De Rham epsilon factors for flat connections on higher local fields

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Abstract

This note is a companion to the author’s Higher de Rham epsilon factors. Using Grayson’s binary complexes and the formalism of \(n\)-Tate spaces we develop a formalism of graded epsilon lines, associated to flat connections on a higher local field of characteristic 0. The definition is based on comparing a Higgs complex with a de Rham complex on the same underlying vector bundle.

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1 Linear differential equations on higher local fields

This section is devoted to a study of flat connections \((E, \nabla)\) on the higher local field \(F_n = k((t_1)) \cdots ((t_n))\), where \(k\) denotes as always a field of characteristic 0. A lot of the material will be familiar to the expert. We include the often well-known proofs to demonstrate that the behaviour of differential equations on \(F_n\) is not any different from the theory of \(D\)-modules on algebraic varieties. We do not lay claim to originality in this subsection.

1.1 Definitions and basic properties

Definition 1.1. Let \((E, \nabla)\) be a flat connection on \(F_n\). The corresponding de Rham complex is the chain complex of \(F_n\)-vector spaces \(\Omega^*_{\nabla} = [E \xrightarrow{\nabla} E \otimes \Omega^1_{\nabla} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} E \otimes \Omega^n_{\nabla}]\). For \(i \in \mathbb{Z}\) we denote the \(i\)-th cohomology group of this complex by \(H^i_{\nabla}(E)\).
1.2 Cyclic vectors

The proof of the following lemma follows the lecture notes on algebraic $D$-modules by Braverman-Chmutova [24, Lecture 3]. We denote by $D$ the ring of formal differential operators on $F_1 = k((t))$, which we define to be the free $k$-algebra $F_1(\partial)$ modulo the relation $[\partial, t] = 1$. It is clear that the centre is given by $Z(D) = k$ (relying heavily on the assumption that $k$ has characteristic 0).

Lemma 1.2 (Cyclic Vector Lemma). Let $(E, \nabla)$ be a flat connection on $F_1 = k((t))$. Then, as a $D$-module, there exists a cyclic vector $s \in E$, that is, a generator $DS = E$.

Proof. The proof begins by observing that $D$ does not have any proper two-sided ideals, thus is a simple $k$-algebra. Indeed, let $I \subset D$ be a two-sided ideal, distinct from the trivial ideal $D$. It is clear that the intersection $F_1 \cap I$ must be $\{0\}$. We will use the tautological filtration on $D$ (also known as the order of differential operators) to establish that $I$ must be the zero ideal. If there is a non-zero element in $I$, then there is a minimal integer $r \geq 1$, such that $I$ contains an element $p$ of order $\leq r$. Since $p$ cannot be central in $D$ (as $Z(D) = k$) we obtain the existence of either a differential operator $q$ of order $\leq 1$, such that $[p, q] \neq 0$. The definition of two-sided ideals shows that $[p, q] \in I$. Hence it is a non-zero element of $I$ of order $\leq r - 1$ which contradicts the minimality assumption on $r$.

Since $E$ is a finite-dimensional $F_1$-vector space, it has finite length as a $F_1$-module and therefore also as a $D$-module. We assume by induction that the existence of a cyclic vector has already been established for $D$-modules of smaller length than the one of $E$ (the case of length 1 being automatically true). There exists a short exact sequence of $D$-modules

$$0 \rightarrow F \rightarrow E \xrightarrow{\phi} M \rightarrow 0,$$

such that $F$ has length 1. Let $m \in M$ be a cyclic vector for $M$. We assume by contradiction that $\phi^{-1}(m)$ does not contain a cyclic vector for $E$. Then we have $DS \cap F = \{0\}$. Let $p \in D$ be a differential operator, such that $pm = 0$, and $f \in F$ an arbitrary element of $F$. We have $\phi(s + f) = m$, and therefore also $D(s + f) \cap F = \{0\}$. Since $p(s + f) = pf \in D(s + f) \cap F$, we must have that $pf = 0$. Therefore, the ideal $\text{Ann}(m)$ contains the two-sided ideal given by $\text{Ann}(F)$.

This leaves us with two options to consider, $\text{Ann}(F)$ equal to $\{0\}$ or $D$. The first case is not possible as then $F$ would be isomorphic to $D$ and hence $\text{dim}_F F = \infty$ which is impossible. We conclude that $\text{Ann}(m) = D$ which implies $M = 0$ and therefore simplicity of $E$. \hfill $\square$

Corollary 1.3. Let $\nabla$ be a flat connection on the trivial rank $m$ bundle $F_1^{\otimes m}$. There exist elements $a_0, \ldots, a_{m-1} \in F_1$, such that the $k$-vector space of $\nabla$-flat sections is isomorphic to the $k$-vector space of solutions to the ordinary differential equation $y^{(m)} + a_{m-1}y^{(m-1)} + \cdots + a_0y = 0$.

Proof. Let $s$ be a cyclic vector. The $F_1$-dimension of $E = F_1^{\otimes m}$ is $m$, which implies linear dependence of the $(m + 1)$-tuple $s, \partial s, \cdots \partial^m s$. Therefore we can write $\partial^m s = -\sum_{i=0}^{m-1} a_i \partial^i s$, for certain elements $a_i \in F_1$. We conclude that the $D$-module $E$ is equivalent to the quotient $D/\langle \partial^m + \sum_{i=0}^{m-1} a_i \partial^i \rangle$. \hfill $\square$

Furthermore the cyclic vector lemma for $F_1$ implies the existence of cyclic vectors for $F_n$.

Corollary 1.4. Let $(E, \nabla)$ be a flat connection on $F_n$. Then $E$ is cyclic as a module over $D_n = D_{F_n}$.

Proof. The ring $D_n$ contains a subring $D' \subset D_n$ which is generated by $F_n = F_{n-1}((t_n))$ and $\partial_n = \frac{\partial}{\partial t_n}$. The ring $D'$ is canonically isomorphic to $D_{F_{n-1}((t_n))}/F_{n-1}$. As a $D'$-module, $E$ is cyclic by virtue of Lemma 1.2. This implies that $E$ is cyclic as a $D$-module. \hfill $\square$

1.3 Formal de Rham cohomology is finite-dimensional

Lemma 1.5. An $m$-tuple of formal Laurent series $y_1, \ldots, y_m \in k((t))$ is $k$-linearly independent if and only if the Wronskian $W(y_1, \ldots, y_m) = |(y_{(j-1)})_{1 \leq i,j \leq m}|$ is non-zero.

Proof. It is clear that linear dependence of the $m$-tuple $y_1, \ldots, y_m$ implies vanishing of the Wronskian. We prove the converse by induction on $m$, anchored to the case $m = 1$ which is a tautology. Let us assume that the statement has been proven for $(m - 1)$-tuples and that $y_1 \neq 0$. This allows us to divide every element of the $m$-tuple by $y_1$ (which changes the Wronskian by a factor of $y_1^{-m}$) and does not affect $k$-linear independence. Henceforth we assume without loss of generality $y_1 = 1$ and compute

$$0 = W(1, y_2, \ldots, y_m) = W(y_2', \ldots, y_m').$$
By assumption this shows that the \((m-1)\)-tuple \(y_1', \ldots, y_m'\) is \(k\)-linear independent. Let \(\lambda_2, \ldots, \lambda_m \in k\) be scalars bearing witness to this fact, i.e., \(\sum_{i=2}^{m} \lambda_i y_i' = 0\). Integrating this equation we obtain the existence of a scalar \(\lambda_1 \in k\), such that \(\lambda_1 \cdot 1 + \sum_{i=2}^{m} \lambda_i y_i = 0\). Since we assumed \(y_1 = 1\) this establishes a linear independence and hence concludes the proof.

**Corollary 1.6.** Let \(a_0, \ldots, a_{m-1} \in k((t))\). The k-vector space of solutions to the ordinary differential equation

\[ y^{(m)} + a_{m-1} y^{(m-1)} + \cdots + a_0 y = 0 \]

is of dimension at most \(m\).

**Proof.** Let \(y_1, \ldots, y_{m+1}\) be solutions to the ordinary differential equation above. We claim that they are \(k\)-linearly dependent. As we have seen above this is equivalent to vanishing of the Wronskian \(W(y_1, \ldots, y_{m+1}) = |(y_i^{(j-1)})_{1 \leq i, j \leq m+1}|\). The \(F_1\)-linear relation \(y_i^{(m)} + a_{m-1} y_i^{(m-1)} + \cdots + a_0 y_i = 0\) (which holds for all \(i\) by assumption) describes a \(F_1\)-linear relation between the rows of the matrix \((y_i^{(j-1)})_{1 \leq i, j \leq m+1}\). This implies vanishing of the Wronskian.

**Proposition 1.7.** Let \((E, \nabla)\) be a flat connection on \(F_1 = k((t))\). The k-vector spaces \(\text{ker}(\nabla)\) and \(\text{coker}(\nabla)\) are finite-dimensional.

**Proof.** We begin by showing finite-dimensionality of the kernel. As we have seen in Subsection 1.2 there exists an \(m\)-th order linear differential equation \(y^{(m)} + a_{m-1} y^{(m-1)} + \cdots + a_0 y = 0\) whose solutions form a \(k\)-vector space isomorphic to \(\text{ker}(\nabla)\). Corollary 1.6 shows that this \(k\)-vector space is finite-dimensional.

Finite-dimensionality of the cokernel is shown by dualising. Since the \(k\)-vector spaces \(E\) and \(E \otimes \Omega^1_{F_1}\) are infinite-dimensional, it helps to endow them with the \(t\)-adic topology. This is the topology induced by the natural valuation on \(F_1 = k((t))\) (which is discrete on \(k\)). With respect to this topology, \(E\) is a so-called linearly locally compact \(k\)-vector space in the sense of Lefschetz (also known as Tate \(k\)-vector spaces). The residue pairing yields an isomorphism of the topological dual \(E^\vee\) with the Tate \(k\)-vector space \(E^\vee \otimes \Omega^1_{F_1}\) (where \(E^\vee\) refers to the \(K_1\)-linear dual). With respect to this isomorphism, the dual of the connection \(\nabla\) can be seen to be the map induced by the dual connection. We conclude finite-dimensionality of the cokernel of \(\nabla\) from finite-dimensionality of the kernel of the dual connection \(\nabla^\vee\).

**Corollary 1.8.** Let \((E, \nabla)\) be a flat connection on \(F_n\). For all degrees \(i \in \mathbb{Z}\) the de Rham cohomology groups \(H^i(E)\) are finite-dimensional \(k\)-vector spaces.

Before giving the proof it seems appropriate to summarise the underlying ideas. We argue by induction on \(n\), the case \(n = 1\) being a tautology. Let us assume that the assertion has been verified for flat connections on \(F_{n-1}\). There is a canonical inclusion of fields \(F_{n-1} \hookrightarrow F_n\), whose geometric counterpart is projection along the \(t_n\)-coordinate. The de Rham cohomology groups \(H^i(E)\) can be computed from a variant of the Leray spectral sequence relative to this map. Since \(F_n = F_{n-1}((t_n))\) we can apply the finiteness result of Proposition 1.7 for the \(n = 1\) case and the induction hypothesis to conclude the result.

**Proof.** Let \((E, \nabla)\) be a flat connection on \(F_n = F_{n-1}((t_n))\). We denote by \(\Omega^i_{F_n/F_{n-1}}\) the \(F_n\)-linear subspace of \(\Omega^i_{F_n}\), spanned by the elements \(dt_n\). By definition it is orthogonal to the subspace \((\Omega^i_{F_{n-1}})_{F_n} \hookrightarrow \Omega^i_{F_n}\), generated by the elements \(dt_1, \ldots, dt_{n-1}\). By taking exterior powers and tensor products we obtain \(F_n\)-linear subspaces \((\Omega^i_{F_{n-1}})_{F_n} \subset \Omega^i_{F_n}\) and \((\Omega^i_{F_{n-1}})_{F_n} \otimes_{F_n} \Omega^i_{F_{n-1}} \subset \Omega^{i+1}_{F_n}\).

There are natural differential operators \(\nabla': E \longrightarrow E \otimes (\Omega^1_{F_{n-1}})_{F_n}\), \(\nabla'': E \longrightarrow E \otimes \Omega^1_{F_{n-1}/F_{n-1}}\). More generally we may form the double complex

\[
\begin{array}{c}
E \\
\n\xrightarrow{\nabla'} E \otimes (\Omega^1_{F_{n-1}})_{F_n} \\
\xrightarrow{\nabla''} E \otimes (\Omega^1_{F_{n-1}})_{F_n} \otimes (\Omega^1_{F_{n-1}})_{F_n} \\
\end{array}
\]

Note that \(\nabla = \nabla' + \nabla''\) and \(\nabla' \nabla'' = -\nabla'' \nabla'\). We conclude that the de Rham complex \(\Omega^i(E)\) is isomorphic to the totalisation of this double complex. The associated spectral sequence is

\[E_2^{p,q} = H^p_{\nabla'}(H^q_{\nabla''}(E)) \Rightarrow H^{p+q}_{\nabla}(E).\]
By Proposition 1.7 $H^r_{\nabla,V}(E)$ is finite-dimensional, since $F_n = F_{n-1}(t_i)$. The induction hypothesis implies finite-dimensionality of $H^r_{\nabla,V}(H^q_{\nabla,V}(E))$. Therefore, every object appearing on the second page of our spectral sequence is finite-dimensional over $k$. Convergence of the spectral sequence (see [Sta, Tag 0132]) implies that the formal de Rham cohomology groups $H^r_{\nabla,V}(E)$ are of finite dimension over $k$. □

2 De Rham epsilon-factors: definition and basic properties

Given a flat connection $(E,\nabla)$ on the higher local field $F_n = k((t_1)) \cdots ((t_n))$ we define a formalism of epsilon-factors. Our theory depends on an $F_n$-linearly independent n-tuple of 1-forms $\nu = (\nu_1, \ldots, \nu_n) \in \Omega^1_{F_n}$, and produces a graded (or super) line $\epsilon_{\nu}(E,\nabla)$. This graded line is naturally obtained from a homotopy point of the $K$-theory spectrum $K(k)$.

Our treatment begins with a recapitulation of the case $n = 1$, which is due to Beilinson–Bloch–Esnault [BBE02]. We reformulate their definition in terms of Grayson’s binary complexes [Gra12]. This perspective allows us a glimpse at the higher-dimensional generalisation. We then define epsilon-lines in arbitrary dimension $n$, by using Grayson’s iterated binary complexes, finite-dimensionality of formal de Rham cohomology (Corollary 1.8), and crucially, almost-commutativity of the ring of differential operators.

2.1 Recapitulation on Tate objects: categorical preliminaries

The notion of Tate objects in exact categories goes back to Beilinson’s [Bei87] and Kato’s [Kat00], and is inspired by Lefschetz’s theory of linearly locally compact vector spaces. We refer the reader to [Pre11] and [BGW14] for a detailed introduction to the theory and an overview of the history of Tate objects.

To an idempotent complete exact category $C$ (see [Buh10] for an account of the general theory of exact categories) one defines an exact category of admissible $\text{Ind}$ objects $\text{Ind}^a C$. It is the full subcategory of the category of $\text{Ind}$ objects $\text{Ind} C$, whose objects can be represented as a formal colimit $\text{colim}_K X_k$, where for $F_1 \leq k_2 \in K$ the resulting morphism $X_{F_1} \longrightarrow X_{k_2}$ is an admissible monomorphism in the sense of exact categories. The dual construction yields $\text{Pro}^a C$, an exact category of admissible $\text{Pro}$ objects.

The exact category $\text{Ind}^a \text{Pro}^a C$ contains an extension closed full subcategory $\text{Tate}^a(C)$ whose objects are referred to as elementary Tate objects. By definition, every elementary Tate object $V$ belongs to an admissible short exact sequence

$$L \hookrightarrow V \twoheadrightarrow D,$$

where $L \in \text{Pro}^a C$ and $D \in \text{Ind}^a C$. One also says that $L \subset V$ is a lattice. The exact category of Tate objects $\text{Tate}(C)$ is defined to be the idempotent closure of $\text{Tate}^a(C)$.

The categorical quotient $\text{Ind}^a(C)/C$ was studied by Schlichting in [Sch04] as the suspension category $SC$. We denote its idempotent closure by $\text{Calk}(C)$ and refer to it as the category of Calkin objects. There is a natural functor $\text{Tate}(C) \longrightarrow \text{Calk}(C)$ given by sending $V$ to $V/L$ where $L$ is a lattice in $V$. In fact, one has $\text{Tate}^a(C)/\text{Pro}^a(C) \cong \text{Ind}^a(C)/C$.

Remark 2.1. The most important input category $C$ in this paper is $P_f(R)$, that is, the exact category of finitely generated projective $R$-modules. Here we denote by $R$ a commutative ring (with unit). In this case we will simply write $\text{Tate}(R)$, in reference to $\text{Tate}(P_f(R))$.

2.2 A reformulation of BBE’s theory

For the purpose of this subsection we let $(E,\nabla)$ denote a flat connection on $E = F_1 = k((t))$. Recall that $E$ is an $F$-vector space and that $\nabla : E \longrightarrow E \otimes \Omega^1_{F}$ is a $k$-linear map, satisfying the Leibniz identity. Since $F$ has a natural structure of a Tate vector space over $k$, so does $E$, and $\nabla$ can be easily seen to be a morphism in this category. Furthermore this almost defines an isomorphism of linearly locally compact vector spaces (in a technical sense).

Lemma 2.2. A connection $\nabla$ as above induces an isomorphism in the localised category $\text{Tate}^a(k)/\text{Pro}^a(k)$.

Proof. Consider the diagram of abstract $k$-vector spaces

$$\begin{array}{ccc}
E & \xrightarrow{\nabla} & E \otimes \Omega^1_F \\
\downarrow & \downarrow & \downarrow \\
K & \rightarrow & E/K \\
\downarrow & \downarrow & \downarrow \\
C & & C
\end{array}$$

(1)
where the diagonals are short exact sequences. The kernel $K$ and cokernel $C$ are finite-dimensional by virtue of Proposition 1.7. Furthermore, the $k$-vector spaces $E$ and $E \otimes \Omega^1_k$ are endowed with a linearly locally compact topology (that is, they are Tate vector spaces). We claim that the diagram $\square$ is a diagram in category of Tate vector spaces. This amounts to showing that all arrows represent continuous maps of topological vector spaces. Continuity of $\nabla$ has already been asserted and can be checked using an explicit presentation in terms of coordinates. It remains to verify that the diagonal morphisms are continuous.

The $k$-linear map $i : K \hookrightarrow E$ is certainly continuous when endowing $K$ with the discrete topology. Since $K$ is finite-dimensional, this is the only possibility within the category of linearly locally compact $k$-vector spaces. Furthermore we assert that $i$ has a continuous retraction $r$. To construct $r$ one chooses a lattice $L \subset E$, such that $i(K) \subset L$. By definition of the category of linearly locally compact vector spaces there exists a retraction $r' : E \to L$. Moreover we know that the inclusion $K \hookrightarrow L$ has a retraction $r''$, since the category of linearly compact $k$-vector spaces is contravariantly equivalent to the category of discrete $k$-vector spaces (which is a split abelian category). We define $r = r'' \circ r'$. This shows that $E/K$ is a direct summand of $E$ and hence establishes continuity of the morphism $E \to E/K$.

Using the observation of the proof of Proposition 1.7 that the $k$-linear topological dual of the map $E \to E \otimes \Omega^1_k$ is equivalent to the dual connection $\nabla^*$, we obtain continuity of the maps on the right hand side of diagram $\square$. □

In the category $\text{Tate}^+(k)/\text{Pro}^+(k)$ we have that morphisms with finite-dimensional kernel or cokernel are isomorphisms. In particular we see that $E \to E/K$ and $E/K \to E \otimes \Omega^1_k$ are isomorphisms in this localisation. Commutativity of the triangle in $\square$ implies the assertion.

In the article $\text{[Gra12]}$, Grayson approaches higher algebraic $K$-groups using the notion of (iterated) binary complexes. A binary complex in an additive category $\mathcal{C}$ is a complex with two differentials, respectively a pair of complexes sharing the same objects. More precisely it is defined to be a collection of objects $(X^i)_{i \in \mathbb{Z}}$ and morphisms $d_{i,j}, d_{2,j} : X^j \to X^{j+1}$ for all $j \in \mathbb{Z}$, such that for $i = 1, 2$ and all $j \in \mathbb{Z}$ we have $d_{i,j+1} \circ d_{i,j} = 0$.

**Definition 2.3.** Let $\mathcal{C}$ denote an additive category. We denote by $\text{BinCh}(\mathcal{C})$ the category whose objects are binary complexes $((X^i; d_{i,j}, d_{2,j})_{i \in \mathbb{Z}})$, and morphisms $\{(X^i; d_{i,j}, d_{2,j})_{i \in \mathbb{Z}} \to (Y^i; d'_{i,j}, d'_{2,j})_{i \in \mathbb{Z}}\}$ are given by tuples $(f^i)_{i \in \mathbb{Z}}$, such that $d'_{i,j} \circ f^j = f^{j+1} \circ d_{i,j}$ for all $i = 1, 2$ and $j \in \mathbb{Z}$.

**Definition 2.4.** Let $(E, \nabla)$ be a flat connection on $F = k((t))$ and $\nu \in \Omega^1_k$. We define $\Omega^\cdot_{\nabla, \nu}(E)$ to be the length 2 binary complex, given by

\[
\begin{array}{ccc}
E & \xrightarrow{\nabla} & E \otimes \Omega^1_k \\
\nu & \downarrow & \\
\end{array}
\]

in the exact category $\text{Calk}(k)$. For fixed $\nu$ this defines an exact functor $\Omega^\cdot_{\nabla, \nu} : \text{Loc}(F) \to \text{BinCh}(\text{Tate}(k))$.

One says that a binary complex $(X^*; d_1, d_2)$ in an exact category $\mathcal{C}$ is exact (or synonymously acyclic), if both differentials define an exact complex in $\mathcal{C}$ in the sense of $\text{[Bib10]}$ Definition 8.8. We denote the corresponding extension-closed subcategory by $\text{BinCh}_{ex}(\mathcal{C})$. Grayson shows in $\text{[Gra12]}$ that there exists a canonical morphism of connective spectra $K(\text{BinCh}_{ex}(\mathcal{C})) \to \Omega K(\mathcal{C})$. In the article he assumes that $\mathcal{C}$ admits a calculus of long exact sequences, but this assumption is not needed to define the aforementioned morphism but only in proving the main result $\text{[Gra12]}$ Theorem 4.3 that $K(\text{Ch}_{\geq 0}^{ex}(\mathcal{C})) \to K(\text{BinCh}_{ex}(\mathcal{C})) \to \Omega K(\mathcal{C})$ is a fibre sequence of connected spectra.

**Lemma 2.5.** Assume that $\nu$ is a non-zero 1-form on $F$. Then for every flat connection $(E, \nabla)$, the binary complex $\Omega^\cdot_{\nabla, \nu}(E)$ is exact in the Calkin category. In particular we have an exact functor

\[\Omega^\cdot_{\nabla, \nu} : \text{Loc}(F) \to \text{BinCh}_{ex}(\text{Calk}(k)).\]

Consequently we have for $\nu \in \Omega^1_k \setminus \{0\}$ a well-defined map of $K$-theory spectra

\[\varepsilon_{\nu} : K(\text{Loc}_{F_1}) \to K(k),\]

which we call the spectral $\varepsilon$-factor.
2.3 Higher local fields and closed 1-forms

Let us denote by \( \text{Loc}_{F_n} \), the exact category of pairs \((E, \nabla)\), where \( E \) is a finite rank \( F_n \)-vector space, and \( \nabla \) a formal flat connection on \( E \). In this subsection we will define for a \( F_n \)-linearly independent tuple of closed 1-forms \( \nu = (\nu_1, \ldots, \nu_n) \) on \( E \) a map of spectra

\[
\varepsilon_\nu : K(\text{Loc}_{F_n}) \longrightarrow K(k)
\]

which generalises the epsilon-factor formalism for the 1-local field \( k(t) \). One could argue that the assumption that the forms \( \nu_1, \ldots, \nu_n \) are closed is a feature of the higher-dimensional case, since it is automatically true for \( n = 1 \). However we believe that it should not be needed in order to have a well-defined \( \varepsilon_\nu \) (see the companion article [Gro]).

**Definition 2.6.** Let \((E, \nabla)\) be a de Rham local system on \( F_n \) and \( \nu \in (\Omega^1_{F_n})^n \) denote an \( F_n \)-linearly independent tuple of closed 1-forms. We define a multi-complex (whose totalisation is the formal de Rham complex) which is supported on the cube \( \{0,1\}^n \) which we identify with the power set \( \mathcal{P}(\{1, \ldots, n\}) \). For \( M \subset \{1, \ldots, n\} \) we use the notation

\[
\Omega^M \mathcal{L}_\nu(E) = E \otimes E_n(\nu_1 \wedge \cdots \wedge \nu_q) \subset E \otimes \Omega^q_{F_n},
\]

where \( M = \{i_1 < \cdots < i_q\} \). Furthermore, for every inclusion \( j : M \hookrightarrow N = M \cup \{i\} \), we let

\[
\nabla_j : \Omega^M \mathcal{L}_\nu(E) \longrightarrow \Omega^N \mathcal{L}_\nu(E)
\]

be the component of the connection \( \nabla : E \otimes \Omega^q_{F_n} \longrightarrow E \otimes \Omega_q^{q+1} \) with respect to the direct summands \( \Omega^M \mathcal{L}_\nu(E) \) and \( \Omega^N \mathcal{L}_\nu(E) \).

It is important to emphasise that without the assumption that \( \nu \) is an \( n \)-tuple of closed forms, this definition would not produce a multicomplex. Indeed anti-commutativity of the resulting squares is guaranteed by this assumption.

The proof of the following lemma is based on an observation on \( n \)-Tate objects. There is a natural embedding

\[
e : \mathbb{C} \hookrightarrow \text{Tate}(\mathbb{C}).
\]

This yields \( n \) distinct ways to embed \( \text{Tate}^{n-1}(\mathbb{C}) \) into \( \text{Tate}^n(\mathbb{C}) \) which we denote by \( e_1, \ldots, e_n \).

**Lemma 2.7.** The multicomplex of Definition 2.6 is acyclic in the category \( \text{Calk}^n(k) \).

Proof. A direct computation involving writing out power series in the variables \( t_1, \ldots, t_n \) allows one to infer the following claim.

**Claim 2.8.** Let \( V = \sum_j a_j \frac{\partial}{\partial t_j} \), where \( a_j \in F_n \), and \( a_i \neq 0 \). Then the kernel and cokernel of the map

\[
\nabla_V : E \longrightarrow E
\]

belongs to the full subcategory

\[
e_i(\text{Tate}^{n-1}) \hookrightarrow \text{Tate}^n.
\]

Applying this lemma to the vector fields given by the dual basis of \( \nu \) we deduce a similar statement for the differentials of the multicomplex refining the de Rham complex of \( E \). In particular, since \( e_i \) factors through the kernel of \( \text{Tate}^n \longrightarrow \text{Calk}^n \), we deduce the lemma.

**Definition 2.9.** For \((E, \nabla) \in \text{Loc}_{F_n} \) and \( \nu \in (\Omega^1_{F_n})^n \) a linearly independent \( n \)-tuple of closed 1-forms, we define a binary multicomplex supported on \( \{0,1\}^n \) in the sense of Grayson. It is constructed by adding extra differentials to the multicomplex \( \Omega^\nu_{\mathcal{L}}(E) \). For every inclusion \( j : M \hookrightarrow N = M \cup \{i\} \), we let

\[
\nu_i : \Omega^M \mathcal{L}_\nu(E) \longrightarrow \Omega^N \mathcal{L}_\nu(E)
\]

be the component of the morphism \( \wedge \nabla_i : E \otimes \Omega^q_{F_n} \longrightarrow E \otimes \Omega_q^{q+1} \) with respect to the direct summands \( \Omega^M \mathcal{L}_\nu(E) \) and \( \Omega^N \mathcal{L}_\nu(E) \). This binary multi-complex will be denoted by \( \mathcal{E}^\nu_{\mathcal{L}}(E) \).

**Definition 2.10.** We have an exact functor \( \varepsilon_\nu : \text{Loc}_{F_n} \longrightarrow (\text{BinCh}_{ex})((\text{Calk}^n(k))) \), sending \((E, \nabla)\) to the binary multicomplex \( \mathcal{E}^\nu_{\mathcal{L}}(E) \).

**Definition 2.11.** We define the map of spectra \( \varepsilon_\nu : K(\text{Loc}_n) \longrightarrow K(k) \) as the composition

\[
K(\text{Loc}_n) \xrightarrow{K(\varepsilon_\nu)} K((\text{BinCh}_{ex})((\text{Calk}^n(k)))) \longrightarrow K(k),
\]

where the second map combines Grayson’s \( K((\text{BinCh}_{ex})((\text{Calk}^n(k)))) \longrightarrow \Omega^n K(\text{Calk}^n(k)) \) with Saito’s equivalence \( K(\text{Tate}^n(k)) \simeq K(k) \) (see [Sai13]).
2.4 Induction

Let $F_n^n/F_n$ be a finite extension of $n$-fields with last residue fields $k'/k$, and $y \in (\Omega^n_{F_n})^n$ be an $E_n$-linearly independent $n$-tuple. Observe that we have $\Omega^n_{F_n} \subseteq \Omega^n_{F_n}(\nu)$. A de Rham local system $(E, \nabla)$ on $F_n^n$ gives rise to a de Rham local system on $F_n$. The resulting exact functor is denoted by $\text{Ind}: \text{Loc}_{F_n^n} \to \text{Loc}_{F_n}$ and will be referred to as induction with respect to $F_n^n/F_n$.

**Proposition 2.12.** There is a commutative diagram of spectra

\[
\begin{array}{ccc}
K(\text{Loc}_{F_n^n}) & \xrightarrow{\text{Ind}} & K(\text{Loc}_{F_n}) \\
\varepsilon \downarrow & & \varepsilon \downarrow \\
K(k') & \to & K(k).
\end{array}
\]

**Proof.** Note that every finite $F_n^n$-vector space $E$ gives rise to an $n$-Tate $k'$-vector space. However, since $k'/k$ is a finite field extension, we can also view $E$ as an $n$-Tate $k$-vector space. Furthermore, the diagram of exact categories

\[
P_f(F_n^n) \xrightarrow{\text{Ind}} P_f(F_n) \\
\text{Tate}^n(k') \to \text{Tate}^n(k)
\]

commutes strictly. We obtain the commutative diagram above by applying the $K$-theory functor $K(-)$ to the following strictly commutative diagram of exact categories

\[
\begin{array}{ccc}
\text{Loc}_{F_n^n} & \xrightarrow{\text{Ind}} & \text{Loc}_{F_n} \\
\varepsilon \downarrow & & \varepsilon \downarrow \\
\text{BinCh}_{\text{ex}}^n \text{Calk}^n(k') & \to & \text{BinCh}_{\text{ex}}^n \text{Calk}^n(k)
\end{array}
\]

and using the natural map $K(\text{BinCh}_{\text{ex}}^n \text{Calk}^n(k)) \to K(k).$ \qed

2.5 Duality

In this subsection we will prove the proposition below. We refer the reader to [BCHW18] for the definition of duality for higher Tate objects.

**Proposition 2.13.** There is a commutative diagram of spectra

\[
\begin{array}{ccc}
\text{Loc}_{F_n} & \xrightarrow{(\cdot)^\vee} & \text{Loc}_{F_n} \\
\varepsilon \downarrow & & \varepsilon \downarrow \\
K(k) & \to & K(k).
\end{array}
\]

Let $E$ be a finite-dimensional $F_n$-vector space. We denote by $E^\vee$ its $F_n$-linear dual. Since $E$ can be seen as an $n$-Tate $k$-vector space, it is also possible to consider the $k$-linear dual in the sense of $n$-Tate spaces, which we denote by $E'$. The following lemma relates the $F_n$-linear and $n$-Tate dual in a canonical fashion. It is the analogue of Serre duality for $n$-fields. Its proof is a straightforward extension of the well-known case where $n = 1$.

**Lemma 2.14.** For every finite-dimensional $F_n$-vector space $E$ there is a natural isomorphism of $n$-Tate $k$-vector spaces $E \simeq E' \otimes \Omega^n_{F_n}$ which is induced by the higher residue pairing $\text{Res}(\cdot, \cdot): E \times (E^\vee \otimes \Omega^n_{F_n}) \to k$.

For $p > 0$ we define $\Omega^n_{F_n} = \Omega^{n-p}_{F_n} \otimes (\Omega^n_{F_n})^\vee$. It is easy to see that the $F_n$-linear dual of $\Omega^n_{F_n}$ is canonically isomorphic to $\Omega^n_{F_n}$, by virtue of the twisted pairing $\wedge: \Omega^n_{F_n} \otimes \Omega^{n-p}_{F_n} \to \Omega^n_{F_n}$. With respect to this notation we obtain the following isomorphisms from Lemma 2.14 as an immediate consequence.

**Corollary 2.15.** We have natural isomorphisms of $n$-Tate vector spaces $(E \otimes \Omega^n_{F_n})^\vee \simeq E' \otimes \Omega^n_{F_n}$.

With this isomorphism at hand we ask ourselves what the $n$-Tate dual of the map $E \otimes \Omega^n_{F_n} \xrightarrow{\nabla} E \otimes \Omega^{n+1}_{F_n}$ is. The answer to this question is given by the next corollary. We denote by $\nabla^\vee$ the dual connection on $E'$.
Corollary 2.16. With respect to the isomorphism of Corollary 2.15 we have
\[ \nabla' = - \nabla^\vee : E^\vee \otimes \Omega^{n-p-1}_{F_p} \to E^\vee \otimes \Omega^n_{F_p}. \]

Proof. For \( s \in E \) and \( t \in E^\vee \) we have the relation \( (\nabla s, t) + (s, \nabla^\vee t) = d(s, t) \). Applying the higher residue to both sides, we obtain \( \text{Res}(\nabla s, t) + \text{Res}(s, \nabla^\vee t) = 0 \), since the right hand side is exact. This shows \( \nabla t = - \nabla^\vee t \) for all \( t \in E^\vee \).

For a complex \( A^\bullet \) in an exact category \( \mathcal{C} \) we denote by \( \iota^* A^\bullet \) the complex obtained by replacing all differentials \( d_i : A^i \to A^{i+1} \) by \( -d^i \). We can construct an isomorphism \( A^\bullet \cong \iota^* A^\bullet \) in terms of a commutative ladder.

\[ \cdots \to d_i : A^i \to A^{i+1} \to \to \cdot \cdot \cdot \]

However there is a second choice of an isomorphism, which replaces all vertical arrows \( (-1)^i \) by \( (-1)^{i+1} \). So we see that there is a \( \mu_2 \)-torsor of natural isomorphisms \( A^\bullet \cong \iota^* A^\bullet \). The same remark applies to binary complexes. As a consequence one sees that the proof of the proposition below produces not just one commutative square as in [3], but two (we remind the reader that in the context of \( \infty \)-categories, commutativity of diagrams is an exact structure and not a property).

Proof of Proposition 2.13. Corollary 2.16 implies that the \( n \)-Tate dual of the binary multicomplex \( E_{\nabla, \mathcal{C}}(E) \) is given by \( \iota^* E_{\nabla, \mathcal{C}}(E^\vee) \). Using one the natural isomorphisms above we obtain a commutative diagram of exact categories

\[ \begin{array}{ccc}
\text{Loc}_{F_n} & \xrightarrow{(-)^\vee} & \text{Loc}_{F_n} \\
\downarrow E_{\nabla} & & \downarrow E_{-\nabla} \\
\text{BinCh}_{\text{ex}}^n \text{Calc}^n(k) & \xrightarrow{(-)^\vee} & \text{BinCh}_{\text{ex}}^n \text{Calc}^n(k) 
\end{array} \]

Using the natural map of spectra \( K(\text{BinCh}_{\text{ex}}^n \text{Calc}^n(k)) \to K(k) \) we conclude the proof of the proposition.

3 The variation of epsilon-factors in families

3.1 Epsilon-factors for epsilon-nice families

As before we let \( k \) be a field of characteristic 0. We denote by \( R \) a commutative \( k \)-algebra, and use \( F^R_n \) as shorthand for the commutative ring \( R((t_1)) \cdots ((t_n)) \). As an \( R \)-module it carries the structure of an \( n \)-Tate \( R \)-module, and hence every finitely generated projective \( F^R_n \)-module admits a natural realisation in the exact category \( \text{Tate}^n(R) \). Given \( E \in P_f(F^R_n) \), and a relative connection \( \nabla : E \to E \otimes _F \Omega_{F/R} \), we investigate when it is possible to define an epsilon-factor \( \varepsilon(E, \nabla) \) in the \( K \)-theory spectrum \( K(R) \).

Definition 3.1. Let \( (E, \nabla) \in \text{Loc}_R(F_n) \) be a de Rham local system, such that the formal de Rham multi-complex \( \Omega^R_{F} (E) \) is an acyclic multi-complex in \( \text{Calc}^n(R) \). We say that \( (E, \nabla) \) is an epsilon-nice (or blissful) \( R \)-family of flat connections. The corresponding full subcategory of \( \text{Loc}^R(F_n) \) will be denoted by \( \text{Loc}_{\text{bliss}}^R(F_n) \).

Since formation of the de Rham complex \( \Omega^R_{F} (E) \) is an exact functor, and acyclic complexes form a fully exact and idempotent complete subcategory, we see that \( \text{Loc}_{\text{bliss}}^R(F_n) \) is an exact and idempotent complete category.

Remark 3.2. (a) In [BBE02] such \( R \)-families of flat connections are called nice.

(b) Intuitively speaking the irregularity type of the connection stays constant in epsilon-nice \( R \)-families. However, just as in loc. cit. we prefer to define epsilon-nice families in the abstract manner above.

For epsilon-nice families one can define a (spectral) de Rham epsilon factor, for every \( R \)-linearly tuple of closed relative 1-forms \( (\nu_1, \ldots, \nu_n) \in \Omega^1_{F_n} \otimes_R R \).

Definition 3.3. We denote by \( \varepsilon_{\text{bliss}}^R \) the exact functor \( \text{Loc}_{\text{bliss}}^R(F_n) \to \text{BinCh}_{\text{ex}}^n \text{Calc}^n(R) \). Applying the \( K \)-theory functor we obtain a morphism of spectra \( \varepsilon_{\text{bliss}}^R : K(\text{Loc}_{\text{bliss}}^R(F_n)) \to K(R) \).
3.2 The epsilon-crystal

Epsilon-factors in families are naturally endowed with a partial connection (meaning that we cannot covariantly derive along all vector fields, but only along fields belonging to a subbundle). Since we are working with homotopy points of $K$-theory spectra this is best formalised using a crystalline approach. Before stating the main result of this subsection we have to introduce some notation.

**Definition 3.4.** (a) For a scheme $S$ we denote by $S^{dR}$ the presheaf on the category of schemes given by $T \mapsto S(T^{red})$. We have a natural transformation $S \to S^{red}$.

(b) A functor $H : \text{Sch} \to \text{Sp}$ extends to a functor $H : \text{PrSh}(\text{Sch}_{Nis}) \to \text{Sp}$, such that for a sheaf $F$, we have

$$H(F) \simeq \lim_{T/F} H(T).$$

In particular we get a well-defined spectrum $H(S^{dR})$ for every scheme $S$. We call it the spectrum of $F$-crystals over $S$.

(c) We refer to the spectrum $K(S^{dR})$ as the spectrum of $K$-crystals of $S$. Homotopy points thereof will be referred to as $K$-crystals defined over $S$.

Using the natural map of Nisnevich sheaves $K \to B^\infty \mathbb{G}_m$ we see that every $K$-crystal on $S$ gives rise to a (graded) line with a flat connection on $S$. We will now show that the formalism of $\epsilon$-factors has a natural crystalline refinement. This is the higher rank generalisation of the epsilon connection defined in section 3 of [BBE02] (furthermore they did not consider $K$-crystals). The proof is omitted, as it is based on the $(P^1, \infty)$-invariance property of algebraic $K$-theory detailed in the companion paper [Gro] to construct the epsilon connection.

**Proposition 3.5.** Let $(E, \nabla)$ be an object of $\text{Loc}(F_n)$, $S$ an affine $k$-scheme and $\nu = (\nu_1, \ldots, \nu_n)$ a basis of $\Omega^1_{F^n/S}$. There exists a map $\epsilon^{\nu} : K(\text{Loc}(F_n)) \to K(S^{dR})$, which renders the diagram

$$\begin{array}{ccc}
K(\text{Loc}(F_n)) & \xrightarrow{\epsilon^{\nu}} & K(S^{dR}) \\
\downarrow & & \downarrow \\
K(S) & & K(S)
\end{array}$$

commutative.

**Concluding remarks**

In [Gro] the author introduced a theory of de Rham epsilon factors for $D$-modules on higher-dimensional varieties which is supported on points and gives rise to an epsilon-crystal. The definition in loc. cit. is a continuation of Patel’s theory of de Rham epsilon factors introduced in [Pat12]. We expect these two notions of de Rham epsilon factors to be equivalent.

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