THE GLOBAL ATTRACTOR FOR A CLASS OF EXTENSIBLE BEAMS WITH NONLOCAL WEAK DAMPING

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Abstract. The goal of this paper is to study the long-time behavior of a class of extensible beams equation with the nonlocal weak damping

\[ u_{tt} + \Delta^2 u - m(\|\nabla u\|^2)\Delta u + \|u_t\|^p u_t + f(u) = h, \]

on a bounded smooth domain \( \Omega \subset \mathbb{R}^n \) with hinged (clamped) boundary condition. Under some suitable conditions on the Kirchhoff coefficient \( m(\|\nabla u\|^2) \) and the nonlinear term \( f(u) \), the well-posedness is established by means of the monotone operator theory and the existence of a global attractor is obtained in the subcritical case, where the asymptotic smoothness of the semigroup is verified by the energy reconstruction method.

1. Introduction. In this paper, we study the long-time behavior of solutions to the following extensible beam equation with nonlocal weak damping

\[ u_{tt} + \Delta^2 u - m(\|\nabla u\|^2)\Delta u + \|u_t\|^p u_t + f(u) = h, \]

where \( \Omega \subset \mathbb{R}^n \) is a bounded domain with smooth boundary \( \partial \Omega \), \( u = u(x,t) : \Omega \times [0,\infty) \to \mathbb{R} \) is an unknown function and \( h \in L^2(\Omega) \). The assumption on \( m(s), f(u) \) will be given in Sections 2.

We consider two kinds of boundary conditions, namely, clamped or hinged boundary conditions

\[ u \big|_{\partial \Omega} = \frac{\partial u}{\partial \nu} \big|_{\partial \Omega} = 0 \text{ or } u \big|_{\partial \Omega} = \Delta u \big|_{\partial \Omega} = 0, \]

where \( \nu \) is the unit outward normal on \( \partial \Omega \).

The initial conditions associated with (1.1) are given by

\[ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x). \]

The extensible beam model arises from continuous dynamics. In 1950, Woinowsky-Krieger [36] firstly proposed Eq.(1.1) in the one-dimensional case, which describes the transverse deflection of an extensible beam by taking the damping term is vanished, \( f \equiv 0, h \equiv 0 \) and \( m(s) = m_1 + m_2 s \), where \( m_1, m_2 \) are positive

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constants related to the forces applied on the system. Later, Berger [19] established
the equation
\[ u_{tt} + \Delta^2 u - (Q + \int_\Omega |\nabla u|^2 dx)\Delta u = p(u, u_t, x), \] (1.4)
which was called Berger plate model [9, 10], as a simplification of the von Karman
plate equation which describes large deflection of plate in the two-dimensional case,
where the parameter \( Q \) describes in-plane forces applied to the plate and the func-
tion \( p \) represents transverse loads which may depend on the displacement \( u \) and the
velocity \( u_t \). See also the papers by Eisle [15], Dickey [13] and Ball [3] to obtain
more physical interpretations on the extensible beam model.

Concerning to mathematical analysis for existence, uniqueness, stability and de-
cay properties of global solutions to the extensible beam model, see for instance the
papers by Ball [4], Medeiros [21], Brito [2], Biler [6], Vasconcellos and Teixeira [35]
and Cavalcanti [8] for the pioneering studies.

Global attractor is a basic concept in the study of long-time dynamics of nonlinear
evolution equations with different kinds of dissipation. For the research on the global
attractor of the extensible beam model, a series of important results have been
obtained in these years. The earliest study of the long-time behaviors of solutions
to (1.1) with the linear weak damping \( u_t, m(s) = \beta + s, f = 0 \) and \( h = 0 \) in the
one-dimensional case was Hale [16], and then later, Eden and Milani [14] established
the existence of global and exponential attractor of Eq.(1.1) with a linear function
\( m(s) \) and \( f(u) = 0 \). With respect to the damping term \( (-\Delta)^\theta u_t, 0 < \theta \leq 1 \), we can
refer to Biazutti and Crippa [1], Chueshov and Kolbasin [12], Coti Zelati[11], Jorge
Silva and Narciso [28] etc.. In regard to the nonlinear damping \( g(u_t) \), many people
have also extensively studied, see e.g. Koumou Patcheu [24], Ma and Narciso [20],
Cavalcanti [8], Yang [39], Yang and Wang [37], Meng and Wu et al. [22] etc.,
moreover, it is worth mentioning that Chueshov and Lasiecka [10] have considered
systematically the long-time behaviour of the second-order abstract equation
\[
\begin{align*}
&Mu_{tt}(t) + Au(t) + kD(u_t(t)) = F(u(t), u_t(t)), \\
&u_{|t=0} = u_0 \in D(A^{1/2}), \quad u_t_{|t=0} = u_1 \in D(M^{1/2}),
\end{align*}
\] (1.5)
where \( A \) and \( M \) be linear positive, selfadjoint operators densely defined on a Hilbert
space \( H \), \( k \) is a positive parameter, and operators \( D \) and \( F \) satisfy suitable hypothe-
ses formulated in Assumption 1.1 in [10]. The focus of the monograph [10] is on
existence and properties of global attractors, finite dimensionality of attractors,
exponential attractors and determining functionals are of particular interest. The
emphasis is paid to flows generated by equations with nonlinear dissipation \( D \)
and a non-compact nonlinear term \( F \).

Based on the above series of results, our aim is to investigate the long-time
dynamics of the model (1.1) with the following nonlocal weak damping
\[ \|u_t\|^p u_t, \quad p \geq 0, \] (1.6)
which is inspired by the nonlocal damping term
\[ N\left(\int_\Omega |\nabla u|^2 dx\right)(-\Delta)^\theta u_t, \] (1.7)
where \( N > 0 \) is a nonlinear function and \( 0 \leq \theta \leq 1 \). For the nonlocal damping (1.7)
which arises as a type of nonlocal Kirchhoff damping , it was firstly used in Lange
and Perla Menzala [17] as a type of nonlocal beam equations in \( \Omega = \mathbb{R}^n \), with \( \theta = 0 \).
and \( m \equiv 0 \). Chueshov and Kolbasin [11] studied long-time dynamics for a class of abstract equations, as application they considered a Berger plate model in two dimensions (\( m \) is linear) with damping which covers the range \( 0 < \theta \leq 1 \). Jorge Silva and Narciso [28] studied well-posedness and asymptotic behavior to extensible beam equations with damping which covers the range \( 0 \leq \theta \leq 1 \). We also refer to Jorge Silva and Narciso [27, 29] where the authors studied long-time dynamics for a class of extensible beam and plate models with other forms of nonlocal damping.

To the best of our knowledge, few people have studied the damping (1.6). And we will find an interesting phenomenon: the existence of a global attractor of problem (1.1) is independent of \( p \).

Our main objective in this paper is to study the long-time dynamics of problem (1.1)-(1.3). In our opinion, the main features of the present paper are the following:

- The well-posedness of Eq.(1.1) is established by the monotone operator theory with locally Lipschitz perturbation(cf. Chueshov and Lasiecka [10], Showalter [32]). Unlike many other literatures, the reason why the the standard Fatou-Galerkin method is not used to prove the well-posedness is that when estimating the energy boundedness, we can only get the boundedness of \( u_t \) in the \( L^2(\Omega) \)-norm, and then get the weak convergence of \( u_t \) in the \( L^2(\Omega) \)-norm, which can not get that the nonlocal coefficients \( \|u_t\|^p \) converges to the same limit;

- Thanks to Simon [30] and Perai [26], where the monotone inequality of p-Laplacian operator in n-dimension case was given, we will extend the monotone inequality to Hilbert space (Lemma 2.2) to obtain the monotonicity of the damping term, which is useful to prove the well-posedness, dissipativity and the asymptotic smoothness to problem (1.1);

- We will obtain the asymptotic smoothness by taking advantage of the energy reconstruction method given by Chueshov and Lasiecka [10], which was used to prove the existence of global attractors in (1.5) with nonlinear dissipation \( g(u_t) \) and a subcritical nonlinear term \( F \). The main reason is that when the velocity \( u_t \) is very small, the nonlocal damping \( \|u_t\|^pu_t \) is weaker than the linear damping \( u_t \), it is more difficult to obtain the asymptotic smoothness by utilizing the decomposition of semigroup or contractive functions method than in the case of linear damping \( u_t \) (see [34, 33, 38, 23] ).

The outline of our paper is given below. In the next section, we recall several notations, present the assumptions and obtain the well-posedness result, showing that the Eq.(1.1) generates a dynamical system \((\mathcal{H}, S_t)\) in the space \( \mathcal{H} = V \times H \). Dissipativity is given in Section 3. Section 4 is dedicated to the proof of results on asymptotically smoothness and existence of global attractor for problem (1.1)-(1.3).

2. Preliminaries and well-posedness.

2.1. Some notations. We consider \( H = L^2(\Omega) \) with usual inner product \( (\cdot, \cdot) \) and norm \( \|\cdot\| \), and \( L^p(\Omega) \) with norm \( \|\cdot\|_p \). We also consider the space \( V_0 = H^1_0(\Omega) \) and

\[
V = \mathcal{D}(A^{\frac{1}{2}}) = \begin{cases} H^2_0(\Omega), \text{ for clamped boundary condition}, \\ H^2(\Omega) \cap H^1_0(\Omega), \text{ for hinged boundary condition}. \end{cases}
\]

with norm \( \|\nabla \cdot\| \) and \( \|\Delta \cdot\| \) respectively, where the operator \( \mathcal{A} = \Delta^2 \).

Let \( \lambda_1 > 0 \) be the first eigenvalue of the bi-harmonic operator \( \Delta^2 \) with boundary condition (1.2), then it holds

\[
\|\Delta u\|^2 \geq \lambda_1 \|u\|^2, \|\Delta u\|^2 \geq \lambda_1^2 \|\nabla u\|^2, \forall u \in V.
\]
Finally, we define the space

\[ \mathcal{H} = V \times H \]

endowed with norm

\[ \|(u, v)\|_{\mathcal{H}}^2 = \|\Delta u\|^2 + \|v\|^2. \]

Let \( C \) denote any positive constant which may be different from line to line and even in the same line. We also denote the different positive constants such as by \( c_i, C_i, i \in \mathbb{N} \) et al.

### 2.2. Assumptions.

Now, we introduce the assumptions on the functions \( m \) and \( f \) as follows:

(\( A_1 \)) \( m : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is a function of class \( C^1 \), satisfying

\[ m(s)s \geq \frac{1}{2}M(s) - \theta s, \text{ where } M(s) = \int_0^s m(\tau)d\tau, \quad (2.2) \]

where \( 0 \leq \theta \leq \frac{1}{2}\lambda_1^2 \);

(\( A_2 \)) \( f \in C^1(\mathbb{R}) \) satisfies the growth condition

\[ |f'(s)| \leq C(1 + |s|^{\varrho}), \quad (2.3) \]

with \( 1 \leq \varrho < \infty \) if \( n \leq 4 \) and \( 1 \leq \varrho < \frac{4}{n-4} \) if \( n \geq 5 \), and the dissipation condition

\[ \liminf_{|s| \to \infty} f'(s) > -\lambda_1. \quad (2.4) \]

Condition (2.3) and Mean Value Theorem imply that there exists some \( k_0 \) such that

\[ |f(u) - f(v)| \leq k_0(1 + |u|^{\varrho} + |v|^{\varrho})|u - v|, \quad \forall u, v \in \mathbb{R}. \quad (2.5) \]

Set

\[ F(s) = \int_0^s f(\tau)d\tau. \]

Assumption (2.4) yields that

\[ \int_\Omega F(u)dx \geq -\frac{\lambda}{2}\|u\|^2 - C, \quad (2.6) \]

\[ (f(u), u) \geq \int_\Omega F(u)dx - \frac{\lambda}{2}\|u\|^2 - C, \quad (2.7) \]

for some \( \lambda < \lambda_1 \) (see [25]).

(\( A_3 \)) \( \theta \) and \( \lambda \) are chosen so that

\[ 1 - \frac{\lambda}{\lambda_1} - \frac{2\theta}{\sqrt{\lambda_1}} > 0. \]

### 2.3. Well-posedness.

**Definition 2.1.** ([10]) A function \( u(t) \in C([0, T]; V) \cap C([0, T]; H) \) possessing the properties \( u(0) = u_0 \) and \( u_t(0) = u_1 \) is said to be

(\( S \)) a strong solution to problem (1.1) on the interval \([0, T]\), iff

(i) \( u \in W^{1,1}(a, b; V) \) and \( u_t \in W^{1,1}(a, b; H) \) for any \( 0 < a < b < T \);

(ii) \( Au(t) + Du_t(t) \in H' \) for almost all \( t \in [0, T] \);

(iii) Eq.(1.1) is satisfied in \( H' \) for almost all \( t \in [0, T] \).
By homogeneity we can assume that it holds for every $\psi \in V$ and for almost all $t \in [0, T]$.

Theorem 1. ([10]) Let the hypotheses (A$_1$) and (A$_2$) hold. Assume additionally that the damping operator $D$ maps $H$ into $H'$ and is a monotone hemicontinuous operator which is bounded on bounded sets, i.e.

Then every generalized solution is also weak, i.e., the relation

\[
(u(t), \psi) = (u_1, \psi) - \int_0^t ((Au(\tau), \psi) - (Du(\tau), \psi)) + ((h - f(u) + m(\int_\Omega |\nabla u|^2 dx)\Delta u, u_1, \psi) d\tau
\]

holds for every $\psi \in V$ and for almost all $t \in [0, T]$.

Before giving the well-posedness, we will show a monotone inequality of p-Laplacian operator in Hilbert space, which will be used later.

Lemma 2.2. Let $u, v \in H, H$ is a Hilbert space with inner product $(\cdot, \cdot)$ and norm $\|\cdot\|_H$. Then there exists some constant $C_{\gamma}$ which depends on $\gamma$ such that

\[
\left(\|u\|_H^{\gamma - 2}u - \|v\|_H^{\gamma - 2}v, u - v\right) \geq \begin{cases} 
C_{\gamma}\|u - v\|_H^{\gamma}, & \text{if } \gamma \geq 2, \\
C_{\gamma}\frac{\|u - v\|_H^{\gamma}}{\|u\|_H + \|v\|_H}^{\gamma}, & \text{if } 1 \leq \gamma \leq 2.
\end{cases}
\]  

Proof. By homogeneity we can assume that $\|u\|_H = 1$ and $\|v\|_H \leq 1$.

In case of $\gamma \geq 2$, the inequality (2.11) is equivalent to prove that

\[
1 - \left(\frac{\|u\|_H}{\|u\|_H}\right)^{\gamma - 1}\frac{(u,v)}{\|u\|_H\|v\|_H} - \frac{\|v\|_H}{\|u\|_H\|v\|_H} + \left(\frac{\|v\|_H}{\|u\|_H}\right)^{\gamma} \geq C_{\gamma}\left(1 - \frac{2\|v\|_H}{\|u\|_H}\frac{(u,v)}{\|u\|_H\|v\|_H} + \left(\frac{\|v\|_H}{\|u\|_H}\right)^{2}\right).
\]  

Denoting $t = \|v\|_H \in [0, 1]$ and $s = \frac{(u,v)}{\|u\|_H\|v\|_H} \in [-1, 1]$, we rewrite (2.12) as follows,

\[
1 - (t^\gamma - 1)s + t^\gamma \geq C_{\gamma}(1 - 2ts + t^2)\frac{2}{2}.
\]  

Note the fact that

\[
1 - 2ts + t^2 = (t - s)^2 + 1 - s^2 = 0 \iff s = t = 1,
\]

and when $s = t = 1$, (2.13) holds, therefore, to prove (2.13), it is enough to show the following inequality

\[
f(t, s) = \frac{1 - (t^\gamma - 1)s + t^\gamma}{(1 - 2ts + t^2)\frac{2}{2}} \geq C_{\gamma} > 0, \ t \in [0, 1], \ s \in [-1, 1].
\]  

By the direct calculation, we can show that for fixed $t$,

\[
\frac{\partial f}{\partial s} = \frac{t[(1 + t^\gamma)(\gamma - t^\gamma + 1) - (\gamma - 2)(t^\gamma - 1 + t)s]}{(1 - 2st + t^2)\frac{2}{2} + 1}.
\]

It is clear that

\[
t[(1 + t^\gamma)(\gamma - t^\gamma + 1) - (\gamma - 2)(t^\gamma - 1 + t)s] = 0 \iff \frac{\partial f}{\partial s} = 0.
\]
When $t = 0$, $f(0, s) \equiv 1$, (2.14) holds.
When $t \neq 0$, in case of $\gamma = 2$, $f(t, s) \equiv 1$, (2.14) holds.
In case of $\gamma > 2$, we have

$$s_0 := s_0(t) = \frac{(1 + t^\gamma)\gamma - (1 + t^2)(t^{\gamma - 2} + 1)}{(\gamma - 2)(t^{\gamma - 1} + t)},$$

and

$$\frac{\partial s_0(t)}{\partial t} = \frac{(\gamma^2 - \gamma - 2)(t^{\gamma + 2} - t^\gamma) + (1 - \gamma)(t^2 - t^{2\gamma}) - t^4}{(\gamma - 2)(t^4 + t^{\gamma + 2} + t^{2\gamma})} < 0, \forall \ t \in (0, 1],$$

then

$$s_0(t) \geq s_0(0) = 1, \forall \ t \in (0, 1].$$

Hence, we can conclude that

$$\text{if } s \geq s_0, \text{then } \frac{\partial f}{\partial s}|_{s = s_0} \leq 0; \text{ and if } s \leq s_0, \text{then } \frac{\partial f}{\partial s}|_{s = s_0} \geq 0.$$ 

Therefore, $f(t, s)$ is an increasing function with respect to $s \in [-1, 1)$ and

$$\min_{s \in [-1, 1)} f(t, s) = f(t, -1).$$

Moreover, we have

$$\frac{df(t, -1)}{dt} = \frac{(\gamma - 1)(t + 1)(t^{\gamma - 2} - 1)}{(1 + t)^{\gamma + 1}} < 0,$$

then

$$\min_{t \in [0, 1]} f(t, -1) = f(1, -1) = \frac{1}{2^{\gamma - 2}}.$$ 

Therefore,

$$f(t, s) \geq \min\{\frac{1}{2^{\gamma - 2}}, 1\} = \frac{1}{2^{\gamma - 2}} = C_\gamma, \ \gamma \geq 2.$$ 

And similarly, we can prove the case $1 \leq \gamma \leq 2$.  

**Remark 1.** The Lemma 2.2 generalizes the result that the monotone inequality of $p$-Laplacian operator in n-dimension case in [26, 30] to Hilbert space.

**Remark 2.** Denote $D(u_t) = \|u_t\|^pu_t$ and by Lemma 2.2, we obtain

$$(D(u_t) - D(v_t), u_t - v_t)$$

$$= (\|u_t\|^p u_t - \|v_t\|^p v_t, u_t - v_t)$$

$$\geq C_p\|u_t - v_t\|^{p + 2}, \ p \geq 0, \ u_t, v_t \in L^2(\Omega),$$

i.e. the damping operator $D$ is strong monotone, which is an important property for proving well-posedness, dissipativity and the asymptotic smoothness to problem (1.1). Moreover, the damping operator $D$ satisfies the Assumption 1.1 in [10].

The well-posedness of problem (1.1) is given by the following result.

**Theorem 2.3.** Let $T > 0$ be arbitrary. Under the assumption $(A_1)$ and $(A_2)$ the following statements hold.
(i) For every \((u_0, u_1) \in V \times V\), such that \(Au_0 + Du_1 \in L^2(\Omega)\) there exists a unique strong solution to problem (1.1) on the interval \([0, T]\) such that
\[
\begin{align*}
(u_t, u_{tt}) & \in L^\infty(0, T; V \times H), \\
\|u_t\| & \in C_r([0, T]; V), \\
u_{tt} & \in C_r([0, T]; H) \text{ and } Au(t) + Du_1(t) \in C_r([0, T]; H'),
\end{align*}
\] (2.16)
where we denote \(C_r\) the space of right continuous functions. This solution satisfies the energy relation
\[
\mathcal{E}(t) + \int_0^t (\|u_t\|^p u_t, u_t) d\tau = \mathcal{E}(0).
\] (2.17)
where
\[
\mathcal{E}(t) = \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\Delta u(t)\|^2 + \frac{1}{2}M(\|\nabla u\|^2) + \int_{\Omega} F(u) dx - \int_{\Omega} hudx,
\]
\[
E_0(t) = \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\Delta u(t)\|^2.
\]
(ii) For any initial data \((u_0, u_1) \in \mathcal{H} = V \times H\) there exists a unique generalized solution such that
\[
(u, u_t) \in C([0, T]; \mathcal{H}).
\]
In particular, the generalized solution is also weak.

Proof. By Remark 2, we know the damping operator \(D\) possesses strong monotonicity, i.e.
\[
\langle D(u_t) - D(u_v), u_t - u_v \rangle = (\|u_t\|^p u_t - \|u_v\|^p u_v, u_t - u_v) \\
\geq C_p\|u_t - u_v\|^{p+2}, \quad p \geq 0, \quad u_t, u_v \in L^2(\Omega),
\]
and it's easy to verify that \(D\) satisfies the Assumption 1.1 in [10]. Therefore, combining with the assumption (A_1) and (A_2), similar to the proof Theorem 1.5 in [10] which uses the rather standard application of the monotone operator theory with locally Lipschitz perturbations (see e.g. [7, 32]), we obtain the global existence (and uniqueness) of strong and generalized solutions.

Furthermore, thanks to the damping term \(D(u_t) = \|u_t\|^p u_t\), we know the operator \(D\) maps \(L^2(\Omega)\) into \(L^2(\Omega)\) and satisfies
\[
\|D(u_t)\| = \|u_t\|^{p+1} \leq C_p, \quad \text{for all } u_t \in L^2(\Omega), \|u_t\| \leq \rho, \rho > 0,
\] (2.18)
i.e. the monotone hemicontinuous operator \(D\) is bounded on bounded sets. It follows from Proposition 1, we conclude that the generalized solution is also a weak solution.

Remark 3. The reason why we consider using monotone operator theory to obtain the well-posedness to problem (1.1) is that when estimating energy, we can only get the boundedness of \(u_t\) in the \(L^2(\Omega)\)-norm, and then get the weak convergence of \(u_t\) in the \(L^2(\Omega)\)-norm, but not get that the nonlocal coefficients \(\|u_t\|^p\) converges to the same limit, which lead to makes it seem difficult to take advantage of the Fatou Galerkin standard.

In view of Theorem 2.3, problem (1.1) generates a dynamical system \((\mathcal{H}, S_t)\) in the space \(\mathcal{H} = V \times H\). The corresponding evolution operator \(S_t\) is given by the formula
\[
S_t(u_0; u_1) = (u(t); u_t(t)),
\]
where \(u(t)\) solves (1.1) with the initial data \((u_0; u_1)\).
3. Dissipativity. In this section, we are concerned with the dissipative properties of the semigroup \( \{S_t\}_{t \geq 0} \) corresponding to \( (1.1) \).

**Theorem 3.1.** Under the assumptions \((A_1)\) and \((A_2)\), the system \((\mathcal{H}, S_t)\) generated by \((1.1)\) in the space \( \mathcal{H} = V \times H \) is dissipative, i.e. there exists \( R > 0 \) possessing the property: for any bounded set \( B \subset \mathcal{H} \) there exists \( t_0 = t(B) \) such that \( \|S_t y\|_\mathcal{H} = \|(u(t), u_t(t))\|_\mathcal{H} \leq R \) for all \( y \in B \) and \( t \geq t_0 \). In particular, the set

\[
B_0 = \{(u, v) \in \mathcal{H}; \| (u, v) \|_\mathcal{H} \leq R \}
\]

is a bounded absorbing set for the system \((\mathcal{H}, S_t)\).

**Proof.** Multiplying \((1.1)\) by \( u_t \), and integrating over \( \Omega \), we obtain

\[
\frac{d}{dt} \left( \frac{1}{2} \| u_t(t) \|^2 + \frac{1}{2} \| \Delta u(t) \|^2 + \frac{1}{2} M(\| \nabla u \|^2) + \int_\Omega F(u) dx - \int_\Omega h u dx \right) + (\| u_t \|^p u_t, u_t) = 0.
\]

Denote

\[
E_0(t) = \frac{1}{2} \| u_t(t) \|^2 + \frac{1}{2} \| \Delta u(t) \|^2,
\]

and

\[
\mathcal{E}(t) = \frac{1}{2} \| u_t(t) \|^2 + \frac{1}{2} \| \Delta u(t) \|^2 + \frac{1}{2} M(\| \nabla u \|^2) + \int_\Omega F(u) dx - \int_\Omega h u dx.
\]

From \((2.6)\), and combining with \((2.1)\), we can infer that

\[
\int_\Omega F(u) dx \geq -\frac{\lambda}{2} \| u \|^2 - C \geq -\frac{\lambda}{2 \lambda_1} \| \Delta u \|^2 - C.
\]

By Hölder’s inequality, Young’s inequality and \((2.1)\), one has

\[
\left| \int_\Omega h u dx \right| \leq \frac{1}{4 \lambda_1} \| h \|^2 + \beta \lambda_1 \| u \|^2 \leq \frac{1}{4 \lambda_1} \| h \|^2 + \beta \| \Delta u \|^2.
\]

By using the assumption \((A_1)\) and combining with \((3.2)\) and \((3.3)\) with \( \beta = \frac{1}{2} (\frac{1}{2} - \frac{1}{2 \lambda_1}) \), we get

\[
\mathcal{E}(t) \geq c_0 E_0(t) - C_0,
\]

where \( 0 < c_0 < 1 \).

Multiplying \((1.1)\) by \( \alpha u \) and integrating over \( \Omega \), we infer that

\[
\frac{d}{dt} (u_t, \alpha u) - (u_t, \alpha u_t) + \alpha \| u_t \|^2 + \alpha m(\| \nabla u \|^2) \| \nabla u \|^2
\]

\[
+ \alpha (f(u), u) - \alpha (h, u) + (\| u_t \|^p u_t, \alpha u) = 0,
\]

by combining \((3.1)\) with \((3.5)\), implies that

\[
\frac{d}{dt} (\mathcal{E}(t) + (u_t, \alpha u) + (\| u_t \|^p u_t, u_t) + (\| u_t \|^p u_t, \alpha u) - \alpha \| u_t \|^2
\]

\[
+ \alpha \| \Delta u \|^2 + \alpha m(\| \nabla u \|^2) \| \nabla u \|^2 + \alpha (f(u), u) - \alpha (h, u) = 0.
\]

Denote

\[
V(t) = \mathcal{E}(t) + \alpha(u_t, u).
\]

By Hölder’s inequality and Young’s inequality, we have

\[
\alpha \| (u_t, u) \| \leq \alpha \| u_t \| \| u \| \leq \frac{c_0}{4} \| u_t \|^2 + \frac{\alpha^2}{c_0 \lambda_1} \| \Delta u \|^2.
\]

Combining \((3.4)\) with \((3.7)\), there exists \( \alpha_0 > 0 \) such that

\[
V(t) \geq c_1 E_0(t) - C_1, \text{ for all } 0 < \alpha < \alpha_0,
\]

where \( c_1 \) and \( C_1 \) are positive constants. This completes the proof.
where $0 < c_1 < 1$, $C_1$ is some positive constant.

Now, we rewrite (3.6) as follows
\[
\frac{d}{dt} V(t) + \alpha V(t) + \Gamma = 0, \quad (3.9)
\]
where
\[
\Gamma = (\|u_t\|^p u_t, u_t) + \left( \|u_t\|^p u_t, \alpha u \right) - \frac{3\alpha}{2} \|u_t\|^2 + \frac{\alpha}{2} \|\Delta u\|^2 \\
+ \alpha m(\|\nabla u\|^2) \|\nabla u\|^2 - \frac{\alpha}{2} M(\|\nabla u\|^2) \\
+ \alpha (f(u), u) - \alpha \int_{\Omega} F(u) dx - \alpha^2 (u_t, u).
\]

By (2.1) and (2.2), we can get that
\[
\alpha m(\|\nabla u\|^2) \|\nabla u\|^2 - \frac{\alpha}{2} M(\|\nabla u\|^2) \geq -\alpha \theta \|\nabla u\|^2 \geq -\alpha \theta \sqrt{\lambda_1} \|\Delta u\|^2. \quad (3.10)
\]

From (2.7) and combining with (2.1), it is easy to see that
\[
\alpha (f(u), u) - \alpha \int_{\Omega} F(u) dx \geq -\frac{\alpha \lambda_2}{2} \|u\|^2 - \alpha C \geq -\frac{\alpha \lambda_2}{2} \|\Delta u\|^2 - \alpha C. \quad (3.11)
\]

By Young’s inequality, we infer that there exist constants $c_2, c_3 > 0$ such that
\[
(u_t, u_t) = \|u_t\|^2 \leq c_2 + c_3 \|u_t\|^p + 2. \quad (3.12)
\]

By Cauchy inequality, Young’s inequality and (2.1), we infer
\[
\left| \left( \|u_t\|^p u_t, \alpha u \right) \right| = |\alpha \|u_t\|^p (u_t, u)| \\
\leq \alpha \|u_t\|^p \left( \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|u\|^2 \right) \\
\leq \frac{\alpha}{2} \|u_t\|^p + \frac{\alpha}{2} \|u_t\|^p \|u\|^2 \\
\leq \frac{\alpha}{2} \|u_t\|^p + \frac{\alpha}{2} (C_\delta \|u_t\|^p + \delta) \|u\|^2 \\
\leq \frac{\alpha}{2} \|u_t\|^p + \frac{\alpha C_\delta}{2 \lambda_1} \|\Delta u\|^2 \cdot \|u_t\|^p + \frac{\alpha \delta}{2 \lambda_1} \|\Delta u\|^2 \\
\leq \frac{\alpha}{2} \|u_t\|^p + \frac{\alpha C_\delta}{2 \lambda_1} E_0(t) \cdot \|u_t\|^p + \frac{\alpha \delta}{\lambda_1} E_0(t). \quad (3.13)
\]

It follows from (3.4) and energy relation (2.17) that,
\[
E_0(t) \leq C(1 + E(t)) \leq C(1 + E(0)) \leq C_B.
\]

Therefore, we have
\[
\left| \left( \|u_t\|^p u_t, \alpha u \right) \right| \leq \frac{\alpha}{2} \|u_t\|^p + \frac{\alpha C_B C_\delta}{2 \lambda_1} \cdot \|u_t\|^p + \alpha C_2. \quad (3.14)
\]

In view of (3.7) and (3.10)-(3.14), it follows that
exists a compact set asymptotically smooth iff for any bounded set $D$

**Definition 4.1.** \((10)\) Let \((X, S_t)\) be a dynamical system. \((X, S_t)\) is said to be asymptotically smooth iff for any bounded set $D$ such that $S_tD \subset D$ for $t > 0$ there exists a compact set $K$ in the closure $D$ of $D$, such that

$$\lim_{t \to +\infty} d_X \{S_tD|K\} = 0.$$ 

Here $d_X \{A|B\} = \sup_{x \in A} \text{dist}_X(x, B)$ is the Hausdorff semidistance.

From the above proof, we can see that dissipativity of the semigroup

**Remark 4.** From the above proof, we can see that dissipativity of the semigroup is independent of $p$ and the growth exponent of $f(u)$.

### 4. The global attractor.

#### 4.1. Some abstract results. For the reader’s convenience, we recall here some abstract results on the global attractors.

**Definition 4.1.** \((10)\) Let $(X, S_t)$ be a dynamical system. $(X, S_t)$ is said to be asymptotically smooth iff for any bounded set $D$ such that $S_tD \subset D$ for $t > 0$ there exists a compact set $K$ in the closure $D$ of $D$, such that

$$\lim_{t \to +\infty} d_X \{S_tD|K\} = 0.$$ 

Here $d_X \{A|B\} = \sup_{x \in A} \text{dist}_X(x, B)$ is the Hausdorff semidistance.
Definition 4.2. ([10]) A bounded closed set $A \subset X$ is said to be a global attractor of the dynamical system $(X, S_t)$ iff

(i) $A$ is an invariant set, i.e. $S_tA = A$ for $t \geq 0$;

(ii) $A$ is uniformly attracting, i.e. for all bounded set $D \subset X$

$$\lim_{t \to +\infty} d_X\{S_tD|A\} = 0.$$ 

Theorem 4.3. ([10]) Let $(X, S_t)$ be a dynamical system on a complete metric space $X$ endowed with a metric $d$. Assume that for any bounded positively invariant set $B$ in $X$ there exist $T > 0$, a continuous non-decreasing function $q : \mathbb{R}^+ \to \mathbb{R}^+$ and a pseudometric $\varrho_B$ on the $C(0, T; X)$ such that

(i) $q(0) = 0$ and $q(s) < s$ for $s > 0$;

(ii) the pseudometric $\varrho_B$ is precompact (with respect to $X$) in the following sense: any sequence $\{x_n\} \subset B$ has a subsequence $\{x_{n_k}\}$ such that the sequence $\{y_k\} \subset C(0, T; X)$ of elements $y_k(\tau) = S_{\tau}x_{n_k}$ is Cauchy with respect to $\varrho_B$;

(iii) the following inequality holds

$$d(S_{\tau}y_1, S_{\tau}y_2) \leq q\left(d(y_1, y_2) + \varrho_B^*(\{S_{\tau}y_1\}, \{S_{\tau}y_2\})\right),$$

for every $y_1, y_2 \in B$, where we denote by $\{S_{\tau}y_k\}$ the element in the space $C(0, T; X)$ given by function $y_i(\tau) = S_{\tau}y_i$.

Then $(X, S_t)$ is an asymptotically smooth dynamical system.

Theorem 4.4. ([10]) Let $(X, S_t)$ be a dissipative dynamical system in a complete metric space $X$. Then $(X, S_t)$ possesses a compact global attractor $A$ if and only if $(X, S_t)$ is asymptotically smooth.

4.2. A-Priori estimates. Some a-priori estimates are established which will be used to obtain the asymptotic smoothness of the semigroup.

Lemma 4.5. Under the assumptions $(A_1)$ and $(A_2)$, there exist $T_0 > 0$ and a constant $C > 0$ independent of $T$ such that for any pair $w$ and $v$ of strong solutions to (1.1) we have the following relation

$$TE_m(T) + \int_0^T E_m(t)dt \leq C(R)\left\{\int_0^T \|z_t(t)\|^2dt + \int_0^T (D(t, z_t), z_t)dt + \int_0^T |(D(t, z_t), z)|dt + \int_0^T \|\nabla z\|^2dt + \int_0^T dt \int_0^T \|\nabla z(\tau)\|^2d\tau + \int_0^T dt \int_t^T \|\nabla z(\tau)\| \cdot \|z_t(\tau)\|d\tau + \int_0^T (f(w(t)) - f(v(t)), z(t))dt \right\}$$

for every $T \geq T_0$, where $z(t) = w(t) - v(t)$, $((w_0, w_1), (v_0, v_1)) \in V \times V$ and we use the following notations:

$$E_m(t) = \frac{1}{2}(\|z_t(t)\|^2 + \|\Delta z(t)\|^2 + m(\|\nabla w\|^2)\|\nabla z(t)\|^2),$$

$$D(t, z_t) = \|w_t\|^p w_t - \|v_t\|^p v_t.$$
Proof. We start by noting the following equivalence
\[
E_m(t) \sim E_z(t) = \frac{1}{2}(\|z_t(t)\|^2 + \|\Delta z(t)\|^2) = \frac{1}{2}(\|z(t), z_t(t)\|_H^2).
\] (4.3)

Indeed, according to Theorem 3.1 and \(m \in C^1(\mathbb{R}^+)\), there exists a constant \(C (R, \|\nabla w_0\|)\) such that
\[
m(\|\nabla w\|^2)\|\nabla z\|^2 \leq C(R, \|\nabla w_0\|)\|\nabla z\|^2,
\]
moreover, by the interpolation inequality, we have
\[
\|\nabla z\|^2 \leq \frac{1}{2}\|\Delta z\|^2 + C\|z\|^2.
\]
Therefore, we obtain
\[E_z(t) \leq E_m(t) \leq C(R, \|\nabla w_0\|)E_z(t).\]

Note that \(z(t) = w(t) - v(t)\) satisfies the following equality
\[
z_{tt} + \Delta^2 z - m(\|\nabla w\|^2)\Delta z - (m(\|\nabla w\|^2) - m(\|\nabla v\|^2))\Delta v
+ D(t, z_t) + f(w) - f(v) = 0.
\] (4.4)

Multiplying (4.4) by \(z_t(t)\), and integrating over \(\Omega\), we obtain
\[
(z_{tt}, z_t) + (\Delta^2 z, z_t) - (m(\|\nabla w\|^2)\Delta z, z_t) + (D(t, z_t), z_t)
= ((m(\|\nabla w\|^2) - m(\|\nabla v\|^2))\Delta v, z_t) - (f(w) - f(v), z_t).
\] (4.5)

In view of
\[
m(\|\nabla w\|^2)(\Delta z, z_t) = -\frac{1}{2} \frac{d}{dt} [m(\|\nabla w\|^2)\|\nabla z\|^2 - m'(\|\nabla w\|^2)\|\nabla z\|^2(\Delta w, w_t)],
\]
we rewrite (4.5) as
\[
\frac{1}{2} \frac{d}{dt} \left(\|z_t(t)\|^2 + \|\Delta z(t)\|^2 + m(\|\nabla w\|^2)\|\nabla z\|^2\right) + (D(t, z_t), z_t)
= -m'(\|\nabla w\|^2)\|\nabla z\|^2(\Delta w, w_t) + ((m(\|\nabla w\|^2) - m(\|\nabla v\|^2))\Delta v, z_t)
\] (4.6)
\[- (f(w) - f(v), z_t),
\]
from which, by integrating over \([t, T]\), we obtain
\[
E_m(T) + \int_t^T (D(t, z_t), z_t)d\tau = E_m(t) - \int_t^T m'(\|\nabla w\|^2)\|\nabla z\|^2(\Delta w, w_t)d\tau
+ \int_t^T (m(\|\nabla w\|^2) - m(\|\nabla v\|^2))\Delta v, z_t)d\tau
\] (4.7)
\[- \int_t^T (f(w) - f(v), z_t)d\tau.
\]

Multiplying (4.4) by \(z(t)\), and integrating over \(\Omega\), we obtain
\[
\frac{d}{dt}(z_t, z) - \|z_t\|^2 + \|\Delta z\|^2 + m(\|\nabla w\|^2)\|\nabla z\|^2 + (D(t, z_t), z)
= ((m(\|\nabla w\|^2) - m(\|\nabla v\|^2))\Delta v, z) - (f(w) - f(v), z),\] (4.8)
which implies

\[
2 \int_0^T E_m(t) dt - 2 \int_0^T \| z_t \|^2 dt + \int_0^T (D(t, z_t), z) dt + (z_t, z)_{\text{H},0}^T \\
= \int_0^T ((m(\| \nabla w \|^2) - m(\| \nabla v \|^2)) \Delta v, z) dt - \int_0^T (f(w) - f(v), z) dt.
\] (4.9)

By using the fact that \( V \) is continuously embedded into \( H \) and combining with (2.1), we have

\[
|(z_t, z)| \leq \|z_t\| \|z\| \leq \frac{1}{2}(\|z_t\|^2 + \|z\|^2) \leq C E_m(t).
\]

Therefore,

\[
2 \int_0^T E_m(t) dt \leq C_0(E_m(0) + E_m(T)) + 2 \int_0^T \| z_t \|^2 dt - \int_0^T (D(t, z_t), z) dt \\
+ \int_0^T ((m(\| \nabla w \|^2) - m(\| \nabla v \|^2)) \Delta v, z) dt \\
- \int_0^T (f(w) - f(v), z) dt.
\] (4.10)

Setting \( t = 0 \) in (4.7), we have

\[
E_m(0) = E_m(T) + \int_0^T (D(t, z_t), z_t) dt + \int_0^T m(\| \nabla w \|^2) \| \nabla z \|^2 (\Delta w, w_t) dt \\
- \int_0^T ((m(\| \nabla w \|^2) - m(\| \nabla v \|^2)) \Delta v, z_t) dt + \int_0^T (f(w) - f(v), z_t) dt.
\] (4.11)

Moreover, thanks to the monotonicity of \( D \), integrating (4.7) from 0 to \( T \) gives,

\[
T E_m(T) \leq \int_0^T E_m(t) dt - \int_0^T dt \int_t^T m'(\| \nabla w \|^2) \| \nabla z \|^2 (\Delta w, w_t) d\tau \\
+ \int_0^T dt \int_t^T ((m(\| \nabla w \|^2) - m(\| \nabla v \|^2)) \Delta v, z_t) d\tau \\
- \int_0^T dt \int_t^T (f(w) - f(v), z_t) d\tau.
\] (4.12)

Then it follows from (4.10), (4.11) and (4.12) that
\[ T E_m(T) + \int_0^T E_m(t) dt \]
\[ \leq C \left\{ \int_0^T \| z_t \|^2 + \int_0^T (D(t, z_t), z_t) dt + \int_0^T |(D(t, z_t), z)| dt \right\} + \left| \int_0^T \left( (m(\| \nabla w \|^2) - m(\| \nabla v \|^2)) \Delta v, z_t \right) dt \right| \]
\[ + \left| \int_0^T \left( (m(\| \nabla w \|^2) - m(\| \nabla v \|^2)) \Delta v, z \right) dt \right| \]
\[ + \int_0^T m'(\| \nabla w \|^2) \| \nabla z \|^2 \| \Delta w, w_t \| dt \]
\[ + \int_0^T dt \int_t^T \left| (m(\| \nabla w \|^2) - m(\| \nabla v \|^2)) \Delta v, z_t \right| d\tau \]
\[ + \int_0^T \left| (f(w) - f(v), z_t) \right| dt + \left| \int_0^T (f(w) - f(v), z) dt \right| + \left( \int_0^T dt \int_t^T (f(w) - f(v), z_t) d\tau \right) \} \tag{4.13} \]

According to Theorem 3.1 and noting that \( E_0(t) \) is bounded, we have
\[ \| u_t(t) \|^2 + \| \Delta u(t) \|^2 \leq C(R), \quad \forall t \geq 0, \forall \| u(0), u_t(0) \|_H \leq R. \tag{4.14} \]

Since \( m \in C^1(\mathbb{R}^+) \), by using estimate (4.14), the Mean Value Theory, and embedding \( V \hookrightarrow V_0 \), we have
\[ m(\| \nabla w \|^2) \leq C(R), \tag{4.15} \]
\[ m'(\| \nabla w \|^2) \| \nabla z \|^2 \| \Delta w, w_t \| \leq C(R) \| \nabla z \|^2, \tag{4.16} \]
\[ m(\| \nabla w \|^2) - m(\| \nabla v \|^2) \leq C(R) \| \nabla z \|, \tag{4.17} \]
\[ |m(\| \nabla w \|^2) - m(\| \nabla v \|^2)) \| \Delta v, z \| \leq C(R) \| \nabla z \|^2, \tag{4.18} \]
\[ |(m(\| \nabla w \|^2) - m(\| \nabla v \|^2)) \| \Delta v, z_t \| \leq C(R) \| \nabla z \| \| z_t \|. \tag{4.19} \]

From (4.13) and (4.15)-(4.19), we obtain (4.2). \( \square \)

4.3. Asymptotically smooth. In this subsection, we will show the semigroup corresponding to problem (1.1) is asymptotically smooth.

**Proposition 2.** Let assumptions \((A_1)\) and \((A_2)\) hold true, then the system \((\mathcal{H}, S_t)\) generated by (1.1) in the space \( \mathcal{H} \) is asymptotically smooth.

**Proof.** Since \( B_0 \) be a bounded absorbing set for \((\mathcal{H}, S_t)\). By the definition there exists \( t_0 \geq 0 \) such that \( S_t B_0 \subset B_0 \) for all \( t \geq t_0 \). Let \( B = \cup_{t \geq t_0} S_t B_0 \). It is clear that \( B \) is a closed bounded forward invariant set for this system. Since for any bounded set \( B \) we have \( S_t B \subset B_0 \) for all \( t \geq t(B) \), we obtain that \( S_t B \subset B \) for all \( t \geq t_0 + t(B) \). Hence \( B \) is also an absorbing set for this system. Let \( w(t) \) and \( v(t) \) be two weak solutions to (1.1) corresponding to two different initial datas in the invariant set \( B \):
\[ (w(t), w_t(t)) \equiv S_t y_0, \quad (v(t), v_t(t)) \equiv S_t y_1, \quad y_0, y_1 \in B. \tag{4.20} \]
We shall prove that the requirements imposed by Theorem 4.3 are satisfied. Since all term of (4.1) are continuous with respect to the distance $d$ given by the energy norm $\| \cdot \|$. Let $T > 0$. Since $B$ is a bounded forward invariant set, from energy equality (2.17) and continuity of $f, m$, we always have

$$\int_0^T (D(w_t), w_t) dt + \int_0^T (D(v_t), v_t) dt \leq C_B. \tag{4.21}$$

We denote $z(t) = w(t) - v(t)$. Recall that $z$ satisfies

$$z_{tt} + \Delta^2 z - m(\|\nabla w\|^2) \Delta z - (m(\|\nabla w\|^2) - m(\|\nabla v\|^2)) \Delta v + D(t, z_t) + f(w) - f(v) = 0, \tag{4.22}$$

where

$$D(t, z_t) = \|w_t\|^p w_t - \|v_t\|^p v_t.$$  

The standard energy method see (4.7) gives for any $t \in [0, T]$

$$E_m(T) + \int_t^T (D(t, z_t), z_t) dt = E_m(t) - \int_t^T m'(\|\nabla w\|^2) \|\nabla z\|^2 (\Delta w, w_t) dt$$

$$+ \int_t^T (m(\|\nabla w\|^2) - m(\|\nabla v\|^2)) \Delta v, z_t) dt \tag{4.23}$$

$$- \int_t^T (f(w) - f(v), z_t) dt.$$  

**Step 1: Energy reconstruction.** We denote

$$\Psi_T(w, v) = \int_0^T \|\nabla z\|^2 dt + \int_0^T \|\nabla z(\tau)\|^2 d\tau$$

$$+ \int_0^T dt \int_0^T \|\nabla z(\tau)\| \cdot \|z_t(\tau)\| d\tau + \int_0^T (f(w) - f(v), z) dt$$

$$+ \int_0^T (f(w) - f(v), z_t) dt \tag{4.24}$$

from which, we rewrite (4.2) as

$$TE_m(T) + \int_0^T E_m(t) dt \leq C(R) \left\{ \int_0^T \|z_t\|^2 dt + \int_0^T (D(t, z_t), z_t) dt \right.$$  

$$+ \int_0^T (D(t, z_t), z) dt + \Psi_T(w, v) \right\}. \tag{4.25}$$

It follows from the definition of $\Psi_T$ that

$$\Psi_T(w, v) \leq C_{B,T} \left\{ \int_0^T \|\nabla z\|^2 dt + \int_0^T \|\nabla z\| \cdot \|z_t\| dt$$

$$+ \int_0^T \|f(w) - f(v)\| \cdot \|z\| dt + \int_0^T \|f(w) - f(v)\| \cdot \|z_t\| dt \right\}. \tag{4.26}$$
By Cauchy’s inequality and compact embedding theorem, there exists a small constant $0 < \beta < \frac{1}{2}$ such that

$$
\int_0^T \|\nabla z\|^2 dt + \int_0^T \|\nabla z\| \cdot \|z_t\| dt
\leq \int_0^T \|\nabla z\|^2 dt + C_\varepsilon \int_0^T \|\nabla z\|^2 dt + \frac{\varepsilon}{2} \int_0^T \|z_t\|^2 dt
$$

(4.27)

$$
\leq C_{B,\varepsilon} \int_0^T \|A^{\frac{1}{2}-\beta} z\|^2 dt + \varepsilon \int_0^T E_m(t) dt.
$$

By Theorem 3.1 and the restriction (2.3) on the growth of $f$ in (A2) along with Sobolev’s embedding theorems, for $n \geq 5$, let $r = \frac{n}{(n-2)\beta}$ and $\bar{r} = \frac{n}{n-(n-4)\beta}$ are Hölder’s conjugate exponents and for $n \leq 4$, taking $r$ large enough, then we have

$$
\|f(w) - f(v)\|^2 = \int_\Omega |f(w) - f(v)|^2 dx
= \int_\Omega |f'(v + \theta_1(w - v))(w - v)|^2 dx
\leq C \int_\Omega (1 + |v + \theta_1(w - v)|^\beta)^2 |w - v|^2 dx
\leq C \int_\Omega (1 + |w|^{2\bar{r}} + |v|^{2\bar{r}})|w - v|^2 dx
\leq C \int_\Omega (1 + |w|^{2\bar{r}} + |v|^{2\bar{r}})|w - v|^2 dx
\leq C \left(\int_\Omega |w|^{2\bar{r}} + |v|^{2\bar{r}}\right)^\frac{1}{\bar{r}} \left(\int_\Omega |w - v|^{2\bar{r}} dx\right)^\frac{1}{\bar{r}}
\leq C(R) \|w - v\|_{2\bar{r}}^2
\leq C(R) \|A^{\frac{1}{2}-\delta}(w - v)\|^2,
$$

where $0 < \theta_1 < 1$ and $\delta$ is a suitably small constant.

Therefore, from (4.28), we have

$$
\int_0^T \|f(w) - f(v)\| \cdot \|z\| dt + \int_0^T \|f(w) - f(v)\| \cdot \|z_t\| dt
\leq C_\varepsilon \int_0^T \|f(w) - f(v)\|^2 dt + \frac{\varepsilon}{2} \int_0^T E_m(t) dt
$$

(4.29)

$$
\leq C_{B,\varepsilon} \int_0^T \|A^{\frac{1}{2}-\delta} z\|^2 dt + \varepsilon \int_0^T E_m(t) dt.
$$

From (4.26)-(4.29), we get

$$
\Psi_T(w, v) \leq C_{B,\varepsilon}(T) \int_0^T \|A^{\frac{1}{2}-\beta} z\|^2 dt + 2\varepsilon \int_0^T E_m(t) dt,
$$

(4.30)

for every $\varepsilon > 0$, $\beta = \min\{\beta, \delta\}$.

According to Lemma 2.2, taking $H_0(s) = \frac{C_p}{p^{\frac{2}{p}}} s^{\frac{2}{p}}$, $p \geq 0$, which is a strictly increasing, concave function, and $H_0 \in C(\mathbb{R}^+)$ with the property $H_0(0) = 0$ such that

$$
H_0((u + v)^p(u + v) - \|u\|^p u, v)) \geq H_0(C_p\|v\|^{p+2}) = \|v\|^2, \ u, v \in V.
$$

(4.31)
which, together with Jensen’s inequality, yields that
\[
\int_0^T \|z_t\|^2 dt \leq \int_0^T H_0(D(t, z_t), z_t) dt \\
\leq T H_0\left(\frac{1}{T} \int_0^T (D(t, z_t), z_t) dt\right) \\
= H_0\left(\int_0^T (D(t, z_t), z_t) dt\right),
\]
(4.32)
where $H_0(s) = TH_0\left(\frac{s}{T}\right)$. Therefore, applying Lemma 4.5 and using (4.30)-(4.32) with $\varepsilon > 0$ small enough, we obtain that
\[
TE_m(T) + \frac{1}{2} \int_0^T E_m(t) dt \leq C_B \left\{ H_0 + I \left(\int_0^T (D(t, z_t), z_t) dt\right) \right\} \\
+ \int_0^T |(D(t, z_t), z)| dt + C_{B,T} \int_0^T \|A^{\frac{1}{2}-\tilde{\beta}} z\|^2 dt,
\]
(4.33)
for every $T \geq T_0$.

Furthermore, in view of Cauchy’s inequality along with Sobolev’s embedding theorems, there exists a small constant $0 < \eta < \frac{1}{4}$ such that
\[
|(D(t, z_t), z)| = \left| \int_\Omega (\|w_t\|^p w_t - \|v_t\|^p v_t)z dx \right| \\
\leq \|z\| \left( \int_\Omega (\|w_t\|^p w_t - \|v_t\|^p v_t)^2 dx \right)^{\frac{1}{2}} \\
\leq C \|z\| \left( \|w_t\|^{2p} \|w_t\|^2 + \|v_t\|^{2p} \|v_t\|^2 \right)^{\frac{1}{2}} \\
\leq C_B \|z\| \leq C_B \|A^{\frac{1}{2}-\eta} z\|.
\]
Inserting (4.34) into (4.33), we obtain
\[
TE_m(T) + \frac{1}{2} \int_0^T E_m(t) dt \leq C_B \left\{ H_0 + I \left(\int_0^T (D(t, z_t), z_t) dt\right) \right\} \\
+ \int_0^T \|A^{\frac{1}{2}-\eta} z\| dt + C_{B,T} \int_0^T \|A^{\frac{1}{2}-\tilde{\beta}} z\|^2 dt.
\]
(4.35)
Step 2: Handling of the damping. By denoting $\tilde{\eta} = \min\{\eta, \tilde{\beta}\}$, we rewrite (4.35) into
\[
E_m(T) \leq C_{B,T} (H_0 + I) \left(\int_0^T (D(t, z_t), z_t) dt\right) + C_{B,T} \int_0^T \|A^{\frac{1}{2}-\tilde{\eta}} z\| dt \\
\leq C_{B,T} (H_0 + I) \left(\int_0^T (D(t, z_t), z_t) dt\right) + C_{B,T} \sup_{t \in [0,T]} \|A^{\frac{1}{2}-\tilde{\eta}} z(t)\|.
\]
(4.36)
Let $G_0(s) = (H_0 + I)^{-1} \left(\frac{\varepsilon}{2C_{B,T}}\right)$. Since $G_0(s)$ is a strictly increasing, convex function and $(H_0 + I)^{-1}(s) \leq s$, for $s \geq 0$,
and from (4.35) we obtain that
\[ G_0(E_m(T)) = (\mathcal{H}_0 + I)^{-1} \left( \frac{E_m(T)}{2C_{B,T}} \right) \]
\[ \leq (\mathcal{H}_0 + I)^{-1} \left\{ \frac{1}{2} \mathcal{H}_0 + I \right\} \left( \int_0^T (D(t, z_t), z_t) dt \right) + \frac{1}{2} \sup_{t \in [0, T]} \|A^{1-\tilde{\eta}}z(t)\| \]
\[ \leq \frac{1}{2} \int_0^T (D(t, z_t), z_t) dt + \frac{1}{2} (\mathcal{H}_0 + I)^{-1} \left\{ \sup_{t \in [0, T]} \|A^{1-\tilde{\eta}}z(t)\| \right\} \quad (4.37) \]
\[ \leq \frac{1}{2} \int_0^T (D(t, z_t), z_t) dt + \frac{1}{2} \sup_{t \in [0, T]} \|A^{1-\tilde{\eta}}z(t)\|. \]

From energy relation (4.23) with \( t = 0 \), (4.16), (4.19) and (4.28) with compact embedding theorem, we have
\[ \int_0^T (D(t, z_t), z_t) dt \]
\[ = E_m(0) - E_m(T) + \int_0^T m'(\|\nabla w\|^2)\|\nabla z\|^2(\Delta w, w_t) dt \]
\[ + \int_0^T (m(\|\nabla w\|^2) - m(\|\nabla v\|^2))\Delta v, z_t) dt - \int_0^T (f(w) - f(v), z_t) dt \]
\[ \leq E_m(0) - E_m(T) + C_B \left( \int_0^T \|\nabla z\|^2 dt + \int_0^T \|\nabla z\| \|z_t\| dt \right) \]
\[ + \int_0^T (f(w) - f(v), z_t) dt \)
\[ \leq E_m(0) - E_m(T) + C_B \left( \int_0^T \|\nabla z\|^2 dt + \int_0^T \|\nabla z\| dt \right) \]
\[ + \int_0^T \|f(w) - f(v)\| dt \]
\[ \leq E_m(0) - E_m(T) + C_{B,T} \sup_{t \in [0, T]} \|A^{1-\tilde{\eta}}z(t)\|. \]

Thus from (4.37) we obtain
\[ E_m(T) + 2G_0(E_m(T)) \leq E_m(0) + C_{B,T} \sup_{t \in [0, T]} \|A^{1-\tilde{\eta}}z(t)\|. \quad (4.39) \]

Since \( z(t) \) is uniformly bounded in \( V = \mathcal{D}(A^{1/2}) \) with \( \mathcal{D}(A^{1/2}) \hookrightarrow \mathcal{D}(A^{1-\tilde{\eta}}) \leftrightarrow H = L^2(\Omega) \), by interpolation we have that
\[ \|A^{1-\tilde{\eta}}z\| \leq \|z(t)\|^\theta \|D(A^{1/2})\|^1 \|z(t)\|^{1-\theta} \leq C_R \|z(t)\|^{1-\theta} \rightarrow \theta \in (0, 1). \]

Therefore
\[ E_m(T) + 2G_0(E_m(T)) \leq E_m(0) + C_{B,T} \sup_{t \in [0, T]} \|z(t)\|^\kappa, \quad (4.40) \]
for some \( \kappa \in (0, 1] \). Since
\[ E_z(t) = \frac{1}{2}(\|z_t(t)\|^2 + \|\Delta z(t)\|^2) = \|S_T y_1 - S_T y_2\|_{H^1}^2 \leq E_m(t), \]
Ascoli Theorem in [31], we obtain the compactness of embedding

\[ \rho \]

On the other hand, i.e.

This implies that

we have

\[
\left\| S_T y_1 - S_T y_2 \right\|_{\mathcal{H}} \leq 2[I + 2G_0]^{-1} \left\{ \frac{1}{2} \left\| y_1 - y_2 \right\| + C_{B,T} \sup_{t \in [0,T]} \| z(t) \|^\kappa \right\} \]

\[ (4.41) \]

\[
\leq 2[I + 2G_0]^{-1} \left\{ \frac{1}{2} \left( \left\| y_1 - y_2 \right\| + (C_{B,T} \sup_{t \in [0,T]} \| z(t) \|^\kappa)^{\frac{1}{2}} \right) \right\}^2.
\]

This implies that

\[
\left\| S_T y_1 - S_T y_2 \right\|_{\mathcal{H}} \leq \sqrt{2} \left[ I + 2G_0 \right]^{-1} \left\{ \frac{1}{2} \left\| y_1 - y_2 \right\| + C_{B,T} \sup_{t \in [0,T]} \| z(t) \|^\kappa \right\}^2 \]

\[ (4.42) \]

i.e.

\[
\left\| S_T y_1 - S_T y_2 \right\|_{\mathcal{H}} \leq q \left( \left\| y_1 - y_2 \right\| + \rho_B^T(\{S_T y_1, \{S_T y_2\}) \right),
\]

\[ (4.43) \]

where \( q(s) = \sqrt{2} \left[ I + 2G_0 \right]^{-1} (s^2) \) and

\[
\rho_B^T(\{S_T y_1, \{S_T y_2\}) = C_{B,T} \sup_{t \in [0,T]} \| w(t) - v(t) \|^\kappa,
\]

for some \( k^* \in (0, \frac{1}{2}] \).

Clearly, the function \( q \) satisfies all the requirements of Theorem 4.3. At last, we just need to verify the pseudometric \( \rho_B^T \) is precompact on the set \( \mathcal{L}_{B,T} \) of all solutions to (1.1) on \([0, T]\) with initial data in \( B \).

For every bounded set \( F \) of \( C([0, T]; D(A^{\frac{1}{2}})) \cap C^1([0, T]; L^2(\Omega)) \), that is to say, there exists a constant \( C \) such that

\[
\| u(t) \|_{D(A^{\frac{1}{2}})} + \| u_t(t) \| \leq C, \ \forall u(t) \in F(t) = \{ u(t) : u \in F \}.
\]

Since \( D(A^{\frac{1}{2}}) \hookrightarrow L^2(\Omega) \), we infer that

\( F(t) \) is relatively compact in \( L^2(\Omega), \ \forall 0 < t < T. \)

On the other hand, \( \forall \varepsilon > 0, u \in F, \) we have

\[
\| u(t) - u(t_1) \| = \int_{t_1}^t | u_t(s) | ds \leq \int_{t_1}^t \| u_t(s) \| ds
\]

\[
\leq (t - t_1)^{\frac{1}{2}} \left( \int_{t_1}^t \| u_t(s) \|^2 ds \right)^{\frac{1}{2}}
\]

\[
\leq C(t - t_1)^{\frac{1}{2}}
\]

\[
\leq C\varepsilon,
\]

\( \forall 0 \leq t \leq t_1 \leq T \) satisfying \( |t - t_1| \leq \varepsilon \) i.e. \( F \) is uniformly equicontinuous. By the Ascoli Theorem in [31], we obtain the compactness of embedding

\[
C([0, T]; D(A^{\frac{1}{2}})) \cap C^1([0, T]; L^2(\Omega)) \subset C([0, T]; L^2(\Omega))
\]

Therefore, the pseudometric \( \rho_B^T = C_{B,T} \sup_{t \in [0,T]} \| w(t) - v(t) \|^\kappa \) is precompact on the set \( \mathcal{L}_{B,T} \).
Then we can apply Theorem 4.3 to obtain the asymptotic smoothness of \((H, S_t)\).

4.4. The main result.

**Theorem 4.6.** Let assumptions \((A_1)\) and \((A_2)\) be in force, then the system \((H, S_t)\) generated by (1.1) in the space \(H\) possesses a compact global attractor \(A\).

**Proof.** The conclusion follows directly from Theorem 3.1 and Proposition 2.

**Remark 5.** Throughout this paper, when the damping term is 

\[ g(\|u_t\|)u_t, \]

where damping coefficient is a polynomial form of \(\|u_t\|\), we can obtain the existence of global attractor in a completely similar way.

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