Nonlinear sigma models with AdS supersymmetry in three dimensions

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Abstract

In three-dimensional anti-de Sitter (AdS) space, there exist several realizations of $\mathcal{N}$-extended supersymmetry, which are traditionally labelled by two non-negative integers $p \geq q$ such that $p + q = \mathcal{N}$. Different choices of $p$ and $q$, with $\mathcal{N}$ fixed, prove to lead to different restrictions on the target space geometry of supersymmetric nonlinear $\sigma$-models. We classify all possible types of hyperkähler target spaces for the cases $\mathcal{N} = 3$ and $\mathcal{N} = 4$ by making use of two different realizations for the most general $(p,q)$ supersymmetric $\sigma$-models: (i) off-shell formulations in terms of $\mathcal{N} = 3$ and $\mathcal{N} = 4$ projective supermultiplets; and (ii) on-shell formulations in terms of covariantly chiral scalar superfields in (2,0) AdS superspace. Depending on the type of $\mathcal{N} = 3$, 4 AdS supersymmetry, nonlinear $\sigma$-models can support one of the following target space geometries: (i) hyperkähler cones; (ii) non-compact hyperkähler manifolds with a U(1) isometry group which acts non-trivially on the two-sphere of complex structures; (iii) arbitrary hyperkähler manifolds including compact ones. The option (iii) is realized only in the case of critical (4,0) AdS supersymmetry.

As an application of the (4,0) AdS techniques developed, we also construct the most general nonlinear $\sigma$-model in Minkowski space with a non-centrally extended $\mathcal{N} = 4$ Poincaré supersymmetry. Its target space is a hyperkähler cone (which is characteristic of $\mathcal{N} = 4$ superconformal $\sigma$-models), but the $\sigma$-model is massive. The Lagrangian includes a positive potential constructed in terms of the homothetic conformal Killing vector the target space is endowed with. This mechanism of mass generation differs from the standard one which corresponds to a $\sigma$-model with the ordinary $\mathcal{N} = 4$ Poincaré supersymmetry and which makes use of a tri-holomorphic Killing vector.
# Contents

1 Introduction

2 Nonlinear sigma models with four supercharges
   2.1 Nonlinear sigma models in (2,0) AdS superspace
   2.2 Nonlinear sigma models in (1,1) AdS superspace

3 Sigma models with (3,0) AdS supersymmetry: Off-shell approach

4 Sigma models with (3,0) AdS supersymmetry: On-shell approach

5 Sigma models with (2,1) AdS supersymmetry: Off-shell approach
   5.1 Formulation in (2,0) AdS superspace
   5.2 Formulation in (1,1) AdS superspace
   5.3 Sigma model gaugings with a frozen vector multiplet

6 Sigma models with (2,1) AdS supersymmetry: On-shell approach
   6.1 Formulation in (2,0) AdS superspace
   6.2 Formulation in (1,1) AdS superspace

7 $\mathcal{N} = 4$ AdS superspaces
   7.1 Geometry of $\mathcal{N} = 4$ AdS superspaces
   7.2 From $\mathcal{N} = 4$ to $\mathcal{N} = 2$ AdS superspaces
      7.2.1 AdS superspace reduction (4,0) → (2,0)
      7.2.2 AdS superspace reduction (3,1) → (2,0)
      7.2.3 AdS superspace reduction (2,2) → (2,0)
   7.3 From $\mathcal{N} = 4$ to $\mathcal{N} = 3$ AdS superspaces
8 Rigid $\mathcal{N} = 4$ supersymmetric field theories in AdS: Off-shell multiplets and invariant actions

8.1 Covariant projective supermultiplets ............................................. 54
8.2 Reduction to $\mathcal{N} = 2$ AdS superspaces ........................................ 56
8.3 Reduction to $\mathcal{N} = 3$ AdS superspaces ........................................ 57
8.4 $\mathcal{N} = 4$ supersymmetric actions .............................................. 59

8.4.1 $\mathcal{N} = 4$ supersymmetric action in $\mathcal{N} = 2$ AdS ................. 60
8.4.2 $\mathcal{N} = 4$ supersymmetric action in $\mathcal{N} = 3$ AdS ................. 61
8.4.3 $\mathcal{N} = 4$ supersymmetric action in components ............................. 62

9 Relating $\mathcal{N} = 3$ and $\mathcal{N} = 4$ supersymmetric sigma models

9.1 (2,0) AdS superspace approach: Formulation in terms of $\mathcal{N} = 3$ polar supermultiplets ......................................................... 65

9.1.1 (3,0) AdS supersymmetry implies (4,0) AdS supersymmetry ........... 65
9.1.2 (3,0) AdS supersymmetry implies (3,1) AdS supersymmetry .......... 66
9.1.3 (2,1) AdS supersymmetry implies (2,2) AdS supersymmetry .......... 67

9.2 (2,0) AdS superspace approach: Formulation in terms of $\mathcal{N} = 2$ chiral superfields ................................................................. 67

9.2.1 (3,0) AdS supersymmetry implies (4,0) AdS supersymmetry ........... 68
9.2.2 (3,0) AdS supersymmetry implies (3,1) AdS supersymmetry .......... 69
9.2.3 (2,1) AdS supersymmetry implies (2,2) AdS supersymmetry .......... 69

9.3 $\mathcal{N} = 3$ AdS superspace approach .............................................. 69

9.3.1 (3,0) AdS supersymmetry implies (4,0) and (3,1) AdS supersymmetry 70
9.3.2 (2,1) AdS supersymmetry implies (2,2) AdS supersymmetry .......... 71

10 (4,0) supersymmetric sigma models with $X \neq 0$: Off-shell approach

10.1 Off-shell non-critical (4,0) sigma models ....................................... 73
10.2 Off-shell critical (4,0) sigma models ............................................. 75
11 (4,0) supersymmetric sigma models with $X \neq 0$: On-shell approach

11.1 General features of $\mathcal{N} = 4$ supersymmetry

11.2 Formulation of (4,0) supersymmetric sigma models in (2,0) superspace

12 Sigma models with non-centrally extended $\mathcal{N} = 4$ Poincaré supersymmetry

13 Conclusion

A (Hyper) Kähler cones

B Derivation of the action (8.40)

C Deriving conditions for extended supersymmetry

C.1 (3,0) AdS supersymmetry in (2,0) AdS superspace

C.2 (2,1) AdS supersymmetry in (2,0) AdS superspace

C.3 (2,1) AdS supersymmetry in (1,1) AdS superspace

C.4 (4,0) AdS supersymmetry in (2,0) AdS superspace

D $\mathcal{N} = 4 \rightarrow (1,1)$ AdS superspace reduction

1 Introduction

In maximally symmetric spacetimes of dimension $3 \leq d \leq 5$, rigid supersymmetry with six ($d = 3$) or eight ($d = 3, 4, 5$) supercharges requires hyperkähler geometry for the target spaces of nonlinear $\sigma$-models (more exotic geometries can originate in two spacetime dimensions \[1\]). Arbitrary hyperkähler target spaces are allowed by ordinary Poincaré supersymmetry corresponding to Minkowski spacetime \[2\].\footnote{In three dimensions, there exist non-central extensions of the $\mathcal{N} \geq 4$ Poincaré superalgebras \[3\] which have no higher-dimensional analogs. These superalgebras have appeared in various string- and field-theoretic applications, see \[4, 5, 6, 7, 8, 9\] and references therein. The non-central extension of}
in four dimensions (4D) and hyperkähler manifolds, see [11] for a nice derivation of this result. The situation is completely different in the cases of 4D $\mathcal{N} = 2$ and 5D $\mathcal{N} = 1$ anti-de Sitter (AdS) supersymmetries, which enforce nontrivial restrictions on the hyperkähler target spaces of supersymmetric $\sigma$-models [12, 13, 14, 15]. Since these restrictions are identical in AdS$_4$ [12, 13] and AdS$_5$ [14, 15], it suffices to discuss the former case only. The most general $\mathcal{N} = 2$ supersymmetric $\sigma$-model in AdS$_4$ was constructed in [12, 13] using a formulation in terms of $\mathcal{N} = 1$ covariantly chiral superfields. As demonstrated in [12, 13], the target space of such a $\sigma$-model is a non-compact hyperkähler manifold possessing a special Killing vector field which generates an SO(2) group of rotations on the two-sphere of complex structures and necessarily leaves one of them, $J$, invariant. This implies that each of the complex structures that are orthogonal to $J$ is characterized by an exact Kähler two-form, and therefore the target space is non-compact. The existence of such hyperkähler spaces was pointed out twenty-five years ago by Hitchin et al. [16] without addressing their physical significance for supersymmetric $\sigma$-models in AdS.

From the point of view of supersymmetry, the space AdS$_3$ is rather special as compared with AdS$_4$ and AdS$_5$. Here $\mathcal{N}$-extended AdS supersymmetry exists in several incarnations which are labelled by two non-negative integers $p \geq q$ such that $p + q = \mathcal{N}$. The reason is that the 3D anti-de Sitter group is reducible,

$$SO_0(2, 2) \cong \left( SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \right) / \mathbb{Z}_2,$$

and so are its supersymmetric extensions, OSp($p|2; \mathbb{R}$) $\times$ OSp($q|2; \mathbb{R}$), which are known as $(p, q)$ AdS supergroups. This implies that there are several versions of $\mathcal{N}$-extended AdS supergravity [17], known as the $(p, q)$ AdS supergravity theories. These theories can naturally be described in superspace using the off-shell formulation for $\mathcal{N}$-extended conformal supergravity [18, 19]. The supergeometry of $\mathcal{N}$-extended conformal supergravity developed in [19] allows maximally symmetric backgrounds with non-zero covariantly constant curvature, which were classified in [10] and called the $(p, q)$ AdS superspaces. These superspaces have AdS$_3$ as the bosonic body, and their isometry groups are generated by the $(p, q)$ AdS superalgebras [10]. It turns out that different choices of $p$ and $q$, for fixed $\mathcal{N} = p + q$, lead to supersymmetric field theories with drastically different properties.

The $\mathcal{N} = 4$ Poincaré supergroup originates geometrically as the isometry group of the deformed $\mathcal{N} = 4$ Minkowski superspace introduced in [10]. As will be shown below, this supersymmetry type requires the target space of any $\sigma$-model to be a hyperkähler cone.

A general setting to determine the (conformal) isometries of a given curved superspace background in off-shell supergravity was developed long ago in [20], with the goal of constructing rigid supersymmetric theories in curved superspace. Later on, it was applied to various supersymmetric backgrounds in five, four and three dimensions [21, 22, 23, 10]. More recently, an equivalent construction in the compo-
This has been demonstrated for the cases $\mathcal{N} = 2$ and $\mathcal{N} = 3$ by studying the nonlinear $\sigma$-models with (2,0) and (1,1) AdS supersymmetry $^{23}$ and with (3,0) and (2,1) AdS supersymmetry $^{10}$ respectively. The main goal of the present paper is to provide a thorough study of the last nontrivial case $\mathcal{N} = 4$ allowing nonlinear $\sigma$-models of sufficiently general functional form. Specifically, we construct the most general nonlinear $\sigma$-models with (4,0), (3,1) and (2,2) AdS supersymmetries. We also provide a formulation in terms of $\mathcal{N} = 4$ chiral superfields for the most general nonlinear $\sigma$-models with (3,0) and (2,1) AdS supersymmetry. Our analysis is based on the use of two different realizations for $\mathcal{N} = 3$ and $\mathcal{N} = 4$ supersymmetric nonlinear $\sigma$-models in AdS$_3$: (i) off-shell formulations in terms of $\mathcal{N} = 3$ and $\mathcal{N} = 4$ projective supermultiplets $^{19, 10}$; and (ii) on-shell formulations in terms of covariantly chiral scalar superfields in (2,0) AdS superspace and (1,1) AdS superspace (the latter formulation exists in special cases only). We will also heavily build on the results of $^{47, 40, 48}$. The realization (ii) proves to be a very convenient tool to study the target space geometry (this is similar to the $\mathcal{N} = 1$ superfield formulation for $\mathcal{N} = 2$ supersymmetric $\sigma$-models in four dimensions $^{11}$). Therefore, we would like to briefly discuss the reduction from $\mathcal{N} > 2$ to (2,0) AdS superspace.

Consider a supersymmetric field theory formulated in a given $(p, q)$ AdS superspace with $\mathcal{N} = p + q \geq 3$ and $p \geq q$. Such a dynamical system can always be reformulated as a supersymmetric theory realized in the (2,0) AdS superspace, with $(p + q - 2)$ supersymmetries hidden. This observation will be important for the subsequent analysis, and therefore we would like to elaborate on it. The conceptual possibility for a $(p, q) \to (2,0)$ AdS reformulation follows from the explicit structure of the algebra of $(p, q)$ AdS covariant derivatives

$$D_A = (D_a, D^I_A) = E_A + \frac{1}{2} \Omega^{\beta\gamma} M_{\beta\gamma} + \frac{1}{2} \Phi^{KL} N_{KL}, \quad I = 1, \ldots, \mathcal{N},$$  

(1.1)
given in [10]:

\[ \{ D^I_\alpha, D^J_\beta \} = 2i\delta^{IJ} D_{\alpha\beta} - 4i S^{IJ} M_{\alpha\beta} + i\varepsilon_{\alpha\beta} \left( X^{IKL} - 4S^K[I\delta^L] \right) N_{KL}, \] (1.2a)

\[ [D_a, D^I_\beta] = S^I_K (\gamma_a)_\beta \gamma_D^I, \] (1.2b)

\[ [D_a, D_b] = -4 S^2 M_{ab}, \] (1.2c)

with $M_{\alpha\beta} = M_{\beta\alpha}$ (or, using the three-vector notation, $M_{ab} = -M_{ba}$) the Lorentz generators and $N_{KL} = -N_{LK}$ the SO(N) generators. In general, $S^{IJ} = S^{JI}$ is a non-vanishing covariantly constant tensor such that $S = \sqrt{(S^{IJ} S_{IJ})/N} > 0$ is a positive constant parameter of unit mass dimension. It can be chosen, by applying a local SO(N) transformation, in the diagonal form

\[ S^{IJ} = S^{I,J}_{\text{diag}}(p, \ldots, +1, -1, \ldots, -1). \] (1.3)

The other component of the SO(N) curvature, $X^{IJKL} = X^{IJKL}$, is a completely anti-symmetric and covariantly constant tensor which may exist only in the cases $q = 0$ and $p \geq 4$. If present, this tensor in the gauge (1.3) obeys the quadratic constraint

\[ X_N^{I,J[K} X^{L,P]N} = 0. \] (1.4)

Splitting each SO(N) index as $I = (\hat{i}, \hat{I})$, where $\hat{i} = 1, 2$ and $\hat{I} = 3, \ldots, N$, one may see from (1.2) that the operators $(D_a, D^\hat{i}_\alpha)$ form a closed subalgebra,

\[ \{ D^\hat{i}_\alpha, D^\hat{j}_\beta \} = 2i\delta^{\hat{i}\hat{j}} D_{\alpha\beta} - 4i S^{\hat{i}\hat{j}} M_{\alpha\beta} - 4i S\varepsilon_{\alpha\beta\hat{k}\hat{l}} \tilde{N}^{\hat{i}\hat{j}}_{\hat{k}\hat{l}}, \] (1.5a)

\[ [D_a, D^\hat{i}_\beta] = S(\gamma_a)_\beta \gamma D^\hat{i}_\gamma, \] (1.5b)

\[ [D_a, D_b] = -4 S^2 M_{ab}, \] (1.5c)

where we have defined a modified SO(2) generator

\[ \tilde{N}^{\hat{i}\hat{j}} := N^{\hat{i}\hat{j}} - \frac{1}{4S} X^{i\hat{j}j\hat{k}\hat{l}} N_{\hat{k}\hat{l}}, \quad [\tilde{N}^{\hat{k}\hat{l}}, D^\hat{i}_\alpha] = 2\delta^{\hat{k}\hat{l}} D^\hat{i}_\alpha, \quad [\tilde{N}^{\hat{k}\hat{l}}, D_a] = 0. \] (1.6)

The anti-commutation relations (1.5) correspond to the (2,0) AdS superspace [23, 10] (in a real basis for the spinor covariant derivatives). Due to (1.5), the SO(N) connection corresponding to the covariant derivatives $(D_a, D^\hat{i}_\alpha)$ can be reduced to an SO(2) connection associated with $\tilde{N}^{\hat{i}\hat{j}}$ by applying an SO(N) gauge transformation.\(^5\) The (2,0) AdS superspace can be embedded into our $(p,q)$ AdS superspace as a surface defined by

\[ \theta^\mu_i = 0, \quad \hat{i} = 3, \ldots, N, \] (1.7)

\(^5\)In such a gauge, the covariantly constant tensor $X^{ij\hat{k}\hat{l}}$ becomes constant with respect to $(D_a, D^\hat{i}_\alpha)$.\]
for a certain local parametrization of the superspace Grassmann variables $\theta^\mu_I = (\theta^\mu_\hat{i}, \theta^\mu_\hat{I})$.

Now consider the case $p + q \geq 3$ and $p \geq q > 0$, and therefore $X^{IJKL} = 0$, and assume that $S^{IJ}$ is given in the form

$$S^{IJ} = S \text{diag}(+1,-1,+1,\cdots,+1,-1,\cdots,-1),$$

(compare with (1.3)). Then any supersymmetric field theory in the $(p,q)$ AdS superspace can be reformulated as a theory in the (1,1) AdS superspace. Indeed, if we split again each $\text{SO}(\mathcal{N})$ index as $I = (\hat{i}, \hat{I})$, where $\hat{i} = 1, 2$ and $\hat{I} = 3, \ldots, \mathcal{N}$, the anti-commutation relations (1.2) imply that the operators $(D_a, D^\hat{i}_\alpha)$ obey a closed algebra of the form:

$$\{D^\hat{i}_\alpha, D^{\hat{j}}_{\beta}\} = 2i\delta^{\hat{i}\hat{j}}D_{\alpha\beta} + 4iS(-1)^{\hat{i}\hat{j}}\delta^{\hat{i}\hat{j}}M_{\alpha\beta},$$

(1.9a)

$$[D_a, D^{\hat{j}}_{\beta}] = -S(-1)^{\hat{j}}(\gamma_a)_\beta\gamma D^{\hat{j}}_{\gamma},$$

(1.9b)

$$[D_a, D_b] = -4S^2M_{ab}.$$  

(1.9c)

This algebra corresponds to the (1,1) AdS superspace \[23, 10\] (in a real basis for the spinor covariant derivatives).

Another interesting possibility occurs in the case $p + q \geq 4$ and $X^{IJKL} = 0$. It may be seen that any supersymmetric field theory in the $(p,q)$ AdS superspace can be reformulated as a theory in the (3,0) or (2,1) AdS superspace. This will be further explored in the main body of the present paper.

This paper is organized as follows. Section 2 provides review material on the three-dimensional nonlinear $\sigma$-models with (2,0) and (1,1) AdS supersymmetry. The $\sigma$-models with (3,0) AdS supersymmetry are studied in sections 3 and 4. In section 3, we start from the most general off-shell (3,0) supersymmetric $\sigma$-model and then reformulate it in terms of chiral superfields in the (2,0) AdS superspace. In section 4 we start from a general nonlinear $\sigma$-model in the (2,0) AdS superspace and derive the conditions on the target space geometry for the $\sigma$-model to possess (3,0) AdS supersymmetry. Sections 5 and 6 extend the analysis of sections 3 and 4 to the case of (2,1) AdS supersymmetric $\sigma$-models. Various aspects of the $\mathcal{N} = 4$ AdS superspaces are discussed in section 7. Section 8 is devoted to the projective-superspace techniques to formulate off-shell $\mathcal{N} = 4$ supersymmetric $\sigma$-models in AdS$_3$ as well as to carry out their reduction to $\mathcal{N} = 2$ and $\mathcal{N} = 3$ AdS superspaces. In section 9 we demonstrate that any nonlinear $\sigma$-model with $(p,q)$ AdS supersymmetry for $p + q = 3$, in fact, possesses a larger $(p',q')$ AdS supersymmetry with $p' + q' = 4$ and $p' \geq p$, $q' \geq q$. Sections 10 and 11 concern the formulation of the most general $\sigma$-model with (4,0) AdS supersymmetry in terms of chiral
superfields on the (2,0) AdS superspace. In section 12, the results of the two previous sections are used to construct the most general $\sigma$-model with the non-centrally extended $\mathcal{N} = 4$ Poincaré supersymmetry. A brief discussion of the results obtained is given in section 13. The main body of the paper is accompanied by four technical appendices.

2 Nonlinear sigma models with four supercharges

As discussed in the introduction, for $\mathcal{N} = p + q \geq 3$ and $p \geq q$, the most general nonlinear $\sigma$-models with $(p,q)$ AdS supersymmetry can be realized in the (2,0) AdS superspace. If in addition $X^{IJKL} = 0$ and $q > 0$, there also exists a realization in the (1,1) AdS superspace. It is therefore of special importance to understand the structure of supersymmetric field theories in the (2,0) and (1,1) AdS superspaces. The off-shell nonlinear $\sigma$-models in AdS$_3$ with (2,0) and (1,1) supersymmetry were thoroughly studied in [23]. In this section we review the $\sigma$-model results of [23].

2.1 Nonlinear sigma models in (2,0) AdS superspace

The geometry of (2,0) AdS superspace is encoded in its covariant derivatives

$$\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}^\alpha) = E_A^M \partial_M + \frac{1}{2} \Omega_A^{cd} M_{cd} + i \Phi_A J$$

obeying the following (anti) commutation relations:

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = \{\bar{\mathcal{D}}^\alpha, \bar{\mathcal{D}}^\beta\} = 0 , \quad \{\mathcal{D}_\alpha, \bar{\mathcal{D}}^\beta\} = -2i \mathcal{D}_a \mathcal{D}_\alpha - 4i S \varepsilon_{\alpha\beta} J + 4i S M_{\alpha\beta} , \quad [\mathcal{D}_a, \mathcal{D}_\beta] = S (\gamma_a)_{\beta} \gamma^\gamma \mathcal{D}_\gamma , \quad [\mathcal{D}_a, \bar{\mathcal{D}}^\beta] = S (\gamma_a)_{\beta} \gamma^\gamma \bar{\mathcal{D}}^\gamma , \quad [\mathcal{D}_a, \mathcal{D}_b] = -4 S^2 M_{ab} . \quad (2.2)$$

The generator $J$ in (2.2) corresponds to the gauged $R$-symmetry group, $U(1)_R$, and acts on the covariant derivatives as

$$[J, \mathcal{D}_\alpha] = \mathcal{D}_\alpha , \quad [J, \bar{\mathcal{D}}^\alpha] = -\bar{\mathcal{D}}^\alpha . \quad (2.3)$$

The constant parameter $S$ in (2.2) is a square root of the curvature of AdS$_3$. Unlike (1.5), in this section we use the complex basis for the $\mathcal{N} = 2$ spinor covariant derivatives introduced in [19].

6The most general $\sigma$-model couplings to (1,1) and (2,0) AdS supergravity theories were constructed in [23] from first principles. These results generalize those obtained earlier [49, 50] within the Chern-Simons approach [17].
The isometries of the (2,0) AdS superspace are described by Killing vector fields, \( \tau = \tau^a \mathcal{D}_a + \tau^\alpha \mathcal{D}_\alpha + \bar{\tau}_a \bar{\mathcal{D}}^a \), obeying the equation
\[
\left[ \tau + it \mathcal{J} + \frac{1}{2} t^{bc} \mathcal{M}_{bc}, \mathcal{D}_A \right] = 0 ,
\] (2.4)
for some parameters \( t \) and \( t^{ab} \). Choosing \( \mathcal{D}_A = \mathcal{D}_a \) in (2.4) gives
\[
\begin{align*}
\mathcal{D}_a t &= 0 , \quad (2.5a) \\
\mathcal{D}_a \tau_b &= t_{ab} , \quad (2.5b) \\
\mathcal{D}_a \tau^\beta &= -S \tau^\gamma (\gamma_a)^\gamma^\beta , \quad (2.5c) \\
\mathcal{D}_a t^{bc} &= 4S^2 (\delta^b_a \tau^c - \delta^c_a \tau^b) . \quad (2.5d)
\end{align*}
\]
Eq. (2.5b) implies the standard Killing equation
\[
\mathcal{D}_a \tau_b + \mathcal{D}_b \tau_a = 0 , \quad (2.6)
\]
while (2.5c) is a Killing spinor equation. From (2.5b) and (2.5d) it follows that
\[
\begin{align*}
\mathcal{D}_a \mathcal{D}_b \tau_c &= 4S^2 (\eta_{ab} \tau_c - \eta_{ac} \tau_b) . \quad (2.7)
\end{align*}
\]
Next, choosing \( \mathcal{D}_A = \mathcal{D}_\alpha \) in (2.4) gives
\[
\begin{align*}
\mathcal{D}_\alpha \bar{\tau}_\beta &= 0 , \quad (2.8a) \\
\mathcal{D}_\alpha t &= 4S \bar{\tau}_\alpha , \quad (2.8b) \\
\mathcal{D}_\alpha t^{\beta\gamma} &= -4iS (\delta^\beta_\alpha \tau^\gamma + \delta^\gamma_\alpha \bar{\tau}^\beta) , \quad (2.8c) \\
\mathcal{D}_\alpha \tau^{\beta\gamma} &= -2i(\delta^\beta_\alpha \bar{\tau}^\gamma + \delta^\gamma_\alpha \bar{\tau}^\beta) , \quad (2.8d) \\
\mathcal{D}_\alpha \tau^\beta &= \frac{1}{2} t^\beta_\alpha + S \tau^\beta_\alpha + i \delta^\beta_\alpha t . \quad (2.8e)
\end{align*}
\]
These equations have a number of nontrivial implications including the following:
\[
\begin{align*}
\mathcal{D}_{(\alpha \tau_\beta \gamma)} &= \mathcal{D}_{(\alpha t_\beta \gamma)} = 0 , \quad (2.9a) \\
\mathcal{D}_{(\alpha \tau_\beta)} &= -\mathcal{D}_{(\alpha \bar{\tau}_\beta)} = \frac{1}{2} t_{\alpha \beta} + S \tau_{\alpha \beta} , \quad (2.9b) \\
\tau_\alpha &= \frac{i}{6} \mathcal{D}^\gamma \tau_\alpha \beta = \frac{i}{12S} \mathcal{D}^\beta t_{\alpha \beta} , \quad (2.9c) \\
\mathcal{D}_\gamma \tau^\gamma &= -\mathcal{D}^\gamma \bar{\tau}_\gamma = 2it . \quad (2.9d)
\end{align*}
\]
It follows from the above equations that the Killing superfields \( \tau^a \), \( t \) and \( t^{ab} \) are given in terms of the vector parameter \( \tau^a \). Its components defined by \( \tau^a|_{\theta=0} \) and \( (-\mathcal{D}^b \tau^a)|_{\theta=0} \) describe the isometries of AdS\( _{3} \). The other isometry transformations of the (2,0) AdS
superspace are contained not only in $\tau^a$ but also, e.g., in the real scalar $t$ subject to the following equations:

$$D^2 t = \bar{D}^2 t = 0 \quad , \quad (iD^\alpha \bar{D}_\alpha - 8S)t = 0 \quad , \quad D_a t = 0 \quad .$$

(2.10)

At the component level, $t$ contains the real constant parameter $t|_{\theta=0}$ and the complex Killing spinor $D_\alpha t|_{\theta=0}$, which generate the $R$-symmetry and supersymmetry transformations of the (2,0) AdS superspace respectively.

Given a matter tensor superfield $\mathcal{V}$ (with all indices suppressed), its (2,0) AdS transformation law is

$$\delta \mathcal{V} = (\tau + it \mathcal{J} + \frac{1}{2}t^{bc}M_{bc}) \mathcal{V} \quad .$$

(2.11)

Supersymmetric actions can be constructed either by integrating a real scalar $\mathcal{L}$ over the full (2,0) AdS superspace,

$$\int d^3x \, d^4\theta \mathcal{E} \mathcal{L} = \int d^3x \, e \left( \frac{1}{16}D^\alpha \bar{D}^2 D_\alpha + iS\bar{D}^\alpha D_\alpha \right) \mathcal{L} \bigg|_{\theta=0}$$

$$= \int d^3x \, e \left( \frac{1}{16}D_\alpha \bar{D}^2 D^\alpha + iS\bar{D}^\alpha D_\alpha \right) \mathcal{L} \bigg|_{\theta=0} \quad ,$$

(2.12)

with $E^{-1} = \text{Ber}(E_A^M)$, or by integrating a covariantly chiral scalar $\mathcal{L}_c$ over the chiral subspace of the (2,0) AdS superspace,

$$\int d^3x \, d^2\theta \mathcal{E} \mathcal{L}_c = -\frac{1}{4} \int d^3x \, e \bar{D}^2 \mathcal{L}_c \bigg|_{\theta=0} \quad , \quad \bar{D}^\alpha \mathcal{L}_c = 0 \quad ,$$

(2.13)

with $\mathcal{E}$ being the chiral density. The Lagrangians $\mathcal{L}$ and $\mathcal{L}_c$ are required to possess certain $U(1)_R$ charges:

$$\mathcal{J} \mathcal{L} = 0 \quad , \quad \mathcal{J} \mathcal{L}_c = -2\mathcal{L}_c \quad .$$

(2.14)

The two types of supersymmetric actions are related to each other by the rule

$$\int d^3x \, d^4\theta \mathcal{E} \mathcal{L} = -\frac{1}{4} \int d^3x \, d^2\theta \mathcal{E} \bar{D}^2 \mathcal{L} \quad .$$

(2.15)

Using the AdS transformation law of the Lagrangian in [2.12], $\delta \mathcal{L} = \tau \mathcal{L}$, and the Killing equation (2.4), one may check explicitly that the component action defined by the right-hand side of (2.12) is invariant under the (2,0) AdS isometry group. A similar consideration may be given in the case of the chiral action (2.13), with the only difference that the AdS transformation of the chiral Lagrangian is $\delta \mathcal{L}_c = (\tau - 2it)\mathcal{L}_c$.

7The component inverse vierbein is defined as usual, $e_a^m(x) = E_a^m|_{\theta=0}$, with $e^{-1} = \text{det}(e_a^m)$.
For component reduction, it may be useful to choose a Wess-Zumino gauge such that
\[ D_a|_{\theta=0} = e_a^m(x)\partial_m + \frac{1}{2}\omega_a^{bc}(x)\mathcal{M}_{bc} . \] (2.16)
In this gauge, we will use the same symbol \( D_a \) for the space-time covariant derivative in the right-hand side of (2.16).

Given an Abelian vector multiplet, we can introduce gauge covariant derivatives
\[ D_A := D_A + i V_A \mathcal{Z} , \quad [\mathcal{Z}, D_A] = [\mathcal{Z}, D_A] = [\mathcal{Z}, \mathcal{J}] = 0 , \] (2.17)
where \( V_A \) is the gauge connection associated with the U(1) generator \( \mathcal{Z} \). The gauge covariant derivatives are required to obey the relations
\[ \{D_{\alpha}, D_{\beta}\} = 0 , \quad \{D_{\alpha}, \bar{D}_{\beta}\} = -2iD_{\alpha\beta} - 4i\varepsilon_{\alpha\beta}(S\mathcal{J} + G\mathcal{Z}) + 4iS\mathcal{M}_{\alpha\beta} , \] (2.18)
and the other (anti) commutators can be restored by making use of the Bianchi identities and complex conjugation. The gauge invariant field strength \( G \) is real, \( G = \bar{G} \), and covariantly linear,
\[ D^2G = \bar{D}^2G = 0 . \] (2.19)

Suppose the vector multiplet under consideration is intrinsic, that is its field strength \( G \) is a non-zero constant, which can be conveniently normalized as
\[ G = S . \] (2.20)
The name ‘intrinsic’ is due to the fact that such a vector multiplet generates a unique super-Weyl transformation which conformally relates the (2,0) AdS superspace to a flat one [23]. In the case of the intrinsic vector multiplet, the second relation in (2.18) can be rewritten in the form
\[ \{D_{\alpha}, \bar{D}_{\beta}\} = -2iD_{\alpha\beta} - 4i\varepsilon_{\alpha\beta}S\mathcal{J} + 4iS\mathcal{M}_{\alpha\beta} , \quad \mathcal{J} := \mathcal{J} + \mathcal{Z} . \] (2.21)
We see that the gauge covariant derivatives \( D_A \), which are associated with the intrinsic vector multiplet, describe (2,0) AdS superspace with a deformed U(1)_R generator.

The supersymmetric nonlinear \( \sigma \)-models with (2,0) AdS supersymmetry were studied in [23]. In general, each \( \sigma \)-model can be described in terms of chiral scalar superfields \( \phi^a \), \( \bar{D}_a\phi^a = 0 \), taking their values in a Kähler manifold \( \mathcal{M} \). There are two different cases to consider.
The first option corresponds to the situation where the chiral variables $\phi^a$ are neutral with respect to the $R$-symmetry group $U(1)_R$,

$$J \phi^a = 0 . \quad (2.22)$$

In this case, no superpotential is allowed, and the most general $\sigma$-model action is

$$S = \int d^3 x \, d^4 \theta \, E \, K(\phi^a, \bar{\phi}^\dagger) , \quad (2.23)$$

where $K(\phi, \bar{\phi})$ is the Kähler potential of $\mathcal{M}$. The target space, $\mathcal{M}$, may be an arbitrary Kähler manifold. The action (2.23) is invariant under the $(2,0)$ AdS isometry supergroup, $\text{OSp}(2|2; \mathbb{R}) \times \text{Sp}(2, \mathbb{R})$. As follows from (2.12), the above $\sigma$-model possesses the Kähler symmetry

$$K(\phi, \bar{\phi}) \rightarrow K(\phi, \bar{\phi}) + F(\phi) + \bar{F}(\bar{\phi}) , \quad (2.24)$$

where $F(\phi)$ is an arbitrary holomorphic function.

The second option is that a superpotential is allowed\footnote{We will show below that no superpotential is allowed for those $\sigma$-models in (2,0) AdS superspace which have additional supersymmetries, except the case of critical (4,0) supersymmetry.}

$$S = \int d^3 x \, d^4 \theta \, E \, K(\phi^a, \bar{\phi}^\dagger) + \left( \int d^3 x \, d^2 \theta \, \mathcal{E} \, W(\phi^a) + \text{c.c.} \right) , \quad (2.25)$$

for some holomorphic function $W(\phi)$. In this case, the target space must have a $U(1)$ isometry group generated by a holomorphic Killing vector $J^a(\phi)$ defined by

$$J = J^a(\phi) \partial_a + \bar{J}^\dagger(\bar{\phi}) \partial_{\bar{a}} , \quad J^a(\phi) := iJ \phi^a . \quad (2.26)$$

As is known, the Killing condition amounts to

$$J^a K_a + \bar{J}^\dagger \bar{K}_{\bar{a}} = F + \bar{F} ,$$

for some holomorphic function $F(\phi)$. However, since the Lagrangian in (2.12) has to be a scalar superfield, the Kähler potential must be neutral under $U(1)_R$ and hence $F = 0$,

$$J^a K_a + \bar{J}^\dagger \bar{K}_{\bar{a}} = 0 . \quad (2.27)$$

The infinitesimal transformation of $\phi^a$ under the $(2,0)$ AdS isometry supergroup is

$$\delta \phi^a = \tau \phi^a + t J^a(\phi) . \quad (2.28)$$
The action \( \text{2.25} \) is invariant under this transformation if the superpotential \( W(\phi) \) obeys the condition \[23\]

\[
J^a W_a = -2i W .
\] (2.29)

Suppose the target space of the \( \sigma \)-model \( \text{2.25} \) possesses a holomorphic Killing vector field

\[
Z = Z^a(\phi) \partial_a + \bar{Z}^{\bar{a}}(\bar{\phi}) \bar{\partial}_{\bar{a}} , \quad \bar{Z}^{\bar{a}}(\phi) := iZ^a ,
\] (2.30)

which commutes with the Killing vector field \( \text{2.26} \),

\[
[J, Z] = 0 .
\] (2.31)

Without loss of generality, we can assume that\[9\]

\[
Z^a K_a + \bar{Z}^{\bar{a}} K_{\bar{a}} = 0 .
\] (2.32)

We then can gauge the \( \text{U}(1) \) isometry generated by the Killing vector \( Z^a(\phi) \) by means of introducing gauge covariant derivatives \( \mathcal{D}_A \), eq. \( \text{2.17} \), and replacing the chiral superfields \( \phi^a \) in \( \text{2.23} \) with gauge covariantly chiral ones \( \phi^a \),

\[
\mathcal{D}_a \phi^a = 0 .
\] (2.33)

The algebra of covariant derivatives remains unchanged except that we replace \( J \) with \( \mathcal{J} \) and identify \( J^a = J^a + Z^a \). In what follows, we often will not distinguish between these cases.

Using the component reduction rule, one can show that

\[
S = \int \text{d}^3x \text{d}^4\theta \mathcal{E} K + \left( \int \text{d}^3x \text{d}^2\theta \mathcal{E} W + \text{c.c.} \right) = \int \text{d}^3x e L ,
\] (2.34)

where

\[
L = -g_{a\bar{a}} \mathcal{D}_m \phi^a \mathcal{D}^m \bar{\phi}^{\bar{a}} - ig_{a\alpha} \bar{\psi}_\alpha \mathcal{D}\alpha \beta \psi^\beta + F^a F_{a} + \frac{1}{4} R_{a\alpha\bar{a}\bar{\beta}} (\bar{\psi}^\alpha \psi^\beta)(\bar{\psi}^{\bar{\alpha}} \psi^{\bar{\beta}}) + S (\psi^a \bar{\psi}^{\bar{a}}) (ig_{a\bar{a}} + \nabla_a J_{\bar{a}} - \nabla_{\bar{a}} J_a) - 4S^2 (J^a \bar{J}_{a} g_{a\bar{a}} - D) + W_a F^a + \bar{W}_{\bar{a}} \bar{F}^{\bar{a}} - \frac{1}{2} \nabla_a W_b (\psi^a \psi^b) - \frac{1}{2} \nabla_{\bar{a}} \bar{W}_{\bar{b}} (\bar{\psi}^{\bar{a}} \bar{\psi}^{\bar{b}}).
\] (2.35)

\[9\] In the general case that \( Z^a K_a + \bar{Z}^{\bar{a}} K_{\bar{a}} = H + \bar{H} \), for some holomorphic function \( H(\phi) \), we can introduce, following \[11\], a new chiral superfield \( \phi^0 \) and Lagrangian \( K' = K - \phi^0 - \bar{\phi}^0 \), where \( \phi^0 \) transforms as \( iZ \phi^0 = H(\phi^a) \). The Lagrangian \( K' \) possesses the required property \( \text{2.32} \). The field \( \phi^0 \) is a purely gauge degree of freedom.

13
We use the shorthand \((\psi^a\psi^b) := \psi^{\alpha_a}\psi^{\beta_b}, (\bar{\psi}^a\bar{\psi}^b) := \bar{\psi}^{\alpha_a}\bar{\psi}^{\beta_b}\), and \((\bar{\psi}^a\bar{\psi}^b) := \bar{\psi}^{\alpha_a}\bar{\psi}^{\alpha_b}\). The Killing potential \(D\) is defined by \(J^a K_a = -iD/2\). Here and below, \(\nabla_a\) and \(\nabla_{\bar{a}}\) denote the target-space covariant derivatives.

We have defined the components of \(\phi^a\) as
\[
\varphi^a := \phi^a|, \quad \psi^a := \frac{1}{\sqrt{2}}D_\alpha \phi^a|, \tag{2.36a}
\]
\[
F^a := -\frac{1}{4}g^{ab}D^2 K_b| = -\frac{1}{4}(D^2 \phi^a + \Gamma^a_{bc}D^a \phi^b D^c \phi^c)|. \tag{2.36b}
\]

In particular, the auxiliary field \(F^a\) transforms covariantly under target space reparametrizations. The vector derivative on the fermion is similarly reparametrization covariant,
\[
\hat{D}_m \psi^a := D_m \psi^a + \Gamma^a_{bc}D_m \varphi^b \psi^c \tag{2.37}
\]
and the action of \(J\) on the physical fields is defined as
\[
iJ^a \varphi^a = J^a(\varphi), \quad iJ^a \psi^a = \psi^b \partial_b J^a(\varphi) + i\psi^a. \tag{2.38}
\]

The fermion mass terms are given in terms of covariant field derivatives
\[
\nabla_a W_b := \partial_a \partial_b W - g^{ac} \partial_c W \tag{2.39}
\]
Eliminating the auxiliary fields \(F^a\) leads to the component Lagrangian
\[
L = -g_{aa} D_m \varphi^a D^m \varphi^a - ig_{aa} \bar{\psi}^a D^a \bar{\psi}^a + \frac{1}{4}R_{aabb}(\psi^a \psi^b)(\bar{\psi}^a \bar{\psi}^b) + S(\psi^a \bar{\psi}^a)(i\partial_a J_a - \partial_{\bar{a}} \bar{J}_{\bar{a}}) - 4S^2(J^a \bar{J}^a - D) - \frac{1}{2} \nabla_a W_b(\psi^a \psi^b) - \frac{1}{2} \nabla_{\bar{a}} W_{\bar{b}}(\bar{\psi}^{\bar{a}} \bar{\psi}^{\bar{b}}) - g^{a\bar{a}} W_a \bar{W}_{\bar{a}}. \tag{2.40}
\]

One broad class of interest is when the target space is a Kähler cone, see Appendix A. Then the target space admits a homothetic conformal Killing vector \(\chi\), obeying the conditions (A.2) and (A.3). For a superconformal \(\sigma\)-model, the \(U(1)_R\) Killing vector fields \(J\) should commute with \(\chi\), \([\chi, J^a] = \chi^b \nabla_b J^a - J^b \nabla_b \chi^a = 0\), (since these vector fields generate the \(U(1)_R\) and scale transformations), which is equivalent to \(\chi^b \nabla_b J^a = J^a\). It follows that
\[
J^a \chi_a + J^\bar{a} \chi_{\bar{a}} = 0. \tag{2.41}
\]
The superpotential must obey (2.29); if the action is additionally superconformal, it must also obey
\[
W = \frac{1}{4} \chi^a W_a. \tag{2.42}
\]
It is natural to decompose $J^a$ as

$$J^a = -\frac{i}{2} \chi^a + Z^a,$$

where (by construction) $Z^a$ is a Killing vector which leaves the superpotential invariant, $Z^a W_a = 0$, and commutes with $\chi$. In light of our discussion about gauged $\sigma$-models, one may interpret the $\chi^a$ term in (2.43) as the natural part of the $U(1)_R$ Killing vector and the $Z^a$ term as arising from gauging the Kähler cone with a frozen vector multiplet. Introducing a Killing potential $D^{(z)}$ for $Z$ using $Z^a \chi_a = -i D^{(z)}/2$, one finds that the component Lagrangian reduces to

$$L = -g_{a\bar{b}} \mathcal{D}_a \psi^a \mathcal{D}^{\alpha \bar{a}} \bar{\psi}^{\bar{a}} - ig_{a\bar{a}} \bar{\psi}_{\alpha} \mathcal{D}^{a \alpha \beta} \psi_{\beta} + \frac{1}{4} R_{a\bar{b}b\bar{c}} (\psi^a \psi^b)(\bar{\psi}^\alpha \bar{\psi}^\beta)$$

$$+ S (\psi^a \bar{\psi}^{\bar{a}}) (\nabla_a Z_{\bar{a}} - \nabla_{\bar{a}} Z_a) + 3 S^2 K - 4 S^2 (Z^a \bar{Z}_{\bar{a}} g_{a\bar{a}} - \frac{1}{2} D^{(z)})$$

$$- \frac{1}{2} \nabla_a W_b (\psi^a \psi^b) - \frac{1}{2} \nabla_{\bar{a}} \bar{W}_b (\bar{\psi}^{\bar{a}} \bar{\psi}^{\bar{b}}) - g^{a\bar{a}} W_a \bar{W}_{\bar{a}}.$$ (2.44)

The potential-like term $3 S^2 K$ may be combined with the scalar kinetic term to give

$$L = g_{a\bar{b}} \chi^a (\mathcal{D}_a \mathcal{D}^a - \frac{1}{8} \mathcal{R}) \chi^\beta - ig_{a\bar{a}} \bar{\psi}_{\alpha} \mathcal{D}^{a \alpha \beta} \psi_{\beta} + \frac{1}{4} R_{a\bar{b}b\bar{c}} (\psi^a \psi^b)(\bar{\psi}^\alpha \bar{\psi}^\beta)$$

$$+ S (\psi^a \bar{\psi}^{\bar{a}}) (\nabla_a Z_{\bar{a}} - \nabla_{\bar{a}} Z_a) - 4 S^2 (Z^a \bar{Z}_{\bar{a}} g_{a\bar{a}} - \frac{1}{2} D^{(z)})$$

$$- \frac{1}{2} \nabla_a W_b (\psi^a \psi^b) - \frac{1}{2} \nabla_{\bar{a}} \bar{W}_b (\bar{\psi}^{\bar{a}} \bar{\psi}^{\bar{b}}) - g^{a\bar{a}} W_a \bar{W}_{\bar{a}}.$$ (2.45)

after identifying the scalar curvature of AdS as $\mathcal{R} = -24 S^2$ and discarding a total derivative. The scalar kinetic operator in the first term is the conformal d’Alembertian in three dimensions. The actual mass terms are confined to the second and third lines which arise respectively from gauging a $U(1)$ isometry with the intrinsic vector multiplet and from introducing a superpotential.

### 2.2 Nonlinear sigma models in (1,1) AdS superspace

The geometry of (1,1) AdS superspace is described in terms of covariant derivatives

$$\mathcal{D}_A = (\mathcal{D}_a, \bar{\mathcal{D}}_a, \tilde{\mathcal{D}}^a) = E_A^M \partial_M + \frac{1}{2} \Omega_A^{cd} \mathcal{M}_{cd}$$

which obey the following (anti) commutation relations:

$$\{ \mathcal{D}_a, \mathcal{D}_\beta \} = -4 \bar{\mu} \mathcal{M}_{a\beta}, \quad \{ \mathcal{D}_a, \bar{\mathcal{D}}_\beta \} = 4 \mu \mathcal{M}_{a\beta}, \quad \{ \mathcal{D}_a, \tilde{\mathcal{D}}_\beta \} = -2i \mathcal{D}_{a\beta},$$

$$\{ \mathcal{D}_a, \bar{\mathcal{D}}_\beta \} = i \bar{\mu} (\gamma_a)_\beta \gamma \mathcal{D}_\gamma, \quad \{ \mathcal{D}_a, \tilde{\mathcal{D}}_\beta \} = -i \mu (\gamma_a)_\beta \gamma \mathcal{D}_\gamma,$$

$$\{ \mathcal{D}_a, \mathcal{D}_\beta \} = -4 |\mu|^2 \mathcal{M}_{ab}.$$ (2.47a, 2.47b, 2.47c)
Unlike (1.9), here and below we use the complex basis for $N = 2$ covariant derivatives introduced in [19].

The isometries of the (1,1) AdS superspace are described by Killing vector fields, 
\[ l = l^a \mathcal{D}_a + l^\alpha \mathcal{D}_\alpha + \bar{l}_\alpha \bar{\mathcal{D}}^\alpha, \]
which are defined to obey the Killing equation
\[ \left[ l + \frac{1}{2} \lambda^{ab} \mathcal{M}_{ab}, \mathcal{D}_C \right] = 0 , \]
for a certain Lorentz parameter $\lambda^{ab} = -\lambda^{ba}$. This is equivalent to the following set of equations:

\begin{align*}
\mathcal{D}_a \bar{l}_\beta &= -i \bar{\mu} l_{\alpha\beta} , \quad (2.49a) \\
\mathcal{D}_a l_{\beta\gamma} &= 4i \varepsilon_{\alpha(\beta} \bar{l}_{\gamma)} , \quad (2.49b) \\
\mathcal{D}_\alpha \lambda_{\beta\gamma} &= 8 \bar{\mu} \varepsilon_{\alpha(\beta} l_{\gamma)} , \quad (2.49c) \\
\mathcal{D}_a l_\beta &= \frac{1}{2} \lambda_{\alpha\beta} , \quad (2.49d)
\end{align*}

and

\begin{align*}
\mathcal{D}_a l_b &= \lambda_{ab} , \quad (2.50a) \\
\mathcal{D}_a l^\beta &= i \mu \bar{l}_\gamma (\gamma_a)^\gamma_\beta , \quad (2.50b) \\
\mathcal{D}_a \lambda^{bc} &= 4 |\mu|^2 \left( \delta_a^{(b} l^{c)} - \delta_a^{c} l^{b} \right) . \quad (2.50c)
\end{align*}

The above equations are actually equivalent to the following relations

\begin{align*}
l_\alpha &= \frac{i}{6} \bar{\mathcal{D}}^\beta l_{\alpha\beta} , \quad \mathcal{D}_\gamma (\alpha l_{\beta\gamma}) = 0 , \quad \mathcal{D}_\alpha l^\alpha = \bar{\mathcal{D}}^\alpha l_\alpha = 0 , \quad \mathcal{D}_\alpha l_\beta + i \mu l_{\alpha\beta} = 0 , \quad (2.51a) \\
\lambda_{\alpha\beta} &= 2 \mathcal{D}_\gamma (\alpha l_{\beta\gamma}) = 0 , \quad \mathcal{D}_\beta \lambda_{\alpha\gamma} - 12 \bar{\mu} l_\alpha = 0 . \quad (2.51b)
\end{align*}

It can be seen that the parameters $l^a$ and $\lambda^{ab}$ are determined in terms of the vector parameter $l^a$ obeying several constraints including the ordinary Killing equation\footnote{It may be seen that only the equation $\mathcal{D}_\gamma (\alpha l_{\beta\gamma}) = 0$ is critical (along with the definitions of $l_\alpha$ and $\lambda_{\alpha\beta}$ in terms of $l_{\alpha\beta}$). This is the “master equation” from which all the other constraints can be derived, compare with the 4D $N = 1$ case [20].}

\[ \mathcal{D}_a l_b + \mathcal{D}_b l_a = 0 . \quad (2.52) \]

Its bosonic components defined by $l^a|_{\theta=0}$ and $\lambda^{ab}|_{\theta=0} = (-\mathcal{D}^{b\alpha})|_{\theta=0}$ describe the isometries of AdS$_3$. The only independent complex fermionic parameters are $l_\alpha|_{\theta=0} = \frac{i}{6} \bar{\mathcal{D}}^\beta l_{\alpha\beta}|_{\theta=0}$ and its conjugate. The Killing vector fields introduced generate the supergroup OSp(1|2; $\mathbb{R}$) × OSp(1|2; $\mathbb{R}$), the isometry group of the (1,1) AdS superspace.
In the (1,1) AdS superspace, supersymmetric actions can be constructed either by integrating a real function $\mathcal{L}$ over the full superspace,

$$
\int d^3x \, d^4\theta \, E \, \mathcal{L} = \int d^3x \, e \left( \frac{1}{16} \mathcal{D}^\alpha (\bar{\mathcal{D}}^2 - 6\mu) \mathcal{D}_\alpha - \frac{\mu}{4} \mathcal{D}^2 - \frac{\bar{\mu}}{4} \bar{\mathcal{D}}^2 + 4\mu\bar{\mu} \right) \mathcal{L} \bigg|_{\theta = 0} ,
$$

or by integrating a chiral function $\mathcal{L}_c$ over the chiral superspace,

$$
\int d^3x \, d^2\theta \, E \, \mathcal{L}_c = -\frac{1}{4} \int d^3x \, e \left( \mathcal{D}^2 - 16\bar{\mu} \right) \mathcal{L}_c , \quad \bar{\mathcal{D}}^\alpha \mathcal{L}_c = 0 .
$$

These two types of superspace integrals are related to each other by the chiral action rule

$$
\int d^3x \, d^4\theta \, E \, \mathcal{L} = -\frac{1}{4} \int d^3x \, d^2\theta \, E \left( \mathcal{D}^2 - 4\mu \right) \mathcal{L} ,
$$

and its inverse

$$
\int d^3x \, d^2\theta \, E \, \mathcal{L}_c = \int d^3x \, d^4\theta \, E \frac{\mathcal{L}_c}{\mu} .
$$

Eq. (2.56) has no analogue in the (2,0) AdS superspace.

The general form of a (1,1) supersymmetric $\sigma$-model in AdS$_3$ is the single term

$$
\int d^3x \, d^4\theta \, E \, \mathcal{K}(\phi^a, \bar{\phi}^\bar{a}) , \quad \bar{\mathcal{D}}^\alpha \phi^a = 0 ,
$$

where $\mathcal{K}(\phi, \bar{\phi})$ is a real function of chiral superfields $\phi^a$ and their conjugates $\bar{\phi}^\bar{a}$. Since

$$
\int d^3x \, d^4\theta \, E \, F = \int d^3x \, d^2\theta \, E \, \mu F ,
$$

for a holomorphic function $F = F(\phi)$, the model (2.57) does not possess the usual Kähler symmetry. Because the Lagrangian in (2.57) corresponds to the Kähler potential of some Kähler manifold $\mathcal{M}$, we conclude that $\mathcal{K}(\phi, \bar{\phi})$ should be a globally defined function on $\mathcal{M}$. This immediately implies that the Kähler two-form, $\Omega = 2i g_{a\bar{b}} d\phi^a \wedge d\bar{\phi}^\bar{b}$, which is associated with the Kähler metric $g_{a\bar{b}} := \partial_a \partial_{\bar{b}} \mathcal{K}$, is exact and hence $\mathcal{M}$ is necessarily non-compact. We see that the $\sigma$-model couplings with (1,1) AdS supersymmetry are more restrictive than in the Minkowski case, which is completely analogous to the observations made in [52, 24, 12] regarding the four-dimensional $\sigma$-models with $\mathcal{N} = 1$ AdS supersymmetry. We also see that in three dimensions the $\sigma$-model couplings with (2,0) and (1,1) AdS supersymmetry types are rather different. In particular, compact target spaces are allowed in the (2,0) case, while the (1,1) AdS supersymmetry is consistent only with non-compact Kähler manifolds.
It is worth noting that one may reintroduce Kähler symmetry by separating the function $K$ into a Kähler potential $K$ and a holomorphic superpotential $W$, 

$$K = K + \frac{W}{\mu} + \frac{\bar{W}}{\bar{\mu}}. \quad (2.59)$$

Then the action may be written in a familiar way

$$\int d^3x \, d^4\theta \, E \, K + \left( \int d^3x \, d^2\theta \, E W + \text{c.c.} \right). \quad (2.60)$$

However, in this case, the Kähler symmetry manifests itself as

$$K \to K + F + \bar{F}, \quad W \to W - \mu F, \quad (2.61)$$

where $F = F(\phi)$ is a holomorphic function. This implies that $K$ is the only physically meaningful quantity.

Using the component reduction rule (2.53), one can show that

$$\int d^3x \, d^4\theta \, E K = \int d^3x \, e \, L_K \quad (2.62)$$

where

$$L_K = -g_{ab} \mathcal{D}_m \varphi^a \mathcal{D}^m \varphi^b - ig_{ab} \bar{\psi}^b \mathcal{D}^{a\beta} \psi^\beta + F^a \bar{F}^{\bar{b}} g_{ab} + \frac{1}{4} R_{abcd}(\psi^a \psi^c)(\bar{\psi}^b \bar{\psi}^d) - \frac{\mu}{2} \nabla_a K_b \psi^a \psi^b - \frac{\bar{\mu}}{2} \nabla_{\bar{a}} K_{\bar{b}} (\bar{\psi}^a \bar{\psi}^b) + \mu K_a F^a + \bar{\mu} K_{\bar{a}} \bar{F}^{\bar{a}} + 4 \mu \bar{\mu} K. \quad (2.63)$$

We have defined the components of $\phi^a$ as

$$\phi^a := \phi^a|, \quad \psi^a := \frac{1}{\sqrt{2}} \mathcal{D}_a \phi^a|, \quad (2.64a)$$

$$F^a := -\frac{1}{4} g^{ab} \mathcal{D}^2 K^a = -\frac{1}{4} (\mathcal{D}^2 \phi^a + \Gamma^a_{bc} \mathcal{D}^b \phi^c \mathcal{D}_c \phi^a)|. \quad (2.64b)$$

In particular, the auxiliary field $F^a$ transforms covariantly under reparametrizations. The vector derivative on the fermion is similarly reparametrization covariant,

$$\mathcal{D}_m \psi^a := \mathcal{D}_m \psi^a + \Gamma^a_{bc} \mathcal{D}_m \varphi^b \psi^c \quad (2.65)$$

The fermion mass terms are given in terms of covariant field derivatives of the Kähler potential,

$$\nabla_{\alpha} K_{\beta} := \partial_{\alpha} \partial_{\beta} K - \Gamma^c_{\alpha \beta} \partial_{\alpha} K \quad (2.66)$$
Eliminating the auxiliary field $F^a$ and its conjugate $\bar{F}^a$ leads to

$$L = -g_{ab} \mathcal{D}_m \varphi^a \mathcal{D}^m \varphi^b - ig_{ab} \bar{\psi}^\beta_a \hat{\mathcal{D}}^a \psi_\beta^a + \frac{1}{4} R_{abcd} (\psi^a \psi^c) (\bar{\psi}^b \bar{\psi}^d)$$
$$- \frac{\mu}{2} \nabla_a \mathcal{K}_b (\psi^a \psi^b) - \frac{\bar{\mu}}{2} \nabla_a \bar{\mathcal{K}}_b (\bar{\psi}^a \bar{\psi}^b) - \mu g^{ab} \mathcal{K}_a \bar{\mathcal{K}}_b + 4 \mu \bar{\mu} \mathcal{K}.$$  \hspace{1cm} (2.67)

Note that there is generally a scalar potential in AdS.

Let us again consider the case where the target space is a Kähler cone. This implies that there exists a homothetic conformal Killing vector $\chi$ from which one may construct a Kähler potential as $K = g_{ab} \chi^a \chi^b$. However, $\mathcal{K}$ may differ from this choice of $K$ by the real part of a holomorphic field, which, inspired by (2.59), we can choose to parametrize as

$$\mathcal{K} = K + \frac{W}{\mu} + \frac{\bar{W}}{\bar{\mu}}.$$  \hspace{1cm} (2.68)

If the action is superconformal, the holomorphic function $W$ must obey (2.42) and so $\mathcal{K}$ consequently obeys

$$\chi^a \mathcal{K}_a = K + \frac{3}{\mu} W - \frac{1}{\bar{\mu}} \bar{W}.$$  \hspace{1cm} (2.69)

and the component action takes the form

$$L = -g_{ab} \mathcal{D}_m \varphi^a \mathcal{D}^m \varphi^b - ig_{ab} \bar{\psi}^\beta_a \hat{\mathcal{D}}^a \psi_\beta^a + \frac{1}{4} R_{abcd} (\psi^a \psi^c) (\bar{\psi}^b \bar{\psi}^d)$$
$$- \frac{1}{2} \nabla_a W_b (\psi^a \psi^b) - \frac{1}{2} \nabla_a \bar{W}_b (\bar{\psi}^a \bar{\psi}^b) + 3 \mu \bar{\mu} \mathcal{K} - g^{ab} W_a W_b,$$  \hspace{1cm} (2.70)

which, as in the $(2,0)$ case, can be rewritten as

$$L = g_{ab} \chi^a (\hat{\mathcal{D}}_m \hat{\mathcal{D}}^m - \frac{1}{8} \mathcal{R}) \chi^b - ig_{ab} \bar{\psi}^\beta_a \hat{\mathcal{D}}^a \psi_\beta^a + \frac{1}{4} R_{abcd} (\psi^a \psi^c) (\bar{\psi}^b \bar{\psi}^d)$$
$$- \frac{1}{2} \nabla_a W_b (\psi^a \psi^b) - \frac{1}{2} \nabla_a \bar{W}_b (\bar{\psi}^a \bar{\psi}^b) - g^{ab} W_a W_b.$$  \hspace{1cm} (2.71)

This reveals that the mass terms arise solely from the holomorphic function $W$.

### 3 Sigma models with $(3,0)$ AdS supersymmetry: Off-shell approach

In this and the next sections we provide a detailed study of the nonlinear $\sigma$-models with $(3,0)$ AdS supersymmetry.
The off-shell (3,0) supersymmetric $\sigma$-model in the (2,0) AdS superspace considered in \cite{[19]} has the form

$$S = \oint \frac{d\zeta}{2\pi i} \int d^3x \, d^4\theta \, E \, K(\Upsilon^I, \bar{\Upsilon}^J),$$

(3.1a)

with the contour integral being evaluated along a closed path $\gamma$ around the origin in $\mathbb{C}$. Here $K(\Phi^I, \bar{\Phi}^J)$ is a real analytic function subject to the homogeneity conditions

$$\Phi^I \frac{\partial}{\partial \Phi^I} K(\Phi, \bar{\Phi}) = K(\Phi, \bar{\Phi}), \quad \bar{\Phi}^I \frac{\partial}{\partial \bar{\Phi}^I} K(\Phi, \bar{\Phi}) = K(\Phi, \bar{\Phi}).$$

(3.1b)

This condition means that $K(\Phi, \bar{\Phi})$ can be interpreted as the Kähler potential of a Kähler cone $X$, see Appendix A. The dynamical variables in (3.1a) are covariant weight-one arctic multiplets

$$\Upsilon^I(\zeta) = \sum_{n=0}^{\infty} \zeta^n \Upsilon^I_n = \Phi^I + \zeta \Sigma^I + \ldots,$$

(3.2)

and their smile-conjugate weight-one antarctic multiplets

$$\bar{\Upsilon}^I(\zeta) = \sum_{n=0}^{\infty} (-\zeta)^{-n} \bar{\Upsilon}^I_n.$$

(3.3)

Here the component superfields $\Phi^I := \Upsilon^I_0$ and $\Sigma^I := \Upsilon^I_1$ are chiral and complex linear respectively,

$$\mathcal{D}_\alpha \Phi^I = 0, \quad \mathcal{D}^2 \Sigma^I = 0,$$

(3.4)

while the remaining Taylor coefficients in (3.2), $\Upsilon^I_2, \Upsilon^I_3, \ldots$, are complex unconstrained superfields. The latter superfields are auxiliary, since they are unconstrained and appear in the action without derivatives. The superfields $\Phi^I$ and $\Sigma^I$ are physical.

The theory defined by the action (3.1a) and the homogeneity condition (3.1b) is not the most general off-shell nonlinear $\sigma$-model \cite{[19]} with (3,0) AdS supersymmetry. The latter is given by an action of the form:

$$S = \oint \frac{d\zeta}{2\pi i} \int d^3x \, d^4\theta \, E \, \mathcal{L}(\Upsilon^I, \bar{\Upsilon}^J, \zeta), \quad \mathcal{L}(\Upsilon^I, \bar{\Upsilon}^J, \zeta) := \frac{1}{\zeta} \Re(\Upsilon^I, \zeta \bar{\Upsilon}^J),$$

(3.5a)

where $\Re(\Phi, \bar{\Omega})$ is a homogeneous function of $2n$ complex variables $\Phi^I$ and $\bar{\Omega}^I$,

$$\left( \Phi^I \frac{\partial}{\partial \Phi^I} + \bar{\Omega}^I \frac{\partial}{\partial \bar{\Omega}^I} \right) \Re(\Phi, \bar{\Omega}) = 2 \Re(\Phi, \bar{\Omega}),$$

(3.5b)
under the reality condition
\[ \bar{\mathcal{R}}(\bar{\Phi}, -\Omega) = -\mathcal{R}(\Phi, \bar{\Omega}) , \] (3.5c)
where \( \bar{\mathcal{R}}(\bar{\Phi}, \Omega) \) denotes the complex conjugate of \( \mathcal{R}(\Phi, \bar{\Omega}) \). In the case that \( \mathcal{R}(\Phi, \bar{\Omega}) \) is also homogeneous with respect to \( \Phi \) (or, equivalently, with respect to \( \bar{\Omega} \)),
\[ \Phi^I \frac{\partial}{\partial \Phi^I} \mathcal{R}(\Phi, \bar{\Omega}) = \mathcal{R}(\Phi, \bar{\Omega}) , \quad \bar{\Omega}^I \frac{\partial}{\partial \bar{\Omega}^I} \mathcal{R}(\Phi, \bar{\Omega}) = \mathcal{R}(\Phi, \bar{\Omega}) , \] (3.6)
the reality condition (3.5c) is equivalent to \( \bar{\mathcal{R}}(\bar{\Phi}, \bar{\Omega}) = \mathcal{R}(\Phi, \bar{\Omega}) \), that is \( \mathcal{R}(\Phi, \bar{\Omega}) \) is real.

The (3,0) supersymmetric \( \sigma \)-model (3.1) has a simple geometric interpretation. This theory is associated with a Kähler cone \( X \) for which \( K(\Phi, \bar{\Omega}) \) is the preferred Kähler potential (see Appendix A). The \( \sigma \)-model target space turns out to be the cotangent bundle \( T^*X \) which is a hyperkähler cone [17]. In the case of the most general (3,0) supersymmetric \( \sigma \)-model (3.5), a geometric interpretation of \( \mathcal{R}(\Phi, \bar{\Omega}) \) is unclear to us. However, as will be shown below, the \( \sigma \)-model target space is a hyperkähler cone provided the off-shell theory (3.5) leads to a non-degenerate metric for the target space. We will come back to a general discussion of the \( \sigma \)-model (3.1) at the end of this section. Right now, we turn to eliminating the auxiliary superfields in the theory (3.5) in a formal way (that is, we assume that the function \( \mathcal{R}(\Phi, \bar{\Omega}) \) is properly chosen such that all the auxiliary superfields can be eliminated in a unique way).

The fact that \( \Upsilon^I(\zeta) \) is a covariant weight-one arctic multiplet is encoded in its transformation law under the (3,0) AdS isometry supergroup, OSp(3\vert2;\mathbb{R}) \times Sp(2,\mathbb{R}) \) given in [10]. When realized in the (2,0) AdS superspace, the most general (3,0) isometry transformation of any (3,0) supermultiplet splits into two different transformations: (i) a (2,0) AdS isometry transformation generated by superfield parameters specified in eqs. (2.4) and (2.9); (ii) an extended supersymmetry transformation generated by a chiral superfield parameter \( \rho \) and its conjugate \( \bar{\rho} \) which are subject to the constraints
\[ \mathcal{D}_a \rho = 0 , \quad \mathcal{D}_a \bar{\rho} = \bar{\mathcal{D}}_a \bar{\rho} \equiv \rho_a , \quad \mathcal{D}^2 \rho = -8iS\bar{\rho} \quad \implies \quad \mathcal{D}_{a\beta} \rho = 0 . \] (3.7)
These constraints prove to imply that the only independent components of \( \rho \) are the following: (a) the constant complex parameter \( \rho|_{\theta=\bar{\theta}=0} \) which describes an \( R \)-symmetry transformation from the coset \( SU(2)/U(1) \); (b) the Killing spinor parameter \( \mathcal{D}_a \rho|_{\theta=\bar{\theta}=0} \) which describes a third supersymmetry transformation. The (2,0) AdS isometry transformation of \( \Upsilon^I(\zeta) \) is
\[ \delta_\tau \Upsilon^I = (\tau + i\mathcal{J})\Upsilon^I , \quad \mathcal{J} = \left( \zeta \frac{\partial}{\partial \zeta} - \frac{1}{2} \right) \quad \iff \quad \mathcal{J} \Upsilon^I_n = (n - \frac{1}{2}) \Upsilon^I_n . \] (3.8)

\[ \text{Ref. [10] used a different parameter, denoted } \varepsilon , \text{ which is related to } \rho \text{ as } \varepsilon = -8S\rho. \]
In particular, the $U(1)_R$ charges of the dynamical superfields are
$$J\Phi^I = -\frac{1}{2}\Phi^I , \quad J\Sigma^I = \frac{1}{2}\Sigma^I .$$
(3.9)

A finite transformation generated by $J$ is
$$\Upsilon(\zeta) \rightarrow \Upsilon'(\zeta) = e^{-(i/2)\alpha} \Upsilon(e^{i\alpha}\zeta) , \quad \alpha \in \mathbb{R} ,$$
(3.10)
or in components
$$\Phi \rightarrow e^{-i\alpha/2}\Phi , \quad \Sigma \rightarrow e^{i\alpha/2}\Sigma , \quad \Upsilon_2 \rightarrow e^{3i\alpha/2}\Upsilon_2 , \quad \ldots$$
(3.11)

This transformation coincides with the so-called shadow chiral rotation introduced in the context of 4D $\mathcal{N} = 2$ superconformal $\sigma$-models [47]. It is an instructive exercise to show that the transformation (3.10) is a symmetry of the theory (3.5).

The extended supersymmetry transformation of $\Upsilon^I$ is
$$\delta_\rho \Upsilon^I = \{ i\zeta^\rho^\alpha D_\alpha + \frac{i}{\zeta}\rho_\alpha D^\alpha - 4S(\zeta\bar{\rho} + \frac{1}{\zeta}\rho)\zeta \frac{\partial}{\partial \zeta} + 4S\zeta\bar{\rho} \} \Upsilon^I .$$
(3.12)

For the physical superfields, this transformation law leads to
$$\delta_\rho \Phi^I = (i\rho_\alpha \bar{D}^\alpha - 4S\rho)\Sigma^I = \frac{1}{2}D^2(\bar{\rho}\Sigma^I) ,$$
$$\delta_\rho \Sigma^I = (i\rho^\alpha D_\alpha + 4S\bar{\rho})\Phi^I + (i\rho_\alpha \bar{D}^\alpha - 8S\rho)\Upsilon_2^I = i\bar{D}_\alpha(\rho^\alpha \Upsilon_2^I - \bar{\rho}D^\alpha \Phi^I) .$$
(3.13a)

It is seen that the variations $\delta_\rho \Phi^I$ and $\delta_\rho \Sigma^I$ are chiral and complex linear respectively.

As mentioned earlier, the complex unconstrained superfields $\Upsilon_2^I, \Upsilon_3^I, \ldots$, and their conjugates appear in the action without derivatives, and therefore they are auxiliary. These superfields can in principle be eliminated using their algebraic nonlinear equations of motion,
$$\frac{\partial \mathcal{L}}{\partial \Upsilon_n^I} = \oint_\gamma \frac{d\zeta}{2\pi i\zeta} \frac{\partial \mathcal{L}}{\partial \Upsilon_n^I} \zeta^n = 0 , \quad n \geq 2 ;$$
$$\frac{\partial \mathcal{L}}{\partial \bar{\Upsilon}_n^I} = \oint_\gamma \frac{d\zeta}{2\pi i\zeta} \frac{\partial \mathcal{L}}{\partial \bar{\Upsilon}_n^I} (-\zeta)^{-n} = 0 , \quad n \geq 2 .$$
(3.14a)

Here we have introduced the Lagrangian
$$\mathcal{L}(\Upsilon_n, \bar{\Upsilon}_n) = \oint_\gamma \frac{d\zeta}{2\pi i\zeta} \mathcal{L}(\Upsilon, \bar{\Upsilon}; \zeta) = \oint_\gamma \frac{d\zeta}{2\pi i\zeta} \frac{1}{\zeta} \mathcal{R}(\Upsilon, \zeta \bar{\Upsilon}) .$$
(3.15)

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12In the case that $\mathcal{R}$ obeys the stronger homogeneity conditions (3.6), the equations (3.14) coincide with the auxiliary superfield equations corresponding to the 4D $\mathcal{N} = 2$ supersymmetric $\sigma$-models on cotangent bundles of Kähler cones [47], see also [53, 54].
If \( \Upsilon^I(\zeta) \) and \( \tilde{\Upsilon}^I(\zeta) \) give a solution of the auxiliary superfield equations (3.14), then \( r\Upsilon^I(\zeta) \) and \( r\tilde{\Upsilon}^I(\zeta) \) is also a solution for any \( r \in \mathbb{R} - \{0\} \), as a consequence of the homogeneity condition (3.5b).

Once the equations (3.14) have been solved and all the auxiliary superfields are expressed in terms of the physical ones, the Lagrangian (3.15) becomes a function of the physical superfields, \( \mathbb{L}(\Phi^I, \Sigma^I, \bar{\Phi}^J, \bar{\Sigma}^J) \), and the action reads

\[
S = \int d^3x \ d^4\theta \mathbb{L}(\Phi^I, \Sigma^I, \bar{\Phi}^J, \bar{\Sigma}^J) .
\]

(3.16)

The homogeneity condition (3.5b) can be recast in the form

\[
\sum_{k=0}^{\infty} \left( \Upsilon^I_k \frac{\partial}{\partial \Upsilon^I_k} + \tilde{\Upsilon}^I_k \frac{\partial}{\partial \tilde{\Upsilon}^I_k} \right) \mathbb{L}(\Upsilon_n, \tilde{\Upsilon}_n) = 2 \mathbb{L}(\Upsilon_n, \tilde{\Upsilon}_n) .
\]

(3.17)

If the auxiliary superfield equations (3.14) hold, then this condition simplifies

\[
\left( \Phi^I \frac{\partial}{\partial \Phi^I} + \Sigma^I \frac{\partial}{\partial \Sigma^I} + \bar{\Phi}^J \frac{\partial}{\partial \bar{\Phi}^J} + \bar{\Sigma}^J \frac{\partial}{\partial \bar{\Sigma}^J} \right) \mathbb{L} = 2 \mathbb{L} , \quad \mathbb{L} = \mathbb{L}(\Phi, \Sigma, \bar{\Phi}, \bar{\Sigma}) .
\]

(3.18)

Under the equations (3.14), the property that the theory has the \( U(1)_R \) symmetry (3.11) means

\[
\left( \Phi^I \frac{\partial}{\partial \Phi^I} - \Sigma^I \frac{\partial}{\partial \Sigma^I} - \bar{\Phi}^J \frac{\partial}{\partial \bar{\Phi}^J} + \bar{\Sigma}^J \frac{\partial}{\partial \bar{\Sigma}^J} \right) \mathbb{L} = 0 .
\]

(3.19)

Combining these two results gives the homogeneity conditions

\[
\left( \Phi^I \frac{\partial}{\partial \Phi^I} + \Sigma^I \frac{\partial}{\partial \Sigma^I} \right) \mathbb{L} = \mathbb{L} , \quad \left( \Phi^I \frac{\partial}{\partial \Phi^I} + \Sigma^I \frac{\partial}{\partial \Sigma^I} \right) \mathbb{L} = \mathbb{L} .
\]

(3.20)

By construction, the action (3.16) must be invariant under the extended supersymmetry transformation (3.13) in which \( \Upsilon^I_2 = \Upsilon^I_2(\Phi, \Sigma, \bar{\Phi}, \bar{\Sigma}) \) is part of the solution of the auxiliary equations (3.14). In practice, the functions \( \Upsilon^I_2 \) are known only for special manifolds. However, supersymmetry considerations [47, 55] allow one to develop a self-consistent scheme to determine both the Lagrangian \( \mathbb{L}(\Phi, \Sigma, \bar{\Phi}, \bar{\Sigma}) \) and the functions \( \Upsilon^I_2(\Phi, \Sigma, \bar{\Phi}, \bar{\Sigma}) \) appearing in the supersymmetry transformation law (3.13). Requiring the action (3.16) to be invariant under (3.13) leads to several equations

\[
\frac{\partial \mathbb{L}}{\partial \Phi^I} + \frac{\partial \mathbb{L}}{\partial \Sigma^I} \frac{\partial \Upsilon^I_2}{\partial \Sigma^I} = \frac{\partial \mathbb{L}}{\partial \Phi^I} ,
\]

(3.21a)

\[
- \frac{\partial \mathbb{L}}{\partial \Phi^I} + \frac{\partial \mathbb{L}}{\partial \Sigma^I} \frac{\partial \Upsilon^I_2}{\partial \Phi^I} = \frac{\partial \mathbb{L}}{\partial \Phi^I} ,
\]

(3.21b)

\[
\frac{\partial \mathbb{L}}{\partial \Sigma^I} \frac{\partial \Upsilon^I_2}{\partial \Sigma^I} = \frac{\partial \mathbb{L}}{\partial \Sigma^I} .
\]

(3.21c)
as well as

$$2\Xi = -\Phi^I \frac{\partial L}{\partial \Phi^J} + \Sigma^I \frac{\partial L}{\partial \Phi^J} + 2\Upsilon^I \frac{\partial L}{\partial \Sigma^J}, \quad (3.22)$$

for some function \( \Xi(\Phi, \Sigma, \bar{\Phi}, \bar{\Sigma}) \). The existence of \( \Xi \) satisfying the three requirements \((3.21a) - (3.21c)\) can be proved by using the contour integral definition of \( L \), eq. \((3.15)\). It is an instructive exercise to check that the function \( \Xi \) defined as (compare with [48])

$$\Xi := \oint_\gamma \frac{d\zeta}{2\pi i \zeta} - \Upsilon(\zeta) \quad (3.23)$$

does obey the conditions \((3.21)\).

Eq. \((3.22)\) can actually be deduced from the conditions \((3.21)\) in conjunction with some additional observations. The first observation is that (i) \( \Upsilon^I(\zeta) \) is a homogeneous function of \( \Upsilon^I(\zeta) \) and \( \bar{\Upsilon}(\zeta) \) of degree one; and (ii) if \( \Upsilon^I(\zeta) \) is a solution of the auxiliary equations \((3.14)\), then \( r \Upsilon^I(\zeta) \) is also a solution for any \( r \in \mathbb{R} - \{0\} \), as a consequence of the homogeneity condition \((3.5b)\). Upon elimination of the auxiliary superfields, this means

$$\left( \Phi^I \frac{\partial}{\partial \Phi^J} + \Sigma^I \frac{\partial}{\partial \Sigma^J} + \bar{\Phi}^I \frac{\partial}{\partial \bar{\Phi}^J} + \bar{\Sigma}^I \frac{\partial}{\partial \bar{\Sigma}^J} \right) \Upsilon^J = \Upsilon^J, \quad (3.24a)$$

where \( \Upsilon^J = \Upsilon^J(\Phi, \Sigma, \bar{\Phi}, \bar{\Sigma}) \). The second observation is that the symmetry transformation \((3.11)\) acts on \( \Upsilon^J(\Phi, \Sigma, \bar{\Phi}, \bar{\Sigma}) \) exactly as in \((3.11)\), and thus

$$\left( \Phi^I \frac{\partial}{\partial \Phi^J} - \Phi^I \frac{\partial}{\partial \bar{\Phi}^J} - \Sigma^I \frac{\partial}{\partial \Sigma^J} + \bar{\Sigma}^I \frac{\partial}{\partial \bar{\Sigma}^J} \right) \Upsilon^J = -3\Upsilon^J. \quad (3.24b)$$

Combining these two results gives

$$\left( \Phi^I \frac{\partial}{\partial \Phi^J} + \Sigma^I \frac{\partial}{\partial \Sigma^J} \right) \Upsilon^J = -\Upsilon^J, \quad \left( \Phi^I \frac{\partial}{\partial \Phi^J} + \Sigma^I \frac{\partial}{\partial \Sigma^J} \right) \Upsilon^J = 2\Upsilon^J. \quad (3.25)$$

The third observation is that analogs of eqs. \((3.24a)\) and \((3.24b)\) may be derived for the function \( \Xi \) by making use of the contour-integral representation \((3.23)\). Specifically, one derives

$$\left( \Phi^I \frac{\partial}{\partial \Phi^J} + \Sigma^I \frac{\partial}{\partial \Sigma^J} + \bar{\Phi}^I \frac{\partial}{\partial \bar{\Phi}^J} + \bar{\Sigma}^I \frac{\partial}{\partial \bar{\Sigma}^J} \right) \Xi = -2\Xi, \quad (3.26a)$$

$$\left( \Phi^I \frac{\partial}{\partial \Phi^J} - \Phi^I \frac{\partial}{\partial \bar{\Phi}^J} - \Sigma^I \frac{\partial}{\partial \Sigma^J} + \bar{\Sigma}^I \frac{\partial}{\partial \bar{\Sigma}^J} \right) \Xi = -2\Xi. \quad (3.26b)$$

Combining these two results gives

$$\left( \Phi^I \frac{\partial}{\partial \Phi^J} + \Sigma^I \frac{\partial}{\partial \Sigma^J} \right) \Xi = 0, \quad \left( \Phi^I \frac{\partial}{\partial \Phi^J} + \Sigma^I \frac{\partial}{\partial \Sigma^J} \right) \Xi = 2\Xi. \quad (3.27)$$
Now, applying the relations \((3.21)\), \((3.25)\) and \((3.27)\) gives (with all indices suppressed)

\[
2\Xi = \Phi \frac{\partial \Xi}{\partial \Phi} + \Sigma \frac{\partial \Xi}{\partial \Sigma} = \Phi \left( -\frac{\partial \mathcal{L}}{\partial \Sigma} + \frac{\partial \mathcal{L}}{\partial \Phi} \frac{\partial \mathcal{Y}_2}{\partial \Sigma} \right) + \Sigma \left( \frac{\partial \mathcal{L}}{\partial \Phi} + \frac{\partial \mathcal{L}}{\partial \Sigma} \frac{\partial \mathcal{Y}_2}{\partial \Sigma} \right)
\]

\[
= -\Phi \frac{\partial \mathcal{L}}{\partial \Sigma} + \Sigma \left( \frac{\partial \mathcal{L}}{\partial \Phi} \frac{\partial \mathcal{Y}_2}{\partial \Sigma} + \frac{\partial \mathcal{Y}_2}{\partial \Sigma} \frac{\partial \mathcal{L}}{\partial \Phi} \right) = -\Phi \frac{\partial \mathcal{L}}{\partial \Sigma} + \sum \left( \frac{\partial \mathcal{L}}{\partial \Phi} + \frac{\partial \mathcal{L}}{\partial \Sigma} \frac{\partial \mathcal{Y}_2}{\partial \Sigma} \right),
\]

which is exactly the equation \((3.22)\).

In order to understand the target space geometry of the \(\sigma\)-model \((3.16)\), we have to construct a dual formulation for this theory in which the complex linear superfield \(\Sigma^I\) is traded for a covariantly chiral superfield \(\Psi^I\), \(\bar{D}_\alpha \Psi^I = 0\). For this, we introduce the first-order action

\[
S_{\text{first-order}} = \int d^3x \, d^4\theta E \left( \mathcal{L} + \Sigma^I \Psi_I + \bar{\Psi}^I \bar{\Psi}_J \right), \quad \mathcal{J}_I = -\frac{1}{2} \Psi_I, \quad (3.28)
\]

in which the superfields \(\Sigma^I\) are complex unconstrained. The choice of the \(U(1)_R\) charge of \(\Psi^I\) follows from \((3.9)\). The first-order model introduced is equivalent to the original \(\sigma\)-model \((3.16)\). Indeed, varying \((3.28)\) with respect to \(\Psi_I\) enforces complex linearity of \(\Sigma^I\), and then \(S_{\text{first-order}}\) reduces to the original action. On the other hand, if we apply the equation of motion for the unconstrained \(\Sigma^I\), we find

\[
S_{\text{dual}} = \int d^3x \, d^4\theta E \mathcal{K}(\Phi^I, \Psi_I, \bar{\Phi}_I, \bar{\Psi}_I), \quad (3.29)
\]

where \(\mathcal{K}\) is defined as

\[
\mathcal{K} := \mathcal{L} + \Sigma^I \Psi_I + \bar{\Psi}^I \bar{\Psi}_J \quad (3.30)
\]

and \(\Sigma^I\) is chosen to obey

\[
\frac{\partial \mathcal{L}}{\partial \Sigma^I} = -\Psi_I. \quad (3.31)
\]

The homogeneity conditions \((3.20)\) turn into

\[
\left( \Phi^I \frac{\partial}{\partial \Phi_I} + \Psi_I \frac{\partial}{\partial \Psi_I} \right) \mathcal{K} = \mathcal{K}, \quad \left( \Phi^I \frac{\partial}{\partial \Phi_I} + \bar{\Psi}^I \frac{\partial}{\partial \bar{\Psi}_I} \right) \mathcal{K} = \mathcal{K}. \quad (3.32)
\]

It is easy to work out the extended supersymmetry transformation of the first-order action \((3.28)\) starting from the original transformation \((3.13)\). Since \(\Sigma^I\) is complex unconstrained in \((3.13)\), the supersymmetry transformation of \(\Phi^I\) turns into

\[
\delta_\rho \Phi^I = \frac{i}{2} \bar{D}^2 (\rho \Sigma^I) = (i \rho_\alpha \bar{D}^\alpha - 4 S \rho) \Sigma^I + \frac{i}{2} \bar{\rho} \bar{D}^2 \Sigma^I, \quad (3.33)
\]

25
We keep the transformation law of $\Sigma^I$ unchanged, eq. (3.13b). Then, it may be seen that $S_{\text{first-order}}$ is invariant provided $\Psi_I$ transforms as follows:

$$\delta_\rho \Psi_I = -\frac{i}{2} \overline{D}^2 \left( \overline{\rho} \frac{\partial K}{\partial \Psi_I} \right).$$  \hfill (3.34)

From (3.33) and (3.34) we read off the extended supersymmetry of the dual action (3.29):

$$\delta_\rho \Phi^I = \frac{i}{2} \overline{D}^2 \left( \overline{\rho} \frac{\partial K}{\partial \Phi^I} \right), \quad \delta_\rho \Psi_I = -\frac{i}{2} \overline{D}^2 \left( \overline{\rho} \frac{\partial K}{\partial \Phi_I} \right).$$  \hfill (3.35)

If we introduce the condensed notation

$$\phi^a := (\Phi^I, \Psi_I), \quad \overline{\phi}^\bar{a} = (\overline{\Phi}^\bar{I}, \overline{\Psi}^\bar{I}),$$  \hfill (3.36)

the above transformation law can be rewritten as

$$\delta_\rho \phi^a = \frac{i}{2} \overline{D}^2 \left( \overline{\rho} \omega^{ab} \frac{\partial K}{\partial \phi^b} \right),$$  \hfill (3.37)

where

$$\omega^{ab} = \begin{pmatrix} 0 & \delta^I_J \\ -\delta_I^J & 0 \end{pmatrix}, \quad \omega_{ab} = \begin{pmatrix} 0 & \delta_I^J \\ -\delta_I^J & 0 \end{pmatrix}.$$  \hfill (3.38)

In accordance with the analysis in the next section, the $\sigma$-model target space is a hyperkähler manifold. The structure of the supersymmetry transformation (3.37) shows that $\omega^{(2,0)} = \omega_{ab} \, d\phi^a \wedge d\phi^b$ is a covariantly constant holomorphic two-form associated with the two complex structures orthogonal to the diagonal one. The explicit form of $\omega_{ab}$ shows that the coordinates $\Phi^I$ and $\Psi_I$ are holomorphic Darboux coordinates for the target space.

The relation (3.32) means that the target space possesses a homothetic conformal Killing vector $\chi$ which looks like

$$\chi = \phi^a \frac{\partial}{\partial \phi^a} + \overline{\phi}^{\bar{a}} \frac{\partial}{\partial \phi^{\bar{a}}}. \hfill (3.39)$$

The holomorphic Killing vector $J$ associated with the $R$-symmetry generator $J$ is

$$J^a := iJ \phi^a = -\frac{i}{2} \phi^a. \hfill (3.40)$$

It is obvious that the vector fields $\chi$ and $J$ commute, $[\chi, J] = 0$.

Let us now discuss the family of $(3,0)$ supersymmetric $\sigma$-models described by the action (3.1a) and the homogeneity condition (3.1b). In terms of the most general off-shell
action (3.5a), these theories are characterized by $\mathcal{A}(\Upsilon, \zeta, \bar{\Upsilon}) = \zeta K(\Upsilon, \bar{\Upsilon})$. The specific feature of such $\sigma$-models is the rigid $U(1)$ symmetry

$$\Upsilon(\zeta) \rightarrow \Upsilon'(\zeta) = e^{-(i/2)\alpha} \Upsilon(\zeta), \quad \alpha \in \mathbb{R},$$

(3.41)
in addition to the $U(1)_R$ symmetry (3.10). This internal symmetry has a number of nontrivial implications. To uncover them, it is useful to consider a combination of the symmetry transformations (3.10) and (3.41) given by $\Upsilon(\zeta) \rightarrow \Upsilon'(\zeta) = \Upsilon(e^{i\alpha}\zeta)$. This invariance implies that

$$\Sigma^I \frac{\partial L}{\partial \Sigma^I} = \Sigma^I \frac{\partial L}{\partial \Sigma^I}. \quad (3.42)$$

The same invariance enforces certain conditions on the functions $\Upsilon^I_2$ and $\Xi$ [47]:

$$\Sigma^I \frac{\partial \Upsilon^I_2}{\partial \Sigma^J} = \Sigma^J \frac{\partial \Upsilon^I_2}{\partial \Sigma^J} + 2 \Upsilon^I_2, \quad (3.43a)$$

$$\Sigma^I \frac{\partial \Xi}{\partial \Sigma^J} = \Sigma^J \frac{\partial \Xi}{\partial \Sigma^J} + \Xi. \quad (3.43b)$$

These relations in conjunction with eqs. (3.21a) and (3.21c) give

$$\Xi = \Sigma^I \frac{\partial L}{\partial \Phi^I} + 2 \Upsilon^I_2 \frac{\partial L}{\partial \Sigma^I}. \quad (3.44)$$

This is compatible with (3.22) provided an alternative representation for $\Xi$ holds:

$$\Xi = -\bar{\Phi}^I \frac{\partial L}{\partial \bar{\Sigma}^I}. \quad (3.45)$$

In the dual formulation, the condition of $U(1)$ invariance, eq. (3.42), turns into

$$\Psi_I \frac{\partial K}{\partial \Psi_I} = \bar{\Psi}_I \frac{\partial K}{\partial \Psi_I}. \quad (3.46)$$

Using this along with the hyperkähler cone conditions (3.32), we can show that

$$\Phi^I \frac{\partial K}{\partial \Phi^I} - \Psi_I \frac{\partial K}{\partial \Psi_I} + c.c. = 0 \quad (3.47)$$

which, one can check, is a tri-holomorphic isometry,

$$X^a = (i\Phi^I, -i\Psi_I), \quad X^a K_a + \bar{X}^\dot{a} \bar{K}^\dot{a} = 0, \quad \mathcal{L}_X \omega_{ab} = 0, \quad (3.48)$$

of the hyperkähler cone under consideration.
In summary, the characteristic feature of all (3,0) supersymmetric \( \sigma \)-models, which are defined by the action (3.1a) and the homogeneity condition (3.1b), is that the corresponding target space is a hyperkähler cone with a U(1) tri-holomorphic isometry. These models are a subclass of the general case (3.5), whose target space is required merely to be a hyperläher cone, but whether they are a \textit{proper} subclass remains unclear to us.

In the case of the \( \sigma \)-model (3.1), it is not difficult to see that the target space metric is non-degenerate in a neighborhood of the zero section of \( T^*X \). This follows from the explicit structure of the corresponding Kähler potential [47]:

\[
K(\phi^a, \bar{\phi}^b) \equiv K(\Phi^I, \bar{\Phi}^\bar{I}) = K(\Phi, \bar{\Phi}) + H(\Phi, \Psi, \bar{\Phi}, \bar{\Psi}),
\]

(3.49)

where

\[
H(\Phi, \Psi, \bar{\Phi}, \bar{\Psi}) = g_{I\bar{J}}(\Phi, \bar{\Phi})\Psi^I\bar{\Psi}^\bar{J} + \sum_{n=2}^{\infty} H^{I_1\ldots I_n\bar{J}_1\ldots \bar{J}_n}(\Phi, \bar{\Phi})\Psi_{I_1} \ldots \Psi_{I_n} \bar{\Psi}_{\bar{J}_1} \ldots \bar{\Psi}_{\bar{J}_n}.
\]

(3.50)

Here the coefficients \( H^{I_1\ldots I_n\bar{J}_1\ldots \bar{J}_n} \), with \( n = 2, 3, \ldots \), are real tensor functions of the Kähler metric \( g_{I\bar{J}}(\Phi, \bar{\Phi}) = \partial_I \partial_{\bar{J}}K(\Phi, \bar{\Phi}) \) on \( X \), the Riemann curvature \( R_{I\bar{J}KL}(\Phi, \bar{\Phi}) \) and its covariant derivatives. In accordance with (3.32), \( H^{I_1\ldots I_n\bar{J}_1\ldots \bar{J}_n} \) is a homogeneous function of \( \Phi^K \) of degree \((1 - n)\).

If we turn to the more general \( \sigma \)-model (3.5), the requirement that the target space metric be non-singular should be satisfied only under some restrictions on the function \( \mathcal{R}(\Phi, \bar{\Omega}) \) appearing in the off-shell action. The explicit form of such restrictions remains unclear to us. Even if such restrictions are met, different choices of \( \mathcal{R}(\Phi, \bar{\Omega}) \) may lead to the same target space geometry. As an illustration, we discuss the case of a single superconformal hypermultiplet. It is known that any four-dimensional hyperkähler cone is a flat space, for its Riemann curvature identically vanishes [51]. To describe such a hyperkähler cone, it suffices to use the off-shell \( \sigma \)-model (3.1) with \( K(\Upsilon, \bar{\Upsilon}) = \bar{\Upsilon}\Upsilon \). The corresponding hyperkähler potential is \( \mathcal{K}(\Phi, \Psi, \bar{\Phi}, \bar{\Psi}) = \bar{\Psi}\Phi + \bar{\Phi}\Psi \), and the hyperkähler metric is \textit{manifestly} flat, \( g_{ab} = \delta_{ab} \). On the other hand, for a single hypermultiplet there exist infinitely many \( \sigma \)-models (3.5) of the form

\[
\mathcal{L}(\Upsilon, \bar{\Upsilon}; \zeta) = \frac{1}{\zeta} \mathcal{R}(\Upsilon, \zeta\bar{\Upsilon}) = \bar{\Upsilon}\Upsilon f\left(\frac{\zeta}{\bar{\Upsilon}}\right),
\]

(3.51)

with \( f(z) \) a function satisfying the reality condition \( \bar{f}(z) = f(-1/z) \). The corresponding target space is always flat, but the hyperkähler metric generated in complex coordinates \( \Phi \) and \( \Psi \) is \textit{not manifestly} flat.
4 Sigma models with (3,0) AdS supersymmetry: On-shell approach

In the previous section, it was shown that off-shell (3,0) $\sigma$-models lead to a formulation in (2,0) superspace involving the Kähler potential $K(\Phi, \bar{\Phi}, \Psi, \bar{\Psi})$ of a hyperkähler cone $\mathcal{M}$, where the chiral superfields $\Phi^I$ and $\Psi^I$ correspond to a choice of complex Darboux coordinates on $\mathcal{M}$. We may proceed instead “from the ground up” and consider the most general (2,0) models which exhibit a (3,0) extended supersymmetry.

A (2,0) supersymmetric $\sigma$-model without a superpotential is given by the action

$$S = \int d^3x \, d^4\theta \, E \, K$$

for a Kähler potential $K = K(\phi, \bar{\phi})$. The chiral superfield $\phi^a$ must transform under (2,0) AdS supersymmetry as

$$\delta \phi^a = \tau \phi^a + tJ^a, \quad J^a := iJ^{\phi^a},$$

where $J^a$ is a holomorphic Killing vector.

Suppose now that the action possesses a full (3,0) supersymmetry, with $\phi^a$ transforming under the extended supersymmetry as

$$\delta \phi^a = \frac{i}{2} \bar{D}^2(\bar{\rho} \Omega^a)$$

for some function $\Omega^a(\phi, \bar{\phi})$, where $\bar{\rho}$ obeys the same conditions as in the previous section. The variation of the action must be zero,

$$0 = \delta S = \int d^3x \, d^4\theta \, E \left( \frac{i}{2} K_a \bar{D}^2(\bar{\rho} \Omega^a) - \frac{i}{2} K_a \bar{D}^2(\rho \Omega^a) \right).$$

This condition, together with the requirement that the algebra should close, imposes a number of conditions on the function $\Omega^a$ as well as the target space. The analysis is somewhat technical, so we delay the discussion to appendix C. It turns out that the following properties must hold:

1. The target space possesses a homothetic conformal Killing vector $\chi^a$ (see Appendix A) which is related to the U(1) Killing vector by $J^a = -\frac{i}{2} \chi^a$.

2. The quantity $\omega_{ab} := g_{a\bar{a}} \partial_b \Omega^\bar{a}$ is a covariantly constant holomorphic two-form, $\nabla_c \omega_{ab} = \nabla_c \omega_{ab} = 0$, obeying the equation $\omega^{ab} \omega_{bc} = -\delta^a_c$, where $\omega^{ab} := g^{ac} g^{bd} \omega_{cd}$.
One may also check that $L_{J} \omega_{ab} = -i \omega_{ab}$. In terms of these geometric objects, it holds that $\Omega^a = \omega^{ab} \chi_b$, and the extended supersymmetry transformation may equivalently be written

$$\delta \phi^a = i \bar{\rho}_a \omega_a^b \mathcal{D}^a \phi^b - 4 S \rho \omega^{ab} \chi_b + \frac{i}{2} \bar{\rho} \omega^{ab} \mathcal{D}^2 \chi_b ,$$

with the extended supersymmetry algebra closing only on-shell. Note that the last term in (4.5) vanishes on-shell as a consequence of the absence of a superpotential.

Together these conditions imply that the target space geometry is a hyperkähler cone. The three covariantly constant complex structures can be taken as

$$(J_1)_{\mu}^a = \left( \begin{array}{c} 0 & \omega^a_{,b} \\ \omega^a_{,b} & 0 \end{array} \right) , \quad (J_2)_{\mu}^a = \left( \begin{array}{c} 0 & i \omega^a_{,b} \\ -i \omega^a_{,b} & 0 \end{array} \right) , \quad (J_3)_{\mu}^a = \left( \begin{array}{c} i \delta^a_{,b} & 0 \\ 0 & -i \delta^a_{,b} \end{array} \right) .$$

(4.6)

Using $J_A$ and $\chi$, we may construct three SU(2) Killing vectors

$$V_A^\mu = \frac{1}{2} (J_A)^{\mu}_{\nu} \chi^\nu ,$$

(4.7)

with $V_3^\mu$ coinciding with the U(1) Killing vector $J^\mu$ required by (2,0) supersymmetry. These vectors commute with the homothetic conformal Killing vector, obey an SU(2) algebra amongst themselves,

$$[V_A, \chi] = 0 , \quad [V_A, V_B] = \epsilon_{ABC} V_C ,$$

(4.8)

and act as an SU(2) rotation on the complex structures

$$\mathcal{L}_{V_A} J_B = \epsilon_{ABC} J_C .$$

(4.9)

The component action may be easily derived using (2.40) and applying what we have learned of the target space geometry:

$$L = -g_{ab} \mathcal{D}_m \varphi^a \mathcal{D}^m \varphi^b - i g_{ab} \bar{\psi}_a^b \mathcal{D}^a \psi_b^a + \frac{1}{4} R_{abcd} (\psi^a \psi^c) (\bar{\psi}^b \bar{\psi}^d) + 3 S^2 K$$

(4.10)

where we have used $K = \chi^a \bar{\chi}^b g_{ab}$. The apparent scalar potential actually corresponds to a massless model in AdS, confirmed by the absence of fermionic mass terms. This can be made apparent by rewriting the Lagrangian as

$$L = g_{ab} \chi^a (\mathcal{D}_a \mathcal{D}^a - \frac{1}{8} \mathcal{R}) \bar{\chi}^b - i g_{ab} \bar{\psi}_a^b \mathcal{D}^a \psi_b^a + \frac{1}{4} R_{abcd} (\psi^a \psi^c) (\bar{\psi}^b \bar{\psi}^d)$$

(4.11)

after identifying the scalar curvature of AdS as $\mathcal{R} = -24 S^2$. The scalar kinetic operator in the first term is indeed the conformal d’Alembertian in three dimensions.

We note that it is not possible to introduce any mass deformations, either via a superpotential or by deforming the Killing vectors. We will see that the (2,1) situation is quite different.
5 Sigma models with (2,1) AdS supersymmetry: Off-shell approach

In the case of (2,1) AdS supersymmetry, general off-shell σ-models can be realized in terms of weight-zero polar hypermultiplets living in the (2,1) AdS superspace \([10]\). The geometry of this superspace is encoded in a covariantly constant real isotriplet \(w^{ij} = w^{ji}\), conventionally normalized by \(w^{ij}w_{ij} = 2\), which can be interpreted as the field strength of a frozen vector multiplet (called the *intrinsic vector multiplet* in \([10]\)). The local SU(2) gauge freedom can partially be fixed by choosing a gauge \(w^{ij} = \text{const}\) and then mapping \(w^{ij}\) to any particular position on the two-sphere \(w^{ij}w_{ij} = 2\). Depending on the explicit choice of \(w^{ij}\) made, the manifestly (2,1) supersymmetric σ-models can be reduced to either (2,0) or (1,1) AdS superspace \([10]\). Here we will use these off-shell realizations in order to reformulate the (2,1) supersymmetric σ-models in terms of covariantly chiral superfields on (2,0) or (1,1) AdS superspace.

5.1 Formulation in (2,0) AdS superspace

The off-shell (2,1) supersymmetric σ-model in the (2,0) AdS superspace \([10]\) is

\[
S = \int \frac{d\zeta}{2\pi i} \int d^3 x \, d^4 \theta \, E \, K(\Upsilon^I, \bar{\Upsilon}^J). \tag{5.1}
\]

Formally this coincides with the (3,0) supersymmetric σ-model action \((3.1a)\). However, the conceptual difference between the two cases is that the Lagrangian in \((5.1)\) is not required to obey any homogeneity condition like \((3.1b)\). The only conditions on the Lagrangian in \((5.1)\) are that (i) \(K(\Phi^I, \bar{\Phi}^J)\) is a real analytic function of \(n\) complex variables \(\Phi^I\) and their conjugates; and (ii) the matrix \(g_{IJ} := \partial_I \partial_J K\) is nonsingular. One can consistently interpret \(K(\Phi, \bar{\Phi})\) as the Kähler potential of a Kähler manifold \(\mathcal{X}\), since the action \((5.1)\) may be seen to be invariant under Kähler transformations of the form

\[
K(\Upsilon, \bar{\Upsilon}) \to K(\Upsilon, \bar{\Upsilon}) + \Lambda(\Upsilon) + \bar{\Lambda}(\bar{\Upsilon}), \tag{5.2}
\]

with \(\Lambda(\Phi^I)\) a holomorphic function.

The dynamical variables \(\Upsilon^I(\zeta)\) and \(\bar{\Upsilon}^J(\zeta)\) in \((5.1)\) are *covariant weight-zero arctic* and *antarctic* multiplets. Considered as (2,0) AdS superfields, they are completely specified by eqs. \((3.2) - (3.4)\). The information that \(\Upsilon^I(\zeta)\) is a covariant weight-zero arctic multiplet is encoded in its transformation law under the (2,1) AdS isometry supergroup,
OSp(2|2; \mathbb{R}) \times \text{OSp}(1|2, \mathbb{R})$, given in [10]. Upon reduction to the (2,0) AdS superspace, the most general (2,1) isometry transformation of any (2,1) supermultiplet splits into two different transformations: (i) a (2,0) AdS isometry transformation generated by superfield parameters specified in eqs. (2.4) – (2.9); (ii) a third supersymmetry transformation generated by a real spinor parameter \( \rho \) obeying the constraints

\[
\mathcal{D}_\beta \rho_\alpha = \bar{\mathcal{D}}_\beta \rho_\alpha = 0 . \tag{5.3}
\]

These conditions mean that \( \rho_\alpha \) is an ordinary Killing spinor,

\[
\mathcal{D}_\beta \rho_\alpha = S(\varepsilon_{\alpha\beta} \rho_\gamma + \varepsilon_{\alpha \gamma} \rho_\beta) . \tag{5.4}
\]

The (2,0) AdS isometry transformation of \( \Upsilon^I(\zeta) \) is

\[
\delta_{\tau} \Upsilon^I = (\tau + itJ) \Upsilon^I, \quad J = \zeta \frac{\partial}{\partial \zeta} \iff J \Upsilon^I_n = n \Upsilon^I_n . \tag{5.5}
\]

A finite transformation generated by \( J \),

\[
\Upsilon(\zeta) \rightarrow \Upsilon'(\zeta) = \Upsilon(e^{i\alpha} \zeta) , \quad \alpha \in \mathbb{R} , \tag{5.6}
\]

is a symmetry of the \( \sigma \)-model (5.1). It coincides with the U(1) symmetry of the 4D \( \mathcal{N} = 2 \) supersymmetric \( \sigma \)-models on cotangent bundles of Kähler manifolds [53, 54].

The third supersymmetry transformation of \( \Upsilon^I \) is

\[
\delta_{\rho} \Upsilon^I = \left\{ i\zeta \rho^\alpha D_\alpha + i \frac{\zeta}{\rho} \bar{D}^\alpha \right\} \Upsilon^I . \tag{5.7}
\]

It is useful to represent

\[
\rho_\alpha = D_\alpha \rho = \bar{D}_\alpha \bar{\rho} , \quad J \rho = -\rho \tag{5.8}
\]

for some complex superfield parameter \( \rho \) defined modulo arbitrary antichiral shifts

\[
\rho \rightarrow \rho + \bar{\lambda} , \quad \mathcal{D}_\alpha \bar{\lambda} = 0 . \tag{5.9}
\]

Due to (5.3), this scalar parameter is subject to the constraints

\[
\mathcal{D}^2 \rho = 0 , \quad \mathcal{D}_\alpha \mathcal{D}_\beta \rho = 0 \tag{5.10}
\]

in addition to the reality condition \( \mathcal{D}_\alpha \rho = \bar{D}_\alpha \bar{\rho} \). For the physical superfields, the transformation law (5.7) leads to

\[
\begin{align*}
\delta_{\rho} \Phi^I &= i \rho_\alpha D^\alpha \Sigma^I = \frac{i}{2} \mathcal{D}^2 (\rho \Sigma^I) , \\
\delta_{\rho} \Sigma^I &= i \rho^\alpha D_\alpha \Phi^I + i \rho_\alpha \bar{D}^\alpha \Upsilon^I = i \mathcal{D}_\alpha (\rho^\alpha \Upsilon^I_2 - \bar{\rho} \bar{D}^\alpha \Phi^I) .
\end{align*}
\tag{5.11a,b}
\]
The variations \( \delta_\rho \Phi^I \) and \( \delta_\rho \Sigma^I \) are chiral and complex linear respectively. One should remember that the U(1) charges of the physical superfields are

\[
\mathcal{J}_\Phi = 0, \quad \mathcal{J}_\Sigma = \Sigma^I. \tag{5.12}
\]

Similar to the (3,0) case considered earlier, the (2,1) supersymmetric \( \sigma \)-model (5.1) can be reformulated solely in terms of the physical superfields and then, upon performing a duality transformation, in terms of (2,0) chiral superfields (the procedure is almost identical to that described in [47, 48] for 4D \( \mathcal{N} = 2 \) supersymmetric \( \sigma \)-models). Most aspects of these procedures are completely analogous to the (3,0) case, but some differences also occur. Upon elimination of the auxiliary superfields \( \Upsilon_I^2, \Upsilon_I^3, \ldots \), the action takes the form

\[
S = \int d^3x d^4\theta E \mathbb{L}(\Phi^I, \Sigma^I, \bar{\Phi}^I, \bar{\Sigma}^I). \tag{5.13}
\]

The complex variables \((\Phi^I, \Sigma^I)\) parametrize the holomorphic tangent bundle \( T\mathcal{X} \) of the Kähler manifold. In complete analogy with the four-dimensional analysis in [56], this follows from the observation that a holomorphic reparametrization of the Kähler manifold, \( \Phi^I \rightarrow \Phi'^I = f^I(\Phi) \), has the following counterpart

\[
\Upsilon^I(\zeta) \rightarrow \Upsilon'^I(\zeta) = f^I(\Upsilon(\zeta)) \tag{5.14}
\]

for the (2,1) arctic multiplets. Therefore, the physical superfields

\[
\Upsilon^I(\zeta) \bigg|_{\zeta=0} = \Phi^I, \quad \frac{d\Upsilon^I(\zeta)}{d\zeta} \bigg|_{\zeta=0} = \Sigma^I, \tag{5.15}
\]

should be regarded, respectively, as coordinates of a point in \( \mathcal{X} \) and a tangent vector at the same point. The general form of the Lagrangian in (5.13) is (compare with [53, 54])

\[
\mathbb{L}(\Phi, \Sigma, \bar{\Phi}, \bar{\Sigma}) = K(\Phi, \bar{\Phi}) + \sum_{n=1}^{\infty} \mathbb{L}_{I_1 \ldots I_n J_1 \ldots J_n}(\Phi, \bar{\Phi}) \Sigma^{I_1} \ldots \Sigma^{I_n} \bar{\Sigma}^{J_1} \ldots \bar{\Sigma}^{J_n}, \tag{5.16}
\]

where \( \mathbb{L}_{I,J} = -g_{IJ}(\Phi, \bar{\Phi}) \) and the Taylor coefficients \( \mathbb{L}_{I_1 \ldots I_n J_1 \ldots J_n} \), for \( n > 1 \), are tensor functions of the Kähler metric \( g_{IJ}(\Phi, \bar{\Phi}) = \partial_I \partial_J K(\Phi, \bar{\Phi}) \), the Riemann curvature \( R_{IJKL}(\Phi, \bar{\Phi}) \) and its covariant derivatives. Each term in the action contains equal powers of \( \Sigma \) and \( \bar{\Sigma} \), since the off-shell action (5.1) is invariant under the U(1) transformation (5.6).

By construction, the tangent-bundle action (5.13) must be invariant under the third supersymmetry transformation (5.11) in which \( \Upsilon_2^I \) has to be the function \( \Upsilon_2^I(\Phi, \Sigma, \bar{\Phi}, \bar{\Sigma}) \).
obtained by solving the equations of motion for the auxiliary superfields, eq. (3.14). Requiring the action (5.13) to be invariant under (5.11) leads to the consistency conditions (3.21), for some function $\Xi$ given by eq. (3.23). Using the contour integral representation (3.23), one may prove that all conditions (3.21) hold identically. Unlike the (3,0) $\sigma$-model, eq. (3.22) does not appear in the present case.

A novel feature occurs when we turn to the first-order formulation of the $\sigma$-model (5.13) given by

$$S_{\text{first-order}} = \int \! d^3x \, d^4\theta \, E \left( \mathcal{L} + \Sigma^I \Psi_I + \bar{\Sigma}^J \bar{\Psi}_J \right), \quad \mathcal{J} \Psi_I = -\Psi_I, \quad (5.17)$$

in which the superfields $\Sigma^I$ are complex unconstrained, while the Lagrange multipliers $\Psi_I$ are chiral, $\mathcal{D}_\alpha \Psi_I = 0$. Since the action must be invariant under holomorphic reparametrizations of the Kähler manifold $\mathcal{X}$, the variables $\Psi_I$ describe the components of a (1,0) form at the point $\Phi$ of $\mathcal{X}$. Using the invariance of (5.13) under (5.11), this action may be seen to be invariant under the following supersymmetry transformation:

$$\delta_\rho \Phi^I = \frac{i}{2} \bar{\mathcal{D}}^2 (\bar{\rho} \Sigma^I) = i \rho_\alpha \bar{\mathcal{D}}^\alpha \Sigma^I + \frac{i}{2} \bar{\rho} \bar{\mathcal{D}}^2 \Sigma^I, \quad (5.18a)$$

$$\delta_\rho \Psi_I = -\frac{i}{2} \bar{\mathcal{D}}^2 \left( \bar{\rho} \frac{\partial \mathcal{L}}{\partial \Phi^I} \right). \quad (5.18b)$$

It should be pointed out that the original supersymmetry transformation (5.11) involved the complex parameter $\bar{\rho}$, defined modulo the gauge freedom (5.9), only via its spinor derivative $\rho_\alpha = \mathcal{D}_\alpha \bar{\rho}$. On the contrary, the supersymmetry transformation of $S_{\text{first-order}}$, (5.18) involves the naked parameter $\bar{\rho}$. One may see that eq. (5.18) describes not only the third supersymmetry, but also a gauge symmetry obtained by choosing $\bar{\rho}$ to be a chiral superfield $\lambda$, compare with (5.9). It is not difficult to understand that this gauge invariance is a trivial gauge symmetry of the theory with action (5.17).

Starting from the first-order action (5.17) and integrating out the auxiliary superfields $\Sigma^I$ and $\bar{\Sigma}^J$ leads to the dual action

$$S_{\text{dual}} = \int \! d^3x \, d^4\theta \, E \, \mathbb{K}(\Phi^I, \Psi_I, \bar{\Phi}^J, \bar{\Psi}_J). \quad (5.19)$$

Its target space is (an open domain of the zero section of) the cotangent bundle $T^*\mathcal{X}$ (compare with [53]). The extended supersymmetry of this $\sigma$-model is given by eq. (3.35) or equivalently, using the condensed notation (3.36), by eq. (3.37).
5.2 Formulation in (1,1) AdS superspace

The off-shell (2,1) supersymmetric σ-model in the (1,1) AdS superspace \([10]\) is

\[
S = \frac{1}{2} \oint \frac{d\zeta}{2\pi i\zeta} \int d^3x \, d^4\theta \, E \, w^{[2]} \, K(\Upsilon, \bar{\Upsilon}) , \quad w^{[2]} := \frac{i}{|\mu|} \left( \frac{\mu}{\zeta} + \bar{\mu}\zeta \right) . \tag{5.20}
\]

The dynamical variables \(\Upsilon^I(\zeta)\) and \(\bar{\Upsilon}^I(\zeta)\) have the functional form

\[
\Upsilon^I(\zeta) = \sum_{n=0}^{\infty} \zeta^n \Upsilon_n^I = \Phi^I + \zeta \Sigma^I + \ldots , \quad \bar{\Upsilon}^I(\zeta) = \sum_{n=0}^{\infty} (-\zeta)^{-n} \bar{\Upsilon}_n^I , \tag{5.21}
\]

where \(\Phi^I\) and \(\Sigma^I\) are chiral and complex linear superfields respectively,

\[
\bar{D}_\alpha \Phi^I = 0 , \quad (\bar{D}^2 - 4\mu) \Sigma^I = 0 , \tag{5.22}
\]

and the other components \(\Upsilon^I_2, \Upsilon^I_3, \ldots\), are complex unconstrained superfields.\(^{13}\)

The (1,1) AdS isometry transformation of \(\Upsilon^I(\zeta)\) is very simple

\[
\delta_l \Upsilon^I = l \Upsilon^I . \tag{5.23}
\]

The third supersymmetry transformation of \(\Upsilon^I\) is

\[
\delta_\varepsilon \Upsilon^I = -\left\{ \zeta (\bar{D}_\alpha \varepsilon) D_\alpha^I - \frac{1}{\zeta} (\bar{D}_\alpha \varepsilon) D^{\alpha I} + 2S\varepsilon \left( \zeta w + \frac{1}{\zeta w} \right) \zeta \partial / \partial \zeta \right\} \Upsilon^I . \tag{5.24}
\]

Here \(\varepsilon\) is a real parameter constrained by

\[
\bar{D}_\alpha \varepsilon = -i \frac{\mu}{|\mu|} D_\alpha \varepsilon , \quad (\bar{D}^2 - 4\bar{\mu}) \varepsilon = (\bar{D}^2 - 4\mu) \varepsilon = 0 , \tag{5.25}
\]

and hence

\[
\mathcal{D}_\alpha \mathcal{D}_\beta \varepsilon = \bar{\mathcal{D}}_\alpha \mathcal{D}_\beta \varepsilon = -2i|\mu| \varepsilon_{\alpha\beta\varepsilon} \quad \rightarrow \quad \mathcal{D}_\alpha \varepsilon_{\alpha\beta\varepsilon} = 0 . \tag{5.26}
\]

For the physical fields this transformation law gives

\[
\delta_\varepsilon \Phi^I = (\bar{D}_\alpha \varepsilon) \bar{D}_\alpha^I + 2\mu \varepsilon \Sigma^I = \frac{1}{2} (\bar{D}^2 - 4\mu) (\varepsilon \Sigma^I) , \tag{5.27a}
\]

\[
\delta_\varepsilon \Sigma^I = -(\bar{D}^{\alpha I} \bar{D}_\alpha \varepsilon) \Phi^I + ((\bar{D}_\alpha \varepsilon) \bar{D}^\alpha + 4\mu \varepsilon) \Upsilon_2^I = \bar{D}_\alpha \left( \frac{\mu}{|\mu|} \varepsilon \bar{D}^\alpha \Phi^I + \Upsilon_2^I \bar{D}_\alpha \varepsilon \right) . \tag{5.27b}
\]

\(^{13}\)The spinor covariant derivatives of (1,1) AdS superspace, \(\mathcal{D}_\alpha\) and \(\bar{\mathcal{D}}_\alpha\), are related to those used in \([10]\), \(\nabla_\alpha\) and \(\bar{\nabla}_\alpha\), as follows: \(\mathcal{D}_\alpha = \sqrt{|\mu|/|\mu|} \nabla_\alpha\) and \(\bar{\mathcal{D}}_\alpha = \sqrt{-i\bar{\mu}|\mu|} \bar{\nabla}_\alpha\).
The off-shell action (5.20) and the constraints obeyed by the physical superfields, eq. (5.22), are similar to those describing the most general 4D \( \mathcal{N} = 2 \) supersymmetric \( \sigma \)-model in AdS\(_4\) [48]. We therefore can apply the four-dimensional results obtained in [48] to the \( \sigma \)-model under consideration without any additional calculation. We summarize the results and refer the interested reader to [48] for the technical details. Upon elimination of the auxiliary superfields \( \Upsilon^I_2, \Upsilon^I_3, \ldots \), from the action (5.20) and subsequent dualization of the complex linear superfield \( \Sigma^I \) and its conjugate \( \bar{\Sigma}^I \) into a chiral scalar \( \Psi^I \) and its conjugate \( \bar{\Psi}^I \), \( \bar{\partial}^{\alpha} \Psi^I = 0 \), we end up with a \( \sigma \)-model action of the form

\[
S_{\text{dual}} = \int d^3x \ d^4\theta E K(\Phi^I, \Psi^I, \bar{\Phi}^J, \bar{\Psi}^J),
\]

where the Kähler potential \( K(\Phi, \Psi, \bar{\Phi}, \bar{\Psi}) \) is a globally defined function on the target space. This action is invariant under supersymmetry transformations

\[
\delta \Phi^I = \frac{1}{2}(\bar{\partial}^2 - 4\mu) \left( \epsilon \frac{\partial K}{\partial \Psi^I} \right), \quad \delta \Psi^I = -\frac{1}{2}(\bar{\partial}^2 - 4\mu) \left( \epsilon \frac{\partial K}{\partial \Phi^I} \right).
\]

Using the concise notation \( \phi^a = (\Phi^I, \Psi^I) \), this transformation can be rewritten as

\[
\delta \phi^a = \frac{1}{2}(\bar{\partial}^2 - 4\mu)(\epsilon \omega^{ab} \mathcal{K}_b),
\]

where we have introduced the symplectic matrices

\[
\omega^{ab} = \begin{pmatrix} 0 & \delta^I J \\ -\delta_I J & 0 \end{pmatrix}, \quad \omega_{ab} = \begin{pmatrix} 0 & \delta_I J \\ -\delta_I J & 0 \end{pmatrix}.
\]

5.3 Sigma model gaugings with a frozen vector multiplet

An important feature that distinguishes the (2,1) AdS superspace from the (3,0) one is that the former allows the existence of a frozen vector multiplet with the property that its field strength is covariantly constant. Following the four-dimensional terminology [22, 48], such a vector multiplet is called intrinsic since it is intimately connected to the geometry of the (2,1) AdS superspace. It can be employed in the context of gauged supersymmetric \( \sigma \)-models in the (2,1) AdS superspace.

Let us consider a U(1) vector multiplet in the (2,1) AdS superspace. It can be described in terms of gauge covariant derivatives

\[
\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}^{ij}_a) = \partial_A + i V_A \mathcal{Z}, \quad [\mathcal{Z}, \mathcal{D}_A] = [\mathcal{Z}, \mathcal{D}_A] = [\mathcal{Z}, \mathcal{J}_{ij}] = 0,
\]

36
where $\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}^{ij}_a)$ are the covariant derivatives of the (2,1) AdS superspace (see [10] for more details), $V_A$ and $Z$ are the U(1) gauge connection and generator respectively. The anti-commutator of two spinor gauge covariant derivatives is

$$\{\mathcal{D}^{ij}_\alpha, \mathcal{D}^{kl}_\beta\} = -2i \varepsilon^{ik}(\varepsilon^{j\ell}j) w^{pq} \mathcal{J}_{pq} - 2\varepsilon_{\alpha\beta}(\varepsilon^{i(k} W^{j)l} + \varepsilon^{j(k} W^{i)l}) Z , \quad (5.33)$$

where the gauge invariant field strength $W^{ij} = W^{ji}$ is real, $W_{ij} = \varepsilon_{ik} \varepsilon_{jl} W^{kl}$, and obeys the Bianchi identity

$$\mathcal{D}^{(ij} W^{kl)} = 0 . \quad (5.34)$$

Suppose the field strength $W^{ij}$ is covariantly constant, $\mathcal{D}^{ij}_a W^{kl} = 0$. In accordance with (5.33), the integrability conditions for this constraint are $\mathcal{D}_a W^{ij} = 0$ and $W^{ij} = G w^{ij}$, with $G$ a real constant parameter. Without loss of generality, we can normalize $G = S$, and thus

$$W^{ij} = S w^{ij} . \quad (5.35)$$

The field strength is determined by the superspace curvature. This is why the vector multiplet under consideration is called intrinsic. Now eq. (5.33) takes the form

$$\{\mathcal{D}^{ij}_\alpha, \mathcal{D}^{kl}_\beta\} = -2i \varepsilon^{ik}(\varepsilon^{j\ell}j) w^{pq} \mathcal{J}_{pq} - 2\varepsilon_{\alpha\beta}(\varepsilon^{i(k} W^{j)l} + \varepsilon^{j(k} W^{i)l}) j , \quad (5.36)$$

where

$$j := j + Z , \quad j := -\frac{i}{2} w^{pq} \mathcal{J}_{pq} . \quad (5.37)$$

A universal procedure to construct gauged supersymmetric $\sigma$-models with (2,1) AdS supersymmetry makes use of a real analytic Kähler manifold $\mathcal{X}$ possessing a U(1) isometry group. Such a transformation group is generated by a holomorphic Killing vector field

$$Z = Z^I(\Phi) \partial_I + \bar{Z}^I(\bar{\Phi}) \partial_{ar{I}} , \quad (5.38)$$

defined by

$$Z^I(\Phi) := i \mathcal{Z} \Phi^I , \quad (5.39)$$

where $\Phi^I$ are local complex coordinates for $\mathcal{X}$. We assume that the Kähler potential $K(\Phi, \bar{\Phi})$ is invariant under the U(1) isometry group,

$$Z^I K_I + \bar{Z}^I K_{\bar{I}} = 0 . \quad (5.40)$$
Associated with \( X \) is an off-shell supersymmetric \( \sigma \)-model in the (2,1) AdS superspace described by the Lagrangian
\[
L^{(2)} = w^{(2)}K(Y^I, \bar{Y}^J), \quad w^{(2)} := v_1v_2w^{ij}, \tag{5.41}
\]
with \( v^i \in \mathbb{C}^2 - \{0\} \) the homogeneous coordinate for \( \mathbb{C}P^1 \). The dynamical variables \( Y^I \) and \( \bar{Y}^J \) are gauge covariantly arctic and antarctic multiplets, respectively. They obey the analyticity constraints
\[
\mathcal{D}^{(2)}_\alpha Y^I = 0, \quad \mathcal{D}^{(2)}_\alpha \bar{Y}^J = 0, \quad \mathcal{D}^{(2)}_\alpha := v_1v_2 \mathcal{D}^{ij}_\alpha \tag{5.42}
\]
and have the following functional form:
\[
Y^I(v) = \sum_{k=0}^{\infty} Y^I_k \zeta^k = \Phi^I + \Sigma^I \zeta + \ldots, \quad \zeta = v_2/v_1, \tag{5.43a}
\]
\[
\bar{Y}^J(v) = \sum_{k=0}^{\infty} \bar{Y}^J_k (-\zeta)^{-k}. \tag{5.43b}
\]
The antarctic multiplet \( \bar{Y}^J(v) \) is said to be the smile-conjugate of \( Y^I(v) \). If the background vector multiplet is switched off, the Lagrangian (5.41) reduces to that considered in [10].

In accordance with [10], the AdS superspace reduction \( (2,1) \rightarrow (2,0) \) can be performed by choosing \( w^{ij} \) in the form: \( w^{11} = w^{22} = 0 \) and \( w^{12} = -w^{21} = i \). The spinor covariant derivatives for (2,0) AdS superspace can be chosen as \( \mathcal{D}_\alpha := \mathcal{D}^{11}_\alpha \) and \( \bar{\mathcal{D}}_\alpha := -\mathcal{D}^{22}_\alpha \). These operators obey the anti-commutation relation (2.21). When projected to (2,0) AdS superspace, the physical superfields \( \Phi^I \) and \( \Sigma^I \) are constrained as
\[
\mathcal{D}_\alpha \Phi^I = 0, \quad \bar{\mathcal{D}}^2 \Sigma^I = 0, \tag{5.44}
\]
as a consequence of the analyticity constraints (5.42). When projected to (2,0) AdS superspace, the \( \sigma \)-model action generated by the Lagrangian (5.41) proves to be
\[
S = \int \frac{d\zeta}{2\pi i\zeta} \int d^3x d^4\theta E K(Y^I, \bar{Y}^J). \tag{5.45}
\]
If the background (2,0) vector multiplet is switched off, this action reduces to (5.1). The only difference between the \( \sigma \)-model (5.45) and the \( \sigma \)-model (5.1) studied earlier is a different choice of the \( U(1)_R \) generator; specifically, it is \( J \) for the \( \sigma \)-model (5.1) and \( J = J + Z \) the \( \sigma \)-model (5.45). Therefore, all the results obtained for the \( \sigma \)-model (5.1) can naturally be extended to the theory under consideration.
As shown in [10], the AdS superspace reduction $(2,1) \rightarrow (1,1)$ is carried out by choosing $w^{ij}$ in the form: $w^{12} = 0$, $w^{11} = -\bar{\mu}/|\mu|$ and $w^{22} = -\mu/|\mu|$. The spinor covariant derivatives for $(1,1)$ AdS superspace can be chosen as

$$D_\alpha := \sqrt{i \frac{\mu}{|\mu|}} D_{\alpha 1}, \quad \bar{D}_\alpha := -\sqrt{i \frac{\bar{\mu}}{|\mu|}} D_{\alpha 2}. \quad (5.46)$$

As follows from (5.36), the operators $D_\alpha$ and $\bar{D}_\alpha$ obey the $(1,1)$ AdS (anti)commutation relations, eq. (2.47), which do not involve any $U(1)_R$ curvature. This means that the $U(1)_R$ connection associated with the covariant derivatives $\mathcal{D}_A = (D_a, D_\alpha, \bar{D}_\alpha)$ can be completely gauged away, ending up with standard $(1,1)$ AdS covariant derivatives $\mathcal{D}_A = (D_a, D_\alpha, \bar{D}_\alpha)$. When projected to $(1,1)$ AdS superspace, the physical superfields $\Phi^I$ and $\Sigma^I$ can be seen to be constrained as

$$\bar{D}_\alpha \Phi^I = 0, \quad -\frac{1}{4} (\bar{D}^2 - 4\mu) \Sigma^I = 2Z^I(\Phi). \quad (5.47)$$

Thus $\Phi^I$ is an ordinary chiral superfield in $(1,1)$ AdS superspace (and this can simply be denoted as $\Phi^I$), while $\Sigma^I$ obeys a modified linear constraint. When projected to $(1,1)$ AdS superspace, the $\sigma$-model action generated by the Lagrangian (5.41) proves to be

$$S = \frac{1}{2} \int d\gamma \frac{d\zeta}{2\pi i\zeta} \int d^3 x d^4 \theta E w^{[2]}(\mathcal{Y}, \bar{\mathcal{Y}}), \quad w^{[2]} := i \left( \frac{\mu}{\zeta} + \bar{\mu} \zeta \right). \quad (5.48)$$

Setting $Z^I = 0$ in (5.47) reduces the $\sigma$-model action (5.48) to (5.20). The off-shell action (5.48) and the constraints obeyed by the physical superfields, eq. (5.47), are similar to those describing the gauged 4D $\mathcal{N} = 2$ supersymmetric $\sigma$-model in AdS$_4$ [48]. We therefore can apply the four-dimensional results obtained in [48] to the $\sigma$-model under consideration without any additional calculation. Upon elimination of the auxiliary superfields from the action (5.48) and subsequent dualization of the deformed complex linear superfield $\Sigma^I$ and its conjugate $\bar{\Sigma}^I$ into a chiral scalar $\Psi_I$ and its conjugate $\bar{\Psi}_I$, $\bar{D}_\alpha \Psi_I = 0$, we end up with the $\sigma$-model

$$S_{\text{dual}} = \int d^3 x d^4 \theta E \left( \mathcal{K}(\Phi, \Psi, \bar{\Phi}, \bar{\Psi}) + \frac{1}{\mu} W(\Phi, \Psi) + \frac{1}{\bar{\mu}} \bar{W}(\bar{\Phi}, \bar{\Psi}) \right), \quad (5.49)$$

where we have introduced the superpotential

$$W(\Phi, \Psi) = -2\Psi_I X^I(\Phi). \quad (5.50)$$

The $\sigma$-model action is invariant under the extended supersymmetry transformation (5.29).
6 Sigma models with (2,1) AdS supersymmetry: On-shell approach

In the previous section, off-shell (2,1) \( \sigma \)-models were shown to lead to formulations either in (2,0) or (1,1) superspace. In this section, we will again analyze the situation in reverse and consider the most general (2,0) and (1,1) models possessing a full (2,1) symmetry.

6.1 Formulation in (2,0) AdS superspace

We take the (2,0) action involving a Kähler potential

\[
S = \int d^3x \, d^4\theta \, E \mathbb{K}(\phi^a, \bar{\phi}^\bar{b})
\]  

and postulate the extended supersymmetry transformation law

\[
\delta \phi^a = \frac{i}{2} \bar{\mathcal{D}}^2(\bar{\rho} \Omega^a) \]  

where \( \Omega^a = \Omega^a(\phi, \bar{\phi}) \). The complex parameter \( \rho \) obeys the conditions (5.8) and (5.10). We require the action to be invariant. We delay the technical analysis to Appendix C and present the result: in order for the action to be invariant, the combination

\[
\omega_{ab} := g_{a\bar{a}} \partial \bar{b} \Omega^\bar{a}
\]

must be a covariantly constant holomorphic two-form,

\[
\nabla_c \omega_{ab} = \nabla_{\bar{c}} \omega_{ab} = 0.
\]

A further analysis of the closure of the algebra dictates that

\[
\omega_{ab} \omega_{bc} = -\delta^a_c,
\]

and that the U(1) Killing vector \( J^a \) obeys

\[
\mathcal{L}_J \omega_{ab} = -i \omega_{ab}.
\]

In contrast to the (3,0) situation, the target space is not in general a hyperkähler cone. Rather, it is a hyperkähler manifold with the single additional constraint that it possess a U(1) Killing vector \( J^a \) which rotates the complex structures; choosing these as in (4.6) leads to

\[
\mathcal{L}_J J_1 = J_2, \quad \mathcal{L}_J J_2 = -J_1, \quad \mathcal{L}_J J_3 = 0.
\]

An analogous result has recently been established for \( \mathcal{N} = 2 \) \( \sigma \)-models in AdS\(_4\) [12, 13] and for \( \mathcal{N} = 1 \) models in AdS\(_5\) [14, 15]. The component Lagrangian (with auxiliaries eliminated) can be calculated using (2.40):

\[
\mathcal{L} = -g_{a\bar{a}} \mathcal{D}_m \psi^a \mathcal{D}^m \bar{\psi}^\bar{a} - ig_{a\bar{a}} \bar{\psi}^\bar{a} \bar{\mathcal{D}}^\alpha \psi^a + \frac{1}{4} R_{a\bar{a}b\bar{b}} (\psi^a \psi^b)(\bar{\psi}^\bar{a} \bar{\psi}^\bar{b})
\]

\[
+ S(\psi^a \bar{\psi}^\bar{a})(ig_{a\bar{a}} + \nabla_a J_{\bar{a}} - \nabla_{\bar{a}} \bar{J}_a) - 4S^2(J^a \bar{J}^\bar{a} g_{a\bar{a}} - D).
\]
It is natural to ask what happens if we want to introduce additional mass terms while maintaining the extended supersymmetry. Just as in the (3, 0) case, this cannot be done with a superpotential while maintaining the extended supersymmetry. However, there is an alternative way to introduce masses (or, more accurately, to deform the mass terms already present): this is to deform the U(1) Killing vector. We have already discussed in section 2.1 that the inclusion of a frozen vector multiplet deforms the (2, 0) supersymmetry algebra by replacing the U(1) generator $J$ with $J = J + Z$ where $Z$ is the generator associated with the frozen vector multiplet. For a $\sigma$-model, this amounts to the replacement of the U(1) Killing vector $J^a$ with $J^a = J^a + Z^a$. Since the effective U(1) Killing vector $J^a$ generates the masses in the component Lagrangian, this provides an alternative way to introduce massive deformations.

When extended supersymmetry is taken into account, this procedure still holds with one additional restriction: the Killing vector $Z^a$ associated with the gauging must be a tri-holomorphic isometry, obeying $L_Z \omega_{ab} = 0$. Its addition to $J^a$ then leads to a U(1) Killing vector $J^a$ that still rotates the complex structures as in (6.3). Deforming the masses simply involves deforming the U(1) Killing vector. Since the inclusion of a massive deformation only deforms the U(1) Killing vector, the form of the component action remains unchanged. One simply replaces $J^a \rightarrow J^a$ and $D \rightarrow D$.

6.2 Formulation in (1,1) AdS superspace

Let us now consider the most general $(2, 1)$ model written in (1,1) superspace. We begin with the most general $\sigma$-model in (1,1) AdS,

$$S = \int d^3x \, d^4\theta \, E \mathcal{K}(\phi^a, \bar{\phi}^\b).$$

Inspired by the solution in projective superspace, we postulate

$$\delta \phi^a = \frac{1}{2}(\bar{\partial}^2 - 4\mu)(\varepsilon \Omega^a) \quad (6.5)$$

where the real parameter $\varepsilon$ obeys (5.25) and (5.26). We find as usual that

$$\omega_{ab} = g_{ab} \partial_a \Omega^\b \quad (6.6)$$

is a covariantly constant holomorphic two-form; closure of the algebra further imposes that $\omega^{ab} \omega_{bc} = -\delta^a_c$. This demonstrates that the target space is hyperkähler. In addition, we discover that there exists a U(1) Killing vector

$$V^a = \frac{\mu}{2S} \Omega^{ab} \mathcal{K}_b \quad (6.7)$$
Introducing the complex structures as in (4.6), one can check that $V^a$ acts as a rotation:

\[ \mathcal{L}_V J_1 = \text{Im} \frac{\mu}{S} J_3 , \quad \mathcal{L}_V J_2 = -\text{Re} \frac{\mu}{S} J_3 , \quad \mathcal{L}_V J_3 = \text{Re} \frac{\mu}{S} J_2 - \text{Im} \frac{\mu}{S} J_1 , \tag{6.8} \]

and leaves invariant one particular combination of complex structure

\[ J_{\text{AdS}} = -\text{Re} \frac{\mu}{S} J_1 - \text{Im} \frac{\mu}{S} J_2 . \tag{6.9} \]

In fact, we may rewrite $V^\mu$ as

\[ V^\mu = -\frac{1}{2} (J_{\text{AdS}})^{\mu \nu} \nabla^\nu K \tag{6.10} \]

which identities $K$ as the Killing potential for $V^\mu$, with respect to the complex structure $J_{\text{AdS}}$. The component Lagrangian can be derived using (2.67):

\[
\begin{align*}
\mathcal{L} &= -g_{ab} \nabla_m \varphi^a \nabla^m \varphi^b - ig_{ab} \bar{\psi}^b \bar{\psi}^d \nabla_{cd} \psi^a + \frac{1}{4} R_{abcp} (\psi^a \psi^c)(\bar{\psi}^b \bar{\psi}^d) \\
&\quad - \frac{\mu}{2} \nabla_a K_b (\psi^a \psi^b) - \frac{\bar{\mu}}{2} \nabla_\bar{a} K_\bar{b} (\bar{\psi}^a \bar{\psi}^b) - 4 S^2 (g_{ab} V_a V_b - K) . \tag{6.11}
\end{align*}
\]

As in the (2,0) formulation, we should inquire about introducing mass terms. Because of the extremely close relationship between the presentation here and that of AdS$_4$ [13, 48], the answer is quite apparent. The introduction of a superpotential amounts to the replacement in the action of

\[ \mathcal{K} \to \mathcal{K} = \mathcal{K} + \frac{W}{\mu} + \frac{\bar{W}}{\bar{\mu}} . \tag{6.12} \]

In order for this to be invariant under the extended supersymmetry (6.5), the superpotential must obey

\[ \mathcal{K}_a \omega^{ab} W_b + \mathcal{K}_a \omega^{ab} \bar{W}_b = F + \bar{F} \tag{6.13} \]

where $F = F(\phi)$ is a holomorphic function. This means that the superpotential $W$ is associated with a holomorphic isometry, which we may denote

\[ Z^a := \omega^{ab} W_b . \tag{6.14} \]

In fact, one may check that $Z^a$ is tri-holomorphic, obeying $\mathcal{L}_Z \omega_{ab} = 0$. Examining the definition (6.7) of the U(1) Killing vector $V^a$, we see that the replacement (6.12) leads to

\[ V^a := V^a + Z^a . \tag{6.15} \]

where the new U(1) Killing vector $V^a$ still obeys the same conditions (6.8) as before. This is precisely the same physical situation we observed in the (2,0) formulation: the allowed deformation of the masses corresponds to deforming the U(1) Killing vector by the addition of a tri-holomorphic piece.
7 \( \mathcal{N} = 4 \) AdS superspaces

In the remainder of this paper, we will focus on nonlinear \( \sigma \)-models with \( \mathcal{N} = 4 \) supersymmetry in AdS\(_3\). We have previously described \( \mathcal{N} = 3 \) supersymmetric \( \sigma \)-models using only two manifest supersymmetries. In the \( \mathcal{N} = 4 \) case, we will likewise consider formulations involving a smaller amount of manifestly realized supersymmetry. Our first task, which is the focus of this section, will be to analyze how the various \( \mathcal{N} = 4 \) AdS superspaces (and their isometries) can be projected to AdS superspaces with a smaller number of Grassmann variables.

7.1 Geometry of \( \mathcal{N} = 4 \) AdS superspaces

We begin by reviewing the geometry of the \( \mathcal{N} = 4 \) AdS superspaces constructed in [10]. Consistent with the analysis of [17], there are three types of \( \mathcal{N} = 4 \) AdS supersymmetry in three dimensions, and therefore three inequivalent superspaces. These three cases, which we call the \((4,0)\), \((3,1)\) and \((2,2)\) AdS superspaces, are special backgrounds allowed by the superspace geometry of three-dimensional \( \mathcal{N} = 4 \) conformal supergravity of [19].

All the three \( \mathcal{N} = 4 \) AdS supergeometries have covariant derivatives of the form [10]

\[
D_A = (D_a, D_{\bar{a}}) = E_A^M \partial_M + \frac{1}{2} \Omega_A^{cd} M_{cd} + \Phi_A^{kl} L_{kl} + \Phi_{\bar{A}}^{\bar{k}\bar{l}} R_{\bar{k}\bar{l}}. \tag{7.1}
\]

The operators \( L_{kl} \) and \( R_{\bar{k}\bar{l}} \) generate the \( R \)-symmetry group \( SU(2)_L \times SU(2)_R \) and act on the covariant derivatives as

\[
[L_{kl}, D_{\bar{a}}] = \epsilon^{(k} D_{\bar{a}}^{l)} , \quad [R_{\bar{k}\bar{l}}, D_{\bar{a}}] = \epsilon^{(\bar{k}} D_{\bar{a}}^{\bar{l})} . \tag{7.2}
\]

For each of the \( \mathcal{N} = 4 \) AdS superspaces, the algebra of covariant derivatives is

\[
\{D_{\alpha}^{ij}, D_{\beta}^{\bar{j}\bar{k}}\} = 2i \delta_{\alpha\beta} \{S_{ij}^{\bar{j}\bar{k}} \} + 4i (S_{ij}^{\bar{j}\bar{k}} + \epsilon_{ij} S_{\bar{j}\bar{k}}) M_{\alpha\beta} - 2i \epsilon_{\alpha\beta} \epsilon^{ij} \epsilon^{\bar{j}\bar{k}} L_{kl} + 2i \epsilon_{\alpha\beta} \epsilon^{ij} \epsilon^{\bar{j}\bar{k}} S_{ij}^{\bar{j}\bar{k}} L_{kl} , \tag{7.3a}
\]

\[
[D_{\alpha\beta}, D_{\gamma}^{k\bar{k}}] = -2 \left( \delta_{\gamma}^{k} \delta_{\beta}^{\bar{k}} S + S_{\gamma}^{\bar{k}} S_{\beta}^{k} \right) \epsilon_{(\alpha} D_{\beta)}^{\bar{k}} , \tag{7.3b}
\]

\[
[D_{\alpha}, D_{\bar{a}}] = -4 S^2 M_{ab} , \tag{7.3c}
\]

where the real tensor \( S_{ij}^{\bar{j}\bar{k}} = S^{(ij)(\bar{j}\bar{k})} \) is covariantly constant, and the real scalars \( S \), \( X \) and \( S \) are constant. The parameter \( S \) determines the curvature scale. Depending on the
different superspace geometries, the parameters $S, S^{ijj}$ and $X$ are

\begin{align}
(4, 0) & : \quad S = S , \quad S^{ijj} = 0 , \quad X \text{ arbitrary } ; \\
(3, 1) & : \quad S = \frac{1}{2} S , \quad S^{ijj} = \left( \frac{1}{2} \varepsilon^{ij} \varepsilon^{jj} - w^{i} w^{j} \right) S , \quad X = 0 ; \\
(2, 2) & : \quad S = 0 , \quad S^{ijj} = l^{ij} r^{jj} S , \quad X = 0 .
\end{align}

In the (3,1) case, the covariantly constant tensor $w^{i}$ is real, $w^{i} = w^{i} = \varepsilon_{ij} \varepsilon_{ij} w^{ij}$, and normalized as

\begin{equation}
w^{i} w^{ij} = \delta^{i} , \quad w^{k} w^{k} = \delta^{j} .
\end{equation}

In the (2,2) case, the real iso-triplets $l^{ij} = l^{ij}$ and $r^{ij} = r^{ij}$ are covariantly constant and normalized as

\begin{equation}
l^{ij} l^{ij} = \delta^{i} , \quad r^{ij} r^{ij} = \delta^{j} .
\end{equation}

We emphasize that $X$ can appear in the algebra only in the (4,0) case. The (4,0) AdS superspace is conformally flat if and only if $X = 0$ [10]. For general values of $X$, the tangent space group of the (4,0) AdS supergeometry is the full $R$-symmetry group $SU(2)_{L} \times SU(2)_{R}$. For the two critical values, $X = 2S$ and $X = -2S$, the $SU(2)_{R}$ or $SU(2)_{L}$ group respectively can be gauged away.

For each of the (3,1) and (2,2) geometries, the $R$-symmetry sector of the superspace holonomy group is a subgroup of $SU(2)_{L} \times SU(2)_{R}$ [10]. For the (3,1) supergeometry, the relevant subgroup is $SU(2)_{\mathcal{I}}$ generated by

\begin{equation}
\mathcal{I}_{kl} = L_{kl} + w^{i} w^{j} R_{kl} , \quad \text{or} \quad \mathcal{I}_{kl} = w^{i} w^{j} L_{kl} + R_{kl} = w^{i} w^{j} \mathcal{I}_{kl} .
\end{equation}

Indeed, the anti-commutator \((7.3a)\) in the (3,1) case can be rewritten in the following equivalent forms:

\begin{align}
\{ D^{ij}_{\alpha} , D^{j\beta} \} & = 2 i \varepsilon^{ij} \varepsilon^{j\beta} D_{\alpha \beta} + 2 i \varepsilon_{\alpha \beta} S \left( \varepsilon^{ij} \delta^{i} \delta^{j} + \varepsilon^{ij} w^{k} w^{j} \right) \mathcal{J}^{kl} \\
& \quad - 4 i S \left( \varepsilon^{ij} \varepsilon^{j} - w^{i} w^{j} \right) M_{\alpha \beta} .
\end{align}

The generators $\mathcal{I}_{kl}$ and $\mathcal{I}_{kl}$ leave $w^{i}$ invariant, $\mathcal{I}_{kl} w^{i} = \mathcal{I}_{kl} w^{i} = 0$. Since the $R$-symmetry curvature is spanned by the generators of $SU(2)_{\mathcal{I}}$, it is possible to choose a gauge in which
the $R$ symmetry connection takes its values in the Lie algebra of $SU(2)_J$; in this gauge, the parameter $w^\tilde{i}$ is constant. This gauge choice is always assumed in the remainder of the paper.

In the (2,2) case, the $R$-symmetry sector of the superspace holonomy group is the Abelian subgroup $U(1)_L \times U(1)_R$ of $SU(2)_L \times SU(2)_R$ generated by

$$L := l^{kl} L_{kl} , \quad R := r^{\tilde{k}\tilde{l}} R_{\tilde{k}\tilde{l}} .$$

(7.9)

This subgroup leaves invariant the covariantly constant parameters $l^{kl}$ and $r^{\tilde{k}\tilde{l}}$. In the remainder of the paper, we choose a gauge in which only this subgroup appears in the (2,2) covariant derivatives; then the parameters $l^{kl}$ and $r^{\tilde{k}\tilde{l}}$ are constant.

Given a particular $\mathcal{N} = 4$ AdS superspace, its isometry group is generated by Killing vector fields, $\xi = \xi^a \partial_a + \xi^{\tilde{i}} \partial_{\tilde{i}}$, obeying the equation

$$0 = \left[ \xi + \frac{1}{2} \Lambda^{\gamma\delta} \mathcal{M}_{\gamma\delta} + \Lambda^{kl} L_{kl} + \Lambda^{\tilde{k}\tilde{l}} R_{\tilde{k}\tilde{l}}, \mathcal{D}_A \right] .$$

(7.10)

This Killing equation is equivalent to

$$\mathcal{D}^i_{\alpha} \xi_{\beta\gamma} = 4i \varepsilon_{\alpha(\beta} \xi_{\gamma)}^i ,$$

(7.11a)

$$\mathcal{D}^i_{\alpha} \xi^{ij} = \xi_{\alpha(\beta} (\varepsilon^{ij} \varepsilon^{kl} S + S^{ijkl}) + \frac{1}{2} \Lambda_{\alpha\beta} \varepsilon^{ij} \varepsilon^{kl} + \Lambda^{ij} \varepsilon^{kl} \varepsilon_{\alpha\beta} ,$$

(7.11b)

$$\mathcal{D}^i \Lambda_{\beta\gamma} = 8i \varepsilon_{\alpha(\beta} \xi_{\gamma)}^i (S^{ijkl} + \varepsilon^{ijkl} S) ,$$

(7.11c)

$$\mathcal{D}^i \Lambda^{kl} = -2i \varepsilon^{lijk} (2S + X) - 2i \varepsilon^{lijk} S^{ijkl} ,$$

(7.11d)

$$\mathcal{D}^i \Lambda^{\tilde{k}\tilde{l}} = -2i \varepsilon^{i\tilde{k}\tilde{l}} (2S - X) - 2i \varepsilon^{i\tilde{k}\tilde{l}} S^{ijkl} ,$$

(7.11e)

and

$$\mathcal{D}_a \xi_b = \Lambda_{ab} ,$$

(7.12a)

$$\mathcal{D}_a \xi^{\beta} = -S \mathcal{E}_{\beta}^{\gamma\delta} \mathcal{E}_{\gamma\delta}^{\delta\gamma} ,$$

(7.12b)

$$\mathcal{D}_a \Lambda^{bc} = 4S^2 (\delta^b \mathcal{E}^c - \delta^c \mathcal{E}^b) ,$$

(7.12c)

$$\mathcal{D}_a \Lambda^{kl} = \mathcal{D}_a \Lambda^{\tilde{k}\tilde{l}} = 0 .$$

(7.12d)

Some useful implications of the above equations are

$$\mathcal{D}^{i}_{(\alpha} \xi_{\beta\gamma)} = \mathcal{D}^{i}_{(\alpha} \Lambda_{\beta\gamma)} = 0 ,$$

(7.13a)

$$\mathcal{D}^{i\tilde{i}\tilde{i}} \xi_{\alpha\beta} = 6i \xi^{i}_{\alpha} , \quad \mathcal{D}^{i\tilde{i}} \Lambda_{\alpha\beta} = 12i \xi_{\alpha\beta} (S^{ijkl} + \varepsilon^{ijkl} S) ,$$

(7.13b)

$$\mathcal{D}^{i\tilde{i}} \xi_{\alpha\beta} = \mathcal{D}^{i\tilde{i}} (\alpha} \xi_{\beta)_{i}^{\tilde{j}} = 0 , \quad \mathcal{D}^{(i\tilde{i}\tilde{i})} \xi_{\alpha\beta} = \xi_{\alpha\beta} S^{ijkl} , \quad \mathcal{D}^{i\tilde{i}} \xi_{\alpha\beta} = 4i \varepsilon_{\alpha\beta} S + 2\Lambda_{\alpha\beta} ,$$

(7.13c)

$$\mathcal{D}^{i\tilde{i}} \xi_{\alpha\tilde{a}} = \mathcal{D}^{(i\tilde{i}) \tilde{a}} \xi_{\alpha\beta} = 0 , \quad \mathcal{D}^{i\tilde{i}} \xi_{\alpha\tilde{a}} = -4i \varepsilon^{ijkl} , \quad \mathcal{D}^{i\tilde{i}} \xi_{\alpha\tilde{a}} = -4\Lambda^{ijkl} .$$

(7.13d)
Here we have written the results in a form valid for the (4,0), (3,1) and (2,2) cases. Depending on the \( N = 4 \) AdS superspace under consideration, \( S, X \) and \( S^{ij,j} \) are constrained by \((7.4a)-(7.4c)\), while the \( SU(2)_L \) and \( SU(2)_R \) parameters \( \Lambda^{kl} \) and \( \Lambda^{k\bar{l}} \) are restricted by

\[
(4,0) \text{ with } X = 2S: \quad \Lambda^{k\bar{l}} = 0 ; \quad (7.14a)
\]

\[
(4,0) \text{ with } X = -2S: \quad \Lambda^{kl} = 0 ; \quad (7.14b)
\]

\[
(3,1) : \quad \Lambda^{k\bar{l}} = w_k^k w_l^l \Lambda^{kl} ; \quad (7.14c)
\]

\[
(2,2) : \quad \Lambda^{kl} = k^{kl} \Lambda_L , \quad \Lambda^{k\bar{l}} = r^{k\bar{l}} \Lambda_R , \quad (\Lambda_L) = \Lambda_L , \quad (\Lambda_R) = \Lambda_R . \quad (7.14d)
\]

### 7.2 From \( N = 4 \) to \( N = 2 \) AdS superspaces

It was argued in the introduction that any \( N = 4 \) AdS superspace can be reduced to the (2,0) AdS superspace. Here we elaborate on the details of such a reduction.

Let us fix a certain \( N = 4 \) AdS superspace. We start by showing that its algebra of covariant derivatives possesses an \( N = 2 \) AdS subalgebra associated with the covariant derivatives \( \mathcal{D}_a, \mathcal{D}_a^{11} \) and \((\mathcal{D}_a^{22})\)\(^\text{14}\). These operators obey the (anti) commutation relations

\[
\{\mathcal{D}_a^{11}, \mathcal{D}_b^{11}\} = -4iS^{11\bar{1}\bar{1}} \mathcal{M}_{a\beta} , \quad (7.15a)
\]

\[
\{\mathcal{D}_a^{11}, (-\mathcal{D}_b^{22})\} = -2i \mathcal{D}_{a\beta} + 4i(S^{12\bar{1}2} + S) \mathcal{M}_{a\beta}
- 2i\epsilon_{a\beta}(2S + X) L^{12} + 2i\epsilon_{a\beta} S^{kJ12} L_{kl}
- 2i\epsilon_{a\beta}(2S - X) R^{1\bar{2}} + 2i\epsilon_{a\beta} S^{12k\bar{l}} R_{k\bar{l}} , \quad (7.15b)
\]

\[
[\mathcal{D}_{a\beta}, \mathcal{D}_a^{11}] = -2 \left( \delta_l^i \delta_l^l S + S^{11\bar{1}\bar{1}} \right) \epsilon_{\gamma(\alpha} \mathcal{D}_b^{1\bar{l}}) , \quad (7.15c)
\]

\[
[\mathcal{D}_a, \mathcal{D}_b] = -4 S^{2i} \mathcal{M}_{ab} . \quad (7.15d)
\]

For the right-hand side of \((7.15c)\) not to involve \( \mathcal{D}_a^{12} \) and \( \mathcal{D}_a^{21} \), we must require

\[
S^{11\bar{1}\bar{1}} = S^{12\bar{1}\bar{2}} = 0 , \quad (7.16)
\]

which leads to

\[
\{\mathcal{D}_a^{11}, \mathcal{D}_b^{11}\} = -4iS^{11\bar{1}\bar{1}} \mathcal{M}_{a\beta} , \quad (7.17a)
\]

\[
\{\mathcal{D}_a^{11}, (-\mathcal{D}_b^{22})\} = -2i \mathcal{D}_{a\beta} + 4i(S + S^{12\bar{1}2}) \mathcal{M}_{a\beta}
- 2i\epsilon_{a\beta}(2(S + S^{12\bar{1}2}) \hat{J} + X \hat{\mathcal{E}}) , \quad (7.17b)
\]

\[
[\mathcal{D}_{a\beta}, \mathcal{D}_a^{11}] = -2 \left( S + S^{12\bar{1}2} \right) \epsilon_{\gamma(\alpha} \mathcal{D}_b^{1\bar{l}}) - 2S^{11\bar{1}\bar{1}} \epsilon_{\gamma(\alpha} \mathcal{D}_b^{2\bar{2}}) , \quad (7.17c)
\]

\[
[\mathcal{D}_a, \mathcal{D}_b] = -4 S^{2i} \mathcal{M}_{ab} , \quad (7.17d)
\]

\(^{14}\)Given a tensor superfield \( U \) of Grassmann parity \( \epsilon(U) \), the operation of complex conjugation maps \( \mathcal{D}_a^{11} U \) to \( \mathcal{D}_a^{11} \bar{U} = (-1)^{\epsilon(U)} \mathcal{D}_a^{11} \bar{U} = (-1)^{\epsilon(U)} \mathcal{D}_a^{22} \bar{U} \).
where we have introduced two U(1) generators
\[
\mathcal{J} := (L^{12} + R^{12}) , \quad [\mathcal{J}, D^{11}_\alpha] = D^{11}_\alpha , \quad [\mathcal{J}, (-D^{22}_\alpha)] = -(-D^{22}_\alpha) , \quad (7.18a)
\]
\[
\mathcal{Z} := (L^{12} - R^{12}) , \quad [\mathcal{Z}, D^{11}_\alpha] = [\mathcal{Z}, (-D^{22}_\alpha)] = 0 , \quad (7.18b)
\]
such that
\[
[\mathcal{J}, \mathcal{Z}] = 0 . \quad (7.19)
\]
As will be shown shortly, the condition (7.16) can always be satisfied.

In the (4,0) AdS superspace, \( S^{ij\bar{j}\bar{i}} = 0 \) and thus the condition (7.16) holds identically. Since in this case we also have \( S = S \), the algebra (7.17) can be seen to coincide with that defining the (2,0) AdS superspace, eq. (2.2), if the U(1)\(_R\) generator is identified with
\[
\mathcal{J} := \mathcal{J} + \frac{X}{2S} \mathcal{Z} . \quad (7.20)
\]
This identification is not unique. Instead of (7.20), one could have chosen \( \mathcal{J} = \mathcal{J} + \xi \mathcal{Z} \) as the U(1)\(_R\) generator and \( \mathcal{Z} = (\frac{X}{2S} - \xi) \mathcal{Z} \) as the central charge, for some real parameter \( \xi \). Such a choice would have led to the ‘gauged’ realization of the (2,0) AdS algebra, eq. (2.21). In what follows, we will use the identification (7.20).

In the case of the (3,1) and (2,2) AdS superspaces, it follows from the relations (7.4b) and (7.4c) that the condition (7.16) has only two solutions: either \( S^{111} = 0 \) and then the resulting algebra (7.17) is of the (2,0) AdS type; or \( S^{1212} = 0 \) and then the (anti)commutation relations (7.17) are of the (1,1) AdS type. We will mainly be concerned with reductions to (2,0) AdS superspace and consider that case in the remainder of this subsection. The reductions to (1,1) AdS superspace are included for completeness in Appendix D.

For any \( \mathcal{N} = 4 \) tensor superfield \( U(x, \theta_{ij}) \), we define its \( \mathcal{N} = 2 \) projection by
\[
U| := U(x, \theta_{ij})|_{\theta_{1\bar{1}} = \theta_{2\bar{2}} = 0} . \quad (7.21)
\]
By definition, \( U| \) depends on the Grassmann coordinates \( \theta^\mu := \theta^{11}_\mu \) and their complex conjugates, \( \bar{\theta}^\mu = \theta^{22}_\mu \). For the \( \mathcal{N} = 4 \) AdS covariant derivative,
\[
D_A = E_A^M \partial_M + \frac{1}{2} \Omega_A^{bc} M_{bc} + \Phi_A^{kl} L_{kl} + \Phi_A^{\bar{k}\bar{l}} R_{\bar{k}\bar{l}} , \quad (7.22)
\]
15The reader should bear in mind that, depending on the choice of parameters \( S, S^{ij\bar{j}\bar{i}}, X \), the R-symmetry connection may take values only in a subgroup of SU(2)_L×SU(2)_R.
the projection is defined as
\[ \mathcal{D}_A = E_A^M|\partial_M + \frac{1}{2}\Omega_A^{bc}|\mathcal{M}_{bc} + \Phi_A^{kl}|\mathbf{L}_{kl} + \Phi_A^{\bar{kl}}|\mathbf{R}_{\bar{kl}}. \] (7.23)

Since the operators \( \mathcal{D}_a, \mathcal{D}_a^{11}, -\mathcal{D}_a^{22} \) form a closed algebra isomorphic to that of the (2,0) AdS superspace, one can use the freedom to perform general coordinate, local Lorentz and SU(2) transformations to choose a gauge in which
\[ \mathcal{D}_a^{11} = \mathcal{D}_a, \quad -\mathcal{D}_a^{22} = \bar{\mathcal{D}}_a, \] (7.24)
where
\[ \mathcal{D}_a = (\mathcal{D}_a, \mathcal{D}_\alpha, \bar{\mathcal{D}}^\alpha) = E_A^M|\partial_M + \frac{1}{2}\Omega_A^{cd}|\mathcal{M}_{cd} + i\Phi_A^J \] (7.25)
denote the covariant derivatives of the (2,0) AdS superspace.

Note that the U(1) generator \( J \) is defined by (7.20) and coincides with \( \hat{J} \) in the cases (3,1), (2,2) and (4,0) with \( X = 0 \). We recall that the covariant derivatives of the (2,0) AdS superspace obey the (anti) commutation relations (2.2).

In the coordinate system defined by (7.24), the operators \( \mathcal{D}_a^{11} \) and \( \mathcal{D}_a^{22} \) involve no partial derivative with respect to \( \theta_{1\bar{2}}, \theta_{2\bar{1}}, \) and therefore, for any positive integer \( k \), it holds that \( (\mathcal{D}_a \cdots \mathcal{D}_a U)| = \mathcal{D}_a| \cdots \mathcal{D}_a| U, \) where \( \mathcal{D}_a := (\mathcal{D}_a^{11}, -\mathcal{D}_a^{22}) \) and \( U \) is a tensor superfield. This implies that \( \mathcal{D}_a = \mathcal{D}_a \) in (7.25).

Let us now consider a Killing vector field \( \xi \) of one of the \( \mathcal{N} = 4 \) AdS superspaces, as specified by eqs. (7.10) – (7.12). We introduce \( \mathcal{N} = 2 \) projections of the \( \mathcal{N} = 4 \) Killing parameters:
\[ \tau^a := \xi^a, \quad \tau^\alpha := \xi^\alpha_{11}, \quad \bar{\tau}^\alpha = \xi^\alpha_{22}, \quad t := i(\Lambda^{12} + \Lambda^{12}) = \bar{t}, \quad t^{ab} := \Lambda^{ab}; \] (7.26a)
\[ \varepsilon^\alpha := -\xi^\alpha_{12}, \quad \bar{\varepsilon}^\alpha = \xi^\alpha_{21}, \quad \sigma := i(\Lambda^{12} - \bar{\Lambda}^{12}) = \bar{\sigma}; \] (7.26b)
\[ \bar{\varepsilon}_L := -\frac{1}{4S}\Lambda^{11}, \quad \varepsilon_L = -\frac{1}{4S}\Lambda^{22}, \quad \bar{\varepsilon}_R = -\frac{1}{4S}\Lambda^{11}, \quad \varepsilon_R = -\frac{1}{4S}\Lambda^{22}. \] (7.26c)

The parameters \( (\tau^a, \tau^\alpha, \bar{\tau}_\alpha, t^{\alpha\beta}, t) \) describe the infinitesimal isometries of the (2,0) AdS superspace. This can be easily proven by \( \mathcal{N} = 2 \) projection of the equations (7.10)–(7.11e). The remaining parameters \( (\varepsilon^\alpha, \bar{\varepsilon}_\alpha, \sigma, \varepsilon_L, \bar{\varepsilon}_L, \varepsilon_R, \bar{\varepsilon}_R) \) are associated with the remaining two supersymmetries and the residual R-symmetry. Depending on the initial \( \mathcal{N} = 4 \) AdS superspace, these parameters are constrained in different ways.

\[ ^{16}\text{We hope the reader will not be confused by the use of the same notation } \mathcal{D}_A \text{ in (7.22) and (7.25) for the covariant derivative of the } \mathcal{N} = 4 \text{ and (2,0) AdS superspaces respectively.} \]
7.2.1 AdS superspace reduction \((4,0) \rightarrow (2,0)\)

For the \((2,0)\) reduction of the \((4,0)\) Killing vectors, we find a set of differential relations between \(\varepsilon_\alpha, \varepsilon_L, \varepsilon_R\) and their complex conjugates:

\[
\mathcal{D}_\alpha \bar{\varepsilon}_\beta = 4S\varepsilon_{\alpha\beta}\bar{\varepsilon}_L, \quad \mathcal{D}_\alpha \varepsilon_\beta = -4S\varepsilon_{\alpha\beta}\varepsilon_L, \quad (7.27a)
\]

\[
\mathcal{D}_\alpha \varepsilon_\beta = -4S\varepsilon_{\alpha\beta}\bar{\varepsilon}_R, \quad \mathcal{D}_\alpha \bar{\varepsilon}_\beta = 4S\varepsilon_{\alpha\beta}\varepsilon_R, \quad (7.27b)
\]

\[
\mathcal{D}_\alpha \varepsilon_L = \mathcal{D}_\alpha \varepsilon_R = 0, \quad \mathcal{D}_\alpha \bar{\varepsilon}_L = i\varepsilon_\alpha \left(1 + \frac{X}{2S}\right), \quad \mathcal{D}_\alpha \bar{\varepsilon}_R = -i\bar{\varepsilon}_\alpha \left(1 - \frac{X}{2S}\right). \quad (7.27c)
\]

The action of the \(U(1)_R\) generator \((7.20)\) on these parameters is

\[
\mathcal{J}_\varepsilon_\alpha = -\frac{X}{2S}\varepsilon_\alpha, \quad \mathcal{J}_\varepsilon_L = -\left(1 + \frac{X}{2S}\right)\varepsilon_L, \quad \mathcal{J}_\varepsilon_R = -\left(1 - \frac{X}{2S}\right)\varepsilon_R. \quad (7.28)
\]

The real parameter \(\sigma\), corresponding to one of the residual \(R\)-symmetries, can be shown to obey

\[
\sigma - \frac{X}{2S} t = \text{const}. \quad (7.29)
\]

A finite \(U(1)\) transformation generated by the constant parameter \((\sigma - tX/2S)\) does not act on the \((2,0)\) AdS superspace, and thus it is analogous to the so-called shadow chiral rotation introduced in the context of 4D \(\mathcal{N} = 2\) superconformal \(\sigma\)-models \([47]\).

In the critical cases the parameters are further constrained to satisfy

\[
X = 2S : \quad \Lambda^{k\bar{l}} = \varepsilon_R = 0, \quad \mathcal{D}_\alpha \varepsilon_\beta = \mathcal{D}_\alpha \bar{\varepsilon}_\beta = 0; \quad (7.30a)
\]

\[
X = -2S : \quad \Lambda^{k\bar{l}} = \varepsilon_L = 0, \quad \mathcal{D}_\alpha \varepsilon_\beta = \mathcal{D}_\alpha \bar{\varepsilon}_\beta = 0. \quad (7.30b)
\]

7.2.2 AdS superspace reduction \((3,1) \rightarrow (2,0)\)

For the reduction from \((3,1)\) to \((2,0)\) superspace, a local \(R\)-symmetry transformation can be applied to bring \(w^{i\bar{j}}\) to look like

\[
w^{1\bar{1}} = w^{2\bar{2}} = 0, \quad w^{1\bar{2}} = 1, \quad w^{2\bar{1}} = -(w^{1\bar{2}})^* = -1, \quad (7.31)
\]

and then the condition \((7.16)\) holds. This implies

\[
\Lambda^{k\bar{l}} = \delta^{k\bar{l}} \varepsilon = \varepsilon_L = \varepsilon_R := \varepsilon. \quad (7.32)
\]

The \((2,0)\) projection of \((7.10)\)–\((7.11e)\) gives

\[
\mathcal{D}_\alpha \varepsilon_\beta = -\mathcal{D}_\alpha \bar{\varepsilon}_\beta = 4S\varepsilon_{\alpha\beta}\bar{\varepsilon}, \quad \mathcal{D}_\alpha \bar{\varepsilon}_\beta = -\mathcal{D}_\alpha \varepsilon_\beta = -4S\varepsilon_{\alpha\beta}\varepsilon, \quad (7.33a)
\]

\[
\mathcal{D}_\alpha \varepsilon = 0, \quad \mathcal{D}_\alpha \varepsilon = \frac{1}{2}(\varepsilon_\alpha - \bar{\varepsilon}_\alpha). \quad (7.33b)
\]
These imply
\[ D_\alpha (\varepsilon_\beta + \bar{\varepsilon}_\beta) = \bar{D}_\alpha (\varepsilon_\beta + \bar{\varepsilon}_\beta) = 0 . \] (7.34)

The real parameter \( \sigma \) turns out to vanish.

### 7.2.3 AdS superspace reduction \((2,2) \to (2,0)\)

For the final case of the \((2,2)\) to \((2,0)\) reduction, a local \(R\)-symmetry transformation can be applied to bring \( t^{ij} \) and \( r^{i\bar{j}} \) to the form:
\[
\begin{align*}
l^{11} &= l^{22} = r^{1\bar{1}} = r^{2\bar{2}} = 0, & l^{12} = -i, & r^{1\bar{2}} = i, \\
\varepsilon_L &= \Lambda^{22} = 0, & \varepsilon_R &= \Lambda^{\bar{2}\bar{2}} = 0, 
\end{align*}
\] (7.35a, 7.35b)

and then the condition (7.16) holds. The \((2,0)\) projection of (7.10)–(7.11e) gives
\[ D_\alpha \varepsilon_\beta = \bar{D}_\alpha \varepsilon_\beta = 0 . \] (7.36)

The real parameter \( \sigma \) must be constant.

### 7.3 From \( N = 4 \) to \( N = 3 \) AdS superspaces

Rather than reduce the \( N = 4 \) AdS superspaces to \( N = 2 \), we can choose instead to reduce to \( N = 3 \). This will turn out to have a very interesting consequence for \( N = 4 \) supersymmetric \( \sigma \)-models: it will be possible to directly relate an \( N = 4 \) \( \sigma \)-model in projective superspace to one in \( N = 3 \) (and vice-versa) with very little work. However, in contrast to the \( N = 2 \) reductions considered above, a restriction must be imposed on the \( N = 4 \) geometry: \( X = 0 \). The point is that all \( N = 3 \) AdS superspaces are conformally flat \([10]\). The conformally flat \( N = 4 \) AdS superspaces are those for which \( X = 0 \) \([10]\).

To start with, let us look for an \( N = 3 \) subalgebra of the \( N = 4 \) algebra. For this we break the \( R \)-symmetry group \( SU(2)_L \times SU(2)_R \) to its central subgroup \( SU(2)_J \) generated by
\[ J^{ij} := L^{ij} + \delta^i_{\bar{i}} \delta^j_{\bar{j}} R^{\bar{i}\bar{j}} . \] (7.37)

Here we identify the \( \bar{i}, \bar{j} \) indices with the \( i, j \) ones. By choosing \( J^{ij} \) to be the \( SU(2) \) generator of the \( N = 3 \) structure group we naturally split the \( N = 4 \) derivatives as
\[ D_\alpha^{\bar{i}} = D_\alpha^{(\bar{i})} - \frac{1}{2} \varepsilon^{\bar{i}\bar{j}} \varepsilon_{\bar{j}\bar{k}} D_\alpha^{\bar{k}} . \] (7.38)
The second term is invariant under the action of $\mathcal{J}^{ij}$ while the first one, symmetric in $i$ and $\bar{i}$, transforms as

$$[\mathcal{J}^{ij}, \mathcal{D}_\alpha^{(kk)}] = \frac{1}{2} \varepsilon^{k\bar{i}} \mathcal{D}_\alpha^{(\bar{k})} + \frac{1}{2} \varepsilon^{k\bar{i}} \mathcal{D}_\alpha^{(\bar{k})}.$$  

(7.39)

The remainder of the $\mathcal{N} = 4$ $R$-symmetry group is generated by

$$\Delta^{ij} := \mathbf{L}^{ij} - \delta^i_\bar{j} \delta^j_i \mathbf{R}.$$  

(7.40)

From the $\mathcal{N} = 4$ AdS algebra \[7.3\], we derive the following vector-spinor commutator

$$[\mathcal{D}_\alpha^{(i)}, \mathcal{D}_\gamma^{(kk)}] = -2 \varepsilon^{i(\bar{\gamma})}_{\delta} \delta_j^i \left( \delta_k^j \mathcal{S} + \mathcal{S}_i^{(k)l} \right) \mathcal{D}_\delta^{jl}.$$  

(7.41)

By imposing the closure of the algebra of the operators \((\mathcal{D}_\alpha^{(i)}, \mathcal{D}_a, \mathcal{J}^{kl})\), the following necessary condition arises

$$\varepsilon_{ij} \mathcal{S}^{ij\bar{j}j} = 0,$$  

(7.42)

together with the requirement that $X \equiv 0$. The equation \[7.42\] implies

$$\mathcal{S}^{ij\bar{j}j} = \mathcal{S}^{ij\bar{j}j} - \frac{1}{3} \varepsilon^{i(\bar{\gamma})j} \varepsilon_{kk} \varepsilon_{\bar{\ell}l} \mathcal{S}^{k\bar{k}\bar{k}l},$$  

(7.43)

as well as $\mathcal{S}^{ij\bar{j}j} = \mathcal{S}^{ij\bar{j}j}$. Then the algebra of the covariant derivatives $\mathcal{D}_a$ and $\mathcal{D}_a^{(i)}$ becomes

$$\{\mathcal{D}_\alpha^{(i)}, \mathcal{D}_\beta^{(j)}\} = -2i \varepsilon^{i(\bar{\gamma})}(\mathcal{D}_\alpha^{(i)} - \varepsilon^{i(\bar{\gamma})j}(S - \frac{1}{6} \varepsilon_{kk} \varepsilon_{\bar{\ell}l} \mathcal{S}^{k\bar{k}\bar{k}l})\mathcal{M}_{\alpha\beta}$$

$$-i \varepsilon_{\alpha\beta} \left( \varepsilon^{ij} \mathcal{S}^{ij} \mathcal{J}_{kl} + \varepsilon^{ij} \mathcal{S}^{ij} \mathcal{J}_{kl} \right)$$

$$+ 2i \varepsilon_{\alpha\beta} \left( S - \frac{1}{6} \varepsilon_{kk} \varepsilon_{\bar{\ell}l} \mathcal{S}^{k\bar{k}\bar{k}l} \right) \left( \varepsilon^{ij} \mathcal{J}_{ij} + \varepsilon^{ij} \mathcal{J}_{ij} \right),$$

(7.44a)

$$[\mathcal{D}_\alpha^{(k)}, \mathcal{D}_\gamma^{(kk)}] = -2 \varepsilon^{(k)}_{(\alpha}(S - \frac{1}{6} \varepsilon_{kk} \varepsilon_{\bar{\ell}l} \mathcal{S}^{k\bar{k}\bar{k}l}) \mathcal{D}_\beta^{(k)} + \mathcal{S}^{(k)l} \mathcal{D}_\beta^{(k)} \mathcal{D}_\beta^{(k)}),$$

(7.44b)

$$[\mathcal{D}_a, \mathcal{D}_b] = -4 S^2 \mathcal{M}_{ab}. $$

(7.44c)

This has the form of a general $\mathcal{N} = 3$ AdS algebra \[10\] once we identify the SU(2)$_L$ and SU(2)$_R$ indices.

There are two possibilities for the $\mathcal{N} = 3$ AdS algebra:

$$(3, 0) : \quad S = S, \quad S^{(ij)k} = 0$$  

(7.45a)

$$(2, 1) : \quad S = \frac{1}{3} S, \quad S^{(ij)k} = -S w^{ij} w^{kl}, \quad w^{ij} w^{ij} = 2.$$  

(7.45b)
Here $w^{ij} = w^{ji}$ is a real covariantly constant isotriplet which can be identified with the field strength of a frozen vector multiplet on the (2,1) AdS superspace, see subsection 5.3 for more details. We immediately conclude that if we begin with the (4,0) algebra, then $S^{ijij} = 0$ and only a (3,0) reduction is possible. Similarly, for the (2,2) case, one finds $S^{ijij} \neq 0$, which always leads to a (2,1) truncation. For the (3,1) case, we find two distinct classes of solution to (7.42): one with $S^{ijij} = 0$ and a (3,0) geometry; and another with $S^{ijij} \neq 0$ and a (2,1) geometry.

Now we sketch some of the technical details of the reduction. Given an $N = 4$ tensor superfield $U(x, \theta_{ij})$, we define its $N = 3$ projection as

$$U| := U(x, \theta_{ij})|_{\varepsilon_{ii} \theta_{ii} = 0}. \quad (7.46)$$

The superfield $U|$ depends on only the six Grassmann coordinates $\theta^{\mu}_{(ii)}$. The projection of the $N = 4$ covariant derivatives to the $N = 3$ subspace formally goes along the same lines of the reduction from $N = 4$ to $N = 2$ described in the previous subsection and here we skip the technicalities. The only important point is this: since in the different consistent $N = 3$ truncations the derivatives $D_a$ and $D^{(ii)}_{\alpha}$ form a closed algebra, one can use the freedom to perform general coordinate, local Lorentz and $SU(2)_L \times SU(2)_R$ transformations to choose a gauge where

$$D_A| = (D_a|, D^{(ii)}_{\alpha}|) = E_A^M \partial_M + \frac{1}{2} \Omega_A^{cd} M_{cd} + \frac{1}{2} \Phi_A^{kl} J_{kl} \quad (7.47)$$

denote the covariant derivatives of either (3,0) or (2,1) AdS. In the (2,1) case $\Phi_A^{kl} = \Phi_A w^{kl}$ since the $SU(2)_J$ is broken to a U(1) subgroup.

The reduction of an $N = 4$ Killing vector $\xi$ is more interesting. The natural decomposition of $\xi$ is

$$\xi = \xi^a D_a + \xi_{(ii)}^a D^{(ii)}_a = \xi^a D_a + \xi_{(ii)}^a D^{(ii)}_a + \Xi^a \varepsilon_{ii} D^{(ii)}_a , \quad \Xi^a := -\frac{1}{2} \varepsilon^{ii} \xi_{ii}^a. \quad (7.48)$$

The parameters $\xi^a|$ and $\xi_{(ii)}^a$ will play the role of the $N = 3$ Killing vectors while $\Xi^a|$ will yield the extra supersymmetry.

For the $SU(2)_L \times SU(2)_R$ symmetry parameters, we must be a little careful. Because we are identifying the two types of $SU(2)$ indices, we will affix an additional label to the parameters, denoting them $\Lambda_{L}^{ij}$ and $\Lambda_{R}^{ij}$. Then we may decompose as

$$\Lambda^{ij} := \Lambda_{L}^{ij} + \Lambda_{R}^{ij}, \quad \Lambda^{\tilde{i}j} := \Lambda_{L}^{ij} - \Lambda_{R}^{ij}. \quad (7.49)$$

The superfields $\Lambda^{ij}$ and $\Lambda^{\tilde{i}j}$ are associated respectively with the $\mathcal{J}^{ij}$ and $\Delta^{ij}$ generators.
By consistently reducing to $\mathcal{N} = 3$ AdS the $\mathcal{N} = 4$ Killing equations (7.11a)-(7.12d), it can be proven that the superfields $(\xi^a|, \xi^\alpha|, \Lambda^{ab}|, \Lambda^{ij}|)$ parametrize a general isometry of $(3,0)$ or $(2,1)$ AdS (see [10] for the $\mathcal{N} = 3$ AdS Killing equations). Similarly, one can derive the differential constraints satisfied by the extra superfields $\Xi^\alpha|$ and $\tilde{\Lambda}^{ij}|$ that parametrize the extra supersymmetry and the remaining $R$-symmetry. The most important equations are

$$
D^a(\bar{i})\Xi^\beta = -\frac{1}{2}\varepsilon_{\alpha\beta}\tilde{\Lambda}^{\bar{i}}, \quad D^a\Xi^\beta = \left(\frac{1}{2}\varepsilon_{\alpha\beta}\varepsilon_{\bar{j}j}S^{\bar{i}ij} - S\right)\Xi^\gamma(\gamma_a)\gamma^\beta, \quad (7.50a)
$$

$$
D^a(\bar{i})\tilde{\Lambda}^{kl} = i\left(8\varepsilon^{k(i}\varepsilon^{\bar{j}l)}(S + \frac{1}{6}\varepsilon_{kk}\varepsilon_{ll}S^{k\bar{i}l}) - 4S^{(\bar{i}ikl)}\right)\Xi^\alpha, \quad D^a\tilde{\Lambda}^{kl} = 0. \quad (7.50b)
$$

Let us now address each case specifically.

**$(4,0) \rightarrow (3,0)$** This is the easiest case since $S^{\bar{i}ij} \equiv 0$ and $S = S$. There is no reduction of the structure group $\text{SU}(2)_L \times \text{SU}(2)_R$: half of it is manifest in the $(3,0)$ structure group $\text{SU}(2)_J$ and parametrized by $\Lambda^{ij}$, while the non-manifest half is generated by $\Delta^{ij}$ and parametrized by $\tilde{\Lambda}^{ij}$.

**$(3,1) \rightarrow (3,0)$** In the $(3,1)$ case, we have a covariantly constant isovector $w^{\bar{i}i}$, which reduces the structure group to an $\text{SU}(2)$ generated by $L^{ij} + w^{\bar{i}k}w^{ij}\text{R}^{\bar{k}\bar{l}}$. A $(3,0)$ reduction arises when this generator can be identified with the $(3,0)$ SU(2) generator $J^{ij}$ (7.37). This occurs when $w^{\bar{i}k}$ is antisymmetric; without loss of generality, we can choose $w^{\bar{i}k} = \varepsilon_{ik}$ and $w^{ik} = -\varepsilon^{ik}$. Using the expressions for $S$ and $S^{\bar{i}ij}$ in the $(3,1)$ case, one easily finds

$$
S^{(\bar{i}ij)} = 0, \quad S - \frac{1}{6}\varepsilon_{kk}\varepsilon_{ll}S^{k\bar{i}l} = S, \quad (7.51)
$$

which is a necessary condition for the $(3,1) \rightarrow (3,0)$ reduction to be consistent.

**$(3,1) \rightarrow (2,1)$** The other possible reduction of $(3,1)$ is to $(2,1)$. Recall that $(2,1)$ is equipped with a symmetric $w_{ij}$ tensor. The obvious thing to do here is to choose $w_{ik}$ to be symmetric and identify it with $w_{ij}$. One can check that

$$
S - \frac{1}{6}\varepsilon_{kk}\varepsilon_{ll}S^{k\bar{i}l} = \frac{1}{3}S, \quad S^{(\bar{i}ij)} = -Sw^{(\bar{i}ij)}w^{\bar{i}j}, \quad (7.52)
$$

which is the appropriate condition for a $(2,1)$ AdS superspace. To understand what happens to the $R$-symmetry group, we observe that the $(3,1)$ $R$-symmetry parameter is constrained by $\Lambda^{kl} = w^{k\bar{k}}w^{i\bar{i}}\text{R}^{\bar{k}\bar{l}}$. By choosing $w^{\bar{i}i}$ to be symmetric, one can show that

$$
\Lambda^{ij} = \Lambda^{\bar{i}j} + \Lambda^{\bar{j}i} = w^{\bar{i}j}\Lambda \quad (7.53)
$$
for some parameter $\Lambda$. This is what we expect since the SU(2)$_J$ in the (2,1) case is reduced to a U(1) subgroup. The other parameter $\tilde{\Lambda}^{ij}$ obeys

$$\tilde{\Lambda}^{ij} w_{ij} = 0 . \quad (7.54)$$

The unbroken SU(2) of the (3, 1) algebra has decomposed into a U(1) appearing in the (2, 1) algebra, with the rest relegated to the extended supersymmetry.

**(2,2) → (2,1)** For the (2,2) geometry, there is a single possibility: reduction to (2,1). We recall that the (2,2) geometry is characterized by two covariantly constant symmetric tensors: $l^{ij}$ and $r^{k\bar{l}}$. In reducing to (2,1), however, we should end up with a single tensor $w^{ij}$. We can always choose $l^{ij} = -r^{ij} = \pm w^{ij}$ and check that (keeping in mind the fact that $S = 0$ in the (2,2) case)

$$-\frac{1}{6} \varepsilon_{k\bar{k}} \varepsilon_{\bar{l}l} S^{k\bar{l}\bar{m}} = \frac{1}{3} S , \quad S^{(ij)\bar{j}} = -S w^{(ij)\bar{j}} \quad (7.55)$$

as expected. Then it can be easily proven that the $R$-symmetry parameter $\Lambda^{ij}$ becomes

$$\Lambda^{ij} = w^{ij} \Lambda , \quad (7.56)$$

for some real parameter $\Lambda$. We also find $\tilde{\Lambda}^{ij} = w^{ij} \tilde{\Lambda}$, for some real parameter $\tilde{\Lambda}$. These results are completely natural, since the $R$-symmetry part of the (2,2) structure group is $U(1)_L \times U(1)_R$, in accordance with eq. (7.14d). One U(1) subgroup turns into the U(1) $R$-symmetry of the (2,1) AdS superspace, and the other becomes part of the extended supersymmetry.

### 8 Rigid $\mathcal{N} = 4$ supersymmetric field theories in AdS: Off-shell multiplets and invariant actions

In this and subsequent sections, our goal is to apply the supergravity techniques of [19] to describe general nonlinear $\sigma$-models in AdS$_3$ possessing $\mathcal{N} = 4$ supersymmetry. Our discussion here is a generalization of the $\mathcal{N} = 3$ analysis given in [10].

#### 8.1 Covariant projective supermultiplets

In complete analogy with matter couplings in $\mathcal{N} = 4$ supergravity [19], a large class of rigid supersymmetric theories in (4,0), (3,1) and (2,2) AdS superspaces can be formulated
in terms of covariant projective supermultiplets. As described in [19], because the supergravity structure group includes the factor SU(2)_L \times SU(2)_R, it is natural to introduce left and right isotwistors, \( v_L = v^i \), \( v_R = v^\bar{i} \), and to define inequivalent left and right projective multiplets. Here we will focus on the properties of left projective multiplets; analogous results for right projective multiplets can be obtained by applying a mirror map [19].

A \textit{covariant left projective supermultiplet} of weight \( n \), \( Q_L^{(n)}(z^M, v^i) \), is defined to be a Lorentz scalar and SU(2)_R invariant superfield that lives on the appropriate \( \mathcal{N} = 4 \) AdS superspace, is holomorphic with respect to isospinor variables \( v^i \) on an open domain of \( \mathbb{C}^2 \setminus \{0\} \), and is characterised by the following conditions:

(i) \( Q_L^{(n)} \) is a homogeneous function of \( v \) of degree \( n \),
\[
Q_L^{(n)}(z, cv_L) = c^n Q_L^{(n)}(z, v_L) , \quad c \in \mathbb{C}^* \equiv \mathbb{C} \setminus \{0\} ;
\]  
(8.1)

(ii) \( Q_L^{(n)} \) transforms under the AdS isometry supergroup as
\[
\delta_\xi Q_L^{(n)} = \left( \xi + \Lambda^{ij} L_{ij}\right) Q_L^{(n)} , \quad \Lambda^{ij} L_{ij} Q_L^{(n)} = - \left( \Lambda^{(2)} \partial_L^{(-2)} - n \Lambda^{(0)} \right) Q_L^{(n)} , \quad \partial_L^{(-2)} := \frac{1}{(v_L, u_L)} u_i \partial v^i ;
\]  
(8.2)
where \( \xi \) denotes an arbitrary AdS Killing vector field, eq. (7.10), and \( \Lambda^{ij} \) the associated SU(2)_L parameter;

(iii) \( Q_L^{(n)} \) obeys the analyticity constraint
\[
\mathcal{D}^{(1)} \bar{\alpha} Q_L^{(n)} = 0 , \quad \mathcal{D}^{(1)} \bar{\alpha} := v_i \mathcal{D}_i \bar{\alpha} .
\]  
(8.3)

Some comments are necessary. The homogeneity condition (8.1) and the analyticity constraint (8.3) are consistent with the interpretation that the isospinor \( v^i \in \mathbb{C}^2 \setminus \{0\} \) is defined modulo the equivalence relation \( v^i \sim cv^i \), with \( c \in \mathbb{C}^* \). Therefore, the projective multiplets live in \( \mathcal{M}^{3|6} \times \mathbb{C}P^1 \). On the other hand, the transformation law (8.2) and the parameters
\[
\Lambda^{(2)} := \Lambda^{ij} v_i v_j , \quad \Lambda^{(0)} := \frac{v_i u_j}{(v_L, u_L)} \Lambda^{ij} , \quad (v_L, u_L) := v^i u_i
\]  
(8.4)
depend on an additional isotwistor \( u_i \), which is subject only to the condition \( (v_L, u_L) \neq 0 \) and otherwise is completely arbitrary. Nevertheless, both \( Q_L^{(n)} \) and \( \delta_\xi Q_L^{(n)} \) are independent of \( u_i \).
The analyticity condition (8.3) is quite powerful. Requiring the consistency condition \( \{ \mathcal{D}_a^{(1)i}, \mathcal{D}_\beta^{(1)j} \} Q_L^{(n)} = 0 \), we conclude that all left projective multiplets must be Lorentz and SU(2)_R scalars. If instead we require only the conditions (i) and (ii) to be satisfied, we find the so-called left isotwistor supermultiplets, which may belong to nontrivial representations of the Lorentz and SU(2)_R groups. These isotwistor superfields can be used to construct projective ones with the aid of the left analytic projection operator \( \Delta_L^{(4)} \) [19], which in AdS is given by

\[
\Delta_L^{(4)} = \frac{1}{48} (D^{(2)k\bar{l}} - 4iS^{(2)k\bar{l}}) D_{ki}^{(2)} , \quad D_{ij}^{(2)} := D_{\gamma i}^{(1)} D_{\gamma j}^{(1)} .
\]

In fact, given a Lorentz and SU(2)_R invariant weight-\((n-4)\) left isotwistor superfield, \( T_L^{(n-4)} \), a covariant left projective multiplet \( Q_L^{(n)}(v_L) \) of weight \( n \) can be constructed as

\[
Q_L^{(n)} = \Delta_L^{(4)} T_L^{(n-4)} .
\]

General off-shell \( \mathcal{N} = 4 \) \( \sigma \)-models can be described by left arctic weight-\( n \) projective multiplets \( \Upsilon_L^{(n)}(v) \). These are defined to be holomorphic in the north chart of \( \mathbb{C}P^1 \), and so can be represented as

\[
\Upsilon_L^{(n)}(v_L) = (v^1)^n \Upsilon_L^{[n]}(\zeta) , \quad \Upsilon_L^{[n]}(\zeta) = \sum_{k=0}^{\infty} \Upsilon_k \zeta^k .
\]

Their smile-conjugates are left antarctic multiplets \( \breve{\Upsilon}_L^{(n)}(v_L) \),

\[
\breve{\Upsilon}_L^{(n)}(v_L) = (v^2)^n \breve{\Upsilon}_L^{[n]}(\zeta) = (v^1 \zeta)^n \breve{\Upsilon}_L^{[n]}(\zeta) , \quad \breve{\Upsilon}_L^{[n]}(\zeta) = \sum_{k=0}^{\infty} \breve{\Upsilon}_k \zeta^k .
\]

In complete analogy to the \( \mathcal{N} = 3 \) case, we have introduced the inhomogeneous complex coordinate \( \zeta = v^2/v^1 \) on the north chart of \( \mathbb{C}P^1 \). The pair of fields \( \Upsilon_L^{[n]}(\zeta) \) and \( \breve{\Upsilon}_L^{[n]}(\zeta) \) constitute the so-called left polar weight-n multiplet.

For further details on \( \mathcal{N} = 4 \) multiplets, the reader can refer to the full supergravity treatment in [19] and restrict to the appropriate \( \mathcal{N} = 4 \) AdS background.

### 8.2 Reduction to \( \mathcal{N} = 2 \) AdS superspaces

It will be useful to reduce \( \mathcal{N} = 4 \) AdS projective multiplets to \( \mathcal{N} = 2 \) AdS superfields. According to the analysis of section 7, one can choose to reduce \( \mathcal{N} = 4 \) AdS either to (1,1) or (2,0) AdS. The (2,0) reduction is more interesting because it is more general – it is available for all \( \mathcal{N} = 4 \) AdS geometries – so we will restrict our attention to it.
Actions involving the polar multiplets \( \Upsilon_L^{[n]}(z, \zeta) \) and \( \tilde{\Upsilon}_L^{[n]}(z, \zeta) \) yield the most general \( \sigma \)-models, so it is sufficient to discuss these alone. We work in the north chart where the analyticity condition is equivalent to

\[
D^2_{\alpha} \Upsilon_L^{[n]} = \frac{1}{\zeta} D^2_{\alpha} \Upsilon_L^{[n]} , \quad D^2_{\alpha} \tilde{\Upsilon}_L^{[n]} = \zeta D^2_{\alpha} \tilde{\Upsilon}_L^{[n]} .
\] (8.9)

These equations ensure that the dependence of \( \Upsilon_L^{[n]}(x, \theta, \bar{\theta}, \zeta) \) on the Grassmann coordinates \( \theta_{12}^\mu \) and \( \theta_{22}^\mu \) is entirely determined by its dependence on the other coordinates \( \theta_{11}^\mu \) and \( \theta_{21}^\mu \). In other words, all the information about \( \Upsilon_L^{[n]}(\zeta) \) is encoded in its \( N = 2 \) projection \( \Upsilon_L^{[n]}(\zeta) \). The same holds true for all projective multiplets written in the north chart.

The isometry transformations for the polar multiplet, reduced to (2,0) \( \text{AdS} \), are

\[
\delta \xi \Upsilon_L^{[n]} = \left[ \tau + i t \left( \frac{\partial}{\partial \zeta} - \frac{n}{2} \right) + \zeta \bar{\varepsilon} D_\alpha - \frac{1}{\zeta} \varepsilon D^\alpha + i \sigma \left( \frac{\partial}{\partial \zeta} - \frac{n}{2} \right) \right] \Upsilon_L^{[n]} ,
\] (8.10a)

\[
\delta \xi \tilde{\Upsilon}_L^{[n]} = \left[ \tau + i t \left( \frac{\partial}{\partial \zeta} + \frac{n}{2} \right) + \zeta \bar{\varepsilon} D_\alpha - \frac{1}{\zeta} \varepsilon D^\alpha + i \sigma \left( \frac{\partial}{\partial \zeta} + \frac{n}{2} \right) \right] \tilde{\Upsilon}_L^{[n]} .
\] (8.10b)

Note that \( \tau = \tau^a D_a + \tau^\alpha D_\alpha + \bar{\tau}_\alpha \bar{D}^\alpha \) is a (2,0) Killing vector. The additional parameters appearing above are defined in (7.26a)–(7.26c).

We will also need the (2,0) transformation for the projective superspace Lagrangian, which is a real weight-two left projective superfield \( L_L^{(2)} \). In the north chart, we represent it as

\[
L_L^{(2)}(v_L) = i (v^1)^2 \zeta L_L^{[2]}(\zeta) ,
\] (8.11)

and we find

\[
\delta \xi L_L^{[2]} = \left[ \tau + i t \left( \frac{\partial}{\partial \zeta} + \frac{n}{2} \right) + \zeta \bar{\varepsilon} D_\alpha - \frac{1}{\zeta} \varepsilon D^\alpha + i \sigma \left( \frac{\partial}{\partial \zeta} + \frac{n}{2} \right) \right] L_L^{[2]} ,
\] (8.12)

8.3 Reduction to \( \mathcal{N} = 3 \) \( \text{AdS} \) superspaces

An interesting alternative reduction procedure is to take \( \mathcal{N} = 4 \) projective superfields to \( \mathcal{N} = 3 \). As discussed already in section 7, there are several distinct cases to consider.

\footnote{Within the 3D harmonic superspace approach, the reduction \( \mathcal{N} = 4 \rightarrow \mathcal{N} = 3 \) was worked out by Zupnik for the case of Poincaré supersymmetry [57].}
since neither (3,0) nor (2,1) AdS is universal in the same sense that (2,0) is. Fortunately, the general features of the reduction are independent of which exact reduction is being considered.

We have previously described, in subsection 7.3, how the $\mathcal{N}=3$ reduction of a general $\mathcal{N}=4$ superfield proceeds. In the case of a left projective multiplet, $Q^{(n)}_L$, a slight elaboration is needed since the multiplet also depends on the isotwistor $v^i_L := v^i$ parametrizing the manifold $\mathbb{CP}^1$. Recall that such a multiplet must satisfy the analyticity condition $\mathcal{D}^{(1)i\bar{j}} Q^{(n)}_L = 0$. Decomposing the $\mathcal{N}=4$ covariant derivative as in (7.38), one finds

$$v_i \mathcal{D}^{i\bar{j}} Q^{(n)}_L = 0 \iff \mathcal{D}^{(2)}_\alpha Q^{(n)}_L = 0 , \quad \varepsilon_{i\bar{j}} \mathcal{D}^{i\bar{j}} Q^{(n)}_L = -2\mathcal{D}^{(0)}_\alpha Q^{(n)}_L , \quad (8.13)$$

where we have defined

$$\mathcal{D}^{(2)}_\alpha := v_i v_j \delta^{i\bar{j}} \mathcal{D}^{(i\bar{j})}_\alpha , \quad \mathcal{D}^{(0)}_\alpha := \frac{v_i u_j}{(v_L, u_L)} \delta^{i\bar{j}} \mathcal{D}^{(i\bar{j})}_\alpha . \quad (8.14)$$

We have introduced an auxiliary isotwistor $u_i$ above, but the multiplet $Q^{(n)}_L$ doesn’t depend on it.

In the $\mathcal{N}=3$ reduction introduced in subsection 7.3, we identified the left and right $\text{SU}(2)$ indices. It follows that we should also identify the left and right isotwistors. In other words, we take $v^i_L := v^i$ and $v^i_R := v^\bar{i}$ to be equivalent, $v^i = \delta^i_\bar{i} v^\bar{i}$. Similarly, we take the auxiliary isotwistors to obey $u_i = \delta^i_\bar{i} u^\bar{i}$. The meaning of the two constraints in (8.13) then becomes clear. The first constraint $\mathcal{D}^{(2)}_\alpha Q^{(n)}_L = 0$, when projected to $\mathcal{N}=3$ superspace, is simply the analyticity condition of an $\mathcal{N}=3$ projective superfield \cite{10}. The second constraint then fixes the dependence of $Q^{(n)}_L(x, \theta_{ij}, v)$ on the Grassmann coordinate $\varepsilon^{i\bar{j}} \theta_{ij}$ in terms of its dependence on the other coordinates $\theta_{ij}$. In other words, all the information about $Q^{(n)}_L$ is encoded in its $\mathcal{N}=3$ projection $Q^{(n)}_L |_{\varepsilon^{i\bar{j}} \theta_{ij}=0}$.

We can now rewrite the isometry transformation (8.2) for a weight-$n$ left projective superfield projected to $\mathcal{N}=3$:

$$\delta_\xi Q^{(n)}_L = \left( \xi^a \mathcal{D}_a + \xi^a_{(ij)} \mathcal{D}^{(i\bar{j})}_\alpha - \Lambda^{(2)} a (-2) + n \Lambda^{(0)} \right) Q^{(n)}_L - 2\Xi^a \mathcal{D}^{(0)}_\alpha - \tilde{\Lambda}^{(2)} a (-2) + n \tilde{\Lambda}^{(0)} Q^{(n)}_L \right) . \quad (8.15)$$

The first line corresponds to an $\mathcal{N}=3$ Killing isometry, with parameters $(\xi^a, \xi_{i\bar{j}}^a, \Lambda^{i\bar{j}})$, as described in \cite{10}. The second line, involving the parameters $(\Xi^a, \tilde{\Lambda}^{i\bar{j}})$, corresponds to an extended supersymmetry and $R$-symmetry transformation associated with the rest of the $\mathcal{N}=4$ AdS isometry group. Naturally, the precise relations these parameters satisfy depends on the case in question.
In order for the transformation law (8.15) to be sensible, \( \delta \xi Q_L^{(n)} \) must remain a weight-\( n \) projective multiplet in \( \mathcal{N} = 3 \) superspace. The first line of (8.15) satisfies this condition automatically since it is an \( \mathcal{N} = 3 \) Killing transformation. To verify that the second line of (8.15) is sensible, we must check two conditions: it must be annihilated by \( D_\alpha^{(2)} \) and it must be independent of the auxiliary isotwistor \( u_i \).

Both conditions may be checked by exploiting an alternative parametrization of the parameters \( \Xi_\alpha \) and \( \tilde{\Lambda}^{ij} \). Using eqs. (7.50a) and (7.50b), one can show that

\[
\Xi_\alpha = -\frac{i}{2} D_\alpha^{(2)} \Omega^{(-2)} , \quad \tilde{\Lambda}^{(2)} := v_i v_j \tilde{\Lambda}^{ij} = \frac{i}{2} D^{(4)} \Omega^{(-2)} , \quad D^{(4)} := D^{\alpha(2)} D_\alpha^{(2)} ,
\]

where \( \Omega^{(-2)} \) is some weight-\( (-2) \) isotwistor superfield depending on \( v^i \) but not \( u_i \). Naturally, it must be chosen so that \( \Xi_\alpha \) is independent of \( v^i \), \( \partial^{(-2)} \Xi_\alpha = 0 \); in addition, a number of other conditions are encoded in (7.50a)–(7.50b). We will not attempt a comprehensive analysis here, since the representation (8.16) is sufficient for our needs.

In terms of the parameter \( \Omega^{(-2)} \), the transformation (8.15) can be rewritten

\[
\delta \xi Q_L^{(n)} \bigg| = \delta \xi^{\mathcal{N} = 3} Q_L^{(n)} \bigg| + \Delta^{(4)} \left( \Omega^{(-2)} \partial^{(-2)} + \frac{n}{2} (\partial^{(-2)} \Omega^{(-2)}) \right) Q_L^{(n)} \bigg| ,
\]

where we have introduced the \( \mathcal{N} = 3 \) analytic projector operator [10]

\[
\Delta^{(4)} := \frac{i}{4} \left( D^{(4)} - 4i S^{(4)} \right) , \quad S^{(4)} := v_i v_j v_k v_l S^{ijkl} .
\]

Its presence guarantees that the second term in (8.17) is analytic. It is an instructive exercise to check that this expression is also independent of \( u_i \). In the form given above, this is straightforward.

Although our analysis has focused on left projective superfields, it turns out that the only difference for the case of an \( \mathcal{N} = 4 \) right projective superfield is the overall sign of the extended supersymmetry transformation:

\[
\delta \xi Q_R^{(n)} \bigg| = \delta \xi^{\mathcal{N} = 3} Q_R^{(n)} \bigg| - \Delta^{(4)} \left( \Omega^{(-2)} \partial^{(-2)} + \frac{n}{2} (\partial^{(-2)} \Omega^{(-2)}) \right) Q_R^{(n)} \bigg| .
\]

### 8.4 \( \mathcal{N} = 4 \) supersymmetric actions

In order to formulate the dynamics of rigid \( \mathcal{N} = 4 \) supersymmetric field theories in AdS\(_3\), a manifestly supersymmetric action principle is required. We can readily construct
such an action principle by restricting to AdS the locally supersymmetric action introduced in [19]. The action is generated by a real Lagrangian $\mathcal{L}^{(2)}_L(z,v_L)$, which is a covariant weight-two left projective multiplet, and has the form:

$$S[\mathcal{L}^{(2)}_L] = \frac{1}{2\pi i} \oint_\gamma (v_L, d v_L) \int d^3 x \, d^8 \theta \, E \, C^{(-4)}_L \, \mathcal{L}^{(2)}_L, \quad E^{-1} = \text{Ber}(E A^M). \quad (8.20)$$

Here the line integral is carried out over a closed contour $\gamma = \{v^i(t)\}$ in $\mathbb{C}P^1$. The action involves a left isotwistor superfield $C^{(-4)}_L(z,v_L)$ which can be defined as

$$C^{(-4)}_L := \frac{U^{(n)}_L}{\Delta^{(4)}_L U^{(n)}_L}, \quad (8.21)$$

for some left isotwistor multiplet $U^{(n)}_L$ such that $1/\Delta^{(4)}_L U^{(n)}_L$ is well defined. The superfield $C^{(-4)}_L$ is required to write the action as an integral over the full AdS superspace. It can be proven that (8.20) is independent of the explicit choice of $U^{(n)}_L$. For a proof the reader may consult [19] or follow the analogous derivation given for the $\mathcal{N} = 3$ case in [10].

The crucial feature of the action (8.20) is that it is manifestly invariant under isometry transformations of the appropriate $\mathcal{N} = 4$ AdS superspace. On the other hand the action involves the superfield $C^{(-4)}$, which is a purely gauge degree of freedom, and is given by an integral over all the eight Grassmann variables of the AdS superspaces even though, due to the analyticity constraint, the Lagrangian $\mathcal{L}^{(2)}_L$ depends only on four fermionic variables. For this reason, (8.20) is not the most practical version of the projective action principle.

By integrating out four, two or all the fermionic directions it is possible to write (8.20) in more useful ways: as (i) an integral in $\mathcal{N} = 2$ AdS, (ii) an integral in $\mathcal{N} = 3$ AdS, or (iii) as a component expression. The price is that we lose manifest invariance under the full $\mathcal{N} = 4$ isometry group. Let us demonstrate each case in turn.

8.4.1 $\mathcal{N} = 4$ supersymmetric action in $\mathcal{N} = 2$ AdS

Here we present the $\mathcal{N} = 4$ supersymmetric action reduced to (2,0) superspace. We focus on (2,0) only to simplify the presentation, but the reader should keep in mind that the same results hold for any consistent (1,1) reduction.

Recall that the Lagrangian $\mathcal{L}^{(2)}_L(v_L)$ is a real weight-two left projective superfield, and in the north chart it is associated with the superfield $\mathcal{L}^{[2]}_L(\zeta)$ defined by $\mathcal{L}^{(2)}_L(v) =$
\( i(v^1)^2 \zeta L^{[2]}_L(\zeta) \). We shall prove that for any \( \mathcal{N} = 4 \) AdS superspace the supersymmetric action (8.20) takes the following form in (2,0) AdS superspace:

\[
S(L^{(2)}_L) = \int d^3x \, d^2\theta d^2\bar{\theta} \, E \oint_{C} \frac{d\zeta}{2\pi i} \zeta L^{[2]}_L, \quad E^{-1} := \text{Ber}(E_A^M). \tag{8.22}
\]

To do this, we must check explicitly that (8.22) is invariant under the full isometry group. By making use of the transformation rule (8.12), the variation of (8.22) is

\[
\delta\xi S(L^{(2)}_L) = \int d^3x \, d^2\theta d^2\bar{\theta} \, E \oint_{C} \frac{d\zeta}{2\pi i} \left[ \tau + \imath \eta \zeta \frac{\partial}{\partial \zeta} + \zeta \bar{\epsilon}^\alpha D_{\alpha} - \frac{1}{\zeta} \epsilon_{\alpha} \bar{D}^\alpha + \imath \sigma \zeta \frac{\partial}{\partial \zeta} - 4S \left( \zeta \bar{\epsilon}^\alpha + \frac{1}{\zeta} \epsilon_{\alpha} \right) \frac{\partial}{\partial \zeta} - 4S \left( \frac{1}{\zeta} \epsilon_{\alpha} - \zeta \bar{\epsilon}^\alpha \right) \right] L^{[2]}_L . \tag{8.23}
\]

The first term corresponds to the variation under an infinitesimal isometry transformation of the (2,0) AdS superspace. Since the action is manifestly invariant under the (2,0) AdS isometry group, this variation vanishes. The second and fifth terms are total derivatives in \( \zeta \) and vanish under the contour integral. Integrating by parts the remaining terms, we find

\[
\delta\xi S(L^{(2)}_L) = \int d^3x \, d^2\theta d^2\bar{\theta} \, E \oint_{C} \frac{d\zeta}{2\pi i} \left[ \frac{1}{\zeta} (D_{\alpha} \bar{\epsilon}^\alpha - 8S \epsilon_{\alpha}) + \zeta (\bar{D}^\alpha \epsilon_{\alpha} + 8S \bar{\epsilon}^\alpha) \right] L^{[2]}_L . \tag{8.24}
\]

As a consequence of the equations (7.27a), (7.32)–(7.33a) and (7.35b)–(7.36), the terms in parentheses vanish and the action is invariant for all the \( \mathcal{N} = 4 \) geometries and consistent (2,0) reductions.

### 8.4.2 \( \mathcal{N} = 4 \) supersymmetric action in \( \mathcal{N} = 3 \) AdS

Next we consider the reduction of the \( \mathcal{N} = 4 \) action to \( \mathcal{N} = 3 \) AdS. Begin by recalling the form of the \( \mathcal{N} = 3 \) action principle found by restricting the locally supersymmetric \( \mathcal{N} = 3 \) action introduced in [19] to the appropriate \( \mathcal{N} = 3 \) AdS superspace. The action is generated by a Lagrangian \( L^{(2)}(z, v) \), which is a real weight-two \( \mathcal{N} = 3 \) projective multiplet, and has the form

\[
S[L^{(2)}] = \frac{1}{2\pi i} \oint_{\gamma} (v, dv) \int d^3x \, d^6\theta E C^{(-4)} L^{(2)} , \quad E^{-1} = \text{Ber}(E_A^M) . \tag{8.25}
\]

where as usual the contour integral is carried out over a closed contour \( \gamma = \{v^i(t)\} \) in \( \mathbb{C}P^1 \). The isotwistor superfield \( C^{(-4)}(z, v) \) appearing above is given by

\[
C^{(-4)} := \frac{U^{(n)}}{\Delta^{(n)} U^{(n)}} . \tag{8.26}
\]
The action looks the same for both (3,0) and (2,1) AdS except for the precise definition of the $\mathcal{N} = 3$ analytic projector $\Delta^{(4)}$. Just as in the $\mathcal{N} = 4$ case, the superfield $\mathcal{C}^{(-4)}$ is actually a pure gauge degree of freedom in the sense that (8.25) is independent of the explicit choice of $\mathcal{U}^{(n)}$.

We must now relate the $\mathcal{N} = 4$ action (8.20) to an $\mathcal{N} = 3$ action (8.25) for some choice of $\mathcal{L}^{(2)}$. It turns out that simply reducing the Lagrangian in the obvious way

$$\mathcal{L}^{(2)} = \mathcal{L}^{(2)}_{L} \mid_{\epsilon^{ij} \theta_{ij} = 0}$$

(8.27)

gives the correct answer. To prove this, we must check that the action (8.25) is invariant under the full $\mathcal{N} = 4$ AdS isometry for this choice. Using (8.17) for the case $n = 2$, we have

$$\delta_{\xi} \mathcal{L}^{(2)}_{L} = \delta_{\xi}^{N=3} \mathcal{L}^{(2)}_{L} + \Delta^{(4)} \mathcal{C}^{(-2)} \mathcal{O}^{-2} \mathcal{L}^{(2)}_{L} .$$

(8.28)

The first term automatically leaves the action (8.25) invariant, while the second term leads to

$$\delta_{\xi} S[\mathcal{L}^{(2)}] = \frac{1}{2\pi i} \oint_{\gamma} (v, dv) \int d^{3}x d^{6} \theta E \mathcal{C}^{(-4)} \Delta^{(4)} \mathcal{O}^{(-2)} \mathcal{L}^{(2)}_{L} .$$

(8.29)

Integrating the analytic projector by parts, we find

$$\delta_{\xi} S[\mathcal{L}^{(2)}] = \frac{1}{2\pi i} \oint_{\gamma} (v, dv) \mathcal{O}^{(-2)} \int d^{3}x d^{6} \theta E \mathcal{O}^{(-2)} \mathcal{L}^{(2)}_{L} ,$$

(8.30)

which vanishes identically upon integrating around the contour.

### 8.4.3 $\mathcal{N} = 4$ supersymmetric action in components

Finally, we present the evaluation of (8.20) upon integration of all the Grassmann coordinates. We will apply the technique first used in [21] to derive the $\mathcal{N} = 1$ supersymmetric action in AdS$_{5}$ and similarly used in [10] for the $\mathcal{N} = 3$ supersymmetric action in AdS$_{3}$.

We start with the $\mathcal{N} = 4$ left projective superspace action in three-dimensional Minkowski space, which was introduced in [40]. It has the form

$$S[\mathcal{L}^{(2)}_{L}] = \frac{1}{2\pi} \oint_{\gamma} v_{i} dv^{i} \int d^{3}x D^{(-4)}_{L} \mathcal{L}^{(2)}_{L} (z, v_{L}) \bigg|_{\theta = 0} .$$

(8.31)
where the Lagrangian $L^{(2)}_L(z, v_L)$ is a real left weight-two projective multiplet, and the operator $D^{(-4)}_L$ is defined in terms of the flat spinor covariant derivatives $D^i_{\alpha}$ as

$$D^{(-4)}_L := \frac{1}{48} D^{(-2) kl} D^{(-2)}_{kl} , \quad D^{(-2)}_{kl} := D^{(-1)}_k D^{(-1)}_l , \quad D^{(-1)i}_{\alpha} := \frac{1}{(v_L, u_L)} u_i D^i_{\alpha} . \quad (8.32)$$

The $D^{(-4)}_L$ operator depends on the isotwistor $v^i(t)$, which varies along the integration contour, but also on a constant ($t$-independent) isotwistor $u_i$ which is chosen such that $v_i(t)$ and $u_i$ are everywhere linearly independent along the contour $\gamma$, that is $(v_L(t), u_L) \neq 0$. The flat action $\{8.31\}$ is actually independent of $u_i$, since it proves to be invariant under arbitrary projective transformations of the form

$$\left( u_i , v_i(t) \right) \to \left( u_i , v_i(t) \right) R_L(t) , \quad R_L(t) = \begin{pmatrix} a_L(t) & 0 \\ b_L(t) & c_L(t) \end{pmatrix} \in \text{GL}(2, \mathbb{C}) , \quad (8.33)$$

where $a_L(t)$ and $b_L(t)$ obey the first-order differential equations

$$\dot{a}_L = b_L \frac{(\dot{v}_L, v_L)}{(v_L, u_L)} , \quad \dot{b}_L = -b_L \frac{\dot{v}_L, u_L)}{(v_L, u_L)} , \quad (8.34)$$

with $\dot{f} := df(t)/dt$ for any function $f(t)$. This invariance follows from the fact that the Lagrangian $L^{(2)}_L(v_L)$ has the following properties: (i) it is a homogeneous function of $v^i$ of degree two; and (ii) it satisfies the analyticity condition $D^{(1)i}_{\alpha} L^{(2)}_L(v_L) = 0$. Due to the property (ii) it turns out that the action $\{8.31\}$ is invariant under the standard $\mathcal{N} = 4$ super-Poincaré transformations in three dimensions $[40]$

Our goal is to generalize the above construction to the $\mathcal{N} = 4$ AdS case. Let $z^M = (x^m, \theta^i_{ij})$ be local coordinates of the AdS superspace. Given a tensor superfield $U(x, \theta)$, we define its restriction to the body of the superspace, $\theta^i_{ij} = 0$, according to the rule

$$U|| := U(x, \theta)|_{\theta^i_{ij} = 0} . \quad (8.35)$$

We also define the double-bar projection of the covariant derivatives

$$D_A|| := E_A^M|| \partial_M + \frac{1}{2} \Omega_A^{bc}|| M_{bc} + \Phi_A^{kl}|| L_{kl} + \Phi_A^{\bar{k}\bar{l}}|| R_{\bar{k}\bar{l}} . \quad (8.36)$$

Recall that for all the AdS geometries, $[D_a, D_b] = -4 S^2 M_{ab}$. This allows us to use general coordinate and local structure group transformations to choose a (Wess-Zumino) gauge in which

$$D_a|| = \nabla_a = e^m_a (x) \partial_m + \frac{1}{2} \omega_a^{bc} (x) M_{bc} , \quad (8.37)$$

The $D^{(-4)}_L$ operator depends on the isotwistor $v^i(t)$, which varies along the integration contour, but also on a constant ($t$-independent) isotwistor $u_i$ which is chosen such that $v_i(t)$ and $u_i$ are everywhere linearly independent along the contour $\gamma$, that is $(v_L(t), u_L) \neq 0$. The flat action $\{8.31\}$ is actually independent of $u_i$, since it proves to be invariant under arbitrary projective transformations of the form

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where $a_L(t)$ and $b_L(t)$ obey the first-order differential equations

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with $\dot{f} := df(t)/dt$ for any function $f(t)$. This invariance follows from the fact that the Lagrangian $L^{(2)}_L(v_L)$ has the following properties: (i) it is a homogeneous function of $v^i$ of degree two; and (ii) it satisfies the analyticity condition $D^{(1)i}_{\alpha} L^{(2)}_L(v_L) = 0$. Due to the property (ii) it turns out that the action $\{8.31\}$ is invariant under the standard $\mathcal{N} = 4$ super-Poincaré transformations in three dimensions $[40]$

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$$D_a|| = \nabla_a = e^m_a (x) \partial_m + \frac{1}{2} \omega_a^{bc} (x) M_{bc} , \quad (8.37)$$
where \( \nabla_a \) stands for the covariant derivative of anti-de Sitter space \( \text{AdS}_3 \),

\[
[\nabla_a, \nabla_b] = -4S^2 \mathcal{M}_{ab} .
\]

We would like to construct an AdS generalization of the action (8.31) that describes (8.20) in a form where all the Grassmann variables have been integrated out. We expect

\[
S[\mathcal{L}_L^{(2)}] = S_0 + \cdots , \quad S_0 = \frac{1}{2\pi} \oint \gamma v_i dv^i \int d^3x e \mathcal{D}_L \mathcal{D}_L^{(2)} \, ,
\]

where \( S_0 \) is the covariantization of the Minkowski result (8.31), with \( e := \det(e^a_m) \). The ellipsis stands for curvature-dependent corrections that are necessary for the action to be invariant under the symmetries of its parent action (8.20). Note that both the flat action (8.31) and the parent curved full superspace action (8.20) are manifestly projective invariant (8.33). On the other hand, \( S_0 \) is not. Projective invariance can be used as a tool to iteratively complete \( S_0 \) to \( S[\mathcal{L}_L^{(2)}] \). We describe this procedure in Appendix B, which the interested reader may consult. The final form is

\[
S[\mathcal{L}_L^{(2)}] = \oint \gamma \frac{(dv_L, v_L)}{2\pi} \int d^3x e \left[ \mathcal{D}_L \mathcal{D}_L^{(2)} - \frac{1}{3} S^{(-2)kl} \mathcal{D}_k \mathcal{D}_l^{(-2)} - 2S^{(-2)kl} S_{kl}^{(-2)} \right] \mathcal{L}_L^{(2)} \, .
\]

Here we have used

\[
S^{(-2)ij} := \frac{u_i u_j}{(v_L, u_L)^2} S^{ij} ,
\]

and presented the action in a form valid for all \( N = 4 \) AdS superspaces. To pick up a specific AdS background, one should choose the appropriate value of the curvature \( S^{ij} \) according to (7.4a)–(7.4c).

One can obtain the action principle for right projective Lagrangians by applying the mirror map of [19] to (8.40).

9 Relating \( N = 3 \) and \( N = 4 \) supersymmetric sigma models

In three-dimensional Minkowski space, \( N = 3 \) supersymmetry for \( \sigma \)-models is known to imply \( N = 4 \) supersymmetry [2]. The standard argument (see, e.g., [55]) goes as follows: \( N \)-extended supersymmetry requires \( N - 1 \) anti-commuting complex structures. In the case \( N = 3 \), the target space has two such complex structures, \( I \) and \( J \). Their
product \( K := I J \) is a third complex structure which anticommutes with \( I \) and \( J \), and therefore the \( \sigma \)-model is \( \mathcal{N} = 4 \) supersymmetric. This argument tells us nothing about off-shell supersymmetry. There have been developed alternative proofs \cite{40} which are intrinsically off-shell and make use of the formulations for general \( \mathcal{N} = 3 \) supersymmetric \( \sigma \)-models in terms of (i) \( \mathcal{N} = 2 \) chiral superfields; and (ii) \( \mathcal{N} = 3 \) polar supermultiplets.

The goal of this section is to understand whether a \((p,q)\) supersymmetric \( \sigma \)-model in \( \text{AdS}_3 \) with \( \mathcal{N} = p + q = 3 \) actually possesses an enhanced \((p',q')\) supersymmetry with \( \mathcal{N}' = p' + q' = 4 \). We will see that this is always the case, and the necessary (but not sufficient) conditions are \( p' \geq p \) and \( q' \geq q \). To prove that, in this section we will use both \( \mathcal{N} = 2 \) and \( \mathcal{N} = 3 \) superfields.

### 9.1 (2,0) AdS superspace approach: Formulation in terms of \( \mathcal{N} = 3 \) polar supermultiplets

The starting point of our analysis is the transformation law of the \( \mathcal{N} = 4 \) left arctic supermultiplet of weight-\( n \) given in eq. (8.10a). We split it into two different transformations:

\[
\delta_\tau \Upsilon^{[n]}_L = \left\{ \tau + i t \left( \zeta \frac{\partial}{\partial \zeta} - \frac{n}{2} \right) \right\} \Upsilon^{[n]}_L ;
\]

\[
\delta_\varepsilon \Upsilon^{[n]}_L = \left\{ \zeta \varepsilon^\alpha \mathcal{D}_\alpha - \frac{1}{\zeta} \varepsilon_\alpha \mathcal{D}^\alpha - 4S \left( \varepsilon_L \zeta + \frac{1}{\zeta} \varepsilon_L \right) \zeta \frac{\partial}{\partial \zeta} + 4Sn \zeta \varepsilon_L \right\} \Upsilon^{[n]}_L
+ i \sigma \left( \zeta \frac{\partial}{\partial \zeta} - \frac{n}{2} \right) \Upsilon^{[n]}_L .
\]

We recall that the parameters \( \tau = \tau^a \mathcal{D}_a + \tau^a \mathcal{D}_a + \bar{\tau}_a \bar{\mathcal{D}}^a \) and \( t \) describe the isometry transformations of the (2,0) AdS superspace. The other parameters \( (\varepsilon^a, \varepsilon_\alpha, \sigma, \varepsilon_L, \varepsilon_R) \) as well as \( (\varepsilon_R, \bar{\varepsilon}_R) \), which appear in the transformation laws of right projective supermultiplets, are associated with the remaining two supersymmetries and the residual \( R \)-symmetry.

#### 9.1.1 (3,0) AdS supersymmetry implies (4,0) AdS supersymmetry

In the case of (4,0) AdS supersymmetry, the equations obeyed by the parameters \( (\varepsilon^a, \varepsilon_\alpha, \sigma, \varepsilon_L, \varepsilon_R, \varepsilon_L, \varepsilon_R) \) follow from (7.27) by setting \( X = 0 \):

\[
\mathcal{D}_a \varepsilon_\beta = 4S \varepsilon_{a\alpha} \varepsilon_L , \quad \bar{\mathcal{D}}_a \varepsilon_\beta = -4S \varepsilon_{a\alpha} \varepsilon_L , \quad (9.2a)
\]

\[
\mathcal{D}_a \varepsilon_\beta = -4S \varepsilon_{a\alpha} \varepsilon_R , \quad \bar{\mathcal{D}}_a \varepsilon_\beta = 4S \varepsilon_{a\alpha} \varepsilon_R , \quad (9.2b)
\]

\[
\mathcal{D}_a \varepsilon_L = \mathcal{D}_a \varepsilon_R = 0 , \quad \mathcal{D}_a \varepsilon_L = i \varepsilon_\alpha , \quad \mathcal{D}_a \varepsilon_R = -i \varepsilon_\alpha . \quad (9.2c)
\]
In accordance with eq. (7.28), the action of the U(1) generator (7.20) on these parameters is as follows:

\[ J\varepsilon_\alpha = 0 , \quad J\varepsilon_L = -\varepsilon_L , \quad J\varepsilon_R = -\varepsilon_R . \]  

Finally, the relation (7.29) tells us that the parameter \( \sigma \) is constant.

In general, the supersymmetry parameters \( \varepsilon_\alpha \) and \( \bar{\varepsilon}_\alpha \) are linearly independent. Let us now consider a special choice

\[ \bar{\varepsilon}_\alpha = -\varepsilon_\alpha = i\rho_\alpha , \quad \bar{\rho}_\alpha = \rho_\alpha . \]  

For this specific case we also denote

\[ \rho \equiv \varepsilon_L , \quad \bar{\rho} \equiv \bar{\varepsilon}_L . \]  

Then it follows from the relations (9.2) that the (3,0) AdS identities (3.7) hold. If we also choose

\[ \sigma = 0 , \]  

then the (4,0) AdS supersymmetry transformation law (9.1b) reduces to the (3,0) one, eq. (3.12), if the hypermultiplet weight is \( n = 1 \). Moreover, for \( n = 1 \) it is easy to see that a finite U(1) transformation generated by the constant parameter \( \sigma \) in (9.1b) coincides with the U(1) symmetry (3.10) of the off-shell (3,0) supersymmetric \( \sigma \)-model (3.5). Now, it is an instructive exercise to check explicitly that the (3,0) supersymmetric \( \sigma \)-model (3.5) is invariant under the (4,0) supersymmetry transformation (9.1b) with \( n = 1 \). The solution to this problem is actually given in section 8.4.1. We conclude that any off-shell (3,0) supersymmetric \( \sigma \)-model in AdS\(_3\) possesses (4,0) off-shell supersymmetry. This theory is actually \( \mathcal{N} = 4 \) superconformal.

### 9.1.2 (3,0) AdS supersymmetry implies (3,1) AdS supersymmetry

In the case of (3,1) AdS supersymmetry, the equations obeyed by the parameters \( (\varepsilon^0, \bar{\varepsilon}_\alpha, \varepsilon_L, \bar{\varepsilon}_L, \varepsilon_R, \bar{\varepsilon}_R) \) are given in section 7.2.2. We recall that the parameter \( \sigma \) is absent in this case.

We can consider a special (3,1) supersymmetry transformations defined by the conditions (9.4). Using the identification (9.5), we then deduce from the relations (7.33) that the (3,0) AdS identities (3.7) hold. As a result, the (3,1) AdS supersymmetry transformation law (9.1b) reduces to the (3,0) one, eq. (3.12), if the hypermultiplet weight is
9.1.3 (2,1) AdS supersymmetry implies (2,2) AdS supersymmetry

In the case of (2,2) AdS supersymmetry, the parameters $\epsilon_L$ and $\epsilon_R$ vanish, while the complex supersymmetry parameter $\epsilon_\alpha$ is almost constant, that is, it obeys the constraints \((7.36)\). Finally, the $R$-symmetry parameter $\sigma$ is constant.

We consider a special (2,2) supersymmetry transformation defined by the conditions \((9.4)\). In accordance with \((7.36)\), the real spinor parameter $\rho_\alpha$ obeys the (2,1) constraints \((5.3)\). Let us also choose $\sigma = 0$. Then the (2,2) AdS supersymmetry transformation law \((9.1b)\) reduces to the (2,1) one, eq. \((5.7)\), in the case of weight-zero hypermultiplets, $n = 0$. For $n = 0$, the finite U(1) transformation of $\Upsilon(\zeta)$ generated by the parameter $\sigma$ in \((9.1b)\) coincides with the U(1) symmetry \((5.6)\) of the most general off-shell (2,1) supersymmetric $\sigma$-model \((5.1)\). It is a simple exercise to show that this $\sigma$-model action is invariant under the (2,2) AdS supersymmetry transformation \((9.1b)\). We conclude that any off-shell (2,1) supersymmetric $\sigma$-model in AdS$_3$ possesses (2,2) off-shell supersymmetry.

We recall that the most general off-shell $\sigma$-model with (2,1) AdS supersymmetry is described by the action \((5.1)\), with $K(\Phi, \bar{\Phi})$ being a general real analytic function. An important subclass of these theories corresponds to the case that $K(\Phi, \bar{\Phi})$ obeys the homogeneity conditions

$$
\Phi^I \frac{\partial}{\partial \Phi^I} K(\Phi, \bar{\Phi}) = K(\Phi, \bar{\Phi}) , \quad \bar{\Phi}^{\bar{I}} \frac{\partial}{\partial \bar{\Phi}^{\bar{I}}} K(\Phi, \bar{\Phi}) = K(\Phi, \bar{\Phi}) .
$$

The resulting $\sigma$-model is $\mathcal{N} = 3$ superconformal. In fact, it possesses $\mathcal{N} = 4$ superconformal invariance. The latter observation means that such a $\sigma$-model can support each of the three $\mathcal{N} = 4$ AdS supersymmetries: (4,0), (3,1) and (2,2).

9.2 (2,0) AdS superspace approach: Formulation in terms of $\mathcal{N} = 2$ chiral superfields

Next we turn to the most general $\sigma$-models in (2,0) AdS superspace, eq. \((2.25)\), possessing two additional supersymmetries. We have already described in section \(7\) how
these two supersymmetries and the residual $R$-symmetry transformations are described in terms of the parameters $(\varepsilon^a, \varepsilon_\alpha, \sigma, \varepsilon_L, \varepsilon_R)$. Similar to the discussion in the previous subsection, the parameter $\sigma$ can be neglected in our discussion. It is useful to decompose the complex spinor parameter $\varepsilon_\alpha$ as

$$\varepsilon_\alpha = -i\rho_\alpha + \rho'_\alpha, \quad \bar{\varepsilon}_\alpha = i\rho_\alpha + \rho'_\alpha,$$

where $\rho_\alpha$ and $\rho'_\alpha$ are both real spinors. Due to the constraints on $\rho_\alpha$ and $\rho'_\alpha$, we may introduce complex parameters $\rho$ and $\rho'$ which obey

$$\rho_\alpha = \mathcal{D}_\alpha \rho = \bar{\mathcal{D}}_\alpha \bar{\rho}, \quad \rho'_\alpha = \mathcal{D}_\alpha \rho' = \bar{\mathcal{D}}_\alpha \bar{\rho}' .$$

(9.9)

When $\rho'_\alpha$ vanishes, an $\mathcal{N} = 4$ Killing transformation reduces to an $\mathcal{N} = 3$ transformation with spinor parameter $\rho_\alpha$.

We have already shown that any (2,0) $\sigma$-model in AdS$_3$ with the $\mathcal{N} = 3$ supersymmetry transformation

$$\delta \phi^a = \frac{i}{2} \bar{\mathcal{D}}^2 (\bar{\rho} \Omega^a)$$

(9.10)

imposes the requirement that the target space be hyperkähler. In the (3,0) case, the target space must be a hyperkähler cone. In the (2,1) case, the target space must possess a U(1) isometry which acts as a rotation on the complex structures. In both cases, we may choose $\Omega^a = \omega^{ab} K_b$.

It turns out that, just as in the off-shell models, it is always possible to impose $\mathcal{N} = 4$ supersymmetry without any further restrictions on the target space geometry. In particular, we may lift (3,0) to either (4,0) or (3,1), while (2,1) may be lifted only to (2,2). In each case, the full extended supersymmetry transformation is simply

$$\delta \phi^a = \frac{i}{2} \bar{\mathcal{D}}^2 \left( (\bar{\rho} \pm i\bar{\rho}') \Omega^a \right), \quad \Omega^a = \omega^{ab} K_b ,$$

(9.11)

where a plus sign is associated with a left $\mathcal{N} = 4$ hypermultiplet and a minus sign with a right $\mathcal{N} = 4$ hypermultiplet. Let us now prove this for each case.

### 9.2.1 (3,0) AdS supersymmetry implies (4,0) AdS supersymmetry

We begin with the lift of (3,0) AdS supersymmetry to (4,0). The parameters $\rho$ and $\rho'$ for a (4,0) isometry transformation can be taken to obey

$$\rho = \frac{1}{2} (\varepsilon_L + \varepsilon_R), \quad \rho' = -i \frac{1}{2} (\varepsilon_L - \varepsilon_R).$$

(9.12)
One can then check that $\rho$ obeys the conditions (3.7) consistent with an extended (3,0) parameter. Remarkably, $\rho'$ obeys the same conditions. To prove invariance of the action under (9.11), we need only recycle the proof already given for the (3,0) case. The only change is the phase of the extended supersymmetry transformation, which corresponds to a phase rotation of the complex structure. We immediately conclude that the action must remain invariant under the full (4,0) AdS supersymmetry.

9.2.2 (3,0) AdS supersymmetry implies (3,1) AdS supersymmetry

Next, we address the (3,1) lift of (3,0) AdS supersymmetry. The (3,1) Killing parameters obey $\varepsilon_L = \varepsilon_R = \varepsilon$, and we may choose

$$\rho = \frac{1}{2} \varepsilon . \quad (9.13)$$

Once again, $\rho$ obeys the (3,0) conditions (3.7). However, $\rho'_\alpha$ is nearly constant, so $\rho'$ only needs to obey the conditions

$$\mathcal{D}^2 \rho' = 0 , \quad \mathcal{D}_\alpha \mathcal{D}_\beta \rho' = 0 . \quad (9.14)$$

Since $\rho'$ is no longer quite the same sort of parameter as $\rho$, it is a minor task to check that the action is indeed invariant under (9.11), which we leave as an exercise to the reader. We conclude any on-shell $\sigma$-model with (3,0) supersymmetry can be lifted to (3,1).

9.2.3 (2,1) AdS supersymmetry implies (2,2) AdS supersymmetry

Our last case to consider is the lift of (2,1) to (2,2). For a (2,2) isometry transformation, the parameters $\varepsilon_L$ and $\varepsilon_R$ both vanish with $\rho$ and $\rho'$ obeying the identical conditions (5.10) and (9.14). To prove invariance of the action, we again need only recycle the proof given for the (2,1) case since the only modification is a change in phase of the complex structure. We conclude that an on-shell $\sigma$-model with (2,1) AdS supersymmetry may always be lifted to (2,2).

9.3 $\mathcal{N} = 3$ AdS superspace approach

Let us now consider a theory with manifest off-shell $\mathcal{N} = 3$ AdS supersymmetry written in projective superspace. Depending on whether the action possesses (3,0) or (2,1) supersymmetry, the natural weight for the $\mathcal{N} = 3$ multiplets and the conditions on
the Lagrangian will be different. For the moment, we will remain with a general case: a weight-two projective Lagrangian $\mathcal{L}^{(2)}$ depending on weight-$n$ projective multiplets $Q^{(n)}$, their smile-conjugates, and possibly the isotwistor $v^i$ as well.

We have already seen in section 8.3 that an $\mathcal{N} = 4$ theory constructed out of a Lagrangian $\mathcal{L}_L^{(2)}$ built from left projective multiplets $Q_L^{(n)}$ can be reduced to $\mathcal{N} = 3$ via a projection operation

$$Q^{(n)} = Q_L^{(n)}|_{e^{\theta_j} = 0}, \quad \mathcal{L}^{(2)} = \mathcal{L}_L^{(2)}|_{e^{\theta_j} = 0}. \quad (9.15)$$

The $\mathcal{N} = 4$ isometry transformation on $Q^{(n)}$ is decomposed into the sum of an $\mathcal{N} = 3$ Killing transformation and an extended supersymmetry transformation, the latter being

$$\delta_\Omega Q^{(n)} = \Delta^{(4)} \left( \Omega^{(-2)} \partial^{(-2)} Q^{(n)} + \frac{n}{2} (\partial^{(-2)} \Omega^{(-2)}) Q^{(n)} \right). \quad (9.16)$$

The $v^i$-dependent parameter $\Omega^{(-2)}$ was required to obey a number of conditions encoded in (8.16) and (7.50a)–(7.50b).

It is a simple exercise to reverse this procedure. Given any $\mathcal{N} = 3$ theory, we may simply impose (9.16) as the extended supersymmetry transformation for the multiplet $Q^{(n)}$. This constitutes its $\mathcal{N} = 4$ lift and completely determines the left projective multiplet $Q_L^{(n)}$ into which $Q^{(n)}$ may be encoded.\footnote{One can also perform a lift to a right projective multiplet. The only difference is the overall sign of the transformation law \ref{9.16}.} Naturally, we expect only specific lifts to be possible: (3,0) certainly cannot be lifted to (2,2), nor can (2,1) be lifted to (4,0), but most other possibilities are allowed. We discuss (and demonstrate) these possibilities below.

### 9.3.1 (3,0) AdS supersymmetry implies (4,0) and (3,1) AdS supersymmetry

A $\sigma$-model with off-shell (3,0) supersymmetry is described in projective superspace by a weight-two Lagrangian $\mathcal{L}^{(2)}$ that is a homogeneous function of weight-one polar multiplets, $\Upsilon^{(1)I}$ and $\hat{\Upsilon}^{(1)J}$:

$$\left( \Upsilon^{(1)I} \frac{\partial}{\partial \Upsilon^{(1)I}} + \hat{\Upsilon}^{(1)J} \frac{\partial}{\partial \hat{\Upsilon}^{(1)J}} \right) \mathcal{L}^{(2)} = 2 \mathcal{L}^{(2)}. \quad (9.17)$$

We impose

$$\delta_\Omega \Upsilon^{(1)I} = \Delta^{(4)} \left( \Omega^{(-2)} \partial^{(-2)} \Upsilon^{(1)I} + \frac{1}{2} (\partial^{(-2)} \Omega^{(-2)}) \right) \Upsilon^{(1)I}. \quad (9.18)$$
The parameter $\Omega^{-2}$ may correspond to either a (4,0) or a (3,1) extended supersymmetry. Depending on the situation, the parameters $\Xi_\alpha$ and $\tilde{\Lambda}^{ij}$ defined in (8.16) obey different conditions \((7.50a)-(7.50b)\). Regardless of the case in question, it is easy to prove that the off-shell $\mathcal{N} = 3$ action possesses an extended supersymmetry. Due to the homogeneity condition, the Lagrangian $\mathcal{L}^{(2)}$ must obey

$$
\delta_\Omega \mathcal{L}^{(2)} = \Delta^{(4)} \left( \Omega^{-2} \partial^{(-2)} + (\partial^{(-2)} \Omega^{-2}) \right) \mathcal{L}^{(2)} = \Delta^{(4)} \partial^{(-2)} \left( \Omega^{-2} \mathcal{L}^{(2)} \right)
$$

and one can prove invariance of the action along the same lines as in section 8.3. We therefore conclude that any off-shell (3,0) Lagrangian is actually invariant under both off-shell (4,0) and (3,1) supersymmetries.

9.3.2 (2,1) AdS supersymmetry implies (2,2) AdS supersymmetry

A general off-shell $\sigma$-model with (2,1) supersymmetry is given by a Lagrangian

$$
\mathcal{L}^{(2)} = w^{(2)} \mathcal{L}(\Upsilon^I, \tilde{\Upsilon}^j)
$$

where $\Upsilon^I$ and $\tilde{\Upsilon}^j$ are weight-zero polar multiplets. No homogeneity condition is imposed on $\mathcal{L}$, and so the action need not be superconformal. To perform a (2,2) lift, we require the extended supersymmetry transformation

$$
\delta_\Omega \Upsilon^I = \Delta^{(4)} \left( \Omega^{-2} \partial^{(-2)} \Upsilon^I \right),
$$

which leads to

$$
\delta_\Omega \mathcal{L}^{(2)} = w^{(2)} \delta_\Omega \mathcal{L} = \Delta^{(4)} \partial^{(-2)} \left( \Omega^{-2} \mathcal{L}^{(2)} \right) - v_i v_j \tilde{\Lambda}^{ijk}_k \mathcal{L}.
$$

The first term involving the analytic projector vanishes under the $\mathcal{N} = 3$ measure as before. The second term, which involves $w^{ij}$ explicitly, vanishes for the (2,2) case alone where $\tilde{\Lambda}^{ij} \propto w^{ij}$.

10 (4,0) supersymmetric sigma models with $X \neq 0$: Off-shell approach

The results of section 9 imply that there is only one class of $\mathcal{N} = 4$ supersymmetric $\sigma$-models in AdS$_3$ which requires a separate study. It consists of all $\sigma$-models possessing
(4,0) AdS supersymmetry with $X \neq 0$. These theories cannot be realized as $\mathcal{N} = 3$ supersymmetric $\sigma$-models in AdS$_3$, and therefore the analysis of sections 3–6 is not applicable. In this and subsequent sections we provide a detailed study of the $\sigma$-models possessing (4,0) AdS supersymmetry with $X \neq 0$.

Let us begin by recalling the algebra of covariant derivatives for the (4,0) AdS superspace with $X \neq 0$:

$$\{\mathcal{D}_\alpha^i, \mathcal{D}_\beta^j\} = 2i \varepsilon^{ij} \varepsilon^{\bar{ij}} \mathcal{D}_\alpha \beta - 4i \varepsilon^{ij} \varepsilon^{\bar{ij}} S M_{\alpha \beta}$$

$$+ 2i \varepsilon_{\alpha \beta} \varepsilon^{\bar{ij}} (2S + X) \mathbf{L}^{ij} + 2i \varepsilon_{\alpha \beta} \varepsilon^{ij} (2S - X) \mathbf{R}^{ij},$$

(10.1a)

$$[\mathcal{D}_\alpha, \mathcal{D}_\beta^k] = -2S \varepsilon_\gamma (\alpha \mathcal{D}_\beta^k),$$

(10.1b)

$$[\mathcal{D}_a, \mathcal{D}_b] = -4S^2 M_{\alpha \beta}.$$

(10.1c)

For $|X| \neq 2S$, the $R$-symmetry part of the superspace holonomy group (or simply the $R$-holonomy group in what follows) is $\text{SU}(2)_L \times \text{SU}(2)_R$. However, when $X = \pm 2S$, which we will refer to as the critical cases, either the $\text{SU}(2)_L$ or the $\text{SU}(2)_R$ curvature vanishes and one may adopt a gauge where the corresponding connection is zero.

This algebra possesses a (2,0) AdS subalgebra corresponding to the choice $\mathcal{D}_\alpha = \mathcal{D}_\alpha^{11}$ and $\mathcal{D}_a = -\mathcal{D}_a^{22}$:

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = \{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = 0, \\
\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = -2i \mathcal{D}_\alpha \beta + 4i S M_{\alpha \beta} - 4i S \varepsilon_{\alpha \beta} \mathcal{J},$$

(10.2a)

$$[\mathcal{D}_\alpha, \mathcal{D}_\beta] = -2S \varepsilon_\gamma (\alpha \mathcal{D}_\beta), \\
[\mathcal{D}_a, \mathcal{D}_b] = -4S^2 M_{\alpha \beta}.$$

(10.2b)

where the $\text{U}(1)_R$ generator $\mathcal{J}$ is given by (7.20), which we repeat here in a slightly modified form:

$$\mathcal{J} = \left(1 + \frac{X}{2S}\right) \mathbf{L}^{12} + \left(1 - \frac{X}{2S}\right) \mathbf{R}^{12}.$$

(10.3)

For later convenience, we will introduce operators

$$\mathcal{J}_L := \mathbf{L}^{12}, \\
\mathcal{J}_R := \mathbf{R}^{12}.$$

(10.4)

In the critical cases of $X = \pm 2S$, we find the particularly simple results $\mathcal{J} = 2\mathcal{J}_L$ or $\mathcal{J} = 2\mathcal{J}_R$.

Because of the potential truncation of the $R$-holonomy group, it is clear that there are essentially two cases of interest. The first is the general non-critical case where $|X| \neq 2S$ and the full $R$-holonomy group is $\text{SU}(2)_L \times \text{SU}(2)_R$, and the second is when $X = 2S$ and only $\text{SU}(2)_L$ remains. (The case $X = -2S$ is found by applying the mirror map.)
10.1 Off-shell non-critical (4,0) sigma models

Let us first discuss the non-critical case, where $|X| \neq 2S$. The general off-shell (4,0) supersymmetric $\sigma$-model can be written in (2,0) AdS superspace as

$$S = \int d^3x \, d^4\theta \, E (\mathbb{L}_L + \mathbb{L}_R).$$  \hspace{1cm} (10.5)

The term $\mathbb{L}_L$ is a Lagrangian constructed via a contour integral

$$\mathbb{L}_L = \oint_C \frac{d\zeta}{2\pi i} \mathcal{L}_L^{(2)},$$  \hspace{1cm} (10.6)

from the left Lagrangian $\mathcal{L}_L^{(2)}(\Upsilon_L, \check{\Upsilon}_L; \zeta)$ which depends on left arctic *weight-one* superfields $\Upsilon_L^I(\zeta)$ and their smile-conjugates $\check{\Upsilon}_L^I(\zeta)$. In some cases, $\mathcal{L}_L^{(2)}$ may also explicitly depend on the complex variable $\zeta$. Similar statements pertain to the right Lagrangian $\mathbb{L}_R$. The left and right sectors are completely decoupled (without higher-derivative couplings); we may even choose one of the sectors to vanish. For this reason, we can focus our analysis for the moment on the left sector.

We have to add some projective-superspace details regarding the left $\sigma$-model sector. The dynamical variables are weight-one arctic multiplets $\Upsilon_L^{(1)I}(v_L)$ and their smile-conjugate antarctic multiplets $\check{\Upsilon}_L^{(1)I}(v_L)$. The $\sigma$-model Lagrangian $\mathcal{L}_L^{(2)}$ defining the (4,0) supersymmetric action (8.20) is

$$\mathcal{L}_L^{(2)} = i \mathfrak{R}_L (\Upsilon_L^{(1)}, \check{\Upsilon}_L^{(1)}),$$  \hspace{1cm} (10.7)

where the function $\mathfrak{R}_L$ obeys the homogeneity condition

$$\left( \Upsilon_L^{(1)I} \frac{\partial}{\partial \Upsilon_L^{(1)I}} + \check{\Upsilon}_L^{(1)I} \frac{\partial}{\partial \check{\Upsilon}_L^{(1)I}} \right) \mathfrak{R}_L = 2 \mathfrak{R}_L$$  \hspace{1cm} (10.8)

which guarantees that $\mathcal{L}_L^{(2)}$ has weight two. The Lagrangian $\mathcal{L}_L^{(2)}$ has to be real under the smile-conjugation, which restricts $\mathfrak{R}_L$ to obey a reality condition of the type (3.5c).

To reformulate the (4,0) supersymmetric action (8.20) in (2,0) AdS superspace, as in equation (8.22), we recall that all projective supermultiplets should be recast as functions of $\zeta$, using a prescription $Q_L^{(n)}(v_L) \to Q_L^{(n)}(\zeta) \propto Q_L^{(n)}(v_L)$. For the Lagrangian $\mathcal{L}_L^{(2)}(v_L)$, we have to use the rule (8.11). For the dynamical superfields $\Upsilon_L^{(1)}(v_L)$ and $\check{\Upsilon}_L^{(1)}(v_L)$, we have to use the rules (8.7) and (8.8) respectively. Since we are going to work only with weight-one hypermultiplets, we will denote $\Upsilon_L^{(1)}(\zeta)$ and $\check{\Upsilon}_L^{(1)}(\zeta)$ simply as $\Upsilon_L(\zeta)$ and

73
As a result, for the Lagrangian $\L^{[2]}(\zeta)$ we end up with the following expression:

$$\L^{[2]}(\Upsilon_L, \Tilde{\Upsilon}_L; \zeta) = \frac{1}{\zeta} \mathbf{R}_L(\Upsilon_L, \zeta \Tilde{\Upsilon}_L) .$$  \hspace{1cm} (10.9)

The arctic $\Upsilon^I_L(\zeta)$ and antarctic $\Tilde{\Upsilon}^I_L(\zeta)$ multiplets are given by Taylor and Laurent series in $\zeta$ respectively, with the coefficients being $\mathcal{N} = 2$ superfields,

$$\Upsilon^I_L(\zeta) = \sum_{n=0}^{\infty} \zeta^n \Upsilon^I_{Ln} = \Phi^I_L + \zeta \Sigma^I_L + \cdots , \quad \Tilde{\Upsilon}^I_L(\zeta) = \sum_{n=0}^{\infty} (-\zeta)^{-n} \Tilde{\Upsilon}^I_{Ln} .$$  \hspace{1cm} (10.10)

The component superfields $\Phi^I_L := \Upsilon^I_{L0}$ and $\Sigma^I_L := \Upsilon^I_{L1}$ are chiral and complex linear respectively. These correspond to the physical fields while the remaining superfields are auxiliary. The extended supersymmetry transformation of $\Upsilon^I_L$ (8.10a) implies the transformations

$$\delta \Phi^I_L = -\epsilon_\alpha \bar{D}^\alpha \Sigma^I_L - 4S \bar{\epsilon}_L \Sigma^I_L = -\frac{1}{2} \bar{D}^2(\bar{\rho}_L \Sigma^I_L) ,$$  \hspace{1cm} (10.11a)

$$\delta \Sigma^I_L = (\epsilon^\alpha D_\alpha + 4S \epsilon_L) \Phi^I_L - \bar{D}_\alpha (\epsilon^\alpha \Upsilon^I_{L2})$$  \hspace{1cm} (10.11b)

on the physical fields. For convenience, we have introduced the parameter $\bar{\rho}_L$ which is defined by the condition

$$\epsilon_\alpha = \bar{D}_\alpha \bar{\rho}_L , \quad \bar{\epsilon}_\alpha = D_\alpha \rho_L .$$  \hspace{1cm} (10.12)

In the non-critical case we are discussing here, due to eq. (7.27c) we can always choose

$$\rho_L = \frac{2i}{2 - X/S} \bar{\epsilon}_R , \quad \bar{\rho}_L = -\frac{2i}{2 - X/S} \epsilon_R .$$  \hspace{1cm} (10.13)

This off-shell formulation with an infinite number of auxiliary fields is rather elaborate. We can simplify the theory by expressing the auxiliary superfields $\Upsilon^I_{L2}, \Upsilon^I_{L3}, \ldots$ in terms of the physical ones, using their equations of motion

$$\frac{\partial \mathbb{L}}{\partial \Upsilon^I_{Ln}} = 0 , \quad n = 2, 3, \ldots$$  \hspace{1cm} (10.14)

The price for such a simplification is that $\mathcal{N} = 4$ supersymmetry is no longer off-shell. This leaves the Lagrangian $\mathbb{L}_L$ depending only on $\Phi_L, \Sigma_L$ and their complex conjugates. As a consequence of the homogeneity condition (10.8),

$$\left( \Phi^I_L \frac{\partial}{\partial \Phi^I_L} + \Sigma^I_L \frac{\partial}{\partial \Sigma^I_L} \right) \mathbb{L}_L = \mathbb{L}_L .$$  \hspace{1cm} (10.15)
Next, we dualize the complex linear superfields $\Sigma^I_L$ and their conjugates $\bar{\Sigma}^{\bar{I}}_L$ into chiral superfields $\Psi^L_I$ and their conjugates $\bar{\Psi}^{\bar{L}}_{\bar{I}}$ via a Legendre transformation and arrive at the dual action
\[
S_{\text{dual}} = \int d^3x \ d^4\theta \ E_L (\Phi_L, \Psi_L, \bar{\Phi}_L, \bar{\Psi}_L) , \quad K_L = L_L + \Sigma^I_L \Psi_{L}^I + \bar{\Sigma}^{\bar{I}}_L \bar{\Psi}^{\bar{L}}_{\bar{I}} . \tag{10.16}
\]

The target space of this $\sigma$-model is a hyperkähler cone. The cone condition follows from the fact that Kähler potential $K_L$ obeys the homogeneity condition
\[
\chi^a_L \frac{\partial}{\partial \phi^a_L} K_L = K_L , \quad \chi^a_L = \phi^a_L = (\Phi^I_L, \Psi^L_I) , \tag{10.17}
\]
as a consequence of (10.15). The target space is hyperkähler since it possesses a covariantly constant holomorphic two-form $\omega_{ab}$ given by
\[
\omega_{ab} = \begin{pmatrix} 0 & \delta_I^j \\ -\delta_J^I & 0 \end{pmatrix} \tag{10.18}
\]
with the property $\omega^{ab} \omega_{bc} = -\delta^a_c$, where $\omega^{ab} = g^{ac} g^{bd} \bar{\omega}_{cd}$. The action (10.16) proves to be invariant under the extended supersymmetry transformation
\[
\delta \Phi^I_L = -\frac{1}{2} \mathcal{D}^2 \left( \bar{\rho}_L \frac{\partial K_L}{\partial \Phi^I_L} \right) , \quad \delta \Psi^L_I = \frac{1}{2} \mathcal{D}^2 \left( \bar{\rho}_L \frac{\partial K_L}{\partial \Phi^I_L} \right) , \tag{10.19}
\]
which we can cast in the target-space reparametrization-covariant form
\[
\delta \phi^a_L = -\frac{1}{2} \mathcal{D}^2 \left( \bar{\rho}_L \omega^{ab} \partial_b K_L \right) . \tag{10.20}
\]

A similar calculation may be carried out for the right sector; one finds an identical expression with the parameters $\rho_R$ and $\bar{\rho}_R$ given by
\[
\rho_R = \frac{2i}{2 + X/S} \varepsilon_L , \quad \bar{\rho}_R = -\frac{2i}{2 + X/S} \bar{\varepsilon}_L . \tag{10.21}
\]

### 10.2 Off-shell critical (4,0) sigma models

Now we turn to the critical case of $X = 2S$. Because the SU(2)$_R$ factor in the structure group has vanished, it is possible to introduce a frozen vector multiplet.

Let us consider a left $U(1)$ vector multiplet in (4,0) AdS superspace\(^{19}\) It is described by the covariant derivatives
\[
\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}^\dagger_a) = \mathcal{D}_A + i V_A Z , \quad [Z, \mathcal{D}_A] = [Z, \mathcal{D}_A] = [Z, \mathcal{J}_{ij}] = 0 , \tag{10.22}
\]

\(^{19}\)There are two types of vector multiplets in $\mathcal{N} = 4$ conformal supergravity, left and right ones [19]. The left vector multiplets are used to gauge left hypermultiplets and do not couple to right hypermultiplets. The field strength of a left Abelian vector multiplet is a right $\mathcal{O}(2)$ multiplet. The right vector multiplets are obtained from the left ones by applying the mirror map.
where \( V_A \) and \( Z \) are the U(1) gauge connection and generator respectively. The anticommutator of two spinor derivatives is given by

\[
\{ \mathcal{D}_\alpha^{ij}, \mathcal{D}_\beta^{\bar{ij}} \} = 2i\varepsilon^{ij}\varepsilon^{\bar{ij}}\mathcal{D}_{\alpha\beta} - 4iS\varepsilon^{ij}\varepsilon^{\bar{ij}}M_{\alpha\beta} + 8iS\varepsilon_{\alpha\beta}\varepsilon^{ij}L^{ij} + 2\varepsilon_{\alpha\beta}\varepsilon^{ij}W^{\bar{ij}}Z_L
\]  

(10.23)

where the gauge invariant field strength \( W^{\bar{ij}} = W^{\bar{ij}} \) is real, \( \mathcal{D}_\alpha^{ij} = W^{\bar{ij}} = \varepsilon_{ik}\varepsilon_{\bar{j}l}W^{k\bar{l}} \), and obeys the Bianchi identity

\[
\mathcal{D}_\alpha^{ij}\mathcal{D}_\beta^{\bar{ij}} = 0 .
\]  

(10.24)

We may take the field strength \( W^{\bar{ij}} \) to be covariantly constant, \( \mathcal{D}_\alpha^{ij}W^{k\bar{l}} = 0 \). Remarkably, unlike the situation in (2,1) AdS, there is no integrability condition to be satisfied since the SU(2)\(_R\) factor has been removed from the structure group. Thus \( W^{\bar{ij}} \) can be chosen completely arbitrarily. In fact, since we can assume that we have gauged away the SU(2)\(_R\) connection, we can take \( W^{\bar{ij}} \) to be any constant isotriplet. The remainder of the (4,0) algebra is then simply

\[
[\mathcal{D}_{\alpha\beta}, \mathcal{D}_\gamma^{ij}] = -2S\varepsilon_{\gamma\epsilon}(\mathcal{D}_\beta^{ij}) , \quad [\mathcal{D}_a, \mathcal{D}_b] = -4S^2M_{ab} .
\]  

(10.25)

The presence of the gauging requires a minor alteration to the (4,0) isometry,

\[
\delta = \xi^A\mathcal{D}_A + \frac{1}{2}\Lambda^{ab}M_{ab} + \Lambda^{ij}L_{ij} + i\lambda Z
\]  

(10.26)

where \( \lambda \) is the U(1) parameter. All the other Killing parameters obey the same conditions while \( \lambda \) obeys

\[
\mathcal{D}_a^{ij}\lambda = 2i\xi^i_{\alpha}W^{k\bar{j}} .
\]  

(10.27)

As before, we can recover a (2,0) subalgebra, which possesses the spinor anticommutator

\[
[\mathcal{D}_a, \mathcal{D}_b] = -2i\mathcal{D}_{a\beta} + 4iSM_{a\beta} - 8i\varepsilon_{a\beta}L^{12} - 2\varepsilon_{a\beta}W^{\bar{1}2}Z
\]  

(10.28)

where the effective U(1) generator \( \mathcal{J} \) is given by

\[
\mathcal{J} = 2\mathcal{J}_L - \frac{i}{2S}W^{\bar{1}2}Z
\]  

(10.29)

The U(1) parameter \( \lambda \) decomposes as

\[
\lambda = -\frac{i}{2S}W^{\bar{1}2}t + \lambda' .
\]  

(10.30)
where $t$ is the $U(1)_R$ parameter of the $(2,0)$ AdS isometry group and $\lambda'$ is associated with the extended supersymmetry. The parameter $\lambda'$ obeys

$$D_\alpha\lambda' = -2i\bar{\varepsilon}_\alpha W^{11}, \quad D_\alpha\lambda' = 2i\varepsilon_\alpha W^{22}. \quad (10.31)$$

One may check that $D_{\alpha\beta}\lambda' = 0$.

Let us now consider the general off-shell $\sigma$-models in this superspace. In contrast to the non-critical case, there are two conceptually new features which occur. The first is that we may introduce gauge-covariant left arctic weight-one multiplets $\Upsilon_L^{(1)}(v)$ obeying the covariant analyticity condition

$$D^{(1)}_\alpha \Upsilon_L^{(1)} = 0, \quad D^{(1)}_\alpha := v_i D_i^{(1)} \Upsilon_L^{(1)} \quad (10.32)$$

However, if we try to do the same for the right arctic weight-one multiplets, we encounter a nontrivial integrability condition

$$0 = \{D^{(1)}_\alpha, D^{(1)}_\beta\} \Upsilon_R^{(1)} = 2\varepsilon_{\alpha\beta} \varepsilon^{ij} W^{(2)}_{RZ} \Upsilon_R^{(1)} \implies Z \Upsilon_R^{(1)} = 0. \quad (10.33)$$

In other words, we can take the left arctic multiplets to be charged under the frozen vector multiplet $U(1)$, but the right arctic multiplets must be neutral. The second elaboration is that the nonzero vector multiplet field strength $W_{ij}$ can be used to construct a covariantly constant $\mathcal{O}(2)$ multiplet $W_R^{(2)}$, upon which the right Lagrangian $\mathcal{L}_R^{(2)}$ may depend.

Let us see how these modifications alter the analysis. We begin with the left sector, where the Lagrangian is given by

$$\mathcal{L}_L^{[2]}(\Upsilon_L, \bar{\Upsilon}_L; \zeta) = \frac{1}{\zeta} \hat{\mathcal{R}}_L(\Upsilon_L, \zeta \bar{\Upsilon}_L), \quad (10.34)$$

where $\hat{\mathcal{R}}_L$ still obeys the homogeneity condition (10.8). However, now $\Upsilon_L^I$ is charged under the $U(1)$ gauge group associated with the frozen vector multiplet. We require the Lagrangian to be invariant under $U(1)$ gauge transformations. Let us denote

$$Z \Upsilon_L^I = -i Z^I(\Upsilon_L) \quad (10.35)$$

where $Z^I(\Upsilon_L)$ is some function of the left arctic multiplets. Thus, the condition of gauge invariance is

$$\left(Z^I(\Upsilon_L) \frac{\partial}{\partial \Upsilon_L^I} + \bar{Z}^\bar{I}(\bar{\Upsilon}_L) \frac{\partial}{\partial \bar{\Upsilon}_L^I}\right) \mathcal{L}_L^{[2]} = 0. \quad (10.36)$$

An important special case corresponds to the situation when $\hat{\mathcal{R}}_L$ obeys a homogeneity conditions of the form (3.6). In this case the Lagrangian (10.34) turns into

$$\mathcal{L}_L^{[2]}(\Upsilon_L, \bar{\Upsilon}_L) = K_L(\Upsilon_L, \bar{\Upsilon}_L), \quad (10.37)$$
where $K_L^I(\Phi^I, \bar{\Phi}^J)$ is the preferred Kähler potential of a Kähler cone $\mathcal{X}$ (see appendix A). Then the invariance condition (10.36) means that the holomorphic vector field (10.35) is a Killing vector field on $\mathcal{X}$.

Because $\gamma^I_L$ is a covariant arctic multiplet, its lowest two components $\Phi^I_L$ and $\Sigma^I_L$ turn out to obey
\begin{equation}
\bar{\mathcal{D}}_\alpha \Phi^I_L = 0, \quad \mathcal{D}^2 \Sigma^I_L = -4iW^{22}Z^I(\Phi^I_L).
\end{equation}
That is, $\Phi^I_L$ is covariantly chiral while $\Sigma^I_L$ obeys a modified complex linearity condition. They inherit from (10.35) the gauge transformations
\begin{equation}
\mathcal{Z} \Phi^I_L = -iZ^I(\Phi^I_L), \quad \mathcal{Z} \Sigma^I_L = -i\Sigma^I_L \partial_j Z^I(\Phi^I_L).
\end{equation}
Their transformation rules differ slightly from before. The extended supersymmetry transformation of $\Phi^I_L$ is
\begin{equation}
\delta \Phi^I_L = -\varepsilon_\alpha \bar{\mathcal{D}}^\alpha \Sigma^I_L - 4S\varepsilon_L \Sigma_L + \lambda'Z^I(\Phi^I_L).
\end{equation}
For the critical case $X = 2S$, recall that we have
\begin{equation}
\mathcal{D}_\alpha \varepsilon_L = 2i\varepsilon_\alpha, \quad \bar{\mathcal{D}}_\alpha \bar{\varepsilon}_L = -2i\bar{\varepsilon}_\alpha
\end{equation}
(the parameter $\varepsilon_R$ may be consistently set to zero) while we introduce $\bar{\rho}_L$ and $\bar{\rho}_R$ via the equations
\begin{equation}
\varepsilon_\alpha = \bar{\mathcal{D}}_\alpha \bar{\rho}_L = -\mathcal{D}_\alpha \rho_R, \quad \bar{\varepsilon}_\alpha = \mathcal{D}_\alpha \rho_L = -\bar{\mathcal{D}}_\alpha \bar{\rho}_R.
\end{equation}
We may choose $\bar{\rho}_R = -\frac{i}{2}\bar{\varepsilon}_L$ and choose $\bar{\rho}_L$ to be antichiral. It follows that (up to a constant)
\begin{equation}
\lambda' = -2i\rho_L W^{11} + 2i\bar{\rho}_L W^{22}.
\end{equation}
Then the extended supersymmetry transformation becomes
\begin{equation}
\delta \Phi^I_L = -\frac{1}{2} \mathcal{D}^2(\bar{\rho}_L \Sigma_L) - 2i\rho_L W^{11} Z^I(\Phi^I_L),
\end{equation}
\begin{equation}
\delta \Sigma^I_L = -\mathcal{D}_\alpha \left(\frac{i}{2} \bar{\varepsilon}_L \mathcal{D}^\alpha \Phi^I_L + \varepsilon^\alpha \gamma^I_L\right) - 2\bar{\varepsilon}_L W^{12}Z^I(\Phi^I_L) + \lambda' \Sigma^J_L \partial_j Z^I(\Phi^I_L).
\end{equation}
Because of the modified complex linearity condition $\Sigma_L$ obeys, the dual action receives a superpotential contribution:
\begin{equation}
S^\text{dual} = \int d^3x \, d^4\theta \, E \mathbb{K}_L + \left( \int d^3x \, d^2\theta \, \mathcal{E} \, W_L + \text{c.c.} \right), \quad 
\mathbb{K}_L = \mathbb{L}_L + \Sigma^I_L \Psi_{LI} + \bar{\Sigma}^I_L \bar{\Psi}_{IJ}, \quad W_L = -i W^{22} \Psi_{LI} Z^I(\Phi^I_L).
\end{equation}
Here $\Psi_{LI}$ is gauge covariantly chiral, $\mathbf{D}_a \Psi_{LI} = 0$. The action of $Z$ on $\Psi_{LI}$ is

$$Z\Psi_{LI} = i \partial_I Z^I(\Phi_L) \Psi_{LI} .$$  \hspace{1cm} (10.44)

As before, the left target space comes equipped with a holomorphic two-form which identifies the target space as hyperkähler. Moreover, it is a hyperkähler cone, since the Kähler potential $K_L$ obeys the homogeneity condition (10.17). However, the $\sigma$-model is not superconformal due to the presence of the superpotential with the property

$$\chi^a L W_a = 2 W_L, \quad \chi^a = (\Phi^I_L, \Psi^{LI}).$$  \hspace{1cm} (10.45)

The fact that the $\sigma$-model is not superconformal becomes obvious if we interpret $W^{22}$ as a frozen $N = 2$ chiral superfield carrying dimension.

The extended supersymmetry is also modified:

$$\delta \Phi^I_L = - \frac{1}{2} D^2 (\bar{\rho}_L \partial K_L \pm \bar{\Phi}^I_L) - 2i \rho_L W^{I\bar{I}} Z^I ,$$  \hspace{1cm} (10.46a)

$$\delta \Psi_{LI} = + \frac{1}{2} D^2 (\bar{\rho}_L \partial K_L \mp \bar{\Phi}^I_L) - 2i \rho_L W^{I\bar{I}} \partial_I Z^J(\Phi_L) \Psi_{LJ} ,$$  \hspace{1cm} (10.46b)

which can be written as

$$\delta \phi^a = - \frac{1}{2} D^2 (\bar{\rho}_L \omega^{ab} K_b) - 2i \rho_L W^{I\bar{I}} Z_a ,$$  \hspace{1cm} (10.47)

where $\omega_{ab}$ is given by (10.18) and $Z^a = i Z \phi^a = (Z^I, -\partial_I Z^J(\Phi_L) \Psi_{LJ})$ is a tri-holomorphic Killing vector with the properties

$$Z^a K_L a + Z^\bar{a} K_L \bar{a} = 0 , \quad \mathcal{L}_Z \omega_{ab} = 0 .$$  \hspace{1cm} (10.48)

Now we turn to the right sector. The corresponding Lagrangian $\mathcal{L}^{(2)}_R$ may possess an additional $\zeta$-dependence through a frozen hypermultiplet $q^{(1)}$ and its smile-conjugate $\bar{q}^{(1)}$,

$$q^{(1)}_R := q_i \bar{v}^i , \quad \bar{q}^{(1)}_R := \bar{q}_i \bar{v}^i , \quad q_i = \text{const} .$$  \hspace{1cm} (10.49)

Since in the critical case there is no SU(2)$_R$ factor in the structure group, the conditions $\mathbf{D}_A q_i = 0$ are integrable. The most general $\sigma$-model Lagrangian is

$$\mathcal{L}^{(2)}_R = \mathcal{L}_R(\Upsilon^{(1)}_R, \bar{\Upsilon}^{(1)}_R, q^{(1)}_R, \bar{q}^{(1)}_R) .$$  \hspace{1cm} (10.50)

The Lagrangian must be of weight two, which means

$$\left( \Upsilon^I_R \frac{\partial}{\partial \Upsilon^I_R} + \bar{\Upsilon}^I_R \frac{\partial}{\partial \bar{\Upsilon}^I_R} + q^{(1)}_R \frac{\partial}{\partial q^{(1)}_R} + \bar{q}^{(1)}_R \frac{\partial}{\partial \bar{q}^{(1)}_R} \right) \mathcal{L}_R = 2 \mathcal{L}_R .$$  \hspace{1cm} (10.51)
Here we have omitted the weight superscripts of the physical and frozen hypermultiplets, to make the equations less cluttered. The relation (10.51) is a generalization of the homogeneity condition (10.8). The right multiplets $\Upsilon_R^{(1)}$ and $\tilde{\Upsilon}_R^{(1)}$ are neutral under the $\text{U}(1)$ gauge group and so they possess the same transformation laws and obey the same constraints as before. We find $\Phi^I_R$ and $\Sigma^I_R$ to be chiral and complex linear, respectively, and to possess the extended supersymmetry transformations

$$
\delta \Phi^I_R = \bar{\epsilon} \alpha \bar{D}^\alpha \Sigma^I_R - 4 S \bar{\varepsilon}_R \Sigma^I_R = -\frac{1}{2} \bar{D}^2 (\bar{\rho}_R \Sigma^I_R),
$$
(10.52a)

$$
\delta \Sigma^I_R = -(\varepsilon^\alpha \mathcal{D}_\alpha - 4 S \bar{\varepsilon}_R) \Phi^I_R - \mathcal{D}_\alpha (\varepsilon^\alpha \Upsilon^I_R),
$$
(10.52b)

We may perform the duality transformation as before, with the result

$$
S_{\text{dual}} = \int d^3x \ d^4\theta E \mathcal{K}_R(\Phi_R, \bar{\Phi}_R, \Psi_R, \bar{\Psi}_R),
\quad \mathcal{K}_R = L_R + \Sigma^I_R \Psi_{RI} + \bar{\Sigma}^J_{RI} \bar{\Psi}_{RJ}.
$$
(10.53)

The target space is an arbitrary hyperkähler manifold, with a covariantly constant holomorphic two-form $\omega_{ab}$ given by

$$
\omega_{ab} = \begin{pmatrix} 0 & \delta^I_J \\ -\delta^J_I & 0 \end{pmatrix},
$$
(10.54)

obeying $\omega_{ab} \omega_{bc} = -\delta^a_c$. The Kähler potential need no longer satisfy a homogeneity condition. The extended supersymmetry transformations of the fields are as before

$$
\delta \Phi^I_R = -\frac{1}{2} \bar{D}^2 (\bar{\rho}_R \partial \mathcal{K}_R \partial \Psi_{RI}),
\quad \delta \Psi_{RI} = \frac{1}{2} \bar{D}^2 \left( \bar{\rho}_R \partial \mathcal{K}_R \right),
$$
(10.55)

with $\bar{\rho}_R = \frac{-i}{2} \bar{\varepsilon}_L$.

11 (4,0) supersymmetric sigma models with $X \neq 0$: On-shell approach

In the previous section, we discussed the off-shell approach to general (4,0) supersymmetric $\sigma$-models in AdS$_3$. In this section, we turn to developing the on-shell formulation for these $\sigma$-models in terms of chiral superfields in (2,0) AdS superspace.

Before diving directly into the specifics of the (4,0) situation, we begin with a brief discussion of the universal details we will encounter, which are valid for any $\mathcal{N} = 4$ supersymmetric $\sigma$-model in either AdS or Minkowski. After that, we will focus on the (4,0) case specifically.
11.1 General features of $\mathcal{N} = 4$ supersymmetry

From the discussion of the off-shell supersymmetric $\sigma$-models in projective superspace both in the previous section and in prior papers \[40, 19\] any $\mathcal{N} = 4$ supersymmetric $\sigma$-model naturally involves two sectors: a left sector constructed of left analytic multiplets and a right sector involving right analytic multiplets. When the auxiliary fields are eliminated, one recovers two separate sectors involving left and right hypermultiplets which are transformed into each other under mirror symmetry\[20\]. Interactions between the two sectors can be mediated by vector multiplets, but we will avoid discussing these.

The existence of these decoupled sectors can be deduced from the $\sigma$-model by the presence of two copies of the covariantly constant holomorphic two-form $\omega_{ab}$, which we may denote $\omega_{L ab}$ and $\omega_{R ab}$. They obey both an orthogonality condition and a completeness condition

$$\omega_{L ab} \omega_{R bc} = 0, \quad \omega_{L ab} \omega_{L bc} + \omega_{R ab} \omega_{R bc} = -\delta^a_c.$$  \hspace{1cm} (11.1)

These conditions allow us to construct covariantly constant projection operators

$$(P_L)^a_b = -\omega^{ac}_{L} \omega_{L cb}, \quad (P_R)^a_b = -\omega^{ac}_{R} \omega_{R cb},$$

$$P_L P_R = 0, \quad P_L + P_R = 1.$$  \hspace{1cm} (11.2)

Naturally, one may adopt a coordinate system where $P_L = \text{diag}(1, \cdots, 1, 0, \cdots, 0)$ and similarly for $P_R$, which allows us to separate the fields into a “left sector” and a “right sector.” The covariant constancy of both operators then allows one to prove that the Kähler potential should decouple into two sectors. Naturally, each sector is separately hyperkähler. One may interpret $\omega_L$ and $\omega_R$ as arising from the covariantly constant $\omega_{ab} := \omega_{L ab} + \omega_{R ab}$ which can be easily seen to obey $\omega_{ab} \omega_{bc} = -\delta^a_c$. One finds $\omega_{L ab} = (P_L)^c_b \omega_{cb}$ and similarly for $\omega_{R ab}$.

We note that these same conditions (rephrased in slightly different language) were found by de Wit, Tollstén and Nicolai in the context of locally supersymmetric $\mathcal{N} = 4$ $\sigma$-models \[59\]. There, the target space was found to be the product of two quaternionic Kähler manifolds; this naturally reduces to a product of two hyperkähler manifolds when supergravity is turned off.

\[20\]The two classes of hypermultiplets have been distinguished in the literature by referring to one type as a “twisted hypermultiplet.” From an AdS perspective, the left / right nomenclature is more precise.
11.2 Formulation of (4,0) supersymmetric sigma models in (2,0) superspace

When the Killing vectors of (4,0) AdS superspace are recast in (2,0) language, one recovers the usual (2,0) Killing vectors plus additional parameters associated with the extended supersymmetry. These parameters are the complex spinor $\varepsilon_\alpha$, whose real and imaginary parts correspond to the extra two supersymmetries, and the complex chiral parameters $\varepsilon_L$ and $\varepsilon_R$ which correspond to the off-diagonal SU(2)$_L$ and SU(2)$_R$ transformations. As in the previous section, it is useful to introduce the antichiral parameters $\bar{\rho}_L$ and $\bar{\rho}_R$ which obey

$$0 = \bar{\mathcal{D}}_a \rho_L = \mathcal{D}_a \bar{\rho}_L = \mathcal{D}_a \bar{\rho}_R ,$$

(11.3a)

$$\varepsilon_\alpha = \bar{\mathcal{D}}_a \bar{\rho}_L = - \mathcal{D}_a \rho_R , \quad \bar{\varepsilon}_\alpha = - \bar{\mathcal{D}}_a \bar{\rho}_R = \mathcal{D}_a \rho_L ,$$

(11.3b)

$$8S\varepsilon_L = \mathcal{D}^2 \bar{\rho}_L , \quad 8S\varepsilon_R = \mathcal{D}^2 \bar{\rho}_R .$$

(11.3c)

For the noncritical cases where $|X| \neq 2S$, we can choose

$$\bar{\rho}_L = \frac{2i}{2 - X/S} \bar{\varepsilon}_R , \quad \bar{\rho}_R = \frac{2i}{2 + X/S} \varepsilon_L .$$

(11.4)

Such a choice is not possible when $|X| = 2S$. For the case $X = 2S$ (the case $X = -2S$ is similar), the elimination of the SU(2)$_R$ factor in the structure group means one may take $\varepsilon_R = \bar{\varepsilon}_R = 0$. This means that $\bar{\rho}_L$ cannot be given explicitly in terms of any other parameters, but only implicitly through the equations (11.3). However, the choice $\bar{\rho}_R = - \frac{1}{2} \bar{\varepsilon}_L$ remains possible.

Let us make an ansatz for the extended supersymmetry which is consistent with the off-shell analysis in the previous section:

$$\delta \phi^a = - \frac{1}{2} \mathcal{D}^2 (\bar{\rho}_L \Omega^a_L) - \frac{1}{2} \mathcal{D}^2 (\bar{\rho}_R \Omega^a_R)$$

(11.5)

where $\Omega^a_L$ and $\Omega^a_R$ are functions of $\phi$ and $\bar{\phi}$. Requiring that the action be invariant under this transformation (we leave the details again to appendix C), we recover the following feature common to both critical and non-critical cases: the objects $\omega_{Lab} = g_{aa} \bar{\partial}_b \Omega^a_L$ and $\omega_{Rab} = g_{aa} \bar{\partial}_b \Omega^a_R$ prove to be covariantly constant holomorphic two-forms obeying the conditions (11.1). As discussed in the previous subsection, these allow us to define the projector operators $P_L$ and $P_R$, obeying (11.2), which separate the hyperkähler target space into distinct left and right sectors. Now let us address the features which distinguish the noncritical and critical cases.
Noncritical case: $|X| \neq 2S$

We first observe that the structure group possesses the full $\text{SU}(2)_L \times \text{SU}(2)_R$. This ensures that both the left and right sectors of the hyperkähler manifold will possess the full $\text{SO}(3)$ Killing vectors. Moreover, it turns out that both sectors are hyperkähler cones. Indeed, the explicit analysis reveals that there must exist a homothetic conformal Killing vector $\chi^a$, which can be decomposed into left and right sectors via

$$
\chi^a_L = (P_L)^a{}_b \chi^b, \quad \chi^a_R = (P_R)^a{}_b \chi^b,
$$

so that the Kähler potential is given by

$$
K = \chi^a \chi_a = \chi^a_L \chi^a_L + \chi^a_R \chi^a_R.
$$

The term $\chi^a_L \chi^a_L$ is the hyperkähler potential for the left sector as is $\chi^a_R \chi^a_R$ for the right sector. The vector $J^a$ on the target space turns out to be given by

$$
J^a = \left( 1 + \frac{X}{2S} \right) J^a_L + \left( 1 - \frac{X}{2S} \right) J^a_R,
$$

$$
J^a_L = -\frac{i}{2} \chi^a_L, \quad J^a_R = -\frac{i}{2} \chi^a_R.
$$

Of course, the Kähler cone is also hyperkähler, possessing a covariantly constant holomorphic two-form $\omega_{ab}$ obeying the condition $\omega^{ab} \omega_{bc} = -\delta^a_c$. Constructing the complex structures as usual (4.6), one finds a full set of $\text{SU}(2)$ Killing vectors

$$
V_A^\mu = -\frac{1}{2} (J_A)^\mu \chi^\nu.
$$

These decompose, using the projection operators, into $\text{SU}(2)_L$ and $\text{SU}(2)_R$ Killing vectors which act separately on the left and right sectors.

As in the $(3,0)$ case, the component Lagrangian is quite constrained, and is given by

$$
\mathcal{L} = -g_{aa} \mathcal{D}_m \varphi^a \mathcal{D}^m \tilde{\varphi}^a - ig_{aa} \bar{\psi}_a^\alpha \mathcal{D}^\alpha \psi_b^\beta + \frac{1}{4} R_{aabb} (\psi^a \psi^b)(\bar{\psi}^\alpha \bar{\psi}^\beta)
$$

$$
- \frac{i}{2} X (\psi^a \bar{\psi}^b)(P_L - P_R)_{ab} + (3S^2 - \frac{1}{4} X^2)(K_L + K_R) + XS(K_L - K_R).
$$

It is not possible to deform the mass terms.
Critical case: $|X| = 2S$

The critical case proves to possess a richer structure. Without loss of generality, we take $X = 2S$. For this choice, we can consistently remove SU(2)$_R$ from the structure group, and so we consider only the Killing parameters $\varepsilon_\alpha$, $\Lambda^{22}$ and their complex conjugates.

Since the structure group is simply SU(2)$_L$ it follows that we should expect stringent conditions on the target space geometry only for the left case. Indeed, this is what we find. The left sector, but only the left sector, must be a hyperkähler cone. That is, there exists a holomorphic vector $\chi^a_L$ obeying

\[(P_L\chi_L)^a = \chi^a_L, \quad \nabla_b \chi^a_L = (P_L)^a_b, \quad \nabla_b \chi^a_L = 0.\]  

These conditions imply that the Kähler potential in the left sector is given by $K_L = \chi^a_L \chi_L^a$ as usual for a cone.

Similarly, the U(1) Killing vector $J^a$ required by the (2,0) algebra turns out to be given by $J^a = 2J^a_L$, where $J^a_L$ obeys

\[(P_L J_L)^a = J^a_L, \quad \mathcal{L}_{J_L} \omega_{Lab} = -i \omega_{Lab}.\]  

This implies that $J^a_L$ can be decomposed as

\[J^a_L = -\frac{i}{2} \chi^a_L + r Z^a_L,\]  

where $Z^a_L$ is a tri-holomorphic Killing vector. We have inserted a real parameter $r$ for later convenience. No further restrictions are imposed on $Z^a_L$, and its presence represents a consistent deformation of the mass parameters of the theory. The three SU(2)$_L$ Killing vectors remain defined as in (11.9) but with $\chi$ replaced by $\chi_L$.

Remarkably, the (4,0) critical case also allows a superpotential to be introduced. Recall that a tri-holomorphic isometry $Z^a_L$ can be written (at least locally) as $Z^a_L = \omega^{ab} \partial_b \Lambda_L$ where $\Lambda_L$ is a holomorphic function depending only on the left sector. We may introduce a superpotential $W = w \Lambda_L$ where $w$ is some complex number. One can check that the action is invariant if we modify the transformation law of the fields as

\[\delta \phi^a = -\frac{1}{2} \mathcal{D}^2(\tilde{\rho}_L \Omega^a_L) - \frac{1}{2} \mathcal{D}^2(\tilde{\rho}_R \Omega^a_R) - 2 \rho_L \bar{w} Z^a_L.\]  

On-shell, this transformation reduces to

\[\begin{align*}
(P_L)^a_b \delta \phi^b &= -\rho_a (\omega_L)^a_b \mathcal{D}^a \phi^b - 4S \varepsilon_L \omega^{ab} \chi_{Lb} - 2(\bar{w} \rho_L + w \tilde{\rho}_L) Z^a_L, \\
(P_R)^a_b \delta \phi^b &= \tilde{\rho}_a (\omega_R)^a_b \mathcal{D}^a \phi^b
\end{align*}\]  

(11.15a)

(11.15b)
for the left and the right sectors.

From our experience with off-shell \((4,0)\) models, we may identify

\[
  r = -\frac{i}{4S} W_{12} , \quad w = -i W^{22} , \quad \bar{w} = i W^{11} ,
\]

where \(W^{i\bar{j}}\) can be interpreted as a frozen right vector multiplet.

The component Lagrangian is

\[
  L = -g_{aa} \mathcal{D}_m \phi^a \mathcal{D}^m \tilde{\phi}^a - i g_{aa} \tilde{\phi}^a \mathcal{D}^a \phi^a + \frac{1}{4} \mathcal{R}_{aabb} (\psi^a \bar{\psi}^b) (\bar{\psi}^a \bar{\psi}^b)
  - i S (\psi^a \bar{\psi}^b) (P_L - P_R)_{ab} + 4 S^2 K_L
  - \frac{1}{2} W^{ij} W^{\bar{i}\bar{j}} Z^a_L Z^b_{\bar{L}} g_{ab} - \frac{i}{2} W_{12} (\psi^a \bar{\psi}^b) (\nabla_a Z_{L\bar{b}} - \nabla_{\bar{b}} Z_{La})
  - \frac{i}{2} W^{22} (\psi^a \bar{\psi}^b) \omega_{\bar{b}}^c \nabla_a Z_{Lc} + \frac{i}{2} W^{11} (\psi^a \bar{\psi}^b) \omega_{\bar{b}}^c \nabla_{\bar{a}} Z_{Lc} .
\]

\[\text{(11.17)}\]

\[\text{12 Sigma models with non-centrally extended } \mathcal{N} = 4 \text{ Poincaré supersymmetry}\]

As reviewed in the introduction, following \[10\], the \((p,q)\) AdS supergeometry \[1.2\] is completely determined in terms of the scalar parameter \(S\) defined as \(S = \sqrt{(S^I S_{IJ})/N}\) (we recall that \(\mathcal{N} = p+q\) and \(p \geq q \geq 0\) provided (i) \(p+q < 4\); or (ii) \(p+q \geq 4\) and \(q > 0\).

In the limit \(S \to 0\), this supergeometry reduces to that of ordinary \(\mathcal{N}\)-extended Minkowski superspace. However, the situation is different in the case of \((\mathcal{N},0)\) AdS supergeometry with \(\mathcal{N} \geq 4\), which allows for a second parameter – the v.e.v. of the supersymmetric Cotton tensor \(X^{IJKL} = X^{[IJKL]}\). In the limit \(S \to 0\), this supergeometry reduces to that of the deformed \(\mathcal{N}\)-extended Minkowski superspace \[10\]

\[
  \{\mathcal{D}^I_{\alpha}, \mathcal{D}^J_{\beta}\} = 2i \delta^{IJ} \mathcal{D}_{\alpha\beta} + i \varepsilon_{\alpha\beta} X^{IJKL} \mathcal{N}_{KL} , \quad \mathcal{N} = \frac{1}{\sqrt{S^I S_{IJ}}/N} ,
\]

\[\text{(12.1a)}\]

\[
  [\mathcal{D}_a, \mathcal{D}_b] = 0 , \quad [\mathcal{D}_a, \mathcal{D}_b] = 0 ,
\]

\[\text{(12.1b)}\]

where the constant antisymmetric tensor \(X^{IJKL}\) is constrained by

\[
  X_{N}^{IJKL} X^{LPQ} = 0 .
\]

\[\text{(12.2)}\]

If \(\mathcal{N} = 4\) we simply have \(X^{IJKL} = X^I e^{JIKL}\), and eq. \[12.2\] is identically satisfied. This solution can trivially be generalized to the case of \(\mathcal{N} = 4n\), with \(n\) an integer, by considering \(n\) copies of the \(\mathcal{N} = 4\) superalgebra.
For \( X^{IJKL} \neq 0 \), the isometry group of the superspace \((12.1)\) is a deformation of the super Poincaré group with \( \mathcal{N} \geq 4 \). The corresponding superalgebra is a non-central extension of the standard Poincaré superalgebra in three dimensions. Such non-centrally extended superalgebras do not exist in four and higher dimensions. Although the existence of these superalgebras was pointed out by Nahm [3] many years ago, only in the last decade have they appeared explicitly in various string- and field-theoretic applications [4, 5, 6, 7, 8, 9]. From the point of view of extended conformal supergravity in three dimensions [18, 19], the deformation parameter \( X^{IJKL} \) in \((12.1)\) is an expectation value of the supersymmetric Cotton tensor.

Unitary representations of the non-centrally extended Poincaré superalgebras for certain choices of \( X^{IJKL} \) have been studied, e.g., in [9]. In general, the presence of \( X^{IJKL} \neq 0 \) makes supermultiplets massive. In particular, in the case \( \mathcal{N} = 4 \) there are no massless representations if \( X \neq 0 \). Here we study the most general nonlinear \( \sigma \)-models possessing the non-centrally extended \( \mathcal{N} = 4 \) Poincaré supersymmetry. We will show that the non-central extension has the physical effect of introducing a massive deformation proportional to \( X \) when one begins with an otherwise massless superspace action; equivalently, any component action invariant under deformed Minkowski supersymmetry must have an \( X \)-dependent mass term.

We first rewrite the algebra of covariant derivatives \((12.1)\) for \( \mathcal{N} = 4 \) in a form compatible with the notation used in the previous sections. This is found by taking the formal \( S = 0 \) limit of the \((4,0)\) AdS algebra given in section 7.1:

\[
\{D_{\alpha}^{ij}, D_{\beta}^{ji}\} = 2i \varepsilon^{ij} \varepsilon^{\tilde{ij}} D_{\alpha\beta} + 2i \varepsilon_{\alpha\beta} X(\varepsilon_{\tilde{ij}} L^{ij} - \varepsilon^{ij} R_{\tilde{ij}}),
\]

\[
[D_{\alpha\beta}, D_{k\bar{k}}] = 0, \quad [D_{\alpha}, D_{\beta}] = 0.
\]

(12.3a)

(12.3b)

Embedded in this \( \mathcal{N} = 4 \) superspace is a centrally extended \( \mathcal{N} = 2 \) Minkowski superspace.

The latter is characterized by spinor covariant derivatives which are obtained by \( \mathcal{N} = 2 \) projection from the operators \( D_{\alpha} := D_{\alpha}^{11} \) and \( \tilde{D}_{\alpha} := -D_{\alpha}^{22} \) obeying the algebra

\[
\{D_{\alpha}, \tilde{D}_{\beta}\} = -2i D_{\alpha\beta} - 2 \varepsilon_{\alpha\beta} X \Delta, \quad \{D_{\alpha}, D_{\beta}\} = 0, \quad \{D_{\alpha}, \tilde{D}_{\beta}\} = 0,
\]

\[
[D_{\alpha}, D_{\beta}] = 0, \quad [D_{\alpha}, \tilde{D}_{\beta}] = 0.
\]

(12.4a)

(12.4b)

Here we have introduced the central charge operator \( \Delta := i(L^{12} - R^{22}) \). The existence of the superspace reduction \( \mathcal{N} = 4 \rightarrow \mathcal{N} = 2 \) implies that any nonlinear \( \sigma \)-model in the deformed \( \mathcal{N} = 4 \) Minkowski superspace \((12.3)\) can be reformulated as a certain \( \sigma \)-model in \( \mathcal{N} = 2 \) Minkowski superspace with a central charge.

---

\(^{21}\)We recall that the \((4,0)\) AdS algebra is given by the relations 7.3 with \( S = S \) and \( S^{ij\tilde{ij}} = 0 \).
It is pertinent to recall that the most general $\mathcal{N} = 2$ supersymmetric $\sigma$-model with a central charge involves a Kähler potential $K(\phi, \bar{\phi})$ and a superpotential $W(\phi)$, with a superspace action formally identical to eq. (2.25). Both are required to be invariant under the central charge

$$\Delta^a K_a + \bar{\Delta}^\bar{a} K_{\bar{a}} = 0, \quad \Delta^a W_a = 0, \quad \Delta^a(\phi) := \Delta \phi^a. \quad (12.5)$$

The component Lagrangian is

$$L = -g^{a\bar{a}} D_m \phi^a D^m \bar{\phi}^\bar{a} - ig^{a\bar{a}} \bar{\psi}^\bar{a} \bar{D}^{\alpha\beta} \psi^a + \frac{1}{4} R_{a\bar{a}b\bar{b}} (\psi^a \psi^b)(\bar{\psi}^\bar{a} \bar{\psi}^\bar{b})$$

$$- g_{a\bar{b}} X^2 \Delta^a \Delta^\bar{b} + \frac{1}{2} X (\psi^b \bar{\psi}^\bar{b})(\nabla_b \Delta_{\bar{b}} - \nabla_{\bar{b}} \Delta_b)$$

$$- g^{a\bar{b}} W_a W_{\bar{b}} - \frac{1}{2} \nabla_a W_b (\psi^a \psi^b) - \frac{1}{2} \nabla_{\bar{a}} \bar{W}_{\bar{b}} (\bar{\psi}^{\bar{a}} \bar{\psi}^{\bar{b}}) \quad (12.6)$$

after eliminating the auxiliary fields. Note that masses for the fermions and a scalar potential are generated from a nonzero constant $X$ as well as from the superpotential.

Since the deformed Minkowski algebra is simply the $S = 0$ limit of the $(4,0)$ AdS algebra with $X \neq 0$ discussed in the previous section, we already know the form the extended supersymmetry should take and its consequences on the target space. We must have $\delta \phi^a$ given by (11.5) where the parameters $\tilde{\rho}_L$ and $\tilde{\rho}_R$ obey (11.3). Since $S = 0$ but $X \neq 0$, the target space must obey the same constraints as the non-critical AdS case. Hence, it must be a hyperkähler cone with separate left and right sectors. By comparing the deformed Minkowski algebra to the $(4,0)$ AdS algebra, we see that the Killing vector $J^a$ must be related to $\Delta^a$ by $2SJ^a = X \Delta^a$ in the $S = 0$ limit. Using (11.8), the Killing vector $\Delta^a$ is given by

$$\Delta^a = -\frac{i}{2} \chi^a_L + \frac{i}{2} \chi^a_R. \quad (12.7)$$

The hyperkähler cone structure dictates that one actually has a full set of SU(2) Killing vectors $V_A$ given by (11.9); the central charge Killing vector $\Delta^a$ is related to one of these by $\Delta^a = (P_L - P_R)^a_b V^b_3$. One also finds that the superpotential $W$ must vanish. The component action is

$$L = -g^{a\bar{a}} D_m \phi^a D^m \bar{\phi}^\bar{a} - ig^{a\bar{a}} \bar{\psi}^\bar{a} \bar{D}^{\alpha\beta} \psi^a + \frac{1}{4} R_{a\bar{a}b\bar{b}} (\psi^a \psi^b)(\bar{\psi}^\bar{a} \bar{\psi}^\bar{b})$$

$$- \frac{i}{2} X (\psi^a \bar{\psi}^\bar{b})(P_L - P_R)^a_b - \frac{1}{4} X^2 (K_L + K_R). \quad (12.8)$$

We emphasize that as in the $(4,0)$ case with $X \neq 0$, the action is not superconformal even though the target space is a cone.
The Lagrangian (12.8) describes the most general nonlinear $\sigma$-models with the non-centrally extended $\mathcal{N} = 4$ Poincaré supersymmetry. Setting $X = 0$ in (12.8) gives the most general $\mathcal{N} = 4$ superconformal $\sigma$-model, with its target space being a hyperkähler cone $\mathcal{M}_L \times \mathcal{M}_R$. The deformation parameter $X$ appears in both terms in the second line of (12.8). The first structure constitutes the fermionic mass term, while the second gives the scalar potential

$$V = \frac{1}{4} X^2 (K_L + K_R) .$$

(12.9)

For each of the left and right $\sigma$-model sectors, the scalar potential is constructed in terms of the homothetic conformal Killing vector associated with the corresponding target space. This follows from the general result that for any hyperkähler cone the preferred Kähler potential is given in terms of the homothetic conformal Killing vector $\chi$ as follows:

$$K(\phi, \bar{\phi}) := g_{\bar{a}b}(\phi, \bar{\phi}) \chi^{\bar{a}}(\phi) \bar{\chi}^{b}(\bar{\phi}) ,$$

(12.10)

see Appendix A. Therefore the scalar potential (12.9) is positive except at the tip of the left and right cones. Eq. (12.9) provides a new mechanism to generate massive $\sigma$-models. In the case of 4D $\mathcal{N} = 2$ or 5D $\mathcal{N} = 1$ Poincaré supersymmetries, the standard mechanism to construct massive $\sigma$-models [60, 11, 61, 62, 63, 64] makes use of a tri-holomorphic Killing vector $Z^a(\phi)$ on the hyperkähler target space (provided such a tri-holomorphic Killing vector exists). The superpotential generated, $W(\phi)$, is related to the tri-holomorphic Killing vector as $\partial_a W \propto \omega_{ab} Z^b$, with $\omega_{ab}(\phi)$ the holomorphic two-form on the hyperkähler target space (we assume that the formulation in terms 4D $\mathcal{N} = 1$ chiral superfields is used); the corresponding scalar potential is $V \propto g_{\bar{a}b} Z^a \bar{Z}^b$. Technically, any massive $\sigma$-model can be obtained by gauging an off-shell massless $\sigma$-model in projective superspace and then freezing the background vector multiplet to have a constant field strength [63, 64]. In the case of non-centrally extended 3D $\mathcal{N} = 4$ supersymmetry, the scalar potential (12.9) is constructed solely in terms of the homothetic conformal Killing vector each hyperkähler cone possesses; no tri-holomorphic Killing vector is involved. Technically, any $\sigma$-model with the deformed $\mathcal{N} = 4$ supersymmetry can be obtained by coupling an $\mathcal{N} = 4$ superconformal $\sigma$-model to $\mathcal{N} = 4$ conformal supergravity and then freezing the Weyl multiplet to have a constant supersymmetric Cotton tensor and zero values for the other components of the supergravity torsion and curvature.
13 Conclusion

In this paper we have thoroughly studied the nonlinear σ-models in AdS$_3$ with six and eight supercharges. With the exception of two critical (4,0) AdS supersymmetries, for which $X = \pm 2S$, all σ-model target spaces belong to the following large families of non-compact hyperkähler manifolds: (i) hyperkähler cones; and (ii) hyperkähler spaces with a U(1) isometry group which acts non-trivially on the two-sphere of complex structures (and necessarily leaves one complex structure invariant). It is obvious that all hyperkähler cones belong to the family (ii). The target spaces of arbitrary nonlinear σ-models with (3,0), (3,1) and non-critical (4,0) supersymmetries in AdS$_3$ are hyperkähler cones. The main reason for this is the property that the $R$-symmetry group of such supersymmetric σ-models includes SU(2) as a subgroup, see the discussion in [10]. The target spaces of arbitrary nonlinear σ-models with (2,1) and (2,2) supersymmetries in AdS$_3$ are hyperkähler manifolds belonging to the family (ii).

We have demonstrated that compact target spaces are allowed only for those nonlinear σ-models in AdS$_3$ which possess critical (4,0) supersymmetry such that $X = \pm 2S$. For concreteness, let us choose $X = 2S$. Then the target space of any supersymmetric σ-model has the form

$$\mathcal{M}_L \times \mathcal{M}_R,$$

where $\mathcal{M}_L$ is a hyperkähler cone, while $\mathcal{M}_R$ is an arbitrary hyperkähler manifold.

It is well known that the target spaces of superconformal σ-models with six and eight supercharges are hyperkähler cones, see e.g. [51, 65]. All nonlinear σ-models in AdS$_3$ with (3,0) and (3,1) supersymmetries are actually $\mathcal{N} = 4$ superconformal. As concerns the nonlinear σ-models in AdS$_3$ with (4,0) supersymmetry, they are superconformal only in the case $X = 0$.

We have constructed the most general nonlinear σ-model in Minkowski space with a non-centrally extended $\mathcal{N} = 4$ Poincaré supersymmetry. Its target space is a hyperkähler cone, but the σ-model is massive. The Lagrangian includes a positive potential proportional to the norm squared of the homothetic conformal Killing vector the target space is endowed with. This mechanism of mass generation differs from the standard one which corresponds to σ-model with the ordinary $\mathcal{N} = 4$ Poincaré supersymmetry and which makes use of a tri-holomorphic Killing vector.
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A  (Hyper) Kähler cones

Consider a Kähler manifold \((\mathcal{M}, g_{\mu\nu}, J^{\mu\nu})\), where \(\mu, \nu = 1, \ldots, 2n\), and introduce local complex coordinates \(\phi^a\) and their conjugates \(\bar{\phi}^a\), in which the complex structure \(J^{\mu\nu}\) is diagonal. It is called a Kähler cone \([66]\) if it possesses a homothetic conformal Killing vector \(\chi\)

\[
\chi = \chi^a \frac{\partial}{\partial \phi^a} + \bar{\chi}^a \frac{\partial}{\partial \bar{\phi}^a} \equiv \chi^\mu \frac{\partial}{\partial \phi^\mu}
\]

which is the gradient of a function. These conditions mean that

\[
\nabla_\nu \chi^\mu = \delta_\nu^\mu \iff \nabla_b \chi^a = \delta_b^a, \quad \nabla_b \chi^a = \partial_b \chi^a = 0.
\]

In particular, \(\chi\) is holomorphic. The properties of \(\chi\) include the following:

\[
\chi_a := g_{ab} \chi^b = \partial_a K \quad \Rightarrow \quad \chi^a K_a = K,
\]

where

\[
K := g_{ab} \chi^a \chi^b
\]

can be used as a Kähler potential, \(g_{ab} = \partial_a \partial_b K\). Associated with \(\chi\) is the U(1) Killing vector field

\[
V^\mu = -\frac{1}{2} J^{\mu\nu} \chi^\nu, \quad \nabla_\mu V_\nu + \nabla_\nu V_\mu = 0.
\]

Local complex coordinates for \(\mathcal{M}\) can always be chosen such that

\[
\chi = \phi^a \frac{\partial}{\partial \phi^a} + \bar{\phi}^a \frac{\partial}{\partial \bar{\phi}^a},
\]
and then the second relation in (A.3) turns into the homogeneity condition
\[ \phi^a K_a(\phi, \bar{\phi}) = K(\phi, \bar{\phi}). \] (A.7)

A hyperkähler manifold \((\mathcal{M}, g_{\mu\nu}, (J_A)^{\mu}_{\nu})\), where \(\mu, \nu = 1, \ldots, 4n\) and \(A = 1, 2, 3\), is called a hyperkähler cone [65] if it is a Kähler cone with respect to each complex structure. Using \(J_A\) and \(\chi\), we may construct three SU(2) Killing vectors
\[ V_A^{\mu} := -\frac{1}{2}(J_A)^{\mu}_{\nu}\chi^{\nu}. \] (A.8)

These vectors commute with \(\chi\) and obey an SU(2) algebra amongst themselves,
\[ [V_A, \chi] = 0, \quad [V_A, V_B] = \epsilon_{ABC}V_C. \] (A.9)

They generate a transitive action of SO(3) on the two-sphere of complex structures,
\[ \mathcal{L}_{V_A}J_B = \epsilon_{ABC}J_C. \] (A.10)

More information about the hyperkähler cones can be found in [65].

B Derivation of the action (8.40)

Here we sketch the derivation of the action (8.40) by requiring its invariance under the projective transformations (8.33). The derivation is analogous to the analysis given in [21, 67, 41] and more recently in [10] for the 3D \(\mathcal{N} = 3\) AdS case. The interested reader is referred to those papers for more technical details regarding the general procedure.

To derive the action we start from the term \(S_0\) in (8.39). As a first step, we vary \(S_0\) with respect to the infinitesimal transformation (8.33). Then, we iteratively add to the action extra terms which cancel the variation order by order such that the final action is invariant. The \(a_L\) and \(c_L\) variations do not give important informations. The non-trivial terms are generated by the \(b_L\) variation in (8.33),
\[ \delta u_i = b_L v_i. \] (B.1)

The transformation (B.1) induces the variations:
\[ \delta D^{(-1)i}_a = \frac{b_L}{(v_L, u_L)}D^{(1)i}_a, \quad \delta S^{(-2)i\bar{j}} = \frac{2b_L}{(v_L, u_L)}S^{(0)i\bar{j}}, \quad \delta S^{(0)i\bar{j}} = \frac{b_L}{(v_L, u_L)}S^{(2)i\bar{j}}, \] (B.2)
with \( S^{(0)ij} := (v_i u_j S^{ij})/(v_L, u_L) \).

Let us compute the variation of \( S_0 \) defined by (8.39). By making use of (B.2) and the analyticity condition \( \mathcal{D}_a^{(2)} L^{(2)} = 0 \) we obtain

\[
\delta S_0 = \frac{1}{96\pi} \oint_{\gamma} \frac{v_i dv^i}{(v_L, u_L)} \int d^3 x e b_L \left\{ D^{(1)}_{\gamma k} D^{(-1)\overline{ij}}_{k \overline{l}} D^{(2)}_{\overline{kl}} + D^{(2)}_{\overline{kl}} \{ D^{(1)}_{\gamma k}, D^{(-1)\overline{ij}}_{\overline{l}} \} \right. \\
+ D^{(-1)\gamma \delta} \left( D^{(1)}_{\gamma k}, D^{(-1)\overline{ij}}_{\overline{l}} \right) + D^{(-1)\gamma \delta} \left( D^{(1)l}, D^{(-1)\overline{ij}}_{\overline{l}} \right) D^{(-1)\delta}_{\overline{l}} \\
+ \left. D^{(-1)\gamma \delta} \left( D^{(1)l}, D^{(-1)\overline{ij}} \right) \right\} L^{(2)}_{L} (z, v_L) || . \tag{B.3}
\]

Next we use the anti-commutation relations (7.3a)-(7.3b) to transform the expression in the right-hand side of (B.3). As a result we obtain a number of terms dependent on the Lorentz and SU(2) generators as well as terms containing the vector covariant derivative \( \mathcal{D}_a \). The next step is to push all the Lorentz and SU(2) generators to the right. Once they hit \( L^{(2)}_{L} \) we use the identities \( \mathcal{M}_{a\beta} L^{(2)}_{L} = R^{ij} L^{(2)}_{L} = v_i v_j L^{ij} L^{(2)}_{L} = 0 \) and \( v_i u_j L^{ij} L^{(2)}_{L} = -(v_L, u_L) L^{(2)}_{L} \). To compute the contributions coming from \( u_i u_j L^{ij} L^{(2)}_{L} \) one has to use the following formula

\[
\oint_{\gamma} \frac{v_i dv^i}{(v_L, u_L)} \oint_{\gamma} \frac{v_i dv^i}{(v_L, u_L)} \{ b_L \left( u^k \partial_k T^{(3)} \right) L^{(2)}_{L} \} \tag{B.4}
\]

which follows from the analysis of [11, 67]. The equation (B.4) holds for any operator \( T^{(3)} \) which is a function of \( v_L \) and \( u_L \) and homogeneous in \( v_L \) of degree three: \( T^{(3)}(c v_L) = c^3 T^{(3)}(v_L) \). There are also terms containing a vector derivative \( \mathcal{D}_a \). To simplify those terms, we push to the left all the vector derivatives obtaining a total derivative, that can be ignored, plus terms involving commutator of spinor and vector derivatives. The final result of the procedure sketched is

\[
\delta S_0 = \frac{1}{2\pi} \oint_{\gamma} \frac{v_i dv^i}{(v_L, u_L)} \int d^3 x e b_L \left[ \frac{2i}{3} S^{(0)k\overline{i}} D^{(-2)}_{\overline{kl}} + \frac{8}{3} S^{(-2)\overline{k}l} S^{(0)k\overline{i}} \right] L^{(2)}_{L} (z, v_L) || . \tag{B.5}
\]

The only possible functional that can be added to \( S_0 \) to cancel the above variation has the form

\[
S_{\text{extra}} = \frac{1}{2\pi} \oint_{\gamma} v_i dv^i \int d^3 x e \left[ a_1 S^{(-2)\overline{k}l} D^{(-2)}_{\overline{kl}} + a_2 S^{(-2)\overline{k}l} S^{(-2)}_{\overline{kl}} \right] L^{(2)}_{L} || , \tag{B.6}
\]

with \( a_1 \) and \( a_2 \) some numerical coefficients. By using the same procedure described for the computation of \( \delta S_0 \), we derive

\[
\delta S_{\text{extra}} = \int d^3 x e \oint_{\gamma} \frac{v_i dv^i}{2\pi} \frac{b_L}{(v_L, u_L)} \left[ 2ia_1 S^{(0)k\overline{i}} D^{(-2)}_{\overline{kl}} + (4a_2 - 16a_1) S^{(-2)\overline{k}l} S^{(0)k\overline{i}} \right] L^{(2)}_{L} || . \tag{B.7}
\]
If we impose the condition $\delta S_0 + \delta S_{\text{extra}} = 0$, we fix the coefficients to be
\[
a_1 = -\frac{1}{3}, \quad a_2 = -2.
\] (B.8)

The functional given by $S = S_0 + S_{\text{extra}}$ is the invariant action \[8.40\].

C Deriving conditions for extended supersymmetry

In this appendix we briefly sketch how to establish the conditions imposed by extended supersymmetry on the target spaces of $\mathcal{N} = 2$ $\sigma$-models.

C.1 (3,0) AdS supersymmetry in (2,0) AdS superspace

In section [4] we addressed what conditions $\sigma$-models in (2,0) AdS superspace must obey in order to possess (3,0) AdS supersymmetry. We summarize below how one goes about establishing the properties of the target space.

Deriving the conditions

Let us first derive a set of necessary conditions for the variation of the action, eq. (4.4), to vanish. Instead of analyzing it directly, we can consider its functional variation with respect to the chiral superfield $\phi^a$:

\[
0 = \delta_\phi \delta S = \frac{1}{2} \int d^3 x d^4 \theta E \delta \phi^a \left( K_{ab} \bar{D}^2 (\bar{\rho} \Omega^b) + K_{b} \bar{D}^2 (\bar{\rho} \Omega^b \rho^a) \\
- K_{ab} \bar{D}^2 (\rho \Omega^b) - \bar{D}^2 K_b (\rho \Omega^b \rho^a) \right).
\]

It turns out that this condition is far simpler to analyze. We must impose
\[
0 = -\frac{i}{8} \bar{D}^2 \left( K_{ab} \bar{D}^2 (\bar{\rho} \Omega^b) + K_{b} \bar{D}^2 (\bar{\rho} \Omega^b \rho^a) - K_{ab} \bar{D}^2 (\rho \Omega^b) - \bar{D}^2 K_b (\rho \Omega^b \rho^a) \right).
\] (C.1)

Let us consider several classes of terms. Those involving $D^{\alpha \beta} D_{\alpha \beta} \phi^b$ will be proportional to $(K_{ab} \Omega^b \rho^a + K_{b} \Omega^b \rho^a)$ and so $\omega_{ab} := g_{ac} \Omega^c \rho^b$ must be antisymmetric. Making use of this condition,
\[
\frac{i}{8} \bar{D}^2 \left( K_{ab} \bar{D}^2 (\bar{\rho} \Omega^b) + D^2 K_b (\rho \Omega^b \rho^a) \right) = \frac{i}{8} \bar{D}^2 \left( D^2 \rho K_{ab} \Omega^b + 2K_{ab} D^2 \rho \omega_b \phi^b + D^2 K_b (\rho \Omega^b \rho^a) \right)
+ K_{ab} \rho \partial c \omega^b \bar{D}^2 \rho \phi^b + \partial c K_{b} \bar{D}^2 \rho \phi^b \rho^a \omega^c_a.
\]
The third and fourth terms are unique: these will give terms involving two vector derivatives of $\phi$. They are cancelled only if $\nabla_c \omega_{ba} = \partial_c \omega^{\bar{b}a} = 0$. Imposing both of these conditions, we are left with

$$0 = -\frac{i}{8} \bar{D}^2 \left( K_{ab} \bar{D}^2 (\rho \Omega^b) + K_b \bar{D}^2 (\rho \Omega^{b,a}) + 8i S \rho \Omega_a - 2 \mathcal{D}^a \rho \mathcal{D}_b \phi^b \omega_{ab} \right).$$

Taking now all terms proportional to $\bar{D}^2 \bar{\phi}^b \bar{D}^2 \bar{\phi}^c$, one finds that $\partial_a \omega_{b\bar{c}} = 0$. This establishes the existence of a covariantly constant two-form $\omega_{ab}$.

Exploiting this result, we can rewrite the condition (C.1) as

$$0 = -\frac{i}{8} \bar{D}^2 \left( 8i S \rho \Omega_a - \bar{D}_b \bar{\rho} \bar{D}^\beta \bar{\phi}^\alpha \partial_\alpha \Omega_b - 2 \mathcal{D}^a \rho \mathcal{D}_b \phi^b \omega_{ab} \right). \quad (C.2)$$

The only term involving $\rho \bar{D}^2 \bar{\phi}^b$ is proportional to $\partial_a \Omega_b$, so we must require $\partial_a \Omega_b = 0$. Now we observe that $\Omega_a$ obeys the following properties:

$$\nabla_b \Omega_a = 0, \quad \nabla_b \Omega_a = \omega_{ab}. \quad (C.3)$$

We will shortly find that $\omega_{ab}$ is invertible. Assuming this now, we immediately observe that the conditions (C.3) are satisfied if and only if

$$\Omega_a = \omega_{ab} \chi^b \quad (C.4)$$

where $\chi^b$ is a homothetic conformal Killing vector. The remainder of the condition (C.2) amounts to

$$0 = 8i S^2 \rho \Omega_a + 16 \rho^2 S^2 J^b \omega_{ab}$$

and so we conclude that $J^a = -\frac{i}{2} \chi^a$.

**Closure of the algebra**

To complete our analysis, we require an additional condition: we must enforce that the algebra of two extended supersymmetries closes on-shell. Examining

$$[\delta_2, \delta_1] \phi^a = i \bar{D}^2 (\bar{\rho} [1 \delta_2] \Omega^a), \quad (C.5)$$

and using the constraint

$$0 = [\nabla_a, \nabla_{\bar{a}}] (\omega^b_{\bar{b}c} \chi^\bar{b}) = R_{a\bar{a}}^b c \omega^c_{\bar{b}c} \chi^\bar{b} = R_{a\bar{a}}^b c \Omega^c, \quad (C.6)$$

94
one can straightforwardly check that
\[ [\delta_2, \delta_1] \phi^a = -\rho_{[2\alpha} D^\alpha \bar{\phi}^b \omega^{bc} \left( \bar{\rho}_{1]} R_{\bar{b}a c d} \omega^{\bar{b}d} D_\beta \bar{\phi}^\beta \bar{D}^\bar{c} \phi^c \right) - i \bar{D}^2 (\bar{\rho}_{[2\delta_1]} \bar{\phi}^b \omega^{a b}) . \]  
(C.7)

The first term vanishes since \( \omega^{c}_{\bar{a}} \omega^{b}_{\bar{b}} R_{\bar{a} \bar{b} \bar{c} d} \) is totally symmetric in the indices \( \bar{c}, \bar{b} \) and \( \bar{d} \) and the symmetrized product of the three fermionic fields vanishes. For the remaining term, we get
\[ [\delta_2, \delta_1] \phi^a = -\frac{1}{2} D^2 \left( \bar{\rho}_{[2} \omega^b_{\bar{c}} \bar{D}^2 (\rho_{1]} \bar{\chi}_c) \right) = -\frac{1}{2} \omega^{ab} \omega_{bc} \left( -2i \bar{\varepsilon}_{[2} \rho^a_{1]} \bar{D}_a \phi^c - 4i \rho^a_{[2} \rho^b_{1]} \bar{D}_{ab} \phi^c - i \bar{\varepsilon}_{[2} \varepsilon_1] J^c + \frac{1}{2} \bar{D}^2 (\bar{\rho}_{[2} \rho^c_{1]} \bar{D}^2 \bar{\chi}_c) \right) \]  
(C.8)
where we have used \( \varepsilon = -8 S \rho. \)

We may check this against the extended supersymmetry algebra. Let \( \Psi \) be some \( \mathcal{N} = 3 \) superfield. Under an extended supersymmetry transformation,
\[ \delta \Psi = 2i \rho^a D^a_2 \Psi + \frac{1}{2} \bar{\varepsilon} J^a_1 \Psi + \frac{1}{2} \varepsilon J^a_2 \Psi , \]  
(C.9)
and a straightforward calculation leads to
\[ [\delta_2, \delta_1] \Psi = -4i \rho^a_{[2} \rho^a_{1]} \bar{D}_a \Psi - 2i \varepsilon_{[2} \rho^a_{1]} \bar{D}_a \Psi + \bar{\varepsilon} \varepsilon_{[2} J^a_1 \Psi . \]  
(C.10)
In order for the \( \mathcal{N} = 2 \) projection of this expression to match (C.8) for \( \phi^a = \Psi \), we find that \( \phi^a \) must be on-shell, \( D^2 \chi_a = D^2 K_a = 0 \), and that the holomorphic two-form \( \omega^{ab} \) must obey \( \omega^{ab} \omega_{bc} = -\delta^a_c. \)

**Invariance of the full action**

We must still check that the conditions we have derived are sufficient to ensure the invariance of the action. First note that \( \delta \phi^a \) can be rewritten
\[ \delta \phi^a = \frac{i}{2} \bar{D}^2 (\bar{\rho}_a) = \frac{i}{2} \bar{D}^2 (\bar{\omega}^{ab} \chi_b) , \quad \chi_b = K_b . \]  
(C.11)
This allows the variation of the action to be written
\[ \delta S = \int d^3 x d^4 \theta E \left( -\frac{i}{2} \rho_a \tilde{A}^a + \frac{i}{2} \rho^a A_a \right) , \quad A_a = D_a \phi^b \omega_{bc} \chi^c . \]  
(C.12)
Now note that
\[ \bar{D}_\beta A_a = -2i D_{a\beta} \phi^b \omega_{bc} \chi^c - 4 \varepsilon_{a\beta} S J^b \omega_{bc} \chi^c = -2i D_{a\beta} \phi^b \omega_{bc} \chi^c . \]  
(C.13)
In particular, \( \bar{D}^a A_a = 0 \). It follows that \( \delta S = 0 \) by writing \( \rho^a A_a = \bar{D}^a \tilde{\rho} A_a \) and integrating by parts.
C.2 (2,1) AdS supersymmetry in (2,0) AdS superspace

Our next case involves a (2,1)-supersymmetric \( \sigma \)-model in (2,0) AdS superspace.

Derivation of conditions and invariance of the action

We proceed as in our analysis of the (3,0) case. Instead of requiring \( \delta S = 0 \) directly, we analyze \( \delta \phi \delta S = 0 \). This condition amounts to

\[
0 = -\frac{1}{4} \mathcal{D}^2 \left( \frac{i}{2} K_{ab} \mathcal{D}^2 (\bar{\rho} \Omega^b) + \frac{i}{2} K_b \mathcal{D}^2 (\bar{\rho} \partial_a \Omega^b) - \frac{i}{2} g_{ab} \mathcal{D}^2 (\rho \Omega^b) - \frac{i}{2} \mathcal{D}^2 K_b \rho \partial_a \Omega^b \right). \tag{C.14}
\]

Analyzing (C.14) and focusing our attention on terms involving the highest number of derivatives, we immediately deduce as before that \( \omega_{ab} := g_{aa} \partial_b \Omega^a \) is antisymmetric and covariantly constant. Then (C.14) simplifies to

\[
0 = -\frac{1}{4} \mathcal{D}^2 \left( -i \omega_{ab} \rho^a \mathcal{D}_a \phi^b - \frac{i}{2} \rho_a \bar{\mathcal{D}}^a \bar{\phi}^b \partial_a \Omega_b \right). \tag{C.15}
\]

The first term gives zero when we apply \( \bar{\mathcal{D}}^2 \) since \( \rho^a \) is constant, \( \omega_{ab} \) is chiral, and \( \mathcal{D}^2 \mathcal{D}_a \phi^b = 0 \). The remaining term is

\[
0 = \frac{i}{8} \rho_a (\bar{\mathcal{D}}^a \phi^b \mathcal{D}^2 \phi^c (\partial_c \partial_a \Omega_b) - \mathcal{D}^2 \phi^b \bar{\mathcal{D}}_a \bar{\phi}^c \partial_a \partial_c \Omega_b + \bar{\mathcal{D}}^a \bar{\phi}^b \bar{\mathcal{D}}_a \bar{\phi}^c \mathcal{D}^2 \phi^c (\partial_c \partial_a \partial_b \Omega_b)). \tag{C.16}
\]

The first two terms cancel since they amount to

\[
\partial_a \partial_c \partial_b \Omega_b - \partial_a \partial_b \partial_c \Omega_b = \partial_a (2 \omega_{bc}) = 0. \tag{C.17}
\]

The third term cancels since the product of the three fermionic fields vanishes if totally symmetrized in \( \bar{b}, \bar{c} \) and \( \bar{d} \), and the quantity \( (\partial_{\bar{d}} \partial_{\bar{c}} \partial_{\bar{a}} \Omega_{\bar{b}}) \) is indeed totally symmetric in these indices. To prove this, we first note that it is already symmetric in \( \bar{c} \) and \( \bar{d} \) by construction. Then writing

\[
\partial_{\bar{a}} \partial_{\bar{c}} \partial_{\bar{b}} \Omega_{\bar{b}} = \partial_{\bar{d}} \partial_{\bar{a}} (\omega_{\bar{b} \bar{c}} + \Gamma_{\bar{c} \bar{b}} \partial_{\bar{d}} \Omega_{\bar{d}}) = \partial_{\bar{a}} \partial_{\bar{a}} (\Gamma_{\bar{c} \bar{b}} \partial_{\bar{d}} \Omega_{\bar{d}}) \tag{C.18}
\]

we discover it is symmetric in \( \bar{c} \) and \( \bar{b} \) as well.

Now let us prove these conditions are sufficient. The proof is very similar to that in AdS4 [13]. We begin by writing

\[
\delta S = \int d^3 x d^4 \theta \, E \left( \frac{i}{2} \rho^a A_a - \frac{i}{2} \rho_a \bar{A}^a \right), \quad A_a = \mathcal{D}_a \phi^b \omega_{ab} \bar{K}_b. \tag{C.19}
\]
One can check that
\[ D_\alpha A_\beta + D_\beta A_\alpha = -2D_\alpha \phi^b D_\beta \phi^c \omega_{bc} . \]  
(C.20)

Since \( \omega_{bc} \) is closed, we may take (at least locally) \( \omega_{bc} = \partial_b \Gamma_c - \partial_c \Gamma_b \) for some holomorphic one-form \( \Gamma_a \), and rewrite the integrand of \( \delta S \) as
\[ \frac{i}{2} \rho^a (A_\alpha + 2D_\alpha \phi^b \Gamma_b) + \text{c.c.} \]  
(C.21)

The additional term we have added is annihilated by \( \bar{D}^2 \). The term in parentheses can be denoted \( B_\alpha \) and obeys \( D(\alpha B_\beta) = 0 \), implying that \( B_\alpha = D_\alpha B \) for some complex quantity \( B \). Making this substitution and integrating by parts, we find \( \delta S = 0 \).

So far we have established only the existence of the covariantly constant two-form. We must still establish that \( \omega^{ab} \omega_{bc} = -\delta^a_c \), and that the U(1)\(_R\) isometry rotates \( \omega_{ab} \). The first condition is completely straightforward to prove and proceeds analogously as in the \((3,0)\) case from an analysis of closure. The second condition is slightly more involved, and arises by considering the algebra of an extended supersymmetry transformation with a \((2,0)\) supersymmetry transformation – in particular with a U(1)\(_R\) transformation. We note that the U(1)\(_R\) generator acts as \( J \phi^a = -i J^a \) and \( J \phi^b = -i J^b \). It follows that
\[ J \delta \phi^a = \frac{i}{2} J \bar{D}^2 (\bar{\rho} \Omega^a) = \frac{1}{2} \bar{D}^2 (\bar{\rho} J^b \partial_b \Omega^a + \bar{\rho} J^b \Omega^a \bar{b} - i \bar{\rho} \Omega^a) \]  
(C.22)

after accounting for the nontrivial U(1)\(_R\) transformation properties of \( D_\alpha \) and \( \bar{\rho} \). We next note that
\[ \delta J \phi^a = -i \delta J^a = \frac{1}{2} \bar{D}^2 (\bar{\rho} \Omega^b \partial_b J^a) \]  
(C.23)

In order for the extended supersymmetry transformation to be an isometry, it must commute with the generator \( J \). In other words,
\[ 0 = [J, \delta] \phi^a = \frac{1}{2} \bar{D}^2 (\bar{\rho} J^b \partial_b \Omega^a + \bar{\rho} J^b \Omega^a \bar{b} - i \bar{\rho} \Omega^a - \bar{\rho} \Omega^b \partial_b J^a) \]  
(C.24)

This holds if and only if
\[ \mathcal{L}_J \Omega^{ab} = -i \omega_{ab} , \quad \mathcal{L}_J \omega_{ab} = i \omega_{ab} . \]  
(C.25)

In contrast to the \((3,0)\) \(\sigma\)-model, we see here only a single U(1) Killing vector exists in the hyperkähler target space – coinciding with that generated by the U(1)\(_R\) of the \((2,0)\) algebra – rather than the full triplet of SU(2) Killing vectors which the \((3,0)\) \(\sigma\)-model possesses. This is completely consistent with the off-shell description and reflects simply the fact that the \((2,1)\) algebra possesses only a single U(1) isometry.
C.3 (2,1) AdS supersymmetry in (1,1) AdS superspace

Now we derive the consequences of imposing (2,1) AdS supersymmetry on a \( \sigma \)-model in (1,1) AdS superspace.

Analysis of constraints and invariance of the action

Recall the postulated transformation law

\[
\delta \phi^a = \frac{1}{2} (\bar{\partial}^2 - 4\mu)(\varepsilon \Omega^a) .
\]  

(C.26)

As in AdS\(_4\) [12, 13], this is the most general ansatz available. Also as in AdS\(_4\), the most general Lagrangian is just the full superspace integral of a real function \( K \). We derive the conditions necessary for \( \delta S = 0 \) by analyzing the weaker condition \( \delta \phi \delta S = 0 \). Observe that

\[
\delta \phi \delta S = \int d^3x \, d^4\theta \, E \, \delta \phi^a \left( \frac{1}{2} K_{ab}(\bar{\partial}^2 - 4\mu)(\varepsilon \Omega^b) + \frac{1}{2} K_b(\bar{\partial}^2 - 4\mu)(\varepsilon \partial_a \Omega^b) \right)
+ \frac{1}{2} g_{ab}(\bar{\partial}^2 - 4\bar{\mu})(\varepsilon \Omega^b) + \frac{1}{2}(\bar{\partial}^2 - 4\bar{\mu})K_b \varepsilon \partial_a \Omega^b \right) .
\]  

(C.27)

The vanishing of the terms with the highest number of derivatives indicates that \( \omega_{ab} := g_{aa} \partial_b \Omega^a \) is a covariantly constant two-form. Without loss of generality, we can then choose \( \Omega^a = \omega^{ab} K_b \) and one finds that

\[
\delta \phi \delta S = \int d^3x \, d^4\theta \, E \, \delta \phi^a \left( \frac{1}{2} \nabla_a \nabla_b \bar{K}_b \omega^b \bar{D}_a \varepsilon \bar{D}_a \varepsilon \bar{D}_a \varepsilon \bar{D}_a \phi^b \omega_{ab} + 2\mu \varepsilon \omega^b \bar{K}_b + \bar{D}_a \varepsilon \bar{D}_a \phi^b \omega_{ab} \right) .
\]  

(C.28)

Let us define \( V^a = \frac{\mu}{2S} \omega^{ab} K_b \) for shorthand. Then we have

\[
\delta \phi \delta S = \int d^3x \, d^4\theta \, E \, \delta \phi^a \left( - \frac{S}{\mu} \bar{D}_a \varepsilon \bar{D}_a \varepsilon \bar{D}_a \varepsilon \bar{D}_a \phi^b \nabla_a V_b + 4S \varepsilon V_a + \bar{D}_a \varepsilon \bar{D}_a \phi^b \omega_{ab} \right) .
\]  

(C.29)

The third term is zero since we can rewrite \( \bar{D}_a \varepsilon = i\sqrt{\mu/\bar{\mu}} \bar{D}_a \varepsilon \) and then integrate \( \bar{D}_a \varepsilon \) by parts to give zero.

The vanishing of the remaining terms requires

\[
0 = -\frac{1}{4} (\bar{\partial}^2 - 4\mu) \left( 4S \varepsilon V_a - \frac{S}{\mu} \bar{D}_a \varepsilon \bar{D}_a \phi^b \nabla_a V_b \right) .
\]  

(C.30)

Examining all the terms involving \( \bar{D}^2 \phi^b \), we find that these arise from

\[
\frac{S}{4\mu} \bar{D}^2 (\bar{D}_a \varepsilon \bar{D}_a \phi^b) \nabla_a V_b - S \varepsilon \bar{D}^2 \phi^b \nabla_b V_a = -S \varepsilon \bar{D}^2 \phi^b \nabla_a V_b - S \varepsilon \bar{D}^2 \phi^b \nabla_b V_a
\]  

(C.31)
so we conclude that $V_a$ obeys $\nabla_a V_b + \nabla_b V_a = 0$. Since it obeys $\nabla_a V_b + \nabla_b V_a = 0$ by construction, $V^a$ must be a Killing vector.

This turns out to be the final condition we require to prove $\delta S = 0$. The variation of the action can be written

$$\delta S = \int d^3 x \, d^4 \theta \left( - \frac{1}{2} g_{a\bar{b}} \bar{\mathcal{D}}_a \phi^b \bar{\mathcal{D}}^a \omega^{ab} \mathcal{K} + \text{c.c.} \right)$$

$$= \int d^3 x \, d^4 \theta \left( i \mathcal{D}_a \bar{\mathcal{D}}^a \phi^b V_b + \text{c.c.} \right)$$

$$= \int d^3 x \, d^4 \theta \left( - i \mathcal{D}_a \phi^b \bar{\mathcal{D}}^a \phi^b \nabla_b V_b + \text{c.c.} \right) = 0$$  \hspace{1cm} (C.32)

using the Killing condition.

A straightforward analysis of closure of the algebra reveals that $\omega^{ab} \omega_{bc} = -\delta_c^a$.

### C.4 (4,0) AdS supersymmetry in (2,0) AdS superspace

Our ansatz for the second supersymmetry is

$$\delta \phi^a = -\frac{1}{2} \bar{\mathcal{D}}^2 (\bar{\rho}_L \Omega^a_L) - \frac{1}{2} \bar{\mathcal{D}}^2 (\bar{\rho}_R \Omega^a_R)$$  \hspace{1cm} (C.33)

where $\bar{\rho}_L$ and $\bar{\rho}_R$ are antichiral and obey certain conditions discussed in section 11. Let's require as usual that $\delta \phi \delta S = 0$. This is quite similar to the (3,0) case we have already addressed. We find immediately that $\omega_{a\bar{b}} := g_{a\bar{c}} \Omega^c_{\bar{L}},b$ and $\omega_{a\bar{b}} := g_{a\bar{c}} \Omega^c_{\bar{R}},b$ must be antisymmetric and covariantly constant. This implies that

$$\delta \phi \delta S = \int d^3 x \, d^2 \theta \, \delta \phi^a \left( \frac{1}{2} \bar{\mathcal{D}}^a \phi^b \partial_a \Omega_{\bar{L}b} - 4S \bar{\mathcal{E}} \partial_a \omega_{\bar{L}a} + \text{m.c.} \right)$$  \hspace{1cm} (C.34)

where m.c. denotes mirror conjugate. Now let us perform the $\bar{\theta}$ integrals to give

$$\delta \phi \delta S = \int d^3 x \, d^2 \theta \, \mathcal{E} \delta \phi^a \left( S \bar{\mathcal{E}} \partial_a \partial_b \Omega_{\bar{L}b} + 2S \bar{\mathcal{E}} \partial_a \omega_{\bar{L}a} ight)$$

$$+ 4i S (2S + X) \bar{\mathcal{E}} \omega_{\bar{L}a} - i (2S + X) \bar{\mathcal{E}} \partial_b \Omega_{\bar{L}b} + S \bar{\mathcal{E}} \partial_a \omega_{\bar{L}a} + \text{m.c.} \right) \hspace{1cm} (C.35)

Before analyzing this further, we need a result from analyzing closure of the algebra. As we discussed in section 11.1, the holomorphic two-forms $\omega_{a\bar{b}}$ and $\omega_{a\bar{b}}$ must obey $\omega_{a\bar{b}}$, which implies the existence of covariantly constant projection operators $(P_L)^a_b$ and $(P_R)^a_b$. The Kähler metric obeys

$$g_{a\bar{b}} = (P_L)^a_{\bar{b}} + (P_R)^a_{\bar{b}} \equiv (g_L)_{a\bar{b}} + (g_R)_{a\bar{b}}$$  \hspace{1cm} (C.36)

with $g_{a\bar{b}} = \partial_a \partial_b K_L$ and $g_{a\bar{b}} = \partial_a \partial_b K_R$, with a Kähler potential given by the sum of two decoupled sectors, $K = K_L + K_R$. 

99
Deriving conditions for the non-critical case

In the non-critical case, both \( \varepsilon_L \) and \( \varepsilon_R \) are non-zero, so we must arrange for their coefficients in \((C.35)\) to vanish. This requires two conditions to be satisfied. The first condition,

\[ \partial_a \Omega_{Lb} = \partial_a \Omega_{Rb} = 0 \tag{C.37} \]

along with \( \nabla_a \Omega_{Lb} = \omega_{Lba} \), implies that

\[ \Omega^a_L = \omega^{ab}_L \chi^b_L, \quad \Omega^a_R = \omega^{ab}_R \chi^b_R, \tag{C.38} \]

where \( \chi^a \) is a homothetic conformal Killing vector. Hence, the target space must be a hyperkähler cone, which decomposes, due to the projection operators, into a left cone and a right cone. The second condition

\[ (2S + X)\Omega_{La} = 4iSJ^b_L \omega_{Lab}, \quad (2S - X)\Omega_{Ra} = 4iSJ^b_R \omega_{Rab}, \tag{C.39} \]

implies that

\[ J^a = \left( 1 + \frac{X}{2S} \right) J^a_L + \left( 1 - \frac{X}{2S} \right) J^a_R, \tag{C.40} \]

where

\[ J^a_L = -\frac{i}{2} (P_L \chi)^a, \quad J^a_R = -\frac{i}{2} (P_R \chi)^a. \tag{C.41} \]

Deriving conditions for the critical case

In the critical case \( X = 2S \), we can consistently take \( \varepsilon_R = \bar{\varepsilon}_R = 0 \). Then \((C.35)\) leads to

\[ \delta \phi \delta S = \int d^3x \ d^2 \theta \ E \ \delta \phi^a \left( S_{\varepsilon L} \bar{D}_a (\bar{D}^\alpha \phi^b \partial_\alpha \Omega_{Lb}) + 2S_{\varepsilon L} \bar{D}^a (D^b \phi^b \omega_{Rab}) \ight. \]

\[ - 4S \bar{\varepsilon}_L \bar{D}^b \phi^b \partial_\alpha \Omega_{La} + S_{\varepsilon L} \bar{D}^2 \Omega_{La} \bigg). \tag{C.42} \]

Invariance requires that \( \partial_a \Omega_{Lb} = 0 \). Introducing

\[ \chi^a_L = \omega^{ab}_L \Omega_{Lb} = \omega^{ab}_L \Omega_{Lb}^\dagger, \tag{C.43} \]

we find that

\[ \nabla_b \chi^a_L = 0, \quad \nabla_b \chi^a_L = (P_L)^a_b. \tag{C.44} \]
This means that the left sector is a hyperkähler cone. We can define its Kähler potential by

\[ K_L := \chi^a_L \chi_{La}, \quad \chi_{La} = \nabla_a K_L. \tag{C.45} \]

The remaining variation of the action is

\[ \delta \phi \delta S = \int d^3x d^2 \theta \mathcal{E} \delta \phi^a \left( 2S \bar{\varepsilon}_L \bar{\mathcal{D}}^a (\mathcal{D}_a \phi^b \omega_{Rab}) \right). \tag{C.46} \]

This vanishes only if \( \omega_{Rab} \jmath^b = 0 \), so we conclude that

\[ \jmath^a = 2 \jmath^a_L \tag{C.47} \]

where \( \jmath^a_L \) is some Killing vector in the left sector. The normalization is chosen to match (C.40) and for later convenience.

Further information about the Killing vector \( \jmath^a_L \) can be gleaned by requiring consistency of the extended supersymmetry transformation with the U(1) isometry. In other words, we require

\[ \delta (i \jmath_L \phi^a) = \delta \jmath^a_L = -\frac{1}{2} \bar{D}^2 \left( \bar{\rho}_L \Omega^b_L \partial_b \jmath^a_L + \bar{\rho}_R \Omega^b_R \partial_b \jmath^a_L \right) \tag{C.48} \]

to match

\[ i \jmath_L \delta \phi^a = -\frac{1}{2} \bar{\jmath}_L \bar{D}^2 (\bar{\rho}_L \Omega^a_L) - \frac{1}{2} \jmath_L \bar{D}^2 (\bar{\rho}_R \Omega^a_R) \]
\[ = \frac{1}{2} \bar{D}^2 (\bar{\rho}_L \Omega^a_L) - \frac{1}{2} \bar{D}^2 (\bar{\rho}_L \jmath_L \Omega^a_L) - \frac{1}{2} \bar{D}^2 (\bar{\rho}_R \jmath_L \Omega^a_R). \tag{C.49} \]

This forces

\[ 0 = \frac{1}{2} \bar{D}^2 \left( \bar{\rho}_L (\jmath^b_L \nabla_b \Omega^a_L - \Omega^b_L \nabla_b \jmath^a_L) - i \bar{\rho}_L \Omega^a_L + \bar{\rho}_L \jmath^b_L (\omega_L)^a_{b} \right) \]
\[ + \bar{\rho}_R \jmath^b_L \nabla_b \Omega^a_R + \bar{\rho}_R \jmath^b_L (\omega_R)^a_{b} - \bar{\rho}_R \Omega^b_R \nabla_b \jmath^a_L \right). \tag{C.50} \]

All the terms involving \( \bar{\rho}_R \) cancel using the left and right projection operators. (Here we take \( \Omega^a_R = \omega^a_{Rb} \kappa_b \) so that \( \Omega^a_R \) is a vector in the right sector.) For the remaining terms, we use \( \Omega^a_L = \omega^a_{Lb} \chi_L b \) and find that a certain combination of terms must vanish,

\[ 0 = -i \omega^a_{Lb} \chi_L b - \omega^a_{Lc} \chi_L c \nabla_b \jmath^a_L + (\omega_L)^a_{b} \bar{\jmath}^b_L. \tag{C.51} \]

We can decompose \( \jmath^a_L \) as

\[ \jmath^a_L = -\frac{i}{2} \chi^a_L + Z^a_L. \tag{C.52} \]
Since $J^a_L$ is a Killing vector by assumption and $\chi^a_L$ is Killing by construction, $Z^a_L$ must also be a Killing vector. Using this expression, one can prove that
\[
\mathcal{L}_{Z^a_L} \Omega^a_L = 0 \implies \mathcal{L}_{Z^a_L} (\omega_L)^{a\bar{b}} = 0 ,
\] (C.53)
and so $Z^a_L$ is a tri-holomorphic Killing vector. Since $\Omega^a_L = \omega^{ab}_L \chi^b_L$, we find the additional condition that $Z^a_L$ commutes with the homothetic conformal Killing vector.

All the relevant features of the target space geometries in the critical and non-critical cases have now been determined. The interested reader can straightforwardly check that the action is invariant, $\delta S = 0$, by applying techniques similar to those we have used elsewhere in this appendix.

D $\mathcal{N} = 4 \rightarrow (1,1)$ AdS superspace reduction

In subsection 7.2 we studied the reduction of all $\mathcal{N} = 4$ AdS superspaces to the (2,0) one. It was observed that the (4,0) case admits only the reduction to (2,0) AdS superspace. The (3,1) and (2,2) superspaces also admit consistent reductions to (1,1) AdS. Here we elaborate on the details of the (1,1) reduction procedures. This analysis is parallel to that of the (2,0) reduction while some differences occur.

We start by noting that, for the (3,1) and (2,2) AdS supergeometries, the conditions $S^{11\bar{1}\bar{2}} = S^{12\bar{1}} = S^{12\bar{2}} = 0$ make the algebra (7.17a)–(7.17d) isomorphic to that of the (1,1) AdS superspace. Let us now see how to project the (3,1) and (2,2) AdS superspaces to (1,1).

The $\mathcal{N} = 2$ projection of a tensor field and of the $\mathcal{N} = 4$ covariant derivatives is defined as in eqs. (7.21)–(7.23). Since the (3,1) and (2,2) derivatives $(\mathcal{D}_a, \mathcal{D}_{\alpha}^{11}, -\mathcal{D}_{\alpha}^{22})$ form a closed algebra, which is isomorphic to that of the covariant derivatives of the (1,1) AdS superspace, one can use the freedom to perform general coordinate, local Lorentz and SU(2) transformations to chose a gauge in which
\[
\mathcal{D}^{11}_{\alpha} = \sqrt{-i \frac{\mu}{|\mu|}} \mathcal{D}_\alpha , \quad \mathcal{D}^{22}_{\bar{\alpha}} = -\sqrt{i \frac{\mu}{|\mu|}} \bar{\mathcal{D}}_{\bar{\alpha}} .
\] (D.1)
Here $\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_{\alpha}, \bar{\mathcal{D}}^{\bar{\alpha}})$ denote the covariant derivatives of the (1,1) AdS superspace, eq. (2.2). They obey the (anti) commutation relations (2.47).

Next consider a Killing vector field for one of the $\mathcal{N} = 4$ AdS superspaces,
\[
\xi = \xi^a \mathcal{D}_a + \xi^{\alpha} \mathcal{D}_{\alpha}^{\bar{\alpha}} .
\] (D.2)
We introduce the $\mathcal{N} = 2$ projections of the transformation parameters involved

$$l^a := \xi^a|, \quad l^\alpha := \sqrt{-i|\mu| \xi^\alpha_{12}}, \quad \bar{\alpha} = \sqrt{i|\mu| \xi_{21}}, \quad \chi^{ab} := \Lambda^{ab} ; \quad (D.3a)$$

$$\varepsilon^\alpha := -\sqrt{i|\mu| \xi^\alpha_{12}}, \quad \bar{\varepsilon}^\alpha := \sqrt{-i|\mu| \xi_{21}}, \quad \varepsilon_L = -\frac{1}{4S} \Lambda^{22}, \quad T_L := -i\Lambda^{22}, \quad T_R := -\frac{1}{4S} \Lambda^{22} \quad (D.3b)$$

The parameters $(l^a, l^\alpha, \bar{\alpha}, \chi^{ab})$ describe the infinitesimal isometries of the $(1,1) \text{AdS}$ superspace. This can be easily proven by $\mathcal{N} = 2$ projection of the equations (7.10)–(7.11e).

The parameters $(\varepsilon^\alpha, \bar{\varepsilon}^\alpha, \bar{\varepsilon}^\alpha_L, \varepsilon_R, \bar{\varepsilon}^r, T_L, T_R)$ describe the extra supersymmetry and $R$-symmetry transformations of either the (3,1) or (2,2) superspace. The (1,1) projection of the relations (7.10)–(7.11e) and (7.14c)–(7.14d) gives certain constraints on the parameters in (D.3b)–(D.3c). Such constraints are different in the (3,1) and (2,2) cases as we are going to describe now.

$(3,1) \rightarrow (1,1)$ For the reduction from (3,1) to (1,1) AdS superspace, we can always choose $w^{ij}$ to be

$$w^{12} = w^{21} = 0, \quad \mu := iS(w^{22})^2, \quad \bar{\mu} = -iS(w^{11})^2 \quad (D.4)$$

The constraint on the (3,1) structure group, eq. (7.14c), implies that

$$\varepsilon := \varepsilon_L, \quad \varepsilon_R = -\frac{i\mu}{|\mu|} \bar{\varepsilon}, \quad T_L = T_R := T \quad (D.5)$$

In the (3,1) case, the (1,1) projection of eqs. (7.10)–(7.11e) gives

$$\bar{\mathcal{D}}_a \varepsilon_{\beta} = -4|\mu| \varepsilon_{a\beta} \varepsilon, \quad \mathcal{D}_a \varepsilon_{\beta} = 4|\mu| \varepsilon_{a\beta} \varepsilon, \quad (D.6a)$$

$$\mathcal{D}_a \varepsilon = \frac{1}{2} \varepsilon_a, \quad \bar{\mathcal{D}}_a \varepsilon = -\frac{1}{2} \bar{\varepsilon}_a, \quad \mathcal{D}_a T = \bar{\mathcal{D}}_a T = 0 \quad (D.6b)$$

These imply the conditions

$$\mathcal{D}_a \left( \varepsilon_{\beta} - \bar{\varepsilon}_{\beta} \right) = \bar{\mathcal{D}}_a \left( \varepsilon_{\beta} - \bar{\varepsilon}_{\beta} \right) = 0 \quad (D.7)$$

$(2,2) \rightarrow (1,1)$ For the reduction from (2,2) to (1,1) superspace we can always choose the constant parameters $l^{ij}$ and $r^{ij}$ to be

$$l^{12} = r^{12} = 0, \quad l := l^{11}, \quad r := r^{11}, \quad |l| = |r| = 1, \quad \mu := -iSl, \quad \bar{\mu} = iSr \quad (D.8)$$
The constraint on the \((2,2)\) structure group, eq. (7.14d), implies that
\[
\varepsilon_L = \bar{l}\varepsilon_L, \quad (\varepsilon_L)^* = \varepsilon_L, \quad \varepsilon_R = \bar{r}\varepsilon_R, \quad (\varepsilon_R)^* = \varepsilon_R, \quad T_L = T_R = 0. \tag{D.9}
\]

In the \((2,2)\) case, the \((1,1)\) projection of eqs. (7.10)–(7.11e) gives
\[
\begin{align*}
\mathcal{D}_a \bar{\varepsilon}_\beta &= -4\varepsilon_{a\beta}|l|\varepsilon_L, \quad \mathcal{D}_a \varepsilon_\beta = -4\varepsilon_{a\beta}|\mu|\bar{l}\varepsilon_R, \tag{D.10a} \\
\mathcal{D}_a \varepsilon_L &= \frac{i}{2} \bar{l}\varepsilon_\alpha, \quad \mathcal{D}_a \varepsilon_R = \frac{i}{2} l\varepsilon_\alpha. \tag{D.10b}
\end{align*}
\]

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