A Fractional Power for Dunkl Transforms in $L^2(\mathbb{R}^N, \omega_k(x)dx)$.

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Abstract

A new fractional version of the Dunkl transform for real order $\alpha$ is obtained. An integral representation, a Bochner type identity and a Master formula for this transform are derived.

Keywords: Fractional Fourier transform, fractional Hankel transform, Fractional Dunkl transform, Generalized Hermite polynomials and functions, semigroups of operators.

1 Introduction

In recent years, there has been considerable interest in fractional versions of classical integral transform. For example the Fractional Fourier Transform (FFT), which may be considered as a fractional generalization of the classical Fourier Transform, becomes a remarkably powerful tool in signal processing, optics and quantum mechanics [1], [2], [3], [4] and [18]. The idea of Fractional powers of the Fourier Transform operator appeared firstly in the Wiener’s works [26]. The development of a wide-ranging modern theory, including operational formula, stems from a paper by Namias [18]. As Namias's innovative ideas and results were developed in a formal manner, they were later revisited by McBride and Kerr [17], where a mathematically rigorous account is presented for the FFT on the space $S(\mathbb{R}^N)$ of rapid descent functions. More recently, Zayed [28] used the same approach as Namias to produce more general method in order to defining fractional versions of a wider transforms class such that the fractional Hankel transform [15], [16], [19], the fractional integration and differentiation operators.

This paper deals with the construction of a fractional power of the Dunkl transform called the fractional Dunkl transform (FDT), using the Zayed’s approach [28] and the multivariable generalized Hermite function introduced by Rösler [21]. The resulting family of operators $\{D^\alpha_k\}_{\alpha \in \mathbb{R}}$ was proved to be a $C_0$-group of unitary operators on $L^2(\mathbb{R}^N, \omega_k(x)\,dx)$, with infinitesimal generator $T$. The spectral properties of $T$ is studied using the semigroup techniques. The FDT given in this paper has an integral representation which used with the analogue of the Funk-Hecke formula for $k$-spherical harmonics [27] to derive a Bochner type identity for the FDT. The Master formula of the FDT is proved and founded to be generalizing the one given by Rösler [21] in Proposition 3.10. This Master formula is used to develop a new proof of the statement (2) of the 3.4 Rösler’s theorem [21]. The contents of the present paper are as follows. In section 2, some basic definitions and results about harmonic analysis associated with Dunkl operators are collected. In section 3, the fractional Dunkl transform definition is given and then some elementary properties of this transformation are listed. The spectral properties of $T$ is studied. In section 4, the integral representation of the fractional Dunkl transform as well as the Bochner type identity and Master formula are given. In section 5, we find a subspace of $L^2(\mathbb{R}^N, \omega_k(x)dx)$ in which we define $T$ explicitly.

2 Background: Dunkl operator

In this section, we recall some notations and results on Dunkl operators, Dunkl transform, and generalized Hermite functions (see, [7], [8], [11], [20], [23]). In $\mathbb{R}^N$, we consider the standard inner product

$$\langle x, y \rangle = \sum_{k=1}^{N} x_k y_k$$

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and the norm $|x| = \sqrt{(x, x)}$. For $u \in \mathbb{R}^N \setminus \{0\}$, let $\sigma_u$ be the reflection on the hyperplane $(\mathbb{R}u)^\perp$ orthogonal to $u$. It is given by

$$\sigma_u(x) = x - 2\frac{(u, x)}{|u|^2}u.$$ \hspace{2cm} (2.1)

A root system is a finite set $R$ of nonzero vectors in $\mathbb{R}^N$ such that

$$R \cap \mathbb{R}u = \{ \pm u \} \text{ and } \sigma_u R = R,$$

for all $u \in R$. For a given root system $R$ normalized such that $\langle u, u \rangle = 2$ for all $u \in R$, we denote by $W$ the subgroup of $O(N, \mathbb{R})$ generated by the reflections $\{ \sigma_u : u \in R \}$ and for $v \in \mathbb{R}^N \setminus \bigcup_{u \in R} u > 1$, we fix the positive subsystem $R_+ = \{ u \in R/\langle u, v \rangle > 0 \}$. A function $k : R \rightarrow \mathbb{C}$ on the root system $R$ is called a multiplicity function, if for every $u \in R$ and $\sigma \in W$, $k(\sigma u) = k(u)$.

The Dunkl operators $T_\xi := T_\xi(k), \xi \in \mathbb{R}^N$, associated with the reflection group $W$ and the multiplicity function $k$ are given by

$$T_\xi f(x) := \partial_\xi f(x) + \sum_{\eta \in R_+} k(\eta) \langle \eta, x \rangle > 0 \frac{f(x) - f(\sigma_\eta x)}{\langle \eta, x \rangle},$$ \hspace{2cm} (2.2)

where $\partial_\xi$ denotes the derivative in the direction of $\xi$. Thanks to the $W$-invariance of the function $k$, this definition is independent of the choice of the positive subsystem $R_+$. For the $i$-th standard basis vector $\xi = e_i \in \mathbb{R}^N$, we use the abbreviation $T_i = T_{e_i}$. The most striking property of Dunkl operators $T_\xi$, which is the foundation for rich analytic structures with them, is the following \cite{[7]}:

**Theorem 2.1** For fixed $k$, the associated $T_\xi = T_\xi(k), \xi \in \mathbb{R}^N$ commute.

For fixed non-negative multiplicity functions $k$, let $\omega_k$ be the weight function on $\mathbb{R}^N$ defined by

$$\omega_k(x) = \prod_{\eta \in R_+} |\langle \eta, x \rangle|^{2k(\eta)}.$$ 

It is $W$-invariant and homogeneous of degree $2\gamma$, with the index

$$\gamma = \gamma(k) = \sum_{\eta \in R_+} k(\eta).$$

**Notation:** We denote by $\mathbb{Z}_+$ the set of non-negative integers. For a multi-index $\nu = (\nu_1, \ldots, \nu_N) \in \mathbb{Z}_+^N$, we write $|\nu| = \nu_1 + \cdots + \nu_N$. The $\mathbb{C}$-algebra of polynomial functions on $\mathbb{R}^N$ is denoted by $\mathcal{P} = \mathbb{C}[\mathbb{R}^N]$. It has a natural grading

$$\mathcal{P} = \bigoplus_{n \geq 0} \mathcal{P}_n,$$

where $\mathcal{P}_n$ is the subspace of homogenous polynomials of (total) degree $n$. $\mathcal{S}(\mathbb{R}^N)$ is the Schwartz space of rapidly decreasing functions on $\mathbb{R}^N$. Finally, $L^p(\mathbb{R}^N, \omega_k(x) \, dx)$ is the space of measurable functions on $\mathbb{R}^N$ such that

$$\|f\|_p = \left( \int_{\mathbb{R}^N} |f(y)|^p \omega_k(y) \, dy \right)^{\frac{1}{p}} < +\infty, \quad \text{if } 1 \leq p < +\infty.$$

The generalized Laplacian associated with $W$ and $k$, is defined by $\Delta_k := \sum_{i=1}^N T_i^2$. It is homogeneous of degree $-2$ on $\mathcal{P}$ and given explicitly by

$$\Delta_k f(x) = \Delta f(x) + 2 \sum_{\alpha \in R} k(\alpha) \left[ \frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle^2} \right].$$ \hspace{2cm} (2.3)

(Here $\Delta$ and $\nabla$ denote the usual Laplacian and gradient respectively).
For $y \in \mathbb{C}^N$, the system
\[
\begin{cases}
    T_\xi f(x,y) = \langle \xi, y \rangle f(x,y), & \text{for all } \xi \in \mathbb{R}^N, \\
    f(0,y) = 1,
\end{cases}
\]
(2.4)
admits a unique analytic solution $K(\cdot, y)$ on $\mathbb{R}^N$, which is called the Dunkl kernel. Moreover, $K(\cdot, y)$ extends to holomorphic function on $\mathbb{C}^N \times \mathbb{C}^N$ and possesses the following properties:
1) For every $x, y \in \mathbb{C}^N$, we have
\[
K(x, y) = K(y, x) \quad \text{and} \quad K(\lambda x, y) = K(x, \lambda y) \quad \text{for all } \lambda \in \mathbb{C}.
\]
(2.5)
2) For every $x, y \in \mathbb{C}^N$ and $g \in W$, we have
\[
\overline{K(x, y)} = K(x, \overline{y}) \quad \text{and} \quad K(gx, gy) = K(x, y).
\]
(2.6)
3) For all $\nu = (\nu_1, \ldots, \nu_N) \in \mathbb{Z}^N_+$, $x \in \mathbb{R}^N$ and $y \in \mathbb{C}^N$, we have
\[
|D^\nu_y K(x, y)| \leq |x|^{|\nu|_2} |x||y||\mathbb{R}|,
\]
(2.7)
and for all $x, y \in \mathbb{R}^N$:
\[
|K(ix, y)| \leq 1,
\]
(2.8)
with $D^\nu_y = \frac{\partial^\nu}{\partial y_1^{\nu_1} \cdots \partial y_N^{\nu_N}}$.

The Dunkl transform associated with $R$ and $k \geq 0$ is defined on $L^1(\mathbb{R}^N, \omega_k(x) \, dx)$ by
\[
D_k f(x) = \frac{c_k}{2^{1+N/2}} \int_{\mathbb{R}^N} f(y) K(-ix, y) w_k(y) dy,
\]
where the Mehta-type constant $c_k$ is given by
\[
c_k = \left( \int_{\mathbb{R}^N} e^{-|x|^2} w_k(x) dx \right)^{-1}.
\]
Some of the properties of the Dunkl transform are collected below [11].

**Theorem 2.2**

a) **(Riemann-Lebesgue lemma)** For all $f \in L^1(\mathbb{R}^N, \omega_k(x) \, dx)$, the Dunkl transform $D_k f$ belongs to $C_0(\mathbb{R}^N)$.

b) **(L1-INVERSION)** For all $f \in L^1(\mathbb{R}^N, \omega_k(x) \, dx)$ with $D_k f \in L^1(\mathbb{R}^N, \omega_k(x) \, dx)$,
\[
D_k^2 f = \hat{f}, \ a.e., \ where \ \hat{f}(x) = f(-x).
\]
(2.9)
c) The Dunkl transform $f \rightarrow D_k f$ is an automorphism of $\mathcal{S}(\mathbb{R}^N)$.

d) **(Plancherel Theorem)**
i) If $f \in L^1(\mathbb{R}^N, \omega_k(x) \, dx) \cap L^2(\mathbb{R}^N, \omega_k(x) \, dx)$, then $D_k f \in L^2(\mathbb{R}^N, \omega_k(x) \, dx)$ and $\|D_k f\|_2 = \|f\|_2$.

ii) The Dunkl transform has a unique extension to an isometric isomorphism of $L^2(\mathbb{R}^N, \omega_k(x) \, dx)$. The extension is also denoted by $f \rightarrow D_k f$.

In [21], Rösler constructed systems of naturally associated multivariable generalized Hermite polynomials \{\text{H}_\nu, \nu \in \mathbb{Z}^N_+\} and Hermite functions \{\text{h}_\nu, \nu \in \mathbb{Z}^N_+\},
\[
\begin{align*}
\text{H}_\nu(x) &= 2^{|\nu|} e^{-\Delta_k/4} \varphi_\nu(x), \\
\text{h}_\nu(x) &= \frac{\sqrt{c_k}}{2^{|\nu|/2}} e^{-|x|^2/2} \text{H}_\nu(x),
\end{align*}
\]
(2.10)
(2.11)
such that $\varphi_\nu \in \mathcal{P}_{|\nu|}$ and the coefficients of $\varphi_\nu$’s are real. He proved that the generalized Hermite functions \{\text{h}_\nu, \nu \in \mathbb{Z}^N_+\} form an orthonormal basis of eigenfunctions for the Dunkl operator on $L^2(\mathbb{R}^N, \omega_k(x) \, dx)$ with
\[
D_k \text{h}_\nu = (-i)^{|\nu|} \text{h}_\nu.
\]
(2.12)

### 3 Fractional Dunkl transforms in $L^2(\mathbb{R}^N, \omega_k(x) \, dx)$.

In the remainder of this paper, we denote by $R$ a root system in $\mathbb{R}^N$, $R_+$ a fixed positive subsystem of $R$ and $k$ a nonnegative multiplicity function defined on $R$. 
3.1 Definition and properties.

Consider the Hilbert space \( L^2(\mathbb{R}^N, \omega_k(x)dx) \) equipped with the orthonormal basis \( \{ h_\nu, \nu \in \mathbb{Z}_+^N \} \).

Let \( f \in L^2(\mathbb{R}^N, \omega_k(x)dx) \), then we can expand \( f \) in the orthonormal basis \( \{ h_\nu, \nu \in \mathbb{Z}_+^N \} \) as follows:

\[
f = \sum_{\nu \in \mathbb{Z}_+^N} \langle f, h_\nu \rangle h_\nu = \sum_{\nu \in \mathbb{Z}_+^N} \hat{f}_\nu h_\nu,
\]

where

\[
\hat{f}_\nu = \langle f, h_\nu \rangle = \int_{\mathbb{R}^N} f(x) \overline{h_\nu(x)} \omega_k(x) dx.
\]

From Theorem 2.2, (d), (ii) and (2.12), the identity

\[
D_k f = \sum_{\nu \in \mathbb{Z}_+^N} (-i)^{\nu} \langle f, h_\nu \rangle h_\nu
\]

holds for every \( f \in L^2(\mathbb{R}^N, \omega_k(x)dx) \), which allows us to give the following definition:

**Definition 3.1** Let \( \alpha \in \mathbb{R} \), we define the fractional Dunkl transform \( D_k^\alpha \) by

\[
D_k^\alpha f = \sum_{\nu \in \mathbb{Z}_+^N} e^{i\nu \cdot \alpha} \langle f, h_\nu \rangle h_\nu.
\]

**Remark 3.1** The limit in the second member of (3.2) exists. In fact, let us denote by \( \mathcal{P}_0(\mathbb{Z}_+^N) \) the set of finite subsets of \( \mathbb{Z}_+^N \). For every \( J \subset \mathcal{P}_0(\mathbb{Z}_+^N) \), we have

\[
\left\| \sum_{\nu \in J} e^{i\nu \cdot \alpha} \langle f, h_\nu \rangle h_\nu \right\|^2 = \sum_{\nu \in J} |\langle f, h_\nu \rangle|^2.
\]

The convergence in \( L^2(\mathbb{R}^N, \omega_k(x)dx) \) of the series

\[
\sum_{\nu \in \mathbb{Z}_+^N} e^{i\nu \cdot \alpha} \langle f, h_\nu \rangle h_\nu,
\]

is obtained by means of the Cauchy convergence test and (3.3).

We summarize the elementary properties of \( D_k^\alpha \) in the next Proposition.

**Proposition 3.1** Let \( \alpha, \beta \in \mathbb{R} \). The fractional Dunkl transform \( D_k^\alpha \) satisfies the following properties:

1) \( D_k^0 = I \), which is the identity operator;
2) \( D_k^{\pi/2} = D_k \);
3) \( D_k^\alpha \circ D_k^\beta = D_k^{\alpha+\beta} \);
4) \( D_k^{\alpha+2\pi} = D_k^\alpha \);
5) \( D_k^\alpha = \hat{I} \), where \( \hat{f}(x) = f(-x) \);
6) For all \( f \) and \( g \in L^2(\mathbb{R}^N, \omega_k(x)dx) \), \( \langle D_k^\alpha f, g \rangle = \langle f, D_k^{-\alpha} g \rangle \).

**Proof.**

1) and 4) follow immediately from (3.2).
2) Follows immediately from (3.1).
3) From (3.2), we have

\[
D_k^\alpha (D_k^\beta f) = \sum_{\nu \in \mathbb{Z}_+^N} e^{i\nu \cdot \alpha} \langle D_k^\beta f, h_\nu \rangle h_\nu
\]

\[
= \sum_{\nu \in \mathbb{Z}_+^N} e^{i\nu \cdot (\alpha+\beta)} \langle f, h_\nu \rangle h_\nu = D_k^{\alpha+\beta} f.
\]
5) From (Lemma 3.11, [21]), we conclude that
\[ e^{-|x|^2/2} h_\nu(x) = \sqrt{c_k 2^{|
u|}} \int_{\mathbb{R}^N} K(x, -2iy) \varphi_\nu(y) d_\mu(y), \]
where \( \varphi_\nu \) is a homogenous polynomials of order \(|\nu|\). Thus
\[ h_\nu(-x) = e^{i|x|^2/2} \sqrt{c_k 2^{|
u|}} \int_{\mathbb{R}^N} K(x, -2iy) \varphi_\nu(y) d_\mu(y), \]
\[ = e^{i|x|^2/2} \sqrt{c_k 2^{|
u|}} \int_{\mathbb{R}^N} K(x, 2iy) \varphi_\nu(y) d_\mu(y), \]
\[ = (-1)^{|
u|} e^{i|x|^2/2} \sqrt{c_k 2^{|
u|}} \int_{\mathbb{R}^N} K(x, 2iy) \varphi_\nu(y) d_\mu(y), \]
\[ = (-1)^{|
u|} h_\nu(x). \]

Hence,
\[ D_k^{-\pi} f = \sum_{\nu \in \mathbb{Z}_+^N} e^{-i|\nu|\pi} \langle f, h_\nu \rangle h_\nu = \sum_{\nu \in \mathbb{Z}_+^N} (-1)^{|
u|} \langle f, h_\nu \rangle h_\nu, \]
\[ = I f. \]

6) Let \( f \) and \( g \in L^2(\mathbb{R}^N, \omega_k(x) \, dx) \). It is easy to check that
\[ \langle D_k^\alpha f, g \rangle = \sum_{\nu \in \mathbb{Z}_+^N} e^{i|\nu|\alpha} \langle f, h_\nu \rangle \overline{\langle g, h_\nu \rangle} = \sum_{\nu \in \mathbb{Z}_+^N} \langle f, h_\nu \rangle e^{-i|\nu|\alpha} \langle g, h_\nu \rangle \]
\[ = \langle f, D_k^{-\alpha} g \rangle. \]

**Theorem 3.1** The family of operators \( \{D_k^\alpha\}_{\alpha \in \mathbb{R}} \) is a \( C_0 \)-group of unitary operators on \( L^2(\mathbb{R}^N, \omega_k(x) \, dx) \).

**Proof.**
From Proposition 3.1 we deduce that the family \( \{D_k^\alpha\}_{\alpha \in \mathbb{R}} \) satisfies the algebraic properties of a group:
\[ D_k^0 = I, \quad D_k^\alpha \circ D_k^\beta = D_k^{\alpha + \beta} = D_k^\beta \circ D_k^\alpha; \quad \alpha, \beta \in \mathbb{R}. \]

For the strong continuity, assume that \( f \in L^2(\mathbb{R}^N, \omega_k(x) \, dx) \). Then
\[ \|D_k^\alpha f - f\|_2^2 = \sum_{\nu \in \mathbb{Z}_+^N} |e^{i|\nu|\alpha} - 1|^2 |\langle f, h_\nu \rangle|^2. \]

For each \( \nu \in \mathbb{Z}_+^N \), we have
\[ \lim_{\alpha \to 0} |e^{i|\nu|\alpha} - 1|^2 |\langle f, h_\nu \rangle|^2 = 0, \]
\[ |e^{i|\nu|\alpha} - 1|^2 |\langle f, h_\nu \rangle|^2 \leq 4 |\langle f, h_\nu \rangle|^2. \]

Since
\[ \sum_{\nu \in \mathbb{Z}_+^N} |\langle f, h_\nu \rangle|^2 = \|f\|_2^2 < \infty, \]

then we can interchange limits and sum to get:
\[ \lim_{\alpha \to 0} \|D_k^\alpha f - f\|_2^2 = 0. \]

Hence \( \{D_k^\alpha\}_{\alpha \in \mathbb{R}} \) is a strongly continuous group of operators on \( L^2(\mathbb{R}^N, \omega_k(x) \, dx) \). In addition, by Proposition 3.1 we have for all \( f, g \in L^2(\mathbb{R}^N, \omega_k(x) \, dx) \),
\[ \langle D_k^\alpha f, g \rangle = \langle f, D_k^{-\alpha} g \rangle, \]
and therefore \((D_k^\alpha)^* = D_k^{-\alpha} = (D_k^\alpha)^{-1}\), establishing that each \(D_k^\alpha\) is unitary.

We have therefore shown that \(\{D_k^\alpha\}_{\alpha \in \mathbb{R}}\) is a \(C_0\)-group of unitary operators on \(L^2(\mathbb{R}^N, \omega_k(x)dx)\).

The infinitesimal generator \(T\) of \(\{D_k^\alpha\}_{\alpha \in \mathbb{R}}\) is defined by

\[
T : L^2(\mathbb{R}^N, \omega_k(x)dx) \ni D(T) \quad \to \quad L^2(\mathbb{R}^N, \omega_k(x)dx),
\]

\[
f \quad \to \quad Tf,
\]

where

\[
D(T) = \left\{ f \in L^2(\mathbb{R}^N, \omega_k(x)dx) : \lim_{\alpha \to 0} (1/\alpha)[D_k^\alpha f - f] \in L^2(\mathbb{R}^N, \omega_k(x)dx) \right\},
\]

\[
Tf = \lim_{\alpha \to 0} (1/\alpha)[D_k^\alpha f - f], \quad f \in D(T).
\]

### 3.2 Spectral properties of the operator \(T\).

We denote by \(\mathcal{B}(L^2(\mathbb{R}^N, \omega_k(x)dx))\), the set of all linear bounded operator in \(L^2(\mathbb{R}^N, \omega_k(x)dx)\). The resolvent set of \(T\) is the set \(\rho(T)\) consisting of all scalars \(\lambda\) for which the linear operator \(\lambda I - T\) is a 1-1 mapping from its domain \(D(\lambda I - T) = D(T)\) on to the Hilbert space \(L^2(\mathbb{R}^N, \omega_k(x)dx)\) with \((\lambda I - T)^{-1} \in \mathcal{B}(L^2(\mathbb{R}^N, \omega_k(x)dx))\).

The spectrum of \(T\) is the set \(\sigma(T)\) that is the complement of \(\rho(T)\) in \(\mathbb{C}\). The function \(R(\lambda, T) = (\lambda I - T)^{-1}\) from \(\rho(T)\) into \(\mathcal{B}(L^2(\mathbb{R}^N, \omega_k(x)dx))\) is the resolvent of \(T\).

As \(T\) is the generator of the \(C_0\)-group \(\{D_k^\alpha\}_{\alpha \in \mathbb{R}}\), some elementary properties of \(T\) and \(D_k^\alpha\) are listed in the following proposition (see [12, 13]).

**Proposition 3.2** Let \(\alpha \in \mathbb{R}\). The following properties hold.

i) If \(f \in D(T)\), then \(D_k^\alpha f \in D(T)\) and

\[
\frac{d}{d\alpha}D_k^\alpha f = D_k^\alpha T f = TD_k^\alpha f. \tag{3.4}
\]

ii) For every \(t \in \mathbb{R}\) and \(f \in L^2(\mathbb{R}^N, \omega_k(x)dx)\), one has

\[
\int_0^t D_k^\alpha f \, d\alpha \in D(T).
\]

iii) For every \(\alpha \in \mathbb{R}\), one has

\[
D_k^\alpha f - f = T \int_0^\alpha D_k^s f \, ds, \quad \text{if} \quad f \in L^2(\mathbb{R}^N, \omega_k(x)dx) \tag{3.5}
\]

\[
= \int_0^\alpha D_k^\alpha Tf \, ds, \quad \text{if} \quad f \in D(T). \tag{3.6}
\]

**Remark 3.2** If we apply the Proposition 3.2 iii) to the rescaled semigroup

\[
S(\alpha) := e^{-\lambda \alpha} D_k^\alpha; \quad \alpha \in \mathbb{R}
\]

whose generator is \(B := T - \lambda I\) with domain \(D(B) = D(T)\), we obtain for every \(\lambda \in \mathbb{C}\) and \(\alpha \in \mathbb{R}\),

\[
-e^{-\lambda \alpha} D_k^\alpha f + f = (\lambda I - T) \int_0^\alpha e^{-\lambda s} D_k^s f \, ds; \quad f \in L^2(\mathbb{R}^N, \omega_k(x)dx), \tag{3.7}
\]

\[
= \int_0^\alpha e^{-\lambda s} D_k^\alpha (\lambda I - T) f \, ds; \quad f \in D(T). \tag{3.8}
\]

Now we are interesting with the eigenvalues of \(T\) by giving an important formula relating the semigroup \(\{D_k^\alpha\}_{\alpha \in \mathbb{R}}\), to the resolvent of its generator \(T\).

**Proposition 3.3** For the operator \(T\), the following properties hold:

1) \(T\) is closed and densely defined.

2) The operator \(iT\) is self-adjoint.

3) \(\sigma(T) = \sigma_p(T) \subset i\mathbb{Z}\), and for each \(\lambda \in \mathbb{C}\setminus i\mathbb{Z}\) and for all \(f \in L^2(\mathbb{R}^N, \omega_k(x)dx)\),

\[
R(\lambda, T)f = (1 - e^{-2\pi \lambda})^{-1} \int_0^{2\pi} e^{-\lambda s} D_k^\alpha f \, ds. \tag{3.9}
\]

Here \(\sigma_p(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not injective}\}\).
The previous Proposition indicates that every point in the spectrum of \( T \) is an element of the spectrum of \( D^n \). The fact that \( T \) is closed and densely defined follows from the Hille-Yosida Theorem (see [13], p. 15).

Proof.  
1) The fact that \( T \) is closed and densely defined follows from the Hille-Yosida Theorem [13], p. 32] that \( T \) is skew-adjoint (\( T^* = -T \)) and therefore \( iT \) is self-adjoint.

2) Since \( \{ \alpha \} \) is a Jordan path in the complement of \( \sigma(T) \), \( \alpha \) is a periodic \( C_0 \)-group with period \( 2\pi \), we get

\[
(1 - e^{-2\pi \lambda}) f = (\lambda - T) \int_0^{2\pi} e^{-\lambda s} D_k^i f \, ds; \quad f \in L^2(\mathbb{R}^N, \omega_k(x) dx) \tag{3.10}
\]

\[
= \int_0^{2\pi} e^{-\lambda s} D_k^i (\lambda - T) f \, ds; \quad f \in D(T). \tag{3.11}
\]

Let \( \lambda \notin i\mathbb{Z} \). Then \( 1 - e^{-2\pi \lambda} \neq 0 \). By the use of (3.10) and (3.11), \( \lambda - T \) is invertible \( (\lambda \in \rho(T)) \) and

\[
(\lambda - T)^{-1} f = R(\lambda, T) f = (1 - e^{-2\pi \lambda})^{-1} \int_0^{2\pi} e^{-\lambda s} D_k^i f \, ds.
\]

The previous Proposition indicates that every point in the spectrum of \( T \) is an isolated point of the set \( i\mathbb{Z} \). Let \( \nu \) be an element of the spectrum of \( T \) and

\[
P_n = \frac{1}{2\pi} \int_\Gamma R(\lambda, T) \, d\lambda,
\]

the associated spectral projection, where \( \Gamma \) is a Jordan path in the complement of \( i\mathbb{Z} \setminus \{ \nu \} \) and enclosing \( \nu \).

The function \( \lambda \mapsto R(\lambda, T) \) can be expanded as a Laurent series

\[
R(\lambda, T) = \sum_{k=-\infty}^{+\infty} (\lambda - \nu)^k B_k
\]

for \( 0 < |\lambda - \nu| < \delta \) and some sufficiently small \( \delta > 0 \). The coefficients \( B_k \) of this series are bounded operators given by the formulas

\[
B_k = \frac{1}{2i\pi} \int_\Gamma \frac{R(\lambda, T)}{(\lambda - \nu)^{k+1}} \, d\lambda, \quad k \in \mathbb{Z}.
\]

The coefficient \( B_{-1} \) is exactly the spectral projection \( P_n \) corresponding to the decomposition \( \sigma(T) = \{ \nu \} \cup \{ i\mathbb{Z} \setminus \{ \nu \} \} \) of the spectrum of \( T \). From (3.13), one deduces the identity

\[
P_n = B_{-1} = \lim_{\lambda \to \nu} (\lambda - \nu) R(\lambda, T)
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} e^{-\nu s} D_k^i \, ds, \tag{3.12}
\]

which allows us to interpret \( P_n \) as the \( n \)th Fourier coefficient of the \( 2\pi \)-periodic function \( s \mapsto D_k^i \).

In the following Proposition we gather some properties of the operator \( D_k^i \).

**Proposition 3.4** Let \( n, m \in \mathbb{Z} \) such that \( n \neq m \) and \( f, g \in L^2(\mathbb{R}^N, \omega_k(x) dx) \). Then

i) \( T P_n = i \nu P_n \),

ii) \( D_k^i P_n = e^{i\nu s} P_n \),

iii) \( P_n P_m = 0 \),

iv) \( \langle P_n f, g \rangle = \langle f, P_m g \rangle \). In particular \( \langle P_n f, P_m g \rangle = 0 \).

v) The linear span

\[
\text{lin} \bigcup_{n \in \mathbb{Z}} P_n L^2(\mathbb{R}^N, \omega_k(x) dx)
\]

is dense in \( L^2(\mathbb{R}^N, \omega_k(x) dx) \).

Proof.  
1) It follows directly from (3.10) applied to \( \lambda = i \nu \).   
2) Applying \( D_k^i \) to each member of (3.12) then according to Proposition 3.1, 3), we obtain

\[
D_k^i P_n f = \frac{1}{2\pi} \int_0^{2\pi} e^{-i\nu t} D_k^i D_k^i f \, dt
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} e^{-i\nu t} D_k^{i+1} f \, dt.
\]
The change of variables \( u = s + t \) gives the desired result.

\( \text{iii) From (3.12) and ii), we have} \)

\[
P_n P_m f = \frac{1}{2\pi} \int_0^{2\pi} e^{-in s} D_k^*(P_m f) \, ds
\]

\[
= \left( \frac{1}{2\pi} \int_0^{2\pi} e^{i(m-n) s} \, ds \right) P_m f
\]

\[
= 0.
\]

\( \text{iii) Obvious.} \)

\( \text{iv) Assume that the linear span} \)

\[
\text{lin} \bigcup_{n \in \mathbb{Z}} P_n L^2(\mathbb{R}^N, \omega_k(x)dx)
\]

is not dense in \( L^2(\mathbb{R}^N, \omega_k(x)dx) \). By the Hahn-Banach theorem there exists a nonzero linear functional

\[
\varphi : L^2(\mathbb{R}^N, \omega_k(x)dx) \to \mathbb{C}
\]

vanishing on each \( P_n L^2(\mathbb{R}^N, \omega_k(x)dx), \ n \in \mathbb{Z} \). By the Riesz representation theorem, there exists a unique vector \( g \in L^2(\mathbb{R}^N, \omega_k(x)dx) \setminus \{0\} \) such that

\[
\varphi(f) = \langle f, g \rangle \text{ for all } f \in L^2(\mathbb{R}^N, \omega_k(x)dx).
\]

Hence for all \( n \in \mathbb{Z} \) and \( f \in L^2(\mathbb{R}^N, \omega_k(x)dx), \)

\[
0 = \langle P_n f, g \rangle = \left\langle \frac{1}{2\pi} \int_0^{2\pi} e^{-in s} D_k^* f \, ds, g \right\rangle
\]

\[
= \frac{1}{2\pi} \int_0^{2\pi} e^{-in s} \langle D_k^* f, g \rangle \, ds.
\]

For each \( f \in L^2(\mathbb{R}^N, \omega_k(x)dx) \), the function \( s \mapsto \langle D_k^* f, g \rangle \) has all its Fourier coefficients equal to zero, then it vanishes. This cannot be true, since if we take \( f = g \) and \( s = 0 \),

\[
\langle D_k^0 f, g \rangle = \|g\|_2^2 > 0.
\]

**Proposition 3.5** Let \( f \in D(T) \). Then

\[
f = \sum_{n=-\infty}^{+\infty} P_n f, \tag{3.13}
\]

and therefore, if \( f \in D(T^2) \)

\[
T f = \sum_{n=-\infty}^{+\infty} in P_n f. \tag{3.14}
\]

**Proof.** We are going to show that the series \( \sum_{n \in \mathbb{Z}} P_n f \) is summable for all \( f \in D(T) \). For this, let \( f \in D(T) \) and put \( g = T f \). The commutativity of \( T \) and \( P_n \) together with Proposition 3.4 gives:

\[
P_n g = P_n T f = T P_n f = in P_n f.
\]

By the Cauchy-Schwartz inequality, it follows that

\[
\left| \sum_{n \in H} \langle P_n f, h \rangle \right| = \left| \sum_{n \in H} (in)^{-1} \langle P_n g, h \rangle \right|
\]

\[
\leq \left( \sum_{n \in H} n^{-2} \right)^{1/2} \left( \sum_{n \in H} |\langle P_n g, h \rangle|^2 \right)^{1/2},
\]
where \( h \in L^2(\mathbb{R}^N, \omega_k(x)dx) \) and \( H \) be a finite subset of \( \mathbb{Z} \setminus \{0\} \). The function \( s \mapsto \langle D^s_k g, h \rangle \) belongs to \( L^2([0, 2\pi]) \), then we obtain from Bessel’s inequality
\[
\sum_{n \in H} |\langle P_n g, h \rangle|^2 \leq \frac{1}{2\pi} \int_0^{2\pi} |\langle D^s_k g, h \rangle|^2 \, ds \\
\leq \frac{\|h\|^2_2}{2\pi} \int_0^{2\pi} \|D^s_k g\|^2_2 \, ds = \|h\|_2^2 \|g\|_2^2.
\]
Therefore, for any \( h \in L^2(\mathbb{R}^N, \omega_k(x)dx) \),
\[
\left| \sum_{n \in H} P_n f, h \right| = \left| \sum_{n \in H} \langle P_n f, h \rangle \right| \leq \|h\|_2 \|g\|_2 \left( \sum_{n \in H} n^{-2} \right)^{1/2}.
\]
Taking supremum over \( h \in L^2(\mathbb{R}^N, \omega_k(x)dx) \) with \( \|h\|_2 \leq 1 \), we get
\[
\left\| \sum_{n \in H} P_n f \right\|_2 \leq \|g\|_2 \left( \sum_{n \in H} n^{-2} \right)^{1/2},
\]
which means that the series \( \sum_{n \in \mathbb{Z}} P_n f \) converges in \( L^2(\mathbb{R}^N, \omega_k(x)dx) \).

Set
\[
f_1 = \sum_{n=-\infty}^{+\infty} P_n f
\]
and let \( g \in L^2(\mathbb{R}^N, \omega_k(x)dx) \). As the Fourier coefficients of the continuous, 2\( \pi \)-periodic functions
\[
s \mapsto \langle D^s_k f_1, g \rangle \quad \text{and} \quad s \mapsto \langle D^s_k f, g \rangle
\]
coincide. Then, for all \( s \in \mathbb{R} \),
\[
\langle D^s_k f_1, g \rangle = \langle D^s_k f, g \rangle.
\]
In particular, for \( s = 0 \), \( \langle f_1, g \rangle = \langle f, g \rangle \) and therefore \( f_1 = f \).

Replacing \( f \) in \( \text{(3.13)} \) by \( Tf \), then we get \( \text{(3.14)} \).

At the end of the section 4, we will show that \( P_n = 0 \), for any negative integer \( n \neq 0 \).

### 4 The fractional Dunkl transform in \( L^1(\mathbb{R}^N, \omega_k(x)dx) \cap L^2(\mathbb{R}^N, \omega_k(x)dx) \).

In this section, we shall derive an integral representation for the fractional Dunkl transform \( D^\alpha_k \) defined by \( \text{(3.2)} \), for suitable function \( f \).

#### 4.1 Integral representation.

Following Zayed’s approach \( \text{(28)} \), we define the operator \( D^\alpha_{k,r} \) as
\[
D^\alpha_{k,r} f := \sum_{\nu \in \mathbb{Z}^N_+} r^{2|\nu|} e^{i\nu \cdot \alpha} \langle f, h_\nu \rangle h_\nu,
\]
where \( 0 < r \leq 1 \) and so \( D^\alpha_{k,1} = D^\alpha_{k,1} \).

In the next proposition, we collect some properties of \( D^\alpha_{k,r} \).

**Proposition 4.1** Let \( \alpha \in \mathbb{R} \) and \( r \in [0, 1] \). Then
1) \( D^\alpha_{k,r} \) is a bounded operator on \( L^2(\mathbb{R}^N, \omega_k(x)dx) \) satisfying \( \|D^\alpha_{k,r} f\|_2 \leq \|f\|_2 \).
2) For all \( f \in L^2(\mathbb{R}^N, \omega_k(x)dx) \), \( D^\alpha_{k,r} f \to D^\alpha_k f \) in \( L^2(\mathbb{R}^N, \omega_k(x)dx) \) as \( r \to 1^- \).

**Proof.** Let \( f \in L^2(\mathbb{R}^N, \omega_k(x)dx) \).
1) According to Parseval’s formula, we have
\[
\|D^\alpha_{k,r} f\|_2^2 = \sum_{\nu \in \mathbb{Z}^N_+} r^{2|\nu|} |\langle f, h_\nu \rangle|^2 \\
\leq \sum_{\nu \in \mathbb{Z}^N_+} |\langle f, h_\nu \rangle|^2 = \|f\|_2^2.
\]
2) It is easy to see that
\[ D_{k,r}^\alpha f - D_k^\alpha f = \sum_{\nu \in \mathbb{Z}_+^N} (r|\nu| - 1)e^{i|\nu|\alpha} \langle f, h_\nu \rangle h_\nu. \]

Then
\[
\|D_{k,r}^\alpha f - D_k^\alpha f\|_2^2 = \sum_{\nu \in \mathbb{Z}_+^N} |r|^{2|\nu|} - 1|^2 |\langle f, h_\nu \rangle|^2.
\]

By the dominated convergence theorem it follows that \( \lim_{r \to 1^-} \|D_{k,r}^\alpha f - D_k^\alpha f\|_2^2 = 0. \)

**Corollary 4.1** For each fixed \( f \in L^2(\mathbb{R}^N, \omega_k(x)dx) \), there exists \( \{r_j\}_{j=1}^\infty \), with \( r_j \to 1^- \) as \( j \to \infty \), such that
\[
D_k^\alpha f(x) = \lim_{j \to \infty} D_{k,r_j}^\alpha f(x)
\]
for almost all \( x \in \mathbb{R}^N. \)

**Proof.**
This is a consequence of a standard result that if a sequence \( \{f_n\} \) converges in \( L^2(\mathbb{R}^N, \omega_k(x)dx) \) to \( f \), then there exists a subsequence \( \{f_{n_k}\} \) that converges pointwise almost everywhere to \( f \).

The operator \( D_{k,r}^\alpha \) defined above have the integral representation given in the next lemma.

**Lemma 4.1** For \( f \in L^2(\mathbb{R}^N, \omega_k(x)dx) \) and \( 0 < r < 1 \), we have
\[
D_{k,r}^\alpha f(x) = \int_{\mathbb{R}^N} K_\alpha(r, x, y)f(y)\omega_k(y) \, dy,
\]
where
\[
K_\alpha(r, x, y) = \sum_{\nu \in \mathbb{Z}_+^N} r|\nu| e^{i|\nu|\alpha} h_\nu(x)\overline{h_\nu(y)}.
\]

**Proof.** Let \( x \in \mathbb{R}^N \) and \( H \) be a finite subset of \( \mathbb{Z}_+^N \). Then
\[
\left\| \sum_{\nu \in H} h_\nu(x)h_\nu(y)(re^{i\alpha})^{|
u|} \right\|_2^2 = \sum_{\nu \in H} |h_\nu(x)|^2 |\nu|^{2|\nu|}.
\]

Since the series (see Theorem 3.12 in \[21\])
\[
\sum_{\nu \in \mathbb{Z}_+^N} h_\nu(x)h_\nu(y)(re^{i\alpha})^{|
u|}
\]
converges absolutely for all \( x, y \in \mathbb{R}^N \), then according to (4.3), the series
\[
\sum_{\nu \in H} h_\nu(x)h_\nu(y)(re^{i\alpha})^{|
u|}
\]
converges in \( L^2(\mathbb{R}^N, \omega_k(x)dx) \) to a function denoted by \( K_\alpha(r, x, \cdot) \).

By the use of Cauchy-Schwartz inequalities, we obtain
\[
D_{k,r}^\alpha f(x) = \sum_{\nu \in \mathbb{Z}_+^N} (re^{i\alpha})^{|
u|} h_\nu(x) \int_{\mathbb{R}^N} f(y)h_\nu(y)\omega_k(y) \, dy
\]
\[
= \int_{\mathbb{R}^N} f(y) \sum_{\nu \in \mathbb{Z}_+^N} h_\nu(x)h_\nu(y)(re^{i\alpha})^{|
u|} \omega_k(y) \, dy
\]
\[
= \int_{\mathbb{R}^N} K_\alpha(r, x, y)f(y)\omega_k(y) \, dy.
\]

Now, we summarize some properties of the kernel \( K_\alpha(r, x, y) \).
**Proposition 4.2** Let \( x, y, r \in \mathbb{R}^N \), \( \alpha, r \in \mathbb{R} \) such that 0 < |\( \alpha \) | < \( \pi \) and 0 < \( r < 1 \), then we have

1) \[
K_\alpha(r, x, y) = c_k e^{- \frac{1 + r^2 e^{2i\alpha}}{2(1 - r^2 e^{2i\alpha})} (|x|^2 + |y|^2)} \left( \frac{2r e^{i\alpha} x}{1 - r^2 e^{2i\alpha}} - y \right),
\]

2) \[
\lim_{r \to 1^-} K_\alpha(r, x, y) = A_\alpha K_\alpha(x, y),
\]

where \( K_\alpha(x, y) = e^{-\frac{i}{2} \cot(\alpha)(|x|^2 + |y|^2)} \left( \frac{ix}{\sin \alpha} \right) \), \( A_\alpha = \frac{c_k e^{i(\gamma + N/2)(\bar{\alpha}\pi/2 - \alpha)}}{2|\sin \alpha|^{\gamma + N/2}} \) and \( \bar{\alpha} = \text{sgn}(\sin \alpha) \).

3) \[
\left| e^{- \frac{(1 + r^2 e^{2i\alpha}) |z|^2}{2(1 - r^2 e^{2i\alpha})}} K \left( \frac{2r e^{i\alpha} x}{1 - r^2 e^{2i\alpha}} - y \right) \right| \leq e^{\frac{r^2 (1 - \alpha^2) \cos^2(\alpha) |x|^2}{2(1 - r^2 e^{2i\alpha})^{\gamma + N/2}}},
\]

**Proof.** 1) According to [11], we have

\[
K_\alpha(r, x, y) = c_k e^{-\frac{(|x|^2 + |y|^2)/2}{2|\alpha|^2}} \sum_{\nu \in \mathbb{Z}^N} (\nu e^{i\alpha}) |\nu| \frac{H_\nu(x) H_\nu(y)}{2|\nu|}.
\]

Using Mehler’s formula for the generalized Hermite polynomials (see Theorem 3.12 in [21]) which says that for all \( x, y \in \mathbb{R}^N \) and \( z \in \mathbb{C} \) with |\( z \) | < 1,

\[
\sum_{\nu \in \mathbb{Z}^N} H_\nu(x) H_\nu(y) = \frac{e^{-\frac{1}{2} (|z|^2 + |1 - z|^2)} K \left( \frac{2zx}{1 - z^2} \right)}{\left( \frac{1 - z^2}{\gamma + N/2} \right)},
\]

and setting \( z = re^{i\alpha} \) with |\( z \) | = \( r < 1 \), we obtain the desired result.

2) Clearly \[
\lim_{r \to 1^-} \frac{1 + r^2 e^{2i\alpha}}{1 - r^2 e^{2i\alpha}} = i \cot \alpha,
\]

\[
\lim_{r \to 1^-} \frac{r e^{i\alpha}}{1 - r^2 e^{2i\alpha}} = \frac{i}{2 \sin \alpha},
\]

\[
\lim_{r \to 1^-} \left( 1 - r^2 e^{2i\alpha} \right)^{-(\gamma + \frac{N}{2})} = \frac{e^{i(\gamma + N/2)(\bar{\alpha}\pi/2 - \alpha)}}{(2|\sin \alpha|)^{\gamma + N/2}}, \] where \( \bar{\alpha} = \text{sgn}(\sin \alpha) \).

Then, for 0 < |\( \alpha \) | < \( \pi \),

\[
\lim_{r \to 1^-} K_\alpha(r, x, y) = A_\alpha K_\alpha(x, y),
\]

where \( K_\alpha(x, y) \) and \( A_\alpha \) are defined respectively in \( (4.7) \) and \( (4.8) \).

3) It is straightforward to show that

\[
a_r = \Re \left( \frac{1 + r^2 e^{2i\alpha}}{1 - r^2 e^{2i\alpha}} \right) = \frac{(1 - r^4)}{(1 + r^4) - 2r^2 \cos 2\alpha} > 0,
\]

\[
b_r = \Re \left( \frac{2r e^{i\alpha}}{1 - r^2 e^{2i\alpha}} \right) = \frac{2(r - r^3) \cos \alpha}{1 + r^4 - 2r^2 \cos 2\alpha}.
\]
From (2.7) and (4.10), we deduce the following majorization:

$$
\left| K\left( \frac{2re^{i\alpha}x}{1 - r^2e^{2i\alpha}} , y \right) \right| \leq e^{\|x\| |y|}.
$$

Hence,

$$
\left| e^{-\frac{2(1+r^2e^{2i\alpha})|y|^2}{2(1-r^2e^{2i\alpha})}} K\left( \frac{2re^{i\alpha}x}{1 - r^2e^{2i\alpha}} , y \right) \right| \leq e^{-a_r |y|^2 + |b_r| |x||y|}.
$$

(4.11)

As $a_r > 0$, we deduce that

$$
\sup_{s \geq 0} (-a_r s^2 + |b_r| |x| s) = -\frac{b^2 |x|^2}{4a_r}.
$$

(4.12)

Combining (4.11) and (4.12), we see that

$$
\left| e^{-\frac{2(1+r^2e^{2i\alpha})|y|^2}{2(1-r^2e^{2i\alpha})}} K\left( \frac{2re^{i\alpha}x}{1 - r^2e^{2i\alpha}} , y \right) \right| \leq e^{\frac{-2(1+r^2e^{2i\alpha})|y|^2}{2(1-r^2e^{2i\alpha})}}.
$$

Proposition 4.3 Let $\alpha \in \mathbb{R} \setminus \pi \mathbb{Z}$ and $f \in L^1 (\mathbb{R}^N, \omega_k(x) \, dx) \cap L^2 (\mathbb{R}^N, \omega_k(x) \, dx)$. Then the fractional Dunkl transform $D^\alpha_k$ have the following integral representation

$$
D^\alpha_k f(x) = A_\alpha \int_{\mathbb{R}^N} f(y) K_\alpha(x,y) \omega_k(y) \, dy, \text{a.e.}
$$

(4.13)

where

$$
K_\alpha(x,y) = e^{-\frac{1}{2} \cot(\alpha) (|x|^2 + |y|^2)} K\left( \frac{i x}{\sin \alpha}, y \right)
$$

and

$$
A_\alpha = c_\alpha e^{\frac{(\gamma + N/2)(\gamma - 2 - \alpha)}{\sin \alpha}}.
$$

Proof. $D^\alpha_k$ is periodic in $\alpha$ with period $2\pi$, we can assume that $0 < |\alpha| < \pi$. Let $f \in L^1 (\mathbb{R}^N, \omega_k(x) \, dx) \cap L^2 (\mathbb{R}^N, \omega_k(x) \, dx)$. From Corollary 4.1

$$
D^\alpha_k f(x) = \lim_{j \to \infty} \int_{\mathbb{R}^N} K_\alpha(r_j, x, y) f(y) \omega_k(y) \, dy, \text{a.e.}
$$

From Proposition 4.2 2) we see that

$$
\lim_{j \to \infty} K_\alpha(r_j, x, y) f(y) = A_\alpha K_\alpha(x,y) f(y).
$$

Using again Proposition 4.2 3), we obtain

$$
\left| e^{-\frac{2(1+r_j^2e^{2i\alpha})|y|^2}{2(1-r_j^2e^{2i\alpha})}} K\left( \frac{2r_j e^{i\alpha}x}{1 - r_j^2 e^{2i\alpha}}, y \right) f(y) \right| \leq M_x |f(y)|,
$$

where $M_x = \sup_{0 \leq r < 1} e^{\frac{2\pi^2 (1-r^2) \cos^2(\alpha)|y|^2}{2\pi^2 (1-r^2) \cos^2(\alpha)+ 1}}.$

Hence, the dominated convergence theorem gives

$$
D^\alpha_k f(x) = A_\alpha \int_{\mathbb{R}^N} f(y) K_\alpha(x,y) \omega_k(y) \, dy, \text{a.e.}
$$
Definition 4.1 We define the fractional Dunkl transform $D_k^\alpha$ for $f \in L^1(\mathbb{R}^N, \omega_k(x) \, dx) \cap L^2(\mathbb{R}^N, \omega_k(x) \, dx)$ by

$$D_k^\alpha f(x) = A_\alpha \int_{\mathbb{R}^N} f(y) K_\alpha(x,y) \omega_k(y) \, dy.$$  

Remark 4.1

- For $\alpha = -\frac{\pi}{2}$, the fractional Dunkl transform $D_k^\alpha$ is reduced to the Dunkl transform $D_k$ and when the multiplicity function $k \equiv 0$, $D_k^\alpha$ coincides with the fractional Fourier transform $\mathcal{F}^\alpha$.

$$\mathcal{F}^\alpha f(x) = \frac{e^{i(\alpha/2-\alpha)\pi/2}}{(2\pi \sin \alpha)^{N/2}} \int_{\mathbb{R}^N} e^{-\frac{i}{\alpha}(x^2 + y^2) \cot \alpha} + \frac{x}{\sin(x,y)} f(y) \, dy.$$  

- In the one-dimensional case ($N = 1$), the corresponding reflection group $W$ is $\mathbb{Z}_2$ and the multiplicity function $k$ is equal to $\nu + 1/2 > 0$. The kernel $K_\alpha(x,y)$ defined by (4.7) becomes

$$K_\alpha(x,y) = e^{-\frac{i}{\alpha} \cot \alpha (x^2 + y^2)} E_{\nu} \left( \frac{ix}{\sin \alpha}, y \right),$$  

where $E_{\nu}(x,y)$ is the Dunkl kernel of type $A_2$ given by (see [23])

$$K(ix,y) = j_{\nu}(xy) + \frac{ixy}{2(\nu + 1)} j_{\nu+1}(xy),$$  

and $j_{\nu}$ denotes the normalized spherical Bessel function

$$j_{\nu}(x) := 2^\nu \Gamma(\nu + 1) \frac{J_\nu(x)}{x^\nu} = \Gamma(\nu + 1) \sum_{n=0}^{+\infty} \frac{(-1)^n (x/2)^{2n}}{n! \Gamma(n + \nu + 1)}.$$  

Here $J_\nu$ is the classical Bessel function (see, Watson [22]). The related fractional Dunkl transform $D_k^\alpha$ in rank-one case takes the form

$$D_k^\alpha f(x) = B_\nu \int_{-\infty}^{+\infty} K_\alpha(x,y) f(y) |y|^{2\nu + 1} \, dy,$$  

where

$$B_\nu = \frac{e^{i(\nu+1)(\alpha/2-\alpha)}}{(\Gamma(\nu + 1)(2 \sin \alpha))^{\nu + 1}}.$$  

Note that if $f$ is an even function then, the fractional Dunkl transform (4.14) coincides with the fractional Hankel transform (19)

$$H_k^\alpha f(x) = 2B_\nu \int_{0}^{+\infty} e^{-\frac{i}{\alpha}(x^2 + y^2) \cot \alpha} J_{\nu} \left( \frac{xy}{\sin \alpha} \right) f(y) y^{2\nu + 1} \, dy.$$  

- More generally, for $W = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ and the multiplicity function $k = (\nu_1, \ldots, \nu_N)$, the kernel $K_\alpha(x,y)$ defined by (4.7) is given explicitly by

$$K_\alpha(x,y) = e^{-\frac{i}{\alpha} \cot \alpha (|x|^2 + |y|^2)} \prod_{j=1}^{N} E_{\nu_j} \left( \frac{ix_j}{\sin \alpha}, y_j \right),$$  

where $x = (x_1, \ldots, x_N)$, $y = (y_1, \ldots, y_N) \in \mathbb{R}^N$ and $E_{\nu_j}(x_j, y_j)$ is the function defined by (4.14). In this case the fractional Dunkl transform will be

$$D_k^\alpha f(x) = A_\alpha \int_{\mathbb{R}^N} f(y) K_\alpha(x,y) \omega_k(y) \, dy,$$  

where

$$A_\alpha = \frac{e^{i(\gamma + N/2)(\alpha/2 - \alpha)}}{\Gamma(\nu_1 + 1) \cdots \Gamma(\nu_N + 1)(2 \sin \alpha)^{\gamma + N/2}}$$  

and

$$\omega_k(y) = \prod_{j=1}^{N} |x_j|^{2\nu_j}.$$
4.2 Bochner type identity for the fractional Dunkl transform.

In this subsection, we start with a brief summary on the theory of k-spherical harmonics. An introduction to this subject can be found in the monograph [10]. The space of \( k \)-spherical harmonics of degree \( n \geq 0 \) is defined by

\[
\mathcal{H}^k_n = \text{ker} \Delta_k \cap \mathcal{P}_n.
\]

Let \( S^{N-1} = \{ x \in \mathbb{R}^N ; \ |x| = 1 \} \) be the unit sphere in \( \mathbb{R}^N \) with normalized Lebesgue surface measure \( d\sigma \) and \( L^2(S^{N-1}, \omega_k(x) \, d\sigma(x)) \) be the Hilbert space with the following inner product given by

\[
\langle f, g \rangle_k = \int_{S^{N-1}} f(\omega) \overline{g(\omega)} \omega_k(\omega) \, d\sigma(\omega).
\]

As in the theory of ordinary spherical harmonics, the space \( L^2(S^{N-1}, \omega_k(x) \, d\sigma(x)) \) decomposes as an orthogonal Hilbert space sum

\[
L^2(S^{N-1}, \omega_k(x) \, d\sigma(x)) = \bigoplus_{n=0}^{\infty} \mathcal{H}^k_n.
\]

In [27], Y. Xu gives an analogue of the Funk-Hecke formula for \( k \)-spherical harmonics. The well-known special case of the Dunkl-type Funk-Hecke formula is the following (see [24]):

**Proposition 4.4** Let \( N \geq 2 \) and put \( \lambda = \gamma + (N/2) - 1 \). Then for all \( Y \in \mathcal{H}^k_n \) and \( x \in \mathbb{R}^N \),

\[
\frac{1}{d_k} \int_{S^{N-1}} K(ix, y) Y(y) \omega_k(y) \, d\sigma(y) = \frac{\Gamma(\lambda + 1)}{2^n \Gamma(n + \lambda + 1)} j_{n+\lambda}(|x|) Y(ix),
\]

where

\[
d_k = \int_{S^{N-1}} \omega_k(y) \, d\sigma(y).
\]

In particular

\[
\frac{1}{d_k} \int_{S^{N-1}} K(ix, y) \omega_k(y) \, d\sigma(y) = j_{\lambda}(|x|).
\]

An application of the Dunkl-type Funk-Hecke formula is the following:

**Theorem 4.1** (Bochner type identity) If \( f \in L^1(\mathbb{R}^N, \omega_k(x) \, dx) \cap L^2(\mathbb{R}^N, \omega_k(x) \, dx) \) is of the form \( f(x) = p(x)\psi(|x|) \) for some \( p \in \mathcal{H}^k_n \) and a one-variable \( \psi \) on \( \mathbb{R}_+ \), then

\[
D^\alpha_k f(x) = e^{i\alpha x} p(x) H^\alpha_{n+\gamma+(N/2)-1} \psi(|x|).
\]

In particular, if \( f \) is radial, then

\[
D^\alpha_k f(x) = H^\alpha_{n+\gamma+(N/2)-1} \psi(|x|).
\]

**Proof.** Since \( D^\alpha_k \) is periodic in \( \alpha \) with period \( 2\pi \), we can assume that \(-\pi < \alpha \leq \pi\). We see immediately that

\[
D^\alpha_k f(x) = f(x), \\
D^\alpha_k f(x) = f(-x) = p(-x) \psi(-x) = (-1)^n p(x) \psi(x).
\]

Now, let \( 0 < |\alpha| < \pi \). By spherical polar coordinates, we have

\[
D^\alpha_k f(x) = A_\alpha \int_{\mathbb{R}^N} f(y) K_\alpha(x, y) \omega_k(y) \, dy \\
= A_\alpha \int_0^{\infty} r^{N-1} F(r, x) \, dr,
\]

where

\[
F(r, x) = \frac{2\pi^N}{\Gamma(N/2)} \int_{S^{N-1}} K_\alpha(x, ry) p(ry) \psi(|ry|) \omega_k(ry) \, d\sigma(y).
\]
From (4.17) and the homogeneity of $\omega_k$ and $p$, we obtain

$$F(r, x) = \frac{2\pi^{N/2}}{\Gamma(N/2)} e^{-\frac{1}{2}(|x|^2 + r^2) \cot(\alpha)} \psi(r) r^{2\gamma + n} \int_{S^{N-1}} p(y) K \left( \frac{irx}{\sin(\alpha)} , y \right) \omega_k(y) \, d\sigma(y).$$

Using (4.17), we get

$$F(r, x) = \frac{2\pi^{N/2} d_k}{\Gamma(N/2)} \frac{\Gamma(\lambda + 1)}{2^n \Gamma(\lambda + n + 1)} e^{-\frac{1}{2}(|x|^2 + r^2) \cot(\alpha)} \psi(r) r^{2\gamma + n} \int_{S^{N-1}} p(y) \frac{ix}{2\sin(\alpha)} j_{\lambda + n} \left( \frac{r|x|}{\sin(\alpha)} \right) \, d\sigma(y).$$

Now we can express a relationship between $d_k$ and $c_k$. In fact

$$c_k^{-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)} \int_{\mathbb{R}^N} e^{-|y|^2} \omega_k(y) \, dy = \frac{2\pi^{N/2}}{\Gamma(N/2)} \int_0^{+\infty} r^{N-1} e^{-r^2} \int_{S^{N-1}} \omega_k(ry) \, d\sigma(y) \, dr = \frac{2\pi^{N/2}}{\Gamma(N/2)} \int_0^{+\infty} r^{2\gamma + n - 1} e^{-r^2} \int_{S^{N-1}} \omega_k(y) \, d\sigma(y) \, dr = \pi^{N/2} \Gamma(\lambda + 1) d_k \frac{\Gamma(\lambda + 1)}{\Gamma(N/2)} = \frac{\pi^{N/2} \Gamma(\lambda + 1) d_k}{\Gamma(N/2)}. \tag{4.21}$$

Recall that

$$A_\alpha = c_k \left( \frac{ie^{-i\alpha}}{2\sin(\alpha)} \right)^{\gamma + (N/2)},$$

then by the use of (4.21), we obtain

$$A_\alpha 2\pi^{N/2} d_k \frac{\Gamma(\lambda + 1)}{\Gamma(N/2)} \frac{\Gamma(\lambda + 1)}{2^n \Gamma(\lambda + n + 1)} \left( \frac{i}{2\sin(\alpha)} \right)^n = 2 \frac{\left( \frac{ie^{-i\alpha}}{2\sin(\alpha)} \right)^{\lambda + n + 1}}{\Gamma(\lambda + n + 1)} e^{i\alpha} = 2B_\alpha e^{i\alpha}.$$

Hence

$$F(r, x) = 2B_\alpha e^{i\alpha} e^{-\frac{1}{2}(|x|^2 + r^2) \cot(\alpha)} \psi(r) r^{2\gamma + 2n} p(x) j_{\lambda + n} \left( \frac{r|x|}{\sin(\alpha)} \right). \tag{4.22}$$

Substituting (4.22) in (4.20) to get

$$D_\alpha^k f(x) = 2B_\alpha e^{i\alpha} p(x)$$

$$\times \int_0^{+\infty} e^{-\frac{1}{2}(|x|^2 + r^2) \cot(\alpha)} \psi(r) r^{2(\lambda + n) + 1} j_{\lambda + n} \left( \frac{r|x|}{\sin(\alpha)} \right) \, dr$$

$$= e^{i\alpha} p(x) H_{n+\lambda}^\alpha(|x|)$$

$$= e^{i\alpha} p(x) H_{n+\gamma+(N/2)-1}^\alpha(|x|).$$
Now, we give the material needed for an application of Bochner type identity. Let \( \{p_{n,j}\}_{j \in J_n} \) be an orthonormal basis of \( \mathcal{H}_n^k \). Let \( m, n \) be non-negative integers and \( j \in J_n \). Define
\[
c_{m,n} = \left( \frac{m! \Gamma(N/2)}{\pi^{N/2} \Gamma((N/2) + \gamma + n + m)} \right)^{1/2}
\]
and
\[
\psi_{m,n,j}(x) = c_{m,n} p_{n,j}(x) L_m^{(n+\gamma+N/2-1)}(\beta \frac{r}{|x|}); e^{-\beta |x|^2/2}, \tag{4.23}
\]
where \( L_m^{(\alpha)} \) denote the Laguerre polynomial defined by
\[
I_m^{(\alpha)}(x) = \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n} (e^{-x} x^{\alpha}).
\]
It follows from Proposition 2.4 and Theorem 2.5 of Dunkl [9] that
\[
\{\psi_{m,n,j} : m, n = 0, 1, 2, \ldots, j \in J_n\}
\]
forms an orthonormal basis of \( L^2(\mathbb{R}^N, \omega_k(x)dx) \).

**Theorem 4.2** The family \( \{\psi_{m,n,j} : m, n = 0, 1, 2, \ldots, j \in J_n\} \) are a basis of eigenfunctions of the fractional Dunkl transform \( D_k^\alpha \) on \( L^2(\mathbb{R}^N, \omega_k(x)dx) \), satisfying
\[
D_k^\alpha \psi_{m,n,j} = e^{i\alpha(n+2m)} \psi_{m,n,j}. \tag{4.24}
\]

**Proof.** We need only to prove [4.24]. Applying Theorem 4.3 with \( p \) replaced by \( p_{n,j} \) and with \( \psi(r) = L_m^{(n+\gamma+N/2-1)}(r^2) e^{-r^2/2} \), we obtain
\[
D_k^\alpha \psi_{m,n,j}(x) = c_{m,n} e^{i\alpha} p_{n,j}(x) \mathcal{H}_\nu^\alpha \psi(|x|),
\]
where
\[
\nu = n + \gamma + (N/2) - 1,
\]
and
\[
\mathcal{H}_\nu^\alpha \psi(|x|) = 2B_\nu \int_{0}^{+\infty} e^{-\frac{x}{\sin \alpha} \cot(\alpha) \gamma (\beta \frac{r}{|x|} + 2)} \mu \ \mu (\frac{r |x|}{\sin \alpha}) I_m^{(\mu)}(r^2) e^{-r^2/\beta} dr.
\]
Observe that
\[
\mathcal{H}_\nu^\alpha \psi(|x|) = 2B_\nu e^{-\frac{x}{\sin \alpha} \cot(\alpha) |x|^2} I_\nu,
\]
where
\[
I_\nu = \int_{0}^{+\infty} r^{2\nu+1} L_m^{(\mu)}(r^2) e^{-\frac{x}{\sin \alpha} \cot(\alpha) \gamma (\beta \frac{r}{|x|} + 2)} J_\nu (\frac{|x|}{\sin \alpha}) dr
\]
\[
= 2^\nu \Gamma(\nu + 1) \left( \frac{\sin \alpha}{|x|} \right) \int_{0}^{+\infty} r^{2\nu+1} L_m^{(\mu)}(r^2) e^{-\frac{x}{\sin \alpha} \cot(\alpha) \gamma (\beta \frac{r}{|x|} + 2)} J_\nu (\frac{|x|}{\sin \alpha}) dr.
\]
To compute \( I_\nu \), we need the following formulas (see 7.4.21 (4) in [14])
\[
\frac{\int_{0}^{+\infty} y^{\nu+1} e^{-\beta y^2} L_m^{(\nu \gamma))(a \gamma^2) J_\nu (a \gamma z)} dy}{d_m z^{\nu} e^{-z^2/(4\beta) L_m^{(\nu \gamma))(a \gamma^2) J_\nu (a \gamma z)}}
\]
where \( d_m = ((\beta - a)^m / (2^{\nu+1} \beta^{\nu+1} + m + 1)) \), \( a, \beta > 0, \Re \nu > -1 \).
Let us take \( \beta = \frac{1}{2} + \frac{1}{2} \cot(\alpha) = \frac{e^{-i \alpha}}{2 \sin \alpha} \), \( a = 1 \) and \( z = \frac{|x|}{\sin \alpha} \), then
\[
d_m = \frac{e^{2i \alpha m}}{2^{\nu+1} A_\alpha \Gamma(\nu + 1)},
\]
\[
\frac{az^2}{4 \beta (a - \beta)} = \frac{|x|^2}{2},
\]
\[
\frac{z^2}{4 \beta} = -\frac{|x|^2}{2} + \frac{i}{2} \cot(\alpha) |x|^2.
\]
Hence
\[ \int_0^{+\infty} r^{\nu+1} L_m^{(\nu)}(r^2) e^{-\frac{1}{2} r^2 + \frac{1}{2} \cot(\alpha) r^2} J_\nu \left( \frac{r|x|}{\sin \alpha} \right) \, dr = \frac{e^{2i\alpha m} e^{-\frac{1}{2} r^2 - \frac{1}{2} \cot(\alpha) r^2}}{2^{\nu+1} A_n \Gamma(\nu+1)} \left( \frac{|x|}{\sin \alpha} \right)^\nu L_m^{(\nu)}(|x|^2), \]
and therefore
\[ H_\nu^\alpha \psi(|x|) = e^{2i\alpha m} L_m^{(\nu)}(|x|^2) e^{-|x|^2/2}, \]
which finishes the proof.

### 4.3 Master Formula for the fractional Dunkl transform.

In this subsection, we are interesting with a master formula for the fractional Dunkl transform. For this we need the following lemma

**Lemma 4.2** Let \( p \in \mathcal{P}_n \) and \( x = (x_1, \ldots, x_N) \in \mathbb{C}^N \). Then for \( \omega \in \mathbb{C} \) and \( \Re(\omega) > 0 \),
\[
ck \int_{\mathbb{R}^N} p(y) K(x, 2y) e^{-\omega|y|^2} \omega_k(y) \, dy = \frac{e^{i(x)} \omega^{-n+\frac{1}{2}}}{\omega^{\gamma+n+(n+1)/2}} e^{\frac{\Delta k}{4} p(x)},
\]
where \( l(x) = \sum_{j=1}^N x_j \).

**Proof.** First compute the above integral when \( \omega > 0 \).
\[
ck \int_{\mathbb{R}^N} p(y) K(x, 2y) e^{-\omega|y|^2} \omega_k(y) \, dy = ck \int_{\mathbb{R}^N} p(y) K(x, 2y) e^{-|y|^2} \omega_k(y) \, dy.
\]
By the change of variables \( u = \sqrt{\omega} y \) and the homogeneity of \( \omega_k \) and \( p \), we obtain
\[
ck \int_{\mathbb{R}^N} p(y) K(x, 2y) e^{-\omega|y|^2} \omega_k(y) \, dy = \frac{ck}{\omega^{\gamma+n+(n+1)/2}} \int_{\mathbb{R}^N} p(y) K \left( \frac{x}{\sqrt{\omega}}, 2y \right) e^{-|y|^2} \omega_k(y) \, dy.
\]
Using Proposition 2.1 from [9], which says that for all \( p \in \mathcal{P} \) and \( x \in \mathbb{C}^N \)
\[
ck \int_{\mathbb{R}^N} e^{-\frac{\Delta k}{4} p(y)} K(x, 2y) e^{-|y|^2} \omega_k(y) \, dy = e^{\frac{1}{4} l(x)} p(x),
\]
we deduce the following identity:
\[
ck \int_{\mathbb{R}^N} p(y) K(x, 2y) e^{-|y|^2} \omega_k(y) \, dy = e^{\frac{1}{4} l(x)} e^{\frac{1}{4} \Delta k} p(x).
\]
Combining (4.26) and (4.27) to get
\[
ck \int_{\mathbb{R}^N} p(y) K(x, 2y) e^{-\omega|y|^2} \omega_k(y) \, dy = \frac{e^{i(x)} \omega^{-n+\frac{1}{2}}}{\omega^{\gamma+n+(n+1)/2}} e^{\frac{\Delta k}{4} p(x)}.
\]
Now use Lemma 2.1 from [21] to obtain
\[
e^{\frac{\Delta k}{4} p \left( \frac{x}{\sqrt{\omega}} \right)} = \frac{1}{\omega^{n/2}} e^{\frac{\Delta k}{4} p}(x).
\]
Hence, we find the equality (4.25) for \( \omega > 0 \). By analytic continuation, this holds for \( \{\omega \in \mathbb{C} : \Re(\omega) > 0\} \).

We are now in a position to give the master formula.

**Theorem 4.3** Let \( p \in \mathcal{P}_n \) and \( x \in \mathbb{R}^N \). Then
\[
D_k^\alpha \left[ e^{-\frac{|y|^2}{2}} e^{-\frac{\Delta k}{4} p(y)} \right](x) = e^{i\alpha x} e^{-\frac{|x|^2}{2}} e^{-\frac{\Delta k}{4} p(x)}.
\]
Proof. It follows easily from (4.13) that
\[
D_k^x \left[ e^{-\frac{ix}{2}} e^{-\Delta_k} p(y) \right] (x) = A_\alpha e^{-\frac{i}{2} \cot(\alpha) |x|^2} \int_{\mathbb{R}^N} e^{-\Delta_k} p(y) K \left( \frac{ix}{\sin \alpha}, \gamma \right) e^{-\omega |y|^2} \omega_k(y) \, dy,
\]
where
\[
\omega = \frac{1}{2} + i \frac{1}{2} \cot(\alpha) = \frac{i e^{-i \alpha}}{2 \sin \alpha}.
\]
Since
\[
e^{-\Delta_k} p(y) = \sum_{s=0}^{\left[ \frac{d}{4} \right]} (-1)^s \Delta_k^s p(y),
\]
we conclude that
\[
\int_{\mathbb{R}^N} e^{-\Delta_k} p(y) K \left( \frac{ix}{\sin \alpha}, \gamma \right) e^{-\omega |y|^2} \omega_k(y) \, dy = \sum_{s=0}^{\left[ \frac{d}{4} \right]} (-1)^s \int_0^{\frac{N}{2}} \Delta_k^s p(y) K \left( \frac{ix}{\sin \alpha}, \gamma \right) e^{-\omega |y|^2} \omega_k(y) \, dy.
\]
(4.30)

For \( s \in \mathbb{Z}_+ \) with \( 2s \leq n \), the polynomial \( \Delta_k^s p \) is homogeneous of degree \( n - 2s \). Hence by the previous Lemma, we obtain
\[
c_k \int_{\mathbb{R}^N} \Delta_k^s p(y) K \left( \frac{ix}{\sin \alpha}, \gamma \right) e^{-\omega |y|^2} \omega_k(y) \, dy = \frac{e^{i \Delta_k}}{\omega^{\gamma + n + (N/2)}} e^{-\frac{n}{2} \Delta_k} \left[ \omega^{2s} \Delta_k^s p \right] (X_\alpha),
\]
where
\[
X_\alpha = \frac{ix}{2 \sin \alpha}.
\]
(4.32)

Substituting (4.31) in (4.30) to get
\[
c_k \int_{\mathbb{R}^N} e^{-\Delta_k} p(y) K \left( \frac{ix}{\sin \alpha}, \gamma \right) e^{-\omega |y|^2} \omega_k(y) \, dy = \frac{e^{i \Delta_k}}{\omega^{\gamma + n + (N/2)}} e^{-\frac{n}{2} \Delta_k} \sum_{s=0}^{\left[ \frac{d}{4} \right]} (-1)^s \omega^{2s} \Delta_k^s p(X_\alpha)
\]
\[
= \frac{e^{i \Delta_k}}{\omega^{\gamma + n + (N/2)}} e^{-\frac{n}{2} \Delta_k} e^{-\frac{n}{2} \Delta_k} p(X_\alpha)
\]
\[
= \frac{e^{i \Delta_k}}{\omega^{\gamma + n + (N/2)}} e^{-\frac{n}{2} \Delta_k} p(X_\alpha).
\]

Replacing \( \omega \) and \( X_\alpha \) by their values given in (4.29) and (4.32) and use Lemma 2.1 in [21], we obtain
\[
e^{\frac{n-2}{2} \Delta_k} p(X_\alpha) = \frac{i^n}{2^n} \sin^n \alpha e^{-\sin^2(\alpha)(\omega - \omega^2) \Delta_k} p(x)
\]
\[
= \frac{i^n}{2^n} \sin^n \alpha e^{-\Delta_k} p(x).
\]

Also,
\[
\omega^{\gamma + n + (N/2)} = \left( \frac{ie^{-i \alpha}}{2 \sin \alpha} \right)^{n + \gamma + (N/2)} = \frac{i^n e^{-i \alpha}}{2^n \sin^n \alpha} e^{i \gamma + n + (N/2)}
\]
\[
e^{\frac{i \Delta_k}{2}} = e^{\frac{\sin \alpha}{2 \sin \alpha} |x|^2}.
\]

Then
\[
\int_{\mathbb{R}^N} e^{-\Delta_k} p(y) K \left( \frac{ix}{\sin \alpha}, \gamma \right) e^{-\omega |y|^2} \omega_k(y) \, dy = A_\alpha^{-1} e^{i \alpha} e^{\frac{i \alpha}{2 \sin \alpha} |x|^2} e^{-\Delta_k} p(x).
\]
(4.33)

Finally, if we multiply equation (4.33) by \( A_\alpha e^{-\frac{i}{2} \cot(\alpha) |x|^2} \), we obtain the desired result.

A consequence of the Master formula (4.28) is
Corollary 4.2 (Hecke type identity) If in addition to the assumption in Theorem 4.3 the polynomial \( p \in \mathcal{H}_n \), then (4.28) becomes

\[
D_k^n \left[ e^{-\frac{|x|^2}{2}} p \right](x) = e^{i\alpha} e^{-\frac{|x|^2}{2}} p(x).
\]

(4.34)

Now, we are interesting to complete the spectral study of \( T \) started in proposition 3.5 by means of the Master formula. In fact we have the following

Corollary 4.3

\( L^2(\mathbb{R}^N, \omega_k(x) \, dx) \) decomposes as an orthogonal Hilbert space sum according to

\[
L^2(\mathbb{R}^N, \omega_k(x) \, dx) = \bigoplus_{n \in \mathbb{Z}_+} V_n,
\]

where

\[
V_n = \left\{ e^{-\frac{|x|^2}{2}} e^{-\frac{\Delta_k}{2}} p(x) ; \quad p \in \mathcal{P}_n \right\}
\]

is the eigenspace of \( T \) corresponding to the eigenvalue \( in \). In particular, \( T \) is essentially self-adjoint. The spectrum of its closure is purely discrete and given by

\[
\sigma(T) = i\mathbb{Z}_+.
\]

Proof.

Let \( f \) be an element of the subspace \( V_n \) defined by

\[
f(x) = e^{-\frac{|x|^2}{2}} e^{-\frac{\Delta_k}{2}} p(x),
\]

where \( p \in \mathcal{P}_n \). From (4.28), the limits

\[
\lim_{\alpha \to 0} \frac{D_k^n f - f}{\alpha} = \lim_{\alpha \to 0} \frac{e^{i\alpha} - 1}{\alpha} f
\]

exists in \( L^2(\mathbb{R}^N, \omega_k(x)dx) \) and equals \( \inf \). Then

\[
f \in D(T) \quad \text{and} \quad T(f) = \inf.
\]

Hence, \( V_n \) is the eigenspace of \( T \) corresponding to the eigenvalue \( in \).

5 Realization of the operator \( T \).

The aim of the following is to find a subspace \( \mathcal{W} \subset D(T) \) of \( L^2(\mathbb{R}^N, \omega_k(x)dx) \) in which we define \( T \) explicitly.

Lemma 5.1 For \( z \in \mathbb{C}^N \) set \( l(z) = \sum_{i=1}^{N} z_i^2 \). Then for all \( z, \omega \in \mathbb{C}^N \),

\[
k_c \int_{\mathbb{R}^N} K(2z, x)K(2\omega, x)e^{-A|x|^2} \omega_k(x) \, dx = \frac{e^{l(z)+l(\omega)}}{A^{1+N/2}} \frac{1}{A^{1+N/2}} K(2z/A, \omega),
\]

(5.1)

where \( A \) is a complex number such that \( \Re(A) > 0 \).

Proof. The result is obtained by means of a similar technic used in the proof of Lemma 4.2 and the following formula (see [9])

\[
k_c \int_{\mathbb{R}^N} K(2z, x)K(2\omega, x)e^{-|x|^2} \omega_k(x) \, dx = e^{l(z)+l(\omega)} K(2z, \omega).
\]

Theorem 5.1 Let \( f \in L^1(\mathbb{R}^N, \omega_k(x)dx) \cap L^2(\mathbb{R}^N, \omega_k(x)dx) \) such that \( D_k f \in L^1(\mathbb{R}^N, \omega_k(x)dx) \) and \( \alpha \not\in \left\{ \frac{1}{2} + k\pi , \quad k \in \mathbb{Z} \right\} \). Then

\[
D_k^n f(x) = k_c \left( \frac{e^{-i\alpha}}{2 \cos \alpha} \right)^{\frac{1}{2}} \int_{\mathbb{R}^N} e^{\frac{i\tan(\alpha)(|x|^2+|y|^2)}{2}} K \left( \frac{i\alpha}{\cos \alpha}, y \right) D_k f(y) \omega_k(y) dy.
\]

(5.2)
Proof. Let \( f \in L^1(\mathbb{R}^N, \omega_k(x)dx) \cap L^2(\mathbb{R}^N, \omega_k(x)dx) \) such that \( D_k f \in L^1(\mathbb{R}^N, \omega_k(x)dx) \). Let \( \epsilon \) be an arbitrary positive number and put

\[
F_\epsilon(x) = \int_{\mathbb{R}^N} f(y) g_\epsilon(y) \omega_k(y) \, dy,
\]

where \( g_\epsilon(y) = e^{-\epsilon + \frac{1}{\epsilon} \cot \alpha |y|^2} K \left( \frac{i x}{\sin \alpha}, y \right) \).

From (2.8), we deduce that

\[
|f(y) g_\epsilon(y)| \leq |f(y)|,
\]

so the dominated convergence theorem can be invoked again to give

\[
\lim_{\epsilon \to 0} F_\epsilon(x) = \frac{e^{\frac{1}{2} t^2 \cot \alpha}}{A_\alpha} D_k^0 f(x).
\]

Using Lemma [5.1] we can show

\[
D_k g_\epsilon(x) = e^{-\frac{|x|^2}{4 \epsilon \sin^2 \alpha + \cos^2 \alpha}} e^{-\frac{|x|^2}{4 \epsilon \cot \alpha}} \times K \left( \frac{x}{2 \epsilon \sin \alpha + i \cos \alpha}, \xi \right).
\]

Now applying the Parseval formula for the Dunkl transform (see Lemma 4.25, [11]) and using (5.4), we obtain

\[
F_\epsilon(x) = \frac{e^{-\frac{|x|^2}{4 \epsilon \sin^2 \alpha + \cos^2 \alpha}}}{(2 \epsilon + i \cot \alpha)^{\gamma + N/2}} \times \int_{\mathbb{R}^N} e^{-\frac{|x|^2}{4 \epsilon \cot \alpha}} K \left( \frac{x}{2 \epsilon \sin \alpha + i \cos \alpha}, \xi \right) D_k f(-\xi) \omega_k(\xi) \, d\xi.
\]

(2.7) gives again the following majorization:

\[
\left| K \left( \frac{x}{2 \epsilon \sin \alpha + i \cos \alpha}, \xi \right) \right| \leq e^{-\frac{2 \epsilon \sin \alpha}{4 \epsilon \sin^2 \alpha + \cos^2 \alpha} |x| |\xi|}.
\]

Hence,

\[
e^{-\frac{1}{4 \epsilon \cot \alpha}} K \left( \frac{x}{2 \epsilon \sin \alpha + i \cos \alpha}, \xi \right) \leq e^{-p_\epsilon |\xi|^2 + q_\epsilon |\xi|},
\]

where

\[
p_\epsilon = \frac{\epsilon}{4 \epsilon^2 + \cot^2 \alpha}
\]

and

\[
q_\epsilon = \frac{2 \epsilon \sin(\alpha) |x|}{4 \epsilon^2 \sin^2(\alpha) + \cos^2(\alpha)}.
\]

As \( p_\epsilon > 0 \), we deduce that

\[
\sup_{s \geq 0} (-p_\epsilon s^2 + q_\epsilon s) = -\frac{q_\epsilon^2}{4 p_\epsilon}.
\]

Applying formula (5.5) and (5.6), we obtain

\[
\left| e^{-\frac{1}{4 \epsilon \cot \alpha} K \left( \frac{x}{2 \epsilon \sin \alpha + i \cos \alpha}, \xi \right)} D_k f(-\xi) \right| \leq e^{-\frac{q_\epsilon |x|}{4 \epsilon^2 \sin^2 \alpha + \cos^2 \alpha}} |D_k f(-\xi)| \leq B_\epsilon |D_k f(-\xi)|,
\]

where \( B_\epsilon = \sup_{\epsilon \in [0,1]} e^{-\frac{q_\epsilon}{4 \epsilon^2 \sin^2 \alpha + \cos^2 \alpha} |x|} \). The function \( \xi \mapsto D_k f(-\xi) \) is in \( L^1(\mathbb{R}^N, \omega_k(x)dx) \), then the dominated convergence theorem implies

\[
\lim_{\epsilon \to 0} F_\epsilon(x) = \frac{e^{|x|^2}}{(i \epsilon)^{\gamma + N/2}} \times \int_{\mathbb{R}^N} \frac{e^{i |\xi|^2 \tan \alpha}}{2} K \left( -\frac{i x}{\cos \alpha}, \xi \right) D_k f(-\xi) \omega_k(\xi) \, d\xi.
\]
Hence, (5.3) and (5.7) gives after simplification

\[ D_k^a f(x) = c_k \left( \frac{e^{-ia \alpha}}{2 \cos \alpha} \right)^{\gamma + \frac{i}{2}} e^{\frac{1}{2} |x|^2 \tan \alpha} \times \int_{\mathbb{R}^N} e^{\frac{i}{2} |\xi|^2 \tan \alpha} K \left( -\frac{i}{\cos \alpha}, \xi \right) D_k f (-\xi) \omega_k (\xi) \, d\xi. \]  

Finally, if we make the change of variables \( u = -y \) in (5.3), then we find (5.2).

**Remark 5.1** Using (5.2) together with the dominated convergence theorem, we get

\[ \lim_{\alpha \to 0^+} D_k^a f(x) = \lim_{\alpha \to 0^-} D_k^a f(x) = D_k^a f(-x) = f(x), \ a.e., \]

\[ \lim_{\alpha \to -\pi^+} D_k^a f(x) = \lim_{\alpha \to -\pi^-} D_k^a f(x) = D_k^a f(x) = f(-x) \ a.e. \]

**Corollary 5.1** Under the assumptions of Theorem 5.1, we have

\[ \frac{D_k^a f(x) - f(x)}{\alpha} = r_1 (\alpha) \frac{c_k}{2^{\gamma + \frac{N}{2}}} \int_{\mathbb{R}^N} e^{\frac{i}{2} \tan(\alpha)(|x|^2 + |y|^2)} K \left( \frac{i}{\cos \alpha}, y \right) D_k f(y) \omega_k (y) \, dy \]

\[ + \frac{c_k}{2^{\gamma + \frac{N}{2}}} \int_{\mathbb{R}^N} r_2 (\alpha, x, y) D_k f(y) \omega_k (y) \, dy, \ a.e., \]  

where

\[ r_1 (\alpha) = \left( \frac{e^{-ia \alpha}}{2 \cos \alpha} \right)^{\gamma + \frac{i}{2}} - 1 \]

and

\[ r_2 (\alpha, x, y) = e^{\frac{i}{2} \tan(\alpha)(|x|^2 + |y|^2)} K \left( \frac{i}{\cos \alpha}, y \right) - K \left( i, y \right). \]

**Proof.** The result is consequence of (5.2) and (2.9).

**Lemma 5.2** Let \( \alpha_0 \in ]0, \frac{\pi}{2} [ \) and \( x, y \in \mathbb{R}^N \). Then

\[ |r_2 (\alpha, x, y)| \leq \frac{1}{2} (1 + \tan^2 \alpha_0)(|x|^2 + |y|^2) + \left| \frac{\sin(\alpha_0)}{\cos^2(\alpha_0)} \right| \sqrt{N} |x||y|, \]  

where \( \alpha \in ]0, \alpha_0 [ \).

**Proof.** By the mean value theorem, we have

\[ |r_2 (\alpha, x, y)| \leq \sup_{\alpha \in ]0, \alpha_0 [} \left| \frac{\partial}{\partial \alpha} r_3 (\alpha, x, y) \right|, \]

where

\[ r_3 (\alpha, x, y) = e^{\frac{i}{2} \tan(\alpha)(|x|^2 + |y|^2)} K \left( \frac{i}{\cos \alpha}, y \right). \]

From (2.5), we get

\[ K \left( \frac{i}{\cos \alpha}, y \right) = K \left( x, \frac{iy}{\cos \alpha} \right). \]

Therefore,

\[ r_3 (\alpha, x, y) = e^{\frac{i}{2} \tan(\alpha)(|x|^2 + |y|^2)} K \left( x, \frac{iy}{\cos \alpha} \right). \]

A simple calculations shows that

\[ \frac{\partial}{\partial \alpha} r_3 (\alpha, x, y) = \frac{i}{2} (1 + \tan^2 \alpha)(|x|^2 + |y|^2) r_3 (\alpha, x, y) \]

\[ + \frac{i \sin(\alpha)}{\cos^2(\alpha)} e^{\frac{i}{2} \tan(\alpha)(|x|^2 + |y|^2)} \sum_{j=1}^{N} y_j \frac{\partial}{\partial y_j} K \left( x, \frac{iy}{\cos \alpha} \right). \]  

(5.11)
From (2.7), the inequality
\[ \left| \frac{\partial}{\partial y_j} K \left( x, \frac{iy}{\cos \alpha} \right) \right| \leq |x| \]
holds and hence
\[ \left| \frac{\partial}{\partial \alpha} r_3(\alpha, x, y) \right| \leq \frac{1}{2} (1 + \tan^2 \alpha)(|x|^2 + |y|^2) + \frac{|\sin(\alpha)|}{\cos^2(\alpha)} |x| \sum_{j=1}^N |y_j| \]
\[ \leq \frac{1}{2} (1 + \tan^2 \alpha)(|x|^2 + |y|^2) + \frac{|\sin(\alpha)|}{\cos^2(\alpha)} \sqrt{N} |x||y| \]
\[ \leq \frac{1}{2} (1 + \tan^2 \alpha_0)(|x|^2 + |y|^2) + \frac{|\sin(\alpha_0)|}{\cos^2(\alpha_0)} \sqrt{N} |x||y|. \]

Which finishes the proof.

**Theorem 5.2** Let
\[ \mathcal{W} = \{ f \in L^1(\mathbb{R}^N, \omega_k(x)dx) \cap L^2(\mathbb{R}^N, \omega_k(x)dx) \mid |y|^2 f \in L^2(\mathbb{R}^N, \omega_k(x)dx) \}
\]
and \[ |y|^2 D_k f \in L^1(\mathbb{R}^N, \omega_k(x)dx) \cap L^2(\mathbb{R}^N, \omega_k(x)dx) \} . \]

Then for all \( f \in \mathcal{W} \),
\[ Tf(x) = -i(\gamma + (N/2))f(x) + \frac{i}{2} |x|^2 f(x) + \frac{i}{2} D_k \left[ |y|^2 D_k(y) \right] (-x) \ a.e. \quad (5.12) \]

**Proof.** It is clear that
\[ \lim_{\alpha \to 0} r_1(\alpha) = -i(\gamma + (N/2)). \]
In view of (2.3), we deduce
\[ \left| K \left( \frac{ix}{\cos \alpha}, y \right) \right| \leq 1. \]
Then
\[ \left| e^{\frac{i}{2} \tan(\alpha)(|x|^2 + |y|^2)} K \left( \frac{ix}{\cos \alpha}, y \right) D_k f(y) \right| \leq |D_k f(y)|. \]
Let \( y \in \mathbb{R}^N \) such that \( |y| > 1 \). Then
\[ |D_k f(y)| \leq |y|^2 |D_k f(y)|. \]
Since \( y \mapsto |y|^2 D_k f \in L^1(\mathbb{R}^N, \omega_k(x)dx) \), it follows that \( D_k f \in L^1(\mathbb{R}^N, \omega_k(x)dx) \) and the dominated convergence theorem implies
\[ \lim_{\alpha \to 0} r_1(\alpha) \frac{c_k}{2^{7/2}} \int_{\mathbb{R}^N} e^{\frac{i}{2} \tan(\alpha)(|x|^2 + |y|^2)} K \left( \frac{ix}{\cos \alpha}, y \right) D_k f(y) \omega_k(y) dy \]
\[ = -i(\gamma + (N/2)) \frac{c_k}{2^{7/2}} \int_{\mathbb{R}^N} K(ix, y) D_k f(y) \omega_k(y) dy \]
\[ = -i(\gamma + (N/2)) D_k f(-x) \]
\[ = -i(\gamma + (N/2)) f(x), \ a.e. \]

From (5.11), we deduce
\[ \lim_{\alpha \to 0} r_2(\alpha, x, y) = \frac{i}{2} (|x|^2 + |y|^2) K(ix, y). \]

By the previous Lemma, we have the following majorization:
\[ |r_2(\alpha, x, y) D_k f(y)| \leq f_1(y) + f_2(y) + f_3(y), \]
where
\[ f_1(y) = \frac{1}{2} (1 + \tan^2 \alpha_0)|x|^2 |D_k f(y)|, \]
\[ f_2(y) = \frac{1}{2} (1 + \tan^2 \alpha_0)|y|^2 |D_k f(y)|, \]
\[ f_3(y) = \frac{|\sin(\alpha_0)|}{\cos^2(\alpha_0)} \sqrt{N} |x||y| |D_k f(y)|. \]
Corollary 5.2
1) $S(\mathbb{R}^N) \subset W \subset D(T)$.
2) For all $f \in S(\mathbb{R}^N)$,
\[
-iTf = - (\gamma + (N/2))f + \frac{1}{2}(|x|^2 - \Delta_k)f
\]

Proof.
1) Obvious.
2) Let $f \in S(\mathbb{R}^N)$. From Corollary 2.11 in [9], we deduce
\[
- y_j^2 D_k f(y) = D_k [T_j^2 f](y),
\]
where $j \in \{1, 2, \ldots, N\}$. Then
\[
- |y|^2 D_k f(y) = D_k [\Delta_k f](y).
\]
Therefore
\[
- D_k \left[|y|^2 D_k(y)\right](-x) = D_k^2[\Delta_k f(y)](-x) = \Delta_k f(x).
\]
(5.13)

Finally, from (5.12) and (5.13) we obtain the desired result.

Remark 5.2 It is clear that the operator $2iT - (2\gamma + N)$ is an extension on $W$ of the Hermite operator $\mathcal{H}_k = \Delta_k - |x|^2$ studied by Rösler [21] where it used another approach based on the notion of Lie algebra. In the same context, we give a new proof of the following result established in [21]

Corollary 5.3 For $n \in \mathbb{N}$ and $p \in \mathcal{P}_n$, the function $f = e^{-\frac{|x|^2}{2}} e^{-\frac{\Delta_k}{2}} p(x)$ satisfies
\[
(\Delta_k - |x|^2)f = -(2n + 2\gamma + N)f.
\]
(5.14)

In particular
\[
(\Delta_k - |x|^2)h_\nu = -(2|\nu| + 2\gamma + N) h_\nu.
\]
(5.15)

Proof. Since $f \in S(\mathbb{R}^N)$, (5.14) is obtained by the use of the previous Corollary and (5.35)

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