Non-existence of Higgs fields on Calabi-Yau Manifolds

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Abstract

In this article, we study the Higgs $G$-bundles $(E, \theta)$ on a compact Calabi-Yau manifolds $X$. Our main result is that there is non-existence Higgs fields $\theta$ on a semistable Higgs $G$-bundle over a compact connected Calabi-Yau surface by using Yang-Mills-Higgs flow. The vanish theorem can extends to higher dimensional Calabi-Yau compact $n$-folds, but we should add the principal $E$ with vanishing Chern-classes. In particular, the $G$-bundle $E$ must be a semistable bundle in both cases.

Keywords. semistable Higgs bundle, semistable bundle, Calabi-Yau manifold

1 Introduction

Let $(X, \omega)$ be a compact Kähler manifold and $E$ a holomorphic principal $G$-bundle over $X$. A Higgs $G$-bundle on $X$ is a holomorphic bundle $E$ on $X$ equipped with a holomorphic section $\theta$ of $\text{End}(E) \otimes \Omega^{1,0}(X)$ such $\theta \wedge \theta = 0$. Higgs bundle first emerged twenty years ago in Hitchin’s study of self-dual equation on a Riemann surface (one can see [9]). Furthermore, Simpson using Higgs bundle to study non-abelian Hodge theory (see [18]).

A Higgs bundle $(E, \theta)$ is called stable (semistable) if the usual stability condition:

$$
\mu(E') = \frac{\text{deg}(E')}{\text{rank}E'} < (\leq) \mu(E) = \frac{\text{deg}(E)}{\text{rank}E},
$$

hold for all proper $\theta$-invariant sub-sheaves, where $\mu(E)$ is called the slop of $E$. A Hermitian metric $h$ in Higgs bundle $(E, \theta)$ is said to be Hermitian-Einstein if the curvature of the Hitchin-Simpson connection $d_{A_h} + \theta + \theta^* h$ satisfies the Einstein condition:

$$
\sqrt{-1} \Lambda_{\omega} (F_{A_h} + [\theta, \theta^* h]) - \lambda I d_E = 0,
$$

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where $F_{A_h}$ is the curvature of Chern connection $A_h$, $\theta^{*h}$ is the adjoint of $\theta$ with respect to the metric $h$. In [9] and [18], it is proved that a Higgs bundle admits the Hermitian-Einstein if only if it is polystable. In [4, 5, 6], it shows that a Higgs bundle is semistable if only if it is admits approximate Hermitian-Einstein structure, i.e., for every positive $\varepsilon$, there is a Hermitian metric $h$ on $E$ such that

$$\max |\sqrt{-1}\Lambda_\omega (F_{A_h} + [\theta, \theta^{*h}]) - \lambda Id_E| < \varepsilon.$$  

The existence of stable (semistable) Higgs bundles is depends on the geometry of the underlying manifold. In [2], it shows that the existence of semitable Higgs bundles with a non-trivial Higgs fields on a compact Kähler manifolds $X$ constrains the geometry of $X$. In particular, it was show that if $X$ is a compact Kähler-Einstein manifold with $c_1(TX) \geq 0$, then it is necessary Calabi-Yau, i.e., $c_1(TX) = 0$. In [3], they extend the analysis of the interplay between the existence of semistable Higgs bundles and the geometry of the underlying manifold. If $X$ is Calabi-Yau manifold and $(E, \theta)$ is a semistable Higgs bundle over $X$, they proved that the underlying principal $G$-bundle $E$ is semistable. However, these articles don’t study about the Higgs fields.

In this article, we consider the Higgs fields of the Higgs bundle $E$ over a compact Calabi-Yau manifold $X$. Our main theorem 1.1 says that if $X$ is a compact smooth Calabi-Yau surface with fully holonomy, $(E, \theta)$ is a semistable Higgs $G$-bundle on $X$, then the Higgs fields $\theta$ are vanish. In natural, the Higgs bundle $E$ is a semistable bundle.

**Theorem 1.1.** Let $G$ be a compact Lie group and $E$ a holomorphic principal $G$-bundle with Hermitian metric $H_0$ over a compact smooth Calabi-Yau surface $X$ has fully holonomy and $A \in A_1^{1,1}$ be a connection on $E$. If $(E, \overline{\partial}A, \theta)$ is a semistable Higgs bundle on $X$, then the Higgs fields $\theta$ are vanish.

On the case of higher dimensions Calabi-Yau manifolds, we need add the condition principal bundle $E$ with vanish Chern-class. At first, thanks to Li-Zhang’s work [11, 12, 13], we observe that the global solutions $(A(t), \theta(t))$ of Yang-Mills-Higgs flow (4.1) with initial data $(A_0, \theta_0)$ on a semistable Higgs bundle $(E, \theta)$ with vanish Chern-classes over a compact Kähler manifold will converges to the pair $(A_\infty, \theta_\infty)$ which satisfies the Hermitian-Yang-Mills equations (See Theorem 4.10). In [14], Nie-Zhang also proved a similar theorem in the case of higher dimensional Kähler manifold and the bundle $E$ with vanish Chern classes. We extend the idea of Theorem 1.1 to the case of higher dimensions Calabi-Yau manifolds, hence we also obtain a vanish theorem as follow.

**Theorem 1.2.** Let $G$ be a compact Lie group and $E$ a holomorphic principal $G$-bundle with Hermitian metric $H_0$ over a compact smooth Calabi-Yau $n$-fold $X$ has fully holonomy and $A \in A_{H_0}^{1,1}$ be a connection on $E$. If $(E, \overline{\partial}A, \theta)$ is a semistable Higgs bundle on $X$ and $c_1(E) = c_2(E) = 0$, then the Higgs fields $\theta$ are vanish.
The organization of this paper is as follows. In section 2, we first review the basics about Higgs bundles \((E, \theta)\) and give the Weitzenböck formula for Higgs fields \(\theta\). In Section 3, we define the least eigenvalue \(\lambda(A)\) of \(\nabla_A^* \nabla_A \mid_{\Omega^{1,0}(X, \text{ad}(E))}\) with respect to the connection \(A\). We prove that \(\lambda(A)\) has a lower bound that is uniform with respect to \([A]\) obeying Hermitian-Yang-Mills equation and under the compact Calabi-Yau surface \(X\) has fully holonomy. The last section, we used Yang-Mills-Higgs flow to construction a family \(A\) pair \((A(t), \theta(t))\), then the least eigenvalue \(\lambda[\cdot]\) with respect to \(A(t)\) will has a contradict unless the Higgs field is vanish. We extends the idea to higher dimension Calabi-Yau \(n\)-fold, but in this case we should addition the principal \(E\) with vanishing Chern. At last, we also prove that if \((E, \theta)\) be a polystable Higgs \(G\)-bundle on a compact Calabi-Yau manifold with fully holonomy or \((E, \theta)\) is a semistable Higgs \(G\)-bundle on a compact Kähler-Einstein manifold with \(c_1(TX) > 0\), then \(\theta \equiv 0\). These results had been proved in [2].

2 Preliminaries

In this section, we recall some basics about principal Higgs bundles. Let \(X\) be a smooth complex Kähler manifold, and \(G\) a compact Lie group. If \(E\) is a principal \(G\)-bundle on \(X\), \(Ad(E)\) is its adjoint bundle, and \(\alpha, \beta\) are global sections of \(Ad(E) \otimes \Omega^1_X\), we can define a section \([\alpha, \beta]\) of \(Ad(E) \otimes \Omega^2_X\) by combining the bracket \([\cdot, \cdot]\) : \(Ad(E) \otimes Ad(E) \rightarrow Ad(E)\) with the natural morphism \(\Omega^1_X \otimes \Omega^1_X \rightarrow \Omega^2_X\). Suppose that there is a Hermitian structure \(H_0\) on \(E\). We denote \(A^{1,1}_{H_0}\) by the space of unitary integrable connection on \(E\). Given a unitary integrable connection \(A\) on \((E, H_0)\), \(\bar{\partial}_A = \bar{\partial}_E\) defines a holomorphic structure on \(E\). Conversely, given a Hermitian metric \(H_0\) on a holomorphic bundle \((E, \bar{\partial}_E)\), there is a unique \(H_0\)-unitary connection \(A\) on \(E\) satisfying \(\bar{\partial}_A = \bar{\partial}_A\). One also can see [7] Section 2.1.5.

**Definition 2.1.** A pair \((A, \theta) \in A^{1,1}_{H_0} \otimes \Omega^{1,0}(X, \text{ad}(E))\) is called a Higgs pair if
\[
\bar{\partial}_A \theta = 0 \text{ and } \theta \wedge \theta = 0.
\]

A Higgs \(G\)-bundle on \(X\) is a pair of the form \((E, \theta)\), where \(E\) is holomorphic principal \(G\)-bundle on \(X\) and \(\theta\) is a Higgs filed on \(E\). We denote \(B_{E, H_0}\) by the space of all Higgs pairs on the Hermitian bundle \((E, H_0)\). We recall a energy identity for the Yang-Mills-Higgs functional which defined on \(B_{E, H_0}\) as follow:

\[
YMH(A, \theta) = \int_X \left( |F_A + [\theta, \theta^*]|^2 + 2|\bar{\partial}_A \theta|^2 \right) \frac{\omega^n}{n!}
\]

\[
= \int_X \left| \sqrt{-1} \Lambda_\omega (F_A + [\theta, \theta^*] - \lambda Id_{E}) \right|^2 \frac{\omega^n}{n!} + \lambda^2 \text{rank}(E) \int_X \frac{\omega^n}{n!}
\]

\[
+ 4\pi^2 \int_X (2c_2(E) - c_1^2(E)) \wedge \frac{\omega^{n-2}}{(n-2)!},
\]
where
\[ \lambda = \frac{2\pi \int_X c_1(E) \wedge \omega^{n-1}}{\text{rank}(E) \int_X \omega^n}. \]

From the above identity, we see that if \((A, \theta)\) satisfies the Hermitian-Einstein equation
\[ \sqrt{-1} \Lambda \omega(F_A + [\theta, \theta^*]) = \lambda I_{E}, \]
then it is the absolute minimum of the above Yang-Mills-Higgs functional. Equivalently, if \((A, \theta)\) satisfies the above Hermitian-Einstein equation, then \(h\) must be Hermitian-Einstein metric on the Higgs bundle \((E, \bar{\partial}_A, \theta)\), studied by Hitchin [9] and Simpson [18]. In [18], it is proved that a Higgs bundle admits the Hermitian-Einstein metric if only if the bundle \(E\) is polystable Higgs bundle.

**Lemma 2.2.** (Weitzenböck formula for Higgs field). Let \(X\) be a compact Kähler \(n\)-manifold, let \((A, \theta)\) be a Higgs pair over \(X\), then
\[ \int_X |\nabla_A \theta|^2 + \int_X \langle \text{Ric}_X \circ \theta, \theta \rangle + \int_X |[\theta, \theta^*]|^2 = \text{Re} \int_X \langle [\sqrt{-1} \Lambda \omega(F_A + [\theta, \theta^*]), \theta], \theta \rangle. \]

**Proof.** Since \(\theta \wedge \theta = \theta_i dz^i \wedge \theta_j dz^j = 0\), we have \([\theta_i, \theta_j] = 0\). Then
\[ |[\theta, \theta^*]|^2 = \sum_{i,j} |[\theta_i, \theta_j^*]|^2 = Tr(\theta_i \theta_j^* - \theta_j \theta_i^*) = Tr(\theta_i \theta_j - \theta_j \theta_i^*) \]
\[ = Tr(\theta_i^* \theta_j^* - \theta_j^* \theta_i) = |\sqrt{-1} \Lambda \omega(\theta, \theta^*)|^2. \]

Noting that the \(\text{End}(E)\)-value \((1, 0)\)-from \(\theta\) can be seen as a section of the bundle \(\text{End}(E) \otimes \Omega^{1,0}(X)\), and denoting the induced connection on the bundle \(\text{End}(E) \otimes \Omega^{1,0}(X)\) also by \(\nabla_A\) for simplicity, we have
\[ \int_X \langle \nabla_A \theta, \nabla_A \theta \rangle = \int_X \langle \nabla^*_A \nabla_A \theta, \theta \rangle \]
\[ = \int_X \langle \sqrt{-1} \Lambda \omega F_A \circ \theta - \theta \circ (\sqrt{-1} \Lambda \omega F_A \otimes \text{Id}_{T^{1,0}X} + \text{Id}_E \otimes \text{Ric}_X), \theta \rangle \]
\[ = \text{Re} \int_X \langle [\sqrt{-1} \Lambda \omega(F_A + [\theta, \theta^*]), \theta], \theta \rangle - \int_X \langle \text{Ric}_X \circ \theta, \theta \rangle - \int_X |[\theta, \theta^*]|^2. \]

where \(\text{Ric}_X\) denotes the Ricci transformation of the Kähler manifold \((X, \omega)\). \(\square\)

### 3 Continuity of the least eigenvalue with respect to connection

At first, we define a subset of \(\Omega^{1,0}(X, ad(E))\) as follow:
\[ \tilde{\Omega}^{1,0}(X, ad(E)) = \{ \theta \in \Omega^{1,0}(X, ad(E)) : \theta \wedge \theta = 0 \}. \]
One can see $v$ takes values in an abelian subalgebra of $ad(E)$ for $v \in \tilde{\Omega}^{1,0}(X, ad(E))$. The Higgs fields are also belong to $\tilde{\Omega}^{1,0}(X, ad(E))$. Now, we begin to define the least eigenvalue of $\nabla^*_A \nabla_A$ on $L^2(X, \tilde{\Omega}^{1,0}(ad(E)))$.

**Definition 3.1.** Let $E$ be a principal $G$-bundle over a compact smooth four-manifold $X$ and endowed with a smooth Riemannian metric, $g$. Let $A$ be a connection of Sobolev class $L^2_1$ on $E$. The least eigenvalue of $\nabla^*_A \nabla_A$ on $L^2(X, \tilde{\Omega}^{1,0}(ad(E)))$ is

$$\lambda(A) := \inf_{v \in \tilde{\Omega}^{1,0}(X, ad(E)) \setminus \{0\}} \frac{\langle \nabla^*_A \nabla_A v, v \rangle_{L^2}}{\|v\|_{L^2}^2}. \quad (3.1)$$

The Sobolev norms $L^p_{k,A}$, where $1 \leq p < \infty$ and $k$ is an integer, with respect to the connections defined as:

$$\|u\|_{L^p_{k,A}(X)} := \left( \sum_{j=0}^k \int_X |\nabla^j_A u|^p dv_g \right)^{1/p}, \forall u \in L^p_{k,A}(X, ad(E)),$$

where $\nabla^j_A := \nabla_A \circ \ldots \circ \nabla_A$ (repeated $j$ times for $j \geq 0$).

**Lemma 3.2.** ([22] Lemma 2.4, [7] Lemma 7.2.10) There is a universal constant $C$ and for any $N \geq 2$, $R > 0$, a smooth radial function $\beta = \beta_{N,R}$ on $\mathbb{R}^4$, with

$$0 \leq \beta(x) \leq 1$$

$$\beta(x) = \begin{cases} 1 & |x| \leq R/N \\ 0 & |x| \geq R \end{cases}$$

and

$$\|\nabla \beta\|_{L^4} + \|\nabla^2 \beta\|_{L^2} < \frac{C}{\sqrt{\log N}}.$$

Assuming $R < R_0$, the same holds for $\beta(x - x_0)$ on any geodesic ball $B_R(x_0) \subset X$.

**Proof.** We take

$$\beta(x) = \psi\left( \frac{\log \frac{N}{R} |x|}{\log N} \right),$$

where

$$\psi(s) = \begin{cases} 1 & s \leq 0 \\ 0 & s \geq 1 \end{cases}$$

is a standard cutoff function, with respect to the cylindrical coordinate $s$. \hfill \Box

**Proposition 3.3.** Let $X$ be a compact smooth four-manifold with Riemannian metric, $g$. Let $\Sigma = \{x_1, x_2, \ldots, x_L\} \subset X$ ($L \in \mathbb{N}^+$) and $0 < \rho \leq \min_{i \neq j} dist_g(x_i, x_j)$, let $U \subset X$ be the open subset give by

$$U := X \setminus \bigcup_{i=1}^L \overline{B}_{\rho/2}(x_i).$$
Let $A_0, A$ are connections of class $L^2_1$ on the principal bundles $E_0$ and $E$ over $X$. There is an isomorphism $u : E \upharpoonright X \backslash \Sigma \cong E_0 \upharpoonright X \backslash \Sigma$, and identify $E \upharpoonright X \backslash \Sigma$ with $E_0 \upharpoonright X \backslash \Sigma$ using this isomorphism. Then there are positive constants $C = C(\rho, g) \in (0, 1]$, $c \in (1, \infty)$ and $\delta \in (0, 1]$ with the following significance. If $A, A_0$ are $C^\infty$ connections on $E$, $E_0$ and

$$\|A - A_0\|_{L^4(U)} \leq \delta,$$

then for $\forall \eta \in (0, \infty), \lambda(A)$ and $\lambda(A_0)$ satisfy

$$\lambda(A) \leq (1 + \eta)\lambda(A_0) + c(C + \delta^2 + (1 + \frac{1}{\eta})L\rho^2\lambda(A))(1 + \lambda(A_0)) \quad (3.2)$$

and

$$\lambda(A_0) \leq (1 + \eta)\lambda(A) + c(C + \delta^2 + (1 + \frac{1}{\eta})L\rho^2\lambda(A_0))(1 + \lambda(A)) \quad (3.3)$$

Proof. Assume first that $\text{supp}(v) \subset U$, write $a := A - A_0$. We the have

$$\|\nabla_A v\|^2 = \|\nabla_{A_0} v + [a, v]\|^2$$

and

$$\|\nabla_A v\|^2 - \|\nabla_{A_0} v\|^2 \leq 2\|a\|^2_{L^4} \|v\|^2_{L^4}.$$

On the other hand, if $\text{supp}(v) \subset \bigcup_{l=1}^L \overline{B}_{\rho/2}(x_l)$, then

$$\|v\|^2_{L^2} \leq cL\rho^2\|v\|^2_{L^4}.$$

Let $\psi = \sum \beta_{N, \rho}(x - x_i)$ be a sum of the logarithmic cutoffs of Lemma 3.3 and $\bar{\psi} = 1 - \psi$. At last, we observe that $\bar{\psi} v \wedge \bar{\psi} v = 0$ i.e. $\bar{\psi} v \in \Omega^{1,0}$, we have

$$\lambda(A)\|\bar{\psi} v\|^2 \leq \|\nabla_A (\bar{\psi} v)\|^2.$$
Combining the above observations, we have
\[
\lambda(A)\|v\|^2_{L^2(X)} \leq \lambda(A)(\|\psi v\|^2_{L^2(X)} + \|\bar{\psi} v\|^2_{L^2(X)} + 2(\psi v, \bar{\psi} v)_{L^2(X)}) \\
\leq \lambda(A)(1 + \eta)\|\psi v\|^2_{L^2(X)} + \lambda(A)(1 + \frac{1}{\eta}\|\bar{\psi} v\|^2_{L^2(X)}) \\
\leq (1 + \eta)\|\nabla_A(\bar{\psi} v)\|^2_{L^2(X)} + c(1 + \frac{1}{\eta})L\rho^2\lambda(A)\|v\|^2_{L^4(X)} \\
\leq (1 + \eta)(\|\nabla_A(\bar{\psi} v)\|^2_{L^2(X)} + \|\nabla\bar{\psi} v\|^2_{L^2(X)} + \|a\|^2_{L^4(U)}\|v\|^2_{L^4(X)}) \\
+ c(1 + \frac{1}{\eta})L\rho^2\lambda(A)\|v\|^2_{L^4(X)} \\
\leq (1 + \eta)\|\nabla_A v\|^2_{L^2(X)} + (2(\|\nabla\bar{\psi} \|^2_{L^4(X)} + \|a\|^2_{L^4(U)}) \\
+ c(1 + \frac{1}{\eta})L\rho^2\lambda(A))\|v\|^2_{L^4(X)} \\
\leq (1 + \eta)\|\nabla_A v\|^2_{L^2(X)} + c(\|\nabla\bar{\psi} \|^2_{L^4(X)} + \|a\|^2_{L^4(U)} + (1 + \frac{1}{\eta})L\rho^2\lambda(A)) \\
\times (\|v\|^2_{L^2(X)} + \|\nabla_A v\|^2_{L^2(X)}).
\]

In the space \(\tilde{\Omega}^{1,0}(X, ad(E))\), although we can’t get \(v \in \tilde{\Omega}^{1,0}\) such that \(\langle \nabla_A^*\nabla_A v, v \rangle = \lambda(A_0)\|v\|^2\), but for any \(\tilde{\varepsilon} \in (0, 1)\), we choose \(v_{\tilde{\varepsilon}}\) such that

\[
\langle \nabla_A^*\nabla_A v_{\tilde{\varepsilon}}, v_{\tilde{\varepsilon}} \rangle = (\lambda(A_0) + \tilde{\varepsilon})\|v_{\tilde{\varepsilon}}\|^2 and \|v_{\tilde{\varepsilon}}\|^2 = 1
\]

hence

\[
\lambda(A) \leq c(\|\nabla\bar{\psi} \|^2_{L^4(X)} + \|a\|^2_{L^4(U)} + (1 + \frac{1}{\eta})L\rho^2\lambda(A))(1 + \lambda(A_0) + \tilde{\varepsilon)) \\
+ (1 + \eta)(\lambda(A_0) + \tilde{\varepsilon)) \forall \tilde{\varepsilon} \in (0, 1),
\]

We let \(\tilde{\varepsilon} \to 0^+\), we have

\[
\lambda(A) \leq (1 + \eta)\lambda(A_0) + c(\|\nabla\bar{\psi} \|^2_{L^4(X)} + \|a\|^2_{L^4(U)} + (1 + \frac{1}{\eta})L\rho^2\lambda(A))(1 + \lambda(A_0)) \forall \eta \in (0, \infty),
\]

Since \(\|\nabla\bar{\psi} \|^2_{L^4(X)} \leq \frac{C}{\log \lambda}\), then we have

\[
\lambda(A) \leq (1 + \eta)\lambda(A_0) + c(C + \delta^2 + (1 + \frac{1}{\eta})L\rho^2\lambda(A))(1 + \lambda(A_0))
\]

Therefore, exchange the roles of \(A\) and \(A_0\) in the preceding derivation yields the inequality (3.3) for \(\lambda(A)\) and \(\lambda(A_0)\).
We consider a sequence of $C^\infty$ connection on $E$ such that $\sup \|F_A\|_{L^2(X)} < \infty$. We denote
\[ \Sigma = \{ x \in X : \lim_{r \to 0} \lim_{n \to \infty} \|F_{A_n}\|_{L^2(B_r(x))}^2 \geq \varepsilon \} \]
the constant $\varepsilon \in (0, 1]$ as in [17] Theorem 3.2. We can see $\Sigma$ is a finite points $\{x_1, \ldots, x_L\}$ in $X$. We recall a result due to Sedlacek.

**Theorem 3.4.** ([17] Theorem 3.1 and [8] Theorem 35.15) Let $G$ be a compact Lie group and $E$ a principal $G$-bundle over a close, smooth four-dimensional $X$ with Riemannian metric, $g$. If $\{A_i\}_{i \in \mathbb{N}}$ is a sequence of $C^\infty$ connection on $E$ such that $\sup \|F_{A_i}\|_{L^2(X)} < \infty$, there are sequences of (a) geodesic ball $\{B_{\alpha}\}_{\alpha \in \mathbb{N}} \subset X \setminus \Sigma$ such that $\cup_{\alpha \in \mathbb{N}} B_{\alpha} = X \setminus \Sigma$; (b) section $\sigma_{\alpha,i} : B_{\alpha} \to E$; (c) local connection one-forms, $a_{\alpha,i} := \sigma_{\alpha,i}^* A_i \in L^2_1(B_{\alpha}, \Omega^1 \otimes g)$ and; (d) transition functions $g_{\alpha\beta,i} \in L^1_1(B_{\alpha \cap B_{\beta}}; G)$, such that the following hold for each $\alpha, \beta \in \mathbb{N}$,
1. $d^*_\Gamma a_{\alpha,i} = 0$, for all $i$ sufficiently large;
2. $d^*_\Gamma a_{\alpha} = 0$,
3. $g_{\alpha\beta,i} \to g_{\alpha\beta}$ weakly in $L^1_1(B_{\alpha \beta}; G)$,
4. $F_{\alpha,i} \to F_\alpha$ weakly in $L^2(B_{\alpha}, \Omega^2 \otimes g)$,
5. the sequence $\{a_{\alpha,i}\}_{i \in \mathbb{N}}$ obeys
   (a) $\{a_{\alpha,i}\}_{i \in \mathbb{N}} \subset L^2_1(B_{\alpha}, \Omega^1 \otimes g)$ is bounded,
   (b) $a_{\alpha,i} \to a_\alpha$ weakly in $L^1_1(B_{\alpha}, \Omega^1 \otimes g)$ and
   (c) $a_{\alpha,i} \to a_\alpha$ strongly in $L^p(B_{\alpha}, \Omega^1 \otimes g)$ for $1 \leq p < 4$,
where $F_{\alpha,i} = da_{\alpha,i} + [a_{\alpha,i}, a_{\alpha,i}] = \sigma_{\alpha,i}^* F_{A_i}$ and $F_\alpha := da_\alpha + [a_\alpha, a_\alpha]$, and $d^*$ is the formal adjoint of $d$ with respect to the flat metric defined by a choice of geodesic normal coordinates on $B_{\alpha}$.

We denotes $YM(A) = \|F_A\|_{L^2(X)}^2$ by the energy of a $C^\infty$-connection on $E$. We denote $m(E) := \inf \{YM(A)\}$ by the minimum Yang-Mills energy. In our article, we consider $E$ is a holomorphic $G$-bundle on a compact Kähler surface, then
\[ YM(A) = \int_X |\sqrt{-1} \Lambda_\omega F_A - \lambda Id_E|^2 + \lambda^2 \text{rank}(E) \text{vol}(X) + 4\pi^2 \int_X (2c_2(E) - c_1^2(E)) \]
From the above identity, we see that if $A$ satisfies the Hermitian-Einstein equation
\[ \sqrt{-1} \Lambda_\omega F_A = \lambda Id_E, \]
then it is the absolute minimum
\[ m(E) = \lambda^2 \text{rank}(E) \text{vol}(X) + 4\pi^2 \int_X (2c_2(E) - c_1^2(E)) \]
of the above Yang-Mills energy.
Following Taubes’ definition ([21] Definition 4.1), we called a sequence of connections \( \{A_i\}_{i \in \mathbb{N}} \) on a principal \( G \)-bundle \( P \) is good sequence if
\[
\sup_{i \in \mathbb{N}} YM(A_i) < \infty \text{ and } d_{A_i}^* F_{A_i} \to 0 \text{ as } i \to \infty.
\]
A sequence \( \{A_i\}_{i \in \mathbb{N}} \) is called a convergent Palais-Smale sequence if
\[
YM(A_i) \to YM(A_\infty) \text{ and } d_{A_i}^* F_{A_i} \to 0 \text{ as } i \to \infty,
\]
where \( A_\infty \) is a Yang-Mills connection on \( P_\infty \), \( P_\infty \) is as in [8] Theorem 35.17 and [17] Theorem 4.3. One can see, a convergent Palais-Smale sequence of connections is good. In [8], Feehan observed that there existence of a convergent Palais-Smale sequence with non-increasing energies for the Yang-Mills functional. It’s constructed by the way of Yang-Mills gradient flow. One also can see Kozono-Maeda-Naito [15] or Schlatter [16] and Struwe [20]. Here, we give a proof in detail for the readers convenience.

**Proposition 3.5.** ([8] Proposition 35.19) Let \( G \) be a compact Lie group and \( P \) a principal \( G \)-bundle over a compact smooth four-manifold \( X \) with Riemannian metric \( g \). Then there exists a finite subset, \( \Sigma \subset X \), a sequence \( \{A_i\}_{i \in \mathbb{N}} \) of connections of Sobolev class \( L^p_k \) with \( k \geq 1 \) and \( p \geq 2 \) obeying \( kp > 4 \) on a finite sequence of principal \( G \)-bundles, \( P_0 = P, P_1, \ldots, P_K \) with obstructions \( \eta(P_k) = \eta(P) \) for \( k = 0, 1, \ldots, K \), where \( A_i \) is defined on \( P_k \) for \( i_k \leq i < i_{k+1} \), with \( k = 0, 1, \ldots, K \) and \( i_{K+1} = \infty \) that obeying (3.4), for some Yang-Mills connection \( A_\infty \) on a principal \( G \)-bundle \( P_\infty \) over \( X \) with obstruction \( \eta(P_\infty) = \eta(P) \) and the sequence \( \{YM(A_i)\}_{i \in \mathbb{N}} \) is non-increasing.

**Proof.** We consider the Yang-Mills gradient flow with the initial connection \( A_0 \) on \( P \) of Sobolev class \( L^p_k \):
\[
\begin{align*}
\frac{dA(t)}{dt} &= -d_{A(t)}^* F_{A(t)} \\
A(0) &= A_0
\end{align*}
\]
From [15] Theorem A, [16] Theorem 1.2 and 1.3 and [20] Theorem 3.4, modulo singularities occurring at finitely many times \( \{T_1, \ldots, T_K\} \subset (0, \infty) \), the flow \( A(t) \) may be extended to a global weak solution on a finite sequence of principal \( G \)-bundles \( P = P_0, P_1, \ldots, P_K \) with \( A(t) \) defined on \( [T_k, T_{k+1}] \) for \( k = 0, 1, \ldots, K \) and \( T_{K+1} = \infty \). In particular, [16] Theorem 1.2 and 1.3 and Equations (56) or (58) provide a sequence of times \( \{t_i\}_{i \in \mathbb{N}} \subset [0, \infty) \) such that \( YM(A_i) \to YM(A_\infty) \) and \( d_{A_i}^* F_{A_i} \to 0 \) as \( i \to \infty \) and \( \{YM(A_i)\}_{i \in \mathbb{N}} \) is non-increasing, which completes the proof by taking \( A_i := A(t_i) \) for \( i \in \mathbb{N} \).

**Remark 3.6.** The subset \( \Sigma = \{x_1, \ldots, x_L\} \subset X \) appear in Proposition [3.5] is the union of the sets of bubble points \( \Sigma_1, \Sigma_K, \Sigma_\infty \) arising in the application of [16] Theorem 1.2 and 1.3 in the proof of Proposition [3.5].
Thanks to Taubes’ result [21], Proposition 4.5, Feehan observed the mode of convergence in Proposition 3.5 may be improved to give,

**Proposition 3.7.** ([8] Proposition 35.20) Let $G$ be a compact Lie group and $P$ a principal $G$-bundle over a compact smooth four-manifold $X$ with Riemannian metric $g$ and $\{A_i\}_{i \in \mathbb{N}}$ a good sequence of connections of Sobolev class $L^2$ on a finite sequence of principal $G$-bundles, $P_0 = P, P_1, \ldots, P_K$ with obstructions $\eta(P) = \eta(P)$ for $k = 0, 1, \ldots, K$, where $A_i$ is defined on $P_k$ for $i_k \leq i < i_{k+1}$, with $k = 0, 1, \ldots, K$ and $i_{K+1} = \infty$. After passing to a subsequence of $\{A_i\}_{i \in \mathbb{N}}$ and relabeling there is a sequence $\{g_i\}_{i \in \mathbb{N}} : P_\infty \uparrow X \setminus \Sigma \cong P \uparrow X \setminus \Sigma$ of class $L^3_{1,loc}(X \setminus \Sigma)$ such that

$$\|g_i^*A_i - A_\infty\|_{L^2_{1,\infty}(U)} \to 0 \text{ strongly in } L^2_{1,\infty}(U, \Omega^1 \times g_{P_\infty}) \text{ as } i \to \infty,$$

for every $U \Subset X \setminus \Sigma$ for some Yang-Mills connection $A_\infty$ on a principal $G$-bundle $P_\infty$ over $X$ with obstruction $\eta(P_\infty) = \eta(P)$.

We then have the convergence of the least eigenvalue of $\Delta_{A_i}|_{\Omega^1}$ for a convergent Palais-Smale sequence of connection $\{A_i\}_{i \in \mathbb{N}}$ converging strongly in $L^1_{1,loc}(X \setminus \Sigma)$.

**Corollary 3.8.** Let $G$ be a compact Lie group and $P$ a principal $G$-bundle over a compact smooth four-manifold $X$ with Riemannian metric $g$ and $\{A_i\}_{i \in \mathbb{N}}$ a convergent Palais-Smale sequence of connections of Sobolev class $L^2$ on $P$ that converges strongly in $L^1_{1,loc}(X \setminus \Sigma)$, modulo a sequence $\{g_i\}_{i \in \mathbb{N}} : P_\infty \uparrow X \setminus \Sigma \cong P \uparrow X \setminus \Sigma$ of class $L^3_{1,loc}(X \setminus \Sigma)$ to a Yang-Mills connection $A_\infty$ on a principal $G$-bundle $P_\infty$ over $X$. Then

$$\lim_{i \to \infty} \lambda(A_i) = \lambda(A_\infty),$$

where $\lambda(A)$ is as in Definition 3.1.

**Proof.** By the Sobolev embedding $L^1 \hookrightarrow L^4$ and Kato inequality, Proposition 3.7 implies that

$$\|g_i^*A_i - A_\infty\|_{L^4(U)} \to 0 \text{ strongly in } L^2_{1,\infty}(U, \Omega^1 \times g_{P_\infty}) \text{ as } i \to \infty.$$

Hence from the inequalities on Proposition 3.5 for $\forall \eta \in (0, \infty)$, we have

$$\lambda(A_\infty) \leq (1 + \eta) \liminf_{i \to \infty} \lambda(A_i) + c(C + (1 + \frac{1}{\eta})L\rho^2 \lambda(A_\infty))(1 + \liminf_{i \to \infty} \lambda(A_i)) \quad (3.5)$$

and

$$\limsup_{i \to \infty} \lambda(A_i) \leq (1 + \eta)\lambda(A_\infty) + c(C + (1 + \frac{1}{\eta})L\rho^2 \limsup_{i \to \infty} \lambda(A_i))(1 + \lambda(A_\infty)) \quad (3.6)$$

Because the inequalities (3.5) and (3.6) about $\liminf_{i \to \infty} \lambda(A_i)$ and $\limsup_{i \to \infty} \lambda(A_i)$ hold for every $\rho \in (0, \rho_0]$ and $\eta \in (0, \infty)$, It’s easy to see $C \to 0^+$ while $\rho \to 0^+$. So at first, we let $\rho \to 0^+$, hence

$$\lambda(A_\infty) \leq (1 + \eta) \liminf_{i \to \infty} \lambda(A_i) \leq (1 + \eta) \limsup_{i \to \infty} \lambda(A_i) \leq (1 + \eta)^2 \lambda(A_\infty).$$

Next, we let $\eta \to 0^+$, hence the conclusion follows. \(\square\)
4 A vanish theorem of Higgs fields

4.1 Calabi-Yau surface

At first, we prove a vanish theorem about the space $\tilde{\Omega}^1(X, ad(E))$ when $X$ is a Calabi-Yau $n$-fold. The proof is similar to Theorem 6.21 in [19].

**Lemma 4.1.** Let $G$ be a compact Lie group, $E$ a principal $G$-bundle over a compact Calabi-Yau $n$-fold $(n \geq 2)$ $X$ has fully holonomy and $A$ a $C^\infty$ connection on $E$. If $\theta \in \Omega^{1,0}(X, ad(E))$ satisfy

$$\nabla_A \theta = 0 \text{ and } \theta \wedge \theta = 0$$

then $\theta$ is vanish.

**Proof.** Let $R_{ij} dx^i \wedge dx^j$ denote the Riemann curvature tensor viewed as an $ad(T^*X)$ valued 2-form, The vanishing of $\nabla_A \theta$ implies

$$0 = [\nabla_i, \nabla_j] \theta = ad((F_{ij}) + R_{ij}) \theta$$

for all $i, j$. Because $\theta \wedge \theta = 0$, hence $\theta$ takes values in an abelian subalgebra of $ad(E)$, $[F_{ij}, \theta] \perp R_{ij} \theta$. Hence $R_{ij} \theta = 0$, and the components of $\theta$ are in the kernel of the Riemannian curvature operator. This reduces the Riemannian holonomy group, unless $\theta = 0$. □

**Corollary 4.2.** Let $G$ be a compact Lie group, $E$ a principal $G$-bundle over a compact Calabi-Yau $n$-fold $X$ with fully holonomy, let $A$ be a $C^\infty$-connection on $E$. If $\lambda(A)$ is as in Definition 3.1 then $\lambda(A) > 0$.

**Proof.** Suppose that the the constant $\lambda(A) = 0$. We may then choose a sequence $\{v_i\}_{i \in \mathbb{N}} \subset \tilde{\Omega}^{1,0}$ such that

$$\|\nabla_A v_i\|_{L^2(X)}^2 = \lambda_i \|v_i\|_{L^2(X)}^2$$

and

$$\lambda(A_i) \to 0 \text{ as } i \to \infty.$$ 

Since $\frac{v}{\|v\|_{L^2}} \wedge \frac{v}{\|v\|_{L^2}} = 0$, for $v \in \Omega^{1,0}(X, ad(E))$, then we can noting

$$\|v_i\|_{L^2(X)} = 1, \text{ for } i \in \mathbb{N},$$

hence

$$\|v_i\|_{L^2_1}^2 \leq (1 + \lambda_i) < \infty,$$

then there exist a subsequence $\Xi \subset \mathbb{N}$ such that $\{v_i\}_{i \in \Xi}$ weakly convergence to $v_\infty$ in $L^2_1$, hence

$$\nabla_A v_\infty = 0.$$
On the other hand, $L^2_1 \hookrightarrow L^p$, for $2 \leq p \leq \frac{2n}{n-1}$, hence we set $p = 2$, 

$$
\|v_{\infty} \wedge v_{\infty}\|_{L^1(X)} = \|(v_{\infty} - v_i) \wedge v_i + (v_{\infty} - v_i) \wedge v_{\infty}\|_{L^1(X)} 
\leq \|v_i - v_{\infty}\|_{L^2(X)}^2(\|v\|_{L^2(X)}^2 + \|v_i\|_{L^2(X)}^2),
$$

\rightarrow 0 \text{ as } i \rightarrow \infty,

then we get 

$$v_{\infty} \wedge v_{\infty} = 0.$$

From Lemma 4.1, $v_{\infty}$ is vanish. It’s contradicting to $\|v_{\infty}\|_{L^2(X)} = 1$. In particular, the preceding arguments shows that the $\lambda(A) > 0$.

Now, we denote 

$$M_{HYM} = \{[A] \in A^{1,1} | \sqrt{-1}A_\omega F_A = \lambda Id_E\}$$

by the moduli space of Hermitian-Yang-Mills connection on $E$. We denote $\bar{M}_{HYM}$ by the Uhlenbeck compactification of $M_{HYM}$.

**Proposition 4.3.** Let $G$ be a compact Lie group and $E$ a holomorphic principal $G$-bundle over a compact smooth Calabi-Yau surface $X$ has fully holonomy. If $A$ is the Hermitian-Yang-Mills connection on $E$, then there exists a positive constant $\lambda > 0$ such that 

$$\lambda(A) \geq \lambda,$$

where $\lambda(A)$ is as in Definition 3.1.

**Proof.** The conclusion is a consequence of the fact that $\bar{M}_{HYM}$ is compact, 

$$\lambda[\cdot] : \bar{M}_{HYM} \ni [A] \rightarrow \lambda(A) \in [0, \infty),$$

to $\bar{M}_{HYM}$ is continuous by Corollary 3.8 the fact that $\lambda(A) > 0$ for $[A] \in \bar{M}_{HYM}$ (see Corollary 4.2).

### 4.2 Yang-Mills-Higgs flow

In this section, we used the Yang-Mills-Higgs flow to construct a sequence is a convergent Palais-Smale sequence. We recall some basic results about Yang-Mills-Higgs flow, one also can see [23, 12, 11]. We called a family $(A(t, x), \theta(t, x))$ is a Yang-Mills-Higgs flow, if $(A(t, x), \theta(t, x))$ satisfy

$$
\left\{ \begin{array}{l}
\frac{\partial A}{\partial t} = d_A^* F_A - \sqrt{-1}(\partial_A \Lambda_\omega - \partial_A \Lambda_\omega)[\theta, \theta^*], \\
\frac{\partial \theta}{\partial t} = [\sqrt{-1} \Lambda_\omega (F_A + [\theta, \theta^*]), \theta].
\end{array} \right.
$$

(4.1)
Non-existence of Higgs fields on Calabi-Yau Manifolds

Let \((A_0, \theta_0)\) be an initial Higgs pair on \((E, H_0)\). Denote the complex gauge group of the Hermitian bundle \((E, H_0)\) by \(G_C\), (denote \(G = \{ \sigma \in G_C : \sigma \sigma^* H_0 = Id \}\)). The group \(G_C\) acts on the space \(A_{H_0}^{1,1} \times \Omega^{1,0}(ad(E))\) as follow (see [7] Equ (6.1.4)):

\[
\begin{align*}
\partial_{\sigma^* (A)} = \sigma \circ \bar{\partial}_A \circ \sigma^{-1} &= \bar{\partial}_A - (\bar{\partial}_A \sigma) \sigma^{-1}, \\
\partial_{(A^* H_0)} = (\sigma^* H_0)^{-1} \circ \partial_A \sigma^* H_0 &= \partial_A + [((\bar{\partial}_A \sigma) \sigma^{-1})^* H_0, \\
\sigma^*(\theta) &= \sigma \circ \theta \circ \sigma^{-1}.
\end{align*}
\]

Let \(\sigma(t) \in G_C\) satisfy \(\sigma^*(\sigma) = h(t)\), where \(h(t) := H_0 H^{-1}(t)\) satisfies

\[
\frac{\partial h}{\partial t} = -2 \sqrt{-1} h \Lambda_\omega (F_{A_0} + \bar{\partial}_{A_0} (h^{-1} \partial_{A_0} h) + [\theta_0, h^{-1} \theta^* H_0 h]) + 2 \lambda h. \tag{4.3}
\]

Let \(\tilde{A}(t) = \sigma^*(A_0)\) and \(\tilde{\theta} = \sigma^*(\theta_0)\). Denote \(\alpha = -\frac{1}{2} (\frac{\partial \sigma}{\partial t} \sigma^{-1} - (\sigma^* H_0)^{-1} \frac{\partial \sigma^* H_0}{\partial t})\), hence we have

\[
\begin{align*}
\frac{\partial \bar{\partial}}{\partial t} &= -\sqrt{-1} (\bar{\partial} - \bar{\partial} \Lambda_\omega (F_{\tilde{A}} + [\tilde{\theta}, \tilde{\theta}^* \theta]) + \alpha, \\
\frac{\partial \tilde{\theta}}{\partial t} &= \tilde{\theta} \sqrt{-1} \Lambda_\omega (F_{\tilde{A}} + [\tilde{\theta}, \tilde{\theta}^* \theta]) + \alpha. \tag{4.4}
\end{align*}
\]

It is easy to see \(\alpha(t) \in Lie(G)\). Now let \(S(t) \in G\) be the unique solution to the linear ODE

\[
\frac{dS}{dt} = S\alpha, \quad S(0) = Id.
\]

Then, the pair \((A = S^*(\tilde{A}), \theta = S^*(\tilde{\theta}))\) satisfies the Yang-Mills-Higgs flow \((4.1)\). Hence, we have a identity

\[
|\sqrt{-1} \Lambda_\omega (F_{H(t)} + [\theta, \theta^* H(t)])| - \lambda Id_E|_{H(t)} = |\sqrt{-1} \Lambda_\omega (F_{\tilde{A}(t)} + [\tilde{\theta}(t), \tilde{\theta}(t)^* H_0])| - \lambda Id_E|_{H_0}.
\]

On the other hand, one can check

\[
\tilde{\theta}(t) \wedge \tilde{\theta}(t) = 0 \text{ and } \bar{\partial}_{\tilde{A}(t)} \tilde{\theta}(t) = 0,
\]

and

\[
\theta(t) \wedge \theta(t) = 0 \text{ and } \bar{\partial}_{A(t)} \theta(t) = 0.
\]

In [23], Wilkin studied the global existence and uniqueness of the solution for the above gradient flow on Riemann surface ([23] Proposition 3.2). In [11], Li-Zhang extended the results of Wilkin’s to the case of Kähler surface ([11] Theorem 2.1), they also studied the asymptotic behaviour of a regular solution at infinity.
Theorem 4.4. ([11] Theorem 3.2 and Corollary 3.9). Let \((A(t), \theta(t))\) be a global smooth solution of the gradient flow (4.1) on a Kähler surface \(X\) with smooth initial data \((A_0, \theta_0)\). Then there exists a sequence \(t_j \to \infty\) such that \((A_{t_j}, \theta_{t_j})\) converges, modulo gauge transformations, to a Yang-Mills Higgs pair \((A_\infty, \theta_\infty)\) outside finite points \(\Sigma\), and \((A_\infty, \theta_\infty)\) can be extended smoothly to a vector bundle \(E_\infty\) over a Kähler surface.

Denote \(\Theta(A, \theta) = \sqrt{-1} \Lambda_\omega(F_A + [\theta, \theta^*H_0] - \lambda Id_E)\), we have

Proposition 4.5. ([11] Corollary 3.12). Let \((A_i, \theta_i)\) be a sequence of Higgs pairs along the Yang-Mills-Higgs gradient flow (4.1) with Uhlenbeck limit \((A_\infty, \theta_\infty)\). Then

\[ \Theta(A_i, \theta_i) \to \Theta(A_\infty, \theta_\infty) \text{ in } L^p \text{ for } 1 \leq p < \infty, \]

and

\[ \lim_{t \to \infty} \|\Theta(A_i, \theta_i)\|_{L^2(X)} = \|\Theta(A_\infty, \theta_\infty)\|_{L^2(X)}. \]

Now, we assume the Higgs bundle \((E, \theta)\) is a semistable Higgs bundle. In [12], they proved that if \((A(t), \theta(t))\) is a global smooth solution of the gradient flow (4.1) on a compact Kähler manifold, then

\[ \|\Theta(H(t), \theta)\|_{L^2(X)}^2 = \|\sqrt{-1} \Lambda_\omega(F_{H(t)} + [\theta, \theta^*H(t)] - \lambda Id_E)\|_{L^2(X)} \]

\[ = \|\sqrt{-1} \Lambda_\omega(F_{A(t)} + [\theta(t), \theta(t)^*H_0] - \lambda Id_E)\|_{L^2(X)} \]

\[ \to 0 \text{ as } t \to \infty. \]

Hence, We have

Corollary 4.6. Let \(G\) be a compact Lie group and \(E\) a holomorphic principal \(G\)-bundle over a compact smooth Kähler surface \(X\), \((E, \theta)\) a semistable Higgs bundle on \(X\). If \((A(t), \theta(t))\) is a global smooth solution of the gradient flow (4.1) on a Kähler surface \(X\) with smooth initial data \((A_0, \theta_0)\), there are (a) a sequence \(t_j \to \infty\) (b) a set \(\Sigma := \{x_1, \ldots, x_L\} \subset X\), (c) the Uhlenbeck limit \((A_\infty, \theta_\infty)\) satisfies Hermitian-Yang-Higgs-Higgs Equation, such that \((A_{t_j}, \theta_{t_j}) \to (A_\infty, \theta_\infty)\) as \(t_j \to \infty\) on \(X \setminus \Sigma\), under modulo gauge transformations, in the sense of \(C^\infty\).

**Prove Main Theorem [1.1]**

Proof. We consider the Yang-Mills-Higgs gradient flow (4.1) with the initial data \((A, \theta)\). We denote \(A = A + \theta + \theta^*\) by the Hitchin-Simpson connection on \(E\). Then from Corollary 4.6, we have a sequence \(\{t_j\}\) such that \((A(t_j), \theta(t_j) \to (A_\infty, \theta_\infty)\) on \(X \setminus \Sigma\) in the sense of \(C^\infty\) and \(A_\infty\) is a Hermitian-Yang-Mills connection on \(E\) i.e.

\[ \sqrt{-1} \Lambda_\omega(F_{A_\infty} + [\theta_\infty, \theta_\infty^{*H_0}] - \lambda Id_E) = 0. \]
Since \((A(t), \theta(A))\) is a Higgs pair, the Weitzenböck formula for Higgs field gives
\[
\|\nabla_{A(t)} \theta(t)\|^2 + \|[(\theta(t), \theta^*(t))]\|^2 = Re([\Theta(A(t), \theta(t)), \theta(t)])_{L^2(\mathcal{X})}.
\]
Hence we have
\[
\|\nabla_{A(t)} \theta(t)\|^2_{L^2(\mathcal{X})} \leq \|\Theta(A(t), \theta(t))\|_{L^2(\mathcal{X})} \|\theta(t)\|^2_{L^4(\mathcal{X})} \\
\leq C \|\Theta(A(t), \theta(t))\|_{L^2(\mathcal{X})} (\|\theta(t)\|^2_{L^2(\mathcal{X})} + \|\nabla_{A(t)} \theta(t)\|^2_{L^2(\mathcal{X})}).
\]
For large enough \(t_j\), we can choose \(\|\Theta(A(t), \theta(t))\|_{L^2(\mathcal{X})} \leq \frac{1}{2C}\), then we obtain we have
\[
\|\nabla_{A(t)} \theta(t)\|^2_{L^2(\mathcal{X})} \leq 4 \|\Theta(A(t), \theta(t))\|_{L^2(\mathcal{X})} \|\theta(t)\|^2_{L^2(\mathcal{X})}.
\]
If suppose the Higgs field \(\theta\) is non-zero, then from the third identity on (4.2), then \(\theta(t) = S^*(t)\sigma^*(t)(\theta)\) is also non-zero. Then by the definition of \(\lambda[\cdot]\), we have
\[
\lambda(A(t)) \leq \frac{\|\nabla_{A(t)} \theta(t)\|^2}{\|\theta(t)\|^2}.
\]
Hence, from Corollary 3.8 and the fact \(\|\Theta(A(t), \theta(t))\|_{L^2(\mathcal{X})} \to 0\), we get
\[
\lambda(A_{\infty}) = \lim_{t_j \to \infty} \lambda(A(t_j)) = 0. \tag{4.6}
\]
But by Corollary 4.2 \(\lambda(A_{\infty}) > 0\). It’s contradicting to our initial assumption regarding the Higgs field is non-zero. In particular, the Higgs field \(\theta\) must be vanish. 

### 4.3 Higher dimensional Kähler manifolds

In this section we consider the Higgs bundle \((E, \theta)\) on higher dimensional Kähler manifolds. We recall some estimates for Yang-Mills-Higgs flow \([11\text{][13}]\). Let \((A(t), \theta(t))\) be the solutions of the heat flow on Hermitian bundle \((E, H_0)\) with initial Higgs pair \((A_0, \theta_0)\). For a fixed point \(u_0 = (x_0, t_0) \in X \times \mathbb{R}^+\), we denote
\[
P_r(u_0) = B_r(x_0) \times [t_0 - r^2, t_0 + r^2].
\]
We also set
\[
e(A, \theta)(x, t) = |F_A + [\theta, \theta^*H_0]|^2 + 2|\partial_A \theta|^2.
\]
We have a \(\varepsilon\)-regularity theorem for the Yang-Mills-Higgs flow. The argument is very similar with the Kähler surface \([11\text{]}\).

**Theorem 4.7.** (\([13]\text{][23]\text{ Theorem 2.3}]) Let \((A(t), \theta(t))\) be a smooth solutions of Yang-Mills-Higgs flow over an \(n\)-dimension compact Kähler manifold \((X, \omega)\) with initial value \((A_0, \theta_0)\).
There exist positive constants $\varepsilon_0, \delta_0 < \frac{1}{4}$, such that, if for some $0 < R < \min_{x \in X} \frac{\sqrt{n}}{4}$, the inequality
\[
R^{2-2n} \int_{P_R(x_0,t_0)} e(A(t), \theta(t)) \, dvol_g \, dt \leq \varepsilon_0,
\]
holds, then for any $\delta \in (0, \delta_0)$, we have
\[
\sup_{P_{\delta R}(x_0,t_0)} e(A(t), \theta(t)) \leq 16(\delta R)^{-4},
\]
and
\[
\sup_{P_{\delta R}(x_0,t_0)} |\nabla A(t) \theta(t)|^2 \leq C,
\]
where $C$ is a constant depending only on the geometry of $(X, \omega)$, the initial data $(A_0, \theta_0)$, $\delta_0$ and $R$.

For any pair $(A, \theta)$, we denote
\[
\Sigma = \bigcap_{\rho > r > 0} \{ x \in X : \liminf_{j \to \infty} r^{4-2n} \int_{B_r(x)} e(A, \theta)(\cdot, t_j) \, dvol_g \geq \bar{\varepsilon}, \quad (4.7) \}
\]
where $\bar{\varepsilon}$ is a constant. In [10] Proposition or [11] Theorem 3.2., they give a way to determined the constant. Following the argument of Hong-Tian [10] for the Yang-Mills flow case, Li-Zhang [13] using the above $\varepsilon$-regularity theorem, they can analyze the limiting behavior the Yang-Mills-Higgs flow.

**Theorem 4.8.** Let $(A(t), \theta(t))$ be a smooth solutions of Yang-Mills-Higgs flow over an $n$-dimension compact Kähler manifold $(X, \omega)$ with initial value $(A_0, \theta_0)$. Then for every sequence $t_i \to \infty$, there exist a sequence $\{t_j\}$ such that $t_i \to \infty$, $(A(t_j), \theta(t_j))$ converges, moduli gauge transformation, to a Higgs pair $(A_\infty, \theta_\infty)$ on a principal $G$-bundle $E_\infty$ with Hermitian metric $H_\infty$ in $C^\infty_{loc}$-topology outside a closet set $\Sigma \subset X$, where $\Sigma$ a closed set of Hausdorff codimension at least 4 and there exist an isometry between $(E, H_0)$ and $(E_\infty, H_\infty)$ outside $\Sigma$.

Further more, we assumption the Higgs bundle $(E, \theta)$ is a semistable Higgs bundle over $(X, \omega)$. We denote $\Theta(A(t), \theta(t)) = \sqrt{-1} \Lambda_\omega (F_{A(t)} + [\theta, \theta^* H_0]) - \lambda Id_E$. Then from (4.8), we have
\[
\left( \frac{\partial}{\partial t} - \Delta \right) |\Theta(H(t), \theta)|_{H(t)} \leq 0.
\]
In [12], Li-Zhang had proved that

**Theorem 4.9.** Let $(E, \theta)$ be a semistable Higgs bundle with Hermitian metric $H_0$ on Kähler manifold $(X, \omega)$. Then the Yang-Mills-Higgs flow with initial data $(A_0, \theta_0)$ has a global smooth solution $(A(t), \theta(t))$ and
\[
\max_{x \in X} |\Theta(A(t), \theta(t))|_{H_0} \to 0 \text{ as } t \to \infty.
\]
We are finally ready to use the above results to in following theorem. The argument is similar to Nie-Zhang’s Theorem 1.1 [14].

**Theorem 4.10.** Let \((E, \theta)\) be a semistable Higgs bundle with Hermitian metric \(H_0\) on Kähler manifold \((X, \omega)\). If \(c_1(E) = c_2(E) = 0\), then the Yang-Mills-Higgs flow with initial data \((A_0, \theta_0)\) has a global smooth solution \((A(t), \theta(t))\) and for every sequence \(t_i \to \infty\), there exist a sequence \(\{t_j\}\) such that \(t_j \to \infty\), \((A(t_j), \theta(t_j))\) converges, moduli gauge transformation, to a Higgs pair \((A_\infty, \theta_\infty)\) on a principal \(G\)-bundle \(E_\infty\) with Hermitian metric \(H_\infty\) in \(C^\infty\)-topology over all \(X\) and \((E_\infty, H_\infty) \cong (E, H_0)\).

**Proof.** From the Energy identity about YMH-functional, under the condition \(c_1(E) = c_2(E)\), we have

\[
YMH(A, \theta) = \int_X (|F_A + [\theta, \theta^*]|^2 + 2|\partial_A \theta|^2) \frac{\omega^n}{n!} \leq \int_X |\sqrt{-1} \Lambda_\omega (F_A + [\theta, \theta^*] - \lambda Id_E)|^2 \frac{\omega^n}{n!}
\]

For any point \((x_0, t_0) \in X \times \mathbb{R}^+\), we have

\[
\int_{P_r(x_0, t_0)} e(A(t), \theta(t)) \leq \int_{t_0-r^2}^{t_0+r^2} \int_X e(A(t), \theta(t)) dv_{g} dt
\]

\[
\leq 2r^2 \int_X |\Theta(A(t), \theta(t))|^2.
\]

From Theorem 4.7, for \(0 < r < \rho\), we can choose \(T\) large enough such that

\[
r^{4-2n} \int_{P_r(x_0, t_0)} e(A(t), \theta(t)) \leq \varepsilon_0,
\]

where \(\varepsilon_0\) is constant in Theorem 4.7. By \(\varepsilon\)-regularity theorem, we have

\[
e(A, \theta)(x, t) \leq \sup_{P_{3r}(x_0, t_0)} e(A, \theta) \leq 16(\delta R)^{-4}
\]

and

\[
|\nabla_A \theta|^2_{H_0}(x_0, t) \leq \sup_{P_{3r}(x_0, t)} |\nabla_A \theta|^2_{H_0} \leq C.
\]

In addition, one can see \(\sup_X e(A, \theta)(x, t)\) and \(\sup_X |\nabla_A \theta|^2_{H_0}\) are bounded in \([0, t_0]\).

We using the differential inequality (2.12) about \(e(A(t), \theta(x))\) in [11],

\[
(\Delta - \frac{\partial}{\partial t})(e(A(t), \theta(A(t)))) \geq -C(n)(|F_{A(t)} + [\theta, \theta^*H_0]|_{H_0}) + |\nabla_{A(t)} \theta(t)|_g + |Riem|_g e(A(t), \theta(t)).
\]

Hence there exist a positive constant \(C\) such that

\[
(\Delta - \frac{\partial}{\partial t})(e(A(t), \theta(A(t)))) \geq -Ce(A(t), \theta(t)).
\]
From the mean value inequality Theorem, we have
\[
\sup_X e(A(t), \theta(t)) \leq C \int_X e(A(t-1), \theta(t-1)) d\text{vol}_g.
\]
Hence we get
\[
\sup_X e(A(t), \theta(t)) \to 0 \text{ as } t \to \infty.
\]
Then the set \( \Sigma \) defined as Equation (4.7) is empty. Now, We complete our proof. \( \square \)

Lemma 4.11. \((L^n\text{-}continuity \ of \ least \ eigenvalue \ of \ \nabla A \ with \ respect \ to \ the \ connection)\)

Let \( G \) be a compact Lie group, \( E \) be a \( G \)-bundle over a closed, smooth manifold \( X \) of dimension \( n \geq 4 \) and endowed with a smooth Riemannian metric, \( g \). If \( A_0, A \) are \( C^\infty \)-connections on \( E \) such that
\[
\| A - A_0 \|_{L^n(X)} \leq \varepsilon
\]
then, we denote \( a := A - A_0 \),
\[
(1 - c\| a \|_{L^n(X)}) \lambda(A_0) - c\| a \|_{L^n(X)} \leq \lambda(A) \leq (1 - c\| a \|_{L^n(X)})^{-1}(\lambda(A_0) + c\| a \|_{L^n(X)}).
\]

Proof. For convenience, write \( a := A - A_0 \in \Omega^1(X, g_P) \). For \( v \in L^2_1(X, \Omega^0 \otimes g_P) \), we have \( d_A v = d_{A_0} v + [a, v] \) and
\[
\| \nabla_A v \|^2_{L^2(X)} = \| \nabla_{A_0} v + [a, v] \|^2_{L^2(X)} \\
\geq \| \nabla_{A_0} v \|^2_{L^2(X)} - \| [a, v] \|^2_{L^2(X)} \\
\geq \| \nabla_{A_0} v \|^2_{L^2(X)} - 2\| a \|^2_{L^n(X)} \| v \|^2_{L^{2n/(n-2)}(X)} \\
\geq \| \nabla_{A_0} v \|^2_{L^2(X)} - 2c_1\| a \|^2_{L^n(X)} \| v \|^2_{L^2_{1,A_0}(X)},
\]
where \( c_1 = c_1(g) \) is the Sobolev embedding constant for \( L^2_1 \hookrightarrow L^{2n/(n-2)} \). Now for a constant \( \varepsilon \in (0, 1) \), we take \( v_\varepsilon \) such that
\[
\langle \nabla_A^* \nabla_A v_\varepsilon, v_\varepsilon \rangle = (\lambda(A_0) + \varepsilon)\| v_\varepsilon \|^2 \text{ and } \| v_\varepsilon \|_{L^2(X)} = 1
\]
and also suppose that \( \| A - A_0 \|^2_{L^n(X)} \) is small enough that \( 2c_1\| a \|^2_{L^n(X)} \leq 1/2 \). The preceding inequality then gives
\[
\lambda(A) + \varepsilon \geq (1 - 2c_1\| a \|^2_{L^n(X)} )\| \nabla_{A_0} v \|^2_{L^2(X)} - 2c_1\| a \|^2_{L^n(X)}.
\]
Since \( \| v_\varepsilon \|_{L^2(X)} = 1 \), we have \( \| \nabla_{A_0} v_\varepsilon \|^2_{L^2(X)} \geq \lambda(A_0) \), hence
\[
\lambda(A) + \varepsilon \geq (1 - 2c_1\| a \|^2_{L^n(X)} )\lambda(A_0) - 2c_1\| a \|^2_{L^n(X)} \forall \varepsilon \in (0, 1).
\]
We let \( \varepsilon \to 0^+ \), hence
\[
\lambda(A) \geq (1 - 2c_1\| a \|^2_{L^n(X)} )\lambda(A_0) - 2c_1\| a \|^2_{L^n(X)}.
\]
To obtain the upper bounded for $\lambda(A)$, we only exchange the roles of $A$ and $A_0$ yields the inequality,

$$
\lambda(A_0) \geq (1 - 2c_1 \|a\|^2_{L^\infty(X)}) \lambda(A) - 2c_1 c \|a\|^2_{L^\infty(X)}.
$$

\[\square\]

**Corollary 4.12.** Let $G$ be a compact Lie group and $P$ a principal $G$-bundle over a compact smooth Riemannian manifold $X$ of dimension $n \geq 4$ with Riemannian metric $g$ and $\{A_i\}_{i \in \mathbb{N}}$ is a sequence $C^\infty$-connections on $P$ that converges strongly in $L^n(X)$, moduli gauge transformation, to a connection $A_\infty$, then

$$
\lim_{i \to \infty} \lambda(A_i) = \lambda(A_\infty),
$$

where $\lambda(A)$ is as in Definition 3.1.

**Proof Theorem 1.2**

*Proof.* The proof is similar to Theorem 1.1. We consider the Yang-Mills-Higgs gradient flow (4.1) with the initial data $(A, \theta)$. We denote $A = A + \theta + \theta^*$ by the Hitchin-Simpson connection on $E$. Then from Corollary 4.6, we have a sequence $\{t_j\}$ such that $(A(t_j), \theta(t_j)) \to (A_\infty, \theta_\infty)$ on $X$ in the sense of $C^\infty$ and $A_\infty$ is a Hermitian-Yang-Mills connection on $E_\infty$. Since $(A(t), \theta(A))$ is a Higgs pair, we also using the Weitzenböck formula for Higgs field to gives

$$
\|\nabla_{A(t)} \theta(t)\|_{L^2(X)}^2 \leq \|\Theta(A(t), \theta(t))\|_{L^\infty(X)} \|\theta(t)\|_{L^2(X)}^2.
$$

If suppose the Higgs field $\theta$ is non-zero, then $\theta(t) = S^*(t)\sigma^*(t)(\theta)$ is also non-zero. Then by the definition of $\lambda([\cdot])$, we have

$$
\lambda(A(t)) \leq \frac{\|\nabla_{A(t)} \theta(t)\|^2}{\|\theta(t)\|^2}.
$$

Hence, from Corollary 4.12 and the fact $\sup_X |\Theta(A(t), \theta(t))| \to 0$, we get

$$
\lambda(A_\infty) = \lim_{t_j \to \infty} \lambda(A(t_j)) = 0.
$$

(4.8)

But by Corollary 4.2 $\lambda(A_\infty) > 0$. It’s contradicting to our initial assumption regarding the Higgs field is non-zero. In particular, the Higgs field $\theta$ must be vanish.

\[\square\]

**Remark 4.13.** On the case of semisable principal bundle $(E, \theta)$ with any Chern-class on a compact Calabi-Yau $n$-fold, let $(A(t), \theta(t))$ be a smooth solutions of Yang-Mills-Higgs flow over an $n$-dimension compact Calabi-Yau manifold $(X, \omega)$ with initial value $(A_0, \theta_0)$. We also obtain

$$
\lim_{t_j \to \infty} \lambda(A(t_j)) = 0.
$$
But for any sequence \( \{t_i\} \), there is a subsequence such \( (A(t_i), \theta(t_i)) \) converges to \( (A_\infty, \theta_\infty) \) in \( C^\infty_{\text{loc}} \)-topology outside a closet set \( \Sigma \subset X \). In the case of Kähler surface, the limit \( (E_\infty, A_\infty, \theta_\infty) \) can be extend to the whole \( X \) as a Higgs bundle. But on higher dimensional, the limiting \( (E_\infty, A_\infty, \theta_\infty) \) only can be extended to the whole \( X \) as a reflexive Higgs sheaf. Higgs sheaf.

At last, we using our method to re-prove some results had been proved by Biswas-Bruzzo-Otero-Giudice [2]. We recall that a Kähler metric is called Kähler-Einstein if its Ricci curvature is a constant real multiple of the Kähler form. Let \( X \) be a compact connected Kähler manifold admitting a Kähler-Einstein metric. We assume that \( c_1(TX) > 0 \); this is equivalent to the condition that the above mentioned scalar factor is positive. Fix a Kähler-Einstein form \( \omega \) on \( X \).

**Theorem 4.14.** ([2] Proposition 2.1) Let \( G \) be a compact Lie group and \( E \) a holomorphic principal \( G \)-bundle with Hermitian metric \( H_0 \) over a compact connected Kähler-Einstein manifold \( X \) with \( c_1(TX) > 0 \) and \( A \in \mathcal{A}_{H_0}^{1,1} \) be a connection on \( E \). If \( (E, \partial_A, \theta) \) is a semistable Higgs bundle on \( X \), then the Higgs fields \( \theta \) are vanish.

**Proof.** We also consider the Yang-Mills-Higgs gradient flow (4.1) with the initial data \((A, \theta)\). We have

\[
\max |\Theta(H(t), \theta)|_{H(t)} \to 0 \text{ as } t \to \infty.
\]

We can choose \( t \) sufficiently large such that

\[
\max |\Theta(A(t), \theta(t))|_{H_0} = \max |\Theta(H(t), \theta)|_{H(t)} \leq \frac{1}{4} \text{Ric}_X.
\]

Since \( (A(t), \theta(A)) \) is a Higgs pair, the Weitzenböck formula for Higgs field gives

\[
0 = \|\nabla_{A(t)} \theta(t)\|^2_{L^2(X)} + \|\theta(t) - \theta^*(t)\|^2_{L^2(X)} + (\text{Ric}_X \circ \theta(t), \theta(t))_{L^2(X)}
\]

\[
- 2\text{Re}([\sqrt{-1} \Lambda_{\omega}(F_{A(t)} + [\theta(t), \theta^*(t)] - \lambda \text{Id}_E, \theta(t), \theta(t))]_{L^2(X)}
\]

\[
\geq \text{Ric}_X \|\theta(t)\|^2_{L^2(X)} - 2 \max |\Theta(A(t), \theta(t))| \cdot \|\theta(t)\|^2_{L^2(X)}
\]

\[
\geq \frac{\text{Ric}_X}{2} \|\theta(t)\|^2_{L^2(X)}
\]

Hence \( \theta(t) \equiv 0 \), since \( \theta = (S(t)\sigma(t))^{-1} \circ \theta(t) \circ (S(t)\sigma(t)) \), then the Higgs field \( \theta \) is vanish. \( \square \)

In [1, 2], they proved that for a polystable Higgs \( G \)-bundle \( (E, \theta) \) on a compact connected Calabi-Yau manifold, the underlying principal \( G \)-bundle \( E_G \) is polystable.

**Theorem 4.15.** ([2] Theorem 3.3) Let \( G \) be a compact Lie group and \( E \) a holomorphic principal \( G \)-bundle with Hermitian metric \( H_0 \) over a compact Calabi-Yau \( n \)-fold \( X \) has fully holonomy and \( A \in \mathcal{A}_{H_0}^{1,1} \) be a connection on \( E \). If \( (E, \partial_A, \theta) \) is a polystable Higgs bundle on \( X \), then the Higgs fields \( \theta \) are vanish.
Proof. We also consider the Yang-Mills-Higgs gradient flow (4.1) with the initial data $(A, \theta)$. Since $(E, \theta)$ is a polystable Higgs bundle, then the solution of the gradient flow, $(A(t),\theta(t))$ satisfies

$$(A(t),\theta(t)) \rightarrow (A_{\infty},\theta_{\infty}) \text{ over } X \text{ in } C^\infty - \text{ topology},$$

where $(A_{\infty},\theta_{\infty})$ satisfies the Hermitian-Yang-Mills equation

$$\sqrt{-1} \Lambda_\omega (F_{A_{\infty}} + [\theta, \theta^* H_0]) = \lambda Id_E.$$ 

Since $(A_{\infty},\theta_{\infty})$ is also a Higgs pair, the Weitzenböck formula for Higgs field gives,

$$0 = \| \nabla_{A_{\infty}} \theta_{\infty} \|^2 + \|[\theta_{\infty}, \theta^* H_0]\|^2.$$ 

here we have used the vanishing of the Ricci-curvature on Calabi-Yau manifolds. Hence

$$\theta_{\infty} \wedge \theta^*_{\infty} H_0 = 0 \text{ and } \nabla_{A_{\infty}} \theta_{\infty} = 0.$$ 

From Lemma 4.1 we have $\theta_{\infty} = 0$, and $\theta = (S(\infty)\sigma(\infty))^{-1} \circ \theta_{\infty} \circ (S(\infty)\sigma(\infty)) = 0$. 

\[ \square \]

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