THREE-DIMENSIONAL HOMOGENEOUS
GENERALIZED RICCI SOLITONS

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ABSTRACT. We study three-dimensional generalized Ricci solitons, both in Riemannian and Lorentzian settings. We shall determine their homogeneous models, classifying left-invariant generalized Ricci solitons on three-dimensional Lie groups.

1. Introduction

Generalized Ricci solitons were recently introduced in [12]. A generalized Ricci soliton is a pseudo-Riemannian manifold \((M, g)\) admitting a smooth vector field \(X\), such that

\[
L_X g + 2\alpha X^\flat \otimes X^\flat + 2\beta \text{Ric} = 2\lambda g,
\]

for some real constants \(\alpha, \beta, \lambda\), where \(L_X\) denotes the Lie derivative in the direction of \(X\), \(X^\flat\) denotes a 1-form such that \(X(Y) = g(X, Y)\) and \(\text{Ric}\) is the Ricci tensor.

For particular values of the constants \(\alpha, \beta, \lambda\), several important equations occur as special cases of equation (1.1). In particular, one has:

- (K) the Killing vector field equation when \(\alpha = \beta = \lambda = 0\);
- (H) the homothetic vector field equation when \(\alpha = \beta = 0\);
- (RS) the Ricci soliton equation when \(\alpha = 0\) and \(\beta = 1\) [6];
- (E-W) a special case of the Einstein-Weyl equation in conformal geometry when \(\alpha = 1\) and \(\beta = -\frac{1}{n-2} (n > 2)\) [2];
- (PS) the equation for a metric projective structure with a skew-symmetric Ricci tensor representative in the projective class when \(\alpha = 1\), \(\beta = -\frac{1}{n-1}\) and \(\lambda = 0\) [14];
- (VN-H) the vacuum near-horizon geometry equation of a spacetime when \(\alpha = 1\) and \(\beta = \frac{1}{2}\), with \(\lambda\) playing the role of the cosmological constant [9].

Equation (1.1) corresponds to an overdetermined system of partial differential equations of finite type. The study of this system was undertaken in the fundamental paper [12]. Explicit solutions were determined in [12] in the two-dimensional case. For the three-dimensional case, the authors restricted in [12] to the case with \(\alpha = 0\). Note that as already pointed out in [12], for \(\alpha = 0 \neq \beta\), rescaling the vector field \(X\) to \(-\frac{1}{\beta} X\), equation (1.1)
reduces to the Ricci soliton equation. We also observe that a trivial solution of (1.1) is given by $X = 0$ and $\beta = \lambda = 0$, so we shall always exclude this solution.

The aim of this paper is to determine the three-dimensional homogeneous models of generalized Ricci solitons. A connected, complete and simply connected three-dimensional homogeneous manifold, if not symmetric, is isometric to some Lie group equipped with a left-invariant metric (see [15] for the Riemannian case and [3] for the Lorentzian one). Moreover, with the obvious exceptions of $\mathbb{R} \times S^2$ (Riemannian) and $\mathbb{R}_1 \times S^2$ (Lorentzian), three-dimensional connected simply connected symmetric spaces are also realized in terms of suitable left-invariant metrics on Lie groups [1]. For this reason, we shall consider three-dimensional Lie groups, equipped with a left-invariant metric (either Riemannian or Lorentzian). We shall specify our study to solutions of (1.1) determined by a left-invariant vector field $X$. In this way, (1.1) will transform into a system of algebraic equations, which we can solve, obtaining a complete classification of three-dimensional left-invariant generalized Ricci solitons, and determining several new solutions of (1.1). We recall that the study of three-dimensional Ricci solitons already showed some interesting differences arising between the Riemannian case (for which left-invariant solutions do not occur [7]) and the Lorentzian one, where several left-invariant solutions exist [1].

2. 3D Riemannian left-invariant generalized Ricci solitons

Three-dimensional Riemannian Lie groups were classified in [11]. We shall treat separately the unimodular and non-unimodular cases.

2.1. Unimodular case. Let $G$ be a connected three-dimensional Lie group with a left-invariant Riemannian metric. Choose an orientation for the Lie algebra $\mathfrak{g}$ of $G$, so that the cross product $\times$ is defined on $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is unimodular if and only if the endomorphism $L$, defined by $[Z,Y] = L(Z \times Y)$, is self-adjoint [11]. Therefore, $\mathfrak{g}$ admits an orthonormal basis $\{e_1, e_2, e_3\}$ of eigenvectors for $L$, so that

\begin{align}
[e_1, e_2] = Ce_3, & \quad [e_2, e_3] = Ae_1, & \quad [e_3, e_1] = Be_2,
\end{align}

for some real constants $A, B, C$. Explicitly, depending on the sign of $A, B, C$, the Lie group $G$ is isomorphic to one of the cases listed in the following Table I.
In the above Table I and throughout the paper, $\tilde{SL}(2, \mathbb{R})$ will denote the universal covering of $SL(2, \mathbb{R})$, $\tilde{E}(2)$ the universal covering of the group of rigid motions in the Euclidean two-space, $E(1, 1)$ the group of rigid motions of the Minkowski two-space and $H_3$ the Heisenberg group.

The description of the Ricci curvature with respect to the basis $\{e_1, e_2, e_3\}$ is well known (see again [11]). We have:

$$Ric = \begin{pmatrix}
\frac{1}{2}(A^2 - B^2 - C^2) + BC & 0 & 0 \\
0 & \frac{1}{2}(B^2 - A^2 - C^2) + AC & 0 \\
0 & 0 & \frac{1}{2}(C^2 - A^2 - B^2) + AB
\end{pmatrix}.$$ 

In particular, it is easily seen that the left-invariant metric is of constant sectional curvature if and only if either $A = B = C$, $A - B = C = 0$, $A = B - C = 0$ or $A - C = B = 0$.

We now consider an arbitrary vector field $X$, that is, $X = X_i e_i \in \mathfrak{g}$, for some real constants $X_1, X_2, X_3$. Then, by (2.1), we get

$$\mathcal{L}_X g = \begin{pmatrix}
0 & X_3(A - B) & -X_2(A - C) \\
X_3(A - B) & 0 & X_1(B - C) \\
-X_2(A - C) & X_1(B - C) & 0
\end{pmatrix}$$

with respect to the basis $\{e_1, e_2, e_3\}$. Moreover, since $\{e_1, e_2, e_3\}$ is orthonormal, for any vector $X = X^i e_i \in \mathfrak{g}$ we have $X^i \otimes X^j(e_i, e_j) = X_i X_j$. Therefore, equation (1.1) becomes the following system of algebraic equations:
Theorem 2.1. Let \( g \) denote a three-dimensional unimodular Riemannian Lie algebra, as described by \cite{2}, with respect to a suitable orthonormal basis \( \{e_1, e_2, e_3\} \). Then, up to a renumeration of \( e_1, e_2, e_3 \), the nontrivial left-invariant generalized Ricci solitons on \( g \) are the following:

\[
\begin{align*}
(1) & \quad A = B = C, \quad \alpha = 0, \quad \lambda = -\frac{1}{2} \beta A^2, \quad \text{for all } \beta \text{ and } X_1, X_2, X_3, \text{ showing that when } A = B = C \text{ (case of constant sectional curvature on } SU(2)), \text{ all vectors in } g \text{ are Killing.} \\
(2) & \quad A = B = C, \quad \lambda = -\frac{1}{2} \beta A^2, \quad X = 0, \quad \text{for all } \alpha, \beta: \text{ when } A = B = C, \text{ the metric is Einstein.} \\
(3) & \quad A = B - C = 0, \quad \lambda = 0, \quad X = 0, \quad \text{for all } \alpha, \beta: \text{ the metric is flat.} \\
(4) & \quad A = B \neq C, \quad \lambda = \frac{1}{2} \beta A(A - C), \quad X_1 = \pm \sqrt{\frac{\beta A(A - C)}{\alpha}}, \quad X_2 = X_3 = 0, \quad \text{for any } \alpha, \beta \text{ such that } \alpha \beta A(A - C) > 0.
\end{align*}
\]

Solutions (1)-(3) are somewhat “trivial”, as they correspond to cases of metrics of constant sectional curvature and could be expected. On the other hand, by (4) we have solutions when just two between \( A, B, C \) coincide. By Table I, this yields nontrivial left-invariant generalized Ricci solitons on \( SU(2), \tilde{S}L(2, \mathbb{R}), E(2), H_3 \). Observe that \( \alpha \neq 0 \) in case (4), while cases (1)-(3) are Einstein. Thus, no (nontrivial) left-invariant Ricci solitons occur in three-dimensional Riemannian Lie groups, coherently with the results of \cite{12}.

Finally, we remark that \( \lambda \neq 0 \) in solution (4). This excludes the possibility of solutions of \( \text{(PS)} \). On the other hand, if \((\text{E-W})\) holds for a three-dimensional manifold, then \( \alpha \beta = -1 \). Therefore, condition \( \alpha \beta A(A - C) > 0 \) in (4) yields \( A(A - C) < 0 \). Similarly, in the case of the vacuum near-horizon geometry equation (VN-H) (considered in \cite{12} both for Riemannian and Lorentzian two-manifolds), we have \( \alpha \beta = 1 \), so that condition \( \alpha \beta A(A - C) > 0 \) in (4) yields \( A(A - C) > 0 \). Taking into account the above Table I, we then have the following:

Corollary 2.2. Three-dimensional Riemannian Lie group \( SU(2) \) gives solutions to the special Einstein-Weyl equation (E-W). Three-dimensional Riemannian Lie groups \( SU(2), \tilde{S}L(2, \mathbb{R}), E(2) \) give solutions to the vacuum near-horizon geometry equation (VN-H).
2.2. Non-unimodular case. Let now \( g \) denote a three-dimensional non-unimodular Riemannian Lie algebra. Then, its unimodular kernel \( u \) is two-dimensional. Choosing an orthonormal basis \( \{e_1, e_2, e_3\} \) so that \( e_1 \) is orthogonal to \( u \) and \( [e_1, e_2], [e_1, e_3] \) are mutually orthogonal \([11]\), the bracket product is described by (2.3)

\[
[e_1, e_2] = Ae_2 + Be_3, \quad [e_1, e_3] = Ce_2 + De_3, \quad [e_2, e_3] = 0, \quad A + D \neq 0, \quad AC + BD = 0,
\]

for some real constants \( A, B, C, D \).

With respect to the basis \( \{e_1, e_2, e_3\} \), the Ricci curvature is described by (see [11])

\[
Ric = \begin{pmatrix}
-A^2 - \frac{1}{2}B^2 - \frac{1}{2}C^2 - D^2 - BC & 0 & 0 \\
0 & -A^2 - \frac{1}{2}B^2 + \frac{1}{2}C^2 - AD & 0 \\
0 & 0 & -D^2 + \frac{1}{2}B^2 - \frac{1}{2}C^2 - AD
\end{pmatrix}.
\]

In particular, the left-invariant metric is of constant sectional curvature if and only if \( A - D = B + C = 0 \).

For an arbitrary left-invariant vector field \( X = X_i e_i \in g \), we have

\[
\mathcal{L}_X g = \begin{pmatrix}
0 & AX_2 + CX_3 & BX_2 + DX_3 \\
AX_2 + CX_3 & -2AX_1 & -(B + C)X_1 \\
BX_2 + DX_3 & -(B + C)X_1 & -2DX_1
\end{pmatrix}
\]

with respect to the basis \( \{e_1, e_2, e_3\} \), and we have again \( X^a \otimes X^b (e_i, e_j) = X_i X_j \). Hence, equation (1.1) now gives

\[
\begin{cases}
2\alpha X_i^2 + \beta(2A^2 + B^2 + C^2 + 2D^2 + 2BC) = 2\lambda, \\
-2AX_1 + 2\alpha X_2^2 + \beta(2A^2 + B^2 - C^2 + 2AD) = 2\lambda, \\
-2DX_1 + 2\alpha X_3^2 + \beta(2D^2 - B^2 + C^2 + 2AD) = 2\lambda, \\
AX_2 + CX_3 + 2\alpha X_1 X_2 = 0, \\
BX_2 + DX_3 + 2\alpha X_1 X_3 = 0, \\
-(B + C)X_1 + 2\alpha X_2 X_3 = 0.
\end{cases}
\]

We now solve (2.4) and list its different solutions, proving the following result.

**Theorem 2.3.** Let \( g \) denote a three-dimensional non-unimodular Riemannian Lie algebra, as described by (2.3) with respect to a suitable orthonormal basis \( \{e_1, e_2, e_3\} \). Then, up to a renumerated \( e_1, e_2, e_3 \), the nontrivial left-invariant generalized Ricci solitons on \( g \) are the following:

1. \( A - D = B + C = 0, \ \lambda = (2\alpha^2 \beta + \alpha)X_1^2, \ X_1 = -\frac{A}{\alpha}, \ X_2 = X_3 = 0, \) for all \( \alpha \neq 0 \) and \( \beta \) (constant sectional curvature).

2. \( C = D = 0, \ \lambda = \frac{1}{2}\beta(2A^2 + B^2), \ X_1 = X_2 = 0 \) and \( X_3 = \pm \sqrt{\frac{\beta(A^2 + B^2)}{\alpha}} \), for all \( \alpha \) and \( \beta \) satisfying \( \alpha \beta > 0 \).

3. \( A - D = B + C = 0, \ \lambda = 2\beta A^2, \ X = 0, \) for all \( \alpha, \beta \): the metric is Einstein.
(4) \( B = C = 0, \ \alpha = -\frac{A^2 + D^2}{\beta(A + D)}, \neq 0, \ \lambda = 0, \ \lambda_X = X_3 = 0, \) for any \( \beta \neq 0 \) and \( X_1 = \beta(A + D). \)

(5) \( A = D, \ B = C = 0, \ \lambda = A^2(\frac{1}{\alpha} + 2\beta), \) for any \( \alpha \neq 0 \) and \( \beta, \) with \( X_2 = X_3 = 0 \) and \( X_1 = -\frac{A}{\alpha} \) (constant sectional curvature).

It is easy to check that the above solutions (1) and (5) are compatible with (PS) (since for \( \alpha = -1 \) and \( \beta = -\frac{1}{2} \) we have \( \lambda = 0 \) in both (1) and (5)), while all solutions (1), (4), (5) are compatible with (E-W). Thus, we have the following.

**Corollary 2.4.** Three-dimensional non-unimodular Riemannian Lie groups give solutions to the special Einstein-Weyl equation (E-W) and (in the case of constant sectional curvature) to the equation (PS) for a metric projective structure with a skew-symmetric Ricci tensor representative.

### 3. 3D Lorentzian left-invariant unimodular generalized Ricci solitons

Let now \( \times \) denote the Lorentzian vector product on the Minkowski space \( \mathbb{R}^3_1, \) induced by the product of the para-quaternions \( (e_1 \times e_2 = e_3, e_2 \times e_3 = e_1, e_3 \times e_1 = e_2, \) for a pseudo-orthonormal basis \( e_1, e_2, e_3, \) with \( e_3 \) time-like. The Lie bracket \([\ , \ ]\) defines the corresponding Lie algebra \( g, \) which is unimodular if and only if the endomorphism \( L, \) defined by \([Z, Y] = L(Z \times Y),\) is self-adjoint \( \{13\}. \) Differently from the Riemannian case, in Lorentzian settings \( L \) can assume four different standard forms (Segre types), giving rise to four classes of three-dimensional unimodular Lorentzian Lie algebras:

- \( g_1: \) \( L \) is of Segre type \( \{3\}, \) that is, its minimal polynomial has a triple root.
- \( g_2: \) \( L \) is of Segre type \( \{1z\bar{z}\}, \) that is, it has two complex conjugate eigenvalues.
- \( g_3: \) \( L \) is of Segre type \( \{11, 1\}, \) that is, diagonalizable.
- \( g_4: \) \( L \) is of Segre type \( \{21\}, \) that is, its minimal polynomial has a double root.

We shall treat these cases separately.

#### 3.1. Lie algebra \( g_1. \) There exists a pseudo-orthonormal basis \( \{e_1, e_2, e_3\}, \) with \( e_3 \) time-like, such that

\[
\begin{align*}
[e_1, e_2] &= Ae_1 - Be_3, \\
[e_1, e_3] &= -Ae_1 - Be_2, \\
[e_2, e_3] &= Be_1 + Ae_2 + Ae_3 \\
A &\neq 0.
\end{align*}
\]

If \( B \neq 0, \) then \( G = \widetilde{SL}(2, \mathbb{R}), \) while \( G = E(1, 1) \) when \( B = 0. \)

The curvature of Lorentzian Lie algebra \( g_1 \) was completely determined in \( \{1\}. \) In particular, with respect to \( \{e_i\}, \) the Ricci tensor is described by

\[
Ric = \begin{pmatrix}
-\frac{1}{4}B^2 & -AB & AB \\
-AB & -2A^2 - \frac{1}{2}B^2 & 2A^2 \\
AB & 2A^2 & -2A^2 + \frac{1}{2}B^2
\end{pmatrix},
\]
and the left-invariant metric is never Einstein.

For a left-invariant vector field $X = X_i e_i \in g$, we have

\[
\mathcal{L}_X g = \begin{pmatrix}
2A(X_2 - X_3) & -AX_1 & AX_1 \\
-AX_1 & 2AX_3 & -A(X_2 + X_3) \\
AX_1 & -A(X_2 + X_3) & 2AX_2
\end{pmatrix}
\]

with respect to the basis \{e_1, e_2, e_3\}, and $X^b \circ X^b(e_i, e_j) = \varepsilon_i \varepsilon_j X_i X_j$ for all indices $i, j$, where $\varepsilon_1 = \varepsilon_2 = -\varepsilon_3 = 1$ corresponds to the causal character of $e_1, e_2, e_3$. Therefore, equation (1.1) now becomes

\[
\begin{align*}
2A(X_2 - X_3) + 2\alpha X_1^2 + \beta B^2 &= 2\lambda, \\
2AX_3 + 2\alpha X_2^2 + \beta(4A^2 + B^2) &= 2\lambda, \\
2AX_2 + 2\alpha X_3^2 + \beta(4A^2 - B^2) &= -2\lambda, \\
-AX_1 + 2\alpha X_1 X_2 + 2\beta AB &= 0, \\
AX_1 - 2\alpha X_1 X_3 - 2\beta AB &= 0, \\
-A(X_2 + X_3) - 2\alpha X_2 X_3 - 4\beta A^2 &= 0.
\end{align*}
\]

Solving (3.2), we obtain the following.

**Theorem 3.1.** Consider the three-dimensional unimodular Lorentzian Lie algebra $g_1$, as described by (3.1) with respect to a suitable pseudo-orthonormal basis \{e_1, e_2, e_3\}, with $e_3$ timelike. Then, the nontrivial left-invariant generalized Ricci solitons on $g_1$ are the following:

1. \(\beta = 0, \lambda = 0, X_1 = 0, X_2 = X_3 = -\frac{A}{\alpha}, \) for all \(A \neq 0\), \(B\) and \(\alpha \neq 0\).
2. \(\alpha = 0, \lambda = \frac{1}{2} \beta B^2, X_1 = 2\beta B, X_2 = X_3 = -2\beta A, \) for all \(A \neq 0\) and \(B \neq 0\).
3. \(B = 0, \lambda = 0, X_1 = 0, X_2 = X_3 = -\frac{1}{2} \sqrt{\frac{A}{B} - \varepsilon} , \) for all \(\alpha, \beta\) satisfying \(\alpha \beta \leq \frac{1}{8}\).

Solution (2) in Theorem 3.1 corresponds to the existence of Ricci solitons on this class of Lorentzian Lie algebras [11]. Solution (3), requiring that \(\alpha \beta \leq \frac{1}{8}\) and \(\lambda = 0\), is incompatible with (VN-II), but compatible with (E-W) and (PS). It is also worth to remark that for $B = 0$, vector $X = X_2(e_2 + e_3)$ occurring in solution (3) spans a parallel degenerate line field, so showing that the Lorentzian Lie group is then a three-dimensional Walker manifold [8]. These observations yield the following result.

**Corollary 3.2.** Three-dimensional Lorentzian Lie group $E(1,1)$, with Lie algebra described by (3.1), gives solutions to the special Einstein-Weyl equation (E-W) and the equation (PS) for a metric projective structure with a skew-symmetric Ricci tensor representative.
3.2. Lie algebra \( g_2 \). There exists a pseudo-orthonormal basis \( \{e_1, e_2, e_3\} \), with \( e_3 \) time-like, such that

\[
\begin{align*}
[e_1, e_2] &= -Ce_2 - Be_3, \\
\{e_1, e_3\} &= -Be_2 + Ce_3, \quad C \neq 0, \\
[e_2, e_3] &= Ae_1.
\end{align*}
\]

(3.3)

In this case, \( G = SU(2, \mathbb{R}) \) if \( A \neq 0 \), while \( G = E(1, 1) \) if \( A = 0 \). With respect to \( \{e_i\} \), the Ricci tensor is given by (see [4])

\[
Ric = \begin{pmatrix}
-\frac{1}{2}A^2 - 2B^2 & 0 & 0 \\
0 & \frac{1}{2}A^2 - AB & C(A - 2B) \\
0 & C(A - 2B) & -\frac{1}{2}A^2 + AB
\end{pmatrix},
\]

and the left-invariant metric is never Einstein.

The Lie derivative \( \mathcal{L}_X g \) with respect to a vector \( X = X_i e_i \in g \) is described by

\[
\mathcal{L}_X g = \begin{pmatrix}
0 & -CX_2 + (A - B)X_3 & (B - A)X_2 - CX_3 \\
-CX_2 + (A - B)X_3 & 2CX_1 & 0 \\
(B - A)X_2 - CX_3 & 0 & 2CX_1
\end{pmatrix}
\]

with respect to the basis \( \{e_1, e_2, e_3\} \), and again \( X^i \otimes X^j(e_i, e_j) = \varepsilon_i \varepsilon_j X_i X_j \) for all indices \( i, j \). Thus, equation (1.1) becomes

\[
\begin{cases}
2\alpha X_1^2 + \beta(A^2 + 4B^2) = 2\lambda, \\
2CX_1 + 2\alpha X_2^2 - \beta(A^2 - 2AB) = 2\lambda, \\
2CX_1 + 2\alpha X_3^2 + \beta(A^2 - 2AB) = -2\lambda, \\
-CX_2 + (A - B)X_3 + 2\alpha X_1 X_2 = 0, \\
(B - A)X_2 - CX_3 - 2\alpha X_1 X_3 = 0, \\
-2\alpha X_2 X_3 - 2\beta C(A - 2B) = 0.
\end{cases}
\]

(3.4)

We then solve (3.4), obtaining the following classification result.

**Theorem 3.3.** Consider the three-dimensional unimodular Lorentzian Lie algebra \( g_2 \), as described by (3.3) with respect to a suitable pseudo-orthonormal basis \( \{e_1, e_2, e_3\} \), with \( e_3 \) time-like. Then, the nontrivial left-invariant generalized Ricci solitons on \( g_2 \) are given by

1. \( A + 2B = 0, \beta = -\frac{3}{8\alpha}, \lambda = \frac{\alpha(3X_2^4 - 10X_2^2X_3^2 + 3X_3^4)}{X_2^2 - X_3^2}, \ X_1 = \pm \frac{2\alpha^2 C^2 X_2^2 - 9B^2 C^2}{\sqrt{\alpha^2 C^2 X_2^2}}, \ X_3 = -\frac{3BC}{2\alpha^2 X_2}, \) for all \( \alpha \neq 0 \), where \( X_2 \) is a real solution of

\[
4\alpha^4 X_2^4 + 4\alpha^2 C^2 X_2^2 - 9B^2 C^2 = 0.
\]

(3.5)

Note that equation (3.5) admits real solutions for any value of \( \alpha \neq 0 \).
3.3. **Lie algebra** $g_3$. For a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$, with $e_3$ time-like, of eigenvector of $L$, we have

\[(3.6) \quad g_3 : \quad [e_1, e_2] = -Ce_3, \quad [e_1, e_3] = -Be_2, \quad [e_2, e_3] = Ae_1.\]

The following Table II lists all the Lie groups $G$ which admit a Lie algebra $g_3$, according to the different possibilities for $A$, $B$ and $C$:

| Lie group     | $A$ | $B$ | $C$ |
|---------------|-----|-----|-----|
| $\tilde{SL}(2, \mathbb{R})$ | +   | +   | +   |
| $\tilde{SL}(2, \mathbb{R})$ | +   | −   | −   |
| $SU(2)$       | +   | +   | −   |
| $\tilde{E}(2)$ | +   | +   | 0   |
| $E(2)$        | +   | 0   | −   |
| $E(1,1)$      | +   | −   | 0   |
| $H_3$         | +   | 0   | 0   |
| $H_3$         | 0   | 0   | −   |
| $\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ | 0   | 0   | 0   |

Table II: 3D Lorentzian Lie groups with Lie algebra $g_3$

Following [11] (or by direct calculation), the Ricci curvature of Lorentzian Lie algebra $g_3$, with respect to $\{e_i\}$, is described by

\[
Ric = \begin{pmatrix}
\frac{1}{2}(B^2 - A^2 + C^2) - BC & 0 & 0 \\
0 & \frac{1}{2}(A^2 - B^2 + C^2) - AC & 0 \\
0 & 0 & \frac{1}{2}(C^2 - A^2 - B^2) + AB
\end{pmatrix},
\]

and the left-invariant metric is Einstein (equivalently, of constant sectional curvature) if and only if either $A = B = C$, $A - B = C = 0$, $A - C = B = 0$ or $A = B - C = 0$. In the last three cases, the metric is flat.

For a left-invariant vector field $X = X_i e_i \in \mathfrak{g}$, we have

\[
\mathcal{L}_X g = \begin{pmatrix}
0 & X_3(A - B) & -X_2(A - C) \\
X_3(A - B) & 0 & X_1(B - C) \\
-X_2(A - C) & X_1(B - C) & 0
\end{pmatrix}
\]
Theorem 3.4. Consider the three-dimensional unimodular Lorentzian Lie algebra $\mathfrak{g}_3$, as described by (3.6) with respect to a suitable pseudo-orthonormal basis $\{e_1, e_2, e_3\}$, with $e_3$ time-like. Then, the nontrivial left-invariant generalized Ricci solitons on $\mathfrak{g}_3$ are the following:

1. $A = B = C$, $\alpha = 0$, $\lambda = \frac{1}{2} \beta A^2$, for all $X_1, X_2, X_3$, showing that when $A = B = C$, all left-invariant vector fields are Killing.

2. $A = B - C = 0$, $\alpha = \beta = \lambda = 0$, $X_2 = X_3 = 0$: $X = X_1 e_1$ is Killing for the flat metric obtained when $A = B - C = 0$ (corresponding solutions occur for $A - B = C = 0$ and $A - C = B = 0$).

3. $A = B = C = 0$, $\lambda = 0$, $X = 0$, for all $\alpha, \beta$: the metric is flat (corresponding solutions occur for $A - B = C = 0$ and $A - C = B = 0$).

4. $A = B = C$, $\lambda = \frac{1}{2} \beta A^2$, $X = 0$, for all $\alpha, \beta$: when $A = B = C$, the metric is Einstein.

5. $B = C$, $\lambda = \frac{1}{2} \beta A(2C - A)$, $X_1 = \pm \sqrt{\frac{\beta A(C - A)}{\alpha}}$, $X_2 = X_3 = 0$, for all $\alpha, \beta$, with $\alpha \beta A(C - A) > 0$.

6. $A = B$, $\lambda = \frac{1}{2} \beta C(2B - C)$, $X_1 = X_2 = 0$, $X_3 = \pm \sqrt{\frac{\beta C(C - B)}{\alpha}}$, for all $\alpha, \beta$, with $\alpha \beta C(C - B) > 0$.

7. $A + B = C$, $\beta = -\frac{3}{8 \alpha}$, $\lambda = \frac{A^2}{2 \alpha}$, $X_1 = -X_2 = \frac{A}{\pm \sqrt{2 \alpha}}$, $X_3 = \pm \frac{A}{2 \alpha}$, for all $\alpha \neq 0$.

8. $C = \pm \frac{X^2 - X_1^2}{4 \sqrt{X_1^2 + X_2^2}}$, $\alpha = -\frac{3}{8 \alpha}$, $\lambda = -\frac{X_1^4 + X_1^2 X_2^2 + X_1^4}{4 \beta (X_1^2 + X_2^2)}$, $X_3 = -\frac{X_1 X_2}{(X_1^2 + X_2^2)^{3/2}}$, where $X_1$ is a real root of the equation

$$3X_1^4 - 4^2(5A^2 - 2B^2)X_1^2 + 32\beta^4 A^2(A^2 - B^2) = 0,$$
and \(X_2 = \pm \sqrt{X_1^2 - \frac{16}{3} \beta^2 (A^2 - B^2)}\), for all \(\beta \neq 0\).

With regard to solution (8), we observe that equation (3.8) admits real solutions for all values of \(A, B\).

Because of condition \(\alpha \beta = -\frac{3}{8}\), solutions (7) and (8) are not compatible with none of equations (RS), (E-W), (PS) and (VN-H). We now focus our attention to solutions (5) and (6). Note that these solutions are very similar to one another. However, we listed both of them, since for (5) the vector satisfying (3.7) is space-like, while for (6) it is time-like. Moreover, we observe that conditions \(A = B\) and \(B = C\) are also related to the classification of three-dimensional naturally reductive Lorentzian Lie groups [5].

Solutions (5) and (6) are compatible with (E-W), (PS) and (VN-H). More precisely, for any choice of \(A\) and \(C \neq A\):

- If \(A(A - C) > 0\), then by (5) we have that the same left-invariant Lorentzian metric on \(g_3\) is solution to both (E-W) and (PS).
- If \(A(A - C) < 0\), then (5) describes a left-invariant Lorentzian metric on \(g_3\), which is a solution to (VN-H).

Similar observations hold for solution (6), discussing the cases when \(\alpha \beta C(C - B) > 0\).

Hence, taking into account Table II, we have the following result.

**Corollary 3.5.** Three-dimensional Lorentzian Lie groups \(\tilde{S}L(2, \mathbb{R})\) and \(H_3\), with Lie algebra described by (3.6), give solutions to the special Einstein-Weyl equation (E-W), the equation (PS) for a metric projective structure with a skew-symmetric Ricci tensor representative and to the vacuum near-horizon geometry equation (VN-H). Three-dimensional Lorentzian Lie group \(SU(2)\), with Lie algebra described by (3.6), gives solutions to (VN-H).

3.4. Lie algebra \(g_4\). There exists a pseudo-orthonormal basis \(\{e_1, e_2, e_3\}\), with \(e_3\) time-like, such that

\[
[e_1, e_2] = -e_2 + (2\eta - \beta)e_3, \quad \eta = \pm 1, \\
 \quad [e_1, e_3] = -\beta e_2 + e_3, \\
 \quad [e_2, e_3] = \alpha e_1.
\]

(3.9)

\(g_4\) : 

| Lie group \(|\mathbb{R}| \) | \(\eta A\) | \(B\) |
|-----------------|----------|------|
| \(\tilde{S}L(2, \mathbb{R})\) | \(\neq 0\) | \(\neq \eta\) |
| \(E(1, 1)\) | 0 | \(\neq \eta\) |
| \(E(1, 1)\) | < 0 | \(\eta\) |
| \(\tilde{E}(2)\) | > 0 | \(\eta\) |
| \(H_3\) | 0 | \(\eta\) |

*Table III: 3D Lorentzian Lie groups with Lie algebra \(g_4\)*
With respect to \( \{e_i\} \), the Ricci tensor is described by (see [4])

\[
Ric = \begin{pmatrix}
-\frac{1}{2}A^2 & 0 & 0 \\
0 & \frac{1}{2}A^2 + 2\eta(A - B) - AB + 2A & A + 2(\eta - B) \\
0 & 0 & -A^2 + AB + 2 - 2\eta B
\end{pmatrix},
\]

and the left-invariant metric is Einstein if and only if \( A = B - \eta = 0 \).

For a left-invariant vector field \( X = \xi e_i \in \mathfrak{g} \), we have

\[
\mathcal{L}_{X}g = \begin{pmatrix}
0 & -X_2 + (A - B)X_3 & (B - A - 2\eta)X_2 - X_3 \\
-X_2 + (A - B)X_3 & 2X_1 & 2\eta X_1 \\
(B - A - 2\eta)X_2 - X_3 & 2\eta X_1 & 2X_1
\end{pmatrix}
\]

with respect to the basis \( \{e_1, e_2, e_3\} \), and \( X^b \circ X^j(e_i, e_j) = \varepsilon_i\varepsilon_jX_iX_j \). Therefore, equation (1.1) now becomes

\[
\begin{align*}
2\alpha X_1^2 + \beta A^2 &= 2\lambda, \\
2X_1 + 2\alpha X_2^2 - \beta(A^2 + 4\eta(A - B) - 2AB + 4) &= 2\lambda, \\
2X_1 + 2\alpha X_3^2 + \beta(A^2 - 2AB - 4 + 4\eta B) &= -2\lambda, \\
-X_2 + (A - B)X_3 + 2\alpha X_1X_2 &= 0, \\
(B - A - 2\eta)X_2 - X_3 - 2\alpha X_1X_3 &= 0, \\
2\eta X_1 - 2\alpha X_2X_3 - 2\beta(A + 2(\eta - B)) &= 0.
\end{align*}
\]

(3.10)

We then solve (3.10) and prove the following.

**Theorem 3.6.** Consider the three-dimensional unimodular Lorentzian Lie algebra \( \mathfrak{g}_4 \), as described by (5.9) with respect to a suitable pseudo-orthonormal basis \( \{e_1, e_2, e_3\} \), with \( e_3 \) time-like. Then, the nontrivial left-invariant generalized Ricci solitons on \( \mathfrak{g}_4 \) are the following:

1. \( A = B - \eta, \lambda = \frac{1}{2}\beta A^2, X_1 = 0, X_2 = -\eta X_3, X_3 = \pm \sqrt{-\frac{\eta A}{4\alpha^2}}, \) for all \( \alpha, \beta \) satisfying \( \eta A \alpha \beta < 0 \).
2. \( A = B - \eta, \alpha = 0, \lambda = \frac{1}{2}\beta A^2, X_1 = -\beta A, X_2 = -\eta X_3, \) for any value of \( \beta \) and \( X_3 \).
3. \( B = \frac{1}{2}A + \eta, \beta = -\frac{1}{8}\alpha, \lambda = 0, X_1 = -\frac{A}{4\alpha}, X_3 = -\eta X_2, X_2 = \pm \sqrt{-\frac{\eta A}{4\alpha^2}}, \) for any \( \alpha \neq 0 \) and \( \eta A < 0 \).
4. \( \beta = -\frac{A(A - B + \eta)}{\alpha(A - 2B + 2\eta)^2}, \lambda = -\frac{1}{2}\beta A(A - 2B + 2\eta), X_1 = \beta(A - 2B + 2\eta), X_2 = X_3 = 0, \) for any \( \alpha \neq 0 \) and \( A - 2B + 2\eta \neq 0 \).
In the above Theorem 3.6 solution (2) corresponds to the existence of Ricci solitons for Lorentzian Lie algebras of the form $\mathfrak{g}_4$. Solution (1) requires that the sign of $\alpha \beta$ is opposite to the one of $\eta A$. Consequently, taking into account the above Table III, it yields solutions to (E-W) for Lorentzian Lie groups $\widetilde{SL}(2, \mathbb{R})$ and $\widetilde{E}(2)$, and solutions to (VN-H) for $\widetilde{SL}(2, \mathbb{R})$ and $E(1, 1)$.

In solution (4), we have $\alpha \beta = -\frac{A(A - B + \eta)}{(A - 2B + 2\eta)^2}$. For (E-W), as $\alpha \beta = -1$, this equation becomes $4B^2 + (5A + 8\eta)B + 3\eta A = 0$, which admits real solutions for any value of $A$. On the other hand, for (VN-H) we have $\alpha \beta = \frac{1}{2}$. Hence, we find $3A^2 - 6(B - \eta)A + 4(B - \eta)^2 = 0$, which has not real solutions. Finally, it is easily seen that solution (4) is not compatible with (PS).

By similar arguments, from solution (5) we find again $\widetilde{SL}(2, \mathbb{R})$ as solution to (E-W) and (VN-H). Thus, we proved the following.

**Corollary 3.7.** Three-dimensional Lorentzian Lie group $\widetilde{SL}(2, \mathbb{R})$, with Lie algebra described by (3.9), gives solutions to the special Einstein-Weyl equation (E-W) and the vacuum near-horizon equation (VN-H). Moreover, Lorentzian Lie groups $\widetilde{E}(2)$ and $E(1, 1)$, with Lie algebra described by (3.9), give solutions to (E-W) and (VN-H) respectively.

### 4. 3D Lorentzian Left-invariant Non-Unimodular Generalized Ricci Solitons

Let now $\mathfrak{g}$ denote a three-dimensional non-unimodular Lorentzian Lie algebra. Differently from the Riemannian case, we must now consider three distinct cases, depending on whether the two-dimensional unimodular kernel $u$ is either space-like, time-like or degenerate [10]. These three cases were listed in [3] as Lorentzian Lie algebras $\mathfrak{g}_5$, $\mathfrak{g}_6$ and $\mathfrak{g}_7$.

#### 4.1. Lie algebra $\mathfrak{g}_5$.

There exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$, with $e_3$ time-like, such that

\[(4.1) \quad [e_1, e_2] = 0, \quad [e_1, e_3] = Ae_1 + Be_2, \quad [e_2, e_3] = Ce_1 + De_2, \quad A + D \neq 0, \quad AC + BD = 0,
\]

for some real constants $A, B, C, D$.

With respect to the basis $\{e_1, e_2, e_3\}$, the Ricci curvature is described by (see [1])

\[
Ric = \begin{pmatrix}
A^2 + \frac{1}{2}B^2 - \frac{1}{2}C^2 + AD & 0 & 0 \\
0 & AD - \frac{1}{2}B^2 + \frac{1}{2}C^2 + D^2 & 0 \\
0 & 0 & -A^2 - \frac{1}{2}B^2 - \frac{1}{2}C^2 - D^2 - BC
\end{pmatrix}.
\]
In particular, the left-invariant metric is of constant sectional curvature if and only if $A - D = B + C = 0$.

For an arbitrary left-invariant vector field $X = X_i e_i \in \mathfrak{g}$, we have

$$L_X g = \begin{pmatrix}
2AX_3 & (B + C)X_3 & -AX_1 - CX_2 \\
(B + C)X_3 & 2DX_3 & -BX_1 - DY_2 \\
-AX_1 - CX_2 & -BX_1 - DY_2 & 0
\end{pmatrix}. $$

With respect to the basis \{e_1, e_2, e_3\}, we have again $X^3 \circ X^3(e_i, e_j) = \varepsilon_i \varepsilon_j X_i X_j$. Hence, equation (4.1) now gives

$$
\begin{cases}
2AX_3 + 2\alpha X_1^2 - \beta(2A^2 + B^2 - C^2 + 2AD) = 2\lambda, \\
2DX_3 + 2\alpha X_2^2 - \beta(2AD - B^2 + C^2 + 2D^2) = 2\lambda, \\
2\alpha X_3^2 + \beta(2A^2 + B^2 + C^2 + 2D^2 + 2BC) = -2\lambda, \\
(B + C)X_3 + 2\alpha X_1 X_2 = 0, \\
-AX_1 - CX_2 - 2\alpha X_1 X_3 = 0, \\
-BX_1 - DX_2 - 2\alpha X_2 X_3 = 0.
\end{cases}
$$ (4.2)

We then solve (4.2) and obtain the following.

**Theorem 4.1.** Let $\mathfrak{g}$ denote a three-dimensional non-unimodular Lorentzian Lie algebra $\mathfrak{g}_5$, as described by (1.1) with respect to a suitable pseudo-orthonormal basis \{e_1, e_2, e_3\}, with $e_3$ time-like. Then, the nontrivial left-invariant generalized Ricci solitons on $\mathfrak{g}_5$ are the following:

1. $A - D = B + C = 0$, $\lambda = -\left(\frac{1}{\alpha} + 2\beta\right)A^2$, $X_1 = X_2 = 0$, $X_3 = -\frac{A}{\alpha}$, for all $\alpha \neq 0$ and $\beta$ (constant sectional curvature).

2. $C = D = 0$, $\beta = -\frac{1}{4\alpha}$, $\lambda = \frac{B^2}{8\alpha}$, $X_1 = \varepsilon X_3$, $X_2 = \frac{\varepsilon B}{2\alpha}$, $X_3 = -\frac{A}{2\alpha}$, $\varepsilon = \pm 1$, for all $\alpha \neq 0$.

3. $A - D = B + C = 0$, $\lambda = -2\beta A^2$, $X = 0$, for all $\alpha, \beta$: the metric is of constant sectional curvature.

4. $C = D = 0$, $\lambda = -\frac{1}{2}\beta(2A^2 + B^2)$, $X_1 = X_3 = 0$, $X_2 = \pm \sqrt{-\frac{\beta(2A^2 + B^2)}{\alpha}}$, for any $\alpha$ and $\beta$ satisfying $\alpha \beta < 0$.

5. $B = C = D = 0$, $\beta = -\frac{1}{\alpha}$, $\lambda = 0$, $X_1 = X_2 = 0$, $X_3 = -\frac{A}{\alpha}$, for any $\alpha \neq 0$.

6. $A - D = B = C = 0$, $\lambda = -\left(\frac{1}{\alpha} + 2\beta\right)D^2$, $X_1 = X_2 = 0$, $X_3 = -\frac{D}{\alpha}$, for any $\alpha \neq 0$ and $\beta$.

7. $B = C = 0$, $\lambda = \frac{A^2(32\alpha^3\beta^3 + 28\alpha^2\beta^2 + 9\alpha\beta + 1)}{4\alpha(2\alpha\beta + 1)^2}$, $X_1 = \pm \frac{A\sqrt{8\alpha^2\beta^2 + 5\alpha\beta + 1}}{2\alpha(2\alpha\beta + 1)}$, $X_2 = 0$, $X_3 = -\frac{A}{2\alpha}$, for any $\alpha \neq 0$ and $\beta$, with $2\alpha\beta + 1 \neq 0$ (note that $a^2\beta^2 + 5\alpha\beta + 1 > 0$).
\((8)\) \(B = C = 0, \beta = -\frac{2A - D}{4\alpha(A-D)}, \lambda = -\frac{A(2A^2 - AD + D^2)}{4\alpha(A-D)}, X_1 = 0, X_2 = \pm \frac{1}{\alpha} \sqrt{A^2 - \frac{1}{2}AD + \frac{1}{2}D^2}, X_3 = -\frac{D}{2\alpha}\), for any \(\alpha \neq 0\) and \(A \neq D\) (note that \(2A^2 - AD + D^2 > 0\)).

\((9)\) \(A - D = B - C = 0, \beta = -\frac{3}{8\alpha}, X_2 = -kX_1, X_3 = k\alpha X_1^2, \) where \(k := \pm \frac{1}{4\alpha^2 X_1^2 - C^2}\) and \(X_1\) is a real solution (when it exists) of

\[2\alpha^3 X_1^4 - 4\alpha^2 \lambda Y_1^2 + \lambda C^2 = 0.\]

It is easy to check that several of the above solutions are compatible with (E-W),(PS) and (VN-H). In particular, solution (6) yields solutions to all these equations. Hence, we have the following.

**Corollary 4.2.** Three-dimensional non-unimodular Lorentzian Lie groups with Lie algebra \(g_5\) give solutions to the special Einstein-Weyl equation (E-W), to the equation (PS) for a metric projective structure with a skew-symmetric Ricci tensor representative and to the vacuum near-horizon geometry equation (VN-H).

4.2. **Lie algebra \(g_6\).** There exists a pseudo-orthonormal basis \(\{e_1, e_2, e_3\}\), with \(e_3\) time-like, such that

\[(4.3)\]

\[[e_1, e_2] = Ae_2 + Be_3, \quad [e_1, e_3] = Ce_2 + De_3, \quad [e_2, e_3] = 0, \quad A + D \neq 0, \quad AC - BD = 0,\]

for some real constants \(A, B, C, D\).

Following [4], with respect to the basis \(\{e_1, e_2, e_3\}\) we have

\[Ric = \begin{pmatrix}
\frac{1}{2}B^2 - A^2 + \frac{1}{2}C^2 - D^2 + BC & 0 & 0 \\
0 & \frac{1}{2}B^2 - A^2 - \frac{1}{2}C^2 - AD & 0 \\
0 & 0 & AD + \frac{1}{2}B^2 - \frac{1}{2}C^2 + D^2
\end{pmatrix}.
\]

In particular, the left-invariant metric is of constant sectional curvature if and only if either \(A - D = B - C = 0, A + B = C + D = 0\) or \(A - B = C - D = 0\).

For an arbitrary left-invariant vector field \(X = X_i e_i \in g\), we have

\[\mathcal{L}_X g = \begin{pmatrix}
0 & AX_2 + CX_3 & -BX_2 - DX_3 \\
AX_2 + CX_3 & -2AX_1 & (B - C)X_1 \\
-BX_2 - DX_3 & (B - C)X_1 & 2DX_1
\end{pmatrix}.
\]
Theorem 4.3. We solve (4.4) and prove the following.

We solve (4.4) and prove the following:

\[ \begin{align*}
2\alpha X_2^2 + \beta (2A^2 - B^2 - C^2 + 2D^2 - 2BC) &= 2\lambda, \\
-2AX_1 + 2\alpha X_2^2 + \beta (2A^2 - B^2 + C^2 + 2AD) &= 2\lambda, \\
2DX_1 + 2\alpha X_2^2 - \beta (2AD + B^2 - C^2 + 2D^2) &= -2\lambda, \\
AX_2 + CX_3 + 2\alpha X_1 X_2 &= 0, \\
-BX_2 - DX_3 - 2\alpha X_1 X_3 &= 0, \\
(B-C)X_1 - 2\alpha X_2 X_3 &= 0.
\end{align*} \] (4.4)

We solve (4.4) and prove the following.

**Theorem 4.3.** Let \( \mathfrak{g} \) denote a three-dimensional non-unimodular Lorentzian Lie algebra \( \mathfrak{g}_6 \), as described by (1.3) with respect to a suitable pseudo-orthonormal basis \( \{e_1, e_2, e_3\} \), with \( e_3 \) time-like. Then, the nontrivial left-invariant generalized Ricci solitons on \( \mathfrak{g}_6 \) are the following:

1. \( A - D = B - C = 0, \lambda = \frac{1}{2}(A^2 + D^2), X_1 = X_2 = 0, X_3 = 0, \) for all \( \alpha \neq 0 \) and \( \beta, \) (constant sectional curvature).
2. \( A - D = B - C = 0, \lambda = 2\beta A^2, X = 0, \) for all \( \alpha \neq 0 \) and \( \beta, \) (constant sectional curvature).
3. \( B = \pm A, C = \pm U, \lambda = \frac{1}{2}\beta(A^2 + D^2), X = 0, \) for all \( \alpha, \beta; \) the metric is Einstein.
4. \( C = D = 0, \lambda = \frac{1}{2}\beta(2A^2 - B^2), X_1 = X_2 = 0, X_3 = \pm \sqrt{\frac{\beta(B^2 - A^2)}{\alpha}}, \) for any \( \alpha \) and \( \beta \) satisfying \( \alpha\beta(B^2 - A^2) > 0. \)
5. \( B = C = D = 0, \beta = \frac{1}{2}, \lambda = 0, X_1 = -\frac{A}{\alpha}, X_2 = X_3 = 0, \) for any \( \alpha \neq 0. \)
6. \( B = C = 0, \beta = -\frac{A^2 + D^2}{\alpha(A + D)}, \lambda = 0, X_1 = \beta(A + D), X_2 = X_3 = 0, \) for any \( \alpha \neq 0. \)
7. \( B = C = 0, \beta = -\frac{2A-D}{4\alpha(A-D)}, \lambda = \frac{A(2A^2 - AD + D^2)}{4\alpha(A-D)}, X_1 = -\frac{D}{2\alpha}, X_2 = 0, X_3 = \pm \frac{1}{2} \sqrt{A^2 - \frac{1}{2}AD + \frac{1}{2}D^2}, X_3 = 0, \) for any \( \alpha \neq 0 \) and \( \beta \neq 0. \)
8. \( A = B = C = 0, \beta = -\frac{1}{2\alpha}, \lambda = 0, X_1 = -\frac{D}{2\alpha}, X_2 = 0, X_3 = \pm X_1, \) for any \( \alpha \neq 0 \) and \( \beta. \)
9. \( A = B = 0, \lambda = \frac{1}{2}\beta(2D^2 - C^2), X_1 = 0, X_2 = \pm \sqrt{\frac{\beta(D^2 - C^2)}{\alpha}}, X_3 = 0, \) for all \( \alpha, \beta \) with \( \alpha\beta(D^2 - C^2) > 0. \)

By the same argument used in the previous case, it is easy to check that several of the above solutions are compatible with (E-W), (PS) and (VN-H). Thus, we have the following.
Corollary 4.4. Three-dimensional non-unimodular Lorentzian Lie groups with Lie algebra \( g_6 \) give solutions to the special Einstein-Weyl equation (E-W), to the equation (PS) for a metric projective structure with a skew-symmetric Ricci tensor representative and to the vacuum near-horizon geometry equation (VN-H).

4.3. Lie algebra \( g_7 \). There exists a pseudo-orthonormal basis \( \{e_1, e_2, e_3\} \), with \( e_3 \) timelike, such that
\[
[e_1, e_2] = -[e_1, e_3] = -Ae_1 - Be_2 - De_3, \quad [e_2, e_3] = e_1 + De_2 + De_3, \quad A + D \neq 0, \quad AC = 0,
\]
for some real constants \( A, B, C, D \). The Ricci curvature is then described by (see [4])
\[
\begin{pmatrix}
-\frac{1}{2}C^2 & 0 & 0 \\
0 & AD - A^2 + \frac{1}{2}C^2 - BC & A^2 - AD + BC \\
0 & A^2 - AD + BC & AD - A^2 - \frac{1}{2}C^2 - BC
\end{pmatrix}
\]
and the left-invariant metric is of constant sectional curvature (flat) if and only if either \( A = C = 0 \) or \( A - D = C = 0 \). For a vector \( X = X_i e_i \in g \), we have
\[
\mathcal{L}_X g = \begin{pmatrix}
-2A(X_2 - X_3) & AX_1 - BX_2 + (B + C)X_3 & -AX_1 + (B - C)X_2 - BX_3 \\
AX_1 - BX_2 + (B + C)X_3 & 2BX_1 + 2DX_3 & -2BX_1 - DX_2 - DX_3 \\
-AX_1 + (B - C)X_2 - BX_3 & -2BX_1 - DX_2 - DX_3 & 2BX_1 + 2DX_2
\end{pmatrix}.
\]
Thus, equation (1.1) becomes
\[
\begin{align*}
-2A(X_2 - X_3) + 2\alpha X_1^2 + \beta C^2 &= 2\lambda, \\
2BX_1 + 2DX_3 + 2\alpha X_2^2 + \beta(2A^2 - 2AD - C^2 + 2BC) &= 2\lambda, \\
2BX_1 + 2DX_3 + 2\alpha X_2^2 + \beta(2A^2 - 2AD + C^2 + 2BC) &= -2\lambda, \\
AX_1 - BX_2 + (B + C)X_3 + 2\alpha X_1 X_2 &= 0, \\
-AX_1 + (B - C)X_2 - BX_3 - 2\alpha X_1 X_3 &= 0, \\
-2BX_1 - DX_2 - DX_3 - 2\alpha X_2 X_3 - 2\beta(A^2 - AD + BC) &= 0.
\end{align*}
\]
Solving (4.6), we prove the following.

Theorem 4.5. Let \( g \) denote a three-dimensional non-unimodular Lorentzian Lie algebra \( g_7 \), as described by [4.5] with respect to a suitable pseudo-orthonormal basis \( \{e_1, e_2, e_3\} \), with \( e_3 \) timelike. Then, the nontrivial left-invariant generalized Ricci solitons on \( g_7 \) are the following:

1. \( A = C = 0, \alpha = 0, \lambda = 0, X_2 = X_3 = -\frac{B}{D}X_1 \), for all \( \beta \) and \( X_1 \) (flat).

2. \( A = \frac{1}{2}D, C = 0, \alpha = 0, X_1 = -\frac{4B\lambda}{D^2}, X_2 = \frac{16\lambda B - 4\lambda D^2 + \beta D^4}{4D^3}, X_3 = \frac{16\lambda B + 4\lambda D^2 + \beta D^4}{4D^3} \), for all \( \beta \).
Corollary 4.6. Three-dimensional non-unimodular Lorentzian Lie groups with Lie algebra \( g_7 \) give solutions to the special Einstein-Weyl equation (E-W), to the equation (PS) for a metric projective structure with a skew-symmetric Ricci tensor representative and to the vacuum near-horizon geometry equation (VN-H).

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