Abstract

Given a link in a 3-manifold such that the complement is hyperbolic, we provide two modifications to the link, called the chain move and the switch move, that preserve hyperbolicity of the complement, with only a relatively small number of manifold-link pair exceptions, which are also classified. These modifications provide a substantial increase in the number of known hyperbolic links in the 3-sphere and other 3-manifolds.

1 Introduction.

Thurston proved that every knot in the 3-sphere $S^3$ is either a torus knot, a satellite knot or a hyperbolic knot, by which we mean that its complement in $S^3$ admits a complete hyperbolic metric. By the Mostow-Prasad Rigidity Theorem, the complement of a hyperbolic knot in $S^3$ has a unique hyperbolic metric, which must have finite volume; hence, a hyperbolic knot in $S^3$ has associated to it a well-defined set of hyperbolic invariants such as volume, cusp volume, cusp shape, etcetera. More generally, Thurston proved that a link in a closed, orientable 3-manifold has hyperbolic complement (necessarily of finite volume) if and only if the exterior of the link contains no properly embedded essential disks, spheres, tori or annuli, terms that are described in Definition 2.1.

One would like to be able to identify link complements that satisfy Thurston’s criteria, and that therefore possess a hyperbolic metric. In [9], Menasco proved

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that every non-2-braid prime alternating link in $S^3$ is hyperbolic. In [6], Adams extended this result to augmented alternating links, where additional non-parallel trivial components wrapping around two adjacent strands in the alternating projection were added to the link. These additional components bound twice-punctured disks, which are totally geodesic in the hyperbolic structure of the complement. By [2], the link complement can be cut open along such a twice-punctured disk, twisted a half-twist and re-glued to obtain another hyperbolic link complement, with identical volume. This operation adds one crossing to the link projection. In many hyperbolic link complements, twice-punctured disks are particularly useful, because they are totally geodesic; see for instance [11] and the references therein.

We consider two moves that one can perform on a link in a 3-manifold with hyperbolic complement. The first move we consider is called the chain move. Here, we start with a trivial component bounding a twice-punctured disk in a ball $B$ as in Figure 1, and we replace the tangle on the left with the tangle on the right in Figure 1, where $k$ is any integer. Assuming that the rest of the manifold outside $B$ is not the complement of a rational tangle in a 3-ball (see Chapter 2 of [7] for this definition), the result is hyperbolic.

![Figure 1: Replacing (a) with (b) preserves hyperbolicity of the complement.](image)

There are counterexamples to extending the result to the case where the manifold outside $B$ is a rational tangle complement in a 3-ball as demonstrated by the hyperbolic link in the 3-sphere appearing in Figure 2. When the chain move is applied with $k = 3$, the resultant 3-component link is $6_3^3$ in Alexander-Rolfsen notation, which is not hyperbolic. However, in Lemma 3.4, we delineate explicitly the only possible exceptions.

The second move is called the switch move. Suppose we have a 3-manifold $M$ and a link $L$ in $M$ with hyperbolic complement. Let $\alpha$ be an embedded arc
Figure 2: Applying the chain move to this hyperbolic link with $k = 3$ yields the non-hyperbolic link complement $6^3_3$.

Figure 3: The trace of a geodesic $\alpha$ of $(M \setminus L, h)$ connects one or two components of $L$ to one another, and a neighborhood $B$ of $\alpha$ intersects $L$ in two arcs.

that runs from $L$ to $L$ with interior that is isotopic to an embedded geodesic in the complement, as in Figure 3.

Such a geodesic always exists since we could take one with minimal length outside fixed cusp boundaries. We consider the possibility that the arc runs from one component of $L$ back to the same component or from one component to a second component. Let $B$ be a neighborhood of $\alpha$. Then $B$ intersects $L$ in two arcs, as in Figure 4 (a). The switch move allows us to surger the link and add in a trivial component as in Figure 4 (b) while preserving hyperbolicity.

Remark 1.1. Note that the projection depicted in Figure 3 is not well-defined, since if the two arcs are skew inside the ball, there are two different projections,
Figure 4: The switch move replaces the arcs $g$ and $g'$ by the tangle $\gamma_1 \cup \gamma_2 \cup C$.

depending on point of view. So in fact, for each such geodesic $\alpha$, there are two
switch moves possible. This is equivalent to cutting along the twice-punctured
disk $D$ bounded by $C$ and twisting a half-twist in either direction before regluing.
Once we prove that the switch move depicted in Figure 4 preserves hyperbol-
icity, the hyperbolicity of the half-twisted version follows immediately from the
previously mentioned results of [2]. Further twists give link complements home-
omorphic to the original or the half-twisted version.

These moves show that many additional link complements in 3-manifolds are
hyperbolic. In [4], the chain move and the switch move, together with the related
switch move gluing operation described in Section 5, are utilized in the proof that,
for any given surface $S$ of finite topology and negative Euler characteristic and
any $H \in [0, 1)$, there exists a proper, totally umbilic embedding of $S$ into some
hyperbolic 3-manifold of finite volume with image surface having constant mean
curvature $H$.

In [3], the chain move is used in the proof that a virtual link obtained by taking
a reduced classical prime alternating link projection and changing one crossing to
be virtual yields a non-classical virtual link. See that paper for details.

We can also use the chain move and the switch move to obtain straightforward
proofs of hyperbolicity of well-known classes of links.

**Example 1.2.** We can show that every chain link of five or more components,
no matter how twisted, is hyperbolic. This was first proved in [10] using explicit
hyperbolic structures for manifolds covered by these link complements.
Start with the alternating 4-chain, known to be hyperbolic by Menasco’s work in [9]. Then apply the chain move repeatedly. This proves hyperbolicity of any chain link of five or more components with an arbitrary amount of twisting in the chain.

We note that the chain and switch moves apply more broadly than is apparent from Figures 1 and 4. In the case of the chain move, instead of specifying a hyperbolic link complement $M \setminus L$, we can start with a cusped hyperbolic 3-manifold $M'$ containing a two-sided essential embedded thrice-punctured sphere $S$. Treating two of the boundary curves on the cusps as the meridianal punctures of the disk in Figure 4 and the third as the longitudinal boundary of the disk, we can apply the chain move, removing the cusp that contains the longitude by doing a Dehn filling along a curve that crosses the longitude once and adding in the additional two components within a neighborhood of $S$. In the case that two of the boundaries of $S$ are on the same cusp, they must play the role of the meridional punctures. (Note that a two-sided thrice punctured sphere cannot have all three boundaries on the same cusp.)

In the case of the switch move, we can again begin with a cusped hyperbolic 3-manifold $M'$. For two cusps connected by an embedded geodesic, we can choose a nontrivial simple closed curve on each torus corresponding to each cusp. Then by Dehn filling along those curves we obtain a 3-manifold $M$ for which $M'$ is a link complement and the switch move applies.

The same procedure holds for a geodesic from a cusp back to the same cusp, and a specification of a nontrivial simple closed curve on the torus corresponding to the cusp, two copies of which play the role of the meridians around $\gamma_1$ and $\gamma_2$. Note that when applied to a link complement $M \setminus L$, but with a choice of curve other than meridians, the end result is not a new link complement in the same manifold.

Finally, we point out that there is a variant of the chain move called the augmented chain move as in Figure 5 wherein the two new components of the chain move are added in but the previous trivial component is not removed. We prove here that this move also preserves hyperbolicity.

To see this, we consider the link appearing in Figure 6, which is a twisted five-chain.

All five-chains are hyperbolic, as we just proved, so it has a hyperbolic complement. Now we apply the idea of a walnut as in [1]. We can cut the manifold $M$
open along the twice-punctured disk bounded by $C$, cut the 5-chain link complement open along the twice punctured disk $E$, and then glue copies of the twice-punctured disks to one another to insert the cut-open link complement into $M$. As in [2], since a twice-punctured disk is totally geodesic with a rigid unique hyperbolic structure, the gluings are isometries and the resulting manifold is hyperbolic with volume the sum of the volumes of the two manifolds.

Next, we explain the organization of the paper. First, we remark that it suffices to demonstrate our results when the ambient manifold is orientable. This property is proved by showing that the oriented cover of a related non-orientable link complement admits a hyperbolic metric and then one applies the Mostow-Prasad Rigidity Theorem to conclude that the associated order-two covering transformation is an isometry, which in turn implies that the hyperbolic metric on the oriented covering descends. In Section 2, we present some of the background material necessary to the proofs of our main results in the orientable setting. In Section 3, we prove the Chain Move Theorem, stated there as Theorem 3.1. In Section 4, we prove the Switch Move Theorem, see Theorem 4.1. In Section 5 we prove the Switch Move Gluing Operation, which constructs from a pair of hyperbolic 3-
manifolds of finite volume, an infinite number of new hyperbolic 3-manifolds of finite volume, see Theorem 5.1. In the appendix at the end of the paper, we classify the exceptional links for which the chain move fails to produce hyperbolic 3-manifolds of finite volume.

2 Preliminaries.

In this section, we recall some definitions and results that are needed to understand hyperbolic 3-manifolds of finite volume and certain embedded surfaces in such ambient spaces. Our first goal is to understand the statement of Thurston’s hyperbolization theorem in our setting. Before stating this result, we first explain some of the definitions and notations we use. Throughout this discussion, $P$ will denote a connected, orientable, compact 3-manifold with nonempty boundary $\partial P$ consisting of tori and such that $P$ is not the product of a two-torus with an interval, and $\text{int}(P)$ will denote the interior of $P$. Moreover, a surface $\Sigma$ in $P$ means a properly embedded surface $\Sigma \subset P$, i.e., $\Sigma$ is embedded in $P$ with $\partial \Sigma = \Sigma \cap \partial P$.

Definition 2.1.

1. Given a surface $\Sigma$ in $P$, a compression disk for $\Sigma$ is a disk $E \subset P$ with $\partial E = E \cap \Sigma$ such that $\partial E$ is homotopically nontrivial in $\Sigma$. If $\Sigma$ does not admit any compression disk, we say $\Sigma$ is incompressible.

2. Given a surface $\Sigma$ in $P$, a boundary-compression disk for $\Sigma$ is a disk $E \subset P$ with $\partial E = E \cap (\Sigma \cup \partial P)$ such that $\partial E = \alpha \cup \beta$, where $\alpha$ and $\beta$ are arcs intersecting only in their endpoints such that $\alpha = E \cap \Sigma$ and $\beta = E \cap \partial P$ and $\alpha$ does not cut a disk from $\Sigma$. If $\Sigma$ does not admit any boundary-compression disk, we say $\Sigma$ is boundary-incompressible.

3. A torus $T$ in $P$ is boundary parallel if $T$ is isotopic to a boundary component of $P$.

4. An annulus $A$ in $P$ is boundary parallel if $A$ is isotopic, relative to $\partial A$, to an annulus $A' \subset \partial P$.

5. A sphere $S$ in $P$ is essential if $S$ does not bound a ball in $P$.

6. A disk $E$ in $P$ is essential if $\partial E$ is homotopically nontrivial in $\partial P$.
A torus $T$ (respectively an annulus $A$) is essential in $P$ if $T$ (resp. $A$) is incompressible and not boundary parallel.

Using the above definitions, Thurston’s hyperbolization theorem implies that a connected, orientable, noncompact 3-manifold $N$ admits a hyperbolic metric of finite volume if and only if $N$ is diffeomorphic to $\text{int}(P)$ as above and there are no essential spheres, disks, tori or annuli properly embedded in $P$. In this case, we shall say that $N$ is hyperbolic. When a link $L$ in a 3-manifold $M$ has hyperbolic complement, we will say either $M \setminus L$ is hyperbolic, or $L$ is hyperbolic.

A useful fact is that if $\alpha$ is an arc with endpoints in a link $L$ in a 3-manifold $M$ such that $\alpha$ corresponds to a geodesic in the hyperbolic link complement $M \setminus L$, then $\alpha$ cannot be homotoped through $M \setminus L$ into $L$ while fixing its endpoints on $L$. This follows from the fact any such geodesic will lift to geodesics connecting distinct horospheres in the universal cover $\mathbb{H}^3$, whereas an arc that is homotopic into $L$ will lift to arcs, each of which connects one and the same horosphere.

In the case that a manifold $M$ has no essential disks, we say it is boundary-irreducible. In the case that a manifold $M$ has no essential spheres, we say it is irreducible. Note that if $M$ has only toroidal boundaries and it is not a solid torus, which is the situation we will consider, irreducibility implies boundary-irreducibility. This is because if there exists an essential disk $D$ with boundary in a torus $T \subset \partial M$, then $\partial N(D \cup T)$ is a sphere which must bound a ball to the non-$D$ side, implying $M$ is a solid torus. Here and elsewhere, $N(G)$ denotes a regular neighborhood of a set $G \subset M$.

Given an annulus $A$ properly embedded in an irreducible manifold $M$ with toroidal boundary, we note that if $A$ is boundary-compressible, it is boundary-parallel. This follows because we can surger the annulus along the boundary-compressing disk to obtain a properly embedded disk $D$, with trivial boundary on $\partial M$. Then $\partial D$ bounds a disk $D'$ in $\partial M$, and $D \cup D'$ is a sphere bounding a ball. This allows us to isotope $A$ relative $\partial A$ into $\partial M$.

## 3 The Chain Move Theorem

Let $L$ be a hyperbolic link in a 3-manifold $M$ and let $B \subset M$ be a ball in $M$ that intersects $L$ as in Figure 1 (a). In this section we prove the Chain Move Theorem, as stated by Theorem 3.1 below. This proof breaks up into two cases depending on whether or not the pair $(M \setminus B, L \setminus B)$ is a rational tangle in a 3-ball, see [7, Chapter 2] for this definition and for the representation of a rational tangle by a
sequence of integers. Since the proof of the chain move in this specific setting uses a specialized knowledge of classical knot theory, it will be presented in the Appendix of this paper.

**Theorem 3.1** (Chain Move Theorem). Let \( L \) be a link in a 3-manifold \( M \) such that the link complement \( M \setminus L \) admits a complete hyperbolic metric of finite volume. Suppose that there is a sphere \( S \) in \( M \) bounding a ball \( B \) that intersects \( L \) as in Figure 1 (a). Let \( L' \) be the resulting link obtained by replacing \( L \cap B \) by the components as appear in Figure 1 (b). Then if \( (M \setminus B, L \setminus (B \cap L)) \) is not any of the rational tangles \(-k\), \(-(k + 1)\), or \(-2 - k\) in a 3-ball, then \( M \setminus L' \) admits a complete hyperbolic metric of finite volume.

In Figure 7(a), we see the new link components that are inserted into the ball \( B \). In Figures 7(b), (c) and (d), we see, for any fixed integer \( k \), the three cases of rational tangles in the exterior 3-ball that do not yield a hyperbolic link complement.

![Figure 7](image)

*Figure 7: The link components we are inserting in \( B \) and the three rational tangles in an exterior ball that do not generate a hyperbolic link complement.*
Remark 3.2. The crossings around the single trivial component need not be non-alternating for Theorem 3.1 to apply. If the crossings alternate (as shown in Figure 8 (a)), we could add a crossing to $\gamma_2$ and work in a sub-ball as in Figure 8 (b) so that the crossings are those shown in Figure 1(a).

![Figure 8: Using a Type I Reidemeister move to obtain a sub-ball where the Chain Move Theorem applies.](image)

Remark 3.3. Note that repeated application of the Chain Move Theorem allows us to create a hyperbolic link complement with an arbitrarily long chain of trivial components and with any amount of twist. Moreover, if the original exterior tangle is assumed not to be rational, the subsequent exterior tangles to which we apply the move cannot be rational either, so all resulting link complements are hyperbolic. In fact, even if the initial exterior tangle is rational, if our first application of the move results in a hyperbolic link complement, all repeated applications will also be hyperbolic.

Proof of Theorem 3.1. Let $X = M \setminus L$ and, for $i = 1, 2$, let $\Gamma_i$ be the connected component of $L$ containing the arc $\gamma_i$ (note that possibly $\Gamma_1 = \Gamma_2$). First, we assume that $M$ is orientable. We let $L'$ be the link formed by the replacement stated in Theorem 3.1. As stated in the introduction of this section, we will assume that $(M \setminus B, L \setminus (B \cap L))$ is not a rational tangle in a 3-ball; this special case can be described by the following result, which is proved in the Appendix.
Lemma 3.4. Let $L$ be a link in the 3-sphere such that the tangle $R = L \setminus B$ is a rational tangle and the tangle $L \cap B$ is the tangle $T_k$ appearing in Figure 1(b), for some integer $k$. If $R$ is none of the rational tangles $\infty$, $-k$, $-(k + 1)$ or $-2 - k$, the link complement is hyperbolic.

Note that we do not include the rational tangle $\infty$ as a tangle to exclude in the statement of Theorem 3.1 since, in the case of this tangle, the original link $L$ is not prime and hence $X$ is not hyperbolic. We prove Theorem 3.1 when $M$ is orientable by showing that the resulting link complement $Y = M \setminus L'$ does not admit essential disks, spheres, tori or annuli. In order to do so, we first prove the following.

Claim 3.5. The four-punctured sphere $Q = S \setminus L$ is incompressible and boundary-incompressible in $X$ and also in $Y$.

Proof. We prove that if $Q$ is compressible in $X$ or in $Y$, then $(M \setminus B, L \setminus (B \cap L))$ is a rational tangle in a 3-ball. We first prove this property in $X$. Let $\gamma$ be a nontrivial simple closed curve in $Q$ and assume that there is a compact disk $E \subset X$ with $\partial E = \gamma = E \cap Q$. Then, each of the two disks $E_1$ and $E_2$ bounded by $\gamma$ in $S$ must contain exactly two of the punctures of $Q$, otherwise we could attach a one-punctured disk in $Q$ to $E$ to find an essential disk in $X$, contradicting its hyperbolicity.

If $E$ were contained in $B$, then $E \cup E_1$ and $E \cup E_2$ are two spheres in $B$, each punctured twice by $L$. Since both punctures in each sphere cannot come from distinct arcs in $L \cap B$, $E$ separates $B$ into two balls $B_1, B_2$, where $\gamma_1 \subset B_1$ and $\gamma_2 \subset B_2$, and it then follows that $C$ cannot link $\gamma_1$ and $\gamma_2$ simultaneously, a contradiction.

Next assume that $E \cap \text{int}(B) = \emptyset$. Let, for $i = 1, 2$, $A_i = E \cup E_1 \setminus L$; then each $A_i$ is an annulus in $X$. Since each $A_i$ is incompressible and $X$ is hyperbolic, $A_i$ is boundary parallel. Therefore, the closure of $A_i$ in $M$ bounds a closed ball $B_i \subset M \setminus \text{int}(B)$ with $\partial B_i = E_i \cup E$ and such that $B_i \cap L$ is an unknotted arc in $B_i$. Hence, we can isotope $L \cap B_i$ through $B_i$ to the surface $S$. Then, after the isotopy, $S = \partial N(B)$ is a sphere in $X$. Since $X$ is hyperbolic, $S$ bounds a ball which is disjoint from $B$, and this is a contradiction unless $M = \mathbb{S}^3$.

If $M = \mathbb{S}^3$, then the fact $L \cap B_i$ can be isoted through $B_i$ to the surface $S$ implies $L \setminus (B \cap L)$ can be isotoped to be two disjoint embedded arcs on $S$. Hence, $(M \setminus B, L \setminus (B \cap L))$ is a rational tangle determined by $\gamma$, and up to isotopy, $E$ is the only compression disk for $Q$ in $X$. 
Note that the above argument implies that \( Q \) is also incompressible in \( Y \), as we next explain. If \( E \subset Y \) was a compressing disk with \( \partial E = E \cap Q \), then \( E \) necessarily is contained in \( B \), otherwise \( E \subset Y \setminus B \) and \( Y \setminus B = X \setminus B \) would give a compression disk for \( Q \) in \( X \). Once again, \( E \subset B \) gives that \( E \) separates \( B \) into two balls \( B_1, B_2 \) such that \( \gamma_1 \subset B_1 \) and \( \gamma_2 \subset B_2 \). Then, since \( C_1 \) links \( \gamma_1 \), \( C_1 \subset B_1 \). And since \( C_2 \) links \( \gamma_2 \), \( C_2 \subset B_2 \). But then \( E \) separates \( C_1 \) from \( C_2 \) in \( B \), a contradiction to the fact they are linked in \( B \).

To prove boundary-incompressibility of \( Q \) in either \( X \) or \( Y \), suppose \( E \) is a boundary-compression disk such that \( \partial E = \alpha \cup \beta \) with \( \alpha = E \cap Q \). If \( \alpha \) connects two distinct punctures of \( Q \) and \( N(E) \) is a small neighborhood of \( E \) in \( M \), then \( \partial N(E) \setminus (\partial N(E) \cap B) \) is a compression disk for \( Q \), a contradiction.

If both endpoints of \( \alpha \) are at the same puncture, then, since the interior of \( \beta \) is disjoint from \( Q \), then \( \beta \), together with an arc in \( Q \), bound a disk \( \tilde{E} \) in \( \partial N(L) \). Then, \( E \cup \tilde{E} \) is a compression disk for \( Q \), a contradiction. \( \square \)

For the next arguments in the proof, let \( D_1, D_2 \subset Y \cap B \) denote two twice-punctured disks bounded respectively by \( C_1, C_2 \subset L' \) and let \( \overline{D}_i \) denote the closure of \( D_i \) in \( M \); thus each \( \overline{D}_i \) is a disk in \( B \). Then, we prove the following.

**Claim 3.6.** The twice punctured disks \( D_1 \) and \( D_2 \) are incompressible and boundary-incompressible in \( Y \).

**Proof.** Suppose there were a disk \( E \subset Y \), \( \text{int}(E) \cap \overline{D}_i = \emptyset \) with nontrivial boundary in \( D_i \). Since \( Q \) is incompressible and we may assume general position, any component in \( E \cap Q \) is a simple closed curve that is trivial both in \( E \) and in \( Q \). Choose an innermost curve \( \alpha \subset E \cap Q \) in the sense that the interior of the disk \( E' \subset E \) bounded by \( \alpha \) does not intersect \( Q \) and let \( E'' \) be the disk bounded by \( \alpha \) in \( Q \). Then \( E' \cup E'' \) is a sphere that is either in the hyperbolic manifold \( X \) or in \( Y \cap B \). In either case, \( E' \cup E'' \) bounds a ball in \( Y \) that can be used to isotope \( E' \) to \( E'' \) and further to remove \( \alpha \) from the intersection \( E \cap Q \). After repeating this disk replacement argument a finite number of times, we may assume that \( E \subset B \).

Let \( E' \) be the disk in \( \overline{D}_i \) bounded by \( \partial E \). Then \( E \cup E' \) is a sphere in \( B \) which is punctured only once by at least one of the components in \( L' \cap B \), a contradiction that shows that \( D_1 \) and \( \overline{D}_2 \) are incompressible in \( Y \).

To finish the proof of Claim 3.6, we argue by contradiction and assume that, for some \( i \in \{1, 2\} \), \( D_i \) is boundary-compressible. Then, there exists a disk \( E \) such that \( \partial E = \alpha \cup \beta \), where \( \alpha \) and \( \beta \) are arcs, \( \alpha = E \cap D_i \) and \( \beta \subset \partial N(L') \).

For simplicity, we assume that \( i = 1 \) and note that the case \( i = 2 \) can be treated...
analogously. We also notice that Claim 3.5 allows us to isotope $E$ in $Y$ to lie in $B$.

Since $\beta$ cannot join points in distinct components of $\partial N(L')$, there are three cases to consider. First, $\alpha$ could connect $\partial N(C_1)$ to itself and separate the two punctures on $D_1$. Let $E'$ be one of the two disks obtained by removing $\alpha$ from $D_1$, and we may choose $E'$ in such a way that $E'' = E \cup E'$ is a disk in $B$ with nontrivial boundary in $\partial N(C_1)$. But then $\partial E''$ must be a longitude of $\partial N(C_1)$ and any such disk would be punctured twice by $L' \cap B$, a contradiction since $E \cap L' = \emptyset$ and $E'$ is only punctured once by $L'$.

The second possibility is that both endpoints of $\alpha$ are in $\partial N(\gamma_1)$, with $\alpha$ going around the puncture of $D_1$ that comes from $C_2$. Let $E'$ be the punctured disk cut by $\alpha$ from $D_1$ and let $E'' = E \cup E'$. So $E''$ is a once-punctured disk in $Y$. If $E''$ has trivial boundary in the boundary of $N(\gamma_1)$, then we have a sphere in $B$ punctured once by $C_2$, which cannot happen. Thus, $E''$ has nontrivial boundary in $\partial N(\gamma_1)$. Since $E \subset B$, the boundary of $E''$ is isotopic to a meridian curve on the boundary of $N(\gamma_1)$. Thus, after adding to $E''$ a meridional disk in $N(\gamma_1)$, we have a sphere in $B$ punctured once by each of $\gamma_1$ and $C_2$, a contradiction, since any component entering a sphere in $B$ must also leave the sphere.

The last possibility is that $\alpha$ connects $\partial N(C_2)$ to itself and goes around the puncture $\gamma_1$ creates in $D_1$. Once again, let $E'$ be the once-punctured disk $\alpha$ cuts from $\Delta_1$ and let $E'' = E \cup E'$. Then, the closure of $E''$ is a disk in $B$ that is not punctured by $C_1$, and the fact that $\partial E'' \subset \partial N(C_2)$ and the linking property between $C_1$ and $C_2$ implies that either $\partial E''$ is trivial or it is a meridian in $\partial N(C_2)$ and the proof continues as previously, finishing the proof of Claim 3.6. 

Claim 3.7. $Y$ does not admit any essential spheres or essential disks.

Proof. We argue by contradiction and first suppose that there is an essential sphere $S$ in $Y$. If $S$ intersects $Q$, then, by incompressibility of $Q$, we can exchange disks on $S$ for disks on $Q$ in order to obtain an essential sphere $S'$ in $Y$ that does not intersect $Q$. If $S' \subset Y \setminus B$, then $S' \subset X$, which implies $S'$ is the boundary of a ball $B \subset X$. In this case, $B$ must be disjoint from $B$, since $C \subset B$; hence, $B \subset Y$ which contradicts that $S'$ is essential in $Y$. Thus, we may assume that $S'$ is contained in $B$, and so it bounds a sub-ball $B$ of $B$. If $B$ intersects $C_1 \cup C_2$, then, by the linking properties of these circles, $C_1 \cup C_2$ must be contained in $B$. As $\gamma_1$ links $C_1$ in $B$, we arrive at a contradiction because the endpoints of $\gamma_1$ lie outside of $B$. This contradiction implies that $L' \cap B$ is disjoint from $B$, which in turn implies that $B \subset Y$, contradicting that $S'$ is essential in $Y$. 

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Suppose now that there is an essential disk $D$ with boundary in $\partial N(L')$. Then, there is a component $J$ of $L'$ such that $\partial D \subset \partial N(J)$, and we let $S = \partial N(D \cup N(J))$. It then follows that $S$ is an essential sphere, as it splits $J$ from the other components of $L'$, contradicting the nonexistence of such spheres.

**Claim 3.8.** $Y$ does not admit essential annuli.

**Proof.** Arguing by contradiction, assume there exists an essential annulus $A$ in $M \setminus N(L')$. Let $\alpha_1, \alpha_2$ denote the two boundary components for $A$. Then, there are $J_1, J_2$ components of $L'$ such that $\alpha_1 \subset \partial N(J_1)$ and $\alpha_2 \subset \partial N(J_2)$. After an isotopy of $A$ we will assume without loss of generality that both $\alpha_1$ and $\alpha_2$ are taut in the respective tori $\partial N(J_1), \partial N(J_2)$, in the sense that, in the product structure generated by respective meridional curves in $\partial N(J_i)$, each $\alpha_i$ is transverse to all meridians and also to all longitudes, unless $\alpha_i$ is one of them.

We next rule out the various possibilities for $A$, starting with the assumption that $A$ does not intersect $D_1 \cup D_2$.

In this case, we may use the fact that $\partial N(D_1 \cup D_2) \setminus N(\gamma_1 \cup \gamma_2)$ is isotopic to $Q$ to isotope $A$ in $M \setminus N(L')$ to lie outside of $B$. Thus, $A$ is an annulus in $X$, and the fact that $X$ is hyperbolic implies that $A$ is either compressible or boundary parallel in $X$. If $A$ is compressible in $X$, then we may use the fact that $Q$ is incompressible in $Y$ and a disk replacement argument to show that $A$ is compressible in $Y$, a contradiction.

Next, we treat the case when $A$ is boundary parallel in $M \setminus N(L)$. In this case, $A$ defines a product region $W \subset M \setminus N(L)$ through which $A$ is parallel to a subannulus in $\partial N(L)$. Since $C$ lies outside of $W$, separation properties imply that $B$ is disjoint from $W$; hence, $W \subset M \setminus N(L')$ from where it follows that $A$ is boundary parallel in $Y$, a contradiction.

Now suppose that $A$ intersects $D_1 \cup D_2$ and assume that $A$ has the fewest number of intersection components in $A \cap (D_1 \cup D_2)$ for an essential annulus in $Y$. Note that for $i = 1, 2$, the intersection curves which may appear in $A \cap D_i$ are either simple closed curves or arcs with endpoints in $\partial A$.

We next eliminate the possibility that $A \cap D_i$ contains a simple closed curve. Since $D_i$ is incompressible, by minimality of intersection curves, any simple closed curve in the intersection $A \cap D_i$ is nontrivial in $A$. Note that if $A \cap D_i$ contains a simple closed curve that circles one puncture, we may take an innermost such curve and use the once-punctured disk on $D_i$ that it bounds to surger $A$ to obtain two annuli, each with fewer intersection curves and at least one of them must be essential. So we may assume that all simple closed curves in $A \cap D_i$ circle
both punctures of $D_i$. But then, the outermost of such intersection curves bounds an annulus that again allows us to surger $A$ to obtain an essential annulus with fewer intersection curves. Hence, all curves in $A \cap D_i$ are arcs with endpoints in $\partial A$.

Next, we show that there are no arcs in $A \cap D_i$ that have endpoints on the same boundary component of $A$. Assume that $\alpha$ is such an arc and let $E_1$ be the disk defined by $\alpha$ in $A$. We assume that $\alpha$ is innermost in the sense that the interior of $E_1$ is disjoint from $D_i$. Since Claim 3.6 implies that $D_i$ is boundary-incompressible, it follows that $\alpha$ must cut a disk $E_2$ from $D_i$. Then, $E = E_1 \cup E_2$ is a disk with boundary $\partial E \subset \partial N(J)$. Since $Y$ does not admit essential disks, it follows that $\partial E$ is trivial in $\partial N(J)$, and we may use the fact that all spheres in $Y$ bound balls to isotopy $A$ so that $E_1$ moves past $E_2$, thus eliminating the intersection curve $\alpha$ and contradicting minimality of the number of intersection components.

In particular, if $A$ intersects $D_i$, both $\alpha_1$ and $\alpha_2$ must intersect $D_i$, and none of the intersection arcs on $A \cap D_i$ can cut a disk off $D_i$, as if they did, $A$ would be boundary-compressible and hence boundary-parallel since $Y$ is irreducible. Note that because there is at least one arc of intersection of $A$ with a $D_i$, and such arc goes from $\alpha_1$ to $\alpha_2$, then $\partial A \subset (\partial N(C_1) \cup \partial N(C_2) \cup \partial N(\Gamma_1) \cup \partial N(\Gamma_2))$. Moreover, since both $\alpha_1$ and $\alpha_2$ intersect $D_1 \cup D_2$ and we assume minimality of intersection components in $\partial A \cap (D_1 \cup D_2)$, no component of $\partial A$ can be a meridian in $\partial N(\Gamma_1)$ or in $\partial N(\Gamma_2)$, hence any closed curve in $A \cap Q$ must be trivial in $A$, and, consequently, trivial in $Q$.

We next consider the case that $\partial A \subset \partial N(C_1) \cup \partial N(C_2)$. Then by incompressibility of $Q$, we can isotopy $A$ to lie inside $B$. Moreover, $C_1 \cup C_2$ is a Hopf link with complement in the 3-sphere that is a thickened torus $T \times [0, 1]$. Thus, $B \setminus (N(C_1) \cup N(C_2))$ is the complement of a ball $B$ in $T \times [0, 1]$, where we identify $\partial N(C_1)$ with $T \times \{0\}$ and $\partial N(C_2)$ with $T \times \{1\}$.

Assume that both boundary components of $A$ are on $\partial N(C_1)$. Then, $A$ is an annulus in $(T \times [0, 1]) \setminus B$ with both boundaries on $T \times \{0\}$. In particular, in $T \times [0, 1]$, $A$ is boundary-parallel through a solid torus $V$ that $A$ cuts from $T \times [0, 1]$. Since $\partial V$ is a closed surface in the interior of the three-ball $B$, it defines a unique compact region disjoint from $\partial B = \partial V$, from where it follows that $B$ must be disjoint from $V$. But then both the arcs $\gamma_1$ and $\gamma_2$, which have endpoints on $\partial B$, must also be disjoint from $V$, meaning that $V \subset Y$, and then $A$ is boundary-parallel in $Y$, a contradiction. By symmetry, the same argument also proves that $A$ cannot have both boundary components on $\partial N(C_2)$.

Next, suppose that one boundary of $A$ is on $\partial N(C_1)$ and the other is on
∂N(C₂). Then again, A can be seen as an annulus in (T × [0, 1]) \ B, but now its boundary is a pair of nontrivial parallel curves on T × {0} and T × {1}. These curves are respectively realized as a (p, q)-curve on ∂N(C₁) and a (q, p)-curve on ∂N(C₂). But there exist arcs γ₁ and γ₂ on Q such that the closed curve γ₁ ∪ γ₂ wraps meridianally around C₁ and the closed curve γ₂ ∪ γ₁ wraps meridianally around C₂, where in T × [0, 1], a meridian of ∂N(C₂) corresponds to a longitude of N(C₁). Hence, when we add γ₁ and γ₂ to T × [0, 1] \ B, one wrapping meridianally around T × [0, 1] and the other wrapping longitudinally, at least one will puncture A, a contradiction.

Thus, at least one boundary component of A, say α₁, must be on ∂N(Γ_i), for some i ∈ {1, 2}. As already explained, α₁ is not a meridian on ∂N(Γ_i).

Next, assume that α₂ is on ∂N(C₁) or ∂N(C₂). Since Q is incompressible and Y is irreducible, after performing a disk replacement argument, we may assume that A ∩ Q is a family of pairwise disjoint arcs, each with both endpoints in α₁. Let a be one of such arcs and assume that a cuts an innermost disk D from A, in the sense that D ∩ Q = φ. If D ⊂ B, then, if we let b = D \ a, it follows that b ⊂ (∂N(γ_i)) \ B and our assumptions on α₁ being taut imply that b joins two distinct punctures of Q. But then it follows that ∂D links C₁ on B, and D must be punctured by C₁, a contradiction. Hence, it follows that D is to the outside of B. Once again, our assumptions on α₁ imply that a joins two distinct punctures of Q, from where it follows that D is a boundary-compression disk for Q, which contradicts Claim 3.5.

It remains to rule out the case where α₁ ∪ α₂ ⊂ ∂N(Γ₁) ∪ ∂N(Γ₂). Let a be an arc of intersection A ∩ (D₁ ∪ D₂). Then our previous arguments give that a joins α₁ and α₂ and that a cannot cut a disk off D₁. In particular, a must necessarily intersect the disk D_j for j ≠ i and that creates another arc b ⊂ A ∩ D_j which meets a transversely at a point p and joins α₁ and α₂. In particular, Γ₁ = Γ₂. The point p separates both arcs a and b, and that defines a unique disk D ⊂ A with boundary given by one arc in a, one arc in b and one arc c in α₁. Note that D ∩ D_i ⊂ a ∪ b, since any arc in A ∩ D_i must join α₁ and α₂. Let E be a connected component of D \ B that contains a subarc of c in its boundary. Such component exists because the endpoints of a and b on D are on distinct disks D₁, D₂ and hence c cannot be contained in B. Once again, the fact that α₁ is taut gives that ∂E ∩ Q is an arc joining two distinct punctures of Q. But then, E is a boundary-compression disk for Q, a contradiction.

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¹For given relatively prime integers p and q, a (p, q)-curve is a torus knot that winds p times around the meridian of the torus and q times around its longitude.
The cases treated above rule out the possibility that \( Y \) admits an essential annulus, thereby proving Claim 3.8.

\[ \square \]

**Claim 3.9.** \( Y \) does not admit essential tori.

*Proof.* We argue by contradiction and suppose that \( T \subset Y \) is a torus which is incompressible and not boundary-parallel in \( Y \). First, suppose that \( T \) does not intersect \( D_1 \cup D_2 \). Then we can isotope \( T \) in \( Y \) to assume that \( T \cap B = \emptyset \), and then \( T \subset X \). Since \( X \) is hyperbolic, either \( T \) admits a compression disk in \( X \) or \( T \) is boundary parallel in \( X \).

First assume that \( E \subset X \) is a compression disk for \( T \subset X \). Since \( Q \) is incompressible in \( X \), after disk replacements, we may assume that \( E \subset X \setminus B \subset Y \), which is a contradiction.

Next, suppose that \( T \) is parallel to the boundary of a neighborhood of one of the components \( J \) of \( L \), and let \( W \subset X \) be the related proper product region with boundary \( T \). We claim that \( Q \) must be disjoint from \( W \). Otherwise, \( Q \subset W \) which would imply that \( B \setminus (\gamma_1 \cup \gamma_2 \cup C) \subset W \); this is a contradiction because \( W \) has only one end corresponding to a single component of \( L \). Since \( Q \) separates \( X \) and is disjoint from \( W \), then \( W \subset Y \), which means \( T \) is boundary parallel in \( Y \). This proves that any essential torus in \( Y \) must intersect \( D_1 \cup D_2 \).

Let \( T \subset Y \) be an essential torus that intersects \( D_i \), for some \( i \in \{1, 2\} \). Next, we prove that \( Y \) must contain an essential annulus, which contradicts Claim 3.8. After possibly replacing disks in \( T \) by disks in the incompressible surface \( D_i \), we may assume that any component in \( T \cap D_i \) is homotopically nontrivial in \( D_i \); let \( \gamma \subset T \cap D_i \) be one such components. First assume that \( \gamma \) encircles a single puncture in \( D_i \) and choose it to be an innermost such curve in \( T \cap D_i \). Using the once-punctured disk bounded by \( \gamma \) in \( D_i \) to surger \( T \), we obtain an essential annulus in \( Y \), as claimed. Next, assume that \( \gamma \) encircles both punctures of \( D_i \) and that it is an outermost such curve on \( T \cap D_i \). In this case, we may use the outer annulus on \( D_i \) to surger \( T \) in order to obtain an essential annulus in \( Y \), thereby proving Claim 3.9.

Having proved that there are no essential disks, spheres, tori or annuli in \( Y \), then \( Y \) satisfies Thurston’s conditions for hyperbolicity, proving Theorem 3.1 when \( M \) is orientable.

The case when \( M \) is nonorientable can be proved using the orientable case as we next explain. Suppose that \( M \) is nonorientable and that \( L, L' \) and \( B \) are as stated. Let \( \Pi: \hat{M} \to M \) be the oriented two-sheeted covering of \( M \) and let \( \hat{L} = \Pi^{-1}(L) \) and \( B_1, B_2 \) be the two connected components of \( \Pi^{-1}(B) \). Then, \( \hat{L} \) is
a hyperbolic link in $\hat{M}$ and $\hat{L} \setminus B_1$ is not a rational tangle in a 3-ball, since $\hat{L} \cap B_2$ is diffeomorphic to $L \cap B$. Then, we may use the chain move to modify $\hat{L}$ in $B_1$, replacing $\hat{L} \cap B_1$ by a tangle diffeomorphic to $L' \cap B$, which creates a hyperbolic link $\hat{L}'$ in $\hat{M}$. Then, since $\hat{L}' \cap B_2 = \hat{L} \cap B_2$ and $\hat{M} \setminus \hat{L}'$ is hyperbolic, we can use the chain move in $B_2$ to replace $\hat{L}' \cap B_2$ by a tangle diffeomorphic to $L' \cap B$ and create another hyperbolic link $\hat{L}''$ in $\hat{M}$. Since we may do this second replacement in an equivariant manner with respect to the nontrivial covering transformation $\sigma$ defined by $\Pi$, the restriction of $\Pi$ to the hyperbolic manifold $\hat{M} \setminus \hat{L}''$ is the two-sheeted covering space of $M \setminus L'$. Since $\sigma$ is an order-two diffeomorphism of $\hat{M} \setminus \hat{L}''$, the Mostow-Prasad rigidity theorem implies that we may consider $\sigma$ to be an isometry of the hyperbolic metric of $\hat{M} \setminus \hat{L}''$. Hence, the hyperbolic metric of $\hat{M} \setminus \hat{L}''$ descends to $M \setminus L'$ via $\Pi$, which finishes the proof of Theorem 3.1. □

4 The Switch Move Theorem

**Theorem 4.1** (Switch Move Theorem). Let $L$ be a link in a 3-manifold $M$ such that $M \setminus L$ admits a complete hyperbolic metric of finite volume. Let $\alpha \subset M$ be a compact arc which intersects $L$ transversely in its two distinct endpoints, and such that $\text{int}(\alpha)$ is a complete, properly embedded geodesic of $M \setminus L$. Let $B$ be a closed ball in $M$ containing $\alpha$ in its interior and such that $B \cap L$ is composed of two arcs in $L$, as in Figure 3. Let $L'$ be the resulting link in $M$ obtained by replacing $L \cap B$ by the components as appearing in Figure 4 (b). Then $M \setminus L'$ admits a complete hyperbolic metric of finite volume.

**Proof.** We begin the proof by setting the notation. Let $G$ and $G'$ be the connected components of $L$ containing the arcs $g$ and $g'$ respectively, as in Figure 4(a). Note that it can be the case $G = G'$. Let $L'$ be the link formed by replacing $g \cup g'$ in $B$ by $\gamma_1 \cup \gamma_2 \cup C$. For $i = 1, 2$, let $\Gamma_i$ be the component of $L'$ containing $\gamma_i$. Note that possibly $\Gamma_1 = \Gamma_2$ and let $\Gamma = \Gamma_1 \cup \Gamma_2$.

We split the proof into two cases, depending on whether or not $(M \setminus B, L \setminus (B \cap L))$ is a rational tangle in a 3-ball.

**Claim 4.2.** If $(M \setminus B, L \setminus (B \cap L))$ is a rational tangle in a 3-ball, then $M \setminus L'$ is hyperbolic.

**Proof.** A rational tangle in a 3-ball always has a projection that is alternating (see for instance [8]). Then, $L$ is a rational, alternating link in $S^3$ that is prime, since $M \setminus L$ is hyperbolic. By Corollary 2 of [9], a rational, alternating link in $S^3$ that is
prime is hyperbolic if and only if it is nontrivial and not a 2-braid. After forming $L'$, we consider the link $L''$ obtained from $L'$ by doing a half-twist on the twice-punctured disk bounded by $C$ to add a crossing so that $L'' \setminus C$ has an alternating projection, as in Figure 9. Then, $L''$ is in an augmented alternating link projection obtained from a prime, non-split reduced alternating projection. If $L'' \setminus C$ is neither trivial nor a 2-braid, $L''$ is hyperbolic by [6]. However, if $L'' \setminus C$ is trivial, then $L$ is a 2-braid and hence it does not satisfy the hypothesis that $M \setminus L$ is hyperbolic. And if $L'' \setminus C$ is a 2-braid, then $L$ is a trivial knot, again not satisfying the same hypothesis. So $L''$ is a hyperbolic link in $S^3$. But from Theorem 4.1 of [2], $L''$ is hyperbolic if and only if $L'$ is hyperbolic.

**Remark 4.3.** If $M \setminus L$ is hyperbolic and $(M \setminus B, L \setminus (B \cap L))$ is a rational tangle in a 3-ball, then $L$ is either a rational link or a rational knot in $S^3$ which is hyperbolic. In this case, there is always an arc $\alpha$ as depicted in Figure 3 which is isotopic to a geodesic and hence the switch move can be applied. This follows because $\alpha$ is part of the fixed point set of an involution of the complement, which is realized by an isometry, and fixed-point sets of isometries must be geodesics (see [5] for the details).

From now on, we assume that $(M \setminus B, L \setminus (B \cap L))$ is not a rational tangle in a 3-ball. As in the proof of the Chain Move Theorem (Theorem 3.1), we first assume that $M$ is orientable. We also let $X = M \setminus L$ and $Y = M \setminus L'$ and we will prove that $Y$ is hyperbolic by showing that there are no essential disks, spheres, tori or annuli in $Y$. Once again, we let $S = \partial B$, $Q = S \setminus L = S \setminus L'$ and notice that the same arguments used to prove Claim 3.5 can be used to prove that
Q is incompressible and boundary-incompressible in Y; the details are left to the reader.

Let $D$ be the interior of a twice-punctured disk in $B \setminus L'$ bounded by $C$ and let $\overline{D}$ be its closure in $M$.

**Claim 4.4.** $D$ is incompressible and boundary-incompressible in $Y$.

**Proof.** Using the facts that $Q$ is incompressible in $Y$, that $X$ is hyperbolic and $Y \setminus B = X \setminus B$, we may use a disk replacement argument to assume that any compression disk for $D$ is contained in $B \setminus L'$. Arguing by contradiction, assume that $E \subset B \setminus L'$ is a disk with $\partial E = E \cap D$, and that $\partial E$ is nontrivial in $D$. Let $E_1 \subset \overline{D}$ be the subdisk bounded by $\partial E$ in $\overline{D}$. Let $S = E_1 \cup E$. Then, $S$ is a two-sphere in the ball $B$ which is punctured only once by at least one of the arcs $\gamma_1$ or $\gamma_2$, which is impossible.

Next, we prove that $D$ is boundary-incompressible. Arguing by contradiction, let $E$ be a boundary-compression disk for $D$. By incompressibility and boundary-incompressibility of $Q$, we can assume that $E \subset B$.

First, consider the case when $E \cap D$ is an arc $\beta$ with endpoints in $\partial N(C)$ separating the punctures in $D$. Let $D'$ be the closure in $\overline{D}$ of one of the punctured disks that results from that separation. Then $E' = E \cup D'$ is a disk in $B \setminus N(C)$ with boundary in $\partial N(C)$ and $\partial E'$ must either be trivial or longitudinal on $\partial N(C)$. If $\partial E'$ is trivial, we can form a sphere by adding to $E'$ the disk on $\partial N(C)$ also bounded by $\partial E'$, thereby creating a once-punctured sphere in $B \setminus L'$, a contradiction. If, on the other hand, $\partial E'$ is a longitude on $\partial N(C)$, then the fact $E'$ is only punctured once by $L'$ is a contradiction to its construction in $B$.

Suppose now that $E \cap D$ is an arc $\beta$ from one puncture of $D$ to the other. Then, $\partial E \setminus \beta$ is an arc that cannot be contained in $B$, a contradiction.

Suppose now that $E \cap D$ is an arc $\beta$ on $D$ that begins and ends at the same puncture and goes around the second puncture. Then $\partial E$ cannot link $C$ since $E$ is unpunctured. So $\partial E \cap \partial N(\Gamma)$ can be isotoped on $\partial N(\Gamma)$ into $D$. But then, $E$ becomes a compression disk for $D$, a contradiction.

Using that both $D$ and $Q$ are incompressible and boundary incompressible, we next proceed with the proof of Theorem 4.1.

**Claim 4.5.** $Y$ does not admit essential spheres or essential disks.

**Proof.** We first show that there are no essential spheres in $Y$. Suppose that $S \subset Y$ is a sphere and first assume that $S \cap B = \emptyset$. Then, $S \subset X$, and, since there are no
Claim 4.6. There are no essential annuli in $M \setminus N(L')$.

Proof of Claim 4.6. Suppose that $A$ is an essential annulus in $M \setminus N(L')$. Consider first the case $A \cap B = \emptyset$. Then, $A \subset M \setminus N(L)$ and it must either compress or be boundary-parallel in $M \setminus N(L)$. First, let us assume that $E \subset M \setminus N(L)$ is a compression disk to $A$ with boundary $\beta$. Then $\beta$ separates $A$ into two sub annuli $A_1$ and $A_2$, and $E \cup A_1$ and $E \cup A_2$ give rise to two essential disks in $X$, which contradicts its hyperbolicity.

Hence, we may assume that $A$ is boundary-parallel in $M \setminus N(L)$. Then, there is a component $J$ of $L$ and an annulus $A' \subset \partial N(J)$ such that $\partial A' = \partial A$ and $A \cup A'$ bounds a solid torus $W$ in $M \setminus N(L)$, through which $A$ is parallel to $A'$. If $B \cap W = \emptyset$, then $A$ is boundary parallel in $Y$, a contradiction. Hence, we may assume that $B \cap W \neq \emptyset$. Since $A \cap B = \emptyset$ and $L \cap W = \emptyset$, then $A'$ must intersect $B$ and $J$ must be either $G$ or $G'$, which could be the same component. Suppose first that $G$ and $G'$ are distinct. Then if $\lambda$ is an arc in $B \setminus N(L)$ with an endpoint in $\partial N(G)$ and the other in $\partial N(G')$, at least one endpoint of $\lambda$ is not in $W$. Since $\text{int}(\lambda)$ cannot intersect $A$, it follows that $G' \subset W$ a contradiction.

Suppose now $G$ and $G'$ are the same component $J$. Since $A \cap B = \emptyset$, $\partial A$ is a pair of meridians on $\partial N(J)$. Then, there is a ball $B'$ in $N(J)$ bounded by $A'$ and two meridinal disks in $N(J) \setminus B$ bounded by $\partial A$. Then $W' = W \cup B'$ is a ball in $M$, and $J \cap W'$ is an unknotted properly embedded arc within it. Since $B \cap W \neq \emptyset$, then $B \cap W' \neq \emptyset$. But then, the fact that $\partial W' \cap B = \emptyset$ implies that $B \subset W'$. Hence $\alpha$ can be homotoped into $\partial N(J)$, contradicting the fact it is a geodesic with endpoints on $L$. So $A \cap B \neq \emptyset$.}

essential spheres in $X$, it follows that $S$ bounds a ball $B \subset X$. Since $L$ intersects $B$, this gives that $B \cap B = \emptyset$, hence $B \subset Y$, proving that $S$ is not essential in $Y$.

Next, we treat the case where $S$ intersects $B$. We can take $S$ to have the least number of intersection curves in $S \cap Q$ over all essential spheres. If $S$ were contained in $B$, it bounds a ball in $B$ which is also a ball in $Y = M \setminus L'$, since $S \cap L' = \emptyset$. Next, we assume that $S \cap Q \neq \emptyset$. Then, there exists a disk $E \subset S$ with $\partial E = E \cap Q$. After a standard disk replacement argument using that $Q$ is incompressible and that there are no essential spheres that do not intersect $Q$, we isotope $S$ to lower the number of components in $S \cap Q$, which proves that there are no essential spheres in $Y$.

To prove that there are no essential disks in $Y$, we argue by contradiction and assume that $E$ is such a disk with boundary on a regular neighborhood of a component $J$ of $L'$. Then, $S = \partial N(E \cup N(J))$ is an essential sphere in $Y$, as it splits $J$ from the other components of $L'$, a contradiction.

Claim 4.6. There are no essential annuli in $M \setminus N(L')$. 

Proof of Claim 4.6. Suppose that $A$ is an essential annulus in $M \setminus N(L')$. Consider first the case $A \cap B = \emptyset$. Then, $A \subset M \setminus N(L)$ and it must either compress or be boundary-parallel in $M \setminus N(L)$. First, let us assume that $E \subset M \setminus N(L)$ is a compression disk to $A$ with boundary $\beta$. Then $\beta$ separates $A$ into two sub annuli $A_1$ and $A_2$, and $E \cup A_1$ and $E \cup A_2$ give rise to two essential disks in $X$, which contradicts its hyperbolicity.

Hence, we may assume that $A$ is boundary-parallel in $M \setminus N(L)$. Then, there is a component $J$ of $L$ and an annulus $A' \subset \partial N(J)$ such that $\partial A' = \partial A$ and $A \cup A'$ bounds a solid torus $W$ in $M \setminus N(L)$, through which $A$ is parallel to $A'$. If $B \cap W = \emptyset$, then $A$ is boundary parallel in $Y$, a contradiction. Hence, we may assume that $B \cap W \neq \emptyset$. Since $A \cap B = \emptyset$ and $L \cap W = \emptyset$, then $A'$ must intersect $B$ and $J$ must be either $G$ or $G'$, which could be the same component. Suppose first that $G$ and $G'$ are distinct. Then if $\lambda$ is an arc in $B \setminus N(L)$ with an endpoint in $\partial N(G)$ and the other in $\partial N(G')$, at least one endpoint of $\lambda$ is not in $W$. Since $\text{int}(\lambda)$ cannot intersect $A$, it follows that $G' \subset W$ a contradiction.

Suppose now $G$ and $G'$ are the same component $J$. Since $A \cap B = \emptyset$, $\partial A$ is a pair of meridians on $\partial N(J)$. Then, there is a ball $B'$ in $N(J)$ bounded by $A'$ and two meridinal disks in $N(J) \setminus B$ bounded by $\partial A$. Then $W' = W \cup B'$ is a ball in $M$, and $J \cap W'$ is an unknotted properly embedded arc within it. Since $B \cap W \neq \emptyset$, then $B \cap W' \neq \emptyset$. But then, the fact that $\partial W' \cap B = \emptyset$ implies that $B \subset W'$. Hence $\alpha$ can be homotoped into $\partial N(J)$, contradicting the fact it is a geodesic with endpoints on $L$. So $A \cap B \neq \emptyset$. 

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Next, we treat the case $A \subset B$. Let $\alpha_1, \alpha_2$ denote the two components of $\partial A$. First, we assume that $\alpha_1 \subset \partial N(\Gamma)$ and $\alpha_2 \subset \partial N(C)$. Since $A \subset B \setminus N(L')$, $\alpha_1$ is either a meridian of $\partial N(\gamma_1)$ or a meridian of $\partial N(\gamma_2)$, and the symmetry between $\gamma_1$ and $\gamma_2$ allows us to assume $\alpha_1 \subset \partial N(\gamma_1)$. Take a meridional disk $E_1$ in $N(\gamma_1) \cap B$ with $\partial E_1 = \alpha_1$. Then, $E = A \cup E_1$ is a disk in $B \setminus N(C)$ with $\partial E = \alpha_2 \subset \partial N(C)$. Hence, $\alpha_2$ is a longitude of $\partial N(C)$. In particular, $\alpha_2$ links $\gamma_2$ in $B$, and hence $\gamma_2$ must puncture $E$, which is a contradiction. This contradiction shows that if $A \subset B \setminus N(L')$ is an essential annulus, then $\alpha_1$ and $\alpha_2$ are either both parallel curves on $\partial N(C)$ or both meridians on $\partial N(\Gamma)$.

Assume that $A$ is an essential annulus in $M \setminus N(L')$ such that $A \subset B$ and $\alpha_1$ and $\alpha_2$ are meridians on $\partial N(\Gamma)$. Let $E_1, E_2$ be two meridional disks in $N(\Gamma)$ with respective boundaries $\alpha_1, \alpha_2$. Then, $A \cup E_1 \cup E_2$ is a sphere in $B$ that bounds a ball $B \subset B$, which is either punctured once by each $\gamma_1$ and $\gamma_2$, which is not possible, or twice by one of them, say $\gamma_1$. Since $A$ is not boundary parallel, then $C \subset B$. However, since $C$ links both $\gamma_1$ and $\gamma_2$, $\gamma_2$ must be contained in $B$, which is a contradiction.

Still assuming that $A \subset B$, it remains to obtain a contradiction when both $\alpha_1$ and $\alpha_2$ are $(p, q)$-curves on $\partial N(C)$. In this case, $B \setminus (N(\gamma_1) \cup N(C))$ is diffeomorphic to $T \times [0, 1]$, where $T = S^1 \times S^1$ is a torus, and we identify $\partial N(C)$ with $T \times \{0\}$. Since any annulus in $T \times [0, 1]$ with boundary in $T \times \{0\}$ is parallel to an annulus in $T \times \{1\}$, it follows that $A$ is parallel to an annulus $A' \subset \partial N(C)$ with $\partial A' = \alpha_1 \cup \alpha_2$, in the sense that there is a solid torus region $W \subset T \times [0, 1]$ with $\partial W = A \cup A'$. Since $N(\gamma_2) \subset T \times [0, 1]$ and does not intersect $\partial W$, the fact that the endpoints of $\gamma_2$ lie in $T \times \{1\}$ implies that $N(\gamma_2)$ is disjoint from $W$. Therefore, $A$ is boundary parallel in $B \setminus N(L')$, contradicting the assumption that $A$ was essential.

From now on, we will assume that $A$ intersects $Q$. We will also assume that $A$ minimizes the number of intersection curves of an essential annulus of $M \setminus N(L')$ with $Q$. In particular, since $Q$ is incompressible, the connected components of $A \setminus Q$ are either annuli or disks whose boundary intersect $\partial A$.

Suppose first that there is an intersection arc $a$ in $A \cap Q$ that cuts a disk $E$ from $A$. Then both endpoints of $a$ are on the same boundary component of $A$ and $E \cap Q \subset \partial E$. Because $Q$ is boundary-incompressible, it must be the case that $a$ cuts a disk $E_1$ from $Q$. Then $E_2 = E \cup E_1$ is a disk properly embedded in $M \setminus N(L')$. Since there are no essential disks in $M \setminus N(L')$, then $\partial E_2$ bounds a disk $E_3$ in $\partial N(L')$. Then, $E_2 \cup E_3$ is a sphere that bounds a ball in $M \setminus N(L')$, through which $E$ can be isotoped to $E_1$, and just beyond to eliminate $a$ from $A \cap Q$, contradicting that we assumed a minimal number of intersection components.
Thus, we now know that there are only two possibilities for the intersection curves in $A \cap Q$. Either they are all parallel nontrivial closed curves on $A$ or they are all arcs with endpoints on distinct boundary components of $A$.

If there are no arcs in $A \cap Q$, then $\partial A \cap Q = \emptyset$. Since $A$ and $Q$ are incompressible in $M \setminus L'$, the minimality condition on the number of curves in $A \cap Q$ implies that any curve in $A \cap Q$ is nontrivial on both $A$ and on $Q$.

Next, we prove that any curve in $A \cap Q$ must encircle two of the punctures of $Q$. Arguing by contradiction, assume that $a$ is a simple closed curve in $A \cap Q$ and assume that $a$ bounds a once-punctured disk $E$ in $Q$. Without loss of generality, we may assume that $E$ is innermost in the sense that $E \cap A = a$. Using $E$ to surger $A$, we obtain two annuli in $M \setminus L'$, where at least one is still essential, and, after a small isotopy, with a lesser number of intersection components with $Q$, which is a contradiction. Thus, any curve in $A \cap Q$ encircles two of the punctures of $Q$ and all intersection curves must be parallel on $Q$, separating one pair of punctures from the other pair.

Still under the assumption that $\partial A \cap Q = \emptyset$ and $A \cap Q \neq \emptyset$, we next rule out the case where at least one boundary component of $A$, say $\alpha_1$, lies in $\partial N(C)$. In this case, let $A_1$ be the connected component of $A \cap B$ containing $\alpha_1$ and let $a = \partial A_1 \setminus \alpha_1$ denote the other boundary component of the annulus $A_1$. Let $E$ be one of the two disks defined by $a$ in $S$. Then, $A_1 \cup E$ is a disk in $B \setminus N(C)$ which has nontrivial boundary in $\partial N(C)$; hence, $\alpha_1$ is a longitude. After an isotopy on $A_1$, we may assume that $\alpha_1 \cap D = \emptyset$, and thus $\partial A_1 \cap D = \emptyset$. Since $D$ is incompressible, we may isotope $A_1$ in $B \setminus L'$ to assume that $A_1 \cap D$ does not contain any trivial curves. Moreover, if $\beta \subset A_1 \cap D$ is a nontrivial simple closed curve both in $D$ and in $A_1$, then $\beta$ cannot encircle one puncture in $D$, since this would generate a thrice-punctured sphere in $B \setminus L'$, a contradiction. Hence, any curve in $A_1 \cap D$ encircles both punctures of $D$; this gives rise to solid tori regions in $B \setminus N(L')$ that can be used to further isotope $A_1$ in $B \setminus L'$ to assume that $A_1 \cap D = \emptyset$. In particular, after capping $\alpha_1$ with a longitudinal disk in $B \setminus (N(C) \cup A_1)$, it follows that $a$ is the boundary of a disk in $B \setminus L$.

Since any other curve in $A \cap Q$ must be parallel to $a$, $A \cap Q$ is a family $\{a_1, a_2, \ldots, a_n\}$ of pairwise disjoint simple closed curves, all parallel to each other both in $Q$ and in $A$. In particular, for each $i \in \{1, 2, \ldots, n\}$, $a_i$ generates $\pi_1(A)$ and bounds a disk $E_i \subset B \setminus L$, punctured once by the arc $a$. Note that $n \geq 2$, since otherwise $\alpha_2 \subset \partial N(J)$, where $J$ is a component of $L$ and then capping $A$ with a disk in $B \setminus L$ bounded by $\alpha_1$ would yield an essential disk in $X$. This implies that there exists a subannulus $A_2 \subset A \setminus B$ with boundary $\partial A_2 \subset Q$. Let us assume that $\partial A_2 = a_1 \cup a_2$. Then (after possibly isotoping the disks $E_1, E_2$
in \( B \setminus L \) so they become disjoint) \( S = A_2 \cup E_1 \cup E_2 \) is a sphere in \( X \), which bounds a ball \( B \subset X \). Let \( V = B \setminus B \), then \( V \) is a solid torus in \( X \setminus B = Y \setminus B \) and we may use \( V \) to isotope \( A \) in \( Y \) to reduce the number of intersection components in \( A \cap Q \), a contradiction.

In this point on the proof, assuming that \( \partial A \cap Q = \emptyset \) and that \( A \cap Q \neq \emptyset \), it remains to rule out the case where no boundary component of \( A \) is on \( \partial N(C) \). Then, \( \partial A \cap B = \emptyset \), since otherwise a boundary component of \( A \) would be a meridian in \( \partial N(\Gamma) \) and we could isotope \( A \) to reduce the number of intersection components in \( A \cap Q \). Next, we show that, after an isotopy, \( A \cap D = \emptyset \). Indeed, since \( D \) and \( A \) are both incompressible, after a disk replacement argument we may assume that any curve in \( A \cap D \) is a simple closed curve that generates \( \pi_1(A) \) and either encircles one or two of the punctures of \( D \). If there is a curve \( a \subset A \cap D \), we may assume that either \( a \) encircles one puncture of \( D \) and is innermost or that \( a \) encircles the two punctures of \( D \) and is outermost. In either case, we can surger \( A \) to obtain two annuli in \( Y \), where at least one is still essential in \( Y \) and with less intersection components with \( Q \), a contradiction that proves that \( A \cap D = \emptyset \).

Next, the identification \( B \setminus N(D \cup L') \equiv Q \times [0, 1] \) gives us that we may isotope \( A \) in \( Y \) to make \( A \) disjoint from \( B \). Since we already showed that there are no essential annuli in \( Y \) disjoint from \( B \), this is a contradiction.

This shows that if \( A \) is an essential annulus in \( Y \), then all intersection curves in \( A \cap Q \) are arcs, the endpoints of each of which lie on distinct boundary components of \( A \). Such arcs cut \( A \) into a collection of disks. Because \( S \) separates \( M \), there must be an even number of such arcs and hence such disks, and the disks must alternate between lying inside and outside \( B \).

There are no such arcs that cut a disk from \( Q \). Indeed, if there were such a disk, by choosing an innermost one, we could surger \( A \) along this disk to obtain a disk \( \Delta \) with boundary in \( \partial N(L') \). Since there are no essential disks in \( M \setminus N(L') \), \( \partial \Delta \) must bound a disk \( \Delta' \) on \( \partial N(L') \). Then \( \Delta \cup \Delta' \) is a sphere bounding a ball in \( M \setminus N(L') \). Hence, we can isotope \( \Delta \) to \( \Delta' \) through the ball, and hence isotope \( A \) to an annulus in \( \partial N(L') \), contradicting the fact that \( A \) is not boundary parallel in \( M \setminus N(L') \).

Let \( E \) be a connected component of \( A \cap B \), which necessarily is a disk. Next, we show that there are two possibilities for \( E \) up to isotopy and switching the roles of \( \gamma_1 \) and \( \gamma_2 \). These two possibilities are depicted in Figure 10.

Let \( \partial E = \beta_1 \cup \mu_1 \cup \beta_2 \cup \mu_2 \) where \( \beta_1 \) and \( \beta_2 \) lie in \( Q \) and \( \mu_1 \) and \( \mu_2 \) lie in \( \partial N(L') \). Note that each of \( \mu_1 \) and \( \mu_2 \) must begin and end at distinct components of \( \partial N(\Gamma) \cap Q \), since otherwise we could lower the number of intersection curves
Figure 10: Possibilities for $E$, a connected component of $A \cap B$ when all intersections of $A \cap Q$ are arcs.

of $A$ with $Q$.

For the arguments that follow, we set coordinates and consider $B = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$, $D$ a horizontal disk in $\{z = 0\}$ and the two arcs $\gamma_1, \gamma_2$ parallel to the $z$-axis. Let $A'$ be the annular connected component of $(B \setminus N(C)) \cap \{z = 0\}$. Then, $A'$ is annulus with one boundary component in $Q$ and the other boundary component a longitude on $\partial N(C)$.

We assume that we have isotoped $E$ in $B \setminus N(L')$ to minimize the number of intersection curves in $E \cap A'$, and next we prove that $E \cap A' = \emptyset$. First, we claim that $E \cap A'$ does not contain any arc. Indeed, if there were an arc $\phi \subset E \cap A'$, since $\partial E \cap \partial N(C) = \emptyset$, $\phi$ would cut a disk $H_1$ from $A'$ and a disk $H_2$ from $E$. Let $H_3 = H_1 \cup H_2$. If $\phi$ has both endpoints in the same $\beta_i$, then $\partial H_3 \subset Q$, which, by incompressibility of $Q$, implies that $\partial H_3$ is a trivial curve bounding a disk $H_4 \subset Q$. Then $H_3 \cup H_4$ is a sphere bounding a ball, through which we can isotope $H_1$ through $H_2$, and lower the number of intersection curves in $E \cap A'$, a contradiction.

If $\phi$ has one endpoint in $\beta_1$ and the other in $\beta_2$, then $\partial H_3$ consists of one arc in $Q$ and one taut arc on $\partial N(\gamma_i)$. Then, we can use $H_3$ to isotope $\gamma_i$ to $Q$, a contradiction to the fact that $C$ links $\gamma_i$ in $B$. So $E \cap A'$ can only contain simple closed curves.

If $\phi \subset E \cap A'$ is a simple closed curve, then there is a disk $E' \subset E$ with $\partial E' = \phi$. In particular, $\phi$ is nontrivial in $A'$, since otherwise we could use a disk
replacement argument to isotope $E$ removing $\phi$ from $E \cap A'$. Since $\phi$ is isotopic to $C$ through $A'$, we could obtain a disk in $Y$ with nontrivial boundary in $\partial N(C)$, a contradiction. Thus, $A' \cap E = \emptyset$.

If $\mu_1$ and $\mu_2$ lie on $\partial N(\gamma_1)$ and $\partial N(\gamma_2)$ respectively, then by an isotopy on $Q \cup \partial N(\gamma_1 \cup \gamma_2)$, we can assume that $\mu_1$ and $\mu_2$ are vertical arcs that do not wind around $\partial N(\gamma_1)$ or $\partial N(\gamma_2)$. Then because $\beta_1$ and $\beta_2$ cannot cross the equator $\partial A' \cap Q$, after possibly reindexing, $\beta_1$ connects the top two punctures of $Q$ and $\beta_2$ connects the bottom two punctures. Since $\partial E$ must be trivial as an element of the fundamental group of the handlebody $B \setminus N(\gamma_1 \cup \gamma_2)$, there can be no twisting around the punctures, and $E$ must appear as in Figure 10 (a).

If $\mu_1$ and $\mu_2$ both lie on $\partial N(\gamma_1)$, then $\beta_1$ and $\beta_2$ are loops on $Q$ based at a puncture and restricted to the upper and lower hemisphere. Since no arcs in $A \cap Q$ can cut disks off $Q$, then each $\beta_1$ and $\beta_2$ circle a puncture in $Q$. Hence, $E$ must appear as in Figure 10 (b). A similar case occurs when $\mu_1$ and $\mu_2$ both lie on $\partial N(\gamma_2)$.

This argument allows us to introduce the following language. If $\beta \subset A \cap Q$ is any arc, then there is a unique disk $E \subset A \cap B$ with $\beta \subset \partial E$. If $E$ is a Type I disk, we shall say that $\beta$ is a Type I arc. Otherwise, we will say that $\beta$ is a Type II arc.

Next, we show that all intersection arcs in $A \cap Q$ are of the same type. If $\Gamma_1 \neq \Gamma_2$, then, if there exists a Type I disk, the two boundaries of $A$ are on different components and only Type I disks can occur. If there is not a Type I disk, then all disks are Type II. On the other hand, if $\Gamma_1 = \Gamma_2$, both components of $\partial A$ intersect $\partial N(\gamma_1)$ and $\partial N(\gamma_2)$ the same equal number of times. In this case, the existence of a Type II disk $E_1$ with boundary intersecting $\partial N(\gamma_1)$ in two arcs, implies that there exists a Type II disk $E_2$ intersecting $\partial N(\gamma_2)$. But $E_1$ and $E_2$ would then intersect, a contradiction.

Assume that all arcs in $A \cap Q$ are of Type I and let $E$ be a connected component of $A \setminus B$. Then, when we switch from $L'$ to $L$, $E$ can be extended to a disk properly embedded in $M \setminus N(L)$. Thus, there is a trivial component in $L$, a contradiction to its hyperbolicity.

Our next argument eliminates the last case when all intersections of $A \cap Q$ are of Type II, and $\Gamma_1 \neq \Gamma_2$, since we cannot mix the two types of Type II intersections. Until the end of the proof we will assume that $\partial A \subset \partial N(\Gamma_1)$. Let $E \subset A$ be a connected component of $A \setminus B$, and we label $\partial E = \beta_1 \cup \mu_1 \cup \beta_2 \cup \mu_2$, where $\beta_1$ and $\beta_2$ lie in $Q$ and $\mu_1$ and $\mu_2$ lie in $\partial N(\Gamma_1)$. Then, $\mu_1$ and $\mu_2$ define two disks $\Delta, \tilde{\Delta}$ in the annulus $\partial N(\Gamma_1) \setminus B$. We assume that the disk $\Delta$ is the one that makes
\(\tilde{A} = E \cup \Delta\) an annulus in \(Y \setminus B\) with both boundary components in \(Q\) parallel to the punctures that come from \(\Gamma_2\).

After capping \(\tilde{A}\) with the two once-punctured disks bounded by \(\partial \tilde{A}\) in \(Q\), we create an incompressible annulus \(\tilde{A}\) in \(Y \setminus B\) which also lives and is incompressible in \(X \setminus B\). Since \(X\) is hyperbolic, it follows that \(\tilde{A}\) must be boundary-parallel to \(\Gamma_2\). But this implies that \(\mu_1\) is parallel in \(X \setminus B\) to the arc \(j_2 = \Gamma_2 \setminus B\), and there exists a disk \(E' \subset X \setminus B\) with \(\partial E' = \mu_1 \cup \nu_1 \cup j_2 \cup \nu_2\), where \(\nu_1 \cup \nu_2 = E' \cap Q\) are two arcs joining the respective two upper punctures and the two lower punctures of \(Q\) which avoid the equator of \(Q\). It then follows that \(\nu_1 \cup g\) and \(\nu_2 \cup g'\) bound two respective disks in \(B \setminus L\), and the union of those disks with \(E'\) is an essential disk in \(M \setminus L\), contradicting hyperbolicity of \(X\) and finishing the proof of Claim 4.6. \(\square\)

**Claim 4.7.** \(Y\) does not admit essential tori.

*Proof.* We argue by contradiction and suppose that \(T\) is a torus which is incompressible and not boundary-parallel in \(Y\). First, suppose that \(T \cap D = \emptyset\). Then, after an isotopy in \(Y\), we may assume that \(T \cap B = \emptyset\). Hence, \(T \subset X\) and, since \(X\) is hyperbolic, \(T\) is either compressible or boundary parallel in \(X\). If \(T\) is boundary parallel, since both \(G\) and \(G'\) intersect \(B\), \(T\) must be parallel to a component \(J\) of \(L\) which lives in \(L'\), contradicting that \(T\) is essential in \(Y\).

Next, we treat the case where \(T\) is compressible in \(X\); let \(E \subset X\) be a compression disk for \(T\) and assume that \(E\) has the least number of intersection curves with \(Q\) among compression disks for \(T\). Since \(T\) is incompressible in \(Y\), \(E\) intersects \(B \cap L'\) and the arc \(\alpha\), which is a complete geodesic in the hyperbolic metric of \(X\). Let \(\overline{N}(E) \subset X\) be a closed neighborhood of \(E\) with coordinates \(E \times [0, 1]\) and such that \((\partial E \times [0, 1]) \subset T\). Since \(\alpha\) is transverse to \(E\), we may choose such a coordinate system on \(\overline{N}(E)\) in such a way that, for each \(t \in [0, 1]\), each component of \(\alpha \cap \overline{N}(E)\) intersects \(E \times \{t\}\) transversely in a single point.

Let \(S = (T \setminus (\partial E \times [0, 1])) \cup (E \times \{0\}) \cup (E \times \{1\})\). Then, \(S\) is a sphere in \(X\) and \(T \setminus S = \partial E \times [0, 1]\). Since \(X\) is hyperbolic, \(S\) separates and must bound a closed ball \(B \subset X\). Let \(\alpha_1, \alpha_2, \ldots, \alpha_n\) be the arcs in \(\alpha \cap \overline{N}(E)\). We claim that each \(\alpha_i\) is contained in \(B\). This follows because the endpoints of \(\alpha\) are in \(L\), \(L \cap B = \emptyset\) and \(T \cap B = \emptyset\). In particular, \(\overline{N}(E) \subset B\).

Let \(W = B \setminus \overline{N}(E)\). Then \(\partial W = T\) and \(W\) is a knot exterior in \(B\) bounded by \(T\) (in fact, we think of \(W\) as obtained from \(B\) by removing a potentially knotted hole, see Figure 11). Since \(T \cap L' = \emptyset\) and \(L'\) intersects \(\overline{N}(E)\), then \(W \subset Y\). Our next argument is to show that \(\overline{N}(E)\) is unknotted in \(B\); thus \(W\) is a solid torus bounded by \(T\), which contradicts the essentiality of \(T\) in \(Y\).
Let $\Pi: \mathbb{H}^3 \to X$ be the Riemannian universal covering map of $X$. By appropriately choosing a neighborhood $\mathcal{N}(L)$, it follows that $\Pi^{-1}(\partial \mathcal{N}(L))$ is a collection of horospheres in $\mathbb{H}^3$. Moreover, $\Pi^{-1}(\alpha)$ is a collection of geodesics connecting these horospheres. On the other hand, $B$ lifts to a collection of balls, one of which is a ball $\tilde{B}$, containing a lift of $W$, denoted $\tilde{W}$.

In order for $T$ to be incompressible in $Y$, a lift of $\alpha$, which we denote by $\tilde{\alpha}$, must pass through the hole $\tilde{B} \setminus \tilde{W}$ in $\tilde{B}$. Since $\tilde{\alpha}$ is a geodesic in $\mathbb{H}^3$, it follows that $\tilde{\alpha}$ is unknotted, which implies that $\tilde{W}$ is a solid torus. Since $W$ is homeomorphic to $\tilde{W}$, this gives a contradiction, as previously explained.

It remains to prove that there are no essential tori in $Y$ which intersect $D$. Arguing by contradiction, assume that $T$ is such a torus. Since $D$ is incompressible in $Y$, a disk replacement argument allows us to further assume that any curve in $T \cap D$ is nontrivial both in $T$ and in $D$. Let $\beta$ be a curve in $T \cap D$. If $\beta$ encircles one puncture of $D$, take an innermost curve in such intersection and use the one-punctured disk it bounds in $D$ to surger $T$ and obtain an essential annulus in $Y$ with boundary in $\partial N(\Gamma_i)$. If $\beta$ encircles both punctures of $D$, take an outermost curve on $T \cap D$ and use the outer annulus on $D$ to surger $T$ and obtain an essential annulus with boundary in $\partial N(C)$. Since Claim 4.6 gives that $Y$ does not admit essential annuli, this proves Claim 4.7.

Thus, having proved that $Y$ satisfies Thurston’s hyperbolicity conditions, Theorem 4.1 follows when $M$ is orientable.
Next, we assume that $M$ is nonorientable and that $L$, $L'$, $\alpha$ and $B$ are as before. Let $\Pi: \hat{M} \to M$ be the two-sheeted oriented covering map of $M$. Then, $\hat{L} = \Pi^{-1}(L)$ is a hyperbolic link in $\hat{M}$. We also let $\hat{L}'' = \Pi^{-1}(L')$, $B_1$, $B_2$ be the connected components of $\Pi^{-1}(B)$ and $\alpha_1$, $\alpha_2$ be the connected components of $\Pi^{-1}(\alpha)$. We claim that $\hat{Y} = \hat{M} \setminus \hat{L}''$ is also hyperbolic. Note that, as explained in the proof of the nonorientable setting for the chain move, the fact that $\hat{Y}$ is hyperbolic implies that $Y = M \setminus L'$ is hyperbolic.

Since $\alpha$ is a complete geodesic in the hyperbolic metric of $M \setminus L$, then both $\alpha_1$ and $\alpha_2$ are complete geodesics in $\hat{M} \setminus \hat{L}$. In particular, since $\hat{M}$ is orientable, the switch move allows us to replace $\hat{L} \cap B_1$ by a tangle diffeomorphic to $L' \cap \hat{B}$ to create a new hyperbolic link $\hat{L}'$ in $\hat{M}$. Note that $\hat{L}''$ may be obtained from $\hat{L}'$ by replacing the tangle $\hat{L}' \cap B_2 = \hat{L} \cap B_2$ by a tangle diffeomorphic to $L' \cap B$.

Since it might be the case that $\alpha_2$ is not isotopic to a geodesic in the hyperbolic metric of $\hat{X} = \hat{M} \setminus \hat{L}'$, one cannot directly apply the switch move a second time. However, most of the arguments in its proof can be repeated without change for this setting. We next guide the reader over the steps in the proof that need some adaptation.

First, the arguments in the proof of the orientable case for the switch move can be used to prove that the four-punctured sphere $\hat{Q}_1 = \partial B_1 \setminus \hat{L}''$ is incompressible and boundary-incompressible in $\hat{X}$ and in $\hat{Y} = \hat{M} \setminus \hat{L}''$, that $\hat{Q}_2 = \partial B_2 \setminus \hat{L}''$ is incompressible and boundary-incompressible in $\hat{Y}$ and that $\hat{Y}$ does not admit any essential disks and essential spheres.

To prove that $\hat{Y}$ does not admit any essential annuli, the arguments in Claim 4.6 apply to show that if $A$ is an essential annulus in $\hat{Y}$, then both boundary components of $A$ are meridians in a component $\hat{G}'$ of $\hat{L}'$ that intersects $B_2$ and that we may isotope $A$ in $\hat{Y}$ to assume that $A \cap B_2 = \emptyset$. Since $\hat{X} \setminus B_2 = \hat{Y} \setminus B_2$, then $A$ is an incompressible annulus in $\hat{X}$, and $A$ must be boundary-parallel in $\hat{M} \setminus N(\hat{L}')$. In particular, after an isotopy in $\hat{Y}$ that does not change the property $A \cap B_2 = \emptyset$, we may assume that $\partial A \cap B_1 = \emptyset$ and that if $A$ intersects $B_1$, then each connected component of $A \cap B_1$ is an annulus parallel to one of the two arcs in the tangle $\hat{L}' \cap B_1$.

If $A \cap B_1 = \emptyset$, $A$ is an incompressible annulus in the hyperbolic manifold $\hat{M} \setminus \hat{L}$, and the same arguments in the proof of Claim 4.6 apply to show that the neighborhood through which $A$ is parallel to an annulus $A'$ in $\partial N(\hat{L})$ can be capped off by meridional disks to define a ball $W'$ in $\hat{M}$ that contains both $B_1$ and $B_2$ and may be used to homotope the arcs $\alpha_1$ and $\alpha_2$ to $\partial N(\hat{L})$, a contradiction.
with the fact that both $\alpha_1$ and $\alpha_2$ are geodesics in the hyperbolic metric of $\hat{M} \setminus \hat{L}$.

Hence, there must exist $A_0$ a connected component of $A \cap B_1$. We assume that $A_0$ is innermost in the sense that no other component of $A \cap B_1$ lies in the ball region defined by $A_0$ in $B_1$. Then, each boundary component of $A_0$ is a curve in $Q_1$ that encircles one puncture, defining a once-punctured disk in $Q_1$. Using these two once-punctured disks to surger $A$ gives two incompressible annuli in $\hat{M} \setminus N(\hat{L'})$, both disjoint from $B_2$ and at least one of them must be essential in $\hat{\mathcal{Y}}$. By induction on the number of components in $A \cap B_1$, this argument yields an essential annulus $\hat{A}$ in $\hat{\mathcal{Y}}$, with both boundary components being meridians, and that is disjoint both from $B_1$ and from $B_2$. As already shown, this is a contradiction that proves that $\hat{\mathcal{Y}}$ does not admit any essential annuli.

The proof that $\hat{\mathcal{Y}}$ does not admit any essential tori uses the arguments in Claim 4.7. Among all possible essential tori, the only case that still needs an adaptation is when $V$ is an essential torus in $\hat{\mathcal{Y}}$ that can be isotoped to be disjoint from $B_2$. Let $D_1$ be a twice punctured disk in $B_1 \setminus \hat{L}$ bounded by the trivial component $\hat{C}_1$ of $\hat{L'} \cap B_1$. Then, $D_1$ is incompressible and we may isotope $V$ in $\hat{\mathcal{Y}}$ to assume that there are no trivial curves in $V \cap D_1$. Hence, $V \cap D_1 = \emptyset$, since the existence of a nontrivial curve in $V \cap D_1$ allows us to surger $V$ to produce an essential annulus in $\hat{\mathcal{Y}}$, which we already proved that cannot exist. In particular, $V$ can also be isotoped in $\hat{\mathcal{Y}}$ to be disjoint from $B_1$, and then $V$ is a torus in the hyperbolic manifold $\hat{M} \setminus \hat{L}$. Since $V$ cannot be boundary parallel in $\hat{M} \setminus \hat{L}$, there exists a compressing disk $E$ for $V$ in $\hat{M} \setminus \hat{L}$, and the fact that $V$ is incompressible in $\hat{\mathcal{Y}}$ implies that $E$ must necessarily intersect the arcs $\alpha_1$ and $\alpha_2$, which are geodesics in the hyperbolic metric of $\hat{M} \setminus \hat{L}$. Now, the same arguments in the proof of Claim 4.7 apply to show that $V$ bounds a unknotted solid region $W$ in $\hat{\mathcal{Y}}$, contradicting the fact that $V$ is essential in $\hat{\mathcal{Y}}$. This argument finishes the proof that $\hat{\mathcal{Y}}$ satisfies Thurston’s hyperbolicity conditions and, as already explained, proves the Switch Move Theorem for the nonorientable case.

\section{The Switch Move Gluing Operation.}

We describe in Theorem 5.1 below a method to obtain new hyperbolic 3-manifolds of finite volume from two previously given ones; this method uses a variant of the switch move (Theorem 4.1). Before stating this result, we set the notation. Let $M_1, M_2$ be compact 3-manifolds with nonempty boundary with zero Euler characteristic. Let $L_1 \subset M_1$ and $L_2 \subset M_2$ be links such that $\text{int}(M_1 \setminus L_1)$
and int($M_2 \setminus L_2$) admit hyperbolic metrics of finite volume. Let $T_1 \subset \partial M_1$ and $T_2 \subset \partial M_2$ be boundary components that are either both tori or both Klein bottles. For each $i \in \{1, 2\}$, let $\alpha_i$ be a complete geodesic in the hyperbolic metric of int($M_i \setminus L_i$) with one endpoint in $T_i$ and another endpoint in a component $J_i$ of $L_i$.

Choose a gluing diffeomorphism $\phi: T_1 \to T_2$ which maps the endpoint of $\alpha_1$ in $T_1$ to the endpoint of $\alpha_2$ in $T_2$ and let $M$ be the quotient manifold obtained by gluing $M_1$ and $M_2$ along $\phi$. Then, we may consider $M_1$ and $M_2$ as subsets of $M$, which are separated by the compact surface $T$ that comes from the identification of $T_1$ and $T_2$.

**Theorem 5.1 (Switch Move Gluing Operation).** With the above notation, let $\alpha$ be the concatenation of $\alpha_1$ and $\alpha_1^{-1}$ in $M$. Let $B$ be a ball neighborhood of $\alpha$ in $M$ that intersects $L = L_1 \cup L_2$ in two arcs $g, g'$ as in Figure 3 and intersects $T$ in a disk $\Delta$. Let $L'$ be the resulting link in $M$ obtained by replacing $g \cup g'$ by the tangle $\gamma_1 \cup \gamma_2 \cup C$ as in Figure 4 (b), where $C \subset \Delta$. Then, the manifold $Y = M \setminus L'$ is hyperbolic.

After choosing $\phi$ as above, like in the case of the switch move, the operation described above may yield two distinct hyperbolic 3-manifolds depending on the projection of the strands $g, g'$, see Remark 1.1.

**Proof of Theorem 5.1.** We first prove the theorem in the case when $M$ is orientable. Let $X = M \setminus L$. Then, the setting in Theorem 5.1 is the same as in the Switch Move Theorem, Theorem 4.1, with the exception that $X$ is no longer hyperbolic. However, $X$ is close to being hyperbolic in the following sense:

**Claim 5.2.** $X$ does not admit any essential spheres, essential disks and essential annuli. Moreover, any essential torus in $X$ is isotopic to $T$.

**Proof of Claim 5.2.** Suppose there were an essential disk $E$ in $M \setminus L$. Since both int($M_1 \setminus L_1$) and int($M_2 \setminus L_2$) are hyperbolic, it follows that $E$ must intersect $T$. But because $T$ is incompressible and $\partial E$ is disjoint from $T$, we may replace subdisks in $E$ by disks in $T$ to obtain an essential disk in either int($M_1 \setminus L_1$) or int($M_2 \setminus L_2$), a contradiction. The same argument shows that an essential sphere in $M \setminus L$ would generate an essential sphere in either int($M_1 \setminus L_1$) or int($M_2 \setminus L_2$), also a contradiction.

Since $T$ separates, an essential torus in $M \setminus L$ that does not intersect $T$ must be parallel to $T$, and hence isotopic to $T$. To prove that $T$ is the only possible essential torus up to isotopy, we argue by contradiction and assume that $\tilde{T}$ is an
essential torus in $M \setminus L$ that is not isotopic to $T$ and has the fewest number of intersection components with $T$. Then, any curve in $\hat{T} \cap T$ is nontrivial both in $T$ and in $\hat{T}$. Then, it follows that there is a component of $\hat{T} \setminus T$ that is an essential annulus either in $\text{int}(M_1 \setminus L_1)$ or in $\text{int}(M_2 \setminus L_2)$, a contradiction. Analogously, we may show that $M \setminus L$ does not admit any essential annuli, and this proves Claim 5.2.

Having proved Claim 5.2, observing that $L \setminus B$ is not a rational tangle in a 3-ball, we notice that the arguments in the proof of the Switch Move Theorem apply directly to show that $Y$ does not admit any essential spheres and any essential disks and that the four-punctured sphere $Q = \partial B \setminus L = \partial B \setminus L'$ and the twice-punctured disk $D$ bounded by $C$ on $T$ are incompressible and boundary-incompressible in $Y$.

To prove that $Y$ does not admit any essential annuli, the proof of Claim 4.6 applies directly, since the arcs $g$ and $g'$ in $L$ that intersect $B$ are on distinct components of $L$. Hence, to prove Theorem 5.1 when $M$ is orientable, it remains to show that $Y$ does not admit any essential tori.

We argue by contradiction and assume that $V$ is an essential torus in $M \setminus L'$ that has the least number of intersection components with $T$ among all essential tori in $Y$. Then, after assuming general position, $T \cap V$ is a finite collection of pairwise disjoint simple closed curves. Let $\gamma$ be one of such intersection components. If $\gamma \subset D$, then it does not bound a disk in $T$ and either encircles one or two of the punctures of $D$. Then, we can choose a component $\gamma'$ in $V \cap D$ (if $\gamma$ encircles one puncture, we choose $\gamma'$ as an innermost curve, otherwise we choose $\gamma'$ as an outermost curve) and surger $T$ to obtain an essential annulus in $Y$, a contradiction. It then follows that $V \cap D = \emptyset$, and then $V$ can be isotoped through $Y$ to be disjoint from $B$, without increasing the number of intersection components in $V \cap T$. Hence, $V$ is a torus that is contained in $X$.

In $X$, $V$ is not isotopic to $T$, since $V \subset M \setminus L'$ and any torus isotopic to $T$ is punctured by $L'$. We claim that $V \cap T = \emptyset$. Argue by contradiction and assume that there exists a curve $\gamma$ in $V \cap T$. Then, $\gamma$ does not intersect $D$ and there are two possibilities: either $\gamma$ is a nontrivial curve in $T$ or $\gamma$, together with $C$, bounds an annulus in $T \setminus D$. In the latter case, we may use this annulus to surger $V$ and obtain an essential annulus in $Y$. Since $Y$ does not admit essential annuli, $\gamma$ is nontrivial in $T$. Then, there is a component of $V \setminus T$ that is an essential annulus in either $M_1 \setminus L_1$ or in $M_2 \setminus L_2$, which proves that $V \cap T = \emptyset$.

Since $T$ separates $M$, there exists $i \in \{1, 2\}$ such that $V$ is a torus in the hyperbolic manifold $M_i \setminus L_i$. For convenience, we will assume that $i = 1$. Note
that \( V \) cannot be boundary parallel in \( M_1 \setminus L_1 \), since this either contradicts its essentiality in \( Y \) or the fact that it is not isotopic to \( T \). Hence, it must be the case that \( V \) is compressible in \( M_1 \setminus L_1 \). Let \( E \subset M_1 \setminus L_1 \) be a compression disk for \( V \). Then, since \( V \) is incompressible in \( Y \), the geodesic \( \alpha_1 \) must intersect \( E \). Now, the same arguments used in the proof of Claim 4.7 for the case when \( T \) was an essential torus in \( Y \), disjoint from \( B \) and compressible in \( X \) apply to obtain that \( V \) is compressible in \( Y \), a contradiction that proves Theorem 5.1 when \( M \) is orientable.

Next, we sketch the arguments that prove Theorem 5.1 when \( M \) is nonorientable, using the notation already introduced. There are two cases to consider. If \( T \) is a torus in \( M \) and \( \Pi: \hat{M} \to M \) is the oriented two-sheeted covering map, then \( T \) lifts to a separating torus \( \hat{T} \) in \( \hat{M} \) and the orientable case can be applied directly. On the other hand, if \( T \) is a Klein bottle in \( M \), then \( T \) does not lift, but \( \hat{T} = \Pi^{-1}(T) \) is again a separating torus in \( \hat{M} \). Now, there are two balls \( B_1, B_2 \) which are the connected components of \( \Pi^{-1}(B) \) and the proof of the Switch Move Theorem when \( M \) is nonorientable can be repeated to prove Theorem 5.1 when \( M \) is nonorientable.

### Appendix: Rational tangle case for the chain move

In this section we consider the special case relevant to the chain move when the underlying manifold \( M \) is the 3-sphere \( S^3 \) and the tangle outside of the ball \( B \) is a rational tangle. We represent rational tangles by a sequence of integers, all nonpositive or all nonnegative, as in Chapter 2 of [7].

**Proof of Lemma 3.4.** We first untwist the \( k \) half-twists of \( C_2 \) in \( B \) and thereby twist \(-k\) half-twists in \( R \) to move this integer tangle into \( R \), which will remain a rational tangle. Let \( B' = S^3 \setminus B \) be the ball containing the rational tangle \( R \) to the outside of \( B \). We let \( C_3 \) be the component of \( L \) containing \( \gamma_1 \) and \( C_4 \) be the component containing \( \gamma_2 \), keeping in mind that it may be the case that \( C_3 = C_4 \). Since \( R \) is a rational tangle, there are no other components in \( L \). Our goal will be to show that, after this twisting, \( S^3 \setminus L \) is hyperbolic unless \( R \) is one of the four tangles depicted in Figure 12.

We first show that there are no essential spheres in \( S^3 \setminus L \). Let us assume that \( E \) is an essential sphere in \( S^3 \setminus L \). Then, \( E \) separates \( S^3 \) into two balls, each of which contains at least one component of \( L \). Since \( C_1 \) is linked with \( C_2 \) and also
with $C_3$, these three components must be on the same side $V$ of $E$. But $C_2$ is also linked with $C_4$ so $C_4 \subset V$, contradicting the need for components to lie on both sides of $E$.

Next, we prove that there are no essential disks. If there were such a disk $D$ with boundary in a component $C$ of $L$, then the boundary of $N(D \cap C)$ is a sphere that separates $C$ from the other components of $L$, contradicting that there are no essential spheres.

We now show that the existence of an essential annulus in $S^3 \setminus L$ gives that $L$ is one of the four exceptions in Figure 12. Suppose such an annulus $A$ exists. If the boundary curves of $A$ lie on distinct components $C_i, C_j$ of $L$, then we may take $A' = \partial N(A \cup \partial N(C_j))$, which is an essential annulus in $S^3 \setminus L$ with boundary curves in $\partial N(C_i)$. Hence, it suffices to show that there are no essential annuli that have both their boundary components on the same $\partial N(L)$.

Let $D_1$ and $D_2$ be the twice-punctured disks bounded by $C_1$ and $C_2$ respectively and let $Q$ be the four-punctured sphere $\partial B \setminus L$. We assume that $A$ is an essential annulus with both boundary curves on the same component $\partial N(L)$ with the minimal number $n \geq 0$ of intersections with $D_1, D_2$ and $Q$ among all such annuli.

We first prove that $A$ can be chosen so that $A \cap (D_1 \cup D_2)$ does not contain any simple closed curve. Let us assume that $\alpha$ is a simple closed curve in $A \cap D_i$ and that $\alpha$ encloses only one puncture of $D_i$, which comes from a component $C_j$ of $L$. In this case, after possibly replacing $\alpha$ by an innermost curve, we could surger $A$ to obtain two annuli $A_1, A_2$, at least one of which, is essential and with fewer intersection components with $D_1 \cup D_2 \cup Q$. Hence, the minimality property defining $A$ implies that the boundary curves of $A_1$ lie on distinct components of $\partial N(L)$. In particular, $A_2$ cannot be boundary parallel, from where it follows that $A_2$ is also essential. Hence, the roles of $A_1$ and $A_2$ are identical and at least one of them, say $A_1$, satisfies that the number of intersection curves with $D_1 \cup D_2 \cup Q$ is less than $n/2$. Let $A' = \partial N(A_1 \cup \partial N(C_j))$. Then, $A'$ is an essential annulus in

Figure 12: Non-hyperbolic link complements.
$S^3 \setminus L$ with both boundary curves in the same component of $\partial N(L)$. However, the number of intersection curves in $A' \cap (D_1 \cup D_2 \cup \mathcal{Q})$ is less than $n$, a contradiction. On the other hand, if $\alpha$ encircles both punctures of $D_i$ and it is outermost, we could again surger $A$ to obtain two annuli with longitudinal boundaries, at least one of which would be essential, and the same argument as before applies to give a contradiction.

We next prove that any intersection arc $\alpha$ in $A \cap D_i$ is not boundary-parallel on either $A$ or $D_i$. If it were boundary-parallel on both, we could form a disk $D$ from the disk $\alpha$ cuts from $A$ and the disk $\alpha$ cuts from $D_i$. Then the boundary of $D$ must be a trivial curve in $\partial N(C_i)$, so we can add to it a disk from $\partial N(C_i)$ to obtain a sphere that must bound a ball. Hence we can isotope $A$ through this ball to remove the intersection arc $\alpha$, a contradiction to the minimality of intersection arcs.

Next, consider $A$ in $V = S^3 \setminus N(C_1)$, which is a solid torus. Since $\partial A$ is nontrivial on $\partial V$ and $A$ is properly embedded on $V$, then $A$ must be boundary-parallel on $V$ and define a solid torus region $V' \subset V$. In particular, since we have minimized the intersections of $A$ with $D_1$, which is a meridional disk for $V$, then $V' \cap D_1$ has $|p|$ components, each of which is a disk with boundary given by two arcs, one in $\partial D_1$ and the other on $A$. In particular, the fact that all such arcs are parallel on $D_1$ implies that $|p| \leq 2$.

We now treat the case when $|p| = 2$. In this case, any closed curve in $V'$ has even linking number with $C_1$, which implies that both $C_2$ and $C_3$ lie outside of $V'$. But, since $A$ is essential in $S^3 \setminus L$, $V'$ must contain at least one of the components of $L \setminus C_1$, hence $C_4$ is distinct from $C_3$ and $C_4 \subset V'$. But then, the fact that $C_2$ links $C_4$ and $C_4$ is separable from $C_1$ implies that $C_2 \subset V'$, a contradiction.

Next assume that $|p| = 1$, and we will show that $q = 0$ or $|q| = 1$. Arguing by contradiction, suppose $|q| > 1$. As before, at least one component of $L \setminus C_1$ must
be inside $V'$ to prevent $A$ from being boundary-parallel in $S^3 \setminus L$. First assume that $C_2$ is in $V'$ and we will show that this implies that $L \setminus C_1 \subset V'$. Since the linking number of $C_1$ with $C_2$ is one, $C_2$ must intersect every meridional disk of $V'$. But then, again by linking number, the winding number of $C_2$ in $V'$ must be 1. Suppose $C_3 \not\subset V'$. Since $C_3$ links $C_1$ once and $V'$ is a solid torus glued to $N(C_1)$ along a $(1, q)$-curve, $C_3$ links $C_2$ $q$ times, a contradiction that shows that $C_3$ is also in $V'$. Suppose $C_4 \neq C_3$ and $C_4 \not\subset V'$. Then, since $C_4$ does not link $C_1$, once again one can prove that it cannot link $C_2$, a contradiction. Hence, $L \setminus C_1 \subset V'$.

An analogous argument also shows that starting from any component of $L \setminus C_1$ inside $V'$, all of them must be inside $V'$. But then the single arc of intersection of $A$ in $D_1$ cannot separate the punctures, as both components corresponding to the punctures are not separated by $A$, a contradiction that shows that $|q| \leq 1$.

If $q = 0$, both boundary components of $A$ are meridians on $\partial N(C_1)$ and $A \cap D_1$ is a single arc that must split the two punctures. We may consider $A$ as a sphere punctured twice by $C_1$, with $C_3$ to one side and $C_2$ to the other, and since $C_4$ is linked with $C_2$, and therefore must be to that same side, it must be the case that $C_3$ and $C_4$ are distinct and separable. Hence the rational tangle must be the tangle denoted by $\infty$ appearing on the left in Figure 12.

Now, we consider the case for $\partial A \subset \partial N(C_1)$ when $|p| = 1$ and $|q| = 1$. Let $V''$ be the closure of the complement of $V'$ in $V$ and note that, in $V$, $A$ is parallel to both $V' \cap \partial N(C_1)$ and to $V'' \cap \partial N(C_1)$. Without loss of generality, we will assume that $C_2 \subset V'$. Then, since $C_1 \cup C_2$ is a Hopf link, it must be that $C_2$ is isotopic to the core curve of $V'$. In particular, $C_2$ is isotopic, in $V'$, to either component of $\partial A$ on $\partial N(C_1)$, and $C_2$ must hit every meridional disk of $V'$ and have winding number 1. Note that $V'' = V' \cup N(C_1)$ is a solid torus such that each of $C_1$ or $C_2$ can be taken to be a core curve.

We next treat the case where $C_3 \neq C_4$. Since $C_4$ links $C_2$ but not $C_1$, it cannot be in $V''$. Since $C_3$ links $C_1$ but not $C_2$, it also cannot be in $V''$. But then $A$ is isotopic to $\partial N(C_1)$ through $V''$, a contradiction.

Thus, it must be the case that $C_3 = C_4$. Since $C_2 \subset V'$, then $C_3 \subset V''$, since otherwise $A$ would be boundary-compressible. Moreover, $C_3$ also must hit every meridional disk of $V''$ and have winding number 1 so it links with $C_1$ once and with $C_2$ once. In particular, because $C_3$ is prime (every rational knot is a 2-bridge knot, therefore prime), it must be that $C_3$ is isotopic to the core curve of $V''$ and is in particular, trivial in $S^3$. This forces the rational tangle $L \setminus B$ to be an integer tangle. But for every such tangle other than 0 and -1 (cases excluded in Figure 12), the resultant 3-chain link is alternating and hence hyperbolic by [9].

In the final case, when $p = 0$ and $|q| = 1$, both boundary components of $A$
are longitudes on $\partial N(C_1)$. Neither $C_2$ nor $C_3$ can be in $V'$ since they both link $C_1$ once. Hence, it must be the case that $C_3 \neq C_4$, and $C_4 \subset V'$. Since $C_4$ must prevent $A$ from being boundary-parallel, it must hit every meridional disk in $V'$. But since $C_2$ links each of $C_1$ and $C_4$ once, $C_4$ must have winding number 1 in $V'$. In order that $C_3 \cup C_4$ be a rational and hence a 2-bridge link, it must be the case that $C_4$ intersects a meridional disk of $V'$ exactly once, implying that in fact $C_4$ is parallel to the core curve of $V'$ and therefore parallel to $C_1$, which yields an annulus $E$ with $\partial E = C_1 \cup C_4$. Therefore, $C_3$ and $C_4$ are also linked once, and in the same way that $C_3$ is linked with $C_1$. But then, the fact that $C_3$ is not linked with $C_2$ implies that $C_3$ and $C_2$ are also parallel to each other, hence the rational tangle $L \setminus \mathcal{B}$ is $-2, 0$, as in the fourth case depicted in Figure 12. This completes the case when $\partial A \subset \partial N(C_1)$.

Our next argument analyzes the situation when both boundary components of $A$ are $(p, q)$-curves on $\partial N(C_3)$. Then, there are two cases to treat according to whether $C_3 \neq C_4$ or $C_3 = C_4$.

When $C_3 \neq C_4$, then both $C_3$ and $C_4$ are trivial as knots in $S^3$. Once again, $A$ must be boundary-parallel in $V = S^3 \setminus N(C_3)$ and we let $V'$ denote the solid torus region that $A$ cuts from $V$. Since $A$ is assumed to be essential, one of the other components $C_1, C_2, C_4$ of $L$ must be contained in $V'$. Next, we assume $|p| \geq 1$.

If $C_1 \subset V'$, then $C_1$ must hit every meridional disk of $V'$ since it has linking number 1 with $C_3$. But since this linking number with $C_3$ is $pk$, where $k$ is the winding number of $C_1$ in $V'$, it follows that $|p| = 1$ and $k = 1$. Since $C_2$ links $C_1$, and $C_2$ is separable from $C_3$, it must be the case that $C_2 \subset V'$ also. But $C_2$ does not hit every meridional disk of $V'$ and hence $C_2$ is in a ball contained in $V'$. But $C_4$ must link $C_2$, so $C_4 \subset V'$. But then $L \setminus C_3 \subset V'$ and $|p| = 1$, so the annulus $A$ is boundary-compressible through the disk bounded by $C_3$, a contradiction.

The case when $C_2 \subset V'$ implies that $C_4 \subset V'$, as shown in the previous paragraph. Hence, it suffices to get a contradiction assuming $C_4 \subset V'$ and $C_1 \not\subset V'$. Assuming we are not in the case of the rational tangle being $\infty$, $C_4$ must intersect all meridional disks of $V'$. Since $C_2$ links $C_4$ but is separable from $C_3$, then $C_2 \subset V'$. But $C_2$ cannot hit every meridional disk of $V'$, and hence $C_2$ lives in a ball in $V'$. Then, because $C_1$ links $C_2, C_1$ must also be in $V'$, a contradiction.

Thus it must be the case that $|p| = 0$, and $\partial A$ is a pair of longitudes on $\partial N(C_3)$. Then, because $C_1$ links $C_3$, it follows that $C_1$ cannot be in $V'$. If $C_2 \subset V'$, then, if $C_2$ misses a meridional disk, it is contained in a ball in $V'$, making it impossible for $C_1$ to link with it. Hence, $C_2$ must hit every meridional disk of $V'$. Since $C_1$ links each of $C_3$ and $C_2$ once, $C_2$ must have winding number 1 in $V'$. In fact,
because $C_1 \cup C_2$ is the Hopf link, $C_2$ must be isotopic to the core curve of $V'$, and hence isotopic to $C_3$. In particular, $C_4$ links $C_3$ the same way it links $C_2$, and therefore the rational tangle is -2 0.

Next, assume that $\partial A \subset \partial N(C_3)$ and $C_3 = C_4$. Then, $C_3$ is a rational knot that is hyperbolic unless $C_3$ is trivial or is a 2-braid knot. This follows from the fact all rational knots are alternating and the work in [9].

If $C_3$ is a nontrivial 2-braid knot, then, up to isotopy, there is a unique annulus that is essential in $S^3 \setminus C_3$ corresponding to the complement of $C_3$ in the torus upon which a 2-braid can be pictured to sit. But because both $C_1$ and $C_2$ are linked once with $C_3$, they both must puncture this annulus. Thus any essential annulus in $S^3 \setminus N(L)$ must be boundary-parallel in $S^3 \setminus N(C_3)$.

If $C_3$ is nontrivial and is not a 2-braid knot, it has hyperbolic complement. Therefore, in this situation also, since $\partial A$ is nontrivial on $\partial N(C_3)$, $A$ must be boundary-parallel in $S^3 \setminus N(C_3)$.

In either case, one of $C_1$ or $C_2$, say $C_1$, must be contained in the solid torus $V'$ that $A$ cuts from $S^3 \setminus N(C_3)$. Because $C_1$ links $C_3$ once, then $C_1$ must hit every meridional disk of $V'$. But then, it must be the case that $|p| = 1$, and the winding number of $C_1$ in $V'$ is 1. But if $|q| \geq 1$, then $C_1$ is a satellite knot of $C_3$, which because $C_3$ is nontrivial, must be nontrivial as well, a contradiction.

So it must be the case that if $C_3$ is not trivial, both boundary components of $A$ are meridians of $C_3$. Since it is assumed that there are no simple closed curves of intersection of $A$ with $D_1$ or with $D_2$, then $A$ is disjoint from $D_1 \cup D_2$.

But once $A$ does not intersect $D_1 \cup D_2$, we can push $A$ out of $B$ entirely. But all twice-punctured spheres in a rational tangle are boundary-parallel. So $A$ is boundary-parallel in the complement of $B$, a contradiction.

Thus, $C_3$ is a trivial knot. Hence, the rational tangle must be represented as an integer tangle. But this makes $L$ into a 3-chain and as in [10] (or as it also follows from [9]), a 3-chain is non-hyperbolic if and only if it is non-alternating, hence $L \setminus B$ is one of the tangles 0 or -1 represented by the two middle figures in Figure 12. This finishes the proof that if $S^3 \setminus L$ admits an essential annulus, then $L$ is one of the links shown in Figure 12.

To finish the proof of Lemma 3.4, suppose $T$ is an essential torus in $S^3 \setminus L$. We may assume that $T$ has been isotoped to minimize the number of intersection curves $Q$, and also with the disks $D_1$ and $D_2$ bounded by $C_1$ and $C_2$. If $T$ intersects either of these disks, then in either case it does so in simple closed curves that either surround a single puncture, or surround both punctures. In the first case, we can take an innermost such curve on $D_i$ and surger the torus to obtain an
essential annulus, both boundary-components of which become meridians in the link complement. In the second case, we can take an outermost such curve and surger $T$ to obtain an essential annulus, both boundary components of which are longitudes on $\partial N(C_i)$. Thus, in either case $L$ is one of the exceptions stated by the lemma.

Otherwise, if $T$ does not intersect $D_1 \cup D_2$, we can assume that the torus is outside the ball $N(D_1 \cup D_2)$. Since $\partial N(D_1 \cup D_2)$ is a 4-punctured sphere isotopic to the 4-punctured sphere $Q$, the torus can be pushed out of $B$ entirely. But then the incompressible torus $T$ lies in $B'$ in the complement of the two unknotted arcs that make up the rational tangle. If $T$ cuts a solid torus from $B'$, it will compress in that solid torus. So it must be that the torus is a cube-with-knotted-hole. But then to prevent compression in the knotted hole, the two arcs must form a knot that passes through the hole, contradicting the fact this is a rational tangle, each arc of which is trivial. 

\[ \square \]

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