The mode solution of the wave equation in Kasner spacetimes and redshift

Oliver Lindblad Petersen

Universität Potsdam, Institut für Mathematik
Am Neuen Palais 10
14469 Potsdam, Germany
E-mail: lindblad@uni-potsdam.de

Abstract

We study the mode solution to the Cauchy problem of the scalar wave equation $\Box \phi = 0$ in Kasner spacetimes. As a first result, we give the explicit mode solution in axisymmetric Kasner spacetimes, which is a special case of the general Kasner spacetimes. Furthermore, we give the small and large time asymptotics of the modes in general Kasner spacetimes. Generically, the modes in non-flat Kasner spacetimes grow logarithmically for small times, while the modes in flat Kasner spacetimes stay bounded for small times. For large times, however, the modes in general Kasner spacetimes oscillate with a polynomially decreasing amplitude. This gives a notion of large time frequency of the modes, which we use to model the wavelength of light rays in Kasner spacetimes. We show that the redshift one obtains in this way actually coincides with the usual cosmological redshift.

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1 Introduction

The scalar wave equation $\Box \phi = 0$ has in the last decades been studied extensively on black hole spacetimes. See for example [8] and [9] for some important recent results and [7] with reference list for an overview of the work done up to 2008. The same wave equation has been studied in Robertson-Walker spacetimes, see e.g. [1, 6, 12] and in de Sitter and anti-de Sitter spacetimes, see e.g. [16]. However, publications concerning the wave equation in Kasner spacetimes, which are one of the most simple non-trivial vacuum solutions to Einstein’s equation, seem hard to find. Only recently, the wave equation in Kasner spacetimes was studied in [4], where the authors
asymptotics are given by the formulation of the small time asymptotics in Kasner spacetimes. We show that the large time asymptotics for a generic choice of \( \omega \), therefore calculate the small and large time asymptotics of the modes \( \alpha_\omega \). However, in general Kasner spacetimes, the ODE does not seem to be explicitly solvable. We see Theorem 4.3 and Theorem 4.5. In fact, the explicit solution in flat Kasner spacetimes are, we give the exact solution in axisymmetric Kasner spacetimes, i.e., where two of the \( p_j \) are equal, see Theorem 6.1. In particular, we conclude that the amplitude of the modes decays (since \( p_j \leq 1 \)) as

\[
|\alpha_\omega(t)| \leq \frac{|c_1| + |c_2| + 1}{\left( \sum_{j=1}^{3} \omega_j^2 t^{2-2p_j} \right)^{1/4}}
\]

for large times \( t \). The fact that the modes oscillate (up to a decreasing amplitude) for large times, naturally gives a notion of 'large time frequency' of the modes, namely the function \( f_\omega \) above. As an application, we show that if we interpret this as the frequency of a light ray sent out in the spatial direction \( \omega \in \mathbb{R}^3 \), we obtain a notion of redshift in Kasner spacetimes which coincides with the usual notion of cosmological redshift. See Section 7 for a discussion of this and Theorem 7.4 for the precise statement.

Before we state and prove the results, we review the necessary global and uniqueness theorem of linear wave equations on globally hyperbolic manifolds and continue by giving a rather thorough introduction to Kasner spacetimes as the non-trivial Bianchi type I vacuum solutions to Einstein’s equation. The experienced reader is therefore recommended skip to Section 3, where we define the mode solution in Kasner spacetime.
Acknowledgements

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2 Preliminaries

2.1 Wave equations on globally hyperbolic manifolds

The well-posedness of the Cauchy problem for the linear wave equations on globally hyperbolic manifolds is known, see [5] for a thorough treatment on this topic. In this paper, we only need the existence and uniqueness of a global solution and will therefore cite the existence and uniqueness theorem proven in [5]. For this, we define so called ‘normally hyperbolic operators’.

Definition 2.1 (Normally hyperbolic operator). Let \((M, g)\) be a Lorentz manifold and let \(E \to M\) be a real or a complex vector bundle. A linear differential operator \(P : C^\infty(M, E) \to C^\infty(M, E)\) of second order is called normally hyperbolic if its principal symbol is given by the metric, i.e. if

\[
\sigma_P(\xi) = -g(\xi, \xi) \cdot \text{id}_E
\]

for all \(x \in M\) and all \(\xi \in T^*_x M\). In other words, if we choose local coordinates \(x^1, \ldots, x^n\) on \(M\) and a local trivialization of \(E\), then

\[
P = -\sum_{i,j=1}^n g^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{j=1}^n A_j(x) \frac{\partial}{\partial x^j} + B(x)
\]

where \(A_j\) and \(B\) are matrix-valued coefficients, depending smoothly on \(x\).

The following theorem is essential for this paper.

Theorem 2.2. On a globally hyperbolic Lorentz manifold \(M\), let \(S \subset M\) be a spacelike Cauchy hypersurface. Let \(\nu\) be the future directed timelike unit normal field along \(S\). Let \(E\) be a vector bundle over \(M\), with connection \(\nabla\), and let \(P\) be a normally hyperbolic operator acting on sections in \(E\), then

\[
\text{supp}(u) \subset J^+_M(K), \quad \text{where } K = \text{supp}(u_0) \cup \text{supp}(u_n) \cup \text{supp}(f) \text{ and } J^+_M(K) \text{ is the causal future of } K.
\]

Proof. See [5, Theorem 3.2.11].

Definition 2.3 (The d’Alembert operator). We define the d’Alembert operator as

\[
\Box := -\text{div} \circ \text{grad} : C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R}).
\]

Remark 2.4 (The wave equation \(\Box \varphi = 0\)). We claim that the d’Alembert operator is normally hyperbolic. Indeed, the principal symbol is given by

\[
\sigma_{\Box}(\xi) = -g(\xi, \xi),
\]

which is exactly the requirement for a normally hyperbolic operator.
2.2 Bianchi type I spacetimes

The Kasner spacetimes are the non-trivial (non-Minkowski space) Bianchi type I-solutions to Einsteins vacuum equations. Therefore, it is natural to introduce the Bianchi type I spacetimes first.

**Definition 2.5** (Bianchi type I spacetimes). Let \( I \subset \mathbb{R} \) be an open interval. A **Bianchi type I spacetime** is the manifold 
\[ M := I \times \mathbb{R}^3, \]
equipped with a metric of the form
\[ g := -dt^2 + \sum_{j=1}^{3} a_j(t)^2(dx^j)^2, \]
where \( a_j : I \rightarrow (0, \infty) \) are smooth.

We define the natural orthonormal frame Bianchi type I spacetimes.

**Definition 2.6.** Define the orthonormal frame
\[ E_0 := \partial_t, \]
\[ E_j := \frac{1}{a_j} \partial_j, \quad \text{for } j = 1, 2, 3. \]

**Remark 2.7.** The only non-zero components of the Levi-Civita connection with respect to this frame are
\[ \nabla_{E_j} E_0 = a_j' \frac{a_j}{a_j} E_j, \]
\[ \nabla_{E_j} E_j = a_j' \frac{1}{a_j} E_0, \]
where \( a_j' := \partial_t a_j \).

Using this, we can calculate the d’Alembert operator in Bianchi type I spacetimes.

**Lemma 2.8.** In Bianchi type I spacetimes, the d’Alembert operator is given by
\[ \square = \partial_t^2 - \sum_{j=1}^{3} a_j' \partial_j - \sum_{j=1}^{3} a_j^2 \partial_j^2. \]

In general relativity, the *cosmological principle* is said to hold in a spacetime if the following two symmetry properties hold.

**Definition 2.9** (Spatial homogeneity and spatial isotropy). Let \((M, g)\) be a globally hyperbolic Lorentz manifold with preferred foliation \( M = J \times S \) with \( J \subset \mathbb{R} \) and \( g = -\beta dt^2 + g_t \) where \( \beta \in C^\infty(M) \) and such that \((S, g_t)\) is Riemannian. Then \((M, g)\) is called

- **spatially homogeneous** if for every \( t \in J \) and every \( p, q \in \{t\} \times S \) there exists an foliation preserving isometry \( f \) of \( M \) such that \( f(p) = q \).
- **spatially isotropic** if for every \((t, p) \in M\) and every \( v, w \in T_{(t, p)}((t) \times S)\) with \( g_t(v, v) = g_t(w, w) \), there exists a foliation preserving isometry \( \phi \) defined on a neighbourhood of \( p \) such that \( d\phi(v) = w \).

Note that all Bianchi type I spacetimes are spatially homogeneous, in fact the coordinate vector fields \( \partial_j \), for \( j = 1, 2, 3 \) are Killing vector fields. However, the cosmological principle does in general not hold, since the spatial isotropy is violated for general Bianchi type I spacetimes. We will give the exact condition in the next proposition.
Proposition 2.10. A Bianchi type I spacetime \((M, g)\) is spatially isotropic, if and only if there exist constants \(c_1, c_2 \in \mathbb{R}_+\), such that
\[
c_1 a_1(t) = c_2 a_2(t) = a_3(t),
\]
for all \(t \in I\).

Proof. It is easy to see that if such constants exist, then \((M, g)\) is isotropic. Assume therefore that \((M, g)\) is isotropic. Since the induced metric on each time slice \(\{t\} \times \mathbb{R}^3\) is flat, the second fundamental form is given by
\[
\mathbb{I}(E_j, E_j) = \nabla_{E_j} E_j = \frac{a_j'}{a_j} E_0.
\]
By assumption, for each point \(p \in M\) there exists an isometry \(\phi\) defined on a neighbourhood of \(p\) preserving the foliation and the point \(p\) and such that
\[
d\phi_p(E_1) = E_2(p).
\]
Moreover, by changing the time orientation if necessary, which is an isometry, we can assume that
\[
d\phi_p(E_0) = E_0(p).
\]
But an isometry also preserves the second fundamental form, and hence
\[
\frac{a_1'}{a_1} E_0(p) = d\phi_p \left( \frac{a_1'}{a_1} E_0 \right) = d\phi_p(\mathbb{I}(E_1, E_1)) = \mathbb{I}(d\phi_p(E_1), d\phi_p(E_1)) = \mathbb{I}(E_2, E_2) = \frac{a_2'}{a_2} E_0(p).
\]
Hence, by the analogous arguments,
\[
\frac{a_1'}{a_1} = \frac{a_2'}{a_2} = \frac{a_3'}{a_3},
\]
which proves the statement by integrating. \(\Box\)

2.3 Kasner spacetimes

As mentioned in the previous section, the Kasner spacetimes are the non-trivial Bianchi type I spacetimes satisfying Einstein's vacuum equations.

Definition 2.11 (Kasner spacetime). Let
\[
M := \mathbb{R}_+ \times \mathbb{R}^3,
\]
be equipped with a Kasner metric
\[
g = -dt^2 + \sum_{j=1}^3 t^{2p_j} (dx^j)^2,
\]
where the \(p_j \in \mathbb{R}\) satisfy the following relations:
\[
\sum_{j=1}^3 p_j^2 = 1, \quad (1)
\]
\[
\sum_{j=1}^3 p_j = 1, \quad (2)
\]
Then \((M, g)\) is called a Kasner spacetime.
Remark 2.12. By Lemma 2.8, the d’Alembert operator in Kasner spacetime is

\[ \Box = \partial^2_t - \frac{1}{t} \partial_t - \sum_{j=1}^{3} \frac{1}{t^{2p_j}} \partial^2_{j} \]

Theorem 2.13. Kasner spacetimes solve Einstein’s vacuum equations, i.e.

\[ \text{Ric}(g) = 0. \]

Moreover, the only Bianchi type I solutions (up to isometry and choice of time orientation) to Einstein’s vacuum equations are the Kasner spacetimes and the Minkowski space or submanifolds thereof.

Proof. See e.g. [15, Chapter 3].

Remark 2.14 (Spatial homogeneity/isotropy). Since Kasner spacetimes are special cases of Bianchi type I spacetimes, they are spatially homogeneous. The conditions (1) and (2) together with Proposition 2.10 imply that Kasner spacetimes are not spatially isotropic. In other words, Kasner spacetimes are spatially anisotropic.

It is actually natural to divide the Kasner spacetimes into two subclasses, depending on the values of the \( p_j \). We see immediately that at least one \( p_j \) must satisfy

\[ 0 \leq p_j \leq 1. \]

A simple calculation gives the formula for the other two.

Lemma 2.15. Assume that \( 0 \leq p_j \leq 1 \). Then

\[ p_k = 1 - p_j \pm \sqrt{1 - p_j^2 - \left( \frac{1 - p_j}{2} \right)^2} \]

for \( k \neq j \). Moreover, one \( p_j \in [-\frac{1}{3}, 0] \) and the other two are contained in \([0, 1]\). The lemma implies that if some \( p_j = 1 \), then the other two must be 0. On the other hand, if all \( p_j \neq 1 \), then all \( p_j \) are non-zero. We distinguish between these cases.

Definition 2.16 (Flat and non-flat Kasner spacetimes). A Kasner spacetime \((M, g)\) is called a flat Kasner spacetime if one \( p_j = 1 \) and it is called non-flat Kasner spacetime if all \( p_j \neq 1 \).

The names suggest that Kasner spacetimes are flat precisely when some \( p_j = 1 \). This statement follows from Lemma 2.18 and Proposition 2.19 below.

In the rest of this section we discuss interesting features of Kasner spacetimes, which will however not be used any further in this paper. The reader interested only in the wave equation on Kasner spacetimes is recommended to continue reading in Section 3.

Proposition 2.17 (The timelike geodesics). Let \((M, g)\) be a Kasner spacetime and \( \gamma : J \subset \mathbb{R} \to M \) a timelike future pointing geodesic, with maximal interval of existence \( J \). Then \( J = (a, \infty) \),

where \( a \in \mathbb{R} \). Moreover, if \( t : M \to I \) is the canonical time coordinate, then

\[ t \circ \gamma(s) \to 0 \]

as \( s \to a \).
Idea of proof. If we write \( \gamma(s) = (t(s), x^1(s), x^2(s), x^3(s)) \) and note that since \( \partial_j \) are Killing vector fields,
\[
g(\partial_j, \gamma'(s)) = c_j,
\]
where \( c_j \in \mathbb{R} \) are constant. Hence
\[
(x^j)'(s) = c_j t(s)^{-2p_j}.
\]
Since \( \gamma \) is timelike, there exists a \( c_0 > 0 \) such that
\[
-c_0 = g(\gamma'(s), \gamma'(s)) = -t'(s)^2 + \sum_{j=1}^{3} t(s)^{2p_j} ((x^j)')^2 = -t'(s)^2 + \sum_{j=1}^{3} c_j^2 t(s)^{-2p_j}.
\]
The problem reduces to the study of the above ODE. For the rest of the proof, see e.g. [15, Chapter 4].

Lemma 2.18. The Kretschmann scalar on a Kasner spacetime is
\[
R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{4}{t^4} \left( p_1^2 p_2^2 + p_1^2 p_3^2 + p_2^2 p_3^2 + \sum_{i=1}^{3} (p_i^2 - p_i) \right).
\]
Consequently, the Kretschmann scalar vanishes for flat Kasner spacetimes, but is nonzero for non-flat Kasner spacetimes. In particular, the Riemannian curvature tensor does not vanish in non-flat Kasner spacetimes.

Proof. For the calculation, see e.g. [15, Chapter 5].

Since the Kretschmann scalar does not ‘blow up’ for \( t \to 0 \) in the case of flat Kasner spacetime, one could hope for an extension of the spacetime ‘over \( t = 0 \)’. Indeed, an extension exists.

Proposition 2.19 (Extendibility of flat Kasner spacetimes). Let \( (M, g) \) be a flat Kasner spacetime with \( p_1 = 1 \) and \( p_2 = p_3 = 0 \) (the other cases are of course similar). Then the map
\[
f : (M, g) \to (\mathbb{R}^4_1, \eta)
f(t, x_1, x_2, x_3) := (t \cosh(x_1), t \sinh(x_1), x_2, x_3)
\]
is a non-surjective isometric embedding of \( (M, g) \) into Minkowski space. In particular, the Riemannian curvature tensor vanishes in flat Kasner spacetimes.

Proof. A simple calculation.

For non-flat Kasner spacetimes, the Kretschmann scalar does ‘blow up’ as \( t \to 0 \). This is exactly what is needed to show that non-flat Kasner spacetimes are inextendible.

Proposition 2.20 (Non-extendibility of non-flat Kasner spacetimes). Non-flat Kasner spacetimes are not extendible, i.e. there exist no non-surjective isometric embedding of a non-flat Kasner spacetime into another spacetime.

Idea of proof. One needs to show that if there exists an extension, then there is a timelike geodesic with image intersecting both the embedded manifold and the complement. This is a standard result in Lorentz geometry, see e.g. [15, Lemma 38]. By Proposition 2.17, we conclude that a timelike geodesic must be ‘extended over \( t = 0 \). But this contradicts the ‘blow up’ of the Kretschmann scalar at \( t = 0 \). For the details, see e.g. [15, Theorem 39].
3 Definition of the mode solution in Kasner spacetimes

In order to fix the notation, we first define the Fourier transform.

**Definition 3.1 (Fourier transform).** For \( f \in L^1(\mathbb{R}^n) \), define the Fourier transform of \( f \) as

\[
F(f)(\omega) := \int_{\mathbb{R}^n} f(x)e^{2\pi i x \cdot \omega} dx,
\]

for all \( \omega \in \mathbb{R}^3 \).

The following theorem is the Fourier decomposition of the solution in Kasner spacetimes.

**Theorem 3.2 (The Fourier decomposition).** Let \( (M = \mathbb{R}_+ \times \mathbb{R}^3, g) \) be a Kasner spacetime. Let \( t_0 \in \mathbb{R}_+ \) and \( \varphi_0, \varphi_n \in C_c^\infty(\{t_0\} \times \mathbb{R}^3) \) be given. The unique solution \( \varphi : M \to \mathbb{R} \) to the Cauchy problem

\[
\Box \varphi = 0, \; \varphi|_{\{t_0\} \times \mathbb{R}^3} = \varphi_0, \; \partial_t \varphi|_{\{t_0\} \times \mathbb{R}^3} = \varphi_n.
\]

is given by

\[
\varphi(t, x) = \int_{\mathbb{R}^3} \alpha_\omega(t) e^{-2\pi i x \cdot \omega} d\omega
\]

for all \( (t, x) \in M \), where \( \alpha_\omega : \mathbb{R}_+ \to \mathbb{C} \) is the unique solution to

\[
\alpha''_\omega(t) + \frac{\alpha'_\omega(t)}{t} + \alpha_\omega(t) 4\pi^2 \sum_{j=1}^3 \frac{\omega_j^2}{t^{2p_j}} = 0, \; \forall t \in \mathbb{R}_+, \tag{3}
\]

\[
\alpha_\omega(t_0) = \int_{\mathbb{R}^3} \varphi_0(x) e^{2\pi i \omega \cdot x} dx, \tag{4}
\]

\[
\alpha'_\omega(t_0) = \int_{\mathbb{R}^3} \varphi_n(x) e^{2\pi i \omega \cdot x} dx. \tag{5}
\]

**Definition 3.3.** The set of all \( \alpha_\omega \) in the above theorem, i.e. the solutions to (3 - 5) for different \( \omega \in \mathbb{R}^3 \), is called the mode solution to the wave equation in Kasner spacetimes. For a fixed \( \omega \in \mathbb{R}^3 \), the solution \( \alpha_\omega \) is called a mode.

From now on, we will study the modes \( \alpha_\omega \) for a fixed \( \omega \in \mathbb{R}^3 \). Note that since we are studying a linear differential equation with real coefficients, the real and the imaginary part will be solutions satisfying the corresponding real and imaginary initial data. We will however, for convenience of notation, study the complex solution. The results of this paper can easily be translated to results for the real and imaginary parts.

**Proof of Theorem 3.2.** Recall from Remark 2.12 that

\[
\Box = -\text{div} (\text{grad}(\varphi)) = \partial_t^2 + \frac{1}{t} \partial_t - \sum_{j=1}^3 \frac{1}{t^{2p_j}} \partial_j^2.
\]

Theorem 2.2 implies that for each \( t \in \mathbb{R}_+ \), the solution \( \varphi(t, \cdot) \) will be compactly supported and therefore the Fourier transform (in the space variable) is well-defined. For a fixed \( t \in \mathbb{R}_+ \), define

\[
\alpha_\omega(t) := F(\varphi(t, \cdot))(\omega).
\]

Note by the dominated convergence theorem that

\[
\Box \varphi(t, x) = \Box F^{-1}(\varphi(t, \cdot))(x) = \int_{\mathbb{R}^3} \Box (\alpha_\omega(t) e^{-2\pi i x \cdot \omega}) d\omega
\]

\[
= \int_{\mathbb{R}^3} \left( \alpha''_\omega(t) + \frac{\alpha'_\omega(t)}{t} + \alpha_\omega(t) 4\pi^2 \sum_{j=1}^3 \frac{\omega_j^2}{t^{2p_j}} \right) e^{-2\pi i x \cdot \omega} d\omega.
\]
This shows that if (3) is fullfilled, then □ϕ = 0. The initial conditions translate into initial conditions for \( \alpha_\omega \) as

\[
\begin{align*}
\alpha_{\omega}(t_0) &= F(\varphi(t_0, \cdot))(\omega) = F(\varphi_0)(\omega), \\
\alpha'_{\omega}(t_0) &= \partial_{t | t=t_0}F(\varphi(t, \cdot))(\omega) = F(\partial_{t | t=t_0}\varphi(t, \cdot))(\omega) = F(\varphi_n)(\omega).
\end{align*}
\]

This completes the proof. \( \square \)

For the proofs of the theorems on the asymptotics, in sections 5 and 6, the following rewritten version of equations (3 - 5) will be useful.

**Lemma 3.4.** The solution \( \alpha_\omega \) in the previous theorem can be written as

\[
\alpha_{\omega}(t) =: \beta_{\omega}(\ln(t))
\]

where \( \beta_{\omega} : \mathbb{R} \to \mathbb{C} \) is the unique solution to

\[
\beta''_{\omega}(s) + \beta_{\omega}(s)\frac{4\pi^2}{3} \sum_{j=1}^{3} \omega_j^2 e^{(2-2p_j)s} = 0, \tag{6}
\]

\[
\beta_{\omega}(\ln(t_0)) = \int_{\mathbb{R}^3} \varphi_0(x)e^{2\pi i \omega \cdot x} dx, \tag{7}
\]

\[
\beta'_{\omega}(\ln(t_0)) = t_0 \int_{\mathbb{R}^3} \varphi_n(x)e^{2\pi i \omega \cdot x} dx. \tag{8}
\]

**Proof.** A simple verification. \( \square \)

## 4 The explicit modes in axisymmetric Kasner spacetimes

In this section we give the explicit solutions to equations (3 - 5) for axisymmetric Kasner spacetimes. This subclass of Kasner spacetimes include the flat Kasner spacetimes.

**Definition 4.1.** We define the axisymmetric Kasner spacetimes to be the Kasner spacetimes where two \( p_j \) are equal. Up to permutation of the indices, there are two possibilities.

**Remark 4.2.** If \( p = (p_1, p_2, p_3) \) satisfies

\[
\sum_{i=1}^{3} p_i = \sum_{i=1}^{3} p_i^2 = 1,
\]

and two of the \( p_i \) are equal, then

\[
\{p_1, p_2, p_3\} = \{1, 0, 0\} \text{ (flat Kasner metric), or } \{p_1, p_2, p_3\} = \left\{-\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right\}.
\]

### 4.1 The explicit mode solution in flat Kasner spacetimes

We start by presenting the explicit mode solution in the flat Kasner spacetimes. Choose

\[
p_1 = 1, \quad p_2 = p_3 = 0,
\]

the other cases are similar. Equation (3) becomes

\[
\alpha''_{\omega}(t) + \frac{\alpha'_{\omega}(t)}{t} + 4\pi^2 \alpha_{\omega}(t) \left(\omega_1^2 t^{-2} + \omega_2^2 + \omega_3^2\right) = 0,
\]

This shows that if (3) is fullfilled, then □ϕ = 0. The initial conditions translate into initial conditions for \( \alpha_\omega \) as

\[
\begin{align*}
\alpha_{\omega}(t_0) &= F(\varphi(t_0, \cdot))(\omega) = F(\varphi_0)(\omega), \\
\alpha'_{\omega}(t_0) &= \partial_{t | t=t_0}F(\varphi(t, \cdot))(\omega) = F(\partial_{t | t=t_0}\varphi(t, \cdot))(\omega) = F(\varphi_n)(\omega).
\end{align*}
\]

This completes the proof. \( \square \)

For the proofs of the theorems on the asymptotics, in sections 5 and 6, the following rewritten version of equations (3 - 5) will be useful.

**Lemma 3.4.** The solution \( \alpha_\omega \) in the previous theorem can be written as

\[
\alpha_{\omega}(t) =: \beta_{\omega}(\ln(t))
\]

where \( \beta_{\omega} : \mathbb{R} \to \mathbb{C} \) is the unique solution to

\[
\beta''_{\omega}(s) + \beta_{\omega}(s)\frac{4\pi^2}{3} \sum_{j=1}^{3} \omega_j^2 e^{(2-2p_j)s} = 0, \tag{6}
\]

\[
\beta_{\omega}(\ln(t_0)) = \int_{\mathbb{R}^3} \varphi_0(x)e^{2\pi i \omega \cdot x} dx, \tag{7}
\]

\[
\beta'_{\omega}(\ln(t_0)) = t_0 \int_{\mathbb{R}^3} \varphi_n(x)e^{2\pi i \omega \cdot x} dx. \tag{8}
\]

**Proof.** A simple verification. \( \square \)
for all \( t \in \mathbb{R}_+ \). We will see that for generic \( \omega \in \mathbb{R}^3 \), the solution will be expressed by Bessel functions. The Bessel functions of first and second kind, usually denoted by \( J_\nu, Y_\nu : \mathbb{C} \to \mathbb{C} \) for \( \nu \in \mathbb{C} \) are linearly independent solutions to the Bessel equation

\[
x^2 y''(x) + xy'(x) + (-\nu^2 + x^2)y(x) = 0.
\]  

See e.g. [2] for the definitions and basic properties of Bessel functions.

The solution for \( \omega_2^2 + \omega_3^2 \neq 0 \) has recently been published in [4, Chapter II]. We complete the list by adding the solution when \( \omega_2 = \omega_3 = 0 \).

**Theorem 4.3** (The explicit solution in flat Kasner spacetimes). Let \((M, g)\) be a flat Kasner spacetime with \( p_1 = 1, p_2 = p_3 = 0 \). The solution to equations (3 - 5) is given by the following, for the different cases of \( \omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3 \):

- \( \omega_1 = \omega_2 = \omega_3 = 0 \):
  \[ \alpha_\omega(t) = \alpha'_\omega(t_0) \ln \left( \frac{t}{t_0} \right) t_0 + \alpha_\omega(t_0), \]

- \( \omega_1 \neq 0, \omega_2 = \omega_3 = 0 \):
  \[ \alpha_\omega(t) = c_1 e^{2 \pi i \omega_1 \ln(t)} + c_2 e^{-2 \pi i \omega_1 \ln(t)}, \]

- \( \omega_2^2 + \omega_3^2 \neq 0 \):
  \[
  \alpha_\omega(t) = c_1 J_{2 \pi i \omega_1} \left( 2 \pi t \sqrt{\omega_2^2 + \omega_3^2} \right) + c_2 Y_{2 \pi i \omega_1} \left( 2 \pi t \sqrt{\omega_2^2 + \omega_3^2} \right),
  \]

where \( c_1, c_2 \in \mathbb{C} \) are constants depending on the initial data given in (4) and (5).

**Proof.** The proof is a simple verification.

\[ \square \]

### 4.2 The explicit mode solution in axisymmetric non-flat Kasner spacetimes

We now present the explicit mode solution in axisymmetric non-flat Kasner spacetimes. Let

\[ p_1 = -\frac{1}{3}, \quad p_2 = p_3 = \frac{2}{3}, \]

the other cases are similar. Equation (3) becomes

\[
\alpha''_\omega(t) + \frac{\alpha'_\omega(t)}{t} + 4 \pi^2 \alpha_\omega(t) \left( (\omega_1^2 t^{2/3} + (\omega_2^2 + \omega_3^2) t^{-4/3} \right) = 0,
\]

for all \( t \in \mathbb{R}_+ \).

In addition to the Bessel functions, the solution will be expressed in terms of a so called "Heun Biconfluent function". The definition of the Heun Biconfluent functions follows the documentation of the computer algebra system Maple, which in turn refers to [11]. See also [3] for a discussion of the Heun Biconfluent function. The literature list of [3] includes several examples from physics and mathematics where this function is used and its properties discussed.

**Definition 4.4** (Heun Biconfluent function). We define the **Heun Biconfluent function**

\[ \text{HeunB}(\alpha, \beta, \gamma, \delta, \cdot) : \mathbb{C} \to \mathbb{C} \]

of the fixed constants \( \alpha, \beta, \gamma, \delta \in \mathbb{C} \) as the unique solution to

\[
xy''(x) - (\beta x + 2x^2 - \alpha - 1) y'(x) - \frac{1}{2}((2\alpha - 2\gamma + 4)x + \beta \alpha + \beta + \delta)y(x) = 0,
\]

such that

\[ y(0) = 1, \quad y'(0) = \frac{\alpha \beta + \beta + \delta}{2 \alpha + 2}. \]
We will use the Heun Biconcluent function only in the case where all constants vanish except δ. In this case
\[ \text{HeunB}(0, 0, 0, \delta, \cdot) : \mathbb{C} \to \mathbb{C} \]
is the unique solution to
\[ xy''(x) + (1 - 2x^2) y'(x) - \left(2x + \frac{\delta}{2}\right)y(x) = 0, \]
such that
\[ y(0) = 1, \quad y'(0) = \frac{\delta}{2}. \]

**Theorem 4.5** (The explicit solution in non-flat axisymmetric Kasner spacetimes). Let \((M, g)\) be a (non-flat) Kasner spacetime with \(p_1 = -\frac{1}{3}, p_2 = \frac{2}{3}\). The solution to equations (3 - 5) is given by the following, for the different cases of \(\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3\):

- \(\omega_1 = \omega_2 = \omega_3 = 0\):
  \[ \alpha_{\omega}(t) = \alpha_{\omega}'(t_0) \ln \left(\frac{t}{t_0}\right) t_0 + \alpha_{\omega}(t_0), \]
- \(\omega_1 \neq 0, \omega_2 = \omega_3 = 0\):
  \[ \alpha_{\omega}(t) = c_1 J_0 \left(\frac{3}{2} \pi \omega_1 t^{4/3}\right) + c_2 Y_0 \left(\frac{3}{2} \pi \omega_1 t^{4/3}\right), \]
- \(\omega_1 = 0, \omega_2^2 + \omega_3^2 \neq 0\):
  \[ \alpha_{\omega}(t) = c_1 J_0 \left(6\pi \sqrt{\omega_2^2 + \omega_3^2} t^{1/3}\right) + c_2 Y_0 \left(6\pi \sqrt{\omega_2^2 + \omega_3^2} t^{1/3}\right), \]
- \(\omega_1 \neq 0, \omega_2^2 + \omega_3^2 \neq 0\):
  \[ \alpha_{\omega}(t) = e^{-\frac{x(t)^2}{2}} \text{HeunB}(0, 0, 0, \delta_{\omega}, x(t)) \left(\int_{t_0}^{t} \frac{e^{x(u)^2}}{\text{HeunB}(0, 0, 0, \delta_{\omega}, x(u))^2} du \cdot c_1 + c_2\right), \]
where
\[ x(t) := L_{\omega} t^{2/3}, \]
\[ \delta_{\omega} := -18\pi^2 \omega_2^2 + \omega_3^2 \]
\[ L_{\omega} := \sqrt{\frac{3}{2} \pi \omega_1 (1 + i)}, \]
and \(c_1, c_2 \in \mathbb{C}\) are constants depending on initial data (4) and (5).

**Proof.** The proof is a straightforward verification. \(\square\)

5 **Small time asymptotics of the modes in general Kasner spacetimes**

**Theorem 5.1** (The asymptotics for small times). Let \(\omega \in \mathbb{R}^3\). Assume that \((M, g)\) is a non-flat Kasner spacetime or a flat Kasner spacetime with \(\omega_j = 0\) for the index \(j\) such that \(p_j = 1\). Then there exist constants \(c_1, c_2 \in \mathbb{C}\), depending on \(\omega\), such that
\[ \alpha_{\omega}(t) - (c_1 \ln(t) + c_2) \to 0 \]
as \( t \to 0 \). Moreover, if \((M, g)\) is a flat Kasner spacetime with \( \omega_j \neq 0 \) for the index \( j \) such that \( p_j = 1 \), then
\[
\alpha_\omega(t) - \left( c_1 e^{2\pi i \omega_1 \ln(t)} + c_2 e^{-2\pi i \omega_1 \ln(t)} \right) \to 0
\]
as \( t \to 0 \), for some constants \( c_1, c_2 \in \mathbb{C} \) depending on \( \omega \).

**Remark 5.2** (Optimal bound on the growth for small times). We claim that the constant \( c_1 \) and \( c_2 \) in the previous theorem cannot be set to zero in general. Theorem 4.5 implies that if \( p_1 = -\frac{1}{3} \) and \( p_2 = p_3 = \frac{2}{3} \), and \( \omega_2 = \omega_3 = 0 \), the solution is given by
\[
\alpha_\omega(t) = \tilde{c}_1 J_0 \left( \frac{3}{2} \pi \omega_1 t^{4/3} \right) + \tilde{c}_2 Y_0 \left( \frac{3}{2} \pi \omega_1 t^{4/3} \right),
\]
where \( \tilde{c}_1, \tilde{c}_2 \in \mathbb{C} \). If we choose initial data so that \( \tilde{c}_1 = 0 \neq \tilde{c}_2 \), then \( |\alpha_\omega(t)| \to \infty \) as \( t \to 0 \), so we cannot set \( c_1 = 0 \) in general. If we choose the initial data so that \( \tilde{c}_1 \neq 0 = \tilde{c}_2 \), then \( \alpha_\omega(t) \to \tilde{c}_1 \) as \( t \to 0 \), so \( c_2 \) does not vanish in general. An analogous argument holds in the flat Kasner case, using Theorem 4.3. Hence the result of Theorem 5.1 is optimal.

We proceed by proving the theorem.

**Proof of Theorem 5.1.** We first observe that if \( \omega = 0 \), the statement is clear since the explicit solution is given by
\[
\alpha_\omega(t) = \alpha'_\omega(t_0) \ln \left( \frac{t}{t_0} \right) t_0 + \alpha_\omega(t_0).
\]
Assume therefore that \( \omega \neq 0 \) in the rest of the proof. Assume first that \((M, g)\) is a non-flat Kasner spacetime or that \((M, g)\) is a flat Kasner spacetime with \( \omega_j = 0 \) for the index \( j \) such that \( p_j = 1 \). Note that these assumptions are exactly what is needed for
\[
K_\omega(s) := 4\pi^2 \sum_{j=1}^{3} \omega_j^2 e^{(2-2p_j)s}
\]
to decay exponentially as \( s \to -\infty \). The crucial observation in the proof is that
\[
\frac{d}{ds} \left( \beta'_\omega(s)^2 + K_\omega(s) \beta_\omega(s)^2 \right) = 2 \beta'_\omega(s) \left( \beta''_\omega(s) + K_\omega(s) \beta_\omega(s) \right) + K'_\omega(s) \beta_\omega(s)^2
\]
\[
= K'_\omega(s) \beta_\omega(s)^2 \geq 0,
\]
since \( K_\omega \) is everywhere increasing. Hence
\[
\beta'_\omega(s_1)^2 + K_\omega(s_1) \beta_\omega(s_1)^2 \leq \beta'_\omega(s_2)^2 + K_\omega(s_2) \beta_\omega(s_2)^2
\]
for any \( s_1 \leq s_2 \). Fix \( s_c \in \mathbb{R} \), and define
\[
C := \beta'_\omega(s_c)^2 + K_\omega(s_c) \beta_\omega(s_c)^2.
\]
The estimate (10) implies that
\[
K_\omega(s) \beta_\omega(s)^2 \leq C
\]
for all \( s \leq s_c \) and hence
\[
|\beta''_\omega(s)| \leq K_\omega(s) |\beta_\omega(s)| \leq \sqrt{K_\omega(s)} \sqrt{C},
\]
for all \( s \leq s_c \). Since \( K_\omega \) decays exponentially for \( s \to -\infty \), this implies that \( \beta''_\omega \in L^1(-\infty, s_c) \). Therefore,
\[
\beta'_\omega(s) = - \int_{s_c}^{s} \beta''_\omega(u) du + \beta'_\omega(s_c) \to - \int_{-\infty}^{s_c} \beta''_\omega(u) du + \beta'_\omega(s_c) =: c_1 \in \mathbb{R}.
\]
We now claim that there exists a $c_2 \in \mathbb{R}$ such that
\[ \beta_\omega(s) - sc_1 \to c_2 \]
as $s \to -\infty$. To see this, note that
\[ \beta_\omega'(s) - c_1 = \int_s^{-\infty} \beta_\omega''(u) du \leq \int_s^{-\infty} |\beta_\omega''(u)| du \leq \sqrt{C} \int_s^{-\infty} \sqrt{K_\omega(u)} du \]
decays exponentially as $s \to -\infty$. In particular,
\[ \beta_\omega'(s) - c_1 \in L^1(-\infty, s_c) \]
and hence
\[ \beta_\omega(s) - c_1 s = - \int_s^{s_c} \beta_\omega'(u) du + \beta_\omega(s_c) - c_1 s \]
\[ = - \int_s^{s_c} [\beta_\omega'(u) - c_1] du - c_1 s_c + \beta_\omega(s_c) \]
\[ \to - \int_{-\infty}^{s_c} [\beta_\omega'(u) - c_1] du - c_1 s_c + \beta_\omega(s_c) =: c_2, \]
as $s \to -\infty$.

Reformulating the convergence using
\[ \beta_\omega(ln(t)) = \alpha_\omega(t) \]
implies that
\[ \alpha_\omega(t) - c_1 ln(t) \to c_2 \]
as $t \to 0$.

For the second statement, it is enough to check the result in Theorem 4.3 and to recall the small time asymptotics for Bessel functions with imaginary parameter $\nu$, see e.g. [2].

6 Large time asymptotics of the modes in general Kasner spacetimes

**Theorem 6.1 (The asymptotics for large times).** Let $(M, g)$ be a Kasner spacetime and let $\omega \neq 0 \in \mathbb{R}^3$. Then there exist constants $c_1, c_2 \in \mathbb{C}$, depending on $\omega$, such that
\[ \alpha_\omega(t) \left( \sum_{j=1}^{3} \omega_j^2 t^{2 - 2p_j} \right)^{1/4} - \left[ c_1 e^{2\pi i \int_0^t f_\omega(u) du} + c_2 e^{-2\pi i \int_0^t f_\omega(u) du} \right] \to 0, \]
as $t \to \infty$, where
\[ f_\omega(t) := \left( \sum_{j=1}^{3} \omega_j^2 \right)^{1/2}. \]

In particular, there exists a $T \geq 0$ such that
\[ |\alpha_\omega(t)| \leq \frac{|c_1| + |c_2| + 1}{\left( \sum_{j=1}^{3} \omega_j^2 t^{2 - 2p_j} \right)^{1/4}} \]
for all $t \geq T$. 

Theorem 6.1 implies that the amplitudes of the modes are monotone decreasing, but also that the modes start to oscillate as trigonometric functions. For trigonometric functions, we have a natural definition of frequency. We define this frequency as the 'large time frequency' of the modes.

**Definition 6.2** (Large time frequency of $\alpha_\omega$). We define the large time frequency of a mode $\alpha_\omega$ to be $f_\omega$ as in Theorem 6.1.

**Corollary 6.3** (Large time frequency in flat and non-flat Kasner spacetimes).

- Let $(M, g)$ be a flat Kasner spacetime with $p_j = 1$ and $p_k = p_l = 0$ for distinct indices $j, k$ and $l$. Then
  
  $$f_\omega(t) \to \sqrt{\omega_k^2 + \omega_l^2}$$

  as $t \to \infty$.

- Let $(M, g)$ be a non-flat Kasner spacetime, with $-\frac{1}{3} \leq p_k < 0$. If $\omega_k \neq 0$, then
  
  $$f_\omega(t) \to \infty$$

  as $t \to \infty$, otherwise
  
  $$f_\omega(t) \to 0$$

  as $t \to \infty$.

It follows that the large time frequency, for a generic choice of $\omega \in \mathbb{R}^3$, converges to a constant in flat Kasner spacetimes and goes to infinity in non-flat Kasner spacetimes. We proceed by proving the theorem. For this we will use the following lemma.

**Lemma 6.4.** Assume that $K : (a, \infty) \to \mathbb{R}_+$ is smooth and $v : (a, \infty) \to \mathbb{R}$ satisfies

$$v''(s) + K(s)v(s) = 0, \quad \forall s \in (a, \infty).$$

Assume furthermore that

$$\psi_K := K^{-\frac{1}{4}} \frac{d^2}{ds^2} \left(K^{\frac{1}{4}}\right) \in L^1(a, \infty)$$

and that

$$K^{1/2} \notin L^1(a, \infty).$$

Then there exist solutions $v_1, v_2 : (a, \infty) \to \mathbb{C}$ to equation (12) such that

$$v_1(s)e^{-i \int_a^s K(u)^{1/2} du} K(s)^{1/4} \to 1$$

$$v_2(s)e^{i \int_a^s K(u)^{1/2} du} K(s)^{1/4} \to 1$$

when $s \to \infty$.

**Proof.** This is a special case of [13, Proposition 3.2].

We apply this result to our case.

**Proof of Theorem 6.1.** Recall, by Lemma 3.4, that equation (3) is equivalent to

$$\beta''_\omega(s) + K_\omega(s)\beta_\omega(s) = 0$$

with

$$K_\omega(s) := 4\pi^2 \sum_{j=1}^{3} \omega_j^2 e^{(2-2p_j)s}.$$

where $\beta_\omega(\ln(t)) := \alpha_\omega(t)$. We claim that the assumptions in Lemma 6.4 are satisfied for this equation. Since $\omega \neq 0$, note that $K_\omega$ is nonzero and constant or grows exponentially as $s \to \infty$. If
\(K_{\omega}\) is a nonzero constant, the assumptions in Lemma 6.4 are trivially satisfied. Assume therefore that \(K_{\omega}\) is exponentially growing. Note that
\[
\psi_{K_{\omega}}(s) = K_{\omega}(s)^{-1/4} \frac{d^2}{ds^2} \left( K_{\omega}(s)^{-1/4} \right) = \frac{1}{4 K_{\omega}(s)^{1/2}} \left( \frac{5 K_{\omega}'(s)^2}{4 K_{\omega}(s)^2} - \frac{K_{\omega}''(s)}{K_{\omega}(s)} \right).
\]
We claim that the term \(\frac{K_{\omega}''(s)}{K_{\omega}(s)}\) is bounded. Let \(k\) be an index such that \(\omega_k \neq 0\) and \(p_k \leq p_j\) for all \(j\) such that \(\omega_j \neq 0\). By factoring out \(e^{-(2p_k)s}\) from both numerator and denominator, we see that
\[
\frac{K_{\omega}''(s)}{K_{\omega}(s)} = \frac{\sum_{j=1}^{3} (2 - 2p_j)^2 \omega_j^2 e^{(2-2p_j)s}}{\sum_{j=1}^{3} \omega_j^2 e^{(2-2p_j)s}} = \frac{\sum_{j=1}^{3} (2 - 2p_j)^2 \omega_j^2 e^{2(p_k-p_j)s}}{\sum_{j=1}^{3} \omega_j^2 e^{2(p_k-p_j)s}}.
\]
Since \(p_k - p_j \leq 0\), we see that both denominator and numerator converge as \(s \to \infty\). Furthermore, the denominator is bounded from below by \(\omega_k^2\). This implies that the quotient converges when \(s \to \infty\). Analogously, one proves that the term \(\frac{K_{\omega}'(s)^2}{K_{\omega}(s)^2}\) converges and it follows that
\[
\frac{5 K_{\omega}'(s)^2}{4 K_{\omega}(s)^2} - \frac{K_{\omega}''(s)}{K_{\omega}(s)}\]
is bounded as \(s \to \infty\). Hence it suffices to show that \(K_{\omega}^{-1/2} \in L^1(a, \infty)\), in order to prove that \(\psi_{K_{\omega}} \in L^1(a, \infty)\). But this is clear, since \(K_{\omega}\) grows exponentially as \(s \to \infty\). What remains in order to apply Lemma 6.4 is to show that \(K_{\omega}^{1/2} \notin L^1(a, \infty)\). Again, this is clear, since \(K_{\omega}\) grows exponentially.

We can therefore apply Lemma 6.4 and conclude that there exist solutions \(\beta_{\omega}^1, \beta_{\omega}^2 : (a, \infty) \to \mathbb{C}\) to equation (6) such that
\[
\begin{align*}
\beta_{\omega}^1(s) &e^{i \int_a^s K_{\omega}(u)^{1/2} du} K_{\omega}(s)^{1/4} \to 1, \\
\beta_{\omega}^2(s) &e^{-i \int_a^s K_{\omega}(u)^{1/2} du} K_{\omega}(s)^{1/4} \to 1,
\end{align*}
\]
which is equivalent to
\[
\begin{align}
\beta_{\omega}^1(s) K_{\omega}(s)^{1/4} - e^{-i \int_a^s K_{\omega}(u)^{1/2} du} &\to 0, \quad (13) \\
\beta_{\omega}^2(s) K_{\omega}(s)^{1/4} - e^{i \int_a^s K_{\omega}(u)^{1/2} du} &\to 0. \quad (14)
\end{align}
\]
It follows that \(\beta_{\omega}^1\) and \(\beta_{\omega}^2\) are linearly independent, therefore there exist constants \(c_1, c_2 \in \mathbb{C}\) such that
\[
\beta_{\omega} = c_1 \beta_{\omega}^1 + c_2 \beta_{\omega}^2.
\]
We now substitute back using \(\alpha_{\omega}(t) = \beta_{\omega}(\ln(t))\). Note that
\[
K_{\omega}(\ln(t)) = 4\pi^2 \sum_{j=1}^{3} \omega_j^2 2^{2-2p_j},
\]
\[
\int_a^{\ln(t)} K_{\omega}(u)^{1/2} du = \int_{e^a}^{t} K_{\omega}(\ln(u))^{1/2} \frac{1}{u} du
\]
\[
= 2\pi \int_{e^a}^{e^t} \left( \sum_{j=1}^{3} \frac{\omega_j^2}{u^{2p_j}} \right)^{1/2} du.
\]
Note that we can, by change of the parameters \(c_1, c_2\) if necessary, choose \(a = \ln(t_0)\). Inserting this in equations (13) and (14) implies the theorem. \(\square\)
7 Application: Redshift in Kasner spacetimes

Light rays in general relativity are described by lightlike geodesics. It is well known that the wavelength of a light ray changes under the impact of a gravitational field. This gives rise to the so called redshift. In general relativity, one usually models the wavelength of a lightlike geodesic as being proportional to the energy of the lightlike geodesic. The question for this section is, can we instead model the wavelength of light using a mode of the wave equation, similar to electrodynamics in Minkowski space? One important difference between Kasner spacetimes and Minkowski spacetimes is the ‘Big Bang’ at $t = 0$. Therefore, one can only expect a well-defined notion of wavelength, modeled by the wave equation, for large times. We will use a notion of ‘large time wavelength’, simply the inverse of the ‘large time frequency’ defined in the previous chapter. We calculate the redshift one obtains using this definition of wavelength and show that it coincides with the classical notion of cosmological redshift.

For notational convenience, we start by fixing a lightlike geodesic. Let us specify the initial data for the lightlike geodesic on the Cauchy hypersurface $\{t_0\} \times \mathbb{R}^3$. Informally, we want to consider a light ray $\gamma$ sent out at time $t_0$ at an arbitrary point $x_0$ in space with the initial (spatial) direction given by an arbitrary vector $v$. Fixing an arbitrary lightlike geodesic

Let $\gamma : J \to M$ be the future pointing lightlike geodesic, written in coordinates as

$$\gamma(s) = (t(s), x^1(s), x^2(s), x^3(s)),$$

with the initial data

$$\gamma(s_0) = (t_0, x_0),$$

$$\gamma'(s_0) = \left(\sum_{j=1}^{3} v_j^2 t_0^{2p_j}, v\right) \in T_{(t_0, x_0)} M,$$

for fixed $x_0, v \in \mathbb{R}^3$. Note that $\gamma$ is indeed lightlike.

The two notions of wavelength

Let us first present the usual way of describing wavelength of a lightlike geodesic. The redshift using this definition of wavelength is called the cosmological redshift and is presented for Robertson-Walker spacetimes in [14, p. 353]. We generalize this notion to Kasner spacetimes by defining the wavelength analogously.

**Definition 7.1** (Wavelength proportional to the energy). Let $\gamma : J \to M$ be the lightlike geodesic we fixed above. The wavelength using the energy $\lambda^E_\gamma : J \to \mathbb{R}_+$ of $\gamma$, measured by $\partial_t$ is defined as

$$\lambda^E_\gamma(s) := \frac{h}{E_\gamma(s)},$$

for all $s \in J$, where $h$ is the Planck constant and $E_\gamma(s)$ is the energy measured by $\partial_t$. The redshift obtained using this definition of wavelength is called the cosmological redshift.

We now turn to our model of the wavelength of $\gamma$ as a mode $\alpha_\omega$ of the wave equation, for some $\omega \in \mathbb{R}^3$. The question is: what $\omega$ should be chosen? Recalling the theory of electromagnetic waves in Minkowski space, the naive choice of mode to model the wave of $\gamma$ is the initial spatial direction $v \in \mathbb{R}^3 \cong T_{(t_0, x_0)} \{t_0\} \times \mathbb{R}^3$. However, there is a subtle but important detail to be noted at this point. The vector $v$ was interpreted above as the initial spatial direction of $\gamma$ in the Cauchy hypersurface $\{t_0\} \times \mathbb{R}^3$ in the Kasner spacetime, i.e. with induced metric

$$g_{t_0} := \sum_{j=1}^{3} t_0^{2p_j} (dx^j)^2.$$
But the mode \( \alpha_\omega(t) \) at \( t = t_0 \), was obtained by taking the Fourier transform of \( \varphi(t_0, \cdot) \) on the hypersurface \( \{t_0\} \times \mathbb{R}^3 \) with the Euclidean metric. Hence choosing the vector \( v \) would be a plausible choice only if \( t_0 = 1 \). In general, we need to pull back \( v \) along the canonical isometry of the hypersurfaces \( \{t_0\} \times \mathbb{R}^3 \) in Minkowski spacetime and the Kasner spacetime. The correct vector is therefore

\[
\tilde{v} := (t_0^{p_1} v_1, t_0^{p_2} v_2, t_0^{p_3} v_3) \in T_{(t_0, x_0)} \{t_0\} \times \mathbb{R}^3 \cong \mathbb{R}^3.
\]

**Definition 7.2** (The large time wavelength). Let \( \gamma : J \subset \mathbb{R} \to M \) be the lightlike geodesic we fixed above. The **large time wavelength** \( \lambda^L_T : J \to \mathbb{R}_+ \) of \( \gamma \), measured by \( \partial_t \), is defined as

\[
\lambda^L_T(s) := \frac{1}{f^\prime(t(s))},
\]

for all \( s \in J \), where \( t : J \to \mathbb{R}_+ \) is the time component of \( \gamma \) and \( f^\prime(t) \) is the large time frequency (see Definition 6.2) of the mode \( \alpha_\omega \), where

\[
\tilde{v} := (t_0^{p_1} v_1, t_0^{p_2} v_2, t_0^{p_3} v_3).
\]

**Comparison of the obtained redshifts**

**Definition 7.3** (Redshift of the lightlike geodesic \( \gamma \)). Let \( \gamma : J \subset \mathbb{R} \to M \) be the lightlike geodesic we fixed above. Let \( p \) and \( q \) be in the image of \( \gamma \), and let \( s_p < s_q \in J \) be such that \( \gamma(s_p) = q \) and \( \gamma(s_q) = q \). Assume that the wavelength of \( \gamma \) observed by \( \partial_t \) is given by a function \( \lambda_\gamma : J \to \mathbb{R}_+ \). Then the redshift between \( p \) and \( q \) relative to the observer field \( \partial_t \) is defined by

\[
z_\gamma(p, q) := \frac{\lambda_\gamma(s_q) - \lambda_\gamma(s_p)}{\lambda_\gamma(s_p)}.
\]

We show that the two notions of redshift are equivalent in Kasner spacetimes.

**Theorem 7.4.** Let \((M, g)\) be a Kasner spacetime and let \( \gamma \) be the above fixed geodesic with initial spatial direction \( v \). Let \( p \) and \( q \) be in the image of \( \gamma \), and let \( s_p < s_q \in J \) be such that \( \gamma(s_p) = p \) and \( \gamma(s_q) = q \). The redshift obtained by using the large time wavelength coincides with the cosmological redshift and equals

\[
z_\gamma(p, q) = \left(\frac{\sum_{j=1}^3 u_j^2 \left(\frac{t_0}{t(s_p)}\right)^{2p_j}}{\sum_{j=1}^3 u_j^2 \left(\frac{t_0}{t(s_q)}\right)^{2p_j}}\right)^{1/2} - 1,
\]

where \( t : J \to \mathbb{R}_+ \) is the time coordinate of \( \gamma \).

**Proof.** Theorem 6.1 implies that

\[
f^\prime(t) = \left(\sum_{j=1}^3 u_j^2 \left(\frac{t_0}{t}\right)^{2p_j}\right)^{1/2},
\]

and therefore

\[
\lambda^L_T(s) = \frac{1}{f^\prime(t(s))} = \frac{1}{\left(\sum_{j=1}^3 u_j^2 \left(\frac{t_0}{t(s)}\right)^{2p_j}\right)^{1/2}}.
\]

Inserting this into Definition 7.3 proves the first part. By definition of the geodesic being lightlike,

\[
0 = \langle \gamma'(s), \gamma'(s) \rangle = -(t'(s))^2 + \sum_{j=1}^3 t(s)^{2p_j} \left(x'_j(s)\right)^2,
\]

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for all \( s \in J \). Since \( \partial_j \) for \( j = 1, 2, 3 \) are Killing fields, \([14, \text{Lemma 9.27}]\) implies that
\[
v_j t_0^{2p_j} = \langle \gamma'(s), \partial_j \rangle = x_j^2(s)t(s)^{2p_j},
\]
for all \( s \in J \), where \( v = (v_1, v_2, v_3) \) is as above. Altogether, this implies that the energy is given by
\[
E_\gamma(s) = -\langle \gamma'(s), \partial_t \rangle = t'(s) = \left( \sum_{j=1}^3 v_j^2 \left( \frac{t_0}{t(s)} \right)^{2p_j} \right)^{1/2}
\]
and hence
\[
\lambda^E_\gamma(s) = \frac{\hbar}{E_\gamma(s)} = \frac{\hbar}{\left( \sum_{j=1}^3 v_j^2 \left( \frac{t_0}{t(s)} \right)^{2p_j} \right)^{1/2}},
\]
for all \( s \in J \). Inserting into Definition 7.3 finishes the proof. \( \square \)

**Remark 7.5.** Studying the proof of the theorem, one observes that the wavelengths coincide up to multiplication by the Planck constant. This is not relevant for our purposes here, therefore it is convenient to study the redshift, where such constants cancel.

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