Local Cohomological Dimension and Rectified $\mathbb{Q}$-Homological Depth
of Complex Analytic spaces

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Abstract. We show that the sum of the local cohomological dimension and the rectified $\mathbb{Q}$-homological depth of a closed analytic subspace of a complex manifold coincide with the dimension of the ambient manifold. In the algebraic case this is equivalent to the coincidence of the rectified $\mathbb{Q}$-homological depth with the de Rham depth studied by Ogus, and follows essentially from his work.

Introduction

Let $X$ be a smooth complex algebraic variety with $\{i: Y \hookrightarrow X$ a reduced closed subvariety. The local cohomological dimension of $Y \subset X$ ([Og 73]) can be defined by

$$\text{lcd}(Y, X) := \max \{ j \in \mathbb{Z} \mid H^j_Y(X, \mathcal{O}_X) \neq 0 \}.$$  

This condition is considered for any quasicoherent sheaves on $X$, but it can be reduced to the case of the structure sheaf $\mathcal{O}_X$ (using inductive limit, local free resolutions of coherent sheaves, and a spectral sequence, see (1.4) below). This definition can be extended to the analytic case where $Y$ is a closed analytic subspace of a complex manifold $X$, using the algebraic local cohomology $H^j_Y$, see [Gr 68], [Mc 77], [Ka 78] (and (1.3) below). From now on, we assume that $X, Y$ are analytic.

We can calculate $\text{lcd}(Y, X)$ in a purely topological way using the $t$-structure on $D^b_c(X, \mathbb{C})$ constructed in [BBD 82]. It is then not very difficult to prove the following (see (2.1) below and also [RSW 21]).

**Theorem 1.** In the above notation, we have the equality

$$\text{lcd}(Y, X) = \text{dim} X - \text{RHD}_\mathbb{Q}(Y)$$  

with

$$\text{RHD}_\mathbb{Q}(Y) := \min \{ j \in \mathbb{Z} \mid p\mathcal{H}^j(Y) \neq 0 \}.$$  

Here $p\mathcal{H}^\ast$ is the cohomological functor associated to the $t$-structure on $D^b_c(X, \mathbb{C})$ (see [BBD 82]), and $\text{RHD}_\mathbb{Q}(Y)$ is called the rectified $\mathbb{Q}$-homological depth, see [HL 91, 1.1.3] and also [Sc 01, p. 387]. Note that the notion of rectified homotopical depth, denoted by $\text{rhd}(Y)$, was introduced in [Gr 68].

The following is already known.

**Theorem 2** ([Hamm, Lê [HL 91, Thm. 1.9]). For a complex analytic space $Y$ and $k \in \mathbb{Z}_{>0}$, the following conditions are equivalent.

(i) $\text{RHD}_\mathbb{Q}(Y) \geq k$,

(ii) For any locally closed irreducible analytic subset $Z \subset Y$, we have

$$H^j_Z \mathcal{Q}_Y = 0 \quad (\forall j < k - \text{dim} Z).$$  

The last condition should be compared with (1.1.2) below (where we may assume that the Whitney stratification is compatible with $Z$ locally on $Y$) and also with the following definition of the de Rham depth $\text{DRD}(Y)$ of $Y$ in the algebraic setting (see [Og 73, 2.12]):

**Definition 1.** We have $\text{DRD}(Y) \geq k \iff$ For any (not necessarily closed) point $y \in Y$,

$$H^j_y(Y) = 0 \quad (\forall j < k - \text{dim} \{ y \}).$$  

Indeed, in the algebraic case, it is shown (see [Og 73, Thm. 2.13]) that
\[
\text{lcd}(Y, X) = \dim X - \text{DRD}(Y).
\]
Theorem 1 is then equivalent to the following.

**Corollary 1.** For a complex algebraic variety $Y$, we have the equality
\[
\text{DRD}(Y) = \text{RHD}_\mathbb{Q}(Y_{\text{an}}).
\]

Note that the definition of $H^j_{\text{dR}}(Y)$ in [Ha 75] employs the completion along $Y$ of the de Rham complex of a smooth ambient variety $X$. Consequently the translation to our setting is not quite trivial. However, the comparison with singular cohomology is shown in [Ha 75, Ch. 4, Thm. 1.1] when $y$ is a closed point (as is informed from T. Reichelt and U. Walther). Moreover the restriction to the closed points is allowed by the constructibility of the subset $S_Y$ in [Og 73, 2.15.1]. (Note that the latter subset corresponds to $S^\nu_y(\mathcal{Q}_Y)$ in (1.1.7) below.) So Theorem 1 in the algebraic case follows essentially from [Og 73, Thm. 2.13 combined with 2.15.2].

Using our method, it is rather easy to show the following (which should be known to specialists).

**Theorem 3.** For a reduced complex analytic space $Y$, we have $\text{RHD}_\mathbb{Q}(Y) \geq 1$ if and only if $Y$ does not contain an isolated point.

(This corresponds to a special case of a theorem of Hartshorne-Lichtenbaum [Ha 68, Thm. 3.1], [Og 73, Cor. 2.10], see also [MP 21, Cor. 1.19].)

We have the equality $\text{RHD}_\mathbb{Q}(Y) = 1$ if $Y$ contains a 1-dimensional irreducible component or if $Y = Y_1 \cup Y_2$ with $Y_1, Y_2$ smooth, $d_{Y_1}, d_{Y_2} \geq 2$, and $Y' := Y_1 \cap Y_2$ is non-empty and 0-dimensional, where the intersection is not assumed transversal. For the last case, we use the distinguished triangle
\[
\mathcal{Q}_Y \to \mathcal{Q}_{Y_1} \oplus \mathcal{Q}_{Y_2} \to \mathcal{Q}_{Y'}, +1,
\]
and the long exact sequence associated to the cohomological functor $\mathcal{P}_\mathcal{H}^\bullet$. The last case follows also from Theorem 4 just below (which corresponds to [Og 73, Cor. 2.11]).

**Theorem 4.** For a reduced complex analytic space $Y$, the following conditions are equivalent:

(i) $\text{RHD}_\mathbb{Q}(Y) \geq 2$ (that is, $\text{lcd}(Y, X) \leq d_X - 2$),

(ii) Any irreducible component of $Y$ has dimension at least 2, and $S_y \cap Y$ is connected for any $y \in Y$, where $S_y$ is a sufficiently small sphere in an ambient space with center $y$.

The argument is similar to the proof of Theorem 3 with certain modifications (here the topological cone theorem is used), see (2.4) below for some generalizations.

In Section 1 we review some basics of $t$-structure, holonomic $\mathcal{D}$-modules, algebraic local cohomology, and local cohomological dimension. In Section 2 we prove Theorems 1 and 3–4 together with Theorems 2.4a–b.

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1. Preliminaries

In this section we review some basics of $t$-structure, holonomic $\mathcal{D}$-modules, algebraic local cohomology, and local cohomological dimension.
1.1. t-structure. Let $X$ be a reduced complex analytic space. We denote by $D^b_c(X, \mathbb{Q})$ the derived category of bounded $\mathbb{Q}$-complexes on $X$ with constructible cohomology sheaves. For $k \in \mathbb{Z}$ we have the full subcategories $D^b_c(X, \mathbb{Q})_{\leq k}$, $D^b_c(X, \mathbb{Q})_{\geq k}$ defined by the following conditions for $\mathcal{F} \in D^b_c(X, \mathbb{Q})$ respectively:

(1.1.1) $\mathcal{H}^j \mathcal{F}|_S = 0$ \quad ($j > k - d_S$),

(1.1.2) $\mathcal{H}_S^j \mathcal{F} = 0$ \quad ($j < k - d_S$).

Here $S$ runs over strata of a Whitney stratification $\mathcal{I}$ of $X$ compatible with $\mathcal{F}$, and $d_S := \dim S$, see [BBD 82 2.1.2] (where $p(S) = -d_S$) and [Di 04], [KS 90], [Sc 01], etc. This definition is independent of the Whitney stratification (as is seen just below). Recall that

(1.1.3) $\mathcal{H}_S^j \mathcal{F} = \mathcal{H}_S^j i_S^* \mathcal{F}$ \quad ($j \in \mathbb{Z}$),

with $i_S : S \hookrightarrow X$ the inclusion, see (1.1.10) below. Notice that a shift of indices by $d_S$ occurs in the last conditions of (1.1.1–2) if we restrict $\mathcal{F}$ to a local transversal slice to each stratum $S \in \mathcal{I}$ as in the proof of Theorem 3, see (2.2) below. There is also a shift by $2d_S$ (or $-2d_S$) if we apply $R\Gamma (x)$ (or the dual functor $D$) to a local system on $S$ with $x \in S$. (This is closely related to [Og 73 2.14].)

We then see that the above two conditions are respectively equivalent to the following support and cosupport conditions (implying the independence of stratification):

(1.1.4) $\dim S_i(\mathcal{F}) \leq k - i$ \quad ($\forall i \in \mathbb{Z}$),

(1.1.5) $\dim S'_i(\mathcal{F}) \leq i - k$ \quad ($\forall i \in \mathbb{Z}$),

with

(1.1.6) $S_i(\mathcal{F}) := \{ x \in X \mid \mathcal{H}^i \mathcal{F}_x \neq 0 \}$,

(1.1.7) $S'_i(\mathcal{F}) := \{ x \in X \mid \mathcal{H}^i(\mathcal{F}) \neq 0 \}$,

see [Di 04], [KS 90], [Sc 01], etc. (The cosupport condition (1.1.5) seems to be related closely to [Og 73 2.14–15].)

For $k \in \mathbb{Z}$, set

$$D^b_c(X, \mathbb{Q})^{[k]} := D^b_c(X, \mathbb{Q})_{\leq k} \cap D^b_c(X, \mathbb{Q})_{\geq k}.$$ 

These are abelian full categories of $D^b_c(X, \mathbb{Q})$, see [BBD 82]. By definition this $t$-structure is self-dual, that is,

(1.1.8) $D(\mathcal{F})^{[k]} = \mathcal{H}^k(\mathcal{F})_{\leq -k} = \mathcal{H}^k(\mathcal{F})_{\geq -k}$,

in particular, $D^b_c(X, \mathbb{Q})^{[0]}$ is stable by the functor $D$ (which assigns the dual). Recall that, in the $X$ smooth case, we have (omitting the Tate twist)

(1.1.9) $D(\mathcal{F}) = R\mathcal{H}om_{\mathbb{Q}}(\mathcal{F}, Q^\bullet_{X}[2d_X]).$

Remark 1.1 (i). The above argument is valid with $\mathbb{Q}$ replaced by any subfield of $\mathbb{C}$.

Remark 1.1 (ii). For a closed analytic subset $Y \subset X$, we have the isomorphisms

(1.1.10) $R\Gamma_Y = (i_Y)_* i_Y^*$, \quad $i_Y^* = D i_Y^* D$,

with $i_Y : Y \hookrightarrow X$ the canonical inclusion. Indeed, if $j' : X \setminus Y \hookrightarrow X$ denotes the inclusion, we have the distinguished triangles

(1.1.11) $(i_Y)_* i_Y^* \to \text{id} \to Rj'_* j'^{-1} ±$,

$$R\Gamma_Y \to \text{id} \to Rj'_* j'^{-1} ±.$$
These imply the first isomorphism in (1.1.10) (non-canonically). The second isomorphism can be reduced (non-canonically) to

\begin{equation}
\delta Y = D \delta Y D,
\end{equation}

since $D^2 = \text{id}$. (Some more argument is required to get the canonical isomorphisms due to the ambiguity of mapping cones.)

**Remark 1.1** (iii). In the case $\mathcal{D} = \mathcal{Q}_Y$ with $Y$ a closed analytic subset of $X$, we have

\begin{equation}
S_0^Y(Y) = \{ x \in Y \mid H^{1-i}(S_x \cap Y, \mathbb{Q}) \neq 0 \},
\end{equation}

where $S_x$ is a sufficiently small sphere in $X$ with center $y$ (using a topological cone theorem together with (1.1.11)). This may be related to [Mi 68] in [Og 73].

**Remark 1.1** (iv). The topological cone theorem means a stratified homeomorphism

\begin{equation}
Y \cap B_y \cong \text{Cone}(Y \cap S_y).
\end{equation}

Here $S_y := \partial B_y$ with $B_y$ a sufficiently small closed ball in an ambient smooth space with center $y$, and $\text{Cone}(E)$ for a topological space $E$ means a topological cone, which is obtained by collapsing $E \times \{0\}$ in $E \times [0,1]$. Moreover the distance function $\delta_y$ from $y$ on the left-hand side coincides with the second projection on the right-hand side up to a constant multiple. (This is proved by using the integral curves of a controlled $C^\infty$ vector field $\xi$ on $B_y \setminus \{y\}$ satisfying $\langle \xi, d\delta_y \rangle = 1$.)

**Remark 1.1** (v). For a closed analytic subset $Y \subset X$ and $k \in \mathbb{Z}_{>0}$, we have by using (1.1.2)

\begin{equation}
\text{RHD}_Y(Y) \geq k \iff \mathcal{Q}_Y \in D^b(X, \mathbb{Q})^{\geq k}.
\end{equation}

### 1.2. Holonomic $\mathcal{D}$-modules

Let $X$ be a complex manifold, and $\mathcal{D}_X$ be the sheaf of holomorphic differential operators. It has the filtration $F$ by order of differential operators such that the associated graded ring $\text{Gr}_F \mathcal{D}_X$ is locally isomorphic to the polynomial ring over $\mathcal{O}_X$, and $\text{Specan}_X \text{Gr}_F \mathcal{D}_X$ is naturally identified with the cotangent bundle $T^*X$, that is, $\text{Gr}_F \mathcal{D}_X$ is identified with the direct image to $X$ of the sheaf of holomorphic functions on $T^*X$ whose restriction to each fiber is a polynomial.

A coherent left $\mathcal{D}_X$-module $\mathcal{M}$ is called holonomic if there is locally an increasing filtration $F$ on $\mathcal{M}$ such that $(\mathcal{M}, F)$ is a filtered $\mathcal{D}_X$-module, $F_p \mathcal{M} = 0$ for $p \ll 0$, $\text{Gr}_F \mathcal{M}$ is a coherent $\text{Gr}_F \mathcal{D}_X$-module, and $\text{Supp} \text{Gr}_F \mathcal{M} \subset T^*X$ has dimension $d_X := \dim X$.

We denote by $M_{\text{hol}}(\mathcal{D}_X)$ the category of holonomic left $\mathcal{D}_X$-modules. This is an abelian subcategory of the category of left $\mathcal{D}_X$-modules $M(\mathcal{D}_X)$, which is closed by subquotients and extensions in $M(\mathcal{D}_X)$. Moreover it is stable by the dual functor $D$.

**Remark 1.2** (i). For a holonomic $\mathcal{D}_X$-module $\mathcal{M}$, let $\text{DR}_X(\mathcal{M})$ be the de Rham complex shifted by $d_X$. It is well-known that

\begin{equation}
\text{DR}_X(\mathcal{M}) \in D^b_c(X, \mathbb{C})^{[0]}.
\end{equation}

Indeed, it is shown in [Ka 74] that

\begin{equation}
\text{DR}_X(\mathcal{M}) \in D^b_c(X, \mathbb{C})^{<0}.
\end{equation}

This can be proved by induction on strata of a Whitney stratification $\mathcal{S}$. If one can show that the $\mathcal{H}^j \text{DR}_X(\mathcal{M})|_S$ are locally constant on each $S \in \mathcal{S}$, the finiteness follows from the theory of compact operators as in the case of the cohomology of vector bundles on compact complex manifolds. Indeed, the hypercohomology on a sufficiently small ball does not depend on the radius of the ball using the transversality of the boundary spheres with the strata of
the stratification. Note that there is a shift by $d_S$ if we restrict to a local transversal slice to $S$. This implies (1.1.1) with $j = 0$.

The assertion (1.2.1) then follows from (1.2.2) using (1.1.8), since the de Rham functor $\text{DR}_X$ is compatible with the dual functor $\text{D}$ (see [Sa.88, Prop. 2.4.12] for the filtered case).

As a corollary of (1.2.1), we get the functorial isomorphisms

\[(1.2.3)\quad \text{DR}_X \circ \mathcal{H}^j = \mathcal{P} \mathcal{H}^j \circ \text{DR}_X \quad (j \in \mathbb{Z}),\]

where $\mathcal{P} \mathcal{H}^\bullet$ is the cohomological functor associated with the $t$-structure on $D^b_c(X, \mathbb{C})$ which is constructed in [BBD.82].

**Remark 1.2** (ii). In the case $\mathcal{M}$ is a *regular holonomic* $\mathcal{D}_X$-module, it is also possible to prove (1.2.1) by induction on $d = \dim \text{Supp} \mathcal{M}$ using [De.70] and the distinguished triangle

\[(1.2.4)\quad \text{R} \Gamma_{[Y]} \mathcal{M} \to \mathcal{M} \to \text{R} \Gamma_{[X|Y]} \mathcal{M} \to \]

(see (1.3) just below) together with an embedded resolution of $Y \subset \text{Supp} \mathcal{M}$. Here $Y$ is a closed analytic subset of dimension at most $d-1$ such that the complement $\text{Supp} \mathcal{M} \setminus Y$ is smooth and purely $d$-dimensional, and the restriction of $\text{DR}_X(\mathcal{M})$ to this complement is a shifted local system. (We may assume $Y$ is a divisor shrinking $X$ if necessary.)

It is also possible to define *regular holonomic* $\mathcal{D}$-modules by induction on $\dim \text{Supp} \mathcal{M}$ using (1.2.4) and [De.70], see [Sa.22].

**1.3. Algebraic local cohomology.** Let $X$ be a complex manifold with $Y \subset X$ a closed analytic subset. The *algebraic local cohomology* $\mathcal{H}^j_{[Y]} \mathcal{M}$ for an $\mathcal{O}_X$-module $\mathcal{M}$ is defined by

\[(1.3.1)\quad \mathcal{H}^j_{[Y]} \mathcal{M} := \lim_k \text{Ext}^j_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^k_Y, \mathcal{M}) \quad (j \in \mathbb{Z}),\]

with $\mathcal{I}_Y \subset \mathcal{O}_X$ the ideal of $Y$, see [Gr.68]. This can be extended to the case of a $\mathcal{D}_X$-module $\mathcal{M}$ using an injective resolution of $\mathcal{M}$, since

\[(1.3.2)\quad \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^k_Y, \mathcal{M}) = \mathcal{M}^{\mathcal{I}^k_Y},\]

with right-hand side the subsheaf of $\mathcal{M}$ annihilated by $\mathcal{I}_Y^k$ (and $\partial_x \mathcal{I}_Y^k \subset \mathcal{I}_Y^{k-1}$ with $x_i$ local coordinates), see also [Ka.78].

It is known that the $\mathcal{H}^j_{[Y]} \mathcal{M}$ are holonomic if so is $\mathcal{M}$ (where regularity is not assumed), see [Ka.78] (where $b$-functions in a generalized sense are used).

**Remark 1.3** (i). We can also define

\[\text{R} \Gamma_{[Y]}, \quad \text{R} \Gamma_{[X|Y]}, \quad \mathcal{H}^j_{[X|Y]},\]

by replacing $\text{Ext}^j_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^k_Y, \mathcal{M})$ in (1.3.1) respectively with

\[\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{I}^k_Y, \mathcal{J}^\bullet), \quad \mathcal{H}om_{\mathcal{O}_X}(\mathcal{J}^k_Y, \mathcal{J}^\bullet), \quad \text{Ext}^j_{\mathcal{O}_X}(\mathcal{J}^k_Y, \mathcal{M}),\]

where $\mathcal{J}^\bullet$ is an injective resolution of $\mathcal{M}$. (Recall that an injective $\mathcal{D}_X$-module is an injective $\mathcal{O}_X$-module, since $\mathcal{D}_X$ is flat over $\mathcal{O}_X$.) Note that $\mathcal{H}^0_{[X|Y]}$ is the usual localization if $Y$ is a hypersurface.

**Remark 1.3** (ii). It is well-known (see for instance [Me.77]) that

\[(1.3.3)\quad \text{DR}_X \text{R} \Gamma_{[Y]} \mathcal{O}_X = \text{R} \Gamma_{[X|Y]} \text{DR}_X \mathcal{O}_X,\]

as a consequence of [AH.55, Lem. 17] (applied to an embedded resolution), see also [Gr.66, Thm. 2]. This is closely related to regularity (although the equivalence of categories is not needed for our purpose).
Remark 1.3 (iii). For the calculation of the local cohomology, we have locally a Čech-type complex associated to local generators $g_j$ of the ideal of $Y$ and defined by using the $\mathcal{H}^0_{[X|Z]}$ with $Z = \bigcup_{j \in J} Z_j$, where the $Z_j$ are hypersurfaces defined by $g_j$, see [Ei05, Thm. A1.3] for stalks. This can be applied to show the commutativity of the local cohomology with the direct image of $\mathcal{D}$-modules by an embedded resolution of $Y$.

Using this argument, we can prove also the holonomicity of the local cohomology in the structure sheaf case by reducing to the normal crossing divisor case.

1.4. Local cohomological dimension. For a closed subvariety $Y$ of a smooth variety $X$, the local cohomological dimension ([Og 73]) is defined by

\begin{equation}
\text{lcd}(X, Y) := \max \{ j \in \mathbb{Z} \mid \mathcal{H}^j_Y(\mathcal{F}) \neq 0 \ (\exists \mathcal{F} \text{ quasicoherent}) \}.
\end{equation}

Here it is enough to consider the case $\mathcal{F} = \mathcal{O}_X$. Indeed, the assertion is essentially local, and we may assume $X = \text{Spec} \ A$, where a quasicoherent sheaf $\mathcal{F}$ corresponds to an $A$-module $M$. Since the local cohomology commutes with inductive limit, and any $A$-module is the inductive limit (or union) of its finite $A$-submodules, it is enough to consider the case of coherent sheaves.

Let $\mathcal{L}^i \to \mathcal{F}$ be a free resolution, where the $\mathcal{L}^p$ are free sheaves of finite length, and vanish for $p > 0$. We have the spectral sequence

\begin{equation}
E^{p,q}_1 = \mathcal{H}^q_Y \mathcal{L}^p \implies \mathcal{H}^{p+q}_Y \mathcal{F}.
\end{equation}

This implies that it is sufficient to consider the structure sheaf case. The above argument may be called in [Og 73] “dévissage” (which is sometimes used in the Grothendieck school).

2. Proof of the main theorems

In this section we prove Theorems 1 and 3–4 together with Theorems 2.4a–b.

2.1. Proof of Theorem 1. There are isomorphisms in $D^b_c(X, \mathbb{C})$:

\begin{equation}
\begin{aligned}
\text{DR}_X \Gamma_Y \mathcal{O}_X &= \text{R} \Gamma_Y \text{DR}_X \mathcal{O}_X = \text{R} \Gamma_Y \mathcal{O}_X[d_X] \\
&= i_Y^* \mathcal{C}_X[d_X] = D i_Y^* D(\mathcal{C}_X[d_X]) = (D \mathcal{C}_Y)[-d_X],
\end{aligned}
\end{equation}

where the zero extension $(i_Y)_*$ is omitted to simplify the notation. Indeed, the first, third and fourth, and last isomorphisms respectively follow from (1.3.3), (1.1.10), and (1.1.9). Since the de Rham functor $\text{DR}_X$ is faithful on holonomic $\mathcal{D}_X$-modules, the assertion then follows from (1.1.8) and (1.2.3), see also [RSW 21, Section 1.1]. This finishes the proof of Theorem 1.

2.2. Proof of Theorem 3. If $Y$ contains an isolated point $y$, we have $\text{RHD}_Q(\{y\}) = 0$ by definition. Assume $Y$ does not contain such a point. By induction on $d_Y := \text{dim} \ Y$ we show that

\begin{equation}
Q_Y \in D^b_c(Y, \mathbb{Q})^{>1}.
\end{equation}

Let $\mathcal{S}$ be a Whitney stratification of $Y$. If there is no 0-dimensional stratum, the assertion (2.2.1) is shown by taking a local transversal slice to each stratum and applying the inductive hypothesis. Indeed, we have the local topological triviality along the stratum. This means that we have locally a homeomorphism $X \cong T_S \times S$, where $T_S$ denotes a local transversal slice to $S$ (and $S$ is replaced by a small open ball). Hence $Q_X$ is locally the pull-back of $Q_{T_S}$ by the smooth projection $T_S \times S \to T$. (Note that, under the pull-back by a smooth projection with relative real dimension $2r$ in general, the indices in the last conditions of
(1.1.1-2) are shifted by \( r \), since \( p(S) \) in \([BBD82, 2.1.2]\) is given by the real dimension of \( S \) divided by 2 up to sign, see also [Di04, p.131].)

In general, let \( y \) be a point contained in a 0-dimensional stratum \( S \in \mathcal{S} \). We may assume there are no other 0-dimensional stratum, since the assertion is local. Set \( U := Y \setminus \{ y \} \) with natural inclusion \( j_U : U \hookrightarrow Y \). There is a distinguished triangle

\[
(2.2.2) \quad Q_Y \to R(j_U)_*Q_U \to \mathcal{C}^\bullet, \quad \mathcal{C}^\bullet \cong R(j_U)_*Q_U/Q_Y.
\]

By the above assertion, \( Q_U \in D_c^b(U, \mathbb{Q})^{>1} \). We then get that

\[
(2.2.3) \quad R(j_U)_*Q_U \in D_c^b(Y, \mathbb{Q})^{>1},
\]

using the Čech-type complex defined locally as in Remark (1.3) (iii) by choosing generators of the maximal ideal of \( \mathcal{O}_{Y,y} \) (see \([BBD82]\) for the algebraic case). On the other hand, \( \mathcal{C}^\bullet \) is supported on \( \{ y \} \), and \( \mathcal{C}^\bullet[-1] \in D_c^b(Y, \mathbb{Q})^{>1} \). So we get the assertion (2.2.1) using the long exact sequence associated to the cohomological functor \( p\mathcal{H}^\bullet \). This finishes the proof of Theorem 3.

2.3. Proof of Theorem 4. The argument is similar to the proof of Theorem 3 using the distinguished triangle (2.2.2) and applying Theorem 3 to a local transversal slice to each 1-dimensional strata. (For a higher-dimensional stratum, the shift of indices associated with the restriction to a local transversal slice is sufficiently large.) Note that \( \mathcal{C}^\bullet \in D_c^b(Y, \mathbb{Q})^{>1} \) if and only if the last condition of (ii) holds. This finishes the proof of Theorem 4.

2.4. Generalizations of Theorems 3–4. It is not difficult to generalize Theorems 3–4 as follows. (This answers a question of T. Reichelt and U. Walther.)

**Theorem 2.4a.** For a reduced complex analytic space \( Y \) and for \( k \in \mathbb{Z}_{>0} \), the following conditions are equivalent.

(i) \( \text{RHD}_Q(Y) \geq k \),

(ii) \( \text{RHD}_Q(T_S) \geq k - d_S \) for a local transversal slice \( T_S \) to each stratum \( S \) of a Whitney stratification \( \mathcal{S} \) of \( Y \) with \( d_S \in [1, k-1] \), and \( \mathcal{H}^j_{\{y\}}Q_Y = 0 \) for \( j < k \) if \( y \) belongs to a 0-dimensional stratum of \( \mathcal{S} \).

**Proof.** It is easy to see that (i) \( \implies \) (ii). Assume (ii) holds. Let \( U \) be the complement of the union of 0-dimensional strata. Then the first condition of (ii) implies that \( \text{RHD}_Q(U) \geq k \). (Here we have to consider all the strata \( S \) with \( d_S \in [1, k-1] \), since it might happen that there is no 1-dimensional stratum.) The assertion then follows from the theory of gluing \( t \)-structure, see \([BBD82, 1.4.10]\). (It is also possible to verify (1.1.5).) This finishes the proof of Theorem 2.4a.

By a similar argument, we get the following.

**Theorem 2.4b.** For a reduced quasi-projective complex variety \( Y \) and \( k \in \mathbb{Z}_{>0} \), the following conditions are equivalent.

(i) \( \text{RHD}_Q(Y^{an}) \geq k \),

(ii) \( \text{RHD}_Q(Y'^{an}) \geq k - 1 \) for a general hyperplane section \( Y' \subset Y \), and \( \mathcal{H}^j_{\{y\}}Q_{Y^{an}} = 0 \) for \( j < k \) and for any \( y \in Y^{an} \).

**Proof.** A general hyperplane section means that \( Y' \) is transversal to any stratum of a Whitney stratification \( \mathcal{S} \) (hence \( Y' \) does not meet any 0-dimensional stratum). Note that a general hyperplane section intersects any positive-dimensional stratum \( S \). (Indeed, it must intersect the closure of \( S \), and the intersection \( Y' \cap S \) cannot be contained in the boundary of \( S \) using the transversality for the strata \( S' \subset S \) with \( d_{S'} = d_S - 1 \), where we may assume \( Y \)
is projective.) So the argument is essentially the same as above. (For the proof of the implication $(i) \implies (ii)$ where $y \in Y^{\text{an}}$ is arbitrary, we can employ the independence of stratification.) This finishes the proof of Theorem 2.4b.

**Remark 2.4.** In the last condition of $(ii)$ in Theorem 2.4b, we may restrict to the $y$ belonging to the union of $0$-dimensional strata of some Whitney stratification of $Y$.

**References**

[AH55] Atiyah, M. F., Hodge, W. V. D., Integrals of the second kind on an algebraic variety, Ann. Math. 62 (1955), 56–91.

[BBD82] Beilinson, A., Bernstein, J., Deligne, P., Faisceaux pervers, Astérisque 100, Soc. Math. France, Paris, 1982.

[De70] Deligne, P., Equations différentielles à points singuliers réguliers, Lect. Notes in Math. 163, Springer, Berlin, 1970.

[Di04] Dimca, A., Sheaves in Topology, Universitext, Springer, Berlin, 2004.

[El05] Eisenbud, D., The Geometry of Syzygies – A Second Course in Commutative Algebra and Algebraic Geometry, Springer, Berlin, 2005.

[Gr66] Grothendieck, A., On the de Rham cohomology of algebraic varieties, Publ. Math. IHES 29 (1966), 95–103.

[Gr68] Grothendieck, A., Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux (SGA2), North-Holland, Amsterdam, 1968.

[Ha68] Hartshorne, R., Cohomological dimension of algebraic varieties, Ann. Math. 88 (1968), 403–450.

[Ha75] Hartshorne, R., On the de Rham cohomology of algebraic varieties, Publ. Math. IHES 45 (1975), 5–99.

[Ka74] Kashiwara, M., On the maximally overdetermined system of linear differential equations I, Publ. RIMS 10 (1974/75), 563–579.

[Ka78] Kashiwara, M., On the holonomic systems of linear differential equations II, Inv. Math. 49 (1978), 121–135.

[KS90] Kashiwara, M., Schapira, P., Sheaves on Manifolds, Springer, Berlin, 1990.

[Me77] Mebkhout, Z., Local cohomology of analytic spaces, Publ. RIMS 12 (1977), 247–256.

[Mi68] Milnor, J., Singular Points of Complex Hypersurfaces, Annals of Math. Studies No. 61, Princeton University Press, 1968.

[MP21] Mustaţă, M., Popa, M., Hodge filtration on local cohomology, Du Bois complex, and local cohomological dimension (arXiv:2108.05192).

[Og73] Ogus, A., Local cohomological dimension of algebraic varieties, Ann. Math. 98 (1973), 327–365.

[RSW21] Reichelt, T., Saito, M., Walther, U., Dependence of Lyubeznik numbers of cones of projective schemes on projective embeddings, Selecta Math. (N.S.) 27:6 (2021), 22 pp.

[Sa88] Saito, M., Modules de Hodge polarisables, Publ. RIMS, Kyoto Univ. 24 (1988), 849–995.

[Sa22] Saito, M., Notes on regular holonomic $\mathcal{D}$-modules for algebraic geometers (preprint).

[Sc01] Schürmann, J., Topology of Singular Spaces and Constructible Sheaves, Monografie Matematyczne, vol. 63, Birkhäuser, Basel, 2001.

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