TIME-CONSISTENT INVESTMENT-REINSURANCE STRATEGY WITH A DEFAULTABLE SECURITY UNDER AMBIGUOUS ENVIRONMENT

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Abstract. This paper considers an investment and reinsurance problem with a defaultable security for an insurer in an environment with parameter uncertainties. Suppose that the insurer is ambiguous about the insurance claims. Specifically, the insurance claim is exponentially distributed and the rate parameter is uncertain. The insurer is allowed to invest in a financial market consisting of a risk-free bond, a stock whose price process satisfies the Heston’s SV model and a defaultable bond. Moreover, the insurer is allowed to purchase proportional reinsurance and aims to maximize the smooth ambiguity utility proposed in Klibanoff et al. [15]. By applying stochastic control approach, we establish the extended HJB system and derive the time-consistent investment-reinsurance strategy for the post-default case and the pre-default case, respectively. Finally, a sensitivity analysis is provided to illustrate the effects of model parameters on the equilibrium reinsurance-investment strategy under the smooth ambiguity.

1. Introduction. Research on optimal reinsurance and investment problems has long been a spotlight in actuarial science. For example, Browne [4] considers a diffusion risk model and obtains optimal investment strategies for exponential utility maximization and ruin probability minimization. Yang and Zhang [23] study the same investment problem for an insurer with jump-diffusion risk model. Hipp and Plum [13] assume the insurer can invest in a risky asset and obtain the optimal investment strategy for ruin probability minimization. Liu and Yang [19] extend the model of Hipp and Plum [13] to incorporate a non-zero interest rate. But in this case, a closed-form solution cannot be obtained. In [22], optimal investment strategies of different utilities maximization are obtained for an insurer via the martingale approach. With regard to reinsurance, Promislow and Young [20] obtain the optimal reinsurance-investment strategy for an insurer to minimize the ruin probability. Bai and Guo [1] consider an optimal proportional reinsurance and investment problem with multiple risky assets for a diffusion risk model. Cao and

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Wan [5] investigate the proportional reinsurance and investment problem of utility maximization for an insurer. In [25], time-consistent investment and proportional reinsurance strategies are obtained under the mean-variance criterion for an insurer.

However, the above mentioned researches generally assume that the risky asset’s prices are driven by geometric Brownian motions (GBMs), which implies the volatilities of risky asset’s prices are constant and deterministic. But empirical analysis shows that the volatility is not constant and a model with stochastic volatility will be more practical, see e.g. [14] and the references therein. Therefore, many works propose various stochastic volatility (SV) models, such as constant elasticity of variance (CEV) model ([7]) and Heston’s SV model ([12]), etc. Gu et al. [10], Lin and Li [18] and Liang et al. [17] investigate the optimal reinsurance and investment problem for an insurer under the CEV model. Li et al. [16] begin to consider the Heston’s SV model and study the reinsurance and investment problem under the mean-variance criterion. Zhao et al. [27] investigate the optimal excess-of-loss reinsurance and investment problem for an insurer under the Heston’s SV model. Yi et al. [24] derive the robust optimal strategy for an insurer under Heston’s SV model.

Although optimal reinsurance and investment problems have been extensively studied, most previous works assume that the financial market consists of one risk-free asset and one stock/multiple stocks. The credit or default risk is rarely considered in the modeling framework. But in recent years, institutional investors, such as insurers and pension funds, are actively participating in trading high yield bonds with default risk, say, corporate bonds. Thus it is of great relevance to investigate the impact of adding defaultable securities into portfolio selection problems. Zhu et al. [30] focus on the optimal proportional reinsurance and investment problem in a defaultable market to maximize the expected exponential utility of the terminal wealth. Zhao et al. [28] study an optimal investment and reinsurance problem involving a defaultable security for an insurer under the mean-variance criterion in a jump-diffusion risk model. Sun et al. [21] extend the work of Zhao et al. [28] to incorporate the model uncertainty. Deng et al. [8] investigate non-zero-sum stochastic differential reinsurance and investment games for two CARA insurers with default risk under the Heston’s SV model.

Moreover, the parameters in the financial model are difficult to estimate with precision, it is desirable to take model uncertainty into account. Recently, many works investigate the impact of model uncertainty on portfolio selection problem. The general formulation of ambiguity involves a class of equivalent probability measures \( Q = \{ Q \mid Q \sim P \} \) and searches solving the following two-step max-min problem

\[
\sup_{\pi} \inf_{Q \in \mathcal{Q}} E^Q[U(X^\pi(T))].
\]

Usually, in order to characterize the deviation from the reference model, a term of penalty is involved in the optimization objective, see Yi et al. [24]. Zheng et al. [29] study an optimal proportional reinsurance and investment problem with ambiguity under the CEV model. Zeng et al. [26] derive the robust proportional reinsurance-investment strategy under the mean-variance criterion for an ambiguity-averse insurer (AAI). More recent works about the AAI can refer to Sun et al. [21], Gu et al. [9] and the references therein.

In general, the robust strategy for the AAI is made under the worst case. However, the worst case solution under the equivalent probability measures often results
in significant utility loss. Therefore, we are concerned with the strategies under average case, which is formulated in [15]. In [15], investor is uncertain about the real probability law and assigns a subjective measurement of the possibility for different laws. Klibanoff et al. [15] assume that the investor is risk averse (the known probability distribution) towards ambiguity and try to find the average case solution, they propose the following decision making model under smooth ambiguity

$$\sup_{\pi, Q \in \mathcal{Q}} E^\mu \phi(E^Q[U(X^\pi(T))]),$$

where $\phi$ depicts the attitude towards ambiguity and $\mu$ is a subjective probability over the set $\mathcal{Q}$. Chen et al. [6] investigate the effects of ambiguity and ambiguity aversion (the unknown probability distribution) on prices of mortality-linked securities with smooth ambiguity. Guan et al. [11] study the equilibrium proportional reinsurance and investment strategies for an insurer with the smooth ambiguity utility. However, since the smooth ambiguity model involves two expectations and two utility functions, there is few research about investor’s behavior under smooth ambiguity.

In this paper, we study the optimal reinsurance and investment problem for an insurer with smooth ambiguity utility under default risk. The insurer’s surplus process is described by the classical Cramér-Lundberg risk model and the insurance claim is supposed to be exponential distributed. However, the insurer holds uncertain beliefs over the rate parameter of the exponential distribution. Moreover, the insurer is allowed to purchase proportional reinsurance and invest in a risk-free bond, a stock and a defaultable bond. Besides, the price process of the stock follows the Heston’s SV model. The insurer aims to maximize the smooth ambiguity utility function proposed in Klibanoff et al. [15]. Since the utility function does not satisfy the general assumption in traditional dynamic programming, the optimization problem for the insurer in this situation is not time-consistent. Therefore, we introduce the concept of Nash equilibrium and define the equilibrium strategy in this paper. The extended HJB systems of equations are derived for the post-default case and the pre-default case, respectively. Using ansatz and variable separation techniques, we obtain the time-consistent equilibrium reinsurance-investment strategies and the corresponding value functions in both cases. Finally, we present a detailed sensitivity analysis to illustrate the effects of model parameters on the equilibrium reinsurance-investment strategy.

The main contributions of this paper are: (i) We formulate an analytical framework for an insurer in a smooth ambiguity model, which takes model uncertainty into account and derives the strategy under average case. (ii) Results show that the uncertain beliefs (the parameters of financial and insurance markets are difficult to estimate with precision) play an important role in the equilibrium reinsurance-investment strategy. When the insurer is more risk averse towards ambiguity, the insurer will undertake fewer reinsurance risk. (iii) Default risk is considered and we find that the optimal investment strategy in the defaultable bond depends on the parameters of stock and risk model, but the optimal investment in the stock is independent with the parameters of defaultable bond.

The remainder of this paper is organized as follows. Section 2 describes the formulation of the model. Section 3 introduces the optimization problem for the insurer with smooth ambiguity utility. Section 4 derives the time-consistent equilibrium reinsurance-investment strategy for the post-default and the pre-default cases,
respectively. In Section 5, we provide sensitivity analyses to illustrate our results. Section 6 concludes the paper.

2. Model formulation. Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0,T]}, P)\) is a filtered complete probability space and \(\mathbb{F} := \{\mathcal{F}_t\}_{t \in [0,T]}\), where \(\mathcal{F}_t\) is the information of the market available to time \(t\). \([0,T]\) is a fixed transaction time horizon. All the processes below are assumed to be well defined and adapted to \(\{\mathcal{F}_t\}_{t \in [0,T]}\). Denote \(\mathbb{H} := \{\mathcal{H}_t\}_{t \in [0,T]}\) the filtration of a default process \(\{H(t)\}_{t \in [0,T]}\). Let \(\mathbb{G} := \{\mathcal{G}_t\}_{t \in [0,T]}\) be the enlarged filtration of \(\mathbb{F}\) and \(\mathbb{H}\), i.e., \(\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t\), satisfying the usual conditions of right-continuity and \(P\)-completeness. We will specify \(\mathbb{F}\) and \(\mathbb{H}\) later in this section.

As in the previous literature on credit risk modeling (e.g., Bielecki and Rutkowski [3]), we assume that the H-hypothesis holds, that is, every \(F\)-martingale is also a \(\mathbb{G}\)-martingale. Throughout this paper, \(P\) is a reference probability measure in the real world and \(Q\) denotes the risk-neutral probability measure.

2.1. Surplus process. Suppose that the surplus process of the insurer is described by the classical Cramér-Lundberg model, i.e.,

\[
dR(t) = cd t + \beta dW(t) - dC(t),
\]

where \(c\) is the premium rate, \(\{W(t)\}\) is a standard Brownian motion, \(\beta dW(t)\) stands for the uncertainty associated with the insurer’s surplus at time \(t\) and \(C(t)\) represents the cumulative claims up to time \(t\). We assume that \(C(t) = \sum_{i=1}^{N(t)} Y_i\), which is a compound Poisson process, where \(Y_i\) represents the size of the \(i\)th claim and \(\{N(t)\}\) is a homogeneous Poisson process with intensity \(\lambda > 0\). All the claims \(Y_i, i = 1, 2, 3, \cdots\) are assumed to be independent and identically distributed positive random variables. \(N(t)\) is supposed to be independent of \(\{Y_i\}_{i \geq 1}\) and \(\sum_{i=1}^{N(t)} Y_i\) is independent of \(\{W(t)\}\). In addition, we assume that the insurer is ambiguous about the distribution of \(Y_i\). Let \(\nu\) be a random variable on probability space \((\Omega, \mathcal{F}, P)\). In the insurer’s belief, the distribution of \(Y_i\) is given by \(F^\nu(y)\). Specifically, we assume that \(F^\nu(y)\) is exponentially distributed with parameter \(\nu\), i.e.,

\[
F^\nu(y) = P\{Y_i \leq y\mid \nu = x\} = 1 - e^{-xy}, \quad y \geq 0.
\]

We suppose that the support set of \(\nu\) is \(Y \subset \mathbb{R}\) and \(Y \subset [\nu_0, +\infty)\), \(\nu_0 > 0\). Denote the first moment of \(Y_i\) by \(E[Y_i] = \mu_1\) and the premium is calculated according to the expected value principle, i.e., \(c = (1 + \theta)\lambda\mu_1\), where \(\eta > 0\) is the insurer’s safety loading.

The insurer is allowed to purchase proportional reinsurance or acquire new business to adjust the exposure to insurance risk. Let \(a(t) \geq 0\) be the risk exposure of the insurer at time \(t\). When \(a(t) \in [0, 1]\), it corresponds to a proportional reinsurance cover, the insurer disperses \(100(1 - a(t))\%\) of the insurance risk to the reinsurer at time \(t\). The reinsurance premium is \((1 + \theta)\lambda\mu_1(1 - a(t))\) at time \(t\) according to the expected value premium principle, where \(\theta\) is the reinsurer’s relative safety loading satisfying \(\theta > \eta\). When \(a(t) \in (1, +\infty)\), it indicates that the insurer acquires new business. Then the surplus process of the insurer after purchasing reinsurance is governed by

\[
dR^a(t) = [(1 + \eta)\lambda\mu_1 - (1 + \theta)(1 - a(t))\lambda\mu_1]dt + \beta dW(t) - a(t)\left(\sum_{i=1}^{N(t)} Y_i\right)
\]
of the risk-free bond is given by

\[ \lambda u_1 [a(t)(1 + \theta) - (\theta - \eta)]dt + \beta d\bar{W}(t) - a(t)d\left(\sum_{i=1}^{N(t)} Y_i\right). \]  \hfill (2.2)

2.2. Financial market. We consider a financial market consisting of one risk-free bond, one stock and one defaultable corporate zero-coupon bond. The price process of the risk-free bond is given by

\[ dB(t) = rB(t)dt, \quad B(0) = b_0 > 0, \]  \hfill (2.3)

where \( r > 0 \) is the risk-free interest rate. The price process of the stock is described by the Heston's SV model

\[
\begin{aligned}
    dS(t) &= S(t) \left[ (r + \xi L(t))dt + \sqrt{L(t)}dW_1(t) \right], \quad S(0) = s_0 > 0, \\
    dL(t) &= \gamma(k - L(t))dt + \sigma \sqrt{L(t)}dW_2(t), \quad L(0) = l_0 > 0,
\end{aligned}
\]  \hfill (2.4)

where \( \xi > 0 \) is the premium for volatility; \( \gamma > 0 \) denotes the mean-reversion rate; \( k > 0 \) represents the long-run mean and \( \sigma > 0 \) is the volatility of the volatility parameter. \( \{W_1(t)\} \) and \( \{W_2(t)\} \) are two standard Brownian motions with \( E[W_1(t)W_2(t)] = \rho t \) in which \( \rho \in [-1, 1] \). Moreover, we require \( 2\gamma k \geq \sigma^2 \) to ensure that \( L(t) \) is almost surely nonnegative and we assume that \( \{\bar{W}(t)\} \) is independent of \( \{W_1(t)\} \) and \( \{W_2(t)\} \).

We now consider a zero-coupon defaultable bond with a one-unit nominal value and a maturity date \( T_1 > T \). Let \( \tau \) be a nonnegative random variable defined on \( (\Omega, \mathcal{F}, \mathbb{P}) \), representing the default time of the company issuing the bond. Define a default indicator process by

\[ H(t) := 1_{\{\tau \leq t\}}, \]

where \( 1 \) denotes the indicator function that equals one if there is a jump and zero otherwise. Then, \( H(t) = 0 \) and \( H(t) = 1 \) correspond to the pre-default case \( \tau > t \) and the post-default case \( \tau \leq t \), respectively. This implies that the default process makes discrete jumps at the random time \( \tau \). Moreover, we assume that the default process \( \{H(t)\} \) has a constant intensity \( h^P > 0 \) under probability measure \( \mathbb{P} \). Let \( \mathbb{H} \) be the filtration generated by the default process \( H(t) \). By definition, \( \tau \) is naturally a \( \mathbb{H} \)-stopping time and a \( \mathbb{G} \)-stopping time. Then the process \( M^P(t) \) defined by

\[ M^P(t) := H(t) - \int_0^t (1 - H(u^-))h^P du \]

is a \( \mathbb{G} \)-martingale under \( \mathbb{P} \). Suppose that if a default occurs, the investor recovers a fraction of the market value of the defaultable bond just prior to default and then the post-default value of the defaultable bond is zero. We use \( 0 \leq \zeta \leq 1 \) to denote the constant loss rate when a default occurs, and \( 1 - \zeta \) is the default recovery rate.

As in Bielecki and Jang [2], we denote by \( 1/\Delta := h^Q/h^P \) the default risk premium. According to Bielecki and Jang [2], the dynamics of the defaultable bond under \( \mathbb{P} \) is given by

\[ dp(t, T_1) = p(t, T_1) \left[ rdt + (1 - H(t))(1 - \Delta)\delta dt - (1 - H(t^-))\zeta dM^P(t) \right], \]  \hfill (2.5)

where \( \delta = h^Q/\zeta \) is the risk neutral credit spread.

Besides purchasing proportional reinsurance, the insurer is allowed to invest in the financial market described above. Denoting the money amount invested in the stock and the defaultable bond at time \( t \) by \( \pi_s(t) \) and \( \pi_p(t) \), respectively. Considering the proportional reinsurance strategy \( a(t) \), we denote the trading strategy by
Substituting (2.2), (2.3), (2.4) and (2.5) into (2.6), we have follows

$$
\begin{aligned}
\pi = \{(a(t), \pi_a(t), \pi_p(t)) : 0 \leq t \leq T\}. \text{ Then the surplus process associated with } \pi \\
\text{follows}
\end{aligned}
$$

$$
dX^\pi(t) = dR^\pi(t) + (X^\pi(t) - \pi_a(t) - \pi_p(t)) \frac{dB(t)}{B(t)} + \pi_a(t) \frac{dS(t)}{S(t)} + \pi_p(t) \frac{dp(t, T_1)}{p(t, T_1)}. \tag{2.6}
$$

Substituting (2.2), (2.3), (2.4) and (2.5) into (2.6), we have

$$
egin{cases}
\pi \mu_1[(1 + \theta) \alpha(t) - (\theta - \eta)] + rX^\pi(t) + \xi \pi_a(t)L(t) \\
\pi_p(t)[1 - H(t)](1 - \Delta)\delta dt + \beta d\bar{W}(t) - a(t)d\left(\sum_{i=1}^{N(t)} Y_i\right) \\
\pi \mu_1[(1 + \theta) \alpha(t) - (\theta - \eta)] + rX^\pi(t) + \xi \pi_a(t)L(t) \\
\pi_p(t)[1 - H(t)](1 - \Delta)\delta dt + \beta d\bar{W}(t) - a(t)d\left(\sum_{i=1}^{N(t)} Y_i\right) \\
\pi \mu_1[(1 + \theta) \alpha(t) - (\theta - \eta)] + rX^\pi(t) + \xi \pi_a(t)L(t) \\
\pi_p(t)[1 - H(t)]\delta dt + \beta d\bar{W}(t) - a(t)d\left(\sum_{i=1}^{N(t)} Y_i\right) \\
\pi_a(t)\sqrt{L(t)}dW_1(t) - \pi_p(t)\zeta dM^\pi(t) \\
\pi \mu_1[(1 + \theta) \alpha(t) - (\theta - \eta)] + rX^\pi(t) + \xi \pi_a(t)L(t) \\
\pi_p(t)[1 - H(t)]\delta dt + \beta d\bar{W}(t) - a(t)d\left(\sum_{i=1}^{N(t)} Y_i\right) \\
\pi \mu_1[(1 + \theta) \alpha(t) - (\theta - \eta)] + rX^\pi(t) + \xi \pi_a(t)L(t) \\
\pi_p(t)[1 - H(t)]\delta dt + \beta d\bar{W}(t) - a(t)d\left(\sum_{i=1}^{N(t)} Y_i\right) \\
\pi_a(t)\sqrt{L(t)}dW_1(t) - \pi_p(t)\zeta dH(t) \\
\end{cases}
\tag{2.7}
$$

$$
X^\pi(0) = x_0
$$

with $$(1 - H(t-))dM^\pi(t) = dM^\eta(t)$$ and under the convention that $$0/0 = 0$$. This convention is needed to deal with the post-default case, i.e., $$\tau \leq t$$, so that $$p(t-, T_1) = 0$$ and we fix $$\pi_p(t) = 0$$ afterwards.

**Definition 2.1.** A reinsurance-investment strategy $$\pi = \{(a(t), \pi_a(t), \pi_p(t)), t \in [0, T]\}$$ is said to be admissible if

(i) $$\pi(t)$$ is a $$\mathbb{G}$$-progressively measurable process;

(ii) $$\forall t \in [0, T], a(t) \geq 0$$ and $$E\left[\int_0^t (a^2(u) + \pi_a^2(u) + \pi_p^2(u))du\right] < +\infty$$;

(iii) $$(\pi, X^\pi(t))$$ is the unique strong solution to the stochastic differential equation (2.7).

For any initial condition $$(t, x, l, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^+ \times \{0, 1\}$$, the corresponding set of all admissible strategies is denoted by $$\Pi$$. Here $$z$$ denotes the default state with $$z = 0$$ and $$z = 1$$ corresponding to the pre-default case ($$\tau > t$$) and the post-default case ($$\tau \leq t$$), respectively.

### 3. Reinsurance and investment problem under ambiguous environment.

Inspired by Klibanoff et al. [15] and Guan et al. [11], we assume that the insurer makes decision under ambiguity in a smooth framework. Denote $$\text{P}^\nu$$ the probability generated by $$\nu$$ and then the expectation under $$\text{P}^\nu$$ is denoted by $$E^\text{P}^\nu$$. And the expectation w.r.t. $$\nu$$ is denoted by $$E^\nu$$. Under probability measure $$\text{P}^\nu$$, $$\nu$$ can be viewed as a constant and the ambiguity over $$\nu$$ is depicted by distribution $$F^\nu(x)$$. 

Suppose that the insurer aims to maximize the expected utility of terminal wealth under ambiguous environment, i.e.,
\[
\sup_{\pi \in \Pi} \left\{ E^\nu \phi \left( E^{P^\nu}_{t,x,l,z} [U(X(t))] \right) \right\},
\]
where \( U(\cdot) \) is a von Neumann-Morgenstern utility function, \( \phi(\cdot) \) is an increasing transformation and \( U(\cdot), \phi(\cdot) \) are both utility functions. As mentioned in Klibanoff et al. [15], a key feature of this model is that it achieves a separation between ambiguity and ambiguity aversion. Ambiguity is depicted by \((\nu,F^\nu(x))\) and ambiguity aversion is depicted by \(\phi(\cdot)\). \(\phi(x) = x\) indicates that the insurer is ambiguity neutral and if \(\phi(x)\) is concave, the insurer is ambiguity averse.

If there is only one element in \(\Upsilon\), the above problem is equivalent to the traditional optimization problem with no ambiguity. However, when \(\Upsilon\) has at least two elements, the optimization problem is time-inconsistent caused by the nonlinear function \(\phi(\cdot)\). Therefore, the traditional Bellman’s stochastic principle of optimality cannot be applied here. In order to solve the time-inconsistent problem, we define the equilibrium strategy for our model according to game theory and derive the extended HJB system of equations for post-default and pre-default cases, respectively.

The insurer’s objective is
\[
\sup_{\pi \in \Pi} \left\{ E^\nu \phi \left( E^{P^\nu}_{t,x,l,z} [U(X(t))] \right| X(t) = x, L(t) = l, H(t) = z \right\}
\]
and define
\[
J(t,x,l,z,\pi) = E^\nu \phi \left( E^{P^\nu}_{t,x,l,z} [U(X(t))] \right| X(t) = x, L(t) = l, H(t) = z \right).
\]
In the following, we provide the definition of equilibrium strategy.

**Definition 3.1.** For any fixed initial state \((t,x,l,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^+ \times \{0,1\}\), consider an admissible strategy \(\pi^*\) (a candidate equilibrium control). For any fixed admissible strategy \(\pi\) and \(\varepsilon > 0\), we define a new strategy \(\pi_\varepsilon\) by
\[
\pi_\varepsilon(s,x,l,z) = \begin{cases} 
\pi(s,x,l,z), & \text{for } (s,x,l,z) \in [t,T] \times \mathbb{R} \times \mathbb{R}^+ \times \{0,1\}, \\
\pi^*(s,x,l,z), & \text{for } (s,x,l,z) \in [t+\varepsilon,T] \times \mathbb{R} \times \mathbb{R}^+ \times \{0,1\}.
\end{cases}
\]
If
\[
\lim_{\varepsilon \to 0^+} \inf_{\pi \in \Pi} \frac{J(t,x,l,z,\pi^*) - J(t,x,l,z,\pi_\varepsilon)}{\varepsilon} \geq 0
\]
for all \(\pi \in \Pi\), then \(\pi^*\) is called an equilibrium strategy and the equilibrium value function is
\[
V(t,x,l,z) = J(t,x,l,z,\pi^*).
\]
Firstly, we specify the form of $U(z)$ in the post-default case. Then, we derive the equilibrium reinsurance-strategy is given by $\pi$. Here the random variable $\nu$ is denoted as a generic claim size with the same distribution as $Y_1$.

The following theorem provides verification theorems for the post-default case ($z = 1$) and the pre-default case ($z = 0$), respectively.

**Theorem 3.2.** (Verification Theorem). For the post-default case ($z = 1$) and pre-default case ($z = 0$), if there is a function $g(t, x, l, \nu, z) \in C^{1,2,0}_a([0, T] \times \mathbb{R} \times \mathbb{R}^+ \times \Upsilon \times \{0, 1\})$ and a related process defined by $F(t, x, l, z) = E^\nu F_\nu g(t, x, l, \nu, z) \in C^{1,2}_a([0, T] \times \mathbb{R} \times \mathbb{R}^+ \times \{0, 1\})$ satisfying the following conditions: $\forall(t, x, l, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^+ \times \{0, 1\},$

$$\sup_{\pi \in \Pi} \{ E^\nu A^\nu_\pi F(t, x, l, z) + E^\nu E^\nu [\phi'(g(t, x, l, \bar{\nu}, z)) A^\nu_\pi g(t, x, l, \bar{\nu}, z) - A^\nu_\pi \phi \circ g(t, x, l, \bar{\nu}, z)] \} = 0,$$

$$F(T, x, l, z) = \phi(U(x)), \quad \text{(3.5)}$$

$$A^\nu_\pi g(t, x, l, \nu, z) = 0, \quad \nu \in \Upsilon, \quad \text{(3.6)}$$

$$g(T, x, l, \nu, z) = U(x), \quad \nu \in \Upsilon, \quad \text{(3.7)}$$

where $\bar{\nu}$ has the distribution $F^\nu(x)$ and independent of $\nu$, let

$$\pi^* := \arg \sup_{\pi \in \Pi} \{ E^\nu A^\nu_\pi F(t, x, l, z) + E^\nu E^\nu [\phi'(g(t, x, l, \bar{\nu}, z)) A^\nu_\pi g(t, x, l, \bar{\nu}, z) - A^\nu_\pi \phi \circ g(t, x, l, \bar{\nu}, z)] \},$$

then $V(t, x, l, z) = F(t, x, l, z), \quad E^\nu_\Pi \{ U(X^{\pi^*}(T)) \} = g(t, x, l, \nu, z)$ and the equilibrium strategy is given by $\pi^*$.

**Proof.** The proof is similar to Theorem 3.1 in Guan et al. [11], thus we omit it. \qed

4. Solution to the model. In this section, we derive the equilibrium reinsurance-investment strategies and the corresponding equilibrium value functions for problem (3.1) in the post-default case ($z = 1$) and pre-default case ($z = 0$), respectively. Firstly, we specify the form of $U(x)$ and $\phi(x)$. Chen et al. [6] use an exponential-power specification for $(U(x), \phi(x))$. Following their work, the utility functions $U(x)$
In post-default case, the equilibrium investment strategy is

\[ \pi^*_\nu(t) = \frac{\xi}{b(t)} + \frac{n(t)\sigma\rho}{b(t)} \]  

and the equilibrium proportional reinsurance strategy \( a^*(t) \) satisfies

\[ \mu_1(1 + \theta)\mathbb{E}^\nu \exp(\alpha m(t, \nu)) = \mathbb{E}^\nu \left[ \frac{\nu}{(\nu - a^*(t)b(t))^2} \right]. \]  

The value function is given by

\[ V(t, x, l, 1) = F(t, x, l, 1) = \mathbb{E}^\nu \phi \circ g(t, x, l, \nu, 1), \]  

where \( g(t, x, l, \nu, 1) = -\frac{1}{\xi} \exp[-b(t)x + m(t, \nu) + n(t)l]. \)

Moreover, \( b(t) = k \exp(r(T - t)) \),

\[ n(t) = \begin{cases} \frac{\xi^2}{2(\gamma + \sigma\xi)} [e^{-\gamma + \sigma\xi}(T - t) - 1], & \rho = 1, \\ \frac{1}{2} \xi^2(T - t), & \rho = -1 \text{ and } \gamma = \xi\sigma, \\ \frac{\xi^2}{2(\gamma - \sigma\xi)} [e^{-\gamma - \sigma\xi}(T - t) - 1], & \rho = -1 \text{ and } \gamma \neq \xi\sigma, \\ \frac{q_1 q_2}{q_1 \exp(\sqrt{\Delta}(T - t) - q_1 q_2)} - \frac{q_1 q_2}{q_1 \exp(\sqrt{\Delta}(T - t) - q_2)}, & \rho \neq \pm 1, \end{cases} \]  

and \( \phi(x) \) are given by

\[ U(x) = -\frac{1}{k} e^{-kx}, \quad k > 0, \]

\[ \phi(x) = -(x)^{\alpha}, \quad \alpha > 1, \]

where \( k \) and \( \alpha \) are the relative risk aversions. \( U(x) \) is the standard CARA utility function and \( \phi(x) \) is the standard CRRA utility function. In this formulation, \( k \) characterizes the insurer’s risk aversion over the market risks while \( \alpha \) measures the risk aversion over ambiguity.

Since \( F(t, x, l, z) = \mathbb{E}^\nu \phi \circ g(t, x, l, \nu, z), \nu \) and \( \tilde{\nu} \) have the same distribution, we have \( F(t, x, l, z) = \mathbb{E}^\nu \phi \circ g(t, x, l, \tilde{\nu}, z). \) Thus,

\[ \mathbb{E}^\nu \mathbb{E}^\nu \mathcal{A}_\nu^+ \phi \circ g(t, x, l, \tilde{\nu}, z) = \mathbb{E}^\nu \mathcal{A}_\nu^+ \mathbb{E}^\nu \phi \circ g(t, x, l, \tilde{\nu}, z) = \mathbb{E}^\nu \mathcal{A}_\nu^+ F(t, x, l, z). \]

Therefore the first and the third term in (3.4) can be cancelled out and the second term in (3.4) can be rewritten as follows

\[ \mathbb{E}^\nu \mathbb{E}^\nu \phi' \left(g(t, x, l, \tilde{\nu}, z)\right) \mathcal{A}_\nu^+ g(t, x, l, \tilde{\nu}, z) = \mathbb{E}^\nu \phi' \left(g(t, x, l, \tilde{\nu}, z)\right) \mathcal{A}_\nu^+ g(t, x, l, \tilde{\nu}, z). \]  

We omit \( \mathbb{E}^\nu \) in (4.1) because the function inside has no relationship with \( \nu \). Furthermore, since we can replace \( \tilde{\nu} \) by \( \nu \), HJB equation (3.4) can be reduced to

\[ \sup_{\pi \in \Pi} \{ \mathbb{E}^\nu \phi' \left(g(t, x, l, \nu, z)\right) \mathcal{A}_\nu^+ g(t, x, l, \nu, z) \} = 0. \]  

4.1. Post-default case: \( z = 1 \). In the post-default case, we have that \( p(t, T_1) = 0, t \leq T \). Thus \( \pi_\rho(t) = 0, \) for \( t \leq T \). The following theorem provides the equilibrium strategy and the corresponding value function in post-default case.

**Theorem 4.1.** In post-default case, the equilibrium investment strategy is

\[ \pi^*_\nu(t) = \frac{\xi}{b(t)} + \frac{n(t)\sigma\rho}{b(t)} \]  

and the equilibrium proportional reinsurance strategy \( a^*(t) \) satisfies

\[ \mu_1(1 + \theta)\mathbb{E}^\nu \exp(\alpha m(t, \nu)) = \mathbb{E}^\nu \left[ \exp(\alpha m(t, \nu)) \frac{\nu}{(\nu - a^*(t)b(t))^2} \right]. \]  

The value function is given by

\[ V(t, x, l, 1) = F(t, x, l, 1) = \mathbb{E}^\nu \phi \circ g(t, x, l, \nu, 1), \]  

where \( g(t, x, l, \nu, 1) = -\frac{1}{\xi} \exp[-b(t)x + m(t, \nu) + n(t)l]. \)
where \( \Delta = (\gamma + \sigma \rho \xi)^2 + \sigma^2 \xi^2 (1 - \rho^2) \), \( q_{1,2} = \frac{(\gamma + \sigma \rho \xi) \pm \sqrt{\Delta}}{\sigma^2 (1 - \rho^2)} \). And

\[
m(t, \nu) = \int_t^T b(s) \{ \lambda m_1[\theta - \eta] - (1 + \theta) a^*(s) \} ds + \frac{1}{2} \beta^2 \int_t^T b^2(s) ds \\
+ \gamma \kappa \int_t^T n(s) ds + \lambda \int_t^T \frac{a^*(s)b(s)}{\nu - a^*(s)b(s)} ds.
\]

(4.7)

Furthermore, according to expression of \( n(t) \), \( m(t, \nu) \) are given by (A.13), (A.15), (A.17), (A.20).

Proof. Refer to Appendix A.

Remark 4.2. If equilibrium reinsurance strategy \( a^*(t) \) satisfies \( a^*(t)b(t) - \nu < 0 \), then

\[
E^\nu \exp[a^*(t)b(t)Y] - 1 = \frac{a^*(t)b(t)}{\nu - a^*(t)b(t)},
\]

\[
E^\nu \left[ \exp[a^*(t)b(t)Y] \right] = \frac{\nu}{(\nu - a^*(t)b(t))^2}.
\]

Proof. Refer to Appendix B.

4.2. Pre-default case: \( z = 0 \). The following theorem provides the equilibrium reinsurance-investment strategy and the corresponding value function in pre-default case.

Theorem 4.3. \(^1\) In pre-default case, the equilibrium investment strategy is

\[
\pi^*_s(t) = \frac{\xi}{ke^{r(T-t)}} + \tilde{n}(t) \rho, \quad \pi^*_p(t) = \frac{1}{ke^{r(T-t)}} \left( \ln \frac{E^\nu \exp[\alpha \tilde{m}(t, \nu)]}{\ln \frac{\delta}{k}} + \ln \frac{\delta}{k} \right)
\]

(4.8)

(4.9)

and the equilibrium proportional reinsurance strategy \( a^*(t) \) satisfies

\[
\mu_1 (1 + \theta) E^\nu \exp[\alpha \tilde{m}(t, \nu)] = E^\nu \left[ \exp[\alpha \tilde{m}(t, \nu)] \frac{\nu}{(\nu - ke^{r(T-t)} a^*(t))^2} \right].
\]

(4.10)

The value function is given by

\[
V(t, x, l, 0) = F(t, x, l, 0) = E^\nu \phi \circ g(t, x, l, \nu, 0),
\]

(4.11)

where \( g(t, x, l, \nu, 0) = -\frac{1}{2} \exp[-ke^{r(T-t)} x + \tilde{m}(t, \nu) + \tilde{n}(t) l]. \)

Moreover

\[
\tilde{n}(t, \nu) = n(t), \quad \tilde{m}(t, \nu) = \ln \left( 1 + \int_t^T \frac{Z(s) \delta e^{m(s, \nu)}}{\zeta} e^{\int_t^s A(s, \nu) ds} ds \right) - \int_t^T A(s, \nu) ds,
\]

(4.12)

(4.13)

where \( Z(t) = \frac{E^\nu \exp[\alpha \tilde{m}(t, \nu)]}{E^\nu \exp[\alpha \tilde{m}(t, \nu)] + \tilde{n}(t, \nu)} \) and \( n(t), m(t, \nu), A(t, \nu) \) are given by (4.6), (4.7), (C.19), respectively.

\(^1\)From Eqs. (4.3), (4.8) and (4.9), we find that the optimal investment strategies are deterministic functions of \( t \). However, we cannot show that \( a^*(t) > 0 \) from Eqs. (4.4) and (4.10) due to the complexity of coefficients equations. But in numerical example, for appropriate parameter values, we have that \( a^*(t) > 0 \) and it is bounded. Thus the optimal strategies satisfy the conditions of admissible strategies.
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Proof. Refer to Appendix C. \hfill \square

Remark 4.4. Note that the solutions to Eqs.(4.4) and (4.10) may not be positive. If the solutions of Eqs.(4.4) and (4.10) is negative, then the optimal reinsurance strategy \( a^*(t) = 0 \).

Remark 4.5. From Theorem 4.1 and Theorem 4.3, we find that the optimal amount invested in the stock are the same for the pre-default case and post-default case. This indicates that the parameters of defaultable bond have no effect on the investment in the stock. However, the parameters of stock do influence the investment in the defaultable bond.

Remark 4.6. The optimal investment strategy of the stock does not depend on the volatility process of the stock. However, the correlation coefficient between the stock’s price and its volatility has an influence on the investment in the stock.

Remark 4.7. If the insurer is ambiguity neutral, \( \Upsilon \) has only one element, the optimal investment strategy in post-default case reduces to

\[
\tilde{\pi}^*_s(t) = \frac{\xi}{\tilde{b}(t)} + \frac{\tilde{n}(t)\sigma \rho}{\tilde{b}(t)},
\]

and the optimal reinsurance strategy satisfies

\[
\mu_1 (1 + \theta) E^\nu \exp(\tilde{m}(t, \nu)) = E^\nu \left[ \exp(\tilde{m}(t, \nu)) \frac{\nu}{(\nu - \tilde{a}^*(t)\tilde{b}(t))^2} \right],
\]

where \( \tilde{b}(t) = b(t), \tilde{n}(t) = n(t) \) and

\[
\tilde{m}(t, \nu) = \int_t^T \tilde{b}(s) \{ \lambda \mu_1 [(\theta - \eta) - (1 + \theta)\tilde{a}^*(s)] \} ds + \frac{1}{2} \beta^2 \int_t^T \tilde{b}^2(s) ds
\]

\[+ \gamma \kappa \int_t^T \tilde{n}(s) ds + \lambda \int_t^T \frac{\tilde{a}^*(s)\tilde{b}(s)}{\nu - \tilde{a}^*(s)\tilde{b}(s)} ds.
\]

For the pre-default case, the optimal investment strategy is

\[
\tilde{\pi}^*_s(t) = \frac{\xi}{ke^{\rho(T-\tilde{t})}} + \frac{\tilde{n}(t)\sigma \rho}{ke^{\rho(T-\tilde{t})}},
\]

\[
\tilde{\pi}^*_p(t) = \frac{1}{ke^{\rho(T-\tilde{t})}\xi} \left( \ln E^\nu \exp(\tilde{m}(t, \nu)) + \ln \frac{\delta}{\tilde{h}^\nu \xi} \right)
\]

and the optimal reinsurance strategy satisfies

\[
\mu_1 (1 + \theta) E^\nu \exp(\tilde{m}(t, \nu)) = E^\nu \left[ \exp(\tilde{m}(t, \nu)) \frac{\nu}{(\nu - ke^{\rho(T-\nu)}\tilde{a}^*(t))^2} \right],
\]

where \( \tilde{n}(t) = \tilde{n}(t) \),

\[
\tilde{m}(t, \nu) = \ln \left( 1 + \int_t^T \tilde{Z}(t) \frac{\delta e^{\tilde{m}(s, \nu)}}{\xi} e^{\tilde{A}(s, \nu) ds} ds \right) - \int_t^T \tilde{A}(s, \nu) ds,
\]

\( \tilde{Z}(t) \) and \( \tilde{A}(t, \nu) \) are given by (D.26) and (D.27).

Proof. Refer to Appendix D. \hfill \square

Remark 4.8. From Theorems 4.1, 4.3 and Remark 4.7, we find that: (i) The optimal investment in the stock is not changed by ignoring model uncertainty. This is because that the risk model and the stock’s price are independent and the insurer is only ambiguous about insurance claims. (ii) The optimal demand for the
Remark 4.9. If the price process of the stock follows the geometric Brownian motion (GBM) model
\[ dS(t) = S(t)(\mu dt + \sigma_1 dW(t)), \quad S(0) = S_0 > 0, \]
where \( \mu \) denotes the appreciation rate, \( \sigma_1 \) is the volatility and \( \{W(t)\} \) is a standard Brownian motion independent of \( \{\mathbb{W}(t)\} \). Then the equilibrium investment strategy is
\[
\pi^*_s(t) = \frac{\mu - r}{ke^{r(T-t)}\sigma_1^2}, \quad (4.19)
\]
and the equilibrium reinsurance strategy satisfies (4.4) and (4.10) for the post-default case and pre-default case, respectively. Where

\[
m(t, \nu) = \int_t^T ke^{r(T-s)} \left\{ \lambda \mu_1[(\theta - \eta) - (1 + \theta)a^*(s)] \right\} ds + \frac{\beta^2 k^2}{4r}(e^{2r(T-t)} - 1)
- \frac{(\mu - r)^2}{2\sigma_1^2}(T - t) + \lambda \int_t^T k e^{r(T-s)} a^*(s) \frac{\nu - ke^{r(T-s)}a^*(s)}{\nu - ke^{r(T-s)}a^*(s)} ds,
\]
\[
\bar{m}(t, \nu) = \ln \left( 1 + \int_t^T Z(s) \delta e^{m(s, \nu)} \frac{A(s, \nu)ds}{\kappa A(s, \nu)ds} \right) - \int_t^T A(s, \nu)ds,
\]
\[
A(t, \nu) = -ke^{r(T-t)} \left\{ \lambda \mu_1[(1 + \theta)a^*(t) - (\theta - \eta)] \right\} + \frac{1}{2} \beta^2 k^2 e^{2r(T-t)}
- \frac{(\mu - r)^2}{2\sigma_1^2} + \lambda \frac{ke^{r(T-t)}a^*(t)}{\nu - ke^{r(T-t)}a^*(t)} - \frac{\delta}{\kappa} \ln \left( \frac{\bar{Z}(t)\delta}{k^p} \right) - h^p.
\]

Remark 4.10. In post-default case, if there is no defaultable bond in the financial market and the stock’s price process follows the GBM model, the equilibrium investment strategy reduces to

\[
\pi^*_s(t) = \frac{\mu - r}{ke^{r(T-t)}\sigma_1^2},
\]
and the equilibrium reinsurance strategy satisfies

\[
\mu_1(1 + \theta)E^\nu \exp(\alpha m(t, \nu)) = E^\nu \left[ \frac{\nu}{(\nu - ke^{r(T-t)}a^*(t))^2} \left( \exp(\alpha m(t, \nu)) \right) \right],
\]
where \( m(t, \nu) \) is given by (4.21). The reinsurance-investment strategy here is the strategy in Guan et al. [11] by only considering the ambiguous about insurance claims. If we also assume that the insurer is ambiguous about the stock, then the equilibrium investment strategy is given by

\[
\pi^*_s = \frac{E^\omega \omega \exp(\alpha \omega p(t))}{E^\omega \exp(\alpha \omega p(t))ke^{r(T-t)}\sigma_1},
\]
and the robust equilibrium proportional reinsurance strategy $a^*(t)$ satisfies
\[ \mu_1(1 + \theta)E^{\cdot,\nu}(\alpha m^2(t, \nu)) = E^{\cdot,\nu}\left[\exp(\alpha m^2(t, \nu))(\nu - k \epsilon(T-t) a^*(t))^2\right], \]
where $p(t)$ satisfies (E.13) and
\[ m^2(t, \nu) = \lambda \int_T^t \frac{k \epsilon(T-s)a^*(s)}{\nu - k \epsilon(T-s)a^*(s)}ds - \int_t^T k \epsilon(T-s)\lambda \mu_1(1 + \theta) a^*(s)ds. \]

The strategy is the same as the strategy invested in the ambiguous stock in Theorem 4.1 of Guan et al. [11].

**Proof.** Refer to Appendix E. \(\square\)

The following remark presents the robust optimal strategies of the ambiguity averse insurer (AAI) under the worst-case scenario.

**Remark 4.11.** In the post-default case, the robust equilibrium investment strategy is
\[ \pi^*_s(t) = \frac{\xi}{b(t)} + \frac{n(t)\sigma\rho}{b(t)}, \] \[ (4.24) \]
and the robust equilibrium proportional reinsurance strategy $a^*(t)$ satisfies
\[ -\mu_1(1 + \theta) + E^{Q^\nu}\left[\exp(a^*(t)b(t)Y)\right] \exp\left[\frac{\mu}{k} E^{Q^\nu}\left[\exp(a^*(t)b(t)Y) - 1\right]\right] = 0. \] \[ (4.25) \]
The corresponding equilibrium value function is given by
\[ V(t, x, l, 1) = -\frac{1}{k} \exp(-b(t)x + m(t) + n(t)l), \]
where $b(t) = k \exp(r(T-t))$,
\[ n(t) = \begin{cases} \frac{\xi^2}{2(\gamma + \sigma \xi)}[e^{-\gamma s}(T-t) - 1], & \rho = 1, \\ \frac{1}{2}\xi^2(t - T), & \rho = -1 \text{ and } \gamma = \xi \sigma, \\ \frac{\xi^2}{2(\gamma - \sigma \xi)}[e^{-\gamma s}(T-t) - 1], & \rho = -1 \text{ and } \gamma \neq \xi \sigma, \\ \frac{\sigma q_2 \exp(\sqrt{\gamma s}(T-t)) - q_1 q_2}{q_1 \exp(\sqrt{\gamma s}(T-t)) - q_2}, & \rho \neq \pm 1, \end{cases} \] \[ (4.26) \]
\[ m(t) = \lambda \mu_1(\theta - \eta) \int_T^t b(s)ds + \gamma \lambda \int_T^t n(s)ds - \frac{\lambda}{\mu} k (T-t) \]
\[ + \frac{1}{2} \beta^2 \int_T^t b^2(s)ds + \int_T^t q(a^*(s))ds. \] \[ (4.27) \]
And the worst-case measure is determined by
\[ \psi^*(t) = \exp\left[\frac{\mu}{k} E^{Q^\nu}\left[\exp(a^*(t)b(t)Y) - 1\right]\right]. \] \[ (4.28) \]

In the pre-default case, the robust equilibrium investment strategy is
\[ \pi^*_s(t) = \frac{\xi}{k \epsilon(T-t)} + \frac{\tilde{n}(t)\sigma\rho}{k \epsilon(T-t)}, \] \[ (4.29) \]
\[ \pi^*_p(t) = \frac{1}{k \epsilon(T-t)\zeta} \left[\left(\frac{\bar{h}^\nu \zeta}{\delta} - 1 + \ln\left(\frac{\delta}{h^\nu \zeta}\right)\right)e^{-\frac{\xi}{\delta}(T-t)} - \frac{h^\nu \zeta}{\delta} + 1\right], \] \[ (4.30) \]
and the robust equilibrium proportional reinsurance strategy $a^*(t)$ satisfies Eq. (4.25) The corresponding equilibrium value function is given by

$$V(t,x,l,1) = -\frac{1}{k} \exp[-ke^{\gamma(T-t)}x + \tilde{m}(t) + \bar{n}(t)]$$

where

$$\bar{n}(t) = n(t),$$

$$\tilde{m}(t) = \left(\frac{h^P\zeta}{\delta} - 1 + \ln \frac{\delta}{h^P\zeta}\right)\left(e^{-\gamma(T-t)} - 1\right) + m(t).$$

And the worst-case measure is determined by Eq. (4.28).

From Theorems 4.1, 4.3 and Eqs. (4.24), (4.29), we can find that the optimal amount invested in the stock are the same under the worst case and the average case. This is because that the risk model and the stock’s price are independent and the insurer is only ambiguous about insurance claims. Therefore, whether considering the two-step max-min problem or the smooth ambiguity model, it does not affect the investment in the stock.

From Eq. (4.9) and Eq. (4.30), we find that the ambiguity aversion to the claim do not impact the optimal investment in the defaultable bond under the worst case. However, risk aversion over ambiguity has an effect on the optimal amount invested in the defaultable bond under the average case.

For the reinsurance strategy, the insurer’s ambiguity aversion has an effect on the strategies under the worst case and average case.

Proof. Refer to Appendix F.

Remark 4.12. There exists a unique $a^*(t) \in (0, +\infty)$ such that Eq. (4.25) holds.

Proof. Refer to Appendix F.

5. Sensitivity analysis. In this section, we present a sensitivity analysis to show the impact of different parameters on the equilibrium reinsurance-investment strategy. Suppose that the claim size $Y_i$ follows the exponential distribution and $\nu$ denotes the parameter of this exponential distribution, i.e., $F^\nu(y) = 1 - e^{-\nu x}, x \geq 0$. For simplification, the distribution of $\nu$ is assumed to be discretely distributed, i.e., the support set of $\nu$ is $\Upsilon = \{\nu_1, \nu_2\}$ and $P(\nu = \nu_1) = q_1^\nu, P(\nu = \nu_2) = 1 - q_1^\nu$. For numerical illustrations, unless otherwise stated, the basic parameters are given as follows: $\beta = 0.5, r = 0.05, k = 2, \alpha = 1.5, T = 10, \lambda = 3, \eta = 0.05, \xi = 1.5, \gamma = 2, \kappa = 0.3, \rho = 0.5, \sigma = 1, \mu_1 = 1, \nu_1 = 0.8, \nu_2 = 1.2, q_1^\nu = 0.5, q_2^\nu = 0.5, \theta = 1.5, \zeta = 0.5, \delta = 0.01, h^P = 0.01$.

5.1. Effects of parameters on equilibrium reinsurance strategy. In this subsection, we study the effects of parameters on the equilibrium reinsurance strategy $a^*$. Without generality of loss, we consider the post-default case.

Figure 1 (a) shows that the equilibrium reinsurance strategy $a^*$ decreases with the insurer’s risk aversion coefficient over the market risks $k$. Since $k$ represents the insurer’s risk aversion, the insurer is more sensitive to risk when $k$ increases. Thus, the reinsurance strategy decreases with $k$. As indicated in Figure 1 (b), when the insurer’s risk aversion coefficient over ambiguity $\alpha$ increases, the optimal retention level of reinsurance decreases gradually. This is because that the insurer is more risk-averse with larger $\alpha$ and then will disperses more risk to the reinsurer.
The distribution of $\nu$ depicts the ambiguity and $\nu$ is assumed to take values $\nu_1$ or $\nu_2$. In Figure 2 (a), we find that $a^*$ increases with $\nu_1$. Furthermore, we see that $a^*$ is a decreasing function of $q_1$, which is displayed in Figure 2 (b). The two figures indicate that if the insurer may encounter larger claims or have a greater probability to suffer large claims, he/she will reduce risk by moving the risk to the reinsurer.

Figure 3 (a) plots the relationship between $\lambda$ and the optimal reinsurance strategy $a^*$. The insurer will tend to encounter more claims when $\lambda$ increases and thus he/she will reduce retention level of reinsurance to take less risk by himself/herself. In Figure 3 (b), it is shown that $a^*$ increases with the reinsurer’s safety loading $\theta$. When $\theta$ increases, the proportional reinsurance is more expensive and thus the insurer will purchase less reinsurance.

5.2. Effects of parameters on equilibrium investment strategy. In this subsection, we study the effects of parameters on the equilibrium investment strategies $\pi^*_s$ and $\pi^*_p$.

In Figure 4 (a), we find that insurer’s risk aversion towards market risks $k$ exerts a negative effect on the optimal investment in the stock $\pi^*_s$. Note that the larger
$k$ is, the more risk averse the insurer is. Thus as $k$ increases, the insurer will reduce investment in the stock to avoid risks. Figure 4 (b) shows that $\pi^*_s$ increases with premium for volatility $\xi$. This is because that the appreciation rate of stock increases with respect to $\xi$ and thus a larger $\xi$ leads to more investment in the stock.

Figure 5 plots the effects of parameter $\rho$ on $\pi^*_s$. We see that $\pi^*_s$ decreases with $\rho$ when $\rho > 0$ and increases with $\rho$ when $\rho < 0$. Since $\rho$ reflects the correlation between the stock’s price and its volatility, the uncertainties of the two processes change in the same sense when $\rho$ is positive and change in different ways when $\rho$ is negative. Then the insurer will reduce investment in the stock as $\rho$ increases when $\rho > 0$ and invest more in the stock as $\rho$ increases when $\rho < 0$ to hedge the risk, which is consistent with intuition.

For the Heston’s SV model, $L(t)$ reflects the volatility of the stock’s price and we analyze the effects of parameters of $L(t)$ on the optimal amount invested in the stock $\pi^*_s$ in Figures 6 and 7. From Figure 6, we find that $\pi^*_s$ increases as $\gamma$ increases for $\rho > 0$ while $\pi^*_s$ decreases with $\gamma$ for $\rho < 0$. If $\rho > 0$, the uncertainties of the stock’s price and its volatility change in the same way. $\gamma$ reflects the speed of $L(t)$
toward $\kappa$ when it wanders away. A larger $\gamma$ means a more stable volatility of the stock, and then the insurer will increase the investment in the stock. For $\rho < 0$, the uncertainties of $L(t)$ and $S(t)$ change in different ways. When $\gamma$ increases, $L(t)$ will be more stable. Then due to the negative correlation between $L(t)$ and $S(t)$, there is an increased probability of a decrease in the stock’s price. Thus $\pi^*_s$ decreases with $\gamma$ when $\rho < 0$.

Figure 5 demonstrates that $\pi^*_s$ decreases with respect to $\sigma$ for $\rho > 0$ and increases with $\sigma$ when $\rho < 0$. $\sigma$ denotes the volatility of the volatility of the stock’s price and this is consistent with our intuition. If $\sigma$ increases, the volatility of stock will fluctuate a little drastically and then the insurer would like to invest less in the stock when $\rho > 0$. But for $\rho < 0$, a more volatile volatility leads to an expected increase in stock’s price due to the negative correlation between the stock’s volatility and its price. Hence the insurer should increase investment in the stock as $\sigma$ increases if $\rho < 0$.

As shown in Figure 8 (a), the money amount invested in the defaultable bond changed slightly when the risk aversion coefficient over ambiguity $\alpha$ increases. This indicates that the ambiguity aversion coefficient of the claim has no effect on the
The effect of $\sigma$ on $\pi^*$ for $\nu(\sigma; \mu; 0.5) > 0$. (\(= 0.5\)) = 0.6, = 1, = 1.5

(Figure 7. The Effect of parameter $\sigma$ on $\pi^*$.)

investment in the defaultable bond. Figure 8 (b) implies that the optimal amount invested in the defaultable bond decreases with respect to the risk aversion coefficient towards market risks $k$ and the default intensity $h^P$. When $k$ increases, the insurer is more sensitive to risk and then will invest less in the defaultable bond. As the default intensity $h^P$ increases, the probability of a default of the company issuing the defaultable bond becomes larger. In that case, the counterparty risk of the defaultable bond will undermine its investment grade and thus make it less attractive to the insurer. Observing Figure 8 (c), we find that the insurer will invest more in the defaultable bond with smaller loss rate $\zeta$ and higher credit spread $\delta$. This illustrates the intuitive observation that the defaultable bond will be more attractive when its default recovery rate $1 - \zeta$ is larger and the credit spread is higher.

(Figure 8. Effects of parameters $\alpha, k, h^P, \zeta$ and $\delta$ on $\pi^*_p$.)

6. Conclusion. In this paper, we study the equilibrium reinsurance-investment strategy for the insurer. The insurer can purchase proportional reinsurance and invest in a risk-free bond, a stock whose price process follows the Heston’s SV model and a defaultable bond. The insurer is uncertain about the insurance claims. In practice, the insurer is risk averse towards the ambiguity in the market. To capture this phenomenon, we introduce the smooth ambiguity utility function proposed in Klibanoff et al. [15] in our work. There are two risk aversion parameters in
our model. By applying stochastic control approach, we derive the equilibrium reinsurance-investment strategy for the post-default case and the pre-default case, respectively. Different from the worst case concern, we consider the average case solution for ambiguity. Finally, the effects of parameters on the equilibrium reinsurance-investment strategy are provided. From sensitivity analysis, we find that (i) the default event has no impact on the optimal money amount invested in the stock, while parameters of both stock’s price and risk model have impacts on the insurer’s optimal demand for the defaultable bond; (ii) the uncertain beliefs play an important role in the equilibrium reinsurance-investment strategy.

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Appendix A. Proof of Theorem 4.1.

Proof. For the post-default case, i.e., \( z = 1 \), by the expression of \( A_\nu^z f(t,x,l,z) \) in (3.3), (4.2) becomes

\[
\sup_{\pi \in \Pi} \left\{ E_\nu^{z} \phi'(g(t,x,l,\nu,1)) \left[ g_x(t,x,l,\nu,1) + \{ \lambda \mu_1[(1 + \theta)a(t) - (\theta - \eta)] \\
+ rx + \xi \pi \sigma \} \right] \\
+ g_{xx}(t,x,l,\nu,1) + \frac{1}{2} \sigma^2 \pi \sigma \rho g_{x}(t,x,l,\nu,1) \\
+ \lambda E^\nu \left[ g(t,x-aY,l,\nu,1) - g(t,x,l,\nu,1) \right] \right\} = 0.
\]

(A.1)

In order to derive the equilibrium strategy, we conjecture \( F(t,x,l,1) \) and \( g(t,x,l,\nu,1) \) in the following:

\[
g(t,x,l,\nu,1) = -\frac{1}{k} \exp[-b(t)x + m(t,\nu) + n(t)]
\]

\[
b(T) = k, \quad m(T,\nu) = 0, \quad n(T) = 0, \quad (A.2)
\]

\[
F(t,x,l,1) = E^\nu \phi \circ g(t,x,l,\nu,1), \quad F(T,x,l,1) = E^\nu \phi \circ \left( -\frac{1}{k} \exp(-kx) \right). \quad (A.3)
\]

Then we have

\[
g_t(t,x,l,\nu,1) = [-b_1 x + m_1(t,\nu) + n_1] g(t,x,l,\nu,1),
\]

\[
g_x(t,x,l,\nu,1) = -b(t) g(t,x,l,\nu,1),
\]

\[
g_{xx}(t,x,l,\nu,1) = b^2(t) g(t,x,l,\nu,1),
\]

\[
g_t(t,x,l,\nu,1) = n(t) g(t,x,l,\nu,1),
\]

\[
g_{tt}(t,x,l,\nu,1) = n^2(t) g(t,x,l,\nu,1),
\]

\[
g_{tx}(t,x,l,\nu,1) = -b(t) n(t) g(t,x,l,\nu,1),
\]

\[
g(t,x-aY,l,\nu,1) - g(t,x,l,\nu,1) = [\exp(ab(t)Y) - 1] g(t,x,l,\nu,1).
\]

Substituting the above differentials into (A.1), we have

\[
\sup_{\pi \in \Pi} \left\{ E^\nu \phi(-g(t,x,l,\nu,1)) \phi' \left[ b_1 x - m_1(t,\nu) - n_1 + \{ \lambda \mu_1[(1 + \theta)a(t) - (\theta - \eta)] \right] \\
+ r x + \xi \sigma \pi \right\} = 0.
\]

(A.4)
Putting (A.5), (A.6) into (3.6) and arranging the equation by the order of \( x \), we obtain the following two partial differential equations

\[
-\left[(\theta - \eta) + r x + \xi \pi s l\right] b(t) - n(t)\gamma (\kappa - l) - \frac{1}{2} b^2(t)(\beta^2 + l\pi^2) \nonumber
- \frac{1}{2} \sigma^2 ln^2(t) + \pi_s \sigma l b(t) n(t) - \lambda E^{P^R} \left[ \exp(ab(t)Y) - 1 \right] \} = 0. \tag{A.4}
\]

Differentiating (A.4) w.r.t. \( \pi_s \), we have

\[
\pi_s^*(t) = \frac{\xi}{b(t)} + \frac{n(t)\sigma}{b(t)} \quad \tag{A.5}
\]

and the equilibrium reinsurance strategy satisfies

\[
a^*(t) = \arg \max_{a \geq 0} \{ E^\nu \alpha(-g)\lambda \mu_1(1+\theta)a(t)b(t) - \lambda E^\nu \alpha(-g)E^{P^R} \left[ \exp(a(t)b(t)Y) - 1 \right] \}. \tag{A.6}
\]

Putting (A.5), (A.6) into (3.6) and arranging the equation by the order of \( x \), we obtain the following two partial differential equations

\[
b_t + rb(t) = 0, \quad b(T) = k, \quad \tag{A.7}
\]

\[
m_t(t, \nu) + n_t l - b(t)\left\{ \lambda \mu_1[(1+\theta)a^*(t) - (\theta - \eta)] \right\} - \frac{1}{2} \xi^2 l 
+ \gamma(\kappa - l)n(t) + \frac{1}{2} \beta^2 b^2(t) - \frac{1}{2} \ln^2(t)\sigma^2 + \frac{1}{2} \sigma^2 n^2(t) \nonumber 
- \xi \sigma l m(t) + \lambda E^{P^R} \left[ \exp(a^*(t)b(t)Y) - 1 \right] = 0. \quad \tag{A.8}
\]

In terms of (A.7), \( b(t) \) can be explicitly obtained

\[
b(t) = k \exp(r(T - t)). \tag{A.9}
\]

To solve (A.8), we separate the variables with and without \( l \), respectively. Then, we derive the following system of ODEs

\[
m_t - b(t)\left\{ \lambda \mu_1[(1+\theta)a^*(t) - (\theta - \eta)] \right\} + \gamma \kappa n(t) 
+ \frac{1}{2} \beta^2 b^2(t) + \lambda E^{P^R} \left[ \exp(a^*(t)b(t)Y) - 1 \right] = 0, \quad \tag{A.10}
\]

\[
n_t - \frac{1}{2} \xi^2 - (\gamma + \sigma \xi) n(t) + \frac{1}{2} \sigma^2 (1 - \rho^2)n^2(t) = 0. \tag{A.11}
\]

According to the boundary condition \( m(T, \nu) = 0, \ n(T) = 0 \) and \( \nu \) is assumed to be exponentially distributed, we solve (A.10) and (A.11) in the following cases.

(i) If \( \rho = 1 \), (A.11) reduces into

\[
n_t - \frac{1}{2} \xi^2 - (\gamma + \sigma \xi) n(t) = 0.
\]

Considering the boundary condition, we obtain

\[
n(t) = \frac{\xi^2}{2(\gamma + \sigma \xi)} \left[ e^{-\gamma (\sigma \xi) (T-t)} - 1 \right], \quad \tag{A.12}
\]

\[
m(t, \nu) = \int_t^T \lambda \mu_1[(\theta - \eta) - (1+\theta)a^*(s)] ds + \frac{1}{2} \beta^2 \int_t^T b^2(s) ds 
+ \gamma \kappa \left[ \frac{\xi^2}{2(\gamma + \sigma \xi)} (1 - e^{-\gamma (\sigma \xi)(T-t)}) - \frac{\xi^2}{2(\gamma + \sigma \xi)} (T-t) \right] 
+ \lambda \int_t^T \frac{a^*(s)b(s)}{\nu - a^*(s)b(s)} ds. \tag{A.13}
\]

(i)
(ii) If $\rho = -1$ and $\gamma = \xi \sigma$, (A.11) becomes to

$$n_t - \frac{1}{2} \xi^2 = 0$$

and take the boundary condition into account, we obtain

$$n(t) = \frac{1}{2} \xi^2 (t - T), \quad (A.14)$$

$$m(t, \nu) = \int_t^T b(s) \{ \lambda \mu_1 [\theta - \eta] - (1 + \theta) a^*(s) \} ds + \frac{1}{2} \beta^2 \int_t^T b^2(s) ds$$

$$- \frac{1}{4} \gamma \kappa \xi^2 (T - t)^2 + \lambda \int_t^T \frac{a^*(s)b(s)}{\nu - a^*(s)b(s)} ds, \quad (A.15)$$

(iii) If $\rho = -1$ and $\gamma \neq \xi \sigma$, by the similar derivation, we have

$$n(t) = \frac{\xi^2}{2(\gamma - \sigma \xi)} [e^{-(\gamma - \sigma \xi)(T - t)} - 1], \quad (A.16)$$

$$m(t, \nu) = \int_t^T b(s) \{ \lambda \mu_1 [\theta - \eta] - (1 + \theta) a^*(s) \} ds + \frac{1}{2} \beta^2 \int_t^T b^2(s) ds$$

$$+ \gamma \kappa \left[ \frac{\xi^2}{2(\gamma - \sigma \xi)^2} (1 - e^{-(\gamma - \sigma \xi)(T - t)}) - \frac{\xi^2}{2(\gamma - \sigma \xi)} (T - t) \right]$$

$$+ \lambda \int_t^T \frac{a^*(s)b(s)}{\nu - a^*(s)b(s)} ds. \quad (A.17)$$

(iv) If $\rho \neq \pm 1$, let $\Delta = (\gamma + \sigma \rho \xi)^2 + \sigma^2 \xi^2 (1 - \rho^2) > 0$, then the solution to the following equation

$$- \frac{1}{2} \sigma^2 (1 - \rho^2) n^2(t) + (\gamma + \sigma \rho \xi) n(t) + \frac{1}{2} \xi^2 = 0$$

is

$$q_{1,2} = \frac{(\gamma + \sigma \rho \xi) \pm \sqrt{\Delta}}{\sigma^2 (1 - \rho^2)}.$$

Rewrite (A.11) as

$$\frac{n_t}{\frac{1}{2} \xi^2 + (\gamma + \sigma \rho \xi) n(t)} - \frac{n_t}{\frac{1}{2} \sigma^2 (1 - \rho^2) n^2(t)} = 1,$$

then simplify as

$$\frac{n_t}{n(t) - q_1} - \frac{n_t}{n(t) - q_2} = -\sqrt{\Delta}. \quad (A.18)$$

Integrating (A.18) from $t$ to $T$, we obtain

$$\int_t^T \frac{n_t}{n(t) - q_1} dt - \int_t^T \frac{n_t}{n(t) - q_2} dt = -\sqrt{\Delta} (T - t).$$

Considering the boundary condition, we have

$$\ln \left( \frac{q_1}{q_2} \right) - \ln \left( \frac{n(t) - q_1}{n(t) - q_2} \right) = -\sqrt{\Delta} (T - t),$$
Proof. Since the distribution of $\Delta(T-t)$ is exponential, we have

$$n(t) = \frac{q_1q_2\exp(\sqrt{\Delta}(T-t)) - q_1q_2}{q_1\exp(\sqrt{\Delta}(T-t)) - q_2}$$

$$= \frac{\exp(\sqrt{\Delta}(T-t)) - 1}{2\sqrt{\Delta + (\sqrt{\Delta - \gamma - \sigma\rho\xi})(\exp(\sqrt{\Delta}(T-t)) - 1)}}\xi^2, \quad (A.19)$$

$$m(t, \nu) = \int_t^T b(s)\{\lambda\mu_1[(\theta - \eta) - (1 + \theta)a^*(s)]\}ds + \frac{1}{2}\beta^2 \int_t^T b^2(s)ds$$

$$- \frac{2\gamma\kappa}{\sigma^2(1-\rho^2)}\ln\left(\frac{2\sqrt{\Delta}\exp((\sqrt{\Delta - \gamma - \sigma\rho\xi})(T-t)/2)}{2\sqrt{\Delta + (\sqrt{\Delta - \gamma - \sigma\rho\xi})(\exp(\sqrt{\Delta}(T-t)) - 1)}\right)$$

$$+ \lambda \int_t^T \frac{a^*(s)\nu}{\nu - a^*(s)b(s)}ds. \quad (A.20)$$

Based on (A.6), we have

$$\mathbb{E}^\nu(-g(t, x, l, \nu, 1))\mu_1(1 + \theta) - \mathbb{E}^\nu(-g(t, x, l, \nu, 1))\mathbb{E}^{\nu^*}[\exp(a^*(t)b(t)Y)] = 0.$$}

Noticing the distribution of $\nu$, we obtain that the equilibrium reinsurance strategy satisfies

$$\mu_1(1 + \theta)\mathbb{E}^\nu\exp(\alpha m(t, \nu)) - \mathbb{E}^\nu\left[\exp(\alpha m(t, \nu))\frac{\nu}{(\nu - a^*(t)b(t))^2}\right] = 0, \quad (A.21)$$

where the expressions of $m(t, \nu)$ are given by (A.13), (A.15), (A.17) and (A.20). 

Appendix B. Proof of Remark 4.2.

Proof. Since the distribution of $Y$ is given by $F^\nu(y)$ and $F^\nu(y)$ is assumed to be exponentially distributed, we have

$$\mathbb{E}^{\nu^*}\left[\exp(a^*(t)b(t)Y) - 1\right] = \int_0^{+\infty} \left(e^{a^*(t)b(t)x} - 1\right) \cdot x e^{-\nu x}dx$$

$$= \frac{\nu}{a^*(t)b(t) - \nu} \left(x e^{a^*(t)b(t)x} - \nu e^{-\nu x}\right)\bigg|_0^{+\infty} - 1,$$

which is convergent when $a^*(t)b(t) - \nu < 0$.

If $a^*(t)b(t) - \nu < 0$, $\mathbb{E}^{\nu^*}\left[\exp(a^*(t)b(t)Y) - 1\right] = \frac{\nu}{a^*(t)b(t) - \nu} - 1 = \frac{a^*(t)b(t)}{\nu - a^*(t)b(t)}$.

Similarly,

$$\mathbb{E}^{\nu^*}\left[\exp(a^*(t)b(t)Y)\right] = \int_0^{+\infty} e^{a^*(t)b(t)x} \cdot x e^{-\nu x}dx$$

$$= \frac{\nu}{a^*(t)b(t) - \nu} \cdot x e^{a^*(t)b(t)x}\bigg|_0^{+\infty} - \int_0^{+\infty} \frac{\nu}{a^*(t)b(t) - \nu} e^{a^*(t)b(t)x}dx.$$ 

If $a^*(t)b(t) - \nu < 0$,

$$\mathbb{E}^{\nu^*}\left[\exp(a^*(t)b(t)Y)\right] = \int_0^{+\infty} \frac{-\nu}{a^*(t)b(t) - \nu} e^{a^*(t)b(t)x}dx = \frac{\nu}{(\nu - a^*(t)b(t))^2}.$$
Appendix C. Proof of Theorem 4.3.

Proof. For the pre-default case, i.e., $z = 0$, by the expression of $\mathcal{A}^+ f(t,x,l,z)$ in (3.3), (4.2) becomes

$$\sup_{\pi \in \Pi} \left\{ \mathbb{E}^\nu \phi'(g(t,x,l,\nu,0)) \left[ g_t(t,x,l,\nu,0) + \{\lambda \mu_1[(1 + \theta)\alpha(t) - (\theta - \eta)] + rx + \xi \pi_x l \right. \right.$$

$$+ \pi_p \delta \} g_x(t,x,l,\nu,0) + \gamma (\kappa - l) g_l(t,x,l,\nu,0) + \frac{1}{2} (\beta^2 + l^2 \Sigma) g_{xx}(t,x,l,\nu,0)$$

$$+ \pi_s \sigma l g_{x}(t,x,l,\nu,0) + \lambda \mathbb{E}^\nu \{ g(t, x - aY, l, \nu, 0) - g(t, x, l, \nu, 0) \}$$

$$\left. \left. + \frac{1}{2} \sigma^2 l g_{ll}(t,x,l,\nu,0) + [g(t, x - \zeta \pi_p, l, \nu, 1) - g(t, x, l, \nu, 0)] h^p \right] \right\} = 0.$$  \hspace{1cm} (C.1)

In order to derive the equilibrium strategy, we conjecture $g(t, x, l, \nu, 0) = \mathbb{E}^\nu \phi \circ g(t, x, l, \nu, 0)$ in the following:

$$g(t, x, l, \nu, 0) = -\frac{1}{k} \exp[-ke^{r(T-t)} x + \bar{m}(t, \nu) + \bar{n}(t)]l, \quad \bar{m}(T, \nu) = 0, \quad \bar{n}(T) = 0,$$

$$F(t, x, l, 0) = \mathbb{E}^\nu \phi \circ g(t, x, l, \nu, 0), \quad F(T, x, l, 0) = \mathbb{E}^\nu \phi \circ \left(-\frac{1}{k} \exp(-ke^{r}) \right).$$  \hspace{1cm} (C.2)

Then we have

$$g_t(t,x,l,\nu,0) = [k e^{r(T-t)} x + \bar{m}(t, \nu) + \bar{n}(t)] g_t(t,x,l,\nu,0),$$

$$g_x(t,x,l,\nu,0) = -k e^{r(T-t)} g(t,x,l,\nu,0),$$

$$g_{xx}(t,x,l,\nu,0) = k^2 e^{2r(T-t)} g(t,x,l,\nu,0),$$

$$g_l(t,x,l,\nu,0) = \bar{n}(t) g(t,x,l,\nu,0),$$

$$g_{ll}(t,x,l,\nu,0) = \bar{n}^2(t) g(t,x,l,\nu,0),$$

$$g_{xl}(t,x,l,\nu,0) = -k e^{r(T-t)} \bar{n}(t) g(t,x,l,\nu,0),$$

$$g(t,x - aY,l,\nu,0) = g(t,x,l,\nu,0) = \left[ \exp(k e^{r(T-t)} aY) - 1 \right] g(t,x,l,\nu,0),$$

$$g(t,x - \zeta \pi_p,l,\nu,1) - g(t,x,l,\nu,0) = \left[ \exp(k e^{r(T-t)} \zeta \pi_p + (m(t,\nu) - \bar{m}(t,\nu)) + (n(t) - \bar{n}(t)) l - 1 \right] g(t,x,l,\nu,0).$$

Substituting the above differentials into (C.1), we have

$$\sup_{\pi \in \Pi} \left\{ \mathbb{E}^\nu \alpha(-g(t,x,l,\nu,0)) \mathbb{E}^\nu \right.$$

$$\cdot \left[ -k e^{r(T-t)} x - \bar{m}(t, \nu) - \bar{n}l + ke^{r(T-t)} \{\lambda \mu_1[(1 + \theta)\alpha(t) - (\theta - \eta)] + rx + \xi \pi_x l + \pi_p \delta \} \right.$$

$$- \gamma (\kappa - l) \bar{n}(t) - \frac{1}{2} k^2 e^{2r(T-t)} (\beta^2 + l^2 \Sigma)$$

$$\left. - \frac{1}{2} \sigma^2 \bar{n}^2(t) + \lambda \mathbb{E}^\nu \{ \exp(k e^{r(T-t)} aY) - 1 \} \right. \left. \mathbb{E}^\nu \left[ \exp(k e^{r(T-t)} \zeta \pi_p + (m(t,\nu) - \bar{m}(t,\nu)) + (n(t) - \bar{n}(t)) l - 1 \right] h^p \right\} = 0.$$  \hspace{1cm} (C.4)
Differentiating (C.4) w.r.t. \( \pi_s \) and \( \pi_p \), we have
\[
\pi_s^*(t) = \frac{\xi}{ke^{r(T-t)}} + \frac{\tilde{n}(t)\sigma}{ke^{r(T-t)}}, \tag{C.5}
\]
\[
\pi_p^*(t) = \frac{1}{ke^{r(T-t)}\zeta} \left( \ln E^\nu E^\nu \exp[(\alpha - 1)\tilde{m}(t, \nu) + m(t, \nu)] + (\tilde{n}(t) - n(t))l + \ln \frac{\delta}{h^p\zeta} \right), \tag{C.6}
\]
and the equilibrium reinsurance strategy satisfies
\[
a^*(t) = \arg\max_{a \geq 0} \{E^\nu \alpha(-g(t, x, l, \nu, 0))\lambda \mu_1(1 + \theta)\alpha(t) \cdot ke^{r(T-t)} - E^\nu \alpha(-g(t, x, l, \nu, 0))\lambda E^{P^\nu} [\exp(ke^{r(T-t)} a(t)Y) - 1] \}. \tag{C.7}
\]
Putting (C.5), (C.6) and (C.7) into (3.6), we obtain the following partial differential equation
\[
\tilde{m}_t(t, \nu) + \tilde{n}_t(t) - ke^{r(T-t)}\{\lambda \mu_1[(1 + \theta)\alpha^*(t) - (\theta - \eta)]\} - \frac{1}{2}\xi^2 l
+ \gamma(k - l)\tilde{n}(t) + \frac{1}{2}\beta^2 k^2 e^{2r(T-t)} - \frac{1}{2}ln^2(t)\sigma^2 \rho^2 + \frac{1}{2}\sigma^2 \tilde{n}^2(t)l
+ \lambda E^{P^\nu} [\exp(ke^{r(T-t)} a^*(t)Y) - 1] - \xi \sigma l \tilde{m}(t)
- \delta \left( \ln E^\nu E^\nu \exp[(\alpha - 1)\tilde{m}(t, \nu) + m(t, \nu)] + (\tilde{n}(t) - n(t))l + \ln \frac{\delta}{h^p\zeta} \right)
+ h^P \left( \frac{E^\nu \exp(\alpha \tilde{m}(t, \nu))}{E^\nu E^\nu \exp[(\alpha - 1)\tilde{m}(t, \nu) + m(t, \nu)]} \frac{e^{m(t, \nu)}}{e^{m(t, \nu)} - 1} \right) = 0. \tag{C.8}
\]
To solve (C.8), we separate the variables with and without \( l \), respectively. Then, we derive the following system of ODEs
\[
\tilde{m}_t(t, \nu) - ke^{r(T-t)}\{\lambda \mu_1[(1 + \theta)\alpha^*(t) - (\theta - \eta)]\} + \lambda E^{P^\nu} [\exp(ke^{r(T-t)} a^*(t)Y) - 1]
+ \gamma \kappa \tilde{n}(t) + \frac{1}{2}\beta^2 k^2 e^{2r(T-t)} + \frac{E^\nu \exp(\alpha \tilde{m}(t, \nu))}{E^\nu E^\nu \exp[(\alpha - 1)\tilde{m}(t, \nu) + m(t, \nu)]} \frac{e^{m(t, \nu)}}{e^{m(t, \nu)} - 1}
- \delta \left( \ln E^\nu E^\nu \exp[(\alpha - 1)\tilde{m}(t, \nu) + m(t, \nu)] + \ln \frac{\delta}{h^p\zeta} \right)
- h^P = 0, \tag{C.9}
\]
\[
\tilde{n}_t - \frac{1}{2}\xi^2 l + \frac{\delta}{\zeta} \tilde{n}(t) - (\gamma + \sigma \rho \xi + \delta \frac{\delta}{\zeta})\tilde{n}(t) + \frac{1}{2}\sigma^2 (1 - \rho^2)\tilde{n}^2(t) = 0. \tag{C.10}
\]
Firstly, according to the boundary condition \( \tilde{n}(T) = 0 \), we solve (C.10) in the following cases.
(i) If \( \rho = 1 \), (C.10) reduces into
\[
\tilde{n}_t - \frac{1}{2}\xi^2 l + \frac{\delta}{\zeta} \tilde{n}(t) - (\gamma + \sigma \xi + \delta \frac{\delta}{\zeta})\tilde{n}(t) = 0.
\]
Considering the boundary condition \( \tilde{n}(T) = 0 \) and (A.12), we obtain
\[
\tilde{n}(t) = e^{(\gamma + \sigma \xi + \frac{\delta}{\zeta})t} \int_0^T e^{-(\gamma + \sigma \xi + \frac{\delta}{\zeta})s} \left( \frac{\delta}{\zeta} \tilde{n}(s) - \frac{1}{2}\xi^2 \right) ds
= \frac{\xi^2}{2(\gamma + \sigma \xi)} [e^{-(\gamma + \sigma \xi)(T-t)} - 1]. \tag{C.11}
\]
(ii) If \( \rho = -1 \) and \( \gamma + \frac{\delta}{\zeta} = \xi \sigma \), (C.10) becomes to

\[
\bar{n}_t - \frac{1}{2} \xi^2 + \frac{\delta}{\zeta} n(t) = 0
\]

and in terms of (A.16) we know that \( n(t) = -\frac{\xi^2}{2\delta} e^{\xi(T-t)} - 1 \). Taking the boundary condition \( \bar{n}(T) = 0 \) into account, we obtain

\[
\bar{n}(t) = -\frac{\xi^2 \zeta}{2\delta} \left( e^{\xi(T-t)} - 1 \right). \tag{C.12}
\]

(iii) If \( \rho = -1 \) and \( \gamma + \frac{\delta}{\zeta} \neq \xi \sigma \),

1. If \( \gamma = \sigma \xi \), (C.10) reduces into

\[
\bar{n}_t - \frac{1}{2} \xi^2 + \frac{\delta}{\zeta} n(t) - \frac{\delta}{\zeta} \bar{n}(t) = 0.
\]

Considering the boundary condition \( \bar{n}(T) = 0 \) and (A.14), we obtain

\[
\bar{n}(t) = e^{\frac{\xi}{\zeta} t} \int_{t}^{T} e^{-\frac{\xi}{\zeta} s} \left( \frac{\delta}{\zeta} n(s) - \frac{1}{2} \xi^2 \right) ds = \frac{1}{2} \xi^2 (t - T). \tag{C.13}
\]

2. If \( \gamma \neq \sigma \xi \), by the similar derivation, we have

\[
\bar{n}(t) = e^{(\gamma - \sigma \xi + \frac{\delta}{\zeta}) t} \int_{t}^{T} e^{-(\gamma - \sigma \xi + \frac{\delta}{\zeta}) s} \left( \frac{\delta}{\zeta} n(s) - \frac{1}{2} \xi^2 \right) ds
\]

\[
= \frac{\xi^2}{2(\gamma - \sigma \xi)} \left[ e^{-(\gamma - \sigma \xi)(T-t)} - 1 \right]. \tag{C.14}
\]

(iv) If \( \rho \neq \pm 1 \), according to Eqs.(A.11) and (C.10), we obtain

\[
\bar{n}_t - n_t - (\gamma + \sigma \rho \xi + \frac{\delta}{\zeta})(\bar{n}(t) - n(t)) + \frac{1}{2} \sigma^2 (1 - \rho^2) (\bar{n}^2(t) - n^2(t)) = 0. \tag{C.15}
\]

Define \( N(t) = \bar{n}(t) - n(t) \) and Eq.(C.15) becomes

\[
N_t - (\gamma + \sigma \rho \xi + \frac{\delta}{\zeta}) N(t) + \frac{1}{2} \sigma^2 (1 - \rho^2) N^2(t) + \sigma^2 (1 - \rho^2) n(t) N(t) = 0. \tag{C.16}
\]

Considering the boundary condition \( N(T) = 0 \), we have \( N(t) = 0 \).

So

\[
\bar{n}(t) = n(t) = \frac{q_1 q_2}{q_1} \exp(\sqrt{\Delta} (T-t)) - q_1 q_2, \tag{C.17}
\]

where \( \Delta = (\gamma + \sigma \rho \xi)^2 + \sigma^2 \xi^2 (1 - \rho^2) \).

From (A.12), (A.14), (A.16), (A.19) and (C.11), (C.12), (C.13), (C.14), (C.17), we find that \( \bar{n}(t) = n(t) \). Thus, (C.6) becomes

\[
\pi^*(t) = \frac{1}{\ker(T-t) \zeta} \left( \ln \frac{E^\nu \exp(\alpha \bar{m}(t, \nu))}{E^\nu \exp((\alpha - 1) \bar{m}(t, \nu) + m(t, \nu))} + \ln \frac{\delta}{K^\nu \zeta} \right).
\]

Secondly, according to the boundary condition \( \bar{m}(T, \nu) = 0 \) and \( \nu \) is assumed to be exponentially distributed, we solve (C.9) in the following. Defining

\[
Z(t) = \frac{E^\nu \exp(\alpha \bar{m}(t, \nu))}{E^\nu \exp((\alpha - 1) \bar{m}(t, \nu) + m(t, \nu))}, \quad Z(T) = 1,
\]

then (C.9) becomes

\[
\bar{m}_t(t, \nu) + A(t, \nu) + Z(t) \frac{\delta e^{m(t, \nu)}}{\zeta} \frac{1}{e^m(t, \nu)} = 0, \tag{C.18}
\]
We obtain

\[ A(t, \nu) = -ke^{r(T-t)}(\lambda \mu_2[(1+\theta)a^*(t) - (\theta-\eta)]) + \frac{1}{2}\beta^2k^2e^{2r(T-t)} \]

\[ + \gamma \kappa \bar{m}(t) + \lambda \frac{ke^{r(T-t)}a^*(t)}{\nu - ke^{r(T-t)}a^*(t)} - \frac{\delta}{\zeta} \left( \ln Z(t) - \frac{\delta}{h^*} \right) - h^*. \]  

(C.19)

We obtain

\[ \bar{m}(t, \nu) = \ln \left( 1 + \int_t^T \frac{Z(s)\delta e^{m(s, \nu)}}{\zeta} e^{f(t, s, \nu)A(s, \nu)ds} ds \right) - \int_t^T A(s, \nu) ds. \]  

(C.20)

Based on (C.7), we have

\[ E^\nu(-g(t, x, l, \nu, 0))^\alpha \mu_1(1+\theta) = E^\nu((-g(t, x, l, \nu, 0))^\alpha E^{P^\nu}[\exp(ke^{r(T-t)}a^*(t))Y]). \]  

(C.21)

Noticing the distribution of \( \nu \), we obtain that the optimal proportional reinsurance strategy satisfies

\[ \mu_1(1+\theta)E^\nu \exp(\alpha \bar{m}(t, \nu)) - E^\nu \left[ \exp(\alpha \bar{m}(t, \nu)) \frac{\nu}{\nu - ke^{r(T-t)}a^*(t)} \right] = 0, \]  

where the expression of \( \bar{m}(t, \nu) \) given by (C.20).

\[ \square \]

### Appendix D. Proof of Remark 4.7.

**Proof.** If the insurer is ambiguity neutral, the optimization problem of the insurer becomes

\[ \sup_{\hat{\pi} \in \Pi} \left\{ E^\nu \left[ E_{t,x,l,z}^{P^\nu} U(X^\hat{\pi}(T)) \right] \right\}, \]  

(D.1)

where \( \hat{\pi} \) is an equilibrium strategy and \( \Pi \) is the set of all admissible strategies.

The following theorem provides verification theorems for the post-default case (\( z = 1 \)) and the pre-default case (\( z = 0 \)) with no ambiguity, respectively.

**Theorem D.1.** (Verification Theorem with no ambiguity). For the post-default case (\( z = 1 \)) and pre-default case (\( z = 0 \)), if there is a function \( \tilde{g}(t, x, l, \nu, z) \in C^{1,2,0}(\{0, T\} \times \mathbb{R} \times \mathbb{R}^+ \times \mathcal{Y} \times \{0, 1\}) \) and a related process defined by \( \tilde{F}(t, x, l, z) = E^\nu \tilde{g}(t, x, l, \nu, z) \in C^{1,2,2}(\{0, T\} \times \mathbb{R} \times \mathbb{R}^+ \times \{0, 1\}) \) satisfying the following conditions:

\[ \forall (t, x, l, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^+ \times \{0, 1\}, \]

\[ \sup_{\hat{\pi} \in \Pi} \left\{ E^\nu A^\nu_{t,x,l,z} \tilde{g}(t, x, l, \nu, z) \right\} = 0, \]  

(D.2)

\[ \tilde{F}(T, x, l, z) = U(x), \]  

(D.3)

\[ A^\nu_{t,x,l,z} \tilde{g}(t, x, l, \nu, z) = 0, \]  

(D.4)

\[ \tilde{g}(T, x, l, \nu, z) = U(x), \]  

(D.5)

let \( \tilde{\pi}^* := \arg \sup_{\hat{\pi} \in \Pi} \left\{ E^\nu A^\nu_{t,x,l,z} \tilde{g}(t, x, l, \nu, z) \right\} \), then \( \tilde{V}(t, x, l, z) = \tilde{F}(t, x, l, z), E^\nu_{t,x,l,z}[U(X^{\tilde{\pi}^*}(T))] = \tilde{g}(t, x, l, \nu, z) \) and \( \tilde{\pi}^* \) is the equilibrium strategy.
In the post-default case, \( \tilde{\pi}_p(t) = 0 \) and (D.2) can be rewritten as
\[
\sup_{\tilde{\pi} \in \Pi} \left\{ E^\prime \left[ \tilde{g}_t(t, x, l, \nu) + \lambda \mu_1 [(1 + \theta) \tilde{\alpha}(t) - (\theta - \eta)] + rx + \xi \tilde{\pi}_s l \right] \cdot \tilde{g}_x(t, x, l, \nu) + \gamma (\kappa - l) \tilde{g}_t(t, x, l, \nu) + \frac{1}{2} (\beta^2 + l \tilde{\pi}_s^2) \tilde{g}_{xx}(t, x, l, \nu, 1) + \frac{1}{2} \sigma^2 l \tilde{g}_{tl}(t, x, l, \nu, 1) + \tilde{\pi}_s \lambda \rho \tilde{g}_{xl}(t, x, l, \nu, 1) + \lambda E^{\nu} \left[ \tilde{g}(t, x - \tilde{\alpha} Y, l, \nu, 1) - \tilde{g}(t, x, l, \nu, 1) \right] \right\} = 0. \tag{D.6}
\]
In order to derive the equilibrium strategy, we conjecture \( \tilde{g}(t, x, l, \nu, 1) \) and \( \tilde{F}(t, x, l, 1) \) in the following:
\[
\tilde{g}(t, x, l, \nu, 1) = -\frac{1}{k} \exp[-\tilde{b}(t)x + \tilde{m}(t, \nu) + \tilde{n}(t)l], \quad \tilde{b}(T) = k, \quad \tilde{m}(T, \nu) = 0, \quad \tilde{n}(T) = 0, \tag{D.7}
\]
\[
\tilde{F}(t, x, l, 1) = E^\prime \tilde{g}(t, x, l, \nu, 1), \quad \tilde{F}(T, x, l, 1) = -\frac{1}{k} e^{-kx}. \tag{D.8}
\]
Substituting the partial differentials of \( \tilde{g}(t, x, l, \nu, 1) \) into (D.6), we have
\[
\sup_{\tilde{\pi} \in \Pi} \left\{ E^\prime \tilde{g}(t, x, l, \nu) \left[ -\tilde{b}_x x + \tilde{m}_x(t, \nu) + \tilde{n}_l l - \left\{ \lambda \mu_1 [(1 + \theta) \tilde{\alpha}(t) - (\theta - \eta)] + rx + \xi \tilde{\pi}_s l \right\} \tilde{b}(t) + \gamma (\kappa - l) \tilde{n}(t) + \frac{1}{2} (\beta^2 + l \tilde{\pi}_s^2) \tilde{b}^2(t) + \frac{1}{2} \sigma^2 l \tilde{n}^2(t) - \tilde{\pi}_s \lambda \rho \tilde{b}(t) \tilde{n}(t) + \lambda E^{\nu} \left[ \exp(\tilde{\alpha}(t) Y) - 1 \right] \right\} = 0. \tag{D.9}
\]
Differentiating (D.9) with respect to \( \tilde{\pi}_s \), we have
\[
\tilde{\pi}_s^*(t) = \frac{\xi}{\tilde{b}(t)} + \frac{\tilde{n}(t) \sigma \rho}{\tilde{b}(t)}, \tag{D.10}
\]
and the equilibrium reinsurance strategy \( \tilde{\alpha}^*(t) \) satisfies
\[
\mu_1 [(1 + \theta) E^\prime \exp(\tilde{m}(t, \nu)) - (\nu - \tilde{\alpha}^*(t) \tilde{b}(t))^2] = 0. \tag{D.11}
\]
Putting (D.10), (D.11) into (D.4), arranging the equation by the order of \( x \) and separating the variables with and without \( l \), respectively. We have
\[
\tilde{\alpha}_t + r \tilde{b}(t) = 0, \quad \tilde{b}(T) = k, \tag{D.12}
\]
\[
\tilde{m}_t - \tilde{b}(t) \left\{ \lambda \mu_1 [(1 + \theta) \tilde{\alpha}(t) - (\theta - \eta)] \right\} + \gamma \tilde{n}(t) + \frac{1}{2} \beta^2 \tilde{b}^2(t) + \frac{1}{2} E^{\nu} \left[ \exp(\tilde{\alpha}(t) \tilde{b}(t) Y) - 1 \right] = 0, \tag{D.13}
\]
\[
\tilde{n}_t - \frac{1}{2} \xi^2 + (\gamma + \sigma \rho \xi) \tilde{n}(t) + \frac{1}{2} \sigma^2 (1 - \rho^2) \tilde{n}^2(t) = 0. \tag{D.14}
\]
Solving the above equations, we have
\[
\tilde{b}(t) = k \exp(r(T - t)), \tag{D.15}
\]
\[
\tilde{n}(t) = n(t), \tag{D.16}
\]
\[ \tilde{m}(t, \nu) = \int_{t}^{T} \tilde{b}(s) \{ \lambda \mu_{1} \langle [\theta - \eta] - (1 + \theta) \tilde{a}^* (s) \rangle \} ds + \frac{1}{2} \beta^{2} \int_{t}^{T} \tilde{b}^2 (s) ds + \gamma \kappa \int_{t}^{T} \tilde{n}(s) ds + \lambda \int_{t}^{T} \frac{\tilde{a}^* (s) \tilde{b} (s)}{\nu - \tilde{a}^* (s) \tilde{b} (s)} ds, \]

where \( n(t) \) is given by (4.6).

Similar to the post-default case, we conjecture \( \tilde{F}(t, x, l, 0) \) and \( \tilde{g}(t, x, l, \nu, 0) \) in the following:

\[ \tilde{g}(t, x, l, \nu, 0) = - \frac{1}{k} \exp [-ke^{r(T-t)}x + \tilde{m}(t, \nu) + \tilde{n}(t)] \], \( \tilde{m}(T, \nu) = 0 \), \( \tilde{n}(T) = 0 \),

\[ \tilde{F}(t, x, l, 0) = E^{\nu} \tilde{g}(t, x, l, \nu, 0), \quad \tilde{F}(T, x, l, 1) = - \frac{1}{k} e^{-kx}. \] 

Substituting the partial differentials of \( \tilde{g}(t, x, l, \nu, 1) \) into (D.2), we have

\[ \sup_{\tilde{\pi} \in \tilde{\Pi}} \left\{ E^{\nu} \tilde{g}(t, x, l, \nu, 1) \left[ k e^{r(T-t)} x + \tilde{m}(t, \nu) + \tilde{n}(t) l - k e^{r(T-t)} \{ \lambda \mu_{1} \langle (1 + \theta) \tilde{a} (t) \rangle - (\theta - \eta) \} + \gamma (\kappa - l) \tilde{n}(t) + \frac{1}{2} k^{2} e^{2r(T-t)} (\beta^{2} + l \tilde{\pi}^{2}) \right. \right. \\
\left. \left. + \frac{1}{2} \sigma^{2} l \tilde{n}^{2} (t) - k e^{r(T-t)} \tilde{\pi}_{s} \sigma \rho \tilde{n}(t) + \lambda E^{\nu} \left[ \exp (k e^{r(T-t)} \tilde{a} Y) - 1 \right] \right] \left. \left. + \left[ \exp [k e^{r(T-t)} \tilde{\pi}_{p} + (\tilde{m}(t, \nu) - \tilde{m}(t, \nu)) + (\tilde{n}(t) - \tilde{n}(t)) l] - 1 \right] h^{p} \right) \right\} = 0. \]

Differentiating (D.20) with respect to \( \tilde{\pi}_{s} \) and \( \tilde{\pi}_{p} \), we have

\[ \tilde{\pi}_{s}^* (t) = \frac{\xi}{k e^{r(T-t)}} + \frac{\tilde{n}(t) \rho}{k e^{r(T-t)}}, \]

\[ \tilde{\pi}_{p}^* (t) = \frac{1}{k e^{r(T-t)}} \left( \ln \frac{E^{\nu} \exp (\tilde{m}(t, \nu))}{E^{\nu} \exp (\tilde{m}(t, \nu))} + (\tilde{n}(t) - \tilde{n}(t)) l + \ln \frac{\delta}{h^{p} \zeta} \right), \]

and the equilibrium reinsurance strategy satisfies

\[ \mu_{1} (1 + \theta) E^{\nu} \exp (\tilde{m}(t, \nu)) = E^{\nu} \left[ \exp (\tilde{m}(t, \nu)) \frac{\nu}{\nu - k e^{r(T-t)} \tilde{a}^* (t)} \right] . \]

Putting (D.21), (D.22) and (D.23) into (D.4), separating the variables with and without \( l \), respectively. We derive

\[ \tilde{m}_t - k e^{r(T-t)} \{ \lambda \mu_{1} \langle (1 + \theta) \tilde{a}^* (t) - (\theta - \eta) \} + \frac{1}{2} \beta^{2} k^{2} e^{2r(T-t)} \]

\[ + \lambda E^{\nu} [\exp (k e^{r(T-t)} \tilde{a}^* (t) Y) - 1] + \frac{E^{\nu} \exp (\tilde{m}(t, \nu)) \tilde{\pi}_{p}^* (t)}{E^{\nu} \exp (\tilde{m}(t, \nu))} \frac{\delta}{h^{p} \zeta} \]

\[ - \frac{\delta}{\zeta} \left( \ln \frac{E^{\nu} \exp (\tilde{m}(t, \nu))}{E^{\nu} \exp (\tilde{m}(t, \nu))} + \ln \frac{\delta}{h^{p} \zeta} \right) + \gamma \kappa \tilde{n}(t) - h^{p} = 0, \]

\[ \tilde{n}_t - \frac{1}{2} \sigma^{2} + \frac{\delta}{\zeta} \tilde{n}(t) - (\gamma + \sigma \rho \xi + \frac{\delta}{\zeta}) \tilde{n}(t) + \frac{1}{2} \sigma^{2} (1 - \rho^{2}) \tilde{n}^{2} (t) = 0. \]

Solving (D.25), we obtain \( \tilde{n}(t) = \tilde{n}(t) \). Thus

\[ \tilde{\pi}_{p}^* (t) = \frac{1}{k e^{r(T-t)}} \left( \ln \frac{E^{\nu} \exp (\tilde{m}(t, \nu))}{E^{\nu} \exp (\tilde{m}(t, \nu))} + \ln \frac{\delta}{h^{p} \zeta} \right). \]
Next, we define
\[
\bar{Z}(t) = \frac{E^\nu \exp(\bar{m}(t, \nu))}{E^\nu \exp(\bar{m}(t, \nu))}, \quad \bar{Z}(T) = 1,
\] (D.26)
\[
\bar{A}(t, \nu) = -ke^{\nu(T-t)}\{\lambda_1[(1 + \theta)a(t) - (\theta - \eta)]\} + \frac{1}{2}\beta^2 k^2 e^{\nu(T-t)}
\]
\[
+ \gamma \kappa \bar{m}(t) + \frac{ke^{\nu(T-t)} \bar{a}^+(t)}{\nu - ke^{\nu(T-t)} \bar{a}^+(t)} - \frac{\delta}{\zeta} \left( \ln \bar{Z}(t) \delta \right) \bar{h}_\nu^\nu - \bar{h}_\nu^\nu,
\] (D.27)
then we have
\[
\bar{m}(t, \nu) = \ln \left( 1 + \int_t^T \frac{-\bar{Z}(s) \delta e^{\bar{m}(s, \nu)}}{\zeta} e^{\int_t^s \bar{A}(s, \nu) ds} ds \right) - \int_t^T \bar{A}(s, \nu) ds.
\] (D.28)

\[\Box\]

**Appendix E. Proof of Remark 4.10.**

**Proof.** We consider that the insurer is uncertain about the stock yield, i.e., we model this phenomenon by assuming that there exists a random variable \( \omega \) with distribution function \( F^\omega_\nu(x) \). The support set of \( \omega \) is denoted by \( \Lambda \in \mathbb{R} \). The stock’s price process follows
\[
dS(t) = S(t)(r dt + \sigma_1(\omega dt + dW(t)), \quad S(0) = S_0 > 0.
\] (E.1)

Then we can obtain the wealth process of the insurer
\[
dX^\pi(t) = \{\lambda_1[(1 + \theta)a(t) - (\theta - \eta)] + rX^\pi(t) + \sigma_1 \omega \pi_s(t)\} dt
\]
\[
+ \beta dW(t) - a(t)d\left( \sum_{i=1}^{N(t)} Y_i \right) + \pi_s(t) \sigma_1 dW(t).
\] (E.2)

For any state \((t, x) \in [0, T] \times \mathbb{R} \), we have the following variational operator
\[
A^\pi_{\omega, \nu} f(t, x) = f_1(t, x) + f_x(t, x) \{\lambda_1[(1 + \theta)a - (\theta - \eta)] + rx + \sigma_1 \omega \pi_s\}
\]
\[
+ \frac{1}{2}(\beta^2 + \sigma_1^2 \pi_s^2)f_{xx}(t, x) + \lambda E^{\nu'} \exp^\nu \left[ f(t, x - aY) - f(t, x) \right].
\] (E.3)

By the expression of \( A^\pi_{\omega, \nu} f(t, x) \) in (E.3), (4.2) becomes
\[
\sup_{\pi \in \Pi} \left\{ E^{\nu' \phi'(g(t, x, \omega, \nu))} \left[ g(t, x, \omega, \nu) + \{\lambda_1[(1 + \theta)a - (\theta - \eta)]
\right.
\]
\[
+ \lambda E^{\nu' \exp^\nu \left[ g(t, x - aY, \omega, \nu) - g(t, x, \omega, \nu) \right]}
\]
\[
\left. + \frac{1}{2}(\beta^2 + \sigma_1^2 \pi_s^2)g_{xx}(t, x, \omega, \nu) \right\} = 0.
\] (E.4)

In order to derive the equilibrium strategy, we conjecture \( F(t, x) \) and \( g(t, x, \omega, \nu) \) in the following:
\[
g(t, x, \omega, \nu) = -\frac{1}{k} \exp[-ke^{\nu(T-t)}x + m(t, \omega, \nu)], \quad m(T, \omega, \nu) = 0,
\] (E.5)
\[
F(t, x) = E^{\nu' \phi \circ g(t, x, \omega, \nu)}, \quad F(T, x) = E^{\nu' \phi \circ \left( -\frac{1}{k} \exp(-kx) \right)}.
\] (E.6)
Differentiating (E.4) w.r.t. \( \pi_s \) and substituting the partial differentials of \( g(t, x, \omega, \nu) \), we have

\[
\pi_s^* = \frac{E^{\omega,\nu} \exp(\alpha m(t, \omega, \nu))}{E^{\omega,\nu} \exp(\alpha m(t, \omega, \nu)) ke^{r(T-t)} \sigma_1},
\]  

(E.7)

and the equilibrium reinsurance strategy satisfies

\[
a^*(t) = \arg \max_{\alpha \geq 0} \left[ E^{\omega,\nu} \alpha (-g(t, x, \omega, \nu)) \lambda \mu_1 (1 + \theta) a(t) \cdot ke^{r(T-t)} \right. \\
- E^{\omega,\nu} \alpha (-g(t, x, \omega, \nu)) \lambda \mu_1 (1 + \theta) a^*(t) \left. \right] \exp(ke^{r(T-t)} a(t) Y - 1).
\]  

(E.8)

Since \( \omega \) and \( \nu \) are dependent, we suppose that

\[
m(t, \omega, \nu) = m^1(t, \omega) + m^2(t, \nu).
\]  

(E.9)

Putting (E.7) and (E.8) into (3.6), we obtain the following equations

\[
m^1_t(t, \omega) - \omega E^{\omega} \exp(\alpha m^1(t, \omega)) + \frac{1}{2} E^{\omega} \exp(\alpha m^1(t, \omega))^2 \\
+ ke^{r(T-t)} \lambda_1 (\theta - \eta) + \frac{1}{2} \beta^2 k^2 e^{2r(T-t)} = 0, \quad m^1(T, \omega) = 0,
\]  

(E.10)

\[
m^2_t(t, \nu) + \lambda E^{\nu} \exp(\alpha m^1(t, \nu) + 1 \lambda \mu_1 (1 + \theta) a^*(t) Y - 1) \\
- ke^{r(T-t)} \lambda_1 (1 + \theta) a^*(t) = 0, \quad m^2(T, \nu) = 0.
\]  

(E.11)

Then we define

\[
h(t) = \frac{E^{\omega} \exp(\alpha m^1(t, \omega))}{E^{\omega} \exp(\alpha m^1(t, \omega))}, \quad h(T) = E^{\omega} \omega.
\]

By (E.10), for any fixed \( \omega_1 \in \Lambda \) and different \( \omega \in \Lambda \), we have the following two equations

\[
m^1(t, \omega) - \omega h(t) + \frac{1}{2} h^2(t) + ke^{r(T-t)} \lambda_1 (\theta - \eta) \\
+ \frac{1}{2} \beta^2 k^2 e^{2r(T-t)} = 0, \quad m^1(T, \omega) = 0,
\]

\[
m^1(t, \omega_1) - \omega_1 h(t) + \frac{1}{2} h^2(t) + ke^{r(T-t)} \lambda_1 (\theta - \eta) \\
+ \frac{1}{2} \beta^2 k^2 e^{2r(T-t)} = 0, \quad m^1(T, \omega_1) = 0.
\]

Subtracting the above two equations, we have

\[
m^1(t, \omega) - m^1(t, \omega_1) = (\omega - \omega_1) h(t), \quad m^1(T, \omega) - m^1(T, \omega_1) = 0.
\]  

(E.12)

Integrating (E.12) from \( t \) to \( T \) and considering the boundary condition, we obtain

\[
m^1(t, \omega) - m^1(t, \omega_1) = -(\omega - \omega_1) \int_t^T h(s) ds.
\]

Using the definition of \( h(t) \), we have

\[
h(t) = \frac{\int_t^T h(s) ds}{\int_t^T h(s) ds}.
\]

Denote

\[
p(t) = - \int_t^T h(s) ds,
\]
then \( p(t) \) satisfies
\[
p'(t) = \frac{E^\omega \exp(\omega p(t))}{E^\omega \exp(\omega p(t))}, \quad p(T) = 0. \tag{E.13}
\]
As such, (E.7) becomes to
\[
\pi^*_s = \frac{E^\omega \exp(\omega p(t))}{E^\omega \exp(\omega p(t))ke^{(T-t)}\sigma_1}. \tag{E.14}
\]
Based on (E.8), we have
\[
E^{\omega,\nu}(-g(t, x, \omega, \nu))^\alpha \mu_1(1 + \theta) = E^{\omega,\nu}(-g(t, x, \omega, \nu))^\alpha E^{h^{\omega,\nu}}[\exp(a^*(t)b(t)Y)].
\]
Noticing the distribution of \( \nu \), we obtain that the equilibrium reinsurance strategy satisfies
\[
\mu_1(1 + \theta)E^{\omega,\nu} \exp(\alpha m^2(t, \nu)) = E^{\omega,\nu} \left[ \exp(\alpha m^2(t, \nu)) \frac{\nu}{(\nu - ke^{(T-t)}a^*(t))^2} \right],
\]
where the expressions of \( m^2(t, \nu) \) is given by (4.23). \( \square \)

**Appendix F. Proof of Remark 4.11.**

*Proof.* To simplify our presentation, we rewrite the compound Poisson process
\[
N(t) \sum_{i=1}^{N(t)} Y_i \text{ by using a Poisson random measure } N(\cdot, \cdot) \text{ on } \Omega \times [0, T] \times [0, +\infty)
\]
as follows:
\[
\sum_{i=1}^{N(t)} Y_i = \int_0^t \int_0^{+\infty} yN(ds, dy), \quad \forall t \in [0, T].
\]
If we denote by \( \mu(dt, dy) := \lambda dt F^{\nu}(y) \), then
\[
E \left[ \sum_{i=1}^{N(t)} Y_i \right] = \int_0^t \int_0^{+\infty} y\mu(ds, dy), \quad \forall t \in [0, T],
\]
and in fact, \( \mu(\cdot, \cdot) \) is called the compensator of the random measure \( N(\cdot, \cdot) \). So, the insurer’s surplus process (2.7) can be rewritten as
\[
\begin{cases}
    dX^\pi(t) = \{\lambda \mu_1(1 + \theta)(a(t) - (\theta - \eta)) + rX^\pi(t) + \xi \pi_s(t)L(t) \\
    \quad + \pi_p(t)(1 - H(t))\delta\} dt - a(t) \int_0^{+\infty} yN(ds, dy) \\
    \quad + \beta d\bar{W}(t) + \pi_s(t)\sqrt{L(t)}dW_1(t) - \pi_p(t)\zeta dH(t), \\
    X^\pi(0) = x_0.
\end{cases} \tag{F.1}
\]

The ambiguity aversive insurer recognizes that the models under the reference probability \( P \) only approximate the true models and takes into account some alternative models, which can be defined via a family of probability measures equivalent to \( P \) as follows
\[
\mathcal{Q} := \{Q | Q \sim P\}.
\]
Suppose that \( \{\psi(t)\}_{t \in [0, T]} \) satisfies the following two conditions:
(1) \( \psi(t) > 0 \) for a.s. \( (t, \omega) \in [0, T] \times \Omega \), and \( \{\psi(t)\}_{t \in [0, T]} \) is \( \mathcal{G} \)-predictable;
(2) \( E \left[ \exp \left( \int_0^T (\psi(t) \ln \psi(t) - \psi(t) + 1) \lambda dt \right) \right] < +\infty.\)
For each \( \psi := \{\psi(t)\}_{t \in [0,T]} \), we define a real-valued, G-adapted process \( \{\Lambda^\psi(t)\}_{t \in [0,T]} \) as

\[
\Lambda^\psi(t) := \exp \left\{ \int_0^t \int_0^{+\infty} \ln(\psi(s)) N(ds, dy) + \int_0^t \int_0^{+\infty} (1 - \psi(s)) \mu(ds, dy) \right\},
\]

and construct a new alternative measure \( Q \) equivalent to \( P \) as follows:

\[
\frac{dQ}{dP} \bigg|\bigg|_{\mathcal{F}_T} := \Lambda^\psi(T).
\]

Whenever necessary, we write \( Q^\psi \) as the equivalent probability measure to highlight its dependence on \( \psi \). According to Girsanov’s theorem, the Poisson processes \( N(t) \) has intensity \( \lambda \psi(t) \) under \( Q \) and is denoted by \( N_Q(t) \). Then the insurer’s surplus process under \( Q \) is governed by

\[
dX^\pi(t) = \left\{ \lambda \mu_1[(1 + \theta)a(t) - (\theta - \eta)] + rX^\pi(t) + \xi\pi_s(t)L(t) + \pi_p(t)(1 - H(t))\delta \right\} dt - a(t) \int_0^{+\infty} yN_Q(ds, dy) + \beta dW(t) + \pi_s(t)\sqrt{L(t)}dW_1(t) - \pi_p(t)\xi dH(t).
\]

Next, in order to derive the robust optimal investment–reinsurance strategy under the worst-case scenario, we first introduce a penalty function. To embody the ambiguity arising from jump risk (risks caused by the compound Poisson process), we adopt the following penalty function:

\[
\Psi(t) = \frac{\lambda \psi(t) \ln \psi(t) - \psi(t) + 1}{u(t)},
\]

where \( u(t) \) is nonnegative and stands for the preference parameters for ambiguity aversion, which measures the degree of confidence to the reference probability \( P \). The larger \( u(t) \) is, the more ambiguity averse the insurer is.

On the basis of the penalty function \( \Psi(t) \), we formulate a robust optimal investment–reinsurance problem as follows:

\[
\sup_{\pi \in \Pi} \inf_{Q^\psi \in \mathcal{Q}} E_{t,x,l,z}^{Q^\psi} \left[ U(X^\pi(T)) + \int_t^T \Psi(s)ds \right],
\]

where \( E_{t,x,l,z}[\cdot] = E[\cdot|X^\pi(t) = x, L(t) = l, H(t) = z] \).

To solve problem (F.4), we define the optimal value function as

\[
V(t, x, l, z) = \sup_{\pi \in \Pi} \inf_{Q^\psi \in \mathcal{Q}} E_{t,x,l,z}^{Q^\psi} \left[ U(X^\pi(T)) + \int_t^T \Psi(s)ds \right].
\]

Let \( C^{1,2}(0, T] \times \mathbb{R} \times \mathbb{R}^+ \) denotes the space of \( V(t, x, l) \), such that \( V(t, x, l) \) and its derivatives \( V_t(t, x, l), V_x(t, x, l), V_{xx}(t, x, l), V_l(t, x, l), V_{ll}(t, x, l) \) and \( V_{zl}(t, x, l) \) are continuous on \( [0, T] \times \mathbb{R} \times \mathbb{R}^+ \). For any state \( (t, x, l, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^+ \times \{0, 1\} \),
we define the infinitesimal generator for the surplus process in (F.3) is defined as
\[ A^{\pi,\psi} f(t, x, l, z) = \begin{cases} 
    f_t(t, x, l, 1) + \{ \lambda \mu_1[(1 + \theta)a - (\theta - \eta)] + rx + \xi \pi_s l \} 
    \cdot f_x(t, x, l, 1) + \gamma(\kappa - l) f_t(t, x, l, 1) + \frac{1}{2} \beta^2 + ln^2 
    \cdot f_{xx}(t, x, l, 1) + \frac{1}{2} \sigma^2 f_t(t, x, l, 1) + \pi_s \sigma_l f_{x}(t, x, l, 1) 
    + \lambda \psi(t) E^{\psi}[f(t, x - aY, l, 1) - f(t, x, l, 1)] \quad z = 1; 
    f_t(t, x, l, 0) + \{ \mu_1[(1 + \theta)a - (\theta - \eta)] + rx + \xi \pi_s l + \pi_p \delta \} 
    \cdot f_x(t, x, l, 0) + \gamma(\kappa - l) f_t(t, x, l, 0) + \frac{1}{2} \beta^2 + ln^2 
    \cdot f_{xx}(t, x, l, 0) + \frac{1}{2} \sigma^2 f_t(t, x, l, 0) + \pi_s \sigma_l f_{x}(t, x, l, 0) 
    + \lambda \psi(t) E^{\psi}[f(t, x - aY, l, 0) - f(t, x, l, 0)] 
    + [f(t, x - \zeta \pi_p, l, 1) - f(t, x, l, 0)] h^\psi, \quad z = 0. 
\] 
(A.6)

According to the optimality principle of dynamic programming, we can derive the HJBI equation for problem (F.4) as
\[ \sup_{\pi \in \Pi} \inf_{Q \in \Omega} \{ A^{\pi,\psi} V(t, x, l, z) + \Psi(t) \} = 0, \quad (F.7) \]
with the boundary condition being \( V(T, x, l, z) = U(x) \).

Following the Section 4, we assume that the AAI has an exponential utility function, i.e.,
\[ U(x) = -\frac{1}{k} e^{-kx}, \quad k > 0. \]
In addition, we define the ambiguity preference functions as
\[ u(t) = -\frac{\mu}{k V(t, x, l, z)}, \quad (F.8) \]
where \( \mu \) is positive constant and stands for the ambiguity aversion parameter.

**Lemma F.1.** Let
\[ q(a(t)) = -b(t) \mu_1(1 + \theta)a(t) + \frac{\lambda k}{\mu} \exp \left[ \frac{\mu}{k} E^{\psi}[\exp(ab(t)Y) - 1] \right], \quad (F.9) \]
then \( q'(a(t)) = 0 \) has a unique positive root \( a^*(t) \in [0, +\infty) \).

**Proof.** We can rewrite \( q'(a(t)) = 0 \) as
\[ -\mu_1(1 + \theta) + E^{\psi}[\exp(ab(t)Y)Y'] \exp \left[ \frac{\mu}{k} E^{\psi}[\exp(ab(t)Y) - 1] \right] = 0. \]
Denote
\[ h(a(t)) = E^{\psi}[\exp(ab(t)Y)Y] \exp \left[ \frac{\mu}{k} E^{\psi}[\exp(ab(t)Y) - 1] \right], \]
we have
\[ h'(a(t)) = b(t) E^{\psi}[\exp(ab(t)Y)Y^2] \exp \left[ \frac{\mu}{k} E^{\psi}[\exp(ab(t)Y) - 1] \right] 
    + \left[ E^{\psi}[\exp(ab(t)Y)Y] \right] \exp \left[ \frac{2\mu}{k} b(t) \exp \left[ \frac{\mu}{k} E^{\psi}[\exp(ab(t)Y) - 1] \right] \right] > 0, \]
so \( h(a(t)) \) is an increasing function with respect to \( a(t) \). In addition, we have \( h(0) = E^{\psi}[Y] = \mu_1 < \mu_1(1 + \theta) \) and \( h(+\infty) = +\infty \). Therefore, \( q'(a(t)) = 0 \) has a unique positive root \( a^*(t) \in [0, +\infty) \).
Post-default case. In the post-default case, \( \pi_p(t) = 0 \) for \( \tau \leq t \leq T \) as the default has already occurred. By observing the expression of \( A^{\pi,\psi} \) in (F.6), we can rewrite (F.7) as

\[
\sup_{\pi \in \Pi} \inf_{Q^\pi \in Q} \left\{ V_i(t, x, l, 1) + \{ \lambda \mu_1[(1 + \theta)a - (\theta - \eta)] + \rho x + \xi \pi_1 \} \right. \\
\cdot V_x(t, x, l, 1) + \gamma(\kappa - l)V_i(t, x, l, 1) + \frac{1}{2}(\beta^2 + l\pi_s^2)V_{xx}(t, x, l, 1) \\
+ \pi_s \sigma l \rho V_{x}(t, x, l, 1) + \lambda \psi(t) E^{Q^\pi} \left[ V(t, x - aY, l, 1) - V(t, x, l, 1) \right] \\
+ \frac{1}{2}\sigma^2 l^2 V_{ll}(t, x, l, 1) + \frac{\lambda [\psi(t) \ln \psi(t) - \psi(t) + 1]}{u(t)} \right\} = 0. \tag{F.10}
\]

Conjecturing that the value function has the following form

\[
V(t, x, l, 1) = -\frac{1}{k} \exp[-b(t)x + m(t) + n(t)l], \tag{F.11}
\]

and a direct calculation yields

\[
\left\{ \begin{array}{l}
V_i(t, x, l, 1) = (-b_i x + m_i + n_i l)V(t, x, l, 1), \quad V_i(t, x, l, 1) = n(t)V(t, x, l, 1), \\
V_x(t, x, l, 1) = -b(t)V(t, x, l, 1), \quad V_{xx}(t, x, l, 1) = b^2(t)V(t, x, l, 1), \\
V_{il}(t, x, l, 1) = n^2(t)V(t, x, l, 1), \quad V_{xi}(t, x, l, 1) = -b(t)n(t)V(t, x, l, 1), \\
V(t, x - aY, l, 1) - V(t, x, l, 1) = [\exp(ab(t)Y) - 1]V(t, x, l, 1). 
\end{array} \right. \tag{F.12}
\]

Differentiating (F.10) with respect to \( \psi(t) \) gives

\[
\psi^*(t) = \exp \left[ -u(t) E^{Q^\pi} \left[ V(t, x - aY, l, 1) - V(t, x, l, 1) \right] \right], \tag{F.13}
\]

then substituting (F.8) and (F.11), we obtain

\[
\psi^*(t) = \exp \left[ \frac{\mu E^{Q^\pi}}{k} \left[ \exp(ab(t)Y) - 1 \right] \right]. \tag{F.14}
\]

Then, plugging (F.11), (F.12) and (F.14) into (F.10) yields

\[
\sup_{\pi \in \Pi} \inf_{Q^\pi \in Q} \left\{ -b_i x + m_i + n_i l - \{ \lambda \mu_1[(1 + \theta)a - (\theta - \eta)] + \rho x + \xi \pi_s l \} b(t) \\
+ \gamma(\kappa - l)n(t) + \frac{1}{2}(\beta^2 + l\pi_s^2)b^2(t) + \frac{1}{2}\sigma^2 n^2(t) - \pi_s \sigma l \rho b(t)n(t) \\
+ \frac{\lambda k}{\mu} \exp \left[ \frac{\mu E^{Q^\pi}}{k} \left[ \exp(ab(t)Y) - 1 \right] \right] - \frac{\lambda k}{\mu} \right\} = 0. \tag{F.15}
\]

Differentiating (F.15) w.r.t. \( \pi_s \) and \( a \), we have

\[
\pi_s^*(t) = \frac{\xi}{b(t)} + \frac{n(t)\sigma l}{b(t)}, \tag{F.16}
\]

and \( a^*(t) \) satisfies

\[
- \mu_1(1 + \theta) + E^{Q^\pi} \left[ \exp(a^*b(t)Y)Y \right] \exp \left[ \frac{\mu E^{Q^\pi}}{k} \left[ \exp(ab(t)Y) - 1 \right] \right] = 0. \tag{F.17}
\]
Putting (F.16), (F.17) into (F.15) and arranging the equation by the order of \( x \), we obtain the following two partial differential equations

\[
b_t + rb(t) = 0, \quad b(T) = k, \tag{F.18}
\]

\[
m_t + n_t + b(t)\lambda_1(\theta - \eta) - \frac{1}{2}\xi^2l + \gamma kn(t) - (\gamma + \sigma \rho \xi)n(t)l
+ \frac{1}{2}\beta^2b^2(t) + \frac{1}{2}ln^2(t)\sigma^2(1 - \rho^2) - \frac{\lambda_k}{\mu} + q(a^*(t)) = 0. \tag{F.19}
\]

In terms of (F.18), \( b(t) \) can be explicitly obtained

\[
b(t) = k \exp(r(T - t)). \tag{F.20}
\]

To solve (F.19), we separate the variables with and without \( l \), respectively. Then, we derive the following system of ODEs

\[
m_t + b(t)\lambda_1(\theta - \eta) + \gamma kn(t) + \frac{1}{2}\beta^2b^2(t) - \frac{\lambda_k}{\mu} + q(a^*(t)) = 0, \tag{F.21}
\]

\[
n_t - \frac{1}{2}\xi^2 - (\gamma + \sigma \rho \xi)n(t) + \frac{1}{2}\sigma^2(1 - \rho^2)n^2(t) = 0. \tag{F.22}
\]

According to the boundary condition \( m(T) = 0 \) and \( n(T) = 0 \), we obtain

\[
n(t) = \begin{cases} \frac{\sigma^2}{2(\gamma + \sigma \xi)^2}[e^{-(\gamma + \sigma \xi)(T - t)} - 1], & \rho = 1, \\ \frac{\xi^2}{2(\gamma - \sigma \xi)}(T - t), & \rho = -1 \text{ and } \gamma = \xi \sigma, \\ \frac{\xi^2}{2(\gamma - \sigma \xi)}[e^{-(\gamma - \sigma \xi)(T - t)} - 1], & \rho = -1 \text{ and } \gamma \neq \xi \sigma, \\ \frac{q_1\alpha_2\exp(\sqrt{\Delta}(T - t)) - q_1\alpha_2}{q_1\exp(\sqrt{\Delta}(T - t)) - q_2}, & \rho \neq \pm 1, \end{cases} \tag{F.23}
\]

\[
m(t) = \lambda_1(\theta - \eta)\int_t^T b(s)ds + \gamma k\int_t^T n(s)ds - \frac{\lambda_k}{\mu}(T - t)
+ \frac{1}{2}\beta^2\int_t^T b^2(s)ds + \int_t^T q(a^*(s))ds. \tag{F.24}
\]

where \( \Delta = (\gamma + \sigma \rho \xi)^2 + \sigma^2(1 - \rho^2)^2 \), \( q_{1,2} = \frac{(\gamma + \sigma \rho \xi)\pm \sqrt{\Delta}}{\sigma^2(1 - \rho^2)} \).

### F.2. Pre-default case.

In the pre-default case, i.e., \( z = 0 \), by the expression of \( A^{\pi,\psi} \) in (F.6), (F.7) becomes

\[
\sup_{\pi \in \Pi} \inf_{Q^\psi \in Q} \left\{ V_t(t, x, l, 0) + (\lambda_1[(1 + \theta)a - (\theta - \eta)] + rx + \xi \pi s l + \pi_p \delta)V_x(t, x, l, 0)
+ \gamma(\kappa - l)V(t, x, l, 0) + \frac{1}{2}(\beta^2 + l\pi^2)V_{xx}(t, x, l, 0) + \frac{1}{2}\sigma^2 l V_{ll}(t, x, l, 0)
+ \pi_s\sigma l V_x(t, x, l, 0) + \lambda \psi(t) V(t, x - aY, l, 0) - V(t, x, l, 0)
+ [V(t, x - \zeta, x, l, 1) - V(t, x, l, 0)]\bar{h}^p + \frac{\lambda \psi(t) \ln \psi(t)}{u(t)} \right\} = 0. \tag{F.25}
\]

Conjecturing that the value function has the following form

\[
V(t, x, l, 1) = -\frac{1}{\bar{k}} \exp[-k e^{(T-t)}x + \bar{m}(t) + \bar{n}(t)]l, \tag{F.26}
\]
and a direct calculation yields
\[
\begin{align*}
\frac{\partial V}{\partial t}(t, x, l, 0) &= (kre^{r(T-t)}x + \bar{m}_t + \bar{n}_t l) V(t, x, l, 0), \\
\frac{\partial V}{\partial x}(t, x, l, 0) &= -ke^{r(T-t)}V(t, x, l, 0), \\
\frac{\partial^2 V}{\partial x^2}(t, x, l, 0) &= k^2 e^{2r(T-t)}V(t, x, l, 0), \\
V(t, x, l, 0) &= \bar{n}(t) V(t, x, l, 0), \\
V_x(t, x, l, 0) &= \bar{n}_t V(t, x, l, 0), \\
V_{xx}(t, x, l, 0) &= -ke^{r(T-t)}\bar{n}(t) V(t, x, l, 0), \\
V(t, x - aY, l, 0) - V(t, x, l, 0) &= \left[\exp(ke^{r(T-t)}aY) - 1\right] V(t, x, l, 0), \\
V(t, x - \zeta \pi_p, l, 1) - V(t, x, l, 0) &= \left[\exp[ke^{r(T-t)}\zeta \pi_p + (m(t) - \bar{m}(t)) + (n(t) - \bar{n}(t))l] - 1\right] V(t, x, l, 0).
\end{align*}
\] (F.27)

Differentiating (F.25) with respect to \(\psi(t)\) gives
\[
\psi^{*}(t) = \exp \left[ -u(t) E^{Q^*} \left[ V(t, x - aY, l, 0) - V(t, x, l, 0) \right] \right],
\] (F.28)
then substituting (F.8) and (F.27), we obtain
\[
\psi^{*}(t) = \exp \left( \frac{\mu}{k} E^{Q^*} \left[ \exp(ke^{r(T-t)}aY) - 1 \right] \right).
\] (F.29)

Plugging (F.26), (F.27) and (F.29) into (F.25) yields
\[
\sup_{\pi \in \Pi \forall l \in \mathbb{Q}} \inf \left\{ \frac{kre^{r(T-t)}x + \bar{m}_t + \bar{n}_t l - \{\lambda \mu_1(1 + \theta)a - \theta - \eta\} + rx + \xi \pi_s l + \pi_p \delta}{ke^{r(T-t)} + \gamma(k - l)\bar{n}(t) + \frac{1}{2}(\beta^2 + l\pi_s^2)k^2 e^{2r(T-t)} - \pi_s \sigma \rho ke^{r(T-t)}\bar{n}(t) + \frac{1}{2}\frac{\lambda k}{\mu}\exp \left( \frac{\mu}{k} E^{Q^*} \left[ \exp(ke^{r(T-t)}aY) - 1 \right] \right) - \frac{\lambda k}{\mu}} \right\} = 0.
\] (F.30)

Differentiating (F.30) w.r.t. \(\pi_s, \pi_p\) and \(a\), we have
\[
\pi^{*}_s(t) = \frac{\xi}{ke^{r(T-t)}} + \frac{\bar{n}(t)\sigma \rho}{ke^{r(T-t)}},
\] (F.31)
\[
\pi^{*}_p(t) = \frac{1}{ke^{r(T-t)}\zeta} \left( \bar{m}(t) - m(t) + (\bar{n}(t) - n(t))l + \ln \frac{\delta}{h^P \zeta} \right),
\] (F.32)
and \(a^{*}(t)\) satisfies
\[
- \mu_1(1 + \theta) + E^{Q^*} \left[ \exp(ke^{r(T-t)}a^{*}Y) \right] \exp \left( \frac{\mu}{k} E^{Q^*} \left[ \exp(ke^{r(T-t)}a^{*}Y) - 1 \right] \right) = 0.
\] (F.33)

Putting (F.31), (F.32) and (F.33) into (F.30), then separating the variables with and without \(l\), respectively. We obtain the following system of ODEs
\[
\begin{align*}
\bar{m}_t + ke^{r(T-t)}\lambda \mu_1(1 + \theta) - \gamma \kappa \bar{n}(t) + \frac{1}{2}\beta^2 k^2 e^{2r(T-t)} - \frac{\lambda k}{\mu} + q(a^{*}(t)) - \frac{\delta}{\zeta} \left( \bar{m}(t) - m(t) + \ln \frac{\delta}{h^P \zeta} \right) + \frac{\delta}{\zeta} - h^P = 0, \\
\bar{n}_t - \frac{1}{2}\xi^2 - (\gamma + \sigma \rho \xi + \frac{\delta}{\zeta})\bar{n}(t) = \frac{\delta}{\zeta} \bar{n}(t) + \frac{1}{2}\sigma^2(1 - \rho^2)\bar{n}_t(t) = 0.
\end{align*}
\] (F.34) (F.35)

According to the boundary condition \(\bar{n}(T) = 0\), we obtain
\[
\bar{n}(t) = n(t).
\]
Then, according to the boundary condition $\bar{m}(T) = 0$, we solve (F.34) in the following. Since $m(t)$ satisfies equation (F.21), subtracting (F.34) and (F.21), we have

$$m_t - m_t - \frac{\delta}{\zeta} \left( \bar{m}(t) - m(t) + \ln \frac{\delta}{h^P\zeta} \right) + \frac{\delta}{\zeta} - h^P = 0.$$  \hfill (F.36)

Defining $M(t) = \bar{m}(t) - m(t)$, then (F.36) becomes

$$M_t - \frac{\delta}{\zeta} \left( M(t) + \ln \frac{\delta}{h^P\zeta} \right) + \frac{\delta}{\zeta} - h^P = 0.$$  \hfill (F.37)

We obtain

$$M(t) = \left( \frac{h^P\zeta}{\delta} - 1 + \ln \frac{\delta}{h^P\zeta} \right) \left( e^{-\frac{\zeta}{\delta}(T-t)} - 1 \right),$$  \hfill (F.38)

so

$$\bar{m}(t) = M(t) + m(t),$$  \hfill (F.39)

where $m(t)$ satisfies Eq.(F.24).

Thus, Eq.(F.32) becomes

$$\pi^*_P(t) = \frac{1}{k_{\zeta}(T-t)\zeta} \left[ \left( \frac{h^P\zeta}{\delta} - 1 + \ln \frac{\delta}{h^P\zeta} \right) e^{-\frac{\zeta}{\delta}(T-t)} - h^P\zeta \delta + 1 \right].$$  \hfill (F.40)

\[\square\]

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