ON THE DIRECT PRODUCT OF PARTIAL BURNSIDE RINGS

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Abstract. In this paper, we describe the structure of the direct product of partial Burnside rings relative to the collection of a finite group. In particular, we determine the unit group of the partial Burnside ring relative to the set of all parabolic subgroups of a reducible finite Coxeter group is isomorphic to the direct product of unit groups of the partial Burnside ring of irreducible Coxeter group.

1. Introduction

Let $G$ be a finite group. The unit group of the Burnside ring of $G$ has been studied (see [1], [3], [9], [11]). Yoshida introduced a theory of generalized Burnside ring (GBR for short) with respect to a family of subgroups of $G$ (see [10]). In 2015, the study of the unit group of the GBR relative to a collection of $G$ was started by Idei and Oda (see [2]). Also it is discussed by [4] and [5].

This paper presents some results of the unit group of the GBR of a reducible finite Coxeter group.

The Burnside ring of a finite group $G$ is defined to be the Grothendieck ring of the semi-ring generated by isomorphism classes of finite (left) $G$-sets where the addition and multiplication are given by disjoint unions and Cartesian products. Denote by $\Omega(G)$ the Burnside ring of $G$. If a family $\mathcal{D}$ of subgroups of $G$ contains $G$ and it is closed under taking conjugation and intersection, then $\mathcal{D}$ is called a collection of $G$. We call the Grothendieck ring of the semi-ring generated by isomorphism classes of finite (left) $G$-sets whose stabilizers of any element lie in a collection $\mathcal{D}$ of $G$ a partial Burnside ring (PBR for short) relative to $\mathcal{D}$ of $G$. We denote the PBR relative to a collection $\mathcal{D}$ of $G$ by $\Omega(G, \mathcal{D})$. Let $W$ be a finite Coxeter group with Coxeter system $(W, S)$. Then a subgroup $P$ of $W$ is called parabolic subgroup if there exists $J \subseteq S$ and $g \in W$ such that $P = g^{-1} \langle J \rangle g$. If $W$ is a finite Coxeter group, then the set of all parabolic subgroups of $W$ is a collection of $W$ (see [8]). Denote by $\mathcal{P}_W$ the set of all parabolic subgroups of a Coxeter group $W$.

The paper is organized as follows. In Section 2, we recall the basic definitions and results from GBR and PBR. In Section 3, we consider the relationship between the direct product of PBR, tensor product of PBR, and PBR of direct product. Section 4 determines the structure of the unit group of the PBR relative to the set of all parabolic subgroups of a reducible finite Coxeter group.

For a unitary ring $R$, write $R^\times$ for the unit group of $R$. For the PBR of a finite Coxeter group relative to the set of all parabolic subgroups, the following lemma is well known.

Key words and phrases. Burnside ring; Coxeter group; unit group; parabolic subgroups.
Lemma 4.2 \(^{[7]}\). Let \(W\) be a finite Coxeter group with Coxeter system \((W, S)\). Then the element
\[
\varepsilon_W := \sum_{J \subseteq S} (-1)^{|J|} [W/ \langle J \rangle]
\]
lies in \(\Omega(W, P_W)^\times\).

We call \(\varepsilon_W\) in Lemma 4.2 a sign unit of \(\Omega(W, P)\).

The cardinality of the unit group of the PBR relative to \(P_W\) of a finite Coxeter group \(W\) is 4 if \(W\) is an irreducible Coxeter group with type A, B, D, E_6, E_7, or E_8 (see [2], [5]).

The main result of the paper is the following:

Theorem 4.3. Let \(W\) be a finite group and let \((W, S)\) be a reducible Coxeter system with \(\ell\)-irreducible Coxeter system \((W_t, S_t)\) for \(t = 1, \ldots, \ell\), namely \(W = \prod_{i=1}^{\ell} W_i\) and \(S = \bigsqcup_{i=1}^{\ell} S_i\). Then the following holds:

1. \(|\Omega(W, P_W)\times| = \frac{1}{2^{2\ell-1}} \prod_{i=1}^{\ell} |\Omega(W_i, P_{W_i})\times|.
2. If \(\varepsilon_W\) is the sign unit of \(\Omega(W, S)\), then \(\varepsilon_W = \prod_{i=1}^{\ell} f_i(\varepsilon_{W_i})\).

Theorem 4.3 is special case of key result (Lemma 3.3), and the result is extension of [6].

2. Preliminaries

2.1. Notation. Let \(G\) be a finite group. Denote by \(S(G)\) the set of subgroups of \(G\). For a family \(\mathcal{D}\) of subgroups of \(G\) with closed under \(G\)-conjugation, write \(\mathcal{D}^c\) for the set of the conjugacy classes of \(\mathcal{D}\). Denote by \([X]\) the isomorphism class of finite \(G\)-set \(X\). For a family \(\mathcal{D}\) of subgroups of \(G\), a \(G\)-set \(X\) is called a \((G, \mathcal{D})\)-set if the stabilizer of any element of \(X\) lies in \(\mathcal{D}\). If \(X\) is a finite set, write \(|X|\) for the cardinality of \(X\). Denote by \(\bigsqcup\) the disjoint union of sets.

2.2. Partial Burnside rings. Let \(G\) be a finite group. Then the Burnside ring \(\Omega(G)\) of \(G\) can be regarded as a free abelian group with basis \(\{ [G/H] \mid (H) \in S(G)^c \}\). The multiplication in the ring is given by
\[
[G/H] \cdot [G/K] = \sum_{HgK \in \mathcal{H}(G/K)} [G/(H \cap gKg^{-1})].
\]

For a family \(\mathcal{D}\) of subgroups of \(G\) with closed under taking conjugation, we put
\[
\Omega(G, \mathcal{D}) := \{ [G/H] \mid (H) \in \mathcal{D}^c \}_\mathbb{Z}.
\]

Definition 2.1. Let \(G\) be a finite group and let \(\mathcal{D}\) be a family of subgroup of \(G\). We call \(\mathcal{D}\) a collection of \(G\) if \(\mathcal{D}\) satisfies the following 3 conditions:

- \(G \in \mathcal{D}\)
- \(H, K \in \mathcal{D} \Rightarrow H \cap K \in \mathcal{D}\)
- \(H \in \mathcal{D}, g \in G \Rightarrow gHg^{-1} \in \mathcal{D}\)

If \(\mathcal{D}\) is a collection of \(G\), then \(\Omega(G, \mathcal{D})\) is a subring of \(\Omega(G)\), hence \(\Omega(G, \mathcal{D})\) is called a partial Burnside ring (PBR for short) relative to \(\mathcal{D}\) of \(G\).

For \(K \leq G\) and a \(G\)-set \(X\), we set
\[
\text{inv}_K(X) := \{ x \in X \mid kx = x \text{ for all } k \in K \}.
\]
If $D$ is a collection of $G$, then for each $K \in D$, the $\mathbb{Z}$-linear map $\varphi_K^G : \Omega(G, D) \to \mathbb{Z}$ which is induced by $[G/H] \mapsto \#\text{inv}_K(G/H)$ is a ring homomorphism. By [10 Theorem 3.10], the map $\varphi^G = (\varphi_K^G : \Omega(G, D) \to \prod_{(K) \in D} \mathbb{Z})$ is an injective ring homomorphism, where $\varphi^G$ is called the Burnside homomorphism. Hence the unit group of the Burnside ring of a finite group is an elementary abelian 2-group.

3. Direct product of PBRs

Let $G_t$ be a finite group, and let $D_t$ be a collection of $G_t$ for $t \in \{1, \cdots, \ell\}$. Then $\prod_{t=1}^\ell D_t$ is a collection of $\prod_{t=1}^\ell G_t$.

**Lemma 3.1.** Let $G_t$ be a finite group, and let $D_t$ be a collection of $G_t$ for $t = 1, 2$. Then

$$\Omega(G_1, D_1) \otimes_{\mathbb{Z}} \Omega(G_2, D_2) \simeq \Omega(G_1 \times G_2, D_1 \times D_2)$$

as rings.

**Proof.** We put $G := G_1 \times G_2$, $D := D_1 \times D_2$. Let $X_t$ be a $(G_t, D_t)$-set for $t = 1, 2$. The correspondence $(X_1, X_2) \mapsto X_1 \times X_2$ induces a bilinear map from $\Omega(G_1, D_1) \times \Omega(G_2, D_2)$ to $\Omega(G_1 \times G_2, D_1 \times D_2)$, hence there exists an injective module homomorphism

$$\pi_2 : \Omega(G_1, D_1) \otimes_{\mathbb{Z}} \Omega(G_2, D_2) \longrightarrow \Omega(G_1 \times G_2, D_1 \times D_2).$$

If $X_t$ is a $(G_t, D_t)$-set of cardinality 1, then $X_1 \times X_2$ is a $(G, D)$-set of cardinality 1. Moreover, for any $(G_t, D_t)$-set $X_1, X_2$, $(G_2, D_2)$-set $X_2, X_2'$, the map

$$(X_1 \times X_1') \times (X_2 \times X_2') \longrightarrow (X_1 \times X_2) \times (X_1' \times X_2')$$

defined by

$$((x_1, x_1'), (x_2, x_2')) \longmapsto ((x_1, x_2), (x_1', x_2'))$$

is an isomorphism of $G$-sets. This shows the map $\pi_2$ is a ring homomorphism. The partial Burnside ring $\Omega(G_t, D_t)$ is a free abelian group with the basis $\{[G_t/H_t] | (H_t) \in D_t^t \}$, hence the $\Omega(G_1, D_1) \otimes_{\mathbb{Z}} \Omega(G_2, D_2)$ is a free abelian group with basis $\{[G_1/H_1] \otimes [G_2/H_2] | (H_t) \in D_t^t, t = 1, 2\}$. Moreover, it is easy to see that $\pi_2([G_t/H_t] \otimes [G_2/H_2]) = ([G_1 \times G_2]/(H_1 \times H_2))$, so the map $\pi_2$ is surjective. Therefore the map $\pi_2$ is a ring isomorphism. \qed

In the rest of this section, let $G_t$ be a finite group, and let $D_t$ be a collection of $G_t$ for $t = 1, \cdots, \ell$.

**Lemma 3.1** shows the following.

**Corollary 3.2.** The map

$$\pi_2 : \bigotimes_{i=1}^{\ell} \Omega(G_i, D_i) \longrightarrow \Omega(\prod_{i=1}^{\ell} G_i, \prod_{i=1}^{\ell} D_i)$$

induced by a bilinear map $\prod_{i=1}^{\ell} \Omega(G_i, D_i) \rightarrow \Omega(\prod_{i=1}^{\ell} G_i, \prod_{i=1}^{\ell} D_i)$ given by $(X_1, \cdots, X_\ell) \rightarrow X_1 \times \cdots \times X_\ell$ is a ring isomorphism.

The following lemma is well known.

**Lemma 3.3.** If the map

$$\rho_2 : \prod_{i=1}^{\ell} \Omega(G_i, D_i) \longrightarrow \bigotimes_{i=1}^{\ell} \Omega(G_i, D_i)$$

is a canonical $\mathbb{Z}$-linear map, then the map $\rho_\ell$ is a surjective ring homomorphism.

For each $i \in \{1, \cdots, \ell\}$ let $G_i$ be a finite group and let $\mathcal{D}_i$ be a collection of $G_i$. Then for each $j \in \{1, \cdots, \ell\}$ it is easy to see that the ring homomorphism

$$f_j : \Omega(G_j, \mathcal{D}_j) \rightarrow \Omega(\prod_{i=1}^{\ell} G_i, \prod_{i=1}^{\ell} \mathcal{D}_i)$$

which is induced by $[G_j/H] \mapsto [\prod_{i=1}^{\ell} G_i/\hat{H}_j]$ is injective, where $\hat{H}_j = G_1 \times \cdots \times G_{j-1} \times H \times G_{j+1} \times \cdots \times G_{\ell}$.

**Lemma 3.4.** If $x_t \in \Omega(G_t, \mathcal{D}_t)$ and $H_t \in \mathcal{D}_t$ for $t = 1, \cdots, \ell$, then

$$\phi^G_{\prod_{i=1}^{\ell}} H_t(f_j(x)) = \phi^G_{H_j}(x_j) \quad (j = 1, \cdots, \ell),$$

where $G = \prod_{i=1}^{\ell} G_i$ and $\hat{H}_j = G_1 \times \cdots \times G_{j-1} \times H_j \times G_{j+1} \times \cdots \times G_{\ell}$.

**Proof.** For $K \in \mathcal{D}_j$,

$$[G/\hat{K}_j] = [\prod_{i=1}^{\ell} G_i/1 \times \cdots \times (G_{j-1}/G_{j-1}) \times (G_j/K) \times (G_{j+1}/G_{j+1}) \times \cdots \times (G_{\ell}/G_{\ell})]$$

in $\Omega(G, \prod_{i=1}^{\ell} \mathcal{D}_i)$. Hence

$$\phi^G_{\prod_{i=1}^{\ell}} H_t(f_j([G/K])) = \left| \text{inv}_{\prod_{i=1}^{\ell}} H_t(G/\hat{K}_j) \right| = \phi^G_{H_j}([G_j/K_j]).$$

Since $\phi^G$ and $f_j$ are ring homomorphisms, this completed the proof of the lemma.

**Lemma 3.5.** For each $t \in \{1, \cdots, \ell\}$ let $G_t$ be a finite group and let $\mathcal{D}_t$ be a collection of $G_t$. Then,

$$\left| \prod_{i=1}^{\ell} \Omega(G_i, \mathcal{D}_i)^\times \right| = 2^{\ell-1} \left| \Omega(\prod_{i=1}^{\ell} G_i, \prod_{i=1}^{\ell} \mathcal{D}_i)^\times \right|.$$

In particular, if $|\Omega(G_i, \mathcal{D}_i)^\times| = 2^{\gamma_i+1}$ and $\Omega(G_i, \mathcal{D}_i)^\times = \langle -1, u_1^{(i)}, \cdots, u_{\gamma_i}^{(i)} \rangle$ for each $i \in \{1, \cdots, \ell\}$, then

$$\Omega(\prod_{i=1}^{\ell} G_i, \prod_{i=1}^{\ell} \mathcal{D}_i)^\times = \left\langle -1, \Omega(G_i)^\times \cup \bigcup_{i=1}^{\ell} \{f_i(u_1^{(i)}, \cdots, u_{\gamma_i}^{(i)})\} \right\rangle.$$

**Proof.** Now, $\bigotimes_{i=1}^{\ell} \Omega(G_i, \mathcal{D}_i)$ is isomorphic to $\Omega(\prod_{i=1}^{\ell} G_i, \prod_{i=1}^{\ell} \mathcal{D}_i)$ by Corollary 3.2. The map

$$\rho_\ell^\times : \prod_{i=1}^{\ell} \Omega(G_i, \mathcal{D}_i)^\times \rightarrow \left( \bigotimes_{i=1}^{\ell} \Omega(G_i, \mathcal{D}_i) \right)^\times,$$
which obtained by $\rho_\ell$ in Lemma 3.3 is a group homomorphism. It suffices to show that $\ker \rho_\ell^x = 2^{\ell-1}$. We put $G := \prod_{i=1}^\ell G_i$, $\tilde{H}_j = G_1 \times \cdots \times G_{j-1} \times H \times G_{j+1} \times \cdots \times G_\ell \leq G$ for $H \in \mathcal{D}_j$. Since if $K_t \in \mathcal{D}_t$ for each $t = 1, \ldots, \ell$, then
\[
\pi_\ell([G_1/K_1] \otimes \cdots \otimes [G_\ell/K_\ell]) = \prod_{i=1}^\ell [G/G_{i-1} \times K_i \times G_{i+1} \times \cdots \times G_\ell],
\]
Lemma 3.4 shows that for any $H, H' \in \mathcal{D}_j$ and $(x_1, \ldots, x_\ell) \in \ker \rho_\ell^x$,
\[
\varphi_{H_j}^G \circ \pi_\ell(x_1 \otimes \cdots \otimes x_\ell)
= \varphi_{H_j}^G \left( \prod_{i=1}^\ell f_\ell(x_i) \right)
= \varphi_{G_1}^G(x_1) \cdots \varphi_{H}^G(x_j) \cdots \varphi_{G_\ell}^G(x_\ell)
= \varphi_{H_j}^G \circ \pi_\ell(1_{\Omega(G_1)} \otimes \cdots \otimes 1_{\Omega(G_\ell)})
= 1,
\]
and
\[
\varphi_{H_j}^G \circ \pi_\ell(x_1 \otimes \cdots \otimes x_\ell)
= \varphi_{H_j}^G \left( \prod_{i=1}^\ell f_\ell(x_i) \right)
= \varphi_{G_1}^G(x_1) \cdots \varphi_{H}^G(x_j) \cdots \varphi_{G_\ell}^G(x_\ell)
= \varphi_{H_j}^G \circ \pi_\ell(1_{\Omega(G_1)} \otimes \cdots \otimes 1_{\Omega(G_\ell)})
= 1,
\]
for $j = 1, \ldots, \ell$. Thus we obtain $\varphi_{H}^G(x_j) = \varphi_{H}^G(x_j)$ and $\varphi_{H}^G(x_j)$ has a value of 1 or $-1$. Furthermore by injectivity of the Burnside homomorphism $\varphi_{G_j}, x_j = 1_{\Omega(G_i)}$ or $-1_{\Omega(G_i)}$ for $j = 1, \ldots, \ell$. Therefore
\[
|\ker \rho_\ell^x| = \sum_{k=0}^{\ell} \binom{\ell}{2k} = 2^{\ell-1}.
\]
Next we assume that $|\Omega(G_i, \mathcal{D}_i)| = 2^{\ell+1}$ and $\Omega(G_i, \mathcal{D}_i) = \{-1_{\Omega(G_i)}, u_1^{(i)}, \ldots, u_r^{(i)}\}$ for each $i \in \{1, \ldots, \ell\}$. We put
\[
L := \{-1_{\Omega(G_i)}\} \cup \bigcup_{i=1}^{\ell} \{f_i(u_1^{(i)}), \ldots, f_i(u_r^{(i)})\}.
\]
Since for $H \in \mathcal{D}_j, K \in \mathcal{D}_k (j < k)$,
\[
[G/H_j] : [G/K_k] = [G/G_1 \times \cdots \times H \times \cdots \times K \times \cdots G_\ell] = [G/H_j],
\]
and $f_j([G/H_j]) = [G/H_j]$, it is easy to see that
\[
f_j(u_k^{(j)}) \notin \langle L \setminus f_j(u_k^{(j)}) \rangle
\]
for any $j, k \in \{1, \ldots, \ell\}$. Therefore
This completes the proof. □

4. PBR of a reducible finite Coxeter group

Definition 4.1. Let $W$ be a finite Coxeter group with Coxeter system $(W, S)$. Then a subgroup $P$ is called parabolic subgroup if there exists $J \subseteq S$ such that $(P) = \langle J \rangle$. The $P_W$ denote the set of all parabolic subgroups of $W$.

Let $W$ be a finite Coxeter group. Then the set $P_W$ of all parabolic subgroups of $W$ becomes a collection (see [8]). So $\Omega(W, P_W)$ is the PBR of $W$ relative to $P_W$. For the PBR of a finite Coxeter group relative to the set of all parabolic subgroups, the following lemma is well known.

Lemma 4.2 ([7]). Let $W$ be a finite Coxeter group with Coxeter system $(W, S)$. Then the element $\varepsilon_W := \sum_{J \subseteq S} (-1)^{|J|} [W/\langle J \rangle]$ lies in $\Omega(W, P_W)^\times$. Moreover, for $P \in P_W$ with $(P) = \langle J \rangle$,

$$\varphi_P(\varepsilon_W) = (-1)^{|J|}.$$ 

We call $\varepsilon_W$ in Lemma 4.2 a sign unit of $\Omega(W, P)$.

Theorem 4.3. Let $W$ be a finite group and let $(W, S)$ be a reducible Coxeter system with $\ell$-irreducible Coxeter system $(W_t, S_t)$ for $t = 1, \ldots, \ell$, namely $W = \prod_{i=1}^{\ell} W_i$ and $S = \bigsqcup_{i=1}^{\ell} S_i$. Then the following holds:

1. $|\Omega(W, P_W)^\times| = \frac{1}{2^r} \prod_{i=1}^{\ell} |\Omega(W_i, P_{W_i})^\times|$, where $P_{W_i}$ is the set of all parabolic subgroups of $W_i$ for $i = 1, \ldots, \ell$.
2. If $\varepsilon_W$ is the sign unit of $\Omega(W, S)$, then $\varepsilon_W = \prod_{i=1}^{\ell} f_i(\varepsilon_{W_i})$.

Proof. Since $P_W = \prod_{i=1}^{\ell} P_{W_i}$, we obtain (1) by Lemma 4.2. Let $J$ be a subset of $S$ with $J = \bigcup_{i=1}^{\ell} J_i$ and $J_i \subseteq S_i$ for each $i$, then $(J) = \langle \bigcup_{i=1}^{\ell} J_i \rangle = \prod_{i=1}^{\ell} \langle J_i \rangle$. 

$$\left| < L > \right| = 2^{1+r_1+\cdots+r_\ell}$$

$$= \frac{1}{2^{r-1}} \left| \prod_{i=1}^{\ell} \Omega(G_i, D_i)^\times \right|$$

$$= \left| \Omega(\prod_{i=1}^{\ell} G_i, \prod_{i=1}^{\ell} D_i)^\times \right|.$$
Therefore for \( P \in \mathcal{P}_W \) if \((P) = ((J))\), then we have
\[
\varphi^W_P \left( \prod_{i=1}^\ell f_i(\varepsilon_{W_i}) \right) = \varphi^W_{\prod_{i=1}^\ell (J_i)} \left( \prod_{i=1}^\ell f_i(\varepsilon_{W_i}) \right)
\]
\[
= \prod_{i=1}^\ell \varphi^W_{(J_i)}(\varepsilon_{W_i})
\]
\[
= (-1)^{\sum_{i=1}^\ell |J_i|}
\]
\[
= (-1)^{|J|}
\]
\[
= \varphi^W_P (\varepsilon_W)
\]
by Lemma 3.4. Therefore we obtain (2) by injectivity of the Burnside homomorphism \( \varphi^W \). □

For a unit group of the PBR relative \( \mathcal{P}_W \) of an irreducible finite Coxeter group, there are the following theorems.

**Theorem 4.4 (2).** Let \( W = W(A) \) is a Coxeter group of type A. Then,
\[
|\Omega(W, \mathcal{P}_W)^\times| = 4.
\]

**Theorem 4.5 (5).** Let \( W \) be an irreducible finite Coxeter group. If the type of \( W \) is \( B, D, E_6, E_7 \) or \( E_8 \), then
\[
|\Omega(W, \mathcal{P}_W)^\times| = 4.
\]

It is easy to see that the following by calculation.

**Theorem 4.6.** Let \( W = W(I_2(m)) \) is a Coxeter group of type \( I_2(m) \). Then,
\[
|\Omega(W, \mathcal{P}_W)^\times| = 4.
\]

Also Theorem 1.3 shows the following.

**Corollary 4.7.** Let \( W \) be a finite group and let \((W, S)\) be a reducible Coxeter system with \( \ell \)-irreducible Coxeter system \((W_t, S_t)\) for \( t = 1, \cdots, \ell \). If \(|\Omega(W_t, \mathcal{P}_{W_t})^\times| = 4\) for any \( t \in \{1, \cdots, \ell\} \), then the following holds:

(i) \(|\Omega(W, \mathcal{P}_W)^\times| = 2^{\ell+1}\).
(ii) \(|\Omega(W, \mathcal{P}_W)^\times| = \langle -1_{\Omega(W)}, f_1(\varepsilon_{W_1}), \cdots, f_\ell(\varepsilon_{W_\ell}) \rangle\), where \( \varepsilon_{W_i} \) is a sign unit of \( \Omega(W_i, \mathcal{P}_{W_i}) \) for each \( i \in \{1, \cdots, \ell\} \).

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