Combinatorics of canonical bases revisited: type A

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Abstract
We initiate a new approach to the study of the combinatorics of several parametrizations of canonical bases. In this work we deal with Lie algebras of type $A$. Using geometric objects called rhombic tilings we derive a “Crossing Formula” for the action of the crystal operators on Lusztig data for an arbitrary reduced word of the longest Weyl group element. We provide the following three applications of this result. Using the tropical Chamber Ansatz of Berenstein–Fomin–Zelevinsky we prove an enhanced version of the Anderson–Mirković conjecture for the crystal structure on MV polytopes. We establish a duality between Kashiwara’s string and Lusztig’s parametrization, revealing that each of them is controlled by the crystal structure of the other. We identify the potential functions of the unipotent radical of a Borel subgroup of $SL_n$ defined by Berenstein–Kazhdan and Gross–Hacking–Keel–Kontsevich, respectively, with a function arising from the crystal structure on Lusztig data.

Keywords
Rhombic tilings · Lusztig’s parametrization · Crystal bases · String parametrization · MV-polytopes · Double Bruhat cells · Cluster algebras · Berenstein–Kazhdan potential · Gross–Hacking–Keel–Kontsevich potential

Mathematics Subject Classification 17B10 · 17B37
1 Introduction

Let $\mathfrak{g}$ be a complex simple Lie algebra and $n^-$ its negative part in the Cartan decomposition. Bases of PBW type for the universal enveloping algebra of $n^-$ play an important role in the study of representations of $\mathfrak{g}$. Passing to the negative part $U_q^{-}$ of the quantized enveloping algebra of $\mathfrak{g}$, Lusztig constructed PBW-type bases by introducing so-called braid group operators. These bases consist of ordered monomials of root vectors $F_\beta$, one for each positive root $\beta$, depending on a choice of convex ordering of the positive roots of $\mathfrak{g}$.

Lusztig discovered in [21] a basis $B$ of $U_q^{-}$, called the canonical basis, with various favorable properties. The canonical basis is, by construction, in natural bijection with any PBW-type basis which leads to remarkable piecewise linear combinatorics.

Fixing a convex ordering $\leq$ of the positive roots we can identify a PBW-type basis element $b$ with the exponent vector of the corresponding monomial in the $\leq$-ordered root vectors. This element of $\mathbb{N}^N$, where $N$ is the number of positive roots of $\mathfrak{g}$, is called the $\leq$-Lusztig datum of $b$ (or equivalent the $\leq$-Lusztig datum of the element $b \in B$ to which $b$ maps under the natural bijection of the PBW-type basis with $B$). On the set of all $\leq$-Lusztig data we have a crystal structure in the sense of [18] inducing a colored graph on the canonical basis. The vertices of this graph, called the crystal graph, are given by the $\leq$-Lusztig data and the arrows are induced by Kashiwara operators $f_\alpha$, one for every simple root $\alpha$. The action of $f_\alpha$ on a $\leq$-Lusztig datum is given explicitly if $\alpha$ is the $\leq$-minimal positive root. For arbitrary convex orderings and simple roots
the operator $f_a$ is defined as a composition of piecewise linear maps and no explicit description is known.

One of the aims of this paper is to give an explicit description of $f_a$ that works for all choices of convex orderings. We solve this problem for simple Lie algebras of type $A$.

We assume from now on that $\mathfrak{g} = \text{sl}_n(\mathbb{C})$. Let $I$ be the index set of simple roots of $\mathfrak{g}$. For each $a \in I$ and each convex order $\leq$ of the positive roots of $\mathfrak{g}$ we define in Sect. 3.1 finitely many sequences $\gamma = (\gamma_j)$ of positive roots of $\mathfrak{g}$ with certain properties which we call $a$-crossings. These sequences come with an order relation $\preceq$. To each $a$-crossing $\gamma$ we associate a vector $r_\gamma \in \mathbb{Z}^N$ and linear form $s_\gamma \in \text{Hom}(\mathbb{Z}^N, \mathbb{Z})$. Our main result reads as follows.

**Theorem 1.1** [Crossing formula (Theorem 3.15)]. For a $\preceq$-Lusztig datum $x$ let $\gamma$ be $\preceq$ maximal such that $(s_\gamma)x$ is maximal. Then $f_a x = x + r_\gamma$. Furthermore, the set of all $\gamma$ with $r_\gamma = f_a x - x$ can be described explicitly.

Our work was inspired by the results of [28]. Here Reineke obtained an analogue of the Crossing Formula for convex orderings adapted to Dynkin quivers satisfying a homological condition. This was achieved using Ringel’s Hall algebra which is isomorphic to $U_q^{-}$. Under this isomorphism the PBW-type bases for convex ordering are given naturally in terms of representations of quivers. In [31] it is shown that Reineke’s construction has a natural interpretation in the geometric construction of crystal bases using quiver varieties. However, for PBW-type bases corresponding to convex orderings that are not adapted to a quiver there are no comparable techniques to [28,31] available. The Crossing Formula is a generalization of [28] to all convex orderings in type $A$ obtained by purely combinatorial methods.

A particular case of the Crossing Formula for convex orderings corresponding to so-called good enumerations of Dynkin diagrams was obtained in [30].

Since a $\preceq$-Lusztig datum is given by a vector $x \in \mathbb{N}^N$, the action of the Kashiwara operator $f_a$ is given by adding a vector in $f_a x - x \in \mathbb{Z}^N$. We call vectors of the form $f_a x - x$ Reineke vectors. Surprisingly, these vectors play an important role in the interplay of various parametrizations of canonical bases. The rest of this work is concerned with this relationship.

In Sect. 4 we deal with Mirković Vilonen (MV) polytopes. These are momentum polytopes associated to Mirković Vilonen cycles, certain subvarieties of the affine Grassmannian. Each MV polytope packs together all Lusztig data for varying convex orderings $\preceq$ that are in bijection with a fixed canonical basis element. Each Lusztig datum can be read of from a distinguished path in the 1-skeleton of the polytope. This defines a crystal isomorphism between the crystal structure on MV polytopes and the crystal structure on $\preceq$-Lusztig data ([16]).

An explicit description of the action of the Kashiwara operators on MV polytopes is desirable and in particular would lead to an inductive procedure to generate all such polytopes. Anderson and Mirković conjectured such a description. This conjecture was proved in type $A$ by Kamnitzer [16] and subsequently Saito [29]. While Kamnitzer gave a counterexample [16, Section 5.4] in type $C$ it remains an open question in simply-laced types. Using our description of Reineke vectors in Theorem 3.15 we prove a
stronger statement and obtain a new proof of the Anderson–Mirković conjecture for type $A$, which provides a conceptual explanation.

In [26] a modified version of the Anderson–Mirković conjecture for the types $B$ and $C$ is proved. It would be interesting to see if their result can also be deduced from our methods using folding techniques.

Let $N$ be the unipotent radical of a Borel subgroup of $\text{SL}_n(\mathbb{C})$. There are two natural ways to realize an irreducible highest weight representation $V$ of $g$. The representation $V$ appears as a quotient of $U(n^-)$ but also canonically as a subspace of the coordinate ring $\mathbb{C}[N]$ which can be identified with the graded dual $U(n^-)^*$. This leads to another parametrizations of $B$ in terms of string data following work of Kashiwara, Berenstein-Zelevinsky and Littelmann [4,17,20]. This string parametrization, encoded in the string cone, can be thought of more naturally as a parametrization of the dual basis $B^*$.

On the one hand the action of the Kashiwara operators on the string cone is given by an explicit formula while the inequalities of this cone seem intricate. On the other hand the cone corresponding to Lusztig data is the standard cone given by the positive orthant while the action of the Kashiwara operators seems mysterious. In the Crossing Formula (Theorem 3.15) we establish a formula for the action of the Kashiwara operators on any $\leq$-Lusztig datum in the flavor of the description on string cones.

In [32] Zelikson obtained the remarkable result that for convex orderings $\leq$ adapted to quivers of type $A$ the Reineke vectors associated to the set of $\leq$-Lusztig data yield defining inequalities of the string cone associated to the reversed order of $\leq$. In Sect. 5 we extend Zelikson’s result to all convex orderings in type $A$. This is obtained by attaching a Laurent polynomial $r_{\leq} \in \mathbb{Z}[x_\beta \mid \beta \in \Phi^+]$ to the set of Reineke vectors associated to a convex ordering $\leq$. Surprisingly, the functions $r_{\leq}$ transform under the geometric lifting of the piecewise-linear transformation of the string cone defined by Berenstein and Zelevinsky in [5]. By this result we recover the string cone inequalities for type $A$ of Gleizer–Postnikov [14]. We show that also the dual statement holds: The set of vectors describing the crystal structure on the string cone give the defining inequalities of the positive orthant and therefore the cone of $\leq$-Lusztig data.

In Sect. 6 we recover the duality between Lusztig’s parametrization and the string cone in the setup of cluster algebras and mirror symmetry. We identify $r_{\leq}$ with two potential functions on the unipotent radical $N$ appearing in the current literature. The first potential function $f_{\chi}$ was defined by Berenstein and Kazhdan in the context of geometric crystals in [3] and the other potential function $W$ by Gross, Hacking, Keel and Kontsevich in the context of mirror duality for cluster varieties in [15]. Each convex ordering $\leq$ on $\Phi^+$ defines a cluster in the coordinate ring of $N$ and thereby an open embedding of a torus $T_{\leq}$ into $N$. We show that $r_{\leq}$ coincides (up to an explicit isomorphism) with the restrictions of $f_{\chi}$ and $W$ to $T_{\leq}$. Using this identification and the representation theoretical interpretation of $r_{\leq}$ via the Crossing Formula, we obtain explicit formulas for $W$ and $f$ in the corresponding cluster coordinates. By this description we prove structural results about the potential functions (see Theorems 6.6 and 6.13). In particular the enhanced version of the Anderson–Mirković conjecture proved in Sect. 4 implies that the Landau–Ginzburg potential function defined by Gross, Hacking, Keel and Kontsevich restricted to any $T_{\leq}$ is a Laurent polynomial without constant term, all coefficients in $\{0, 1\}$ and exponents in $\{0, -1\}$. As another
consequence we obtain that the cones attached to the tropicalizations of the potential functions coincide with the string cones (up to an explicit change of coordinates).

2 Rhombic tilings and Lusztig data

2.1 Notations

Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ be the Lie algebra of type $A_{n-1}$ and $\mathfrak{h} \subset \mathfrak{g}$ the Cartan subalgebra consisting of the diagonal matrices in $\mathfrak{g}$. For $s \in [n] := \{1, \ldots, n\}$ let $\epsilon_s \in \mathfrak{h}^*$ be the functional with $\epsilon_s(\text{diag}(h_1, \ldots, h_n)) = h_s$. Let $\Phi^+$ be the set of positive roots in $\mathfrak{g}$ given by

$$\Phi^+ = \{\epsilon_s - \epsilon_t | 1 \leq s < t \leq n\}.$$ 

We abbreviate by $N = \frac{n(n-1)}{2}$ the cardinality of $\Phi^+$. We denote by $\alpha_a := \epsilon_a - \epsilon_a + 1$ ($a \in [n-1]$) the simple roots of $\mathfrak{g}$ and write $\alpha_{s,t} := \epsilon_s - \epsilon_t + 1 \in \Phi^+$. The fundamental weights of $\mathfrak{g}$ are given by

$$\omega_a := \sum_{s \in [a]} \epsilon_s.$$ 

We denote by $U_q^{-}$ the negative part of the quantized enveloping algebra of $\mathfrak{g}$ with generators $F_a$, by $B(\infty)$ its crystal basis and by $f_a$ and $e_a$ the corresponding Kashiwara-operators (see [18]).

Let $W$ be the Weyl group of $\mathfrak{g}$ which is naturally isomorphic to $S_n$, the symmetric group in $n$ letters. The group $W$ is generated by the simple reflections $\sigma_i$ ($i \in [n-1]$) interchanging $i$ and $i + 1$ with relations

$$\sigma_i^2 = id,$$

$$\sigma_{i_1}\sigma_{i_2} = \sigma_{i_2}\sigma_{i_1} \quad \text{for } |i_1 - i_2| \geq 2 \quad \text{(commutation relation)},$$

$$\sigma_{i_1}\sigma_{i_2}\sigma_{i_1} = \sigma_{i_2}\sigma_{i_1}\sigma_{i_2} \quad \text{for } |i_1 - i_2| = 1 \quad \text{(braid relation).}$$

The Weyl group $W$ has a unique longest element $w_0$ of length $N$. For a reduced expression $\sigma_{i_1} \cdots \sigma_{i_N}$ of $w_0$ we write $i := (i_1, \ldots, i_N)$ and call $i$ a reduced word (for $w_0$). We call a total ordering $\leq$ on $\Phi^+$ convex if whenever $\beta_1, \beta_2, \beta_1 + \beta_2 \in \Phi^+$ we have either $\beta_1 \leq \beta_1 + \beta_2 \leq \beta_2$ or $\beta_2 \leq \beta_1 + \beta_2 \leq \beta_1$. The set of total convex ordering is in natural bijection with the set of reduced words. Namely, for a reduced word $i = (i_1, \ldots, i_N)$ the total ordering $\leq_i$ on $\Phi^+$ given by

$$\alpha_{i_1} \leq_i s_{i_1}(\alpha_{i_2}) \leq_i \cdots \leq_i s_{i_1} \cdots s_{i_{N-1}}(\alpha_{i_N})$$

(3)

is convex and every convex ordering on $\Phi^+$ arises that way.

2.2 Rhombic tilings

Our main combinatorial tool are rhombic tilings, which are geometric objects corresponding to commutation equivalence classes of convex orderings on $\Phi^+$. In the following we recall from [9] how rhombic tilings illustrate convex orderings. We then introduce some notions using rhombic tilings which are essential for the Crossing Formula derived in this work.
Let \( P_0 \) be a regular \( 2n \)-gon with side length 1. By a (rhombic) tiling \( T \) we mean a tiling of \( P_0 \) into rhombi (tiles) with side length 1. We label the edges on the left boundary of \( P_0 \) consecutively starting with the lowest vertex \( v_0 \) of \( P_0 \) and proceeding clockwise. This induces a labeling of all edges in \( T \) such that parallel edges have the same label.

**Example 2.1** We give an example for \( n = 5 \). Since all other edge labels are determined by the labels of the edges lying on the left boundary of \( P_0 \) we omit them.

We call a connected set of edges in \( T \) a border if it contains exactly one edge with each label. Fixing \( s \in [2n] \) we define a partition

\[
\bigcup_k \mathcal{T}_k = T
\]

of the tiles in \( T \) as follows. Let \( b_1, \ldots, b_{2n} \) be the edges of the boundary of \( P_0 \) in clockwise order starting from \( v_0 \). Let \( B_1 \) be the border consisting of the edges \( b_{n+s+1}, \ldots, b_{2n+s} \) where we read the indices modulo \( 2n \). Let \( T_1 \) be the set of all tiles in \( T \) intersecting \( B_1 \) in two edges (a tile of this form always exists by [9, Lemma 2.1]). We change the border \( B_1 \) into the new border \( B_2 \) by replacing for every tile \( T \in T_1 \) the two edges intersecting \( B_1 \) by the other two edges of \( T \). We define \( T_2 \) as the set of all tiles in \( T \setminus T_1 \) intersecting \( B_2 \) with two edges. We proceed inductively until \( B_k \) consists only of edges lying on the boundary of \( P_0 \).

**Example 2.2** We give the decomposition \( \bigcup_k \mathcal{T}_k = T \) for the tiling from Example 2.1. For this we denote by \([t, u]\) with \( 1 \leq t < u \leq n \) the tile \( T \) which has edges with labels in the set \([t, u]\).

The partition of the tiles with respect to \( s = 5 \) is

\[
\begin{align*}
\mathcal{T}_1 &= \{[2, 3]\}, & \mathcal{T}_2 &= \{[1, 3]\}, & \mathcal{T}_3 &= \{[1, 2]\}, & \mathcal{T}_4 &= \{[1, 4]\}, & \mathcal{T}_5 &= \{[1, 5]\}, \\
\mathcal{T}_6 &= \{[4, 5]\}, & \mathcal{T}_7 &= \{[2, 5]\}, & \mathcal{T}_8 &= \{[3, 5], [2, 4]\}, & \mathcal{T}_9 &= \{[3, 4]\}.
\end{align*}
\]

The partition of the tiles with respect to \( s = 3 \) is

\[
\begin{align*}
\mathcal{T}_1 &= \{[2, 3]\}, & \mathcal{T}_2 &= \{[1, 3]\}, & \mathcal{T}_3 &= \{[1, 2], [3, 5]\}, & \mathcal{T}_4 &= \{[2, 5], [3, 4]\} \\
\mathcal{T}_5 &= \{[2, 4]\}, & \mathcal{T}_6 &= \{[4, 5]\}, & \mathcal{T}_7 &= \{[1, 4]\}, & \mathcal{T}_8 &= \{[1, 5]\}.
\end{align*}
\]
Definition 2.3 A sequence $\gamma = (\gamma_i)_{1 \leq i \leq m}$ of tiles in $\mathcal{T}$ is called neighbour sequence if for all $i$ with $1 \leq i < m - 1$ the tiles $\gamma_i$ and $\gamma_{i+1}$ share an edge. For $s \in [n]$ we define the $s$-strip $L^s$ to be the neighbour sequence $\gamma = (\gamma_i)_{1 \leq i \leq n-1}$ consisting of all tiles with edges labeled by $s$ oriented by requiring that $\gamma_1$ meets the left boundary of $P_0$.

Example 2.4 We continue with Example 2.1 and depict the 2-strip $L^2$.

Definition 2.5 For $s \in [2n]$ a neighbour sequence $(\gamma_1, \ldots, \gamma_m) \subset \mathcal{T}$ is called $s$-ascending if for $1 \leq i < m - 1$

$$\kappa(\gamma_i) < \kappa(\gamma_{i+1}),$$

where $\kappa(\gamma_i)$ is defined by $\gamma_i \in \mathcal{T}_k(\gamma_i)$ for the partition $\mathcal{T} = \sqcup \mathcal{T}_k$ defined in (4).

Definition 2.6 For $s \in [2n]$ we define a partial ordering $\leq_s$ on $\mathcal{T}$ by setting for $T_1, T_2 \in \mathcal{T}$

$$T_1 \leq_s T_2 \iff \exists s\text{-ascending neighbour sequence } (\gamma_i)_{1 \leq i \leq m} \text{ with } T_1 = \gamma_1, T_2 = \gamma_m.$$ Given two neighbour sequences $\gamma = (\gamma_i)_{1 \leq i \leq m}$ and $\lambda = (\lambda_i)_{1 \leq i \leq m'}$ we denote by $\gamma \circ \lambda$ the neighbour sequence given by

$$\gamma \circ \lambda = \begin{cases} (\rho_i)_{1 \leq i \leq m+m'-1} & \text{if } \gamma_1 = \lambda_{m'}, \\ \emptyset & \text{else,} \end{cases}$$

$$\rho_i = \begin{cases} \lambda_i & \text{if } i \leq m', \\ \gamma_{i-m'+1} & \text{if } i > m'. \end{cases}$$

Setting

$$-\gamma := (\gamma_{m-i+1})_{1 \leq i \leq m}$$

we obtain the following characterization of $a$-ascending neighbour sequences.
Remark 2.7 An \(a\)-ascending neighbour sequence is composed of segments of \(\text{sgn}(a - s)\mathcal{L}_{j}^{s}\) with \(s \in [n]\), i.e. the \(a\)-ascending neighbour sequences are the sequences in \(T\) of the form

\[
\text{sgn}(a - s_1)\left(\mathcal{L}_{j}^{s_1}\right)_{m_1 \leq j \leq m_1'} \circ \cdots \circ \text{sgn}(a - s_M)\left(\mathcal{L}_{j}^{s_M}\right)_{m_M \leq j \leq m_M'}.
\]

2.3 Tilings associated to reduced words

For a vertex \(v\) of \(T\) we consider a minimal path of edges connecting the lowest vertex \(v_0\) and \(v\). The set \(S\) containing the labels of the edges of such a minimal path is independent of the path and we write \(v = v_S\). We denote a tile \(T\) with vertices \(v_S, v_{S \cup \{t\}}, v_{S \cup \{u\}}, v_{S \cup \{t, u\}}\) by \(T = [t, u; S]\). Since \(\mathcal{L}_{k}\) and \(\mathcal{L}_{u}\) intersect and since there are as many tiles in \(T\) as there are pairs \((t, u)\) with \(1 \leq t < u \leq n\), the set \(S\) is uniquely determined by the pair \((t, u)\) and we write \(T = [t, u; S]_T = [t, u][T]\). We suppress the index \(T\) when it is clear from the context which tiling \(T\) we refer to.

Example 2.8 We continue with Example 2.1 and depict the vertex \(v_{\{2, 3, 4\}}\) of \(T\) and the tile \(T = [4, 5; \{2, 3\}]\).

![Diagram of a tiling with vertices and a tile labeled as T]

We consider two reduced words \(i\) and \(j\) for \(w_0\) as equivalent if they differ by a sequence of commutation relations (1). The set of equivalence classes of reduced words can then by [9, Theorem 2.2] be identified with the set of rhombic tilings \(T\) of the \(2n\)-gon as follows.

Let \(T = \{T_1, T_2, \ldots, T_m\}\) be a tiling and \(\leq'_n\) any refinement of \(\leq_n\) (see Definition 2.6) to a total order on the tiles of \(T\):

\[
T_1 = [t_1, u_1; S_1] \leq'_n T_2 = [t_2, u_2; S_2] \leq'_n \cdots \leq'_n T_N = [t_N, u_N; S_N].
\]

Then the sequence \(i := (i_{|S_1|+1}, \ldots, i_{|S_N|+1})\) is a reduced word for \(w_0\) corresponding to \(T\) and we write \(T = T_i\). In particular, we have \(T_i = T_j\) if \(i\) and \(j\) are related by a commutation relation (1). Furthermore, identifying the tiles of \(T\) with the set of positive roots \(\Phi^+\) using the rule

\[
[s, t]_{T_i} \mapsto \alpha_{s,t-1} \quad (s < t)
\]

we have \(\leq' = \leq_i\), where the later is given in (3).
Example 2.9 For $\mathcal{T}$ given in Example 2.1 there are two refinements of $\leq_n$ to a total ordering (see also Example 2.2)

$[2, 3] \leq_n [1, 3] \leq [1, 2] \leq [1, 4] \leq [2, 5] \leq [3, 5] \leq [2, 4] \leq [3, 4]$,

$[2, 3] \leq''_n [1, 3] \leq'' [1, 2] \leq'' [1, 4] \leq'' [2, 5] \leq'' [3, 5] \leq'' [2, 4] \leq'' [3, 4]$.

Under the identification (5) the orderings $\leq'$ and $\leq''$ translate to the convex orderings $\leq_i$ and $\leq_j$ on $\Phi^+$, respectively, where

$$i = (2, 1, 2, 3, 4, 3, 2, 1, 3, 2),$$

$$j = (2, 1, 2, 3, 4, 3, 2, 3, 1, 2).$$

Thus $\mathcal{T} = \mathcal{T}_i = \mathcal{T}_j$. Note that $i$ and $j$ are related by a commutation relation.

Assume next that $i = (i_1, \ldots, i_N)$ and $j = (j_1, \ldots, j_N)$ are two reduced words such their corresponding reduced expressions differ by a braid relation $\sigma_{i_k} \sigma_{i_{k+1}} \sigma_{i_{k+2}} = \sigma_{j_k} \sigma_{j_{k+1}} \sigma_{j_{k+2}}$ as in (2). Then $\mathcal{T}_i$ and $\mathcal{T}_j$ are obtained from each other by interchanging the two subtilings

$$[[s, t; S], [s, u; S \cup \{t\}], [t, u; S]] \leftrightarrow [[t, u; S \cup \{s\}], [s, u; S], [s, t; S \cup \{u\}]]$$

(6)

of a regular 6-gon $[s, t; S] \cup [s, u; S \cup \{t\}] \cup [t, u; S] = [t, u; S \cup \{s\}] \cup [s, u; S] \cup [s, t; S \cup \{u\}]$ with $s < t < u$. This can be illustrated as follows.

We call a tiling $\mathcal{H}$ of the regular 6-gon a hexagon. We call the replacement of $\mathcal{H}$ in $\mathcal{T}$ with the unique different tiling of the 6-gon a flip of $\mathcal{T}$ at $\mathcal{H}$.

Example 2.10 We depict a flip of the tiling $\mathcal{T}_{(1,2,3,1,2,1)}$ at the hexagon given by the tiles $[2, 3], [2, 4], [3, 4]$ yielding the tiling $\mathcal{T}_{(1,2,3,2,1,2)}$. 

![Diagram of a hexagon with tiles]
2.4 Lusztig’s parametrization of the canonical basis

Using braid group operators Lusztig defined in [21] for each reduced word \( \mathbf{i} = (i_1, \ldots, i_N) \) a PBW-type basis of \( U_q^- \)

\[
B_i = \left\{ F_{i_1}^{(x_{\beta_1})} \cdots F_{i_N}^{(x_{\beta_N})} \mid (x_{\beta_1}, \ldots, x_{\beta_N}) \in \mathbb{N}^N \right\},
\]

where \( \{\beta_1, \ldots, \beta_N\} = \Phi^+, \beta_1 \leq \cdots \leq \beta_N, F_{i_1,\beta_1} = T_{i_1} \cdots T_{i_{j-1}} F_j \) is given via the braid group action \( T_i \) defined in [22, Section 1.3], \( X^{(m)} \) is the \( q \)-divided power defined by \( X^{(m)} := [x_1, \ldots, x_m]^m \) and \( [m] := q^{m-1} + \cdots + q^{-m+1} \).

**Definition 2.11** For \( x = (x_1, \ldots, x_N) \in \mathbb{N}^N \), we denote the element \( F_{i_1,\beta_1}^{(x_1)} \cdots F_{i_N,\beta_N}^{(x_N)} \) by \( F^x \) and call \( x \) its \( i \)-Lusztig datum. Using the identification of the positive roots with tiles in \( T_i \) as in (5), we write \( x = (x_T) \in \mathbb{N}^{T_i} \).

The sets of \( i \)- and \( j \)-Lusztig data are related via piecewise linear bijections

\[
\phi^i_j : \mathbb{N}^{T_i} \to \mathbb{N}^{T_j}
\]

as follows. If \( j \) is obtained from \( i \) by a commutation move replacing the subword \( (i_k, i_{k+1}) \) by \( (i_{k+1}, i_k) \) with \( \sigma_{i_k} \sigma_{i_{k+1}} = \sigma_{i_{k+1}} \sigma_{i_k} \) we set

\[
(\phi^i_j x)_{[s,t]} = x_{[s,t]}.
\]

If \( j \) is obtained from \( i \) by a braid move that corresponds to a flip at

\[
\mathcal{H} := \{ [s, t], [s, u], [t, u] \} \subset T_i
\]

with \( s < t < u \), we set \( y = \phi^i_j x \), where \( y_T = x_T \) for \( T \notin \{ [s, t], [s, u], [t, u] \} \) and

\[
\begin{align*}
y_{[s,t]} &= x_{[s,t]} + x_{[s,u]} - \min(x_{[s,t]}, x_{[t,u]}), \\
y_{[s,u]} &= \min(x_{[s,t]}, x_{[t,u]}), \\
y_{[t,u]} &= x_{[t,u]} + x_{[s,u]} - \min(x_{[s,t]}, x_{[t,u]}). \tag{8}
\end{align*}
\]

Any reduced word \( j \) can be obtained from any other reduced word \( i \) by a sequence of commutation and braid moves. We thus obtain a definition of \( \phi^i_j \) for \( i \) and \( j \) arbitrary.

The \( \mathbb{Z}[q^-] \)-lattice \( \mathcal{L} \) spanned by \( B_i \) is independent of the choice of reduced expression \( i \), as is the induced basis \( B := \pi(B_i) \) of \( \mathcal{L}/q \mathcal{L} \), where \( \pi : \mathcal{L} \to \mathcal{L}/q \mathcal{L} \) is the canonical projection. There exists a unique basis \( B \) of \( \mathcal{L} \) whose image under \( \pi \) is \( B \) and which is stable under the \( \mathbb{Q} \)-algebra automorphism preserving the generators of \( U_q^- \) and sending \( q \) to \( q^{-1} \). The resulting parametrizations of \( B \) by \( \mathbb{N}^{T_i} \) are intertwined by the maps \( \phi^i_j \) and \( \mathbb{N}^{T_i} \) has a crystal structure isomorphic to \( B(\infty) \) (see [5,23]). It is given by
**Definition 2.12** Let $a \in [n - 1]$ and let $i = (i_1, \ldots, i_N)$. We define maps

$$f_a : \mathbb{N}^T_i \to \mathbb{N}^T_i,$$

$$e_a : \mathbb{N}^T_i \to \mathbb{N}^T_i \sqcup \{0\},$$

$$\varepsilon_a : \mathbb{N}^T_i \to \mathbb{N}$$

as follows. If $a = i_1$ we set for $x \in \mathbb{N}^T_i$ and $T \in T_i$

$$(f_a x)_T = \begin{cases} x_T + 1 & \text{if } T = [a, a + 1] \\ x_T & \text{else,} \end{cases}$$

$$(e_a x)_T = \begin{cases} 0 & \text{if } x_{[a,a+1]} = 0, \\ x_T - 1 & \text{if } T = [a, a + 1] \text{ and } x_{[a,a+1]} > 0, \\ x_T & \text{else,} \end{cases}$$

$$\varepsilon_a(x) = x_{[a,a+1]}.$$ 

Generally, we define $f_a$, $e_a$ and $\varepsilon_a$ by requiring for all reduced words $i$ and $j$

$$f_a = \phi_i^j \circ f_a \circ \phi_i^j,$$

$$e_a = \phi_i^j \circ e_a \circ \phi_i^j,$$

$$\varepsilon_a = \varepsilon_a \circ \phi_i^j.$$

To ease notation we suppress the dependency of $f_a$, $e_a$ and $\varepsilon_a$ on $i$.

### 2.5 Kashiwara involution on Lusztig’s parametrization

Let $*$ be the anti-automorphism of $U_q^-$ preserving the generators $F_a$. By [18, Theorem 8.1], we have $B(\infty)^* = B(\infty)$ as sets. Therefore this anti-automorphism yields an involution on the crystal basis called **Kashiwara involution**.

For $x \in \mathbb{N}^T_i$ and $a \in [n - 1]$ we define the operators $f_a^* x = (f_a x^*)^*$, $e_a^* x = (e_a x^*)^*$ and the function $\varepsilon_a^*(x) = \varepsilon_a(x^*)$. This equips the set $\mathbb{N}^T_i$ with a second crystal structure isomorphic to $B(\infty)$. Explicitly, the $*$-crystal structure is given as follows (see e.g. [5, Proposition 3.3 (iii)]).
Definition 2.13 Let \( a \in [n - 1] \) and \( i = (i_1, \ldots, i_N) \). If \( n - a = i_N \) then we set for \( x \in \mathbb{N}^T \) and \( T \in \mathcal{T}_i \)

\[
(f_a^* x)_T = \begin{cases} 
  x_T + 1 & \text{if } T = [a, a + 1] \\
  x_T & \text{else}, 
\end{cases}
\]

\[
(e_a^* x)_T = \begin{cases} 
  0 & \text{if } x_{[a,a+1]} = 0, \\
  x_T - 1 & \text{if } T = [a, a + 1] \text{ and } x_{[a,a+1]} > 0, \\
  x_T & \text{else}, 
\end{cases}
\]

\[
(\varepsilon_a^* x)_T = x_{[a,a+1]}.
\]

Furthermore, for two reduced words \( i \) and \( j \) we require

\[
f_a^* = \phi_1^j \circ f_a^* \circ \phi_1^i \\
e_a^* = \phi_1^j \circ e_a^* \circ \phi_1^i \\
\varepsilon_a^* = \varepsilon_a^* \circ \phi_1^i.
\]

3 Crossing formula

In this section we derive a “Crossing Formula” for the crystal structure \((f_a, e_a, \varepsilon_a)\) on Lusztig data \(\mathbb{N}^T\). For this we introduce in Sect. 3.1 a certain poset of crossings related to the Kashiwara operators. In Sect. 3.2 we provide preparatory results needed in the proof of the formula. In Sect. 3.3 we state and prove the Crossing Formula. In Sect. 3.4 we prove a \( \ast \)-Crossing Formula for the crystal structure \((f_a^\ast, e_a^\ast, \varepsilon_a^\ast)\) on Lusztig data and describe the explicit embedding of the highest weight crystal \(B(\lambda)\) into \(B(\infty)\) in terms of Lusztig data.

3.1 Poset of \( a \)-crossings

Let \( a \in [n - 1] \) and \( T \) be a tiling of the regular \( 2n \)-gon. We keep the notation of Definition 2.5 and introduce the main objects encoding the operation of the Kashiwara operators.

Definition 3.1 An \( a \)-ascending neighbour sequences \((\gamma_i)_{1 \leq i \leq m} \subset \mathcal{T}\) starting at \( \gamma_1 = L_1^a \) and ending at \( \gamma_m = L_1^{a+1} \) is called an \( a \)-crossings. We denote the set of \( a \)-crossings by \( \Gamma_a \).

Any such neighbour sequence is uniquely determined by its associated strip sequence \((s_1, \ldots, s_M)\), which is defined by

\[
\gamma = \gamma_1 \circ \cdots \circ \gamma_M, \quad \gamma^i \subset L_i^a, \quad s_i \neq s_{i+1}, \quad s_1 = a, \quad s_M = a + 1.
\]

We therefore use the notations \((\gamma_i) \subset \mathcal{T}\) and \((s_i) \subset [n]\) interchangeably to denote an \( a \)-crossing.
**Example 3.2** We continue with Example 2.1 and depict the 3-crossing

\[ \gamma = ([2, 3], [1, 3], [1, 2], [2, 5], [2, 4], [4, 5], [1, 4]), \]

which is given by the tiles surrounding the line in the picture below. The associated strip sequence is \( \gamma = (3, 1, 2, 4) \).

We introduce a relation \( \preceq \) on the set \( \Gamma_a \) of \( a \)-crossings as follows. Since \( \preceq_a \) defined in Definition 2.6 is anti-symmetric, \( \gamma \in \Gamma_a \) cannot contain cycles. Consequently, the set \( T \setminus \{ T \mid T \in \gamma \} \) is partitioned into the set of tiles lying on the left of \( \gamma \) and the set of tiles lying on the right of \( \gamma \). We denote the set consisting of those \( T \in T \) which do not lie right of \( \gamma \) by \( \overline{\gamma} \) and define for \( \gamma, \lambda \in \Gamma_a \)

\[ \hat{\gamma} := \overline{\gamma} - \gamma, \]

\[ \gamma \preceq \lambda :\Leftrightarrow (\overline{\gamma} \subseteq \overline{\lambda}) \land (\hat{\gamma} \subseteq \hat{\lambda}). \] (10)

We show that \( \preceq \) turns \( \Gamma_a \) into a poset:

**Proposition 3.3** The relation \( \preceq \) on \( \Gamma_a \) is an order relation.

**Proof** Clearly, \( \preceq \) is reflexive and transitive. For \( \gamma, \lambda \in \Gamma_a \) we have

\[ \gamma \preceq \lambda \Leftrightarrow (\overline{\gamma} = \overline{\lambda}) \land (\hat{\gamma} = \hat{\lambda}) \Rightarrow \gamma = \overline{\gamma} - \hat{\gamma} = \overline{\lambda} - \hat{\lambda} = \lambda, \]

where the equalities are equalities of sets. Since \( \gamma \) and \( \lambda \) are \( a \)-ascending neighbour sequences, \( \gamma = \lambda \) also holds as an equality of sequences. Thus, \( \preceq \) is anti-symmetric. \( \square \)

We illustrate the poset \( \Gamma_a \) in

**Example 3.4** We continue with our running example and depict the poset of 3-crossings in \( \mathcal{T}_i \) for \( i = (2, 1, 2, 3, 4, 3, 2, 3, 1, 2) \) given in Example 2.1. The crossings are given by the tiles surrounding the blue lines.
There is a unique maximal element in $\Gamma_a$. It is given by the strip sequence $(a, a + 1)$. Thus any $\gamma \in \Gamma_a$ consists of tiles in

$$W_a := (a, a + 1) \subset T.$$  \hfill (11)

We call $W_a$ the $a$-comb.

**Example 3.5** We continue with Example 2.1 and depict $W_1$ and $W_3$ in $T$.

### 3.2 Degeneration of combs

In the following we explain an inductive way to transform $W_a$ into a singleton by successive flips and how this reflects on the associated posets of $a$-crossings.

**Lemma 3.6** If $\# W_a > 1$ then there exists a hexagon $H \subset W_a$ with

$$H = \{[a, s], [a, t], [s, t]\}.$$

**Proof** The set $W_a - L^a$ is non-empty for $\# W_a > 1$. Let $T = [t, s] \in W_a - L^a$ be $\leq_a$ minimal. From $T \in W_a$ we obtain that $L^s \cap L^a \cap W_a$ and $L^t \cap L^a \cap W_a$ are non-empty. By the $\leq_a$ minimality of $T \in W_a - L^a$ both edges of $T$ are edges of $L^a \cap W_a$ and $H := \{[a, s], [a, t], [s, t]\} \subseteq W_a$ is a hexagon. $\square$

**Example 3.7** We depict the Proof of Lemma 3.6 for the reduced word $i = [1, 3, 4, 3, 2, 3, 4, 5, 1, 2, 3, 4, 3, 2, 1]$ and $a = 2$: 

![Diagram showing the proof of Lemma 3.6](image-url)
The red area illustrates $\mathcal{W}_a - \mathcal{L}^a$, which is $\leq_a$ ordered as follows:

\[
\begin{align*}
[1, 3] & \quad [1, 5] \quad [1, 4] \\
[3, 5] & \quad [3, 4] \\
[4, 5] &
\end{align*}
\]

Thus, $\mathcal{L}^1$ and $\mathcal{L}^5$, illustrated by the blue and the green line, respectively, pass through the unique $\leq_a$ minimal rhombus $[1, 5]$ in $\mathcal{W}_a - \mathcal{L}^a$. We obtain the hexagon $\mathcal{H} = \{[1, 2], [2, 5], [1, 5]\} \subset \mathcal{W}_a$, which is painted bold.

Let $\mathcal{H} = \{[a, s], [a, t], [s, t]\} \subset \mathcal{W}_a$ be a hexagon as in Lemma 3.6 and let $S$ be the tiling obtained from $T$ by flipping $\mathcal{H}$. Using Definition 2.6 we assume that $s, t$ are chosen such that $[a, s]_T < [a, t]_T$ (i.e. $s < a < t$ or $a < t < s$ or $t < s < a$). Then $\mathcal{V}_a := (a, a + 1) \subset S$ satisfies

\[
\mathcal{V}_a = \begin{cases} 
\{[u_1, u_2]_S \mid [u_1, u_2]_T \in \mathcal{W}_a\} - \{[s, t]_S\} & \text{if } t \neq a + 1, \\
\{[u_1, u_2]_S \mid [u_1, u_2]_T \in \mathcal{W}_a\} - \{[s, t]_S, [a, s]_S\} & \text{if } t = a + 1. 
\end{cases}
\]

(12)

In particular $\#\mathcal{V}_a < \#\mathcal{W}_a$.

We denote the set of $a$-crossing in $S$ by $\Lambda_a$ and define the map

\[
\pi = \pi_\mathcal{H} : \Gamma_a \rightarrow \Lambda_a \\
\pi_\mathcal{H}(u_1, \ldots, u_m) := \begin{cases} 
(u_1, u_3, u_4, \ldots, u_m) & \text{if } u_2, u_3 \in \{s, t\}, \\
(u_1, \ldots, u_m) & \text{else.}
\end{cases}
\]

(13)

Furthermore, we introduce the following subsets of crossings in $\Gamma_a$. 
Definition 3.8 The subset $\Gamma_{-} \subset \Gamma_{a}$, resp. $\Gamma_{+} \subset \Gamma_{a}$, is the set of crossings whose first turn is at a tile $[a, u_2]$ which appears before $[a, s]$ in $\mathcal{L}^a$, resp. whose first turn is at a tile $[a, u_2]$ which appears after $[a, t]$ in $\mathcal{L}^a$, i.e.

$$\Gamma_{-} := \{(u_i)_{1 \leq i \leq m} \in \Gamma_a \mid [u_1, u_2]_T < [a, s]_T\}.$$  
$$\Gamma_{+} := \{(u_i)_{1 \leq i \leq m} \in \Gamma_a \mid [u_1, u_2]_T > [a, t]_T\}.$$  

The subset $\Gamma_{st} \subset \Gamma_{a}$, resp. $\Gamma_{s} \subset \Gamma_{a}$, is the set of crossings whose first turn is at the tile $[a, s]$ and whose second turn is at the tile $[s, t]$, resp. whose second turn is at a tile $[s, t'] > [s, t]$ on $\mathcal{L}^s$, i.e.

$$\Gamma_{st} := \{(u_i)_{1 \leq i \leq m} \in \Gamma_a \mid u_2 = s, u_3 = t\}, \quad \Gamma_{s} := \{(u_i)_{1 \leq i \leq m} \in \Gamma_a \mid u_2 = s, u_3 \neq t\}.$$  

The subset $\Gamma_{ts} \subset \Gamma_{a}$, resp. $\Gamma_{t} \subset \Gamma_{a}$, is the set of crossings whose first turn is at the tile $[a, t]$ and whose second turn is at the tile $[s, t]$, resp. whose second turn is at a tile $[s', t'] > [s, t]$ on $\mathcal{L}^s$, i.e.

$$\Gamma_{ts} := \{(u_i)_{1 \leq i \leq m} \in \Gamma_a \mid u_2 = t, u_3 = s\}, \quad \Gamma_{t} := \{(u_i)_{1 \leq i \leq m} \in \Gamma_a \mid u_2 = t, u_3 \neq s\}.$$  

Analogously we define the subsets $\Lambda_{-}$ and $\Lambda_{+}$ of $\Lambda_a$ by

$$\Lambda_{-} := \{(u_i)_{1 \leq i \leq m} \in \Lambda_a \mid [u_1, u_2]_S < [a, t]_S\}.$$  
$$\Lambda_{+} := \{(u_i)_{1 \leq i \leq m} \in \Lambda_a \mid [u_1, u_2]_S > [a, s]_S\}.$$  

and the subsets $\Lambda_{s}$ and $\Lambda_{t}$ of $\Lambda_a$ by

$$\Lambda_{s} := \{(u_i)_{1 \leq i \leq m} \in \Lambda_a \mid u_2 = s\} \quad \Lambda_{t} := \{(u_i)_{1 \leq i \leq m} \in \Lambda_a \mid u_2 = t\}.$$
We further define

\[ X := \{ \Gamma^-, \Gamma^s, \Gamma^{st}, \Gamma^t, \Gamma^{ts}, \Gamma^+ \} - \{\emptyset\}, \quad Y := \{ \Lambda^-, \Lambda^s, \Lambda^t, \Lambda^+ \} - \{\emptyset\}. \]

Since \( \Gamma_a = \cup_{\Gamma \in X} \Gamma^* \) and \( \Lambda_a = \cup_{\Lambda \in Y} \Lambda^* \) we obtain quotient maps \( p_X : \Gamma_a \rightarrow X \) and \( p_Y : \Lambda_a \rightarrow Y \). The partial order induced from \( \Gamma_a \) on \( X \) via \( p_X \) is

\[
\begin{array}{ccc}
\Gamma^- & \rightarrow & \Gamma^{st} \\
\downarrow & & \downarrow \\
\Gamma^t & \rightarrow & \Gamma^+ \\
\end{array}
\]

The partial order induced from \( \Lambda_a \) on \( Y \) via \( p_Y \) is

\[ \Lambda^- < \Lambda^t < \Lambda^s < \Lambda^+. \]

The map \( f : X \rightarrow Y \), defined by \( f(\Gamma^-) = \Lambda^- \), \( f(\Gamma^{st}) = f(\Gamma^t) = \Lambda^t \), \( f(\Gamma^{ts}) = f(\Gamma^s) = \Lambda^s \), is order preserving and

\[
\begin{array}{ccc}
\Gamma_a & \overset{\pi_H}{\rightarrow} & \Lambda_a \\
p_X \downarrow & & \downarrow p_Y \\
X & \overset{f}{\rightarrow} & Y,
\end{array}
\]

is a Cartesian diagram of sets, i.e. \( \Gamma_a = \Lambda_a \times_Y X \). We show:

**Proposition 3.9** The diagram (15) is a Cartesian diagram of distributive lattices.

**Proof** We first show that (15) is a Cartesian diagram of posets, i.e. for \( \gamma_1 = (\lambda_1, \xi_1), \gamma_2 = (\lambda_2, \xi_2) \in \Gamma_a = \Lambda_a \times_Y X \) we have

\[ \gamma_1 \leq \gamma_2 \iff (\lambda_1 \leq \lambda_2) \land (\xi_1 \leq \xi_2). \]
Using the definition of $\leq$ given in (10) we write

\[
\begin{align*}
\gamma_1 \leq \gamma_2 & \iff \underbrace{\underbrace{\overline{\gamma_1} \cap \mathcal{H} \subseteq \overline{\gamma_2} \cap \mathcal{H} \wedge \overline{\gamma_1} - \mathcal{H} \subseteq \overline{\gamma_2} - \mathcal{H}}_{(i)} \wedge \underbrace{\gamma_1 \cap \mathcal{H} \subseteq \gamma_2 \cap \mathcal{H} \wedge \gamma_1 - \mathcal{H} \subseteq \gamma_2 - \mathcal{H}}_{(ii)} \wedge \underbrace{\underbrace{\overline{\gamma_1} \cap \mathcal{H} \subseteq \overline{\gamma_2} \cap \mathcal{H} \wedge \overline{\gamma_1} - \mathcal{H} \subseteq \overline{\gamma_2} - \mathcal{H}}_{(iii)} \wedge \underbrace{\gamma_1 \cap \mathcal{H} \subseteq \gamma_2 \cap \mathcal{H} \wedge \gamma_1 - \mathcal{H} \subseteq \gamma_2 - \mathcal{H}}_{(iv)}}, \\
\lambda_1 \leq \lambda_2 & \iff \underbrace{\underbrace{\underbrace{\overline{\lambda_1} \cap \mathcal{H} \subseteq \overline{\lambda_2} \cap \mathcal{H} \wedge \overline{\lambda_1} - \mathcal{H} \subseteq \overline{\lambda_2} - \mathcal{H}}_{(i')} \wedge \underbrace{\lambda_1 \cap \mathcal{H} \subseteq \lambda_2 \cap \mathcal{H} \wedge \lambda_1 - \mathcal{H} \subseteq \lambda_2 - \mathcal{H}}_{(ii')} \wedge \underbrace{\underbrace{\overline{\lambda_1} \cap \mathcal{H} \subseteq \overline{\lambda_2} \cap \mathcal{H} \wedge \overline{\lambda_1} - \mathcal{H} \subseteq \overline{\lambda_2} - \mathcal{H}}_{(iii')} \wedge \underbrace{\lambda_1 \cap \mathcal{H} \subseteq \lambda_2 \cap \mathcal{H} \wedge \lambda_1 - \mathcal{H} \subseteq \lambda_2 - \mathcal{H}}_{(iv')}},
\end{align*}
\]

Since $\lambda_1 = \pi_1 \gamma_1$ and $\lambda_2 = \pi_2 \gamma_2$ we have $(ii) \iff (ii')$ and $(iv) \iff (iv')$.

For an arbitrary $\gamma \in \Gamma_a$ we have

\[
\begin{align*}
\hat{\gamma} \cap \mathcal{H} &= \begin{cases} 
\{[s, t]\} & \text{if } \gamma \in \Gamma^+, \\
\emptyset & \text{else},
\end{cases} \\
\overline{\gamma} \cap \mathcal{H} &= \begin{cases} 
\emptyset & \text{if } \gamma \in \Gamma^-, \\
\{[a, s], [s, t]\} & \text{if } \gamma \in \Gamma^{st} \cup \Gamma^t, \\
\mathcal{H} & \text{else}.
\end{cases}
\end{align*}
\]

Thus, $\xi_1 \leq \xi_2$ implies $(i)$ and $(iii)$.

Condition $(iii')$ is empty, since for any $\lambda \in \Lambda_a$ one has $\overline{\lambda} \cap \mathcal{H} = \emptyset$. To establish (16) it remains to show $\gamma_1 \leq \gamma_2 \Rightarrow (i')$. This holds since $f \circ p_X$ is order preserving and for any $\gamma \in \Gamma_a$:

\[
\begin{align*}
\overline{\gamma} \cap \mathcal{H} &= \begin{cases} 
\emptyset & f \circ p_X(\gamma) = \Lambda^-, \\
\{[a, t]\} & f \circ p_X(\gamma) = \Lambda^t, \\
\{[a, t], [a, s]\} & \text{else}.
\end{cases}
\end{align*}
\]

One can show that $\Gamma_a$ and $\Lambda_a$ are lattices by explicitly constructing suprema and infima. We give a different proof using induction on $\# \mathcal{W}_a$.

We order $\mathcal{L}^a$ via $\mathcal{L}_i^a \leq \mathcal{L}_j^a :\iff i \leq j$ and define maps

\[
\begin{align*}
\overline{p}_X : \Gamma_a &\rightarrow \mathcal{L}^a, \quad (a = u_1, \ldots, u_m) \mapsto [a, u_2], \\
\overline{p}_Y : \Lambda_a &\rightarrow \mathcal{L}^a, \quad (a = u_1, \ldots, u_m) \mapsto [a, u_2].
\end{align*}
\]

If $\overline{p}_Y$ is a morphism of distributive lattices, then so is $p_Y$. This in turn implies using that $f$ is a morphism of distributive lattices that the Cartesian diagram of posets (15) is a Cartesian diagram of distributive lattices. From this we obtain that $\overline{p}_X$ is a morphism of distributive lattices. Using induction we can assume by Lemma 3.6 and (12) that $\Lambda_a$ is a singleton and the claim follows.

\[\square\]

### 3.3 Crossing formula

Let $a \in [n-1]$. To state the formula for the crystal structure $(f_a, e_a, c_a)$ on $\mathbb{N}^T$ we introduce the following notation. For $\gamma \in \Gamma_a$ with strip sequence $(s_1 = a, s_2, \ldots, s_M = \ldots)$
$a + 1$ as in (9) we define $\tau \gamma \in \mathbb{Z}^T$ by

$$
(\tau \gamma)_T := \begin{cases} 
\text{sgn} \ (s_{i+1} - s_i) & \text{if } T = [s_i, s_{i+1}] \text{ for some } i, \\
0 & \text{else}
\end{cases}
$$

and $\gamma \in \text{Hom}(\mathbb{Z}^T, \mathbb{Z})$ by

$$
\epsilon_a ([s, t]) := \begin{cases} 
1 & \text{if } s \leq a < a + 1 \leq t, \\
-1 & \text{else}
\end{cases} \quad \text{for } [s, t] \in T,
$$

and

$$
(\gamma)(x) = \sum_{T \in \gamma, \epsilon_a(T) = 1} x_T - \sum_{T \in \gamma, \epsilon(T) = -1, (\tau \gamma)_T = 0} x_T \quad \text{for } x \in \mathbb{Z}^T. 
$$

**Example 3.10** Below is depicted $\tau \gamma$, where $\gamma = (3, 1, 2, 4)$ is one of the elements of $\Gamma_3$ displayed in Example 3.4. Here $(\tau \gamma)_{[1,2]} = (\tau \gamma)_{[2,4]} = 1$, $(\tau \gamma)_{[1,3]} = -1$ and $(\tau \gamma)_T = 0$ for $T \notin \{[1, 2], [2, 4], [1, 3]\}$.

![Diagram](image)

Furthermore, $(\gamma)(x) = -x_{[2,3]} + x_{[2,5]} + x_{[2,4]} - x_{[4,5]} + x_{[1,4]}$.

**Definition 3.11** We say $\gamma \in \Gamma_a$ is a *Reineke a-crossing* if it satisfies the following condition: For any $\gamma_i = [s, t]$ such that $\gamma_{i-1}, \gamma_i$ and $\gamma_{i+1}$ lie on the same strip $L^s$, we have

$$
s > t \quad \text{if } t \leq a,
$$

$$
s < t \quad \text{if } t \geq a + 1.
$$

We denote the set of all Reineke $a$-crossings by $\mathcal{R}_a$.

**Remark 3.12** Using the relation between rhombic tilings and wiring diagrams (see [7, Section 2]) the notion of Reineke crossings translates into the notion of rigorous paths which appear in the work [14] of Gleizer and Postnikov in the description of string cone inequalities, see also Remark 5.5.
Remark 3.13 Using the notation of Sect. 3.2 we have that $\gamma \in \Gamma_a$ is a Reineke crossing precisely if $\pi(\gamma) \in \Lambda_a$ is a Reineke crossing and

\[
\begin{align*}
\gamma & \not\in \Gamma^t & & \text{if } t \leq a, \\
\gamma & \not\in \Gamma^s & & \text{if } s \geq a + 1.
\end{align*}
\]

Thus, by induction we obtain from Proposition 3.9 that $\mathcal{R}_a \subseteq \Gamma_a$ is a sublattice.

Example 3.14 In the poset $\Gamma_3$ given in Example 3.4 all 3-crossings are Reineke 3-crossings except $\gamma = (3, 1, 4)$ since we have $1 < 2 \leq 3$.

We prove the following Crossing Formula for the action of the Kashiwara operators on $i$-Lusztig data.

Theorem 3.15 (Crossing Formula). Let $i$ be a reduced word and $T$ the tiling associated to $i$. For $a \in [n - 1]$ and an $i$-Lusztig datum $x \in \mathbb{N}^T$ we have

\[
f_a x = x + r\gamma^x,
\]

where $\gamma^x$ is the $\preceq$-maximal element in $\Gamma_a$ satisfying

\[
\left( s\gamma^x \right)(x) = \max_{\gamma \in \Gamma_a} (s\gamma)(x). \quad (19)
\]

Furthermore, we have $\gamma^x \in \mathcal{R}_a$ and

\[
\varepsilon_a(x) = \max_{\gamma \in \Gamma_a} (s\gamma)(x) = \max_{\gamma \in \mathcal{R}_a} (s\gamma)(x).
\]

Before proving the Crossing Formula we provide the following

Example 3.16 Recall from Example 3.14 that the lattice of Reineke 3-crossings for $i = (2, 1, 2, 3, 4, 3, 2, 3, 1, 2)$ is obtained by removing $\gamma = (3, 1, 4)$ from the poset $\Gamma_3$ depicted in Example 3.4. We name the Reineke 3-crossings from $\Gamma_3$ as in Example 3.4 as follows:
Denoting by $1_T$ the characteristic function of the tile $T$ we compute

\[
\begin{align*}
\gamma_1 &= -1_{[2,3]} - 1_{[1,2]} + 1_{[1,4]}, \\
\gamma_2 &= -1_{[2,3]} + 1_{[2,5]} - 1_{[4,5]}, \\
\gamma_3 &= -1_{[2,3]} + 1_{[2,4]}, \\
\gamma_4 &= -1_{[1,3]} + 1_{[1,2]} + 1_{[2,5]} - 1_{[4,5]}, \\
\gamma_5 &= -1_{[1,3]} + 1_{[1,2]} + 1_{[2,4]}, \\
\gamma_6 &= 1_{[3,5]} - 1_{[4,5]}, \\
\gamma_7 &= 1_{[3,5]} - 1_{[2,5]} + 1_{[2,4]}, \\
\gamma_8 &= 1_{[3,4]},
\end{align*}
\]

For $x = 1_{[2,3]} + 1_{[4,5]} \in \mathbb{N}^T$ we thus obtain

\[
\begin{align*}
(\sigma \gamma_1)(x) &= (\sigma \gamma_2)(x) = 0, \\
(\sigma \gamma_3)(x) &= (\sigma \gamma_4)(x) = (\sigma \gamma_6)(x) = -1, \\
(\sigma \gamma_5)(x) &= (\sigma \gamma_7)(x) = (\sigma \gamma_8)(x) = -2.
\end{align*}
\]

Hence $\gamma^x = \gamma_2$ and $f_3 x = x + \gamma_2$.

**Proof of the Crossing Formula** We proceed by induction on $\#W_a$. For $\#W_a = 1$ the statement follows by the definition of $f_a$ and $\epsilon_a$ given in Definition 2.12. Assuming
now $\# \mathcal{W}_a \geq 2$ by Lemma 3.6 there exists a hexagon $\mathcal{H} = \{[a, s], [a, t], [s, t]\} \subset \mathcal{W}_a$. We assume without loss of generality that the labels $s$ and $t$ are chosen such that $[a, s] <_a [a, t]$. We denote by $S = T_j$ the tiling obtained from by $T$ by flipping $\mathcal{H}$. The reduced word $j$ is obtained from $i$ by the braid move corresponding to the flip of $\mathcal{H}$ (see Sect. 2.4). Further by (12) $\mathcal{V}_a := (a, a + 1) \subset S$ satisfies $\# \mathcal{V}_a < \# \mathcal{V}_a$.

We write $\Lambda_a$ for the set of $a$-crossings in $S$ and define $\Gamma^-$, $\Gamma^s$, $\Gamma^f$, $\Gamma^t$, $\Gamma^{st}$, $\Gamma^+$, $X$, $\Lambda^-$, $\Lambda^s$, $\Lambda^t$, $\Lambda^+$, $Y$ as in Definition 3.8. Since (14) is Cartesian we can identify $\Gamma_a$ with $\Lambda_a \times Y X$. Writing with the notations of (8)

$$y = \phi^j_y x,$$

by the induction hypothesis there exists a unique $\preceq$-maximal element $\lambda^y \in \Lambda_a$ satisfying

$$\varepsilon_a(x) = (s\lambda^y)(x) = \max_{\lambda \in \Lambda_a} (s\lambda)(y).$$

Let $\gamma^x \in \Gamma_a$ be a $\preceq$-maximal element satisfying (19). Since a neighbour sequence is determined by its associated strip sequence, the vector $\gamma' \in \mathbb{Z}^T$ determines $\gamma \in \Gamma_a$.

By Definition 2.12 we have $\varepsilon_a(x) = f_a x = \phi^j_y f_a \phi^j_y x$. Thus using the induction hypothesis it is enough to show

$$\phi^j_y (y + r\lambda^y) = x + ty^x,$$  \hspace{1cm} (20)

$$\forall \lambda \in \Lambda_a : \max_{\gamma' \in \pi^{-1}(\lambda)} (s\gamma')(x) = (s\lambda)(y),$$  \hspace{1cm} (21)

$$\gamma^x \in \mathcal{R}_a.$$  \hspace{1cm} (22)

We consider the three cases 1) $a < t < s$, 2) $s < a < t$ and 3) $t < s < a$ separately.

**First case:** We first assume $a < t < s$. Then by (8) we have for $x_{s,t} := x_{[s,t]}$, $y_{s,t} := y_{[s,t]}$ and $\gamma \in \Gamma_a$

$$(s\gamma)(x) - (s\pi \gamma)(y) = \begin{cases} 0 & \gamma \in \Gamma^-, \\ x_{a,s} - y_{a,t} = \min(0, y_{s,t} - y_{a,t}) & \gamma \in \Gamma^s, \\ x_{a,s} + x_{a,t} - x_{s,t} - y_{a,t} = \min(0, y_{a,t} - y_{s,t}) & \gamma \in \Gamma^t, \\ x_{a,s} - x_{s,t} - y_{a,t} - y_{a,s} = -x_{s,t} - x_{a,t} & \gamma \in \Gamma^t, \\ x_{a,s} + x_{a,t} - y_{a,t} - y_{a,s} = 0 & \gamma \in \Gamma^{st} \cup \Gamma^+.
\end{cases}$$  \hspace{1cm} (23)

Since $\pi$ is surjective, $\pi(\lambda, \Gamma^s) = \pi(\lambda, \Gamma^{st})$ and $\pi(\lambda, \Gamma^t) = \pi(\lambda, \Gamma^{st})$ we obtain from (23) the equality (21) and

$$(s\gamma^x)(x) = \max_{\gamma \in \Gamma_a} (s\gamma)(x) = \max_{\lambda \in \Lambda_a} (s\lambda)(y) = (s\lambda^y)(y).$$  \hspace{1cm} (24)

Writing $\gamma^x = (\lambda^x, \xi^x) \in \Lambda_a \times Y X = \Gamma_a$ we obtain from (21) to (24)

$$\lambda^x = \pi \gamma^x \preceq \lambda^y.$$  \hspace{1cm} (25)
By (21) and (24) we can choose $\xi^y \in X$ maximal with $(s(\lambda^y, \xi^y))(x) = (s\gamma^x)(x)$. Then, by $\Gamma^s < \Gamma^t$ and (23), we have $\xi^y \not\in \Gamma^s$. Since $X - \{\Gamma^s\}$ is linearly ordered we conclude from the $\leq$-maximality of $\gamma^x$, (18) and Proposition 3.9 that $\Gamma^x = \xi^y$. Thus, using Proposition 3.9 again and (25) we obtain $\gamma^x \leq (\lambda^y, \xi^y)$. By the $\leq$-maximality of $\gamma^x$ we conclude $\gamma^x = (\lambda^y, \xi^y)$. Using the induction hypothesis, $\gamma^x = (\lambda^y, \xi^y)$ implies (22). Furthermore, we obtain

$$\pi \gamma^x = \lambda^y. \quad (26)$$

By (8) we have

$$\phi^y_1 (y + r\lambda^y) - x = \begin{cases} r(\lambda^y, \Gamma^s) & \text{if } \lambda^y \in \Lambda^t \land y_{a,t} < y_{s,t}, \\ r(\lambda^y, \Gamma^t) & \text{if } \lambda^y \in \Lambda^t \land y_{a,t} \geq y_{s,t}, \\ r(\lambda^y, \Gamma^s) & \text{if } \lambda^y \in \Lambda^s, \\ r(\lambda^y, \Gamma^r) & \text{if } \lambda^y \in \Lambda^r, \\ r(\lambda^y, \Gamma^+) & \text{if } \lambda^y \in \Lambda^+. \end{cases} \quad (27)$$

If $\lambda^y \not\in \Lambda^t$ then $\#\pi^{-1}(\lambda^y) \cap R_a = 1$ and (20) follows from (22), (26) and (27).

If $\lambda^y \in \Lambda^t$, we have $\pi^{-1}(\lambda^y) = \{(\lambda^y, \Gamma^s), (\lambda^y, \Gamma^t)\}$. Furthermore, from Proposition 3.9 we obtain $(\lambda^y, \Gamma^s) \leq (\lambda^y, \Gamma^t)$. Thus, (20) follows from (23), (26), (27) and (24).

**Second Case:** We next consider the case $t < s < a$. By (8), we have for $x_{s,t} := x_{[s,t]}$, $y_{s,t} := y_{[s,t]}$ and $\gamma \in \Gamma_a$

$$(s\gamma)(x) - (s\gamma)(y) = \begin{cases} 0 & \gamma \in \Gamma^- \cup \Gamma^s, \\ -x_{s,t} - x_{s,a} & \gamma \in \Gamma^t, \\ -x_{s,t} + y_{t,a} = \min(0, y_{s,a} - y_{s,t}) & \gamma \in \Gamma^s, \\ -x_{s,a} + y_{t,a} = \min(0, y_{s,t} - y_{s,a}) & \gamma \in \Gamma^t, \\ -x_{s,a} + x_{t,a} + y_{t,a} + y_{s,a} = 0 & \gamma \in \Gamma^+. \end{cases} \quad (28)$$

Since $\pi$ is surjective, $\pi(\lambda, \Gamma^s) = \pi(\lambda, \Gamma^t)$ and $\pi(\lambda, \Gamma^t) = \pi(\lambda, \Gamma^s)$ we obtain from (28) the equality (21) and

$$\left(s\gamma^x\right)(x) = \max_{\gamma \in \Gamma_a} (s\gamma)(x) = \max_{\lambda \in \Lambda_a} (s\lambda)(y) = \left(s\lambda^y\right)(y). \quad (29)$$

Writing $\gamma^x = (\lambda^x, \Gamma^x) \in \Lambda_a \times X = \Gamma_a$ we obtain from (21) to (29)

$$\lambda^x = \pi \gamma^x \leq \lambda^y. \quad (30)$$

By (21) and (29) we can choose $\xi^y \in X$ maximal with $(s(\lambda^y, \xi^y))(x) = (s\gamma^x)(x)$. Then $\xi^y \not\in \Gamma^t$, since otherwise by (28) we have $y_{a,t} = \min(x_{a,s}, x_{s,t}) = 0$ and using the induction hypothesis we obtain the contradiction $f_{a,y} = y + t\lambda^y \not\in N^S$. Since $X - \{\Gamma^t\}$ is linearly ordered we conclude from the $\leq$-maximality of $\gamma^x$, (18) and
Proposition 3.9 that $\Gamma^x = \xi^y$. Thus, using Proposition 3.9 again and (30) we obtain $\gamma^x \leq (\lambda^y, \xi^y)$. By the $\leq$-maximality of $\gamma^x$ we conclude $\gamma^x = (\lambda^y, \xi^y)$. Using the induction hypothesis, $\gamma^x = (\lambda^y, \xi^y)$ implies (22). Furthermore, we obtain

$$\pi \gamma^x = \lambda^y.$$  \hspace{1cm} (31)

By (8) we have

$$\phi^y_1 (y + r \lambda^y) - x = \begin{cases} 
    r(\lambda^y, \Gamma^{ix}) & \text{if } \lambda^y \in \Lambda^s \land y_{s,a} \leq y_{s,t}, \\
    r(\lambda^y, \Gamma^y) & \text{if } \lambda^y \in \Lambda^s \land y_{s,a} > y_{s,t}, \\
    r(\lambda^y, \Gamma^t) & \text{if } \lambda^y \in \Lambda^t, \\
    r(\lambda^y, \Gamma^\gamma) & \text{if } \lambda^y \in \Lambda^-, \\
    r(\lambda^y, \Gamma^\gamma) & \text{if } \lambda^y \in \Lambda^+. 
\end{cases} \hspace{1cm} (32)$$

If $\lambda^y \notin \Lambda^s$ then $\#\pi^{-1}(\lambda^y) \cap \mathcal{R}_a = 1$ and (20) follows from (22), (31) and (32).

If $\lambda^y \in \Lambda^s$, we have $\pi^{-1}(\lambda^y) = \{(\lambda^y, \Gamma^s), (\lambda^y, \Gamma^{ix})\}$. Furthermore, from Proposition 3.9 we obtain $(\lambda^y, \Gamma^s) \leq (\lambda^y, \Gamma^{ix})$. Thus, (20) follows from (28), (31), (32) and (29).

Third Case: We finally consider the case $s < a < t$. In this case (22) follows directly from the induction hypothesis. Furthermore, by (8) we have for $x_{s,t} := x_{[s,t]}$, $y_{s,t} := y_{[s,t]}$ and $\gamma \in \Gamma_a$

$$(s\gamma)(x) - (s\pi \gamma)(y) = \begin{cases} 
    0 & \gamma \in \Gamma^-, \\
    x_{s,t} - y_{a,t} = \min(0, y_{s,a} - y_{a,t}) & \gamma \in \Gamma^t \cup \Gamma^s, \\
    -x_{s,a} + x_{a,t} + x_{s,t} - y_{a,t} = \min(0, y_{a,t} - y_{s,a}) & \gamma \in \Gamma^t \cup \Gamma^{ix}, \\
    -x_{s,a} + x_{a,t} - y_{a,t} + y_{s,a} = 0 & \gamma \in \Gamma^+. 
\end{cases} \hspace{1cm} (33)$$

Since $\pi$ is surjective, $\pi(\lambda, \Gamma^s) = \pi(\lambda, \Gamma^{ix})$ and $\pi(\lambda, \Gamma^s) = \pi(\lambda, \Gamma^{ix})$ we obtain from (33) the equality (21) and

$$(s\gamma^x)(x) = \max_{\gamma \in \Gamma_a} (s\gamma)(x) = \max_{\lambda \in \Lambda_a} (s\lambda^y)(y) = (s\lambda^y)(y). \hspace{1cm} (34)$$

Writing $\gamma^x = (\lambda^x, \Gamma^x) \in \Lambda_a \times \gamma X = \Gamma_a$ we obtain from (21) to (34)

$$\lambda^x = \pi \gamma^x \leq \lambda^y. \hspace{1cm} (35)$$

By (21) and (34) we can choose $\xi^y \in X$ maximal with $(s(\lambda^y, \xi^y))(x) = (s\gamma^x)(x)$. Then $[\xi^y, \Gamma^s] \neq \{\Gamma^s, \Gamma^t\}$, since otherwise by (33) we obtain $y_{s,a} = y_{a,t}$ and the contradiction $(s(\lambda^y, \Gamma^{ix}))(x) = (s\gamma^x)(x)$ to $\Gamma^s, \Gamma^t < \Gamma^{ix}$. Since $X - \{\Gamma^s\}$ and $X - \{\Gamma^t\}$ are linearly ordered we conclude from the $\leq$-maximality of $\gamma^x$, (18) and Proposition 3.9 that $\Gamma^x = \xi^y$. Thus, using Proposition 3.9 and (35) we obtain $\gamma^x \leq (\lambda^x, \xi^y)$ and by the $\leq$-maximality of $\gamma^x$ we conclude

$$\pi \gamma^x = \lambda^y. \hspace{1cm} (36)$$
By (8) we have

\[ \phi_1^j(y + r\lambda) - x = \begin{cases} 
\tau(\lambda^y, \Gamma^s) & \text{if } \lambda^y \in \Lambda^s \land y_{s,a} \leq y_{a,t}, \\
\tau(\lambda^y, \Gamma^t) & \text{if } \lambda^y \in \Lambda^t \land y_{s,a} > y_{a,t}, \\
\tau(\lambda^y, \Gamma') & \text{if } \lambda^y \in \Lambda^t \land y_{s,a} \leq y_{a,t}, \\
\tau(\lambda^y, \Gamma^s) & \text{if } \lambda^y \in \Lambda^t \land y_{s,a} > y_{a,t}, \\
\tau(\lambda^y, \Gamma^-) & \text{if } \lambda^y \in \Lambda^-, \\
\tau(\lambda^y, \Gamma^+) & \text{if } \lambda^y \in \Lambda^+. 
\end{cases} \]  

(37)

If \( \lambda^y \notin \Lambda^s \cup \Lambda^t \) then \#\pi^{-1}(\lambda^y) = 1 and (20) follows from (36) to (37).

If \( \lambda^y \in \Lambda^s \), we have \( \pi^{-1}(\lambda^y) = (\lambda^y, \Gamma^s) \). Furthermore, from Proposition 3.9 we obtain \( (\lambda^y, \Gamma^s) \leq (\lambda^y, \Gamma^{s+}) \). Thus, (20) follows from (33), (36), (37) and (34).

If \( \lambda^y \in \Lambda^t \), we have \( \pi^{-1}(\lambda^y) = (\lambda^y, \Gamma^s) \). Furthermore, from Proposition 3.9 we obtain \( (\lambda^y, \Gamma^s) \leq (\lambda^y, \Gamma^t) \). Thus, (20) follows from (33), (36), (37) and (34).

For \( x \in \mathbb{Z}^T \) we define \([x]^- := (\min(x_T, 0))_{T \in \mathbb{T}} \). Then we have

**Proposition 3.17** For \( \gamma \in \mathcal{R}_a \) and \( x = [r\gamma]^- \) we have

\[ \gamma^x = \gamma. \]

**Proof** By the Crossing Formula (Theorem 3.15) the statement is equivalent to

\[ f_a[r\gamma]^- = [r\gamma]^- + r\gamma. \]

(38)

We prove (38) by induction on \#\mathcal{V}_a. For \#\mathcal{V}_a = 1 the claim is true by the definition of \( f_a \) given in Definition 2.12. Assuming now \#\mathcal{V}_a \geq 2 by Lemma 3.6 there exists a hexagon \( \mathcal{H} = \{[a, s], [a, t], [s, t]\} \subseteq \mathcal{V}_a \). We assume that \([a, s] <_a [a, t]\) and denote by \( S = T_j \) the tiling obtained from by \( T \) by flipping \( \mathcal{H} \). The reduced word \( j \) is obtained from \( i \) by the braid move corresponding to the flip of \( \mathcal{H} \) (see Sect. 2.4). Further by (12) \( \mathcal{V}_a := ([a, a + 1]) \subseteq S \) satisfies \#\mathcal{V}_a < \#\mathcal{V}_a.

We write \( \Lambda_a \) for the set of \( a \)-crossings in \( S \) and define \( \Gamma^-, \Gamma^s, \Gamma^{s+}, \Gamma^t, \Gamma^{t+}, \Gamma^+, X, \Lambda^-, \Lambda^s, \Lambda^t, \Lambda^+, Y \) as in Definition 3.8.

We first assume that either \( \gamma \in \Gamma^s \cup \Gamma^t \cup \Gamma^- \cup \Gamma^+ \) or \( t < s < a \) and \( \gamma \in \Gamma^{s+} \) or \( a < t < s \) and \( \gamma \in \Gamma^{t+} \). One verifies case by case that

\[ \phi_1^j([r\gamma]^-) = [r\tau \gamma]^- , \]

(39)

\[ \phi_1^j([r\tau \gamma]^- + r\tau \gamma) = [r\gamma]^- + r\gamma. \]

(40)

Since by definition \( f_a = \phi_1^j f_a \phi_1^j \) we conclude

\[ f_a[r\gamma]^- = \phi_1^j(f_a[r\tau \gamma]^-) = \phi_1^j([r\tau \gamma]^- + r\tau \gamma) = r\gamma + [r\gamma]^- , \]
where the first equality follows from (39), the second equality follows from the induction hypothesis and the third equality from (40).

We now consider the remaining cases and fix $\gamma \in \tilde{\Gamma}$ where

$$\tilde{\Gamma} = \begin{cases} 
\Gamma_{st} & \text{if } a < t < s \lor s < a < t, \\
\Gamma_{ts} & \text{if } t < s < a \lor s < a < t. 
\end{cases}$$

In these cases we have for all $\lambda \in \Lambda_a$

$$(s\lambda)(\phi^1_j [r\gamma]^-) \leq \begin{cases} 
(s\lambda) \left( [t\pi \gamma]^- \right) + 1 & \text{if } s < a < t \lor \gamma \in \Gamma_{ts}, \\
(s\lambda) \left( [t\pi \gamma]^- \right) & \text{else,} 
\end{cases}$$

with equality for $\lambda = \pi \gamma$. Thus, using the induction hypothesis and the Crossing Formula we obtain

$$f_a \phi^1_j [r\gamma]^- = \phi^1_j [r\gamma]^- + t\pi \gamma. \quad (41)$$

Using (41) and the definition of $f_a$ one directly computes

$$f_a [r\gamma]^- = \phi^1_j \left( f_a \phi^1_j [r\gamma]^- \right) = \phi^1_j \left( \phi^1_j [r\gamma]^- + t\pi \gamma \right) = [r\gamma]^- + t\gamma. \qed$$

By the Crossing Formula (Theorem 3.15) and Proposition 3.17 we obtain

**Theorem 3.18** We have

$$R_a = R_a(T) := \{ f_a x - x \mid x \in \mathbb{N}^T \} = \{ r\gamma \mid \gamma \in R_a \}.$$  

**Definition 3.19** The subset $R_a \subset \mathbb{Z}^T$ is called the set of $a$-Reineke vectors.

**Remark 3.20** For reduced words $i$ adapted to quivers of type $A$, the set of vectors of the form $\{ f_a x - x \mid x \in \mathbb{N}^T \}$ was already studied in the work of Zelikson [32] in the setup of pseudoline arrangements. Here the term Lusztig moves was used and the set of such moves was shown to give defining inequalities of a certain string cone (see Remark 5.6).

The following result can be proved analogously to the Crossing Formula.

**Theorem 3.21** For $a \in [n-1]$ and $x \in \mathbb{N}^T$ with $\varepsilon_a(x) \geq 1$ we have

$$e_a x = x - t\gamma_\lambda,$$

where $\gamma_\lambda$ is the $\preceq$-minimal element in $\Gamma_a$ satisfying

$$(s\gamma_\lambda)(x) = \max_{\gamma \in \Gamma_a} (s\gamma)(x).$$
Example 3.22  We continue with Example 3.16 and keep the notation. For $x = 1_{[2,3]} + 1_{[4,5]} \in \mathbb{N}_T$ we have computed $f_3 x = x + \gamma_2 = 1_{[2,5]} := y$. Let us now compute $e_3 y$ using Theorem 3.21. We have $(s \gamma_1)(y) = (s \gamma_8)(y) = 0$ and $(s \gamma_j)(y) = 1$ for $j \in \{2, 3, 4, 5, 6, 7\}$. Hence $\gamma_x = \gamma_2$ and $e_3 y = y - r \gamma_2 = x$.

3.4 *-Crossing formula and highest weight crystals

Let $T_i$ be the tiling associated to the reduced word $i$ of $w_0$ as defined in Sect. 2.3. For $\lambda$ a dominant integral weight of $\mathfrak{sl}_n(\mathbb{C})$, the crystal graph of the highest weight crystal $B(\lambda)$ is a full subgraph of $B(\infty)$ (with weights appropriately shifted). In this section we give an explicit description of the *-crystal structure on $\mathbb{N}_T^i$ and the explicit embedding of the highest weight crystals into $\mathbb{N}_T^i$ where * denotes the Kashiwara involution introduced in Sect. 2.5.

We introduce the *-notions of those used in the Crossing Formula (Theorem 3.15). Using the notation of Definition 2.5 we define the set $\Gamma^*_1$ of $a$-*crossings as the set of $n + a$-ascending neighbour sequences $(\gamma_i)_i \in \mathbb{T} \subset T_i$ starting at $\gamma_1 = L_n - 1$ and ending at $\gamma_m = L_n + 1$.

We introduce a relation $\leq^*$ on $\Gamma^*_a$ as follows. Since $\leq_{n+a}$ defined in Definition 2.6 is anti-symmetric, $\gamma \in \Gamma^*_a$ cannot contain cycles. Consequently, the set $T \setminus \{T \mid T \in \gamma\}$ is partitioned into the set of tiles lying on the left of $\gamma$ and the set of tiles lying on the right of $\gamma$. We denote the set consisting of those $T \in T$ which do not lie left of $\gamma$ by $\overline{\gamma}^*$ and define for $\gamma, \lambda \in \Gamma^*_a$

$$\gamma^* := \overline{\gamma}^* - \gamma,$$

$$\gamma \leq^* \lambda : \iff (\gamma^* \subseteq \overline{\lambda}^*) \wedge (\gamma^* \subseteq \overline{\lambda}^*).$$

The proof of Proposition 3.3 shows that $\leq^*$ is an order relation on $\Gamma^*_a$.

To $\gamma \in \Gamma^*_a$ with strip sequence $(s_1 = a, s_2, \ldots, s_M = a + 1)$ (see (9)) we associate $r \gamma \in \mathbb{Z}_T^i$ and $s \gamma \in \text{Hom}(\mathbb{Z}_T^i, \mathbb{Z})$ by (17) and (18), respectively.

Definition 3.23 We say $\gamma \in \Gamma^*_a$ is a Reineke $a$-*crossing if it satisfies the following condition: For any $\gamma_i = [s, t]$ such that $\gamma_{i-1}, \gamma_i$ and $\gamma_{i+1}$ lie in the same strip sequence $\mathcal{L}^s$ we have

$$s > t \quad \text{if} \quad t \leq a$$

$$s < t \quad \text{if} \quad t \geq a + 1.$$ 

We denote the set of all Reineke $a$-*crossings by $\mathcal{R}_{a}^*$. 

We obtain the following *-Crossing Formula.
Theorem 3.24 (⋆-Crossing Formula). Let \( i \) be a reduced word, \( a \in [n-1] \) and \( x \in \mathbb{N}^{\ast}_{a} \). Then we have

\[
\begin{align*}
  f_{a}^{\ast}x - x &= \tau y^{x}, \\
  e_{a}^{\ast}x - x &= \begin{cases} \tau y^{x} & \text{if } e_{a}^{\ast}(x) \geq 1, \\ 0 & \text{else}, \end{cases}
\end{align*}
\]

where \( y^{x} \) (resp. \( y_{x} \)) is the \( \leq^{\ast} \)-maximal (resp. minimal) element \( \tilde{\gamma} \) in \( \Gamma_{a}^{\ast} \) satisfying

\[
(\tilde{\gamma})^{x}(x) = \max_{\gamma \in \Gamma_{a}^{\ast}} (\tilde{\gamma})(x).
\]

Furthermore, we have \( y^{x}, y_{x} \in \mathcal{R}_{a}^{\ast} \) and

\[
\varepsilon_{a}^{\ast}(x) = \max_{\gamma \in \Gamma_{a}^{\ast}} (\tilde{\gamma})(x) = \max_{\gamma \in \mathcal{R}_{a}^{\ast}} (\tilde{\gamma})(x).
\]

Moreover,

\[
\{ \tau y \mid y \in \mathcal{R}_{a}^{\ast} \} = \{ f_{a}^{\ast}x - x \mid x \in \mathbb{N}^{\ast}_{a} \}.
\]

Proof To \( i = (i_{1}, \ldots, i_{N}) \) we associate the reduced word \( i \cdot \text{op} = (i_{1} \cdot \text{op}, \ldots, i_{N} \cdot \text{op}) \) with \( i_{k} \cdot \text{op} = n - i_{N+1-k} \). We denote by \( \Gamma_{a}^{\ast}(\mathcal{T}_{i}) \) the set of \( a \cdot \ast \)-crossings in \( \mathcal{T}_{i} \) and by \( \Gamma_{a}(\mathcal{T}_{\text{op}}) \) the set of \( a \cdot \ast \)-crossings in \( \mathcal{T}_{\text{op}} \). We note that \( \mathcal{T}_{i} \) is the unique tiling such that under the identification of tiles with positive roots as in (5) the ordering \( \leq_{\text{op}} \) is a refinement of \( \leq_{2n} \) to a total order. As a consequence, the map

\[
\Gamma_{a}(\mathcal{T}_{\text{op}}) \rightarrow \Gamma_{a}^{\ast}(\mathcal{T}_{i}) , \quad \gamma = ([k_{1}, \ell_{1}], \ldots, [k_{m}, \ell_{m}]) \mapsto \gamma^{\ast} := ([k_{1}, \ell_{1}], \ldots, [k_{m}, \ell_{m}])
\]

is well defined and an order isomorphism.

For \( x \in \mathbb{Z}^{\mathcal{T}_{i}} \) let \( x^{\circ} \in \mathbb{Z}^{\mathcal{T}_{\text{op}}} \) be defined by \( x_{[k, \ell]}^{\circ} = x_{[k, \ell]} \). For \( y \in \text{Hom}(\mathbb{Z}^{\mathcal{T}_{i}}, \mathbb{Z}) \) we denote the map \( x \mapsto y(x^{\circ}) \) by \( y^{\circ} \). Then for \( \gamma \in \Gamma_{a}(\mathcal{T}_{\text{op}}) \) we have \( \tau(y)^{\circ} = \tau(y^{\circ}) \) and \( s(y)^{\circ} = s(y^{\circ}) \). Furthermore, by Definition 2.13 we have \( f_{a}^{\ast}x = (f_{a}x^{\circ})^{\circ} \), The statement now follows from Theorems 3.15, 3.18 and 3.21.

Definition 3.25 The subset \( \mathcal{R}_{a}^{\ast} := \{ f_{a}^{\ast}x - x \mid x \in \mathbb{N}^{\ast}_{a} \} \subset \mathbb{Z}^{\mathcal{T}_{i}} \) is called the set of \( a \cdot \ast \)-Reineke vectors.

By [18, Proposition 8.2] we have the following description of highest weight crystals analogously to [28, Proposition 7.4].

Proposition 3.26 Let \( \lambda = \sum_{a \in [n-1]} \lambda_{a} \omega_{a} \) be a dominant integral weight of \( \mathfrak{g} \). The corresponding crystal graph \( B(\lambda) \) is the full subgraph of the crystal graph of \( \mathbb{N}^{\mathcal{T}_{i}} \) given by all Lusztig data \( x \in \mathbb{N}^{\mathcal{T}_{i}} \) such that \( (\tilde{\gamma})(x) \leq \lambda_{a} \) for all \( a \in [n-1] \) and for all \( \gamma \in \Gamma_{a}^{\ast} \).
In this section we apply the Crossing Formula (Theorem 3.15) to MV-polytopes. By this we obtain a proof of a stronger version of the Anderson–Mirković (AM) conjecture. The AM conjecture was originally proved by Kamnitzer in [16] and subsequently by Saito in [29].

### 4 Application 1: The crossing formula applied to MV-polytopes

In this section we apply the Crossing Formula (Theorem 3.15) to MV-polytopes. By this we obtain a proof of a stronger version of the Anderson–Mirković (AM) conjecture. The AM conjecture was originally proved by Kamnitzer in [16] and subsequently by Saito in [29].

#### 4.1 MV polytopes and BZ data

As in Sect. 2.1 let \( \{ \alpha_a \}_{a \in [n-1]} \) be the simple roots and \( \{ \omega_a \}_{a \in [n-1]} \) the fundamental weights. Furthermore denote by \( h^\mathbb{R} \) the real span of the roots.

Recall the notation \( v_S \) for a vertex of a tiling \( T \) from Sect. 2.3. We denote by \( \mathcal{P}([n]) \) the power set of \([n]\) and identify the vertices of a tiling with a subset of \( \mathcal{P}([n]) \) via \( v_S \mapsto S \).

Recall that the chamber weights of \( g \) are the elements of \( \{ \sigma \omega_a \mid \sigma \in W, a \in [n-1] \} \). For \( \emptyset \neq S \subset [n] \) we denote the chamber weight \( \sum_{s \in S} \epsilon_s \) also by \( v_S \), where \( \epsilon_s \) was defined in Sect. 2.1.

We denote the set of \( w_0 \)-normalized BZ-data by \( \mathcal{P} \).

For each \( (z_S)_{S \in \mathcal{P}^0([n])} \in \mathcal{P} \) the polytope

\[
P(z) = \left\{ h \in h^\mathbb{R} \mid \langle h, v_S \rangle \geq z_S \quad \forall S \in \mathcal{P}^0([n]) \right\},
\]

is called a \((w_0\text{-normalized})\) Mirković-Vilonen \((MV)\) polytope. For a reduced word \( i \) of \( w_0 \) let \( T_i \) be the tiling associated to \( i \) as in Sect. 2.3. We denote the set of MV polytopes by \( \mathcal{M}V \). By [16] the set \( \mathcal{M}V \) has a crystal structure isomorphic to \( B(\infty) \) and for each reduced word \( i \) the map \([CA_i]_{\text{trop}} : \mathcal{M}V \to \mathbb{Z}^{\mathbb{T}_i}_{\geq 0} \) induces a crystal isomorphism, where \([CA_i]_{\text{trop}} \) is defined as follows. Let \( P(z) \) be an MV-polytope with corresponding BZ-datum \( z \), then

\[
[CA_i]_{\text{trop}} (P(z)) = (z_o(T) + z_u(T) - z_l(T) - z_r(T))_{T \in T_i},
\]

#### 4.2 Tropical Plücker relations

The tropical Plücker relations are:

\[
z_{\sigma \sigma_a[a]} + z_{\sigma \sigma_b[b]} = \min (z_{\sigma[a]} + z_{\sigma \sigma_a[b]}, z_{\sigma[b]} + z_{\sigma \sigma_b[a]}).
\]
where for $T = [s, t; S] \in \mathcal{T}_I$ in the notation of Sect. 2.3 we set

$$
\begin{aligned}
u(T) &:= S, & o(T) &:= S \cup \{s, t\}, \\
\ell(T) &:= S \cup \{\min(s, t)\}, & r(T) &:= S \cup \{\max(s, t)\}.
\end{aligned}
$$

(44)

The notation is illustrated in the following picture.

\begin{center}
\begin{tikzpicture}
\node (oT) at (0,0) {o(T)};
\node (lT) at (-2,-2) {\ell(T)};
\node (rT) at (2,-2) {r(T)};
\node (uT) at (0,-4) {u(T)};
\node (T) at (0,-1) {$T$};
\draw (oT) -- (T);
\draw (lT) -- (T);
\draw (rT) -- (T);
\draw (uT) -- (T);
\end{tikzpicture}
\end{center}

Remark 4.1 The naming $[CA_I]_{trop}$ comes from the fact that this map is given by the tropicalization of the Chamber Ansatz (see [1]).

4.2 The Anderson–Mirković conjecture in type A

For $a \in [n - 1]$ and $S \subset [n]$ we set $\sigma_a S := (S \setminus \{a\}) \cup \{a + 1\}$. In Sect. 3.1 we introduced the poset $(\Gamma_a, \preceq)$ of $a$-crossings. By translating $(\gamma(y))(x)$ defined in (18) for the $\preceq$-maximal element $\gamma = (a, a + 1) \in \Gamma_a$ to the language of BZ-data we obtain:

Theorem 4.2 Let $z, z'$ be BZ-data such that $f_a P(z) = P(z')$. Then we have for $S \subset [n]$

$$
(z_S - z'_S) = \begin{cases} 
1 & \text{if } a \in S, a + 1 \notin S \text{ and } z_S - z_{\sigma_a S} \geq z[a] - z_{\sigma_a[a]}, \\
0 & \text{else.}
\end{cases}
$$

(45)

Furthermore, if $a \in S$ and $a + 1 \notin S$, then

$$
z_S - z_{\sigma_a S} \leq z[a] - z_{\sigma_a[a]}.
$$

Proof For any $S \subset [n]$ there exists a tiling $T$ such that $v_S$ is a vertex of $T$. If $a \notin S$ or $a + 1 \notin S$ then $v_S$ is a vertex of a tile $T$ which is not in $\mathcal{W}_a$ and using Lemma 3.6 one finds by induction that

$$
z_S - z'_S = 0.
$$

(46)

Thus, (45) holds if $a \notin S$ or $a + 1 \notin S$.

From now on we assume $a \in S$ and $a + 1 \notin S$. We choose a tiling $\mathcal{T}_I$ such that $v_{\sigma_a[a]}$ is a vertex of $\mathcal{T}_I$. Then the tile $[a, a + 1]$ has vertices $v_{[a-1]}$, $v_{[a]}$, $v_{[a+1]}$, $v_{\sigma_a[a]}$ and $\gamma = (a, a + 1)$ is the only $a$-crossing since $[a, a + 1]$ intersects the left boundary of $\mathcal{T}_I$ with two edges. Thus, using (43) we obtain from the Crossing Formula (Theorem 3.15)

$$
\varepsilon_a \left(P(z)\right) = \varepsilon_a \left([CA_I]_{trop} \left(P(z)\right)\right) = (s(a, a + 1)) \left([CA_I]_{trop} \left(P(z)\right)\right) = (\left([CA_I]_{trop} \left(P(z)\right)\right)_{[a,a+1]} = z_{[a+1]} + z_{[a-1]} - z[a] - z_{\sigma_a[a]}.
$$

(47)
From [7, Theorem 9.3] (see also [19]) we deduce the existence of a tiling $T_j$ such that both $v_S$ and $v_{\sigma_a S}$ are vertices of $T_j$ (since $S$ and $\sigma_a S$ form a strongly separated collection). For $T = [a, a + 1] \in T_j$ we have in the notation (44) that $\ell(T) = S$, $r(t) = \sigma_a S, u(T) = S\setminus[a], o(T) = S \cup [a+1]$. Thus, by (46) we have $z_o(T) - z_o'(T) = z_u(T) - z_u'(T) = z_r(T) - z_r'(T) = 0$. Setting $x = [CA]_{\text{trop}}(P(z)) \in \mathbb{N}T_j$ we obtain from the crossing formula that

$$z_S - z_S' = - (x_{[a,a+1]} - f_a x_{[a,a+1]}) = \begin{cases} 1 & \text{if } (s(a,a+1))(x) \geq \varepsilon_a(P(z)) = \varepsilon_a(x), \\ 0 & \text{else}. \end{cases}$$  

(48)

Since by the Crossing Formula we have $(s(a,a+1))(x) \leq \varepsilon_a(x)$, the claim follows from (47), (48) and $(s(a,a+1))(x) = z_S - z_{\sigma_a S} + z_{[a+1]} - z_{[a-1]} - 2z_{[a]}$.

\[\square\]

We obtain the Anderson–Mirković conjecture in type $A$ as an immediate consequence of Theorem 4.2:

**Corollary 4.3** [16, Corollary 5.6] Let $z, z'$ be $BZ$-data such that $f_a P(z) = P(z')$. Then we have for $S \subset [n]$

$$z_S - z'_S = \max \left(0, z_S - z_{\sigma_a S} + z_{[a+1]} - z_{[a-1]} + 1\right) \quad \text{if } a \in S \text{ and } a + 1 \notin S,$$

$$0 \quad \text{else.}$$

\[5\text{ Application 2: A duality between Lusztig’s and Kashiwara’s parametrization}\]

Let $T_i$ be the tiling associated to the reduced word $i$ for $w_0$ as in Sect. 2.3. In Definition 3.25 we defined the set $R^*_a = \{f^*_a x - x \ | \ x \in \mathbb{N}T_i\}$ of $a$-$*$-Reineke vectors associated $i$ where $*$ is the Kashiwara involution on the set of $i$-Lusztig data (see Sect. 2.5). Let $G_m$ be the multiplicative group. In this section we associate to $R^*_a$ a function $r_a$ on the torus $\hat{L}_i := G^{T_i}_m$ and show that it transforms under a certain geometric lift of the transition maps between the string parametrizations defined by Berenstein and Zelevinsky in [5]. As a consequence we obtain that the cone corresponding to Kashiwara’s string parametrization of the dual canonical basis is dual to the cone spanned by $R^*_a$.

**5.1 String parametrizations**

We recall the string parametrization of the dual canonical basis $B^{\text{dual}}$ corresponding to the reduced word $i = (i_1, \ldots, i_N)$. The parametrizing set is the set of $i$-string data of elements of the crystal $B(\infty)$. An $i$-string datum $s_i(b)$ of $b \in B(\infty)$ is a tuple
\( s_i(b) = (x_1, \ldots, x_N) \in \mathbb{N}^N \) defined inductively as follows.

\[
x_1 = \max \left\{ k \in \mathbb{N} \mid (e_{i_1}^*)^k b \neq 0 \right\},
\]
\[
x_2 = \max \left\{ k \in \mathbb{N} \mid (e_{i_2}^*)^k (e_{i_1}^*)^{x_1} b \neq 0 \right\},
\]
\[
\vdots
\]
\[
x_N = \max \left\{ k \in \mathbb{N} \mid (e_{i_N}^*)^k (e_{i_{N-1}}^*)^{x_{N-1}} \cdots (e_{i_1}^*)^{x_1} b \neq 0 \right\}.
\]

By [4,20] the set
\[
S_i := \{ s_i(b) \mid b \in B(\infty) \} \subset \mathbb{N}^N
\]
(49) is a polyhedral cone called the string cone associated to \( i \).

We equip the cone \( S_i \) with a crystal structure following [17,27]. First we define for \( a \in [n - 1] \) the operator \( f_a \) on \( \mathbb{Z}^N \) as follows. We set for \( x = (x_1, \ldots, x_N) \in \mathbb{Z}^N \) and \( a \in [n - 1] \)
\[
v_\ell(x) := x_\ell + \sum_{\ell < j \leq N} \langle h_\ell, \alpha_{ij} \rangle x_j,
\]
\[
\varepsilon_a(x) := \max \{ v_\ell(x) \mid \ell \in [N], i_\ell = a \}.
\]

We let \( \ell^x \) be minimal with \( i_{\ell^x} = a \) and \( v_\ell^x(x) = \varepsilon_a(x) \) and define
\[
f_a x := x + (\delta_{k, \ell^x})_{k \in [N]} \in \mathbb{Z}^N.
\]
(50)

By [27] the map
\[
B(\infty) \hookrightarrow \mathbb{Z}^N_{\geq 0} \subset \mathbb{Z}^N, \quad b \mapsto s_i(b)
\]
is an embedding of crystals.

**Example 5.1** Let \( n = 3 \) and \( i = (1, 2, 1) \). Recall the associated tiling \( \mathcal{T}_i \):

Writing \( x \in \mathbb{N}^{\mathcal{T}_i} \) as \( x = (x_1, x_2, x_3) = (x_{\{1,2\}}, x_{\{1,3\}}, x_{\{2,3\}}) \) we compute using the Crossing Formula and *-Crossing Formula (Theorem 3.15, Theorem 3.24) and (50):

\[
\begin{align*}
  s_i(f_2(0, 0, 0)) &= s_i(0, 0, 1) = (0, 1, 0) = f_2(0, 0, 0), \\
  s_i(f_1f_2(0, 0, 0)) &= s_i(1, 0, 1) = (0, 1, 1) = f_1f_2(0, 0, 0), \\
  s_i\left( f_1^2f_2(0, 0, 0) \right) &= s_i(2, 0, 0) = (1, 1, 1) = f_1^2f_2(0, 0, 0).
\end{align*}
\]
Analogously to the definition of the set of ∗-Reineke vectors $R^*(T)$ (see (54)) for $T$ we associate to the string cone $S$ the sets

$$R_a(S) := \bigcup_{a} R_a(S),$$

Using (50) we obtain the following description of $R_a(S)$:

**Lemma 5.2** For a reduced word $i = (i_1, \ldots, i_N)$ and $a \in [n-1]$ we have

$$R_a(S) = \{ e_k := (\delta_{k, \ell})_{\ell \in [N]} \mid i_k = a \}.$$

**Proof** By (50) we have $R_a(S) \subset \{ e_k \mid i_k = a \}$. We fix $k \in [N]$ with $i_k = a$. Since $\dim S = N$, there exists $x = (x_1, \ldots, x_N) \in S$ with $x_k > 0$. Writing $x = f_{a_1} \cdots f_{a_M}(0, \ldots, 0)$ we obtain from (50)

$$e_k \in \{ f_a y - y \mid a \ell = a, y = f_{a_{\ell+1}} \cdots f_{a_M}(0, \ldots, 0) \} \subset R_a(S).$$

☐

### 5.2 Geometric liftings

As in [5, Chapter 5] we associate to a reduced word $i$ tori $L_i := \text{Hom}(\text{Hom}(\mathbb{Z}^{T}, \mathbb{Z}), \mathbb{G}_m)$, $\hat{L}_i := \text{Hom}(\mathbb{Z}^{T}, \mathbb{G}_m)$ and define maps

$$\Psi_i : L_i \to \hat{L}_i,$$

$$\Phi_i : L_i \to \hat{L}_i$$

as follows. If $j$ is obtained from $i$ by replacing the subword $(i_k, i_{k+1})$ by $(i_{k+1}, i_k)$, where $|i_k - i_{k+1}| > 1$ (commutation move), then

$$(\Psi_j x)_{[s, t]} = (\Phi_j x)_{[s, t]} = x_{[s, t]},$$

If $j$ is obtained from $i$ by a braid move that corresponds to a flip at

$$\mathcal{H} := \{(s, t), [s, u], [t, u]\} \subset T$$

with $s < t < u$, we set $y = \Psi_j x$, where $y_T = x_T$ for $T \notin \{(s, t), [s, u], [t, u]\}$ and

$$y_{[s, t]} = \frac{x_{[s, t]} x_{[t, u]} + x_{[s, u]}}{x_{[t, u]}},$$

$$y_{[s, u]} = x_{[s, t]} x_{[t, u]},$$

$$y_{[t, u]} = \frac{x_{[s, u]} x_{[t, u]} + x_{[s, u]}}{x_{[s, t]} x_{[t, u]} + x_{[s, u]}}.$$
Furthermore, we set \( y = \Phi_j^i x \), where \( y_T = x_T \) for \( T \notin \{ [s, t], [s, u], [t, u] \} \) and

\[
\begin{align*}
    y_{[s, t]} &= \frac{x_{[s, t]}x_{[s, u]}}{x_{[s, t]} + x_{[t, u]}}, \\
    y_{[s, u]} &= x_{[s, t]} + x_{[t, u]}, \\
    y_{[t, u]} &= \frac{x_{[s, u]}x_{[t, u]}}{x_{[s, t]} + x_{[t, u]}}.
\end{align*}
\] (53)

We thus obtain a definition of \( \Psi_j^i \) for \( i, j \) arbitrary. Similarly, we obtain for reduced words \( i \) and \( j \) maps \( \Phi_j^i \) by composing maps of type (51) and (53).

By [4, Theorem 2.2] the tropicalizations of \( \Psi_j^i \) intertwine the string parametrizations, i.e.

\[ s_i = s_j \circ \left[ \Psi_j^i \right]_{\text{trop}}. \]

Similarly, since the tropicalization of (53) is given by (8), we have that \( [\Phi_j^i]_{\text{trop}} = \phi_j^i \) intertwines the Lusztig parametrizations associated to \( i \) and \( j \).

### 5.3 Transformation behaviour of Reineke vectors

In Definition 3.25 we defined the set of \( a \)-s-Reineke vectors

\[ \mathbf{R}_a^* (T_i) = \left\{ f_a^* x - x \mid x \in \mathbb{N}^{T_i} \right\} \] (54)

associated to a reduced word \( i \) and \( a \in [n - 1] \). With the notation of Sect. 3.4 we obtain by the \( \ast \)-Crossing Formula (Theorem 3.24)

\[ \mathbf{R}_a^* (T_i) = \left\{ \gamma y \mid \gamma \in \mathbf{R}_a^* \right\}. \] (55)

Similarly, by Lemma 5.2 we have \( \mathbf{R}_a (S_i) := \{ f_a x - x \mid x \in S_i \} = \{ e_k | i_k = a \} \).

Setting

\[ x^y = \prod_{T \in T_i} x_T^{y_T} \]

we define the functions

\[ \varrho_a = \varrho_{a, i} : \widehat{\mathbb{L}}_i \rightarrow \mathbb{A}, \quad x \mapsto \sum_{y \in \mathbf{R}_a^* (T_i)} x^y, \] (56)

\[ \widehat{\varrho}_a = \widehat{\varrho}_{a, i} : \mathbb{L}_i \rightarrow \mathbb{A}, \quad x \mapsto \sum_{y \in \mathbf{R}_a (S_i)} x^y. \] (57)

Under a change of reduced words the functions \( \varrho_a \) and \( \widehat{\varrho}_a \) transform the following way.
Theorem 5.3  For $a \in [n-1]$ and reduced words $i$ and $j$ we have:

$$\varrho_{a,i} = \varrho_{a,j} \circ \Psi^j_i,$$  \hfill (58)

$$\hat{\varrho}_{a,i} = \hat{\varrho}_{a,j} \circ \Phi^j_i.$$  \hfill (59)

**Proof**  There is nothing show if $j$ is obtained by a commutation move or a braid move replacing $(i_k-1, i_k, i_k+1)$ by $(i_k, i_k-1, i_k)$ with $a \notin \{i_k-1, i_k\}$. Thus, by induction and exchanging $i$ and $j$ if necessary we can assume $j$ is obtained from $i$ by a braid move replacing $(a, a+1, a)$ by $(a+1, a, a+1)$. We denote the corresponding hexagon (see (6)) by $\mathcal{H} = \{[s, t], [s, u], [t, u]\} \subset T_i$ and set $y = \Phi^j_i x$. Equality (59) follows, since by (57) and (53) we have

$$\hat{\varrho}_{a,i}(x) - \hat{\varrho}_{a,j}(y) = x[s, t] + x[t, u] - y[s, u] = 0.$$

We next show (58). Using induction over $\#W_a$ (see (11)) we can assume by Lemma 3.6 that $\mathcal{T_j}$ is obtained from $\mathcal{T_i}$ by flipping a hexagon $H = \{[a, s], [a, t], [s, t]\} \subset W_a$. In Sect. 3.4 we introduced the set of $a$-crossings $\Gamma_a^*(\mathcal{T_i})$ and the set of $a$-Reineke crossings $\mathcal{R}_a^*(\mathcal{T_i})$. Let $\pi : \Gamma_a^*(\mathcal{T_i}) \to \Gamma_a^*(\mathcal{T_j})$ be the map defined by (13). Then we have for $\lambda \in \mathcal{R}_a^*(\mathcal{T_j})$ and $R := \pi^{-1}(\lambda) \cap \mathcal{R}_a^*(\mathcal{T_i})$

$$\sum_{\gamma \in R} x^{\gamma^*} = \left(\Psi^j_i(x)\right)^{\gamma^*}. \hfill (60)$$

Since $\pi \mathcal{R}_a^*(\mathcal{T_i}) = \mathcal{R}_a^*(\mathcal{T_j})$, we obtain from (60) to (55)

$$\varrho_{a,i}(x) = \sum_{\gamma \in \mathcal{R}_a^*(\mathcal{T_i})} x^{\gamma^*} = \sum_{\lambda \in \mathcal{R}_a^*(\mathcal{T_j})} \left(\Psi^j_i(x)\right)^{\gamma^*} = \varrho_{a,j} \circ \Psi^j_i(x).$$

$\square$

### 5.4 Duality of cones

As an application of Theorem 5.3 we derive the following duality between Lusztig’s parametrizations of $B(\infty)$ and the string parametrizations.

For $\emptyset \neq C \subset \mathbb{R}^N$ we define the **dual cone** $C^D$ by

$$C^D := \{x \in \mathbb{R}^N \mid \forall y \in C : \langle x, y \rangle \geq 0\}.$$ 

We have the following duality.

**Theorem 5.4**  Let $i$ be a reduced word for $w_0$ and $\mathcal{T_i}$ the tiling associated to $i$. Then

$$\mathcal{S}_i = \mathbb{R}^*(\mathcal{T_i})^D \quad \text{and} \quad \mathbb{N}^{\mathcal{T_i}} = \mathbb{R}(\mathcal{S}_i)^D$$
Applying Theorem 3.24 to the tiling associated to the lexicographically minimal word $i_0 = (1, 2, 1, \ldots, n, n-1, \ldots, 1)$ and comparing $R^* (T_{i_0})$ with the defining inequalities of the string cone $S_i$ as computed in [4,20] one obtains $S_{i_0} = R^* (T_{i_0})^D$. Tropicalizing (58) and using $[\Psi^1_j]_{\text{trop}}S_i = S_j$ the statement $S_j = R^* (T_j)^D$ now follows for all $j$. The statement $N_{T_i} = R(S_i)^D$ follows from Lemma 5.2.

$\square$

Remark 5.5 As noted in Remark 3.12, Reineke crossings translate into the notion of rigorous paths which appear in the work [14]. Here Gleizer and Postnikov associate vectors to rigorous paths, which can be seen to coincide with the $*_\text{Reineke}$ vectors $R^* (T_i) = \cup_a R^* a (T_i)$. As a consequence of the transformation behaviour of certain tropical functions $M_{i,a+1}^a$ under the tropicalization of the map (52) Gleizer and Postnikov show in loc. cit. that the vectors associated to rigorous paths are the defining inequalities of the string cones $S_i$. One finds that $M_{i,a+1}^a$ coincides with the tropicalization of $\varrho_{a,i}$ and thus by Theorem 3.24 and [14, Theorem 5.11] we obtain the first part of Theorem 5.4 stating $S_i = R^* (T_i)^D$.

Remark 5.6 For the special case of reduced words adapted to quivers the equality

$$S_i = R^* (T_i)^D$$

was already obtained by Zelikson in [32] using the results of [14,28]. Zelikson conjectured this relation for simply laced Lie algebras and reduced words adapted to quivers satisfying a certain homological condition. We conjecture that this relation holds for all simple Lie algebras and arbitrary reduced words, i.e. the string cone $S_i$ is dual to the cone spanned by the vectors of the form $f_{x,i}^* x - x$ for an $i$-Lusztig datum $x$ (see [32, Section 7] for an example in type $D$).

Example 5.7 We give an example for $n = 3$. Consider the following tiling corresponding to the reduced word $i = (2, 1, 2)$.

The ordering of the tiles in the 3-order is as follows:

$$[2, 3] \leq_3 [1, 3] \leq_3 [1, 2].$$

We label the coordinates with respect to that order. The $*_\text{Reineke}$ vectors are given by

$$R^* (T_i) = \{(1, 0, 0), (0, 1, -1), (0, 0, 1)\}.$$

By Theorem 5.4 we obtain the inequalities for the string cone $S_i$:

$$v_{2,3} \geq 0 \quad v_{1,3} \geq v_{1,2} \geq 0.$$
On the other hand, we have
\[ R(S_i) = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}. \]

By Theorem 5.4 we obtain hereby inequalities for the cone \( N^T \) of Lusztig data:
\[ v_{1, 3} \geq 0, \ v_{2, 3} \geq 0, \ v_{1, 2} \geq 0. \]

6 Application 3: Potential functions for cluster varieties and crystal operations

In Sect. 5 we associated to a reduced word \( i \) and \( a \in [n - 1] \) the function 
\[ \varrho_{a}(x) = \varrho_{a, i}(x) = \sum_{y \in R^*_T(T_i)} x^y \]
on the torus \( \mathbb{T}_i \) encoding the crystal operations on \( i \)-Lusztig data. In this section we relate \( \varrho_{a} \) to potential functions arising from the theory of geometric crystals in [3] and from mirror symmetry for cluster varieties in [15].

The relation is given by regular changes of coordinates provided by maps introduced by Berenstein and Zelevinsky (see [5, Theorem 4.8]) to solve certain factorization problems.

As a consequence we obtain explicit descriptions of these potential functions written in certain torus coordinates associated to reduced words \( i \) on the reduced double Bruhat cell \( L_{e, w_0} \) as well as the coordinate transformation translating one potential function to the other. Furthermore, for an arbitrary reduced word \( i \) we obtain unimodular isomorphisms of the cones defined by the tropicalizations of the potential functions with the string cone \( S_i \) defined in (49).

6.1 Unipotent radicals as cluster varieties

Let \( B_+ \) denote the upper triangular matrices, \( B_- \) the lower triangular matrices in \( G = \text{SL}_n(\mathbb{C}) \) and \( \mathcal{N} \subset B_+ \) be the unipotent radical of \( B_+ \). The reduced double Bruhat cell \( L_{e, w_0} \) associated to \( e \) and \( w_0 \) is the non-vanishing locus of the upper right minors in \( \mathcal{N} \):
\[ L_{e, w_0} = B_- w_0 B_- \cap \mathcal{N}. \]

Following Berenstein–Fomin–Zelevinsky, Fomin-Zelevinsky and Fock-Goncharov [2,10–12] we endow \( L_{e, w_0} \) with an \( A \)-cluster structure \((\mathbb{T}_\Sigma)\) and a mirror dual \( X \)-cluster structure \((\hat{\mathbb{T}}_\Sigma)\). The families \((\mathbb{T}_\Sigma)\) and \((\hat{\mathbb{T}}_\Sigma)\) are up to codimension 2 open coverings of \( L_{e, w_0} \) by tori
\[ \mathbb{T}_\Sigma := \mathbb{G}_m^\Sigma \quad \text{and} \quad \hat{\mathbb{T}}_\Sigma := \text{Hom}(\mathbb{G}_m, \mathbb{T}_\Sigma, \mathbb{G}_m). \]

The birational transition maps \( \mu_{\Sigma} \) and \( \hat{\mu}_{\Sigma} \), called \( A \)- and \( X \)-cluster mutation, respectively, are given as follows. Assume that the exchange graph \( Q_{\Sigma'} \) of the seed 6' is obtained from the exchange graph \( Q_{\Sigma} \) of the seed \( \Sigma \) by cluster mutation at a vertex \( k \).
Let $I$ be a set containing $k$ which consistently labels both the vertices $Q_\Sigma$ and $Q_{\Sigma'}$. Then (see [10, Equations (13) and (14)])

$$
\mu_{\Sigma'} : T_{\Sigma'} \to T_{\Sigma}, \quad (\mu_{\Sigma'}x)_i = \begin{cases} 
  x_i^{-1} \left( \prod_{j : \varepsilon_{j,k} > 0} x_j^{\varepsilon_{j,k}} + \prod_{j : \varepsilon_{j,k} < 0} x_j^{-\varepsilon_{j,k}} \right) & \text{if } i = k, \\
  x_i & \text{else},
\end{cases}
$$

$$
\hat{\mu}_{\Sigma} : \hat{T}_{\Sigma'} \to \hat{T}_{\Sigma}, \quad (\hat{\mu}_{\Sigma}x)_i = \begin{cases} 
  x_i^{-1} \left( 1 + x_i^{\varepsilon_{i,k}} \right)^{-\varepsilon_{i,k}} & \text{if } i = k, \\
  x_i & \text{else},
\end{cases}
$$

where $\varepsilon_{i,k}$ denotes the number of arrows in $Q$ from $i$ to $k$.

We refer to the set of coordinate functions on a given torus $T_{\Sigma}$ or $\hat{T}_{\Sigma}$ together with the exchange graph $Q_{\Sigma}$ as an $\mathcal{A}$- and $\mathcal{X}$-cluster seed, respectively.

As initial seed we may choose the seed $\Sigma_i$ associated to a reduced word $i$ in [2, Section 2.2]. We recall from [8] the construction of the $\mathcal{A}$-cluster seed $\Sigma_1$ associated to a tiling $T = T_i$. Let $V(T)$ be the set of vertices of $T$ that do not lie on the left boundary of $T$. As in (42) we identify a vertex $v = v_S$ of $T$ with a subset of $S \subset P(n)$.

To a vertex $v_S \in V(T)$ we associate the chamber minor $\Delta_S = \Delta_S^{[\#S]}$, which is the regular function on $L^{e,w_0}$ taking the determinant of the submatrix consisting of the first $\#S$ rows and columns labeled by $S$. The chamber minors $\{\Delta_S \mid v_S \in V(T)\}$ parametrize $T_i := T_\Sigma$ via

$$
L^{e,w_0} \rightarrow \mathbb{G}^V_m(T) = T_\Sigma, \quad x \mapsto (\Delta_S(x))_{S \in V(T)}.
$$

The chamber minors corresponding to the vertices on the right boundary are the frozen variables.

The $\mathcal{X}$-cluster torus $\hat{T}_i$ is the dual torus of $T_i$, i.e. $\hat{T}_i := \text{Hom}(\text{Hom}(G_m, T_i), G_m)$.

The exchange graph for $\Sigma_i$ can be obtained from $T = T_i$ in the following way. The vertices of the exchange graph are given by the elements in $V(T)$. Using the notation of Sect. 4 we draw for each tile $T \in T$ with $\ell(T) \in V(T)$ an arrow pointing from $\ell(T)$ to $r(T)$. The edges of $T$ get oriented such that we have oriented cycles $\ell(T) \rightarrow r(T) \rightarrow o(T) \rightarrow \ell(T)$ and $\ell(T) \rightarrow r(T) \rightarrow u(T) \rightarrow \ell(T)$ for all $T \in T$. If this procedure does not yield a unique orientation for an edge of $T$ this edge gets deleted. Furthermore we delete all arrows between vertices on the boundary.

**Example 6.1** We describe the $\mathcal{A}$-cluster seed $\Sigma_i$ for the reduced word $i = (1, 2, 1)$ in the case $n = 3$. The chamber minors $\{\Delta_{(2)}, \Delta_{(2,3)}, \Delta_{(3)}\}$ are associated to the vertices of $T_i$ as follows.

![Diagram with chamber minors](image-url)
The exchange graph $Q_1$ is given by:

$$
\Delta_{(2,3)} \\
\Delta_{[2]} \\
\Delta_{(3)}
$$

**Remark 6.2** Our convention differs slightly from the one used in [2] in that we associate the chamber minor $\Delta_S$ to a vertex, to which $\Delta_{w_0 S}$ was associated in [2, Equation (2.11)].

The following notion is crucial for the definition of the two potential functions appearing in this section.

**Definition 6.3** An $\mathcal{A}$-cluster (resp. $\mathcal{X}$-cluster) seed $\Sigma$ with coordinate functions $\{x_1, \ldots, x_N\}$ is *optimized* for a frozen variable $x_i$ if there is no arrow in $Q_\Sigma$ pointing from $x_i$ towards a non-frozen variable.

If $\Sigma$ corresponds to a reduced word $i$ we can reformulate this definition as follows. Denoting the coordinate function on $T_i$ or $\hat{T}_i$ attached to the vertex $v_S$ of $T_i$ by $x_S$ we have:

**Lemma 6.4** Let $i$ be a reduced word and $T_i$ be the corresponding tiling. Then the seed $\Sigma_i$ corresponding to $i$ is optimized for $x_{[a+1, \ldots, n]}$ if and only if the tile $[a, a+1]$ intersects the right boundary of $T_i$ in two edges. In other words $\Sigma_i$ is optimized for $x_{[a+1, \ldots, n]}$ if and only if $i$ is commutation equivalent to a reduced word ending with $n-a$.

**Proof** This follows directly from the construction of the exchange graph of $\Sigma_i$ and the definition of $T_i$ given in (5).



## 6.2 Gross–Hacking–Keel–Kontsevich potential functions and Reineke vectors

The unipotent radical $\mathcal{N}$ is the partial compactification of the reduced double Bruhat cell $L^{e,w_0}$:

$$
\mathcal{N} = L^{e,w_0} \cup \bigcup_{a \in [n-1]} D_a,
$$

where $D_a$ is the divisor of zeros of the frozen variable $\Delta_{[n-a+1, \ldots, n]}$.

In [15] a Landau–Ginzburg potential $W = \sum_{a \in [n-1]} W_a$ on the $\mathcal{X}$ cluster variety associated to $\mathcal{N}$ is defined as the sum of certain global monomials $W_a$ attached to the divisors $D_a$. The function $W_a$ can be defined as follows. Let $i = (i_1, \ldots, i_N)$ be a reduced word with $i_N = n-a$ and let $x_S$ denote the coordinate function on $\hat{T}_i$ attached to the vertex $v_S$ of $T_i$. Then we have by Lemma 6.4 and [15, Corollary 9.17]

$$
W_a \big|_{\hat{T}_i} = x_{[a+1, \ldots, n]}^{-1}.
$$

(61)
We call $W$ the GHKK-potential.

The potential $W_a$ is related to the function $\varrho_{a,i}$ arising from crystal operations (see (56)) via the map $\tilde{CA}_i \in \text{Hom}(T_i, T_i)$ defined by

$$\tilde{CA}_i : \hat{L}_i = \mathbb{G}_m \to \mathbb{G}_m^{\nu(T_i)} = \hat{T}_i$$

$$(\tilde{CA}_ix)_v = \prod_T x_T^{\varepsilon(v,T)}, \quad \varepsilon(v,T) = \begin{cases} -1 & \text{if } v \in [\ell(T), r(T)], \\ 1 & \text{if } v \in [o(T), u(T)], \\ 0 & \text{else}, \end{cases}$$

where $o(T), u(T), \ell(T)$ and $r(T)$ are defined in (44). The map $\tilde{CA}_i$ is the dual map of the Chamber Ansatz $CA_i \in \text{Hom}(T_i, T_i)$ of Berenstein, Fomin and Zelevinsky [1]. The family $(\tilde{CA}_i)$ is compatible with $\lambda$-mutations in the following sense:

**Proposition 6.5** For two reduced words $i$ and $j$ for $w_0$ the following diagram commutes.

Proof: By induction we can assume that $T_i$ and $T_j$ are obtained from each other by braid move corresponding to a flip of a hexagon $H \subset T_i$ as in (6). By direct computation it can be checked in this case that $\tilde{\mu}_{ij}$ is given by mutation at the inner vertex of $H$ and that the diagram commutes. $\Box$

Using Theorem 5.3 and the enhanced AM-formula (Theorem 4.2) we obtain:

**Theorem 6.6** Let $i$ be a reduced word for $w_0$ and $a \in [n-1]$. Then we have

$$\varrho_{a,i} = W_a \Big|_{\hat{T}_i} \circ \tilde{CA}_i. \quad (62)$$

Furthermore, the GHKK-potential $W \Big|_{\hat{T}_i} = \sum_{a \in [n-1]} \varrho_{a,i} \circ (\tilde{CA}_i)^{-1}$ is a Laurent polynomial without constant term, coefficients in $\{0, 1\}$ and exponents in $\{0, -1\}$.

Proof: In order to prove (62) it suffices by Theorem 5.3 and Proposition 6.5 to show the statement for the case that the tile $[a, a+1]$ intersects the right boundary of $T = T_i$ in two edges (i.e. $i$ is commutation equivalent to a reduced word ending with $i_N = n-a$). In this case we have by the definition of $\varrho_{a,i}$ and (61) that $\varrho_{a,i}(x) = x_{[a,a+1]} = W_a \Big|_{\hat{T}_i} \circ \tilde{CA}_i(x)$. By the duality of $\tilde{CA}_i$ and $CA_i$ we obtain

$$W_a \Big|_{\hat{T}_i} (x) = \varrho_{a,i} \circ (\tilde{CA}_i)^{-1}(x) = \sum_{y \in R_\nu(T_i)} x^{[CA_i]^{-1}}(y).$$
The map $[\text{CA}_i]_{\text{trop}}$ is explicitly given in (43) (we identify $P(z)$ with the corresponding point $(z_v)_{v \in V(\mathcal{T}_i)} \in \mathbb{Z}^{V(\mathcal{T}_i)}$). For $y \in \mathbb{R}_+^* (\mathcal{T}_i)$ there exists, by definition, $z \in \mathbb{N}^{\mathcal{T}_i}$ with $y = f_a z - z$, where $f_a$ is the Kashiwara operator associated to the simple root $\alpha_a$. By Theorem 4.2 we conclude for $S \subset [n]$

$$
\left((\text{CA}_i)_{\text{trop}} y\right)_S = \left(f_a \text{CA}_i^{-1}\text{trop} z - \text{CA}_i^{-1}\text{trop} z\right)_S \in \{0, -1\}.
$$

$\square$

Gross, Hacking, Keel and Kontsevich construct in [15] a canonical basis $\mathbb{B}^{\text{GHKK}}$ of $\mathbb{C}[\mathcal{N}]$ and parametrizations of $\mathbb{B}^{\text{GHKK}}$ by the cones

$$
\mathcal{C}^{\text{GHKK}}_{\Sigma} := \left\{ z \mid \left[ W_{|\Sigma} \right]_{\text{trop}} (z) \geq 0 \right\} \subset \text{Hom}(\mathbb{G}_m, \hat{\mathcal{T}}_{\Sigma}).
$$

As a consequence of Theorems 5.3 and 6.6 we obtain:

**Corollary 6.7** For any reduced word $i$ for $w_0$ we have

$$\mathcal{C}^{\text{GHKK}}_{\Sigma_i} = (\text{CA}_i)_{\text{trop}} (S_i).$$

**Remark 6.8** Let $\mathcal{T}_0$ be the tiling associated to the lexicographically minimal reduced word $i_0$. In [24, Corollary 21] the function $W_{|\mathcal{T}_0}$ is explicitly computed. This description agrees with the one given by Theorem 6.6 in this case. Furthermore, in [6] an unimodular isomorphism between the cone cut out by the GHKK potential function for the partial compactification $SL_n/N$ of the double Bruhat cell $G^{e,w_0} = B - \cap B w_0 B$ and the weighted string cone was independently discovered by Bossinger and Fourier.

### 6.3 Berenstein–Kazhdan potential functions and Reineke vectors

In analogy to (61) we define a potential function on the $\mathcal{A}$-cluster variety associated to $\mathcal{N}$ as follows.

**Definition 6.9** For $a \in [n - 1]$ the function $f_{\chi,a}$ on $\mathcal{N}$ is defined by requiring for reduced words $i = (i_1, \ldots, i_N)$ of $w_0$ with $i_N = n - a$ that

$$
f_{\chi,a} = \frac{\Delta_{\ell[a,a+1]i}}{\Delta_{r[a,a+1]i}}.
$$

We further define

$$
f_{\chi} := \sum_{a \in [n-1]} f_{\chi,a}
$$

and call $f_{\chi}$ the Berenstein–Kazhdan (BK) potential function.
Note that, by Lemma 6.4, the function $f_{\chi,a}$ is well-defined and we have

$$f_{\chi,a} = \frac{\Delta_{\{a,a+2,\ldots,n\}}}{\Delta_{\{a+1,\ldots,n\}}}.$$  

**Remark 6.10** The naming comes from the fact that $f_\chi$ appears in the *decoration function* $f_{G,\chi}$ introduced by Berenstein and Kazhdan as part of the data of a unipotent crystal structure on $G = \text{SL}_n$. Namely by [3, Example 1.10], we have

$$f_{G,\chi}(g) = \sum_{a \in [n-1]} f_{\chi,a}(g) + \frac{\Delta_{\{1,\ldots,n-a-1,n-a+1\}}(g)}{\Delta_{\{a+1,\ldots,n\}}(g)},$$

where $\Delta_{\{1,\ldots,n-a-1,n-a+1\}}(g)$ is the determinant of the submatrix of $g$ with rows labeled by $\{1,\ldots,n-a-1,n-a+1\}$ and columns labeled by $\{a+1,\ldots,n\}$ and $\Delta_S = \Delta_{[#S]}$.

Note that in contrary to [3] we write $f_\chi$ as a Laurent polynomial in the coordinates of the $A$-cluster torus $T_1$.

The BK potential $f_\chi$ is related to the function $\varphi_{a,i}$ arising from the crystal operations (see (56)) via the maps $N_{A_i} \in \text{Hom}(T_1, \hat{T}_i)$ given by

$$N_{A_i} : T_1 = \mathbb{C}_{\text{m}}^V(T_i) \rightarrow \mathbb{C}_{\text{m}}^V(T_i) = \hat{T}_i,$$

$$x \mapsto \left(\frac{x_{\ell(T)}}{x_{r(T)}}\right)_{T \in T_i},$$

(64)

where $\ell(T)$ and $r(T)$ are defined in (44).

**Remark 6.11** The map $N_{A_i}$ appears in [5, Equation (4.14)] as a crucial part of the solution of a factorization problem. In analogy to the naming of Chamber Ansatz we call $N_{A_i}$ *Neighbour Ansatz*.

The family $(N_{A_i})$ is compatible with $A$-mutations in the following sense:

**Proposition 6.12** For two reduced words $i$ and $j$ for $w_0$ the following diagram commutes.

$$\begin{array}{cccc}
T_1 & \overset{N_{A_i}}{\longrightarrow} & \hat{T}_i & \\
\downarrow & & \downarrow & \\
\mu^{-1}_j & & \psi_j & \\
\downarrow & & \downarrow & \\
T_j & \overset{N_{A_j}}{\longrightarrow} & \hat{T}_j & \\
\end{array}$$
Proof By induction we can assume that $T_i$ and $T_j$ are obtained from each other by braid move corresponding to a flip of a hexagon $H \subset T_i$ as in (6). By direct computation it can be checked in this case that $\mu^1_j$ is given by mutation at the inner vertex of $H$ and that the diagram commutes. \qed

Using the $\ast$-Crossing Formula (Theorems 3.24) and 5.3 we obtain:

**Theorem 6.13** Let $i$ be a reduced word for $w_0$ and $a \in [n-1]$. We have

$$f_{\chi,a}|_{T_i} = \varrho_{a,i} \circ N\!A_i$$

Furthermore, the BK-potential $f_{\chi}|_{T_i} = \sum_{a \in [n-1]} \varrho_{a,i} \circ N\!A_i$ is a Laurent polynomial with coefficients in $\{0,1\}$ and exponents in $\{-1,0,1\}$ under the change of variables (64).

Proof In order to prove (65) it suffices by Theorem 5.3 and Proposition 6.12 to show the statement for the case that the tile $[a,a+1]$ intersects the right boundary of $T = T_i$ in two edges (i.e. $i$ is commutation equivalent to a reduced word ending with $i_N = a$). In this case we have by the definition of $\varrho_{a,i}$ and Lemma 6.4 $\varrho_{a,i} = x_{[a,a+1]} = f_{\chi,a}|_{T_i} \circ N\!A_i^{-1}(x)$. By the $\ast$-Crossing Formula we obtain that the exponents of $\varrho_{a,i}$ are contained in $\{-1,0,1\}$. \qed

In analogy to (63) we define the polyhedral cones

$$C_{\Sigma}^{BK} := \left\{ z \left| f_{\chi,a}|_{T_i} \right|_{trop}(z) \geq 0 \right\} \subset \text{Hom}(\mathbb{G}_m, T_{\Sigma}).$$

From Theorems 5.4 and 6.13 we conclude:

**Corollary 6.14** For any reduced word $i$ for $w_0$ we have

$$S_i = [N\!A_i]_{trop} \left( C_{\Sigma}^{BK} \right).$$

From Theorems 6.13 and 6.6 we obtain the following relation between the potential functions $f_{\chi}$ arising from the theory of geometric crystals and the potential function $W$ arising from the partial compactification of $L^{e,\hat{w}_0}$:

**Corollary 6.15** Let $i$ be a reduced word for $w_0$. Then

$$f_{\chi,a}|_{T_i} = W_a|_{T_i} \circ \hat{C}A_i \circ N\!A_i.$$

**Remark 6.16** In [15] a result about the relation of the BK-potential function and $W$ is announced to appear in [25].

**Remark 6.17** The relation given in Corollary 6.15 does not arise from the coordinate change $p$ between $A-$ and $X-$cluster coordinates given in [10, Equation (6)], i.e. we have $f_{\chi,a}|_{T_i} \neq W_a|_{T_i} \circ p_i$. The reason for this is that the equality $p_i = \hat{C}A_i \circ N\!A_i$ only
holds restricted to vertices which do not correspond to frozen variables. Furthermore, the map $p_i$ is not canonically defined for frozen variables.

In [13] the first author introduces a canonical modification $\bar{p}_i$ of $p_i$ and shows $\bar{p}_i = \overline{CA_i^{-1}} \circ NA_i$. The canonical $p$-map $\bar{p}_i$ furthermore relates the two main ingredients $\tau_i : R^a_*(T_i) \to Z^T_i$ and $s_i : R^a_*(T_i) \to \text{Hom}(Z^T_i, Z)$ of the $*$-Crossing Formula as follows: In [13] the first author shows $s_i = [NA_i \circ CA_i^{-1}]_{\text{trop}} \circ \tau_i$. Consequently, after change of coordinates by $CA_i$ the linear form $s_i\gamma$ is obtained from the Reineke vector $\tau_i\gamma$ by applying the tropicalized canonical $p$-map $[\bar{p}]_{\text{trop}}$.

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