Two Proofs of a Conjecture of Hori and Vafa

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March 29, 2022

§0. Introduction. The (small) quantum cohomology $QH^*(X)$ of a complex projective manifold $X$ can be thought of either as a deformation of the even degree cohomology $V := H^{2*}(X, \mathbb{C})$ or as a family of associative, commutative products on $V$ indexed by $H^2(X, \mathbb{C})$. Indeed, if $T_1, ..., T_m \in H^2(X, \mathbb{C})$ are a basis, which we assume for simplicity to consist of nef divisors, (with dual basis $t_1, ..., t_m$), then the small quantum product is a map:

$$* : V \times V \to V[[e^{t_1}, ..., e^{t_m}]]$$

(reducing to the cup product when $t_i \to -\infty$) determined by the enumerative geometry of rational curves (the genus-zero Gromov-Witten invariants) on $X$. If $X$ is a Fano manifold then $*$ is polynomial-valued in the exponentials.

The complex Grassmannian $G := G(r, n)$ has one of the most-studied and best-understood small quantum cohomology rings. Recall that the ordinary cohomology of $G$ has presentation:

$$H^*(G) \cong \mathbb{C}[[\sigma_1, ..., \sigma_r]]/\langle h_{n-r+1}, ..., h_n \rangle$$

where the $\sigma_i$ are the elementary symmetric polynomials in degree 2 variables $x_1, ..., x_r$ (the Chern roots of the dual to the universal bundle) and the $h_i$ are the complete symmetric polynomials (sums of all monomials of degree $i$) in $x_1, ..., x_r$.

The quantum cohomology of $G$, on the other hand, has presentation:

$$QH^*(G) \cong \mathbb{C}[[\sigma_1, ..., \sigma_r, e^T]]/\langle h_{n-r+1}, ..., h_n + (-1)^re^T \rangle$$

where $T = \sigma_1$ is the basis for $H^2(G, \mathbb{C})$ (see [Wit], [ST]).

Quantum cohomology has a mathematical partner that is frequently better suited for applications. Here one regards $*$ as an $O$-linear product of $TV$-valued vector fields over $H^2(X, \mathbb{C})$. That is, if $T_0, T_1, ..., T_n$ is a basis for $V$, extending the basis of $H^2(X, \mathbb{C})$, with $T_0 = 1$ and $T_n$ its Poincaré dual, and if:

$$T_i * T_j = \sum_k \Phi_{ij}^k (e^{t_1}, ..., e^{t_m}) T_k$$
\[ \ast : TV|_{H^2} \otimes TV|_{H^2} \to TV|_{H^2} \]

is defined by \( \partial_i \ast \partial_j = \sum_k \Phi^k_{ij}(e^1, \ldots, e^m) \partial_k \), where \( \partial_i := \frac{\partial}{\partial t_i} \).

If we reinterpret \( \ast \) once more as:

\[ \ast : TV|_{H^2} \to TV|_{H^2} \otimes T^*V|_{H^2} \]

then the \( \mathcal{O} \)-linearity means that there is a family of connections:

\[ \nabla_h : TV|_{H^2} \to TV|_{H^2} \otimes T^*V|_{H^2}; \nabla_h = d + \frac{1}{\hbar} \ast \]

deforming \( d \), and the associativity of \( \ast \) translates into the flatness of the \( \nabla_h \), i.e. existence of flat sections \( F(t, h) = F_0(t, h)\partial_0 + \cdots + F_n(t, h)\partial_n \) satisfying:

\[ \hbar \frac{\partial F}{\partial t_i} = \partial_i \ast F \text{ for } i = 1, \ldots, m \]

(see [Dub]).

Givental [Giv1] computed the fundamental \((n+1) \times (n+1)\) matrix of solutions in terms of Gromov-Witten invariants “with gravitational descendents.” The solution is unique of the following form with suitable initial conditions:

\[ (F_{i,j}(t, e^i, \hbar)\partial_j)_{0 \leq i,j \leq n} \]

(polyomial in the \( t = (t_1, \ldots, t_m) \)). It is very useful to regard the columns as cohomology-valued, and the last column in particular:

\[ J^X = \sum_{i=0}^{n} F_{i,n}(t, e^i, \hbar)\tilde{T}_i \]

is very functorial and plays an important role in mirror symmetry (here \( \{\tilde{T}_i\} \) is the basis dual to \( \{T_i\} \) with respect to the intersection pairing on \( X \)).

In the simplest case, let \( x \) be the hyperplane class in \( H^2(P^{n-1}) \). Then:

\[ J^{P^{n-1}} = e^{\frac{t}{x^n}} \sum_{d=0}^{\infty} \frac{e^{d t}}{\prod_{i=1}^{d} (x + \hbar)^n} \text{ (mod } x^n)\).\]

Our goal here is to compute the \( J \)-function of the Grassmannian \( G \) and explore various applications that result. One of the nice functorial properties of \( J \)-functions guarantees that:

\[ J^P = \prod_{i=1}^{r} J^{P^{n-1}} = e^{\frac{t_{11} + \ldots + t_{rr} }{x^n}} \sum_{(d_1, \ldots, d_r)} \frac{e^{d_1 t_1 + \ldots + d_r t_r}}{\prod_{i=1}^{d_i} \prod_{l=1}^{d_l} (x_i + \hbar)^{n}} \]
where \( P = \prod_{i=1}^{r} P^{n-1} \) with hyperplane classes \( x_i \) on the factors.

We will give two proofs of the following additional property of \( J \):

**Conjecture (Hori-Vafa, [HV, Appendix A]):** Let:

\[
\Delta = \prod_{i<j} (x_i - x_j) \quad \text{and} \quad D_\Delta = \prod_{i<j} \left( \hat{h} \frac{\partial}{\partial t_i} - \hat{h} \frac{\partial}{\partial t_j} \right)
\]

denote the Vandermonde determinant and “Vandermonde operator”, respectively. Then:

\[
J^G = \left( e^{-\sigma_1 (r-1) \pi \sqrt{-1}/\hbar} \right) \frac{D_\Delta (J^P)}{\Delta} \bigg|_{t_i = t + (r-1) \pi \sqrt{-1}}
\]

i.e. the \( J \)-function of the Grassmannian is obtained by applying \( D_\Delta \) to \( J^P \) and then “symmetrizing” (and translating).

We will see in the second proof of the conjecture, in particular, that this can be thought of as another nice functorial property of \( J \)-functions, which looks as though it ought to generalize to wider classes of geometric invariant theory quotients.

We should note that Hori and Vafa formulate the conjecture in terms of “period” integrals, rather than \( J \)-functions. The connection between the coefficients of the \( J \)-function and these integrals is explained in Section 3. As an application, we then use the (equivariant) integral representation form of the Hori-Vafa conjecture to give a proof of Givental’s “R-Conjecture”, and hence of the Virasoro conjecture for Grassmannians. Finally, we give (with Dennis Stanton) a proof for \( G(2,n) \) of another formula for one of the coefficients of the \( J \)-function, which was conjectured in [BCKS1] from different considerations.

**Acknowledgements:** The authors are aware of at least two previous attempts to compute the \( J \)-function of the Grassmannian (or an equivalent formulation). In particular, recent work of [LLLY] contains Lemmas 1.1 and 1.2 of this paper, and then produces a lengthy algorithm, but not a closed formula, for the coefficients of the \( J \)-function. Our first proof of the conjecture avoids many of the technical difficulties of that paper (and of an earlier paper of [Kim1] along similar lines) by using Euler sequences rather than relying on the equivariant cohomology ring of the Grassmannian.

Our second proof relies upon a quantum cohomology interpretation of the Hori-Vafa conjecture that was suggested to the second author by Sheldon Katz in the Fall of 2000.

We thank Dennis Stanton for providing the proof of Proposition 3.5, and for allowing us to include it in this paper. We also thank Alexander Givental, Dosang Joe, J. Park, and Alexander Yong for useful discussions.

A. Bertram and I. Ciocan-Fontanine have been partially supported by NSF grants DMS-0200895 and DMS-0196209, respectively. B. Kim has been supported by KOSEF 1999-2-102-003-5, R03-2001-00001-0, R02-2002-00-00134-0.
§1. A Localization Proof of the Hori-Vafa Conjecture. Let:

\[ \text{Quot}_{\mathbb{P}^1, d}(C^n, n - r) \]

be the Grothendieck quot scheme parametrizing the coherent-sheaf quotients \( C^n \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{Q} \) with Hilbert polynomial \( d + (n - r)(t + 1) \) (see e.g. [BDW]) (this is the Hilbert polynomial of a locally free sheaf on \( \mathbb{P}^1 \) of rank \( n - r \) and degree \( d \)). The quot scheme is a smooth, projective variety and if \( n > r \), it is a compactification of the Hilbert scheme of maps \( g: \mathbb{P}^1 \to G \) of degree \( d \) since a locally free quotient of \( C^n \otimes \mathcal{O}_{\mathbb{P}^1} \) of rank \( n - r \) and degree \( d \) is a map to \( G \).

Consider on the other hand the quot scheme:

\[ \text{Quot}_{\mathbb{P}^1, d}(C^r, 0) \]

defining torsion quotients of \( C^r \otimes \mathcal{O}_{\mathbb{P}^1} \) of length \( d \). These were considered by Weil, as higher rank versions of the symmetric product \( \mathbb{P}^d = \text{Sym}^d \mathbb{P}^1 = \text{Quot}_{\mathbb{P}^1, d}(C, 0) \). Indeed, every such quot scheme maps to the symmetric product:

\[ \Lambda^r: \text{Quot}_{\mathbb{P}^1, d}(C^r, 0) \to \text{Quot}_{\mathbb{P}^1, d}(C, 0) = \mathbb{P}^d \]

via the top exterior power of the kernel vector bundle:

\[ \Lambda^r(K \subset C^r \otimes \mathcal{O}_{\mathbb{P}^1}) := (\Lambda^rK \subset C \otimes \mathcal{O}_{\mathbb{P}^1}) \]

Each such kernel is a locally free sheaf on \( \mathbb{P}^1 \) with a splitting type:

\[ K \cong \mathcal{O}_{\mathbb{P}^1}(-d_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(-d_r) \]

such that \( d_1 + \cdots + d_r = d \). This is unique when we require \( 0 \leq d_1 \leq d_2 \leq \cdots \leq d_r \). The action of \( \text{PGL}(2, \mathbb{C}) \) on \( \mathbb{P}^1 \) determines an action on the quot schemes by pulling back kernels. For such actions, \( \Lambda^r \) is equivariant and the splitting type of the kernel \( K \) is evidently invariant.

We will consider the diagonal action of \( \mathbb{C}^* \subset \text{PGL}(2, \mathbb{C}) \) on \( \mathbb{P}^1 \):

\[ \sigma(\zeta, (x : y)) = (\zeta x : y) \]

with fixed points at \( 0 = (0 : 1) \) and \( \infty = (1 : 0) \). If \( C^n \otimes \mathcal{O}_{\mathbb{P}^1} \to \mathcal{Q} \) is a fixed point for the induced action of \( \mathbb{C}^* \) on \( \text{Quot}_{\mathbb{P}^1, d}(C^r, 0) \), then it is immediate that the (reduced) support of \( \mathcal{Q} \) is contained in \( \{0, \infty\} \); the fixed points of the action on \( \mathbb{P}^1 \). What is less immediate is the following:

**Lemma 1.1:** For each splitting type \( \{d_i\} \) as above, let \( m_1, m_2, \ldots, m_k \) denote the jumping indices (i.e. \( 0 \leq d_1 = \cdots = d_{m_1} < d_{m_1+1} = \cdots = d_{m_2} < \cdots \)). Then there is an embedding:

\[ i_{\{d_i\}}: \text{Fl}(m_1, m_2, \ldots, m_k, r) \hookrightarrow \text{Quot}_{\mathbb{P}^1, d}(C^r, 0) \]
with the property that each fixed point of the \( \mathbb{C}^* \)-action with \( \text{supp}(Q) = \{0\} \) and with kernel splitting type \( \{d_i\} \) corresponds to a point of the flag variety.

**Proof:** Start with the universal flag on \( Fl := Fl(m_1, m_2, ..., m_k, r) \)

\[
0 \subseteq S_{m_1} \subset S_{m_2} \subset ... \subset S_{m_k} \subset S_{m_{k+1}} = \mathbb{C}^r \otimes O_{Fl}
\]

let \( \pi : \mathbb{P}^1 \times Fl \rightarrow Fl \) be the projection and let \( z = 0 \times Fl \subset \mathbb{P}^1 \times Fl \). Then we construct a modified flag of subsheaves of \( \mathbb{C}^r \otimes O_{\mathbb{P}^1 \times Fl} \) as follows. First, construct \( S_i \) for \( i > 1 \) via an elementary modification:

\[
\begin{align*}
0 &\rightarrow \pi^* S_{m_1} \rightarrow S'_{m_i} \rightarrow \pi^* S_{m_i}/S_{m_1}(-d_{m_i} z) \rightarrow 0 \\
0 &\rightarrow \pi^* S_{m_1} \rightarrow \pi^* S_{m_i} \rightarrow \pi^* S_{m_i}/S_{m_1} \rightarrow 0
\end{align*}
\]

and note that this gives a flag: \( \pi^* S_{m_1} \subset S'_{m_2} \subset ... \subset S'_{m_{k+1}} \subset \mathbb{C}^r \otimes O_{\mathbb{P}^1 \times Fl} \).

Then, inductively, define \( S_{m_i}^{(j)} \) for \( i > j \) by:

\[
\begin{align*}
0 &\rightarrow S_{m_i}^{(j-1)} \rightarrow S_{m_i}^{(j)} \rightarrow S_{m_i}^{(j-1)}/S_{m_i}^{(j-2)}(-d_{m_i-1} z) \rightarrow 0 \\
0 &\rightarrow S_{m_i}^{(j-1)} \rightarrow S_{m_i}^{(j-1)} \rightarrow S_{m_i}^{(j-1)}/S_{m_i}^{(j-2)} \rightarrow 0
\end{align*}
\]

This process yields, in the end, a flag of sheaf-inclusions of vector bundles:

\[
0 \subset \pi^* S_{m_1} \subset S_{m_2}^{(1)} \subset ... \subset S_{m_i}^{(i)} \subset ... \subset S_{m_{k+1}}^{(k)} \subset \mathbb{C}^r \otimes O_{\mathbb{P}^1 \times Fl}
\]

with the property that

\[
S_{m_i}^{(i-1)}/S_{m_i-1}^{(i-2)} \cong \pi^* (S_{m_i}/S_{m_i-1})(-d_{m_i} z)
\]

and we define \( K := S_{m_{k+1}}^{(k)} \) with \( Fl \)-valued quotient

\[
\mathbb{C}^r \otimes O_{\mathbb{P}^1 \times Fl} \rightarrow Q = \mathbb{C}^r \otimes O_{\mathbb{P}^1 \times Fl}/K
\]

This quotient is flat over \( Fl \), of (relative) length \( d \) supported on \( z \). If we restrict to a point \( f \) of the flag variety, we get a quotient \( \mathbb{C}^r \otimes O_{\mathbb{P}^1} \rightarrow Q|_{\mathbb{P}^1 \times f} \) with kernel \( K_f \) of splitting type \( \{d_i\} \).

It follows from the global description of the kernel that \( K_f \rightarrow \mathbb{C}^r \otimes O_{\mathbb{P}^1} \) can be expressed in matrix (block) form as:

\[
\begin{bmatrix}
A_1x^{d_1} & A_2x^{d_2} & \cdots & A_{k+1}x^{d_{k+1}}
\end{bmatrix} : K \rightarrow \mathbb{C}^r \otimes O_{\mathbb{P}^1}
\]

where each \( A_i \) is an \( r - m_{i-1} \times m_i - m_{i-1} \) (block) matrix of scalars, augmented below by zeroes (we set \( m_0 = 0 \)). The \( A_i \) represent the (un-modified) flag of subspaces corresponding to \( f \), and the \( x^{d_i} \) factors are produced by the elementary modifications.
On the other hand, if $Q$ is a quotient of length $d$ supported at 0 and a fixed point for the action of $C^*$, it follows that every entry in the matrix for the map $K \to C^r \otimes \mathcal{O}_{P^1}$ is a multiple of a power of $x$. It follows that after an automorphism of $K$, the matrix can be put into the block form above, hence is in the image of $i_{\{d_i\}}$ where $\{d_i\}$ is the splitting type of $K$. Finally, the ambiguity in the block form of the matrix is the same on both sides, namely the action of $\times GL(m_i - m_i - 1, C)$. Thus the map $i_{\{d_i\}}$ is an embedding of smooth varieties and surjects onto the desired fixed loci. This concludes the proof of Lemma 1.1.

If $X$ is a scheme equipped with a vector bundle $E$ of rank $r$, then there is a relative version of Lemma 1.1. Namely, the relative quot scheme over $X$:

$$\text{Quot}_{P^1,d}(E,0) \to X$$

represents the functor: "quotients $\pi^*E \to Q$ (for $\pi : P^1 \times T \to T \to X$) that are flat of relative length $d$ over $T$". By a theorem of Grothendieck, the fibers of the relative quot scheme over $X$ are isomorphic to $\text{Quot}_{P^1,d}(C^n,0)$. Moreover, the $C^*$ action globalizes, and we obtain morphisms of $X$-schemes:

$$i_{\{d_i\}} : Fl(m_1, m_2, ..., m_k, E) \to \text{Quot}_{P^1,d}(E,0)$$

characterizing the fixed loci of the $C^*$ action with $\text{supp}(Q) = \{0 \times X\}$.

We apply this version for the universal sub-bundle $S$ on $G(r,n)$ to get:

**Lemma 1.2:** There is a natural $C^*$-equivariant embedding:

$$i : \text{Quot}_{P^1,d}(S,0) \hookrightarrow \text{Quot}_{P^1,d}(C^n, n - r)$$

such that all the fixed points of the $C^*$ action on $\text{Quot}_{P^1,d}(C^n, n - r)$ are contained in the image. The fixed points of $\text{Quot}_{P^1,d}(C^n, n - r)$ that also satisfy $\text{supp}(\text{tor}(Q)) = \{0\}$ (the support of the torsion part of $Q$) are precisely the images of flag manifolds:

$$F(m_1, m_2, ..., m_k, r, n) = F(m_1, m_2, ..., m_k, S) \hookrightarrow \text{Quot}_{P^1,d}(S,0)$$

embedded by the relative version of Lemma 1.1.

**Proof:** The map $i$ is defined as follows. The kernel of the universal quotient:

$$K \hookrightarrow \pi^*S \to Q$$

on $P^1 \times \text{Quot}_{P^1,d}(S,0)$ can be thought of as a subsheaf of $C^n \otimes \mathcal{O}_{P^1 \times \text{Quot}_{P^1,d}(S,0)}$ by composing with the inclusion $\pi^*S \to \pi^*C^n \otimes \mathcal{O}_G$. Then the quotient:

$$C^n \otimes \mathcal{O}_{P^1 \times \text{Quot}_{P^1,d}(S,0)} \to Q' = C^n \otimes \mathcal{O}_{P^1 \times \text{Quot}_{P^1,d}(S,0)}/K$$

is flat of the desired Hilbert polynomial. This gives the map $i$. 
The image of i is the set of quotients $C^n \otimes O_{P^1} \to Q'$ such that:

$$K \hookrightarrow C^r \otimes O_{P^1} \to C^n \otimes O_{P^1}$$

with $Q = C^r \otimes O_{P^1}/K$ and $Q' = C^n \otimes O_{P^1}/K$.

Now suppose that a quotient $C^n \otimes O_{P^1} \to Q'$ in Quot${}_{P^1/d}(C^n, n-r)$ is fixed under the action of $C^*$. Then the matrix of the map $K \to C^n \otimes O_{P^1}$ has ith column $\vec{v} = y_i y' \cdot \vec{v}$, where $\vec{v}$ is a vector of scalars, and $b_i + c_i = d_i$. From this it follows that $K$ factors through the subspace $C^r \hookrightarrow C^n$ with matrix given by the $\vec{v}_i$. Finally, it is clear from the construction that $\text{supp}(\text{tor}(Q')) = \text{supp}(Q)$, hence that the fixed loci with $\text{supp}(\text{tor}(Q')) = \{0\}$ are the flag manifolds of the global version on Lemma 1.1. This completes the proof of Lemma 1.2.

Next, we need to relate these quot schemes to Kontsevich-Manin stacks. Recall the:

**Definition:** A map $f : C \to X$ from a curve $C$ with marked points $p_1, ..., p_n \in C$ is **pre-stable** if:

(i) $C$ is connected and projective, with at worst ordinary nodes

(ii) the marked points are smooth points of $C$

and $f$ is **stable** if, in addition:

(iii) the automorphisms of $C$ fixing the $p_i$ and commuting with $f$ are finite.

Assuming that $X$ is homogeneous, there is a smooth proper stack $\overline{M}_{0,n}(X, \beta)$ for each class $\beta \in H_2(X, \mathbb{Z})$ and each $n$ representing the functor “flat families of stable maps of genus zero, $n$-pointed curves of class $\beta.$” Of special interest is the “graph space:”

$$\overline{M}_{0,0}(X \times P^1, (\beta, 1))$$

which can be thought of as a compactification of the Hilbert scheme of maps $g : P^1 \to X$ of class $\beta$ since a stable map $f : C \to X \times P^1$ of bidegree $(\beta, 1)$ is the graph of a map $g : P^1 \to X$ whenever $C$ is irreducible. We will compare this compactification (when $X = G$ and $\beta = d$) with the quot scheme. But first:

The $C^*$ action on $P^1$ induces an action on the graph spaces, and one of the connected components of the fixed point locus is

$$\overline{M}_{0,1}(X, \beta) \cong F \subset \overline{M}_{0,0}(X \times P^1, (\beta, 1))$$

corresponding to the stable maps $f : C \to X \times P^1$ with the property that $C = C_0 \cup C_1$, $n = C_0 \cap C_1$ with $f(n) = (p, 0)$,

$$f|_{C_0} : C_0 \to X \times \{0\} \text{ of class } \beta \text{ and } f|_{C_1} : C_1 \to p \times P^1$$

This defines a stable map to $X \times P^1$ exactly when the map $f_0$ is stable as a map from the pointed curve $(C_0, n)$. 
**Theorem (Givental):** The coefficients of the $J$-function of $X$ satisfy:

$$J^X = e^{\sum_{i=1}^m t_i T_i} \sum_{\beta} e^{\sum_{i=1}^m t_i \int_{\beta} T_i} J_\beta(h),$$

where

$$J_\beta(h) = ev_* \frac{1}{e_{C^*}(F)}.$$ 

$e_{C^*}(F)$ is the equivariant Euler class of $F$, computed in the ring:

$$H_{C^*}(F, \mathbb{Q}) = H^*(F, \mathbb{Q}) \otimes \mathbb{Q}[\bar{h}]$$

(i.e. we interpret $\bar{h}$ geometrically via $H^*(BC^*, \mathbb{Q}) = \mathbb{Q}[\bar{h}]$) and

$$ev : F = \overline{M}_{0,1}(X, \beta) \to X$$

is the evaluation map discussed above.

Our task now is to compute this with the quot scheme when $X = G$. For this, the following diagram (denoted (†)) is the key:

$$\begin{array}{cccc}
\text{Quot}_{P^1, d}(C^n, n - r) & \overset{\wedge^r}{\hookrightarrow} & P^\left(^{(n)}\right)_{d} & \overset{\Phi}{\hookleftarrow} & \overline{M}_{0,0}(G(r, n) \times P^1, (d, 1)) \\
& \cup & & \cup & \\
\prod_{i(d_i)}^{i(d)}(Fl) & \overset{\beta}{\to} & P(\left(^n\right)_{d}) & \overset{\xi}{\to} & F \\
\downarrow \rho & \cup & \quad ev \quad & \downarrow \psi & \\
G(r, n) & & & & \\
\end{array}$$

We need to explain this diagram. First, $G(r, n)$ is embedded by Plücker:

$$\wedge^r : G(r, n) \subset P\left(^{(n)}\right)_{d}^{-1}$$

Next, $P\left(^{(n)}\right)_{d}^{-1} = \text{Quot}_{P^1, d}(C^n, \left(^n\right)_{d} - 1)$ is a projective space, more simply:

$$P\left(^{(n)}\right)_{d}^{-1} = P(\text{Hom}_d(C^2, C^n))$$

is the projectivized space of $d$-linear maps $\text{Hom}_d(C^2, C^n) = \text{Sym}^d(C^2)^* \otimes C^n$.

Thus the second row of (†) consists of components of the fixed locus for the $C^*$-actions on the first row, with respect to which $\wedge^r$ and $\Phi$ are equivariant. By Lemma 1.2, the morphisms $p$ and $q$ have, as their domains, all the fixed points
contained in $(\wedge^r)^{-1}(P^n_r)^{-1}$ and $\Phi^{-1}(P^n_r)^{-1}$ respectively. The restriction of $\rho$ to each component $i_{\{d_i\}}(Fl)$ is the natural projection from the flag bundle $Fl(m_1,...,m_k,S) \to G(r,n)$. It then follows from the localization theorem of Atiyah-Bott (see [Ber3]) that:

$$\wedge^r J_d(h) = q_* \frac{1}{e_{C^*}(F_0)} = \sum_{\{d_i\}} p_* \left( \frac{1}{e_{C^*}(i_{\{d_i\}}(Fl))} \right)$$

We claim that more is true:

**Lemma 1.3:** The $J$-function on $G(r,n)$ satisfies:

$$J_d(h) = \sum_{\{d_i\}} \rho_* \left( \frac{1}{e_{C^*}(i_{\{d_i\}}(Fl))} \right)$$

**Proof:** If the push-forward $\wedge^r : H^*(G(r,n),\mathbb{Q}) \to H^*(P^n_r)^{-1},\mathbb{Q}$ were injective, this would follow from (*). Of course this is not the case, but the push-forward of equivariant cohomology rings is injective. That is, the “big” torus $T = C^* \times C^*$ acting diagonally on $C^n$, with $H^*(BT,\mathbb{Q}) = \mathbb{Q}[\lambda_1,...,\lambda_n]$ induces an action of the product $C^* \times T$ on each of the spaces in (†). All the maps are equivariant for this product action, and it follows from the localization theorem that the Plücker embedding induces an injective map:

$$\wedge^r : H^r_T(G(r,n),\mathbb{Q}) \to H^r_T(P^n_r)^{-1},\mathbb{Q}$$

in the equivariant cohomology rings (for the $T$-action), and that moreover:

$$(†)_T q_{T*} \frac{1}{e_{C^* \times T}(F_0)} = \sum_{\{d_i\}} p_{T*} \frac{1}{e_{C^* \times T}(i_{\{d_i\}}(Fl))} \in H^r_T(G(r,n),\mathbb{Q})$$

for the $C^*$-equivariant Euler classes with values in $T$-equivariant cohomology, and $T$-equivariant push-forwards $p_{T*}$ and $q_{T*}$. Thus:

$$ev_{T*} \frac{1}{e_{C^* \times T}(F_0)} = \sum_{\{d_i\}} p_{T*} \frac{1}{e_{C^* \times T}(i_{\{d_i\}}(Fl))}$$

and since

$$\lim_{\lambda_i \to 0} ev_{T*} \frac{1}{e_{C^* \times T}(F_0)} = ev_* \frac{1}{e_{C^*}(F_0)} = J_d(h)$$

and likewise for each $p_{T*}(1/e_{C^* \times T}(i_{\{d_i\}}(Fl)))$, the Lemma follows.

Thus we need to compute the equivariant Euler classes:

$$e_{C^*}(i_{\{d_i\}}(Fl))$$

for each of the embeddings $i_{\{d_i\}} : Fl(m_1,...,m_k,S) \to Quot_{P^n_d}(C^n,n-r)$. 

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Let:

$$x_{m_i-1+s} \text{ for } i = 1, ..., k + 1 \text{ and } s = 1, ..., m_i - m_{i-1}$$

denote the Chern roots of the universal vector bundles $$(S_{m_i}/S_{m_{i-1}})^*$$ on the flag bundle (recall that in our notation $m_0 = 0$, $m_{k+1} = r$, and $S_{m_{k+1}} = \rho^*(S)$, with $\rho : Fl(m_1, ..., m_k, S) \to G(r, n)$ the projection).

That is, the $x_{m_i-1+s}$ (formally) factor the Chern polynomials:

$$\prod_{s=1}^{m_i-m_{i-1}} (t + x_{m_i-1+s}) = c_t((S_{m_i}/S_{m_{i-1}})^*).$$

and in particular the $x_1, ..., x_r$ are the Chern roots of $\rho^*(S^*)$.

Recall that $S^* \otimes Q$ is the tangent bundle to the Grassmannian, where:

$$0 \to S \to \mathbb{C}^n \otimes O_G \to Q \to 0$$

is the universal sequence of bundles on $G(r, n)$ and similarly, $\pi_*(K^* \otimes Q)$ is the tangent bundle to $\text{Quot}_{\mathbb{P}^1, d}(\mathbb{C}^n, n - r)$, where

$$0 \to K \to \mathbb{C}^n \otimes O_{\mathbb{P}^1 \times \text{Quot}} \to Q \to 0$$

is the universal sequence of sheaves on $\mathbb{P}^1 \times \text{Quot}_{\mathbb{P}^1, d}(\mathbb{C}^n, n - r)$.

From these, we obtain “Euler sequences”:

$$0 \to S^* \otimes S \to S^* \otimes \mathbb{C}^n \to TG(r, n) \to 0$$

and

$$0 \to \pi_*(K^* \otimes K) \to \pi_*(K^* \otimes \mathbb{C}^n) \to T\text{Quot} \to R^1 \pi_*(K^* \otimes K) \to 0$$

As for the flag bundle $Fl(m_1, ..., m_k, S)$, its tangent bundle is given by:

$$0 \to K \to \rho^*S^* \otimes \mathbb{C}^n \to TFl(m_1, ..., m_k, S) \to 0$$

where $K$ has an increasing filtration $0 = K_0 \subset ... \subset K_k \subset K_{k+1} = K$ with $K_i/K_{i-1} \cong (S_{m_i}/S_{m_{i-1}})^* \otimes S_{m_i}$, and the quotients $K_i/K_{i-1}$ filter further:

$$(S_{m_i}/S_{m_{i-1}})^* \otimes S_{m_i} \subset ... \subset (S_{m_i}/S_{m_{i-1}})^* \otimes S_{m_{i-1}} \subset (S_{m_i}/S_{m_{i-1}})^* \otimes S_{m_i}$$

so that in the Grothendieck group of sheaves on $Fl(m_1, ..., m_k, S)$:

$$[TFl] \sim n[\rho^*S^*] - \sum_{i \leq j} [(S_{m_j}/S_{m_{j-1}})^* \otimes (S_{m_i}/S_{m_{i-1}})]$$

The restriction of $K$ from $\mathbb{P}^1 \times \text{Quot}$ to $\mathbb{P}^1 \times Fl$ (also denoted by $K$) also filters by $K_i/S_{m_{i-1}}^{(i-1)}$ in the proof of Lemma 1.1 with

$$K_i/K_{i-1} \equiv \pi^*(S_{m_i}/S_{m_{i-1}})(-d_{m_i})$$

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Lemma 1.4:

To complete the calculations, we need to compute the push-forwards.

and now we can compute

so that in the Grothendieck group of sheaves on \( \mathbb{P}^1 \times Fl \):

\[
[K^* \otimes K] \sim \sum_{i,j=1}^{k+1} [\pi^*((S_{m_i}/S_{m_i-1})^* \otimes (S_{m_j}/S_{m_j-1})) (d_{m_i} - d_{m_j})]
\]

and now we can compute \( \epsilon_{C^*}(i_{(d_i)} Fl) = c^*_\text{top}(T \text{Quot}|_{i_{(d_i)} Fl}/TFl) \) using

\[
(\ast \ast) \quad c^*_\text{top}(T \text{Quot}|_{i_{(d_i)} Fl}/TFl) = \frac{c^*_\text{top}(\pi_*K^*/\pi_*S^*)^n \cdot c^*_\text{top}(R^1\pi_*(K^* \otimes K))}{c^*_\text{top}(\pi_*(K^* \otimes K)/K)}
\]

and replacing \( K \) and \( K \) by their equivalent forms in the Grothendieck group.

Note that the ratio has many cancellations:

\[
(b)/(c) = \frac{(-1)^{s_f(d-1)}}{\prod_{i,j}(y_j - x_i)(y_j - x_i + dh)}
\]

Putting all of this together, we obtain:

\[
\epsilon_{C^*}(i_{(d_i)} Fl) = \frac{\prod_{i=1}^{k+1} \prod_{s=1}^{r_i} \prod_{t=1}^{d_{m_i}} (x_{m_i+s} + dh)^n}{\prod_{1 \leq j < i \leq k+1} (-1)^{r_j(r_j+1)/2} \prod_{s,t}(x_{m_j+s} - x_{m_j+t} + d_{ij} h)}
\]

where \( d_{ij} = d_{m_i} - d_{m_j} \) and \( r_i = m_i - m_{i-1} \), and with Lemma 1.3:

\[
J_{d}(h) = \sum_{\{d_{i}\}} \rho_\ast \left( \frac{\prod_{1 \leq j < i \leq k+1} (-1)^{r_j(r_j+1)/2} \prod_{s,t}(x_{m_j+s} - x_{m_j+t} + d_{ij} h)}{\prod_{i=1}^{k+1} \prod_{s=1}^{r_i} \prod_{t=1}^{d_{m_i}} (x_{m_i+s} + dh)^n} \right)
\]

To complete the calculations, we need to compute the push-forwards.

The following lemma is a special case of a general formula of Brion ([Bri]):

**Lemma 1.4:** Let \( S \) be a rank \( r \) vector bundle on a smooth variety \( X \), let \( F := Fl(m_1, \ldots, m_k, S) \) be the associated flag bundle with projection \( \rho : F \to X \), and let \( x_1, \ldots, x_r \) denote the Chern roots of \( S^* \) (and of \( \rho^*(S^*) \)). Let \( \rho_\ast \) be the
push-forward map on Chow groups (or rational (co)homology). Then for any polynomial $P \in \mathbb{Q}[X_1, \ldots, X_r]$,
\[
\rho_* P(x_1, \ldots, x_r) = \sum_w w \left[ \prod_{1 \leq j < i \leq k+1} \frac{P(x_1, \ldots, x_r)}{\prod_{s, t} (x_{m_{i-1}+t} - x_{m_{j-1}+s})} \right],
\]
the sum over cosets $w \in S_r/(S_r \times \cdots \times S_{r+k+1})$, where $S_r$ is the symmetric group, and the coset representatives are chosen to be the permutations $w \in S_r$ with descents in $\{m_1, \ldots, m_k\}$.

When we apply the Lemma to the formula for $J_d$, we obtain:
\[
J_d = \sum_{\{d_i\}} \sum_w \left[ \frac{(-1)^{\sum_{j<i} r_j r_i d_{ij}} \prod_{j<i} \prod_{s, t} (x_{m_{i-1}+s} - x_{m_{j-1}+t} + d_{ij} h)}{\prod_{j<i} \prod_{s, t} (x_{m_{i-1}+s} - x_{m_{j-1}+t}) \prod_{i} \prod_{l=1}^{d_{ii}} (x_{m_{i-1}+s} + l h)^n} \right] \]

Notice first that the sign of each term in the sum depends only on $r$ and $d$. Indeed,
\[
\sum_{1 \leq j < i \leq k+1} r_i r_j (d_{mi} - d_{mj}) \equiv (r - 1) d \pmod{2}
\]

Second, we can simplify the formula for $J_d$ by replacing the double sum with a single sum over $r$-tuples $(d_1, \ldots, d_r)$ giving a partition $d_1 + \ldots + d_r = d$. Namely, given such an $r$-tuple, let $k+1$ be the number of distinct $d_i$’s, with $r_1$ the multiplicity of the smallest part, $r_2$ the multiplicity of the next smallest, etc.

Then there is an unique $w \in S_r/(S_{r_1} \times \cdots \times S_{r_{k+1}})$ whose inverse $w^{-1}$ arranges $(d_1, \ldots, d_r)$ in nondecreasing order $d_1 \leq d_2 \leq \ldots \leq d_r$ and we have:
\[
\frac{\prod_{1 \leq j < i \leq r} (x_i - x_j + (d_i - d_j) h)}{\prod_{1 \leq j < i \leq r} (x_i - x_j) \prod_{i=1}^{r-1} \prod_{l=1}^{d_{ii}} (x_i + l h)^n} = w \left[ \frac{\prod_{1 \leq j < i \leq k+1} \prod_{s, t} (x_{m_{i-1}+s} - x_{m_{j-1}+t} + d_{ij} h)}{\prod_{1 \leq j < i \leq k+1} \prod_{s, t} (x_{m_{i-1}+s} - x_{m_{j-1}+t}) \prod_{i=1}^{k+1} \prod_{s=1}^{d_{ii}} (x_{m_{i-1}+s} + l h)^n} \right],
\]
so that, putting all this together, we arrive at:

**Theorem 1.5:** The $J$-function of the Grassmannian $G = G(r, n)$ is
\[
J^G = e^{\frac{r+1}{2}} \sum_{d \geq 0} e^{dt} J_d(h), \text{ where}
\]
\[
J_d(h) = (-1)^{(r-1)d} \sum_{(d_1, \ldots, d_r)} \frac{\prod_{1 \leq j < i \leq r} (x_i - x_j + (d_i - d_j) h)}{\prod_{1 \leq j < i \leq r} (x_i - x_j) \prod_{i=1}^{r} \prod_{l=1}^{d_{ii}} (x_i + l h)^n},
\]
and $x_1, \ldots, x_r$ are the Chern roots of $S^*$, the dual of the tautological subbundle.
Remark: Theorem 1.5 specializes to Givental’s formula for $\mathbb{P}^{n-1} \cong G(1, n)$:

$$J^{\mathbb{P}^{n-1}} = e^\frac{x}{\kappa} \sum_{d \geq 0} \frac{e^{dt}}{\prod_{l=1}^{d}(x + lh)^n}$$

**First Proof of the Hori-Vafa Conjecture:** Simply compute:

$$J^P = \prod_{i=1}^{r} J^{\mathbb{P}^{n-1}} = e^\frac{t_1 + \ldots + t_r}{\kappa} \sum_{(d_1, \ldots, d_r)} \frac{e^{d_1 t_1 + \ldots + d_r t_r}}{\prod_{i=1}^{r} \prod_{l=1}^{d_i}(x_i + lh)^n}$$

so

$$\frac{D_\Delta (J^P)}{\Delta} = e^\frac{t_1 + \ldots + t_r}{\kappa} \sum_{(d_1, \ldots, d_r)} \frac{e^{d_1 t_1 + \ldots + d_r t_r}}{\prod_{i=1}^{r} \prod_{l=1}^{d_i}(x_i + lh)^n} \prod_{1 \leq i < j \leq r} (x_i - x_j + (d_i - d_j)h),$$

and then the conjecture immediately follows.

Finally, there is an analogous formula for the $J$-function in $T$-equivariant cohomology (the action of $T = (\mathbb{C}^*)^n$ was described in the proof Lemma 1.3):

$$J^T_d(h, \lambda_1, \ldots, \lambda_n) = \frac{1}{e^{c_{T \times T}(F_0)}} \in H^*_T(G(r, n), \mathbb{Q}).$$

Indeed, a computation analogous to the proof of Theorem 1.5 gives:

**Theorem 1.5':** The equivariant $J$-function of the Grassmannian $G(r, n)$ is

$$J^{G} = e^\frac{t_d}{\kappa} \sum_{d \geq 0} e^{dt} J^T_d,$$

where

$$J^T_d = (-1)^{(r-1)d} \sum_{(d_1, \ldots, d_r) \in \mathbb{Z}_+^{r}} \frac{\prod_{1 \leq i < j \leq r} (x_i - x_j + (d_i - d_j)h)}{\prod_{i=1}^{r} \prod_{l=1}^{d_i}(x_i - \lambda_j + lh)}.$$

The substitution of $\prod_{l=1}^{d_i} \prod_{j=1}^{n}(x_i - \lambda_j + lh)$ in place of $\prod_{l=1}^{d_i}(x_i + lh)^n$ corresponds to taking into account the $T$-action on the trivial bundles $\mathbb{C}^n$ in the Euler sequences. Under this action, the trivial bundle decomposes into a direct sum of representations $\mathbb{C}^n = \bigoplus_{i=1}^{n} C_i$, with $c^T_i (C_i) = -\lambda_i$. Consequently, the $T$-equivariant version of formula (***) is

$$c_{top}(TQuot/TFl) = \frac{(\prod_{i=1}^{n} c_{top}(\pi_* (K^* / n^* S^*) \otimes C_i)) \cdot c_{top}(R^1 \pi_* (K^* \otimes K))}{c_{top}(\pi_* (K^* \otimes K) / K)},$$

where the Chern classes are now $\mathbb{C}^* \times T$-equivariant.
§2. A Quantum Cohomology Proof of the Hori-Vafa Conjecture. This proof follows from “quantum” versions of some results of [ES] and [Mar], on the classical cohomology. Consider the rational map:

\[ \Phi : P = \prod_{i=1}^r P^{n-1} \dashrightarrow G; (p_1, ..., p_r) \mapsto \text{span}\{p_1, ..., p_r\} \subset C^n \]

Recall the presentation of the cohomology ring of \( G \):

\[ H^*(G) \cong C[x_1, ..., x_r]^{S_r}/(h_{n-r+1}, ..., h_n), \]

and the presentation of the cohomology ring of \( P \):

\[ H^*(P) \cong C[x_1, ..., x_r]/(x_1^n, ..., x_r^n). \]

Thus each class \( \gamma \in H^*(G) \) together with a symmetric polynomial \( P(x_1, ..., x_r) \) representing \( \gamma \) lifts to \( \tilde{\gamma} \in H^*(P) \) by evaluating \( P \) in \( H^*(P) \).

For a partition \( \mu = (n-r \geq \mu_1 \geq ... \geq \mu_r \geq 0) \) one defines \( \sigma_\mu \) to be the corresponding Schur polynomial:

\[ \sigma_\mu = \det \begin{pmatrix} x_1^{\mu_1+r-1} & x_1^{\mu_2+r-2} & ... & x_1^{\mu_r} \\ x_2^{\mu_1+r-1} & x_2^{\mu_2+r-2} & ... & x_2^{\mu_r} \\ \vdots & \vdots & \ddots & \vdots \\ x_r^{\mu_1+r-1} & x_r^{\mu_2+r-2} & ... & x_r^{\mu_r} \end{pmatrix} \]

\[ \det \begin{pmatrix} x_1^{r-1} & x_1^{r-2} & ... & 1 \\ x_2^{r-1} & x_2^{r-2} & ... & 1 \\ \vdots & \vdots & \ddots & \vdots \\ x_r^{r-1} & x_r^{r-2} & ... & 1 \end{pmatrix} \]

In the cohomology of \( G \) this represents a “Schubert” class, and such classes form an additive basis of \( H^*(G) \) as \( \mu \) runs over the set of partitions. Since each \( x_i \) appears in the Schur polynomial \( \sigma_\mu \) with exponent at most \( n-r \), the lift of a Schubert class to \( H^*(P) \) determines the Schur polynomial. We will often identify a partition with its Young diagram. The partitions corresponding to Schubert classes on \( G \) are those whose Young diagrams fit in an \( r \times (n-r) \) rectangle.

Finally, recall that the denominator of \( \sigma_\mu \) is the Vandermonde determinant:

\[ \Delta = \prod_{1 \leq i < j \leq r} (x_i - x_j) \]

The relation between the cohomology of \( G \) and that of \( P \) is encoded in the following “integration formula”
**Theorem 2.1 (Martin):** For any cohomology class $\gamma \in H^*(G)$,
\[
\int_G \gamma = (-1)^{\frac{r(r-1)}{2}} r! \int_P \tilde{\gamma} \cup P \Delta^2.
\]

**Corollary 2.2 (Ellingsrud-Strømme):** The linear map
\[
\theta : H^*(G) \rightarrow H^*(P); \quad \theta(\gamma) = \tilde{\gamma} \cup P \Delta
\]
is injective, and its image $V \subset H^*(P)$ is the subspace of anti-symmetric classes.

**Proof:** If $\theta(\gamma) = 0$, then
\[
\int_P (\tilde{\gamma} \cup P \Delta) \cup P b = 0, \quad \forall \ b \in H^*(P).
\]
In particular
\[
\int_G \gamma \cup_G \gamma' = \int_P (\tilde{\gamma} \cup P \Delta) \cup_P (\tilde{\gamma}' \cup P \Delta) = 0, \quad \forall \ \gamma' \in H^*(G),
\]
by the integration formula. Hence $\gamma = 0$. Finally, it is evident that each class $\tilde{\gamma} \cup_P \Delta$ in the image of $\theta$ is anti-symmetric, but conversely, an anti-symmetric class is always of the form $\tilde{\gamma} \cup_P \Delta$, with $\tilde{\gamma} \in \mathbb{C}[x_1, ..., x_r]^{S_r}$.

Thus $V = \text{span} \{ \{ \sigma_{\mu} \cup_P \Delta \mid \mu = (n-r \geq \mu_1 \geq \ldots \geq \mu_r \geq 0) \} \}$. In addition, let $W = V^\perp$ be its orthogonal complement with respect to the intersection form. Note that the polynomial $\sigma_{\mu} \Delta$ represents the class $\sigma_{\mu} \cup_P \Delta$.

If $\tilde{\mu}$ is the dual partition (the complement of $\mu$ in the $r \times (n-r)$ rectangle), then it follows from the integration formula that
\[
\int_P (\sigma_{\mu} \Delta) \cup (\sigma_{\nu} \Delta) = (-1)^{\frac{r(r-1)}{2}} r! \delta_{\nu \tilde{\mu}}
\]
Choose a basis $\{b_i\}$ of $W$ that extends $\{\sigma_{\mu} \Delta\}$ to a basis $\mathcal{B}$ of $H^*(P)$ with intersection form $((-1)^{\frac{r(r-1)}{2}} r!) \text{Id}$ and let
\[
p_V : H^*(P) \rightarrow V
\]
be the orthogonal projection onto $V$.

**Corollary 2.3.** For any two partitions $\mu, \nu$,
\[
\theta(\sigma_{\mu} \cup_G \sigma_{\nu}) = \theta(\sigma_{\mu}) \cup_P \sigma_{\nu}.
\]

**Proof:** Since the right hand side is antisymmetric, it lies in $V$, hence
\[
\theta(\sigma_{\mu}) \cup_P \sigma_{\nu} = \sum_\rho \left( \int_P (\sigma_{\mu} \Delta) \cup \sigma_{\nu} \cup (\sigma_{\rho} \Delta) \right) \frac{(-1)^{\frac{r(r-1)}{2}} r!}{r!} \sigma_{\rho} \Delta
\]
so continuing with Martin’s formula gives:

$$\theta(\sigma_\mu) \cup_p \sigma_\nu = \sum_\rho \left( \int_G \sigma_\mu \cup \sigma_\nu \cup \sigma_\rho \right) \theta(\sigma_\rho) = \theta(\sigma_\mu \cup_G \sigma_\nu).$$

This corollary has a natural generalization to quantum cohomology.

Recall that the 3-point genus zero Gromov-Witten invariants of a smooth projective variety $X$ define the small quantum cohomology ring as a deformation of the usual cup-product on $X$. Specifically, if $\{\gamma_i\}_{i=1}^n$ is a $\mathbb{C}$-basis of $H^2(X)$ and $\{\tilde{\gamma}_i\}_{i=1}^n$ is the dual basis (with respect to the intersection form) then the quantum product of two cohomology classes is:

$$\gamma \ast \delta = \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_k e^\beta \langle \gamma, \delta, \gamma_k \rangle_X \tilde{\gamma}_k,$$

We think of $\beta$ either as the class of an algebraic curve, and compute:

$$\langle \gamma, \gamma', \gamma'' \rangle_X = \int_{\overline{M}_{0,3}(X,\beta)} e_1(\gamma) \cup e_2(\gamma') \cup e_3(\gamma'')$$

where

$$ev_i: \overline{M}_{0,n}(X,\beta) \to X; \ ev([f]) = f(p_i)$$

and $[\overline{M}_{0,3}(X,\beta)] \in A_*(\overline{M}_{0,3}(X,\beta))\mathbb{Q}$ is the “virtual fundamental class” in the Chow group of the Kontsevich-Manin stack of stable maps (see [BF]). Otherwise, we think of $\beta = \sum_{i=1}^m d_i t_i$ as an element of the dual space to $H^2(X,\mathbb{C})$ so that:

$$e^\beta = e^{d_1 t_1 + \ldots + d_m t_m}$$

If we assume that $H^2(X)$ has a basis consisting of nef divisors, this deformation defines small quantum cohomology as an algebra over $\mathbb{C}[[e^{t_1}, \ldots, e^{t_m}]]$ ($m = \dim(H^2(X))$. In the cases $X = G$ and $X = P$, set:

$$QH^*(G) := H^*(G) \otimes_{\mathbb{C}} \mathbb{C}[e^t]$$

with multiplication defined on the Schubert basis by

$$\sigma_\mu \ast_G \sigma_\nu = \sum_{d \geq 0} \sum_\rho e^{dt}(\sigma_\mu, \sigma_\nu, \sigma_\rho)_d^G \sigma_\rho$$

and similarly:

$$QH^*(P) = H^*(P) \otimes_{\mathbb{C}} \mathbb{C}[e^{t_1}, \ldots, e^{t_r}]$$

with multiplication given on a basis $\{\gamma_K\}$ of $H^*(P)$ by

$$\gamma_I \ast_P \gamma_J = \sum_{d_1, \ldots, d_r \geq 0} \sum_K e^{d_1 t_1 + \ldots + d_r t_r} \langle \gamma_I, \gamma_J, \tilde{\gamma}_K \rangle \langle d_1, \ldots, d_r \rangle \tilde{\gamma}_K.$$
where \((d_1, ..., d_r)\) is the multi-degree of a curve in \(P = (P^{n-1})^r\).

We’ve already mentioned the well-known presentations:

\[
\mathcal{Q}H^*(G) \cong \mathbb{C}[x_1, ..., x_r] \otimes_{\mathbb{C}} \mathbb{C}(e^t)/\langle h_{n-r+1}, ..., h_n - (-1)^{r-1} e^t \rangle,
\]
and:

\[
\mathcal{Q}H^*(P) \cong \mathbb{C}[x_1, ..., x_r][e^{t_1}, ..., e^{t_r}]/\langle x_{1}^{n} - e^{t_1}, ..., x_{r}^{n} - e^{t_r} \rangle.
\]

We will use two bases for \(\mathcal{Q}H^*(P)\) (over \(\mathbb{C}[e^{t_1}, ..., e^{t_r}]\)). One is the basis \(B\) above. The other is the natural one consisting of monomials \(X^I = x_{i_1}^{j_1} \cdots x_{i_r}^{j_r}\), \(0 \leq j_i \leq n - 1\). In the latter basis, the quantum product \(X^I \cdot X^J\) is obtained simply by multiplying \(X^I \cdot X^J\) and then substituting:

\[x_i^n = e^{t_i}\]

As a consequence of this we have

**Lemma 2.4:** For any partition \(\mu = (n - r \geq \mu_1 \geq \mu_2 \geq ... \geq \mu_r \geq 0)\)

\[\sigma_\mu \ast \mathfrak{P} \Delta = \sigma_\mu \cup \mathfrak{P} \Delta = \sigma_\mu \Delta.\]

**Proof:** The highest power of each \(x_i\) that appears in \(\sigma_\mu \Delta\) is at most \(n - 1\), as is evident from the Schur polynomial.

The action of the symmetric group \(S_r\) on polynomials in the \(x_i\)’s and \(e^{t_i}\)’s gives an action on \(\mathcal{Q}H^*(P)\). Let \(\mathcal{Q}H^*_{e^{t}}(P)\) denote the quotient obtained by substituting

\[e^{t_i} = (-1)^{r-1} e^{t_i}; \quad i = 1, ..., r\]

The \(S_r\)-action descends to an action on \(\mathcal{Q}H^*_{e^{t}}(P)\) (permuting only the \(x_i\)’s) and the anti-invariants are \(V \otimes \mathbb{C}[e^t]\). Now extend \(\theta\) by linearity over \(\mathbb{C}[e^t]\) to:

\[\overline{\theta}: \mathcal{Q}H^*(G) \longrightarrow \mathcal{Q}H^*_{e^{t}}(P),\]

which obviously continues to be injective with image equal to \(V \otimes \mathbb{C}[e^t]\).

The quantum analogue of Corollary 2.3 is the main result of this section:

**Theorem 2.5:** For any two partitions \(\mu, \nu\):

\[\overline{\theta}(\sigma_\mu \ast \sigma_\nu) = (\theta(\sigma_\mu) \ast \sigma_\nu)|_{e^t = (-1)^{r-1} e^t}.\]

In the spirit of this paper we will give two proofs.

**First Proof:** For a sequence of non-negative integers \(\alpha = (\alpha_1, \alpha_2, ..., \alpha_r)\) define \(D_\alpha = \text{det}(x_{i,j}^{n})_{1 \leq i, j \leq r}\) so if we let \(\delta = (r - 1, r - 2, ..., 0)\), then:

\[D_\delta = \Delta, \quad \sigma_\mu(x_1, ..., x_r) = \frac{D_{\mu+\delta}}{D_\delta} \quad \text{for all partitions } \mu,\]
and in general \( D_\alpha \neq 0 \) if and only if the \( \alpha_j \)'s are all distinct.

The product of two Schur polynomials in \( r \) variables is given by

\[
\sigma_\mu \sigma_\nu = \sum_\rho N^\rho_{\mu,\nu} \sigma_\rho,
\]

(1)

where \( N^\rho_{\mu,\nu} \) are the Littlewood-Richardson coefficients and \( \rho \) is constrained to be of the form \( \rho_1 \geq \rho_2 \geq \ldots \geq \rho_r \geq 0 \), but we do not require \( \rho_1 \leq n-r \). Thus:

\[
\sigma_\mu \Delta \sigma_\nu = \sum_\rho N^\rho_{\mu,\nu} D^\rho \delta.
\]

(2)

Let

\[
I_G = \langle h_{r+1}, \ldots, h_n - (-1)^{r-1} e^t \rangle
\]

and

\[
I_P = \langle x_1^n - (-1)^{r-1} e^t, \ldots, x_r^n - (-1)^{r-1} e^t \rangle
\]

be the ideals of relations from the presentations of \( QH^*(G) \) and of \( \overline{QH}^*(P) \). Then \( \theta(\sigma_\mu \ast_G \sigma_\nu) \) is obtained by reducing the right-hand side of (1) modulo \( I_G \) and multiplying the result by \( \Delta \), while \( \theta(\sigma_\mu \ast_P \sigma_\nu) \) is obtained by reducing the right-hand side of (2) modulo \( I_P \).

The reduction of a Schur polynomial modulo \( I_G \) can be done using the rim-hook algorithm of [BCF], which goes as follows. The rim of a partition \( \rho = (\rho_1, \ldots, \rho_r) \) consists of the boxes on the border of its Young diagram from the northeast (upper-right) to the southwest (lower-left) corners. An \( n \)-rim hook is a contiguous collection of \( n \)-boxes in the rim with the property that when removed from the Young diagram of \( \rho \) the remaining shape is again the diagram of a partition. If we start at the end of any row and count \( n \) boxes along the rim, moving down and to the left, then we either obtain an \( n \)-rim hook, or the process ends in a box directly above the last box in some row. In the latter case, the set of \( n \) boxes is called an illegal \( n \)-rim. With these definitions, the Main Lemma from [BCF] gives:

(i) If \( \rho_1 > n-r \) and either \( \rho \) contains no \( n \)-rim hook, or \( \rho \) contains an illegal \( n \)-rim, then:

\[
\sigma_\rho \equiv 0 \pmod{I_G}
\]

(ii) If \( \pi = (\pi_1 \geq \pi_2 \geq \ldots \geq \pi_r \geq 0) \) is obtained from \( \rho \) by removing an \( n \)-rim hook, then

\[
\sigma_\rho \equiv (-1)^{r-s} e^t \sigma_\pi \pmod{I_G}
\]

where \( s \) is the height of the rim hook, i.e., the number of rows it occupies.

On the other hand, it is obvious that if \( \alpha_j \geq n \), then

\[
D_\alpha \equiv (-1)^{r-1} e^t D_\alpha' \pmod{I_P},
\]

(3)
where $\alpha' = (\alpha_1, \ldots, \alpha_j - n, \ldots, \alpha_r)$. It is easy to see that if $\rho_1 > n - r$ and $\rho$ contains no $n$-rim, then $\rho + \delta - (n, 0, \ldots, 0)$ has two equal entries, hence $D_{\rho + \delta} \equiv 0 \pmod{I_P}$. Assume now that $\rho$ has an $n$-rim, starting in row $j$ and ending in row $k$ for some $1 \leq j < k \leq r$ (so that its height is $s := k - j + 1$). Then we must have $\rho_j + \delta_j > n$. Moreover, if the rim is illegal, then $k < r$ and $\rho_j = \rho_{k+1} + (n - (k + 1) + j).

This implies that $\rho_j + \delta_j - n = \rho_{k+1} + \delta_{k+1},$

so again $D_{\rho + \delta} \equiv 0 \pmod{I_P}$.

Finally, if the $n$-rim is a rim hook, let $\pi$ be the partition obtained by removing it from $\rho$. Then one gets $\pi + \delta$ from $\rho + \delta - (0, \ldots, 0, n, 0, \ldots, 0)$ by taking its $j$th entry $\rho_j + \delta_j - n$ and moving it past the next $s - 1$ entries. This gives

$$D_{\rho + \delta} = (-1)^{s-1} (-1)^{r-1} e^t D_{\pi + \delta} \equiv (-1)^{r-s} e^t D_{\pi + \delta} \pmod{I_P},$$

and finishes the first proof of the Theorem.

**Second Proof of Theorem 2.5:** In the basis $B = \{\sigma, \Delta\} \cup \{b_i\}$:

$$\theta(\sigma_{\mu}) \ast_P \sigma_{\nu} = \frac{(-1)^{\binom{n}{2}}}{r!} \left( \sum_{\rho, \{d_i\}} e^{\sum_{t=1}^r d_t \sigma_{\mu} \Delta} \sum_{\rho', \{d'_i\}} e^{\sum_{t=1}^r d'_t \sigma_{\nu} \Delta} \langle \sigma_{\mu} \Delta, \sigma_{\nu} \Delta \rangle_{\{d_i\}} \sigma_{\rho' \Delta} \right)$$

$$+ \sum_{j, \{d_i\}} e^{\sum_{t=1}^r d_t \sigma_{\mu} \Delta} \langle \sigma_{\mu} \Delta, \sigma_{\nu}, b_j \rangle_{\{d_i\}} \sigma_{\rho' \Delta}$$

After substituting for the $e^t$’s this becomes

$$\frac{(-1)^{\binom{n}{2}}}{r!} \left( \sum_{\rho, d} \sum_{d_1 + \ldots + d_r = d} (-1)^{d(r-1)} e^{dt} \langle \sigma_{\mu} \Delta, \sigma_{\nu}, \sigma_{\rho' \Delta} \rangle_{\{d_i\}} \sigma_{\rho \Delta} \right)$$

$$+ \sum_{j, d} \sum_{d_1 + \ldots + d_r = d} (-1)^{d(r-1)} e^{dt} \langle \sigma_{\mu} \Delta, \sigma_{\nu}, b_j \rangle_{\{d_i\}} \sigma_{\rho \Delta} \right).$$

This is antisymmetric in the $x_i$’s, so each:

$$\sum_{d_1 + \ldots + d_r = d} \langle \sigma_{\mu} \Delta, \sigma_{\nu}, b_j \rangle_{\{d_i\}} = 0$$
Moreover, equating the coefficients for the basis elements $\sigma_\beta \Delta$ gives us the following equivalent formulation of Theorem 2.5:

$$\langle \sigma_\mu, \sigma_\nu, \sigma_\rho \rangle^G_d = \frac{(-1)^{\frac{r^2}{2}}}{r!} \sum_{d_1 + \ldots + d_r = d} (-1)^{d(r-1)} \langle \sigma_\mu \Delta, \sigma_\nu, \sigma_\rho \Delta \rangle^P_{\{d_i\}}$$  \hspace{1cm} (3)$$

which is a “quantum” analogue of the integration formula (Theorem 2.1).

This can be proved directly with the Vafa - Intriligator residue formula. Given classes $\sigma_\mu, \sigma_\nu, \sigma_\rho$ with $|\mu| + |\nu| + |\rho| = \dim(G)$, then the “classical” version of the formula gives:

$$\int_G \sigma_\mu \sigma_\nu \sigma_\rho = \frac{(-1)^{\frac{r^2}{2}}}{r! n^r} \sum_{\{\epsilon_1, \ldots, \epsilon_r\}} \sigma_\mu \sigma_\nu \sigma_\rho \prod_{i} \epsilon_i \prod_{i \neq j} (\epsilon_i - \epsilon_j)$$

where $\sigma_\mu = \sigma_\mu(\epsilon_1, \ldots, \epsilon_r)$ on the right side, and likewise for $\sigma_\nu, \sigma_\rho$. The quantum formula is essentially identical. If $|\mu| + |\nu| + |\rho| = nd + \dim(G)$ then:

$$\langle \sigma_\mu, \sigma_\nu, \sigma_\rho \rangle^G_d = \frac{(-1)^{\frac{r^2}{2}}}{r! n^r} \sum_{\{\epsilon_1, \ldots, \epsilon_r\}} \sigma_\mu \sigma_\nu \sigma_\rho \prod_{i} \epsilon_i \prod_{i \neq j} (\epsilon_i - \epsilon_j)$$

This formula is proved (see [Ber1]) by taking the potential function:

$$P = \sum_{i=1}^{r} \frac{x_i^{n+1}}{n+1} - (-1)^{r-1} x_i$$

satisfying $\langle \frac{\partial P}{\partial \sigma_1}, \ldots, \frac{\partial P}{\partial \sigma_n} \rangle = I_G$ and summing over the residues of the $r$-form:

$$\sigma_\mu \sigma_\nu \sigma_\rho \frac{d\sigma_1 \wedge \ldots \wedge d\sigma_r}{\prod_i \frac{\partial P}{\partial \sigma_i}}$$

On the other hand, this same $P$ satisfies $\langle \frac{\partial P}{\partial x_1}, \ldots, \frac{\partial P}{\partial x_n} \rangle = I_P$, and it follows from Lemma 2.4 and summing over residues of the $r$-form:

$$\sigma_\mu \Delta \sigma_\nu \sigma_\rho \Delta \frac{dx_1 \wedge \ldots \wedge dx_r}{\prod_i \frac{\partial P}{\partial x_i}}$$

(just as in [Ber1]) that:

$$\sum_{d_1 + \ldots + d_r = d} \langle \sigma_\mu \Delta, \sigma_\nu, \sigma_\rho \Delta \rangle^P_{\{d_i\}} = \frac{(-1)^{\frac{r^2}{2}}}{n^r} \sum_{\{\epsilon_1, \ldots, \epsilon_r\}} \sigma_\mu \Delta \sigma_\nu \sigma_\rho \Delta \prod_{i} \epsilon_i$$

and this, plus $\Delta^2 = \prod_{i \neq j} (x_i - x_j)$, proves the Theorem.
We next want to consider the flat sections for $P$, that is, solutions to:

$$h \frac{\partial}{\partial t_i} f = x_i \ast_P f, \quad i = 1, \ldots, r.$$ 

with

$$f(t_1, \ldots, t_r, \bar{h}) = \sum_{\mu} f_{\mu}(\sigma_{\mu} \Delta) + \sum_j f_j b_j$$

a vector field (over $H^2(P)$) written in terms of the basis $B$. (Here, and for the rest of the paper, we abuse the notation slightly by writing vector fields as cohomology-valued, via the canonical identification of the tangent space at a point in $H^*(X)$ with $H^*(X)$ itself.) Let:

$$pV(f) = \sum_{\mu} f_{\mu}(\sigma_{\mu} \Delta) \text{ with } pV(f)^\perp = \sum_j f_j b_j$$

for arbitrary vector fields. Then:

**Lemma 2.6:** For all $\nu = (n - r \geq \nu_1 \geq \ldots \geq \nu_r \geq 0)$ and all $f = f(t_1, \ldots, t_r, \bar{h})$:

$$pV \left( (\sigma_{\nu} \ast_P f) \big|_{t_i = t + \pi \sqrt{-1}(r - 1)} \right) = (\sigma_{\nu} \ast_P pV(f)) \big|_{t_i = t + \pi \sqrt{-1}(r - 1)}$$

**Proof:** We need to show that:

$$pV \left( (\sigma_{\nu} \ast_P pV(f)^\perp) \big|_{t_i = t + \pi \sqrt{-1}(r - 1)} \right) = 0.$$

But $\sigma_{\nu} \ast_P pV(f)^\perp = \sum_j f_j (\sigma_{\nu} \ast_P b_j)$ and after specializing the $e^{t_i}$’s, the coefficient of $\sigma_{\nu} \Delta$ in $\sigma_{\nu} \ast_P b_j$ is

$$\sum_{d_1 + \ldots + d_r = d} \langle \sigma_{\nu}, b_j, \sigma_{\nu} \Delta \rangle_{\{d_i\}}^P$$

which vanishes, as noted earlier.

Now let $f$ be a flat section for $P$. Then:

$$\delta f = \sigma_1 \ast_P f \text{ where } \delta = h \sum_{i=1}^r \frac{\partial}{\partial t_i} \text{ and } \sigma_1 = x_1 + \ldots + x_r$$

Since $pV(\delta f) = \delta pV(f)$, we get $\delta pV(f) = pV(\sigma_1 \ast_P f)$ and:

$$(\delta pV(f))_{|_{t_i = t + \pi \sqrt{-1}(r - 1)}} = (\sigma_1 \ast_P pV(f))_{|_{t_i = t + \pi \sqrt{-1}(r - 1)}}$$

from Lemma 2.6 applied to $\nu = (1 \geq 0 \geq 0 \ldots \geq 0)$. 

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Theorem 2.7: If \( f = \sum_{\mu} f_{\mu}(\sigma_{\mu}\Delta) + \sum_j f_j b_j \) is a flat section for \( P \), then
\[
\left. \frac{pv(f)}{\Delta} \right|_{t_i = t + \pi \sqrt{-1}(r-1)} = \sum_{\mu} \left( f_{\mu}\big|_{t_i = t + \pi \sqrt{-1}(r-1)} \right) \sigma_{\mu}
\]
is a flat section for \( G \).

Proof: We need to show that
\[
\left. \hbar \frac{d}{dt} \left( \frac{pv(f)}{\Delta} \right) \right|_{t_i = t + \pi \sqrt{-1}(r-1)} - \sigma_1 *_{G} \left( \left. \frac{pv(f)}{\Delta} \right|_{t_i = t + \pi \sqrt{-1}(r-1)} \right) = 0.
\]
Think of this as an \( H^*(G) \)-valued function. By Corollary 2.2, it suffices to show
\[
\left( \sigma_1 \Delta \right) = \sigma_1 *_{P} \Delta \quad \text{and}
\]
so that we may rewrite the desired relation once again (using the chain rule) as:
\[
\left( \delta pv(f) \right)|_{t_i = t + \pi \sqrt{-1}(r-1)} - \left( \sigma_1 *_{P} pv(f) \right)|_{t_i = t + \pi \sqrt{-1}(r-1)} = 0
\]
and the validity of this relation, as noted above, follows from Lemma 2.6.

Finally, we are ready for the:

Quantum Cohomology Proof of the Hori-Vafa Conjecture:

The conjecture will follow from Theorem 2.7 and from basic properties of the explicit fundamental matrices of flat sections for \( P \) and \( G \) given by two-point gravitational descendants, which we now recall.

For a smooth projective \( X \), the two-point invariants involved will be of the form
\[
\langle \psi^m \gamma, \gamma' \rangle^X_{\beta} = \int_{\mathcal{M}_{0,2}(X, \beta)} \psi^m ev_1^*(\gamma) \cup ev_2^*(\gamma'),
\]

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where \( \gamma, \gamma' \) are (even) cohomology classes on \( X \) and \( \psi \) is the first Chern class of the line bundle \( L_1 \) on \( \overline{M}_{0,2}(X, \beta) \) with fiber over the point \([C, p_1, p_2, f]\) the cotangent line to \( C \) at \( p_1 \). More formally, \( L_1 = s_1^*(\omega_x) \), with \( s_1 \) the natural section – corresponding to the first marked point – of the projection \( \pi \) from the universal curve to \( \overline{M}_{0,2}(X, \beta) \), and \( \omega_x \) the relative dualizing sheaf. If, as before, \( \{T_i\}_{i=0}^s \) and \( \{\hat{T}_i\}_{i=0}^s \) are dual bases of \( H^{2*}(X) \), with \( \{T_1, \ldots, T_m\} \) a basis of \( H^2(X) \), then for each \( \gamma \in H^{2*}(X) \), the cohomology-valued function

\[
F^X_\gamma(t, e^t, h) := \sum_{k=0}^s \langle \langle \frac{e^{(t_1T_1+\ldots+t_mT_m)/h}}{h-\psi}, T_k \rangle \rangle^X T_k
\]

is a flat section, where we used the customary notation

\[
\langle \langle \frac{e^{(t_1T_1+\ldots+t_mT_m)/h}}{h-\psi}, T_k \rangle \rangle^X = \sum_{\beta} e^\beta \langle \langle \frac{e^{(t_1T_1+\ldots+t_mT_m)/h}}{h-\psi}, T_k \rangle \rangle^X.
\]

When \( \gamma \) runs over the basis \( \{T_i\} \) these sections form the rows of a fundamental matrix of flat sections, and the \( J \)-function of \( X \) is by definition

\[
J^X := \sum_{i=0}^s \left( \int_X F^X_{T_i} \right) \hat{T}_i = \sum_{i=0}^s \langle \langle \frac{e^{(t_1T_1+\ldots+t_mT_m)/h}}{h-\psi}, 1 \rangle \rangle^X T_i.
\]

We now consider these matrices corresponding to the standard basis of \( H^*(G) \), and the basis \( B \) of \( H^*(\mathbb{P}) \). By Theorem 2.7,

\[
\frac{pv(F^P_{\sigma \Delta})}{\Delta} \bigg|_{t_i = t+(r-1)\pi \sqrt{-1}} = \sigma_r e^{(t+(r-1)\pi \sqrt{-1})\sigma_1/h} + o(e^t)
\]

is a flat section for \( G \). However, we also have

\[
F^G_{\sigma' e^{(r-1)\pi \sqrt{-1}} \sigma_1} = \sigma_r e^{(t+(r-1)\pi \sqrt{-1})\sigma_1/h} + o(e^t).
\]

By the uniqueness of flat sections with given intial term in the \( e^t \)-expansion we conclude that, for any two partitions \( \mu, \nu \),

\[
\frac{(-1)^{\ell(\gamma)}}{r!} \langle \langle \sigma_\mu \Delta e^{\sum t_i x_i/h} \sigma_\nu \Delta \rangle \rangle^P \bigg|_{t_i = t+(r-1)\pi \sqrt{-1}} = \langle \langle \sigma_\mu e^{(t+(r-1)\pi \sqrt{-1})\sigma_1/h} \sigma_\nu \rangle \rangle^G.
\]

On the other hand, the same argument as in the proof of Lemma 4.2 of [JK] shows that

\[
\int_{\mathbb{P}} F^P_{\sigma \Delta} \cup \Delta = D_\Delta \int_{\mathbb{P}} F^P_{\sigma \Delta}
\]

(5)
since, for any decomposition \( \Delta = \prod (x_i - x_j) \prod (x_k - x_l) \), there are no quantum correction terms in the product \( \prod (x_i - x_j) \ast \prod (x_k - x_l) \).

Next observe that multiplication by the class \( e^{\sigma_1 (r-1) \pi \sqrt{-1}/\hbar} \) is an invertible linear operator on \( H^* (G) \) (with inverse the operator of multiplication by \( e^{-\sigma_1 (r-1) \pi \sqrt{-1}/\hbar} \)), therefore

\[
e^{\sigma_1 (r-1) \pi \sqrt{-1}/\hbar} G = \sum_\nu \left( \frac{\sigma_\nu e^{(t+(r-1) \pi \sqrt{-1})/\hbar}}{\psi}, 1 \right) G \sigma_\nu.
\]

It follows then from (4) (applied to \( \mu = \) the empty partition) that

\[
e^{\sigma_1 (r-1) \pi \sqrt{-1}/\hbar} G = \sum_\nu \frac{(-1)^{\nu}}{\nu!} \left( \int_P F_{\sigma_\nu \Delta} \right) G \sigma_\nu \bigg|_{t_i = t + (r-1) \pi \sqrt{-1}}.
\]

From this and (5) we get

\[
e^{\sigma_1 (r-1) \pi \sqrt{-1}/\hbar} G = D_\Delta \sum_\nu \frac{(-1)^{\nu}}{\nu!} \left( \int_P F_{\sigma_\nu \Delta} \right) G \sigma_\nu \bigg|_{t_i = t + (r-1) \pi \sqrt{-1}}.
\]

To finish the proof we simply note that

\[
\frac{1}{\Delta} \rho_\nu \left( D_\Delta (J^P) \bigg|_{t_i = t + (r-1) \pi \sqrt{-1}} \right) = \frac{1}{\Delta} D_\Delta (J^P) \bigg|_{t_i = t + (r-1) \pi \sqrt{-1}}
\]

since \( D_\Delta (J^P) \bigg|_{t_i = t + (r-1) \pi \sqrt{-1}} \) is already antisymmetric.

**Remark:** Since multiplication by \( e^{t \sigma_1} \) is an invertible operator on \( H^* (P) \), equation (4) gives

\[
\frac{(-1)^{\nu}}{\nu!} \left( \int_P F_{\sigma_\nu \Delta} \right) G \sigma_\nu \bigg|_{t_i = t + (r-1) \pi \sqrt{-1}} = \left( \frac{\sigma_\nu}{\psi}, \sigma_\mu \right)_G,
\]

for any two partitions \( \mu, \nu \). Just as the formula (3) in the case of three-point Gromov-Witten invariants, this can be viewed as an analogue of Martin’s integration formula for two-point descendants.
§3. Applications.

The first application we discuss is a proof (for Grassmannians) of the “R-conjecture” of Givental. According to Givental’s program [Giv3], once the R-conjecture is established, the Virasoro conjecture for Grassmannians follows. For this we will need the Hori-Vafa conjecture in its original formulation [HV], which we now explain.

There is an extension of Mirror Symmetry to Fano manifolds (see [EHX1], [Giv1], [HV]). If $X$ is Fano, the general Physics formulation of it says that the nonlinear sigma model for $X$ is equivalent to the Landau-Ginzburg theory with a certain potential function $W$. For $X = \mathbf{P}$ Hori and Vafa ([HV]) argue on physical grounds that the mirror symmetric Landau-Ginzburg theory is given by

$$W = \sum_{i=1}^{r} W_i,$$

where the variables $y_{ij}$ satisfy the relations $\prod_{j=1}^{n} y_{ij} = e^{t_i}, \ i = 1, \ldots, r.$

While the equivalence cannot be expressed in this generality in rigorous mathematical terms, a precise statement about integral representations for the coefficients of the $J$-function can be extracted from it. This statement has been formulated and proved by Givental, several years before [HV] was written.

**Theorem 3.1 ([Giv1]):** For each $\bar{h} \neq 0$, the coefficients of $J^P$ – with respect to every basis of $H^*(\mathbf{P})$ – span the same $C$-subspace (of dimension $n^r = \text{rank}(H^*(\mathbf{P}))$ in the vector space $C[t_1, \ldots, t_r][[e^{t_1}, \ldots, e^{t_r}]]$ as do the functions

$$I_K = \int_{\prod_{i=1}^{n} \Gamma_{i,k_i}} e^{W/\bar{h}} \frac{\wedge y_{ij} \wedge d(\prod_{j} y_{ij})}{\wedge d(\prod_{j} y_{ij})}, \ K = (k_1, \ldots, k_r), \ 1 \leq k_i \leq n,$$

where, for each $i$, $\Gamma_{i,k_i} \subset Y_{t_i} := \{\prod_{j=1}^{n} y_{ij} = e^{t_i}\}$ is cycle representing a relative homology class in $H_{n-1}(Y_{t_i}, \text{Re}(W_i/\bar{h}) = -\infty)$.

One gets a basis of this subspace by choosing for each $1 \leq i \leq r$ the basis $\{\Gamma_{i,1}, \ldots, \Gamma_{i,n}\}$ of $H_{n-1}(Y_{t_i}, \text{Re}(W_i/\bar{h}) = -\infty)$ consisting of the descending Morse cycles corresponding to the $n$ non-degenerate critical points of $W_i$. In what follows we will always consider this choice of $\Gamma_{i,k_i}$’s, and denote $\Gamma_K := \prod_{i=1}^{n} \Gamma_{i,k_i}$.

Furthermore, in the same paper Hori and Vafa conjecture that the Landau-Ginzburg theory mirror-symmetric to the nonlinear sigma model on the Grassmannian $G$ is obtained by a kind of symmetrization from the Landau-Ginzburg theory mirror to $\mathbf{P}$. As a consequence, they conjecture

**Theorem 3.2 (Hori-Vafa Conjecture - integral representation form):** For each $\bar{h} \neq 0$, the functions

$$D_{\Delta}I_K|_{t_i = t + (r-1)\pi \sqrt{-1}}, \ K = (k_1, \ldots, k_r), \ 1 \leq k_i \leq n$$
span the same \( \mathbb{C} \)-subspace of \( \mathbb{C}[t][[e^t]] \) as do the coefficients of \( J^G \).

**Proof:** The \( J \)-function version proved in this paper gives

\[
J^G = e^{-\sigma_1(r-1)\pi \sqrt{-1}T/h} D_{\Delta} J^P \bigg|_{t_i = t + (r-1)\pi \sqrt{-1}T/h}.
\]

Since multiplication by \( e^{-\sigma_1(r-1)\pi \sqrt{-1}T/h} \) is an invertible linear transformation on \( H^*(G) \), it follows that the overall factor \( e^{-\sigma_1(r-1)\pi \sqrt{-1}T/h} \) can be discarded from \( J^G \) without changing the span of the coefficients. The statement follows now immediately from Theorem 3.1.

**Remark:** Note that \( D_{\Delta} I_K |_{t_i = t + (r-1)\pi \sqrt{-1}T} \) vanishes whenever we have \( k_i = k_j \) for some \( i \neq j \). Therefore we can restrict in the statement of the theorem to those \( K \)'s with distinct entries.

Recall that \( G \) has a natural action of the torus \( T = (\mathbb{C}^*)^n \). The torus also acts diagonally on \( P \) and there are \( T \)-equivariant versions of \( J^P \) and of Givental's theorem above. Explicitly, it is shown in [Giv1] that the formula

\[
J^P_T = e^{\sum_{i=1}^n x_i} \prod_{i=1}^n \prod_{j=1}^n (x_i - y_j + lh)
\]

holds, and in [Giv3] that, for fixed \( h \neq 0, \lambda_i \neq 0, i = 1, \ldots, n \), the coefficients of \( J^P_T \) (in any \( \mathbb{C}[\lambda_1, \ldots, \lambda_n] \)-basis of the equivariant cohomology ring \( H^*_T(P) \)) generate the same \( \mathbb{C} \)-subspace of \( \mathbb{C}[t][[e^t]] \) as do the integrals

\[
I^T_K = \int_{T^T} e^{W_T/h} \prod_{i=1}^n e^{\lambda_i y_{ij} \cdot \partial_{y_{ij}}}, \quad K = (k_1, \ldots, k_n), \quad 1 \leq k_i \leq n.
\]

The only difference between the integrals \( I_K \) and \( I^T_K \) is in the “phase function” \( W_T \), which is now given by

\[
W_T = \sum_{i=1}^r W^T_i, \quad \text{with} \quad W^T_i = W_i + \sum_{j=1}^n \lambda_j \ln y_{ij} = \sum_{j=1}^n y_{ij} + \lambda_j \ln y_{ij}.
\]

Consequently, the cycles \( \Gamma^T_{t,k_i} \subset Y_{t_i} := \{ \prod_{j=1}^n y_{ij} = e^{t_i} \} \) represent relative homology classes in \( H_{n-1}(Y_{t_i}, \text{Re}(W^T_i/h) = -\infty) \).

The same argument as above, but using Theorem 1.5' gives

**Theorem 3.2’:** For fixed \( h \neq 0, \lambda_i \neq 0, i = 1, \ldots, n \), the functions

\[
D_{\Delta} I^T_K |_{t_i = t + (r-1)\pi \sqrt{-1}T}, \quad K = (k_1, \ldots, k_r), \quad 1 \leq k_i \leq n
\]

span the same \( \mathbb{C} \)-subspace of \( \mathbb{C}[t][[e^t]] \) as do the coefficients of \( J^{G_T} \).
Theorems 3.2 and 3.2’ are “mirror theorems” for the Grassmannian, in the sense of Givental.

We now turn to Givental’s R-Conjecture and first explain briefly the statement, refering to [Giv3], [JK] for details.

Let $X$ be a Fano projective manifold with an action of a torus $T = (\mathbb{C}^*)^n$. As before, we denote by $\lambda_1, \ldots, \lambda_n$ the generators of $H^*_T(BT)$. Assume that there are $T$-equivariant nef divisor classes $\gamma_i$, $i = 1, \ldots, m$ forming a basis of $H^2_T(X)$. As usual, introduce dual parameters $t_i$ and set $q_i = e^{t_i}$. All structures discussed in the previous sections of the paper have equivariant analogues: the $T$-equivariant Gromov-Witten invariants of $X$ ([Giv1], [Kim2]) can be used to construct a (small) equivariant quantum product $\ast_T$ on $H^*_T(X) \otimes \mathbb{C}[e^{t_1}, \ldots, e^{t_m}]$, and we have the associated flat connection $\nabla_h$ defined by the same formula.

Choose a $\mathbb{C}[\lambda_1, \ldots, \lambda_n]$-basis $\{\phi_j\}$ of $H^*_T(X)$. A fundamental matrix of flat sections is constructed using two-point descendants, and its last column gives the equivariant $S$-function.

Let $A_i$ be the matrix in the basis $\{\phi_j\}$ of the operator on $QH^*_T(X)$ given by quantum multiplication by $\gamma_i$. Assume furthermore that the $T$-fixed points on $X$ are isolated and the 1-dimensional orbits are isolated as well. Then the equivariant cohomology ring $H^*_T(X)$, and therefore also the equivariant small quantum cohomology ring $QH^*_T(X)$ are semi-simple. In this case there are canonical co-ordinates $u_j$ on $H^*_T(X)$, unique up to addition of terms constant in the $t_i$’s, such that the connection matrix of 1-forms matrix has $N := \text{rank}(H^*_T(X))$ distinct eigenvalues $\partial U_j$ for general $\lambda$ and $q$.

Denote by $\Psi$ the eigenmatrix of $\sum_i A_i dt_i$ normalized so that the row-vectors $\sum_j \Psi_{jk} \phi_j$ are unit vectors with respect to the Poincaré metric. We then have

$$(\sum_i A_i dt_i) \Psi = \Psi dU,$$

where $U = \text{diag}(u_1, \ldots, u_N)$.

It is shown in [Giv2], [Giv3] that there is an asymptotic fundamental solution matrix of the connection $\nabla_h$ of the form $\Psi Re^{U/h}$, with

$$R = 1 + R_1 h + R_2 h^2 + \ldots$$

and $R_i$ analytic in $q$ and $\lambda$ near any point $(q, \lambda)$ at which $QH^*_T(X)$ is semi-simple (i.e., $\frac{\partial}{\partial q} (\Psi Re^{U/h}) \sim A_i \Psi Re^{U/h}$ asymptotically in $h e^{U/h}$, $l \to \infty$).

The solution is unique if the following two conditions are imposed:

1) the orthogonal condition: $R^T(h) R(-h) = 1$,

2) the classical limit condition: for generic values of $\lambda_1, \ldots, \lambda_n$

$$\lim_{q \to 0} R = e^{\text{diag}(b_1, \ldots, b_N)},$$

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where
\[ b_i(h) = \sum_{k=1}^{\infty} N_{2k-1}^{(i)} \frac{B_{2k} h^{2k-1}}{2k 2k - 1}. \]
Here \( B_{2k} \) are Bernoulli numbers defined by
\[ x/(1 - e^{-x}) = 1 + x/2 + \sum_{k=1}^{\infty} B_{2k} \frac{x^{2k}}{(2k)!}. \]

Finally, the \( N^{(i)}_j \) are defined as follows: There is a natural 1-1 correspondence between the \( T \)-fixed points \( \{v_i | i = 1, \ldots, N\} \) in \( X \) and idempotents \( \frac{\partial}{\partial v_i} \) in \( H^*_{T}(X) \otimes_{C[\lambda_1, \ldots, \lambda_n]} C(\lambda_1, \ldots, \lambda_n) \). Let \( w_{ij} \) denote the weights of the \( T \) action on the tangent spaces \( T_{v_i}X \). With this notation,
\[ N^{(i)}_j = \sum_j \frac{1}{w_{ij}}. \]

**Remark:** In [JK] an additional requirement (the “equivariant homogeneity condition”) is listed as necessary to get the uniqueness of \( R \). In fact, this condition is always satisfied once 1) and 2) above are imposed, as can be seen from the inductive construction of \( R \) in [Giv2].

We are ready to state Givental’s \( R \)-conjecture (in fact, the “\( R \)-conjecture restricted to \( H^{2n} \), in the terminology of [JK]).

**\( R \)-Conjecture ([Giv3]):** The non-equivariant limit \( \lambda_1 \to 0, \ldots, \lambda_n \to 0 \) of the matrix \( R \) exists.

Givental [Giv3] proves the conjecture in the case \( X \) is a projective space, and in fact the same proof works in the case of a product of projective spaces. In particular, the conjecture holds for our \( P \). The main ingredient of the proof is the existence of a fundamental solution matrix of the flat connection whose entries are the integrals from Theorem 3.1 and their derivatives.

The validity of the \( R \)-conjecture for \( G \) would follow immediately from its validity for \( P \) if we would have at our disposal the equivariant version of Theorem 2.7. In turn, this would follow from an equivariant version of the “quantum integration formula” (3). However, neither of the two proofs we gave to this formula are readily generalized. Indeed, there is no analogue of the rim-hook algorithm for equivariant quantum cohomology, while to establish the equivariant Vafa-Intriligator formula one needs an equivariant version of the quantum Giambelli formula of [Ber2], which is not known. In short, although we believe that the equivariant quantum integration formula is true, lacking this, we give a direct proof to the \( R \)-conjecture for \( G \) which uses Theorem 3.2’ and parallels the proofs in [Giv3] and [JK].
Theorem 3.3: The R-conjecture holds for $G = G(r, n)$.

Proof: We start by noting that it follows from the localization theorem that the powers

$$\{ \sigma_i^1 \mid i = 0, 1, \ldots, \mathrm{rank}(H^*(G)) - 1 \}$$

form a basis over $C(\lambda_1, \ldots, \lambda_n)$ for the localized equivariant cohomology ring

$$H^*_T(G) \otimes_{C[\lambda_1, \ldots, \lambda_n]} C(\lambda_1, \ldots, \lambda_n)$$

(cf. the proof of Lemma 1.3).

By Theorem 3.2' and Lemma 4.2 of [JK], there exist differential operators $D_i$, $i = 1, \ldots, N$, in $C(\lambda_1, \ldots, \lambda_n)[e^t, \bar{h}, \bar{h} d/dt]$ such that

$$s_K := \sum_i \sigma_i^1 D_i(D\Delta^T_K|_{t_j = t + (r-1)\pi \sqrt{-1}})$$

form a fundamental solution to the flat connection associated to the equivariant small quantum cohomology of $G$. By clearing denominators, we may assume that the differential operators are in fact in $C(\lambda_1, \ldots, \lambda_n)e^t, \bar{h}, \bar{h} d/dt]$. Now write $\sigma_i^1$ as a linear combination of $\phi_j$ over $C[\lambda_1, \ldots, \lambda_n]$. It follows that

$$s_K = \sum_i s_{K,i} \phi_i = \sum_i \phi_i \left( \int_{\Gamma^T_K} e^{W^T/\hbar} \frac{\lambda_{ij} dy_{ij}}{\prod_j(y_{ij})} \right) |_{t_j = t + (r-1)\pi \sqrt{-1}}$$

form a fundamental solution, where the functions $\varphi_{i,t,h,(\lambda_j)}$ depend analytically on $q = e^t, \bar{h}$, and the $\lambda_j$'s. The stationary phase approximation of the integrals $s_{K,i}$ near the non-degenerate critical point corresponding to the cycle $\Gamma^T_K$ gives an asymptotic expansion of the form

$$(s_{K,i}) \sim \Psi' Re^{U/\hbar},$$

for some matrices $\Psi'$ and $R = 1 + R_1 \hbar + R_2 \hbar^2 + \ldots$.

Since the integrand depends analytically on the $\lambda_j$'s (and the usual small quantum cohomology ring $QH^*(G)$ is semi-simple [Abr]), it follows that for the matrix $R$ constructed this way the limit $\lim_{\lambda_j \to 0} R$ exists; moreover, since the $s_{K,i}$'s are the entries of a fundamental solution to the flat connection, one can show that the eigenmatrices $\Psi'$ and $\Psi$ are related by $\Psi' = C\Psi$, where $C$ is a matrix constant with respect to $t$ (see e.g., the proof of Theorem 4.4 in [JK] for details). Therefore $\Psi' Re^{U/\hbar}$ is a fundamental solution matrix. It is now routine to check that the orthogonality and classical limit conditions hold for $R$ (cf. [Giv3], [JK]). Hence the matrix $R$ constructed in this manner is indeed the matrix in the $R$-conjecture.

According to [Giv3], if the (non-equivariant) small quantum cohomology ring $QH^*(X)$ is also semi-simple, then the Virasoro Conjecture of Eguchi-Hori-Xiong [EHX2], and S. Katz follows from the $R$-conjecture. Therefore, Theorem 3.3 has the following...
**Corollary 3.4:** The Virasoro conjecture is true for $G$.

As our second application, we present a proof for $G(2, n)$ of a conjecture made in [BCKS1]. In that paper, another “mirror construction” for Grassmannians (indeed, for all partial flag manifolds, see [BCKS2]) was proposed. The construction is via toric degenerations, and leads to integral representations for the coefficients of the $J$-function that are completely different than the ones in Theorem 3.2 above. As a consequence, an explicit formula for the coefficient of the cohomology class 1 in $J_G^{G(r, n)}$ was conjectured. We were able at the time to verify it by brute force computation only in a few cases ($G(2, n), n \leq 7$, and $G(3, 6)$) and then used it, together with the “quantum Lefschetz hyperplane theorem” to calculate the genus 0 Gromov-Witten invariants of some complete intersection Calabi-Yau 3-folds.

Since in this paper we give a formula for the full $J$-function, the proof of the conjecture reduces to checking the following (highly nontrivial) combinatorial identity:

**Conjecture:** The constant term with respect to the $x_i$’s from $J_G^{G(r, n)}$ in Theorem 1.5 equals $A_{G(r, n)}(e^t)$ in [BCKS1], Conjecture 5.2.3.

We now make this more explicit in the case $r = 2$. First, the formula in [BCKS1] is

$$A_{G(2, n)}(e^t) = \sum_{d \geq 0} \frac{e^{dt}}{(d!)^n} \sum_{j_{n-3} \geq \cdots \geq j_1} \left( \frac{d}{j_{n-3}} \right)^2 \left( \frac{d}{j_{n-4}} \right) \cdots \left( \frac{d}{j_1} \right) \left( \frac{j_{n-3}}{j_1} \right) \cdots \left( \frac{j_2}{j_1} \right),$$

which may be rewritten

$$\sum_{d \geq 0} \frac{e^{dt}}{(d!)^n} \sum_{j_{n-3} \geq j_{n-4} \geq \cdots \geq j_1 \geq j_0 = 0} \frac{d^{n-2}}{(d-j_{n-3})! \prod_{i=2}^{n-2} (d-j_{i-1})! (j_{i-1} - j_{i-2})! j_{i-1}!}.$$

On the other hand, extracting the constant term with respect to the $x_i$’s from $J_G^{G(2, n)}$ (and setting $\hbar = 1$) in Theorem 1.5 we get

$$\sum_{d \geq 0} \frac{e^{dt}}{(d!)^n} \frac{(-1)^d}{2} \sum_{m=0}^{d} \binom{d}{m} n (d-2m)(\gamma(m) - \gamma(d-m)) + 2,$$

where for a positive integer $m$

$$\gamma(m) = \sum_{j=1}^{m} \frac{1}{j}$$

is the partial sum of the harmonic series, and $\gamma(0) = 0$.

The conjecture for $G(2, n)$ follows then from
Proposition 3.5: If \( d \) are \( n \) are nonnegative integers, \( n \geq 3 \), then

\[
\sum_{j_{n-3} \geq j_{n-4} \geq \cdots \geq j_1 \geq j_0 = 0}^{d} \frac{d!^{n-2}}{(d-j_{n-3})! \prod_{i=2}^{n-2} (d-j_{i-1})! (j_{i-1} - j_{i-2})! j_{i-1}!} = \frac{(-1)^d 2^d}{d} \sum_{m=0}^{d} \left( \frac{d}{m} \right) \left( n(d - 2m)(\gamma(m) - \gamma(d - m)) + 2 \right).
\]

Proof (Dennis Stanton): A formula expressing a \( k \)-fold sum as a sum over a single summation index is given by the iterate of Bailey’s lemma. Explicitly, if in [And], p. 30, Theorem 3.4 we change notation by setting \( n = m, N = d, k = n - 2, \) and \( m = j_{i-1} \), then we choose

\[
\beta_m = \begin{cases} 1 & \text{if} \ m = 0 \\ 0 & \text{if} \ m > 0, \end{cases} \quad \alpha_m = (-1)^m q \left( \frac{(m)}{2} \right) \frac{(a; q)_m}{(q; q)_m} \frac{1 - aq^{2m}}{1 - a},
\]

all \( c_i = q^{-d} \), and let all \( b_i \to \infty \), the identity in loc. cit. becomes

\[
\frac{(aq; q)_d}{(aq^{1+d}; q)_d} \sum_{j_{n-3} \geq j_{n-4} \geq \cdots \geq j_1 \geq j_0 = 0}^{d} \frac{(q^{-d}; q)_{j_{n-3}} \prod_{i=2}^{n-2} (q^{-d}; q)_{j_{i-1}}}{\prod_{i=2}^{n-2} (aq^{1+d}; q)_{j_{i-1}} (q; q)_{j_{i-1} - j_{i-2}}} \times (q^{-2d}/a)^{j_{n-3}} (-1)^{j_{n-3}} q^{-\left( \frac{j_{n-3}}{2} \right)} \prod_{i=2}^{n-2} (-1)^{j_{i-1}} q^{d_{i-1}} a^{j_{i-1}} q^{\left( \frac{j_{i-1} + 1}{2} \right)} = \sum_{m=0}^{d} \frac{(q^{-d}; q)_{m+1}}{(aq^{1+d}; q)_m} (-1)^{(m+1)} q^{-\left( \frac{(m+1)}{2} \right)} a^{(m+1)} q^{(m+1) + dm(m-1)} q^{-\left( \frac{(m)}{2} \right)} \alpha_m,
\]

where for a nonnegative integer \( m \)

\[
(a; q)_m = \prod_{t=0}^{\infty} \frac{(1 - aq^t)}{(1 - aq^{t+m})}.
\]

The \( q \to 1 \) limit of this equation when \( a = q^{-d} \) is

\[
\sum_{j_{n-3} \geq j_{n-4} \geq \cdots \geq j_1 \geq j_0 = 0}^{d} \frac{d!^{n-2}}{(d-j_{n-3})! \prod_{i=2}^{n-2} (d-j_{i-1})! (j_{i-1} - j_{i-2})! j_{i-1}!} = \lim_{a \to q^{-d}} \frac{d!}{(a + 1)_d} \sum_{m=0}^{d} \left( \frac{d}{m} \right) \left( \frac{m!^{n-1} (a)_m a + 2m}{(a + 1 + d)^{m-1} m!} \right) (-1)^m,
\]

where \((a)_m\) is the generalized factorial

\[
(a)_m = a(a+1)...(a+m-1).
\]
Notice that the left-hand side of the last identity coincides with the left
hand-side of the identity in the Proposition, while the right-hand side is equal to

\[(*) \quad d(-1)^{d-1} F'(-d),\]

where

\[F(a) = \sum_{m=0}^{d} \left( \frac{d}{m} \right)^{n-1} \frac{m!^{n-1}}{(a + 1 + d)^{n-1}} \frac{(a)_m}{m!} \frac{a + 2m}{a} (-1)^m.\]

However,

\[(** \quad F'(-d) = \sum_{m=0}^{d} \left( \frac{d}{m} \right)^{n-1} \left( -(n-1)\gamma(m) \left( \frac{d}{m} \right)^{-d + 2m \over -d} + \left( \frac{d}{m} \right)^{-d + 2m \over d^2} \right) +\]

\[= \frac{-1}{d} \sum_{m=0}^{d} \left( \frac{d}{m} \right)^{n} \left( (n-1)\gamma(m) + \gamma(d) - \gamma(d-m) \right) \frac{(d-2m) + 2m/d}{(d-2m) + 2m/d}.\]

Replacing \(m\) by \(d - m\) in this sum and adding to \((**)\) yields

\[F'(-d) = \frac{-1}{2d} \sum_{m=0}^{d} \left( \frac{d}{m} \right)^{n} \left( n\gamma(m) + \gamma(d-m) \right) (d - 2m + 2) \].

If we substitute now this expression for \(F'(-d)\) into \((*)\), the right-hand side of the identity in the Proposition is obtained. This finishes the proof.
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