EQUIVALENCE POSTULATE AND THE QUANTUM POTENTIAL OF TWO FREE PARTICLES

MARCO MATONE

Department of Physics "G. Galilei" – Istituto Nazionale di Fisica Nucleare
University of Padova, Italy

Commutativity of the diagram of the maps connecting three one–particle state, implied by the Equivalence Postulate (EP), gives a cocycle condition which unequivocally leads to the quantum Hamilton–Jacobi equation. Energy quantization is a direct consequence of the local homeomorphicity of the trivializing map. We review the EP and show that the quantum potential for two free particles, which depends on constants which may have a geometrical interpretation, plays the role of interaction term that admits solutions which do not vanish in the classical limit.

1. The Equivalence Postulate

Let us consider two one–dimensional one–particle state. The Equivalence Postulate (EP) is the condition that the coordinate transformation

$$S^b_0(q^b) = S^a_0(q^a),$$

(1)

be well–defined for any pair of states. $S_0$ is also characterized by the condition that in a suitable limit reduces to the Hamiltonian characteristic function (also called reduced action) $S^{cl}_0$. Eq.(1) implies that $\mathcal{W}(q) \equiv V(q) - E$

$$\mathcal{W}^a(q^a) \rightarrow \mathcal{W}^b(q^b) = \left(\partial q^a q^b\right)^2 \mathcal{W}^a(q^a) + (q^a; q^b),$$

(2)

where, due to the commutative diagram of maps

\[ a \quad \rightarrow \quad \downarrow \]
\[ \quad \downarrow \quad \rightarrow \quad c \]

(3)

the unknown term $(q^a; q^b)$ must satisfy the cocycle condition

$$(q^a; q^c) = \left(\partial q^a q^b\right)^2 (q^a; q^b) + (q^b; q^c).$$

(4)

It is well–known that this is satisfied by the Schwarzian derivative. However, it turns out that it is essentially the unique solution, that is we have

**Theorem 1.** Eq.(4) defines the Schwarzian derivative up to a multiplicative constant and a coboundary term.
Since the differential equation for $S_0$ should depend only on $\partial^k_q S_0$, $k \geq 1$, it follows that the coboundary term must be zero, so that

$$ (q^a ; q^b) = \frac{-\beta^2}{4m} (q^a , q^b) ,$$

(5)

where $\{ f(q), q \} = f''' / f' - 3(f''/f')^2 / 2$ is the Schwarzian derivative and $\beta$ is a nonvanishing constant that we identify with $\hbar$. As a consequence, $S_0$ satisfies the Quantum Stationary Hamilton–Jacobi Equation (QSHJE)

$$ \frac{1}{2m} \left( \frac{\partial S_0(q)}{\partial q} \right)^2 + V(q) - E + \frac{\hbar^2}{4m} \{ S_0, q \} = 0.$$

(6)

\[ \psi = S_0^{-1/2} \left( A e^{-S_0} + B e^{i S_0} \right) \] solves the Schrödinger Equation (SE)

$$ \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial q^2} + V \right) \psi = E \psi .$$

(7)

The ratio $w = \psi^D / \psi$ of two real linearly independent solutions of (7) is, in deep analogy with uniformization theory, the *trivializing map* transforming any $W$ to $W_0 \equiv 0$. This formulation, proposed in collaboration with Faraggi, extends to higher dimension and to the relativistic case as well.

Let $q_{-/+}$ be the lowest/highest $q$ for which $W(q)$ changes sign, we have

\[ \text{Theorem 2. If} \]

$$ V(q) - E \geq \begin{cases} P^2 > 0, & q < q_-, \\ P^2 > 0, & q > q_+ , \end{cases},$$

(8)

then $w$ is a local self–homeomorphism of $\hat{\mathbb{R}}$ iff Eq.(7) has an $L^2(\mathbb{R})$ solution.

Since the QSHJE is defined iff $w$ is a local self–homeomorphism of $\hat{\mathbb{R}}$, it follows that energy quantization is implied by the QSHJE itself.

Let us now review the derivation of Eqs.(1)–(6). We first look for a coordinate transformation identifying two classical one–dimensional one–particle state. We could tentatively impose the apparently harmless condition for the classical reduced action to transform as a scalar

$$ S_{0}^{c \, b}(q^b) = S_{0}^{c \, a}(q^a) ,$$

(9)

so that $q^a \rightarrow q^b = S_{0}^{d \, b \, -1} \circ S_{0}^{c \, a}(q^a)$. However, there is an inconsistency which arises for all possible pairs of states. In particular, since for a free particle of vanishing energy the reduced action is a constant, it follows that even if the transformation is well–defined, the same becomes inconsistent once the two actions are described in a frame which is at rest with respect to one of them. Hence, in Classical Mechanics (CM), the equivalence under
coordinate transformations requires choosing a frame in which no particle is at rest. In order to make coordinate equivalence a frame independent concept, we postulate that Eq. (1) is always defined. That is given two one-particle state with reduced actions \( S_0 \) and \( S'_0 \), it always exists the “\( v \)-map” \( q \to q' \) defined by \( S'_0(q') = S_0(q) \). Let \( \mathbf{H} \) be the space of all possible \( \mathcal{W} \). This condition is essentially the same of imposing the EP\(^1\).

For each pair \( \mathcal{W}^a, \mathcal{W}^b \in \mathbf{H} \), there exists a \( v \)-transformation such that
\[
\mathcal{W}^a(q) \rightarrow \mathcal{W}^{av}(q') = \mathcal{W}^b(q').
\]

The fact that one point cannot be diffeomorphic to a line is essentially the reason why the EP excludes the existence of states corresponding to a point in phase space; that is the EP implies a sort of nonlocalization in phase space which is reminiscent of the Heisenberg uncertainty relation. Another way to see that the EP cannot be implemented in CM is to note that the two Classical Stationary HJ Equations (CSHJE)
\[
\frac{1}{2m} \left( \frac{\partial S^{\mathcal{W}a}_0(q^a)}{\partial q^a} \right)^2 + \mathcal{W}^a(q^a) = 0, \quad \frac{1}{2m} \left( \frac{\partial S^{\mathcal{W}b}_0(q^b)}{\partial q^b} \right)^2 + \mathcal{W}^b(q^b) = 0,
\]
and Eq.(9) give \( \mathcal{W}^b(q^b) = (\partial_{q^b} q^a)^2 \mathcal{W}^a(q^a) \). It follows that the state corresponding to \( \mathcal{W}^0 \equiv 0 \) is a fixed a point in \( \mathbf{H} \), that is \( \mathcal{W}^0 \to \partial_{q^b} q^a \mathcal{W}^0 \equiv 0 \).

The only way to eliminate this fixed point is to admit an inhomogeneous term in the transformation properties of \( \mathcal{W} \), as in Eq.(2). On the other hand, we saw that the transformation properties of \( \mathcal{W} \) are fixed by the CSHJE, so to make (1) consistent with (2), we must modify the CSHJE by adding a term \( Q(q) \) that for the time being is completely arbitrary, that is
\[
\frac{1}{2m} \left( \frac{\partial S_0(q)}{\partial q} \right)^2 + \mathcal{W}(q) + Q(q) = 0.
\]

Eqs.(1)(2) and (12) give \( \mathcal{W}^b(q^b) + Q^b(q^b) = (\partial_{q^b} q^a)^2 \mathcal{W}^a(q^a) + Q^a(q^a) \), and
\[
Q^a(q^a) \rightarrow Q^b(q^b) = (\partial_{q^b} q^a)^2 Q^a(q^a) - (q^a; q^b).
\]

The main steps in deriving theorem 1 are the two lemmas\(^1\)

- If \( \gamma(q) \equiv \frac{\partial S_0}{\partial q} \), then, up to a coboundary term, Eq.(4) implies
  \[
  (\gamma(q^a); q^b) = (q^a; q^b), \quad (q^a; \gamma(q^b)) = \left( \partial_{q^b} \gamma(q^b) \right)^{-2} (q^a; q^b).
  \]

- If \( q^a = q^b + \epsilon^{ab}(q^b) \), the unique solution of Eq.(4), depending only on the first and higher derivatives of \( q^a \), is
  \[
  (q^a; q^b) = c_1 \epsilon^{ab} q^m(q^b) + O^{ab}(\epsilon^2), \quad c_1 \neq 0.
  \]
Let us review the proof of the second lemma and theorem 1. Since \( (q^a; q^b) \) should depend only on \( \partial_q^b q^a \), \( k \geq 1 \), we have

\[
(q + \epsilon f(q); q) = c_1 \epsilon f^{(k)}(q) + O(\epsilon^2),
\]

(16)

where \( q^a = q + \epsilon f(q), q \equiv q^b \) and \( f^{(k)} \equiv \partial_q^b f, k \geq 1 \). Note that by (14)

\[
(Aq + \epsilon Af(q); Aq) = (q + \epsilon f(q); Aq) = A^{-2}(q + \epsilon f(q); q),
\]

(17)

on the other hand, setting \( F(Aq) = Af(q) \), by (16) \( (Aq + \epsilon Af(q); Aq) = (Aq + \epsilon F(Aq); Aq) = c_1 \epsilon \partial^k_{Aq} F(Aq) = A^{1-k} c_1 \epsilon f^{(k)}(q), \) so that \( k = 3 \). One sees that \( (Aq + \epsilon Af(q); Aq) \) at order \( \epsilon^n \) is a sum of terms of the form

\[
c_{i_1\ldots i_n} \partial^{i_1}_{Aq} \epsilon F(Aq) \cdots \partial^{i_n}_{Aq} \epsilon F(Aq) = c_{i_1\ldots i_n} \epsilon^n A^{n-k} \sum f^{(i_1)}(q) \cdots f^{(i_n)}(q),
\]

and by (17) \( \sum_{k=1}^n i_k = n + 2 \). On the other hand, since \( (q^a; q^b) \) depends only on \( \partial_q^b q^a \), \( k \geq 1 \), we have \( i_k \geq 1, k \in [1, n] \), so that either \( i_k = 3, i_j = 1, j \in [1, n] j \neq k \), or \( i_k = i_j = 2, i_i = 1, l \in [1, n] l \neq k, l \neq j \). Hence

\[
(q + \epsilon f(q); q) = \sum_{n=1}^{\infty} \epsilon^n \left( c_n f^{(3)} f^{(1)^{n-1}} + d_n f^{(2)^2} f^{(1)^{n-2}} \right), \quad d_1 = 0.
\]

Inserting the expansion (18) in (4), we obtain

\[
c_n = (-1)^{n-1} c_1, \quad d_n = \frac{3}{2} (-1)^{n-1} (n - 1) c_1,
\]

(19)

which are the coefficients in the power expansion of \( c_1 \{q + \epsilon f(q), q\} \).

In deriving the equivalence of states we considered the case of one–
particle states with identical masses.\(^1\) The generalization to the case with
different masses is straightforward. In particular, the right hand side of
Eq.(2) gets multiplied by \( m_b/m_a \), so that the cocycle condition becomes

\[
m_a(q^a; q^c) = m_a \left( \partial_q^c q^b \right)^2 (q^a; q^b) + m_b(q^b; q^c),
\]

(20)

explicitly showing that the mass appears in the denominator and that it
refers to the label in the first entry of \( (\cdot; \cdot) \), that is

\[
(q^a; q^b) = - \frac{\hbar^2}{4m_a} (q^a; q^b),
\]

(21)

from which the QSHJE (6) follows almost immediately.\(^1\)

The EP leads to the introduction of length scales,\(^1,2\) a fact related to
the nontriviality of the quantum potential, even in the case of \( \mathcal{W}^0 \). We note
that also \( \mathcal{S}_0 \), as follows by the EP, is never trivial, in particular

\[
\mathcal{S}_0 \neq \text{const}, \quad \forall \mathcal{W} \in H.
\]

(22)

The QSHJE (6), first investigated by Floyd in a series of important
papers,\(^3\) has been recently studied and reviewed by several authors.\(^4,5\)
2. The two–particle model

The real solution of the QSHJE (6) is $e^{\frac{i}{\hbar}S_0(\delta)} = e^{i\alpha \frac{w+i\ell}{w-i\alpha}} \delta \equiv \{\alpha, \ell\}$, with $\alpha \in \mathbb{R}$ and $\ell = \ell_1 + i\ell_2$, $\ell_1 \neq 0$, are integration constants. The condition $\ell_1 \neq 0$ is necessary for $S_0$ and the quantum potential $Q$ to be well–defined.

The formulation has a manifest duality between real pairs of linearly independent solutions, i.e. the wave–function itself, in our formulation the relevant formulas contain the linear combination $\psi^D + i\ell\psi$. Since $\ell_1 \neq 0$, $\psi^D$ and $\psi$ appear always in pair. So, Legendre duality, nontriviality of $S_0$ and $Q$ are deeply related features which are direct consequences of the EP. In turn, these properties imply the appearance of fundamental constants such as the Planck length. The simplest way to see this is to consider the SE in the trivial case, that is $\partial^2_{q^0}\psi = 0$, so that $\psi^D = q^0$, $\psi = 1$ and the typical combination reads $q^0 + i\ell_0$, implying that $\ell_0 \equiv \ell$ should have the dimension of a length. The fact that $\ell$ has the dimension of a length is true for any state. Since $\ell_0$ appears in the QSHJE with $W^0 \equiv 0$, the system does not provide any dimensional quantity, so that we have to introduce some fundamental length. The appearance of fundamental constants also arises in considering the limits $\hbar \to 0$ and $E \to 0$ in the case of a free particle. So, for example, a consistent expression for the quantum potential associated to the trivial state $W^0$, which vanishes as $\hbar \to 0$, is

$$Q^0 = \frac{\hbar}{4m} \{S_0^0, q^0\} = -\frac{\hbar^3 G}{2mc^3} \frac{1}{|q^0 - i\lambda_p|^4}, \quad (23)$$

where $\lambda_p = \sqrt{\hbar G/c^3}$ is the Planck length. However, in considering the classical limit of the reduced action one should include the gravitational contribution. So, for example, it is clear that also at the classical level the reduced action for a pair of “free” particles should include the Newton potential. This may be related to the above mentioned appearance of fundamental constants in the QSHJE. Related to this is the Floyd observation that in the classical limit there is a residual indeterminacy depending on the integration constants. Thus we see that the classical limit may in fact lead to some effect which is of quantum origin even if $\hbar$ does not appear explicitly. We also note that, in principle, the Planck constant may appear in macroscopic phenomena. This indicates that it is worth studying the structure of the quantum potential also at large scales.

It seems that the fundamental properties of $Q$ have not yet fully been investigated because the usual solutions one finds are essentially trivial. This is due to a clearly unsatisfactory identification, that may lead to some
inconsistency, of \( \text{Re}e^{\mp S_0} \) with the wave–function. As noticed by Floyd,\(^3\) if \( \text{Re}e^{\mp S_0} \) solves the SE, then the wave–function will in general have the form \( R(Ae^{-\imath \hbar S_0} + Be^{\imath \hbar S_0}) \). This simple remark has important consequences. So, for example, note that a real wave–function, such as the one for bound states, simply implies \(|A| = |B|\) rather than \( S_0 = 0 \). As Einstein noticed in a letter to Bohm, the latter would imply that a quantum particle in a box is at rest and starts moving in the classical limit. Therefore, besides the mathematical consistency, identifying the wave–function with \( \text{Re}e^{\mp S_0} \) cannot in general lead to a quantum analog of the reduced action. This change in the definition of \( S_0 \) implies a new view of \( Q \) which needs to be further investigated. In this respect we note that \( Q \) provides particle’s response to an external perturbation. For example, in the case of tunnelling, where according to the standard definition \( S_0 \) would be vanishing, the attractive nature of \( Q \) guarantees the reality of the conjugate momentum.\(^1\) As a consequence, the role of this intrinsic energy, which is a property of all forms of matter, should manifest itself through effective interactions depending on the above fundamental constants.

It is therefore natural to consider the so called two–particle model\(^2\) consisting of two free particles in the three dimensional space. This provides a simple physical model to investigate the structure of the interaction provided by \( Q \). The QSHJE decomposes in equations for the center of mass and for the relative motion. The latter is the QSHJE

\[
\frac{1}{2m}(\nabla S_0)^2 - E - \frac{\hbar^2}{2m} \frac{\Delta R}{R} = 0, \quad \nabla \cdot (R\nabla S_0) = 0, \tag{24}
\]

where \( r = r_1 - r_2 \) and \( m = \frac{m_1 m_2}{m_1 + m_2} \). Due to the quantum potential the QSHJE has solutions in which the relative motion is not free as in the classical case. Also note that the quantum potential is negative definite.

Since \( \psi = \text{Re}e^{\mp S_0} \) is a solution of the SE, we have \( S_0 = \frac{\hbar}{2\imath} \ln(\psi/\psi) \), so that

\[
(\nabla S_0)^2 = -\frac{\hbar^2}{4|\psi|^4} (\bar{\psi} \nabla \psi - \psi \nabla \bar{\psi})^2, \tag{25}
\]

where \( \psi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \sum_{j=1}^{2} c_{lmj} R_{klj}(r) Y_{lm}(\theta, \phi) \), with \( Y_{lm} \) the spherical harmonics and \( R_{klj} \) the solutions of the radial part of the SE.\(^2\)

As \( m = l \to \infty \) \( P^l_i \propto \sin^l \theta \) vanishes unless \( \theta = \pi/2 \), and the motion is on a plane as in the classical orbits. However, since \( \lim_{l \to \infty} \partial_\theta P^l_i(\cos \theta) = 0 \), we have \( \lim_{m \to \infty} \partial_\theta P^m_l(\cos \theta) = 0 \). Thus, considering solutions with \( c_{lmj} \neq 0 \) only for sufficiently large \( m \) and \( l \), we have

\[
\nabla \psi = \sum_{\{lmj\}} \left( c_{lmj} R_{klj}' Y_{lm}, 0, \frac{i}{r} c_{lmj} R_{klj} m Y_{lm} \right). \tag{26}
\]
Depending on the coefficients \( c_{lmj}, (\nabla S_0)^2/2m \) may contain nontrivial terms which do not cancel as \( \hbar \to 0 \). These may arise as a deformation of the classical kinetic term, which includes the centrifugal term. The \( c_{lmj} \), which may depend on fundamental constants, fix the structure of the possible interaction in the two-particle model. The \( c_{lmj} \) may be related to some boundary conditions implied by the geometry and the matter content of the three-dimensional space. This would relate the fundamental constants to possible collective effects which may depend on cosmological aspects.

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