THE MV FORMALISM FOR IBL$_\infty$- AND BV$_\infty$-ALGEBRAS

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Abstract. We develop a new formalism for the Quantum Master Equation $\Delta e^{S/\hbar} = 0$ and the category of IBL$_\infty$-algebras and simplify some homotopical algebra arising in the context of oriented surfaces with boundary. We introduce and study a category of MV-algebras, which, on the one hand, contains such important categories as those of IBL$_\infty$-algebras and $L_\infty$-algebras, and on the other hand, is homotopically trivial, in particular allowing for a simple solution of the quantum master equation. We also present geometric interpretation of our results.

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1. Introduction

Recent developments in String Topology, Symplectic Field Theory, and Lagrangian Floer Theory have led to a new wave of homotopical algebra, heavily burdened by formulas that seem overwhelming to the eye of an unpretentious mathematician, see [3, 3]. The algebra

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is associated to a homotopy version of involutive Lie bialgebras, or IBL\(_\infty\)-algebras. The algebraic structure in question is governed by surfaces with boundary. It arises from multi-linear operators that correspond to diffeomorphism classes of connected, oriented surfaces of various genera and with various numbers of labelled boundary components, separated into “inputs” and “outputs,” like in Topological Quantum Field Theory (TQFT). Since such a diffeomorphism class is determined by the genus and the number of inputs and outputs, the algebraic structure is determined by linear operators:

\[
l^g_{m,n} : S^m(U) \to S^n(U), \quad g \geq 0, \quad m, n \geq 1,
\]

between symmetric powers of a graded vector space \(U\). However, unlike in TQFT, the correspondence is not supposed to be a functor: \(l^0_{1,1} : U \to U\) has to be a differential on \(U\) rather than the identity, and \(l^g_{m,n}\) must be a null-homotopy for the sum of all possible gluings of two surfaces along part of their inputs and outputs resulting in a surface of genus \(g\) with \(m\) inputs and \(n\) outputs, see [7, 19, 5]. These operations, \(l^g_{m,n}\), may be collected nicely into a generating function \(\Delta : S(U)[[\hbar]] \to S(U)[[\hbar]]\), called a BV (or BV\(_\infty\)) operator, which is in fact an odd differential operator such that \(\Delta^2 = 0\), see [7, 5, 17]. From the operadic perspective, we are talking about the differential graded (dg) dual notion to the notion of a TQFT, namely an algebra over the dg dual properad to the Frobenius algebra properad, which is Koszul dual to the involutive Lie bialgebra properad, see [7, 14]. The operadic yoga applies and shows that the algebraic structure is a canonical homotopy version of the structure of an involutive Lie bialgebra, producing an (almost) canonical name: that of an IBL\(_\infty\)-algebra.

In concrete geometric contexts, the construction of an IBL\(_\infty\) structure involves a number of choices of geometric data, such as an almost complex structure on the target space, a Riemannian metric on the surface, &c., and showing the independence of the algebraic structure on the choices invokes the notion of an IBL\(_\infty\)-morphism. That notion is quite elaborate algebraically, see [3], but may be packed nicely in a generating function \(\Delta : S(U)[[\hbar]] \to S(U)[[\hbar]]\), called a BV (or BV\(_\infty\)) operator, which is in fact an odd differential operator such that \(\Delta^2 = 0\), see [7, 5, 17]. From the operadic perspective, we are talking about the differential graded (dg) dual notion to the notion of a TQFT, namely an algebra over the dg dual properad to the Frobenius algebra properad, which is Koszul dual to the involutive Lie bialgebra properad, see [7, 14]. The operadic yoga applies and shows that the algebraic structure is a canonical homotopy version of the structure of an involutive Lie bialgebra, producing an (almost) canonical name: that of an IBL\(_\infty\)-algebra.

The current work came out of the authors’ realization that the exponential of a map was a true exponential in a (graded) commutative algebra given by the convolution product, see Equations (16) and (17). This led to a significant simplification of dealings with the category of IBL\(_\infty\)-algebras as well as to a “larger” category of MV-algebras, which we present in this paper. We show that the category of MV-algebras is equivalent to a certain category of pointed complexes. Thus, the category of MV-algebras is rather trivial as a category, but this does not prevent it from having room for such highly nontrivial subcategories as those of L\(_\infty\)- and IBL\(_\infty\)-algebras. This categorical triviality may be regarded as homotopy triviality, presenting itself through the observation that the quantum master equation (1) in an MV-algebra is just a cocycle condition. We also show that IBL\(_\infty\)-morphisms are closed under composition within the category of MV-algebras, which seems to be a nontrivial property, cf. [3].

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The theme of the current article may also be viewed as the study of the quantum master equation (QME)

$$\Delta e^{S/\hbar} = 0,$$

which originates in the BV formalism, see e.g. [13], and takes place in a dg BV- or, more generally, (hereafter, commutative) BV$_\infty$-algebra. IBL$_\infty$-algebras provide an important, but not exhaustive class of examples of BV$_\infty$-algebras: those for which the underlying commutative algebra is free, see Examples 9 and 10. The QME may be viewed as an $\hbar$-deformation of the Maurer-Cartan equation $dS + \frac{1}{2}[S, S] = 0$, cf. Theorem 38. Thus, the study of solutions of the QME must be related to a quantization of deformation theory, cf. [19], in which the question of homotopy invariance of solutions with respect to weak equivalences would be one of the primary problems. The homotopy category of dg BV- and BV$_\infty$-algebras should be one of the working notions of quantized deformation theory. Cieliebak and Latschev [6] defined the notion of a BV$_\infty$-morphism, only when the source was free as a commutative algebra, i.e. derived from an IBL$_\infty$-algebra. Such a mismatch of sources and targets prevents BV$_\infty$-algebras from forming a category. MV-morphisms described in this paper include such BV$_\infty$-morphisms and, on the other hand, form a category. Moreover, the QME makes perfect sense in the more general context of MV-algebras, see Section 7, which makes them arguably a better candidate for providing a background for quantum deformation theory.

The main results of the paper in this direction are Theorem 39, which may be interpreted as a representability theorem for the functor of solutions of the QME, see Remark 40, and a description of the transfer of solutions of QME, Theorem 44. We also present geometric interpretation of our results.

**Disclaimer.** It may appear that our choice of terminology goes against the good old mathematical tradition of being modestly egocentric. We should assure the reader that, on the contrary, we have chosen to be modest about the mathematical content of the paper and, at the same time, allowed ourselves to be somewhat egocentric in such a minor, cosmetic issue as terminology.

**Conventions.** The symbol $k$ will denote a fixed field $k$ of characteristic zero and $\otimes$ the tensor product over $k$. We will denote by $id_X$ or simply by $id$ when $X$ is understood, the identity endomorphism of an object $X$ (vector space, algebra, &c.). We will sometimes denote the product of elements $a$ and $b$ of an algebra using the explicit name of the multiplication (typically $\mu(a, b)$), sometimes, when the meaning is clear from the context, by $a \cdot b$, or simply by $ab$.

For a graded $k$-vector vector space $V$ and a complete local commutative ring $R$ with the residue field $k$ and the maximal ideal $m$ we denote by $V \otimes R$ the completed tensor product $\lim_{\leftarrow n} V \otimes R/V \otimes m^n$. In the particular case when $R$ is the formal power series ring $k[[\hbar]]$ in $\hbar$ we abbreviate, as usual, $V \hat{\otimes} k[[\hbar]]$ by $V[[\hbar]]$. We also abbreviate $k((\hbar)) := k[[\hbar]][\hbar^{-1}]$ and $V((\hbar)) := V \hat{\otimes} k((\hbar))$. The degree of a homogeneous element $v \in V$ is denoted by $|v|$.
Recall that, for graded indeterminates $u_1, \ldots, u_n$ and a permutation $\sigma \in \Sigma_n$, the Koszul sign $\epsilon(\sigma) = \epsilon(\sigma; u_1, \ldots, u_n) \in \{+1, -1\}$ is defined by

$$u_1 \odot \cdots \odot u_n = \epsilon(\sigma; u_1, \ldots, u_n) \cdot u_{\sigma(1)} \odot \cdots \odot u_{\sigma(n)}$$

which has to be satisfied in the free graded commutative algebra $S(u_1, \ldots, u_n)$ generated by $u_1, \ldots, u_n$. If $k \geq 1$ and $a_1, \ldots, a_k$ are non-negative integers such that $a_1 + \cdots + a_k = n$, then an $(a_1, \ldots, a_k)$-unshuffle of $n$ elements is a permutation $\sigma \in \Sigma_n$ such that

$$\sigma(1) < \cdots < \sigma(a_1), \quad \sigma(a_1 + 1) < \cdots < \sigma(a_1 + a_2), \quad \ldots, \quad \sigma(n - a_k + 1) < \cdots < \sigma(n).$$

The subset of all $(a_1, \ldots, a_k)$-unshuffles will be denoted by $\text{Sh}_{a_1, \ldots, a_k}^{-1} \subset \Sigma_n$.

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## 2. MV-algebras

**Definition 1.** Let $R$ be a complete local Noetherian commutative ring with the residue field $k$ and a maximal ideal $m$. An **MV-algebra** over $R$ is a quadruple $V = (V, \mu, \delta, \Delta)$ consisting of

(i) a unital graded associative commutative $k$-algebra $(V, \mu)$,

(ii) a conilpotent counital graded coassociative cocommutative $k$-coalgebra $(V, \delta)$, and

(iii) a continuous degree $+1$ $R$-linear map $\Delta : V \hat{\otimes} R \to V \hat{\otimes} R$ such that $\Delta^2 = 0$ and $\Delta(1) = 0$.

We require that the unit map $\eta : k \to V$ for $(V, \mu)$ be a coaugmentation for $(V, \delta)$ and the counit $\epsilon : V \to k$ be an augmentation for $(V, \mu)$. These conditions imply that these maps are nontrivial and $V \neq 0$. In most situations, $R$ will either be the ground field $k$ or a power series ring $k[[\hbar]]$.

**Remark 2.** Let $I = \text{Ker}(\epsilon)$ be the augmentation ideal of $(V, \mu)$. The **reduced diagonal** $\bar{\delta}$ is defined by the standard formula

$$\bar{\delta}(v) := \delta(v) - (v \otimes 1) - (1 \otimes v)$$

for $v \in I$, while $\bar{\delta}(1) := 0$. We iterate $\bar{\delta}$ by putting $\bar{\delta}^0(v) := v$ and

$$\bar{\delta}^k(v) := (\bar{\delta} \otimes id^{\otimes (k-1)})\bar{\delta}^{k-1}(v)$$

*Abbreviating Markl-Voronov. Not to be mistaken with MV-algebras occurring in many-valued logic.*
for \( k \geq 1 \). The conilpotency of \((V, \delta)\) means that for each \( v \in V \), \( \bar{\delta}^{[k]}(v) = 0 \) for \( k \) large enough. Notice that we do not require any compatibility between \( \mu \) and \( \delta \), which would be required of \((V, \mu, \delta)\) to be a bialgebra, a Frobenius algebra, &c.

**Remark 3.** Each MV-algebra over \( k \) can be considered as an MV-algebra over \( R \) after the \( R \)-linear extension of \( \Delta \). In this manner we obtain precisely those MV-algebras over \( R \) for which \( \Delta(V) \subset V \).

**Example 4.** Every augmented unital commutative associative \( k \)-algebra \( A \) is an MV-algebra over \( k \) with the comultiplication defined by \( \delta(1) := 1 \otimes 1 \), while \( \delta(v) := v \otimes 1 + 1 \otimes v \) for \( v \) in the augmentation ideal of \( A \), and \( \Delta := 0 \).

**Example 5.** Dually, a conilpotent cocommutative coassociative \( k \)-coalgebra \((V, \delta)\) with a counit \( \epsilon : V \to k \) and a coaugmentation \( \eta : k \to V \) is an MV-algebra over \( k \) with a unit \( \eta \), augmentation \( \epsilon \) and multiplication given by \( 1 \cdot v = v \cdot 1 := v \) for each \( v \in V \), and \( v' \cdot v'' = 0 \) for \( v', v'' \in \text{Ker}(\epsilon) \). We again put \( \Delta := 0 \).

**Example 6.** Each graded \( k \)-vector space \( V \) equipped with linear maps \( \epsilon : V \to k \) and \( \eta : k \to V \) such that \( \epsilon \circ \eta = \text{id} \) bears the ‘supertrivial’ unital algebra and counital coalgebra structures defined similarly as in the above two examples. Such \( V \) is therefore a ‘supertrivial’ MV-algebra with any \( R \)-linear \( \Delta : \widehat{V} \otimes R \to \widehat{V} \otimes R \) satisfying \( \Delta^2 = 0 \) and \( \Delta \circ \eta = 0 \).

**Example 7.** Let \( R \) be such that the quotient \( R/\mathfrak{m}^p \) is a finite-dimensional \( k \)-vector space for each \( p \geq 1 \). The continuous \( k \)-linear dual \( R^* \), with the comultiplication given by the dual of the multiplication in \( R \), is conilpotent by the completeness of \( R \). In particular, \( R^* \) itself with \( \Delta := 0 \) and trivial multiplication, as in Example 3, is an MV-algebra over \( k \).

**Example 8.** A less trivial example of an MV-algebra over \( k \) is obtained by taking \( V := S(U) \), the (graded) symmetric, or polynomial, algebra generated by a graded vector space \( U \), with the standard bialgebra structure, and \( \Delta \) a degree 1 coderivation with \( \Delta^2 = 0 \) and \( \Delta(1) = 0 \). Such a structure is a disguise of a strongly homotopy Lie (L\(_\infty\)-) algebra [12, Theorem 2.3].

Let \( A \) be a unital associative commutative algebra and \( \Delta : A \to A \) be a \( k \)-linear map. For \( n \geq 0 \), consider the iterated graded commutators
\[
[[ \cdots [\Delta, L_{a_1}], \ldots], L_{a_n}],
\]
\( L_a \) denoting the operator of left multiplication by \( a \in A \). By convention, we just set the commutator of \( \Delta \) with \( n = 0 \) left-multiplication operators to be \( \Delta \). We call an operator \( \Delta \) an order \( \leq k \) differential operator if the iterated commutator with any \( k + 1 \) left-multiplication operators vanishes.

Now suppose that \( \Delta(1) = 0 \). Define
\[
\Phi^\Delta_n(a_1, \ldots, a_n) := [[ \cdots [\Delta, L_{a_1}], \ldots], L_{a_n}](1).
\]
In particular, \( \Phi^\Delta_0 = 0 \) and \( \Phi^\Delta_1 = \Delta \). If \( \Phi^\Delta_n = 0 \) for \( n > k \), the operator \( \Delta \) is called an order \( k \) derivation [13, Section 1.2)]. Notice that first-order derivations are vector fields (derivations).
Example 9. Recall [11, Definition 7] that a (commutative) BV\(_\infty\)-algebra consists of a unital graded commutative associative algebra \(A\) and a \(k[[ℏ]]\)-linear degree 1 map \(Δ : A[[ℏ]] → A[[ℏ]]\) such that \(Δ^2 = 0\) and \(Δ(1) = 0\). One moreover requires that \(Δ\) decomposes into a sum (4) \[Δ = Δ_1 + ℏΔ_2 + ℏ^2Δ_3 + \cdots,\]
where \(Δ_k : A → A\) is a \(k\)-linear order \(≤ k\) differential operator on \(A\). If we assume that \(A\) has an augmentation as a graded commutative algebra and equip \(A\) with the comultiplication \(δ\) constructed in Example 4, then \(A = (A, µ, δ, Δ)\) becomes an MV-algebra over \(k[[ℏ]]\). The most common degree convention is \(|ℏ| = 2\), which is implicit in [11] and explicit in [2], but one can consider \(ℏ\) of degree zero, as in [17], or arbitrary even degree, as in [6, 5]. To comply with our convention that the ground ring \(R\) is not graded, we set \(|ℏ| = 0\).

Example 10. Let \(A := S(U)\) be the symmetric algebra generated by a graded vector space \(U\). An IBL\(_\infty\)-algebra structure on \(U\) [3], [17, §4.2] is given by a degree 1, \(k[[ℏ]]\)-linear map \(Δ : A[[ℏ]] → A[[ℏ]]\) satisfying \(Δ^2 = 0\), \(Δ(1) = 0\), which decomposes as in (4). In other words, an IBL\(_\infty\)-algebra is a BV\(_\infty\)-algebra whose underlying augmented commutative algebra is \(S(U)\). If we equip \(S(U)\) with the standard bialgebra structure, \((S(U), µ, δ, Δ)\) will be another example of an MV-algebra over \(k[[ℏ]]\).

Remark 11 (Geometric interpretation, see Table 1). One can think of an MV-algebra \(V\) over \(R\) as the algebra of functions on a family, that is to say a fiber bundle, \(X → B\) of graded manifolds \(B = \text{Spec}(R), X = \text{Spec}(V ˆ⊗ R)\), endowed with a square-zero differential operator \(Δ\) of degree one on \(X\) over \(B\), thereby, generalizing the notion of a differential graded manifold, when the operator happens to be a derivation linear over functions on \(B\). This fiber bundle has a section \(B → X\), thought of as a family of basepoints, and the space of distributions on \(X\) over \(B\) is provided with a graded commutative product, not necessarily related to multiplication of functions on \(X\), cf. Table 1 for the origin of these geometric data. The only relationship between the products on functions and distributions we place is that the projection \(X → B\) and the section \(B → X\) must provide an augmentation and a unit, respectively, for multiplication of distributions. This geometric object may be called an MV-manifold. A typical situation is when the fiber bundle is trivial: \(F × B → B\), and one can think of an MV-manifold as a \(B\)-parameterized family of differential operators on \(F\). See more on geometric analogies in Table 1.

Since the definition is essentially symmetric with respect to the algebra-coalgebra structures, we can dually consider the graded algebra structure on the \(k\)-linear dual \(V^*\) defined by the graded coalgebra structure on \(V\) and think of \(V^*\) as the algebra of functions on a family \(X^* → B\) of “formal pointed graded manifolds” endowed with a square-zero, odd “differential operator” over \(B\) and the structure of a graded commutative algebra on the space of distributions on \(X^*\), along with similar compatibility conditions between the units and counits. We will call such geometric objects dual MV-manifolds. Depending on the situation, either

\[\text{February 20, 2017} \]\[\text{[ibl.tex]}\]
interpretation could be preferable. For instance, in Example 8, the MV-algebra $V = S(U)$ corresponding to an $L_\infty$-algebra $U[-1]$, where $U[-1]$ denotes an appropriate degree shift, is usually interpreted as a formal differential graded manifold $U$. This is an example of a dual MV-manifold $X = \text{Spec } V^*$. On the other hand, in Example 9, the MV-algebra $A$ may rather be interpreted as a $BV_\infty$-manifold $\text{Spec } A$, i.e. a family of graded manifolds with an odd, square-zero differential operator over $\text{Spec } k[[\hbar]]$. This is an example of an MV-manifold $\text{Spec } A$.

Geometric objects of this nature may arise in various situations, starting from a graded manifold or graded scheme $X$ provided with extra structure, such as those in the following examples:

- $X$ is a graded “abelian Lie group,” see Examples 8 and 10, in which $X$ is moreover a vector space (or a vector bundle over $B$).
- $X$ has a “volume density” inducing a graded Frobenius algebra structure on the space of functions on $X$.
- $X$ has a basepoint $x_0$ and is provided with a “trivial” multiplication law on distributions so that distributions vanishing on constants multiply to zero and the delta function $\delta_{x_0}$ serves as a unit, see Examples 4 and 9. In particular, the delta functions of points multiply as follows: $\delta_x \cdot \delta_y := \delta_x + \delta_y - \delta_{x_0}$ for $x, y \in X$. In this case $X^*$ is a point with a ring $V^*$ of functions such that the maximal ideal $I^*$ of this point has trivial multiplication, $(I^*)^2 = 0$.
- Dually, $X$ may have the “infinitesimal” geometry of a point enriched with a ring $V$ of functions such that the maximal ideal $I$ has trivial multiplication $I^2 = 0$, whereas the space $V^*$ of distributions may have an interesting multiplication, reflected in some nontrivial geometry of $X^*$, as in Example 5.

### 3. The arithmetic of convolution product

Let $V' = (V', \mu', \delta', \Delta')$, $V'' = (V'', \mu'', \delta'', \Delta'')$ be two MV-algebras, and let $\text{Lin}_k(V', V'')$ be the set of degree-0 $k$-linear maps between their underlying spaces. It is well-known, see [ibl.tex]
e.g. \cite{MarklVoronov02} \S III.3, that the comultiplication $\delta'$ together with the multiplication $\mu''$ induces on $\text{Lin}_R(V', V'')$ the structure of a unital commutative associative augmented algebra, via the convolution product $\ast$. Explicitly

\begin{equation}
\label{eq:convolution_product}
f \ast g := \mu''(f \otimes g)\bar{\delta}'
\end{equation}

for $f, g \in \text{Lin}_R(V', V'')$. The unit $e = e_{V', V''}$ for $\ast$ is the composition $\eta'' \circ \epsilon'$ of the augmentation of $V'$ with the unit of $V''$. The algebra $\text{Lin}_R(V', V'')$ is augmented by the map that sends $f \in \text{Lin}_R(V', V'')$ to $\epsilon''(f(1)) \in k$. Notice that

\begin{equation}
\label{eq:augmentation_comultiplication}
e_{V'', V''} \circ e_{V', V''} = e_{V', V''}, \quad e_{V', V''} \circ \eta' = \eta'', \quad \epsilon'' \circ e_{V', V''} = \epsilon'.
\end{equation}

The above constructions clearly extend by $R$-linearity to the space $\text{Lin}_R(V' \otimes R, V'' \otimes R)$ of continuous $R$-linear maps which we will, for brevity, denote by $\text{Lin}_R(V', V'')$ believing that the reader will not be too confused by this shorthand.

Let $m$ be the maximal ideal of $R$. Denote by $\text{Lin}^0_R(V', V'')$ the subset of $\text{Lin}_R(V', V'')$ consisting of continuous $R$-linear maps $f : V' \otimes R \to V'' \otimes R$ such that

\begin{equation}
\label{eq:image}
\text{Im}(f \circ \eta') \subset V'' \hat{\otimes} m.
\end{equation}

Notice that for $f, g \in \text{Lin}^0_R(V', V'')$,

\begin{equation}
\label{eq:algebra_structure}
f \ast g \equiv \mu''(f \otimes g)\bar{\delta}' \mod V'' \hat{\otimes} m,
\end{equation}

where $\bar{\delta}'$ is the reduced diagonal associated to $\delta'$, and moreover $\text{Lin}^0_R(V', V'')$ is an ideal in $\text{Lin}_R(V', V'')$. We leave to prove as an exercise that the conilpotency of $\delta'$ together with the completeness of $R$ implies:

**Lemma 12.** All power series in elements of $\text{Lin}^0_R(V', V'')$ converge.$^\dagger$

In particular, for $f \in \text{Lin}^0_R(V', V'')$ it makes sense to take

\begin{equation}
\label{eq:exponential}
\exp(f) := e + f + \frac{f^2}{2!} + \frac{f^3}{3!} + \cdots \in \text{Lin}_R(V', V'') \quad (\text{powers w.r. to the } \ast\text{-product}).
\end{equation}

Notice that, while $\exp(f) \notin \text{Lin}^0_R(V', V'')$, clearly $\exp(f) - e \in \text{Lin}^0_R(V', V'')$.

If $g \in \text{Lin}^0_R(V', V'')$, then $e + g \in \text{Lin}_R(V', V'')$ and, thanks to \eqref{eq:augmentation_comultiplication}, we have

$$
(e + g) \circ \eta' \equiv \eta'' \mod V'' \hat{\otimes} m
$$

and the power series

$$
\log(e + g) := g - \frac{g^2}{2} + \frac{g^3}{3} - \cdots \in \text{Lin}^0_R(V', V'')
$$

converges. For $f, g \in \text{Lin}^0_R(V', V'')$, one clearly has the expected equalities

$$
\log(\exp(f)) = f, \quad \exp(\log(e + g)) = e + g.
$$

\[\dagger\] We used the same symbol for the map $\eta'$ and its $R$-linear extensions. We keep this convention throughout the paper.

\[\dagger\] Convergence is always understood as convergence in the $m$-adic topology.

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Example 13. Let $G(V') := \{ v \in V' \mid \delta'(v) = v \otimes v \}$ be the subset of group-like elements in $V'$, and $v \in G(V') \otimes R$. Then, for any $f, g \in \text{Lin}_R(V', V'')$, 
\[(f \ast g)(v) = \mu''(f \otimes_R g)\delta'(v) = \mu''(f \otimes_R g)(v \otimes_R v) = \mu''(f(v), g(v)),\]
so $(f \ast g)(v)$ is the ‘actual’ product of the elements $f(v)$ and $g(v)$ in the algebra $V''$. Consequently, for $f \in \text{Lin}_R^0(V', V'')$ and $v \in G(V') \otimes R$,
\[\exp(f)(v) = e^{f(v)},\]
the ‘actual’ exponential of $f(v)$ in the algebra $V''$; similarly for the logarithm.

Proposition 14. Given two MV-algebras $V'$ and $V''$ and $f \in \text{Lin}_R^0(V', V'')$, suppose that $V'$ is in fact a bialgebra, i.e. $\delta'$ is an algebra morphism. Then, for any $v_1, v_2 \in V'$,
\[\exp(f)(v_1v_2) \equiv \exp(f)(v_1)\exp(f)(v_2) \mod (V'' \otimes \mathfrak{m}, f(I^2)),\]
where $I'$ is the augmentation ideal of $V'$. In other words, $\exp(f)$ is an algebra morphism modulo the ideal in $V'' \otimes R$ generated by $V'' \otimes \mathfrak{m}$ and $f(I^2)$.

Proof. Since $[I]$ obviously holds if $v_1$ or $v_2$ are proportional to $1 \in V'$, we will assume that $v_1, v_2 \in I'$. In this case, since $\epsilon'(v_1) = \epsilon'(v_2) = 0$, $\exp(f)(v_1)\exp(f)(v_2)$ is the product
\[(f(v_1) + \frac{1}{2!}f^2(v_1) + \frac{1}{3!}f^3(v_1) + \cdots)(f(v_2) + \frac{1}{2!}f^2(v_2) + \frac{1}{3!}f^3(v_2) + \cdots)\]
while
\[\exp(f)(v_1v_2) = f(v_1v_2) + \frac{1}{2!}f^2(v_1v_2) + \frac{1}{3!}f^3(v_1v_2) + \cdots.\]
As $\delta'$ is, by assumption, an algebra morphism, one has
\[\delta'(v_1v_2) = \delta'(v_1)\delta'(v_2) = (v_1 \otimes 1 + \overline{\delta}(v_1) + 1 \otimes v_1)(v_2 \otimes 1 + \overline{\delta}(v_2) + 1 \otimes v_2)\]
\[= v_1 \otimes v_2 + (-1)^{|v_1||v_2|}v_2 \otimes v_1 + v_1 v_2 \otimes 1 + 1 \otimes v_1 v_2 + \overline{\delta}(v_1)\delta'(v_2)\]
\[+ (1 \otimes v_1)\delta'(v_2) + (v_1 \otimes 1)\delta'(v_2) + \overline{\delta}(v_1)(1 \otimes v_2) + \delta'(v_1)(v_2 \otimes 1).\]
Notice that $\mu'' \circ (f \otimes f)$ applied to the terms on the right-hand side of the above equation vanishes modulo $(V'' \otimes \mathfrak{m}, f(I^2))$ everywhere except at the terms $v_1 \otimes v_2$ and $v_2 \otimes v_1$, thus
\[f^2(v_1v_2) = \mu''(f \otimes f)\delta'(v_1v_2) \equiv \mu''(f(v_1) \otimes f(v_2) + (-1)^{|v_1||v_2|}f(v_2) \otimes f(v_1)) = 2f(v_1)f(v_2)\]
modulo $(V'' \otimes \mathfrak{m}, f(I^2))$. Similarly we obtain that
\[f^3(v_1v_2) = \mu''(f^2 \otimes f)\delta'(v_1v_2) \equiv 3f^2(v_1)f(v_2) + 3f(v_1)f^2(v_2) \mod (V'' \otimes \mathfrak{m}, f(I^2))\]
and, inductively,
\[f^n(v_1v_2) \equiv \sum_{1 \leq i \leq n-1} \binom{n}{i} f^i(v_1)f^{n-i}(v_2) \mod (V'' \otimes \mathfrak{m}, f(I^2)).\]
This formula makes the verification that the product $[\square]$ equals $[\square]$ modulo $(V'' \otimes \mathfrak{m}, f(I^2))$ obvious. \hfill \Box
Corollary 15. Under the assumptions of Proposition 14, suppose also that \( f(I^2) \subset V'' \hat{\otimes} \mathfrak{m} \). Then, for any \( v_1, v_2 \in V' \),

\[
\exp(f)(v_1v_2) \equiv \exp(f)(v_1)\exp(f)(v_2) \mod V'' \hat{\otimes} \mathfrak{m}.
\]

Thus, \( \exp(f) \) is an algebra morphism modulo \( V'' \hat{\otimes} \mathfrak{m} \).

Example 16. Let us show that Proposition 14 cannot be strengthened. Take \( V' = V'' = \mathbb{k} \) with the obvious bialgebra structure. Then \( f \in \text{Lin}_R^0(\mathbb{k}, \mathbb{k}) \) is determined by \( \alpha := f(1) \in \mathfrak{m} \). By definition, \( \exp(f)(v) = ve^\alpha \) for \( v \in \mathbb{k} \), so we have

\[
\exp(f)(v_1v_2) = v_1v_2e^\alpha
\]

while

\[
\exp(f)(v_1)\exp(f)(v_2) = v_1v_2e^{2\alpha}.
\]

The map \( \exp(f) : \mathbb{k} \rightarrow \mathbb{k} \) is in this case an algebra morphism only modulo the ideal generated by \( \alpha \in \mathfrak{m} \).

As the second example, take \( V' = V'' = S(U) \) with the standard bialgebra structure, \( R := \mathbb{k} \), and assume that \( f : S(U) \rightarrow S(U) \) is such that \( f|_{S^n(U)} \neq 0 \) only for \( n = 2 \). One then easily calculates that, for \( v_1, v_2 \in U \),

\[
\exp(f)(v_1) = \exp(f)(v_2) = 0
\]

while

\[
\exp(f)(v_1v_2) = f(v_1v_2).
\]

The map \( \exp(f) : S(U) \rightarrow S(U) \) is an algebra morphism only modulo \( f(S^{\geq 2}(U)) \).

Assume now that \( f|_{S^n(U)} \neq 0 \) only for \( n = 1 \). Then \( f(S^{\geq 2}(U)) = 0 \) and, since \( R = \mathbb{k} \), also \( \mathfrak{m} = 0 \). Formula (11) therefore asserts that \( \exp(f) \) is an algebra morphism. We leave as an exercise to verify that indeed \( \exp(f) \) provides the unique extension of a linear map \( U \rightarrow S(U) \) into an algebra morphism.

Remark 17. Proposition 14 and Corollary 15 provide sufficient conditions for \( \exp(f) \) to be a perturbation of an algebra morphism. See also Examples 25 and 26 in which \( \exp(f) \) will be an \( h \)-perturbation of an algebra morphism. In the general case, one can view \( \exp(f) \) as a generalized algebra morphism.

Example 18. Let us describe the exponential of an \( R \)-linear map \( f \in \text{Lin}_R^0(S(U), B) \), where \( S(U) = (S(U), \delta, \eta) \) is the symmetric algebra generated by a graded vector space \( U \) with the standard cocommutative coassociative counital comultiplication, and \( B = (B, \mu, 1) \) a commutative associative unital algebra.

Denoting, as usual, by \( \delta^{[k-1]} : S(U) \rightarrow S(U)^{\otimes k} \) the diagonal iterated \((k - 1)\)-times and, likewise, by \( \mu^{[k-1]} : B^{\otimes k} \rightarrow B \) the \( R \)-linear extension of the iterated multiplication in \( B \), we have, for \( k \geq 1 \) and \( f \) as above

\[
f^k = \mu^{[k-1]} \circ f^{\otimes k} \circ \delta^{[k-1]}.
\]
It is easy to verify that the $(k-1)$-times iterated diagonal on the product $u_1 \odot \cdots \odot u_n \in S^n(U)$ of $u_1, \ldots, u_n \in U$ equals

\begin{equation}
\delta^{[k-1]}(u_1 \odot \cdots \odot u_n) = \sum_{a_1! \cdots a_k!} \frac{\epsilon(\sigma)}{a_1! \cdots a_k!} [u_{\sigma(1)} \odot \cdots \odot u_{\sigma(a_1)}] \otimes \cdots \otimes [u_{\sigma(n-a_k+1)} \odot \cdots \odot u_{\sigma(n)}],
\end{equation}

where the summation runs over all permutations $\sigma \in \Sigma_k$ and integers $a_1, \ldots, a_k \geq 0$ such that $a_1 + \cdots + a_k = n$, and $\epsilon(\sigma)$ is the Koszul sign. Here we also use the convention that if $a_i = 0$, then

\[ [u_{\sigma(a_1+\cdots+a_i-1)} \odot \cdots \odot u_{\sigma(a_1+\cdots+a_i)}] = 1. \]

Evaluating $\delta^{[k-1]}$ in (14) using (15) gives, for $u_1 \odot \cdots \odot u_n \in S^n(U)$ and $n \geq 1$,

\begin{equation}
\exp(f)(u_1 \odot \cdots \odot u_n) = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{a_1! \cdots a_k!} \frac{\epsilon(\sigma)}{a_1! \cdots a_k!} f(u_{\sigma(1)} \odot \cdots \odot u_{\sigma(a_1)}) \cdots f(u_{\sigma(n-a_k+1)} \odot \cdots \odot u_{\sigma(n)}).
\end{equation}

We recognize a formula in [3, Section 5]. Notice that, thanks to the commutativity of the multiplication $\odot$ in $S(U)$, this formula can be rewritten as

\begin{equation}
\exp(f)(u_1 \odot \cdots \odot u_n) = \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{a_1! \cdots a_k!} \frac{\epsilon(\sigma)}{a_1! \cdots a_k!} f(u_{\sigma(1)} \odot \cdots \odot u_{\sigma(a_1)}) \cdots f(u_{\sigma(n-a_k+1)} \odot \cdots \odot u_{\sigma(n)})
\end{equation}

where $\sigma$ runs now over all $(a_1, \ldots, a_k)$-unshuffles of $n$ elements. The calculation is completed by

\[ \exp(f)(1) = e^{f(1)}, \]

the ‘actual’ exponential of $f(1)$ in $B \hat{\otimes} \mathfrak{m}$.

4. THE CATEGORY OF MV-ALGEBRAS

In this section we define the category of MV-algebras over $R$. Firstly, we introduce morphisms:

**Definition 19.** The space of (MV-)morphisms between MV-algebras $V' = (V', \mu', \delta', \Delta')$ and $V'' = (V'', \mu'', \delta'', \Delta'')$ is given by

\[ \mathcal{MV}_R(V', V'') := \{ f \in \text{Lin}_R^0(V', V'') \mid \Delta'' \circ \exp(f) = \exp(f) \circ \Delta' \}. \]

The categorical composition of $f \in \mathcal{MV}_R(V'', V''')$ with $g \in \mathcal{MV}_R(V', V'')$ is defined as

\[ f \circ g := \log(\exp(f) \circ \exp(g)). \]

The unit endomorphism of an MV-algebra $V$ over $R$ is $1_V := \log(id_{V \hat{\otimes} R})$, which is defined because $(id_{V \hat{\otimes} R} - e) \circ \eta = \eta - \eta = 0$, whence $id_{V \hat{\otimes} R} - e \in \text{Lin}_R^0(V, V)$.

**Theorem 20.** $\mathcal{MV}_R$ with the above notion of morphisms forms a category.
Proof. Let us verify the associativity of the categorical composition. By definition,

\[(f \circ g) \circ h = \log(\exp(f \circ g) \circ \exp h) = \log(\exp(f) \circ \exp(g) \circ \exp(h))\]

while

\[f \circ (g \circ h) = \log(\exp f \circ (g \circ h)) = \log(\exp(f) \circ \exp(g) \circ \exp(h)),\]

so \((f \circ g) \circ h = f \circ (g \circ h)\) as required. To verify the axiom for the categorical identity is also easy; one has

\[f \circ 1_V = \log(\exp(f) \circ id_{V \otimes R}) = \log(\exp(f)) = f.\]

The identity \(1_V \circ f = f\) is verified in a similar fashion. The last thing which remains to be verified is that \(\Delta''' \circ \exp(f \circ g) = \exp(f \circ g) \circ \Delta'\) which follows from

\[\Delta''' \circ \exp(f \circ g) = \Delta''' \circ \exp(f) \circ \exp(g) = \exp(f) \circ \exp(g) \circ \Delta' = \exp(f \circ g) \circ \Delta'.\]

\[\square\]

Example 21. Let us show that, as stated in [2], the unit endomorphism

\[1_{S(U)} \in \text{Lin}_R^0(S(U), S(U))\]

of the IBL\(_\infty\)-algebra \(S(U)\) recalled in Example [10] is the projection \(\pi_1: S(U) \to U\) to the space of algebra generators. Since \(\pi_1\) is the projection to \(U\), it follows from formula (13) for the iterated diagonal that \(\pi_1^{\otimes k} \circ \delta^{[k-1]}(u_1 \odot \cdots \odot u_n) \neq 0\) only when \(k = n\), in which case (13) gives

\[
\mu^{[k-1]} \circ \pi_1^{\otimes k} \circ \delta^{[k-1]}(u_1 \odot \cdots \odot u_n) = \begin{cases} 
n! (u_1 \odot \cdots \odot u_n) & \text{if } n = k \\
0 & \text{otherwise.} \end{cases}
\]

This readily implies that \(\exp(\pi_1) = id_{S(U)}\) i.e. that \(1_{S(U)} = \pi_1 = \log(id_{S(U)})\) as claimed.

We recommend as an exercise to verify that \(\pi_1 = \log(id_{S(U)})\) directly. It turns out that this equation leads to an interesting combinatorial formula for the unshuffles.

Remark 22 (Geometric interpretation). We can interpret a morphism in \(\text{MV}_R(V', V'')\) geometrically as a generalized, as in Remark [17], morphism \(X'' \to X'\) of MV-manifolds. Dually, we can think of it as a morphism \(X^* \to X^{**}\) of dual MV-manifolds.

Example 23. Notice that if the reduced diagonal in \(V'\) is trivial,

\[\exp(f) \equiv e + f \mod V'' \otimes m, \text{ and } \log(e + g) = g \mod V'' \otimes m.\]

The category \(\text{MV}_k\) has a full subcategory consisting of MV-algebras over \(k\) with trivial reduced diagonals. The composition rule in this subcategory is given by

\[f \circ g = \log(\exp(f) \circ \exp(g)) = \log((e + f) \circ (e + g)) = \log(e \circ e + e \circ g + f \circ e + f \circ g) = f \circ g + e \circ g + f \circ e.\]

We used the fact that \(e \circ e = e\) by (3). It is an instructive exercise to verify that the categorical unit endomorphism is \(id - e\). Notice that the ‘expected’ unit \(id\) is not even an element of \(\text{Lin}_R^0(V, V)\).
Restricting to an even smaller subcategory whose morphisms \( f \in \mathcal{MV}_R(V', V'') \) satisfy the stronger condition
\[
f \circ \eta' = 0, \quad \epsilon'' \circ f = 0,
\]
the composition rule \( f \circ g \) becomes the standard composition of morphisms.

**Example 24.** The category \( \mathcal{MV}_k \) contains the subcategory \( L_\infty \) whose objects are \( L_\infty \)-algebras recalled in Example 8 and morphisms are maps \( f : S(U') \to S(U'') \) such that
\[
f(1) = 0, \quad \Delta'' \circ \exp(f) = \exp(f) \circ \Delta', \quad \text{and} \quad \text{Im}(f) \subset U''.
\]
Such a map automatically belongs to \( \text{Lin}_k^0(S(U'), S(U'')) \). We leave as an interesting exercise to prove that
\[
\exp(f) : S(U') \to S(U'')
\]
is the unique coextension of \( f \) into a morphism of counital coalgebras. We conclude that \( L_\infty \) is isomorphic to the category of \( L_\infty \)-algebras and their (weak) \( L_\infty \)-morphisms \([12, \text{Remark 5.3}]\).

**Example 25.** Let us consider IBL\(_\infty\)-algebras recalled in Example 10 with \( \mathbb{k}[[\hbar]] \)-linear maps
\[
f : S(U')[[\hbar]] \to S(U'')[[\hbar]]
\]
of the form
\[
f = f^{(1)} + \hbar f^{(2)} + \hbar^2 f^{(3)} + \cdots
\]
such that
\[
f^{(1)}(1) = 0, \quad \Delta'' \circ \exp(f) = \exp(f) \circ \Delta', \quad \text{and} \quad \bigoplus_{n \geq k} S^n(U') \subset \text{Ker}(f^{(k)}).
\]
In Corollary 33 below we prove that the above structure forms a subcategory IBL\(_\infty\) of the category \( \mathcal{MV}_{\mathbb{k}[[\hbar]]} \) of MV-algebras over \( \mathbb{k}[[\hbar]] \), c.f. also \([3, \text{§5}]\) and \([4, \text{Definition 2.8}]\). One can consider a version of this subcategory with the condition (19) replaced with a “dual” condition:
\[
\text{Im}(f^{(k)}) \subset \bigoplus_{1 \leq n \leq k} S^n(U'').
\]
This modified subcategory of \( \mathcal{MV}_{\mathbb{k}[[\hbar]]} \) may be called the category of IBL\(_\infty\)-algebras in the sense of \([17, \text{§4.3}]\).

**Example 26** (Cieliebak-Latschev \([5]\)). A BV\(_\infty\)-morphism from a BV\(_\infty\)-algebra \((S(U), \Delta')\) of Example 10 to a BV\(_\infty\)-algebra \((A, \Delta'')\) of Example 3 is an MV-morphism given by a \( \mathbb{k}[[\hbar]] \)-linear map
\[
f : S(U)[[\hbar]] \to A[[\hbar]]
\]
of the form
\[
f = f^{(1)} + \hbar f^{(2)} + \hbar^2 f^{(3)} + \cdots,
\]
ibl.tex
such that
\[ f^{(1)}(1) = 0, \quad \Delta'' \circ \exp(f) = \exp(f) \circ \Delta', \quad \text{and} \]
\[ \bigoplus_{n>k} S^n(U) \subset \ker(f^{(k)}). \]
This is a generalization of the notion of an IBL\(_{\infty}\)-morphism of the type \((\mathbb{I})\).

We are going to define a product \(V' \odot V''\) of two MV-algebras \(V' = (V', \mu', \delta', \Delta')\) and \(V'' = (V'', \mu'', \delta'', \Delta'')\) over \(R\) as follows. Its underlying graded vector space is \(V' \otimes V''\) and the structure operator is given as \(\Delta' \otimes_R id + id \otimes_R \Delta''\). The multiplication is defined in the standard way:
\[ (v'_1 \otimes v''_1) \cdot (v'_2 \otimes v''_2) := (-1)^{|v'_2||v''_1|} v'_1 \cdot v'_2 \otimes v''_1 \cdot v''_2, \quad v'_1, v'_2 \in V', \quad v''_1, v''_2 \in V'', \]
with the unit given by the map \(\eta' \otimes \eta' : \mathbb{k} \cong \mathbb{k} \otimes \mathbb{k} \to V' \otimes V''\). The comultiplication is defined as
\[ \delta(v' \otimes v'') := \tau_{23}((\delta'(v') \otimes \delta''(v'')), \quad v' \in V', \quad v'' \in V'', \]
where \(\tau_{23}\) permutes the second and the third factors with the Koszul sign and, finally, \(\epsilon' \otimes \epsilon'' : V' \otimes V'' \to \mathbb{k} \otimes \mathbb{k} \cong \mathbb{k}\) is the counit. The \(\odot\)-product of morphisms \(f \in \text{MV}_R(V'_1, V'_2)\) and \(g \in \text{MV}_R(V''_1, V''_2)\) is given by the formula
\[ f \odot g := \log \left( \exp(f) \otimes_R \exp(g) \right). \]

**Proposition 27.** The \(\odot\)-product equips \(\text{MV}_R\) with a symmetric monoidal structure.

**Proof.** Direct verification. \(\square\)

**Proposition 28.** The category \(\text{MV}_R\) of MV-algebras over \(R\) is isomorphic to the category \(\widehat{\text{MV}}_R\) with the same objects and morphisms
\[ \widehat{\text{MV}}_R(V', V'') := \{ \varphi \in \text{Lin}_R(V', V'') \mid \varphi \circ \eta' \equiv \eta'' \mod V'' \otimes \mathfrak{m} \quad \text{and} \quad \Delta'' \circ \varphi = \varphi \circ \Delta' \}. \]
The categorical composition is the usual composition of maps, and the unit \(1_V \in \widehat{\text{MV}}_R(V, V)\) is the identity \(\text{id} : V \to V\).

**Proof.** The isomorphism between \(\text{MV}_R\) and \(\widehat{\text{MV}}_R\) is given by the mutually inverse functors identical on objects and taking a morphism \(f \in \text{MV}_R(V', V'')\) to \(\exp(f) \in \widehat{\text{MV}}_R(V', V'')\) and \(\varphi \in \widehat{\text{MV}}_R(V', V'')\) to \(\log(\varphi) \in \text{MV}_R(V', V'')\). Let us check that this isomorphism is well-defined.

Notice first that for \(\varphi \in \text{Lin}_R(V', V'')\), the condition \(\varphi \circ \eta' \equiv \eta'' \mod V'' \otimes \mathfrak{m}\) is equivalent to the condition \((\varphi - \mathfrak{e}) \circ \eta' \equiv 0 \mod V'' \otimes \mathfrak{m}\). To see it, recall that \(\mathfrak{e} = \eta'' \circ \epsilon'\), thus
\[ (\varphi - \mathfrak{e}) \circ \eta' = \varphi \circ \eta' - \eta'' \circ \epsilon' \circ \eta' = \varphi \circ \eta' - \eta''. \]
Likewise, for \(f \in \text{Lin}_R(V', V'')\), \(f \circ \eta' \equiv 0 \mod V'' \otimes \mathfrak{m}\) is equivalent to \((f + \mathfrak{e}) \circ \eta' \equiv \eta'' \mod V'' \otimes \mathfrak{m}\).

It is now easy to verify, using the definitions of \(\exp\) and \(\log\), that \(\exp(f)\) indeed belongs to \(\widehat{\text{MV}}_R(V', V'')\) and \(\log(\varphi)\) to \(\text{MV}_R(V', V'')\). The fact that the above correspondence converts the \(\circ\)-composition to the usual one is clear. \(\square\)

[February 20, 2017]
Corollary 29. The category $\mathcal{MV}_R$ of MV-algebras over $R$ is equivalent to the category $\mathcal{Chn}_R^\circ$ of pointed complexes over $R$ whose objects are graded vector spaces $V$ with a continuous degree $+1$ $R$-linear differential $\Delta : V \otimes R \to V \otimes R$ and a $\mathbb{k}$-linear monomorphism $\eta : \mathbb{k} \to V$ such that $\Delta \circ \eta = 0$. A morphism between $V'$ and $V''$ is a chain map $\varphi \in \text{Lin}_R(V', V'')$ such that $\varphi \circ \eta' \equiv \eta'' \mod V'' \otimes \mathfrak{m}$.

Proof. By Proposition 28, it is enough to establish an equivalence between the categories $\widehat{\mathcal{MV}}_R$ and $\mathcal{Chn}_R^\circ$. Let us construct mutual weak inverses $\Box : \widehat{\mathcal{MV}}_R \to \mathcal{Chn}_R^\circ$ and $F : \mathcal{Chn}_R^\circ \to \widehat{\mathcal{MV}}_R$.

On objects, the functor $\Box$ forgets everything except the structure operator $\Delta$ and the unit map $\eta$. For $(V, \Delta, \eta) \in \mathcal{Chn}_R^\circ$ choose a right inverse $\epsilon$ of $\eta$ and define $F(V, \Delta, \eta)$ the supertrivial MV-algebra as in Example 6. Notice that, for $V', V'' \in \widehat{\mathcal{MV}}_R$,

$$\mathcal{Chn}_R^\circ(\Box V', \Box V'') = \widehat{\mathcal{MV}}_R(V', V'')$$

and, likewise, for $V', V'' \in \mathcal{Chn}_R^\circ$,

$$\widehat{\mathcal{MV}}_R(FV', FV'') = \mathcal{Chn}_R^\circ(V', V'').$$

We define $\Box$ and $F$ to be the identities on morphisms. It is simple to verify that we have constructed mutual weak inverses. \hfill \Box

Remark 30. The definition of the category of MV-algebras can be modified. For instance, we may leave the conilpotency of $\delta$ out, but instead of (7) require that $\text{Im}(f) \in V'' \otimes \mathfrak{m}$. Likewise, we need not require $R$ to be complete, but then (7) must be replaced by $f \circ \eta' = 0$. In both cases the above constructions remain valid. We may also require $\epsilon \circ \Delta = 0$, or require both conditions simultaneously. We may allow $R$ to be differential graded, which could be useful in some contexts.

5. Generalizations to other algebra types

MV-algebras were defined as spaces that are simultaneously commutative associative algebras and cocommutative coassociative coalgebras. In this mildly speculative section we discuss possible generalizations to structures other than commutative associative (co-)algebras. We will assume basic knowledge of operads as it can be gained for example from [18].

As preparation we view the exponential (1) from a different angle. Namely, we describe the isomorphism

$$\exp - \mathfrak{e} : \text{Lin}_R^0(V', V'') \xrightarrow{\cong} \text{Lin}_R^0(V', V''),$$

which was the core of our construction, in terms of universal algebra, assuming for simplicity that $R = \mathbb{k}$. Let us denote by $\,^c S(V'')$ the symmetric algebra $S(V'')$ considered as a coalgebra with the standard coalgebra structure. Since $\,^c S(V'')$ with the projection $\,^c S(V'') \to V''$ realizes the cofree conilpotent coassociative cocommutative coalgebra cogenerated by $V''$ [18, Example II.3.79], each $f : V' \to V'' \in \text{Lin}_R^0(V', V'')$ uniquely coextends to a coalgebra map $u_f : V' \to \,^c S(V'')$.

\*Here our assumption of the conilpotency of $V'$ resurfaces again.
On the other hand, the multiplication of $V''$ determines a linear map $m : S(V'') \to V''$. Expressing the coextension $u_f : V' \to ^cS(V'')$ using e.g. formula (3.66) in Section II.3.7 of [18] with $P$ the operad for commutative associative algebras, we easily see that $\exp(f) - e$ equals the composition

$$(20) \quad V' \xrightarrow{u_f} ^cS(V'') \xrightarrow{\text{can}} S(V'') \xrightarrow{m} V''$$

in which

$$\text{can} : ^cS(V'') \xrightarrow{\cong} S(V'')$$

is the identity of the underlying graded vector spaces.

Let us try to generalize the composed map (20) to the case when $V'$ is a $P$-coalgebra and $V''$ a $Q$-algebra, for some $k$-linear operads $P$ and $Q$. We may assume from the very beginning that $P$ has finite-dimensional pieces, as most operads relevant for physical applications have this property. We certainly have again the canonical morphisms $u_f$ and $m$ in the sequence

$$(21) \quad V' \xrightarrow{u_f} ^cF_P(V'') \xrightarrow{?} F_Q(V'') \xrightarrow{m} V''$$

in which $^cF_P(V'')$ is the cofree conilpotent $P$-coalgebra on $V''$ and $F_Q(V'')$ the free $Q$-algebra on $V''$. The only datum which is not automatic is an isomorphism

$$(22) \quad ? : ^cF_P(V'') \xrightarrow{\cong} F_Q(V'').$$

Its existence must therefore be accepted as an assumption, i.e. the operads $P$ and $Q$ must be such that the graded spaces $^cF_P(V'')$ and $F_Q(V'')$ are isomorphic via an isomorphism natural in $V''$.

To formulate this assumption solely in terms of the operads $P$ and $Q$, we invoke from [18, Definitions II.1.24 and II.3.74] the formulas

$$^cF_P(V'') = \bigoplus_{n \geq 1} (P(n)^* \otimes V'' \otimes \Sigma_n)^{\Sigma_n} \quad \text{and} \quad F_Q(V'') = \bigoplus_{n \geq 1} Q(n) \otimes \Sigma_n V'' \otimes \Sigma_n,$$

where $P(n)^*$ is the linear dual of the vector space $P(n)$. It is easy to see now that if a functorial isomorphism in (22) exists then one has for each $n \geq 1$ an isomorphism

$$(23) \quad P(n)^* \cong Q(n).$$

It must moreover, very crucially, be ‘nice’ and explicit enough so that we could express the composition (21) by a formula involving the convolution product in $Lin_d^0(V', V'')$.

The existence of (23) is already very restrictive. Since we assumed that the pieces of the operad $P$ are finite-dimensional, it implies that $P(n) \cong Q(n)$ for each $n$, so the generating series of the operads $P$ and $Q$ are the same. We do not know about any couple of different operads relevant for applications with the same generating series. We are thus led to the assumption $P = Q$, supported by the natural requirement of essential self-duality of the definition of MV-algebras.

\[\|\text{‘Nice’ means in particular that the isomorphism explicitly relates the cooperad structure of } P^* \text{ with the operad structure of } Q. \text{ Paragraph 2.5 of [15] shall give a more concrete idea what we mean by it when } P = Q = \mathcal{L}ie, \text{ the operad for Lie algebras.}\]
Finding interesting operads \( \mathcal{P} \) admitting a nice isomorphisms \( \mathcal{P}(n)^* \cong \mathcal{P}(n) \), \( n \geq 1 \), is however a difficult task. For instance, an explicit canonical isomorphism
\[
\mathcal{L}ie(n)^* \cong \mathcal{L}ie(n)
\]
for the operad \( \mathcal{L}ie \) governing Lie algebras is known only for small \( n \), and finding one is closely related to the problem of Eulerian idempotents, see the discussion in §2.5 and Remark 2.9 of [15].

On the other hand, a nice canonical isomorphism as in (23) always exists when \( \mathcal{P} = \mathcal{Q} \) are \( k \)-linearizations of an operad \( \mathcal{P} \) defined in the category of sets, as then \( \mathcal{P}(n) \) has for each \( n \geq 1 \) a canonical basis spanned by the elements of \( \mathcal{P}(n) \). There are two prominent examples of this situation. The first one is \( \mathcal{P} = \mathcal{Q} = \text{Com} \), the operad for commutative associative algebras which is the linearization of the terminal set-operad. The corresponding MV-algebras are the ones discussed in this paper.

The second outstanding example is \( \mathcal{P} = \mathcal{Q} = \text{Ass} \), the operad for associative algebras which is the linearization of the terminal non-\( \Sigma \) set-operad. The corresponding theory should be that of an \( \text{A}_\infty \)-version of MV-algebras. We expect that it has a similar flavor as the commutative one, with the notable difference that the exponential (9) shall be replaced by the series
\[
e + f + f^2 + f^3 + \cdots = (e - f)^{-1}
\]
and the logarithm by its functional inverse \((g - e)^{-1}\).

Let us close this section by a remark about the convolution product. In general it equips, for \( V' \) a \( \mathcal{P} \)-coalgebra and \( V'' \) a \( \mathcal{Q} \)-algebra, \( \text{Lin}^0_R(V', V'') \) only with a structure of a \((\mathcal{P} \otimes \mathcal{Q})\)-algebra. A special feature of the cases \((\mathcal{P}, \mathcal{Q}) = (\text{Com}, \text{Com})\) or \((\text{Ass}, \text{Ass})\) is that both \( \text{Com} \) and \( \text{Ass} \) are Hopf operads [13, Definition II.3.135], i.e. the ones equipped with the diagonals
\[
\text{Com} \longrightarrow \text{Com} \otimes \text{Com} \quad \text{and} \quad \text{Ass} \longrightarrow \text{Ass} \otimes \text{Ass},
\]
which make the space \( \text{Lin}^0_R(V', V'') \) actually a commutative associative algebra, respectively an associative algebra. Since each operad which is a linearization of a set-theoretic one is a Hopf operad [16, Proposition 11], such a property of the convolution product holds for all operads of this type.

We conclude that sensible generalizations of MV-algebras may exist for couples of the form \((\mathcal{P}, \mathcal{P})\), where \( \mathcal{P} \) is a linearization of a set-theoretic operad. We however think that working out the details would make sense only when a relevant motivating example appears.

6. A composition formula and IBL\(_\infty\)-algebras

Let us consider morphisms \( g \in \text{Lin}^0_R(S(U'), S(U'')) \) and \( f \in \text{Lin}^0_R(S(U'''), S(U'''')) \), where \( S(U'), S(U'') \) and \( S(U''') \) are symmetric algebras with the standard coalgebra structures. The aim of this section is to give an explicit formula for
\[
f \circ g = \log \left( \exp(f) \circ \exp(g) \right) \in \text{Lin}^0_R(S(U'), S(U'''')).
\]
Further, using this formula, we prove that $\text{IBL}_\infty$-algebras with morphisms \cite{IBL} form a subcategory of $\text{MV}_k[[h]]$.

Let us formulate some preparatory observations. Each $R$-linear map
\[
 h : S(V') \otimes R \to S(V'') \otimes R
\]
determines a family
\[
 h^n : S^n(V') \to S^n(V'') \otimes R, \quad m, n \geq 0,
\]
with $h^n_m$ the composition
\[
 S^n(V') \hookrightarrow S(V'') \otimes R \xrightarrow{h_{|S(V')}} S(V'') \otimes R \xrightarrow{\Psi} S(V'') \otimes R
\]
where $\hookrightarrow$ resp. $\rightarrow$ is the canonical inclusion resp. the canonical projection. Vice versa, each family $\{h^n_m\}_{m,n \geq 0}$ as above such that the sum
\[
 h_{|S(V')}(x) := \sum_{m,n \geq 0} h^n_m(x)
\]
converges in $S(V'') \otimes R$ for each fixed $n \geq 0$ and $x \in S^n(V')$, determines an $R$-linear map $h : S(V') \otimes R \to S(V'') \otimes R$. We will call $h^n_m$ the $(m,n)$-component of $h$. We will describe $f \circ g$ in terms of its $(m,n)$-components.

For natural numbers $k, l$ and non-negative integers $r, s_1, \ldots, s_l, j_1, \ldots, j_k$ such that
\[
 j_1 + \cdots + j_k = s_1 + \cdots + s_l = r
\]
we define
\[
 (u_1'' \circ \cdots \circ u_{j_1}'') \cdots \circ (u_r'' \circ \cdots \circ u_{r+j_k+1}'') \in S^{j_1}(U'') \otimes \cdots \otimes S^{j_k}(U'')
\]
to be the $k\langle k \rangle$-linear map that sends
\[
 \sum_{\kappa \in \text{Sh}_{s_1, \ldots, s_l}^{-1}} e(\kappa) (u''_{\kappa(1)} \circ \cdots \circ u''_{\kappa(s_1)}) \otimes \cdots \otimes (u''_{\kappa(r-s_l+1)} \circ \cdots \circ u''_{\kappa(r)}) \in S^{s_1}(U'') \otimes \cdots \otimes S^{s_l}(U'')
\]
over the set $\text{Sh}^{-1}_{s_1, \ldots, s_l}$ of all $(s_1, \ldots, s_l)$-unshuffles $\kappa$ of $r = s_1 + \cdots + s_l$ elements. It is easy to check that \cite{26} is well-defined, i.e. that \cite{26} is invariant under permutations of generators inside the groups
\[
 \{u''_{j_1}, \ldots, u''_{j_k}\}, \ldots, \{u''_{r-j_k+1}, \ldots, u''_r\}.
\]

We associate to each $\kappa \in \text{Sh}^{-1}_{s_1, \ldots, s_l}$ in \cite{26} a graph $\Gamma$ with two types of vertices, the upper and lower ones. The upper ones are labelled by $1, \ldots, l$, the lower ones by $1, \ldots, k$. The upper vertex labelled by $b \in \{1, \ldots, l\}$ is connected to the lower vertex labelled by $a \in \{1, \ldots, k\}$ if and only if $\kappa$ maps some element of the $b$th segment of the decomposition
\[
 \{1, \ldots, r\} = \{1, \ldots, s_1\} \cup \cdots \cup \{r - s_l + 1, \ldots, r\}
\]*We use the convention that $S^0(V') = S^0(V'') = k$. [ibl.tex]
to some element of the $a$th segment of the decomposition

$$
\{1, \ldots, r\} = \{1, \ldots, j_1\} \cup \cdots \cup \{r - j_k + 1, \ldots, r\}.
$$

Let $c(\kappa)$ be the number of connected components of $\Gamma$. We will call $c(\kappa)$ the connectivity of $\kappa$ and say that $\kappa$ is connected if $c(\kappa) = 1$. Denote by $c\Sigma_{j_1, \ldots, j_k}$ the subset of all $\kappa \in Sh^{-1}_{j_1, \ldots, j_k}$ of connectivity $c$ and let finally

$$
c \Psi^{s_1, \ldots, s_l}_{j_1, \ldots, j_k} : S^{j_1}(U'') \otimes \cdots \otimes S^{j_k}(U'') \rightarrow S^{s_1}(U'') \otimes \cdots \otimes S^{s_l}(U'')
$$

be the map defined as $\Psi^{s_1, \ldots, s_l}_{j_1, \ldots, j_k}$ but with the sum (27) restricted to connected $\kappa \in c\Sigma_{j_1, \ldots, j_k}$.

**Proposition 31.** The maps $c \Psi^{s_1, \ldots, s_l}_{j_1, \ldots, j_k}, c \geq 1$, are well-defined and

$$
(28) \quad \Psi^{s_1, \ldots, s_l}_{j_1, \ldots, j_k} = 1 \Psi^{s_1, \ldots, s_l}_{j_1, \ldots, j_k} + 2 \Psi^{s_1, \ldots, s_l}_{j_1, \ldots, j_k} + 3 \Psi^{s_1, \ldots, s_l}_{j_1, \ldots, j_k} + \cdots.
$$

**Proof.** Since composing with permutations inside the groups (27) does not change the connectivity of $\kappa$, each individual $c \Psi^{s_1, \ldots, s_l}_{j_1, \ldots, j_k}$ is well-defined. Formula (28) is obvious. \qed

In what follows we use the same symbols both for the maps (28) and for their $R$-linear extensions. Therefore, for instance, $1 \Psi^{s_1, \ldots, s_l}_{j_1, \ldots, j_k}$ will also denote an $R$-linear map

$$
(29) \quad 1 \Psi^{s_1, \ldots, s_l}_{j_1, \ldots, j_k} : [S^{j_1}(U'') \otimes \cdots \otimes S^{j_k}(U'')] \otimes R \rightarrow [S^{s_1}(U'') \otimes \cdots \otimes S^{s_l}(U'')] \otimes R.
$$

Let $g^m_n : S^m(U') \rightarrow S^m(U'') \otimes R$ be, for $m, n \geq 0$, the $(m \choose n)$-components of $g$ and let

$$
f^m_n : S^n(U') \otimes R \rightarrow S^m(U'') \otimes R
$$

be the $R$-linear extensions of the $(m \choose n)$-components of $f$. For given natural numbers $k, l$ and non-negative integers $i_1, \ldots, i_k, t_1, \ldots, t_l$ we define an auxiliary map

$$
(f \circ g)_{i_1, \ldots, i_k}^{t_1, \ldots, t_l} : S^{i_1}(U'') \otimes \cdots \otimes S^{i_k}(U'') \rightarrow [S^{s_1}(U'') \otimes \cdots \otimes S^{s_l}(U'')] \otimes R
$$

as the sum

$$
(30) \quad \sum \frac{1}{k! \, l!} (f_{s_1} \otimes \cdots \otimes f_{s_l}) \circ 1 \Psi^{s_1, \ldots, s_l}_{j_1, \ldots, j_k} \circ (g_{i_1} \otimes \cdots \otimes g_{i_k})
$$

over all non-negative integers $s_1, \ldots, s_l, j_1, \ldots, j_k$ such that $s_1 + \cdots + s_l = j_1 + \cdots + j_k$, where $1 \Psi^{s_1, \ldots, s_l}_{j_1, \ldots, j_k}$ is the $R$-linear extension (29) of the connected component of the map $\Psi^{s_1, \ldots, s_l}_{j_1, \ldots, j_k}$ in (27).

In the proof of Theorem 32 below we write (30) in the form

$$
(31) \quad \sum \frac{1}{k! \, l!} \left( f_{s_1} \otimes \cdots \otimes f_{s_l} \right) \left( 1 \Psi^{s_1, \ldots, s_l}_{j_1, \ldots, j_k} \right) \left( g_{i_1} \otimes \cdots \otimes g_{i_k} \right)
$$

which would enable us to squeeze formula (34) into a line of finite length. Finally, for $m, n > 0$ define a $k$-linear map

$$
(f \circ g)^m_n : S^n(U') \rightarrow S^m(U'') \otimes R
$$

by the formula

$$
(32) \quad (f \circ g)^m_n (u'_1 \otimes \cdots \otimes u'_n) := \sum \mu^{l-1} \epsilon(\sigma) (f \circ g)_{i_1, \ldots, i_k}^{t_1, \ldots, t_l} (u'_{\sigma(1)} \otimes \cdots \otimes u'_{\sigma(i_1)}) \otimes \cdots \otimes (u'_{\sigma(n-ik+1)} \otimes \cdots \otimes u'_{\sigma(n)}),
$$
where the summation runs over all natural numbers \( k, l \), non-negative integers \( i_1, \ldots, i_k \), \( t_1, \ldots, t_l \) such that
\[
i_1 + \cdots + i_k = n \quad \text{and} \quad t_1 + \cdots + t_l = m,
\]
and over all \((i_1, \ldots, i_k)-\text{unshuffles} \sigma \in \text{Sh}_{i_1, \ldots, i_k}^{-1} \). In (32),
\[
\mu^{[l-1]} : \left[ S^{t_1}(U^m) \otimes \cdots \otimes S^{t_l}(U^m) \right] \otimes R \rightarrow S^m(U^m) \otimes R
\]
is the \( R \)-linear extension of the multiplication in \( S(U^m) \) iterated \((l-1)\)-times.

**Theorem 32.** The \((m)_n\)-part of the composition \( f \circ g \in \text{Lin}_R^0(S(U'), S(U''')) \) is given by formula (32).

*Proof.* Our strategy will be to show that the exponential of the map whose \((m)_n\)-parts are given by (32) equals \( \exp(f) \circ \exp(g) \). Using (17) we find the following expression for the \((m)_n\)-components of the composition \( \exp(f) \circ \exp(g) \):
\[
(\exp(f) \circ \exp(g))^{m}_{n}(u'_1 \circ \cdots \circ u'_n) = \sum_{(33)} \mu^{[l-1]}(\epsilon(\sigma)) (\exp(f) \circ \exp(g))^{t_1, \ldots, t_l}_{i_1, \ldots, i_k}(u'_{\sigma(1)} \circ \cdots \circ u'_{\sigma(i_1)}) \cdots \circ (u'_{\sigma(n-k+1)} \circ \cdots \circ u'_{\sigma(n)}),
\]
with the summation as in (31). The crucial difference against (31) is however that (33) involves the entire \( \Psi_{j_1, \ldots, j_k}^{s_1, \ldots, s_l} \) not only its connected part.

Now the theorem follows from the principle standard in the theory of Feynman diagrams\[1\] that the logarithm singles out connected components of graphs or, equivalently, that the exponential assembles graphs from their connected components. Let us explain what this principle says in our case. Calculating the exponential of \( f \circ g \) using formula (32) involves, instead of (33), expressions like
\[
(34) \sum_{u!k_1! \cdots k_u!l_u!} \frac{1}{u!k_1! \cdots k_u!l_u!} \left( f_{s_1}^{t_1} \otimes \cdots \otimes f_{s_l}^{t_l} \right) \otimes \cdots \otimes \left( f_{s_u}^{t_u} \otimes \cdots \otimes f_{s_u}^{t_u} \right)
\]
with some \( u \geq 1, l_1 + \cdots + l_u = l, k_1 + \cdots + k_u = k \), and
\[
t_1^1 + \cdots + t_1^1 + \cdots + t_u^1 + \cdots + t_u^1 = m \quad \text{and} \quad i_1^1 + \cdots + i_1^k + \cdots + i_u^k = n.
\]

In fact, formula (34) expresses the right hand side of (33) via products of its ‘connected’ components. The seeming discrepancy between the coefficients
\[
(35) \frac{1}{k!!} \text{ in (33)} \quad \text{and} \quad \frac{1}{u!k_1! \cdots k_u!l_u!} \text{ in (34)}
\]
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is explained as follows. While in (33) the connected components are entangled, in (34) they are separated. However, thanks to the commutativity of symmetric algebras, the entangled components can be separated using suitable permutations on the output and input sides. It is straightforward though tedious to check that, with the coefficients (35), the corresponding terms appear with the same multiplicity.

Corollary 33. IBL∞-algebras with morphisms (18) form a subcategory IBL∞ of MVk[[ℏ]].

Proof. It is simple to verify that a map as in (18) satisfies (19) if and only if its (m,n)-component

\[ f^m_n = ℏ^{n-1}φ^m_n, \quad m \geq 0, \quad n \geq 1, \]

for some \( φ^m_n : S^m(U') \to S^m(U'')[[ℏ]]. \) Let therefore

\[ f : S(U'')[[ℏ]] \to S(U''')[[ℏ]] \quad \text{and} \quad g : S(U')[[ℏ]] \to S(U'')[[ℏ]] \]

be such maps. We must prove that each (m,n)-component \((f \circ g)^m_n\) of their MV-composition (24) with \( n \geq 1 \) is divisible by \( ℏ^{n-1} \). This clearly happens if the expression (31) is, for \( n = i_1 + \cdots + i_k \), divisible by \( ℏ^{n-1} \).

Since each \( f^l_{s_b} \) is divisible by \( ℏ^{s_b-1} \) if \( s_b \geq 1, \quad 1 \leq b \leq l \), and each \( g^l_{i_a} \) is divisible by \( ℏ^{i_a-1} \) if \( i_a \geq 1, \quad 1 \leq a \leq k \) by assumption, (31) is divisible by

\[ h^{i_1 + \cdots + i_k + s_1 + \cdots + s_l - (k + l)} = ℏ^{n + s_1 + \cdots + s_l - (k + l)}, \]

where \( k \) is the number of \( a \)'s for which \( i_a \neq 0 \) (resp. \( \overline{I} \) the number of \( b \)'s for which \( s_b \neq 0 \)). Thus (31) is divisible by \( ℏ^{n-1} \) if

\[ n + s_1 + \cdots + s_l - (k + \overline{I}) \geq n - 1. \]

Since \( \overline{k} \leq k \) and \( \overline{I} \leq l \), (36) would follow from

\[ n + s_1 + \cdots + s_l - (k + l) \geq n - 1, \]

which is the same as

\[ s_1 + \cdots + s_l - (k + l) + 1 \geq 0. \]

Notice that (31) is the sum over unshuffles \( \kappa \) whose associated graphs \( \Gamma \) are connected. The number of vertices \( V \) of such a graph clearly equals \( k + l \) while its number \( E \) of edges is \( s_1 + \cdots + s_l \). We therefore need to prove that

\[ E - V + 1 \geq 0. \]

Since \( \Gamma \) connected, \( V - E = 1 - b_1(\Gamma) \) by Euler’s theorem or, equivalently, \( E - V + 1 = b_1(\Gamma) \), where \( b_1(\Gamma) \) is the first Betti number of \( \Gamma \). As \( b_1(\Gamma) \geq 0 \), (37) immediately follows. □
7. The Quantum Master Equation

Definition 34. Let $V$ be an MV-algebra over $R$ with a maximal ideal $m$ as in Definition 1. The quantum master equation (QME) in $V$ is the equation

$$\Delta e^S = 0,$$

for a degree-0 element $S \in V \hat{\otimes} m$. We denote by $\text{QME}_V(R)$ the set of all solutions of the quantum master equation in $V$.

In (38), $e^S$ is the exponential in the graded commutative associative algebra $V \hat{\otimes} R$. The existence of the exponential is guaranteed by the completeness of $R$.

Example 35 (Münster-Sachs’ formulation). Let $A$ be an IBL$_\infty$-algebra as in Example 10. Münster and Sachs consider $S \in S(U)[[\hbar]]$ of the form

$$S = \sum_{k \geq 2} \hbar^{k-1} c_k$$

with some $c_k \in \bigoplus_{1 \leq n \leq k} S^n(U)$.

Remark 36. Münster and Sachs allowed $k = 1$ in (39), which would be inaccurate in our setup, unless there were reasons to guarantee convergence of $e^S$ – see the next example.

Example 37 (Formulation of [2]). Let $A$ be a BV$_\infty$-algebra as in Example 9. Consider the functional field

$$k := k((\hbar)) = k[[\hbar]][[\lambda]].$$

For an independent symbol $\lambda$, $k[[\hbar]][[\lambda]] = k[[\hbar, \lambda]]$ is a complete local ring with residue field $k$ and $k[[\lambda]] := k((\hbar))[[\lambda]]$ is a complete local ring with residue field $k$. Extending Remark 3, we may consider $A$ as an MV-algebra over $k[[\hbar, \lambda]]$ or over $k((\hbar))[[\lambda]]$.

The work [2] treated the quantum master equation for $S$ of the form $S = \tilde{S}/\hbar$, with some $\tilde{S} \in \lambda A[[\hbar]][[\lambda]]$ or $\tilde{S} \in \lambda A((\hbar))[[\lambda]]$. Here we work in our more general setup and consider $\tilde{S} \in A((\hbar)) \hat{\otimes} n$, where $n$ is the maximal ideal in a complete local ring $T$ with residue field $k$.

Theorem 38 ([3]). In the situation of Example 37, the equation $\Delta e^{\tilde{S}/\hbar} = 0$ for $\tilde{S} \in A((\hbar)) \hat{\otimes} n$ of degree zero is equivalent to

$$\Delta \tilde{S} + \frac{1}{2!} l_2(\tilde{S}, \tilde{S}) + \frac{1}{3!} l_3(\tilde{S}, \tilde{S}, \tilde{S}) + \cdots = 0,$$

where $l_n$ for each $n \geq 2$ is the higher derived bracket:

$$l_n(a_1, \ldots, a_n) := \frac{1}{\hbar^{n-1}} \Phi_n^\Delta(a_1, \ldots, a_n) = \sum_{k=1}^{\infty} \hbar^{k-1} \Phi_n^{\Delta_{k+n-1}}(a_1, \ldots, a_n),$$

with $\Phi_n^2$ being defined in (3).

[February 20, 2017] [ibl.tex]
Proof. Let \( \text{Ad} \) denote the adjoint action \( \text{Ad}_g Y := gYg^{-1} \) of the group \( \text{GL}(A) \) of invertible, degree-zero linear maps \( g : A \rightarrow A \) on the Lie algebra \( \mathfrak{gl}(A) := \text{Lin}_k(A, A) \) of all degree-zero linear maps \( Y : A \rightarrow A \). Let \( \text{ad} \) be the adjoint action \( \text{ad}_X Y := [X, Y] \) of \( \mathfrak{gl}(A) \) on itself. Then, for \( X \in \mathfrak{gl}(A) \otimes n \), we have

\[
\text{Ad}_e X = e^{\text{ad}_X}.
\]

Given a degree-zero element \( \tilde{S} \in A((\hbar)) \otimes n \), apply Equation (40) to the operator \( X = L(-\tilde{S}/\hbar) \) of left multiplication by \( -\tilde{S}/\hbar \). We get

\[
\Delta \circ e^{\tilde{S}/\hbar} = e^{\tilde{S}/\hbar} \sum_{n=0}^{\infty} \frac{\text{ad}_{(-\tilde{S}/\hbar)}^n}{n!} \Delta = e^{\tilde{S}/\hbar} \sum_{n=0}^{\infty} \left[ \left[ \cdots \left[ \Delta, L\tilde{S}/\hbar, \cdots \right], L\tilde{S}/\hbar \right], \cdots \right] \frac{1}{n!},
\]

where in the last formula there are exactly \( n \) iterated commutators with \( L\tilde{S}/\hbar \). Applying both sides of this equation to \( 1 \in A \) and using Equation (3), we obtain

\[
\Delta(e^{\tilde{S}/\hbar}) = e^{\tilde{S}/\hbar} \left( \Delta \tilde{S} + \frac{1}{2!} l_2(\tilde{S}, \tilde{S}) + \frac{1}{3!} l_3(\tilde{S}, \tilde{S}, \tilde{S}) + \cdots \right),
\]

whence the result. \( \square \)

**Theorem 39.** There is a natural one-to-one correspondence between solutions \( S \) of the QME (38) in \( V \) and \( MV \)-morphisms \( s \in \text{MV}_R(k, V) \), i.e. one has a natural isomorphism

\[
\text{QME}_V(R) \cong \text{MV}_R(k, V),
\]

given by \( s(1) = S \). In (41), the MV structure on \( k \) is defined by \( \delta(1) := 1 \otimes 1 \) and \( \Delta = 0 \), as in Example 4.

**Remark 40** (Geometric interpretation). Using the geometric interpretation of MV-morphisms in Remark 22, we can interpret the proposition as stating that a solution of the QME is a family \( X \) of MV-manifolds over \( B = \text{Spec} R \), or rather a family of MV-manifold structures on the trivial fiber bundle \( X = \text{Spec} V \hat{\otimes} B \). One can also view a solution as a \( B \)-point \( B \rightarrow X^* \) of the dual MV-manifold.

**Proof of Theorem 39.** By definition, \( \text{Lin}_R^0(k, V) \) consists of \( R \)-linear maps

\[
s : k \otimes R \cong R \rightarrow V \hat{\otimes} R
\]

such that \( s(1) \in V \hat{\otimes} m \). Under the canonical isomorphism

\[
\text{Lin}_R(k, V) \cong V \hat{\otimes} R, \ s \mapsto \text{Lin}_R(k, V) \mapsto s(1) \in V \hat{\otimes} R,
\]

maps with this property tautologically correspond to \( V \hat{\otimes} m \).

A map \( s \) in (12) is an MV-morphism if and only if \( \Delta \circ \exp(s) = 0 \). Since \( \Delta \circ \exp(s) \) is \( R \)-linear, this happens if and only if \( \Delta \exp(s)(1) = 0 \). As \( 1 \in k \) is group-like,

\[
\Delta \exp(s)(1) = \Delta e^{s(1)} = \Delta e^S
\]

by Example 13. \( \square \)
Let $V$ be an MV-algebra over $k[[\hbar]]$ and $T$ a complete local ring with the maximal ideal $n$ and residue field $k$. Extend the MV structure on $V$ linearly, as in Remark 3, to an MV structure on the same $V$ over $k[[\hbar]] \otimes T$ and $k((\hbar)) \otimes T$, regarded as complete local rings with residue fields $k$ and $K = k((\hbar))$, respectively.

In the BV formalism of theoretical physics [13] and applications to deformation theory [8, 10, 20], it is important to consider the QME for elements $S$ of the form $\tilde{S} = S/\hbar$ for $\tilde{S} \in V((\hbar)) \otimes n$ or $\tilde{S} \in V[[\hbar]] \otimes n$. We relate these types of solutions to MV-morphisms below.

Corollary 41. (i) There is a natural one-to-one correspondence between solutions $\tilde{S} \in V((\hbar)) \otimes n$ of the quantum master equation

$$
\Delta e^{\tilde{S}/\hbar} = 0
$$
and MV-morphisms $s \in \text{MV}_{K \otimes T}(K, V)$, i.e. one has a natural isomorphism

$$
\{ \tilde{S} \in V((\hbar)) \otimes n \mid \Delta e^{\tilde{S}/\hbar} = 0 \} \sim \text{MV}_{K \otimes T}(K, V((\hbar))),
$$
given by $s(1) = \tilde{S}/\hbar$.

(ii) There is a natural one-to-one correspondence between solutions $\tilde{S} \in V[[\hbar]] \otimes n$ of the quantum master equation

$$
\Delta e^{\tilde{S}/\hbar} = 0
$$
and a certain subset of MV-morphisms $s \in \text{MV}_{K \otimes T}(K, V((\hbar)))$, namely one has a natural isomorphism

$$
\{ \tilde{S} \in V[[\hbar]] \otimes n \mid \Delta e^{\tilde{S}/\hbar} = 0 \} \sim \text{MV}^*_s_{K \otimes T}(K, V((\hbar))),
$$
given by $s(1) = \tilde{S}/\hbar$, where $\text{MV}^*_s_{K \otimes T}(K, V((\hbar)))$ denotes the set of MV-morphisms $s$ whose value $s(1)$ at $1 \in K \otimes T$ has at most a simple pole at $\hbar = 0$.

Remark 42 (Geometric interpretation). Now, as we have two local algebras, geometric interpretation becomes subtler. Let $D$ denote $\text{Spec } k[[\hbar]]$, often called the formal disk, $\hat{D} = \text{Spec } K$ be the deleted formal disk, and $C = \text{Spec } T$. An MV-algebra over $k[[\hbar]]$ has been interpreted as an MV-manifold over $D$, which may be restricted to an MV-manifold over $\hat{D}$. Let us denote it $X_0 \to \hat{D}$. Then the isomorphism (44) interprets a solution of the modified QME (43) as a family of MV-manifolds extending the family $X_0 \to \hat{D}$ to the base $\hat{D} \times C$:

$$
\begin{array}{ccc}
X_0 & \to & X \\
\downarrow & & \downarrow \\
\hat{D} & \to & \hat{D} \times C.
\end{array}
$$

In the dual interpretation, the isomorphism (44) asserts that the dual MV-manifold $X_0^* \to \hat{D}$ is an MV-manifold representing the functor solutions of QME (43) or the “moduli MV-space”.

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of solutions of the QME. Indeed, a solution of (43) may be interpreted as an MV-morphism
\[
\tilde{D} \times C \to X_0^* \to \tilde{D}.
\]

Theorem 39 treats solutions of the QME (38) as morphisms in the category of MV-algebras over \(R\). Thus, composition of MV-morphisms can be used to transfer these solutions over MV-morphisms. More specifically, let \(f \in \text{MV}_R(V', V'')\) be a morphism and \(S \in V' \hat{\otimes} m\) a solution of the QME in \(V'\). Theorem 39 translates \(S\) into a morphism \(s \in \text{MV}_R(k, V')\). Then \(f \circ s\) will be a morphism in \(\text{MV}_R(k, V'')\) therefore, by Theorem 39 again, \((f \circ s)(1)\) will solve the QME in \(V''\). Since \(1 \in k\) is group-like, one moreover has

\[(f \circ s)(1) = \log \left( \exp(f) \circ \exp(s) \right)(1) = \log \left( \exp(f)(e^{s(1)}) \right) = \log \left( \exp(f)(e^{S}) \right),\]

the ‘actual’ logarithm of \(\exp(f)(e^{S}) \in V'' \hat{\otimes} m\).

**Definition 43.** The element

\[f_i(S) := \log \left( \exp(f)(e^{S}) \right) \in \text{QME}_{V''}(R)\]

is the push-forward of the solution \(S \in \text{QME}_{V'}(R)\).

**Theorem 44** (Cf. [2] for the case \(V' = S(U')\)). Assume that \(V'\) is a bialgebra, i.e. the standard compatibility between the multiplication and the comultiplication is fulfilled. Suppose moreover that \(S \in P(V') \hat{\otimes} m\), where \(P(V')\) is the subspace of primitive elements in \(V'\). Then

\[f_i(S) = f(e^{S}).\]

**Proof.** Observe that \(e^{S}\) is a group-like element in \(V' \hat{\otimes} R\). Indeed,

\[\delta'(e^{S}) = \exp(S \otimes 1 + 1 \otimes R^1) = e^{S \otimes 1} e^{1 \otimes R^1} = e^{S} \otimes R e^{S},\]

where we used the fact that \(\delta'\) is an algebra morphism and that \(S \otimes R 1\) commutes with \(1 \otimes R S\) in \((V' \hat{\otimes} R) \otimes_R (V' \hat{\otimes} R)\). The rest follows from the observations in Example 13. \qed

**References**

[1] D. Bessis, C. Itzykson, and J. B. Zuber. Quantum field theory techniques in graphical enumeration. *Adv. in Appl. Math.*, 1(2):109–157, 1980.

[2] D. Bashkirov and A. A. Voronov. The BV formalism for \(L_{\infty}\)-algebras. Preprint IHES/M/14/36, *arXiv:1410.6432 [math.QA]*, to appear in *J. Homotopy Relat. Struct.*, 2017.

[3] Ch. Braun and A. Lazarev. Homotopy BV algebras in Poisson geometry. *Trans. Moscow Math. Soc.*, pages 217–227, 2013.

[4] R. Campos, S. Merkulov, and T. Willwacher. The Frobenius properad is Koszul. *Duke Math. J.*, 156(4):2921–2989.

[5] K. Cieliebak, K. Fukaya, and J. Latschev. Homological algebra related to surfaces with boundaries. Preprint *arXiv:1508.02741 [math.QA]*.

[6] K. Cieliebak and J. Latschev. The role of string topology in symplectic field theory. In *New perspectives and challenges in symplectic field theory*, volume 49 of *CRM Proc. Lecture Notes*, pages 113–146. Amer. Math. Soc., Providence, RI, 2009.
[7] G. C. Drummond-Cole, J. Terilla, and T. Tradler. Algebras over $\Omega(\text{coFrob})$. *J. Homotopy Relat. Struct.*, 5(1):15–36, 2010.

[8] D. Iacono. Deformations and obstructions of pairs $(X,D)$. Preprint arXiv:1302.1149 [math.AG].

[9] Ch. Kassel. *Quantum groups*, volume 155 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.

[10] L. Katzarkov, M. Kontsevich, and T. Pantev. Hodge theoretic aspects of mirror symmetry. In *From Hodge theory to integrability and TQFT tt*-geometry*, volume 78 of *Proc. Sympos. Pure Math.*, pages 87–174. Amer. Math. Soc., Providence, RI, 2008.

[11] O. Kravchenko. Deformations of Batalin-Vilkovisky algebras. In *Poisson geometry (Warsaw, 1998)*, volume 51 of *Banach Center Publ.*, pages 131–139. Polish Acad. Sci., Warsaw, 2000.

[12] T. Lada and M. Markl. Strongly homotopy Lie algebras. *Comm. Algebra*, 23(6):2147–2161, 1995.

[13] A. Losev. From Berezin integral to Batalin-Vilkovisky formalism: a mathematical physicist’s point of view. In *Felix Berezin. Life and death of the mastermind of supermathematics*, pages 3–30. Edited by M. Shifman. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007.

[14] M. Markl. *Deformation theory of algebras and their diagrams*, volume 116 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2012.

[15] M. Markl. On the origin of higher braces and higher-order derivations. *J. Homotopy Relat. Struct.*, 10(3):637–667, 2015.

[16] M. Markl and E. Remm. Algebras with one operation including Poisson and other Lie-admissible algebras. *J. Algebra*, 299:171–189, 2006.

[17] K. Münster and I. Sachs. Quantum open-closed homotopy algebra and string field theory. *Comm. Math. Phys.*, 321(3):769–801, 2013.

[18] M. Markl, S. Shnider, and J.D. Stasheff. *Operads in algebra, topology and physics*, volume 96 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.

[19] J. Terilla. Quantizing deformation theory. In *Deformation spaces*, volume E40 of *Aspects Math.*, pages 135–141. Vieweg + Teubner, Wiesbaden, 2010.

[20] J. Terilla. Smoothness theorem for differential BV algebras. *J. Topol.*, 1(3):693–702, 2008.

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