DIFFERENTIAL CALCULUS ON GRAPHON SPACE

PETER DIAO, DOMINIQUE GUILLOT, APOORVA KHARE, AND BALA RAJARATNAM
STANFORD UNIVERSITY

Abstract. Recently, the theory of dense graph limits has received attention from multiple disciplines including graph theory, computer science, statistical physics, probability, statistics, and group theory. In this paper we initiate the study of the general structure of differentiable graphon parameters \( F \). We derive consistency conditions among the higher Gâteaux derivatives of \( F \) when restricted to the subspace of edge weighted graphs \( \mathcal{W}_p \). Surprisingly, these constraints are rigid enough to imply that the multilinear functionals \( \Lambda : \mathcal{W}_p^n \to \mathbb{R} \) satisfying the constraints are determined by a finite set of constants indexed by isomorphism classes of multigraphs with \( n \) edges and no isolated vertices. Using this structure theory, we explain the central role that homomorphism densities play in the analysis of graphons, by way of a new combinatorial interpretation of their derivatives. In particular, homomorphism densities serve as the monomials in a polynomial algebra that can be used to approximate differential graphon parameters as Taylor polynomials. These ideas are summarized by our main theorem, which asserts that homomorphism densities \( t(H, -) \) where \( H \) has at most \( N \) edges form a basis for the space of smooth graphon parameters whose \((N + 1)\)st derivatives vanish. As a consequence of this theory, we also extend and derive new proofs of linear independence of multigraph homomorphism densities, and characterize homomorphism densities. In addition, we develop a theory of series expansions, including Taylor’s theorem for graph parameters and a uniqueness principle for series. We use this theory to analyze questions raised by Lovász, including studying infinite quantum algebras and the connection between right- and left-homomorphism densities. Our approach provides a unifying framework for differential calculus on graphon space, thus providing further links between combinatorics and analysis.

Contents

1. Introduction 2
1.1. Organization of the paper 4
2. Preliminaries 4
2.1. Continuity and homomorphism densities 4
2.2. Differentiation on \( \mathcal{W}_{[0,1]} \) 8
3. Derivatives of \( C^N \) class functions 12
3.1. Symmetric \( S_{[0,1]} \)-invariant multilinear functionals 13
3.2. Bases of consistent vectors 18
3.3. Characterizing homomorphism densities 22
4. Power series and Taylor series 24
4.1. The algebra of partially labelled multigraphs 25
4.2. Series 27
4.3. Taylor series 31
4.4. Uniqueness of Taylor series and linear independence 33

Date: March 18, 2014.
1. Introduction

The theory of dense graphs and their limits introduced in [23] has attracted much attention recently (see e.g. [5, 6, 2, 8, 9, 13]). It has also been observed that limit theories developed in the context of (i) graphons, (ii) exchangeable arrays of random variables ([17]), and (iii) metric measure spaces ([15, Chapter 3] and [31]) can often be translated into each other (see [11, 12, 1]). Several questions have benefited from reformulation in this language - see for instance [16, 26, 21] and [22, Chapter 16]. The monograph [22] covers many aspects of this development, including topology and analysis on the space of graph limits. Since graphs have become a central abstraction for the modern analysis of complex systems, the theory has also been used to address applied questions in the study of estimable graph parameters [25], machine learning ([19]), and statistical modelling of networks [3, 10, 27]. It seems that such a language was needed as much for mathematical theory as for practical application.

The present paper begins the study of functional analysis of dense graph limits. More specifically, we exploit the linear structure of the space of graphons to build a theory of differential calculus. In order to explain our motivation and results, we first set some notation. Let \( W_{[0,1]} \) denote the space of graphons, i.e., bounded symmetric measurable functions \( f : [0,1]^2 \to [0,1] \). We denote weakly equivalent graphons \( f, g \in W_{[0,1]} \) by saying that \( f \sim g \). We refer to functions \( F : W_{[0,1]} \to \mathbb{R} \) which factor through \( W_{[0,1]}/\sim \) as class functions. (These are also sometimes referred to in the literature as “graphon parameters”.)

We say that a function \( F : W_{[0,1]} \to \mathbb{R} \) is continuous if it is continuous with respect to the cut norm on \( W_{[0,1]} \) (unless a different topology is specified). We say that \( F : W_{[0,1]}/\sim \to \mathbb{R} \) is continuous if the lift of \( F \) to \( W_{[0,1]} \) is continuous.

The most fundamental class functions are the homomorphism densities. These are defined for a graphon \( f \) and a multigraph \( H = (\{1, \ldots, k\}, E(H)) \) by

\[
t(H,f) := \int_{[0,1]^k} \prod_{(i,j) \in E(H)} f(x_i, x_j) \, dx_1 \ldots dx_k.
\]

Note that if \( H \) is a graph with no edges, then we define \( t(H,f) \equiv 1 \).

Let \( d^n F(f; g_1, \ldots, g_n) \) denote the (higher order) Gâteaux derivative of the function \( F : W_{[0,1]} \to \mathbb{R} \) at \( f \in W_{[0,1]} \) along the directions \( g_1, \ldots, g_n \in W_{[0,1]} \). (See Definition 2.13.) The main result of the present paper shows that homomorphism densities span the complete space of solutions to fundamental differential equations on \( W_{[0,1]} \).

**Theorem 1.1.** Let \( F : W_{[0,1]} \to \mathbb{R} \) be a class function which is continuous with respect to the \( L^1 \) norm and \((N + 1)\) times Gâteaux differentiable for some \( N \geq 0 \). Then \( F \) satisfies:

\[
d^{N+1} F(f; g_1, \ldots, g_{N+1}) = 0, \quad \forall f \in W_{[0,1]}, \ g_1, \ldots, g_{N+1} \in \text{Adm}(f), \quad (1.2)
\]
if and only if there exist constants \(a_H\) such that

\[
F(f) = \sum_{H \in \mathcal{H} \leq N} a_H t(H, f).
\]

Moreover, the constants \(a_H\) are unique. (Here, \(\mathcal{H} \leq N\) denotes the set of isomorphism classes of multigraphs with no isolated vertices and at most \(N\) edges, and \(\text{Adm}(f)\) is the set of admissible directions for computing the Gâteaux derivative; see Definition 2.10.)

If in addition \(F\) is continuous with respect to the cut-norm, then \(a_H = 0\) if \(H \in \mathcal{H} \leq N\) is not a simple graph.

To explain why the theorem is surprising, consider removing the condition that \(F\) is a class function on \(\mathcal{W}_{[0,1]}\). In that case, consider any multilinear \(\Lambda : \mathcal{W}_{[0,1]}^n \rightarrow \mathbb{R}\) with \(n \leq N\). Then \(F(f) := \Lambda(f, f, \ldots, f)\) would be a Gâteaux-smooth solution to (1.2). Since the “tangent space” to \(\mathcal{W}_{[0,1]}\) at the origin is infinite-dimensional, the space of such solutions is also infinite-dimensional. Theorem 1.1 shows that imposing the symmetry condition on \(F\) (i.e., assuming that \(F\) is a class function) collapses the set of solutions to a finite-dimensional space.

The proof of this theorem requires several steps which we now outline (see beginning of Section 3 for a more detailed outline). It begins with the observation that the differential equation (1.2) has solutions \(F\) that satisfy:

\[
F(g) = F(0) + dF(0; g) + \frac{d^2 F(0; g, g)}{2!} + \cdots + \frac{d^n F(0; g, \ldots, g)}{n!}.
\]

The proof of the theorem then proceeds by first understanding the structure of the functional \(\Lambda(g_1, \ldots, g_n) := d^n F(0; g_1, \ldots, g_n)\). In fact, the maps \(\Lambda : \mathcal{W}_{[0,1]}^n \rightarrow \mathbb{R}\) are multilinear functionals satisfying rigid symmetries. We exploit these symmetries to show that \(\Lambda\) restricted to weighted graphs are determined by constants indexed by the isomorphism classes \(\mathcal{H}_n\) of graphs with \(n\) edges and no isolated vertices. This surprising local structure of derivatives of smooth class functions at the origin is developed in Section 3.1. Computing the constants of Section 3.1 for homomorphism densities shows that the derivatives of any smooth class function at zero restricted to weighted graphs can be written uniquely in terms of the derivatives of homomorphism densities. In order to prove this, we give a new combinatorial interpretation of higher Gâteaux derivatives of homomorphism densities in Section 3.2.

Although the local structure theory of derivatives is interesting in its own right, we show that it yields rich rewards.

**Taylor polynomials.** The finite-dimensionality of the solution spaces to (1.2) allows us to develop the theory of Taylor polynomials. In particular, we prove that every smooth continuous class function \(F\) has a unique Taylor expansion where homomorphism densities play the role of monomials. We also give sufficient conditions for when this sequence of Taylor polynomials converges to \(F\) (see Theorem 4.32).

**Linear independence of homomorphism densities.** Our techniques allow us to prove the linear independence of homomorphism densities for partially labelled multigraphs (see Theorem 4.33). Such linear independence results go back to Whitney [32] for simple graph homomorphism densities (see also [14]). The proof technique depends on a combinatorial interpretation of the formula for derivatives of homomorphism densities.
Partially labelled multigraphs and infinite series. Theorem 1.1 shows that homomorphism densities $t(H, -)$ can be seen as monomials with degree $|E(H)|$. We investigate infinite power series of such monomials in the general setting of partially labelled graphs. This allows us to provide an answer to a question of Lovász about infinite quantum algebras (see Theorem 4.42).

Characterizing homomorphism densities. As another application of our theory, in Theorem 3.29 we characterize homomorphism densities $t(H, -)$ as the continuous maps on $W_{[0,1]} / \sim$ which are multiplicative with respect to tensor products. This complements previous characterizations of $\text{hom}(H, -)$ and $\text{hom}(-, H)$ (see [22, §5.6]).

1.1. Organization of the paper. The rest of this paper is organized as follows.

In Section 2.1 we review some basic properties of the cut-norm on $W_{[0,1]}$ and characterize the continuous homomorphism densities. The main theme is how to effectively exploit the density of the finite simple graphs in $W_{[0,1]}$. In Section 2.2 we develop the general theory of differentiation on $W_{[0,1]}$ that is used to prove the main theorem in the present paper. We explain what kind of smoothness assumptions are required, as well as why we use the Gâteaux derivative in favor of the Fréchet derivative.

In Section 3 we prove the main theorem by investigating the derivatives of smooth class functions in detail. In particular, we find that the Gâteaux derivatives $d^n F(0; g_1, \ldots, g_n)$ for a continuous class function satisfy relations that allow us to extract combinatorial data indexed by isomorphism classes of graphs to characterize the function. We also give a new proof of the linear independence of homomorphism densities using this structure theory, as well as a combinatorial interpretation of derivatives of homomorphism densities. Using the results of this section and the previous sections, we prove Theorem 1.1. As an application, we obtain an analytic characterization of homomorphism densities $t(H, -)$.

In Section 4 we consider partially labelled multigraphs and form algebras of linear combinations of them. We define weighted homomorphism densities for such graphs and develop a general analytic theory of infinite series of such functions. As an application, we investigate whether right homomorphism densities can be expanded in terms of left homomorphism densities (see [20, Problem 16]). We also explain the uniqueness and existence of Taylor series of homomorphism densities of smooth class functions and give sufficient conditions for their convergence. Finally, we generalize linear independence of homomorphism densities to partially labelled multigraphs, and explain how to construct an analytic theory of infinite quantum algebras. The theory allows us to address another of Lovász’s questions ([20, Problem 7]).

2. Preliminaries

In this section we review the topology on $W_{[0,1]}$ and its basic properties. We then introduce the general notions of differentiability on $W_{[0,1]}$.

2.1. Continuity and homomorphism densities. Recall that a finite simple labelled graph $G$ with vertices $V = \{1, 2, \ldots, n\}$ is identified with the graphon $f^G$, defined as follows:

$$f^G(x, y) = 1_{([nx], [ny]) \in E} = \begin{cases} 1, & \text{if } ([nx], [ny]) \text{ is an edge in } G, \\ 0, & \text{otherwise}. \end{cases}$$
The space $\mathcal{W}_{[0,1]}$ of all graphons sits inside $\mathcal{W}$, the vector space of bounded symmetric measurable functions $f : [0,1]^2 \to \mathbb{R}$. The space $\mathcal{W}$ has a seminorm

$$\|f\|_{\text{cut}} := \sup_{S,T \subseteq [0,1]} \left| \int_{S \times T} f(x,y) \, dx \, dy \right|$$

called the cut-norm, which is computed by taking the supremum over all pairs of Lebesgue measurable subsets $S, T$ of $[0,1]$. The group of invertible measure-preserving maps $S_{[0,1]}$ acts on $\mathcal{W}_{[0,1]}$ by $f^\sigma(x,y) := f(\sigma(x), \sigma(y))$ for $\sigma \in S_{[0,1]}$. Now define

$$\delta_{\square}(f,g) = \inf_{\psi \in S_{[0,1]}} \|f - g^\psi\|_{\text{cut}}.$$

Note that $\delta_{\square}$ is a pseudo-metric on $\mathcal{W}_{[0,1]}$. Say that two graphons $f \sim g$ are weakly equivalent if $\delta_{\square}(f,g) = 0$. For example, if $G, G'$ are isomorphic finite simple graphs, then $f^G$ and $f^{G'}$ are weakly equivalent because we can simply consider the map $\sigma$ obtained from the appropriate permutation of the vertices of $G$ to obtain $G'$. This shows that there is a map from isomorphism classes of simple graphs into $\mathcal{W}_{[0,1]}/\sim$. In addition, $\delta_{\square}$ descends to a metric on $\mathcal{W}_{[0,1]}/\sim$ forming a complete metric space (see [5]).

We say that a function $F : \mathcal{W}_{[0,1]} \to \mathbb{R}$ is a class function if it factors through $\mathcal{W}_{[0,1]}/\sim$. Note that a class function $F$ is continuous with respect to the cut-norm if and only if the induced function on $\mathcal{W}_{[0,1]}/\sim$ is continuous with respect to $\delta_{\square}$.

The metric on $\mathcal{W}_{[0,1]}/\sim$ defines a distance between isomorphism classes of graphs. When are two such graphs close? The answer lies in the homomorphism densities. It turns out that a sequence of graphs $f^{G_n}$ converges to a graphon $f$ in $\mathcal{W}_{[0,1]}/\sim$ if and only if $t(H, f^{G_n}) \to t(H, f)$ for every finite simple graph $H$. Intuitively, a sequence of graphs converges if their edge densities, triangle densities, etc. all converge when suitably normalized. More precisely, consider two finite simple graphs $G = (V(G), E(G)), H = (V(H), E(H))$. Let $\text{hom}(H,G)$ denote the number of edge-preserving maps : $V(H) \to V(G)$. Then, $t(H, f^G) = \text{hom}(H,G)/|V(G)||V(H)|$. Sometimes we will write $t(H, G)$ for $t(H, f^G)$.

The basic properties of the metric space $(\mathcal{W}_{[0,1]}/\sim, \delta_{\square})$ are as follows.

1. The countable family of graphons $f^G$ associated with simple graphs $G$ is dense in $(\mathcal{W}_{[0,1]}/\sim, \delta_{\square})$ (see [23]).
2. As a consequence of the Weak Regularity Lemma in graph theory, $(\mathcal{W}_{[0,1]}/\sim, \delta_{\square})$ is a compact metric space (see [24]).

Another viewpoint on the metric is that two graphs are close in the $\delta_{\square}$-topology if finite random subgraphs of them have similar distributions. As a consequence, continuous class functions on graphon space are precisely the estimable (or “testable”) graph parameters [22, Theorem 15.1]. Informally, these are the functions of isomorphism classes of graphs, that can be estimated at a graph from a random induced subgraph.

Our goal is to study continuous functions on $(\mathcal{W}_{[0,1]}/\sim, \delta_{\square})$, and by extension, on $\mathcal{W}_{[0,1]}$. As we now explain, the homomorphism densities $t(H, -)$ are fundamental amongst such continuous functions. Let $\mathcal{S}_t$ denote the linear span of homomorphism densities $t(H, -)$ for $H$ a simple graph. Since for any two disjoint finite simple graphs $H_1, H_2$, one has

$$t(H_1 \coprod H_2, f) = t(H_1, f) \times t(H_2, f),$$

(2.1)

$\mathcal{S}_t$ is actually an algebra. We now prove a Stone-Weierstrass-type theorem for this algebra.
Theorem 2.2 (Density theorem). The linear span of homomorphism densities $\mathcal{S}$ is dense in $C(\mathcal{W}_{[0,1]}/\sim, \delta_\infty)$, the space of continuous functions on $\mathcal{W}_{[0,1]}/\sim$, under the topology of uniform convergence.

Proof. The space of functions $\mathcal{S}$ is an algebra of functions which contains the constant function. The space $\mathcal{W}_{[0,1]}/\sim$ is compact [24] and Hausdorff and so it suffices to check that the set of functions separates points [28, Theorem 7.32]. We need only recall that $f \sim g$ if and only if $t(H, f) = t(H, g)$ for all finite simple $H$.

Similarly, we show that function values can be interpolated using functions from $\mathcal{S}$.

Theorem 2.3 (Lagrange Interpolation). Let $f_1, \ldots, f_k \in \mathcal{W}_{[0,1]}/\sim$ be distinct graphons and $a_1, \ldots, a_k \in \mathbb{R}$ be arbitrary real numbers. Then there exist finite simple graphs $H_i$ and scalars $c_i$ such that $\sum_i c_i t(H_i, f_j) = a_j$ for all $1 \leq j \leq k$. In other words, there exists an element $F \in \mathcal{S}$ such that $F(f_j) = a_j$.

Proof. It suffices to show the result for $a_1 = 1$ and $a_j = 0$ for all $2 \leq j \leq k$. Since the functions $t(H, –)$ separate points in $\mathcal{W}_{[0,1]}/\sim$, for each $j \geq 2$ there exists a graph $H_j$ such that $t(H_j, f_j) \neq t(H_j, f_1)$. In particular there exist $b_j$ and $c_j$ such that $b_j t(H_j, f_j) + c_j = 1$ and $b_j t(H_j, f_1) + c_j = 0$. Recalling that the linear span of homomorphism densities form an algebra, the function $\prod_{j=2} b_j(t(H_j, f_j) + c_j)$ works.

Homomorphism densities as monomials.

The above results suggest that $\mathcal{S}$ may play an important role in the functional analysis of $(\mathcal{W}_{[0,1]}/\sim)$, for several reasons. For instance, in addition to spanning an algebra of functions, the homomorphism densities $t(H, f)$ naturally have a notion of degree. To elaborate, let $G_n$ be the set of isomorphism classes of unlabelled simple graphs with $n$ edges and no isolated vertices. Now define

$$G_{\leq n} := \bigcup_{j \leq n} G_j, \quad G := \bigcup_{j \in \mathbb{N}} G_j.$$  \hspace{1cm} (2.4)

Equation (2.1) clearly shows that if $H_1 \in G_n$ and $H_2 \in G_m$ then $t(H_1, f) \times t(H_2, f) = t(H_3, f)$ for $H_3 \in G_{n+m}$. Therefore, the number of edges in the graph $H$ naturally serves as a degree (i.e., a $\mathbb{Z}_+$-grading) for the function $t(H, f)$. Combined with Theorems 2.2 and 2.3, this suggests that homomorphism densities may play the role of monomials in the algebra $\mathcal{S}$.

To carry this analogy further, recall that polynomials of degree at most $N$ could be defined as solutions to the differential equation

$$\frac{d^{N+1}}{dx^{N+1}} F \equiv 0.$$  \hspace{1cm} (2.5)

A question of interest in the graphon setting would thus be to ask which functions satisfy the system of differential equations

$$d^{N+1} F(f; g_1, \ldots, g_{N+1}) \equiv 0, \quad \forall f \in \mathcal{W}_{[0,1]}, \ g_i \in Adm(f).$$  \hspace{1cm} (2.5)

It is not hard to check that all homomorphism densities $F(f) := t(H, f)$ are solutions of (2.5), for $H \in G_{\leq N}$. Therefore Theorem 2.1 first of all gives further weight to our notion of degree. Any function $\phi : G \rightarrow \mathbb{N}$ satisfying $\phi(H_1 \bigcup H_2) = \phi(H_1) + \phi(H_2)$ might be a candidate. However, we also show later that taking $N$+1 derivatives annihilates $t(H, –)$ for $H$ with at most $N$ edges (see Proposition 3.20).

What is more surprising is the fact that the homomorphism densities $t(H, f)$ for $H \in G_{\leq N}$ span all solutions of (2.5). Therefore homomorphism densities will play a fundamental role
in the differential calculus of class functions on $\mathcal{W}_{[0,1]}$. We will make this analogy much more precise at the end of Section 4.

**Continuity of multigraph homomorphism densities.**

Our main result, Theorem 1.1, refers to homomorphism densities $t(H,f)$ for $H$ a general multigraph without loops. For this paper, an (undirected) multigraph $G$ is given by the data of the set of vertices $V(G)$, set of undirected edges $E(G)$, and a map sending an edge $e$ to its endpoints $\{e_s,e_t\} \subset V(G)$. We allow multiple edges to have the same endpoints $\{e_s,e_t\}$, and the graph is undirected so $e_s$ and $e_t$ could just as easily be interchanged.

Recall [22, Section 5.2.1] that a *(node-and-edge)* homomorphism of multigraphs $f : H \to G$ is defined by the data of a map of vertices $V_f : V(H) \to V(G)$ and a map of edges $E_f : E(H) \to E(G)$. The maps $E_f$ and $V_f$ must be compatible in the sense that

$$\{E_f(e)_s, E_f(e)_t\} = \{V_f(e)_s, V_f(e)_t\} \forall e \in E(H). \quad (2.6)$$

Note that when $G$ is a multigraph, $E_f$ is not completely determined by $V_f$. However, if $G$ is simple, we will identify $e \in E(G)$ with its endpoints $\{e_s,e_t\}$. Moreover, we say that $f$ is (respectively) injective, surjective, or bijective, when both $V_f$ and $E_f$ have the same property.

Now let $\mathcal{H}_n$ denote the isomorphism classes of graphs with $n$ edges, no isolated vertices, and no self loops but possible multi-edges. Also let $\mathcal{H}_{\leq n} = \bigcup_{j \leq n} \mathcal{H}_j$. Clearly $\mathcal{G}_n \subset \mathcal{H}_n$ for all $n$. Then we have already defined

$$t(H,f) := \int_{[0,1]^{V(H)}} \prod_{(i,j) \in E(H)} f(x_i, x_j) dx$$

for an arbitrary multigraph $H \in \mathcal{H}_n$, where $E(H)$ now denotes the multiset of edges in $H$ and is independent of the choice of representative $H$. When it is important to be more explicit about naming the vertices and edges of $H$ we will write

$$t(H,f) = \int_{[0,1]^{V(H)}} \prod_{e \in E(H)} f(x_{e_s}, x_{e_t}) \prod_{i \in V(H)} dx_i.$$

There is no consensus on the definition of a graph morphism between multigraphs. One advantage of using the node-and-edge notion of homomorphism is that if we define for multigraphs $H, G$ the combinatorial quantity $t(H,G) := \text{hom}(H,G)/|V(G)|^{\vert V(H)\vert}$, then

$$t(H,G) = t(H,f^G)$$

where $f^G$ is defined as for simple graphs, but weighted according to the multiplicity of the edge. So these multigraph homomorphism densities are class functions and behave similarly to simple graph homomorphism densities. We now show that multigraph homomorphism densities are no longer continuous in the cut-norm topology unless they lie in $\mathcal{S}_1$. The following proposition collects together the continuity properties of homomorphism densities that are needed for the proof of Theorem 1.1. The proof exploits the fact that $\{0,1\}$-valued graphons are dense in $\mathcal{W}_{[0,1]}$.

**Proposition 2.7.** Fix $n \geq 0$ and consider $F : \mathcal{W} \to \mathbb{R}$ of the form

$$F(f) := \sum_{H \in \mathcal{H}_{\leq n}} a_H t(H,f) \quad (2.8)$$

for some constants $a_H$.

(i) Then $F : \mathcal{W} \to \mathbb{R}$ is continuous in the $L^1$ topology.
(ii) If moreover $F$ is continuous in the cut-norm topology, then

$$F(f) = \sum_{H \in \mathcal{H}_{\leq n}} a_H t(H^{\text{simp}}, f),$$

where $H^{\text{simp}}$ is the simple graph obtained from $H$ by replacing each set of repeated edges between a pair of vertices by one edge.

Proof.

(i) It suffices to consider a single multigraph homomorphism density $t(H, -)$. Consider the multilinear functional $\Lambda : W^m_{[0,1]} \to \mathbb{R}$ defined by

$$\Lambda((f_e)_{e \in E(H)}) := \int_{[0,1]^{|V(H)|}} \prod_{e \in E(H)} f_e(x_{e_s}, x_{e_t}) \prod_{i \in V(H)} dx_i.$$

It is not difficult to see, by replacing $f_e$ by $g_e$ one term at time, that for $f_e, g_e \in W_{[0,1]}$,

$$|\Lambda((f_e)_{e \in E(H)}) - \Lambda((g_e)_{e \in E(H)})| \leq \sum_{e \in E(H)} \|f_e - g_e\|_1$$

since $\|f_e\|, \|g_e\|_{\infty} \leq 1$ for $f_e, g_e \in W_{[0,1]}$. It follows that $|t(H, f) - t(H, g)| \leq |E(H)| \cdot \|f_e - g_e\|_1$ and so is continuous with respect to the $L^1$ topology.

(ii) We can rewrite the function $t(H, f)$ as

$$t(H, f) = \int_{[0,1]^k} \prod_{(i, j) \in E(H^{\text{simp}})} \varphi_{ij}(f(x_i, x_j)) \prod_{i=1}^k dx_i$$

where $\varphi_{ij}(x) = x^{m_{ij}}$, with $m_{ij}$ the multiplicity of the edge $(i, j)$ in $H$. From this expression, it is clear that $t(H, f) = t(H^{\text{simp}}, f)$ for $\{0,1\}$-valued graphons $f$. In particular, $F(f) = \sum_{H \in \mathcal{H}_{\leq n}} a_H t(H^{\text{simp}}, f)$ for $\{0,1\}$-valued graphons $f$. The result now follows from the continuity of both sides in the cut-norm and the density of such graphons in $W_{[0,1]}$.

\[\square\]

Remark 2.9. In fact we show below that the homomorphism densities over all multigraphs are linearly independent as functions on $W_{[0,1]}$ - see Corollary 3.24. As a consequence, if $F$ is given as in (2.8) and is continuous in the cut-norm, then $a_H = 0$ for all $H$ that are not simple.

2.2. Differentiation on $W_{[0,1]}$. In this section we develop a general theory of differentiating functions on graphon space. There are two standard notions of derivatives in such a setting: the Gâteaux derivative and the Fréchet derivative. We show in this section that taking the Fréchet derivative is a very restrictive notion and is not appropriate for our analysis, in that most homomorphism densities are not Fréchet differentiable. We proceed to develop some technical machinery to refine the Gâteaux theory on $W_{[0,1]}$ to helps bypass the fact that $W_{[0,1]}$ is not a vector space (so one cannot take Gâteaux derivatives at all points along all directions). We also make precise our notion of (sufficiently) smooth Gâteaux differentiable functions on $W_{[0,1]}$. Finally, we illustrate our analytic methods by providing a new proof of Sidorenko’s Theorem for star graphs.
Gâteaux derivatives, admissibility, and smoothness

The Gâteaux derivative is usually defined in the context of a real linear space $E$ and a map $F : E \to \mathbb{R}$. In such settings the Gâteaux derivative of $F$ at $f \in E$ in the direction of $g \in E$ is defined to be the limit

$$dF(f; g) := \lim_{\lambda \to 0} \frac{F(f + \lambda g) - F(f)}{\lambda},$$

if such a limit exists. However, in this paper we have to differentiate functions $F : \mathcal{W}_{[0,1]} \to \mathbb{R}$ where $\mathcal{W}_{[0,1]}$ is not a vector space. In that case, if $f \in \mathcal{W}_{[0,1]}$ and $g \in E = \mathcal{W} \supset \mathcal{W}_{[0,1]}$, then $f + \lambda g$ need not always lie in $\mathcal{W}_{[0,1]}$, so we need to clarify what we mean by the Gâteaux derivative for functions on $\mathcal{W}_{[0,1]}$. For instance, if $f \equiv 0$ and $g$ is defined to be $-1$ and $1$ on disjoint complementary subsets of $[0,1]^2$, then $f + \lambda g \notin \mathcal{W}_{[0,1]}$ for any $\lambda \neq 0$. In this paper, we use the following notion to deal with this issue.

**Definition 2.10.** Given a nonempty convex subset $U$ in a real linear space $E$ and $f \in U$, define the admissible directions at $f$ to be

$$\text{Adm}_U(f) := \{ g \in E : f + \epsilon g \in U \text{ for all sufficiently small } 0 \leq \epsilon < 1 \}. \quad (2.11)$$

**Remark 2.12.** In this paper, unless otherwise specified, admissibility is always assumed to be with respect to $E = \mathcal{W}$ and $U = \mathcal{W}_{[0,1]}$. Thus we always write $\text{Adm}(f)$ to mean $\text{Adm}_{\mathcal{W}_{[0,1]}}(f)$. Also note that $\text{Adm}_U(f)$ always contains the origin and is itself a convex subset and a cone in $E$, since $U$ is convex.

We now explain how the notion of admissibility applies to (higher) Gâteaux derivatives.

**Definition 2.13.** Let $E$ be a real linear space, $U \subset E$ a nonempty convex subset, and $F : U \to \mathbb{R}$. We say that the Gâteaux derivative exists at $f \in U$ in the direction $g \in \text{Adm}_U(f)$ if the limit

$$dF(f; g) := \lim_{\lambda \to 0, \ f + \lambda g \in U} \frac{F(f + \lambda g) - F(f)}{\lambda}$$

exists. Note that this limit is one-sided if $-g \notin \text{Adm}_U(f)$.

Similarly, we say that $F$ is $n$-times Gâteaux differentiable at $f \in Y$ in the directions $g_1, \ldots, g_n \in \text{Adm}_U(f)$, if the higher mixed Gâteaux derivatives $d^{n-1}F(f + \lambda g_n; g_1, \ldots, g_{n-1})$ exist for all $\lambda$ small enough such that $f + \lambda g_n \in U$, and the limit

$$d^n F(f; g_1, \ldots, g_n) := \lim_{\lambda \to 0, \ f + \lambda g_n \in U} \frac{d^{n-1}F(f + \lambda g_n; g_1, \ldots, g_{n-1}) - d^{n-1}F(f; g_1, \ldots, g_{n-1})}{\lambda}$$

exists.

**Remark 2.14.** In the formula for defining higher Gâteaux derivatives, one might suspect that admissibility issues arise - for instance, that $g_1 \in \text{Adm}_U(f + \lambda g_n)$ needs to hold for all small $\lambda$. However, these issues are immediately bypassed since we assume $U$ to be convex. Indeed, this is easily verified by induction on $n$, using the fact that if $g_i \in \text{Adm}_U(f)$ for all $i$, then $f + \sum_{i=1}^n \lambda_i g_i \in U$ for all sufficiently small $0 \leq \lambda_i < 1$.

In the remainder of this paper, we will need the notion of a continuously differentiable function. This is made precise in the following definition.

**Definition 2.15.** Fix a convex subset $U \subset E$, and a function $F : U \to \mathbb{R}$. Given $f \in U$, $m \in \mathbb{N}$, and $g_1, \ldots, g_m \in \text{Adm}_U(f)$, define an auxiliary function $F_{f,g}(\lambda_1, \ldots, \lambda_m) := F(f + \lambda_1 g_1 + \cdots + \lambda_m g_m)$. Now given an integer $n \geq 0$, we say that $F$ is $C^n$ at $f$ if $F_{f,g}$ has
continuous $n$th partial derivatives in a neighborhood of 0 (possibly with boundary) for any choice of $g_1, \ldots, g_n \in \text{Adm}_U(f)$ for all $m \in \mathbb{N}$ (it is equivalent to assume it only for $m = n$). We say that $F$ is smooth at $f$ if it is $C^n$ at $f$ for every $n \geq 0$. We say that $F$ is $C^n$ (or smooth) if it is $C^n$ (or smooth) everywhere.

In Definition 2.15 we could also require that the Gâteaux derivatives $d^n F(f; g_1, \ldots, g_n)$ be continuous in $f$. This would imply that $F$ is $C^n$, but is difficult to verify in practice. The notion of being $C^n$ is essentially multivariable calculus in $\mathbb{R}^n$, and so $d^n F(f; g_1, \ldots, g_n)$ is multilinear in the $g_i$. Some caution must be exercised here as the $g_i$ are necessarily in $\text{Adm}_U(f)$; the following definition makes precise what we mean.

**Definition 2.16.** Suppose $V \subset E$ is a subset of a real vector space $E$. A function $\Lambda : V^n \to \mathbb{R}$ is said to be multilinear if $\Lambda$ extends (uniquely) to a multilinear functional : $(\text{span}_\mathbb{R} V)^n \to \mathbb{R}$.

We now write down a precise result about the multilinearity of the (higher) Gâteaux derivatives. This result is crucial for the rest of the paper.

**Theorem 2.17.** Suppose $U \subset E$ is convex, and the function $F : U \to \mathbb{R}$ is $C^n$ at $f \in U$. Also fix $g_i \in \text{Adm}_U(f)$ for $1 \leq i \leq n$.

(i) For any permutation $\tau \in S_n$, we have $d^n F(f; g_1, \ldots, g_n) = d^n F(f; g_{\tau(1)}, \ldots, g_{\tau(n)})$.

(ii) The Gâteaux derivatives $d^n F(f; g_1, \ldots, g_n)$ are multilinear in the $g_i$.

The proof of Theorem 2.17 is somewhat involved and is deferred to Appendix A.

In later sections, we will almost always assume that our functions are Gâteaux smooth or $C^n$. Indeed, we need these properties (including Theorem 2.17) in order to prove Theorem 1.1 and other main results.

A stronger notion of differentiability of functions $F : E \to \mathbb{R}$ which is often used on normed linear spaces $E$ is the Fréchet derivative. It is natural to ask if such a notion can be used to study functions on $\mathcal{W}$ equipped with the seminorm $\| \cdot \|_{\text{cut}}$. However, even homomorphism densities are generally not Fréchet differentiable.

**Theorem 2.18.** Let $A_n$ denote the graph with vertex set $\{1, 2, 3, \ldots, n\}$ and edges $(i, i+1)$ for $1 \leq i \leq n-1$, and $H$ denote an arbitrary multigraph. The following are equivalent:

(i) The homomorphism density $t(H, f)$ is Fréchet differentiable.

(ii) The graph $H$ is a disjoint union of copies of $A_2$ and $A_1$.

The proof of Theorem 2.18 is provided in Appendix B. Theorem 2.18 shows that Fréchet differentiability is too restrictive for our purposes. We therefore work with Gâteaux derivatives in the rest of the paper.

Gâteaux derivatives and other variational techniques have been used to investigate problems in combinatorics and graph theory in the literature; see for instance [7, 22, Chapter 16.2]. We illustrate how the Gâteaux derivative can be used to solve optimization problems via a simple case of Sidorenko’s conjecture - namely, for star graphs. This case was solved in Sidorenko’s original paper [29].

**Theorem 2.19.** Let $S_k$ be the star graph with $k+1$ vertices $\{0, 1, 2, \ldots, k\}$ and $k$ edges from $0$ to all $i > 0$.

(i) If $f \in \mathcal{W}_{[0,1]}$ with edge density $c$ then $t(S_k, f) \geq t(S_k, c)$.

(ii) If $f \in \mathcal{W}_{[0,1]}$ with edge density $c$ then $t(S_k, f) = t(S_k, c)$ if and only if $\int_x \int_y f(x, y) = c$ for almost every $y$. 
Proof. (i) Let $W(0) \subset W$ denote the linear subspace of all $f \in W$ with edge density 0. Either by direct computation or by Proposition 3.20 we can compute the higher derivatives of $t(S_k, f)$ and see that it is Gâteaux smooth on $W_{[0,1]}$. Let $g := f - c$ and note that $g \in W(0)$. The first derivative of $t(S_k, -)$ at $c$ is

$$
 dt(S_k, c; g) = \sum_{(i_1, j_1) \in E(S_k)} \int_{[0,1]^{k+1}} g(x_{i_1}, x_{j_1}) \prod_{(i,j) \neq (i_1, j_1)} cdx = 0,
$$

(2.20)

since $g \in W(0)$. The second derivative at any $f \in W_{[0,1]}$ and $g \in Adm(f)$ is

$$
 d^2t(S_k, f; g, g) = 2 \sum_{1 \leq i < j \leq k} \int_{[0,1]^k} g(x_0, x_i)g(x_0, x_j) \prod_{l \in \{0, i, j\}} f(x_0, x_l)dx
$$

$$
 = 2 \sum_{1 \leq i < j \leq k} \int_{[0,1]} \left( \int_{[0,1]} g(x_0, z)dz \right)^2 \left( \int_{[0,1]} f(x_0, y)dy \right)^{k-2} dx_0.
$$

(2.21)

(2.22)

Since $f \geq 0$ and $(\int_{[0,1]} g(x_0, z)dz)^2 \geq 0$, we conclude that

$$
 d^2t(S_k, f; g, g) \geq 0.
$$

(2.23)

Now consider the set

$$
 I(f, c) := \{ \lambda c + (1 - \lambda) f : \lambda \in [0, 1] \} \subset c + W(0).
$$

By Equation (2.20), we conclude that the constant graphon $c$ is a local minimum of $t(S_k, f)$ on $I(f, c)$. The star density $t(S_k, -)$ is convex on $I(f, c)$ by equation (2.23), therefore $t(S_k, f) \geq t(S_k, c)$.

(ii) Assume now that $\int_x f(x, y)dx = c$ for almost every $y$. We compute

$$
 t(S_k, f) = \int_{[0,1]} \prod_{1 \leq i \leq k} \left( \int_{[0,1]} f(x_0, x_i)dx_i \right) dx_0
$$

$$
 = \int_{[0,1]} c^k dx_0 = t(S_k, c).
$$

Assume now that $\int_x f(x, y)dy$ is not equal to a constant for almost every $y$. Let $g := f - c$. Using Equation (2.22) we see that the second derivative is given by

$$
 d^2t(S_k, c; g, g) = 2 \sum_{1 \leq i < j \leq k} \int_{[0,1]} \left( \int_{[0,1]} g(x_0, z)dz \right)^2 \left( \int_{[0,1]} cdy \right)^{k-2} dx_0
$$

$$
 = 2c^{k-2} \sum_{1 \leq i < j \leq k} \int_{[0,1]} \left( \int_{[0,1]} g(x_0, z)dz \right)^2 dx_0.
$$

This last integral is positive because $\int_x f(x, y)$ is not equal to a constant for almost every $y$. It follows that $t(S_k, -)$ is strictly convex on $I(f, c)$ so $t(S_k, f) > t(S_k, c)$. □
3. Derivatives of $C^N$ Class Functions

The main goal of this section is to prove Theorem 1.1. As the proof of Theorem 1.1 is long and technical, we begin with an overview of the ingredients that will be used to prove it. The main ingredients have been separated out into subsections for ease of presentation. We begin by investigating the derivatives of smooth class functions as developed in Section 2.2.

To explain that connection, consider the differential equation of Theorem 1.1. One direction begins by investigating the derivatives of smooth class functions as developed in Section 2.2. We begin with an overview of the ingredients that will be used to prove it.

Definition 3.1. Fix $n \in \mathbb{N}$. A functional $\Lambda : W_{[0,1]}^m \rightarrow \mathbb{R}$ is said to be

- symmetric if for all permutations $\tau \in S_n$, $\Lambda(g_1, \ldots, g_n) = \Lambda(g_{\tau(1)}, \ldots, g_{\tau(n)})$;
- $S_{[0,1]}$-invariant if $\Lambda(g_{\sigma_1}, \ldots, g_{\sigma_n}) = \Lambda(g_1, \ldots, g_n)$ for all Lebesgue measure preserving bijections $\sigma : [0, 1] \rightarrow [0, 1]$.

Proposition 3.2. If $F : W_{[0,1]} \rightarrow \mathbb{R}$ is a $C^n$ class function for some integer $n > 0$, then $d^n F(0; g_1, \ldots, g_n)$ is a symmetric $S_{[0,1]}$-invariant multilinear functional.

Proof. By part (i) of Theorem 2.17, we know for $C^n$ functions $F : W_{[0,1]} \rightarrow \mathbb{R}$ that mixed $n$th partial Gâteaux derivatives are equal. Therefore, $d^n F(0; g_1, \ldots, g_n)$ is symmetric. By part (ii) of Theorem 2.17, we also get that the derivative is multilinear.

Next, note that if $F : W_{[0,1]} \rightarrow \mathbb{R}$ is an $n$-times Gâteaux differentiable class function, $\sigma \in S_{[0,1]}$, and $g_i \in \text{Adm}(f)$ for $f \in W_{[0,1]}$, then $g_i^\sigma \in \text{Adm}(f^\sigma)$ and

$$d^n F(f^\sigma; g_1^\sigma, \ldots, g_n^\sigma) = d^n F(f; g_1, \ldots, g_n).$$

(3.3)

Applying this equation to $f = 0$, we obtain that $d^n F(0; g_1, \ldots, g_n)$ is $S_{[0,1]}$-invariant. □

Let $X_n$ denote the vector space of symmetric $S_{[0,1]}$-invariant multilinear functionals $\Lambda : W_{[0,1]}^m \rightarrow \mathbb{R}$ for $n \geq 1$. By Proposition 3.2, that the derivatives at zero of the solutions of the differential equation in Theorem 1.1 all lie in $X_n$. In Section 3.1, we study the space $X_n$ via its image under linear maps $C_{n,p} : X_n \rightarrow X_{n,p}$ for $p \geq 2$. Here, $X_{n,p} = \mathbb{R}^{|\mathcal{H}_{n,p}|}$ with $\mathcal{H}_{n,p}$ the set of isomorphism classes of multigraphs with $n$ edges and $p$ vertices. Notice that $\mathcal{H}_{n,p}$ is not necessarily a subset of $\mathcal{H}_n$ because it allows for isolated vertices. We show for each $\Lambda \in X_n$ that the value of $\Lambda$ restricted to $n$-tuples of edge-weighted graphs is determined by $C_n(\Lambda) := (C_{n,p}(\Lambda))_{p \geq 2}$. In addition, we show for $p|q$ that there are linear relations $\pi_{n,q \rightarrow p}$ called the consistency constraints mapping $X_{n,q}$ to $X_{n,p}$, which send $C_{n,q}(\Lambda)$ to $C_{n,p}(\Lambda)$. The upshot is that the image $C_n(X_n)$ has dimension at most $|\mathcal{H}_n|$.

In Section 3.2, we first note that the $n$th Gâteaux derivatives at 0 of $\{t(H, -)\}_{H \in \mathcal{H}_n}$ are in $X_n$. Next we show that $C_n(d^n t(H, -)(0; -))$ are linearly independent for $H \in \mathcal{H}_n$. Therefore, by counting dimensions, the image $C_n(X_n)$ is spanned by the $n$th Gâteaux derivatives of homomorphism densities for $H \in \mathcal{H}_n$. Finally, we collect the different solutions for $n \leq N$ and use the continuity assumptions as in Proposition 2.7 to conclude the proof.
3.1. Symmetric $S_{[0,1]}$-invariant multilinear functionals. The main goal of this section is to investigate symmetric $S_{[0,1]}$-invariant multilinear functionals $\Lambda : \mathcal{W}_{[0,1]}^n \to \mathbb{R}$, where $n \in \mathbb{N}$.

**Definition 3.4.** For integers $1 \leq a < b \leq p$, define

$$e_{(a,b)}^p := \begin{cases} 1 & \text{if } a < p \text{ or } b < p, \\ 0 & \text{otherwise.} \end{cases}$$

Now define $E_p := \{ e_{(a,b)}^p : 1 \leq a < b \leq p \}$, and

$$\mathcal{W}_p := \mathcal{W}_{[0,1]} \cap \text{span}_\mathbb{R} E_p, \quad \mathcal{W}_p := \bigcup_{p=1}^{\infty} \mathcal{W}_p. \quad (3.6)$$

We classify the different symmetric $S_{[0,1]}$-invariant multilinear functionals restricted to $\mathcal{W}_p^n$ by defining constants that determine them. Let $\Lambda : \mathcal{W}_{[0,1]}^n \to \mathbb{R}$ be a multilinear functional. By multilinearity, the restriction of $\Lambda$ to $S$ is to investigate symmetric relations between the $e$ determined by only a finite number of these constants. To prove this, we investigate the isolated vertices of $H$.

Definition 3.1. Symmetric $S_{[0,1]}$-invariant multilinear functionals. The main goal of this section is to investigate symmetric $S_{[0,1]}$-invariant multilinear functionals $\Lambda : \mathcal{W}_{[0,1]}^n \to \mathbb{R}$, where $n \in \mathbb{N}$.

**Definition 3.4.** For integers $1 \leq a < b \leq p$, define

$$e_{(a,b)}^p := \begin{cases} 1 & \text{if } a < p \text{ or } b < p, \\ 0 & \text{otherwise.} \end{cases}$$

Now define $E_p := \{ e_{(a,b)}^p : 1 \leq a < b \leq p \}$, and

$$\mathcal{W}_p := \mathcal{W}_{[0,1]} \cap \text{span}_\mathbb{R} E_p, \quad \mathcal{W}_p := \bigcup_{p=1}^{\infty} \mathcal{W}_p. \quad (3.6)$$

We classify the different symmetric $S_{[0,1]}$-invariant multilinear functionals restricted to $\mathcal{W}_p^n$ by defining constants that determine them. Let $\Lambda : \mathcal{W}_{[0,1]}^n \to \mathbb{R}$ be a multilinear functional. By multilinearity, the restriction of $\Lambda$ to $S$ is determined by the infinite set of constants $(\Lambda(e))_{e \in E_p^n, \ p \geq 2}$. Surprisingly, once we assume that $\Lambda$ is symmetric and $S_{[0,1]}$-invariant, $\Lambda$ is determined by only a finite number of these constants. To prove this, we investigate the relations between the $\Lambda(e)$ for $e \in E_p^n$.

We begin by defining and explaining some basic notation that is used in the proof of Theorem[4]. First note that there is a group action of the symmetric group $S_p$ on $E_p$ and therefore on $E_p^n$, defined by $\sigma(e_{(a,b)}^p) := e_{(\sigma(a),\sigma(b))}^p$. There is also an $S_n$ action on $E_p^n$ defined by permuting the coordinates of $(e_{(a_l,b_l)}^p)_{l=1}^n$. The $S_p$ and $S_n$ actions commute so together they define an $S_p \times S_n$ action on $E_p^n$.

There is a natural map way to associate a multigraph to any tuple $x = (e_{(a_l,b_l)}^p)_{l=1}^n \in E_p^n$, with vertex set $\{1, \ldots, p\}$ and edges $\{x(l) : (x(l)_s, x(l)_t) = (a_l, b_l), 1 \leq l \leq n\}$. Denote this multigraph by $\Gamma_{n,p}(e_{(a_l,b_l)}^p)_{l=1}^n$. Given any multigraph $G$, let $[G]$ denote the equivalence class of multigraphs isomorphic to $G$.

For simplicity, we will often drop either of the subscripts in the notation for $\Gamma$ when there is no chance of confusion. If $h \in \mathcal{H}_n^{(p)}$, let $\overline{h} \in \mathcal{H}_n$ be the graph obtained by removing the isolated vertices of $h$. Similarly, if $H$ is any multigraph, then denote by $\overline{H}$ the multigraph obtained by removing the isolated vertices of $H$. The following proposition summarizes the basic properties of the map $\Gamma_{n,p}$.

**Proposition 3.7 (Properties of $\Gamma_{n,p}$).**

(i) The map sending $x \in E_p^n$ to $[\Gamma_{n,p}(x)] \in \mathcal{H}_n^{(p)}$ is surjective.

(ii) The fibers of the map sending $x \in E_p^n$ to $[\Gamma_{n,p}(x)] \in \mathcal{H}_n^{(p)}$ are precisely the $S_p \times S_n$-orbits.

(iii) The map $h \mapsto \overline{h}$ sending an element of $\mathcal{H}_n^{(p)}$ to $\mathcal{H}_n$ by removing the isolated vertices is injective. In addition, it is bijective if and only if $p \geq 2n$.

**Proof.**

(i) Let $G$ denote a representative of a class in $\mathcal{H}_n^{(p)}$, and fix bijections

$$\phi : V(G) \to \{1, \ldots, p\}, \quad \psi : E(G) \to \{1, \ldots, n\}.$$

Now define $x = (e_{(a_l,b_l)}^p)_{l=1}^n \in E_p^n$ via: $a_i := \phi(\psi^{-1}(i)_s), b_i := \phi(\psi^{-1}(i)_t)$. Clearly $\Gamma_{n,p}(x)$ and $G$ are isomorphic so $\Gamma_{n,p}$ is surjective onto $\mathcal{H}_n^{(p)}$. 

[4] An anonymous referee suggested the following notation.
(ii) Let \( x \in E^n_p \), \( \sigma \in S_p \), and \( \tau \in S_n \). Then \( \Gamma_{n,p}(x) \) and \( \Gamma_{n,p}(\tau(x)) \) are the same multigraph and \( \Gamma_{n,p}(x) \) and \( \Gamma_{n,p}(\sigma(x)) \) are clearly isomorphic multigraphs.

Conversely, if \( \Gamma_{n,p}(x_1) \) is isomorphic to \( \Gamma_{n,p}(x_2) \) then there exist two maps \( V_f : V(\Gamma_{n,p}(x_1)) \to V(\Gamma_{n,p}(x_2)) \) and \( E_f : E(\Gamma_{n,p}(x_1)) \to E(\Gamma_{n,p}(x_2)) \) that form an isomorphism of multigraphs \( f \). Note that \( V_f \in S_p \) because it is a bijection \( \{1, \ldots, p\} \to \{1, \ldots, p\} \). If \( x_1 = (e^p_{(a_l,b_l)})_{l=1}^n \) and \( x_2 = (e^p_{(c_l,d_l)})_{l=1}^n \) then the bijection \( E_f \) defines a bijective map \( \tau : \{1, \ldots, n\} \to \{1, \ldots, n\} \) by sending \( i \to j \) if the \( i \)th edge \( (a_l, b_l) \) maps to the \( j \)th edge \( (c_l, d_l) \). Then, \( (V_f, \tau) \in S_p \times S_n \) and \( (V_f, \tau)(x_1) = x_2 \).

(iii) The map \( h \to \overline{h} \) is clearly bijective. The graph with the most number of vertices in \( \mathcal{H}_n \) is the one with \( n \) disjoint edges - i.e., \( A_{2}^{\Pi n} \). This has \( 2n \) vertices and so when \( p \geq 2n \), \( h \to \overline{h} \) surjects onto \( \mathcal{H}_n \), while \( A_{2}^{\Pi n} \) does not lie in the image when \( p < 2n \).

For fixed \( p \geq 2 \), we now show how Proposition 3.7 allows us to define constants associated to \( \Lambda \in \mathfrak{X}_n \) indexed by \( h \in \mathcal{H}_n^{(p)} \), that carry all of the information of \( (\Lambda(x))_{x \in E^n_p} \). In particular, since \( \Lambda \) is symmetric and \( S_{[0,1]} \)-invariant, it is invariant under the \( S_n \) and \( S_p \) actions respectively.

**Definition 3.8.** Let \( \Lambda : \mathcal{H}_n^{(p)} \to \mathbb{R} \) be a symmetric \( S_{[0,1]} \)-invariant multilinear functional, i.e., \( \Lambda \in \mathfrak{X}_n \). Then for any \( p \geq 2 \) and \( h \in \mathcal{H}_n^{(p)} \), pick by Proposition 3.7(ii) an \( x \in E^n_p \) such that \( h = [\Gamma_{n,p}(x)] \), and define

\[
C_{n,p}(\Lambda)(h) := \Lambda(x).
\]

The value of \( C_{n,p}(\Lambda)(h) \) does not depend on the choice of \( x \) by Proposition 3.7(ii).

Also define the map \( C_n : \mathfrak{X}_n \to \prod_{p \geq 2} X_{n,p} \) where \( X_{n,p} = \mathbb{R} \mathcal{H}_n^{(p)} \) by

\[
C_n(\Lambda) := (C_{n,p}(\Lambda))_{p \geq 2}, \quad \text{where} \quad C_{n,p}(\Lambda) := (C_{n,p}(\Lambda)(h))_{h \in \mathcal{H}_n^{(p)}}.
\]

The following theorem reveals the relations between the vectors \( C_{n,p}(\Lambda) \). We shall see that the vectors necessarily satisfy certain compatibility conditions, for a fixed \( \Lambda \) and varying \( p \in \mathbb{N} \). More surprisingly, we now show that for each \( n, k \in \mathbb{N} \), there exists a single matrix that determines the compatibility constraints, across all \( \Lambda \in \mathfrak{X}_n \) and all \( p \geq 2 \).

**Theorem 3.9 (Consistency Relations).** Fix \( n, k \in \mathbb{N} \). There exists a fixed matrix \( \pi_{n,k} \in \mathbb{Z}_{\geq 0}^{\mathcal{H}_n \times \mathcal{H}_n} \) such that for any \( p \geq 2 \), \( \Lambda \in \mathfrak{X}_n \), and \( g \in \mathcal{H}_n^{(p)} \), we have:

\[
C_{n,p}(\Lambda)(g) = \sum_{h \in \mathcal{H}_n^{(kp)}} \pi_{n,k}(\overline{h}, h) C_{n,kp}(\Lambda)(h), \quad \forall g \in \mathcal{H}_n^{(p)}.
\]

In addition, \( \pi_{n,k}(\overline{h}, h) \) is nonzero only if there exists \( H \in \overline{h} \) that surjects onto \( G \in \overline{g} \) as a multigraph, and \( \pi_{n,k}(\overline{g}, \overline{h}) > 0 \).

**Proof.** Write the basis elements of \( E_p \) in terms of \( k^2 \) basis elements in \( E_{kp} \) as follows:

\[
e^p_{(a,b)} = \sum_{i,j=1}^k e^{kp}_{(k(a-1)+i,k(b-1)+j)}.
\]

Now choose any \( x = (e^p_{(a_l,b_l)})_{l=1}^n \in E^n_p \) such that \( g = [\Gamma_{n,p}(x)] \), and expand

\[
C_{n,p}(\Lambda)(g) = \Lambda((e^p_{(a_l,b_l)})_{1 \leq l \leq n}) = \sum_{(i_1,j_1) \in \{1, \ldots, k\}^{2n}} \Lambda((e^p_{(k(a_l-1)+i_l,k(b_l-1)+j_l)})_{1 \leq l \leq n})
\]
by splitting up each basis element using equation (3.11) and multilinearity. For every choice of \((i_l,j_l)\in\{1,\ldots,k\}^{2n}\), define a graph

\[
H((i_l,j_l)_{l=1}^n) = \Gamma_{n,kp}(e_{(k(a_l-1)+i_l,k(b_l-1)+j_l)_l}^{n})_{1\leq l\leq n}).
\]

We can then rewrite Equation (3.12) as

\[
C_{n,p}(\Lambda)(g) = \Lambda((e_{(a_l,b_l)}^{p})_{l=1}^{n})_{1\leq l\leq n} = \sum_{(i_l,j_l)\in\{1,\ldots,k\}^{2n}} C_{n,kp}(\Lambda)((H((i_l,j_l)_{l=1}^n))].
\]

Let the map \(\alpha_{n,k,p,x} : \{1, \ldots k\}^{2n} \to \mathcal{H}_n\) be defined by sending \((i_l,j_l)_{l=1}^n\) to \([H((i_l,j_l)_{l=1}^n)]\) and let \(M(n,k,p,x,h)\) be the size of the fiber of \(\alpha_{n,k,p,x}\) over \(h \in \mathcal{H}_n\). Then

\[
C_{n,p}(\Lambda)(g) = \sum_{h\in\mathcal{H}_n^{(kp)}} C_{n,kp}(\Lambda)(h) \cdot M(n,k,p,x,h).
\]

Using the \(S_p\times S_n\) action on \(\alpha_{n,k,p,x}\), one verifies that \(M(n,k,p,x,h) = M(n,k,p,\sigma,\tau(x),h)\) for all \((\sigma,\tau) \in S_p \times S_n\). Hence by Proposition 3.7(ii), \(M(n,k,p,\Gamma_{n,p}(x)),h) := M(n,k,p,x,h)\) is well-defined, and

\[
C_{n,p}(\Lambda)(g) = \sum_{h\in\mathcal{H}_n^{(kp)}} C_{n,kp}(\Lambda)(h) \cdot M(n,k,p,g,h).
\]

We now claim that for any \(g \in \mathcal{H}_n^{(p)}\) and \(h \in \mathcal{H}_n\), \(M(n,k,p,g,h) = \pi_{n,k}(\mathcal{H},h)\) for some fixed matrix \(\pi_{n,k} \in \mathbb{Z}^{\mathcal{H}_n \times \mathcal{H}_n}\) independent of \(p\) and \(\Lambda\). Indeed, given integers \(2 \leq p \leq p'\) and \(g \in \mathcal{H}_n^{(p)}\), choose \(x := (e_{(a_l,b_l)}^{p})_{l=1}^{n}\) with \([\Gamma_{n,p}(x)] = g\). Now define \(x' := (e_{(a_l,b_l)}^{p})_{l=1}^{n}\) and \(g' := [\Gamma_{n,p'}(x')]\); then \([g] = [g']\). Moreover, \(\alpha_{n,k,p,x}((i_l,j_l)_{l=1}^n) = \alpha_{n,k,p',x}(x)_{(i_l,j_l)_{l=1}^n}\) for all \(((i_l,j_l)_{l=1}^n) \in \{1, \ldots, k\}^{2n}\). Therefore since \(M(n,k,p,g,h)\) is the size of the fiber of \(\alpha_{n,k,p,x}\) and \(M(n,k,p',g',h)\) is the size of the fiber of \(\alpha_{n,k,p',x}\) over \(h \in \mathcal{H}_n\), \(M(n,k,p,g,h) = M(n,k,p,g',h) =: \pi_{n,k}(\mathcal{H},h)\).

It remains to show the last sentence of the result. Suppose \(\pi_{n,k}(g,h) > 0\) for \(g,h \in \mathcal{H}_n\). Pick arbitrary fixed \(p \geq 2n\) and \(x = (e_{(a_l,b_l)}^{p})_{l=1}^{n}\) such that \([\Gamma_{n,p}(x)] = g\) by Proposition 3.7. Then there exists \((i_l,j_l)_{l=1}^n) \in \{1, \ldots, k\}^{2n}\) such that \(h = [H((i_l,j_l)_{l=1}^n)]\) (see Equation (3.13)) by the above analysis. There is an obvious surjective map from \(H((i_l,j_l)_{l=1}^n)\) to \(\Gamma_{n,p}(x)\) given by sending the vertex \(a\) to the vertex \([(a-1)/k] + 1\) and sending the \(l\)th edge of \(H((i_l,j_l)_{l=1}^n)\) to the \(l\)th edge of \(\Gamma_{n,p}(x)\). Therefore, \(\pi_{n,k}(g,h) > 0\) implies that there exists a surjective map from a multigraph \(H \in h\) to a multigraph \(G \in g\). In addition, picking \((i_l,j_l) = (1,1)\) for \(1 \leq l \leq n\) shows that \(\pi_{n,k}(g,g) > 0\).

**Definition 3.14.** Given \(n, k, 2 \leq p \in \mathbb{N}\), define the map \(\pi_{n,k,p} : X_{n,p} \to X_{n,p}\) as follows: \(\pi_{n,k,p} : X_{n,p} \to X_{n,p}\) sends the vector \(A_{n,kp} = (A_{n,kp}(h))_{h\in\mathcal{H}_n^{(kp)}}\) to the vector \(A_{n,p} = (A_{n,p}(g))_{g\in\mathcal{H}_n^{(p)}}\), where

\[
A_{n,p}(g) = \sum_{h\in\mathcal{H}_n^{(kp)}} \pi_{n,k}(\mathcal{H},h) A_{n,kp}(h).
\]

We call the linear maps \(\pi_{n,k,p}\) the *consistency constraints*. We also say that any vector \(A = (A_{n,p})_{p\geq 2} \in \prod_{p\geq 2} X_{n,p}\) satisfying the constraints

\[
A_{n,p} = \pi_{n,k,p} (A_{n,kp})
\]

is linearly consistent.
Using the consistency constraints, we now prove that multilinear functionals $\Lambda \in \mathfrak{x}_n$ restricted to $\mathcal{W}_p$ are determined by $|\mathcal{H}_n|$ constants.

**Theorem 3.15.** Fix $n, k \in \mathbb{N}$. Then the following hold.

(i) The matrix $\pi_{n,k}$ is triangular with positive diagonal entries when $\mathcal{H}_n$ is partially ordered by the existence of a surjective map of multigraphs.

(ii) The $\pi_{n,k,p} \to p$ are surjective maps that are invertible for $p \geq 2n$, and compatible in the following sense: given positive integers $n, k_1, k_2, 2 \leq p$,

$$\pi_{n,k_1k_2,p} = \pi_{n,k_2,p} \circ \pi_{n,k_1k_2 \rightarrow k_2,p}. \tag{3.16}$$

(iii) For each $n \in \mathbb{N}$, the subspace $LC_n \subset \prod_{p \geq 2} X_{n,p}$ of linearly consistent vectors

$$(A_{n,p}(h))_{h \in \mathcal{H}_n^{(p)}, p \geq 2} \in \prod_{p \geq 2} X_{n,p}$$

has dimension $|\mathcal{H}_n|$.

(iv) If $\Lambda \in \mathfrak{x}_n$, the $|\mathcal{H}_n|$ components of $C_{n,p_0}(\Lambda) \in X_{n,p_0}$ for any $p_0 \geq 2n$ determine the value of $\Lambda$ on $\sigma(\mathcal{W}_p^{n})$ for all $\sigma \in S_{[0,1]}$.

**Proof.**

(i) Note that the graphs in $\mathcal{H}_n$ are partially ordered by the existence of a surjective map of multigraphs. By Theorem 3.9, $\pi_{n,k}([g], [h]) > 0$ for any two multigraphs $g, h$ with $n$ edges only if there exists a surjective map from $h$ to $g$. Thus $\pi_{n,k}$ is triangular when $\mathcal{H}_n$ is ordered with respect to the existence of a surjective map. In addition, Theorem 3.9 states that $\pi_{n,k}([g], [g]) > 0$, so the diagonal entries of $\pi_{n,k}$ are positive.

(ii) It is easy to see that $\mathcal{H}_n^{(p)} \rightarrow \mathcal{H}_n^{(kp)}$. Let the corresponding subspace of $\mathbb{R}^{\mathcal{H}_n^{(kp)}} = X_{n,kp}$ be called $Y_{n,p,k}$. We now claim that $\pi_{n,k,p} \rightarrow : Y_{n,p,k} \rightarrow X_{n,p}$ is an isomorphism - in particular, it is surjective. Indeed, this is obvious since the restriction of $\pi_{n,k,p} \rightarrow$ to $Y_{n,p,k}$ is given by a principal submatrix of $\pi_{n,k}$, which is itself triangular with nonzero diagonal entries. Now if $p \geq 2n$, the maps $\pi_{n,k,p} \rightarrow$ are invertible because $Y_{n,p,k} = X_{n,kp}$.

Finally, to show that $\pi_{n,k,p} \rightarrow \circ \pi_{n,k_1k_2,p} \rightarrow = \pi_{n,k_1k_2,p} \rightarrow$, note that expanding basis elements in $E_p$ by Equation (3.11) into basis elements elements in $E_{k_1k_2,p}$, via

$$e^p_{(a,b)} = \sum_{i,j=1}^{k_1k_2} e^{k_1k_2p} (k_1k_2(a-1)+i,k_1k_2(b-1)+j)$$

is the same as expanding $e^p_{(a,b)}$ into basis elements in $E_{k_2p}$ and then splitting those basis elements into basis elements in $E_{k_1k_2,p}$. The proof follows by using counting arguments as in the proof of Theorem 3.9.

(iii) Fix $p_0 \geq 2n$. We show that the map $P : LC_n \rightarrow X_{n,p_0}$ sending $A = (A_{n,p})_{p \geq 2}$ to $A_{n,p_0}$ is a linear isomorphism. Indeed, $P$ is injective because if $A_{n,p_0} = 0$, then by (ii),

$$A_{n,p} = \pi_{n,p_0,p} \circ \pi_{n,p_0,p_0}^{-1} (A_{n,p_0}) = 0, \quad \forall p \geq 2.$$  

We now show that $P : LC_n \rightarrow X_{n,p_0}$ is surjective. Fix $A_{n,p_0} \in \mathbb{R}^{\mathcal{H}_n^{(p_0)}}$, and define for $p \geq 2$:

$$A_{n,p} := \pi_{n,p_0,p} \circ \pi_{n,p_0,p_0}^{-1} (A_{n,p_0}) \in \mathbb{R}^{\mathcal{H}_n^{(p)}}.$$
It remains to show that \( \pi_{n,kp} \circ (A_{n,kp}) = A_{n,p} \) for all \( p \geq 2 \) and \( k \geq 1 \). This follows by diagram chasing in the following diagram, which commutes by Equation \((3.16)\).

(iv) Recall by \( S_{[0,1]} \)-invariance, that \( \Lambda \) restricted to \( \sigma(W^n_p) \) is determined by \( C_n(\Lambda) = (C_{n,p}(\Lambda))_{p \geq 2} \). In turn, \( C_n(\Lambda) \in LC_n \) is determined by \( C_{n,p_0}(\Lambda) \in X_{n,p_0} \) for any \( p_0 \geq 2n \), by the previous part.

\[ \square \]

Though the main goal of this subsection was to prove Theorem \[3.15\] (along the way to proving Theorem \[1.1\]), a question of independent interest is to explicitly compute all entries of the \( n \)-tuples \((i_1,\ldots,i_n)\). A related question is to explicitely compute all entries of the matrix \( \Gamma_{n,kg} \), where \( g \) is a fixed graph.

Consider fixed multigraphs \( G, H \) without isolated vertices such that \([G],[H] \in \mathcal{H}_n\). Denote \( g = [G] \) and \( h = [H] \). Also fix \( p \geq 2n \). Then by Proposition \[3.7\] there exists \( x = (e_p^{g})_{i=1}^{n} \in E_n^{p} \) such that \([\Gamma_{n,p}(x)] = g\). For the remainder of this proof, we fix such an \( x \) as well as an isomorphism \( \phi : G \to \Gamma_{n,p}(x) \).

Now define a map \( y_{x} : \{1,\ldots,k\}^{2n} \to E_n^{kp} \) by:

\[ y_{x}((i_1,j_1),\ldots,(i_n,j_n)) := (e_{(k(a_1-1)+i_1,k(b_1-1)+j_1)})_{1 \leq l \leq n}. \]

In this new notation, our aim is to compute the quantity \( \pi_{n,k}(g,h) \), which is the number of tuples \((i_1,j_1),\ldots,(i_n,j_n)\) such that \([\Gamma_{n,kp}(y_{x}((i_1,j_1),\ldots,(i_n,j_n)))]) = h\). To do so, recall from the proof of Theorem \[3.9\] that such a tuple also yields a graph surjection from \( \Gamma_{n,kp}(y_{x}((i_1,j_1),\ldots,(i_n,j_n))) \) to \( \Gamma_{n,p}(x) \) as follows.

Given a tuple \((i_1,j_1),\ldots,(i_n,j_n)\) such that \( y_{x}((i_1,j_1),\ldots,(i_n,j_n)) \in E_n^{kp} \). As in the proof of Theorem \[3.9\], there exists a surjective map \( \Gamma_{n,kp}(y) \to \Gamma_{n,p}(x) \) defined by sending the vertex \( v \in \{1,\ldots,kp\} \) to \([v-1]/k) + 1 \in \{1,\ldots,p\} \) and sending the \( l \)th edge \( y(l) \in E(\Gamma_{n,kp}(y)) \) to the \( l \)th edge \( x(l) \in E(\Gamma_{n,p}(x)) \). Composing this surjection with the map \( \phi^{-1} \) yields a surjective map \( \Gamma_{n,kp}(y) \to G \) that we call \( P_y \). Since the map \( y_{x} \) is an injection there is no ambiguity in using the \( P_y \) notation.

We would like to compute the quantity \( \pi_{n,k}(g,h) \) by summing over all possible surjective maps \( \Gamma_{n,kp}(y) \to G \) that arise in the above manner. However, to deal with the fact that the \( \Gamma_{n,kp}(y) \) are distinct graphs, we instead begin by associating to surjective maps from a fixed graph \( H \) to \( G \) different tuples \( y \in \text{Im}(h_{x}) \) such that the surjective map can be factored through \( P_y \). More precisely, we define a map \( A : \text{Surj}(H,G) \to 2^{E_n^{kp}} \) as follows. Given any
surjective map of multigraphs $\psi : H \to G$, the set $A(\psi) \subseteq E^n_{kp}$ will consist of the distinct $y \in \text{Im}(y_x) \subseteq E^n_{kp}$ such that there exists an isomorphism $\beta : H \to \Gamma_{n,kp}(y)$ with $\psi = P_y \circ \beta$.

We would like to compute the number of $y \in \text{Im}(y_x) \subseteq E^n_{kp}$ such that $[\Gamma_{n,p}(y)] = h$. We claim that any such $y$ is in $\cup_{\psi:H\to G} A(\psi)$. Indeed, one can take an arbitrary isomorphism $\beta^{-1} : \Gamma_{n,p}(y) \to H$, and then define $\psi := P_y \circ \beta$. Then $\psi : H \to G$ is a surjective map and $y \in A(\psi)$. Conversely, if $y \in A(\psi)$ then by the above analysis, $\Gamma_{n,p}(y)$ is isomorphic to $H$ through $\beta$ and $y \in \text{Im}(y_x)$ by definition. Therefore, to compute $\pi_{n,k}$ it suffices to compute the size of $\cup_{\psi:H\to G} A(\psi)$.

To do this, we first show that $|A(\psi)| = \prod_{v \in V(G)} k(k - 1) \cdots (k - |\psi^{-1}(v)| + 1)$. Indeed, for every $v \in V(G)$ consider the $k(k - 1) \cdots (k - |\psi^{-1}(v)| + 1)$ distinct ways of sending the vertices of $u \in \psi^{-1}(v)$ to distinct vertices $y_{v}(v) - 1 + i_{u}$ for $i_{u} \in \{1, \ldots, k\}$.

Each choice defines an injective map $V_{\beta} : V(H) \to \{1, \ldots, kp\}$. We can now pick a tuple $(i_{t}, j_{t})^{n}_{t=1} \in \{1, \ldots, k\}^{2n}$ and $E_{\beta} : E(H) \to \{1, \ldots, n\}$ such that $(V_{\beta}(e_{s}), V_{\beta}(e_{t})) = (y(E_{\beta}(e_{s})), y(E_{\beta}(e_{t})))$ where $y = y_{v}(i_{t}, j_{t})^{n}_{t=1}$. Then $V_{\beta}$ and $E_{\beta}$ together define an isomorphism $\beta : H \to \Gamma_{n,kp}(y)$ such that $\psi = P_y \circ \beta$. It is not hard to see that every element of $y \in A_\psi$ arises in this way.

To show that counting every $y \in A_\psi$ overcounts by a factor of $|\text{Aut}(H)|$. Note that each automorphism $\alpha \in \text{Aut}(H)$ yields a distinct surjective map $\psi \circ \alpha : H \to G$ and $A(\psi) = A(\psi \circ \alpha)$. Conversely, assume that there exists $y \in E^n_{kp}$ such that $y \in A(\psi_{1}) \cap A(\psi_{2})$. Then there are isomorphisms $\beta_{1}, \beta_{2} : H \to \Gamma_{n,kp}(y)$ satisfying $\psi_{1} = P_{y} \circ \beta_{1}$ and $\psi_{2} = P_{y} \circ \beta_{2}$. Therefore $\psi_{1} \circ \beta_{1}^{-1} \circ \beta_{2} = \psi_{2}$, and $\beta_{1}^{-1} \circ \beta_{2} \in \text{Aut}(H)$. We conclude that the sets $A(\psi)$ are either disjoint or equal. In addition, $A(\psi_{1}) = A(\psi_{2})$ if and only if there exists $\alpha \in \text{Aut}(H)$ such that $\psi_{1} = \psi_{2} \circ \alpha$. Therefore we finally obtain:

$$
\pi_{n,k}([G],[H]) := \frac{1}{|\text{Aut}(H)|} \sum_{\psi : H \to G} \prod_{v \in V(G)} k(k - 1) \cdots (k - |\psi^{-1}(v)| + 1).
$$

\[\square\]

Remark 3.18. In certain special cases, the formula of Proposition 3.17 is easy to evaluate. For instance when $[G] = [H]$, we have

$$
\pi_{n,k}([G],[H]) = \frac{1}{|\text{Aut}(G)|} \sum_{\psi : H \to G} \prod_{v \in V(G)} k(k - 1) \cdots (k - |\psi^{-1}(v)| + 1).
$$

However, $|\psi^{-1}(v)| = 1$ always in this case and $|\text{Surj}(H,G)| = |\text{Aut}(G)|$, so this formula reduces to: $\pi_{n,k}([G],[H]) = k^{|V(G)|}$.

If instead $[H] = A_2^{1n}$, then

$$
\pi_{n,k}([G],[A_2^{1n}]) = \frac{1}{|\text{Aut}(H)|} \sum_{\psi : H \to G} \prod_{v \in V(G)} k(k - 1) \cdots (k - |\psi^{-1}(v)| + 1).
$$

In this case, $|\text{Aut}(H)| = 2^{n}n!$ and each of the $2^{n}n!$ maps $\psi \in \text{Surj}(H,G)$ satisfy $|\psi^{-1}(v)| = \deg(v)$, so we obtain: $\pi_{n,k}([G],[A_2^{1n}]) = \prod_{v \in V(G)} k(k - 1) \cdots (k - \deg(v) + 1)$.

3.2. Bases of consistent vectors. Given a $C^n$ class function $F$, note by Proposition 3.2 that $d^nF(0; g_1, \ldots, g_n) \in X_n$. Now define

$$
T_{n,p}(F)(h) := C_{n,p}(d^nF(0,-))(h), \quad p \geq 2, \; h \in H_n^{(p)}.
$$
By Theorem 3.9, we obtain a linearly consistent vector \( T_n(F) := (T_{n,p}(F))_{p \geq 2} \in \prod_{p \geq 2} X_{n,p} \), where \( T_{n,p}(F) := (T_{n,p}(F)(h))_{h \in \mathcal{H}_n(p)} \). Define \( T_0(F) := F(0) \). If \( F \) is a \( C^N \) class function, then \( T_n(F) \) is defined for integers \( n \in \{0, \ldots, N\} \). We will sometimes write

\[
T(F) := (T_n(F))_{n \geq 0} \in \prod_{n \geq 0} \prod_{p \geq 2} X_{n,p}
\]

for the entire collection if \( F \) is smooth.

Theorem 3.15 then asserts that the values of the derivatives \( d^n F(0; g_1, \ldots, g_n) \) for directions \( g_1, \ldots, g_n \in \mathcal{W}_p \) are determined by \( T(F) \). In the case of solutions \( F \) to the differential equation of Theorem 1.1, \( T(F) \) determines \( F \) restricted to \( \mathcal{W}_p \) - and by continuity, on all of \( \mathcal{W}_{[0,1]} \). Therefore, Theorem 3.15 in fact already shows that the space of solutions in Theorem 1.1 has dimension at most \( \sum_{n \leq N} |\mathcal{H}_n| \).

To complete the proof we now show that the \( t(H, -) \) form a linearly independent family of solutions. In order to do so, we obtain a general formula for the derivatives of \( t(H, -) \) in Proposition 3.20. We then use that result in Theorem 3.21 to give a combinatorial formula for the \( T(t(H, -)) \) from which linear independence follows.

We begin by proving the following useful lemma.

**Lemma 3.19.** Let \( \Lambda : \mathcal{W}^n_{[0,1]} \to \mathbb{R} \) be a multilinear functional. Then the function \( F(g) := \Lambda(g, g, \ldots, g) \) is Gâteaux smooth. The Gâteaux derivatives of \( F \) are

\[
d^k F(f; g_1, g_2, \ldots, g_k) = \frac{1}{(n - k)!} \sum_{\sigma \in S_n} \Lambda(f, \sigma(f), g_1, \ldots, g_k) \quad \forall 0 \leq k \leq n,
\]

where \( f \in \mathcal{W}_{[0,1]}, g_i \in \text{Adm}(f), \) and \( (f, \ldots, f, g_1, \ldots, g_k) \in \mathcal{W}^n \).

**Proof.** By multilinearity,

\[
dF(f; g_1) = \lim_{h \to 0} \frac{\Lambda(f + hg_1, f + hg_1, \ldots, f + hg_1) - \Lambda(f, f, \ldots, f)}{h}
\]

\[
= \lim_{h \to 0} \left[ \Lambda(g_1, f, \ldots, f) + \Lambda(f, g_1, f, \ldots, f) + \cdots + \Lambda(f, \ldots, f, g_1) + O(h) \right]
\]

\[
= \Lambda(g_1, f, \ldots, f) + \Lambda(f, g_1, f, \ldots, f) + \cdots + \Lambda(f, \ldots, f, g_1)
\]

\[
= \frac{1}{(n - 1)!} \sum_{\sigma \in S_n} \Lambda(f, \sigma(f), g_1).
\]

Note that each term in the sum is also multilinear so the result follows by induction. Finally, all Gâteaux derivatives of order \( n + 1 \) are zero, so all Gâteaux derivatives of all orders are continuous. \( \square \)

We now apply Lemma 3.19 to compute derivatives of homomorphism densities \( t(H, f) \).

**Proposition 3.20.** Given \( H \in \mathcal{H}_n \), the functions \( t(H, -) \) are Gâteaux smooth. Their Gâteaux derivatives \( d^n t(H, f; g_1, g_2, \ldots, g_n) \) are all zero if \( n > |E(H)| \), while if \( n \leq |E(H)| \), then

\[
d^n t(H, f; g_1, g_2, \ldots, g_n)
\]

\[
= \sum_{\lambda \subseteq E(H)} \sum_{|\lambda| = n} \int_{[0,1]^{\lambda}} \prod_{i=1}^{n} g_i(x_{\lambda(i)_1}, x_{\lambda(i)_2}) \prod_{e \in E(H) \setminus A} f(x_{e_1}, x_{e_2}) \prod_{i \in V(H)} dx_i.
\]
Theorem 3.21.

(i) Let \( n \in \mathbb{N} \) and \( p \geq 2 \) be integers. If \( H \in \mathcal{H} \) and \( h \in \mathcal{H}_n^{(p)} \), then \( T_{n,p}(t(H,-))(h) = 0 \) if \( |E(H)| \neq n \).

(ii) Let \( H \in \mathcal{H}_n \) and \( h \in \mathcal{H}_n^{(p)} \). Then

\[
T_{n,p}(t(H,-))(h) = |\text{Surj}(H, H)|^{V(H)}.
\]

Therefore, \( T_{n,p}(t(H,-))(h) > 0 \) for \( H \in \mathcal{H}_n \) and \( h \in \mathcal{H}_n^{(p)} \) if and only if there exists a surjective map from \( H \) to \( \overline{h} \).

(iii) The vectors \( \{T_{n,p}(t(H,-))(h)\}_{h \in \mathcal{H}_n^{(p)}} \) for fixed \( p \geq 2n \) and \( H \) varying over all of \( \mathcal{H}_n \) are linearly independent.

(iv) For all \( n \in \mathbb{N} \), the vector space \( LC_n \) of linearly consistent vectors \( (A_{n,p}(h))_{h \in \mathcal{H}_n,p \geq 2} \in \prod_{p \geq 2} X_{n,p} \) has a basis given by

\[
\{(T_{n,p}(t(H,-))(h))_{h \in \mathcal{H}_n^{(p)},p \geq 2} : H \in \mathcal{H}_n\}.
\]

Proof.

(i) For \( |E(h)| > |E(H)| \), \( T_{n,p}(t(H,-))(h) = 0 \) because higher derivatives are zero by Proposition 3.20. For \( |E(h)| < |E(H)| \), \( T_{n,p}(t(H,-))(h) = 0 \) by Proposition 3.20 because we are evaluating a lower-order derivative at 0.

(ii) We claim that \( T_{n,p}(t(H,-))(h) = |\text{Surj}(H, h)|^{V(H)} \) where \( \text{Surj}(H, h) \) is the set of surjective maps from \( H \) to \( h \). To prove this claim, let \( H \in \mathcal{H}_n \) be fixed. For every fixed \( h \in \mathcal{H}_n^{(p)} \), fix a bijective map \( \phi : V(h) \to \{1, \ldots, p\} \). We now define the tuples \( g(h) \) by:

\[
g(h)_e := e_{\phi(e_{\sigma(e)}^e)}^{(\phi(e_{\sigma(e)}^e))} \quad \text{for } e \in E(h).
\]

Then for all \( H \in \mathcal{H}_n \),

\[
T_{n,p}(t(H,-))(h) = d^n t(H, 0; (g(h)_e)_{e \in E(h)})
\]

\[
= \sum_{\sigma : E(H) \to E(h)} \int_{[0,1]^0} \prod_{e \in E(H)} \int_{[0,1]} g(h)_e(x_{e_l}, x_{e_t}) \prod_{i \in V_0} dx_i,
\]

where the last equality follows by Proposition 3.20 (Note that the order of Gâteaux differentiation does not matter since mixed partials are equal.)
To prove the claim, consider an arbitrary term in (3.22). Then \( g(h) \) is constant on each “sub-rectangle” in \([0, 1]^{\mathcal{V}(H)}\) of size \(1/p^{\mathcal{V}(H)}\). Hence

\[
\int_{[0,1]^{\mathcal{V}(H)}} \prod_{e \in E(H)} g(h)_{\sigma(e)}(x_{e_s}, x_{e_t}) \prod_{i \in V(H)} dx_i = \frac{1}{p^{\mathcal{V}(H)}} \sum_{\tau:V(H) \rightarrow \{1, \ldots, p\}} \prod_{e \in E(H)} g(h)_{\sigma(e)} \left( \frac{\tau(e_s) - 0.5}{p}, \frac{\tau(e_t) - 0.5}{p} \right). \tag{3.23}
\]

By our choice of \( g(h) \), we have

\[
g(h)_{\sigma(e)} \left( \frac{\tau(e_s) - 0.5}{p}, \frac{\tau(e_t) - 0.5}{p} \right) = \begin{cases} 
1 & \text{if } \{\tau(e_s), \tau(e_t)\} = \{\phi(\sigma(e)_s), \phi(\sigma(e)_t)\}, \\
0 & \text{otherwise}.
\end{cases}
\]

Since \( \phi \) is bijective we can define for such \( \tau \) vertex maps \( \phi^{-1}\tau : V(H) \rightarrow V(h) \). We recognize the expression on the right-hand side of Equation (3.23) to be equal to \( \frac{1}{p^{\mathcal{V}(H)}} \) times the number of vertex maps \( \phi^{-1}\tau : V(H) \rightarrow V(h) \) that form a map of multigraphs \( H \rightarrow h \), when combined with the edge map \( \sigma : E(H) \rightarrow E(h) \). In addition, Equation (3.22) sums over all surjective maps \( \sigma : E(H) \rightarrow E(h) \) so \( T_{n,p}(t(H, -))(h) = |\text{Surj}(H, h)|/p^{\mathcal{V}(H)} \). Since \( H \) has no isolated vertices, there is a natural bijection between \( \text{Surj}(H, h) \) and \( \text{Surj}(\tilde{H}, h) \); thus the result follows.

(iii) The vectors \( (T_{n,p}(t(H, -))(h))_{h \in \mathcal{H}_n} \) for fixed \( p \geq 2n \) and \( H \) varying over all of \( \mathcal{H}_n \) form an upper triangular matrix with non-zero diagonal if ordered consistently with the existence of a surjection. Therefore, the matrix is non-singular and the assertion of linear independence follows.

(iv) The vector space of linearly consistent vectors \( (A_{n,p}(h))_{h \in \mathcal{H}_n}^{[p]} \) has dimension \( |\mathcal{H}_n| \) by Theorem 3.15. On the other hand, \( T(t(H, -)) \) for \( H \in \mathcal{H}_n \) are a linearly independent sets of size \( |\mathcal{H}_n| \) so they form a basis.

\[ \square \]

As a consequence of Theorem 3.21 we show the linear independence of \( t(H, -) \) for multigraphs \( H \in \mathcal{H} \). Such linear independence results go back to Whitney [32] for \( H \in \mathcal{G} \).

**Corollary 3.24.** The homomorphism densities \( t(H, -) \) for \( H \in \mathcal{H} \) are linearly independent as functions on \( \mathcal{W}_p := \bigcup_{p=1}^{\infty} \mathcal{W}_p \), and hence on \( \mathcal{W} \). In particular, the \( t(H, -) \) are also linearly independent for \( H \in \mathcal{G} \).

**Proof.** Assume that there is a finite linear relation \( \sum_{H \in \mathcal{H}} a_H t(H, f) = 0 \) (for all \( f \)). Now by taking derivatives at zero, it is clear by Theorem 3.21(i) that \( \sum_{H \in \mathcal{H}_n} a_H T_n(t(H, -)) = 0 \) for every \( n \in \mathbb{N} \). By Theorem 3.21(iv), we conclude that \( a_H = 0 \) for all \( H \in \mathcal{H}_n \) and all \( n \). \( \square \)

**Remark 3.25.** The proof shows that the linear independence of functions \( t(H, -) \) for \( H \in \mathcal{H}_{\leq n} \) holds even when restricted to their values on \( \mathcal{W}_{2n} \).

We now bring together the results of the previous sections and this section to prove the main theorem.

**Proof of Theorem 1.1.** We associated to any smooth continuous class solution \( F \), the linearly consistent vector \( T(F) \in \prod_{n \geq 0} \prod_{p \geq 2} X_{n,p} \). By parts (i),(iv) of Theorem 3.21 the space
of linearly consistent vectors are spanned by those arising from homomorphism densities. It follows that there exist constants \(a_H\) so that

\[
T(F) = \sum_{H \in \mathcal{H}_{\leq N}} a_H t(H, -)).
\] (3.26)

Note that for any fixed direction \(f \in \mathcal{W}_p\), the one-dimensional differential equation is solved by

\[
F(f) = F(0) + dF(0; f) + \frac{d^2 F(0; f, f)}{2!} + \cdots + \frac{d^N F(0; f, \ldots, f)}{N!}.
\]

Since the derivatives are all multilinear, the value of \(F\) on \(\mathcal{W}_p\) is therefore determined by \(T(F)\) by this formula. The same applies to \(t(H, -)\) for all \(H \in \mathcal{H}_{\leq N}\), so we see that \(F(f) = \sum_{H \in \mathcal{H}_{\leq N}} a_H t(H, f)\) on \(f \in \mathcal{W}_p\).

Since both sides are continuous in the \(L^1\) topology by Proposition 2.7 and since \(\mathcal{W}_p\) is dense in \(\mathcal{W}_{[0,1]}\) in the \(L^1\) topology, the two sides are equal on all \(f \in \mathcal{W}_{[0,1]}\). In addition, by Corollary 3.24 the coefficients \(a_H\) are unique. This shows the first part of the result. If we assume further that \(F\) is continuous with respect to the cut-norm, then Proposition 2.7 shows that

\[
F(f) = \sum_{H \in \mathcal{H}_{\leq N}} a_H t(H^{\text{simp}}, f)
\]

where \(H^{\text{simp}}\) is the simple graph obtained from \(H\) by retaining only a single edge between two vertices of \(H\) if there are one or more edges connecting the same two vertices.

On the other hand, \(F(f) = \sum_{H \in \mathcal{H}_{\leq N}} a_H t(H, f)\) for all \(f \in \mathcal{W}_{[0,1]}\) from above. Using Corollary 3.24 this is possible if and only if \(a_H = 0\) for all \(H \in \mathcal{H}_{\leq N} \setminus \mathcal{G}_{\leq N}\), and the second part of the result follows.

**Remark 3.27.** Note that it is enough in the statement of the theorem to only assume that

\[
d^{N+1}(F)(cg; g, g, \ldots, g) \equiv 0, \quad \forall g \in \mathcal{W}_p, \ c \in (0, 1),
\]

since the above proof only uses this assumption.

Moreover, the first part of Theorem 1.1 holds for any topology on \(\mathcal{W}\) such that \(\mathcal{W}_p\) is dense in \(\mathcal{W}\) and such that \(t(H, -)\) for \(H \in \mathcal{H}\) is continuous with respect to the topology.

### 3.3. Characterizing homomorphism densities.

As we have seen, homomorphism densities \(t(H, -)\) are fundamental continuous class functions on \(\mathcal{W}_{[0,1]}\). It is natural to characterize homomorphism densities amongst all continuous class functions. Indeed, similar characterizations exist ([22, Section 5.6]) for functions \(F : \mathcal{G} \to \mathbb{R}\) of the form \(\text{Hom}(-, H)\) and \(\text{Hom}(H, -)\).

Our characterization is based on the work of the previous sections and the notion of the tensor product of two graphons. Recall from [22, Section 7.4] that given graphons \(f, g \in \mathcal{W}\), their tensor product \(f \otimes g : [0, 1]^4 \to [0, 1]\) is defined to be the map:

\[
(f \otimes g)(x_1, x_2, y_1, y_2) := f(x_1, y_1)g(x_2, y_2).
\]

Given an arbitrary measure preserving map \(\phi : [0, 1] \to [0, 1]^2\), the tensor product can be associated with a graphon as follows:

\[
(f \otimes g)^\phi(x, y) := (f \otimes g)(\phi(x), \phi(y)).
\]
Different choices of $\phi$ yield weakly equivalent graphons. With a small abuse of notation, we shall write $f \otimes g$ to denote the graphon $(f \otimes g)^\phi$, where $\phi : [0, 1] \to [0, 1]^2$ is fixed for the rest of this section.

The tensor product has the property ([22, Section 7.4]) that for all multigraphs $H$ and graphons $f, g$,

$$t(H, f \otimes g) = t(H, f)t(H, g).$$

(3.28)

In particular, for every $m \in \mathbb{N}$ and multigraph $H \in \mathcal{H}$, we have: $t(H, f^{\otimes m}) = t(H, f)^m$.

We now characterize homomorphism densities in terms of tensor products and Gâteaux derivatives.

**Theorem 3.29.** A class function $F : \mathcal{W}_{[0,1]} \to \mathbb{R}$ is a homomorphism density $t(H, -)$ for $H \in \mathcal{H}$ (resp. $H \in \mathcal{G}$), if and only if it satisfies the following properties:

(i) $F$ is continuous in the $L^1$-topology (resp. cut topology) on $\mathcal{W}_{[0,1]}$;

(ii) there exists $n > 0$ so that $d^n F(f; g_1, \ldots, g_n) = 0$ for all $f \in \mathcal{W}_{[0,1]}$ and $g_1, \ldots, g_n \in \text{Adm}(f)$; and

(iii) $F(f)^m = F(f^{\otimes m})$ for some even integer $m \in \mathbb{N}$, for all $f \in \mathcal{W}_{[0,1]}$.

**Proof.** Suppose first that $F$ is of the form $F(f) = \sum_{i=1}^n a_i t(H_i, f)$ with $a_i \neq 0$ for all $i$ and $H_i \in \mathcal{H}$ pairwise distinct. To prove the result in this case, we define a well-ordering $\leq$ on $\mathcal{H}$ as follows. First define a well-ordering on the set of connected multigraphs $H \in \mathcal{H}$ by setting $H < G$ if $|E(H)| < |E(G)|$, and picking an arbitrary total ordering on the finite set of connected multigraphs with a fixed number of edges. Now any finite multigraph has a unique decomposition into finitely many connected multigraphs of the form $H = \bigsqcup_{i=1}^k H_i$.

The set $\mathcal{H}$ can therefore be put into bijection with sequences of non-negative integers $e_H$, where $H$ ranges over connected finite multigraphs and only finitely many $e_H$ are non-zero. Therefore $\mathcal{H}$ can be ordered lexicographically since we have already put a total order on the connected multigraphs in $\mathcal{H}$. This ordering on $\mathcal{H}$ is now a well-ordering, whose unique minimum is given by the graph with a single vertex and no edges. It satisfies the property that if $H_1 \leq H_2$ and $G_1 \leq G_2$, then $H_1 H_2 \leq G_1 G_2$, with equality if and only if $H_1 = H_2$ and $G_1 = G_2$.

Now by assumption,

$$\left[ \sum_{i=1}^n a_i t(H_i, f) \right]^m = \sum_{i=1}^n a_i (t(H_i, f))^m$$

for some even $m > 1$. By the linear independence of multigraph homomorphism densities (Corollary 3.24) we must have equality termwise in $\mathcal{H}$. Assume without loss of generality that $H_1 < \cdots < H_n$ in the total ordering $<$, and suppose for contradiction that $n > 1$. Then the cross term $na_1^{n-1}a_2 t(H_1^{n-1} H_2, f)$ on the left hand side is non-zero but does not appear on the right hand side. We conclude that there is at most one non-zero $a_i$. In that case, $a_i = a_i^n$ so $a_i \in \{0, 1\}$.

Finally, suppose $F$ is any function satisfying the hypotheses. By conditions (i) and (ii) and Theorem 1.1, $F$ is of the form $F(f) = \sum_{i=1}^n a_i t(H_i, f)$ with $H_i \in \mathcal{H}$ or $\mathcal{G}$ for all $i$, depending on the topology used in (i). Therefore the theorem follows from the above analysis. \hfill $\square$

**Remark 3.30.** We now explain how condition (iii) above can be replaced by a purely combinatorial condition. Note that by the assumed continuity of $F$, $F(f)^m$ is continuous in $f$. Also note that if $f_k \to f$ then $t(H, f_k^{\otimes m}) \to t(H, f^{\otimes m})$ for every simple graph $H$ by
Equation (3.28). Thus $f_k \otimes m \rightarrow f \otimes m$ - i.e., $f \rightarrow f \otimes m$ is continuous - and hence, $F(f \otimes m)$ is continuous with respect to $f$ as well. Therefore, it suffices to assume condition (iii) for simple graphs $f^H$ in place of general $f$. Moreover, the tensor product of two graphs $f^{H_1} \otimes f^{H_2}$ is weakly equivalent to a graph $(on)$ corresponding to the Kronecker product of the adjacency matrices of $H_1$ and $H_2$. Thus, the third condition can be replaced by a purely combinatorial condition involving only finite simple graphs - i.e., Kronecker powers of their adjacency matrices.

**Remark 3.31.** A natural question that now arises is whether all class functions $F: W \rightarrow \mathbb{R}$ that are multiplicative (with respect to $\otimes$) are of the form $t(H, -)$ for some $H \in \mathcal{H}$. The answer turns out to be negative. For instance, consider a finite set of graphs $S := \{H_1, \ldots, H_k\} \subset \mathcal{G}$, and define $M_S(f) := \max_{1 \leq i \leq k} t(H_i, f)$. This is clearly multiplicative, but we claim that it is not $C^1$ if there are two graphs in $S$ with unequal number of edges. For in this case, consider the behavior of $M_S$ along the “constant” line $f \equiv c$. Then $t(H_i, c) = c^{\lvert E(H_i) \rvert}$ for all $i$. So below $c = 1$, the function is $M_S(c) = c^{\min \lvert E(H_i) \rvert}$ while above 1, it equals $c^{\max \lvert E(H_i) \rvert}$. Hence $dM_S(1; 1)$ does not exist. This counterexample shows us why additional hypotheses are required in order to characterize homomorphism densities.

### 4. Power series and Taylor series

The previous sections demonstrate that $t(H, -)$ with $\lvert E(H) \rvert = n$ can be seen as the analogue of monomials of degree $n$ in the graphon space. By analogy to single variable Taylor series, we study expansion of smooth class functions on $W_{[0,1]}$ in terms of infinite series of homomorphism densities. We give sufficient conditions for the existence of such series in Section 4.3 and prove their uniqueness in Section 4.4. The proofs of these results rely heavily on the work of Section 3. In particular, in Section 4.3 we use Theorem 1.1 to show that the Taylor expansion of a smooth class function can be written in terms of homomorphism densities. In Section 4.4 we generalize the linear independence result of Section 3 to prove uniqueness and explain the general philosophy behind the proofs of both the linear independence results.

In order to do this we first investigate general facts about differentiation and convergence of series of homomorphism densities in Section 4.2. The general theory also includes an algebra structure on such series, that is obtained from the algebra structure on formal linear combinations of graphs $Q_0$. The algebra structure can be extend to formal linear combinations of $k$-labelled multigraphs $Q_k$, so we develop the general properties to include their homomorphism densities.

In Section 4.1 we explain how to package all of the $Q_k$ into a single algebra $Q_N$ and then define *weighted homomorphism densities*. These simultaneously generalize multigraph homomorphism densities, partially labelled graph homomorphism densities, and their derivatives. We develop the theory of series in this generality in order that the space of series is closed under differentiation and has an algebra structure.

Along the way, we address in Section 4.2 one of Lovász’s questions [20, Problem 16] about whether it is possible to expand right homomorphism densities in terms of left homomorphism densities. Finally, in Section 4.3 we apply this theory to give a proposal for an analytic theory of infinite quantum algebras, thereby addressing another of Lovász’s questions [20, Problem
This last application gives a second motivation for developing the properties of series in the generality we do.

4.1. The algebra of partially labelled multigraphs. In this section we recall how to equip the space of formal linear combinations of partially labelled multigraphs with an algebra structure. In addition, we define generalizations of homomorphism densities indexed by such graphs with weights so that the functions are closed under differentiation. These functions will form the individual terms of the infinite series we investigate in this paper.

Definition 4.1. Given an integer \( k \geq 0 \), a \( k \)-labelled multigraph is a multigraph \( H \) with an injective label map \( l_H : \{1, \ldots, k\} \to V(H) \). (If \( k = 0 \) then \( H \) is unlabelled.) If \( G, H \) are \( k \)-labelled multigraphs for \( k > 0 \), then a map \( f \) of such multigraphs is a multigraph node-and-edge map (satisfying Equation (2.6)), for which \( V_f \) sends the labelled vertices of \( H \) to the corresponding labelled vertices of \( G \), i.e. \( V_f(l_H(a)) = l_G(a) \) for \( a \in \{1, \ldots, k\} \).

For all integers \( k, n \geq 0 \), define \( \mathcal{H}_{n,k} \) to be the finite set of isomorphism classes of \( k \)-labelled multigraphs with \( n \) edges and no unlabelled isolated nodes. Finally, following [22, Section 6.1], define

\[
\mathcal{Q}_k := \text{span}_\mathbb{R} \bigcup_{n \geq 0} \mathcal{H}_{n,k}
\]

to be the vector space with basis given by the disjoint union of the sets \( \{\mathcal{H}_{n,k} : n \geq 0\} \).

Note that \( \mathcal{Q}_k \) can be given the structure of a commutative algebra by defining the product of two labelled multigraphs \( F_1 \) and \( F_2 \) to be the multigraph obtained by taking their disjoint union and identifying equivalently labelled nodes. In fact, we note that for all \( n \), the set of graphs \( \mathcal{H}_{n,k} \) embeds into \( \mathcal{H}_{n,k+1} \) by attaching an additional isolated node labelled \( k + 1 \) to any graph \( H \in \mathcal{H}_{n,k} \). This induces the obvious linear injection \( \mathcal{Q}_k \hookrightarrow \mathcal{Q}_{k+1} \) for all \( k \). Thus, define (following [22, Section 6.1]) their directed limit to be the algebra of partially labelled multigraphs:

\[
\mathcal{Q}_\infty := \bigcup_{k \geq 0} \mathcal{Q}_k = \text{span}_\mathbb{R} \mathcal{H}', \quad \mathcal{H}' := \bigcup_{k \geq 0} \prod_{n \geq 0} \mathcal{H}_{n,k}.
\]

The space \( \mathcal{Q}_0 = \text{span}_\mathbb{R} \mathcal{H} \) is called the space of quantum graphs [22, Section 6.1]. There is an algebra map \( \alpha \) from \( \mathcal{Q}_0 \) to the space of class functions on \( \mathcal{W}_{[0,1]} \), defined by \( \alpha(H) := t(H, -) \) and extending linearly. This is a map of algebras because the product on \( \mathcal{Q}_0 \) is just disjoint union and \( t(H_1, f) t(H_2, f) = t(H_1 \prod H_2, f) \).

The map \( \alpha \) has been generalized to define homomorphism densities from partially labelled multigraphs \( H \in \bigcup_{n \geq 0} \mathcal{H}_{n,k} \) for any \( k \geq 0 \) - see [22, Section 7.2]. In the language of this paper, fix \( H \in \mathcal{H}_{n,k} \) and \( x_1, \ldots, x_k \in [0,1] \), and let \( V_0 := V \setminus l_H(\{1, \ldots, k\}) \) denote the set of unlabelled vertices. We now define

\[
f \mapsto t_x(H,f) = t_{(x_1,\ldots,x_k)}(H,f) := \int_{[0,1]^V} \prod_{e \in E(H)} f(x_{e_1},x_{e_2}) \prod_{i \in V_0} dx_i \quad (4.2)
\]

where \( x_v = x_{j_i}^{(i)}(v) \) for labelled vertices \( v \in V(H) \setminus V_0 \). It is not difficult to see that this notion only depends on the isomorphism class of \( H \).

As in the \( k = 0 \) case, we have

\[
t_x(H_1, f)t_x(H_2, f) = t_x(H_1 H_2, f) \quad (4.3)
\]
for all $k$-labelled multigraphs $H_1, H_2$. Hence $\alpha_\mathbf{x} : \mathcal{Q}_k \to \text{Func}(\mathcal{W}, \mathbb{R})$ is an algebra map for any choice of $\mathbf{x} \in [0, 1]^k$. More precisely, fix $\mathbf{x}_n \in [0, 1]$ for all $n \in \mathbb{N}$, and define $\mathbf{x} := (\mathbf{x}_n) \in [0, 1]^\mathbb{N}$. Then one can define the map $\alpha_\mathbf{x} : \mathcal{Q}_\mathbb{N} \to \text{Func}(\mathcal{W}, \mathbb{R})$, given by

$$
\alpha_\mathbf{x}(H) := t_\mathbf{x}(H, -), \quad \forall \mathbf{x} \in [0, 1]^\mathbb{N}, \ H \in \bigcup_{k \geq 0} \bigcup_{n \geq 0} \mathcal{H}_{n,k},
$$

and extending by linearity. Here if $H \in \mathcal{H}_{n,k}$ then we define $t_\mathbf{x}(H, -) := t_{(\mathbf{x}_1, \ldots, \mathbf{x}_k)}(H, -)$. The following result is then immediate.

**Lemma 4.5.** For all $\mathbf{x} \in [0, 1]^\mathbb{N}$, the map $\alpha_\mathbf{x} : \mathcal{Q}_\mathbb{N} \to \text{Func}(\mathcal{W}, \mathbb{R})$ is an algebra homomorphism.

**Weighted homomorphism densities.**

The space of functions $\mathcal{S}$ can be extended in several different directions. It is the aim of this section to extend it to contain multigraph homomorphism densities, the image of $\alpha_\mathbf{x}$, infinite convergent series, and weighted series that arise naturally after taking derivatives. To that end, we make the following definition.

**Definition 4.6.** Given a multigraph $H \in \mathcal{H}_{n,k}$ with unlabelled nodes given by $V_0 := \{l_H(\{1, \ldots, k\})\}$, together with numbers $x_1, \ldots, x_k \in [0, 1]$, and a bounded kernel $g_H : [0, 1]^V(H) \to \mathbb{R}$, define the corresponding weighted partially labelled multigraph homomorphism density to be:

$$
t_{(x_1, \ldots, x_k)}(H, f; g_H) := \int_{[0,1]^{V_0}} g_H((x_i)_{i \in V(H)}) \prod_{e \in E(H)} f(x_{e_{x}}, x_{e_{1}}) \prod_{i \in V_0} dx_i,
$$

where $x_v = x_{i_{H}^{-1}(v)}$ for labelled vertices $v \in V(H) \setminus V_0$.

In particular, $t_{\mathbf{x}}(H, f, 1) = t_{\mathbf{x}}(H, f)$ in the notation of (4.2). We use this without further reference in the remainder of the paper.

We now generalize Equations (2.1) and (1.3) to weighted, partially labelled multigraphs.

**Proposition 4.8.** We have for $H \in \mathcal{H}_{n,k}$ and $H' \in \mathcal{H}_{m,k}$:

$$
t_{\mathbf{x}}(H, f, g_H) t_{\mathbf{x}}(H', f, g'_{H'}) = t_{\mathbf{x}}(H H', f, g_H g'_{H'}).$$

Here, $g_H g'_{H'} : [0, 1]^V(H H') \to \mathbb{R}$ denotes the function $g_H((x_i)_{i \in V(H)}) g'_{H'}((x_i)_{i \in V(H')})$.

**Proof.** Let $V_0$ and $V_0'$ be the unlabelled vertices of $H$ and $H'$ respectively. Then,

$$
t_{\mathbf{x}}(H, f, g_H) t_{\mathbf{x}}(H', f, g'_{H'})
$$

$$= \left( \int_{[0,1]^{V_0}} g_H((x_i)_{i \in V(H)}) \prod_{e \in E(H)} f(x_{e_{x}}, x_{e_{1}}) \prod_{i \in V_0} dx_i \right) \times
$$

$$\left( \int_{[0,1]^{V_0'}} g'_{H'}((x_i)_{i \in V(H')}) \prod_{e \in E(H')} f(x_{e_{x}}, x_{e_{1}}) \prod_{i \in V_0'} dx_i \right)
$$

$$= \int_{[0,1]^{V_0} \times [0,1]^{V_0'}} g_H((x_i)_{i \in V(H)}) g'_{H'}((x_i)_{i \in V(H')}) \prod_{e \in E(H H')} f(x_{e_{x}}, x_{e_{1}}) \prod_{i \in V_0} dx_i \prod_{i \in V_0'} dx_i
$$

$$= t_{\mathbf{x}}(H H', f, g_H g'_{H'}).$$

\qed
Remark 4.9. One can take (higher) Gâteaux derivatives of \( t_x(H, f, g_H) \) along arbitrary directions, just as for ordinary homomorphism densities \( t(H, f) \). More precisely,

\[
d(t_x(H, f, g_H); g) = \sum_{e_1 \in E(H)} \int_{[0,1]^6} g_H((x_i)_{i \in V(H)}) g((x_{(e_1)_1}, x_{(e_1)_r})) \prod_{e \in E(H) \setminus \{e_1\}} f(x_e, x_e) \prod_{i \in V_0} dx_i.
\]

where:

- \( H_{e_1} \in \mathcal{H}_{n-1,k} \) is obtained from \( H \in \mathcal{H}_{n,k} \) by removing the edge \( e_1 \in E(H) \) and then further removing all unlabelled isolated nodes.

- The new weights are:

\[
g_{e_1}((x_i)_{i \in V(H_{e_1})}) := \int_{[0,1]^{V(H) \setminus V(H_{e_1})}} g_H((x_i)_{i \in V(H)}) g((x_{(e_1)_1}, x_{(e_1)_r})) \prod_{i \in V(H) \setminus V(H_{e_1})} dx_i.
\]

Equation (4.10) implies that the linear span of \( t_x(H, -, g_H) \) is closed under differentiation, whereas the span of the unweighted (partially labelled multigraph) homomorphism densities is not. Further note that \( t_x(H, -, g_H) \) is \( C^0 \) in the sense of Definition 2.15. Therefore the functions \( t_x(H, -, g_H) \) are Gâteaux smooth.

4.2. Series. In the previous subsection, we defined the weighted homomorphism densities, simultaneously generalizing simple graph homomorphism densities, multigraph homomorphism densities, and partially labelled graph homomorphism densities. In addition we found that the span of such functions is closed under addition, multiplication, and differentiation. In this subsection we develop the basic convergence properties of infinite series of such functions. We show that such “absolutely convergent” series are closed under addition, multiplication, and differentiation. In addition, we use this theory to examine whether right homomorphism densities can be expanded as formal series of homomorphism densities.

Definition 4.11. For \( k \geq 0 \) an integer and \( \mathbf{x} = (x_1, \ldots, x_k) \in [0,1]^k \), define the weighted \((k, \mathbf{x})\)-labelled power series to be the set of formal series of the form

\[
\sum_{n=0}^{\infty} \sum_{H \in \mathcal{H}_{n,k}} t_x(H, -, g_H).
\]

The subset of \((k, \mathbf{x})\)-labelled power series consists of those formal series for which \( g_H \) is constant for all \( H \). Given \( \mathbf{x} \in [0,1]^N \), a (weighted) \( \mathbf{x} \)-labelled power series is a (weighted) \((k, (x_1, \ldots, x_k))\)-labelled power series for some \( k \geq 0 \).

We first discuss the structure of the set of such series.

Proposition 4.12. Given \( \mathbf{x} \in [0,1]^N \), the set of weighted \( \mathbf{x} \)-labelled power series is a (unital) commutative graded \( \mathbb{R} \)-algebra, with termwise addition, and with multiplication given by:

\[
(FG)(-) := \sum_{n \geq 0} \sum_{H \in \mathcal{H}_{n,\max(k,k')}} t_x(H, -, \sum_{(H_1,H_2):H_1H_2=H} f_{H_1}g_{H_2}).
\]
where \( F(-) = \sum_{n \geq 0} \sum_{H \in \mathcal{H}_{\max}} t_x(H, -, f_H) \) and \( G(-) = \sum_{n \geq 0} \sum_{H \in \mathcal{H}_{\max}} t_x(H, -, g_H) \). The weighted \((k, x)\)-labelled power series form an increasing family of subalgebras in \( k \geq 0 \) with the same properties.

Note that the subspaces of unweighted power series form subalgebras of the above algebras. Also note that in defining the product, with a slight abuse of notation, we continue to denote the weightings for \( \max(k, k') \) by \( f_{H_1} \) and \( g_{H_2} \), but it is clear what this means.

**Proof.** We first discuss the multiplication operation. It is not hard to show that the binary operations on functions and on \( \mathcal{H}' \), given respectively by

\[
(g_{H'}, f'_{H'}) \mapsto g_H f_{H'}, \quad (H, H') \mapsto HH',
\]

are each associative as well as commutative. This easily shows the commutativity of the product. For associativity, given formal power series \( E, F, G \) with the obvious formal expansions, we compute using the associativity of the operations in (4.13) that \((EF)G\) and \(E(FG)\) both equal

\[
\sum_{n \geq 0} \sum_{H \in \mathcal{H}_{\max}(k_E, k_F, k_G)} t_x(H, -, \sum_{(H_1, H_2, H_3): H_1 H_2 H_3 = H} e_{H_1} f_{H_2} g_{H_3}),
\]

where \( k_E, k_F, k_G \) are part of the data of \( E, F, G \) respectively. Next, the addition, distributivity, existence of a unit \( U \), and \( \mathbb{R} \)-algebra structure are also easy to show. The unit is given by \( O_x := t_x(H, -, 1) \), where \( H \) is the unique graph in \( \mathcal{H}_{0,k} = \mathcal{H}_{0,0} \). Finally, the \( n \)th graded piece is precisely the span of all \( t_x(H, -, g_H) \) for \( H \in \mathcal{H}_{n,k} \) (for all \( k \geq 0 \)) and all \( g_H \). The corresponding results for \((k, x)\)-labelled series now follow easily. \( \square \)

We now study the convergence properties of formal power series. Given Proposition 4.12, it suffices to study weighted \((k, x)\)-labelled power series for any fixed \( k \geq 0 \). Using the analogy to monomials discussed in Section 2.1, we arrange weighted \((k, x)\)-labelled power series according to their “degree”, and say that such a series **converges at** \( f \in \mathcal{W} \) if

\[
\sum_{n=0}^{\infty} \sum_{H \in \mathcal{H}_{n,k}} t_x(H, f, g_H) := \lim_{N \to \infty} \sum_{n=0}^{N} \sum_{H \in \mathcal{H}_{n,k}} t_x(H, f, g_H)
\]

exists. Similarly, a weighted \((k, x)\)-labelled power series **converges absolutely at** \( f \in \mathcal{W} \) if

\[
\sum_{n=0}^{\infty} \sum_{H \in \mathcal{H}_{n,k}} t_x(H, |f|, |g_H|) < \infty.
\]

As for power series of one variable (in real analysis), we will interchangeably use \( F(-) = \sum_{n=0}^{\infty} \sum_{H \in \mathcal{H}_{n,k}} t_x(H, -, g_H) \) and \( F(f) = \sum_{n=0}^{\infty} \sum_{H \in \mathcal{H}_{n,k}} t_x(H, f, g_H) \), which denote (respectively) the formal weighted power series and the function that it defines.

We now define a family of distinguished subsets of \( \mathcal{W} \).

**Definition 4.15.** Given \( I \subset \mathbb{R} \), define \( \mathcal{W}_I \) to be the set of all \( f \in \mathcal{W} \) with image in \( I \). Also define

\[
\mathcal{W}_{I^r} := \bigcup_{0 \leq s < r} \mathcal{W}_{(-s, s)} = \bigcup_{0 \leq s < r} \mathcal{W}_{[-s, s]}, \quad 0 < r \leq \infty.
\]
We have the useful observation that if \( I = [a, b] \) with \(|a| \leq b\), then a weighted \((k, \mathbf{x})\)-labelled power series \( F \) is absolutely convergent on \( \mathcal{W}_I \) if and only if \( F \) is absolutely convergent on \( \mathcal{W}_{[-b,b]} \). Moreover, \( F \) is absolutely convergent on \( \mathcal{W}_{\tau_I} \) if and only if \( F \) is absolutely convergent on the constant graphons \((0,r)\).

**Remark 4.17.** Consider the special case where \( k = 0 \), \( g_H \equiv a_H \) is constant for all \( H \), and \( a_H = 0 \) if \( H \notin G \). It is clear by the theory of rearrangements of series that if \( \sum_n s^n \sum_{H \in \Gamma_n} |a_H| \) converges then so does \( \sum_{H \in \mathcal{G}} a_H t(H, f) \) for all \( f \in \mathcal{W} \) with image in \([-s,s]\). On the other hand, convergence at all constant graphons need not guarantee convergence on all of \( \mathcal{W} \). For instance, suppose \( a_{K_3 \uparrow n} = 2^{-n} = -a_{K_1,3n} \) for the triangle \( K_3 \) and all (bipartite) star graphs \( K_{1,3n} \) with \( n \geq 1 \), and all other \( a_H \) are zero. Then the corresponding power series is given by:

\[
\sum_{n \in \mathbb{N}} 2^{-n} (t(K_3, f)^n - t(K_{1,3n}, f)).
\]

(4.18)

It is clear that the series (4.18) converges at all \( f \equiv s \in \mathbb{R} \). However, note that \( t(K_3, K_2) = 0 < 2^{-3n} = t(K_{1,3n}, K_2) \) since \( K_{1,3n} \) and \( K_2 = K_{1,1} \) are bipartite while \( K_3 \) is not. Hence the series (4.18) diverges to \(-\infty\) at \( sf^{K_2} \), for all \( s \geq 2^{4/3} \).

In light of Remark 4.17 we introduce the following notation.

**Definition 4.19.**

1. Given \( \mathbf{x} \in [0,1]^k \) and \( g_H \) a bounded measurable function on \([0,1]^{V(H)}\), define the semi-norm

\[
\|g_H\|_{1,\mathbf{x}} := \int_{[0,1]^{V_0}} |g_H| \prod_{i \in V_0} dx_i = t_{\mathbf{x}}(H,1,|g_H|).
\]

(4.20)

2. Given a weighted power series \( F(-) := \sum_{n \geq 0} \sum_{H \in \mathcal{H}_{n,k}} t_{\mathbf{x}}(H, -, g_H) \), define its radius of convergence via:

\[
R_F^{-1} := \limsup_{n \to \infty} \left( \sum_{H \in \mathcal{H}_{n,k}} \|g_H\|_{1,\mathbf{x}} \right)^{1/n}.
\]

(4.21)

We now establish several fundamental properties of weighted \((k, \mathbf{x})\)-labelled power series. The proofs combine standard analysis arguments while keeping track of the combinatorics of multigraphs that arises in the present setting. To simplify the exposition, we defer the proofs of the next two results to Appendix C.

Our first result justifies the use of the name “radius of convergence”.

**Proposition 4.22.** Every weighted power series \( F(-) := \sum_n \sum_{H \in \mathcal{H}_{n,k}} t_{\mathbf{x}}(H, -, g_H) \) converges absolutely on \( \mathcal{W}_{\tau_{R_F}} \). Moreover, this convergence is uniform on \( \mathcal{W}_{[-s,s]} \) for any \( 0 \leq s < R_F \). For all \( r \in (R_F, \infty) \), there exist functions \( f \in \mathcal{W}_r \), such that \( F(f) \) is not absolutely convergent.

Our second result shows that convergent weighted \((k, \mathbf{x})\)-labelled power series are closed under addition, multiplication, and differentiation.

**Theorem 4.23.** Suppose \( c, d \in \mathbb{R} \),

\[
F(-) = \sum_{n \geq 0} \sum_{H \in \mathcal{H}_{n,k}} t_{\mathbf{x}}(H, -, g_H), \quad G(-) = \sum_{n \geq 0} \sum_{H \in \mathcal{H}_{n,k}} t_{\mathbf{x}}(H, -, g'_H)
\]

Then

\[
F(-) + G(-) = \sum_{n \geq 0} \sum_{H \in \mathcal{H}_{n,k}} t_{\mathbf{x}}(H, -, (g_H + g'_H)),
\]

\[
FG(-) = \sum_{k \geq 0} \sum_{H \in \mathcal{H}_{n,k}} t_{\mathbf{x}}(H, -, (g_H f_H)),
\]

\[
F'G(-) = \sum_{k \geq 0} \sum_{H \in \mathcal{H}_{n,k}} t_{\mathbf{x}}(H, -, (g_H f'_H)).
\]
are two weighted \((k, x)\)-labelled power series, and \(R := \min(R_F, R_G)\) is positive. Then the following three properties hold.

(i) For all \(f \in \mathcal{W}_{1R}\),

\[
(cF + dG)(f) = \sum_{n \geq 0} \sum_{H \in \mathcal{H}_{n,k}} t_{x}(H, f, cg_{H} + dg'_{H})
\]  

(4.24)

where the right-hand side is a weighted \((k, x)\)-labelled power series such that \(R_{cF+dG} \geq R\).

(ii) For all \(f \in \mathcal{W}_{1R}\),

\[
(FG)(f) = \sum_{n \geq 0} \sum_{H \in \mathcal{H}_{n,k}} t_{x}(H, f, \sum_{H_{1}, H_{2} = H} g_{H_{1}}g'_{H_{2}})
\]  

(4.25)

where the right-hand side is a weighted \((k, x)\)-labelled power series such that \(R_{FG} \geq R\).

(iii) Moreover, \(F\) is Gâteaux-smooth on \(\mathcal{W}_{1R_{F}}\). More precisely, for all \(f_{0} \in \mathcal{W}_{1R_{F}}\),

\[
d^{m}F(f_{0}; f_{1}, \ldots, f_{m}) = \sum_{n \geq m} \sum_{H \in \mathcal{H}_{n,k}} d^{m}t_{x}(H, -, g_{H})(f_{0}; f_{1}, \ldots, f_{m}), \quad \forall f_{1}, \ldots, f_{m} \in \mathcal{W},
\]  

(4.26)

where every summand on the right-hand side is a linear combination of terms of the form \(t_{x}(H', -, g_{H})\), and together they form a weighted \((k, x)\)-labelled power series such that \(R_{d^{m}F(-; f_{1}, \ldots, f_{m})} \geq R_{F}\).

**Application 3: Right homomorphism densities.**

As an application of this work on series, we address a question posed by Lovász in his list of open problems. Namely, Lovász asks in [20, Problem 16] if there is a way to find a overlay functional convergence proposes several closely related natural remedies (see [22, Chapter 12]). The theory of right-convergence proposes several closely related natural remedies (see [22, Chapter 12]). The only proposal that is a continuous function on graphons is the overlay functional

\[
C(U, W) := \sup_{\phi \in S_{[0,1]}} \int_{[0,1]^{2}} U(x, y)W(\phi(x), \phi(y))dx\,dy, \quad U, W \in \mathcal{W}.
\]  

(See [22, Lemma 12.7].) We now show that \(C(U, W)\) cannot be expanded as an absolutely convergent sequence in general.

**Proposition 4.27.** There exists \(W \in \mathcal{W}_{[0,1]}\) for which there is no sequence of constants \(a_{H}\) such that the series \(\sum_{H \in \mathcal{H}} a_{H}t(H, -)\) is absolutely convergent and equals \(C(-, W)\) on \(\mathcal{W}_{[0,1]}\).

**Proof.** From the definitions, note that \(dC(-, W)(0; U) = C(U, W)\) for all \(U, W \in \mathcal{W}_{[0,1]}\). Since \(C(U, W)\) is not bilinear, there exists \(W \in \mathcal{W}_{[0,1]}\) such that \(dC(-, W)(0; U)\) is not linear in \(U\). Therefore it cannot be \(C^{1}\) at 0. If \(C(-, W)\) were expandable in terms of an absolutely convergent series of multigraph homomorphism densities \(t(H, f)\), then it would be Gâteaux smooth by Theorem 4.23, so such an expansion is impossible. \(\square\)

\(^{1}\)In fact, \(t(-, G)\) is not even well defined. For example \(n^{2}t(H, c) = t(H[n], c)\) where \(H[n]\) is the \(n\)-fold blowup of \(H\), but \(f^{H} = f^{H[n]}\) for all \(n\).
4.3. **Taylor series.** The Stone-Weierstrass type Theorem 2.2 implies that the linear span \( \mathcal{S}_t \) of homomorphism densities \( t(H, -) \) is a dense subalgebra of \( C(\mathcal{W}_{[0,1]}/ \sim, \mathbb{R}) \). In this (non-constructive) sense, homomorphism densities can be used to approximate continuous class functions on \( \mathcal{W}_{[0,1]} \). The goal of this subsection is to show how to write Taylor expansions for smooth functions on \( \mathcal{W}_{[0,1]}/ \sim \) around 0 in terms of homomorphism densities. In addition, we provide sufficient conditions for the convergence of the Taylor series of a function \( F \) to the function \( F \). The main idea is that Theorem 1.1 shows that the multilinear derivatives of smooth class functions can be represented by those of \( t(H, f) \). The following proposition demonstrates this principle in action.

**Proposition 4.28.** Let \( n \geq 1 \) and \( \Lambda : \mathcal{W}^n_{[0,1]} \to \mathbb{R} \) be a symmetric \( S_{[0,1]} \)-invariant multilinear functional. If in addition \( \Lambda(g, g, \ldots, g) \) is continuous with respect to the \( L^1 \) topology on \( \mathcal{W}_{[0,1]} \) then there exist unique coefficients \( \{a_H\}_{H \in \mathcal{H}_n} \) such that

\[
\Lambda(g_1, \ldots, g_n) = \sum_{H \in \mathcal{H}_n} a_H \cdot d^n t(H, 0; g_1, \ldots, g_n), \quad \forall g_1, \ldots, g_n \in \mathcal{W}_{[0,1]}.
\]

**Proof.** Given such a \( \Lambda \), define \( F : \mathcal{W}_{[0,1]} \to \mathbb{R} \) by \( F(g) := \Lambda(g, g, \ldots, g) \). By Lemma 3.19, \( F \) is Gâteaux smooth and its derivatives are

\[
d^n F(f; g_1, \ldots, g_n) = n! \Lambda(g_1, g_2, \ldots, g_n), \quad \forall g_i \in \text{Adm}(f).
\]

Therefore \( d^{n+1} F(f; g_1, \ldots, g_{n+1}) = 0 \) for \( f \in \mathcal{W}_{[0,1]} \) and \( g_i \in \text{Adm}(f) \). In addition, \( F \) is continuous in the \( L^1 \) topology by assumption, as well as a class function (because \( \Lambda \) is \( S_{[0,1]} \)-invariant). We conclude by Theorem 1.1 that \( F(g) = \sum_{H \in \mathcal{H}_n} a_H t(H, g) \) for unique coefficients \( a_H \). \( \square \)

We now apply the above ideas to define Taylor polynomials for smooth class functions and prove a Taylor’s theorem for them.

**Theorem 4.30.** Suppose \( F : \mathcal{W}_{[0,1]} \to \mathbb{R} \) is a \( C^n \) class function. Then:

(i) \( F \) has a Taylor polynomial \( P_n(f) := \sum_{m=0}^n \frac{1}{m!} d^m F(0; f, f, \ldots, f) \). The remainder is:

\[
R_n(f) := F(f) - P_n(f) = \frac{d^{n+1} F(cf; f, \ldots, f)}{(n+1)!}
\]

for some \( c_f \in [0, 1] \).

(ii) From \( F \) one can uniquely define scalars \( a_H \) for all \( H \in \mathcal{H}_{\leq n} \), such that for all \( f \in \bigcup_{\sigma \in S_{[0,1]}} \sigma(\mathcal{W}_p) \),

\[
\frac{1}{m!} d^m F(0; f, f, \ldots, f) = \sum_{H \in \mathcal{H}_m} a_H t(H, f)
\]

for all integers \( 0 \leq m \leq n \).

(iii) If in addition the higher derivatives \( \frac{1}{m!} d^m F(0; f, f, \ldots, f) \) are continuous in the \( L^1 \) topology, then

\[
\frac{1}{m!} d^m F(0; f, f, \ldots, f) = \sum_{H \in \mathcal{H}_m} a_H t(H, f)
\]

for all integers \( 0 \leq m \leq n \) and all \( f \in \mathcal{W}_{[0,1]} \).
Proof.

(i) Define the function $F(t) := F(tf)$ for a fixed direction $f \in \mathcal{W}_{[0,1]}$. Then $F$ is $C^n$ and this result follows from the one variable Taylor’s theorem for $F$.

(ii) Since $F$ is a $C^n$ class function, we know by Proposition 3.2 that for $0 \leq m \leq n$, $\frac{1}{m!}d^m F(0; f_1, f_2, \ldots, f_m)$ is a symmetric $S_{[0,1]}$-invariant multilinear function. The result now follows from Theorem 3.2(i).

(iii) Applying Proposition 3.2 yields that for $0 \leq m \leq n$, $\frac{1}{m!}d^m F(0; f_1, \ldots, f_m)$ is a symmetric $S_{[0,1]}$-invariant multilinear function of the $f_i$. The result now follows by applying Proposition 4.28 to $\frac{1}{m!}d^m F(0; f_1, \ldots, f_m)$.

Thus, given a smooth class function $F : \mathcal{W}_{[0,1]} \to \mathbb{R}$, we can use Theorem 4.30 to define an infinite Taylor series

$$P(F)(-):=\sum_{m=0}^{\infty} \sum_{H \in \mathcal{H}_m} a_H t(H, -), \quad (4.31)$$

where

$$\sum_{H \in \mathcal{H}_m} a_H t(H, f) = \frac{1}{m!}d^m F(0; f, f, \ldots, f), \quad \forall m \geq 0, \ f \in \bigcup_{\sigma \in S_{[0,1]}} \sigma(\mathcal{W}_\mu).$$

Note that $P(F)$ is in fact a weighted power series as in Definition 4.11 (with $m = 0$ and $g_H \equiv a_H$ constant for all $H \in \mathcal{H}$).

Given a smooth function $F : \mathcal{W}_{[0,1]} \to \mathbb{R}$, a natural question to ask is if the Taylor series defined above converges to $F$. Recall from the one-variable Taylor theory that there exist nonzero smooth functions $F$ on $\mathbb{R}$, all of whose derivatives vanish at the origin (and so $F$ has trivial Taylor polynomials). We now show that a similar phenomenon occurs for graphons. Namely, consider the function $F(f) := e^{-1/t(H,f)}$ for a finite simple graph $H$ with at least one edge (with $F(f) := 0$ if $t(H, f) = 0$). Then all higher Gâteaux derivatives of $F$ vanish at the origin. Indeed, fix a direction $g \in \mathcal{W}_{[0,1]}$ and set $F(c) := e^{-1/t(H,c)} = e^{-1/c}e^{t(H, g)}$. Now the Gâteaux derivatives of $e^{-1/t(H,f)}$ at $0$ in the $g$ direction are just one-sided derivatives of $F(c)$. Moreover, $F(c) = e^{-A/c^n}$, where $A = 1/t(H, g)$ and $n = |E(H)|$. Taking the derivative yields $F'(c) = e^{-A/c^n} \frac{n}{c^{n+1}}$. Higher derivatives are all of the form $F^{(m)}(c) = e^{-A/c^n} R_m(c)$ where $R_m$ is a rational function. Moreover, in all cases $F^{(m)}(0) = 0$ for all $m$. However, $F$ is not the zero function.

We now provide a sufficient condition under which the Taylor series of a smooth function $F$ converges to $F$.

Theorem 4.32. Suppose $F : \mathcal{W} \to \mathbb{R}$ satisfies the following assumptions:

1. $F$ is Gâteaux smooth, continuous in the cut-norm, and a class function.
2. For all $\{0,1\}$-valued graphons $f \in \mathcal{W}_{[0,1]}$, the Taylor polynomials

$$P_n(f) := \sum_{m=0}^{n} \frac{1}{m!}d^m F(0; f, f, \ldots, f)$$

converge to $F(f)$ as $n \to \infty$.
3. The power series $P(F)$ given by (4.31) is absolutely convergent on $\mathcal{W}_{[0,1]}$. 

Then \( a_H = 0 \) for all \( H \in \mathcal{H} \setminus \mathcal{G} \), and

\[
F(f) = P(F)(f) = \sum_{m \geq 0} \sum_{H \in \mathcal{G}_m} a_H t(H, f), \quad \forall f \in \mathcal{W}_{[0,1]}.
\]

In other words, the Taylor series of \( F \) converges to \( F \) on all of \( \mathcal{W}_{[0,1]} \).

**Proof.** Define the weighted power series

\[
\bar{P}(F)(-) := \sum_{m \geq 0} \sum_{H \in \mathcal{H}_m} a_H t(H^{\text{simp}}, -),
\]

where \( H^{\text{simp}} \) is the simple graph obtained from \( H \) by replacing each set of repeated edges between a pair of vertices by one edge. Then \( \bar{P}(F) \) is also absolutely convergent on \( \mathcal{W}_{[0,1]} \). Indeed, this is equivalent to

\[
\sum_{m \geq 0} \sum_{H \in \mathcal{H}_m} |a_H| t(H^{\text{simp}}, 1)
\]

converging and that follows from rearranging, by absolute convergence, the terms of the convergent sequence

\[
\sum_{m \geq 0} \sum_{H \in \mathcal{G}_m} |a_H| t(H, 1).
\]

In addition, \( \bar{P}(F) \) is continuous in the cut-norm because it is a uniform limit of continuous functions on \( \mathcal{W}_{[0,1]} \). Moreover, \( \bar{P}(F)(f) = P(F)(f) \) for all \( \{0,1\} \)-valued graphons \( f \in \mathcal{W}_{[0,1]} \) since \( t(H^{\text{simp}}, f) = t(H, f) \) for such \( f \). By assumption, \( F(f) = P(F)(f) \) for \( f \) a \( \{0,1\} \)-valued graphon, so \( F(f) = \bar{P}(F)(f) \) for such \( f \). Since both \( F \) and \( \bar{P}(F) \) are continuous on \( \mathcal{W}_{[0,1]} \) and equal on all finite simple graphs, we conclude that they are equal on all \( f \in \mathcal{W}_{[0,1]} \).

Now by Theorem 4.30(ii), \( P(F) \) and \( \bar{P}(F) \) have the same Gâteaux derivatives at 0 along the directions in \( \mathcal{W}_p \). We will show in the next subsection (Theorem 4.33) that a power series with a positive radius of convergence is uniquely determined by its Gâteaux derivatives at 0 along the directions in \( \mathcal{W}_p \). We conclude that \( \bar{P}(F) = P(F) \) as weighted power series. \( \square \)

### 4.4. Uniqueness of Taylor series and linear independence

In the previous section, we showed how to represent the Taylor series of a smooth class function \( F \) around 0, in terms of homomorphism densities. We also provided sufficient conditions for when such a Taylor series expansion is absolutely convergent, and converges to \( F \) on all of \( \mathcal{W}_{[0,1]} \). We left open the question of whether or not this expansion is unique. The next theorem shows that such an expansion is indeed unique. In fact, we prove this is true for arbitrary \((k, \mathbf{x})\)-labelled power series when the entries of \( \mathbf{x} \) are distinct. The main ingredient is a generalization of the crucial linear independence result of Theorem 3.21 to functions \( t_{\mathbf{x}}(H, -) \).

**Theorem 4.33.** Fix an integer \( k \geq 0 \) and a vector \( \mathbf{x} \in [0,1]^k \) with distinct entries. Suppose the formal series \( \sum_{n=0}^{\infty} \sum_{H \in \mathcal{H}_{n,k}} t_{\mathbf{x}}(H, -, a_H) \) has a positive radius of convergence. Then the coefficients \( a_H \) for \( \bigcup_{n=0}^{\infty} \mathcal{H}_{n,k} \) are uniquely determined by the function \( F(f) := \sum_{n=0}^{\infty} \sum_{H \in \mathcal{H}_{n,k}} t_{\mathbf{x}}(H, f, a_H) \). More precisely, the coefficients \( a_H \) can be recovered from the derivatives \( d^{n}F(0; f, \ldots, f) \) for \( f \in \mathcal{W}_p \) and \( n \geq 0 \).
Proof of Theorem 4.33. By part (iii) of Theorem 4.23 and Lemma 3.19 note that for all \( n \geq 0 \),
\[
\frac{1}{n!}(d^n F)(0; f, f, \ldots, f) = \sum_{H \in \mathcal{H}_{n,k}} t_x(H, f, a_H).
\]
Therefore, the theorem follows from the following generalization of Corollary 3.24.

Let \( n, k \geq 0 \) be fixed integers, and \( x \) be a vector of \( k \) distinct constants \( x_1, \ldots, x_k \in [0, 1] \).
Then the functions \( t_x(H, -1) \) are linearly independent for \( H \in \mathcal{H}_{n,k} \).
To prove this claim, note that if the functions \( t_x(H_i, -) = t_x(H_i, -1) \) are linearly dependent, then so are \( d^m t_x(H_i, -; g_1, \ldots, g_m) \) for all \( m \) and all choices of tuples \( g = (g_1, \ldots, g_m) \).
Thus we will set \( m = n \) and produce tuples \( \{g(G) : G \in \mathcal{H}_{n,k}\} \) such that the matrix
\[
M := ((d^n t_x(H, 0; g(G))))_{H,G \in \mathcal{H}_{n,k}}
\]
is nonsingular. Indeed, recall that in the \( k = 0 \) case, we showed linear independence, in Corollary 3.24, by picking the \( g(G) \) such that \( d^n t(H, 0; g(G)) = T(t(H, -))_{n}(G, p) \). In that case, the matrix \( M \) was triangular.

We now generalize this argument to \( \mathcal{H}_{n,k} \). To that end, fix an integer
\[
p > 2n + \frac{2}{\min_{1 \leq i < j \leq k} |x_i - x_j|}
\]
with \( p \) relatively prime to the denominators of any \( x_i \) that are rational. Then \( p \) is finite because the \( x_i \) are distinct. We will now study the Gâteaux derivatives of \( t_x(H, -) \) along the directions
\[
f = e^p_{(a,b)} = \mathbf{1}_{(\frac{a}{p}, \frac{b}{p}) \times (\frac{a}{p}, \frac{b}{p})} + \mathbf{1}_{(\frac{a}{p} + \frac{1}{p}, \frac{b}{p}) \times (\frac{a}{p} + \frac{1}{p}, \frac{b}{p})}.
\]
By choice of \( p \), the \( x_i \) lie in the interiors of distinct intervals of the form \([a_i/p, (a_i + 1)/p]\).

Now for every fixed \( G \in \mathcal{H}_{n,k} \), fix an injective map \( \phi : V(G) \to \{1, \ldots, p\} \) such that for each labelled vertex \( v \in V(G) \), \( \phi(v) := [px_v] \). This is possible by choice of \( p \). We now define the tuples \( g(G) \) by: \( g(G)_e := e^p_{(\phi(e_s), \phi(e_t))} \) for \( e \in E(G) \). Then for all \( H \in \mathcal{H}_{n,k} \),
\[
M(H, G) := d^n t_x(H, 0; (g(G)_e)_{e \in E(G)}) = \sum_{\sigma : E(H) \to E(G)} \int_{[0,1]^V_0} \prod_{e \in E(H)} g(G)_{\sigma(e)}(x_{e_s}, x_{e_t}) \prod_{i \in V_0} dx_i,
\]
where the last equality follows by Lemma 3.19 (Note that the order of Gâteaux differentiation does not matter since mixed partials are equal.)

We now claim that \( M(H, G) = |\text{Surj}(H, G)|/|p|^{V_0} \), where \( \text{Surj}(H, G) \) is the set of node-and-edge maps from \( H \) to \( G \) (in the sense of Definition 4.11) that are surjective. To prove the claim, let \( \tau' : V(H) \to V_0 \to \{1, \ldots, p\} \) be defined for labelled vertices \( v \in V(H) \) by:
\[
\tau'(v) := [px_v].
\]
Consider an arbitrary term in (4.34). Then \( g(G)_e \) is constant on each “sub-rectangle” in \([0,1]^V_0 \) of size \( 1/p^{V_0} \), hence
\[
\int_{[0,1]^V_0} \prod_{e \in E(H)} g(G)_{\sigma(e)}(x_{e_s}, x_{e_t}) \prod_{i \in V_0} dx_i
\]
\[
= \frac{1}{p^{V_0}} \sum_{\tau : V(H) \to \{1, \ldots, p\}} \prod_{e \in E(H)} g(G)_{\sigma(e)} \left( \frac{\tau(e_s) - 0.5}{p}, \frac{\tau(e_t) - 0.5}{p} \right).
\]
By our choice of $g(G)_e$ we have

$$g(G)_{\sigma(e)} \left( \frac{\tau(e_s) - 0.5}{p}, \frac{\tau(e_l) - 0.5}{p} \right) = \begin{cases} 1 & \text{if } \{\tau(e_s), \tau(e_l)\} = \{\phi(\sigma(e)_s), \phi(\sigma(e)_l)\}, \\ 0 & \text{otherwise}. \end{cases}$$

Note that the product of the above expression (over $e \in E(H)$) is zero unless $\tau(V(H)) \subset \phi(V(G))$. Therefore, the right side of Equation (4.35) can be written as

$$\frac{1}{p|V_0|} \sum_{\tau: V(H) \rightarrow \phi(V(G))} \prod_{e \in E(H)} g(G)_{\sigma(e)} \left( \frac{\tau(e_s) - 0.5}{p}, \frac{\tau(e_l) - 0.5}{p} \right).$$

Since $\phi$ is injective, we can define vertex maps $\phi^{-1}\tau : V(H) \rightarrow V(G)$ for every such $\tau$. Note that $\phi^{-1}\tau$ sends labelled vertices of $H$ to the corresponding labelled vertices in $G$, because $\phi(v) := [px_{i_H^{-1}(v)}]$ and $\tau$ extends $\tau'$. We recognize the sum on the right hand side of Equation (4.35) to be equal to $\frac{1}{p|V_0|}$ times the number of vertex maps $\phi^{-1}\tau : V(H) \rightarrow V(G)$ that form a map of multigraphs $H \rightarrow G$, when combined with the edge map $\sigma : E(H) \rightarrow E(G)$. In addition, Equation (4.34) sums over all surjective maps $\sigma : E(H) \rightarrow E(G)$ so $M(H,G) = |\text{Surj}(H,G)|/p|V_0|$ as claimed.

Now note that $(M(H,G))_{H,G \in \mathcal{H}_{n,k}}$ is triangular with nonzero diagonal entries, when $\mathcal{H}_{n,k}$ is partially ordered consistent with the existence of surjections. Hence $(M(H,G))_{H,G \in \mathcal{H}_{n,k}}$ is an invertible matrix, which concludes the proof. □

**Remark 4.36.** We now explain more generally why several of the integral formulas we have examined above, can be interpreted as combinatorial quantities. Let $H$ be a $k$-labelled multigraph with unlabelled vertices $V_0$, and say $x \in [0,1]^k$ are fixed irrational numbers, such that $x_v = x_{i_H^{-1}(v)}$, as above. For fixed $p \in \mathbb{N}$, define $\tau'(v) = [px_{i_H^{-1}(v)}]$ for labelled vertices $v$.

Now consider the expression

$$\int_{[0,1]^{V_0}} \prod_{e \in E(H)} g_e(x_{e_s}, x_{e_l}) \prod_{i \in V_0} dx_i$$

$$= \frac{1}{p|V_0|} \sum_{\tau: V(H) \rightarrow \{1,\ldots,p\}, \tau \text{ extends } \tau'} \prod_{e \in E(H)} g_e \left( \frac{\tau(e_s) - 0.5}{p}, \frac{\tau(e_l) - 0.5}{p} \right)$$

(4.37)

where $g_e = f^{G_e}$, and $G_e$ are simple graphs on the vertex set $\{1,\ldots,p\}$ for each edge $e \in E(H)$.

Just as in the proof of Theorem 4.33

$$\prod_{e \in E(H)} g_e \left( \frac{\tau(e_s) - 0.5}{p}, \frac{\tau(e_l) - 0.5}{p} \right) = \begin{cases} 1 & \text{if } \{\tau(e_s), \tau(e_l)\} \in E(G_e) \forall e \in E(H), \\ 0 & \text{otherwise}. \end{cases}$$

Therefore the quantity in Equation (4.37) is equal to $\frac{1}{p|V_0|}$ times the number of maps $\tau : V(H) \rightarrow \{1,\ldots,p\}$ with $\tau$ extending $\tau'$ such that $\{\tau(e_s), \tau(e_l)\} \in E(G_e)$ for each $e \in E(H)$.

In the special case $G_e = G$ for a single graph $G$ (for all $e$), the above analysis shows that $t(H, f^G) = \text{hom}(H,G)/|V(G)||E(H)|$. A similar formula can be obtained by extending this analysis to multigraphs $G_e$, by weighting the edges of $G_e$ according to the multiplicity of edges found.
Sometimes it is useful to consider multiple sets of graphs \( \{G_e : e \in E(H)\} \) simultaneously. In particular, one can combinatorially interpret the derivatives

\[
d^m t_x(H, f^G; g_1, g_2, \ldots, g_m) =
\sum_{A \subseteq E(H)} \sum_{|A| = m} \int_{[0,1]^{V_0}} \prod_{e \in A} g_1(x_{\sigma(l)} x_{\sigma(t)}) \prod_{e \in E(H) \setminus A} f^G(x_e, x_e) \prod_{i \in V_0} dx_i, \; \forall f^G, g_i \in \mathcal{W}_p
\]

for \( 0 \leq m \leq n, 0 \leq k, H \in \mathcal{H}_{n,k} \). In this paper we have specialized to the case \( m = n \), for the proof of Theorem 4.33. In that proof we picked the \( G_e \) to each be a single edge, and let the \( G_e \) vary over all edges \( e \in E(H) \) to form a multigraph \( G \). Since \( G_e \) was a single edge, it forced \( \{\tau(e_s), \tau(e_t)\} = G_e \). The sum over all \( \sigma \) in the above equation allowed us to interpret this derivative in terms of the number of surjective maps \( H \to G \). We were thus able to obtain linear independence results about homomorphism densities in an analytic way.

4.5. Infinite quantum algebras. We now explore a question raised by Lovász regarding the algebras \( \mathcal{Q}_k \). Lovász asks in [20, Problem 7] if it is possible to extend the definition of \( \mathcal{Q}_k \) to infinite sums of \( k \)-labelled multigraphs. One answer to this question is to interpret \( \mathcal{Q}_k \) as the graded vector space \( \bigoplus_{n \geq 0} \mathbb{R}^{\mathcal{H}_{n,k}} \); then an extension to infinite sums would simply be the larger space \( \hat{\mathcal{Q}}_k := \prod_{n \geq 0} \mathbb{R}^{\mathcal{H}_{n,k}} \).

Let \( \hat{\mathcal{Q}}_{k,x} \) denote the set of all weighted \((k, x)\)-labelled power series with constant coefficients \( g_H \equiv a_H \) (for all \( H \)). Then note that \( \hat{\mathcal{Q}}_{k,x} \) is a unital \( \mathbb{R} \)-algebra of the commutative algebra studied in Proposition 4.12. Moreover, the obvious map \( \hat{\mathcal{Q}}_k \to \hat{\mathcal{Q}}_{k,x} \) sending the tuple \( \{a_H : H \in \bigcup_n \mathcal{H}_{n,k}\} \) to \( \sum_{n \geq 0} \sum_{H \in \mathcal{H}_{n,k}} t_x(H, -, a_H) \) is a vector space isomorphism. Therefore \( \hat{\mathcal{Q}}_k \) inherits an algebra structure from \( \hat{\mathcal{Q}}_{k,x} \), which we note is independent of \( x \) and extends the algebra structure on \( \mathcal{Q}_k \).

**Remark 4.38.** Note that \( \hat{\mathcal{Q}}_k \) can also be interpreted as the completion of the topological graded vector space \( \mathcal{Q}_k \) under the metrizable topology - in fact, the translation-invariant metric - defined by the grading. In this topology, the subspaces \( \bigoplus_{n \geq N} \text{Span}_\mathbb{R} \mathcal{H}_{n,k} \) - which are in fact graded ideals - form a fundamental system of neighborhoods of 0. Moreover, it is not hard to show that the algebra operations on \( \hat{\mathcal{Q}}_k \) (defined by Proposition 4.12) are continuous in this topology. In this sense the grading introduced in this paper provides an algebraic candidate to Lovász’s question. This candidate is essentially unique by the universality of completions.

**Remark 4.39.** One can show that the span \( \mathcal{Q}_N \) of \( \mathcal{H} \) is a polynomial algebra \( \mathbb{R}[X] \), whose set of generators \( X \) consists of all (isomorphism classes of) connected multigraphs \( H \in \mathcal{H} \) of one of two kinds: (1) \( H \) contains exactly two vertices, both of which are labelled and adjacent with a unique edge; or (2) the subset of labelled vertices in \( V(H) \) is independent, and removing this subset (and all edges adjacent to it) from \( H \) does not disconnect the resulting induced sub-multigraph. Similarly for each \( k \geq 0 \), \( \mathcal{Q}_k = \mathbb{R}[X_k] \) is also a polynomial algebra, with generators \( X_k := X \cap \mathcal{Q}_k \). Note that the completion \( \hat{\mathcal{Q}}_k \) can be identified with “formal power series” in \( X_k \).

Having explored Lovász’s question algebraically, we now explore how to apply analytical techniques to infinite series of algebras. Note that this can be done for elements of \( \mathcal{Q}_k \) by using the map \( \alpha_x \). For instance if \( k = 0 \), then by Corollary 3.24, the map \( \alpha : \mathcal{Q}_0 \to \mathcal{S}_t \) is an
algebra isomorphism that identifies multigraphs with their homomorphism densities, which are functions on \( \mathcal{W}_{[0,1]} \) and hence amenable to analytical treatment.

Thus the immediate goal is to try extending the map \( \alpha_x : Q_k \to \text{Func}(\mathcal{W}_{[0,1]}, \mathbb{R}) \) to convergent \((k,x)\)-labelled power series in \( \hat{Q}_k \).

**Definition 4.40.** Given \( 0 < R \leq \infty \), define \( Q_{k,R} \) to be the set of formal \((k,x)\)-labelled power series of the form \( \sum_{n=0}^{\infty} \sum_{H \in \mathcal{H}_{n,k}} t_x(H, -, a_H) \) (for some \( x \in [0,1]^k \)) whose radius of convergence is greater than \( R \). Now define

\[
Q_{N,R} := \bigcup_{k \geq 0} Q_{k,R}, \quad R \in (0, \infty].
\]

**Remark 4.41.** Note that all values of \( x \in [0,1]^k \) yield the same set of power series in \( Q_{k,R} \), since the radius of convergence defined in [4.21] does not depend on \( x \) if all \( a_H \) are constant. Also note that the \( Q_{k,R} \) constitute a two-parameter family of commutative unital graded \( \mathbb{R} \)-algebras that is decreasing in \( 0 < R \leq \infty \) and increasing in \( k \geq 0 \), by Theorem [4.23].

We now have the following result which shows when infinite formal series of graphs can be embedded into spaces of functions amenable to analytic treatment.

**Theorem 4.42.** Fix \( x \in [0,1]^N \), and \( 0 < s < R \leq \infty \). The algebra map \( \alpha_x : Q_N \to \text{Func}(\mathcal{W}_{[-s,s]}, \mathbb{R}) \) of \( \mathbb{R} \)-algebras can be extended continuously to \( Q_{N,R} \) (with respect to the topology of \( \bigcup_{k \geq 0} \hat{Q}_k \)) and uniform convergence in \( \text{Func}(\mathcal{W}_{[-s,s]}, \mathbb{R}) \).

Furthermore, \( \alpha_x \) is an embedding of \( \mathbb{R} \)-algebras if and only if the \( x_i \) are distinct.

Note that continuously extending \( \alpha_x \) from \( Q_N \) to \( Q_{N,R} \) is equivalent to continuously extending \( \alpha_x \) from \( Q_k \) to \( Q_{k,R} \) for each \( k \geq 0 \).

**Proof.** The map \( \alpha_x \) can be extended by Proposition [4.22] from \( Q_k \) to \( Q_{k,R} \). The extension is continuous because Proposition [4.22] guarantees uniform convergence of the series. Now the first two parts of Theorem [4.23] show that \( \alpha_x : Q_{N,R} \to \text{Func}(\mathcal{W}_{[-s,s]}, \mathbb{R}) \) is an algebra map for each \( k \geq 0 \). The result for \( Q_{N,R} \) follows by compatibility across \( k \geq 0 \).

By Theorem [4.33] \( \alpha_x \) is injective when the \( x_i \) are distinct. Now if \( H_0 \in Q_{N,R} \) is in \( \ker \alpha_x \), then \( H_0 \in Q_{k,R} \) for some \( k \), whence \( H_0 = 0 \).

Finally, assume that for \( x \in [0,1]^k \), there exist two \( x_i \) that are equal. Without loss of generality, we can assume that \( x_1 = x_2 \). Consider now any \( k \)-labelled graph \( H \) and the graph \( H' \) that swaps the vertex labelled 1 with the vertex labelled 2. Then

\[
t_x(H, f) = t_x(H', f)
\]

for all \( f \in \mathcal{W}_{[0,1]} \) so \( \alpha_x \) is not injective on \( Q_{k,R} \). The result for \( Q_{N,R} \) follows immediately. \( \square \)

Note that the images of the maps \( \alpha_x \) are inter-related for different \( x \) as follows. Given \( k \in \mathbb{N} \) and \( x \in [0,1]^k \), and \( H \in \mathcal{H}_{n,k} \),

\[
t_x(H, f) = t_{\sigma(x)}(H, f^{\sigma^{-1}}) =: \sigma(t_{\sigma(x)}(H, -))(f), \quad \forall \sigma \in S_{[0,1]}, f \in \mathcal{W}.
\]

Now if \( x \) and \( y \) both have pairwise distinct elements, then for any \( \sigma \in S_{[0,1]} \) such that \( \sigma(x) = y \), we get that \( \sigma(\text{Im} \alpha_x) = \text{Im} \alpha_y \). Thus, the image of the maps \( \alpha_x \) are the same up to the action of \( S_{[0,1]} \) on \( \text{Func}(\mathcal{W}_{[-s,s]}, \mathbb{R}) \), with \( s \) as in Theorem [4.32].
Concluding remarks.

It is now possible to concretely explain the analogy in Section 2.1 between homomorphism densities and monomials, with degree the number of edges. Namely, using Remark 4.39 it is clear that the set of (unlabelled) multigraphs $\mathcal{H}$ spans the polynomial algebra $\mathbb{Q}_0 = \mathbb{R}[X_0]$ - and hence serves as a family of monomials in the generators $X_0$, with degree given by the number of edges. Now Theorem 4.42 provides a canonical (up to the $S_{[0,1]}$ action) way of embedding a subalgebra of infinite formal series of $k$-labelled graphs into $\text{Func}(\mathcal{W}_{[-s,s]}, \mathbb{R})$, with $s$ as in Theorem 4.42. The homomorphism densities $t(H, -)$ are simply the images of the monomials in $X_0$, under the algebra embedding $\alpha$.

Additionally, we have found that our notion of degree in this polynomial algebra interacts well with Gâteaux differentiation. Theorem 1.1 shows us that degree $N$ homomorphism densities are precisely the continuous class functions that vanish after taking $N+1$ derivatives. All of this suggests that the $\alpha$ map is a good starting point for further investigation into the analytic theory of infinite quantum algebras.

Acknowledgements. We would like to thank Professor Amir Dembo for valuable discussions.

References

[1] David J. Aldous. Exchangeability and continuum limits of discrete random structures. In Proceedings of the International Congress of Mathematicians. Volume I, pages 141–153, New Delhi, 2010. Hindustan Book Agency.

[2] Tim Austin and Terence Tao. Testability and repair of hereditary hypergraph properties. Random Structures Algorithms, 36(4):373–463, 2010.

[3] Peter J. Bickel and Aiyou Chen. A nonparametric view of network models and Newman-Girvan and other modularities. Proceedings of the National Academy of Sciences, 106(50):21068–21073, 2009.

[4] Christian Borgs, Jennifer T. Chayes, and László Lovász. Moments of two-variable functions and the uniqueness of graph limits. Geom. Funct. Anal., 19(6):1597–1619, 2010.

[5] Christian Borgs, Jennifer T. Chayes, László Lovász, Vera T. Sós, and Katalin Vesztergombi. Convergent sequences of dense graphs. I. Subgraph frequencies, metric properties and testing. Adv. Math., 219(6):1801–1851, 2008.

[6] Christian Borgs, Jennifer T. Chayes, László Lovász, Vera T. Sós, and Katalin Vesztergombi. Convergent sequences of dense graphs II. Multiway cuts and statistical physics. Ann. of Math. (2), 176(1):151–219, 2012.

[7] Sourav Chatterjee and Persi Diaconis. Estimating and understanding exponential random graph models. Ann. Statist., 41(5):2428–2461, 2013.

[8] Sourav Chatterjee and S.R.S. Varadhan. The large deviation principle for the Erdős-Rényi random graph. European J. Combin., 32(7):1000–1017, 2011.

[9] Sourav Chatterjee and S.R.S. Varadhan. Large deviations for random matrices. Commun. Stoch. Anal., 6(1):1–13, 2012.

[10] Persi Diaconis, Susan Holmes, and Svante Janson. Interval graph limits. Ann. Comb., 17(1):27–52, 2013.

[11] Persi Diaconis and Svante Janson. Graph limits and exchangeable random graphs. Rend. Mat. Appl. (7), 28(1):33–61, 2008.

[12] Gábor Elek. Samplings and observables. Invariants of metric measure spaces. arXiv:1205.6936, 2012.

[13] Gábor Elek and Balázs Szegedy. A measure-theoretic approach to the theory of dense hypergraphs. Adv. Math., 231(3-4):1731–1772, 2012.

[14] Paul Erdős, László Lovász, and Joel Spencer. Strong independence of graphcopy functions. In Graph theory and related topics (Proc. Conf., Univ. Waterloo, Waterloo, Ont., 1977), pages 165–172. Academic Press, New York-London, 1979.

[15] Mikhail Gromov. Metric structures for Riemannian and non-Riemannian spaces. Springer, 2001.

[16] Hamed Hatami. Graph norms and Sidorenko’s conjecture. Israel J. Math., 175:125–150, 2010.
In this section we prove Theorem 2.17. We require a couple of preliminary results in order to do so.

Lemma A.1. Suppose $V$ is a convex subset of a real vector space $E$, with $0 \in V$. Let $\Lambda : V \to \mathbb{R}$. Then the following are equivalent:

1. $\Lambda$ is linear (in the sense of Definition 2.16).
2. For all $m \in \mathbb{N}$, $g_1, \ldots, g_m \in V$, and $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ such that $v_m := \sum_{i=1}^{m} \alpha_i g_i \in V$,
   \[ \Lambda(v_m) = \sum_{i=1}^{m} \alpha_i \Lambda(g_i). \]
3. Given $\alpha_1, \alpha_2 \geq 0$ and $g_1, g_2 \in V$ such that $v := \alpha_1 g_1 + \alpha_2 g_2 \in V$, $\Lambda(v) = \alpha_1 \Lambda(g_1) + \alpha_2 \Lambda(g_2)$.

Proof. We may assume that $E$ is spanned by $V$. Clearly, (1) $\implies$ (2) $\implies$ (3). We now show that (2) $\implies$ (1). Clearly $V$ contains a basis of $E$, which uniquely extends $\Lambda$ to all of $E$. This extension agrees with $\Lambda$ on $V$ by the condition in (2).
Finally, we show that (3) ⇒ (2). We first claim that

\[ \alpha_i \geq 0, \ g_i \in V, \ v_m := \sum_{i=1}^{m} \alpha_i g_i \in V \implies \Lambda(v_m) = \sum_{i=1}^{m} \alpha_i \Lambda(g_i). \] (A.2)

Indeed, since \( V \) is convex and contains the origin and all \( g_i \), hence there exists a sufficiently small \( 0 < \epsilon < 1 \) such that \( \sum_{i=1}^{m} \beta_i g_i \in V \) for all \( 0 \leq \beta_i \leq \epsilon \). Now define \( \delta := \epsilon / \sum_{j=1}^{m} \alpha_j \) and \( \beta_i := \delta \alpha_i \), and consider the partial sums \( \sum_{j=1}^{i} \beta_j g_j \in V \). Applying condition (3) inductively, we obtain that

\[ \Lambda(\sum_{i=1}^{m} \beta_i g_i) = \sum_{i=1}^{m} \beta_i \Lambda(g_i). \]

Using this and once again applying condition (3),

\[ \Lambda(v_m) = \Lambda(\delta^{-1} \sum_{i=1}^{m} \beta_i g_i + 0) = \delta^{-1} \Lambda(\sum_{i=1}^{m} \beta_i g_i) + 0 = \delta^{-1} \sum_{i=1}^{m} \beta_i \Lambda(g_i) = \sum_{i=1}^{m} \alpha_i \Lambda(g_i). \]

This proves the claim (A.2). Now to prove the general case of (2), let \( \alpha_i, \beta_j \geq 0 \) and \( g_i, h_j \in V \), and define:

\[ v^+ := \sum_{i=1}^{m} \alpha_i g_i, \quad v^- := \sum_{j=1}^{k} \beta_j h_j, \quad v := v^+ - v^- . \]

Since \( V \) is convex and \( 0 \in V \), we know that for small \( 0 < \delta \ll 1, \delta v, \delta v^+, \delta v^- \in V \). Now by (3),

\[ \Lambda(\delta v^+) = \Lambda(\delta v) + \Lambda(\delta v^-). \]

Since we also know that \( v \in V \), hence this implies:

\[ \delta \Lambda(v) = \Lambda(\delta v) = \Lambda(\delta v^+) - \Lambda(\delta v^-). \]

Now using (A.2) for \( v^\pm \), we get:

\[ \delta \Lambda(v) = \sum_{i=1}^{m} \delta \alpha_i \Lambda(g_i) - \sum_{j=1}^{k} \delta \beta_j \Lambda(h_j), \]

and (2) follows. \( \square \)

The next result relates linearity and multilinearity in this restricted setting.

**Lemma A.3.** Suppose \( V \) is a convex subset of a real vector space \( E \), with \( 0 \in V \). Let \( \Lambda : V^n \rightarrow \mathbb{R} \) for some \( n \in \mathbb{N} \). Then the following are equivalent:

1. \( \Lambda \) is multi-linear (in the sense of Definition 2.16).
2. For all \( m_1, \ldots, m_n \in \mathbb{N} \) and \( \alpha_{ij} \in \mathbb{R}, g_{ij} \in V \) (with \( 1 \leq i \leq m_j \)) such that \( h_j := \sum_{i=1}^{m_j} \alpha_{ij} g_{ij} \in V \) for all \( j \),

\[ \Lambda(h_1, \ldots, h_n) = \sum_{i_1, \ldots, i_n} \prod_{j=1}^{n} \alpha_{i_j,j} \cdot \Lambda(g_{i_1,1}, \ldots, g_{i_n,n}). \]
Proof. That (1) $\iff$ (2) $\implies$ (3) $\implies$ (4) are standard or as in the proof of Lemma A.1. That (3) $\implies$ (2) is also standard, since we expand each factor $h_j$ out by linearity and the terms $g_{ij}$ still lie in $V$. Finally assume that (4) holds. Then (3) holds by applying Lemma A.1 to the functional $\Lambda(h_1, \ldots, h_j-1, -, h_{j+1}, \ldots, h_n)$ for all $h_j \in V$. 

We can now prove Theorem 2.17.

Proof of Theorem 2.17. For the first part, since $F$ is $C^n$ at $f \in U$, it suffices to show part (1) for $\tau$ a transposition. This further reduces the problem to assuming that $F$ is $C^2$. In this case, a similar proof to that of [28, Theorem 9.41] yields the result.

For the second part, first note that since $U \subset E$ is convex, $V := \text{Adm}_U(f)$ satisfies the hypotheses of Lemma A.3. Therefore to prove multilinearity, it suffices to show that part (4) of Lemma A.3 is satisfied. Now by the previous part of this theorem, it suffices to show that $dF(f, -)$ satisfies part (3) of Lemma A.1 (as we can take higher Gâteaux derivatives beyond that point). Finally, the proof of [28, Theorem 9.21] can be adapted to one-sided directional derivatives in $\mathbb{R}^2$ to show that Lemma A.1(3) is satisfied. 

Appendix B. Proof of Theorem 2.18

In this section we prove Theorem 2.18. First recall the definition of the Fréchet derivative and its relation with the Gâteaux derivative.

Definition B.1. Let $U$ be open in a normed linear space $E$, $x \in U$, and $F$ be a normed linear space. Let $f : U \to F$ be a map. We say that $f$ is Fréchet differentiable at $x$ if there exists a continuous linear map $A_x : E \to F$ such that

$$\lim_{h \to 0} \frac{\|f(x + h) - f(x) - A_x(h)\|_F}{\|h\|_E} = 0.$$ 

The linear map $A_x$ is unique and is called the Fréchet derivative of $f$ at $x$.

Note that if $F$ is Fréchet differentiable at $f \in \mathcal{W}$, then:

1. $F$ is also Gâteaux differentiable at $f$;
2. The Gâteaux derivative $dF(f, h)$ is linear (in $h$);
3. The Fréchet derivative coincides with the linear operator $h \mapsto dF(f, h)$.

The Fréchet derivative may fail to exist even if the Gâteaux derivative exists and is linear. Nevertheless, if $F$ is Gâteaux differentiable at $f$ and the derivative is linear, the Gâteaux derivative is the only possible candidate for the Fréchet derivative. See [30, Section 5.6] for further details.

In our discussion of the Gâteaux derivative, we defined $\text{Adm}(f)$ in order to deal with the fact that $\mathcal{W}_{[0,1]}$ is not a vector space. To avoid this issue for the Fréchet derivative, we consider
functions \( f \in \mathcal{W}_{[0,1]} \) which are bounded away from both 0 and 1. For such \( f \) it is possible to define the Gâteaux derivative with respect to every direction in \( \mathcal{W} \) - i.e., \( \text{Adm}(f) = \mathcal{W} \). In that case, we say that \( F \) is Fréchet differentiable at \( f \) if those derivatives form a linear operator \( A_f \) that is a continuous linear map. Now say that a function \( F : \mathcal{W}_{[0,1]} \to \mathbb{R} \) is Fréchet differentiable if it is Fréchet differentiable at every \( f \in \mathcal{W}_{[0,1]} \) bounded away from 0 and 1. Using this terminology, we now prove Theorem \ref{thm:frechet_diff}.

**Proof of Theorem \ref{thm:frechet_diff}** We begin by analyzing three simple cases: \( t(A_1, -) \), \( t(A_2, -) \), and \( t(A_3, -) \). Clearly, \( t(A_1, -) \equiv 1 \) and thus is Fréchet differentiable. For \( t(A_2, -) \), the computation of the Gâteaux derivative

\[
d(t(H, f); g) = \int_{[0,1]^k} \sum_{(i,j)\in E(H)} g(x_{i1}, x_{j1}) \prod_{(i,j)\in E(H)\setminus (i_1,j_1)} f(x_i, x_j) \, dx_1 \cdots dx_k
\]

yields the unique candidate for the Fréchet derivative:

\[
g \mapsto d(t(A_2, f); g) = \int_{[0,1]^2} g(x_1, x_2) \, dx_1 \, dx_2.
\]

We verify that this linear map is the Fréchet derivative by noting that

\[
\lim_{g \to 0} \frac{|t(H, f + g) - t(H, f) - \int_{[0,1]^2} g(x_1, x_2) \, dx_1 \, dx_2|}{\|g\|_\Box} = \lim_{g \to 0} \frac{0}{\|g\|_\Box} = 0.
\]

For \( t(A_3, -) \) we have the following unique candidate for the Fréchet derivative:

\[
g \mapsto d(t(A_3, f); g) = \int_{[0,1]^3} g(x_1, x_2) f(x_2, x_3) + f(x_1, x_2) g(x_2, x_3) \, dx_1 \, dx_2 \, dx_3
\]

\[
= 2 \int_{[0,1]^3} f(x_1, x_2) g(x_2, x_3) \, dx_1 \, dx_2 \, dx_3.
\]

Thus \( t(A_3, -) \) is Fréchet differentiable if and only if the following limit exists and equals 0:

\[
\lim_{g \to 0} \frac{|t(A_3, f + g) - t(A_3, f) - d(t(A_3, f); g)|}{\|g\|_\Box} = \lim_{g \to 0} \frac{\int_{[0,1]^3} g(x_1, x_2) g(x_2, x_3) \, dx_1 \, dx_2 \, dx_3}{\|g\|_\Box}.
\]

(B.2)

Now define \( g_n \in \mathcal{W}_{[0,1]} \) via:

\[
g_n(x_1, x_2) = 1(\min(x_1, x_2) < 1/n).
\]

(B.3)

We claim that \( g_n \to 0 \) in the cut-norm, but the limit (B.2) does not go to zero along \( g_n \). Indeed, \( \|g_n\|_\Box \to 0 \) since \( \|g_n\|_1 = \frac{2}{n} - \frac{1}{n^2} \). However, the numerator in (B.2) is bounded below by

\[
\left| \int_{[0,1]^3} g_n(x_1, x_2) g_n(x_2, x_3) \, dx_1 \, dx_2 \, dx_3 \right| \geq \int_{[0,1]^3, x_2 \in [0,1/n]} g_n(x_1, x_2) g_n(x_2, x_3) \, dx_1 \, dx_2 \, dx_3
\]

\[
= \int_{[0,1]^3, x_2 \in [0,1/n]} 1 \, dx_1 \, dx_2 \, dx_3 = 1/n.
\]

We conclude that \( t(A_3, -) \) is not Fréchet differentiable.

We now prove the result. First assume that (i) holds. If \( H \) is a disjoint union of copies of \( A_2 \) and \( A_1 \) then by Equation (2.1), \( t(H, -) \) is a product of \( t(A_2, -) \) and \( t(A_1, -) \). We have just shown that both are Fréchet differentiable, so by the product rule (18), \( t(H, -) \) is Fréchet differentiable as well.
We show that \((i) \Rightarrow (ii)\) by proving its contrapositive.

A finite simple graph \(H\) is a disjoint union of \(A_2\) and \(A_1\) if and only if it does not admit an injective graph homomorphism \(A_3 \rightarrow H\). Thus, assume that \(H\) is not a disjoint union of copies of \(A_2\) and \(A_1\), and that vertices 1, 3 in \(H\) are adjacent to vertex 2 without loss of generality. Now for \(f\) bounded below by \(c > 0\), we can perform the same calculation as for \(A_3\) for the same sequence \(g_n\) as in \(\text{[B.3]}\). Since all terms in the Gâteaux derivative computation are nonnegative, hence

\[
t(H, f + g) - t(H, f) - d(t(H, f); g_n) \\
\geq \int_{[0,1]^2 \setminus V(H)} g(x_1, x_2)g(x_2, x_3) \prod_{(ij) \in E(H) \setminus \{(1,2),(2,3)\}} f(x_i, x_j) \, dx \\
\geq c|E(H)|^{-2} \int_{[0,1]^3} g(x_1, x_2)g(x_2, x_3) \, dx_1dx_2dx_3.
\]

Now by the same argument as for \(H = A_3\),

\[
\lim_{n \to \infty} \frac{|t(H, f + g_n) - t(H, f) - d(t(H, f); g_n)|}{\|g_n\|} \\
\geq \lim_{n \to \infty} \frac{c|E(H)|^{-2} \int_{[0,1]^2} g_n(x_1, x_2)g_n(x_2, x_3) \, dx_1dx_2dx_3}{\|g_n\|} \\
\geq \frac{c|E(H)|^{-2} / n}{2/n} = \frac{c|E(H)|^{-2}}{2} > 0.
\]

Therefore \(t(H, -)\) is not Fréchet differentiable at graphons \(f > c\).

Similarly, if \(H\) is not a simple graph then we can assume that vertices 1 and 2 of \(H\) have at least two edges between them without loss of generality. Now for \(f\) bounded below by \(c > 0\) and the sequence \(g_n\) we can make the same calculation as above. Hence,

\[
t(H, f + g) - t(H, f) - d(t(H, f); g_n) \geq c|E(H)|^{-2} \int_{[0,1]^2} g(x_1, x_2)g(x_1, x_2) \, dx_1dx_2.
\]

Now,

\[
\lim_{n \to \infty} \frac{|t(H, f + g_n) - t(H, f) - d(t(H, f); g_n)|}{\|g_n\|} \\
\geq \lim_{n \to \infty} \frac{c|E(H)|^{-2} \int_{[0,1]^2} g_n(x_1, x_2)^2 \, dx_1dx_2}{\|g_n\|} \\
= c|E(H)|^{-2} > 0.
\]

Therefore \(t(H, -)\) is not Fréchet differentiable at graphons \(f > c\) for \(H\) not simple. \(\square\)

**Appendix C. Proofs of Series Properties**

In this section we write down the proofs of the results in Section \(\text{[4.2]}\) that were deferred until later.

**Proof of Proposition \(\text{[4.22]}\)** If \(f \in \mathcal{W}_{[-s,s]}\) for some \(0 \leq s < R_F\), then

\[
|t_x(H, f; g_H)| \leq \|g_H\|_{1,s} |E(H)| \quad \forall H \in \mathcal{H}_{n,k}.
\]
Choose \( s < s_1 < R_F \); then \( \sum_{H \in \mathcal{H}_{n,k}} \|g_H\|_{1,x} \leq s_1^n \) for all \( n \) sufficiently large, say \( n \geq N \). Therefore as in the usual Root Test for one-variable power series \([28]\),

\[
\sum_{n \geq N} \sum_{H \in \mathcal{H}_{n,k}} |t_x(H, f, g_H)| \leq s \sum_{n \geq N} \sum_{H \in \mathcal{H}_{n,k}} \|g_H\|_{1,x} \leq \sum_{n \geq N} \left( \frac{s}{s_1} \right)^n < \infty.
\]

This proves the first part of the theorem. Next, fix \( 0 \leq s < R_F \) and consider \( f \in \mathcal{W}_{[-s,s]} \). Then by the Weierstrass M-test \([28\text{ Theorem } 7.10]\), since \( |t_x(H, |f|, |g_H|)| \leq \|g_H\|_{1,x} s^{|E(H)|} \), the uniform convergence of \( F \) on \( \mathcal{W}_{[-s,s]} \) follows. Finally, suppose \( r > R_F \) and let \( f \equiv \frac{-s \pm \sqrt{s^2 - 4}}{2} \). Straightforward computations now show that \( F \) is not absolutely convergent at \( f \). \( \square \)

**Proof of Theorem 4.23**

(i) Note that the radius of convergence of \( cF + dG \) is bounded by

\[
R_{cF+dG}^{-1} = \limsup_{n \to \infty} \left( \sum_{H \in \mathcal{H}_{n,k}} \|cg_H + dg'_H\|_{1,x} \right)^{1/n} \leq \limsup_{n \to \infty} \left( \sum_{H \in \mathcal{H}_{n,k}} \|cg_H\|_{1,x} + \|dg'_H\|_{1,x} \right)^{1/n} \leq R^{-1}.
\]

Since \( t_x(H, f, cg_H) + t_x(H, f, dg'_H) = t_x(H, f, cg_H + dg'_H) \), the partial sums on the left hand side and the right hand side are equal and both converge to the same quantity for \( f \in \mathcal{W}_{1,R} \).

(ii) Begin by defining

\[
A_n := \sum_{H \in \mathcal{H}_{n,k}} \|g_H\|_{1,x}, \quad B_n := \sum_{H \in \mathcal{H}_{n,k}} \|g'_H\|_{1,x},
\]

where we know that \( R_F^{-1} = \limsup A_n^{1/n} \) and \( R_G^{-1} = \limsup B_n^{1/n} \) are both finite. Then,

\[
\sum_{H \in \mathcal{H}_{n,k}} \|g_{H_1}g'_{H_2}\|_{1,x} \leq \sum_{H \in \mathcal{H}_{n,k}} \sum_{H_1H_2=H} \|g_{H_1}g'_{H_2}\|_{1,x} = \sum_{l=0}^{|E(H)|} A_l B_{|E(H)|-l},
\]

since \( \|g_{H_1}g'_{H_2}\|_{1,x} = \|g_{H_1}\|_{1,x} \|g'_{H_2}\|_{1,x} \) for all \( g_{H_1}, g'_{H_2} \) by Proposition 4.8. The analogous result for one-variable power series applied to the product of the power series \( \sum_n A_n x^n \) and \( \sum_n B_n x^n \) shows that \( R_{FG} \geq R \).

Since \( t_x(H, f, g_H) t_x(H', f, g'_{H'}) = t_x(HH', f, g_Hg'_{H'}) \) by Proposition 4.8 the partial sums on the left hand side and the right hand side are equal and both converge to the same quantity for \( f \in \mathcal{W}_{1,R} \).

(iii) Note that the Gâteaux derivative \( d^m t(H, -, g_H)(f_0; f_1, \ldots, f_m) \) can be expressed as a sum \( \sum_{H'} t_x(H', - g'_{H'}) \) where \( H' \in \mathcal{H}_{n-m,k} \) varies over subgraphs \( H' \) of \( H \) with \( m \) edges removed. If \( n = |E(H)| \) then there are \( n(n-1) \ldots (n-m+1) \) terms in the derivative,
each with $|g'_H| \leq |g_H| \prod_{i=1}^{m} |f_i|$. Therefore, the series for the derivative

$$
\sum_{n=0}^{\infty} \sum_{H \in \mathcal{H}_{n,k}} d^m t_x(H, -, g_H)(f_0; f_1, \ldots, f_m)
$$

is a weighted $(k, x)$-labelled power series. We bound its radius of convergence by noting that if we call its coefficients $g''_H$, then

$$
\limsup_{n \to \infty} \left( \sum_{H \in \mathcal{H}_{n,k}} \|g''_H\|_{1,x} \right)^{1/n} \leq \limsup_{n \to \infty} \left( \frac{(n + m)!}{n!} \sum_{H \in \mathcal{H}_{n+m,k}} \|g_H\|_{1,x} \prod_{i=1}^{m} \|f_i\|_{\infty} \right)^{1/n}
$$

$$
\leq \limsup_{n \to \infty} \left( \sum_{H \in \mathcal{H}_{n+m,k}} \|g_H\|_{1,x} \right)^{1/(n+m)} = R_F^{-1},
$$

so the radius of convergence of the derivative is finite and at least $R_F$.

Now for all $0 \leq r < R_F$, the series for the derivative converges uniformly on $\mathcal{W}_{[-r, r]}$ by Proposition 4.22. Thus we can apply [28, Theorem 7.17] and conclude that the derivatives of the partial sums converge on $\mathcal{W}_{R_F}$. Therefore for $f_0 \in \mathcal{W}_{R_F}$,

$$
d^m F(f_0; f_1, \ldots, f_m) = \sum_{n=0}^{\infty} \sum_{H \in \mathcal{H}_{n,k}} d^m t_x(H, -, g_H)(f_0; f_1, \ldots, f_m), \quad \forall f_1, \ldots, f_m \in \mathcal{W}. \quad (C.1)
$$

In addition, the individual terms above are $C^0$ on $\mathcal{W}_{R_F}$. Since the uniform convergence of continuous functions is continuous, $F$ is Gâteaux smooth.

\[\square\]

Department of Mathematics, Stanford University