LOCAL GENERALIZATIONS OF DERIVATIVES

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Abstract. Derivatives can be viewed as mathematical idealizations of the linear growth. The linear growth condition has special properties, which make it preferred. The manuscript investigates some general properties of the local generalizations of derivatives. The concept of derivative is generalized in terms of the class of the modulus of continuity of the primitive function. This definition focuses on applications involving continuous but possibly non-absolutely continuous functions of a real variable. The main applications of the approach are a generalization of the Lebesgue monotone differentiation theorem. On the second place, the conditions of continuity of generalized derivative are also stated.

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1. Introduction

Since the time of Newton, it is accepted that celestial mechanics and physical phenomena are, by and large, described by smooth and continuous functions. The second law of Newton demands that the velocity is a differentiable function of time. This ensures mathematical modeling in terms of differentiable equations. Ampere even tried to prove that all functions are almost everywhere differentiable. Now we know that this attempt was doomed to fail.

Various non-differentiable functions have been constructed in the XIXth century and regarded with a mixture of wonder and horror. The interest in fractal and non-differentiable functions was rekindled with the works of Mandelbrot in fractals [12]. Now we know that non-differentiable functions can not be avoided when modeling nature. For instance, it is easy to establish that stochastic paths of the classical Wiener process are non-differentiable. In a closely related manner, almost all, in the measure sense, paths in the formulation of the Feynman path-integral are non-differentiable [8].

The derivatives can be generalized in several ways. If continuity is perceived as an essential property such generalization leads to various integro-differential operators. The best known examples here are the Riemann-Liouville and Caputo operators. However, such operators lead to non-local (interval) functions. Application of a subsequent localization operation can lead to a local operator. An example of this is the local fractional derivative introduced by Kolwankar and Gangal [10]:

$$\lim_{x \to a} \frac{1}{\Gamma(1 - \beta)} \frac{d}{dx} \int_a^x \frac{f(t)}{(x - t)\beta}dt$$

However, such localization can only lead to trivially continuous functions - that is the result of the localization is zero where the function is continuous [16].
Point-wise, the derivatives can be generalized by formal "fractionalization" – i.e. by replacement \( \epsilon \rightarrow \epsilon^{\beta} \) as
\[
\lim_{\epsilon \to 0} \frac{f(x + \epsilon) - f(x)}{\epsilon^{\beta}}
\]
The quantity in this definition is called fractional velocity. Such an approach has been considered for the first time by du Bois-Reymond and Faber in their studies of the point-wise differentiability of functions \([6, 7]\). In the late XX\textsuperscript{th} century, Cherbit introduced the same quantity under the name fractional velocity as a tool to study the fractal phenomena and physical processes for which instantaneous velocity was not well defined \([5]\). The properties of fractional velocity have been extensively studied in \([11]\) and \([14]\). The special choice of the function \( \epsilon^{\beta} \) can be justified from the theory of the fractional calculus as the limit of the regularized Riemann-Liouville differ-integral.

As can be expected, the overlap of the definitions of the Cherbit’s fractional velocity and the Kolwankar-Gangal local fractional derivative is not complete. The precise equivalence conditions have been established elsewhere \([2, 16]\). Both definitions are closely linked with conditions for the growth of the functions. Notably, Kolwankar-Gangal fractional derivatives are sensitive to the critical local Hölder exponents, while the fractional velocities are sensitive to the critical point-wise Hölder exponents and there is no complete equivalence between those quantities \([11]\).

On the other hand, mathematically, there is no reason to limit the choice of the function in the denominator to a power function. In such way, more general limit objects can be studied. This is the objective of the present paper. In the present paper, derivatives are generalized in terms of the class of the modulus of continuity of the primitive function. Such definition focuses on applications involving continuous but possibly non-absolutely continuous functions of a real variable.

The paper is structured as follows: Section 2 introduces the notational conventions. Section 3 introduces point-wise oscillation of functions. Section 4 characterizes some of the properties of the moduli of continuity. Section 5 introduces the concept of generalized \( \omega \)-derivatives, define from the maximal modulus of continuity. Section 6 introduces the concept of modular derivatives. Section 7 discusses the continuity sets of derivatives form the perspective of the theory developed in Sec. 3.

2. General definitions and conventions

The term \textit{variable} denotes an indefinite number taken from a predefined set, usually the real numbers. Sets are denoted by capital letters, while variables taking values in sets are denoted by lowercase.

The term \textit{function} denotes a mapping from one number to another and the action of the function is denoted as \( f(x) = y \). Implicitly the mapping acts on the real numbers: \( f : \mathbb{R} \mapsto \mathbb{R} \). If a statement of a function \( f \) fulfills a certain predicate with argument \( A \) (i.e. \( \text{Pred}[A] \)) the following short-hand notation will be used \( f \cong \text{Pred}[A] \).

The co-domain of the function \( f : X \mapsto Y \) is denoted as \( f[X] = Y \). The term \textit{operator} denotes the mapping from a functional expression to functional expression. The term \textit{functional} denotes the mapping from a functional expression to a number. Square brackets are used for the arguments of operators and functionals, while
round brackets are used for the arguments of functions. The term Cauchy sequence will be always interpreted as a null sequence.

**Definition 1** (Asymptotic $O$ notation). The notation $O(x^\alpha)$ is interpreted as the convention that
\[ \lim_{x \to 0} \frac{O(x^\alpha)}{x^\alpha} = 0 \]
for $\alpha > 0$. Or in general
\[ O(g(x)) \Rightarrow \lim_{x \to 0} \frac{O(g(x))}{g(x)} = 0 \]
The notation $O_x$ will be interpreted to indicate a Cauchy-null sequence with possible dependence in the variable $x$.

**Definition 2.** Define the parametrized difference operators acting on the function $f(x)$ as
\[ \Delta^+_\epsilon [f](x) := f(x + \epsilon) - f(x), \]
\[ \Delta^-_\epsilon [f](x) := f(x) - f(x - \epsilon), \]
\[ \Delta^2_\epsilon [f](x) := f(x + \epsilon) - 2f(x) + f(x - \epsilon), \]
where $\epsilon > 0$. The three operators are referred to as forward difference, backward difference and 2nd order difference operators, respectively.

**Definition 3** (Anonymous function notation). The notation for the pair $\mu :: \epsilon$ will be interpreted as the implication that if LHS is fixed then RHS is fixed by the value chosen on the left, i.e. as an anonymous functional dependency $\epsilon = \epsilon(\mu)$.

**Definition 4.** Consider the interval $I = [a, b]$. A partition of $I$ is a set of $n$ numbers $\mathcal{P}[I] := (a < x_1 \ldots x_{n-1} < b)$. The function $f : \mathbb{R} \to \mathbb{R}$ is said to be of bounded variation on $I$ if and only if there is a constant $M > 0$ such that
\[ V_{\mathcal{P}}[I] := (\mathcal{P}) \sum_{i=1}^{n} |f(x_i - x_{i-1})| \leq M \]
for all partitions $\mathcal{P}$. The total variation of the function is defined
\[ \text{Var}(f, I) := \sup_{\mathcal{P}} V_{\mathcal{P}}[I] \]
The class of function of bonded variation in a compact interval $I$ will be denoted as $BV[I]$.

The notation $|I|$ for an interval $I$ will mean its length.

2.1. Null sets.

**Definition 5** (Null sets and Null functions, [3]).

- A **null set** $Z \subset \mathbb{R}$ (or a set of measure 0) is called a set, such that for every $0 < \epsilon < 1$ there is a countable collection of sub-intervals $\{I_k\}_{k=1}^\infty$,
\[ Z \subseteq \bigcup_{k=1}^{\infty} I_k \]
\[ \text{assumed to hold throughout the paper for the variable } \epsilon. \]
such that $\sum_{k=1}^{\infty} |I_k| \leq \epsilon$ where $|.|$ is the interval length. Then we write $|Z| = 0$.

- A null function is a function which is non zero on a null set. That is for $f : X \rightarrow Y; X, Y \subset \mathbb{R}$ holds
  \[ |\{x : f(x) \neq 0\}| = 0 \]

**Theorem 1** (Null set disconnectedness). Suppose that $E$ is a null set. Then $E$ is
totally disconnected. Conversely, suppose that $E$ is totally disconnected and countable. Then $E$ is a null set.

**Proof.**  
**Forward statement:** Suppose that $Z \subset E$ is connected and open. Then there exist 3 numbers $x_1 < z < x_2$, such that $[x_1, x_2] \subset Z$. Then $|[x_1, x_2]| = x_2 - x_1 > 0$. Therefore, $3\epsilon$, such that $0 < \epsilon < z - x_1 < x_2 - x_1$; so that $\epsilon < |Z| \leq |E|$, which is a contradiction. Therefore, $x_2 = x_1$ and hence $Z$ is singleton. Therefore, by induction $E$ is totally disconnected.

**Converse statement:** Since $E$ is totally disconnected for every $z, w \in E$ there is a $h$, such that $[z - h/2, z + h/2] \cap [w - h/2, w + h/2] = \emptyset$. Therefore, there is a collection of such intervals, such that

\[ I_k = [z_k - h/2, z_k + h/2]/2^k \]

Therefore, $\sum_{k=1}^{\infty} |I_k| = h$. 

\[ \square \]

**Remark 1.** There are sets that are totally disconnected, uncountable and non-null. An example of such sets is the Smith-Volterra-Cantor set (i.e. the so-called fat Cantor set), which is of measure $1/2$.

**Example 1.** The construction of the Smith-Volterra-Cantor set is given as follows: The set is constructed by iteratively removing certain intervals from the unit interval $I_0 = [0,1]$. At each step $k$, the length that is removed $p_{k+1} = p_k/4$ from the middle of each of the remaining intervals. That is, starting from $I_0$ and $p_0 = 1/4$ on every step

\[ I_k = [u, v] \rightarrow I_{k+1}^c = [u, (u + v)/2 - p_k/2], \quad I_{k+1} = [(u + v)/2 + p_k/2, v] \]

\[ p_k \rightarrow p_{k+1} = p_k/4 \]

For example,

\[ k = 1: \quad I_1 = [0, \frac{3}{8}], \quad I_2 = [\frac{5}{8}, 1] \]

\[ k = 2: \quad I_{21} = [0, \frac{5}{16}], \quad I_{22} = [\frac{7}{16}, \frac{3}{8}], \quad I_{23} = [\frac{5}{8}, \frac{15}{16}], \quad I_{24} = [\frac{15}{16}, 1] \]

\[ k = 3: \quad I_{31} = [0, \frac{9}{32}], \quad I_{32} = [\frac{11}{32}, \frac{9}{16}], \quad I_{33} = [\frac{11}{16}, \frac{27}{32}], \quad I_{34} = [\frac{27}{32}, \frac{33}{32}], \quad I_{35} = [\frac{33}{32}, 1] \]

\[ I_{36} = [\frac{33}{32}, \frac{89}{128}], \quad I_{37} = [\frac{89}{128}, \frac{97}{128}], \quad I_{38} = [\frac{97}{128}, 1] \]

During the process, disjoint intervals of total length

\[ L = \sum_{k=0}^{\infty} \frac{1}{4 \cdot 2^k} = \frac{1}{2} \]
are removed so that the resulting set is of measure 1/2. The Smith-Volterra-Cantor set is closed as it is an intersection of closed sets. Furthermore, at step \( n \) the length of each closed subinterval is \( l_n = \frac{1}{2} (l_{n-1} - p_{n-1}) \). Starting from \( l_0 = 1 \) one gets

\[
l_n = \frac{1}{2} \left( \frac{1}{2^n} + \frac{1}{4^n} \right)
\]

Therefore, by the Nested Interval Theorem the SVC set is totally disconnected and contains no intervals.

The set presented in the above example can be used to construct a singular function, resembling by some of its properties the famous Cantor-Lebesgue ”Devil’s staircase” function (see Fig. 2).

3. Point-wise oscillation of functions

The concept of point-wise oscillation is used to characterize the set of continuity of a function. To this end I build further on a technical result, which is presented as a Theorem 3.5.2 in Trench [17][p. 173]. Here the proof is slightly modified to account for separate treatment of right- and left- continuity.

**Definition 6.** Define forward oscillation and its limit as the operators

\[
osc^+_\epsilon[f](x) := \sup_{[x, x+\epsilon]} f - \inf_{[x, x+\epsilon]} f
\]

\[
osc_\epsilon^+[f](x) := \lim_{\epsilon \to 0} \left( \sup_{[x, x+\epsilon]} f - \inf_{[x, x+\epsilon]} f \right) = \lim_{\epsilon \to 0} osc^+_\epsilon[f](x)
\]

and backward oscillation and its limit as the operators

\[
osc^-_\epsilon[f](x) := \sup_{[x-\epsilon, x]} f - \inf_{[x-\epsilon, x]} f
\]

\[
osc^-_\epsilon[f](x) := \lim_{\epsilon \to 0} \left( \sup_{[x-\epsilon, x]} f - \inf_{[x-\epsilon, x]} f \right) = \lim_{\epsilon \to 0} osc^-_\epsilon[f](x)
\]

according to previously introduced notation [14]. If the context of use is known the short-hand notation for the supremum and the infimum will be used as follows

\[
\sup_{[x, x \pm \epsilon]} f \equiv \sup_{\epsilon} f(x)
\]

\[
\inf_{[x, x \pm \epsilon]} f \equiv \inf_{\epsilon} f(x)
\]

**Lemma 1** (Oscillation lemma). Consider the function \( f : X \mapsto Y \subseteq \mathbb{R} \). Suppose that \( I_+ = [x, x + \epsilon] \subseteq X \), \( I_- = [x - \epsilon, x] \subseteq X \), respectively.

If \( f \) is right-continuous in \( I_+ \) then \( osc^+_\epsilon[f](x) = 0 \). Conversely, if \( osc^+_\epsilon[f](x) = 0 \) then \( f \) is right-continuous in \( I_+ \).

If \( f \) is left-continuous in \( I_- \) then \( osc^-_\epsilon[f](x) = 0 \). Conversely, if \( osc^-_\epsilon[f](x) = 0 \) then \( f \) is left-continuous in \( I_- \). That is,

\[
\lim_{\epsilon \to 0} osc^+_\epsilon[f](x) = 0 \iff \lim_{\epsilon \to 0} f(x \pm \epsilon) = f(x)
\]

**Proof.** **Forward case:** Suppose that \( osc^+_\epsilon[f](x) = 0 \). Then there exists a pair \( \mu : \delta, \delta \leq \epsilon \), such that \( osc^+_\epsilon[f](x) \leq \mu \). Therefore, \( f \) is bounded in \( I_+ \). Since \( \mu \) is arbitrary we select \( x' \), such that

\[
|f(x') - f(x)| = \mu' \leq \mu
\]
and set \(|x - x'| = \delta'\). Since \(\mu\) can be made arbitrary small so does \(\mu'\).

Therefore, \(f\) is (right)-continuous at \(x\).

**Reverse case:** If \(f\) is (right-) continuous on \(x\) then there exist a pair \(\mu :: \delta\) such that

\[
|f(x') - f(x)| < \mu/2, \quad |x' - x| < \delta/2 \\
|f(x) - f(x'')| < \mu/2, \quad |x - x''| < \delta/2
\]

Then we add the inequalities and by the triangle inequality we have

\[
|f(x') - f(x'')| \leq |f(x') - f(x)| + |f(x) - f(x'')| < \mu \\
|x' - x''| \leq |x' - x| + |x - x''| < \delta.
\]

However, since \(x'\) and \(x''\) are arbitrary we can set the former to correspond to the minimum and the latter to the maximum of \(f\) in the interval. therefore, by the least-upper-bond property we can identify \(f(x') \mapsto \inf_x f(x), f(x'') \mapsto \sup_x f(x)\). Therefore, \(\text{osc}^+ f(x) < \mu\) for \(|x' - x''| < \delta\) (for the pair \(\mu :: \delta\)). Therefore, the limit is \(\text{osc}^+ f(x) = 0\).

The left case follows by applying the right case, just proved, to the mirrored image of the function: \(f(-x)\). \(\square\)

Then the negation of the statement is also true:

**Corollary 1.** The following are equivalent

\[
\lim_{\epsilon \to 0} \text{osc}^\pm f(x) > 0 \iff \lim_{\epsilon \to 0} f(x \pm \epsilon) \neq f(x)
\]

The so-stated lemma is of a fundamental importance for it opens up the possibility to characterize the properties of the functions based on their oscillation on intervals. The oscillation of a function can be viewed in two ways: as a functional having the interval of study fixed; or alternatively as a function of the interval having the function under study fixed. There is no ambiguity as in fact both aspects are complementary as will be demonstrated.

As a first task we demonstrate the properties of the oscillation function.

### 3.1. The Oscillation function.

**Definition 7.** Consider a bounded function \(f\) defined on an arbitrary real domain. Define the interval oscillation function as

\[
\omega^+_x(\epsilon) := \text{osc}^+_I f(x) \\
\omega^-_x(\epsilon) := \text{osc}^-_I f(x)
\]

for \(I = [x, x+\epsilon]\) and

for \(I = [x-\epsilon, x]\). \(I\) is omitted for clarity of presentation but is given in the context. If the context is clear the superscript will be omitted.

**Theorem 2** (Sub-additivity of oscillation). Consider a bounded function \(f\) on an interval \(I\). Then \(\omega_x\) is a non-decreasing non-negative function. For \(a + b \leq |I|\)

\[
\omega_x(a + b) \leq \omega_x(a) + \omega_x(b) \tag{1}
\]

For a real \(\lambda \geq 1\)

\[
\omega_x(\lambda a) \leq \lambda \omega_x(a) \tag{2}
\]

\[
\omega_x(a + \lambda b) \leq \omega_x(a) + \lambda \omega_x(b) \tag{3}
\]
For a real $\lambda$, such that $0 \leq \lambda \leq 1$,

\[
\omega_x(\lambda a) \geq \lambda \omega_x(a) \tag{4}
\]

\[
\omega_x(a + \lambda b) \geq \omega_x(a) + \lambda \omega_x(b) \tag{5}
\]

Moreover, $\omega_x$ is concave:

\[
\omega_x(\lambda a + (1 - \lambda) b) \geq \lambda \omega_x(a) + (1 - \lambda) \omega_x(b) \tag{6}
\]

Proof. Consider the interval $I = [x, x + \epsilon]$. Trivially,

\[
\text{osc}_x^+[f](x) = \sup_{u,v \in I, u \neq v} |f(u) - f(v)| \geq 0 \tag{7}
\]

Therefore, without loss of generality we can set $\inf_{x \in I} f(x) = 0$ and consider only

the properties of the supremum. Since the supremum function is non-decreasing so

is $\text{osc}_x^+[f](x)$ for the argument $\epsilon$. Let $f$ attain an maximum at $x + \epsilon_2$. Then for $\epsilon \geq \epsilon_2$

$sup\epsilon_2 f(x) = sup\epsilon f(x)$, which we add to the inequality $sup\epsilon f(x) \leq sup\epsilon f(x)$

so that

\[
sup f(x) + sup f(x) \leq 2 sup f(x) \tag{8}
\]

Then

\[
0 \leq sup f(x) + sup f(x) - sup f(x) \leq sup f(x) \tag{9}
\]

since either $sup f(x) = sup\epsilon f(x)$ or $sup f(x) = sup\epsilon f(x)$:

\[
\sup f(x) \leq sup f(x) + sup f(x) \tag{10}
\]

so that under the above hypothesis

\[
\text{osc}_x^+[f](x) \leq \text{osc}_x^+[f](x) + \text{osc}_x^-[f](x) \tag{11}
\]

Therefore, under change of notation $\epsilon_1 = \epsilon - \epsilon_2$ the inequality transforms as

\[
\omega_x(\epsilon_2 + \epsilon_1) \leq \omega_x(\epsilon_2) + \omega_x(\epsilon_1) \tag{12}
\]

In particular, observe that $\omega_x(0) = \text{osc}_x^+[f](x) \geq 0$.

The second part of the claim relies on implicit type conversion. Consider the

integer $k \geq 1$. Then for some real $a : \omega_x(ka) \leq k \omega_x(a)$. Suppose that $a = b/k$ for

some $b$. Then $\omega_x(b)/k \leq \omega_x(b/k)$ so that combining for a rational $q = p/k \geq 1 :$

$\omega_x(qa) \leq q \omega_x(a)$. Let $r = 1/q$ and $b = a/r$ then $\omega_x(qa) \leq q \omega_x(a) \rightarrow r \omega_x(r) \leq$

$\omega_x(b/r)$, $r \leq 1$. Since $a$ is arbitrary then there the inequality is valid for any $b > 0$.

Letting $a = \lambda/q$, $\lambda \in \mathbb{R}$ it follows that $\omega_x(\lambda) \leq \lambda/a \omega_x(a)$ for $\lambda \geq 1$. Since

now both variables are real the entire domain is real. For the third inequality we

combine the first two inequalities:

\[
\omega_x(a + \lambda b) \leq \omega_x(a) + \omega_x(\mu b) \leq \omega_x(a) + \mu \omega_x(b) \tag{13}
\]

The concavity follows as: Let $\lambda = a/(a + b) \leq 1$ and the opposite be true.

\[
\omega_x(a) + \omega_x(b) = \omega_x \left( \frac{a}{a + b} (a + b) \right) + \omega_x \left( \frac{b}{a + b} (a + b) \right) =
\]

\[
\omega_x (\lambda(a + b)) + \omega_x ((1 - \lambda)(a + b)) \tag{14}
\]

So that

\[
\omega_x (\lambda(a + b)) + \omega_x ((1 - \lambda)(a + b)) \leq \lambda \omega_x (a + b) + (1 - \lambda) \omega_x (a + b) = \omega_x (a + b) \tag{15}
\]

Therefore, $\omega_x(a) + \omega_x(b) \leq \omega_x(a + b)$, which is a contradiction. Therefore, $\omega_x$ is

concave. The fifth inequality follows from concavity. □
Corollary 2. Under the same hypotheses if $f$ is either increasing or decreasing in $I = [x, x + h]$ then $\omega_x(h) = |f(x + h) - f(x)| = |\Delta_h f(x)|$.

Corollary 3. Under the same hypotheses $\omega_x \cong BV[I]$.

Proof. Consider an increasing collection of intervals $U(n) = \{[x, x + a_k]\}_{k=1}^n$ such that $a_1 < \cdots < a_n$. Then these form a partition $P[x, x + a_k]$ over $I = \bigcup_{k=1}^n [x, x + a_k]$. Then $\text{Var}[\omega_x, I] = \omega_x(a_n) - \omega_x(a_1)$, which is bounded.

The sub-additivity is an interval property, which imposes a large extent of regularity on the oscillation function.

Theorem 3 (Continuity of oscillation). Consider a bounded function $f$ defined on an interval $I$ with length $h = |I|$. The continuity set of the oscillation $\omega_x$ can be written as

$$C_\omega[h] = \bigcup_{k=1}^{\infty} (a_k, b_k), \quad b_k \leq a_{k+1}$$

The discontinuity set $\Delta_\omega[h]$ is a null set.

Proof. Let $h = |I| = [x, x + h]|$ and denote $J_h = (0, h]$ then $\exists q = Q \cap J_h$. Therefore, there is a map $J_h \mapsto q$. Since $\omega_x(h)$ is non-decreasing it has only jump discontinuities. Indeed by LUB and GLB

$$\left| \sup_{\epsilon} \omega_x(h + \epsilon) - L_1 \right| \leq \mu/2, \quad \epsilon \::\: \mu$$
$$\left| \inf_{\epsilon} \omega_x(h + \epsilon) - L_2 \right| \leq \mu/2$$
$$\left| \sup_{\epsilon} \omega_x(h + \epsilon) - \inf_{\epsilon} \omega_x(h + \epsilon) - L_1 + L_2 \right| \leq \mu$$
$$\left| \omega_x(h + \epsilon) - \omega_x(h) - L_1 + L_2 \right| \leq \mu \rightarrow \left| \omega_x(h + \epsilon) - \omega_x(h + 0) - L_1 + L_2 \right| \leq \mu$$

Therefore, if $h \in C_\omega$

$$|A| = |\omega_x + h(\epsilon) - \omega_x + h(0)| \leq \nu, \quad \epsilon :: \nu$$
$$|B| = |\omega_x + h(\epsilon) - \omega_x + h(0) - L_1 + L_2| \leq \mu$$
$$|A + B| = |L_1 - L_2| \leq |A| + |B| \leq \mu + \nu$$

Therefore, in limit $|L_1 - L_2| = 0$.

Conversely, if $L_1 \neq L_2$

$$\omega_x(h) \geq \omega_x(h(\epsilon)) - \omega_x(h(0)) \geq L_1 - L_2 > 0$$

and there is a jump discontinuity at $h$. In such a case, $\exists p \in Q \cap [L_1, L_2]$ so that $[L_1, L_2] = L_p$ for a uniquely chosen $p$. Since $\omega_x$ is non-decreasing then for $p \neq p' \rightarrow L_p \cap L_p' = \emptyset$. Therefore, there is an isomorphism $p \leftrightarrow h$. Therefore, the set of continuity of $\omega_x$ is $J_h \{h\}$ and it is open and countable. Hence the claim follows.

Proposition 1. Consider a function $f \cong C[I]$, where $|I| = h$. Then $\omega_x(h)$ is continuous in its domain and $\omega_x(0) = 0$.

Proof. By sub-additivity

$$\omega_x(\mu b) \geq \omega_x(a + \mu b) - \omega_x(a) \geq \mu \omega_x(b)$$
Then taking the limit in $\mu$ establishes
\[
\omega_x(0^+) \geq \omega_x(a^+) - \omega_x(a) \geq 0
\]
Then under the hypothesis of the corollary, by Lemma 1 $\omega_x(0) = 0$. Therefore, $\omega_x(a^+) = \omega_x(a)$ and the claim follows.

Under the hypothesis of Prop. 1 the oscillation function will be analytically continued on the entire real line as $\omega_x(h) := 0$ for $h < 0$.

Having established these properties, we will characterize the discontinuities of a function using the definition:

**Definition 8.** Define the set of discontinuity for the function $F$ in the compact interval $I$ as
\[
\Delta[F, I] := \{x : \text{osc}[F](x) > 0, x \in I\}
\]
or if the context is known $\Delta[F, I] \equiv \Delta[I]$. In particular, under this definition $\text{osc}[F](x) = \infty$ is admissible.

The Darboux–Froda’s theorem states that the set of discontinuities of a monotone function is at most countable. Hence, by Th. 1 it is also totally disconnected. In fact, the latter inference can be strengthened to arbitrary functions.

**Theorem 4** (Disconnected discontinuity set). Consider a function $F$ defined on closed interval $I$. Then its set of discontinuity $\Delta[F, I]$ is totally disconnected in $I$.

**Proof.** Consider a decreasing collection of closed nested intervals $\{I_k = [x, x + a_k]\}_{k=1}^n, I_{k+1} \subset I_k \subset \ldots \subset I_1 = [x + h]$ for some $h$.

Since the case when the function is locally constant is trivial we consider only two cases: increasing and decreasing.

Let $E_n = I_n \cap I_{n-1}$ and set
\[
\Delta_n = \sup_{I_n} F - \inf_{I_n} F
\]
\[
\Delta_{n-1} = \sup_{I_{n-1}} F - \inf_{I_{n-1}} F
\]
\[
\Delta_E = \sup_{E_n} F - \inf_{E_n} F
\]

Notations are indicated in the diagram (Fig. 3.1) below:

**Increasing case:** Suppose that $F$ is increasing in $I_{n-1}$ then
\[
\Delta_{n-1} - \Delta_n = \sup_{I_{n-1}} F - \inf_{I_{n-1}} F - \sup_{I_n} F + \inf_{I_n} F = \\
\sup_{E_n} F - \inf_{I_{n-1}} F - \inf_{I_n} F + \inf_{I_{n-1}} F = \Delta_E = \omega_x(|E_n|)
\]

**Decreasing case:** Suppose that $F$ is decreasing in $I_{n-1}$ then
\[
\Delta_{n-1} - \Delta_n = \sup_{I_n} F - \inf_{E_n} F - \sup_{E_n} F + \inf_{E_n} F = \Delta_E = \omega_x(|E_n|)
\]

On the other hand, $\omega_x(|I_n|) = \Delta_n$ and $\omega_x(|I_{n-1}|) = \Delta_{n-1}$. Therefore, by Th. 3 we take such $h$ that
\[
\lim_{n \to \infty} \omega_x(|I_n|) - \omega_x(|I_{n-1}|) = 0 \Rightarrow \lim_{|E_n| \to 0} \omega_x(|E_n|) = 0
\]
By the Nested Interval Theorem \( \{x\} = \bigcap_{k=1}^{\infty} I_k \). Therefore, if \( \Delta[F, I] \neq \emptyset \) then it is totally disconnected.

\[\square\]

4. MODULI OF CONTINUITY

The moduli of continuity are second-order properties of the preimage functions.

**Definition 9 (Modulus of continuity).** A modulus of continuity \( g_x : \mathbb{R} \mapsto \mathbb{R} \) is a

1. non-decreasing continuous function, such that
2. \( g_x(0) = 0 \) and
3. \( |\Delta^\pm f(x)| \leq K g_x(\epsilon) \) holds in the interval \( I = [x, x \pm \epsilon] \) for some constant \( K \).

\( I \) is assumed to be in the domain of the function \( f \). In addition, a regular modulus is such that \( g_x(1) = 1 \).

Under this definition every continuous function admits a modulus of continuity:

**Theorem 5 (Modulus characterization theorem).** Every continuous function admits a modulus of continuity on an interval \( I \), which is a subset of its domain. Any modulus of continuity is \( BVC[I] \).

**Proof.** Consider the oscillation function \( \omega_x(\epsilon) \). Then \( \omega_x \) is non-decreasing. Trivially, \( |\Delta^\pm f(x)| \leq \omega_x(\epsilon) \) holds. Finally, \( \omega_x \) is continuous and \( \omega_x(0) = \text{osc}_+ f(x) = 0 \) by Prop. [\[1\]]. Then \( g_x(\epsilon) = \omega_x(\epsilon)/\omega_x(1) \). \[\square\]

The oscillation function used is the proof of Th. [\[5\]] will be called a canonical modulus of continuity of a continuous function.

4.1. Classification of the moduli of continuity.

**Proposition 2.** Suppose that \( \omega_x \) is strictly sub-additive. Then \( \omega'_x(0) = \infty \). Suppose that \( \omega_x \) is additive. Then \( \omega_x \) is linear and homogeneous.
Proof.  **Strictly sub-additive case:** Suppose that the derivative exists finitely and let \( M > \omega_x'(0) \geq 0 \). By sub-additivity there is \( h \), such that
\[
2 \omega_x(h/2) > \omega_x(h) \Rightarrow M > \frac{2}{h} \omega_x \left( \frac{h}{2} \right) > \frac{1}{h} \omega_x(h) \geq m
\]
Then by induction:
\[
M > \frac{2^n}{h} \omega_x \left( \frac{h}{2^n} \right) \geq m \Rightarrow M > \frac{1}{h} \omega_x \left( \frac{h}{2^n} \right) \geq m
\]
Taking the limit in \( n \to \infty \) leads to
\[
0 > \omega_x'(0) > 0
\]
which is a contradiction. Therefore, the limit does not exist finitely and \( \omega_x'(0) = \infty \).

**Additive-case:** By additivity, for all integer \( k \):
\[
\omega_x(kh) = k \omega_x(h)
\]
Then by change of variables \( z = kh \). \( \omega_x(z) = k \omega_x(z/h) \). Therefore, \( \omega_x(qh) = q \omega_x(h) \) for all rational \( q \). Then by continuity, \( \omega_x(h) = Kh \) for some \( K > 0 \).

Based on this result it is useful to apply the following definition.

**Definition 10 (g-continuous class).** Define the growth class \( C^g[I] \) induced by the modulus of continuity \( g(|I|) \) by the conditions: If \( f \equiv C^g[I] \) on the compact interval \( I \) then
\[
(1) \quad C_x = \lim_{\epsilon \to 0} \frac{\omega_x(\epsilon)}{g(\epsilon)} \text{ exists finitely and}
(2) \quad |\Delta^+ f(x)| \leq C_x g(\epsilon).
\]
for \( \epsilon = |I| \). To emphasize the dependence on \( x \) we may write \( C^g_x \) and skip \( I \) if it is known.

This definition encompasses the definitions of Hölder and Lipschitz functions. So that
\[
L \equiv \mathbb{H}^1 \equiv C^g, \quad g(x) = x
\]
or
\[
\mathbb{H}^\alpha \equiv C^g, \quad g(x) = x^\alpha, \quad 0 < \alpha < 1
\]
By Prop. 2 we can classify modular functions into two distinct types

**Linear:** for which \( \lim_{\epsilon \to 0} \frac{\omega_x(\epsilon)}{\epsilon} < L \) for some \( L \). Therefore, \( \omega'_x(0) < L \) by L'Hôpital's rule and this function is linear by Prop[2]

**Singular:** (or strongly non-linear) for which the ratio \( \omega_x(\epsilon)/\epsilon \) diverges and \( \omega_x'(0) = \infty \) by L'Hôpital's rule.

5. **Generalized maximal \( \omega \) derivatives**

**Definition 11.** For a function \( f \) define superior and inferior, and respectively forward and backward, maximal \( \omega \) modular derivatives, as the limit numbers
\[
\left. \sup_{\epsilon} \frac{\Delta^+ f(x)}{\omega_x(\epsilon)} - L \right| < \mu \Rightarrow D^+_\omega f(x) = L
\]
\[
\left. \inf_{\epsilon} \frac{\Delta^+ f(x)}{\omega_x(\epsilon)} - L \right| < \mu \Rightarrow D^-_\omega f(x) = L
\]
for all \( \epsilon :: \mu, \epsilon > 0 \).
Remark 2. These derivative functions obviously generalize the concept of Dini derivatives, given below for convenience: Define the Dini derivatives as the functions

\[ D^\pm f(x) = \limsup_{\epsilon \to 0} \frac{\Delta^\pm \epsilon [f](x)}{\epsilon} \]

\[ D^\pm f(x) = \liminf_{\epsilon \to 0} \frac{\Delta^\pm \epsilon [f](x)}{\epsilon} \]

For the function \( f \).

Equipped with the above definition we can state the first existence result:

**Theorem 6** (Bounded \( \omega \)-derivatives). For a continuous function the four derivative functions exist as real numbers. Moreover, if \( f \) is non-decreasing about \( x^+ \)

\[ D^\pm_{\omega} f(x) = 1 \]

\[ 0 \leq D^\pm_{\omega} f(x) \leq 1 \]

while if \( f \) is non-increasing about \( x^- \)

\[ D^\pm_{\omega} f(x) = -1 \]

\[ 0 \geq D^\pm_{\omega} f(x) \geq -1 \]

**Proof.** Let \( I = [x, x \pm \epsilon] \) be given and \( x \) is fixed but we can vary \( \epsilon \). Consider the auxiliary function

\[ \nu^\pm_{\omega} [f](x) := \frac{\Delta^\pm \epsilon [f](x)}{\omega_x(\epsilon)} \]

The supremum definitions are restatements of the LUB property for \( \nu^\pm_{\omega} [f](x) \) in terms of the variable \( \epsilon \), while the infimum derivatives are restatements with the GLB property of the reals again for the same variable. Therefore, all four numbers exist for a given argument \( x \) and, therefore, under the above hypothesis. Moreover, since \( |\Delta^\pm [f](x)| \leq \omega_x(\epsilon) \) then for an non-decreasing function \( |D^\pm_{\omega} f(x)| \leq 1 \). Therefore, by the supremum property \( \sup_{\epsilon} \frac{\Delta^\pm [f](x)}{\omega_x(\epsilon)} = 1 \). For a decreasing function \( f \) it is sufficient to consider \(-f\) and apply the same arguments. \( \square \)

**Corollary 4.** Suppose that \( f \) is monotone and continuous. Then if it is increasing \( D^\pm_{\omega} f(x) = D^\pm_{\omega} f(x) = 1 \). If it is decreasing \( D^\pm_{\omega} f(x) = D^\pm_{\omega} f(x) = -1 \).

**Proof.** Fix \( x \) and consider \( I = [x, \pm \epsilon] \). The proof follows from the fact that in both cases \( |\Delta^\pm [f](x)| = \omega_x(\epsilon) \). \( \square \)

We can give generalized definition of local differentiability (called \( \omega \)-differentiability) as follows

**Definition 12.** A function \( f \) is \( \omega \)-differentiable at \( x \) if at least one of the two limits exist

\[ \left| \frac{\Delta^\pm [f](x)}{\omega_x(\epsilon)} - L \right| < \mu \implies D^\pm_{\omega} f(x) = L \]

where the conventions for \( L, \mu \) and \( \epsilon \) are as above. Moreover,

\[ D^\pm_{\omega} f(x) \neq D^\pm_{\omega} f(x) \]

is admissible.
The definition only says that the one-sided limits of the increments, that is \( \mathcal{D}^+ \omega f(x) \) (respectively \( \mathcal{D}^- \omega f(x) \)) exist as real numbers. This is the minimal statement that can be given for the limit of an increment of a function. Nevertheless, based on two strong properties – monotonicity and continuity – it can be claimed that

**Proposition 3** (Monotone \( \omega \)-differentiation). If a function \( f \) is monotone and continuous in a closed interval \( I \) then it is continuously \( \omega \)-differentiable everywhere in the opening \( I^\circ \).

*Proof.* The continuity follows directly from Corr. 4 while the restriction comes from the fact that at the boundary only one of the increments can be defined without further hypothesis for the values of \( f \) outside of \( I \). \qed

**Proposition 4** (BVC \( \omega \)-differentiation). If the function \( f \) is BVC \([I]\) in a closed interval \( I \) then it is \( \omega \)-differentiable everywhere in the opening \( I^\circ \).

*Proof.* The proof follows from the Jordan theorem, since a BV\([I]\) function can be decomposed into a difference of two non-decreasing functions. On the other hand, without further hypotheses we can not claim anything about eventual equality of \( \mathcal{D}^+ \omega f(x) \) and \( \mathcal{D}^- \omega f(x) \) since

\[
J_x = [x - \epsilon, x] \cap [x, x + \epsilon] = \{x\}
\]

so that we can form only the trivial map \( x \mapsto \{x\} \), which without further restrictions of the domain of \( x \) (i.e. topological obstructions) is uncountable. \qed

We can further utilize the concept of oscillation to give a concise general differentiability condition

\[
\lim_{\epsilon \to 0} \text{osc}_\omega \frac{\Delta^\pm [f](x)}{\omega_x(\epsilon)} = 0
\]  \hspace{1cm} (8)

**Theorem 7** (Characterization of \( \omega \)-derivative). The following implications hold

\[
\mathcal{D}^\pm \omega f(x) = \mathcal{D}^\pm \omega f(x) = \mathcal{D}^\pm \omega f(x) \implies f = C[\pm]
\]

\[
\lim_{\epsilon \to 0} \text{osc}_\omega \frac{\Delta^\pm [f](x)}{\omega_x(\epsilon)} = 0 \iff \mathcal{D}^\pm \omega f(x) = \mathcal{D}^\pm \omega f(x) = \mathcal{D}^\pm \omega f(x)
\]

so that if Eq. 8 holds at \( x \) then \( f \) is \( \omega \)-differentiable (and hence continuous) at \( x \).

*Proof.* Continuity implication: Consider the inequality

\[
\mathcal{D}^\pm \omega f(x) = \mathcal{D}^\pm \omega f(x) \implies \left| \frac{\Delta^\pm [f](x)}{\omega_x(\epsilon)} - L \right| \leq \mu/2, \ \epsilon : \mu
\]

so that

\[
L - \mu/2 \leq \frac{\Delta^\pm [f](x)}{\omega_x(\epsilon)} \leq L + \mu/2 \implies \sup_{\epsilon} \frac{\Delta^\pm [f](x)}{\omega_x(\epsilon)} \leq (L + \mu/2) \omega_x(\epsilon)
\]

\[
(L - \mu/2) \omega_x(\epsilon) \leq \inf_{\epsilon} \Delta^\pm [f](x) \implies
\]

\[
\left| \frac{\sup_{\epsilon} \Delta^\pm [f](x)}{\omega_x(\epsilon)} - L \right| \leq \mu/2
\]

\[
\left| \frac{\inf_{\epsilon} \Delta^\pm [f](x)}{\omega_x(\epsilon)} - L \right| \leq \mu/2
\]
Let \( \sup_{\epsilon} \Delta_{\epsilon}^+[f](x) = M \) and \( \inf_{\epsilon} \Delta_{\epsilon}^+[f](x) = m \). Then by triangle inequality

\[
\frac{\sup_{\epsilon} \Delta_{\epsilon}^+[f](x)}{\omega_x(\epsilon)} - \frac{\inf_{\epsilon} \Delta_{\epsilon}^+[f](x)}{\omega_x(\epsilon)} = \frac{M - m}{\omega_x(\epsilon)} \leq \mu
\]

\[
M - m \leq \mu \omega_x(\epsilon)
\]

Therefore, in limit \( M - m \leq 0 \), hence \( M = m \) and \( f \) is continuous. This sequence of operations reminds the fact that real numbers are constructed by a limiting process.

**Forward statement:** Suppose that \( \bar{D}_{\omega} f(x) = L_1 \) and \( \bar{D}_{\omega} f(x) = L_2 \). Then by LUB

\[
\left| \sup_{\epsilon} \frac{\Delta_{\epsilon}^+[f](x)}{\omega_x(\epsilon)} - L_1 \right| \leq \mu/2
\]

\[
\left| \inf_{\epsilon} \frac{\Delta_{\epsilon}^+[f](x)}{\omega_x(\epsilon)} - L_2 \right| \leq \mu/2
\]

so that

\[
\left| \sup_{\epsilon} \frac{\Delta_{\epsilon}^+[f](x)}{\omega_x(\epsilon)} - L_1 \right| + \left| \inf_{\epsilon} \frac{\Delta_{\epsilon}^+[f](x)}{\omega_x(\epsilon)} - L_2 \right| \leq \mu
\]

Then by the triangle inequality

\[
\left| \sup_{\epsilon} \frac{\Delta_{\epsilon}^+[f](x)}{\omega_x(\epsilon)} - \inf_{\epsilon} \frac{\Delta_{\epsilon}^+[f](x)}{\omega_x(\epsilon)} \right| + L_1 - L_2 \]

\[
\leq \mu
\]

Then in limit by Lemma. [1]

\[
|L_1 - L_2| \leq 0 \implies L_1 = L_2
\]

Further, starting from

\[
\inf_{\epsilon} \frac{\Delta_{\epsilon}^+[f](x)}{\omega_x(\epsilon)} \leq \frac{\Delta_{\epsilon}^+[f](x)}{\omega_x(\epsilon)} \leq \sup_{\epsilon} \frac{\Delta_{\epsilon}^+[f](x)}{\omega_x(\epsilon)} \implies
\]

\[
0 \leq \frac{\Delta_{\epsilon}^+[f](x)}{\omega_x(\epsilon)} - \inf_{\epsilon} \frac{\Delta_{\epsilon}^+[f](x)}{\omega_x(\epsilon)} \leq \sup_{\epsilon} \frac{\Delta_{\epsilon}^+[f](x)}{\omega_x(\epsilon)} - \inf_{\epsilon} \frac{\Delta_{\epsilon}^+[f](x)}{\omega_x(\epsilon)} = \text{osc} \frac{\Delta_{\epsilon}^+[f](x)}{\omega_x(\epsilon)}
\]

Therefore,

\[
\frac{\Delta_{\epsilon}^+[f](x)}{\omega_x(\epsilon)} - \inf_{\epsilon} \frac{\Delta_{\epsilon}^+[f](x)}{\omega_x(\epsilon)} \leq \mu \implies \left| \sup_{\epsilon} \frac{\Delta_{\epsilon}^+[f](x)}{\omega_x(\epsilon)} - \Delta_{\epsilon}^+[f](x) \right| \leq \mu
\]

Therefore, all three limits coincide.

**Converse statement:** Suppose that

\[
\bar{D}_{\omega} f(x) = \bar{D}_{\omega} f(x) = L > 0
\]
By hypothesis

\[
L - \inf_{\epsilon} \frac{\Delta_{x}^{\pm} [f](x)}{\omega_{x}(\epsilon)} \leq \frac{\mu}{2}
\]

\[
\sup_{\epsilon} \frac{\Delta_{x}^{\pm} [f](x)}{\omega_{x}(\epsilon)} - L \leq \frac{\mu}{2}
\]

\[
\sup_{\epsilon} \frac{\Delta_{x}^{\pm} [f](x)}{\omega_{x}(\epsilon)} - \inf_{\epsilon} \frac{\Delta_{x}^{\pm} [f](x)}{\omega_{x}(\epsilon)} \leq |A_{x}| + |B_{x}| \leq \mu
\]

osc_{x} \frac{\Delta_{x}^{\pm} [f](x)}{\omega_{x}(\epsilon)}

Therefore, in limit

\[
0 \leq \text{osc}_{x} \frac{\Delta_{x}^{\pm} [f](x)}{\omega_{x}(\epsilon)} \leq 0
\]

so that \( \lim_{\epsilon \to 0} \text{osc}_{x} \frac{\Delta_{x}^{\pm} [f](x)}{\omega_{x}(\epsilon)} = 0 \).

\[\square\]

**Corollary 5 (Range of \( D_{x}^{\pm} \)).** The range of \( D_{x}^{\pm} \) is given by

\[
D_{x}^{\pm} f(x) = \{-1, 0, +1\}
\]

**Proof.** Let \( I = [x, x + \epsilon] \) be given. If \( f \) is constant in \( I \) trivially \( D_{x}^{\pm} f(x) = 0 \). If \( f \) is increasing in \( I \) then \( D_{x}^{\pm} f(x) = 1 \) and by duality if \( f \) is decreasing in \( I \) then \( D_{x}^{\pm} f(x) = -1 \).

\[\square\]

**Theorem 8 (Non-differentiability set).** Consider the bounded function \( f \equiv C[I] \). Then the sets

\[
\Delta_{x}^{\pm} [I] := \{ x : D_{x}^{\pm} f(x) > D_{x}^{\pm} f(x) \} \cap I
\]

are null sets.

**Proof.** Consider the case where the right \( \omega \)-derivative does not exist. That is the defining quotient oscillates without a limit. Then for \( 0 < u, v \leq \delta \)

\[
\left| \frac{\Delta_{u}^{\pm} [f](x)}{\omega_{x}(u)} - \frac{\Delta_{v}^{\pm} [f](x)}{\omega_{x}(v)} \right| > \mu
\]

(D1) for some \( \mu > 0 \). We can consider a variable \( \xi \in [x, x + u] \cap [x, x + v] = [x, x + \min(u, v)] = J \). There is a rational \( r = Q \cap J \). Associate \( (r, J) \equiv J_{r} \) so that \( J_{r} \) can be counted by an enumeration of the rationals and index \( \delta :: r \). Therefore, the set

\[
\Delta_{\omega} := \bigcup_{k=1}^{\infty} \{ z : [D1] \equiv \text{true}, z \in J_{k} \}
\]

is countable \( \forall \delta > 0 \). Since \( \Delta_{\omega} \) is totally disconnected by Th. 4 we can select \( \delta_{k} = \delta/2^{k} \) and \( J_{k} \subset J_{r} \). Therefore,

\[
|\Delta_{\omega}| = \sum_{k=1}^{\infty} |J_{k}| \leq \sum_{k=1}^{\infty} \frac{\delta}{2^{k}} = \delta
\]
and $\Delta_{\omega}$ is a null set. The left derivative case holds by duality.

This is the best possible result for the local -type of derivatives.

5.1. The Lebesgue monotone differentiation theorem. In the following we re-state the classical result of the Lebesgue differentiation theorem. The proof is given using the machinery of $\omega$-differentiation.

NB! In the following argument I reserve the term "monotone function" to mean only a strictly increasing or strictly decreasing function in an interval.

**Theorem 9** (Lebesgue monotone differentiation theorem). *Suppose that $f$ is monotone and continuous in the compact interval $I$. Then $f$ is continuously differentiable almost everywhere. The set $\Delta f[I] := \{ x : f'_+(x) \neq f'_-(x) \} \cap I$ is a null set.*

**Proof.** Let $D_{\omega}f(x) = L > 0$. By Corr. 4 for $\epsilon :: \mu$

\[
\left| \frac{L - \Delta^+_x[f](x) \omega_x(\epsilon)^+}{\omega_x(\epsilon)^+} \right| \leq \frac{\mu}{2}
\]

\[
\left| \frac{\Delta^-_x[f](x) \omega_x(\epsilon)^-}{\omega_x(\epsilon)^-} \right| \leq \frac{\mu}{2}
\]

\[
\left| \frac{\Delta^+_x[f](x) \omega_x(\epsilon)^+ - \Delta^-_x[f](x) \omega_x(\epsilon)^-}{\epsilon} \right| \leq |A| + |B| \leq \mu
\]

\[
\left| \frac{\omega_x(\epsilon)^+ - \omega_x(\epsilon)^-}{\epsilon} \right| \leq \mu
\]

Therefore, by monotonicity in the original notation

\[
\left| \frac{\Delta^+_x[f](x)}{\epsilon} - \frac{\Delta^-_x[f](x)}{\epsilon} \right| = \left| \frac{\Delta^2_x[f](x)}{\epsilon} \right| \leq \mu
\]

hence $f'_+(x) = f'_-(x)$ and $\Delta f[I] = \emptyset$.

Recall the definitions of nowhere monotone functions:

**Definition 13.** A function $f$ is non-decreasing (non-increasing) on $I = [a, b]$ if given any $a < x < y < b$

\[ f(y) - f(x) \geq 0 \quad (f(y) - f(x) \leq 0) \]

A function, which is neither non-decreasing nor non-increasing changes direction of growth in $I$. A function is nowhere monotone (NM[I]) if given any $a < x < y < z < b$

\[ (f(y) - f(x))(f(z) - f(y)) \leq 0 \]

so that NM[I] function is neither non-decreasing nor non-increasing on any sub-interval of $I$. A function, which is nowhere monotone at a point (NM[y]), is treated as above while $y$ is fixed.
From the Lebesgue monotone differentiation theorem it follows that a nowhere differentiable function on an open interval $I$ is simultaneously nowhere monotone on $I$. Brown et al. establish that no continuous function of bounded variation (CBV) is $\text{MN}[y]$. [4][Th. 12. Corr. 3]. That is to say $\text{NM}[x]$ for $x \in I$ as above. Therefore, it is of interest to establish the following result.

**Theorem 10** (NM continuous $\omega$-differentiability). Suppose that $f \equiv \mathcal{C}[I]$ and $f \equiv \text{NM}[I]$. Then

$$D^\pm_\omega f(x) \equiv \mathcal{C}[I] \implies D^\pm_\omega f(x) = 0$$

**Proof.** The set $\{x : D^+_\omega f(x) = 1\}$ is totally disconnected. By duality, the set $\{x : D^-_\omega f(x) = -1\}$ is totally disconnected. Hence only $\{x : D^\pm_\omega f(x) = 0\}$ has connected components. \hfill $\square$

### 6. Modular derivatives

We recall here for convenience the definition of Baire functions:

**Definition 14.** Let $X$ be a metric space. A set $E \subseteq X$ is of first category if it can be written as a countable union of nowhere dense sets, and is of second category if $E$ is not of first category.

For example $\mathbb{Q}$ and $\emptyset$ are I category, while the class of continuous functions is of category 0.

**Definition 15.** The function $f : \mathbb{R} \mapsto \mathbb{R}$ is called Baire-class I if there is a sequence of continuous functions converging to $f$ point-wise.

For convenience the reader is recalled with the definitions of $G_\delta$ and $F_\sigma$ meager sets:

**Definition 16.** Let $X$ be a metric space.

- The set $E \subseteq X$ is $G_\delta$ if it is countable intersection of open sets, and it is $F_\sigma$ if it is countable union of closed sets.
- The set $E \subseteq X$ is meager if it can be expressed as the union of countably many nowhere dense subsets of $X$.
- Dually, a co-meager set is one whose complement is meager, or equivalently, the intersection of countably many sets with dense interiors.

Derivatives can be generalized in several directions. The most natural way is to replace the assumption of local linear growth with less restricted modular-bound growth. In such way one can generalize, for example, the fractional velocity of Cherbit [5]. Let us first use an auxiliary notation.

**Definition 17.** Define $g$-variation operators as

$$v^+_g[f](x) := \frac{\Delta^+_g[f](x)}{g(\epsilon)} = \frac{f(x + \epsilon) - f(x)}{g(\epsilon)} \quad (9)$$

$$v^-_g[f](x) := \frac{\Delta^-_g[f](x)}{g(\epsilon)} = \frac{f(x) - f(x - \epsilon)}{g(\epsilon)} \quad (10)$$

for a positive $\epsilon$ and a modular function $g$. 
Condition 1 (Modulus-bound growth condition). For given $x$ and a modular function $g$.

\[ \text{osc}^\pm \epsilon f(x) \leq C g(\epsilon) \] (C1)

for some $C \geq 0$ and $\epsilon > 0$.

Condition 2 (Quotient oscillation condition). For given $x$ and $\epsilon > 0$

\[ \text{osc}^\pm \nu_g^\pm [f](x) = 0 \] (C2)

where the limit is taken in $\epsilon$.

Define the modular derivative as:

Definition 18 (Modular derivative, g-derivative). Consider an interval $[x, x \pm \epsilon]$ and define

\[ D^\pm_g f(x) := \lim_{\epsilon \to 0} \frac{\Delta^\pm_l [f](x)}{g(\epsilon)} \] (11)

for a modulus of continuity $g(\epsilon)$. The last limit will be called modular derivative or a g-derivative.

NB! We do not demand equality of $D^+_g f(x)$ and $D^-_g f(x)$.

We are ready to establish the existence conditions of the g-derivative.

Theorem 11 (Conditions for existence of g-derivative). If $D^+_g f(x)$ exists (finitely), then $f$ is right-continuous at $x$ and $C_1$ holds, and the analogous result holds for $D^-_g f(x)$ and left-continuity.

Conversely, if $C_2$ holds then $D^\pm_g f(x)$ exists finitely. Moreover, $C_2$ implies $C_1$.

Proof. We will first prove the case for right continuity. Condition $C_1$ trivially implies the g-continuity, which according to our notation is given as $\nu_g^\pm [f](x) \leq C g(\epsilon)$.

Forward statement:

Without loss of generality suppose that $L > 0$ is the value of the limit. Then by hypothesis

\[ \left| \frac{\Delta^+_l [f](x)}{g(\epsilon)} - L \right| < \mu \]

holds for every $\mu : \delta, \epsilon < \delta$. Straightforward rearrangement gives

\[ |f(x + \epsilon) - f(x) - L g(\epsilon)| < \mu g(\epsilon) \cdot \]

Then by the reverse triangle inequality

\[ |f(x + \epsilon) - f(x)| - L g(\epsilon) \leq |f(x + \epsilon) - f(x) - L g(\epsilon)| < \mu g(\epsilon) \cdot \]

so that $|f(x + \epsilon) - f(x)| < (\mu + L) g(\epsilon)$. Further, by the least-upper-bound property there exists a number $C \leq \mu + L$, such that

\[ |f(x + \epsilon) - f(x)| \leq C g(\epsilon) \cdot \]

which is precisely the Modulus bound growth condition. The left continuity can be proven in the same way.

\[ ^2 \text{Alternatively, we can also assign a Cauchy sequence to } \delta \text{ and demand that RHS approaches arbitrary close to } 0 \text{ implying also } \text{osc}^+_f[f](x) = 0. \]
Converse statement:
In order to prove the converse statement we can observe that the condition $C_2$ implies that $\text{osc}_+ v_+ \frac{f(x)}{g(\epsilon)} = 0$ so that
$$\text{osc}_+ \frac{\Delta_+^+ f(x)}{g(\epsilon)} \leq \mu$$
for $\mu :: \epsilon$ (and in particular for a Cauchy null-sequence $\mu$) so that
$$\left| \sup_\epsilon \frac{\Delta_+^+ f(x)}{g(\epsilon)} - \inf_\epsilon \frac{\Delta_+^+ f(x)}{g(\epsilon)} \right| \leq \mu$$
by Lemma 1 and
$$\sup_\epsilon \frac{\Delta_+^+ f(x)}{g(\epsilon)} \leq \mu + \inf_\epsilon \frac{\Delta_+^+ f(x)}{g(\epsilon)}$$
so that taking the limits in $\mu$ (and hence $\epsilon$) implies
$$\limsup_{\epsilon \to 0} \frac{\Delta_+^+ f(x)}{g(\epsilon)} = \liminf_{\epsilon \to 0} \frac{\Delta_+^+ f(x)}{g(\epsilon)}$$
Hence $\lim_{\epsilon \to 0} v_+^+ \frac{f(x)}{g(\epsilon)} = L = D_+^+ f(x)$ for some real number $L$.

However, the latter limit can be rewritten from its definition as
$$\left| \frac{\Delta_+^+ f(x)}{g(\epsilon)} - Lg(\epsilon) \right| < \mu$$
for an arbitrary $\mu :: \epsilon$. Then since $\mu$ is arbitrary by the least upper bound property there is $\epsilon'$, such that
$$|\Delta_+^+ f(x)| = \text{osc}_+ [f(x)] \leq (\mu + L)g(\epsilon')$$
for $\mu :: \epsilon'$ and we identify condition $C_1$.

The left case follows by applying the right case, just proved, to the reflected function $f(-x)$.

6.1. Generalized Taylor-Lagrange property.

Proposition 5 (Taylor-Lagrange property). The existence of $D_+^+ f(x) \neq 0$ implies that
$$f(x \pm \epsilon) = f(x) \pm D_+^+ f(x) g(\epsilon) + o(g(\epsilon))$$
for the modular function $g$. While if
$$f(x \pm \epsilon) = f(x) \pm K g(\epsilon) + \gamma g(\epsilon)$$
uniformly in the interval $x \in [x, x + \epsilon]$ for some Cauchy sequence $\gamma_\epsilon = o(x)$ and $K \neq 0$ is constant in $\epsilon$ then $D_+^+ f(x) = K$.

Proof. We prove only for the forward modular derivative. The case for the backward modular derivative is proven in the same way following a reflection of the argument.

Forward statement: By the definition of fractional velocity $\exists \gamma$, such that
$$f(x + \epsilon) = f(x) + D_+^+ f(x) g(\epsilon) + \gamma.$$ Moreover, $\gamma = o(g(\epsilon))$. 
Converse statement: Suppose that
\[ f(x + \epsilon) = f(x) + Kg(\epsilon) + \gamma \epsilon g(\epsilon) , \]
uniformly in the interval \( x \in [x, x + \epsilon] \) for some number \( K \) and \( \gamma \epsilon = o_x \).
Then this fulfills both Hölder growth and vanishing oscillation conditions. Therefore, \( K = D^+_g f (x) \beta \) observing that \( \lim_{\epsilon \to 0} \gamma \epsilon = 0 \).

\[ \square \]

6.2. Characterization by \( \omega \)-derivatives.

Proposition 6. Consider the modular function \( g \). Then
\[ D^\pm_g f (x) = K D^\pm_g f (x) \]
for some constant \( K \) wherever all limits exist.

Proof. Let
\[ K = \lim_{\epsilon \to 0} \frac{\omega_x (\epsilon)}{g(\epsilon)} \]
and suppose that the limit is finite.
\[ \frac{\Delta^+_g [f] (x)}{g(\epsilon)} = \frac{\Delta^+_g [f] (x) \omega_x (\epsilon)}{\omega_x (\epsilon) g(\epsilon)} \]
Therefore, the statement of the result follows.

\[ \square \]
In view of Prop. 3 this means that a function can change its modulus of continuity point-wise. Since the cases of Hölder and Lipschitz functions have been treated extensively in literature we will consider only the general case.

6.3. Continuity of \( g \)-derivatives. Gleyzal [9] established that a function is Baire class I if and only if it is the limit of an interval function. Therefore, \( D^\pm_g f (x) \) are Baire class I from which it follows that \( D^\pm_g f (x) \) must be continuous on a dense set. Moreover, since the continuity set of a function is a \( G_\delta \) set, (i.e. an intersection of at most countably many open sets), from the Osgood-Baire Category Theorem it follows that the set of points of discontinuity of \( D^\pm_g f (x) \) is \( F_\sigma \) meager (i.e. a union of at most countably many nowhere dense sets or else it has empty interior).

Since in the previous sections it was established that the modulus of continuity can be conveniently classified as used conventionally in applied literature we are ready to state an important result concerning the continuity of \( g \)-derivatives. First, we have the following Theorem:

Theorem 12 (Continuity of \( g \)-derivatives). Suppose that \( g \) is a strictly sub-additive modular function on the compact interval \( I \). Then wherever \( D^\pm_g f (x) \) is continuous it is zero.

Proof. Let \( D^+_g f (x) = K > 0 \).
\[ \frac{\Delta^+_g [f] (x)}{g(\epsilon)} = \frac{f(x + \epsilon) - f(x + \epsilon/2)}{g(\epsilon)} + \frac{f(x) - f(x - \epsilon/2)}{g(\epsilon)} = \]
\[ = \frac{f(x + \epsilon) - f(x + \epsilon/2)}{g(\epsilon/2)} \frac{g(\epsilon/2)}{g(\epsilon)} + \frac{f(x) - f(x - \epsilon/2)}{g(\epsilon/2)} \frac{g(\epsilon/2)}{g(\epsilon)} \]
Therefore, in limit supremum and by hypothesis of continuity
\[ K = K \limsup_{\epsilon \to 0} \frac{2g(\epsilon/2)}{g(\epsilon)} \]

By strict sub-additivity \(2g(\epsilon/2)/g(\epsilon) < 1\) therefore, the limit \(G\) exists. So it is established that \(K = GK < K\), which is a contradiction. Therefore, \(K = 0\) on the first place. The case for the left derivative follows by duality. \(\square\)

**Corollary 6.** The continuity requirement is equivalent to requiring that
\[ \lim_{\epsilon \to 0} \frac{2g(\epsilon/2)}{g(\epsilon)} = 1 \]

**7. Continuity sets of derivatives**

**Theorem 13** (Continuity of derivatives). Consider a bounded and continuous function \(f\) on a compact interval \(I\). Suppose that \(f'_+(x)\) and \(f'_-(x)\) are separately continuous then the following holds:

1. \(f'_+(x) = f'_-(x) = f'(x)\)
2. \(\Delta_{f,I} := \{x : f' \notin C, x \in I\}\) is totally disconnected with empty interior.
3. The total discontinuity set can be written as \(\Delta_{f,I} = \Delta_{1,I} \cup \Delta_{2,I}\), where \(\Delta_{1,I}\) is \(F_\sigma\) and \(\Delta_{2,I}\) is a null set.
4. The continuity set is written as \(C_f = \bigcup_{k=1}^\infty (a_k, b_k), b_k \leq a_{k+1}\) and is thus \(G_\delta\).

**Proof.** Consider the interval \(I = [u, v]\). Then there is rational \(r \in \mathbb{Q} \cap I\).

Associate \((r, I) \equiv I_r\) so that \(I_r\) can be counted by an enumeration of the rationals.

Assume that \(f'_+(x)\) and \(f'_-(x)\) are separately continuous on the opening of \(I_r^c = I_r - \{u\} - \{v\}\). Fix \(x\), such that \(u \geq x > v\).

\[
\begin{align*}
\frac{f(u) - f(v)}{u - v} &= f(u) - f(x) + f(x) - f(v) \\
&= \frac{f(u) - f(x)}{u - x} \cdot \frac{u - x}{u - v} + \frac{f(x) - f(v)}{x - v} \cdot \frac{x - v}{u - v}
\end{align*}
\]

\[
\begin{align*}
&= \frac{f(u) - f(x)}{u - x} \cdot (1 - \lambda) + \frac{f(x) - f(v)}{x - v} \cdot \lambda \\
&\downarrow \lim_{u \to x} \quad \downarrow \lim_{v \to x}
\end{align*}
\]

\[
(1 - \lambda)f'_+(x) + \lambda f'_-(x) = f'_+(x) - \lambda (f'_+(x) - f'_-(x))
\]

By continuity
\[
\lim_{v \to x} f'_+(v) = f'_+(x) = f'_+(x) - \lambda (f'_+(x) - f'_-(x))
\]

However, since \(x\) and hence \(\lambda \neq 0\) is arbitrary \(f'_+(x) = f'_-(x)\) must hold \(\forall x \in I_r^c\).

Hence, \(f'\) is continuous on \(I_r^c\).

By this argument we establish that the set \(\Delta_{1,I} := \{x : f' \notin C\} \cap I\) is \(F_\sigma\), where we also assume that whenever \(f'(x)\) does not exist it is replaced by a value that makes \(f'\) discontinuous. By Th. 4 the discontinuity set is totally disconnected and with empty interior.
Let us further consider the case where left and right derivatives do not exist (either diverge or oscillate without a limit). It is enough to consider the right derivative. Then we have that for $0 < u, v \leq \delta$

$$\left| \frac{\Delta^+_u [f] (x)}{u} - \frac{\Delta^+_v [f] (x)}{v} \right| > \epsilon > 0$$

(D2)

for some $\epsilon$. We can consider a variable $\xi \in [x, x+u] \cap [x, x+v] = [x, x+\min(u, v)] = J$. There is a rational $r = \mathbb{Q} \cap J$. Associate $(r, J)$ so that $J_r$ can be counted by an enumeration of the rationals and index $\delta :: r$. Therefore, the set

$$\Delta_{2,f} := \bigcup_{k=1}^{\infty} \{ z : (D2) \iff true, z \in J_k \}$$

is countable $\forall \delta > 0$. Since it is totally disconnected by Th. 4 we can select $\delta_k = \delta/2^k$. Therefore,

$$|\Delta_{2,f}| = \sum_{k=1}^{\infty} |J_k| \leq \sum_{k=1}^{\infty} \frac{\delta}{2^k} = \delta$$

and $\Delta_{2,f}$ is a null set.

The same argument can be applied to the left derivative considering $f(x)$. The total discontinuity set can be written as

$$\Delta_{f,I} = \Delta_{1,f} \cup \Delta_{2,f}$$

Therefore, the continuity set can be written as

$$C_f = \Delta_c^f = \bigcup_{k=1}^{\infty} (a_k, b_k), \ a_k \leq b_{k+1}, \ b_k \leq b_{k+1},$$

□

**Theorem 14.** Consider a function $f$ having a strictly sub-additive modulus function $g$ on the compact interval $I$. Then the set

$$\chi_g^\pm (f) := \{ x : D^\pm_g f (x) \neq 0 \} \cap I$$

is totally disconnected and of measure zero, that is $|\chi_g^\pm (f)| = 0$. The set $\chi_g^\pm$ will be called the set of change of $f$.

**Proof.** Using the same argument as in the proof of Th. 12 we establish that either $K = 0$ allowing for continuity of $D^\pm_g f (x)$ or $K \neq 0$ but then $D^\pm_g f (x)$ can not be continuous. Furthermore, by Th. 13 it follows that $|\chi_g (f)| = 0$. □

**Corollary 7.** Under the same notation, let $g(\epsilon) = \epsilon^\beta$, for $\beta \in (0, 1]$. If $|\chi_g (f)| > 0$ then $\beta = 1$ and $f$ is Lipschitz.

**Corollary 8.** Under the same hypotheses the image set $D^\pm_g f$ is totally disconnected.

**Remark 3.** Suppose that $f \equiv BV[I]$. Then $\Delta_{2,f}$ defined as above is a null set. This is the most that can be said about the set of non-differentiability if differentiability is interpreted only as existence of a continuous derivative $f'(x)$.

The remark will be illustrated in the following example.
Example 2. Define the SVC function as the map between the SVC set and the dyadic rationals \( D \setminus \{1/2\} \) in the following construction. Let \( S_n \) be the sequence of the end-points of the interval in the \( n \)-th step in the construction of the SVC set. Let \( D_n \) be the sequence of dyadic rationals with denominator \( 2^n \) excluding \( 1/2 \).

Define the sequence of continuous piece-wise linear functions (see Fig. 2) \( F_n : [0,1] \to [0,1] \), such that

\[
F_n(S_n) = D_n
\]

and the limit \( F_C(x) := \lim_{n \to \infty} F_n(x) \). Then by construction, the set \( C = \{x : F'_C(x) = 0\} \) has measure 1/2. On the other hand, for \( u \in S_n \) and \( h_0 = 1 \)

\[
h_k = \frac{1}{2} \left( h_{k-1} - \frac{1}{4^k} \right)
\]

\[
q_k = 2^k h_k = 2^{k-1} h_{k-1} - \frac{1}{2^{k+1}} = q_{k-1} - \frac{1}{2^{k+1}}
\]

Then by induction

\[
q_n = 1 - \frac{1}{2} \sum_{k=1}^{n} \frac{1}{2^k}
\]

Therefore, in limit

\[
q^* = 1 - \frac{1}{2} \left( \frac{1}{1 - 1/2} - 1 \right) = \frac{1}{2}
\]

Then for the derivative

\[
\frac{\Delta_{h_n} F_n(u)}{h_n} = \frac{1}{q_n}
\]

Therefore, in limit

\[
\lim_{n \to \infty} \frac{\Delta_{h} F_n(u)}{h} = F'_C^r(u) = 2
\]

while \( F'_C^l(u) = 0 \). Therefore, \( |\Delta_{l,F_C}| = 1/2 \).

![Figure 1. Approximations of the SVC and Cantor’s functions](image)
The relaxation of the differentiability assumption opens new avenues in describing physical phenomena, for example, using stochastic calculus or the scale relativity theory developed by Nottale [13], which assumes fractality of geodesics in spacetime and hence of quantum-mechanical paths.

In contrast to usual fractional derivative, the geometrical, and hence physical, interpretation of modular derivative is easier to establish due to its local character and the demonstrated generalized Taylor-Lagrange property. That is, the $g$-derivative provides the best possible local non-linear approximation for the given modulus function.

From the perspective of approximation, derivatives can be viewed as mathematical idealizations of the linear growth. The linear growth, i.e., the Lipschitz condition has special properties, which make it preferred. The desirable properties of the derivatives, such as their continuity, are established from the more general setting of the moduli of continuity. Importantly, the statements of the Th. 12 and 14 give further insight on why linear ordinary derivatives are so useful for describing physical phenomena in terms of differential equations.

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