Some properties of catalog of (3, g) Hamiltonian bipartite graphs: orders, non-existence and infiniteness

Vivek S. Nittoor
vivek@nittoor.com
Independent Consultant & Researcher *

Abstract

The focus of this paper is on discussion of a catalog of a class of (3, g) graphs for even girth g. A (k, g) graph is a graph with regular degree k and girth g. This catalog is compared with other known lists of (3, g) graphs such as the enumerations of trivalent symmetric graphs and enumerations of trivalent vertex-transitive graphs, to conclude that this catalog has graphs for more orders than these lists. This catalog also specifies a list of orders, rotational symmetry and girth for which the class of (3, g) graphs do not exist. It is also shown that this catalog of graphs extends infinitely.

1 Background

A catalog of (3, g) graphs for even girth g has been listed in [1]. A (k, g) graph is a graph with regular degree k and girth g. This catalog has been listed for the Hamiltonian bipartite class of trivalent graphs. Notation D3 for representing this class of trivalent graphs has been described in [2].

A detailed discussion on the properties of this catalog is in [2]. Quoting from [2], “The enumeration of trivalent symmetric graphs began with the Foster census and was expanded by Conder et al. up to order 10,000. In 2012, Potocnik et al. extended the list to trivalent vertex-transitive graphs up to order 1280. Sophisticated techniques in these developments are based upon the classification of finite simple groups. These lists provide useful data to various areas of graph theory.

Quoting from [2], “The important steps in the approach to find graphs of high girth can be described as follows.

1. The search space for computer search is restricted to Hamiltonian trivalent bipartite class of trivalent graphs.

2. An efficient representation for Hamiltonian trivalent bipartite graphs with a specified level of rotational symmetry.

*formerly with the University of Tokyo
3. A range of rotational symmetries are chosen wisely for each value of $g$ such that a $(3, g)$ HBG with that level of rotational symmetry could be found by computer search.

4. We treat $(3, g)$ graphs of a particular level of rotational symmetry within the identified class of trivalent graphs as a subclass, and seek to list at least one representative from each subclass.

The significance of identifying a class of $(3, g)$ graphs and creating a catalog for this class can be explained as follows. Firstly, the inherent difficulty of finding trivalent graphs of large girth outside the vertex-transitive class. Quoting from [2], “The difficulty in finding the smallest $(3, g)$ graph in general is illustrated in the following historical example. The $(3, 14)$ vertex-transitive graph with order 406 was found by Hoare [3] in 1981, and it was only in 2002 that a smaller $(3, 14)$ graph with order 384 was found by Exoo [4] outside the vertex-transitive class.”

Secondly, the enumeration of all non-isomorphic trivalent graphs for orders greater than 32 is considered infeasible. Thirdly, there is currently no other known listing of $(3, g)$ graphs, where the minimum $(3, g)$ graph within the class is also the smallest known $(3, g)$ graph for $g > 8$, and there do not exist $(3, g)$ vertex-transitive graphs for many orders. This work has been motivated by enumeration of trivalent graphs with high girth, with a goal of obtaining a catalog of graphs that is wider than other known enumerations of classes of trivalent graphs. Trivalent graphs are graphs with regular degree 3. Trivalent graphs are also referred to in the literature as cubic graphs. Only graphs that are undirected and without multiple edges between any two vertices are considered in this paper.”

The focus of this paper is as follows.

- A detailed comparison with existing works, i.e., the enumerations of trivalent symmetric graphs and trivalent vertex-transitive graphs in section 3.

- A discussion on other methods to construct graphs of high girth has been provided in section 2. A more detailed survey on catalogs of graphs can be found in [2].

- A detailed list of orders, rotational symmetry and girth for which the class of $(3, g)$ graphs do not exist has been provided in section 4.

- Infiniteness of our catalog of $(3, g)$ graphs in section 5.

1.0.1 D3

The notation $D_3$ refers to $D_3$ chord indices $l_1, l_2, \ldots, l_b$ for a Hamiltonian trivalent bipartite graph with symmetry factor $b$ and $2m$ vertices where $b|m$. As mentioned above, a detailed description of $D_3$ can be found in [2].
2 Related Works

The cage problem is related to construction of regular graphs of high girth. A related
question is construction of graphs of regular degree with high girth. Many high girth
graph individual construction techniques have been listed in [5].

Some of the important constructions for trivalent graphs are as follows. The voltage
graph construction method has been used by Exoo in [4] for the (3, 14) record graph
with order 384, Exoo in [6] for (3, 16) record graph with order 960, Exoo in [7] for (3, 18)
record graph with order 2560, Exoo in [8] for (3, 20) record graph with order 5376, and
Exoo and Jajcay in [7] for (3, 30) record graph with order 1143408.

Quoting from [5]: "Bray, Parker and Rowley constructed a number of current record
holders for degree three by factoring out the 3-cycles in trivalent Cayley graphs." Several
of the current trivalent record holder graphs described in [9] that are constructed by this
method and are (3, 24) record graph of order 49608, (3, 26) record graph of order 109200,
(3, 28) record graph of order 415104, (3, 32) record graph of order 3650304.
Even though this method produces record graphs for higher girth than our approach, it
is not as general as our approach. Cayley graphs are a very small subset of graphs of
our interest and hence this approach is not as general as ours even though this method
is more useful in a practical sense to find graphs of large girth.

The Foster census to enumerate all trivalent symmetric graphs was initiated by Ronald
M. Foster in the 1930s. The Foster census was published as a book, Bouwer et al 1988 [10]
with trivalent symmetric graphs until order 512. The Extended Foster Census until order
768 was published in Conder et al 2002 [11]. Conder et al have extended this list to order
2048 and more recently to order 10000. Conder’s list of trivalent symmetric graphs up to
order 2048 is available at the link, \texttt{http://www.math.auckland.ac.nz/~conder/symmcubic2048list.txt}.
The more recent list of trivalent symmetric graphs up to order 10000 by Conder is avail-
alable at the link, \texttt{http://www.math.auckland.ac.nz/~conder/symmcubic10000list.txt}.
A more recent effort to enumerate trivalent vertex-transitive graphs by Primož Potočnik,
Pablo Spiga and Gabriel Verret in 2012 [12] and [13]. This is a generalization of the
enumeration of trivalent symmetric graphs, and the current method works until order
1280. Quoting from [12]: “Let $\Gamma$ be a cubic $G$-vertex-transitive graph, let $v$ be a vertex
of $\Gamma$ and let $m$ be the number of orbits of the vertex-stabiliser $G_v$ in its action on the
neighbourhood $(v)$. It is an easy observation that, since $\Gamma$ is $G$-vertex-transitive, $m$
 is equal to the number of orbits of $G$ in its action on the arcs of $\Gamma$ (and, in particular,
does not depend on the choice of $v$). Since $\Gamma$ is cubic, it follows that $m = 1, 2, 3$ and there is
a natural split into three cases, according to the value of $m$.”

The case $m = 1$ corresponds to that of trivalent symmetric graphs for which $|G| \leq
48|V(\Gamma)|$, based on a celebrated theorem from Tutte 1947 [14], Tutte 1959 [15], that says
that the vertex-stabilizer has order of at most 48. Quoting further from [12]: "Since the
order of the groups involved grows at most linearly with the order of the graphs and

1Orbits are equivalence classes under the relation, $x \equiv y$ if and only if there exists $h \in H$ with $h.x = y$
2For $x \in X$, the stabilizer subgroup of $x$, is the set of all elements in $H$ that are fixed-points of $x.
H_x = \{ h \in H | h.x = x \}$. Given group $H$ acting on a set $X$, and given $h$ in $H$ and $x$ in $X$ with $h.x = h$,
then $x$ is a fixed point of $h$. 
the groups have a particular structure [12], a computer algebra system can find all the graphs up to a certain order rather efficiently (by using the LowIndexNormalSubgroups algorithm in Magma for example).

For \( m = 3 \), quoting from [12]: "If \( m = 3 \), then \( G_v \) fixes the neighbours of \( v \) pointwise and, by connectedness, it is easily seen that \( G_v = 1 \). This lack of structure of the vertex-stabiliser makes it difficult to use the method that was successfully used in the arc-transitive case. On the other hand, since \( G_v = 1 \), it follows that \( |G| = |V(\Gamma)| \leq 1280 \)."

Thus, \( |G| = |V(\Gamma)| \leq 1280 \) for \( m = 3 \). Quoting further from [12]: "This allows us to use the SmallGroups database in Magma to find all possibilities for \( G \) (and then for \( \Gamma \))."

For \( m = 2 \), quoting from [12]: "Therefore, in order to find all cubic \( G \)-vertex-transitive graphs with \( m = 2 \) up to \( n \) vertices, it suffices to construct the list of all tetravalent arc-transitive graphs of order at most \( n/2 \)." Quoting further from [12]: "This allows us to use a method similar to the one used in the cubic arc-transitive case to construct a list of all tetravalent arc-transitive graphs of order at most \( 640 \)."

The information from enumeration of vertex-transitive graphs from [12] and [13] is also available at the link, http://www.matapp.unimib.it/~spiga/TableLineByLine.html.

Quoting from Lubotzky, Phillips and Sarnak [17], "Ramanujan graphs \( X_{p,q} \) are \( p+1 \)-regular Cayley graphs of the group \( \text{PSL}(2,\mathbb{Z}/q\mathbb{Z}) \) if the Legendre symbol \( (\frac{p}{q}) = 1 \) and of \( \text{PGL}(2,\mathbb{Z}/q\mathbb{Z}) \) if the Legendre symbol \( (\frac{p}{q}) = -1 \). \( X_{p,q} \) is bipartite of order \( n = |X_{p,q}| = q \ast (q^2 - 1) \) and a bound on the girth is given by the equation, \( g(X_{p,q}) \geq 4 \log_q(q) - \log_4(4) \)."

For \( q \) being a power of a prime \( k \geq 3 \), Lazebnik in [18] describes explicit construction of a \( q \)-regular bipartite graph on \( v = 2q^k \) vertices with girth \( g \geq k + 5 \).

Chandran in [19] describes a high graph construction method that constructs a graph with girth \( \log(n) \) with order \( n \). The research on improving lower bound for \( (k, g) \) consists of proving the non-existence of a \( (k, g) \) graph with a given number of vertices. This approach has been used to find the correct values of the lower bound for \( n(3,11) \) and \( (4,7) \) in [19]. Extensive computer searches have already been used for improving lower bounds for the cage problem.

Quoting from [5]: "Such proofs are organized by splitting the problem into a large number of subproblems, which can then be handled independently, and the work can be done in parallel on many different computers. The computation can begin by selecting a root vertex and constructing a rooted \( k \)-nary tree of radius \( \frac{q-1}{2} \). The actual computation proceeds in two phases. First, all non-isomorphic ways to add sets of \( m \) edges to the tree are determined (for some experimentally determined value of \( m \)). This phase involves extensive isomorphism checking. The second phase is the one that is more easily distributed across a large number of computers. Each of the isomorphism classes found in the first phase becomes an independent starting point for an exhaustive search to determine whether the desired graph can be completed. Of all possible edges that could be added to the graph at this point, those that would violate the degree or girth conditions are eliminated. The order in which the remaining edges are considered is then determined by heuristics." O’Keefe and Wong detailed case analysis: These methods are for lower bound improvement by checking and establishing a \( (k, g) \) cage, and different
Table 1: Bounds for trivalent cages for even values of girth from [5]

| Girth $g$ | $n(3, g)$ | Number of cages | Due to |
|-----------|------------|-----------------|--------|
| 6         | 14         | 1               | Heawood |
| 8         | 30         | 1               | Tutte  |
| 10        | 70         | 3               | O’Keefe-Wong |
| 12        | 126        | 1               | Benson |
| 14        | $258 \leq n(3, 14) \leq 384$ | 1 | Exoo; (Lower Bound -McKay) |
| 16        | $512 \leq n(3, 16) \leq 960$ | 1 | Exoo |
| 18        | $1024 \leq n(3, 18) \leq 2560$ | 1 | Exoo |
| 20        | $2048 \leq n(3, 20) \leq 5376$ | 1 | Exoo |
| 22        | $4096 \leq n(3, 22) \leq 16206$ | 1 | Biggs-Hoare |
| 24        | $8192 \leq n(3, 24) \leq 49608$ | 1 | Bray-Parker-Rowley |
| 26        | $16384 \leq n(3, 26) \leq 109200$ | 1 | Bray-Parker-Rowley |
| 28        | $32768 \leq n(3, 28) \leq 415104$ | 1 | Bray-Parker-Rowley |
| 30        | $65536 \leq n(3, 30) \leq 1143408$ | 1 | Exoo-Jajcay |
| 32        | $131072 \leq n(3, 32) \leq 3650304$ | 1 | Bray-Parker-Rowley |

from our approach in terms of focus.

- (3, 10): 1980, M. O’Keefe, P.K. Wong [20].
- (6, 5): 1979, M. O’Keefe, P.K. Wong [21].
- (7, 6): 1981, M. O’Keefe, P.K. Wong [22].
- Girth 5: 1984, M. O’Keefe, P.K. Wong [23]. The following methods are general, but are different from our approach since they are more focussed on improving the lower bound and showing non-existence and then establishing cages.
  - (3, 9): 1995, Brinkmann, [24]; (3, 11): 1998, McKay [25] and (4, 7): 2011 Exoo [16]
  - Largest case for elimination of symmetry assumption $n(3, 9) = 58$ and $n(4, 7) = 67$: Our methods do not work for odd girth, and (3, 8) is the largest case that works for full symmetry factor.
Table 2: Known trivalent cages from [5]

| Girth $g$ | Order $n(3,g)$ | Number of Cages |
|-----------|----------------|-----------------|
| 5         | 10             | 1               |
| 6         | 14             | 1               |
| 7         | 24             | 1               |
| 8         | 30             | 1               |
| 9         | 58             | 18              |
| 10        | 70             | 3               |
| 11        | 112            | 1               |
| 12        | 126            | 1               |

3 Existence results

Our catalog of $(3,g)$ Hamiltonian bipartite graphs has more orders than lists of $(3,g)$ symmetric graphs and $(3,g)$ vertex-transitive graphs. The comparison of our list of $(3,g)$ Hamiltonian bipartite graphs with lists of $(3,g)$ vertex-transitive and $(3,g)$ symmetric graphs is summarized in Table 3 from [2] for even values of girth $g$ between 6 and 14. As shown in Table 4, our lists are exhaustive for $(3,6)$ and $(3,8)$ Hamiltonian bipartite graphs and partial for $(3,10), (3,12)$ and $(3,14)$ Hamiltonian bipartite graphs.

1. $(3,6)$ Hamiltonian bipartite graphs until 50
   Our enumeration of distinct orders for which $(3,6)$ Hamiltonian bipartite graphs exist until 50 is exhaustive. $(3,6)$ Hamiltonian bipartite graphs exist for all even orders greater than or equal to 14. As shown in Table 5, $(3,6)$ Hamiltonian bipartite graphs exist for 19 distinct orders until 50 and in Table 6, $(3,6)$ vertex-transitive graphs exist for 19 orders until 50 and $(3,6)$ symmetric graphs exist for 19 orders until 50.

2. $(3,8)$ Hamiltonian bipartite graphs until 90
   Our enumeration of distinct orders for which $(3,8)$ Hamiltonian bipartite graphs

Table 3: Comparison of orders of $(3,g)$ lists from [2]

| $(3,g)$ graphs | Until order | Hamiltonian bipartite | Vertex-transitive | Symmetric |
|----------------|-------------|-----------------------|-------------------|-----------|
| $(3,6)$        | 50          | 19                    | 19                | 10        |
| $(3,8)$        | 90          | 29                    | 21                | 6         |
| $(3,10)$       | 160         | 29                    | 15                | 7         |
| $(3,12)$       | 400         | 84                    | 26                | 16        |
| $(3,14)$       | 1000        | 164                   | 35                | 11        |
exist until 90 is exhaustive, since (3,8) Hamiltonian bipartite graphs have been found to exist for all distinct orders between 30 and 90, with the exception of 32, for which it is shown that a (3,8) Hamiltonian bipartite graph cannot exist in Table 17. As shown in Table 16 (3,8) Hamiltonian bipartite graphs exist for 28 orders until 90 and in Table 8 (3,8) vertex-transitive graphs exist for 21 orders until 90 and (3,8) symmetric graphs exist for 6 orders until 90. It is observed in Table 4 and Table 3 that for each (3,8) vertex-transitive graph on the vertex-transitive list, there exists a (3,8) Hamiltonian bipartite graph on our list until order 90.

Remark 1. \( \exists \) (3,8) HBG for even orders satisfying \( [34,90] \cup \{30\} \).

3. **(3,10) Hamiltonian bipartite graphs until 160**
   Our enumeration of distinct orders for which (3,10) Hamiltonian bipartite graphs exist until 160 is partial, since our conclusion on existence of (3,10) Hamiltonian bipartite graphs for some orders is inconclusive. As shown in Table 9 (3,10) Hamiltonian bipartite graphs exist for 29 orders until 160 and in Table 10 (3,10) vertex-transitive graphs exist for 15 orders until 160 and (3,10) symmetric graphs exist for 7 orders until 160. It is observed in Table 9 and Table 10 that for each (3,10) vertex-transitive graph on the vertex-transitive list, there exists a (3,10) Hamiltonian bipartite graph on our list until order 160. It is observed in Table 9 and Table 10 that for each (3,10) vertex-transitive graph on the vertex-transitive list, there exists a (3,10) Hamiltonian bipartite graph on our list until order 160.

4. **(3,12) Hamiltonian bipartite graphs until 400**
   Our enumeration of distinct orders for which (3,12) Hamiltonian bipartite graphs exist until 400 is partial, since our conclusion on existence of (3,12) Hamiltonian bipartite graphs for some orders is inconclusive. As shown in Table 11 (3,12) Hamiltonian bipartite graphs exist for 84 orders until 400 and in Table 12 (3,12) vertex-transitive graphs exist for 26 orders until 400 and (3,12) symmetric graphs exist for 16 orders until 400. It is observed in Table 11 and Table 12 that for each (3,12) vertex-transitive graph on the vertex-transitive list, there exists a (3,12) Hamiltonian bipartite graph on our list until order 400. It is observed in Table 11 and Table 12 that for each (3,12) vertex-transitive graph on the vertex-transitive list, there exists a (3,12) Hamiltonian bipartite graph on our list until order 400. It is observed in Table 11 and Table 12 that for each (3,12) vertex-transitive graph on the vertex-transitive list, there exists a (3,12) Hamiltonian bipartite graph on our list until order 400.

5. **(3,14) Hamiltonian bipartite graphs until 1000**
   Our enumeration of distinct orders for which (3,14) Hamiltonian bipartite graphs exist until 1000 is partial, since our conclusion on existence of (3,14) Hamiltonian bipartite graphs for some orders is inconclusive. As shown in Table 13 (3,14) Hamiltonian bipartite graphs exist for 84 orders until 400 and in Table 14 (3,14) vertex-transitive graphs exist for 35 orders until 1000 and (3,14) symmetric graphs exist for 11 orders until 400. It is observed in Table 13 and Table 14 that for each
Table 4: Catalog of \((3, g)\) Hamiltonian bipartite graphs

| \((3, g)\) | Until Coverage | Table | Upper bound |
|------------|----------------|-------|-------------|
| \((3, 6)\) | 50 | Exhaustive | | (3, 6) cage included |
| \((3, 8)\) | 90 | Exhaustive | | (3, 8) cage included |
| \((3, 10)\) | 160 | Partial | | (3, 10) cage included |
| \((3, 12)\) | 400 | Partial | | (3, 12) cage included |
| \((3, 14)\) | 1000 | Partial | | (3, 14) record graph included |

Table 5: \((3, 6)\) lists

| Order of \((3, 6)\) Hamiltonian bipartite graph |
|------------------------------------------------|
| 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50 |

\((3, 14)\) vertex-transitive graph on the vertex-transitive list, there exists a \((3, 14)\) Hamiltonian bipartite graph on our list until order 1000.

**Note 2. Observation** \((3, g)\) Hamiltonian bipartite graphs exist for each distinct order for which \((3, g)\) vertex-transitive graphs exist for considered ranges, whether bipartite or non-bipartite exist for even girth until \(g = 14\).

Quoting from [2] on outcomes of listing of \((3, g)\) Hamiltonian bipartite graphs as follows.

- **Exhaustive**: Outcome of listing of \((3, g)\) Hamiltonian bipartite graphs is exhaustive to the extent that all orders in specified range that have a \((3, g)\) Hamiltonian bipartite graph are listed, with proof for non-existence for orders not listed.

- **Partial**: Outcome of listing of \((3, g)\) Hamiltonian bipartite graphs is partial if results on existence \((3, g)\) Hamiltonian bipartite graph for some orders in specified range are inconclusive.

Table 6: Other \((3, 6)\) lists

| Class of \((3, 6)\) graph | Orders for specified class of \((3, 6)\) graph |
|--------------------------|-----------------------------------|
| \((3, 6)\) symmetric   | 14, 16, 18, 20, 24, 26, 32, 38, 42, 50 |
| \((3, 6)\) vertex-transitive | 14, 16, 18, 20, 22, 24, 26, 28, 30, 32, 34, 36, 38, 40, 42, 44, 46, 48, 50 |
Table 7: (3, 8) lists

| Order of (3, 8) Hamiltonian bipartite graph |
|--------------------------------------------|
| 30, 34, 36, 38, 40, 42, 44, 48, 50, 52, 54, 56, 58, 60, 62, 64, 66, 68, 70, 72, 74, 78, 80, 82, 84, 86, 88, 90 |

Table 8: Other (3, 8) lists

| Class of (3, 8) graph | Orders for specified class of (3, 8) graph |
|-----------------------|-------------------------------------------|
| (3, 8) symmetric      | 30, 40, 48, 56, 64, 96                     |
| (3, 8) vertex-transitive | 30, 40, 42, 48, 50, 52, 54, 56, 58, 60, 64, 66, 68, 70, 72, 74, 78, 80, 82, 84, 90 |

Table 9: (3, 10) lists

| Symmetry factor | Order of (3, 10) Hamiltonian bipartite graph |
|-----------------|---------------------------------------------|
| 4               | 72, 80, 88, 96, 104, 112, 120, 128, 136, 144, 152, 160 |
| 5               | 80, 90, 100, 110, 120, 130, 140, 150, 160 |
| 6               | 84, 96, 108, 120, 132, 144, 156, 168 |
| 7               | 70, 98, 112, 126, 140, 154, 168 |
| 8               | 80, 96, 112, 128, 144, 160 |
| 9               | 90, 108, 126, 144, 162 |
| 10              | 80, 100, 120, 140, 160 |
| 11              | 88, 110, 132, 154 |
| 12              | 72, 96, 120, 144, 168 |

Table 10: Other (3, 10) lists

| Class of (3, 10) graph | Orders for specified class of (3, 10) graph |
|------------------------|---------------------------------------------|
| (3, 10) symmetric      | 80, 90, 110, 112, 120, 128, 144 |
| (3, 10) vertex-transitive | 80, 90, 96, 100, 108, 110, 112, 120, 126, 128, 132, 140, 144, 156, 160 |
Table 11: \((3, 12)\) lists

| Symmetry factor | Order of \((3, 12)\) Hamiltonian bipartite graph |
|-----------------|-------------------------------------------------|
| 3               | 162, 180, 186, 192, 198, 204, 216, 222, 228, 234, 240, 246, 252, 258, 264 |
|                 | 270, 276, 282, 288, 294, 300, 306, 312, 318, 324, 330, 336, 342, 348, 354 |
|                 | 360, 366, 372, 378, 384, 390, 396 |
| 4               | 216, 224, 232, 240, 248, 256, 264, 272, 280, 288, 296, 304, 312, 320, 328 |
|                 | 336, 344, 352, 360, 368, 376, 384, 392, 400 |
| 5               | 190, 210, 220, 230, 240, 250, 260, 270, 280, 290, 300, 310, 320, 330, 340 |
|                 | 350, 360, 370, 380, 390, 400 |
| 6               | 168, 180, 192, 204, 216, 228, 240, 252, 264, 276, 288, 300, 312, 324, 336 |
|                 | 348, 360, 372, 384, 396 |
| 7               | 182, 196, 210, 224, 238, 252, 266, 280, 294, 308, 322, 336, 350, 364, 378, 392 |
| 8               | 208, 224, 240, 256, 272, 288, 304, 320, 336, 352, 368, 400 |
| 9               | 126 |
| 10              | 200, 220, 240, 260, 280, 300, 320, 340, 360, 380, 400 |
| 12              | 216, 240, 264, 288, 312, 336, 360, 384 |

Table 12: Other \((3, 12)\) lists

| Class of \((3, 12)\) graph | Orders for specified class of \((3, 12)\) graph |
|-----------------------------|------------------------------------------------|
| \((3, 12)\) symmetric      | 162, 168, 182, 192, 204, 216, 224, 234, 256, 288, 312, 336 |
|                             | 360, 364, 378, 384 |
| \((3, 12)\) vertex-transitive | 162, 168, 182, 192, 204, 216, 224, 234, 240, 256, 264, 270, 272 |
|                             | 288, 312, 320, 324, 330, 336, 342, 360, 364, 378, 384, 390, 392 |
Table 13: (3,14) lists

| Symmetry factor | Order of (3,14) Hamiltonian bipartite graph |
|-----------------|---------------------------------------------|
| 4               | 440, 448, 464, 472, 480, 488, 496, 504, 512, 520, 528, 536, 544, 552, 560 |
|                 | 568, 576, 584, 592, 608, 616, 624, 632, 640, 648, 656, 664, 672, 680, 688 |
|                 | 696, 704, 712, 720, 728, 736, 744, 752, 760, 768, 776, 784, 792, 800, 808 |
|                 | 816, 824, 832, 840, 848, 856, 864, 872, 880, 888, 896, 904, 912, 920, 928 |
|                 | 936, 944, 952, 960, 968, 976, 984, 992, 1000 |
| 5               | 460, 490, 500, 510, 520, 530, 540, 550, 560, 570, 580, 590, 600, 610, 620 |
|                 | 630, 640, 650, 660, 670, 680, 690, 700, 710, 720, 730, 740, 750, 760, 770 |
|                 | 780, 790, 800, 810, 820, 830, 840, 850, 860, 870, 880, 890, 900, 910, 920 |
|                 | 930, 940, 950, 960, 970, 980, 990, 1000 |
| 6               | 456, 468, 480, 492, 504, 516, 528, 540, 552, 564, 576, 588 |
|                 | 600, 612, 624, 636, 648, 660, 672, 684, 696, 708, 720, 732, 744, 756, 768 |
|                 | 780, 792, 804, 816, 828, 840, 852, 864, 876, 888, 900, 912, 924, 936, 948 |
|                 | 960, 972, 984, 996 |
| 7               | 406, 462, 476, 490, 504, 518, 532, 546, 560, 574, 588, 602 |
|                 | 616, 630, 644, 658, 672, 686, 700, 714, 728, 742, 756, 770, 784, 798, 812 |
|                 | 826, 840, 854, 868, 882, 896, 910, 924, 938, 952, 966, 980, 994 |
| 8               | 384, 464, 480, 496, 512, 528, 544, 560, 576, 592, 608, 624 |
|                 | 640, 656, 672, 688, 704, 720, 736, 752, 768, 784, 800, 816, 832, 848, 864 |
|                 | 880, 896, 912, 928, 944, 960, 976, 992 |
| 9               | 504, 522, 540, 558, 576, 594, 612, 630, 648, 666, 684, 702 |
|                 | 720, 738, 756, 774, 792, 810, 828, 846, 864, 882, 900, 918, 936, 954, 972, 990 |
| 10              | 520, 540, 560, 580, 600, 620, 640, 660, 680, 700, 720, 740 |
|                 | 760, 780, 800, 820, 840, 860, 880, 900, 920, 940, 960, 980, 1000 |
| 11              | 506, 528, 550, 572, 594, 616, 638, 660, 682, 704, 726, 748, 770 |
|                 | 792, 814, 836, 858, 880, 902, 924, 946, 968, 990 |
| 12              | 576, 600, 624, 672, 696, 720, 744, 768, 792, 816, 840, 864 |
|                 | 888, 912, 936, 960, 984 |
| 13              | 572, 598, 624, 650, 676, 702, 728, 754, 780, 806, 832, 858 |
|                 | 884, 910, 936, 962, 988 |
| 14              | 588, 616, 644, 672, 700, 728, 756, 784, 812, 840, 868, 896 |
|                 | 924, 952, 980 |
| 15              | 600, 630, 660, 690, 720, 750, 780, 810, 840, 870, 900, 930 |
|                 | 960, 990 |
| 16              | 576, 608, 640, 672, 704, 736, 800, 832, 864, 896, 928, 960 |
|                 | 992 |
### Table 14: Other (3,14) lists

| Class of (3,14) graph | Orders for specified class of (3,14) graph |
|-----------------------|------------------------------------------|
| (3,14) symmetric      | 448, 504, 506, 624, 672, 768, 816, 880, 896, 912, 960 |
| (3,14) vertex-transitive | 406, 448, 480, 486, 504, 506, 512, 544, 576, 600, 602, 624, 640, 648, 660, 672, 720, 750, 768, 784, 800, 812, 820, 840, 880, 896, 900, 912, 930, 960, 972, 984, 990, 994, 1000 |

Note 3. Sub-problems introduced by our approach

The sub-problems for the cage problem that are discussed in [5] are subproblems for Cayley graphs and vertex-transitive graphs.
Table 15: Sub-problems: Finding $(3, g)$ HBGs of minimum order with symmetry factor $b$ from [2]

| $b$ | $(3, 6)$ | $(3, 8)$ | $(3, 10)$ | $(3, 12)$ | $(3, 14)$ | $(3, 16)$ |
|-----|----------|----------|----------|----------|----------|----------|
| 1   | 14, Cage |          |          |          |          |          |
| 2   | 36       | 80       |          |          |          |          |
| 3   | 30, Cage | 90       | 162      |          |          |          |
| 4   | 40       | 72       | 216      | 440      |          |          |
| 5   | 40       | 80       | 190      | 460      |          |          |
| 6   | 36       | 84       | 168      | 456      |          |          |
| 7   | 14, Cage | 42       | 70, Cage | 182      | 406#     |          |
| 8   | 48       | 80       | 208#     |          |          |          |
| 9   | 54       | 90       | 126, Cage| 504#     |          |          |
| 10  | 40       | 80       | 200#     |          |          |          |
| 11  | 44       | 88       |          | 506#     |          |          |
| 12  | 48       | 72       | 216#     |          |          |          |
| 13  | 52       |          |          | 572#     |          |          |
| 14  | 56       |          |          | 588#     |          |          |
| 15  | 30, Cage |          |          |          | 600#     |          |
| 16  | 64       |          |          |          |          | 576#     |

Color Significance

- # (3, g) sub-problem resolved between known lower and upper bound
- * (3, g) sub-problem for HBGs found, but open to improvement
- ** (3, g) sub-problem for HBGs resolved
- - Does Not Exist
- * * (3, g) sub-problem for HBGs is inconclusive
Table 16: (3, 14) sub-problems lower and upper bounds from [2]

| Symmetry factor $b$ | $lb(3, 14, b)$ | Lower bound (3, 14) HBG for symmetry factor $b$ | Upper bound (3, 14) HBG for symmetry factor $b$ |
|---------------------|----------------|----------------------------------|----------------------------------|
| 3                   | 258            | 300                              | 400                              |
| 4                   | 260            | 440                              | 440                              |
| 5                   | 260            | 460                              | 460                              |
| 6                   | 264            | 456                              | 456                              |
| 7                   | 266            | 364                              | 406                              |
| 8                   | 272            | 304                              | 384, Record                      |
| 9                   | 270            | 288                              | 504                              |
| 10                  | 260            | 260                              | 520                              |
| 11                  | 264            | 264                              | 506                              |
| 12                  | 264            | 264                              | 576                              |
| 13                  | 260            | 260                              | 572                              |
| 14                  | 280            | 280                              | 588                              |
| 15                  | 270            | 270                              | 600                              |
| 16                  | 288            | 288                              | 576                              |

Color Significance:
- **Red**: Found to not exist
- **Gray**: Lower bound equals upper bound
- **Orange**: Lower bound improved
- **Blue**: Scope for potentially improving bounds
4 Non-existence Lists

The cases for which a conclusive result has been reached for non-existence of a graph with a specified symmetry factor and girth are referred to as "Non-existence List".

1. There does not exist a $(3, 14)$ Hamiltonian bipartite graph with symmetry factors 4, 5, 6 between orders 258 and 384.

2. Non-existence of orders of $(3, 6)$ and $(3, 8)$ Hamiltonian bipartite graphs are given in Table 17 for full symmetry factors.

3. Non-existence of orders of $(3, 8)$ Hamiltonian bipartite graphs for various symmetry factors, are given in Table 18.

4. Non-existence of orders of $(3, 10)$ Hamiltonian bipartite graphs for various symmetry factors, are given in Table 19.

5. Non-existence of orders of $(3, 12)$ Hamiltonian bipartite graphs for various symmetry factors, are given in Table 20.

6. Non-existence of orders of $(3, 14)$ Hamiltonian bipartite graphs for various symmetry factors, are given in Table 21.

7. Non-existence of orders of $(3, 16)$ Hamiltonian bipartite graphs for various symmetry factors, are given in Table 22.

8. Non-existence of orders of $(3, 18)$ Hamiltonian bipartite graphs for various symmetry factors, are given in Table 23.

Table 17: Non-existence of $(3, g)$ Hamiltonian bipartite graphs for the following number of even vertices

| $(3, g)$ | Orders for non-existence of $(3, g)$ Hamiltonian bipartite graph |
|----------|---------------------------------------------------------------|
| $(3, 6)$ | 10, 12                                                        |
| $(3, 8)$ | 20, 22, 24, 26, 28, 32                                       |
Table 18: Non-existence of \((3, 8)\) Hamiltonian bipartite graphs for various symmetry factors for the following orders

| Symmetry factor \(b\) | Orders for non-existence of \((3, 8)\) Hamiltonian bipartite graph with symmetry factor \(b\) |
|------------------------|-------------------------------------------------------------------------------------------------|
| 3                      | 36                                                                                              |
| 4                      | 32                                                                                              |
| 5                      | 30                                                                                              |
| 6                      | 24                                                                                              |
| 7                      | 28                                                                                              |
| 8                      | 32                                                                                              |
| 9                      | 36                                                                                              |
| 10                     | 20                                                                                              |

Table 19: Non-existence of \((3, 10)\) Hamiltonian bipartite graphs for various symmetry factors for the following orders

| Symmetry factor \(b\) | Orders for non-existence of \((3, 10)\) Hamiltonian bipartite graph with symmetry factor \(b\) |
|------------------------|-------------------------------------------------------------------------------------------------|
| 4                      | 64                                                                                              |
| 5                      | 50, 60, 70                                                                                      |
| 6                      | 60, 72                                                                                          |
| 7                      | 56, 84                                                                                          |
| 8                      | 64                                                                                              |
| 9                      | 54, 72                                                                                          |
| 10                     | 60                                                                                              |
| 12                     | 24, 48                                                                                          |
Table 20: Non-existence of \((3, 12)\) Hamiltonian bipartite graphs for various symmetry factors for the following orders

| Symmetry factor \(b\) | Orders for non-existence of \((3, 12)\) Hamiltonian bipartite graph with symmetry factor \(b\) |
|------------------------|---------------------------------------------------------------------------------------------------|
| 2                      | 60 – 512, in steps of 4                                                                         |
| 3                      | 132, 138, 144, 150, 156, 168, 174                                                               |
| 4                      | 56 – 208, in steps of 8                                                                         |
| 5                      | 60 – 180, in steps of 10, 200                                                                   |
| 6                      | 60, 72, 84, 96, 108, 120, 132, 144, 156                                                        |
| 7                      | 140, 154, 168                                                                                   |
| 8                      | 128                                                                                             |
| 9                      | 144                                                                                             |

Table 21: Non-existence of \((3, 14)\) Hamiltonian bipartite graphs for various symmetry factors for the following orders

| Symmetry factor \(b\) | Orders for non-existence of \((3, 14)\) Hamiltonian bipartite graph with symmetry factor \(b\) |
|------------------------|---------------------------------------------------------------------------------------------------|
| 4                      | 272, 280, 288, 296, 304, 312, 320, 328, 336, 344, 352, 360, 368, 376, 384, 392, 400, 408, 416, 424, 432, 456 |
| 5                      | 260, 270, 280, 290, 300, 310, 320, 330, 340, 350, 360, 370, 380, 390, 400, 410, 420, 430, 440, 450, 470, 480 |
| 6                      | 264, 274, 288, 300, 312, 324, 336, 348, 360, 372, 384, 396, 408, 420, 432, 444                     |
| 7                      | 266, 280, 294, 308, 322, 336, 350                                                                 |
| 8                      | 272                                                                                             |
| 9                      | 270                                                                                             |
| 19                     | 380                                                                                             |

Table 22: Non-existence of \((3, 16)\) Hamiltonian bipartite graphs for various symmetry factors for the following orders

| Symmetry factor \(b\) | Orders for non-existence of \((3, 16)\) Hamiltonian bipartite graph with symmetry factor \(b\) |
|------------------------|---------------------------------------------------------------------------------------------------|
| 5                      | 950                                                                                             |

Table 23: Non-existence of \((3, 18)\) Hamiltonian bipartite graphs for various symmetry factors for the following orders

| Symmetry factor \(b\) | \((3, 18)\) Non-existence |
|------------------------|----------------------------|
| 4                      | 1920                       |
5 Infinite family of graphs

The D3 chord index notation can specify an infinite family of graphs.

**Definition 1.** Extent of a path

Let $G$ be a graph represented by a D3 chord index, and $P$ be a path in $G$. The extent of $P$ is the maximum distance of pairs of vertices in $P$ along with the Hamiltonian cycle of $G$.

**Lemma 1.** Let $G$ be a $(k, g)$ graph represented by a D3 chord index, $P$ be a path of length $g$, and $l$ be the length of the longest chord in $P$. Then the extent of $P$ is not more than $(l + 1)g/2$.

**Proof.** First observe that the edges adjacent to a chord in $P$ are edges in the Hamiltonian edges, because each vertex has only one chord edge. Thus the number of chords in $P$ is not more than $g/2$. Consider a path of length $g$ in $G$, and the order of $G$ is large enough. Then the maximal extent is attained when all the edges in $P$ are in the same direction along with the Hamiltonian path, and when chords of length $l$ and Hamiltonian edges are interleaved each other. Thus the extent of $(l + 1)g/2$ is attained. If the order of $G$ is smaller, the maximal extent can be less than $(l + 1)g/2$, but not more than that. QED

**Lemma 2.** Let $G$ be a graph represented by a D3 chord index, and $C$ be a cycle of length $g$. Then the extent of $C$ is not more than $(l + 1)g/4$.

**Proof.** First assume that the order of $G$ is large enough. Because a cycle is a path, the previous lemma applies. In a cycle, the first vertex is the same as the last vertex, so the edges cannot direct in the same direction, and thus the attainable extent is a half of that of paths, which is $(l + 1)g/4$. If the order of $G$ is smaller, then the extent can be smaller, but not larger. QED

**Theorem 4.** Let $G$ be a graph represented by a D3 chord index, of order $2m$, of symmetry factor $b$, any chord is no longer than $l$, and $(l + 1)g/4 <= m$. Let $D$ be the D3 chord index of graph $G$. Consider a graph of order $2m + 2b$, supplied with the same D3 chord index as $G$. This gives proper trivalent Hamiltonian bipartite graph, and with the same girth $g$.

**Proof.** Because the cycles of length $g$ have extents of no more than $(l + 1)g/2 <= m$, the extents are less than half of the Hamiltonian cycle. When $b$ vertices are added without changing D3 chord index, one more repeat of the repeated structure is added, thus the shortest path length is not changed. Thus it corresponds to a graph with girth $g$.

**Note 5.** This is not always true for smaller graphs. If the cycle of length $g$ is around the Hamiltonian path, then inserting more vertices without changing D3 chord indices make the cycle longer. Thus the girth may not be kept unchanged.
**Corollary 1.** Let $G$ be a graph represented by a $D^3$ chord index, of order $2m$, of symmetry factor $b$, maximum chord length be $l$, and $(l + 1)g/4 \leq m$. Then the $D^3$ chord index gives $(3, g)$ graphs for orders $2m' = 2(m + b), 2(m + 2b), ...$

**References**

[1] Nittoor V. S. A catalog of $(3, g)$ Hamiltonian bipartite graphs. *draft*.

[2] Nittoor V. S. A new approach to catalog small graphs of high girth. *arXiv:1601.02880 [math.CO]*.

[3] Hoare M. J. On the girth of trivalent Cayley graphs. graphs and other combinatorial topics. In *Proceedings of the Third Czechoslovak Symposium on Graph Theory, Prague 1982, Teubner, Leipzig*, pages 109–114, "" 1983.

[4] Exoo G. A small trivalent graph of girth 14. *The Electronic Journal Of Combinatorics*, 9, 2002.

[5] Exoo G. and Jajcay R. Dynamic cage survey. *Electronic Journal of Combinatorics*, 18(DS16), 2011.

[6] Exoo G. Voltage graphs, group presentations and cages. *Electronic Journal of Combinatorics*, 11 (1)(N2), 2004.

[7] Jajcay R. Exoo G. On the girth of voltage graph lift. *European Journal of Combinatorics*, 32:554–562, 2011.

[8] Exoo G. New small trivalent graphs for girths 17, 18 and 20. *preprint*.

[9] Bray J. and Parker C. and Rowley P. Cayley type graphs and cubic graphs of large girth. *Discrete Mathematics*, 214:113–121, 2000.

[10] Bouwer Z., Chernoff W. W., Monson B., and Star Z. The Foster census. Technical report, Charles Babbage Research Centre, 1988.

[11] Conder M. and Dobcsanyi P. Trivalent symmetric graphs on up to 768 vertices. *J. Combinatorial Mathematics & Combinatorial Computing*, 40:41–63, 2002.

[12] Potočnik P., Spiga P., and Verret G. Cubic vertex-transitive graphs on up to 1280 vertices. *arXiv:1201.5317v1 [math.CO]*.

[13] Potočnik P., Spiga P., and Verret G. Bounding the order of the vertex-stabiliser in 3-valent vertex-transitive and 4-valent arc-transitive graphs. *arXiv:1010.2546v1 [math.CO]*.

[14] Tutte W. T. A family of cubical graphs. *Proceedings of the Cambridge Philosophical Society*, 43:459–474, 1947.
[15] Tutte W. T. On the symmetry of cubic graphs. *Canadian Journal of Mathematics*, 11:621–624, 1959.

[16] Exoo G., McKay B. D., Myrvold W. J., and Nadon J. Computational determination of (3, 11) and (4, 7) cages. *J. Discrete Algorithms*, 9(2):166–169, 2011.

[17] Lubotzky A., Philips R., and Sarnak P. Ramanujan graphs. *Combinatorica*, 8(3):261–277, 1988.

[18] Lazebnik F. and Ustimenko V. A. Explicit construction of graphs with arbitrary large girth and of large size. *Discrete Applied Mathematics*, 60:275–284, 1995.

[19] Chandran L. S. A high girth graph construction. *SIAM Journal of Discrete Mathematics*, 16:366–370, 2003.

[20] O’Keefe M. and Wong P. K. A smallest graph of girth 10 and valency 3. *Combin. Theory*, 29:91–105, 1980.

[21] O’Keefe M. and Wong P. K. A smallest graph of girth 5 and valency 6. *J. Combin. Theory Ser. B*, 26:145–149, 1979.

[22] O’Keefe M. and Wong P. K. The smallest graph of girth 6 and valency 7. *J. Graph Theory*, 5:79–85, 1981.

[23] O’Keefe M. and Wong P. K. On certain regular graphs of girth 5. *International Journal of Mathematics and Mathematical Sciences*, 7:785–791, 1984.

[24] Brinkmann G., McKay B. D., and Saager C. The smallest cubic graphs of girth nine. *Combin. Probab. Comput.*, 5:1–13, 1995.

[25] McKay B. D., Myrvold W., and Nadon J. Fast backtracking principles applied to find new cages. In *Proceedings of the ninth annual ACM-SIAM symposium on Discrete algorithms, SODA ’98. Society for Industrial and Applied Mathematics Philadelphia, PA, USA*, pages 188–191, 1998.