 1 \end{bmatrix}, \\
H_4^{-1} &= \begin{bmatrix} I & 0 \\ v^T & -1 \end{bmatrix}
\end{align}

where $v = -\frac{2}{||\tilde{c}||^2}R^Tt$. These have the following inverses.
\begin{align}
H_1 &= \begin{bmatrix} I & 0 \\ 0 & 1 \end{bmatrix}, \\
H_2 &= \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix}, \\
H_3 &= \begin{bmatrix} I & 0 \\ -v^T & 1 \end{bmatrix}, \\
H_4 &= \begin{bmatrix} I & 0 \\ v^T & -1 \end{bmatrix}
\end{align}

A Euclidean reconstruction $(\{A_1 = [I \ 0], A_2 = [R \ t]\}, Q)$ can be made chiral if and only if one of $(AH_1^{-1}, H_1Q)$ is chiral. Just as in the projective case, we assume we start with a signed reconstruction. Let $h_i$ be the last row of $H_i$. From Theorem 6 we know we need only check if one of $h_i$ lies in the cone intersection $K_0^\oplus \cap (\text{int } K_{\sigma C}^\oplus \cup \text{int } K_{-\sigma C}^\oplus)$. As the following lemma shows, the special structure of $h_i$ causes the cone conditions to simplify.

**Lemma 8.** Let $(\{A_1 = [I \ 0], A_2 = [R \ t]\}, Q)$ be a signed Euclidean reconstruction of $\mathcal{P}$ such that $t \neq 0$.

1. If $\sigma_1 = \sigma_2$, then $h_1, h_2 \in \text{int } K_{\sigma C}^\oplus \cup \text{int } K_{-\sigma C}^\oplus$ and $h_3, h_4 \notin \text{int } K_{\sigma C}^\oplus \cup \text{int } K_{-\sigma C}^\oplus$.
2. If $\sigma_1 \neq \sigma_2$ then $h_3, h_4 \in \text{int } K_{\sigma C}^\oplus \cup \text{int } K_{-\sigma C}^\oplus$ and $h_1, h_2 \notin \text{int } K_{\sigma C}^\oplus \cup \text{int } K_{-\sigma C}^\oplus$.

**Proof.** We first compute $h_i^T\sigma_jc_j$ for all $i, j$:
\begin{align}
(31) & \quad h_1^T\sigma_1c_1 = \sigma_1, \quad h_1^T\sigma_2c_2 = \sigma_2 \\
(32) & \quad h_2^T\sigma_1c_1 = -\sigma_1, \quad h_2^T\sigma_2c_2 = -\sigma_2 \\
(33) & \quad h_3^T\sigma_1c_1 = \sigma_1, \quad h_3^T\sigma_2c_2 = (-v^T(-R^Tt) + 1)\sigma_2 = -\sigma_2 \\
(34) & \quad h_4^T\sigma_1c_1 = -\sigma_1, \quad h_4^T\sigma_2c_2 = (v^T(-R^Tt) - 1)\sigma_2 = \sigma_2
\end{align}

The vectors $h_1$ and $h_2$ make the same sign inner product with $\sigma_1c_1$ and $\sigma_2c_2$ if and only if $\sigma_1 = \sigma_2$. Similarly the vectors $h_3$ and $h_4$ make the same sign inner product with $\sigma_1c_1$ and $\sigma_2c_2$ if and only if $\sigma_1 = -\sigma_2$.

**Theorem 5** Let $(\{A_1 = [I \ 0], A_2 = [R \ t]\}, Q)$ be a signed Euclidean reconstruction of $\mathcal{P}$ with distinct centers. There exists a chiral Euclidean reconstruction of $\mathcal{P}$ if and only if $n_\infty \in K_0^\oplus \cup K_{-\sigma C}^\oplus$ or $\left[-\frac{2}{||\tilde{c}||^2}R^Tt, 1\right] \in K_0^\oplus \cup K_{-\sigma C}^\oplus$. 

□
Theorem 9. Let \((A, Q)\) be a signed Euclidean reconstruction of \(\mathcal{P}\) with \(m > 2\) cameras, distinct centers, and \(A_1 = \begin{bmatrix} I & 0 \end{bmatrix}\). There exists a chiral Euclidean reconstruction of \(\mathcal{P}\) if and only if \(n_\infty \in K_\mathcal{Q}^* \cup K_\mathcal{Q}^*\).

Proof. Since the cameras have distinct centers, the vectors \(\mathbf{h}_1 = n_\infty \in K_\mathcal{Q}^* \cup \mathbb{R}^\perp\) will not coincide, so by Lemma 7, the only homographies we can consider are \(H_1\) and \(H_2\). As in Lemma 8, \(\mathbf{h}_1 = n_\infty, \mathbf{h}_2 = -n_\infty \in \text{int} K_\mathcal{Q}^* \cup \text{int} K_\mathcal{Q}^*\), if and only if \(\sigma_i = \sigma_j\) for all \(i, j\). When this is the case, a chiral reconstruction exists if and only if \(n_\infty \in K_\mathcal{Q}^* \cup K_\mathcal{Q}^*\), proving the statement.

Theorem 10. \(\overline{\varphi_A(\mathbb{P}^3)}^Z_{ar} = \overline{\varphi_A(\mathbb{P}^3)}\).

Proof. Recall that \(\mathbf{p} = (\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_m) \in E_j\) is of the form \((e_{1j}, \ldots, \mathbf{p}_j, \ldots, e_{mj})\) for some \(\mathbf{p}_j \in \mathbb{P}^2\). So all coordinates of \(\mathbf{p}\) except \(\mathbf{p}_j\) are the images of \(e_j = \begin{bmatrix} -G_j^{-1} & t_j \end{bmatrix}\).

Consider now the curve
\[
\mathbf{v}(s) = \begin{bmatrix} sG_j^{-1} & -G_j^{-1} & t_j \\ 1 & 0 & 1 \end{bmatrix}
\]
as \(s\) varies over \(\mathbb{R}\). Then \(\lim_{s \to 0} \varphi_A(\mathbf{v}(s)) = \mathbf{p}\), since for \(i \neq j\), \(A_i \mathbf{v}(s) = sG_iG_j^{-1} \mathbf{p}_j + e_{ij}\) and \(A_j \mathbf{v}(s) = s \mathbf{p}_j + e_j\). So \(\mathbf{p} \in \overline{\varphi_A(\mathbb{P}^3)}\). Hence \(E_j \subseteq \overline{\varphi_A(\mathbb{P}^3)}\). Therefore, by Theorem 9, \(\overline{\varphi_A(\mathbb{P}^3)}^Z_{ar} \subseteq \overline{\varphi_A(\mathbb{P}^3)}\). This means that
\[
\overline{\varphi_A(\mathbb{P}^3)} \subseteq \overline{\varphi_A(\mathbb{P}^3)}^Z_{ar} \subseteq \overline{\varphi_A(\mathbb{P}^3)}
\]
and taking Euclidean closure throughout and noting that Zariski closed sets are also closed in the Euclidean topology, we get the needed equality.

Lemma 9. Let \(A = \{A_1, A_2\}\) be a pair of finite cameras \(A_i = [G_i, \mathbf{t}_i]\) with distinct centers. Fix \(\mathbf{q} \in \mathbb{P}^3\) and write \((\mathbf{p}_1, \mathbf{p}_2) \sim \varphi_A(\mathbf{q})\) with \(A_i \mathbf{q} = \lambda_i \mathbf{p}_i\). Then, either \(a_1, a_2\) and \(b_{12}\) are collinear, in which case \(\mathbf{q}\) lies on the baseline of the camera pair so that \(\mathbf{p}_1 = e_{12}\) and \(\mathbf{p}_2 = e_{21}\), or the following conditions hold:

\[
\mathbf{b}_{12}^\top (a_1 \times a_2) = 0,
\]
\[
\text{sign}(\lambda_1 a_4) = \text{sign} \left( (a_1 \times a_2)^\top (b_{12} \times a_2) \right),
\]
\[
\text{sign}(\lambda_2 a_4) = \text{sign} \left( (a_1 \times a_2)^\top (b_{12} \times a_1) \right),
\]
\[
\text{sign}(\lambda_1 \lambda_2) = \text{sign} \left( (b_{12} \times a_1)^\top (b_{12} \times a_2) \right).
\]
We now consider two cases:

1. \( q_4 \neq 0 \). Since \( \lambda_1 \) and \( \lambda_2 \) are nonzero if \( q_4 \neq 0 \), \( 40 \) shows that the three vectors are coplanar and \( 36 \) is satisfied. Further, it is straightforward to see that either \( a_1 \times a_2 \), \( b \times a_1 \) and \( b \times a_2 \) are all equal to zero or not, i.e. either \( a_1, a_2, \) and \( b \) are all pairwise collinear or not. If they are not, then \( 37, 38, 39 \) follow from multiplying each equality above by the transpose of the left hand side.

2. \( q_4 = 0 \) implies that \( a_1 \) and \( a_2 \) are collinear and \( a_1 \times a_2 = 0 \). This proves \( 36, 37 \) and \( 38 \). If \( b \) is not collinear with \( a_1 \) and \( a_2 \) \( 39 \) follows by multiplying the third equality above by its right hand side.

\[ \det(G_1) p_{13} (a_1 \times a_2)^\top (b \times a_2) \geq 0. \]

Plugging \( a_1 = \lambda_1 b \) and \( p_1 = e_{12} = \lambda_1 s_{13} \) in the above we get

\[ \det(G_1) \lambda_1 s_{13} (a_1 \times a_2)^\top (b \times a_2) \geq 0 \]

\[ \det(G_1) \lambda_1^2 s_{13} \| b \times a_2 \|^2 \geq 0 \]

\[ \det(G_1) s_{13} \| b \times a_2 \|^2 \geq 0 \]

This can be satisfied in two ways, namely \( b \times a_2 = 0 \) or \( \det(G_1) s_{13} \geq 0 \).

Suppose \( b \times a_2 = 0 \). Then since

\[ b \times a_2 = 0 \iff a_2 \sim b \iff p_2 \sim G_2 b \sim e_{21}, \]

the condition \( b \times a_2 = 0 \) is the same as \( p_2 \sim e_{21} \).

Now suppose \( \det(G_1) s_{13} \geq 0 \). Observe that the depth of \( e_2 \) in \( A_1 \) is

\[ \text{depth}(e_2; A_1) = \frac{1}{|\det(G_1)| \| G_1^s \| \det(G_1) s_{13}}. \]

Therefore, \( \det(G_1) s_{13} \geq 0 \) if and only if \( e_2 \) has nonnegative depth in \( A_1 \). In this case, the inequality \( 44 \) imposes no constraints on \( p_2 \) as claimed.
Lemma 3. Let \( A = \{A_1, A_2, \ldots, A_m\} \) be an arrangement of finite cameras. If the centers are not collinear, we have \( E_A \cap B_A = \emptyset \). If the centers are collinear, we have \( E_A \cap B_A = B_A = \{(e_1, e_2, \ldots, e_m)\} \).

Proof. By definition, \( B_A \) is contained in \( \varphi_A(\mathbb{P}^3) \). On the other hand, if the centers are not collinear, \( E_A \cap \varphi_A(\mathbb{P}^3) = \emptyset \). Indeed, given \( j \in \{1, 2, \ldots, m\} \), we can find \( k, \ell \in \{1, 2, \ldots, m\} \) such that \( c_j, c_k, \) and \( c_\ell \) are not collinear. Since we have \( p_k = e_{kj} \) and \( p_\ell = e_{\ell j} \) for all \( p = (p_1, p_2, \ldots, p_m) \in E_j \), the only possible preimage of \( p \) under \( \varphi_A \) would have to be \( c_j \), where the rational map \( \varphi_A \) is not defined. If the centers are collinear, then \( B_A = \{(e_1, e_2, \ldots, e_m)\} \), where \( e_i \) is the only epipole in the image of \( A_i \). This point lies in \( E_j \) for all \( j \in \{1, 2, \ldots, m\} \) as well, finishing the proof.

Lemma 4. Let \( A = \{A_1, A_2, \ldots, A_m\} \) be an arrangement of finite cameras. Let \( E_A^+ := \bigcup_{j | \epsilon_j \in \mathbb{P}_A^3} E_j \). If the centers \( \epsilon_j \) are not collinear then \( C_A \cap E_A = E_A^+ \). Otherwise, \( C_A \cap E_A = E_A^+ \cup \{(e_1, \ldots, e_m)\} \).

Proof. We first show that \( E_A^+ \) is in \( C_A \). Since the inequalities defining \( C_A \) only depend on pairs of cameras, we can restrict to the case of every pair \( \{A_k, A_\ell\} \). If none of the indices are equal to \( j \), then the cameras see the center of camera \( A_j \) and the inequalities are satisfied if \( \epsilon_j \in \mathbb{P}_A^3 \). If one of the indices is equal to \( j \), we use the previous Lemma 10. So we conclude \( E_j \subseteq C_A \) in case \( \epsilon_j \in \mathbb{P}_A^3 \).

Now we consider the case that \( \epsilon_j \) has negative depth in a camera \( A_k \in A \). If the camera centers are not collinear, we can choose a camera \( A_\ell \) with \( \ell \neq j \) such that \( \epsilon_j, \epsilon_k, \) and \( \epsilon_\ell \) do not lie on a line. For any point in \( E_j \), the cameras \( A_k \) and \( A_\ell \) see the image of \( \epsilon_j \). So the inequality \( \det(G_k)p_{k\ell}(a_k \times a_\ell)^\top(b_{k\ell} \times a_\ell) \) is violated by Lemma 5, which shows that \( E_j \cap C_A = \emptyset \).

If the camera centers are collinear so that \( (e_1, e_2, \ldots, e_m) \) exists, this point trivially lies in \( C_A \) because all defining inequalities evaluate to 0 on this point. Again, let \( A_k \) be a camera such that \( \epsilon_j \) has negative depth in \( A_k \). Lemma 10 shows that \( (e_1, e_k) \) is the only point in \( E_i \) that lies in \( C_{\{A_i, A_k\}} \).

Lemma 5. Let \( A = \{A_1, A_2, \ldots, A_m\} \) be an arrangement of finite cameras with distinct centers. Let \( E_A^{++} \) be the union of the sets \( E_j \) such that \( \epsilon_j \) has positive depth in every camera \( A_i \in A \setminus \{A_j\} \), then \( E_A^{++} \subseteq \varphi_A(\mathbb{P}_A^3) \).

Proof. We can approach the point \( p = (p_1, p_2, \ldots, p_m) \in E_j \) by \( \varphi_A(\tilde{v}(s)) \), where \( \tilde{v}(s) = sG_j^{-1}p_j - G_j^{-1}t_j \) as \( s \) goes to 0. Since \( \epsilon_j \) has positive depth in the other cameras, the point \( \tilde{v}(s) \) is in \( \mathbb{P}_A^3 \) for sufficiently small positive or negative \( s \), depending on the depth of \( (G_j^{-1}p_j, 1) \in \mathbb{P}_A^3 \) in camera \( A_j \). So \( p \) lies in the closure of \( \varphi_A(\mathbb{P}_A^3) \).

Lemma 6. Let \( A = \{A_1, \ldots, A_m\} \) be an arrangement of finite cameras such that \( \mathbb{P}_A^3 \) is nonempty. Then,

\[
(\varphi_A(\mathbb{P}_A^3) \setminus B_A) \cap C_A = \varphi_A(\mathbb{P}_A^3) \setminus B_A.
\]
In particular, \( \varphi_A(\mathbb{P}^3_A) \subseteq C_A \).

**Proof.** Suppose \( p = (p_1, p_2, \ldots, p_m) = \varphi_A(q) \) for a \( q = (r, q_4) \) in \( \mathbb{P}^3 \) where \( p \notin B_A \). As before, we write \( b_{ij} = G^{-1}_i t_i - G^{-1}_j t_j \) and \( a_i = G^{-1}_i p_i \). Fix a pair of indices \( i \) and \( j \). Observe that if \( a_i, a_j, \) and \( b_{ij} \) are all collinear, then \( q \) lies on the baseline \( l_{ij} \) of the pair \( \{A_i, A_j\} \) (Lemma 9). The assumption \( p \notin B_A \) exactly says that the three vectors \( b_{ij}, a_i, \) and \( a_j \) are not collinear. To prove Equation (50), it suffices to show that \( p \in \varphi_A(\mathbb{P}^3) \cap C_A \) if and only if \( p \in \varphi_A(\mathbb{P}^3_A) \).

Recall that the principal ray of camera \( A_i \) is given by \( \det(G_i)(A_i)_3 \) and \( p_{i3} = (A_i)_3 \cdot q \). Thus by Theorem 2 we know \( p \in \varphi_A(\mathbb{P}^3_A) \) if and only if for all \( i, j \),

\[
\det(G_i) \lambda_i p_{i3} q_4 \geq 0 \\
\det(G_j) \lambda_j p_{j3} q_4 \geq 0 \\
\det(G_i) \det(G_j) \lambda_i \lambda_j p_{i3} p_{j3} \geq 0.
\]

Since \( a_i, a_j, \) and \( b_{ij} \) are not collinear, by Lemma 9 we can replace \( \lambda_i q_4 \) by \( (a_i \times a_j)^\top (b_{ij} \times a_j) \), \( \lambda_j q_4 \) by \( (a_i \times a_j)^\top (b_{ij} \times a_i) \), and \( \lambda_i \lambda_j \) by \( (b_{ij} \times a_i)^\top (b_{ij} \times a_j) \) without changing the sign of the inequality, giving us the inequalities defining \( C_A \) involving only the indices \( i \) and \( j \). Repeating the same arguments for every pair of indices, we get the equality in Equation (50).

Since \( \mathbb{P}^3 \setminus \bigcup_{i,j} l_{ij} \) is dense in \( \mathbb{P}^3 \), \( \varphi_A \) is continuous, and \( C_A \) is closed, it follows from \( \varphi_A(q) \) for \( q \in \mathbb{P}^3 \setminus \bigcup_{i,j} l_{ij} \) that \( \varphi_A(\mathbb{P}^3_A) \subseteq C_A \).

\( \square \)
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