Additivity of entanglement of formation for some special cases

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The proof of additivity of entanglement of formation for some special cases is given. The strong
concavity of von Neumann entropy due to strong subadditivity of von Neumann entropy is presented.
Some general relations concerning about the entanglement of formation are proposed.

I. INTRODUCTION AND SOME GENERAL RESULTS

Entanglement of formation (EoF) is a widely accepted measurement of entanglement [1]. For a bipartite state $\rho_{AB}$ in Hilbert space $H_A \otimes H_B$, the entanglement of formation is defined as

$$E_f(\rho_{AB}) = \min \sum_i p_i S\left(\rho\left|_A^i\right|\right),$$

where the minimization is over all possible ensembles such that $\rho_{AB} = \sum_i p_i |\Psi_i^A_B\rangle\langle\Psi_i^A_B|$, $S(\rho) = -\text{Tr}\rho \log_2 \rho$ is the von Neumann entropy.

It is a long-standing conjecture that for product density matrix $\rho_{AB} \otimes \rho_{A'B'}$ in Hilbert space $H_A \otimes H_B \otimes H_A' \otimes H_B'$ the EoF is additive:

$$E_f(\rho_{AB} \otimes \rho_{A'B'}) = E_f(\rho_{AB}) + E_f(\rho_{A'B'}).$$

But only a few cases are proved.

From the definition we know that EoF is weak additive, i.e., the following inequality holds:

$$E_f(\rho_{AB} \otimes \rho_{A'B'}) \leq E_f(\rho_{AB}) + E_f(\rho_{A'B'}).$$

In order to prove the additivity of EoF, we just need to prove the opposite inequality.

First, we present some relations concerning about the EoF, trivial or not:

$$\min \sum_i p_i \left(S(\rho_A^i) + S(\rho_{A'}^i)\right) = \min \sum_i p_i \left(S(\rho_A^i) + E_f(\rho_{A'B'}^i)\right) = \min \sum_i p_i \left(E_f(\rho_{AB}^i) + S(\rho_{A'}^i)\right) = \min \sum_i p_i \left(E_f(\rho_{AB}^i) + E_f(\rho_{AB'}^i)\right) = E_f(\rho_{AB}) + E_f(\rho_{A'B'}),$$

where we have pure state decomposition

$$\rho_{AB} \otimes \rho_{A'B'} = \sum_i p_i |\Psi_i^A_{ABA'}\rangle\langle\Psi_i^A_{ABA'}|,$$

Here we denote $\rho_A^i = Tr_{BA'B'}(|\Psi_i^A_{ABA'}\rangle\langle\Psi_i^A_{ABA'}|)$, and similar notations are also used for other states with partial traces. We remark that optimal pure state decomposition in one minimization is not necessarily the optimal decomposition in another minimization. The proof of these relations are straightforward.

We can find from these relations that in order to prove the additivity of EoF, we need to prove the EoF

$$E_f(\rho_{AB} \otimes \rho_{A'B'}) = \sum_i p_i S(\rho_{AA'}^i),$$
is at least one quantities appeared in the minization in relations (4,5,6,7), where we suppose (8) is optimal pure states decomposition for EoF. However, it seems unlikely to use directly the first quantities since we have subadditivity inequality,

$$S(\rho_{AA'}^i) \leq S(\rho_A^i) + S(\rho_{A'}^i),$$  \hspace{1cm} (10)

where the equality holds if and only if $\rho_{AA'}^i = \rho_A^i \otimes \rho_{A'}^i$, which can not be satisfied in general in (9). In this paper, we will use the second or the third quantities in (4,5) to prove the additivity of EoF for some special cases.

II. STRONG CONCAVITY OF VON NEUMANN ENTROPY

Before we proceed, we give some useful relations \([4]\),

$$S(\sum_i p_i \rho' \otimes |i\rangle\langle i|) = H(p_i) + \sum_i p_i S(\rho').$$ \hspace{1cm} (11)

where $H(p_i)$ is Shannon entropy. We also have the strong concavity of von Neumann entropy

$$S(\sum_i p_i \rho_1^i \otimes \rho_2^i) \geq \sum_i p_i S(\rho_1^i) + S(\sum_i p_i \rho_2^i),$$ \hspace{1cm} (12)

$$S(\sum_i p_i \rho_1^i \otimes \rho_2^i) \geq \sum_i p_i S(\rho_1^i) + \sum_i p_i S(\rho_2^i),$$ \hspace{1cm} (13)

due to the strong subadditivity of von Neumann entropy \([8]\)

$$S(\rho_{123}) + S(\rho_2) \leq S(\rho_{12}) + S(\rho_{23}).$$ \hspace{1cm} (14)

We define $\rho_{123} = \sum_i p_i \rho_1^i \otimes \rho_2^i \otimes |i\rangle\langle i|$, and use the relation \([13]\), and we have

$$S(\rho_{12}) \geq S(\rho_{123}) + S(\rho_2) - S(\rho_{23}) = H(p_i) + \sum_i p_i S(\rho_1^i \otimes \rho_2^i) + S(\sum_i p_i \rho_2^i) - H(p_i) - \sum_i p_i S(\rho_2^i)$$

$$= \sum_i p_i S(\rho_1^i) + S(\sum_i p_i \rho_2^i).$$ \hspace{1cm} (15)

So, we proved the relation \([12]\), similarly for \([13]\). Relations \([12,13]\) is stronger than the relation due to concavity of von Neumann entropy,

$$S(\sum_i p_i \rho_1^i \otimes \rho_2^i) \geq \sum_i p_i (S(\rho_1^i) + S(\rho_2^i)).$$ \hspace{1cm} (16)

Comparing \([14]\) and \([11]\), we know

$$H(p_i) \geq S(\sum_i p_i \rho_i).$$ \hspace{1cm} (17)

This was proved in ref. \([8]\), and reproved by a different method in ref. \([7]\), here we provide a different proof. The strong subadditivity of von Neumann entropy \([8]\) was recently used to prove the additivity of entanglement breaking channel \([5]\) and entanglement cost \([9]\) where essentially the strong concavity of von Neumann entropy \([12,13]\) were used. Since the relations \([12,13]\) are very useful and deserve an independent name, we call them the strong concavity of von Neumann entropy in this paper and use them directly without tracking back to the strong subadditivity of von Neumann entropy.
III. ADDITIVITY OF EOF FOR SPECIAL CASE I

Next we consider a special class of pure states,

$$|\Psi_{ABA'B'}\rangle = \sum_{\alpha\beta} \sqrt{\lambda_{\alpha\beta}}|\alpha\rangle_A|\alpha\rangle_B|\beta\rangle_{A'}|\beta\rangle_{B'}.$$  \hspace{1cm} (18)

we know

$$\rho_{AB} = Tr_{A'B'}(|\Psi_{ABA'B'}\rangle\langle\Psi_{ABA'B'}|)$$

$$= \sum_{\beta} \left( \sum_{\alpha} \sqrt{\lambda_{\alpha\beta}}|\alpha\rangle_A\langle\alpha| \right) \left( \sum_{\alpha'} \sqrt{\lambda_{\alpha'\beta}_A}|\alpha'\rangle_B\langle\alpha'| \right)$$

$$= \sum_{\beta} \lambda_{\beta}|\Psi_{AB}^\beta\rangle\langle\Psi_{AB}^\beta|,$$  \hspace{1cm} (19)

where we use the notation

$$|\Psi_{AB}^\beta\rangle \equiv \frac{1}{\sqrt{\lambda_{\beta}}} \left( \sum_{\alpha} \sqrt{\lambda_{\alpha\beta}}|\alpha\rangle_A\langle\alpha| \right)$$,  \hspace{1cm} (20)

Similarly, we have the result for \(\rho_{A'B'}\)

$$\rho_{A'B'} = \sum_{\alpha} \left( \sum_{\beta} \sqrt{\lambda_{\alpha\beta}}|\beta\rangle_{A'}\langle\beta| \right) \left( \sum_{\beta'} \sqrt{\lambda_{\beta\beta'}_{A'}}|\beta\rangle_{B'}\langle\beta'| \right)$$

$$= \sum_{\alpha} \lambda_{\alpha}|\Psi_{A'B'}^\alpha\rangle\langle\Psi_{A'B'}^\alpha|,$$  \hspace{1cm} (21)

and we use the notations

$$|\Psi_{A'B'}^\alpha\rangle \equiv \frac{1}{\sqrt{\lambda_{\alpha}}} \left( \sum_{\beta} \sqrt{\lambda_{\alpha\beta}}|\beta\rangle_{A'}\langle\beta| \right)$$,  \hspace{1cm} (22)

We also know the reduced density operator

$$\rho_{AA'} = \sum_{\alpha\beta} \lambda_{\alpha\beta}|\alpha\rangle_A\langle\alpha| \otimes |\beta\rangle_{A'}\langle\beta|$$

$$= \sum_{\alpha} \lambda_{\alpha}|\alpha\rangle_A\langle\alpha| \otimes Tr_{B'}(|\Psi_{A'B'}^\alpha\rangle\langle\Psi_{A'B'}^\alpha|).$$  \hspace{1cm} (23)

Then by using the relation \([11]\), we obtain

$$S(\rho_{AA'}) = H(\lambda_{\alpha}) + \sum_{\alpha} \lambda_{\alpha} S(Tr_{B'}(|\Psi_{A'B'}^\alpha\rangle\langle\Psi_{A'B'}^\alpha|))$$

$$= S(\rho_A) + \sum_{\alpha} \lambda_{\alpha} S(Tr_{B'}(|\Psi_{A'B'}^\alpha\rangle\langle\Psi_{A'B'}^\alpha|)) \hspace{1cm} (24)$$

Due to pure state decomposition of \(\rho_{A'B'}\) presented in \([21]\), we have

$$\sum_{\alpha} \lambda_{\alpha} S(Tr_{B'}(|\Psi_{A'B'}^\alpha\rangle\langle\Psi_{A'B'}^\alpha|)) \geq E_f(\rho_{A'B'}).$$  \hspace{1cm} (25)

Thus

$$S(\rho_{AA'}) \geq S(\rho_A) + E_f(\rho_{A'B'}).$$  \hspace{1cm} (26)

Similar to relation \([24]\), we have
\[ S(\rho_{AA'}) = S(\rho_{A'}) + \sum_{\beta} \lambda_{\beta} S \left( Tr_B (|\Psi_{AB}^{\beta}\rangle\langle \Psi_{AB}^{\beta}|) \right). \] (27)

We conclude the superadditivity of EoF is true for pure state \(|\Psi_{ABA'B'}\rangle\). The relations (24) and (26) are even stronger than the superadditivity of entanglement of formation. We summarize

\[ S(\rho_{AA'}) \geq E_f(\rho_{AB}) + E_f(\rho_{A'B'}). \] (28)

In case all pure states in the decomposition of \(\rho_{AB} \otimes \rho_{A'B'}\) can be written as the form (18) by independent Schmidt decomposition on \(H_A \otimes H_B\) and \(H_{A'} \otimes H_{B'}\), the additivity of EoF holds since we have

\[ E_f(\rho_{AB} \otimes \rho_{A'B'}) = \sum_{i} p_i E_f(|\Psi_{AB}^{i}\rangle\langle \Psi_{AB}^{i}|) \geq \sum_{i} p_i \left( E_f(\rho_{AB}) + E_f(\rho_{A'B'}) \right) \geq E_f(\rho_{AB}) + E_f(\rho_{A'B'}), \] (29)

where we assume (8) is the optimal decomposition for EoF. We remark that the additivity of EoF \([2,9]\) deduced from additivity of entanglement breaking channel \([5]\) belong to the class \([18]\).

For an arbitrary pure state \(|\tilde{\Psi}_{ABA'B'}\rangle\), we can always find unitary transformations \(U_{AA'}\) and \(V_{BB'}\) in \(H_A \otimes H_{A'}\) and \(H_B \otimes H_{B'}\) respectively, and transfer \(|\tilde{\Psi}_{ABA'B'}\rangle\) to (18),

\[ U_{AA'} \otimes V_{BB'} |\tilde{\Psi}_{ABA'B'}\rangle = |\Psi_{ABA'B'}\rangle = \sum_{\alpha \beta} \sqrt{\lambda_{\alpha \beta}} |\alphaangle_A |\alpha\rangle_B |\beta\rangle_A' |\beta\rangle_{B'}. \] (30)

We know

\[ S(\tilde{\rho}_{AA'}) = S(\rho_{AA'}) = S(\rho_A) + \sum_{\alpha} \lambda_{\alpha} S \left( Tr_B (|\Psi_{AB}^{\alpha}\rangle\langle \Psi_{AB}^{\alpha}|) \right), \] (31)

but it is not clear whether in general the relation

\[ S(\tilde{\rho}_{AA'}) \geq E_f(\rho_{AB}) + E_f(\tilde{\rho}_{A'B'}) \] (32)

holds or not, this relation is called superadditivity of EoF \([3,2]\). If the superadditivity of EoF holds for arbitrary pure states, the additivity of EoF follows directly.

**IV. ADDITIVITY OF EOF FOR SPECIAL CASE II**

In what follows, we restate the HJW theorem presented in ref. \([7]\): For any density matrix \(\rho\) having the diagonal form

\[ \rho = \sum_i \lambda_i |e_i\rangle\langle e_i|, \] (33)

can be written as the mixed states of \(|\psi_i\rangle\) with probability \(p_i\)

\[ \rho = \sum_i p_i |\psi_i\rangle\langle \psi_i|, \] (34)

iff there exists a unitary transformation \(U\) such that

\[ |\psi_i\rangle = \frac{1}{\sqrt{p_i}} \sum_j U_{ij} \sqrt{\lambda_j} |e_j\rangle. \] (35)
Now we present another class of states for which the additivity of EoF holds. Consider about two density matrices in $H_A \otimes H_B$ and $H_{A'} \otimes H_{B'}$,

$$\rho_{AB} = \sum_j \lambda_j |J\rangle_{AB}\langle J|,$$

$$\rho_{A'B'} = \sum_K \lambda_K |K\rangle_{A'B'}\langle K|,$$

where $\lambda_j$ and $|J\rangle_{AB}$ are eigenvalues and eigenvectors of density operator $\rho_{AB}$, and similarly for $\rho_{A'B'}$.

Let’s assume that the eigenvectors $|J\rangle_{AB}$ have some special properties. Suppose $|J\rangle_{AB} = \sum a_{j_1j_2}|j_1\rangle_A|j_2\rangle_B$, $|J'\rangle_{AB} = \sum a'_{j_1'j_2'}|j_1'\rangle_A|j_2'\rangle_B$, $J \neq J'$, we assume $|j_1\rangle_A \neq |j_1'\rangle_A$, $|j_2\rangle_B \neq |j_2'\rangle_B$. For example, $\rho_{AB} = \lambda|00\rangle\langle 00| + (1 - \lambda)\frac{1}{2}(|11\rangle\langle 11| + |22\rangle\langle 22|)$. We assume $|K\rangle_{A'B'}$ also has this property.

The Schmidt decomposition of states $|J\rangle_{AB}$ and $|K\rangle_{A'B'}$ have the following form:

$$|J\rangle_{AB} = \sum_{\alpha_j} \sqrt{\eta_{\alpha_j}}|\alpha_j\rangle_A|\alpha_j\rangle_B,$$

$$|K\rangle_{A'B'} = \sum_{\beta_K} \sqrt{\xi_{\beta_K}}|\beta_K\rangle_{A'}|\beta_K\rangle_{B'},$$

Since $|J\rangle_{AB}$ and $|J'\rangle_{AB}$ have the property presented above, we have the following relation

$$\langle \alpha_j | \alpha'_{j'} \rangle = \delta_{j,j'} \delta_{\alpha_j, \alpha'_{j'}}.$$  

This can be understood that $\rho_{AB}$ is block diagonal, so besides $\langle \alpha_j | \alpha'_{j} \rangle = \delta_{\alpha_j, \alpha'_{j}}$ inside one block, we also have orthogonal relation for different blocks. Similarly, we have

$$\langle \beta_K | \beta'_{K'} \rangle = \delta_{K,K'} \delta_{\beta_K, \beta'_{K'}}.$$  

Now, let’s consider the pure states decomposition,

$$\rho_{AB} \otimes \rho_{A'B'} = \sum_i p_i |\Psi_{ABA'B'}^i\rangle\langle \Psi_{ABA'B'}^i|.$$  

Due to HJW theorem [2] we can find a unitary matrix $U_{i,JK}$ and also considering about the Schmidt decomposition, we have

$$|\Psi_{ABA'B'}^i\rangle = \frac{1}{\sqrt{p_i}} \sum_{JK} U_{i,JK} \sqrt{\lambda_J \lambda_K} |J\rangle_{AB} \otimes |K\rangle_{A'B'},$$

$$= \frac{1}{\sqrt{p_i}} \sum_{JK} U_{i,JK} \sqrt{\lambda_J \lambda_K} \sum_{\alpha_j} \sqrt{\eta_{\alpha_j}}|\alpha_j\rangle_A|\alpha_j\rangle_B \otimes (\sum_{\beta_K} \sqrt{\xi_{\beta_K}}|\beta_K\rangle_{A'}|\beta_K\rangle_{B'}).$$

We remark that the properties of eigenvectors already used here. So, the reduced density operator in $H_A \otimes H_{A'}$ can be obtained as

$$\rho_{AA'} = Tr_{BB'} (|\Psi_{ABA'B'}^i\rangle\langle \Psi_{ABA'B'}^i|$$

$$= \frac{1}{p_i} \sum_{JK} U_{i,JK}^* U_{i,JK} \sqrt{\lambda_J \lambda_K \lambda_{J'} \lambda_{K'}} \sqrt{\eta_{\alpha_j} \eta_{\alpha'_{j'}}} \sum_{\beta_K} \xi_{\beta_K} \delta_{\beta_K, \beta'_{K'}} |\alpha_j\rangle_A \langle \alpha_{j'}| \otimes |\beta_K\rangle_{A'} \langle \beta_{K'}|.$$  

where we have used the relations [10][11] when we take trace in $H_B \otimes H_{B'}$. With the help of strong cancavity relation [22], the von Neumann entropy of $\rho_{AA'}$ has the following form,
\[ S(\rho_{AA}') = S \left( \sum_K \lambda_K \left[ \frac{1}{p_i} \sum_j U_{iJK} U_{iJK}^* \sum_j \eta_{\alpha_j} |\alpha_j\rangle_A \langle \alpha_j| \otimes [\sum_{\beta_K} \xi_{\beta_K} |\beta_K\rangle_{A'} \langle \beta_K|] \right) \right. \]

\[ \geq \sum_K \lambda_K S \left( \frac{1}{p_i} \sum_j U_{iJK} U_{iJK}^* \sum_j \eta_{\alpha_j} |\alpha_j\rangle_A \langle \alpha_j| \right) + S \left( \sum_K \lambda_K \sum_{\beta_K} \xi_{\beta_K} |\beta_K\rangle_{A'} \langle \beta_K| \right) \]

\[ = \sum_K \lambda_K S \left( \rho^{iK} \right) + S \left( \rho_{A}' \right), \tag{47} \]

where we used the following notations and relations

\[ \rho^{iK} = \frac{1}{p_i} \sum_j U_{iJK} U_{iJK}^* \sum_j \eta_{\alpha_j} |\alpha_j\rangle_A \langle \alpha_j|, \tag{48} \]

\[ \rho_{A}' = \sum_K \lambda_K \sum_{\beta_K} \xi_{\beta_K} |\beta_K\rangle_{A'} \langle \beta_K| \tag{49} \]

\[ = Tr_A(\rho_{AA'}). \tag{50} \]

In the same time, we can identify

\[ \rho^{iK} = Tr_B \frac{1}{\sqrt{p_i}} \sum_{jj'} U_{iJK} U_{iJK}^* \sqrt{\lambda_J \lambda_{J'}} \sum_{\alpha_j \alpha'_{jj'}} \sqrt{\eta_{\alpha_j} \eta_{\alpha'_{jj'}}} |\alpha_j\rangle_A \langle \alpha_j| \otimes |\alpha'_{jj'}\rangle_B \langle \alpha'_{jj'}| \]

\[ = Tr_B |\Psi^{iK}_{AB}\rangle \langle \Psi^{iK}_{AB}|, \tag{51} \]

where we use the definition

\[ |\Psi^{iK}_{AB}\rangle \equiv \frac{1}{\sqrt{p_i}} \sum_j U_{iJK} \sqrt{\lambda_J} \sum_{\alpha_j} \sqrt{\eta_{\alpha_j}} |\alpha_j\rangle_A \langle \alpha_j|_B. \tag{52} \]

We can find the reduced density operator \( \rho_{AB} = Tr_{A'B'}(|\Psi_{ABA'B'}\rangle \langle \Psi_{ABA'B'}|) \) has the pure state decomposition

\[ \rho_{AB} = \sum_K \lambda_K |\Psi^{iK}_{AB}\rangle \langle \Psi^{iK}_{AB}|. \tag{53} \]

Thus we obtain the result from relation (47),

\[ E_f(|\Psi_{ABA'B'}\rangle) = S(\rho_{A'}) \geq E_f(\rho_{AB}) + E_f(\rho_{A'B'}). \tag{54} \]

Here we use the fact

\[ E_f(\rho_{AB}) = E_f(\sum_K \lambda_K |\Psi^{iK}_{AB}\rangle \langle \Psi^{iK}_{AB}|) \leq \sum_K \lambda_K S(\rho^{iK}), \tag{55} \]

\[ E_f(\rho_{A'B'}) \leq S(\rho_{A'}). \tag{56} \]

Here we summarize that the EoF of pure state \(|\Psi_{ABA'B'}\rangle\) is at least the sum of entanglement of formation of \(\rho_{AB}\) and \(\rho_{A'B'}\) provided the pure state decomposition relation (42),

\[ E_f(|\Psi_{ABA'B'}\rangle) \geq E_f(\rho_{AB}) + E_f(\rho_{A'B'}). \tag{57} \]

What follows seems straightforward. Suppose (42) is the optimal decomposition for EoF, we have

\[ E_f(\rho_{AB} \otimes \rho_{A'B'}) = \sum_i p_i E_f(|\Psi_{ABA'B'}\rangle) \]

\[ \geq \sum_i p_i (E_f(\rho_{AB}) + E_f(\rho_{A'B'})) \]

\[ \geq E_f(\rho_{AB}) + E_f(\rho_{A'B'}), \tag{58} \]
the last inequality is due to the fact
\[ \rho_{AB} = \sum_i p_i \rho_{AB}^i, \quad \rho_{A'B'} = \sum_i p_i \rho_{A'B'}^i. \] (59)

We already know the EoF of \( \rho_{AB} \otimes \rho_{A'B'} \) is at most the sum of entanglement of formation of \( \rho_{AB} \) and \( \rho_{A'B'} \)
\[ E_f(\rho_{AB} \otimes \rho_{A'B'}) \leq E_f(\rho_{AB}) + E_f(\rho_{A'B'}), \] (60)
thus we have the additivity of EoF for this special case.
\[ E_f(\rho_{AB} \otimes \rho_{A'B'}) = E_f(\rho_{AB}) + E_f(\rho_{A'B'}). \] (61)

V. SUMMARY AND DISCUSSION

Though we present here the additivity of EoF for some special cases, the problem of additivity of EoF for general case is still open.

For pure state in the decomposition,
\[ |\Psi_{ABA'B'}\rangle = \frac{1}{\sqrt{p_i}} \sum_{J,K} U_{i,J,K} \sqrt{\lambda_K} \lambda_K |J\rangle_{AB} \otimes |K\rangle_{A'B'}, \] (62)
we denote
\[ |\Psi_{iK}^{iK}\rangle = \frac{1}{\sqrt{p_i}} \sum_{J} U_{i,J,K} \sqrt{\lambda_K} |J\rangle_{AB}, \quad \rho_A^{iK} = Tr_B|\Psi_{iK}^{iK}\rangle \langle \Psi_{iK}^{iK}|; \] (63)
\[ |\Psi_{iJ}^{iJ}\rangle = \frac{1}{\sqrt{p_i}} \sum_{K} U_{i,J,K} \sqrt{\lambda_K} |K\rangle_{A'B'}, \quad \rho_{A'}^{iJ} = Tr_B|\Psi_{iJ}^{iJ}\rangle \langle \Psi_{iJ}^{iJ}|; \] (64)
\[ \rho_{AA'}^{i} = Tr_{BB'}|\Psi_{ABA'B'}\rangle \langle \Psi_{ABA'B'}|. \] (65)

Here, similar as the superadditivity of EoF, we may ask the question: whether the following relation holds:
\[ S(\rho_{A\overline{A}}^{i}) \geq \sum_K \lambda_K S(\rho_A^{iK}) + \sum_J \lambda_J S(\rho_{A'}^{iJ})? \] (66)

One possible method to answer this question is to know whether the following relation is true:
\[ S(\rho_{A\overline{A}}^{i}) \geq S\left( \sum_K \lambda_K \rho_A^{iK} \otimes (Tr_B|K\rangle_{A'B'}\langle K|) \right) + S\left( \sum_J \lambda_J (Tr_B|J\rangle_{AB}\langle J|) \otimes \rho_{A'}^{iJ} \right) \]
\[ -S\left( \sum_{JK} \frac{1}{p_i} \lambda_J \lambda_K |U_{i,J,K}|^2 (Tr_B|J\rangle_{AB}\langle J|) \otimes (Tr_B|K\rangle_{A'B'}\langle K|) \right)? \] (67)

We can obtain (63) from (67) by using strong concavity of von Neumann entropy for the first two terms and the subadditivity for the third term.

We summarize our idea here: a, relation (67) is a sufficient condition of relation (66); b, relation (66) is a sufficient condition for the additivity of EoF. But both a or b may not be necessary conditions.

Werner state is an interesting state to check the superadditivity of the EoF since the EoF of Werner state is known. Suppose we have a Werner state in \( H_A \otimes H_B \otimes H_A' \otimes H_B' \) with dimension 2 for each Hilbert space. The reduced density operators in \( H_A \otimes H_B \) and \( H_A' \otimes H_B' \) are still Werner states, and EoF are known. The result shows that the superadditivity of the EoF is correct for this case.

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