Ranking pages and the topology of the web

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Abstract

This paper presents our studies on the rearrangement of links in the structure of websites for the purpose of improving the valuation of a page or group of pages as established by a ranking function as Google’s PageRank. We build our topological taxonomy starting from unidirectional and bidirectional rooted trees, and up to more complex hierarchical structures as cyclical rooted trees (obtained by closing cycles on bidirectional trees) and PR–digraph rooted trees (digraphs whose condensation digraph is a rooted tree that behave like cyclical rooted trees). We give different modifications on the structure of these trees and its effect on the valuation given by the PageRank function. We derive closed formulas for the PageRank of the root of various types of trees, and establish a hierarchy of these topologies in terms of PageRank.

Keywords: PageRank, world wide web topology, link structure.

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1 Introduction

Google is still today’s most popular search engine for the World Wide Web, and the key to its success has been its PageRank algorithm [1], which ranks documents based primarily on the link structure of the web. Simply put, PageRank considers a link from a page \( H \) to another page \( J \) as a weighted vote from \( H \) in favour of the importance of \( J \), where the weight of the vote of \( H \) is itself determined by the number of links (or voters) to \( H \). Therefore, part of the game of the electronic business today is to find ways of lifting a page’s link popularity, and specifically the PageRank, by either obtaining the vote of a very important page (which is unlikely) or manufacturing a large set of pages that would be “willing” to link to a client’s page. For the latter solution, known as link farms in the jargon of the Search Engine Optimisation (SEO) community, much care must be taken since it is widely believed that Google had tuned

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up its original PageRank algorithm to detect fictitious linking and similar forms of spamming (e.g. the 2003 “Florida” update, see [9]).

At the heart of the challenge of improving a page PageRank value is the role played by the topology of the web. This is a widely recognised fact as there can be found in the internet many SEO analyses of link patterns, together with tips on how to rearrange these to raise the PageRank of specific pages. On the theoretical side, having acknowledged that the World Wide Web should be treated as a directed graph, there are various publications that propose different graph decompositions on regular patterns, as a way to improve PageRank computation (e.g. [2], [11]), and news that suggests that newly acquired technology by Google, in the hope to enhance PageRank, is based on localisation of the computations on certain tree structures underlying the Web ([10], [13]).

Motivated by these graph combinatorial challenges particular to the Web, we have studied the PageRank formula from a mathematical perspective, and its relation with the web site’s topology, with the twofold goal of accelerating the computation of PageRank and maximising its value for an specific page or set of pages. We summarise here all our findings starting from (unidirectional) rooted trees and up to more complex hierarchical structures. Ultimately, our academic goals are to disclose some of the graph combinatorics underlying the World Wide Web and this popular ranking function, and to contribute to the mathematical foundation of many heuristics and ad hoc rules in used by the SEO community in its attempt to tweak the valuations assigned by PageRank.

2 Some preliminaries on Graph Theory

In this paper we will use some standard concepts and results about directed graphs, which we detail in this section in order to fix our notation.

By a digraph $\mathcal{D}$ we mean a pair $\mathcal{D} = (V, A)$ where $V$ is a finite nonempty set and $A \subset V \times V \setminus \{(v, v) : v \in V\}$. Elements in $V$ and $A$ are called vertices and arcs respectively. For an arc $(u, v)$ we will say that $u$ is adjacent to $v$, and we may sometimes also use $uv$ to denote an arc $(u, v)$. The order and the size of $\mathcal{D}$ are, respectively, $\text{Card}(V)$ and $\text{Card}(A)$. If $v$ is a vertex, the in-degree, $\text{id}(v)$, of $v$ is the number of arcs $(u, v)$ in $A$. Similarly, the out-degree, $\text{od}(v)$, of $v$ is the number of arcs $(v, u)$ in $A$.

A sequence of vertices $v_1 v_2 \ldots v_q, q \geq 2$, such that $(v_i, v_{i+1}) \in A$ for $i = 1, 2, \ldots, q-1$ is a walk of length $q - 1$ joining $v_1$ with $v_q$ or more simply a $v_1v_q$ walk. If the vertices of $v_1 v_2 \ldots v_q$ are distinct the walk is called a path. A cycle of length $q$ or a $q$-cycle is a path $v_1 v_2 \ldots v_q$ closed by the arc $v_q v_1$. A digraph is acyclic if it has no cycle. By a semipath joining $v_1$ with $v_q$ we mean a sequence of distinct vertices $v_1 v_2 \ldots v_q, q \geq 2$, such that $(v_i, v_{i+1}) \in A$ or $(v_{i+1}, v_i) \in A$ for $i = 1, 2, \ldots, q - 1$.

A digraph is connected if for each pair $u$ and $v$ of distinct vertices, there is a semipath joining $u$ with $v$. By a subdigraph of the digraph $(V, A)$ we mean a digraph $(W, B)$ such that $W \subset V$ and $B \subset A$. The subdigraph is called a partial digraph when $W = V$. The induced subdigraph by the digraph $(V, A)$ on $W \subset V$ is the digraph $(W, A/W)$ where $A/W = A \cap (W \times W)$.

For an acyclic digraph there exists at least one vertex $v$ (resp. $u$) such that $\text{od}(v) = 0$ (resp. $\text{id}(u) = 0$). Such vertex will be called a maximal (resp. minimal) in the
digraph. Moreover, the vertices in an acyclic digraph \((V, A)\) can be distributed by levels \(N_0, N_1, \ldots\), where \(N_0 = \{v \in V : v \text{ is maximal in } (V, A)\}\) and, recursively for \(p > 0\),

\[
N_p = \{v \in V \setminus \bigcup_{i=0}^{p-1} N_i : v \text{ is maximal in the induced subdigraph on } V \setminus \bigcup_{i=0}^{p-1} N_i\}
\]

Thus one has a partition of \(V, V = N_0 \cup N_1 \cup \cdots \cup N_h\), \(h\) being the height of the digraph, i.e. the last index such that \(N_h \neq \emptyset\).

## 3 Short Introduction on PageRank

The mathematical view of the World Wide Web is as a digraph \(W = (V, A)\), where a vertex represents any document posted on the web (a page), and an arc \((b, a)\) indicates that there is a link from page \(b\) to page \(a\). In this setting, Brin and Page proposed in [4] to evaluate each page in the Web with a positive real number, which they named its PageRank, given by the formula (in its refined version from [5]):

\[
\mathcal{P}(a) = \frac{1 - \alpha}{N} + \alpha \sum_{(b,a) \in A} \frac{\mathcal{P}(b)}{\text{od}(b)}
\]  

(1)

where \(\mathcal{P}(a)\) is the PageRank of page \(a\), \(\text{od}(b)\) is the number of links going out of page \(b\) (the out-degree of \(b\)), \(\alpha\) is a constant that can take any real value in the interval \((0, 1)\) (although Brin and Page always prefer to set it to 0.85), \(N\) is the total number of pages of the Web, and the sum is taken over all pages \(b\) that have a link to \(a\).

The motivation, given by the authors, is that formula (1) models the behaviour of a random surfer of the Web who, being at a certain page \(b\), either follows one of the links shown in that page with probability \(\alpha\), or jumps to any other page with probability \(1 - \alpha\), disregarding the contents of the pages. The probability of choosing a link in \(b\) that takes him to page \(a\) depends on the number \(\text{od}(b)\) of links out of \(b\); so \(\mathcal{P}(b)/\text{od}(b)\) is the contribution of \(b\) to the PageRank of \(a\) amortised by \(\alpha\). In this setting, the PageRank of \(a\) is the probability of a user reaching page \(a\) directly or after following all appropriate links, and the sum of the PageRank of all the pages is 1, and so, forms a probability distribution over the Web (see [3] and [12]).

Yet another view of PageRank is the analytical formulation given by Brinkmeier (see [7]), who conceived the PageRank function as a power series. In this setting, a formula is given that highlights the fact that the ranking of a vertex \(v\), as assigned by PageRank, depends on the weighted contributions of each vertex in every walk that leads into \(v\), being these contributions higher in value for vertices that are nearer in distance from \(v\).

For a given walk \(\rho = v_1 v_2 \ldots v_n\) in the graph \((V, A)\), define the branching factor of \(\rho\) by the formula

\[
D(\rho) = \frac{1}{\text{od}(v_1)\text{od}(v_2) \cdots \text{od}(v_{n-1})}
\]

Then, for any vertex \(a \in V\), we have

\[
\mathcal{P}(a) = \frac{1 - \alpha}{N} \sum_{w \in V} \sum_{\rho : w \rightarrow a} \alpha^{|\rho|} D(\rho)
\]  

(2)

3
where $\rho : w \rightarrow a$ denotes a walk $\rho$ starting at vertex $w$ and ending in vertex $a$, and $l(\rho)$ is the length of this walk $\rho$.

## 4 Ranking vertices on trees

Our starting case study is the set of **rooted trees**, where a **tree with root** is an acyclic digraph for which there exists a maximal vertex $r$ (the root), such that for every vertex $v \neq r$ there is a unique $v-r$ path. We denote a tree with root $r$ as $T^r$. Thus, a tree $T^r$ is a connected graph, its root $r$ is unique and all vertices distinct from $r$ have out-degree 1, whilst the in-degree may vary. Vertices with in-degree 0 are called **leaves**. The root is the targeted page for improving its PageRank valuation. The **height** of a vertex in a rooted tree is the length of the path from the vertex to the root. The **level** $k$ of a rooted tree is the set of vertices with height $k$; the root is at level $N_0$. The **height** of a rooted tree is the length of the longest path from a leaf to the root.

**Remark 4.1** Since we are interested in studying the behaviour of PageRank when localised in certain subdigraphs of the Web digraph, we think, in particular, of our trees as local closed web sites. This means that the value of $N$ in formula (1) is the number of vertices in the tree. □

Our first result shows that to compute the PageRank of the root of a tree all we need to do is count the number of vertices at each level of the tree.

**Theorem 4.2** If a rooted tree has $N$ vertices and height $h$, then the PageRank of its root $r$ is given by the formula

$$P(r) = \frac{1 - \alpha}{N} \sum_{k=0}^{h} \alpha^k n_k$$

where $n_k := |N_k|$ is the number of vertices of the $k$-th level, $N_k$, of the tree.

**Proof:** Below we use $b \in N_k : b \to a$ to indicate that vertex $b$ at level $N_k$ has a link to $a$. Assume the first level of the tree $N_1 = \{a_1, \ldots, a_{n_1}\}$. Then, according to equation (1)

$$P(r) = \frac{1 - \alpha}{N} + \alpha \sum_{a \in N_1} P(a) = \frac{1 - \alpha}{N} + \alpha \left( \frac{1 - \alpha}{N} + \alpha \sum_{b \in N_2 : b \to a_1} P(b) \right) + \ldots + \left( \frac{1 - \alpha}{N} + \alpha \sum_{b \in N_2 : b \to a_{n_1}} P(b) \right)$$

The index sets $\{b \in N_2 : b \to a_i\}$, for $i = 1, \ldots, n_1$, are pairwise disjoint; therefore,

$$P(r) = \frac{1 - \alpha}{N} (1 + \alpha n_1) + \alpha^2 \sum_{b \in N_2} P(b)$$

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Repeating the above manipulations on levels $N_2$, $N_3$, and up to level $N_{h-1}$, we have

$$\mathcal{P}(r) = \frac{1 - \alpha}{N} \sum_{k=0}^{h-1} \alpha^k n_k + \alpha^h \sum_{b \in N_h} \mathcal{P}(b)$$

At the last level $N_h$ all vertices are leaves, which have no in-coming arcs, hence the PageRank of any $b \in N_h$ is $(1 - \alpha)/N$. Then

$$\alpha^h \sum_{b \in N_h} \mathcal{P}(b) = \frac{1 - \alpha}{N} \alpha^h n_h$$

and the result follows. □

**Remark 4.3** Theorem 4.2 shows that we can do any rearrangements of links between two consecutive levels of a web set up as a rooted tree, and the PageRank of the root will be the same. □

**Remark 4.4** Due to Theorem 4.2, we will from now on describe a rooted tree $T^r$, with root $r$ and $h \geq 0$ levels, each of cardinality $n_0 = 1, n_1, \ldots, n_h$, as the string $T^r = 1n_1 \ldots n_h$. Also the PageRank for the root $r$ of $T^r$, or for any other vertex seemed as the root of a subtree in $T^r$, will depend on the height and the number of vertices at each level of $T^r$. Henceforth, we write PageRank of $r$ in the tree $T^r$ as a function of the height $h$, and denote it $\mathcal{P}(h)$. □

For some regular topologies we can have nice closed formulas for their PageRank. Some examples follow below.

### 4.1 $m$-ary trees

For $m, h \geq 1$, let $T^r_m(h)$ be the full $m$–ary tree of height $h$, i.e. a tree of height $h$ whose vertices, except by the leaves, have in–degree $m$. The 1–ary tree of height $h$, $T^r_1(h)$, is a path of length $h$. For $m > 1$, $T^r_m(h)$ has $m^k$ vertices at each level $k = 0, 1, \ldots, h$, and the total number of vertices is $(m^h - 1)/(m - 1)$. Using Theorem 4.2 we can quickly calculate the PageRank for the root $r$ (which depends on the height $h$ and fixed arity $m$), and so we denote $\mathcal{P}_m(h))$. This is

$$\mathcal{P}_m(h) = (1 - \alpha) \frac{m - 1}{m^h - 1} \sum_{k=0}^{h} m^k \alpha^k = (1 - \alpha) \left( \frac{m - 1}{m^h - 1} \right) \frac{(m\alpha)^{h+1} - 1}{m\alpha - 1}$$

and for the 1–ary tree

$$\mathcal{P}_1(h) = \frac{1 - \alpha^{h+1}}{h + 1}$$
4.2 Binomial trees

A binomial tree is at the core of fundamental data structures such as heaps, and hence, it qualifies as a good candidate for a website’s topology. We use $T^r_b(h)$ to denote the full binomial tree of height $h$. We recall from §9.1 that $T^r_b(0)$ consists of only one vertex—the root—and, inductively, $T^r_b(h+1)$ is two copies of $T^r_b(h)$ joint with an arc from the root of one of the $T^r_b(h)$ to the root of the other. At each level $k = 0, 1, \ldots, h$, $T^r_b(h)$ has $\binom{h}{k}$ vertices, and the total number of vertices in $T^r_b(h)$ is $\sum_{k=0}^{h} \binom{h}{k} = 2^h$. Using Theorem 4.2 we get a nice formula to easily calculate the PageRank for the root $r$ of $T^r_b(h)$, namely

$$P_b(h) = 1 - \alpha 2^h \sum_{k=0}^{h} \binom{h}{k} \alpha^k = (1 - \alpha) \left(1 + \alpha \right)^h$$

5 Rearrangements of vertices

We begin our explorations on the possible modifications on the tree structure that will improve the valuation of PageRank. Our first result shows that completely erasing the vertices farthest away from the root improves the PageRank. This corroborates the known fact that the optimal configuration is a star, i.e. a rooted tree of height 1 (see e.g. [3], [12]).

**Theorem 5.1** If in a tree $T^r = 1n_1 \ldots n_h$ of height $h \geq 1$, the last level $N_h$ is completely erased, then the PageRank of its root $r$, $P(h)$, increases its value.

**Proof:** After passing from the tree $T^r = 1n_1 \ldots n_h$, with $N = 1+n_1+\ldots+n_h$ vertices and PageRank $P(h)$, to the tree $T'^r = 1n_1 \ldots n_{h-1}$ with $N-n_h$ vertices and PageRank $P'(h)$, we get

$$P'(h) - P(h) = \frac{1-\alpha}{N-n_h}N\left(1+n_1 \alpha + \ldots + n_{h-1} \alpha^{h-1} - (N-n_h) \alpha^h\right)$$

$$= \frac{(1-\alpha)n_h}{N-n_h}N\left(1+n_1 \alpha + \ldots + n_{h-1} \alpha^{h-1} - (1+n_1+\ldots+n_{h-1}) \alpha^h\right)$$

$$= \frac{1-\alpha}{N-n_h}N\left((1-\alpha^h) + n_1(\alpha - \alpha^h) + \ldots + n_{h-1}(\alpha^{h-1} - \alpha^h)\right) > 0$$

because $0 < \alpha < 1$ and $h \geq 1$. □

**Remark 5.2** Thus, in order to improve the PageRank of the root of a tree one can delete as many levels, from highest to lowest, as the context permits. Conversely, if a new level of vertices is added to a tree, then the PageRank of its root decreases. □

If it were the case that for practical, or any other reason, we were obliged to keep certain height, then a natural question is how much can we prune the tree to improve on PageRank. The extreme situation is to prune all but one arc at each level, so we take that structure as benchmark and called it queue tree.

\[1\]Goodness as always is understood in terms of PageRank.
Definition 5.3 The queue tree of a tree $T^r = 1n_1 \ldots n_h$ is the tree

$$T_q^r = 1n_1 \ldots n_{\lfloor \frac{h-1}{2} \rfloor} \overbrace{1 \ldots 1}^{\lfloor \frac{h}{2} \rfloor +1}$$

Theorem 5.4 The PageRank of the root of a tree is smaller than the PageRank of the root of its queue tree.

Proof: We proceed recursively from the last level down to $\lfloor \frac{h-1}{2} \rfloor$.

(a) The PageRank $\mathcal{P}(h)$ of the root $r$ of $T^r = 1n_1 \ldots n_{h-1}n_h$ is smaller than the PageRank $\mathcal{P}'(h)$ of $T'^r = 1n_1 \ldots n_{h-1}1$. Indeed, let $N = 1 + n_1 + \ldots + n_h$, then

$$\mathcal{P}'(h) - \mathcal{P}(h) = \frac{(n_h - 1)(1 - \alpha)}{(N - (n_h - 1))N} \left( \sum_{k=0}^{h-1} n_k \alpha^k - (N - n_h)\alpha^h \right)$$

$$= \frac{(n_h - 1)(1 - \alpha)}{(N - (n_h - 1))N} \sum_{k=0}^{h-1} n_k (\alpha^k - \alpha^h) > 0$$

Apply the same methodology for $T^r = 1n_1 \ldots n_{h-2}n_{h-1}1$ and $T'^r = 1n_1 \ldots n_{h-2}11$, and so on, up to $\lfloor \frac{h}{2} \rfloor$. At this last step we have

(b) $T^r = 1n_1 \ldots n_{\lfloor \frac{h-1}{2} \rfloor} \overbrace{1 \ldots 1}^{\lfloor \frac{h}{2} \rfloor}$, and we shall see that its PageRank is less than that of the root of $T'^r = 1n_1 \ldots n_{\lfloor \frac{h}{2} \rfloor}1$. We work separately the cases of $h$ even or $h$ odd.

(b.i) If $h = 2p - 1$ then $T^r = 1n_1 \ldots n_{2p-1}n_p \overbrace{1 \ldots 1}^{p-1}$, $T'^r = 1n_1 \ldots n_{p-1}1 \ldots 1$ and $N = n_1 + \ldots + n_p + p$. Let $M = \frac{\alpha^{p}(1-\alpha)}{(N - (n_p - 1))N}$. Then

$$\mathcal{P}'(h) - \mathcal{P}(h) = M \left( 1 + \sum_{k=1}^{p-1} n_k \alpha^k - (N - n_p)\alpha^p + \sum_{k=p+1}^{2p-1} \alpha^k \right)$$

$$= M \left( (1 - \alpha^p) + \sum_{k=1}^{p-1} n_k (\alpha^k - \alpha^p) + \sum_{k=p+1}^{2p-1} \alpha^k \right)$$

$$= M \left( (1 - \alpha^p) + \sum_{k=1}^{p-1} (n_k - \alpha^{p-k})(\alpha^k - \alpha^p) \right) > 0$$

(b.ii) If $h = 2p$ then $T^r = 1n_1 \ldots n_{2p-1}n_p \overbrace{1 \ldots 1}^{p-1}$, $T'^r = 1n_1 \ldots n_{p-1}1 \ldots 1$ and $N = n_1 + \ldots + n_p + p + 1$. One then shows $\mathcal{P}'(h) - \mathcal{P}(h) > 0$ by a similar argument as in (b.i). □

Remark 5.5 Theorem 5.4 cannot be improved, in the sense that deleting further vertices (but keeping the height) in a queue tree may or may not improve the PageRank.
of the root. For small values of $h$, the queue tree is the optimal pruning of a tree for increasing PageRank. For example, if $h = 4$ the corresponding queue tree is $T^r_q = 1n_1111$ with PageRank $P(h)$, and if $n_1 > 1$ and we remove a vertex from level $N_1$, we get the tree $T^{r'} = 1(n_1 - 1)111$ with PageRank $P(h)$, and their difference is

$$P^{r'}(h) - P(h) = \frac{1 - \alpha}{(n_1 + 3)(n_1 + 4)}(1 - 4\alpha + \alpha^2 + \alpha^3 + \alpha^4) < 0$$

for any $\alpha$ such that $0.27568 < \alpha < 1$.

For larger values of $h$, an improvement of PageRank will depend on $\alpha$ and on the cardinalities of the levels $N_1, \ldots, N_{\lfloor \frac{h}{2} \rfloor}$. There are also some improvements that can be done on queue trees of particular trees, such as $m$-ary and binomial. □

6 Hierarchies of trees by height and size

In what follows we assume that $\frac{1}{2} < \alpha < 1$, an interval of useful values for $\alpha$ in practice (see the analysis on this subject in [12]). We want to order the $m$-ary and binomial trees with respect to their PageRank. Which tree structure is best for PageRank? Our first result on this theme gives a hierarchy with respect to the height.

**Theorem 6.1** For values of the height $h$ sufficiently large, we have

$$P_1(h) > P_b(h) > P_2(h) > P_3(h) > \ldots > P_m(h)$$

**Proof:** We have to compute the appropriate limits:

1. For $1 < k < m$, $\lim_{h \to \infty} \frac{P_k(h)}{P_m(h)} = \frac{(k-1)(m\alpha - 1)}{(m-1)(k\alpha - 1)} > 1$, from where we conclude that $P_k(h) > P_m(h)$.

2. $\frac{P_2(h)}{P_b(h)} = \frac{\alpha + 1}{2(2\alpha - 1)} \frac{(2\alpha)^{h+1} - 1}{(\alpha + 1)^{h+1}} \frac{2^{h+1}}{2^{h+1} - 1} \xrightarrow{h \to \infty} 0$.

3. $\frac{P_b(h)}{P_1(h)} = \frac{(1 - \alpha)(h + 1)}{1 - \alpha^{h+1}} \frac{(\frac{1+\alpha}{2})^h}{(\frac{1+\alpha}{2})^h} \xrightarrow{h \to \infty} 0$. □

Next we classify the PageRank of the $m$-ary and binomial trees with respect to their order. (Beware that we will express the order in terms of the height, and therefore we keep the notation $P_m(h)$ and $P_b(h)$ that remarks the dependency of PageRank on the height of the tree.)

**Theorem 6.2** For sufficiently large order $N$ and $\alpha \geq 0.58$, we have

$$\ldots P_m(h) > \ldots > P_5(h) > P_4(h) > P_3(h) > P_2(h) > P_1(h)$$

**Proof:** The proof splits into three cases.

(a) If $1 < k < m$ and $N >> 0$ then $P_m(h) > P_k(h)$: A $k$-ary tree of height $h$ has $N = \frac{k^{h+1} - 1}{k - 1}$ many vertices. An $m$-ary tree of height $h'$ has the same number of
vertices as in a \(k\)-ary tree if, and only if, \(\frac{k^{h+1} - 1}{k - 1} = \frac{m^{h' + 1} - 1}{m - 1}\), or equivalently \(h' + 1 = \log_m \left( \frac{m - 1}{k - 1} (k^{h+1} - 1) + 1 \right) \approx \log_m \left( \frac{m - 1}{k - 1} (k^{h+1}) \right)\), where the last relation indicates an equivalence among infinite large quantities. Then

\[
\lim_{h \to \infty} \frac{\mathcal{P}_k(h)}{\mathcal{P}_m(h)} = \lim_{h \to \infty} \frac{(m\alpha - 1) (k\alpha)^{h+1} - 1}{(k\alpha - 1) (m\alpha)^{h'+1} - 1} = \lim_{h \to \infty} \frac{(m\alpha - 1) (1 - \alpha \frac{m^{h+1} - 1}{m - 1})}{(m\alpha)^{h+1} - 1} \to 0
\]

since \(k < m\) implies \(\log_m k < 1\), and in consequence \(\frac{\alpha}{\alpha \log_m k} < 1\).

(b) If \(m \geq 2\) and \(N \gg 0\) then \(\mathcal{P}_m(h) > \mathcal{P}_1(h)\):

\[
\frac{\mathcal{P}_1(h)}{\mathcal{P}_m(h)} = \frac{\frac{1 - \alpha^{h+1}}{h+1}}{(m - 1)(1 - \alpha) \frac{m^{h+1} - 1}{m\alpha - 1}} = \frac{(m\alpha - 1) (1 - \alpha \frac{m^{h+1} - 1}{m - 1})}{(m\alpha)^{h+1} - 1} \to 0
\]

(c) If \(N \gg 0\) then \(\mathcal{P}_5(h) > \mathcal{P}_6(h) > \mathcal{P}_4(h)\): A binomial tree of height \(h'\) has \(2^{h'}\) vertices. For \(m \geq 2\), an \(m\)-ary tree of height \(h\) has same cardinality of a binomial tree of height \(h'\) if, and only if, \(\frac{m^{h+1} - 1}{m - 1} = 2^{h'}\), or equivalently,

\[
h' = \log_2 \frac{m^{h+1} - 1}{m - 1} \approx \log_2 \frac{m^{h+1}}{m - 1}.
\]

We then show that

\[
\lim_{h \to \infty} \frac{\mathcal{P}_m(h)}{\mathcal{P}_6(h)} = \frac{(1 + \alpha)^{\log_2 (m - 1)}}{m\alpha - 1} \left( \frac{m\alpha}{(1 + \alpha)^{\log_2 m}} \right)^{h+1} = L
\]

The limit \(L\) is 0 or \(\infty\) depending on \(\frac{m\alpha}{(1 + \alpha)^{\log_2 m}}\) being less than or greater than 1, respectively. We showed \(L = 0\) if \(m \leq 4\) or \(m = 5\) and \(\alpha < \alpha_0\), where \(\alpha_0\) is the irrational number solution of \(5\alpha_0 = (1 + \alpha_0)^{\log_2 5}\), namely, \(\alpha_0 = 0.57016\ldots\). On the other hand, \(L = \infty\) if \(m \geq 6\) or \(m = 5\) and \(\alpha > \alpha_0\). \(\square\)

### 6.1 Refining the hierarchy

According to our results there are many non uniform ways in which one can improve the PageRank in our binomial or \(m\)-ary trees, yet keeping the height as a constraint for maintaining a hierarchically organised website: it is sufficient to remove vertices at farther distance from the root. However, there are also non trivial uniform trees with better PageRank than the binomial, with respect to height (Theorem 6.1). We present one such possibility which can be seen as a basic backbone for a website with different levels. Our intention with this example is to illustrate the following fact: to optimise the PageRank values of certain pages of a web site is in general a hard task, as there could be exponentially many possible rearrangements. We will come back to this point in section 7.

We define the **path tree of height** \(h\), denoted \(\mathcal{T}_h^p\), as a tree with a root from which hangs \(h\) paths of \(h, h - 1, \ldots, 2,\) and \(1\) vertices. Observe that \(\mathcal{T}_h^p\) has \(h\)
vertices at level 1 (connected to the root), \( h - 1 \) vertices at level 2, \( h - 2 \) vertices at level 3, \ldots, one vertex at level \( h \). Hence, \( |T_p^q(h)| = 1 + h(h + 1)/2 \) and if \( r \) is the root of \( T_p^q(h) \), we have its PageRank, \( P_\rho(h) \), is given by

\[
P_\rho(h) = \frac{2(1 - \alpha)}{2 + (h + 1)h} \left( 1 + h\alpha + (h - 1)\alpha^2 + (h - 2)\alpha^3 + \ldots + 2\alpha^{h-1} + \alpha^h \right)
\]

\[
= \frac{2(1 - \alpha)}{2 + (h + 1)h} \left( 1 + \frac{\alpha^{h+1}}{h} \sum_{k=1}^{h} \left( \frac{1}{\alpha} \right)^k \right)
\]

\[
= \frac{2(1 - \alpha)}{2 + (h + 1)h} \left( 1 + \frac{\alpha^{h+2}}{(1 - \alpha)^2} \left( 1 - (h + 1) \left( \frac{1}{\alpha} \right)^h + h \left( \frac{1}{\alpha} \right)^{h+1} \right) \right)
\]

We then show that

\[
\frac{P_\rho(h)}{P_b(h)} = \frac{2}{2 + (h + 1)h} \left( \frac{2}{1 + \alpha} \right)^h \left( 1 + \frac{\alpha^{h+2} + \alpha(1 - \alpha)h - \alpha^2}{(1 - \alpha)^2} \right) \xrightarrow{h \to \infty} 2\alpha
\]

Thus \( P_\rho(h) > P_b(h) \) for sufficiently large \( h \), although we have checked the inequality computationally for values of \( h \geq 3 \) (for \( h < 3 \) the binomial tree and the chain tree are the same).

On the other hand, one can show that \( \frac{P_\rho(h)}{P_1(h)} \xrightarrow{h \to \infty} 2\alpha \), and hence the relative position of \( P_\rho(h) \) and \( P_1(h) \) in the hierarchy depends on whether \( \alpha \) is > 1/2 or < 1/2.

### 6.2 Hierarchies for queue trees

Surprisingly, for the queue trees of \( m \)-ary and binomial trees, the same ordering of their PageRank with respect to height and order holds. For the \( m \)-ary (resp. binomial) tree of height \( h \), we use \( P_{q,m}(h) \) (resp. \( P_{q,b}(h) \)) to denote the PageRank of the root of its queue tree. These values will depend on the parity of the height \( h \), since by Definition 5.3, if \( h = 2p-1 \) then \( T_q^r = n_1 \ldots n_{p-1} 1 \ldots 1 \), and if \( h = 2p \) then

\( T_q^r = n_1 \ldots n_{p-1} 1 \ldots 1 \). Thus, for \( m > 1 \),

\[
P_{q,m}(2p-1) = \frac{1 - \alpha}{m\alpha - 1 + p} \left( \frac{(m\alpha)^p - 1}{m\alpha - 1} + \alpha^p \frac{\alpha^p - 1}{\alpha - 1} \right)
\]

\[
P_{q,m}(2p) = \frac{1 - \alpha}{m\alpha - 1 + p + 1} \left( \frac{(m\alpha)^p - 1}{m\alpha - 1} + \alpha^p \frac{\alpha^{p+1} - 1}{\alpha - 1} \right)
\]

(the case \( m = 1 \) is trivial since the 1-ary queue tree coincides with the 1-ary tree). And,

\[
P_{q,b}(2p-1) = \frac{1 - \alpha}{2^{p-2} + p} \left( \sum_{k=0}^{p-1} \binom{2p - 1}{k} \alpha^k + \alpha^p \frac{\alpha^p - 1}{\alpha - 1} \right)
\]

\[
P_{q,b}(2p) = \frac{1 - \alpha}{2^{p-1} + p + \frac{1}{2}(2p)} \left( \sum_{k=0}^{p-1} \binom{2p}{k} \alpha^k + \alpha^p \frac{\alpha^{p+1} - 1}{\alpha - 1} \right)
\]
The proofs of (i) Lemma 6.4 observation is: Part (i): We need to study separately the cases of \( h \) being even or odd. The crucial observation is:

**Lemma 6.4** If \( 0 < \alpha < 1 \) then

\[
(1 + \alpha)^{2p-2} \leq \sum_{k=0}^{p-1} \binom{2p-1}{k} \alpha^k + \alpha^p \frac{p^2 - 1}{\alpha - 1} \leq (1 + \alpha)^{2p-1}
\]

(To show this observe that for \( 0 < \alpha < 1 \) then the quotient of \( (1 + \alpha)^{2p-1} \) and \( \sum_{k=0}^{p-1} \binom{2p-1}{k} \alpha^k + \alpha^p \frac{p^2 - 1}{\alpha - 1} \) tends to a constant \( L \), with \( 1 \leq L \leq 1 + \alpha \), as \( p \) grows.)

Using this lemma, we obtain the same limits 1), 2) and 3) in Theorem 6.1 for \( h = 2p \) and for \( h = 2p - 1 \).

**Part (ii):** As in (i) we need to study separately the cases of \( h \) being even or odd. Observe that for the same type \( \tau \) of queue tree (\( \tau \) being \( m \)-ary, binomial, etc.), by Theorem 5.1 the queue tree of height \( 2p - 1 \) is obtained by deleting the level \( N_{2p} \) of the queue tree of height \( 2p \), and hence,

\[
P_{q,\tau}(2p - 1) > P_{q,\tau}(2p)
\]

Therefore, for any two queue trees of types \( \tau_1 \) and \( \tau_2 \) we have

\[
\frac{P_{q,\tau_1}(2p - 1)}{P_{q,\tau_2}(2p)} > \max \left\{ \frac{P_{q,\tau_1}(2p)}{P_{q,\tau_2}(2p)}, \frac{P_{q,\tau_1}(2p - 1)}{P_{q,\tau_2}(2p - 1)} \right\} > \frac{P_{q,\tau_1}(2p)}{P_{q,\tau_2}(2p - 1)}
\]

and since all these quotients are positive, if any of them reduces to zero then all those that are to the right hand side (the ones that are smaller) will also reduce to zero. Hence, to prove that \( P_{q,\tau_1}(h) < P_{q,\tau_2}(h) \), it will be enough to prove that \( \frac{P_{q,\tau_1}(2p - 1)}{P_{q,\tau_2}(2p)} \xrightarrow{p \to \infty} 0 \).

Now to obtain the inequalities claimed in (ii) follow the scheme of Theorem 6.2

(a) If \( 1 < k < m \) then show \( \frac{P_{q,k}(2p - 1)}{P_{q,m}(2p)} \xrightarrow{p \to \infty} 0 \).

(b) If \( m > 1 \) then show \( \frac{P_{q,1}(h)}{P_{q,m}(2p - 1)} \xrightarrow{p \to \infty} 0 \), where \( h = |P_{q,m}(2p - 1)| \) (so this \( h \) depends on \( p \)).

(c) To show \( P_{q,5}(h) > P_{q,4}(h) \), due to the observation that \( P_{q,\tau}(2p - 1) > P_{q,\tau}(2p) \), it will be enough to show that \( P_{q,5}(2p) > P_{q,4}(2p - 1) \) and \( P_{q,4}(2p) > P_{q,4}(2p - 1) \). These two inequalities are shown by taking analogous limits as in the proof of part (c) in Theorem 6.2 and applying the same constraints about \( \alpha \).
7 The problem of optimising the link structure

As mentioned in the introduction, a theoretically as well as commercially important problem is to find a scheme for modifying the link structure of a local web in order to improve its ranking, as set by PageRank or any other ranking function. In this paper we have presented the most fundamental goal of designing a local web (or fixing an already existing one) with a tree-like structure, where the PageRank of the main page, located at the root of the tree, should have the highest possible value, but at the same time the overall structure of the web should satisfy certain conditions given by the context. We shall not make precise the details of the context, but are the general conditions imposed by design. Let us refer to the context as Π. By virtue of Theorem 4.2 this translates into the following optimisation problem.

Main Objective: Given a certain context Π, to maximise the function

\[ P(h, 1, n_1, \ldots, n_h) = \frac{1 - \alpha}{1 + n_1 + \ldots + n_h} \sum_{k=0}^{h} \alpha^k n_k \]

for fixed \( \alpha \), such that \( 0 < \alpha < 1 \), and all trees \( T = 1n_1 \ldots n_h \) with integer values \( h, n_i \geq 1, 1 \leq i \leq h \). If the total number \( N \) of vertices is bounded then we can assure that the maximum exists. The complexity of the problem depends mostly on the conditions imposed by the context Π. This justifies approaching the solution through heuristics. Here we give an ad hoc list of rules that clearly stem from our theorems.

Rule 1: Due to Theorem 5.1 the first action to take is to reduce the height as much as the context allows.

Rule 2: Keep in mind that while applying Rule 1 (and deleting levels), links between consecutive levels can be rearranged in any way you like, as long as the context is kept consistent, and this has no effect on the root’s PageRank value (by Theorem 4.2).

Rule 3: Once the optimal height \( h > 1 \) is attained\(^2\), we delete (as much as possible) vertices from levels in the upper half of the tree, trying to get it close to its underlying queue tree (Theorem 5.4), and those vertices that cannot be deleted should be moved as closer to level 1 as possible (by Theorem 4.2).

The above rules of general nature can be complemented by next working on the particularities of the queue tree structure. For example, if the applications of rules 1 to 3 give as a final result a 5–ary queue tree, then Theorem \( 6.3(i) \) tell us that pruning more vertices to convert this tree into a binary queue tree, or binomial queue tree (of same height) improves PageRank. The caveat is that we have proved Theorem 6.3 using continuous calculus and, therefore, cannot be applied without doubt for small values of the height. To remedy this deficiency, we have computed \( P_{q,b}(h) \) and \( P_{q,m}(h) \) for various \( m \) and many small integer values of \( h \), and concluded the following facts, which strengthen Theorem 6.3–(i):

1. \( P_{q,1}(h) > P_{q,b}(h) > P_{q,2}(h) \), for \( h \geq 17 \).
2. \( P_{q,b}(h) > P_{q,1}(h) \), for \( 2 \leq h \leq 16 \).

\(^2\)Optimality here again depends on maintaining the context consistent. This height could mean the minimal levels of a hierarchy that we need to reflect in the web site; say, for example, of a corporation or a hypertext.
3. $P_{q,b}(h) > P_{q,2}(h)$, for all $h > 1$.

4. $P_{q,1}(h) > P_{q,2}(h)$, for $h \geq 15$.

5. $P_{q,2}(h) > P_{q,1}(h)$, for $2 \leq h \leq 14$.

6. $P_{q,2}(h) > P_{q,3}(h) > P_{q,4}(h) > \ldots > P_{q,m}(h)\ldots$, for $h \geq 9$.

7. $P_{q,2}(h) < P_{q,3}(h) < P_{q,4}(h) < \ldots < P_{q,m}(h)\ldots$, for $h = 3, 4$.

8. Theorem 6.3–(i) is “almost” true for $h = 5, 6, 7, 8$ (all but except some arity $m$ from 2 to 6).

Now, depending on the value of the height of the queue tree obtained by rules 1 to 3, we use the appropriate inequality from the above list to guide our pruning correctly and raise the root’s PageRank. For example, if we had arrived to an $m$–ary queue tree of height 17, and $m \geq 2$, we can delete and move vertices, shaping the tree like a $k$–ary queue tree, for some $k < m$, or like a binomial tree.

8 The bidirectional case

We turn now to trees with bidirectional as well as unidirectional arcs. We use $B^r$ to denote a tree rooted at $r$ with both unidirectional and bidirectional arcs, where all unidirectional arcs points towards the root $r$. Formally, a digraph $B^r = (V, A)$ is a bidirectional tree with root $r$ if its set of arcs $A$ can be partitioned in two disjoint sets $A_1$ and $A_2$ such that:

- $(V, A_1)$ is a partial tree with root $r$ (the underlying tree of $B^r$), and
- if $uv \in A_2$ then $vu \in A_1$, and in this case we say that $v$ is the origin of the bidirectional arc $vuv$. (Intuitively think of a bidirectional arc as a 2-cycle.)

Observe that for each arc $uv \in A_2$ the corresponding bidirectional arc $vuv$ defines an infinite number of walks ending at the root $r$ (just as would do any cycle within a tree). Henceforth, to the effect of computing the PageRank of $r$ with equation (2), we can view each arc $uv \in A_2$ as a path of infinite length hanging from the vertex $v$, and containing alternatively copies of vertices $u$ and $v$, where at each $v$ hangs a copy of the tree rooted at $v$, $T^v$, and at each $u$ hangs a copy of the remainder of the tree rooted at $u$ after removing from it the sub–tree $T^v$, that is, $T^u \setminus T^v$. Note that $T^u$ (and $T^v$) may contain bidirectional arcs. Extending this idea through all bidirectional arcs, we can view the bidirectional tree $B^r$ as an infinite tree. Figure 1 shows a bidirectional tree $B^r$ with two bidirectional arcs, $vuv$ and $v'uv'$ (leftmost tree); next to it the bidirectional tree with an infinite branch corresponding to $vuv$; and the rightmost tree is the full infinite tree associated to $B^r$.

This view of $B^r$ as an infinite tree makes it easier to understand the interpretations we do below of equation (2) adapted to our trees. To be clear, what we mean by the infinite tree associated to $B^r$ is the tree rooted at $r$, which contains the underlying tree previously defined (i.e. the partial tree rooted at $r$, $(V, A_1)$), and such that for each vertex $v \neq r$ that is the origin of a bidirectional arc $vuv$ in $B^r$, substitute the arc $uv$ by a countable infinite path rooted at $v$, containing alternatively copies of vertices $u$
and \( v \), where at each \( v \) hangs a copy of the tree rooted at \( v \), \( T^v \), and at each \( u \) hangs a copy of the remainder of the tree rooted at \( u \) after removing from it the sub-tree \( T^v \), that is \( T^u \setminus T^v \).

Now, let us recall equation (2): 
\[
\mathcal{P}(a) = \frac{1 - \alpha}{N} \sum_{v \in V} \sum_{\rho : v \xrightarrow{*} a} \alpha^{l(\rho)} D(\rho).
\]
In it, the sum is taken over all vertices \( v \) connected through a walk to \( a \). In an associated infinite tree this walk is a unique path \( \rho \) connecting \( v \) with \( a \). This path could have various incidence of bidirectional arcs. On the other hand, each bidirectional arc \( uvu \), with \( u \neq r \) and \( od(u) = 2 \), produces an infinite number of walks: \( u, uvu, uvuvu, \ldots \), with branching factors \( D(u) = 1, D(uvu) = 1/2, D(uvuvu) = 1/2^2, \ldots \); hence, summing over all these walks we get

\[
\sum_{\rho : u \xrightarrow{a} u} \alpha^{l(\rho)} D(\rho) = 1 + \frac{\alpha^2}{2} + \frac{\alpha^4}{2^2} + \cdots = \frac{1}{1 - \alpha^2/2}
\]

Therefore, if the path \( \rho : v \xrightarrow{*} a \) contains \( q \) vertices, each meeting a bidirectional arc, the contribution to \( \mathcal{P}(a) \) of the possible walks produced on \( \rho \) is \( \frac{1}{(1 - \alpha^2/2)^q} \).

If the bidirectional arc is \( rvr \), with \( od(r) = 1 \), and hence \( D(rvr \ldots vr) = 1 \) for any walk on this arc, we get that the contribution to \( \mathcal{P}(a) \) is \( \frac{1}{(1 - \alpha^2)} \).

All the above observations lead to the following result on computing the PageRank on bidirectional trees.

**Theorem 8.1** Let \( B^r = (V, A) \) be a bidirectional tree rooted at \( r \).

1. If \( od(r) = 0 \), then for all \( a \in V \),

\[
\mathcal{P}(a) = \frac{1 - \alpha}{N} \sum_{v \in V} \frac{\alpha^{l(\rho)}}{2^n(1 - \alpha^2/2)^q}
\]

2. If \( od(r) = 1 \) with bidirectional arc \( rvr \), then

\[
\mathcal{P}(a) = \frac{1 - \alpha}{N} \sum_{v \in V} \frac{\alpha^{l(\rho)}}{2^n(1 - \alpha^2/2)^q}, \quad \text{for} \ a \notin \{r, u\}
\]
and
\[ P(a) = \frac{1 - \alpha}{N} \sum_{v \in V} \frac{\alpha^{l(\rho)}}{2^n (1 - \alpha^2/2)^{q-1} (1 - \alpha^2)}, \quad \text{for} \ a \in \{r, u\} \]  
(6)

where in all cases, \( \rho : v \to a \) is the unique path from \( v \) to \( a \), and \( l(\rho) \) is the length of this path; \( n \) is the number of bidirectional vertices (i.e. with \( od(u) = 2 \)) not being an end-vertex in \( \rho \); \( q \) is the number of bidirectional arcs meeting \( \rho \). \[ \square \]

In particular, if \( od(r) = 0 \),
\[ P(r) = \frac{1 - \alpha}{N} \sum_{v \in V} \frac{\alpha^{l(\rho)}}{(2 - \alpha^2)^q} \]  
(7)

since \( n = q \) for this case. And if \( od(r) = 1 \),
\[ P(r) = \frac{1 - \alpha}{N} \sum_{v \in V} \frac{\alpha^{l(\rho)}}{(2 - \alpha^2)^{q-1} (1 - \alpha^2)} \]  
(8)

since \( n = q - 1 \) for this case.

At this point we would like to make a digression into the nature of the formulas for PageRank we have just deduced. These have their origin in Brinkmeier’s equation (eq. (2)), which in essence computes the contributions of vertices to the value \( P(a) \) by a depth-first search exploration. Our proposed equation for computing the PageRank of the root in the case of unidirectional trees (section 4, equation (3)) is founded on the complementary tree-search routine, namely, breadth-first search; and we would like to have a result on the same spirit of counting by levels for the case of bidirectional trees.

For a breadth-first search type of computation of PageRank on a bidirectional tree, we must classify somehow the vertices by levels of the tree. For each \( k > 0 \), the vertices at level \( N_k = \{v_{k1}, \ldots, v_{kn_k}\} \) are characterised by the number of bidirectional arcs met by their paths which ends in the root, \( v_{ki} \to r \). Hence, \( n_k = n_k^0 + \cdots + n_k^{k+1} \), where \( n_k^q \) denotes the number of vertices at level \( N_k \) having \( q \) bidirectional arcs meeting their paths to \( r \). Some of these \( n_k^q \) could be null. The non-null \( n_k^q \) many vertices contribute to the summation in equations (7) and (8) the quantities \( \frac{n_k^q \alpha^k}{(2 - \alpha^2)^q} \) and \( \frac{n_k^q \alpha^k}{(2 - \alpha^2)^{q-1} (1 - \alpha^2)} \) according to either case of \( od(r) = 0 \) or \( od(r) = 1 \). Thus, we have the following result.

**Theorem 8.2** Let \( B^r \) be a bidirectional tree rooted at \( r \), with \( N \) vertices and height \( h > 0 \).

1. If \( od(r) = 0 \),
\[ P(r) = \frac{1 - \alpha}{N} \sum_{k=0}^{h} \sum_{q=0}^{k} \frac{n_k^q \alpha^k}{(2 - \alpha^2)^q} \]
2. If \( od(r) = 1 \),
\[ P(r) = \frac{1 - \alpha}{N} \sum_{k=0}^{h} \sum_{q=0}^{k} \frac{n_k^{q+1} \alpha^k}{(2 - \alpha^2)^q (1 - \alpha^2)} \]

where \( q \) is the number of bidirectional arcs met by the path ending in \( r \), but distinct from the bidirectional arc incidence with \( r \), if such bidirectional arc exists. \[ \square \]
We can give a more succinct vectorial formulation of the previous result, if we
develop the sums “by rows” (outmost sum) and group column terms in a vector.

**Theorem 8.3** Let $B^r$ be a bidirectional tree rooted at $r$ with $N$ vertices and height $h > 0$. If $od(r) = 0$, then $P(r) = \frac{1 - \alpha}{N} \sum_{q=0}^{h} \Delta_q \cdot \Lambda_q$, where $\Delta_q = (n_q, n_{q+1}, \ldots, n_h)$ and $\Lambda_q = (\alpha_q, \alpha_{q+1}, \ldots, \alpha_h)$. Similarly, if $od(r) = 1$, then $P(r) = \frac{1 - \alpha}{N} \sum_{q=0}^{h} \frac{\Delta'_q \cdot \Lambda_q}{(2 - \alpha^2)^q(1 - \alpha^2)}$, where $\Delta'_q = (n_q^{q+1}, n_{q+1}^{q+1}, \ldots, n_h^{q+1})$. □

8.1 Case of s-cycles

In this section we generalise the computation of PageRank to bidirectional trees of
height $h > 1$ on which we close permissible cycles of any length obtained by joining
vertices from level $N_j$ with vertices from level $N_k$, for $0 \leq j < k \leq h$. In this way we
can transform bidirectional arcs $vuw$ into cycles $vuw_1 \ldots v_1 v$ of longer length, where
the arc $uv_n$ close the new cycle inserted in the rooted tree. Also the arc $uv$ of the
bidirectional arc $uvw$ can be substituted by a new arc $ut$ closing a larger path $t \ldots vu$
in the tree. In Figure 2 we exhibit some examples of these transformations.

Let us call these classes of digraphs obtained by closing cycles on bidirectional trees
as **cyclical trees**. Formally we define a digraph $C^r = (V, A)$ as a **cyclical tree with
root** $r$, if its set of arcs $A$ can be partitioned in two disjoint sets $A_1$ and $A_2$ such that:

- $(V, A_1)$ is a partial tree with root $r$ (the underlying tree of $C^r$), and
- if $uv \in A_2$ then there is a path $v_1v_2 \ldots v_{s-1}v_s$, beginning at $v_1 = v$, ending at
  $v_s = u$ and with intermediate vertices and arcs $v_i;v_{i+1}$ in $A_1$, and in this case we
  say that $v$ is the **origin** of the cycle $vu2 \ldots vs-1uv$. 

Figure 2: Examples of cyclical trees.
We proceed to compute the PageRank of these cyclical trees. Similarly to the bidirectional case (which is no other than a 2-cycle), we have that each cycle \( uv \ldots u \) of length \( l > 2 \) and \( \text{od}(u) = 2 \), produces an infinite number of walks: \( u, uv \ldots u, uv \ldots uv \ldots u, \ldots \), with branching factors \( D(u) = 1, D(uv \ldots u) = 1/2, D(uv \ldots uv \ldots u) = 1/2^2, \ldots \); hence, summing over all these walks we get

\[
\sum_{\rho: u^* \rightarrow u} \alpha^{l(\rho)} D(\rho) = 1 + \frac{\alpha^l}{2} + \frac{\alpha^{2l}}{2^2} + \cdots = \frac{1}{1 - \alpha^l/2}
\]

Therefore, if the path \( \rho : v \xrightarrow{*} a \) contains \( q \) vertices, meeting \( q \) cycles of length \( l_1, l_2, \ldots, l_q \), respectively, then the contribution to \( P(a) \) of the possible walks produced on \( \rho \) is

\[
\frac{1}{1 - \alpha^{l_1}/2} \cdot \frac{1}{1 - \alpha^{l_2}/2} \cdots \frac{1}{1 - \alpha^{l_q}/2}
\]

If the cycle is \( rv \ldots r \), with \( \text{od}(r) = 1 \), and hence \( D(rv \ldots r) = 1 \), we get that the contribution to \( P(a) \) is \( \frac{1}{1 - \alpha^l} \).

**Theorem 8.4** Let \( C' = (V, A) \) be a cyclical tree rooted at \( r \).

1. If \( \text{od}(r) = 0 \), then for all \( a \in V \),

\[
P(a) = \frac{1 - \alpha}{N} \sum_{v \in V} \frac{\alpha^{l(\rho)}}{2^n(1 - \alpha^{l_1}/2) \cdots (1 - \alpha^{l_q}/2)}
\]

2. If \( \text{od}(r) = 1 \) in the cycle \( rv_1 \ldots v_{q-1} r \), then

\[
P(a) = \frac{1 - \alpha}{N} \sum_{v \in V} \frac{\alpha^{l(\rho)}}{2^n(1 - \alpha^{l_1}/2) \cdots (1 - \alpha^{l_q}/2)}, \quad \text{for } a \notin \{r, v_1, \ldots, v_{q-1}\}
\]

and

\[
P(a) = \frac{1 - \alpha}{N} \sum_{v \in V} \frac{\alpha^{l(\rho)}}{2^n(1 - \alpha^{l_1}/2) \cdots (1 - \alpha^{l_{q-1}}/2)(1 - \alpha^{l_q})}, \quad \text{for } a \in \{r, v_1, \ldots, v_{q-1}\}
\]

where in all cases, \( \rho : v \xrightarrow{*} a \) is the unique path from \( v \) to \( a \), and \( l(\rho) \) is the length of this path; \( n \) is the number of bidirectional vertices (i.e. with \( \text{od}(u) = 2 \)) not being an end-vertex in \( \rho \); \( q \) is the number of cycles meeting \( \rho \) and of lengths \( l_1, l_2, \ldots, l_q \).

In particular, if \( \text{od}(r) = 0 \), \( n = q \), and

\[
P(r) = \frac{1 - \alpha}{N} \sum_{v \in V} \frac{\alpha^{l(\rho)}}{(2 - \alpha^{l_1}) \cdots (2 - \alpha^{l_q})} \tag{9}
\]

And if \( \text{od}(r) = 1 \), \( n = q - 1 \), and

\[
P(r) = \frac{1 - \alpha}{N} \sum_{v \in V} \frac{\alpha^{l(\rho)}}{(2 - \alpha^{l_1}) \cdots (2 - \alpha^{l_{q-1}})(1 - \alpha^{l_q})} \tag{10}
\]
9 Properties of bidirectional and cyclical trees

Analogously to the case of unidirectional trees we shall analyse in this section the behaviour of PageRank on bidirectional, and more general, cyclical trees when their topology is modified. Our first result shows that on a unidirectional tree changing unidirectional arcs to bidirectional enhance the PageRank value of the end-vertices of the transformed arc, but reduces the PageRank of the root of the tree.

Theorem 9.1 If in a unidirectional tree $T^r$ an arc $vu$, with $u \neq r$, is changed to a bidirectional arc $uvu$, then $P(u)$ and $P(v)$ both increase, but $P(r)$ decreases.

Proof: We introduce some notation first. $P_x(T_y)$ denotes the PageRank of vertex $x$ in the tree $T_y$ with root $y$; $n_p(T_y)$ denotes the number of vertices at level $N_p$ in the tree $T_y$. Now, assume that $u$ is at level $N_k$ in the tree $T_r$ (and, hence, $v \in N_{k+1}$).

Then, we have that

$$P_r(T_r) = \frac{1 - \alpha}{N} \sum_{p=0}^{h} n_p(T_r) \alpha^p$$

and, therefore, if $B^r$ is the bidirectional tree obtained from $T^r$ by just adding the bidirectional arc $uvu$, we have

$$P_r(B^r) = \frac{1 - \alpha}{N} \left( \sum_{p=0}^{h} n_p(T^r - T^u) \alpha^p + \frac{1}{2} \sum_{p=k}^{h} n_p(T^u) \alpha^p \right) < P_r(T^r)$$

which shows that the PageRank of the root $r$ decreases. On the other hand, the PageRanks of $u$ and $v$ are given by the equations:

$$P_u(B^u) = \frac{1 - \alpha}{N(1 - \alpha^2/2)} \sum_{p=k}^{h} n_p(T^u) \alpha^{p-k} = \frac{1}{1 - \alpha^2/2} P_u(T^r) > P_u(T^r)$$

and

$$P_v(B^v) = \frac{1 - \alpha}{N(1 - \alpha^2/2)} \left( \frac{\alpha}{2} \sum_{p=k}^{h} n_p(T^u - T^v) \alpha^{p-k} + \sum_{p=k+1}^{h} n_p(T^v) \alpha^{p-(k+1)} \right)$$

$$> P_v(T^v) \quad \square$$

Using same arguments as given for the previous theorem, we can generalized the result to the case where the original tree is bidirectional, and some of its unidirectional arc (if any) is promoted to being bidirectional.

Theorem 9.2 Let $B^r$ be a bidirectional tree, and let $B'^r$ be the tree resulting from $B^r$ when one of its arcs $vu$, with $u \neq r$ is transformed into bidirectional arc $uvu$. Then
1. \( P_u(B^u) = \frac{1}{1 - \alpha^2/2} P_u(B^u) > P_u(B^v) \).

2. \( P_v(B^u) > P_v(B^u) \).

3. If \( uv'v'u' \) is a bidirectional arc intersecting the path \( uv_1 \ldots v_k = r \), then
   \( P_u(B^v) < P_u(B^v) \) and \( P_v(B^v) < P_v(B^v) \).

4. \( P_x(B^v) < P_x(B^v) \) for all vertex \( x \) in the path \( v_1 \ldots v_k = r \).

5. In particular, \( P_r(B^v) < P_r(B^v) \).

6. The vertices which are neither contained in the path \( uv_1 \ldots v_k = r \) nor in the bidirectional arcs intersecting this path preserve their original PageRank. □

Theorems 9.1 and 9.2 suggest that in order to increase the PageRank of the root \( r \) of a tree we have to directly promote to bidirectional the arc's incidence to \( r \). The consequences of this manipulation is summarized in the following theorem, which is a direct consequence of the two previous results.

**Theorem 9.3** Let \( B^v \) be a bidirectional tree, with \( od(r) = 0 \), and let \( B^v \) be the tree resulting from \( B^v \) when one of its arcs \( vr \) is transformed into bidirectional arc \( rvr \). Then

1. \( P_r(B^v) = \frac{P_r(B^v)}{1 - \alpha^2} \). (Note that for \( \alpha = 0.85 \) this increment is \( \approx 3.6 P_r(B^v) \).)

2. \( P_v(B^v) = P_v(B^v) + \frac{\alpha P_r(B^v)}{1 - \alpha^2} \).

3. \( P_r(B^v) \geq P_r(B^v) \iff P_r(B^v) \geq (1 + \alpha) P_v(B^v) \).

4. All other vertices (different from \( r \) and \( v \)) preserve their PageRank. □

For cyclical trees we have results similar to Theorems 9.1 9.3 but factoring out by \( 1/(1 - \alpha^2) \) in place of \( 1/(1 - \alpha^2) \).

Now, the pruning of the lower levels of a bidirectional tree has mix consequences for the PageRank of the root, as opposed to the positive results obtained for unidirectional trees in section 5. We illustrate the possible outcomes of pruning lower levels of a bidirectional trees in the figures below.

In the tree shown in Figure 3 for \( n \leq 75 \) and for all \( m \geq 1 \), successive removal of the \( m \) vertices of the last level increments the PageRank of the root, \( P(1) \). For \( n \geq 76 \) and for all \( m \geq 1 \), successive removal of the \( m \) vertices of the last level decrements \( P(1) \). On the other hand, in the tree shown in Figure 4 for \( n \leq 31 \) and for all \( m \geq 1 \), successive removal of the \( m \) vertices of the last level increments \( P(1) \). For \( n \geq 32 \) and for all \( m \geq 1 \), successive removal of the \( m \) vertices of the last level decrements \( P(1) \).
The previous results give us some clues on ways of optimising PageRank of tree-like organised sites. Obviously these rules for rearrangement should apply insofar as the context allows.

**Rule 1** To augment the PageRank of the root transform incoming arcs bidirectional. Furthermore, link the root with vertices below in the tree (so that cycles passing by the root are build).

**Rule 2** To augment the PageRank of a vertex $u$ different from the root, link $u$ with a bidirectional arc to each one of the vertices on the subtree with root $u$ (hence obtaining a cyclical tree). Keep in mind that this enhances the PageRank of $u$ but reduces the PageRank of the root. One may interpret this action as linking an individual with all its subordinates in a hierarchical organisation.

### 10 More complex topologies

The next natural step is to upgrade the preceding results on bidirectional and cyclical trees to finite cyclic structures which can be modelled by our infinite trees. In order to achieve this further extensions we should then visualise an arbitrary digraph through its *condensation digraph* as the acyclic digraph consisting of its *strongly connected components*.

A digraph $D = (V, A)$ is **strongly connected** if for each pair $u$ and $v$ of distinct vertices, there is a path joining $u$ with $v$ and a path joining $v$ with $u$. Define $u \equiv v$ provided there are paths joining $u$ with $v$ and $v$ with $u$. This is an equivalence relation and, in consequence, $V$ is partitioned into equivalence classes $V_1, \ldots, V_p$. The $p$ subdigraphs $D_i = (V_i, A/V_i)$ induced on the sets $V_i$, $i = 1, \ldots, p$, are the **strong connected components** of $D$. The digraph $D$ is strongly connected if and only if it has exactly one strong component. The **condensation digraph** of the digraph $D$ is the acyclic digraph whose vertices are the strong connected components, or SCC, of $D$, and there is an arc from one SCC $D_i$ to another $D_j$, $i \neq j$, if a vertex of $V_i$ is adjacent in $D$ to a vertex of $V_j$.

Now, the extension of our techniques and procedures to a more general digraph requires that its corresponding condensation digraph be a rooted tree whose SCC behave like the cyclical rooted trees. Not all SCC may have the required behaviour, but many do so, and the key is that each SCC should have a root through which it

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3We are aware that some of the rules listed here (and more that could be derived from our results) are, to some extend, already in use by web masters and SEO analysts, but as far as we have seen, without much mathematical justification.
connects to the rest of the tree (and by no other vertex), and two such SCC which are adjacent in the condensation digraph are linked by just one arc in the original digraph. The SCC which have these properties we shall called PR–digraph.

Formally, a **PR–digraph with root** $r$ is a strongly connected digraph with at least one vertex $r$ (the root of the PR–digraph) such that for all vertex $v$ ($v \neq r$), there is a unique path joining $v$ with $r$. In this structure we would say that a vertex $v$ is at level $N_k$ if the path that connects $v$ with $r$ is of length $k$. Now, in essence, a PR–digraph is much like a cyclical tree in as much as it can be seen as a tree with cycles formed by adding arcs from one level up to another level down. The key is that a PR–digraph admits a corresponding infinite tree due to the fact that each vertex has been assigned a unique level of the graph, or in other words, a unique path to $r$. Note that the root as the rest of the vertices may have out–degree greater or equal to 1. As an illustration of structures that can be PR–digraphs or not see Figure 5. There, in the strongly connected digraph $D$ the vertex 1 can not be the root of a PR–digraph since vertex 2 has two paths towards 1. On the contrary, the vertices 2 and 3 can be roots of a PR–digraph $D$.

Also the complete digraph $\{(1,2), (2,1), (1,3), (3,1), (2,3), (3,2)\}$ can not be a PR–digraph for any of its vertices. But, the strongly connected digraph $\{(1,2), (2,1), (2,3), (3,2)\}$ is a PR–digraph rooted at any of its three vertices.

The condition characterising a PR–digraph must also apply to the connections among SCC which are PR–digraphs. It can not be the case that in a tree of SCCs, which are PR–digraphs, one such SCC connects to another SCC in the tree through two arcs or more; that is, in the original digraph there must be a root (which itself could be the root of a PR–digraph) and it must be the case that each vertex $v$ connects to the root by a unique path. On the other hand, we must admit the possibility of producing cycles of length $s > 1$ in this structure by connecting a vertex at the level $N_{k+s}-s$ with a vertex at level $N_k$.

We shall then define a **PR–digraph tree with root** $r$, as a digraph $D = (V, A)$ whose set of arcs $A$ can be partitioned in two disjoint sets $A_1$ and $A_2$ such that:

- $(V, A_1)$ is a partial digraph whose condensation digraph is a tree of SCCs which are PR–digraphs, each pair of adjacent PR–digraphs are linked by a unique arc and the maximal PR–digraph contains the root $r$ (the underlying digraph of $D$); and
- if $uv \in A_2$ then there is a path $v_1v_2 \ldots v_{s-1}v_s$, beginning at $v = v_1$, ending at $u = v_s$ and with intermediate vertices and arcs $v_iv_{i+1}$ in $A_1$, and in this case we

![Figure 5: Digraph $D$ and the PR-digraphs that can be derived from it.](image)
say that \( v \) is the \textbf{origin} of the cycle \( vv_2 \ldots v_{s-1}uv \).

Note that this time the arc \( uv \), as well as the cycle \( vv_2 \ldots v_{s-1}uv \), could be in the partial digraph \((V, A_1)\). We show in Figure 6 a digraph \( D \) that can not be a PR–digraph tree. This is due to the fact that the left–side SCC of \( D \) is not compatible with a PR–digraph tree, since the vertex \( u \) has out degree 2 towards the root \( r \). Also in the SCC on the right branch of \( D \) either one of the arcs \( xy \) or \( zw \) represents a surplus that forbids \( D \) from being a PR–digraph tree.

**Theorem 10.1** A PR–digraph tree with root \( r \) is a cyclical tree with root \( r \).

**Proof:** Let \( D = (V, A) \) be a PR–digraph tree with root \( r \). It is sufficient to prove that the underlying digraph, \( C = (V, A_1) \), of \( D \) is a cyclical tree with root \( r \). The set of arcs \( A_1 \) can be partitioned in two disjoint sets: \( A_{11} = \{ vu : \text{there is a path } vu \ldots r \text{ in } C \} \) and \( A_{12} = A_1 \setminus A_{11} \). The digraph \( (V, A_{11}) \) is a tree rooted at \( r \) because, by the definition of the underlying digraph \( C \), each vertex of \( V \) is joined with the root \( r \) by a unique path. Then \( (V, A_{11}) \) is the underlying directed tree of \( C \) and, moreover, if \( uv \) belongs to \( A_{12} \) then \( uv \) is in a SCC that is a PR–digraph with some root \( r' \). By the strong connection, there is a path joining \( v \) to \( u \), and thus \( C \) is a cyclical tree. \( \square \)

As a consequence of this theorem we can compute the PageRank of the root \( r \) of a PR–digraph tree by a similar formula as given in section 8.1 for cyclical trees. In Figure 7 we exhibit a PR–digraph tree \( D \) and its representation as a cyclical tree.

The PR–digraph trees are the most general cyclical structures which can be interpreted as unidirectional infinite trees, and on which we can apply the optimisation techniques displayed in this article by treating each SCC as one unit. This could also revert on a speed up on the PageRank calculation. More explicitly, the last point we want to call attention to is the following. There are several approaches in the literature to the task of speeding up the calculation of PageRank, based upon the following general scheme (see, for example, [11, 2, 6]):

- Partition the web into local subwebs; then compute some independent ranking for each local subweb, which will apply to the whole subweb treated as a unit; and then compute the ranking of the graph of subwebs.

In [2] and [6] the local splitting of the web is done in strongly connected components, and further in [6] Thm 2.1, it is shown that the PageRank can be calculated
independently on each SCC, provided we know the PageRank of all vertices outside the SCC, but directly linking to vertices in the SCC. Our PR–digraph tree is the most simple splitting of the web in the way of [2] and [6], namely as SCC, with the additional strongest condition of having a single link between components, which by the previously mentioned result of [6] can have PageRank computed independently on each SCC, and on a very simple way, provided we know the PageRank of their descendants in the topological structure of the tree. This suggests computing PageRank in parallel and through layers, as it is proposed in [6 §3], following an iterated process on the tree from a top level \( N_h \) down to the root at \( N_0 \). The PR–digraph is a suitable structure for the application of this process.

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