We consider the problem of restoring a two-dimensional convex source of gravity based on values of its potential function on a union of half-planes. With that aim we prove a Paley–Wiener type theorem for functions of exponential type in a sector. In particular, we find the maximal set of analytic continuation whose complement is convex inside a sector. Our result serves as a sectorial analog of G. Polya’s indicator theorem. It corrects, simplifies, and extends a result of M. Morimoto. It also improves a related result of M. Dzhrbashyan and A. Avetisyan.

**KEYWORDS**
analytic continuation, analytic functionals, exponential functions, geophysics, Paley–Wiener theorem

**MSC CLASSIFICATION**
30E20, 30D10, 30D15, 86A20

## 1 | INTRODUCTION

Potential field data play an important role in the study of geologic structure and resource exploration. Analytic continuation is the most common and effective method to interpret potential field data and to locate anomalies of gravity. We refer the reader to previous articles [1–3] for a review of the role of analytic continuation in geophysics.

Let us discuss the following particular problem of geophysics. Let a two-dimensional convex body $K$ be the source of gravity. Let $\Gamma$ be an arbitrary curve that encompasses the source $K$. Assume that we are able to evaluate the potential function $h : \Gamma \to \mathbb{R}$ induced by the source of gravity $K$ at any point of the curve $\Gamma$. Being given the function $h$, we can consider the analytic function $g$ by stating that $\text{Re}(g) = h$. By Polya’s indicator theorem [4], the set $K$ is the smallest convex set, such that the function $g$ is analytic in its complement. Accordingly, we are able to restore the set $K$ from the function $g$ by means of the same theorem 1.15.

Now consider the more general case, when we are not able to evaluate the potential function $h$ on a curve that encompasses the whole convex set $K$; either because such evaluations may not be carried out in practice or because the set $K$ may not be bounded. Instead, assume that we are able to evaluate the potential function $h$ on a single half-plane or, more generally, on a union of half-planes. In that case, one might not be able to restore the convex set $K$ completely. However, by means of Theorem 1.10, we are able to claim that

- on the one hand, the set $K$ does not contain any point of the union of half-planes $\Omega$ defined by formula (1.7),
- on the other hand, for any half-plane $\Omega'$ that properly contains the half-plane $\Omega_0$ defined by (1.5), the set $K$ does contain a point of the half-plane $\Omega'$.  

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Our main results 1.7, 1.10 constitute a Paley–Wiener type theorem. We recall that generally speaking a Paley–Wiener type theorem is any theorem that relates decay properties of a function or distribution at infinity with analyticity of its Fourier transform. The first of such theorems was proved by Paley and Wiener [5] for entire functions of exponential type:

**Theorem 1.1** (Paley–Wiener). The set of entire functions $f$ of exponential type at most $h$, that are square-integrable on the real axis, coincides with the set of functions $f$ that allow the integral representation

$$f(z) = \int_{-h}^{h} e^{iz\omega} g(\omega) d\omega, \quad \text{for } g \in L_2(-h, h).$$

Our main results - the Fourier inversion formula 1.7 and the description 1.10 of the domain of analyticity - constitute a Paley–Wiener type theorem for functions of exponential type in a sector. We complement these results by Remarks 1.11, 1.12, and 1.13 that correct, simplify, and extend a result of Morimoto; a result that was originally used to derive representations of analytic functionals. We then proceed to compare our results to those of Polya (see Remarks 1.14 and 1.16) and Dzhrbashyan and Avetisyan (see Remark 1.17). We conclude the introduction by showing some theoretical applications of our results (see Remark 1.18).

In order to phrase the results of the article, we need some auxiliary definitions.

**Definition 1.2.** For an angle $\alpha$ in the first quadrant,

$$0 < \alpha < \pi/2,$$

(1.1)

denote by $\Delta_{\alpha} \subset \mathbb{C}$ the open sector determined by the angle $\alpha$, namely,

$$\Delta_{\alpha} := \{ z \in \mathbb{C} \setminus \{0\} : -\alpha < \arg(z) < \alpha \}.$$

(1.2)

**Definition 1.3.** For an open sector $\Delta_{\alpha}$ introduced by definition 1.2 and a non-negative number $h$, denote by $\text{Exp}((\Delta_{\alpha}, h))$ the class of functions $f$ that are analytic in $\Delta_{\alpha}$ and are of exponential type at most $h$ in $\Delta_{\alpha}$, that is, for any $\epsilon > 0$, there exists a constant $C_{\epsilon} \geq 0$ such that

$$|f(z)| \leq C_{\epsilon} e^{(h+\epsilon)|z|}, \quad \text{for } z \in \Delta_{\alpha}.$$

(1.3)

In this article, for a function $f \in \text{Exp}((\Delta_{\alpha}, h))$ and a boundary point $\zeta \in \partial \Delta_{\alpha}$, we will denote by $f(\zeta)$ the non-tangential limit of $f$ at $\zeta$ (see Remark 2.1 on further discussion of non-tangential limits).

**Definition 1.4.** Let $f \in \text{Exp}((\Delta_{\alpha}, h))$. For an angle $|\theta| \leq \alpha$, denote by $I_f(\theta)$ the indicator of $f$ in direction $\theta$, namely,

$$I_f(\theta) := \limsup_{s \to +\infty} \frac{\ln |f(se^{i\theta})|}{s}.$$

(1.4)

**Definition 1.5.** Let $f \in \text{Exp}((\Delta_{\alpha}, h))$. Define $f$’s directional Laplace transform $g_{\theta}$ on the half-plane

$$\Omega_{\theta} := \{ \omega : \text{Re} \left( \omega e^{i\theta} \right) < -I_f(\theta) \}$$

(1.5)

by the formula

$$g_{\theta}(\omega) := \frac{1}{2\pi i} \int_{e^{i\theta}(0, +\infty)} f(\zeta) e^{\omega \zeta} d\zeta, \quad \text{for } \omega \in \Omega_{\theta}.$$

(1.6)

Further, define $g$ to be the concatenated Laplace transform of $f$. Namely, define the function $g$ on the union $\Omega$ of all half-planes $\Omega_{\theta}$,

$$\Omega := \bigcup_{\theta : e^{i\theta}(0, +\infty) \subset \Delta_{\alpha}} \Omega_{\theta}.$$

(1.7)
in terms of directional Laplace transforms $g_\theta$, by the following formula:

$$g(\omega) := g_\theta(\omega), \quad \text{for } \omega \in \Omega_\theta \subset \Omega.$$  \hfill (1.8)

As proved in Lemma 4.1, the function $g$ is well-defined by formula (1.8).

**Definition 1.6.** For a class of functions $\text{Exp}(\Delta_\alpha, h)$, denote by $\Gamma$ the curve given by the following parametrization:

$$\begin{align*}
\gamma &: \mathbb{R} \to \mathbb{C}, \\
\gamma(t) &= p - |t|e^{i\alpha}, \quad \text{for } t \in (-\infty, 0], \\
\gamma(t) &= p + |t|e^{-i\alpha}, \quad \text{for } t \in (0, +\infty],
\end{align*}$$  \hfill (1.9)

where $p$ is an arbitrary real number satisfying

$$p \cos(\alpha) < -h$$  \hfill (1.10)

(see Figure 1).

**Theorem 1.7.** For a function $f \in \text{Exp}(\Delta_\alpha, h)$, the following Fourier inversion formula holds:

$$f(z) = \int_{\Gamma} g(\omega)e^{-\omega z}d\omega,$$  \hfill (1.11)

where the concatenated Laplace transform $g$ is introduced by Definition 1.5.

The outline of proof of Theorem 1.7 is as follows: We first express $f$ via a Cauchy integral formula (2.4); then we rewrite the Cauchy integral formula (2.4) as a Fourier inversion formula (3.7); finally, in Remark 4.2, we rewrite the internal integrals appearing in the Fourier inversion formula (3.7) in a unified way, namely, as values of the concatenated Laplace transform $g$.

**Remark 1.8.** We remark on the formulation of Theorem 1.7 that the reasons to take the set $\Delta_\alpha$ to be a sector are the following: with each point $e^{i\theta} \in \Delta_\alpha$ the set $\Delta_\alpha$ contains the whole ray $e^{i\theta}(0, +\infty)$, so that we can introduce the notion.

![Figure 1](image-url)
of indicator (1.4) properly; additionally, the set of angles

\[ \{ \theta : e^{i\theta}(0, +\infty) \subset \Delta_\alpha \} \]

is connected, so that we can apply the Cauchy integral formula (2.3).

**Remark 1.9.** One can eliminate restriction (1.1) on the angle \( \alpha \) and obtain results similar to Theorem 1.7 for wider, more general sectors \( \Delta_\alpha \) if one splits the sector into narrower ones.

We complement Theorem 1.7 by the following theorem that relates the indicator of \( f \) to the natural domain of analyticity of \( g \):

**Theorem 1.10.** For a function \( f \in \text{Exp}(\Delta_\alpha, h) \) and an angle \( |\theta| < \alpha \), the concatenated Laplace transform \( g \), introduced by Definition 1.5, may not be analytically continued to any half-plane \( \Omega' \) that contains the half-plane \( \Omega_\theta \) properly,

\[ \Omega_\theta \subset \subset \Omega' \].

As shown in Section 5, Theorem 1.10 follows from Theorem 1.7.

This paper was motivated by the following remark:

**Remark 1.11 (Yoshino and Suwa [6]).** The authors claim that for a function \( f \) of exponential type in the right open half-plane, the following Paley–Wiener type representation holds:

\[ f(z) = \int_C e^{-\omega z} d\mu(\omega), \quad (1.12) \]

where \( \mu \) is a measure defined in the complex plane \( \mathbb{C} \), whose support is bounded from above, below, and the right. Their claim is based on proposition 5.2 page 95 of Morimoto’s article [7]. In turn, in lemma 5.2 page 93 of the same article [7], the following is claimed: If a function is holomorphic on the set

\[ W(1) = \{ \omega : \text{Im}(\omega) > k_2 + \epsilon \} , \]

then it is holomorphic on the larger set

\[ \{ \omega : \text{Im}(\omega) > k_2 \} ; \]

an obviously wrong claim. This mistake affects the construction of the Laplace transform in definition 5.3 page 95 of Morimoto [7] and ultimately affects the proof of proposition 5.2 of Morimoto [7].

However, a closer inspection of the article [7] shows that under proper conditions (1.3) imposed on the function \( f \) its boundary values exist almost everywhere (see Remark 2.1), and consequently the Laplace transform may be defined explicitly (see formula 1.6) instead of having to rely on the elaborate technique of factor spaces of holomorphic functions.

In this paper, we use this latter approach to derive a representation of the function \( f \) in terms of an integral of exponents.

The following two remarks compare our Theorems 1.7 and 1.10 to those of Morimoto (1.12):

**Remark 1.12.** Theorem 1.7 extends the result (1.12) in the following ways: Our claim holds for a general sector not only for half-plane; and instead of a measure \( \mu \) integrated over the complex plane \( \mathbb{C} \), we use a function \( g \) integrated over a curve \( \Gamma \).

**Remark 1.13.** We extend the result (1.12) in yet another way: we complement Theorem 1.7 by Theorem 1.10 regarding the natural domain of analyticity of the function \( g \).

**Remark 1.14** We remark that theorems 1.7, 1.10 can be viewed as an analog of Polya’s indicator theorem (see the literature [4, 8] or theorem 5.5 in Leont’ev [9]) for a sector.
Theorem 1.15 (Polya). Let \( f \) be an entire function of exponential type. Denote by \( K \subset \mathbb{C} \) the convex set whose support function \( k(\theta) \) is determined by \( f \)'s indicator as follows:

\[
k(\theta) = I_f(\theta)
\]

Then \( f \) can be restored by

\[
f(z) = \frac{1}{2\pi i} \int_{\Gamma} g(t)e^{-zt}dt,
\]

where \( \Gamma \) is a closed contour containing the set \( K \), and \( g \) is the Laplace transform of \( f \). Additionally, \( K \) is the smallest convex set such that \( g \) is analytic in \( \mathbb{C}\setminus K \).

In fact, Polya's indicator theorem implies the Paley–Wiener theorem (see Levin [10], section 10.1).

Remark 1.16. The peculiarity of our results is as follows: As stated in the very name of article [7], unlike the setting discussed in Theorem 1.1 of Paley–Wiener or in Theorem 1.15 of Polya, the curve \( \Gamma \), that appears in the Fourier inversion formula (1.11), is unbounded.

By virtue of being unbounded, the integration contour \( \Gamma \) in (1.11) resembles the Bromwich inversion formula, which involves a contour integral over a vertical line in the complex plane:

\[
f(z) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} g(\omega)e^{\omega z}d\omega
\]

(see, e.g., Widder [11], p. 67).

Additionally, unlike in the case of remark 4.1.2 in Andersson et al. [12], we may not express the directional Laplace transform (1.6) in terms of \( f \)'s Taylor series expansion at 0, simply because \( f \) may not be analytic at 0.

Remark 1.17. After this article was completed, the author came across the article of Dzhrbashyan and Avetisyan [13] where Theorem 1.7 was proved in quite a similar way to our proof, some eighteen years before Morimoto's article [7].

Specifically, Dzhrbashyan and Avetisyan treat a more general class of functions than the class of functions \( \text{Exp}(\Delta_\alpha, h) \) considered by us. Namely, they consider a class \( A(\alpha)[\rho_1, \sigma_1] \) of functions \( F \) who are holomorphic inside the angle \( \Delta_\alpha \) and who satisfy the inequality

\[
|F(z)| \leq M_F e^{\sigma_1|z|}, \quad \text{for all } z \in \Delta_\alpha
\]

(see formula 1.1 on page 384 of Dzhrbashyan and Avetisyan [13]).

They introduce a properly defined Laplace transform (see formula (1.2) on page 384 of Dzhrbashyan and Avetisyan [13]). If specified for the class of functions \( \text{Exp}(\Delta_\alpha, h) \), it coincides with the concatenated Laplace transform \( g \) that we introduced by Definition 1.5.

They prove an analog of the Fourier inversion formula for the class of functions \( A(\alpha)[\rho_1, \sigma_1] \) (see formula (1.30) of theorem 1 on page 380 of Dzhrbashyan and Avetisyan [13]). If specified for the class \( \text{Exp}(\Delta_\alpha, h) \), their inversion formula coincides with our inversion formula (1.11).

At the same time, in Theorem 1.10, we improve the results of Dzhrbashyan and Avetisyan by proving the following additional fact that doesn't appear in Dzhrbashyan and Avetisyan [13]: The concatenated Laplace transform \( g \) may not be analytically continued to any half-plane \( \Omega' \) that contains the half-plane \( \Omega_\theta \) properly.

Remark 1.18. Theorem 1.7 has proved to have several applications, for example, in a result of the author on refining Carlson's uniqueness theorem (see Vagharshakyan [14]) and in a result of the author and of A. Mkrtchyan on proving trigonometric convexity for the multidimensional indicator after Ivanov (see Mkrtchyan and Vagharshakyan [15]).

2 | CAUCHY FORMULA

Remark 2.1. Note that, due to condition (2.1), the function \( f \) is bounded in a vicinity of each of the points of \( \partial \Delta_\alpha \). Hence, by Fatou's theorem, the non-tangential limit of \( f \) exists at a.e. point \( z \in \partial \Delta_\alpha \). Extend the function \( f \) from the
open sector $\Delta_\alpha$ to the closed sector $\overline{\Delta_\alpha}$ in the following way: for $z \in \partial \Delta_\alpha$ define $f(z)$ to be the non-tangential limit provided by Fatou’s theorem if the limit exists and define $f(z)$ to be 0 if the limit doesn’t exist. Cauchy integral formula is then applicable to any bounded domain in $\Delta_\alpha$. Additionally, the extension of the function $f$ to $\overline{\Delta_\alpha}$, defined in the aforementioned way, satisfies the same estimate (1.3) on the whole closed sector $\overline{\Delta_\alpha}$: For any $\epsilon > 0$, there exists a constant $C_\epsilon \geq 0$ such that

$$|f(z)| \leq C_\epsilon e^{(h+\epsilon)|z|}, \quad \text{for } z \in \overline{\Delta_\alpha}. \quad (2.1)$$

Due to (2.1),(1.2) and the fact that $p$ is a negative number, we can estimate

$$|f(z)e^{iep}| \leq C_\epsilon e^{(h+\epsilon+p\cos(\alpha))|z|}, \quad \text{for } z \in \overline{\Delta_\alpha}. \quad (2.2)$$

In particular, due to the choice (1.10) of point $p$, we can take $\epsilon$ small enough to guarantee that the coefficient

$$h + \epsilon + p \cos(\alpha)$$

in the power of the last exponent of (2.2) is negative. This is why the Phragmen–Lindelöf maximum principle applies to the Cauchy integral formula for the function $f(z)e^{iep}$ in the sector $\overline{\Delta_\alpha}$. Consequently, we have

$$f(z)e^{iep} = \frac{1}{2\pi i} \int_{\partial \Delta_\alpha} \frac{f(\zeta)e^{iep}}{\zeta - z} d\zeta, \quad \text{for } z \in \Delta_\alpha, \quad (2.3)$$

where the curve $\partial \Delta_\alpha$ is oriented counterclockwise, that is, it is oriented with respect to the sector $\Delta_\alpha$. Rewrite the latter formula as

$$2\pi i \cdot f(z) = \int_{\partial \Delta_\alpha} \frac{f(\zeta)e^{iep(\zeta - z)}}{\zeta - z} d\zeta, \quad \text{for } z \in \Delta_\alpha.$$  

Split integration over the oriented curve $\partial \Delta_\alpha$ into integration over its two rays: $e^{-ia}[0, +\infty)$ and $e^{ia}[0, +\infty)$, to get

$$2\pi i \cdot f(z) = \int_{e^{-ia}[0, +\infty)} \frac{f(\zeta)e^{iep(\zeta - z)}}{\zeta - z} d\zeta - \int_{e^{ia}[0, +\infty)} \frac{f(\zeta)e^{iep(\zeta - z)}}{\zeta - z} d\zeta, \quad (2.4)$$

for $z \in \Delta_\alpha$.

### 3 | FOURIER INVERSION

Note the following elementary formula that recovers the Cauchy kernel $1/z$ from exponential functions $z \to e^{i\omega}$ by a proper integration over parameter $\omega$:

$$\frac{1}{z} = -\int_{0}^{+\infty} e^{i\omega} d\omega, \quad \text{for } \text{Re}(z) < 0. \quad (3.1)$$

Now let $\zeta \in e^{ia}[0, +\infty)$ and $z \in \Delta_\alpha$. Then, by definition (1.2) of the sector $\Delta_\alpha$ and restriction (1.1) on the angle $a$, we have $\text{arg}(\zeta - z) \in (a, a + \pi)$. Hence, $\pi/2 - a + \text{arg}(\zeta - z) \in (\pi/2, 3\pi/2)$, or in other words

$$\text{Re} \left( i e^{-ia}(\zeta - z) \right) < 0.$$  

Hence, similarly to formula (3.1), we have

$$\frac{e^{i\omega(\zeta - z)}}{\zeta - z} = -\int_{p + ie^{-ia}[0, +\infty)} e^{i\omega(\zeta - z)} d\omega, \quad \text{for } \zeta \in e^{ia}[0, +\infty), \ z \in \Delta_\alpha. \quad (3.2)$$

By an analogous argument, we can also prove that

$$\frac{e^{i\omega(\zeta - z)}}{\zeta - z} = -\int_{p - ie^{ia}[0, +\infty)} e^{i\omega(\zeta - z)} d\omega, \quad \text{for } \zeta \in e^{-ia}[0, +\infty), \ z \in \Delta_\alpha. \quad (3.3)$$
Formulas (3.2) and (3.3) allow us to rewrite Cauchy’s integral formula (2.4) as

\[ 2\pi i f(z) = - \int_{e^{-i\alpha}} f(\zeta) \int_{p-i\epsilon\alpha} e^{\omega(z-\zeta)} d\omega d\zeta + \int_{e^{i\alpha}} f(\zeta) \int_{p+i\epsilon\alpha} e^{\omega(z-\zeta)} d\omega d\zeta, \quad \text{for } z \in \Delta_a. \]  

(3.4)

The following two lemmas justify changing the order of integration in (3.4):

**Lemma 3.1.** Under assumptions of Theorem 1.7, we have

\[ \int_{e^{-i\alpha}} \left| f(\zeta) \right| e^{\omega \zeta} |d\zeta| \leq \frac{-C_{\epsilon}}{h + \epsilon + p \cos(\alpha)}, \quad \text{for } \omega \in p - ie^{i\alpha}[0, +\infty). \]

**Proof.** Denote

\[ \omega' = p - \omega. \]  

(3.5)

Then conditions \( \omega \in p - ie^{i\alpha}[0, +\infty) \) and \( \zeta \in e^{-i\alpha}[0, +\infty) \) imply that

\[ |e^{-\omega' \zeta}| = 1. \]  

(3.6)

We estimate

\[
\int_{e^{-i\alpha}} \left| f(\zeta) \right| e^{\omega \zeta} |d\zeta| \leq C_{\epsilon} \int_{e^{-i\alpha}} e^{(h+\epsilon)\zeta} |d\zeta| = C_{\epsilon} \int_{e^{-i\alpha}} e^{(h+\epsilon + p \cos(\alpha))\zeta} |d\zeta|.
\]

From this estimate, Lemma 3.1 follows. \( \square \)

**Lemma 3.2.** We have

\[
\int_{p-i\epsilon\alpha} |e^{-\omega \zeta}| |d\omega| = \frac{-|e^{-\omega \zeta}|}{Re \left( i e^{i\alpha} \right)}, \quad \text{for } z \in \Delta_a.
\]

**Proof.** We evaluate

\[
\int_{p-i\epsilon\alpha} |e^{-\omega \zeta}| |d\omega| = \frac{-|e^{-\omega \zeta}|}{Re \left( i e^{i\alpha} \right)} < +\infty, \quad \text{for } -\alpha < \arg(z) < \pi - \alpha.
\]

At the same time, due to (1.1), condition \(-\alpha < \arg(z) < \pi - \alpha \) holds for all \( z \in \Delta_a. \) \( \square \)

By Lemmas 3.1 and 3.2, the condition of Fubini’s theorem for changing the order of integration in the first term of (3.4) holds. Similarly, one can check that the condition of Fubini’s theorem for changing the order of integration in the second term of (3.4) holds, too. We apply Fubini’s theorem to both terms of (3.4) to get the following Fourier inversion formula:

\[ 2\pi i f(z) = - \int_{p-i\epsilon\alpha} \left( \int_{e^{-i\alpha}} f(\zeta) e^{\omega \zeta} d\zeta \right) e^{-\omega \zeta} d\omega + \int_{p+i\epsilon\alpha} \left( \int_{e^{i\alpha}} f(\zeta) e^{\omega \zeta} d\zeta \right) e^{-\omega \zeta} d\omega. \]  

(3.7)
We have the following estimate for the indicator:

\[ I_f(\theta) = \limsup_{s \to +\infty} \frac{\ln |f(se^{i\theta})|}{s} \leq h, \quad \text{for } |\theta| \leq \alpha. \]

Hence, the integral in (1.6) is absolutely convergent, and the function \( g_\theta \) is well-defined on the set \( \Omega_\theta \). Further, from Morera's theorem, it follows that \( g_\theta \) is holomorphic in \( \Omega_\theta \).

\[ g_\theta \in \text{Hol}(\Omega_\theta). \quad (4.1) \]

**Lemma 4.1.** Under notations of Theorem 1.7, the concatenated Laplace transform \( g \) is well-defined by formula (1.8): that is, in (1.8), the choice of a particular direction \( \omega \) satisfying condition \( \omega \in \Omega_\theta \) is irrelevant.

**Proof.** Let

\[ -\alpha \leq \theta_1 \leq \theta \leq \theta_2 \leq \alpha, \quad (4.2) \]

\[ \omega(s) = -se^{-(\theta_1+\theta_2)/2}, \quad s \geq 0. \quad (4.3) \]

We estimate

\[ \text{Re} \left( \omega(s)e^{i\theta} \right) \stackrel{(4.3),(4.4)}{=} -s \text{Re} \left( e^{i(\theta_1/2-\theta_2/2)} \right) = -s \cos \left( \theta - \theta_2/2 - \theta_1/2 \right) \leq -s \cos(\alpha). \quad (4.5) \]

By taking \( \theta = \theta_1 \) and \( \theta = \theta_2 \) in the estimate (4.5), due to definition (1.5), we have

\[ \omega(s) \in \Omega_{\theta_1} \cap \Omega_{\theta_2}, \quad \text{for } s \gg 1. \]

For \( \arg(\zeta) \in [\theta_1, \theta_2] \), we estimate

\[ \left| f(\zeta)e^{i\omega(s)\zeta} \right| \leq Ce^{(h+e)s|\arg(\omega(s))|} = Ce^{(h+e+\text{Re}(\omega(s)\arg(\zeta)))|\zeta|} \leq Ce^{(h+e-s\cos(\alpha))|\zeta|}. \quad (4.6) \]

For \( s \gg 1 \), the coefficient

\[ h + e - s \cos(\alpha) \]

in front of \( |\zeta| \) in the last exponent of (4.6) is negative. Consequently, the Phragmen–Lindelöf maximum principle applies to the Cauchy integral theorem for the function

\[ \zeta \to f(\zeta)e^{i\omega(s)\zeta} \]

in the sector

\[ \{ \zeta \in \mathbb{C} : \arg(\zeta) \in [\theta_1, \theta_2] \}; \]

so that

\[ \int_{e^{\theta_1}[0, +\infty)} f(\zeta)e^{i\omega(s)\zeta} d\zeta = \int_{e^{\theta_2}[0, +\infty)} f(\zeta)e^{i\omega(s)\zeta} d\zeta. \]

Or equivalently

\[ g_{\theta_1}(\omega(s)) = g_{\theta_2}(\omega(s)), \quad \text{for } s \gg 1. \]

Hence, by uniqueness of analytic functions,

\[ g_{\theta_1}(\omega) = g_{\theta_2}(\omega), \quad \text{for } \omega \in \Omega_{\theta_1} \cap \Omega_{\theta_2}. \]

\[ \square \]
Due to Claim (4.1) and Lemma 4.1, we have

\[ g \in \text{Hol}(\Omega). \]

**Remark 4.2.** We can rewrite the Fourier inversion formula (3.7) in terms of the concatenated Laplace transform \( g \) (defined by (1.8) and (1.6)) and in terms of the curve \( \Gamma \) (parametrized by (1.9)) as formula (1.11).

**Remark 4.3.** By (1.9), the curve \( \Gamma \) consists of two subsets: \( p - ie^{i\alpha}[0, +\infty) \) and \( p + ie^{i\alpha}[0, +\infty) \). Lemma 3.1 provides an estimate on the function \( g \) along the subset \( p - ie^{i\alpha}[0, +\infty) \). One can prove a similar estimate for the other subset of \( \Gamma \) in order to obtain the following unified estimate:

\[ |g(\omega)| \leq \frac{-C_c}{h + e + p \cos(\alpha)}, \quad \text{for } \omega \in \Gamma. \]

### 5 | DOMAIN OF ANALYTICITY

Under assumptions of Theorem 1.10, suppose that the function \( g \) may be analytically continued to a half-plane \( \Omega' \) that contains the half-plane \( \Omega_\theta \) properly, \( \Omega_\theta \subset \Omega' \). We will show that this supposition leads to contradiction (5.6).

Indeed, on this supposition, we can define the curve \( \Gamma' \) as in Figure 2. That is, the difference between curves \( \Gamma \) and \( \Gamma' \) is that the curve \( \Gamma' \) passes directly from point \( q \) to point \( r \) instead of passing from \( q \) to \( p \) and then from \( p \) to \( r \), like the curve \( \Gamma \) does.

\[ \text{Figure 2} \quad \text{The curves } \Gamma \text{ and } \Gamma'. \text{ The difference between those curves is that } \Gamma \text{ first passes from } q \text{ to } p \text{ and then passes from } p \text{ to } r, \text{ whereas } \Gamma' \text{ passes directly from } q \text{ to } r \text{ and otherwise coincides with } \Gamma. \]
The curve $\Gamma'$ is bounded away from the half space $\Omega_\theta$. Hence, the set

$$\{ \text{Re} (\omega e^{i\theta}) : \omega \in \Gamma' \},$$

being the projection of the curve $\Gamma'$ onto the ray $e^{i(\pi-\theta)}[0, +\infty)$, is bounded away from $-I_f(\theta)$,

$$\inf_{\omega \in \Gamma'} \text{Re} (\omega e^{i\theta}) > -I_f(\theta). \quad (5.1)$$

Inequality (5.1) that may not hold for the original curve $\Gamma$.

Since in Theorem 1.10 the function $g$ is assumed to be analytic in the whole half-plane $\Omega'$, the Cauchy integral theorem applies in $\Omega'$, and consequently, the Fourier inversion formula (1.11) stays valid if in that formula we substitute the curve $\Gamma$ by the curve $\Gamma'$,

$$f(z) = \int_{\Gamma'} g(\omega) e^{-\omega z} d\omega, \quad \text{for } z \in \Delta_\alpha.$$ 

We split the curve $\Gamma'$ into three subcurves: $[q, r]$, $r + i e^{-i \alpha} [0, +\infty)$ and $q - i e^{i \alpha} [0, +\infty)$, and estimate

$$|f(se^{i\theta})| \leq \int_{\Gamma'} |e^{-\omega e^{i\theta}}| |g(\omega)| |d\omega| = \int_{[q, r]} |e^{-\omega e^{i\theta}}| |g(\omega)| |d\omega| +$$

$$\int_{r + i e^{-i \alpha} [0, +\infty)} |e^{-\omega e^{i\theta}}| |g(\omega)| |d\omega| +$$

$$\int_{q - i e^{i \alpha} [0, +\infty)} |e^{-\omega e^{i\theta}}| |g(\omega)| |d\omega| = J_1(s) + J_2(s) + J_3(s), \quad \text{for } s > 0. \quad (5.2)$$

We further estimate

$$J_1(s) \leq e^{-s} \inf_{\omega \in \Gamma'} \text{Re} (\omega e^{i\theta}) \int_{[q, r]} |g(\omega)| |d\omega|. \quad \text{for } s > 0.$$ 

So that

$$\limsup_{s \to +\infty} \frac{\ln |J_1(s)|}{s} \leq -\inf_{\omega \in \Gamma'} \text{Re} (\omega e^{i\theta}). \quad (5.3)$$

Using Remark 4.3, we also estimate

$$J_3(s) \leq \frac{4.3}{h + \epsilon + p \cos(\alpha)} \int_{q - i e^{i \alpha} [0, +\infty)} |e^{\omega(-se^{i\theta})}| |d\omega| \leq$$

$$\frac{4.3}{h + \epsilon + p \cos(\alpha)} \cdot \frac{-e^{-q e^{i\theta}}}{\text{Re} (i e^{i\theta} s e^{i\theta})}, \quad \text{for } s > 0$$

(The last inequality is obtained by repeating the arguments of Lemma 3.2 for $q$ instead of $p$). So that

$$\limsup_{s \to +\infty} \frac{\ln |J_3(s)|}{s} \leq -\text{Re} (qe^{i\theta}) = -\inf_{\omega \in \Gamma'} \text{Re} (\omega e^{i\theta}). \quad (5.4)$$

Similarly, one can prove that

$$\limsup_{s \to +\infty} \frac{\ln |J_2(s)|}{s} \leq -\text{Re} (re^{i\theta}) = -\inf_{\omega \in \Gamma'} \text{Re} (\omega e^{i\theta}). \quad (5.5)$$

Using the estimates (5.3), (5.5), and (5.4) in (5.2), we get

$$I_f(\theta) \leq -\inf_{\omega \in \Gamma'} \text{Re} (\omega e^{i\theta}) < I_f(\theta). \quad (5.6)$$

A contradiction.
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CONFLICT OF INTEREST STATEMENT
This work does not have any conflicts of interest.

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