Trapped surfaces and horizons in static massless scalar field spacetimes

Swastik Bhattacharya\(^1\) and Pankaj S. Joshi\(^1\)

\(^1\)Tata Institute for Fundamental Research, Homi Bhabha Road, Mumbai 400005, India

We consider here the existence and structure of trapped surfaces, horizons and singularities in spherically symmetric static massless scalar field spacetimes. Earlier studies have shown that there exists no event horizon in such spacetimes if the scalar field is asymptotically flat. We extend this result here to show that this is true in general for spherically symmetric static massless scalar field spacetimes, whether the scalar field is asymptotically flat or not. Other general properties and certain important features of these models are also discussed.

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I. INTRODUCTION

Spherically symmetric solutions of Einstein equations for static massless scalar field configurations have been investigated in considerable detail in the past. There are two different ways in which this has been done. The first approach has been to look for solutions of the Einstein equations in this case and thus gain more understanding on the structure of the solutions. Bergmann and Leipnik \(^1\) were among the first to construct such spherically symmetric static solutions for massless scalar fields. They had, however, only a limited success due to an inappropriate choice of coordinates. Around the same time Buchdahl \(^2\) developed techniques to generate solutions for this system, and also Yilmaz \(^3\) and Szekeres \(^4\) found some classes of solutions for the static massless scalar field configurations in general relativity. Subsequently, Wyman \(^5\) systematically discussed these solutions and showed a general method to obtain solutions in the case when the scalar field was allowed to have no time dependence. This gave a unified method to obtain most of the solutions obtained earlier. Also, Xanthopoulos and Zannias \(^6\) gave a class of solutions for time independent scalar fields in arbitrary dimensions where the spacetime metric was static.

Further, the scalar fields conformally coupled to gravity have been a subject of immense interest to many researchers \(^7\). Static massless scalar fields have also been investigated in settings more general as compared to spherical symmetry (see e.g. \(^8\)). Thus, there have been many investigations on special cases of static solutions of the Einstein equations for the massless scalar field system, as indicated above.

It is clear, however, from the analysis of Wyman \(^5\), that there is still a large class of solutions that as yet remains unexplored. In fact, Wyman identifies that class to be the solutions of a particular non-linear ordinary differential equation, which is difficult to solve analytically. It turns out nevertheless that even in these cases when it is not possible to solve the Einstein equations explicitly or completely, there are still some general properties of such solutions that can be deduced which are of physical interest. This is the second way of approaching this problem. In that direction, some interesting general properties of static massless scalar field spacetimes have been found by Chase \(^9\). For example, it was found that every massless scalar field, which is gravitationally coupled and asymptotically flat, becomes singular at a simply-connected event horizon. In fact, the result obtained by Chase is more general in the sense that it does not assume spherical symmetry of the spacetime. However, for this result to hold true, the scalar field has to be asymptotically flat necessarily, which means that the scalar field goes to zero in the limit of going to spatial infinity. But there can be solutions, where this condition does not apply. In fact, we shall show later that such solutions exist for static spherically symmetric massless scalar field spacetimes.

One of the main interest for the study of these solutions comes from the fact that for simple models for gravitational collapse in general relativity, these settle to a Schwarzschild solution where the final singularity of collapse is hidden within an event horizon of gravity. But the analogous static scalar field solutions do not have such event horizons, as was pointed out by many of the above works. Of course, such models cannot be considered to be counter-example to the cosmic censorship conjecture, which states that naked singularities do not develop as final state of gravitational collapse. That is because these are static solutions and do not develop from collapse from a regular initial data. However, the cosmic censorship hypothesis continues to be unproved and without even a definite mathematical formulation, despite serious attempts in that direction for past many decades. Therefore a study of models such as undertaken here would be of interest as this may provide us with an understanding into the nature and structure of the singularities, horizons, and trapped surfaces that can take place or exist in general relativistic spacetime models.

From such a perspective, in this note we extend the result found by Chase in the case of static spherically symmetric massless scalar field spacetimes, by showing that in such spacetimes, event horizon cannot exist, whether the scalar field is asymptotically flat or not. However, our result holds true and extends the findings by Chase only
for spherically symmetric spacetimes, whereas Chase had not assumed any such symmetry of the spacetime, apart from its staticity. Apart from this result, we also discuss briefly the existence of singularities for these models and their visibility.

In section II, we write down the Einstein equations for the massless scalar fields in comoving coordinates. As explained there, the spacetimes in this case are divided into two classes, namely, \( \phi = \phi(t) \) and \( \phi = \phi(r) \). This coordinate system has been used to analyse the static massless scalar field models by Wyman [5] and also others. In section III, we shall consider the general properties of both these classes of models. Specifically, we shall show that if the spacetime is non-empty, then event horizon does not exist. In the final Section we give and summarize the conclusions.

II. THE EINSTEIN EQUATIONS

In our analysis here, we consider a four-dimensional spacetime manifold which has spherical symmetry. The massless scalar field \( \phi(x^a) \) on such a spacetime \((M, g_{ab})\) is described by the Lagrangian,

\[
\mathcal{L} = -\frac{1}{2} \dot{\phi}^a g_{ab} \phi^b.
\]

The corresponding Euler-Lagrange equation is then given by,

\[
\dot{\phi}^a g_{ab} = 0,
\]

and the energy-momentum tensor for the scalar field, as calculated from the above Lagrangian, is given as

\[
T_{ab} = \phi_c \phi^c g_{ab} - \frac{1}{2} g_{ab} \left( \phi_c \phi^c g^{cd} \phi_d \right).
\]

The massless scalar field is a Type I matter field [10], i.e., it admits one timelike and three spacelike eigenvectors. At each point \( q \in M \), we can express the tensor \( T_{ab} \) in terms of an orthonormal basis \((E_0, E_1, E_2, E_3)\), where \( E_0 \) is a timelike eigenvector with the eigenvalue \( \rho \), and \( E_\alpha \) \( (\alpha = 1, 2, 3) \) are three spacelike eigenvectors with eigenvalues \( p_\alpha \). The eigenvalue \( \rho \) represents the energy density of the scalar field as measured by an observer whose world line at \( q \) has an unit tangent vector \( E_0 \), and the eigenvalues \( p_\alpha \) represent the principal pressures in three spacelike directions \( E_\alpha \).

We now choose the spherically symmetric coordinates \((t, r, \theta, \phi)\) along the eigenvectors \((E_0, E_\alpha)\), such that the reference frame is comoving. This coordinate system has also been used by Wyman [3] to discuss the static massless scalar field spacetimes. As discussed in [11], the general spherically symmetric metric in comoving coordinates can be written as,

\[
d\Omega^2 = e^{2\nu(t,r)} dt^2 - e^{2\psi(t,r)} dr^2 - R^2(t,r)d\Omega^2,
\]

where \( d\Omega^2 \) is the metric on a unit two-sphere and we have used the two gauge freedoms of two variables, namely, \( t' = f(t, r) \) and \( r' = g(t, r) \), to make the \( g_{tr} \) term in the metric and the radial velocity of the matter field to vanish. That means that the energy-momentum tensor is necessarily diagonal in such a coordinate system. We note that we still have two scaling freedoms of one variable available, namely \( t \to f(t) \) and \( r \to g(r) \). We note here that the metric function \( R \) is sometimes called the physical radius, especially in gravitational collapse situations.

As we are considering here spherically symmetric spacetimes, we have \( \phi = \phi(t, r) \) necessarily. Furthermore, from equation (4) we can easily see that in the comoving reference frame with the metric given by (4), \( T_{t0} = \phi \dot{\phi} \). It follows therefore that we must have here necessarily \( \phi(t, r) = \phi(t) \) or \( \phi(t, r) = \phi(r) \), where the energy-momentum tensor is necessarily diagonal.

For the metric (4), and using the following definitions,

\[
G(t, r) = e^{-2\psi(R')}^2, \quad H(t, r) = e^{-2\nu}(\dot{R})^2,
\]

\[
F = R(1 - G + H),
\]

we can write the independent Einstein equations for the spherical massless scalar field (in the units \( 8\pi G = c = 1 \)) as below (see [12]),

\[
\rho = \frac{F'}{R^2 R'},
\]

\[
P_r = -\frac{\dot{F}}{R^2 R'},
\]

\[
u'(\rho + P_r) = 2(P_\theta - P_r) \frac{R'}{R} - P_r,
\]

\[-2\dot{R}' + R' \frac{G'}{G} + \frac{\dot{R} H'}{H} = 0,
\]

In the above, the function \( F(t, r) \) has the interpretation of the mass function for the matter field, in that it represents the total mass contained within the sphere of coordinate radius \( r \) at any given time \( t \). As noted above, in the static case the metric components \( g_{\mu\nu} \) are functions of the radial coordinate \( r \) only necessarily, but the scalar field \( \phi \) itself can still be either \( r \) or \( t \) dependent. In either of these cases, namely \( \phi = \phi(r) \) or \( \phi = \phi(t) \), the equation of state relating the energy density and pressure for the scalar field are different as we shall find here.

We note that the condition of staticity for the spacetime metric implies that the metric components are not functions of time. There is no such restriction, however, on the scalar field itself, which can still be time-dependent.
III. GENERAL PROPERTIES OF THE SPACETIMES

We shall now analyse the Einstein equations given above for the massless scalar field, and point out several properties of these solutions in general. Many of these relate to the nature of the singularity and the trapped surfaces in these models. The Einstein equations are not fully solved here but a general analysis of their properties is carried out which implies these conclusions, which thus hold true for all solutions of this system.

A. The \( \phi = \phi(r) \) class of models

In this case for static spacetimes, when \( \phi = \phi(r) \) and \( g_{\mu\nu} = g_{\mu\nu}(r) \), the Einstein equations given in the previous section reduce to the following set of equations,

\[
\frac{1}{2} e^{-2\psi} \partial'^2 - \frac{F'}{R^2 R'} = 0, \quad (11)
\]

\[
\frac{1}{2} e^{-2\psi} \partial'^2 = e^{-2\psi} \left( \frac{R'^2}{R^2} + \frac{2R'\nu'}{R} \right) - \frac{1}{R^2}, \quad (12)
\]

\[
\phi'' = (\psi' - \frac{2R'}{R} - \nu') \phi'. \quad (13)
\]

\[
e^{-2\psi} R'^2 = 1 - \frac{F}{R}. \quad (14)
\]

In the above, the equation (13) can be integrated once with respect to \( r \) to give

\[
\phi' = \frac{e^{-\psi - \nu + a}}{R^2}. \quad (15)
\]

Eliminating now \( \phi' \) from these equations gives,

\[
\frac{1}{2} e^{-2\nu + 2a} \frac{R'^2}{R^2} = F', \quad (16)
\]

\[
\frac{1}{2} e^{-2\nu + 2a} \frac{R'}{R^4} = e^{-2\psi} \left( \frac{R'^2}{R^2} + \frac{2R'\nu'}{R} \right) - \frac{1}{R^2}, \quad (17)
\]

\[
e^{-2\psi} R'^2 = 1 - \frac{F}{R}. \quad (18)
\]

We note that there is still a freedom left to transform the coordinate \( R \), and thus the number of unknown variables is reduced to three in the three equations above. This freedom is just a coordinate transformation of the form

\[
r \to g(r), \quad (19)
\]

which is allowed by the spherical symmetry of the spacetime.

The implications of the Einstein equations above will be investigated now. In what follows, we do not make any further assumptions or special choices and therefore the conclusions apply in generality. We define a function \( f(R) \) by,

\[
f(R) = \frac{e^{-2\nu}}{2R^2}. \quad (20)
\]

Using this in (16), we get,

\[
F = e^{2a} f(R) + C_1 \quad (21)
\]

We can take \( C_1 = 0 \), which gives \( F = e^{2a} f(R) \). Using this in (18), we get

\[
e^{-2\psi} R'^2 = 1 - \frac{e^{2a} f(R)}{R} \quad (22)
\]

Also, from (20) we get,

\[
-2\nu' = \frac{f_{,RR}}{f_{,R}} + \frac{2}{R^2} R' \quad (23)
\]

Taking \( e^{2a} = 1 \), \( C_1 = 0 \) for simplicity and clarity of presentation, the last two equations, together with (17) give

\[
R(f_{,R})^2 = (f - R)(f_{,R} + Rf_{,RR}) - Rf_{,R} \quad (24)
\]

The equation above holds true in generality and where we have not made any special coordinate choices. This represents clearly the main Einstein equation in the case when \( \phi = \phi(r) \), and the solutions to the same give the classes of allowed solutions in this case for the static massless scalar field in general relativity. The above is a non-linear ordinary differential equation of second order which is in general difficult to solve fully. It is, however, possible to draw some general consequences from this, as we now show below. In particular, we point out the implications of the above towards the existence and nature of the spacetime singularity and trapped surfaces in these spacetime models.

1. No trapped surfaces in the spacetime

For spherically symmetric spacetimes, the apparent horizon surface is given in general by the equation,

\[
g^{\mu\nu} R_{\mu\nu} = 0 \quad (25)
\]

We note here that for the static case, the apparent horizon and the event horizon are the same. In the static case, the above becomes \( e^{-2\psi} R'^2 = 0 \). Using (15), this condition can be rewritten as \( f(R) = R e^{-2a} \). When \( e^{2a} = 1 \), we have

\[
f(R) = R. \quad (26)
\]
We note here that the results derived here do not depend upon changing the value of the constant $a$ and their qualitative nature remains the same. Substituting the above equation in (24), we get $R_{f,R} (f, R + 1) = 0$. We note that because of the energy condition ensuring the positivity of the mass energy density, namely $\rho \geq 0$, $f_{,R}$ cannot be negative. Further, in a non-empty spacetime, the density of the scalar field is non-zero everywhere. This implies that $f_{,R} \neq 0$ for $R > 0$. It follows that the apparent horizon condition $f(R) = R$ can be satisfied only at $R = 0$. Since the physical area is zero at $R = 0$, in this case the event horizon or trapped surfaces do not therefore exist in the spacetime.

2. All solutions are singular at $R = 0$

In order to examine the existence of spacetime singularity in these models, we investigate the behaviour of the Ricci scalar, which is given here by the expression $R_c = -e^{-2\nu+2\alpha}/R^4$. This gives,

$$R_c = -\frac{2f_{,R} e^{2\alpha}}{R^2} \quad (27)$$

We can see from the expression for the Ricci scalar, that if $R_c$ is to remain finite as $\lim R \to 0$, then clearly $\lim_{R \to 0} f_{,R}$ should go to zero at least as fast as $R^2$. Therefore, let us consider the case when $\lim_{R \to 0} f(R) \sim R^\alpha$, where $\alpha \geq 3$. Then $f_{,R} \sim R^{\alpha-1}$ and $f_{,RR} \sim R^{\alpha-2}$. Keeping only largest terms as $\lim R \to 0$ in (24), we get

$$-R_{f,RR} + 2f_{,R} = 0 \quad (28)$$

From this, we get $f_{,R} \sim \frac{1}{R^2}$, which contradicts the initial assumption about $\lim_{R \to 0} f(R)$. This conclusion remains the same even if $f$ goes to zero faster than $R^2$, even in the cases when it is not necessarily a power law dependence on $R$. This means that (24) cannot have any solutions such that $\lim_{R \to 0} f_{,R}$ goes to zero as $R^2$ or faster. This in turn implies that there will always be a spacetime singularity occurring at $R = 0$, for all the solutions in the static class of massless scalar field models with $\phi = \phi(r)$.

3. Radial outgoing null geodesics from $R = 0$

In what follows, we show that there exist radial null geodesics coming out from the singularity at $R = 0$, whenever the condition for apparent horizon is satisfied only at $R = 0$ i.e. there is no apparent or event horizon in the spacetime, which is the case for all non-empty spacetimes as we pointed out above. The radial null geodesics in the spacetime are given by the equation, $ds^2 = 0 = e^{2\nu}dt^2 - e^{2\nu}dr^2$. This gives $dt^2(1 - \frac{\nu}{f}) = 2f_{,R} R^2 dr^2$. Rewriting it we get,

$$dt^2 = \frac{2R^3 f_{,R}}{(R - f)} dR^2 \quad (29)$$

The equation for outgoing radial null geodesics is now given by $dt = (\text{mod } \frac{2R^3 f_{,R}}{(R - f)^{\frac{1}{2}}}) dR$

In the limit $R \to 0$, if the coefficient of the $dR$ term above is finite or zero, then the outgoing null geodesics from $R = 0$ singularity will exist, and the time taken for a light ray coming out from $R = 0$ to a very small value of $R$ is bounded. This can be seen in the following way. The geodesic equation in this case can be written as,

$$dt = c(R) dR \quad (30)$$

If $\lim_{R \to 0} c(R) = c_0, c_0 > 0$, then (30) can be integrated to give $t = c_0 R + a_2$, which is the equation of outgoing null geodesic close to the center where $a_2$ is some constant. If on the other hand, $\lim_{R \to 0} c(R) = c_1 R^n$ where $n > 0$, then (30) can be integrated to give $t = c_n (R^{n+1}) + a_3$. Therefore, in both the cases radial outgoing null geodesics coming out of the central singularity exist.

If in the limit $R \to 0$, either of the quantities $f$ or $f_{,R}$ has a divergence, then for a broad class of functions $f$, and for very small values of $R$ we can write $O(f_{,R}) = O(\frac{1}{R})O(f)$. So, for small values of $R$, we have $O(R - f) = O(f)$ and $O(R^2 f_{,R}) = O(R^2)O(f)$. In that case, the coefficient of $dR$ is bounded because $\lim R \to 0$. This implies, as per our discussion above, that outgoing null geodesics do come out from the singularity.

This shows that, for a wide class of static massless scalar field spacetimes, the eternal singularity present at $R = 0$, the location where the physical radius vanishes, is always a visible naked singularity which is not hidden within an event horizon.

B. The $\phi = \phi(t)$ class of models

In the comoving frame, the components of the energy-momentum tensor in the case when $\phi = \phi(t)$ are given as,

$$T^t_t = -T^r_r = -T^\theta_\theta = -T^\phi_\phi = \frac{1}{2} e^{-2\nu(t,r)} \dot{\phi}^2 \quad (31)$$

Thus, we see that the massless scalar field behaves in this frame like a stiff isentropic perfect fluid with the equation of state

$$p(t, r) = \rho(t, r) = \frac{1}{2} e^{-2\nu(t,r)} \dot{\phi}^2 \quad (32)$$

In this case, we can easily see that for any real valued function $\phi(t)$, all energy conditions ensuring positivity of mass and energy density of the field are satisfied by the energy momentum tensor.

We now have here $\phi = t$, and $g_{\mu\nu} = g_{\mu\nu}(r)$, as we are considering the static class of solutions. We note that the Chase theorem does not include this case, because for a non-empty model in this case, the scalar field does not go to a vanishing value faraway at large values of the
radial coordinate, and the scalar field is not asymptotically flat. To avoid any confusion, we emphasize that the condition of staticity for the spacetime implies that the metric components are not functions of time. The scalar field, however, can be time-dependent in such a way so that the physical quantities like energy density and pressure are functions of $r$ only.

From the Einstein equations then it is seen that $F = F(r)$ necessarily. From the Klein-Gordon equation we have $\phi(t) = \text{constant}$, and we can normalise this constant to 1. Then the Einstein equations become

$$\frac{1}{2}e^{-2\nu} = \frac{F'}{R^2R'}$$

$$\frac{1}{2}e^{-2\nu} = e^{-2\psi}(\frac{R^2}{R^2} + 2\rho'\nu' - 1 - \frac{1}{R^2})$$

and

$$e^{-2\psi}R'^2 = 1 - \frac{F}{R}.$$ 

Since both $e^{-2\nu}$ and $R$ are functions of $r$ only, therefore $e^{-2\nu}$ can always be written in the following form,

$$\frac{1}{2}e^{-2\nu} = \frac{f_{,R}}{R^2}$$

Putting this back into (33), we get

$$F = f + C_1$$

We take $C_1 = 0$ and then that gives

$$e^{-2\nu}R'^2 = 1 - \frac{f}{R}$$

From (35), we have

$$\nu' = (\frac{f_{,RR}}{2f_{,R}} + \frac{1}{R})R'$$

Using (39) and (36) and (38) in (34), we get

$$Rf_{,R}^2 = (R - f)(3f_{,R} - Rf_{,RR}) - Rf_{,R}$$

The solutions to the above Einstein equation gives the classes of models for the case $\phi = \phi(t)$.

We note here that the class $\phi = \phi(t)$ for the static massless scalar models is non-empty, and one solution for this class was given by Wyman [3]. Apart from that particular solution, it may be possible to argue that there exist other solutions as well in this class. To consider this, the Einstein equations in this case can be reduced to a single second order ordinary differential equation [10].

Given the values of $f(R)$ and $f(R)_{,R}$ at $R = 0$ as initial conditions, from the existence theorems, a solution of [10] exists. While this requires further investigation which we do not go into presently, this indicates the existence of other solutions for this class as well.

We can now work out the Ricci scalar in this case, which is given by,

$$R_c = e^{-2\nu}\phi'^2 = \frac{2f_{,R}}{R^2}. \quad (41)$$

It is clear that unless $f_{,R}$ goes to zero as $R$ goes to zero at least as fast as $R^2$, $R_c$ will blow up at $R = 0$. Unlike the $\phi = \phi(r)$ case, however, [10] does not forbid such a behaviour of $f$. So it is possible that singularity-free solutions may exist in this case where the spacetime has no central singularity at $R = 0$ in all cases.

Again, the equation of horizon [25] reduces to $e^{-2\psi}R'^2 = 0$ in this case also. Like the $\phi = \phi(r)$ static case, this relation gives us the apparent horizon. From the Einstein equations then, we have $F = R$. This implies $f = R$ through a procedure similar to that we followed earlier in the case of $\phi = \phi(r)$. Putting this in (40), we get $Rf_{,R}^2(f_{,R} + 1) = 0$. So $f(R) = R$ can be satisfied only at $R = 0$, or if $f_{,R} = 0$ at some positive value of $R$. We note here that because of energy conditions, $f_{,R}$ cannot be negative. Therefore the relation $f = R$ can be satisfied only at $R = 0$. As in the case of $\phi = \phi(r)$, this implies that there is no event horizon if the spacetime is non-empty. We note that this is exactly the same conclusion that we reached earlier for the $\phi = \phi(r)$ case.

IV. DISCUSSION

We briefly summarize here our main results about the spacetime singularity and the trapped surfaces in the case of massless scalar field spacetimes.

1) For static spherically symmetric massless scalar field spacetimes, there is no event horizon if the spacetime is non-empty (i.e. the density does not go to zero at any comoving coordinate radius $r$.)

2) For the solutions where the scalar field is asymptotically flat, this agrees with the result by Chase [3]. For the class of solutions $\phi = \phi(t)$, the scalar field does not go to zero at spatial infinity, and therefore the scalar field is not asymptotically flat. So Chase’s theorem does not hold for that class of solutions. However, we still found that event horizon does not exist for this class of spherically symmetric spacetimes, independently of the asymptotic flatness condition.

3) For the $\phi = \phi(r)$ class of scalar field models, there is always a curvature singularity present at $R = 0$. It is seen that this singularity is visible in the sense that future directed null geodesics come out from the same. This result agrees with that found by Xanthopoulos and Zannias [6], as they considered spherically symmetric massless scalar fields which are static and asymptotically flat, in an arbitrary number of spacetime dimensions, and found a visible curvature singularity at the physical radius $R = 0$.

4) We note that the issue of visibility of the central singularity for the solution given by Newman, Janis and Winicour [13], has been considered by Virbhadra, Jhingan and Joshi [14]. They found that the singularity is visible
in this case. It is to be further noted that for the class \( \phi = \phi(t) \), there may or may not be a singularity at the center \( R = 0 \), though this point requires some further examination.

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