Rotational Dragging Effect on Statistical Thermodynamics of (2+1)-dimensional Rotating de Sitter Space

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Abstract

Brief comments on a plausible holographic relationship between the opposite rotational dragging effect of a (2+1)-dimensional rotating de Sitter space and the non-unitarity of a boundary conformal field theory are given. In addition to the comments, we study how the opposite rotational dragging effect affects the statistical-mechanical quantities in the rotating de Sitter space in comparison with a BTZ black hole.
Gibbons and Hawking [1] have shown that an inertial observer in de Sitter (dS) space detects thermal radiation at temperature $\kappa/2\pi$ coming from the cosmological horizon and the entropy of dS space satisfies the Bekenstein-Hawking area law given by

$$S = \frac{A}{4G},$$

where $\kappa$, $A$ are the surface gravity and the area of the cosmological horizon, respectively. This result extended the validity of classical and semiclassical thermodynamic properties of black holes to cosmological spaces. Recently, Gibbons and Hawking’s work has been reviving at a microscopic level; guided by analogies to the anti-de Sitter (AdS)/conformal field theory (CFT) correspondence [2], one has obtained the entropy of dS space (1) by naively using the Cardy formula under the assumption of a boundary CFT, which is dual to quantum gravity on dS space [3].

In spite of many fruitful observations of the conjectured dS/CFT correspondence in analogy to the AdS/CFT, there are strict and important differences between the two correspondences. One is the appearance of complex conformal weights in the dS/CFT, which reflects non-unitarity of the dual CFT [4]. It has been shown that in the case of (2+1)-dimensional rotating dS (RdS) spaces non-unitarity of the dual CFT may be related to rotational motion of the bulk space [5][6]. Strictly speaking, imaginary part of eigenvalues of conformal generators $L_0$ and $\overline{L}_0$ in stationary coordinates is given by the angular momentum of the RdS space. Thus, taking the limit of zero angular momentum, the unitarity of the dual CFT should be recovered. While on the other, the boundary CFT, which is dual to a BTZ black hole [7], has real eigenvalues for $L_0$ and $\overline{L}_0$ even for non-zero angular momentum [8]. Therefore, it is suspected that somehow the rotational effect of (2+1)-dimensional RdS space holographically triggers non-unitarity in the dual CFT.

What is the geometrical difference between RdS spaces and rotating AdS black holes? The most distinct geometrical aspect of spacetimes around rotating objects is the rotational dragging effect on inertial frames around the object. Inside the cosmological horizon of RdS spaces, in contrast to spacetimes outside rotating AdS black holes (and asymptotically flat rotating black holes), the rotational dragging effect is exerting in the opposite direction of space rotation.\footnote{A detailed comment is given in the paragraph between Eq.(23) and Eq.(24).} Thus, it is likely that the
non-unitarity of the boundary CFT, which is dual to (2+1)-dimensional RdS space, may be holographically related to the opposite rotational dragging effect (inside the cosmological horizon) of the bulk space.

An objection may be raised to the suggestion. Since the conjectured dual CFT lives in the future and/or past infinity outside the cosmological horizon, the rotational dragging effect inside the horizon cannot affect the boundary CFT. A plausible answer to this objection is directly related to one of the main issues of holographic principle, non-locality: according to the holographic principle, an event in a bulk spacetime, which is causally disconnected to the boundary from the bulk viewpoint, has to be encoded in the boundary dual theory in a non-local way. The boundary variables in which such events are encoded are called ‘precursors’ [10]. Thus, it seems that the opposite rotational dragging effect inside the cosmological horizon may be encoded in the boundary theory in a non-local way and triggers the non-unitarity of the boundary theory. We leave the proof of the idea to a future work.

Adopting a strong interpretation of non-locality, thermodynamics of the cosmological horizon, which has been formulated inside the horizon [1], may be reconstructed outside the horizon by requiring some appropriate prescriptions. In fact, an analogue of the Gibbs-Duhem relation was obtained outside the cosmological horizon in [1]. This statement is closely related to the concept of entanglement entropy, which is a strong candidate for the statistical-mechanical origin of the black hole entropy. (As a matter of fact, non-locality is a consequence of quantum entanglement.) For instance, if a bipartite system is a pure state, its two subsystems (e.g. inside and outside the horizon) have the same entanglement entropy [11].

In this sense, it seems reasonable to study how the opposite rotational dragging effect affects the statistical-mechanical quantities in a (2+1)-dimensional RdS space comparing with the case of a BTZ black hole [12]. For that purpose, we shall use the brick wall model [13], which is strongly supported by the concept of entanglement entropy. The statistical-mechanical entropy of quantum fields in a (2+1)-dimensional non-rotating de Sitter space has been calculated in [14].

The (2+1)-dimensional RdS space is described by the metric

\[ ds^2 = -N^2 dt^2 + N^{-2} dr^2 + r^2 (d\varphi + N^\varphi dt)^2, \] (2)
where
\[ N^2 = M - \frac{r^2}{l^2} + \frac{J^2}{4r^2} = \frac{(r^2 - r_+^2)(r^2 + r_+^2)}{r^2 l^2}, \quad N\varphi = -\frac{J}{2r^2}, \quad (3) \]
and \( r_+ = l\sqrt{M/2(1 + (1 + J^2/M^2 l^2)^{1/2})^{1/2}} \) denotes the cosmological horizon. We also introduced a positive parameter \( r_- \) defined by
\[ r_-^2 \equiv -(Ml^2/2)(1 - (1 + J^2/M^2 l^2)^{1/2}). \]
In terms of \( r_\pm \), the mass and angular momentum \[ \text{[6]} \] can be rewritten by \( M = (r_+^2 - r_-^2)/l^2 \) and \( J = 2r_+r_-/l \), respectively. Note that an ergoregion exists inside the horizon \( r_{\text{erg}} = l\sqrt{M} < r_+ \).

In the original brick wall model, a field is confined to a shell region between inner and outer walls and satisfies the periodic boundary condition
\[ \Phi(r_+ - h) = \Phi(L). \quad (4) \]
Here, \( h \) is the brick wall cutoff, and \( r_+ - h \) the ‘outer’ wall and \( L (\leq r_+ - h) \) the ‘inner’ wall. In the dS space the volume inside the horizon is finite, so there are no infrared divergences. Thus, the infrared cutoff \( L \) is not necessary in our calculation \[ \text{[14]} \]. We consider a massless scalar field satisfying the Klein-Gordon equation \( \Box \Phi = 0 \). The mode solutions of the Klein-Gordon equation is given by \( \Phi(t, r, \varphi) = R_{Em}(r)e^{-iEt+i\omega \varphi} \).

The radial part \( R_{Em}(r) \) satisfies
\[ \frac{1}{\sqrt{-g}} \frac{d}{dr} \left( \sqrt{-g} g^{rr} \frac{dR_{Em}}{dr} \right) + g^{rr} k^2(r; E, m) R_{Em} = 0, \quad (5) \]
where
\[ k^2(r; m, E) \equiv g_{rr} \left( \frac{g_{\varphi\varphi}}{-D} E^2 + 2 \frac{g_{t\varphi}}{-D} mE + \frac{g_{tt}}{-D} m^2 \right), \quad (6) \]
and \( D \equiv g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2 \). In the WKB approximation, the radial quantum number \( n(E, m) \) with energy \( E \) and angular momentum \( m \) is given by the radial wave number \( k(r; E, m) \) in \[ \text{[8]} \]
\[ \pi n(E, m) = \int_0^{r_+ - h} dr' k'(r'; E, m), \quad (7) \]
where \( k'(r; E, m) \) is set to be zero if \( k^2(r; E, m) \) becomes negative for given \( (E, m) \). \[ \text{[13]} \]. Then, from the definition of the density function \( g(E, m) = \partial n(E, m)/\partial E \), \( g(E, m)dE \) represents the number of single-particle states whose energy lies between \( E \) and \( E + dE \), and whose angular momentum is \( m \).

The free energy is obtained by using the single-particle spectrum. Due to the presence of ergoregion, scalar fields near the rotating horizon have superradiant (SR)
mode solutions. In this case, the free energy of the system can be decomposed into the non-superradiant (NSR) modes part \( F_{\text{NSR}} \) and the SR modes part \( F_{\text{SR}} \), \( F = F_{\text{NSR}} + F_{\text{SR}} \) [12]. Following the formulation given in [12], we write the NSR and SR parts of the free energy as following:

\[
F_{\text{NSR}} = \sum_{\text{NSR}} \int dE g(E, m) \ln \left( 1 - e^{-\beta (E - \Omega_H m)} \right),
\]

\[
F_{\text{SR}} = \sum_{\text{SR}} \int dE g(E, m) \ln \left( 1 - e^{\beta (E - \Omega_H m)} \right),
\]

where \( \Omega_H \) is the angular speed of the horizon. Here, we assume that the scalar field is in a thermal equilibrium state at temperature \( \beta^{-1} \) and all states with \( E - \Omega_H m < 0 \) belong to the SR modes and others to the NSR modes.

Performing the integration, we obtain the NSR part of the free energy given by

\[
F_{\text{NSR}} = F_{\text{NSR}}^{(m>0)} + F_{\text{NSR}}^{(m<0)},
\]

where

\[
F_{\text{NSR}}^{(m>0)} = -\frac{\Gamma(3) \zeta(3)}{\pi \beta^3} \int_{r_{\text{er}}^+}^{r_{\text{er}}^-} dr \frac{g_{rr} g_{\varphi\varphi}}{-D} \left( \sin^{-1} \frac{g_{t\varphi} + \Omega_H g_{\varphi\varphi}}{(-D)^{1/2}} + \frac{\pi}{2} \right),
\]

\[
F_{\text{NSR}}^{(m<0)} = -\frac{1}{\pi \beta} \int_{r_{\text{er}}^-}^{r_{\text{er}}^+} dr \frac{g_{rr} g_{tt}}{-D} \left( \frac{g_{t\varphi} + \Omega_H g_{\varphi\varphi}}{(-D)^{1/2}} - \frac{\pi}{2} \right) + \frac{\Gamma(3) \zeta(3)}{\pi \beta^3} \int_{r_{\text{er}}^-}^{r_{\text{er}}^+} dr \frac{g_{rr} g_{\varphi\varphi}}{-D} \left( \sin^{-1} \frac{g_{t\varphi} + \Omega_H g_{\varphi\varphi}}{(-D)^{1/2}} - \frac{\pi}{2} \right),
\]

where the functions \( K_{\text{NSR}} \) are defined by

\[
K_{\text{NSR}}^{(m>0)} = \frac{g_{t\varphi} + \Omega_H g_{\varphi\varphi}}{-2\tilde{g}} + \frac{-D}{2(-\tilde{g})^{3/2} g_{\varphi\varphi}^{1/2}} \left( \sin^{-1} \frac{g_{t\varphi} + \Omega_H g_{\varphi\varphi}}{(-D)^{1/2}} + \frac{\pi}{2} \right),
\]

\[
K_{\text{NSR}}^{(m<0)} = \left( g_{t\varphi} + \Omega_H g_{\varphi\varphi} \right) \left( \Omega_H - \frac{g_{t\varphi} + \Omega_H g_{\varphi\varphi}}{(-D)^{1/2}} \right) + \left( g_{t\varphi} + \Omega_H g_{\varphi\varphi} \right)^{1/2} \left( \frac{g_{t\varphi} + \Omega_H g_{\varphi\varphi}}{(-D)^{1/2}} \right) - \frac{1}{2\Omega_H^2},
\]

where \( \tilde{g} \equiv g_{\varphi\varphi} \Omega_H^2 + 2g_{t\varphi} \Omega_H + g_{tt} \). The superradiant part of the free energy becomes

\[
F_{\text{SR}} = +\frac{1}{\pi \beta} \int_{r_{\text{er}}^-}^{r_{\text{er}}^+} dr \frac{g_{rr} g_{tt}}{-D} \left( \frac{g_{t\varphi} + \Omega_H g_{\varphi\varphi}}{(-D)^{1/2}} - \frac{\pi}{2} \right) + \frac{\Gamma(3) \zeta(3)}{\pi \beta^3} \int_{r_{\text{er}}^-}^{r_{\text{er}}^+} dr \frac{g_{rr} g_{\varphi\varphi}}{-D} \left( \sin^{-1} \frac{g_{t\varphi} + \Omega_H g_{\varphi\varphi}}{(-D)^{1/2}} - \frac{\pi}{2} \right),
\]

\[
F_{\text{SR}} = +\frac{1}{\pi \beta} \int_{r_{\text{er}}^-}^{r_{\text{er}}^+} dr \frac{g_{rr} g_{tt}}{-D} \left( \frac{g_{t\varphi} + \Omega_H g_{\varphi\varphi}}{(-D)^{1/2}} - \frac{\pi}{2} \right) + \frac{\Gamma(3) \zeta(3)}{\pi \beta^3} \int_{r_{\text{er}}^-}^{r_{\text{er}}^+} dr \frac{g_{rr} g_{\varphi\varphi}}{-D} \left( \sin^{-1} \frac{g_{t\varphi} + \Omega_H g_{\varphi\varphi}}{(-D)^{1/2}} - \frac{\pi}{2} \right).
\]
\[
K_{\text{SR}} = \frac{\Omega_H g_{tt} + g_{tt}}{-2\tilde{g}\Omega_H^2} \left( \frac{g_{tt}}{g_{\phi\phi}} \right)^{1/2} + \frac{-D}{2(-\tilde{g})^{3/2}g_{\phi\phi}^{1/2}} \left( \sin^{-1} \frac{\Omega_H g_{tt} + g_{tt}}{(-D)^{1/2}\Omega_H^2 + \frac{\pi}{2}} \right).
\]

(17)

Then, neglecting the terms including the integration range from 0 to \( r_{\text{erg}} \), the total free energy is given by a simple form

\[
F = F_{\text{NSR}} + F_{\text{SR}}
= -\frac{\zeta(3)}{\beta^3} \int_{r_{\text{erg}}}^{r_h} dr \frac{(-g_{rr}D)^{1/2}}{(-\tilde{g})^{3/2}}.
\]

(18)

Now, we are ready to derive thermodynamic quantities of the scalar field in thermal equilibrium with the RdS space. At first, the entropy of the scalar field is obtained by

\[
S = \beta^2 \frac{\partial F}{\partial \beta} \bigg|_{\beta = \beta_H} = \left( \frac{3\zeta(3)}{16\pi^3\epsilon} \right) (2 \cdot 2\pi r_+),
\]

(19)

where \( \beta_H = 2\pi l^2 r_+/(r_+^2 + r_-^2) \) is the inverse Hawking temperature of the RdS space and \( \epsilon \) is the proper distance of the brick wall from the horizon given by

\[
\epsilon = \int_{r_{\text{erg}}}^{r_h} dr N^{-1}(r) \approx \left( \frac{2l^2 r_+}{r_+^2 + r_-^2} \right)^{1/2} \sqrt{h}.
\]

(20)

Thus, choosing the cutoff of the proper distance as

\[
\epsilon = 3\zeta(3)/(16\pi^3),
\]

(21)

the entropy given by (19) satisfies the Bekenstein-Hawking area law. Note that the brick wall cutoff in (21) is equal to the cutoff of the BTZ black hole [12]. In addition, as in the case of BTZ black hole, the SR contribution makes the cutoff twice as much as that of non-rotating (2+1)-dimensional dS space. (The free energy of SR modes (9) is equal to that of NSR modes (8) up to the leading term. See [12].)

Using the cutoff value given by (21), the angular momentum \( J_m \) and internal energy \( U_m \) of the scalar field are obtained as

\[
J_m = -\frac{\partial F}{\partial \Omega} \bigg|_{\beta = \beta_H, \Omega = \Omega_H} = -J,
\]

(22)

\[
U_m = \frac{\partial}{\partial \beta} (\beta F) \bigg|_{\beta = \beta_H} + \Omega_H J_m = \frac{4}{3} M + \frac{1}{3} \Omega_H J.
\]

(23)
In fact, these thermal wall contributions, which is interpreted as the backreaction of background space, lead a fatal flaw to the brick wall model in which the background geometry is fixed \[15\]. This problem, however, can be resolved by taking the Boulware state as a ground state in the model \[16\]. In this ‘topped-up’ Boulware description of the brick wall model, the thermal contributions given by \(22\) and \(23\) are to be canceled by the Boulware energy up to an appropriate order of mass and angular momentum. However, since our purpose in this report is to examine the thermal properties of quantum fields, the regularization scheme is not considered here.

First of all, the angular momentum in Eq.\((22)\) is the negative value of the dS space angular momentum. Note that the angular momentum of quantum fields on the BTZ black hole is the positive value of the BTZ black hole angular momentum, \(J_{m}^{\text{BTZ}} = J_{m}^{\text{BTZ}} \[12\]\. The negative sign in \(22\) can be interpreted as following; inside RdS spaces the angular speed \(\Omega = J/(2r^2)\) decreases as approaching to the cosmological horizon \(r \to r_+ (r < r_+).\) This means that the rotational dragging effect of the space reduces the angular speed of a test particle as approaching to the horizon, and entering the ergoregion \(r_{\text{erg}} < r < r_+\) the particle cannot rotate with angular speed greater than the angular speed of the horizon. In other words, the direction of the dragging effect is opposite to the direction of the horizon angular velocity. Therefore, the minus sign in \(22\) indicates the opposite rotational dragging effect of the RdS space.

The opposite rotational dragging effect also arises in the internal energy given by Eq.\((23)\). The internal energy is enhanced with the positive term \(+\Omega_{H}J/3\), while in the case of BTZ (and Kerr) black hole, rotation reduces internal energy, \(U_{m}^{\text{BTZ}} = 4M^{\text{BTZ}}/3 - \Omega_{H}J^{\text{BTZ}}/3 \[12\]\. This argument looks strange, because the rotational energy \(\Omega_{H}J_m\) in \(23\) is negative in the RdS space and positive in the BTZ black hole. In order to interpret this result, consider the internal energy of the system with respect to a zero-angular-momentum-observer (ZAMO) \(U_{m}^{\text{ZAMO}}\). Since the rotation of the dS space enlarges the radius of the cosmological horizon \(r_+\) (thus, increases entropy), the ZAMO’s internal energy receives a positive rotational contribution

\[
U_{m}^{\text{ZAMO}} = \frac{\partial}{\partial \beta} (\beta F)\bigg|_{\beta=\beta_{H}} = \frac{2}{3} \frac{S}{\beta_{H}} = \frac{4}{3} M + \frac{4}{3} \Omega_{H} J. \tag{24}
\]

Note that Eq.\((24)\) (and Eq.\((23)\)) is equivalent to the Gibbs-Duhem relation and the factor \(2/3\) of the term \(S/\beta_{H}\) is related to the equation of state \(\rho_{m} \simeq 2P_{m}\), where
\( \rho_m \) and \( P_m \) denote the energy density and pressure, respectively. Then, turning to Eq. (23), the negative rotational energy \( \Omega_H J_m = -\Omega_H J \) does not eliminate the whole amount of the positive rotational effect in the internal energy of ZAMO, \( +4\Omega_H J/3 \).

In the case of BTZ (Kerr) black hole, the rotational effect decreases entropy and the positive rotational energy \( \Omega_{BTZ} J_{m}^{BTZ} = +\Omega_{BTZ} J^{BTZ} \) compensates partially the negative rotational effect in the internal energy of ZAMO, \( -4\Omega_{BTZ} J^{BTZ}/3 \).

In summary, we have studied how the opposite rotational dragging effect of the RdS space affects thermodynamic quantities of quantum fields on the RdS space. The effect arising in the internal energy and angular momentum of thermal excitations is opposite to the case of BTZ black hole. If the conjectured dS/CFT is true, these statistical-mechanical properties inside the cosmological horizon should be encoded in the boundary theory. Our expectation is that these properties would be closely related to the non-unitarity of the boundary CFT. As a byproduct we have shown that even though the appearance of SR modes is due to the presence of ergoregion, the direction of rotational dragging effect does not affect to the SR contribution to the brick wall cutoff, i.e. the SR contribution makes the cutoff twice as much as that of non-rotating dS space as in the case of BTZ black hole.

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