Functional Integration on
Constrained Function Spaces II: Applications

J. LaChapelle

Abstract

Some well-known examples of constrained quantum systems commonly quan-
tized via Feynman path integrals are re-examined using the notion of conditional
integrators introduced in [1]. The examples yield some new perspectives on old
results. As an interesting new application, the formalism is used to construct
a physical model of average prime counting functions modeled as a constrained
gamma process.

Keywords: Constrained dynamical systems, constrained path integrals, constraints in
quantum mechanics, prime number counting functions.

MSC: Primary 81Q35, 46N50, 35Q40; Secondary 11N05.

1 Introduction

A basis for functional integration on constrained function spaces was proposed in [1] in
analogy with Bayesian inference theory. Needless to say, the proposed formalism must
reproduce known results. So here some implications and applications of various integral
representations are checked against four archetypical classes of constrained systems that
were heuristically reviewed in [1]. The four classes can be roughly characterized as: Kine-
matical; 1) fixed end-points, 2) bounded configuration space, 3) segmented configuration
space, and Dynamical; 4) functional constraints. The subsequent efficient derivation of
old results illustrates the utility of the new techniques introduced.

This exercise requires the development of some new tools. In particular, kinemat-
ic constraints suggest the notion of a Dirac delta functional on the topological dual
of the constraint space. It turns out that these delta functionals are particular types
of gamma functional integrals. In fact, gamma functional integrals can be used to de-
fine ‘step functionals’ and all their (Gateaux) derivatives. Presumably, one could use
such tools to construct a functional analog of distribution theory. Similarly, dynamical
constraints suggest the notion of Dirac integrators characterized by Dirac delta function-
als on the constraint space (as opposed to the dual constraint space). Their heuristic
equivalents have long been used to enforce functional constraints — the most notable
example being the Faddeev-Popov method in quantum field theory. Here we use them
to treat topologically nontrivial systems that can be modeled as quotient spaces: Their
associated functional constraints can be efficiently encoded using paths in an appropri-
ate fiber bundle that are constrained to be horizontal relative to a given connection.
Finally, discontinuities in configuration space can be handled using the idea of “path decomposition” \[3\]–\[5\]. The resulting functional integral tools lead to a recursive process to calculate propagators on bounded and/or segmented configuration spaces. The recursive process can be evaluated iteratively, but “path decomposition” suggests a new and potentially useful approximation technique rooted in boundary Green’s functions. More importantly, the functional integral construction shows that the propagators associated with such systems are intimately related to Poisson integrators and, hence, gamma integrators.

As a new application of the formalism, we conjecture functional integral representations of some prime counting functions. The functional integrals represent average counting functions, and they give excellent approximations to the exact counting functions. According to the construction, the prime counting functions are modeled as a constrained gamma process (as opposed to constrained Gaussian processes of the quantum examples) — a perspective that might lead to a deeper understanding of the distribution of prime numbers. This is a nice realization of the often observed interplay between physics and number theory.

We should alert the reader that results from \[1\], particularly notation and details of integrator families presented in appendix B \[1\], will be used without explanation.

2 Two conditional integrators

The value of the formalism proposed in \[1\] is perhaps best appreciated by application to the familiar examples outlined in §2 \[1\] in the context of quantum mechanics (QM). Recall that the task is to represent constrained function spaces as enlarged unconstrained function spaces— including non-dynamical degrees of freedom — equipped with conditional and conjugate integrators.

Now, in simple QM, eigenfunctions of position observables for the semiclassical approximation are characterized by their mean and covariance. When viewed as functions of time, they become paths in some configuration space. Consequently, the semiclassical QM can be described in terms of functional integrals with Gaussian integrators on the space of paths characterized by an assumed mean and covariance.

Applying §3 \[1\] to the case \[Y \equiv C\], it is clear that the imposition of constraints in QM can change the quadratic form and/or the mean associated with an unconstrained Gaussian integrator. The notion of conjugate integrators enables one to construct the ‘marginal’ and ‘conditional’ integrators associated with those constraints, and hence make sense of constrained functional integrals. We will see that the type of conjugate integrator family depends on the type of constraint.

\[1\]Recall that \(C\) is a Banach space of non-dynamical degrees of freedom induced by constraints on a dynamical system whose degrees of freedom are elements of a Banach space \(X\).
2.1 Delta functional on \( C' \)

**Claim 2.1** Let the space of \( L^2,1 \) pointed paths \( x : [t_a, t_b] \to \mathbb{X} \) with \( x(t_a) = x_a \in \mathbb{X} \) (denoted \( X_a \)) be endowed with a Gaussian integrator \( D \omega \bar{x}, Q(x) \). A uniform functional on \( C \) represents a constraint that can alter the covariance but not the mean of \( x \in X_a \) relative to \( D \omega \bar{x}, Q(x) \).

According to the Bayesian analogy, we consult a table of conjugate prior probability distributions and find the conjugate family associated with a Gaussian distribution of known mean and unknown covariance is a gamma distribution. So the integrator on \( C \) associated with a uniform functional constraint is expected to be a gamma integrator.

Now that we know the ‘probability’ nature of a uniform constraint on a Gaussian QM system through its marginal integrator family, the task is to understand the integrator family parameters and the relevant sufficient statistics.

As motivation, consider the lower gamma integrator family (see Appendix B, \([1]\)). Put \( \alpha = 1 \), and let \( L : T_0 \to i\mathbb{R}^n \) with \( \langle \beta', L(\tau) \rangle \mapsto 2\pi \lambda \cdot i u \) and \( \lambda = \lambda^* \). Then,

\[
\int_{T_0} D_{\gamma_1, \beta', \tau_o}(\tau) \xrightarrow{L} \int_{\mathbb{R}^n} e^{-2\pi i \lambda \cdot u} du = \delta(\lambda). \quad (2.1)
\]

with \( \lambda \in \mathbb{R}^n \) and the integral over \( \mathbb{R}^n \) understood as an inverse Fourier transform. On the other hand,

\[
\int_{T_0} D_{\gamma_1, \beta', \tau_o}(\tau) := \frac{\gamma(1, \tau_o)}{\text{Det}(\beta')} = \frac{1 - e^{-\tau_o}}{\text{Det}(\beta')}, \quad (2.2)
\]

and so the integrator \( D_{\gamma_1, \beta', \tau_o}(\tau) \) can be understood as a limit;

\[
\int_{T_0} D_{\gamma_1, \beta', \tau_o}(\tau) := \lim_{|\tau_o| \to \infty} \int_{T_0} D_{\gamma_1, \beta', \tau_o}(\tau). \quad (2.3)
\]

Consequently, when \( \tau_o \) is strictly imaginary, \( D_{\gamma_1, i\beta', \tau_o}(\tau) \) can be interpreted as the functional analog of a two-sided Laplace transform implying

\[
\int_{T_0} D_{\gamma_1, i\beta', \tau_o}(\tau) = \lim_{|\tau_o| \to \infty} e^{\tau_o} - e^{-\tau_o}; \quad (2.4)
\]

which formally vanishes except when \( \text{Det}(i\beta') = 0 \). In fact, this can be interpreted as the functional analog of a delta function. In particular, this integrator can be used to localize onto a subset of the dual constraint space \( C' \).

This justifies the definition:

**Definition 2.1** Suppose \( i\langle c', c \rangle \in \mathbb{I} := i\mathbb{R} \) and \( S_s(c') \) is degenerate on \( C \). A delta functional\(^3\) on \( C' \) is defined by

\[
\delta_{S_s}(c') := \frac{1}{\Gamma(1)} \int_C D_{\gamma_1, i\beta', \tau_o}(c) \quad (2.5)
\]

\(^2\)The gamma family is also conjugate for exponential-type distributions so the following analysis holds for action functionals that are not necessarily quadratic.

\(^3\)Since \( C' \) is a polish space, the delta functional can be interpreted in a measure theoretic sense as the complex Borel measure associated with the identity element in the Banach space \( F(C) \), i.e. \( \text{Id} = F_{\mu_s}(c) = \int_C e^{i\langle c', c \rangle} d\mu_s(c') \forall c \in C \) — hence the term ‘uniform functional’ in Claim 2.1.
and a Heaviside step functional by

$$\theta_{S_s}(c') := \frac{-i}{\Gamma(0)} \int_C D_{\gamma_{0,ic',\infty}}(c') . \quad (2.6)$$

The subscript $S_s$ reminds that the functionals are evaluated on the sufficient statistic subspace. Consequently, the delta functional vanishes unless $\text{Det}_{S_s}(c') = 0$, and the step functional vanishes unless $\text{Det}_{S_s}(c') \in \mathbb{C}_+$. 

Remark that this definition suggests the characterization

$$\delta_{S_s}^{(\alpha-1)'}(c') = \frac{i^{\alpha-1}}{\Gamma(\alpha)} \int_C D_{\gamma_{\alpha,ic',\infty}}(c) \quad (2.7)$$

when $i\langle c', c \rangle \in \mathbb{I}$ and $\text{Det}_{S_s}(c') = 0$. The characterization is “good” in the sense that $\delta_{S_s}^{(\alpha-1)'}(c')$ reduces to the usual Dirac delta function under linear maps $L : C \to \mathbb{R}^n$ for any $n$, and for $\alpha = m \geq 1$ with $m \in \mathbb{N}$ we have

$$\delta_{S_s}^{(m-1)'}(c')(t) = i^{m-1} \frac{\Gamma(m-1)}{\Gamma(m)} \int_C \frac{\delta^m}{\delta c'(t)} D_{\gamma_{0,ic',\infty}}(c) \quad (2.8)$$

Evidently gamma integrators and their associated functional integrals could be used as a basis for a theory of what might be called ‘distributionals’.

### 2.2 Delta functional on $C$

**Claim 2.2** Let the space of $L^{2,1}$ pointed paths $x : [t_a, t_b] \to \mathbb{X}$ be endowed with a Gaussian integrator $D_{\omega, Q}(x)$. A delta functional on $C$ represents a constraint that can alter the mean but not the covariance of $x \in X_a$ relative to $D_{\omega, Q}(x)$.

Again consulting a table of conjugate priors, the conjugate family associated with a Gaussian distribution of known covariance and unknown mean is again a Gaussian distribution so the integrator associated with a delta functional constraint is expected to be a Gaussian. Essentially this means that the unknown mean of the conditional integrator on $X_a$ is normally distributed with respect to the marginal and conditional integrators on $C$.

**Definition 2.2** A Dirac integrator is defined by

$$D\delta_{\epsilon}(c) := D_{\omega,\epsilon,\infty}(c) := \delta(c - \epsilon)Dc \text{ such that } \lim_{W(c') \to 0} \text{Det}(W)^{1/2}e^{-\pi W(c')} := 1 . \quad (2.9)$$

---

4Evidently dynamical constraints, which are encoded as delta functionals on $C$, influence mean paths but do nothing to covariances. This explains the often observed mismatch in QM between enforcing topological/geometrical constraints (e.g. confinement to $S^1$) on a quantum system via Lagrange multipliers in the action versus a free quantum system on a manifold representing the topology/geometry. Classically the two are equivalent because the constraints produce identical mean paths, but their covariances differ so their associated quantum mechanics are different.

5The Dirac integrator is improper in the sense that it is a limit of a Gaussian that requires regularization to achieve a sensible normalization.
Proposition 2.1 The Dirac integrator is normalized \( \int_C D\delta_\epsilon(c) = 1 \), translation invariant \( D\delta_\epsilon(c - c_0) = D\delta_\epsilon(c) \), and furnishes the functional analog of a Dirac measure

\[
\int_C F_\mu(c) D\delta_\epsilon(c) = F_\mu(\bar{c}) .
\]

**Proof.** The normalization is obvious and the translation invariance follows from the translation invariance of the primitive integrator. The third relation follows trivially from definitions and translation invariance of the primitive Gaussian integrator;

\[
\int_C F_\mu(c) D\delta_0(c) = \int_{C'} Z(c') \, d\mu(c')
= \int_{C'} 1 \, d\mu(c')
= F_\mu(0)
\]

where the last line follows from the definition of \( F_\mu(c) \) and \( c = \bar{c} \Rightarrow e^{\pi i(c',c-\bar{c})} = 1 \forall c' \in C' \).

The justification for the term ‘delta functional’ is obvious: Similar to the gamma integrator with \( \alpha = 1 \) characterizing a uniform functional on \( C \), the Dirac integrator characterizes a uniform functional on \( C' \) in the sense that \( \int_C e^{-2\pi i(c',c-\bar{c})} D\delta_\epsilon(c) = \text{Id} \) for all \( c' \in C' \).

To see that Dirac integrators are conjugate integrators, use eq. (B.18) \( \blacksquare \) in the context of sufficient statistics. To make this concrete we stipulate that

\[
G^{-1}_{xc} = D_{xc} : S_a(X_a) \to C' ; \quad G^{-1}_{xc} = D_{xc} : C \to S_a(X_a)' .
\]

Insofar as the sufficient statistics characterize ‘classical’ observables in a QM context, this means the constraints are only correlated with classical observables. Now taking the \( G_{xc} \to \infty \) limit gives

\[
e^{-\pi/s}[Q(c-m_{c|x}) - B(m_{c|x})] \to e^{-\pi/s}[\tilde{Q}(c-m_{c|x}) - B(m_{c|x})] \delta(c - m_{c|x})
\]

where \( \tilde{Q}(c_1, c_2) = \langle (G_{xc}D_{xx}G_{xc})^{-1}c_1, c_2 \rangle = \langle \tilde{G}^{-1}c_1, c_2 \rangle \) and \( m_{c|x} = \bar{c} + G_{xc}D_{xx}(x - \bar{x}) \).

Clearly, \( \tilde{Q}(c_1, c_2) = Q_X(x_1, x_2)|_{S_a(X_a)} \) because of (2.12) so the exponential can be viewed as a likelihood functional

\[
\Theta_{X|C}(S_a(x)|c, \cdot) = e^{-\pi/s}[Q_X(c-\bar{x}) - B(\bar{x})]
\]

where now \( \bar{x} := (D_{xx}G_{xc})^{-1}m_{c|x} \). It follows that

\[
\Theta_{C|X}(c|x, \cdot) \sim e^{-\pi/s}[Q_X(c-\bar{x}) - B(\bar{x})] \delta_\epsilon(c)
\]

and, hence,

\[
D\omega_{m_{c|x}, Q_X} (c|x) \propto e^{-\pi/s}[Q_X(c-\bar{x}) - B(\bar{x})] \delta_\epsilon(c) D(x, c) .
\]
Since the mean is altered but not the covariance, the pair \( (\mathcal{D}\omega_{m_{\epsilon|x,Q_{C|X}}}(c), \mathcal{D}\omega_{e,\infty}(c)) \) are conjugate integrators and \( \int_G \delta(c - \bar{c}) \mathcal{D}c \) enforces a delta functional constraint.

More generally, the Dirac integrator yields
\[
\int_G F_\mu(c) \mathcal{D}\delta_c(M(c)) = \sum_{c_0} \frac{F_\mu(c_0)}{\text{Det} M'_c(c_0)}
\]
with \( M : C \to C \) a diffeomorphism, \( M(c_0 - \bar{c}) = 0 \), and \( \text{Det} M'_c(c_0) \) non-vanishing and appropriately regularized. Nothing is altered if we allow general action functionals. That is, we can define Dirac integrators as the limit \( S(c) \to 0 \) for exponential-type integrators. The quintessential example of this type (which, however, is outside the scope of this article) is gauge fixing in quantum field theory. Assuming well-defined functional integrals for fields, a Dirac integrator on the space of gauge transformations \( G \) is just the Faddeev-Popov ‘trick’
\[
\int_G \mathcal{D}\delta_g(M(g)) = \sum_{g_0} \frac{1}{\text{Det} M'_{g_0}}
\]
where \( M : G \to G \) and \( M(g_0 - \bar{g})(a) = 0 \) with \( a \in A \). Here \( A \) is the space of connections on some principal bundle. Of course, an admissible gauge fixing condition requires a single \( g_0 \), and it is standard to average the delta functional over \( \bar{g} \) with respect to some (usually Gaussian) integrator.

The whole Faddeev-Popov procedure can be readily interpreted from a conditional integrator viewpoint (eq. (3.14)). Formally,
\[
\int_{\tilde{A}} F(\tilde{a}) e^{\text{S}(\tilde{a})} \mathcal{D}(\tilde{a}) = \int_{A \times G} \left[ F(a) \text{Det} (M'_{g_0}(a)) \delta(M(g_0(a))) \mathcal{D}(g) \right] e^{\text{S}(a)} \mathcal{D}(a)
\]
\[
= \int_{A} \tilde{G}(a) e^{\text{S}(a)} \mathcal{D}(a).
\]
where \( \tilde{G}(a) \) absorbs the gauge group volume.

3 Examples revisited

3.1 Fixed end-points

Return to the free QM point-to-point propagator in \( \mathbb{R}^n \) discussed in §2. The constraint is clearly gamma-type. Define
\[
i\langle c'(x), c \rangle := 2\pi i \int_{t_a}^{t_b} |x(t) - \bar{x}(t)| c(t) \, dt
\]
where $c(t) \in \mathbb{R}$. Impose the constraint $\delta(x(t_b) - x_b)$ by choosing the sufficient statistics so that

\[-i\langle S_s(c'(x)), c \rangle := 2\pi i \int_{t_a}^{t_b} |x(t_b) - \bar{x}(t_b)| c(t) \, dt\]
\[= 2\pi i |x(t_b) - \bar{x}(t_b)| \int_{t_a}^{t_b} c(t) \, dt\]
\[= 2\pi i |x(t_b) - x_b| \bar{c} \equiv 0 \; \forall c \in C. \quad (3.2)\]

Being a Banach space, the vanishing of $|x(t_b) - \bar{x}(t_b)|$ ensures coincidence of the endpoints $x(t_b) = \bar{x}(t_b)$.

We have $\bar{x}(t_b) = x_a + \dot{x}_b(t_b - t_a) = x_b$ in the third line because the mean (actually critical in this case) path in $X_a$ is given by

$$\bar{x}(t) = \frac{x_a(t_b - t)}{(t_b - t_a)} + \dot{x}_b(t - t_a). \quad (3.3)$$

On the other hand, the space of point-to-point paths $X_{a,b}$ has boundary conditions $\bar{x}(t_a) = x_a$ and $\bar{x}(t_b) = x_b$. That is, the mean path is alternatively parametrized as

$$\bar{x}(t) = \frac{x_a(t_b - t) + x_b(t - t_a)}{(t_b - t_a)}. \quad (3.4)$$

Clearly, the integral in Definition 2.1 reduces to a simple delta function $\delta(|x(t_b) - x_b|)$ on $\mathbb{R}$ for this choice of sufficient statistics since the integrand is a function of $\bar{c}$, and we can write

$$\int_{X_{a,b}} D\omega_{\bar{x},Q(a,b)}(x) := \int_{X_a \times C} e^{-i\langle S_s(c'(x)), c \rangle} D_{1,0,\infty}(c) D\omega_{\bar{x},Q}(x)$$
\[= \int_{X_a \times \mathbb{R}} e^{2\pi i |x(t_b) - x_b|} \bar{c} \; D\omega_{\bar{x},Q}(x)\]
\[= \int_{X_a} \delta(x(t_b), x_b) \; D\omega_{\bar{x},Q}(x) \quad (3.5)\]

where the right-hand side is to be interpreted as the integral of a conditional integrator on $X_a \times C$. 

7
Now, instead of solving the constraint first, we will do the integral over $X_a$ first:

$$
\int_{X_{a,b}} \mathcal{D} \omega_{x,Q(a,b)}(x) = \int_{X_a \times C} c e^{-i(S_c(c'(x)),c)} \mathcal{D} c \mathcal{D} \omega_{x,Q}(x) \\
= \int_X \int_{X_a} c e^{2\pi i \langle \langle \delta_{t_b} \bar{x} \rangle \rangle, c} \mathcal{D} \omega_{x,Q}(x) \mathcal{D} c \\
= \int_c \left[ \int_{X_0} e^{2\pi i \langle [\langle \delta_{t_b} \bar{x} \rangle \rangle, c} \mathcal{D} \omega_{0,Q}(\bar{x}) \right] \mathcal{D} c \\
= e^{\pi i B(\bar{x})} \sqrt{\text{det} [i G(t_b,t_b)]} \int_c e^{\pi i \langle [W(\delta_{t_b}), c} \mathcal{D} c \\
= \frac{e^{\pi i B(\bar{x})}}{\sqrt{\text{det} [i G(t_b,t_b)]}} \\
= e^{\pi i B(\bar{x})} \sqrt{\text{det} [i G(t_b,t_b)]} \right) \mathcal{D} c (3.6)
$$

where the second line uses functional Fubini, the third line follows using results from Appendix B [1] to shift the integration variable, and the fourth line follows because $\text{Det}(i W(\delta_{t_b})) = \text{det} [i G(t_b,t_b)] \neq 0$.

It should be emphasized that $G(t_b,t_b)$ is the covariance matrix associated to paths with $x(t_a) = 0$, and similarly for the boundary form $B(\bar{x})$. The boundary form, of course, evaluates to $B(\bar{x}) = \dot{x}_b^2(t_b - t_a)$ which becomes $B(\bar{x}) = (x_b - x_a)^2 / (t_b - t_a)$ when expressed in terms of $\bar{x} \in X_{a,b}$.

In the presence of boundary conditions or for less trivial geometry, there will be more than one critical path with the appropriate boundary conditions. In that case, it follows from eq. (B.7) [1] that

$$
K_Q(x_a, x_b) = \sum_{\bar{x}} \frac{e^{\pi i B(\bar{x})}}{\sqrt{\text{det} [i G(t_b,t_b)]}}. (3.7)
$$

The subscript Q has been included here to emphasize that the propagator is determined by a covariance associated with specific boundary conditions, and it is a sum over $\bar{x}$ of all Gaussian integrators with $\bar{x}$ having the appropriate boundary conditions. This will

---

6 Although this calculation has been done many times by time slicing, semi-classical, or linear mapping techniques; the point of repeating it here is to demonstrate that the as-defined functional integral tools allow the calculation to be carried out entirely at the level of the function space — the target manifold only makes an appearance through the boundary conditions imposed on the mean and covariance and the regularization/normalization of the functional determinant.

7 Significantly, we do not have to expand about the critical path to do the calculation. Including the mean and boundary form in the definition of the integrator automatically handles this for us. But it does more. It tells us that, when $Q \rightarrow S$ (a non-quadratic action), the ‘sufficient statistic’ of import is not the critical path but the mean path. So, for example, the semi-classical expansion in terms of the mean path automatically accounts for self-interactions. In other words, once the mean is known, the Feynman diagram procedure (now without loop diagrams) is a way to estimate the moments of the integrator $\mathcal{D} \omega_{x,S}$. This is the essence of the effective action approach in quantum field theory. Of course, the catch is $\bar{x}$ is hard to find for generic $S$. 

---
be a recurring theme: propagators are represented by a sum over relevant parameters of an integrator family.

The free QM point-to-point propagator is a specialization of the more general integral

$$K_S(x_a, x_b) := \int_{X_a} \delta_{S_a}(c'(x)) \mathcal{D} \omega_{x,S}(x) := \sum_x \int_{X_{x,a}} \delta_{S_x}(c'(x)) \mathcal{D} \omega_{x,S}(x)$$

(3.8)

where $\mathcal{D} \omega$ is characterized by an action functional $S$ that is generically not quadratic, and the mean paths satisfy associated boundary conditions enforced by sufficient statistics on $C'$. It is to be understood as an integral over a conditional integrator on $X_a \times C$.

An explicit example of this type of integral comes from the fixed-energy propagator

$$G_{ps}(q_b, q_a; E) = \int_{(Q,P)_{a,b}} \delta(H(q, p) - E) \mathcal{D} \omega^{(a,b)}(q, p)$$

(3.9)

where $\mathcal{D} \omega^{(a,b)}(q, p)$ is an appropriately defined point-to-point integrator on some phase space and $H(q, p) := \int_{t_a}^{t_b} h(q(t), p(t)) dt$.

According to the functional Fubini, the order of integration can be interchanged for product integrators (the analog of independent joint distributions). Then, as we saw in the example, since the gamma family is conjugate for Gaussian likelihood functionals, the integral over $X_a$ will yield another gamma integrator — in which case the integral with respect to $\mathcal{D}c$ is well defined. Explicitly,

$$\int_{X_a} \delta_{S_a}(c'(x)) \mathcal{D} \omega_{x,Q}(x) = \int_{B_a} \mathcal{D} \omega_{x,Q}(x) \mathcal{D} \gamma_{1,i(c'(x)),\infty}(c)$$

$$= \int_{C} \langle \exp\{-i\langle S_a(c'(x)), c \rangle\} \rangle_{\omega_{x,Q}} \mathcal{D} \gamma_{1,0,\infty}(c)$$

(3.10)

and the Gaussian expectation of $\exp\{-i\langle S_a(c'(x)), c \rangle\}$ must lie in the family of gamma integrators. Likewise, $\langle \exp\{-\pi/s Q(x)\} \rangle_{\gamma_{1,i(c'(x)),\infty}}$ must lie in the family of Gaussian integrators — which brings us to the next subsection where the constraint is Gaussian-type.

3.2 Quotient spaces

Let $M$ be a Riemannian manifold without boundary and $\pi_G : P \to M$ a principal fiber bundle endowed with a connection. We wish to define the functional integral

$$\int_{M_a} F_\mu(m) Dm$$

where $M_a \ni m : [t_a, t_b] \to M$. The problem is that the base space is complicated in general: It may be very difficult or impossible to directly define an integrator on $M$.

On the other hand, the covering space is usually easier to handle, and we assume that we can define an integrator so that the integral $\int_{P_a} F_\mu(p) Dp$ is well-defined. We also assume that $F_\mu(p)(t)$ furnishes a representation of $\mathbb{G}$ and is equivariant so that

$$F_\mu(p \cdot g)(t) = \rho(g^{-1})F_\mu(p)(t)$$

(3.11)
where \( \rho \) is a possibly non-faithful representation of \( G \).

Now, in the CDM scheme expressions like \( p(t) \) are shorthand for a parametrized curve: \( p(b)(t) = p_a \cdot \Sigma(b, t) \) where \( b \in B_a \) and \( \Sigma(t, b) : \mathbb{P} \to \mathbb{P} \) is a global transformation such that \( p_a \cdot \Sigma(t_a, \cdot) = p_a \). This parametrization allows integrals over the generally non-Banach space \( P_a \) to be expressed as well-defined integrals over \( B_a \).

The first point to make is that the parametrization for quotient spaces is gleaned from the local structure of the bundle \( U_i \times G \) where \( U_i \subset M \) and a local trivialization is given so that \( p(b)(t) = (m(b), g(b))(t) \). The parametrization for the first component is dictated by the manifold structure of \( M \). The parametrization for the second component is fixed by requiring parallel transport of \( p(t) \) — since this will restrict paths to \( M \) if that is where they start. Consider an open set \( U_i \subset M \), and let \( A_i \) denote the local gauge potential relative to the canonical local section \( s_i : \mathbb{M} \to \mathbb{P} \). The equation for parallel transport,

\[
dg_i(t) = -g_i(t)A_i(\dot{m}(t))dt
\]

clearly indicates the conditional relation between \( m \) and \( g \), and it emphasizes the interplay between constraints and conditionals.

To attack this problem we need to be more explicit about the parametrization of \( P_a \).

**Definition 3.1** Let \( \{ \omega_i = 0 \} \) where \( \omega_i \in \Lambda T^*P_a \) and \( i \in \{1, \ldots, r\} \) be an exterior differential system with integral manifold \((B, P_a)\). This system defines a parametrization \( \mathcal{P} : B \to P_a \) by

\[
\mathcal{P}^*\omega_i = 0 \forall i.
\]

In particular, if \( B = X \times Y \) and \( i = 2 \), the parametrization can be written locally on \( \mathbb{P} \) as

\[
\begin{cases}
  dm(a)(t) = X_{(a)}(p(t))dx(t)^a \\
  dg(b)(t) = Y_{(b)}(p(t))dy(t)^b
\end{cases}
\]

\[ m(t_a) = m_a \quad g(t_a) = g_a \quad (3.14) \]

where \( a \in \{1, \ldots, p_a\} \), \( b \in \{1, \ldots, p_b\} \), \( p_a + p_b \leq \dim \mathbb{P} \), and the set \( \{X_{(a)}, Y_{(b)}\} \) generates a vector sub-bundle \( \nabla \subseteq TP \) of the tangent bundle. The solution of \((3.14)\) will be denoted \( p(x, y)(t) = p_a \cdot \Sigma(t, x, y) \) where \( \Sigma(t, x, y) : \mathbb{P} \to \mathbb{P} \) is a global transformation on the covering space such that \( \Sigma(t_a, \cdot, \cdot) = \text{Id} \).

The parallel transport equation implies \( d\log(g(t)) \sim A \cdot dm(t) \) which implicitly encodes the constraint through \((3.14)\). This particular parametrization yields (within \( U_i \))

\[
p(x)(t) = (m(x), g(m(x)))(t) = (m_a \cdot \sigma(t, x), g_a \cdot T e^{-\int_{t_a}^{t} A \cdot \dd m dt})
\]

\[ (3.15) \]

where \( \sigma = \pi(\Sigma) \). On the other hand, since \( p \) is a horizontal lift it can be represented as

\[
p(x)(t) = s_i(m(x)(t)) \cdot g(m(x))(t).
\]

\[ (3.16) \]

It is clear that the restriction expressed by \((3.12)\) implies \( g(m(x))(t) \in G \forall \{t, x\} \), and \( p(x) : [t_a, t_b] \to \mathbb{P}(p_a) \) where \( \mathbb{H}_{(p_a)} \to \mathbb{P}(p_a) \to \mathbb{G} \) is the holonomy bundle.
We have learned that, given \( m(x) \) and a local trivialization, the functional constraint 
\[ \delta(M(g)) = \delta(\dot{g} - g \cdot \dot{m}) \]
will lead to a path confined to an open neighborhood of a section that is isomorphic to the base space. And since the constraint only shifts the path along fibers, we expect it to be represented by a delta functional on \( G_a \) which implies a Dirac integrator \( D\delta_{g_a}(M(g)) \) should be used. Moreover, \( M \) just effects translation on \( G_a \) so the functional determinant of \( M' \) is trivial and the zero locus coincides with the holonomy group \( \mathbb{H}(p_a) \).

So, for a Gaussian integrator,
\[
\int_{M_a} F_\mu(m) D\omega(m) := \int_{X_0} \int_{G_a} F_\mu(P(x, y))\delta(M(g)) \ Dg \ D\omega_{\mathbb{E},Q}(P(x)) \\
= \int_{\mathbb{E}(p_a)} \int_{X_0} F_\mu(P(x) \cdot h) \ D\omega_{\mathbb{E},Q}(P(x)) \ dh \\
= \int_{\mathbb{E}(p_a)} \int_{X_0} \rho(h^{-1})F_\mu(P(x)) \ D\omega_{\mathbb{E},Q}(P(x)) \ dh
\]
(3.17)
where \( P(x)(t) \in \mathbb{P}(p_a) \).

In particular, the point-to-point propagator on \( \mathbb{M} \) (in the \( \rho^r \) representation) obtains for the familiar choice \( F_\mu(P(x))(t_b) = \delta(p(x)(t_b), p_b) \):
\[
K^r(m_a, m_b) := \int_{M_a} \delta(m(t_b), m_b) \ D\omega(m) \\
= \int_{\mathbb{E}(p_a)} \rho^r(h^{-1}) \int_{X_0} \delta(p(x)(t_b), p_b) \ D\omega_{\mathbb{E},Q}(P(x)) \ dh \\
=: \int_{\mathbb{E}(p_a)} \rho^r(h^{-1})K_{h_b}(m_a, m_b) \ dh
\]
(3.18)
where the propagator \( K_{h_b}(m_a, m_b) \) is the point-to-point propagator on \( \mathbb{P}(p_a) \) — associated with the homotopy class of \( h_b \) — pulled back to \( \mathbb{M}_8 \).

In most dynamical systems of interest, the integrator \( D\omega_{\mathbb{E},Q}(P(x)) \) is invariant under the restricted holonomy group \( \mathbb{H}^0(p_a) \). Then \( K(p_a, p_b \cdot h^0) = K(p_a, p_b) \) where \( h^0 \in \mathbb{H}^0(p_a) \). In particular, for a one-dimensional representation, the propagator on the base space reduces to the well-known result (for \( \chi : G \to \mathbb{T} \subset \mathbb{C} \))
\[
K^{\Lambda_{p_a}}(m_a, m_b) = \sum_{g \in G} \chi^{\Lambda_{p_a}}(g) K_{[h_b]}(m_a, m_b)
\]
(3.19)
where \( \Lambda_{p_a} \) labels inequivalent one-dimensional unitary representations of the monodromy group \( G = \mathbb{H}(p_a)/\mathbb{H}^0(p_a) \) at the point \( p_a \) and the equivalence classes \([h_b]\) depend on \( g \in G \).

---

\( ^8 \)It is legitimate to write \( K_{h_b}(m_a, m_b) \) instead of \( K(p_a, p_b) = K(s_i(m_a), s_i(m_b) \cdot h_b) \) because the horizontal lifting does not depend on the trivialization and, hence, the particular canonical section \( s_i \). So we are free to choose the trivial section.
Using (3.7), we get what can be interpreted as a semi-classical trace formula for point-to-point transitions on $M$ in terms of the monodromy group

$$\sum_{\tilde{m}} \frac{e^{\pi i B(\tilde{m}(t_b))}}{\sqrt{\det [i \tilde{G}(t_b, t_b)]}} = \sum_{\Lambda_{m_a}} \sum_{g \in G} d_{\Lambda_{m_a}} \chi^{\Lambda_{m_a}}(g) K_{[t_b]}(m_a, m_b)$$

(3.20)

with $d_{\Lambda_{m_a}}$ the multiplicity of $\Lambda_{m_a}$.

### 3.3 Bounded configuration space

This type of system is interesting, because it requires a gamma integrator both for the non-dynamical degree of freedom $\tau : [t_a, t_b] \to \mathbb{R}_+$ associated with fixed boundaries and for the constraint that enforces the boundary conditions.

We wish to define an integrator for a space of pointed paths $M^0_a$ with $m : [\tau_a, \tau_b] \to M$ and $\partial M \neq \emptyset$ sufficiently regular. Experience suggests we consider a product manifold $N = M \times \mathbb{R}_+$ and take $B_a = X_a \times T_0 \times C$ to impose the required constraints. The task is to make sense of an integral of the form $\int_{M^0_a} F_\mu(m) Dm$.

The parametrization of $n := (m, \tau) : [\tau_a, \tau_b] \times [t_a, t_b] \to N$ can be written

$$\begin{cases}
    dm(x)(\tau) = X_{(a)}(m(x)(\tau)) dx(\tau) & m(x)(\tau_a) = m_a \\
    d\tau(t) = Y(\tau(t)) dt & \tau(t_a) = \tau_a
\end{cases} \quad \text{(3.21)}$$

To make the notation manageable, let $Y_a$ stand for $X_a \times T_0$. Then we can write simply $y \equiv (x, \tau)$. We also put $D\Omega_{g, Q}(y) := D\omega_{x, Q}(x) D\gamma_{0, 0, \infty}(\tau)$. Note that for the $D\Omega$ integrator it is consistent to use a gamma integrator to account for the boundary constraints since it is conjugate to $D\Omega$.

Define the integral by

$$\int_{M^0_a} F_\mu(m) Dm := \int_{Y_a \times C} F_\mu(n(y)|c) \Theta_Y|C((n(y)|c, \cdot) D\Theta_{Y|C} Z_{Y|C} n(y)|c$$

$$=: \int_{Y_a \times C} \tilde{F}_\mu(S_s(n(y)), c, \cdot) D\Omega_{g, Q}(n(y)) D\gamma_{0, 0, \infty}(c)$$

$$= \int_{C} \left\langle \tilde{F}_\mu(S_s(n(y)), c) \right\rangle_{g, Q} D\gamma_{0, 0, \infty}(c)$$

$$=: \int_{C} \tilde{H}_\mu(c) D\gamma_{0, 0, \infty}(c). \quad \text{(3.22)}$$

It remains to infer the nature of $\tilde{F}_\mu(S_s(n(y)), c)$ and the associated integrator parameters.

From the variational principle, we learned that the constraints impose transversality conditions on critical paths. We also learned that $\tau$ should be viewed as a non-dynamical

---

9. We claim that $D\gamma_{0, 0, \infty}(\tau)$ is the correct integrator to use on $T_0$ because $\tau(t) \in \mathbb{R}_+$ and $\tau$ is non-dynamical.

10. We won’t motivate this definition, but the reader is invited to attach interpretations and intuitions to the various representations presented in the definition.
degree of freedom that reparametrizes \( m(x) \). So the plan is to search for a conditional integrator for which

\[
\Theta_{C|X}(c|x, \cdot) \propto \Theta_{S_{s}(X)|C}(S_{s}(x), \cdot) \Theta_{C}(c, \cdot)
\]  
(3.23)

where \( S_{s}(X) \) is determined by mean paths. We replace critical with mean paths for the sufficient statistic because the quantum analog of the variational principle is

\[
\frac{\delta \Gamma(x)}{\delta x(t)} \sim x'(t)
\]  
(3.24)

with \( \Gamma \) defined in eq. (B.10) \[1\]. Furthermore, we restrict to the case \( \tau(t) \) real.

Having identified the relevant sufficient statistics for the paths, we now switch solution strategies and work instead with \( \bar{G}_{\mu}(S_{s}(c), n(y)) \). To simplify matters, only the two limiting cases of transversal intersection and fixed energy discussed in §2 \[1\] will be considered. Recall that these cases correspond to point-to-boundary and point-to-point paths respectively. Let \( \bar{n} = (\bar{m}, \bar{\tau}) \) represent a mean path. Define the “first exit time” \( \tau_{0} \) implicitly by \( \bar{n}(t_{b}) = (\bar{m}(\tau_{0}), \bar{\tau}(t_{b})) \) such that \( \bar{m}(\tau_{0}) \in \partial \mathcal{M} \) where \( \bar{\tau}(t_{b}) := \tau_{0} \) characterizes the mean gamma process. Recall the mean path \( \bar{n} \) is also critical since \( Q \) is quadratic, and there may be more than one critical path.

The functional integral for a functional \( F_{\mu}^{\theta} \) that takes its values on \( \partial \mathcal{M} \) is defined by

\[
\Phi^{\theta}(m_{a}) := \int_{Y_{a}} \langle \tilde{F}_{\mu}^{\theta}(m(x)) \rangle_{\gamma_{1, \varsigma'(\tau)} \infty} \mathcal{D} \Omega_{\tilde{q}, q}(n(y))
\]  
(3.25)

where

\[
\langle \tilde{F}_{\mu}^{\theta}(m(x)) \rangle_{\gamma_{1, \varsigma'(\tau)} \infty} := \int_{\tau_{0} \times C} \tilde{F}_{\mu}^{\theta}(m(x)(\tau(t_{b}))) \mathcal{D} \gamma_{1, \varsigma'(\tau)} \mathcal{D} \gamma_{1, \varsigma'(\tau)} \infty(c)
\]  
(3.26)

with \( i(S_{s}(c'(\tau), c) = -2\pi i[\tau(t_{b}) - \bar{\tau}(t_{b})] \cdot \bar{e} \) and \( \bar{\tau}(t_{b}) = \tau_{0} \).

As we have seen before, the constraint is just a delta functional on \( C' \) and the integral reduces; (restoring \( x \to (x, \tau) \) for clarity)

\[
\Phi^{\theta}(m_{a}) = \int_{X_{a} \times T_{0}} \langle \tilde{F}_{\mu}^{\theta}(m(x)) \rangle_{\gamma_{1, \varsigma'(\tau)} \infty} \mathcal{D} \omega_{\bar{m}, \bar{q}}(x) \mathcal{D} \gamma_{0, 0, \infty}(\tau)
\]  
\[
= \int_{X_{a} \times \mathbb{R}^{+}} \tilde{F}_{\mu}^{\theta}(m(x)(\tau_{0})) \mathcal{D} \omega_{\bar{m}, \bar{q}}(x) d(ln(\tau_{0}))
\]  
\[
= \int_{\mathbb{R}^{+}} \langle \tilde{F}_{\mu}^{\theta}(m(x)(\tau_{0})) \rangle_{\bar{m}, \bar{q}} d(ln(\tau_{0}))
\]  
(3.27)

where \( \bar{Q} := Q \circ m \).

Similarly, for a functional \( F_{\mu}^{\gamma} \) that takes its values in \( \mathcal{M} \backslash \partial \mathcal{M} \),

\[
\Phi^{\gamma}(m_{a}) := \int_{Y_{a}} \langle \tilde{F}_{\mu}^{\gamma}(m(x)) \rangle_{\gamma_{0, \varsigma'(\tau)} \infty} \mathcal{D} \Omega_{\tilde{q}, q}(n(y))
\]  
(3.28)

\[1\]The same idea was implicit in the quotient space analysis. There, \( \Theta_{X|G} \) was determined by \[3.12\].
where

$$\langle \tilde{F}_\mu^\vartheta (m(x)) \rangle_{\gamma_0,\vartheta(\tau),\infty} := \int_{T_0 \times C} \tilde{F}_\mu^\vartheta (m(x)(\tau_0)) \mathcal{D}\gamma_1, \mathcal{D}\gamma_{\gamma_0,\vartheta(\tau),\infty}(c)$$

$$= \int_{\mathbb{R}^+} \theta(\tau_o - \tau_b) \tilde{F}_\mu^\vartheta (m(x)(\tau_0)) \, d\tau_b \quad (3.29)$$

Note the step functional constraint in this case.

These definitions also hold for $Q \rightarrow S$, but the mean paths are no longer critical. Of course, the suitability of these definitions rests on their ability to reproduce known results (see [2] for some examples).

Like the quotient space example, the two integrals simplify for propagators. For the point-to-boundary propagator$^{[2]}$,

$$K_\vartheta(m_a, m_B) = \mathcal{N}(m_a) \int_{Y_o} \delta(m(x)(\tau_0), m_B) \mathcal{D}\gamma_{\vartheta, Q}(n(y))$$

$$= \mathcal{N}(m_a) \sum_{m(\tau_0)} \int_{\mathbb{R}^+} \frac{\pi B(\hat{m}(\tau_0))}{\sqrt{\text{det} \, G(\tau_o, \tau_b)}} \, d(\ln \tau_o) \quad (3.30)$$

where $m_B \in \partial M$ denotes an end-point on the boundary and the normalization $\mathcal{N}(m_a)$ enforces $\int_B K_\vartheta(m_a, m_B) \, dm_B = 1$. For example, if $M \subset \mathbb{R}^n$, then the boundary term goes like $B(\hat{m}(\tau_0)) \sim |m_B - m_a|^2/\tau_o$. If there is more than one critical (or mean) path, care must be taken to split the integral over the boundary into regions associated with a particular critical path.

For the point-to-point propagator, it is convenient to first integrate with respect to $\mathcal{D}\omega$;

$$K(m_a, m_b) = \int_{Y_a \times C} \langle \delta(m(x), m_b) \rangle_{\gamma_0, \vartheta(\tau), \infty} \mathcal{D}\gamma_{\vartheta, \omega}(n(y))$$

$$= \sum_{m_E} \int_{\tau_0} \int_{\mathbb{R}^+} \theta(\tau_o - \tau_b) \frac{e^{\pi B(\hat{m}(\tau))}}{\sqrt{\text{det} \, G(\tau_o, \tau_b)}} \, d\tau_b \mathcal{D}\gamma_{0, \infty}(\tau) \quad (3.31)$$

where $m_b := m(\tau_b)$. The theta function makes this propagator fairly difficult to handle since $\tau_o$ depends on $m_a$. One way around the difficulty is to write the inner integral as

$$\int_{\mathbb{R}^+} \theta(\tau_o - \tau_b) \frac{e^{\pi B(\hat{m}(\tau))}}{\sqrt{\text{det} \, G(\tau_o, \tau_b)}} \, d\tau_b = \left( \int_{\mathbb{R}^+} - \int_{\tau_o}^{\infty} \right) \frac{e^{\pi B(\hat{m}(\tau))}}{\sqrt{\text{det} \, G(\tau_o, \tau_b)}} \, d\tau_b \quad (3.32)$$

and then find a point transformation on $M$ that takes $m_a$ to the boundary (see [2]).$^{[3]}$

$^{[2]}$The need for the normalization constant $\mathcal{N}(m_a)$ can be established from dimensional analysis.

$^{[3]}$The point transformation then implies $\tau_o \rightarrow 0$ and $m^E$ is transformed accordingly in the second integral on the right. Since there is no longer any $\tau_o$ dependence in this case, the outer integral in (3.31) just contributes its normalization (which we have set by the choice $\int_{T_0} \mathcal{D}\tau = 1$). The same thing happens for the unbounded case when $\tau_o \rightarrow \infty$ in the theta function.
These point-to-point and point-to-boundary propagators are *restricted* in the sense that the paths are not allowed to penetrate the boundary. However, there are cases of interest when the boundary represents a discontinuity, and the paths are defined on both sides of the boundary.

### 3.4 Segmented configuration space

Often the target space of paths will have codimension-1 submanifolds that induce a decomposition of path space \( \bigoplus_i X_a^{(i)} = X_a \) in the sense that each \( X_a^{(i)} \) has its own integrator. Such is the case, for example, when a Gaussian integrator is defined in terms of an action functional and the potential in the action is discontinuous. More explicitly, \( x_a^{(i)} \in X_a^{(i)} \) is the pointed path \( x_a^{(i)} : [t_a, t_b] \rightarrow (M^{(i)}_a, m_a^{(i)}) \) where \( M = \bigcup M^{(i)} \) such that the intersection \( M^{(i)} \cap M^{(j)} = \partial M^{(ij)} \) is a submanifold of codimension-1.

The objects of interest in this case are the propagators from the previous subsection. Since it is known that the propagators are kernels for certain differential operators, Green’s theorem provides a convenient starting point for the analysis.

For an operator \( L \) defined in a bounded open region \( U \subset M \) acting on complex scalar functions from an appropriate function space,

\[
\int_U (L\phi) \bar{\varphi} - \int_U \phi (L^* \varphi) = \int_{\partial U} B(\phi, \varphi) \tag{3.33}
\]

where \( L^* \) is the formal adjoint of \( L \), and the functional form of \( B \) is determined from Stoke’s theorem and the particular boundary conditions associated with the function space.

In particular, let \( U_1 = U^{(1)} \cup U^{(2)} \) be a bounded open region in \( \mathbb{R}^3 \) with one surface \( S = U^{(1)} \cap U^{(2)} \) of discontinuity. Choose \( \varphi \) to be the Green’s function of \( \nabla^* \) in \( U_1 \) with vanishing Dirichlet boundary conditions on \( \partial U_1 \) and \( \phi \) the Green’s function of \( \nabla \) in \( U^{(1)} \) with vanishing Dirichlet conditions on \( S \). Then the theorem gives the Green’s function \( \varphi \) of \( \nabla \) in \( U^{(1)} \) with non-vanishing boundary conditions on \( S \)

\[
\varphi = \phi + \int_S \varphi \nabla \phi \cdot d\sigma \tag{3.34}
\]

and in \( U_1 \setminus U^{(1)} \)

\[
\varphi = \int_S \varphi \nabla \phi \cdot d\sigma \tag{3.35}
\]

This theorem has a simple interpretation in terms of functional integral representations of propagators: For bounded regions that allow paths to penetrate the boundary, the total point-to-point propagator includes the *restricted* point-to-point propagator (which does not allow paths to leave the region) plus the potential on the bounding surface induced by all sources accessible to paths that are allowed to leave the region. This prescription is equivalent to the “path decomposition technique” used in [3]–[5].

In other words, Green’s theorem can be used to partition the space of paths taking their values in a segmented configuration space into restricted and unrestricted sets. This
is useful because the paths in the partitioned sets have particularly convenient boundary conditions and their associated propagators are relatively easy to calculate.

Let $K_\U^{(D)}_{i(j)}$ be the restricted point-to-point propagators with Dirichlet boundary conditions on $S$, and $K_\partial(i)^{(D)}$ the restricted point-to-boundary homogenous propagators for their respective regions $\U^{(i)}$. These are the propagators derived in the previous subsection. According to Green’s theorem, the *unrestricted* point-to-point propagator from $m_a^{(i)}$ to $m_b^{(j)}$ in $\U_1$ with one surface of discontinuity can be written

$$K_\U^{(D)}_{i,j}(m_a^{(i)}, m_b^{(j)}) := \delta_{ij}K_\U^{(D)}_{i}\left(m_a^{(i)}, m_b^{(j)}\right) + \int_S K_\partial(i)^{(D)}(m_a^{(i)}, \sigma)K_\U^{(D)}_{j}(\sigma, m_b^{(j)}) d\sigma \quad (3.36)$$

where $m^{(i)} \in \U^{(i)}$ and $\sigma \in S$. Intuitively, the unrestricted point-to-point propagator within a bounded region $\U^{(i)}$ is implicitly determined by the restricted point-to-point and point-to-boundary propagators in $\U^{(i)}$. Similarly, the point-to-point propagator from $\U^{(i)}$ to $\U^{(j)}$ is implicitly determined by the restricted point-to-boundary propagator in $\U^{(i)}$. Note that $K_\U^{(D)}_{i}$ has non-trivial boundary conditions on $S$, but it still satisfies Dirichlet boundary conditions on $\partial\U_1$.

Equation $(3.36)$ is a familiar expression, and it is often solved by iteration. However, the path decomposition idea along with the fact that $K_\U^{(D)}_{i}$ *inside the integral is evaluated on the surface*, suggests that we replace $K_\U^{(D)}_{i}$ (inside the integral) with $\tilde{K}_\U^{(D)}_{i}$ defined by

$$\tilde{K}_\U^{(D)}_{i}(\sigma, m_b^{(j)}):=\left\{egin{array}{ll}
r_{(-i)}(\sigma)K_\U^{(D)}_{i}(m_a^{(i)}, m_b^{(j)})|_{m_a^{(i)}=\sigma} & \text{if } i=j \\
t_{(-j)}(\sigma)K_\U^{(D)}_{j}(m_a^{(i)}, m_b^{(j)})|_{m_a^{(i)}=\sigma} & \text{if } i \neq j
\end{array}\right.. \quad (3.37)$$

where $K_\U^{(D)}_{i}$ is the unrestricted propagator evaluated in $\U^{(j)}$, and

$$r_{(-i)}(\sigma) = \int_S K_\U^{(D)}_{i\rightarrow j}(\sigma, \sigma')K_\partial(i)^{(D)}(\sigma', \sigma) \, d\sigma'$$
$$t_{(-j)}(\sigma) = \int_S K_\U^{(D)}_{i\rightarrow j}(\sigma, \sigma')K_\partial(j)^{(D)}(\sigma', \sigma) \, d\sigma' \quad (3.38)$$

are evaluated in the region on the other side $(-j)$ of $m_b^{(j)}$. According to the path decomposition picture, $r(\sigma)$ and $t(\sigma)$ measure the (probability amplitude) contribution of pointed loops based at $\sigma$ that lie on either side of $S$ and so $|r|^2 + |t|^2 = 1$.

Usually it is much simpler to find $\tilde{K}_\U^{(D)}_{i}$ than to iterate $(3.36)$. For example, consider the standard elementary text book example of the fixed energy propagator in $\mathbb{R}$ with a step potential $V(x) = V_0 \theta(x)$. Let $k_0$ be the wave vector to the left and $k_{V_0}$ to the right of $x = 0$ respectively. Then it is easy to see from the surface integral that $t \sim 2\sqrt{k_0 k_{V_0}}/(k_0 + k_{V_0})$ since this must be symmetric under $k_0 \leftrightarrow k_{V_0}$; and, hence, $r \sim (k_0 - k_{V_0})/(k_0 + k_{V_0})$ from the normalization condition.

Now let $\U$ be divided into three regions. We can use the results for a single surface of discontinuity by covering $\U$ with two overlapping copies of $\U_1$. That is, each region

---

14*This argument is admittedly highly heuristic.*
contains only one surface of discontinuity. There are now six relevant propagators with appropriate boundary conditions. Their nature depends on whether or not the boundaries intersect \( \partial \mathbb{U} \). Since we require Dirichlet boundary conditions for point-to-point transitions and \( K_{U_1}^{(D)} \) can only propagate across a single discontinuity, care must be taken to use the appropriate \( U_1 \) for any given transition.

To simplify, specialize to a planar geometry, and order the regions \( \{1, 2, 3\} \). There will be two classes of propagators; half-space-type in regions 1 and 3 defined by \( U_1 \cap U_3 \), and unit-strip-type in region 2 defined by \( U_1 \cap U_2 \). Choose the partition \( U = U_1 \cup U_2 \) so that \( U_1 \equiv U_1^{(j)} \) contains the initial point \( x_a^{(j)} \). By combining the single surface result appropriately, the approximate point-to-point propagator for two surfaces of discontinuity can be written

\[
K_{U_2}^{(D)}(x_a^{(i)}, x_a^{(j)}) \simeq \delta_{ij} K_{U_1}^{(D)}(x_a^{(i)}, x_a^{(j)}) + \int_{S_2} K_{\partial(j)}^{(D)}(x_a^{(i)}, \sigma^{(j)}) \tilde{K}_{U_1}^{(D)}(\sigma^{(j)}, x_a^{(j)}) d\sigma^{(j)}
\]

(3.39)

where \( K_{U_1}^{(D)} \) is the restricted point-to-point propagator and \( K_{\partial(j)}^{(D)} \) the restricted point-to-boundary propagator derived from \( (3.36) \).

It is important to remember that the propagator depends on all critical paths. For regions bounded by two planes, there is obviously a sum over all ‘bounces’ within the bounded region\(^\text{15}\). These bounce transitions are encoded in the \( K_{\partial(j)}^{(D)}(x_a^{(i)}, \sigma^{(j)}) \) propagator. Hence, poles of the convolution of relevant bounce propagators yield spectral information for their corresponding regions.

In general, then, the approximate point-to-point propagator for \( U = \bigcup_{i=1}^{n+1} U^{(i)} \) is determined recursively from \( K_{U_0}^{(D)} \), which is the standard unrestricted point-to-point elementary kernel in the region \( U^{(j)} \) with vanishing Dirichlet boundary conditions on \( \partial \mathbb{U} \), and

\[
K_{U_n}^{(D)}(x_a^{(i)}, x_a^{(j)}) \simeq \delta_{ij} K_{U_1}^{(D)}(x_a^{(i)}, x_a^{(j)}) + \int_{S_{n-1}} K_{\partial(j)}^{(D)}(x_a^{(i)}, \sigma^{(j)}) \tilde{K}_{U_{n-1}}^{(D)}(\sigma^{(j)}, x_a^{(j)}) d\sigma^{(j)}
\]

(3.40)

where the region \( U_{n-1} \) is chosen to contain the initial point. This represents an alternative to the iteration approximation that, in particular, may be useful in numerical applications.

There are special cases when the iteration of \( (3.36) \) can be summed explicitly. The method (see e.g. [6]) essentially boils down to finding a Poisson integrator that is valid everywhere in \( X_a \). To see this, let \( S(x) = Q(x) + V(x) \) describe some general (action) functional, and suppose the kernel \( K_Q \) for the time-independent Schrödinger operator has been found in each \( U^{(i)} \). Define \( V_Q = K_Q \circ V \circ K_Q \) by its evaluation on

\(^{15}\)To the extent that the boundaries are exactly parallel and/or the planes extend to infinity, this gives an infinite sum which can be written analytically in the usual way as an inverse propagator.
ordered graphs in $\mathbb{M}$, i.e. under the linear maps $L_n : T_0 \rightarrow i\mathbb{R}^n$ we have $m(x)(\tau) \mapsto (m(x(\tau_1)), \ldots, m(x(\tau_n))) =: (m_1, \ldots, m_n) \in \mathbb{M}^n$. Then

$$V_Q(m(x(\tau))) \mapsto K_Q(m_b, m_k)K_Q(m_k, m_{k-1}), \ldots, V(m_1)K_Q(m_1, m_a)$$

(3.41)

where $m_a \leq m_1 < \ldots < m_b$ are the time-ordered graph nodes. A time-slicing analysis [6] when $\mathbb{M} = \mathbb{R}^m$ shows the kernel has the form

$$K_S(m_a, m_b) = \langle m_a | \langle \mathcal{V}_Q \rangle (t_b - t_a) | m_b \rangle =: \langle m_a | K_S((t_b - t_a)) | m_b \rangle$$

(3.42)

where $\langle \cdot \rangle_c$ is the Poisson expectation defined in subsection B.0.4 [1]. But the time-slicing analysis is only straightforward in $\mathbb{R}^m$; otherwise there are well-known pitfalls.

However, the target manifold independence of the right-hand side of (3.42) suggests that the definition is correct for any manifold $\mathbb{M}$. The point is, the right-hand side is more general than a perturbation expansion: It is defined in the function space rather than on the target manifold so it offers potentially new insight and calculational techniques. Consequently, an obvious proposal is to represent $K_S(c)$ by

$$K_S(c) \sim \int_C \left[ \int_{T_0} \mathcal{D}\gamma_{\alpha, S, c} (\tau) \right] d\alpha$$

(3.43)

for an appropriate contour $C \subset \mathbb{C}$. This is a substantial generalization of the Poisson expectation because now the target space is $\mathbb{C}_+$ rather than $i\mathbb{R}$. Not only do we get phase information not carried by the Poisson integrators, but we don’t have to restrict to $\langle \beta', \tau \rangle = \lambda(\text{Id}', \tau)$.

Here again is the thematic idea connecting propagators to summation over an integrator family. According to the proposal, the analytic structure of the propagator could be encoded in the integral defined by $K_S(c; \alpha) := \int_{T_0} \mathcal{D}\gamma_{\alpha, S, c} (\tau)$, and $K_S(m_a, m_b)$ would be determined by the residues of $K_S(c; \alpha)$.

### 4 Prime examples

So far we have used the proposed formalism to re-derive more-or-less known results using new tools. The goal in this section is to derive something new; functional integral representations of the average prime counting function and the average twin prime counting function. We formulate the counting functions in the spirit of quantum mechanical expectation values in the sense that they will represent the sum over ‘paths’ with certain attributes. Specifically, the paths are conjectured to follow gamma rather than Gaussian statistics, and restricting to prime events/numbers imposes a non-homogeneous scaling factor.

To see how to proceed, let’s calculate the expected number of integers occurring up to some cut-off $x \in \mathbb{R}_+$ by defining a suitable $\alpha$-trace applied to the simple case of an homogenous process. That is, we take $\beta' = \text{Id}'$ in the lower gamma integral and impose

\[16\text{Of course, } K_S(m_a, m_b) \text{ generally will also have singularities associated with its spectrum.}\]
the sufficient statistic associated with a cut-off by restricting the domain of paths via the linear map $L: \tau \to \tau(t_b)$. In this case, the functional integral can be explicitly evaluated and we get

$$N(x) := \text{tr}_\alpha \int_{T_0} (-1)^\alpha D_{\gamma_{\alpha,Id',x}}(\tau) = \int_{\mathcal{C}} \frac{\Gamma(1 - \alpha)}{2\pi i} (-1)^\alpha (1 - \alpha) d\alpha = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{(n-1)!} \gamma(n, x) = \sum_{n=1}^{\infty} P(n, x) = \Gamma(1, -\log(x)) = x$$

(4.1)

where the contour encircles the positive real axis. The result lends credence to the choice of $\text{tr}_\alpha$.

### 4.1 Prime counting

Postulate that the prime counting function is the expectation of a gamma process with unknown scale parameter due to the constraint associated with counting only primes. The functional integral that enforces the constraint must be a gamma integral because the conjugate prior of a gamma distribution with unknown scaling parameter is again a gamma distribution. Therefore, according to the general construction, the constrained functional integral that represents the expectation value can be written as a constrained functional that is integrable with respect to two marginal gamma integrators. Analogous to the QM point-to-point free propagator example, we put $\langle c'(\tau), c \rangle = \langle (\tau - \lambda(\tau)), c \rangle$. Then let us define the average number of primes up to some cut-off integer $x$ to be

$$\overline{\pi_1(x)} := \text{tr}_\alpha \int_{T_0} (-1)^\alpha D_{\gamma_{\alpha,Id',\bar{x}(\tau)}}(\bar{\tau}) = \text{tr}_\alpha \int_{T_0 \times C} (-1)^\alpha D_{\gamma_{\alpha,Id',x}(\tau)} D_{\gamma_{1,\bar{c}'(\tau),\infty}(c)} = \text{tr}_\alpha \int_{T_0} (-1)^\alpha D_{\gamma_{\alpha,Id',\lambda(x)}(\tau)}$$

(4.2)

where $\langle S_s(c'(\tau)), c \rangle = (x - \lambda(x)) \cdot \overline{c}$ such that $\lambda(x)$ represents an unknown possibly non-homogenous scaling factor, and the $\alpha$-trace is defined below.\footnote{Contrary to what was done in §3.3 here we do not integrate over $x$ because it is obviously fixed.} Note that the constraint imposes the non-homogeneous scaling factor on the cut-off.

We are counting primes so the counting should begin with the second event (since primes start with $p = 2$). Recalling the counting of integers from the previous section
suggests the proposal

\[ \overline{\pi_1(x)} = \text{tr}_\alpha \int_{T_0} (-1)^\alpha D\gamma_{\alpha, \text{Id}^*, \lambda(x)}(\tau) \]

\[ = \text{tr}_\alpha [(-1)^\alpha \gamma(\alpha, \lambda(x))] \]

\[ := \frac{1}{2\pi i} \int_{C+1} \frac{\pi \csc(\pi(\alpha + 1))}{\Gamma(\alpha + 1)} (-1)^\alpha \gamma(\alpha, \lambda(x)) \, d\alpha \]

\[ = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n!} \gamma(n, \lambda(x)) \]

\[ = -\sum_{n=1}^{\infty} \frac{\Gamma(n)}{\Gamma(n+1)} P(n, \lambda(x)) \]  \hspace{1cm} (4.3)

where the new contour begins at \( \infty \) above the real axis, circles the point \( \{1\} \) counterclockwise, and returns to \( \infty \) below the real axis. Roughly speaking, this calculation simply sums the positive integers appropriately adjusted with a non-homogenous scaling factor and weighted by \( \frac{\Gamma(n)}{\Gamma(n+1)} = 1/n \) (which motivated the choice of \( \text{tr}_\alpha \)).

The series converges absolutely since

\[ \lim_{n \to \infty} \left| \frac{n!}{(n+1)!} \frac{\gamma(n+1, \lambda(x))}{\gamma(n, \lambda(x))} \right| = \lim_{n \to \infty} \left| \frac{1}{(n+1)} \right| \lambda(x) = 0 . \]  \hspace{1cm} (4.4)

And notice that

\[ \overline{\pi_1(x+1)} - \overline{\pi_1(x)} = -\sum_{n=1}^{\infty} \frac{1}{n!} \int_{\lambda(x)}^{\lambda(x+1)} e^{-t} t^{n-1} \, dt \sim \frac{-1}{\lambda(x)} \]  \hspace{1cm} (4.5)

is supposed to represent the average density of primes at \( x \). Accordingly, a good and obvious choice for the scaling factor is \( \lambda(x) = -\log(x) \) yielding the hypothesis

\[ \overline{\pi_1(x)} = -\sum_{n=1}^{\infty} \frac{\Gamma(n)}{\Gamma(n+1)} P(n, -\log(x)) . \]  \hspace{1cm} (4.6)

Graphics show that the exact number of primes \( \pi(x) \) oscillates about \( \sum_n \frac{\mu(n)}{n} \overline{\pi_1(x^{1/n})} \) within a fairly narrow bound — at least up to \( x \sim O(10^{14}) \). Indeed, the Moebius inversion of \( \overline{\pi_1(x)} \) gives essentially the same estimate as Riemann’s R-function. Apparently, \( P(n, -\log(x)) \simeq \pi(x^{1/n}) \) in the sense that

\[ -\sum_{n=1}^{\infty} \frac{1}{n} P(n, -\log(x)) \simeq \sum_{p^k \leq x} \frac{1}{k} \]  \hspace{1cm} (4.7)

where \( p^k \) is a prime power.

\[ ^{18}\text{The numerical calculations were performed using Mathematica 9.0 which doesn’t support prime counting beyond this order, but isolated points up to order } x \sim O(10^{24}) \text{ are available in tables [10].} \]
4.2 Twin prime counting

The constrained gamma process postulate can be applied to twin prime counting as well. Our reasoning remains heuristic.

We maintain the hypothesis that the occurrence of twin prime numbers is a constrained gamma process. But now, taking pairs whose difference is \( n = 2 \) will incur the twin prime constant \( C_2 \) normalization according to the standard probabilistic argument. Also, events between twin primes should be excluded from the \( \alpha \)-trace so we should only count every other event. The proposed twin prime integrator is

\[
\frac{C_2}{\Gamma(\alpha + 1)} (-1)^{2\alpha - 1} D \gamma_{2\alpha - 1, \lambda(x)}(\tau) .
\]  

(4.8)

Following the example of single primes suggests

\[
\pi_2(x) := \frac{C_2}{2\pi i} \int_{c+i} \frac{\pi \csc(\pi(\alpha + 1))}{\Gamma(\alpha + 1)} (-1)^{2\alpha - 1} \gamma(2\alpha - 1, \lambda(x)) \, d\alpha
\]

\[
= C_2 \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \gamma(2n - 1, \lambda(x)) .
\]  

(4.9)

Continuing this heuristic for arbitrary prime doubles \( \pi_{2i}(x) \) within an interval \( 2i \leq x - 2 \), it is known that the normalizing constant becomes

\[
C_{2i} = C_2 \prod_{p|i} \frac{p - 1}{p - 2}
\]

(4.10)

for prime numbers \( p > 2 \), and this is the only adjustment to the joint integrator. In general then, the prime double hypothesis is

\[
\pi_{2i}(x) = C_{2i} \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \gamma(2n - 1, -\log(x)) , \quad x - 2 > 2i \in \mathbb{N}_+ .
\]  

(4.11)

Note that only the normalizing constant depends on \( i \), and the series converges absolutely for finite \( x \);

\[
\lim_{n \to \infty} \left| \frac{(n!)^2}{((n + 1)!)^2} \right| \left| \frac{\gamma(2n + 1, -\log(x))}{\gamma(2n - 1, -\log(x))} \right| = \lim_{n \to \infty} \left| \frac{1}{(n + 1)^2} \right| \log(x)^2 = 0 .
\]  

(4.12)

However, since \( \lim_{x \to \infty} \gamma(2n - 1, -\log(x)) = -\Gamma(2n - 1) \), and the sequence

\[
\{a_n\} := \frac{(-1)^n \Gamma(2n - 1)}{\Gamma(n + 1)^2}
\]

(4.13)

does not converge to zero, then \( \pi_{2i}(x) \) diverges as \( x \to \infty \). Therefore, given the hypothesis of the constrained gamma process for locating joint prime numbers, we conclude that

\[
\lim_{x \to \infty} \pi_{2i}(x) \to \infty \quad \forall i \in \mathbb{N}_+ .
\]  

(4.14)

\[\text{19The reader is invited to compare this average against tabulated twin primes.}\]
This is reasonable for small \(i\), but it seems questionable when \(2i \to x - 2\). Intuitively, it is hard to believe that there are an infinite number of prime doubles \((p, p + 2i)\) if \(i\) gets too large. On the other hand, technically \(i\) must remain finite while \(x\) is allowed to go to infinity. So (4.14) is possible if \(C_{2i}\) remains relatively constant for all \(i \in \mathbb{N}_+\), because there are an infinite number of primes available for pairing. In other words, in the limit \(x \to \infty\), there are an infinite number of points that prime doubles \((p, p + 2i)\) (which are separated by a finite distance) can straddle. Of course intuition is often flawed, and one should rigorously examine the assumption of simple normalization by \(C_{2i}\).

Owing to its probabilistic foundation, the prime double hypothesis cannot be confirmed unconditionally. However, given the success of the average prime counting function \(\pi_1(x)\), it appears plausible that the hypothesis is correct; \(C_{2i}\) notwithstanding. In particular, if we accept it for at least \(i = 1\), then verification of the twin prime counting conjecture follows immediately — albeit conditionally — since the sum diverges with \(x\).

Of course, one might argue that the hypothesis is just an alternative to the Hardy-Littlewood twin prime conjecture. However, the hypothesis is not asymptotic. With it we can statistically verify the Goldbach conjecture:

**Proposition 4.1** If the occurrence of prime doubles is a constrained gamma process, then every even number greater than 2 is asymptotically almost surely the sum of two primes.

*Proof outline:* Assume the contrary. Then there exists a \(2x\) that is not the sum of two primes. Clearly, \(x\) cannot be prime. Further, \(x\) cannot be ‘straddled’ by a prime double \((p, p + 2i)\) with \(p + i = x\) for some \(i \in \{1, \ldots, x - 1\}\) since otherwise \(p + [p + 2i] = (x - i) + [(x - i) + 2i] = 2x\).

But according to the constrained gamma process hypothesis, the probability density of prime doubles straddling the point \(x\) is given by the absolutely converging series

\[
P_i(x) = \frac{C_{2i}}{2i} \sum_{n=1}^{\infty} (-1)^n \frac{\Delta_+(n, x) + \Delta_-(n, x)}{\Gamma(n + 1)^2} \tag{4.15}
\]

where

\[
\Delta_{\pm}(n, x) := [\pm \gamma(2n - 1, -\log(x \pm 1))] \mp \gamma(2n - 1, -\log(x)) . \tag{4.16}
\]

So the expected number of prime doubles that straddle \(x\) is given by

\[
S(x) := \sum_{i=1}^{x-1} \pi_{2i}(2x) P_i(x) . \tag{4.17}
\]

Now, for sufficiently large \(x\) the expected number goes like \(S(x) \sim x/\log^4(x)\). Moreover, since \(x/\log^4(x)\) is monotonically increasing for sufficiently large \(x\), it only takes calculating \(S(x)\) for a few small \(x\) to see that \(S(x) > 1\) for all \(x\) sufficiently large. Since the probability that \(x\) is straddled by at least one prime double is \(1 - e^{-S(x)}\), we have a contradiction asymptotically almost surely. \(\square\)
One can check explicitly up to some sufficiently large cut-off that the probability of a contradiction is essentially almost sure. For example at \( x = 10^9 \) we find the expected number of straddling prime doubles \( S(x) > 29000 \) where we used the under-estimate \( \sum_i C_{2i}/i \approx 1 \) for simplicity. So the probability that the next even integer is not the sum of two primes is less than about \( 10^{-12500} \). It is perhaps disconcerting that the conjecture cannot be settled with certainty by this argument, but it is comforting that the probability that it is false — beyond where one is willing to explicitly check — decreases exponentially like \( e^{-\epsilon x/(\log(x))^4} \) with \( \epsilon = O(1) \) a positive constant.

One final implication: Since the probability associated with prime doubles only depends on the gap between them through \( C_{2i} \), the probability of twin primes at an interval \([x - 1, x + 1]/2\) is the joint distribution to use for the conditional probability of two primes being separated by a gap. So the expected gap between prime \( p_1 \) and \( p_2 \) given \( p_1 \) is \( P(p_1 + 1)^{-1} \), and it is easy to establish that \( P(p)^{-1} \sim \log(p)^2 \). Hence Cramér’s conjecture is true on average — given the constrained gamma hypothesis.

5 Conclusions

Constrained dynamical systems were studied from a function space perspective using newly developed functional integration tools. The tools rely on the notions of conditional and conjugate integrators — the analogs of conditional and conjugate probability distributions in Bayesian inference theory. These notions show the well-known Gaussian functional integrals to be only part of the picture: To describe constrained systems, one must be able to manipulate functional integrals over constrained function spaces using conjugate integrator families.

Applying the constrained functional integral concepts, well-known results were re-derived efficiently at the functional level. Additionally, the framework allowed construction of a model for various counting functions associated with prime numbers that give excellent numerical estimates and, hopefully, a basis for better understanding prime distributions. The examples analyzed here point to the utility of gamma and Poisson integrator families, but it is likely that other probability distribution analogs will be useful.

No attempt was made to develop methods to calculate non-trivial gamma functional integrals. That \( Z(\tau') \) is comprised of the incomplete gamma function and it is defined for complex parameters, points to considerable complexity. Evidently the study of \( D_\gamma \) is an involved but important project. The perturbation expansion notwithstanding, the gamma functional integral can be expected to yield new calculation techniques.

It would be fruitful to extend the concepts developed in this article beyond simple QM. In particular, the domain of \( x \) and \( \tau \) can be altered in obvious ways to include quantum fields and loops. Similarly, the domain of \( X_a \) can be extended to include matrix-valued functions — opening the door to matrix QM. Together with the complex Gaussian integrator and the complex nature of the gamma integrator, such extensions would appear to offer broad applicability and significant potential.
References

[1] J. LaChapelle, Functional Integration on Constrained Function Spaces I: Foundations, arXiv:math-ph/1212.0502 (2012)

[2] J. LaChapelle, Path integral solution of linear second order partial differential equations: I and II. Ann. Phys. 314, 362–424 (2004).

[3] A. Auerbach adn L.S. Schulman, A path decomposition exapnsion proof for the method of images J. Phys. A 30, 5993–5995 (1997).

[4] P. van Baal, Tunneling and the path decomposition expansion Lectures on Path Integration (Trieste, 1991) ed H A Cerdeira, S Lundqvist, D Mugnai, A Ranfagni, V Sa-yakanit and L S Schulman (Singapore: World Scientific).

[5] J.J. Halliwell, An operator derivation of the path decomposition expansion, Phys. Lett. A, 207(5), 237–242 (1995).

[6] R.E. Crandall, Combinatorial approach to Feynman path integration, J. Phys. A, 26, 3627–3648 (1993).

[7] C. Garrod, Hamiltonian Path Integral Methods. Rev. Mod. Phys. 38, 483 (1966).

[8] J. Korevaar, and H.J.J. te Riele, Average prime-pair counting formula, Math. Comp., 79, 1209–1229 (2010).

[9] A. Granville, Refinements of Goldbach’s conjecture, and the generalized Riemann hypothesis, Funct. Approx. Comment. Math., 37(1), 159–173 (2007).

[10] Table of $\pi(x)$ in Prime-counting function, retrieved 8/2012 from Wikipedia.