Spectral Theory for Discrete Laplacians

Dorin Ervin Dutkay and Palle E. T. Jorgensen

Abstract. We give the spectral representation for a class of selfadjoint discrete graph Laplacians $\Delta$, with $\Delta$ depending on a chosen graph $G$ and a conductance function $c$ defined on the edges of $G$. We show that the spectral representations for $\Delta$ fall in two model classes, (1) tree-graphs with $N$-adic branching laws, and (2) lattice graphs. We show that the spectral theory of the first class may be computed with the use of rank-one perturbations of the real part of the unilateral shift, while the second is analogously built up with the use of the bilateral shift. We further analyze the effect on spectra of the conductance function $c$: How the spectral representation of $\Delta$ depends on $c$.

Using $\Delta_G$, we introduce a resistance metric, and we show that it embeds isometrically into an energy Hilbert space. We introduce an associated random walk and we calculate return probabilities, and a path counting number.

Mathematics Subject Classification (2000). 34B45, 46E22, 47L30, 54E70, 60J10, 81S30.

Keywords. Operators in Hilbert space, discrete Laplacians, infinite graphs, spectral representation, spectral measures, multiplicity tables, semicircle laws, rank-one perturbations, spectrum, absolutely continuous, graph Laplacian, electrical network.

1. Introduction

One fascination with operator theory is its connections to other areas such as geometry and discrete analysis: Because of applications to electrical networks, to statistical mechanics, and to fractals (see e.g., [37, 41, 42, 50, 51, 62]), there is a recent increased interest in detailed spectral representation for operators on infinite graphs. In addition to the applications, these connections further suggest a search for more a direct link between, on the one hand, metric geometry of infinite graphs, and on the other, a spectral analysis of associated families of operators.

Research supported in part by a grant from the National Science Foundation DMS-0704191.
However as we see, classical methods break down in infinite discrete models: With Fourier analysis in classical potential theory, it is often possible to represent Laplace operators by multiplication, and hence realize the spectral representation this way. The analysis then breaks up into the study of discrete and continuous parts. However we show that analogues of this that adapt to the discrete case have strong limitations. New tools from operator theory are needed: For example in the discrete case, typically there is not a natural Fourier duality available, and the graph may not even be associated with a group in a way that facilitates computation of spectral representations. Further, in the study of Laplacians on infinite graphs $G$, there are several Hilbert spaces in the picture. Choices must be made: There are Hilbert space completions of functions on the vertices in $G$, and similarly for functions on the edges in $G$ (Definitions 2, 4.1), the energy Hilbert space.

Here we focus on classes of graphs $G$ that require new tools. Our conclusions (Theorem 3.26, Corollary 4.6, and Theorem 4.11) imply that not only is there a direct connection between the spectrum of the graph Laplacian $\Delta_G$ and the metric geometry of $G$; but this connection carries over to the detailed fine-structure of the multiplicity configurations for $\Delta_G$.

Our proofs rely on a mixture of operator theory (Section 3) and complex analysis (Section 2). Since we address three different audiences, for the convenience of readers, we have included a few details which may be known to operator theorists but not to graph theorists, and vice versa.

In this paper we study the operator theory of infinite graphs $G$, and especially a natural family of Laplace operators directly associated with the graph in question. These operators depend not only on $G$, but also on a chosen positive real valued function $c$ defined on the edges in $G$. In electrical network models, the function $c$ will determine a conductance number for each edge $e$: the conductance being the reciprocal of the resistance between the endpoint vertices in the edge. Specifically if $e = (xy)$ connects vertices $x$ and $y$ in $G$, the number $c(e)$ is the reciprocal of the resistance between $x$ and $y$. Hence prescribing a conductance leads to classes of admissible flows in $G$ determined from Ohm’s law and Kirchhoff’s laws of electrical networks. This leads to a measure of energy directly associated with the graph Laplacian. There are Hilbert spaces $H(G)$ which offer a useful spectral theory, and our main results concern the spectral theory of these operators. In a recent paper [36] it was proved that the graph Laplacians are automatically essentially selfadjoint, i.e., that the associated operator closures are selfadjoint operators in $H(G)$.

Here we give a spectral analysis of the graph Laplacians. We are motivated by a pioneering paper [50] which in an exciting way applies graphs and resistor networks to a problem in quantum statistical mechanics.

There are many benefits from having a detailed spectral picture of graph Laplacians: We get a spectral representation realization of the graph Laplacians, the operators $\Delta_{G,c}$, i.e., a unitarily equivalent form of these operators which may arise in a variety of applications. See e.g., [6, 48].