Dunkl–Schrödinger operators

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Abstract

In this paper, we consider the Schrödinger operators $L_k = -\Delta_k + V$, where $\Delta_k$ is the Dunkl-Laplace operator and $V$ is a non-negative potential on $\mathbb{R}^d$. We establish that $L_k$ is essentially self-adjoint on $C_0^\infty$. In particular, we develop a bounded $H^\infty$-calculus on $L^p$ spaces for the Dunkl harmonic oscillator operator.

Keywords. Self-adjoint operator, Schrödinger operator, Dunkl operators.

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1 Introduction

During the recent decades the ordinary Schrödinger operators $-\Delta + V$ have been generalized in a domains where a family of a Laplace type operators are given, for example, in Heisenberg group [12], nilpotent Lie groups [11], and spaces of homogeneous type [4]. In a similar way, in the area of harmonic analysis, functional calculus for self-adjoint operators and a number of important applications are developed [8, 9, 13].

In this paper we consider Schrödinger operators $L_k = -\Delta_k + V$ associated to the Dunkl Laplace operator on $\mathbb{R}^d$ given by $\Delta_k = \sum_{j=1}^d T_j^2$ where $T_j$ are a family of differential-difference operators associated to a finite reflection group, which are called Dunkl operators. It is well known that Dunkl theory provides a generalization of the Fourier Analysis. There are many classical results in Fourier analysis that are extended to Dunkl setting and the present work fits within this framework. Arise from the work of Simon [19], we study the problem of essential selfadjointness of $L_k$ and the correspondent heat semi group $e^{-tL_k}$. We investigate the spectral theory, complex analysis and theory of holomorphic functional calculus for the generator of an holomorphic semi group, as present in [8, 9, 13] to develop an $L^p$- boundedness of holomorphic functional calculus for the Dunkl harmonic oscillator $H_k = -\Delta_k + |x|^2$, where the use of the heat kernel being the most powerful tools.

The paper is outline as follows. In the next Section we give Backgrounds form Dunkl’s theory. The Sections III and IV are devoted to study the essential selfadjointness of the operators $\Delta_k$ and $L_k$. The Section V treat the $L^p$- boundedness of holomorphic functional calculus for $H_k$. 
2 Basics of the Dunkl theory

For details, we refer to [6, 7, 5, 17] and the references cited there.

\( \mathbb{R}^d \) is equipped with a scalar product \( \langle x, y \rangle = \sum_{j=1}^{d} x_j y_j \) which induces the Euclidean norm \( |x| = \langle x, x \rangle^{1/2} \).

Let \( G \subset O(\mathbb{R}^d) \) be a finite reflection group associated to a reduced root system \( R \) and \( k : R \to [0, +\infty) \) be a \( G \)-invariant function (called multiplicity function). Let \( R^+ \) be a positive root subsystem. The Dunkl operators \( T_\xi \) on \( \mathbb{R}^d \) are the following \( k \)-deformations of directional derivatives \( \partial_\xi \) by difference operators:

\[
T_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in R^+} k(\alpha) \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle},
\]

where \( \sigma_\alpha \) denotes the reflection with respect to the hyperplane orthogonal to \( \alpha \). The Dunkl operators are antisymmetric with respect to the measure \( w_k(x) \, dx \) with density

\[
w_k(x) = \prod_{\alpha \in R^+} |\langle \alpha, x \rangle|^{2k(\alpha)}.
\]

The operators \( \partial_\xi \) and \( T_\xi \) are intertwined by a Laplace–type operator

\[
V_k f(x) = \int_{\mathbb{R}^d} f(y) \, d\nu_x(y),
\]

associated to a family of compactly supported probability measures \( \{ \nu_x | x \in \mathbb{R}^d \} \).

Specifically, \( \nu_x \) is supported in the the convex hull \( \text{co}(G.x) \).

For every \( y \in \mathbb{C}^d \), the simultaneous eigenfunction problem

\[
T_\xi f = \langle y, \xi \rangle f \quad \forall \xi \in \mathbb{R}^d
\]

has a unique solution \( f(x) = E_k(x, y) \) such that \( E_k(0, y) = 1 \), called the Dunkl kernel and is given by

\[
E_k(x, y) = V(e^{\langle \lambda, \cdot \rangle})(x) = \int_{\mathbb{R}^d} e^{\langle z, y \rangle} \, d\nu_x(z) \quad \forall x \in \mathbb{R}^d.
\]

Furthermore this kernel has a holomorphic extension to \( \mathbb{C}^d \times \mathbb{C}^d \) and the following estimate hold: for \( x, y \in \mathbb{C}^d \),

\[(i)\quad E_k(x, y) = E_k(y, x),\]

\[(ii)\quad E_k(\lambda x, y) = E_k(x, \lambda y), \text{ for } \lambda \in \mathbb{C}\]

\[(iii)\quad E_k(g.x, g.y) = E_k(x, y), \text{ for } g \in G.\]

In dimension \( d = 1 \), these functions can be expressed in terms of Bessel functions. Specifically,

\[
E_k(x, y) = J_{k+\frac{1}{4}}(xy) + \frac{x^n}{2^{n+1}} J_{k+\frac{n}{2}}(xy),
\]

where

\[
J_\nu(z) = \Gamma(\nu+1) \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!(\nu+n+1)} \left( \frac{z}{2} \right)^{2n}
\]

are normalized Bessel functions.
The Dunkl transform is defined on \( L^1(\mathbb{R}^d, w_k(x)dx) \) by

\[
\mathcal{F}_k f(\xi) = c_k \int_{\mathbb{R}^d} f(x) E_k(x, -i \xi) w_k(x) \, dx,
\]

where

\[
c_k = \int_{\mathbb{R}^d} e^{-|\xi|^2/2} w(x) \, dx.
\]

We list some known properties of this transform:

(i) The Dunkl transform is a topological automorphism of the Schwartz space \( \mathcal{S}(\mathbb{R}^d) \).

(ii) (Plancherel Theorem) The Dunkl transform extends to an isometric automorphism of \( L^2(\mathbb{R}^d, w_k(x)dx) \).

(iii) (Inversion formula) For every \( f \in \mathcal{S}(\mathbb{R}^d) \), and more generally for every \( f \in L^1(\mathbb{R}^d, w_k(x)dx) \) such that \( \mathcal{F}_k f \in L^1(\mathbb{R}^d, w_k(\xi)d\xi) \), we have

\[
f(x) = \mathcal{F}_k^2 f(-x) \quad \forall \, x \in \mathbb{R}^d.
\]

(iv) if \( f \) is a radial function in \( L^1(\mathbb{R}^d, w_k(\xi)d\xi) \) such that \( f(x) = \widehat{f}(|x|) \), then \( \mathcal{F}_k(f) \) is also radial and one has

\[
\mathcal{F}_k(f)(x) = b_k \int_0^\infty \widehat{f}(s) J_{\gamma_k+d/2-1}(s|x|) s^{2\gamma_k+d} \, ds.
\]

where \( b_k = 2^{-(\gamma_k+d/2-1)}/\Gamma(\gamma_k + d/2) \).

Let \( x \in \mathbb{R}^d \), the Dunkl translation operator \( \tau_x \) is given for \( f \in L_k^2(\mathbb{R}^d, \mathbb{C}) \) by

\[
\mathcal{F}_k(\tau_x(f))(y) = \mathcal{F}_k f(y) E_k(x, iy), \quad y \in \mathbb{R}^d.
\]

Using the Dunkl’s intertwining operator \( V_k \), the operator \( \tau_x \) is related to the usual translation by

\[
\tau_x(f)(y) = (V_k)_x (V_k)_y (V_k)^{-1}(f)(x + y).
\]

In the case when \( f(x) = \widehat{f}(|x|) \) is a radial function in \( \mathcal{S}(\mathbb{R}^d) \), the Dunkl translation is represented by the following integral

\[
\tau_x(f)(y) = \int_{\mathbb{R}^d} \widehat{f}(\sqrt{|y|^2 + |x|^2 + 2 < y, \eta >}) \, d\nu_x(\eta).
\]

This formula shows that the Dunkl translation operators can be extended to all radial functions \( f \) in \( L^p(\mathbb{R}^d, w_k(x)dx) \), \( 1 \leq p \leq \infty \) and the following holds

\[
||\tau_x(f)||_{p,k} \leq ||f||_{p,k},
\]

where \( ||.||_{p,k} \) is the usual norm of \( L^p(\mathbb{R}^d, w_k(x)dx) \).

We define the Dunkl convolution product for suitable functions \( f \) and \( g \) by

\[
f \ast_k g(x) = \int_{\mathbb{R}^d} \tau_x(f)(-y)g(y) \, d\mu_k(y), \quad x \in \mathbb{R}^d
\]
We note that it is commutative and satisfies the following property:

\[ \mathcal{F}_k(f * k g) = \mathcal{F}_k(f) \mathcal{F}_k(g), \quad f, g \in L^2(\mathbb{R}^d, w_k(x)dx). \]  

(2.5)

Moreover, the operator \( f \to f * k g \) is bounded on \( L^p(\mathbb{R}^d, w_k(x)dx) \) provide \( g \) is a bounded radial function in \( L^1(\mathbb{R}^d, w_k(x)dx) \). In particular we have the following Young's inequality:

\[ \| f * k g \|_{p,k} \leq \| g \|_{1,k} \| f \|_{p,k}. \]  

(2.6)

3 The Dunkl Laplacian operator

In this section and in what follows, we often use the language of spectral theory for unbounded operators. Our main reference is [15].

Let \((e_1, e_2, ..., e_d)\) be an orthonormal basis of \(\mathbb{R}^d\). The Dunkl Laplacian operator is defined by

\[ \Delta_k = \sum_{j=1}^{d} T_j^2, \]

where \(T_j = T_{e_j}\). We consider \(-\Delta_k\) as a densely defined operator on the Hilbert space \(L^2(\mathbb{R}^d, w_k(x)dx)\) with domain \(D(-\Delta_k) = \mathcal{S}(\mathbb{R}^d)\), the Schwartz space of rapidly decreasing functions. By means of the Dunkl transform one can prove that, for \(f, g \in \mathcal{S}(\mathbb{R}^d)\),

\[ -\Delta_k(f)(x) = \mathcal{F}_k^{-1}(|x|^2 \mathcal{F}_k(f)), \]

\[ \langle -\Delta_k f, g \rangle = \langle f, -\Delta_k g \rangle, \]

\[ \langle -\Delta_k f, f \rangle = \sum_{j=1}^{d} \|T_j(f)\|_{2,k}^2, \]

which show that \(-\Delta_k\) is a densely defined, symmetric and positive operator on \(L^2(\mathbb{R}^d, w_k(x)dx)\). Thus, the Friedrichs extension theorem tells us that there is a positive self-adjoint extension of \(-\Delta_k\). Define the linear operator \(A_k\) as extension of \(-\Delta_k\) by

\[ D(A_k) = H_k^2(\mathbb{R}^d) = \{ f \in L^2(\mathbb{R}^d, w_k(x)dx); \ |x|^2 \mathcal{F}_k(f) \in L^2(\mathbb{R}^d, w_k(x)dx) \}; \]

\[ A_k(f) = \mathcal{F}_k^{-1}(|x|^2 \mathcal{F}_k(f)); \quad f \in D(A_k). \]

Clearly \(A_k\) is symmetric and positive.

**Theorem 3.1.** The operator \(A_k\) is self-adjoint and that is the unique positive self-adjoint extension operator of \(-\Delta_k\).

**Proof.** Recall first that the adjoint operator \(A_k^*\) is defined on the domain \(D(A_k^*)\) consisting of the function \(f \in L^2(\mathbb{R}^d, w_k(x)dx)\) for which the functional \(g \to \langle A_k g, f \rangle\) is bounded on \(D(A_k)\) and by Riesz representation theorem there exists a unique \(f^* \in L^2(\mathbb{R}^d, w_k(x)dx)\) such that \(\langle A_k(g), f \rangle = \langle g, f^* \rangle\), since \(D(A_k)\) is dense in \(L^2(\mathbb{R}^d, w_k(x)dx)\).
We define $A_k^*$ on $D(A_k^*)$ by $A_k^*(f) = f^*$. Since $A_k$ is symmetric, then $A_k$ is self-adjoint if and only if $D(A_k^*) = D(A_k)$. Noting that $D(A_k) \subset D(A_k^*)$ is obvious. Let $f \in D(A_k^*)$, then there exists a constant $C > 0$ such that for all $g \in D(A_k)$ we have

$$\langle A_k g, f \rangle \leq C \|g\|_{2,k}. \quad (3.1)$$

For $r > 0$ define the function $g_r \in L^2(\mathbb{R}^d, w_k(x)dx)$ by

$$\mathcal{F}_k(g_r)(x) = |x|^2 \mathcal{F}_k(f)(x) \chi_{|x|<r}$$

where $\chi$ denotes the characteristic function. Clearly we have $g_r \in D(A_k)$ and in view of (3.1)

$$|\langle f, A_k g_r \rangle| = |\langle \mathcal{F}_k(f), \mathcal{F}_k(A_k g_r) \rangle| = \int_{|x|<r} |x|^4 |\mathcal{F}_k(f)(x)|^2 w_k(x)dx \leq C \left( \int_{|x|<r} |x|^4 |\mathcal{F}_k(f)(x)|^2 w_k(x)dx \right)^{1/2}.$$  

It yields that

$$\int_{|x|<r} |x|^4 |\mathcal{F}_k(f)(x)|^2 w_k(x)dx \leq C^2.$$

Therefore by letting $r \to \infty$ we deduce that $f \in D(A_k)$ and conclude that $D(A_k^*) \subset D(A_k)$.

Let us now prove that $-\Delta_k$ is essentially self-adjoint, this means that $-\Delta_k$ admits an unique self-adjoint extension and that is equal to $A_k$, since we have proved that $A_k$ is self-adjoint. From the general theory of unbounded operators, see for example the chapter VIII of [15], it suffices to prove that $(-\Delta_k \pm i)D(-\Delta_k)$ is dense, which is equivalent to

$$\left( (-\Delta_k \pm i)D(-\Delta_k) \right) ^{\perp} = \{0\}.$$  

In fact, let $g \in \left( (-\Delta_k \pm i)D(-\Delta_k) \right) ^{\perp}$. Then for any $f \in D(-\Delta_k) = \mathcal{S}(\mathbb{R}^d)$

$$0 = \langle (-\Delta_k \pm i)f, g \rangle = \langle (|.|^2 \pm i)\mathcal{F}_k(f), \mathcal{F}_k(g) \rangle = \langle (\mathcal{F}_k(f), (|.|^2 \pm i)\mathcal{F}_k(g) \rangle,$$

Since $\mathcal{F}_k(\mathcal{S}(\mathbb{R}^d)) = \mathcal{S}(\mathbb{R}^d)$, this implies by density argument that $\mathcal{F}_k(g) = 0$ and so $g = 0$, as desired.

The operator $A_k$ is generator of strongly continuous one parameter semi group $(e^{-tA_k})_{t \geq 0}$ where the operator $e^{-tA_k}$ is given by

$$e^{-tA_k} f = \mathcal{F}_k^{-1}(e^{-t|.|^2} \mathcal{F}_k(f))$$

for all $t > 0$ and $f \in L^2(\mathbb{R}^d, w_k(x)dx)$. It follows that $e^{-tA_k}$ is an integral operator given by

$$e^{-tA_k} f(x) = k_t * f = \int_{\mathbb{R}^d} K_t(x,y) f(y) w_k(y)dy \quad (3.2)$$

where

$$k_t(x) = \mathcal{F}_k^{-1}(e^{-t|.|^2})(x) = t^{-\gamma_d/2}e^{-|x|^2/t} \quad (3.3)$$
and from (2.1) and (2.3)

\[ K_t(x, y) = \tau_x(k_t)(-y) = t^{-\kappa} e^{-|x|^2/2} E(2x/t, y). \] (3.4)

Now using Hölder’s Inequality, one can state the following

**Corollary 3.2.** \( e^{-tA_k} \) can be extended to a bounded operator from \( L^p(\mathbb{R}^d, w_k(x)dx) \) to \( L^\infty(\mathbb{R}^d, w_k(x)dx) \), for \( 1 \leq p \leq \infty \).

## 4 Dunkl Schrödinger operator

In this section, we use arguments analogous to those used in \[19\].

Let \( V \) be a nonnegative measurable function on \( \mathbb{R}^d \) that is finite almost everywhere. In the Hilbert space \( L^2(\mathbb{R}^d, w_k(x)dx) \) we consider the operator

\[ \mathcal{L}_k = A_k + V \]

with domain \( D(\mathcal{L}_k) = D(A_k) \cap D(V) \) where

\[ D(A_k) = H^2_k(\mathbb{R}^d) = \{ f \in L^2(\mathbb{R}^d, w_k(x)dx); |x|^2 \mathcal{F}_k(f) \in L^2(\mathbb{R}^d, w_k(x)dx) \} \]

and

\[ D(V) = \{ f \in L^2(\mathbb{R}^d, w_k(x)dx); Vf \in L^2(\mathbb{R}^d, w_k(x)dx) \}. \]

We call this operator the Dunkl Schrödinger operator. We have already proven that \( A_k \) is positive self-adjoint operator, we should add here that the multiplication operator \( f \to Vf \) is a positive self-adjoint, see \[15\], VIII.3, Proposition 1. The important fact that we shall use comes from the theory of the quadratic form. Define the form \( q_k \) by

\[
D(q_k) = \{ f \in L^2(\mathbb{R}^d, w_k(x)dx); \left( \sum_{j=1}^n |T_j f|^2 \right)^{1/2} , V^{1/2} f \in L^2(\mathbb{R}^d, w_k(x)dx) \}
\]

\[ q_k(f) = \sum_{j=1}^n \| T_j f \|_{2,k}^2 + \| V^{1/2} f \|_{2,k}^2 \]

Clearly \( C_0^\infty \subset D(q_k) \), so \( q_k \) is densely defined.

**Lemma 4.1.** \( q_k \) is closed, that is if \( (\varphi_n) \subset D(q_k) \), \( \| \varphi_n - \varphi \|_{2,k} \to 0 \) and \( q_k(\varphi_n - \varphi_m) \to 0 \) then \( \varphi \in D(q_k) \) and \( q_k(\varphi_n - \varphi) \to 0 \).

**Proof.** Since \( q_k(\varphi_n - \varphi_m) \to 0 \) then \( (V^{1/2} \varphi_n) \) and \( (T_j \varphi_n) \) are Cauchy sequences in \( L^2(\mathbb{R}^d, w(x)dx) \) and so are convergent. Let \( g_j = \lim T_j \varphi_n \) and \( h = \lim V^{1/2} \varphi_n \). For any function \( \psi \in C_0^\infty \),

\[ \langle g_j, \psi \rangle = \lim \langle T_j \varphi_n, \psi \rangle = - \lim \langle \varphi_n, T_j \psi \rangle = - \langle \varphi, T_j \psi \rangle. \]

This yields that \( T_j \varphi = g_j \in L^2(\mathbb{R}^d, w(x)dx) \). Similarly,

\[ \langle h, \psi \rangle = \lim \langle V^{1/2} \varphi_n, \psi \rangle = \lim \langle \varphi_n, V^{1/2} \psi \rangle = \langle \varphi, V^{1/2} \psi \rangle = \langle V^{1/2} \varphi, \psi \rangle \]

and so \( V^{1/2} \varphi = h \in L^2(\mathbb{R}^d, w(x)dx) \). Therefore \( \varphi \in D(q_k) \) and \( q_k(\varphi_n - \varphi) \to 0 \) \( \square \)
Let $B_{q_k}$ be the associated sesquilinear form on $D(q_k)$. It follows that $D(q_k)$ is a Hilbert space with the inner product:

$$\langle \varphi, \psi \rangle_{q_k} = \langle \varphi, \psi \rangle + B_{q_k}(\varphi, \psi), \quad \varphi, \psi \in D(q_k).$$

The corresponding norm is given by

$$\|\varphi\|_{q_k} = \sqrt{\|\varphi\|_{L_k}^2 + q_k(\varphi)}.$$  

The important consequence of the above lemma is that there exist a unique positive self-adjoint operator $L_k$ such that

$$q_k(\varphi) = \langle L_k(\varphi), \varphi \rangle, \quad \varphi \in D(q_k),$$

and is defined as follows

$$D(L_k) = \{ \varphi \in D(q_k) : \exists \tilde{\varphi} \in L^2(\mathbb{R}^d, w(x)dx), B_{q_k}(\varphi, \psi) = \langle \tilde{\varphi}, \psi \rangle \forall \psi \in D(q_k) \}$$

$$L_k(\varphi) = \tilde{\varphi}, \quad \forall \varphi \in D(L_k). \tag{4.1}$$

Moreover,

$$D(q_k) = D(L_k^{1/2}) \quad \text{and} \quad q_k(\varphi) = \|L_k^{1/2}(\varphi)\|_{2,k}.$$  

**Theorem 4.2.** Assume that $V \in L^1_{loc}(\mathbb{R}^d, w_k(x)dx)$ and $V \geq 0$, then $C_0^\infty(\mathbb{R}^d)$ is dense in $D(q_k)$ in the norm $\|\varphi\|_{q_k} = \sqrt{\|\varphi\|_{L_k}^2 + q_k(\varphi)}$.

For the proof we will need the following lemmas.

**Lemma 4.3.** The range of $e^{-L_k}$ is dense in the Hilbert space $(D(q_k), \|\cdot\|_{q_k})$.

**Proof.** Let $\varphi \in D(q_k)$ such that

$$\langle e^{-L_k}\psi, \varphi \rangle + B_{q_k}(e^{-L_k}\psi, \varphi) = \langle e^{-L_k}\psi, \varphi \rangle + \langle L_k e^{-L_k}\psi, \varphi \rangle = 0, \quad \forall \psi \in L^2(\mathbb{R}^d, w_k(x)dx)$$

This implies that

$$\langle (L_k + 1)e^{-L_k}\varphi, \psi \rangle = 0, \quad \forall \psi \in L^2(\mathbb{R}^d, w_k(x)dx)$$

Since $L_k + 1$ is invertible, then $e^{-L_k}\varphi = 0$ which implies that $\varphi = 0$. The density of $\text{Ran}(e^{-L_k})$ follows. \hfill \square

**Lemma 4.4.** $L^\infty \cap D(q_k)$ is dense in the Hilbert space $(D(q_k), \|\cdot\|_{q_k})$.

**Proof.** Using the Kato’s strong Trotter product formula, see Theorem S.21 of [15], we have in the strong convergence

$$s - \lim \left(e^{-tA_k/n} e^{-tV/n}\right)^n = e^{-tL_k}, \quad t \geq 0. \tag{4.2}$$

By the fact that $|e^{-tV/n}f| \leq |f|$ and $|e^{-tA_k}(f)| \leq e^{-tA_k}(|f|)$, which can be seen from (3.2), it follows that

$$|e^{-tL_k}(f)| \leq e^{-tA_k}(|f|). \tag{4.3}$$
In particular we have
\[ \| e^{-L_k}(f) \|_\infty \leq \| e^{-A_k}(|f|) \|_\infty \leq c \| f \|_{2,k}, \]  
(4.4)
where the second inequality follows from (3.2), by using Cauchy-Schwarz Inequality and (2.4). From (4.4) we have that
\[ \operatorname{Ran}(e^{-L_k}) \subset L_\infty. \]  
(4.5)
Since the function \( \lambda \mapsto \lambda^{1/2} e^{-\lambda} \) is bounded on \((0, \infty)\) then by spectral theorem the operator \( L_1^{1/2} e^{-L_k} \) is bounded on \( L^2(\mathbb{R}^d, w_k(x)dx) \). We deduce that
\[ \operatorname{Ran}(e^{-L_k}) \subset D(L_1^{1/2}) = D(q_k). \]  
(4.6)
Similarly we have
\[ \operatorname{Ran}(e^{-L_k}) \subset D(L_k) \]
and in view of (4.5) and (4.6) we have that
\[ \operatorname{Ran}(e^{-L_k}) \subset L_\infty \cap D(q_k). \]

We then conclude Lemma 4.4 from the result of Lemma 4.3.

**Proof of Theorem 4.2.** Let
\[ S = \{ \varphi \in L^\infty \cap D(q_k); \operatorname{supp}(\varphi) \text{ is compact} \}. \]

We claim that \( S \) is dense in \( (D(q_k), \| \cdot \|_{q_k}) \). Observe that for \( \varphi \in L^\infty \cap D(q_k) \) and \( \psi \in C_0^\infty(\mathbb{R}^d) \) be a radial function, we have that \( V^{1/2} \varphi \psi \in L^2(\mathbb{R}^d, w_k(x)dx) \) and in the distributional sense
\[ T_j(\varphi \psi) = T_j(\varphi) \psi + \varphi T_j(\psi), \quad j = 1, 2 \ldots d \]  
(4.7)
which gives that \( T_j(\varphi \psi) \in L^2(\mathbb{R}^d, w_k(x)dx) \). Hence \( \varphi \psi \in S \). From this fact if we choose a radial function \( \psi \in C_0^\infty(\mathbb{R}^d) \) with \( \psi(x) = 1 \) near 0 and we set \( \varphi_n = \psi(. / n) \varphi \) then by Lebesgue dominated convergence theorem and (4.7) the following hold
\begin{itemize}
  \item \( \| \varphi_n - \varphi \|_{2,k} \to 0 \),
  \item \( \| V^{1/2} \varphi_n - V^{1/2} \varphi \|_{2,k} \to 0 \),
  \item \( \| T_j \varphi_n - T_j \varphi \|_{2,k} \to 0 \).
\end{itemize}
This implies that \( \varphi_n \to \varphi \) in norm \( \| \cdot \|_{q_k} \).

We now claim that \( C_0^\infty(\mathbb{R}^d) \) is dense in \( (D(q_k), \| \cdot \|_{q_k}) \). Take a radial function \( \rho \in C_0^\infty(\mathbb{R}^d) \) with
\[ \int_{\mathbb{R}^d} \rho(x) w_k(x) dx = 1 \]
For \( \varphi \in S \) we define \( (\varphi_n)_n \) by \( \varphi_n = \rho_n * \varphi \) where \( \rho_n = n^{-2\gamma_k - d} \rho(x/n) \). Let us observe that \( \varphi_n \in C_0^\infty(\mathbb{R}^d) \) and
\[ T_j \varphi_n = \rho_n * T_j \varphi, \quad j = 1, 2 \ldots d, \]
which can be seen as following: for \( \psi \in C_0^\infty(\mathbb{R}^d) \),

\[
\int_{\mathbb{R}^d} \varphi_n T_j \psi = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(y) \tau_x(\rho_n)(-y) T_j(\psi)(x) w_k(y) w_k(x) dy dx
\]

\[
= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \tau_y(\rho_n)(x) T_j(\psi)(x) w_k(x) dx \right) \varphi(y) w_k(y) dy
\]

\[
= -\int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \tau_y(T_j(\rho_n))(x) \psi(x) w_k(x) dx \right) \varphi(y) w_k(y) dy
\]

\[
= -\int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \tau_x(T_j(\rho_n))(-y) \varphi(y) w_k(y) dy \right) \psi(x) w_k(x) dx
\]

\[
= -\int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} T_j \tau_x(\rho_n)(-y) \varphi(y) w_k(y) dy \right) \psi(x) w_k(x) dx
\]

\[
= -\int_{\mathbb{R}^d} \rho_n * k \ T_j \varphi(x) \psi(x) w_k(x) dx
\]

Therefore as convergent in \( L^2(\mathbb{R}^d, w_k(x) dx) \) we obtain

- \( \varphi_n \to \varphi \),
- \( V^{1/2} \varphi_n \to V^{1/2} \varphi \),
- \( T_j \varphi_n \to T_j \varphi \)

and thus \( \varphi_n \to \varphi \) in the norm \( \| \cdot \|_{q_k} \). This conclude the proof of the density of \( C_0^\infty(\mathbb{R}^d) \).

Next we define in the distributional way \( \mathcal{L}_{k, \text{dist}} = A_k + V \), that is for \( \varphi \in L^2(\mathbb{R}^d, w_k(x) dx) \),

\[
\int_{\mathbb{R}^d} \mathcal{L}_{k, \text{dist}} \varphi(x) \psi(x) w_k(x) dx = \int_{\mathbb{R}^d} \varphi(x) (A_k + V) \psi(x) w_k(x) dx, \quad \forall \psi \in C_0^\infty(\mathbb{R}^d).
\]

Clearly \( \mathcal{L}_{k, \text{dist}} = \mathcal{L}_k \) on \( C_0^\infty(\mathbb{R}^d) \).

**Corollary 4.5.** We have that

\[
D(L_k) = \{ \varphi \in D(q_k); \ \mathcal{L}_{k, \text{dist}}(\varphi) \in L^2(\mathbb{R}^d, w_k(x) dx) \}
\]

**Proof.** For \( \varphi \in D(q_k) \) and \( \psi \in C_0^\infty(\mathbb{R}^d) \) we have

\[
B_{q_k}(\varphi, \psi) = \sum_{j=1}^d \langle T_j(\varphi), T_j(\psi) \rangle + \langle V \varphi, \psi \rangle
\]

\[
= -\sum_{j=1}^d \langle \varphi, T_j^2(\psi) \rangle + \langle \varphi, V \psi \rangle
\]

\[
= \langle \varphi, (A_k + V) \psi \rangle
\]

\[
= \langle \mathcal{L}_{k, \text{dist}} \varphi, \psi \rangle
\]

\[
= \langle \mathcal{L}_k \varphi, \psi \rangle
\]
We conclude the corollary by the definition (14.1) of the domain $D(L_k)$ and the density of $C_0^\infty(\mathbb{R}^d)$. We add here that when $\mathcal{L}_{k,\text{dist}}(\varphi) \in L^2(\mathbb{R}^d, w_k(x)dx)$,

$$H(\varphi) = \mathcal{L}_{k,\text{dist}}(\varphi)$$

(4.8)

This fact will be used in the proof of the next theorem.

\begin{proof}[Theorem 4.6] Assume that $V \in L^2_{\text{loc}}(\mathbb{R}^d, w_k(x)dx)$ and $V \geq 0$, then $\mathcal{L}_k$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^d)$ and its closure is $L_k$.

The proof of this theorem is the same as the proof of Theorem 4.2. Recall that $D \subset D(L_k)$ is a core of $L_k$ if for all $\varphi \in D(L_k)$ there exist in $D$ a sequence $(\varphi_n)$ such that $\Vert \varphi_n - \varphi \Vert_{2,k} \to 0$ and $\Vert L_k(\varphi_n) - L_k(\varphi) \Vert_{2,k} \to 0$.

\begin{lemma} $D(L_k) \cap L^\infty$ is a core of $L_k$.

\begin{proof} Notice that we have already proved that $\text{Ran}(e^{-tL_k}) \subset D(L_k) \cap L^\infty$. Let $\varphi \in D(L_k)$, by the spectral theorem

$$\Vert e^{-tL_k}(\varphi) - \varphi \Vert_{2,k}^2 = \int_0^\infty |e^{-t\lambda} - 1|^2 \, d(\text{tr} \lambda, \varphi),$$

where $\text{tr} \lambda$ is the projection-valued measure with respect to $L_k$. So using the dominated convergence theorem we obtain that $\Vert e^{-tL_k}(\varphi) - \varphi \Vert_{2,k} \to 0$. Similarly,

$$\Vert L_k e^{-tL_k}(\varphi) - L_k(\varphi) \Vert_{2,k} = \Vert e^{-tL_k} L_k(\varphi) - L_k(\varphi) \Vert_{2,k} \to 0.$$

Thus the lemma follows from (4.9).
\end{proof}
\end{lemma}

\begin{proof}[Proof of Theorem 4.6] We first show for $\varphi \in D(L_k)$ and $\psi \in C_0^\infty(\mathbb{R}^d)$ be a radial function we have that $\varphi \psi \in D(L_k)$. Indeed, since $D(L_k) \subset D(q_k)$ then we already have $\varphi \psi \in D(q_k)$ and by a direct calculation

$$\mathcal{L}_{k,\text{dist}}(\varphi \psi) = \mathcal{L}_{k,\text{dist}}(\varphi \psi) - 2 \sum_{j=1}^d T_j \varphi T_j \psi - \varphi \Delta_k \psi$$

$$+ \sum_{j=1}^d \sum_{\alpha \in \mathbb{R}_+} k(\alpha) \alpha_j \frac{\varphi(x) - \varphi(\varphi, x)}{\varphi \sigma \alpha} \frac{\psi(\varphi, x) - \psi(\varphi, \varphi, x)}{\psi(x, \alpha)}$$

(4.10)

This is proven by showing that both sides have the same inner product with a function in $C_0^\infty$ and using the density of $C_0^\infty$ in form norm. Since the function

$$x \to \frac{T_j(\psi)(x) - T_j(\psi)(\varphi, x)}{\varphi \sigma \alpha}$$

is in $C_0^\infty(\mathbb{R}^d)$, it follows that $\mathcal{L}_{k,\text{dist}}(\varphi \psi) \in L^2(\mathbb{R}^d, w_k(x)dx)$ and from Corollary 4.5 we have $\varphi \psi \in D(L_k)$.
\end{proof}
Let $\psi \in C_c^\infty(\mathbb{R}^d)$ be a radial function with $\psi(x) = 1$ near 0 and set $\varphi_n = \psi(. /n)\varphi$. In view of (4.8) and (4.10) we get that $\|H(\varphi_n) - H(\varphi)\|_{2,k} \to 0$. This yields that

$$S' = \{ \varphi \in D(L_k) \cap L^\infty / \text{supp}(\varphi) \text{ is compact} \}$$

is a core for $L_k$.

Now we proceed as follows. Let $\varphi \in S'$ then $A_k\varphi + V\varphi \in L^2(\mathbb{R}^d, w_k(x)dx)$ and $V\varphi \in L^2(\mathbb{R}^d, w_k(x)dx)$, since $V \in L^2_{loc}(\mathbb{R}^d, w_k(x)dx)$ and $\varphi \in L^\infty$. It follows that $A_k\varphi \in L^2(\mathbb{R}^d, w_k(x)dx)$. However, if $\varphi_n = \rho_n * k \varphi \in C_c^\infty(\mathbb{R}^d)$, where $(\rho_n)_n$ is defined in the proof of Theorem 3.1 and as $\varphi \in L^\infty$ and supp$(\varphi)$ is compact then $\|V\varphi_n - V\varphi\|_{2,k} \to 0$. But by means of Dunkl transform we see that the positive kernel, it is positivity preserving. Thus using the Trotter product formula we have that $e^{tL_k}$ is the closure of $L_k$ on $C_c^\infty(\mathbb{R}^d)$.

Proof. Recall that from (4.3) we have that $e^{tL_k}$ is a kernel operator with

$$0 \leq W_t(x, y) \leq K_t(x, y) = \frac{1}{(2t)^{\gamma_d/2}c_k} e^{-\langle|\langle x \rangle|, |y|\rangle^2/4t} E_k\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right).$$

(4.11)

Thus by Corollary 3.2, $W_t$ is a bounded operator from $L^p(\mathbb{R}^d, w_k(x)dx)$ to $L^\infty$ for all $1 \leq p \leq \infty$. The theorem of Dunford and Pettis (see for example theorem 4.2 of [18]) asserts that such operator is a kernel operator. Since $e^{tA_k}$ is an integral operator with positive kernel, it is positivity preserving. Thus using the Trotter product formula (4.2) we have that $W_t$ is positivity preserving which implies that $W_t(x, y) \geq 0$. The second inequality of (4.11) follows from (4.3) and from [18], Theorems 2.2 and 4.3.

5 Dunkl harmonic oscillator

We first recall some known facts about the Dunkl harmonic oscillator. The reader is referred to [2, 3, 14, 16].

The Dunkl harmonic oscillator is the Schrödinger operator $H_k = -\Delta_k + |x|^2$. It can be expressed in terms of generalized Hermite functions $h_n^k$,

$$H_k(f) = \sum_{n \in \mathbb{N}^d} (2|n| + \gamma_k + d)\langle f, h_n^k \rangle^2 h_n^k$$
where $|n| = n_1 + \ldots + n_d$. The functions $h_k^k$ are eigenfunctions of $H_k$ with
\[ H(h_k^k) = (2|n| + \gamma_k + d) h_k^k \]
and form an orthonormal basis of $L^2(\mathbb{R}^d, w_k(x)dx)$.

The holomorphic Hermite semi-group $e^{-zH_k}$, $\text{Re}(z) > 0$ is given by
\[ e^{-zH_k}(f) = \sum_{n \in \mathbb{N}^d} e^{-z(2|n| + \gamma_k + d)} \langle f, h_n^k \rangle h_n^k. \quad (5.1) \]

It has the following integral representation
\[ e^{-zH_k}(f)(x) = \int_{\mathbb{R}^d} \mathcal{H}_z(x, y) f(y) w_k(y) dy, \]
where from the generalized Mehler-formula,
\begin{align*}
\mathcal{H}_z(x, y) &= \sum_{n \in \mathbb{N}^d} e^{-z(2|n| + \gamma_k + d)} h_n^k(x) h_n^k(y) \\
&= c_k \left( \frac{\sinh(2z)}{2} \right)^{-\gamma_k} E_k \left( \frac{x}{\sinh(2z)} y \right) e^{-\coth(2z)(|x|^2 + |y|^2)}.
\end{align*}

It can be written as
\[ \mathcal{H}_z(x, y) = c_k \sinh(2z)^{-\gamma_k} \int_{\mathbb{R}^d} \mathcal{H}^0_z(\eta, y) e^{-\coth(2z)(|\eta|^2 + |y|^2)} d\nu_x(\eta). \]

where $\mathcal{H}^0_z$ is the kernel of the classical Hermite semi-group given by
\begin{align*}
\mathcal{H}^0_z(x, y) &= (2\pi \sinh(2z))^{-\frac{d}{2}} e^{-\coth(2z)(|x-y|^2 + \tanh(t))(x, y)} \\
&= (2\pi \sinh(2z))^{-\frac{d}{2}} e^{-\frac{1}{4}(\coth(z)(|x-y|^2 + \tanh(z)|x-y|^2)). \quad (5.2) \]

**Proposition 5.1.** For all $z \in \mathbb{C}$, $0 \leq \arg(z) \leq \omega < \pi/2$ there exist $c > 0$ and $C > 0$ such that
\[ |\mathcal{H}_z(x, y)| \leq \mathcal{H}_{\text{Re}(z)}(cx, cy). \]

Let us first prove the following lemma

**Lemma 5.2.** If $0 \leq \arg(z) \leq \omega < \pi/2$ then there exist $c > 0$ and $C > 0$ such that
\[ c \coth(\text{Re}(z)) \leq \text{Re}(\coth(z)) \leq C \coth(\text{Re}(z)) \]

**Proof.** Note first that for $z = t + iu$
\[ \text{Re}(\coth(z)) = \frac{e^{4t} - 1}{(e^{2t} - 1)^2 + 2e^{2t}(1 - \cos(2u))} \leq \frac{e^{2t} + 1}{e^{2t} - 1} = \coth(t). \]

Now if $0 \leq \arg(z) \leq \omega < \pi/2$, then
\[ |u| \leq \tan(\omega) t. \]
Choosing $a = \pi/(4\tan(\omega))$, it follows that for $t \in (0, a]$, we have $2|u| \leq \pi/2$ and $\cos(2u) \geq \cos(2\tan(\omega)t)$. Then we get

$$Re(\coth(z)) \geq \frac{e^{4t} - 1}{(e^{2t} - 1)^2 + 2e^{2t}(1 - \cos(2\tan(\omega)t))}.$$ 

If we take the function

$$\varphi(t) = \left( \frac{e^{4t} - 1}{(e^{2t} - 1)^2 + 2e^{2t}(1 - \cos(2\tan(\omega)t))} \right) \left( \frac{e^{2t} - 1}{e^{2t} + 1} \right)$$ 

we see that

$$\lim_{t \to 0} \varphi(t) = 2 \cos^2(\omega) > 0$$

and $\varphi$ define a positive continuous function on the interval $[0, a]$, thus $\inf_{y \in (0, a]} \varphi(t) = c > 0$. Therefore, for $0 < t \leq a$

$$Re(\coth(z)) \geq c \coth(t).$$

For $t \geq a > 0$ there exit $c > 0$ so that

$$2e^{2t}(1 - \cos(2u)) \leq 4e^{2t} \leq c(e^{2t} - 1)^2.$$ 

It follows that,

$$Re(\coth(z)) \geq \frac{e^{4t} - 1}{(1 + c)(e^{2t} - 1)^2} = c' \coth(t).$$

The lemma follows. \qed

**Proof of Proposition 5.1** This follows from (5.2), Lemma 5.2 and the fact that

$$|\sinh(z)| = \sqrt{\sinh^2(t) + \sin^2(u)} \geq \sinh(t),$$

for $z = t + iu$. Indeed,

$$|\mathcal{H}_z(x, y)| = |(2\pi \sinh(2z))^{-1/2} e^{-\frac{1}{2} \text{coth}(z)|x-y|^2 + \tanh(z)|x-y|^2}|$$

$$\leq (2\pi \sinh(2t))^{-1/2} e^{-\frac{1}{2} \text{coth}(t)|x-y|^2 + \tanh(t)|x-y|^2}$$

and

$$|\mathcal{H}_z(x, y)| \leq c_k \sinh(2t)^{-\gamma_k - d/2} \int_{\mathbb{R}^d} e^{-\frac{1}{2} \text{coth}(t)|\eta-y|^2 + \tanh(t)|\eta-y|^2} \frac{\text{coth}(2t)}{2} (|x| - |\eta|)^2 \ dv_x(\eta)$$

$$= c_k \sinh(2t)^{-\gamma_k - d/2} \int_{\mathbb{R}^d} e^{-\frac{1}{2} \text{coth}(2t)|\eta-y|^2 + \tanh(t)|\eta-y|^2} - 2 \tanh(t)(y, \eta) \ dv_x(\eta)$$

$$= c_k \sinh(2t)^{-\gamma_k - d/2} e^{-\frac{c \text{coth}(2t)}{2} (|x|^2 + |y|^2)} \int_{\mathbb{R}^d} e^{c \text{sinh}(2t)(y, \eta)} \ dv_x(\eta)$$

$$= c_k \sinh(2t)^{-\gamma_k - d/2} e^{-\frac{c \text{coth}(2t)}{2} (|x|^2 + |y|^2)} E_k(\frac{c}{\sinh(2t)}, x, y)$$

$$= \mathcal{H}_z(\sqrt{c} x, \sqrt{c} y),$$

which is the desired inequality. \qed
Now, since from (4.11)
\[0 \leq H_t(x, y) \leq K_t(x, y), \quad t > 0,\] (5.3)
then we can state

**Corollary 5.3.** For all \(z \in \mathbb{C}, \) \(0 \leq \arg(z) \leq \omega < \pi/2\) there exist \(c > 0\) and \(C > 0\) such that
\[|H_z(x, y)| \leq K_{\text{Re}(z)} \left(cx, cy\right).\]

### 5.1 \(H^\infty\)-functional calculus on \(L^p(\mathbb{R}^d, w_k(x)dx)\) for Dunkl oscillator operator

We briefly recall the definition of sectorial operators and their holomorphic functional calculus. More on basic properties of sectorial operators can be found in [1, 10, 13].

A closed operator \(T\) on complex Hilbert space is said to be sectorial of type \(\omega \in [0, \pi]\) if the following hold

(i) The spectrum \(\sigma(T) \subset S_\omega = \{z \in \mathbb{C}^*, \ |\arg(z)\| < \omega\} \cup \{0\}\)

(ii) For each \(\mu > \omega\) there exists \(C_\mu\) such that \(\|(T - zI)^{-1}\| \leq C_\mu|z|^{-1}\) for \(z \notin S_\mu\).

A non-negative self-adjoint operator in a Hilbert space is an operator of type \(S_\omega\) for all \(\omega > 0\).

Let \(H^\infty(S_\mu^o)\) be the space of bounded holomorphic functions in the open sector \(S_\mu^o,\) the interior of \(S_\mu\) equipped with the norm \(\|f\|_\infty = \sup_{z \in S_\mu^o} |f(z)|\) and let
\[\Psi(S_\mu^o) = \{\xi \in H^\infty(S_\mu^o); \exists s > 0; |\xi(z)| \leq C|z|^s/(1 + |z|)^{2s}\}.

For \(\xi \in \Psi(S_\mu^o)\) we define the operator \(\xi(T)\)
\[\xi(T) = \frac{1}{2\pi i} \int_\gamma \xi(z)(T - zI)^{-1}dz\] (5.4)
where \(\gamma\) is the unbounded contour

\[\gamma(t) = \begin{cases} te^{i\theta}, & t \geq 0 \\ -te^{-i\theta}, & t \leq 0 \end{cases}\] (5.5)

When \(T\) is a one-one operator of type \(\omega\) then one can define a holomorphic functional calculus as follows: Let \(\psi\) the function defined on \(\mathbb{C}\setminus\{-1\}\) by \(\psi(z) = z/(1+z)^2\).

For each \(\mu > \omega\) and for each \(f \in H^\infty(S_\mu^o)\) we have that \(\psi, f\psi \in \Psi(S_\mu^o)\) and \(\psi(T)\) is one-one. So \(f\psi(T)\) is a bounded operator and \(\psi(T)^{-1}\) is a closed operator. Define \(f(T)\) by
\[f(T) = \psi(T)^{-1}(f\psi)(T) = (I + T)^2T^{-1}(f\psi)(T).\] (5.6)

The definitions given by (5.4) and (5.6) are consistent with the usual definition of polynomials of an operator.

We say that \(T\) has bounded \(H^\infty\)-functional calculus if further for all \(f \in H^\infty(S_\mu^o)\) the operator \(f(T)\) is bounded and
\[\|f(T)\| \leq c_\mu\|f\|_\infty,
\]
for some constant \(c_\mu\). An interesting result is given by the following
Proposition 5.4 (see [1], p.101). If $T$ is a positive self-adjoint operator then it has a bounded $H^\infty$-functional calculus for all $\mu > 0$ and

$$\|\xi(T)\| \leq \|\xi\|_\infty,$$

for all $\xi \in H^\infty(S^o_\mu)$.

Theorem 5.5 (Convergence theorem, Th. D of [1]). Let $T$ be a one-one operator of type $\omega$ on a Hilbert space and $\{\xi_s\}$ be a uniformly bounded sequence in $H^\infty(S^o_\mu)$, $\mu > \omega$, which converges to a function $\xi \in H^\infty(S^o_\mu)$ uniformly on compact subsets of $S^o_\mu$, such that $\{\xi_s(T)\}$ is a uniformly bounded set in $L(H)$. Then $\xi(T) \in L(H)$, $\xi_s(T)(u) \to \xi(T)(u)$ for all $u \in H$, and $\|\xi(T)\| \leq \sup_s \|\xi_s(T)\|$.

Now assume that an operator $T$ has a bounded $H^\infty$-functional calculus on the Hilbert space $L^2(\mathbb{R}^d, w_k(x)dx)$. We say that $T$ has a bounded $H^\infty$-functional calculus on $L^p(\mathbb{R}^d, w_k(x)dx)$ for $1 < p < \infty$, if for all $\xi \in H^\infty(S^o_\mu)$ the operator $\xi(T)$ can be extended to a bounded operator on $L^p(\mathbb{R}^d, w_k(x)dx)$ that is,

$$\|\xi(T)(u)\|_p \leq c\|u\|_p,$$

for all $u \in L^p(\mathbb{R}^d, w_k(x)dx)$.

We conclude with the following remark.

Remark 5.6. A function $\xi \in H^\infty(S^o_\mu)$ is the limit of a uniformly bounded sequence of functions in $\Psi(S^o_\mu)$ in the sense of uniform convergence on compact subsets of $S^o_\mu$.

5.2 The main result

Our main result in this section is the following theorem.

Theorem 5.7. The Dunkl harmonic oscillator operator $H_k$ has a bounded $H^\infty$-functional calculus on $L^p(\mathbb{R}^d, w_k(x)dx)$ for $1 < p < \infty$. Moreover, for each $\xi \in H^\infty(S^o_\mu)$, $0 < \mu < \pi$, the operator $\xi(L_k)$ is of weak type $(1, 1)$.

From this theorem we recover and extend the result of [3] where a particular case is dealt with $\xi(z) = z^a$, $a \in \mathbb{R}$. For the proof we follow the elegant approach of [8].

Let $k_t$ and $K_t$ the kernels given by (3.3) and (3.4). Recall that

$$K_t(x, y) = \tau_x(k_t)(-y).$$

Lemma 5.8. There exists $c > 0$ such that for all $t > 0$ and $z \in \mathbb{R}^d$,

$$\sup_{y \in B(z, \sqrt{t})} K_t(x, y) \leq c \inf_{y \in B(z, \sqrt{t})} K_{2t}(x, y) \quad (5.7)$$

Proof. Recall that

$$\tau_x(k_t)(-y) = t^{-\gamma_d-d/2} \int_{\mathbb{R}^d} e^{-\langle y-\eta \rangle^2 + |x|^2 - |\eta|^2}/t d\nu_x(\eta)$$
Let \( y_1, y_2 \in B(z, \sqrt{t}) \) and \( \eta \in \mathbb{R}^d \) we have
\[
\frac{|y_2 - \eta|^2}{2t} \leq \frac{|y_1 - \eta|^2 + |y_2 - y_1|^2}{t} \leq \frac{|y_1 - \eta|^2}{t} + 4.
\]

It follows that
\[
t^{-\gamma_k d^2} e^{-|y_1 - \eta|^2 / t} \leq e^{4} t^{-\gamma_k d^2} e^{-|y_2 - \eta|^2 / 2t},
\]
Now, since we have
\[
e^{-(|x|^2 - |\eta|^2) / t} \leq e^{-(|x|^2 - |\eta|^2) / 2t}
\]
thus for all \( y_1, y_2 \in B(z, \sqrt{t}) \)
\[
\tau_x(k_t)(-y_1) \leq c \tau_x(k_{2t})(-y_2),
\]
which yields the desired inequality.

For \( f \in L^1_{loc}(\mathbb{R}^d, w_k(x)dx) \), the Dunkl maximal function \( M_k f \) is defined by
\[
M_k f(x) = \sup_{r>0} \frac{1}{d_k r^{2k + d}} |f \ast_k \chi_{B_r}|
\]
where \( \chi_{B_r} \) is the characteristic function of the ball \( B_r \) of radius \( r \) centered at \( 0 \).

According to the theorem 6.2 of [21].

**Lemma 5.9.** There exists \( c > 0 \) such that
\[
\sup_{t>0} |k_t \ast f(x)| \leq c M_k f(x).
\]

**Lemma 5.10.** For all \( \delta > 0 \) there exists a constant \( c > 0 \) so that for all \( r > 0 \) and \( x \in \mathbb{R}^d \)
\[
\sup_{\min_{g \in G} |y - g.x| > r} K_t(x, y) w_k(y) dy \leq c(1 + r^2 t^{-1})^{-\delta}.
\]

**Proof.** Let us note that for all \( \eta \in co(G.x) \)
\[
\min_{g \in G} |y - g.x| \leq |x|^2 + |y|^2 - 2(y, \eta) \leq \max_{g \in G} |y - g.x|.
\]

Then we have
\[
K_t(x, y) = \tau_x(k_t)(-y) = t^{-\gamma_k d^2} \int_{\mathbb{R}^d} e^{(|x|^2 + |y|^2 - 2(y, \eta))/t} d\nu_x(\eta)
\]
\[
\leq e^{-(\min_{g \in G} |y - g.x|^2)/2t} \tau_x(k_{2t})(-y)
\]
and
\[
\int_{\min_{g \in G} |y - g.x| > r} K_t(x, y) w_k(y) dy \leq ce^{-r^2/t} \tau_x(k_{2t}) ||1_{1,k} = c\|k_{2t}||_{1,k} e^{-r^2/t} = ce^{-r^2/t}
\]
\[
\leq c(1 + r^2 t^{-1})^{-\delta}
\]
which is the desired inequality. \( \Box \)
Lemma 5.11. Let $T$ be a bounded operator on $L^2(\mathbb{R}^d, w_k(x)dx)$. Suppose that $(T_n)_n$ is a sequence of bounded operators satisfy the condition: there exists $c > 0$ such that for each $f \in L^2(\mathbb{R}^d, w_k(x)dx) \cap L^1(\mathbb{R}^d, w_k(x)dx)$ and $\lambda > 0$

$$\mu_k\{x \in \mathbb{R}^d, \ T_n(f)(x) > \lambda\} \leq c \frac{\|f\|_{k,1}}{\lambda}, \ n \in \mathbb{N}$$

where the measure $d\mu_x = w_k(x)dx$. Assume that for each $f \in L^2(\mathbb{R}^d, w_k(x)dx) \cap L^1(\mathbb{R}^d, w_k(x)dx)$ and $\lambda > 0$

$$T(f)(x) = \lim_{j \to \infty} T_{n_j}(f)(x); \text{ a.e. } x \in \mathbb{R}^d. \quad (5.8)$$

Then $T$ is of weak type $(1, 1)$, i.e., there exists $c > 0$ such that

$$\mu_k\{x \in \mathbb{R}^d, \ T(f)(x) > \lambda\} \leq c \frac{\|f\|_{k,1}}{\lambda}$$

for each $f \in L^2(\mathbb{R}^d, w_k(x)dx) \cap L^1(\mathbb{R}^d, w_k(x)dx)$ and $\lambda > 0$.

Proof. Let $f \in L^2(\mathbb{R}^d, w_k(x)dx) \cap L^1(\mathbb{R}^d, w_k(x)dx)$ and $\lambda > 0$. Put

$$A_j = \{x \in \mathbb{R}^d, \ T_{n_j}(f)(x) > \lambda\}$$

and

$$C_j = \bigcap_{t \geq j} A_{n_t}.$$ 

Then, we have $C_j \subset C_{j+1}$. From (5.8) we see that

$$\mu_k(\{x \in \mathbb{R}^d, \ T(f)(x) > \lambda\}) \leq \mu_k\left(\bigcup_j C_j\right) = \lim_{j \to \infty} \mu_k(C_j)$$

But,

$$\mu_k(C_j) \leq \mu_k(A_{n_j}) \leq c \frac{\|f\|_{k,1}}{\lambda}.$$ 

Therefore

$$\mu_k(\{x \in \mathbb{R}^d, \ T(f)(x) > \lambda\}) \leq c \frac{\|f\|_{k,1}}{\lambda}$$

which is the required inequality.

The next lemma show that the Euclidian $\mathbb{R}^d$ endowed with measure $\mu_k$ is a space of homogeneous type. Recall that a metric measure space $(X, d, \varrho)$ is said to be of homogeneous type if $\varrho$ is a doubling measure on $X$, i.e., there exists $c > 0$ such that such that for all $x_0 \in \mathbb{R}^d$ and $r > 0$,

$$\varrho(B(x_0, 2r)) \leq c\varrho(B(x_0, r)).$$

Where $B(x_0, r) = \{x \in X, \ d(x, x_0) < r\}$.

Lemma 5.12. $\mu_k$ is a doubling measure.
Proof. It is enough to check that the weight \( w_k \) belongs to a Muckenhoupt class \( A_p \) for some \( p > 1 \). Indeed, it is known ([20], Ch V, 6.5) that when \( P \) is a polynomial on \( \mathbb{R}^d \) having degree \( \ell \) then \( |P|^{a} \) belongs to \( A_p \) whenever \( -1 < \ell a < p - 1 \). Applying this fact to the polynomial \( P_\alpha(x) = \langle x, \alpha \rangle \) for \( \alpha \in \mathbb{R}^+ \) and taking \( p > 2N\gamma_k + 1 \) where \( N \) is the cardinality of \( \mathbb{R}^+ \), we see that \( |P_\alpha|^{2\gamma_k} \in A_p \). Then according to ([20], Ch V, 6.1) and the fact that

\[
 w_k = \left( \prod_{\alpha \in \mathbb{R}^+} |P_\alpha|^{2\gamma_k(\alpha)} \right)^{\frac{1}{\gamma_k}}
\]

we obtain, with this choice of \( p \), that \( w_k \in A_p \).

We are now able to prove Theorem 5.7.

**Proof of the Theorem 5.7.** Let \( \xi \in H^\infty(S^0_\mu) \), \( \mu > 0 \). It suffices by Marcinkiewicz interpolation and duality to prove that \( \xi(H_k) \) is a weak-type \((1,1)\). Before beginning we note first that in view of the convergence theorem 5.5 and lemma 5.11 one can assume that \( \xi \in \Psi(S^0_\mu) \).

Let \( f \in L^2(\mathbb{R}^d), w_k(x)dx \cap L^1(\mathbb{R}^d), w_k(x)dx \) and \( \lambda > 0 \). From the Calderon-Zygmund decomposition, there exist a functions \( g \) and \( f_j \) and balls \( B_j = B(x_j, r_j) \) such that

(i) \( f = g + h \) with \( h = \sum_j f_j \),

(ii) \( \|g\|_\infty \leq c\lambda \),

(iii) \( \text{supp}(f_j) \subset B_j \), and

(iv) \( \|f_j\|_{1,k} \leq c\mu_k(B_j) \),

(v) \( \sum_j \mu_k(B_j) \leq \frac{c}{\lambda} \|f\|_{1,k} \),

(vi) Each point of \( \mathbb{R}^d \) is contained in at most a finite number \( M \) of the balls \( B_j \).

Note that (iv) and (v) imply that \( \|h\|_{1,k} \leq c\|f\|_{1,k} \). Hence

\[
\|g\|_{1,k} \leq (1 + c)\|f\|_{1,k} .
\] (5.9)

Let \( S_t = e^{-th_k} \), we split \( h = h_1 + h_2 \) with

\[
 h_1 = \sum_j S_{t_j}(f_j); \quad h_2 = h - h_1 = \sum_j (I - S_{t_j})f_j
\]

where \( t_j = r_j^2 \). Then

\[
 \mu_k\{x; |\xi(H_k)(f)(x)| > \lambda\} \leq \mu_k\left\{ x; |\xi(H_k)(g)(x)| > \frac{\lambda}{3} \right\} + \mu_k\left\{ x; |\xi(H_k)(h_1)(x)| > \frac{\lambda}{3} \right\} + \mu_k\left\{ x; |\xi(H_k)(h_2)(x)| > \frac{\lambda}{3} \right\}
\] (5.10)
For the first term of the right hand side of (5.10) we use the \( L^2 \)-boundedness of \( \xi(H_k) \), (5.9) and (ii) to get
\[
\mu_k \left\{ x; \xi(H_k)(g) > \frac{\lambda}{3} \right\} \leq \frac{9}{\lambda^2} \| \xi(H_k)(g) \|_{2,k} \leq \frac{c}{\lambda^2} \| g \|_{2,k} \leq \frac{c}{\lambda^2} \| g \|_{1,k} \| g \|_{\infty} \leq \frac{c}{\lambda} \| f \|_{1,k}.
\]
For the second term we have
\[
\mu_k \left\{ x; |\xi(H_k)(h_1)(x)| > \frac{\lambda}{3} \right\} \leq \frac{9}{\lambda^2} \| \xi(H_k)(h_1) \|_{2,k} \leq \frac{9}{\lambda^2} \sum_j \| S_{t_j}(f_j) \|_{2,k}^2.
\]
Using (5.3) and Lemma 5.8 we get
\[
|S_{t_j}(f_j)(x)| \leq \int_{\mathbb{R}^d} \tau_x(k_{t_j})(-y) |f_j(y)| w_k(y) dy \leq \sup_{y \in B(z,\sqrt{t_j})} \tau_x(k_{t_j})(-y) \| f_j \|_{1,k} \leq c\lambda \mu(B_j) \inf_{y \in B(z,\sqrt{t_j})} \tau_x(k_{2t_j})(-y) \leq c\lambda \int_{\mathbb{R}^d} \tau_x(k_{2t_j})(-y) \chi_{B_j} w_k(y) dy
\]
and in view of Lemma 5.9 it follows that for any \( \varphi \in L^2(\mathbb{R}^d, w_k(x)dx) \),
\[
|\langle S_{t_j}(f_j), \varphi \rangle_k| \leq c\lambda (k_{2t_j} * \varphi | \chi_{B_j})_k \leq c\lambda (M_k |\varphi|, \chi_{B_j})_k.
\]
Thus the \( L^2 \)-boundedness of \( M_k \) yield
\[
\left\| \sum_j S_{t_j}(f_j) \right\|_{2,k} = \sup \left\{ \left\| \sum_j S_{t_j}(f_j), \varphi \right\|_k, \| \varphi \|_{2,k} \leq 1 \right\} \leq c\lambda \sup \left\{ \left\| M_k |\varphi|, \sum_j \chi_{B_j} \right\|_k, \| \varphi \|_{2,k} \leq 1 \right\} \leq c\lambda \left\| \sum_j \chi_{B_j} \right\|_{2,k}.
\]
We now use properties (vi) of the Calderon-Zygmund decomposition to obtain the estimate
\[
\left\| \sum_j \chi_{B_j} \right\|_{2,k} \leq M \left( \sum_j \mu(B_j) \right)^{\frac{1}{2}}.
\]
Thus in view (v)
\[
\left\| \sum_j S_{t_j}(f_j) \right\|_{2,k}^2 \leq c\lambda \| f \|_{1,k}
\]
and we conclude that
\[
\mu_k \left\{ x; |\xi(H_k)(h_1)(x)| > \frac{\lambda}{3} \right\} \leq \frac{c}{\lambda} \| f \|_{1,k}.
\]
Now consider the third term of the right hand side of (5.10). Putting
\[ B_j = \bigcup_{g \in G} g(B(y_j, 2r_j)) = \bigcup_{g \in G} B(gy_j, 2r_j) \]
and write
\[ \mu_k \left\{ x; |\xi(H_k)(h_2)(x)| > \frac{\lambda}{3} \right\} \leq \sum_j \mu_k(B_j) + \mu_k \left\{ x \notin \bigcup_j B_j; |\xi(H_k)(h_2)(x)| > \frac{\lambda}{3} \right\}. \]

By the doubling property of the measure \( \mu_k \) and \((iii)\) we have
\[ \sum_j \mu_k(B_j) \leq |G| \sum_j \mu_k(B(y_j, 2r_j)) \leq |G| \sum_j \mu_k(B_j) \leq c \lambda \|f\|_{1,k}, \]
since \( w_k \) is a \( G \)-invariant function. So it remains to prove that
\[ \mu_k \left\{ x \notin \bigcup_j B_j; |\xi(H_k)(h_2)(x)| > \frac{\lambda}{3} \right\} \leq c \lambda \|f\|_{1,k}. \] (5.11)

As in [8] define \( \xi_j(v) = \xi(v)(1 - e^{-t_j v}) \) and write
\[ \xi(H_k)(h_2) = \sum_j \xi_j(H_k)f_j. \]

We represent the operator \( \xi_j(H_k) \) by
\[ \xi_j(H_k) = \frac{1}{2i\pi} \int_\gamma \xi_j(v)(H_k - vI)^{-1}dv = \xi_j^+(H_k) + \xi_j^-(H_k) \]
where
\[ \xi_j^\pm(H_k) = \frac{1}{2i\pi} \int_{\gamma^\pm} \xi_j(v)(H_k - vI)^{-1}dv \]
and the contour \( \gamma = \gamma^+ \cup \gamma^- \) with \( \gamma^+(t) = te^{i\theta} \) for \( t \geq 0 \), \( \gamma^-(t) = -te^{i\theta} \) for \( t < 0 \) and with \( 0 < \theta < \pi/2 \). Consider \( \xi_j^+(H_k) \), for \( v \in \gamma^+ \) we substitute
\[ (H_k - vI)^{-1} = \int_{\Gamma^+} e^{vz}e^{-zH_k}dz \]
where the curve \( \Gamma^+ \) is defined by \( \Gamma^+ = te^{i\beta} \) with \( \pi/2 - \theta < \beta < \pi/2 \). It follows that
\[ \xi_j^+(H_k) = \int_{\Gamma^+} n^+(z)e^{-zH_k}dz, \]
where
\[ n^+(z) = \int_{\gamma^+} \xi_j(v)e^{vz}dv. \]

Here we have used Fubini’s theorem to change the order of integration. Therefore we have
\[\int_{x \in B_j^c} |\xi_j^+(H_k)(f_j)(x)| w_k(x) dx \leq c \int_0^{+\infty} \int_0^{+\infty} |1 - e^{-t_j v}| e^{-|v||z|\sigma} \left( \int_{\mathbb{R}^d} \int_{x \in B_j^c} |H_z(x, y)||f_j(y)| w_k(x) w_k(y) dx dy \right) d|v|d|z| \]

where \( v = |v|e^{i\theta} \) and \( \sigma = \cos(\theta + \beta) \). Let us noting that \( x \in B_j^c \) is equivalent to the condition \( \min_{g \in G} |x - gy| > 2r_j \).

Then using Proposition 5.1 and Lemma 5.10

\[\int_{x \in B_j^c} |H_z(x, y)| w_k(x) dx \leq \int_{\min_{y \in G} |x - gy| > 2r_j} K_{Re(z)}(cx, cy) w_k(x) dx \]

\[= c^{2r_j} \int_{\min_{y \in G} |x - gy| > 2r_j} K_{Re(z)}(x, cy) w_k(x) dx \]

\[\leq C \left( 1 + \frac{r_j}{Re(z)} \right)^{-\delta} \leq C \left( 1 + \frac{r_j}{|z|} \right)^{-\delta}, \]

from which it follows that

\[\int_{x \in B_j^c} |\xi_j^+(H_k)(f_j)(x)| w_k(x) dx \leq C \|f_j\|_{1, k} \int_0^{+\infty} \int_0^{+\infty} |1 - e^{-t_j v}| e^{-|v||z|\sigma} \left( 1 + r_j |z|^{-1} \right)^{-\delta} d|v|d|z|. \]

This integral is treated in [8] by splitting it into two parts, \( I_1 \) and \( I_2 \), corresponding to integration over \( t_j |v| > 1 \) and \( t_j |v| \leq 1 \) and gives

\[\int_{x \in B_j^c} |\xi_j^+(H_k)(f_j)(x)| w_k(x) dx \leq C \|f_j\|_{1, k}. \]

We also obtain similar estimates for \( \xi_j^-(H_k) \) by the same argument. Therefore

\[\mu_k \left\{ x \notin \bigcup_j B_j; \ |\xi(H_k)(h_2)(x)| > \frac{\lambda}{3} \right\} \leq \frac{3}{\lambda} \sum_j \int_{x \in \bigcup_i B_i} |\xi_j(H_k)(f_j)(x)| w_k(x) dx \]

\[\leq \frac{3}{\lambda} \sum_j \int_{x \in B_j^c} |\xi_j(H_k)(f_j)(x)| w_k(x) dx \]

\[\leq c \sum_j \|f\|_{1, k} \leq \frac{c}{\lambda} \|f\|_{1, k}. \]

This archives the proof of the weak type estimates \((1, 1)\) for \( \xi(H_k) \).
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