From configuration to dynamics
Emergence of Lorentz signature in classical field theory

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The Lorentzian metric structure used in any field theory allows one to implement the relativistic notion of causality and to define a notion of time dimension. This article investigates the possibility that at the microscopic level the metric is Riemannian, i.e. locally Euclidean, and that the Lorentzian structure, that we usually consider as fundamental, is in fact an effective property that emerges in some regions of a 4-dimensional space with a positive definite metric. In such a model, there is no dynamics nor signature flip across some hypersurface: instead, all the fields develop a Lorentzian dynamics in these regions because they propagate in an effective metric. It is shown that one can construct a decent classical field theory for scalars, vectors and (Dirac) spinors in flat spacetime. It is then shown that gravity can be included but that the theory for the effective Lorentzian metric is not general relativity but of the covariant Galileon type. The constraints arising from stability, the equivalence principle and the constancy of fundamental constants are detailed and a phenomenological picture of the emergence of the Lorentzian metric is also given. The construction, while restricted to classical fields in this article, offers a new view on the notion of time.

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I. INTRODUCTION

When constructing a physical theory, there is a large freedom in the choice of the mathematical structures. The developments of theoretical physics taught us that some of these structures are well-suited to describe some classes of phenomena (e.g. the use of a vector field for electromagnetism, of spinors for some class of particles, the use of some symmetries, etc.). However, these choices can only be validated by the mathematical consistency of the theory and the agreement between the consequences of these structures and experiments. It may even be that different structures are possible to reproduce what we know about physics and one may choose one over the other on the basis of less well-defined criteria such as simplicity and economy.

At each step, some properties such as the topology of space [1], the number of spatial dimensions or the numerical values of the free parameters that are the fundamental constants [2], may remain a priori free in a given framework, or imposed in another framework (e.g. the number of space dimensions is fixed in string theory [3]).

Among all these structures, and in the framework of metric theories of gravitation, the signature of the metric is in principle arbitrary. Indeed, it seems that on the scales that have been probed so far there is the need for only one time dimension and three spatial dimensions. In special and general relativity, time and space are geometrically different because the geometry of spacetime is locally Minkowskian, i.e. it enjoys a Lorentzian metric with signature ($-, +, +, +$), i.e. the line element is $ds^2 = -dt^2 + d\ell^2 \equiv \eta_{\mu\nu}dx^\mu dx^\nu$. While the existence of two time directions may lead to confusion [4], it is not clear if there are theoretical obstacles to have more than one time directions, as even suggested by some framework [5] (see the argument for such possibilities made by Ref. [6] and detailed further in Ref. [7]). Several models for the birth of the universe [8] are based on a change of signature via an instanton in which a Riemannian and a Lorentzian manifolds are joined across a hypersurface which may be thought of as the origin of time. While there is no time in the Euclidean region, where the signature is ($+, +, +, +$), it flips to ($-, +, +, +$). Eddington even suggested [9] that it can flip across some surface to ($-, -, +, +$). Signature flip also arises in brane-world scenarios [10] (see Ref. [7] for a review of these possibilities) or in loop quantum cosmology [11]. These discussions however let the problem of the origin of the time direction open [12].

In Newtonian theory, time is a fundamental concept. It is assumed to flow and is described by a real variable. It
can be measured by good clocks and any observers shall, irrespective of their motion, agree on the time elapsed between two events [13]. The laws of dynamics describe the change of configurations of a system with time. In relativity, first the notions of space and time are set on the same footing and second, the notion of time is no more unique. One has to distinguish between a coordinate time, with no physical meaning, and the proper time that can be measured by an observer. Quantum mechanics offers another insight on time: there, while there may be operators or observables corresponding to spatial positions, time is not an observable, and thus not an operator [10]. As detailed in Ref. [7], by an argument going back to Pauli, commutation relations like $[\hat{\mathbf{x}}^\mu, \hat{\mathbf{P}}_\nu] = i\delta^\mu_\nu$ are incompatible with the spectrum of $\hat{\mathbf{P}}^\mu$ lying in the future lightcone and the notion of time is intimately related to the complex (Hilbert Space) structure of quantum mechanics [7].

The question of whether time does actually “exist” has been widely debated in the context of classical physics [14], relativity [15] and quantum mechanics [16]. The debate on the nature of time has shifted with quantum gravity where the recovery of a classical notion of time is considered as a problem. In that case, the Schrödinger equation becomes the Wheeler-de Witt equation, of the form $\frac{d}{d\tau}\hat{\Psi} = 0$, so that the allowed states are those for which the Hamiltonian vanishes. Thus, it determines in which states the universe can be but does not give any evolution through time. We refer to Refs. [17–19] for general discussions on the nature of time. This has led to numerous works on the emergence of time in different versions of quantum gravity [20–25] (and indeed the reverse opinion has been argued [26]). Also, the thermodynamical aspects of gravity, the existence of dualities between gauge theories and gravity theories [27], and holography [28] have led to the idea that the metric itself may have to be thought of as the result of a coarse-graining of underlying more fundamental degrees of freedom [29].

The local Minkowski structure is an efficient way to implement the notion of causality in realistic theories and is today accepted as a central ingredient of the construction of the relativistic theory of fields. When gravity is included, the equivalence principle implies (this is not a theoretical requirement, but just an experimental fact, required at a given accuracy) that all the fields are universally coupled to the same Lorentzian metric. From the previous discussion, we may wonder whether the nature of this metric is only a convenient way to implement causality or whether it is just a property of an effective description of a microscopic theory in which there is no such notion.

This article proposes the view according to which the fundamental physical theory is intrinsically purely Euclidean so that its field equations determine a static 4-dimensional field configuration. The Lorentzian dynamics that we can observe in our universe has then to be thought of as an emergent property, that is as an illusion holding in a small patch of a Euclidean mathematical space. This is thus an attempt to go further than early proposals [30–32] and see to which extent this can be an open possibility. We emphasize that it is different from the models discussed above involving a signature change across a boundary or obtained by rotating to an Euclidean space. We consider it important to take the freedom to see how far one can go in such a direction. As we shall later discuss, if possible, such a setting may shed a new light on several theoretical issues from the nature of singularities to quantum gravity.

Our attitude is however more modest and we want to start by constructing a decent classical field theory under this hypothesis. Section II explains the basics of our mechanism and then describes the construction of the scalar, vector and spinor sectors in flat spacetime. We show that the whole standard model of particle physics can be constructed from a Euclidean theory, at the classical level. Section III addresses the more difficult question of gravity. While general relativity is not recovered in general, it shows that an extended $K$-essence theory of gravity called covariant Galileon can be obtained. We then show in Section IV that the dynamics of scalar and vector in curved spacetime can also be obtained. We then discuss the experimental and theoretical constraint on our construction in § V and also propose a way to understand phenomenologically the emergence of the effective Lorentzian dynamics. It is however to be remembered that there is no dynamics at the fundamental level and that this illusion is restricted to a domain of a large Euclidean space.

II. FIELD THEORY IN FLAT SPACE

This section introduces the mechanism in the simple case of a flat space (§ II A). It shows how scalars (§ II B), vectors (§ II C and § II D) and spinors (§ II E) defined in Euclidean space can have an apparent Lorentzian dynamics. We finish by pointing out the properties and limits of this mechanism in § II F many of them being discussed in a more realistic version in the following sections.

A. Clock field

In order to understand the basics of our model, let us consider a 4-dimensional Riemannian manifold $\mathcal{M}$ with a positive definite Euclidean metric $g^{\mu\nu}_E = \delta_{\mu\nu}$ in a Cartesian coordinate system. As a consequence, the theory we shall consider on this manifold does not have a natural concept of time. In order to make such a notion emerge locally, we introduce a scalar field $\phi$ and assume that its derivative has a non-vanishing vacuum expectation value (vev) in a region $\mathcal{M}_0$ of the Riemannian space (see Fig. 1). To be more precise, we assume that
\[ \partial_{\mu}\phi = \text{const.} \neq 0 \text{ in } \mathcal{M}_0. \] It follows that we can set
\[ \partial_{\mu}\phi = M^2 n_{\mu} \text{ in } \mathcal{M}_0 \quad (2.1) \]
with \( n_{\mu} \) a unit constant vector \((\delta^{\mu\nu} n_{\mu} n_{\nu} = 1)\). We have introduced a mass scale \( M \) so that \( n_{\mu} \) is dimensionless. By construction, its norm \( X_E = \delta^{\mu\nu} \partial_{\mu}\phi \partial_{\nu}\phi = M^4 \) is constant and satisfies
\[ X_E > 0 \text{ in } \mathcal{M}_0. \quad (2.2) \]
Now, under this assumption, one of the coordinates can be chosen as
\[ dt = n_{\mu} dx^\mu. \quad (2.3) \]
This accounts for choosing
\[ t \equiv \frac{\phi}{M^2} \quad (2.4) \]
up to a constant that can be set to zero without loss of generality. The metric of the 4-dimensional Riemannian space (with Euclidean geometry) can be rewritten as
\[ ds_E^2 = \delta_{\mu\nu} dx^\mu dx^\nu = (n_{\mu} dx^\mu)^2 + (\delta_{\mu\nu} - n_{\mu} n_{\nu}) dx^\mu dx^\nu = dt^2 + \delta_{ij} dx^i dx^j, \quad (2.5) \]
by introducing a set of three independent coordinates \( x^i \) \((i = 1 \ldots 3)\) on the hypersurfaces \( \Sigma_t \) normal to \( n^\mu \). Note that the geometry on \( \Sigma_t \) would not be Euclidean if \( n_{\mu} \) were not constant. As we shall now discuss, the scalar field \( \phi \) will be related to what we usually call “time”, so that we shall call such a scalar field a clock field.

**FIG. 1:** Example of a spatial configuration of the clock field. Locally, one can define regions such as \( \mathcal{M}_0, \mathcal{M}_0' \) and \( \mathcal{M}_0'' \), in each of which a time direction emerges. Indeed this direction does not preexist at the microscopic level and can be different from patches to patches.

### B. Scalar field

The Euclidean configuration of a scalar field \( \chi \) can be obtained by combining the usual action for a scalar field, with a kinetic term and a potential,
\[
- \int d^4x \left[ \frac{1}{2} \delta^{\mu\nu} \partial_{\mu}\chi \partial_{\nu}\chi + V(\chi) \right],
\]
with a coupling to the clock field \( \phi \) as
\[
\int d^4x \left( \delta^{\mu\nu} \partial_{\mu}\phi \partial_{\nu}\chi \right)^2. \]
Let us consider the action obtained by the following combination
\[
S_\chi = \int d^4x \left[ -\frac{1}{2} \delta^{\mu\nu} \partial_{\mu}\chi \partial_{\nu}\chi - V(\chi) \right] + \frac{1}{M^4} \left( \delta^{\mu\nu} \partial_{\mu}\phi \partial_{\nu}\chi \right)^2. \quad (2.6)
\]
It is straightforward to conclude that since \( \delta^{\mu\nu} \partial_{\mu}\chi \partial_{\nu}\chi = (\partial_t \chi)^2 + \delta^{ij} \partial_i \chi \partial_j \chi \) and, when restricted to \( \mathcal{M}_0 \), \( (\delta^{\mu\nu} \partial_{\mu}\phi \partial_{\nu}\chi)^2 = M^4 (\partial_t \chi)^2 \), the action (2.6) reduces to
\[
S_\chi = \int dt d^3x \left[ -\frac{1}{2} (\partial_t \chi)^2 - \frac{1}{2} \delta^{ij} \partial_i \chi \partial_j \chi - V \right]. \quad (2.7)
\]
in \( \mathcal{M}_0 \). This can indeed be rewritten as
\[
S_\chi = \int dt d^3x \left[ -\frac{1}{2} \delta^{\mu\nu} \partial_{\mu}\chi \partial_{\nu}\chi - V \right]. \quad (2.8)
\]
The action (2.6) thus describes, when restricted to \( \mathcal{M}_0 \), the dynamics of a scalar field propagating in a 4-dimensional Minkowski spacetime with metric \( \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \). The apparent Lorentzian dynamics, with a preferred time direction, is thus the result of the coupling to the scalar clock field.

### C. Vector field

Usually, the dynamics of a vector field \( A_{\mu} \) is dictated by the action \( F_{\mu\nu} F_{\nu}^{\mu} \) where \( F_{\mu\nu} \) is the Faraday tensor defined as \( F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \) and where the subscript \( E \) indicates that the indices are raised with the Euclidean metric \( \delta^{\mu\nu} \).

The standard action of the vector field can be extended to include a coupling to the clock field of the form \( F_{\mu\nu} F_{\nu}^{\mu} \partial_{\mu}\phi \partial_{\nu}\phi \) so that the action for the vector field we consider is
\[
S_A = \frac{1}{4} \int d^4x \left[ -F_{\mu\nu} F_{\nu}^{\mu} + \frac{4}{M^4} F_{\mu\nu} F_{\nu}^{\mu} \partial_{\mu}\phi \partial_{\nu}\phi \right]. \quad (2.9)
\]
Since \( F_{\mu\nu} F_{\nu}^{\mu} = 2 \delta^{ij} F_{\mu i} F_{\mu j} + \delta^{ij} \delta^{kl} F_{ij} F_{kl} \) and since \( F_{\mu\nu} F_{\nu}^{\mu} \partial_{\mu}\phi \partial_{\nu}\phi = M^4 \delta^{ij} F_{0i} F_{0j} \) in \( \mathcal{M}_0 \), it is easily concluded that this action can be rewritten as
\[
S_A = \frac{1}{4} \int dt d^3x \left[ 2 \delta^{ij} F_{0i} F_{0j} - \delta^{ik} \delta^{jl} F_{ij} F_{kl} \right]. \quad (2.10)
\]
or more simply as

\[ S_A = -\frac{1}{4} \int dt d^3x \eta^{\mu \alpha} \eta^{\nu \beta} F_{\mu \alpha} F_{\nu \beta}. \quad (2.11) \]

Because of the coupling of the Faraday tensor to the clock field in the Euclidean theory, the vector field propagates effectively in a Minkowski metric and we recover the standard Maxwell action for a vector field. The generalization to an non-Abelian group is straightforward.

D. Charged scalar field

The construction of II E can easily be generalized to a complex scalar field charged under a \( U(1) \). Considering a complex scalar field \( \omega \), we add to the standard kinetic term \( \delta^{\mu \nu} \langle D_{\mu} \omega \rangle^*(D_{\nu} \omega) \) a coupling to the clock field of the form \( \delta^{\mu \nu} \partial_{\mu} \phi \langle D_{\nu} \omega \rangle^2 \), where \( D_{\mu} \equiv \partial_{\mu} - iq A_{\mu} \). The Euclidean action is then chosen to be

\[ S_\omega = \int d^4x \left[ -\frac{1}{2} \delta^{\mu \nu} \langle D_{\mu} \omega \rangle^* \langle D_{\nu} \omega \rangle - U(|\omega|^2) \right. \]

\[ + \left. \frac{1}{M^4} \delta^{\mu \nu} |\partial_{\mu} \phi |D_{\nu} \omega|^2 \right]. \quad (2.12) \]

Following the same arguments as for the real scalar field \( \chi \), this action takes the form

\[ S_\omega = \int dt d^3x \left[ -\frac{1}{2} \eta^{\mu \nu} \langle D_{\mu} \omega \rangle^* \langle D_{\nu} \omega \rangle - U \right]. \quad (2.13) \]

Again, the coupling to the clock field implies that the Euclidean dynamics leads to an effectively Minkowskian dynamics for \( \omega \).

E. Spinor fields

The next step is to include fermions in such a way that the standard Dirac dynamics emerges from an Euclidean action. Let us start by comparing the standard Dirac algebra in Minkowski spacetime (II E 1) and that in Euclidean space (II E 2) before we propose a choice of Euclidean action for the fermions (II E 3).

1. Dirac matrices in Minkowski spacetime

In a Minkowski spacetime with signature \((- + + +)\), Dirac matrices are \( 4 \times 4 \) matrices satisfying the anti-commutation relation

\[ \{ \gamma^\mu, \gamma^\nu \} = -2 \eta^{\mu \nu}. \quad (2.14) \]

For concreteness, throughout this section we shall adopt the following form of the Dirac matrices in Minkowski spacetime

\[ \gamma^0 = \boldsymbol{\sigma}_0 \otimes \sigma_1 = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}, \]

\[ \gamma^i = i \sigma_i \otimes \sigma_2 = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad (2.15) \]

where \( \sigma_0 \) is the \( 2 \times 2 \) unit matrix and \( \sigma_i \) \((i = 1, 2, 3)\) are Pauli matrices,

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.16) \]

While \( \gamma^0 \) is Hermitian, \( \gamma^i \) are anti-Hermitian. One then defines \( \gamma^5 \) by

\[ \gamma^5 = -i \gamma^0 \gamma^1 \gamma^2 \gamma^3 = \sigma_0 \otimes \sigma_3 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \quad (2.17) \]

which satisfies

\[ (\gamma^5)^2 = 1, \quad \{ \gamma^5, \gamma^\mu \} = 0 \quad (\mu = 0, \cdots, 3). \quad (2.18) \]

The matrices

\[ S^{\mu \nu} = \frac{i}{4} \left\{ \gamma^\mu, \gamma^\nu \right\} \quad (2.19) \]

satisfy the algebra of Lorentz generators

\[ [S^{\mu \nu}, S^{\rho \sigma}] = i(\eta^{\nu \rho} S^{\sigma \mu} - \eta^{\mu \rho} S^{\sigma \nu} - \eta^{\nu \sigma} S^{\sigma \mu} + \eta^{\mu \sigma} S^{\sigma \nu}) \quad (2.20) \]

Hence, the Lorentz transformation for a Dirac field \( \psi \) is

\[ \psi \rightarrow \Lambda_{\frac{1}{2}} \psi, \quad \Lambda_{\frac{1}{2}} = \exp \left[ -\frac{i}{2} \omega_{\mu \nu} S^{\mu \nu} \right], \quad (2.21) \]

where \( \omega_{\mu \nu} \) are real numbers. Concretely,

\[ S^{0i} = -\frac{i}{2} \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}, \]

\[ S^{ij} = \frac{1}{2} \sum_{k=1}^{3} \epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}. \quad (2.22) \]

While \( S^{ij} \) are Hermitian, \( S^{0i} \) are anti-Hermitian. As a consequence, \( \Lambda_{\frac{1}{2}} \) is not unitary in general. In particular this means that

\[ \psi^\dagger \rightarrow \psi^\dagger \Lambda_{\frac{1}{2}}^\dagger \neq \psi^\dagger \Lambda_{\frac{1}{2}}^{-1} \quad (2.23) \]

and that \( \psi^\dagger \psi \) is not a scalar under Lorentz transformation. However, it is easy to check that

\[ \tilde{\psi} \rightarrow \tilde{\psi} \Lambda_{\frac{1}{2}}^{-1}, \quad \tilde{\psi} \equiv \psi^\dagger \gamma^0 \quad (2.24) \]

so that \( \tilde{\psi} \psi \) is a scalar under Lorentz transformations. This is the reason why the Dirac action in Minkowski spacetime is usually constructed as

\[ S^M_\psi = \int dt d^3x \left( i \gamma^\mu \partial_{\mu} m - m \right) \psi. \quad (2.25) \]
2. $\gamma$ matrices in Euclidean space

In a 4-dimensional Euclidean space with metric $\delta_{\mu\nu}$, one can also define matrices $\gamma^\mu_E$ according to

$$\gamma^0_E = i\gamma^5, \quad \gamma^i_E = \gamma^i$$

so that they obey the anti-commutation relation

$$\{\gamma^\mu_E, \gamma^\nu_E\} = -2\delta^\mu\nu.$$  \hspace{1cm} (2.26)

Then, we can define

$$\gamma_5^E \equiv \gamma^0_E \gamma^1_E \gamma^2_E \gamma^3_E = \gamma^0,$$  \hspace{1cm} (2.27)

which satisfies

$$(\gamma_5^E)^2 = 1, \quad \{\gamma_5^E, \gamma^\mu_E\} = 0 \quad (\mu = 0, \ldots, 3).$$  \hspace{1cm} (2.29)

It follows that the matrices

$$S^\mu_\nu^E \equiv \frac{i}{4} [\gamma^\mu_E, \gamma^\nu_E]$$

satisfy the algebra of $SO(4)$ rotation generators

$$[S^\mu_\nu^E, S^{\rho\sigma}_E] = i(\delta^{\mu\rho} S^{\nu\sigma}_E - \delta^{\nu\rho} S^{\mu\sigma}_E - \delta^{\mu\sigma} S^{\nu\rho}_E + \delta^{\nu\sigma} S^{\mu\rho}_E).$$  \hspace{1cm} (2.30)

Hence, the $SO(4)$ rotation for the Dirac field $\psi$ is

$$\psi \rightarrow \Lambda_{E, 1/2} \psi, \quad \Lambda_{E, 1/2} = \exp \left[-\frac{i}{2} \omega^E_{\mu\nu} S^{\mu\nu}_E \right],$$  \hspace{1cm} (2.32)

where $\omega^E_{\mu\nu}$ are real numbers. Since all $S^{\mu\nu}_E$ are Hermitian, $\Lambda_{E, 1/2}$ is unitary. In particular, this implies that

$$\bar{\psi} \rightarrow \bar{\psi} \Lambda_{E, 1/2}^{-1}, \quad \psi^\dagger \rightarrow \psi^\dagger \Lambda_{E, 1/2}^{-1},$$  \hspace{1cm} (2.33)

and that both $\bar{\psi}\gamma^5\gamma^i\psi (= \psi^\dagger\psi)$ are scalars under a $SO(4)$ transformation (see e.g. Refs. \[33, 34\]). Note also that $\psi = \psi^\dagger \gamma^0$ can be written as

$$\bar{\psi} = \psi^\dagger \gamma^5_E.$$  \hspace{1cm} (2.34)

3. Euclidean action and emergence of the Lorentzian Dirac action

As in the previous sections, we will need to couple the spinor field $\psi$ to the clock field $\phi$ in order for the spinor to have an apparent Lorentzian dynamics. Starting from the Euclidean Dirac action in flat space with the metric $\delta_{\mu\nu}$,

$$\int dx^4 \bar{\psi} \left(\frac{i}{2} \gamma_5^E \not\!\! \partial_\mu \gamma^\mu_E - m\right) \psi,$$

and assuming that the clock field $\phi$ has derivative couplings to the Euclidean Dirac field $\psi$ of the form

$$\int dx^4 \delta_{\mu\nu} \left(\partial^\phi \gamma_5^E \gamma^\mu_E \not\!\! \partial_\mu \psi\right) \partial_\nu \phi, \quad \int dx^4 \delta_{\mu\nu} \left(\partial^\phi \gamma_5^E \gamma^\mu_E \not\!\! \partial_\mu \psi\right) \partial_\mu \phi \partial_\nu \phi,$$

we can consider an Euclidean action for the Dirac spinor of the form

$$S_\psi = \int dx^4 \left\{ \bar{\psi} \left(\frac{i}{2} \gamma_5^E \not\!\! \partial_\mu \not\!\! \psi + \frac{i}{2} \gamma_5^E \not\!\! \partial_\mu \gamma^\mu_E \not\!\! \psi \right) \not\!\! \mu - m \psi \right\} \right\}.

\hspace{1cm} (2.35)

As in the previous sections, the action $S_\psi$ reduces to

$$S_\psi = \int dx^4 \bar{\psi} \left[ \frac{i}{2} \gamma^0 \not\!\! \partial_0 + \frac{i}{2} \gamma^5 \not\!\! \partial_0 - m \right] \psi.$$  \hspace{1cm} (2.36)

The coupling to the clock field implies that $\psi$ effectively propagates in an effective Lorentzian metric and we recover the standard Minkowskian Dirac action \[2.25\] with the usual algebra \[2.14\] for the $\gamma$-matrices.

F. Massive point particle

The dynamics of massive object is usually derived from an action defined from the length of their worldline. In order to recover a proper dynamics, we start from the Euclidean action for a point particle

$$\int \left( \frac{1}{2} N^{-1} \delta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - \frac{1}{2} m^2 \right) d\tau$$

to which we add the coupling to the clock field of the form

$$\int N^{-1} \partial_\mu \phi \partial_\nu \phi \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} d\tau.$$

The Euclidean action for a point particle is thus given by

$$S_{pp} = \int \left[ \frac{1}{2} N^{-1} \left( \delta_{\mu\nu} - \frac{2}{M^2} \partial_\mu \phi \partial_\nu \phi \right) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} d\tau - \frac{1}{2} N m^2 \right] d\tau.$$  \hspace{1cm} (2.37)

The equation of motion is thus simply given by the geodesic equation for the effective metric $\tilde{g}^{(m)}_{\mu\nu} = \delta_{\mu\nu} - \frac{2}{M^2} \partial_\mu \phi \partial_\nu \phi$. It is obvious that in $M_0$ this effective metric reduces to the Minkowski metric $\eta_{\mu\nu}$.

G. Discussion

This section has provided the general construction of a mechanism that allows for scalar, vector, and spinor fields to actually propagate in an effective Lorentzian metric even though the underlying theory is purely Euclidean and written in terms of the Euclidean metric $\delta_{\mu\nu}$. This general construction assumes the existence of a scalar field $\phi$, called clock field, that couples to all fields (scalar, vector and spinor fields). In particular, this implies that we can construct the whole standard model of particle physics.

Let us now discuss some properties and limitations of such a construction.
1. It requires that the clock field satisfies \( \partial_\mu \phi = \text{const.} \neq 0 \) in a region \( \mathcal{M}_0 \) of the Euclidean space. It follows that the effective Lorentzian description is local and holds in \( \mathcal{M}_0 \). The properties of this model when \( \partial_\mu \phi \) is not constant will be discussed in \( \S \) below. As we shall see in the next section the clock field should enjoy a shift symmetry in order for the system to exhibit the time translation symmetry after the emergence of time. In \( \mathcal{M}_0 \), both the shift symmetry and the translational symmetry along the direction of \( \partial_\mu \phi \) are spontaneously broken, but a combination of them remains unbroken and is responsible for the existence of a conserved quantity that reduces in \( \mathcal{M}_0 \) to the usual notion of energy.

2. It is limited to classical field theory in flat space. The extension to curved space is discussed in \( \S \) below and the quantum aspects are left for future investigations.

3. The origin of the effective Lorentzian dynamics in \( \mathcal{M}_0 \) can be intuitively understood for scalars and vectors. For scalars, the action (2.6) is equivalent to the coupling to the effective metric

\[
\hat{g}^{\mu \nu} = g^{\mu \nu} - \frac{2}{M^4} \delta^{\alpha \nu} \delta^{\beta \mu} \partial_\alpha \phi \partial_\beta \phi. \tag{2.38}
\]

For vectors, one could have simply used a coupling to \( \hat{g}^{\mu \nu} \) and a Lagrangian of the form \( \hat{g}^{\mu \nu} \delta^{\alpha \beta} F_{\mu \nu} F_{\alpha \beta} \) since the extra term quartic in \( \partial_\mu \phi \) compared to the action (2.9) is of the form \( 4 M^{-8} \delta^{\alpha \lambda} \delta^{\beta \nu} \delta^{\alpha \gamma} \delta^{\beta \nu} \partial_\lambda \phi \partial_\gamma \phi \partial_\nu \phi \partial_\nu \phi F_{\mu \nu} F_{\alpha \beta} \) and does not contribute (note that it reduces to \( 4 F_{00}^2 = 0 \) in \( \mathcal{M}_0 \)). Hence, the apparent Lorentzian dynamics for scalars and vectors boils down to the fact that \( \hat{g}^{\mu \nu} |_{\mathcal{M}_0} = g^{\mu \nu} \). Massive point particles also propagate in this metric.

4. This interpretation cannot be extended to spinors mostly because of the \( \gamma \)-matrices, at least straightforwardly.

5. It is however important to realize that despite this, when restricted to \( \mathcal{M}_0 \) all fields propagate in the same effective Minkowski metric so that the equivalence principle is safe in first approximation.

6. The couplings to the clock field have been tuned in order to recover the exact Minkowski actions. For instance, the action (2.6) for a scalar field could have been chosen as

\[
S_\chi = \int d^4 x \left[ -\frac{\alpha_\chi}{2} \delta^{\mu \nu} \partial_\mu \chi \partial_\nu \chi - V(\chi) + \frac{\chi_0}{2 M^4} (\delta^{\mu \nu} \partial_\mu \phi \partial_\nu \chi)^2 \right]. \tag{2.39}
\]

In such a case, a Lorentzian signature is recovered only if \( \alpha_\chi > \kappa_\chi > 0 \). In the case where these constants are not tuned, different fields can have different lightcones. This will be discussed in Section IV.

7. In the bosonic sector, since the theory is invariant under the Euclidean parity \( (x^\mu \rightarrow -x^\mu) \) as well as the field parity \( (\phi \rightarrow -\phi) \), both \( P \) and \( T \) invariances in the emergent Lorentzian theory are ensured. Without the field parity invariance, the \( T \) invariance would be spontaneously broken by a non-vanishing vacuum expectation value (vev) of the derivative of the clock field. This explains the reason why we have included only quadratic terms in \( \partial_\mu \phi \) in the actions for scalars and vectors.

8. In the fermionic sector, let us first remark that one could have constructed 16 independent Euclidean \( \gamma \)-matrices, explicitly given by

\[
1, \quad \gamma_5^E \gamma_4^E, \quad \gamma_\mu^E \gamma_4^E, \quad \gamma_5^E \gamma_\mu^E \gamma_4^E, \quad S^{\mu \nu}. \tag{2.40}
\]

From the Dirac spinor \( \psi \), we can thus construct bilinear combinations that transform as scalars under \( SO(4) \) rotations. Among them, Hermitian bilinears that do not include more than one derivative acting on spinors are the following ten possibilities

\[
\begin{align*}
&\bar{\psi} \gamma^\mu \partial_\mu \psi, \quad \bar{\psi} \gamma^\mu \gamma_\mu \partial_\nu \psi, \\
&\bar{\psi} \gamma^\mu \gamma_\mu \gamma_\nu \partial_\nu \psi, \quad \bar{\psi} \gamma^\mu \gamma_\mu \gamma_\nu \gamma_\nu \partial_\nu \psi, \\
&\delta^{\mu \nu} (i \bar{\psi} \gamma^\rho \gamma_\rho \partial_\mu \psi) \partial_\nu \phi, \quad \delta^{\mu \nu} (i \bar{\psi} \gamma^\rho \gamma_\rho \gamma_\nu \partial_\nu \psi) \partial_\mu \phi, \\
&\delta^{\mu \nu} (i \bar{\psi} \gamma^\rho \gamma_\rho \gamma_\nu \gamma_\nu \partial_\nu \psi) \partial_\mu \phi, \quad \delta^{\mu \nu} (i \bar{\psi} \gamma^\rho \gamma_\rho \gamma_\nu \gamma_\nu \gamma_\nu \partial_\nu \psi) \partial_\nu \phi.
\end{align*}
\]

The first two of the left column correspond to the standard mass and kinetic terms while six among the eight others describe possible couplings to the clock field. Among these six couplings, we have only used the two which were sufficient as an existence proof of our mechanism for Dirac spinors, namely \( \delta^{\mu \nu} (i \bar{\psi} \gamma^\rho \gamma_\rho \partial_\mu \psi) \partial_\nu \phi \) and \( \delta^{\mu \nu} (i \bar{\psi} \gamma^\rho \gamma_\rho \gamma_\nu \partial_\nu \psi) \partial_\mu \phi \). It has to be remarked that this second term is not CPT invariant after the clock field has a vev. Hence, unless the coefficient of this term is exactly the value shown in (2.35), the CPT invariance is violated. We also need to emphasize that we have been able to construct Dirac spinor but that we also need to construct Majorana and Weyl spinors. This is an open problem at the moment.

9. The mass scale \( M \) is related to \( \partial_\mu \phi \) and is arbitrary. It is important to realize that it does not appear in the final expressions of the effective Lorentzian actions.

10. \( X_E \) may not be constant if \( g^{\mu \nu}_E \neq \delta^{\mu \nu} \) (curved space) and/or if \( \partial_\mu \phi \) is not strictly constant in \( \mathcal{M}_0 \). This will be discussed in \( \S \) and \( \S \).
III. GRAVITATION AND CURVED SPACE

So far, our description has been restricted to the classical dynamics of standard fields in flat spacetime. The first natural generalisation we must consider is the way classical dynamics of standard fields in flat spacetime. The given by the shift-symmetric generalized Galileon \[36\] as deki theory \[35\] with shift symmetry. Equivalently, it is \[\phi\] for \[\phi\] assumed it enjoys a shift symmetry that, as we have al-

A. Generic couplings to the clock field

In order to minimize the number of physical degrees of freedom, we demand that the equation of motion for \(\phi\) is a second-order differential equation. Hence, the action for \(\phi\) is restricted to the Riemannian version of the Horndeski theory \[35\] with shift symmetry. Equivalently, it is given by the shift-symmetric generalized Galileon \[36\] as

\[ S_g = \int \! dx^4 \sqrt{g_E} \left( L_2 + L_3 + L_4 + L_5 \right), \] (3.1)

where the Lagrangians are explicitly given by

\[ L_2 = K(X_E), \]
\[ L_3 = -G_3(X_E)\nabla_{E\phi}, \]
\[ L_4 = G_4(X_E)R_E - 2G_4'(X_E) \left( (\nabla_{E\phi})^2 - (\nabla^E\nabla_{E\phi})^2 \right), \]
\[ L_5 = -g_E G^\mu\nu_{E} \partial_\mu \phi \partial_\nu \phi + G_5(X_E) G^{\alpha\beta}_{E} \nabla_{E\phi} \nabla_{E\phi} \phi \]
\[ + \frac{1}{3} G_5'(X_E) \left( (\nabla_{E\phi})^3 - 3(\nabla_{E\phi})(\nabla^E\nabla_{E\phi})^2 \right) \]
\[ + 2(\nabla^E_{\mu}\nabla_{E\phi})^3 \]. (3.2)

Here, \(\nabla^E_{\mu}\) \(R_E\) and \(G^\mu\nu_{E}\) are the covariant derivative associated with the Riemannian metric \(g^E_{\mu\nu}\), its Ricci scalar and Einstein tensor. The coefficient \(g_5\) is a constant and \(K(X_E), G_3, G_4, G_5\) are arbitrary functions of \(X_E\) and a prime refers to a derivative with respect to \(X_E\) that is defined as

\[ X_E \equiv g^E_{\mu\nu} \partial_\mu \phi \partial_\nu \phi. \] (3.3)

We use the following short-hand notations

\[ \nabla^2_{E\phi} \equiv g^E_{\mu\nu} \nabla^E_{\mu\nu} \nabla^E_{E\phi}, \]
\[ (\nabla^E_{\mu} \nabla^E_{E\phi})^2 \equiv g^E_{\mu\nu} \nabla^E_{\mu} \nabla^E_{\nu} \phi \nabla^E_{\mu} \nabla^E_{\nu} \phi, \]
\[ (\nabla^E_{\mu} \nabla^E_{E\phi})^3 \equiv g^E_{\mu\nu} \nabla^E_{\mu} \nabla^E_{\nu} \phi \nabla^E_{\mu} \nabla^E_{\nu} \phi \nabla^E_{\mu} \nabla^E_{\nu} \phi, \]

where \(g^E_{\mu\nu}\) is the inverse of \(g_E^{\mu\nu}\).

For the effective equations, i.e., once the Lorentzian structure and the notion of time have emerged, we would like to ensure that the system is invariant not only under time translation but also under CPT \[2\].

For this reason, we require that besides the shift symmetry (\(\phi \rightarrow \phi + \text{const}\)) the theory also enjoys a \(Z_2\) symmetry (\(\phi \rightarrow -\phi\)) for the clock field action. With these symmetries, the action reduces to

\[ S_g = \int \! dx^4 \sqrt{g_E} \left\{ G_4(X_E)R_E - g_5 G^\mu\nu_{E} \partial_\mu \phi \partial_\nu \phi + K(X_E) -2G_4'(X_E) \left[ (\nabla_{E\phi})^2 - (\nabla^E_{\mu}\nabla_{E\phi})^2 \right] \right\}, \] (3.5)

since only \(L_2, L_4\) and the first term of \(L_5\) can contribute. It is easy to show that the constant \(g_5\) in the action \[3.5\] can be absorbed into the redefinition of \(G_4(X_E)\) up to a boundary term. Hence, by setting \(g_5 = 0\), hereafter we consider the Riemannian gravity action of the form

\[ S_g = \int \! dx^4 \sqrt{g_E} \left\{ G_4(X_E)R_E + K(X_E) -2G_4'(X_E) \left[ (\nabla_{E\phi})^2 - (\nabla^E_{\mu}\nabla_{E\phi})^2 \right] \right\}. \] (3.6)

B. Action for the gravitational sector

Following the logic developed in Section \[II\] we restrict our analysis to a region \(\mathcal{M}_0\) in which \(X_E > 0\) so that we can define a preferred direction, that we shall call \(t\), defined as in Eq. \[2.4\].

\[ t \equiv \frac{\phi}{M^2}, \] (3.7)

\[ 1 \text{ We use the term Riemannian for a curved spacetime with a positive definite metric and Lorentzian for a curved spacetime with a Lorentz signature. We keep the terms Euclidean and Minkowskian for the analog in flat space. However, for simplicity, we use the same subscript E for the Riemannian and Euclidean cases.} \]

\[ 2 \text{ As we have seen in the previous section, this requirement is not obviously fulfilled for spinors without fine-tuning. An additional mechanism is needed to naturally ensure the CPT invariance for spinors. In the present article we shall thus focus on the bosonic sector.} \]
that is chosen as one of coordinates of the 4-dimensional Riemannian manifold. We refer to such a coordinate choice \([3.7]\) as unitary gauge.

1. **Decomposition of the Riemannian metric**

One can then introduce a set of three other independent coordinates \(x^i\) \((i = 1, 2, 3)\) so that the Riemannian metric is decomposed as

\[
g^{E}_{\mu\nu} \, dx^\mu \, dx^\nu = N^2_E \, dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt),
\]

\(\text{where the lapse } N_E \text{ is given, thanks to Eq. } [3.7], \text{ by}
\[
N^2_E \equiv \frac{1}{\sqrt{g^E}} = \frac{M^2}{\sqrt{N^2_E}}.
\]

The 3-metric \(\gamma_{ij}\) is given by

\[
\gamma_{ij} = g^E_{ij},
\]

and \(\gamma^{ij}\) is its inverse. To finish, the shift vector \(N^i\) is given by

\[
N^i = \gamma^{ij} g^E_{ij}.
\]

One can then easily check that the inverse Riemannian metric is given by

\[
g^{tt} = \frac{1}{N^2_E},
\]

\[
g^{ti} = g^E_{ti} = -\frac{N^i}{N^2_E},
\]

\[
g^{ij} = \gamma^{ij} + \frac{N^i N^j}{N^2_E}.
\]

2. **Riemannian geometrical quantities**

With the decomposition \([3.8]\), it is straightforward to show that the Einstein-Hilbert term reduces to

\[
\sqrt{g^E} R^E_E = N_E \sqrt{\gamma} (-K^E_{ij} K^E_{ij} + K^2_E + R^{(3)})
\]

\[
-2\partial_i(\sqrt{\gamma} \gamma^{ij} \partial_j N_E) - 2\partial_i(\sqrt{\gamma} K^E_{ij})
\]

\[
+2\partial_i(\sqrt{\gamma} N^i K^E_E),
\]

in terms of the extrinsic curvature of the constant-\(t\) hypersurface, \(K^E_{ij}\), defined by

\[
K^E_{ij} = \frac{1}{2N_E} (\partial_t \gamma_{ij} - D_i N_j - D_j N_i),
\]

where \(D_i\) is the spatial covariant derivative compatible with \(\gamma_{ij}\), and \(R^{(3)}\) is its Ricci scalar. We have used the notations \(K^E_{ij} = \gamma^{ik} \gamma^{jl} K^E_{kl}\), \(K_E \equiv \gamma^{ij} K^E_{ij}\), and \(N_i \equiv \gamma_{ij} N^j\).

3. **Riemannian action in \(\mathcal{M}_0\)**

With the use of the quantities introduced above, the Riemannian action \([3.6]\) takes the form

\[
S_g = \int \sqrt{g^E} \, d^3x \sqrt{\gamma} \left\{ -G_4 (K^E_{ij} K^E_{ij} - K^2_E) + G_4 R^{(3)} + L_\phi \right\},
\]

where the Lagrangian \(L_\phi\) is given by

\[
L_\phi = -2(\partial^E_\mu \partial^E_\nu + D^2) G_4 - 2G_4 \left[ (\nabla^E_{\mu} \phi)^2 - (\nabla^E_{\mu} \phi)(\nabla^E_{\nu} \nabla^E_{\nu} \phi) \right] + K(X_E),
\]

in which the 3-dimensional Laplacian is defined as usual as \(D^2 = \gamma^{ij} D_i D_j\). The perpendicular derivative \(\partial^E_\mu\) is defined in terms of the unit vector normal to the constant \(\phi\) hypersurfaces, \(n^E_\mu = \partial_\mu \phi/\sqrt{X_E}\), as

\[
\partial^E_\mu = n^E_\mu n^E_{\nu} \partial^\nu - \frac{1}{N_E} (\partial_t - N^i \partial_i),
\]

with \(n^E_\mu = g^E_{\mu\nu} n^E_{\nu}\).

In order to further simplify \(L_\phi\), note that \(\nabla^E_{\mu} \nabla^E_{\nu} \phi = -M^2 \Gamma^E_{E\mu\nu}\) in terms of the Christoffel symbols for the metric \(g^E_{\mu\nu}, \Gamma^E_{\mu\nu}\). Its components are explicitly given by

\[
\phi^E_{\mu \mu} = \sqrt{X_E} \, K^E_{ij},
\]

\[
\phi^E_{\mu \mu} = n^E_\mu n^E_{\nu} \partial^\nu \phi = \frac{1}{2} \, \sqrt{X_E} \, \partial^\nu \phi \, \ln X_E,
\]

\[
\phi^E_{\mu \mu} = n^E_\mu n^E_{\nu} \partial^\nu \phi = \frac{1}{2} \, \sqrt{X_E} \, \partial^\nu \phi \, \ln X_E.
\]

It implies that the term \(2(\nabla^E_\mu \nabla^E_\nu \phi)(\nabla^E_{\mu} \nabla^E_{\nu} \phi)\) appearing in Eq. \([3.16]\) takes the form

\[
(\gamma^{ij} \gamma^{kl} - \gamma^{ik} \gamma^{lj}) \phi_{\mu \nu} \psi_{\mu \nu} + 2\gamma^{ij} (\phi^E_{\mu \mu} \phi^E_{\mu \mu} - \phi^E_{\mu \mu} \phi^E_{\mu \mu})
\]

and thus reduces to

\[
-X_E (K^E_{ij} K^E_{ij} - K^2_E) + K_E \partial^E_\mu X_E - \frac{1}{2} X_E D_i \ln X_E)^2.
\]

Inserting this into Eq. \([3.16]\), it follows that \(L_\phi\) takes the form

\[
L_\phi = 2G_4 X_E (K^E_{ij} K^E_{ij} - K^2_E) + K(X_E) + \frac{\Delta \phi}{N_E \sqrt{\gamma}},
\]

where the last term is given by

\[
\Delta \phi = -2 \partial_i (\sqrt{\gamma} \partial^E_\mu G_4) + 2 \partial_i (\sqrt{\gamma} N^i \partial^E_\mu G_4)
\]

\[
-2 \partial_i (\sqrt{\gamma} K^E_{ij} N^i \partial^E_\mu G_4)
\]

and is a total derivative. We finally obtain the expression of the Riemannian action

\[
S_g = \int \sqrt{g^E} \, d^3x \sqrt{\gamma} \left\{ 2G_4 X_E (K^E_{ij} K^E_{ij} - K^2_E) + G_4 R^{(3)} + K(X_E) \right\}.
\]
where it is understood that \( X_E \) defined in Eq. (3.3) is given by
\[
X_E = \frac{M^4}{N^2}
\]
and that the time coordinate is fixed according to the unitary gauge (3.7).

4. Lorentzian metric

We now introduce a Lorentzian metric \( g_{\mu\nu} \) and decompose it as
\[
g_{\mu\nu} dx^\mu dx^\nu = -N^2 dt^2 + \gamma_{ij} (dx^i + N^i dt)(dx^j + N^j dt),
\]
where the lapse \( N \) is defined by
\[
NdN = -N_E dN_E
\]
so that the Riemannian and Lorentzian lapses are related to each other as
\[
N = \sqrt{N_E^2 - N_2^2},
\]
where \( N_2 \) is an arbitrary positive constant. As above, we can define the extrinsic curvature of this metric as
\[
K_{ij} = \frac{1}{2N} (\partial_t \gamma_{ij} - D_i N_j - D_j N_i),
\]
and \( K_{ij} \equiv \gamma^{ik} \gamma^{jl} K_{kl}, \, K \equiv \gamma^{ij} K_{ij}. \) It is related to the Riemannian extrinsic curvature by
\[
K_{ij} = \frac{N_E}{N} K_{ij},
\]
\[
K^{ij} = \frac{N_E}{N} K^{ij},
\]
\[
K = \frac{N_E}{N} K_E.
\]
The Ricci scalar of \( g_{\mu\nu} \) can be expressed in terms of the extrinsic curvature by the well-known formula
\[
\sqrt{-g} R = N \sqrt{\gamma} \left[ K^{ij} K_{ij} - K^2 \right] - \Delta
\]
with \( \Delta = 2 \partial_t (\sqrt{\gamma} K) - 2 \partial_i (\sqrt{\gamma} K) + 2 \partial_t (\sqrt{\gamma} N^i K) \) and \( g = \det g_{\mu\nu}. \)

5. Lorentzian action in unitary gauge

In the unitary gauge we have been using so far, the action (3.21) is now rewritten as
\[
S_g = \int dt dx^3 N \sqrt{\gamma} \left\{ [f(X) - 2X f'(X)] (K^{ij} K_{ij} - K^2)
+ f(X) R^{(3)} + P(X) \right\}, \tag{3.28}
\]
where the functions \( f \) and \( P \) are defined by
\[
f(X) = \frac{N_E G_4(X_E)},
\]
\[
f'(X) = \frac{df(X)}{dX},
\]
\[
P(X) = \frac{N_E}{N} K(X_E),
\]
in terms of
\[
X = \frac{M^4}{N^2}. \tag{3.30}
\]
To show the equivalence between Eq. (3.21) and Eq. (3.28), we have noted that Eq. (3.24) implies that
\[
\frac{1}{X} + \frac{1}{X^2} = \frac{N^2}{M^2} = \frac{dX}{dX_E} = \frac{X^2}{X_E^2},
\]
and that the time coordinate was fixed according to the unitary gauge (3.7).

Now, using the property (3.27), the action (3.28) can be further simplified to
\[
S_g = \int dt dx^3 N \sqrt{\gamma} \left\{ [f(X) - 2X f'(X)] (K^{ij} K_{ij} - K^2)
+ f'(X) \left[ \left( \frac{D_i F}{X} \right)^2 + 2K \partial_\perp X \right] + P(X) \right\}, \tag{3.32}
\]
where the perpendicular derivative \( \partial_\perp \) is defined similarly as Eq. (3.17) in terms of the normal vector to the constant \( \phi \) hypersurfaces, \( n_\mu = \partial_\mu \phi / \sqrt{X} \), as
\[
\partial_\perp = n^\mu \partial_\mu = \frac{1}{N} (\partial_i - N^i \partial_i), \tag{3.33}
\]
with \( n^\mu = g^{\mu\nu} n_\nu \).

6. Covariant expression

In the previous section the action has been derived assuming that the time coordinate was fixed according to the unitary gauge (3.7).

The action (3.32) can be rewritten in a covariant way by noting that \( \nabla_\mu \nabla_\nu \phi = -M^2 \Gamma_\mu^{\nu\rho} \), where \( \nabla_\mu \) is the covariant derivative compatible with the Lorentzian metric \( g_{\mu\nu} \) and \( \Gamma_\mu^{\nu\rho} \) are its Christoffel symbols for \( g_{\mu\nu} \). Concretely, its components are given by
\[
\phi_{\rho;ij} \equiv \nabla_\rho \nabla_\mu \phi = -\sqrt{X} K_{ij},
\]
\[
\phi_{\perp i} \equiv \phi_{\rho i} \equiv n^\rho \nabla_\rho \nabla_\mu \phi = \frac{1}{2} \sqrt{X} \partial_i \ln X,
\]
\[
\phi_{\perp \perp} \equiv n^\rho n^\nu \nabla_\rho \nabla_\nu \phi = \frac{1}{2} \sqrt{X} \partial_\perp \ln X. \tag{3.34}
\]
Hence, the term \( (\nabla^2 \phi)^2 - (\nabla_\mu \nabla_\nu \phi)(\nabla_\mu \nabla_\nu \phi) \) can be expressed as
\[
(g^{ij} \gamma_{kl} - \gamma^{ik} \gamma^{jl}) \phi_{ij} \phi_{kl} - 2 \gamma^{ij} (\phi_{\perp \perp} \phi_{\perp \mu} - \phi_{\rho \perp} \phi_{\rho \mu}),
\]
which reduces to
\[ -X(K^{ij} K_{ij} - K^2) + K \partial_X X + \frac{1}{2} \frac{(D_i X)^2}{X}. \]
Finally, the Lorentzian action takes the form
\[
S_g = \int dx^4 \sqrt{-g} \left\{ f(X) R + 2 f'(X) [\nabla^2 \phi]^2 \\
- \left( \nabla^\mu \nabla^\nu \phi \right) \left( \nabla_\mu \nabla_\nu \phi \right) \right\} + P(X). \tag{3.35}
\]
Whilst this form of the action was derived assuming the unitary gauge (3.7), it can become manifestly covariant by promoting $X$ to a scalar defined by
\[ X = -g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi. \tag{3.36} \]
It is thus well-defined without the unitary gauge condition. Actually, the covariant action (3.35) is a special case of the covariant Galileon considered in Ref. [36] coupled to the Lorentzian metric $g_{\mu \nu}$. In particular, the equations of motion are second order (see Ref. [38] for comparison).

C. Correspondence

The derived Lorentzian theory (3.35) and the parent Riemannian theory (3.6) are related to each other by the following relations.
\[
\begin{align*}
g_{\mu \nu} &= g_E^{\mu \nu} + \frac{\partial_\mu \phi \partial_\nu \phi}{X_c}, \\
g^{\mu \nu} &= g_E^{\mu \nu} + \frac{g_E^{\mu \rho} g_E^{\nu \sigma} \partial_\rho \phi \partial_\sigma \phi}{X_c - X_E}, \\
\frac{1}{X} &= \frac{1}{X_c} - \frac{1}{X_E}, \\
f(X) &= G_4(X_E), \\
\sqrt{X} &= \sqrt{X_E}, \\
P(X) &= \mathcal{K}(X_E), \tag{3.37}
\end{align*}
\]
where $X_c$ is an arbitrary positive constant given by
\[ X_c = \frac{M^4}{N_c^2}. \tag{3.38} \]
These relations are well-defined even without the unitary gauge condition as far as $X_E/X_c$ is large enough. It is straightforward to express various quantities defined in the Lorentzian theory in terms of those in the Riemannian theory.

D. Stability analysis

We now analyze the stability of a general flat Friedmann-Lemaître (FL) background using the Lorentzian action (3.35) with the Lorentzian ADM decomposition (3.22).

I. Cosmological background

We consider a flat FL background spacetime for which the metric in cosmic time reduces to
\[ N = 1, \quad \dot{N} = 0, \quad \gamma_{ij} = a(t)^2 \delta_{ij}, \tag{3.39} \]
where $a$ is the scale factor, and for which the clock field $\phi = \phi_0(t)$.

The action (3.22) being invariant under a constant shift of the clock field $\phi$, there is a conserved current associated with the shift symmetry so that the equation of motion for $\phi$ takes the form
\[ \dot{\phi} + 3H \phi = 0, \tag{3.40} \]
where $H = \dot{a}/a$ is the Hubble function and
\[ J_\phi \equiv [P_0' + 6H^2(2X_0 f_0'' + f_0')] \phi_0. \tag{3.41} \]
We are using notations according to which
\[ X_0 = \phi_0^2, \quad P^{(n)}_0 = P^{(n)}(X_0), \quad f^{(n)}_0 = f^{(n)}(X_0), \tag{3.42} \]
where $n$ stands for the order of the derivative. Thus, Eq. (3.40) implies that $J_\phi$ decays as $J_\phi \propto 1/a^3$.

By using the correspondence (3.37), $J_\phi$ can be expressed in the language of the Riemannian theory as
\[
J_\phi \dot{\phi}_0 = \left\{ [4G^{\mu \nu}X_E^2 + 4G^\mu \nabla_X - G_4] r^{3/2} \\
+ [2G^\mu_X - G_4] r^{1/2} \right\} X_E \mathcal{K}^{1/2} + \frac{1}{2} \left[ \mathcal{K} - 2\mathcal{K} X_E r^{1/2} + \frac{\mathcal{K}}{r^{1/2}} \right], \tag{3.43}
\]
where a prime in the right hand side represents derivative with respect to $X_E$, and where the ratio $r$ is defined by
\[ r \equiv \frac{X_E}{X_c} = \frac{X_E}{X_c} - 1. \tag{3.44} \]
From Eq. (3.24), we have that $N_c^2 > (N_E^2, N_c^2)$ which implies that $r > 0$.

The equation of motion for the metric reduces, as usual, to the Friedmann equation that takes the form
\[ 3M_{\text{eff}}^2 H^2 = 2J_\phi \dot{\phi}_0 - P_0, \tag{3.45} \]
where the effective mass scale is defined by
\[ M_{\text{eff}}^2 = 2(f_0 - 2X_0 f_0'). \tag{3.46} \]
To understand the qualitative behaviour of the system, let us suppose that $H^2/M^2 \ll 1$ and Taylor expand $P(X)$ and $f'(X)$ around a local minimum of $P(X)$ (which we denote as $X \equiv qM^4$) as
\[ P(X) = p_2 \delta + \mathcal{O}(\delta^2), \quad f'(X) = \frac{f_1 + f_2 \delta}{M^2} + \mathcal{O}(\delta^2), \tag{3.47} \]
where \( q, p_2 \) and \( f_{1, 2} \) are dimensionless constants of order unity, and \( \delta = \frac{X}{M^2} - q \) is a small quantity. Accordingly,

\[
J_\phi = \left[p_2 \delta + 6 \frac{H^2}{M^2} (2f_2q + f_1) + \mathcal{O}\left(\frac{H^2}{M^2} \delta, \delta^2\right)\right] \phi_0.
\]

(3.48)

As already stated above, \( J_\phi \) behaves as \( \propto 1/a^3 \to 0 \) \((a \to \infty)\). Hence, apart from the trivial behavior with \( \phi_0 \to 0 \), the system has a non-trivial attractor with \( \delta + \mathcal{O}(H^2/M^2) \propto 1/a^3 \to 0 \). This implies that \( \phi_0 \to \sqrt{q}M^2(1 + \mathcal{O}(H^2/M^2)) \) and that \( M_{\text{eff}}^2 \) and \( P_0 \) approach constant values up to \( \mathcal{O}(H^2/M^2) \) corrections. Therefore, Eq. (3.45) is no more than the standard Friedmann equation for a universe containing a pressureless fluid (from \( J_\phi \propto 1/a^3 \)) and a cosmological constant (from \( P_0 \to \text{const} \)). This behavior is similar to the one obtained in ghost condensate models [39, 40]. To be consistent with the cosmic expansion as understood today, we need to have

\[
P_0 < 0.
\]

(3.49)

More precisely, we even need \( P_0 \) to be tuned so that

\[
P_0 \sim -3\Omega_{00}M_{\text{eff}}^2H_0^2 \sim -2.1M_{\text{eff}}^2H_0^2.
\]

(3.50)

where \( \Omega_{00} \approx 0.7 \) is the standard density parameter for the cosmological constant. Note also that since \( J_\phi \) contributes to the dark matter component, it has to be bounded so that we shall require

\[
\frac{2}{3} \frac{J_\phi}{M_{\text{eff}}^2} \sqrt{q} \frac{M^2}{H_0^3} \leq \Omega_{m0} \approx 0.3,
\]

(3.51)

today. Note that the term can even be negative at the expense of introducing more dark matter. These two last bounds are the only indicative form that can be derived from cosmology, but a full cosmological analysis will be presented elsewhere.

2. Tensor perturbations

We now consider tensor (T) perturbations around the FL background so that the metric is given by

\[
N = 1 + \alpha, \quad N_i = \partial_i \beta, \quad \gamma_{ij} = a(t)^2 e^{2\zeta} \delta_{ij},
\]

(3.52)

where \( h_{ij} \) is transverse and traceless (i.e. \( \partial_j h_k^j = 0 = \delta^j h_{ij} \)). We still have that \( \phi = \phi_0(t) \).

In Fourier space, the quadratic action for each polarization of the tensor mode is given by

\[
\delta S_{T,k}^{(2)} = \frac{1}{8} \int dt a^3 \left[ M_{\text{eff}}^2 h_k^2 - 2f_0 \frac{k^2}{a^2} h_k^2 \right].
\]

(3.53)

Note that this result can be easily inferred from the expression (3.28). Hence, the stability of the tensor sector requires that

\[
M_{\text{eff}}^2 > 0, \quad f_0 > 0.
\]

(3.54)

By using the correspondence \([3.37]\), \( M_{\text{eff}}^2 \) and \( f_0 \) are expressed in the language of the Riemannian theory as

\[
M_{\text{eff}}^2 = 2(2G'_4X_E - G_4)\sqrt{r},
\]

\[
f_0 = \frac{1}{\sqrt{r}} G_4.
\]

(3.55)

where a prime in the right hand side of these expressions represents derivative with respect to \( X_E \) and \( r \) is defined in Eq. (3.44). Thus the stability of the tensor sector gives the constraint

\[
2G'_4X_E > G_4 > 0.
\]

(3.56)

3. Scalar perturbations

For scalar perturbations around the FL background, the metric in the unitary gauge is given by

\[
N = 1 + \alpha, \quad N_i = \partial_i \beta, \quad \gamma_{ij} = a(t)^2 e^{2\zeta} \delta_{ij},
\]

(3.57)

and, by definition, \( \phi = \phi_0(t) \).

It is then straightforward to calculate the quadratic perturbation action since the time derivatives of \( \alpha \) and \( \beta \) do not appear in the action. Thus, the equations of motion for \( \alpha \) and \( \beta \) become constraint equations. After solving for those constraint equations with respect to \( \alpha \) and \( \beta \), one gets that the perturbed action for \( \zeta \), in Fourier space, is

\[
\delta S_{S,k}^{(2)} = \frac{1}{2} \int dt a^3 \left[ A \zeta_k^2 - B \frac{k^2}{a^2} \zeta_k^2 \right].
\]

(3.58)

where \( A \) and \( B \) are given by

\[
A = \frac{M_{\text{eff}}^2}{H^2} \left( 6 + M_{\text{eff}}^2 \mathcal{F} \right),
\]

\[
B = \frac{1}{a} \frac{d}{dt} \left( \frac{aM_{\text{eff}}^2}{HG^2} \right) + 4f_0,
\]

(3.59)

with \( \mathcal{F} \) and \( \mathcal{G} \) given by

\[
\mathcal{F} = \mathcal{P}_0'X_0^2 + \frac{1}{2} J_\phi \dot{\phi}_0
\]

\[
\mathcal{G} = 4f_0'X_0^2 + 4f_0X_0 - f_0.
\]

(3.60)

The quadratic action (3.58) agrees with a special case of the action derived in Ref. [11]. The stability of scalar perturbations requires that

\[
A > 0, \quad B > 0.
\]

(3.61)

By using the correspondence (3.37), this sets 2 other constraints on the Riemannian theory since \( \mathcal{F} \) and \( \mathcal{G} \) can be
expressed in the language of the Riemannian theory as
\[
F = \left\{ 4G''_{4} X_{E}^{4} + 18G''_{4} X_{E}^{2} + 9G'_{4} X_{E} - \frac{3}{2} G_{4} \right\} r^{5/2}
+ \left[ 2G''_{4} X_{E}^{2} + 2G'_{4} X_{E} - \frac{1}{2} G_{4} \right] r^{3/2} \times 3H^{2}
\]
\[
+ \left( K'' X_{E}^{2} + K' X_{E} - \frac{4}{3} K \right) r^{3/2}
- \frac{1}{4} (K - 2K' X_{E}) r^{1/2},
\]
\[
G = \left[ 4G''_{4} X_{E}^{3} + 4G'_{4} X_{E} - G_{4} \right] r^{3/2},
\]
where a prime in the right hand side of these expressions represents derivative with respect to \(X_{E}\) and \(r\) is defined in Eq. (3.34).

E. Summary

Starting from the Riemannian action (3.6) for a positive definite metric \(g_{\mu\nu}^{E}\), we have been able to derive an action for a Lorentzian metric \(g_{\mu\nu}\). The key ingredient is the coupling of the Einstein tensor of \(g_{\mu\nu}^{E}\) to the clock field. In \(\mathcal{M}_{0}\), the dynamics of \(g_{\mu\nu}\) is dictated by the covariant action (3.35), which is a special case of the covariant Galileon considered in Ref. [36]. (See Ref. [37] for the original Galileon theory.)

The theory has 2 free functions, \(\mathcal{K}\) and \(G_{4}\), and we have shown that the stability of the FL spacetime with respect of both scalar and tensor perturbations at linear level sets 4 constraints on these quantities. An extra constraint appears from the requirement that the constant term entering the Friedmann equation reproduces a positive cosmological constant.

IV. BOSONIC MATTER FIELDS IN CURVED SPACE

Given the formulation of gravity described in the previous section, we shall now extend the constructions presented in § II C to describe the proper dynamics of the matter fields in curved spacetime.

A. Scalar field

Following § II C we assume that the clock field \(\phi\) has a derivative coupling to a real scalar field \(\chi\) of the form
\[
S_{\chi} = \int dx^{4} \sqrt{|g_{E}|} \left[ \frac{-\kappa_{\chi}}{2} g_{E}^{\mu\nu} \partial_{\mu} \chi \partial_{\nu} \chi - V(\chi) \right]
+ \frac{\alpha_{\chi}}{2M^{2}} (g_{E}^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \chi)^{2}. \tag{4.2}
\]
It involves two dimensionless constants, \(\kappa_{\chi}\) and \(\alpha_{\chi}\). There is a freedom to rescale \(\chi\) and thus we can set \(\kappa_{\chi} = \pm 1\) if needed.

With the decomposition (3.8), the Riemannian kinetic and the derivative coupling terms are correspondingly rewritten as
\[
g_{E}^{\mu\nu} \partial_{\mu} \chi \partial_{\nu} \chi = (\partial_{\mu} \chi)^{2} + \gamma^{ij} \partial_{i} \chi \partial_{j} \chi,
(\partial_{\mu} \phi \partial_{\nu} \chi)^{2} = \frac{M^{4}}{N_{E}^{2}} (\partial_{\mu} \chi)^{2}, \tag{4.3}
\]
where \(\partial_{\mu} \chi\) is defined in Eq. (3.17). Therefore, the action of the scalar field \(\chi\) reduces to
\[
S_{\chi} = \int dt dx^{3} N_{E} \sqrt{|\gamma|} \left[ \frac{1}{2} \left( \frac{\alpha_{\chi}}{N_{E}^{2}} - \kappa_{\chi} \right) (\partial_{\mu} \chi)^{2} - V(\chi) \right]
+ \frac{\kappa_{\chi}}{2} \gamma^{ij} \partial_{i} \chi \partial_{j} \chi. \tag{4.4}
\]

If
\[
\frac{\alpha_{\chi}}{N_{E}^{2}} > \kappa_{\chi} > 0, \tag{4.5}
\]
then \(S_{\chi}\) describes a scalar field propagating in a Lorentzian spacetime. To see this explicitly let us define a Lorentzian effective metric \(g_{E}^{\chi\mu\nu}\) by
\[
g_{E}^{\chi\mu\nu} dx^{\mu} dx^{\nu} = -N_{\chi}^{2} dt^{2} + \Omega_{\chi}^{2} dx^{j}(dx^{j} + N^{j} dt)(dx^{j} + N^{j} dt), \tag{4.6}
\]
where
\[
N_{\chi} = N_{E} \left[ \frac{\kappa_{\chi}^{3}}{\frac{\alpha_{\chi}}{N_{E}^{2}} - \kappa_{\chi}} \right]^{1/4}, \quad \Omega_{\chi} = \left[ \kappa_{\chi} \left( \frac{\alpha_{\chi}}{N_{E}^{2}} - \kappa_{\chi} \right) \right]^{1/4}. \tag{4.7}
\]
The scalar field action \(S_{\chi}\) is rewritten as
\[
S_{\chi} = - \int dx^{4} \sqrt{-g_{\chi}} \left[ \frac{1}{2} g_{\chi}^{\mu\nu} \partial_{\mu} \chi \partial_{\nu} \chi + V(\chi, X) \right]. \tag{4.8}
\]
where \(g_{\chi}\) and \(g_{\chi}^{\mu\nu}\) are the determinant and the inverse of \(g_{E}^{\mu\nu}\), and
\[
V(\chi, X) = \tilde{V}(\chi) \left[ \kappa_{\chi}^{3} \left( \frac{\alpha_{\chi}}{N_{E}^{2}} - \kappa_{\chi} \right) \right]^{-1/2}
= \tilde{V}(\chi) \left[ \kappa_{\chi}^{3} \left( \frac{\alpha_{\chi}^{E} X_{E}}{M^{4}} - \kappa_{\chi} \right) \right]^{-1/2}. \tag{4.9}
\]
Note that \(\alpha_{\chi}\) and \(\kappa_{\chi}\) may depend on \(X_{E}\) and that \(X_{E}\) is related to \(X\) via the correspondence (3.37).
to the standard (Riemannian) Maxwell action. Again, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Faraday tensor of $A_\mu$, and we use the notations $F^E_{\mu\nu} = g^E F_{\mu\nu}$ and $F^{\mu\nu}_{\rho\sigma} = g^E F^{\mu\nu}_{E\rho\sigma}$. This leads to the general gauge-invariant action for the vector field

$$S_A = \frac{1}{4} \int d^4 x \sqrt{g_E} \left( -\kappa_A F^{\mu\nu} F_{\mu\nu} + \frac{2 \alpha A}{M^4} F^{\mu\nu}_{E\rho\sigma} \partial_\mu \phi \partial_\nu \phi \right),$$

where $\kappa_A$ and $\alpha_A$ are two dimensionless constants.

With the decomposition (3.8) for the Riemannian metric, the Riemannian kinetic term and the non-minimal coupling term can be respectively written as

$$F^{\mu\nu} F_{\mu\nu} = 2 \gamma^{ij} \tilde{F}_{i\perp j} \tilde{F}_{\perp j} + \gamma^{ik} \gamma^{jl} F_{ij} F_{kl},$$

$$F^E_{\mu\nu} F^{\mu\nu}_{E\rho\sigma} \partial_\mu \phi \partial_\nu \phi = \frac{M^4}{N_E} \gamma^{ij} \tilde{F}_{i\perp j} \tilde{F}_{\perp j},$$

where $\tilde{F}_{i\perp j} \equiv \frac{1}{N_E} (F_{i\perp j} - N^j F_{i\perp})$.

Therefore, the gauge-invariant action takes the form

$$S_A = \frac{1}{4} \int dt d^4 x \sqrt{g_E} \left[ \left( \frac{\alpha_A}{N^2} - \kappa_A \right) \gamma^{ij} \tilde{F}_{i\perp j} \tilde{F}_{\perp j} - \kappa_A \gamma^{ik} \gamma^{jl} F_{ij} F_{kl} \right].$$

If

$$\frac{\alpha_A}{N^2} > \kappa_A > 0,$$

then $S_A$ describes a $U(1)$ gauge field propagating in a Lorentzian spacetime. To see this explicitly let us define a Lorentzian effective metric $g^A_{\mu\nu}$ by

$$g^A_{\mu\nu} d\Omega^A = -N_A^2 dt^2 + \Omega_A^2 \gamma_{ij} (dx^i + N^i dt)(dx^j + N^j dt),$$

where $\Omega_A$ is an arbitrary positive function and

$$N_A = N_E \left[ \frac{\kappa_A}{\alpha_A} \right]^{1/2}.\quad (4.17)$$

The vector field action $S_A$ takes the form of the usual Maxwell action

$$S_A = -\int d^4 x \sqrt{-g^A} \frac{1}{4e^2} g^A_{\mu\nu} g^A_{\rho\sigma} F_{\mu\nu} F_{\rho\sigma},$$

where $g^A$ and $g^A_{\mu\nu}$ are the determinant and the inverse of $g^A_{\mu\nu}$, and the effective coupling constant $e^2$ is given by

$$e^2 = \left[ \frac{\alpha_A}{N^2} \left( \frac{\alpha_A}{N_E} - \kappa_A \right) \right]^{1/2}.\quad (4.19)$$

Note that $\alpha_A$ and $\kappa_A$ may depend on $X_E$ and that $X_E$ is related to $X$ via the correspondence (3.37).

C. Generalization to a complex scalar field

The generalization to a complex scalar field charged under the $U(1)$ is straightforward and follows the construction presented in § 3. Consider the action for a complex scalar $\psi$

$$S_\psi = \int d^4 x \sqrt{-g_E} \left\{ -\kappa_\psi \frac{1}{2} \psi^* \left( \partial_\mu + i q A_\mu \right) \psi \left( \partial_\rho - i q A_\rho \right) \psi + \frac{\alpha_\psi}{2M^4} \left| g^{\mu\nu}_{E} \partial_\mu \phi \partial_\nu \phi \right|^2 - \tilde{U}(|\psi|^2) \right\},$$

where $q$, $\kappa_\psi$ and $\alpha_\psi$ are dimensionless constants and $\tilde{U}(|\psi|^2)$ is a function of $|\psi|^2$.

Supposing that

$$\frac{\alpha_\psi}{N_E} > \kappa_\psi > 0,$$

it is easy to show that

$$S_\psi = -\int d^4 x \sqrt{-g^E} \left( \frac{1}{2} g^{\mu\nu}_{E} \left( \partial_\mu + i q A_\mu \right) \psi^* \left( \partial_\rho - i q A_\rho \right) \psi + U(|\psi|^2, X) \right),\quad (4.22)$$

where we have introduced a Lorentzian metric $g^E_{\mu\nu}$ by

$$g^E_{\mu\nu} d\Omega^E = -N^2_E dt^2 + \Omega_E^2 \gamma_{ij} (dx^i + N^i dt)(dx^j + N^j dt),$$

and $\kappa_\psi$ and $g^{\mu\nu}_{E}$ are the determinant and the inverse of $g^{\mu\nu}_{E}$, and

$$U(|\psi|^2, X) = \tilde{U}(|\psi|^2) \left[ \kappa_\psi \left( \frac{\alpha_\psi}{N_E} - \kappa_\psi \right) \right]^{-1/2},$$

Note that $\alpha_\psi$ and $\kappa_\psi$ may depend on $X_E$ and that $X_E$ is related to $X$ via the correspondence (3.37).

Generalization to a non-Abelian group is trivial.
D. Massive point particle

For a massive point particle, we assume that the action is given by

\[ S_{pp} = \frac{1}{2} \int \left[ \mathcal{N}^{-1} \left( \tilde{\kappa}_{pp} g_{\mu\nu} - \frac{\bar{\alpha}_{pp}}{M^{2}} \partial_{\mu} \phi \partial_{\nu} \phi \right) \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} - N m^{2} \right] d\tau, \]  

(4.26)

where \( \mathcal{N} \) is a function of \( \tau \). Then, a test particle propagates in the effective metric \( g^{\text{eff}}_{\mu\nu} = \tilde{\kappa}_{pp} g_{\mu\nu} - \frac{\bar{\alpha}_{pp}}{M^{2}} \partial_{\mu} \phi \partial_{\nu} \phi \) so that its equation of motion is simply a geodesic equation for this metric

\[ u^{\mu} \nabla_{\mu} u^{\nu} = 0, \]  

(4.27)

with \( u^{\mu} = dx^{\mu} / d\lambda \), where \( \lambda \) is an affine parameter defined by \( d\lambda = \mathcal{N} d\tau \). Using the decomposition (3.8) of the Riemannian metric, we obtain that

\[ g^{\text{eff}}_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} = - (\bar{\alpha}_{pp} - \tilde{\kappa}_{pp} N_{E}^{2}) \frac{d\tau^{2}}{\left( d\tau \right)^{2}} + \kappa_{pp} \gamma_{ij} (dx^{i} + N^{i} d\tau)(dx^{j} + N^{j} d\tau). \]  

(4.28)

This effective metric has Lorentzian signature if

\[ \frac{\bar{\alpha}_{pp}}{N_{E}^{2}} > \kappa_{pp} > 0. \]  

(4.29)

V. PHENOMENOLOGY

A. Summary

We have proposed a Riemannian field theory for gravity, vector and scalar fields that, with the expense of the introduction of a scalar field \( \phi \) called a clock field, leads to an effective Lorentzian dynamics.

This construction involves a set of free parameters:

- For the gravitational sector, we have two free functions of \( X_{E}, K \) and \( G_{4} \) in terms of which the two free functions of the Lorentzian theory \( f(X) \) and \( P(X) \) are defined; see the correspondence between the two sets given in Eq. (3.37).

- For the matter sector, we have derived the actions for scalar and vector fields. Each action depends on 2 parameters \((\kappa, \alpha)\) that are allowed to be functions of \( X_{E} \), or equivalently \( X \), in general but may as well be assumed constant.

- Besides, there is an environmental parameter which characterizes the clock field configuration on the patch \( \mathcal{M}_{0}, N_{E} \), [see Eqs. (3.3) and (3.9)] and the associated integration constant \( N_{c} \) [see Eq. (3.24)]. They combine in the parameter \( r \) [see Eq. (3.41)].

With these parameters the actions for gravity, scalar and vector fields are respectively given by Eqs. (3.35), (4.8) and (4.18).

We have already shown that these parameters are subject to a series of constraints.

- For the gravitational sector, we have two sets of constraints. The first one arises from the stability analysis and is given by Eqs. (3.56) and (3.61). The second is related to the dynamics of the homogeneous model. Interestingly, the model induces two components which respectively behave as dark matter and dark energy. This sets the two constraints (3.50) and (3.51) in order for the cosmology to be consistent with the standard cosmology [42], at least at the background level.

- For the matter sector, the constants \( \alpha \) have to satisfy [see Eqs. (4.5) and (4.15)]

\[ \alpha_{\chi} > N_{E}^{2} \kappa_{\chi}, \quad \alpha_{A} > N_{E}^{2} \kappa_{A}. \]  

(5.1)

From the effective Lagrangians (4.8) and (4.18), we see that scalars and vectors propagate in 2 different effective metrics. In order for the weak equivalence principle to hold, we have to impose that these two metrics coincide. This can be obtained by imposing that \( c_{A}^{2} = c_{1}^{2} \), with \( c_{A}^{2} = \Omega_{A}^{-2} N_{A}^{2} / N_{E}^{2} \) and \( c_{\chi}^{2} = \Omega_{\chi}^{-2} N_{\chi}^{2} / N_{E}^{2} \). This sets the following constraints

\[ \frac{\kappa_{A}}{\alpha_{A}} = \frac{\kappa_{\chi}}{\alpha_{\chi}}. \]  

(5.2)

In the simplest situation in which the coefficients \((\kappa, \alpha)\) are assumed to be constant, we can always set \( \kappa = 1 \) in both sectors so that we are left with the constraint \( \alpha_{A} = \alpha_{\chi} \) for the two coupling constants. This is similar to what we performed in §III in which the couplings to the clock field were chosen a priori so that effectively all fields propagate in the same effective Minkowski metric. Interestingly, in this class of models, one requires a tuning on the parameters of the Lagrangians, but once it is done, it is satisfied whatever the configuration of the clock field, that is whatever \( N_{E} \) or \( X_{E} \). In this sense the tuning is not worse than the one usually does by assuming that all the fields propagate in the same metric. This conclusion holds even if \( \kappa \)'s and \( \alpha \)'s are functions of \( X_{E} \) as long as their ratios agree between different sectors. Again, once this condition is satisfied, it holds whatever the field configuration.

The Lorentzian effective metric (4.28) for a point particle coincides with that for the vector if \( \bar{\alpha}_{pp} / \tilde{\kappa}_{pp} - \kappa_{pp} = N_{E}^{2} N_{A}^{2} / \Omega_{A}^{2} \), that is if

\[ \frac{\bar{\alpha}_{pp}}{\kappa_{pp} N_{E}^{2}} - 1 = \left[ \frac{\alpha_{A}}{\kappa_{A} N_{E}^{2}} - 1 \right]^{-1}. \]  

(5.3)

This may look as a functional fine-tuning depending on the local value of \( X_{E} \). Actually, this arises
from the fact that \( \alpha_{pp} \) and \( \kappa_{pp} \) have been introduced with reference to \( g_{\mu\nu} \) while \( \alpha_{AX} \) and \( \kappa_{AX} \) have been introduced with reference to \( g_{\mu\nu}^{AX} \). Shifting to the inverse metric and redefining these coefficients lead to a constraint similar to Eq. (5.2).

Only the condition [5.1] for the emergence of the Lorentzian signature is environmentally dependent so that there are regions in the configuration space where the dynamics is effectively Lorentzian while other regions remain Riemannian. This could drive us toward a multiverse description in the configuration space but we do not have any anthropic reasons associated.

Note that without such a tuning so that all fields propagate in the same effective Lorentzian metric, one can choose one metric as reference so that the equations of motion of other fields will exhibit an explicit coupling to the clock field.

### B. Other constraints

The fundamental parameters entering our effective Lorentzian actions are environmentally determined. This means that if \( X_E \) is not strictly constant on \( M_0 \), fundamental constants may be spacetime dependent, which can induce a violation of the equivalence principle [2].

- From Eq. (4.19), it can be deduced that the coupling constant of any gauge field will be environment-dependent. The first implication is that the coupling constants of the three non-gravitational interactions have to be space-time varying. There exist strong constraints on such a possibility [2].

- The action for the gravitational sector implies that the Newton constant is also expected to be space-time dependent.

- Furthermore, even if \( c_A = c_\gamma \) so that scalars and vectors propagate on the same lightcone, we have to compare the propagation speeds of gravity waves and photons. The first is given by

\[
c_\gamma^2 = c_A^2 = \frac{N_A^2}{\Omega_A^2 N_e} = \left[ \frac{\alpha_{AX} X_E}{\kappa_{AX} M^4} - 1 \right]^{-1}. \quad (5.4)
\]

The propagation speed of the gravity waves can be obtained from the action (3.53) rewritten as

\[
\delta S^{(2)}_{T,k} = \frac{1}{8} \int dt a^3 N_E \left[ M_{eff}^2 N_E \left( \frac{h_k}{N_E} \right)^2 - 2f_0 N E \frac{k_2 a h_k^2}{a^2} \right],
\]

from which we read off

\[
c_{GW}^2 = \frac{2f_0 N^2}{M_{eff}^2 N_E} = \frac{2f_0}{M_{eff}^2}. \quad (5.5)
\]

It can be rewritten in terms of the function \( G_4(X_E) \) entering the Euclidean gravitational action (3.6) as

\[
c_{GW}^2 = \left[ \frac{2G_1 X_E}{G_4} - 1 \right]^{-1}. \quad (5.6)
\]

As long as both lightcones are non-degenerate, there is no a priori intrinsic problem even if these two propagation speeds are different [43] [14] and similar features indeed appear in many bimetric theories such as TeVeS [45] or many other extensions of general relativity [43]. This difference can be tested by future experiments by comparing e.g. the arrival time of gravity waves and light emitted during the explosion of supernovae; see e.g. Ref. [46]. Models in which \( c_{GW}^2 < c_A^2 \) are very constrained by the observations of cosmic rays [47] because particles propagating faster than the gravity waves emit gravi-Cerenkov radiation. They lead to the constraint [48]

\[
\frac{c_\gamma - c_{GW}}{c_\gamma} < 2 \times 10^{-15}. \quad (5.7)
\]

### C. Emergence of Lorentz symmetry on intermediate scales

Let us first consider our mechanism in flat space. As discussed in §11, \( M \) is not the most important mass scale of the problem. More important is the scale characterizing the variation of \( X_E/M^4 \). In order to illustrate this and to capture the way the Lorentz dynamics appear on relevant scales, let us assume that the clock field configuration is given by

\[
\phi(x^\mu) = M [m_x x + \cos(m_y y) + \beta \cos(M_y y)]
\]

with \( m_y \ll M_y \) two mass scales that characterize respectively large- and small-scale variations of \( \phi \) and \( \beta \) a dimensionless number. We neglect the two other dimensions for simplicity. With such a form it is obvious that \( \partial_\mu \phi \) is not constant.

Now, assume that we are smoothing the dynamics at a scale \( R \sim m_x^{-1} \) with e.g. a top-hat window function. On the scale \( R \), the clock field is given by

\[
\phi = M \left[ m_x x + 2 \frac{J_1(m_y R)}{m_y R} \cos(m_y y) \right]
\]

\[
+ 2 \beta \frac{J_1(M_y R)}{M_y R} \cos(M_y y)
\]

where \( J_1 \) is the Bessel function of order 1. It follows that \( \partial_\mu \phi \) is given by

\[
\frac{\partial_\mu \phi}{M^2} = \left( -2 \frac{J_1(m_y R)}{m_y R} \sin(m_y y) - 2 \beta \frac{J_1(M_y R)}{M_y R} \sin(M_y y) \right).
\]

This can be considered as constant only if \( 2J_1(m_y R) \ll m_x R \) and \( 2\beta J_1(M_y R) \ll m_x R \). By choosing \( R \sim m_x^{-1} \),
the first condition reduces to
\[ \frac{m_y}{m_x} \ll 1, \]
since \( J_1(x) \sim x/2 \) at small \( x \), while the second gives
\[ 2\sqrt{2} \pi \beta \cos \left( \frac{M_y}{m_x} + \frac{\pi}{4} \right) \sqrt{\frac{m_x}{M_y}} \ll 1. \]

This condition is clearly fulfilled if \( m_y \ll m_x \ll \beta^{-2} M_y \). For example, if we assume that \( M_y \) is of the order of the Planck mass, \( M_y \sim M_0 \sim 10^{19} \) GeV and that the large-scale variations appear on Hubble scales, \( m_y \sim H_0 \sim 10^{-41} \) GeV, then we end up with the conclusion that \( \partial_{\mu} \phi \) can be considered as constant at the level \( 10^{-n} \) on scales
\[ 10^{n-41} \text{GeV} < m_x < 10^{21-2n} \left( \frac{\beta}{0.1} \right)^{-2} \text{GeV}. \quad (5.8) \]

For \( n = 9 \) and \( \beta = \mathcal{O}(0.1) \), this means that we can work with scales
\[ 10^{-19} \text{m} < m_x^{-1} < 10^{16} \text{m}. \quad (5.9) \]

In such a range of scales, we expect no deviations larger than \( 10^{-9} \) to the standard field theory. On cosmological scales, we can probably relax the bound to deviations of order \( 10^{-1} - 10^{-2} \) so that our model may be compatible with standard cosmology on scales of the order of the observable universe.

VI. SUMMARY AND DISCUSSION

We have followed the idea that the apparent Lorentzian dynamics of the usual field theories is an emergent property and that the underlying field theory is in fact strictly Riemannian. This requires the introduction of the clock field, a scalar field playing the role of the physical time. We emphasize that the microscopic theory is Euclidean, and that time evolution is just an effective and emergent property, that holds on some energy scales, and in some regions of the Euclidean space. We have thus to think of time and dynamics as illusions in our local patch \( M_0 \). This has to be distinguished from the mathematical trick of a Wick rotation used to effectively study genuine Lorentzian theories in a Euclidean space.

We have been able to perform such a construction in flat spacetime for scalar, vector and spinor, hence allowing for the construction of the standard model of particle physics. Our construction is however restricted to classical field theory and the spinor sector suffers from the severe fine-tuning to ensure the CPT invariance. (See e.g. Ref. [49] and references therein for recent constraints on CPT violation.)

We have then generalized our construction to curved spacetimes. This generalizes an early attempt [31] that ended up with a Nordström theory of gravity. Our construction leads to an extended \( K \)-essence model for gravity called covariant Galileon, which can be close enough to general relativity to be experimentally acceptable. We have then generalized the scalar and vector sectors to curved spacetimes. This requires the introduction of \( 4 \) arbitrary functions. Again, so far we have not generalized
the construction of spinors to curved spacetime. We have expressed the effective Lorentzian action that can emerge in a patch $\mathcal{M}_0$ of the Euclidean space. It allowed us to list a set of constraints arising from the stability of the cosmological solution, and the requirement for the different test fields to propagate in the same metric, in order for the weak equivalence principle to hold. The effective fundamental constants, such as the three non-gravitational coupling constants and the gravitational constant, are spacetime dependent and the difference of the propagation speeds of gravity and electromagnetic waves can also set constraints on our model. We have proposed an heuristic description on the way the Lorentz symmetry can emerge on a band of energy scales.

From a theoretical point of view, our construction gives a new insight into the need for Lorentzian metric as a fundamental entity. As we have shown, this is not a mandatory requirement and a decent field theory, at least at the classical level, can be constructed from a Riemannian metric. Such a formalism may be fruitful in the development of quantum gravity.

It also opens up a series of questions and possibilities that will be addressed in a companion article [50]. We can list (1) the construction of Majorana and Weyl spinors, (2) the development of a quantum theory and study of particle creation [51], (3) the possibility from the classical viewpoint that singularities in our local Lorentzian region may be related to singularities in the clock field (e.g. similar to topological defects) and not in the metric of the Euclidean theory (see Ref. [52] for a similar idea in a totally different setup), (4) the possibility that a de Sitter spacetime may be an “illusion” in an anti-de Sitter Riemannian space. It then follows that a Euclidean AdS/CFT correspondence at the microscopic level would reveal itself as a dS/CFT correspondence in our effective Lorentzian universe.

All these are indeed, for now, bold speculations but they illustrate that this framework may be fruitful for extending our current field theories, including general relativity.

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[1] J.-P. Luminet, et al., “Dodecahedral space topology as an explanation for weak wide-angle temperature correlations in the cosmic microwave background”, Nature (London) 425, 593 (2003), astro-ph/0310253.

[2] J.-P. Uzan, “The fundamental constants and their variation: observational status and theoretical motivations”, Rev. Mod. Phys. 75, 403 (2003), hep-ph/0205340; J.-P. Uzan, “Variation of the constants in the late and early universe”, AIP Conf. Proc. 736, 3 (2004), astro-ph/0409424; J.-P. Uzan, “Varying constants, Gravitation and Cosmology”, Living Rev. Relat. 4, 2 (2011). arXiv:1009.5514.

[3] J. Polchinski, String theory (Cambridge University Press, 1998).

[4] I. Bars, “Survey of two time physics”, Class. Quant. Grav. 18, 3113 (2001). hep-th/0008164; G B Halsted, “Four-Fold Space and Two-Fold Time”, Science 19, 319 (1892).

[5] C.M. Hull, “Duality and the signature of space-time”, JHEP 9811, 017 (1998), hep-th/9807127.

[6] J. Dorling, “The Dimensionality of Time”, Am. J. Phys. 38, 539 (1970).

[7] G.W. Gibbons, in Fundamental Theories of Physics, 164, “The Arrows of Time: A debate in Cosmology” eds L.Mersini-Houghton and R. Vass, pp 107-146 Springer 2012. arXiv:1111.0457.

[8] J.B. Hartle, and S.W. Hawking, “Wave function of the Universe”, Phys. Rev. D 28, 2960 (1983); J. L. Friedman, “Lorentzian universes from nothing”, Class. Quant. Grav. 15, 2639 (1998); J.R. Gott, J.R., and X.I. Li, “Can the universe create itself?“, Phys. Rev. D 58, 023501 (1998). arXiv:astro-ph/9712344; G.W. Gibbons, and J.B. Hartle, “Real Tunneling Geometries and the Large Scale Topology of the Universe”, Phys. Rev. D 42, 2458 (1990).

[9] A.S. Eddington, The Mathematical Theory of Relativity (Cambridge University Press, 1923), p. 25.

[10] G. W. Gibbons, and A. Ishibashi, “Topology and signature changes in braneworlds”, Class. Quant. Grav. 21, 2919 (2004). arXiv:hep-th/0402024; M. Mars, J.M.M. Senovilla, and R. Vera, “Lorentzian and signature chang- ing branes”, Phys. Rev. D 76, 044029 (2007). arXiv:0705.3380 [hep-th]; M. Mars, J.M.M. Senovilla, and R. Vera, “Is the accelerated expansion evidence of a forthcoming change of signature?”, arXiv:0710.0820 [gr-qc].

[11] J. Mielezarek, “Signature change in loop quantum cosmology”, arXiv:1207.4657 [gr-qc].

[12] H.D. Zeh, The physical basis of the direction of time (Springer-Verlag, Berlin, 2001).

[13] N. Deruelle, and J.-P. Uzan, M´ecanique et gravitation newtonienne (Vuibert, Paris, 2006).
eral second-order scalar-tensor models from gravitational Cherenkov radiation”, JCAP **1207**, 050 (2012), [arXiv:1112.4284](https://arxiv.org/abs/1112.4284).

[49] K. Toma, S. Mukohyama, D. Yonetoku, T. Murakami, S. Gunji and T. Mihara et al., “Strict Limit on CPT Violation from Polarization of Gamma-Ray Burst,” Phys. Rev. Lett. **109**, 241104 (2012) [arXiv:1208.5288 [astro-ph.HE]].

[50] Work in progress.

[51] A. White, S. Weinfurtner and M. Visser, “Signature change events: A Challenge for quantum gravity?,” Class. Quant. Grav. **27**, 045007 (2010) [arXiv:0812.3744 [gr-qc]]; S. Weinfurtner, A. White and M. Visser, “Trans-Planckian physics and signature change events in Bose gas hydrodynamics,” Phys. Rev. D **76**, 124008 (2007) [gr-qc/0703117 [GR-QC]].

[52] V. P. Frolov and S. Mukohyama, “Brane Holes,” Phys. Rev. D **83**, 044052 (2011) [arXiv:1012.4541 [hep-th]].