IDEMPOTENT MEASURES: ABSOLUTE RETRACTS AND SOFT MAPS

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Abstract. We investigate under which conditions the space of idempotent measures is an absolute retract and the idempotent barycenter map is soft.

1. INTRODUCTION

The notion of idempotent (Maslov) measure finds important applications in different part of mathematics, mathematical physics and economics (see the survey article [8] and the bibliography therein). Topological and categorical properties of the functor of idempotent measures were studied in [19]. Although idempotent measures are not additive and corresponding functionals are not linear, there are some parallels between topological properties of the functor of probability measures and the functor of idempotent measures (see for example [19] and [12]) which are based on existence of natural equiconnectedness structure on both functors.

However, some differences appear when the problem of the openness of the barycentre map was studied. The problem of the openness of the barycentre map of probability measures was investigated in [5], [6], [4], [10] and [11]. In particular, it is proved in [10] that the barycentre map for a compact convex set in a locally convex space is open if the map \((x, y) \mapsto 1/2(x + y)\) is open. Zarichnyj defined in [19] the idempotent barycentre map for idempotent measures and asked if the analogous characterization is true. A negative answer to this question was given in [13].

We investigate the problem when the space of idempotent measures is absolute retract (shortly AR). It is shown in [19] that the space of idempotent measures \(I([0, 1]^\tau)\) on Tychonov cube \([0, 1]^\tau\) is not an absolute retract for any \(\tau > \omega_1\). It follows from the results of [12] that the space of idempotent measures \(IX\) is an absolute retract for each openly generated compactum \(X\) of the weight \(\leq \omega_1\). We will show in this paper that the space of idempotent measures \(IX\) is an absolute retract iff \(X\) is an openly generated compactum of the weight \(\leq \omega_1\). Let us remark that it is an idempotent analogue of Ditor-Haydon Theorem for probability measures [3].

The problem of the softness of the barycentre map of probability measures was investigated in [6], [14] and [15]. Fedorchuk proved in [6] that each product of \(\omega_1\) barycentrically open convex metrizable compacta (i.e. convex metrizable compacta for which the barycentre map is open) is barycentrically soft and asked two questions: if each barycentrically open convex compactum of the weight \(\leq \omega_1\) is barycentrically soft and if there exists a barycentrically soft convex compactum of the weight \(\geq \omega_2\). The first question was answered in negative in [12], showing that barycentric softness of the space of probability measures \(PX\) implies metrizability of the compactum \(X\). The second question was answered in negative in [15].

In this paper we discuss analogous problems for the space of idempotent measures and idempotent barycenter map.

2. IDEMPOTENT MEASURES: PRELIMINARIES

In the sequel, all maps will be assumed to be continuous. Let \(X\) be a compact Hausdorff space. We shall denote by \(C(X)\) the Banach space of continuous functions on \(X\) endowed with the sup-norm. For any \(c \in R\) we shall denote by \(c_X\) the constant function on \(X\) taking the value \(c\).

Let \(R_{max} = R \cup \{-\infty\}\) be the metric space endowed with the metric \(g\) defined by \(g(x, y) = |c^x - c^y|\). Following the notation of idempotent mathematics (see e.g., [9]) we use the notations \(\oplus\) and \(\ominus\) in \(R\) as alternatives for max and + respectively. The convention \(-\infty \ominus x = -\infty\) allows us to extend \(\ominus\) and \(\oplus\) over \(R_{max}\).

Max-Plus convex sets were introduced in [22]. Let \(\tau\) be a cardinal number. Given \(x, y \in R^\tau\) and \(\lambda \in R_{max}\), we denote by \(y \oplus x\) the coordinatewise maximum of \(x\) and \(y\) and by \(\lambda \ominus x\) the vector obtained from \(x\) by adding \(\lambda\) to each of its coordinates. A subset \(A\) in \(R^\tau\) is said to be Max-Plus convex if \(\alpha \ominus a \ominus b \in A\) for all \(a, b \in A\) and \(\alpha \in R_{max}\) with \(\alpha \leq 0\). It is easy to check that \(A\) is Max-Plus convex iff \(\bigoplus_{i=1}^n \lambda_i \ominus \delta_{x_i} \in A\) for all \(x_1, \ldots, x_n \in A\) and \(\lambda_1, \ldots, \lambda_n \in R_{max}\) such that \(\bigoplus_{i=1}^n \lambda_i = 0\). In the following by Max-Plus convex compactum we mean a Max-Plus convex compact subset of \(R^\tau\).

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We denote by \( \odot : \mathbb{R} \times C(X) \to C(X) \) the map acting by \((\lambda, \varphi) \mapsto \lambda X + \varphi\), and by \( \oplus : C(X) \times C(X) \to C(X) \) the map acting by \((\psi, \varphi) \mapsto \max\{\psi, \varphi\}\).

**Definition 2.1.** \(^{[19]}\) A functional \( \mu : C(X) \to \mathbb{R} \) is called an idempotent measure (a Maslov measure) if

1. \( \mu(1_X) = 1\);
2. \( \mu(\lambda \odot \varphi) = \lambda \odot \mu(\varphi) \) for each \( \lambda \in \mathbb{R} \) and \( \varphi \in C(X)\);
3. \( \mu(\psi \oplus \varphi) = \mu(\psi) \oplus \mu(\varphi) \) for each \( \psi, \varphi \in C(X)\).

Let \( IX \) denote the set of all idempotent measures on a compactum \( X \). We consider \( IX \) as a subspace of \( \mathbb{R}^{C(X)} \).

It is shown in \(^{[19]}\) that \( IX \) is a compact Max-Plus subset of \( \mathbb{R}^{C(X)} \). The construction \( I \) is functorial what means that for each continuous map \( f : X \to Y \) we can consider a continuous map \( If : IX \to IY \) defined as follows.\(^{[19]}\) A functional \( \mu \) is upper semicontinuous and is called the density of \( \mu \) if \( \mu(\lambda \odot \varphi) = \lambda \odot \mu(\varphi) \) for each \( \lambda \in \mathbb{R} \) and \( \varphi \in C(X)\). Let \( \beta_A : IA \to A \) be a function defined by the formula \( \beta_A(\mu) = \mu(\cdot) \) for a compact Max-Plus convex subset. It is easy to see that the map \( \beta_A : IA \to A \) is continuous. The map \( \beta_A \) is called the idempotent barycenter map.

For a function \( \varphi \in C(X) \) by \( \tilde{\varphi} \in C(IX) \) we denote the function defined by the formula \( \tilde{\varphi}(\nu) = \nu(\varphi) \) for \( \nu \in IX \). Diagonal product \( (\tilde{\varphi})_{\varphi \in C(X)} \) embeds \( IX \) into \( \mathbb{R}^{C(X)} \) as a Max-Plus convex subset. It is easy to see that the map \( \beta_{IX} \) satisfies the equality \( \beta_{IX}(\mathcal{M})(\varphi) = M(\tilde{\varphi}) \) for any \( \mathcal{M} \in \mathcal{P}^2 X = I(X) \) and \( \varphi \in C(X) \). Particularly we have \( \beta_{IX} \circ I(\delta_X) = 1_{ix} \) for each compactum \( X \).

A map \( f : X \to Y \) between Max-Plus convex compacta \( X \) and \( Y \) is called Max-Plus affine if for each \( a, b \in X \) and \( \alpha \in [-\infty, 0] \) we have \( f(\alpha \odot a \oplus b) = \alpha \odot f(a) \oplus f(b) \). It is easy to check that the diagram

\[
\begin{array}{ccc}
IX & \xrightarrow{If} & IY \\
\downarrow{\beta_X} & & \downarrow{\beta_Y} \\
X & \xrightarrow{f} & Y
\end{array}
\]

is commutative provided \( f \) is Max-Plus affine. It is also easy to check that the map \( b_X : X \to Y \) is Max-Plus affine for each Max-Plus convex compactum \( X \) and the map \( If \) is Max-Plus affine for each continuous map \( f : X \to Y \) between compacta \( X \) and \( Y \).

The notion of density for an idempotent measure was introduced in \(^{[1]}\). Let \( \mu \in IX \). Then we can define a function \( d_\mu : X \to [-\infty, 0] \) by the formula \( d_\mu(x) = \inf\{\mu(\varphi) : \varphi \in C(X) \} \) such that \( \varphi \leq 0 \) and \( \varphi(x) = 0 \), \( x \in X \). The function \( d_\mu \) is upper semicontinuous and is called the density of \( \mu \). Conversely, each upper semicontinuous function \( f : X \to [-\infty, 0] \) with \( \max f = 0 \) determines an idempotent measure \( \nu_f \) by the formula \( \nu_f(\varphi) = \max\{f(x) \odot \varphi(x) : x \in X\} \), for \( \varphi \in C(X) \).

Let \( A \) be a closed subset of a compactum \( X \). It is easy to check that \( \nu \in IA \) iff \( \{x \in X : d_\nu(x) > -\infty\} \subset A \).

**Lemma 2.2.** Let \( A \) be a closed subset of a compactum \( X \). Then \( \beta^{-1}_{IA}(IA) \subset \mathcal{P}^2 A \).

**Proof.** Suppose the contrary. Then there exists \( \mathcal{M} \in \beta^{-1}_{IX}(IA) \setminus \mathcal{P}^2 A \). Hence there exists \( \mu \in IX \) such that \( d_{\mathcal{M}}(\mu) = s > -\infty \) and \( d_{\nu}(x) = t > -\infty \) for some point \( x \in X \setminus A \). Choose a function \( \varphi \in C(X) \) such that \( \varphi(x) = 1 - s - t \) and \( \varphi(A) \subset \{0\} \). Then we have \( \mathcal{M}(\tilde{\varphi}) \geq \tilde{\varphi}(\mu) + s = \mu(\varphi) + s \geq \varphi(x) + t + s = 1 \). On the other hand \( \beta_{IX}(\mathcal{M}) \in IA \) implies \( \beta_{IX}(\mathcal{M})(\varphi) = 0 \). But \( \mathcal{M}(\tilde{\varphi}) = \beta_{IX}(\mathcal{M})(\varphi) \) and we obtain a contradiction.

**Lemma 2.3.** Let \( f : X \to Y \) be a continuous map, \( \nu \in IX \). Then \( d_{If(\nu)}(y) = \max\{d_{\nu}(x) : x \in f^{-1}(y)\} \) for each \( y \in Y \).

**Proof.** Let \( d : Y \to [-\infty, 0] \) be a function defined by the formula \( d(y) = \max\{d_{\nu}(x) : x \in f^{-1}(y)\} \) for \( y \in Y \). It is easy to see that the function \( d \) is upper semicontinuous with \( \max d = 0 \). Let \( \mu \) be an idempotent measure generated by \( d \). Then we have

\[
\mu(\varphi) = \max\{d(y) + \varphi(y) : y \in Y\} = \max\{\varphi(y) + \max\{d_{\nu}(x) : x \in f^{-1}(y)\} : y \in Y\} = \\
= \max\{\varphi \circ f(x) + d_{\nu}(x) : x \in X\} = \nu(\varphi \circ f) = If(\nu)(\varphi)
\]

for each \( \varphi \in C(Y) \). Hence \( \mu = If(\nu) \).
Lemma 2.4. Let $f : X \to Y$ be a continuous map, $A$ and $B$ are disjoint closed subsets of $Y$ and $\mu \in IX$ such that $I\mu = s \circ \nu \circ \pi$ where $\nu \in IA$ and $\pi \in IB$. Then there exist $\nu' \in I(f^{-1}(A))$ and $\pi' \in I(f^{-1}(B))$ such that $\mu = s \circ \nu' \circ \pi'$.

Proof. Consider the density $d_\mu$ of $\mu$. We have that $\max\{d_\mu(x) | x \in f^{-1}(A)\} = s$ and $\max\{d_\mu(x) | x \in f^{-1}(B)\} = 1$ by Lemma [23]. Consider functions $d_1, d_2 : X \to [-\infty,0]$ defined by the formulas

$$d_1(x) = \begin{cases} d_\mu(x) - c, & x \in A, \\ -\infty, & x \notin A \end{cases}$$

and

$$d_2(x) = \begin{cases} d_\mu(x), & x \in B, \\ -\infty, & x \notin B \end{cases}$$

and idempotent measures $\nu'$ and $\pi'$ generated by function $d_1$ and $d_2$. Then $\nu'$ and $\pi'$ are the measures we are looking for.

3. IDEMPOTENT MEASURES AND ABSOLUTE RETRACTS

By $w(X)$ we denote the weight of the space $X$ and by $\chi(X)$ the character of the space $X$.

We will need some notions and facts from the theory of non-metrizable compacta. See [17] for more details. Let $\tau$ be an infinite cardinal number. A partially ordered set $A$ is called $\tau$-complete, if every subset of cardinality $\leq \tau$ has a least upper bound in $A$. An inverse system consisting of compacta and surjective bonding maps over a $\tau$-complete indexing set is called $\tau$-complete. A continuous $\tau$-complete system consisting of compacta of weight $\leq \tau$ is called a $\tau$-system.

As usual, by $\omega$ we denote the countable cardinal number, by $\omega_1$ we denote the first uncountable cardinal number and so on.

A compactum $X$ is called openly generated if $X$ can be represented as the limit of an $\omega$-system with open bonding maps. We have $w(X) = \chi(X)$ for each openly generated compactum $X$ (see for example Lemma 4 from [16]). A compactum $X$ is called absolute extensor in the class of 0-dimensional compacta (shortly AE(0)) if for any 0-dimensional compactum $Z$, any closed subspace $A$ of $Z$ and a continuous map $\varphi : A \to X$ there exists a continuous map $\Phi : Z \to X$ such that $\Phi|A = \varphi$. Evidently each absolute retract is AE(0). Let us also remark that each AE(0) is openly generated and these classes coincide for compacta of the weight $\leq \omega_1$.

By $D$ we denote the two-point set with discrete topology.

Lemma 3.1. The compactum $I(D^\tau)$ is not an absolute retract for each $\tau \geq \omega_2$.

Proof. Suppose the contrary: there exists $\tau \geq \omega_2$ such that $I(D^\tau)$ is an absolute retract. Choose a continuous onto map $f : D^\tau \to [0,1]^\tau$ such that there exists a continuous map $s : [0,1]^\tau \to I(D^\tau)$ such that $If \circ s = \delta[0,1]^\tau$. Existence of such map follows from Theorem 2.1 [12].

Then we have $I(If) \circ s = s$ and $I(If) \circ s = \delta[0,1]^\tau$. Hence the map $I(If) : I(D^\tau) \to I([0,1]^\tau)$ is a retraction and the compactum $I([0,1]^\tau)$ is an absolute retract. We obtain a contradiction to the above mentioned Zarichnyi result.

Theorem 3.2. The compactum $IX$ is an absolute retract iff $X$ is an openly generated compactum of the weight $\leq \omega_1$.

Proof. The sufficiency follows from Corollary 3.5 [12] and the fact that the functor of idempotent measures preserves weight of infinite compacta, open maps and preimages [19].

Let us prove the necessity. Consider any compactum $X$ such that the compactum $IX$ is an absolute retract. Since the functor $I$ is normal [19], the compactum $X$ is AE(0) [17], Corollary 4.2. Let us show that $w(X) \leq \omega_1$. Suppose the contrary $w(X) > \omega_1$, then by Theorem 5.6 and Proposition 6.3 from [7], there exists an embedding $s : D^{\omega_2} \to X$. It follows from results of [2] that there exists a continuous map $f : X \to I(D^{\omega_2})$ such that $Is(f(x)) = \delta_s$ for each $x \in s(D^{\omega_2})$. Since the map $Is$ is an embedding, we have $f \circ s = \delta D^{\omega_2}$.

Define a map $u : C(D^{\omega_2}) \to C(X)$ by the formula $u(\varphi)(x) = f(x)(\varphi)$ for $\varphi \in C(D^{\omega_2})$ and $x \in X$. It is easy to check that $u$ is well-defined, continuous and preserves operations $\circ$, $\circ$ and constant functions. The equality $f \circ s = \delta D^{\omega_2}$ implies $u(\varphi) \circ s = \varphi$.

Define a map $\phi : IX \to I(D^{\omega_2})$ by the formula $\phi(\nu)(\varphi) = \nu(u(\varphi))$ for $\varphi \in C(D^{\omega_2})$ and $\nu \in IX$. Since $u$ preserves operations $\circ$, $\circ$ and constant functions, $\phi(\nu) \in I(D^{\omega_2})$ for each $\nu \in IX$. It is easy check that $\phi$ is continuous.

Finally, for each $\varphi \in C(D^{\omega_2})$ and $\nu \in I(D^{\omega_2})$ we have $(\phi \circ Is)(\nu)(\varphi) = Is(\nu)(u(\varphi)) = \nu(u(\varphi) \circ s) = \nu(\varphi)$. Hence the map $\phi$ is a retraction and $I(D^{\omega_2}$ is an absolute retract. We obtain a contradiction to Lemma 3.1.

□
4. On the Softness of the Idempotent Barycenter Map

A map \( f : X \to Y \) is said to be (0)-soft if for any (0-dimensional) paracompact space \( Z \), any closed subspace \( A \) of \( Z \) and maps \( \Phi : A \to X \) and \( \Psi : Z \to Y \) with \( \Psi|A = f \circ \Phi \) there exists a map \( G : Z \to X \) such that \( G|A = \Phi \) and \( \Psi = f \circ G \). This notion is introduced by E. Shchepin \[18\]. Let us remark that each 0-soft map is open and 0-softness is equivalent to the openness for all the maps between metrizable compacta.

Let

\[
\begin{array}{ccc}
X_1 & \xrightarrow{p} & X_2 \\
\downarrow f_1 & & \downarrow f_2 \\
Y_1 & \xrightarrow{q} & Y_2
\end{array}
\]

be a commutative diagram. The map \( \chi : X_1 \to X_2 \times Y_2 Y_1 = \{(x, y) \in X_2 \times Y_1 \mid f_2(x) = q(y)\} \) defined by \( \chi(x) = (p(x), f_1(x)) \) is called a characteristic map of this diagram. The diagram is called open (0-soft, soft) if the map \( \chi \) is open (0-soft, soft).

The following theorem from \[21\] gives a characterization of 0-soft maps:

**Theorem A.** \[21\] A map \( f : X \to Y \) is 0-soft if and only if there exist \( \omega \)-systems \( S_X \) and \( S_Y \) with the limits \( X \) and \( Y \) respectively and a morphism \( \{f_\alpha\} : S_X \to S_Y \) with the limit \( f \) such that 1) \( f_\alpha \) is 0-soft for every \( \alpha \); 2) every limit square diagram is 0-soft.

**Theorem 4.1.** Let \( X \subset \mathbb{R}^\omega \) be a compact Max-Plus convex subset and \( f : X \to Y \) be an open map onto a compact metrizable space with Max-Plus convex preimages. Then the map \( f \) is soft.

**Proof.** The theorem can be proved using the same arguments as in \[20\], where the statement of the theorem was proved for finite-dimensional \( X \).

A Max-Plus convex compactum \( K \) is said to be I-barycentrically soft (open), if the idempotent barycenter map \( b_K \) is soft (open). It is easy to see that the idempotent barycenter map has Max-Plus convex preimages. Hence we obtain the following corollary.

**Corollary 4.2.** Each metrizable I-barycentrically open compactum is I-barycentrically soft.

Now we investigate non-metrizable compacta. It was proved in \[19\] that the functor \( I \) preserves open maps, i.e. the openness of a map \( f : X \to Y \) implies the openness of the map \( If : IX \to IY \). We will need the converse statement.

**Lemma 4.3.** Let \( f : X \to Y \) be a continuous map such that the map \( If : IX \to IY \) is open. Then \( f \) is open.

**Proof.** Suppose the contrary. Then there exists a point \( x \in X \), a neighborhood \( U \) of \( x \) and a net \( (y_\alpha)_{\alpha \in A} \) converging to \( f(x) \) such that \( f^{-1}(y_\alpha) \cap U = \emptyset \). Then we have that the net \( (\delta_{y_\alpha})_{\alpha \in A} \) converges to \( \delta_{f(x)} = Ff(\delta_x) \). Take a function \( \psi \in C(X) \) such that \( \psi(x) = 1 \) and \( \psi(X \setminus U) \subset \{0\} \). We have \( \delta_2(\psi) = 1 \). Since the functor \( I \) preserves preimages \[19\], we have \( (If)^{-1}(\delta_{y_\alpha}) \subset I(X \setminus U) \) for each \( \alpha \in A \). Hence \( \nu(\psi) = 0 \) for each \( \nu \in (If)^{-1}(\delta_{y_\alpha}) \) and \( \alpha \in A \). We obtain a contradiction to openness of the map \( If \).

**Corollary 4.4.** The compactum \( IX \) is openly generated if and only if \( X \) is openly generated.

**Lemma 4.5.** Let \( f : X \to Y \) be a Max-Plus affine surjective map between Max-Plus convex compacta \( X \) and \( Y \) such that the diagram

\[
\begin{array}{ccc}
IX & \xrightarrow{If} & IY \\
\downarrow \beta_X & & \downarrow \beta_Y \\
X & \xrightarrow{f} & Y
\end{array}
\]

is open. Then \( f \) is open.

**Proof.** Suppose the contrary. Then there exists a point \( x \in X \), a neighborhood \( U \) of \( x \) and a net \( (y_\alpha)_{\alpha \in A} \) converging to \( f(x) = y \) such that \( f^{-1}(y_\alpha) \cap U = \emptyset \). (By exp \( X \) we denote the hyperspace of \( X \), i.e., the set of nonempty closed subsets of \( X \) endowed with Vietoris topology). We can assume that \( f^{-1}(y_\alpha) \) converges to \( A \) in \( \exp X \). Since \( f \) is a closed map we have that \( A \) is a subset in \( f^{-1}(y) \). Evidently, \( x \notin A \). Choose a point \( x_1 \in A \). There exists \( s \in (\neg, 0) \) such that \( s \odot x_1 \notin A \). Consider any open set \( V \supset A \) such that \( s \odot x_1 \notin \text{CLV} \). We can assume that \( f^{-1}(y_\alpha) \subset V \) for every \( \alpha \in \mathcal{B} \). Consider \( \delta_{s \odot x_1} \in IX \). Then \( \chi(\delta_{s \odot x_1} \odot y_\alpha) = (s \odot x_1 \odot y; \delta_y) \).\]
For each $\alpha \in \mathcal{B}$ choose a point $x_\alpha \in f^{-1}(y_\alpha)$ such that the net $x_\alpha$ converges to $x_1$. We have that the net $s \circ x_\alpha \oplus x$ converges to $s \circ x_1 \oplus x$ in $X$ and the net $s \circ \delta_{y_\alpha} \oplus \delta_y$ converges to $\delta_y$ in $I^2Y$. Moreover, $(s \circ x_\alpha \oplus x; s \circ \delta_{y_\alpha} \oplus \delta_y) \in X \times TIT$.

Choose a function $\varphi \in C(X)$ such that $\varphi(C\text{IV}) \subset \{-s + 1\}$ and $\varphi(s \circ x_1 \oplus x) = 0$. Consider the neighborhood $O = \{\nu \in IX | \nu(\varphi) < \frac{1}{2}\}$ of $s \circ x_1 \oplus x$. Let $\mu = I^{-1}(s \circ \delta_{y_\alpha} \oplus \delta_y)$. By Lemma 2.2 we have $\mu = s \circ \eta \oplus \nu$ where $\eta \in I(f^{-1}(y_\alpha))$ and $\nu \in I(f^{-1}(y))$. Hence $\mu(\varphi) = 1 > \frac{1}{2}$ and $\mu \notin O$. We obtain a contradiction to openness of the characteristic map $\chi$.

**Theorem 4.6.** Let $K$ be a Max-Plus convex compactum such that the map $\beta_K$ is $0$-soft. Then $K$ is openly generated.

**Proof.** Present $K$ as a limit of an $\omega$-system $S_K = \{K_\alpha, p_\alpha, A\}$ where $K_\alpha$ are Max-Plus convex metrizable compacta and bonding maps $p_\alpha$ are Max-Plus affine for every $\alpha \in A$. If the map $b_K : I(K) \to K$ is $0$-soft, then, using the spectral theorem of E.V. Shchepin [17] and theorem A, we obtain that there exists a closed cofinal subset $B \subset A$ such that for each $\alpha \in B$ the diagram

$$
\begin{array}{ccc}
I(K) & \xrightarrow{I(p_\alpha)} & I(K_\alpha) \\
\downarrow b_K & & \downarrow b_{K_\alpha} \\
K & \xrightarrow{p_\alpha} & K_\alpha
\end{array}
$$

is $0$-soft and therefore open. It follows from Lemma 4.3 that the map $p_\alpha$ is open for each $\alpha \in B$. But since $K = \lim \{K_\alpha, p_\alpha, B\}$, the compactum $K$ is openly generated. The theorem is proved.

**Lemma 4.7.** Let compactum $X$ be a limit space of an $\omega$-system $S_X = \{X_\alpha, p_\alpha, A\}$ and $x \in X$ has uncountable character. Then there exists $\alpha \in A$ such that the point $p_\alpha(x)$ is non-isolated in $X_\alpha$.

**Proof.** Take any $\alpha_1 \in A$. Since $X_{\alpha_1}$ is metrizable, the set $p_{\alpha_1}^{-1}(p_{\alpha_1}(x))$ contains more than one point. There exists $\alpha_2 \in A$ such that $p_{\alpha_1}^{-1}(p_{\alpha_1}(x))$ contains more than one point. Inductively we can find a sequence $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_i \leq \cdots$ such that $p_{\alpha_i}^{-1}(p_{\alpha_i}(x))$ contains more than one point for each $i \in \mathbb{N}$. Since the system $S_X$ is $\omega$-complete, there exists $\alpha = \sup_{i \in \mathbb{N}} \alpha_i$. Since the system $S_X$ is continuous, $X_\alpha = \lim \{X_{\alpha_i}, p_{\alpha_i}^\alpha, \mathbb{N}\}$ and the point $p_\alpha(x)$ is non-isolated in $X_\alpha$.

**Theorem 4.8.** Let $X$ be a compactum such that the map $\beta_{I^2X}$ is $0$-soft. Then $X$ is metrizable.

**Proof.** Theorem 4.6 implies that $IX$ is openly generated. Then $X$ is openly generated by Corollary 4.4.

Suppose that $X$ is non-metrizable. Present $X$ as a limit of an $\omega$-system $S_X = \{X_\alpha, p_\alpha, A\}$ with open surjective maps $p_\alpha$. If the map $b_{I^2X} : I^2(X) \to IX$ is $0$-soft, then, using the spectral theorem of E.V. Shchepin [17] and theorem A, we obtain that there exists a closed cofinal subset $B \subset A$ such that for each $\alpha \in B$ the diagram

$$
\begin{array}{ccc}
I^2(X) & \xrightarrow{I^2(p_\alpha)} & I^2(X_\alpha) \\
\downarrow b_{I^2X} & & \downarrow b_{I(X_\alpha)} \\
IX & \xrightarrow{I(p_\alpha)} & I(X_\alpha)
\end{array}
$$

is $0$-soft and therefore open.

Since $X$ is non-metrizable, there exists a point $x \in X$ with uncountable character. By Lemma 4.7 there exists $\alpha \in B$ such that the point $y = p_\alpha(x)$ is non-isolated. Since $X_\alpha$ is metrizable, the set $p_\alpha^{-1}(y)$ is not a one-point set. Then there exists $\beta \in A$ such that $\beta \geq \alpha$ and the set $(p_\beta)_{\alpha}^{-1}(y)$ is not a one-point set. The characteristic map $\chi : I^2(X_\beta) \to I(X_\beta) \times X_\alpha I^2(X_\alpha)$ of the diagram

$$
\begin{array}{ccc}
I^2(X_\beta) & \xrightarrow{I^2(p_\alpha)} & I^2(X_\alpha) \\
\downarrow b_{I(X_\beta)} & & \downarrow b_{I(X_\alpha)} \\
I(X_\beta) & \xrightarrow{I(p_\alpha)} & I(X_\alpha)
\end{array}
$$

is open being a left divisor of the open map $(I(p_\beta) \times \text{id}_{I^2(X_\alpha)})((I(X_\beta) \times X_\alpha I^2(X_\alpha))) \circ \chi'$ where $\chi'$ is the characteristic map of the diagram

$$
\begin{array}{ccc}
I^2(X) & \xrightarrow{I^2(p_\alpha)} & I^2(X_\alpha) \\
\downarrow b_{I^2X} & & \downarrow b_{I(X_\alpha)} \\
IX & \xrightarrow{I(p_\alpha)} & I(X_\alpha).
\end{array}
$$
Choose two distinct point $x_1$ and $x_2 \in (p^\beta_n)^{-1}(y)$. Consider $\delta_{x_1} \oplus \delta_{x_2} \in \Pi(X_\beta)$. Then we have $\chi(\delta_{x_1} \oplus \delta_{x_2}) = (\delta_{x_1} \oplus \delta_{x_2} : \delta_y)$.

Choose any converging to $y$ sequence $(y_i)$ such that $y_i \neq y$ for each $i \in \mathbb{N}$. Since the map $p^\beta_n$ is open, there exists a sequence $(x_i)$ converging to $x_2$ such that $(p^\beta_n(x_i)) = y_i$. Then the sequence $\delta_{x_1} \oplus \delta_{x_2} \oplus \delta_{x_i}$ converges to $\delta_{x_1} \oplus \delta_{x_2}$ and the sequence $\delta_i \oplus \delta_{x_i}$ converges to $\delta_y$. Moreover, $I(p^\beta_n)(\delta_{x_1} \oplus \delta_{x_2} \oplus \delta_{x_i}) = (\delta_y \oplus \delta_{y_i}) = b_{I(\alpha)}(\delta_i \oplus \delta_y)$, hence we have $\chi(\delta_{x_1} \oplus \delta_{x_2} \oplus \delta_{x_i}) = (\delta_{x_1} \oplus \delta_{x_2} : \delta_y)$, for each $i \in \mathbb{N}$.

Consider any $\mathcal{M}_i \in \chi^{-1}(\delta_{x_1} \oplus \delta_{x_2} \oplus \delta_{x_i} : \delta_y)$, since $\mathcal{M}_i \in (\Pi(\alpha))^{-1}(\delta_{x_1} \oplus \delta_{x_2} \oplus \delta_{x_i})$, we obtain $\mathcal{M}_i \in I^2((x_1, x_2, x_i))$ by Lemma 2.2. Since $\mathcal{M}_i \in (I^2(p^\beta_n))^{-1}(\delta_{x_1} \oplus \delta_y)$ and the functor $I$ preserves preimages, we have $\mathcal{M}_i \in IS$ where $S = \{\nu \in I((x_1, x_2, x_i))| \nu = s \circ \delta_{x_i} \circ t \circ \delta_{x_2} \circ \delta_{x_1} \circ \delta_y\}$, where $t, s \in [-\infty, 0]$ with $s \neq t = 0$, by Lemma 2.4.

Choose a function $\varphi \in C(X_\beta)$ such that $\varphi(x_1) = 0$ and $\varphi(x_2) > 1$. We can assume that $\varphi(x_i) > 1$ for each $i \in \mathbb{N}$. Consider open sets $O_1 = \{\nu \in I(X_\beta)|\nu(\varphi) < \frac{1}{2}\}$ and $O_2 = \{\nu \in I(X_\beta)|\nu(\varphi) > \frac{3}{2}\}$. Then we have $\delta_{x_i} \in O_1$, $S \subset O_2$ and $\text{Cl}O_1 \cap \text{Cl}O_2$. Choose a function $\psi \in C(I(X_\beta))$ such that $\psi(O_1) \subset \{1\}$ and $\psi(O_2) \subset \{0\}$. Then we have $\delta_{x_i} \oplus \delta_{x_2} = (\psi) = 1$ and $\mathcal{M}_i(\psi) = 1$ for each $i \in \mathbb{N}$. We obtain a contradiction to openness of $\chi$. The theorem is proved.

Let us remark that an analogous theorem for probability measures was proved in [14]. Fedorchuk proved in [0] that each product of $\omega_1$ barycentrically open convex metrizable compacta is barycentrically soft. The following theorem demonstrates that the situation is different in the case of idempotent probability measures.

**Theorem 4.9.** The map $\beta_{(0,1]}$ is not $0$-soft.

**Proof.** Suppose the contrary. Then using the same arguments as in the proof of Theorem 4.8 we obtain that the diagram

$$
\begin{align*}
I([0,1]^\omega \times [0,1]^\omega) \xrightarrow{\pi_\omega} I([0,1]^\omega) \\
\downarrow b_{[0,1]^\omega \times [0,1]^\omega} \quad \quad \quad \quad \quad \quad \downarrow b_{[0,1]^\omega} \\
[0,1]^\omega \times [0,1]^\omega \xrightarrow{p} [0,1]^\omega
\end{align*}
$$

is open (by $p : [0,1]^\omega \times [0,1]^\omega \to [0,1]^\omega$ we denote the natural projection to the second coordinate). As before by $\chi$ we denote the characteristic map.

For $t \in [0,1]$ we put $\overline{t} = (t, 0, 0, \ldots) \in [0,1]^\omega$. Consider $\delta_{(\overline{0,T})} \oplus \delta_{(\overline{1,T})} \in I((0,1)^\omega \times (0,1)^\omega)$. Then we have $\chi(\delta_{(\overline{0,T})} \oplus \delta_{(\overline{1,T})}) = (\delta_{(0,T)} \oplus \delta_{(1,T)})$.

The sequence $(\overline{0}; \overline{1} - \frac{1}{2})$ converges to $(\overline{1}; \overline{1})$ and the sequence $\delta_{(\overline{0}; \overline{1})} \oplus (-\frac{1}{2}) \circ \delta_{(\overline{1}; \overline{1})}$ converges to $\delta_{(\overline{0}; \overline{1})} \oplus \delta_{(\overline{1}; \overline{1})}$. Moreover, $(\delta_{(\overline{0}; \overline{1})} \oplus (-\frac{1}{2}) \circ \delta_{(\overline{1}; \overline{1})}) = ([0,1]^\omega \times [0,1]^\omega) \times (0,1]^\omega \to I((0,1)^\omega)$ for each $i \in \mathbb{N}$.

Consider any $\pi_i \in \chi^{-1}(\overline{0}; \overline{1} - \frac{1}{2}, \delta_{(\overline{0}; \overline{1})} \oplus (-\frac{1}{2}) \circ \delta_{(\overline{1}; \overline{1})})$. Since $\pi_i \in (Ip)^{-1}(\delta_{(\overline{0}; \overline{1})} \oplus (-\frac{1}{2}) \circ \delta_{(\overline{1}; \overline{1})})$, we have by Lemma 2.4 that $\pi_i = \nu_i \oplus (\frac{1}{2}) \circ \mu_i$, where $\nu_i \in I([0,1]^\omega \times \{\overline{0}\})$ and $\mu_i \in I([0,1]^\omega \times \{\overline{1}\})$ for each $i \in \mathbb{N}$. Since $\pi_i \in (b_{[0,1]^\omega \times [0,1]^\omega})^{-1}(\overline{0}; \overline{1} - \frac{1}{2})$, we have $\nu_i(\overline{0}) = 1$ for each $i \in \mathbb{N}$ where $\nu_i$ is the density of $\nu_i$.

Choose a function $\varphi \in C([0,1]^\omega \times [0,1]^\omega)$ such that $\varphi(\overline{0}; \overline{0}) = 1$ and $\varphi(\overline{1}; \overline{1}) = \varphi(\overline{0}; \overline{1}) = 0$. Then we have $\delta_{(\overline{0}; \overline{0})} \oplus \delta_{(\overline{1}; \overline{1})}(\varphi) = 0$ and $\pi_i(\varphi) \geq \nu_i(\varphi) \geq 1$ for each $i \in \mathbb{N}$. We obtain a contradiction to openness of $\chi$. The theorem is proved.

**Question 1.** If there exists a non-metrizable $I$-barycentrically soft compactum?

**REFERENCES**

[1] M.Akian, Densities of idempotent measures and large deviations, Trans. of Amer.Math.Soc. 351 (1999), no. 11, 4515-4543.

[2] T.Banakh, T.Radul, F-Dugundji spaces, F-Milutin spaces and absolute F-valued retracts, Topology Appl. 179 (2015), 34-50.

[3] S.Ditor,R.Haydon, On absolute retracts, $P(S)$ and complemented subspaces of $C(D^{\omega_1})$, Studia Math., 56 (1976), 243-251.

[4] L.O. Eifler, Openness of convex averaging, Glasnik Mat. Ser. III, 32 (1977), no. 1, 67-72.

[5] V.V. Fedorchuk, On a barycentric map of probability measures, Vestn. Mosk. Univ, Ser. I, No 1, (1992), 42-47.

[6] V.V. Fedorchuk, On barycentrically open bicompacta, Siberian Mathematical Journal, 33 (1992), 1135–1139.

[7] R.Haydon, Embedding $D^{\omega}$ in Dugundji spaces, with an application to linear topological classification of spaces of continuous function, Studia Math., 56 (1976), 31-44.

[8] G. L.Litvinov, The Maslov dequantization, idempotent and tropical mathematics: a very brief introduction, Idempotent mathematics and mathematical physics, 117, Contemp. Math., 377, Amer. Math. Soc., Providence, RI, 2005.

[9] V.P. Maslov, S.N. Samborski, Idempotent Analysis, Adv. Soviet Math., vol. 13. Amer. Math. Soc., Providence, 1992.

[10] R.C. O’Brien, On the openness of the barycentric map, Math. Ann., 223 (1976), 207–212.

[11] S. Papadopoulos, On the geometry of stable compact convex sets, Math. Ann., 229 (1977), 193–200.
[12] T. Radul, Absolutes retracts and equiconnected monads, Topology Appl. 202 (2016), 1–6.
[13] T. Radul, On the openness of the idempotent barycenter map, Topology Appl. (submitted).
[14] T. Radul, On the baricentric map of probability measures, Vestn. Mosk. Univ., Ser. I (1994), No.1, 3–6.
[15] T. Radul, On baricentrically soft compacta, Fund.Math. 148 (1995), 27–33.
[16] T. Radul, Topology of the space of ordering-preserving functionals, Bull. Pol. Acad. Sci. 47 (1999), 53–60.
[17] E.V. Shchepin, Functors and uncountable powers of compacta, Uspekhi Mat. Nauk 36 (1981), 3–62.
[18] E.V. Shchepin, Topology of limit spaces of uncountable inverse spectra, Russian Mathematical Surveys 31 (1976), 155–191.
[19] M. Zarichnyi, Spaces and mappings of idempotent measures, Izv. Ross. Akad. Nauk Ser. Mat. 74 (2010), 45–64.
[20] M. Zarichnyi, Michael selection theorem for max-plus compact convex sets, Topology Proceedings. 31 (2007), 677–681.
[21] M. Zarichnyi, Absolute extensors and the geometry of multiplication of monads in the category of compacta, Mat. Sbornik. 182 (1991), 1261–1288.
[22] K. Zimmermann, A general separation theorem in extremal algebras, Ekon.-Mat. Obz. 13 (1977) 179–201.

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