Analytic derivation of the non-linear gluon distribution function

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In the present article, two analytical solutions based on the Laplace transforms method for the linear and non-linear gluon distribution functions have been presented at low values of $x$. These linear and non-linear methods are presented based on the solutions of the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equation and the Gribov-Levin-Ryskin Mueller-Qiu (GLR-MQ) equation at the leading-order accuracy in perturbative QCD respectively. The gluon distributions are obtained directly in terms of the parametrization of structure function $F_2(x, Q^2)$ and its derivative and compared with the results from the parametrization models. The $n_f$ changes at the threshold are considered in the numerical results. The effects of the non-linear corrections are visible as $Q^2$ decreases and vanish as $Q^2$ increases. The non-linear corrections tame the behavior of the gluon distribution function at low $x$ and $Q^2$ in comparison with the parametrization models.

I. Introduction

In recent years, the study and consider of solutions of the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equations [1] based on Laplace transforms method have been considered by many authors [2-4]. Firstly authors in Ref.[5] showed that it is possible to solve the leading order (LO) DGLAP evolution equations directly from the parametrization of the proton structure function $F_2(x, Q^2)$ based on Laplace transform method. These evolution equations in Ref.[5] are a set of integro-differential equations which can be used to evolve the quark and gluon distributions. The proton distribution functions in hadrons play a key role in the Standard Model processes and searches for new physics in future accelerators. The structure function of the proton measured experimentally in deep inelastic scattering processes then traditionally the gluon and quark distribution functions where have been determined simultaneously by fitting experimental data on the proton structure function at small values of the Bjorken variable $x$. At LO approximation the proton structure function is expressed through the quark density as $F_2(x, Q^2) = \sum_{i=1}^{n_f} e_i^2 x [q(x, Q^2) + \bar{q}(x, Q^2)]$, where $n_f$ is the number of flavors.

Authors in Ref.[5] derived an explicit expression for the gluon distribution function $G(x, Q^2) = xg(x, Q^2)$ in the proton in terms of the proton structure function $F_2(x, Q^2)$ by solving the LO DGLAP equation for the $Q^2$ evolution of $F_2(x, Q^2)$ analytically. In particular, accurate knowledge of gluon distribution functions at small $x$ and small $Q^2$ will play a vital role in the electron-proton future colliders. Indeed, the non-linear corrections (NLC) play an important role in the small $x$ and small $Q^2$ regions at the Large Hadron electron Collider (LHeC) and Future Circular Collider hadron-electron (FCC-he) [6]. The non-linear corrections of the gluon recombination to the parton distributions have been calculated by Gribov-Levin-Ryskin (GLR) and Mueller-Qiu (MQ) in [7] based on the Abramovsky-Gribov-Kancheli (AGK) cutting rules in the double leading logarithmic approximation (DLLA). It is known that the gluon recombination effects reduce the growth of the gluon distribution function, therefore cannot be negligible at the small $x$ and $Q^2$ regions. Indeed all possible $g + g \rightarrow g$ ladder recombinations are resummed to leading order of the parameter $\alpha_s \ln(1/x) \ln(Q^2/Q_0^2)$ where leads to saturation of the gluon density at this region.

In the following sections, we present two analytic methods where determine $G(x, Q^2)$ directly from $F_2^p(x, Q^2)$ and its derivative into $\ln Q^2$. The first one is a review for the linear evolution equations based on the Laplace transforms method for the DGLAP evolution equation. The second one is the same method for the GLR-MQ equation which presents the non-linear corrections to the gluon distribution function directly from the proton structure function and its derivative. These methods lead to equivalent results without the intervening differential equation.

The non-linear corrections emerge from the recombination of two gluon ladders where modify the evolution equation of singlet quark distribution by an extra non-linear term. This non-linear term add to the linear DGLAP evolution equation by the following form

$$\frac{\partial F_2(x, Q^2)}{\partial \ln Q^2} = \frac{\partial F_2(x, Q^2)}{\partial \ln Q^2}|_{DGLAP}$$

$$- < e^2 > \frac{27 \alpha_s^3(Q^2)}{160 R^3 Q^2} [xg(x, Q^2)]^2 + HT,$$
where
\[
\frac{dF_2(x,Q^2)}{d\ln Q^2}|_{\text{DGLAP}} = \frac{\alpha_s(Q^2)}{4\pi} [P_{qq}(x)\otimes F_2(x,Q^2) + <e^2>P_{gq}(x)\otimes G(x,Q^2)](2)
\]
The splitting functions $P_{ij}$ are the Altarelli-Parisi splitting kernels at one loop correction, and $<e^2>$ is the average of the charge $e^2$ for the active quark flavors, $<e^2> = n_f^{-1}\sum_{i=1}^{n_f}e_i^2$. We take the $n_f = 3$ for $\mu^2 < m^2$, $n_f = 4$ for $m^2 < \mu^2 < m^2$ and $n_f = 5$ for $\mu^2 > m^2$ and adjust the QCD parameter $\Lambda$ at each heavy quark mass threshold. The correlation length $\mathcal{R}$ determine the size of the non-linear term. This value depends on how the gluon ladders are coupled to the nucleon or on how the gluons are distributed within the nucleon. The $\mathcal{R}$ is approximately equal to $\approx 5$ GeV$^{-1}$ if gluons are populated across the proton and it is equal to $\approx 2$ GeV$^{-1}$ if gluons have the hotspot like structure. Here the higher dimensional gluon distribution(i.e., higher twist) is assumed to be zero.

Now we review the method of extracting the gluon distribution from the parametrization of the proton structure function and its derivative in the linear and non-linear corrections using the Laplace transforms method in next sections respectively.

II. Linear Formalism

By considering the variable changes $\nu \equiv \ln(1/x)$ and $w \equiv \ln(1/y)$, one can rewrite the DGLAP evolution equation (i.e., Eq.2) in $s$-space as
\[
\frac{\partial f_2(s,Q^2)}{\partial \ln Q^2} = \Phi_f(s,Q^2)f_2(s,Q^2) + <e^2>\Theta_f(s,Q^2)g(s,Q^2), \quad (3)
\]
where the Laplace-transform of the distribution functions read
\[
\mathcal{L}[\hat{F}_2(v,Q^2);s] = f_2(s,Q^2), \quad \mathcal{L}[\hat{G}(v,Q^2);s] = g(s,Q^2) \quad (4)
\]
where
\[
\hat{F}_2(v,Q^2) = F_2(e^{-v},Q^2), \quad \hat{G}(v,Q^2) = G(e^{-v},Q^2). \quad (5)
\]
and the coefficient functions $\Phi$ and $\Theta$ in $s$-space are given by
\[
\Theta_f(s,Q^2) = n_f\frac{\alpha_s(Q^2)}{2\pi}\left(\frac{1}{1+s} - \frac{2}{2+s} + \frac{2}{3+s}\right),
\]
\[
\Phi_f(s,Q^2) = \frac{\alpha_s(Q^2)}{4\pi}[4 - \frac{8}{3}\frac{1}{1+s} + \frac{1}{2+s} + 2S(s)].\quad (6)
\]

In the above equation the quantity $S(s)$ is related with the Euler $\Psi(s+1)$ function as $S(s) = \Psi(s+1) + \gamma_E$ where $\Psi(s)$ is defined by $\Psi(s) = \frac{d}{ds}\ln\Gamma(s)$. Here $\Psi(x)$ is the digamma function and $\gamma_E = 0.5772156..$ is Euler constant. We introduce the notion of the so-called nested sums [8] throughout the rest of the paper where the function $S(s)$ is defined
\[
S(s) = -\ln(2) - \sum_{l=0}^{\infty} (-1)^{l+1}\frac{1}{s + l + 1}. \quad (7)
\]

In Fig.1 we consider the expansion of Eq.(7) by using different points. We observe that for $l \geq 10$, the results are almost equivalent and have the smooth behavior for $s > 10$. In this article we shall widely use the notation $l = 10$.

The LO solution of the gluon distribution in $s$-space in Eq.(3) reads
\[
g(s,Q^2) = k(s,Q^2)Df_2(s,Q^2) - h(s,Q^2)f_2(s,Q^2)\quad (8)
\]
where
\[
Df_2(s,Q^2) = \partial f_2(s,Q^2)/\partial \ln Q^2,
\]
\[
k(s,Q^2) = 1/(<e^2>\Theta_f(s,Q^2)),
\]
\[
h(s,Q^2) = \Phi_f(s,Q^2)/(<e^2>\Theta_f(s,Q^2)). \quad (9)
\]

The inverse Laplace transform of coefficients $k$ and $h$ in Eq.(9) are defined by the following forms

\[\text{FIG. 1: Sensitivity of the function } S(s) \text{ verses } s \text{ for different values of } l.\]
\[ k(\nu, Q^2) \equiv \mathcal{L}^{-1}[k(s, Q^2); \nu] = \frac{\pi}{2 < e^2 >} \alpha_s \{ \delta'(\nu) + 3\delta(\nu) - \exp\left(-\frac{3}{2}\nu\right)[2\cos\left(\frac{1}{2}\sqrt{7}\nu\right) + \frac{6}{7}\sqrt{7}\sin\left(\frac{1}{2}\sqrt{7}\nu\right)] \}, \]

\[ h(\nu, Q^2) \equiv \mathcal{L}^{-1}[h(s, Q^2); \nu] = \frac{1}{2 < e^2 >} \{ -(\frac{1}{2} + \frac{2}{3}\ln 2)\delta'(\nu) - \frac{1}{6} + 2\ln 2\delta(\nu) + \exp\left(-\frac{3}{2}\nu\right)[3.606\cos\left(\frac{1}{2}\sqrt{7}\nu\right) \]

\[ + 1.371\sin\left(\frac{1}{2}\sqrt{7}\nu\right)] + \frac{1}{7}\exp(-5\nu) + \frac{20}{11}\exp(-6\nu) - \frac{5}{2}\exp(-7\nu) + \frac{35}{11}\exp(-8\nu) \]

\[ - \frac{112}{29}\exp(-9\nu) + \frac{168}{37}\exp(-10\nu) - \frac{120}{23}\exp(-11\nu) \}. \] (10)

Finally the gluon distribution function directly is obtained from the parameterization of the structure function \( F_2(x, Q^2) \) and its derivatives by the following form

\[ G(x, Q^2) = \frac{\pi}{2 < e^2 >} \alpha_s \{ \frac{\partial DF_2(x, Q^2)}{\partial \ln x} + 3DF_2(x, Q^2) - \int_x^1 \frac{dy}{y} \frac{DF_2(y, Q^2)}{\partial \ln x} \} \]

\[ - \frac{1}{2 < e^2 >} \{ (\frac{1}{2} + \frac{2}{3}\ln 2) \frac{\partial F_2(x, Q^2)}{\partial \ln x} + (\frac{1}{6} + 2\ln 2)F_2(x, Q^2) - \int_x^1 \frac{dy}{y} F_2(y, Q^2) \} \]

\[ + 1.371\sin\left(\frac{1}{2}\sqrt{7}\ln \frac{y}{x}\right)] - \int_x^1 \frac{dy}{y} F_2(y, Q^2) \frac{1}{2} \left( \frac{x}{y} \right)^4 - \frac{8}{7} \left( \frac{x}{y} \right)^5 + \ldots - \frac{120}{23} \left( \frac{x}{y} \right)^{11} \}. \] (11)

The parameterization of the structure function \( F_2(x, Q^2) \), which describes fairly well the available experimental data [9] on the reduced cross sections in a full accordance with the Froissart predictions in a range of the kinematical variables \( x \) and \( Q^2 \), \( x \leq 0.1 \) and \( 0.1 \text{ GeV}^2 < Q^2 < 3000 \text{ GeV}^2 \), suggested by authors in Ref.[10]. This parametrization and its derivative read

\[ F_2(x, Q^2) = D(Q^2)(1 - x)^n \sum_{m=0}^{2} A_m(Q^2) L^m, \] (12)

and

\[ DF_2(x, Q^2) = \frac{\partial F_2(x, Q^2)}{\partial \ln Q^2} = F_2(x, Q^2) \left[ \frac{\partial \ln D(Q^2)}{\partial \ln Q^2} + \frac{\partial \ln \sum_{m=0}^{2} A_m(Q^2) L^m}{\partial \ln Q^2} \right], \]

where the effective parameters are defined in Appendix A and Table I. Consequently, one can obtain the gluon distribution into the effective coefficients obtained from a combined fit of the H1 and ZEUS collaborations data.

### III. Non-Linear Formalism

Now, the above procedure is used to derive the non-linear corrections to the gluon distribution function directly from the parametrization of the proton structure function and its derivatives. The resulting modified structure function should now be driven by non-linear DGLAP evolution in \( s \)-space in a limited approach. In \( \nu \)-space, we have defined the Laplace transform of the \( \mathcal{L}[\hat{G}(\nu, Q^2); s] \) to be less than \( g^2(s, Q^2) \equiv |G(s, Q^2)|^2 \). Indeed \( \mathcal{L}[\hat{G}(\nu, Q^2); s] < \mathcal{L}[\hat{G}(\nu, Q^2); s]^2 \). Therefore in this limit, we take the Laplace transform of (1), by follow

\[ \frac{\partial f_2(s, Q^2)}{\partial \ln Q^2} \sim \Phi_f(s)f_2(s, Q^2) + < e^2 > \Theta_f(s)g(s, Q^2) \]

\[ < e^2 > < \zeta g^2(s, Q^2), \] (13)

1 The standard parametrization of the gluon distribution function at low \( x \) introduced by

\[ G(x, Q^2) = f(Q^2)x^{-\delta} \]

where the low \( x \) behavior could well be more singular. By considering the variable change \( \nu = \ln(1/x) \), one can rewrite the gluon distribution in \( s \)-space as

\[ \mathcal{L}[\hat{G}(\nu, Q^2); s] = \frac{f(Q^2)^2}{(s - 25)}, \]

\[ \mathcal{L}[\hat{G}(\nu, Q^2); s]^2 = \frac{f(Q^2)^2}{(s - 25)}. \]

We observe that the function \( \mathcal{L}[\hat{G}(\nu, Q^2); s] \) is always lower than \( \mathcal{L}[\hat{G}(\nu, Q^2); s]^2 \) for low \( s \) values in a wide range of \( Q^2 \) values. According to this result, we use from this limited approach for solving the quadratic equation in \( s \)-space.
where $\zeta = \frac{27a_s^2(Q^2)}{560RQ^2}$. The non-linear gluon distribution function is defined by a quadratic equation in $s$-space in the following form

$$g^2(s, Q^2) - k(s, Q^2)g(s, Q^2) + h(s, Q^2) = 0, \quad (14)$$

where

$$h(s, Q^2) = \frac{1}{< e^2 > \zeta} [Df_2(s, Q^2) - \Phi_f(s, Q^2)f_2(s, Q^2)],$$

$$k(s, Q^2) = \Theta_f(s, Q^2)/\zeta. \quad (15)$$

One can easily solve this equation and extract the non-linear gluon distribution in $s$-space as

$$g(s, Q^2) = \frac{1}{2} k(s, Q^2)[1\pm (1 - \frac{4h(s, Q^2)}{k^2(s, Q^2)})^{1/2}]. \quad (16)$$

The quadratic equation (16) has two roots, which the negative root reads

$$g(s, Q^2) = \frac{h(s, Q^2)}{k(s, Q^2)} [\text{Linear Term}] + \{\frac{h^2(s, Q^2)}{k^3(s, Q^2)} + 2\frac{h^3(s, Q^2)}{k^5(s, Q^2)} + \ldots\} [\text{Non - Linear Terms}]. \quad (17)$$

The non-linear terms in Eq.(17) are coefficients in the form $\sum n=1 \zeta^n$, where $\zeta$ is around the order $O(\sim 10^{-3})$ at $Q = 1$ GeV and $R = 2$ GeV$^{-1}$, so this series is convergent when $n \to \infty$.

We now re-derive our analytic solution using the inverse Laplace transforms method for $g(s, Q^2)$ (i.e., Eq.(17)). The inverse Laplace transform of terms in Eq.(17) are given by the following form

$$G^{NLC}(x, Q^2) = \text{Eq.}(10) + L^{-1}\left[\frac{h^2(s, Q^2)}{k^3(s, Q^2)}\right]$$

$$+ \frac{h^3(s, Q^2)}{k^5(s, Q^2)} + \ldots; \nu. \quad (18)$$

Therefore the non-linear corrections to the gluon distribution function due to the Laplace transforms method is defined directly from the parametrization of the structure function $F_2$ and its derivative. Then we consider the positive roots for the non-linear corrections to the gluon distribution in Eq.(16). For $Q^2 < 2 \text{ GeV}^2$ in the range $10^{-5} < x < 10^{-1}$, the positive roots are dominant in Eq.(16). In this domain, the non-linear gluon distribution is defined by

$$G^{NLC}(x, Q^2) = \mathcal{L}^{-1}[\frac{h(s, Q^2)}{k(s, Q^2)} - \frac{h^2(s, Q^2)}{k^3(s, Q^2)} - 2\frac{h^3(s, Q^2)}{k^5(s, Q^2)} + \ldots; \nu]$$

$$= \frac{2\alpha_s}{\pi\zeta} \int_x^1 \frac{dy}{y} [(\frac{x}{y}) - 2(\frac{x}{y})^2 + 2(\frac{x}{y})^3]$$

$$- \text{Eq.}(10) - \mathcal{L}^{-1}[\frac{h^2(s, Q^2)}{k^3(s, Q^2)} + \frac{h^3(s, Q^2)}{k^5(s, Q^2)} + \ldots; \nu], \quad (19)$$

In next section, we will describe the numerical results for $G^{NLC}(x, Q^2)$ by using the analytical solutions, (i.e., Eqs.(18) and (19), at low and high $Q^2$ values.

**IV. Results and Discussions**

Now we use the LO approximation of $\alpha_s(Q^2)$ which is defined by

$$\alpha_s(Q^2) = \frac{4\pi}{(11 - \frac{2}{3}n_f)\ln(Q^2/\Lambda^2)},$$

with $n_f = 5$ and $\Lambda_5 = 80.80$ MeV for $Q > m_b$, $n_f = 4$ and $\Lambda_4 = 136.80$ MeV for $m_c < Q < m_b$, and $n_f = 3$ and $\Lambda_3 = 136.80$ MeV for $Q < m_c$. Where $\Lambda$’s have been extracted with $\alpha_s(m_c^2) = 0.1166$ at the Z-boson mass. By using the Eqs.(11), (18) and (19), we can extract numerically the linear and non-linear gluon distribution inside the proton from the parameterization of the proton structure function and its derivatives.
compare them with the results in Refs.[11] and [12]. The results of the calculations based on the linear gluon
distribution (i.e., Eq.(11)) are shown in Fig.3. In this figure the straight lines represent the solutions resulted
from the Laplace transform technique. We take the
3 flavors for $Q < 1.3$ GeV, $n_f = 4$ for $Q < 4.5$ GeV and
$n_f = 5$ for $Q > 4.5$ GeV as the gluon distribution function
$G(x, Q^2)$ depends on $n_f$. In Fig.4, the results for
the linear gluon distributions have been shown and com-
pared with the parametrization methods in Refs.[11,12]
for $Q^2 = 10$ and 100 GeV$^2$. For $Q^2 = 10$ GeV$^2$ these
results compared with $G_{n_f=4}(x, Q^2)$ in Refs.[11] and
[12]. For $Q^2 = 100$ GeV$^2$ where the active flavor is
$n_f = 5$, we compared our results with those obtained in
Ref.[11]3. This figure indicate that the obtained results
from the present analysis, based on the Laplace
transform technique using the number of active flavors,
are in good agreements with the ones obtained from the
parametrization methods.

In Fig.5 we plot the $Q^2$ dependence of the non-linear
corrections to the gluon distribution for $R = 2$ GeV$^{-1}$
at some representative $x$ and check the compatibility of
the non-linear results with the linear gluon distributions.

In this figure (i.e., Fig.5) we can see that at large scales,
the nonlinear corrections (due to the $1/Q^2$ dependence)
relax into the linear gluon distribution functions. Also
the non-linear corrections play an important role on
gluon distribution as $x$ and $Q^2$ decreases. In Fig.5 we
observe that the dot line separates linear and non-linear
behaviors of the gluon distributions. It is seen that these
distributions change discontinuously at each threshold
but remain constant between thresholds. These results are
comparable with Eskola-Honkanen-Kolhinen-Qiu-
Salgado (EHKSQ), which obtained [14-15] the parton
distribution functions using the CTEQ6L [16] with
respect to the non-linear GLRMQ evolution equations.

At low scales $Q^2 < 2$ GeV$^2$, where the positive roots are
dominate in the non-linear distributions, we obtained
the non-linear behavior of the gluon distribution in
Fig.6. In this figure (i.e., Fig.6), we show the non-linear
corrections to the gluon distribution determined from
Eqs.(18) and (19) as a function of $x$ for two different
values of $Q$, namely $Q = 1.14$ GeV and 1.30 GeV with
respect to the Laplace transforms method. In these
calculations, we use the non-linear gluon distribution
with $n_f = 3$ for $Q = 1.14$ GeV and compared the
obtained results with the parametrization method in
Ref.[11] at the same active flavor number. Also the
charm threshold is shown for $Q = 1.30$ GeV with $n_f = 3$
and 4 in Fig.6 and compared the obtained results with
the parametrization method in Ref.[11] at each active
flavor number. A depletion occurs at $x \leq 10^{-4}$ where
these results show that the non-linear behavior of the
gluon distribution function is tamed with respect to
the positive roots. This taming behavior of non-linear
gluon distribution function towards low $x$ at low $Q^2$
values become significant at the hot spot point. Further
the computed values of the gluon distribution with
non-linear effects play an increasingly important role at
low $x$ and low $Q^2$ values.

In conclusion, we have presented a certain theoretical
model to describe the non-linear corrections to the gluon
distribution function based on the Laplace transforms
method at low values of $x$ and $Q^2$ in a limited approach.
A detailed analysis has been performed to find an
analytical solution of the linear and non-linear gluon
distribution functions from the proton structure function
and its derivative. The effect of non-linear corrections
on the behavior of $G(x, Q^2)$ with decreasing $Q^2$ become
significant at the hot spot point. At high $Q^2$ values
the non-linear corrections relax into the linear gluon
distribution function. The nonlinear corrections have
been tamed the behavior of the gluon distribution function at $Q^2 < 2$ GeV$^2$ in comparison with the linear
behavior.

2 In Ref.[11] the gluon distribution for $n_f = 4$ is just
$G_{n_f=4}(x, Q^2)$, where $G_{n_f=3}(x, Q^2)$ is obtained from a fit to
ZEUS data [13] into an expression in both ln($Q^2$) and ln(1/$x$)
to include the effects of heavy-quark masses. In Ref.[12] authors
obtained an analytical solution for $G(x, Q^2)$ using a Froissart
bounded structure function for $0 < x \leq 0.09$. Those obtained a
simple quadratic polynomial in ln(1/$x$) with quadratic polyno-
mial coefficients in ln($Q^2$).

3 In Ref.[11] authors obtained the gluon distribution $G(x, Q^2)$
for 5 active quarks (for massless $u$, $d$, $s$ and massive $c$, $b$
quarks) into the massless gluon distribution $G_{n_f=3}(x, Q^2)$, as
$G_{n_f=5}(x, Q^2) = \frac{N}{5} G_{n_f=3}(x, Q^2)$. Also authors obtained an
excellent fit to the gluon distribution for $n_f = 5$ using a quadratic
expression in ln(1/$x$) and a much more complicated power series
in ln($Q^2$) for $x \leq 0.05$.

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Appendix A

The explicit expression for the proton structure function suggested in Ref.[10] is defined by the following form

\[ F_2^{p}(x,Q^2) = D(Q^2)(1 - x)^a [C(Q^2) + A(Q^2) \ln \left( \frac{1}{x} \frac{Q^2}{Q^2 + \mu^2} \right)+ B(Q^2) \ln^2 \left( \frac{1}{x} \frac{Q^2}{Q^2 + \mu^2} \right)], \tag{20} \]

where

\[ A(Q^2) = a_0 + a_1 \ln(1 + \frac{Q^2}{\mu^2}) + a_2 \ln^2(1 + \frac{Q^2}{\mu^2}), \]
\[ B(Q^2) = b_0 + b_1 \ln(1 + \frac{Q^2}{\mu^2}) + b_2 \ln^2(1 + \frac{Q^2}{\mu^2}), \]
\[ C(Q^2) = c_0 + c_1 \ln(1 + \frac{Q^2}{\mu^2}), \]
\[ D(Q^2) = \frac{Q^2(Q^2 + \lambda M^2)}{(Q^2 + M^2)^2}. \tag{21} \]

Here \( M \) is the effective mass and \( \mu^2 \) is a scale factor. The additional parameters with their statistical errors are given in Table I.

| parameters | value |
|------------|-------|
| \( a_0 \)  | \( 8.265 \times 10^{-4} \pm 4.62 \times 10^{-4} \) |
| \( a_1 \)  | \( -5.148 \times 10^{-2} \pm 8.19 \times 10^{-3} \) |
| \( a_2 \)  | \( -4.725 \times 10^{-3} \pm 1.01 \times 10^{-3} \) |
| \( b_0 \)  | \( 2.217 \times 10^{-3} \pm 1.42 \times 10^{-4} \) |
| \( b_1 \)  | \( 1.244 \times 10^{-2} \pm 8.56 \times 10^{-4} \) |
| \( b_2 \)  | \( 5.958 \times 10^{-4} \pm 2.32 \times 10^{-4} \) |
| \( c_1 \)  | \( 1.475 \times 10^{-1} \pm 3.025 \times 10^{-2} \) |
| \( \alpha \) | \( 11.49 \pm 0.99 \) |
| \( \lambda \) | \( 2.430 \pm 0.153 \) |
| \( \chi^2 \) (goodness of fit) | 0.95 |

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FIG. 5: Linear and non-linear behavior of the gluon distributions for several fixed values of \( x \) in a wide range of \( Q^2 \) shows that the effect of the nonlinear terms vanishes as \( Q^2 \) increases.

FIG. 6: Non-linear gluon distribution at low \( Q \) values due to the positive and negative roots in Eqs.(18) and (19) compared with the linear gluon distribution functions (dashed lines) obtained in Ref.[11] at each active flavor number.