Aspects of D-brane Dynamics in Supergravity Backgrounds with Fluxes, Kappa–symmetry and Equations of Motion. Part IIB

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Abstract

We derive and carry out a detailed analysis of the equations of motion of the type IIB D branes in generic supergravity backgrounds with fluxes making account of the worldvolume Born–Infeld gauge field and putting a special emphasis on the structure of the Dirac equation for Dp brane fermionic modes. We present an explicit form of the worldvolume field equations for each of the Dp branes ($p = 1, 3, 5, 7, 9$) in the cases in which the Neveu–Schwarz flux and the Ramond–Ramond p–form flux along the Dp–brane worldvolume are zero and the supergravity backgrounds do not necessarily induce the worldvolume Born–Infeld flux. We then give several examples of D3, D5 and D7 brane configurations in which the worldvolume Born–Infeld flux is intrinsically non–zero and therefore must be taken into account in studying problems where such branes are involved. The examples include D3 and D5 brane instantons carrying (self–dual) worldvolume gauge fields in warped compactification backgrounds.
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1 Introduction

The actions for the M2 [1] and M5 branes [2] in the $D = 11$ supergravity background and those for the Dirichlet branes in type II $D = 10$ supergravity backgrounds [3]–[6], as well as their equations of motion [1, 7, 8, 9] have been used for studying numerous problems of string theory and M theory. The superbrane actions and equations of motion are constructed in curved target superspaces in terms of the pullbacks to the brane worldvolume of bulk supervielbeins and tensor gauge field superforms subject to superfield constraints. The constraints (which actually put the $D = 10, 11$ supergravity theories on the mass shell) ensure that the superbranes possess local fermionic kappa–symmetry. In addition to the physical fields of the $D = 10, 11$ supergravity supermultiplets the supervielbeins and the tensor gauge field superforms contain an enormous amount of other fields which upon imposing the supergravity constraints can either be gauged away by local superdiffeomorphisms and gauge symmetries or can be expressed in terms of the physical fields via algebraic equations. Hence, the superbrane actions and equations of motion only implicitly describe the interaction of the worldvolume modes of the superbranes with the physical supergravity fields, especially as far as the worldvolume fermions $\theta(\xi)$ are concerned. These fermionic modes are Goldstone fields associated with brane fluctuations in the 32 Grassmann–odd directions $\theta$ of the target superspace.

For various applications of the M–branes and the D–branes it is necessary to know the explicit dependence of their actions and equations of motion on the physical fields in the bulk. To obtain the explicit and entire expressions one should eliminate the auxiliary fields by imposing on the supergravity superfields a Wess–Zumino gauge $^1$, expand these superfields in power series of the Grassmann–odd coordinates $\theta$ (up to the 32-nd order in $D = 10, 11$ superspaces) and insert these polynomials into the superbrane actions and/or field equations. One can perform such an expansion using the gauge completion method [12] or a much more systematic method of superspace normal coordinates [13, 14, 15, 11, 16].

Though it might be technically possible to arrive at the final expressions, in the general case they will be unpracticable even if by gauge fixing the $\kappa$–symmetry one eliminates half of the 32 $\theta(\xi)$ components and reduces the polynomial order down to 16. In some cases of particular branes and backgrounds, such as flat space–time (see e.g. [1, 3, 17]) and $AdS_d \times S^{D-d}$ [18], gauge fixed M2, M5 and Dp brane actions can be constructed of the supervielbeins which are polynomials of up to the second or fourth order in the physical modes of $\theta(\xi)$.

In some applications, for instance, for studying brane–world scenarios and non–perturbative effects due to brane instantons on the generation of superpotentials in string/M–theory compactifications ([19]–[22] and references therein) it is sufficient, at a first stage, to know the explicit form of the brane actions up to the second order in worldvolume fermions, and the equations of motion linear in fermions. This approximation is admissible, e.g. in situations where the Dirac operator on the brane has exactly two fermion zero modes which one integrates over to calculate instanton contributions to the effective action.

$^1$This conventional procedure deals with the background field approximation to supergravity–superbrane interactions. For further discussion and an approach to account the back reaction of branes see [10, 11].
Quadratic actions for fermions on branes coupled to a generic supergravity background with fluxes were derived in [15] for the M2 brane, in [23] for a D3 brane, in [24, 25] for the Dp–branes, and in [26] for the M5–brane. The back reaction of branes on the compactification setup has been discussed in [27] and in a general context of interacting superbrane–supergravity systems in [28].

Using these results, the zero modes of the Dirac operators of brane fermions interacting with bulk fluxes have been analyzed in [29, 30] for M5–brane instantons in M–theory on $M_3 \times X_8$, and in [31, 32, 33] for D3 brane instantons in type IIB String Theory on $M_4 \times X_6$ (where $M$ is the effective physical space–time and $X$ are compactified subspaces of String/M–theory). It has been shown that in some cases including the $K3 \times T^2 / Z^2$ orientifold compactification of type IIB String Theory, brane instantons coupled to bulk fluxes can produce non–perturbative corrections to the superpotential and hence should be taken into account when carrying out the analysis in search for phenomenologically relevant models of particle interactions and cosmology.

A role of the worldvolume flux of D–branes in producing D- and F- terms in effective $D = 4$ theory and thus giving the possibility of stabilizing moduli in type IIB String Theory compactified on Calabi–Yau orientifolds has been discussed e.g. in [34, 35, 36] and references therein. The analysis of the latter paper has been based on the relation with supersymmetry conditions for D branes with general worldvolume fluxes on general $N = 1$ flux backgrounds studied in detail in [37].

Recently, effects of a worldvolume flux on solutions of the Dirac equation for fermions on an M5 brane instanton wrapping a Calabi–Yau space have been considered in [38].

It seems of interest to look for more examples of brane configurations generating scalar field potentials and to analyze whether also the worldvolume fluxes on D branes and the M5 brane may produce or suppress some non–trivial effects. To this end one should analyze the brane actions and equations of motion in the presence of worldvolume Born–Infeld gauge fields. This can also be useful for other brane applications such as brane worlds and AdS/CFT correspondence. A main goal of this paper is to carry out such an analysis for the Dp–branes of type IIB string theory, so $p = 1, 3, 5, 7$ and 9. A reason why we restrict the consideration to the type IIB D branes is the desire to present the results (at least where it is possible) in an explicit form. The actions and equations of motion for the type IIA branes can be analyzed in a similar way elsewhere.

Though the explicit and geometrically suggestive quadratic actions for the type IIB (and IIA) Dp–brane fermions in an arbitrary bosonic supergravity background with the bulk and worldvolume fluxes have already been given in [25], one should still extract from these actions the Dirac equations and present them in a hopefully tractable form. For instance, though the kappa–symmetry gauge fixed actions for the Dp–branes in the string frame do not contain terms with dilaton derivatives ([25]), these terms reappear in the fermion equations of motion, and they should be taken into account in the analysis in the situations of a non–trivial dilaton–axion. Also the consequences of different gauge fixings of kappa–symmetry should be understood in more detail. For this we shall elucidate the role of the kappa–symmetry projector and present the fermionic equations in a gauge independent covariant form. As a valuable byproduct of this analysis we observe an interesting feature, which for
some reason has not been emphasized in the literature, that a kind of the Dirac equation for the Dp–brane kappa–symmetry projector reproduces in a concise form the full set of the D–brane bosonic field equations (for a concise form of the $D = 11$ supergravity equations see [39] and references therein and for a concise form of the (massive) type IIA supergravity equations see [40]).

Let us stress that when analyzing the fermionic equation in a certain setup, one should take into consideration the consistency of the setup with the Dp–brane bosonic equations. For instance, if one ignores the contribution of the worldvolume flux this should be in agreement with the BI equation for the worldvolume gauge field and with the field equation for the worldvolume scalars. This also concerns static brane configurations which do not fluctuate in the transverse directions.

We also find it instructive and useful for further applications to collect in one place and review the complete set of the Dp–brane equations of motion in an arbitrary superspace supergravity background before truncating them to the linear approximation in fermions. With this instructive purpose we present the equations both in the Einstein and string frame.

The paper is organized as follows. In Sections 2, 3 and 4 we review the generic structure of the actions and of the worldvolume field equations for the Dp–branes coupled to the Neveu–Schwarz and Ramond–Ramond fields in curved type IIB superspace using techniques similar to those in the superembedding approach (see [52, 53, 54, 55] for a review). This simplifies the derivation and the analysis of the Dp–brane equations. In Section 4 we study the properties of the D–brane kappa–symmetry and of its projector and show that the projector matrix takes values in the group $Spin(1, p) \subset Spin(1, 9)$ and that it is related via the spinor–vector representation correspondence to the $SO(1, p)$–valued matrix $k_a^b = (\eta + F)_{ac}(\eta - F)^{-1}cb$ [41], the so called Cayley image of the worldvolume gauge field strength $F_{ab}$. Using this property we then show that a first–order differential equation for the $\kappa$–symmetry projection matrix amounts to the full set of equations and Bianchi identities for the bosonic worldvolume fields on the brane. In Subsection 4.2 we linearize the D–brane equations for the fermion field $\theta(\xi)$ in purely bosonic supergravity backgrounds and in Section 6 we present them in a $\kappa$–symmetry covariant form making use of the concise form of the bosonic equations derived in Section 5 in terms of the kappa–symmetry projector. The covariant (gauge independent) form of the fermionic equations allows one to see how different kappa–symmetry gauge choices are related to each other.

In Subsection 6.1 we give the explicit form of the fermionic equations for each of the type IIB Dp–branes ($p = 1, 3, 5, 7, 9$) in the supergravity backgrounds which do not induce a worldvolume flux. In Subsection 6.2.2 we consider a $D3$ brane instanton wrapping $K3$ and carrying an (anti)–self–dual Born–Infeld instanton field. We show that if the compactified space–time is warped by the $R_5$ flux, the consistency of the equations of motion of the $D3$ brane instanton wrapping a four–cycle of the Calabi–Yau manifold can require the presence of the worldvolume flux.

In Subsection 7 we give more examples of situations in which the worldvolume BI field must be taken into account in the study of the dynamics of D–branes in certain setups. For instance, a D5 brane wrapping the compact manifold $X_5$ in an $AdS_5 \times X_5$ background (where $X_5$ is an $S^5$ sphere or a Sasaki–Einstein space) carries an effective electric charge
proportional to the inverse $X_5$ radius which induces an electric BI field in its worldvolume. The D5 brane wrapping $S^5$ is associated via AdS/CFT correspondence with the baryon vertex of the effective $D = 4$ gauge theory [42, 43] and has been extensively studied from various perspectives [44]–[49]. We add to the known results the explicit form of the linearized Dirac equation for the fermionic modes on the baryonic D5 brane.

Other examples of branes with worldvolume fluxes are D5 and D7 branes in warped compactification backgrounds and, in particular, a D5 brane instanton wrapping a Calabi–Yau manifold with non–zero 5–form and/or 3–form background fluxes. For all examples we give an explicit form of the Dirac equation for the Dp–brane fermionic modes.

We have tried to present the examples in a self–contained form so that the reader interested in applications could directly use them without the necessity of ploughing through the generic part of this paper. Basic notation and conventions are given in the Appendix.

2 The action for the super-Dp-branes in $D = 10$ type IIB superspace

The $p+1$ dimensional worldvolume $W^{p+1}$ of a D–brane is parametrized by the coordinates $\xi^m (m = 0, 1 \ldots, p)$. On its worldvolume the D–brane carries a vector gauge field $A_m(\xi)$. The dynamics of the D–brane is described by how its worldvolume is embedded into $D = 10$ type IIB superspace whose supercoordinates $Z^M = (x_m, \theta^{1\mu}, \theta^{2\nu})$ include the space–time coordinates $x_m (m = 0, 1, \ldots, 9)$ and the fermionic coordinates $\theta^{1\mu}, \theta^{2\nu} (\mu, \nu = 1, \ldots, 16)$. The D–brane interacts with fields of the $D = 10$ supergravity multiplet. The type IIB supergravity fields are contained in the dilaton superfield $\Phi(Z)$, the Neveu–Schwarz–Neveu–Schwarz two–form superfield $B_2(Z)$, the Ramond–Ramond superforms $C_{2n}(Z) (n = 0, 1, 2)$ and their duals $C_{2n}(Z) (n = 4, 3, 2)$, and in the supervielbein

$$E^A(Z) = dZ^M E^A = (E^a, E^{1\alpha}, E^{2\beta}),$$

where $a = 0, 1, \ldots, 9$ are $D = 10$ vector tangent space indices and $\alpha, \beta = 1, \ldots, 16$ are the Majorana–Weyl spinor indices, $E^{1\alpha}$ and $E^{2\beta}$ being the two Majorana–Weyl spinors of the same chirality.

The above listed superfields are subject to a certain set of constraints (which are required to eliminate redundant and, in particular, higher spin degrees of freedom) [50].

2.1 Type IIB supergravity constraints in the Einstein frame

We shall deal with the following set of the type IIB supergravity constraints in the Einstein frame. Below and in all places where it will not cause confusion we shall skip the wedge

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2 We use the conventions of the superembedding approach proposed in [58] such that all underlined bosonic indices correspond to the target superspace and the not underlined ones from different parts of the Latin alphabet correspond to both the D–brane worldvolume and to directions orthogonal to it.
product symbol. The superspace torsion constraint is
\[ T^a = DE^a - E^b \Omega^a_{b \alpha} = -i E^a \gamma^\alpha_{\alpha \beta} - i E^a \gamma^\beta_{\alpha \beta} = -i E^1 \gamma^1 E^1 - i E^2 \gamma^2 E^2 \]  \tag{2.2}

(where \( \Omega^a_{b \alpha} \) is the \( SO(1, 9) \) spin connection) and the NS–NS field strength constraint has the form
\[ H_3 = dB_2 = -i e^{\Phi} E^a (E^1 \gamma^1 E^1 - E^2 \gamma^2 E^2) - \frac{1}{2} e^{\Phi} (E^1 \gamma^{(2)} \nabla_1 \Phi - E^2 \gamma^{(2)} \nabla_2 \Phi) + \frac{1}{3!} E^a E^b E^c H_{a \beta \gamma} \]  \tag{2.3}

where the general structure of the first term is determined by the superstring Wess–Zumino term in flat superspace, the dilaton contribution \( e^{\Phi} \) can be recovered considering the superstring in curved type IIB superspace, while the second term can be then derived by studying Bianchi identities.

In eqs. (2.2) and (2.3) \( \nabla_{1, \alpha} = E^M_{1, \alpha \beta} \partial_M \), \( \gamma^{(2)} = \frac{1}{2} E^b \wedge E^a \gamma^a_{\beta} \) and \( \gamma^\alpha_{\alpha \beta} \) and \( \tilde{\gamma}^\alpha_{\alpha \beta} \) are \( 16 \times 16 \) (symmetric) matrix counterparts of the Pauli matrices satisfying the relations
\[ \gamma^a_{\beta} \gamma^b_{\alpha} + \gamma^b_{\alpha} \gamma^a_{\beta} = 2 \eta^{ab}, \quad \gamma^{[a_1 \cdots a_n]} = \frac{1}{n!} \epsilon^{a_1 \cdots a_n b_1 \cdots b_n} \gamma^b_{a_1} \cdots \gamma^b_{a_n} \]  \tag{2.4}

We use the mostly minus convention for the Minkowski metric \( \eta^{ab} = diag(+, - \cdots, -) \), \( \epsilon^{0123456789} = 1 \) and \( \gamma^0 \gamma^1 \cdots \gamma^9 = -\gamma^0 \gamma^1 \cdots \gamma^9 = 1 \). Note that there is no “charge conjugation” matrix in \( D = 10 \) which would lower or rise 16-component Majorana–Weyl spinor indices \( \alpha, \beta \). This is why one should distinguish between \( \gamma^a_{\alpha} \) and \( \tilde{\gamma}^a_{\alpha \beta} \).

The constraints on the field strengths of the RR superforms \( C_{2n} \) \((n = 1, \cdots, 5)\) are (see [5])
\[ R_{2n+1} = -2i e^{\Phi} E^2 \gamma^{(2n-1)} \nabla_1 \Phi + \frac{n-2}{2} e^{\Phi} \left( E^2 \gamma^{(2n)} \nabla_1 \Phi - (-)^n E^1 \gamma^{(2n)} \nabla_2 \Phi \right) + \frac{1}{(2n+1)!} E^{a_2 a_1} \cdots E^{a_n} R_{a_1 \cdots a_{n+1}} \]  \tag{2.5}

where
\[ \gamma^{(2n)} = \frac{1}{2n!} E^{a_2 a_n} \cdots E^{a_2 a_1} \gamma_{a_1} \gamma_{a_2} \cdots \gamma_{a_{2n}}, \]  \tag{2.6}

\[ \gamma^{(2n-1)} = \frac{1}{(2n-1)!} E^{a_2 a_{n-1}} \cdots E^{a_2 a_1} \gamma_{a_1} \gamma_{a_2} \cdots \gamma_{a_{2n-1}}. \]

Note that RR 10–form \( C_{10} \) does not carry any independent degrees of freedom. Its 11–form field strength is non–trivial only because of the presence of the Grassmann–odd directions in type IIB D=10 superspace, while the purely bosonic part of its field strength is identically zero.
2.2 Type IIB supergravity constraints in the string frame

Let us now present how the type IIB supergravity constraints look like in the string frame. This frame is related to the Einstein frame by the following (conformal) redefinition of the supervielbeins and the spin connection

\[ E_{a}^{\text{str}} = e^{1\Phi} E_a, \quad E_{I\alpha}^{\text{str}} = e^{\Phi} \left( E_{I\alpha} - \frac{i}{8} E_{a}^{\gamma} \gamma_{a}^{\alpha \beta} \nabla_{I\alpha} \Phi \right) \quad (I = 1, 2) \]

\[ \Omega_{ab}^{\text{str}} = \Omega_{ab} + \frac{1}{2} E_{a} \nabla_{b} \Phi - \frac{1}{4} E_{I} \gamma_{a}^{\alpha \beta} \nabla_{I\alpha} \Phi + \frac{i}{16} \gamma_{a}^{\alpha \beta} \nabla_{I\alpha} \Phi E_{b} , \]

\[ \frac{1}{3!} \left( E_{\alpha}^{2}\ E_{\beta}^{2}\ E_{\gamma}^{2}\ H_{\alpha \beta \gamma} \right)_{\text{str}} = \frac{1}{3!} E_{\alpha}^{2}\ E_{\beta}^{2}\ E_{\gamma}^{2}\ H_{\alpha \beta \gamma} - i \frac{3!}{16} e^{\Phi} \left( \nabla_{1} \Phi \gamma^{(3)} \nabla_{1} \Phi - \nabla_{2} \Phi \gamma^{(3)} \nabla_{2} \Phi \right) , \]

\[ \frac{1}{(2n+1)!} \left( E_{a_{2n+1}} \cdots E_{a_{1}} R_{a_{1} \cdots a_{2n+1}} \right)_{\text{str}} \right. \]

\[ = \frac{1}{16} \left( \frac{n-4}{(2n+1)!} \Phi \cdot \nabla_{2} \Phi \gamma^{(2n+1)} \nabla_{1} \Phi \right) \]

\[ R_{2n+1} = -2i e^{-\Phi} E_{1}^{2} \nabla_{1} \Phi - e^{-\Phi} \left( E_{2}^{2} \nabla_{2} \Phi - (-)^{n} E_{1}^{2} \nabla_{2} \Phi \right) + \]

\[ + \frac{1}{(2n+1)!} E_{a_{2n+1}} \cdots E_{a_{1}} R_{a_{1} \cdots a_{2n+1}} . \]

The superspace constraints imply the self–duality of the five–form field strength and the duality relations between the lower and the higher form field strengths

\[ R_{a_{1} \cdots a_{5}} = \frac{1}{5!} \epsilon_{a_{1} \cdots a_{5}} b_{a_{1} \cdots b_{5}} R_{b_{1} \cdots b_{5}} , \]

\[ R_{a_{1} \cdots a_{2n-2n}} = -(-)^{n(n-1)/2} \frac{1}{(2n+1)!} \epsilon_{a_{1} \cdots a_{2n-2n}} b_{a_{1} \cdots b_{2n+1}} R_{b_{1} \cdots b_{2n+1}} \quad (n = 0, 1, \cdots, 4) . \]

The proper action for the physical fields of type IIB supergravity producing (self)–duality relations (2.10) and (2.11) as equations of motion was constructed in [51].

2.3 The Dp–brane action

The super–Dp–brane action is constructed using the pullbacks to the brane worldvolume of the dilaton superfield, the NS–NS two–form, the RR 2n–forms and the supervielbein vector
components (2.1). We shall denote the pullbacks of forms to the worldvolume by the same letters as the forms themselves. This should not produce any confusion since in this paper we mainly deal with the pullbacks.

We shall consider the D–brane action in a form proposed in [9] which is equivalent to the original action of [3]–[6] but at the same time can be directly uplifted to the superembedding approach via the generalized action principle and similar techniques [56, 57, 9]. We find this form of the action more convenient for the derivation and the analysis of the D–brane equations of motion.

In such a formulation the D–brane Lagrangian density is constructed as a worldvolume differential $(p + 1)$–form. In particular, we should determine what is the worldvolume area density. To this end we have to define a metric or a vielbein in the worldvolume induced by its embedding into the target superspace. This can be done in the following way. Let us take the pullback of the vector part of the supervielbein (2.1)

$$E^a(Z(\xi)) = d\xi^m \partial_m Z^M E^a_M. \quad (2.12)$$

On the $(p + 1)$–dimensional non–degenerate surface there exist $p + 1$ linearly independent vectors. So, using an appropriate $SO(1, 9)$ transformation $u^a_b(\xi)$ in the ten–dimensional tangent space with local parameters on the worldvolume we can always transform the vielbein (2.12) in such a way that $(p + 1)$ of its components will be parallel to the worldvolume and $(9 - p)$ components will be orthogonal to the worldvolume

$$E^a(Z(\xi)) \Rightarrow E^a_b u^a_b(\xi) := (E^a, E^i) \quad a' = (a, i), \quad a = 0, 1, \ldots, p, \quad i = 1, \ldots, 9 - p \quad (2.13)$$

such that

$$E^i = E^a_b u^a_i = 0. \quad (2.14)$$

Eq. (2.14) is called the embedding condition (see e.g. [52, 56, 9]). It simply implies that the pullback onto the worldvolume of a vector orthogonal to it is zero.

In eqs. (2.13) and (2.14) the ten–dimensional vector index $a$ is split into the $(p + 1)$–dimensional worldvolume tangent space index $a$ and the index $i$ labeling the $9 - p$ directions orthogonal to the brane. This splitting reflects the fact that the presence of the brane spontaneously breaks the target space $SO(1, 9)$ Lorentz symmetry down to its subgroup $SO(1, p) \times SO(9 - p)$. The components of the local Lorentz matrix

$$u^a_b(\xi) = (u^a_b(\xi), u^i_b(\xi)), \quad (2.15)$$

satisfying the orthogonality conditions

$$u^a_b u^b_a = \eta^{ab} = diag(+1, -1, \ldots, -1), \quad u^a_i u^i_a = 0, \quad u^a_i u^i_j = -\delta_{ij}, \quad (2.16)$$

$$u^a_b u^b_a \eta_{ab} - u^i_b u^i_a \delta_{ij} = \eta_{ab}. \quad (2.17)$$
play the role of the Goldstone fields associated with this symmetry breaking. They parameterize the coset space $SO(1,9)/SO(1,p) \times SO(9-p)$ and are called Lorentz harmonics. $u_\perp^a(\xi)$ and $u_i^b(\xi)$ will be auxiliary fields of our construction (see [59, 60] for more details about Lorentz harmonics techniques). The variation of $u_\perp^a(\xi)$ in the D–brane action will produce the embedding condition (2.14).

Using the splitting (2.13) of the supervielbein components, we can associate the pullback $E^a(Z(\xi))$ with the induced vielbein of the worldvolume. Such a choice is in accordance with the conventional definition of the induced metric on the brane worldvolume

$$g_{mn} = \partial_m Z^M E^a_M \partial_n Z^N E^b_N \eta_{ab} \quad (m, n = 0, 1, \cdots p).$$

(2.17)

It is easy to see, using the embedding equations (2.12)–(2.16), that the metric (2.17) is expressed as the product of the components of the induced worldvolume vielbein $E^a$ and the metric (2.17), (2.18) the worldvolume area density is

$$\frac{1}{(p+1)!} \epsilon^{a_1 \cdots a_{p+1}} E^{a_1} \cdots E^{a_{p+1}} = d^{p+1} \xi \sqrt{|\det g_{mn}|}.$$

(2.19)

Let us now show following [59, 52] that when the embedding condition (2.14) is satisfied the Lorentz harmonics can be expressed in terms of components of the pullback of $E^a$ and hence are indeed auxiliary fields. As in the case of the frame formulation of gravity, using local $SO(1,p)$ transformations in the worldvolume tangent space (acting on the indices $a, b$) and the definition (2.17), (2.18) of the induced metric we can reduce $E_m^a \eta_{ab}$ to a symmetric matrix. The components of this matrix are completely determined by the components of the induced worldvolume vielbein $E^a_m$ and the metric (2.17), (2.18) the worldvolume area density is

$$E^a = d\xi^m \partial_m Z^M E^a_M = d\xi^m E^a_m \Rightarrow u_\perp^a = E^m_a \partial_m Z^M E^a_M,$$

(2.20)

where $u_\perp^a = \eta^{ab} u_\perp^b \eta_{ab}$ and $E_m^a$ is inverse of the induced worldvolume vielbein $E_m^a$ associated with the induced metric $g_{mn}$ in (2.18).

Therefore, the components of $u_\perp^a$ are expressed solely through components of the supervielbein pullback $E^a_M = \partial_m Z^M E^a_M$. For instance, in the case of the bosonic brane in a flat background one finds that $u_\perp^a$ reduces to $u_\perp^a = E^m_a \partial_m x^a$.

Finally, from the orthogonality conditions (2.16) it follows that, up to local $SO(9-p)$ rotations in the directions orthogonal to the brane, the components of $u_i^b$ are determined by the components of $u_\perp^a$. This completes the proof that the Lorentz harmonics are auxiliary variables. Their use in the D–brane action simplifies the derivation and the analysis of its equations of motion.

Using the induced worldvolume vielbein $E^a = E^a \perp u_\perp^a$ and the pullbacks of the type IIB supergravity fields in the Einstein frame we construct the Dp–brane action in the following
form [9]

\[ S_{Dp} = \int_{\mathbb{R}^{p+1}} L = \int_{\mathbb{R}^{p+1}} (L_{DBI}^{Dp} + L_{LM}^{Dp} + L_{WZ}^{Dp}) = \]

\[ = \int_{\mathbb{R}^{p+1}} \frac{e^{\frac{p+3}{2}}}{(p+1)!} \epsilon_{a_1\ldots a_{p+1}} E^{a_1} \cdots E^{a_{p+1}} \sqrt{\det(\eta_{ab} + F_{ab})} + Q_{p-1}(e^{-\frac{1}{2}\Phi} F_2 - F_2) + e^{F_2} \wedge \mathbb{C}|_{p+1}, \]

where in \( L_{WZ}^{Dp} = e^{F_2} \wedge \mathbb{C}|_{p+1}, \) which is the conventional Wess–Zumino (WZ) term [3], [61, 62], \( |_{p+1} \) means that we pick only the terms which are the \((p + 1)\)-forms in the external product of the formal sums of the forms of different order

\[ e^{F_2} = 1 + F_2 + \frac{1}{2} F_2 F_2 + \frac{1}{3!} F_2 F_2 F_2 + \frac{1}{4!} F_2 F_2 F_2 F_2 + \frac{1}{5!} F_2 F_2 F_2 F_2 F_2 = \sum_{k=0}^{5} \frac{1}{n!} (F_2)^n, \]

\[ \mathbb{C} = C_0 + C_2 + C_4 + C_6 + C_8 + C_{10} = \sum_{n=0}^{5} C_{2n}. \quad (2.22) \]

The structure of the Wess–Zumino term tells us, in particular, that the Dp–brane minimally couples to the RR field potential \( C_{p+1} \) and also couples to lower order RR forms. Remember also that the bosonic parts of the RR field strengths \( R_{2k+1} \) and \( R_{2k-1} \) \((k = 0, 1)\) are dual to each other and \( R_5 \) is self–dual \((2.11)^3\).

The first term in (2.21) is the Dirac–Born–Infeld–like part of the D–brane action, where \( F_{ab}(\xi) \) is an auxiliary antisymmetric tensor field taking values in the worldvolume tangent space. This field is related to the extended field strength of the Born–Infeld (BI) gauge field \( A = d\xi^m A_m(\xi) \)

\[ F_2 = dA - B_2 \quad (2.23) \]

via the second term of the action \((2.21)\)

\[ L_{LM}^{Dp} = Q_{p-1} \left( e^{-\frac{1}{2}\Phi} F_2 - F_2 \right) = Q_{p-1} \left( e^{-\frac{1}{2}\Phi} (dA - B_2) - \frac{1}{2} E^a E^b F_{ab} \right), \quad (2.24) \]

where \( Q_{p-1}(\xi) \) is a \((p - 1)\)–form Lagrange multiplier.

Indeed, varying \((2.24)\) with respect to \( Q_{p-1}(\xi) \) we get the algebraic relation

\[ \frac{\delta L_{LM}^{Dp}}{\delta Q_{p-1}} = 0 \quad \Rightarrow \quad F_2 = e^{-\frac{1}{2}\Phi} F_2 \quad i.e. \quad \frac{1}{2} E^a E^b F_{ba} = e^{-\frac{1}{2}\Phi} (dA - B_2). \quad (2.25) \]

Then substituting eq. \((2.25)\) into the first term of \((2.21)\) and using the definition of the induced metric \((2.17), (2.18)\) one can reduce the frame–like Dp–brane action to the conventional form [3]–[6] consisting of the Dirac–Born–Infeld action and the WZ term

\[ S = \int d^{p+1} \xi e^{\frac{p+3}{2}} \sqrt{|g_{mn} + e^{-\frac{1}{2}\Phi} F_{mn}|} + \int e^{F_2} \wedge \mathbb{C}|_{p+1}, \quad (2.26) \]

\(^3\) Actually, the form of the action \((2.21)\) is generic and also describes the Dp–branes in type IIA supergravity. In that case one should consider the Dp–branes with even values of \( p \) coupled to \((2k + 1)\)-form RR fields.
or in the string frame (2.7)

\[ S_{str} = \int d^{p+1}\xi \, e^{-\Phi} \sqrt{|g_{mn} + \mathcal{F}_{mn}|} + \int e^{\mathcal{F}_2} \wedge \mathcal{C}_{p+1}. \]  
\[ (2.27) \]

Note that in the string frame relation (2.25) takes the form

\[ F_2 = \mathcal{F}_2 \quad (= dA - B_2). \]  
\[ (2.28) \]

A reason why we introduced the auxiliary field \( F_{ab}(\xi) \) is that it is in some sense frame independent, that simplifies a bit the derivation of the field equations and their analysis.

By construction, the action (2.21) is invariant under target space superdiffeomorphisms and the local \( SO(1,9) \) symmetry acting on the background superfields, under the worldvolume diffeomorphisms and under local worldvolume \( SO(1,p-1) \) transformations acting on the fields carrying worldvolume tangent space indices \( a, b, \ldots \), such as \( E^a \) and \( F_{ab} \). It is also invariant under local worldvolume fermionic \( \kappa \)–symmetry whose form and properties will be discussed in detail in Section 4.

Having constructed the D–brane action and described its properties we now turn to the derivation and the consideration of the D–brane equations of motion in an arbitrary type IIB supergravity background.

## 3 Bosonic equations

### 3.1 Equations for the auxiliary fields and the embedding condition

As we have already discussed, the variation of the action (2.21) with respect to the Lagrange multiplier \( Q_{p-1} \) expresses the auxiliary field \( F_{ab} \) in terms of the BI field strength (2.25)

\[ F_2 = e^{-\frac{1}{2}\Phi} F_2 = e^{-\frac{1}{2}\Phi}(dA - B_2). \]  
\[ (3.1) \]

The variation with respect to \( F_{ab} \) gives the expression for the Lagrange multiplier in terms of other fields

\[ Q_{p-1} = \frac{e^{\frac{p-1}{4}\Phi}}{2 \cdot (p-1)!} \epsilon_{ab} E^{c_1 \cdot \cdot \cdot c_{p-1}} E^{c_{p-1}} \cdot \cdot \cdot E^{c_{p-1}} (\eta + F)^{-1} \epsilon^{c_1 \cdot \cdot \cdot c_{p-1}} (\eta + F)^{-1} \epsilon_{ab} \sqrt{|\eta + F|}, \]  
\[ (3.2) \]

where \((\eta + F)^{-1} \) is the inverse matrix of \( \eta_{ab} + F_{ab} \) and \(|\eta + F| = |det(\eta_{ab} + F_{ab})| \). To arrive at eq. (3.2) we used the identity

\[ \epsilon_{ab} E^{c_1 \cdot \cdot \cdot c_{p-1}} E^{c_{p-1}} E^{d} = \frac{2}{p \cdot (p+1)} \epsilon_{a_1 \cdot \cdot \cdot a_{p+1}} E^{a_1} \cdot \cdot \cdot E^{a_{p+1}} \delta_{c}^{d} \delta_{b}. \]

To perform the variation of the action (2.21) with respect to the auxiliary Lorentz harmonic variables \( u_{a}^{c} \) contained in \( E^{a} = E_{b}^{a} u_{c}^{b} \) one should take into account that they are constrained to be \( SO(1,9) \) matrices (2.16). As discussed in detail e. g. in [59, 60, 52, 9] the independent
variations of \( u^a_\perp \) subject to the orthogonality conditions (2.16) are those projected along the directions orthogonal to the worldvolume, i.e. \( u^i_\perp \frac{\delta}{\delta u^a_\perp} \). With this in mind one gets the following equations of motion of \( u^a_\perp(\xi) \)

\[
u^i_\perp \frac{\delta}{\delta u^a_\perp} S = 0 \implies E^i \land \left( \frac{e^{-\frac{1}{2}\Phi}}{p!} \epsilon_{a_1 b_1 \ldots b_p} E^{b_1} \cdots E^{b_p} \sqrt{|\eta + F| + Q_{p-1} F_{ab} E^b} \right) = 0 \, ,
\]

(3.3)

where \( Q_{p-1} \) is defined in (3.2). Substituting (3.2) into (3.3), and defining the worldvolume form \( E^a \) as \( E^a = E^a_\perp u^i_\perp \) we get

\[
E^i_b (\delta - FF)^{-1}_a b \sqrt{|\eta + F|} \left( \text{det} E^a_m \right) d^{p+1} \xi = 0 \, .
\]

(3.4)

Assuming that the brane worldvolume metric is non degenerate, i.e. that \( \text{det} E^a_m \) is non–zero, and that the matrix \( (\delta - FF)^{-1}_a b \sqrt{|\eta + F|} \) is also non–degenerate 4 we find that eq. (3.3) reduces to the embedding condition (2.14)

\[
E^i_a = 0 \implies E^i = E^a u^i_\perp = 0 \, .
\]

(3.5)

Thus, the embedding condition (2.14) naturally arises in our formulation as the equation of motion (3.5) of the Lorentz–harmonic variables and ensures that the latter are purely auxiliary fields, as demonstrated in the previous Section. The use of eq. (3.5) simplifies the derivation of the equations of motion of the worldvolume fields \( Z^M(\xi) \) which we shall consider in Subsection 3.3.

### 3.2 Equations of motion of the worldvolume BI gauge field

The variation of the Lagrangian (2.21) with respect to the BI field \( A = d\xi^m A_m(\xi) \) is

\[
\delta A \mathcal{L}_{p+1} = \left[ d \left( e^{-\frac{1}{2}\Phi} Q_{p-1} \right) + \Re e^{(dA^1 - B_2)} \right] \delta A + d \left[ e^{-\frac{1}{2}\Phi} Q_{p-1} \delta A + \mathbb{C} e^{(dA^1 - B_2)} \right] p \, ,
\]

(3.6)

where

\[
\Re := \sum_{k=0}^4 R_{2k+1} = d\mathbb{C} - \mathbb{C} H_3 \equiv e^{B_2} d(\mathbb{C} e^{-B_2}) \, ,
\]

(3.7)

the form \( \mathbb{C} \) was defined in (2.22) and \( |_k \) means that only the \( k \)–form terms are retained in the product of formal sums of the pullbacks of the differential forms.

Then, neglecting in (3.6) the second, total derivative term \( \text{(i.e. possible boundary contributions)} \) and using eqs. (3.1) and (3.2) one arrives at the BI field equation in an arbitrary type IIB supergravity background \textbf{in the Einstein frame}

\[
d \left( \frac{1}{(p-1)!} \epsilon_{a_1 a_2 \ldots a_p} E^{c_1} \cdots E^{c_{p-1}} (\eta + e^{-\frac{1}{2}\Phi} F)^{-1} ab \sqrt{|\eta + e^{-\frac{1}{2}\Phi} F|} e^{\frac{p-2}{4}\Phi} \right) = -2 \Re e^{F_1} |_p \, ,
\]

(3.8)

4The singularity of \( (\delta - FF)^{-1}_a b \sqrt{|\eta + F|} \sim (1 - (F_{0m})^2) \) occurs at the critical value \( (F_{0m})^2 = 1 \) of the electric field \( F_{0m} \).
the right hand side of which can further be specified using the definition of $R$ (3.7) and the superspace constraints (2.5).

In terms of the induced metric (2.18), the equation (3.8) takes the form

$$\frac{1}{p!} e_{m_1 \ldots m_p}^l d\xi^{m_1} \ldots d\xi^{m_p} \partial_m \left( (g + e^{-\frac{1}{2}\Phi} \mathcal{F})^{-1} |_{[ml]} \sqrt{|g_{ns} + e^{-\frac{1}{2}\Phi} \mathcal{F}_{ns}|} e^{\frac{\Phi}{4}} \right) = R e^\mathcal{F}_2 |_{p} \quad (3.9)$$

or, equivalently,

$$\partial_m \left( (g + e^{-\frac{1}{2}\Phi} \mathcal{F})^{-1} |_{[ml]} \sqrt{|g + e^{-\frac{1}{2}\Phi} \mathcal{F}|} e^{\frac{\Phi}{4}} \right) = \quad (3.10)$$

where $R_{m_1 \ldots m_n} = E^{A_1}_{m_1} \ldots E^{A_n}_{m_n} R_{A_1 \ldots A_n}$ are the worldvolume pullbacks of the RR field strengths (2.5). Note that eqs. (3.9) and (3.10) do not contain the Lorentz harmonic variables.

In the string frame the equation (3.10) simplifies a bit

$$\partial_m \left( (g + \mathcal{F})^{-1} |_{[ml]} \sqrt{|g + \mathcal{F}|} e^{-\Phi} \right) = \quad (3.11)$$

$$(-) \frac{p-1}{2} \sum_{n=0}^{p-1} \frac{1}{2n! (p-2n)!} e^{im_1 \ldots m_{p-2n} r_1 s_1 \ldots r_n s_n} R_{m_1 \ldots m_{p-2n}} \mathcal{F}_{r_1 s_1} \ldots \mathcal{F}_{r_n s_n},$$

The equations (3.8) - (3.11), which are the differential equations for the BI field strength $\mathcal{F}_2$, should be accompanied by the Bianchi identities which follow from the definition (2.23) of $\mathcal{F}_2$

$$d\mathcal{F}_2 = -H_3 |_{p+1}. \quad (3.12)$$

From the form of (3.12) we conclude that the worldvolume flux $\mathcal{F}_2$ cannot be zero unless the pullback of the NS–NS flux $H_{m_1 m_2 m_3}$ is zero, while eqs. (3.8), (3.9) and (3.10) imply that $\mathcal{F}_2$ also cannot be zero if the pullback of the RR flux $R_{m_1 \ldots m_n}$ is non–zero. This should be taken into account in the analysis of branes in the supergravity backgrounds with fluxes and, in particular, of brane instantons wrapping compactified submanifolds. Several examples will be considered in Subsection 7.

### 3.3 Equations of motion of the coordinate fields $Z(\xi) = (x(\xi), \theta(\xi))$

#### Equation for the bosonic fields $x(\xi)$

It is convenient to consider the variation of the D–brane action (2.21) with respect to the worldvolume fields $Z^M(\xi) = (x^m(\xi), \theta^{\mu}(\xi), \theta^{2\nu}(\xi))$ and the corresponding field equations by projecting them to the tangent space of the embedding superspace, i.e.

$$\delta Z^M \frac{\delta \mathcal{L}_{p+1}}{\delta Z^M} = \delta Z^M E^A_M E^N_A \frac{\delta \mathcal{L}_{p+1}}{\delta Z^N} = (\delta Z^M E^A_M) \frac{\delta \mathcal{L}_{p+1}}{\delta Z^N E^A_N}. \quad (3.13)$$
The bosonic field equations are then identified with
\[ \delta_a \mathcal{L}_{p+1} = \frac{\delta \mathcal{L}_{p+1}}{\delta Z^N E^a_N} = 0 \] (3.14)
and the fermionic field equations are
\[ \delta_{\hat{\alpha}} \mathcal{L}_{p+1} = \frac{\delta \mathcal{L}_{p+1}}{\delta Z^N E^\alpha_N} = 0 , \] (3.15)
where instead of two 16–component spinor indices 1\(\alpha\) and 2\(\alpha\) we have introduced a single 32–component index \(\hat{\alpha}\), so that \(E^\hat{\alpha} = (E^{1\alpha}, E^{2\alpha})\). To get the explicit form of the \(Z^M\) field equations it is useful to recall that the variation of a generic \(n\)–super form \(G_n(Z)\) produced by a variation \(Z^M\) is
\[ \delta G_n := d (i_\delta G_n) + i_\delta dG_n \]
\[ = \frac{1}{(n-1)!} d(dZ^M_1 \cdots dZ^M_n \delta Z^M_1 G_{M_1 \cdots M_n}) + \frac{1}{n!} dZ^M_{n+1} \cdots dZ^M_n \delta Z^M_1 \partial_{[M_1} G_{M_2 \cdots M_{n+1}]} . \] (3.16)

However, the D–brane Lagrangian (2.21) is a \((p+1)\)–form which depends not only on \(Z^M(\xi)\) and their differentials but also on the purely worldvolume fields \(u^a_b(\xi), F_{ab}(\xi), Q_{p-1}(\xi)\) and \(dA(\xi)\). So to apply eq. (3.16) to the calculation of the \(Z^M\)–variation of the Lagrangian (3.13) one takes into account that the formal external differential \(dG_n\) on an auxiliary \((p+2)\)–dimensional surface also contains terms of the form
\[ dZ^M_{n+1} dZ^M_n \cdots dZ^M_1 dw(\xi) \frac{\partial G_{M_1 \cdots M_n M_{n+1}}}{\partial w(\xi)} , \]
where \(w(\xi)\) stands for the worldvolume fields which are treated as independent coordinates.

Thus, up to boundary terms and taking into account eqs. (3.1)–(3.5), the field equations (3.14) and (3.15) can be given in the following generic form
\[ \delta_a \mathcal{L}_{p+1} = \frac{1}{(p+1)!} E^{b_{p+1}} \cdots E^{b_{1}} (d \mathcal{L}_{p+1})_{a b_{1} \cdots b_{p+1}} + \frac{1}{p!} E^{\hat{\alpha}} E^{b_{p}} \cdots E^{b_{1}} (d \mathcal{L}_{p+1})_{a \hat{\alpha} b_{1} \cdots b_{p}} \]
\[ + \frac{1}{2(p-1)!} E^{\hat{\beta}} E^{\hat{\alpha}} E^{b_{p-1}} \cdots E^{b_{1}} (d \mathcal{L}_{p+1})_{a \hat{\beta} \hat{\alpha} b_{1} \cdots b_{p-1}} = 0 , \quad (b = 0, 1, \cdots, p) \] (3.17)
\[ \delta_{\hat{\alpha}} \mathcal{L}_{p+1} = \frac{1}{(p+1)!} E^{b_{p+1}} \cdots E^{b_{1}} (d \mathcal{L}_{p+1})_{\hat{\alpha} b_{1} \cdots b_{p+1}} + \frac{1}{p!} E^{\hat{\beta}} E^{b_{p}} \cdots E^{b_{1}} (d \mathcal{L}_{p+1})_{\hat{\beta} \hat{\alpha} b_{1} \cdots b_{p}} = 0 , \] (3.18)
where \(E^b = E^c u^a_c b\). Note that the fermionic equation (3.18) is linear in the fermionic super-vielbein \(E^{\hat{\alpha}} = (E^{1\alpha}, E^{2\beta})\) and the bosonic equation (3.17) is only a second order polynomial.
in \( E^\alpha \). This is due to the structure of the D–brane action (2.21) and the form of the supergravity constraints (2.2), (2.3) and (2.5).

Now let us remember that, since the theory is invariant under the local worldvolume diffeomorphisms (in particular, \( \delta_{\text{diff}} Z^M = \delta \xi^m \partial_m Z^M \)), among \( x^m(\xi) \) there are only \( 9 - p \) independent bosonic degrees of freedom corresponding to the normal fluctuations of the brane in target space. Therefore, among ten eqs. (3.14), (3.17) there are only \( 9 - p \) independent bosonic field equations. To identify them in a Lorentz covariant way it is again convenient to use the Lorentz harmonics \( u_a^\lambda (\xi) = (u_a^\hat{\beta}, u_a^\hat{\bar{\beta}}) \) (2.15), (2.16).

One can notice that if (3.17) is multiplied by \( u_a^\hat{\beta} \), the first term vanishes since the indices \( a \) and \( b \) take \( p + 1 \) values while \( (d\mathcal{L})_{ab_1\cdots b_{p+1}} \) is an antisymmetric tensor of rank \( p + 2 \). At the same time the projection of the second and the third term of (3.17) along \( u_a^\hat{\beta} \) are proportional to the Born–Infeld (3.10) and fermionic equation (3.18). This reflects the worldvolume diffeomorphism invariance of the D–brane theory.

Thus, the \( 9 - p \) independent bosonic equations are those obtained from eq. (3.17) multiplying it by \( u_a^\hat{\beta} \) \((i = 1, \cdots, 9 - p)\), \text{i.e.} projecting (3.17) along the directions orthogonal to the brane worldvolume

\[
u_a^\lambda \left[ \frac{1}{(p + 1)!} E_b^{p+1} \cdots E_b^1 (d\mathcal{L}_{p+1})_{a_1\cdots a_{p+1}} + \frac{1}{p!} E_{\hat{\alpha}}^a E_b^{p+1} \cdots E_b^1 (d\mathcal{L}_{p+1})_{a\hat{\alpha}_1\cdots a\hat{\alpha}_p} \right] = 0.
\]

\textbf{In the Einstein frame}, in the conventional (induced metric) form, the D–brane bosonic equation in an arbitrary type IIB supergravity background look as follows

\[
D_m \left( e^{\frac{\Phi}{2}} E_{\hat{\alpha}} (g + F)^{-1(mn)} \sqrt{|g + F|} \right) - \frac{1}{2} e^{\frac{\Phi}{2}} H_{\hat{\alpha}\hat{\beta}\hat{\gamma}} (g + F)^{-1mn} \sqrt{|g + F|} -
\]

\[
- e^{\frac{\Phi}{2}} E^M_{\hat{\alpha}} \partial_M \Phi \left( \frac{p - 3}{4} - \frac{1}{4} F_{mn} (g + F)^{-1mn} \right) \sqrt{|g + F|} = \frac{(-e^{\frac{\Phi}{2}})_{a_1\cdots a_{p+1}}}{(p + 1)!} e^{m_1\cdots m_{p+1}} (\mathbb{R} e^{F_2})_{a_1\cdots a_{p+1}},
\]

where \( \partial_m \xi^m \mathcal{D}_m \) is the pullback of the target superspace covariant derivative \( \mathcal{D}_m = \partial_M - \Omega_M \), \( H_{\hat{\alpha}\hat{\beta}\hat{\gamma}} = E^B_{\hat{\alpha}} E^A_{\hat{\beta}} H_{\hat{\alpha} \hat{\beta} \hat{\gamma}} \) is the pullback of the NS–NS superform (2.3) and \( F_2 = e^{-\frac{\Phi}{2}} F_2 \).

\textbf{In the string frame}, in which the supervielbein is \( E_{\hat{\alpha}} = e^{\frac{\Phi}{2}} E^\hat{\alpha} \) and \( F_2 = F_2 = dA - B_2 \), the bosonic equation has even a bit more compact form (in which we again skip the subscript string)

\[
D_m \left( e^{-\Phi} E_{\hat{\alpha}} (g + F)^{-1(mn)} \sqrt{|g + F|} \right) - \frac{1}{2} e^{-\Phi} H_{\hat{\alpha}\hat{\beta}\hat{\gamma}} (g + F)^{-1mn} \sqrt{|g + F|} +
\]

\[
e^{-\Phi} E^M_{\hat{\alpha}} \partial_M \Phi \sqrt{|g + F|} = \frac{(-e^{\frac{\Phi}{2}})_{a_1\cdots a_{p+1}}}{(p + 1)!} e^{m_1\cdots m_{p+1}} (\mathbb{R} e^{F_2})_{a_1\cdots a_{p+1}},
\]
where
\[
(\mathcal{R} e^{\mathcal{F}_2})_{\underline{a} m_1 \cdots m_{p+1}} = \sum_{n=0}^{p+1} \frac{(p+1)!}{2^n n! (p+1-2n)!} R_{\underline{a} [m_1 \cdots m_{p+1-2n} \mathcal{F} \cdots \mathcal{F}_{a p+1]}}
\] (3.22)

One should keep in mind that in eq. (3.21) the superforms \( H_3 \) and \( \mathbb{R} \) are subject to supergravity constraints in the string frame (2.8) and (2.9), which are different from the Einstein frame constraints (2.3) and (2.5) used to derive (3.20).

The linearly independent part of (3.21) (or equivalently of (3.20)) is singled out with the help of \( u^a_i \) as in eq. (3.19), while the projection of eq. (3.21) along \( u^a_i \) is zero modulo the BI field equations (3.11), the Bianchi identities (3.12) and the fermionic equations considered in Section 4.

### 3.4 The bosonic equations and the second fundamental form of the embedding

For further analysis to be carried out in Sections 5-7 we would like to present the bosonic equation (3.21) in a different but equivalent form. This subsection is complimentary, so the reader who is not interested in the details about the relation of the brane equations to the geometric properties of the embedding such as the second fundamental form (or the extrinsic curvature) can jump over to Section 4 and return here later on.

Let us first rewrite the string frame equation (3.21) in the worldvolume tangent frame and single out its independent components by projecting it along \( u^a_i \)
\[
\frac{1}{\sqrt{|\eta + \mathcal{F}|}} D_a \left( e^{-\Phi} E^a_b \left( \eta + \mathcal{F} \right)^{-1(ab)} \sqrt{|\eta + \mathcal{F}|} \right) u^a_i =
\] (3.23)
\[
\frac{1}{2} e^{-\Phi} u^a_i H_{ab} \left( \eta + \mathcal{F} \right)^{-1ab} - e^{-\Phi} u^a_i E^M_a \partial_M \Phi - \frac{(-)^{p+1}}{(p+1)! \sqrt{|\eta + \mathcal{F}|}} \epsilon^{a_1 \cdots a_{p+1}} \left( \mathbb{R} e^{\mathcal{F}_2} \right)_{a_1 \cdots a_{p+1}} u^a_i .
\]

We now notice that because of the embedding condition \( E^a_b u^a_i = 0, E^a_b = u^a_i \) (see Subsection 3.1) the left hand side of (3.23) reduces to
\[
\frac{1}{\sqrt{|\eta + \mathcal{F}|}} D_a \left( e^{-\Phi} E^a_b \left( \eta + \mathcal{F} \right)^{-1(ab)} \sqrt{|\eta + \mathcal{F}|} \right) u^a_i = (D_a E^a_b) u^a_i (\eta + \mathcal{F})^{-1(ab)} e^{-\Phi}. \] (3.24)

In the right hand side of (3.24) one can recognize the second fundamental form
\[
K^i_b = E^a_b K^i_a = -D E^a_b u^a_i = -D u^a_b u^i_a = -d u^a_b u^i_a + u^b_k \Omega^a_{b \underline{a}} u^i_{\underline{a}}
\] (3.25)

characterizing the embedding of the \((p+1)\)-dimensional worldvolume into a curved target type IIB D=10 superspace (see [52, 53, 54, 55] for a review of geometrical grounds of the embedding and superembedding).

Note that in the static gauge, in which \( p + 1 \) components of the 10D target space–time coordinates \( x^m = (x^m, x^i) \) are identified with the worldvolume coordinates \( x^m = \xi^m \) and the
physical modes are associated with $x^i(\xi^m)\ (i = 1, \ldots, 9 - p)$, the second fundamental form looks as follows

$$K_{ab}^i = -E_a^n E_b^m (\partial_m x^i E_i^i + \Gamma_{mn}^l E_l^i + \partial_m x^i \Gamma_{ni}^l E_l^i) + \text{fermion contributions}, \quad (3.26)$$

where $\Gamma_{mn}^l$ and $\Gamma_{ni}^l$ are components of the target space Christoffel symbol $\Gamma_{mn}^l = D_m (\eta^a E^n) a^M E_M^{ab} \ (l, m, n = 0, 1, \ldots, 9; \ m = 0, 1, \ldots, p; \ i = 1, \ldots, 9 - p)$.

We see that if in a pure bosonic background the brane does not fluctuate in the transverse directions, i.e. $\partial_m x^i = 0$, its second fundamental form is non-zero if the pullback of the target-space Christoffel symbol components with one index corresponding to orthogonal directions is non-zero. This occurs, e.g. in warped compactifications considered in Section 7.

In terms of the second fundamental form the Dp–brane bosonic equation takes the following form in the string frame

$$K_{ba}^i (\eta + \mathcal{F})^{-1(ab)} = -\frac{1}{2} H^i_{ab} (\eta + \mathcal{F})^{-1(ab)} + D^i \Phi + \frac{(-)^{p+1}}{(p+1)! \sqrt{\eta + \mathcal{F}}} e^{a_1 \cdots a_{p+1}} (\mathbb{R} e^{\mathcal{F}})_{a_1 \cdots a_{p+1}} (3.27)$$

where (as defined in Subsection 2.3) the index $i$ in the right hand side denotes the component of the pullback of a superform in the directions orthogonal to the brane, e.g. $H_{iab} = u_i^a E^M H_{Mab}$, etc.

From the geometrical point of view the Dp–brane bosonic equation (3.27) is an embedding condition which tells us that its “trace” $K_{ba}^i (\eta + \mathcal{F})^{-1(ab)}$ is expressed in terms of the pullbacks of background fields. In particular, when the worldvolume gauge field, the axion–dilaton, the background gauge fields and all the fermionic fields are zero on the worldvolume the trace of the second fundamental form (which is also called the extrinsic curvature) is zero,

$$K_{ba}^i \eta^{ba} = 0 , \quad (3.28)$$

implying that the embedding of the surface in superspace is minimal in the sense of its induced area. The presence of other fields on the worldvolume and in the background modifies the minimal embedding condition (3.28) in accordance with eq. (3.27).

Let us now reveal another property of the second fundamental form (3.25) which follows from the embedding condition (2.14)

$$E^i = 0 \quad \Rightarrow \quad 0 = dE^i = (DE^{a \dot{a}}) u^{i}_{\dot{a}} + E^{a \dot{a}} D u^{i}_{\dot{a}} = T^{a \dot{a}} u^{i}_{\dot{a}} - E^a K_{a}^i . \quad (3.29)$$

In view of the torsion constraint (2.2), from the form of eq. (3.29) we conclude that the antisymmetric part of the second fundamental form can only be non–zero due to the presence of the worldvolume and/or background fermionic fields

$$K_{[ab]}^i = -iE^{1 \dot{a}} \gamma^i E_{1}^{\dot{a}} - iE^{2 \dot{a}} \gamma^i E_{2}^{\dot{a}} . \quad (3.30)$$
4 Fermionic equation, its concise form and $\kappa$–symmetry

Let us now present the explicit form of the equation of motion (3.18) for the worldvolume fermions $\theta(\xi)$. After some algebra and taking into account the bosonic equations one finds [9] the following variation of the D–brane Lagrangian with respect to $i_\delta E^\alpha = \delta Z^N E^\alpha_N$ in the Einstein frame

\[
\frac{e^{-\frac{\Phi}{2}}}{\sqrt{|\eta + F|}} \delta_\alpha \mathcal{L}_{p+1} = \frac{1}{(p+1)!} \epsilon_{b_1 \cdots b_{p+1}} E^{b_1} \cdots E^{b_{p+1}} i_\delta E^\alpha (I - \tilde{\Gamma})^\alpha_\beta \nabla_\beta \Phi
\]

where $\tilde{\Gamma}$ is the $\kappa$–symmetry projector ($\tilde{\Gamma}^2 = 1$) whose properties will be discussed a bit later,

\[
E^\alpha = E^{\alpha I} = (E^{\alpha 1}, E^{\alpha 2}) , \quad \nabla_\beta \Phi := \left( \frac{\nabla_{\beta 1} \Phi}{\nabla_{\beta 2} \Phi} \right)
\]

(4.1)

\[
i(-)^{p+1} \epsilon_a b_1 \cdots b_p E^{b_1} \cdots E^{b_p} i_\delta E (I - \tilde{\Gamma}) (\eta - \sigma_3 F)^{-1 ba} \left( E - \frac{i}{8} \tilde{\Gamma}^{(1)} \nabla \Phi \right).
\]

(4.2)

where $\tilde{\Gamma}$ is the $\kappa$–symmetry projector ($\tilde{\Gamma}^2 = 1$) whose properties will be discussed a bit later,

\[
E^\alpha = E^{\alpha I} = (E^{\alpha 1}, E^{\alpha 2})
\]

(4.3)

\[
\tilde{\Gamma}^{(1)} := E^a \tilde{\Gamma}_a = \left( \begin{array}{cc} E^{a \gamma} & 0 \\ 0 & E^{a \gamma} \end{array} \right)
\]

(4.4)

the Pauli matrix $\sigma_3$ acts on the indices $I = 1, 2$ of the spinor $E^\alpha = E^{I \alpha}$ and $E^m_a(\xi)$ is the inverse induced worldvolume vielbein introduced in (2.18)–(2.20).

From the variation (4.1) we find the concise form of the fermionic equations for the Dp–branes in an arbitrary type IIB supergravity background in the Einstein frame [9] \footnote{In [9] the fermionic equation was given in the differential form}

\[
(I - \tilde{\Gamma}) \left[ (\eta - \sigma_3 F)^{-1 ba} \right] \left( \mathcal{E}_a - \frac{i}{8} \tilde{\Gamma}_a \nabla \Phi \right) + \frac{i}{2} \nabla \Phi = 0,
\]

(4.5)

where the calligraphic $\mathcal{E}_a$ denotes the pullback

\[
\mathcal{E}^\alpha_a = E^{m}_a(\xi) \partial_m Z^M E^\alpha_M
\]

(4.6)

of the spinorial supervielbein to avoid the confusion with the induced worldvolume vielbein $E^a = dZ^M E^b_M u^a_\xi = d\xi^m E^a_m$. Note that in the form (4.5) the D–brane fermionic equation does not contain the Lorentz harmonics. Their role is taken up by the induced worldvolume vielbein $E^a_m$ and its inverse.
In the string frame the fermionic equation simplifies to

\[
(I - \tilde{\Gamma}) \left[ (\gamma_b \otimes (\eta - \sigma_3 F)^{-1} ba) \, \mathcal{E}_a + \frac{i}{2} \nabla \Phi \right] = 0,
\]

where \(\gamma_b \otimes (\eta - \sigma_3 F)^{-1} ba\) is defined in (4.4) and \(F_2 = dA - B_2\).

4.1 The \(\kappa\)–symmetry projector and the worldvolume Lorentz group

The appearance of the projector \(\frac{1}{2} (I - \Gamma)\) in the fermionic variation (4.1) of the Dp–brane action implies the well known fact that the actions of the superbranes possess worldvolume fermionic \(\kappa\)–symmetry. This means that the superbrane action is invariant under the following variation of the worldvolume scalars \(Z^M(\xi)\)

\[
\delta_\kappa Z^M E^{\hat{\alpha}}_M = \kappa^{\hat{\beta}}(\xi)(I + \tilde{\Gamma})^{\hat{\alpha}}_{\hat{\beta}}, \quad \delta_\kappa Z^M E^{\alpha}_M = 0
\]

accompanied by an appropriate variation of the BI field \(A_m(\xi)\) and of the auxiliary fields \(u^{\alpha}_{a}(\xi)\) and \(F_{ab}(\xi)\) which we shall not present here. Note only that \(A_m(\xi)\) transforms in such a way (\(\delta_\kappa A = i\delta_\kappa B_2\)) that the \(\kappa\)–transformation of its extended field strength \(F_2 = dA - B_2\) is \(\delta_\kappa F_2 = -i\delta_\kappa H_3\). If one substitutes the variation (4.8) into eq. (4.1) one finds that it vanishes identically since \((I - \Gamma)(I + \Gamma) = 0\), thus indicating the presence of a symmetry.

In the case under consideration the \(\kappa\)–symmetry projector \(\tilde{\Gamma}\) is a block–anti-diagonal \(32 \times 32\) matrix (see [6, 63] and also [64])

\[
\tilde{\Gamma}^{\hat{\alpha}}_{\hat{\beta}} = \begin{pmatrix} 0 & h_{\beta}^{\alpha} \\ (h^{-1})_{\beta}^{\alpha} & 0 \end{pmatrix}.
\]

It can be written in the following differential form (see [4, 5] and more recent [25])

\[
\frac{1}{(p+1)!} \epsilon_{a_1 \cdots a_{p+1}} E^{a_1} \cdots E^{a_{p+1}} \tilde{\Gamma} = \frac{1}{\sqrt{|\eta + F|}} \sum_{n=0}^{p+1} \left( \begin{array}{cc} 0 & (-)^n \gamma^{(2n)}_{\beta} \alpha \\ \gamma^{(2n)}_{\alpha} \beta & 0 \end{array} \right) \wedge F_{2, \frac{p+1}{2}-n} \frac{(2n)!}{(p+1)!}.
\]

where the matrix differential forms \(\gamma^{(2n)}\) have been defined in (2.6).

Eq. (4.10) implies that the matrix \(h_{a}^{\beta}\) in (4.9) is defined by the following relation

\[
\frac{1}{(p+1)!} \epsilon_{a_1 \cdots a_{p+1}} E^{a_1} \cdots E^{a_{p+1}} h^{\alpha}_{\beta} = \frac{1}{\sqrt{|\eta + F|}} \sum_{n=0}^{p+1} (-)^n \gamma^{(2n)}_{\beta} \alpha \wedge F_{2, \frac{p+1}{2}-n} \frac{(2n)!}{(p+1)!}.
\]

and its inverse is

\[
\frac{1}{(p+1)!} \epsilon_{a_1 \cdots a_{p+1}} E^{a_1} \cdots E^{a_{p+1}} (h^{-1})_{\beta}^{\alpha} = -\frac{1}{\sqrt{|\eta + F|}} \sum_{n=0}^{p+1} \gamma^{(2n)}_{\beta} \alpha \wedge F_{2, \frac{p+1}{2}-n} \frac{(2n)!}{(p+1)!}.
\]
The form of the matrix \( h^a_\beta \) drastically simplifies when the worldvolume field strength \( F^2 \) vanishes. Then, it reduces to the antisymmetrized product of the \((p+1)\) gamma–matrices along the brane worldvolume,

\[
h^a_\beta \bigg|_{F^2=0} = -\frac{1}{(p+1)!} \epsilon_{a_0 \cdots a_p} (\gamma^a_0 \gamma^a_1 \gamma^a_2 \cdots \gamma^a_p) \beta^a =: \bar{\gamma}^{T-1} \beta^a, \quad \bar{\gamma}^{-1} = (-)^{\frac{p+1}{2}} \gamma,
\]

(4.13)

One can also check, using the properties (2.16) of the Lorentz harmonics, that the following relations hold

\[
\bar{\gamma}^{T} \gamma^a = (-)^{\frac{p+1}{2}} \gamma^a \bar{\gamma} = (-)^{\frac{p-1}{2}} \gamma_{a_1 \cdots a_p} \gamma^{b_1 \cdots b_p},
\]

(4.14)

\[
\bar{\gamma}^{T} \gamma^i = (-)^{\frac{p+1}{2}} \gamma^i \bar{\gamma}, \quad \text{where} \quad \gamma^i = \gamma^a u^i_a, \quad \gamma^a = \gamma^a u^a_a,
\]

(4.15)

and hence

\[
\bar{\gamma}^{T} \gamma^a \bar{\gamma} = \gamma^a, \quad \bar{\gamma}^{T} \gamma^i \bar{\gamma} = -\gamma^i \quad (a = 0, \cdots, p, \quad i = 1, \cdots, D - p - 1).
\]

(4.16)

In the general case, in which the worldvolume field strength \( F^2 \) is non–zero, the relations (4.16) get modified and take the following form

\[
h \gamma^a h^T = \gamma^b k^a_b = \gamma^b (\eta + F)_{bc} (\eta - F)^{-1} c^a, \quad h \gamma^i h^T = -\gamma^i,
\]

(4.17)

where the matrix \( k^a_b = (\eta + F)_{bc} (\eta - F)^{-1} c^a \) is orthogonal

\[
k^a_b = (\eta + F)_{bc} (\eta - F)^{-1} c^a \quad \Rightarrow \quad k^a_b k^c_d \eta_{ac} = \eta_{bd}, \quad \text{det} k = 1.
\]

(4.18)

(note that \( \text{det} (\eta + F)_{bc} = \text{det} (\eta - F)_{bc} \)).

Therefore, \( k^a_b \) belongs to the worldvolume Lorentz group \( SO(1, p) \subset SO(1, 9) \) and is called the Cayley image of the antisymmetric tensor \( F_{ab} \). Because of the relations (4.17) and of the fact that \( k^a_b \in SO(1, p) \) we observe an interesting property of the Dp–brane \( \kappa \)–symmetry projector: \( h^a_\beta \) belongs to a \( 16 \times 16 \) real matrix representation of the group \( Spin(1, p) \subset Spin(1, 9) \). The relations (4.17) and (4.18) are very useful for the analysis of the fermionic equations (4.5), (4.7). Note that the form of the \( \kappa \)–symmetry projector and all the above relations are the same in the Einstein and the string frame.

If we substitute the form (4.9) of the \( \kappa \)–symmetry projector into the fermionic equation (e.g. in the string frame (4.7)) we find the following linearly independent 16–component equation

\[
(\eta + F)^{-1} \gamma_b (\mathcal{E}_a^2 - \mathcal{E}_a^1 h) + i \frac{1}{2} (\nabla_2 - h^{-1} \nabla_1) \Phi = 0.
\]

(4.19)

\(^7\)The \( \kappa \)–symmetry projector and the BI field strength of the type IIA D–branes have the same group–theoretical meaning.
Notice that the fermionic supervielbeins $E_1^\alpha, E_2^\alpha$ (remember the notation (4.6)) and their inverse (encoded in $\nabla_\alpha I$) enter the equation (4.19) only through the single 16–component combination $E^2 - E^1 h$, which manifests the fact that due to $\kappa$–symmetry only half of the fermionic degrees of freedom of the superbrane are dynamical. Note that so far we have not imposed any $\kappa$–symmetry gauge. Therefore, in the above form the equation for the physical fermionic modes on the superbrane is gauge independent up to the bosonic equations, as will be explained in the next Subsection and Section 5.

### 4.2 The linearized Dp–brane fermionic equation

Let us now consider the fermionic equation of motion of the Dp–branes in the linear approximation in $\theta(\xi)$. This will allow us to obtain an explicit coupling of the worldvolume fermions to the background and worldvolume fluxes, which may be useful for various applications including brane instanton calculations.

To obtain the linearized fermionic equation we use the following trick [28]. It is based on the fact that the Dp–brane action and equations of motion are invariant under the target space superdiffeomorphisms $Z^M \rightarrow f^M(Z)$ and that $\theta^\mu(\xi)$ are the Goldstone fermions on the brane associated with the spontaneous breaking of half of the target space supersymmetry. Using the background diffeomorphisms, we can choose a supercoordinate system in which the brane does not fluctuate along the Grassmann–odd directions in superspace, i.e. $\theta^\mu(\xi) = 0$ (see [10] for a discussion of dynamical supergravity–superbrane systems with the target space superdiffeomorphisms being a gauge symmetry). Then the fermionic equation, e.g. in the string frame (4.7), reduces to the algebraic relations on the components of the pullbacks of the gravitino and the dilatino fields $\psi^\hat{\alpha}_m(x) = E^\hat{\alpha}_m |_{\theta=0}$ and $\lambda^\hat{\alpha}(x) = -\frac{i}{2} \nabla^{\hat{\alpha}} \Phi |_{\theta=0}$:

$$ (I - \bar{\Gamma}) \left[ (\gamma_b \otimes (\eta - \sigma_3 F)^{-1} ba) \partial_a x^m \psi_m - \lambda \right] = 0, $$

where the values of the $\kappa$–symmetry projector $\bar{\Gamma}$ and of the pullback of the bosonic vielbein $E^\hat{\alpha}_m$ are taken at $\theta = 0$. I.e. in (4.20) $E^\hat{\alpha}_m = \partial_m x^\alpha E^\alpha_m (x)$ is just the pullback of the $D = 10$ gravitational field.

Let us now recall that $\theta(\xi)$ are Goldstone fermions for the broken target space supersymmetry. Therefore their presence in the fermionic equation at linear order can be restored by performing in (4.20) an infinitesimal local supersymmetry transformation of the pullbacks of the type IIB supergravity fields $E^\alpha_m (x)$, $\psi^\alpha_m (x)$ and $\lambda^\alpha(x)$ whose supersymmetry parameter $e^\hat{\alpha}(x(\xi))$, we identify (in the Wess–Zumino gauge) with $\theta^\alpha(\xi) = \theta^\mu(\xi) E^\hat{\alpha}_\mu (x, \theta)$ so that $d e^\hat{\alpha}(x(\xi)) = d \theta^\alpha(\xi)$. We thus get

$$ (I - \bar{\Gamma}) \left[ (\gamma_b \otimes (\eta - \sigma_3 F)^{-1} ba) \partial_a x^m \psi_m - \lambda \right] + $$

$$ + \delta \left[ (I - \bar{\Gamma}) (\gamma_b \otimes (\eta - \sigma_3 F)^{-1} ba) E^m_a \right] \psi_m + \delta \lambda + $$

$$ + (I - \bar{\Gamma}) \left[ (\gamma_b \otimes (\eta - \sigma_3 F)^{-1} ba) E^m_a \delta \psi_m - \delta \lambda \right] = 0, $$

21
where the second and the third term contains the supersymmetry variation of the pullback of the gravitational field $E_a^\alpha(x)$ entering $\Gamma$, $\gamma_b$ and $\mathcal{F}_2 = \frac{1}{2} E^b E^a F_{ab} = dA - B_2$.

Ignoring the three–fermion contributions, the pullbacks $\delta \psi_m^\alpha = \delta \psi_m^{I\alpha}$ and $\delta \lambda_\alpha = \delta \lambda_{I\alpha}$ of the local supersymmetry variations of the type IIB gravitino and dilatino have the following form in the string frame

\[
\delta \psi_m = D_m \theta + \frac{1}{4 \cdot 2!} H_{mab} \tilde{\gamma}^{ab} \gamma_3 \theta + \frac{1}{8} e^\Phi [R_a \tilde{\gamma}^a (i \sigma_2) - \frac{1}{3!} R_{abc} \tilde{\gamma}^{abc} \sigma_1 + \frac{1}{2 \cdot 5!} R_{abcde} \tilde{\gamma}^{abcde} (i \sigma_2)] \gamma_m \theta ,
\]

\[
\delta \lambda = \frac{1}{2} [D_a \Phi + e^\Phi R_a (i \sigma_2)] \gamma^a \theta - \frac{1}{4!} [H_{abc} - e^\Phi R_{abc} (i \sigma_2)] \gamma^{abc} \gamma_3 \theta ,
\]

where $\gamma_m = E_a^m \gamma_a$, all $\tilde{\gamma}^{abc\cdots}$ start with $\tilde{\gamma}^{a\beta}$, $D_m = \partial_m - \frac{1}{4} \partial_m x^a \Omega_{Ia} \gamma_a$ is the covariant derivative with the spin connection pulled back onto the worldvolume and the Pauli matrices act on the index $I = 1, 2$ of the spinor $\theta^{abI} = (\theta^{a1}, \theta^{a2})$.

The type IIB supersymmetry variation of the graviton field $E_a^\alpha(x)$ is

\[
\delta E_a^\alpha(x) = -2 i \psi_m^\alpha \gamma^a \theta .
\]

Substituting eqs. (4.22)–(4.24) and using the definition of $\tilde{\Gamma}$ one can find the linear fermion equations for any Dp–brane in an arbitrary type IIB supergravity background with non–zero graviton, gravitino and dilatino. An explicit form of such an equation for the M2–brane has been obtained in [28].

If we are interested in purely bosonic supergravity backgrounds (in which the gravitino and the dilatino are zero) the first, second and third term of eq. (4.21) vanish and the equation reduces to the Dirac–like equation which can be derived from the quadratic action of [25]

\[
(I - \Gamma) \left[ (\gamma_b \otimes (\eta - \sigma_3 F)^{-1} ba) E_a^m \delta \psi_m - \delta \lambda \right] = 0 ,
\]

or

\[
(I - \Gamma) (\gamma_b \otimes (\eta - \sigma_3 F)^{-1} ba) D_a \theta = 0 ,
\]

where the generalized covariant derivative $D_a$ contains the pullback to the worldvolume of the $Spin(1,9)$ connection and all contributions of the worldvolume and backgroup gauge field strengths in accordance with the form of the supersymmetry transformations (4.22) and (4.23)

\[
D_a = D_a + \frac{1}{4 \cdot 2!} H_{abc} \tilde{\gamma}^{abc} \gamma_3 + \frac{1}{8} e^\Phi (i R_a \tilde{\gamma}^a \sigma_2 - \frac{1}{3!} R_{abc} \tilde{\gamma}^{abc} \sigma_1 + \frac{i}{2 \cdot 5!} R_{abcde} \tilde{\gamma}^{abcde} \sigma_2) \gamma_a
\]

\[
- \frac{1}{2^{(p+1)}} \tilde{\gamma}^b \otimes (\eta + \sigma_3 F)_{ba} \left[ (D_a \Phi + ie^\Phi R_a \sigma_2) \gamma^a - \frac{1}{2 \cdot 5!} (H_{abc} - i e^\Phi R_{abc} \sigma_2) \gamma^{abc} \gamma_3 \right] .
\]

\[\text{In eqs. (4.22), (4.23) and in what follows we skip } \otimes \text{ product symbol between } \gamma \text{ and } \sigma \text{ where this does not cause confusion.}\]
where $\frac{1}{p+1} \tilde{z}^b \otimes (\eta + \sigma_3 F)_{ba}$ is inverse of (4.4).

One can notice that the covariant derivative (4.26) entering (4.25) is the covariant derivative of the generalized holonomy which determines the gravitino supersymmetry variation (4.22) extended by the terms from the supersymmetry variation of the dilatino (4.23).

In the case of zero worldvolume field strength $F_2$ the equations (4.25) and (4.26) reduce, respectively, to

$$\left( I - \bar{\Gamma} \right) \gamma^a D_a \theta = 0 \quad \text{(4.27)}$$

and

$$D_a = D_a + \frac{1}{4 \cdot 2!} H_{abc} \tilde{\gamma}^{abc} \sigma_3 + \frac{1}{3} e^\Phi (i R_{\alpha \gamma} \tilde{\gamma}^a \gamma_2 - \frac{1}{3} R_{abc} \tilde{\gamma}^{abc} \sigma_1 + \frac{1}{2 \cdot 3!} R_{abcde} \tilde{\gamma}^{abcde} \sigma_2) \gamma_a$$

$$- \frac{1}{2 (p+1)} \tilde{\gamma}_a \left[ (D_\phi \Phi + i e^\Phi R_{\alpha \gamma} \gamma_2) \gamma^a - \frac{1}{2 \cdot 3!} (H_{abc} - i e^\Phi R_{abc} \sigma_2) \gamma^{abc} \sigma_3 \right]. \quad \text{(4.28)}$$

They have been studied in detail for the D3 brane in particular backgrounds with fluxes in [31, 32, 33].

Let us now proceed with the analysis of the general structure of the fermionic equation (4.25). Using the transposition identity

$$\left( I - \bar{\Gamma} \right) \left( \gamma_a \otimes (\eta - \sigma_3 F)^{-1} \right) \equiv \left( \gamma_a \otimes (\eta - \sigma_3 F)^{-1} \right) \left( I - \bar{\Gamma} \right)^T \quad \text{(4.29)}$$

we can rewrite (4.25) in the form

$$D_b \theta \left( I - \bar{\Gamma} \right) \left( \gamma_a \otimes (\eta - \sigma_3 F)^{-1} \right) = 0. \quad \text{(4.30)}$$

Then using the identity $(I - \bar{\Gamma}) \equiv \frac{1}{2} (I - \bar{\Gamma}) (I - \bar{\Gamma})$, the Leibnitz rule and the identity $\mathcal{D} \bar{\Gamma} (I - \bar{\Gamma}) = (I + \bar{\Gamma}) \mathcal{D} \bar{\Gamma}$, one can equivalently write eq. (4.30) as

$$D_b (\theta (I - \bar{\Gamma})) (I - \bar{\Gamma}) \left( \gamma_a \otimes (\eta - \sigma_3 F)^{-1} \right) +$$

$$+ 2 \theta (I + \bar{\Gamma}) D_b \bar{\Gamma} \left( \gamma_a \otimes (\eta - \sigma_3 F)^{-1} \right) = 0. \quad \text{(4.31)}$$

Now we observe that the first and the second terms should vanish separately, i.e.

$$D_b (\theta (I - \bar{\Gamma})) (I - \bar{\Gamma}) \left( \gamma_a \otimes (\eta - \sigma_3 F)^{-1} \right) = 0 \quad \text{(4.32)}$$

and

$$\theta (I + \bar{\Gamma}) D_b \bar{\Gamma} \left( \gamma_a \otimes (\eta - \sigma_3 F)^{-1} \right) = 0. \quad \text{(4.33)}$$

The reason is that at the leading order in $\theta$ which we are interested in at the moment, the $\kappa$-symmetry transformation of $\theta$ which follows from eq. (4.8) is

$$\delta_\kappa \theta = \kappa (\xi) (1 + \bar{\Gamma})|_{\theta = 0}. \quad \text{(4.34)}$$
Therefore, $\theta(\xi) (I + \bar{\Gamma})$ is a pure gauge degree of freedom. It can be put to zero as a $\kappa$-symmetry gauge fixing condition. In this gauge the second term in (4.31) vanishes. Now we notice that eq. (4.32) is $\kappa$-invariant at the linear order in $\theta$, because $\theta (I - \bar{\Gamma})$ is $\kappa$-invariant in this order. Hence the second term (4.33) of (4.31) must vanish in any gauge which is only possible if the following equation holds

\[ (I + \bar{\Gamma}) D_b \bar{\Gamma} \left( \gamma_a \otimes (\eta - \sigma_3 F)^{-1} \right) = 0 \]  

(4.34)

at least for the purely bosonic supergravity backgrounds in which all fermionic fields are set to zero. Eq. (4.34) on the $\kappa$-symmetry projector is a differential equation for bosonic fields on the Dp–brane. As such it cannot be an independent equation but only a concise form of the bosonic equations which we have already found by varying the Dp–brane action.

5 The $\kappa$–symmetry projector and the concise form of the Dp–brane bosonic equations

Let us study in detail eq. (4.34) and its relation to the Dp–brane bosonic equations (3.10), (3.11), (3.20) and (3.21) and to the Bianchi identity (3.12). Since eq. (4.34) contains the projector $I + \bar{\Gamma}$ it is actually a $16 \times 16$ component equation for the matrix $h^{\alpha \beta}$ that determines $\bar{\Gamma}$, eq. (4.9). Upon some algebra with the use of eqs. (4.17) it can be reduced to the following form

\[ D_a h \gamma_b (\eta + F)^{-1} = 0 \Leftrightarrow D_a h h^{-1} \gamma_b (\eta - F)^{-1} = 0 \]  

(5.1)

Note that in (5.1) $D_a h h^{-1}$ is a counterpart of the Cartan form $dh h^{-1}$ for the group $Spin(1,p)$. Eq. (5.1) can thus be regarded as a Dirac–like equation on $h$ where the generalized covariant derivative $D_a$ is the following $16 \times 16$ counterpart of (4.26) in the sense that

\[ (1 + \bar{\Gamma}) D \bar{\Gamma} = \left( \begin{array}{ccc} hDh^{-1} & Dh \\ D(h^{-1}) & h^{-1}Dh \end{array} \right) \]  

(5.2)

(the explicit expression for $Dh$ is given in the Appendix).

To reveal the contents of eq. (5.1) let us first simplify things and consider the case when the dilaton and all the background fluxes are zero. Then the generalized covariant derivative $D_a$ (5.2) reduces to the $Spin(1,9)$ covariant derivative $D_a$ and eq. (5.1) takes the form

\[ D_a h h^{-1} \gamma_b (\eta - F)^{-1} = 0 \]  

(5.3)

We now notice that eq. (5.3) can be obtained by differentiating the relations (4.17) as follows. Taking the covariant differential of (4.17) we get

\[ (Dh h^{-1})^a + \gamma^a (Dh h^{-1})^T = \gamma^b (Dk k^{-1})^b a + 2\gamma^i K^i b (\eta + F)^{-1} \]  

(5.4)

\[ (Dh h^{-1})^i + \gamma^i (Dh h^{-1})^T = 2K^{ai} (\eta + F)^{-1} a b \gamma^b \]  

(5.5)
where $k^b_a = (\eta + F)\omega_c (\eta - F)^{-1} \gamma^a$ is the Cayley image defined in (4.18) and $K^i_a = E^b K^i_{ba}$, with $K^i_{ba}$ being the second fundamental form (3.25). To derive eqs. (5.4) and (5.5) we have used that

$$\gamma^a = \gamma^a \gamma^i, \quad \gamma^i = \gamma^a \gamma^i,$$

(see eqs. (2.15), (2.16) and (2.20) for the definition of the harmonics $u^a = \gamma^a \gamma^i$), the identities

$$\delta^a_b + k^a_b \equiv 2\eta^c (\eta - F)^{-1} \gamma^a, \quad \delta^a_b + k^{-1} a \equiv \delta^a_b + k^a_b \equiv 2\eta^c (\eta + F)^{-1} \gamma^a,$$

and the definition of the second fundamental form (3.25) which, due to the harmonic relations (2.16), implies that

$$Du^a = u^i \gamma^i, \quad Du^a = u^c \gamma^c,$$

(5.7)

$$\Rightarrow \quad D\gamma^a = \gamma^i \gamma^i, \quad D\gamma^i = \gamma^a \gamma^a.$$

(5.8)

The general solution of eqs. (5.4) and (5.5) is

$$(Dh^{-1})_{\beta}^\alpha = \frac{1}{4} (Dk^{-1})^{ab} (\gamma^a \gamma^b)_{\beta}^\alpha - K^a (\eta + F)^{-1} \gamma^a F_{cd} (\gamma^b \gamma^c)_{\beta}^\alpha,$$

(5.9)

where the first term can be further written as follows (see [64] for the D9 brane case)

$$(Dk^{-1})^{ab} = 2(\eta - F)^{-1} ac DF_{cd} (\eta + F)^{-1} db = 2DF_{cd} (\eta + F)^{-1} ca (\eta + F)^{-1} db.$$ (5.10)

Multiplying the left and the right hand side of (5.9) with $\gamma^a (\eta - F)^{-1} ab$ we get

$$Dh^{-1} \gamma^a (\eta - F)^{-1} ab =$$

$$K^a (\delta - FF)^{-1} a b \gamma^i - K^i (\eta + F)^{-1} a c (\eta + F)^{-1} b d \gamma^c d^i -$$

$$- D^a F_{bc} (\delta - FF)^{-1} a b (\eta + F)^{-1} c d \gamma^d$$

$$+ \frac{1}{2} D[a, F_{bc}] (\eta + F)^{-1} a b (\eta + F)^{-1} b c \gamma^d.$$ (5.11)

We see that the right hand side of eq. (5.11) is expressed as a polynomial in the linearly independent products of the matrices $\gamma^a$ and $\gamma^i$. Therefore, if eq. (5.3) holds, each term in the right hand side of (5.11) must vanish separately. But these are nothing but the bosonic equations and the Bianchi identities of the Dp–brane in backgrounds with vanishing dilaton and vanishing background fluxes. Indeed, the first term in the r.h.s. of (5.11) is the l.h.s. of the equation of motion of the bosonic field $x^a$ written in terms of the (generalized) trace of the second fundamental form (3.24)–(3.27). The second term in (5.11) contains the antisymmetric part of the second fundamental form $K_{[ab]}^i$ (3.30) which is zero in the pure bosonic background. The fourth term is the l.h.s. of the Bianchi identity (3.12) which is zero when
the pullback of $H_3$ is zero and the third term contains the BI equation (3.11) (in which the dilaton and the background fluxes are set to zero) and the Bianchi identity (3.12). To see this one should use the identity

$$D_b \left( (\eta + F)^{-[bc]} \sqrt{|\eta + F|} \right) (\delta - F)_c^a \equiv -\sqrt{|\eta + F|} \left[ D^b F_{cd} (\delta - FF)^{-1}_b (\eta + F)^{-1da} + \right. $$

$$+ \frac{2}{7} D_b F_{cd} (\eta + F)^{-1[bc]} (\eta - F)^{-1[de]} (\delta - F)_e^a \right]. \quad (5.12)$$

In the presence of the dilaton and the background fluxes the bosonic equations (3.11), (3.12) and (3.23) acquire the corresponding contributions. Hence, the r.h.s. of (5.11) is not zero anymore and eq. (5.3) is not valid. But, as one can show using the form of the bosonic equations of motion (3.23) and (3.12), the dilaton and background flux contributions arrange themselves in such a way that the matrix $h$ satisfies the Dirac–like equation (5.1) (see the Appendix for its explicit form).

We have thus demonstrated that eqs. (4.34), (5.1) for the $\kappa$–symmetry projector contain in a compact form all the bosonic field equations of the superbrane when the supergravity fermionic fields are zero. When the supergravity background contains the gravitino and the dilatino, the equations (4.34), (5.1) get modified by the corresponding contributions of the fermionic fields. In this paper we shall not elaborate this case in detail.

6 Linearized fermionic equation with couplings to world-volume and background fluxes. Further analysis and examples

We now go back to the consideration of the linearized fermionic equations (4.32) and rewrite them explicitly in terms of a single physical 16–component fermionic field $\theta$ ($1 - \Gamma$) ⇒ $\Theta^\alpha (\xi)$ = $(\theta^2 - \theta^1 h)^\alpha$, similar to the generic equation (4.19), without imposing any $\kappa$–symmetry gauge fixing condition. Using the definition of $\Gamma$ (4.9), the generalized covariant derivative (4.28) and the Dp–brane concise bosonic equations (5.1) we get the following Dirac–like equation for 16–component fermionic field in the Einstein frame

$$D_b \Theta \gamma_a (\eta + F)^{-1ab} = -\frac{e^\Phi}{8} \Theta \left[ (p - 3) \left( e^{-\Phi} \partial \Phi - R_{1h} \eta^{T-1} \right) + 2 R_{b\gamma_a}(\eta - F)^{-1ab} h^{T-1} \right.$$

$$- e^{-\Phi} \partial \Phi \gamma_b \gamma_a (F(\eta + F)^{-1})^{ab} - \gamma_{1h} \gamma_a (F(\eta - F)^{-1})^{ab} h^{T-1} \right]$$

$$+ \frac{e^\frac{1}{2}}{4} \Theta \left[ e^{-\Phi} \partial h^{T-1} - \frac{p}{2} R^3 h^{T-1} \right] - \frac{1}{2} \gamma^{cd} \gamma_a (\eta - F)^{-1ab} \left( e^{-\Phi} H_bcd - R_{bcd} h^{T-1} \right) -$$

$$- \frac{1}{2} R_{3 \gamma_{1h} \gamma_a (F(\eta - F)^{-1})^{ab} h^{T-1}} - \frac{1}{8 \cdot 4!} \Theta R_{\dot{b} \xi \dot{c} \dot{d}} \gamma^{\dot{a} \dot{b} \dot{c} \dot{d}} \gamma_a ((\eta - F)^{-1})^{ab} h^{T-1}, \quad (6.1)$$

$$\frac{1}{2} R_{3 \gamma_{1h} \gamma_a (F(\eta - F)^{-1})^{ab} h^{T-1}} - \frac{1}{8 \cdot 4!} \Theta R_{\dot{b} \xi \dot{c} \dot{d}} \gamma^{\dot{a} \dot{b} \dot{c} \dot{d}} \gamma_a ((\eta - F)^{-1})^{ab} h^{T-1}, \quad (6.1)$$
where $\Theta^\alpha(\xi) = (\theta^2 - \theta^1 h)^\alpha$ is a single 16-component combination of the two worldvolume fermions and the “slashed” objects are

$$\hat{\partial} \Phi = \gamma^a D_a \Phi, \quad R_1 = \gamma^a R_a, \quad R_3 = \frac{1}{3!} \gamma^{abc} R_{abc}, \quad H = \frac{1}{3!} \gamma^{abc} H_{abc}$$

$(a, b, c = 0, 1, \cdots, 9)$.

**In the string frame** (see Subsection 2.2) $\Theta|_{str} = e^{-\Phi} \Theta$ and the fermionic equation takes the form (where the overall dilaton factor $e^{-\Phi}$ is brought to the left hand side)$^9$

$$e^{-\Phi} D_b \Theta \gamma_a (\eta + F)^{-1}{}^{ab} = -\frac{1}{8} \Theta \left[ -4 e^{-\Phi} \hat{\partial} \Phi - (p - 3) R_1 h^{T^{-1}}ight. + 2 R_b \gamma_a (\eta - F)^{-1}{}^{ab} h^{T^{-1}} - R_1 \tilde{\gamma}_b \gamma_a (F(\eta - F)^{-1})^{ab} h^{T^{-1}} \bigg]$$

$$+ \frac{1}{4} \Theta \left[ \left( e^{-\Phi} \hat{H} - \frac{p-1}{2} \tilde{\gamma}_b h^{T^{-1}} \right) - \frac{1}{2} \Theta \gamma_a (\eta - F)^{-1}{}^{ab} \left( e^{-\Phi} H_{bcd} - R_{bcd} h^{T^{-1}} \right) \right. - \frac{1}{8} \Theta R_{bc1 \cdots c4} \gamma^a \tilde{\gamma}_b \gamma_a (\eta - F)^{-1}{}^{ab} h^{T^{-1}}].$$

Equations (6.1) and (6.3) contain all possible couplings of the Dp–brane fermions to the worldvolume and type IIB background fluxes linear in fermions.

This is however not the end of the story, because in its left hand side the covariant derivative $D_a = E_a^b \partial_b - \frac{1}{4} \Omega^{bc}_a \gamma_{bc}$ is not a proper induced covariant derivative in the worldvolume. It contains the pullback of the $Spin(1,9)$ connection rather than the induced $Spin(1,9)$ connection of the worldvolume theory. Therefore, to complete the derivation of the explicit form of the Dirac equation for the brane fermions we should extract from $D_a$ the proper worldvolume covariant derivative. To this end, using the harmonic relations (2.15) and (2.16) we rewrite $D$ as follows$^{10}$

$$D = d - \frac{1}{4} \Omega^{bc} \gamma_{bc} = d - \frac{1}{4} \Omega^{bc} \gamma_{bc} - \frac{1}{4} \Omega^{ij} \gamma_{ij} + \frac{1}{2} \Omega^{bi} \gamma_b \tilde{\gamma}^i$$

$$= D + \frac{1}{2} K^{bi} \gamma_b \tilde{\gamma}^i + \frac{1}{4} u^{ab} d u_a \gamma_{bc},$$

where $K^{bi} = u^a D u^i = u^a du^i + \Omega^{bi}$ is the second fundamental form defined in eq. (3.25), $\gamma^a = \gamma^a u^a_\omega$, $\tilde{\gamma}^i = \gamma^a u^a_\omega i$ and

$$D_a = E_a^b \partial_b - \frac{1}{4} \omega^{bc}_a \gamma_{bc} - \frac{1}{4} A^{ij}_a \gamma_{ij}$$

$^9$To arrive at (6.3) we used the relation

$[D_b \Theta \gamma_a (\eta + F)^{-1}{}^{ab}]_{str} = e^{-\frac{3}{8}} \left( D_b \Theta \gamma_a (\eta + F)^{-1}{}^{ab} + \frac{3}{8} \Theta \hat{\partial} \Phi - \frac{1}{8} \Theta \hat{\partial} \Phi \tilde{\gamma}_b \gamma_a (F(\eta + F)^{-1})^{ab} \right).$

$^{10}$We recall that in our mostly minus metric convention $\eta_{ab} = (\eta_{ab}, -\delta_{ij}).$
is the proper worldvolume covariant derivative with the $SO(1,p)$ spin connection $\omega^{bc}$ and the R--symmetry $SO(9-p)$ connection $A^{ij}$ induced by the embedding. They are related to the pullback of the $SO(1,9)$ connection as follows (see [52, 55] for details)

$$\omega^{bc} = \Omega^{bc} u^b u^c - u^b du^c, \quad A^{ij} = \Omega^{bc} u^j u^c - u^i du^j. \quad (6.6)$$

The inhomogeneous term $u d u$ in (6.4) is due to the $SO(1,9)$ Lorentz transformation of the vector vielbein $E^a$ performed to adapt it to the brane worldvolume, as was explained in Subsection 2.3. At the same time, the spinorial objects like $E^\alpha$ and $\Theta^\alpha$ have not yet been subject to the corresponding $Spin(1,9)$ transformation associated with $u^a u^b$. Such a transformation is performed by a $16 \times 16$ matrix $v^a_{\alpha}$ of $Spin(1,9)$ (which defines spinor Lorentz harmonic [59, 60, 52]) and which is determined by the conventional relation between the vector and the spinor representations of the Lorentz group

$$v^a v^T u^b = \gamma^a, \quad v^a v^T u^i = \gamma^i. \quad (6.7)$$

From this relation it follows, in particular, that

$$\frac{1}{4} u^a d u^c = v d v. \quad (6.8)$$

Now, let us multiply both sides of eq. (6.3) by $v^T$ from the right, use the relations (6.7), (6.8) and make the following redefinitions

$$\Theta v^{-1} \Rightarrow \Theta, \quad v^a v^T u^a = \gamma^a, \quad v^a v^T u^i = \gamma^i, \quad v^{-1} \Rightarrow h^a. \quad (6.9)$$

Using the Clifford algebra, the second term in the r.h.s. of eq. (6.9) can be decomposed as follows

$$K^c_{ab} \Theta \gamma^c (\eta + F)^{-1} \Rightarrow e^{-\Phi} D_b \Theta g_a (\eta + F)^{-1} - \frac{1}{2} e^{-\Phi} K^c_{ab} \Theta \gamma^c \gamma_a (\eta + F)^{-1}. \quad (6.10)$$

where in the right hand side we split $K^c_{ab}$ into symmetric and antisymmetric parts and took into account that $K^c_{bc} = 0$ in a background with zero bulk fermions and in the leading order in $\Theta$ (see eq. (3.30)). Note that the pair of upper indices $i$ implies the contraction with the unit matrix $\delta^i$. One can notice that the last term in (6.10) is the left hand side of the worldvolume scalar equation (3.27) whose right hand side contains the contribution of fluxes. Thus this term
brings additional couplings of the worldvolume fermion to fluxes. Actually, as we shall see on examples these additional terms cancel the non–Lagrangian flux terms of the fermion equation (6.3) in which the products of the gamma–matrices are symmetric $S_{\alpha\beta} = S_{\beta\alpha}$, i.e. the terms which (since $\Theta S \Theta \equiv 0$) cannot be obtained from the quadratic Lagrangian of [24, 25], besides the terms with the dilaton derivatives and those proportional to the BI field equation and the Bianchi identity which result from integrating by parts the kinetic term $L_{\text{kin}} = \sqrt{|\eta + F|} e^{-\Phi} \Theta D_b \gamma_a (\eta + F)^{-1} ab$.

Finally, taking into account the bosonic equation (3.27), the linear fermion Dirac equation which includes all possible interactions of D–brane fermions with the worldvolume fields and the background gravity and fluxes takes the following form

\[
e^{-\Phi} D_b \Theta \gamma_a (\eta + F)^{-1} ab + \frac{1}{2} e^{-\Phi} K_{bc} i \Theta \gamma^{i} \gamma^{b} \gamma^{c} [F(\eta + F)^{-1}]^{ab} = \]

\[
= -\frac{1}{8} \Theta \left[ -4 e^{-\Phi} \gamma^{a} D_{a} \Phi - (p - 3) R_{1} h^{T-1}
+ 2 R_{b} \gamma_a (\eta - F)^{-1} ab h^{T-1} - R_{1} \tilde{\gamma}_{b} \gamma_a (F(\eta - F)^{-1})^{ab} h^{T-1} \right]
\]

\[
+ \frac{1}{2} \Theta \left[ (e^{-\Phi} H - \frac{p-1}{2} R_{3} h^{T-1}) - \frac{1}{2} \gamma^{c d} \gamma_{a} (\eta - F)^{-1} ab \left( e^{-\Phi} H_{b d} - R_{b d} h^{T-1} \right) - \right.
\]

\[
- \frac{1}{2} R_{b} \tilde{\gamma}_{b} \gamma_a (F(\eta - F)^{-1})^{ab} h^{T-1} \right] - \frac{1}{8} \Theta R_{b_{1} \cdots b_{p+1}} \gamma^{b_{1} \cdots b_{p+1}} \gamma_{a} (\eta - F)^{-1} ab h^{T-1}
\]

\[
- \frac{1}{4} e^{-\Phi} H_{1} (\eta + F)^{-1} ab \Theta \gamma_{i} + \frac{(-)^{p+1}}{2 (p+1)! \sqrt{|\eta + F|}} e^{a_{1} \cdots a_{p+1} \cdots p} \Theta \gamma_{i} (\Re e^{F_{2}})_{a_{1} \cdots a_{p+1}},
\]

where the slashed quantities have been defined in (6.2).

Let us repeat once again that the above Dirac equations for a single Majorana–Weyl spinor $\Theta(\xi)$ are $\kappa$–gauge independent, since they were derived without gauge fixing $\kappa$–symmetry.

### 6.1 On $\kappa$–symmetry gauge fixing

Some comments about gauge fixing kappa–symmetry are now in order. As we have already noted, the fermionic field $\Theta^{\alpha}(\xi) = (\theta^{2} - \theta^{1} h)^{\alpha}$ is $\kappa$–invariant in the leading order in $\theta$, while another linearly independent combination of $\theta^{1}$ and $\theta^{2}$, e.g. $\eta^{\alpha}(\xi) = (\theta^{2} + \theta^{1} h)^{\alpha}$ transforms under the $\kappa$–symmetry (4.8) as a Goldstone fermion

\[
\delta_{\kappa} \eta^{\alpha}(\xi) = \delta_{\kappa} (\theta^{2} + \theta^{1} h)^{\alpha} = \kappa^{\alpha}(\xi) := 2(\kappa^{1} + \kappa^{2} h)^{\alpha} + \mathcal{O}(\theta \theta).
\]

We can use (6.12) to gauge fix $\eta^{\alpha}(\xi)$ to zero. This is the most natural choice of the $\kappa$–symmetry gauge fixing condition which is always consistent with the superbackground where the brane moves, since the properties of the background are encoded in the form of the matrix $h_{\alpha}^{\beta}$, as we discussed in Section 5. Such a gauge fixing was used for the analysis of the Dirac equation for M5 and D3 branes with fluxes in [26, 29, 30, 32].

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One also often uses an alternative D=10 covariant gauge fixing condition [65], e.g. \( \theta^1 = 0 \Rightarrow \Theta = \theta^2 \), however, in this case one should be careful and check whether this condition is consistent with the background chosen, e.g. in [32] it was shown that such a condition can be inconsistent with orientifold compactifications.

We shall now analyze the structure of the Dp–brane fermion equations in different cases. We shall work in the string frame only.

### 6.2 Equations for all type IIB Dp–branes in the absence of the worldvolume flux. String frame

Putting to zero the worldvolume field strength \( F_2 \) must be compatible with the equation of motion of the BI field (3.11) and the Bianchi identity (3.12). The Bianchi identity implies that \( \mathcal{F}_2 \) (which in the string frame coincides with \( F_2 \)) can be zero if the brane is placed into the background in such a way that the pullback of the NS–NS flux, along the brane worldvolume is zero

\[
H_{abc} = 0 , \quad (6.13)
\]

while the BI equation of the Dp–brane implies that the pullback of the RR p–form flux \( R_p \) along the brane worldvolume, which is a source for \( F_{ab} \) must also be zero

\[
R_{a_1 \cdots a_p} = 0 . \quad (6.14)
\]

In such a case the bosonic scalar field equation (3.21) is

\[
\frac{1}{\sqrt{|g|}} D_m \left( e^{-\Phi} \sqrt{|g|} g^{mn} E_{na} \right) + e^{-\Phi} E^M_a \partial_M \Phi = - \frac{(-)^{p+1}}{(p+1)! \sqrt{|g|}} \epsilon^{m_1 \cdots m_{p+1}} R_{a m_1 \cdots m_{p+1}} \quad (6.15)
\]

or in terms of the second fundamental form (3.25)

\[
K_{a}^{a i} = D^i \Phi + \frac{(-)^{p+1}}{(p+1)!} \epsilon^{a_1 \cdots a_{p+1}} R_{a_1 \cdots a_{p+1}}^{i} .
\]

Taking into account the simplification of the \( \kappa \)–symmetry projector (4.13), the Dirac equation (6.11) reduces to

\[
e^{-\Phi} D_a \Theta \gamma^a = \frac{1}{2} \Theta \left( e^{-\Phi} \gamma^a D_a \Phi + \frac{p-3}{4} R_2 \gamma^a \bar{\gamma} - \frac{1}{2} R_2 \gamma^a \bar{\gamma} \right) + \]

\[
+ \frac{1}{4} \Theta \left[ \frac{1}{3!} \gamma^{abc} (e^{-\Phi} H_{abc} - \frac{p-1}{2} R_{abc} \bar{\gamma}) \right] - \frac{1}{2} \gamma^{cd} \left( e^{-\Phi} H_{cd} - R_{cd} \bar{\gamma} \right) \]

\[
- \frac{1}{8} \cdot 4! \Theta R_{a_1 \cdots a_4} \gamma^{a_1 \cdots a_4} \bar{\gamma} + \frac{(-)^{p+1}}{2 \cdot (p+1)!} \Theta \epsilon^{a_1 \cdots a_{p+1}} R_{i a_1 \cdots a_{p+1}} \gamma_i .
\]

We recall that \( \bar{\gamma} = \frac{1}{(p+1)!} \epsilon_{a_0 a_1 \cdots a_p} \gamma^{a_0} \cdots \gamma^{a_p} \), as defined in (4.13), \( a = (a, i) \), the indices \( a \) are those of the worldvolume and the index \( i \) corresponds to the directions orthogonal to the
brane. The pair of indices \( i \) which are both lower (or upper) ones denotes the contraction with the unit matrix \( \delta_{ij} \), as in the last term of (6.16).

Note that in agreement with our arguments below eq. (6.10), except for the dilaton derivative term, all products of the gamma matrices in the right hand side of (6.16) are antisymmetric, as we shall now see on the examples.

### 6.2.1 D1 brane

In the case of the D–string (see e.g. [66]) the worldvolume field strength has only one component \( F_{ab} = \epsilon_{ab} F \) and the Born–Infeld equation simplifies to

\[
\partial_b \left( e^{-\Phi} \frac{F}{\sqrt{1 - F^2}} \right) = R_b = \partial_b C_0 \quad \Rightarrow \quad F = e^\Phi \frac{c + C_0}{\sqrt{1 + (c + C_0)^2}} e^{2\Phi},
\]

where \( C_0(x) \) is the axion and \( c \) is an integration constant.

The worldvolume scalar field equation (3.24), (3.27) is

\[
2 e^{-\Phi} K_a^{ai} = 2 e^{-\Phi} (1 - F^2) D^i \Phi + e^{-\Phi} F H_{ab}^i \epsilon^{ab} - \sqrt{1 - F^2} R_{ab}^i \epsilon^{ab} + 2 F \sqrt{1 - F^2} R^i \quad (i = 1, \cdots 8).
\]

One can see that \( F = 0 \) requires that \( R_a = 0 \).

The Dirac equation for a D1 brane with \( F_{ab} = 0 \) is

\[
e^{-\Phi} D_a \Theta \gamma^a = \frac{1}{2} \Theta \left( e^{-\Phi} \gamma^a D_a \Phi + \frac{1}{2} R_i \gamma_i \tilde{\gamma} \right) - \frac{1}{2} 2! \Theta \epsilon^{ab} R_{ab}^{i_1 i_2 i_3} \gamma^{i_1 i_2 i_3}
\]

\[
- \frac{1}{8} \Theta \left( e^{-\Phi} \epsilon^{ab} H_{abi} \gamma_i \tilde{\gamma} + \frac{1}{3} e^{-\Phi} H_{ijk} \gamma_{ijk} - R_{a ij} \epsilon^{ab} \gamma_{ij} \right), \quad \tilde{\gamma} = \frac{1}{2} \epsilon_{ab} \tilde{\gamma}^{ab}.
\]

### 6.2.2 D3 brane (without and with a topological Born–Infeld flux on K3)

The D3 brane worldvolume field strength can be zero if \( H_{abc} \) and \( R_{abc} \) are zero along the brane. The Dirac equation for the D3 brane, which in the case of the constant dilaton–axion coincides with the equation derived in [31, 32], is

\[
e^{-\Phi} D_a \Theta \gamma^a = \frac{1}{2} \Theta \gamma^a \left( e^{-\Phi} D_a \Phi - \frac{1}{2} R_a \tilde{\gamma} \right) - \frac{1}{2} 4! \Theta R_{a i_1 \cdots i_4} \gamma^{a i_1 \cdots i_4} \tilde{\gamma}
\]

\[
+ \Theta \left( \frac{1}{8} \gamma_{ab} \gamma_i (e^{-\Phi} H_{abi} - R_{abi} \tilde{\gamma}) - \frac{1}{4!} \gamma_{ijk} (e^{-\Phi} H_{ijk} - R_{ijk} \tilde{\gamma}) \right), \quad \tilde{\gamma} = \frac{1}{8} \epsilon_{abcd} \tilde{\gamma}^{abcd}.
\]

Though, even if the background sources for the Born–Infeld field i.e. \( H_{abc} \) and \( R_{abc} \) are zero, in general, the BI field strength \( F_2 \) may acquire non–zero values when D3 branes wrap a topologically nontrivial manifold.

Consider, e.g. a D3–brane instanton wrapping the 4–dimensional manifold \( K3 \) in type IIB orientifold compactification on \( M_4 \times K3 \times T^2/Z^2 \). Without the worldvolume flux this
example was considered in [32, 33]. The Dirac equation derived therein has (in a constant
dilaton–axion background and in the absence of the $R_5$ flux) the form

$$D_a \Theta \gamma^a = \frac{1}{8} \Theta \tilde{G}_{abI} \gamma^{ab} \gamma_I,$$

(6.20)

where $I = 1, 2$ is the index corresponding to $T^2$ and $a, b = 1, 2, 3, 4$ are the indices of the D3
brane worldvolume which coincides with $K3$ (we consider the case of a static D3 which does
not oscillate in the directions of the physical space $M_4$) and

$$\tilde{G}_{abI} \equiv H_{abI} - i R_{abI} \bar{\gamma}.$$  

(6.21)

The imaginary unit $i$ appeared in (6.21) because of the Wick rotation to Euclidean space $^{11}$.

In a complex coordinate system, the flux on $K3 \times T^2/Z^2$ has the following form [67]

$$G_3 = R_3 - i H_3 = c \Omega_2 \wedge d\bar{z} + G_z \wedge dz,$$

(6.22)

where $c$ is a constant, $\Omega_2$ is the holomorphic self–dual (2,0)–form on $K3$, $G_z$ is a closed anti–
self–dual (1,1)–form on $K3$ and $(z, \bar{z})$ are complex coordinates on $T^2$. In [67] it was shown
that for the compactification of $K3 \times T^2/Z^2$ with the flux $G_3$ to preserve supersymmetry,
either $G_z = 0$ or $c = 0$, so that $G_3$ is either self–dual or anti–self–dual. This prompts us
also to consider a Born–Infeld instanton field on the D3–brane instanton wrapping $K3$. This
BI instanton can be probably related to the presence of a worldvolume on D7 branes
(filling $M_4$ and wrapping $K3$) which participate in the tadpole cancelation and gaugino
condensation [67, 34].

In the presence of a non–zero worldvolume field $F_2$ the D3–brane worldvolume scalar
field equation (3.27) is

$$K_{ab} (\delta + F)^{-1} = \delta_I \left( \frac{1}{2} H^I_{ab} (\delta - F)^{-1} + \frac{i}{4 \sqrt{|\delta + F|}} e^{aI} e^{bJ} R^I_{ab} F_{ab} \right),$$

(6.23)

where the imaginary unit $i$ appeared because of the Wick rotation, the indices $i = 1, \cdots, 6$
are those of the directions orthogonal to D3 (and $K3$) and $I = 1, 2$ are those of $T^2$.

Remember that in the static gauge in which $p + 1$ coordinates of the target space are
identified with those of the worldvolume, $x^m = \xi^m$, the second fundamental form has the form (3.26)

$$K_{ab}^i = - E_a^m E_b^n (\partial_m \partial_n x^i \tilde{E}^i + \Gamma_{mn}^\xi E^i \tilde{E}^i + \partial_m x^i \Gamma_{ni}^\xi E^i \tilde{E}^i),$$

(6.24)

where $\Gamma_{mn}^\xi$ and $\Gamma_{ni}^\xi$ are components of the target space Christoffel symbol and
$dx^i \tilde{E}^i = (dx^m E^i_m, dx^i E^i)$ are components of the target space vielbein in the $i$–directions
orthogonal to the brane worldvolume.

If we assume that, like in the case of $F_2 = 0$, the D3–brane wraps $K3$ smoothly, i.e. it
is static and does not fluctuate in transverse directions, $\partial_m x^j = 0$, the second fundamental

$^{11}$Note that in contrast to [32] in eq. (6.21) $R_3$ enters with the minus sign. This is because in our present
convention for the Wick rotation to Euclidean space $e^{abcd} \rightarrow ie^{abcd}$, $\gamma \rightarrow i \bar{\gamma}$ and Euclidean $\gamma_a$ satisfy the
anti–commutation relations $\gamma_a \gamma_b + \gamma_b \gamma_a = -\delta_{ab}$.
form of the $D3$ worldvolume is zero (since $\Gamma_{mnl}^i E^i_L = 0$ for $M_4 \times K3 \times T^2/Z^2$) so the left hand side of (6.23) must vanish, and we are left with the condition

$$\frac{i}{2 \sqrt{\delta + F}} \epsilon^{a_1 \ldots a_4} R_{a_1a_2 I} F_{a_3a_4} + H_{abl} (\delta - F)^{-1_{ab}} = 0. \tag{6.25}$$

In the case of a weak $F_2$ field we can restrict eq. (6.25) to the linear order in $F_2$

$$\frac{1}{2} \epsilon^{a_1 \ldots a_4} R_{a_1a_2 I} F_{a_3a_4} - i H_{abl} F^{ab} = 0. \tag{6.26}$$

Consider now the case of the compactification with the flux (6.22) be self–dual in $K3$, i.e. $G_3 = 0$. Then both $R_3$ and $H_3$ are self–dual, and eq. (6.26) takes the form

$$(R_{abl} - i H_{abl}) F^{ab} = G_{abl} F^{ab} = 0, \tag{6.27}$$

from which it follows that for eq. (6.27) to be satisfied, $F_2$ must be anti–self–dual, $F_{ab} = -\frac{1}{2} \epsilon_{abcd} F^{cd}$. On the other hand, if $G_3$ in (6.22) is anti–self–dual in $K3$, i.e. $c = 0$, the equation (6.26) takes the form

$$(R_{abl} + i H_{abl}) F^{ab} = \bar{G}_{abl} F^{ab} = 0. \tag{6.28}$$

It is satisfied if $F_2$ is self–dual, $F_{ab} = \frac{1}{2} \epsilon_{abcd} F^{cd}$. In both of the cases, because of (anti)–self–duality of $F_2$ and of the Bianchi identity $dF_2 = 0$, the linearized BI equation on $K3$ is identically satisfied, $\mathcal{D}_a F^{ab} = 0$.

Away from the linear approximation, the (anti)–self–duality condition on $F_2$ seems to become non–linear

$$\pm \frac{1}{2} \epsilon^{abcd} F_{cd} = (\delta + F)^{-1_{[ab]} \sqrt{\delta + F}}, \tag{6.29}$$

which in view of the Bianchi identity $dF_2 = 0$ implies the BI equation

$$\mathcal{D}_a (\sqrt{\delta + F} (\delta + F)^{-1_{[ab]} \sqrt{\delta + F}}) = 0.$$ 

However surprisingly, one can verify that the non–linear self–duality relation (6.29) actually reduces to the linear one $F_{ab} = \pm \frac{1}{2} \epsilon_{abcd} F^{cd}$. The simplest way to see this is to choose in a given point $\xi$ on the $D3$ brane worldvolume a special $SO(4)$ frame in which the only non–zero components of the Euclidean $F_{ab}$ are $F_{12} = -F_{21}$ and $F_{34} = -F_{43}$. In this frame the r.h.s. of eq. (6.29) takes the form

$$(\delta + F)^{-1_{[ab]} \sqrt{\delta + F}} = -\frac{F_{ab} + F_{12} F_{34} F^{*ab}}{\sqrt{1 + F_{12}^2} \sqrt{1 + F_{34}^2}},$$

Substituting this expression into (6.29) we get the conventional linear self–duality condition

$$\pm F_{ab}^* = \frac{F_{ab} + F_{12} F_{34} F_{ab}^*}{\sqrt{1 + F_{12}^2} \sqrt{1 + F_{34}^2}} \Rightarrow F_{12} = \pm F_{34} \Rightarrow F_{ab} = \pm \frac{1}{2} \epsilon_{abcd} F^{cd}. \tag{6.30}$$
In the case of a non-zero generic BI field and in the setup under consideration, eq. (6.25) is not satisfied. This happens, for instance, when both, the \( G_3 = R_3 - iH_3 \) and \( F_{ab} \) are either self-dual or anti-self-dual on \( K3 \). The situation may be cured if in the compactification background there is also a non-zero \( R_5 \) flux.

The presence of \( R_5 \), in general, leads to a warping of the compactified space. We shall discuss more examples of this setup in Section 7. A generic form of the part of \( R_5 \) along the tangent space of the compactified manifold \( X_6 \) (which, by duality, preserves Lorentz-invariance in effective four-dimensional space-time) is

\[
R_{q_1 \ldots q_5} = -\epsilon_{q_1 \ldots q_5 q_6} \partial_{q_6} \ln Z(y),
\]

(6.31)

where \( Z(y^q) \) is a function of \( X_6 \) coordinates whose square root is the warp (conformal) factor \( Z^{1/2}(y) \) of the compactified space metric (see eq. (7.3) of Section 7). In the Wick rotated space time the self-dual \( R_5 \) is complex. We can make a natural assumption that upon Wick rotation the metric warp factor \( Z(y^q) \) and \( R_5 \) remain real in the a priori Euclidean \( X_6 \). If so, by duality, \( R_5 \) becomes pure imaginary in the effective four-dimensional space \( M_4 \)

\[
R_{r_1 r_2 r_3 r_4 q} = -i\epsilon_{r_1 r_2 r_3 r_4} \partial_q \ln Z(y) \quad (r = 1, 2, 3, 4).
\]

(6.32)

In other words, the imaginary unit appears in the above expression because of the Wick rotation of the time coordinate in \( M_4 \).

In the warped \( K3 \times T^2/Z^2 \) background, the non-fluctuating (i.e. \( \partial_a x^i = 0 \)) D3 brane instanton wrapping the conformally warped \( K3 \) has a non-zero second fundamental form (6.24) with the indices \( I, J = 1, 2 \) corresponding to \( T^2 \)

\[
K_a^{bl} = \frac{1}{4} \delta^b_a \partial^l \ln Z.
\]

(6.33)

Then, in accordance with the generic equation (3.27) the D3 brane instanton scalar field equation acquires the contribution from \( R_5 \) and takes the form

\[
\frac{1}{2} \partial_I \ln Z \delta^a_b (\delta + F) \delta_{a}^{-1} b \sqrt{\delta + F} =
\]

\[
= \frac{i}{12} \epsilon^{a_1 \ldots a_4} R_{a_1 \ldots a_4 I} + \frac{i}{2} \epsilon^{a_1 \ldots a_4} R_{a_1 a_2 I} F_{a_3 a_4} + H_{abl} (\delta - F)^{-1} ab \sqrt{\delta - F}.
\]

(6.34)

The assumption about the reality of Wick rotated bosonic fields helps us to find particular solutions of the D3 brane bosonic field equations. Thus, equation (6.34) imposes additional conditions which relate the warp factor and generic bulk and worldvolume fluxes

\[
\frac{1}{2} \partial_I \ln Z(y) \delta^a_b (\delta + F)^{-1} a b \sqrt{\delta + F} = H_{abl} (\delta - F)^{-1} ab \sqrt{\delta - F},
\]

(6.35)

\[
2 \epsilon_{I,J} \partial_J \ln Z(y) = \frac{1}{2} \epsilon^{a_1 \ldots a_4} R_{a_1 a_2 I} F_{a_3 a_4}.
\]
We should note that eqs. (6.35) cannot be regarded as differential equations on \( \ln Z \), since these equations are valid on the D3 brane worldvolume at a certain point in \( T^2/Z_2 \), while \( \partial_I \) is the derivative in \( T^2/Z_2 \) directions orthogonal to the worldvolume.

Consider now the case when \( F_{ab} \) is either self–dual or anti–self–dual, but let us for a moment do not specify \( H_3 \) and \( R_3 \). Now take into account that for a self–dual or anti–self–dual \( F_{ab} \) the equation (6.29) and the following relations hold

\[
\sqrt{|\delta + F|} = 1 + \frac{1}{4} F^2, \quad F_{ac} F_b^c = \frac{1}{4} \delta_{ab} F^2, \quad (\delta \pm F)^{-1ab} = \frac{\delta^{ab} \mp F^{ab}}{1 + \frac{1}{4} F^2}
\]

\[
(\delta + F)^{-1(ab)} (1 + \frac{1}{4} F^2) = \delta^{ab} \quad \Rightarrow \quad \delta^{ab} (\delta + F)^{-1ab} \sqrt{|\delta + F|} = 4,
\]

we reduce eqs. (6.35) to

\[
2 \partial_I \ln Z(y) = H_{abl} F^{ab}, \quad 2 \epsilon_{IJ} \partial_J \ln Z(y) = \frac{1}{2} \epsilon^{a_1...a_4} R_{a_1a_2I} F_{a_3a_4}.
\]  

(6.36)

If \( F_{ab} \) is anti–self–dual, while \( \Omega_2 \) of \( G_3 = R_3 - iH_3 \) is self–dual, then the right hand sides of eqs. (6.36) vanish identically, so their left hand sides must also be zero, \( \partial_I Z = 0 \), and hence the warp factor should not depend on the coordinates on \( T^2 \).

On the other hand, if \( F_{ab} \) is self–dual, e.g. proportional to \( \Omega_2 \), then the r.h.s. of (6.36) do not vanish, and eqs. (6.36) give a relation between the warp factor and the topological background and worldvolume fluxes. In addition, if \( F_{ab} \) is self–dual, the conditions (6.36) require the 3–form flux \( G_3 = R_3 - iH_3 \) to be imaginary self–dual on the whole six–dimensional manifold (being warped \( K3 \times T^2/Z^2 \) in the case under consideration), i.e. \( G_3 = i^*G_3 \) or \( H_{abl} = -\frac{1}{2} \epsilon_{IJ} \epsilon_{abcd} R_{cdIJ} \), which is indeed the case when in (6.22) the term \( G_z \) is zero.

One can turn around the above argument of the necessity to have a non–zero \( R_3 \) flux when a generic worldvolume gauge field is induced. Namely, if one starts from the consideration of the D3 brane instanton on \( K3 \) without the worldvolume field and switches on the \( R_3 \) flux, the letter contributes to the r.h.s. of the D3 brane scalar field equation (3.21), (3.27). Since the Wick rotated \( R_3 \) is complex and the l.h.s. of eqs. (3.21), (3.27) are assumed to remain real, for these equations to have a solution in general case one should also excite a worldvolume gauge field on D3 as in eq. (6.34). The message is that if one considers a D3 brane instanton wrapping a four–fold of a Calabi–Yau manifold in warped compactifications with non–zero \( R_3 \) flux, one should in certain cases take into account the worldvolume gauge field modes of such an instanton.

It is of interest to analyze (elsewhere) in detail whether the above instanton configurations, and in particular, the fluxes defined by eqs. (6.35) are compatible with Wick rotated supergravity solutions determining Calabi–Yau compactification backgrounds.

Finally, let us present the Dirac equation for the fermionic modes on the D3–brane instanton in the presence of the BI instanton or anti–instanton, \( F_2 = \pm i F_2 \), in the case without warping

\[
-D_6 \Theta \gamma_\alpha (\delta^{ab} + F^{ab}) = \frac{1}{8} \Theta \gamma^{ab} \gamma_I \left( (1 - \frac{1}{4} F^2) H_{abl} - i \left( 1 + \frac{1}{4} F^2 \right) R_{abl} h^{T-1} \right) -
\]

\[
-\frac{1}{4} \Theta \gamma_I \gamma^{ab} F_{ca} H_{bcl} + \frac{1}{4} \Theta \gamma_I H_{abl} F^{ab} - \frac{i}{8} \Theta \gamma_I R_{abl} F^{ab} h^{T-1} - \frac{i}{8} \Theta \gamma_I R_{abl} \ast F^{ab} \gamma h^{T-1}.
\]  

(6.37)
In the cases in which \( H_{ab} \) and \( R_{ab} \) have opposite K3 duality in comparison with \( F_2 = \pm^* F_2 \), \( i.e. \ H_{ab} = \mp \epsilon_{abcd} H^{cd} \) and \( R_{ab} = \mp \epsilon_{abcd} R^{cd} \), the last two terms in \( eq. \) \( (6.37) \) vanish and it reduces to

\[
\mathcal{D}_b \Theta \gamma_a (\bar{\gamma}^{ab} + F^{ab}) = \frac{1}{8} \Theta \gamma^{ab} \gamma_I \left( H_{ab} - i R_{ab} h^{T-1} - 2 F_{c[a} H_{b]cl} - \frac{1}{4} F^2 (H_{ab} + i R_{ab} h^{T-1}) \right),
\]

where \( h^{T-1} = \frac{1}{1 + \frac{1}{2} F^2} (\bar{\gamma} - \frac{1}{4} \epsilon^{abcd} F_{ab} \bar{\gamma}_{cd} + \frac{1}{8} \epsilon^{abcd} F_{ab} F_{cd} \cdot \mathbf{1}) \), \( \bar{\gamma} = \frac{1}{4!} \epsilon_{abcd} \bar{\gamma}^{abcd} \) and \( \gamma_a \bar{\gamma}_b + \gamma_b \bar{\gamma}_a = -\delta_{ab} \).

### 6.2.3 D5 brane

\( F_{ab} \) on the D5 brane can be zero if \( H_{abc} \) and \( R_{a_1 \ldots a_5} \) are zero. Thus, the D5 brane Dirac equation reduces to

\[
e^{-\Phi} \mathcal{D}_a \Theta \gamma^a = \frac{1}{2} \Theta e^{-\Phi} \left( \gamma^a D_a \Phi - \frac{1}{2} \gamma_i R_i \bar{\gamma} \right) + \frac{1}{4!} \epsilon^{a_1 \ldots a_4 bc} \Theta R_{i a_1 \ldots a_4} \gamma_i \gamma_{bc} + \frac{1}{8} \Theta \left( e^{-\Phi} H_{ab} \gamma^{ab} \gamma_i - \frac{1}{3} e^{-\Phi} H_{ijk} \gamma_{ijk} - \gamma^{ij} \gamma_a R_{ija} \bar{\gamma} \right), \quad \bar{\gamma} = \frac{1}{6!} \epsilon_{a_0 \ldots a_5} \tilde{\gamma}^{a_0 \ldots a_5}.
\]

### 6.2.4 D7 brane

In this case for the BI field strength \( F_{ab} \) to be zero, it is necessary that \( H_{abc} = 0 \) and \( R_{a_1 \ldots a_7} = -R_{i a_1} = 0 \), and the D7 brane Dirac equation takes the form

\[
e^{-\Phi} \mathcal{D}_a \Theta \gamma^a = \frac{1}{2} \Theta e^{-\Phi} \left( \gamma^a D_a \Phi + \frac{1}{2} \gamma^a R_a \bar{\gamma} \right) + \frac{1}{2 \cdot 4!} \Theta \gamma^{abc} \epsilon_{ij} R_{abcij} + \frac{1}{8} \Theta \gamma^{ab} \gamma_i \left( e^{-\Phi} H_{ab} - \epsilon_{ij} R_{jab} \right), \quad \bar{\gamma} = \frac{1}{8!} \epsilon_{a_0 \ldots a_7} \tilde{\gamma}^{a_0 \ldots a_7}.
\]

### 6.2.5 D9 brane (with and without worldvolume field)

The D9 brane is space filling, so there are no orthogonal directions and the worldvolume indices \( a, b, \ldots \) coincide with target space indices \( \underline{a}, \underline{b}, \ldots \). The D9 brane always carry a non-zero worldvolume field in the backgrounds with a non-zero \( H_3 \) and/or the axion \( R_1 \). Its Dirac equation is

\[
e^{-\Phi} \mathcal{D}_b \Theta \gamma_a (\eta + F)^{-1} = -\frac{1}{8} \Theta \left[ -4 e^{-\Phi} \gamma^a D_a \Phi - 6 \gamma^a R_a h^{T-1} + 2 R_b \gamma_a (\eta - F)^{-1} h^{T-1} - \gamma^c R_c \gamma_b \gamma_a (F (\eta - F)^{-1})^{ab} h^{T-1} \right] +
\]
\[ + \frac{1}{4} \Theta \left( \frac{1}{3!} \gamma^{abc} \left( e^{-\Phi} H_{abc} - 4 R_{abc} h^{T-1} \right) - \frac{1}{2} \gamma^{cd} \gamma_a (\eta - F)^{-1} ab \left( e^{-\Phi} H_{bcd} - R_{bcd} h^{T-1} \right) - \right. \\
\left. - \frac{1}{2 \cdot 3} \gamma^{e_1 e_2 e_3} R_{e_1 e_2 e_3} \bar{\gamma}_b \gamma_a (F(\eta - F)^{-1} ab h^{T-1} \right) - \frac{1}{8 \cdot 4!} \Theta R_{bc_1 \cdots c_4} \gamma^{c_1 \cdots c_4} \gamma_a (\eta - F)^{-1} ab h^{T-1} \right] \]

with \( h^{T-1} \) defined in (4.12).

The Dirac equation for the D9 brane with \( F_{ab} = 0 \) (which requires \( H_{abc} = 0 \) and \( R^*_a = R_{a_1 \cdots a_9} = 0 \)) has the following form

\[ e^{-\Phi} D_a \Theta \gamma^a = \frac{1}{2} \Theta e^{-\Phi} \gamma^a D_a \Phi - \frac{1}{4!} \Theta \gamma^{abc} R_{abc} . \]

Note that in this case \( \bar{\gamma} = 1 \) and the term with \( R_5 \) vanishes because of the self–duality.

### 7 Dp–branes with intrinsically non–zero worldvolume flux

Let us now consider examples of backgrounds with fluxes in which the BI field on the brane cannot be put to zero and thus should be taken into account in the solutions of the Dp–brane bosonic and fermionic equations of motion. As we have already mentioned, this happens when \( H_{abc} \) and/or \( R_{a_1 \cdots a_p} \) are non–zero along the brane worldvolume.

Such a situation may occur in (warped) compactifications of type IIB string theory with \( R_5 \) and/or \( H_3 \) and \( R_3 \) fluxes turned on. These include e.g. the classical \( AdS_5 \times X_5 \) backgrounds [68, 69] with \( X_5 \) being a sphere \( S^5 \) or an Einstein manifold such as the Sasaki–Einstein manifolds of \( S^2 \times S^3 \) topology. In a certain coordinate system the \( AdS_5 \times X_5 \) metric has the following form (in our mostly minus metric signature)

\[ ds^2 = \left( \frac{r}{\rho} \right)^2 \left( dx_0^2 - dx^i dx_i \right) - \left( \frac{r}{\rho} \right)^2 dr^2 - \rho^2 ds^2_{X_5}, \quad i = 1, 2, 3 \]

where \( \rho \) is a characteristic radius of \( AdS_5 \) and \( X_5 \) (the square root of the inverse cosmological constant), \( r \) is the radial coordinate of \( AdS_5 \) and \( dx_0^2 - dx^i dx_i \) is the metric of the Minkowski space boundary of \( AdS_5 \) corresponding to a stack of \( N \) near horizon D3 branes.

The \( AdS_5 \times X_5 \) compactifications are triggered by the non–zero \( R_5 \) flux which in the local Lorentz frame is\(^{12}\)

\[ R^{AdS_5}_{r_1 \cdots r_5} = \frac{4}{\rho} \epsilon_{r_1 \cdots r_5}, \quad R^{X_5}_{q_1 \cdots q_5} = \frac{4}{\rho} \epsilon_{q_1 \cdots q_5} \cdot \]

In addition to the \( R_5 \) flux, \( R_1 \), \( R_3 \) and \( H_3 \) fluxes are switched on in more general type IIB supersymmetry configurations with general fluxes (see [70, 71] for review and references therein).

\(^{12}\)In our mostly minus metric convention \( \epsilon^r_{q_1 \cdots q_5} = \epsilon^r_{q_1 \cdots q_5} \epsilon^{q_1 \cdots q_5}, \epsilon_{r_1 \cdots r_5 q_1 \cdots q_5} = - \epsilon_{r_1 \cdots r_5} \epsilon_{q_1 \cdots q_5} \) and \( \epsilon^{1 \cdots 5} = \epsilon_{1 \cdots 5} = 1 \).

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The generic setup is the following. The D=10 space–time is a warped \( M_4 \times_w X_6 \) (where \( \times_w \) means that because of warping the D=10 manifold is not really the direct product) manifold with a metric (of mostly minus signature)

\[
ds^2 = Z(y)^{-1/2} ds_{M_4}^2 - Z(y)^{1/2} ds_{X_6}^2 \tag{7.3}
\]

and with fluxes \( H_3 \) and \( R_3 \) turned on in the compactified manifold \( X_6 \) (usually taken to be a Calabi–Yau) and with the self–dual \( R_5 \) flux of the following form in a local Lorentz frame

\[
R_{r_1\ldots r_4 q} = -\epsilon_{r_1\ldots r_4} \partial_q \ln Z \quad R_{q_1\ldots q_5} = \epsilon_{q_1\ldots q_6} \partial^{q_6} \ln Z, \tag{7.4}
\]

where \( Z(y) \) is a warp factor depending on the coordinates \( y^q \) of \( X_6 \) \((q = 1, \ldots 6)\) and the indices \( r = 0, 1, 2, 3 \) are those of the space–time \( M_4 \) in a local Lorentz frame. When \( Z(y) = \rho^2, \ r^2 = y_q y_q \), the metric (7.3) reduces to that of \( AdS_5 \times X_5 \). The explicit form of \( H_3, R_3 \) and of the axion–dilaton depends on the compactification solution and will not be given here (see [71] for a review and references).

In what follows we shall work in the static gauge \( \xi^m = x^m \) for the brane worldvolume coordinates. We shall restrict ourselves to situations which allow for static brane configurations, i.e. those whose oscillating modes do not depend on time variable \( \partial_0 x^i(\xi) = 0 \).

### 7.1 D5 branes in \( AdS_5 \times S^5 \) and \( AdS_5 \times X_5 \)

Let us consider the example of a probe D5 brane wrapping \( S^5 \) in the conventional \( AdS_5 \times S^5 \) background, or a (Sasaki–)Einstein space \( X_5 \) in \( AdS_5 \times X_5 \), and having the time direction in \( AdS_5 \). For an \( AdS_5 \) observer it thus looks like a particle. The D5 brane wrapping \( S^5 \) has been under extensive study from various perspectives [44]–[48], in particular, as a manifestation of the Hanany–Witten effect [72] and as a baryon vertex of an effective \( D = 4 \) gauge theory [42, 43].

We add to these results the derivation of the explicit form of the linearized Dirac equation for the fermionic modes on this “baryonic” D5 brane.

Since \( R_5 \) is given by eq. (7.2) and other fluxes are zero, and if the D5 brane is static and if it does not have modes excited in transverse directions, i.e. \( \partial_m x^i(\xi) = 0 \), the D5 brane scalar field equation (3.23), (3.27) reduces to

\[
K_{ab}^{i} (\eta + F)^{-1ba} = 0,
\]

where the second fundamental form \( K_{ab}^{i} \) describing the embedding of the D5 brane worldvolume into target space–time has been defined in (3.26).

The BI equation (3.11) takes the form

\[
\partial_m \left((g + F)^{-1[ml]} \sqrt{|g + F|}\right) = \frac{1}{5!} \epsilon^{lm_1\ldots m_5} R_{m_1\ldots m_5}, \tag{7.5}
\]

where \( g_{mn} \) is the (almost minus signature) metric in the D5 brane worldvolume induced by embedding into \( AdS_5 \times X_5 \) (7.1) which for the static D5 wrapping \( X_5 \) is

\[
ds_{D5}^2 = \left(\frac{r}{\rho}\right)^2 dx_0^2 - \rho^2 ds_{X_5}^2 \tag{7.6}
\]
Taking into account (7.6) one gets the following BI equation for the static D5 brane

$$D_m \left( (g + F)^{-1} \sqrt{\eta + F} \right) = \frac{1}{5!} \sqrt{|g_{D5}|} e^{l_{m_1 \cdots m_5}} R_{m_1 \cdots m_5} = \frac{4 \rho}{r} \delta^{0 \rho} = \frac{4}{r} \delta^{0 \rho}, \quad (7.7)$$

where $r$ is the radial coordinate of $AdS^5$ (see (7.1)). For the static D5 brane $r$ takes a constant value on its worldvolume, i.e. $\partial_m r = 0$.

We notice that the $R_5$ flux (7.4) produces the source term to the electric part of the BI field strength which thus cannot be zero.

In the linear BI field approximation eq. (7.7) reduces to

$$D_q F^{q0} = \frac{4}{r}, \quad D_{q_2} F^{q_2 q_1} - \partial_0 F^{q_1 0} = 0, \quad q = 1, \cdots, 5, \quad (7.8)$$

where $D_q$ is the covariant derivative on $X_5$. We thus get the Maxwell equation for the electric field $F^{q0}$ on the $X_5$ surface of the D5 brane which is electrically charged with the constant charge density $1/r$.

In the case in which the magnetic part $F_{q_1 q_2}$ of the worldvolume field strength is zero $^{13}$ the equations (7.8) reduce to the Poincare equation for the static electric potential $A^0$ on $X_5$ with the constant source

$$D_q D_q A^0 = \triangle A^0 = \frac{4}{r}. \quad (7.9)$$

This equation, as well as the Born–Infeld equation (7.7) and the Maxwell equation (7.8) (which imply that the vector $V^q = (\eta + F)^{-1} q^0 \sqrt{\eta + F}$ must have a constant divergence $D_q V^q = \frac{1}{r}$) do not have solutions on the compact manifold $X_5$ and in particular on $S^5$. The physical explanation of this is that since the D5 brane carries an effective electric charge there should be an electric flow in a direction orthogonal to $S^5$. This flow can be associated with N strings attached to $S^5$, extended along the radial direction of $AdS_5$ and ended on its Minkowski space boundary [42], which is formed at the near horizon limit by N coincident D3 branes. The charge of these N strings compensates N units of the $R_5$ flux on $S_5$.

This points to the well known fact that the brane cannot smoothly wrap $X_5$, i.e. the condition $\partial_m x^i(\xi) = 0$ is not consistent and should be relaxed. In particular, if a brane mode is excited along the radial coordinate $r$ of $AdS_5$ (along which the N strings joining D5 with the Minkowski boundary are stretched), $r(\xi)$ becomes a function of the D5 worldvolume coordinates of $S^5$ and does not depend on $x_0$ ($\partial_0 r = 0$), the source in the BI equation becomes non–constant and the BI equation with a properly behaved $r(\xi)$ has (supersymmetric) solutions [44]–[48].

Indeed, the D5 brane worldvolume metric becomes

$$ds^2_{D5} = \left( \frac{r}{\rho} \right)^2 dx_0^2 - \left( \frac{\rho}{r} \right)^2 \partial_{q_1} r \partial_{q_2} r d\xi^{q_1} d\xi^{q_2} - \rho^2 ds^2_{X_5} \quad (7.10)$$

$^{13}$Effects of a non–zero self–dual magnetic field $F_{q_1 q_2}$ have been recently studied in [49].
(where $\xi^q$ are the coordinates of $X_5$) and the BI equation takes the following form

$$D_m \left( (g + F)^{-1[n]} \sqrt{|\eta + F|} \right) = \frac{1}{6!} \frac{\rho}{r} \frac{1}{\sqrt{1 + (\frac{\rho}{r})^2}} \delta^{01} \epsilon_{01\cdots q_5} R^X_{q_1\cdots q_5},$$

(7.11)

(the indices $q$ in $\partial_q r \partial^q r$ are contracted with the metric $\frac{1}{\rho^2}g^{q_1 \cdots q_2}$ on $X_5$). The right hand side of this equation is non-constant and allows for solutions in $X_5$.

For such a D5–brane configuration the Dirac equation (6.11) takes the form

$$D_b \Theta \gamma_a \left( \eta + F \right)^{-1} - \frac{1}{2} K_{bc}^i \Theta \gamma^i \gamma_a \left[ F \left( \eta + F \right)^{-1} \right] =$$

$$= - \frac{1}{2 \rho \sqrt{1 + (\frac{\rho}{r})^2}} \Theta \left[ (\eta - F)^{-1 \rho a} + \frac{\rho}{r} \gamma_0 \gamma_h (\eta - F)^{-1 \rho a} \right] \gamma_b \bar{\gamma} h^{T-1} +$$

$$+ \frac{1}{2 \rho} \Theta \partial_\xi e^a \epsilon \left( \xi \right) \left[ \gamma_h (\eta - F)^{-1 \rho a} - \gamma^a \gamma_0 \gamma_h (\eta - F)^{-1 \rho a} + \frac{\rho}{r} \gamma^0 \gamma_0 \gamma_h (\eta - F)^{-1 \rho a} \right] \gamma^r \bar{\gamma} h^{T-1},$$

(7.12)

where $\bar{\gamma} = (a, i)$, with $a = (0, q)$ being tangent space worldvolume indices and $q = 1, \cdots, 5$ corresponding to $X_5$, $i = 1, 2, 3, 4$ denote the directions transverse to the D5 brane with the index $r$ being of the radial $AdS_5$ direction, $e^a_{\xi} \left( \xi \right) = (e^a_0, e^a_0), e^0_0 = e^0_0 = 0$, is the inverse vielbein associated with the induced worldvolume metric (7.10), $\bar{\gamma} = \frac{1}{6!} \epsilon_{a_1 \cdots a_6} \gamma^{a_1} \cdots \gamma^{a_6}$ and

$$\bar{\gamma} h^{T-1} = \frac{1}{\sqrt{|\eta + F|}} \left( 1 - \frac{1}{2} \gamma^{a_1 a_2} F_{a_1 a_2} - \frac{1}{8} \gamma^{a_1 a_2 a_3 a_4} F_{a_1 a_2} F_{a_3 a_4} - \frac{1}{2 \cdot 4!} \epsilon^{a_1 \cdots a_6} F_{a_1 a_2} F_{a_3 a_4} F_{a_5 a_6} \bar{\gamma} \right).$$

The equation (7.12) tells us how the worldvolume fermions interact with the non–zero worldvolume flux $F_2$ induced by the background $R_5$ flux.

Analogously, one can consider the dual case, i.e. a D5 brane filling $AdS_5$ and wrapping a cycle in $X_5$. The form of the equations remain the same with only difference that the time component gets replaced with the component corresponding to the cycle in $X_5$ wrapped by the D5 brane. A warp compactification counterpart of this situation will be considered in Subsection 7.2.1.

### 7.2 D5 and D7 branes in warped compactification backgrounds

Let us consider D5 and D7 branes in a more generic, warped $M_4 \times_{\omega} X_6$ compactification background introduced in the beginning of Subsection 7. To simplify things we shall mainly restrict ourselves to backgrounds with a constant dilaton–axion except for the example of a D5–brane instanton. Recall that, to preserve Lorentz invariance in $M_4$ the three form fluxes $H_3$ and $R_3$ along $M_4$ are assumed to be zero.
7.2.1 Space–time filling D5 brane

Consider a D5 brane that fills the effective four–dimensional space–time and wraps a two–cycle in $X_6$. This brane configuration participates e.g. in the tadpole cancelation (see [71] for a review and references). The D5 brane equations take the following form.

The worldvolume scalar field equation is (3.23), (3.26), (3.27)

$$e^{-\Phi} K_{ba} i (\eta + F)^{-1(ab)} = e^{-\Phi} D^i \Phi - \frac{e^{-\Phi}}{2} H_{ab} (\eta + F)^{-1ab} - \frac{1}{6!} \frac{1}{|\eta + F|} \epsilon^{a_1 \cdots a_6} R_{a_1 \cdots a_6}$$

(7.13)

$$\frac{1}{2 \cdot 4!} \frac{1}{|\eta + F|} \epsilon^{a_1 \cdots a_6} R_{a_1 \cdots a_4} F_{a_5 a_6} - \frac{1}{16} \frac{1}{|\eta + F|} \epsilon^{a_1 \cdots a_6} R^{a_1 a_2} F_{a_3 a_4} F_{a_5 a_6}$$

$$\frac{1}{2 \cdot 4!} \frac{1}{|\eta + F|} \epsilon^{a_1 \cdots a_6} F_{a_1 a_2} F_{a_3 a_4} F_{a_5 a_6} R^i ,$$

where the first and the last term vanish in the constant axion–dilaton background.

In the case when the brane is static and does not fluctuate in orthogonal directions, and taking into account that $R_7 = -R_3$ and the form of $R_5$ (7.4), the equation (7.13) in the linear BI field approximation reduces to

$$e^{-\Phi} K_{a i} = e^{-\Phi} D^i \Phi + \frac{e^{-\Phi}}{2} H_{q_1 q_2} F^{q_1 q_2} + \frac{1}{6} \epsilon_{i_2 i_3 i_4} R_{i_2 i_3 i_4} F_{q_1 q_2} \partial^i \ln Z(y) + \mathcal{O}(F F) .$$

(7.14)

Recall that the index $i = 1, 2, 3, 4$ corresponds to the four directions in $X_6$ transversal to the brane, $q_1, q_2 = 1, 2$ correspond to the directions of $X_6$ along the brane and the indices $r$ are those of $M_4$ which is part of the worldvolume, i.e. $\alpha = (r_1, \cdots, r_4, q_1, q_2)$.

The BI equation for the D5 brane is

$$\mathcal{D}_b \left( e^{-\Phi} (\eta + F)^{-1[ba]} \sqrt{|\eta + F|} \right) = \frac{1}{5!} \epsilon^{ab_1 \cdots b_5} R_{b_1 \cdots b_5} + \frac{1}{2 \cdot 3!} \epsilon^{ab_1 \cdots b_5} R_{b_1 b_2 b_3} F_{b_4 b_5}$$

(7.15)

$$+ \frac{1}{8} \epsilon^{abc_1 \cdots c_4} R_c F_{b_1 b_2} F_{c b_3 b_4} ,$$

where again the last term vanishes in the constant axion–dilaton background $R_c = 0 = \Phi$, which we shall further assume.

If the space–time $M_4$ filling D5 brane does not fluctuate in $X_6$, the second term of the right hand side of (7.15) also vanishes (since $R_3$ with any number of indices in $M_4$ is zero) and in the linear approximation eq. (7.15) reduces to the Maxwell equation

$$\mathcal{D}_b F^{ba} = -\frac{1}{5!} \epsilon^{ab_1 \cdots b_5} R_{b_1 \cdots b_5} = -\delta_{q_1}^{a_1} \epsilon^{a_1 q_2} \partial_{q_2} \ln Z \quad \Rightarrow$$

$$\Rightarrow \mathcal{D}_b F^{br} = 0, \quad \mathcal{D}_b F^{bq_1} = -\epsilon^{a_1 q_2} \partial_{q_2} \ln Z(y) ,$$

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We observe that in the directions of \( M_4 \) the worldvolume gauge field equation is source–less, while in the directions of \( X_6 \) along the brane it acquires the non–constant source term. We can thus put to zero all components of \( F_2 \) along \( M_4 \) (in which case the D5–brane configuration preserves \( M_4 \) Lorentz invariance). Remaining Maxwell equation for \( F_{q_1 q_2} \) is

\[
\mathcal{D}_{q_2} F^{q_2 q_1} = -\epsilon^{q_1 q_2} \partial_{q_2} \ln Z(y), \quad \Rightarrow \quad F_{q_1 q_2} = \epsilon_{q_1 q_2} (c + \ln Z(y)),
\]

where \( c \) is an integration constant.

In the full non–linear case, the Born–Infeld equation for the worldvolume field on the \( X_6 \) part of the D5–brane worldvolume is similar to the \( D1 \)–brane equation (6.17) (the difference is only in the sign in the square root because of the Euclidean metric on \( X_6 \)) and is also exactly solvable

\[
\partial_q \left( \frac{F}{\sqrt{1 + F^2}} \right) = \partial_q \ln Z \quad \Rightarrow \quad F = \frac{c + \ln Z}{\sqrt{1 - (c + \ln Z)^2}}, \quad F_{q_1 q_2} = \epsilon_{q_1 q_2} F. \quad (7.17)
\]

We see that the worldvolume gauge field may have a non–trivial flux on a two–cycle of \( X_6 \) wrapped by the D5 brane. The constant part of its field strength can be regarded as the magnetic field of a monopole located in the center of the compact two–cycle of \( X_6 \).

The bosonic equation (7.13) for the static space–time filling D5 brane with the BI field (7.17) in the constant axion–dilaton background takes the following form

\[
\frac{1}{1 + F^2} K^{qi}_q = -\frac{F}{2(1 + F^2)} H^{q_1 q_2} - \frac{1}{6\sqrt{1 + F^2}} \epsilon^{i_1 i_2 i_3 i_4} R_{i_2 i_3 i_4} - (c - 1 + \ln Z) \partial_i \ln Z(y). \quad (7.18)
\]

When deriving (7.18) we took into account that for the static \((\partial_{ni} x_i = 0)\) D3 brane under consideration the trace \( K^{ri}_r \) of the components of the second fundamental form in \( M_4 \) is

\[
K^{ri}_r = -\partial_i \ln Z \quad (7.19)
\]

which follows from the definition (3.26) of the second fundamental form and the form of the warped \( M_4 \times_w X_6 \) metric and connection. Note also that the components \( K^{ri}_r \) of the second fundamental form with the index \( r \) in \( M_4 \) and the index \( q \) in \( X_6 \) are zero.

Finally, for this D5 brane configuration the Dirac equation is

\[
\mathcal{D}_r \Theta \gamma^r + \frac{1}{1 + F^2} (\mathcal{D}_q \Theta \gamma^q - \mathcal{D}_{q_2} \Theta \gamma_{q_2} \epsilon^{q_1 q_2} F) + = \frac{F}{4} \Theta \gamma_i \tilde{\epsilon}^{q_1 q_2} \epsilon_{q_1 q_2} \partial_i \ln Z
\]

\[
+ \frac{1}{4} \Theta \gamma^{ab} \epsilon^{c d e f g} \left[ \frac{1}{3} \left( H_{a b c} - \frac{2 + F^2}{1 + F^2} R_{a b c} \tilde{h}^{T-1} \right) - \frac{1}{2} \eta_{a c d} (\eta - F)^{-1 a d} \left( H_{d a b} - R_{d a b} \tilde{h}^{T-1} \right) \right] +
\]

\[
+ \frac{F}{2(1 + F^2)} R_{a b c} \tilde{\epsilon}^{q_1 q_2} \epsilon^{q_1 q_2} \left( \tilde{h}^{T-1} \right) - \frac{1}{84} \Theta R_{b c d e} \epsilon^{a b c d e} \gamma_a (\eta - F)^{-1 a b} \tilde{h}^{T-1} +
\]

\[
+ \frac{1}{4} \Theta \gamma_i (1 + \frac{F}{2} \tilde{\gamma}^{q_1 q_2} \epsilon_{q_1 q_2}) \left( \frac{F}{1 + F^2} H_{i q_1 q_2} \epsilon^{q_1 q_2} + \frac{1}{3 \sqrt{1 + F^2}} \epsilon_{i_2 i_3 i_4} R_{i_2 i_3 i_4} - \partial_i (c + \ln Z)^2 \right),
\]

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where the underlined indices run (as usual) from 0 to 9; $a = (r, q); r = 0, 1, 2, 3$ are in $M_4$, $q = 1, 2$ and transverse $i = 1, 2, 3, 4$ are in $X_6$,
\[ h^{T^{-1}} = -\frac{1}{\sqrt{1 + F^2}} \left( F - \frac{1}{2} \epsilon_{q_1 q_2} \tilde{\gamma}^{q_1 q_2} \right) \tilde{\gamma}^5 \quad \text{and} \quad \tilde{\gamma}^5 = \frac{1}{4!} \epsilon_{r_1 \cdots r_4} \tilde{\gamma}^{r_1 \cdots r_4}. \]

It seems of interest to study in detail this D5 brane configuration in concrete compactification schemes (see [71] and references therein).

### 7.2.2 D5 brane instanton wrapping Calabi–Yau six-fold

Consider now a Euclidean D5 brane (instanton) wrapping $X_6$ \(^{14}\). Then $H_3, R_3$ and $R_5$, which in the general case are non–zero in $X_6$, can induce a worldvolume flux in the D5 brane worldvolume in virtue of the BI equations.

For the D5–brane instanton on $X_6$, which does not fluctuate in transverse directions, and the background fluxes are non–zero only on $X_6$, the both sides of the worldvolume scalar field equation (7.13) are identically zero separately.

The D5 worldvolume field strength must satisfy the Bianchi identity $d F_2 = - H_3 |_{X_6}$ and obey the BI field equation (3.11). In view of our ansatz (6.31), (6.32) for the Wick rotated $R_5$ flux in $X_6$ the BI equation takes the form
\[ - D_b \left( (\delta - F)^{-1}_{[ba]} \sqrt{|\delta + F|} e^{-\Phi} \right) = \]
\[ = -i \partial^a \ln Z + \frac{i}{2 \cdot 3!} \epsilon^{ab_1 b_2 b_3 b_4 b_5} R_{b_1 b_2 b_3} F_{b_4 b_5} + \frac{i}{8} \epsilon^{ab_1 b_2 b_3 b_4 b_5} R_{b_1 b_2 b_3} F_{b_4 b_5}. \]

Since with our assumption about the Wick rotation, the left hand side of (7.21) is real, while its right hand side is imaginary they should be equal to zero separately. Such an ansatz allows us to find particular solutions of the D5 brane equations of motion. E.g. this imposes the relation between the warp factor and the $R_1, R_3$ and $F_2$ fluxes similar to that for the D3 brane instanton of Subsection 6.2.2 \(^{15}\)

\[ - \partial^a \ln Z + \frac{1}{2 \cdot 3!} \epsilon^{ab_1 b_2 b_3 b_4 b_5} R_{b_1 b_2 b_3} F_{b_4 b_5} + \frac{1}{8} \epsilon^{ab_1 b_2 b_3 b_4 b_5} R_{b_1 b_2 b_3} F_{b_4 b_5} = 0. \]

---

\(^{14}\)NS5 brane instantons wrapping a Calabi–Yau manifold in type IIA theory have been studied in [73] and references therein.

\(^{15}\)Eric Bergshoeff pointed our attention to the fact that since the axion field $C_0$ is a pseudoscalar, it should be replaced with $i C_0$ upon Wick rotation. This, in particular, is required for getting D–instanton solutions in Euclidean type IIB supergravity. In our case this will result in additional factor of $i$ in the term with the $R_1$ flux and hence in a different splitting of eq. (7.21) into the real and imaginary part, namely
\[ D_b \left( (\delta - F)^{-1}_{[ba]} \sqrt{|\delta + F|} e^{-\Phi} \right) = \frac{1}{8} \epsilon^{ab_1 b_2 b_3 b_4 b_5} R_{b_1 b_2 b_3} F_{b_4 b_5}, \]
\[ \partial^a \ln Z = \frac{1}{2 \cdot 3!} \epsilon^{ab_1 b_2 b_3 b_4 b_5} R_{b_1 b_2 b_3} F_{b_4 b_5}. \]
The Dirac equation (6.11) for the D5 brane fermionic modes is (where under the Wick rotation $h^{T-1} \rightarrow i h^{T-1}$)

$$-e^{-\Phi} D_b \Theta \gamma_a (\delta - F)^{-1ab} = -\frac{1}{8} \Theta \left[ -4 e^{-\Phi} \gamma^a D_a \Phi - 2i R_a \gamma^a h^{T-1} - 2i R_b \gamma_a (\delta + F)^{-1ab} h^{T-1} + i R_c \gamma^c \tilde{\gamma}_b \gamma_a (F(\delta + F)^{-1ab} h^{T-1}) \right] + \frac{1}{4} \Theta \left[ \frac{1}{3!} \gamma^{abc} \left( e^{-\Phi} H_{abc} - 2i R_{abc} h^{T-1} \right) + \frac{1}{2} \gamma^{cd} \gamma_a (\delta + F)^{-1ab} \left( e^{-\Phi} H_{bcd} - i R_{bcd} h^{T-1} \right) - \right. \right.$$

$$\left. \left. + \frac{i}{3!} R_{cdf} \gamma^{cdf} \tilde{\gamma}_b \gamma_a (F(\delta + F)^{-1ab} h^{T-1}) \right] \right.$$

$$+ \frac{1}{8} \Theta \gamma^5 \gamma_a \partial_b \ln Z (\delta + F)^{-1ab} h^{T-1} + \frac{i}{8} \Theta \gamma_{cb} \gamma_a \partial^c \ln Z (\delta + F)^{-1ab} \tilde{\gamma} h^{T-1},$$

where $\gamma^5 = \frac{1}{6!} \epsilon_{i_1 \ldots i_6} \gamma^{i_1} \ldots \gamma^{i_6}$ is the gamma–five matrix of $M_4$ orthogonal to the D5 brane worldvolume, $\tilde{\gamma} = \frac{1}{6!} \epsilon_{a_1 \ldots a_6} \gamma^{a_1} \ldots \gamma^{a_6}$ and

$$h^{T-1} = \frac{1}{\sqrt{|\delta + F|}} \left( \gamma - \frac{1}{2 \cdot 4!} \epsilon^{a_1 \ldots a_6} \gamma_{a_1 \ldots a_4} F_{a_5 a_6} + \frac{1}{10} \epsilon^{a_1 \ldots a_6} \gamma_{a_1 a_2} F_{a_3 a_4} F_{a_5 a_6} \right) + \frac{1}{2 \cdot 4!} \epsilon^{a_1 \ldots a_6} F_{a_1 a_2} F_{a_3 a_4} F_{a_5 a_6} \cdot \left( 1 \right).$$

Note that the D5–brane instanton acquires the worldvolume flux also in compactifications on $M_4 \times X_6$ with non–zero fluxes $H_3$ and $R_3$ in $X_6$ but with zero $R_5$ and $R_1$.

In such cases the D5 worldvolume flux is defined by the relations

$$dF_2 = -H_3|_{X_6}, \quad D_b \left( (\delta - F)^{-1ba} \sqrt{|\delta + F|} e^{-\Phi} \right) = -\frac{i}{2 \cdot 3!} \epsilon^{ab_1 b_2 b_3 b_4 b_5} R_{b_1 b_2 b_3} F_{b_4 b_5}. $$

Since, as we assume (upon the Wick rotation) the l.h.s. of the latter equation is real and the r.h.s. is imaginary, they must vanish separately which imposes further condition on $F_{ab}$

$$R_3 F_2 = 0 \quad \Rightarrow \quad d (R_3 F_2) = -R_3 H_3 = 0. \quad \text{(7.23)}$$

The simplest situation when the above equation is satisfied is $R_3 = 0$. This takes place in the compactification solutions of Type A in the Table 3.4 of [71].

In such a background the Dirac equation of the D5 brane instanton is

$$D_b \Theta \gamma_a (\delta - F)^{-1ab} = -\frac{1}{8} \Theta \left( 4 \gamma^a D_a \Phi + \frac{1}{3} \gamma^{abc} H_{abc} + \gamma^{cd} \gamma_a (\delta + F)^{-1ab} H_{bcd} \right). \quad \text{(7.24)}$$

It will be of interest to analyze non–perturbative effects due to the D5 brane instantons in effective four–dimensional theory.
7.2.3 Compact space filling D7 brane

A D7 brane wraps $X_6$ and its worldvolume has time and one space direction along $M_4$. So it looks like a string for an observer living in $M_4$. The D7 brane scalar field equation is (3.23), (3.27)

$$e^{-\Phi} K_{ba}^i (\eta + F)^{-1(ab)} = e^{-\Phi} D^i \Phi - \frac{e^{-\Phi}}{2} H_{ab} (\eta + F)^{-1ab}$$

$$+ \frac{1}{\sqrt{|\eta + F|}} (-\epsilon^{ij} R^j + \frac{1}{16 \cdot 3!} \epsilon^{a_1 \ldots a_8} F_{a_1 a_2} F_{a_3 a_4} F_{a_5 a_6} F_{a_7 a_8} R^i$$

$$- \frac{1}{2} \epsilon^{ij} R^{j a b} F_{a b} + \frac{1}{16 \cdot 3!} \epsilon^{a_1 \ldots a_8} R^i_{a_1 a_2} F_{a_3 a_4} F_{a_5 a_6} F_{a_7 a_8}$$

$$+ \frac{1}{8 \cdot 4!} \epsilon^{a_1 \ldots a_8} R^i_{a_1 a_2 a_3 a_4} F_{a_5 a_6} F_{a_7 a_8}) \right).$$

(7.25)

The second fundamental form $K_{ba}^i$ of the embedded D7 brane worldvolume is defined in (3.26).

If the brane is static and does not fluctuate along the transverse space directions, for a given $M_4$ Lorentz preserving configuration of non-zero fluxes $H_3$, $R_3$ and $R_5$ in $X_6$ and a constant axion–dilaton, the right hand side of (7.25) vanishes. Also $K_{ba}^i = 0$ for such a static embedding into the warped background, due to the definition (3.26) of the second fundamental form and because $x^i$ are transverse directions in $M_4$ on which the warp factor $Z(y)$ does not depend. In this case the BI equation takes the form

$$D_b \left( (\eta + F)^{-1[b]} \sqrt{|\eta + F|} \right) = -\frac{1}{2} \delta^a_q \epsilon^{r_1 r_2} F_{r_1 r_2} \partial_q \ln Z - \delta^a_{r_1} \epsilon^{r_1 r_2} F_{r_2 q} \partial_q \ln Z$$

$$- \frac{1}{4!} \delta^a_q \epsilon^{q_1 q_2 q_3 q_4 q_5} \epsilon^{r_1 r_2} (R_{q_1 q_2 q_3 f q_4 q_5} F_{r_1 r_2} - 2 R_{q_1 q_2 q_3} F_{r_1 q_4} F_{r_2 q_5})$$

$$- \frac{1}{3!} \delta^a_{r_1} \epsilon^{r_1 r_2} \epsilon^{q_1 q_2 q_3 q_4 q_5} R_{q_1 q_2 q_3} F_{r_2 q_4} F_{q_5 q_6}$$

(7.26)

where $a = (r_1, r_2, q)$ is the D7 worldvolume index, the index $q$ corresponds to the $X_6$ part of the D7 brane, the indices $r_1, r_2$ label the two directions of the D7 worldvolume along $M_4$ and the indices $i, j = 1, 2$ correspond to the $M_4$ orthogonal directions. The BI field strength satisfies the Bianchi identity $dF_2 = -H_3 |_{X_6}$ and thus at least $F_{q_1 q_2}$ in $X_6$ is intrinsically non-zero if the pullback of the NS–NS flux is non-zero.

Finally the static D7 brane Dirac equation is

$$D_b \Theta \gamma_a (\eta + F)^{-1(ab} =$$

$$-\frac{1}{8} \Theta \gamma_{q_1 q_2} \gamma_a \left[ \frac{1}{3} \delta^a_{q_3} \left( H_{q_1 q_2 q_3} - 3 R_{q_1 q_2 q_3} h^{r-1} \right) - (\delta - F)^{-1} q_3 \left( H_{q_1 q_2 q_3} - R_{q_1 q_2 q_3} h^{r-1} \right) \right]$$

(7.27)
\[-\frac{1}{3} \delta_{q_3} R_{q_1 q_2 q_3} \tilde{\gamma}_6 \gamma_c \left( F(\eta - F)^{-1} \right) e^{\eta} h^{T-1} \] 
\[-\frac{1}{8^3} \Theta \epsilon^{q_1 \ldots q_6} \gamma_{q_1 \ldots q_6} \partial_{q_6} \ln Z \gamma_a (\eta - F)^{-1} a_{q_6} h^{T-1} \]
where
\[h^{T-1} = \frac{1}{\sqrt{|\eta + F|}} \left( \tilde{\gamma} - \frac{1}{2^6} \epsilon^{q_1 \ldots q_6} \tilde{\gamma}_{q_1 \ldots q_6} F_{a_{q_1} a_{q_2}} + \frac{1}{8^3} \epsilon^{q_1 \ldots q_6} \tilde{\gamma}_{q_1 \ldots q_6} F_{a_{q_1} a_{q_2}} F_{a_{q_3} a_{q_4}} \right) \]
and
\[\tilde{\gamma} = \frac{8!}{8^4} \epsilon^{q_1 \ldots q_6} \tilde{\gamma}_{q_1 \ldots q_6}.\]

\[\text{7.2.4 Space–time filling D7 brane}\]

A D7 brane fills $M_4$ and wraps a four–fold in $X_6$. Such a D7–brane takes part in the tadpole cancelation and gaugino condensation (see [71] for a review and references). The role of D7 brane worldvolume fluxes in the generation of supersymmetry breaking terms in effective four–dimensional theory has been discussed in [34, 36].

If the D7 brane is static and does not fluctuate in the transverse directions of $X_6$ its scalar field equation is similar to (7.25) and in the constant axion–dilaton background takes the form

\[K_{ab} (\eta + F)^{-1(ab)} = -\frac{1}{2} H_{q_1 q_2} (\eta + F)^{-1} q_1 q_2 - \frac{1}{2 \sqrt{|\eta + F|}} \epsilon^{ij} R^{j}_{q_1 q_2} F^{q_1 q_2} \]
\[+ \frac{1}{8 \sqrt{|\eta + F|}} \partial^i \ln Z F_{q_1 q_2} F_{q_3 q_4} \epsilon^{q_1 \ldots q_4} - \frac{1}{8 \sqrt{|\eta + F|}} \epsilon^{ij} \partial^i \ln Z F_{r_1 r_2} F_{r_3 r_4} \epsilon^{r_1 \ldots r_4} \]
\[+ \frac{4!}{4! \sqrt{|\eta + F|}} \epsilon^{q_1 \ldots q_6} R^{i}_{a_{q_1} a_{q_2}} F_{a_{q_3} a_{q_4}} F_{a_{q_5} a_{q_6}} \]

where the second fundamental form $K_{ab} (\eta + F)^{-1(ab)}$ of the embedded worldvolume has been defined in (3.26), $a = (r, q)$ is the D7 worldvolume index, the indices $r$ correspond to the $M_4$ part of the D7 brane and the indices $q = 1, 2, 3, 4$ label the four directions of D7 along $X_6$. The indices $i$ denote two directions of $X_6$ orthogonal to D7. $Z(y)$ is the warp factor defining the $R_5$ flux (7.4).

The equation of motion of the D7–brane BI field (satisfying the Bianchi identity $dF_2 = -H_3$) is

\[D_c \left( (\eta + F)^{-1 [ba]} \sqrt{|\eta + F|} \right) = \frac{1}{2} \delta^a_{r_1} \epsilon_{i_1 i_2} R^{q_1 q_2} F_{q_3 q_4} \partial_{q_4} \ln Z \]
\[-\frac{1}{2 \delta^a_{r_1}} \epsilon^{r_1 r_2 r_3 r_4} q_1 q_2 q_3 q_4 R_{q_1 q_2 q_3} F_{q_4 r_2} F_{r_3 r_4} - \frac{1}{2 \delta^a_{r_1}} \epsilon^{q_1 q_2 q_3 q_4} R_{q_1 q_2 q_3} F_{r_1 r_2} F_{r_3 r_4} \epsilon^{r_1 r_2 r_3 r_4}.\]

We can further assume that for the space–time filling D7–brane configuration to preserve $M_4$ Lorentz invariance, the vacuum value of its Born–Infeld field in $M_4$ is zero, namely
For instance, consider a compactification of type IIB string theory on $AdS_{5}$, then taking into account that the trace of the components of the second fundamental form along warped $M_{4}$ is given by eq. (7.19), the bosonic equations (7.28) and (7.29) reduce, respectively, to
\begin{equation}
K_{q_{1}q_{2}}^{i}(\delta + F)^{-1}q_{1}q_{2}) = -\partial^{i} \ln Z - \frac{1}{2} H_{i}^{1}q_{1}q_{2} (\delta - F)^{-1}q_{1}q_{2} + \frac{1}{2\sqrt{\delta + F}} \epsilon^{ij} R_{q_{1}q_{2}}^{i} F^{q_{1}q_{2}}
\end{equation}
and
\begin{equation}
D_{q_{2}} ((\delta - F)^{-1}q_{1}q_{2}) \sqrt{\delta + F}) = -\frac{1}{2} \epsilon_{i_{1}i_{2}} R^{q_{1}i_{1}i_{2}} + \frac{1}{2} \epsilon^{q_{1}q_{2}q_{3}q_{4}} F_{q_{2}q_{3}} \theta_{q_{4}} \ln Z.
\end{equation}
For $F_{r_{1}r_{2}} = F_{rq} = 0$ in $M_{4}$, the Dirac equation has the following form
\begin{equation}
D_{r} \Theta \gamma^{r} - D_{q_{1}} \Theta \gamma_{q_{2}} (\delta - F)^{-1}q_{1}q_{2} - \frac{1}{2} K_{(q_{2}q_{1})}^{i} \Theta \gamma^{i} \gamma_{q_{3}} [F(\delta - F)^{-1}]q_{3}q_{2} =
\end{equation}
\begin{align*}
-\frac{1}{8} \Theta \gamma_{q_{1}q_{2}} \gamma_{q_{3}} \left[ \frac{1}{3} \left( H_{q_{1}q_{2}q_{3}} - 3 R_{q_{1}q_{2}q_{3}} h^{-1} \right) - (\delta - F)^{-1}q_{3}^{q_{4}} \left( H_{q_{1}q_{2}q_{4}} - R_{q_{1}q_{2}q_{4}} h^{-1} \right) \\
+ \frac{1}{2} R_{q_{1}q_{2}q_{3}} \gamma_{q_{4}} \gamma_{q_{3}} (\delta + F)^{-1}q_{4}q_{4} h^{-1} \right] - \frac{1}{2} \Theta \gamma^{q} (\gamma_{q} \partial_{q} \ln Z + \gamma_{q} \partial_{q} \ln Z) h^{-1} \\
+ \frac{1}{16.3} \Theta \gamma^{q_{1}} \epsilon_{q_{1}q_{2}q_{3}q_{4}} \gamma_{q_{2}q_{4}q_{4}} \gamma_{q_{1}} (\delta - F)^{-1} q_{1} q_{1} h^{-1} \\
+ \frac{1}{64} \Theta \gamma^{q_{1}} \epsilon_{q_{1}q_{2}q_{3}q_{4}} \theta_{q_{4}} \ln Z \gamma_{q_{2}q_{3}} \gamma_{q_{1}} (\delta - F)^{-1} q_{1} q_{1} h^{-1} \\
- \frac{1}{8} \Theta \gamma^{q} \partial_{q} \ln Z (\delta - F)^{-1} q_{1} q_{2} h^{-1} + \frac{1}{4} \Theta \gamma_{i} H_{i q_{1} q_{2}} (\delta - F)^{-1} q_{1} q_{2} \\
- \frac{1}{4\sqrt{\delta + F}} \Theta \gamma_{i} \epsilon_{i j} R_{j q_{1} q_{2}} F^{q_{1} q_{2}} + \frac{1}{16 \sqrt{\delta + F}} \Theta \gamma_{i} \partial_{i} \ln Z F_{q_{1} q_{2}} F_{q_{3} q_{4}} \epsilon_{q_{1} q_{2} q_{3} q_{4}},
\end{align*}
where $\gamma^{5} = \frac{1}{4} \epsilon_{r_{1} r_{2} r_{3} r_{4}} \gamma^{r_{1} r_{2} r_{3} r_{4}}$ is the gamma–five matrix on $M_{4}$ and
\begin{equation}
\gamma^{-1} = \frac{1}{\sqrt{\delta + F}} \epsilon_{q_{1} q_{2} q_{3} q_{4}} \left( \frac{1}{4!} \gamma_{q_{1} q_{2} q_{3} q_{4}} - \frac{1}{4} \gamma_{q_{1} q_{2}} (\gamma_{q_{3} q_{4}} - \frac{1}{2} F_{q_{3} q_{4}}) \right).
\end{equation}
In the case when $F_{2}$ is self–dual or anti–self–dual on the four–cycle of $X_{6}$, the Dirac equation can be simplified using the relations for self–dual $F_{2}$ given in Subsection 6.2.2.

**7.2.5 D3 branes in $AdS_{3} \times S^{3} \times X_{4}$ and $M_{4} \times S^{3} \times S^{3}$**

In the variety of brane configurations and compactification setups considered in the literature one can easily find other examples when the BI worldvolume field on the brane is non–zero. For instance, consider a compactification of type IIB string theory on $AdS_{3} \times S^{3} \times X_{4}$ or $M_{4} \times S^{3} \times S^{3}$ with non–zero $H_{3}$ and $R_{3}$ fluxes in $S^{3}$ and/or $AdS_{3}$. Then a D3 brane wrapping $S^{3}$ or $AdS_{3}$ will carry a non–zero BI field and a corresponding effective charge, like the D5 branes in the examples of Subsection 7.1.
8 Conclusion

We have carried out a detailed analysis of the bosonic and fermionic equations of motion of type IIB D branes in generic supergravity backgrounds with fluxes taking into account the worldvolume BI field. We have presented the explicit form of these equations for each of the Dp branes (\( p = 1, 3, 5, 7, 9 \)) in the cases in which the \( H_3 \) and \( R_p \) fluxes along the brane worldvolume are zero and the supergravity backgrounds do not induce the worldvolume BI gauge field. We then gave several examples of brane configurations in which the worldvolume flux is intrinsically non-zero and thus must be taken into account in studying the problems in which such branes are involved.

It would be of interest to analyze further the brane configurations with background and worldvolume fluxes discussed in the above Subsections and other ones and to study their effects in concrete compactification setups, supersymmetry etc., e.g. within the lines of recent papers considering properties and effects of D3, D5 and D7 branes [36] and Euclidean D3 branes [74] on cycles of warped Calabi-Yau manifolds.

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Appendix

The underlined indices \( a, b, \ldots = 0, 1, \ldots, 9 \) are \( D = 10 \) target space indices. The indices \( a, b, c, d = 0, 1, \ldots, p \) are those of \((p+1)\)-dimensional Dp–brane worldvolume and the indices \( i, j, k = 1, \ldots, 9 – p \) are of the directions orthogonal to the worldvolume.

We use the mostly minus convention for the Minkowski metric \( \eta^{ab} = \text{diag} (+, -, \cdots, -) \). The unit antisymmetric tensor \( \varepsilon^{a_1 \cdots a_9} \) is defined as follows \( \varepsilon^{0123456789} = 1 \) and
\[
\gamma^0 \gamma^1 \cdots \gamma^9 = -\gamma^0 \gamma^1 \cdots \gamma^9 = 1,
\]
where \( \gamma^a_{\alpha\beta} \) and \( \tilde{\gamma}^a_{\alpha\beta} \) are \( 16 \times 16 \) (symmetric) matrix counterparts of the Pauli matrices satisfying the relations
\[
\gamma^a \gamma^b + \gamma^b \gamma^a = 2 \eta^{ab}, \quad \gamma^{[a_1 \cdots a_5]} \gamma^{a_6 \cdots a_{10}} = \frac{1}{5!} \varepsilon^{a_1 \cdots a_5 a_6 \cdots a_{10}} \gamma_{a_6} \gamma_{a_7} \cdots \gamma_{a_{10}}.
\] (8.33)

Note that there is no “charge conjugation” matrix in \( D = 10 \) which would lower or rise 16-component Majorana–Weyl spinor indices \( \alpha, \beta \). This is why one should distinguish between
\( \gamma_{\alpha\beta} \) and \( \tilde{\gamma}^{\alpha\beta} \).

The explicit expression for the generalized covariant derivative acting on the \( \kappa \)-symmetry projector matrix \( h \) in the concise form of the bosonic equations (5.1) in the Einstein frame is

\[
D_b h = D_b h + \frac{1}{8} \left\{ H \tilde{\gamma}_b + \gamma_b H, h \right\} - \frac{1}{4(p+1)} \left\{ H \tilde{\gamma}^c, h \right\} (\eta + F)_{cb} - \\
- \frac{1}{4} \left[ W \Phi \tilde{\gamma}_b , h \right] - \frac{1}{2(p+1)} \left[ \tilde{W} \Phi \tilde{\gamma}^c , h \right] (\eta + F)_{cb} - \\
- \frac{1}{8} e^\Phi \left( \gamma_b \tilde{R}_1 + h \gamma_b \tilde{R}_1 h \right) - \frac{1}{2(p+1)} e^\Phi \left( \tilde{R}_1 \tilde{\gamma}^c (\eta - F)_{cb} - h \tilde{R}_1 \tilde{\gamma}^c h (\eta + F)_{cb} \right) +
\]

\[
+ \frac{1}{8} e^{\frac{1}{2} \Phi} \left( \gamma_b \tilde{R}_3 - h \gamma_b \tilde{R}_3 h \right) + \frac{1}{2(p+1)} e^{\frac{1}{2} \Phi} \left( \tilde{R}_3 \tilde{\gamma}^c (\eta - F)_{cb} + h \tilde{R}_3 \tilde{\gamma}^c h (\eta + F)_{cb} \right) - \\
- \frac{1}{16} \left( \gamma_b \tilde{R}_5 + h \gamma_b \tilde{R}_5 h \right)
\]

where the slashed objects imply the contraction of the components of \( D = 10 \) n-forms \( W_n \) (\( n = 1, 3, 5 \)) with \( \gamma_{\alpha\beta} \) and \( \tilde{\gamma}^{\alpha\beta} \) such that \( \tilde{W}_n := \frac{1}{n!} \gamma^{\alpha_1 \cdots \alpha_n} W_{\alpha_1 \cdots \alpha_n} \) and \( \tilde{W}_n := \frac{1}{n!} \tilde{\gamma}^{\alpha_1 \cdots \alpha_n} W_{\alpha_1 \cdots \alpha_n} \).

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