HOW CAN WE OBSERVE AND DESCRIBE CHAOS?

Andrzej Kossakowski†, Masanori Ohya‡ and Yosio Togawa‡
†Institute of Physics
N. Copernicus University, Grudziadzka 5, 87-100 Torun, Poland
‡Department of Information Sciences
Tokyo University of Science, Noda City, Chiba 278-8510, Japan

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1 Introduction

There exist several trials to describe chaos appeared in classical or quantum dynamical systems [?]. One of the present authors introduced Information Dynamics (ID for short) [18] as a frame to discuss complexity and chaos appeared in various fields, in which he tried to find a common basis by synthesizing the state change (dynamics) and the complexity associated with dynamical systems. Since then, ID has been applied to several different topics [?], among which a chaos degree, a quantity measuring the degree of chaos associated with a dynamics, was introduced by means of the complexities in ID and its entropic version (called Entropic Chaos Degree (EDC for short)) has been computed numerically for rather famous chaotic dynamics such as logistic map, baker’s transformation, Tinkerbel map. It is surprised that the result of the ECD exactly matches to that of Lyapunov exponent in the case that the later can be computed. Moreover the algorithm computing the ECD is much easier than that of Lyapunov exponent, so that the ECD can be almost always computable even when the Lyapunov exponent can not be so. However there are some unclear points in both conceptually and mathematically why the ECD could be so successful for computational experiments. In this paper we study these points and propose a new description of chaos.

In Section 2, we briefly review Information Dynamics and Chaos Degree, and in Section 3 the entropic chaos degree and its algorithm are recalled with a computational result. In Section 4, a new way judging chaos from a given dynamics is discussed based on the ECD, that is, we propose a new view to define chaos of dynamical systems.
2 Information Dynamics and Chaos Degree

We briefly review what ID is. Let \((A, S, \alpha(G))\) be an input (or initial) system and \((\overline{A}, \overline{S}, \overline{\alpha(G)})\) be an output (or final) system. Here \(A\) is a set of some objects to be observed and \(S\) is a set of some means to get the observed value, \(\alpha(G)\) describes a certain evolution of system with a parameter \(g\) in a certain set \(G\). Often we have \(A = \overline{A}, S = \overline{S}, \alpha = \overline{\alpha}, G = \overline{G}\). Therefore it can be said

\[[\text{Giving a mathematical structure to input and output triples}]\quad \equiv \quad \text{Having a theory}\]

The dynamics of state change is described by a channel, that is, a map \(\Lambda^*: \mathcal{S} \to \overline{\mathcal{S}}\) (sometimes \(\mathcal{S} \to \mathcal{S}\)). The fundamental point of ID is that ID contains two complexities in itself. Let \((A_t, S_t, \alpha_t(G_t))\) be the total system of \((A, S, \alpha)\) and \((\overline{A}, \overline{S}, \overline{\alpha})\), and \(\mathcal{S}\) be a subset of \(\mathcal{S}\) in which we are measuring observables (e.g., \(\mathcal{S}\) is the set of all KMS or stationary states in \(C^*-\text{system}\)). Two complexities are denoted by \(C\) and \(T\). \(C\) is the complexity of a state \(\varphi\) measured from a reference system \(\mathcal{S}\), in which we actually observe the objects in \(A\) and \(T\) is the transmitted complexity associated with a state change \(\varphi \to \Lambda^*\varphi\), both of which should satisfy the following properties:

\[\langle \text{Axioms of complexities} \rangle\]

(i) For any \(\varphi \in \mathcal{S} \subset \mathcal{G}\),
\[C^S(\varphi) \geq 0, \quad T^S(\varphi; \Lambda^*) \geq 0\]

(ii) For any orthogonal bijection \(j : ex\mathcal{S} \to ex\overline{\mathcal{S}}\), the set of all extremal points of \(\mathcal{S}\),
\[C^{j(S)}(j(\varphi)) = C^S(\varphi), \quad T^{j(S)}(j(\varphi); \Lambda^*) = T^S(\varphi; \Lambda^*)\]

(iii) For \(\Phi = \varphi \otimes \psi \in \mathcal{S}_t \subset \overline{\mathcal{S}}_t\),
\[C^{\mathcal{S}_t}(\Phi) = C^S(\varphi) + C^{\overline{\mathcal{S}}}(\psi)\]

(iv) \(0 \leq T^S(\varphi; \Lambda^*) \leq C^S(\varphi)\)

(v) \(T^S(\varphi; id) = C^S(\varphi)\), where "id" is an identity map from \(\mathcal{S}\) to \(\mathcal{S}\).

Instead of (iii), when "(iii') \(\Phi \in \mathcal{S}_t \subset \overline{\mathcal{S}}_t\), put \(\varphi \equiv \Phi \upharpoonright \overline{A}\) (i.e., the restriction of \(\Phi\) to \(\overline{A}\)), \(\psi \equiv \Phi \upharpoonright \overline{A}\), \(C^{\mathcal{S}_t}(\Phi) \leq C^S(\varphi) + C^{\overline{\mathcal{S}}}(\psi)\) " is satisfied, \(C\) and \(T\) is called a pair of strong complexity. Therefore ID is defined as follows:

\[\text{Definition 1 : Information Dynamics is described by} \]
\[(A, \mathcal{S}, \alpha(G); \overline{A}, \overline{\mathcal{S}}, \overline{\alpha}(\overline{G}); \Lambda^*; C^S(\varphi), T^S(\varphi; \Lambda^*))\]
\[\text{and some relations } R \text{ among them.} \]
In the framework of ID, we have to

(i) mathematically determine $\mathcal{A}, \mathcal{G}, \alpha(G); \overline{\mathcal{A}}, \overline{\mathcal{G}}, \overline{\alpha}(G)$,

(ii) choose $\Lambda^*$ and $R$, and

(iii) define $C^S(\varphi), T^S(\varphi; \Lambda^*)$.

In ID, several different topics can be treated on a common standing point so that we can find a new clue bridging several fields.

We assume $\mathcal{A} = \mathcal{A}$ for simplicity in the sequel. For a certain subset $\mathcal{S}$ (called the reference space) of $\mathcal{G}$ and a state $\varphi \in \mathcal{S}$, there exists a decomposition of the state $\varphi$ into a mixture of extreme (pure) states such that

$$\varphi = \int_S \omega d\mu$$

This extremal decomposition of $\varphi$ describes the degree of mixture of $\varphi$ in the reference space $\mathcal{S}$. The measure $\mu$ is not always unique, so that the set of all such measures is denoted by $M_\mu(\mathcal{S})$.

For instance, when $(\mathcal{A}, \mathcal{G})$ and is a C*-system containing both classical and quantum systems; that is, $\mathcal{A}$ and is a C* algebra and $\mathcal{G}$ is the set of all states on $\mathcal{A}$, the reference space $\mathcal{S}$ is a weak* compact convex subset of $\mathcal{G}$ and the measure $\mu$ is not uniquely determined unless $\mathcal{S}$ is the Schoen simplex. In this paper we will not go to the details of such general mathematical discussion.

A measure of chaos produced by dynamics $\Lambda^*$ is defined in [21, 22]:

**Definition 2**

(1) $\psi$ is more chaotic than $\varphi$ if $C(\psi) \geq C(\varphi)$.

(2) When $\varphi \in \mathcal{S}$ changes to $\Lambda^* \varphi$, the chaos degree associated to this state change (dynamics) $\Lambda^*$ is given by

$$D^S(\varphi; \Lambda^*) = \inf \left\{ \int_S C^S(\Lambda^* \omega) d\mu; \mu \in M_\mu(\mathcal{S}) \right\}.$$  

**Definition 3** A dynamics $\Lambda^*$ produces chaos iff $D^S(\varphi; \Lambda^*) > 0$.

It is important to note here that the dynamics $\Lambda^*$ in the definition is not necessarily same as original dynamics (channel) but is one reduced from the original one such that it causes an evolution for a certain observed value like orbit. However for simplicity we often use the same notation in this paper. In some cases, the above chaos degree $D^S(\varphi; \Lambda^*)$ can be expressed as

$$D^S(\varphi; \Lambda^*) = C^S(\Lambda^* \varphi) - T^S(\varphi; \Lambda^*).$$
3 Entropic Chaos Degree and its Algorithm

Although there exist several complexities [20], one of the most useful examples of $C$ and $T$ are Shannon's entropy and mutual entropy in classical systems (von Neumann entropy and quantum mutual entropy in quantum systems [23]), respectively.

The concept of entropy was introduced and developed to study the topics such as irreversible behavior, symmetry breaking, amount of information transmission, so that it originally describes a certain chaotic property of state.

Let us recall the simplest case of $C$ and $T$, that is, Shannon's entropy and mutual entropy. In classical communication systems, an input state $\varphi$ is a probability distribution $p = (p_k) = \sum_k p_k \delta_k$ and a channel $\Lambda^*$ is a transition probability $(t_{i,j})$, so that the compound state of $\varphi$ and its output $\overline{p} (\equiv \overline{p} = (\overline{p}_i) = \Lambda^* p)$ is the joint distribution $r = (r_{i,j})$ with $r_{i,j} \equiv t_{i,j} p_j$. Then the complexities $C$ and $T$ are given as

\[
C (p) = S (p) = - \sum_k p_k \log p_k,
\]

\[
T (p; \Lambda^*) = I (p; \Lambda^*) = \sum_{i,j} r_{i,j} \log \frac{r_{i,j}}{p_i \overline{p}_j}.
\]

Thus the entropic chaos degree of the channel $\Lambda^*$ becomes

**Definition 4**

\[
D (p; \Lambda^*) = S (\Lambda^* p) - I (p; \Lambda^*).
\]

Quantum version of the above entropic chaos degree was discussed in [10] [22], on which we will briefly review here in the case of usual Hilbert space expression.

Let $\rho$ be a quantum state, namely, a density operator on a Hilbert space $H$, and $\Lambda^*$ be a channel sending the set $\mathcal{S}$ of all states on $H$ into itself. Then the entropic chaos degree is defined by

\[
D (\rho; \Lambda^*) = \inf \left\{ \sum_k \lambda_k S (\Lambda^* E_k) ; \{ E_k \} \in \mathcal{E} \right\},
\]

where $\mathcal{E}$ is the set of all Schatten decompositions (i.e., one dimensional spectral decompositions) of the state $\rho := \sum_k \lambda_k E_k$, and $S$ is the von Neumann entropy.

3.1 Algorithm Computing Chaos Degree

In order to observe a chaos produced by a dynamics, one often looks at the behavior of orbits made by that dynamics, more generally, looks at the behavior of a certain observed value. Therefore in our scheme we directly compute the chaos degree once a dynamics is explicitly given as a state change of a system. However even when the direct calculation does not show a chaos, a chaos will appear if one focuses to some aspect of the state change, e.g., a certain observed value which may be called orbit as usual. The algorithm computing the chaos degree for a dynamic is the following two cases [21] [22] [12] [10]:
(1) **Dynamics is given by** \( \frac{dx}{dt} = \hat{\xi}(x) \) with \( x \in I = [a, b]^N \subset \mathbb{R}^N \): First find a difference equation \( x_{n+1} = F(x_n) \) with a map \( F \) on \( I \equiv [a, b]^N \subset \mathbb{R}^N \) into itself, secondly let \( I \equiv \bigcup_k A_k \) be a finite partition with \( A_i \cap A_j = \emptyset \) \((i \neq j)\). Then the state \( \varphi^{(n)} \) of the orbit determined by the difference equation is defined by the probability distribution \( \left(p^{(n)}_i\right) \), that is, \( \varphi^{(n)} = \sum_i p^{(n)}_i \delta_i \), where for a given initial value \( x \in I \) and the characteristic function \( 1_{A_i} \)

\[
p^{(n)}_i = \frac{1}{n+1} \sum_{k=m}^{m+n} 1_{A_i} (F^k x).
\]

Now when the initial value \( x \) is distributed due to a measure \( \nu \) on \( I \), the above \( p^{(n)}_i \) is given as

\[
p^{(n)}_i = \frac{1}{n+1} \int_I \sum_{k=m}^{m+n} 1_{A_i} (F^k x) \, d\nu.
\]

The joint distribution \( \left(p^{(n,n+1)}_{ij}\right) \) between the time \( n \) and \( n+1 \) is defined by

\[
p^{(n,n+1)}_{ij} = \frac{1}{n+1} \sum_{k=m}^{m+n} 1_{A_i} (F^k x) 1_{A_j} (F^{k+1} x)
\]
or

\[
p^{(n,n+1)}_{ij} = \frac{1}{n+1} \int_I \sum_{k=m}^{m+n} 1_{A_i} (F^k x) 1_{A_j} (F^{k+1} x) \, d\nu.
\]

Then the channel \( \Lambda^* \) at \( n \) is determined by

\[
\Lambda^*_n \equiv \left( \frac{p^{(n,n+1)}_{ij}}{p^{(n)}_i} \right) \implies \varphi^{(n+1)} = \Lambda^*_n \varphi^{(n)},
\]

and the entropic chaos degree is given by the definition 3.1;

\[
D_A(x; F) = D_A \left(p^{(n)}; \Lambda^*_n\right) = \sum_i p^{(n)}_i S(\Lambda^*_n \delta_i) = \sum_{i,j} p^{(n,n+1)}_{ij} \log \frac{p^{(n)}_i}{p^{(n,n+1)}_{ij}}.
\]

We can judge whether the dynamics causes a chaos or not by the value of \( D \) as the definition 2.2

\[
D > 0 \iff \text{chaotic}
\]

\[
D = 0 \iff \text{stable}.
\]

This chaos degree was applied to several dynamical maps such logistic map, Baker’s transformation and Tinkerbel map, and it could explain their chaotic
characters. This chaos degree has several merits compared with usual measures such as Lyapunov exponent as explained below.

(2) Dynamics is given by $\varphi_t = F^t \varphi_0$ on a Hilbert space: Similarly as making a difference equation for state, the channel $\Lambda^+_n$ at $n$ is first deduced from $F^+_t$, which should satisfy $\varphi^{(n+1)} = \Lambda^+_n \varphi^{(n)}$. By means of this constructed channel (a) we compute the chaos degree $D$ directly according to the definition 3.2 or (b) we take a proper observable $X$ and put $x_n = \varphi^{(n)}(X)$, then go back to the algorithm (1).

Note that the chaos degree $D$ does depend on a partition $A$ taken, which is somehow different from usual degree of chaos (cf., dynamical entropy [1, 4, 3, 14]). This is a key point of our understanding of chaos, which will be discussed in the next section.

3.2 Logistic Map

Let us explain how the entropy chaos degree (ECD) well describes to the chaotic behavior of logistic map.

The logistic map is defined by

$$x_{n+1} = ax_n (1 - x_n), x_n \in [0, 1], 0 \leq a \leq 4$$

The solution of this equation bifurcates as shown in Fig.5.1.

![The bifurcation diagram for logistic map](image)

Fig.1 The bifurcation diagram for logistic map

In order to compare ECD with other measure describing chaos, we take Lyapunov exponent for this comparison and remaind here its definition.

\textbf{Lyapunov exponent $\lambda(f)$}

(1) Let $f$ be a map on $\mathbb{R}$, and let $x_0 \in \mathbb{R}$. Then the Lyapunov exponent $\lambda^O(f)$ for the orbit $O \equiv \{ f^n(x_0) ; n = 0, 1, 2, \cdots \}$ is defined by

$$\lambda^O(f) = \lim_{n \to \infty} \lambda^{(n)}^O(f), \quad \lambda^{(n)}^O(f) = \frac{1}{n} \log \left| \frac{df^n}{dx}(x_0) \right|$$
(2) Let \( f = (f_1, f_2, \cdots, f_m) \) be a map on \( \mathbb{R}^m \), and let \( x_0 \in \mathbb{R}^m \). The Jacobi matrix \( J_n = Df^n (r_0) \) at \( r_0 \) is defined by

\[
J_n = Df^n (r_0) = \begin{pmatrix}
\frac{\partial f_1^n}{\partial x_1} (r_0) & \vdots & \frac{\partial f_m^n}{\partial x_m} (r_0) \\
\vdots & \ddots & \vdots \\
\frac{\partial f_1^n}{\partial x_1} (r_0) & \vdots & \frac{\partial f_m^n}{\partial x_m} (r_0)
\end{pmatrix}.
\]

Then, the Lyapunov exponent \( \lambda_\mathcal{O} (f) \) of \( f \) for the orbit \( \mathcal{O} \equiv \{f^n (x_0); n = 0, 1, 2, \cdots \} \) is defined by

\[
\lambda_\mathcal{O} (f) = \log \tilde{\mu}, \quad \tilde{\mu}_k = \lim_{n \to \infty} (\mu_k^n)^{\frac{1}{n}} (k = 1, \cdots, m).
\]

Here, \( \mu_k^n \) is the \( k \)th largest square root of the \( m \) eigenvalues of the matrix \( J_n J_n^T \).

\[
\lambda_\mathcal{O} (f) > 0 \Rightarrow \text{Orbit } \mathcal{O} \text{ is chaotic.}
\]
\[
\lambda_\mathcal{O} (f) \leq 0 \Rightarrow \text{Orbit } \mathcal{O} \text{ is stable.}
\]

The properties of the logistic map depend on the parameter \( a \). If we take a particular constant \( a \), for example, \( a = 3.71 \), then the Lyapunov exponent and the entropic chaos degree are positive, the trajectory is very sensitive to the initial value and one has the chaotic behavior.

Fig.2. Chaos degree for logistic map
From the above example and some other maps (see the paper [11]), Lyapunov exponent and the entropic chaos degree have clear correspondence, but the ECD can resolve some inconvenient properties of the Lyapunov exponent as follows:

1. Lyapunov exponent takes negative value and sometimes $-\infty$, but the ECD is always positive for any $a \geq 0$.
2. It is difficult to compute the Lyapunov exponent for some maps like Tinkerbell map $f$ because it is difficult to compute $f^n$ for large $n$. On the other hand, the ECD of $f$ is easily computed.
3. Generally, the algorithm for the ECD is much easier than that for the Lyapunov exponent.

4 **New Description of Chaos**

First of all we examine carefully when we say that a certain dynamics produces a chaos. Let us take the logistic map as an example. The original differential equation of the logistic map is

$$\frac{dx}{dt} = ax(1 - x), \ 0 \leq a \leq 4 \quad (2)$$

with initial value $x_0$ in $[0, 1]$. This equation can be easily solved analytically, whose solution (orbit) does not have any chaotic behavior. However once we make the equation above discrete such as

$$x_{n+1} = ax_n(1 - x_n), \ 0 \leq a \leq 4. \quad (3)$$
This difference equation produces a chaos. Taking the discrete time is necessary not only to make a chaos but also to observe the orbits drawn by the dynamics. Similarly as quantum mechanics, it is not possible for human being to understand any object without observing it, for which it will not be possible to trace an orbit continuously in time.

Now let us think about finite partition \( A = \{ A_k; k = 1, \ldots, N \} \) of a proper set \( I \equiv [a, b]^N \subset \mathbb{R}^N \) and equi-partition \( B^e = \{ B^e_k; k = 1, \ldots, N \} \) of \( I \). Here "equi" means that all elements \( B^e_k \) are identical. We denote the set of all partitions by \( \mathcal{P} \) and the set of all equi-partitions by \( \mathcal{P}^e \). In the section 3, we specify a special partition, in particular, an equi-partition for computer experiment calculating the ECD. Such a partition enables to observe the orbit of a given dynamics, and moreover it provides a criterion for observing chaos. There exist several reports saying that one can observe chaos in nature, which are very much related to how one observes the phenomena, for instance, scale, direction, aspect. It has been difficult to find a satisfactory theory (mathematics) to explain such chaotic phenomena. In the difference equation () we take some time interval \( \tau \) between \( n \) and \( n+1 \), if we take \( \tau \to 0 \), then we have a complete different dynamics. If we take coarse graining to the orbit of \( x_t \) in () for time during \( \tau \); \( x_n = \frac{1}{\tau} \int_{[n\tau, (n+1)\tau)} x_t dt \), we again have a very different dynamics. Moreover it is important for mathematical consistency to take the limits \( n \to \infty \) or \( N \) (the number of equi-partitions)\( \to \infty \), i.e., making the partition finer and finer, and consider the limits of some quantities as describing chaos, so that mathematical terminologies such as "lim", "sup", "inf" are very often used to define such quantities. In this paper we take the opposite position, that is, any observation will be unrelated or even contradicted to such limits. Observation of chaos is a result due to taking suitable scales of, for example, time, distance or domain, and it will not be possible in the limiting cases.

We claim in this paper that most of chaos are scale-dependent phenomena, so the definition of a degree measuring chaos should dependes on certain scales taken.

Taking into cosideration of this view we modify the definitions of the chaos degree given in the previous sections as below.

Going back to a triple \( (\mathcal{A}, \mathcal{S}, \alpha (G)) \) considered in Section 2 and we use this triple both for an input and an output systems. Let a dynamics be described by a mapping \( \Gamma_t \) with a parameter \( t \in G \) from \( \mathcal{S} \) to \( \mathcal{S} \) and let an observation be described by a mapping \( \mathcal{O} \) from \( (\mathcal{A}, \mathcal{S}, \alpha (G)) \) to a triple \( (\mathcal{B}, \mathcal{T}, \beta (G)) \). The triple \( (\mathcal{B}, \mathcal{T}, \beta (G)) \) might be same as the original one or its subsystem and the observation map \( \mathcal{O} \) may contains several different types of observations, that is, it can be decomposed as \( \mathcal{O} = \mathcal{O}_m \cdots \mathcal{O}_1 \).Let us list some examples of observations.

For a given dynamics \( \frac{dx}{dt} = F(x) \), equivalently, \( \varphi_t = \Gamma^*_t \varphi \), one can take several observations.

**Example 5 Time Scaling (Discretizing): \( \mathcal{O}_\tau : t \to n, \frac{dx}{dt} (t) \to \varphi_{n+1} \), so that \( \frac{dx}{dt} = F(x) \Rightarrow \varphi_{n+1} = F(\varphi_t) \) and \( \varphi_t = \Gamma^*_t \varphi \Rightarrow \varphi_n = \Gamma^*_n \varphi \). Here \( \tau \) is a unit time needed for the observation.
Example 6 Size Scaling (Conditional Expectation, Partition): Let \((B, \mathcal{X}, \beta (G))\) be a subsystem of \((A, \mathcal{S}, \alpha (G))\), both of which have a certain algebraic structure such as \(C^*\)-algebra or von Neumann algebra. As an example, the subsystem \((B, \mathcal{X}, \beta (G))\) has abelian structure describing a macroscopic world which is a subsystem of a non-abelian (non-commutative) system \((A, \mathcal{S}, \alpha (G))\) describing a micro-world. A mapping \(O_C\) preserving norm (when it is properly defined) from \(A\) to \(B\) is, in some cases, called a conditional expectation. A typical example of this conditional expectation is according to a projection valued measure \(\{P_k; P_k P_j = P_k \delta_{kj} = P_k \delta_{kj} \geq 0, \; \sum_k P_k = I\}\) associated with quantum measurement (von Neumann measurement) such that \(O_C (\rho) = \sum_k P_k \rho P_k\) for any quantum state (density operator) \(\rho\). When \(B\) is a von Neumann algebra generated by \(\{P_k\}\), it is an abelian algebra isometrically isomorphic to \(L^\infty (\Omega)\) with a certain Hausdorff space \(\Omega\), so that in this case \(O_C\) sends a general state \(\varphi\) to a probability measure (or distribution) \(p\). Similar example of \(O_C\) is one coming from a certain representation (selection) of a state such as a Schatten decomposition of \(\rho: \rho = \mathcal{O}_R \rho = \sum_k \lambda_k E_k\) by one-dimensional orthogonal projections \(\{E_k\}\) associated to the eigenvalues of \(\rho\) with \(\sum_k E_k = I\).

Another important example of the size scaling is due to a finite partition of an underlying space \(\Omega\), e.g., space of orbit, defined as \(\mathcal{O}_P (\Omega) = \{P_k; P_k \cap P_j = P_k \delta_{kj}(k, j = 1, \cdots, N), \; \bigcup_{k=1}^N P_k = \Omega\}\).

We go back to the discussion of the entropic chaos degree. Starting from a given dynamics \(\varphi_t = \Gamma_t \varphi\), it becomes \(\varphi_n = \Gamma_n^* \varphi\) after handling the operation \(\mathcal{O}_T\). Then by taking proper combinations \(\mathcal{O}\) of the size scaling operations like \(\mathcal{O}_C, \mathcal{O}_R\) and \(\mathcal{O}_P\), the equation \(\varphi_n = \Gamma_n^* \varphi\) changes to \(\mathcal{O} (\varphi_n) = \mathcal{O} (\Gamma_n^* \varphi)\), which will be written by \(\mathcal{O} \varphi_n = \mathcal{O} \Gamma_n^* \mathcal{O}^{-1} \varphi\) or \(\varphi_n^\mathcal{O} = \Gamma_n^\mathcal{O} \varphi^\mathcal{O}\). Then our entropic chaos degree is redifined as follows:

**Definition 7** The entropic chaos degree of \(\Gamma^*\) with an initial state \(\varphi\) and observation \(\mathcal{O}\) is defined by \(D(\varphi; \Gamma^*) = \int_{\Omega (\mathcal{O})} S(\Gamma_n^* \omega^\mathcal{O}) d\mu^\mathcal{O}\), where \(\mu^\mathcal{O}\) is the measure operated by \(\mathcal{O}\) to a extremal decomposition measure of \(\varphi\) selected by \(\mathcal{O}\) of the observation \(\mathcal{O}\) (its part \(\mathcal{O}_R\)).

**Definition 8** The entropic chaos degree of \(\Gamma^*\) with an initial state \(\varphi\) is defined by \(D(\varphi; \Gamma^*) = \inf \{D(\varphi; \Gamma^*) ; \mathcal{O} \in \mathcal{S}O\}\), where \(\mathcal{S}O\) is a proper set of observations naturally determined by a given dynamics.

Then one judges whether a given dynamics causes a chaos or not by the following way:

**Definition 9** (1) A dynamics \(\Gamma^*\) is chaotic for an initial state \(\varphi\) in an observation \(\mathcal{O}\) iff \(D(\varphi; \Gamma^*) > 0\). (2) A dynamics \(\Gamma^*\) is totally chaotic for an initial state \(\varphi\) iff \(D(\varphi; \Gamma^*) > 0\).

In Definition, \(\mathcal{S}O\) is determined by a given dynamics and some conditions attached to the dynamics, for instance, if we start from a difference equation with a special representation of an initial state, then \(\mathcal{S}O\) excludes \(\mathcal{O}_T\) and \(\mathcal{O}_R\).
The idea introduced in this paper to understand chaos can be applied not only to the entropic chaos degree but also to some other degrees such as dynamical entropy, whose applications and the comparison of several degrees will be discussed in the forthcoming paper.

In the case of logistic map, \( x_{n+1} = ax_n(1 - x_n) \equiv F(x_n) \), we obtain this difference equation by taking the observation \( O_{\tau} \) and take an observation \( O_P \) by equi-partition of the orbit space \( \Omega = \{x_n\} \) so as to define a state (probability distribution). Thus we can compute the entropic chaos degree as is discussed in Section 3.

It is important to notice here that the chaos degree does depend on the choice of observations. As an example, we consider a circle map

\[ \theta_{n+1} = f_\nu(\theta_n) = \theta_n + \omega \pmod{2\pi}, \tag{4} \]

where \( \omega = 2\pi \nu (0 < \nu < 1) \). If \( \nu \) is a rational number \( N/M \), then the orbit \( \{\theta_n\} \) is periodic with the period \( M \). If \( \nu \) is irrational, then the orbit \( \{\theta_n\} \) densely fills the unit circle for any initial value \( \theta_0 \); namely, it is a quasiperiodic motion.

We proved in \([10]\) the following theorem.

**Theorem 10** Let \( I = [0, 2\pi] \) be partitioned into \( L \) disjoint components with equal length; \( I = B_1 \cap B_2 \cap \ldots \cap B_L \).

1. If \( \nu \) is rational number \( N/M \), then the finite equi-partition \( P = \{B_k; k = 1, \ldots, M\} \) implies \( D_{\Omega}(\theta_0; f_\nu) = 0 \).

2. If \( \nu \) is irrational, then \( D_{\Omega}(\theta_0; f_\nu) > 0 \) for any finite partition \( P = \{B_k\} \).

Note that our entropic chaos degree shows a chaos to quasiperiodic circle dynamics by the observation due to a partition of the orbit, which is different from usual understanding of chaos. However usual belief that quasiperiodic circle dynamics will not cause a chaos is not at all obvious, but is realized in a special limiting case as shown in the following proposition.

**Proposition 11** For the above circle map, if \( \nu \) is irrational, then \( D(\theta_0; f_\nu) = 0 \).

**Proof.** Let take an equipartition \( P = \{B_k\} \) as

\[ B_k \equiv \left\{ x; 2\pi \frac{k - 1}{l} \leq x < 2\pi \frac{k}{l} \right\}, \quad k = 1, 2, \ldots, l, \]

where \( l \) is a certain integer and \( B_{k+l} = B_k \). When \( \nu \) is irrational, put \( \nu_0 \equiv |l\nu| \) with Gaussian \(|\cdot|\) . Then \( f_\nu(B_k) \) intersects only two intervals \( B_{k+\nu_0} \) and \( B_{k+\nu_0+1} \), so that denote the ration of the Lebesgue measure of \( f_\nu(B_k) \cap B_{k+\nu_0} \) and that of \( f_\nu(B_k) \cap B_{k+\nu_0+1} \) by \( 1 - s : s \). This \( s \) is equal to \( l\nu - |l\nu| \) and the entropic chaos degree becomes

\[ D_P = -s \log s - (1 - s) \log (1 - s). \]
Take the continued fraction expansion of $\nu$ and denote its $j$-th approximate by $b_j \frac{c_j}{c_j}$. Then it holds

$$\left| \nu - \frac{b_j}{c_j} \right| \leq \frac{1}{c^2_j}.$$ 

For the above equi-partition $B = \{B_k\}$ with $l = c_j$, we find

$$|l\nu - b_j| \leq \frac{1}{k}$$ 

and

$$[l\nu] = \begin{cases} 
b_j & \text{(when } l\nu - b_j > 0) \\
b_{j-1} & \text{(when } l\nu - b_j < 0) \end{cases}.$$ 

It implies

$$D^P := \frac{\log c_j}{c_j},$$

which goes to 0 as $j \to \infty$. Hence $D = \inf \{D^P; P\} = 0$. Such a limiting case will not take place in real observation of natural objects, so that we claim that chaos is a phenomenon depending on observations, which results the definition of chaos above.

In the forthcoming paper [24], we will discuss how to reach to chaos dynamics by starting from general differential dynamics in both classical and quantum systems. That is, it is demonstrated how we can get to chaos dynamics by considering observations introduced in this paper, and we calculate the entropic chaos degrees in each dynamics.

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