A class of logarithmically completely monotonic functions related to the $q$-gamma function and applications

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Abstract In this paper, the logarithmically complete monotonicity property for a functions involving $q$-gamma function is investigated for $q \in (0, 1)$. As applications of this results, some new inequalities for the $q$-gamma function are established. Furthermore, let the sequence $r_n$ be defined by $n! = \sqrt{2\pi n}(n/e)^n e^{r_n}$. We establish new estimates for Stirling’s formula remainder $r_n$.

Keywords Completely monotonic functions · Logarithmically completely monotonic functions · $q$-gamma function · Stirling’s formula · Inequalities

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1 Introduction

A real valued function $f$, defined on an interval $I$, is called completely monotonic, if $f$ has derivatives of all orders and satisfies

\[ (-1)^n f^{(n)}(x) \geq 0, \quad n \in \mathbb{N}_0, \quad x \in I, \quad (1) \]

where $\mathbb{N}$ the set of all positive integers.

A positive function $f$ is said to be logarithmically completely monotonic on an interval $I$ if its logarithm $\log f$ satisfies
\((-1)^n \left( \log f(x) \right)^{(n)}(x) \geq 0,\)

for all \(x \in I\) and \(n \in \mathbb{N}\).

Completely monotonic functions have remarkable applications in different branches of mathematics. For instance, they play a role in potential theory, probability theory, physics, numerical and asymptotic analysis, and combinatorics [see (3) and the references given therein]. The \(q\)-analogue of the gamma function is defined as

\[
\Gamma_q(x) = (1 - q)^{1-x} \prod_{j=0}^{\infty} \frac{1 - q^{j+1}}{1 - q^{j+x}}, \quad 0 < q < 1.
\]

The \(q\)-gamma function \(\Gamma_q(z)\) has the following basic properties:

\[
\lim_{q \to 1^-} \Gamma_q(z) = \lim_{q \to 1^+} \Gamma_q(z) = \Gamma(z),
\]

and

\[
\Gamma_q(z) = q^{\frac{(z-1)(z-2)}{2}} \tilde{\Gamma}_{\frac{1}{q}}(z).
\]

The \(q\)-digamma function \(\psi_q\), the \(q\)-analogue of the psi or digamma function \(\psi\) is defined for \(0 < q < 1\) by

\[
\psi_q(x) = \frac{\Gamma_q'(x)}{\Gamma_q(x)}
= -\log(1 - q) + \log q \sum_{k=0}^{\infty} \frac{q^{k+x}}{1-q^{k+x}}
= -\log(1 - q) + \log q \sum_{k=1}^{\infty} \frac{q^k x}{1-q^k}.
\]

Using the Euler-Maclaurin formula, Moak [(6), p. 409] obtained the following \(q\)-analogue of Stirling formula

\[
\log \Gamma_q(x) \sim \left( x - \frac{1}{2} \right) \log \left( \frac{1 - q^x}{1 - q} \right) + \text{Li}_2(1 - q^x) + \frac{1}{2} H(q - 1) \log q + C_{\hat{q}}
+ \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left( \frac{\log \hat{q}}{\hat{q}^x - 1} \right)^{2k-1} \hat{q}^x P_{2k-3}(\hat{q}^x)
\]

as \(x \to \infty\) where \(H(.)\) denotes the Heaviside step function, \(B_k, \ k = 1, 2, \ldots\) are the Bernoulli numbers,

\[
\hat{q} = \begin{cases} 
q & \text{if } 0 < q < 1 \\
1/q & \text{if } q > 1
\end{cases}
\]
Li₂(z) is the dilogarithm function defined for complex argument z as (1)

$$\text{Li}_2(z) = - \int_0^z \frac{\log(1-t)}{t} dt, \ z \notin [1, \infty)$$  \hspace{1cm} (7)

$P_k$ is a polynomial of degree $k$ satisfying

$$P_k(z) = (z - z^2) P_{k-1}'(z) + (kz + 1) P_{k-1}(z), \ P_0 = P_{-1} = 1, \ k = 1, 2, \ldots$$ \hspace{1cm} (8)

and

$$C_q = \frac{1}{2} \log(2\pi) + \frac{1}{2} \log \left( \frac{q-1}{\log q} \right) - \frac{1}{24} \log q + \frac{1}{\log q} \int_0^{-\log(q)} \frac{udu}{e^u - 1} + \log \left( \sum_{m=-\infty}^{\infty} r^m(6m+1) - r^{(2m+1)(3m+1)} \right),$$

where $r = \exp(4\pi^2 / \log q)$. Simple computation shows that

$$\left( \frac{\text{Li}_2(1 - q^x)}{\log(q)} \right)' = \frac{xq^x \log(q)}{1 - q^x}$$ \hspace{1cm} (9)

On the other hand, we have

$$\lim_{q \to 1} \frac{\text{Li}_2(1 - q^x)}{\log q} = -x.$$ \hspace{1cm} (10)

Indeed, let $U(q) = 1 - q^x$, by using the l’Hospital’s rule and (9) we get

$$\lim_{q \to 1} \frac{\text{Li}_2(1 - q^x)}{\log q} = \lim_{q \to 1} \left[ \left( \frac{q-1}{\log q} \right) \cdot \left( \frac{\text{Li}_2(U(q)) - \text{Li}_2(U(1))}{q-1} \right) \right]$$

$$= \lim_{q \to 1} \left[ \left( \frac{\partial U(q)}{\partial q} \right) \left( \frac{\partial \text{Li}_2}{\partial q} \right)(U(q)) \right]$$

$$= \lim_{q \to 1} \left( \frac{x^2 q^x \log(q)}{q(1 - q^x)} \right) = -x. \hspace{1cm} (11)$$

Stirling’s formula

$$n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n, \ n \in \mathbb{N} \hspace{1cm} (12)$$

has many applications in statistical physics, probability theory and number theory. Actually, it was first discovered in 1733 by the French mathematician Abraham de Moivre (1667–1754) in the form

$$n! \sim \text{constant} \sqrt{n} \left( \frac{n}{e} \right)^n$$

when he was studying the Gaussian distribution and the central limit theorem. Afterwards, the Scottish mathematician James Stirling (1692–1770) found the missing
constant $\sqrt{2\pi}$ when he was trying to give the normal approximation of the binomial distribution.

In 1940 Hummel (4) defined the sequence $r_n$ by
\begin{equation}
\sqrt{2\pi} n / e^n e^{r_n},
\end{equation}
and established
\begin{equation}
\frac{11}{12} < r_n + \log \sqrt{2\pi} < 1
\end{equation}
After the inequality (14) was published, many improvements have been given. For example, Robbins [(8), p. 26] established
\begin{equation}
\frac{1}{12n + 1} < r_n < \frac{1}{12n}
\end{equation}
The main aim of this paper is to investigate the logarithmic complete monotonicity property of the function
\begin{equation}
f_{\alpha, \beta}(q; x) = \frac{\Gamma_q(x + \beta) \exp\left(-\frac{\text{Li}_2(1-qx)}{\log q}\right)}{\left(\frac{1-\log q}{1-q}\right)^{x+\beta-\alpha}}, \quad x > 0,
\end{equation}
for all reals $\alpha, \beta$ and $q$ such that $q \in (0, 1)$. As applications of these results, sharp bounds for the $q$-gamma function are derived. In addition, we present new estimate for Stirling’s formula remainder $r_n$. Some results are shown to be a generalization of results which were obtained by Chen and Qi (2).

2 Useful lemmas

In order to study the function defined by (16) we need the following lemmas which is considered the main tool to arrive at our results.

**Lemma 1** Let $\alpha, \beta, q$ be a reals numbers such that $q \in (0, 1)$. Then
\begin{equation}
(\log f_{\alpha, \beta}(q; x))'' = \sum_{k=1}^{\infty} q^{kx} \log(q) \frac{\log(q)}{1 - q^k} \Phi_{\alpha, \beta}(q^k),
\end{equation}
where
\begin{equation}
\Phi_{\alpha, \beta}(y) = y^\beta \log(y) + (1-y) + (\beta - \alpha) (1-y) \log(y), \quad y = q^k, \ k = 1, 2, \ldots
\end{equation}

**Proof** Taking logarithm of the function $f_{\alpha, \beta}(q; x)$ leads to
\begin{equation}
\log f_{\alpha, \beta}(q; x) = \log \Gamma_q(x + \beta) - (x + \beta - \alpha) \log \left(\frac{1-q^x}{1-q}\right) - \frac{\text{Li}_2(1-qx)}{\log(q)}
\end{equation}
Differentiation yields
\[
(\log f_{\alpha, \beta}(q; x))' = \psi_q(x + \beta) - \log \left( \frac{1 - q^x}{1 - q} \right) + (\beta - \alpha) \frac{q^x \log(q)}{1 - q^x}.
\]
Thus
\[
(\log f_{\alpha, \beta}(q; x))'' = \psi_q'(x + \beta) + \frac{q^x \log(q)}{1 - q^x} + (\beta - \alpha) \frac{q^x (\log(q))^2}{(1 - q^x)^2}. \tag{19}
\]
From the series expansion
\[
\frac{x}{(1 - x)^2} = \sum_{k=1}^{\infty} k x^k, \quad x \in (0, 1),
\]
and (19) we get
\[
(\log f_{\alpha, \beta}(q; x))'' = (\log(q))^2 \sum_{k=1}^{\infty} k q^{k(x + \beta)} \frac{k}{1 - q^k} + \log(q) \sum_{k=1}^{\infty} q^{kx} + (\beta - \alpha) \log(q) + O(q^{x \log^2(q)} (1 - q^{x^2})^2).
\]
Lemma 1 is thus proved. \hfill \Box

**Lemma 2 (6)** The following approximation for the q-digamma function
\[
\psi_q(x) = \log \left( \frac{1 - q^x}{1 - q} \right) + \frac{1}{2} \frac{q^x \log(q)}{1 - q^x} + O \left( \frac{q^x \log^2(q)}{(1 - q^x)^2} \right), \tag{20}
\]
holds for all \( q > 0 \) and \( x > 0 \).

**Lemma 3 (9)** For every \( x, q \in \mathbb{R}_+ \), there exists at least one real number \( a \in [0, 1] \) such that
\[
\psi_q(x) = \log \left( \frac{1 - q^{x+a}}{1 - q} \right) + \frac{q^x \log(q)}{1 - q^x} - \left( \frac{1}{2} - a \right) H(q - 1) \log(q) \tag{21}
\]
where \( H(.) \) is the Heaviside step function.
3 Logarithmically completely monotonic function related to the \( q \)-gamma function

**Theorem 1** Let \( \alpha \) be a real number. The function \( f_{\alpha,1}(q; x) \) is logarithmically completely monotonic on \((0, \infty)\), if and only if \( 2\alpha \leq 1 \).

**Proof** From the Lemma 1, we get

\[
\left( \log f_{\alpha,1}(q; x) \right)^{\prime\prime} = \sum_{k=1}^{\infty} \frac{q^{k} \log(q)}{1-q^{k}} \Phi_{\alpha,1}(q^{k}),
\]

where

\[
\Phi_{\alpha,1}(y) = y \log y + (1 - \alpha)(1 - y) \log y + (1 - y), \ y = q^{k}, \ k = 1, 2, \ldots
\]

In order to determine the sign of the function \( \Phi_{\alpha,1}(y) \), by again using the series expansion

\[
x(e^{x} - 1) = \sum_{k=2}^{\infty} \frac{x^{k}}{(k-1)!}, \text{ with } x = \log(1/y),
\]

we obtain

\[
\Phi_{\alpha,1}(y) = y \left( - \log(1/y) + (1 - \alpha) \log(1/y)(1 - 1/y) + 1/y - 1 \right)
\]

\[
= y \sum_{k=2}^{\infty} \frac{(\log(1/y))^{k}}{(k-1)!} \left[ \alpha - 1 + \frac{1}{k} \right].
\]

Therefore, the function \( \Phi_{\alpha,1}(y) \) is less than zero if \( 2\alpha \leq 1 \). Thus implies that the function \( \left( \log f_{\alpha,1}(q; x) \right)^{\prime\prime} \) is completely monotonic on \((0, \infty)\). This can be rewritten as

\[
(-1)^{n} \left( \log f_{\alpha,1}(q; x) \right)^{(n)} \geq 0, \ n \geq 2.
\]

In particular, \( \left( \log f_{\alpha,1}(q; x) \right)^{\prime\prime} \geq 0 \), so \( \left( \log f_{\alpha,1}(q; x) \right)^{\prime} \) is increasing on \((0, \infty)\), and consequently

\[
\left( \log f_{\alpha,1}(q; x) \right)^{(1)} \leq \lim_{x \to \infty} \left( \log f_{\alpha,1}(q; x) \right)^{(1)}
\]

\[
= \lim_{x \to \infty} \left( \psi_{q}(x + 1) - \log \left( \frac{1-q^{x}}{1-q} \right) + (1 - \alpha) \frac{q^{x} \log(q)}{1-q^{x}} \right)
\]

\[
= 0.
\]

So \( f_{\alpha,1} \) is logarithmically completely monotonic on \((0, \infty)\) if \( 2\alpha \leq 1 \).
Conversely, if the function \( f_{\alpha,1}(q; x) \) is logarithmically completely monotonic on \((0, \infty)\), then for all real \( x > 0 \),

\[
(\log f_{\alpha,1}(q; x))^\prime = \psi_q(x + 1) - \log \left( \frac{1 - q^x}{1 - q} \right) + (1 - \alpha) \frac{q^x \log(q)}{1 - q^x} \leq 0. \tag{24}
\]

From the Eq. (24) and along with the identity

\[
\psi_q(x + 1) = \psi_q(x) - \frac{q^x \log(q)}{1 - q^x}, \tag{25}
\]

we have

\[
(\log f_{\alpha,1}(q; x))^\prime = \psi_q(x) - \log \left( \frac{1 - q^x}{1 - q} \right) - \alpha \log(q) \frac{q^x}{1 - q^x} \leq 0,
\]

which is equivalent to

\[
\psi_q(x) - \log \left( \frac{1 - q^x}{1 - q} \right) \leq \alpha \frac{q^x \log(q)}{1 - q^x}. \tag{26}
\]

According to the result obtained in Lemma 2, we see that \( \psi_q(x) \sim I(q; x) \) on \((0, \infty)\) where

\[
I(q; x) = \log \left( \frac{1 - q^x}{1 - q} \right) + \frac{1}{2} \frac{q^x \log(q)}{1 - q^x}. \tag{27}
\]

Combining (26) and (27) we have

\[
\alpha \leq \frac{1}{2}.
\]

The proof is complete. \( \square \)

**Theorem 2** Let \( \alpha \) be a real number. The function \([f_{\alpha,1}(q; x)]^{-1}\) is logarithmically completely monotonic on \((0, \infty)\), if and only if \( \alpha \geq 1 \).

**Proof** From (23), we conclude that the function \( \Phi_{\alpha,1}(y) \geq 0 \) if \( \alpha \geq 1 \) we conclude that

\[
(-1)^n \left( \log \frac{1}{f_{\alpha,1}(q; x)} \right)^{(n)} \geq 0
\]

for all \( x > 0, \alpha \geq 1, q \in (0, 1) \) and \( n \geq 2 \). So,

\[
\left( \log \frac{1}{f_{\alpha,1}(q; x)} \right)^{(1)} = \log \left( \frac{1 - q^x}{1 - q} \right) - \psi_q(x) + \alpha \frac{q^x \log(q)}{1 - q^x},
\]
is increasing, thus
\[
\left( \log \frac{1}{f_{\alpha,1}(q;x)} \right)^{(1)} < \lim_{x \to \infty} \left( \log \frac{1}{f_{\alpha,1}(q;x)} \right)^{(1)}
= \lim_{x \to \infty} \left( \log \left( \frac{1-q^x}{1-q} \right) - \psi_q(x) + \alpha \frac{q^x \log(q)}{1-q^x} \right)
= 0.
\]

Hence, For \(\alpha \geq 1\) and \(n \in \mathbb{N}\),
\[
(-1)^n \left( \log \frac{1}{f_{\alpha,1}(q;x)} \right)^{(n)} \geq 0,
\]
on \((0, \infty)\). Now, assume that \(\frac{1}{f_{\alpha,1}(q;x)}\) is logarithmically completely monotonic on \((0, \infty)\), by definition, this give us that for all \(q \in (0, 1)\) and \(x > 0\),
\[
\left( \log \frac{1}{f_{\alpha,1}(q;x)} \right)^{(1)} = \log \left( \frac{1-q^x}{1-q} \right) - \psi_q(x) + \alpha \frac{q^x \log(q)}{1-q^x} \leq 0,
\]
which implies that
\[
\alpha \geq \frac{1-q^x}{q^x \log(q)} \left( \psi_q(x) - \log \left( \frac{1-q^x}{1-q} \right) \right). \tag{28}
\]

In view of Lemma 3 and inequality (28), we see that for all \(x > 0\) and \(q \in (0,1)\) there exists at least one real number \(a \in [0, 1]\) such that
\[
\alpha \geq \frac{1-q^x}{q^x \log(q)} \left( \log \left( \frac{1-q^{x+a}}{1-q^x} \right) + \frac{q^x \log(q)}{1-q^x} \right),
\]
and consequently
\[
\alpha \geq 1
\]
as \(x \to \infty\). This ends the proof.

**Theorem 3** Let \(\alpha\) be a real number and \(\beta \geq 0\). Then, the function \(f_{\alpha,\beta}(q;x)\) is logarithmically completely monotonic function on \((0, \infty)\) if \(2\alpha \leq 1 \leq \beta\).

**Proof** In view of Lemma 1 we have
\[
\left( \log f_{\alpha,\beta}(q;x) \right)^{''} = \sum_{k=1}^{\infty} \frac{q^{kx} \log(q)}{1-q^x} \Phi_{\alpha,\beta}(q^k),
\]
where $\Phi_{\alpha,\beta}(y)$ defined as in Lemma 1. Thus

$$\Phi_{\alpha,\beta}(y) = y^\beta \left( \sum_{k=2}^{\infty} \frac{(\log(1/y))^k}{(k-1)!} \left[ \frac{\beta^k - (\beta - 1)^k}{k} + (\beta - \alpha)(\beta - 1)^{k-1} - \beta^{k-1} \right] \right)$$

$$= y^\beta (\log(1/y))^2 \cdot \frac{2\alpha - 1}{2} + y^\beta \left( \sum_{k=3}^{\infty} \frac{(\log(1/y))^k}{(k-1)!} \left[ \frac{\beta^k - (\beta - 1)^k}{k} + (\beta - \alpha)(\beta - 1)^{k-1} - \beta^{k-1} \right] \right).$$

(29)

In [(2), p. 408], the authors proved the following inequality

$$\beta^k - (\beta - 1)^k < k(\beta - \alpha)(\beta^{k-1} - (\beta - 1)^{k-1}),$$

where $k \geq 3$ and $2\alpha \leq 1 \leq \beta$. This in turn together with the (29) implies that $\Phi_{\alpha,\beta}(y) \leq 0$. So, for all $n \geq 2$ we gave

$$(-1)^n \left( \log f_{\alpha,\beta}(q; x) \right)^{(n)} \geq 0 \quad (30)$$

on $(0, \infty)$ for $2\alpha \leq 1 \leq \beta$. As $\left( \log f_{\alpha,\beta}(q; x) \right)^{(2)} \geq 0$, it follows that $\left( \log f_{\alpha,\beta}(q; x) \right)^{(1)}$ is increasing on $(0, \infty)$, and consequently

$$\left( \log f_{\alpha,\beta}(q; x) \right)^{(1)} \leq \lim_{x \to \infty} \left( \log f_{\alpha,\beta}(q; x) \right)^{(1)}$$

$$= \lim_{x \to \infty} \left( \psi_q(x + \beta) - \log \left( \frac{1-q^x}{1-q} \right) + (\beta - \alpha) \frac{q^x \log(q)}{1-q^x} \right)$$

$$= 0.$$

In conclusion, (30) is true also $n = 1$, and we conclude that the function $f_{\alpha,\beta}(q; x)$ is logarithmically completely monotonic on $(0, \infty)$ for $2\alpha \leq 1 \leq \beta$. The proof is now completed.

4 Inequalities

As applications of the logarithmic complete monotonicity properties of the function (16) which are proved in Theorems 1, 2 and 3, we can provide the following inequalities for the $q$-gamma functions.

**Corollary 1** Let $q \in (0, 1)$, $n \in \mathbb{N}$ and $x_k > 0 \ (1 \leq k \leq n)$. Suppose that

$$\sum_{k=1}^{n} p_k = 1 \ (p_k \geq 0).$$
If \(2\alpha \leq 1 \leq \beta\), then
\[
\frac{\Gamma_q \left( \sum_{k=1}^n p_k x_k + \beta \right)}{\prod_{k=1}^n \Gamma_q (x_k + \beta)^{p_k}} \leq \left( \frac{1 - q \sum_{k=1}^n p_k x_k}{1 - q} \right) \frac{\sum_{k=1}^n p_k x_k + \beta - \alpha}{\prod_{k=1}^n (1 - q^{x_k})^{p_k x_k + \beta - \alpha}} K(q; p; x_1, \ldots, x_n)
\]
(31)
where
\[
K(q; p; x_1, \ldots, x_n) = \exp \left( \frac{Li_2 \left( 1 - q \sum_{k=1}^n p_k x_k \right) - \sum_{k=1}^n p_k Li_2 \left( 1 - q^{x_k} \right)}{\log q} \right),
\]
(32)
\[p = (p_1, \ldots, p_n).\]

**Proof** From Theorem 3, \(f_{\alpha, \beta}(q; x)\) is logarithmically completely monotonic on the interval \((0, \infty)\), which also implies that the function \(f_{\alpha, \beta}(q; x)\) is logarithmically convex. Combining this fact with Jensen’s inequality for convex functions yields
\[
\log f_{\alpha, \beta} \left( q; \sum_{k=1}^n p_k x_k \right) \leq \sum_{k=1}^n p_k \log f_{\alpha, \beta}(q; x_k).
\]
(33)
Rearranging (33) can lead to the inequality (31).

**Corollary 2** Let \(q \in (0, 1)\), \(n \in \mathbb{N}\) and \(x_k > 0\) \((1 \leq k \leq n)\). Suppose that
\[
\sum_{k=1}^n p_k = 1 \ (p_k \geq 0).
\]
Then, the following inequalities holds
\[
\left( \frac{1 - q \sum_{k=1}^n p_k x_k}{1 - q} \right) \frac{\sum_{k=1}^n p_k x_k}{\prod_{k=1}^n (1 - q^{x_k})^{p_k x_k}} \leq \frac{\Gamma_q \left( \sum_{k=1}^n p_k x_k + 1 \right)}{\prod_{k=1}^n \Gamma_q (x_k + 1)^{p_k}} \frac{\sum_{k=1}^n p_k x_k + 1/2}{\prod_{k=1}^n (1 - q^{x_k})^{p_k (x_k + 1/2)}} K(q; p; x_1, \ldots, x_n)
\]
(34)
where \(K(q; p; x_1, \ldots, x_n)\) defined as in (32).
Proof The right side inequality of (34) follows by inequality (31). From Theorem 2, the function $f_{1,1}(q; x)$ is logarithmically concave. Combining this fact with Jensen’s inequality for convex functions we obtain the left side inequality of (34).

**Corollary 3** Let $q \in (0, 1)$ and $a, b$ be a reals numbers such that $0 < a < b$. Then the following inequalities

\[
\left( \frac{1-q^b}{1-q} \right)^{b-1} \frac{1-q^a}{1-q}\exp\left( \frac{Li_2(1-q^b) - Li_2(1-q^a)}{\log q} \right) \leq \Gamma_q(b) \leq \frac{1-q^b}{1-q}\exp\left( \frac{Li_2(1-q^b) - Li_2(1-q^a)}{\log q} \right)
\]

holds.

**Proof** From the monotonicity of the functions $f_{1,1/2}(q; x)$ and $[f_{1,1}(q; x)]^{-1}$ and the recurrence formula

\[
\Gamma_q(x + 1) = \frac{1-q^x}{1-q} \Gamma_q(x),
\]

we obtain the inequalities (35).

**Corollary 4** Let $q \in (0, 1)$ the following inequalities

\[
\left( \frac{1-q^x}{1-q} \right)^x \exp\left( - \frac{Li_2(1-q)}{\log q} \right) \exp\left( \frac{Li_2(1-q^x)}{\log q} \right) \leq \Gamma_q(x + 1)
\]

\[
\leq \left( \frac{1-q^x}{1-q} \right)^{x+1/2} \exp\left( - \frac{Li_2(1-q)}{\log q} \right) \exp\left( \frac{Li_2(1-q^x)}{\log q} \right)
\]

holds for all $x \in [1, \infty)$.

**Proof** As the function $f_{1/2,1}(q; x)$ is logarithmically completely monotonic, $f_{1/2,1}(q; x)$ is also decreasing. The following inequality hold true for every $x \geq 1$:

\[
f_{1/2,1}(q; x) \leq f_{1/2,1}(q; 1).
\]

In addition, as $1/f_{1,1}(q; x)$ is logarithmically completely monotonic, we deduce that $f_{1,1}(q; x)$ is increasing. The following inequality hold true for all $x \geq 1$:

\[
f_{1,1}(q; x) \leq f_{1,1}(q; x).
\]

Combining inequalities (38) and (39) we obtain the inequalities (37).
In the next Corollary we present new estimates for Stirling’s formula remainder \( r_n \).

**Corollary 5** The following inequalities hold true for every integer \( n \geq 1 \):

\[
e \cdot \left( \frac{n}{e} \right)^n \leq n! \leq e \cdot \sqrt{n} \left( \frac{n}{e} \right)^n,
\]

and

\[
1 - \frac{\log 2\pi n}{2} \leq r_n \leq 1 - \frac{\log 2\pi}{2}
\]

In each of the above inequalities equality hold if and only if \( n = 1 \).

**Proof** Letting \( q \to 1 \) in (37). Since

\[
\lim_{q \to 1} \left( \frac{1 - q^x}{1 - q} \right)^x = \lim_{q \to 1} \exp \left[ x \log \left( \frac{1 - q^x}{1 - q} \right) \right] = \lim_{q \to 1} \exp \left[ x \log \left( \frac{x \log(q)}{q - 1} \right) \right] = x^x,
\]

and using (3) and (10) we obtain

\[
x^x e^{1-x} \leq \Gamma(x + 1) \leq x^{x+1/2} e^{1-x}.
\]

Replacing \( x \) by \( n \geq 1 \) in (42) we get (40). Finally, from the definition of the sequence \( r_n \) we have

\[
e^{r_n} = \left( \frac{e}{n} \right)^n \cdot \frac{n!}{\sqrt{2\pi n}}
\]

Combining (40) and (43) we conclude that (41) is valid. \( \square \)

**5 Concluding remarks**

In this section we would like to comment the main results of this paper.

1. It is worth mentioning that the inequality (35) when letting \( q \) tends to 1, returns to the inequalities [(2), p. 407]

\[
\frac{b^{b-1}}{a^{a-1}} e^{a-b} \leq \frac{\Gamma(a)}{\Gamma(b)} < \frac{b^{b-1/2}}{a^{a-1/2}} e^{a-b},
\]

for \( b > a > 0 \).
2. Let \( n = 2 \) and \( p_k = 1/2, \ k = 1, 2 \) in inequalities (34) we obtain the lower bounds for the \( q \)-analogue for Gurland’s ratio (5) as follows

\[
\left( \frac{1-q^x}{1-q} \right)^{x+y} \leq \Gamma_q^2 \left( \frac{x+y+2}{2} \right) \frac{\Gamma_q(x+1)\Gamma_q(y+1)}{\exp \left( \text{Li}_2(1-q^x) + \text{Li}_2(1-q^y) - 2 \text{Li}_2(1-q^{x+y/2}) \right)} \log q \leq \left( \frac{1-q^x}{1-q} \right)^{x+y} \left( \frac{1-q^y}{1-q} \right)^{y+1/2} \cdot (45)
\]

where \( x, y \in (0, \infty) \). In particular, let \( q \) tends to 1 in (45) we get

\[
\left( \frac{x+y}{2} \right)^{x+y} \leq \Gamma(x+1)\Gamma(y+1) \leq \left( \frac{x+y}{2} \right)^{x+y} \cdot (46)
\]

The left hand side inequalities of (46) has proved first by Mortici [(7), p. 188] and the right hand side of inequalities (46) is new.

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