de Sitter-covariant Hamiltonian formalism
of Einstein–Cartan gravity

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Abstract

The Hamiltonian formalism of Einstein–Cartan (EC) gravity is a starting point for canonical quantum gravity. The existing formalisms are at most Lorentz covariant, or diffeomorphism covariant. Here we analyze the Hamiltonian EC gravity in a 5d covariant way, with the gauge group being the de Sitter (dS) group, which unifies the Lorentz transformations and translation in an elegant manner, and also coincides with the acceleration of the universe. We reformulate the EC equations into a dS-covariant form, then find out the dS-covariant constraints of the phase space, and make all the constraint functions constitute a closed algebra by constructing a dS-invariant Dirac bracket, for the purpose of quantization.

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1 Introduction

In search of a quantum theory of gravity, generally one should start from a classical theory and then quantize it. As the simplest classical theory of gravity, Einstein’s general relativity (GR) is unnatural in the viewpoint of a gauge theory, since the Lorentz connection as the gauge potential is not an independent variable. By including spacetime torsion, the Lorentz connection becomes independent, then one obtains the Einstein–Cartan (EC) theory of gravity [1].

The EC gravity is usually interpreted as a Poincaré gauge theory of gravity, in which the gauge transformations are the Lorentz and diffeomorphism transformations, acting on the Lorentz connection and co-tetrad field [2, 3]. But there exist alternative interpretations, where the Lorentz connection and co-tetrad field are combined into a 5d connection, valued at the Poincaré/de Sitter/anti-de Sitter (P/dS/AdS) algebra [4–7]. Then the gauge transformations consist of the P/dS/AdS and diffeomorphism transformations, acting on the 5d connection and a 5d vector field $\xi^A$. Actually, $\xi^A$ constitutes a system of local 5d Minkowski coordinates, named the local inertial coordinates (LIC) [5, 8].

In this formulation, there exist some gauges in which the 5d connection reduces to the Lorentz connection and co-tetrad field. These gauges constitute a Lorentz subgroup of the P/dS/AdS symmetry. Also, matter fields in the standard model of particle physics are described by representations of the Lorentz group, other than the complete P/dS/AdS group. For these reasons, it is argued that the 5d connection should be projected into a

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Cartan connection, with the Lorentz group as the stability subgroup \[5, 7, 9\]. Note that the Cartan connection transforms nonlinearly under the complete gauge group, and so the corresponding formalism is called nonlinear realization \[10\].

However, the P/dS/AdS symmetry remains the true symmetry for gravitation. Accordingly, the original linearly realized formulation \[11–14\] is more fundamental than the nonlinear realization. Besides, the linear realization is able to include those matter fields transforming under the full representations of the P/dS/AdS group \[15, 16\]. Such new matter fields might be responsible for new physics \[17, 18\].

Moreover, among the P, dS and AdS groups, the dS group seems to be the best choice for gravity. Firstly, the dS/AdS group provides an elegant unification of the Lorentz transformations and translation, resulting in a 5d covariant theory. For example, the energy-momentum conservation and angular momentum conservation can be united into a 5d covariant conservation in the dS/AdS gravity \[18\]. Secondly, the dS group is consistent with the asymptotic symmetry of the expanding universe.

In this paper, the EC gravity is analyzed in the linearly realized formalism, with the dS group as the gauge group. It is shown that the field equations, consisting of the Einstein equation and Cartan equation, can be united into a 5d covariant equation. Then we go on to the Hamiltonian formalism. Making use of the Legendre transformation, the configuration tangent bundle is transformed into the phase space. Next, with the help of the Arnowitt–Deser–Misner (ADM) decomposition, we find out the first class constraints which generate the dS and diffeomorphism transformations on the constraint surface. Furthermore, the Poisson bracket is modified into a Dirac bracket, such that all the second class constraints become first class, and so all the constraint functions form a closed Poisson algebra. The work paves the way for the canonical quantization of the dS-covariant theory of gravity.

The paper is organized as follows. In section 2, the EC gravity is formulated in a dS-covariant way. In section 3, the Hamiltonian formalism of the theory is calculated. In section 4, we give some remarks on the linear realization.

Here are the conventions to be used. The Greek letters \(\mu, \nu\ldots\) label the spacetime indices and take the values \(t, a\), being lowered or raised by the metric \(g_{\mu\nu}\) or its inverse. The Latin letters \(a, b\ldots\) label the spatial indices and run over \(x, y, z\). Also, the Latin letters \(A, B\ldots\) refer to the \(SO(1, 4)\) indices and take the values \(0, 1\ldots 4\), being lowered or raised by \(\eta_{AB} = \text{diag}(-1, 1\ldots 1)\) or its inverse. The Greek letters \(\alpha, \beta\ldots\) refer to the \(SO(1, 3)\) indices and run over \(0, 1\ldots 3\), being lowered or raised by \(\eta_{\alpha\beta} = \text{diag}(-1, 1\ldots 1)\) or its inverse.

## 2 EC gravity as dS gravity

### 2.1 dS gravity from gauge principle

The dS gravity is a gauge theory of the dS group. In the gauge theory, a global symmetry is localized by introducing a gauge field. For the present case, the global symmetry is the dS group \(SO(1, 4)\). Let us start from a classical matter field with both the global dS invariance and the diffeomorphism invariance. Its action integral reads

\[
S_M = \int_{\Omega} d^4x L_M \sqrt{-g}, \quad L_M = L_M(\psi, \partial_\mu \psi, \text{c.c.}, \xi^A, \partial_\mu \xi^A), \quad (1)
\]
where $Ω$ is an arbitrary domain of the dS spacetime $M_t$, \{x$^\mu$\} is an arbitrary coordinate system on $Ω$, $g$ is the determinant of the dS metric $g_{\mu\nu}$, $ψ$ is the matter field, and $ξ^A$ is the radius vector field of $M_t$, viewed in the 5d ambient Minkowski space, and subject to the condition $η_{AB}ξ^Aξ^B = l^2$. Note that $g_{\mu\nu}$ is considered as a functional of $ξ^A$:

$$g_{\mu\nu} = η_{AB}(∂_\mu ξ^A)(∂_\nu ξ^B),$$

(2)

and so $S_M$ is a functional of $ψ$ and $ξ^A$. The conservation law with respect to the dS and diffeomorphism symmetries of this theory is discussed in Ref. [19]. In order to localize the dS symmetry, introduce a dS connection $Ω^A_{\mu\nu}$ and change the ordinary derivative $D_\mu$ to be a covariant derivative $D_\mu$, e.g., $D_\mu ξ^A = ∂_\mu ξ^A + Ω^A_{\mu\nu}ξ^B$. It follows that [4, 5]

$$g_{\mu\nu} = η_{AB}(D_\mu ξ^A)(D_\nu ξ^B).$$

(3)

Consider the gauges with $ξ^\alpha = 0, ξ^4 = l$. For any dS transformation given by the group element $h^A_B ∈ SO(1, 4)$, $ξ^A$ transforms to $h^A_Bξ^B$. To preserve the gauge condition, there should be $h^\alpha_\beta ∈ SO(1, 3)$, $h^\alpha_4 = h^4_\alpha = 0$, and $h^4_4 = 1$. For this reason, we call the gauges the Lorentz gauges. Then Eq. (3) reduces to $g_{\mu\nu} = η_{αβ}(D_\mu ξ^α)(D_\nu ξ^β)$, implying that $D_μξ^α$ is an orthonormal co-tetrad field, denoted by $e^α_\mu$. Moreover, note that $D_μξ^α = Ω^α_{\mu\nu}ξ^\nu$ and thus $Ω^α_{\mu\nu} = l^{-1}e^α_\mu$. Also, the geometrical meaning of $Ω^α_{\beta\mu}$ can be read off from its transformation property: it is just a Lorentz connection, denoted by $Γ^α_{\beta\mu}$. In conclusion, in the Lorentz gauges, the dS connection [4, 7]

$$Ω^AB_{\mu} = \begin{pmatrix} Γ^α_μ & l^{-1}e^α_μ \\ -l^{-1}e^β_μ & 0 \end{pmatrix}. \tag{4}$$

It is derived from the gauge principle, other than being defined ad hoc. To complete the construction of dS gravity, introduce the action integral of the gravitational field:

$$S_G = ∮Ω x L_G √−g, \quad L_G = L_G(ξ^A, D_μξ^A, F^AB_{\mu\nu}),$$

(5)

where $F^AB_{\mu\nu} = d_\mu Ω^A_{\nu\sigma} + Ω^A_{\mu\nu} ∧ Ω^C_{\nu\sigma}$ is the dS curvature. Define $S = S_M + κS_G$ and the variational derivatives $V^AB_{\mu}$, $V_A$ by $δS = ∮Ω x (∂L_G / ∂F^AB_{\mu\nu}) V^AB_{\mu} + V_A δξ^A) √−g$, where $κ$ is the coupling constant, and $V_A δξ^A ≡ 0$, because $δξ^A$ is constrained by $ξ^Aξ_A = l^2$. Then the gravitational field equations consist of $V^AB_{\mu} = 0$ and $V_A = 0$. Also, the conservation law with respect to the local dS symmetry and diffeomorphism symmetry is discussed in Ref. [18]. With the help of this, we have

$$V^AB_{\mu} = τ^AB_{\mu} + Σ^ν_\mu D^\nu[ξ^A ; ξ_B],$$

(6)

$$V_A = V^BC_{\nu} D^\nu[ξ_A ; F^{BC}_{\nu\mu}],$$

(7)

where $τ^AB_{\mu} = (∂L G / ∂D_\mu ψ) T^{AB}_\mu ψ + c.c. + 2D_\nu D_\mu L / ∂F^{AB}_{\mu\nu}$ is the dS spin current, $Σ^\nu_\mu = −(∂L G / ∂D_\mu ψ) D_\nu ψ + c.c. − 2(∂L G / ∂F^{AB}_{\mu\nu}) F^{AB}_{\mu\nu} + L δ^\mu = 0$, $F^{AB}_{\mu\nu}$ is the energy-momentum tensor, $L = L_M + κL_G$, and $T^{AB}$ are representations of the dS generators. In the special relativity limit with $F^{AB}_{\mu\nu} = 0$, $V_A = 0$ holds automatically. In the general theory of dS gravity, $V_A = 0$ as long as $V^AB_{\mu} = 0$. Hence, the gravitational field equation is only given by $V^AB_{\mu} ≡ δS / δΩ^{AB}_{\mu} = 0$. 

3
2.2 EC theory of gravity revisited

So far the gravitational Lagrangian function (5) is rather arbitrary. To recover the EC gravity (with a cosmological constant), put \( \mathcal{L}_g = R - 2\Lambda \), where \( R = R^{\alpha\beta\mu\nu}e_\alpha^\mu e_\beta^\nu \) is the trace of the Lorentz curvature \( R^{\alpha\beta\mu\nu} = d_\mu \Gamma_\alpha^\beta_\nu + \Gamma_\alpha^\gamma_\mu \land \Gamma_\gamma^\beta_\nu \), and \( \Lambda = 3/l^2 \). Making use of Eq. (4), in the Lorentz gauges,

\[
F^{AB\mu\nu} = \begin{pmatrix}
R^{\alpha\beta\mu\nu} & -l^{-2}e_\alpha^\mu \land e_\beta^\nu & l^{-1}S_\alpha^\beta & 0 \\
-l^{-1}S_\beta^\alpha & 0 & 0 & 0
\end{pmatrix}
\]  

(8)

(c.f. [4, 7]), where \( S_\alpha^\beta = d_\mu \Gamma_\alpha^\beta_\mu + \Gamma_\alpha^\gamma_\mu \land e_\beta^\nu \) is the torsion 1-form. Then the EC Lagrangian function can be rewritten by

\[
R - 2\Lambda = F^{AB\mu\nu}(D^\mu \xi_A)(D^\nu \xi_B) + 2\Lambda,
\]

which is dS invariant, and so valid in any gauge. Now the field equation reads

\[
G^\mu_\nu D^\nu \xi_A \land \xi_B - D_\nu(D^\mu \xi_A \land D^\nu \xi_B) = \kappa^{-1} J_{AB,\mu}^\mu,
\]

(10)

where the first term on the left hand side corresponds to the 5d (orbital) angular momentum current, containing the Einstein tensor \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} \); the second term corresponds to the 5d spin current, containing the torsion and the cosmological constant; and \( J_{AB,\mu}^\mu = \delta S_M/\delta \Omega_{AB,\mu} \) is the material current. Moreover, define the effective energy-momentum tensor \( \xi_{\nu}^\mu = J_{AB,\mu}^\mu(D_\nu \xi_A)(2\xi_B/l^2) \), and the spin tensor \( \tau_{\nu\sigma}^\mu = J_{AB,\mu}^\mu(D_\nu \xi_A)(D_\sigma \xi_B) \). Then the standard form of the EC equations [1–3] can be recovered from Eq. (10): \( R^\mu_\nu - \frac{1}{2}Rg^\mu_\nu + \Lambda \delta^\mu_\nu = (2\kappa)^{-1} \xi_{\nu}^\mu, S^\mu_\nu + 2 \delta^\mu_\nu S^{\rho}_\sigma \rho = -\kappa^{-1} \tau_{\nu\sigma}^\mu \).

3 dS-covariant Hamiltonian formalism

3.1 Consistent constraint surface

To perform the Hamiltonian analysis, suppose that the spacetime region \( \Omega \) has a 3+1 decomposition \( \Omega = \Sigma \times I \), where \( \Sigma \) is a spacelike submanifold, and \( I \) is an open interval on the real line. Define the Lagrangian functional \( L = \int_{\Sigma} d^3x \mathcal{L}[q,v] \), where the Lagrangian density \( \mathcal{L} = (R - 2\Lambda)\sqrt{-g} \), the configuration \( q = y^\mu, \Omega_{AB,\mu} \), and the velocity \( v = \dot{q} \equiv \partial q/\partial t \). Here \( y^\mu = y^\mu(x) \) is a parametrization of the constrained vector field \( \xi^A(x) \), such that \( \xi^A(x) = \xi^A(y^\mu(x)) \).

Moreover, one can calculate \( \delta L/\delta \dot{\mu} = -2G^\nu_\mu D_\nu \xi_A(\partial \xi^A/\partial y^\mu)\sqrt{-g} \), and \( \delta L/\delta \dot{\Omega}_{AB,\mu} = (D^\mu \xi_A \land D^\mu \xi_B)\sqrt{-g} \). Although not explicit, it can be shown that neither of them depends on \( v \), leading to two primary constraints:

\[
\phi_\mu = \pi_\mu + 2G^\nu_\mu D_\nu \xi_A(\partial \xi^A/\partial y^\mu)\sqrt{-g},
\]

(11)

\[
\phi_{AB,\mu} = \pi_{AB,\mu} - (D^\mu \xi_A \land D^\mu \xi_B)\sqrt{-g}.
\]

(12)

where the momenta \( p = \pi^\mu, \pi_{AB,\mu} \) are viewed as new variables. Then the Hamiltonian functional can be written down: \( H = \int_{\Sigma} d^3x \mathcal{H}[q,p,v] \), in which the Hamiltonian density \( \mathcal{H} = p \cdot v - \mathcal{L} \). The consistency condition of a constraint is that its evolution according to the Hamiltonian equations \( \dot{q} = \delta H/\delta p \) and \( \dot{p} = -\delta H/\delta q \) is equal to zero. For the
primary constraints (11)–(12), the consistency conditions lead to two things: The first is the solution of $\tilde{\Omega}^{AB}_a$, as a functional of $q$ and $\dot{y}^\mu$; and the second is the secondary constraint $C_{AB}$. The results are as below:

$$\dot{\Omega}^{AB}_a = D_a \Omega^{AB}_t - t^{-2} D_t \xi^A \wedge D_a \xi^B + R_{\mu\nu\lambda}(D^b \xi^A)(D^c \xi^B) + (g^{tt})^{-1} D^t \xi^A \wedge D^t \xi^B (\Lambda g_{ab} - R^c_{\ a b c} - R_{t b c a} g^{ct}),$$

(13)

$$C_{AB} = -(G_{\nu}^\mu + \Lambda \delta_{\nu}^\mu) D^\nu \xi_A \wedge \xi_B \sqrt{-g} + T_{\nu\sigma}^\nu (D^\nu \xi_A)(D^\rho \xi_B) \sqrt{-g},$$

(14)

where $T^{\sigma}_{\mu \nu} = S^{\sigma}_{\mu \nu} + 2 \delta^{\sigma}_{[\mu} S_{\nu]}$, $S^{\nu}_{\sigma \nu}$. In virtue of the Bianchi identity $D_{[\sigma} F_{\mu \nu]} = 0$, it can be verified that the secondary constraint is consistent already. Consequently, the consistent constraint surface of the phase space $P$ is given by the vanishing of the constraints (11)–(12) and (14).

### 3.2 First-class constraints and symmetries

Next, we shall recombine the above constraints into two classes. A function $F[q, p]$ of the constrained phase space is called first class, if for any constraint $\phi$, the Poisson bracket $\{F, \phi\} \approx 0$, i.e., $\{F, \phi\}$ vanishes on the constraint surface. Otherwise, $F$ is called second class. Here the Poisson bracket is defined by these fundamental relations:

$$\{y^\mu(\bar{x}), \pi_{\nu}(\bar{z})\} = \delta^\mu_\nu \delta(\bar{x} - \bar{z}),$$

(15)

$$\{\Omega^{AB}_\mu(\bar{x}), \pi_{CD}^\nu(\bar{z})\} = \delta^{[A}_{C \ D} \delta^{B]}_{\ D} \delta^\nu_\mu \delta(\bar{x} - \bar{z}),$$

(16)

where $\bar{x}, \bar{z}$ denote the points on the spatial surface $\Sigma$.

The first class constraints can be obtained by analyzing the first-class Hamiltonian, which is defined by inserting the velocity solution (13) into the original Hamiltonian: $H_1[q, p, v_1] \equiv H[q, p, v_1, v_2[q, p, v_1]]$, where $v_1 = \dot{y}^\mu$, $\dot{\Omega}^{AB}_t$ are the unsolvable velocities as free parameters, and $v_2 = \dot{\Omega}^{AB}_a$ is the solvable velocity with the solution $v_2[q, p, v_1] = v_2[q, v_1]$ given by Eq. (13). The consistency of any constraint $\phi$ implies that $\{\phi, H_1\} \approx 0$, and so $H_1$ is a first-class function of $P$. Actually, $H_1$ is a first-class constraint. To see this, first notice that

$$H_1[q, p, v_1] = H[q, p] + \int_\Sigma d^3 x \left( \phi_1 v_1 + \phi_2 v_2[q, v_1] \right),$$

(17)

where $H[q, p] \equiv H[q, p, 0]$, $\phi_1 = \phi^\mu$, $\phi^{AB}_t$, and $\phi_2 = \phi^{AB}_a$. Then it suffices to show that $H[q, p] \approx 0$. By definition, $H[q, p] = -L[q, 0]$. With the help of the Noether identity $(\partial L/\partial D^A_{\mu} \xi^A) D^A_{\nu} \xi^A + 2(\partial L/\partial F_{\mu \nu}^{AB}) F_{\nu \sigma}^{AB} = L \delta^\mu_\nu$ with respect to the diffeomorphism invariance [18], it follows that $L[q, 0] = \int_\Sigma d^3 x \delta L/\delta \Omega^{AB}_t \cdot \Omega^{AB}_t$. Moreover, in virtue of $C^{AB}_t = \delta L/\delta \Omega^{AB}_t$, we have

$$H[q, p] = - \int_\Sigma d^3 x \ C^{AB}_t \Omega^{AB}_t \approx 0,$$

(18)

and hence $H_1$ is a first-class constraint. In $H_1$ given by Eqs. (17)–(18), there are two sets of free parameters: $\Omega^{AB}_t$ and $v_1$, which correspond to two sets of first-class constraints. To find out these constraints, it is convenient to use the ADM decomposition [20] of the
time direction basis vector: \((\partial_t)^\mu = N n^\mu + N^\mu\), where \(N\) is named the lapse function, \(N^\mu\) is named the shift vector, which is tangent to \(\Sigma\), and \(n^\mu\) is normal to \(\Sigma\), with \(n^\mu n_\mu = -1\). Accordingly, \(D_t \xi^A\) can be decomposed as

\[
D_t \xi^A = ND_\perp \xi^A + N^a D_a \xi^A, \tag{19}
\]

where \(D_\perp \xi^A = n^\mu D_\mu \xi^A\) can be solved as the functions of \(D_a \xi^A\). Note that \(\dot{\xi}^A = (\partial y^\mu / \partial \xi^A) (ND_\perp \xi^A + N^a D_a \xi^A - \Omega^A_{Bt} \xi^B\), and so the free parameters \(\dot{\xi}^A\) can be replaced by \(N\) and \(N^a\). Putting this replacement into Eq. (17) results in \(H_1[q, p, v_1] = \int_\Sigma d^3x (N H_\perp + N d H_d + \Omega^A_{Bt} \xi^B + \Omega^A_{Bt} H_{AB})\), where \(H_\perp = \phi_A D_\perp \xi^A + \phi_{AB}^a \mathcal{F}^{AB} d_a\), \(H_d = \phi_A D_\mu \xi^A + \phi_{AB}^a \mathcal{F}^{AB} \mu\), \(\mathcal{F}^{AB} \mu = \mathcal{F}^{AB} \mu a n^\mu\), \(\pi_A = \pi_{\mu} \partial y^\mu / \partial \xi^A\), and likewise, \(\phi_A = \phi_{\mu} \partial y^\mu / \partial \xi^A\).

It can be shown that the constraints \(H_\perp, H_d, H_{AB}\), and \(\phi_{AB}^t\) do not depend on the free parameters \(N, N^a, \Omega_{AB}^t, \) and \(\Omega_{AB}^t\), and so they are first-class constraints according to the above-mentioned expression for \(H_1\). Further, the spatial contractions in \(H_\perp, H_d,\) and \(H_{AB}\) can be replaced by space-time contractions by including \(\phi_{AB}^t\) into the original expressions, leading to

\[
\begin{align*}
H_\perp &= \phi_A D_\perp \xi^A + \phi_{AB}^a \mathcal{F}^{AB} \perp a, \quad (20) \\
H_d &= \phi_A D_\mu \xi^A + \phi_{AB}^a \mathcal{F}^{AB} \mu a, \quad (21) \\
H_{AB} &= -D_\mu \pi_{\mu AB}^a - \pi_{[A} \xi_{B]} , \quad (22)
\end{align*}
\]

all of which are first-class constraints again. They are called the lapse, shift, and dS constraints, respectively. As will be seen, these constraints represent the normal/tangential diffeomorphism invariance, and dS invariance of the system.

Generally, the symmetry transformation of \(\mathcal{P}\) is defined by

\[
\xi^A \rightarrow g^{A}_{B} \phi_{A} \xi^B, \tag{23}
\]

\[
\Omega^{A}_{B\mu} \rightarrow g^{A}_{B C} \phi_{C} \Omega_{C D} \mu (g^{-1})^{B}_{D} + g^{A}_{C} \partial_{\mu} (g^{-1})^{C}_{B}\tag{24},
\]

\[
\pi_{A} \rightarrow \phi_{A} \pi_{B} (g^{-1})^{B}_{A} \det(\partial \phi_{A} x^{\nu} / \partial x^{\sigma}), \tag{25}
\]

\[
\pi_{AB}^\mu \rightarrow \phi_{A} \pi_{C}^{\mu} (g^{-1})^{C}_{A} (g^{-1})^{D}_{B} \det(\partial \phi_{A} x^{\nu} / \partial x^{\sigma}), \tag{26}
\]

where \(g^{A}_{B}\) is an \(SO(1, 4)\)-valued function, and \(\phi_{A}\) is the pushforward by a diffeomorphism transformation \(\phi\). Vary \(g^{A}_{B}\) and \(\phi\) to give the one-parameter local groups \((g_{\lambda})^{A}_{B}\) and \(\phi_{\lambda}\) with the parameter \(\lambda\). Differentiation of Eqs. (23)-(26) with respect to \(\lambda\) gives rise to the infinitesimal transformation:

\[
\begin{align*}
\delta \xi^A &= A^{A}_{B \xi} \xi^B - L_v \xi^A, \quad (27) \\
\delta \Omega^{A}_{B\mu} &= -D_\mu A^{A}_{B} - L_v \Omega^{A}_{B\mu}, \quad (28) \\
\delta \pi_{A} &= -\pi_{B} A^{A}_{B} - L_v \pi_{A} - \pi_{A} \partial_{\nu} v^{\nu}, \quad (29) \\
\delta \pi_{AB}^\mu &= -\pi_{CB}^{\mu} A^{C}_{A} - \pi_{AC}^{\mu} A^{C}_{B} \\
&- L_v \pi_{AB}^\mu - \pi_{AB}^\nu \partial_{\nu} v^{\mu}. \quad (30)
\end{align*}
\]

where \(A^{A}_{B} = \partial / \partial \lambda|_{\lambda=0} (g_{\lambda})^{A}_{B}\), \(v|_{x} = \partial / \partial \lambda|_{\lambda=0} (\phi_{\lambda} x)\), and \(L_v\) is the Lie derivative along \(v\), e.g., \(L_v \pi_{AB}^{\mu} = v^{\nu} \partial_{\nu} \pi_{AB}^{\mu} - \pi_{AB}^{\nu} \partial_{\nu} v^{\mu}\). In the above transformation, putting \(v = 0\) yields
the dS transformation $\delta_A$, while putting $A^A_B = -\Omega^A_{B\mu}v^\mu$ yields the dS-invariant diffeomorphism $\delta_v$ [18]. These transformations can be generated by the first-class constraints (20)–(22) in the following way. Firstly, a function $F$ of the constrained phase space is said to be generating a symmetry, if its Hamiltonian vector field $\chi_F \equiv \{F, \cdot \}$ generates a symmetry. Secondly, define the distributional quantities corresponding to the constraints $\chi_\delta$, the dS transformation $H$, then the constraint algebra becomes closed. For the solution of $\chi_\delta$, it is necessary for the validity of Eq. (30), its existence is supported by the independence of the components of $\phi$.

Also, notice that the inclusion of $\phi_{AB}^i$ in Eqs. (20)–(22) is necessary for the validity of Eq. (31) acting on $\Omega^{AB}$.  

3.3 Second-class constraints and Dirac bracket

According to Dirac’s quantization procedure, the constraints are solved after they are quantized, resulting in a physical Hilbert space. When acting on this Hilbert space, the constraint operators as well as their commutators give zero, and hence the constraint algebra should be closed under the Poisson/Lie bracket [21, 22].

The constraints of EC gravity (11)–(12) and (14) can be recombined into the first-class $\mathcal{H}_\perp, \mathcal{H}_d, \mathcal{H}_{AB}, \phi_{AB}^i$ and the second-class $\phi_{AB}^a$. Because of the existence of second-class constraints, they do not form a closed algebra. To get rid of the second-class constraints, first find out the independent components of them, which would not become first class after some combinations. Let us assume that $C \equiv C^{AB}_c \phi_{AB}^c$ is first class, then $\{C, \phi_{AB}^a\} \approx 0$. In virtue of $\{\phi_{AB}^a(\vec{x}), \phi_{CD}^b(\vec{z})\} = (2 \partial^2 \mathcal{L}/\partial \phi_{CDIJ}^a \partial \phi_{IJ}^b) \delta(\vec{x} - \vec{z})$, one have $C^{AB}_c = (M^{ab}_c + M^{a}[\delta]^b)(D_a \xi^A)(D_b \xi^B)$, where $M^{ab}_c$ is an arbitrary tensor antisymmetric in the $ab$ indices, and $M^a = M^{ca}_c$. Then the independent second-class constraints can be taken by $\phi_I = \phi_{ab}^i, M^{[a}\delta^{b]}c, \phi_{ab}^c, \phi_{\mu\nu}^b$, where $\phi_{\mu\nu}^b = \phi_{AB}^b(D_\mu \xi^A)(D_\nu \xi^B)$ and $\phi_{\mu\nu}^b = \phi_{AB}^b(D_\mu \xi^A)(D_\nu \xi^B)$. Equivalently, we may set $\phi_I = \phi_{ab}^i, \phi_{ad}^a, \phi_{bd}^b$, where the arbitrary $M^a$ is eliminated.

Secondly, modify the Poisson bracket into the Dirac bracket as below:

$$\{F, F'\}_D = \{F, F'\} + \int_\Sigma d^3x C^{IJ}(\vec{x}) \{\phi_I(\vec{x}), F\} \{\phi_J(\vec{x}), F'\},$$

(32)

where $F, F'$ are functions of the phase space $\mathcal{P}$, and $C^{IJ}(\vec{x})$ is antisymmetric, subject to $C^{IJ}(\vec{x}) \{\phi_J(\vec{x}), \phi_K(\vec{z})\} \approx \delta^I_K \delta(\vec{x} - \vec{z})$. The definition is a generalization of the original Dirac bracket [21] from finite degrees to infinite degrees of freedom. For any first-class constraint $\phi$, $\{\phi, \cdot \}_D \approx \{\phi, \cdot \}$, and thus it is still first class under the new bracket. On the other hand, $\{\phi_I, \cdot \}_D \approx 0$, and thus the second-class constraints become first class now.

To conclude, as long as the $C^{IJ}(\vec{x})$ is solved, all the second-class constraints disappear, then the constraint algebra becomes closed. For the solution of $C^{IJ}(\vec{x})$, its existence is supported by the independence of the components of $\phi_I$. Specifically, the solution is

$$C^{\mu_4 b}_{\mu_4 a} = (l^{-2}/\sqrt{-g}) \cdot (\delta^d_b \delta^{\mu_4}_a - 2 \delta^d_a \delta^{\mu_4}_b),$$

$$C^{\mu_4 b}_{\mu_4 a} = (l^{-2}/\sqrt{-g}) \delta^d_b \delta^{\mu_4}_t,$$

(33)

(34)
and other independent components being equal to zero. To rewrite the Dirac bracket in a manifestly dS-invariant way, define

\[ C_{AB}^{CD} \phi_{AB}^a \phi_{CD}^b \equiv C^{ad}_{\mu b} \phi_{d}^a \phi_{\mu}^b + C^{\mu d}_{a b} \phi_{ad}^a \phi_{\mu}^b, \]

and \( C_{AB}^{CD} = -C_{BA}^{CD} = -C_{AB}^{DC}. \) Then one can derive

\[ C_{AB}^{CD} = (l^{-2}/\sqrt{1-g}) \cdot (D_a \xi^A D_b \xi^B) D_t \xi^C \xi^D + D_t \xi^A D_a \xi^B D_b \xi^C \xi^D + 2 D_b \xi^A D_t \xi^B D_a \xi^C \xi^D), \]

(35)

\[ \{F, F'\}_D = \{F, F'\} + \int_{\Sigma} d^3 x \left( C_{AB}^{CD} (\vec{x}) \{\phi_{AB}^a (\vec{x}), F\} \{\phi_{CD}^b (\vec{x}), F'\} - F \leftrightarrow F' \right), \]

(36)

which are dS covariant as expected.

4 Remarks

The present work contributes to the dS-covariant generalization of the Hamiltonian EC gravity. In the Lorentz gauges, our results coincide with those in the Lorentz-covariant formalism [23, 24]. The physical effect associated with our formalism lies in the dS spin, which appears in the gravitational field equation (10). For the geometrical part, the dS spin contains the torsion and the cosmological constant. For the material part, it is a 5d generalization of the Lorentz spin, and should be analyzed in the context of a quantum theory, as well as its semiclassical limit.

The linearly realized formulation also helps us to distinguish translation and diffeomorphism. In this formulation, they are different by definition, with different features as follows. Firstly, the diffeomorphism symmetry is a fundamental symmetry, which does not correspond to any conservation law directly. In fact, the energy-momentum conservation results from both the translation and diffeomorphism invariance, and likewise, the angular momentum conservation results from both the Lorentz and diffeomorphism invariance [8]. Secondly, the distributional dS constraint satisfies \( \{H(\Omega), H(\Omega')\} = -H([\Omega, \Omega']), \) indicating that the localization of the dS group does not deform the dS algebra, including the translation algebra embedded in it. On the other hand, the diffeomorphism algebra deforms the translation algebra, see, e.g. Ref. [25].

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