Topological properties defined in terms of generalized open sets

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Abstract

This paper covers some recent progress in the study of sg-open sets, sg-compact spaces, N-scattered spaces and some related concepts. A subset \( A \) of a topological space \((X, \tau)\) is called sg-closed if the semi-closure of \( A \) is included in every semi-open superset of \( A \). Complements of sg-closed sets are called sg-open. A topological space \((X, \tau)\) is called sg-compact if every cover of \( X \) by sg-open sets has a finite subcover. N-scattered space is a topological spaces in which every nowhere dense subset is scattered.

1 Prelude

Major part of the talk I presented in August 1997 at the Topological Conference in Yatsushiro College of Technology is based on the following three papers:

- J. Dontchev and H. Maki, On sg-closed sets and semi-\( \lambda \)-closed sets, Questions Answers Gen. Topology, (Osaka, Japan), 15 (2) (1997), to appear.
- J. Dontchev and M. Ganster, More on sg-compact spaces, Portugal. Math., 55 (1998), to appear.

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2 Sg-open sets and sg-compact spaces

In 1995, sg-compact spaces were introduced independently by Caldas [2] and by Devi, Balachandran and Maki [4]. A topological space \((X, \tau)\) is called *sg-compact* if every cover of \(X\) by sg-open sets has a finite subcover.

Sg-closed and sg-open sets were introduced for the first time by Bhattacharyya and Lahiri in 1987 [1]. Recall that a subset \(A\) of a topological space \((X, \tau)\) is called *sg-open* [1] if every semi-closed subset of \(A\) is included in the semi-interior of \(A\). A set \(A\) is called *semi-open* if \(A \subseteq \text{Int}A\) and *semi-closed* if \(\text{Int}A \subseteq A\). The *semi-interior* of \(A\), denoted by \(\text{sInt}(A)\), is the union of all semi-open subsets of \(A\), while the *semi-closure* of \(A\), denoted by \(\text{sCl}(A)\), is the intersection of all semi-closed supersets of \(A\). It is well known that \(\text{sInt}(A) = A \cap \text{Int}A\) and \(\text{sCl}(A) = A \cup \text{Int}A\).

Sg-closed sets have been extensively studied in recent years mainly by (in alphabetical order) Balachandran, Caldas, Devi, Dontchev, Ganster, Maki, Noiri and Sundaram (see the references).

In the article [1], where sg-closed sets were introduced for the first time, Bhattacharyya and Lahiri showed that the union of two sg-closed sets is not in general sg-closed. On its behalf, this was rather an unexpected result, since most classes of generalized closed sets are closed under finite unions. Recently, it was proved [5, Dontchev; 1997] that the class of sg-closed sets is properly placed between the classes of semi-closed and semi-preclosed (= \(\beta\)-closed) sets. All that inclines to show that the behavior of sg-closed sets is more like the behavior of semi-open, preopen and semi-preopen sets than the one of ‘generalized closed’ sets (g-closed, gsp-closed, \(\theta\)-closed etc.). Thus, one is more likely to expect that arbitrary intersection of sg-closed sets is a sg-closed set. Indeed, in 1997 Dontchev and Maki [8] solved the first problem of Bhattacharyya and Lahiri in the positive.
Theorem. [3, Dontchev and Maki; 1997]. An arbitrary intersection of sg-closed sets is sg-closed.

Every topological space \((X, \tau)\) has a unique decomposition into two sets \(X_1\) and \(X_2\), where \(X_1 = \{x \in X : \{x\} \text{ is nowhere dense}\}\) and \(X_2 = \{x \in X : \{x\} \text{ is locally dense}\}\). This decomposition is due to Janković and Reilly [19]. Recall that a set \(A\) is said to be locally dense [3, Corson and Michael; 1964] (= preopen) if \(A \subseteq \text{Int}(A)\).

It is a fact that a subset \(A\) of \(X\) is sg-closed (= its complement is sg-open) if and only if \(X_1 \cap \text{sCl}(A) \subseteq A\) [8, Dontchev and Maki; 1997], or equivalently if and only if \(X_1 \cap \text{Int}(A) \subseteq A\). By taking complements one easily observes that \(A\) is sg-open if and only if \(A \cap X_1 \subseteq \text{sInt}(A)\).

Hence every subset of \(X_2\) is sg-open.

Next we consider the bitopological case and utterly \((\tau_i, \tau_j)\)-Baire spaces:

A subset \(A\) of a bitopological space \((X, \tau_1, \tau_2)\) is called \((\tau_i, \tau_j)\)-sg-closed if \(\tau_j\)-Int(\(\tau_i\)-Cl(\(A\))) \(\subseteq U\) whenever \(A \subseteq U\), \(U \in SO(X, \tau_i)\) and \(i, j \in \{1, 2\}\). Clearly every \((\tau_i, \tau_j)\)-rare (= nowhere dense) set is \((\tau_i, \tau_j)\)-sg-closed but not vice versa. A subset \(A\) of a bitopological space \((X, \tau_1, \tau_2)\) is called \((\tau_i, \tau_j)\)-rare [13, Fukutake, 1992] if \(\tau_j\)-Int(\(\tau_i\)-Cl(\(A\))) = \(\emptyset\), where \(i, j \in \{1, 2\}\). \(A\) is called \((\tau_i, \tau_j)\)-meager [14, Fukutake 1992] if \(A\) is a countable union of \((\tau_i, \tau_j)\)-rare sets.

A subset \(A\) of a bitopological space \((X, \tau_1, \tau_2)\) is called \((\tau_i, \tau_j)\)-sg-meager if \(A\) is a countable union of \((\tau_i, \tau_j)\)-sg-closed sets. Clearly, every \((\tau_i, \tau_j)\)-meager set is \((\tau_i, \tau_j)\)-sg-meager but not vice versa.

Definition. [13, Fukutake; 1992]. A bitopological space \((X, \tau_1, \tau_2)\) is called \((\tau_i, \tau_j)\)-Baire if \((\tau_i, \tau_j)\)-M \(\cap \tau_i = \emptyset\), where \(i, j \in \{1, 2\}\).

Definition. A bitopological space \((X, \tau_1, \tau_2)\) is called utterly \((\tau_i, \tau_j)\)-Baire if \((\tau_i, \tau_j)\)-sg-M \(\cap \tau_i = \emptyset\), where \(i, j \in \{1, 2\}\).

Clearly every utterly \((\tau_i, \tau_j)\)-Baire space is a \((\tau_i, \tau_j)\)-Baire space but not conversely.

Question 1. How are utterly \((\tau_i, \tau_j)\)-Baire space and \((\tau_i, \tau_j)\)-semi-Baire spaces related? The class of \((\tau_i, \tau_j)\)-semi-Baire spaces was introduced by Fukutake in 1996 [14]. Under what conditions is a \((\tau_i, \tau_j)\)-Baire space utterly \((\tau_i, \tau_j)\)-Baire?
**Question 2.** Let \((X, \tau_1, \tau_2)\) be a bitopological space such that \(\tau_1 \subseteq \tau_2\) and \(\tau_2\) is metrizable and complete. Under what additional conditions is \((X, \tau_1, \tau_2)\) an utterly \((\tau_i, \tau_j)\)-Baire space? Note that \((X, \tau_1, \tau_2)\) is always a \((\tau_i, \tau_j)\)-Baire space \[\text{[14, Fukutake; 1992].}\]

Observe that sg-open and preopen sets are concepts independent from each other.

**Theorem.** \[\text{[22, Maki, Umehara, Noiri; 1996].}\] Every topological space is pre-\(T_{\frac{1}{2}}\).

**Theorem.** Every topological space is sg-\(T_{\frac{1}{2}}\), i.e., every singleton is either sg-open or sg-closed.

Improved Janković-Reilly Decomposition Theorem. Every topological space \((X, \tau)\) has a unique decomposition into two sets \(X_1\) and \(X_2\), where \(X_1 = \{x \in X: \{x\} \text{ is nowhere dense}\}\) and \(X_2 = \{x \in X: \{x\} \text{ is sg-open and locally dense}\}\).

Let \(A\) be a sg-closed subset of a topological space \((X, \tau)\). If every subset of \(A\) is also sg-closed in \((X, \tau)\), then \(A\) will be called *hereditarily sg-closed* \((= \text{hsg-closed})\) \[\text{[3].}\] Hereditarily sg-open sets are defined in a similar fashion. Observe that every nowhere dense subset is hsg-closed but not vice versa.

**Theorem.** \[\text{[3, Dontchev and Ganster; 1998].}\] For a subset \(A\) of a topological space \((X, \tau)\) the following conditions are equivalent:

1. \(A\) is hsg-closed.
2. \(X_1 \cap \text{Int}\, \overline{A} = \emptyset\).

A topological space \((X, \tau)\) is called a \(C_2\)-space \[\text{[15, Ganster; 1987].}\] (resp. \(C_3\)-space \[\text{[3].}\]) if every nowhere dense (resp. hsg-closed) set is finite. Clearly every \(C_3\)-space is a \(C_2\)-space. Also, a topological space \((X, \tau)\) is indiscrete if and only if every subset of \(X\) is hsg-closed (since in that case \(X_1 = \emptyset\)).

Semi-normal spaces can be characterized via sg-closed sets as follows:

**Theorem.** \[\text{[24, Noiri; 1994].}\] A topological space \((X, \tau)\) is semi-normal if and only for each pair of disjoint semi-closed sets \(A\) and \(B\), there exist disjoint sg-open sets \(U\) and \(V\) such that \(A \subseteq U\) and \(B \subseteq V\).

**Question 3.** How do hsg-open sets characterize properties related to semi-normality?

In terms of sg-closed sets, pre sg-continuous functions and pre sg-closed functions were defined and investigated by Noiri in 1994 \[\text{[24].}\]
Following Hodel [20], we say that a cellular family in a topological space \((X, \tau)\) is a collection of nonempty, pairwise disjoint open sets. The following result reveals an interesting property of \(C_2\)-spaces.

**Theorem.** [6, Dontchev and Ganster; 1998]. Let \((X, \tau)\) be a \(C_2\)-space. Then, every infinite cellular family has an infinite subfamily whose union is contained in \(X_2\).

The \(\alpha\)-topology [23, Njåstad; 1965] on a topological space \((X, \tau)\) is the collection of all sets of the form \(U \setminus N\), where \(U \in \tau\) and \(N\) is nowhere dense in \((X, \tau)\). Recall that topological spaces whose \(\alpha\)-topologies are hereditarily compact have been shown to be semi-compact [17, Ganster, Janković and Reilly, 1990]. The original definition of semi-compactness is in terms of semi-open sets and is due to Dorsett [11]. By definition a topological space \((X, \tau)\) is called semi-compact if every cover of \(X\) by semi-open sets has a finite subcover.

**Remark.** (i) The 1-point-compactification of an infinite discrete space is a \(C_2\)-space having an infinite cellular family.

(ii) [15, Ganster; 1987] A topological space \((X, \tau)\) is semi-compact if and only if \(X\) is a \(C_2\)-space and every cellular family is finite.

(iii) [18, Hanna and Dorsett; 1984] Every subspace of a semi-compact space is semi-compact (as a subspace).

**Theorem.** [6, Dontchev and Ganster; 1998]. (i) Every \(C_3\)-space \((X, \tau)\) is semi-compact.

(ii) Every sg-compact space is semi-compact.

**Remark.** (i) It is known that sg-open sets are \(\beta\)-open, i.e. they are dense in some regular closed subspace. Note that \(\beta\)-compact spaces, i.e. the spaces in which every cover by \(\beta\)-open sets has a finite subcover are finite [16, Ganster, 1992]. However, one can easily find an example of an infinite sg-compact space – the real line with the cofinite topology is such a space.

(ii) In semi-\(T_D\)-spaces the concepts of sg-compactness and semi-compactness coincide. Recall that a topological space \((X, \tau)\) is called a semi-\(T_D\)-space [19, Janković and Reilly; 1985] if each singleton is either open or nowhere dense, i.e. if every sg-closed set is semi-closed.

**Theorem.** [6, Dontchev and Ganster; 1998]. For a topological space \((X, \tau)\) the following conditions are equivalent:
(1) $X$ is sg-compact.

(2) $X$ is a $C_3$-space.

Remark. (i) If $X_1 = X$, then $(X, \tau)$ is sg-compact if and only if $(X, \tau)$ is semi-compact. Observe that in this case sg-closedness and semi-closedness coincide.

(ii) Every infinite set endowed with the cofinite topology is (hereditarily) sg-compact.

As mentioned before, an arbitrary intersection of sg-closed sets is also a sg-closed set [8, Dontchev and Maki; 1997]. The following result provides an answer to the question about the additivity of sg-closed sets.

**Theorem.** [6, Dontchev and Ganster; 1998]. (i) If $A$ is sg-closed and $B$ is closed, then $A \cup B$ is also sg-closed.

(ii) The intersection of a sg-open and an open set is always sg-open.

(iii) The union of a sg-closed and a semi-closed set need not be sg-closed, in particular, even finite union of sg-closed sets need not be sg-closed.

**Problem.** Characterize the spaces, where finite union of sg-closed sets is sg-closed, i.e. the spaces $(X, \tau)$ for which $SGO(X, \tau)$ is a topology. Note: It is known that the spaces where $SO(X, \tau)$ is a topology is precisely the class of extremally disconnected spaces, i.e., the spaces where each regular open set is regular closed.

A result of Bhattacharyya and Lahiri from 1987 [1] states that if $B \subseteq A \subseteq (X, \tau)$ and $A$ is open and sg-closed, then $B$ is sg-closed in the subspace $A$ if and only if $B$ is sg-closed in $X$. Since a subset is regular open if and only if it is $\alpha$-open and sg-closed [9, Dontchev and Przemski; 1996], we obtain the following result:

**Theorem.** [6, Dontchev and Ganster; 1998]. Let $R$ be a regular open subset of a topological space $(X, \tau)$. If $A \subseteq R$ and $A$ is sg-open in $(R, \tau|_R)$, then $A$ is sg-open in $X$.

Recall that a subset $A$ of a topological space $(X, \tau)$ is called $\delta$-open [25, Veličko; 1968] if $A$ is a union of regular open sets. The collection of all $\delta$-open subsets of a topological space $(X, \tau)$ forms the so called semi-regularization topology.

**Corollary.** If $A \subseteq B \subseteq (X, \tau)$ such that $B$ is $\delta$-open in $X$ and $A$ is sg-open in $B$, then $A$ is sg-open in $X$. 

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Theorem. [6, Dontchev and Ganster; 1998]. Every $\delta$-open subset of a sg-compact space $(X, \tau)$ is sg-compact, in particular, sg-compactness is hereditary with respect to regular open sets.

Example. Let $A$ be an infinite set with $p \notin A$. Let $X = A \cup \{p\}$ and $\tau = \{\emptyset, A, X\}$.

(i) Clearly, $X_1 = \{p\}$, $X_2 = A$ and for each infinite $B \subseteq X$, we have $\overline{B} = X$. Hence $X_1 \cap \text{Int}B \neq \emptyset$, so $B$ is not hsg-closed. Thus $(X, \tau)$ is a $C_3$-space, so sg-compact. But the open subspace $A$ is an infinite indiscrete space which is not sg-compact. This shows that hereditary sg-compactness is a strictly stronger concept than sg-compactness and '\(\delta\)-open' cannot be replaced with 'open'.

(ii) Observe that $X \times X$ contains an infinite nowhere dense subset, namely $X \times X \setminus A \times A$. This shows that even the finite product of two sg-compact spaces need not be sg-compact, not even a $C_2$-space.

(iii) [21, Maki, Balachandran and Devi; 1996] If the nonempty product of two spaces is sg-compact $T_{gs}$-space, then each factor space is sg-compact.

Recall that a function $f: (X, \tau) \to (Y, \sigma)$ is called pre-sg-continuous [24, Noiri, 1994] if $f^{-1}(F)$ is sg-closed in $X$ for every semi-closed subset $F \subseteq Y$.

Theorem. [6, Dontchev and Ganster; 1998]. (i) The property 'sg-compact' is topological. (ii) Pre-sg-continuous images of sg-compact spaces are semi-compact.

3 N-scattered spaces

A topological space $(X, \tau)$ is scattered if every nonempty subset of $X$ has an isolated point, i.e. if $X$ has no nonempty dense-in-itself subspace. If $\tau^\alpha = \tau$, then $X$ is said to be an $\alpha$-space [10, Dontchev and Rose; 1996] or a nodec space. All submaximal and all globally disconnected spaces are examples of $\alpha$-spaces. Recall that a space $X$ is submaximal if every dense set is open and globally disconnected [12, El'kin; 1969] if every set which can be placed between an open set and its closure is open, i.e. if every semi-open set is open.

Recently $\alpha$-scattered spaces (= spaces whose $\alpha$-topologies are scattered) were considered by Dontchev, Ganster and Rose [4] and it was proved that a space $X$ is scattered if and only if $X$ is $\alpha$-scattered and N-scattered.
Recall that a topological ideal $\mathcal{I}$, i.e. a nonempty collection of sets of a space $(X, \tau)$ closed under heredity and finite additivity, is $\tau$-local if $\mathcal{I}$ contains all subsets of $X$ which are locally in $\mathcal{I}$, where a subset $A$ is said to be locally in $\mathcal{I}$ if it has an open cover each member of which intersects $A$ in an ideal amount, i.e. each point of $A$ has a neighborhood whose intersection with $A$ is a member of $\mathcal{I}$. This last condition is equivalent to $A$ being disjoint with $A^*(\mathcal{I})$, where $A^*(\mathcal{I}) = \{x \in X : U \cap A \not\in \mathcal{I} \text{ for every } U \in \tau_x\}$ with $\tau_x$ being the open neighborhood system at a point $x \in X$.

**Definition.** [10, Dontchev and Rose; 1996]. A topological space $(X, \tau)$ is called $N$-scattered if every nowhere dense subset of $X$ is scattered.

Clearly every scattered and every $\alpha$-space, i.e. nodec space, is $N$-scattered. In particular, all submaximal spaces are $N$-scattered. The density topology on the real line is an example of an $N$-scattered space that is not scattered. The space $(\omega, L)$ below shows that even scattered spaces need not be $\alpha$-spaces. Another class of spaces that are $N$-scattered (but only along with the $T_0$ separation) is Ganster’s class of $C_2$-spaces.

**Theorem.** [10, Dontchev and Rose; 1996]. If $(X, \tau)$ is a $T_1$ dense-in-itself space, then $X$ is $N$-scattered $\iff N(\tau) = S(\tau)$, where $N(\tau)$ is the ideal of nowhere dense subsets of $X$, and $S(\tau)$ is the ideal of scattered subsets of $X$.

Example. Let $X = \omega$ have the cofinite topology $\tau$. Then $X$ is a $T_1$ dense-in-itself space with $N(\tau) = I_\omega$, where $I_\omega$ is the ideal of all finite sets. Clearly, $X$ is an $N$-scattered space, since $N(\tau) = I_\omega \subseteq S(\tau)$. Note that $N(\tau) = S(\tau)$. Also, $X$ is far from being ($\alpha$)-scattered having no isolated points. It may also be observed that the space of this example is N-scattered being an $\alpha$-space.

Remark. A space $X$ is called (pointwise) homogenous if for any pair of points $x, y \in X$, there is a homeomorphism $h: X \to X$ with $h(x) = y$. Topological groups are such spaces. Further, such a space is either crowded or discrete. For if one isolated point exists, then all points are isolated. However, the above given space $X$ is a crowded homogenous $N$-scattered space.

Noticing that scatteredness and $\alpha$-scatteredness are finitely productive might suggest that $N$-scatteredness is finitely productive. But this is not the case.
Example. The usual space of Reals, \((\mathbb{R}, \mu)\) is rim-scattered but not N-scattered. Certainly, the usual base of bounded open intervals has the property that nonempty boundaries of its members are scattered. However, the nowhere dense Cantor set is dense-in-itself. Another example of a rim-scattered space which is not N-scattered is constructed by Dontchev, Ganster and Rose [7].

Remark. It appears that rim-scatteredness is much weaker than N-scatteredness.

Theorem. [10, Dontchev and Rose; 1996]. N-scatteredness is hereditary.

Theorem. [10, Dontchev and Rose; 1996]. The following are equivalent:

(a) The space \((X, \tau)\) is N-scattered.

(b) Every nonempty nowhere dense subspace contains an isolated point.

(c) Every nowhere dense subset is scattered, i.e., \(N(\tau) \subseteq S(\tau)\).

(d) Every closed nowhere dense subset is scattered.

(e) Every nonempty open subset has a scattered boundary, i.e., \(\text{Bd}(U) \in S(\tau)\) for each \(U \in \tau\).

(f) The \(\tau^\alpha\)-boundary of every \(\alpha\)-open set is \(\tau\)-scattered.

(g) The boundary of every nonempty semi-open set is scattered.

(h) There is a base for the topology consisting of N-scattered open subspaces.

(i) The space has an open cover of N-scattered subspaces.

(j) Every nonempty open subspace is N-scattered.

(k) Every nowhere dense subset is \(\alpha\)-scattered.

Corollary. Any union of open N-scattered subspaces of a space \(X\) is an N-scattered subspace of \(X\).

Remark. The union of all open N-scattered subsets of a space \((X, \tau)\) is the largest open N-scattered subset \(NS(\tau)\). Its complement is closed and if nonempty contains a nonempty crowded nowhere dense set. Moreover, \(X\) is N-scattered if and only if \(NS(\tau) = X\). Since partition spaces are precisely those having no nonempty nowhere dense sets, such spaces are N-scattered. On the other hand we have the following chain of implications. The space \(X\) is discrete \(\Rightarrow\) \(X\) is a partition space \(\Rightarrow\) \(X\) is zero dimensional \(\Rightarrow\) \(X\) is rim-scattered. Also, \(X\) is globally disconnected \(\Rightarrow\) \(X\) is N-scattered. However, this also follows quickly
from the easy to show characterization $X$ is globally disconnected $\iff$ $X$ is an extremally disconnected $\alpha$-space, and the fact that every $\alpha$-space is N-scattered. Actually, something much stronger can be noted. Every $\alpha$-space is N-closed-and-discrete, i.e. $N(\tau) \subseteq CD(\tau)$. Of course, $CD(\tau) \subseteq D(\tau) \subseteq S(\tau)$, where $CD(\tau)$ is the ideal of closed and discrete subsets of $(X, \tau)$, and $D(\tau)$ is the family of all discrete sets. We will show later that for a non-N-scattered space $(X, \tau)$ in which $NS(\tau)$ contains a non-discrete nowhere dense set, $\tau^\alpha$ is not the smallest expansion of $\tau$ for which $X$ is N-scattered, i.e., there exists a topology $\sigma$ strictly intermediate to $\tau$ and $\tau^\alpha$ such that $NS(\sigma) = X$.

Local N-scatteredness is the same as N-scatteredness.

**Theorem.** [10, Dontchev and Rose; 1996]. If every point of a space $(X, \tau)$ has an N-scattered neighborhood, then $X$ itself is N-scattered.

In the absence of separation, a finite union of scattered sets may fail to be scattered. For example, the singleton subsets of a two-point indiscrete space are scattered. But given two disjoint scattered subsets, if one has an open neighborhood disjoint from the other, then their union is scattered.

**Theorem.** [10, Dontchev and Rose; 1996]. In every $T_0$-space $(X, \tau)$, finite sets are scattered, i.e., $I_\omega \subseteq S(\tau)$.

**Theorem.** [10, Dontchev and Rose; 1996]. Let $(X, \tau)$ be a non-N-scattered space, so that $NP(\tau) = X \setminus NS(\tau) \neq \emptyset$. Suppose also that $NS(\tau)$ contains a nonempty non-discrete nowhere dense subset. Then there is a topology $\sigma$ with $\tau \subset \sigma \subset \tau^\alpha$ such that $(X, \sigma)$ is N-scattered.

In search for a smallest expansion of $\sigma$ and $\tau$ for which $(X, \sigma)$ is N-scattered, we have the following:

**Theorem.** [10, Dontchev and Rose; 1996]. Let $(X, \tau)$ be a space and let $I = \{ A \subseteq E: E$ is a perfect (closed and crowded) nowhere dense subset of $(X, \tau) \}$. Then $(X, \gamma)$ is N-scattered, where $\gamma = \tau[I]$, the smallest expansion of $\tau$ for which members of $I$ are closed.

**Theorem.** [10, Dontchev and Rose; 1996]. Every closed lower density topological space $(X, F, I, \phi)$ for which $I$ is a $\sigma$-ideal containing finite subsets of $X$ is N-scattered. Recall that a lower density space $(X, F, I, \phi)$ is closed if $\tau_\phi \subseteq F$. 

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Corollary. If \((X, F, I, \phi)\) is a closed lower density space and \(I\) is a \(\sigma\)-ideal with \(I_\omega \subseteq I\), then \(I = S(\tau_\phi)\).

Corollary. The space of real numbers \(R\) with the density topology \(\tau_d\) is \(N\)-scattered, and moreover, the scattered subsets are precisely the Lebesgue null sets.

The following theorem holds and thus we have another (perhaps new) characterization of \(T_0\) separation. A similar characterization holds for \(T_1\) separation.

Theorem. [10, Dontchev and Rose; 1996]. A space \((X, \tau)\) has \(T_0\) separation if and only if \(I_\omega \subseteq S(\tau)\).

Here is the relation between \(C_2\)-spaces and \(N\)-scattered spaces:

Corollary. Every \(C_2T_0\)-space is \(N\)-scattered.

Theorem. [11]. Dontchev and Rose; 1996]. A space \((X, \tau)\) has \(T_1\) separation if and only if \(I_\omega \subseteq D(\tau)\).

Theorem. [11]. Dontchev and Rose; 1996]. If \((X, \tau)\) is a \(T_0\)-space and if \(S\) is any scattered subset of \(X\) and if \(F\) is any finite subset of \(X\), then \(S \cup F\) is scattered.

Corollary. Every \(T_0\)-space which is the union of two scattered subspaces is scattered.

Corollary. The family of scattered subsets in a \(T_0\)-space is an ideal.

Example. Let \((X, <)\) be any totally ordered set. Then both the left ray and right ray topologies \(L\) and \(R\) respectively, are \(T_0\) topologies. They are not \(T_1\) if \(|X| > 1\). In case \(X = \omega\) with the usual ordinal ordering \(<\), \(L\) and \(R\) are in fact \(T_D\) topologies, i.e. singletons are locally closed. The space \((\omega, L)\), where proper open subsets are finite, is scattered. For if \(\emptyset \neq A \subseteq \omega\) let \(n\) be the least element of \(A\). Then the open ray \([0, n + 1) = [0, n]\) intersects \(A\) only at \(n\). Thus, \(n\) is an isolated point of \(A\). Evidently, \(S(L) = P(\omega)\), the maximum ideal. However, \(S(R) = I_\omega\), the ideal of finite subsets. For if \(A\) is any infinite subset of \(\omega\), \(A\) is crowded. For if \(m \in A\) and if \(U\) is any right directed ray containing \(m\), \((U \setminus \{m\}) \cap A \neq \emptyset\), since \(U\) omits only finitely many points of \(\omega\). But every finite subset is scattered. Of course \(S(L)\) and \(S(R)\) are ideals in the last two examples since both \(L\) and \(R\) are \(T_D\) topologies.

Remark. Note that the space \((\omega, R)\) is a crowded \(T_D\)-space, which is the union of an increasing (countable) chain of scattered subsets. In particular, \(\omega = \cup_{k \in \omega}\{n < k: k \in \omega\}\) and for each \(k\), \(\{n < k: k \in \omega\} \in I_\omega \subseteq S(R)\). This seems to indicate that it is not likely
that an induction argument on the cardinality of a scattered set $F$ can be used similar to
the above argument to show that $S \cup F$ is scattered if $S$ is scattered.

**Question 4.** Characterize the spaces where every hsg-closed set is scattered. How are
they related to other classes of generalized scattered spaces? Note that:

\[ \text{Scattered} \Rightarrow \text{hsg-scattered} \Rightarrow \text{N-scattered} \]

Note that the real line with the cofinite topology is an example of a hsg-scattered space,
which is not scattered, while the real line with the indiscrete topology provides an example
of an $N$-scattered space that is not hsg-scattered.

More generally, if $\mathcal{I}$ is a topological (sub)ideal on a space $(X, \tau)$, investigate the class
of $\mathcal{I}$-scattered spaces, i.e. the spaces satisfying the condition: “Every $I \in \mathcal{I}$ is a scattered
subspace of $(X, \tau)$”.

Note that:
\begin{align*}
\mathcal{F}\text{-scattered} & \Leftrightarrow T_0\text{-space} \\
\mathcal{C}\text{-scattered} & \Leftrightarrow ? \\
\mathcal{N}\text{-scattered} & \Leftrightarrow N\text{-scattered space} \\
\mathcal{M}\text{-scattered} & \Leftrightarrow ? \\
\mathcal{P}(X)\text{-scattered} & \Leftrightarrow \text{Scattered space} \\
\text{Of course, every space is } \mathcal{C}\mathcal{D}\text{-scattered.}
\end{align*}

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