Non-parametric generalised newsvendor model

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Accepted: 28 November 2022 / Published online: 3 December 2022
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Abstract
In the present paper we generalise the classical newsvendor problem for critical perishable commodities having more severe costs than its linear alternative. Piece wise polynomial cost functions are introduced to accommodate the excess severity. Stochastic demand is assumed to follow a completely unknown probability distribution. Non parametric estimator of the optimal order quantity has been developed from an estimating equation using a random sample. Strong consistency of the estimator is proved for unique optimal order quantity and the result is extended for multiple solutions. Simulation results indicate that non parametric estimator is efficient in terms of mean square error. Real life application of the proposed non-parametric estimator has been demonstrated with Avocado demand in the United States of America and Covid-19 test kit demand during second wave of SARS-COV2 pandemic across 86 countries.

Keywords Stochastic programming · Non-parametric estimation · Strong consistency · Monte-Carlo simulation · Newsvendor problem · Non-linear optimisation

1 Introduction
Newsvendor problem deals with determination of optimal order quantity of a perishable commodity by offsetting piece-wise linear shortage and excess costs. We assume a newsvendor model with the following assumptions:

- single perishable item over a single period
- positive random demand with a continuous but unknown probability distribution
- no backlog or pre-booking is allowed
- absence of any influencing factors like marketing efforts, promotions, discounts etc.
- instantaneous replenishment of order quantities

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The decision for a single period problem is taken at the beginning, i.e. before the random demand is realised (see Chernonog & Goldberg, 2018, and the references therein). However, perishable critical resources would often warrant shortage and excess costs to be more severe than linear. In addition, a popular practice is to assume a parametric demand distribution with known parameters. In reality the distribution remains unknown most often. In this paper, we propose non-parametric estimation method of the optimal order quantity in the generalised newsvendor problem, which accommodates the severity of loss through a piece-wise non-linear function (Ghosh et al., 2021).

In many real life situations, shortage and excess costs become more excruciating than a piece-wise linear newsvendor model, which accommodates only the quantity lost. For example, chemotherapy drugs are administered to patients as per a schedule. Shortage of the drug on the scheduled day would result in breaking of the treatment cycle. Here the loss is more severe than merely the quantity lost. Similarly, excess inventory of critical drugs or chemical resources might cause vast environmental and microbial hazards during disposal of the excess material. Piece-wise non-linearity is thus an appropriate choice for shortage and excess cost models. Non-linear newsvendor problem has been studied only recently in the literature. Parlar and Rempala (1992) considered the periodic review inventory problem and derived the solution of a newsvendor problem with a quadratic cost function. Gerchak and Wang (1997) described optimal order quantity determination from a newsvendor problem with linear excess but quadratic shortage cost. Pal et al. (2015) used exponential weight function of order quantity to the holding cost and linear excess cost in a newsvendor set-up. Kyparisis and Koulamas (2018) addressed the newsvendor problem for quadratic utility function. Khouja (1995), Chandra and Mukherjee (2005), among others, considered optimisation of reliability function of the stochastic cost. In this paper, we consider generalisation of the classical newsvendor model of Ghosh et al. (2021). This version of the generalised newsvendor model considers severity of losses by modelling the severity of shortage and excess costs using measurable and continuous non-linear weights and the conditions for existence of the optimal order quantity has been established for exponential and uniform demands.

A critical issue with the optimal order quantity determination in classical newsvendor problem is the lack of knowledge on random demand. Majority of the works assume a completely specified demand distribution, whereas in reality, it is seldom so. In case of unknown demand distribution, parametric and distribution-free estimation of the optimal order quantity has been considered more recently. Parametric estimation of the optimal order quantity has been studied by Nahmias (1994) and more recently, Kevork (2010) for Normal demand. Agrawal and Smith (1996) estimated the order quantity for negative binomial demand. Rossi et al. (2014) has given bounds on the optimal order quantity using confidence interval for parametric demand distributions. Ghosh et al. (2021) estimated optimal order quantity for uniform and exponential demands in non-linear newsvendor problem.

Distribution free estimation of optimal order quantity, on the other hand, has been studied in two parallel ways in the context of classical newsvendor problem. In the first case, the investigator has access to population summary measures like mean, variance etc, but the demand distribution remains unknown (Bai et al., 2020). Scarf (1958) and later Moon and Gallego (1994) studied the min-max optimal order quantity in such cases. The second approach considers the estimation problem based on an uncensored random sample from the unknown demand distribution. Pal (1996), Bookbinder and Lordahl (1989) discussed construction of bootstrap based point and interval estimator of the optimal order quantity using demand data. The sampling average approximation (SAA) method (see Kleywegt et al., 2001; Linderoth et al., 2006), replaces the expected cost by the sample average of the corresponding objective function and then optimises it. Levi et al. (2015) provides bounds...
of the relative bias of estimated optimal cost using SAA based on uncensored demand data. More recently data driven non-parametric approaches has become quite popular in studying different variations of classical newsvendor problems. He et al. (2012) studied the impact of availability of data for newsvendor model using nurse staffing data from a hospital. Ban and Rudin (2019) developed single step machine-learning algorithm for a classical newsvendor problem with historical data on demand and several related features. Punia et al. (2020) proposed machine-learning based solution for multi-item newsvendor model in presence of capacity constraint. Keskin et al. (2021) proposed a data driven estimation method for optimum order quantity when demand is a non-stationary time-series in a classical newsvendor set-up. Lin et al. (2022) discusses the data driven decision making of a risk-averse newsboy by maximising expected profit under the presence of value-at-risk constraint. However, not much work has been done on non-parametric estimation in non-linear newsvendor problems to the best of our knowledge.

In this paper we explain the existence of optimal order quantity and devise a non-parametric technique to estimate it in the generalised newsvendor model. Our study makes two unique contributions to the literature. First, we develop a non-parametric estimator of the optimal order quantity in a generalised newsvendor set-up with non-linear cost function of higher degree. The non-parametric estimator is developed from an estimating equation using an uncensored random sample on stochastic demand. The feasibility of obtaining real positive solutions to the estimating equation has been derived in almost-sure sense. We have studied the strong consistency property of the estimated optimal order quantity when the true solution is unique and its extension to the cases, when true optimal order quantity is not unique or both the true and estimated optimal order quantities are not unique.

Our second contribution is a detailed simulation study of the performance of the non-parametric estimator of optimal order quantity in a generalised newsvendor model. It may be remarked here that analytical measurement of performance of the non-parametric estimator seems to be very difficult for severe cases. Here we present the empirical distributions of the non-parametric estimators of optimal order quantities obtained from the simulation for different severity levels and per-unit costs. We also present comparison between the non-parametric estimator and its parametric counterparts having Uniform and Exponential demands using simulation. For this purpose we have computed the bias and mean square error (MSE) and optimal order quantities (Ghosh et al., 2021) using simulated data from the respective distributions. Asymptotic behaviour is presented through bias and MSE plots. Our simulation study is based on 3.15 million numerical experiments on the mentioned parametric demand distributions. We present the real-life application of the non-parametric method developed in this paper for estimating optimal order quantity in a generalised newsvendor set-up. First real-life scenario describes the amount of avocado imported in United States of America over consecutive weeks starting from January, 2020 to July, 2022. Second one describes the total number of daily tests performed during the second wave of COVID-19 pandemic. In both the cases we have estimated the optimal order quantity using non-parametric and parametric models with uniform and exponential demands. We compared the performance of the non-parametric estimator using percentage savings in estimated optimal cost (see Keskin et al., 2021). The paper concludes with a discussion on the findings.
2 Symmetric generalised newsvendor problem

We begin this section with the note that a table of major notations used in the following sections has been given in the appendix (Table 1) for ready reference of the readers. Our work in this paper considers a case where the severity of the excess and shortage are more than the quantity lost (i.e. the gap between inventory and demand). Let the stochastic demand be represented by a random variable $X$ with a compact support $\mathcal{X} \subseteq \mathbb{R}^+$ defined over the complete probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{F}$ is the $\sigma$-algebra over $\Omega$. Since we do not consider pre-booking, we have $0 \in \mathcal{X}$. Further, let $C_e (\in \mathbb{R}^+)$ and $C_s (\in \mathbb{R}^+)$ be the excess and shortage costs per unit respectively. Then the cost function in classical newsvendor set-up at an inventory level $Q$ is given by

$$C(Q, X) = \begin{cases} C_e(Q - X), & \text{if } X \leq Q \\ C_s(X - Q), & \text{if } X > Q \end{cases}$$

(1)

Related stochastic programming problem under the assumption of existence of $E_G[X]$, is given by

$$\arg\min_{Q \in \mathcal{X}} E_G[C(Q, X)]$$

(2)

where $G(\cdot)$ is the induced probability distribution of $X$ defined over the measurable space $(\mathbb{R}^+, B^+)$, $B^+$ being the corresponding Borel-algebra. We consider generalisation of quadratic cost function by introducing polynomial weights (in $s$ and $Q$) model to demand and inventory dependent cost models, viz. $C_e(Q - X)^{m-1}$ and $C_s(X - Q)^{m-1}$ respectively.

The constant $m$ is integer valued and $m - 1$ could be interpreted as the severity constant. As $m$ increases, more severe is the loss. For $m = 1$, no extra severity is implicated and the problem reduces to the classical newsvendor problem. Thus the new cost function for generalised newsvendor is given by

$$C_m(Q, X) = \begin{cases} C_e(Q - X)^m, & \text{if } X \leq Q \\ C_s(X - Q)^m, & \text{if } X > Q \end{cases}$$

(5)

The new cost functions could also be interpreted as a generalisation of constant costs per unit ($C_e, C_s$) model to demand and inventory dependent cost models, viz. $C_e(Q - X)^{m-1}$ and $C_s(X - Q)^{m-1}$ respectively.

In view of the above weight function structure, we now make the following assumptions about the probability distribution of demand ($X$):
A1. $\mathcal{X}$ is independent of $Q$
A2. $G$ is continuous and strictly increasing over the support $\mathcal{X}$
A3. $X^p$ is $\mathbb{G}$-integrable $\forall \ p \geq 0$

The assumption A1 is required to avoid the trivial solution of zero order quantity, which may arise for certain choices of demand distribution, the degree of severity ($m$) and the costs ($C_e, C_s$). For example, if the demand is $Uniform(0, 2Q)$ then for $C_e = C_s$, the optimum order quantity would become zero. Hence, we make further assumption of $C_e \neq C_s$. The expected cost function in this case can be written as,

$$E_G[C_m(Q, X)] = \int_{S_Q} C_e(Q - x)P_{1,m}(Q, x)d\mathcal{G} + \int_{S'_Q} C_s(x - Q)P_{2,m}(Q, x)d\mathcal{G}$$

(6)

where $S_Q = \{\omega \in \Omega \mid X(\omega) \in (0, Q)\}$, $S'_Q = \mathcal{X}\setminus S_Q$ and $E_G$ denotes expectation with respect to $\mathcal{G}$.

Differentiating Eq. 6 with respect to $Q$ using Leibnitz rule, we get the first order condition for the minimisation problem stated above as follows

$$\frac{\partial E_G[C_m(Q, X)]}{\partial Q} = 0$$

$$\Rightarrow \int_{S_Q} C_e(Q - X)^{m-1}d\mathcal{G} = \int_{S'_Q} C_s(X - Q)^{m-1}d\mathcal{G}$$

$$\Rightarrow C_e\int_{S_Q} (Q - X)^{m-1}d\mathcal{G} = C_s\left[\int_{\mathcal{X}} (X - Q)^{m-1}d\mathcal{G} - \int_{S_Q} (X - Q)^{m-1}d\mathcal{G}\right]$$

$$\Rightarrow \int_{S_Q} (Q - X)^{m-1}d\mathcal{G} = \left[C_s + C_s(-1)^{m-1}\right] \int_{\mathcal{X}} (X - Q)^{m-1}d\mathcal{G}$$

$$\Rightarrow E_G[(Q - X)^{m-1}\mathbb{I}(S_Q)] = k_m$$

(7)

where, $\mathbb{I}(S_Q)$ is an indicator function over the set $S_Q$ and $k_m = \frac{C_s}{C_e + (-1)^{m-1}C_s}$. Denoting $\int_{S_Q} (Q - X)^i d\mathcal{G} = \theta_{1,i}$ and $E(X - Q)^i = \theta_{2,i}, \forall \ i = 1, 2, \ldots$, Eq. 7 can be written as

$$h(\theta, Q) = \frac{\theta_{1,m-1}}{\theta_{2,m-1}} = k_m$$

(8)

Let us define the $j^{th}$ partial raw moment of $X$ as $\delta_j = \int_{S_Q} X^j d\mathcal{G}$ and the $j^{th}$ raw moment of $X$ by $\mu'_j = \int_{\mathcal{X}} X^j d\mathcal{G} \forall j = 1, 2, \ldots$. Further let, the optimal expected cost be denoted by $\varphi_m^*$ and the corresponding set of optimal order quantities by $\mathcal{U}^*$, which are obtained by solving the population stochastic minimisation problem given later in this section (Eq. 9). Next we show that $\mathcal{U}^*$ is non-empty, i.e. at least one feasible solution to Eq. 8 exists.

**Theorem 2.1** Consider the stochastic minimisation problem in a SyGen-NV set-up as follows,

$$\arg\min_{Q \in \mathcal{X}} E_G[C_m(Q, X)]$$

(9)

where $X$ is the positive demand defined over the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $Q$ is order quantity. Under the assumptions A1-A3, at least one positive solution to the stochastic minimisation problem exists.
Proof From the first order condition in Eq. 8, we notice that

\[
\int_{S_Q} (Q - X)^{m-1} d\mathbb{G} = k_m (-1)^{m-1} \int_X (Q - X)^{m-1} d\mathbb{G}, \quad (Q \in \mathcal{X})
\]

\[
\Rightarrow \int_{S_Q} \sum_{j=0}^{m-1} \binom{m-1}{j} Q^{m-1-j} (-1)^j X^j d\mathbb{G} = k_m (-1)^{m-1} \int_X (Q - X)^{m-1} d\mathbb{G}
\]

\[
\Rightarrow \sum_{j=0}^{m-1} \binom{m-1}{j} Q^{m-1-j} (-1)^j \left[ \int_{S_Q} X^j d\mathbb{G} - k_m (-1)^{m-1} \int_X X^j d\mathbb{G} \right] = 0.
\]

\[
\Rightarrow \sum_{j=0}^{m-1} \binom{m-1}{j} Q^{m-1-j} (-1)^j [\delta_j - (-1)^{m-1} k_m \mu_j] = 0
\]

\[
\Rightarrow \sum_{j=0}^{m-1} (-1)^j \beta_j Q^{m-1-j} = 0, \text{ where } \beta_j = \binom{m-1}{j} [\delta_j - (-1)^{m-1} k_m \mu_j] \quad (10)
\]

Odd values of \( m \) \((m = 2d + 1)\) ensures that \( k_m \) lies in the interval \((0, 1)\) and \( \beta_j = \binom{2d}{j} [\delta_j - k_{2d+1} \mu_j]. \) Letting \( Q \to 0, \) it is observed that, \( \delta_{2d} \to 0, \) resulting in \( \lim_{Q \to 0} \beta_{2d} = -k_{2d+1} \mu_{2d} < 0 \) so that \( \lim_{Q \to 0} \sum_{j=0}^{2d}(1) \beta_j Q^{2d-j} = \beta_{2d} < 0. \)

On the other hand, it is possible to choose a large \( Q, \) say \( Q_0, \) so that \( \delta_j \approx \mu_j, \ \forall j = 0, 1, \ldots 2d, \) whenever \( Q \geq Q_0. \) In that case, \( \beta_j \to \tau_j, \) where, \( \tau_j = \binom{2d}{j} \mu_j (1 - k_{2d+1}) > 0, \ \forall j = 0, 1, \ldots 2d. \) Choosing \( Q_0 = \max \left\{ \frac{\tau_{j+1}}{\tau_j} : j = 0, 1, \ldots (d - 1) \right\}, \) we, therefore, obtain

\[
\sum_{j=0}^{2d} (-1)^j \tau_j Q^{2d-j} = \tau_0 Q^{2d} - \tau_1 Q^{2d-1} + \cdots + \tau_{2d-2} Q^2 - \tau_{2d-1} Q + \tau_{2d}
\]

\[
= Q^{2d-1} (\tau_0 Q - \tau_1) + Q^{2d-3} (\tau_2 Q - \tau_3) + \cdots + Q (\tau_{2d-2} Q - \tau_{2d-1}) + \tau_{2d}
\]

\[
> 0, \text{ for } Q > Q_0
\]

Thus, the function in Eq. 10 is negative when \( Q \to 0 \) and is positive for large \( Q \) (i.e. \( Q > Q_0 \)). Hence, presence of a positive solution of Eq. 10 follows from the well known Bolzano’s theorem on zero of continuous functions.

Even values of \( m \) \((m = 2d)\) ensures that either \( k_m > 0 \) or \( k_m < -1. \) The first case is given by \( 0 < k_{2d} \) and \( \beta_j = \binom{2d}{j} [\delta_j + k_{2d} \mu_j]. \) Letting \( Q \to 0, \) it is observed that, \( \delta_{2d-1} \to 0, \) resulting in \( \lim_{Q \to 0} \beta_{2d-1} = k_{2d} \mu_{2d-1} > 0 \) so that \( \lim_{Q \to 0} \sum_{j=0}^{2d}(1) \beta_j Q^{2d-1-j} = (-1)^{2d-1} \beta_{2d-1} = -\beta_{2d-1} < 0. \)

Choosing a large \( Q, \) say \( Q_1, \) implies \( \delta_j \approx \mu_j, \ \forall j = 0, 1, \ldots 2d - 1, \) whenever \( Q \geq Q_1. \) Here, \( \beta_j \to \tau_j = \binom{2d-1}{j} \mu_j (1 - k_{2d}) > 0, \ \forall j = 0, 1, \ldots 2d - 1. \) \( Q_1 \) is selected as \( Q_1 = \max \left\{ \frac{\tau_{j+1}}{\tau_j} : j = 0, 1, \ldots (d - 1) \right\}. \) Therefore the function Eq. 10 is obtained as,

\[
\sum_{j=0}^{2d-1} (-1)^j \tau_j Q^{2d-1-j} = \tau_0 Q^{2d-1} - \tau_1 Q^{2d-2} + \cdots + \tau_{2d-2} Q - \tau_{2d-1}
\]

\[
= Q^{2d-2} (\tau_0 Q - \tau_1) + Q^{2d-4} (\tau_2 Q - \tau_3) + \cdots
\]
similar argument as the previous case guarantees that a positive solution of the stochastic minimisation problem exists.

The second case is given by $k_{2d} < -1$ and $\beta_j = (\frac{d-1}{j})[\delta_j + k_{2d} \mu_j]$. Letting $Q \to 0$, it is observed that, $\delta_{2d-1} \to 0$, resulting in $\lim Q \to 0 \beta_{2d-1} = k_{2d} \mu'_{2d-1} < 0$ so that $\lim Q \to 0 \sum_{j=0}^{2d-1} (-1)^j \beta_j Q^{2d-1-j} = (-1)^{2d-1} \beta_{2d-1} = \beta_{2d-1} > 0$.

A large value of $Q$, say $Q_2$, indicates that $\delta_j \approx \mu_j', \forall j = 0, 1, \ldots 2d - 1$, whenever $Q \geq Q_2$. In that case, $\beta_j \to \tau_j$, where, $\tau_j = (\frac{d-1}{j}) \mu'_j (1 - |k_{2d}|) = -(\frac{d-1}{j}) \mu'_j (|k_{2d}| - 1) = -\kappa_j < 0, \forall j = 0, 1, \ldots 2d - 1$. $\kappa_j$ is obtained as $\kappa_j = (\frac{d-1}{j}) \mu'_j (|k_{2d}| - 1) > 0, \forall j = 0, 1, \ldots 2d - 1$. The function is written as

$$\sum_{j=0}^{2d-1} (-1)^{j+1} \kappa_j Q^{2d-1-j} = -\kappa_0 Q^{2d-1} + \kappa_1 Q^{2d-2} + \cdots - \kappa_{2d-2} Q + \kappa_{2d-1}$$

$$= Q^{2d-2}(\kappa_1 - \kappa_0) + Q^{2d-4}(\kappa_3 - \kappa_2) + \cdots + \kappa_{2d-1} - \kappa_{2d-2} Q$$

$$< 0, \text{ for } Q > Q_2$$

The choice of $Q_2$ is described as $Q_2 = \max \left\{ \frac{k_{2j+1}}{\kappa_j} : j = 0, 1, \ldots (d - 1) \right\}$. Similar argument as previous one establishes the existence of the positive solution.

Since there could be many positive roots, we select the one with maximum magnitude. □

3 Non-parametric optimal order quantity estimation in SyGen-NV

In this section, we present non-parametric estimation of the optimal order quantity, when the demand distribution is completely unknown, but historical uncensored demand data are available. Let us denote an uncensored random sample of size $n$ by $X = (X_1, X_2, \ldots, X_n)'$ drawn from $\mathcal{G}$. We define two statistics $T_{in}(X) : \mathbb{R}^+ \to \mathbb{R}^+$, $(i = 1, 2)$ as $T_{in} = \frac{1}{n} \sum_{i=1}^{n} (Q - X_i)^{m-1} I(X_i \leq Q)$ and $T_{2n} = \frac{1}{n} \sum_{i=1}^{n} (X_i - Q)^{m-1}$. Then the sample version of the first order condition in Eq. 9 can be constructed by replacing $\theta_{i,m-1}$ with corresponding $T_{in}$, $i=1,2$. The estimating equation can be written as

$$h(T_n; Q) = \frac{T_{1n}}{T_{2n}} = k_m$$

(11)

Further, we define sample partial and complete raw moments of order $j$ as $d_j = \frac{1}{n} \sum_{i=1}^{n} X_i^j I(X_i \leq Q)$ and $m'_j = \frac{1}{n} \sum_{i=1}^{n} X_i^j$. It can be easily observed that the sample raw moments $d_j$ and $m'_j$ are unbiased estimators of $\delta_j$ and $\mu'_j$. Hence, $\hat{\beta}_j = (\frac{n-1}{j})[d_j - (-1)^{m-1}k_m m'_j]$ is the unbiased estimator of $\beta_j$. We then construct the sample version of the first order condition provided in Eq. 10 as

$$\sum_{j=0}^{m-1} (-1)^j \hat{\beta}_j Q^{m-1-j} = 0$$

(12)
where $\hat{\beta}_j$ is as defined above. We would refer to $h(T_n; Q)$ as estimating function and the function in the alternative form of the first order condition in Eq. 12 as the random estimating function or simply random function.

### 3.1 Properties of $T_n$\sim

Some important properties of $T_{i,n}$, $i = 1, 2$ are as follows.

P1. $T_{i,n}$ is unbiased for $\theta_{i,m-1}$, $i = 1, 2$.

P2. $T_{i,n} \xrightarrow{a.s.} \theta_{i,m-1}$ as $n \to \infty$.

P3. $\sqrt{n}(T_{i,n} - \theta_{i,m-1}) \xrightarrow{L} N(0, \sigma_{i,n}^2)$, where $n\sigma_{i,n}^2 = \theta_{i,2m-2} - \theta_{i,m-1}^2$, $i = 1, 2$ and the symbol $\xrightarrow{L}$ stands for convergence in distribution.

Proof of P1 is immediate by taking expectation of $T_{i,n}$. P2 follows from Kolmogorov’s strong law of large number (see pp-115 Rao, 1973) and the fact that each of $T_{i,n}$, $i = 1, 2$ is a average of independently and identically distributed (iid) random variables satisfying existence of variance by assumption A3 stated above. P3 is also straightforward from Lindeberg-Levy central limit theorem for iid samples Rao (1973).

### 3.2 Properties of $h(T_n; Q)$

We begin with the statement of the following properties of $h(T_n; Q)$.

P4. $h(T_n; Q)$ is a measurable function over $(\mathbb{R}^{+n}, \mathcal{B}_n)$ for every $Q \in \mathcal{X}$.

P5. $h(T_n; Q)$ is continuously differentiable with respect to $Q$ within the compact set $\mathcal{X}$ a.e $\mathcal{B}_n$.

Property P4 of $h(T_n; Q)$ is straight forward from the fact that it is a ratio of two measurable functions for every $Q \in \mathcal{X}$. The next property follows from the facts that $T_{1,n}$ and $T_{2,n}$ are positive a.e $\mathbb{R}^{+n}$ for every $Q \in \mathcal{X}$ and ratio of non-zero functions are differentiable.

In what follows, we provide the asymptotic distribution of the random function $h(T_n; Q)$ for every $Q \in \mathcal{X}$. First we state an important result, called the delta method for asymptotic normality of a one time differentiable function.

**Theorem 3.1** (Delta Method DasGupta (2008))

Suppose $W_n$ is a sequence of k-dimensional random vectors such that $\sqrt{n}(W_n - \theta) \xrightarrow{L} N_k(0, \Sigma)$. Let $g : \mathbb{R}_k \to \mathbb{R}$ be once differentiable at $\theta$ with the gradient vector $g^{(1)}(\theta)$. Then

$$\sqrt{n}(g(W_n) - g(\theta)) \xrightarrow{L} N(0, g^{(1)}(\theta) \Sigma g^{(1)}(\theta))$$

(13)

We now prove the asymptotic normality of $h(T_n; Q)$ in the following theorem.

**Theorem 3.2** Consider the estimating function $h(T_n; Q)$ in Eq. 11. Then for large $n$

$$\sqrt{n}(h(T_n; Q) - h(\theta; Q)) \xrightarrow{L} N \left( 0, h^{(1)}(\theta) \Sigma h^{(1)}(\theta) \right)$$

(14)
where $\Sigma$ is the dispersion matrix of $T_n$, $h^{(1)}$ is the 1st vector derivative of $h(T_n; Q)$ with respect to $T_n$ evaluated at $\theta$ and

$$h^{(1)}' \Sigma h^{(1)} = h(\theta; Q)^2 \left[ \frac{\theta_{1,2m-2}}{\theta_{2,m-1}^2} + \frac{\theta_{2,2m-2}}{\theta_{2,m-1}^2} + 2(-1)^m \frac{\theta_{1,2m-2}}{\theta_{1,m-1} \theta_{2,m-1}} \right].$$

**Proof** The co-variance between $T_{1n}$ and $T_{2n}$ is

$$\sigma_{12,n} = \text{Cov}(T_{1n}, T_{2n}) = \text{Cov} \left( \frac{1}{n} \sum_{i=1}^{n} (Q - X_i)^{m-1} I(X_i \leq Q), \frac{1}{n} \sum_{i=1}^{n} (X_i - Q)^{m-1} \right)$$

$$= \frac{1}{n^2} \sum_{i=1}^{n} \text{Cov}((Q - X_i)^{m-1} I(X_i \leq Q), (X_i - Q)^{m-1})$$

$$= \frac{1}{n} \left[ (-1)^{m-1} \theta_{1,2m-2} - \theta_{1,m-1} \theta_{2,m-1} \right]$$

(15)

From the property P3 and Eq. 15, it could be easily seen that $\sqrt{n} (T_n - \hat{\theta})$ is asymptotically multivariate normal with dispersion matrix $\Sigma = (\sigma_{ij;n})$, $i, j = 1, 2$ and $\sigma_{ii;n} = \sigma_{i,n}^2$. Also, note that $T_{1n} > 0$ a.e. $\mathbb{R}^{+n}$, $i = 1, 2$ and $h(T_n; Q)$ is once differentiable for every $Q \in \mathcal{X}$. We denote the 1st derivative of $h(T_n; Q)$ by $h^{(1)} = (h^1(\theta; Q), h^2(\theta; Q))'$.

$$h^{(1)}' \Sigma h^{(1)} = h(\theta; Q)^2 \left[ \frac{\sigma_{12,n}}{\theta_{2,m-1}^2} + \frac{\sigma_{2,n}}{\theta_{2,m-1}^2} - 2 \frac{\sigma_{12,n}}{\theta_{1,m-1} \theta_{2,m-1}} \right]$$

$$= \frac{h(\theta; Q)^2}{n} \left[ \frac{\theta_{1,2m-2}}{\theta_{2,m-1}^2} + \frac{\theta_{2,2m-2}}{\theta_{2,m-1}^2} + 2(-1)^m \frac{\theta_{1,2m-2}}{\theta_{1,m-1} \theta_{2,m-1}} \right]$$

The proof of the theorem is then immediate from the delta method (Th. 3.1).

3.3 Solution of the estimating equation

In this section we present the statistical properties of the estimated optimal order quantity and the optimal value function. We denote by $\hat{\mu}_m^n$ the estimated optimal cost function and the corresponding set of estimated optimal order quantities are denoted by $U^*$. In the following theorem we prove that $\hat{U}^*$ is non-empty with probability (wp) 1, i.e. there exists at least one positive solution to Eq. 11 wp 1.

**Theorem 3.3** Under the regularity assumptions A1 – A3, the random function $\sum_{j=0}^{m-1} (-1)^j \hat{\beta}_j Q^{m-1-j}$ will have positive zeroes almost surely.

where $\hat{\beta}_j = d_j - (-1)^{m-1} k_m m_j'$, $\forall j = 1, 2 \ldots m - 1$.

**Proof** Notice that, $d_j \overset{a.s.}{\to} \delta_j$ and $m_j' \overset{a.s.}{\to} \mu_j'$, which implies in turn that $\hat{\beta}_j \overset{a.s.}{\to} \beta_j$. Thus the proof of this theorem is same as that of Th. 2.1 in almost sure sense. We omit the details to avoid repetition.

$\square$
Next we show that any solution to the estimating equation converges to the true optimal order quantity in SyGen-NV problem. Let the solution of the estimating equation Eq. 11 (or Eq. 12) be denoted by \( \hat{Q}^*_n \). We show that the solution is strongly consistent for the solution to the stochastic optimisation problem \( \arg\min_{\hat{Q} \in \mathcal{X}} E_{\mathbb{G}} [C_m(\hat{Q}, X)] \) under mild regularity conditions. First we state the following theorem without proof on existence of optima of a continuous function on a compact set.

**Theorem 3.4** (Extreme value theorem (see Stein & Shakarchi, 2010)) A continuous function on a compact set \( \mathcal{X} \) is bounded and attains a maximum and minimum on \( \mathcal{X} \).

We state the next lemma on the compactness of the complement of an open subset of a compact set.

**Lemma 3.5** Let \( \mathcal{X} \) be a compact set and \( O \) be an open subset of \( \mathcal{X} \). Then \( O' = \mathcal{X} \setminus O \), denoting the complement of \( O \) in \( \mathcal{X} \), is also a compact set.

The proof is a routine exercise in real analysis and hence is omitted.

**Theorem 3.6** Let \( \hat{Q}^*_n \in \mathcal{X} \) be the unique solution to the estimating equation \( h(T_n; Q) = k_m \) and \( Q^* \) uniquely solves the stochastic programming problem

\[
\arg\min_{\hat{Q} \in \mathcal{X}} E_{\mathbb{G}} [C_m(\hat{Q}, X)]
\]

Then

\[
\hat{Q}^*_n \xrightarrow{a.s.} Q^*
\]

**Proof** Let \( O \subseteq \mathcal{X} \) denote an arbitrary open neighbourhood of \( Q^* \). From lemma 3.5, the complement of \( O \), \( O' = \mathcal{X} \setminus O \) is also a compact set. Notice that the expected cost \( E_{\mathbb{G}}[C_m(X, Q)] \) (= \( \varphi_m(Q) \), say), is a continuous function of \( Q \). Hence, from Theorem 3.4, the stochastic optimisation problem \( \arg\min \varphi_m(Q) \) will have a solution in \( O' \) with unique minimum value of \( \varphi_m(Q) \). Let us denote, \( r = \min_{Q \in O'} \varphi_m(Q) - \varphi_m(Q^*) > 0 \).

Also, from property P2 of \( T_n \), \( (i = 1, 2) \) and the continuous mapping theorem, it can be easily seen that \( h(T_n, Q) \xrightarrow{a.s.} h(\theta, Q) \), \( \forall Q \in \mathcal{X} \). Since \( \hat{Q}^*_n \in \mathcal{X} \), there would exist \( n_0(\epsilon) \) for every \( \epsilon > 0 \), such that \( |h(\theta, \hat{Q}^*_n) - k_m| < \epsilon, \forall n \geq n_0(\epsilon), wp 1 \). Therefore \( \exists n > n_0(\epsilon) \) for every \( 0 < \epsilon < \frac{r}{2} \), so that

\[
|h(\theta, \hat{Q}^*_n) - h(\theta, Q^*)| < \epsilon, \forall n > n_0(\epsilon), \ wp 1
\]

This implies \( \hat{Q}^*_n \notin O' \). \( O \) being arbitrary, \( \hat{Q}^*_n \xrightarrow{a.s.} Q^* \).

The roots of the FOC (Eq. 10) may not be unique. Let the set of corresponding distinct roots be denoted by \( Q^* = \{ Q^*_1, Q^*_2, \ldots, Q^*_p \} \). \( k = 1, 2, \ldots, m - 1 \). Similarly, there could be \( p (\geq 1) \) roots of the random function (Eq. 12), say \( \hat{Q}^* = \{ \hat{Q}^*_1, \hat{Q}^*_2, \ldots \hat{Q}^*_p \} \). In the next two corollaries, we extend Theorem 3.6 for multiple roots.

**Corollary 3.6.1** Let \( \hat{Q}^* \) be the set of distinct roots of the random function (Eq. 12) and \( Q^* \) be unique solution to the stochastic minimisation problem (9). Then \( \hat{Q}^*_\text{max} \xrightarrow{a.s.} Q^* \), where \( \hat{Q}^*_\text{max} = \max \{ \hat{Q}^*_n \} \).

**Proof** Notice, the maximum of \( \hat{Q}^* \) is unique. Hence, from Th. 3.6, the proof is immediate. \( \square \)
Corollary 3.6.2 Let $\hat{Q}^*_n$ be the unique solution to the random function Eq. 12 and $Q^*_i$ be the set of distinct solutions to the stochastic minimisation problem (9). Then $\hat{Q}^* \overset{a.s.}{\to} Q^*_i$; for exactly one $i$; $i = 1, 2, \ldots, k$.

Proof Let $O_i$ denote an arbitrary open neighbourhood around $Q^*_i$ selected in such a way that $O_i$’s are disjoint. Then, $O = \bigcup_{i=1}^k O_i$ is also an open set. Implementing the same argument as Theorem 3.6 we ensure that $\hat{Q}^*_n \in O$. Disjoint property of $O_i$ indicates $\hat{Q}^*_n \in O_i$ for exactly one $i$. $\square$

Corollary 3.6.3 Let $\hat{Q}^*$ be the set of distinct solutions to the random function Eq. 12 and $Q^*_i$ is the set of distinct solutions of the FOC Eq. 10, then $\hat{Q}^* \overset{a.s.}{\to} Q^*_i$; for exactly one $i$; $i = 1, 2, \ldots, k$.

Proof Proof immediately follows from previous two corollaries. $\square$

From the above theorem, it can be easily seen that the estimated optimal cost $\hat{\varphi}^*_n = \varphi^*_m(\hat{Q}^*)$ almost surely converges to the true optimal cost $\varphi^*_m$, using the continuity of the cost function $\varphi^*_m(Q)$.

4 Numerical experiments

In this section we present the results of numerical experiments on the non-parametric estimator of the optimal order quantity in SyGen-NV set-up. We use Monte-Carlo simulation as well as real data to gauge the performance of the non-parametric estimator. For comparison purpose, we consider two parametric counter parts, viz. Exponential and Uniform demand distributions for estimating the optimal order quantity in this problem.

4.1 Monte-Carlo simulation

We consider here two known probability distributions for the demand, viz. Uniform(0, 1) and Exponential(1). The severity index $m$ is assumed to be known ($m \in \{2, 3, 4, 5, 10\}$). Further, we take the excess-to-shortage cost ratio, $\eta (= \frac{C_e}{C_s}) \in \{0.25, 0.45, 0.65, 0.85, 1.05, 1.25, 1.45, 1.65, 1.85\}$. For each of the $(m, \eta)$ pairs, we compute numerically the optimal order quantities for both Uniform and Exponential true demands. Further, we conduct 3.15 million Monte-Carlo simulation experiments for each of the demand distributions to understand the small and large sample properties of the non-parametric estimator. In particular, we draw random samples of size $n (= 20, 50, 100, 500, 1000, 5000, 10000)$ for each combination of $(\eta, m)$ and estimate the optimal order quantities $\hat{Q}^*_n$ therefrom. We repeat this process for $M$ times ($M = 5000$). We study the sampling properties of $\hat{Q}^*_n$ from these $M$ estimates.

4.1.1 Uniform demand distribution

The optimal order quantity in the SyGen-NV problem with Uniform(0, 1) demand is given by (Ghosh et al., 2021)

$$Q^*_{uni} = \frac{1}{1 + \eta^\frac{1}{m}}$$
The non-parametric estimator, \(\hat{Q}_n^*\), can be obtained from the estimating equation (Eq. 12).

The probability distribution of the estimated order quantity is presented in the form of box-plots in Fig. 4. For \(\eta < 1\), the probability distributions of \(\hat{Q}_n^*\) are stochastically larger with increasing severity levels, the distribution for \(m = 2\) being centred at the highest value among all others. For \(\eta > 1\), the distributions of estimated order quantity for even \(m\) are different than those of the odd \(m\). Odd severity seems to result in stochastically smaller distribution of \(\hat{Q}_n^*\). The variation, on the other hand, seems to decrease with severity for all \(\eta\).

Next we present the performance study of \(\hat{Q}_n^*\) using the mean square error (MSE) computed from the \(M\) estimates as

\[
MSE = \frac{1}{M} \sum_{i=1}^{M} (\hat{Q}_i^* - Q_n^*)^2.
\]

Figure 5a–i in the appendix presents the MSE’s plotted against sample sizes. It could be seen that for \(\eta < 1\), the MSEs converge to 0 with increasing \(n\) for all \(m\), with worst performance of \(\hat{Q}_n^*\) observed at \(m = 2\). For \(\eta > 1\), however, the convergence is slow in case of even \(m\).

### 4.1.2 Exponential demand distribution

The optimal order quantity in the SyGen-NV problem with \(\text{Exponential}(1)\) demand can be obtained from the random function (Eq. 10) by replacing the partial and full raw moments by those for the \(\text{Exponential}(1)\) distribution. The modified equation is given as (Ghosh et al., 2021)

\[
\sum_{j=0}^{m-1} (-1)^j (Q)^{m-j-1} \frac{1}{(m-j-1)!} = e^{-Q} \left[ \frac{Cs}{Ce} - (-1)^m \right]
\]

As described in the uniform case, \(\hat{Q}_n^*\) can be obtained from the estimating equation (Eq. 12).

Unlike the uniform demand case, probability distribution of the estimated optimal order quantity increases stochastically with severity for all \(\eta\) (see Fig. 6). Not only the location, the scale (or variance) of the distribution also increases with \(m\).

In terms of MSE, \(\hat{Q}_n^*\) performs well asymptotically as the MSE (vs. \(n\)) curve (see Fig. 7a–h) decreases to zero with increasing sample size (for all \(m\) and \(\eta\)), the worst performance being observed for \(m = 10\). The best estimator, in the MSE sense, is obtained for \(m = 2\) when \(\eta < 1\). However, for \(\eta > 1\) performance of \(\hat{Q}_n^*\) for \(m = 2\) worsens in small samples.

### 4.1.3 Relative performance of non-parametric estimator

We compare the performance of the non-parametric estimator with its parametric counterpart in terms of the estimated optimum cost. Following Ghosh et al. (2021), the parametric demand distributions are assumed to be \(\text{Exponential}(\lambda)\) and \(\text{Uniform}(0, b)\), so that the mean remains same (200). Optimum expected cost can be estimated by replacing \(Q\) with \(\hat{Q}^*\) and the parameters by their maximum likelihood estimators. For uniform demand, closed form expression of the optimum cost is

\[
\phi_{m,\text{Unif}}^* = Cs \times \frac{\eta(Q^*)^{m+1} + (b - Q^*)^{m+1}}{b(m+1)}
\]
For exponential demand the optimal cost function could be obtained as

$$
\phi_{m, \text{Exp}}^* = C_s \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} \left[ \eta \Gamma(m-j+1, \lambda) (Q^*_{m-j+1, \lambda})^j (Q^*)^{m-j} + \Gamma(j+1, \lambda) \Gamma Q^*(j+1, \lambda) \right]
$$

(19)

where $\Gamma Q^*(m-j+1, \lambda)$ and $\Gamma Q^*(j+1, \lambda)$ are the cumulative distribution function and survival function of Gamma distribution evaluated at $Q^*$. To estimate the above optimal costs, we replace $Q^*$ by $\hat{Q}^*$ and the parameters by their maximum likelihood estimators (MLE) (see Ghosh et al., 2021) in the above expressions. In the non-parametric case, optimum cost is estimated by replacing $Q$ with respective $\hat{Q}$ in Eq. (6). We use the percentage savings in estimated optimal costs (see Keskin et al., 2021), given by

$$
\hat{\Delta} = \left( \frac{\hat{\phi}_{m, D}}{\hat{\phi}_{m, NP}} - 1 \right) \times 100
$$

to compare the performance of the non-parametric estimator with the parametric alternatives ($D \in \{\text{Unif}, \text{Exp}\}$).

Two parametric demand distributions that we consider here are thin tailed ($\text{Uniform}(0, b)$) and heavy tailed ($\text{Exponential}(\lambda)$). Thin tailed distribution don’t admit extreme observations whereas heavy tailed distributions are more likely to do so. It is also known that parametric inference is more efficient (in the MSE sense) if the data is generated from a correctly specified demand distribution. On the other hand non-parametric methods are more robust with respect to extreme values (or tail fatness) and incorrect specification of the parametric form of the distribution. Noticing these considerations, we have generated the data from $N(200, 65^2)$ distribution for fair comparison between parametric and non-parametric estimates. It is worth mentioning here that Normal is moderately tailed and not too close to either of the parametric models under comparison, viz. Uniform and Exponential. Since $\hat{Q}^*$ is asymptotically unbiased, we have generated a large random sample of size 5000. The estimated percentage savings ($\hat{\Delta}$) has been presented for all $m$ and $\eta$ in the Tables (2a-2b) in appendix A.1. All values in both the tables are negative indicating that non-parametric estimates work better than the parametric ones. Among the parametric models, Uniform is better than the Exponential, which we believe is due to heavier tail of the latter, i.e. due to higher likelihood of observing extreme demand. Variation in $\hat{\Delta}$ with respect to the cost ratio ($\eta$) does not show any pattern. However, percentage savings in cost increases with degree of severity.

### 4.2 Real data applications

In this section we study the impact of different degrees of severity on optimal order quantity and related optimal cost estimation in the SyGen-NV set-up with real data. We use Avocado import data from Haas Avocado Board (2022) and Covid-19 daily new infection rate per day for 86 countries during second corona infection surge. We assume that the data are iid samples from the corresponding demand distributions. We consider the log-transformed data as the observations are quite large. Assuming a SyGen-NV setup, we model the demand as $\text{Uniform}(0, b)$ or $\text{Exponential}(\lambda)$ for parametric estimation of the optimal order quantity. Similarly, we also use the non-parametric method described above to estimate the optimal order quantity using the same data. In both the cases, we study the estimators and compare them for different degrees of severity ($m$) and cost-ratio ($\eta$) detailed in the following two subsections.
4.2.1 Avocado import data

First we consider the import data of Avocado from Haas Avocado Board (2022). The data contains weekly arrival volume of Avocado in the United States of America market over a time-period of two and half years (January, 2020—July, 2022). The volumes (in log-scale) are considered here as iid samples on avocado demand. For parametric inference, we assume that the demand distribution is either Uniform$(0, b)$ or Exponential$(\lambda)$ and estimate the corresponding optimal order quantities.

The MLEs of $b$ and $\lambda$ are 7.89 and 7.71 respectively. Figure 2a–c shows the optimal order quantity plot against the cost ratio for different degrees of severity. Both uniform and exponential demands depict a decreasing trend in estimated optimal order quantity with cost ratio. The estimated optimal order quantity plots in Fig. 2a reveal a tendency of $\hat{Q}^*$ towards 4 (approximately) with increasing severity level (curve corresponding to $m = 10$). Exponential demand exhibits a risk taking choice for the newsvendor as $\hat{Q}^*$ increases with $m$.

The non-parametric estimates of $Q^*$, on the other hand exhibits an overall decreasing trend within a small interval compared to the parametric counterparts. The patterns are not very smooth as in the parametric cases due to restricted sample size. Comparing the percentage gain ($\Delta$) in non-parametric estimator of optimal cost given in tables (3a-3b), we observe that the non-parametric estimators provide lower cost and hence is more useful. Also, among the two parametric demand distributions, Uniform performs better in percentage cost savings.

4.2.2 Covid-19: test kit demand

In this section we analyse data set containing number of tests carried out to detect Corona Virus infection during the second wave of pandemic (01/03/2021–30/07/2021). The data were obtained from Ritchie et al. (2020) and we use the log-transformed number of tests for 86 countries during the said period as iid samples on demand of test kits. We aim to determine the optimal number of test kits that would be required if such a time appears again based on the iid sample. For Uniform$(0, b)$ demand, MLE of $b$ is $\hat{b} = 8.54$. Here, the estimated optimal order quantities show a decreasing pattern in cost ratio with a tendency to $\hat{b}/2 = 4.25$ (approximately) for increasing severity (curve corresponding to $m = 10$). In case of Exponential$(\lambda)$ demand, the estimated optimal order quantity plots show a decreasing pattern with cost ratio, but for higher severity ($m$) the estimates increase sharply compared to the other two cases. In case of non-parametric estimators of the optimal order quantity the parameter estimates show an unsmooth decreasing pattern with asymptotic order quantity estimate seemingly close to 6.25 except for $m = 3$.

Non-parametric estimators of the optimal cost outperforms both the parametric counterparts. The tables (4a–4b) show that percentage gain are all highly negative for every $\eta$ and $m$. In fact for a given $\eta$, the percentage gains improve in favour of non-parametric estimator with $m$. In other words, non-parametric estimator performs better than its parametric counterparts irrespective of the magnitude of cost ratio and severity level.

5 Discussion

In this paper we have discussed non-parametric estimation of the optimal order quantity in case of a general newsvendor problem, where the severity of the losses are much more than merely the quantity lost. Major contributions of this paper are two-fold. First we have
constructed a non-parametric estimation method for the optimal order quantity in the SyGen-NV problem with power type shortage and excess. Secondly, we have studied the properties and performances of the estimators of the optimal order quantities.

Our contribution in the non-parametric estimation of the optimal order quantity starts with formulation of an estimating equation from the first order condition using uncensored demand data. We have presented strong consistency of the estimating function and its asymptotic distribution has been derived. Further, we have established feasibility of solution to the estimating equation by establishing existence of the zeroes of the random function in almost sure sense. We have also proven the strong consistency of the estimated optimal order quantity.

The theoretical results in this paper has been supported by an exhaustive set of simulation experiments and real data analysis. In particular, we have considered known uniform and exponential as true demand distributions. The distribution of the estimated optimal order quantities suggests that odd and even order of severity influences the estimates differently for uniform demand, whereas for exponential demand, the estimate increases uniformly with severity. Comparing the mean square errors for different sample sizes, severity and cost-ratio, it has been found that the estimators perform well in the MSE sense when severity is high in case of uniform demand and the opposite for exponential distribution.

To show how well non-parametric estimators work, we have considered one synthetic data set of 5000 observations simulated from $N(200, 65^2)$ and two real data sets, viz. avocado import to USA and number of Covid-19 tests carried out in 86 countries during second wave of the pandemic (both in log-scale). Analysis of simulated data show that non-parametric method outperforms the parametric alternatives across all cost-ratio and severity values. The results derived from the analysis of real data sets show that estimated optimal order quantities increase with severity in exponential demand whereas the same tends to stabilise to a constant $(b/2)$ with increase in $m$ for uniform demand. In other words, exponential demand leads to an aggressive or risk taking newsvendor whereas uniform demand reflects a risk neutral newsvendor. We argue that this observed differences between uniform and exponential cases (order quantity or cost) happens due to the prospect of selling more in case of exponential model as it admits extreme demand. Uniform demand resembles, on the other hand, a risk neutral choice for the newsvendor since it is equally informative (or non-informative) about higher or lower demand and does not allow extreme observations. $Q^*$ decreases with $m$ in this case and the newsvendor attempts to sell at an average level $(b/2)$ when severity is very high, neglecting the role of $\eta$.

In case of non-parametric analysis, the optimum order quantity estimates for both avocado and Covid-19 data shows not very smooth patterns. This lack of regularity in Figs. 2c and 3c could be argued as an aftermath of poor density estimation caused by the restricted sample size (see Fig. 1a and b).

We conclude the paper with comments on future scope of research. A natural extension of the SyGen-NV problem would be to consider asymmetric weight functions for shortage and excess. Complexity arises due to different dimensions of the two costs as a result of asymmetric weighing. Baraiya and Mukhoti (2019) discussed, in an unpublished manuscript, selection of weights so that the shortage and excess costs remain comparable. However, estimation of optimal order quantity in such asymmetric generalised newsvendor problem remains open.

Acknowledgements The authors would like to thank the anonymous referees for their valuable comments, which has been very helpful in improving the manuscript. Work of first author was supported by INSPIRE Fellowship Grant, Department of Science and Technology, Govt. of India (Grant No. 190728) and work of second author was supported by the Indian Institute of Management Indore SEED grant (Grant No. SM/09/2019-20).
The authors would also like to thank Dr. Abhirup Banerjee, Institute of Biomedical Engineering, University of Oxford for helpful suggestions on the simulation experiments.

Appendix

A.1 Tables

See Tables 1, 2, 3 and 4.

| Table 1 | Table of notations |
|---------|---------------------|
| $X$     | Random demand       |
| $\mathcal{X}$ | Compact support of Demand distribution |
| $G$     | Probability distribution of Demand |
| $E_G$   | expectation with respect to $G$ |
| $\mathcal{B}^+$ | Borel Algebra over $\mathcal{R}^+$ |
| $\mathcal{F}$ | $\sigma$-algebra defined over $\Omega$ |
| $P$     | Probability measure |
| $C_e$   | Excess cost per unit |
| $C_s$   | Shortage cost per unit |
| $Q^*$   | Optimal order quantity |
| $C(Q, X)$ | Cost function for classical newsvendor |
| $P_m(Q, X)$ | Polynomial in $Q$ and $X$ of degree $m$ |
| $a.s. \to$ | Almost sure convergence |
| $\mathcal{L} \to$ | Convergence in Distribution |
| $a.e$   | Almost everywhere |
| $S_Q$   | $\{\omega \in \Omega \mid X(\omega) \in (0, Q)\}$ |
| $S'_Q$  | Complement of $S_Q$ ($\mathcal{X}\setminus S_Q$) |
| $\mathbb{I}(S_Q)$ | an indicator function over the set $S_Q$ |
| $\delta_j$ | $j^{th}$ partial raw moment ($\int_{S_Q} X^{j} dG$) |
| $\mu'_j$ | $j^{th}$ raw moment of $X$ ($\int_{\mathcal{X}} X^{j} dG$) |
| $\mathcal{X}$ | Historical demand data ($X_1, X_2, \ldots, X_n$) |
| $\phi_m$ | Optimal cost function |
| $U^*$   | Set of optimal order quantities |
| $\hat{\phi}_m$ | Estimated optimal cost function |
| $\hat{U}^*$ | Set of estimated optimal order quantities |
| $d_j$   | $j^{th}$ sample partial raw moment ($\frac{1}{n} \sum_{i=1}^{n} X_i^{j} I(X_i \leq Q)$) |
| $m'_j$  | $j^{th}$ sample raw moment ($\frac{1}{n} \sum_{i=1}^{n} X_i^{j}$) |
| $\hat{Q}_n^*$ | Non parametric estimator of optimal order quantity |
| $\hat{Q}^*$ | Set of distinct solutions to the estimating equation |
| $Q^*$   | Set of distinct solutions of First order condition |
### Table 1 continued

| $\hat{Q}^*$ | Largest member in the set $\{\hat{Q}^*\}$ |
|-------------|------------------------------------------|
| $\eta$     | Excess-to-shortage cost ratio $\left(\frac{C_e}{C_s}\right)$ |
| $\gamma Q^*(m-j+1, \lambda)$ | Cumulative Distribution Function (CDF) of Gamma distribution evaluated at $Q^*$ |
| $\Gamma Q^*(j+1, \lambda)$ | Survival function of Gamma distribution evaluated at $Q^*$ |

### Table 2

Percentage gain in non-parametric estimate of optimal cost over Uniform and Exponential alternatives with Normal(200, 65²) data

| $\eta$ | $m = 2$ | $m = 3$ | $m = 4$ | $m = 5$ | $m = 10$ |
|--------|---------|---------|---------|---------|----------|
| 0.25   | -54.98  | -22.62  | -328.48 | -220.94 | -778.49  |
| 0.45   | -40.13  | -77.04  | -249.82 | -167.27 | -11.51   |
| 0.65   | -52.44  | -71.98  | -163.67 | -397.73 | -422.99  |
| 0.85   | -90.43  | -68.11  | -206.91 | -178.43 | -465.26  |
| 1.05   | -34.38  | -166.53 | -248.44 | -129.57 | -461.06  |
| 1.25   | -65.42  | -223.31 | -213.25 | -324.11 | -297.64  |
| 1.45   | -80.11  | -159.23 | -245.71 | -264.08 | -482.44  |
| 1.65   | -28.14  | -65.71  | -174.67 | -322.10 | -657.93  |
| 1.85   | -54.03  | -95.61  | -236.74 | -134.23 | -439.38  |

(a) $\hat{\Delta}$ for Uniform demand

| $\eta$ | $m = 2$ | $m = 3$ | $m = 4$ | $m = 5$ | $m = 10$ |
|--------|---------|---------|---------|---------|----------|
| 0.25   | -818.66 | -207.53 | -1.4E+04| -2.7E+04| -6.9E+08|
| 0.45   | -658.28 | -470.89 | -1.3E+04| -3.6E+04| -1.3E+09|
| 0.65   | -510.55 | -670.18 | -1.6E+04| -3.8E+04| -1.4E+09|
| 0.85   | -438.03 | -581.97 | -1.2E+04| -3.8E+04| -1.4E+09|
| 1.05   | -320.22 | -834.53 | -1.1E+04| -3.5E+04| -1.1E+09|
| 1.25   | -307.7  | -778.99 | -1.1E+04| -2.8E+04| -8.6E+08|
| 1.45   | -311.49 | -661.01 | -1.1E+04| -3.2E+04| -1.0E+09|
| 1.65   | -206.11 | -545.98 | -6.4E+03| -3.0E+04| -1.2E+09|
| 1.85   | -210.83 | -607.93 | -6.2E+03| -2.9E+04| -4.2E+08|

(b) $\hat{\Delta}$ for Exponential demand

### Table 3

Percentage gain in non-parametric estimate of optimal cost over Uniform and Exponential alternatives for AVOCADO data

| $\eta$ | $m = 2$ | $m = 3$ | $m = 4$ | $m = 5$ | $m = 10$ |
|--------|---------|---------|---------|---------|----------|
| 0.25   | -3.8E+04| -2.6E+06| -8.3E+06| -3.8E+07| -8.2E+12|
| 0.45   | -3.7E+04| -2.3E+06| -7.5E+06| -5.1E+07| -1.2E+13|
| 0.65   | -3.3E+04| -2.2E+06| -7.1E+06| -4.9E+07| -2.7E+13|
| 0.85   | -3.1E+04| -2.1E+06| -6.6E+06| -5.1E+07| -4.8E+13|
| 1.05   | -2.8E+04| -2.0E+06| -6.2E+06| -5.2E+07| -5.4E+13|

### A.2 Figures

See Figs. 1, 2, 3, 4, 5, 6 and 7.
Table 3  continued

| $\eta$ | $m = 2$  | $m = 3$  | $m = 4$  | $m = 5$  | $m = 10$ |
|--------|----------|----------|----------|----------|----------|
| 1.25   | -2.7E+04 | -1.7E+06 | -5.3E+06 | -5.3E+07 | -5.8E+13 |
| 1.45   | -2.5E+04 | -1.4E+06 | -4.5E+06 | -4.8E+07 | -5.6E+13 |
| 1.65   | -2.3E+04 | -1.3E+06 | -4.1E+06 | -4.9E+07 | -5.9E+13 |
| 1.85   | -1.7E+04 | -1.2E+06 | -4.0E+06 | -4.9E+07 | -6.0E+13 |

(b) $\hat{\Delta}$ for Avocado data with Exponential demand

| $\eta$ | $m = 2$  | $m = 3$  | $m = 4$  | $m = 5$  | $m = 10$ |
|--------|----------|----------|----------|----------|----------|
| 0.25   | -9.7E+05 | -5.2E+07 | -7.5E+09 | -1.2E+11 | -1.8E+22 |
| 0.45   | -6.7E+05 | -6.7E+07 | -4.9E+09 | -1.6E+11 | -2.2E+22 |
| 0.65   | -4.8E+05 | -6.8E+07 | -3.9E+09 | -1.5E+11 | -4.5E+22 |
| 0.85   | -3.8E+05 | -6.4E+07 | -3.2E+09 | -1.5E+11 | -7.2E+22 |
| 1.05   | -3.0E+05 | -5.9E+07 | -2.7E+09 | -1.5E+11 | -7.7E+22 |
| 1.25   | -2.6E+05 | -5.0E+07 | -2.2E+09 | -1.4E+11 | -7.8E+22 |
| 1.45   | -2.2E+05 | -4.2E+07 | -1.7E+09 | -1.3E+11 | -7.3E+22 |
| 1.65   | -1.9E+05 | -3.8E+07 | -1.5E+09 | -1.3E+11 | -7.3E+22 |
| 1.85   | -1.3E+05 | -3.5E+07 | -1.4E+09 | -1.2E+11 | -7.2E+22 |

Table 4  Percentage gain in non-parametric estimate of optimal cost over Uniform and Exponential alternatives for Covid-19 test data

(a) $\hat{\Delta}$ for Covid-19 data with Uniform demand

| $\eta$ | 2    | 3    | 4    | 5    | 10   |
|--------|------|------|------|------|------|
| 0.25   | -3.0E+02 | -6.0E+02 | -2.1E+03 | -3.0E+03 | -4.5E+04 |
| 0.45   | -3.5E+02 | -7.2E+02 | -2.2E+03 | -3.5E+03 | -9.9E+04 |
| 0.65   | -3.7E+02 | -7.5E+02 | -2.2E+03 | -3.7E+03 | -1.3E+05 |
| 0.85   | -3.6E+02 | -7.8E+02 | -2.2E+03 | -3.8E+03 | -1.4E+05 |
| 1.05   | -3.6E+02 | -6.2E+02 | -2.2E+03 | -3.8E+03 | -1.4E+05 |
| 1.25   | -3.5E+02 | -5.4E+02 | -2.2E+03 | -3.8E+03 | -1.4E+05 |
| 1.45   | -3.5E+02 | -5.5E+02 | -2.1E+03 | -3.7E+03 | -1.4E+05 |
| 1.65   | -3.4E+02 | -5.6E+02 | -2.1E+03 | -3.8E+03 | -1.4E+05 |
| 1.85   | -3.3E+02 | -4.8E+02 | -2.1E+03 | -3.7E+03 | -1.4E+05 |

(b) $\hat{\Delta}$ for Covid-19 data with Exponential demand

| $\eta$ | 2    | 3    | 4    | 5    | 10   |
|--------|------|------|------|------|------|
| 0.25   | -6.3E+03 | -7.2E+03 | -7.5E+05 | -3.1E+06 | -9.3E+12 |
| 0.45   | -4.9E+03 | -1.2E+04 | -5.7E+05 | -3.5E+06 | -1.7E+13 |
| 0.65   | -4.0E+03 | -1.3E+04 | -4.9E+05 | -3.6E+06 | -2.0E+13 |
| 0.85   | -3.4E+03 | -1.3E+04 | -4.4E+05 | -3.5E+06 | -1.9E+13 |
| 1.05   | -2.9E+03 | -1.1E+04 | -3.9E+05 | -3.4E+06 | -1.8E+13 |
| 1.25   | -2.6E+03 | -9.4E+03 | -3.6E+05 | -3.2E+06 | -1.8E+13 |
| 1.45   | -2.3E+03 | -9.4E+03 | -3.3E+05 | -3.1E+06 | -1.7E+13 |
| 1.65   | -2.1E+03 | -9.3E+03 | -3.1E+05 | -3.0E+06 | -1.6E+13 |
| 1.85   | -1.9E+03 | -8.0E+03 | -2.9E+05 | -2.9E+06 | -1.6E+13 |
Fig. 1  Density plots of Avocado and Covid test data (in log scale)

(a) Avocado demand  
(b) Covid tests

Fig. 2  Estimated optimal order quantities ($\hat{Q}^*$) for Avocado between 2020-22

(a) Uniform$(0,b)$ demand  
(b) Exponential$(\lambda)$ demand 
(c) Continuous demand
Fig. 3  Estimated optimal order quantities ($\hat{Q}^*$) for Covid-19 test data
Fig. 4  Boxplot of estimated order quantity for different degrees of severity \( (m) \) for Uniform demand
Fig. 5 MSE of estimated order quantity for different degrees of severity ($\eta$) for Uniform demand
Fig. 6 Boxplot of estimated order quantity for different degrees of severity ($m$) for Exponential demand
Fig. 7 MSE of estimated order quantity for different degrees of severity (m) for Exponential demand
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[Note: This reference list includes a mix of classic and recent works in operations research, covering topics from demand estimation to optimization models and decision theory, showcasing the breadth and depth of the field.]
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