Abstract

The Novikov-Shubin numbers are defined for open manifolds with bounded geometry, the Γ-trace of Atiyah being replaced by a semicontinuous semifinite trace on the C∗-algebra of almost local operators. It is proved that they are invariant under quasi-isometries and, making use of the theory of singular traces for C∗-algebras developed in [29], they are interpreted as asymptotic dimensions since, in analogy with what happens in Connes’ noncommutative geometry, they indicate which power of the Laplacian gives rise to a singular trace. Therefore, as in geometric measure theory, these numbers furnish the order of infinitesimal giving rise to a non trivial measure. The dimensional interpretation is strengthened in the case of the 0-th Novikov-Shubin invariant, which is shown to coincide, under suitable geometric conditions, with the asymptotic counterpart of the box dimension of a metric space. Since this asymptotic dimension coincides with the polynomial growth of a discrete group, the previous equality generalises a result by Varopoulos [52] for covering manifolds.
0 Introduction.

In a celebrated paper [2], Atiyah observed that on covering manifolds $\Gamma \to M \to X$, a trace on $\Gamma$-periodic operators may be defined, called $\Gamma$-trace, with respect to which the Laplace operator has compact resolvent. Replacing the usual trace with the $\Gamma$-trace, he defined the $L^2$-Betti numbers and proved an index theorem for covering manifolds.

Motivated by this paper, Novikov and Shubin [39] observed that, since for noncompact manifolds the spectrum of the Laplacian is not discrete, new global spectral invariants can be defined, which measure the density near zero of the spectrum.

Novikov-Shubin invariants and other $L^2$-invariants have been a very active research field since then, and the interested reader is referred to [3, 5, 25, 34, 35, 36] for recent developments and extensive bibliographies.

Also based on Atiyah’s paper, Roe defined $L^2$-Betti numbers for open manifolds [44] by replacing the $\Gamma$-trace of Atiyah with a trace on a subalgebra of $L^2(M)$, and showed their invariance under quasi-isometries [46].

In this paper, inspired by Roe [47, 44], we define the $C^*$-algebra of almost local operators and a semicontinuous semifinite trace on it, and use this trace to define Novikov-Shubin numbers for open manifolds, proving that they are invariant under quasi-isometries.

The second part of this paper is concerned with a dimensional interpretation of these numbers.

As it is known, a general understanding of the geometric meaning of the Novikov-Shubin invariants is still lacking. To this end, the definition of these numbers in the case of open manifolds corresponds to the idea that interpreting them as global invariants of an open manifold, rather than as homotopy invariants of a compact one, some aspects can be better understood.

The asymptotic character of these numbers is manifest in two parts of their construction.

On the one hand, the trace used to define these numbers is a large scale trace, since, as observed by Roe [44], it is given by an average on the group, in the case of coverings, and by an average on the exhaustion, in the case of open manifolds.

On the other hand these numbers are defined in terms of the low frequency behaviour of the $p$-Laplacians, or the large time behaviour of the $p$-heat kernel.

In this respect, they are the large scale counterpart of the spectral dimension, namely of the dimension as it is recovered by the Weyl asymptotics. Indeed the inverse of the dimension of a manifold coincides with the order of infinitesimal of $\Delta^{-1/2}$, namely with the order of infinitesimal of the eigenvalue sequence $\mu_n(\Delta^{-1/2})$ when $n \to \infty$. The $p$-th Novikov Shubin invariant $\alpha_p$, instead, coincides with the inverse of the order of infinite, when $t \to 0$, of the generalized eigenvalue sequence $\mu_t(\Delta_p^{-1/2})$.

A fundamental observation of Connes is that integration on a compact Riemannian manifold may be reconstructed by making use of the logarithmic trace and the Weyl asymptotics. Indeed if the resolvent of the Dirac operator is
compact and is an infinitesimal of order $1/d$, then $|D|^{-d}$ is (logarithmically) traceable, and the corresponding singular trace reconstructs the integration on the manifold.

This remark goes in the direction of a noncommutative geometric measure theory. In fact, while in geometric measure theory the dimension is the unique exponent to give to the radius of a ball in order to obtain, using Hausdorff procedure, a (possibly) non-trivial measure, in noncommutative geometry the dimension may be defined as the exponent to give to the resolvent of the Dirac operator in order to obtain a non-trivial singular trace, hence a non-trivial measure on the given space.

Also in this respect the Novikov-Shubin numbers may well be considered asymptotic spectral dimensions. In [28] a new type of singular traces, for continuous semifinite von Neumann algebras, were introduced, the so called singular traces at 0, which measure the divergence at 0 of the generalized eigenvalue function $\mu(t)$ introduced by Fack and Kosaki [23]. Such traces were then defined also in the case of C*-algebras via the noncommutative Riemann integration [29]. We show, in analogy with the local results, that the operator $\Delta_p^{1/2}$, raised to the power $\alpha_p$, is singularly traceable, hence a singular trace is naturally attached to these asymptotic spectral dimensions.

The only Novikov-Shubin number for which a clear geometric interpretation has been given is $\alpha_0$, in fact Lott noted in [33] that a result of Varopoulos [52] immediately implies the equality of $\alpha_0$ with the growth of the fundamental group in the case of covering manifolds. We prove a generalization of this result in the case of open manifolds with bounded geometry and satisfying an isoperimetric inequality introduced by Grigor’yan [26]. As a consequence, we compute the range of $\alpha_0$ for such manifolds to be $[1, \infty)$.

First, we associate a number to any metric space, which we call asymptotic dimension since it is the global analogue of the box dimension defined by Kolmogorov and Tihomirov [32], and show that it is invariant under rough isometries. This shows in particular that the asymptotic dimension of a covering manifold coincides with the growth of the fundamental group.

Then, for the mentioned class of open manifolds, we show that such asymptotic dimension coincides with the 0-th Novikov-Shubin number. This result, besides strengthening our dimensional interpretation of $\alpha_0$, shows a stronger invariance property for it. Indeed the Novikov-Shubin numbers depend on a chosen regular exhaustion of the manifold, hence on its geometry in the large. When the mentioned isoperimetric inequality holds, the volume growth of the manifold is subexponential, and in this case Roe proved [44] that there is a regular exhaustion given by balls with a fixed center. With this natural choice, $\alpha_0$ is invariant under rough isometries.

From the technical point of view, a large part of this paper deals with the problem of defining a semicontinuous semifinite trace on the C*-algebra of almost local operators, i.e. on the norm closure of the operators with finite propagation. Such a trace depends on the geometry in the large of the manifold or, more precisely, on the exhaustion $\mathcal{K}$. The semicontinuity and semifiniteness
properties allow us to use the theory of noncommutative Riemann integration developed in [29], hence to extend the trace to a bigger algebra, containing many projections, and eventually to a bimodule of unbounded operators affiliated to it. Then we can associate a positive non-increasing function, the generalized eigenvalue function $\mu_A$, with any operator $A$ in the bimodule, and define the asymptotic spectral dimension of $(M, \mathcal{K}, \Delta_p)$ as the inverse of the “order of infinite” of $\mu_{\Delta_p^{-1/2}}(t)$, when $t \to 0$, showing that, on the one hand, it coincides with the Novikov-Shubin number $\alpha_p$, and, on the other hand, it produces a noncommutative integration procedure, i.e. a singular trace.

The semicontinuity and semifiniteness properties of our trace constitute a technical simplification in the study of Novikov-Shubin numbers, and are crucial for the possibility of defining type $\text{II}_1$ singular traces for $C^*$-algebras. Our proof of the quasi-isometry invariance of the Novikov-Shubin numbers parallels the corresponding invariance for Betti numbers proved in [46].

The definition of singular traces at 0 for $C^*$-algebras is contained in [29], and is briefly described in section 2.2, in the same subsection the singular traceability of $\Delta_p^{-\alpha_p}$ is proved, making use of a general result contained in [30]. Let us remark that we do not need $\mu_{\Delta_p^{-1/2}}(t)$ to have an exact polynomial behaviour, when $t \to 0$, but only to have a polynomial bound from below. The inverse of $\alpha_p$ then coincides with the exponent of the “optimal” bound from below or, more precisely, with the supremum of the $\omega$ such that $t^{-\omega}$ is a lower bound for sufficiently small $t$. The existence of a polynomial bound from below, namely the positivity of $\alpha_p$, guarantees the singular traceability of $\Delta_p^{-\alpha_p/2}$ regardless of the fact that such operator belongs to $L^1(\tau)$ or not, in particular the singular trace is not necessarily the logarithmic trace introduced by Dixmier [19]. This fact corresponds to the idea that the logarithmic behaviour has to be expected only in the regular cases, such as smooth manifolds, which are locally regular, or covering manifolds, which are regular at large scale, but may fail in the general case.

The definition of asymptotic dimension for metric spaces contained in subsection 3.2 is obtained from the definition of dimension given by Kolmogorov and Tihomirov (often called box dimension), simply replacing limits to 0 with limits to $\infty$ and viceversa. More precisely, if $n(r, R)$ denotes the minimum number of balls of radius $r$ necessary to cover a ball of radius $R$ (and given center), the box dimension is the “order of infinite” of $n(r, R)$ when $r \to 0$ (with $R$ fixed, and often independently of $R$), whereas the asymptotic dimension is the “order of infinite” of $n(r, R)$ when $R \to \infty$ (with $r$ fixed, and often independently of $r$).

Our asymptotic dimension enjoys all the formal properties of a dimension, plus the invariance under rough isometries, which corresponds to its large scale character.

The equality between the asymptotic dimension and the 0-th Novikov-Shubin number is based on the strict relation between the asymptotics of the heat kernel $H_0(t, x, x)$ and the asymptotics of the volume of a ball of radius $\sqrt{t}$, for $t \to \infty$. Such relation is known to hold for open manifolds with some kind
of polynomial growth, such as manifolds with positive Ricci curvature (cf. e.g. [17]) or manifolds satisfying the isoperimetric inequality of Grigor'yan [26].

We finally remark that in the definition of box dimension there are in principle two possible choices, corresponding to the lim sup or to the lim inf procedure in the definition of the order of infinite. But only the lim sup gives rise to the correct behaviour for cartesian products, namely the dimension of the product is not greater than the sum of the dimensions (cf. [12] and [24]).

Also in the definition of the Novikov-Shubin numbers two possible choices are available and again the lim sup was chosen since it guarantees the singular traceability property. It is remarkable that these two independent choices agree, giving rise to the equality between the asymptotic dimension and the 0-th Novikov-Shubin invariant.

Some of the results contained in the present paper have been announced in several international conferences. In particular we would like to thank the Erwin Schrödinger Institute in Vienna, where this paper was completed, and the organisers of the “Spectral Geometry Program” for their kind invitation.

1 A trace for open manifolds

This section is devoted to the construction of a trace on (a suitable subalgebra of) the bounded operators on $L^2(\Lambda^p T^* M)$, where $M$ is an open manifold of bounded geometry. The basic idea for this construction is due to Roe [44], and is based on a regular exhaustion for the manifold. We shall regularize this trace, in order to get a semicontinuous semifinite trace on the C*-algebra of almost local operators. As observed by Roe, this trace is strictly related to the trace constructed by Atiyah [2] in the case of covering manifolds. It may therefore be used to define the Novikov-Shubin invariants for open manifolds, as we do in subsection 1.1.

1.1 Open manifolds of bounded geometry

In this subsection we give some preliminary results on open manifolds of bounded geometry that are needed in the sequel.

Several definitions of bounded geometry for an open manifold (i.e. a noncompact complete Riemannian manifold) are usually considered. They all require some uniform bound (either from above or from below) on some geometric objects, such as: injectivity radius, sectional curvature, Ricci curvature, Riemann curvature tensor etc. (For all unexplained notions see e.g. Chavel’s book [3]).

In this paper the following form is used, but see [1] and references therein for a different approach.

**Definition 1.1.** Let $(M, g)$ be a complete Riemannian manifold. We say that $M$ has $C^\infty$-bounded geometry if it has positive injectivity radius, and the curvature tensor is bounded, together with all its covariant derivatives.
Lemma 1.2. Let $M$ be an $n$-dimensional complete Riemannian manifold with
positive injectivity radius, sectional curvature bounded from above by some con-
stant $c_1$, and Ricci curvature bounded from below by $(n-1)c_2g$, in particular $M$
could have $C^\infty$-bounded geometry. Then there are real functions $\beta_1$, $\beta_2$ s.t.
(i) for all $x \in M$, $r > 0$,
\[ 0 < \beta_1(r) \leq \text{vol}(B(x, r)) \leq \beta_2(r), \]
(ii) $\lim_{r \to 0} \frac{\beta_2(r)}{\beta_1(r)} = 1$.

Proof. (i) We can assume $c_2 < 0 < c_1$ without loss of generality. Then, denoting
with $V_\delta(r)$ the volume of a ball of radius $r$ in a manifold of constant sectional
curvature equal to $\delta$, we can set $\beta_1(r) := V_{c_1}(r \wedge r_0)$, and $\beta_2 := V_{c_2}(r)$, where
$r_0 := \min\{\text{inj}(M), \frac{\pi}{\sqrt{|g|}} \}$, and inj($M$) is the injectivity radius of $M$. Then the
result follows from ([6], p.119,123).

(ii) $\lim_{r \to 0} \frac{\beta_2(r)}{\beta_1(r)} = \lim_{r \to 0} \frac{\int_0^r S_{c_2}(t)^{n-1}dt}{\int_0^r S_{c_1}(t)^{n-1}dt} = 1$

where (cfr. [6], formulas (2.48), (3.24), (3.25)) $V_\delta(r) = \frac{n\sqrt{\pi}}{\Gamma(n/2+1) \int_0^r S_\delta(t)^{n-1}dt}$, and
\[ S_\delta(r) := \begin{cases} \sqrt{-\delta} \sinh(r\sqrt{-\delta}) & \delta < 0 \\ r & \delta = 0 \\ \frac{1}{\sqrt{\delta}} \sin(r\sqrt{\delta}) & \delta > 0 \end{cases} \]

Proposition 1.3. Let $M$ be an $n$-dimensional complete Riemannian manifold with
$C^\infty$ bounded geometry, then for all $T > 0$, there are $c, c' > 0$, s.t., for
$0 < t \leq T$,
\[ |H_p(t, x, y)| \leq c t^{-n/2-1} \exp \left( -c'\delta(x, y)^2 \frac{1}{t} \right) \]
\[ |\nabla_x H_p(t, x, y)| \leq c t^{-n/2-3/2} \exp \left( -c'\delta(x, y)^2 \frac{1}{t} \right) \]
where we denoted with $\delta$ the metric induced on $M$ by $g$. As a consequence
$H_p(t, \cdot, \cdot)$ is uniformly continuous on a neighborhood of the diagonal of $M \times M$. 
Proof. The estimates are proved in [4]. For the last statement, for any \( \delta_0 < \min \{1, \text{inj}(M), \frac{1}{\sqrt{|C|}} \} \), \( x \in M, y \in B(x, \delta_0) \), we have \( |H_p(t, x, y) - H_p(t, x, x)| \leq \sup |\nabla_y H_p(t, x, y)| \delta(x, y) \), and we get the uniform continuity. \( \square \)

1.2 The \( C^* \)-algebra of almost local operators

Let \( F \) be a finite dimensional Hermitian vector bundle over \( M \), and let \( L^2(F) \) be the Hilbert space completion of the smooth sections with compact support of \( F \) w.r.t. the scalar product \( \langle s_1, s_2 \rangle := \int_M \langle s_1(x), s_2(x) \rangle dv(x) \).

Recall [17] that an operator \( A \in \mathcal{B}(L^2(F)) \) has finite propagation if there is a constant \( u_A > 0 \) s.t. for any compact subset \( K \) of \( M \), any \( \varphi \in L^2(F) \), \( \text{supp} \varphi \subset K \), we have \( \text{supp} A\varphi \subset \text{Pen}^+(K, u_A) := \{ x \in M : \delta(x, K) \leq u_A \} \).

Let us denote by \( A_0 \equiv A_0(F) \) the set of finite propagation operators. \( A_0 \) may be characterized as follows

Proposition 1.4.

(i) \( A \in A_0 \) iff, for any measurable set \( \Omega \), \( AE_{\Omega} = E_{\text{Pen}^+(\Omega, u_A)}AE \), where \( E_X \) is the multiplication operator by the characteristic function of the set \( X \);

(ii) \( A \in A_0 \) iff, for any functions \( \varphi, \psi \in L^2(F) \) with \( \delta(\text{supp} \psi, \text{supp} \varphi) > u_A \), one has \( (\varphi, A\psi) = 0 \).

Proof. Properties (i) and (ii) \((\Rightarrow)\) are obvious.

(ii) \((\Leftarrow)\) The hypothesis implies that \( \text{supp} A\psi \subset M \setminus \text{supp} \varphi \) for all \( \varphi \) s.t. \( \text{supp} \varphi \subset M \setminus \text{Pen}^+(\text{supp} \psi, u_A) \). The thesis follows. \( \square \)

Proposition 1.5. The set \( A_0(F) \) of finite propagation operators is a \( * \)-algebra with identity.

Proof. Let \( K \) be a compact subset of \( M \), \( \varphi \in L^2(F) \), \( \text{supp} \varphi \subset K \), and \( A, B \in A_0 \). Then \( \text{supp}(A + B) \varphi \subset \text{supp} A \varphi \cup \text{supp} B \varphi \), hence \( u_{A+B} = u_A \vee u_B \) is the requested constant. Moreover \( \text{supp}(AB) \varphi \subset \text{Pen}^+(\text{supp} B \varphi, u_A) \subset \text{Pen}^+(K, u_A + u_B) \), so that we may set \( u_{AB} = u_A + u_B \).

As \( (A^* \psi, \varphi) = (\psi, A \varphi) = 0 \) for all \( \varphi, \psi \in L^2(F) \), with \( \delta(\text{supp} \psi, \text{supp} \varphi) > u_A \), that is \( \text{supp} \varphi \cap \text{Pen}^+(\text{supp} \psi, u_A) = \emptyset \), we get \( u_{A^* \psi} \subset \text{Pen}^+(\text{supp} \psi, u_A) \), which implies \( u_{A^*} \leq u_A \), and exchanging the roles of \( A, A^* \), we get \( u_A = u_{A^*} \). \( \square \)

The norm closure of \( A_0 \) will be denoted by \( A \equiv A(F) \) and will be called the \( C^* \)-algebra of almost local operators on \( L^2(F) \). Now we show that Gaussian decay for the kernel of a positive operator \( A \) is a sufficient condition for \( A \) to belong to \( A \).

Theorem 1.6. Let \( M \) be a complete Riemannian \( n \)-manifold of \( C^\infty \) bounded geometry. If \( A \) is a bounded self-adjoint operator on \( L^2(F) \), with kernel \( a(x, y) \in \)
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$L(F_y, F_x)$, and there are positive constants $c$, $\alpha$, $\delta_0$ s.t.; for $\delta(x, y) \geq \delta_0$, $a(x, y)$ is measurable and

$$|a(x, y)| \leq c \, e^{-\alpha \delta(x, y)^2}$$

then $A \in A$.

In order to prove the theorem, we need some lemmas.

**Lemma 1.7.** Let $A$ be a bounded self-adjoint operator on $L^2(F)$, with measurable kernel. Then

$$\|A\| \leq \sup_{x \in M} \int_M |a(x, y)| dy$$

**Proof.** Since $A$ is self-adjoint, $a(x, y)$ is symmetric, hence

$$\|A\|_{1 \to 1} = \sup \{|(f, Ag) : f \in L^\infty(F), \|f\|_{\infty} = 1, g \in L^1(F), \|g\|_1 = 1\}$$

$$\leq \sup_{x \in M} \int_M |a(y, x)| dy = \|A\|_{\infty \to \infty}$$

The thesis easily follows from Riesz-Thorin interpolation theorem.

**Lemma 1.8.** Let $\varphi : [0, \infty) \to [0, \infty)$ be a non-increasing measurable function. Then, using notation of subsection 1.1,

$$\sup_{x \in M} \int_M \varphi(\delta(x, y)) dy \leq C_n \int_0^\infty \varphi(r) S_{c_2}(r)^{n-1} dr$$

where $C_n := \frac{n \sqrt{\pi}}{\Gamma(n/2 + 1)}$, and $S_{c_2}(r) := \frac{1}{\sqrt{c_2}} \sinh(r \sqrt{-c_2})$.

**Proof.** From Theorem 1.2 we get $V(x, r) \leq C_n \int_0^r S_{c_2}(t)^{n-1} dt$. Then

$$\int_M \varphi(\delta(x, y)) dy = \int_0^\infty \varphi(r) dV(x, r)$$

$$\leq C_n \int_0^\infty \varphi(r) S_{c_2}(r)^{n-1} dr$$

where the equality is in e.g. (31), Theorem 12.46), and the inequality holds because $\varphi$ is non-increasing and positive, and $V(x, 0) = 0$.

**Proof of Theorem 1.4.** Let $\rho > \delta_0$, and decompose $A = A_\rho + A'_\rho$, with $a_\rho(x, y) := a(x, y) \chi_{[0, \rho]}(\delta(x, y))$. Then $A_\rho \in A_0$, and $|a'_\rho(x, y)| \leq c' \varphi(\delta(x, y))$, where

$$\varphi(r) := \begin{cases} e^{-\alpha r^2} & 0 \leq r < \rho \\ e^{-\alpha \rho^2} & r \geq \rho. \end{cases}$$
By Lemmas 1.7, 1.8 we get
\[
\|A - A_0\| = \|A'_0\| \leq \sup_{x \in M} \int_M |a'_0(x,y)| dy
\]
\[
\leq c' \sup_{x \in M} \int_M \varphi(\delta(x,y)) dy
\]
\[
\leq c' \int_0^\infty \varphi(r) S_{e^2}(r)^{n-1} dr
\]
\[
\leq c' e^{-\alpha r^2} \int_0^\rho S_{e^2}(r)^{n-1} dr + c'' \int_\rho^\infty e^{-\alpha r^2 + (n-1) r \sqrt{e^2}} dr
\]
\[
\to 0, \quad \rho \to \infty
\]
and the thesis follows.

Applying the previous Theorem we conclude that $C_0$ functional calculus of the Laplace operator belongs to $\mathcal{A}$.

**Corollary 1.9.** Let $M$ be a complete Riemannian manifold of $C^\infty$ bounded geometry. Then $\varphi(\Delta_p) \in \mathcal{A}(\Lambda^pT^*M)$, for any $\varphi \in C_0([0,\infty))$.

**Proof.** By Proposition 1.3 and Theorem 1.6 we obtain that $e^{-t\Delta_p} \in \mathcal{A}$, for any $t > 0$. Since $\{e^{-t\Delta_p}\}_{t>0}$ generates a $*$-algebra of $C_0([0,\infty))$ which separates points, the thesis follows by Stone-Weierstrass theorem.

**Proposition 1.10.**

(i) $\mathcal{A}$ contains all compact operators, so that $\mathcal{A}'' = \mathcal{B}(\mathcal{H})$

(ii) $\mathcal{A} = \mathcal{B}(\mathcal{H})$ iff $M$ is a closed manifold.

**Proof.** (i) It suffices to prove that $\mathcal{A}$ contains all one-dimensional projections. To begin with, let $e$ be the projection operator onto the multiples of $\varphi \in C_0(\Lambda^pT^*M)$ with $L^2$-norm 1, so that its Schwartz kernel is $K(x,y) := \varphi(x) \otimes \varphi(y)$. Consider the operators $e_n$ with Schwartz kernel $K_n(x,y) := \chi_{B(o,n)}(x)\varphi(x) \otimes \chi_{B(o,n)}(y)\varphi(y)$. Then it follows from Lemma 1.7 that

\[
\|e - e_n\| \leq \sup_{x \in M} \int_M |\varphi(x)\varphi(y) - \chi_{B(o,n)}(x)\varphi(x)\chi_{B(o,n)}(y)\varphi(y)| dy
\]
\[
= \sup_{x \in M} |\varphi(x)| \int_M |\varphi(y)|(1 - \chi_{B(o,n)}(x)\chi_{B(o,n)}(y)) dy
\]
\[
\leq \max_{x \in B(o,n)} |\varphi(x)| \int_{B(o,n)^C} |\varphi(y)| dy, \quad \sup_{x \in B(o,n)^C} |\varphi(x)| \int_M |\varphi(y)| dy \to 0.
\]

Therefore $e \in \mathcal{A}$. Let now $\varphi \in L^2(\Lambda^pT^*M)$, with $\|\varphi\|_2 = 1$, and let $\{\varphi_n\} \subset C_0(\Lambda^pT^*M)$ s.t. $\|\varphi_n\|_2 < 1/n$. Denote by $e$, resp. $e_n$, the projection operator onto the multiples of $\varphi$, resp. $\varphi_n$. Then $e_n \to e$, so that $e \in \mathcal{A}$.

(ii) If $M$ is closed the statement is trivial. Conversely let $M$ be open and $F$
be the trivial bundle, as we may reduce to this case. Then choose a sequence \( \{x_n\} \) in \( M \) s.t. \( \delta(x_i, x_0) \geq 3i, \) and \( \delta(x_i, x_j) \geq 3, \) \( i \neq j. \) Then \( f_i := \frac{\chi_{B(x_i, \delta)}}{V(x_i, \delta)} \) is an orthonormal set in \( L^2(M). \) The operator \( T := (f_0, \cdot) \sum_{i=1}^{\infty} f_i \) has norm 1, but \( T \not\in A, \) because \( \|T - A\| \geq 1 \) for any \( A \in A_0. \)

**1.3 A functional described by J. Roe**

In the rest of this paper \( M \) is a complete Riemannian \( n \)-manifold of \( C^\infty \) bounded geometry as in Definition 1.7, that we assume endowed with a regular exhaustion \( K \) that is an increasing sequence \( \{K_n\} \) of compact subsets of \( M, \) whose union is \( M, \) and s.t., for any \( r > 0 \)

\[
\lim_{n \to \infty} \frac{\text{vol}(K_n(r))}{\text{vol}(K_n(-r))} = 1,
\]

where we set, here and in the following, \( K(r) \equiv \text{Pen}^+(K, r) := \{x \in M : \delta(x, K) \leq r\}, \) and \( K(-r) \equiv \text{Pen}^-(K, r) := \text{the closure of } M \setminus \text{Pen}^+(M \setminus K, r). \)

Observe that, as \( M \) is complete, \( \text{Pen}^+(K, r) \) coincides with the closure of \( \{x \in M : \delta(x, K) < r\}, \) which is the original definition of Roe.

**Lemma 1.11.** Let \( K \) be a compact subset of \( M, \) then

(i) \( K(-r_2) \subset K \subset K(r_1), \) for any \( r_1, r_2 > 0 \)

(ii) \( \{x \in M : \delta(x, M \setminus K) < r\} \subset \text{Interior of } \text{Pen}^+(M \setminus K, r) \equiv M \setminus K(-r) \)

(iii) \( \text{Pen}^+(K(r_1) \setminus K(-r_2), R) \subset K(r_1 + R + \varepsilon) \setminus K(-r_2 - R - \varepsilon), \) for any \( r_1, r_2, R, \varepsilon > 0. \)

**Proof.** (ii) If \( \delta(x, M \setminus K) < r, \) there is \( z \in M \setminus K \) s.t. \( \delta(x, z) < r, \) so that \( x \) belongs to the interior of \( \text{Pen}^+(M \setminus K, r), \) which is the complement of \( K(-r). \)

(iii) Indeed if \( x \in \text{Pen}^+(K(r_1) \setminus K(-r_2), R), \) then for any \( \varepsilon > 0 \) there is \( x_\varepsilon \in K(r_1) \setminus K(-r_2) \) with \( \delta(x, x_\varepsilon) < R + \varepsilon/2. \) Therefore, on the one hand, \( \delta(x, K) \leq R + \varepsilon/2 + r_1, \) which implies \( x \in K(r_1 + R + \varepsilon). \) On the other hand, as \( x_\varepsilon \not\in K(-r_2), \) there is \( y_\varepsilon \in M \setminus K \) s.t. \( \delta(y_\varepsilon, x_\varepsilon) < r_2 + \frac{\varepsilon}{2}, \) hence \( \delta(x, y_\varepsilon) \leq \frac{\varepsilon}{2} + R + r_2 + \frac{\varepsilon}{2} \) and \( x \not\in K(-r_2 - R - \varepsilon). \)

Following Moore-Schochet [37], we recall that an operator \( T \) on \( L^2(F) \) is called locally trace class if, for any compact set \( K \subset M, \) \( E_KTE_K \) is trace class, where \( E_K \) denotes the projection given by the characteristic function of \( K. \) It is known that the functional \( \mu_T(K) := Tr(E_KTE_K) \) extends to a Radon measure on \( M. \) To state next definition we need some preliminary notions.

**Definition 1.12.** Define \( \mathcal{J}_{0+} \equiv \mathcal{J}_{0+}(F) \) as the set of positive locally trace class operators \( T, \) such that

(i) there is \( c > 0 \) s.t. \( \mu_T(K_n) \leq c \text{vol}(K_n), \) asymptotically,

(ii) \( \lim_{n \to \infty} \frac{\mu_T(K_n(r_1) \setminus K_n(-r_2))}{\text{vol}(K_n)} = 0. \)

**Lemma 1.13.** \( \mathcal{J}_{0+} \) is a hereditary (positive) cone in \( \mathcal{B}(L^2(F)). \)
Proof. Linearity follows by $\mu_{A+B} = \mu_A + \mu_B$. If $T \in \mathcal{D}_0^+$, and $0 \leq A \leq T$, then $\text{Tr}(BAB^*) \leq \text{Tr}(TB B^*)$, for any $B \in \mathcal{B}(L^2(F))$, and the thesis follows. 

**Remark 1.14.** The hereditary cone $\mathcal{D}_0^+$ depends on the exhaustion $\mathcal{K}$, however it contains a (hereditary) subcone, given by the operators $T$ for which there is $c > 0$ such that $\mu_T(\Omega) \leq c \text{ vol}(\Omega)$ for any measurable set $\Omega$. Proposition 1.3 implies that the operator $e^{-t\Delta_p}$ belongs to the subcone, hence to $\mathcal{D}_0^+ (\Lambda p T^* M)$.

Recall [44] that $\mathcal{U}_{-\infty}(F)$ is the set of uniform operators of order $-\infty$.

**Proposition 1.15.** $\mathcal{U}_{-\infty}(F)_+ \subset \mathcal{D}_0^+(F)$.

Proof. Let $A \in \mathcal{U}_{-\infty}(F)$, so that $Au(x) = \int_M a(x,y)u(y)dy$, with $a \in C^\infty(F \otimes F)$ is a smoothing kernel, and is uniformly bounded together with all its co-variant derivatives ([44], 2.9). Then for any Borel set $\Omega \subset M$, $\mu_A(\Omega) = \text{Tr}(E_{\Omega} AE_{\Omega}) = \int_\Omega \text{tr}(a(x,x))dx \leq c \text{ vol}(\Omega)$, and the result easily follows.

If $\omega$ is a state on $\ell^\infty(N)$ vanishing on infinitesimal sequences, we use in the following the notation $\text{Lim}_\omega a_n := \omega(\{a_n\})$, for any $\{a_n\} \in \ell^\infty(N)$. Consider the weight $\varphi = \varphi_{\mathcal{K}, \omega}$ on $\mathcal{B}(L^2(F))_+$ given by

$$\varphi(A) := \begin{cases} \text{Lim}_\omega \frac{\mu_A(K_n)}{\text{vol}(K_n)} & A \in \mathcal{D}_0^+ \\ +\infty & A \in \mathcal{B}(L^2(F))_+ \setminus \mathcal{D}_0^+ . \end{cases}$$

Observe that the functional $\varphi$ is the functional defined by Roe in [44], but for the domain.

**Proposition 1.16.** For any $A \in \mathcal{U}_{-\infty}(F)_+$, $\varphi(A) = \text{Lim}_\omega \frac{\int_M \text{tr}(a(x,x))dx}{\text{vol}(K_n)}$, which is Roe’s definition in [44].

Proof. Follows easily from the proof of Proposition 1.15.

**Lemma 1.17.** If $A \in \mathcal{D}_0^+$ then

$$\varphi(A) = \text{Lim}_\omega \frac{\mu_A(K_n(r_1))}{\text{vol}(K_n(r_2))}$$

for any $r_1, r_2 \in \mathbb{R}$.

Proof. Indeed, if $r_1 \geq 0$, we get

$$\text{Lim}_\omega \frac{\mu_A(K_n(r_1))}{\text{vol}(K_n(r_2))} = \text{Lim}_\omega \left( \frac{\mu_A(K_n)}{\text{vol}(K_n)} + \frac{\mu_A(K_n \setminus K_n)}{\text{vol}(K_n)} \right) \frac{\text{vol}(K_n)}{\text{vol}(K_n(r_2))} = \varphi(A)$$

whereas, if $r_1 < 0$, we get

$$\text{Lim}_\omega \frac{\mu_A(K_n(r_1))}{\text{vol}(K_n(r_2))} = \text{Lim}_\omega \left( \frac{\mu_A(K_n)}{\text{vol}(K_n)} - \frac{\mu_A(K_n \setminus K_n(r_1))}{\text{vol}(K_n)} \right) \frac{\text{vol}(K_n)}{\text{vol}(K_n(r_2))} = \varphi(A).$$
The algebra $A$, being a $C^*$-algebra, contains many unitary operators, and is indeed generated by them. The algebra $A_0$ may not, but all unitaries in $A$ may be approximated by elements in $A_0$. Such approximants are $\delta$-unitaries, according to the following

**Definition 1.18.** An operator $U \in \mathcal{B}(L^2(F))$ is called $\delta$-unitary, $\delta > 0$, if $\|U^*U - 1\| < \delta$, and $\|UU^* - 1\| < \delta$.

Let us denote with $U_\delta$ the set of $\delta$-unitaries in $A_0$ and observe that, if $\delta < 1$, $U_\delta$ consists of invertible operators, and $U \in U_\delta$ implies $U^{-1} \in U_{\delta/(1-\delta)}$.

**Proposition 1.19.** The weight $\varphi$ is $\varepsilon$-invariant for $\delta$-unitaries in $A_0$, namely, for any $\varepsilon \in (0, 1)$, there is $\delta > 0$ s.t., for any $U \in U_\delta$, and $A \in A_+$,

$$(1 - \varepsilon)\varphi(A) \leq \varphi(UAU^*) \leq (1 + \varepsilon)\varphi(A).$$

**Lemma 1.20.** If $T \in \mathcal{J}_{0+}$, then $ATA^* \in \mathcal{J}_{0+}$ for all $A \in A_+$.

**Proof.** First observe that for any Borel set $\Omega \subset M$ we have

$$\mu_{ATA^*}(\Omega) = Tr(E_{\Omega}ATA^*E_{\Omega}) = Tr(E_{\Omega}AE_{\Omega(uA)}TE_{\Omega(uA)}A^*E_{\Omega}) \leq \|A^*E_{\Omega}\| Tr(E_{\Omega(uA)}TE_{\Omega(uA)})) \leq \|A\|^2 \mu_T(\Omega(uA))$$

so that

$$\frac{\mu_{ATA^*}(K_n)}{\text{vol}(K_n)} \leq \frac{\|A\|^2 \mu_T(K_n(uA))}{\text{vol}(K_n)} = \frac{\|A\|^2 \mu_T(K_n) + \|A\|^2 \mu_T(K_n \setminus K_n)}{\text{vol}(K_n)}$$

which is asymptotically bounded. Now observe that, by Lemma 1.11 (iii), it follows

$$\frac{\mu_{ATA^*}(K_n(r_1) \setminus K_n(-r_2))}{\text{vol} K_n} \leq \|A\|^2 \frac{\mu_T(Pen^+(K_n(r_1) \setminus K_n(-r_2), u_A))}{\text{vol} K_n} \leq \|A\|^2 \frac{\mu_T(K_n(r_1 + u_A + \varepsilon) \setminus K_n(-r_2 - u_A - \varepsilon))}{\text{vol} K_n} \to 0,$$

the thesis follows.

**Proof of Proposition 1.19.** Assume $A \in \mathcal{J}_{0+} \cap A_+$, then $UAU^* \in \mathcal{J}_{0+}$ and, by Lemma 1.17,

$$\varphi(UAU^*) = \lim_\omega \frac{\mu_{UAU^*}(K_n)}{\text{vol}(K_n)} \leq \|U\|^2 \lim_\omega \left(\frac{\mu_A(K_n(\omega U)))}{\text{vol}(K_n)}\right) \leq (1 + \delta)\varphi(A).$$
Choose now $\delta < \varepsilon/2$, and $U \in \mathcal{U}_\delta$, so that $U^{-1} \in \mathcal{U}_{2\delta}$, and $\varphi(UAU^*) \leq (1+\delta)\varphi(A) < (1+\varepsilon)\varphi(A)$. Replacing $A$ with $UAU^*$, and $U$ with $U^{-1}$, we obtain

$$\varphi(A) \leq (1 + \delta)\varphi(UAU^*) \leq (1 + 2\delta)\varphi(UAU^*) < (1 + \varepsilon)\varphi(UAU^*)$$

and the thesis easily follows.

Finally we observe that, from the proof of Lemma 1.20 the following is immediately obtained

Lemma 1.21. If $A \in \mathcal{A}_0$ and $\|A\| \leq 1$, then $\varphi(ATA^*) \leq \varphi(T)$, for any $T \in \mathcal{J}_0^+$. \\

1.4 A construction of semicontinuous traces on C*-algebras

The purpose of this subsection is to show that the lower-semicontinuous semifinite regularisation of the functional $\varphi|_A$ of the previous subsection gives a trace, namely a unitarily invariant weight on $A$. It turns out that this procedure can be applied to any weight $\tau_0$, on a unital C*-algebra $A$, which is $\varepsilon$-invariant for $\delta$-unitaries of a dense $^*$-subalgebra $A_0$. The particular case of the functional $\varphi|_A$ is treated in the next subsection. First we observe that, with each weight on $A$, namely a functional $\tau_0 : \mathcal{A}_+ \to [0, \infty]$, satisfying the property $\tau_0(\lambda A + B) = \lambda \tau_0(A) + \tau_0(B)$, $\lambda > 0$, $A, B \in \mathcal{A}_+$, we may associate a (lower-)semicontinuous weight $\tau$ with the following procedure

$$\tau(A) := \sup\{\psi(A) : \psi \in \mathcal{A}_+^+, \psi \leq \tau_0\} \tag{1.1}$$

Indeed, it is known that

$$\tau(A) \equiv \sup_{\psi \in \mathcal{F}(\tau_0)} \psi(A)$$

where $\mathcal{F}(\tau_0) := \{\psi \in \mathcal{A}_+^+ : \exists \varepsilon > 0, (1+\varepsilon)\psi < \tau_0\}$. Moreover the following holds

Theorem 1.22. The set $\mathcal{F}(\tau_0)$ is directed, namely, for any $\psi_1, \psi_2 \in \mathcal{F}(\tau_0)$, there is $\psi \in \mathcal{F}(\tau_0)$, s.t. $\psi_1, \psi_2 \leq \psi$.

From this theorem easily follows

Corollary 1.23. Let $\tau_0$ be a weight on the C*-algebra $A$, and $\tau$ be defined as in (1.1). Then

(i) $\tau$ is a semicontinuous weight on $A$

(ii) $\tau = \tau_0$ iff $\tau_0$ is semicontinuous.

(iii) The domain of $\tau$ contains the domain of $\tau_0$. The weight $\tau$ will be called the semicontinuous regularization of $\tau_0$. 

Proof. (i) From Theorem \[\ref{prop:semicontinuous-weight}, \tau(A) = \sup_{\psi \in \mathcal{T}(\tau_0)} \psi(A) = \lim_{\psi \in \mathcal{T}(\tau_0)} \psi(A),\]
whence linearity and semicontinuity of \(\tau\) easily follow.

(ii) is a well known result by Combes \[\ref{combes-1970}].

(iii) Immediately follows from the definition of \(\tau\).

\[\tag*{\Box}\]

**Proposition 1.24.** Let \(\tau_0\) be a weight on \(A\) which is \(\varepsilon\)-invariant by \(\delta\)-unitaries in \(A_0\) (as in Proposition \[\ref{prop:unitary-invariant-weight}\]). Then the associated semicontinuous weight \(\tau\) satisfies the same property.

*Proof.* Fix \(\varepsilon < 1\) and choose \(\delta \in (0, 1/2)\), s.t. \(U \in \mathcal{U}_\delta\) implies \(|\tau_0(UAUT) - \tau_0(A)| < \varepsilon \tau_0(A), \ A \in A_+.\) Then, for any \(U \in \mathcal{U}_\delta\) and any \(\psi \in A_+^*, \psi \leq \tau_0,\) we get

\[
\psi \circ \text{ad}U(A) \leq \tau_0(UAUT) \leq (1 + \varepsilon) \tau_0(A),
\]

for \(A \in A_+,\ i.e. (1 + \varepsilon)^{-1} \psi \circ \text{ad}U \leq \tau_0.\) Then

\[
\tau(UAUT) = (1 + \varepsilon) \sup_{\psi \leq \tau_0} (1 + \varepsilon)^{-1} \psi \circ \text{ad}U(A)
\]

\[
\leq (1 + \varepsilon) \sup_{\psi \leq \tau_0} \psi(A)
\]

\[
= (1 + \varepsilon) \tau(A).
\]

Since \(U^{-1} \in \mathcal{U}_\delta,\) replacing \(U\) with \(U^{-1}\) and \(A\) with \(UAU^*,\) we get \(\tau(A) \leq (1 + \varepsilon) \tau(UAUT).\) Combining the last two inequalities, we get the result. \[\tag*{\Box}\]

**Proposition 1.25.** The semicontinuous weight \(\tau\) is a trace on \(A,\) namely, setting \(\mathcal{J}_+ := \{A \in A_+ : \tau(A) < \infty\},\) and extending \(\tau\) to the linear span \(\mathcal{J}\) of \(\mathcal{J}_+\), we get

(i) \(\mathcal{J}\) is an ideal in \(A\)

(ii) \(\tau(AB) = \tau(BA),\) for all \(A \in \mathcal{J}, B \in A.\)

*Proof.* (i) Let us prove that \(\mathcal{J}_+\) is a unitary invariant face in \(A_+,\) and it suffices to prove that \(A \in \mathcal{J}_+\) implies \(UAU^* \in \mathcal{J}_+,\) for all \(U \in \mathcal{U}(A),\) the set of unitaries in \(A.\) Suppose on the contrary that there is \(U \in \mathcal{U}(A)\) s.t. \(\tau(UAUT) = \infty.\) Then there is \(\psi \in A^*_+\), \(\psi \leq \tau_0,\) s.t. \(\psi(UAUT) > 2\tau(A) + 2.\) Then we choose \(\delta < 3\) s.t. \(V \in \mathcal{U}_\delta\) implies \(\tau(VAV^*) \leq 2\tau(A),\) and an operator \(U_0 \in A_0\) s.t. \(\|U - U_0\| < \min\{\frac{\delta}{4}, \frac{1}{\psi(A)}\}\). The inequalities

\[
\|U_0U_0 - 1\| = \|U_0U_0 - U^*\| \leq \|U^*U_0 - 1\| \|U_0\| + \|U_0^* - U^*\| < \delta
\]

and analogously for \(\|U_0^*U_0 - 1\| < \delta,\) show that \(U_0 \in \mathcal{U}_\delta.\) Then, since \(|\psi(U_0AUT^*) - \psi(UAUT^*)| \leq 3\|\psi\| \|A\| \|U - U_0\| < 1,\) we get

\[
2\tau(A) \geq \tau(U_0AUT_0^*) \geq \psi(U_0AUT_0^*) \geq \psi(UAUT^*) - 1 \geq 2\tau(A) + 1
\]

which is absurd.

(ii) We only have to show that \(\tau\) is unitary invariant. Take \(A \in \mathcal{J}_+, U \in \mathcal{U}(A).\)
For any \( \varepsilon > 0 \) we may find a \( \psi \in \mathcal{A}_+^1, \psi \leq \tau_0 \), s.t. \( \psi(UAU^*) > \tau(UAU^*) - \varepsilon \), as, by \( (i) \), \( \tau(UAU^*) \) is finite. Then, arguing as in the proof of \( (i) \), we may find \( U_0 \in \mathcal{A}_0 \), so close to \( U \) that

\[
|\psi(U_0AU_0^*) - \psi(UAU^*)| < \varepsilon \\
(1 - \varepsilon)\tau(A) \leq \tau(U_0AU_0^*) \leq (1 + \varepsilon)\tau(A).
\]

Then

\[
\tau(A) \geq \frac{1}{1 + \varepsilon} \tau(U_0AU_0^*) \geq \frac{1}{1 + \varepsilon} \psi(U_0AU_0^*) \\
\geq \frac{1}{1 + \varepsilon} (\psi(UAU^*) - \varepsilon) \geq \frac{1}{1 + \varepsilon} (\tau(UAU^*) - 2\varepsilon).
\]

By the arbitrariness of \( \varepsilon \) we get \( \tau(A) \geq \tau(UAU^*) \). Replacing \( A \) with \( UAU^* \), we get the thesis.

The second regularization we need turns \( \tau \) into a (lower semicontinuous) semifinite trace, namely guarantees that

\[
\tau(A) = \sup \{ \tau(B) : 0 \leq B \leq A, B \in \mathcal{J}_+ \}
\]

for all \( A \in \mathcal{A}_+ \). In particular the semifinite regularization coincides with the original trace on the domain of the latter. This regularization is well known (see e.g. [20], Section 6), and amounts to represent \( \mathcal{A} \) via the GNS representation \( \pi \) induced by \( \tau \), define a normal semifinite faithful trace \( \text{tr} \) on \( \pi(A)^\prime\prime \), and finally pull it back on \( \mathcal{A} \), that is \( \text{tr} \circ \pi \). It turns out that \( \text{tr} \circ \pi \) is (lower semicontinuous and) semifinite on \( \mathcal{A} \), \( \text{tr} \circ \pi \leq \tau \), and \( \text{tr} \circ \pi(A) = \tau(A) \) for all \( A \in \mathcal{J}_+ \), that is \( \text{tr} \circ \pi \) is a semifinite extension of \( \tau \), and \( \text{tr} \circ \pi = \tau \) iff \( \tau \) is semifinite.

We still denote by \( \tau \) its semifinite extension. As follows from the construction, semicontinuous semifinite traces are exactly those of the form \( \text{tr} \circ \pi \), where \( \pi \) is a tracial representation, and \( \text{tr} \) is a n.s.f. trace on \( \pi(A)^\prime\prime \).

### 1.5 The regularized trace on the C*-algebra of almost local operators

Now we apply the regularization procedure described in the previous subsection to Roe’s functional. Let us remark that the semicontinuous regularization of the weight \( \varphi|_{\mathcal{A}} \) is a trace in the sense of property \( (ii) \) of Proposition [1.23], which is stronger then the trace property in [14]. First we observe that \( \varphi|_{\mathcal{A}} \) is not semicontinuous.

**Proposition 1.26.** The set \( \mathcal{N}_0 := \{ T \in \mathcal{A}_+ : \varphi(T) = 0 \} \) is not closed. In particular, there are operators \( T \in \mathcal{A}_+ \) s.t. \( \varphi(T) = 1 \) but \( \tau(T) = 0 \) for any (lower-)semicontinuous trace \( \tau \) dominated by \( \varphi|_{\mathcal{A}} \).

**Proof.** Recall from Lemma [1.2](i) that there are positive real functions \( \beta_1, \beta_2 \) s.t. \( 0 < \beta_1(r) \leq V(x,r) \leq \beta_2(r) \), for all \( x \in M, r > 0 \), and \( \lim_{r \to 0} \beta_2(r) = 0. \)
Therefore we can find a sequence $r_n \downarrow 0$ s.t. $\sum_{n=1}^{\infty} \beta_2(r_n) < \infty$. Fix $o \in M$, and set $X_n := \{(x_1, x_2) \in M \times M : n \leq \delta(x_i, o) \leq n + 1, \delta(x_1, x_2) \leq r_n\}$, $Y_n := \bigcup_{k=1}^{n} X_k$, $n \leq \infty$, and finally let $T_n$ be the integral operator whose kernel, a section of $\text{End}(F)$ denoted $k_n$, is the characteristic function of $Y_n$. Since $k_n$ has compact support, if $n < \infty$, $\varphi(T_n) = 0$. On the contrary, since $Y_\infty$ contains the diagonal of $M \times M$, clearly $\varphi(T_\infty) = 1$. Finally

\[ \|T_\infty - T_n\| \leq \sup_{x \in M} \int_M \chi_{\cup_{k=n+1}^{\infty} X_k}(x, y) dy \leq \sup_{x \in M} \sum_{k=n+1}^{\infty} \int_M \chi_{X_k}(x, y) dy \leq \sup_{x \in M} \sum_{k=n+1}^{\infty} V(x, r_k) \leq \sum_{k=n+1}^{\infty} \beta_2(r_k) \to 0. \]

This proves both the assertions.

**Definition 1.27.** Denote by $\text{Tr}_\chi$ the lower-semicontinuous semifinite trace on $A(F)$ obtained by regularising $\varphi|_A$, as in the previous subsection.

**Proposition 1.28.** $\text{Tr}_\chi$ vanishes on compact operators.

**Proof.** If $e$ is the one-dimensional projection onto the multiples of $f \in L^2(F)$, $\varphi(e) = \lim_{\omega} \frac{\text{Tr}(E_{K_n} eE_{K_n})}{\text{vol}_{K_n}} = \lim_{\omega} \frac{\int_{K_n} |\varphi(x)|^2 dx}{\text{vol}_{K_n}} = 0$. Therefore $0 \leq \text{Tr}_\chi(e) \leq \varphi(e) = 0$, and $\text{Tr}_\chi(T) = 0$ for any positive finite rank operator. Let now $T$ be a compact operator, so that $T = U|T|$, and $|T|$ is the norm limit of a sequence $S_n$ of positive finite rank operators. Then $0 \leq |\text{Tr}_\chi(T)| \leq \|U\| |\text{Tr}_\chi(|T|)| \leq \liminf \text{Tr}_\chi(S_n) = 0$.

Finally we give a sufficient criterion for a positive operator $A \in A$ to satisfy $\text{Tr}_\chi(A) = \varphi(A)$.

**Proposition 1.29.** Let $A \in \mathcal{A}_0$ be an integral operator, whose kernel $a(x, y)$ is a section of $\text{End}(F)$ which is uniformly continuous in a neighborhood of the diagonal in $M \times M$, namely

\[ \forall \varepsilon > 0, \exists \delta > 0 : \delta(x, y) < \delta \Rightarrow |a(x, y) - a(x, x)| < \varepsilon. \tag{1.2} \]

Then $\text{Tr}_\chi(A) = \varphi(A)$.

**Proof.** Consider first a family of integral operators $B_\delta$, with kernels, which are sections of $\text{End}(F)$, given by

\[ b_\delta(x, y) := \frac{\beta_1(\delta)}{\beta_2(\delta)} \chi_{\Delta_1}(x, y) \frac{V(x, \delta)}{V(x, \delta)}, \]

where $\chi_{\Delta_1}$ is the characteristic function of the diagonal in $M \times M$. Then $b_\delta(x, y) \to a(x, y)$ uniformly as $\delta \to 0$, and $\text{Tr}_\chi(B_\delta) = \varphi(B_\delta)$. Since $\text{Tr}_\chi(B_\delta) \to \varphi(B_\delta)$ as $\delta \to 0$, we have $\text{Tr}_\chi(A) = \varphi(A)$. 

\[ \Box \]
where $\Delta_\delta := \{(x, y) \in M \times M : \delta(x, y) < \delta\}$. Then $\sup_{x \in M} \int_M b_\delta(x, y)dy = \frac{\beta_1(\delta)}{\beta_2(\delta)} \leq 1$, and $\sup_{y \in M} \int_M b_\delta(x, y)dx \leq \frac{\sup_{x \in M} V(x, \delta)}{\beta_1(\delta)} \leq 1$, which imply $\|B_\delta\| \leq 1$, by Riesz-Thorin theorem.

Set $E_n$ for the multiplication operator by the characteristic function of $K_n$, and observe that

$$
Tr(E_n B_\delta B_\delta^* E_n) = \int_{K_n} dx \int_M b_\delta(x, y)^2 dy
$$

\[
= \frac{\beta_1(\delta)^2}{\beta_2(\delta)^2} \int_{K_n} \frac{dx}{V(x, \delta)} \\
\leq \frac{\beta_1(\delta)}{\beta_2(\delta)^2} \operatorname{vol}(K_n) \leq \frac{\operatorname{vol}(K_n)}{\beta_2(\delta)}
\]

Therefore $\varphi(B_\delta B_\delta^*) \leq \frac{\beta_2(\delta)^{-1}}{\varphi|_{A_+}}$. This implies that $\psi_\delta := \varphi(B_\delta \cdot B_\delta^*)$ belongs to $A_+^*$, and $\psi_\delta \leq \psi|_A$ by Lemma 1.21. By the results of the previous subsection, we have $\psi_\delta(A) \leq Tr_\mathcal{K}(A) \leq \varphi(A)$, for any $A \in A_+$.

Take now $A \in A_+$ satisfying (1.3), for a pair $\varepsilon > 0$, $\delta > 0$, and, setting $\beta(\delta) := \left(\frac{\beta_1(\delta)}{\beta_2(\delta)}\right)^2$ to improve readability, compute

$$
|Tr(E_n B_\delta AB_\delta^* E_n) - Tr(E_n A E_n)|
\leq |Tr(E_n B_\delta AB_\delta^* E_n) - \beta(\delta)Tr(E_n A E_n)| + (1 - \beta(\delta))Tr(E_n A E_n)
\leq \int_{K_n} dx \int_{B(x, \delta) \times B(x, \delta)} b_\delta(x, y)a(y, z) - a(x, x)b_\delta(x, y)dydz
+ (1 - \beta(\delta))Tr(E_n A E_n)
\leq 3\varepsilon \int_{K_n} dx \int_{B(x, \delta) \times B(x, \delta)} b_\delta(x, y)b_\delta(x, z)dydz
+ (1 - \beta(\delta))Tr(E_n A E_n)
\leq 3\varepsilon \beta(\delta)\operatorname{vol}(K_n) + (1 - \beta(\delta))Tr(E_n A E_n)
$$

By the arbitrariness of $\varepsilon$ we get

$$
\frac{|Tr(E_n B_\delta AB_\delta^* E_n) - Tr(E_n A E_n)|}{\operatorname{vol}(K_n)} \leq (1 - \beta(\delta)) \frac{Tr(E_n A E_n)}{\operatorname{vol}(K_n)}.
$$

This implies $|\psi_\delta(A) - \varphi(A)| \leq (1 - \beta(\delta))\varphi(A)$. By Lemma 1.2(ii), we get the thesis. \hfill \square

**Proposition 1.30.** For any $t > 0$, $e^{-t\Delta_p}$ belongs to the domain of $Tr_\mathcal{K}$ and $Tr_\mathcal{K}(e^{-t\Delta_p}) = \varphi(e^{-t\Delta_p})$, where $\Delta_p$ is the $p$-Laplacian operator.

**Proof.** By Remark 1.14 and Corollary 1.3 we have that $e^{-t\Delta_p}$ belongs to $\mathcal{J}_{0+} \cap A$, hence, by Corollary 1.23 (iii), it belongs to the domain of $Tr_\mathcal{K}$. The equality then follows by Proposition 1.29. \hfill \square
and prove their invariance under quasi-isometries. To do this we enlarge the

\[ T : f \in C_c[0, \infty)_+ \rightarrow Tr_\varphi(f(\Delta_p)) \in [0, \infty), \] for any \( f \in C_c[0, \infty)_+ \).

Proof. Let us introduce the positive functionals \( T_0 : f \in C_c[0, \infty)_+ \rightarrow Tr_\varphi(f(\Delta_p)) \in [0, \infty) \), and \( T(f) = \int f \, d\mu \). Then (see e.g. \([29]\), Propositions 5.2, 5.4) \( T_0(f) \geq \int f \, d\mu_0 \), \( T(f) = \int f \, d\mu \), for \( f \in C_0[0, \infty)_+ \). Let us prove that \( \int f \, d\mu_0 \geq \int f \, d\mu \), for \( f \in C_0[0, \infty)_+ \). Indeed, setting \( \tilde{T}_0(f) := \int f \, d\mu_0 \), for \( f \in C_0[0, \infty)_+ \), we get a semicontinuous weight on \( C_0[0, \infty)_+ \), so that \( \tilde{T}_0(f) = \sup\{S(f) : S \in C_0[0, \infty)_+, S \leq T_0\} \). By definition, \( Tr_\varphi(a) = \sup\{\psi(a) : \psi \in \mathcal{A}_+^\tau, \psi \leq \varphi|_A\} \), for any \( a \in \mathcal{A}_+^\tau \) s.t. the right-hand side is finite. Therefore for any \( \psi \in \mathcal{A}_+^\tau \) s.t. \( \psi \leq \varphi|_A \), setting \( S(f) := \psi(f(\Delta_p)) \), we get \( S \in C_0[0, \infty)_+ \), and \( \tilde{T}_0(f) = \sup\{S(f) : S \leq T_0\} \), and, from the arbitrariness of \( \psi \), \( T(f) = Tr_\varphi(f(\Delta_p)) \leq \tilde{T}_0(f) \).

Therefore we conclude that \( \mu_0 - \mu \) is a positive measure on \( [0, \infty) \). As \( \varphi(e^{-t\Delta_p}) = Tr_\varphi(e^{-t\Delta_p}) \), \( t > 0 \), we have \( f \int e^{-t\Delta_p}d(\mu_0 - \mu)(\lambda) = 0 \), so that \( \mu_0 = \mu \), which implies \( Tr_\varphi(f(\Delta_p)) = \varphi(f(\Delta_p)) \), for \( f \in C_c[0, \infty)_+ \).

2 Novikov-Shubin invariants and singular traces

In this section we consider an open manifold with \( C^\infty \) bounded geometry possessing a regular exhaustion, i.e. the same hypotheses assumed in subsection 1.3. Let us fix a \( p \in \{0, \ldots, n\} \) and denote by \( A_p \triangleq A(\Lambda^p T^*M) \) the \( C^* \)-algebra of almost local operators acting on \( L^2(\Lambda^p T^*M) \), the Hilbert space of \( L^2 \)-sections of the vector bundle \( \Lambda^p T^*M \), \( Tr_\varphi \) denotes the lower-semicontinuous semifinite trace on \( A \) obtained in subsection 1.4, and \( \Delta_p \) is the \( p \)-Laplacian, acting as a selfadjoint operator on \( L^2(\Lambda^p T^*M) \).

2.1 Novikov-Shubin numbers for open manifolds and their invariance

In this subsection we define the Novikov-Shubin numbers for such manifolds and prove their invariance under quasi-isometries. To do this we enlarge the \( C^* \)-algebra \( A_p \) in order to include sufficiently many spectral projections of the \( p \)-Laplacian. It turns out that the noncommutative analogue of Riemann integrable functions developed in \([29]\) gives a convenient framework, so we first recall its construction and basic properties.

Let \( \mathcal{A} \) be a general \( C^* \)-algebra with a semicontinuous semifinite trace \( \tau \). A pair of families \( (A^-, A^+) \) in \( \mathcal{A}_{sa} \) is called an \( \mathcal{R} \)-cut in \( \mathcal{A} \) w.r.t. \( \tau \) if they are bounded, separated (i.e. \( a^- \leq a^+ \) for any \( a^\pm \in A^\pm \)) and \( \tau \)-contiguous (i.e. \( \forall \varepsilon > 0 \exists a^\pm \in A^\pm \) s.t. \( \tau(a^+ - a^-) < \varepsilon \)). A selfadjoint element \( x \in \mathcal{A}'' \) is said separating for \( (A^-, A^+) \) if \( a^- \leq x \leq a^+ \) for any \( a^\pm \in A^\pm \). Then the following holds.

Theorem 2.1. \([29]\) The set of separating elements between \( \mathcal{R} \)-cuts in \( \mathcal{A} \) is the selfadjoint part of a \( C^* \)-algebra, denoted by \( \mathcal{A}^\mathcal{R} \), and called the \( C^* \)-algebra of
Riemann measurable operators. The GNS representation \(\pi_\tau\) of \(A\) extends to a \(*\)-homomorphism (still denoted by \(\pi_\tau\)) of \(A^\mathbb{R}\) to \(\pi_\tau(A)'\), hence the trace \(\tau\) extends to a semicontinuous semifinite trace on \(A^\mathbb{R}\). \(A^\mathbb{R}\) contains all the separating elements between \(R\)-cuts in it, and is closed under functional calculus with Riemann integrable functions, in particular almost all spectral projections \(e_{(a,b)}\) of selfadjoint elements of \(A\) belong to \(A^\mathbb{R}\).

Applying this result to \(A_p\), we obtain the \(C^*\)-algebra \(A^\mathbb{R}_p\) with a lower-semicontinuous semifinite trace, still denoted \(\tau\) or \(\tau\). Then \(\chi(0,t)(\Delta_p)\) and \(\chi(\varepsilon,t)(\Delta_p)\) are Riemann measurable spectral projections and belong to \(A^\mathbb{R}_p\) for almost all \(t > \varepsilon > 0\), by the previous Theorem. Denote by \(N_p(t) := \tau(\chi(0,t)(\Delta_p))\), \(\vartheta_p(t) := \tau(\chi(\varepsilon,t)(\Delta_p))\).

**Lemma 2.2.** \(\vartheta_p(t) = \int_0^\infty e^{-t\lambda}dN_p(\lambda)\) so that \(\lim_{t\to0} N_p(t) = \lim_{t\to\infty} \vartheta_p(t)\).

**Proof.** If \(\Delta = \int_0^\infty \lambda de(\lambda)\) denotes the spectral decomposition, then \(e^{-t\Delta} = \int_0^\infty e^{-t\lambda}de(\lambda)\). Since the latter is defined as the norm limit of the Riemann-Stieltjes sums, \(\pi_p(e^{-t\Delta}) = \int_0^\infty e^{-t\lambda}d\pi_p(e(\lambda))\), where \(\pi_p\) denotes the GNS representation of \(A_p\) w.r.t the trace \(\tau\). The result then follows by the normality of the trace in the GNS representation.

**Definition 2.3.** We define \(b_p \equiv b_p(M,\mathcal{X}) := \lim_{t\to0} N_p(t) = \lim_{t\to\infty} \vartheta_p(t)\) to be the \(p\)-th \(L^2\)-Betti number of the open manifold \(M\) endowed with the exhaustion \(\mathcal{X}\). Let us now set \(N^0_p(t) := N_p(t) - b_p \equiv \lim_{t\to0} \tau(\chi(0,t)(\Delta_p))\), and \(\vartheta^0_p(t) := \vartheta_p(t) - b_p = \int_0^\infty e^{-t\lambda}dN^0_p(\lambda)\). The Novikov-Shubin numbers of \((M,\mathcal{X})\) are then defined as

\[
\alpha_p \equiv \alpha_p(M,\mathcal{X}) := 2 \limsup_{t\to0} \frac{\log N^0_p(t)}{\log t},
\]

\[
\alpha'_p \equiv \alpha'_p(M,\mathcal{X}) := 2 \liminf_{t\to0} \frac{\log N^0_p(t)}{\log t},
\]

\[
\alpha_p' \equiv \alpha_p'(M,\mathcal{X}) := 2 \limsup_{t\to\infty} \frac{\log \vartheta^0_p(t)}{\log 1/t},
\]

\[
\alpha'_p \equiv \alpha'_p(M,\mathcal{X}) := 2 \liminf_{t\to\infty} \frac{\log \vartheta^0_p(t)}{\log 1/t}.
\]

It follows from (27) Appendix) that \(\alpha_p = \alpha'_p \leq \alpha'_p \leq \alpha_p\), and \(\alpha'_p = \alpha_p\) if \(\vartheta^0_p(t) = O(t^{-d})\), for \(t \to \infty\), or equivalently \(N^0_p(t) = O(t^d)\), for \(t \to 0\). Observe that \(L^2\)-Betti numbers and Novikov-Shubin numbers depend on the limit procedure \(\omega\) and the exhaustion \(\mathcal{X}\).

**Proposition 2.4.** Let \(M\) be a complete non-compact Riemannian manifold of positive injectivity radius and Ricci curvature bounded from below. Then \(\alpha_0(M,\mathcal{X}) = \alpha'_0(M,\mathcal{X}) \geq 1\) for any regular exhaustion \(\mathcal{X}\).
Proof. Recall that, under the previous assumptions, Varopoulos \[53\] proved that the heat kernel on the diagonal has a uniform inverse-polynomial bound, more precisely, in the strongest form due to \[7\], we have
\[
\sup_{x,y \in M} H_0(t,x,y) \leq Ct^{-1/2}
\]
for a suitable constant \(C\). Then, as
\[
\vartheta(t) = \tau(e^{-t\Delta}) = \lim_{\omega} \int_{B(o,n_k)} H_0(t,x,x) dvol(x) \leq Ct^{-1/2},
\]
it follows from (\[27\], Appendix) that \(\alpha_0 = \alpha'_0\), which concludes the proof. \(\square\)

Remark 2.5. (a) If \(M\) is a covering of a compact manifold \(X\), \(L^2\)-Betti numbers were introduced by Atiyah \[2\] whereas Novikov-Shubin numbers were introduced in \[39\]. They were proved to be \(\Gamma\)-homotopy invariants, where \(\Gamma := \pi_1(X)\) is the fundamental group of \(X\), by Dodziuk \[21\] and Gromov-Shubin \[27\] respectively. \(L^2\)-Betti numbers were subsequently defined for the open manifolds considered in this paper by Roe \[44\], though in a different way, and were proved to be invariant under quasi-isometries (see below) in \[10\].

(b) In the case of coverings, the trace \(Tr_\Gamma\) is normal on the von Neumann algebra of \(\Gamma\)-invariant operators, hence \(\lim_{t \to 0} Tr(e_{[0,t]}(\Delta_p)) = Tr(e_{[0]}(\Delta_p))\). In the case of open manifolds there is no natural von Neumann algebra containing the bounded functional calculi of \(\Delta_p\) on which the trace \(Tr_K\) is normal, hence the previous equality does not necessarily hold. Such phenomenon has been considered by Farber in \[25\] in a context which is similar to ours, and the difference \(\lim_{t \to 0} Tr(e_{[0,t]}(\Delta_p)) - Tr(e_{[0]}(\Delta_p))\) has been called the torsion dimension. We shall denote by \(\text{tordim}(M, \Delta_p)\) such difference, and shall sometimes assume it vanishes. We are not aware of a general vanishing result in our context.

(c) Let us observe that the above definitions for \(L^2\)-Betti numbers and Novikov-Shubin numbers coincide with the classical ones in the case of amenable coverings, if one chooses the exhaustion given by the Følner condition. An explicit argument is given in \[30\].

(d) In the case of coverings there is a well-known conjecture on the positivity of the \(\alpha_p\)’s. A result by Varopoulos \[32\] shows that \(\alpha_0\) is a positive integer, hence \(\alpha_0 \geq 1\). The previous Proposition extends this inequality to the case of open manifolds. Moreover \(\alpha_0\) can assume any value in \([1, \infty)\), as follows from Theorem \[3.35\] and Corollary \[3.27\].

Our first objective is to show that our definition of \(L^2\)-Betti numbers coincides with Roe’s definition. Then we prove that Novikov-Shubin numbers are invariant under quasi-isometries, where a map \(\varphi : M \to \tilde{M}\) between open manifolds of \(C^\infty\)-bounded geometry is a quasi-isometry \[10\] if \(\varphi\) is a diffeomorphism s.t.

(i) there are \(C_1, \ C_2 > 0\) s.t. \(C_1\|v\| \leq \|\varphi_* v\| \leq C_2\|v\|, \ v \in TM\)
(ii) $\nabla - \varphi^*\nabla$ is bounded with all its covariant derivatives, where $\nabla$, $\nabla$ are Levi-Civita connections of $M$ and $\widetilde{M}$.

Recall Roe’s definition of $L^2$-Betti numbers, which we temporarily denote by $b^R_p$, $b^R_p := \inf \{\phi(f(\Delta_p)) : f \in \mathcal{C}\} = \inf \{Tr_{\mathfrak{K}}(f(\Delta_p)) : f \in \mathcal{C}\}$, by Proposition [1.3], where $\mathcal{C} := \{f \in C^\infty([0,\infty)_+ : f(0) = 1\}$. Then

**Proposition 2.6.** $b^R_p = b_p$.

**Proof.** As $b_p = \inf_{t > 0} Tr_{\mathfrak{K}}(\chi(0,t)(\Delta_p))$, and for any $t > 0$ there is $f \in \mathcal{C}$ s.t. $f \leq \chi(0,t)$, we conclude that $b^R_p \leq b_p$. Let now $\varepsilon > 0$ be given, and let $f \in \mathcal{C}$ be s.t. $(1 + \varepsilon)b^R_p > Tr_{\mathfrak{K}}(f(\Delta_p))$. Then there is $\delta = \delta_\varepsilon > 0$ s.t. $f(x) \geq 1 - \varepsilon$, for $x \in [0,\delta)$, so that $\frac{1}{1 + \varepsilon} f \geq \chi(0,\delta)$ and $(1 + \varepsilon)b^R_p \geq (1 - \varepsilon)Tr_{\mathfrak{K}}(\chi(0,\delta)(\Delta_p)) \geq (1 - \varepsilon)b_p$, that is $b^R_p \geq \frac{1 - \varepsilon}{1 + \varepsilon} b_p$, and from the arbitrariness of $\varepsilon$, $b^R_p \geq b_p$, and the thesis follows. $\Box$

**Theorem 2.7.** $L^2$-Betti numbers are invariant under quasi-isometries.

**Theorem 2.8.** Let $(M, \mathfrak{K})$ be an open manifold with a regular exhaustion, and let $\varphi : M \to \widetilde{M}$ be a quasi-isometry. Then $\varphi(\mathfrak{K})$ is a regular exhaustion for $\widetilde{M}$, $\alpha_p(M, \mathfrak{K}) = \alpha_p(\widetilde{M}, \varphi(\mathfrak{K}))$ and the same holds for $\alpha_p'$.

**Proof.** Let us denote by $\Phi \in \mathcal{B}(L^2(\Lambda^*T^*M), L^2(\Lambda^*T^*\widetilde{M}))$ the extension of $(\varphi^{-1})^*$. Then $Tr_{\varphi(\mathfrak{K})} = Tr_{\mathfrak{K}}(\Phi^{-1} \cdot \Phi)$. Also, setting $e_{\varepsilon,t} := \chi([\varepsilon,t])(\Delta_p)$, $q_{\eta,s} := \Phi^{-1}\chi([\varepsilon,t])(\Delta_p)$, we have

$$0 \leq Tr_{\mathfrak{K}}(e_{\varepsilon,t} - e_{\varepsilon,t}q_{\eta,s}e_{\varepsilon,t}) = Tr_{\mathfrak{K}}(e_{\varepsilon,t}(1 - q_{\eta,s})e_{\varepsilon,t})$$

$$= Tr_{\mathfrak{K}}(e_{\varepsilon,t}q_{\eta,s}e_{\varepsilon,t}) + Tr_{\mathfrak{K}}(e_{\varepsilon,t}q_{\eta,s}e_{\varepsilon,t})$$

$$= Tr_{\mathfrak{K}}(q_{\eta,s}e_{\varepsilon,t}q_{\eta,s}) + Tr_{\mathfrak{K}}(e_{\varepsilon,t}q_{\eta,s}e_{\varepsilon,t})$$

$$\leq Tr_{\mathfrak{K}}(q_{\eta,s}e_{\varepsilon,t})q_{\eta,s}e_{\varepsilon,t} + Tr_{\mathfrak{K}}(e_{\varepsilon,t}q_{\eta,s}e_{\varepsilon,t})$$

$$\leq Tr_{\mathfrak{K}}(q_{\eta,s}) C \sqrt{\frac{\eta}{\varepsilon}} + Tr_{\mathfrak{K}}(e_{\varepsilon,t}) C \sqrt{\frac{\varepsilon}{s}},$$

where the last inequality follows from [1.2]. Then

$$Tr_{\mathfrak{K}}(q_{\eta,s}) = Tr_{\mathfrak{K}}(e_{\varepsilon,t}) + Tr_{\mathfrak{K}}(q_{\eta,s} - e_{\varepsilon,t}q_{\eta,s}e_{\varepsilon,t}) - Tr_{\mathfrak{K}}(e_{\varepsilon,t} - e_{\varepsilon,t}q_{\eta,s}e_{\varepsilon,t})$$

$$\geq Tr_{\mathfrak{K}}(e_{\varepsilon,t}) - Tr_{\mathfrak{K}}(q_{\eta,s}) C \sqrt{\frac{\eta}{\varepsilon}} - Tr_{\mathfrak{K}}(e_{\varepsilon,t}) C \sqrt{\frac{\varepsilon}{s}}.$$

Now let $a > 1$ and compute

$$\tilde{N}^0(s) = \lim_{\varepsilon \to 0} Tr_{\mathfrak{K}}(q_{\varepsilon,a} - e_{\varepsilon,t}) = \lim_{\varepsilon \to 0} \left[ Tr_{\mathfrak{K}}(e_{\varepsilon,t}) - Tr_{\mathfrak{K}}(q_{0,a}) + Tr_{\mathfrak{K}}(e_{\varepsilon,t}) C \sqrt{\frac{\eta}{\varepsilon}} - Tr_{\mathfrak{K}}(e_{\varepsilon,t}) C \sqrt{\frac{\varepsilon}{s}} \right]$$

$$= N^0(t) \left[ 1 - C \sqrt{\frac{\eta}{\varepsilon}} \right].$$
Therefore with $\lambda := 4C^2$ we get $\tilde{N}^0(\lambda t) \geq \frac{1}{2} N^0(t)$, and exchanging the roles of $M$ and $\tilde{M}$, we obtain $\frac{1}{2} N^0(\lambda^{-1} t) \leq \tilde{N}^0(t) \leq 2 N^0(\lambda t)$. This means that $N^0$ and $\tilde{N}^0$ are dilatation-equivalent (see [27]) so that the thesis follows from [27].

Remark 2.9. We have chosen Lott’s normalization [33] for the Novikov-Shubin numbers $\alpha_p(M)$ because Laplace operator is a second order differential operator, and this normalization gives the equality between $\alpha_0(M)$ and the asymptotic dimension of $M$, cf. Theorem 3.35.

Our choice of the lim sup in Definition 2.3, in contrast with Lott’s choice [33], is motivated by our interpretation of $\alpha_p$ as a dimension. On the one hand, a noncommutative measure corresponds to $\alpha_p$ via a singular trace, according to Theorem 2.16. On the other hand, this choice implies that $\alpha_0$, being equal to the asymptotic dimension of $M$, possesses the classical properties of a dimension as stated in Theorem 3.11, cf. also Remark 3.12 (b).

2.2 Novikov-Shubin numbers as asymptotic spectral dimensions

In this subsection we discuss a dimensional interpretation for the Novikov-Shubin numbers. In the case of compact manifolds, the dimension may be recovered by the Weyl asymptotics. In particular, the formula $\left(\lim_{t \to 0} \log \frac{\mu_p(t)}{\log 1/t}\right)^{-1}$ gives the dimension of the manifold, where $\mu_p$ refers to $\Delta^{-1/2}$. This formula makes sense also in the non-compact case, if one replaces the eigenvalue sequence with the eigenvalue function as explained below, and clearly recovers the dimension of the manifold. But in this case, the behaviour for $t \to 0$ may be considered also, giving rise to an asymptotic counterpart of the dimension.

In the case of covering manifolds [30], we defined the asymptotic spectral dimension of the triple $(M, \Gamma, \Delta_p)$ as $\left(\liminf_{t \to 0} \frac{\log \mu_p(t)}{\log 1/t}\right)^{-1}$, where $\mu_p$ refers to the operator $\Delta_p^{-1/2}$.

On the one hand, this number is easily shown to coincide with the $p$-th Novikov-Shubin number; on the other hand, it deserves the name of dimension also in the context of noncommutative measure theory. Indeed, Hausdorff dimension determines which power of the radius of a ball gives rise to a non trivial volume on the space. Analogously, the spectral dimension determines which power of the $p$-Laplacian gives rise to a non trivial singular trace on the algebra $A_p$.

We extend this result to the case of open manifolds, using the unbounded Riemann integration and the theory of singular traces for C*-algebras developed in [29]. However, since the trace we use is not normal with respect to the given representation of $A_p$ on the space of $L^2$-differential forms, some assumptions like the vanishing of the torsion dimension introduced in Remark 2.5 (b) are needed.

Let us briefly recall the definition and main properties of unbounded Riemann integrable operators. A linear operator $T$ on $\mathcal{H}$ is said to be affiliated to
a von Neumann algebra $M$ ($T \hat{\in} M$) if all elements of $x \in M'$ send its domain into itself and $Tx\eta = xT\eta$, for any $\eta$ in $D(T)$.

Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a $C^*$-algebra with a lower semicontinuous semifinite trace $\tau$, and $\mathcal{A}^R$ be the $C^*$-algebra of Riemann measurable operators. A sequence $\{e_n\}$ of projections in $\mathcal{B}(\mathcal{H})$ is called a Strongly Dense Domain (SDD) w.r.t. $\mathcal{A}^R$ if $e_n^\perp \in \mathcal{A}^R$ is $\tau$-finite and $\tau(e_n^\perp) \to 0$. We shall denote by $e$ the projection $\sup_n e_n$. Let us remark that, if the trace $\tau$ is not faithful, $e$ is not necessarily 1. Nevertheless it is easy to show that $e^\perp \in \mathcal{A}^R$ and $\tau(e) = 0$.

**Definition 2.10.** We denote by $\overline{\mathcal{A}^R}$ the family of closed, densely defined operators affiliated to $\mathcal{A}''$ for which there exists a SDD $\{e_n\}$ such that

(i) $e_n\mathcal{H} \subset \mathcal{D}(T) \cap \mathcal{D}(T^*)$

(ii) $eTe_n \in \mathcal{A}^R$, $e_nTe \in \mathcal{A}^R$.

We also introduce the relation of $\tau$-a.e. equality, which turns out to be an equivalence relation, among operators in $\overline{\mathcal{A}^R}$, namely $T$ is equal to $S$ $\tau$-a.e. if there exists a common SDD $\{e_n\}$ for $T$ and $S$ such that $eTe_n = eSe_n$ for any $n \in \mathbb{N}$.

In the following we shall denote by $\pi$ the GNS representation of $A$ associated with the trace $\tau$, by $M$ the von Neumann algebra $\pi(A)'$, and by $\overline{M}$ the algebra of $\tau$-measurable operators affiliated to $M$.

**Theorem 2.11.** [29] The set $\overline{\mathcal{A}^R}$ is a $\tau$-a.e. bimodule, namely it is closed under strong sense operations, and the usual properties of a $\tau$-bimodule over $\mathcal{A}^R$ hold $\tau$-almost everywhere. Moreover the GNS representation extends to a map from $\overline{\mathcal{A}^R}$ to $\overline{M}$ which preserves strong sense operations, hence the trace $\tau$ extends to $\overline{\mathcal{A}^R}$. $\overline{\mathcal{A}^R}$ contains the functional calculi of the selfadjoint elements in $\mathcal{A}^R$ under unbounded Riemann integrable functions, and is called the $\tau$-a.e. bimodule of unbounded Riemann integrable elements.

We define the distribution function w.r.t. $\tau$ of an operator $A \in \overline{\mathcal{A}^R}$ via the distribution function of its image under the GNS representation, namely $\lambda_A = \lambda_{\pi(A)}$. If $A \in \overline{\mathcal{A}^R}$ is a positive (unbounded) continuous functional calculus of an element in $A$, then $\chi(\{t, +\infty\})(A)$ belongs to $\overline{\mathcal{A}^R}$ a.e., therefore its distribution function is given by $\lambda_A(t) = \tau(\chi(\{t, +\infty\})(A))$. The non increasing rearrangement is defined as $\mu_A(t) := \inf\{s \geq 0 : \lambda_A(s) \leq t\}$.

Now we come back to our concrete situation, namely to the pair $(\mathcal{A}_p, Tr_{\mathcal{K}})$ associated with a manifold $M$ endowed with a regular exhaustion $\mathcal{K}$. In the following, when the Laplacian $\Delta_p$ has a non trivial kernel, we denote by $\Delta_p^{-\alpha}$, $\alpha > 0$, the (unbounded) functional calculus of $\Delta_p$ w.r.t. the function $\varphi_{\alpha}$ given by $\varphi_{\alpha}(0) = 0$ and $\varphi_{\alpha}(t) = t^{-\alpha}$ when $t > 0$.

**Lemma 2.12.**

(a) The following are equivalent

(a.i) The projection $E_p$ onto the kernel of $\Delta_p$ (also expressed by $\chi(\{1\})(e^{-\Delta_p})$) is Riemann measurable, and the torsion dimension vanishes, namely $Tr_{\mathcal{K}}(E_p)$ is equal to $b_p$. 


(a.ii) \( \chi_{(1)}(\pi(e^{-t\Delta_p})) \) is Riemann integrable in the GNS representation \( \pi \).

(b) The vanishing of the Betti number \( b_p \) implies (a). It is equivalent to (a) if \( \ker(\Delta_p) \) is finite-dimensional.

(c) If (a) is satisfied, \( \Delta_p^{-\alpha} \in \mathcal{A}^\mathbb{R} \) for any \( \alpha > 0 \).

Remark 2.13. The extension of the GNS representation \( \pi \) to \( \mathcal{A}_{\mathbb{R}} \) does not necessarily commute with the Borel functional calculus. In particular \( \chi_{(1)}(\pi(e^{-t\Delta_p})) \) is not necessarily equal to \( \pi(\chi_{(1)}(e^{-t\Delta_p})) \).

Proof. (a). (i) \( \Rightarrow \) (ii). Since the projection \( E_p \equiv \chi_{\{0\}}(\Delta_p) \) is Riemann integrable and less than \( e^{-t\Delta_p} \) for any \( t \), its image in the GNS representation is Riemann integrable and less than \( \pi(e^{-t\Delta_p}) \) for any \( t \). This implies that \( \pi(E_p) = \chi_{\{1\}}(\pi(e^{-t\Delta_p})) \leq \pi(e^{-t\Delta_p}) \) is an \( \mathcal{R} \)-cut, hence the thesis.

(ii) \( \Rightarrow \) (i). By normality of the trace in the GNS representation, \( Tr_\mathcal{R}(e^{-t\Delta_p}) \) converges to \( Tr_\mathcal{R}(\chi_{\{1\}}(\pi(e^{-t\Delta_p}))) \) hence, by hypothesis, for any \( \varepsilon > 0 \) we may find \( a_\varepsilon \in \mathcal{A} \) and \( t_\varepsilon > 0 \) s.t. \( a_\varepsilon \leq \chi_{\{1\}}(\pi(e^{-t\Delta_p})) \) and \( Tr_\mathcal{R}(e^{-t_\varepsilon\Delta_p} - a_\varepsilon) < \varepsilon \). This implies \( a_\varepsilon \leq e^{-t\Delta_p} \) for any \( t \), hence \( a_\varepsilon \leq \chi_{\{0\}}(\Delta_p) \), which means that \( \{a_\varepsilon\}, \{e^{-t\Delta_p}\} \) is an \( \mathcal{R} \)-cut for \( \chi_{\{0\}}(\Delta_p) \), namely this projection is Riemann integrable and \( Tr_\mathcal{R}(e^{-t\Delta_p} - \chi_{\{0\}}(\Delta_p)) \leq \varepsilon \), i.e. the thesis.

(b) If \( \lim_{t \to \infty} Tr_\mathcal{R}(e^{-t\Delta_p}) = 0 \), from \( 0 \leq \chi_{\{1\}}(e^{-t\Delta_p}) \leq e^{-t\Delta_p} \), we have that \( \chi_{\{1\}}(e^{-t\Delta_p}) \) is a separating element for an \( \mathcal{R} \)-cut in \( \mathcal{A} \). The second statement follows from Proposition 2.28.

(c) We have to exhibit an SDD for \( \Delta_p^{-\alpha} \). Indeed, as \( \chi_{\{t,\infty\}}(\Delta_p) \) is Riemann measurable for almost all \( t > 0 \), choose a strictly decreasing sequence \( t_n \to 0 \) of such \( t \), and set \( e_n := \chi_{\{t_n,\infty\}}(\Delta_p) + E_p \). Then \( e_n^{-1} \equiv \chi_{\{0,t_n\}}(\Delta_p) \in \mathcal{A}^\mathbb{R} \), \( Tr_\mathcal{R}(e_n^{-1}) = N^0_p(t_n) \to 0 \), and \( e\Delta_p^{-\alpha} e_n \equiv \int_{t_n}^\infty \lambda^{-\alpha} d\lambda \in \mathcal{A}^\mathbb{R} \).

If hypothesis (a) of the previous Lemma is satisfied, we may define the distribution function \( \lambda_p \) and the eigenvalue function \( \mu_p \) for the operator \( \Delta_p^{-1/2} \), hence the local spectral dimension as the inverse of \( \lim_{t \to \infty} \frac{\log \mu_p(t)}{\log 1/t} \), and it may be shown that such dimension gives the dimension of the manifold for any \( p \).

But, in the case of open manifolds, we may also consider the asymptotics for \( t \to 0 \), which gives rise to an asymptotic counterpart of the dimension. Then we define the asymptotic spectral dimension of the triple \( (M, \mathcal{K}, \Delta_p) \) as

\[
\left( \lim_{t \to 0} \frac{\log \mu_p(t)}{\log 1/t} \right)^{-1}.
\]

It is not difficult to show that such asymptotic spectral dimensions coincide with the Novikov-Shubin numbers.

Theorem 2.14. Let \( (M, \mathcal{K}) \) be an open manifold equipped with a regular exhaustion such that the projection on the kernel of \( \Delta_p \) is Riemann integrable and \( \operatorname{tor} \dim(M, \Delta_p) = 0 \). Then the asymptotic spectral dimension of \( (M, \mathcal{K}, \Delta_p) \) coincides with the Novikov-Shubin number \( \alpha_p(M, \mathcal{K}) \).
\[ \text{Proof.} \text{ By hypothesis, } e_{(0,t)}(\Delta_p) \text{ is Riemann integrable } Tr_{X}-\text{a.e., hence } N^0_p(t) = Tr_X(e_{(0,t)}(\Delta_p)) = Tr_X(e_{(t-1,\infty)}(\Delta_p^{-1})) = Tr_X(e_{(t-1/2,\infty)}(\Delta_p^{-1/2})) = \lambda_p(t^{-1/2}). \]

Then

\[ \alpha_p = 2 \limsup_{s \to 0} \frac{\log N^0_p(s)}{\log s} = 2 \limsup_{s \to 0} \frac{\log \lambda_p(s^{-1/2})}{\log s} = \limsup_{t \to \infty} \frac{\log \lambda_p(t)}{\log \frac{1}{t}}. \quad (2.1) \]

The statement follows if we prove that

\[ \liminf_{t \to 0} \frac{\log \mu(t)}{\log \frac{1}{t}} = \left( \limsup_{s \to \infty} \frac{\log \lambda(s)}{\log \frac{1}{s}} \right)^{-1} \]

for any \( t \), where the values 0 and \( \infty \) are allowed.

Let \( t_n \to 0 \) be a sequence such that \( \lim_{n \to \infty} \frac{\log \mu(t_n)}{\log \frac{1}{t_n}} = \liminf_{t \to 0} \frac{\log \mu(t)}{\log \frac{1}{t}} \), and let \( t_n' := \inf \{s \geq 0 : \mu(s) = \mu(t_n)\} = \min \{s \geq 0 : \mu(s) = \mu(t_n)\} \) where the last equality holds because of right continuity. Then

\[ \liminf_{t \to 0} \frac{\log \mu(t)}{\log \frac{1}{t}} \leq \lim_{n \to \infty} \frac{\log \mu(t_n')}{\log \frac{1}{t_n'}} \leq \lim_{n \to \infty} \frac{\log \mu(t_n)}{\log \frac{1}{t_n}} = \liminf_{t \to 0} \frac{\log \mu(t)}{\log \frac{1}{t}}, \]

namely we may replace \( t_n \) with \( t_n' \) to reach the \( \lim \inf \). Also, \( \lambda(\mu(t_n')) = \inf \{t \geq 0 : \mu(t) \leq \mu(t_n')\} \) therefore

\[ \liminf_{t \to 0} \frac{\log \mu(t)}{\log \frac{1}{t}} \geq \lim_{n \to \infty} \frac{\log s}{\log \frac{1}{\lambda(s)}} = \left( \limsup_{s \to 0} \frac{\log \lambda(s)}{\log \frac{1}{s}} \right)^{-1} \]

For the converse inequality, let \( s_n \to \infty \) be a sequence for which \( \lim_{n \to \infty} \frac{\log \lambda(s_n)}{\log \frac{1}{s_n}} = \limsup_{s \to 0} \frac{\log \lambda(s)}{\log \frac{1}{s}} = \). As before, \( s_n' := \inf \{s \geq 0 : \lambda(s) = \lambda(s_n)\} = \min \{s \geq 0 : \lambda(s) = \lambda(s_n)\} \) still brings to the \( \lim \sup \) and verifies \( \mu(\lambda(s_n')) = s_n' \).

Now we generalise a result proved in [30] in the case of coverings, proving that the spectral dimensions defined above give rise to singular traces, namely select the correct power of the \( p \)-Laplace operator which gives rise to a non trivial singular trace on the \( Tr_X \)-a.e. bimodule \( \mathcal{A}_p \).

Let us recall that an operator \( T \in \mathcal{A}_p \) is called 0-\textit{eccentric} when

\[ \limsup_{t \to 0} \frac{\int_0^t \mu_T(s) ds}{\int_0^t \mu_T(s) ds} = 1, \quad \text{if} \quad \int_0^1 \mu_T(t) dt < \infty, \]

\[ \liminf_{t \to 0} \frac{\int_0^t \mu_T(s) ds}{\int_0^t \mu_T(s) ds} = 1, \quad \text{if} \quad \int_0^1 \mu_T(t) dt = \infty. \]

As in the case of von Neumann algebras, any 0-\textit{eccentric} operator in \( \mathcal{A}_p \) gives rise to a singular trace, more precisely to a trace on \( \mathcal{A}_p \) which vanishes.
on all bounded operators. Singular traces may be described as the pull-back of the singular traces on $M$ via the (extended) GNS representation. On the other hand, explicit formulas may be written in terms of the non-increasing rearrangement. We write these formulas for the sake of completeness.

**Theorem 2.15.** \cite{29} If $T \in \overline{A^\infty}$ is 0-eccentric and $\int_0^1 \mu_T(t)dt < \infty$, there exists a generalized limit $\lim_\omega$ in 0 such that the functional

$$\tau_\omega(A) := \lim_\omega \left( \frac{\int_0^1 \mu_A(s)ds}{\int_0^1 \mu_T(s)ds} \right) \quad A \in X(T)_+$$

extends to a singular trace on the a.e. *-bimodule $X(T)$ over $\overline{A^\infty}$ generated by $T$. If $\int_0^1 \mu_T(t)dt = \infty$, the previous formula should be replaced by

$$\tau_\omega(A) := \lim_\omega \left( \frac{\int_1^t \mu_A(s)ds}{\int_1^t \mu_T(s)ds} \right) \quad A \in X(T)_+.$$

Such traces naturally extend to traces on $X(T) + \overline{A^\infty}$.

Making use of the previous Theorem, we show that Novikov-Shubin numbers are dimensions in the sense of noncommutative measure theory.

**Theorem 2.16.** Let $(M, K)$ be an open manifold with a regular exhaustion such that the projection on the kernel of $\Delta_p$ is Riemann integrable and $\text{tordim}(M, \Delta_p) = 0$. If $\alpha_p$ is finite nonzero, then $\Delta_p^{-\alpha_p/2}$ is 0-eccentric, namely gives rise to a non trivial singular trace on $\overline{A^\infty}$.

**Proof.** The 0-eccentricity of $\Delta_p^{-\alpha_p/2}$ follows by \cite{30}, hence the thesis follows by Theorem 2.15.

---

**3 An asymptotic dimension for metric spaces.**

The main purposes of this section are to introduce an asymptotic dimension for metric spaces, and to show that for a suitable class of manifolds (i.e. open manifolds of $C^\infty$-bounded geometry satisfying Grigor’yan’s isoperimetric inequality) the asymptotic dimension coincides with the 0-th Novikov-Shubin invariant. In particular this shows that, for these manifolds, $\alpha_0$ does not depend on the limit procedure $\omega$ and, in a mild sense, is independent of the regular exhaustion too. More precisely, $\alpha_0$ does not change if we choose $K$ among the exhaustions by balls with a common centre.

To our knowledge, the notion of asymptotic dimension in the general setting of metric dimension theory has not been studied, even though Davies \cite{18} proposed a definition in the case of cylindrical ends of a Riemannian manifold. We shall give a definition of asymptotic dimension for a general metric space, based on the (local) Kolmogorov dimension \cite{32} and state its main properties. We compare our definition with Davies’.
3.1 Kolmogorov-Tihomirov metric dimension

In this subsection we recall a definition of metric dimension due to Kolmogorov and Tihomirov [32] (see also [24] where it is called box dimension). Quoting from their paper, a dimension “corresponds to the possibility of characterizing the “massiveness” of sets in metric spaces by the help of the order of growth of the number of elements of their most economical $\varepsilon$-coverings, as $\varepsilon \to 0$”. Set functions retaining the general properties of a dimension (cf. Theorem 3.3) have been studied by several authors. Our choice of the Kolmogorov dimension is due to the fact that it is suitable for the kind of generalization we need in this paper, namely it quite naturally produces a definition of asymptotic dimension.

In the following, unless otherwise specified, $(X, \delta)$ will denote a metric space, $B_X(x, R)$ the open ball in $X$ with centre $x$ and radius $R$, $n_r(\Omega)$ the least number of open balls of radius $r$ which cover $\Omega \subset X$, and $\nu_r(\Omega)$ the largest number of disjoint open balls of radius $r$ centered in $\Omega$.

The following lemma is proved in [32]. Due to some notational difference, we include a proof.

**Lemma 3.1.** $n_r(\Omega) \geq \nu_r(\Omega) \geq n_{2r}(\Omega)$.

**Proof.** We have only to prove the second inequality when $\nu_r$ is finite. Let us assume that $\{B(x_i, r)\}_{i=1}^{\nu_r(\Omega)}$ are disjoint balls centered in $\Omega$ and observe that, for any $y \in \Omega$, $\delta(y, \bigcup_{i=1}^{\nu_r(\Omega)} B(x_i, r)) < r$, otherwise $B(y, r)$ would be disjoint from $\bigcup_{i=1}^{\nu_r(\Omega)} B(x_i, r)$, contradicting the maximality of $\nu_r$. So for all $y \in \Omega$ there is $j$ s.t. $\delta(y, B(x_j, r)) < r$, that is $\Omega \subset \bigcup_{i=1}^{\nu_r(\Omega)} B(x_i, 2r)$, which implies the thesis. \hfill $\square$

Kolmogorov and Tihomirov [32] defined a dimension for totally bounded metric spaces $X$ as

$$d_0(X) := \limsup_{r \to 0} \frac{\log n_r(X)}{\log(1/r)}.$$  \hfill (3.1)

Then we may give the following definition.

**Definition 3.2.** Let $(X, \delta)$ be a metric space. Then, denoting by $B(X)$ the family of bounded subsets of $X$, the metric Kolmogorov-Tihomirov dimension of $X$ is

$$d_0(X) := \sup_{B \in B(X)} \limsup_{r \to 0} \frac{\log n_r(B)}{\log(1/r)}.$$  \hfill (3.2)

Then the following proposition trivially holds.

**Proposition 3.3.** If $\{B_n\}$ is an exhaustion of $X$ by bounded subsets, namely $B_n$ is increasing and for any bounded $B$ there exists $n$ such that $B \subseteq B_n$, one has $d_0(X) = \lim_n d_0(B_n)$. In particular,

$$d_0(X) = \lim_{R \to \infty} \limsup_{r \to 0} \frac{\log n_r(B_X(x, R))}{\log(1/r)}.$$  \hfill (3.3)
Remark 3.4. If bounded subsets of $X$ are not totally bounded, we could define $d_0(X)$ as the supremum over totally bounded subsets. These two definitions, which agree e.g. on proper spaces, may be different in general. For example an orthonormal basis in an infinite dimensional Hilbert space has infinite dimension according to Definition 3.3, but has zero dimension in the other case. A definition of metric dimension which coincides with $d_0$ on bounded subsets of $\mathbb{R}^p$ has been given by Tricot [51] in terms of rarefaction indices.

Let us now show that this set function satisfies the basic properties of a dimension [12, 51].

Theorem 3.5. The set function $d_0$ is a dimension, namely it satisfies

(i) If $X \subset Y$ then $d_0(X) \leq d_0(Y)$.

(ii) If $X_1, X_2 \subset X$ then $d_0(X_1 \cup X_2) = \text{max}\{d_0(X_1), d_0(X_2)\}$.

(iii) If $X$ and $Y$ are metric spaces, then $d_0(X \times Y) \leq d_0(X) + d_0(Y)$.

Proof. Property (i) easily follows from formula (3.2).

Now we prove (ii). The inequality $d_0(X_1 \cup X_2) \geq \text{max}\{d_0(X_1), d_0(X_2)\}$ follows from (i). For the converse inequality, let $x_i \in X_i$, and set $\delta := \delta(x_1, x_2)$, $d_1 = d_0(X_1)$, $d_2 = d_0(X_2)$, with e.g. $d_1 \geq d_2$. If $d_1 = \infty$ the property is trivial, so we may suppose $d_1 \in \mathbb{R}$. Then

$$B_{X_1 \cup X_2}(x_1, R) \subset B_{X_1}(x_1, R) \cup B_{X_2}(x_2, R + \delta)$$

therefore

$$n_r(B_{X_1 \cup X_2}(x_1, R)) \leq n_r(B_{X_1}(x_1, R)) + n_r(B_{X_2}(x_2, R + \delta)).$$

(3.3)

Now, $\forall R > 0$,

$$\limsup_{r \to 0} \frac{\log n_r(B_{X_1}(x_1, R))}{\log(1/r)} \leq d_i$$

i.e. $\forall R, \varepsilon > 0$ there is $r_0 = r_0(\varepsilon, R)$ s.t. for all $0 < r < r_0$, $n_r(B_{X_1}(x_1, R)) \leq r^{-(d_1 + \varepsilon)}$, and $n_r(B_{X_2}(x_2, R + \delta)) \leq r^{-(d_2 + \varepsilon)}$ hence, by (3.3),

$$n_r(B_{X_1 \cup X_2}(x, R)) \leq r^{-(d_1 + \varepsilon)}(1 + r^{d_1 - d_2}).$$

Finally,

$$\lim_{R \to \infty} \limsup_{r \to 0} \frac{\log n_r(B_{X_1 \cup X_2}(x, R))}{\log(1/r)} \leq d_1 + \varepsilon,$$

that is

$$d_0(X_1 \cup X_2) \leq \text{max}\{d_0(X_1), d_0(X_2)\} + \varepsilon$$

and the thesis follows by the arbitrariness of $\varepsilon$.

The proof of part (iii) is postponed.

Kolmogorov dimension is invariant under bi-Lipschitz maps (also called quasi isometries by some authors), as next proposition shows.
Proposition 3.6. Let \( X, Y \) be metric spaces, and \( f : X \to Y \) a surjective bi-Lipschitz map, namely \( f \) satisfies

\[
c_1 \delta_X(x_1, x_2) \leq \delta_Y(f(x_1), f(x_2)) \leq c_2 \delta_X(x_1, x_2).
\]

Then \( d_0(X) = d_0(Y) \).

Proof. By hypothesis we have \( f(B_X(x, \rho/c_2)) \subset B_Y(f(x), \rho) \subset f(B_X(x, \rho/c_1)) \).

So that, with \( y_j = f(x_j) \), \( n := n_r(B_Y(f(x), R)) \),

\[
f(B_X(x, R/c_2)) \subset B_Y(f(x), R) \subset \bigcup_{j=1}^n B_Y(y_j, r)
\]

\[
\subset \bigcup_{j=1}^n f(B_X(x_j, r/c_1)) = f(\bigcup_{j=1}^n B_X(x_j, r/c_1))
\]

which implies \( n_{r/c_1}(B_X(x, R/c_2)) \leq n_r(B_Y(f(x), R)) \).

Since bi-Lipschitz maps are injective, we may repeat the same argument for \( f^{-1} \), and we get \( n_{r/c_1}(B_Y(f(x), c_1 R)) \leq n_r(B_X(x, R)) \), so that \( n_{r/c_1}(B_X(x, R/c_2)) \leq n_{r/c_1}(B_X(x, R/c_1)) \). Finally

\[
\limsup_{r \to 0} \frac{\log n_{r/c_1}(B_X(x, R/c_2))}{\log(c_1/r) - \log c_1} \leq \limsup_{r \to 0} \frac{\log n_r(B_Y(f(x), R))}{\log(1/r)}
\]

\[
\leq \limsup_{r \to 0} \frac{\log n_{r/c_2}(B_X(x, R/c_1))}{\log(c_2/r) - \log c_2}
\]

and the thesis follows.

Proof of Theorem 3.5 (continued). By the preceding Proposition, we may endow \( X \times Y \) with any metric bi-Lipschitz related to the product metric, i.e.

\[
\delta_{X \times Y}((x_1, y_1), (x_2, y_2)) = \max\{\delta_X(x_1, x_2), \delta_Y(y_1, y_2)\}.
\]

Then, by \( n_r(B_{X \times Y}((x, y), R)) \leq n_r(B_X(x, R)) \cdot n_r(B_Y(y, R)) \), the thesis follows easily.

Remark 3.7. Kolmogorov and Tihomirov assign a metric dimension to a totally bounded metric space \( X \) when \( \exists \lim_{r \to \infty} \text{in equation } (3.1) \), and consider upper and lower metric dimensions in the general case. We observe that if the \( \liminf \) is considered, the classical dimensional inequality \[42\] stated in Theorem 3.5 (iii) is replaced by \( d_0(X \times Y) \geq d_0(X) + d_0(Y) \).

3.2 Asymptotic dimension

The function introduced in the previous subsection can be used to study local properties of metric spaces. In this paper we are mainly interested in the investigation of the large scale behavior of these spaces, so we need a different tool. Looking at equation (3.2), it is natural to set the following
**Definition 3.8.** Let \((X, \delta)\) be a metric space. We call
\[
d_\infty(X) := \lim_{r \to \infty} \limsup_{R \to \infty} \frac{\log n_r(B_X(x, R))}{\log R},
\]
the asymptotic dimension of \(X\).

Let us remark that, as \(n_r(B_X(x, R))\) is nonincreasing in \(r\), the function
\[
r \mapsto \limsup_{R \to \infty} \frac{\log n_r(B_X(x, R))}{\log R}
\]
is nonincreasing too, so the \(\lim_{r \to 0}\) exists.

**Proposition 3.9.** \(d_\infty(X)\) does not depend on \(x\).

**Proof.** Let \(x, y \in X\), and set \(\delta := \delta(x, y)\), so that \(B(x, R) \subset B(y, R + \delta) \subset B(x, R + 2\delta)\). This implies,
\[
\frac{\log n_r(B(x, R))}{\log R} \leq \frac{\log n_r(B(y, R + \delta))}{\log (R + \delta)} \frac{\log (R + \delta)}{\log R}
\]
\[
\leq \frac{\log n_r(B(x, R + 2\delta))}{\log (R + 2\delta)} \frac{\log (R + 2\delta)}{\log R}
\]
so that, taking \(\limsup_{R \to \infty}\) and then \(\lim_{r \to \infty}\) we get the thesis. \(\square\)

**Lemma 3.10.**
\[
d_\infty(X) = \lim_{r \to \infty} \limsup_{R \to \infty} \frac{\log n_r(B_X(x, R))}{\log R}
\]

**Proof.** Follows easily from lemma 3.1. \(\square\)

**Theorem 3.11.** The set function \(d_\infty\) is a dimension, namely it satisfies
(i) If \(X \subset Y\) then \(d_\infty(X) \leq d_\infty(Y)\).
(ii) If \(X_1, X_2 \subset X\) then \(d_\infty(X_1 \cup X_2) = \max\{d_\infty(X_1), d_\infty(X_2)\}\).
(iii) If \(X\) and \(Y\) are metric spaces, then \(d_\infty(X \times Y) \leq d_\infty(X) + d_\infty(Y)\).

**Proof.** (i) Let \(x \in X\), then \(B_X(x, R) \subset B_Y(x, R)\) and the claim follows easily.
(ii) By part (i), we get \(d_\infty(X_1 \cup X_2) \geq \max\{d_\infty(X_1), d_\infty(X_2)\}\). Let us prove the converse inequality.
Let \(x_i \in X_i, i = 1, 2\), and set \(\delta = \delta(x_1, x_2), a = d_\infty(X_1), b = d_\infty(X_2)\), with e.g. \(a \leq b\). Then, \(\forall \varepsilon, r > 0 \exists R_0 = R_0(\varepsilon, r)\) s.t. \(\forall R > R_0\)
\[
n_r(B_{X_1}(x_1, R)) \leq R^{a + \varepsilon}
\]
\[
n_r(B_{X_2}(x_2, R + \delta)) \leq R^{b + \varepsilon},
\]
hence, by inequality (3.3),
\[ n_r(B_{X_1 \cup X_2}(x_1, R)) \leq R^{a+\varepsilon} + R^{b+\varepsilon} = R^{b+\varepsilon}(1 + R^{a-b}). \]

Finally,
\[ \frac{\log n_r(B_{X_1 \cup X_2}(x_1, R))}{\log R} \leq b + \varepsilon + \frac{\log(1 + R^{a-b})}{\log R}. \]

Taking the \( \lim \sup_{R \to \infty} \) and then the \( \lim_{r \to \infty} \) we get
\[ d_\infty(X_1 \cup X_2) \leq \max\{d_\infty(X_1), d_\infty(X_2)\} + \varepsilon \]
and the thesis follows by the arbitrariness of \( \varepsilon \).

The proof of part (iii) is analogous to that of part (iii) in Theorem 3.5, where we may use Proposition 3.15 because bi-Lipschitz maps are rough isometries.

Remark 3.12.

(a) In part (ii) of the previous theorem we considered \( X_1 \) and \( X_2 \) as metric subspaces of \( X \). If \( X \) is a Riemannian manifold and we endow the submanifolds \( X_1, X_2 \) with their geodesic metrics this property does not hold in general. A simple example is the following. Let \( f(t) := (t \cos t, t \sin t) \), \( g(t) := (-t \cos t, -t \sin t) \), \( t \geq 0 \) planar curves, and set \( X, Y \) for the closure in \( \mathbb{R}^2 \) of the two connected components of \( \mathbb{R}^2 \setminus (G_f \cup G_g) \), where \( G_f, G_g \) are the graphs of \( f, g \), and endow \( X, Y \) with the geodesic metric. Then \( X \) and \( Y \) are roughly-isometric to \([0, \infty)\) (see below) so that \( d_\infty(X) = d_\infty(Y) = 1 \), while \( d_\infty(X \cup Y) = 2 \).

(b) As for the local case, the choice of the \( \lim \sup \) in Definition 3.8 is the only one compatible with the classical dimensional inequality stated in Theorem 3.11 (iii). This, together with the singular traceability property 2.16, motivates our choice of the \( \lim \sup \) in Definition 2.3 for the Novikov-Shubin invariants.

Definition 3.13. Let \( X, Y \) be metric spaces, \( f : X \to Y \) is said to be a rough isometry if there are \( a \geq 1, b, \varepsilon \geq 0 \) s.t.

(i) \( a^{-1} \delta_X(x_1, x_2) - b \leq \delta_Y(f(x_1), f(x_2)) \leq a \delta_X(x_1, x_2) + b \), for all \( x_1, x_2 \in X \),

(ii) \( \bigcup_{x \in X} B_Y(f(x), \varepsilon) = Y \)

It is clear that the notion of rough isometry is weaker then the notion of bi-Lipschitz map introduced in the preceding subsection and, since any compact set is roughly isometric to a point, \( d_0 \) is not rough-isometry invariant. We shall show that the asymptotic dimension is indeed invariant under rough isometries.

Lemma 3.14. ([3], Proposition 4.3) If \( f : X \to Y \) is a rough isometry, there is a rough isometry \( f^- : Y \to X \), with constants \( a, b^-, \varepsilon^- \), s.t.

(i) \( \delta_X(f^- \circ f(x), x) < c_X, x \in X \),
(ii) \( \delta_Y(f \circ f^-(y), y) < c_Y, \ y \in Y. \)

**Proposition 3.15.** Let \( X, Y \) be metric spaces, and \( f : X \to Y \) a rough isometry. Then \( d_\infty(X) = d_\infty(Y). \)

**Proof.** Let \( x_0 \in X \), then for all \( x \in B_X(x_0, r) \) we have
\[
\delta_Y(f(x), f(x_0)) \leq a \delta_X(x, x_0) + b < ar + b
\]
so that
\[
f(B_X(x_0, r)) \subset B_Y(f(x_0), ar + b).
\]
Then, with \( n := n_r(B_Y(f(x_0), ar + b)) \),
\[
f(B_X(x_0, R)) \subset \bigcup_{j=1}^{n} B_Y(y_j, r),
\]
which implies
\[
f^- \circ f(B_X(x_0, R)) \subset \bigcup_{j=1}^{n} f^-(B_Y(y_j, r)) \subset \bigcup_{j=1}^{n} B_X(f^-(y_j), ar + b^-).
\]
Let \( x \in B_X(x_0, R) \), and \( j \) be s.t. \( f^- \circ f(x) \in B_X(f^-(y_j), ar + b^-) \), then
\[
\delta_X(x, f^-(y_j)) \leq \delta_X(x, f^- \circ f(x)) + \delta_X(f^- \circ f(x), f^-(y_j)) < c_X + ar + b^-,
\]
so that
\[
B_X(x_0, R) \subset \bigcup_{j=1}^{n} B_X(f^-(y_j), ar + b^- + c_X),
\]
which implies \( n_{ar+b^-+c_X}(B_X(x_0, R)) \leq n_r(B_Y(f(x_0), ar + b)). \)
Finally
\[
d_\infty(X) = \lim_{r \to \infty} \limsup_{R \to \infty} \frac{\log n_r(B_X(x_0, R))}{\log R}
= \lim_{r \to \infty} \limsup_{R \to \infty} \frac{\log n_{ar+b^-+c_X}(B_X(x_0, R))}{\log R}
\leq \lim_{r \to \infty} \limsup_{R \to \infty} \frac{\log n_r(B_Y(f(x_0), ar + b))}{\log R}
= \lim_{r \to \infty} \limsup_{R \to \infty} \frac{\log n_r(B_Y(f(x_0), R))}{\log R}
= d_\infty(Y)
\]
and exchanging the roles of \( X \) and \( Y \) we get the thesis. \( \Box \)
In what follows we show that when $X$ is equipped with a suitable measure, the asymptotic dimension may be recovered in terms of the volume asymptotics for balls of increasing radius, like the local dimension detects the volume asymptotics for balls of infinitesimal radius.

**Definition 3.16.** A Borel measure $\mu$ on $(X, \delta)$ is said to be uniformly bounded if there are functions $\beta_1, \beta_2$, s.t. $0 < \beta_1(r) \leq \mu(B(x, r)) \leq \beta_2(r)$, for all $x \in X$, $r > 0$. That is $\beta_1(r) := \inf_{x \in X} \mu(B(x, r)) > 0$, and $\beta_2(r) := \sup_{x \in X} \mu(B(x, r)) < \infty$.

**Proposition 3.17.** If $(X, \delta)$ has a uniformly bounded measure, then every ball in $X$ is totally bounded (so that if $X$ is complete it is locally compact).

**Proof.** Indeed, if there is a ball $B = B(x, R)$ which is not totally bounded, then there is $r > 0$ s.t. every $r$-net in $B$ is infinite, so $n_r(B)$ is infinite, and $\nu_r(B)$ is infinite too. So that $\beta_2(R) \geq \mu(B) \geq \sum_{i=1}^{\nu_r(B)} \mu(B(x_i, r)) \geq \beta_1(r)\nu_r(B) = \infty$, which is absurd. \qed

**Proposition 3.18.** If $\mu$ is a uniformly bounded Borel measure on $X$ then

$$d_\infty(X) = \limsup_{R \to \infty} \frac{\log \mu(B(x, R))}{\log R}.$$  

**Proof.** As $\bigcup_{i=1}^{\nu_r(B(x, R))} B(x_i, r) \subset B(x, R + r) \subset \bigcup_{j=1}^{n_r(B(x, R+r))} B(y_j, r)$, we get $\beta_2(r)n_r(B(x, R+r)) \geq \mu(B(x, R+r)) \geq \beta_1(r)\nu_r(B(x, R)) \geq \beta_1(r)n_{2r}(B(x, R))$, by Lemma 3.1. So that

$$\beta_1(r/2) \leq \frac{\mu(B(x, R + r/2))}{n_r(B(x, R))}, \quad \frac{\mu(B(x, R))}{n_r(B(x, R))} \leq \beta_2(r),$$

and the thesis follows easily. \qed

Let us now consider the particular case of complete Riemannian manifolds.

**Proposition 3.19.** Let $M, N$ be complete Riemannian manifolds.

(i) If $M$ is non-compact, then $d_\infty(M) \geq 1$

(ii) If $M$ has bounded geometry, then $d_\infty(M) = \lim_{R \to \infty} \frac{\log V(x, R)}{\log R}$

(iii) If $M, N$ have bounded geometry, and admit asymptotic dimension in a strong sense, that is

$$d_\infty(M) = \lim_{R \to \infty} \frac{\log V(x, R)}{\log R},$$

and analogously for $N$, then

$$d_\infty(M \times N) = d_\infty(M) + d_\infty(N).$$
Proof. (i) It follows from Theorem 3.11 (i), and the fact that there is inside $M$ an unbounded geodesic.

(ii) It follows from Lemma 1.2 that the volume is a uniformly bounded measure. Therefore the result follows from Proposition 3.18.

(iii) As $\text{vol}(B_{M \times N}((x, y), R)) = \text{vol}(B_M(x, R))\text{vol}(B_N(y, R))$, we get

$$d_\infty(M \times N) = \lim_{R \to \infty} \frac{\log \text{vol}(B_{M \times N}((x, y), R))}{\log R} = \lim_{R \to \infty} \frac{\log \text{vol}(B_M(x, R))}{\log R} + \lim_{R \to \infty} \frac{\log \text{vol}(B_N(y, R))}{\log R} = d_\infty(M) + d_\infty(N).$$

\[\square\]

Remark 3.20. Conditions under which the inequality in Theorem 3.11 (iii) becomes an equality are often studied in the case of (local) dimension theory (cf. [42, 48]). The previous Proposition gives such a condition for the asymptotic dimension.

As the asymptotic dimension is invariant under rough isometries, it is natural to substitute the continuous space with a coarse graining, which destroys the local structure, but preserves the large scale structure. To state it more precisely, recall ([6], p. 194) that a discretization of a metric space $M$ is a graph $G$ determined by an $\varepsilon$-separated subset $G$ of $M$ for which there is a $R > 0$ s.t. $M = \cup_{x \in G} B_M(x, R)$. The graph structure on $G$ is determined by one oriented edge from any $x \in G$ to any $y \in G$, $y \neq x$, denoted $<x, y>$, precisely when $\delta_M(x, y) < 2R$. Define the combinatorial metric on $G$ by $\delta_c(x, y) := \inf\left\{\sum_{i=0}^{n} \delta(x_i, x_{i+1}) : (x_0, \ldots, x_{n+1}) \in \text{Path}_n(x, y), n \in \mathbb{N}\right\}$, where $\text{Path}_n(x, y) := \{(x_0, \ldots, x_{n+1}) : x_i \in G, x_0 = x, x_{n+1} = y, <x_i, x_{i+1}> \in G\}$.

Proposition 3.21. ([6], Theorem 4.9) Let $M$ be a complete Riemannian manifold with Ricci curvature bounded from below. Then $M$ is roughly isometric to any of its discretizations, endowed with the combinatorial metric. Therefore $M$ has the same asymptotic dimension of any of its discretizations.

The previous result, together with the invariance of the asymptotic dimension under rough isometries, shows that, when $M$ has a discrete group of isometries $\Gamma$ with a compact quotient, the asymptotic dimension of the manifold coincides with the asymptotic dimension of the group, hence with its growth (cf. [30]), hence, by the result of Varopoulos [52], it coincides with the 0-th Novikov-Shubin invariant. We will generalise this result in subsection 3.4.

Let us conclude this subsection with some examples.

Example 3.22.

(i) $\mathbb{R}^n$ has asymptotic dimension $n$.

(ii) Set $X := \cup_{n \in \mathbb{Z}} \{(x, y) \in \mathbb{R}^2 : \delta((x, y), (n, 0)) < \frac{1}{4}\}$, endowed with the Euclidean metric, then $d_0(X) = 2, d_\infty(X) = 1$. 
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(iii) Set $X = \mathbb{Z}$ with the counting measure, then $d_0(X) = 0$, and $d_\infty(X) = 1$.

(iv) Let $X$ be the unit ball in an infinite dimensional Banach space. Then $d_0(X) = +\infty$ while $d_\infty(X) = 0$.

Example 3.23. Set $X := \{(x, y) \in \mathbb{R}^2 : x \geq 0, |y| \leq x^\alpha\}$, endowed with the Euclidean metric, where $\alpha \in (0, 1]$. Then $d_\infty(X) = \alpha + 1$.

Proof. This metric space has a uniformly bounded Borel measure, the Lebesgue area, so we can use Proposition 3.18. Set $x_0 := (0, 0)$, and $B_R := B_X(x_0, R)$. Then, if $R \geq \sqrt[4\alpha]{4+1/\alpha+1} \leq R$, $|y| \leq 4r$, and $Q_2 := \{(x, y) \in \mathbb{R}^2 : (2r)^{1/\alpha} \leq x \leq R, |y| \leq 2x^\alpha\}$. Now, if $x_0 > 0$ is s.t. $x_0^2 + x_0^2 = R^2$, we get

$$
\frac{\alpha + 1}{\log x_R} \leq \lim_{R \to \infty} \frac{\log \text{area}(B_R)}{\log R} \leq \limsup_{R \to \infty} \frac{\log \text{area}(B_R)}{\log R} \leq \alpha + 1
$$

and, as $\lim_{R \to \infty} \frac{\log x_R}{\log R} = \lim_{x \to \infty} \frac{-\log x}{\log \sqrt{x^2 + x^2}} = 1$, we get the thesis. \qed

3.3 Asymptotic dimension of some cylindrical ends

In this subsection we want to compare our work with a recent work of Davies'. In [18] he defines the asymptotic dimension of cylindrical ends of a Riemannian manifold $M$ as follows. Let $E \subset M$ be homeomorphic to $(1, \infty) \times A$, where $A$ is a compact Riemannian manifold. Set $\partial E := \{1\} \times A$, $E_r := \{x \in E : \delta(x, \partial E) < r\}$, where $\delta$ is the restriction of the metric in $M$. Then $E$ has asymptotic dimension $D$ if there is a positive constant $c$ s.t.

$$
c^{-1}r^D \leq \text{vol}(E_r) \leq cr^D;
$$

for all $r \geq 1$. He does not assume bounded geometry for $E$. If one does, the two definitions coincide as in the following

Proposition 3.24. With the above notation, if the volume form on $E$ is a uniformly bounded measure (as in Definition 3.14), or in particular if $E$ has bounded geometry (as in Definition 1.1), and there is $D$ as in (3.4), then $d_\infty(E) = D$. 

Proof. Choose \( o \in E \), and set \( \delta := \delta(o, \partial E), \Delta := \text{diam}(\partial E) \). Then it is easy to prove that \( E_{R-\delta-\Delta} \subset B_E(o, R) \subset E_{R+\delta} \).

Then \( c^{-1}(R-\delta-\Delta)^D \leq \text{vol}(B_E(o, R)) \leq c(R+\delta)^D \), and from 3.18 the thesis follows.

Motivated by ([18], example 16), let us set the following

**Definition 3.25.** \( E \) is a standard end of local dimension \( N \) if it is homeomorphic to \( (1, \infty) \times A \), endowed with the metric \( ds^2 = dx^2 + f(x)^2 d\omega^2 \), and with the volume form \( d\text{vol} = f(x)^{N-1} dxd\omega \), where \( (A, \omega) \) is an \( (N-1) \)-dimensional compact Riemannian manifold, and \( f \) is an increasing smooth function.

**Proposition 3.26.** The volume form on a standard end \( E \) is a uniformly bounded measure. Therefore, if \( E \) satisfies (3.4), we get \( d_{\infty}(E) = D \).

Proof. It is easy to show that, for \( (x_0, p_0) \in E, r < x_0 - 1, \)

\[
[x_0 - r/2, x_0 + r/2] \times B_A \left( p_0, \frac{r/2}{f(x_0 + r/2)} \right) \subset B_E((x_0, p_0), r) \subset [x_0 - r, x_0 + r] \times B_A \left( p_0, \frac{r}{f(x_0 - r)} \right)
\]

So that

\[
\int_{x_0 - r/2}^{x_0 + r/2} f(x)^{N-1} dx V_A \left( p_0, \frac{r/2}{f(x_0 + r/2)} \right) \leq V_E((x_0, p_0), r) \leq \int_{x_0 - r}^{x_0 + r} f(x)^{N-1} dx V_A \left( p_0, \frac{r}{f(x_0 - r)} \right)
\]

which implies

\[
rf(x_0 - r/2)^{N-1} V_A \left( p_0, \frac{r/2}{f(x_0 + r/2)} \right) \leq V_E((x_0, p_0), r) \leq 2rf(x_0 + r)^{N-1} V_A \left( p_0, \frac{r}{f(x_0 - r)} \right)
\]

As for \( x_0 \to \infty, V_A(p_0, \frac{r}{f(x_0 - r)}) \sim c \left( \frac{r}{f(x_0 - r)} \right)^{N-1} \), and the same holds for \( V_A(p_0, \frac{r/2}{f(x_0 + r/2)}), \) we get the thesis.

**Corollary 3.27.** Let \( E \) be the standard end of local dimension \( N \) and asymptotic dimension \( D \) in ([18], example 16), which is homeomorphic to \( (1, \infty) \times S^{N-1} \), endowed with the metric \( ds^2 = dr^2 + r^{2(D-1)/(N-1)} d\omega^2 \), and with the volume form \( d\text{vol} = r^{D-1} dr d\omega^{N-1} \). Then \( d_{\infty}(E) = D \).
Remark 3.28. Observe that \( d_\infty(M) \) makes sense for any metric space, hence for any cylindrical end, while Davies’ asymptotic dimension does not. Indeed let \( E := (1, \infty) \times S^1 \), endowed with the metric \( ds^2 = dr^2 + f(r)^2 d\omega^2 \), and with the volume form \( d\nu = f(r)d\nu d\omega \), where \( f(r) := \frac{d}{dr}(r^2 \log r) \). Then \( d_\infty(E) = 2 \), but \( \nu(E_r) \) does not satisfy one of the inequalities in (3.4).

Before closing this section we observe that the notion of standard end allows us to construct an example which shows that we could obtain quite different results if we used \( \lim \inf \) instead of \( \lim \sup \) in the definition of the asymptotic dimension. It makes use of the following function

\[
 f(x) = \begin{cases} \sqrt{x} & x \in [1, a_1] \\ 2 + b_{n-1} + c_{n-1} + (x - a_{2n-1}) & x \in [a_{2n-1}, a_{2n}] \\ 2 + b_{n-1} + c_n + \sqrt{x - a_{2n} + 1} & x \in [a_{2n}, a_{2n+1}] \end{cases}
\]

where \( a_0 := 0, a_n - a_{n-1} := 2^n, b_n := \sum_{k=1}^{n} \sqrt{2^{2k+1}}, c_n := \sum_{k=1}^{n} (2^{2k} - 1), n \geq 1 \).

Proposition 3.29. Let \( M \) be the Riemannian manifold obtained as a \( C^\infty \) regularization of \( C \cup_\varphi E \), where \( C := \{(x, y, z) \in \mathbb{R}^3 : (x-1)^2 + y^2 + z^2 = 1, x \leq 1 \} \), with the Euclidean metric, \( E := [1, \infty) \times S^1 \), endowed with the metric \( ds^2 = dx^2 + f(x)^2 d\omega^2 \), and with the volume form \( d\nu = f(x)d\nu d\omega \), where \( \varphi \) is the identification of \( \{ y^2 + z^2 = 1 \} \times S^1 \). Then the volume form is a uniformly bounded measure, \( d_\infty(M) \geq 2 \) but \( d_\infty(M) \leq 3/2 \), where \( d_\infty(M) := \lim_{r \to \infty} \liminf_{R \to \infty} \frac{\log n_r(B_M(x_R))}{\log R} \).

Proof. Set \( o := (0,0,0) \in M \), then it is easy to see that, for \( n \to \infty \), \( a_n \sim 2^n \), \( b_n \sim c_n \sim 2^{2^n} \), and

\[
 \text{area}(B_M(o, a_{2n})) \sim \frac{1}{2} a_{2n}^2 \\
 \text{area}(B_M(o, a_{2n-1})) \sim \frac{5}{3} b_{2n-1}^{3/2}
\]

so that, calculating the limit of \( \frac{\log \text{area}(B_M(o,R))}{\log R} \) on the sequence \( R = a_{2n} \), we get 2, while on the sequence \( R = a_{2n-1} \), we get 3/2. The thesis follows easily, using Proposition 3.18. \( \square \)

3.4 The asymptotic dimension and the 0-th Novikov-Shubin invariant

In this subsection we show that, for open manifolds of \( C^\infty \)-bounded geometry and satisfying an isoperimetric inequality due to Grigor’yan [20], the asymptotic dimension coincides with the 0-th Novikov-Shubin invariant. Let us start by recalling a recent result of Coulhon-Grigor’yan which is crucial for the following.

Theorem 3.30. ([14], Corollary 7.3) ([20], Proposition 5.2)

Let \( M \) be a complete Riemannian manifold, and set \( \lambda_1(U) \) for the first Dirichlet eigenvalue of \( \Delta \) in \( U \). Then the following are equivalent.
(i) there are $\alpha, \beta > 0$ s.t. for all $x \in M$, $r > 0$, and all regions $U \subset B(x, r)$,

$$\lambda_1(U) \geq \frac{\alpha}{r^2} \left( \frac{V(x, r)}{vol(U)} \right)^\beta$$  \hspace{1cm} (3.5)

(ii) there are $A, C, C' > 0$ s.t. for all $x \in M$, $r > 0$,

$$V(x, 2r) \leq AV(x, r)$$  \hspace{1cm} (3.6)

$$\frac{C}{V(x, \sqrt{r})} \leq H_0(r, x, x) \leq \frac{C'}{V(x, \sqrt{r})}.$$  \hspace{1cm} (3.7)

Condition (3.5) is introduced in [26] and called isoperimetric inequality, whereas inequality (3.6) is introduced in [14] and called the volume doubling property.

Corollary 3.31. Let $M$ be a complete Riemannian manifold of bounded geometry, and assume one of the equivalent properties of the previous Theorem. Then $d_\infty(M) = \limsup_{t \to \infty} \frac{-\log H_0(t, x_0, x_0)}{\log t}$

Proof. Follows from Theorem 1.2 and estimates (3.7).

Remark 3.32. The previous result shows that there are some connections between the asymptotic dimension of a manifold and the notion of dimension at infinity for semigroups (in our case the heat kernel semigroup) considered by Varopoulos (see [54]).

The volume doubling property is a weak form of polynomial growth condition, but still guarantees the finiteness of the asymptotic dimension (for manifolds of bounded geometry).

Proposition 3.33. Let $M$ be a complete Riemannian manifold of bounded geometry, and suppose the volume doubling property (3.6) holds. Then $M$ has finite asymptotic dimension.

Proof. Let $R > 1$, and $n \in \mathbb{N}$ be s.t. $2^{n-1} < R \leq 2^n$. Then $V(x, R) \leq V(x, 2^n) \leq A^n V(x, 1)$, so that

$$1 \leq \frac{V(x, R)}{V(x, 1)} \leq A^n \leq AR^{\log_2 A}.$$  \hspace{1cm} (3.8)

Therefore $d_\infty(M) = \limsup_{R \to \infty} \frac{\log V(x, R)}{\log R} \leq \log_2 A$.

From now on $M$ is an open manifolds of $C^\infty$-bounded geometry and satisfying the isoperimetric inequality (3.5). Then it has finite asymptotic dimension, which we show to coincide with the 0-th Novikov-Shubin invariant. First we need

Proposition 3.34.
For any $x, y \in M, r > 0$, if $B(x, r) \cap B(y, r) \neq \emptyset$, then
\[ \gamma^{-1} \leq \frac{V(x, r)}{V(y, r)} \leq \gamma. \]

(ii) There is a sequence $n_k \in \mathbb{N}$ s.t. $\{B(x, n_k)\}$ is a regular exhaustion of $M$.

Proof.
(i) The inequality easily follows by a result of Grigor’yan ([20], Proposition 5.2), where it is shown that the isoperimetric inequality above implies the existence of a constant $\gamma$ such that
\[ \gamma^{-1} \left( \frac{R}{r} \right)^{\alpha_1} \leq \frac{V(x, R)}{V(y, r)} \leq \gamma \left( \frac{R}{r} \right)^{\alpha_2} \]
for some positive constants $\alpha_1, \alpha_2$, for any $R \geq r$, and $B(x, R) \cap B(y, r) \neq \emptyset$.

(ii) The statement follows from the fact that the volume doubling property implies subexponential (volume) growth, so that the result is contained in ([44], Proposition 6.2).

Theorem 3.35. Let $M$ be an open manifold of $C^\infty$-bounded geometry and satisfying Grigor’yan’s isoperimetric inequality ([13]), endowed with the regular exhaustion $\mathcal{K}$ given by Proposition 3.34 (ii). Then the asymptotic dimension of $M$ coincides with the $0$-th Novikov-Shubin invariant, namely $d_\infty(M) = \alpha_0(M, \mathcal{K})$.

In particular $\alpha_0$ is independent of the limit procedure $\omega$.

Proof. First, from Theorem 3.30 and the previous Proposition, we get
\[ \frac{C \gamma^{-1}}{V(o, \sqrt{t})} \leq \frac{\int_{B(o, r)} \frac{C}{V(x, \sqrt{t})} \, dvol(x)}{V(o, r)} \leq \frac{\int_{B(o, r)} H_0(t, x, x) \, dvol(x)}{V(o, r)} \leq \frac{C' \gamma}{V(o, \sqrt{t})} \]
therefore, by definition of the trace $Tr_{\mathcal{K}}$,
\[ \frac{C \gamma^{-1}}{V(o, \sqrt{t})} \leq Tr_{\mathcal{K}}(e^{-t \Delta}) \leq \frac{C' \gamma}{V(o, \sqrt{t})} \]

hence, finally,
\[ d_\infty(M) = 2 \limsup_{t \to \infty} \frac{\log(V(o, t))}{2 \log t} = 2 \limsup_{t \to \infty} \frac{\log(C' \gamma V(o, \sqrt{t})^{-1})}{\log \frac{1}{t}} \leq \alpha_0'(M) \equiv 2 \limsup_{t \to \infty} \frac{\log \tau(e^{-t \Delta})}{\log \frac{1}{t}} \leq 2 \limsup_{t \to \infty} \frac{\log(C' \gamma^{-1} V(o, \sqrt{t})^{-1})}{\log \frac{1}{t}} = \alpha_0(M) \]

The thesis then follows from Proposition 2.4.
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