AN APPROACH TO NON-STANDARD ANALYSIS

ELIAHU LEVY

ABSTRACT. This note has two principal aims: to portray an essence of Non-Standard Analysis as a particular structure (which we call lim-rim), noting its interplay with the notion of ultrapower, and to present a construction of Non-Standard Analysis, viewed as a matter of mathematics, where the set *A of non-standard elements of a set A – “the adjunction of all possible limits” is a “good” kind of lim-rim which plays a role analogous to that of the algebraic closure of a field – “the adjunction of all roots of polynomials”. In the same spirit as with algebraic closures, one has uniqueness up to isomorphism, and also universality and homogeneity, provided one has enough General Continuum Hypothesis. The cardinality of *A will be something like 2^{2^{\mid A\mid}} – the same as that of the set of ultrafilters in A, and one has a high degree of saturation.

0. Notations

Denote the cardinality of a set A by \mid A\mid. For a cardinality m, denote by m^+ the successor cardinality.

1. Features of the Conventional Approach to Non-Standard Analysis

Non-standard analysis was constructed by its founder, Abraham Robinson ([R]) as a non-standard model of part of Set Theory. In his and following treatments (see several treatises in the bibliography)\footnote{In another direction, beginning with E. Nelson [N], one constructs a special Set Theory for Non-Standard Analysis.} the idea is to start with a “big” structure – a set V, with the restriction to it of the membership relation \in, which will be big enough so that usual mathematics could be done in it (should contain, e.g. \N (the natural numbers), \R (the real numbers), sets of them, sets of these etc.) V is to be provided with a “mirror” “non-standard” structure *V with a relation *\in, viewed as an “interpretation” of the \in of V (Also, “interpret” “=” in V by “=” in *V), so that

(1) V is embedded in *V – identified with a subset of *V. The members of V (thus considered as members of *V) are referred to as standard (yet by “non-standard” elements we mean any members of *V).

(2) A “Transfer Principle” holds: any (in some constructions one has to say: “bounded”) first-order logical expression with members of V as constants which is a true sentence when quantifiers are interpreted relative to V will remain a true sentence if (the constants – members of V – are replaced by their identified images in *V and) \in is replaced by *\in and the quantifiers are interpreted relative to *V. In this sense *V is a model to the true in V sentences of the first-order theory with \in and all elements of V as constants.

(3) By Transfer, for any standard set A, a standard a \in V will be an *\in-member of it iff it is a usual member, but in *V A may have other, non-standard members. The set of all elements of *V which *\in-belong to A is denoted by *A. Again by Transfer, *(A \times B) is canonically identified with *A \times *B, thus for relations, functions etc. we have the *-relation, *-function etc. among members of *V (which for standard elements coincides
with the original relation, function etc.) When there is no danger of misunderstanding, one often uses the same symbol (e.g. +, \(<\)) for the \(*\)-relation etc. as for the original.

(4) \(*V\) is indeed “non-standard” — it contains elements not in \(V\), and one has features, usually some kind of “saturation” (see below), which will ensure, say, that there is an \(*n \in \*V\) which \(*\in\)-belongs to the set \(\mathbb{N}\) of natural numbers, but such that for every standard \(m \in \mathbb{N}\), \(m \ast \ast n\) (i.e. \((m, \ast n) \in\)-belongs to the graph of \(<\)). Such \(*n\)'s are naturally referred to as “infinite” or “unlimited”\(^2\) and by Transfer they have reciprocals in \(*\mathbb{R}\), referred to as “infinitesimal”.

The “philosophy” here is to develop the theory from these properties, paying less attention to the way such \(*V\) is constructed (or proved to exist). The usual construction is by an ultrapower or a modification of it. Anyway, one has to have a set \(V\) to make the construction, hence only subsets of it will have non-standard members.

We however, choose to put the emphasis on the prescription to define the set (or class) \(*A\) for any set (or class) \(A\) — the relation \(\in\) is viewed as just one instance of this.\(^3\)

2. Ultrapowers

Let \(S\) be a fixed set of indices and \(\mathcal{U}\) a fixed ultrafilter in \(S\). For any set \(A\) we have the ultrapower \(A_\mathcal{U} := A^S/\mathcal{U}\), defined as the power \(A^S\) modulo the equivalence relation: equality modulo \(\mathcal{U}\).

We have here a covariant functor \(A \mapsto A_\mathcal{U}\) (which depends on \(S\) and \(\mathcal{U}\)) from the category of sets to itself. Also, for finite \(I\), \((A^I)_\mathcal{U}\) is canonically identified with \((A_\mathcal{U})^I\), and if \(A' \subset A\) then \((A')_\mathcal{U}\) is canonically identified with a subset of \(A_\mathcal{U}\), and finite unions, finite intersections and complements carry over.

Thus one may view \(A_\mathcal{U}\) as the set \(*A\) of the non-standard elements of \(A\), and the Transfer Principle is readily seen to hold w.r.t. first-order predicate calculus expressions.

Note that we always have the set (or class) \(*A\) of non-standard elements for any set (or class) \(A\). The membership relation goes as just one case. It endows any non-standard element (= “set”) \(*x\) with the set \(\alpha\) of its \(*\)-(non-standard)-members, which by Transfer (of the extensionality of sets) determines \(*x\). The \(\alpha\)'s obtained that way are the internal sets (of non-standard elements). One must emphasize that “standard” or “non-standard” is here just a “role” that usual mathematical (i.e. set-theory) entities play w.r.t. the construction. Sets of non-standard elements will naturally have their own non-standard elements, and that can be used to advantage.

3. Introduction to the Proposed Construction

In our “philosophy”, we insist on viewing Non-Standard Analysis as something totally “part of ordinary mathematics”, where one extends any set by adjoining all (or some) possible limits, this in close analogy in spirit to extending a field to its algebraic closure by adjoining all possible roots of algebraic equations. In the latter case one would like to simply define the adjoined elements as labeled by the irreducible polynomial they satisfy, but that is hampered by such facts as the polynomial whose roots are the sums of the roots of two irreducible

\(^2\)In fact, for \(\mathbb{N}\) the existence of “infinite” members needs only the existence of some non-standard members — one easily proves that all \(*\)-members of a finite standard \(A\) are standard, hence any \(*\)-member of \(\mathbb{N}\) which is not standard is automatically bigger than all standard numbers.

\(^3\)If one wishes to be axiomatic, one may define a Non-Standard Analysis in general as a syntactic procedure to pass from a set-theoretic formula defining \(x \in A\) to a set-theoretic formula defining \(x \ast \in \ast A\) (i.e. \(x \in \ast A\)), so that the required properties are theorems.
polynomials being not irreducible. The algebraic closure thus contains conjugate elements with the same irreducible polynomial and for any such elements there is an automorphism of the algebraic closure exchanging them.

Similarly in our case we would like to label the adjoined “non-standard elements” by the ultrafilters on the original set, but that would not work because the Cartesian product of two ultrafilters is in general not an ultrafilter. We too will have a construction where there will be “conjugate” non-standard elements with the same ultrafilter and any such conjugate elements will be exchangeable by an automorphism, provided one has enough General Continuum Hypothesis (GCH), and then, in the same spirit as with algebraic closures, one has uniqueness (for fixed “basis” B – see below) up to isomorphism, and also universality and homogeneity.

We focus on a structure of a set \(E\) being a “lim-rim” of some other, “basis” set \(B\), a notion defined below. In fact, a lim-rim is a structure obtained in a set \(E\) whenever it serves, in any reasonable way, as a set of non-standard elements of \(B\).

We shall construct the non-standard elements of any set or class using a lim-rim – a set of non-standard elements – on a fixed set \(B\) (i.e. a fixed cardinality) and use that “template” to define the non-standard elements of any set or class \(X\) (including non-standard elements in sets of non-standard elements!) by, in essence, “carrying” them over from \(B\) to \(E\). In fact, we can achieve that goal by defining \(\ast X\) as a “cylindrical” ultrapower of \(X\) w.r.t. to the “cylindrical” ultrafilter given by the lim-rim on \(B\) (see below).

The cardinality of the appropriate lim-rim (set of non-standard elements) which we shall have on \(B\) will be just \(2^{2^{\|B\|}}\) – the same as that of the set of ultrafilters in \(B\). For many purposes it will be convenient to choose \(B\) either countable or with cardinality of the continuum, and different choices of \(B\) can be “merged”.

We shall have the usual features: Transfer Principle, Saturation and also what we call Confinement – every non-standard element belongs to some standard set of a restricted cardinality, where the (conflicting) cardinality restrictions for Saturation and Confinement that are obtained depend on the cardinality of \(B\).

4. Lim-Rims

We say that a set \(E\) has the structure of lim-rim over a set (basis) \(B\) if an ultrafilter \(\mathcal{L}\) in the cylinder Boolean algebra of \(B^E\) is given. The members of the cylinder Boolean algebra are the subsets of \(B^E\) that depend only on a finite number of coordinates. (In case \(E\) is finite this is just the power set of \(B^E\).) What \(\mathcal{L}\) does is deciding, for any (finite) family \(\eta \in E^I\) indexed by a finite \(I\) (pushing \(\mathcal{L}\), using the map \(B^E \to B^I\) induced by \(\eta : I \to E\), to an ultrafilter in \(B^I\)) whether relations – subsets of \(B^I\) – hold or not, thus transferring such \(I\)-relations in \(B\) (= subsets of \(B^I\)) into \(I\)-relations in \(E\), as a non-standard setting should.

To put it otherwise: for an \(n\)-tuple \((e_1, \ldots, e_n)\) of elements of \(E\), a relation \(R \subseteq B^n\) *-holds for \((e_1, \ldots, e_n)\) iff for \(\psi \in B^E\), \((\psi_{e_1}, \ldots, \psi_{e_n}) \in R\) holds modulo \(\mathcal{L}\).

In fact, giving \(\mathcal{L}\) is equivalent to giving a natural mapping between the (contravariant) functors on sets \(I: E^I\) and \((B^I)^\ast\) where the latter denotes the set of ultrafilters in the cylinder Boolean algebra. (Note that these functors and the natural mapping are defined for any \(I\), yet

---

\(^4\)Which may be viewed as the “reason” for the existence of several conjugate roots: the polynomial \(d(t)\) whose roots are the differences of the roots of the irreducible \(f(t)\) has a factor \(t\) but also other irreducible factors, testifying to differences \(\neq 0\).

\(^5\)In the sequel we usually assume \(B\) infinite.
they are determined by giving them for the finite \( I \).) In particular, the “cylindrical” ultrafilter \( \mathcal{L} \) in \( B^E \) can be recovered as the image in \( (B^E)^* \) of the identity family in \( E^E \).

But note that whenever a set \( E \) serves as the set of non-standard element of a set \( B \), with the Transfer Principle holding at least for the Propositional Calculus operations, such a natural mapping between the above functors arises – a family of non-standard elements (i.e. an element of \( E^I \) for finite \( I \)) corresponds to an ultrafilter in \( B^I \) consisting of the “standard” relations it satisfies. Thus the structure of \( E \) a lim-rim over \( B \) captures an essence of the notion of a “set of non-standard elements of \( B \”).

**Remark 1.** There can be sub-lim-rims in two ways (or both combined): Firstly, any subset \( E' \subset E \) has the structure of a lim-rim over \( B \), and secondly for any \( B' \subset B \) the set \( *B' \) of the \( e \in E \) that satisfy the transfer of “ belongs to \( B' \)” (i.e. \( B' \) involved in a lim-rim) form a lim-rim over \( B' \). Instead of talking about sub-lim-rims one may talk about embeddings. Isomorphisms and automorphisms of lim-rims are defined as expected (as those which respect the ultrafilter \( \mathcal{L} \)).

4.1. **Lim-Rims and Ultrapowers.** If \( E \) has the structure of lim-rim over \( B \), any \( e \in E \) defines an evaluation (or coordinate) map \( B^E \to B \). Thus \( E \) is identified with a set of mappings \( B^E \to B \). Also, in \( B^E \) we have the ultrafilter \( \mathcal{L} \) on the cylinder Boolean algebra. \( E \) is thus mapped into the ultrapower \( B^E/\mathcal{U} \), where the ultrafilter \( \mathcal{U} \) is some extension of \( \mathcal{L} \) to all the subsets of \( B^E \). And one easily finds that the notion of a family \( \eta \in E^I \) (finite) \( \ast \)-satisfying a relation \( R \subset B^I \) is the same for this ultrapower as for the original lim-rim \( E \) (whatever ultrafilter extension \( \mathcal{U} \) of \( \mathcal{L} \) to all subsets of \( B^E \) one takes).

Conversely, if \( E \) is a subset of an ultrapower \( B^S/\mathcal{U} \) w.r.t. an ultrafilter \( \mathcal{U} \) in \( S \), then the notion of a family \( \eta : I \to E \) (finite) \( \ast \)-satisfying a relation \( R \subset B^I \) is defined, thus \( E \) is made into a lim-rim over \( B \). To obtain the ultrafilter \( \mathcal{L} \) (on the cylinder Boolean algebra of \( B^E \)) of this lim-rim, lift \( E \) to a subset of \( B^S \), which makes a map \( E \times S \to B \), thus a map \( S \to B^E \), which pushes \( \mathcal{U} \) to an ultrafilter in \( B^E \). Its restriction to the cylinder Boolean algebra does not depend on the choice of the lifting, and will be \( \mathcal{L} \).

Thus, since every reasonable way to endow a set \( B \) with the set of its non-standard elements \( \ast B \) involves a lim-rim, and lim-rims boil down to ultrapowers, one concludes that, in some sense, any reasonable Non-Standard Analysis on a set is, in fact, defined by a lim-rim and by an ultrapower.

4.2. **Exact Lim-Rims.** For any lim-rim, transfer of relations will commute with Propositional Calculus operations, but in general not necessarily with quantifiers, i.e. with projections \( B^I \cup \{i\} \to B^I \ (i \notin I) \).

This, however, will be guaranteed if we impose the requirement that for any \( \xi : I \to J \) (finite) the diagram expressing the naturality of the above mapping between the functors is “exact”, in the sense that a member of \( E^I \) and a member of \( (B^I)^* \) which map to the same member of \( (B^J)^* \) both come from some same member of \( E^J \). For this to hold it suffices that it holds for inclusions “adding one element” \( I \to I \cup \{i\} \) (guaranteeing that the transfer of relations commutes with projections \( B^{I \cup \{i\}} \to B^I \) and for the map \( \{1,2\} \to \{1\} \) (which guarantees that the relation of equality will be transferred to the relation of equality – whenever this is the case for a lim-rim it is called **separated**).

If this exactness requirement is satisfied (for all finite \( I, J \)), we say that the lim-rim is **exact** or is a Non-Standard Analysis **lim-rim**. Since for \( I = \emptyset \) both \( E^\emptyset \) and \( (B^\emptyset)^* \) are singletons with subsets naturally viewed as the truth-values, which the natural mapping always preserves, the above commuting of transfer of relations with logical operations implies
the Transfer Principle: any true sentence made as a first-order logical combination of relations on $B$ will turn, by transferring each of the argument relations, into a sentence true for $E$.

In particular, applying the exactness of the diagram for the map $\emptyset \to \{1\}$ we have, for exact lim-rims, that for every ultrafilter $U$ in $B$ there is an element $e \in E$ whose ultrafilter, i.e. whose image in the natural mapping $E = E^{(1)} \to (B^{(1)})^*$, is $U$, that is, $e$ “belongs” to all members of $U$ (and of course does not belong to non-members of $U$, which are the complements of members).$^6$ If $U$ is fixed (principal) – is the family of all subsets containing some $a \in B$ – than $e$ is unique and is the element of $E$ to be identified with $a$. If $U$ is free (non-principal), however, one proves that $e$ is never unique and we have “conjugate” non-standard elements with the same ultrafilter which, as we shall see below, will be exchangeable by an automorphism with favorable choice of the lim-rim.

But lim-rims may have more: they may have also some saturation. If $m$ is a cardinal, let us say that a lim-rim $E$ is $m$-exact if the above exactness holds for the inclusion $I \to I \cup \{i\}$ if $|I| < m$. This means: for every family $\eta : I \to E$ of elements of $E$, and any ultrafilter $U$ in the cylinders of $B^{U\cup\{i\}}$ which projects on $B^I$ to the ultrafilter pulled-back from $L$ by $\eta$, one can find an element of $E$ as the image of $i$ so that the pull-back of $L$ by the extended $\eta$ will be $U$. By a transfinite process such $m$-exactness will guarantee exactness for any inclusion $I \to M$ for $|I| < m, |M| \leq m$. In particular a lim-rim is exact iff it is separated and $\aleph_0$-exact. It is easily seen that $m$-exactness implies $m$-saturation (see §6), i.e. the property that any family of cardinality $< m$ of internal subsets of $E^n$ (i.e. sets of the form $\{R\}[e]$ where $R \subset B^{n+1}$ and $e \in E$) has non-empty intersection provided any finite subfamily has non-empty intersection. Conversely, $m$-saturation implies $m_1$-exactness if $m' < m_1 \Rightarrow 2^{[B]}|m' < m$ (since, for infinite $B$, the cardinality of the cylinder Boolean algebra of $B^{m'}$ is $2^{[B]}|m'|$). In particular, for $m = (2^{[B]})^+$, $m$-exactness is equivalent to $m$-saturation.

A separated lim-rim $E$ which is $|E|$-exact is universal in the sense that any embedding in $E$ of a sub-lim-rim of cardinality $< |E|$ of a lim-rim $M$ (with basis $B$) of cardinality $\leq |E|$ can be extended to an embedding of all $M$, and using a transfinite back-and-forth construction we conclude that it is homogeneous – any isomorphism between two sub-lim-rims of cardinality $< |E|$ in two separated lim-rims $E$ with the same cardinality $|E|$ which are $|E|$-exact can be extended to an isomorphism between the $E$’s, in particular any two such lim-rims $E$ are isomorphic.

**Remark 2.** (Combining §4.2 and §4.1.) Let $E$ be an exact lim-rim over $B$. By §4.1 one constructs an ultrapower $B^{B^E}/L$. (By abuse of language, we write $L$ for an ultrafilter on all subsets of $B^E$ extending it, since everything will depend only on $L$.) $E$ embeds into the ultrapower by mapping each $e \in E$ to the coordinate map $B^E \to B$. The fact that this is 1-1 follows from $E$ being separated.

It might seem that the whole ultrapower must be much larger than the image of $E$. Yet, consider the fact that $E$ is exact, and the resulting Transfer Principle:

If $f : B^E \to B$ depends on a finite set $I \subset E$ of coordinates, it is essentially a function $f : B^I \to B$. By Transfer, one may plug $I$ into $f$, and the element $e \in E$ so obtained is easily seen to be equal to $f$ modulo $L$. Thus any “family” $f : B^E \to B$ which depends only on a finite set of coordinates is, in the ultrapower, equal modulo $L$ to an element of the embedded $E$.

$^6$This does not hold, in general, for Non-Standard Analysis constructed by an ultrapower (if the power in the ultrapower is countable, a countable set will have only $2^{80}$ non-standard members but has $2^{2^{80}}$ ultrafilters.)
5. Regular Lim-Rims; Furnishing any Set with Non-Standard Elements

For fixed $B$ (whose cardinality $|B|$ we shall call the base) we can construct a separated lim-rim of cardinality $2^{2^{|B|}}$ which is $(2^{|B|})^+$-saturated, i.e. $(2^{|B|})^+$-exact, just by a straightforward transfinite process indexed by the least ordinal of cardinality $(2^{|B|})^+$, in each step adjoining to $E$ all needed elements for the saturation (and extending $\mathcal{L}$ correspondingly), and then taking quotient by the relation among $e, e' \in E$: “the set $\{\xi | \xi_e = \xi_{e'}\} \subset B^E$ is in $\mathcal{L}'''$. We call any separated lim-rim over $B$ with this cardinality and saturation a regular lim-rim with basis $B$. (Note that it was not more difficult to have any saturation than to have $\kappa_0^+$-saturation.) This does not require any Generalized Continuum Hypothesis. But if we have that, namely that $2^{2^{|B|}} = (2^{|B|})^+$ (say, if we work only in the Constructible Universe), then a regular lim-rim $E$ will have $|E|$-exactness, hence is universal and homogeneous as above and is unique up to isomorphism.

Any exact lim-rim $E$ over a basis $B$ induces non-standard elements in any set or class. They are defined as follows and will enjoy $|B|^+$-confinement where $m$-confinement ($m$ a cardinality) means that any non-standard element $^{*}x$-belongs to some standard set of cardinality $< m$. (In fact, as easily seen, these will be, “up to isomorphism”, the non-standard elements, that belong to standard sets of cardinality $\leq |B|$, in any Non-Standard Analysis for which $E$ is the lim-rim of non-standard elements of $B$.)

If we are to have $|B|^+$-confinement, all non-standard elements belong to sets of cardinality $\leq |B|$, so if we know the non-standard elements of such sets (and, by the Transfer Principle, inclusions etc. among such sets will induce inclusions of the sets of non-standard members etc.) we will have non-standard members of any set or class as a direct limit.

One may think of endowing sets $X$ of cardinality $\leq |B|$ with non-standard elements by “carrying” them over from $B$ in some analogy with the way tangent vectors may be defined in any smooth manifold by “carrying” them over from $\mathbb{R}^n$ – just define a non-standard element of $X$ as a mapping which corresponds to each bijection $j : X \simeq B' \subset B$ an element $e_j$ of $E$ which $^{*}$belongs to $B'$, so that for different bijections $j$ the $e_j$ are connected by the transformation that connects the $j$’s. The Transfer Principle holding in the lim-rim $E$ will ensure that everything fits.

But the same effect can be achieved more neatly (see Remark 2) by defining the non-standard elements of any set $X$ as the members of a “cylindrical” ultrapower $X^{B^E}/\mathcal{L}$ where one takes the set of all maps $B^E \to X$ that depend only on a finite number of coordinates and factors by equivalence modulo $\mathcal{L}$.

In this way, for every set (or class) $A$ we will have the “mirror” set (or class) $^*A$ of the non standard elements of $A$ (all that in our usual set theory). We will have the Transfer Principle (i.e. commuting of first-order logical operations with the transfers $A \to ^*A$) in general ($|B|^+$-confinement easily reduced everything to working within a set of cardinality $|B|$). In a similar way, we will have $m$-saturation in general if we have it in $B$ provided $m \leq |B|^+$. When $E$ is a regular lim-rim with basis $B$ and one is working inside a standard set of cardinality $\leq |B|$ (like when one works inside $B$) one has saturation $(2^{|B|})^+$.

A non-standard element has its degree of confinement – how small the cardinality is of a standard set to which it belongs. Note that anyway, for a cardinal $m$, when one works with standard sets of cardinality $\geq m^+$ one cannot have confinement $\leq m^+$ with saturation $\geq (m^+)^+$ – with $(m^+)^+$-saturation there will be a non-standard $^{*}x$ which does not belong to any standard set of cardinality $\leq m$. Indeed, well-order a standard set of cardinality $m^+$ by the set of ordinals with cardinality $\leq m$ and choose (by saturation) a $^{*}x$ greater then all.
standard elements. Thus confinement and saturation are somehow conflicting requirements, and for various needs one may take different bases (see below).

We shall fix a regular lim-rim over a set \( B \) (referred to as the template) with cardinality \(|B| = b\) (called the base) to endow any set (or class) with non-standard elements. Base \( b \) will give confinement \( b^+ \), saturation \( b^+ \) in general but saturation \((2^b)^+\) if one works inside a standard set of cardinality \( \leq b \). The number of non-standard elements of any infinite standard set of cardinality \( \leq 2^b \) will be \( 2^{2b} \). For infinite \( b' < b \), there will be among them \( 2^{2b} \) non-standard elements with “better” confinement \( b' \) (see the cardinality computations in the next §).

For two regular lim-rims with different bases \( b_1 < b \), the base \( b_1 \) lim-rim will have cardinality \( 2^{2b_1} \) while the base \( b \) lim-rim will have saturation, equivalently exactness, \((2^b)^+\). Suppose, e.g. that \( b \geq 2^{b_1} \) (say \( b_1 = \aleph_0 \) and \( b = \text{continuum} \)) then by saturation we can embed the template \( b_1 \)-lim-rim into the template \( b \)-lim-rim and if we have the case \((2^b)^+ = 2^{2b}\) of GCH then any two such embeddings are exchanged by an automorphism of the \( b \)-lim-rim. Thus in this case the \( b_1 \)-non-standard elements can be viewed as identified with part of the \( b \)-non-standard elements (indeed part of those with confinement \( \leq b_1^+ \)). This identification is not unique but with GCH as above all identifications are “equivalent”. In this way working with \( b_1 \)-non-standard elements (in any set or class) merges smoothly with working with \( b \)-non-standard elements.

6. More about Saturation, Some Calculations of Cardinalities

As mentioned above, in Non-Standard Analysis, an internal set of non-standard elements is one obtained as the set \( \alpha \) of non-standard \(*\)-members of some non-standard \((\ast\)-set) \(*x\). \( \alpha \) determines \(*x\) (by transfer of extensionality), and may serve as a substitute when talking about \(*\)-sets: in transferring a statement about sets translate: set (of standard elements) \( \rightarrow \) internal set (of non-standard elements). One easily shows that equivalently an internal set is one obtained as a section (parallel to one of the axes) of \(*R\), for a standard set \( R \) of ordered pairs, through a non-standard element. In general a set of non-standard elements is not internal: indeed, since, by Transfer, every bounded non-empty internal subset of \(*\mathbb{N}\) has a maximal element, the set of \(*\)-natural-numbers that are identified with the standard ones is not internal. A non-internal subset of a set \( \mathbb{A} \) (for some standard \( A \)) is called external.

The assumption of \( m \)-exactness says that if \( X \) is a standard set, \(*X\) the lim-rim over \( X \) of all non-standard members of \( X \), \(|I| < m, i_0 \notin I, \eta : I \rightarrow *X \) and \( U \in (X^{\mu(i_0)})^* \) such that the projection of \( U \) on \( X^I \) coincides with the pullback of \( \mathcal{L} \) by \( \eta \), then \( \exists \) a \(*x \in *X \) so that if we extend \( \eta \) by mapping \( i_0 \rightarrow *x \) then the pullback will be \( U \).

Now, the members of \( U \) depend each on a finite number of coordinates, and as such are standard relations. When we substitute the \( \eta(i) \)'s we get internal properties (=sets) of the sought-for \( \eta(i_0) \). These properties are not empty, since the \((\eta(i))_i \ast\)-belong to the projections of the members of \( U \) (here we use Transfer). Therefore, since \( U \) is an ultrafilter, any finite collection of these properties intersect. The required existence of a suitable \(*x = \eta(i_0) \) means that all intersect. Elaborating this, one connect \( m \)-exactness with \( m \)-saturation, i.e. the property that:

For any family of less than \( m \) internal sets (= internal properties),
if every finite subfamily has non-empty intersection,
then the whole family has non-empty intersection.

As an example for the use of saturation, let us prove

**Proposition 1.** Let us work with a Non-Standard Analysis as above with base \( b \). Then
(i) Let $B$ be a set of cardinality $b$. Then for every internal subset $\beta \subset {}^*B$ which contains all standard elements of $B$, and for every ultrafilter $\mathcal{U}$ on $B$, $\exists x \in \beta$ whose ultrafilter is $\mathcal{U}$. Consequently, $|\beta| = 2^{2^b}$ (recall that $|{}^*B| = 2^{2^b}$).

(ii) If $\alpha$ is internal, then either $\alpha$ is finite or $\exists$ an internal one-one mapping from an internal $\beta$ as in (i) to $\alpha$, hence $|\alpha| \geq 2^{2^b}$.

(iii) If $\alpha$ is $^*$finite (i.e. a $^*$member of a standard family of finite sets) then $|\alpha|$ is finite or $2^{2^b}$.

**Proof.** (i) We use $(2^b)^+$-saturation (which holds when working inside $B$): for any finite family of members of $\mathcal{U}$ $\exists^* x \in \beta$ belonging to all of them (indeed a standard one), and by saturation we are done.

(ii) Here we use $b^+$-saturation: if $\alpha$ is not finite, then for every finite set $F$ of standard members of $B$ $\exists$ an internal set $\beta_F \subset {}^*B$ containing all members of $F$ and an internal one-one mapping $\beta_F \rightarrow \alpha$ (take $\beta_F = F$), hence by saturation $\exists a \beta$ suitable for all standard members of $B$.

(iii) Suppose $\alpha$ not finite. By (ii) $|\alpha| \geq 2^{2^b}$. On the other hand, by $b^+$-confinement $\alpha$ *belongs to a standard set $S$ of cardinality $\leq b$ whose members are finite sets. Then $|\cup S| \leq b$, thus $|^*(\cup S)| \leq 2^{2^b}$, while $\alpha \subset {}^*(\cup S)$. □

**References**

[ChK] C. C. Chang, H. J. Keisler, *Model Theory* (3rd edition), North-Holland, Amsterdam 1992.

[Cu88] N. J. Cutland (ed.), *Nonstandard Analysis and its Applications*. Cambridge Univ. Press, 1988

[Cu01] N. J. Cutland, *Loeb Measures in Practice: Recent Advances*. Lecture Notes in Mathematics Vol. 1751, Springer-Verlag, 2001.

[HL] A. E. Hurd, P. A. Loeb, *An Introduction to Nonstandard Real Analysis*. Academic Press, New York 1985.

[K76] H. J. Keisler, *Foundations of Infinitesimal Calculus*. Prindle, Weber and Schmidt, Boston 1976.

[K94] H. J. Keisler, *The hyperreal line*, in: P. Ehrlich (ed.), Real Numbers, Generalizations of the Reals, and Theories of Continua, pp. 207-237, Kluwer Academic Publishers, 1994.

[Li] T. Lindstrøm, *An invitation to nonstandard analysis*. in: [Cu88], pp. 1-105.

[Lu] W. A. J. Luxemburg, *What is nonstandard analysis?*, Amer. Math. Monthly. 1973 80 (Supplement) pp. 38-67.

[MH] M. Machover, J. Hirschfeld, *Lectures on Non-Standard Analysis*. Lecture Notes in Mathematics, Vol. 94, Springer-Verlag, 1969.

[N] E. Nelson, *Internal set theory: a new approach to nonstandard analysis*, Bull. Amer. Math. Soc. 83 6, pp. 1165-1198 (1977).

[R] A. Robinson, *Non-Standard Analysis*. North-Holland, 1966.

[SL] K. D. Stroyan, W. A. J. Luxemburg, *Introduction to the Theory of Infinitesimals*. Academic Press, New York 1976.

**Department of Mathematics, Technion – Israel Institute of Technology, Haifa 32000, Israel**

*E-mail address: eliahu@techunix.technion.ac.il*