IMPROVED RESULTS FOR KLEIN-GORDON-MAXWELL SYSTEMS WITH GENERAL NONLINEARITY

SITONG CHEN AND XIANHUA TANG∗

School of Mathematics and Statistics, Central South University
Changsha, 410083 Hunan, China

(Communicated by Andrea Malchiodi)

Abstract. This paper is concerned with the following Klein-Gordon-Maxwell system

\[
\begin{align*}
-\Delta u + \left[ m_0^2 - (\omega + \phi)^2 \right] u &= f(u), \quad \text{in } \mathbb{R}^3, \\
\Delta \phi &= (\omega + \phi)u^2, \quad \text{in } \mathbb{R}^3,
\end{align*}
\]

where \(0 < \omega \leq m_0\) and \(f \in C(\mathbb{R}, \mathbb{R})\). By introducing some new tricks, we prove that the above system has 1) a ground state solution in the case when \(0 < \omega < m_0\) and \(f\) is superlinear at infinity; 2) a nontrivial solution in the zero mass case, i.e. \(\omega = m_0\) and \(f\) is super-quadratic at infinity. These results improve the related ones in the literature.

1. Introduction. In this paper, we study the following Klein-Gordon-Maxwell system

\[
\begin{align*}
-\Delta u + \left[ m_0^2 - (\omega + \phi)^2 \right] u &= f(u), \quad \text{in } \mathbb{R}^3, \\
\Delta \phi &= (\omega + \phi)u^2, \quad \text{in } \mathbb{R}^3,
\end{align*}
\]

where \(0 < \omega \leq m_0\), \(u, \phi : \mathbb{R}^3 \to \mathbb{R}\), and \(f : \mathbb{R} \to \mathbb{R}\) satisfies the following basic assumptions:

(F1) \(f \in C(\mathbb{R}, \mathbb{R})\) and there exist constants \(C_0 > 0\) and \(p \in (2, 6)\) such that

\[
|f(t)| \leq C_0 \left(1 + |t|^{p-1}\right), \quad \forall \ t \in \mathbb{R};
\]

(F2) \(f(t) = o(|t|)\) as \(t \to 0\).

Such a system was first introduced by Benci and Fortunato [4] as a model describing solitary waves for the nonlinear Klein-Gordon equation interacting with an electromagnetic field. The presence of the nonlinear term \(f\) simulates the interaction between many particles or external nonlinear perturbations. For more details in the physical aspects, we refer the readers to [4,5].

Under assumptions (F1) and (F2), the weak solutions to (1.1) correspond to the critical points of the energy functional defined in \(H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)\) by

\[
S(u, \phi) = \frac{1}{2} \int_{\mathbb{R}^3} \left\{ |\nabla u|^2 + \left[ m_0^2 - (\omega + \phi)^2 \right] u^2 - |\nabla \phi|^2 \right\} dx - \int_{\mathbb{R}^3} F(u)dx,
\]

2010 Mathematics Subject Classification. 35J10, 35J20.

Key words and phrases. Klein-Gordon-Maxwell system, ground state solutions, zero mass case.

This work is partially supported by the Hunan Provincial Innovation Foundation for Postgraduate (No: CX2017B041) and the National Natural Science Foundation of China (No: 11571370).

* Corresponding author.

2333
where \( F(u) := \int_0^t f(s) \, ds \). We are interested in finding “ground state” solutions of (1.1), that is a solution \((u_0, \phi_0) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)\) which minimizes the functional \( S \) among all the nontrivial solutions of (1.1).

In the past decades, there has been many existence and nonexistence results on Klein-Gordon-Maxwell systems like (1.1) by variational methods, see for examples [2,3,5,7,8,12–15,18,20,27] and the references therein. For related nonlocal problems, we refer to [9–11,22,24–26,29–31] and so on.

Let us recall some previous results that led us to the present research.

The first result is due to Benci and Fortunato. In [5], they proved that the following special form of (1.1)

\[
\begin{align*}
-\Delta u + \left[ m_0^2 - (\omega + \phi)^2 \right] u &= |u|^{q-2} u, \quad \text{in } \mathbb{R}^3, \\
\Delta \phi &= (\omega + \phi) u^2, \quad \text{in } \mathbb{R}^3,
\end{align*}
\]

possesses infinitely many radially symmetric solutions when \( 0 < \omega < m_0 \) and \( 4 < q < 6 \). Afterwards, D’Aprile and Mugnai [13] obtained the same conclusion when \( 0 < \omega < \sqrt{(q-2)/2} m_0 \) and \( 2 < q \leq 4 \). By a Pohozaev-type argument, D’Aprile and Mugnai [14] showed that (1.3) only has a trivial solution when \( 0 < q \leq 2 \) or \( q \geq 6 \). Inspired by [5, 13], Azzollini and Pomponio [2] proved that (1.3) admits a ground state solution if one of the following conditions holds:

(i) \( 4 \leq q < 6 \) and \( 0 < \omega < m_0 \);
(ii) \( 2 < q < 4 \) and \( 0 < \omega < \sqrt{(q-2)/(6-q)} m_0 \).

Their method consisted in minimizing the corresponding functional associated with (1.3) on the Nehari manifold. Following the ideas of [2], Wang [27] further extended the existence range of \((m_0, \omega)\) for \( 2 < q < 4 \) as follows:

\[ 0 < \omega < \sqrt{4(q-2)/(4-q)^2 + 4(q-2)} m_0. \]

Based on the Pohozaev identity and the “monotonicity trick”, developed by Struwe [23] and Jeanjean [17], Azzollini et al. [3] proved that (1.3) has a nontrivial radial solution under one of the following conditions:

(iii) \( 3 \leq q < 4 \) and \( 0 < \omega < m_0 \);
(iv) \( 2 < q < 3 \) and \( 0 < \omega < \sqrt{(q-2)(4-q)} m_0 \),

which gave a little improvement for the existence results on (1.3) when \( 2 < q < 4 \). We point out that the approaches used in [2,3,27] are heavily dependent on the form \( f(t) = |t|^{q-2} t \), which are no longer applicable for (1.1) with general nonlinearity \( f \), even for the case when \( f(t) = (|t|^{\alpha-2} + |t|^{\beta-2}) t \) with \( 2 < \beta < 4 \) and \( \beta < \alpha < 6 \).

Motivated by the aforementioned works, in the present paper, we will use some new tricks to generalize the above results to (1.1) under the following superlinear condition:

(F3) there exist constants \( r_0 > 0 \) and \( \mu > 2 \) such that

\[
F(t) \geq 0, \quad \forall \, |t| \geq r_0, \quad \text{and} \quad f(t) t - \mu F(t) \geq 0, \quad \forall \, t \in \mathbb{R}.
\]

Our first result is as follows.

**Theorem 1.1.** Assume that (F1)-(F3) hold. Then (1.1) has a ground state solution \((\bar{u}, \phi_\bar{u}) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)\) with positive energy if one of the following conditions holds:

(i) \( 3 \leq \mu < \infty \) and \( 0 < \omega < m_0 \);
(ii) $2 < \mu < 3$ and $0 < \omega < \sqrt{(\mu - 2)(4 - \mu)}m_0$.

**Remark 1.2.** If $f(u) = |u|^q u$ with $2 < q < 6$, then one has $\mu = q$ in (F3). In this case, Theorem 1.1 reduces to the best results (iii) and (iv) for (1.3) which was obtained in [3]. Therefore, Theorem 1.1 generalizes and improves the related ones in the literature.

Next, we are mainly interested to study the limit case $\omega = m_0$, when (1.1) becomes

$$
\begin{align*}
-\Delta u - (2\omega \phi + \phi^2) u &= f(u), \quad \text{in } \mathbb{R}^3, \\
\Delta \phi &= (\omega + \phi) u^2, \quad \text{in } \mathbb{R}^3.
\end{align*}
$$

We notice that, in the first equation, besides the interaction term $(2\omega \phi + \phi^2) u$, there is no linear term in $u$. In this sense the situation described by (1.4) is analogous to the zero mass case for nonlinear field equations (see [6]). In contrast to the case $0 < \omega < m_0$, there are few papers dealing with the limit case $\omega = m_0$, we only find one paper [3]. More precisely, under the following assumption:

(AQ) $f \in C(\mathbb{R}, \mathbb{R})$ and there exist constants $4 < q_0 \leq q_1 < 6 < q_2$ and $C_1, C_2 > 0$ such that

$$f(t)t \geq q_0 F(t), \quad F(t) \geq C_1 \min\{ |t|^{q_1}, |t|^{q_2} \}, \quad \forall t \in \mathbb{R},$$

$$|f(t)| \leq C_2 \min\{ |t|^{q_1 - 1}, |t|^{q_2 - 1} \}, \quad \forall t \in \mathbb{R},$$

Azzollini et al. [3] proved that (1.4) has a nontrivial solution in $\mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$. Notice that the condition $q_0 > 4$ in (AQ) plays a crucial role. In the present paper, instead of (AQ), we will use the following weaker condition:

(F4) $f \in C(\mathbb{R}, \mathbb{R})$ and there exist constants $3 < p_0 \leq p_1 < 6 < p_2$ and $\theta_1, \theta_2 > 0$ such that

$$f(t)t \geq p_0 F(t), \quad F(t) \geq \theta_1 \min\{ |t|^{p_1}, |t|^{p_2} \}, \quad \forall t \in \mathbb{R},$$

$$|f(t)| \leq \theta_2 \min\{ |t|^{p_1 - 1}, |t|^{p_2 - 1} \}, \quad \forall t \in \mathbb{R}.$$

This type of condition goes back to the work by Berestycki and Lions in [6], where the authors proved the following Schrödinger equation with zero mass

$$-\Delta u = g(u) \quad \text{in } \mathbb{R}^N, N \geq 3$$

has a nontrivial radial solution if $g$ behaves like $|t|^{p_1 - 2}t$ for small $t$, and like $|t|^{p_2 - 2}t$ for large $t$ with constants $0 < p_1 < 2N/(N - 2) < p_2$.

We are now in a position to state the second result of this paper.

**Theorem 1.3.** Assume that $\omega = m_0$ and (F4) holds. Then (1.1) has a nontrivial solution $(u_0, \phi_0) \in \mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)$.

The paper is organized as follows. In Section 2, we give some preliminary lemmas. We prove Theorems 1.1 and 1.3 in Sections 3 and 4 respectively. Throughout this paper, we let $u(t) := u(tx)$ for $t > 0$, denote the norm of $L^s(\mathbb{R}^3)$ by $\|u\|_s = \left( \int_{\mathbb{R}^3} |u|^s dx \right)^{1/s}$, and positive constants possibly different in different places, by $C_1, C_2, \cdots$. 

2. Preliminary lemmas. Hereafter, $H^1(\mathbb{R}^3)$ is the usual Sobolev space with the standard scalar product and norm

$$(u, v) = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + uv) \, dx, \quad \|u\| = \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \, dx\right)^{1/2}.$$ 

As it has been done by the aforementioned authors, we apply the “reduction method” developed by Benci and Fortunato [5] in order to avoid the difficulty originated by the strongly indefiniteness of the functional $S$ defined by (1.2). Now, we need the following technical results.

**Lemma 2.1.** [13,14] For any $u \in H^1(\mathbb{R}^3)$, there exists a unique $\phi = \phi_u \in D^{1,2}(\mathbb{R}^3)$ which solves the equation

$$-\Delta \phi + u^2 \phi = -\omega u^2. \tag{2.1}$$

Moreover, the map $\Phi : u \in H^1(\mathbb{R}^3) \mapsto \phi_u \in D^{1,2}(\mathbb{R}^3)$ is continuously differentiable, and

$$-\omega \leq \phi_u \leq 0 \text{ on the set } \{x \in \mathbb{R}^3 : u(x) \neq 0\}. \tag{2.2}$$

Multiplying (2.1) by $\phi_u$ and integrating by parts, we obtain

$$\int_{\mathbb{R}^3} |\nabla \phi_u|^2 \, dx = -\int_{\mathbb{R}^3} \omega \phi_u u^2 \, dx - \int_{\mathbb{R}^3} \phi_u^2 u^2 \, dx. \tag{2.3}$$

By the definition of $S$ and using (2.3), the functional $I(u) = S(u, \phi_u)$ has the form

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^3} \left[|\nabla u|^2 + (m_0^2 - \omega^2)u^2\right] \, dx - \int_{\mathbb{R}^3} \left[\frac{1}{2} \omega \phi_u u^2 + F(u)\right] \, dx, \quad \forall u \in H^1(\mathbb{R}^3). \tag{2.4}$$

**Lemma 2.2.** [2, Lemma 2.7] If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then, up to subsequences, $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$. As a consequence, $\Phi'(u_n) \rightharpoonup \Phi'(u)$ in the sense of distributions.

In view of Lemmas 2.1 and 2.2, under (F1) and (F2), we have $I \in C^1(H^1(\mathbb{R}^3), \mathbb{R})$, and

$$\langle I'(u), v \rangle = \int_{\mathbb{R}^3} \left[\nabla u \cdot \nabla v + (m_0^2 - \omega^2)uv\right] \, dx - \int_{\mathbb{R}^3} \left[(2\omega + \phi_u)\phi_u u + f(u)\right] v \, dx. \tag{2.5}$$

Moreover, as in [3, Theorem 2.2], the pair $(u, \phi_u) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ is a solution of (1.1) if and only if $u$ is a critical point of $I$, and $\phi = \phi_u$ which is unique. For the sake of simplicity, in many cases we just say $u \in H^1(\mathbb{R}^3)$, instead of $(u, \phi_u) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$, is a weak solution of (1.1).

In view of [14, (3.25)], any critical point $u$ of $I$ satisfies the following Pohozaev equality:

$$\mathcal{P}(u) = \int_{\mathbb{R}^3} \left[|\nabla u|^2 + 3(m_0^2 - \omega^2)u^2\right] \, dx - \int_{\mathbb{R}^3} (5\omega + 2\phi_u)\phi_u u^2 \, dx$$

$$-6 \int_{\mathbb{R}^3} F(u) \, dx = 0. \tag{2.6}$$
Let
\[ J(u) := \langle I'(u), u \rangle - \frac{1}{2} P(u) \]
\[ = \frac{1}{2} \|\nabla u\|^2_2 - \frac{1}{2} \int_{\mathbb{R}^3} \left( m_0^2 - \omega^2 \right) u^2 \, dx + \frac{\omega}{2} \int_{\mathbb{R}^3} \phi u^2 \, dx \]
\[ + \int_{\mathbb{R}^3} \left[ 3F(u) - f(u)u \right] \, dx \]

(2.7)

and
\[ M := \{ u \in H^1(\mathbb{R}^3) \setminus \{0\} : I'(u) = 0 \}. \]

Then \( J(u) = 0 \), \( \forall u \in M \).

To show \( M \neq \emptyset \), we shall use the following general minimax principle [19, Proposition 2.8], which is a somewhat stronger variant of [28, Theorem 2.8].

Lemma 2.3. Let \( X \) be a Banach space. Let \( M_0 \) be a closed subspace of the metric space \( M \) and \( \Gamma_0 \subset C(M_0, X) \). Define
\[ \Gamma := \left\{ \gamma \in C(M, X) : \gamma|_{M_0} \in \Gamma_0 \right\}. \]

If \( \varphi \in C^1(X, \mathbb{R}) \) satisfies
\[ \infty > c := \inf_{\gamma \in \Gamma} \sup_{u \in M} \varphi(\gamma(u)) > a := \sup_{\gamma_0 \in \Gamma_0} \sup_{u \in M_0} \varphi(\gamma_0(u)), \]

then, for every \( \varepsilon \in (0, (c - a)/2) \), \( \delta > 0 \) and \( \gamma \in \Gamma \) such that
\[ \sup_{M} \varphi \circ \gamma \leq c + \varepsilon, \]

there exists \( u \in X \) such that
\[ a) \quad c - 2\varepsilon \leq \varphi(u) \leq c + 2\varepsilon, \]
\[ b) \quad \text{dist}(u, \gamma(M)) \leq 2\delta, \]
\[ c) \quad \|\varphi'(u)\| \leq 8\varepsilon/\delta. \]

Next, we will apply Lemma 2.3 to obtain a Cerami sequence \( \{u_n\} \) of \( I \) with \( J(u_n) \to 0 \). This idea goes back to Jeanjean [16].

Lemma 2.4. Assume that \( 0 < \omega < m_0 \) and \( f \) satisfies (F1)-(F3). Then there exists a sequence \( \{u_n\} \subset H^1(\mathbb{R}^3) \) satisfying
\[ I(u_n) \to c > 0, \quad \|I'(u_n)/(1 + \|u_n\|)\| \to 0 \quad \text{and} \quad J(u_n) \to 0, \]

(2.8)

where
\[ c := \inf_{\gamma \in \Gamma} \sup_{t \in [0, 1]} C(\gamma(t)), \quad \Gamma := \left\{ \gamma \in C([0, 1], H^1(\mathbb{R}^3)) : \gamma(0) = 0, \; I(\gamma(1)) < 0 \right\}. \]

Proof. First, we prove that \( \Gamma \neq \emptyset \) and \( 0 < c < \infty \). By (F1) and (F2), for every \( \varepsilon > 0 \), there exists a constant \( C(\varepsilon) > 0 \) such that
\[ f(t) \leq \varepsilon t^2 + C(\varepsilon)|t|^p, \quad F(t) \leq \varepsilon t^2 + C(\varepsilon)|t|^p, \quad \forall \; t \in \mathbb{R}. \]

(2.9)

Choosing \( \varepsilon = (m_0^2 - \omega^2)/4 \), by (2.2), (2.4), (2.9), Lemma 2.1 and Sobolev embedding inequality, one has
\[ I(u) \geq \frac{1}{2} \|\nabla u\|_2^2 + \frac{m_0^2 - \omega^2}{4} \|u\|_2^2 - C_1 \|u\|_p^p \]
\[ \geq C_2 \|u\|^2 - C_3 \|u\|_p^p, \quad \forall \; u \in H^1(\mathbb{R}^3). \]

(2.10)
It follows from (2.10) that there exist constants \( \rho_0 > 0 \) and \( a_0 > 0 \) such that
\[
I(u) \geq 0, \quad \forall \|u\| \leq \rho_0 \quad \text{and} \quad I(u) \geq a_0, \quad \forall \|u\| = \rho_0. \tag{2.11}
\]
For any fixed \( u \in H^1(\mathbb{R}^3) \) with \( u \neq 0 \), by (2.2) and (2.4), one has
\[
I(tu) = \frac{t^2}{2} \int_{\mathbb{R}^3} \left[ |\nabla u|^2 + (m_0^2 - \omega^2) u^2 \right] dx - \frac{t^2}{2} \int_{\mathbb{R}^3} \omega \phi_{tu} u^2 dx - \int_{\mathbb{R}^3} F(tu) dx
\leq \frac{t^2}{2} \int_{\mathbb{R}^3} \left[ |\nabla u|^2 + (m_0^2 - \omega^2) u^2 \right] dx + \frac{\omega^2 t^2}{2} \|u\|^2
- \int_{\mathbb{R}^3} F(tu) dx, \quad \forall t > 0,
\]
which, together with (F3), yields
\[
\sup_{t \geq 0} I(tu) < \infty, \quad I(tu) \to -\infty \quad \text{as} \quad t \to +\infty. \tag{2.12}
\]
Using (2.12), we choose \( T > 0 \) be large enough such that \( I(Tu) < 0 \). Let \( \gamma_T(t) = tTu \) for \( t \in [0,1] \). Then \( \gamma_T \in C([0,1], H^1(\mathbb{R}^3)) \) such that \( \gamma_T(0) = 0, I(\gamma_T(1)) < 0 \) and \( \max_{t \in [0,1]} I(\gamma_T(t)) < \infty \). This shows that \( \Gamma \neq \emptyset \) and \( c < \infty \). For every \( \gamma \in \Gamma \), since \( \gamma(0) = 0 \) and \( I(\gamma(1)) < 0 \), then it follows from (2.11) that \( \|\gamma(1)\| > \rho_0 \). By the continuity of \( \gamma(t) \) and the intermediate value theorem, there exists \( t_\gamma \in (0,1) \) such that \( \|\gamma(t_\gamma)\| = \rho_0 \). Thus, we have
\[
\sup_{t \in [0,1]} I(\gamma(t)) \geq I(\gamma(t_\gamma)) \geq a_0 > 0, \quad \forall \gamma \in \Gamma,
\]
which yields
\[
\infty > c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \geq a_0 > 0. \tag{2.13}
\]

Inspired by [16], we define the continuous map
\[
h: \tilde{H} := \mathbb{R} \times H^1(\mathbb{R}^3) \to H^1(\mathbb{R}^3), \quad h(s,v)(x) = e^s v(e^s x)
\]
for \( s \in \mathbb{R} \), \( v \in H^1(\mathbb{R}^3) \) and \( x \in \mathbb{R}^3 \), where \( \tilde{H} \) is a Banach space equipped with the product norm \( \| (s,v) \|_{\tilde{H}} := (|s|^2 + \|v\|^2)^{1/2} \). According to [8, 14], the map \( \Phi \) enjoys the following property:
\[
\Phi[tv] = (\Phi[v])_t, \quad \forall t \in \mathbb{R}^+, \ v \in H^1(\mathbb{R}^3). \tag{2.14}
\]
Then, (2.14) implies
\[
\phi_{h(s,v)}(x) = \phi_v(e^s x), \quad \forall s \in \mathbb{R}, \ v \in H^1(\mathbb{R}^3). \tag{2.15}
\]
We consider the following auxiliary functional:
\[
\Psi(s,v) = I(h(s,v))
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^3} \left[ |\nabla h(s,v)|^2 + (m_0^2 - \omega^2) |h(s,v)|^2 \right] dx
- \frac{1}{2} \int_{\mathbb{R}^3} \omega \phi_{h(s,v)} h(s,v)^2 dx - \int_{\mathbb{R}^3} F(h(s,v)) dx
= \frac{e^s}{2} \int_{\mathbb{R}^3} |\nabla v|^2 dx + \frac{e^{-s}}{2} \int_{\mathbb{R}^3} \left[ (m_0^2 - \omega^2) - \omega \phi_v \right] v^2 dx
- e^{-3s} \int_{\mathbb{R}^3} F(e^s v) dx. \tag{2.16}
\]
It is easy to see that \( \Psi \in C^1(\tilde{H}, \mathbb{R}) \). From (2.7), (2.16) and the definition of \( h \), we deduce that
\[
\partial_s \Psi(s, v) = \frac{e^s}{2} \int_{\mathbb{R}^3} |\nabla v|^2 \, dx - \frac{1}{2e^s} \int_{\mathbb{R}^3} \left[ (m_n^2 - \omega^2) - \omega \phi_c \right] v^2 \, dx \\
+ e^{-3s} \int_{\mathbb{R}^3} [3F(e^sv) - f(e^sv)e^sv] \, dx \\
= J(h(s, v)) \text{ for } s \in \mathbb{R} \text{ and } v \in H^1(\mathbb{R}^3). \tag{2.17}
\]
Moreover, since the map \( v \mapsto h(s, v) \) is linear for fixed \( s \in \mathbb{R} \), we have
\[
\partial_s \Psi(s, v) w = I'(h(s, v)) h(s, w) \text{ for } s \in \mathbb{R} \text{ and } v, w \in H^1(\mathbb{R}^3). \tag{2.18}
\]

Now, we define a minimax value \( \bar{c} \) for \( \Psi \) by
\[
\bar{c} = \inf_{t \in [0, 1]} \max_{\gamma \in \bar{\Gamma}} \Psi(\bar{\gamma}(t)),
\]
where
\[
\bar{\Gamma} = \left\{ \bar{\gamma} \in C([0, 1], \tilde{H}) : \bar{\gamma}(0) = (0, 0), \Psi(\bar{\gamma}(1)) < 0 \right\}.
\]
Since \( \Gamma = \left\{ h \circ \bar{\gamma} : \bar{\gamma} \in \bar{\Gamma} \right\} \), the minimax value of \( I \) and \( \Psi \) coincide, i.e., \( c = \bar{c} \). By the definition of \( c \), for every \( n \in \mathbb{N} \), there exists \( \gamma_n \in \Gamma \) such that
\[
\max_{t \in [0, 1]} \Psi(0, \gamma_n(t)) = \max_{t \in [0, 1]} I(\gamma_n(t)) \leq c + \frac{1}{n^2}.
\]
Next, we apply Lemma 2.3 to \( \Psi, M = [0, 1], M_0 = \{0, 1\} \) and \( \tilde{H}, \bar{\Gamma} \) in place of \( X, \Gamma \). Let \( \varepsilon_n = 1/n^2 \), \( \delta_n = 1/n \) and \( \bar{\gamma}_n(t) = (0, \gamma_n(t)) \). Since (2.13) implies that \( \varepsilon_n = 1/n^2 \in (0, c/2) \) for large \( n \in \mathbb{N} \), Lemma 2.3 yields the existence of \( (s_n, v_n) \in \tilde{H} \) such that, as \( n \to \infty \),
\[
\Psi(s_n, v_n) \to c, \tag{2.19}
\]
\[
\|\Psi'(s_n, v_n)\|_{\tilde{H}'} \cdot (1 + \|(s_n, v_n)\|_{\tilde{H}'}) \to 0, \tag{2.20}
\]
\[
dist((s_n, v_n), \{0\} \times \gamma_n([0, 1])) \to 0. \tag{2.21}
\]
Moreover, (2.21) implies that \( s_n \to 0 \). Since
\[
\langle \Psi'(s_n, v_n), (\tau, w) \rangle = \langle I'(h(s_n, v_n)), h(s_n, w) \rangle + J(h(s_n, v_n)) \tau, \forall (\tau, w) \in \tilde{H}, \tag{2.22}
\]
by (2.17) and (2.18), we can take \( \tau = 1 \) and \( w = 0 \) in (2.22) to obtain
\[
J(h(s_n, v_n)) \to 0 \text{ as } n \to \infty. \tag{2.23}
\]
Let \( u_n := h(s_n, v_n) \). Then it follows from (2.19) and (2.23) that
\[
I(u_n) \to c \text{ and } J(u_n) \to 0 \text{ as } n \to \infty. \tag{2.24}
\]
Finally, for given \( v \in H^1(\mathbb{R}^3) \) we consider \( w_n = e^{-s_n}v(e^{-s_n} \cdot) \in H^1(\mathbb{R}^3) \) and deduce from (2.20) and (2.22) with \( \tau = 0 \) that
\[
(1 + \|u_n\|)(I'(u_n), v) = (1 + \|u_n\|)(I'(u_n), h(s_n, w_n)) = o(1)\|w_n\| \\
= o(1) \left( e^{-s_n/2} \|\nabla v\|_2 + e^{s_n/2} \|v\|_2 \right), \tag{2.25}
\]
which, together with \( s_n \to 0 \), yields
\[
(1 + \|u_n\|)(I'(u_n)) \to 0 \text{ as } n \to \infty.
\]
The proof is thus finished. \( \square \)
Lemma 2.5. Under the assumptions of Theorem 1.1, any sequence \( \{u_n\} \subset E \) satisfying (2.8) is bounded in \( H^1(\mathbb{R}^3) \).

Proof. If \( \mu \geq 4 \) in (F3), then it follows from (F3), (2.2), (2.4), (2.5) and (2.8) that

\[
c + o(1) = I(u_n) - \frac{1}{\mu} I'(u_n), \quad \mu \geq 4 \text{ in (F3)}.
\]

Thus, we deduce from (2.27) that \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^3) \) when \( \mu \geq 4 \) in (F3).

Now, we consider \( 2 < \mu < 4 \) in (F3). By (2.4), (2.5), (2.7) and (2.8), one has

\[
c + o(1) = I(u_n) + \frac{\mu - 4}{6 - \mu} \langle I'(u_n), u_n \rangle + \frac{2 - \mu}{6 - \mu} J(u_n)
\]

Next, we prove that \( \|u_n\|_2 \) is bounded. To this end, we distinguish two cases:

1. \( 3 \leq \mu < 4 \) and \( 0 < \omega < m_0 \).
2. \( 2 < \mu < 3 \) and \( 0 < \omega < \sqrt{(\mu - 2)(4 - \mu)m_0} \).

**Case 1)** \( 3 \leq \mu < 4 \) and \( 0 < \omega < m_0 \). In this case, one has

\[
(4 - \mu)s^2 + 2(3 - \mu)\omega \geq 0, \quad \forall \ - \omega \leq s \leq 0.
\]

Note that

\[
- \omega \leq \phi_{u_n} \leq 0.
\]

Then, from (F3), (2.28), (2.29) and (2.30), we derive that

\[
c + o(1) \geq \frac{\mu - 2}{6 - \mu} \left( m_0^2 - \omega^2 \right) \|u_n\|_2^2.
\]

**Case 2)** \( 2 < \mu < 3 \) and \( 0 < \omega < \sqrt{(\mu - 2)(4 - \mu)m_0} \). An elementary computation shows that

\[
(\mu - 2) \left( m_0^2 - \omega^2 \right) + (4 - \mu)s^2 + 2(3 - \mu)\omega s
\]

\[
\geq (\mu - 2) \left( m_0^2 - \omega^2 \right) - \frac{(\mu - 3)^2}{4 - \mu} \omega^2
\]

\[
= \frac{(\mu - 2)(4 - \mu)m_0^2 - \omega^2}{4 - \mu} > 0 \quad \text{for} \quad - \omega \leq s \leq 0.
\]
Then, from (F3), (2.28), (2.30) and (2.32), we derive that
\[
c + o(1) \geq \frac{1}{(6 - \mu)(4 - \mu)} \left( \mu - 2 \right) (4 - \mu) m^2_0 - \mu^2 \|u_n\|_2^2. \tag{2.33}
\]
Combining (2.31) with (2.33), we obtain the boundedness of \(\{\|u_n\|_2\}\). Then, by (2.30), one has
\[
0 \leq -\int_{\mathbb{R}^3} \omega \phi_{u_n} u_n^2 \, dx \leq \omega^2 \|u_n\|_2^2 \leq C_1. \tag{2.34}
\]
Thus, it follows from (2.4), (2.5), (2.7), (2.8) and (2.34) that
\[
c + o(1) = I(u_n) - \frac{1}{3} \langle I'(u_n), u_n \rangle + \frac{1}{3} J(u_n) = I(u_n) - \frac{1}{6} P(u_n)
\]
\[
= \frac{1}{3} \|\nabla u_n\|_2^2 + \frac{1}{3} \int_{\mathbb{R}^3} [\omega \phi_{u_n} u_n^2 + \phi_{u_n} u_n^2] \, dx
\]
\[
\geq \frac{1}{3} \|\nabla u_n\|_2^2 - \frac{1}{3} C_1, \tag{2.35}
\]
which implies that \(\{\|\nabla u_n\|_2\}\) is also bounded. Hence, \(\{u_n\}\) is bounded in \(H^1(\mathbb{R}^3)\).

**Lemma 2.6.** Under the assumptions of Theorem 1.1, there exists \(v_0 \in \mathcal{M}\) such that \(I(v_0) \leq c\). Moreover, there exists a constant \(\kappa_0 > 0\) such that
\[
\|u\|_p \geq \kappa_0, \quad \forall u \in \mathcal{M}. \tag{2.36}
\]

**Proof.** In view of Lemmas 2.4 and 2.5, there exists a sequence \(\{u_n\} \subset H^1(\mathbb{R}^3)\) satisfying (2.8) and \(\|u_n\|^2 \leq M_1\) for some constant \(M_1 > 0\). If
\[
\delta := \limsup_{n \to \infty} \sup_{y \in \mathbb{R}^3} \int_{B_2(y)} |u_n|^2 \, dx = 0, \tag{2.37}
\]
then by Lion’s concentration compactness principle [28, Lemma 1.21], \(u_n \to 0\) in \(L^s(\mathbb{R}^3)\) for \(2 < s < 6\). By (2.9), for \(\varepsilon = c/3M_1\), there exists \(C_\varepsilon > 0\) such that
\[
\int_{\mathbb{R}^3} \left| \frac{1}{2} f(u_n) u_n - F(u_n) \right| \, dx \leq \frac{3}{2} \varepsilon \|u_n\|_2^2 + C_\varepsilon \|u_n\|_p^p \leq \frac{c}{2} + o(1). \tag{2.38}
\]
Thus, it follows from (2.3), (2.4), (2.5), (2.8) and (2.38) that
\[
c + o(1) = I(u_n) - \frac{1}{2} \langle I'(u_n), u_n \rangle
\]
\[
= -\frac{1}{2} \|\nabla \phi_{u_n}\|_2^2 + \int_{\mathbb{R}^3} \left[ \frac{1}{2} f(u_n) u_n - F(u_n) \right] \, dx
\]
\[
\leq \frac{c}{2} + o(1). \tag{2.39}
\]
This contradiction shows \(\delta > 0\).

Going if necessary to a subsequence, we may assume the existence of \(y_n \in \mathbb{R}^3\) such that
\[
\int_{B_1(y_n)} |u_n|^2 \, dx \geq \frac{\delta}{2}.
\]
Let \(v_n(x) = u_n(x + y_n)\). Then \(\|v_n\| = \|u_n\|\) and
\[
\int_{B_1(0)} |v_n|^2 \, dx \geq \frac{\delta}{2}. \tag{2.40}
\]
Since \( \phi_{u_n}(x + y_n) = \phi_{u_0}(x) \), by (2.4), (2.5), (2.7) and (2.8), we have
\[
I(v_n) \to c, \quad \|I'(v_n)\|(1 + \|v_n\|) \to 0, \quad J(v_n) \to 0. \tag{2.41}
\]

Passing to a subsequence, we have \( v_n \to v_0 \) in \( H^1(\mathbb{R}^3) \), \( v_n \to v_0 \) in \( L^s_{\text{loc}}(\mathbb{R}^3) \) for \( 2 \leq s < 6 \) and \( v_n \to v_0 \) a.e. on \( \mathbb{R}^3 \). Clearly, (2.40) shows that \( v_0 \neq 0 \). Jointly with Lemma 2.2, we can conclude that \( I'(v_0) = 0 \), and so \( v_0 \in \mathcal{M} \). Moreover, using (2.26), (2.28), (2.29), (2.32), (2.41) and Fatou’s Lemma, it is easy to see that \( I(v_0) \leq c \).

Since \( \langle I'(u), u \rangle = 0 \) for \( u \in \mathcal{M} \), by (2.3), (2.5), (2.9) and Sobolev embedding inequality, one has
\[
\min\{1, m_0^2 - \omega^2\}\|u\|^2 \leq \int_{\mathbb{R}^3} \left[ |\nabla u|^2 + (m_0^2 - \omega^2) u^2 + 2|\nabla \phi_{u}|^2 + \phi_{u}^2 u^2 \right] dx
= \int_{\mathbb{R}^3} f(u)udx
\leq \frac{\min\{1, m_0^2 - \omega^2\}}{2\|u\|^2} + C_2\|u\|_p^p
\leq \min\{1, m_0^2 - \omega^2\}\|u\|^2 + C_3\|u\|_p^p, \quad \forall \ u \in \mathcal{M}, \tag{2.42}
\]
which implies that there exists a constant \( \rho_0 > 0 \) such that
\[
\|u\| \geq \rho_0 > 0, \quad \forall \ u \in \mathcal{M}. \tag{2.43}
\]

Thus, it follows from (2.42) and (2.43) that
\[
\|u\|_p \geq \left( \frac{\min\{1, m_0^2 - \omega^2\}}{2C_2} \|u\|^2 \right)^{1/p} \geq \left( \frac{\min\{1, m_0^2 - \omega^2\}}{2C_2} \rho_0^2 \right)^{1/p} := \kappa_0 > 0, \quad \forall \ u \in \mathcal{M}.
\]

3. **Proof of Theorem 1.1.** In this section, we give the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Since \( \langle I'(u), u \rangle = J(u) = 0 \) for \( u \in \mathcal{M} \), if
\[
3 \leq \mu < 4 \quad \text{and} \quad 0 < \omega < m_0, \quad \text{or} \quad 2 < \mu < 3 \quad \text{and} \quad 0 < \omega < \sqrt{\mu - 2} (4 - \mu) m_0, \tag{3.1}
\]
then it follows from (F3), (2.4), (2.5), (2.7), (2.29) and (2.32) that
\[
I(u) = I(u) + \frac{\mu - 4}{6 - \mu} \langle I'(u), u \rangle + \frac{2 - \mu}{6 - \mu} J(u)
= \frac{1}{6 - \mu} \int_{\mathbb{R}^3} \left[ (\mu - 2) (m_0^2 - \omega^2) u^2 + (4 - \mu) \phi_{u}^2 + 2(3 - \mu) \omega \phi_{u} \right] u^2 dx
+ \frac{2}{6 - \mu} \int_{\mathbb{R}^3} [f(u)u - \mu F(u)] dx > 0, \quad \forall \ u \in \mathcal{M}; \tag{3.2}
\]
if \( \mu \geq 4 \), then it follows from (F3), (2.2), (2.4) and (2.5) that
\[
I(u) = I(u) - \frac{1}{\mu} \langle I'(u), u \rangle
\geq \left( \frac{1}{2} - \frac{1}{\mu} \right) \int_{\mathbb{R}^3} \left[ |\nabla u|^2 + (m_0^2 - \omega^2) u^2 \right] dx > 0, \quad \forall \ u \in \mathcal{M}. \tag{3.3}
\]
\]

\[ \square \]
Combining (3.2) with (3.3), we have

\[ m = \inf_{M} I \geq 0. \]  

(3.4)

Now, we let \( \{u_n\} \subset M \) and \( I(u_n) \to m := \inf_{M} I. \) Since \( I'(u_n) = 0 \), we have \( \langle I'(u_n), u_n \rangle = J(u_n) = 0. \) In the same way as Lemma 2.5, we can prove that \( \{u_n\} \) is bounded in \( H^1(\mathbb{R}^3) \). Next, we claim that \( \{u_n\} \) is a non-vanishing sequence. Otherwise, (2.37) holds. Then by Lion’s concentration compactness principle [28, Lemma 1.21], we have \( u_n \to 0 \) in \( L^s(\mathbb{R}^3) \) for \( 2 < s < 6 \), which contradicts (2.36). Hence, \( \{u_n\} \) does not vanish. Going if necessary to a subsequence, we may assume the existence of \( \bar{y}_n \in \mathbb{R}^3 \) such that

\[ \int_{B_1(\bar{y}_n)} |u_n|^2 dx > \frac{\delta}{2}. \]

Let \( \bar{u}_n(x) = u_n(x + \bar{y}_n) \). Then \( \|\bar{u}_n\| = \|u_n\| \) and

\[ \int_{B_1(0)} |\bar{u}_n|^2 dx > \frac{\delta}{2}. \]

(3.5)

Since \( \phi_{u_n}(x + \bar{y}_n) = \phi_{\bar{u}_n}(x) \), by (2.4), (2.5) and (2.7), we have

\[ I(\bar{u}_n) \to m, \quad I'(\bar{u}_n) = 0, \quad J(\bar{u}_n) = 0. \]

(3.6)

Passing to a subsequence, we have \( \bar{u}_n \rightharpoonup \bar{u} \) in \( H^1(\mathbb{R}^3) \), \( \bar{u}_n \to \bar{u} \) in \( L^s_{\text{loc}}(\mathbb{R}^3) \) for \( 1 \leq s < 6 \) and \( \bar{u}_n \to \bar{u} \) a.e. on \( \mathbb{R}^3 \). Thus, (3.5) implies that \( \bar{u} \neq 0 \). Moreover, by Lemma 2.2, one has, up to subsequences, \( \phi_{\bar{u}_n} \rightharpoonup \phi_{\bar{u}} \) in \( D^{1,2}(\mathbb{R}^3) \), \( \phi_{\bar{u}_n} \to \phi_{\bar{u}} \) in \( L^s_{\text{loc}}(\mathbb{R}^3) \) for \( 1 \leq s < 6 \) and \( \phi_{u_n} \to \phi_{\bar{u}} \) a.e. on \( \mathbb{R}^3 \). From this, one can conclude that \( I'(\bar{u}) = 0 \), and so

\[ \langle I'(\bar{u}), \bar{u} \rangle = J(\bar{u}) = 0, \quad I(\bar{u}) \geq m = \inf_{M} I. \]

(3.7)

On the other hand, if (3.1) holds, then by (2.4), (2.6), (2.29), (2.32), (3.1), (3.2), (3.6), (3.7) and Fatou’s Lemma, we have

\[
m = \lim_{n \to \infty} \left[ I(\bar{u}_n) + \frac{\mu - 4}{6 - \mu} \langle I'(\bar{u}_n), \bar{u}_n \rangle + \frac{2 - \mu}{6 - \mu} J(\bar{u}_n) \right]
= \lim_{n \to \infty} \left\{ \frac{1}{6 - \mu} \int_{\mathbb{R}^3} \left[ (\mu - 2) (m_0^2 - \omega^2) \bar{u}_n^2 \right. \right.
+ (4 - \mu) \phi_{u_n}^2 + 2(3 - \mu) \omega \phi_{\bar{u}_n} \left. \bar{u}_n^2 \right\} dx
+ \frac{2}{6 - \mu} \int_{\mathbb{R}^3} |f(\bar{u}_n)\bar{u}_n - \mu F(\bar{u}_n)| dx
\geq \frac{1}{6 - \mu} \int_{\mathbb{R}^3} \left[ (\mu - 2) (m_0^2 - \omega^2) \bar{u}^2 \right. + (4 - \mu) \phi_{\bar{u}}^2 + 2(3 - \mu) \omega \phi_{\bar{u}} \left. \bar{u}^2 \right\} dx
+ \frac{2}{6 - \mu} \int_{\mathbb{R}^3} |f(\bar{u})\bar{u} - \mu F(\bar{u})| dx
= I(\bar{u}) + \frac{\mu - 4}{6 - \mu} \langle I'(\bar{u}), \bar{u} \rangle + \frac{2 - \mu}{6 - \mu} J(\bar{u}) \geq m; \quad (3.8)\]

if \( \mu \geq 4 \), then it follows from (F3), (2.2), (2.4), (2.5), (3.6), (3.7) and Fatou’s Lemma that

\[ m = \lim_{n \to \infty} \left[ I(\bar{u}_n) - \frac{1}{\mu} \langle I'(\bar{u}_n), \bar{u}_n \rangle \right] \]
Combining (3.8) with (3.9), we derive that \( I(\bar{u}) = m = \inf_{\mathcal{M}} I \). Moreover, by (3.2) and (3.3), we have \( I(\bar{u}) = m = \inf_{\mathcal{M}} I > 0 \). Let \((u, \phi) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)\) be a nontrivial solution for (1.1), by Lemma 2.1, we have \( \phi = \phi_u \). Then, it follows from (1.2), (2.3) and (2.4) that

\[
S(u, \phi) = S(u, \phi_u) = I(u) \geq I(\bar{u}) = S(\bar{u}, \phi_{\bar{u}}) = m.
\]

Hence, under assumptions of Theorem 1.1, \((\bar{u}, \phi_{\bar{u}}) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)\) is a ground state solution for (1.1).

\[ \square \]

4. Proof of Theorem 1.3. In this section, we consider the zero mass case, i.e. \( \omega = m_0 \), and give the proof of Theorem 1.3. In what follows, we assume that \( f \) satisfies (F4). By (F4), there exists \( C_0 > 0 \) such that

\[
|f(t)| \leq C_0 |t|^{p_1 - 1} \quad \text{and} \quad |f(t)| \leq C_0 |t|^p, \quad \forall t \in \mathbb{R}.
\]

Inspired by [3], to find nontrivial solutions for (1.4), we will look for a weak limit of solutions for the following approximating problem

\[
\begin{cases}
-\Delta u + [\varepsilon - (2\omega + \phi)\phi] u = f(u), & \text{in } \mathbb{R}^3, \\
\Delta \phi = (\omega + \phi) u^2, & \text{in } \mathbb{R}^3,
\end{cases}
\]

(4.2) where \( \varepsilon > 0 \). For every \( \varepsilon > 0 \), we define \( I_\varepsilon : H^1(\mathbb{R}^3) \rightarrow \mathbb{R} \) as

\[
I_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^3} [\nabla u]^2 + \varepsilon u^2 \, dx - \int_{\mathbb{R}^3} \left[ \frac{1}{2} \phi \nabla u^2 + F(u) \right] \, dx.
\]

(4.3)

Under (F4), we have \( I_\varepsilon \in C^1(H^1(\mathbb{R}^3), \mathbb{R}) \), and

\[
\langle I'_\varepsilon(u), v \rangle = \int_{\mathbb{R}^3} [\nabla u \cdot \nabla v + \varepsilon uv] \, dx - \int_{\mathbb{R}^3} [(2\omega + \phi)\phi u \cdot u + f(u)] v \, dx,
\]

\forall u, v \in H^1(\mathbb{R}^3). \]

(4.4)

Then, any critical point \( u \) of \( I_\varepsilon \) in \( H^1(\mathbb{R}^3) \) satisfies the following Pohozaev identity

\[
P_\varepsilon(u) = \int_{\mathbb{R}^3} (|\nabla u|^2 + 3\varepsilon u^2) \, dx - \int_{\mathbb{R}^3} (5\omega + 2\phi u) \phi_u u^2 \, dx - 6 \int_{\mathbb{R}^3} F(u) \, dx = 0.
\]

(4.5)

For every \( \varepsilon > 0 \), let

\[
\mathcal{M}_\varepsilon := \{ u \in H^1(\mathbb{R}^3) \setminus \{0\} : I'_\varepsilon(u) = 0 \},
\]

\[
c_\varepsilon := \inf_{\gamma \in \Gamma_\varepsilon} \max_{t \in [0,1]} I_\varepsilon(\gamma(t)), \quad \Gamma_\varepsilon := \{ \gamma \in C([0,1], H^1(\mathbb{R}^3)) : \gamma(0) = 0, I_\varepsilon(\gamma(1)) < 0 \}.
\]
Lemma 4.1. For every $\varepsilon > 0$, (4.2) has a ground state solution $(u_\varepsilon, \phi_\varepsilon) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ such that $0 < I_\varepsilon(u_\varepsilon) = \inf_{M_{\varepsilon}} I_\varepsilon \leq c_\varepsilon$. Moreover, there exists a constant $K_0 > 0$ independent of $\varepsilon$ such that $c_\varepsilon \leq K_0$ for all $\varepsilon \in (0, 1]$.

Proof. In view of Lemma 2.6 and Theorem 1.1, for every $\varepsilon > 0$, (4.2) has a ground state solution $(u_\varepsilon, \phi_\varepsilon) \in H^1(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ such that $0 < I_\varepsilon(u_\varepsilon) = \inf_{M_{\varepsilon}} I_\varepsilon \leq c_\varepsilon$. Let $\gamma \in \Gamma_1$, since $I_\varepsilon(u) \leq I_1(u)$ for $u \in H^1(\mathbb{R}^3)$ and $\varepsilon \in (0, 1]$, we have $\gamma \in \Gamma_\varepsilon$ for $\varepsilon \in (0, 1]$, and so

$$c_\varepsilon \leq \max_{t \in [0, 1]} I_\varepsilon(\gamma(t)) = I_\varepsilon(\gamma(t_\varepsilon)) \leq I_1(\gamma(t_\varepsilon)) \leq \max_{t \in [0, 1]} I_1(\gamma(t)) := K_0, \quad \forall \varepsilon \in (0, 1],$$

where $t_\varepsilon \in (0, 1)$.

Lemma 4.2. There exists a constant $\kappa_1 > 0$ independent of $\varepsilon$ such that

$$\int_{\mathbb{R}^3} f(u_\varepsilon)u_\varepsilon \, dx \geq \kappa_1, \quad \forall u_\varepsilon \in M_\varepsilon.$$  \hfill (4.6)

Proof. Since $\langle I_\varepsilon'(u_\varepsilon), u_\varepsilon \rangle = 0$ for $u_\varepsilon \in M_\varepsilon$, by (2.3), (4.1), (4.4) and Sobolev embedding inequality, one has

$$\|\nabla u_\varepsilon\|_2^2 \leq \int_{\mathbb{R}^3} \left[ |\nabla u_\varepsilon|^2 + \varepsilon u_\varepsilon^2 + 2|\nabla \phi_\varepsilon|^2 + \phi_\varepsilon^2 u_\varepsilon^2 \right] \, dx = \int_{\mathbb{R}^3} f(u_\varepsilon)u_\varepsilon \, dx \quad \text{(4.7)}$$

$$\leq C_0 \|u_\varepsilon\|^6_6 \leq C_0 S^{-3}\|\nabla u_\varepsilon\|_2^6, \quad \forall u_\varepsilon \in M_\varepsilon,$$

which implies

$$\|\nabla u_\varepsilon\|_2 \geq S^{3/4} C_0^{-1/4}, \quad \forall u_\varepsilon \in M_\varepsilon.$$  \hfill (4.8)

Thus, it follows from (4.7) and (4.8) that

$$\int_{\mathbb{R}^3} f(u_\varepsilon)u_\varepsilon \, dx \geq \|\nabla u_\varepsilon\|_2^2 \geq S^{3/2} C_0^{-1/2} := \kappa_1, \quad \forall u_\varepsilon \in M_\varepsilon,$$

$\Box$

Similar to the proof of [1, Lemma 2], we can establish the following concentration compactness principle, which goes back to P.L. Lions [21].

Lemma 4.3. Assume that (F4) holds and $\{u_n\}$ is bounded in $D^{1,2}(\mathbb{R}^3)$ and there exists a constant $r > 0$ such that

$$\limsup_{n \to \infty} \sup_{y \in \mathbb{R}^3} \int_{B_r(y)} |u_n|^2 \, dx = 0.$$

Then

$$\lim_{n \to \infty} \int_{\mathbb{R}^3} f(u_n)u_n \, dx = 0.$$

Proof of Theorem 1.3. We choose a sequence $\{\varepsilon_n\} \subset (0, 1]$ such that $\varepsilon_n \searrow 0$. In view of Lemma 4.1, there exists a sequence $\{u_{\varepsilon_n}\} \subset M_{\varepsilon_n}$ such that $0 < I_{\varepsilon_n}(u_{\varepsilon_n}) = \inf_{M_{\varepsilon_n}} I_{\varepsilon_n} \leq c_{\varepsilon_n} \leq K_0$. For the sake of simplicity, we use $u_n$ and $\phi_n$ in place of $u_{\varepsilon_n}$ and $\phi_{\varepsilon_n}$. Now, we prove that $\{(u_n, \phi_n)\}$ is bounded in $D^{1,2}(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$. Since $\langle I_{\varepsilon_n}'(u_n), u_n \rangle = \mathcal{P}_{\varepsilon_n}(u_n) = 0$ for $u_n \in M_{\varepsilon_n}$, it follows from (F4), (4.3), (4.4) and (4.5) that

$$K_0 \geq c_{\varepsilon_n} \geq I_{\varepsilon_n}(u_n) - \frac{2}{3} \langle I_{\varepsilon_n}'(u_n), u_n \rangle - \frac{1}{6} \mathcal{P}_{\varepsilon_n}(u_n)$$

$$= \frac{1}{3} \varepsilon_n \|u_n\|^2 + \frac{1}{3} \int_{\mathbb{R}^3} \phi_n^2 u_n^2 \, dx + \frac{2}{3} \int_{\mathbb{R}^3} [f(u_n)u_n - 3F(u_n)] \, dx$$

$$\geq 2 \left( p_0 - 3 \right) \int_{\mathbb{R}^3} F(u_n) \, dx.$$  \hfill (4.9)
From (4.3) and (4.9), we have
\[
\frac{1}{2}\|\nabla u_n\|^2 - \frac{1}{2} \int_{\mathbb{R}^3} \omega \phi_n u_n^2 \, dx \leq \frac{1}{2}\|\nabla u_n\|^2 + \frac{1}{2} \epsilon_n\|u_n\|^2 - \frac{1}{2} \int_{\mathbb{R}^3} \omega \phi_n u_n^2 \, dx
\]
which, together with (2.3), implies
\[
\|\nabla u_n\|^2 + \|\omega \phi_n\|^2 \leq \|\nabla u_n\|^2 - \int_{\mathbb{R}^3} (\omega + \phi_n) \phi_n u_n^2 \, dx
\]
This shows that \(\{(u_n, \phi_n)\}\) is bounded in \(\mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)\). If
\[
\delta := \limsup_{n \to \infty} \sup_{y \in \mathbb{R}^3} \int_{B_2(y)} |u_n|^2 \, dx = 0,
\]
then by Lemma 4.3, we have
\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} f(u_n) u_n \, dx = 0.
\]
This contradicts Lemma 4.2 due to \(u_n \in \mathcal{M}_{\epsilon_n}\). Thus, we have \(\delta > 0\). Going if necessary to a subsequence, we may assume the existence of \(\tilde{y}_n \in \mathbb{R}^3\) such that
\[
\int_{B_1(\tilde{y}_n)} |\nabla u_n|^2 \, dx > \frac{\delta}{2}.
\]
Let \(\tilde{u}_n(x) = u_n(x + \tilde{y}_n)\). Then \(\|\nabla \tilde{u}_n\|_2 = \|\nabla u_n\|_2\)
and
\[
\int_{B_1(0)} |	ilde{u}_n|^2 \, dx > \frac{\delta}{2}
\]
Since \(\phi_{u_n}(x + \tilde{y}_n) = \phi_{\tilde{u}_n}(x)\), we have \(\|\nabla \phi_{\tilde{u}_n}\|_2 = \|\nabla \phi_{u_n}\|_2\)
and
\[
I'_{\epsilon_n}(\tilde{u}_n) = 0; \quad I_{\epsilon_n}(\tilde{u}_n) = c_{\epsilon_n}.
\]
Passing to a subsequence, we have
\[
(\tilde{u}_n, \phi_{\tilde{u}_n}) \rightarrow (u_0, \phi_{u_0}) \text{ in } \mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3).
\]
Clearly, (4.12) shows that \(u_0 \neq 0\). Similar to the proof of [3, Theorem 1.2], we deduce that \((u_0, \phi_{u_0})\) satisfies
\[
\begin{cases}
\int_{\mathbb{R}^3} (\nabla u_0 \cdot \nabla \psi - 2\omega \phi_0 u_0 \psi - \phi_0^2 u_0^2 \psi) \, dx = \int_{\mathbb{R}^3} f(u_0) \psi \, dx, & \forall \psi \in C_0^\infty(\mathbb{R}^3), \\
\int_{\mathbb{R}^3} (\nabla \phi_0 \cdot \nabla \psi + \phi_0^2 u_0^2 \psi) \, dx = -\int_{\mathbb{R}^3} \omega u_0^2 \psi \, dx, & \forall \psi \in C_0^\infty(\mathbb{R}^3).
\end{cases}
\]
Hence, \((u_0, \phi_{u_0}) \in \mathcal{D}^{1,2}(\mathbb{R}^3) \times \mathcal{D}^{1,2}(\mathbb{R}^3)\) is a nontrivial solution for (1.4). \(\square\)

Acknowledgments. The authors would like to thank the referee for giving valuable comments and suggestions, which make us possible to improve the paper.

REFERENCES

[1] A. Azzollini, V. Benci, T. D’Aprile and D. Fortunato, Existence of static solutions of the semilinear Maxwell equations, Ricerche di Matematica, 55 (2006), 283–297.

[2] A. Azzollini and A. Pomponio, Ground state solutions for the nonlinear Klein-Gordon–Maxwell equations, Topol. Methods Nonlinear Anal., 35 (2010), 33–42.
[3] A. Azzollini, L. Pisani and A. Pomponio, Improved estimates and a limit case for the electrostatic Klein-Gordon-Maxwell system, *Proc. Roy. Soc. Edinburgh Sect. A*, **141** (2011), 449–463.

[4] V. Benci and D. Fortunato, The nonlinear Klein-Gordon equation coupled with the Maxwell equations, *Nonlinear Anal.*, **47** (2001), 6065–6072.

[5] V. Benci and D. Fortunato, Solitary waves of the nonlinear Klein-Gordon equation coupled with the Maxwell equations, *Rev. Math. Phys.*, **14** (2002), 409–420.

[6] H. Berestycki and P. L. Lions, Nonlinear scalar field equations, I - Existence of a ground state, *Arch. Rational Mech. Anal.*, **82** (1983), 313–345.

[7] P. Carrião, P. Cunha and O. Miyagaki, Positive ground state solutions for the critical Klein-Gordon-Maxwell system with potentials, *Nonlinear Anal.*, **75** (2012), 4068–4078.

[8] D. Cassani, Existence and non-existence of solitary waves for the critical Klein-Gordon equation coupled with Maxwell’s equations, *Nonlinear Anal.*, **58** (2004), 733–747.

[9] S. T. Chen and X. H. Tang, Ground state sign-changing solutions for a class of Schrödinger-Poisson type problems in $\mathbb{R}^3$, *Z. Angew. Math. Phys.*, **67** (2016), Art. 102, 18 pp.

[10] S. T. Chen and X. H. Tang, Nehari type ground state solutions for asymptotically periodic Schrödinger-Poisson systems, *Taiwan. J. Math.*, **21** (2017), 363–383.

[11] S. T. Chen and X. H. Tang, Ground state sign-changing solutions for asymptotically cubic or super-cubic Schrödinger-Poisson systems without compact condition, *Comput. Math. Appl.*, **74** (2017), 446–458.

[12] P. L. Cunha, Subcritical and supercritical Klein-Gordon-Maxwell equations without Ambrosetti-Rabinowitz condition, *Differ. Integral Equ.*, **27** (2014), 387–399.

[13] T. D’Aprile and D. Mugnai, Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations, *Proc. R. Soc. Edinb. Sect. A*, **134** (2004), 893–906.

[14] T. D’Aprile and D. Mugnai, Non-existence results for the coupled Klein-Gordon-Maxwell equations, *Adv. Nonlinear Stud.*, **4** (2004), 307–322.

[15] L. Ding and L. Li, Infinitely many standing wave solutions for the nonlinear Klein-Gordon-Maxwell system with sign-changing potential, *Comput. Math. Appl.*, **68** (2014), 589–595.

[16] L. Jeanjean, Existence of solutions with prescribed norm for semilinear elliptic equations, *Nonlinear Anal.*, **28** (1997), 1633–1659.

[17] L. Jeanjean, On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer-type problem set on $\mathbb{R}^N$, *Proc. R. Soc. Edinb., Sect. A*, **129** (1999), 787–809.

[18] W. Jeong and J. Seok, On perturbation of a functional with the mountain pass geometry, *Calc. Var. Partial Differential Equations*, **49** (2014), 649–668.

[19] G. B. Li and C. Wang, The existence of a nontrivial solution to a nonlinear elliptic problem of linking type without the Ambrosetti-Rabinowitz condition, *Ann. Acad. Sci. Fenn. Math.*, **36** (2011), 461–480.

[20] L. Li and C. L. Tang, Infinitely many solutions for a nonlinear Klein-Gordon-Maxwell system, *Nonlinear Anal.*, **110** (2014), 157–169.

[21] P. L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. I & II, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **1** (1984), 109–145.

[22] D. D. Qin, Y. B. He and X. H. Tang, Ground state solutions for Kirchhoff type equations with asymptotically 4-linear nonlinearity, *Comput. Math. Appl.*, **71** (2016), 1524–1536.

[23] M. Struwe, The existence of surfaces of constant mean curvature with free boundaries, *Acta Math.*, **160** (1988), 19–64.

[24] X. H. Tang and B. T. Cheng, Ground state sign-changing solutions for Kirchhoff type problems in bounded domains, *J. Differential Equations*, **261** (2016), 2384–2402.

[25] X. H. Tang and S. T. Chen, Ground state solutions of Nehari-Pohozaev type for Schrödinger-Poisson problems with general potentials, *Disc. Contin. Dyn. Syst.*, **37** (2017), 4973–5002.

[26] X. H. Tang and S. T. Chen, Ground state solutions of Nehari-Pohozaev type for Kirchhoff-type problems with general potentials, *Calc. Var. Partial Differential Equations*, **56** (2017), Art. 110, 25 pp.

[27] F. Wang, Ground-state solutions for the electrostatic nonlinear Klein-Gordon-Maxwell system, *Nonlinear Anal.*, **74** (2011), 4796–4803.

[28] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.

[29] L. Zhang, X. H. Tang and Y. Chen, Infinitely many solutions for a class of perturbed elliptic equations with nonlocal operators, *Commun. Pur. Appl. Anal.*, **16** (2017), 823–842.
[30] J. Zhang, W. Zhang and X. L. Xie, Existence and concentration of semiclassical solutions for Hamiltonian elliptic system, Commun. Pure Appl. Anal., 15 (2016), 599–622.

[31] J. Zhang, W. Zhang and X. H. Tang, Ground state solutions for Hamiltonian elliptic system with inverse square potential, Discrete Contin. Dyn. Syst., 37 (2017), 4565–4583.

Received September 2017; revised December 2017.

E-mail address: mathsitongchen@163.com (Sitong Chen)
E-mail address: tangxh@mail.csu.edu.cn (Xianhua Tang)