Matrix integrals and cluster algebras

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Abstract. Cluster algebras were originated from the study of the canonical basis of quantum groups, as well as the investigation of the total positivity in semi-simple Lie theory. It is well known that the formal matrix integrals with cubic potential are the generating function of discrete surfaces for a given topology. In this presentation, we try to establish certain relationships between matrix integrals and geometric cluster algebras. We propose that these surfaces are also the geometric models of cluster algebras.

1. Introduction
Cluster algebras were originated from the study of the canonical basis of quantum groups, as well as the investigation of the total positivity in semi-simple Lie theory. The further study shows that, the cluster algebras were relevant to various research areas, such as representation theory of quiver [1, 2], T-systems and Y-systems in integrable systems [3, 4] (for details the interested reader may consult A. Kuniba et al. [5] for review and references therein), the study of the Teichmüller space [6, 7], and such that on.

The concept of cluster algebra was first introduced by S. Fomin and A. Zelevinsky [8], and further studied [3, 9, 10]. The definitions of cluster algebras are various with their distinct applications. Generally speaking, a cluster algebra is a commutative subring on an ambient field, formed by seeds and mutations. A seed contains a cluster, a coefficient tuple, and an exchange matrix, different seeds are connected by mutations. Elements in a cluster are called cluster variables, they are the generators of the subring, in which the exchange matrix is skew-symmetric. Starting from any seed, one can get a cluster algebra by seed mutations. The birational mutations of clusters play a key role in seed mutations. Their forms are completely encoded by their exchange matrix and the coefficients. It is much more helpful to describe the concept with a $n$-regular graph. Each vertex denotes a seed, while the decorated edges give mutations among seeds, hence the regular graph frames a cluster algebra. A remarkable property is that any given cluster variable can be expressed as rational functions of other clusters.

There are various kinds of coefficient tuples [3]. If the coefficient tuples are in a semifield, then the correspondence cluster algebras are called geometric type. This kind of cluster algebras plays an important role in diverse applications, hence it is especially investigated. A primary example of geometric cluster algebras is a triangulated polygon in $\mathbb{R}^2$ [11]. The Euclidian lengths of the diagonals are cluster variables, and the lengths of the polygon sides are coefficients. The birational transformations are Ptolemy transformations of the diagonals,
then every triangulation is a cluster and all the triangulations with these relations are a cluster algebra. In fact, following the cluster algebra approach to deal with boarded surfaces were first investigated by V. Fock and A. Goncharov [6], and also considered independently by M. Gekhtman, M. Shapiro and A. Vainshtein [7]. In a decorated Teichmüller theory, the surfaces were relevant to a cluster algebra if their arcs in surfaces were regarded as the Penner coordinates or the lambda lengths of the arcs in the surfaces. If the Penner coordinates become cluster variables, then the property of triangulations are identical with clusters, as well as different triangulations are linked by the Ptolemy transformation in decorated Teichmüller theory.

In another way, S. Fomin and his collaborators [12, 13] studied cluster algebras from the geometric models of boarded surfaces. The essential ingredients are the Penner coordinates of arcs as well. They generalized the concept of ordinary arcs to tagged arcs. A tagged arc is an arc obtained from ordinary arcs by labeling the endpoint, in which each endpoint has been tagged in one of two ways, plain or notched. Then an ordinary triangulation turned into a tagged triangulation. There is no self-fold triangle in tagged triangulations, and the Penner coordinate is generalized for tagged arcs. Picking a point in the decorated Teichmüller space for the plainly labeled arcs, the lambda lengths are defined as usual; for the notched tagged arcs, the lambda lengths are defined by choosing different horocycles at the notched endpoints. Using the generalized Penner coordinates, one found that every tagged triangulation fastened on a cluster, and different clusters are connected by a flipping of tagged arcs. The lambda lengths of boundary segments are the coefficients. In this way, all the tagged triangulations make up a geometric cluster algebra. With elaborate tricks, the results can also be verified for non-normalized cluster algebras [13]. In other words, any cluster algebra can be reformulated by a boarded surface with marked points. In certain sense, one specifically found an intuitive representation for any cluster algebra.

The triangulations of boarded surfaces also appear in the solutions of one-cut Hermitian matrix integrals. These integrals are relevant to 2D gravity [14, 15]. In a one-cut case, the matrix integrals have a similar geometric meaning as cluster algebras, and expansions of the matrix integrals are combinatoric discrete surfaces as well. For the case of multi-matrix integrals, the discrete surfaces will be colored. Furthermore, the boundary segments in colored surfaces will be attached to certain matrices in the integrands of matrix integrals. Solving various matrix integrals are important tasks, and various kinds of methods have been proposed since then. In 2004, a state of art method was given by B. Eynard [16]. With the help of this method, matrix integrals are expressed as combinatoric discrete surfaces [17, 18].

For cluster algebras, there are also lots of properties that we want to investigate. Much more information is needed to be dug out, such as the studying of topological properties [12]. In this presentation, we try to establish certain relationships between matrix integrals and cluster algebras. If the matrix integrals are only involved with cubic potential, then the discrete surfaces are decomposed into triangles. With the help of statement in ref. [13], we know that the tagged triangulations are seeds in a cluster algebra. The equivalent seeds are bridges between matrix integrals and cluster algebras.

This presentation is outlined as follows. At first, in section 2, we will briefly recall the work of S. Fomin and his collaborator [10, 13], in the meanwhile, the matrix integrals are expressed in terms of combinatoric discrete surfaces based on the work of B. Eynard [17]. Then in section 3, we set up the relationships of matrix integrals in the one-cut case and cluster algebras, more accurately, Y patterns. By using the results in [13], we prove the coefficients in the expansions of matrix integrals with cubic potential can be encoded into Y patterns.

### 2. Notations

At first, in this section, we collect basic facts of the cluster algebra for boarded surfaces based on ref. [13], in which one can survey more details, after then matrix integrals in a one-cut case
are briefly recalled, they have a geometric meaning as expected.

2.1. Cluster algebras of geometric type

Let $J$ be a finite set of labels, and let $\text{Trop}(p_j : j \in J)$ be an Abelian group (with multiplication $\cdot$) freely generated by the elements $p_j (j \in J)$. We define the addition $\oplus$ in $\text{Trop}(p_j : j \in J)$ by

$$\bigoplus f_j p_j^a_j \oplus \bigoplus f_j p_j^b_j = \bigoplus f_j p_j^{\min(a_j, b_j)}.$$ 

Then we call $\langle \text{Trop}(p_j : j \in J), \oplus, \cdot \rangle$ a tropical semifield.

A cluster algebra is of geometric type if the coefficients semifield $\mathbb{P}$ is a tropical semifield. For a cluster algebra of geometric type, let $\mathcal{A}$ be an Abelian ground ring and $\mathbf{E}$ be a connected $n$-regular graph. Furthermore, we denote $V(\mathbf{E})$ as the set of vertices and $I(\mathbf{E}) = \{ e, f, g, h \cdots \}$ as the set of edges, then a cluster algebra is a set of triples $\mathcal{A} = \{(x(t), p(t), B(t)) \mid t \in V(\mathbf{E})\}$ together with mutations. For a fixed $t_0$, $x(t_0) = (x_e(t_0))$ is a cluster, and its $n$ elements, i.e. cluster variables are decorated by $n$ edges in $I(\mathbf{E})$ attached with the vertex $t_0$. $p(t_0) = (p_e^+(t_0))$ is a coefficient tuple with $p_e^+(t_0)$ in a subring $\mathbb{P}$ with unit of $\mathcal{A}$. $B(t_0) = (b_{ef}(t_0))$ is a skew-symmetric matrix. If $(p(t))$ are valued in a semifield, the triple $\mathcal{A} = (x(t), p(t), B(t))$ is called a seed for any $t \in V(\mathbf{E})$, then $\mathcal{A}$ is a normalized cluster algebra. For an edge $e$ between vertices $t, \bar{t}$, there are corresponding mutations of $(x(t), p(t), B(t))$ and $(x(\bar{t}), p(\bar{t}), B(\bar{t}))$.

$x(\bar{t}) = x(t) - \{x_e(t) \cup \{x_e(\bar{t})\}, \text{and the variable } x_e(\bar{t}) \text{ is determined by the relation}$

$$x_e(t)x_e(\bar{t}) = p_e^+(t) \prod_{f \in \text{star}(t)} x_f(t)^{b_{fe}(t) + p_e^-(t)} \prod_{f \in \text{star}(t)} x_f(t)^{-b_{fe}(t)}$$

(1)

here $\text{star}(t)$ denotes the set of lines ending on $t$, $b_{fe}$. The expression of the coefficient tuple $p(\bar{t})$ is more complicated.

$$p_e^+(\bar{t})/p_e^+(t) = (p_e^+(t))^{b_{ef}(t)}(p_e^+(t)/p_e^+(t)) \text{ if } b_{ef} \geq 0;$$

$$p_e^+(\bar{t})/p_e^+(t) = (p_e^+(t))^{b_{ef}(t)}(p_e^+(t)/p_e^+(t)) \text{ if } b_{ef} \leq 0;$$

$$p_e^-(\bar{t}) = p_e^-(t) \text{ others.}$$

(2)

respectively. If $f$ in the equation is an edge in $\mathbf{E}$, which is different from $e$, $B(\bar{t}) = (\bar{b}_{gh})$ change as follows

$$\bar{b}_{gh} = \begin{cases} -b_{gh} & \text{if } g = e \text{ or } h = e; \\ b_{gh} + \frac{\|b_{eh} + b_{eh} + b_{eh}\|}{2} & \text{otherwise.} \end{cases}$$

(3)

respectively.

For cluster algebras of geometric type, it is convenient to denote the generators of $\mathbb{P}$ by $x_{n+1}, x_{n+2}, \cdots, x_m$. Then the coefficients at the vertex $t$ have

$$p_e^+(t) = \prod_{i=n+1}^m x_i^{b_{ie}},$$

(4)

where $e \in I(\mathbf{E})$. That is, we extend the exchange matrix $B(t)$ to a larger $m \times n$ matrix $B(t)(1 \leq i \leq m, 1 \leq j \leq n)$, in which it includes the exchange matrix $B(t)$ as its submatrix. In this way, we would call matrix $B(t)(1 \leq i \leq m, 1 \leq j \leq n)$ the extended exchange matrices from now on. The matrix elements $b_{ij}$ with $i > n$ are used to encode the coefficients at the vertex $t$.

A geometric model $\mathcal{A}$ can be constructed as follows. Let $S$ be a connected Riemann surface with a nonempty finite set $M$ of marked points. $\partial S$ is the boundary of $S$, it can be empty, and every connected component of the boundary contains at least one marked point. Assuming that none of the following three cases are $S$: a sphere with one or two punctures, as well as marked points is not on the boundary; unpunctured disks with one, two, or three marked
points separately on the boundary; once-punctured disk with one marked point on the boundary. Hence, following the notes [12], we denote them as \((S, M)\). There are arcs between marked points in \(S\), for any given two marked points \(a, b \in M\), we denote the arcs between them as \(x_{ab}\). Arcs in the same isotopic class are regarded as the same ones, and they do not self-intersect. Two arcs are called compatible if they do not intersect in the interior of \(S\). Therefore, the maximal set of compatible arcs frames an ideal triangulation of the surface, obviously, an ideal triangulation may have self-fold triangles. One can see the case in Figure 1 as an example, there are two self-fold triangles on a sphere.

![Figure 1. An ideal triangulation of a sphere with three punctures.](image1)

![Figure 2. Once-punctured monogon and its tagged graph, \(\triangle\) denotes the notched label.](image2)

We expect that an ideal triangulation will be a cluster. Since each cluster variable can mutate in a cluster algebra, it is reasonable to require that any arc in ideal triangulations can be flipped. However, the self-folded edges in ideal triangulations cannot be flipped. This problem could be solved with the help of tagged arcs.

If we tag endpoints of arcs in two ways, plain or notched, then there are three following rules: The endpoints lying on the boundary of \(S\) are labeled as plain; the endpoints of a loop must be tagged in the same way; the third one is the key rule, such that if an arc is a loop based at \(a\) cutting out a once-punctured monogon with sole puncture \(b\) inside it, then tag the arc plain at \(a\) and notched at \(b\). These tagged arcs have triangulations, while the tagged triangulations have no once-punctured monogon. Therefore, Figure 2 is the illustration.

In order to recover a cluster algebra structure from boarded surfaces, we will consider these tagged arcs in a Teichmüller space. For a given boarded surface \((S, M)\), there is a decorated Teichmüller space \(\mathcal{T}(S, M)\) following R. C. Penner [20], which is the space of hyperbolic structures on finite area of \(S\). Choosing a point in \(\mathcal{T}(S, M)\), then every marked point has a horocycle, and an arc has a unique geodesic in \(S\), then the lambda length of an arc is the finite length of the segment between the two horocycles at the endpoints. A generalized lambda length is obtained by substituting the conjugate horocycles at the notched endpoints for initial horocycles. We say two horocycles \(h, \bar{h}\) are conjugate if the product of their hyperbolic lengths equals 1.

The generalized lambda lengths satisfy transformations which are similar to the Ptolemy transformations. S. Fomin and D. Thurston proved that there was a normalized cluster algebra for \((S, M)\) [13]. The regular graph of the cluster algebra is the exchanged graph of the tagged triangulations, and the cluster variables are the lambda lengths of the tagged arcs. In this way, the cluster mutations are the generalized the Ptolemy transformations, and the exchange matrices are the signed adjacency matrices. The lambda lengths of boundary segments generate the coefficient semifield. Hence, a boarded surface \((S, M)\) definitively gives a cluster algebra. \((S, M)\) becomes a geometric model of a normalized cluster algebra. In fact, the boarded surface
can give a general cluster algebra with certain surgeries [11].

2.2. Formal matrix integrals in combinatoric maps

A polygon can be defined through a set of edges with a fixed cyclic permutation. In a dual manner, the polygon can be seen as a set of half-edges emitted from the center of the polygon to edges with a fixed cyclic permutation. We define a labeled map following the ref. [19]. A labeled map \( G = (B, B^*, \sigma_1, \sigma_2) \) is the data of 2 finite ensembles \( B \) and \( B^* \) of half-edges (\( B^* \) can be empty), such that \( B \cup B^* \) has even cardinality, and two permutations \( \sigma_1 \) and \( \sigma_2 \) of elements of \( B \cup B^* \), such that \( \sigma_2 \) is an involution without fixed points, then every cycle of \( \sigma_1 \) contains at most one element of \( B^* \). The cycles of \( \sigma_1 \) are combined to faces, the cycles of \( \sigma_2 \) are edges, the cycles of \( \sigma_1 \circ \sigma_2 \) are vertices. The point can be easily verified. The automorphism group of a labeled map is the set of bijections \( \phi : B \cup B^* \rightarrow B \cup B^* \), such that \( \phi|_{B^*} = \text{Id} \), \( \sigma_1 = \phi \circ \sigma_1 \circ \phi^{-1} \), and \( \sigma_2 = \phi \circ \sigma_2 \circ \phi^{-1} \).

If a set of connected maps have \( g \) genera (with unmarked faces degree \( \geq 3 \)), \( k \) boundaries (degree of marked faces \( \geq 1 \) with one marked edge), as well as the total number of vertices is \( v \), then we denote the set of maps as \( \mathcal{M}_k^g(v) \). For completeness, we define \( \mathcal{M}_k^{(0)}(1) = \{ \} \), that is to say, a virtual planar map with 1 vertex and 1 boundary is a point, its boundary length is zero.

The maps defined above can be used to compute matrix integrals by applying Wick’s theorem. The relationship between them was first noticed by G. ’t Hooft [21], and then further studied by Brezin-Itzykson-Parisi-Zuber [22].

Consider the set \( H_N \) consisting of all random Hermitian matrices \( M \) of size \( N \) with Gaussian probability measure:

\[
d\mu_0(M) = \frac{1}{Z_0} e^{-\frac{N}{\pi} \text{Tr} M^2} \prod_{i=1}^{N} dM_{ii} \prod_{i<j} d\text{Re} M_{ij} d\text{Im} M_{ij},
\]

in which \( t \) is the ’t Hooft parameter, and \( Z_0 \) is the normalization constant such that \( \int_{H_N} d\mu_0(M) = 1 \). Denote \( \langle x \rangle_0 = \int_{H_N} x d\mu_0(M) \).

For simplicity, we consider one-matrix integral without boundaries as an example. In one-matrix integral, the partition function is [17]:

\[
Z = \int_{H_N} d\mu_0(M) e^{\frac{N}{\pi} \text{tr} V(M)}
\]

where \( H_N \) represent all of Hermitian matrices and \( V(M) = \sum_{j=3}^{d} (t_j/j) M_j \), with \( d \) as an arbitrary positive integer. In the one-cut case, the matrix integral can be expressed by combinatoric maps. Expanding the partition function \( Z \) in \( t \), its connected parts are [19]:

\[
F = \ln Z = \sum_{l} \sum_{g=0}^{l+1} t^l g^2 N^{2g-2} \left( \sum_{S_l, g \in \mathcal{M}} \frac{1}{\# \text{Aut}(S)} \prod_{j=3}^{d} \ell_n(j(S)) \right)
\]

where \( S_{l, g} \) denotes the set of closed discrete surfaces divided by \( n_3 \) triangles such that \( l = \# \text{edges} - \# \text{faces} \), the Euler character is \( 2g - 2 \), \( g \) is the genus of a surface. It is worth noting that \( S \) in \( S_{l, g} \) is a triangulation on a boarded surface \( S_{l, g} \), \( n_j(S) \) is the number of \( j \)-gons, and \( \# \text{Aut}(S) \) is the order of the automorphism group of \( S \). For fixed \( l, g \), the sum is inherently on all inequivalent discrete surfaces with genus \( g \). In other words, the sum is over all possible gluing ways to get genus-\( g \) surfaces. If the potential is cubic, then the sum is over different triangulations of genus-\( g \) surfaces.
3. Formal Matrix Integrals and Cluster Algebras

In this section, we will prove that each extended adjacency matrix and a bordered surface with a fixed triangulation are uniquely determined by each other. If the bordered surfaces have no boundaries, then the free energy of one Hermitian matrix model is encoded by adjacency matrices of cluster algebras. As it has nothing with the lengths of cluster variables, we can also say the free energy is encoded by $Y$ patterns with trivial coefficients. The topological expansion of the correlation functions is recombined by extended adjacency matrices in cluster algebras (or $Y$ patterns).

3.1. Matrix integral with cubic potential and cluster algebras

For any given geometric cluster algebra $A = (x, y, B)$, $B$ is the $n \times m$ extended exchange matrix. We call a transformation of $B$ a row-column transformation if for any integers $1 \leq i, j \leq n$, the rows $i, j$ and the columns $i, j$ are interchanged at the same time.

Given a matrix $B$, for any two row-column transformations $s, t$ of $B$, define the multiplication

$$s \times t \equiv s \circ t.$$  

It is obvious that all the row-column transformations keeping $B$ invariant form a group, this symmetry about $B$ is the automorphism group of $B$, it is denoted as $\text{Aut}(B)$.

As we know in the last section, in a bordered surface with marked points $(S, M)$, there are ideal triangulations with $n$ arcs and $m - n$ segments of boundaries. Corresponding to these triangulations, S. Formin and D. Thurston proved that there is a geometric cluster algebra $A = (x, y, B)$.

Now we can state an equivalence relation, and it will be proved it after then. Given a bordered surface $(S, M)$ and the corresponding cluster algebra $(x, y, B)$ from an extended exchange matrix $\tilde{B}$, we can get a unique ideal triangulation $T$ in $(S, M)$, the order of automorphism group have the relationship

$$\#(\text{Aut}(\tilde{B})) = 2 \times (\#(\text{Aut}(T))).$$  

If the “exchange matrix” does not contain coefficients, one matrix may correspond to many ideal triangulations. It should also be noticed that we have used ideal triangulations to substitute tagged triangulations. A tagged triangulation has the same extended adjacency matrix with its following ideal triangulation. Hence we only need to consider the ideal triangulations.

Given an ideal triangulation $T$, there is a unique matrix $\tilde{B}$. This is following the definition of the adjacency matrix. The extended part is just the boundary adjacency matrix.

In order to prove the bijection between ideal triangulations and extended exchange matrices, we just only prove any adjacency matrix from an oriented bordered surface determines a unique ideal triangulation.

From the definition of adjacency matrices, we know that, with one exception, any ideal triangulation on $(S, M)$ can be decomposed into three kinds of pieces in a certain order. The three self-folded triangles are respectively affixed to a common triangle to form a sphere. The three basic types of puzzle pieces are the normal triangle, the two-sided graph with once-punctured, and the ideal triangulation with a single-sided graph with twice-punctured, as shown in Figure 3.

The first adjacency matrix is straightforward. The adjacency matrices correspond to where in the second matrix, the arcs labeled 3, 4 cannot be contiguous with other arcs. That is, in the extended adjacency matrix of the whole, the values of the two rows and the two columns corresponding to other rows and columns marked with $k$ are zero, $\{b_{ik} = 0|i = 3, 4; k \neq 1, 2, 3, 4\}$. In the same way, if we regarding the rows and columns of the third matrix labeled by 2, 3, 4, 5, and the corresponding matrix values of other ranks labeled $k$ are zero, ie,
\begin{align}
\{b_{ik} = 0|i = 2, 3, 4, 5; k \neq 1, 2, 3, 4, 5\}. \text{ The three matrix blocks also uniquely correspond to the above three ideal ideal triangulation. The corresponding adjacency matrices are} \\
\begin{pmatrix}
0 & 1 & -1 \\
-1 & 0 & 1 \\
1 & -1 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 1 & -1 & -1 \\
-1 & 0 & 1 & 1 \\
1 & -1 & 0 & 0 \\
1 & -1 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 1 & 1 & -1 & -1 \\
-1 & 0 & 0 & 1 & 1 \\
-1 & 0 & 0 & 1 & 1 \\
1 & -1 & -1 & 0 & 0 \\
1 & -1 & -1 & 0 & 0
\end{pmatrix} \tag{10}
\end{align}

respectively. \text{ The only exception of the an ideal triangulation is a spherical surface with four puncture points obtained by gluing three self-folded triangles with sides of ordinary triangles as in Figure 4, the corresponding adjacency matrices for the exception case are} \\
\begin{pmatrix}
0 & 0 & -1 & -1 & 1 & 1 \\
0 & 0 & -1 & -1 & 1 & 1 \\
1 & 1 & 0 & 0 & -1 & -1 \\
1 & 1 & 0 & 0 & -1 & -1 \\
-1 & -1 & 1 & 1 & 0 & 0 \\
-1 & -1 & 1 & 1 & 0 & 0
\end{pmatrix} \tag{11}

\text{ It is easy to note that, the exception case is different from the three basic puzzles that we have mentioned above and it cannot be obtained from them by combination either. In this way, there is a one-to-one correspondence between the ideal decomposition and the extended exchange matrix. For this reason, we just need to prove that there is a unique decomposition for a given extended exchange matrix.} \text{ Now we only need to prove that given an extended adjacency matrix } \tilde{B}, \text{ there is only one path and it is decomposed into the combination of the above three matrix blocks. If two arcs marked } i, j \text{ adjoin, and the value of } b_{ij} \text{ in the adjacency matrix may be zero. Since one edge is in at most two triangles (the boundary line segment is in a triangle), the two sides are in opposite orders in the two triangles, } b_{ij} = 0 \text{ means that the arc } i, j \text{ are in two triangles at the same time, and their taken values cancel each other out, that they have the opposite order in the two triangles. Therefore, the two triangles where arcs } i \text{ and } j \text{ are located either stick to one sphere or stick to one hemisphere. The former is not within the scope of our definition, so the offset is if and only if there is once-punctured two-sided graph, and its triangulation is not self-folded. Let } i = 1, j = 2 \text{ and the other two edges are marked 3, 4 , respectively, then there is an adjacency matrix } \tilde{B}. \text{ Therefore, in the exchange matrix, there is only way that two adjacent edges take value 0. For one edge can occur at most in two triangles. Therefore, it makes sense}
only in two ways, either gluing two triangles into a sphere with one, two and three punctures respectively, or gluing two triangles into a half-sphere, as shown in the first puzzle in Figure 3. We name the hemisphere as a hat. In the first gluing way, it is not allowed to form a cluster algebra, while the second way corresponds the two variable exchange matrix with two non-zero values, i.e.

\[
\begin{pmatrix}
0 & 0 & 1 & -1 \\
0 & 0 & -1 & 1 \\
-1 & 1 & * & * \\
1 & -1 & * & *
\end{pmatrix},
\]

where * denotes that the corresponding value is uncertain, they can take any value, which is different from the matrix of the second puzzle in Figure 3. This matrix can be obtained by combining two of the first puzzle matrix. In the triangles decomposition corresponding to this matrix, the value of the arc with the index 1, 2 and the other arc marked with k is zero, which is \(b_{1k} = b_{2k} = 0\) for \(k \neq 1, 2, 3, 4\).

Given an extended adjacency matrix \(\tilde{B}\) different from the exception one, the number of non-zero values element of each column \(j\) is denoted by \(N_j\). Since it can be formed by the combination of three basic matrix blocks, depending on the combination, the value of \(N_j\) ranges from 2 to 8. We show below that for each case we can uniquely determine which matrix block it is composed of, and thus determine the overall ideal triangulation. It should be pointed out that the boundary line segment can only be in one triangle and the arc line is in two triangles. Thus, for a given extended exchange matrix, we can decompose the matrix uniquely into the three puzzles.

A tagged map is denoted by \(m = (B, B^*, \sigma_1, \sigma_2)\). The cycle of \(\sigma_2\) is an edge in ideal triangulations. Given an ideal triangulation, i.e. a map \(m = (B, B^*, \sigma_1, \sigma_2)\), \(G_m\) is the group of relabeling of unmarked half-edges leaving faces invariant, i.e. \(\phi : B \cup B^* \rightarrow B \cup B^*\) s.t. \(\phi|_{B^*} = \text{Id}\), \(\sigma_1 = \phi \circ \sigma_1 \circ \phi^{-1}\). We have \(G_m = \{\phi|_{\sigma_1} = \phi \circ \sigma_1 \circ \phi^{-1}\}\). The automorphism for the ideal triangulation (map) \(m\) is

\[
\text{Aut}(m) = \{\phi \in G_m|_{\sigma_1} = \phi \circ \sigma_2 \circ \phi^{-1}\}.
\]

In a cluster algebra, a cycle of \(\sigma_2\) is named as a cluster variable, it labels the rows and columns of exchange matrices, and a cycle corresponds to an extended exchange matrix. An extended exchange matrix also gives a \(\sigma_2\) by the previous proof. The automorphism of a map keeping \(\sigma_2\) invariant is equivalent to keep the extended exchange matrix invariant after the row-column transformations. The mutation class of exchange matrices corresponds to the orbit of \(\sigma_2\). Any a row-column transformation which changes the extended exchange matrix gives a gluing. It is the same as the primary gluing without labeling.

Denote a cluster variable \(x_i \equiv (s, t)\), where \(s, t \in B \cup B^*, \sigma_2(s) = t\). For any a row-column transformation \(\varphi\) keeping the extended exchange matrix invariant, assume

\[
\varphi((s, t)) = (p, q).
\]

There are two automorphisms in ideal triangulations,

\[
\begin{align*}
\varphi_1(s) &= p, \quad \varphi_1(t) = q, \quad \varphi_1 \in \text{Aut}(m); \\
\varphi_2(s) &= q, \quad \varphi_2(t) = p, \quad \varphi_2 \in \text{Aut}(m).
\end{align*}
\]

and thus we have \(#(\text{Aut}(B)) = 2 \times (\#(\text{Aut}(T)))\).

For any matrix integral with cubic potential, we have,

\[
Z_N = \int_{H_N} d\mu_0(M) e^{\frac{N^3}{2} V^4 M^3} = \int_{H_N} d\mu_0(M) \sum_{k=0}^{\infty} t^k M^k (\text{Tr} M^3)^k = \sum_{k=0}^{\infty} t^k A_k
\]
For connected parts, we have

\[ \text{In} Z_N = \sum_{g=0}^{\infty} \left( \frac{N}{t} \right)^{2-2g} F_g \]

(16)

where

\[ F_g = \sum_v t^v \sum_{\Sigma \in M_0^v(v)} t^{n(\Sigma)} \frac{1}{\# \text{Aut}(\Sigma)} = \sum_n t^{2-2g+\frac{1}{2}} \sum_{\mathcal{B}} t^{\frac{2n}{3}} \frac{1}{\# \text{Aut}(\mathcal{B})} \]

(17)

The first line of \( F_g \) is encoded by maps while the second line is encoded by cluster algebras.

For any \( n \), a cluster algebra exists, \( \mathcal{B} \) is an extended exchange matrix module row-column transformations.

We consider the corresponding of mutations in maps and formal matrix integrals. In combinatorics maps, mutations in cluster algebras relate to actions of the group \( G_m \) on \( \sigma_2 \). The mutation class corresponds to the orbit of \( \sigma_2 \) under the \( G_m \) action. In Figure 5, the mutation change

\( \sigma_2 : (1, a)(2, b)(3, 4)(5, d)(6, c) \)

to

\( \sigma'_2 = (1, a)(2, 5)(3, d)(6, c)(4, b) \).

\[ \text{Figure 5.} \text{ Changes of maps under mutations.} \]

\[ \text{Figure 6.} \text{ Changes of matrix integrals under mutations.} \]

A formal matrix integral can be expanded as propagator products with the help of Wick’s theorem. The cluster mutations keep the values of products of propagators. In other words, values of propagator products in the same cluster algebra are the same. In Figure 6, the mutation changes the propagator product

\[ \langle M_{ij}M_{ba} \rangle_0 \langle M_{jk}M_{dc} \rangle_0 \langle M_{ki}M_{rho} \rangle_0 \langle M_{sigma}M_{fe} \rangle_0 \langle M_{tau}M_{hg} \rangle_0 \]

to

\[ \langle M_{ij}M_{ba} \rangle_0 \langle M_{jk}M_{tau} \rangle_0 \langle M_{ki}M_{rho} \rangle_0 \langle M_{sigma}M_{dc} \rangle_0 \langle M_{sigma}M_{fe} \rangle_0. \]

The two expressions are different, but they have the same value. Based on the same reason, the mutations keep the values of integrals such as \( A_k \) and \( F_g \).

3.2. Conclusions and Discussions

In this paper, we construct the relationships between cluster algebras and the matrix integrals with cubic potentials in the one-cut case. The expansion of matrix integrals can be recombined into finite type cluster algebras from bordered surfaces. In the manuscript, we prove each bordered surfaces with (without) boundaries can be recovered from an adjacency matrix with boundary parts.
Some problems need to be investigated furthermore. Of course, the matrix integral with cubic potential is just a tiny fraction in the whole story. A nature problem is whether matrix integrals with general polynomial potentials have such algebraic interpretation. They may correspond to generalized cluster algebras or other ones. Following the results in section 3, we can study the topology property of cluster algebra [12] with the help of the results [18]. On the other side, multi-cut matrix integrals are also had geometric meanings of discrete surfaces, it is interesting to developing a cluster kind algebra to relevant those subjects, for which cluster algebras with additional conditions need further consideration.

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