HODGE THEORY OF REPRESENTATION VARIETIES VIA TOPOLOGICAL QUANTUM FIELD THEORIES

ÁNGEL GONZÁLEZ-PRIETO

Abstract. In this paper, we show how lax monoidal TQFTs can be used as an effective computational method of the $K$-theory image of the Hodge structure of representation varieties. In particular, we perform the calculation for parabolic $\text{SL}_2(\mathbb{C})$-representation varieties over a closed orientable surface of arbitrary genus and any number of marked points with holonomies of Jordan type. This technique is based on a building method of lax monoidal TQFTs of physical inspiration that generalizes the construction of González-Prieto, Logares and Muñoz.

1. Introduction

Let $G$ be a complex algebraic group, let $M$ be a compact manifold and let $A \subseteq M$ be a non-empty finite set. The set of representations of the fundamental groupoid $\Pi(M, A)$ into $G$ has a natural structure of complex algebraic variety, called the $G$-representation variety of $(M, A)$ and denoted $X_G(M, A) = \text{Hom} (\Pi(M, A), G)$. In the case that $M$ is connected and $A$ has a single point, this variety is usually shorten $X_G(M) = \text{Hom}(\pi_1(M), G)$.

Evenmore, we can consider a parabolic structure $Q$ on $M$, given by a finite set of codimension two submanifolds $S_1, \ldots, S_r \subseteq M$ and of conjugacy classes $\lambda_1, \ldots, \lambda_r \subseteq G$ (called the holonomies). In that case, we can also define the parabolic $G$-representation variety, $X_G(M, A, Q)$, as the set of representations $\rho : \Pi(M - \bigcup S_i, A) \rightarrow G$ such that $\rho(\gamma_i) \in \lambda_i$ if $\gamma_i$ is a loop around $S_i$. As in the non-parabolic case, it has a natural structure of a complex variety.

There is a natural action of $G$ on $X_G(M, A, Q)$ by conjugation. Hence, if $G$ is also a reductive group, we can also consider the GIT quotient $\mathcal{R}_G(M, A, Q) = X_G(M, A, Q) \diagup G$. This algebraic variety is the so-called parabolic $G$-character variety and it can be shown to be the moduli space of such a parabolic representations.

The understanding of the topological and algebraic structure of these character varieties is an open problem that has been objective of intense study for the last twenty years. A starting point would be to compute their Betti numbers or, equivalently, its Poincaré polynomial for the case $M = \Sigma$, a closed oriented manifold and no parabolic structure. In this direction, the non-abelian Hodge theory becomes a useful tool. Roughly speaking, recall that the non-abelian Hodge theory shows that three different moduli spaces on $\Sigma$ are diffeomorphic: the character variety, the moduli space of flat connections and the moduli space of Higgs bundles. For a precise description of these spaces and the proof of these equivalences, see [65], [66], [64] and [12]. In this way, the topology of $\mathcal{R}_G(\Sigma)$ can be studied via a natural perfect Morse-Bott function on the moduli space of Higgs bundles. For a precise description of these spaces and the proof of these equivalences, see [65], [66], [64] and [12]. In this way, the topology of $\mathcal{R}_G(\Sigma)$ can be studied via a natural perfect Morse-Bott function on the moduli space of Higgs bundles. As a result, the Poincaré polynomial of character varieties has been computed for $G = \text{SL}_2(\mathbb{C})$ [85], for $G = \text{SL}_3(\mathbb{C})$ [82] and for $G = \text{GL}_4(\mathbb{C})$ [24]. In general, in [59] and [63] (see also [49]) a combinatorial formula is given for arbitrary $G = \text{GL}_r(\mathbb{C})$ provided that $n$ and $d$ are coprime. In the parabolic case, it has been computed for $G = \text{SL}_2(\mathbb{C})$ and generic semi-simple conjugacy classes [8] and for $G = \text{GL}_3(\mathbb{C}), \text{SL}_3(\mathbb{C})$ [26].

However, the equivalences from non-abelian Hodge theory are not holomorphic, so the complex structures on the character variety and on the moduli space of flat connections and Higgs bundles do not agree. For this reason, the study of specific algebraic invariants of character

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varieties turns important. An important invariant of a complex variety $X$ is its Deligne-Hodge polynomial, $e(X) \in \mathbb{Z}[u^{\pm 1}, v^{\pm 1}]$, which is an alternating sum of the Hodge numbers of $X$ (with compact support) as a mixture between Euler characteristic and Poincaré polynomial.

In order to compute the Deligne-Hodge polynomials of character varieties over a closed oriented surface, two different approaches have been used. The first one is the so-called arithmetic method, introduced in [37] and based on a theorem of Katz inspired by the Weil conjectures. Using the arithmetic method, several computations of Deligne-Hodge polynomials have been done as in [37] for $G = \text{GL}_n(\mathbb{C})$ and a parabolic structure with a single marked point and holonomy a primitive root of unit and for $G = \text{SL}_n(\mathbb{C})$ in [50]. However, in general, the computations of this method are not explicit and the polynomials are given only in terms of generating functions. Moreover, in the parabolic case, very little has been advanced, being the major achievement the computation of these polynomials for $G = \text{GL}_n(\mathbb{C})$ and a generic parabolic structure in [35].

In order to overcome this problem, a new geometric method was introduced in [43]. The idea of this method is to stratify the representation variety into simpler pieces for which the Deligne-Hodge polynomial can be computed and to add all the contributions. From the Deligne-Hodge polynomial of the representation variety, the one of the character variety follows by analyzing the identifications that take place in the GIT quotient. Using this method, explicit expressions of these polynomials have been computed for the case $G = \text{SL}_2(\mathbb{C})$. They were computed for at most one marked point and closed orientable surfaces of genus $g = 1, 2$ in [43], for $g = 3$ in [47] and for arbitrary genus in [48]. In [42], the case of two marked points is accomplished for $g = 1$. Finally, in [3], a mix between the arithmetic and the geometric method was used, computing explicit expressions of the Deligne-Hodge polynomials for $G = \text{SL}_2(\mathbb{C})$, $\text{SL}_3(\mathbb{C})$ and arbitrary genus with no parabolic structure. Despite its success, this method requires the use of very specific stratifications that are not clear how to generalize to other groups or more general parabolic structures.

Motivated by this problem, in [31], a new categorial approach to this problem was introduced as part of the PhD Thesis [30] of the author under the supervision of M. Logares and V. Muñoz. In that paper, given a complex algebraic group $G$, a lax monoidal Topological Quantum Field Theory (TQFT for short) is constructed that computes the Deligne-Hodge polynomials of $G$-representation varieties. Recall that a lax monoidal TQFT is a lax monoidal symmetric functor $Z_G : \text{Bdp}_n(\Lambda) \to \text{KHS-Mod}$, where $\text{KHS}$ is the Grothendieck ring (aka. the $K$-theory) of the category of Hodge structures, $\text{Bdp}_n(\Lambda)$ is the category of parabolic bordisms of pairs and $\text{KHS-Mod}$ is the category of $\text{KHS}$-modules (see Section 4 for a more precise definition). The important point of this TQFT is that, if $W$ is a closed $n$-dimensional manifold, $A \subseteq W$ is a finite set and $Q$ is a parabolic structure on $W$, then $Z_G(W, A, Q) : \text{KHS} \to \text{KHS}$ is a $\text{KHS}$-linear morphism given by the multiplication (up to a known constant) by the $K$-theory image of the Hodge structure on the cohomology of $X_G(W, A, Q)$.

The objective of this paper is to show that $Z_G$ can be used to give an effective recursive method of computation of the (image on the $K$-theory) of the Hodge structure of parabolic $G$-representation varieties. This recursive nature for character varieties has also been explored in the literature, as in and [51], [36], [15] and [11].

In Section 2 of this paper, we give a brief review of Hodge theory, with special attention to Saito’s theory of mixed Hodge modules (Section 2.2). That will be very useful since a fundamental piece in the construction of $Z_G$ is the category of mixed Hodge modules over an algebraic variety $X$, denoted $\mathcal{M}_X$. In Section 3 we review in deeper detail the relation of mixed Hodge modules with variations of mixed Hodge structures. Using this interplay, we give
a concrete incarnation of some mixed Hodge modules as monodromies of locally trivial fibrations that will be necessary for computational purposes.

In Section 3, we describe a method of construction of TQFTs from two simpler pieces: a field theory and a quantisation. The idea is to consider an auxiliary category $C$ with pullbacks and final object $\ast \in C$, that is going to play the role of a category of fields (in the physical sense). Let $\text{Emb}_c$ be the category whose objects are compact manifolds (maybe with boundary) but only open embeddings between them. A functor $F : \text{Emb}_c \to C$ with the Seifert-van Kampen property (see Definition 1.11) gives rise to a ‘field theory’, which is a functor $F_F \to \text{Span}(C)$.

On the other hand, in Section 4.1 we introduce an algebraic object called a $C$-algebra. Roughly speaking, a $C$-algebra $A$ is a collection of algebras $A_c$ for all $c \in C$ that preserves the functorial structure of $C$. From a $C$-algebra, in Section 4.2 we show how to construct a functor $Q_A : \text{Span}(C) \to A_\ast\text{-Mod}$ that plays the role of a ‘quantisation’ of the fields of $C$. Hence, composing the functors from the field theory part and the quantisation, we obtain a functor $Z = Q_A \circ F_F : \text{Bd}_n \to \text{Span}(C) \to A_\ast\text{-Mod}$ that we show it is a lax monoidal TQFT. The same construction can be done if we consider an extra structure on the category of bordisms given by a sheaf, as finite configurations of points or parabolic structures (see Section 5). Therefore, putting together all this information, we obtain the following result (Theorem 4.13).

**Theorem.** Let $C$ be a category with final object $\ast$ and pullbacks and let $S$ be a sheaf. Given a functor $F : \text{Emb}_c^S \to C$ with Seifert-van Kampen property and a $C$-algebra $A$, there exists a lax monoidal Topological Quantum Field Theory over $S$, $Z_{F,A} : \text{Bd}_n^S \to A_\ast\text{-Mod}_t$.

Section 3 of this paper is devoted to representation varieties. In Section 5.1, we review the construction of the TQFT $Z_G$ in the context of the physical construction. In this case, we take $C = \text{Var}_C$, the category of complex algebraic varieties, $F$ as the representation variety functor and the $\text{Var}_C$-algebra considered is $A_X = K\text{M}_X$, the Grothendieck rings of mixed Hodge modules on the complex variety $X$. This description agrees with the construction of [31].

Even though $Z_G$ computes the Hodge structures of representation varieties, for computational purposes it is better to consider a modified TQFT, called the geometric TQFT and denoted $Z_G^{gm}$, as explained in Section 5.2. It is built by means of a procedure called reduction (see Section 4.5). The advantage of $Z_G^{gm}$ is that, in general, $Z_G(S^1, \ast) = K\text{M}_G$ is not a finitely generated module, but $Z_G^{gm}(S^1, \ast)$ is so.

As an application, in Section 6 we show how $Z_G^{gm}$ can be used to give an effective method of computation of Hodge structures on representation varieties. We focus on the case $G = \text{SL}_2(\mathbb{C})$ and we will allow arbitrary many marked points with holonomies in $\Lambda = \{[J_+], [J_-], \{-1\}\}$. Here, $[J_+], [J_-] \subseteq \text{SL}_2(\mathbb{C})$ are the set of non-diagonalizable matrices of traces 2 and $-2$ respectively.

For that purpose, in Section 6.1 we give an explicit finite set of generators for $Z_{\text{SL}_2(\mathbb{C})}^{gm}(S^1, \ast)$ in terms of monodromies of fibrations (the so-called core submodule). Observe that any closed surface $\Sigma$, endowed with $A \subseteq \Sigma$ finite and $Q$ a parabolic structure, is a composition of the bordisms depicted in the figure below with $\lambda = [J_+], [J_-]$ or $\{-1\}$. Hence, in order to compute de Hodge structure on $\mathfrak{x}_{\text{SL}_2(\mathbb{C})}(\Sigma, A, Q)$, it is enough to compose the morphisms $Z_{\text{SL}_2(\mathbb{C})}^{gm}(D), Z_{\text{SL}_2(\mathbb{C})}^{gm}(D^\dagger), Z_{\text{SL}_2(\mathbb{C})}^{gm}(L)$ and $Z_{\text{SL}_2(\mathbb{C})}^{gm}(L_\Lambda)$.
With that goal, Section 6.2 is devoted to the computation of $Z_{\text{SL}_2(C)}^m(D)$, $Z_{\text{SL}_2(C)}^m(D^\dagger)$ and $Z_{\text{SL}_2(C)}^m(L_{-1})$. In Section 6.3, we compute the more involved homomorphisms $Z_{\text{SL}_2(C)}^m(L_{J_{-1}})$ and $Z_{\text{SL}_2(C)}^m(L_{J_{-1}^*})$. Finally, in Section 6.4, we will show how the results of [28] can be recast to give $Z_{\text{SL}_2(C)}^m(L)$. Putting together all this information, we obtain the main result of this paper.

**Theorem.** The $K$-theory image of the cohomology of $\mathfrak{X}_{\text{SL}_2(C)}(\Sigma_g, Q)$ is

- If $\sigma = 1$, then
  \[
  \left[ H_c^*(\mathfrak{X}_{\text{SL}_2(C)}(\Sigma_g, Q)) \right] = (q^2 - 1)^{2g+r-1}q^{2g-1} + \frac{1}{2}(q-1)^{2g+r-1}q^{2g-1}(q+1)(2^{2g} + q - 3) + \frac{(-1)^r}{2}(q+1)^{2g+r-1}q^{2g-1}(q-1)(2^{2g} + q - 1).
  \]

- If $\sigma = -1$, then
  \[
  \left[ H_c^*(\mathfrak{X}_{\text{SL}_2(C)}(\Sigma_g, Q)) \right] = (q-1)^{2g+r-1}(q+1)q^{2g-1} + (q+1)^{2g+r-2} + 2^{2g-1} - 1 \]
  \[
  + (-1)^{r+1}2^{2g-1}(q+1)^{2g+r-1}(q-1)q^{2g-1}.
  \]

At this point, several research lines open as continuation of this work. The most immediate one is to study how these computations of Hodge structures of representation varieties can be used to give the Deligne-Hodge polynomials of the corresponding character varieties. This gap is filled by the paper [20] (see also [30]).

Moreover, a wider class of holonomies can be considered, as of semi-simple type. However, in that case, new generators of the core submodule appear around the semi-simple points. For this reason, a more subtle analysis is required that goes beyond the scope of the present paper and it is studied in the upcoming paper [28].

A step further, it would be interesting to use the construction method described here to extend the TQFT for character varieties across the non-abelian Hodge correspondence, building analogous TQFTs for the moduli spaces of flat bundles and Higgs bundles. Finally, at long term, we want to study the relation between $Z_G$ and $Z_{L G}$, where $L G$ is the Langlands dual group of $G$. The reason is that, as conjectured in [34] and [37] in the context of the mirror symmetry conjectures and the geometric Landlands program (see [4]), very astonishing symmetries between the Deligne-Hodge polynomials for character varieties over $G$ and $L G$ may be expected.

We hope that the kind of ideas introduced in this paper will help to shed some light over these problems in the future.

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2. Review of Hodge theory

2.1. Mixed Hodge structures. Let $X$ be a complex algebraic variety. The rational cohomology of $X$, $H^\bullet(X;\mathbb{Q})$, carries an additional linear structure, called mixed Hodge structure. It generalizes the so-called pure Hodge structures that appear when $X$ is smooth projective. In this section, we will review some basic facts of Hodge structures that will be useful for this paper. For further information, see [13] and [14], also [54].

Definition 2.1. Let $V$ be a finite dimensional rational vector space and let $k \in \mathbb{Z}$. A pure Hodge structure of weight $k$ on $V$ consists of a finite decreasing filtration $F^\bullet$ of $V_{\mathbb{C}} = V \otimes_{\mathbb{Q}} \mathbb{C}$ such that

$$V_{\mathbb{C}} \supseteq \ldots \supseteq F^p V \supseteq F^p V \supseteq \ldots \supseteq \{0\}$$

such that $F^p V \oplus \overline{F}^{k-p+1} V = V_{\mathbb{C}}$ where conjugation is taken with respect to the induced real structure.

Remark 2.2. An equivalent description of a pure Hodge structure is as a finite decomposition

$$V_{\mathbb{C}} = \bigoplus_{p+q=k} V^{p,q}$$

for some complex vector spaces $V^{p,q}$ such that $V^{p,q} \cong V^{q,p}$ with respect to the natural real structure of $V_{\mathbb{C}}$. From this description, the filtration $F^\bullet$ can be recovered by taking $F^p V = \bigoplus_{r \geq p} V^{r,k-r}$. In this terms, classical Hodge theory shows that the cohomology of a compact Kähler manifold $M$ carries a pure Hodge structure induced by Dolbeault cohomology by $H^k(M;\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(M)$. See [54] for further information.

Example 2.3. Given $m \in \mathbb{Z}$, we define the $m$-th Tate structure, $\mathbb{Q}(m)$, as the pure Hodge structure whose underlying rational vector space is $(2\pi i)^m \mathbb{Q} \subseteq \mathbb{C}$ with a single-piece decomposition $\mathbb{Q}(m) = \mathbb{Q}(m)^{-m,-m}$. Thus, $\mathbb{Q}(m)$ is a pure Hodge structure of weight $-2m$. Moreover, if $V$ is another pure Hodge structure of weight $k$ then $V(m) = V \otimes \mathbb{Q}(m)$ is a pure Hodge structure of weight $k - 2m$, called the Tate twist of $V$. For short, we will denote $\mathbb{Q}_0 = \mathbb{Q}(0)$, the Tate structure of weight 0. Recall that there is a well defined tensor product of pure (and mixed) Hodge structures, see Examples 3.2 of [54] for details.

Definition 2.4. Let $V$ be a finite dimensional rational vector space. A (rational) mixed Hodge structure on $V$ consist of a pair of filtrations:

- An increasing finite filtration $W_\bullet$ of $V$, called the weight filtration.
- A decreasing finite filtration $F^\bullet$ of $V_{\mathbb{C}}$, called the Hodge filtration.

Such that, for any $k \in \mathbb{Z}$, the induced filtration of $F^\bullet$ on the graded complex $(\text{Gr}_k^W V)_{\mathbb{C}} = \left(\frac{W_k V}{W_{k-1} V}\right)_{\mathbb{C}}$ gives a pure Hodge structure of weight $k$. Given two mixed Hodge structures $(V, F, W)$ and $(V', F', W')$, a morphism of mixed Hodge structures is a linear map $f : V \to V'$ preserving both filtrations.

Given a mixed Hodge structure $V$, we define the $(p, q)$-pieces as the complex vector spaces

$$V^{p,q} = \text{Gr}^F_p \left(\text{Gr}^{p+q}_W V\right)_{\mathbb{C}}.$$ 

They can be understood as a kind of generalization of the classical decomposition of pure Hodge structures. A mixed Hodge structure $V$ is said to be balanced or of balanced type if $V^{p,q} = 0$ for $p \neq q$. 

Definition 2.5. Given a weight $k$ pure Hodge structure $V$, a polarization of $V$ is a bilinear form $Q: V \otimes V \to \mathbb{Q}(-k)$, preserving the Hodge structures, such that

- $Q(u, v) = (-1)^k Q(v, u)$ for all $u, v \in V$.
- The bilinear form $(2\pi i)^k Q(Cu, v)$ is an hermitian product on $V_{\mathbb{C}}$, where $C: V_{\mathbb{C}} \to V_{\mathbb{C}}$ is the Weil operator given by $Cv = i^{p-q}v$, for $v \in V_{p,q}$.

A mixed Hodge structure is said to be graded polarizable if each of the pure Hodge structures on $Gr^W_k V$ are polarizable.

Deligne proved in [13] and [14] (for a concise exposition, see also [18]) that, if $X$ is a complex algebraic variety, then $H^k(X; \mathbb{Q})$ carries a natural mixed Hodge structure in a functorial way. More precisely, let $\text{Var}_\mathbb{C}$ be the category of complex algebraic varieties with morphisms given by the regular maps, $\mathbb{Q}$-$\text{Vect}$ the category of $\mathbb{Q}$-vector spaces and $\text{MHS}$ be the category of mixed Hodge structures and filtered linear maps. We have that $\text{MHS}$ is an abelian category (see Théorème 2.3.5 of [13]). Moreover, the cohomology functor $H^k(-; \mathbb{Q}) : \text{Var}_\mathbb{C} \to \mathbb{Q}$-$\text{Vect}$ factorizes through $\text{MHS}$, that is, there exists a commutative diagram

\[
\begin{array}{ccc}
\text{Var}_\mathbb{C} & \xrightarrow{H^k(-; \mathbb{Q})} & \mathbb{Q}$-$\text{Vect} \\
\downarrow & & \downarrow \\
\text{MHS} & \nearrow & \\
\end{array}
\]

where the functor $\text{MHS} \to \mathbb{Q}$-$\text{Vect}$ is the natural forgetful functor. An analogous statement holds for compactly supported cohomology, that is $H^k_c(X; \mathbb{Q})$ has a mixed Hodge structure, for $X \in \text{Var}_\mathbb{C}$, in a functorial way (see section 5.5 of [54] for a complete construction).

Remark 2.6. A pure Hodge structure of weight $k$ is, in particular, a mixed Hodge structure by taking the weight filtration with a single step. When $X$ is a smooth complex projective variety, the induced pure Hodge structure given by Remark 2.2 (as a compact Kähler manifold) corresponds to the mixed Hodge structure given above.

2.2. Mixed Hodge modules. In [58] (see also [56] and [57]), Saito proved that we can assign, to every complex algebraic variety $X$, an abelian category $\mathcal{M}_X$ called the category of mixed Hodge modules on $X$. As described in [62], if $X$ is smooth, the basic elements of $\mathcal{M}_X$ are tuples $M = (\mathcal{M}, F^\bullet, W^\bullet, K, \alpha)$, where $(\mathcal{M}, F^\bullet)$ is a well-filtered regular holonomic $D_X$-module, $K$ is a rational perverse sheaf and $W^\bullet$ is a pair of increasing filtrations of $\mathcal{M}$ and $K$. These filtration have to correspond under the isomorphism

\[
\alpha : \text{DR}(\mathcal{M}) \xrightarrow{\cong} K \otimes_{\mathcal{Q}_X} \mathcal{C}_X,
\]

where $\mathcal{C}_X, \mathcal{Q}_X$ denote the respective constant sheaves on $X$. Here, DR is the Riemann-Hilbert correspondence functor between the category of regular holonomic $D_X$-modules, $D_X$-$\text{Mod}_{rh}$, and the complexification of the category of rational perverse sheaves on $X$, $\text{Perv}(X)$ (for all these concepts, see [54]).

Starting with these basic elements, the category $\mathcal{M}_X$ of mixed Hodge modules on $X$ is a suitable subcategory that can be defined by means of some functors, called the nearby and vanishing cycles functors, that control the behaviour of the filtration. However, the definition is extremely involved by its own. In particular, it is really hard to decide whether an object is a mixed Hodge module or not. For this reason, we will not give here the full definition of this category. For its complete definition, see [58] or [61].
The threatening aspect of this category should not scare us. The accurate point is that it can (and should) be used as a black box with some operations defined. Actually, in Section 3 we will show that some very tangible objects can be automatically interpreted as mixed Hodge modules and this will give us a large amount of elements of this category. In this way, we will never need to check the very definition of mixed Hodge modules.

The category $\mathcal{M}_X$ of mixed Hodge modules on $X$ is an abelian category, as proved in [56] and [58]. By construction, we have functors

$$\text{rat}_X : \mathcal{M}_X \to \text{Perv}(X), \quad \text{Dmod}_X : \mathcal{M}_X \to \mathcal{D}_X\text{-Mod}_{\text{rh}},$$

that just project to the underlying perverse sheaf and $\mathcal{D}_X$-module, respectively. They can be extended to the (bounded) derived categories as functors

$$\text{rat}_X : \mathcal{D}^b\mathcal{M}_X \to \mathcal{D}^b_{\text{cs}}(X; \mathbb{Q}), \quad \text{Dmod}_X : \mathcal{D}^b\mathcal{M}_X \to \mathcal{D}^b_{\text{rh}}(\mathcal{D}_X).$$

Here $\mathcal{D}^b_{\text{cs}}(X; \mathbb{Q})$ is the category of cohomological constructible complexes of sheaves that contains $\text{Perv}(X)$ as a full abelian subcategory, and $\mathcal{D}^b_{\text{rh}}(\mathcal{D}_X)$ is the bounded derived category of coherent $\mathcal{D}_X$-modules whose cohomology sheaves are regular holonomic that contains $\mathcal{D}_X\text{-Mod}_{\text{rh}}$ as a full abelian subcategory.

Moreover, given a regular morphism $f : X \to Y$, we can define the functors

$$f_* : \mathcal{D}^b\mathcal{M}_X \to \mathcal{D}^b\mathcal{M}_Y, \quad f^* : \mathcal{D}^b\mathcal{M}_Y \to \mathcal{D}^b\mathcal{M}_X,$$

by lifting the corresponding functors at the level of constructible sheaves (and their counterparts on $\mathcal{D}$-modules via Riemann-Hilbert correspondence). The fact that they send mixed Hodge modules into mixed Hodge modules is not trivial at all and can be checked in [58].

Finally, the tensor and external product of constructibles complexes (and complexes of $\mathcal{D}$-modules) also lift to give bifunctors

$$\otimes : \mathcal{M}_X \times \mathcal{M}_X \to \mathcal{M}_X, \quad \boxtimes : \mathcal{D}^b\mathcal{M}_X \times \mathcal{D}^b\mathcal{M}_Y \to \mathcal{D}^b\mathcal{M}_{X \times Y}.$$

**Remark 2.7.**

- Recall that $f_*$ and $f_1$ on $\mathcal{D}^b_{\text{cs}}(X; \mathbb{Q})$ are just the usual direct image and proper direct image on sheaves, $f^*$ is the inverse image sheaf and $f^!$ is the adjoint functor of $f_1$ (the so-called extraordinary pullback). See [54], Chapter 13, for a complete definition of these functors.

- The external product can be defined in terms of the usual tensor product by

$$M^* \boxtimes N^* = \pi_1^* M^* \otimes \pi_2^* N^*,$$

for $M^* \in \mathcal{D}^b\mathcal{M}_X$, $N^* \in \mathcal{D}^b\mathcal{M}_Y$ and $\pi_1 : X \times Y \to X$, $\pi_2 : X \times Y \to Y$ the corresponding projections. See also Section 1.1.

- The notation $\mathcal{M}_X$ is not standard. In Saito’s papers, such a category is usually denoted by $\text{MHM}(X)$. However, due to its omnipresence along this paper, we will shorten it.

A very important feature of these induced functors is that they behave in a functorial way, as the following result shows. The proof of this claim is a compendium of Section 4.2 of [62] and Proposition 4.3.2 and Section 4.4 (in particular 4.4.3) of [58].

**Theorem 2.8 (Saito).**

i) Let $f : X \to Y$ be a regular morphism. The morphisms $f_1, f_* : \mathcal{D}^b\mathcal{M}_X \to \mathcal{D}^b\mathcal{M}_Y$ and $f^1, f^! : \mathcal{D}^b\mathcal{M}_Y \to \mathcal{D}^b\mathcal{M}_X$ commute with the external product. Moreover, the morphism $f^* : \mathcal{D}^b\mathcal{M}_Y \to \mathcal{D}^b\mathcal{M}_X$ commutes with the tensor product.

ii) The induced functors commute with composition. More explicitly, let $f : X \to Y$ and $g : Y \to Z$ be regular morphisms of complex algebraic varieties, then

$$(g \circ f)_* = g_* \circ f_* \quad (g \circ f)_1 = g_1 \circ f_1 \quad (g \circ f)^* = f^* \circ g^* \quad (g \circ f)^! = f^! \circ g^!$$
iii) Suppose that we have a cartesian square of complex algebraic varieties (i.e. a pullback diagram in \( \text{Var}_C \))

\[
\begin{array}{ccc}
W & \xrightarrow{g'} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y & \xrightarrow{g} & Z
\end{array}
\]

Then we have a natural isomorphism of functors \( g^* \circ f_! \cong f'_! \circ (g')^* \).

iv) For the singleton variety we have a natural isomorphism of categories \( \mathcal{M}_* = \text{MHS} \).

v) Given a complex algebraic variety we have a natural isomorphism of categories \( \mathcal{M}_* = \text{MHS} \). From the categories explained in the previous section, we can build an extra layer of rings that will be very useful for the constructions of Section 5.1. Consider the Grothendieck group of the category \( \mathcal{M}_X, \mathcal{K}M_X \). It is the abelian group generated by isomorphism classes \([M]\) of objects \( M \in \mathcal{M}_X \) subject to the restriction that \([M] = [M'] + [M'']\) whenever there exists a short exact sequence \( 0 \to M' \to M \to M'' \to 0 \) in \( \mathcal{M}_X \). This abelian group is also endowed with a ring structure by taking \([M] \cdot [N] = [M \otimes N]\) for any \( M, N \in \mathcal{M}_X \). In particular, \( \mathcal{K}M_* = \mathcal{KMHS} = \mathcal{KHS} \), where \( \mathcal{HS} \) is the category of (direct sums of) polarizable pure Hodge structures.

In the same way, as a triangulated category, we can also define the Grothendieck ring of the bounded derived category \( D^b\mathcal{M}_X, \mathcal{KD}^b\mathcal{M}_X \). However, it is not a true gain, since there is a natural identification \( \mathcal{KD}^b\mathcal{M}_X \cong \mathcal{K}M_X \) that assigns, to \([M^*] \in \mathcal{KD}^b\mathcal{M}_X \), the alternating sum \( \chi(M^*) = \sum_k (-1)^k[M^k] \in \mathcal{K}M_X \). Under this identification, the functors of Section 2.2 can be used to define morphisms

\[
f_*, f_! : \mathcal{K}M_X \to \mathcal{K}M_Y, \quad f^*, f^! : \mathcal{K}M_Y \to \mathcal{K}M_X.
\]

All of them are group homomorphisms but, as \( f^* \) commutes with tensor product, it is also a ring homomorphism. Moreover, the external product \( \boxtimes : \mathcal{KD}^b\mathcal{M}_* \times \mathcal{KD}^b\mathcal{M}_* \to \mathcal{KD}^b\mathcal{M}_{* \times \times} = \mathcal{M}_X \) can be used to obtain a map \( \boxtimes : \mathcal{K}M_* \times \mathcal{K}M_X \to \mathcal{K}M_X \) that endows \( \mathcal{K}M_X \) with a \( \mathcal{KHS} \)-module structure. In this context, \( f_*, f_!, f^* \) and \( f^! \) also are \( \mathcal{KHS} \)-module homomorphisms, as they commute with external product.

**Remark 2.9.** Let \( \mathcal{Q}_X \in \mathcal{K}M_X \) be the unit of the ring structure on \( \mathcal{K}M_X \). By Theorem 2.8, if \( c_X : X \to * \) is the projection map with induced map \((c_X)_! : \mathcal{K}M_X \to \mathcal{K}M_* = \mathcal{KHS}\), we have \((c_X)_! \mathcal{Q}_X = [H^*(X; Q)]\), where \([H^*(X; Q)] \in \mathcal{KHS}\) denotes the \( K \)-theory image of the (mixed) Hodge structure on \( H^*(X; Q) \).

Another very useful property of mixed Hodge modules is that they behave well with respect to stratification of the underlying space.

**Proposition 2.10.** Let \( X \) be a complex algebraic variety and suppose that we have a decomposition \( X = Y \sqcup U \) where \( i : Y \leftarrow X \) is a closed subvariety and \( j : U \hookrightarrow X \) is a Zariski open subset. Then, the ring homomorphism

\[
i_i + j_! : \mathcal{K}M_Y \oplus \mathcal{K}M_U \to \mathcal{K}M_X
\]

is an isomorphism. Moreover, \( i_i^* + j_!^* = 1_{\mathcal{K}M_X} \).
2.11 Remark

\[ M = \cdots \] such that:

(\[ \text{Definition 3.1.} \])

Let \( M \) be the monodromy representation introduced in \([43]\). The monodromy actions as mixed Hodge modules. This is the key for putting in context the Hodge special subcategories. That will be very useful since, as we will show, it allows us to formulate the geometric variation of mixed Hodge structures.

\[ i_{!*} (i^* M) + j_!(j^* M) \] with \( i^* M \in \mathcal{K}M_X \) and \( j^* M \in \mathcal{K}M_U \). This finish the proof.

\[ \square \]

Remark 2.11. Given a closed subvariety \( Y \subseteq X \) and \( M \in \mathcal{K}M_X \), for short we will denote \( M|_Y = i^* M \). Observe that, by the previous proposition, if \( X = Y \sqcup U \) we have that \( M = i_!(M|_Y) + j_!(M|_U) \) for all \( M \in \mathcal{K}M_X \).

### 3. Monodromy as mixed Hodge module

Despite the abstract definition of mixed Hodge modules, it is still possible to identify some special subcategories. That will be very useful since, as we will show, it allows us to formulate monodromy actions as mixed Hodge modules. This is the key for putting in context the Hodge monodromy representation introduced in \([43]\).

**Definition 3.1.** Let \( B \) a complex manifold. A *variation of mixed Hodge structures* is a triple \( (\mathcal{V}, W^\bullet, \mathcal{F}^\bullet) \) composed by:

- A local system \( \mathcal{V} \) of \( \mathbb{Q} \)-vector spaces on \( B \) i.e. a locally constant sheaf.
- A finite increasing filtration \( W^\bullet \) of \( \mathcal{V} \) by local subsystems, called the weight filtration.
- A finite decreasing filtration \( \mathcal{F}^\bullet \) of the holomorphic vector bundle \( \mathcal{V} = \mathcal{V} \otimes_{\mathcal{O}^\text{an}_B} \Omega^1_B \) by holomorphic subbundles, called the Hodge filtration.

such that:

1. For any \( b \in B \), the stalks \( W^\bullet_b, \mathcal{F}^\bullet_b \) induce a mixed Hodge structure on \( \mathcal{V}_b \).
2. (Griffiths’ transversality condition) Let \( \nabla : \mathcal{V} \to \mathcal{V} \otimes_{\mathcal{O}^\text{an}_B} \Omega^1_B \) be the induced connection by \( \mathcal{V} \), being \( \Omega^1_B \) the sheaf of holomorphic 1-forms on \( B \). Then, for any \( p \)

\[ \nabla (\mathcal{F}^p) \subseteq \mathcal{F}^{p-1} \otimes \mathcal{O}^\text{an}_B \Omega^1_B. \]

**Remark 3.2.**

- If the weight filtration \( W^\bullet \) is constant and with a single step, the variation is called a variation of (pure) Hodge structures. In that case, the jumping degree is called the weight of the variation.
- The induced connection \( \nabla \) on \( \mathcal{V} \) is the so-called *Gauss-Manin connection* which is an integrable holomorphic connection such that \( \mathcal{V} = \ker \nabla \). For a detailed description see \([54]\), 10.24.
- The Griffiths’ transversality condition appears naturally in the following context. Let \( f : X \to B \) be a regular morphism between complex algebraic varieties with \( B \) smooth which is locally trivial in the analytic topology. We will say that \( f \) is a *nice fibration*. Let \( \mathcal{Q}_X \) denote the constant sheaf on \( X \). In this case, the direct images sheaves \( R^k f_* \mathcal{Q}_X \) have a natural structure of variations of mixed Hodge structures such that, for any \( b \in B \), we have an isomorphism \( (R^k f_* \mathcal{Q}_X)_b \cong H^k(f^{-1}(b); \mathbb{Q}) \) as mixed Hodge structures (see \([62]\), Example 3.11). This construction is the so-called *geometric variation of mixed Hodge structures*. Analogous considerations holds for \( R^k f_* \mathcal{Q}_X \), whose stalks are \( (R^k f_* \mathcal{Q}_X)_b \cong \ldots \)
$H^k(f^{-1}(b); \mathbb{Q})$ as mixed Hodge structures. In this case, the Gauss-Manin connection automatically satisfies Griffiths’ transversality condition (see [54], Corollary 10.31).

**Definition 3.3.** A polarization of a variation $V$ of pure Hodge structures of weight $k$ over $B$ is a morphism $Q: V \otimes V \to \mathbb{Q}(-k)_B$ that induces a polarization of pure Hodge structures on each stalk. A variation of mixed Hodge structures is said to be graded-polarizable if the induced variations of pure Hodge structures $Gr^W V$ are polarizable.

Now, let $B$ a smooth irreducible algebraic variety of dimension $d$ and suppose that, seen as a complex manifold, we have a variation of mixed Hodge structures $(V, W^\bullet, F^\bullet)$ on $B$. Using the Gauss-Manin connection induced by the local system $V$, $V$ becomes a regular holonomic $\mathcal{D}_B$-module and Griffiths’s transversality condition says that $F^\bullet$ is a good filtration. Moreover, taking the shifted de Rham complex then, by the Riemann-Hilbert correspondence (see [9]), $\text{DR}(V)$ is a perverse sheaf on $B$ with an isomorphism $\alpha: \text{DR}(V) \to V[\dim_{\mathbb{C}} B]$, where $\dim_{\mathbb{C}} B$. Shifting $W^\bullet$ to $W^k V[\dim_{\mathbb{C}} B]$ we build a tuple $M_V = (V, F^\bullet, W^\bullet, V[\dim_{\mathbb{C}} B], \alpha)$ as in the definition of a mixed Hodge module.

In general, this tuple $M_V$ does not define a mixed Hodge module if we do not require more conditions to $V$. First, suppose that there exists a compact algebraic variety $\overline{B} \supseteq B$ such that $D = \overline{B} - B$ has smooth irreducible components and looks, locally, like the crossing of coordinate hyperplanes. This is sometimes called a good compactification of $B$ and $D$ is said to be a simple normal crossing divisor.

In that situation, a local system $V$ is said to have quasi-unipotent monodromy at infinity if the monodromy of the loops around $D$ is quasi-unipotent (i.e. some power is unipotent). Moreover, there is a set of technical conditions on a variation of mixed Hodge structures, called admissibility, as described in [68] and [40]. We will not need an explicit formulation of these conditions but they should be thought as the incarnation of the corresponding conditions for mixed Hodge modules in $M_V$.

**Remark 3.4.** To be honest, historically the process went in the other way around. Actually, as Saito explained, he came up the the right definition of admissibility by generalizing the concept for variations of Hodge structures.

**Definition 3.5.** A variation of mixed Hodge structures is said to be good if it is admissible and has quasi-unipotent monodromy at infinity. The full subcategory of graded polarizable good variations of mixed Hodge structures on $B$ is denoted by $\text{VMHS}_0(B)$, which is an abelian category stable under pullbacks and tensor products.

**Example 3.6.** In general, to check whether a variation of mixed Hodge structures is good may be quite hard. However, a large class of such variations are automatically good, as the following results show:

- (Schmid, [60]). A polarizable variation of pure Hodge structures is admissible. Hence, it is good if and only if it has quasi-unipotent monodromy at infinity.
- (Steenbrink-Zucker, [68] and [40]). Geometric variations of mixed Hodge structures (i.e. those coming from nice morphisms, see Remark 3.2, third item) are good.

**Theorem 3.7** (Saito, [58]). Let $B$ a complex algebraic variety. If $V$ is a good variation of mixed Hodge structures on $B$, then $M_V$ is a mixed Hodge module. Moreover, this correspondence is an isomorphism of categories

$$\text{VMHS}_0(B) \xrightarrow{\cong} \mathcal{M}_B^{\text{sm}},$$
where $\mathcal{M}_B^{sm} \subseteq \mathcal{M}_B$ is the full subcategory of smooth mixed Hodge modules, that is, mixed Hodge modules $M \in \mathcal{M}_B$ such that $\text{rat}_B M[-\dim B]$ is a local system.

**Remark 3.8.** Actually, this theorem also identifies the unit of the monoidal structure on $\mathcal{M}_B$: it is precisely the mixed Hodge module associated to the trivial variation of pure Hodge modules of weight zero, $Q_B$, on $B$ that has the constant $\mathbb{Q}$ sheaf as local system and a single step Hodge filtration jumping at $p = 0$. In order to get in touch with the notation of [13], we will also denote this unit as $T_B$ when we want to emphasize its monodromic nature. Finally, if we want to focus on the monoidal structure of $\mathcal{M}_B$, such a unit will be simply denoted by $1 \in \mathcal{M}_B$.

**Corollary 3.9.** Let $f : X \to B$ be a nice morphism and denote the associated variations of mixed Hodge structures as $V^k_f = R^k f_! Q_X$. Then, in $K\mathcal{M}_B$, we have the equality

$$f_! Q_X = \sum_k (-1)^k \mathcal{M}^{V^k_f}.$$

In particular, if $f$ has trivial monodromy with fiber $F$, then $f_! Q_X = [H^\bullet_c(F; \mathbb{Q})] Q_B$.

**Proof.** Let $Q_X$ be the trivial variation of mixed Hodge modules on $X$. By definition, in the Grothendieck ring of variations of mixed Hodge structures on $B$ we have that

$$f_! Q_X = \chi(Rf_! Q_X) = \sum_k (-1)^k V^k_f.$$

In particular, since all the $V^k_f$ are good, $f_! Q_X \in K(\text{VMHS}_0(B))$ and the first part follows by passing to $K\mathcal{M}_B$ via Theorem 3.7. For the second part, just observe that, if $f$ has trivial monodromy, then $V^k_f = H^\bullet_c(F; \mathbb{Q}) Q_B$ as variations of mixed Hodge structures and, thus, as mixed Hodge modules when passing to $\mathcal{M}_B$ via Theorem 3.7. $\square$

**Corollary 3.10** (Logares-Muñoz-Newstead). If $f : X \to B$ is a nice morphism with trivial monodromy and fiber $F$, then

$$[H^\bullet_c(X; \mathbb{Q})] = [H^\bullet_c(F; \mathbb{Q})][H^\bullet_c(B; \mathbb{Q})].$$

In particular, $e(X) = e(F)v(B)$.

**Proof.** Just apply $(c_B)_!$ to both sides of the second part of Corollary 3.9 where $c_B : B \to \ast$ in the projection onto a point, and use Theorem 2.8 (v). $\square$

**Remark 3.11.** As explained in [39], Remark 2.5, there are some cases in which we can automatically know that a nice fibration $f$ has trivial monodromy, as the following:

- If $f$ is locally trivial in the Zariski topology.
- If $f$ is a principal $G$-bundle with $G$ a connected algebraic group.
- If the fiber is $F = \mathbb{P}^N$ for some $N$.

Let $B$ be a smooth algebraic variety and let $\rho : \pi_1(B) \to \text{Aut}(V)$ be a representation where $V$ is a polarized pure Hodge structure preserved by $\rho$. Then, the local system associated to $\rho$ (see [34], Appendix B.3.1), call it $V_\rho$, has a natural structure of polarizable variation of pure Hodge structures just by considering the corresponding subbundles for the Hodge filtration. If $\rho$ is also quasi-unipotent at infinity, then $V_\rho$ is a good variation and, thus, it defines a mixed Hodge module that, for simplicity, we will denote also by $\rho \in \mathcal{M}_B$.

**Example 3.12.**

- If $V$ is a graded-polarizable mixed Hodge structure and $\rho : \pi_1(B) \to \text{Aut}(V)$ is a representation preserving it with quasi-unipotent monodromy at infinity, then, the associated variation of mixed Hodge structures $V_\rho$ is graded-polarizable.
might be not admissible by its own but, in $K$-theory, it decomposes as a sum of polarizable variations of pure Hodge structures. Hence, any case, we can associate to $\rho$ the sum of the associated mixed Hodge modules of each of these summands. The resulting mixed Hodge module is also denoted by $\rho \in KHM_B$.

- If $f : B' \to B$ is a regular morphism and $\rho : \pi_1(B) \to \text{Aut}(V)$ is a representation preserving the graded polarizable mixed Hodge structure $V$, then $f^*\mathcal{V}_\rho = \mathcal{V}_{\rho \circ f}$ as variations of mixed Hodge structures. Hence, as elements of $KHM_B'$, we have $f^*\rho = (\rho \circ f)$ which justifies the notational ambiguity with pullbacks of representations.

- Let $\rho : \pi_1(B) \to \text{Aut}(V)$ be a representation of graded-polarized mixed Hodge structures with finite image. Then, automatically, $\rho$ is quasi-unipotent so, in particular, it is quasi-unipotent at infinity. Hence, it defines a mixed Hodge module.

- Let $f : X \to B$ be a nice fibration with fiber $F$ and associated monodromy representations $h_k : \pi_1(B) \to \text{Aut}(H^k_c(F; \mathbb{Q}))$. If the representations $h_k$ preserve the mixed Hodge structure and have quasi-unipotent monodromy at infinity then, as variations of mixed Hodge modules, we have $R^k f_! Q_X = \mathcal{V}_{h_k}$. So, by Corollary 3.9, we have that, in $KHM_B$

$$R^k f_! Q_X = \sum_k (-1)^k h_k$$

- In particular, if $f : X \to B$ is a finite covering of degree $d$, then $H^0_c(F; \mathbb{Q}) = \mathbb{Q}_0$ and $H^k_c(F; \mathbb{Q}) = 0$ for $k > 0$ as Hodge structures. In that case, the only non-trivial monodromy is $h = h_0 : \pi_1(B) \to \text{Aut}(\mathbb{Q}_0)$ which preserves the Hodge structure and has finite image. Hence, we have that $f_! \mathbb{Q}_X = h$.

**Definition 3.13.** Given a nice morphism $f : X \to B$, we define the Hodge monodromy representation of $X$ on $B$ by

$$R_f(X|B) = f_! \mathbb{Q}_X \in KHM_B.$$  

When the morphism $f$ it clear from the context, we will just write $R(X|B)$.

In order to get in touch with the definition of Hodge monodromy representation of [43], let us denote by $q = \mathbb{Q}(-1)$ the $(-1)$-Tate structure of weight 2. In that case, $KHM_B$ has a natural $\mathbb{Z}[q^\pm]$-module structure inherit from the one as $\text{KHS}$-module. Now, suppose that we have a representation of mixed Hodge modules $\rho : \pi_1(B) \to \text{Aut}(V)$, where $V$ is a balanced mixed Hodge structure. In that case, in $KHM_B$, we have that $\rho = \sum_p \rho^{p,p} q^p$ where $\rho^{p,p} : \pi_1(B) \to \text{Aut}(V^{p,p})$ is the restriction of $\rho$ to the Hodge pieces of $V$.

In particular, for the monodromy representations, $h_k : \pi_1(B) \to \text{Aut}(H^k_c(F; \mathbb{Q}))$, of a nice fibration $X \to B$ with fiber $F$ of balanced type, we have $h_k = \sum_p h^{k,p}_k q^p$ where $h^{k,p}_k : \pi_1(B) \to \text{Aut}(H^k_c(F; \mathbb{Q}))$ are the restrictions. Therefore, adding up we have that

$$R(X|B) = \sum_k (-1)^k h_k = \sum_{k,p} (-1)^k h^{k,p}_k(X) q^p,$$

which can be seen as an element of the representation ring of $\pi_1(B)$ tensorized with $\mathbb{Z}[q^\pm]$. This is precisely the definition of [43] of Hodge monodromy representations. In particular, reinterpreting Hodge monodromy representations as mixed Hodge modules, all the computations of [43], [48] and [46] can be seen as calculations of mixed Hodge modules.

### 3.1. Hodge monodromy representations of covering spaces

Suppose that $X$ and $B$ are smooth complex varieties and that $\pi : X \to B$ is a regular morphism which is a covering space with finite fiber $F$ and degree $d$. In that case, the monodromy of $\pi$ is given by path lifting. More precisely, there is an action of $\pi_1(B)$ on $F$ given as follows. Let $\gamma \in \pi_1(B)$ and $x \in F$ so there
exists an unique path lift $\tilde{\gamma}_x : [0, 1] \to X$ of $\gamma$ such that $\tilde{\gamma}_x(0) = x$. Then, we set $\gamma \cdot x = \tilde{\gamma}_x(1) \in F$. Since $F$ is a finite set of $d$ points, $H^k_c(F; \mathbb{Q}) = 0$ for $k > 0$ and $H^0_c(F; \mathbb{Q}) = \mathbb{Q}^d$. Hence, the only non-trivial action induced on cohomology $h = h_0 : \pi_1(X) \to \text{GL}(H^0_c(F; \mathbb{Q})) = \text{GL}_d(\mathbb{Q})$ is the monodromy action so $R(X|B) = h$.

Using this description, we can compute some examples of monodromy that will be useful in this paper.

**Example 3.14.** Let $X = B = \mathbb{C}^*$ and consider the morphism $\pi : X \to B$ given by $\pi(x) = x^2$. It is a double covering with non-trivial monodromy. It is a straightforward computation to check that the monodromy action $h : \pi_1(\mathbb{C}^*) \to \text{GL}(H^0_c(F; \mathbb{Q})) = \text{GL}_2(\mathbb{Q})$ is given by

$$\gamma \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $\gamma$ is a generator of $\pi_1(\mathbb{C}^*)$. Hence, $h = T_B + S$, where $T_B$ is the trivial action (i.e. the unit of $\mathcal{K}_M(B)$) and $S : \pi_1(\mathbb{C}^*) \to \mathbb{Q}^*$ is the unique action of order two. Thus, the Hodge monodromy representation of $\pi$ is $R(X|B) = T_B + S$.

Given $M \in \mathcal{K}_M(B)$, we will short $\mu(M) = c_0 M \in \mathbb{K}_M(B)$, where $c : B \to \ast$ is the projection onto a point. Observe that $\mu(S) = 0$ because it holds

$$q - 1 = [H^*_c(X)] = \mu(R(X|B)) = \mu(T_B + S) = q - 1 + \mu(S).$$

In general, let us fix $p_0, \ldots, p_s \in \mathbb{C}$ with $p_i \neq \pm p_j$ for $i \neq j$ and consider $B = \mathbb{C} - \{p_0, \ldots, p_s\}$. We have the collection of representations $S_{p_i} : \pi_1(B) \to \mathbb{Q}^*$, for $0 \leq i \leq s$, that are given by $S_{p_i}(\gamma_{p_i}) = 1$ if $i \neq j$ and $S_{p_i}(\gamma_{p_i}) = -1$, where $\gamma_{p_i}$ is a small loop around $p_j$.

The importance of these representations appear when we consider the auxiliary variety $X = \{(x, y) \in B \times \mathbb{C} | y^2 = x - p_0\}$ with projection $\pi : X \to B$, $\pi(x, y) = x$. An analogous analysis to the previous one shows that the Hodge monodromy representation of $\pi$ is $R(X|B) = T_B + S_{p_0}$. However, in this case $X \cong \mathbb{C}^* - \{\pm \sqrt{p_1}, \ldots, \pm \sqrt{p_s}\}$ so $[H^*_c(X)] = q - 2s - 1$. Thus, it holds

$$q - 2s - 1 = [H^*_c(X)] = \mu(R(X|B)) = \mu(T_B + S_{p_0}) = q - (s + 1) + \mu(S_{p_0}).$$

Hence, $\mu(S_{p_0}) = -s$. This agrees with discussion after Theorem 6 of [33]. Actually, a modification of this argument shows that also that $\mu(S_{p_0} \cdots S_{p_{s-1}}) = -s$ for any $r \geq 0$.

**Example 3.15.** Let $X = \mathbb{C}^* - \{\pm 1\}$ and $B = \mathbb{C} - \{\pm 2\}$ with the double cover $t : X \to B$ given by $t(\lambda) = \lambda + \lambda^{-1}$ for $\lambda \in \mathbb{C}^* - \{\pm 1\}$. For the monodromy action of $t$, let $\pi_1(\mathbb{C} - \{\pm 2\}) = \langle \gamma_2, \gamma_{-2} \rangle$, where $\gamma_{\pm 2}$ are positive small loops around $\pm 2$. It can be checked that the path lifting action of $\gamma_{\pm 2}$ interchanges the elements of the fiber. Thus, on cohomology, the monodromy action $h : \pi_1(\mathbb{C} - \{\pm 2\}) \to \text{GL}_2(\mathbb{Q})$ is given by

$$\gamma_2 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_{-2} \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

Therefore, $R(\mathbb{C}^* - \{\pm 1\} | \mathbb{C} - \{\pm 2\}) = h = T + S_{\pm 2}$, where $S_{\pm 2} : \pi_1(\mathbb{C} - \{\pm 2\}) \to \mathbb{Q}^*$ is the action $\gamma_{\pm 2} \mapsto -1$ and $\gamma_{\mp 2} \mapsto 1$, respectively. Observe that, as explained above, $\mu(S_{\pm 2}) = -1$ so

$$[H^*_c(\mathbb{C}^* - \{\pm 1\})] = \mu(R(\mathbb{C}^* - \{\pm 1\} | \mathbb{C} - \{\pm 2\})) = \mu(T_B) + \mu(S_{\pm 2}) = q - 2 - 1 = q - 3,$$

as we already knew.
3.2. **Equivariant Hodge monodromy representations.** Let us look more closely to a special type of coverings coming from group actions. Let $X$ be an algebraic variety with an action of $\mathbb{Z}_2$ on it. We define its equivariant Hodge structures as the elements of KHS given by

$$[H^*_c(X; \mathbb{Q})]^+ = [H^*_c(X/\mathbb{Z}_2; \mathbb{Q})], \quad [H^*_c(X; \mathbb{Q})]^- = [H^*_c(X; \mathbb{Q})] - [H^*_c(X; \mathbb{Q})]^+.$$  

Moreover, if $X \to Z$ is a nice fibration equivariant for the $\mathbb{Z}_2$-action, we denote $R(X|Z)^+ = R(X/\mathbb{Z}_2|Z)$ and $R(X|Z)^- = R(X|Z) - R(X|Z)^+.$

Observe that the action of $\mathbb{Z}_2$ on $X$ induces an action of $\mathbb{Z}_2 = \pi_1(X/\mathbb{Z}_2)/\pi_1(X)$ on the cohomology of $X$. In that case, $[H^*_c(X; \mathbb{Q})]^+$ can also be seen as the fixed part of the cohomology of $X$ by this action (see Proposition 4.3 of [20]). Analogously, $R(X|Z)^+$ can be seen as the invariant part of the monodromy representation by the action of $\mathbb{Z}_2$.

Now, let us consider a nice fibration $f : X \to B$ with fiber $F$ and trivial monodromy. In addition, suppose that there is an action of $\mathbb{Z}_2$ on $X$ and $B$ such that $f$ is equivariant, so it descends to a regular morphism $\tilde{f} : X/\mathbb{Z}_2 \to B/\mathbb{Z}_2$. This means that they fit in a diagram of fibrations

$$\begin{array}{ccc}
F & \longrightarrow & X \\
\downarrow & & \downarrow f \\
F & \longrightarrow & X/\mathbb{Z}_2 \\
\downarrow & & \downarrow \tilde{f} \\
 & & B/\mathbb{Z}_2
\end{array}$$

where the vertical arrows are the quotient maps. In [13], Proposition 2.6, it is proven that, on KHS, we have an equality

$$[H^*_c(X; \mathbb{Q})]^+ = [H^*_c(B; \mathbb{Q})]^+ [H^*_c(F; \mathbb{Q})]^+ + [H^*_c(B; \mathbb{Q})]^- [H^*_c(F; \mathbb{Q})]^-. $$

Here, $[H^*_c(F; \mathbb{Q})]^+$ denotes the invariant part of the mixed Hodge structure on the cohomology of $F$ under the action of $\mathbb{Z}_2$.

The same argument can be adapted to consider the case of Hodge monodromy representations. Suppose that, in addition to previous diagram, we also have a nice morphism $\pi : B \to Z$ which is invariant for the $\mathbb{Z}_2$ action. From this map, we also obtain natural morphisms $X \to Z$, $X/\mathbb{Z}_2 \to Z$ and $B/\mathbb{Z}_2 \to Z$. In that case, we claim that

$$R(X|Z)^+ = R(B|Z)^+ [H^*_c(F; \mathbb{Q})]^+ + R(B|Z)^- [H^*_c(F; \mathbb{Q})]^-. $$

In order to check it, observe that $R(X|Z)^+ = R(X/\mathbb{Z}_2|Z)$ is precisely the invariant part of the monodromy of $X \to Z$ by the action of $\mathbb{Z}_2$. Hence, as $X \to B$ has trivial monodromy, we have $f_!\mathbb{Q}_X = [H^*_c(F; \mathbb{Q})] \mathbb{Q}_B$ and, thus, we can compute

$$R(X|Z)^+ = [\pi_! f_! \mathbb{Q}_X]^+ = [\pi_! \mathbb{Q}_B]^+ = [[H^*_c(F; \mathbb{Q})] \boxtimes R(B|Z)]^+ = [H^*_c(F; \mathbb{Q})]^+ R(B|Z)^+ + [H^*_c(F; \mathbb{Q})]^- R(B|Z)^-. $$

**Remark 3.16.** This kind of arguments deeply reminds to equivariant cohomology. Actually, we expect that, following the ideas of [20], equivariant Hodge representations might be formulated in great generality for equivariant mixed Hodge modules. It is a future work to explore. However, in this paper, we will not need this general case but only for $\mathbb{Z}_2$ actions.
4. Topological Quantum Field Theories

In this section, we will describe a general recipe for constructing lax monoidal TQFTs from two simpler pieces of data: one of geometric nature (a field theory) and one of algebraic nature (a quantisation). It was first described in [31] in the context of representation varieties but, for the purposes of this paper, we need a more general setting.

The idea is a very natural construction that, in fact, has been widely used in the literature in some related form (see, for example, [21], [22], [23], [33], [6] or [5] among others), sometimes referred to as the ‘push-pull construction’. In this paper, we recast this construction to identify the required input data in a simple way that will be useful for applications. For example, it fits perfectly with Saito’s theory of mixed Hodge modules of Section 2.2.

Let us consider the category \( \text{Emb}_c \) whose objects are compact differentiable manifolds, maybe with boundary. Given compact manifolds \( M_1 \) and \( M_2 \) of the same dimension, a morphism \( M_1 \to M_2 \) in \( \text{Emb}_c \) is a class of smooth tame embeddings \( f : M_1 \to M_2 \). The tameness condition means that, for some union of connected components \( N \subset \partial M_1 \), \( f \) extends to a smooth embedding \( \bar{f} : M_1 \cup N (N \times [0, 1]) \to M_2 \) with \( \bar{f}^{-1}(\partial M_2) = \partial M_1 - N \).

Two such a tame embeddings \( f, f' : M_1 \to M_2 \) are declared equivalent if there exists an ambient diffeotopy \( h \) between them i.e. a smooth map \( h : M_2 \times [0, 1] \to M_2 \) such that \( h_t = h(-, t) \) is a diffeomorphism of \( M_2 \) for all \( 0 \leq t \leq 1 \), \( h_0 = \text{Id}_{M_2} \), \( h_t|_{\partial M_2} = \text{Id}_{\partial M_2} \) for all \( t \), and \( h_1 \circ f = f' \).

There are no morphisms in \( \text{Emb}_c \) between manifolds of different dimensions.

Example 4.1. Let \( M \) be a compact manifold and let \( N \subset \partial M \) be a union of connected components of \( \partial M \). Let \( U \subset M \) be an open collar around \( N \), that is, an open subset of \( M \) such that there exists a diffeomorphism \( \varphi : N \times [0, 1) \to U \). In that case, \( \varphi : N \times [0, 1/2] \to U \subset M \) defines a morphism in \( \text{Emb}_c \). Moreover, such a morphism does not depend on the chosen collar as any two collars are ambient diffeotopic. For this reason, we will denote this map \( N \times [0, 1/2] \to M \) in \( \text{Emb}_c \) just by \( N \to M \).

Consider a sheaf \( S : \text{Emb}_c \to \text{Cat} \) with \( S(\emptyset) = \emptyset \). By a sheaf we mean a contravariant functor such that, given a compact manifold \( M \) and a covering \( \{W_\alpha\}_{\alpha \in \Lambda} \) of \( M \) by compact submanifolds, for any collection \( s_\alpha \in S(W_\alpha) \) with \( S(i_{\alpha, \beta})(s_\alpha) = S(i_{\beta, \alpha})(s_\beta) \) for all \( \alpha, \beta \in \Lambda \) (where, \( i_{\alpha, \beta} : W_\alpha \cap W_\beta \to W_\alpha \) is the inclusion) there exists an unique \( s \in S(M) \) such that \( S(i_\alpha)(s) = s_\alpha \), \( i_\alpha : W_\alpha \to M \) being the inclusion maps.

Remark 4.2.

- In contrast with usual sheaves, the coverings considered here are closed.
- However, they also give an open covering since the embeddings \( i_\alpha \) have to extend to a small open set around \( M_\alpha \) by the tameness condition.
- We should think about these sheaves as a way of endowing the bordisms with and geometric extra structure. The formulation of these extra structures as sheaves goes back to [45] (see also [25] and [1]).

Given such a sheaf \( S \), we define the category of embeddings over \( S \), or just \( S \)-embeddings, \( \text{Emb}_c^S \). The objects of this category are pairs \((M, s)\) with \( M \in \text{Emb}_c \) and \( s \in S(M) \) and the initial object \( \emptyset \). Observe that, if \( M \neq \emptyset \) has \( S(M) = \emptyset \), then \( M \) does not appear as object of \( \text{Emb}_c^S \). Given objects \((M_1, s_1), (M_2, s_2) \in \text{Emb}_c^S \), a morphism between them is a pair \((f, \alpha)\) where \( f : M_1 \to M_2 \) is a morphism in \( \text{Emb}_c \) and \( \alpha : s_1 \to S(f)(s_2) \) is a morphism of \( S(M_1) \). If \( \alpha = 1_{s_1} \) (so \( S(f)(s_2) = s_1 \)) we will just denote the morphism by \( f : (M_1, s_1) \to (M_2, s_2) \). The composition of two morphisms \((f, \alpha)\) and \((f', \alpha')\) is \((f' \circ f, S(f)(\alpha') \circ \alpha)\).
Moreover, given \( n \geq 1 \), we define the category of \( n \)-bordisms over \( S \), \( \text{Bd}_n^S \). It is a 2-category (see [7] for a review of 2-categories) given by the following data:

- **Objects**: The objects of \( \text{Bd}_n^S \) is the subclass of objects \((M,s)\in\text{Emb}_c^S\) with \( M \) a \((n-1)\)-dimensional closed manifold.
- **1-morphisms**: Given objects \((M_1,s_1),(M_2,s_2)\) of \( \text{Bd}_n^S \), a morphism \((M_1,s_1)\to(M_2,s_2)\) is a class of pairs \((W,s)\), where \( W \) is a compact \( n \)-dimensional manifold such that \( \partial W = M_1 \sqcup M_2 \) (i.e. a bordism between \( M_1 \) and \( M_2 \)) with \( S(i_1)(s) = s_1 \) and \( S(i_2)(s) = s_2 \), being \( i_k : M_k \to W \) the inclusions for \( k = 1,2 \). Two bordisms \((W,s)\) and \((W',s')\) are in the same class if there exists a boundary preserving diffeomorphism \( f : W \to W' \) such that \( S(f)(s') = s \).

With respect to the composition, given \((W,s) : (M_1,s_1) \to (M_2,s_2)\) and \((W',s') : (M_2,s_2) \to (M_3,s_3)\), we define \((W',s') \circ (W,s)\) as the morphism \((W \cup_{M_2} W',s \cup s') : (M_1,s_1) \to (M_3,s_3)\) where \( W \cup_{M_2} W' \) is the usual gluing of bordisms along \( M_2 \) and \( s \cup s' \in S(W \cup_{M_2} W') \) is the object given by gluing \( s \) and \( s' \) with the sheaf property.

- **2-morphisms**: Given two 1-morphisms \((W,s),(W',s') : (M_1,s_1) \to (M_2,s_2)\), a 2-cell \((W,s) \Rightarrow (W',s')\) is a morphism \((f,\alpha) : (W,s) \to (W',s')\) of \( \text{Emb}_c^S \) with \( f \) a boundary preserving diffeomorphism. Composition of 2-cells is just composition in \( \text{Emb}_c^S \).

In this form, \( \text{Bd}_n^S \) is not exactly a category since, for \((M,s) \in \text{Bd}_n^S\), it may be no unit morphism in the category \( \text{Hom}_{\text{Bd}_n^S}((M,s),(M,s)) \). In that case, it can be solved by weakening slightly the notion of bordism, allowing that \( M \) itself could be seen as a bordism \( M : M \to M \).

With this modification, \((M,s) : (M,s) \to (M,s)\) is the desired unit and it is a straightforward check to see that \( \text{Bd}_n^S \) is a (strict) 2-category. Furthermore, it has a natural monoidal structure by means of disjoint union of objects and bordisms.

Another important category is the category of modules with twists. Let \( R \) be a fixed ring (commutative and with unit). Given two homomorphisms of \( R \)-modules \( f,g : M \to N \), we say that \( g \) is an immediate twist of \( f \) if there exists an \( R \)-module \( L \), homomorphisms \( f_1 : M \\to L, f_2 : L \to N \) and \( \psi : L \to L \) such that \( f = f_2 \circ f_1 \) and \( g = f_2 \circ \psi \circ f_1 \).

In general, given \( f,g : M \to N \) two \( R \)-module homomorphisms, we say that \( g \) is a twist of \( f \) if there exists a finite sequence \( f = f_0,f_1,\ldots,f_r = g : M \to M \) of homomorphisms such that \( f_{i+1} \) is an immediate twist of \( f_i \).

In that case, we define the category of \( R \)-modules with twists, \( R\text{-Mod}_t \), as the category whose objects are \( R \)-modules, its 1-morphisms are \( R \)-modules homomorphisms and, given homomorphisms \( f \) and \( g \), a 2-morphism \( f \Rightarrow g \) is a twist from \( f \) to \( g \) (i.e. a sequence of immediate twists). Composition of 2-cells is juxtaposition of twists. With this definition, \( R\text{-Mod}_t \) has a 2-category structure. Moreover it is a monoidal category with the usual tensor product.

**Definition 4.3.** Let \( S : \text{Emb}_c \to \text{Cat} \) be a sheaf and let \( R \) be a ring. A (lax monoidal) Topological Quantum Field Theory over \( S \), shortened (lax monoidal) \( S\text{-TQFT} \), is a (lax) monoidal symmetric 2-functor

\[
Z : \text{Bd}_n^S \to R\text{-Mod}_t.
\]

**Remark 4.4.** Recall that a functor between monoidal categories \( F : (\mathcal{C}, \otimes, I_\mathcal{C}) \to (\mathcal{D}, \otimes, I_\mathcal{D}) \) is said to be lax monoidal if there exists:
• A morphism $\alpha : I_D \to F(I_C)$.
• A natural transformation $\Delta : F(-) \otimes_D F(-) \Rightarrow F(- \otimes_C -)$.

If $\alpha$ and $\Delta$ are isomorphisms, $F$ is said to be pseudo-monoidal (or just monoidal) and, if they are identity morphisms, $F$ is called strict monoidal.

The important point is that there exists a physical-inspired construction of lax monoidal TQFTs that allows us to construct them from simpler data. The idea of the construction is to consider an auxiliary category $\mathcal{C}$ of pullbacks and final object, that is going to play the role of a category of fields (in the physical sense). Then, we are going to split our functor $Z$ as a composition

$$\text{Bd}_n \xrightarrow{\mathcal{F}} \text{Span}(\mathcal{C}) \xrightarrow{\mathcal{Q}} R\text{-Mod},$$

where $\text{Span}(\mathcal{C})$ is the category of spans of $\mathcal{C}$ (see Section 4.2). The first arrow, $\mathcal{F}$, is the field theory and we will describe how to build it in Section 4.3. The second arrow, $\mathcal{Q}$, is the quantisation part. It is, in some sense, the most subtle piece of data. It will be constructed by means of something called a $\mathcal{C}$-algebra (see Section 4.2). A $\mathcal{C}$-algebra can be thought as collection of rings parametrized by $\mathcal{C}$ with a pair of induced homomorphisms from every morphism of $\mathcal{C}$.

### 4.1. $\mathcal{C}$-algebras

Let $\mathcal{C}$ be a cartesian monoidal category (i.e. a category with finite products where the monoidal structure is precisely to take products) and let $A : \mathcal{C} \to \text{Ring}$ be a contravariant functor. If $\star \in \mathcal{C}$ is the final object, the ring $A(\star)$ plays an special role since, for every $a \in \mathcal{C}$, we have an unique map $c_a : a \to \star$ which gives rise to a ring homomorphism $A(c_a) : A(\star) \to A(a)$. Hence, we can see $A(a)$ as a $A(\star)$-module in a natural way. Such a module structure is the one considered in the first condition of Definition 4.6 below.

Given $a, b, d \in \mathcal{C}$ with morphisms $a \to d$ and $b \to d$, let $p_1 : a \times_d b \to a$ and $p_2 : a \times_d b \to b$ be the corresponding projections. We define the external product over $d$

$$\boxtimes_d : A(a) \otimes_{A(d)} A(b) \to A(a \times_d b),$$

by $z \boxtimes_d w = A(p_1)(z) \cdot A(p_2)(w)$ for $z \in A(a)$ and $w \in A(b)$. The external product over the final object will be denoted just by $\boxtimes = \boxtimes_\star : A(a) \otimes_{A(\star)} A(b) \to A(a \times b)$. It gives a natural transformation $\boxtimes : A(-) \otimes_{A(\star)} A(-) \Rightarrow A(- \times -)$.

**Remark 4.5.** For $b = \star$, we have that $p_1 = \lambda : a \times \star \to a$ is the unital isomorphism of the monoidal structure so it gives raise to an isomorphism $A(\lambda) : A(a) \to A(a \times \star)$ of rings and $A(\star)$-modules. Under this isomorphism, the external product $\boxtimes : A(a) \otimes A(\star) \to A(a \times \star) \cong A(a)$ coincides with the given $A(\star)$-module structure on $A(a)$.

**Definition 4.6.** Let $(\mathcal{C}, \times, \star)$ be cartesian monoidal category. A $\mathcal{C}$-algebra, $A$, is a pair of functors

$$A : \mathcal{C}^{op} \to \text{Ring}, \quad B : \mathcal{C} \to A(\star)\text{-Mod},$$

such that:

• They agree on objects, that is, $A(a) = B(a)$ for all $a \in \mathcal{C}$, as $A(\star)$-modules.
• They satisfy the Beck-Chevalley condition (also known as the base change formula), that is, given $a_1, a_2, b \in \mathcal{C}$ and a pullback diagram

$$\begin{array}{ccc}
d & \xrightarrow{s'} & a_1 \\
f' \downarrow & & \downarrow f \\
a_2 & \xrightarrow{g} & b
\end{array}$$
we have that \( A(g) \circ B(f) = B(f') \circ A(g') \).

- The external product \( \boxtimes : B(-) \otimes_{A(*)} B(-) \Rightarrow B(\times -) \) is a natural transformation with respect to \( B \).

**Remark 4.7.**

- A \( \mathcal{C} \)-algebra can be thought as a collection of rings parametrized by \( \mathcal{C} \). For this reason, we will denote \( \mathcal{A}_a = A(a) \) for \( a \in \mathcal{C} \).
- The Beck-Chevalley condition appears naturally in the context of Grothendieck’s yoga of six functors \( f_* , f^*, f_!, f^! , \otimes \) and \( \mathbb{D} \) in which \( (f_* , f^*) \) and \( (f_!, f^!) \) are adjoints, and \( f^* \) and \( f_! \) satisfy the Beck-Chevalley condition. In this context, we can take \( A \) to be the functor \( f \mapsto f^* \) and \( B \) the functor \( f \mapsto f_! \). Moreover, in order to get in touch with this framework, we will denote \( A(f) = f^* \) and \( B(f) = f_! \). For further information about Grothendieck’ six functors, see for example [19] or [2].

Using the covariant functor \( B \), for every object \( a \in \mathcal{C} \), we obtain a \( \mathcal{A}_a \)-module homomorphism \((c_a)!: \mathcal{A}_a \rightarrow \mathcal{A}_a \). The special element \( \mu(a) = (c_a)!((1) \in \mathcal{A}_a \), where \( 1 \in \mathcal{A}_a \) is the unit of the ring, will be called the *measure* of \( a \).

**Example 4.8.** Given a locally compact Hausdorff topological space \( X \), let us consider the category \( \text{Sh} (X) \) of sheaves (of rational vector spaces) on \( X \) with sheaf transformations between them (i.e. natural transformations). It is an abelian monoidal category with monoidal structure given by tensor product of sheaves. Given a continuous map \( f : X \rightarrow Y \), we can induce two special maps at the level of sheaves. The first one is the inverse image \( f^* : \text{Sh} (Y) \rightarrow \text{Sh} (X) \) and it is an exact functor (see [39], Section II.4). We also have the direct image with compact support functor, \( f_! : \text{Sh} (X) \rightarrow \text{Sh} (Y) \). In this case, \( f_! \) is only left exact, so we can consider its derived functor \( Rf_! : \text{Sh} (X) \rightarrow D^+ \text{Sh} (Y) \) whose stalks are \( (R^nf_!(\mathcal{F}))_y = H^k_c (f^{-1}(y), \mathcal{F}) \) for \( y \in Y \) and \( \mathcal{F} \) a sheaf on \( X \). In this context, the base change theorem with compact support (see [17], Theorem 2.3.27) implies that, for any pullback diagram of locally Hausdorff spaces

\[
\begin{align*}
X \times_Z Y \xrightarrow{g} X \\
\downarrow f' \quad \downarrow f \\
Y \xrightarrow{g} Z
\end{align*}
\]

there is a natural isomorphism \( g^* \circ Rf_! \cong Rf_! \circ g^* \). Even more, \( Rf_! \) preserves the external product.

This situation can be exploited to obtain a \( \mathcal{C} \)-algebra. However, we must surmount the annoying difficulty that, for a general topological space \( X \), \( R^k f_! \mathcal{F} \) may not vanish for arbitrary large \( k \). In order to solve this problem, let us restrict to the full subcategory \( \text{Top}_0 \) of the category of locally compact Hausdorff topological spaces that have finite cohomological dimension (or, even simpler, to the category of smooth manifolds). In this subcategory, we do have \( Rf_! : \text{Sh} (X) \rightarrow D^\mathcal{C} \text{Sh} (Y) \), for \( f : X \rightarrow Y \) continuous, so it induces a map in \( K \)-theory \( f_! : K\text{Sh} (X) \rightarrow K\text{Sh} (Y) \) and analogously for \( f^* : K\text{Sh} (Y) \rightarrow K\text{Sh} (X) \). Even more, \( f^* \) is a ring homomorphism and \( f_! \) is a module homomorphism over \( K\text{Sh} (\ast) = K(\mathbb{Q} \text{-Vect}) = \mathbb{Z} \). By the previous properties, \( K\text{Sh} (\ast) \) is a \( \text{Top}_0 \)-algebra with \( A : f \mapsto f^* \) and \( B : f \mapsto f_! \). Observe that the unit object in \( K\text{Sh} (X) \) is the image of constant sheaf \( \mathbb{Q}_X \) on \( X \) with stalk \( \mathbb{Q} \).

Moreover, given \( X \in \text{Top}_0 \), let \( c_X : X \rightarrow \ast \) the projection onto the singleton set. Then, the measure of \( X \) is the object \( (c_X)_!(\mathbb{Q}_X) \) which is a sheaf whose unique stalk is \( (c_X)_!(\mathbb{Q}_X)_\ast = [H^*_c(X;\mathbb{Q})] = \chi_c(X) \in K(\mathbb{Q} \text{-Vect}) = \mathbb{Z} \). Hence, the measure of \( X \) is nothing but the Euler
characteristic of $X$ (with compact support), which is a kind of data compression of $X$. This justifies the fancy name ‘measure’ of $X$.

4.2. Quantisation. Given a category $\mathcal{C}$ with pullbacks, we can construct the 2-category of spans of $\mathcal{C}$, $\text{Span}(\mathcal{C})$. As described in [7], the objects of $\text{Span}(\mathcal{C})$ are the same as the ones of $\mathcal{C}$. A morphism $a \rightarrow b$ in $\text{Span}(\mathcal{C})$ is a span, that is, a triple $(d, f, g)$ of morphisms

$$
\begin{array}{ccc}
& & d \\
& f & \downarrow g \\
a & \rightarrow & b
\end{array}
$$

where $d \in \mathcal{C}$. Given two spans $(d_1, f_1, g_1) : a \rightarrow b$ and $(d_2, f_2, g_2) : b \rightarrow c$, we define the composition $(d_2, f_2, g_2) \circ (d_1, f_1, g_1) = (d_1 \times_b d_2, f_1 \circ f_2', g_2 \circ g_1')$, where $f_2', g_1'$ are the morphisms in the pullback diagram

$$
\begin{array}{ccc}
& & d_1 \times_b d_2 \\
& f_1 & \downarrow g_1 \\
& & a \\
\downarrow & & \downarrow \\
& f_2 & \downarrow g_2 \\
b & \rightarrow & c
\end{array}
$$

Finally, a 2-morphism $(d, f, g) \Rightarrow (d', f', g')$ between $a \leftarrow d \rightarrow b$ and $a \leftarrow d' \rightarrow b$ is a morphism $\alpha : d' \rightarrow d$ (notice the inverted arrow!) such that the following diagram commutes

$$
\begin{array}{ccc}
& & d' \\
& f' & \downarrow g' \\
a & \rightarrow & b \\
& \downarrow & \downarrow \\
& f & \downarrow g \\
d & \rightarrow & b
\end{array}
$$

Moreover, if $\mathcal{C}$ is a monoidal category, $\text{Span}(\mathcal{C})$ inherits a monoidal structure by tensor product on objects and morphisms.

**Proposition 4.9.** Let $\mathcal{A}$ be a $\mathcal{C}$-algebra, with $\mathcal{C}$ a category with final object $\star$ and pullbacks. Then, there exists a lax monoidal (strict) 2-functor $Q_A : \text{Span}(\mathcal{C}) \rightarrow \mathcal{A}_\star \text{-Mod}_t$ such that

$$
Q_A(a) = \mathcal{A}_a \\
Q_A(d, f, g) = g_! \circ f^* : \mathcal{A}_a \rightarrow \mathcal{A}_b,
$$

for $a, b \in \mathcal{C}$ and a span $a \leftarrow d \rightarrow b$. The functor $Q_A$ is called the quantisation of $\mathcal{A}$.

**Proof.** More detailed, the functor $Q_A : \text{Span}(\mathcal{C}) \rightarrow \mathcal{A}_\star \text{-Mod}_t$ is given as follows:

- For any $a \in \mathcal{C}$ we define $Q_A(a) = \mathcal{A}_a$.
- Fixed $a, b \in \mathcal{C}$, we define the functor

$$(Q_A)_{a,b} : \text{Hom}_{\text{Span}(\mathcal{C})}(a, b) \rightarrow \text{Hom}_{\mathcal{A}_\star \text{-Mod}_t}(\mathcal{A}_a, \mathcal{A}_b)$$

by:

- For a 1-morphism $a \leftarrow d \rightarrow b$ we define $Q_A(d, f, g) = g_! \circ f^* : \mathcal{A}_a \rightarrow \mathcal{A}_d \rightarrow \mathcal{A}_b$. 
Remark 4.10. Let \( \Phi : \mathcal{C} \to \mathcal{S} \) be a 2-valent functor satisfying the six functors formalism (which is the analogous of a \( \mathcal{C} \)-algebra) is equivalent to a functor out of the 2-category of correspondences (Theorem 2.13, Chapter 7).

4.3 Field theory. Let us fix a sheaf \( \mathcal{S} \). Let \( (W_1, s_1) \) and \( (W_2, s_2) \) be two \( n \)-dimensional objects of \( \text{Emb}_\mathcal{S}^n \) and let \( (M, s) \) be a \((n-1)\)-dimensional object. Suppose that there exist tame embeddings \( f_1 : (M, s) \to (W_1, s_2) \) and \( f_2 : (M, s) \to (W_2, s_2) \) of boundaries, as in Example 4.1.
that case, the pushout of $f_1$ and $f_2$ in $\text{Emb}_c^S$, $(W_1 \cup_M W_2, s_1 \cup s_2)$, exists and it is given by the gluing of $W_1$ and $W_2$ along $M$.

This situation will be called a glueing pushout.

**Definition 4.11.** Given a category $C$ with final object, a contravariant functor $F : \text{Emb}_c^S \to C$ is said to have the Seifert-van Kampen property if $F$ sends gluing pushouts in $\text{Emb}_c^S$ into pullbacks of $C$ and sends the initial object of $\text{Emb}_c^S$ (i.e. $\emptyset$) into the final object of $C$.

**Proposition 4.12.** Let $S$ be a sheaf, let $C$ be a category with final object and pullbacks and let $F : \text{Emb}_c^S \to C$ be a contravariant functor satisfying the Seifert-van Kampen property. Then, there exists a monoidal 2-functor $\mathcal{F}_F : \text{Bd}_n^S \to \text{Span}(C)$ such that

$$\mathcal{F}_F(M, s) = F(M, s), \quad F(M_1, s_1) \xleftarrow{F(i_1)} F(W, s) \xrightarrow{F(i_2)} F(M_2, s_2),$$

for all objects $(M, s), (M_1, s_1), (M_2, s_2) \in \text{Bd}_n^S$ and bordisms $(W, s) : (M_1, s_1) \to (M_2, s_2)$ where $i_k : M_k \to W$ are the inclusions as boundaries. In this situation, the functor $F$ is called the geometrisation and $\mathcal{F}_F$ is called the field theory of $F$.

**Proof.** The complete definition of $\mathcal{F}$ is given as follows. Given $(M, s) \in \text{Bd}_n^S$, we just define $\mathcal{F}_F(M, s) = F(M, s)$. With respect to morphisms, given a bordism $(W, s) : (M_1, s_1) \to (M_2, s_2)$ in $\text{Bd}_n^S$, let $i_1 : M_1 \to W$ and $i_2 : M_2 \to W$ be the inclusions of $M_1$ and $M_2$ as boundaries of $W$. The functor $\mathcal{F}_F$ assigns the span

$$F(M_1, s_1) \xleftarrow{F(i_1)} F(W, s) \xrightarrow{F(i_2)} F(M_2, s_2)$$

This assignment is a functor. In order to check it, let $(W, s) : (M_1, s_1) \to (M_2, s_2)$ and $(W', s') : (M_2, s_2) \to (M_3, s_3)$ be two bordisms with inclusions $i_k : M_k \to W$ and $i'_k : M_k \to W'$. By construction, $(W' \circ W, s \cup s')$ is the gluing pushout in $\text{Emb}_c^S$

$$\begin{align*}
(M_2, s_2) & \xrightarrow{i_2} (W, s) \\
(\require{AMScd} \begin{CD}
W' \circ W \rightarrow & W' \cup W \rightarrow (W' \circ W, s \cup s') \rightarrow (W', s') \\
\downarrow{j'} & \downarrow{j} & \downarrow{j}
\end{CD}
\end{align*}$$

Since $F$ sends gluing pushouts into pullbacks, the following diagram is a pullback in $C$

$$\begin{align*}
F(W' \circ W, s \cup s') & \xrightarrow{F(j)} F(W, s) \\
F(j') \downarrow & \downarrow{F(i_2)} \\
F(W', s') & \xrightarrow{F(i_2)} F(M_2, s_2)
\end{align*}$$
Therefore, $\mathcal{F}_F(W', s') \circ \mathcal{F}_F(W, s)$ is given by the span

$$
\begin{array}{ccc}
F(M_1, s_1) & \xrightarrow{F(i_1)} & F(W, s) \\
\downarrow & & \downarrow \quad F(i_2) \\
F(M_2, s_2) & \xrightarrow{F(i_2')} & F(W', s') \\
\downarrow & & \downarrow \quad F(i_3') \\
F(M_3, s_3) & \xrightarrow{F(i_3)} & F(W', s')
\end{array}
$$

Since $i_1 \circ j$ and $i_2' \circ j'$ are the inclusions onto $W' \circ W$ of $M_1$ and $M_3$ respectively, the previous span is also $\mathcal{F}_F(W' \circ W)$, as we wanted.

For the monoidality, let $(M_1, s_1), (M_2, s_2) \in \text{Bd}_n^S$. As the coproduct $(M_1 \sqcup M_2, s_1 \sqcup s_2)$ can be seen as a gluing pushout along $\emptyset$, $F$ also sends coproducts in $\text{Bd}_n^S$ into products on $\mathcal{C}$. Hence, $F(M_1 \sqcup M_2, s_1 \sqcup s_2) = F(M_1, s_1) \times F(M_2, s_2)$ and, since the monoidal structure on $\text{Span}(\mathcal{C})$ is given by products on $\mathcal{C}$, monoidality holds for objects. For morphisms, the argument is analogous.

For the 2-functor structure, suppose that $(f, \alpha)$ is a 2-cell between 1-morphisms $(W, s), (W', s') : (M_1, s_1) \to (M_2, s_2)$ of $\text{Bd}_n^S$. In that case, $(f, \alpha)$ is also a morphism in $\text{Emb}^S$ so we obtain a morphism $F(f, \alpha) : (W', s') \to (W, s)$ fitting in the commutative diagram

$$
\begin{array}{ccc}
F(W', s') & \xrightarrow{F(i_1)} & F(W, s) \\
\downarrow & & \downarrow \quad F(i_2) \\
F(M_1, s_1) & \xrightarrow{F(i_2)} & F(M_2, s_2) \\
\downarrow & & \downarrow \quad F(i_3) \\
F(W', s') & \xrightarrow{F(i_3)} & F(W, s)
\end{array}
$$

This produces the desired 2-morphism in $\text{Span}(\mathcal{C})$. $\square$

Composing together the functors of Proposition 4.9 and Proposition 4.12, we obtain the main theorem of this section.

**Theorem 4.13.** Let $\mathcal{C}$ be a category with final object $*$ and pullbacks. Given a functor $F : \text{Emb}^S \to \mathcal{C}$ with Seifert-van Kampen property and a $\mathcal{C}$-algebra $\mathcal{A}$, there exists a lax monoidal Topological Quantum Field Theory over $S$

$$Z_{F, \mathcal{A}} : \text{Bd}_n^S \to \mathcal{A}_* \text{-Mod}_i.$$ 

**Remark 4.14.** We thank D. Ben-Zvi for suggesting us the name ‘field theory’ based on the physical interpretation of TQFTs.

**Remark 4.15.** From the explicit construction given in the proofs of Propositions 4.9 and 4.12 the functor $Z_{F, \mathcal{A}}$ satisfies:

- For an object $(M, s) \in \text{Bd}_n^S$, it assigns $Z_{F, \mathcal{A}}(M, s) = \mathcal{A}_{F(M, s)}$.
- For a bordism $(W, s) : (M_1, s_1) \to (M_2, s_2)$, it assigns $Z_{F, \mathcal{A}}(W, s) = F(i_2) \circ F(i_1)^* : \mathcal{A}_{F(M_1, s_1)} \to \mathcal{A}_{F(M_2, s_2)}$.
- For a closed $n$-dimensional manifold $(W, s)$, seen as a bordism $(W, s) : \emptyset \to \emptyset$, the homomorphism $Z_{F, \mathcal{A}}(W, s) : \mathcal{A}_s \to \mathcal{A}_s$ is given by multiplication by the measure $\mu(F(W, s)) \in \mathcal{A}_s$. This follows from the fact that, since $F(\emptyset) = *$, the inclusion $i : \emptyset \to (W, s)$ gives the projection $c = F(i) : F(W, s) \to *$. Hence, we have, $Z_{F, \mathcal{A}}(W)(1_s) = c \circ c^*(1_s) =$
Remark 4.16. In the context of bordisms with sheaves, the requirement of the whimsical twisting structure on $\mathcal{A}_*\text{-Mod}_\ell$ becomes evident. The existence of a non-invertible 2-cell $(f, \alpha) : (W, s) \Rightarrow (W, s')$ reflects a non-invertible morphism $\alpha : s \to S(f)(s')$ that can be interpreted as a restriction of the extra structure sheaf (e.g. in the case of unordered configurations of points, as in the following section). As explained in Remark 4.10 these non-invertible cells become non-trivial twists in $\mathcal{A}_*\text{-Mod}_\ell$ that compare how the homomorphism changes under morphisms of sheaves.

Remark 4.17. We can also consider the case in which the geometrisation functor $F : \text{Emb}_c^S \to \mathcal{C}$ no longer has the Seifert-van Kampen property, but it still maps the initial object into the final object. In that case, the image of a gluing pushout under $F$ is not a pullback but, as for any functor, it is a cone. Suppose that we have two bordisms $(W, s) : (M_1, s_1) \to (M_2, s_2)$ and $(W', s') : (M_2, s_2) \to (M_3, s_3)$ that fit in the gluing pushout in $\text{Emb}_c^S$

\[
\begin{array}{ccc}
(M_2, s_2) & \longrightarrow & (W, s) \\
\downarrow & & \downarrow \\
(W', s') & \longrightarrow & (W' \circ W, s \cup s')
\end{array}
\]

By definition of pullback, there exists an unique morphism $\phi : F(W' \circ W, s \cup s') \to F(W', s') \times_{F(M_2, s_2)} F(W, s)$ in $\mathcal{C}$ such that the induced diagram commutes

\[
\begin{array}{ccc}
F(W' \circ W, s \cup s') & \xrightarrow[\phi]{\quad} & F(W', s') \\
\downarrow & & \downarrow \\
F(W', s') \times_{F(M_2, s_2)} F(W, s) & \longrightarrow & F(W', s') \\
\downarrow & & \downarrow \\
F(W, s) & \longrightarrow & F(M_2, s_2)
\end{array}
\]

This morphism $\phi$ induces a 2-morphism $\mathcal{F}_F(W', s') \circ \mathcal{F}_F(W, s) \Rightarrow \mathcal{F}_F(W' \circ W, s \cup s')$. Therefore, in this case, $\mathcal{F} : \text{Bd}_n^S \to \text{Span}(\mathcal{C})$ is no longer a functor but a lax 2-functor (see [7] or [44]). Thus, the induced functor

$$Z_{F,A} : \text{Bd}_n^S \to \mathcal{A}_*\text{-Mod}_\ell$$

is a lax monoidal symmetric lax 2-functor (recall that $Q_A : \text{Span}(\mathcal{C}) \to \mathcal{A}_*\text{-Mod}_\ell$ is a 2-functor). We will call such a functors very lax Topological Quantum Field Theories.

As a final remark, it is customary in the literature to focus on the field theory as a functor $\mathcal{F} : \text{Bd}_n \to \text{Span}(\mathcal{C})$ and on the quantisation as a functor $Q : \text{Span}(\mathcal{C}) \to R\text{-Mod}$, and to forget about the geometrisation and the $\mathcal{C}$-algebra, despite that they underlie the whole construction (see, for example, [22] or [23]). In this form, the Seifert-van Kampen property and the base change property stay encoded in the functoriality of $\mathcal{F}$ of $Q$, respectively, and many properties follow in a simpler way so this description gives a deeper insight in the properties of the TQFT.

However, in this paper we want to emphasize the role of the geometrisation $F$ and the $\mathcal{C}$-algebra $\mathcal{A}$ in the construction. The reason is that we are constructing TQFTs with a view towards the creation of new effective computational methods of algebraic invariants. In this way, $\mathcal{A}$ determines the algebraic invariant under study and $F$ determines the object for which...
we are going to compute the invariant. This principle opens the door to the development of new computational methods based on TQFTs further than the scope of this paper.

4.4. Some useful sheaf structures. In this section, we will describe in detail some examples of extra structures induced by sheaves. The first example is the trivial sheaf $S_{p} : \text{Emb}_{c} \to \text{Cat}$ that assigns, to every compact manifold, the singleton category (i.e. the category with an unique object and only the identity as morphism) and, to every morphism the identity morphism. In that case, $\text{Bd}_{\Lambda}^{n}$ is the usual category of (unoriented) bordisms.

An important example is the sheaf $S_{p} : \text{Emb}_{c} \to \text{Cat}$ of unordered configurations of points (also called the sheaf of pairs). For a compact manifold $M$, the category $S_{p}(M)$ has, as objects, finite subsets $A \subseteq M$ meeting every connected component and every boundary component of $M$. Given two finite subsets $A_{1}, A_{2} \subseteq M$, a morphism $A_{1} \to A_{2}$ in $S_{p}(M)$ is an inclusion $A_{1} \subseteq A_{2}$.

With respect to morphisms, if $f : M \to N$ is an embedding, the functor $S_{p}(f) : S_{p}(N) \to S_{p}(M)$ is given as follows. For an object $A \subseteq N$, it assigns $S_{p}(f)(A) = f^{-1}(A) \in S_{p}(M)$ and, if we have an inclusion $A_{1} \subseteq A_{2}$ then it gives the inclusion $f^{-1}(A_{1}) \subseteq f^{-1}(A_{2})$ as a morphism in $S_{p}(M)$.

It is straightforward to check that $S_{p}$ is a sheaf.

The associated categories $\text{Emb}_{c}^{S_{p}}$ and $\text{Bd}_{\Lambda}^{S_{p}}$ will be called the category of embeddings of pairs and the category of $n$-bordisms of pairs and will be denoted by $\text{Emb}_{c}$ and $\text{Bd}_{\Lambda}$, respectively. A (lax monoidal) $S_{p}$-TQFT will be referred to as a (lax monoidal) Topological Quantum Field Theory of pairs.

Another important sheaf for applications is the so-called sheaf of parabolic structures. The starting point is a fixed set $\Lambda$ that we will call the parabolic data. We define $S_{\Lambda} : \text{Emb}_{c} \to \text{Cat}$ as the following functor. For a compact manifold $W$, $S_{\Lambda}(W)$ is the category whose objects are (maybe empty) finite sets $Q = \{(S_{1}, \lambda_{1}), \ldots, (S_{r}, \lambda_{r})\}$, with $\lambda_{i} \in \Lambda$, called parabolic structures on $W$. The $S_{i}$ are pairwise disjoint compact submanifolds of $W$ of codimension 2 with a co-orientation (i.e. an orientation of their normal bundle) such that $S_{i} \cap \partial M = \partial S_{i}$ transversally. A morphism $Q \to Q'$ between two parabolic structures in $S_{\Lambda}(W)$ is just an inclusion $Q \subseteq Q'$.

Moreover, suppose that we have a tame embedding $f : W_{1} \to W_{2}$ in $\text{Emb}_{c}$ and let $Q \in S_{\Lambda}(W_{2})$. Given $(S, \lambda) \in Q$, if $\partial S$ meets every $\partial W_{i}$ transversally then the intersection has the expected dimension and, thus, $f^{-1}(S)$ is a codimension 2 submanifold of $W_{1}$. Furthermore, the co-orientation of $S$ induces a co-orientation on $f^{-1}(S)$ by pullback. Hence, we can define $S_{\Lambda}(f)(Q)$ as the set of pairs $(f^{-1}(S), \lambda)$ for $(S, \lambda) \in Q$ with $S \cap \partial W_{i}$ transversal. For short, we will denote $Q|_{W_{1}} = S_{\Lambda}(f)(Q)$. Obviously, if $Q \subseteq Q'$ then $Q|_{W_{1}} \subseteq Q'|_{W_{1}}$ so $S_{\Lambda}(f) : S_{\Lambda}(W_{2}) \to S_{\Lambda}(W_{1})$ is a functor. With this definition, $S_{\Lambda}$ is a sheaf, called the sheaf of parabolic structures over $\Lambda$.

**Example 4.18.** In the case of surfaces, a parabolic structure over a surface $M$ is a set $Q = \{(p_{1}, \lambda_{1}), \ldots, (p_{r}, \lambda_{r})\}$ with $\lambda_{i} \in \Lambda$ and $p_{i} \in M$ points with a preferred orientation of a small disc around them.

As for the sheaf of unordered points, the sheaf $S_{\Lambda}$ gives us categories $\text{Emb}_{c}^{S_{\Lambda}}$ and $\text{Bd}_{n}^{S_{\Lambda}}$, that we will shorten $\text{Emb}_{c}(\Lambda)$ and $\text{Bd}_{n}(\Lambda)$. Even more, we can combine the two previous sheaves and to consider the sheaf $S_{p,\Lambda} = (S_{\Lambda} \times S_{p}) \circ \Delta : \text{Emb}_{c} \to \text{Cat}$, where $\Delta : \text{Emb}_{c} \to \text{Emb}_{c} \times \text{Emb}_{c}$ is the diagonal functor. In that case, we will denote by $\text{Emb}_{c}(\Lambda)$ and by $\text{Bd}_{n}(\Lambda)$ the categories of $S_{p,\Lambda}$-embeddings and $S_{p,\Lambda}$-bordisms, respectively. In this case, a (lax monoidal) $S_{p,\Lambda}$-TQFT will be called a (lax monoidal) parabolic Topological Quantum Field Theory of pairs.
4.5. Reduction of a TQFT. Let \( \mathcal{C} \) be a category with final object and pullbacks, let \( F : \text{Emb}_k \to \mathcal{C} \) be a contravariant functor with the Seifert-van Kampen property and let \( \mathcal{A} \) be a \( \mathcal{C} \)-algebra. By Theorem 4.13, these data give rise to a lax monoidal TQFT \( Z = Z_{F,A} : \text{Bd}_n \to \mathcal{A}_*\text{-Mod} \). However, for \( M \in \text{Bd}_n \), the module \( Z(M) = \mathcal{A}_{F(M)} \) may be very complicated (e.g. it might be infinitely generated).

This problem can be mitigated if the objects \( F(M) \) have some kind of symmetry that may be exploited. For example, suppose that there is a natural action of a group \( G \) on \( F(M) \) so that the categorical quotient object \( F'(M) = F(M)/G \in \mathcal{C} \) can be defined. In the general case, such a symmetry can be modelled by an assignment \( \tau \) that, for any \( M \in \text{Bd}_n \), gives a pair \((F'(M),\tau_M)\), where \( F'(M) \in \mathcal{C} \) and \( \tau_M : F(M) \to F'(M) \) is a morphism in \( \mathcal{C} \). Suppose also that \( F'(\emptyset) = F(\emptyset) \).

In that case, we can ‘reduce’ the field theory and to consider \( \mathcal{F}_{F,\tau} : \text{Bd}_n \to \text{Span}(\mathcal{C}) \) such that \( \mathcal{F}_{F,\tau}(M) = F'(M) \), for \( M \in \text{Bd}_n \), and, for a bordism \( W : M_1 \to M_2, \mathcal{F}_{F,\tau}(W) \) is the span
\[
\begin{array}{c}
F'(M_1) \xrightarrow{\tau_M \circ F(1)} F(W) \xrightarrow{\tau_M \circ F(1)} F'(M_2).
\end{array}
\]

With this field theory, we form \( Z_\tau = Q_A \circ \mathcal{F}_{F,\tau} \), called the prereduction of \( Z \) by \( \tau \).

However, even if \( F \) had the Seifert-van Kampen property, \( \mathcal{F}_{F,\tau} \) may not be a functor so \( Z_\tau \) will be, in general, a very lax TQFT. Indeed, the prereduction \( Z_\tau \) is simpler than \( Z \) but, as it is not a functor, we can no longer use it for a computational method. In this section we will show that, under some mild conditions, we can slightly modify \( Z_\tau \) in order to obtain a (strict) almost-TQFT, \( Z_\tau \), called the reduction, with essentially the same complexity as \( Z_\tau \). In this way, \( Z_\tau \) can be used to give an effective computational method.

In order to do so, consider the wide subcategory \( \text{Tb}^S_n \) of \( \text{Bd}^S_n \) of \( S \)-tubes. A morphism \((M,s) \in \text{Bd}^S_n \) is an object of \( \text{Tb}^S_n \) if there exists a compact \( n \)-dimensional manifold \( W \) such that \( \partial W = M \). A bordism \((W,s) : (M_1,s_1) \to (M_2,s_2) \) of \( \text{Bd}^S_n \) is called a strict tube if \( M_1 \) and \( M_2 \) are connected or empty. In this way, the morphisms of \( \text{Tb}^S_n \) are disjoint unions of strict tubes, the so-called tubes. A (strict) monoidal functor \( Z : \text{Tb}^S_n \to \text{R-Mod} \) is called an almost-TQFT over \( S \).

Now, let \( Z_\tau : \text{Bd}^S_n \to \mathcal{A}_*\text{-Mod} \) be a prereduction by \( \tau \). For any \((M,s) \in \text{Tb}^S_n \) connected or empty, we will denote by \( \mathcal{V}_{(M,s)} \subseteq \mathcal{A}_{F(M,s)} \) the submodule generated by the images \( Z_\tau(W_{r,s_r}) \circ \ldots \circ Z_\tau(W_1,s_1)(1) \in \mathcal{A}_{F(M,s)} \) of all the sequences of strict tubes \((W_k,s_k) : (M_{k-1},s_{k-1}) \to (M_k,s_k) \) with \( M_0 = \emptyset \). Notice that, by definition, the submodules \( \mathcal{V}_{(M,s)} \) are invariant for strict tubes.

Lemma 4.19. Let \( \mathcal{C} \) be a category, let \( a,a',b,b' \in \mathcal{C} \) and consider morphisms \( f_1 : a \to a', f^* : a' \to a, g : b \to b' \) and \( h : a \to b \). Suppose that the morphism \( \eta = f_1 \circ f^* : a' \to a' \) is invertible, then there exists a unique morphism \( h' : a' \to b' \) such that \( h' \circ f_1 = g \circ h \).

\[
\begin{array}{ccc}
a & \xrightarrow{h} & b \\
f_1 \downarrow & & g \downarrow \\
\xrightarrow{f^*} & \xrightarrow{\eta} & b' \\
h' \downarrow & & \downarrow \\
a' & \xrightarrow{h'} & b'
\end{array}
\]

Proof. First, let us prove that \( h' \) is unique. Pre-composing with \( f^* \circ \eta^{-1} : a' \to a \) we obtain that
\[
g \circ h \circ f^* \circ \eta^{-1} = h' \circ f_1 \circ f^* \circ \eta^{-1} = h' \circ \eta \circ \eta^{-1} = h'.
\]
Actually, this calculation shows the existence of that morphism since we must take $h' = g \circ h \circ f^* \circ \eta^{-1}$, and it is a straightforward check that $h'$ has the desired property.

**Proposition 4.20.** Let $Z : \mathsf{Bd}^S_n \to \mathcal{A}_*\text{-}\mathsf{Mod}$ be a lax monoidal TQFT and let $\tau$ be a reduction. Suppose that, for all $(M, s) \in \mathsf{Bd}^S_n$, we have $(\tau(M, s))! \circ (\tau(M, s))^* : \mathcal{V}(M, s) \subseteq \mathcal{V}(M, s)$ and the morphisms $\eta(M, s) = (\tau(M, s))! \circ (\tau(M, s))^* : \mathcal{V}(M, s) \to \mathcal{V}(M, s)$ are invertible. Then, there exists an almost-TQFT

$$Z_\tau : \mathsf{Bd}^S_n \to \mathcal{A}_*\text{-}\mathsf{Mod}$$

such that $Z_\tau(M, s) = \mathcal{V}(M, s)$ for all $(M, s) \in \mathsf{Bd}^S_n$ connected or empty and $Z_\tau(W, s) \circ (\tau(M, s))! = (\tau(M, s))! \circ Z(W, s)$ for all strict tubes $(W, s) : (M_1, s_1) \to (M_2, s_2)$. This TQFT is called the reduction of $Z$ via $\tau$.

**Proof.** Recall that, in order to define an almost-TQFT, it is enough to define it on strict tubes and to extend it to a general tube by tensor product (c.f. [31]). Therefore, for $(M, s) \in \mathsf{Bd}^S_n$ connected or empty, we assign $Z_\tau(M, s) = \mathcal{V}(M, s)$. For a strict tube $(W, s) : (M_1, s_1) \to (M_2, s_2)$, by Lemma 4.19 there exists a unique morphism $Z_\tau(W, s) : \mathcal{V}(M_1, s_1) \to \mathcal{V}(M_2, s_2)$ such that the following diagram commute

\[
\begin{array}{ccc}
\tau(M_1, s_1)\mathcal{V}(M_1, s_1) & \xrightarrow{Z(W, s)} & Z(W, s) \left(\tau^*(M_1, s_1)\mathcal{V}(M_1, s_1)\right) \\
(\tau(M_1, s_1))! \downarrow & & \downarrow (\tau(N))! \\
\mathcal{V}(M_1, s_1) & \xrightarrow{Z_\tau(W, s)} & \mathcal{V}(M_2, s_2)
\end{array}
\]

In order to prove that $Z_\tau$ is a functor, suppose that $(W, s) : (M_1, s_1) \to (M_2, s_2)$ and $(W', s') : (M_2, s_2) \to (M_3, s_3)$ are strict tubes. Then, by the previous proposition, we have a commutative diagram

\[
\begin{array}{ccc}
\tau(M_1, s_1)\mathcal{V}(M_1, s_1) & \xrightarrow{Z(W, s)} & Z(W, s) \left(\tau^*(M_1, s_1)\mathcal{V}(M_1, s_1)\right) \\
(\tau(M_1, s_1))! \downarrow & & \downarrow (\tau(M_2, s_2))! \\
\mathcal{V}(M_1, s_1) & \xrightarrow{Z_\tau(W, s)} & \mathcal{V}(M_2, s_2)
\end{array}
\]

Therefore, we have $Z_\tau(W', s') \circ Z_\tau(W, s) \circ (\tau(M_1, s_1))! = (\tau(M_3, s_3))! \circ Z(W', s') \circ Z(W, s) = (\tau(M_3, s_3))! \circ Z(W' \circ W, s \cup s')$ and, by uniqueness, this implies that $Z_\tau(W', s') \circ Z_\tau(W, s) = Z_\tau(W' \circ W, s \cup s')$, as we wanted to prove. 

**Remark 4.21.**

- The almost-TQFT, $Z_\tau : \mathsf{Bd}^S_n \to \mathcal{A}_*\text{-}\mathsf{Mod}$, satisfies that, for all closed $n$-dimensional manifolds $W$ and $s \in \mathcal{S}(W)$

$$Z_\tau(W, s)(1) = Z_\tau(W, s) \circ (\tau_0)! \circ Z(W, s)(1) = Z(W, s)(1),$$

where we have used that $\tau_0 = 1_*$. Hence, $Z_\tau$ and $Z$ compute the same invariant.

- It may happen, and it will be the case for representation varieties, that $\eta(M, s) : \mathcal{V}(M, s) \to \mathcal{V}(M, s)$ are not invertible as $\mathcal{A}_*$-module. However, it could happen that, for some fixed multiplicative system $S \subseteq \mathcal{A}_*$, all the extensions to the localizations $\eta(M, s) : S^{-1}\mathcal{V}(M, s) \to S^{-1}\mathcal{V}(M, s)$ are invertible. In that case, we can fix the problem by localizing all the modules and morphisms of the original TQFT to obtain another TQFT, $Z : \mathsf{Bd}_n \to S^{-1}\mathcal{A}_*\text{-}\mathsf{Mod}$, to which we can apply the previous construction.
Example 4.22. For \( n = 2 \) (surfaces) and \( S = S_1 \) the trivial sheaf, the unique non-empty connected object of \( 
abla \) is \( S^1 \). Hence, it is enough to consider \( \nabla = \mathcal{V}(S^1) \) and \( \tau = \tau_{S^1} : F(S^1) \to F'(S^1) \). Actually, \( \mathcal{V} \) is the submodule generated by the elements \( Z_r(L)^g \circ Z_r(D)(1) \) for \( g \geq 0 \), where \( L : S^1 \to S^1 \) is the holed torus and \( D : \emptyset \to S^1 \) is the disk (i.e. by the image of all the compact surfaces with a disc removed). In that case, the only condition we need to check is that \( \eta = \eta \circ \tau^* : \mathcal{V} \to \mathcal{V} \) is invertible.

Remark 4.23. Instead of the diagram of Lemma 4.19 we could look for a morphism \( Z'_r(W,s) : \mathcal{V}(M_{1,s_1}) \to \mathcal{V}(M_{2,s_2}) \) such the diagram

\[
\begin{array}{c}
\tau^*_{(M_{1,s_1})} \mathcal{V}(M_{1,s_1}) \rightarrow Z(W,s) \mathcal{V}(M_{1,s_1}) \rightarrow \\
\tau^*_{(M_{2,s_2})} \mathcal{V}(M_{2,s_2}) \rightarrow \mathcal{V}(M_{2,s_2})
\end{array}
\]

commutes. In that case, in the same conditions as in Proposition 4.20 \( Z'_r \) exists, it is an almost TQFT and \( Z'_r(W,s) = \eta^{-1}_{(M_{2,s_2})} \circ Z_r(W,s) \). We will call \( Z'_r \) the left \( \tau \)-reduction.

5. TQFTs for representation varieties

In this section we will review the definition of representation and character varieties. They are the central objects of this paper. We will work over an algebraically closed field \( k \).

Definition 5.1. Let \( \Gamma \) be a finitely generated groupoid (i.e. a groupoid with finite many objects and whose vertex groups are all finitely generated, see [31]) and let \( G \) be an algebraic group over \( k \). The set of groupoid representations of \( \Gamma \) into \( G \)

\[ \mathfrak{X}_G(\Gamma) = \text{Hom}(\Gamma, G) \]

is called the representation variety.

As its name suggests, \( \mathfrak{X}_G(\Gamma) \) has a natural algebraic structure. First of all, suppose that \( \Gamma \) is a group i.e. it has a single object. In that case, let \( \Gamma = \langle \gamma_1, \ldots, \gamma_r \mid R_\alpha(\gamma_1, \ldots, \gamma_r) = 1 \rangle \) be a presentation of \( \Gamma \) with finitely many generators, where \( R_\alpha \) are the relations (possibly infinitely many). In that case, we define the injective map \( \psi : \text{Hom}(\Gamma, G) \to G^r \) given by \( \psi(\rho) = (\rho(\gamma_1), \ldots, \rho(\gamma_r)) \). The image of \( \psi \) is the algebraic subvariety of \( G^r \)

\[ \text{Im} \psi = \{ (g_1, \ldots, g_r) \in G^r \mid R_\alpha(g_1, \ldots, g_r) = 1 \} . \]

Hence, we can impose an algebraic structure on \( \text{Hom}(\Gamma, G) \) by declaring that \( \psi \) is a regular isomorphism on its image. Observe that this algebraic structure does not depend on the chosen presentation.

In the general case in which \( \Gamma \) is a groupoid with \( n \) nodes, pick a set \( J = \{ a_1, \ldots, a_k \} \) of objects of \( \Gamma \) such that every connected component of \( \Gamma \) contains exactly one element of \( J \). Moreover, for any object \( a \) of \( \Gamma \), pick a morphism \( f_a : a \to a_i \) where \( a_i \) is the object of \( J \) in the connected component of \( a \). Hence, if \( \rho : \Gamma \to G \) is a groupoid homomorphism, it is uniquely determined by the group representations \( \rho_i : \Gamma_{a_i} \to G \) for \( a_i \in J \), where \( \Gamma_{a_i} = \text{Hom}_\Gamma(a_i, a_i) \) is the vertex group at \( a_i \), together with the elements \( g_a \in G \) corresponding to the morphisms \( f_a \) for any object \( a \). Since the elements \( g_a \) can be chosen without any restriction, we have a natural identification

\[ \text{Hom}(\Gamma, G) \cong \text{Hom}(\Gamma_{a_1}, G) \times \ldots \times \text{Hom}(\Gamma_{a_i}, G) \times G^{n-s} ; \]
and each of these factors has a natural algebraic structure as representation variety. This endows \( \text{Hom}(\Gamma, G) \) with an algebraic structure.

The representation variety \( X_G(\Gamma) \) has a natural action of \( G \) by conjugation i.e. \( g \cdot \rho(\gamma) = g \rho(\gamma) g^{-1} \) for \( g \in G, \rho \in X_G(\Gamma) \) and \( \gamma \) an element of \( \Gamma \). Recall that two representations \( \rho, \rho' \) are said to be isomorphic if and only if \( \rho' = g \cdot \rho \) for some \( g \in G \).

**Definition 5.2.** Let \( \Gamma \) be a finitely generated groupoid and \( G \) an algebraic reductive group. The Geometric Invariant Theory quotient

\[
R_G(\Gamma) = X_G(\Gamma) \sslash G, 
\]

is called the **character variety**.

**Remark 5.3.** The Geometric Invariant Theory quotient, also known as GIT quotient, is a kind of quotient that make sense for algebraic varieties. For a complete introduction to this topic, see [52] or [53].

**Example 5.4.** Let \( M \) be a compact connected manifold with fundamental group \( \Gamma = \pi_1(M) \). The fundamental group of such a manifold is finitely generated since a compact manifold has the homotopy type of a finite CW-complex. Hence, we can form its representation variety, that we will shorten \( X_G(M) = X_G(\pi_1(M)) \). The corresponding character variety is called the character variety of \( M \) and it is denoted by \( R_G(M) = X_G(M) \sslash G \). More generally, if \( A \subseteq M \) is a finite set, the fundamental groupoid \( \Pi(M, A) \) is a finitely generated groupoid. Recall that \( \Pi(M, A) \) is the groupoid whose elements are homotopy classes of paths in \( M \) between points in \( A \). The corresponding representation variety is denoted \( X_G(M, A) \) and the associated character variety \( R_G(M, A) \).

A step further in the construction of character varieties can be done by considering an extra structure on them, called a parabolic structure. A **parabolic structure** on \( X_G(\Gamma) \), \( Q \), is a finite set of pairs \((\gamma, \lambda)\), where \( \gamma \in \Gamma \) and \( \lambda \subseteq G \) is a locally closed subset which is closed under conjugation. Given such a parabolic structure, we define the **parabolic representation variety**, \( X_G(\Gamma, Q) \), as the subset of \( X_G(\Gamma) \)

\[
X_G(\Gamma, Q) = \{ \rho \in X_G(\Gamma) \mid \rho(\gamma) \in \lambda \text{ for all } (\gamma, \lambda) \in Q \}. 
\]

As in the non-parabolic case, \( X_G(\Gamma, Q) \) has a natural algebraic variety structure since, using suitable generators, we have an identification \( X_G(\Gamma, Q) = X_G(\Gamma) \cap (G^r \times \lambda_1 \times \ldots \times \lambda_s) \). The conjugacy action of \( G \) on \( X_G(\Gamma) \) restricts to an action on \( X_G(\Gamma, Q) \) since the subsets \( \lambda_i \) are closed under conjugation. The GIT quotient of the representation variety by this action,

\[
R_G(\Gamma, Q) = X_G(\Gamma, Q) \sslash G, 
\]

is called the **parabolic character variety**.

**Remark 5.5.** As a particular choice for the subset \( \lambda \subseteq G \), we can choose the conjugacy classes of an element \( h \in G \), denoted \([h]\). Observe that \([h]\) is locally closed since, by [53] Lemma 3.7, it is an open subset of its Zariski closure, \([h]\).

**Example 5.6.** Let \( \Sigma = \Sigma_g - \{p_1, \ldots, p_s\} \) with \( p_i \in \Sigma_g \) distinct points, called the punctures or the marked points. In that case, we have a presentation of the fundamental group of \( \Sigma \) given by

\[
\pi_1(\Sigma) = \left\langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \gamma_1, \ldots, \gamma_s \mid \prod_{i=1}^{g}[\alpha_i, \beta_i] \prod_{j=1}^{s} \gamma_j = 1 \right\rangle.
\]
where the $\gamma_i$ are the positive oriented simple loops around the punctures. As parabolic structure, we will take $Q = \{ (\gamma_1, \lambda_1), \ldots, (\gamma_s, \lambda_s) \}$. The corresponding parabolic representation variety is

$$\mathcal{X}_G(\pi_1(\Sigma), Q) = \left\{ (A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_s) \in G^{2g+s} \left| \prod_{i=1}^g [A_i, B_i] \prod_{j=1}^s C_j = 1 \right. \right\}.$$ 

Let $M$ be a compact differentiable manifold and let $S \subseteq M$ be a codimension 2 closed connected submanifold with a co-orientation (i.e. an orientation of its normal bundle). Embed the normal bundle as a small tubular neighbourhood $U \subseteq M$ around $S$. Fixed $s \in S$, consider an oriented local trivialization $\psi : V \times \mathbb{R}^2 \to U$ of the normal bundle around an open neighbourhood $V \subseteq S$ of $s$. In that situation, the loop $\gamma(t) = \psi(s, (\cos t, \sin t)) \in \pi_1(M - S)$ is called the positive meridian around $s$.

![Diagram of a meridian]

**Remark 5.7.**
- Two positive meridians around $s, s' \in S$ are conjugate to each other on $\pi_1(M - S)$.
- If $S$ is not connected, then the conjugacy classes of meridians are in correspondence with the connected components of $S$.
- A loop $\gamma \in \pi_1(M - S)$ is a generalized knot. In that sense, the kernel of $\pi_1(M - S) \to \pi_1(M)$ are the loops in $M - S$ that ‘surround’ $S$. It can be proven (see [67]) that this kernel is the smallest normal subgroup containing all the meridians around the connected components of $S$. Hence, the meridians capture all the information of the loops surrounding $S$.

Take $\Lambda$ to be a collection of locally closed subsets of $G$ that are invariant under conjugation. Suppose that $Q = \{ (S_1, \lambda_1), \ldots, (S_r, \lambda_s) \}$ is a parabolic structure over $\Lambda$, in the sense of Section 4.4. Let us denote $S = \bigcup_i S_i$. To this parabolic structure, we can build the parabolic structure on $\mathcal{X}_G(\Pi(M - S, A))$, where $A \subseteq M$ is any finite set, also denoted by $Q$. It is the parabolic structure $Q = \{ (\gamma_1, \lambda_1), \ldots, (\gamma_{1,m_1}, \lambda_1), \ldots, (\gamma_{r,1}, \lambda_s), \ldots, (\gamma_{r,m_r}, \lambda_s) \}$, where $\gamma_{i,1}, \ldots, \gamma_{i,m_i}$ is a generating set of positive meridians around the connected components of $S_i$ (based on any point of $A$). As for the non-parabolic case, we will shorted the corresponding parabolic character variety by $\mathcal{X}_G(M, A, Q) = \mathcal{X}_G(\Pi(M - S, A), Q)$.

**Remark 5.8.** Suppose that $M = \Sigma_g$ is a closed oriented surface. A parabolic structure is given by $Q = \{ (p_1, \lambda_1), \ldots, (p_s, \lambda_s) \}$, with $p_i \in \Sigma_g$ points with a preferred orientation of a small disk around them (see Example 4.18). In that case, the meridian of $p_i$ is given by a small loop encyling $p_i$ positively with respect to the orientation of the small disk around it. Therefore, the associated parabolic structure of representation variety is

$$\mathcal{X}_G(\Sigma_g, Q) = \left\{ (A_1, B_1, \ldots, A_g, B_g, C_1, \ldots, C_s) \in G^{2g+s} \left| \prod_{i=1}^g [A_i, B_i] \prod_{j=1}^s C_j = 1 \right. \right\},$$
where $\epsilon_j = 1$ if the orientation of the disk around $p_j$ agrees with the global orientation and $\epsilon_j = -1$ if it does not. Notice that, they agree with the ones of Example 5.6.

5.1. **Standard TQFT for representation varieties.** In this section, we sketch the construction of a lax monoidal parabolic TQFT of pairs computing Hodge structures on representation varieties. This construction appeared for the first time in [31]. In this paper, we will focus on several variants. For the sake of completeness, we also include here the original construction. From now on, we will suppose that the ground field is $k = \mathbb{C}$.

Fix $G$ a complex algebraic group and take as parabolic data $\Lambda$ a collection of subvarieties of $G$ that are closed under conjugation. As the category of fields for this construction, we take the category of complex algebraic varieties, $\text{Var}_\mathbb{C}$. The geometrisation functor $\mathcal{X}_G : \text{Emb}_p(\Lambda) \to \text{Var}_\mathbb{C}$ is given as follows. For $(M, A, Q) \in \text{Emb}_p(\Lambda)$, where $M$ is a compact manifold, $A \subseteq M$ is a finite set and $Q$ is a parabolic structure on $M$, it assigns $\mathcal{X}_G(M, A, Q)$, the parabolic representation variety of $(M, A)$ on $G$ with parabolic structure $Q$.

On the other hand, to a morphism $(f, \alpha) : (M, A, Q) \to (M', A', Q')$ in $\text{Emb}_p(\Lambda)$, we associate the regular morphism $\mathcal{X}_G(f) : \mathcal{X}_G(M', A', Q') \to \mathcal{X}_G(M, A, Q)$ induced by the groupoid homomorphism $f_* : \Pi(M, A) \to \Pi(M', A')$. Observe that, as the morphism $\alpha$ is just inclusion (of points and parabolic structures), the morphism $f_*$ preserves the parabolic structures. By means of Seifert-van Kampen theorem for fundamental groupoids ([10], see also [31]), the functor $\mathcal{X}_G$ has the Seifert-van Kampen property so it gives rise to a field theory $\mathcal{F}_{\mathcal{X}_G} : \text{Bdp}_n(\Lambda) \to \text{Span} (\text{Var}_\mathbb{C})$.

For constructing the $\text{Var}_\mathbb{C}$-algebra needed, we will use Saito’s mixed Hodge modules. We are going to take $\text{KM} = (A, B)$, where, on objects, both functors assign, to $X \in \text{Var}_\mathbb{C}$, the Grothendieck group of the category mixed Hodge modules on $X$, $A(X) = B(X) = \text{KM}_X$ (seen as a ring for $A$ and as a $\text{KM}_*$-$\text{KHS}$-module for $B$). With respect to morphisms, given a regular map $f : X \to Y$ between complex algebraic varieties, we assign $A(f) = f^* : \text{KM}_Y \to \text{KM}_X$ and $B(f) = f_! : \text{KM}_X \to \text{KM}_Y$. By the results of Section 2.3, $f^*$ is a ring homomorphism and $f_!$ is a $\text{KHS}$-module homomorphism and they commute with external product, as needed for $\text{Var}_\mathbb{C}$-algebras. Finally, they satisfy the Beck-Chevalley condition by item [iii] of Theorem 2.8. Hence, $\text{KM}$ is a $\text{Var}_\mathbb{C}$-algebra.

Therefore, Theorem 4.13 implies that, for any $n \geq 1$, there exists a lax monoidal TQFT $Z_{\mathcal{X}_G, \text{KM}} : \text{Bdp}_n(\Lambda) \to \text{KHS}$-$\text{Mod}_t$.

For short, we will denote $Z_G = Z_{\mathcal{X}_G, \text{KM}}$ and we will call it the standard TQFT. Using the properties explained in Remark 4.13, this TQFT satisfies that, for any $n$-dimensional closed manifold $W$, any non-empty finite subset $A \subseteq W$ and any parabolic structure $Q$ in $W$, we have

$$Z_G(W, A, Q)(1) = [H^*_c(\mathcal{X}_G(W); \mathbb{Q})] \otimes [H^*_c(\mathcal{G}; \mathbb{Q})]^{[A]-1}$$

where $1 = Q_0 \in \text{KHS}$ is the unit Hodge structure.

**Remark 5.9.** The Hodge structure $[H^*_c(\mathcal{G}; \mathbb{Q})]$ is usually known for the standard groups.

In the case $n = 2$ (surfaces), this TQFT was explicitly described in [31] in Section 4.3. Observe that the morphisms of $\text{Tbp}_2(\Lambda) = \text{Tbp}_2^{S_2, \Lambda}$ are generated by the set $\Delta = \{D, D^\dagger, L, P\} \cup \bigcup_{\lambda \in \Lambda} L_\lambda$, as depicted in Figure [1].

Unraveling the definitions of the TQFT, the associated morphisms of the discs $D$ and $D^\dagger$ under $Z_G$ are

$$Z_G(D) = i_! : \text{KHS} = \text{KM}_1 \to \text{KM}_G \quad Z_G(D^\dagger) = i^* : \text{KM}_G \to \text{KM}_1 = \text{KHS}.$$
31

Figure 1.

where \( i : 1 \rightarrow G \) is the inclusion. For the holed torus \( L : (S^1, \star) \rightarrow (S^1, \star) \) the situation is a bit more complicated. The associated field theory is the span

\[
G \leftarrow \to G^4 \xrightarrow{q} G \xrightarrow{hg[g_1, g_2]h^{-1}}
\]

Hence, the image of \( L \) under the TQFT is the morphism \( Z_G(L) = q \circ p^* : KMG \rightarrow KMG. \) For the morphism \( P \), the associated field theory is the span

\[
G \leftarrow \to G^2 \xrightarrow{v} G \xrightarrow{hg} G
\]

Thus, we have that \( Z_G(P) = v \circ u^* : \text{KMG} \rightarrow \text{KMG}. \) Finally, for the tube with parabolic datum \( \lambda, L_\lambda \), we have that its image under the field theory is the span

\[
G \leftarrow \to G^2 \times \lambda \xrightarrow{h} G \xrightarrow{hg \xi h^{-1}}
\]

Therefore, its image under the TQFT is the morphism \( Z_G(L_\lambda) = s! \circ r^* : \text{KMG} \rightarrow \text{KMG}. \)

With this description, we have proven the following result.

**Theorem 5.10.** Let \( \Sigma_g \) be the closed oriented surface of genus \( g \) and let \( Q \) be a parabolic structure on \( \Sigma_g \) with \( s \) marked points with data \( \lambda_1, \ldots, \lambda_s \in \Lambda \). Then,

\[
[H^*_c(X_G(\Sigma_g, Q))] = \frac{1}{[H^*_c(G)]^{g+s}} Z_G(D^1) \circ Z_G(L_\lambda) \circ \ldots \circ Z_G(L_{\lambda_1}) \circ Z_G(L) \circ Z_G(D)(1).
\]

5.2. **Piecewise algebraic varieties.** In this section, we extend slightly the notion of an algebraic variety in order to allow topological spaces which are not algebraic varieties, but disjoint union of algebraic varieties. This will allow us to consider a variation of the TQFT above that, instead of GIT quotients, uses usual quotients as orbit spaces.

**Definition 5.11.** Let \( k \) be an algebraically closed field. We define the category of piecewise varieties over \( k \), \( \text{PVar}_k \), as the category given by:

- **Objects:** The objects of \( \text{PVar}_k \) is the Grothendieck semi-ring of algebraic varieties over \( k \) (see \[69\]). This is the semi-ring generated by the symbols \([X]\), for \( X \) an algebraic variety, with the relation \([X] = [U] + [Y]\) if \( X = U \sqcup Y \) with \( U \subseteq X \) open and \( Y \subseteq X \) closed. It is a semi-ring with disjoint union as sum and cartesian product as product. For further information, see \[41\].
- **Morphisms:** Given \([X], [Y] \in \text{PVar}_k\) a morphism \( f : [X] \rightarrow [Y] \) is given by a function \( f : X \rightarrow Y \) that decomposes as a disjoint union of regular maps. More precisely, it has
to exist decompositions \([Y] = \sum_i [Y_i]\) and \([X] = \sum_{i,j} [X_{ij}]\) into algebraic varieties such that \(f = \bigsqcup_{i,j} f_{ij}\) with \(f_{ij} : X_{ij} \to Y_i\) a regular morphism.

- Composition is given by the usual composition of maps. Observe that, given \(f : [X] \to [Y]\) and \(g : [Y] \to [Z]\), by Chevalley theorem \([16]\), we can find a common decomposition \(Y = \bigsqcup_i Y_i\) such that

\[
f = \bigsqcup_{i,j} f_{ijk} : \bigsqcup_{i,j,k} X_{ijk} \to \bigsqcup_i Y_i,
\]

\[
g = \bigsqcup_{i,j} g_{ij} : \bigsqcup_{i,j} Y_{ij} \to \bigsqcup_i Z_i
\]

decompose as regular maps. Hence, \(g \circ f = \bigsqcup_{i,j,k} (g_{ij} \circ f_{ijk})\) is a decomposition into regular maps of \(g \circ f\).

**Remark 5.12.**
- There is a functor \(\text{Var}_k \to \text{PVar}_k\) that sends \(X \mapsto [X]\) and analogously for regular maps. On the other way around, there is a forgetful functor \(\text{PVar}_k \to \text{Set}\) that recovers the underlying set of a piecewise variety.
- The functor \(\text{Var}_k \to \text{PVar}_k\) is not an embedding of categories. For example, let \(X = \{y^2 = x^3\}\) be a cuspidal cubic plane curve. Observe that, removing the origin in \(X\), we have a decomposition \([X] = [X - \ast] + [\ast] = [A^1 - \ast] + [\ast] = [A^1]\). Hence, the images of \(X\) and \(A^1\) under this functor agree, even though they are not isomorphic.

**Example 5.13.** The affine line with a double point at the origin \(X\) is not an algebraic variety. However, it is a piecewise variety since we can decompose \([X] = [A^1] + [\ast]\) with both pieces locally closed subsets.

Let \(G\) be an algebraic group acting on a variety \(X\). Maybe after restricting to the open subset of semi-stable points, we can suppose that a (good) GIT quotient \(\pi : X \to X/G\) exists. On \(X\), we find the subset \(X^1 \subseteq X\) of poly-stable points whose orbits have maximum dimension. It is an open subset by an adaptation of Proposition 3.13 of \([13]\) and it is non-empty since the set of poly-stable points is so. On the poly-stable points, the GIT quotient is an orbit space so we have a \(G\)-invariant regular map \(\pi_1 : X_1 \to X_1/G\). Now, let \(Y = X - X_1\). As \(Y\) is closed on \(X\) and the action of \(G\) restricts to an action on \(Y\) and we can repeat the argument to obtain a regular \(G\)-invariant map \(\pi_2 : X_2 \subseteq Y \to X_2/G\), where \(X_2 \subseteq Y\) is the set of poly-stable points of \(Y\) of maximum dimension. Repeating this procedure, we obtain a stratification \(X = X_1 \cup \ldots \cup X_r\) and a set of regular maps \(\pi_i : X_i \to X_i/G\) where each \(X_i/G\) has a natural algebraic structure.

**Definition 5.14.** Let \(G\) be a reductive group and let \(X\) be an algebraic variety. With the decomposition above, the **piecewise quotient** of \(X\) by \(G\), denoted by \([X/G]\) is the object of \(\text{PVar}_k\) given by

\[
[X/G] = [X_1/G] + \ldots + [X_r/G].
\]

We also have a piecewise quotient morphism \(\pi = \bigsqcup_i \pi_i : [X] \to [X/G]\) where \(\pi_i : X_i \to X_i/G\) are the projections on the orbit space of each of the strata.

**Remark 5.15.** Since all the strata considered are made of poly-stable points over the previous statum, no orbits are identified under the quotient. In this sense, the previous construction says that the orbit space of \(X\) by \(G\) has a piecewise variety structure.

**Example 5.16.** Let \(G\) be an algebraic group. Given a pair of topological spaces \((X,A)\) such that \(\Pi(X,A)\) is finitely generated and a parabolic structure \(Q\) on it, we define the **piecewise character variety** as the object of \(\text{PVar}_k\)

\[
\mathcal{R}_G(X,A,Q) = [\mathcal{X}_G(X,A,Q)/G].
\]

Here, \(G\) acts on \(\mathcal{X}_G(X,A,Q)\) by conjugation.
The piecewise quotients have similar properties than GIT quotients but in the category \( \textbf{PVar}_k \). More precisely, let \( G \) be a reductive group acting on a variety \( X \) and let \( f : [X] \to [Y] \) be a morphism of piecewise varieties which is \( G \)-invariant. Using the universal property of categorical quotients (see [53]), we have that there exists an unique piecewise morphism \( \tilde{f} : [X/G] \to [Y] \) such that \( \tilde{f} \circ \pi = f \).

\[
\begin{array}{c}
[X] \xrightarrow{f} [Y] \\
\pi \downarrow \quad \tilde{f} \downarrow \\
[X/G]
\end{array}
\]

The important point is that the category of piecewise algebraic varieties is that the \( \textbf{Var}_C \)-algebra \( K \mathcal{M} \) can be extended in a natural way to a \( \textbf{PVar}_C \)-algebra. That will be very useful in Section 5.3.

**Proposition 5.17.** The \( \textbf{Var}_C \)-algebra \( K \mathcal{M} \) extends to a \( \textbf{PVar}_C \)-algebra.

**Proof.** For an object \( [X] = X_1 + \ldots + X_r \in \textbf{PVar}_C \), we define \( K \mathcal{M}[X] = \bigoplus K \mathcal{M}_{X_i} \). This definition does not depend on the chosen representant of \( [X] \). Indeed, if \( X_i = X_i^1 \sqcup X_i^2 \), with \( X_i^1 \subseteq X_i \) closed, and \( i_k : X_i^k \to X_i \) are the inclusions for \( k = 1, 2 \), then, by Proposition 2.10, the map \( (i_1)_! + (i_2)_! : K \mathcal{M}_{X_i^1} \otimes K \mathcal{M}_{X_i^2} \to K \mathcal{M}_{X_i} \) is an isomorphism.

For a morphism \( f : [X] \to [Y] \), given as a set of regular maps \( f_{i,j} : X_{i,j} \to Y_i \), we define

\[
f^* = \bigoplus_i \left[ \bigoplus_j f_{i,j}^* : K \mathcal{M}_{Y_i} \to \bigoplus_j K \mathcal{M}_{X_{i,j}} \right], \quad f_i = \bigoplus_j \left[ \sum_j (f_{i,j})_! : \bigoplus_j K \mathcal{M}_{X_{i,j}} \to K \mathcal{M}_{Y_i} \right].
\]

Again, they do not depend on the chosen representants. In order to prove it, let us suppose that \( f : X \to Y \) is a morphism between algebraic varieties. If \( X = X_1 \sqcup X_2 \), \( i_k : X_k \to X \) are the inclusions and \( f_k = f|_{X_k} : X_k \to X \) are the corresponding restrictions, we have commutative diagrams

\[
\begin{array}{ccc}
K \mathcal{M}_{X_1} \otimes K \mathcal{M}_{X_2} & \xrightarrow{(f_1)_! + (f_2)_!} & K \mathcal{M}_Y \\
(i_1)_! + (i_2)_! \downarrow & & \downarrow f^* \\
K \mathcal{M}_X & & K \mathcal{M}_X
\end{array}
\]

\[
\begin{array}{ccc}
K \mathcal{M}_Y & \xrightarrow{f^*} & K \mathcal{M}_{X_1} \otimes K \mathcal{M}_{X_2} \\
& & (i_1)_! + (i_2)_! \downarrow \\
& & K \mathcal{M}_X
\end{array}
\]

The commutativity of the first diagram is immediate and the commutativity of the second one follows from the fact that \( (i_1)_! f_1^* + (i_2)_! f_2^* = (i_1)_!(i_1)_! f^* + (i_2)_!(i_2)_! f^* = f^* \) since \( (i_1)_! (i_1)_! + (i_2)_! (i_2)_! = 1_{K \mathcal{M}_X} \) by Proposition 2.10. On the other hand, suppose that we have a decomposition \( Y = Y_1 \sqcup Y_2 \) with inclusions \( j_k : Y_k \to Y \) and we set \( X_k = f^{-1}(Y_k) \) with inclusions \( i_k : X_k \to X \). Setting \( f_k = f|_{X_k} : X_k \to Y_k \), we have commutative diagrams

\[
\begin{array}{ccc}
K \mathcal{M}_{X_1} \otimes K \mathcal{M}_{X_2} & \xrightarrow{(f_1)_! \otimes (f_2)_!} & K \mathcal{M}_{Y_1} \otimes K \mathcal{M}_{Y_2} \\
(i_1)_! + (i_2)_! \downarrow & & \downarrow (j_1)_! \otimes (j_2)_! + (j_1)_! \otimes (j_2)_! \\
K \mathcal{M}_X & & K \mathcal{M}_X
\end{array}
\]

\[
\begin{array}{ccc}
K \mathcal{M}_{Y_1} \otimes K \mathcal{M}_{Y_2} & \xrightarrow{(j_1)_! \otimes (j_2)_!} & K \mathcal{M}_{X_1} \otimes K \mathcal{M}_{X_2} \\
& & (i_1)_! + (i_2)_! \downarrow \\
& & K \mathcal{M}_X
\end{array}
\]

The first one follows from the fact that \( [(j_1)_! + (j_2)_!] \circ [(f_1)_! \otimes (f_2)_!] = (j_1 \circ f_1)_! + (j_2 \circ f_2)_! = (f \circ i_1)_! + (f \circ i_2)_! = f_1 \circ [(i_1)_! + (i_2)_!] \). For the second one, observe that \( [(i_1)_! + (i_2)_!] \circ (f^*_1 \otimes f^*_2) = (i_1)_! f^*_1 + (i_2)_! f^*_2 = f^* \circ [(j_1)_! + (j_2)_!] \), where the last equality follows from the Beck-Chevalley
property of $\text{KM}$ and the fact that, for $k = 1, 2$, the square

$$
\begin{array}{ccc}
X_k & f_k & Y_k \\
\downarrow & & \downarrow \\
X & f & Y
\end{array}
$$

is a pullback.

5.3. Geometric and reduced TQFT. Using the reduction method of Section 4.5 in this section we will describe an step further in the standard TQFT of Section 5.1.

From a complex algebraic group $G$, in Section 5.1 we considered the representation variety geometrisation $X_G : \text{Emb}_c(\Lambda) \to \text{Var}_C$. After the functor $\text{Var}_C \to \text{PVar}_C$ of Remark 5.12 we may consider the reduction $\pi$ that assigns, to $(M, A, Q) \in \text{Bdp}_n(\Lambda)$, the piecewise character variety $\mathcal{R}_G(M, A, Q) = [X_G(M, A, Q)/G]$ together with the piecewise quotient morphism $\pi(M, A, Q) : X_G(M, A, Q) \to \mathcal{R}_G(M, A, Q)$. As explained in Section 4.5 this reduction can be used to modify the standard TQFT, $Z_G$ of Proposition 4.20 in order to obtain a very lax TQFT $Z_G = (Z_G)_\pi : \text{Bdp}_n(\Lambda) \to \text{KHS-Mod}_{\text{t}}$.

Using the notation of Section 4.5 suppose that, for all $(M, A, Q) \in \text{Tbp}_n(\Lambda)$, the morphism $\eta(M, A, Q) = (\pi(M, A, Q))_\pi \circ (\pi(M, A, Q))^* : (\mathcal{V}_{(M, A, Q)})_0 \to (\mathcal{V}_{(M, A, Q)})_0$ is invertible as a $\text{KHS}_0$-module homomorphism. In that case, the morphisms $Z_G^{gm}(W, A, Q) = Z_G^{gm}(W, A, Q) \circ \eta(M, A, Q)$ give rise to a strict TQFT, the $\pi$-reduction of $Z_G$ (Proposition 4.20). Here, $(\mathcal{V}_{(M, A, Q)})_0$ stands for the localization $S^{-1}\mathcal{V}_{(M, A, Q)}$ by an appropriate multiplicative system $S \subseteq \text{KHS}$.

**Definition 5.18.** Let $G$ be a complex algebraic group and $\Lambda$ a collection of conjugacy closed subvarieties of $G$. The Topological Quantum Field Theories

$$Z_G^{gm} : \text{Bdp}_n(\Lambda) \to \text{KHS-Mod}_{\text{t}}, \quad Z_G^{gm} : \text{Bdp}_n(\Lambda) \to \text{KHS}_0-\text{Mod}_{\text{t}},$$

are called the geometric TQFT and the reduced TQFT for representation varieties, respectively.

Such a TQFTs will be extensively used along Section 6 since, as we will see, they allow us to compute Hodge structures on representation varieties in a simple way.

**Example 5.19.** In the case of surfaces ($n = 2$), the morphisms of $Z_G^{gm}$ can be explicitly written down from the description of Section 5.1. Consider the set of generators $\Delta = \{D, D^1, L, P\} \cup \{L_\lambda\}_{\lambda \in \Lambda}$ of $\text{Tbp}_2(\Lambda)$ as depicted in Figure 1. Since, by definition, $Z_G^{gm}(W, A, Q) = \pi_1 \circ Z(W, A, Q) \circ \pi^*$ for any bordism $(W, A, Q)$, then we have that

$$Z_G^{gm}(D) = (\pi \circ \iota) : \text{KHS} \to \text{KM}_{[G/G]}, \quad Z_G^{gm}(D^1) = (\pi \circ \iota)^* : \text{KM}_{[G/G]} \to \text{KHS},$$

$$Z_G^{gm}(L) = \hat{\varphi} \circ \hat{p}^* : \text{KM}_{[G/G]} \to \text{KM}_{[G/G]}, \quad Z_G^{gm}(P) = \hat{\varphi} \circ \hat{u}^* : \text{KM}_{[G/G]} \to \text{KM}_{[G/G]},$$

where $\hat{\varphi} = \pi \circ q, \hat{p} = \pi \circ p : G^4 \to [G/G], \hat{\varphi} = \pi \circ l, \hat{u} = \pi \circ t : G^2 \to [G/G]$ and $\hat{r} = \pi \circ r, \hat{s} = \pi \circ s : G^2 \times \lambda \to [G/G]$. Observe that $\pi_0$ is the identity map since $X_G(\emptyset) = \mathcal{R}_G(\emptyset) = 1$ and that $Z_G^{gm}(S^1, *) = X_G(S^1, *) = [G/G]$.

Finally, as mentioned in Example 4.22 the only relevant submodule is $\mathcal{V} = \mathcal{V}_{(S^1, *)} \subseteq \text{KM}_{\mathcal{R}_G(S^1, *)} = \text{KM}_{[G/G]}$, so we can restrict the previous maps to $\mathcal{V}$. Moreover, if the morphism $\eta = \eta(S^1, *) : \mathcal{V}_0 \to \mathcal{V}_0$ is invertible, the reduction $Z_G^{gm}$ can also be constructed.
6. \(\text{SL}_2(\mathbb{C})\)-representation varieties

In this section, we will focus on the case of surfaces \(n = 2\) and \(G = \text{SL}_2(\mathbb{C})\), the complex special linear group of order two which, for short, will be denoted just by \(\text{SL}_2\). As an application of the previous TQFTs, we will compute the image in \(K\)-theory of the Hodge structures of some parabolic representations varieties for arbitrary genus. For that, we will give explicit expressions of the homomorphisms of Section 5.3.

In \(\text{SL}_2\), there are five special types of elements, the matrices

\[
\begin{align*}
\text{Id} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & -\text{Id} &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, & J_+ &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, & J_- &= \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, & D_\lambda &= \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix},
\end{align*}
\]

with \(\lambda \in \mathbb{C} - \{0, \pm 1\}\). Observe that any element of \(\text{SL}_2\) is conjugated to one of those elements. Such a distinguished representant is unique up to the fact that \(D_\lambda\) and \(D_{\lambda^{-1}}\) are conjugated for all \(\lambda \in \mathbb{C} - \{0, \pm 1\}\).

Hence, if we denote the orbit of \(A \in \text{SL}_2\) under conjugation by \([A]\), we have a stratification

\[
\text{SL}_2 = \{\text{Id}\} \sqcup \{-\text{Id}\} \sqcup [J_+] \sqcup [J_-] \sqcup D,
\]

where

\[
D = \bigcup_{\lambda \in \mathbb{C} - \{0, \pm 1\}} [D_\lambda] = \{A \in \text{SL}_2 \mid \text{tr } A \neq 2\}.
\]

The Hodge structures of these strata can be easily computed. Here, we will sketch the computations but, for more details, we refer to [43]. Recall that \([H^*_c(\text{SL}_2)] = q^3 - q\), where we have shorted \(q = [H^*_c(\mathbb{C})] \in \text{KHS}\). In order to check it, observe that the surjective regular map \(\pi : \text{SL}_2 \to \mathbb{C}^2 - \{(0, 0)\}\) given by \(\pi(A) = A(1, 0)\) defines a fibration \(\mathbb{C} \to \text{SL}_2 \to \mathbb{C}^2 - \{(0, 0)\}\).

Moreover, this fibration is locally trivial in the Zariski topology so, in particular, it has trivial monodromy. Hence, by Corollary 3.10 we have \([H^*_c(\text{SL}_2)] = [H^*_c(\mathbb{C})] \cdot [H^*_c(\mathbb{C}^2 - \{(0, 0)\})] = q(q^2 - 1) = q^3 - q\).

For the orbits of the Jordan type elements \(J_{\pm}\), their stabilizers under the conjugacy action are \(\text{Stab} J_{\pm} \cong \mathbb{C}\) so we have a fibration \(\mathbb{C} = \text{Stab} J_{\pm} \to [J_{\pm}] \times \text{SL}_2 \to [J_{\pm}]\). This fibration is locally trivial in the Zariski topology, so we have \([H^*_c([J_{\pm}])] = [H^*_c(\text{SL}_2)]/[H^*_c(\mathbb{C})] = \frac{q^3 - q}{q} = q^2 - 1\). Analogously, for the orbit of \([D_\lambda]\) we have a locally trivial fibration \(\mathbb{C}^* \to D_\lambda \times \text{SL}_2 \to [D_\lambda]\) so \([H^*_c([D_\lambda])] = (q^2 - q)/(q - 1) = q^2 + q\).

For the computation of \([H^*_c(D)]\), the mixed Hodge modules coming from variations of Hodge structures (see Section 3) are the key ingredients. Observe that the trace map \(\text{tr} : D \to \mathbb{C} - \{\pm 2\}\) defines a double cover with non-trivial monodromy. However, it fits in the diagram of fibrations

\[
\begin{diagram}
\text{SL}_2/\mathbb{C}^* & \longrightarrow & (\mathbb{C}^* - \{\pm 1\}) \times \text{SL}_2/\mathbb{C}^* & \longrightarrow & \mathbb{C}^* - \{\pm 1\} \\
& \downarrow \quad \text{Id} & \downarrow \quad t\ 	ext{tr} & \downarrow \quad \text{tr} & \downarrow \quad \text{tr} \\
\text{SL}_2/\mathbb{C}^* & \longrightarrow & D & \longrightarrow & \mathbb{C} - \{\pm 2\}
\end{diagram}
\]

Here, \(t : \mathbb{C}^* - \{\pm 1\} \to \mathbb{C} - \{\pm 2\}\) is the morphism \(t(\lambda) = \lambda + \lambda^{-1}\). The middle vertical arrow is the quotient by the action of \(\mathbb{Z}_2\) on \((\mathbb{C}^* - \{\pm 1\}) \times \text{SL}_2/\mathbb{C}^*\) given, for \(\lambda \in \mathbb{C}^* - \{\pm 1\}\) and \(P \in \text{SL}_2/\mathbb{C}^*\), by \(-1 \cdot (\lambda, P) = (\lambda^{-1}, P_0 P P_0^{-1})\), where \(P_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\). Since the upper fibration
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has trivial monodromy, by the results of Section 3.2, we have that

\[
R(D|\mathbb{C} - \{\pm 2\}) = [H^\bullet_\ast(SL_2/\mathbb{C}^\ast)]^+ R(\mathbb{C} - \{\pm 2\}|\mathbb{C} - \{\pm 2\}) + [H^\bullet_\ast(SL_2/\mathbb{C}^\ast)]^- (R(\mathbb{C}^\ast - \{\pm 1\}|\mathbb{C} - \{\pm 2\}) - R(\mathbb{C} - \{\pm 2\}|\mathbb{C} - \{\pm 2\}))
\]

\[
= q^2T_{\mathbb{C} - \{\pm 2\}} + q(T_{\mathbb{C} - \{\pm 2\}} + S_2 \otimes S_{-2} - T_{\mathbb{C} - \{\pm 2\}}) = q^2T_{\mathbb{C} - \{\pm 2\}} + qS_2 \otimes S_{-2}.
\]

Here, we have used that \(R(\mathbb{C}^\ast - \{\pm 1\}|\mathbb{C} - \{\pm 2\}) = T_{\mathbb{C} - \{\pm 2\}} + S_2 \otimes S_{-2}\) by Example 3.15. Moreover, observe that \((SL_2/\mathbb{C}^\ast)/Z_2\) is isomorphic to the complex plane minus a conic, so \([H^\bullet_\ast(SL_2/\mathbb{C}^\ast)]^+ = q^2\).

Finally, with respect to the action by conjugation, observe that the GIT quotient is given by \(tr : SL_2 \to SL_2/\mathbb{C} = \mathbb{C}\), where \(tr\) is the trace map. Moreover, the decomposition corresponds, stratum by stratum, with the decomposition used in Example 5.14 for the definition of the piecewise quotient. Hence, as \(D/\mathbb{C} = \mathbb{C} - \{\pm 2\}\), we have that the piecewise quotient is

\[
[SL_2/SL_2] = \{\text{Id}\} + \{-\text{Id}\} + \{[J_+]\} + \{[J_-]\} + (\mathbb{C} - \{\pm 2\}).
\]

6.1. The core submodule. Using the description of the geometric TQFT given in Section 5.3, we have obtained that

\[
Z_G^{gm}(S^1, \ast) = KM_{[SL_2/SL_2]} = KM_{\text{Id}} \oplus KM_{-\text{Id}} \oplus KM_{[J_+]} \oplus KM_{[J_-]} \oplus KM_{\mathbb{C} - \{\pm 2\}},
\]

where the first four summands are mixed Hodge modules over a point and, thus, they are naturally isomorphic to KHS. For short, in the following we will denote \(B_i = \mathbb{C} - \{\pm 2\}\) and we will call it the space of traces, as it is the image of the quotient map \(tr : D \to \mathbb{C} - \{\pm 2\}\).

Let \(T_+ \in KM_{\text{Id}}, T_- \in KM_{-\text{Id}}, T_+ \in KM_{J_+}, T_- \in KM_{J_-}\) be the units of these rings. On \(KM_{B_i}\), we consider the unit \(T_{B_i}\) and \(S_2, S_{-2} \in KM_{B_i}\) the one dimensional representations of \(\pi_1(\mathbb{C} - \{\pm 2\})\) that are non-trivial on small loops \(\gamma_2, \gamma_{-2}\) around 2 and \(-2\), respectively, as in Example 3.15.

Definition 6.1. The set \(S = \{T_1, T_-^1, T_+, T_-, T_{B_i}, S_2, S_{-2}, S_2 \otimes S_{-2}\} \subseteq KM_{[SL_2/SL_2]}\) is called the set of core elements. The submodule of \(KM_{[SL_2/SL_2]}\) generated by \(S\), \(W\), is called the core submodule.

Remark 6.2. The submodule \(W\) will be very important in the incoming computations since we will show that \(W\) is the submodule generated by the elements \(Z_{SL_2}^{gm}(L)^9 \circ Z_{SL_2}^{gm}(D)(1)\) for \(g \geq 0\), that is \(W = V_{(S^1, \ast)}\) using the notations of Section 4.5.

Now, consider the quotient map \(tr : SL_2 \to [SL_2/SL_2]\). From this map, we can build the endomorphism \(\eta = tr_{\ast} \circ tr^\ast : KM_{[SL_2/SL_2]} \to KM_{[SL_2/SL_2]}\). This morphism will be useful in the upcoming sections due to its role in Proposition 4.20.

Proposition 6.3. The core submodule \(W\) is invariant for the morphism \(\eta\). Actually, we have

\[
\begin{align*}
\eta(T_{\pm 1}) &= T_{\pm 1} \\
\eta(T_{B_i}) &= q^2T_{B_i} + qS_2 \otimes S_{-2} \\
\eta(S_2) &= q^2S_2 + qS_{-2} \\
\eta(S_{-2}) &= qS_2 + q^2S_{-2}
\end{align*}
\]

\[
\begin{align*}
\eta(T_{\pm}) &= (q^2 - 1)T_{\pm} \\
\eta(S_2 \otimes S_{-2}) &= qT_{B_i} + q^2S_2 \otimes S_{-2}
\end{align*}
\]
Proof. First of all, observe that, if \( i : X \hookrightarrow \text{SL}_2 / \text{SL}_2 \) is the inclusion of a locally closed subset and \( \bar{X} = \text{tr}^{-1}(X) \subseteq \text{SL}_2 \), they fit in a pullback diagram

\[
\begin{array}{c}
\bar{X} \\
\downarrow \text{tr}_i \\
\text{SL}_2 \\
\downarrow \text{tr} \\
[\text{SL}_2 / \text{SL}_2]
\end{array}
\]

Thus, \( \text{tr}_i \circ \text{tr}^* \circ i_! = (\text{tr} | \bar{X})_! \circ (\text{tr} | \bar{X})^* \) that is, we can compute the image of \( \eta \) within \( \bar{X} \). For this reason, since \( \text{tr} |_{{\pm}1} : \{ \pm 1 \} \to \{ \pm 1 \} \) is the identity map, we have \( \eta(T_{\pm 1}) = T_{\pm 1} \). Analogously, since \( (\text{tr} | J_\pm) \) is a ring homomorphism, it sends units into units so, for \( T_\pm \), we have that \( \eta(T_\pm) = (\text{tr} | J_\pm)(Q_\pm J_\pm) T_\pm = [H^*([J_\pm]]) = (q^2 - 1)T_\pm \).

For the first identity of the second row, we have \( (\text{tr} | D) : (\text{tr} | D)^* T_{B_1} = (\text{tr} | D) T_D = R(D) B_1 \) and the result follows from the computations of the previous section. For \( S_2 \), consider the auxiliary variety \( Y = \{(t, y) \in B_2 \times \mathbb{C}^* \mid y^2 = t - 2 \} \), for which \( R(Y|B_1) = T_{B_1} + S_2 \) (see Section 3.1), and the variety \( Y' = \{(A, y) \in D \times \mathbb{C}^* \mid y^2 = \text{tr}(A) - 2 \} \). They fit in a commutative diagram

\[
\begin{array}{c}
Y' \\
\downarrow \\
Y \\
\downarrow \\
B_1 \\
\downarrow \text{tr}_D \\
D \\
\downarrow \text{tr}_D \\
B_1
\end{array}
\]

where the rightmost square is a pullback. Hence, we have that \( \eta(T_{B_1}) + \eta(S_2) = (\text{tr} | D)(\text{tr} | D)^* (T_{B_1} + S_2) = R(Y'|B_1) \) and, thus, \( \eta(S_2) = R(Y'|B_1) - q^2 T_{B_1} - qS_2 \otimes S_{-2} \) by the computation for \( T_{B_1} \).

In order to compute \( R(Y'|B_1) \) observe that the fiber of the morphism \( Y' \to B_1 \) over a point \( t = [D_A] \times \{ \pm \sqrt{t - 2} \} \), which is isomorphic to \( \left( \mathbb{P}^1 \times \mathbb{P}^1 \right) - \Delta \times \{ \pm \sqrt{t - 2} \} \), being \( \Delta \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \) the diagonal. If we compactify the space \( Y' \) fiberwise, we obtain a variety \( \bar{Y}' \) and a morphism \( \bar{Y}' \to B_1 \) with fiber, \( \bar{F} \), equal to two copies of \( \mathbb{P}^1 \times \mathbb{P}^1 \). Therefore, the Hodge structure on the cohomology of \( \bar{F} \) is pure and its unique non-vanishing pieces are \( \mathbb{H}^{0,0}_c(\bar{F}) = \mathbb{Q} \oplus \mathbb{Q}, \mathbb{H}^{2,1,1}_c(\bar{F}) = \mathbb{Q}^2 \oplus \mathbb{Q}^2 \) and \( \mathbb{H}^{2,2}_c(\bar{F}) = \mathbb{Q} \oplus \mathbb{Q} \). Hence, as mixed Hodge structures, \( \mathbb{H}^k_c(F) = 2\mathbb{Q}_0, \mathbb{H}^k_c(F) = 4q \) and \( \mathbb{H}^k_c(F) = 2q^2 \). Working explicitly with the description above, we obtain that the monodromy action of \( \pi_1(B_1) \) on \( \mathbb{H}^k_c(F) \) is given as in the following table.

\[
\begin{array}{c|c|c|c}
& k = 0 & k = 2 & k = 4 \\
\hline
\gamma_2 & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
\hline
\gamma_{-2} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{array}
\]

Therefore, we have \( R\left( \bar{Y}'|B_1 \right) = T_{B_1} + S_2 + q(T_{B_1} + S_2 + S_{-2} + S_2 \otimes S_{-2}) + q^2(T_{B_1} + S_2) \).

The projection of the complement \( \bar{Y}' - Y' \to B_1 \) has fiber \( \Delta \times \{ \pm \sqrt{t - 2} \} \cong \mathbb{P}^1 \times \{ \pm \sqrt{t - 2} \} \). Hence, for the monodromy action we have that \( \gamma_{-2} \) acts trivially and \( \gamma_2 \) interchanges the two
copies of \( \mathbb{P}^1 \), so \( R \left( \overline{Y} - Y' \mid B_i \right) = T_{B_i} + S_2 + q(T_{B_i} + S_2) \). Thus, we obtain that

\[
R \left( Y' \mid B_i \right) = R \left( \overline{Y} \mid B_i \right) - R \left( \overline{Y} - Y' \mid B_i \right) = q(S_{-2} + S_2 \otimes S_{-2}) + q^2(T_{B_i} + S_2).
\]

From this, it follows that \( \eta(S_2) = q^2S_2 + qS_{-2} \), as claimed. The calculations for \( S_{-2} \) and \( S_2 \otimes S_{-2} \) are analogous.

\[ \square \]

The morphism \( \eta : \mathcal{W} \to \mathcal{W} \) is not invertible since, on \( \text{KHS} \), the elements \( q-1 \) and \( q+1 \) have no inverse. We can solve this problem by considering the localization of \( \text{KHS} \) over multiplicative system generated by \( q-1 \) and \( q+1 \), denoted \( \text{KHS}_0 \). Extending the localization to \( \mathcal{W} \), we obtain the \( \text{KHS}_0 \)-module \( \mathcal{W}_0 \). In that module, by the Proposition above, we have an isomorphism \( \eta : \mathcal{W}_0 \to \mathcal{W}_0 \).

Let us consider the parabolic data \( \Lambda = \{1\}, \{-1\}, [J_\pm], [J_-]\}, \) let \( Z_{\text{SL}_2} : \text{Bdp}_n(\Lambda) \to \text{KHS-Mfd}_i \) be the associated standard TQFT of Section 6.1 and let \( Z_{\text{SL}_2}^{gm} : \text{Bdp}_n(\Lambda) \to \text{KHS-Mfd}_i \) be the geometric TQFT of Section 5.3. As a result of the upcoming computations of Sections 6.3 and 6.4, we will show that \( \mathcal{W} = \mathcal{V}_{\text{SL}_2, \star} \). Hence, applying Proposition 4.20, we obtain the following result.

**Theorem 6.4.** Let \( Z_{\text{SL}_2} : \text{Bdp}_n(\Lambda) \to \text{KHS}_0 \text{-Mfd}_i \) be the TQFT for \( SL_2(\mathbb{C}) \)-representation varieties and let \( Z_{\text{SL}_2}^{gm} : \text{Bdp}_n(\Lambda) \to \text{KHS}_0 \text{-Mfd}_i \) be the geometric TQFT. There exists an almost-TQFT, \( Z_{\text{SL}_2}^{gm} : \text{TB}_2(\Lambda) \to \text{KHS}_0 \text{-Mfd}_i \), such that:

- The image of the circle is \( Z_{\text{SL}_2}^{gm}(\mathbb{S}^1, \star) = \mathcal{W} \subset \text{K} \text{M}_{\text{SL}_2/\text{SL}_2} \).
- For an strict tube \( (W, A, Q) : (\mathbb{S}^1, \star) \to (\mathbb{S}^1, \star) \) it assigns the morphism \( Z_{\text{SL}_2}^{gm}(W, A, Q) = Z_{\text{SL}_2}(W, A, Q) \circ \eta^{-1} : \mathcal{W}_0 \to \mathcal{W}_0 \).
- For a closed surface \( W : \emptyset \to \emptyset \), it gives \( Z_{\text{SL}_2}^{gm}(W, A, Q)(1) = [H^*_e(\mathcal{X}_{\text{SL}_2}(W, Q))] \otimes (q^3 - q)^{|\mathcal{A}|^{-1}} \).

6.2. **Discs and first tube.** As a warm lap, in this section we will compute the morphisms \( Z_{\text{SL}_2}^{gm}(L_{-\text{Id}}), Z_{\text{SL}_2}^{gm}(D) \) and \( Z_{\text{SL}_2}^{gm}(D^1) \). For the first case the image under the field theory of \( L_{-\text{Id}} \) is the span

\[
[\text{SL}_2/\text{SL}_2] \xleftarrow{\text{tr} \circ \pi_1} \text{SL}_2 \xrightarrow{-\text{tr} \circ \pi_1} [\text{SL}_2/\text{SL}_2],
\]

where \( \pi_1 : \text{SL}_2 \to \text{SL}_2 \) is the projection onto the first variable. Observe that \( (\pi_1)_! \circ (\pi_1)^* : \text{KM}_{\text{SL}_2} \to \text{KM}_{\text{SL}_2} \) is just multiplication by \( [H^*_e(\mathcal{X}_{\text{SL}_2})] = q^2 - q \). Hence, we have \( Z_{\text{SL}_2}^{gm}(L_{-\text{Id}}) = (-\text{tr})_! \circ (\pi_1)_! \circ (\pi_1)^* \circ \text{tr}^* = (q^3 - q) \sigma_! \circ \eta, \) where \( \sigma : [\text{SL}_2/\text{SL}_2] \to [\text{SL}_2/\text{SL}_2] \) is the reflection that sends \( \sigma([A]) = [-A] \) for \( A \in \text{SL}_2 \). It is straightforward to check that

\[
\sigma_!(T_{\pm 1}) = T_{\mp 1}, \quad \sigma_!(T_{\pm}) = T_{\mp}, \quad \sigma_!(T_{B_i}) = T_{B_i},
\]

\[
\sigma_!(S_{\pm 2}) = S_{\mp 2}, \quad \sigma_!(S_2 \otimes S_{-2}) = S_2 \otimes S_{-2}.
\]

In particular, \( \sigma_!(\mathcal{W}) \subset \mathcal{W} \) so, by Proposition 6.3, we have that \( Z_{\text{SL}_2}^{gm}(L_{-\text{Id}})(\mathcal{W}) \subset \mathcal{W} \). Hence, \( Z_{\text{SL}_2}^{gm}(L_{-\text{Id}}) = Z_{\text{SL}_2}^{gm}(L_{-\text{Id}}) \circ \eta^{-1} = (q^3 - q) \sigma_! \). Thus, the matrix of \( Z_{\text{SL}_2}^{gm}(L_{-\text{Id}}) \) in the generators \( S \) of \( \mathcal{W} \) is

\[
Z_{\text{SL}_2}^{gm}(L_{-\text{Id}})|_V = (q^3 - q) \cdot \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}.
\]
Finally, with respect to the discs $D : \emptyset \to (S^1, \ast)$ and $D^1 : (S^1, \ast) \to \emptyset$ recall that $Z_{SL_2}^{gm}(D) = i_1$, and $Z_{SL_2}^{gm}(D^1) = i_2^*$, where $i_1 : \{Id\} \hookrightarrow [SL_2/SL_2]$ is the inclusion. The image of the first one is very easy to identify since, by definition $Z_{SL_2}^{gm}(D)(1) = i_1(1) = T_1$. For the second one, observe that $i^*T_1 = 1$ but $i^*T_{-1} = i^*T_\pm = i^*T_B = i^*S_\pm = 0$, so $Z_{SL_2}^{gm}(D^1) : \mathcal{W} \to KHS$ is just the projection onto $T_1$. Since $\eta$ fixes $T_1$, we also have $Z_{SL_2}^{gm}(D) = Z_{SL_2}^{gm}(D)$ and $Z_{SL_2}^{gm}(D^1) = Z_{SL_2}^{gm}(D^1)$.

6.3. The tube with a Jordan type marked point. In this section, we will focus on the computation of the image of the tube $L_{[J_+]} : (S^1, \ast) \to (S^1, \ast)$. Again, the field theory for $Z_{SL_2}^{gm}$ on $L_{[J_+]}$ is the span

$$[SL_2/SL_2] \xrightarrow{\hat{r}} SL_2 \times [J_+] \xrightarrow{\hat{s}} [SL_2/SL_2] \quad \text{tr}(A) \xleftarrow{} (A, B, C) \xleftarrow{} \text{tr}(BACB^{-1}) = \text{tr}(AC)$$

so $Z_{SL_2}^{gm}(L_{[J_+]}) = \hat{s} \circ \hat{r}^* : K\mathbb{M}_{[SL_2/SL_2]} \to K\mathbb{M}_{[SL_2/SL_2]}$.

Notice that, if $i : X \hookrightarrow [SL_2/SL_2]$ is an inclusion of a locally closed subset, then we have a commutative diagram

$$\begin{align*}
\hat{r}^{-1}(X) \xrightarrow{\hat{r}^{-1}(X)} X \xrightarrow{i} \quad \text{SL}_2 \times [J_+] \xrightarrow{\hat{r}} [SL_2/SL_2] \xrightarrow{i} \quad \text{SL}_2 \times [J_+] \xrightarrow{\hat{r}} [SL_2/SL_2]
\end{align*}$$

Hence, for any $M \in K\mathbb{M}_X$, $\hat{s} \circ \hat{r}^* \circ i_* M = \hat{s} \circ (\hat{r}^{-1}(X))^* M$. Moreover, if we decompose $[SL_2/SL_2] = \sum_k Y_k$ with inclusions $j_k : Y_k \hookrightarrow [SL_2/SL_2]$, observe that, by the proof of Proposition 2.10, we have $\sum_k (j_k)_!(jk)_* = 1$ so $\hat{s} = \sum_k (j_k)_!(jk)_* \hat{s} = \sum_k (j_k)_!(\hat{s} \circ \hat{r}^{-1}(X))$. Therefore,

$$Z_{SL_2}^{gm}(L_{[J_+]})(i_* M) = \hat{s} \circ \hat{r}^* \circ i_* M = \sum_k (j_k)_!(\hat{s} \circ \hat{r}^{-1}(X) \circ i_*(\hat{s} \circ \hat{r}^{-1}(X))) = \sum_k (j_k)_!(\hat{s} \circ \hat{r}^{-1}(X) \circ i_*(\hat{s} \circ \hat{r}^{-1}(X)))^* M$$

Notice that we can decompose $\hat{r}^{-1}(X) \cap \hat{s}^{-1}(Y) = \{ (A, B, C) \in \hat{r}^{-1}(X) \times [J_+] | \text{tr}(AC) \in Y \}$

$$\cong \{ A \in \hat{r}^{-1}(X) | \text{tr}(AJ_+) \in Y \} \times (SL_2/C) \times SL_2,$$

where the last isomorphism is given by sending $(A, P, B) \in \hat{r}^{-1}(X) \times (SL_2/C) \times SL_2$, in the variety downstairs, to $(PAP^{-1}, B, PJ_+P^{-1})$, in the variety upstairs. For short, we will denote $Z_{X,Y} = \hat{r}^{-1}(X) \cap \hat{s}^{-1}(Y)$ and $Z^0_{X,Y} = \{ A \in \hat{r}^{-1}(X) | \text{tr}(AJ_+) \in Y \}$. Using as stratification the decomposition $[SL_2/SL_2] = \{ \pm 1 \} + \{ [J_+] \} + B_1$, we can compute:

- For $T_1$, we have $\hat{r}^{-1}(\text{Id}) = Z_{[J_+] Id} = [J_+] \times SL_2$ and $\hat{s}$ is just the projection onto a point $\hat{s} : [J_+] \times SL_2 \to \{ [J_+] \}$. Hence, $Z_{SL_2}^{gm}(L_{[J_+]})(T_1) = (q^2 - 1)(q^3 - q)T_+.$
- For $T_{-1}$, since $[\overline{J_+}] = [J_-]$, we have $\hat{r}^{-1}(-\text{Id}) = Z_{[J_+] \text{Id}} = [J_-] \times SL_2$. Again, $\hat{s}$ is just the projection $[J_-] \times SL_2 \to \{ [J_-] \}$ so $Z_{SL_2}^{gm}(L_{[J_+]})(T_{-1}) = (q^2 - 1)(q^3 - q)T_-$.
- For $T_+$, the situation becomes more involved since we have a non-trivial decomposition of $\hat{r}^{-1}(\{ J_+ \})$. We analyze each stratum separately:
  - For $\text{Id}$, we have $Z_{[J_+] \text{Id}} = [J_+] \times SL_2$ since all the elements of $Z_{[J_+] \text{Id}}$ have the form $(A^{-1}, B, A)$. Thus, $\hat{s} : [J_+] \times SL_2 \to \{ \text{Id} \}$ is the projection onto a point so $Z_{SL_2}^{gm}(L_{[J_+]})(T_+ \text{Id}) = (q^2 - 1)(q^3 - q)T_+$.
  - For $-\text{Id}$, observe that $Z_{[J_+], -\text{Id}} = \emptyset$ so $Z_{SL_2}^{gm}(L_{[J_+]})(T_- \text{Id}) = Z_{SL_2}^{gm}(L_{[J_+]})(T_- \text{Id}) = 0$.
  - For $[J_+]$, a straightforward check shows that

$$Z^0_{[J_+], [J_+]} = \begin{pmatrix}
1 & b \\
0 & 1
\end{pmatrix} \quad b \in \mathbb{C} - \{ 0, -1 \}.$$
Hence, $Z_{[J_+],[J_-]} = (C - \{0, -1\}) \times (\text{SL}_2 / \mathbb{C}) \times \text{SL}_2$ and $\hat{s}$ is a projection onto the singleton $\{[J_+]\}$. Thus, $Z_{\text{SL}_2}^g(L_{[J_+]})(T_+)|_{[J_+]} = (q - 2)(q^2 - 1)(q^3 - q)T_+$.

- For $[J_-]$, we have that

$$Z^0_{[J_+],[J_-]} = \left\{ \begin{pmatrix} 1 + a & a^2 \\ -4 & 1 - a \end{pmatrix} \bigg| a \in \mathbb{C} \right\}. \quad \text{(1)}$$

Therefore, $Z_{[J_+],[J_-]} = \mathbb{C} \times (\text{SL}_2 / \mathbb{C}) \times \text{SL}_2$ so $Z_{\text{SL}_2}^g(L_{[J_+]})(T_+)|_{[J_-]} = q(q^2 - 1)(q^3 - q)T_-$. 

- For $B_t$, we have that

$$Z^0_{[J_+],B_t} = \left\{ \begin{pmatrix} 1 + a & -a^2 \\ c & 1 - a \end{pmatrix} \bigg| a \in \mathbb{C}, c \in \mathbb{C^*} \{-4\} \right\}, \quad \text{(2)}$$

with the projection $Z^0_{[J_+],B_t} \to B_t$ given by $A \mapsto c + 2$. Hence $\hat{s} : Z_{[J_+],B_t} = Z^0_{[J_+],B_t} \times (\text{SL}_2 / \mathbb{C}) \times \text{SL}_2 \to B_t$ applies $\hat{s}(A, B, P) = c + 2$. In particular, $\hat{s}$ has trivial monodromy there, so $Z_{\text{SL}_2}^g(L_{[J_+]})(T_+)|_{B_t} = R\left( Z_{[J_+],B_t} \bigg| B_t \right) = q(q^2 - 1)(q^3 - q)T_{B_t}$. 

Summarizing, we have

$$Z_{\text{SL}_2}^g(L_{[J_+]})(T_+) = (q^3 - q)(q^3 - q) \left[ T_1 + (q - 2)T_+ + qT_- + qT_{B_t} \right].$$

- For $T_-$, the calculation is very similar to the one of $T_+$, since $Z^0_{[J_-],Y} = Z^0_{[J_+],-Y}$ for every stratum $Y$. Hence, we have:

  - For $\text{Id}$, $Z^0_{[J_+],\text{Id}} = \emptyset$, so $Z_{\text{SL}_2}^g(L_{[J_+]})(T_-)|_{\text{Id}} = 0$. Analogously, for $-\text{Id}$ we have $Z_{[J_-],-\text{Id}} = -[J_+] \times \text{SL}_2$, so $Z_{\text{SL}_2}^g(L_{[J_+]})(T_-)|_{-\text{Id}} = (q^2 - 1)(q^3 - q)T_-$. 

  - For $[J_+]$, $Z^0_{[J_+],[J_+]} = Z^0_{[J_+],[J_+]} \cong \mathbb{C}$, so $Z_{\text{SL}_2}^g(L_{[J_+]})(T_-)|_{[J_+]} = q(q^2 - 1)(q^3 - q)T_+$. 

  - For $[J_-]$, we have $Z^0_{[J_-],[J_-]} = Z^0_{[J_+],[J_+]} \cong \mathbb{C} - \{0, -1\}$. Hence, $Z_{\text{SL}_2}^g(L_{[J_+]})(T_-)|_{[J_-]} = (q^2 - 1)(q^3 - q)T_-$. 

  - For $B_t$, since $-B_t = B_t$, we have $Z^0_{[J_+],B_t} = Z^0_{[J_+],-B_t} \cong \mathbb{C} \times (\mathbb{C^*} \{-4\})$. Hence, $\hat{s}$ also has trivial monodromy on $Z_{[J_+],B_t}$, so $Z_{\text{SL}_2}^g(L_{[J_+]})(T_-)|_{B_t} = q(q^2 - 1)(q^3 - q)T_{B_t}$. 

Summarizing, this calculation says that

$$Z_{\text{SL}_2}^g(L_{[J_+]})(T_-) = (q^2 - 1)(q^3 - q) \left[ T_- + qT_+ + (q - 2)T_- + qT_{B_t} \right].$$

- For $T_{B_t}$, observe that we have $Z_{B_t,\pm \text{Id}} = \emptyset$ so $Z_{\text{SL}_2}^g(L_{[J_+]})(T_{B_t})$ has no components on $\pm \text{Id}$. In this section, we will compute the image of $Z_{\text{SL}_2}^g(L_{[J_+]})(T_{B_t})$ on $[J_\pm]$. Its image on $B_t$ is much harder and it will be described later. For $[J_\pm]$ observe that $Z^0_{B_t,[J_\pm]} \cong \mathbb{C} \times (\mathbb{C^*} \{-4\})$. Hence, $Z_{\text{SL}_2}^g(L_{[J_+]})(T_{B_t})|_{[J_\pm]} = q(q^2 - 1)(q^3 - q)T_{B_t}$. 

- For $S_2$, $Z_{\text{SL}_2}^g(L_{[J_+]})(S_2)$ has no components on $\pm \text{Id}$. Again, we will only focus on its image on $[J_\pm]$. We consider the auxiliary variety

$$Y_2 = \left\{ (t, y) \in B_t \times \mathbb{C}^* \mid y^2 = t - 2 \right\}$$

with nice fibration $Y_2 \to B_t$ given by $(t, y) \mapsto t$ whose Hodge monodromy representation is $R(Y_2|B_t) = T_{B_t} + S_2$. In that case, we have a commutative diagram

$$\begin{array}{ccc}
\hat{Y}_2 & \longrightarrow & Y_2 \\
\downarrow & & \downarrow \\
[J_+] & \xrightarrow{\hat{s}} & Z_{B_t,[J_+]} \\
\downarrow & & \downarrow \\
B_t & \xrightarrow{\hat{p}} & B_t
\end{array}$$

where $\hat{Y}_2$ is the pullback of $Z_{B_t,[J_+]}$ and $Y_2$ over $B_t$. Computing directly, we obtain that $\hat{Y}_2 \cong Y_2 \times \mathbb{C} \times (\text{SL}_2 / \mathbb{C}) \times \text{SL}_2$. Hence, $R\left( \hat{Y}_2 \bigg| [J_+] \right) = q(q^2 - 1)(q^3 - q) [H^*_c(Y_2)] T_+$.
\[ q(q^2 - 1)(q^3 - q)(q - 3)T_+ \]. Since
\[ R \left( \tilde{Y}_2 \right| _{|J_+|} = \tilde{s} \tilde{r}^* R \left( Y_2 \right| _{|J_+|} B_t \right) = Z_{SL_2}^{gm} \left( L_{(J_+)} \right) \left( T_{B_t} \right) \left| _{|J_+|} \right) + \tilde{s} \tilde{r}^* S_2, \]
we obtain that \( Z_{SL_2}^{gm} \left( L_{(J_+)} \right) \left( S_2 \right) \left| _{|J_+|} \right) = \tilde{s} \tilde{r}^* S_2 = -q(q^2 - 1)(q^3 - q)T_+ \). Furthermore, as \( Z_{B_t, \left| J_+ \right|} \cong Z_{B_t, \left| J_+ \right|} \), we also have \( Z_{SL_2}^{gm} \left( L_{(J_+)} \right) \left( S_2 \right) \left| _{|J_+|} \right) = -q(q^2 - 1)(q^3 - q)T_- \).

- For \( S_{-2} \) the calculation is analogous to the one of \( Z_{SL_2}^{gm} \left( L_{(J_+)} \right) \left( S_2 \right) \left| _{|J_+|} \right) = -q(q^2 - 1)(q^3 - q)T_+ \) and \( Z_{SL_2}^{gm} \left( L_{(J_+)} \right) \left( S_{-2} \right) \left| _{|J_+|} \right) = -q(q^2 - 1)(q^3 - q)T_- \). The same holds for \( S_2 \otimes S_{-2} \) so \( Z_{SL_2}^{gm} \left( L_{(J_+)} \right) \left( S_2 \otimes S_{-2} \right) \left| _{|J_+|} \right) = -q(q^2 - 1)(q^3 - q)T_+ \) and \( Z_{SL_2}^{gm} \left( L_{(J_+)} \right) \left( S_2 \otimes S_{-2} \right) \left| _{|J_+|} \right) = -q(q^2 - 1)(q^3 - q)T_- \).

The components in \( B_t \). The calculation of the components of the mixed Hodge modules \( T_{B_t}, S_2, S_{-2} \) and \( S_2 \otimes S_{-2} \) on \( B_t \) is harder than the previous ones and it requires a more subtle analysis. For this reason, we compute it separately. First of all, observe that
\[
Z_{B_t, B_t}^0 = \left\{ \begin{pmatrix} \frac{t}{2} + a & b \\ t' - t & \frac{t}{2} - a \end{pmatrix} \in \mathbb{C}^* \times \mathbb{C} \times \mathbb{C}^* \right\}
\]

The projections onto each copy of \( B_t \) are \( \hat{r}(a, b, t, t') = t \) and \( \hat{s}(a, b, t, t') = t' \). We decompose \( Z_{B_t, B_t}^0 = Z_1 \sqcup Z_2 \) where \( Z_1 = Z_{B_t, B_t}^0 \cap \{ t = t' \} \) and \( Z_2 = Z_{B_t, B_t}^0 \cap \{ t \neq t' \} \).

For \( Z_1 \), we have the explicit expression \( Z_1 = \left\{ 4a^2 = (t + 2)(t - 2) \right\} \times \mathbb{C} \). Hence, by the same argument that in Section 3.1 with respect to the projection \( (t, a, b) \mapsto t \) we have \( R \left( Z_1 \right) B_t = q(T_{B_t} + S_2 \otimes S_{-2}) \). Thus, the contribution of \( Z_1 \) to \( Z_{SL_2}^{gm} \left( L_{(J_+)} \right) \left( T_{B_t} \right) \left| _{B_t} \right) = q(q^2 - 1)(q^3 - q)(T_{B_t} + S_2 \otimes S_{-2}) \).

With respect to \( S_2 \), we again consider \( Y_2 = \{ y^2 = t - 2 \} \), for which we have a commutative diagram
\[
\begin{array}{c}
\hat{Y}_2 \downarrow \downarrow \downarrow \\
\downarrow \downarrow \downarrow \\
\tilde{Z}_2 \downarrow \downarrow \downarrow \\
B_t \leftarrow \left| J_+ \right| \rightarrow B_t
\end{array}
\]

where \( \hat{Y}_2 \) is the pullback of \( Z_1 \) and \( Y_2 \) over \( B_t \). Computing directly
\[
\hat{Y}_2 = \left\{ (t, a, y) \in B_t \times \mathbb{C}^* \times \mathbb{C} \times \mathbb{C}^* \right\}
\]
with projection \( \hat{Y}_2 \rightarrow B_t \) given by \( (t, a, y, b) \mapsto t \). The fiber over \( t \in B_t \) is the set of four lines \( F = \left\{ \left( t, \pm \frac{1}{2} \sqrt{(t + 2)(t - 2)}, \pm \sqrt{t - 2} \right) \right\} \times \mathbb{C} \) so the monodromy action of \( \pi_1(B_t) \) on \( H^2_\mathbb{Z}(F) = \mathbb{Q}/(-1)^4 = 4q \) is given by
\[
\gamma_2 \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_{-2} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\]

Therefore, \( R \left( \hat{Y}_2 \right) B_t = q(T_{B_t} + S_2 + S_{-2} + S_2 \otimes S_{-2}) \). Since \( R \left( \hat{Y}_2 \right) B_t \) = \( \tilde{s} \tilde{r}^* R \left( Y_2 \right) B_t \) = \( \tilde{s} \tilde{r}^* (T_{B_t} + S_2) = R \left( Z_1 \right) B_t + \tilde{s} \tilde{r}^* S_2 \), we finally obtain that \( \tilde{s} \tilde{r}^* S_2 = q(S_2 + S_{-2}) \). Therefore, the contribution of \( Z_1 \) to \( Z_{SL_2}^{gm} \left( L_{(J_+)} \right) \left( S_2 \right) \left| _{B_t} \right) = q(q^2 - 1)(q^3 - q)(S_2 + S_{-2}) \). By symmetry, the contribution to \( Z_{SL_2}^{gm} \left( L_{(J_+)} \right) \left( S_{-2} \right) \left| _{B_t} \right) = q(q^2 - 1)(q^3 - q)(S_2 + S_{-2}) \). An analogous computation for \( S_2 \otimes S_{-2} \) shows that the contribution of \( Z_1 \) to \( Z_{SL_2}^{gm} \left( L_{(J_+)} \right) \left( S_2 \otimes S_{-2} \right) \left| _{B_t} \right) = q(q^2 - 1)(q^3 - q)(T_{B_t} + S_2 \otimes S_{-2}) \).
Therefore, we have the explicit expression

\[
Z_2 = \left\{ \left( \frac{t}{2} + a, \frac{t^2 - 4a^2 - 4}{4(t - t') - 2}  \right) \mid t, t' \in B_t, t' \neq t, a \in \mathbb{C} \right\} \cong \mathbb{C} \times \left[ (\mathbb{C} - \{\pm 2\})^2 - \Delta \right],
\]

where \( \Delta \subseteq (\mathbb{C} - \{\pm 2\})^2 \) is the diagonal. Here, the projections are \( \hat{r}(a, t, t') = t \) and \( \hat{s}(a, t, t') = t' \).

In this way, the second factor of this decomposition of \( Z_2 \) is as depicted in Figure 2. In particular, observe that, under the projection \( \hat{s} \), the monodromy is trivial so \( R(Z_2|B_t) = q(q - 3)T_{B_t} \).

Hence, as \( Z_{B_t,B_t} = Z_{20} \times SL_2/\mathbb{C} \times SL_2 \), the contribution of \( Z_2 \) to \( Z_{SL_2}^{gm}(L_{[1]})(T_{B_t})|_{B_t} \) is \( q(q - 3)(q^2 - 1)(q^3 - q)T_{B_t} \).

\[
\text{Figure 2.}
\]

For \( S_2 \), again consider the auxiliar variety \( Y_2 = \{ y^2 = t - 2 \} \) for \( t \neq \pm 2 \). In that case, the pullback of \( Y_2 \) and \( Z_2 \) over \( B_t \) is

\[
\hat{Y}_2 = \mathbb{C} \times \left\{ (t, t', y) \in [ (\mathbb{C} - \{\pm 2\})^2 - \Delta ] \times \mathbb{C}^* \mid y^2 = t - 2 \right\}.
\]

Denote \( X_2 = \left\{ (t, t', y) \in \left[ (\mathbb{C} - \{\pm 2\})^2 - \Delta \right] \times \mathbb{C}^* \mid y^2 = t - 2 \right\} \). Under the projection \( \hat{s} \), the fiber of \( X_2 \), \( F \), is a parabola with five points removed. Compactifying the fibers of \( X_2 \) we obtain a variety \( \bar{X}_2 \) whose fiber \( \overline{F} \) is \( \mathbb{P}^1 \). By Remark 3.11 (see also Remark 2.5 of [43]), a nice fibration whose fiber is \( \mathbb{P}^1 \) has trivial monodromy. Therefore, \( R(\bar{X}_2|B_t) = (q + 1)T_{B_t} \).

Now, consider the difference \( \bar{X}_2 - X_2 \), whose fiber over \( t' \in B_t \) is the set of six points \( \overline{F} - F = \{(2, t', 0), (-2, t', \pm 2t), (t', t', \pm \sqrt{t' - 2}), \infty \} \), where \( \infty \) denotes the (unique) point at infinity of the parabola over \( t' \). From this expression, we obtain that the monodromy action of \( \pi_1(B_t) \) on the fiber of the covering \( \bar{X}_2 - X_2 \) is given by

\[
\gamma_2 \mapsto \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \sim \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
\gamma_{-2} \mapsto \text{Id}
\]

Therefore, \( R(\bar{X}_2 - X_2|B_t) = 5T_{B_t} + S_2 \) so \( R(X_2|B_t) = R(\bar{X}_2|B_t) - R(\bar{X}_2 - X_2|B_t) = (q + 1)T_{B_t} - 5T_{B_t} - S_2 = (q - 4)T_{B_t} - S_2 \). Hence, since \( \hat{Y}_2 = \mathbb{C} \times X_2 \), we finally obtain \( R(\hat{Y}_2|B_t) = qR(X_2|B_t) = q(q - 4)T_{B_t} - S_2 \). With this information at hand, for the restriction to \( Z_2 \) we have

\[
R(\hat{Y}_2|B_t) = (\hat{s}|Z_2)_!(\hat{r}|Z_2)^* R(Y_2|B_t) = (\hat{s}|Z_2)_!(\hat{r}|Z_2)^* (T_{B_t} + S_2) = R(Z_2|B_t) + (\hat{s}|Z_2)_!(\hat{r}|Z_2)^* S_2.
\]

Therefore, the contribution of \( Z_2 \) to \( Z_{SL_2}^{gm}(L_{[1]})(S_2)|_{B_t} \) is \( (q^2 - 1)(q^3 - q)(\hat{s}|Z_2)_!(\hat{r}|Z_2)^* S_2 = (q^2 - 1)(q^3 - q) [-qT_{B_t} - qS_2] \). For \( S_{-2} \), the calculation is absolutely analogous by considering \( Y_{-2} = \{ y^2 = t + 2 \} \), so we obtain that the contribution of \( Z_2 \) to \( Z_{SL_2}^{gm}(L_{[1]})(S_{-2})|_{B_t} \) is \( (q^2 - 1)

Therefore, we have shown that the core submodule, \( W \), is invariant under the morphism \( Z^\text{gm}_{\text{SL}_2}(L_{[J^+]} : \text{KLM}_{\text{SL}_2/\text{SL}_2} \to \text{KLM}_{\text{SL}_2/\text{SL}_2} \). Hence, multiplying by \( \eta^{-1} \) on the right and using the previous expression, we obtain the following theorem.

**Theorem 6.5.** In the set of generators \( S \), the matrix of \( Z^\text{gm}_{\text{SL}_2}(L_{[J^+]} : W_0 \to W_0 \) is

\[
Z^\text{gm}_{\text{SL}_2}(L_{[J^+]} = (q^3 - q) \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ q^2 - 1 & 0 & q - 2 & q (q - 1)^2 & -q + 1 & -q + 1 & -2q + 2 \\ 0 & q^2 - 1 & q & q - 2 & (q - 1)^2 & -q + 1 & -q + 1 & -2q + 2 \\ 0 & 0 & q & q & q^2 - 2q & -q + 1 & -q + 1 & -q + 2 \\ 0 & 0 & 0 & 0 & 0 & -1 & q & 0 \\ 0 & 0 & 0 & 0 & q & -1 & 0 \\ 0 & 0 & 0 & 0 & q & 0 & 0 & -1 \end{pmatrix}
\]

From this computation, it is relatively easy to calculate the image of the tube \( L_{[J^+]} : (S^1, \ast) \to (S^1, \ast) \). The point is that, since \([J^+] = -[J^+] \), the span associated to \( L_{[J^+]} \) fits in a commutative diagram

\[
\begin{array}{ccc}
\text{SL}_2 & \longrightarrow & \text{SL}_2 \\
\text{SL}_2 & \overset{\varphi}{\longrightarrow} & \text{SL}_2 \\
\downarrow & & \downarrow \\
\text{SL}_2 \times [J^-] & \longrightarrow & \text{SL}_2 \times [J^+] \\
\end{array}
\]

where \( \varphi(A, B, C) = (A, B, -C) \) and \( \varsigma : \text{SL}_2 \to \text{SL}_2 \) is given by \( \varsigma(A) = -A \). Hence, as \( \varphi \) is an isomorphism we have \( \varphi \circ \varphi^* = 1 \), so \( Z^\text{gm}_{\text{SL}_2}(L_{[J^+]} = \varsigma \circ s_1 \circ r^* \). This implies that \( Z^\text{gm}_{\text{SL}_2}(L_{[J^-]} = s_1 \circ Z^\text{gm}_{\text{SL}_2}(L_{[J^+]}), \) where \( \sigma : \text{SL}_2/\text{SL}_2 \to [\text{SL}_2/\text{SL}_2] \) is the factorization through the quotient of \( \varsigma \). Hence, using the description of \( \sigma \) of Section 6.2, we obtain that the matrix of \( Z^\text{gm}_{\text{SL}_2}(L_{[J^-]} \) in \( \mathcal{W} \) is

\[
Z^\text{gm}_{\text{SL}_2}(L_{[J^-]} = (q^3 - q) \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ q^2 - 1 & 0 & q - 2 & q^2 - 2q + 1 & -q + 1 & -q + 1 & -2q + 2 \\ 0 & q^2 - 1 & q & q - 2 & q^2 - 2q + 1 & -q + 1 & -q + 1 & -2q + 2 \\ 0 & 0 & q & q & q^2 - 2q & -q + 1 & -q + 1 & -q + 2 \\ 0 & 0 & 0 & 0 & 0 & q & -1 & 0 \\ 0 & 0 & 0 & 0 & q & -1 & q & 0 \\ 0 & 0 & 0 & 0 & q & 0 & 0 & -1 \end{pmatrix}
\]
6.4. The genus tube. In this section, we discuss the case of the holed torus tube \( L : (S^1, *) \to (S^1, *) \). As described there, the standard TQFT induces a morphism \( Z_{SL_2}(L) : \text{KMod}_{SL_2} \to \text{KMod}_{SL_2} \). This morphism has been implicitly studied in the articles of [43] and [48] (see also [47]). However, there, the approach was focused on computing Hodge monodromy representations of representation varieties since the TQFT formalism was not known at that time. Despite of that, as we saw in Section 3, Hodge monodromy representation has a direct interpretation in terms of \( K \)-theory classes of mixed Hodge modules.

Using this interpretation, in this section we will show that the computations of [48] is nothing but the calculation of the geometric TQFT as above, with some peculiarities. For a while, let us consider the general case of an arbitrary (reductive) group \( G \). As explained in [5.1] \( Z_G(L) = q \circ p^* \) with \( p : G^4 \to G \) given by \( p(a, g, h, b) = a \) and \( q : G^4 \to G \) given by \( q(a, g, h, b) = ba[g, h]b^{-1} \) (beware of the change of notation for the arguments). Hence, \( Z_G(L)^q = (q \circ p^*)^q \). In order to compute it explicitly, recall that we have a pullback diagram

\[
\begin{array}{ccc}
G^7 & \longrightarrow & G^4 \\
\downarrow & & \downarrow q \\
G^4 & \longrightarrow & G \\
\downarrow p & & \\
G & & \\
\end{array}
\]

where the upper arrow is \((a, g, h, g', h', b, b') \mapsto (b^{-1}ab, b^{-1}gb, b^{-1}hb, b)\) and the leftmost arrow is \((a, g, h, g', h', b, b') \mapsto (a[g, h], g', h', b')\). Repeating the procedure, we have that \( Z_G(L)^q = (\beta_g \circ (\alpha_g)^*) \) where \( \alpha_g, \beta_g : G^{3g+1} \to G \) are given by

\[
\alpha_g(a, g_1, h_1, g_2, \ldots, g_{g}, h_{g}, b_1, \ldots, b_{g}) = b_g \cdots b_1ab_1^{-1} \cdots b_g^{-1},
\]

\[
\beta_g(a, g_1, h_1, g_2, \ldots, g_{g}, h_{g}, b_1, \ldots, b_{g}) = a \prod_{i=1}^{g} [g_i, h_i].
\]

In particular, \( Z_G(L)^q \circ Z_G(D)(1) = R \left( G^{3g} | G \right) \) under the projection \( \gamma_g(g_1, h_1, \ldots, g_g, h_g, b_1, \ldots, b_g) = \prod_{i \geq 1} [g_i, h_i] \).

Let us come back to the case \( G = SL_2(\mathbb{C}) \). The aim of the paper [48], as initated in [43], is to compute, for \( g \geq 0 \), the Hodge monodromy \( R(Y_g/\mathbb{Z}_2 \mid B_1) \) where

\[
Y_g = \left\{ (A_1, B_1, \ldots, A_g, B_g, \lambda) \in \text{SL}_2^{2g} \times (\mathbb{C}^* - \{\pm 1\}) \ \bigg| \ \prod_{i=1}^{g} [A_i, B_i] = D_{\lambda} \right\},
\]

and the action of \( \mathbb{Z}_2 \) on \( Y_g \) is \(-1 \cdot (A_i, B_i, \lambda) = (P_0A_iP_0^{-1}, P_0B_iP_0^{-1}, \lambda^{-1})\), where \( P_0 \in \text{SL}_2 \) is the matrix of Section 3 and the projection \( w_g : Y_g/\mathbb{Z}_2 \to B_1 \) is \( w_g(A_i, B_i, \lambda) = \lambda + \lambda^{-1} \). In [48], it is proven that \( R(Y_g/\mathbb{Z}_2 | B_1) \subset W \mid B_1 \subseteq \text{KMod}_{[SL_2/SL_2]} \) for all \( g \geq 0 \). Here, \( W \mid B_1 \) denotes the submodule generated by \( T_{B_1}, S_{\pm 2} \) and \( S_2 \otimes \bar{S}_2 \). Moreover, they proved that there exists a module homomorphism \( M : W \mid B_1 \to W \mid B_1 \) such that \( R(Y_g/\mathbb{Z}_2 | B_1) = M(R(Y_{g-1}/\mathbb{Z}_2 | B_1)) \), for all \( g \geq 1 \).

Remark 6.6. In the paper [48], the variety \( Y_g \) is denoted by \( \overline{X}_g \). However, we will not use this notation since it is confusing with ours.

Lemma 6.7. Let \( X_g = \left\{ (A_1, B_1) \in \text{SL}_2^{2g} \mid \prod_{i=1}^{g} [A_i, B_i] \in D \right\} \) and consider \( \omega_g : X_g \to D \) given by \( \omega_g(A_i, B_i) = \prod_{i=1}^{g} [A_i, B_i] \). Then, \( \text{tr}^* R(Y_g | B_1) = R(X_g | D) \).

Proof. Let us denote by \( \bar{Y}_g/\bar{Z}_2 \subseteq Y_g/Z_2 \times D \) the pullback of \( w_g : Y_g/Z_2 \to B_t \) under \( \tr : D \to B_t \), so \( \tr^* (Y_g/Z_2 \vert B_t) = R \left( \frac{Y_g}{Z_2} \right) \). Now, consider the auxiliary varieties

\[
\bar{Y}_g/\bar{Z}_2' = \left\{ (A_i, B_i, \lambda, Q) \in (Y_g \times \SL_2) / Z_2 \mid \prod_{i=1}^g [A_i, B_i] = D_\lambda \right\},
\]

\[
X'_g = \left\{ (A_i, B_i, Q) \in X_g \times \SL_2 / Z_2 \mid Q^{-1} \prod_{i=1}^g [A_i, B_i]Q = D_\lambda \right\},
\]

where \( Z_2 \) acts on \( \SL_2 \) by \(-1 \cdot Q = Q P_0\). These varieties have morphisms \( \bar{Y}_g/\bar{Z}_2' \to \bar{Y}_g/\bar{Z}_2 \) and \( X'_g \to X_g \) given by \((A_i, B_i, \lambda, Q) \mapsto ((A_i, B_i, \lambda), Q D_\lambda Q^{-1})\) and \((A_i, B_i, Q) \mapsto (A_i, B_i)\), both with trivial monodromy and fiber \( \mathbb{C}^* \).

Moreover, we have an isomorphism \( \varphi : \bar{Y}_g/\bar{Z}_2 \to X'_g \) given by \( \varphi(A_i, B_i, \lambda, Q) = (Q A_i Q^{-1}, Q B_i Q^{-1}, Q) \), which fits in a commutative diagram with fibrations as rows

\[
\begin{array}{ccc}
\mathbb{C}^* & \longrightarrow & \bar{Y}_g/\bar{Z}_2' \\
\rotatebox{90}{\varphi} & & \rotatebox{90}{\varphi} \\
\mathbb{C}^* & \longrightarrow & X'_g \quad \longrightarrow \quad X_g
\end{array}
\]

Therefore, since isomorphisms do not change Hodge monodromy representations, we have

\[
\tr^* R (Y_g/Z_2 \vert B_t) = R \left( \frac{Y_g}{Z_2} \right) = \frac{1}{q-1} R \left( R \left( \frac{Y_g}{Z_2} \right) \right) = \frac{1}{q-1} R \left( X_g \right) = R \left( X_g \right),
\]

as we wanted to prove.

Corollary 6.8. For all \( g \geq 0 \), we have \( Z_{\SL_2}^{gm}(L) (R (Y_{g-1}/Z_2 \vert B_t)) \vert B_t = (q^3 - q) \eta (R (Y_g/Z_2 \vert B_t)) \).

Proof. If we compute the right hand side, we have \( \eta (R (Y_g/Z_2 \vert B_t)) = \tr_1 \circ \tr^* (R (Y_g/Z_2 \vert B_t)) = R (X_g \vert B_t) \), by the previous Lemma, where the projection is \( \tr \circ \omega_g : X_g \to B_t \).

On the other hand, the left hand side can be rewritten as \( Z_{\SL_2}^{gm}(L) (R (Y_{g-1}/Z_2 \vert B_t)) \vert B_t = \tr_1 \circ Z_{\SL_2}(L) \circ \tr^* (R (Y_{g-1}/Z_2 \vert B_t)) \vert B_t = \tr_1 \circ Z_{\SL_2}(L) (R (X_{g-1} \vert D)) \vert B_t \). Observe that \( \gamma^{-1}(D) = X_g \times \SL_2^2 \) so \( R (X_g \vert D) = \frac{1}{(q-1)^g} (\gamma)(\gamma^{-1}(D)) \) = \( \frac{1}{(q-1)^g} (Z_{\SL_2}(L)) \) \( \otimes Z(D)(1) \) \( \vert D \). Using that, we finally obtain that

\[
\tr_1 \circ Z_{\SL_2}(L) (R (X_g \vert D)) = \frac{1}{(q^3 - q)^{g-1}} \tr_1 \circ Z_{\SL_2}(L) [Z_{\SL_2}(L) \otimes Z(D)(1)] \vert D
\]

\[
= \frac{1}{(q^3 - q)^{g-1}} \tr_1 [Z_{\SL_2}(L) \otimes Z(D)(1)] \vert D = (q^3 - q) \tr_1 (R (X_g \vert D))
\]

\[
= (q^3 - q) R (X_g \vert B_t).
\]

This proves the desired equality.

Corollary 6.9. The morphism \( M : W \vert B_t \to W \vert B_t \) satisfies that \( (q^3 - q) M = \eta^{-1} \circ Z_{\SL_2}^{gm}(L) \vert B_t \).

Proof. By the results of \( 48 \), the set \( \{ R (Y_g/Z_2 \vert B_t) \}_{g \geq 0} \) generate the same submodule as \( T_{B_t}, S_2, S_{-2} \) and \( S_2 \otimes S_{-2} \). Therefore, since \( R (Y_g/Z_2 \vert B_t) = M (R (Y_{g-1}/Z_2 \vert B_t)) \), from Proposition 6.8, we obtain that \( (q^3 - q) \eta = Z_{\SL_2}^{gm}(L) \) on the submodule generated by \( T_{B_t}, S_2, S_{-2} \) and \( S_2 \otimes S_{-2} \).

Indeed, in \( 48 \), Section 9, a larger homomorphism \( M : W \to W \) is defined for which the previous one is just its restriction to \( W \vert B_t \). Analogous (and simpler) calculations can be done as in Corollary 6.8 in order to show that \( (q^3 - q) \eta \circ M(T_{\pm 1}) = Z_{\SL_2}^{gm}(L)(T_{\pm 1}) \) and \( (q^3 - q) \eta \circ M(T_{\pm 1}) = \)}
$Z_{SL_2}^m(L)(T_±)$. Hence, $(q^3 - q)M = η^{-1} \circ Z_{SL_2}^m(L)$ on the whole $W$. Using the explicit expression of $M$ from [18], we have that, in the set of generators $S$, the matrix of $Z_{SL_2}^m(L)$ is

$$
(q^3 - q)^3 \left( \begin{array}{cccc}
q + 4 & 1 & q^3 - 2q - 3 & q^2 + 3q & q^3 - 2q - 2q - 3 & -q^2 - 4q - 1 & 2q^2 - 7q - 1 & -5q - 1 \\
1 & q + 4 & q^3 - 3q^2 & q^2 - 2q - 3 & q^3 - 2q - 2q - 2 & 2q^2 - 7q - 1 & -q^2 - 4q - 1 & -5q - 1 \\
q^2 - 2q - 3 & q^2 + 3q & q^3 - 3q - 2q - 6q & q^3 - 2q - 3q & q^3 - 2q - 2q - 2q - 2 & 2q^2 - 7q - 1 & -q^2 - 4q - 1 & -5q - 1 \\
q^2 + 3q & q^3 - 3q - 2q - 6q & q^3 - 2q - 3q & q^3 - 2q - 2q - 2q - 2 & 2q^2 - 7q - 1 & -q^2 - 4q - 1 & -5q - 1 \\
q^2 + 1 & q^3 + 1 & q^3 - 2q + 2q & q^3 - 2q + 2q & q^3 - 2q + 2q & q^3 - 2q + 2q & q^3 - 2q + 2q & q^3 - 2q + 2q \\
0 & 3q & 3q & -3q & -3q & -3q & 4q^3 - 6q^2 & -4q^2 & -3q^2 \\
q & q & q & q & q & q & q & q & q \\
q^2 & q^2 & q^3 & q^3 - 2q^2 & -q^2 - q & q^3 - q^2 - q & q^3 - q^2 - q & q^2 - 2q^2 - q
\end{array} \right)
$$

Analogous calculation can be done to obtain $Z_{SL_2}^m(L) = (q^3 - q)\eta M η^{-1}$.

**Remark 6.10.** The previous result can be restated as that $M : W_0 \rightarrow W_0$ is (up to a constant) the left tr-reduction of the standard TQFT, as explained in Remark 4.23. However, we have chosen to consider right tr-reductions in Section 5.3 since they are more natural from the geometric point of view.

Let $Σ_g$ be the closed surface of genus $g ≥ 0$. For the parabolic data $Λ = \{−\text{Id}, [J_+], [J_-]\}$ we consider a parabolic structure on $Σ_g$, $Q = \{(p_1, [C_1]), \ldots, (p_s, [C_s])\}$, where $C_i = J_+, J_-$ or $−\text{Id}$ and $p_1, \ldots, p_s ∈ Σ_g$. Let us denote by $r_+$ be the number of $[J_+]$ in $Q$, $r_-$ the number of $[J_-]$ and $t$ the number of −Id (so that $r_+ + r_- + t = s$). Set $r = r_+ + r_-$ and $σ = (−1)^{r^+ - t}$.

**Theorem 6.11.** The K-theory image of the cohomology of $X_{SL_2(C)}(Σ_g, Q)$ is

- **If** $σ = 1$, **then**

$$
\left[ H^c_*(X_{SL_2(C)}(Σ_g, Q)) \right] = (q^2 - 1)^{2g + r - 1} q^{2g - 1} + \frac{1}{2} (q - 1)^{2g + r - 1} q^{2g - 1} (q + 1) (2q^2 + q - 3)
$$

$$
+ (-1)^{r} (q + 1)^{2g + r - 1} q^{2g - 1} (q - 1) (2q^2 + q - 1).
$$

- **If** $σ = -1$, **then**

$$
\left[ H^c_*(X_{SL_2(C)}(Σ_g, Q)) \right] = (q - 1)^{2g + r - 1} (q + 1) q^{2g - 1} (q + 1) (q^2 + r + 2q^2 + 2q^2 - 1)
$$

$$
+ (-1)^r + 1 2q^2 - 1 (q + 1)^{2g + r - 1} (q - 1) q^{2g - 1}.
$$

**Proof.** By Theorem 5.10, we have that

$$
\left[ H^c_*(X_{SL_2(C)}(Σ_g, Q)) \right] = \frac{1}{(q^3 - q)^{g + r}} Z_{SL_2}^m(D^1) \circ Z_{SL_2}^m(L_{[C_1]}) \circ \ldots \circ Z_{SL_2}^m(L_{[C_s]}) \circ Z_{SL_2}^m(L)^g \circ Z_{SL_2}^m(D)(1).
$$

Moreover observe that, as $Z_{SL_2}^m$ is an almost-TQFT and the strict tubes commute, all these linear morphisms commute. Hence, we can group the $−\text{Id}$ and $J_-$ tubes together and, using that $Z_{SL_2}^m(L_{−\text{Id}}) \circ Z_{SL_2}^m(L_{−\text{Id}}) = (q^3 - q)^2 K_{M,SL_2}$ and $Z_{SL_2}^m(L_{−\text{Id}}) \circ Z_{SL_2}^m(L_{[J_-]}) = (q^3 - q) Z_{SL_2}^m(L_{[J_-]})$, we finally have:

- **If** $σ = 1$, **then all the** $−\text{Id}$ **tubes cancel so we have**

$$
\left[ H^c_*(X_{SL_2(C)}(Σ_g, Q)) \right] = \frac{1}{(q^3 - q)^{g + r}} Z_{SL_2}^m(D^1) \circ Z_{SL_2}^m(L_{[J_-]})^r \circ Z_{SL_2}^m(L)^g \circ Z_{SL_2}^m(D)(1).
$$

Now, observe that, as $Z_{SL_2}^m(L)$ and $Z_{SL_2}^m(L_{[J_-]})$ commute, they can be simultaneously diagonalized. Hence, there exists $P, A, B : W_0 \rightarrow W_0$ such that $PAP^{-1} = Z_{SL_2}^m(L)$ and
\[ PB^{-1} = Z_{SL^2}(L_{J_+}). \] Therefore, we have that

\[ H^*_c (X_{SL^2}(\Sigma g, Q)) = \frac{1}{(q^3 - q)^{g+r}} Z_{SL^2}(D^1) P B^r A^g P Z_{SL^2}(D)(1). \]

From the matrices of \( Z_{SL^2}(L) \) and \( Z_{SL^2}(L_{J+}) \) for the set of generators \( S \), the matrices \( P, A \) and \( B \) can be explicitly calculated with a symbolic computation software (like SageMath [53]). Using that matrices, and recalling that \( Z_{SL^2}(D) \) and \( Z_{SL^2}(D^1) \) are the inclusion and projection onto \( T_1 \) respectively, the calculation follows.

- If \( q = -1 \), then all the \(-\text{Id}\) tubes cancel except one, so we have

\[ H^*_c (X_{SL^2}(\Sigma g, Q)) = \frac{1}{(q^3 - q)^{g+r+1}} Z_{SL^2}(D^1) \circ Z_{SL^2}(L_{-\text{Id}}) \circ Z_{SL^2}(L_{J_+})^r \circ Z_{SL^2}(T)^g \circ Z_{SL^2}(D)(1). \]

Again, decomposing \( P A P^{-1} = Z_{SL^2}(T) \) and \( P B^{-1} = Z_{SL^2}(L_{J+}) \), we obtain that

\[ H^*_c (X_{SL^2}(\Sigma g, Q)) = \frac{1}{(q^3 - q)^{g+r+1}} Z_{SL^2}(D^1) Z_{SL^2}(L_{-\text{Id}}) P B^r A^g P Z_{SL^2}(D)(1). \]

Hence, using a symbolic computation software, the result follows.

\[ \square \]

**Remark 6.12.** When interpreting \( q \) as a variable, the previous polynomials are also the Deligne-Hodge polynomials (see [37] for the definition) of the parabolic representations varieties with \( q = uv \). This is due to the fact that \( e(q) = e([H^*_c (\Sigma)]) = uv \).

Under this point of view, the results of Theorem [6,11] generalize the previous work on Deligne-Hodge polynomials of parabolic representation varieties to the case of an arbitrary number of marked points. For genus \( g = 1, 2 \) and one marked point in \([J_\pm]\) (i.e. \( r_+ = 1 \) or \( r_- = 1 \)), this theorem agrees with the calculations of [43], Sections 4.3, 4.4, 11 and 12. For arbitrary genus and one marked point, this result agrees with [48], Proposition 11. Finally, for the case \( g = 1 \) and two marked points in the classes \([J_\pm]\), this theorem agrees with the computations of [42], Section 3.

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**E-mail address:** angel.gonzalez.prieto@upm.es