Majority problems of large query size

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Abstract

The aim of this paper is twofold: we present improvements of results of De Marco
and Kranakis [6] on majority models, and we also survey recent results concerning the
generalizations of the pairing model with query size \(k\), and also provide bounds on
their non-adaptive versions.

1 Introduction

We are given \(n\) indexed balls - say the set \([n] = \{1, 2, ..., n\}\) - as an input, each colored
in some way unknown to us, and we would like to find a ball of the majority color or show
that there is no majority color by asking subsets of \([n]\), that we call queries. A ball \(i \in [n]\)
is called majority ball if there are more than \(\frac{n}{2}\) balls in the input set that have the same
color as \(i\). We call this combinatorial problem the majority problem. We would like to
determine the minimum number of queries needed in the worst case, when our adversary -
we will call him Adversary in the following -, who tells us the answers for the queries wants

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to postpone the solution of the majority problem. It is possible that all the queries are fixed at the beginning, we call this the non-adaptive version, or the queries may depend on the answers for the previous ones, that we call the adaptive version of the majority problem. A model of the majority problem is given by \([n]\), the number of colors, the size of the queries (that we will denote by \(k\)), the possible answers of Adversary and whether it is adaptive or non-adaptive. Sometimes we will use the hypergraph language, so we will speak about the hypergraph of the queries.

The first majority problem model, the so-called pairing model - that is an adaptive model, when the query size is two, and the answer of Adversary is yes, if the two balls have the same color and no otherwise - was investigated by Fisher and Salzberg [11], who proved that \([3n/2] - 2\) queries are necessary and sufficient for any number of colors to solve the majority problem. If the number of colors is two, then Saks and Werman [16] proved that the minimum number of queries needed is \(n - b(n)\), where \(b(n)\) is the number of 1’s in the diadic form of \(n\) (we note that there are simpler proofs of this result, see [11, 17]). In this paper we deal with a possible generalization of the pairing model, when we ask queries of bigger size. We survey the literature of these models and prove new results about them in the case when \(n\) balls are given, colored with two colors and we ask queries of size \(k\). The models we deal with differ only in the answers for the queries and whether they are adaptive or non-adaptive. The first model of this kind was introduced and investigated by De Marco, Kranakis and Wiener [7], then many results appeared in the literature [3, 6, 10, 14]. We note that for \(k \geq 3\) it is possible that in some model one can not solve the majority problem for small \(n\) or even for any \(n\) (see such a model in [14]).

We note that there are other possibilities to generalize the pairing model of the majority problem, e.g. the plurality problem [1, 7, 13], some random scenarios [3, 9, 12], or investigate the case when Adversary can lie [1, 4], etc.

The rest of the paper is organised as follows: in Section 2 we survey the existing models and the existing bounds prior this paper and state our results, in Section 3 we prove our main result, in Section 4 we pose open questions, while in the Appendix we prove our simpler results and present two tables that summarize the current state of art on majority problems with two colors and large query size.
2 Results

In this section we survey the models of the majority problem, where we are given \( n \) indexed balls, colored with two colors, and we ask queries of size \( k \) (\( k \geq 2 \)) that we denote by \( Q \). We will give each model an abbreviation that we indicate after its name. If the abbreviation is \( X \), then we will denote by \( A(X, k, n) / N(X, k, n) \) the number of queries needed to ask in the worst case of the adaptive/nonadaptive version of that model, respectively. At each model we mention the papers that deal with that model and write the best result that was known.

2.1 Models, known results

- **Output (or Partition) Model = OM, \([6, 10]\):**

  **Answer** : \( \{Q', Q''\} \), a partition of \( Q \), where \( Q' \) is the set of balls of one color and \( Q'' \) of the other color. (Note that no indication is provided to Adversary about the colors of each type.)

**Theorem 1.** *(De Marco, Kranakis \([6]\), Theorem 3.2)* For all \( 2 \leq k \leq n \) we have

\[
A(OM, k, n) \leq \left\lceil \frac{n - \frac{1}{k - 1}}{k - 1} \right\rceil.
\]

**Theorem 2.** *(Eppstein, Hirschberg \([10]\), Theorem 3, Theorem 4)* For all \( 2 \leq k \leq n \) we have

\[
f_1(n, k) \leq A(OM, k, n) \leq f_2(n, k),
\]

where

\[
f_1(n, k) := \begin{cases} 
\left\lceil \frac{n - \frac{1}{k - 1}}{k - 1} \right\rceil & \text{if } k \text{ is odd,} \\
\frac{n}{k - 1} - O(n^{1/3}) & \text{if } k \text{ is even,}
\end{cases}
\]
and

\[
f_2(n, k) := \begin{cases} 
\left\lceil \frac{n-1}{k-1} \right\rceil & \text{if } n \text{ is even}, \\
\left\lceil \frac{n-2}{k-1} \right\rceil & \text{if } n \text{ is odd}.
\end{cases}
\]

• Counting Model = CM. [6 [10]:

**Answer**: is a number \( i \leq k/2 \) such that the query has exactly \( i \) balls of one color and \( k - i \) of the other. (Again, no indication is provided to Adversary about the colors of each type.)

**Theorem 3.** (Eppstein, Hirschberg [10], Theorem 2, Theorem 3, Theorem 4) For all \( 2 \leq k \leq n \) we have

\[
f_3(n, k) \leq A(CM, k, n) \leq n \left\lceil \frac{k}{2} \right\rceil + O(k),
\]

where

\[
f_3(n, k) := \begin{cases} 
\left\lceil \frac{n-1}{k-1} \right\rceil & \text{if } k \text{ is odd}, \\
\frac{n}{k} - O(n^{1/3}) & \text{if } k \text{ is even}.
\end{cases}
\]

• General (or Yes-No) Model = GM, [6]:

**Answer**: yes if all balls have the same color in \( Q \), no otherwise.

**Theorem 4.** (De Marco, Kranakis [10], Theorem 5.1, Theorem 5.4) For all \( 2 \leq k, n \) with \( 2k - 1 \leq n \) we have

\[
\left\lceil \frac{n}{k} \right\rceil \leq A(GM, k, n) \leq n - k + \binom{2k-1}{k}.
\]
Borzyszkowski’s Model = BM. \[7, 3\]:

**Answer:** yes, if there exist two balls of different colors, and a pair is pointed out, no if all balls have the same color.

**Theorem 5.** (Borzyszkowski \([3]\), Theorem 1) For all \(3 \leq k \leq n\) with \(2k - 3 \leq n\) we have

\[
A(BM, k, n) = f_4(n, k),
\]

where

\[
f_4(n, k) := \begin{cases} \frac{n}{2} + k - 2 & \text{if } n \text{ is even,} \\ \lceil \frac{n}{2} \rceil + k - 3 & \text{if } n \text{ is odd.} \end{cases}
\]

**Basic inequalities**

By definition the following inequalities hold for the previous 4 models:

**Fact 6.** For all \(2 \leq k \leq n\) we have the following:

- \(A(OM, k, n) \leq A(CM, k, n) \leq A(GM, k, n)\),
- \(A(OM, k, n) \leq A(BM, k, n) \leq A(GM, k, n)\),
- \(N(OM, k, n) \leq N(CM, k, n) \leq N(GM, k, n)\),
- \(N(OM, k, n) \leq N(BM, k, n) \leq N(GM, k, n)\).

We note that by this observation and Theorem 5 the lower bound on \(A(GM, k, n)\) (given in Theorem 4) can be improved to \(f_4(n, k)\). The main goal of this paper is to prove an even stronger lower bound on \(A(GM, k, n)\).

### 2.2 New results

In this subsection we state our new results and introduce a new model. Before stating our results, we recall some definitions.
A hypergraph has Property B - introduced by Bernstein [2] - if its vertices can be colored with two colors such that there is no monochromatic edge in the hypergraph. Let us denote by $m(k)$ the cardinality of the edge set of a smallest $k$-uniform hypergraph that does not have Property B. We also consider $m(k, n)$ which is the cardinality of a smallest $k$-uniform hypergraph with $n$ vertices that does not have Property B. Obviously if $n$ is large enough we have $m(k, n) = m(k)$. Probably the best known recent upper bound on $m(k)$ is $\Omega(2^k \sqrt{k / \log k})$ due to Radhakrishnan and Srinivasan [15].

We also use a similar notion that we call Property C. We call a coloring of a $k$-set with two colors balanced if the cardinality of the two color classes differ by at most one. We say that a hypergraph has Property C if its vertices can be colored with two colors such that every edge is balanced. Let us denote by $d(k)$ the cardinality of the edge set of a smallest $k$-uniform hypergraph that does not have Property C. We also consider $d(k, n)$, which is the cardinality of the edge set of a smallest $k$-uniform hypergraph with $n$ vertices that does not have Property C. Obviously, if $n$ is large enough we have $d(k, n) = d(k)$. Eppstein and Hirschberg proved the following:

**Theorem 7.** ([10], Theorem 1)
- If $k = 0 \mod 4$, then $d(k) \leq 2 \log k + 1$.
- If $k = 2 \mod 4$, then $d(k) = 2$.
- If $k$ is odd, then $d(k) \leq k + 3 \log k + 4$.

Now we present our results.

**Adaptive results**

**Theorem 8.** For any $2 \leq k \leq n$ we have

$$\frac{3n + 5}{4} \leq A(GM, k, n) \leq n - k + m(k).$$

**Theorem 9.** For any $2 \leq k \leq n$ we have

$$A(CM, k, n) \geq \frac{6n}{5k + 6} - c(k),$$

where $c(k)$ depends only on $k$. 

6
Nonadaptive results

Proposition 10. For $2 \leq k \leq n$ we have

$$N(OM, k, n) = f_5(n, k),$$

where

$$f_5(n, k) := \begin{cases} \left\lceil \frac{n-1}{k-1} \right\rceil & \text{if } n \text{ is even}, \\ \left\lceil \frac{n-2}{k-1} \right\rceil & \text{if } n \text{ is odd.} \end{cases}$$

Proposition 11. For $2 \leq k, n$ with $2k - 1 \leq n$ we have

$$N(CM, k, n) \leq f_6(n, k),$$

where

$$f_6(n, k) := \begin{cases} n - k + 1 & \text{if } k \text{ is even,} \\ (n - k + 1) \times (1 + d(k - 1, n)) & \text{if } k \text{ is odd.} \end{cases}$$

Proposition 12. For $2 \leq k$ and $n_0(k) \leq n$ we have

$$N(CM, k, n) \geq f_7(n, k),$$

where

$$f_7(n, k) := \begin{cases} 2n/(k + 1) & \text{if } k \text{ is even and } n \text{ is even,} \\ (2n - k + 1)/(k + 1) & \text{if } k \text{ is even and } n \text{ is odd,} \\ \left\lceil n/k \times d(k - 1, n - 1) \right\rceil & \text{if } k \text{ is odd and } n \text{ is even,} \\ \left\lceil (n - 1)/2k \times d(k - 1, n - 1) \right\rceil & \text{if } k \text{ is odd and } n \text{ is odd,} \end{cases}$$

Proposition 13. For $2 \leq k$ and $2k - 1 \leq n$ we have

$$N(BM, k, n) \leq N(GM, k, n) \leq (n - k + 1) \times m(k - 1, n - 1).$$
Proposition 14. For $2 \leq k$ and $n_0(k) \leq n$ and $n$ even we have

$$f_8(n, k) \times m(k-1) \leq N(BM, k, n) \leq N(GM, k, n),$$

while for $n$ odd we have

$$f_8(n, k) \times m(k-1) \leq N(GM, k, n) \text{ and } f_8(n, k) \times m(k-2) \leq N(BM, k, n),$$

where

$$f_8(n, k) := \begin{cases} \frac{n}{k} & \text{if } n \text{ is even}, \\ \frac{n-1}{2k} & \text{if } n \text{ is odd}. \end{cases}$$

2.2.1 A new model

To finish this section we introduce a closely related model, where our goal is just to determine the majority color or claim that there is no majority color. (We use the same notation that we used in the description of other models in the previous section.)

- Majority queries model = MM

Answer: majority color in $Q$ or no if there is no majority color.

Here our goal is to tell the majority color in the underlying set, rather than showing a majority ball. Note that if $k = 2$ then the answer tells us the number of blue and the number of red balls in $Q$, thus trivially $A(MM, 2, n) = N(MM, 2, n) = \lceil n/2 \rceil$. If $k > 2$, we prove the following upper and lower bounds for the adaptive and non-adaptive versions for this problem.

Theorem 15. For $3 \leq k$ and $n_0(k) \leq n$ we have

$$\frac{2n - 6k}{k + 4} \leq A(MM, k, n) \leq f_9(n, k),$$

where
\( f_9(n, k) := \begin{cases} \left\lceil \frac{n}{(k+1)} \right\rceil + \left\lceil \frac{n}{2} \right\rceil + \binom{2k}{k} - 2k - 2 & \text{if } n \text{ is even}, \\ \left\lceil \frac{n}{(k+1)} \right\rceil + n + \binom{2k}{k} - 2k - 2 & \text{if } n \text{ is odd}. \end{cases} \)

**Theorem 16.** For \( 3 \leq k \leq n \) we have

\[
f_{10}(n, k) \leq N(MM, k, n) \leq n(n - k).
\]

\[
f_{10}(n, k) := \begin{cases} n^2/2k^2 & \text{if } n \text{ is even}, \\ n(n - 1)/4k^2 & \text{if } n \text{ is odd}. \end{cases}
\]

### 3 Proof of the lower bound in the General model

In this section we prove the lower bound of Theorem 8.

#### 3.1 Introduction, Statements

We present a strategy of Adversary. In this strategy Adversary does not only answer yes or no for a query, but tells also the color of some balls. As it is more information, a lower bound for the number of the asked queries provides a lower bound for the general model.

(We note that the aim of telling the color of some balls is to build extra structure, to be able to control the possible colors of those balls, whose color is yet unknown at a certain step.)

Now we introduce some notions to be able to define Adversary's strategy and tell some words on the meaning of these notions. During the description of the strategy we will use (possibly with some index) small letters for balls, capital letters for sets of balls and calligraphic letters for families of sets of balls.

Let \( Q_i \) be the query asked in the \( i^{th} \) round and the set of queries during the first \( i \) round is \( Q(i) := \{Q_1, Q_1, \ldots, Q_i\} \).

As we mentioned, in each round the answer of Adversary consists of a yes or no response and the color of some (possibly zero) balls. We call a function \( c : [n] \rightarrow \{r, b\} \) a **coloring** of the balls and we say that *we know the color of a ball* after the \( i^{th} \) round if its color is the same for all possible colorings of the balls that are compatible with answers for the first \( i \) round. Let us denote by \( R(i) \) (resp. \( B(i) \)) the set of balls that are known to be red (resp.
blue) after the \( i^{th} \) round. It is easy to see that in this model if we know the color of a ball then it is an element of \( R(i) \cup B(i) \). With these notions we define also the following:

- \( Q_r(i) := \{ Q \in Q(i) : Q \cap R(i) \neq \emptyset, Q \nsubseteq R(i), Q \cap B(i) = \emptyset \} \), the set of queries that contain red but no blue balls.
- \( Q'_r(i) := \{ Q \setminus R(i) : Q \in Q_r(i) \} \), those part of the previous queries that are not necessarily red. Note that by definition each of these contains at least one blue ball (but we do not know which one it is). We call these the reduced versions of the queries.
- \( X_r(i) := \{ x \in [n] : \exists P \in Q'_r(i) \text{ with } x \in P \} (= \cup Q'_r(i)) \)

Now we define the similar notions for color blue:

- \( Q_b(i) := \{ Q \in Q(i) : Q \cap B(i) \neq \emptyset, Q \nsubseteq B(i), Q \cap R(i) = \emptyset \} \)
- \( Q'_b(i) := \{ Q \setminus B(i) : Q \in Q_b \} \)
- \( X_b(i) := \{ x \in [n] : \exists P \in Q'_b(i) \text{ with } x \in P \} (= \cup Q'_b(i)) \)
- \( Q_d(i) := \{ Q \in Q(i) : Q \cap (R(i) \cup B(i)) = \emptyset \} \), the set of queries where all balls are of unknown color.
- \( X_d(i) := \{ x \in [n] : \exists P \in Q_d(i) \text{ with } x \in P \} (= \cup Q_d(i)) \)
- \( Q_0(i) := \{ Q \in Q(i) : Q \cap R(i) \neq \emptyset, Q \cap B(i) \neq \emptyset \text{ or } Q \subset R(i) \text{ or } Q \subset B(i) \} \), the set of queries that do not give any more information once we know \( R(i) \) and \( B(i) \).
- \( X_0(i) := X \setminus (R(i) \cup B(i) \cup X_r(i) \cup X_b(i) \cup X_d(i)) \), the set of remaining balls.
Adversary’s main purpose is ensuring that a majority ball is found only if almost every ball is in \( R(i) \) or \( B(i) \). To do so Adversary makes \( R(i) \) and \( B(i) \), the set of red and blue balls roughly equal size, and maintains strong structure conditions on the sets in \( Q'_b(i) \), \( Q'_r(i) \) and \( Q_d(i) \) to be able to control the balls whose color is unknown. Lemma \[17\] gathers these properties.

The strategy of Adversary consists two parts that we call \( \text{STRATEGY1} \) and \( \text{STRATEGY2} \). Adversary starts with \( \text{STRATEGY1} \), and at one specific point he might switch to \( \text{STRATEGY2} \) and use that till the end of the process. That specific point is described in 2. case of Claim \[18\] If \( \text{STRATEGY1} \) would lead to that point, Adversary immediately aborts \( \text{STRATEGY1} \), picks the opposite answer and switches to \( \text{STRATEGY2} \).

So we state a lemma about \( \text{STRATEGY1} \) and we will define \( \text{STRATEGY1} \) during the proof of Lemma \[17\].

**Lemma 17.** Using \( \text{STRATEGY1} \) Adversary can answer in the first \( i \) rounds such a way that for all \( i \geq 1 \) we have:

1. \( \left| |R(i)| - |B(i)| \right| \leq 1 \),
2. for \( Q \in Q'_r(i) \cup Q'_b(i) \) we have \( |Q| \geq 2 \),
3. either \( Q'_r(i) \) or \( Q'_b(i) \) is empty; if \( |R(i)| > |B(i)| \) then \( Q'_r(i) = \emptyset \) and vice versa,
4. for different \( P, Q \in Q'_r(i) \cup Q'_b(i) \) we have \( P \cap Q = \emptyset \),
5. for \( P \in Q'_r(i) \cup Q'_b(i) \) and \( Q \in Q_d(i) \) we have \( P \cap Q = \emptyset \),
6. for \( Q \in Q_d(i) \) there is \( q \in Q \) such that \( Q \setminus \{ q \} \) is disjoint from other members of \( Q_d \) (or equivalently we can say that each component is a star).

The following claim is an immediate consequence of Lemma \[17\].

**Claim 18.** During \( \text{STRATEGY1} \) we can choose a majority ball only in the following 2 cases:

1. \( |R(i)| + |B(i)| \geq n - 1 \), or
2. \( |R(i)| = |B(i)| + 1 \), all \( Q \in Q'_b(i) \) have \( |Q| = 2 \) and \( |R(i)| + |B(i)| + |X_b(i)| = n \).

**Sketch of the proof of Claim \[18\]** If \( X_0(i) \cup X_d(i) \neq \emptyset \), then Adversary has a lot of freedom to color the balls (in \( X_0(i) \cup X_d(i) \)) in two ways, so using that \( R(i) \) and \( B(i) \) have roughly equal size, it is impossible to choose a majority ball.
If \( R(i) \cup B(i) \cup X_b(i) = \{n\} \) and there is a \( Q \in Q'_b(i) \) with \(|Q| = 3\), then it is also easy to see that Adversary can color the balls in \( Q \) two ways, so that it is impossible to choose a majority ball.

We mention again that case 2 will not actually occur, as Adversary switches to \textsc{Strategy2} in case it would happen. This is what we call the end of \textsc{Strategy1}.

**Lemma 19.** If case 2 of Claim 18 would happen in the \( j \)th round during \textsc{Strategy1}, then from the \( j \)th round Adversary could answer - following \textsc{Strategy2} - in a way that in the \( i \)th round (for \( j \leq i \)) the following hold:

1. \(|R(i)| + 1 = |B(i)|\),
2. for \( Q \in Q'_b(i) \) we have \(|Q| = 2\), and
3. \(|R(i)| + |B(i)| + |X_b(i)| = n\).

The following is immediate by Lemma 19.

**Claim 20.** During \textsc{Strategy2} we can choose a majority ball only in case \(|R(i)| + |B(i)| = n\).

**Definition 21.** We call the \( i \)th round a \((j,k)\)-step, if Adversary puts \( k \) queries into \( Q_0(i) \) (that were not in \( Q_0(i-1) \)) with the coloring of \( j \) balls.

**Lemma 22.** Adversary can answer during \textsc{Strategy1} and \textsc{Strategy2} such that each round is a \((j,k)\)-step with \( j/k \leq 4/3 \), or a \((0,0)\)-step.

Note that if a ball is in \( R(i) \) (or \( B(i) \)), then it is in \( R(j) \) (or \( B(j) \)) for \( j > i \). Lemma 22 with the case 1 of Claim 18 and with Claim 20 immediately implies Theorem 8. Indeed, if step \( s \) is the last one, altogether at least \( n - 1 \) balls must be put into \( R(i) \cup B(i) \) for some \( i \leq s \), thus at least \( 3(n - 1)/4 \) queries must be put into \( Q_0(i) \) for \( i \leq s \) altogether. Note that every query is put into \( Q_0(i) \) for some \( i \) at most once. Thus there are at least \( 3(n - 1)/4 \) queries that do not result in \((0,0)\)-steps. In addition it is going to be obvious from the description of \textsc{Strategy1} that the first two queries are \((0,0)\)-steps.
3.2 Proof of the lemmas

Proof. In this section we will simultaneously prove Lemma 17, Lemma 19 and Lemma 22 by defining STRATEGY1 and STRATEGY2.

We define them by case-by-case analysis of the $i^{th}$ ($i \geq 1$) round. In each case we provide a description of the answer which contains: the answer of Adversary (yes/no), the color of some balls, which queries (of $Q(i)$) go to $Q_0(i)$ after this round, can it be the end of STRATEGY1 and for which $j$ and $k$ it is a $(j, k)$-step.

Recall that we denote the query of the $i^{th}$ round by $Q_i$, and by symmetry we can assume that $Q'_r(i - 1)$ is empty (and $|R(i - 1)| \geq |B(i - 1)|$). Now we describe how Adversary answers to $Q_i$.

About the proof of the lemmas:

• In each case maintaining properties 1,2,4,5 and 6 of Lemma 17 will be obvious from the description.

• After the $i^{th}$ round property 3 of Lemma 17 could be violated by two reasons: the first one is that neither $Q'_b(i)$ nor $Q'_r(i)$ is empty. In this situation we color equal many balls (one in each set) in $Q'_r(i)$ blue and $Q'_b(i)$ red so that either $Q'_b(i)$ or $Q'_r(i)$ becomes empty and so move $2t$ queries to $Q_0(i)$ by coloring $2t$ balls. The other reason is that $|R(i)|$ becomes larger than $|B(i)|$, while $Q'_r(i)$ is not empty. Note that by the above, we can assume that $Q'_b(i)$ is empty. Then, in this situation we can pick an arbitrary member of $Q'_r$ and color one of its elements blue. By repeating this, either $Q'_r(i)$ becomes empty, or $|B(i)|$ becomes at least as large as $|R(i)|$. Here we move $t$ queries to $Q_0(i)$ by coloring $t$ balls. If any of these occurs at a round, this changes a $(j, k)$-step to a $(j + t, k + t)$-step. It is easy to see that if $j/k \leq 4/3$, then $(j + t)/(k + t) \leq 4/3$. Note that this cannot be the end of STRATEGY1, as some balls move to $X_0(i)$.

STRATEGY1

Case 1: $Q_i \cap R(i - i) \neq \emptyset$.

Case 1/A: $Q_i \subset R(i - 1)$.

The answer of Adversary is yes, we do not color any ball (since the color of the balls in $Q_i$ is red), only $Q_i$ becomes a new element of $Q_0(i)$, so it is a $(0, 1)$-step, and it can not
be the end of STRATEGY1.

Case 1/B: $Q_i \cap B(i - 1) \neq \emptyset$.

The answer of Adversary is no, we do not color any ball, only $Q_i$ becomes a new element of $Q_0(i)$, so it is a $(0, 1)$-step, and it can not be the end of STRATEGY1.

Case 1/C: $Q_i \cap B(i - 1) = \emptyset$, $Q_i \cap X_0(i - 1) \neq \emptyset$.

The answer of Adversary is no, we color one ball in $Q_i \cap X_0(i - 1)$ blue, only $Q_i$ becomes a new element of $Q_0(i)$, so it is a $(1, 1)$-step, and it can not be the end of STRATEGY1.

Case 1/D: $Q_i \cap B(i - 1) = \emptyset$, $Q_i \cap X_0(i - 1) = \emptyset$, $Q_i \cap X_b(i - 1) \neq \emptyset$.

In this case we know that $Q_i$ intersects a set $P' \in Q'_b(i - 1)$ that is a subset of a query $P \in Q_b(i - 1)$, and we know that $|P'| \geq 2$.

The answer of Adversary is no, we color a ball in $Q_i \cap P'$ blue and another in $P'$ red (using that $|P'| \geq 2$), thus $Q_i$ and $P$ become new elements of $Q_0(i)$, so it is a $(2, 2)$-step, and it can not be the end of STRATEGY1.

Case 1/E: $Q_i \cap B(i - 1) = \emptyset$, $Q_i \cap X_0(i - 1) = \emptyset$, $Q_i \cap X_b(i - 1) = \emptyset$, $Q_i \cap X_d(i - 1) \neq \emptyset$.

We know that there is $p \in Q_i \cap X_d(i - 1)$. Thus there is $P \in Q_d(i - 1)$ with $p \in P$.

The answer of Adversary is no, we color $p$ blue, choose $q \in P \setminus \{p\}$ and color it red, thus $Q_i$ and $P$ becomes a new element of $Q_0(i)$ (other queries in $Q_d(i - 1)$ can move to either $Q_r(i)$ or $Q_b(i)$), so it is a $(2, 2)$-step, and it can not be the end of STRATEGY1.

Case 2: $Q_i \cap R(i - 1) = \emptyset$, $Q_i \cap B(i - 1) \neq \emptyset$.

Case 2/A: $Q_i \subset B(i - 1)$.

The answer of Adversary is yes, we do not color any new ball, only $Q_i$ becomes a new member of $Q_0(i)$, so it is a $(0, 1)$-step, and it can not be the end of STRATEGY1.

Case 2/B: $Q_i \cap X_b(i - 1) \neq \emptyset$.

We know that $Q_i$ intersects a set $P' \in Q'_b(i - 1)$ that is a subset of a query $P \in Q_b(i - 1)$, with $|P'| \geq 2$ and let $p \in P' \cap Q_i$. 

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The answer of Adversary is no, we color $p$ red and choose a ball $q \in P \setminus \{p\}$ that we color blue, thus $Q_i$ and $P$ becomes a new member of $Q_0(i)$, so it is a $(2, 2)$-step, and it can not be the end of STRATEGY1.

Case 2/C: $Q_i \cap X_b(i - 1) = \emptyset$, $Q_i \cap X_d(i - 1) \neq \emptyset$.

We know that $Q_i$ intersects a query $P \in Q_d(i - 1)$ in a ball $p$, and we choose a ball $q \in P \setminus \{p\}$.

The answer of Adversary is no, we color $p$ red and $q$ blue, thus $Q_i$ and $P$ becomes a new member of $Q_0(i)$ (other queries in $Q_d(i - 1)$ can move to either $Q_r(i)$ or $Q_b(i)$), so it is a $(2, 2)$-step, and it can not be the end of STRATEGY1.

Case 2/D: $Q_i \cap X_b(i - 1) = \emptyset$, $Q_i \cap X_d(i - 1) = \emptyset$, $Q_i \cap X_0(i - 1) \neq \emptyset$.

We have two subcases in this case:

Case 2/D/a: $|Q_i \cap X_0(i - 1)| \geq 2$.

The answer of Adversary is no, we do not color anything, nothing becomes a new member of $Q_0(i)$ ($Q_i$ becomes an element of $Q_b(i)$), so it is a $(0, 0)$-step, and it CAN be the end of STRATEGY1.

If it would be the end of STRATEGY1, then the answer of Adversary is yes, and we continue with STRATEGY2.

Case 2/D/b: $|Q_i \cap X_0(i - 1)| = 1$.

The answer of Adversary is yes (note that this is the only nontrivial case when Adversary answers yes), we color that ball in $Q_i \cap X_0(i - 1)$ blue, thus $Q_i$ becomes a new element of $Q_0(i)$, so it is a $(1, 1)$-step and it can not be the end of STRATEGY1.

Case 3: $Q_i \cap R(i - 1) = \emptyset$, $Q_i \cap B(i - 1) = \emptyset$, $|Q_i \cap X_d(i - 1)| \geq 2$.

Case 3/A: $Q_i$ intersects at least two queries in $Q_d(i - 1)$.

Let $P_1, P_2 \in Q_d(i - 1)$ be these queries and we can choose $p_1 \in P_1 \cap Q_i$, $p_2 \in P_2 \cap Q_i$ with $p_1 \neq p_2$. Let us also choose $q_1 \in P_1 \setminus \{p_1, p_2\}$ and $q_2 \in P_2 \setminus \{p_1, p_2\}$ such that $q_1 \neq q_2$. We can do that since $k \geq 3$. 

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The answer of Adversary is no, we color color $p_1$, $q_2$ red and $p_2$, $q_1$ blue, thus $Q_i$, $P_1$, $P_2$ become new elements of $Q_0(i)$, so it a (4, 3)-step, and it can not be the end of STRATEGY1.

Case 3/B: $Q_i$ intersects only one query in $Q_d(i - 1)$.

Let $P \in Q_d(i - 1)$ be that query and choose $p \in P \cap Q_i$ and $q \in P \setminus \{p\}$ such that $P$ is the only query in $Q_d(i - 1)$ that contains $q$ (this is possible, since $k \geq 3$).

The answer of Adversary is no, we color $p$ red, $q$ blue, thus $Q_i$, $P$ become new elements of $Q_0(i)$, so it a (2,2)-step, and it can not be the end of STRATEGY1.

Case 4: $Q_i \cap R(i - 1) = \emptyset$, $Q_i \cap B(i - 1) = \emptyset$, $|Q_i \cap X_d(i - 1)| \leq 1$, $Q_i \cap X_b(i - 1) \neq \emptyset$.

Let $P' \in Q_b'(i - 1)$ (a reduced version of $P \in Q_b(i - 1)$) for which a ball $p \in Q_i \cap P'$.

Case 4/A: $Q_i \cap X_0(i - 1) \neq \emptyset$.

The answer of Adversary is no, we color $p$ red, a ball from $Q_i \cap X_0(i - 1)$ blue, thus $Q_i$ and $P$ become new elements of $Q_0(i)$, it is a (2,2)-step and it can not be the end of STRATEGY1.

Case 4/B: $Q_i \cap X_0(i - 1) = \emptyset$.

By the above we know that in this case we have $|Q_i \cap X_b(i)| \geq 2$, as the size of the queries is at least 3.

Case 4/B/a: there is a $P \in Q_b(i - 1)$ such that $|Q_i \cap P| \geq 2$, and let $p$ and $q$ be these balls.

The answer of Adversary is no, we color $p$ red and $q$ blue, thus $Q_i$ and $P$ become new elements of $Q_0(i)$, so it is a (2,2)-step and it can not be the end of STRATEGY1.

Case 4/B/b: There are two $P_1, P_2 \in Q_b(i - 1)$ such that there are balls $p_1, p_2$ with $p_1 \in P_1' \cap Q_i$ and $p_2 \in P_2' \cap Q_i$.

In this case choose $q_1 \in P_1' \setminus \{p_1\}$ and $q_2 \in P_2' \setminus Q_i$ (this can be done, since $|P_1'|, |P_2'| \geq 2$).

The answer of Adversary is no, we color $q_1$ and $p_2$ red, $p_1$ and $q_2$ blue, thus $Q_i, P_1, P_2$ become new elements of $Q_0(i)$, it is a (4,3)-step and it can not be the end of STRATEGY1.
Case 5. $Q_i \cap R(i-1) = \emptyset$, $Q_i \cap B(i-1) = \emptyset$, $Q_i \cap X_b(i-1) = \emptyset$, $|Q_i \cap X_d(i-1)| \leq 1$.

Case 5/A: $Q_i$ intersects at least two queries in $Q_d(i-1)$.

The answer of Adversary is no, we do not color any balls, no query becomes a new element of $Q_0(i)$, just $Q_i$ is put into $Q_d(i)$, thus it is a (0,0)-step and it can not be the end of STRATEGY1.

Case 5/B: $Q_i$ intersects one query in $Q_d(i-1)$, that does not intersect any other query in $Q_d(i-1)$.

Like in the previous subcase, the answer of Adversary is no, we do not color any balls, no query becomes a new element of $Q_0(i)$, just $Q_i$ is put into $Q_d(i)$, thus it is a (0,0)-step and it can not be the end of STRATEGY1.

Case 5/C: $Q_i$ intersects only one query $P_1 \in Q_d(i-1)$, that intersect another query $P_2 \in Q_d(i-1)$.

In this case let $\{p_1\} = Q_i \cap P_1$ and $\{p_2\} = P_1 \cap P_2$. We can choose $p_3 \in Q_i \setminus (P_1 \cup P_2)$ and $p_4 \in P_2 \setminus (P_1 \cup Q_i)$.

The answer of Adversary is no, we color $p_1, p_4$ red and $p_2, p_3$ blue, so $Q_i, P_1, P_2$ become new elements of $Q_0(i)$ (other queries containing $p_2$ move from $Q_d(i-1)$ to $Q_b(i)$), thus it is a (4,3)-step and it can not be the end of STRATEGY1.

Case 6: $Q_i \subset X_0(i-1)$.

The answer of Adversary is no, we do not color any ball, thus no query becomes a new member of $Q_0(i)$, so it is a (0,0)-step and it can not be the end of STRATEGY1.

We continue with the description of STRATEGY2:

Note that the only situation, when STRATEGY1 can end is Case 2/D/a. In that situation Adversary answers no and we could choose a majority ball, so the structure of the query hypergraph would satisfy the following:

- $X_b(i) \cup R(i) \cup B(i) = [n]$, 

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• $|R(i)| = |B(i)| + 1$,
• for all $Q \in Q'_b(i)$ we have $|Q| = 2$.

As we mentioned, Adversary answers yes instead, so the query hypergraph (after the 'yes' answer) satisfies the following:
• $X_b(i) \cup R(i) \cup B(i) = [n]$,
• $|R(i)| + 1 = |B(i)|$,
• for all $Q \in Q'_b(i)$ we have $|Q| = 2$.

In the following we describe STRATEGY2 again by case-by-case analysis similar to STRATEGY1. The properties 1,2,3 of Lemma 19 will be easy in each case.

STRATEGY2

Case A: $Q_i \cap X_b(i - 1) \neq \emptyset$.

Let $p \in P' \cap Q_i$ with some $P' \in Q'_b(i - 1)$ (a reduced version of some $P \in Q_b(i - 1)$) and choose $q \in P' \setminus \{p\}$.

Case A/a: $Q_i \cap R(i - 1) \neq \emptyset$.

The answer of Adversary is no, we color $p$ blue and $q$ red, so $P$ and $Q_i$ become new elements of $Q_0(i)$, thus it is (2,2)-step.

Case A/b: $Q_i \cap B(i - 1) \neq \emptyset$.

The answer of Adversary is no, we color $p$ red and $q$ blue, so $P$ and $Q_i$ become new elements of $Q_0(i - 1)$, thus it is (2,2)-step.

Case A/c: $Q_i \cap R(i - 1) = Q_i \cap B(i - 1) = \emptyset$.

In this case $Q_i$ intersects another set $P'_0$ from $Q'_b(i - 1)$ (a reduced version of $P_0 \in Q_b(i - 1)$). Let $p_0 \in P'_0 \cap Q_i$ and choose $q_0 \in P'_0 \setminus \{p_0\}$.

The answer of Adversary is no, we color $p$ and $q_0$ red, $q$ and $p_0$ blue, so $Q_i, P, P_0$ become new elements of $Q_0(i)$, thus it is a (4,3)-step.

Case B: $Q_i \cap X_b(i) = \emptyset$.

Case B/a: $Q_i \subset R(i)$.

The answer of Adversary yes, we do not color any new ball, $Q_i$ becomes a new element
of $Q_0(i)$, thus it is a (0,1)-step.

Case B/b: $Q_i \subset B(i)$.

The answer of Adversary yes, we do not color any new ball, $Q_i$ becomes a new element of $Q_0(i)$, thus it is a (0,1)-step.

Case B/c: $Q_i \cap R(i) \neq \emptyset$, $Q_i \cap B(i) \neq \emptyset$.

The answer of Adversary no, we do not color any new ball, $Q_i$ becomes a new element of $Q_0(i)$, thus it is a (0,1)-step.

4 Concluding remarks

If we replace property 6 of Lemma 17 by the stronger condition that queries in $Q_d(i)$ are disjoint from each other, a similar but simpler case analysis shows that $A(GM, k, n) \geq 2n/3$.

It seems plausible that if we instead replaced it with a weaker condition, allowing a richer structure, a similar but more complicated case analysis would give a stronger lower bound. We conjecture $A(GM, k, n) = (1 - o(n))n$.

Another interesting question is what happens if we allow queries of different sizes. Can it help the questioner if he can pick $k$ at any point? It is especially interesting in the general model, where the length of the algorithm does not seem to depend significantly on $k$. Our adaptive lower bound still holds in this more general version.

Instead of finding a majority ball, a reasonable goal would be to find the two color classes. In fact, our algorithms solve this harder problem almost completely; in all cases a small number of additional queries would be enough to find the color classes. In particular our adaptive algorithm for the General Model identifies the color classes, while our non-adaptive algorithms for the Counting Model and the General Model identify most of the colors. Our non-adaptive algorithm for the Output Model identifies the color classes for odd $n$ and one more query would be needed in the even case. For $k = 2$, in the pairing model, $n - b(n)$ queries are needed to find a majority ball, as mentioned in the introduction, while it is easy to see that $n - 1$ queries are needed solve the harder problem. It seems
that the difference is typically so small that it becomes relevant only when the upper and lower bounds are close enough. However, the adaptive algorithms for the Counting Model by Eppstein and Hirschberg [10] and for Borzyszkowski's Model by Borzyszkowski [3] do not find the color classes. They still find the sizes of the color classes though.

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A Sketch of the remaining proofs

In this section we provide (the sketch of) the proof of the upper bound of Theorem 8, the proof of Theorem 9, the proof of Proposition 10-14, and Theorem 15, 16.

During the proofs we will use the following notation: let $Q$ be the set of asked queries, and let $\deg_Q(i) := |\{Q \in Q : i \in Q\}|$, the degree of the ball $i \in [n]$ in the query hypergraph.
Proof of the upper bound in Theorem 8.

We will improve the upper bound using Property B. So we start with $m(k)$ queries that form a family without Property B. Then we must get a yes answer to one of them. We take a subset of size $k - 1$ of that query, and another ball. This way we find out if that ball is of the same color as the $k - 1$-set. Repeating this for every ball we can identify the color classes.

Proof of Theorem 9.

Now we describe Adversary’s strategy, that will contain the answer and the color of some balls as additional information. We use similar notions as in the proof of Theorem 8. Let us denote by $R(i)$ (resp. $B(i)$) the set of balls that are known to be red (resp. blue) after the $i^{th}$ round, and let $f(i) := ||R(i)| - |B(i)||$ be their difference. Let $x := \lfloor (k - 1)/3 \rfloor$. At any time during the algorithm we say that a query is open if it contains more than $k - x$ balls of degree one (in the query hypergraph till that point) and closed otherwise.

In each round Adversary tells the color of the balls of degree at least two and the color of all the balls in closed queries (i.e. in queries that became closed in that round). Note that a query $Q$ can also become closed in case a query $Q'$ is asked later; at this point Adversary gives the color of all the balls in $Q$, even the ones not in $Q'$. For an open query $Q$ we know the color of some of its balls and also the answer to $Q$, i.e. we know that there are two possible numbers of blue balls among the uncolored balls in $Q$ (which are the balls of degree one in $Q$).

When the $i^{th}$ query $Q$ is asked, it contains some balls already colored (those in $R(i - 1) \cup B(i - 1)$). It also contains some balls that previously had degree one. Adversary will color these balls, so first we describe, how:

- If there are more than $k - x$ such balls, i.e. the query is open, the answer is always $x$. The Adversary does not color the balls that will have degree one after this step. Note that balls that are not colored before the $i^{th}$ round and have degree two after this step come from previously open queries. It is easy to see (using that the intersection of $Q$ with these queries are small), that he can color these balls in such a way that they satisfy the answer for these previously asked queries. He chooses among such colorings one that minimizes $f(i)$.
• If the query $Q$ is closed then Adversary will color all the balls in it. First he colors those balls that had degree one before this round. It is also easy to see that he can do it in such a way that it is consistent with the previous answers, since these balls came from previously open questions. So there is at least such coloring of $Q$. But Adversary chooses one that minimizes $f(i)$.

We claim that at any point $f(i) \leq k - 2x$. Indeed, the only time when colors are picked without the goal of making $f(i)$ as small as possible is when an earlier query $Q'$ becomes closed. In this case there are two possible choices for Adversary: after the coloring there are either $x$ red and $k - x$ blue balls or $k - x$ red and $x$ blue balls in $Q'$. No matter how many red and blue balls are already in $Q'$, the difference between the number of new red balls is $k - 2x$. Another important observation is that one choice colors blue more than half of the newly colored balls, while the other choice colors red more than half of the newly colored balls, but their difference is at most $k - 2x$. Thus if earlier there were more blue balls, or the same as red balls, Adversary colors red, thus either decreases the difference, or pushes it into the other direction, but by no more than $k - 2x$. In other cases the colors are picked without any restriction, with the goal of making $f(i)$ as small as possible, thus it cannot increase (except from 0 to 1).

Now let us assume that the algorithm has finished and let $A$ be the set of balls not appearing in any queries, let $B$ be the set of balls that have degree one in the query hypergraph (i.e. appeared in an open query), and let $C$ be the set of the remaining balls. Note that we know the colors of the balls in $C$.

Consider an open query $Q$, and let $Q_0 := Q \cap B$ (i.e. those part of $Q$ that remained degree one). The answer for $Q$ was $x$. The way we chose $x$ and the definition of an open query shows that both colors appear in $Q_0$, and we do not know which balls are red, thus we cannot claim that a ball in $Q_0$ is a majority ball. It is also easy to see that no matter what colors the balls not in $Q_0$ have, it is consistent with the answers that $Q$ contains $x$ red balls but also that $Q$ contains $x$ blue (and $k - x$ red) balls. It means that inside $Q_0$, independently of the color of other balls, there are two possible colorings, where the difference between the color classes is at least $k - 3x$ and in one of the colorings there are more blue balls, in the other coloring there are more red balls.

So we cannot claim that a ball in $B$ (or in $A$) is a majority ball, thus we have to claim
it about a ball \( i \in C \). We know the color of \( i \), say red. We also know that there are at most \( k - 2x \) more red than blue balls in \( C \). If there are more than \( (k - 2x - |A|)/(k - 3x) \) open queries, then blue can be the majority (if there are \( x \) red balls in each of those queries and \( A \) is blue), a contradiction. We know that \( A \) and \( B \) contains together at most

\[
k(k - 2x - |A|)/(k - 3x) + |A| \leq k(k - 2x)/(k - 3x)
\]

balls. The remaining \( n' = n - k(k - 2x)/(k - 3x) \) balls must be covered by queries that contain at most \( k - x \) balls of degree one. A simple computation shows that at least \( 2n'/(2k - x) \) queries are needed for that, which easily implies the lower bound.

Proof of Proposition 10

It is easy to see that for \( n \) even the query hypergraph \( Q \) gives the solution if and only if it is connected, and for \( n \) odd \( Q \) gives the solution if and only if it consists of at most two connected components, that proves the statement.

Proof of Proposition 11

If \( k \) is even, then we ask all the \( k \)-sets containing a given \( k - 1 \)-set \( A \subset [n] \). Note that - using that \( k \) is even - we know that \( i, j \in [n] \) have the same color if and only if the answers to the queries \( A \cup \{i\} \) and \( A \cup \{j\} \) are the same. So we can partition \( [n] \setminus A \) into two parts so that the balls in each part have the same color. Additionally, if we ever get different answers for two such queries, then we know the number of balls of the corresponding colors inside \( A \) and we can choose a majority ball. If this is not the case, then - using that \( n \geq 2k - 1 \) - we have that all balls in \( [n] \setminus A \) are of the majority color.

If \( k \) is odd, then the previous argument does not work, as the answer to some queries could be that there are \( (k + 1)/2 \) and \( (k - 1)/2 \) balls of the two colors, even if the balls added to \( A \) are of different color. However, it cannot happen if \( A \) contains different number of red and blue balls. So we use unbalanced colorings here. Let us take a \( (k - 1) \)-uniform family \( F \) on \( [n] \) that does not have Property C and has cardinality \( d(k - 1, n) \). Moreover, if every pair of sets in \( F \) have intersection of size at least \( (k - 3)/2 \), then we add another set of size \( k - 1 \) that intersects one of them in a set of size less than \( (k - 3)/2 \). This way we get a family \( F' \), and the query set will be all \( k \)-sets containing members of \( F' \). There is a set \( F \in F' \) that is unbalanced, i.e. it contains at least \( (k + 1)/2 \) ball of the same color,
say blue.

If all the answers (for all the queries) are \{((k - 1)/2)\}, then all the balls not in \(F\) are red. In this case there are exactly \((k + 1)/2\) blue balls and every member of \(F'\) contains at least \((k - 1)/2\) of them. Then their intersection has size at least \((k - 3)/2\), a contradiction. That is why we added the additional set to \(F\).

Thus there is an answer different from \{((k - 1)/2)\}. Now similarly to the case, when \(k\) is even, we can find \(F'\), and then we know the relation of the outside balls to each other, and the number of the balls of the corresponding colors inside \(F\).

Proof of Proposition 12

To prove the lower bound for \(k\) even and \(n\) even, an easy computation shows that it is enough to prove that any query can contain at most one vertex of degree one. We prove it by contradiction, so if \(Q\) would contain \(i\) and \(j\) of degree one, then it is possible that the answer to \(Q\) is \{((k - 2)/2)\} and there are \((k - 2)/2\) red and \((k - 2)/2\) blue balls besides \(i\) and \(j\) in \(Q\). Furthermore, it is possible that altogether there are \(n/2\) blue and \((n - 4)/2\) red balls besides \(i\) and \(j\). We know that \(i\) and \(j\) have the same color, but we do not know if it is blue or red. Thus we do not know if there is a majority color or not.

In case \(k\) is even and \(n\) is odd, similarly it is enough to show that at most one query can contain two elements of degree one. We prove it - similarly to the previous case - by contradiction, since otherwise we can get two monochromatic pairs, and it is possible that there are \((n - 1)/2\) blue and \((n - 7)/2\) red balls besides those four balls. In this case we cannot show a majority ball.

In case \(k\) is odd, \(n\) is even and large enough, it is enough to show that for every \(i \in [n]\) we have \(\text{deg}_Q(i) \geq d(k - 1, n - 1)\). This is true, since otherwise we can color the hypergraph with vertex set \(\cup\{Q : i \in Q\} \setminus \{i\}\) and edge set \(\{Q \setminus \{i\} : i \in Q\}\) (the open neighborhood hypergraph of \(i\)) in a balanced way, thus we do not get any information about the color of \(i\). Then - using that \(n\) is large enough and even - we can color the remaining elements (i.e. \([n] \setminus \cup\{Q : i \in Q\}\)) such that the coloring of all the balls is balanced. But then it depends on the color of \(i\) if there is a majority color or not.

If \(n\) is odd, a similar argument shows that all but one of the balls have degree at least \(d(k - 1, n - 1)/2\) and this implies the statement. Indeed, otherwise there are \(i, j \in [n]\)
such that less than $d(k - 1, n - 1)$ queries contain either $i$ or $j$. Let us remove $i$ and $j$ from them and for those queries containing both we add a new ball $s \notin [n]$ instead. The resulting family (i.e. $Q' = \{Q \setminus \{i, j\} \cup \{s\} : i, j \in Q \in Q\} \cup \{Q \setminus \{i\} : i \in Q, j \notin Q, Q \in Q\} \cup \{Q \setminus \{j\} : j \in Q, i \notin Q, Q \in Q\}$ on the set $[n] \setminus \{i, j\} \cup \{s\}$) is $(k - 1)$-uniform and has Property C. This gives a coloring that is balanced on every set in $Q'$. Let red be the color of $s$. Adversary extends this coloring to all the balls except for $i$ and $j$ such a way that there are $(n - 1)/2$ blue and $(n - 3)/2$ red balls among the balls in $[n] \setminus \{i, j\}$. Then answers according to this coloring to queries containing neither $i$ nor $j$. Moreover, a query containing exactly one of them can also be answered according to this coloring as it is balanced. Finally Adversary answers $\{(k - 1)/2\}$ to queries containing both $i$ and $j$. It is easy to see that if at least one of $i$ and $j$ is red, the answers are consistent with the coloring. Thus any color can be minority (i.e. not majority), hence an other ball cannot be the majority ball, as we know its color. But $i$ can be red and $j$ blue, or the other way around. In that case the red ball is in minority.

Proof of Proposition 13.

We consider a $(k - 1)$-uniform family $F$ of size $m(k - 1, n - 1)$ on $[n - 1]$ that does not have property B. We query each set of size $k$ that contains an $F \in F$. If we get a yes answer, we can identify a monochromatic member $F$ of $F$, which is queried together with every other ball, thus we find out if the other balls have the same color as balls in $F$ or not, i.e. we can identify the color classes. Otherwise, as we know that there is a monochromatic member $F$ of $F$, every ball in $[n] \setminus F$ is the other color, thus majority using that $2k - 1 \leq n$. In particular the ball $n$ is a majority ball. The number of queries is at most $|F|/(n - k + 1)$, as we add one of the balls not in the set to every member of $F$ to get a query.

Proof of Proposition 14.

Let us start with the General Model. We prove that for even $n$ every ball $i$ has degree at least $m(k - 1)$. Indeed, otherwise it is possible that the set $\cup\{Q : i \in Q\} \setminus \{i\}$ is colored by blue and red in such a way, that no edge in the open neighborhood hypergraph of $i$ is monochromatic. Thus Adversary can answer such a way that $i$ does not appear in any
answers. As $n$ is large enough, these less than $m(k-1)$ queries contain less than $n/2$ balls, hence we can color the remaining balls such a way that $n/2$ are red and $n/2 - 1$ are blue. As $i$ can be of any color, we do not know if there is a majority ball or not.

If $n$ is odd, we will use a similar argument to the proof of Proposition 12 to show that all but one balls have degree at least $m(k-1)/2$. Indeed, otherwise there are less than $m(k-1)$ queries containing either $i$ or $j$. Let us remove $i$ and $j$ from them and for those queries containing both we add a new ball $s$. The resulting family has Property B. This gives a coloring on $[n]\{i,j\}\cup\{s\}$, let red be the color of $s$. Adversary extends this coloring to all the balls except for $i$ and $j$ such a way that there are $(n-1)/2$ blue and $(n-3)/2$ red balls among the balls in $[n]\{i,j\}$. Then answers according to this coloring, and answers no to queries containing both $i$ and $j$. It is easy to see that if one of $i$ and $j$ is red, the answers are consistent with the coloring. Thus any color can be minority, hence another ball cannot be the majority ball, as we know its color. But $i$ can be red and $j$ blue, or the other way around. In that case the red ball is in minority.

In Borzyszkowski’s Model if $n$ is even, the same proof works, as we can point out a pair of balls not equal to $i$ in every query. If $n$ is odd, the same proof does not work, as it is possible that only $s$ is of color red in a query. In that case either $i$ or $j$ has to be put in the pair that is pointed out, thus we might find out its color. However, we can show that all but one of the balls have to be contained in at least $m(k-2)/2$ queries. Indeed, otherwise there are less than $m(k-2)$ balls containing either $i$ or $j$. Let us remove $i$ and $j$ from them and for those queries containing only one of them we remove another arbitrary ball. The resulting family has Property B. From here it works as in the case $n$ is odd. 

Proof of Theorem 15

We start with proving the upper bound. Let us assume first $k$ is odd. First we ask disjoint sets till we get two different answers. If all the answers to more than $n/(k+1)$ disjoint queries are the same, that color is the majority. If the answers for $A$ and $B$ are different, then we ask all $k$-subsets of $A\cup B$. This way we can identify a balanced $(k-1)$-set. Indeed, only such sets have the property that adding other balls the answers can be both blue or red. And there are at least $(k+1)/2$ blue and at least $(k+1)/2$ red balls in $A\cup B$, thus there exists a balanced set and both blue and red balls to add. If we query the balanced set with any other ball, we find out the color of that ball. Since there are at least
\((k + 1)/2\) blue and at least \((k + 1)/2\) red balls in \(A \cup B\), we find out the color of every ball \(i\) in \(A \cup B\), as there is a balanced set not containing \(i\). After that, we query the balanced set with every other ball to get the color of those balls.

If \(k\) is even, our goal is to find a balanced \((k - 2)\)-set. First we ask disjoint sets. If all the answers to more than \(n/(k + 1)\) disjoint queries are the same color, that color is the majority. If the answers for \(A\) and \(B\) are different, then we ask all \(k\)-subsets of \(A \cup B\). One can easily see that this way we can identify a balanced \((k - 2)\)-set \(C\) similarly to the previous case. Indeed, these have the property that all three possible answers occur in sets containing them. After that, we query the balanced set with pairs of new balls. The answer to that query tells us the number of balls in each color among the two new balls.

For the lower bound we describe Adversary’s strategy. As usual, let \(Q(i)\) denote the set of queries during the first \(i\) round. When answering a query \(Q\), Adversary also gives the color of some balls (not necessarily in \(Q\)) as additional information and also decides the number \(p(Q)\) of red balls in \(Q\), but does not tell it. Let \(R(i)\) (resp. \(B(i)\)) denote the set of balls that Adversary gave color red (resp. blue) during the first \(i\) rounds. Note that it is possible that the answers to the queries imply the colors of some other balls, but they are not included in \(R(i)\) nor in \(B(i)\). Whenever a ball reaches degree two with a query, its color gets decided. For the balls that appear the first time, Adversary decides how many of them are blue and how many are red. The goal of Adversary is to make the coloring balanced, but the queries unbalanced.

After the \(i^{th}\) round a query \(Q\) asked earlier contains some balls in \(R(i)\) and Adversary has decided \(p(Q)\); let \(r_i(Q)\) denote their difference \(p(Q) - |R(i) \cap Q|\), i.e. the number of balls that should be colored red among the balls of degree one in \(Q\). Let \(b_i(Q) = k - p(Q) - |B(i) \cap Q|\) defined similarly for blue balls. Note that \(r_i(Q) + b_i(Q)\) is equal to the number of balls of degree one in \(Q\). Also note that at any point the number of red balls among all the balls that have appeared in queries is \(r(i) = |R(i)| + \sum_{Q \in Q(i)} r_i(Q)\). Similarly the number of blue balls is \(b(i) = |B(i)| + \sum_{Q \in Q(i)} b_i(Q)\), let \(g(i) = |r(i) - b(i)|\).

When the \(i^{th}\) query \(Q\) is asked, it contains some balls already colored (those in \(R(i - 1) \cup B(i - 1)\)). It also contains some balls that previously had degree one. Adversary will color these balls, so first we describe, how. We partition those according to the other query \(Q'\) that contains them. Adversary colors the balls such a way that neither \(r_i(Q')\) nor \(b_i(Q')\)
becomes zero, if it is possible (clearly it is impossible only if \( r_{i-1}(Q') = 0 \) or \( b_{i-1}(Q') = 0 \) or if there remains only one ball of degree one in \( Q' \)). Note that this does not effect \( g(i) \).

Finally, Adversary should decide \( p(Q) \) (or equivalently the value of \( r_i(Q) \) or \( b_i(Q) \)). If there are less than five balls of degree one, \( p(Q) \) is chosen such a way that makes \( g(i) \) as small as possible. Otherwise there is a choice such that \( r_i(Q) \neq 0 \neq b_i(Q) \) which makes \( Q \) containing at least \( \lceil k/2 \rceil + 2 \) balls of one of the colors (if \( r_i(Q) = 1 \) does not satisfy this, then \( b_i(Q) = 1 \) does). Adversary picks among those choices the one that makes \( g(i) \) as small as possible.

We claim that \( g(i) \) is always at most 4. Let us assume indirectly that \( i \) is the smallest number such that \( g(i) > 4 \). Without loss of generality \( b(i) > r(i) + 4 \) and let \( Q \) be the \( i^{th} \) query. Coloring the balls that previously had degree one does not affect \( g(i) \), thus obviously \( b_i(Q) \geq r_i(Q) \). If Adversary picked \( b_i(Q) - 1 \) instead of \( b_i(Q) \), that would result in smaller \( g(i) \). The only possible reason for Adversary to pick \( b_i(Q) \) is that the other choice would result in exactly \( \lceil k/2 \rceil + 1 \) blue balls in \( Q \). Thus Adversary could pick \( b_i(Q) - 4 \) instead of \( b_i(Q) \), and that would make \( g(i) \) smaller, a contradiction.

Let us assume an algorithm ends with round \( i \) showing that red is the majority color. We claim that there cannot be more than 3 queries containing more than 4 balls of degree one. Indeed, if a query \( Q \) contains at least 5 balls of degree one, Adversary answered such a way that \( r_i(Q) \geq 1 \). However, all the answers would remain valid if \( r_i(Q) \) would be smaller by one (as the number of balls in the majority color in \( Q \) were at least \( \lceil k/2 \rceil + 2 \)). If there were 4 such queries, that would change the majority color. A simple calculation gives the lower bound.

Proof of Theorem 16

Let us start with the upper bound. Suppose that the elements of \([n]\) are around a circle in the natural order and let \( \mathcal{F} \) be the family of the intervals of the circle of length \( k - 1 \). We query each set of size \( k \) that contains an \( F \in \mathcal{F} \). One can easily see that there are \( n(n - k) \) queries. In particular we query each interval of size \( k \). If the answers to those intervals are the same color, then this answer is the majority color. If the answer is that there is no majority in any of these queries, then there is no majority in the whole underlying set. Otherwise there are two neighboring intervals with different answers. Then their intersection is a balanced \((k-1)\)-set. It is queried together with every other ball, thus
we find out the color of those balls, just like in the adaptive case.

For the lower bound if $n$ is even we show that every ball $i$ has degree at least $n/2k$. Indeed, otherwise we can color all the balls that are contained in a query together with $i$ blue, then we color the other balls such that there are $\lfloor n/2 \rfloor$ blue balls, and answer accordingly. Then the answers to the queries containing $i$ are obviously blue, thus we do not know anything about the color of $i$, but the majority color depends on $i$.

If $n$ is odd we show that all but one balls have degree at least $n/4k$. Indeed, otherwise there are less than $n/2k$ queries containing either $i$ or $j$. Then we can color all the balls that are contained in a query together with $i$ or $j$ blue, then, as in the case $n$ even we color the other ball such that there are $\lfloor n/2 \rfloor$ blue balls, and answer accordingly. Then the answers to the queries containing $i$ or $j$ are obviously blue, thus we do not know anything about the color of $i$ or $j$, but the majority color depends on them. 

$\square$
Summary of the results:

| X  | OM | CM | GM | BM |
|----|----|----|----|----|
| \( A(X, k, n) \geq \) | \( f_1(n, k) \) | \( 6n/(5k + 6) - c(k) \) | \( (3n + 5)/4 \) | \( f_4(n, k) \) |
| \( A(X, k, n) \leq \) | \( f_2(n, k) \) | \( n/[\frac{k}{2}] + O(k) \) | \( n - k + m(k) \) | \( f_4(n, k) \) |

Table 1: Adaptive bounds

- \( f_1(n, k) := \begin{cases} \left\lceil \frac{n-1}{k-1} \right\rceil & \text{if } k \text{ is odd}, \\
\frac{n}{k-1} - O(n^{1/3}) & \text{if } k \text{ is even.} \end{cases} \)
- \( f_2(n, k) := \begin{cases} \left\lceil \frac{n+1}{k-1} \right\rceil & \text{if } n \text{ is even}, \\
\frac{n-1}{k-1} & \text{if } n \text{ is odd.} \end{cases} \)
- \( f_4(n, k) := \begin{cases} \left\lceil \frac{n}{2} + k - 2 \right\rceil & \text{if } n \text{ is even}, \\
\left\lceil \frac{n}{2} \right\rceil + k - 3 & \text{if } n \text{ is odd.} \end{cases} \)
Table 2: Nonadaptive bounds

| X     | OM     | CM     | GM     | BM     |
|-------|--------|--------|--------|--------|
| $N(X, k, n)$ $\geq$ | $f_5(n, k)$ | $f_7(n, k)$ | $f_8(n, k) \times m(k - 1)$ | $f_8(n, k) \times m(k - 2)$ |
|       | (Proposition 10) | (Proposition 12) | (Proposition 14) | (Proposition 14) |
| $N(X, k, n)$ $\leq$ | $f_5(n, k)$ | $f_6(n, k)$ | $(n - k + 1) \times m(k - 1, n - 1)$ | $(n - k + 1) \times m(k - 1, n - 1)$ |
|       | (Proposition 10) | (Proposition 11) | (Proposition 13) | (Proposition 13) |

- $f_5(n, k) := \begin{cases} \lceil \frac{n-1}{k-1} \rceil & \text{if } n \text{ is even}, \\ \lceil \frac{n-2}{k-1} \rceil & \text{if } n \text{ is odd}. \end{cases}$

- $f_6(n, k) := \begin{cases} n - k + 1 & \text{if } k \text{ is even}, \\ (n - k + 1)(1 + d(k - 1, n)) & \text{if } k \text{ is odd}. \end{cases}$

- $f_7(n, k) := \begin{cases} 2n/(k + 1) & \text{if } k \text{ is even and } n \text{ is even}, \\ (2n - k + 1)/(k + 1) & \text{if } k \text{ is even and } n \text{ is odd}, \\ [n/k] \times d(k - 1, n - 1] & \text{if } k \text{ is odd and } n \text{ is even}, \\ [(n - 1)/2k \times d(k - 1, n - 1)] & \text{if } k \text{ is odd and } n \text{ is odd}, \end{cases}$

- $f_8(n, k) := \begin{cases} \frac{n}{k} & \text{if } n \text{ is even}, \\ \frac{n-1}{2k} & \text{if } n \text{ is odd}. \end{cases}$