Vacuum polarization of charged massless fermions in Coulomb and Aharonov–Bohm fields

V.R. Khalilov and I.V. Mamsurov

Faculty of Physics, M.V. Lomonosov Moscow State University, 119991, Moscow, Russia

Vacuum polarization of charged massless fermions is investigated in the superposition of Coulomb and Aharonov–Bohm (AB) potentials in 2+1 dimensions. For this purpose we construct the Green function of the two-dimensional Dirac equation with Coulomb and AB potentials (via the regular and irregular solutions of the radial Dirac equation) and calculate the vacuum polarization charge density in these fields in the so-called subcritical and supercritical regimes. The role of the self-adjoint extension parameter is discussed in terms of the physics of problem. We hope that our results will be helpful in the more deep understanding the fundamental problem of quantum electrodynamics and can be applied to the problems of charged impurity screening in graphene with taking into consideration the electron spin.

PACS numbers: 12.20.-m, 03.65.Pm, 73.90.+f
Keywords: Coulomb, Aharonov–Bohm potential; Vacuum polarization; Induced charge density; Subcritical, Supercritical regime
I. INTRODUCTION

The vacuum instability in the supercritical Coulomb field is one of the important problem of quantum electrodynamics that is exhaustively studied in [1–3]. New great interest to the vacuum instability in the supercritical Coulomb field was revived in connection with the Coulomb impurity problem in graphene. The effective fine structure constant in graphene is large, which gives the new possibility to study the two-dimensional quantum electrodynamics in the strong-coupling (in fact, supercritical Coulomb) regime and the existence of charged Fermi quasiparticles in graphene makes experimentally feasible to observe the vacuum polarization in strong Coulomb field.

From the physical point of view there are two (subcritical and supercritical) regimes, depending mainly on the magnitude of Coulomb field charge. Theoretically, charged impurity screening in graphene in terms of vacuum polarization were investigated in [13–24] and in comprehensive reviews [25, 26]. These studies have shown that the vacuum polarization charge density is localized at the potential center in the subcritical Coulomb potential while the vacuum charge density induced by the supercritical Coulomb potential has the form \( c/r^2 \) causing a modification of Coulomb law at large distances. This behavior could be expected on dimensional grounds: \( \delta(r) \) and \( c/r^2 \) are the only dimensionally consistent possibilities due to the scaling invariance of the massless Dirac equation (in the absence of any intrinsic length scale) [26]. It will be remembered that close to the so-called Dirac points, charged quasiparticle excitations in the potential of graphene lattice are massless Dirac-like fermions characterized by a linear dispersion relation and so a single electron dynamics in graphene is described by a massless two-component Dirac equation.

New important results related to screening in graphene were obtained in [15, 20] with taking into account electron-electron interactions in a self-consistent renormalization group treatment. It turns out to be the system self-consistently rearranges itself so that electrons at large distances never feel a supercritical effective coupling and the subcritical (stable) situation is therefore protected. This conclusion agrees with expectations for the corresponding problem in the convenient quantum electrodynamics, where the vacuum polarization charge in super-heavy nuclei behaves in such a way as to reduce the supercritical charge of nucleus to the threshold value [30] (see also [31], where the problem was investigated for super-heavy nuclei in the presence of a superstrong constant uniform magnetic field).

The wonderful quantum phenomenon was revealed in [32]: the induced current density in graphene in the field of a solenoid turns out to be a finite periodic function of the magnetic flux. The induced polarization current in the QED\(_{2+1}\) with an Aharonov–Bohm potential for massive and massless charged fermions was studied in [32]. The induced electric current due to vacuum polarization in the AB potential was observed in [34] in “a quantum-tunneling system using two-dimensional ionic structures in a linear Paul trap”.

The dynamics of charged fermions in the superposition of Coulomb and AB potentials is governed by a singular Dirac Hamiltonian that requires the supplementary definition in order for it to be treated as a self-adjoint quantum-mechanical operator. So we need to determine the self-adjoint Dirac Hamiltonians and then to construct the correct Green function of the Dirac equation in the superposition of Coulomb and AB potentials. In such a superposition the subcritical and supercritical regimes are determined by the magnitudes of parameters (as well as the relations between them) characterizing the Coulomb and AB potentials.

A main feature of the supercritical and (at some magnitudes of parameters) subcritical regimes is a nonuniqueness of the self-adjoint Dirac Hamiltonian; there exists a one-parameter family of self-adjoint Dirac Hamiltonians specified by additional boundary conditions at the origin [35]. This is a manifestation of a nontrivial physics inside the origin and an interpretation of self-adjoint extension parameters is a purely physical problem. For example, in an AB field the magnetic flux within the interior of the vortex determines the effective Hamiltonian outside it; the extensions can be parameterized by nontrivial boundary conditions on the wave functions at the origin and different choices lead to inequivalent physical cases [36]. We can determine the self-adjoint extension parameter in terms of the parameter \( R \) that is the finite radius of a real solenoid. The self-adjoint extension method was used to determine bound states of massive fermions in the Aharonov–Bohm-like fields [37] and of a magnetic dipole moment in electric and magnetic fields generated by an infinitely long charged solenoid, carrying a magnetic field [38].

Here we study the vacuum polarization of charged massless fermions in the superposition of Coulomb and AB potentials in 2+1 dimensions. We calculate the induced charge density in the vacuum in the subcritical and supercritical regimes, for the first time, using the Green function of the two-dimensional
Dirac equation with Coulomb and AB potentials. The presence of AB potential in our model gives the possibility to estimate effects, which are due to the interaction of the electron spin magnetic moment and the Aharonov–Bohm magnetic field. Since the interaction potential is repulsive or attractive for different signs of spin projection this feature must be taken into account in the behavior of wave functions at the origin.

We shall adopt the units where $c = \hbar = 1$.

II. INDUCED VACUUM CHARGE DENSITY

A. Solutions and Green function of the Dirac Hamiltonian

We remind that the Dirac $\gamma^\mu$-matrix algebra is known to be represented in terms of the two-dimensional Pauli matrices $\gamma^0 = \sigma_3$, $\gamma^1 = i\sigma_1$, $\gamma^2 = i\sigma_2$ where the parameter $s = \pm 1$ can be introduced to label two types of fermions in accordance with the signature of the two-dimensional Dirac matrices \cite{39}; for the case of massive fermions it can be applied to characterize two states of the fermion spin (spin "up" and "down") \cite{40}.

We also note that by Coulomb potential in 2+1 dimensions, we mean potential that decreases as $1/r$ with the distance from the source, having in mind that in a physical situation (e.g., in graphene), although the electrons move in a plane, their Coulomb interaction with the external field of the pointlike charge of an impurity occurs in a physical (three-dimensional) space and the electric field strength of the impurity is a three-dimensional (not two-dimensional) vector. Therefore, the potential $A_0(r) \sim 1/r$ (and not $A_0(r) \sim \log r$, as would be the case in 2+1 dimensions) does not satisfy the two-dimensional Poisson equation with a pointlike source at the origin. Besides, in real physical space, because of the existence of finite magnetic flux inside solenoid $\Phi = 2\pi B$ the singular term including the spin parameter appears in the form of an additional delta-function interaction of spin with magnetic field of solenoid $H = (0, 0, H) = \nabla \times A = B \pi \delta(r)$ in the Dirac equation squared. The additional potential $-seB \delta(r)/r$ will be taken into account by boundary conditions. It will be noted that such kind of point interaction also appears in several Aharonov–Bohm-like problems \cite{41, 42}.

The Dirac Hamiltonian for a fermion of the mass $m$ and charge $e = -e_0 < 0$ in an Aharonov–Bohm $A_0 = 0$, $A_r = 0$, $A_\phi = B/r$, $r = \sqrt{x^2 + y^2}$, $\phi = \arctan(y/x)$ and Coulomb $A_0(r) = a/e_0 r$, $A_r = 0$, $A_\phi = 0$, $a > 0$ potentials, is

$$H_D = \sigma_1 P_2 - s\sigma_2 P_1 + \sigma_3 m - e_0 A_0(r), \quad (1)$$

where $P_\mu = -i \partial_\mu - e A_\mu$ is the generalized fermion momentum operator (a three-vector). The Hamiltonian (1) should be defined as a self-adjoint operator in the Hilbert space of square-integrable two-spinors $\Psi(r)$. The total Dirac momentum operator $J = -i \partial/\partial \phi + s\sigma_3/2$ commutes with $H_D$. Eigenfunctions of the Hamiltonian (1) are (see, \cite{43})

$$\Psi(t, r) = \frac{1}{\sqrt{2\pi r}} \left( \begin{array}{c} f(r) \\ g(r)e^{il\varphi} \end{array} \right) \exp(-iEt + il\varphi), \quad (2)$$

where $E$ is the fermion energy, $l$ is the integer quantum number. The wave function $\Psi$ is an eigenfunction of the operator $J$ with eigenvalue $j = \pm(l + s/2)$ in terms of the angular momentum $l$ and

$$\hat{h}F(r) = EF(r), \quad F(r) = \left( \begin{array}{c} f(r) \\ g(r) \end{array} \right), \quad (3)$$

where

$$\hat{h} = is\sigma_2 \frac{d}{dr} + \sigma_1 \frac{l + \mu + s/2}{r} + \sigma_3 m - \frac{a}{r}, \quad \mu \equiv e_0 B. \quad (4)$$

It will be noted that the massless fermions do not have spin degree of freedom in 2+1 dimensions \cite{44}, nevertheless, the Dirac Hamiltonian (1) keeps the introduced spin parameter.

The induced current density due to vacuum polarization is determined by the three-vector $j_\mu(r)$, which is expressed in terms of the single-particle Green function of the Dirac equation as

$$j_\mu(r) = -\frac{e}{2} \int \frac{dE}{2\pi t} \text{Tr}G(r, r'; E) \gamma_\mu, \quad (5)$$
where \( C \) is the path in the complex plane of \( E \) enclosing all the singularities along the real axis \( E \) depending upon the choice of the Fermi level \( E_F \). The Green function \( G \) can be expanded in eigenfunctions of the operator \( J \). Since the induced charge density in the vacuum is divergent and thus needs the renormalization it is helpful first to consider the model with charged massive fermions. For such a model, the radial parts (the doublets) of above eigenvalues must satisfy the two-dimensional Dirac equation. Then the radial partial Green’s function \( G_I(r, r'; E) \) is given by (just as in 3+1 dimensions) \( G_I(r, r'; E) = \frac{1}{W(E)}[\Theta(r' - r)U_I(r') + \Theta(r - r')U_I(r)U_I^*(r')] \),

\[ G_I(r, r'; E)\gamma^0 = \frac{1}{W(E)}[\Theta(r' - r)U_I(r') + \Theta(r - r')U_I(r)U_I^*(r')], \tag{6} \]

where \( W(E) \) is the \((r\text{-independent})\) Wronskian, defined by two doublets \( V \) and \( F \) as \( W(V, F) = Vi\sigma_2F = (v_1f_2 - f_1v_2) \) and \( U_I(r) \) are the regular and irregular solutions of the radial Dirac equation \((\hat{h} - E)U(r) = 0\); the regular (irregular) solutions are integrable at \( r \to 0 \) \((r \to \infty)\). We see that the problem is reduced to constructing the self-adjoint radial Hamiltonian \( \hat{h} \) in the Hilbert space of doublets \( F(r) \) square-integrable on the half-line.

Since the initial radial Dirac operator is not determined as an unique self-adjoint operator the additional specification of its domain, given with the real parameter \( \xi \) (the self-adjoint extension parameter) is required in terms of the self-adjoint boundary conditions. Any correct doublet \( F(r) \) of the Hilbert space must satisfy the self-adjoint boundary condition \( F^\dagger(r)i\sigma_2F(r)|_{r=0} = (\tilde{f}_1\tilde{f}_2 - \tilde{f}_2\tilde{f}_1)|_{r=0} = 0. \tag{7} \)

Physically, the self-adjoint boundary conditions show that the probability current density is equal to zero at the origin.

We shall apply as the solutions of the radial Dirac equation the doublets found in \( D \) the doublets found in \( D \).

\[ F_R = \begin{pmatrix} f_R(r, \gamma, E) \\ g_R(r, \gamma, E) \end{pmatrix}, \quad F_I = \begin{pmatrix} f_I(r, \gamma, E) \\ g_I(r, \gamma, E) \end{pmatrix}, \tag{8} \]

where

\[
\begin{align*}
 f_R(r, \gamma, E) &= \sqrt{\frac{m + E}{x}} \left(A_R M_{aE/\lambda + s/2, \gamma}(x) + C_R M_{aE/\lambda - s/2, \gamma}(x)\right), \\
 g_R(r, \gamma, E) &= \sqrt{\frac{m - E}{x}} \left(A_R M_{aE/\lambda + s/2, \gamma}(x) - C_R M_{aE/\lambda - s/2, \gamma}(x)\right), \\
 f_I(r, \gamma, E) &= \sqrt{\frac{m + E}{x}} \left(A_I W_{aE/\lambda + s/2, \gamma}(x) + C_I W_{aE/\lambda - s/2, \gamma}(x)\right), \\
 g_I(r, \gamma, E) &= \sqrt{\frac{m - E}{x}} \left(A_I W_{aE/\lambda + s/2, \gamma}(x) - C_I W_{aE/\lambda - s/2, \gamma}(x)\right),
\end{align*}
\]

\[
\begin{align*}
 C_R &= \frac{s\gamma - aE/\lambda}{\nu + ma/\lambda}, \\
 C_I &= (ma/\lambda - \nu)^s.
\end{align*}
\]

Here

\[
 x = 2\lambda r, \quad \lambda = \sqrt{m^2 - E^2}, \quad \gamma = \sqrt{\nu^2 - a^2}, \quad \nu = |l + m + s/2|,
\]

\[ A_R, A_I, C_R, C_I \] are numerical coefficients and the Whittaker functions \( M_{a,b}(x) \) and \( W_{c,d}(x) \) represent the regular and irregular solutions.

For \( a^2 < \nu^2 \) \( \gamma \) is real, for \( a^2 > \nu^2 \) \( \gamma = i\sqrt{a^2 - \nu^2} \equiv i\sigma \) is imaginary. The quantities \( q = \sqrt{\nu^2 - \gamma^2} \) and \( q_c = \nu \Leftrightarrow \gamma = 0 \) are called the effective and critical charge, respectively; it is helpful also to determine \( q_u = \sqrt{\nu^2 - 1/4} \Leftrightarrow \gamma = 1/2 \).

### B. Induced charge density in the subcritical range

In the subcritical range for \( q \leq q_u, \gamma \geq 1/2 \), we can chose as the regular solutions only ones \( F_R(r) \) vanishing at \( r = 0 \); for \( 0 < \gamma < 1/2 \) \((q_u < q < q_c)\) the regular solutions \( U_R(r) \) must satisfy the self-adjoint boundary condition \( D \) and should be chosen in the form of linear combination of the functions \( F_R(r) \) and \( F_I(r) \)

\[ U_R(r) = F_R(r) + \xi F_I(r). \tag{12} \]
The \((r\text{-independent})\) Wronskian is easily calculated to be

\[
\text{Wr}(F_R, F_I) \equiv W(E, \gamma) = (g_R f_I - f_R g_I) = -2A_R A_I \frac{\Gamma(2\gamma)}{\Gamma(\gamma + 1/2 - s/2 - aE/\lambda)} \nu + ma/\lambda
\]  

(13)

where \(\Gamma(z)\) is the Gamma function \(47\) and, therefore, in the subcritical range the single-particle Green function is completely determined. One can show that the contribution into the renormalized induced charge density coming from range \(0 < \gamma < 1/2\) is small for any \(\xi\), therefore it is enough to consider the case \(\xi = 0\) in the subcritical range. Thus, we should chose as the regular solutions the functions \(F_R(r)\) for all \(\gamma > 0\) to obtain

\[
\text{tr}G_\nu(r, r'; E)\gamma^0 = \sum_{s=-1}^{+1} \sum_{l=-\infty}^{+\infty} \frac{f_I f_R + g_I g_R}{2\pi s W(E, \gamma)}
\]  

(14)

After some calculations, we obtain

\[
\text{tr}G_\nu(r, r'; E)\gamma^0 = -\frac{1}{2\pi \lambda^2 \gamma^2} \sum_{l=-\infty}^{+\infty} \frac{\Gamma(\gamma - aE/\lambda)}{\Gamma(2\gamma + 1)} [(m^2 a/\lambda + E(x - 2aE/\lambda - 1)) M_{aE/\lambda + 1/2, \gamma}(x) + +m^2 a/\lambda - aE/\lambda) M_{aE/\lambda - 1/2, \gamma}(x) + +E x \frac{d}{dx} (M_{aE/\lambda + 1/2, \gamma}(x) W_{aE/\lambda + 1/2, \gamma}(x))] \]

(15)

where now \(\gamma = \sqrt{\nu^2 - a^2}\), \(\nu = l + \mu + 1/2\).

We note that the singularities of \(G_\nu(r, r'; E)\) can be simple poles associated with the discrete spectrum (in the range \(-m < E < m\)), and two cuts \((-\infty, -m\) and \([m, \infty)\) associated with the continuum spectrum in the ranges \(|E| \geq m \) \(46\).

For the partial Green function in a Coulomb field in 3+1 dimensions, the path \(C\) may be deformed to run along the singularities on the real \(E\) axis as follows: \(C = C_- + C_p + C_+\), where \(C_-\) is the path along the negative real \(E\) axis (\(\text{Re}E < 0\)) from \(-\infty\) to 0 turning around at \(E = 0\) with positive orientation, \(C_p\) is a circle around the bound states singularities with \(-m < E < 0\) (if we chose \(E_F = -m\)), and \(C_+\) is the path along the positive real \(E\) axis (\(\text{Re}E > 0\)) from \(\infty\) to 0 but with negative orientation (i.e. clockwise path) turning around at \(E = 0 \) \(8\). In the considered case in 2+1 dimensions the path \(C\) may be deformed in the similar way \(33\).

One can show that the contour of integration \(C\) with respect to \(E\) can be deformed to coincide with the imaginary axis and we obtain:

\[
\text{tr}G_\nu(r, r'; E)\gamma^0 = -e \int_{-\infty}^{+\infty} \frac{dE}{2\pi} \text{tr}G_\nu(r, r', iE)\gamma^0.
\]

(16)

Applying the following integral representation \(47\)

\[
M_{aE/\lambda \pm 1/2, \gamma}(x) W_{aE/\lambda \pm 1/2, \gamma}(x) = \frac{x \Gamma(2\gamma + 1)}{\Gamma(2\gamma + 1/2 - aE/\lambda + 1/2)} \int_0^\infty e^{-x \cosh s} \sinh(2aE/\lambda) I_{2\gamma}(x \sinh s) ds,
\]

(17)

it is convenient to represent the induced charge density in the form

\[
\text{tr}G_\nu(r, r'; E)\gamma^0 = -e \int_{-\infty}^{+\infty} \frac{dE}{2\pi} \text{tr}G_\nu(r, r', iE)\gamma^0.
\]

(16)

Applying the following integral representation \(47\)

\[
M_{aE/\lambda \pm 1/2, \gamma}(x) W_{aE/\lambda \pm 1/2, \gamma}(x) = \frac{x \Gamma(2\gamma + 1)}{\Gamma(2\gamma + 1/2 - aE/\lambda + 1/2)} \int_0^\infty e^{-x \cosh s} \sinh(2aE/\lambda) I_{2\gamma}(x \sinh s) ds,
\]

(17)

it is convenient to represent the induced charge density in the form

\[
\text{tr}G_\nu(r, r'; E)\gamma^0 = -2e \pi^2 r \sum_{l=-\infty}^{+\infty} \int_0^\infty dE \int_0^\infty \frac{2 x \Gamma(2\gamma + 1)}{\Gamma(2\gamma + 1/2 - aE/\lambda + 1/2)} e^{-x \cosh s} \sinh(2aE/\lambda) I_{2\gamma}(x \sinh s) ds,
\]

(17)

where now \(\lambda = \sqrt{m^2 + E^2}, I_\mu(z)\) is the modified Bessel function of the first kind and the prime (here and below) denotes the derivative of function with respect to argument.

Let us write \(\mu = [\mu] + \alpha \equiv n + \alpha\), where \([\mu]\) \(n\) denotes the largest integer \(\leq \mu\), and \(1 > \alpha \geq 0\). Hence \(n = 0, 1, 2, \ldots\) for \(\mu > 0\) and \(n = -1, -2, -3, \ldots\) for \(\mu < 0\). Since signs of \(e\) and \(B\) are fixed it is enough to consider the only case \(\mu > 0\). Then, denoting \(\nu_+ = l + \alpha + 1/2, \nu_- = l - \alpha + 1/2, \gamma_+ = \sqrt{\nu^2_+ - a^2}, \gamma_- = \sqrt{\nu^2_- - a^2}\).
\[ \gamma_\pm = \sqrt{\nu^2 - a^2}, \] where here and in all formulas below \( l \equiv l + n, \) we rewrite the induced charge density in the form

\[
j_0(r) = -\frac{2e}{\pi^2 r} \sum_{l=0}^{+\infty} \int_0^{\infty} dt \int_0^{\infty} dE \, e^{-2\lambda r \coth t} \left( 2a \cos(2aE/\lambda) \coth t(I_{2\gamma_+}(2\lambda r/\sinh t) + I_{2\gamma_-}(2\lambda r/\sinh t)) - \frac{2E}{\sinh t} \sin(2aE/\lambda)(I'_{2\gamma_+}(2\lambda r/\sinh t) + I'_{2\gamma_-}(2\lambda r/\sinh t)) \right) \quad (19)
\]

emphasized the prime (here and below) denotes We note that \( j_0 \) is odd with respect to charge \( e. \) This expression is similar to the induced charge density in a pure Coulomb field obtained in \( [20]. \) It contains divergence and its renormalization should be carried out using the obvious physical requirement the total induced charge to vanish. Due to the nonzero mass \( m \) the renormalization can be performed by usual way in momentum space:

\[
j_0(\beta) \equiv \rho(\beta) = \int dr e^{iqr} j_0(r) = \frac{2e}{\pi} \sum_{l=0}^{\infty} \int_0^{\infty} dx \int_0^{\infty} dt \int_0^{\infty} dy \, \frac{\sinh t}{2b} e^{-y \cosh t} J_0(\beta y \sinh t/2b)f(y, t),
\]

\[
f(y, t) = \frac{xy}{b} \sin(\mu t)(I_{2\gamma_+}(y) + I_{2\gamma_-}(y)) - 2a \coth t \cos(\mu t)(I_{2\gamma_+}(y) + I_{2\gamma_-}(y)). \quad (20)
\]

Here \( \beta = |q|/m, \) \( x = E/m, \) \( b = \sqrt{1 + x^2}, \) \( y = 2bR/\sinh t, \) \( R = mr, \mu = 2ax/b. \)

Further calculations with this term for \( a < 1/2, \alpha \sim 0 \) are similar to those described in detail in \( [20, 48] \) for the vacuum polarization in a pure Coulomb field in the subcritical range. We introduce the renormalized induced quantity in momentum representation as \( n(\beta) = \lim_{\Lambda \to \infty} [\rho(\beta) - \lim_{\beta \to 0} \rho(\beta)] \) with an ultraviolet cutoff \( |E| < \Lambda \) (see, \( [20, 48] \)). Because the nonzero mass \( m \) is the only dimensionless parameter in the Green function the resulting dimensionless function \( n(\beta) \) can depend only on the ratio \( \beta = q/m. \) Accordingly, it becomes just a constant in the massless limit \( m \to 0, \) which is denoted as \( Q = \lim_{m \to 0} n(\beta). \) It is obvious that \( Q \) is the induced charge density localized in the point \( r = 0 \) in coordinate space. Therefore the induced charge density in coordinate space has the form \( Q = Q(\beta). \) Let us calculate \( Q. \)

As was shown in \( [48] \) the induced charge density (as the series in terms of powers of \( a \)) for small \( a \) contains divergences only in the coefficients of the \( a \) and \( a^3 \) terms. We give \( n_1(\beta) \) with the \( a \) term that reflects the linear one-loop polarization contribution:

\[
n_1(\beta) = \frac{2e}{\pi} \sum_{l=0}^{\infty} \int_0^{\infty} dt \left( \int_0^{\infty} dy \int_0^{\infty} dx \frac{\sinh t}{2b} e^{-y \cosh t} J_0(\beta y \sinh t/2b)f_1(y, t) + a \coth t(e^{-2\nu_+ t} + e^{-2\nu_- t}) \right) \quad (21)
\]

as well as the renormalized induced charge \( Q_1 \) in the first order of \( a: \)

\[
Q_1 = \frac{2ea}{\pi} \sum_{l=0}^{\infty} \int_0^{\infty} dt \left( \int_0^{\infty} dy \sinh t \ln(1/y \sinh t) e^{-y \cosh t} \times 
\right.
\]

\[
\left. \times \left[yt(I'_{2\nu_+}(y) + I'_{2\nu_-}(y)) - \coth t(I_{2\nu_+}(y) + I_{2\nu_-}(y)) \right] + a \coth t(e^{-2\nu_+ t} + e^{-2\nu_- t}) \right) =
\]

\[
= \frac{ea}{\pi} \sum_{l=0}^{\infty} \int_0^{\infty} dt \left( \int_0^{\infty} dy \ln(1/y \sinh t) \left[ t \sinh t \frac{d}{dy}(ye^{-y \cosh t}(I_{2\nu_+}(y) + I_{2\nu_-}(y))) - \right. \right.
\]

\[
\left. \left. - \frac{d}{dt}(t \cosh t e^{-y \cosh t}) (I_{2\nu_+}(y) + I_{2\nu_-}(y)) \right] + a \coth t(e^{-2\nu_+ t} + e^{-2\nu_- t}) \right) \quad (22)
\]

Integrating \( (22), \) we obtain

\[
Q_1 = \frac{2ea}{\pi} \sum_{l=0}^{\infty} \left( (l + 1/2 + \alpha)\psi'(l + 1/2 + \alpha) + (l + 1/2 - \alpha)\psi'(l + 1/2 - \alpha) - 2 - \frac{l + 1/2}{(l + 1/2)^2 - \alpha^2} \right) \quad (23)
\]

where \( \psi(z) \) is the logarithmic derivative of Gamma function \( [47] \). For \( \alpha \ll 1, \) we find

\[
Q_1 = ea \pi/4 + ea \pi (2 \ln 2 + 1 - \pi^2/4) \alpha^2 \approx ea \pi (0.25 - 0.04 \alpha^2). \quad (24)
\]
The first term in Eq. (24) coincides with result obtained in \cite{15,17,20}. We note that the contribution into \(Q_1\) from AB potential arises in the presence of Coulomb field only, is small and has opposite sign compared with a pure Coulomb one.

We have carried out long calculations and got the total exact induced charge in the subcritical range in the form

\[ Q = Q_1 + Q_r, \tag{25} \]

where

\[ Q_r = \frac{2e}{\pi} \sum_{l=0}^{\infty} \text{Im} \left[ \ln(\Gamma(\gamma_+ - ia)\Gamma(\gamma_+ - ia)) + \frac{1}{2} \ln((\gamma_+ - ia)(\gamma_+ - ia)) - 
\] \[ -((\gamma_+ - ia)\psi(\gamma_+ - ia) + (\gamma_+ - ia)\psi(\gamma_+ - ia)) + ia \frac{l + 1/2}{(l + 1/2)^2 - \alpha^2} - 
\] \[ -ia((l + 1/2 + \alpha)\psi'(l + 1/2 + \alpha) + (l + 1/2 - \alpha)\psi'(l + 1/2 - \alpha))] \). \tag{26} \]

This expression at \(\alpha = 0\) is in agreement with result obtained in \cite{20}; the coefficient of the \(a^3\) term at \(\alpha = 0\) was also found in perturbation theory \cite{17}. The induced charge \(Q\) determined by Eq. (25) is negative.

It is worth to note that the vacuum charge density is induced by the homogeneous background magnetic field in the massive and massless QED$_{2+1}$ \cite{49}.

\[ \mathbf{C. \text{ Induced charge density in the supercritical range}} \]

In the supercritical range \(q > q_c(\gamma = ia)\) the stronger singularity of the Coulomb potential at the origin has to be regularized, therefore, we need to determine the self-adjoint Dirac Hamiltonians specified, for example, by self-adjoint boundary conditions \cite{7}. Then, we straightforward construct the Green function in the form \cite{6} in which the regular solutions \(U_R(r)\), satisfying \cite{7}, have to be chosen in the form of linear combination of the functions \(F_R(r)\) and \(F_I(r)\). For this range \((\gamma = ia)\), the above two solutions \(F_R(r)\) and \(F_I(r)\) become oscillatory with the imaginary exponent and it is convenient to use in this range the self-adjoint extension parameter \(\theta\) \cite{43,46}, related to \(\xi\) by

\[ \frac{A_R}{\xi A_I} = e^{2i\theta} \left( \frac{2\lambda}{E_0} \right)^{-2i\sigma} \frac{\nu + a(m + E)/\lambda + i\sigma}{\nu + a(m + E)/\lambda - i\sigma} \frac{\Gamma(2i\sigma)}{\Gamma(1/2 - s/2 - aE/\lambda + i\sigma)} - \frac{\Gamma(-2i\sigma)}{\Gamma(1/2 - s/2 - aE/\lambda - i\sigma)} \tag{27} \]

where \(\pi \geq \theta \geq 0\) and a positive constant \(E_0\) gives an energy scale.

The Green function has a discontinuity, which is solely associated with the appearance of its singularities situated on a second (unphysical) sheet \(\text{Re}E < 0, \text{Im}E < 0\) of the complex plane \(E\) at \(q > q_c\); these singularities are determined by complex roots of equation \(W(E, i\sigma) = 0\) and describe the infinite number of quasistationary (resonant) states with complex "energies" \(E = |E|e^{i\tau}\). For massless fermions \((m = 0)\) and \(\sigma < 1\) their energy spectrum was found in \cite{46}:

\[ E_{k,\theta,s} \equiv \text{Re}E = E_0\cos(\tau)\exp(-k/2\sigma + \theta/\sigma + \pi \coth \pi a/2a), \tag{28} \]

where \(\tau \approx -(1 + s)/4a + \pi\psi(ia) + \pi/2\). Eq. (28) contains an essential singularity. These quasi-localized resonances have negative energies, thus they are situated in the hole sector. For \(\sigma \ll 1\) the imaginary part \(\text{Im}E = \tan \tau E_{k,\theta,s} \ll \text{Re}E\) is very small and, therefore, the resonances are practically stationary states \cite{46}. For example, for \(a = 1/2, s = 1 \approx (1 + 0.04)\pi\).

Physically, the self-adjoint extension parameter can be interpreted in terms of the cutoff radius \(R\) of a Coulomb potential. For this, for example, we can compare Eq. (28) with the spectrum of supercritical resonances in the cutoff Coulomb potential \cite{15,50,51} and approximately derive \(\theta \sim \sigma[c(a) + \ln E_0 R]\), where \(c(a)\) does not depend on \(R\). We note that the cutoff radius \(R\) rather relates to a renormalized critical coupling that is also characterized by a logarithmic singularity at \(mR \ll 1\) in massive case \cite{50,52}.

The simplest way to include these resonances in the induced charge density is to carry out the integral in \(E\) from \(-\infty\) to 0 along the path \(S\) taking into account the singularities on the second sheet. After some calculations, we represent the induced charge (electron) density \cite{43} as the sum of contributions from the
subcritical and supercritical ranges, which have to be treated separately

$$j_0(r) = -\frac{e}{2} \int \frac{dE}{2\pi i} \text{tr} G_\nu(r, r', E) \gamma^0 = -\frac{e}{2} \int \frac{dE}{2\pi i} \sum_{s=1}^{+\infty} f_I(r, \gamma, E) f_R(r, \gamma, E) + g_I(r, \gamma, E) g_R(r, \gamma, E)$$

$$= \frac{e}{2} \int \frac{dE}{2\pi i} \sum_{l,s:|<a} \frac{\xi(f_I^2(r, \nu, E) + g_I^2(r, \nu, E))}{sW(E, \nu)} = j_{sub}(r) + j_{sup}(r).$$  \hspace{1cm} (29)

For the supercritical range $\gamma = \nu$, $0 \geq \theta \geq \pi$, the sum in second term $j_{sup}$ is taken over $l$ of $a^2 > (l + \mu + s/2)^2$. Then the paths $C, S$ can be deformed to coincide with the imaginary axis $E$.

The first term in Eq. (29) was calculated and explicitly represented in previous subsection. The second term is convergent and its contribution to the induced charge density can be directly evaluated at $m = 0$. Having performed simple calculations we leads $j_{sup}$ to

$$j_{sup}(r) = \frac{e}{8\pi^2 r^2} \sum_{l,s:|<a} \frac{s \nu^{s+1} \Gamma(2i\nu)}{\sigma \Gamma(2i\sigma) \Gamma(-2i\sigma)} \int_0^\infty \frac{dE}{E \omega(\nu)} \Gamma(i\sigma + (1-s)/2 - i\nu E/|E|) \times$$

$$\times \Gamma(i\sigma + (1-s)/2 - i\nu E/|E|) W_{iaE}/|E| W_{j\nu + s/2, i\nu}(2E|r|) W_{j\nu - s/2, i\nu}(2E|r|),$$  \hspace{1cm} (30)

where

$$\omega(\sigma) = 1 - e^{2i\sigma} \left( \frac{2E}{E_0} \right)^{2i\sigma} \frac{\nu + i\nu E/|E| + is\nu \Gamma(2i\sigma)}{\nu + i\nu E/|E| - is\nu \Gamma(-2i\sigma) \Gamma(i\sigma + (1-s)/2 - i\nu E/|E|) + is\nu \Gamma(2i\sigma) \Gamma(-2i\sigma) \Gamma(i\sigma + (1-s)/2 - i\nu E/|E|).}$$  \hspace{1cm} (31)

Rewrite $\left( 2E/|E_0| \right)^{2i\sigma}$ as $\exp(-2i\sigma \ln(E/|E_0|))$. As far as the integrand decreases exponentially at $|E| \gg 1/r$ and strongly oscillate at $|E| \rightarrow 0$, the main contribution to the integral $\omega(\sigma)$ is given by the region $|E| \sim 1/r$. So in order to evaluate $j_{sup}$ we replace $|E|$ by $1/r$ in the log-periodic term of the integrand $\omega(\sigma)$ and obtain

$$j_{sup}(r) = -\frac{e}{8\pi^2 r^2} \sum_{l,s:|<a} \frac{s \nu^{s+1} \Gamma(2i\nu)}{\sigma \omega(\nu) \Gamma(2i\nu) \Gamma(-2i\nu)} \Gamma(i\sigma + (1-s)/2 + i\nu E/|E|) \times$$

$$\times \int_0^\infty \frac{dE}{E} W_{-ia+s/2, i\nu}(2E|r|) W_{-ia-s/2, i\nu}(2E|r|),$$  \hspace{1cm} (32)

where

$$\omega(\nu) = 1 - e^{2i\sigma + 2i\sigma \ln(E_0/r)} \frac{\nu + i\nu E/|E| + is\nu \Gamma(2i\sigma) \Gamma(2i\sigma) \Gamma(-2i\sigma) \Gamma(2i\sigma)}{\nu + i\nu E/|E| - is\nu \Gamma(2i\sigma) \Gamma(2i\sigma) \Gamma(2i\sigma) + is\nu \Gamma(2i\sigma) \Gamma(-2i\sigma) \Gamma(2i\sigma) \Gamma(-2i\sigma) \Gamma(2i\sigma) \Gamma(-2i\sigma) \Gamma(2i\sigma)}.$$  \hspace{1cm} (33)

Because of the complex singularities on the unphysical sheet at $q > q_c$, the Green function and $j_{sup}(r)$ are complex though for $\sigma \ll 1$ their imaginary parts are small. In terms of the physics the complex Green function probably reflects the lack of stability of chosen (for constructing Green function) neutral vacuum for $q > q_c$ (see, also [5]).

Now we can integrate in Eq. (32) using formula [47]

$$\int_0^\infty \frac{dE}{E} W_{-ia+s/2, i\nu}(2E|r|) W_{-ia-s/2, i\nu}(2E|r|) = \frac{\pi}{s \sin(2\pi i\sigma)} \times$$

$$\times \left[ \frac{1}{\Gamma((1-s)/2 + ia + i\nu) \Gamma((1+s)/2 + ia - i\nu)} - \frac{1}{\Gamma((1-s)/2 - ia - i\nu) \Gamma((1+s)/2 + ia + i\nu)} \right].$$  \hspace{1cm} (34)

and after simple transformations we finally find the induced charge density in the supercritical range as

$$j_{sup}^r(r) = \frac{e}{2\pi^2 r^2} \sum_{l,s:|<a} \text{Re} \frac{\sigma}{\omega(\nu)} \times$$

$$\times \left[ \frac{1}{\Gamma((1-s)/2 + ia + i\nu) \Gamma((1+s)/2 + ia - i\nu)} - \frac{1}{\Gamma((1-s)/2 - ia - i\nu) \Gamma((1+s)/2 + ia + i\nu)} \right].$$  \hspace{1cm} (35)

The main effect, arising at supercritical regime, is that the induced vacuum polarization for non-interacting massless fermions has a power law form ($\sim c/r^2$) whose coefficient is log-periodic functions with respect to the distance from the origin. In the subcritical regime the induced vacuum
charge is localized at origin and exhibits no long range tail. As an example, we consider Eq. (35) for $1/2 - \alpha < a < 3/2 + \alpha, 1/2 \gg \alpha > 0$, when just the lowest $l + n, s$ channels are supercritical, and find

$$j^r_{\text{sup}}(r) = \frac{e}{2\pi^2 r^2} \sum_{\sigma = \sigma_{\pm}} \sigma \text{Re} \frac{2 - Aze^{2i\theta + 2i\sigma \ln(E_{or})}}{1 - Aze^{2i\theta + 2i\sigma \ln(E_{or})} + A^2[(a - \sigma)/(a + \sigma)]e^{4i\theta + 4i\sigma \ln(E_{or})}},$$  \[(36)\]

where

$$A = \frac{\Gamma(2i\sigma)\Gamma(-i\sigma + ia)}{\Gamma(-2i\sigma)\Gamma(i\sigma + ia)}, \quad z = 2 \frac{a - \sigma}{a}, \quad \nu_{\pm} = 1/2 \pm \alpha, \sigma_{\pm} = a^2 - \nu_{\pm}^2. \quad (37)$$

The induced charge density (36) resembles the local density of states which also exhibits resonances at the negative energies [46].

Using the known representation

$$\text{Arg}\Gamma(x + iy) = y \left[ -C + \sum_{n=1}^{\infty} \left( 1 - \frac{1}{y} \arctan \frac{y}{x + n - 1} \right) \right],$$

where $C = 0.57721$ is Euler’s constant, we finally obtain the induced charge density (36) in the form

$$j^r_{\text{sup}}(r) = \frac{e}{2\pi^2 r^2} \sum_{\sigma = \sigma_{\pm}} \sigma \text{Re} \frac{2 - |A|ze^{2i\theta + 2i\sigma \ln(E_{or})} + \psi}{1 - |A|ze^{2i\theta + 2i\sigma \ln(E_{or})} + \psi + |A|^2[(a - \sigma)/(a + \sigma)]e^{4i\theta + 4i\sigma \ln(E_{or})} + 2i\psi}. \quad (38)$$

Here

$$\psi \equiv \text{Arg}A = -\pi - 2C\sigma + \sum_{n=1}^{\infty} \left( \frac{2\sigma}{n} - 2 \arctan \frac{2\sigma}{n} + \arctan \frac{2n\sigma}{n^2 + \nu^2} \right). \quad (39)$$

For small $\sigma \ll 1$, Eq. (38) takes the simplest form

$$j^r_{\text{sup}}(r) = \frac{e(\sigma_+ + \sigma_-)}{2\pi^2 r^2}, \quad \sigma_{\pm} = \sqrt{a^2 - (1/2 \pm \alpha)^2}. \quad (40)$$

Here “$\pm$” sign before $\alpha$ for fixed sign of $\alpha$, in fact, corresponds to the spin projection sign. One sees that $j^r_{\text{sup}}(r)$ is odd with respect to the fermion charge $e$ and even with respect to $\alpha$. The contribution into the induced charge density due to the AB potential has opposite sign compared with a pure Coulomb one. It is of importance that the induced charge density $j^r_{\text{sup}}(r)$ (10) at $\sigma \ll 1$ does not contain at all the self-adjoint extension parameter $\theta$. From the physical point of view, increased $a$ near the point ($\gamma = 0$) the transition will occur from the subcritical range to the supercritical one, which can be symbolically characterized by $\gamma \to \sigma$ and then a small change in $\sigma$ such that $q > q_c$ leads to a sudden change in the character of a physical phenomenon due to emerging of infinitely many resonances with negative energies. Put another way the character of a physical phenomenon itself must be due only to physical (but not mathematical) reasons. We also note that the expression $j^r_{\text{sup}}(r)$ (at $\alpha = 0$) is in agreement with results obtained in [18] for the problem of vacuum polarization of supercritical impurities in graphene by means of scattering phase analysis.

### III. CONCLUDING REMARKS

In this paper we have studied the vacuum polarization of charged massless fermions in Coulomb and AB potentials in 2+1 dimensions. In particular, we have calculated the induced charge density using the Green’s functions of the Dirac equation with the Coulomb and AB potentials. In subcritical regime the induced vacuum charge $Q$ is localized at the origin and has a screening sign, leading to a decrease of the effective Coulomb charge; the contribution into $Q$ due to the AB potential is small and has opposite sign compared to the Coulomb one. In the supercritical regime the induced vacuum charge, like the subcritical contributions, has a screening sign but it has a power law form, causing a modification of Coulomb law at large distances; the contribution into the induced vacuum charge due to the AB potential has opposite sign compared to the Coulomb one.

Because a single electron dynamics in graphene is described by a massless two-component Dirac equation we hope that our results can be applied in graphene with charged impurities. Furthermore, while the electron-electron interaction has been neglected, results of present paper may be useful to develop
further insight into the screening of the Coulomb impurity with taking into consideration of the electron spin and electron-electron interaction. To approach this problem one can write the self-consistent renormalization group equations in the Hartree approximation in the subcritical range in the same spirit as in [20] and within the Thomas-Fermi method for the supercritical range. We shall defer the self-consistent renormalization group analysis to a future work.

[1] W. Pieper and W. Greiner, Z. Phys., 218, 327 (1969).
[2] S. S. Gershtein and Ya. B. Zeldovich, Sov. Phys. JETP, 30, 358 (1970).
[3] Y.B. Zeldovich and V.S. Popov, Sov. Phys. Uspekhi, 14, 673 (1972).
[4] A. B. Migdal, Fermions and Bosons in Strong Fields (in Russian, Nauka, Moscow, 1978).
[5] J. Rafelski, L. P. Fulcher, and A. Klein, Phys. Rep., 38, 227361 (1978).
[6] M. Soffel, B. Muller, and W. Greiner, Phys. Rep., 85, 51122 (1982).
[7] V.B. Berestetskii, E.M. Lifshitz, and L.P. Pitaevskii, Quantum Electrodynamics, 2nd ed. (Pergamon, New York, 1982).
[8] W. Greiner, J. Reinhardt, Quantum Electrodynamics, 4th ed. (Springer-Verlag, Berlin Heidelberg, 2009).
[9] E. Wichmann and N.M. Kroll, Phys. Rev., 96, 232 (1954); Phys. Rev. 101, 843 (1956).
[10] I.S. Brown, R.N. Cahn, and L.D. McLerran, Phys. Rev. Lett., 32, 562 (1974); Phys. Rev. Lett., 33, 1591 (1974); Phys. Rev., D12, 581 (1975).
[11] M. Gyulassy, Phys. Rev. Lett., 32, 1393 (1974); Phys. Rev. Lett., 33, 921 (1974); Nucl. Phys., A244, 497 (1975).
[12] A.A. Grib, S.G. Mamaev, and V.M. Mostepanenko, Vacuum Quantum Effects in Strong Fields, [in Russian] (Energoatomizdat, Moscow, 1988); (Friedmann Laboratory Publishing, St. Petersburg, 1994).
[13] M.I. Katsnelson, Phys. Rev., B74, 201401(R) (2006).
[14] V. M. Pereira, J. Nilsson, and A.H. Castro Neto, Phys. Rev. Lett., 99, 166802 (2007).
[15] A.V. Shytov, M.I. Katsnelson, and L.S. Levitov, Phys. Rev. Lett., 99, 236801 (2007).
[16] A.V. Shytov, M.I. Katsnelson, and L.S. Levitov, Phys. Rev. Lett., 99, 246802 (2007).
[17] R.R. Biswas, S. Sachdev, and D.T. Son, Phys. Rev., D85, 045021 (2012).
[49] V.R. Khalilov, I.V. Mamsurov, Eur. Phys. J., C75, 167 (2015).
[50] O.V. Gamayun, E.V. Gorbar, and V.P. Gusynin, Phys. Rev., B80, 165429 (2009).
[51] K.S. Gupta and S. Sen, Modern Phys. Lett., A24, 99107 (2009).
[52] V.R. Khalilov and C.-L. Ho, Mod. Phys. Lett., A13, 615 (1998).