A Galois Correspondence
for Compact Groups of Automorphisms
of von Neumann Algebras
with a Generalization to Kac Algebras

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Abstract. Let $M$ be a factor with separable predual and $G$ a compact group of automorphisms of $M$ whose action is minimal, i.e. $M^G \cap M = C$, where $M^G$ denotes the $G$-fixed point subalgebra. Then every intermediate von Neumann algebra $M^G \subset N \subset M$ has the form $N = M^H$ for some closed subgroup $H$ of $G$. An extension of this result to the case of actions of compact Kac algebras on factors is also presented. No assumptions are made on the existence of a normal conditional expectation onto $N$. 

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1. Introduction.

A classical theme in Operator Algebras is the Galois correspondence between groups of automorphisms of a von Neumann algebra and von Neumann subalgebras.

To be more specific, let $M$ be a von Neumann algebra and to each group $G$ of automorphisms of $M$ let associate $M^G$, the von Neumann subalgebra of the $G$–fixed elements $G \to M^G$. \hfill (1.1)

In a dual way to each von Neumann subalgebra $N$ of $M$ we may associate the group $G_N$ of the automorphisms of $M$ leaving $N$ pointwise fixed

$$N \to G_N.$$ \hfill (1.2)

These two maps are in general not one another inverse, but restricting to (closed) subgroups of a given group $G$ and to intermediate von Neumann subalgebras $M^G \subset N \subset M$ they may actually become one another inverse.

Such a Galois correspondence was shown to hold by Nakamura and Takeda [NT] and Suzuki [Su] in the case $M$ be $II_1$–factor and $G$ a finite group whose action on $M$ is minimal, namely $M^G' \cap M = C$.

A different Galois correspondence, between normal closed subgroups of a compact (minimal) group $G$ and globally $G$-invariant intermediate von Neumann algebras, was obtained by Kishimoto [K], following methods in the analysis of the chemical potential in Quantum Statistical Mechanics [AHKT]. Generalizations of this result concerning dual actions of a locally compact group $G$ were dealt by Takesaki, in case of $G$ abelian, and by Nakagami in more generality, see [NTs].

Another kind of Galois correspondence was provided by H. Choda [Ch]. It concerns in particular the crossed product of a factor by an outer action of a discrete group and characterizes the intermediate von Neumann subalgebras that are crossed product by a discrete subgroup. An important assumption here is the existence of a normal conditional expectation onto the intermediate subalgebras.
In this paper we consider any compact group $G$ of automorphisms of a (separable) factor $M$, whose action is minimal, and show that any intermediate von Neumann algebra $M^G \subset N \subset M$ is the fixed-point algebra $N = M^H$ for some closed subgroup $H$ of $G$, namely the general Galois correspondence holds in the compact minimal case. Indeed as a corollary the two maps (1.1) and (1.2) are one another inverse.

A particular case of our result concerning the action of the periodic modular group with minimal spectrum on a type $III_\lambda$ factor, $0 < \lambda < 1$, has been recently obtained in [HS].

Concerning the ingredients in our proof, we mention the spectral analysis for compact group actions, endomorphisms and index theory for infinite factors, arguments based on modular theory, injective subfactors and averaging techniques.

We emphasise that the main step in the proof of our result is showing the existence of a (necessarily unique) normal conditional expectation of $M$ onto any intermediate subfactor between $M^G$ and $M$.

Note that in this way we also obtain a Galois correspondence for intermediate von Neumann algebras in the case of crossed products of factors by outer actions of discrete groups (again, without the a priori existence of normal conditional expectation).

This poses the following question: if $M_1 \subset M_2 \subset M_3$ are von Neumann algebras such that $M'_1 \cap M_3 = C$, with a normal expectation $\epsilon : M_3 \rightarrow M_1$, does there exist a normal expectation of $M_3$ onto $M_2$? In other words, does $\epsilon$ factor through $M_2$? Beside the case dealt in this paper, we know a (positive) answer for example if $M_1 \subset M_3$ has finite index or if $M_3$ is semifinite, but no counter-example is known to us.

At this point we briefly comment on the superselection structure in Particle Physics, that partly motivated our work. As is known the group of the internal symmetries in a Quantum Field Theory is the dual of the tensor C*-category defined by the superselection sectors [DR]. Our result classifies the extensions of the net of the observable algebras made up by field operators. An analysis of further aspects of this structure goes beyond the
purpose of our paper. However we notice that in low dimensional Quantum Field Theory
the internal symmetry is realized by a more general, not yet understood, quantum object
and this suggests to be of interest to extend our result to a wider class of “quantum
groups”.

We take here a first step in this direction providing a version of our result in the
context of actions of compact Kac algebras on factors that turns out to be new even in the
finite-dimensional case. This is included in our last section.

2. Preliminaries.

Throughout this paper, von Neumann algebras have separable preduals.

2-1. Operator valued weights and basic construction. For the theory of operator valued
weights and basic construction, our standard references are [H1][H2][Ko1].

Let $M \supset N$ be an inclusion of von Neumann algebras. We denote by $\mathcal{P}(M,N)$,
$\mathcal{E}(M,N)$ the set of normal semifinite faithful, (abbreviated as n.s.f.), operator valued
weights, and that of normal faithful conditional expectations respectively. We denote
by $\mathcal{P}_0(M,N)$ the set of $T \in \mathcal{P}(M,N)$ whose restriction to $M \cap N'$ is semifinite, (such $T$
is called regular in [Y]). Note that $\mathcal{P}_0(M,N)$ is either empty or $\mathcal{P}(M,N)$ [H2, Theorem 6.6].
For $T \in \mathcal{P}(M,N)$, we use the following standard notations:

$$n_T = \{ x \in M; T(x^* x) < \infty \},$$

$$m_T = n^*_T n_T.$$  

For a n.f.s. weight $\varphi$ on $M$, $H_\varphi$ and $\Lambda_\varphi$ denote the GNS Hilbert space and the canonical
injection $\Lambda_\varphi : n_\varphi \longrightarrow H_\varphi$.

For $M \supset N$ with $E \in \mathcal{E}(M,N)$, we fix a faithful normal state $\omega$ on $N$ and set
$\varphi := \omega \cdot E$. We regard $M$ as a concrete von Neumann algebra acting on $H_\varphi$. Let $e_N$ be the
Jones projection defined by $e_N \Lambda_\varphi(x) = \Lambda_\varphi(E(x))$, which does not depend on $\omega$ but only
on the natural cone of $H_\varphi$ [Ko2, Appendix]. The basic extension of $M$ by $E$ is the von
Neumann algebra generated by $M$ and $e_N$, which coincides with $J_MN'J_M$, where $J_M$ is the modular conjugation for $M$. For $x \in B(H_\varphi)$, we set $j(x) = J_Mx^*J_M$. The dual operator valued weight $\hat{E} \in \mathcal{P}(M_1, M)$ of $E$ is defined by $j \cdot E^{-1} \cdot j|_{M_1}$, where $E^{-1} \in \mathcal{P}(N', M')$ is characterized by spatial derivatives [C1]:

$$\frac{d(\psi \cdot E)}{d\varphi'} = \frac{d\psi}{d(\varphi' \cdot E^{-1})}, \quad \psi \in \mathcal{P}(N, C), \quad \varphi' \in \mathcal{P}(M', C).$$

Since $\hat{E}$ satisfies $\hat{E}(e_N) = 1$ [Ko1, Lemma 3.1], $Me_NM \subset m_{\hat{E}}$. In [Ko1], Kosaki defined the index of $E$ by $\text{Ind}_E = E^{-1}(1)$ in the case where $M$ and $N$ are factors, which is known to coincide with the probabilistic index defined in [PP1].

First, we consider a Pimsner-Popa push down lemma in our setting (c.f. [PP1]).

**Lemma 2.1.** Let $M$ be a von Neumann algebra and $\varphi$ a n.f.s. weight on $M$. Suppose $A$ is a *-subalgebra of $n^*_\varphi \cap n_\varphi$ which is dense in $M$ in weak topology, and globally invariant under the modular automorphism group. Then $\Lambda_\varphi(A)$ is dense in $H_\varphi$.

**Proof.** Let $p$ be the projection onto the closure of $\Lambda_\varphi(A)$. Then $p \in A' = M'$. Thanks to $\sigma^\varphi_t(A) = A$, $p$ commutes with $\Delta^\varphi_t$, and consequently we have $\Delta^\frac{3}{2}_\varphi p \supset p\Delta^\frac{3}{2}_\varphi$. Since $A \subset n_\varphi \cap n^*_\varphi$, $\Lambda_\varphi(x), x \in A$ is in the domains of $S_\varphi$ and $\Delta^\frac{3}{2}_\varphi$. Thus we get the following:

$$J_\varphi\Lambda_\varphi(x) = J_\varphi S_\varphi\Lambda_\varphi(x^*) = \Delta^\frac{3}{2}_\varphi\Lambda_\varphi(x^*) = p\Delta^\frac{3}{2}_\varphi\Lambda_\varphi(x^*) = pJ_\varphi\Lambda_\varphi(x).$$

This means that $p$ commutes with $J_\varphi$, and $p \in M \cap M'$. So we get $\Lambda_\varphi((1 - p)x) = 0$ for $x \in A$. Since $\varphi$ is faithful, this implies $(1 - p)x = 0$, which shows $p = 1$ because $A$ is dense in $M$ in weak topology. Q.E.D.

**Proposition 2.2 (Push down lemma).** Let $M \supset N$ be an inclusion of factors with $E \in \mathcal{E}(M, N)$, and $M_1$ be the basic extension of $M$ by $E$. Then for all $x \in n_E$, $e_N\hat{E}(e_Nx) = e_Nx$ holds.

**Proof.** Let $\varphi$ be as above and $\varphi_1 = \varphi \cdot \hat{E}$. Then $e_Nx$, and $e_N\hat{E}(e_Nx)$ belong to $n_\varphi$. So we
get the following:

$$\|\Lambda \varphi_1(e_N x) - \Lambda \varphi_1(e_N \hat{E}(e_N x))\|^2$$

$$= \varphi_1(x^* e_N x) - \varphi_1(x^* e_N \hat{E}(e_N x)) - \varphi_1(\hat{E}(e_N x)^* e_N x) + \varphi_1(\hat{E}(e_N x)^* e_N \hat{E}(e_N x))$$

$$= \|\Lambda \varphi_1(e_N x)\|^2 - \|\Lambda \varphi_1(e_N \hat{E}(e_N x))\|^2.$$

So, we can define a bounded operator $V$ on $e_N H_\varphi$ by

$$Ve_N \Lambda \varphi_1(x) = \Lambda \varphi_1(e_N \hat{E}(e_N x)), \quad x \in n_E.$$  

By simple computation, one can show that $V$ is identity on $e_N \Lambda \varphi_1(Me_N M)$. So to prove the statement, it suffices to show that $\Lambda \varphi_1(Me_N M)$ is dense in $H_\varphi$. We set $A = Me_N M$ and show that $A$ satisfies the assumption of the previous lemma. Indeed, since $M_1$ is the weak closure of $Me_N M + M$, the weak closure of $A$ is a closed two-sided ideal of $M_1$, and coincides with $M_1$. From the definition of $\varphi_1$, we have $\sigma_t^{\varphi_1}(Me_N M) = \sigma_t^{\varphi}(M) \sigma_t^{\varphi_1}(e_N) \sigma_t^{\varphi}(M) = M \sigma_t^{\varphi}(e_N) M$. Thanks to $j \cdot E^{-1} \cdot j = (j \cdot E \cdot j)^{-1}$ [Ko1, Lemma 1.3], we get

$$\frac{d\varphi_1}{d(\omega \cdot j)} = \frac{d(\varphi \cdot (j \cdot E \cdot j)^{-1})}{d(\omega \cdot j)} = \frac{d\varphi}{d(\omega \cdot E \cdot j)} = \frac{d\varphi}{d(\varphi \cdot j)} = \Delta \varphi.$$  

Since $\Delta \varphi$ commutes with $e_N$, we get $\sigma_t^{\varphi_1}(e_N) = e_N$. Q.E.D.

Remark 2.3. In general, $e_N M_1$ is strictly larger than $e_N n_E$. Indeed, suppose that $M$, $N$ are type III factors and $e_N M_1 = e_N n_E$. Then there is an isometry $v \in n_E$ with $vv^* = e_N$, that implies $1 = v^* v \in m_E$ and $Ind E < \infty$. This also means that $e_N M$ is not necessarily closed in weak topology because of $e_N M_1 = e_N M^\omega$.

The following is a generalization of the abstract characterization of the basic extension in [PP2] to the infinite index case (see also [HK]).

**Lemma 2.4.** Under the same assumption, assume that $R$ is a factor including $M$ and satisfying the following:

(i) There is a projection $e \in R$ such that $R$ is generated by $e$ and $M$, and $exe = E(x)e$ holds for $x \in M$.  

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(ii) There is $T \in \mathcal{P}(R, M)$ satisfying $T(e) = 1$, and $e \in (R \cap N')_{E:T}$.

Then there is an isomorphism $\pi : M_1 \rightarrow R$ satisfying $\pi|_M = id_M$, $\pi(e_N) = e$, and $T \cdot \pi = \pi \cdot \hat{E}$.

Proof. Let $\psi = \varphi \cdot T$. For the same reason as before, $\Lambda_\psi(\text{MeM})$ is dense in $H_\psi$. So we can define a surjective isometry $U : H_{\varphi_1} \rightarrow H_\psi$ and an isomorphism $\pi : M_1 \rightarrow R$ by

$$U \Lambda_{\varphi_1}(\sum x_i e_N y_i) = \Lambda_\psi(\sum x_i e y_i), \quad x_i, y_i, \in M,$$

$$\pi(x) = U x U^*, \quad x \in M_1.$$ 

Clearly, $\pi$ satisfies $\pi|_M = id_M$, $\pi(e_N) = e$. Thanks to $\sigma^E_{i:T}(e) = e$, the modular automorphism groups of $\varphi_1$ and $\psi \cdot \pi$ coincide on $\text{MeN}M$ (and on $M_1$). Since $M_1$ is a factor, this implies that $\varphi_1$ is a scalar multiple of $\psi \cdot \pi$, and consequently that $\hat{E}$ is a scalar multiple of $\pi^{-1} \cdot T \cdot \pi$ [H2, Lemma 4.8]. From $\hat{E}(e_N) = 1$ and $T(e) = 1$, we get the result. Q.E.D.

The following may be a folklore for specialists. However, since the authors cannot find it in the literature, we give a proof.

Lemma 2.5. Let $M \supset N$ be an inclusion of von Neumann algebras (not necessarily with separable pre-dual). Then, there is a unique central projection $z$ of $M \cap N'$ satisfying the following two conditions:

(i) $\mathcal{P}_0(pMp, pN) = \emptyset$ holds for every projection $p \in M \cap N'$, $p \leq 1 - z$.

(ii) $\mathcal{P}_0(zMz, zN) = \mathcal{P}(zMz, zN)$.

Moreover, if $\mathcal{P}(M, N)$ is not empty, then $(1 - z)(M \cap N') \cap m_T = \{0\}$, $z \in (M \cap N')_T$, and $T|_{z(M \cap N')}$ is semifinite for every $T \in \mathcal{P}(M, N)$.

To prove the lemma, we need the following.

Lemma 2.6. The following hold.

(i) Let $\{p_i\}_{i \in I} \subset M \cap N'$ be a family of mutually orthogonal projections, and $p = \sum p_i$.

If $\mathcal{P}_0(p_i Mp_i, p_i N) \neq \emptyset$ for every $i \in I$, then $\mathcal{P}_0(pMp, pN) \neq \emptyset$.

(ii) Let $p \in M \cap N'$ be a projection. If $\mathcal{P}_0(M, N) \neq \emptyset$, then $\mathcal{P}_0(pMp, pN) \neq \emptyset$. 7
(iii) Let \( p \in M \cap N' \) be a projection satisfying \( \mathcal{P}_0(pMp,pN) \neq \emptyset \), and \( c(p) \) the central support of \( p \) in \( M \cap N' \). Then \( \mathcal{P}_0(c(p)Mc(p),c(p)N) \neq \emptyset \).

(iv) Let \( \{p_i\}_{i \in I} \subset M \cap N' \) be a family of projections, and \( p_0 = \vee p_i \). If \( \mathcal{P}_0(p_iMp_i,p_iM) \neq \emptyset \) for every \( i \in I \), then \( \mathcal{P}_0(p_0Mp_0,p_0N) \neq \emptyset \).

**Proof.** (i): This follows from the following easy facts:

\[
\mathcal{P}_0(pMp,\oplus p_iMp_i) \neq \emptyset, \quad \mathcal{P}_0(\oplus p_iMp_i,\oplus p_iN) \neq \emptyset, \quad \mathcal{P}_0(\oplus p_iN,pN) \neq \emptyset.
\]

(ii): Since \( \mathcal{P}_0(M,N) \neq \emptyset \), there is a separating family of normal conditional expectations from \( M \) to \( N \) \{\( E_\alpha \)\} [Ha2, Theorem 6.6]. Then \( \{E_\alpha(p \cdot p)p\} \) is a separating family of bounded normal operator valued weights from \( pMp \) to \( pN \), and \( \mathcal{P}_0(pMp,pN) \neq \emptyset \). (iii): Let \( \{e_j\} \in M \cap N' \) be a family of projections satisfying \( p \succ e_j, \sum e_j = c(p) \). Then \( \mathcal{P}_0(e_jMe_j,e_jN) \neq \emptyset \). So using (i), we get \( \mathcal{P}_0(c(p)Mc(p),c(p)N) \neq \emptyset \). (iv): Let \( z_i = c(p_i) \) and \( z_0 = \vee z_i \). Then thanks to (i)(ii)(iii), \( \mathcal{P}_0(z_0Mz_0,z_0N) \neq \emptyset \). Since \( z_0 \) is the central support of \( p_0 \), we get the statement by using (i). Q.E.D.

**Proof of Lemma 2.5.** Let \( z \) be the supremum of the projections \( p \in M \cap N' \) satisfying \( \mathcal{P}_0(pMp,pN) \neq \emptyset \). Then thanks to Lemma 2.6 (iii)(iv), \( z \) is a central projection satisfying (i)(ii). It is easy to show the uniqueness of such a projection. If \( T \in \mathcal{P}(M,N), \sigma_1^T(z) \) also satisfies (i)(ii) and we get \( z \in (M \cap N')_T \). This implies that \( zT(z \cdot z) \) belongs to \( \mathcal{P}(zMz,zN) \). So due to [Ha2, Theorem 6.6], \( T|_{z(M\cap N')} \) is semifinite. Suppose \( x \) is a non-zero positive element in \( m_T \cap (1 - z)(M \cap N') \). Then there is a non-zero spectral projection \( p \) of \( x \) satisfying \( T(p) < \infty \). This implies \( \mathcal{E}(pMp,pN) \neq \emptyset \), that contradicts Lemma 2.6 (iii). Q.E.D.

To analyze local structure of the inclusions obtained by basic construction in infinite index case, we need the following:

**Lemma 2.7 ([Ko1, Proposition 4.2][Y, Corollary 28]).** Let \( M \supset N \) be an inclusion of factors. Then the following hold:
(i) Let $T \in \mathcal{P}(M,N)$, and $p \in m_T \cap (M \cap N')_T$ a non-zero projection. Then $\text{Ind } T_p = T(p)T^{-1}(p)$, where $T_p \in \mathcal{E}(pMp,pN)$ is defined by $T_p(x) = pT(x)/T(p)$, $x \in pMp$.

(ii) If $\mathcal{P}_0(M,N) \neq \emptyset$, $\mathcal{P}_0(N',M') \neq \emptyset$, then $M \cap N'$ is a direct sum of type I factors and $pMp \supset pN$ has finite index for every finite rank projection in $M \cap N'$.

Proposition 2.8. Let $M \supset N$ be an inclusion of factors with $E \in \mathcal{E}(M,N)$, and $M_1$ the basic extension. Then $M_1 \cap N'$ is direct sum of four subalgebras,

$$M_1 \cap N' = A \oplus B_1 \oplus B_2 \oplus C,$$

satisfying the following:

(i) Each of the four subalgebras is globally invariant under $\{\sigma^E_t\}$.

(ii) $j(A) = A$, $j(B_1) = B_2$, $j(B_2) = B_1$, $j(C) = C$.

(iii) $\hat{E}|_{A \oplus B_1}$ is semifinite.

(iv) $m_{\hat{E}} \cap (B_2 \oplus C) = \{0\}$.

(v) $A$ is direct sum of type I factors and $pM_1p \supset pN$ has finite index for every finite rank projection $p \in A$.

Proof. First, we show $j \cdot \sigma^E_t \cdot \hat{E} \cdot j = \sigma^E_t \cdot \hat{E}$ on $M_1 \cap N'$. Indeed, for $x \in M_1 \cap N'$ we get the following as in the proof of Lemma 2.3:

$$j \cdot \sigma^E_t \cdot \hat{E}(j(x)) = J_M \left( \frac{d(\varphi \cdot \hat{E})}{d(\omega \cdot j)} \right)^{-it} J_M x J_M \left( \frac{d(\varphi \cdot \hat{E})}{d(\omega \cdot j)} \right)^{it} J_M$$

$$= J_M \Delta^it_x J_M \Delta^-it_x J_M = J_M \Delta^it_x J_M \Delta^-it_x = (\frac{d(\varphi \cdot \hat{E})}{d(\omega \cdot j)})^{it} x (\frac{d(\varphi \cdot \hat{E})}{d(\omega \cdot j)})^{-it}$$

$$= \sigma^E_t \cdot \hat{E}(x).$$

Now, let $z$ be the central projection of $M_1 \cap N'$ determined by Lemma 2.5 for $M_1 \supset N$. We set

$$A = zj(z)(M_1 \cap N'), \quad C = (1 - z)j(1 - z)(M_1 \cap N'),$$

$$B_1 = zj(1 - z)(M_1 \cap N'), \quad B_2 = (1 - z)j(z)(M_1 \cap N').$$
Then by construction (ii)(iii)(iv) hold. Since \( j \) commutes with \( \sigma_t^{E\hat{E}} \), \( j(z) \in (M_1 \cap N')_{E\hat{E}} \), and we get (i). Note that for a projection \( p \in M_1 \cap N' \), \( J_M(pM_1p)'J_M = j(p)N \), \( J_M(pN)'J_M = j(p)M_1j(p) \). So \( (pN)' \supset (pM_1p)' \) is anti-conjugate to \( j(p)M_1j(p) \supset j(p)N \). Thus thanks to Lemma 2.7, we get (v). Q.E.D.

2-2. Sectors and simple injective subfactors. Our basic references for the theory of sectors are [L1][L2][I1].

Let \( M \) be an infinite factor. We denote by \( \text{End}(M) \) and \( \text{Sect}(M) \) the set of unital endomorphisms of \( M \) and that of sectors, which is the quotient of \( \text{End}(M) \) by the unitary equivalence. Note that every element in \( \text{End}(M) \) is automatically normal for \( M \) with separable pre-dual. For \( \rho_1, \rho_2 \in \text{End}(M) \), \( (\rho_1, \rho_2) \) denotes the set of intertwiners between \( \rho_1 \) and \( \rho_2 \), i.e.

\[
(\rho_1, \rho_2) = \{ v \in M; v\rho_1(x) = \rho_2(x)v, \ x \in M \}.
\]

If \( \rho_1 \) is irreducible, i.e. \( M \cap \rho_1(M)' = \mathbb{C} \), \( (\rho_1, \rho_2) \) is a Hilbert space with the following inner product:

\[
<V|W> = W^*V, \ V, W \in (\rho_1, \rho_2).
\]

We define the dimension \( d(\rho) \) of \( \rho \) by \( d(\rho) = [M : \rho(M)]_0^{1/2} \), where \( [M : \rho(M)]_0 \) is the minimum index of \( M \supset \rho(M) \) [Hi]. For \( \rho \) with \( d(\rho) < \infty \), we denote by \( E_{\rho} \) and \( \phi_\rho \) the minimal conditional expectation onto \( \rho(M) \) and the standard left inverse of \( \rho \), i.e. \( \phi_\rho = \rho^{-1} \cdot E_{\rho} \).

There are three natural operations in \( \text{Sect}(M) \): the sum, the product and the conjugation. For simplicity, we denote by \( \overline{\rho} \) one of the representatives of the conjugate sector \( \overline{[\rho]} \) of \( [\rho] \). When \( d(\rho) \) is finite, it is known that there are two isometries \( R_\rho \in (id, \overline{\rho}) \), \( \overline{R}_\rho \in (id, \rho \overline{\rho}) \) satisfying

\[
\overline{R}_\rho \rho(R_\rho) = R_\rho \rho(\overline{R}_\rho) = \frac{1}{d(\rho)}.
\]

Although such a pair is not unique, we fix it once and forever in this paper. Unless \( \rho \) is a pseudo-real sector [L1], we can take \( \overline{R}_\rho \) equal to \( R_{\overline{\rho}} \). If it is, we set \( \overline{R}_\rho = -R_\rho \).
Let $M X M$ be a $M - M$ bimodule, and $\rho \in End(M)$. Then we define a new bimodule $M(X_\rho)_M$ (respectively $M(\rho X)_M$) by

$$x \cdot \xi' \cdot y := x \cdot \xi \cdot \rho(y) \quad \text{(respectively } x \cdot \xi' \cdot y := \rho(x) \cdot \xi \cdot y) \quad x, y \in M,$$

where $\xi' = \xi$ as an element of Hilbert space $X$. It is known that there is one-to-one correspondence between $\text{Sect}(M)$ and the set of equivalence classes of $M - M$ bimodules. The correspondence is given by $[\rho] \to [M(L^2(M)_\rho)_M]$, that preserves the three operations. The conjugate sector of $[\rho]$ is characterized by

$$M(L^2(M)_\rho)_M \cong_M (\pi L^2(M))_M.$$

Let $\phi$ be a unital normal completely positive map from $M$ to $M$. Following Connes [C2], there is a natural way to associate a $M - M$ bimodule with $\phi$. Let $\Omega$ be a separating and cyclic vector of $M$. We introduce a positive semi-definite sesquilinear form on the algebraic tensor product $M \otimes_{\text{alg}} M$ as follows:

$$< \sum_i x_i \otimes y_i, \sum_j z_j \otimes w_j > = \sum_{i,j} < \phi(z_j^* x_i) J_M w_j y_i^* J_M \Omega | \Omega >.$$

We denote by $H_\phi$ the Hilbert space completion of the quotient of $M \otimes_{\text{alg}} M$ by the kernel of the sesquilinear form, and by $\Lambda_\phi$ the natural map $\Lambda_\phi : M \otimes_{\text{alg}} M \to H_\phi$. $H_\phi$ is naturally a $M - M$ bimodule by the following action:

$$x \cdot \Lambda_\phi \left( \sum_i z_i \otimes w_i \right) \cdot y = \Lambda_\phi \left( \sum_i x z_i \otimes w_i y \right).$$

Thanks to the one-to-one correspondence stated above, there is an endomorphism $\rho_\phi$ satisfying $M(H_\phi)_M \cong_M (\rho_\phi L^2(M))_M$. Actually, $\rho_\phi$ is Steinspring type dilation of $\phi$. Indeed, let $W : H_\phi \to L^2(M)$ be the intertwining surjective isometry, and set $\xi_0 = W \Lambda_\phi (1 \otimes 1)$. Then we get

$$< \phi(x) \cdot \Omega \cdot y | \Omega > = < x \cdot \Lambda_\phi (1 \otimes 1) \cdot y, \Lambda_\phi (1 \otimes 1) > = < \rho_\phi(x) \cdot \xi_0 \cdot y | \xi_0 >.$$
We define an isometry \( v \) by \( v(\Omega \cdot y) = \xi_0 \cdot y \). Then by definition, \( v \) commutes with the right action of \( M \). So \( v \) belongs to \( M \) and satisfies \( \phi(x) = v^* \rho_\phi(x)v \), \( x \in M \). Note that the support of \( vv^* \) in \( M \cap \rho_\phi(M)' \) is 1. Indeed, suppose \( z \in M \cap \rho_\phi(M)' \) satisfying \( z\xi_0 \cdot y = 0 \), for all \( y \in M \). Then \( z\rho_\phi(x)\xi_0 \cdot y = 0 \) for all \( x, y \in M \). Since \( \rho_\phi(M) \cdot \xi_0 \cdot M = WH_\phi = L^2(M) \), we get \( z = 0 \).

Although the following statements might be found in the literature, we give proofs for readers' convenience.

**Proposition 2.9.** Let \( M \) and \( \phi \) be as above. Then the following hold:

(i) Let \( \sigma \in \text{End}(M) \), and \( v_1 \in M \) be an isometry satisfying \( \phi(x) = v_1^* \sigma(x)v_1 \). If the support of \( v_1v_1^* \) in \( M \cap \sigma(M)' \) is 1, then \([\rho_\phi] = [\sigma]\).

(ii) The equivalence class of \( H_\phi \) does not depend on the choice of the cyclic separating vector \( \Omega \).

(iii) Let \( \mu \) be another unital normal completely positive map from \( M \) to \( M \). If there is a positive constant \( c \) such that \( c\mu - \phi \) is completely positive, then \([\rho_\mu]\) contains \([\rho_\phi]\).

**Proof.**

(i): Let \( \xi_0 \) be as before. Then by assumption, we get the following:

\[
<\sigma(x)v_1\Omega \cdot y|v_1\Omega > =< \rho_\phi(x)\xi_0 \cdot y|\xi_0 >, 
\]

\[
\overline{\sigma(M)v_1\Omega \cdot M} = L^2(M).
\]

So we can define a unitary \( u \in M \) by \( u\sigma(x)v_1\Omega \cdot y = \rho_\phi(x)\xi_0 \cdot y \), and get \( \rho_\phi(x) = u\sigma(x)u^* \).

(ii) follows from (i). (iii): Since \( c\mu - \phi \) is completely positive, we can define a bounded map \( T : H_\mu \rightarrow H_\phi \) by

\[
T\Lambda_\mu(\sum_i x_i \otimes y_i) = \Lambda_\phi(\sum_i x_i \otimes y_i).
\]

Then \( T \) is an \( M - M \) bimodule map whose image is dense in \( H_\phi \). Let \( T = U|T| \) be the polar decomposition of \( T \). Then \( U \) is a coisometry belonging to \( \text{Hom}(M(H_\mu)_{M,M} \to (H_\phi)_M) \). Thus \([\rho_\mu]\) contains \([\rho_\phi]\). Q.E.D.
In [L3], the second author proved that for an arbitrary infinite factor $M$ (with separable pre-dual), there exists an injective subfactor $R \subset M$ satisfying $R' \cap J_M R' J_M = \mathbb{C}$. A subfactor $R$ of $M$ is called simple if $R' \cap J_M R' J_M = \mathbb{C}$. A simple subfactor $R$ determines the automorphisms of $M$ in the following sense; if $\alpha, \beta \in \text{Aut}(M)$ satisfying $\alpha|_R = \beta|_R$, then $\alpha = \beta$. Indeed, let $u$ be the canonical implementation of $\alpha^{-1} \cdot \beta$. Then $u \in R'$, and $u$ commutes with $J_M$. So $u$ is a scalar, that means $\alpha = \beta$. We can generalize this to some class of endomorphisms as follows:

**Proposition 2.10.** Let $M$ be an infinite factor and $R$ a simple subfactor. For every $\rho \in \text{End}(M)$ with $E \in \mathcal{E}(M, \rho(M))$, the following holds:

$$\{ T \in M; Tx = \rho(x)T, \ x \in R \} = (id, \rho). \quad (2.2)$$

**Proof.** First, we show that the general case can be reduced to the case where $(id, \rho) = \{0\}$. Indeed, let $\{V_i\}_i$ be an orthonormal basis of $(id, \rho)$, and $W$ an isometry in $M$ satisfying $WW^* = 1 - \sum V_i V_i^*$. Then $\rho(x) = \sum V_i x V_i^* + W\sigma(x)W^*$, where $\sigma \in \text{End}(M)$ is defined by $\sigma(x) = W^* \rho(x)W$. Note that $(id, \sigma) = \{0\}$ by construction. If $T$ is in the left-hand side of (2.2), then $c_i := V_i^* T \in R' \cap M = \mathbb{C}$, and $W^* T x = \sigma(x) W^* T$, $x \in R$. Since $T = \sum V_i V_i^* T + WW^* T$, if the statement is true for $\sigma$, i.e. $W^* T = 0$, we get $T = \sum c_i V_i \in (id, \rho)$.

Secondly, we construct the “canonical implementation” of $\rho$ as follows. Let $\Omega$ be a separating and cyclic vector for $M$, and $L^2(M, \Omega)_+$ the natural cone with respect to $\Omega$. Then there are unique vectors $\xi_0, \xi_1 \in L^2(M, \Omega)_+$ satisfying

$$< E(x) \Omega | \Omega > = < x \xi_0 | \xi_0 > .$$

$$< \rho(x) \Omega | \Omega > = < x \xi_1 | \xi_1 > .$$

Note that $\xi_0, \xi_1$ are cyclic because they belong to the natural cone and implement faithful states. So we can define an isometry $V_\rho$ by $V_\rho x \xi_1 = \rho(x) \xi_0$. We set $e_\rho = V_\rho V_\rho^*$, which is
the Jones projection of $E$. $V_\rho$ satisfies $V_\rho x = \rho(x)V_\rho$ and $J_M V_\rho = V_\rho J_M$. Indeed, the first equality is obvious. By identifying $e_\rho L^2(M)$ with $L^2(\rho(M),\xi_0)$, we get $J_{\rho(M)} V_\rho = V_\rho J_M$. On the other hand, since $e_\rho$ is the Jones projection, we have $e_\rho J_M = J_M e_\rho = J_{\rho(M)}$. So $V_\rho$ commutes with $J_M$.

Now suppose that $(id, \rho) = \{0\}$ and there exists a non-zero element $T$ in the left-hand side of (2.2). Since $T^* T \in M \cap R' = C$, we may assume that $T$ is an isometry. We set $\hat{T} = T J_M T J_M$, which commutes with $J_M$ and satisfies $\hat{T} x = \rho(x) \hat{T}$, $x \in R$. Then $V_\rho \hat{T} \in R' \cap J_M R' J_M = C$. Let $\lambda = V_\rho^* \hat{T}$, which is not zero because

$$< V_\rho^* \hat{T} \xi_0 | \xi_1 > = < \hat{T} \xi_0 | \xi_0 > = < T \xi_0 | J_M T^* \xi_0 >$$

$$= < T \xi_0 | \Delta^{1/2} T \xi_0 > = ||\Delta^{1/4} T \xi_0||^2,$$

where $\varphi(x) = < E(x) \Omega | \Omega >$, $x \in M$. We define a unital completely positive map $\phi : M \rightarrow M$ by $\phi(x) = T^* \rho(x) T$, $x \in M$, which equals to $\hat{T}^* \rho(x) \hat{T}$. By construction, $[\rho]$ contains $[\rho_\phi]$. So we show that $[\rho_\phi]$ contains $[id]$ and get contradiction. Thanks to Proposition 2.9, it suffices to show that $\phi - |\lambda|^2 id$ is completely positive. In fact,

$$\phi(x) = \hat{T}^* \rho(x) \hat{T} = \hat{T}^* e_\rho \rho(x) \hat{T} + \hat{T}^* (1 - e_\rho) \rho(x) \hat{T}$$

$$= \hat{T}^* V_\rho V_\rho^* \rho(x) \hat{T} + \hat{T}^* (1 - e_\rho) \rho(x) \hat{T} = \hat{T}^* V_\rho x V_\rho^* \hat{T} + \hat{T}^* (1 - e_\rho) \rho(x) \hat{T}$$

$$= |\lambda|^2 x + \hat{T}^* (1 - e_\rho) \rho(x) \hat{T}.$$  

Since $e_\rho$ commutes with $\rho(M)$, $x \mapsto \hat{T}^* (1 - e_\rho) \rho(x) \hat{T}$ is a complete positive map. So $[\rho]$ contains $[id]$ and we get contradiction. Q.E.D.

**Corollary 2.11.** Let $M, R, \rho$ be as above, and $\sigma \in \text{End}(M)$ with $d(\sigma) < \infty$. Then the following hold:

(i) $\{ T \in M; T \sigma(x) = \rho(x) T, \ x \in R \} = (\sigma, \rho)$.

(ii) If $\sigma | R = \rho | R$, then $\sigma = \rho$.

**Proof.** (i): Let $T$ be in the left-hand side of (i), and set $X = \sigma(V) R_\sigma$, where $R_\sigma$ is the isometry in (2.1). Then $X$ satisfies $X x = \sigma \cdot \rho(x) X, x \in R$. So thanks to Proposition 2.10,
we get $X \in (id, \sigma \cdot \rho)$. By simple computation using (2.1), we obtain $V = d(\sigma)\overline{R}_\sigma \sigma(X)$, and $V \in (\sigma, \rho)$. (ii): Thanks to (i), $1 \in (\sigma, \rho)$, that means $\sigma = \rho$. Q.E.D.

Let $\psi$ be a dominant weight on $M$ [CT]. Since every dominant weight is unitary equivalent, for every $\alpha \in Aut(M)$ there is a unitary $u \in M$ satisfying $\psi \cdot \alpha \cdot Ad(u) = \psi$. This fact is used to define the Connes-Takesaki module of $\alpha$. The endomorphism version is given as follows, which will be used in the next section.

**Lemma 2.12.** Let $M$ be an infinite factor. Then the following hold.

(i) For every $\rho \in End(M)$ with $d(\rho) < \infty$, there exist a dominant weight $\psi_\rho$ and a unitary $u \in M$ such that

$$
\psi_\rho \cdot \rho \cdot Ad(u) = d(\rho)\psi_\rho, \quad \psi_\rho \cdot E_\rho = \psi_\rho.
$$

(ii) Let $\psi$ be a dominant weight. Then for every $[\rho] \in Sect(M)$ with $d(\rho) < \infty$, there exists a representative $\rho$ satisfying

$$
\psi \cdot \rho = d(\rho)\psi, \quad \psi \cdot E_\rho = \psi.
$$

**Proof.** (i): Let $\psi_0$ be a dominant weight on $\rho(M)$. Since both $d(\rho)\psi_0 \cdot E_\rho$ and $\psi_0 \cdot \rho$ are dominant weights on $M$, there exists a unitary $u \in M$ satisfying $d(\rho)\psi_0 \cdot E_\rho = \psi_0 \cdot \rho \cdot Ad(u)$. So $\psi_\rho := \psi_0 \cdot E_\rho$ is the desired weight. (ii) follows from (i) and the fact that every dominant weight is unitary equivalent. Q.E.D.

3. **Galois Correspondence.**

In this section, we investigate the structure of irreducible inclusions of factors with normal conditional expectations. We present the ultimate form of the Galois correspondence of outer actions of discrete groups and minimal actions of compact groups on factors, which has been studied by several authors [AHKT][Ch][K][N][NT]. The key argument is how to show the existence of a conditional expectation for every intermediate subfactor.

Let $M \supset N$ be an irreducible inclusion, i.e. $M \cap N' = C$, of infinite factors with a
conditional expectation $E \in \mathcal{E}(M, N)$. For $\rho \in \text{End}(N)$, we set

$$\mathcal{H}_\rho = \{ V \in M; Vx = \rho(x)V, \ x \in N \}.$$

Then thanks to the irreducibility of $M \supset N$, $\mathcal{H}_\rho$ is a Hilbert space with inner product $< V|W > = W^*V$ as usual. We denote by $s(\mathcal{H}_\rho)$ the support of $\mathcal{H}_\rho$, that is $\sum_i V_i V_i^*$ where $\{V_i\}_i$ is an orthonormal basis of $\mathcal{H}_\rho$. Let $M_1$ be the basic extension of $M$ by $N$, and $e_N$ the Jones projection of $E$. Then $\mathcal{H}_\rho^* e_N \mathcal{H}_\rho \subset M_1 \cap N'$.

Let $\gamma: M \to N$ be the canonical endomorphism [L1][L2][L3]. Then it is known that $N L^2(M)_N \cong_N (\gamma|_N L^2(N))_N$. When Ind $E < \infty$, it is easy to show that an irreducible sector $[\rho] \in \text{Sect}(N)$ is contained in $[\gamma|_N]$ if and only if $\mathcal{H}_\rho \neq 0$ (Frobenius reciprocity) [I2]. First, we establish the infinite index version of this statement. For this purpose, it is convenient to give explicit correspondence between submodules of $N L^2(M)_N$ and subsectors of $\gamma|_N$. Let $p \in M_1 \cap N'$ be a non-zero projection. Since both $e_N$ and $p$ are infinite projection in $M_1$, there is a partial isometry $W \in M_1$ satisfying $WW^* = e_N$, $W^*W = p$. Due to $e_N M_1 e_N = e_N N$, we can define $\rho \in \text{End}(N)$ by $WxW^* = e_N \rho(x)$, $x \in N$.

**Lemma 3.1.** Under the above assumption and notation, the following holds:

$$N(p L^2(M))_N \cong_N (p L^2(N))_N.$$

**Proof.** We regard $W$ as a surjective isometry from $p L^2(M)$ to $e_N L^2(M) = L^2(N)$. Since $M_1 = J_M N' J_M$, $W$ commutes with $J_M N J_M$. So for $\xi \in p L^2(M)$, $x, y \in N$, we obtain

$$W(x \cdot \xi \cdot y) = Wx J_M y^* J_M \xi = \rho(x) W J_M y^* J_M \xi = \rho(x) J_M y^* J_M W \xi.$$

By using $e_N J_M = J_M e_N = J_N$, we get $W(x \cdot \xi \cdot y) = \rho(x) J_N y^* J_N W \xi$. Q.E.D.

**Proposition 3.2.** Let $M \supset N$ be an irreducible inclusion of infinite factors with $E \in \mathcal{E}(M, N)$, and $\gamma: M \to N$ the canonical endomorphism. Then for $\rho \in \text{End}(M)$, the following two statements are equivalent:
(i) \( \mathcal{H}_\rho \neq 0 \) and the support of \( E(s(\mathcal{H}_\rho)) \) is 1.

(ii) \( \mathcal{E}(N, \rho(N)) \) is non-empty and \( [\rho] \) is contained in \( [\gamma|N] \) up to multiplicity, i.e. there is decomposition \( [\rho] = \oplus[\rho_a] \) such that each \( [\rho_a] \) is contained in \( [\gamma|N] \).

**Proof.** (i) \( \Rightarrow \) (ii): Assume that \( \rho \) satisfies (i). By a simple argument, one can show that there is decomposition \( [\rho] = \oplus[\rho_a] \) such that for every \( a \) there exists \( V_a \in \mathcal{H}_{\rho_a} \) satisfying \( \mathcal{E}(V_aV_a^*) \geq 1 \). We set \( W_a = e_N E(V_aV_a^*)^{-1/2} V_a \). Then \( W_a \) satisfies \( W_aW_a^* = e_N, p_a := W_a^* W_a \in M_1 \cap N' \). Since \( W_a x W_a^* = e_N \rho_a(x), x \in N \), \( [\rho_a] \) is contained in \( [\gamma|N] \). \( \hat{E}(p_a) = V_a^* E(V_aV_a^*)^{-1} V_a < \infty \) implies \( \mathcal{E}(p_a M_1 p_a, p_a N) \neq \emptyset \), and consequently \( \mathcal{E}(N, \rho_a(N)) \neq \emptyset \). (ii) \( \Rightarrow \) (i): It is easy to show that if \( [\rho] = \oplus[\rho_a] \) and each \( \rho_a \) satisfies (i), then so does \( \rho \). Assume that \( [\rho] \) is contained in \( [\gamma|N] \) and \( \mathcal{E}(N, \rho(N)) \neq \emptyset \). Let \( p \in M_1 \cap N' \) be the projection corresponding to \( [\rho] \). Then \( \mathcal{E}(pM_1 p, pN) \neq \emptyset \), which implies \( p \in A \oplus B \) where \( A \) and \( B \) are as in Lemma 2.7. Let \( z \) be the central support of \( p \) in \( M_1 \cap N' \). Since \( \sigma_t E^E \) is trivial on the center of \( A \oplus B \), \( \hat{E}|_{z(M_1 \cap N')} \) is semifinite. So there are two families of projections \( \{p_a\}, \{q_a\} \) in \( z(M_1 \cap N') \) such that \( p = \sum_a p_a, p_a \sim q_a \) in \( z(M_1 \cap N') \) and \( q_a \in m_E \). Let \( W_a \) be a partial isometry satisfying \( W_a W_a^* = e_N, W_a^* W_a = q_a, \) and \( \rho_a \in End(N) \) defined by \( W_a x W_a^* = e_N \rho_a(x), x \in N \). Then \( [\rho] = \oplus[\rho_a] \). Since \( W_a = e_N W_a q_a \in m_E \), due to Lemma 2.2, there exists \( V_a \in \mathcal{M} \) satisfying \( W_a = e_N V_a \). It is easy to check \( V_a \in \mathcal{H}_{\rho_a} \) and \( E(V_a V_a^*) = 1 \). So \( \rho_a \) satisfies (i). Q.E.D.

Let \( \{[\rho_\xi]\}_{\xi \in \Xi} \) be the set of irreducible sectors with finite dimension contained in \( [\gamma|N] \). We arrange the index set \( \Xi \) such that \( [\rho_\xi] = [\rho_{\bar{\xi}}] \) holds for every \( \xi \in \Xi \). For simplicity, we use notations \( R_\xi, \overline{R}_\xi, \mathcal{H}_\xi, d(\xi), E_\xi \) instead of \( R_{\rho_\xi}, \overline{R}_{\rho_\xi}, \) etc. We define the Frobenius maps \( c_\xi : \mathcal{H}_\xi \rightarrow \mathcal{H}_{\overline{\xi}}, \overline{c}_\xi : \mathcal{H}_{\overline{\xi}} \rightarrow \mathcal{H}_\xi \) by

\[
c_\xi(V) = \sqrt{d(\xi)} V^* \overline{R}_\xi, \quad V \in \mathcal{H}_\xi,
\]

\[
\overline{c}_\xi(V) = \sqrt{d(\xi)} V^* R_\xi, \quad V \in \mathcal{H}_{\overline{\xi}}.
\]

Then thanks to (2.1), \( \overline{c}_\xi c_\xi = 1_{\mathcal{H}_\xi}, c_\xi \overline{c}_\xi = 1_{\mathcal{H}_{\overline{\xi}}} \). So in particular, both \( c_\xi \) and \( \overline{c}_\xi \) are
invertible. We introduce a new inner product to \( \mathcal{H}_\xi \) by
\[
(V_1, V_2) = d(\xi)E(V_1 V_2^*) \in (\rho_\xi, \rho_\xi) = C, \quad V_1, V_2 \in \mathcal{H}_\xi.
\]
Due to the estimate \(|(V_1, V_2)| \leq d(\xi)||V_1||||V_2||\), there is a non-singular positive operator \( a_\xi \in B(\mathcal{H}_\xi) \) satisfying
\[
(V_1, V_2) = \langle a_\xi V_1 | V_2 \rangle.
\]
Let \( \{V_i\} \subset \mathcal{H}_\xi \) be an orthonormal basis of \( \mathcal{H}_\xi \). Since \( \sum V_i V_i^* = s(\mathcal{H}_\xi) \leq 1 \), we get
\[
Tr(a_\xi) = \sum (V_i, V_i) = d(\xi)E(s(\mathcal{H}_\xi)) \leq d(\xi).
\]
So \( a_\xi \) is a trace class operator. By simple computation one can show the following:
\[
\langle c_\xi(V_1) | c_\xi(V_2) \rangle = (V_1, V_2) = \langle a_\xi V_1 | V_2 \rangle,
\]
\[
\langle \overline{c}_\xi(V_1) | \overline{c}_\xi(V_2) \rangle = (\overline{V}_1, \overline{V}_2) = \langle \overline{a}_\xi \overline{V}_1 | \overline{V}_2 \rangle.
\]
Thus we get \( c_\xi^* c_\xi = a_\xi, \overline{c}_\xi^* \overline{c}_\xi = \overline{a}_\xi \). This shows that \( a_\xi \) is an invertible trace class operator, that implies \( n_\xi := \text{dim} \mathcal{H}_\xi < \infty \). Thanks to \( \overline{c}_\xi = c_\xi^{-1} \), we obtain
\[
Tr(\overline{a}_\xi) = Tr(\overline{c}_\xi^* c_\xi) = Tr(\overline{c}_\xi^* \overline{c}_\xi) = Tr(a_\xi^{-1}).
\]
This implies
\[
\frac{1}{d(\xi)} \leq a_\xi \leq d(\xi), \quad n_\xi \leq d(\xi)^2.
\]
If \( a_\xi = 1 \), (this is the case if for instance \( \text{Ind} E < \infty \)), then \( n_\xi \leq d(\xi) \). On the other hand if \( n_\xi = d(\xi) \), then it is easy to show \( a_\xi = 1 \) and \( E(s(\mathcal{H}_\xi)) = 1 \), i.e. \( s(\mathcal{H}_\xi) = 1 \).

**Theorem 3.3.** Let \( M \supset N \) be an irreducible inclusion of infinite factors with \( E \in \mathcal{E}(M, N) \), and \( M_1 \cap N' = A \oplus B_1 \oplus B_2 \oplus C \) the decomposition described in Proposition 2.8. Then with the same notation as above, the following hold:
(i) \( A = \oplus_{\xi \in \Xi} A_\xi, \) where \( A_\xi = \mathcal{H}_\xi^* e_N \mathcal{H}_\xi \cong M(n_\xi, C) \).
(ii) $B_1$ and $B_2$ are of type I.

(iii) For $V_1, V_2 \in \mathcal{H}_\xi$, $\sigma_{E \circ \hat{E}}(V_1^* e_N V_2) = V_1^* a_\xi^{-it} e_N a_\xi^{it} V_2$.

(iv) For $V_1, V_2 \in \mathcal{H}_\xi$, $j(V_1^* e_N V_2) = c_\xi (a_\xi^{1/2} V_2)^* e_N c_\xi (a_\xi^{1/2} V_1)$.

Proof. (i): Thanks to Proposition 2.8, $A$ is direct sum of type I factors. By using the one-to-one correspondence as described just before Lemma 3.1, we can parametrize the direct summands of $A$ by $\Xi$ such that $A = \oplus A_\xi$ and $A_\xi \supset H_\xi e_N H_\xi$ hold. So it suffices to show that $A_\xi$ is of type I, $\forall \xi$. If $A_\xi$ is finite, then $A_\xi \subset m_\hat{E}$ because $\hat{E}|_{A_\xi}$ is semifinite. So we can take matrix units $\{e_{i,j}\}_{1 \leq i, j}$ of $A_\xi$ (with $\sum e_{i,i} = 1_{A_\xi}$) such that $\hat{E}(e_{i,j}) = b_i \delta_{i,j}$. We may assume that there is a partial isometry $W_1 \in M_1$ satisfying $W_1 W_1^* = e_N$, $W_1^* W_1 = 1_{A_\xi}$ and $W_1 x W_1^* = e_N \rho(x)$ for $x \in N$. We set $W_i = W_1 e_{1,i}$. Then thanks to Lemma 2.2, there exists $V_i \in M$ such that $W_i = \sqrt{b_i} e_N V_i$. $\{V_i\}$ is an orthonormal basis of $H_\xi$. Indeed, it is easy to show that it is an orthonormal system. Suppose $V \in H_\xi$ is perpendicular to $\{V_i\}$. Since $e_{i,j} = W_i^* e_N W_j = \sqrt{b_i b_j} V_i^* e_N V_j$, $V^* e_N V$ is an element in $A_\xi$ satisfying $e_{i,j} V^* e_N V = 0$. This means $V^* e_N V = 0$ and $0 = \hat{E}(V^* e_N V) = V^* V$, i.e. $V = 0$. So $\{V_i\}$ is an orthonormal basis of $H_\xi$ and the rank of $A_\xi$ coincides with $n_\xi$. Now suppose $A_\xi$ is of type $I_\infty$. Since $\hat{E}|_{A_\xi}$ is semifinite, there is a matrix unit $\{e_{i,j}\}_{1 \leq i, j < \infty}$ (not necessarily $\sum e_{i,i} = 1$), such that $\hat{E}(e_{i,i}) < \infty$, $\hat{E}(e_{i,j}) = 0$ for $i \neq j$. Then we can define $W_i$ and $V_i$ as before. However, $\{V_i\}_{1 \leq i < \infty}$ is an orthonormal system of $H_\xi$, that contradicts the fact $\dim H_\xi = n_\xi < \infty$.

(ii): Since $\hat{E}|_{B_1}$ is semifinite and $j(B_1) = B_2$, it suffices to show that $pB_1 p$ is of type I for every $p \in B_1$ with $\hat{E}(p) < \infty$. Let $W \in M_1$ be a partial isometry with $WW^* = e_N$, $W^* W = p$, and define $\rho \in \text{End}(M)$ by $W x W^* = e_N \rho(x)$, $x \in N$ as before. Thanks to Lemma 2.2, there exists an isometry $V \in H_\rho$ satisfying $W = \sqrt{c} e_N V$, $c = \hat{E}(p)$. So $E(V V^*) = \frac{1}{c}$ and we get $\frac{1}{c} \leq E(s(H_\rho)) \leq 1$. Let $P = N \cap \rho(N)'$. Then in the same way as in the proof of Lemma 3.1, we can show that $pB_1 p$ is isomorphic to $P$. So we show that $P$ is of type I. Thanks to $PH_\rho = H_\rho$, we can define a normal representation of $P$ on $H_\rho$ by $\pi(x) V = x V$, $x \in P$, $V \in H_\rho$. Note that $\frac{1}{c} \leq E(s(H_\rho))$ implies that $\pi$ is faithful. Thus to
prove that $P$ is of type I, we show that there exists a normal conditional expectation from $B(\mathcal{H}_\rho)$ to $\pi(P)$ [S, Proposition 10.21]. For $\omega \in P_*$ we can define a bilinear form on $\mathcal{H}_\rho$ by $\omega(E(V_1V_2^*))$, $V_1, V_2 \in \mathcal{H}_\rho$ with an estimate $|\omega(E(V_1V_2^*))| \leq ||\omega|| ||V_1|| ||V_2||$. So there exists a unique bounded operator $h_\omega$ satisfying

$$\omega(E(V_1V_2^*)) = h_\omega V_1 V_2.$$  

For $x, y \in P$, $\omega \in P_*$, $h_\omega$ satisfies $h_{x \cdot \omega \cdot y} = \pi(x) h_\omega \pi(y)$. Indeed, by definition we get

$$x \cdot \omega \cdot y(E(V_1V_2^*)) = \omega(yE(V_1V_2^*)x) = \omega(\pi(y)V_1(\pi(x)V_2)^*)$$

$$= h_\omega \pi(y)V_1 \pi(x)V_2 >\pi(x)h_\omega \pi(y)V_1 V_2 >.$$

If $\omega \in P_*$ is positive, we have

$$Tr(h_\omega) = \omega(s(\mathcal{H}_\rho)) \leq \omega(1) = ||\omega||,$$

so by using polar decomposition of linear functionals and the fact just proved above, we get

$$||h_\omega||_1 := Tr(|h_\omega|) = Tr(h_{|\omega|}) \leq ||(|\omega||) = ||\omega||, \quad \omega \in P_*.$$

Hence we can define a bounded order preserving linear map $\theta : P_* \rightarrow B(\mathcal{H}_\rho)_*$ by $\theta(\omega)(a) = Tr(h_\omega a)$, $a \in B(\mathcal{H}_\rho)$. Note that $\theta$ satisfies $\theta(x \cdot \omega \cdot y) = \pi(x) \cdot \theta(\omega) \cdot \pi(y)$, $x, y \in P$.

Let $F_0$ be the transposition of $\theta$. Then $F_0$ is a positive normal map $F_0 : B(\mathcal{H}_\rho) \rightarrow P$ satisfying $F_0(\pi(x)a\pi(y)) = xF_0(a)y$, $x, y \in P$, $a \in B(\mathcal{H}_\rho)$. Note that $F_0(1) = E(s(\mathcal{H}_\rho))$ is a central element of $P$ because $us(\mathcal{H}_\rho)u^* = s(\mathcal{H}_\rho)$ holds for every unitary $u \in P$. Since $E(s(\mathcal{H}_\rho))$ is invertible, we can define a normal conditional expectation $F : B(\mathcal{H}_\rho) \rightarrow \pi(P)$ by

$$F(a) = \pi(E(s(\mathcal{H}_\rho))^{-1/2}F_0(a)E(s(\mathcal{H}_\rho))^{-1/2}), \quad a \in B(\mathcal{H}_\rho).$$

Therefore, $P$ is of type I.

(iii): By a simple argument one can show that unitary perturbation of $\rho_\xi$ does not have any effect on the formulae in (iii)(iv). So thanks to Lemma 2.12, we assume that there is
a dominant weight $\psi$ on $N$ satisfying $\psi \cdot \rho_\xi = d(\xi)\psi$, $\psi \cdot E_\xi = \psi$ for every $\xi \in \Xi$. Then $\sigma_i^{\psi}$ commute with $\rho_\xi$ and we get $\sigma_i^{\psi}(\mathcal{H}_\xi) = \mathcal{H}_\xi$. So we show $\sigma_i^{\psi}(V) = a_\xi^i V$ for $V \in \mathcal{H}_\xi$, that implies the statement. Indeed, since $\dim \mathcal{H}_\xi < \infty$, every element in $\mathcal{H}_\xi$ is analytic for $\{\sigma_i^{\psi}\}$. Let $V \in \mathcal{H}_\xi$ and $x \in m_\psi$. Then by using the KMS condition, we obtain
\[
\psi \cdot E(VxV^*) = \psi \cdot E(xV^*\sigma_i^{\psi}(V)) = <\sigma_i^{\psi}(V)|V > \psi(x).
\]
On the other hand, from $E(VxV^*) = E(\rho_\xi(x)VV^*) = \frac{1}{d(\xi)}(V,V)\rho_\xi(x)$ we get
\[
\psi \cdot E(VxV^*) = \frac{1}{d(\xi)}(V,V)\psi \cdot \rho_\xi(x) = <a_\xi V|V > \psi(x).
\]
So we obtain $\sigma_i^{\psi}(V) = a_\xi^i V$.

(iv): Let $z_\xi$ be the unit of $A_\xi$. Then by using the correspondence between sub-bimodules of $N L^2(M)_N$ and sub-sectors of $\gamma_N$, we get $j(A_\xi) = A_{\bar{\xi}}$ and $j(z_\xi) = \bar{z}_{\bar{\xi}}$. Let $\psi$ be as before. Then due to (i), it is easy to show that $\mathcal{H}_\xi^{*}\Lambda_{\psi,E}(n_\psi \cap n^*_\psi)$ is dense in $z_\xi \mathcal{H}_\psi, E$. Since both $j(V_1^* e_N V_2)$ and $c_\xi(a_\xi^{1/2} V_2)^* e_N c_\xi(a_\xi^{1/2} V_1)$ belong to $A_{\bar{\xi}}$, it suffices to show the equality on $\mathcal{H}_\xi^{*}\Lambda_{\psi,E}(n_\psi \cap n^*_\psi)$. Let $a \in n_\psi \cap n^*_\psi$ and $X \in \mathcal{H}_\xi$. Since $V_1, V_2$ are analytic elements for $\{\sigma_i^{\psi}\}$, we get
\[
j(V_1^* e_N V_2)\Lambda_{\psi,E}(X^*a) = J_M V_2^* J_M e_N J_M V_1 J_M \Lambda_{\psi,E}(X^*a)
\]
\[
= J_M V_2^* J_M e_N \Lambda_{\psi,E}(X^*a) \sigma_i^{\psi}(V_1)^*
\]
\[
= J_M V_2^* J_M e_N \Lambda_{\psi,E}(X^*a) \sigma_i^{\psi}(V_1)^* \rho_\xi(a)
\]
\[
= J_M V_2^* J_M \Lambda_{\psi,E}(X^*a) \sigma_i^{\psi}(V_1)^* \rho_\xi(a)
\]
\[
= \Lambda_{\psi,E}(E(X^*a) \sigma_i^{\psi}(V_1)^* \rho_\xi(a) \sigma_i^{\psi}(V_2))
\]
\[
= \Lambda_{\psi,E}(E(X^*a) \sigma_i^{\psi}(V_1)^* \sigma_i^{\psi}(V_2) a).
\]
By using $X = c_\xi(\bar{\xi}(X)) = d(\xi)^{-1/2} \bar{\xi}(X)^* R_{\bar{\xi}}$, we get
\[
j(V_1^* e_N V_2)\Lambda_{\psi,E}(X^*a) = \sqrt{d(\xi)} \Lambda_{\psi,E}(\bar{\xi}(X) \sigma_i^{\psi}(V_1)^* \sigma_i^{\psi}(V_2))
\]
\[
= \frac{1}{\sqrt{d(\xi)}}(\bar{\xi}(X), \sigma_i^{\psi}(V_1)) \Lambda_{\psi,E}(R_{\bar{\xi}} \sigma_i^{\psi}(V_2) a)
\]
\[
= \frac{1}{d(\xi)}(\bar{\xi}(X), a_\xi^{-1/2} V_1) \Lambda_{\psi,E}(c_\xi(a_\xi^{1/2} V_2)^* a).
\]
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On the other hand, we have
\[ c_\xi(a_\xi^{1/2}V_2)^*e_N c_\xi(a_\xi^{1/2}V_1)\Lambda_{\psi,E}(X^*a) = \frac{1}{d(\xi)}(c_\xi(a_\xi^{1/2}V_1), X)\Lambda_{\psi,E}(c_\xi(a_\xi^{1/2}V_2)^*a), \]
so it suffices to show \((\hat{c}_\xi(X), a_\xi^{-1/2}V_1) = (c_\xi(a_\xi^{1/2}V_1), X)\). Actually,
\[ (c_\xi(a_\xi^{1/2}V_1), X) = < \hat{c}_\xi(X)|a_\xi^{1/2}V_1 > = (\hat{c}_\xi(X), a_\xi^{-1/2}V_1). \]

Q.E.D.

Remark 3.4. Let \(V_1, V_2 \in \mathcal{H}_\xi\). Then we get
\[ \hat{E}(V_1^*e_N V_2) = < V_2|V_1 >. \]
\[ \hat{E}(j(V_1^*e_N V_2)) = < c_\xi(a_\xi^{1/2}V_1)|c_\xi(a_\xi^{1/2}V_2) >= (a_\xi^{1/2}V_2, a_\xi^{1/2}V_1) = < V_2|V_1 >. \]
So \(\hat{E} \cdot j|_A = \hat{E}|_A\) if and only if \(a_\xi = 1\) for all \(\xi \in \Xi\). It is also easy to show that \(\hat{E}|_A\) is a trace if and only if \(a_\xi\) is a scalar for all \(\xi \in \Xi\).

To the best knowledge of the authors there is no known example which violates \(a_\xi = 1\). However, the following example shows that \(\hat{E} \cdot j|_{M_1 \cap N'} = \hat{E}|_{M_1 \cap N'}\) does not hold in general; \(B_i, i = 1, 2\) may not vanish.

Example 3.5.

(i) Let \(G\) be a discrete group and \(H\) a subgroup, and let \(\alpha\) be an outer action of \(G\) on a factor \(L\). We set \(N = L \times_\alpha H\), \(M = L \times_\alpha H\). Then \(M \supset N\) is an irreducible inclusion of factors with a unique conditional expectation \(E\). We identify \(M\) and \(N\) with \((L \otimes \mathbb{C}) \times G\) and \((L \otimes \mathbb{C}) \times H\) acting on \(L_2(L) \otimes \ell^2(G/H) \otimes \ell^2(G)\) in an obvious sense. Let \(f\) be the orthogonal projection onto \(\mathbb{C}\delta_e \subset \ell^2(G/H)\), where \(\delta\) stands for the \(\delta\)-function and \(e\) the class of the neutral element \(e\), and \(F_0 = id \otimes Tr \in \mathcal{P}(L \otimes \ell^2(G/H), L \otimes \mathbb{C})\). Then thanks to Lemma 2.3 we can identify \(M_1\) with \((L \otimes \ell^\infty(G/H)) \times G\) where the action of \(G\) on \(\ell^\infty(G/H)\) is the translation, \(e_N\) with \(1 \otimes f \otimes 1\) and \(\hat{E}\) with the natural extension of \(F_0\) to \((L \otimes \ell^\infty(G/H)) \times G\). So under this identification we get \(M_1 \cap N' = \ell^\infty(H\backslash G/H)\).
For \( \dot{g} \in G/H \), we denote by \( p_{\dot{g}} \in \ell^\infty(H\setminus G/H) \) the projection corresponding to the \( H \)-orbit of \( \dot{g} \). Then \( \hat{E}(p_{\dot{g}}) \) is exactly the length of the orbit, i.e. \( \hat{E}(p_{\dot{g}}) = [H : H_{\dot{g}}] \) where \( H_{\dot{g}} := gHg^{-1} \cap H \). \( j(p_{\dot{g}}) \) can be computed by using bimodules as in \( [KY] \), and we have
\[
j(p_{\dot{g}}) = p_{\dot{g}^{-1}}. \]
So for example if \( g^{-1}Hg \subset H \) and \( g^{-1}Hg \neq H \), then \( \hat{E}(p_{\dot{g}}) = 1 \) although \( \hat{E}(j(p_{\dot{g}})) \neq 1 \). Let \( G \) be the group generated by the finite permutations of \( \mathbb{Z} \) and \( g \) where \( g \) is the translation of \( \mathbb{Z} \), and \( H \) the finite permutations of \( \mathbb{N} \cup \{0\} \). Then \( gHg^{-1} \) is the finite permutation of \( \mathbb{N} \) and we get \( gHg^{-1} \subset H, [H : H_{\dot{g}}] = \infty \). So we obtain \( \hat{E}(p_{\dot{g}}) = \infty \), \( \hat{E}(p_{g^{-1}}) = 1 \). This means \( B_i \neq \{0\}, i = 1, 2 \) in this example.

(ii) Let \( G \supset H \) be a pair of discrete groups with the following property: for every \( g \neq e \in G \; \{hgh^{-1}; h \in H\} \) is an infinite set. Let \( M := L(G) \) be the group von Neumann algebra of \( G \) and \( N := L(H) \) the subfactor of \( M \) generated by \( H \). Then in exactly the same way as one proves that \( M \) is a factor, one can show \( M \cap N' = C \). Although this example looks similar to the previous one, these two have essentially different natures. As before we can identify \( N, M \) and \( M_1 \) with \( C \times H, C \times G \) and \( \ell^\infty(G/H) \times G \) acting on \( \ell^2(G/H) \otimes \ell^2(G) \).

However, we can conclude only \( \ell^\infty(H \setminus G/H) \subset M_1 \cap N' \) because the action of \( G \) on \( G/H \) is not necessarily free. In fact the equility does not hold in general. For example, let \( G = F_3 \) be the free group generated by \( g_1, g_2, g_3 \) and \( H = F_2 = \langle g_1, g_2 \rangle \). Then the \( N \sim N \) bimodule \( N \times N \) generated by \( \delta_{g_3} \in \ell^2(F_3) \) is equivalent to \( N \ell^2(F_2) \otimes \ell^2(F_2) \) where \( \otimes \) is the usual tensor product and the left and the right actions act on each tensor component respectively. So \( \text{End}(N \times N) \simeq N^{op} \otimes N \). This means that \( M_1 \cap N' \) has a type II summand. Actually a little more effort shows that \( A = C e_N, B_1 = B_2 = 0 \) and \( C \) is of type II where \( A, B_1, B_2, \) and \( C \) are as in Proposition 2.8.

**Remark 3.6.** Let \( M_2 \) be the basic extension of \( M_1 \) by \( M \) in the first example, i.e. \( M_2 := J_{M_1} M' J_{M_1} \). Then it is easy to show \( M_2 = L \otimes B(\ell^2(G/H)) \times_{\alpha \otimes \text{Ad}(\pi)} G \) where \( \pi \) is the translation. So \( M' \cap M_2 = \pi(G)' \). In \( [B] \), W. Binder constructs an example of a pair of discrete groups \( G \supset H \) such that \( \pi(G)' \) is a type III factor. This means that the restriction of the unique expectation in \( \mathcal{E}(M_2, M_1) \) to \( M' \cap M_2 \) may fail to be a trace in general.
In [HO] R. Herman and A. Ocneanu called an inclusion of factors $M \supset N$ discrete if $E(M, N)$ is not empty. However, the above examples show that existence of a normal conditional expectation is not strong enough to assure properties resembling those of crossed products by discrete group actions. Therefore, in this paper we use the terminology in the following sense.

**Definition 3.7.** An inclusion of factors is called discrete if $E(M, N)$ is non-empty and $\hat{E}|_{M_1 \cap N'}$ is semifinite for some (and equivalently all) $E \in E(M, N)$.

In what follows we assume that $M \supset N$ is an irreducible discrete inclusion of infinite factors. Note that discreteness is equivalent to $M_1 \cap N' = A$ in the decomposition given in Proposition 2.8, and to $[\gamma|_N] = \oplus n_\xi[p_\xi]$, $d(\xi) < \infty$.

For each $\xi \in \Xi$ choose an orthogonal basis $\{V(\xi)_i\}_{i=1}^{n_\xi}$ consisting of eigenvectors of $a_\xi$ belonging to $a_{\xi,i}$. For $x \in M$ we define the “Fourier coefficient” $x(\xi)_i$ by

$$x(\xi)_i = \frac{d(\xi)}{a_{\xi,i}}E(V(\xi)_i x).$$

Then $x$ has the following formal expansion:

$$x = \sum_{\xi \in \Xi} \sum_{i=1}^{n_\xi} V(\xi)_i^* x(\xi)_i.$$ 

Although the above sum does not converge even in weak topology in general, we can give justification of the expansion as follows. We define $p_{\xi,i} \in M_1 \cap N'$ by

$$p_{\xi,i} = \frac{d(\xi)}{a_{\xi,i}} V(\xi)_i^* e_N V(\xi)_i.$$ 

Then $p_{\xi,i}$ is a projection with $z_\xi = \sum_{i=1}^{n_\xi} p_{\xi,i}$, where $z_\xi$ is the unit of $A_\xi$. By discreteness assumption we have $\sum_{\xi \in \Xi} z_\xi = 1$. Let $\omega$ be a faithful normal state on $N$ and set $\varphi = \omega \cdot E$.

Since $p_{\xi,i} \Lambda_\varphi(x) = \Lambda_\varphi(V(\xi)_i^* x(\xi)_i)$ and $\Lambda_\varphi(x) = \sum_{i=1}^{n_\xi} p_{\xi,i} \Lambda_\varphi(x)$, the sum converges in Hilbert space topology. Note that $\{x(\xi)_i\}$ uniquely determines $x$ while it is difficult to tell when a series $\{x(\xi)_i\}$ is actually the Fourier coefficient of some element $x \in M$.

Although the following lemma might sound trivial, we need to prove it because the expansion does not make sense in any decent operator algebra topology.
Lemma 3.8. Under the above assumption, assume that there is an assignment of subspaces $K_\xi \subset H_\xi$ satisfying the following conditions.

(i) $a_\xi K_\xi \subset K_\xi$.

(ii) $K_\xi^* \subset NK_\xi$.

(iii) Let $\eta, \zeta \in \Xi$ and set $\Xi_{\eta,\zeta} = \{\xi \in \Xi; \rho_\eta \rho_\zeta \succ \rho_\xi\}$. Then, $K_\eta K_\zeta \subset \bigoplus_{\xi \in \Xi_{\eta,\zeta}} NK_\xi$.

Let $L$ be the von Neumann algebra generated by $N$ and $\{K_\xi\}_{\xi \in \Xi}$. Then there exists $E_L \in \mathcal{E}(M, L)$, and $L$ is characterized by

$$L = \{x \in M; E(K_\xi^\perp x) = 0, \quad \xi \in \Xi\},$$

where $K_\xi^\perp$ is the orthogonal complement of $K_\xi$ with respect to $< | >$.

Proof. Let $L_0$ be the direct sum of $K_\xi N$. Thanks to (ii) and (iii), $L_0$ is the *-algebra generated by $N$ and $\{K_\xi\}$, which is dense in $L$. Let $L_1 = \{x \in M; E(K_\xi^\perp x) = 0, \quad \xi \in \Xi\}$ and $K$ the closure of $\Lambda_\varphi(L)$ in $H_\varphi$. First, we claim $L_1 = \{x \in M; \Lambda_\varphi(x) \in K\}$. Indeed, due to (i) we may arrange $\{V(\xi)_i\}$ such that $\{V(\xi)_i\}_{i=1}^{m_\xi}$ is an orthonormal basis of $K_\xi$. Then we get $K = \bigoplus_{\xi \in \Xi} \bigoplus_{i=1}^{m_\xi} H_{\xi,i}$ where $H_{\xi,i} = p_{\xi,i} H_\varphi$, and so

$$L_1 = \{x \in M; p_{\xi,i} \Lambda_\varphi(x) = 0, \quad i > m_\xi\}.$$ 

Thus we get the claim. Secondly, we show that there exists $E_L \in \mathcal{E}(M, L)$ with $\varphi \cdot E_L = \varphi$. Thanks to the Takesaki theorem on conditional expectations [S], it suffices to prove $\sigma_\varphi^E(L) = L$, or in our case $\sigma_\varphi^E(K_\xi) \subset NK_\xi$. As before we may and do assume that there is a dominant weight $\psi$ on $N$ satisfying $\psi \cdot E_\xi = \psi$, $\psi \cdot \rho_\xi = d(\xi) \psi$, so we have $\sigma_\varphi^{\psi}:E(V) = a_\psi^{it} V$ for $V \in H_\xi$. We set $u_\xi^t = [D\omega : D\psi]_t \rho_\xi ([D\omega : D\psi]_t^*) \in N$, where $[D\omega : D\psi]_t$ is the Connes cocycle derivative. Then we get

$$\sigma_\varphi^E(V) = Ad([D\omega \cdot E : D\psi \cdot E]_t) \cdot \sigma_\varphi^{\psi}:E(V) = Ad([D\omega : D\psi]_t) (a_\xi^{it} V) = u_\xi^t a_\xi^{it} V,$$

so due to (i) we get $\sigma_\varphi(K_\xi) \subset NK_\xi$. Now let $e_L$ be the Jones projection for $E_L$, i.e. $e_L \Lambda_\varphi(x) = \Lambda_\varphi(E_L(x)), \quad x \in M$. Then $e_L$ is the orthogonal projection onto $K$. Since $L$ is characterized by $L = \{x \in M; e_L \Lambda_\varphi(x) = \Lambda_\varphi(x)\}$, we get $L = L_1$. Q.E.D.

The following is the main technical result in this paper.

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**Theorem 3.9.** Let $M \supset N$ be an irreducible inclusion of infinite factors with $E \in \mathcal{E}(M,N)$. We assume that the inclusion is of discrete type and $\sigma^E_E$ is trivial. Let $L$ be an intermediate subfactor and $K_\xi = L \cap H_\xi$. Then $\{K_\xi\}$ satisfies the assumption of Lemma 3.8 and $L$ is generated by $N$ and $\{K_\xi\}$. Consequently, there exists $E_L \in \mathcal{E}(M,L)$.

**Proof.** First, we show that the statement can be reduced to the case where $N$ is of type III. Suppose that the statement holds for type III factors. Then we apply the statement to $\hat{M} = M \otimes P$, $\hat{N} = N \otimes P$ and $\hat{L} = L \otimes P$ where $P$ is a type III factor, and get that $\hat{L}$ is generated by $\hat{N}$ and $(H_\xi \otimes C) \cap \hat{L} = K_\xi \otimes C$. $\{K_\xi\}$ satisfies the assumption of Lemma 3.8 because so does $\{K_\xi \otimes 1\}$ by assumption. Thanks to Lemma 3.8 we get

$$\hat{L} = \{x \in M \otimes P; (E \otimes id)((K_\xi^+ \otimes 1)x) = 0, \; \xi \in \Xi\},$$

and so we obtain

$$L = \{x \in M; E(K_\xi^+ x) = 0, \; \xi \in \Xi\}.$$

Therefore, the statement holds for $L$ as well. Now, we assume that $N$ is of type III. Let $\{V(\xi)_i\}$ be as in the proof of Lemma 3.8. Thanks to $H_\xi^* \subset NH_\xi$, $H_\eta H_\zeta \subset \sum_{\xi \in \Xi, \eta, \zeta} NH_\xi$ and the Fourier decomposition, to prove that $\{K_\xi\}$ satisfies the assumption of Lemma 3.8 it suffices to show $x(\xi)_i = 0$ for $x \in L$, $\xi \in \Xi$, $i > m_\xi$, which is actually enough for the statement due to Lemma 3.8. Suppose the converse; there exists $x \in L$ such that $x(\xi)_i \neq 0$ for some $\xi \in \Xi$ and some $i > m_\xi$. Let $y = axb$, $a, b \in N$. Then $E_\xi(y(\xi)_i) = \rho_\xi(a)E_\xi(x(\xi)_i b)$. Since $N$ is a type III factor, we can choose $a, b$ such that $E_\xi(y(\xi)_i) = 1$, so we assume $E_\xi(x(\xi)_i) = 1$ from the beginning. Let $R$ be a simple injective subfactor of $N$ and $U(R)$ the unitary group of $R$. We set $C = \text{conv}\{ux\rho_\xi(u^*); u \in U(R)\}^w$ and define an action $\theta$ of $U(R)$ on $C$ by $\theta_u(w) = uw\rho_\xi(u^*)$, $u \in U(R)$, $w \in C$. We claim that the set of fixed points of $C$ under $\theta$, which is the same as $\{w \in C; aw = w\rho_\xi(a), \; a \in R\}$, is non-empty. Indeed, since $R$ is AFD, there exists an increasing sequence of finite dimensional unital von Neumann-subalgebras $\{R_n\}_{n=1}^\infty$ generating $R$. Let $C_n$ be the fixed points of $C$ under $\theta|_{U(R_n)}$, that is a non-empty compact set because $U(R_n)$ is a compact group. Then $\{C_n\}_{n=1}^\infty$
is a decreasing sequence of non-empty compact sets, and so $C_\infty := \bigcap_{n=1}^\infty C_n$ is non-empty as well. Let $w \in C_\infty$. Then $w$ satisfies $aw = w\rho_\xi(a)$ for $a \in \cup_n R_n$, and for $a \in R$ because $\cup_n R_n$ is dense in $R$. Thus $C_\infty$ is the set of the fixed points. From the definition of the Fourier coefficient of $w \in C_\infty$ we get $\rho_\eta(a)w_j = w_j\rho_\xi(a)$ for $a \in \bigcup_n \mathbb{R}^n$, and for $a \in \mathbb{R}$ because $\bigcup_n \mathbb{R}^n$ is dense in $\mathbb{R}$. Thus $C_\infty$ is the set of the fixed points. From the definition of the Fourier coefficient of $w \in C_\infty$ we get $\rho_\eta(a)w_j = w_j\rho_\xi(a)$ for $a \in \mathbb{R}, \eta \in \Xi$. Applying Corollary 2.11 we obtain $w_j = 0$ for $\eta \neq \xi$ and $w(\xi)_j \in \mathbb{C}$, that means $w^* \in \mathcal{H}_\xi \cap N = \mathcal{K}_\xi$. On the other hand, $\mathcal{E}_\xi(x(\xi)_i) = 1$ implies $\mathcal{E}_\xi((ux\rho_\xi(u^*)))(\xi)_i) = \rho_\xi(u)\mathcal{E}_\xi(x(\xi)_i)\rho_\xi(u^*) = 1$ for $u \in U(R)$, and so $\mathcal{E}_\xi(w(\xi)_i) = 1$ by continuity. Since $w(\xi)_i$ is a scalar $w(\xi)_i = 1$. Hence $w^* \notin \mathcal{K}_\xi$, that is contradiction. Therefore we get $x(\xi)_i = 0$ for $x \in L, \xi \in \Xi, i > m_\xi$. Q.E.D.

**Corollary 3.10.** Let $M, N, \Xi$ be as above and $\Xi_1$ a self-conjugate subset of $\Xi$ with the following properties; whenever $\xi, \eta \in \Xi_1$, $\Xi_{\xi,\eta} \subset \Xi_1$. Then there exists an unique intermediate subfactor $L$ such that if we denote by $\gamma'$ the canonical endomorphism $\gamma' : L \longrightarrow N$, then

$$[\gamma'|_N] = \bigoplus_{\xi \in \Xi_1} n_\xi[\rho_\xi].$$

**Proof.** Set $L = N \vee \{\mathcal{H}_\xi\}_{\xi \in \Xi_1}$. Q.E.D.

**Corollary 3.11.** Let $M \supset N$ be an irreducible inclusion of factors ($N$ is not necessarily infinite) with $E \in \mathcal{E}(M, N)$. We assume that the inclusion is of discrete type and $\sigma_{E}^{E,E}$ is trivial. Then for every intermediate subfactor $L$, $\mathcal{E}(M, L)$ is not empty.

**Proof.** It is enough to prove the statement when $N$ is finite and $M$ is infinite. Let $F$ be a type $I_\infty$ factor. Then thanks to Theorem 3.9 $\mathcal{E}(M \otimes F, L \otimes F)$ is not empty. Since we can identify $M \supset L$ with $e(M \otimes F)e \supset e(L \otimes F)e$ where $e$ is a minimal projection of $F$, $\mathcal{E}(M, L)$ is not empty. Q.E.D.

**Remark 3.12.** Using the same type of argument, we can show the following: for an irreducible discrete inclusion of infinite factors $M \supset N$ and a simple injective subfactor $R$ of $N$, $M \cap R' = \mathbb{C}$ holds. Indeed, suppose $x \in M \cap R'$. Then $x(\xi)_i$ satisfies $x(\xi)_ia = \rho_\xi(a)x(\xi)_i$ for $a \in R$. So we get $x(\xi)_i = 0$ unless $\rho_\xi = id$, and $x \in N \cap R' = \mathbb{C}$.
The above theorem means that when $a_\xi$ is a scalar there is one-to-one correspondence between the set of intermediate subfactors and that of the systems of Hilbert subspaces $\{K_\xi\}$ satisfying (ii) and (iii) of Lemma 3.8. This observation has a lot of useful applications in Galois correspondence of operator algebras as stated below. Although our statements can be unified as that of depth 2 irreducible inclusions of discrete type in the language of Kac algebras (see next section), we first state them in two classical cases: crossed product inclusions of outer actions of discrete groups and fixed point inclusions of minimal actions of compact groups.

**Theorem 3.13.** Let $G$ be a discrete group and $\alpha$ an outer action of $G$ on a factor $N$. Then the map $H \mapsto N \times_\alpha H$ gives one-to-one correspondence between the lattice of all subgroups of $G$ and that of all intermediate subfactors of $N \subset N \times_\alpha G$.

**Proof.** Let $\{\lambda(g)\}$ denote the implementing unitaries of $\alpha$ in $M := N \times_\alpha G$. Then it is easy to see $\Xi = G$ and $\mathcal{H}_g = C\lambda(g)$ where $\Xi$ and $\mathcal{H}_g$ are as in Theorem 3.9. (Note that the argument in Theorem 3.9 makes sense even when $N$ is finite as far as $\{\rho_\xi\}$ are automorphisms.) Let $\{K_g\}_{g \in G}$ be a system of subspaces satisfying (ii) and (iii) of Theorem 3.9. Then there exists a subgroup $H \subset G$ such that $K_g = C\lambda(g)$ if $g \in H$ and $K_g = 0$ if $g \notin H$. This means that for every intermediate subfactor $L$ there exits a subgroup $H$ with $L = N \times_\alpha H$. Q.E.D.

**Remark 3.14.** In [Ch], H. Choda proved that there is one-to-one correspondence between the set of subgroups and the set of intermediate subfactors $L$ with $E(N \times_\alpha G, L)$ non-empty. The above theorem says that the existence of a normal conditional expectation to every intermediate subfactors automatically follows from Theorem 3.9.

Let $G$ be a compact group. We call an action $\alpha$ of $G$ on a factor $M$ minimal if $\alpha$ is faithful and $M \cap M^{G'} = C$ where $M^G$ is the fixed point algebra under $\alpha$. It is known that if $\alpha$ is minimal the crossed product $M \times_\alpha G$ is always a factor (See Remark 4.5). We fix a complete system of representatives of the equivalence classes of the irreducible
representations of $G$ and denote it by $\hat{G}$. If $\alpha$ is minimal and the fixed point algebra $M^G$ is infinite, using the same type of the argument as in [AHKT, Lemma III 3.4], one can show that for every $\pi \in \hat{G}$ there exists a Hilbert space $\mathcal{H}_\pi \in M$ with support 1 such that $\mathcal{H}_\pi$ is globally invariant under $\alpha$ and $\alpha|_{\mathcal{H}_\pi}$ is equivalent to $\pi$. This means that $M$ is the crossed product of $M^G$ and the dual object of $G$ by the corresponding Roberts action [R1]. We fix such a $\mathcal{H}_\pi$ for each $\pi \in \hat{G}$ and choose an orthonormal basis $\{V(\pi)_i\}_{i=1}^{d(\pi)}$ of $\mathcal{H}_\pi$ where $d(\pi)$ is the dimension of $\mathcal{H}_\pi$. Let $N = M^G$ and $E$ the unique element in $\mathcal{E}(M, N)$ obtained by

$$E(x) = \int_G \alpha_g(x)dg, \quad x \in M.$$ 

We define an endomorphism $\rho_\pi \in \text{End}(N)$ by

$$\rho_\pi(x) = \sum_{i=1}^{d(\pi)} V(\pi)_i x V(\pi)_i^*, \quad x \in N.$$ 

Thanks to the minimality of $\alpha$, $\rho_\pi$ is always irreducible with $d(\rho_\pi) = d(\pi)$. It is routine to show that $\rho_\pi$ does not depend on the choice of the basis and that the sector of $\rho_\pi$ does not depend on the choice of $\mathcal{H}_\pi$. Note that $\mathcal{H}_\pi$ is characterized by

$$\mathcal{H}_\pi = \{V \in M; Vx = \rho_\pi(x)V, \quad x \in N\}.$$ 

Let $e_N$ be the Jones projection for $E$. Then using Peter-Weyl theorem we can show

$$\sum_{\pi \in \hat{G}} \sum_{i=1}^{d(\pi)} d(\pi)V(\pi)_i^* e_N V(\pi)_i = 1.$$ 

This means that we can identify $\Xi$ in Theorem 3.3 with $\hat{G}$, and when $\xi \in \Xi$ and $\pi \in \hat{G}$ are identified we can identify $\mathcal{H}_\xi$ with $\mathcal{H}_\pi$ as well. Note that $a_\pi = 1$ because

$$(V(\pi)_i, V(\pi)_j) = d(\pi) \int_G \alpha_g(V(\pi)_i V(\pi)_j^*)dg = d(\pi) \sum_{k,l} \left( \int_G \pi(g)_{k,i} \overline{\pi(g)_{l,j}}dg \right) V(\pi)_k V(\pi)_l^*$$

$$= \delta_{i,j} \sum_k V(\pi)_k V(\pi)_k^* = \delta_{i,j} 1.$$
Theorem 3.15. Let $G$ be a compact group and $\alpha$ a minimal action of $G$ on $M$. Then the map $H \mapsto M^H$ gives one-to-one correspondence between the lattice of all closed subgroups of $G$ and that of all intermediate subfactors of $M \supset M^G$.

To prove the theorem we need the following lemma, which is essentially contained in [R2]. For the sake of completeness we give a proof.

Lemma 3.16. Let $G$ be a compact group and $\text{Rep}(G)$ the category of finite dimensional unitary representations of $G$. For $\pi \in \text{Rep}(G)$ $H_\pi$ denotes the representation space of $\pi$. Suppose we have a Hilbert subspace $K_\pi \subset H_\pi$ for each $\pi \in \text{Rep}(G)$ satisfying the following:

\[ K_\pi \oplus K_\sigma \subset K_{\pi \oplus \sigma}, \quad \pi, \sigma \in \text{Rep}(G), \]

\[ K_\pi \otimes K_\sigma \subset K_{\pi \otimes \sigma}, \quad \pi, \sigma \in \text{Rep}(G), \]

\[ \overline{K_\pi} = K_\overline{\pi}, \quad \pi \in \text{Rep}(G), \]

where $\overline{\pi}$ is the complex conjugate representation and $\overline{K_\pi}$ is the image of $K_\pi$ under the natural map from $H_\pi$ to its complex conjugate Hilbert space. Then there exists a closed subgroup $H \subset G$ such that

\[ K_\pi = \{ \xi \in H_\pi; \pi(h)\xi = \xi, \ h \in H \}. \]

Proof. Let $B_0$ be the linear span of

\[ \{ \langle \pi(\cdot)\xi | \eta \rangle \in C(G); \xi \in K_\pi, \ \eta \in H_\pi, \ \pi \in \text{Rep}(G) \}, \]

where $C(G)$ is the C*-algebra of the continuous functions on $G$. Then by assumption, $B_0$ is a unital *-subalgebra of $C(G)$ that is globally invariant under the left translation by $G$. Let $B$ be the norm closure of $B_0$. Then thanks to [AHKT, Appendix A], there exists a closed subgroup $H \subset G$ such that $B = C(G/H)$. This implies that $K_\pi$ is a subspace of the set of $H$ invariant vectors $L_\pi$. Suppose $\xi \in L_\pi \ominus K_\pi$ and set $f_\eta(g) = \langle \pi(g)\xi | \eta \rangle$ for
η ∈ H, g ∈ G. Then \( f_\eta \in C(G/H) \). On the other hand, Peter-Weyl theorem shows that \( f_\eta \) is perpendicular to \( C(G/H) \) in \( L^2(G) \) because \( B_0 \) is dense in \( C(G/H) \) in uniform norm and consequently in \( L^2(G) \) as well. Thus \( f_\eta = 0 \) for all \( \eta \in H \) and \( \xi = 0 \). This proves the statement. Q.E.D.

**Proof of Theorem 3.15.** We may assume that \( M^G \) is infinite because after getting the result for \( M \otimes B(\ell^2(N)) \) we can remove \( B(\ell^2(N)) \). It easily follows from the existence of \( \{H_\pi\}_{\pi \in \hat{G}} \) that the map is injective. Let \( L \) be an intermediate subfactor and set \( K_\pi = L \cap H_\pi \). We arrange the orthonormal basis \( \{V(\pi)_i\}_{i=1}^{d(\pi)} \) such that \( \{V(\pi)_i\}_{i=1}^{m_\pi} \) is an orthonormal basis of \( K_\pi \). Thanks to Lemma 3.8 and Theorem 3.9 \( L \) is characterized by

\[
L = \{x \in M; E(K_\pi^\bot x) = 0, \quad \pi \in \hat{G}\} = \{x \in M; x(\pi)_i = 0, \quad i > m_\pi, \quad \pi \in \hat{G}\}.
\]

Thus it is enough to show that there exists a closed subgroup \( H \subset G \) such that

\[
K_\pi = \{V \in H_\pi; \alpha_h(V) = V, \quad h \in H\}.
\]

Indeed, since \( \{K_\pi\}_{\pi \in \hat{G}} \) satisfies the assumption of Lemma 3.8, it is routine to show that one can extend the assignment \( \pi \mapsto K_\pi \) to the whole category of representations such that the assumption of Lemma 3.16 is fulfilled. Thus Lemma 3.16 captures the desired closed subgroup \( H \). Q.E.D.

**Remark 3.17.** It follows from [R1, AHKT] that if \( H \) is a closed subgroup of \( G \) as in Theorem 3.15, then \( H \) is equal to the group of all automorphisms of \( M \) leaving \( M^H \) pointwise fixed, therefore we have a complete Galois correspondence.

4. **Kac algebra case.**

In this section we generalize Theorem 3.13 and Theorem 3.15 to the case of minimal actions of compact Kac algebras. It turns out that the Galois correspondence holds between the lattice of intermediate subfactors and that of left coideal von Neumann subalgebras. We also prove a bicommutant type theorem between the left coideal von Neumann subalgebras.
of a compact Kac algebra and right coideal von Neumann subalgebras of its dual Hopf algebras.

Let \( A \) be a compact Kac algebra \( [ES][BS] \) with coproduct \( \delta \), antipode \( \kappa \), and normalized Haar measure \( h \), which is a normal trace state. We regards \( A \) as a concrete von Neumann algebra represented on the G.N.S. Hilbert space \( L^2(A) \) of \( h \) with the G.N.S. cyclic vector \( \Omega_h \). The multiplicative unitary associated with \( A \) is defined by

\[
V(x\Omega_h \otimes \xi) = \delta(x)(\Omega_h \otimes \xi), \quad \xi \in L^2(A), \ x \in A.
\] (4.1)

Following [BS], we adopt the dual Hopf algebra \( [BS] \) rather than the dual Kac algebra \( [ES] \) as the dual object of \( A \); the dual Hopf algebra \( \hat{A} \) of \( A \) is the von Neumann algebra generated by

\[
\{(id \otimes \omega)(V); \omega \in B(L^2(A))_*\}
\]

with the comultiplication and the antipode,

\[
\hat{\delta}(y) = V^*(1 \otimes y)V, \quad \hat{\kappa}(y) = J_A y^* J_A, \quad y \in \hat{A},
\] (4.2)

where \( J_A \) is the canonical conjugation of \( A \) with respect to \( \Omega_h \). Let \( U \in B(L^2(A)) \) be the unitary operator defined by

\[
U x\Omega_h = \kappa(x)\Omega_h, \quad x \in A
\]

and set

\[
\hat{V} = F(U \otimes 1)V(U \otimes 1)F \in A \otimes \hat{A}',
\] (4.3)

\[
\tilde{V} = F(1 \otimes U)V(1 \otimes U)F \in A' \otimes \hat{A},
\] (4.4)

as in [BS] where \( F \) is the flip operator of \( L^2(A) \otimes L^2(A) \). \( \hat{V} \) and \( \tilde{V} \) are multiplicative unitaries satisfying

\[
\hat{V}^*(\xi \otimes x\Omega_h) = \delta(x)(\xi \otimes \Omega_h), \quad \xi \in L^2(A), \ x \in A.
\] (3.5)
A finite dimensional unitary corepresentation \( \pi \) is a pair of a finite dimensional Hilbert space \( H_\pi \) and a linear map \( \Gamma_\pi : H_\pi \rightarrow H_\pi \otimes A \) satisfying
\[
(\Gamma_\pi \otimes id) \cdot \Gamma_\pi = (id \otimes \delta) \cdot \Gamma_\pi
\]
and the following unitarity condition: if \( \{e(\pi)_i\} \) is an orthonormal basis of \( H_\pi \) and
\[
\Gamma_\pi(e(\pi)_j) = \sum_i e(\pi)_i \otimes u(\pi)_{i,j},
\]
then \( u(\pi) = (u(\pi)_{i,j}) \) is unitary as an element in \( M(d(\pi), \mathbb{C}) \otimes A \), where \( d(\pi) \) is the dimension of \( H_\pi \). We abuse the notation and call \( u(\pi) \) a unitary corepresentation as well.

Basic notions such as tensor product, direct sum, complex conjugate corepresentations, and irreducibility are defined by a standard procedure. Note that since \( A \) is a Kac algebra the complex conjugate corepresentation \( u(\pi) = (u(\pi)_{i,j} = u(\pi)_{i,j}^* ) \) of \( u(\pi) \) is always unitary [W]. Let \( \pi, \sigma \) be unitary corepresentations of \( A \). Then the following orthogonality relation holds:
\[
h(u(\pi)_{i,j} u(\sigma)_{k,l}) = \frac{1}{d(\pi)} \delta_{i,k} \delta_{j,l}.
\]
Let \( \Xi \) be a complete system of representatives of the irreducible corepresentations of \( A \).

Then the linear span of \( \{u(\pi)_{i,j}\}_{1 \leq i,j \leq d(\pi), \pi \in \Xi} \) is a dense in \( A \) in weak topology. For \( x \in A \) we define \( x(\pi)_{i,j} \) by
\[
 x(\pi)_{i,j} = d(\pi) h(u(\pi)_{i,j}^* x).
\]
\( \{x(\pi)_{i,j}\} \) determines \( x \) in the sense that \( x = \sum x(\pi)_{i,j} u(\pi)_{i,j} \) holds in Hilbert space topology in \( L^2(A) \).

**Definition 4.1.** A unital von Neumann subalgebra \( B \) of a Kac algebra \( A \) is called a left (right) coideal von Neumann subalgebra if and only if \( \delta(B) \subset A \otimes B \) (respectively \( \delta(B) \subset B \otimes A \)) holds.

Let \( Corep(A) \) be the category of finite dimensional unitary corepresentations of \( A \).
Proposition 4.2. Let $\mathcal{A}$ be a compact Kac algebra. Then there exists one-to-one correspondence between the following two sets.

(i) The sets of left coideal von Neumann subalgebras of $\mathcal{A}$.

(ii) The set of systems of Hilbert subspaces $K_\pi \subset H_\pi$, $\pi \in \text{Corep}(\mathcal{A})$ satisfying the following:

$$K_\pi \oplus K_\sigma \subset K_{\pi \oplus \sigma}, \quad \pi, \sigma \in \text{Corep}(\mathcal{A}).$$

$$K_\pi \otimes K_\sigma \subset K_{\pi \otimes \sigma}, \quad \pi, \sigma \in \text{Corep}(\mathcal{A}).$$

$$K_\pi = K_\pi, \quad \pi \in \text{Corep}(\mathcal{A}).$$

The correspondence is given as follows. Let $\{K_\pi\}$ be a system of subspaces satisfying the condition in (ii) and $\{e(\pi)_i\}^{d(\pi)}_{i=1}$ an orthonormal basis of $H_\pi$ such that $\{e(\pi)_i\}^{m_\pi}_{i=1}$ is an orthonormal basis of $K_\pi$. Then the corresponding left coideal von Neumann subalgebra $B$ is the weak closure of the linear span of $\{u(\pi)_{i,j}\}$ $1 \leq i \leq d(\pi)$, $1 \leq j \leq m_\pi$, $\pi \in \text{Corep}(\mathcal{A})$.

Proof. First we note that two distinct von Neumann subalgebras $B_1$ and $B_2$ give rise to distinct Hilbert subspaces $\overline{B_1\Omega_h}$, $\overline{B_2\Omega_h}$ because $h$ is a faithful normal traces. It is easy to show that the weakly closed linear subspace $B$ defined in the statement is actually a left coideal von Neumann subalgebra, so it suffices to prove that every left coideal von Neumann subalgebra $B$ arises in this way. Let $\{e(\pi)_i\}^{d(\pi)}_{i=1}$ be an orthonormal basis of $H_\pi$ and we set

$$K_\pi = \text{span}\{\sum_{j=1}^{d(\pi)} x(\pi)_{i,j} e(\pi)_{j}; \ x \in B, \ 1 \leq i \leq d(\pi)\}.$$  

Since $K_\pi$ does not depend on the choice of the basis, we may and do assume that $\{e(\pi)_i\}^{m_\pi}_{i=1}$ is an orthonormal basis. Thus $x(\pi)_{i,j} = 0$ for $x \in B$, $j > m_\pi$. We show that $u(\pi)_{i,j} \in B$ for $1 \leq i \leq d(\pi)$, $1 \leq j \leq m_\pi$. By the definition of $K_\pi$, for $j$ with $1 \leq j \leq m_\pi$ there exist $x^1, x^2, \ldots, x^{d(\pi)} \in B$ such that $\sum_{i=1}^{d(\pi)} x^i(\pi)_{i,k} = \delta_{j,k}$. Using unitarity of $u(\pi)$ and $\delta(u(\pi)_{p,q}) = \sum_r u(\pi)_{p,r} \otimes u(\pi)_{r,q}$, we get

$$u(\pi)_{i,k} \otimes 1 = \sum_p (1 \otimes u(\pi)_{k,p})\delta(u(\pi)_{i,p}).$$
Since $\mathcal{B}$ is a left coideal we obtain

$$\mathcal{B} \ni \sum_i (h \otimes \text{id})(u(\pi)_{i,k} \otimes 1)\delta(x^i) = \sum_{i,p} (h \otimes \text{id})((1 \otimes u(\pi)_{k,p})\delta(u(\pi)^*_i u^i))$$

$$= \sum_{i,p} x^i(\pi)_{i,p} u(\pi)_{k,p} = u(\pi)_{k,j}.$$

Thus $\mathcal{B}$ is characterized as

$$\mathcal{B} = \{ x \in \mathcal{A}; \ x(\pi)_{i,j} = 0, \ \pi \in \Xi, \ j > m_{\pi} \}.$$

Since $\mathcal{B}$ is a *-subalgebra, the natural extension of $\{K_{\pi}\}_{\pi \in \Xi}$ to the whole category of unitary corepresentations satisfies the three conditions of (ii). Q.E.D.

**Definition 4.3.** Let $\Gamma : M \to M \otimes A$ be an action of a compact Kac algebra $A$ on a factor $M$.

(i) $\Gamma$ is called minimal if and only if the linear span of $\{ (\omega \otimes \text{id}) \cdot \Gamma(M); \omega \in M_* \}$ is dense in $\mathcal{A}$ and the relative commutant of the fixed point algebra $M^\Gamma = \{ x \in M; \Gamma(x) = x \otimes 1 \}$ in $M$ is trivial.

(ii) Let $\mathcal{B}$ be a left coideal von Neumann subalgebra of $A$. The intermediate subalgebra $M(\mathcal{B})$ of $M^\Gamma \subset M$ associated to $\mathcal{B}$ is defined by

$$M(\mathcal{B}) = \{ x \in M; \Gamma(x) \in M \otimes \mathcal{B} \}.$$

**Theorem 4.4.** Let $\Gamma : M \to M \otimes A$ be a minimal action of a compact Kac algebra $A$ on a factor $M$. Then the map $\mathcal{B} \mapsto M(\mathcal{B})$ gives one-to-one correspondence between the lattice of left coideal von Neumann subalgebras of $A$ and that of the intermediate subfactors of $M^\Gamma \subset M$.

Proof. For the same reason as in the proof of Theorem 3.15 we may assume that $M^\Gamma$ is infinite. Note that there exists a normal conditional expectation $E \in \mathcal{E}(M, M^\Gamma)$ given by

$$E(x) \otimes 1 = (id \otimes h) \cdot \delta(x), \ x \in M.$$
In exactly the same way as in the case of compact group actions, for each \( \pi \in \Xi \) one can find a Hilbert space \( \mathcal{H}_\pi \) in \( M \) with support 1 and its basis \( \{ V(\pi) \}_{i=1}^{d(\pi)} \) satisfying

\[
\delta(V(\pi)_i) = \sum_j V(\pi)_j \otimes u(\pi)_{j,i}.
\]

Thus, as before we can identify our \( \Xi \) with that in Theorem 3.3 and we get \( a_\pi = 1 \) thanks to the orthogonality relation. Let \( L \) be an intermediate subfactor and \( \mathcal{K}_\pi = L \cap \mathcal{H}_\pi \). Thanks to Lemma 3.8 and Theorem 3.9, \( L \) is generated by \( M^\Gamma \) and \( \{ \mathcal{K}_\pi \}_{\pi \in \Gamma} \), and is characterized by

\[
L = \{ x \in M; E(\mathcal{K}_\pi^+ x) = 0, \quad \pi \in \Xi \}.
\]

Therefore, as in the proof of Theorem 3.16 we can conclude \( L = M(\mathcal{B}) \) by using Proposition 4.2, where \( \mathcal{B} \) is the left coideal von Neumann subalgebra corresponding to \( \{ \mathcal{K}_\pi \}_\pi \). The map is injective because 2 distinct systems of subspaces satisfying the assumption of Proposition 4.2 (ii) give rise to 2 distinct intermediate subfactors. Q.E.D.

**Remark 4.5.** The crossed product \( M \times_\Gamma \hat{\mathcal{A}} \) is the von Neumann algebra generated by \( \Gamma(M) \) and \( C \otimes \hat{\mathcal{A}} \). As is expected, we can identify the basic extension \( M_1 \) with \( M \times_\Gamma \hat{\mathcal{A}} \) if the action is minimal as follows. Let \( e_0 \) be the projection in \( \hat{\mathcal{A}} \) corresponding to the trivial corepresentation of \( \mathcal{A} \) and we set \( e = 1 \otimes e_0 \). Since we have the dual operator valued valued weight of the crossed product whose restriction to \( \hat{\mathcal{A}} \) is a semi-finite trace (Plancherel weight), if \( MeM \) is dense in \( M \times_\Gamma \hat{\mathcal{A}} \) we can apply Lemma 2.4 and get the result. Indeed, it is known [BS] that \( \hat{\mathcal{A}} \) is a direct sum of type \( I_{d(\pi)} \) factors \( \hat{\mathcal{A}}_\pi, \pi \in \Xi \) and the multiplicative unitary \( V \) can be expanded as

\[
V = \sum_{\pi \in \Xi} \sum_{1 \leq i,j \leq d(\pi)} e(\pi)_{i,j} \otimes u(\pi)_{i,j},
\]

where \( \{ e(\pi)_{i,j} \} \) are matrix units of \( \hat{\mathcal{A}}_\pi \). Thanks to (4.1), we have

\[
e(\pi)_{i,j} u(\sigma)_{k,l} \Omega_h = \delta_{\pi,\sigma} \delta_{j,l} u(\pi)_{k,i} \Omega_h.
\]

Now, we show

\[
d(\pi) \delta(V(\pi)_i^*) e(\pi)_{j} \delta(V(\pi)_j) = 1 \otimes \hat{\kappa}(e(\pi)_{j,i}).
\]
From $\delta(V(\pi)_i) = \sum_k V(\pi)_k \otimes u(\pi)_{k,i}$ We get

$$\delta(V(\pi)_i^*)e\delta(V(\pi)_j) = \sum_k 1 \otimes u(\pi)_{k,i}^* e_0 u(\pi)_{k,j}.$$ 

Thanks to the orthogonality relation, we obtain

$$d(\pi) \sum_k u(\pi)_k^* e_0 u(\pi)_{k,j} u(\sigma)_p^* \Omega_h = \delta_{\pi,\sigma} \delta_j q u(\pi)_p^* \Omega_h,$$

where we use the fact that $e_0$ is the projection onto the space spanned by $\Omega_h$. On the other hand,

$$\hat{\kappa}(e(\pi)_{j,i}) u(\pi)_p^* \Omega_h = J_A e(\pi)_{i,j} J_A u(\pi)_p^* \Omega_h = J_A e(\pi)_{i,j} u(\sigma)_p^* \Omega_h$$

$$= \delta_{\pi,\sigma} \delta_j q J_A u(\pi)_p^* \Omega_h = \delta_{\pi,\sigma} \delta_j q u(\pi)_p^* \Omega_h.$$

Thus $\delta(M)e\delta(M)$ is dense in $M \times_{\Gamma} \hat{A}$.

The above theorem suggests that it is worth while studying the structure of the lattice of the left coideal von Neumann subalgebras of Kac algebras. For compact and discrete Kac algebras we have the following:

**Theorem 4.6.** Let $A$ be a compact Kac algebra and $\hat{A}$ its dual Hopf algebra represented on $L^2(A)$. Let $B \subset A$ be a left coideal von Neumann subalgebra and $C \in \hat{A}$ a right coideal von Neumann subalgebra. Then the following hold:

(i) $B' \cap \hat{A}$ is a right coideal von Neumann subalgebra of $\hat{A}$ and $(B' \cap \hat{A})' \cap A = B$.

(ii) $C' \cap A$ is a left coideal von Neumann subalgebra of $A$ and $(C' \cap A)' \cap \hat{A} = C$.

(iii) Set $\tilde{B} = \hat{\kappa}(B' \cap \hat{A})$. Then the map given by $B \mapsto \tilde{B}$ is a lattice anti-isomorphism between the set of left coideal von Neumann subalgebras of $A$ and that of $\hat{A}$.

**Proof.** (i): Let $E_B$ be the $h$ preserving conditional expectation in $E(A, B)$ and $e_B$ its Jones projection, i.e. $e_B$ is the projection defined by $e_B x \Omega_h = E_B(x) \Omega_h$, $x \in A$. Note that $e_B \in B'$ and $\{e_B\}' \cap A = B$ hold. Thus to prove $(B' \cap \hat{A})' \cap A = B$ it suffices to show $e_B \in B' \cap \hat{A}$. First, we prove $(id \otimes E_B) \cdot \delta = \delta \cdot E_B$. Let $\{K_\pi\}_{\pi \in \Xi}$ be the system of Hilbert subspaces corresponding to $B$ and $\{e(\pi)_i\}_{i=1}^{d(\pi)}$ an orthonormal basis of $H_\pi$ such
that \( \{e(\pi)_i\}_{i=1}^{m_\pi} \) is a basis of \( K_\pi \). As we saw in the proof of Proposition 4.2, the linear span of \( \{u(\pi)_{i,j}\}, \pi \in \Xi, 1 \leq i \leq d(\pi), 1 \leq j \leq m_\pi \) is dense in \( B \) in weak topology. Let \( x \in \mathcal{A}, 1 \leq j \leq m_\pi \). Then we get

\[
(id \otimes h)((1 \otimes u(\pi)_{i,j})\delta(x)) = \sum_{k=1}^{d(\pi)} u(\pi)_{k,i}^* (id \otimes h)(\delta(u(\pi)_{k,j}x))
\]

\[
= \sum_{k=1}^{d(\pi)} u(\pi)_{k,i}^* h(u(\pi)_{k,j}x)
\]

\[
= \sum_{k=1}^{d(\pi)} u(\pi)_{k,i}^* h(u(\pi)_{k,j}E_B(x))
\]

\[
= (id \otimes h)((1 \otimes u(\pi)_{i,j})\delta(E_B(x))),
\]

which implies \((id \otimes E_B) \cdot \delta(x) = \delta \cdot E_B(x)\). Let \( \hat{V} \) be the multiplicative unitary defined in (4.3). Then thanks to (4.5), for \( \xi \in L^2(\mathcal{A}) \) and \( x \in \mathcal{A} \) we get

\[
\hat{V}^*(1 \otimes e_B)(\xi \otimes x\Omega_h) = \hat{V}^*(\xi \otimes E_B(x)\Omega_h) = \delta(E_B(x))(\xi \otimes \Omega_h)
\]

\[
= (id \otimes E_B) \cdot \delta(x)(\xi \otimes \Omega_h) = (1 \otimes e_B)\hat{V}^*(\xi \otimes x\Omega_h),
\]

and so \((1 \otimes e_B)\) commutes with \( \hat{V} \). Since \( \{(\omega \otimes id)(\hat{V}); \omega \in B(L^2(\mathcal{A}))^*\} \) is dense in \( \hat{A}' \), \( e_B \in \hat{A} \). Let \( x \in B, y \in B' \cap \hat{A} \). Then

\[
\hat{\delta}(y)(x \otimes 1) = V^*(1 \otimes y)V(x \otimes 1) = V^*(1 \otimes y)\delta(x)V = V^*\delta(x)(1 \otimes y)V
\]

\[
= (x \otimes 1)V^*(1 \otimes y)V = (x \otimes 1)\hat{\delta}(y).
\]

Thus \( B' \cap \hat{A} \) is a right coideal von Neumann subalgebra of \( \hat{A} \). (ii): In a similar way as above one can show that \( C' \cap \mathcal{A} \) is a left coideal von Neumann subalgebra of \( \mathcal{A} \). Let \( C_0 = (C' \cap A)' \cap \hat{A} \). Then it is easy to show \( C_0' \cap \mathcal{A} = C' \cap \mathcal{A} \). Thus to prove \( C_0 = C \), it suffices to prove that if \( C_1 \) and \( C_2 \) are distinct right coideal von Neumann subalgebras of \( \hat{A} \), then \( C_1' \cap \mathcal{A} \) and \( C_2' \cap \mathcal{A} \) are distinct. Since the Plancherel weight of \( \hat{A} \) is the restriction of the usual trace on \( B(L^2(\mathcal{A})) \), there exists a trace preserving conditional expectation \( F \in \mathcal{E}(B(L^2(\mathcal{A})), \hat{A}) \). Note that one can identify \( F \) with the dual weight of the crossed product of \( \hat{A} \) and \( \hat{A} = \mathcal{A}' \) when \( \hat{\delta} \) is regarded as an action of \( \hat{A} \) on itself. Thus the
restriction of $F$ to $\mathcal{A}'$ is a trace. We claim that $F((\mathcal{C}' \cap \mathcal{A}')) = \mathcal{C}$ for every right coideal von Neumann subalgebra $\mathcal{C} \subset \hat{\mathcal{A}}$. To prove the claim it is enough to show that $\mathcal{C} \cdot \mathcal{A}'$ is weakly dense in $(\mathcal{C} \cup \mathcal{A}')''$ because of $(\mathcal{C}' \cap \mathcal{A})' = (\mathcal{C} \cup \mathcal{A}')''$. Let $\hat{V}$ be as in (4.4). Thanks to (4.4) and (4.6), for $c \in \mathcal{C}$ and $\omega \in B(L^{2}(\mathcal{A}))_{*}$ we get

$$(id \otimes \omega)(\hat{V})c = (id \otimes \omega)(\hat{V}(c \otimes 1)) = (id \otimes \omega)(\hat{\delta}(c)\hat{V}) \in \overline{\mathcal{C} \cdot \mathcal{A}''},$$

which shows $\overline{\mathcal{C} \cdot \mathcal{A}''} = (\mathcal{C} \cup \mathcal{A}')''$. Using the claim, now we can show that if $\mathcal{C}_1 \neq \mathcal{C}_2$ are right coideal von Neumann subalgebras of $\hat{\mathcal{A}}$, $(\mathcal{C}_1' \cap \mathcal{A})' \neq (\mathcal{C}_2' \cap \mathcal{A})'$, and so $(\mathcal{C}' \cap \mathcal{A})' \cap \hat{\mathcal{A}} = \mathcal{A}$. (iii): This is a direct consequence of (i) and (ii). Q.E.D.

In what follows, we assume $n := \text{dim} \mathcal{A} < \infty$. Let $\epsilon$ and $\hat{\epsilon}$ be the counit of $\mathcal{A}$ and $\hat{\mathcal{A}}$, and $e$ and $\hat{e}$ the integrals of $\mathcal{A}$ and $\hat{\mathcal{A}}$; $e$ and $\hat{e}$ are the minimal central projections satisfying $ex = ee(x), x \in \mathcal{A}$, and $\hat{e}y = \hat{e}\hat{e}(y), y \in \hat{\mathcal{A}}$. It is known that the G.N.S. cyclic vector $\Omega_{\hat{h}}$ of the normalized Haar measure $\hat{h}$ of $\hat{\mathcal{A}}$ can be identified with $\sqrt{n}\epsilon \Omega_{\hat{h}}$ and we have $\sqrt{n}\hat{e}\Omega_{\hat{h}} = \Omega_{\hat{h}}$ as well [KP]. The dual pairing between $\mathcal{A}$ and $\hat{\mathcal{A}}$ can be written in terms of the Hilbert space inner product as follows:

$$<x, y> = \sqrt{n} <x\Omega_{\hat{h}}|y^{*}\Omega_{\hat{h}}>, \ x \in \mathcal{A} \ y \in \hat{\mathcal{A}}. \tag{4.7}$$

The following is a space free description of the anti-isomorphism of the two lattices.

**Proposition 4.8.** Let $\mathcal{A}$ be a finite dimensional Kac algebra and $\mathcal{B}$ a left coideal von Neumann subalgebra of $\mathcal{A}$. We set

$$\tilde{\mathcal{B}} = \{y \in \hat{\mathcal{A}}; <xb, y> = \epsilon(b) <x, y>, \ x \in \mathcal{A}, \ b \in \mathcal{B}\}.$$ 

Then the following hold:

(i) $\tilde{\mathcal{B}}$ is a left coideal von Neumann subalgebra of $\hat{\mathcal{A}}$ with $\text{dim} \mathcal{B} \cdot \text{dim} \tilde{\mathcal{B}} = \text{dim} \mathcal{A}$.

(ii) $\tilde{\mathcal{B}} = \mathcal{B}$.

(iii) $\widetilde{\mathcal{B}} = \hat{\kappa}(\mathcal{B}' \cap \hat{\mathcal{A}})$. 

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Proof. (i): It is routine to show that \( \tilde{B} \) is a left coideal von Neumann subalgebra of \( \hat{A} \).

Using (4.7) for \( x \in A \), \( b \in B \), and \( y \in \hat{A} \) we get the following:

\[
<xb, y> = \sqrt{n} <xb\Omega_h|y^*\Omega_h> = \sqrt{n} <J_A b^*x^*\Omega_h|y^*\Omega_h>
= \sqrt{n} <bJ_A y^*\Omega_h|x^*\Omega_h> = \sqrt{n} <b\hat{\kappa}(y)\Omega_h|x^*\Omega_h>.
\]

On the other hand we have

\[
\epsilon(b) <x, y> = \epsilon(b) \sqrt{n} <x\Omega_h|y^*\Omega_h> = \epsilon(b) \sqrt{n} <\hat{\kappa}(y)\Omega_h|x^*\Omega_h>,
\]

and so \( \tilde{B} \) is characterized as

\[
\tilde{B} = \{ y \in \hat{A}; b\hat{\kappa}(y)\Omega_h = \epsilon(b)\hat{\kappa}(y)\Omega_h, \ b \in B \}.
\]

Let \( E_B \) and \( \hat{E}_B \) be the \( h \) and \( \hat{h} \) preserving conditional expectations onto \( B \) and \( \tilde{B} \) respectively, and \( e_B \) and \( \hat{e}_B \) the corresponding Jones projections. The above characterization shows

\[
\tilde{B} \supset \hat{\kappa}(B' \cap \hat{A}) \ni J_A e_B J_A = e_B.
\]

More specifically we show \( e_B = n\epsilon \cdot E_B(e)E_B(\hat{e}) \). Indeed, using \( \hat{\kappa}(e_B) = J_A e_B J_A = e_B \) we get the following for \( \tilde{b} \in \tilde{B} \):

\[
\hat{h}(e_B \tilde{b}) = \hat{h}(\hat{\kappa}(\tilde{b}) e_B) = <\hat{\kappa}(\tilde{b}) e_B \Omega_h|\Omega_{\hat{h}}>
= \sqrt{n} <e_B e \Omega_h|\hat{\kappa}(\tilde{b}^*) \Omega_{\hat{h}}> = \sqrt{n} <E_B(e) \Omega_h|\hat{\kappa}(\tilde{b}^*) \Omega_{\hat{h}}>
= \sqrt{n} <\Omega_h|E_B(e)\hat{\kappa}(\tilde{b}^*) \Omega_{\hat{h}}> = \epsilon \cdot E_B(e) \sqrt{n} <\Omega_h|\hat{\kappa}(\tilde{b}^*) \Omega_{\hat{h}}>
= \epsilon \cdot E_B(e) \sqrt{n} <\hat{\kappa}(\tilde{b}) \Omega_h|\Omega_{\hat{h}}> = \epsilon \cdot E_B(e) \hat{\epsilon}(\tilde{b}) \sqrt{n} <\Omega_h|\Omega_{\hat{h}}>
= \epsilon \cdot E_B(e) \hat{\epsilon}(\tilde{b}) = n\hat{h}(\hat{e}\tilde{b}) \epsilon \cdot E_B(e).
\]

Thus we obtain the claim. Note that \( \hat{h} \) is the restriction of the normalized trace of \( B(L^2(A)) \), and so \( \hat{h}(e_B) = \frac{\text{dim} B}{n} \). Thus we get

\[
\epsilon \cdot E_B(e) = \frac{\text{dim} B}{n},
\]

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\[ e_B = \text{dim} \mathcal{BE}_B(\hat{e}). \]

In the same way we can get
\[ \hat{e} \cdot \mathcal{E}_B(\hat{e}) = \frac{\text{dim} \tilde{B}}{n}. \]

Since \( e_B \) is a projection,
\[ e_B = e^2_B = (\text{dim} \mathcal{B})^2 \mathcal{E}_B(\hat{e}\mathcal{E}_B(\hat{e})) = (\text{dim} \mathcal{B})^2 \hat{e} \cdot \mathcal{E}_B(\hat{e}) \mathcal{E}_B(\hat{e}) \]
\[ = \text{dim} \mathcal{B} \hat{e} \cdot \mathcal{E}_B(\hat{e}) e_B. \]

Therefore, \( \text{dim} \mathcal{B} \text{dim} \tilde{B} = n \). (ii): It is easy to show \( \mathcal{B} \subset \tilde{\mathcal{B}} \). Since \( \tilde{\mathcal{B}} \) is a left coideal von Neumann subalgebra of \( \hat{\mathcal{A}} \) we also have \( \text{dim} \tilde{B} \cdot \text{dim} \tilde{\mathcal{B}} = n \), and so \( \mathcal{B} = \tilde{\mathcal{B}} \). (iii): In a similar way as in (i), one can show \( \text{dim} \mathcal{B} \cdot \text{dim}(\mathcal{B}' \cap \hat{\mathcal{A}}) = n \). Since we have the inclusion \( \tilde{\mathcal{B}} \supset \hat{\kappa}(\mathcal{B}' \cap \hat{\mathcal{A}}) \) we get the equality. Q.E.D.

Remark 4.9. Let \( \Gamma : M \rightarrow M \otimes \mathcal{A} \) be a minimal action on a factor \( M \) and \( L \) be an intermediate subfactor of \( M^\Gamma \subset M \). Thanks to Theorem 4.6 there exists a left coideal von Neumann subalgebra \( \mathcal{B} \subset \mathcal{A} \) such that \( L = M(\mathcal{B}) \). Let \( L_1 = J_M L' J_M \), which is an intermediate subfactor of \( M \subset M_1 \). Under the identification of \( M_1 \) with \( M \times_\Gamma \hat{\mathcal{A}} \), we actually have \( L_1 = M_1(\tilde{B}) \). Let \( e_L \) be the Jones projection for \( L \). Then,
\[ L_1 = (M \cup \{e_L\})'' = (M \cup (L_1 \cap N'))'' \]
\[ = (M \cup J_M(L' \cap M_1))J_M'' = (M \cup j(L' \cap M_1))'', \]
where \( j \) is the anti-automorphism of \( N' \cap M_1 \) defined by \( j(x) = J_M x^* J_M, x \in N' \cap M \).

It is known [D] that under our identification, \( N' \cap M_1 \) is identified with \( \mathcal{C} \otimes \hat{\mathcal{A}} \) and \( j \) is identified with \( \hat{\kappa} \), so \( j(L' \cap M_1) \) is identified with \( \mathcal{C} \otimes \tilde{B} \). Thus \( L_1 \) is identified with the intermediate subfactor generated by \( \Gamma(M) \) and \( \mathcal{C} \otimes \tilde{B} \), which proves the remark.

References

[AHKT] H. Araki, R. Haag, D. Kastler, M. Takesaki, Extension of KMS states and chemical potential, Commun. Math. Phys. 53 (1977), 97-134.
[B] M. W. Binder, *Induced factor representations of discrete groups and their types*, J. Funct. Anal. **115** (1993), 294-312.

[BS] S. Baaj, G. Skandalis, *Unitaires multiplicatifs et dualité pour les produits croisés de $C^*$-algèbres*, Ann. Scient. Éc. Norm. Sup. **26** (1993), 425-488.

[Ch] H. Choda. *A Galois correspondence in a von Neumann algebra*, Tohoku Math. J. **30** (1978), 491-504.

[C] A. Connes, *On spatial theory of von Neumann algebras*, J. Funct. Anal. **35** (1980), 153-164.

[C2] A. Connes, *Non-commutative Geometry*, Academic Press (1994).

[CT] A. Connes, M. Takesaki, *The flow of weights of on factors of type III*, Tohoku Math. J. **29** (1977), 473-575.

[D] M.-C. David, *Paragroup d’Adrian Ocneanu et algebra de Kac*, Pac. J. Math. (to appear).

[DR] S. Doplicher, J.E. Roberts *Why there is a field algebra with a compact gauge group describing the superselection structure in particle physics* Commun. Math. Phys. **131** 51-107 (1990).

[EN] M. Enock, R. Nest, *Irreducible inclusions of factors and multiplicative unitaries*, preprint.

[ES] M. Enock, J.-M. Schwartz, *Kac algebras and duality of locally compact groups*, Springer, Berlin, 1992.

[GHJ] F. Goodman, P. de la Harpe, V. Jones, *Coxeter graphs and towers of algebras*, MSRI Publications 14, Springer Verlag, 1989.

[H1] U. Haagerup, *Operator valued weights in von Neumann algebras, I*, J. Funct. Anal. **32** (1979), 175-206.

[H1] U. Haagerup, *Operator valued weights in von Neumann algebras, II*, J. Funct. Anal. **33** (1979), 339-361.

[HK] T. Hamachi, H. Kosaki, *Orbital factor map*, Erg. Th. Dyanam. Syst.
[HS] U. Haagerup, E. Størmer, *Subfactors of a factor of type IIIλ which contain a maximal centralizer*, Internat. J. Math. 6 (1995), 273-277.

[Hi] F. Hiai, *Minimizing indices of conditional expectations on a subfactor*, Publ. RIMS, Kyoto Univ. 24 (1988), 673-678.

[HO] R. Hermann, A. Ocneanu, *Index theory and Galois theory for infinite index inclusions of factors*, C.R. Acad. Sci. Paris, 309 (1989), 923-927.

[I1] M. Izumi, *Application of fusion rules to classification of subfactors*, Publ. RIMS, Kyoto Univ. 27 (1991), 953-994.

[I2] M. Izumi, *Subalgebras of infinite C*-algebras with finite Watatani indices. II. Cuntz-Krieger algebras*, preprint.

[J] V. Jones, *Index for subfactors*, Invent. Math. 72 (1983), 1-25.

[K] A. Kishimoto, *Remarks on compact automorphism groups of a certain von Neumann algebra*, Publ. RIMS, Kyoto Univ. 13 (1977), 573-581.

[Ko1] H. Kosaki, *Extension of Jones theory on index to arbitrary factors*, J. Funct. Anal. 66 (1986), 123-140.

[Ko2] H. Kosaki, *Characterization of crossed product (properly infinite case)*, Pacific J. Math. 137 (1989), 159-167.

[KL] H. Kosaki, R. Longo, *A remark on the minimal index of subfactors*, J. Funct. Anal. 107 (1992), 458-470.

[KY] H. Kosaki, S. Yamagami, *Irreducible bimodules associated with crossed product algebras*, Internat. J. Math. 3 (1992), 661-676.

[KP] G.I. Kac, V.G. Pljutkin, *Finite ring groups*, Trans. Moscow Math. Soc. (1966), 251-294, Translated from Trudy Moskov. Mat. Obsc. 15 (1966), 224-261.

[L1] R. Longo, *Index of subfactors and statistics of quantum fields I*, Commun. Math. Phys. 126 (1989), 217-247.

[L2] R. Longo, *Index of subfactors and statistics of quantum fields II*, Commun. Math. Phys. 130 (1990), 285-309.
[L3] R. Longo, *Simple injective subfactors*, Adv. Math. 63 (1987), 152-171.

[L4] R. Longo, *Minimal index and braided subfactors*, J. Funct. Anal. 109 (1992), 98-112.

[N] Y. Nakagami, *Essential spectrum $\Gamma(\beta)$ of a dual action on a von Neumann algebra*, Pacif. J. Math. 70 (1977), 437-478.

[NT] N. Nakamura, Z. Takeda, *A Galois theory for finite factors*, Proc. Japan Acad. 36 (1960), 258-260.

[NTs] Y. Nakagami, M. Takesaki “Duality for crossed product of von Neumann algebras” Lecture Notes in Math. 731, 1979, Springer-Verlag, Berlin-Heidelberg-New York.

[O] A. Ocneanu, *Quantized group string algebra and Galois theory for algebra*, in “Operator algebras and applications, Vol. 2 (Warwick, 1987),” London Math. Soc. Lect. Note Series Vol. 136, Cambridge University Press, 1988, pp. 119-172.

[P] S. Popa, Classification of amenable subfactors of type II, Acta Math. 172 (1994), 352-445.

[PP1] M. Pimsner, S. Popa, *Entropy and index for subfactors*, Ann. scient. Éc. Norm. Sup. 4 57-106, 1986.

[PP2] M. Pimsner, S. Popa, *Iterating basic construction*, Trans. Amer. Math. Soc. 210 (1988), 127-133.

[S] Ş. Strătilă, *Modular theory in operator algebras*, Editra Academiei and Abacus Press 1981.

[R1] J. E. Roberts, *Cross product of von Neumann algebras by group duals*, Symposia Math. Vol. XX (1976).

[R2] J. E. Roberts, *Spontaneously broken gauge symmetries and superselection rules*, Proc. of “School of Mathematical Physics”, Universita’ di Camerino, 1974.

[Su] N. Suzuki *Crossed product of rings of operators* Tôhoku Math. J. 11 113-124 (1960).
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Appendix (Added October 30, 1997)

Related to Theorem 3.3 and Remark 3.4, we give here an example of an irreducible inclusion of factors $N \subset M$ with a normal conditional expectation $E$ such that $N \subset M$ is discrete, $E^{-1}$ is a semifinite trace on $N' \cap M_1$ (so that $B_1 = B_2 = C = \{0\}$) yet $E^{-1} \circ j \neq E^{-1}$. In fact, our factors $N, M$ are hyperfinite of type II$_1$ with $E \in \mathcal{E}(M, N)$ being the unique normal conditional expectation preserving the trace $\tau$ on $M$ and $E^{-1}$ being a semifinite trace on $N' \cap M_1$. Thus, while the irreducibility of an inclusion of (type II$_1$) factors $N \subset M$ with $[M : N] < \infty$ automatically entails its extremality (thus, the trace-preservingness of $j = J_M \cdot J_M$ on $N' \cap M_1$) this is no longer the case when $[M : N] = \infty$, even if $N \subset M$ is discrete.

Our construction is based on Powers binary shifts and their properties ([Po], [PoPr]).

Lemma A.1 ([PoPr]). Let $\sigma$ be a bilateral Powers binary shift acting on $\{u_n\}_{n \in \mathbb{Z}}$ as in [Po], such that each half-line bitstream of $\sigma$ is aperiodic.

Let $P = vN\{u_n\}_{n \in \mathbb{Z}}$ and $N = vN\{u_n\}_{n \in \mathbb{Z}}$. Then the following hold true:

(i) $N$ and $P$ are factors;
(ii) $\sigma(N) \subset N$ and $[N : \sigma(N)] < \infty$.
(iii) $\sigma^n(N)' \cap N = \mathbb{C}, \forall n \geq 0$.
(iv) $\bigcap_{n \geq 1} \sigma^n(N) = \mathbb{C}1$.
(v) $\bigcup_{n \geq 1} \sigma^n(N)$ is a dense $\ast$-subalgebra of $P$.

Proof. All these are well known properties from [Po] [PoPr].
Proposition A.2. Let $P$ be a type $II_1$ factor with an aperiodic automorphism $\sigma \in \text{Aut } P$ and a subfactor $N \subset P$ such that $P, \sigma, N$ satisfy the conditions (i)-(v) of the previous Lemma.

Let $M = P \rtimes_{\sigma} \mathbb{Z}$. Then we have:

a) $N' \cap M = \mathbb{C}1$.

b) $N \subset M$ is discrete, i.e., $L^2(M_1, \text{Tr})$ is generated by $N - N$ sub-bimodules which have finite dimension both as left and right $N$-modules, where $\text{Tr}$ is the unique semifinite trace on $M_1 = \langle N, M \rangle$ such that $\text{Tr} e_N = 1$.

c) $J_M \cdot J_M$ is not $\text{Tr}$-preserving on $N' \cap M_1$, equivalently, there are irreducible $N - N$ sub-bimodules of $L^2(M_1, \text{Tr})$ for which the left dimension over $N$ does not coincide with the right dimension over $N$.

Proof. a). By property (iii) we have $N' \cap \sigma^{-n}(N) = \mathbb{C}1$. Thus, if $a \in N' \cap P$ then $\|E_{\sigma^{-n}(N)}(a) - a\| \to 0$ (by (ii) and (v)) and $E_{\sigma^{-n}(N)}(a) \in N' \cap \sigma^{-n}(N) = \mathbb{C}$ (by commuting squares). Thus $a \in \mathbb{C}1$, showing that $N' \cap P = \mathbb{C}$. Similarly $\sigma^n(N)' \cap P = \mathbb{C}$, $\forall n \in \mathbb{Z}$.

Assume now that $a = \sum_{n \in \mathbb{Z}} b_n u^n$ satisfies $ax = xa$, $\forall x \in N$. Thus, if $b_n \neq 0$ for some $n$ then $xb_n = b_n \sigma^n(x) \forall x \in N$. By using that $\sigma^n(N) \subset N$, it follows that $xb_n b_n^* = b_n \sigma^n(x) b_n^* = b_n b_n^* x, \forall x \in N$. Thus $b_n b_n^* \in \mathbb{C}1$ so that $b_n$ is a (multiple of a) unitary element $v \in P$ satisfying

$$xv = v \sigma^n(x), \forall x \in N. \tag{1}$$

In particular, we have

$$\sigma^n(x)v = v \sigma^{2n}(x), \forall x \in N \tag{2}$$

Also, by applying $\sigma^n$ to both sides of (1) we get

$$\sigma^n(x) \sigma^n(v) = \sigma^n(v) \sigma^{2n}(x), \forall x \in N. \tag{3}$$

From (2) and (3) we get:

$$v^* \sigma^n(x) v = \sigma^n(v^*) \sigma^n(x) \sigma^n(v), \forall x \in N. \tag{4}$$

Thus, $\sigma^n(v) = \alpha v$ for some $\alpha \in \mathbb{C}1$. Let then $m_k \not\to \infty$ be such that $\alpha^{m_k} \to 1$. Then $\|\sigma^{nm_k}(v) - v\| \to 0$ as $k \to \infty$. But $\bigcap_{m \in \mathbb{N}_0} \sigma(N) = \mathbb{C}1$ clearly implies $\sigma$ is mixing, i.e.

$$\lim_{u \to \infty} x \sigma^n(x)y = x \tau(x) y, \forall x, y \in P. A \ contradiction \ unless \ v \in \mathbb{C}1, \ showing \ that \ N' \cap M = \mathbb{C}.$$

(b). Let $K_{n,m} = u^{-n} L^2(N) u^{n+m}$, for $mg0, m \in \mathbb{Z}, mgm$. It is trivial to see that $K_{n,m} \not\to L^2(P) u^m$, as $n \not\to \infty$. Thus $\bigvee \{K_{n,m} | n \geq m, n \geq 0, m \in \mathbb{Z}\} = L^2(M)$, with all $K_{n,m}$ being $N - N$ bimodules.

Also, since as a left $N$ module $K_{n,m} = L^2(\sigma^{-n}(N)) u^m$ is isomorphic to $L^2(\sigma^{-n}(N))$, we have $\dim (K_{n,m}) = [N : \sigma(N)]^n < \infty$. Furthermore, as a right $N$-module $K_{n,m} = u^m L^2(\sigma^{-m-n}(N))$ is isomorphic to $L^2(\sigma^{-m-n}(N))$, so that we have $\dim (K_{n,m}) = [N : \sigma(N)]^{n+m} < \infty$.

This shows that $N \subset M$ is discrete.
c). This part is now clear, since we showed above that there exist sub-bimodules $K \subset L^2(M_1, \text{Tr})$ which are finitely generated both as left and right modules, but with different corresponding dimensions, (e.g., just take $K = K_{n,m}$ for some $ngm, ng0, m \neq 0$.)

Q.E.D.

**Corollary A.3.** There exist irreducible discrete inclusions of hyperfinite type II$_1$ factors $N \subset M$ for which $J_M \cdot J_M$ is not trace preserving on $N' \cap M_1$, equivalently for which $\text{Tr}_{M_1}$ and $\text{Tr}_{N'}$ do not agree on $N' \cap M_1$ or further, for which the local indices $[pM_1p : Np]$ are not equal to $(\text{Tr}p)^2$ for all $p \in N' \cap M_1$.

**References**

[Po] R. T. Powers *An index theory for semigroups of $\ast$-endomorphisms of $B(H)$ and type II$_1$ factors*, Can. J. Math 40 (1988), 86-114.  
[PoPr] R. T. Powers, G. L. Price *Cocycle Conjugacy Classes of Shifts on the Hyperfinite II$_1$ Factor*, J. Funct. Anal. 121 (1994), 275-295.