LARGE DEVIATIONS IN THE SUPREMUM NORM
FOR A REACTION-DIFFUSION SYSTEM

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ABSTRACT. We present large deviations estimates in the supremum norm for a system of independent random walks superposed with a birth-and-death dynamics evolving on the discrete torus with \( N \) sites. The scaling limit considered is the so-called high density limit (see the survey [9] on the subject), where space, time and initial quantity of particles are rescaled. The associated rate functional here obtained is a semi-linearised version of the rate function of [13], which dealt with large deviations of exclusion processes superposed with birth-and-death dynamics. An ingredient in the proof of large deviations consists in providing a limit of a suitable class of perturbations of the original process. This is precisely one of the main contributions of this work: a strategy to extend the original high density approach (as in [1, 4, 5, 10, 15, 16]) to weakly asymmetric systems. Two cases are considered with respect to the initial quantity of particles, the power law and the (at least) exponential growth. In the first case, we present the lower bound only on a certain subset of smooth profiles, while in the second case, additionally assuming concavity of the birth and the death functions and a constant initial profile, we provide a full large deviations principle.

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1. INTRODUCTION

Since the early works of Dobrushin (as [7]) and the seminal paper of Guo, Papanicolau and Varadhan [11], an entire theory on scaling limits of interacting particle systems has been established, see the reference book [14]. Such a subject has its great importance in the context of statistical mechanics, in understanding the behaviour of macroscopic systems by
means of its microscopic interactions, but has also many connections with partial differential equations, probability theory and even combinatorics (see [19]).

At same epoch the hydrodynamic limit (see [14] on the subject) started to be developed, some works were published in a close topic sometimes called high density limit, also in the context of scaling limit of interacting particle systems, as [1, 4, 5, 15, 16] for instance. The main difference between the hydrodynamic limit and the high density limit can be resumed as follows: while in hydrodynamic limit space and time are rescaled in order to obtain a macroscopic limit, in the high density limit, space, time and the initial quantity of particles per site are rescaled, see the survey [9] for a discussion about. Of course, each context requires a different topology. Whilst the hydrodynamic limit usually deals with convergence of measures, Schwartz distributions and Sobolev norms, the high density limit deals with Sobolev norms, but also allows the supremum norm, see [5].

In opposition to the hydrodynamic limit, which has been continuously studied since its beginning, the high density limit felt in disuse for many years. Its was probably due to the following reason: the powerful Varadhan’s Entropy Method allowed the study of systems of non-linear diffusion, while the high density limit approach was restricted to systems of linear diffusion. Basically, independent random walks superposed with some additional dynamics, as the birth-and-death dynamics, for example. Actually, the high density approach is heavily based on the smoothing properties of the discrete heat kernel, which explains this restriction to independent random walks.

On the other hand, despite its symmetric nature, the high density limit offers some particular perspectives, which would be difficult to be followed in the hydrodynamic setting. For example, in [10], it was considered a system exhibiting explosion in finite time. Since the hydrodynamic limit techniques are mainly based on averages, the system of [10] would be a hard topic to be analysed in the hydrodynamic point of view since there is no finite expectation of standard observables. In the intersection, some recent works also rescale the initial quantity of particles per site, which may be interpreted as a kind of high density limit, as [12] for example.

The main result we present here is a large deviations principle for the law of large numbers of [5], which consists in the high density limit in the supremum norm for a system of independent random walks on the discrete torus superposed with a birth and death dynamics. Actually, following some observations of [10], weakening some assumptions on the birth and death rates, we consider a slightly more general system than that in [5], but we may say that the model we consider is essentially that one of [5]. As usual in large deviations, an important ingredient of the proof is a law of large numbers for a class of perturbations of the original model, which is an interesting result by itself. Since the high density limit was originally designated for systems of symmetric diffusion (independent random walks superposed with some extra dynamics), we can say that the more challenging step in our proof is to reach the law of large numbers for the perturbed processes, which are weakly asymmetric systems. Following some remarks from [10] we were also able to assure that the law of large numbers for the perturbed processes takes place in the almost sure sense, which is an important feature.

The rate function we obtain in the large deviations is a spatially linearised version of the rate function of [13], which dealt with large deviations of a superposition of Glauber and Kawasaki dynamics. This fact is quite reasonable since, in some sense, a system of independent random walks is a linearisation of the Glauber dynamics and the Kawasaki dynamics is a birth-and-death dynamics. However, this resemblance is limited to this observation: since [13] works on the hydrodynamic limit while we deal with the high density limit, the technical challenges we face here are very distinct of those in [13].

Due to the strong topological nature of the supremum norm and the obtained almost sure convergence, some usual difficulties when proving large deviations for the hydrodynamic point of view do not appear in this setting, considerably simplifying the upper and lower

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As well as some other methods, as the Yau’s Relative Entropy Method, see [14].
bound arguments, except when achieving the exponential tightness, which demanded some extra effort. For example, no superexponential replacement lemmas are required here. On the other hand, as aforementioned, the convergence of the perturbed processes, which is in general a standard procedure in the hydrodynamic limit (for the exclusion process for instance, see [14, Chapter 10]), here is an obstacle to be overcome.

Apart of the result itself, which is relevant due the broad occurrence of reaction-diffusion partial differential equations and the importance of the supremum norm for simulations, the main novel of the present work consists in providing a strategy to extend the original high density approach (as in [1, 4, 5, 10, 15, 16]), originally developed to systems of symmetric diffusion, to spatially weakly asymmetric systems. Before explaining our strategy for weakly asymmetric systems, let us hand-waving resume the way in [5] of proving the high density limit.

The first ingredient is to show that the solution of a spatially discretized version of the limiting PDE is actually close to that PDE. Next, we must study the martingales associated to the projection at each site. Due to the scale setting of parameters, in opposition to the Entropy Method, showing that the quadratic variation of those martingales vanish does not suffice to lead to the convergence in the supremum norm. From these martingales and the presence of the discrete Laplacian, we obtain integral equations via the Duhamel's Principle, which involve the heat semigroup instead of the Laplacian operator. Then, by providing some suitable estimates on the random term of these equations and recalling smoothing properties of the heat semi-group allows to get the desired convergence in the supremum norm.

For our work we use this same process to get the high density limit for the for weakly asymmetric systems, however, as has been said, the asymmetry in the system causes some difficulties. Having the high density limit for a class of perturbed processes we proceed with the large deviations principle. Before we need to find the expression for the Radon-Nikodym derivative between the original process and the perturbed process. Knowing the existence of the Radon-Nikodym derivative we can prove the large deviations upper bound, here arises the need to show that the sequence of measures of process is exponentially tight. For the lower bound, we separated in two cases. First, we consider that profiles, which are a solution of the differential equation considering the perturbed process, are smooth functions. Finally, we will consider more general profiles but include additional assumptions on the process birth and death rates and about the parameter that indicates the initial average number of particles.

Among open problems which may be considered in future works we may cite: to extend the present result to higher dimensions; to deal with the fluctuations of the system (central limit theorem); in the case the total quantity of particles does not explode, to study the quasi-potential and macroscopic fluctuation theory, see [2] on the subject. The first question is a matter of technicality and mutatis mutandis all arguments here should remain in force. Fluctuations on high density scaling have been addressed before, see [4] for example, but not in the weakly asymmetric version. Finally, the last cited open problem seems to be a challenging and interesting subject to be faced.

The paper’s outline goes as follows. In Section 2, we define the model and state results. In Section 3, we prove the high density limit for the weakly asymmetric perturbation of the original process. In Section 4, we provide the proof of large deviations estimates.

\section*{2. Statements}

\textbf{Notations:} by $g = O(f)$ we mean that the function $g$ is bounded in modulus by a constant times the function $f$, where the constant may change from line to line. The spatial first and second derivates on space will be denoted by $\nabla$ and $\Delta$. However, we sometimes also write $\partial_x$ and $\partial_{xx}$ instead of $\nabla$ and $\Delta$ to better differentiate it of discrete derivatives to be later defined. By $\mathbb{R}_+$ we will mean the set of non-negative real numbers. By $C^{i,j}$ we denote the set of functions which are $C^i$ in the time variable and $C^j$ in the spatial variable.
2.1. The model. Denote by $T_N = \mathbb{Z}/(N\mathbb{Z})$ the discrete torus with $N$ sites and by $T$ denote the continuous torus $\mathbb{R}/\mathbb{Z} = [0, 1)$, where the point 0 is identified with the point 1. Let $b, d : \mathbb{R}_+ \to \mathbb{R}_+$ be two Lipschitz functions such that $d(0) = 0$ and let $\ell = \ell(N)$ be a positive integer parameter. We denote by $(\eta(t))_{t \geq 0}$

the continuous-time Markov chain with state space $\Omega_N = \mathbb{N}^{T_N}$, where $\eta_k(t)$ means the quantity of particles at the site $k$ at the time $t$. Its jump rates are taken as:

- At rate $N^2 \eta_k$, a particle jumps from the site $k$ to the site $k + 1$.
- At rate $N^2 \eta_k$, a particle jumps from the site $k$ to the site $k - 1$.
- At rate $\ell b(\ell^{-1} \eta_k)$, a new particle is created at the site $k$.
- At rate $\ell d(\ell^{-1} \eta_k)$, if $\eta_k \geq 1$, a particle is destroyed at the site $k$.

A time-horizon $T > 0$ will be fixed throughout the paper. Let $\mathcal{D}([0, T], \Omega_N)$ be the path space of càdlàg time trajectories taking values on $\Omega_N$. For short, we will denote this space just by $\mathcal{D}_{\Omega_N}$. Given a measure $\mu_N$ on $\Omega_N$, denote by $\mathbb{P}_N$ the probability measure on $\mathcal{D}_{\Omega_N}$ induced by the initial state $\mu_N$ and the Markov process $\{\eta(t) : t \geq 0\}$. Expectation with respect to $\mathbb{P}_N$ will be denoted by $\mathbb{E}_N$.

The object we are interested in this paper is the spatial density $X^N : T \to \mathbb{R}$ of particles, defined as follows. Keep in mind that $T_N$ is naturally embedded on $T$, and denote $x_k = k/N$ for $k \in T_N$. Let

$$X^N(t, x_k) = \ell^{-1} \eta_k(t)$$

and, for $x_k < x < x_{k+1}$, define $X^N(t, x)$ by means of a linear interpolation, i.e.,

$$X^N(t, x) = (Nx - k)X^N(t, x_{k+1}) + (k + 1 - Nx)X^N(t, x_k).$$

In [5, 10] it was proved the following law of large numbers for the density of particles.

**Theorem 2.1** ([5, 10]). Let $\phi(t, x)$ be the solution of the following initial value problem:

$$
\begin{cases}
\partial_t \phi = \Delta \phi + f(\phi) & (t, x) \in [0, T] \times \mathbb{T}, \\
\phi(0, x) = \gamma(x) \geq 0 & x \in \mathbb{T}.
\end{cases}
$$

Let $b, d : \mathbb{R}_+ \to \mathbb{R}_+$ be Lipschitz $C^1$-functions such that $d(0) = 0$ and $f = b - d$, and let $\gamma : \mathbb{T} \to \mathbb{R}_+$ be a $C^1$ profile. Assume that:

1. $\|X^N(\cdot, 0) - \gamma(\cdot)\|_\infty \to 0$ almost surely as $N \to \infty$,
2. for any $c > 0$, $\ell = \ell(N)$ satisfies $\sum_{N \geq 0} N^3 e^{-c \ell} < \infty$.

Then, for any $T > 0$,

$$\lim_{N \to \infty} \sup_{t \in [0, T]} \|X^N(t, \cdot) - \phi(t, \cdot)\|_\infty = 0 \quad \text{almost surely.}$$

Assumption (1) above and (2.1) allow us to interpret the parameter $\ell$ as the order of particles per site, from where comes the terminology high density limit (see [16]). In contrast with the hydrodynamic limit (see [14]), where only time and space are rescaled, here time, space and the initial quantity of particles per site are rescaled, which permits convergence in the supremum norm.

Some comments: although Theorem 2.1 cannot be found in this exact way in any of the papers [5, 10], it can be deduced from both references together. Since this statement is also a particular case of our Theorem 2.2 to be enunciated ahead, we do not go further into details. Moreover, the Lipschitz assumption on the function $b$ assures growth at most linear, thus preventing the occurrence of explosions in finite time for both microscopic and macroscopic settings. See [10] on the subject of explosions for this kind of reaction-diffusion system.
2.2. High density limit for weakly asymmetric perturbations. In the proof of large deviations estimates, a law of large numbers for a class of perturbations of the original process is naturally required, which is an interesting result by itself. For the reaction-diffusion model we study here, the perturbed process will be the following one, which is inspired by deviations estimates, a law of large numbers for a class of perturbations of the original process.

Given $H \in C^{1,2}$, we define the continuous-time Markov chain $(\eta(t))_{t \geq 0}$ with state space $\Omega_N = \mathbb{N}^{2N}$ by

$$(\eta(t))_{t \geq 0} = (\eta_1(t), \ldots, \eta_N(t))_{t \geq 0},$$

where $\eta_k(t)$ means the quantity of particles at site $k$ at time $t$ as before, and the jump rates of the process are given by:

- a particle jumps from $k$ for $k + 1$ at rate $N^2 \eta_k \exp \left\{ H(t, \frac{k+1}{N}) - H(t, \frac{k}{N}) \right\}$,
- a particle jumps from $k$ for $k - 1$ at rate $N^2 \eta_k \exp \left\{ H(t, \frac{k-1}{N}) - H(t, \frac{k}{N}) \right\}$,
- a new particle is created at site $k$ at rate $\ell b(\ell^{-1} \eta_k) \exp \left\{ H(t, \frac{k}{N}) \right\}$,
- a particle is destroyed at site $k$ at rate $\ell d(\ell^{-1} \eta_k) \exp \left\{ -H(t, \frac{k}{N}) \right\}$, if $\eta_k \geq 1$.

Note that this time inhomogeneous Markov chain actually depends on $H$. However, to not overload notation, this dependence will be dropped. Given a measure $\mu_N$ on $\Omega_N$, denote by $\mathbb{P}_N^H$ the probability measure on $\mathcal{P}_{\Omega_N}$ induced by the initial state $\mu_N$ and the Markov process $\{\eta(t) : t \geq 0\}$ above. Expectation with respect to $\mathbb{P}_N^H$ will be denoted by $\mathbb{E}_N^H$.

Let $\psi : [0, T] \times \mathbb{T} \to \mathbb{R}$ be the solution of the following initial value problem:

$$
\begin{align*}
\partial_t \psi &= \partial_{xx} \psi - 2 \partial_x (\psi \partial_x H) + e^H b(\psi) - e^{-H} d(\psi), \quad (t, x) \in [0, T] \times \mathbb{T}, \\
\psi(0, x) &= \gamma(x), \quad x \in \mathbb{T}.
\end{align*}
$$

Assuming that $H \in C^{1,2}$, $b, d \in C^1$ and $\gamma$ is Holder continuous in $\mathbb{T}$, there exists a unique classical solution of the initial value problem (2.4), which we denote by $\psi$, see [18, Chapter II, Section 2.3]. We point out that the partial differential equation above can be understood as a linearized version of the partial differential equation in [13, (2.11)].

Next, we state the high density limit for the perturbed process. As before, $X^N(t) = X^N(t, x)$ is equal to $\eta_k(t)/\ell$ for $x = k/N$ and linearly interpolated otherwise. Of course, this process depends on $H$, whose dependence is omitted.

**Theorem 2.2** (High density limit for perturbed processes). Let $b, d : \mathbb{R}_+ \to \mathbb{R}_+$ be Lipschitz $C^1$ functions with $d(0) = 0$, let $H \in C^{1,2}$ and let $\gamma : \mathbb{T} \to \mathbb{R}_+$ be a $C^4$ profile. Assume the following conditions:

(A1) The sequence of initial measures $\mu_N$ is such that

$$\|X^N(0, \cdot) - \gamma(\cdot)\|_\infty \to 0, \quad \text{almost surely as } N \to \infty. \quad (2.5)$$

(A2) The parameter $\ell = \ell(N)$ satisfies

$$\frac{N\|\partial_x H\|_\infty^2}{\ell^2 \log N} \to 0, \quad \text{as } N \to \infty. \quad (2.6)$$

Then,

$$\lim_{N \to \infty} \sup_{t \in [0, T]} \|X^N(t, \cdot) - \psi(t, \cdot)\|_\infty = 0, \quad \text{almost surely as } N \to \infty,$$

where $\psi$ is the solution of (2.4).

**Remark 2.3.** There are no further hypothesis on the sequence of initial measures $\mu_N$ aside of (2.5). As an example of a sequence of initial measures, one may consider $\mu_N$ as a product measure of Poisson distributions whose parameter at the site $x \in \mathbb{T}$ is given by $\ell \gamma(x)/N$. However, since we are interested in dynamical large deviations, throughout the paper we assume that $\mu_N$ is a deterministic sequence, that is, each $\mu_N$ is a delta of Dirac on some configuration. This avoids the analysis of static large deviations.
Remark 2.4. Let us discuss the meaning of (A2). Taking \( \ell(N) = N^\alpha \) with \( \alpha > 0 \), condition (2.6) holds once \( \| \partial_s H \|_\infty < \pi \sqrt{\alpha} \). This may look weird at a first glance, but it is not completely unexpected. The role of \( H \) is to introduce an asymmetry in the system. Since the density limit approach is heavily founded on the smoothing properties of the discrete heat kernel (which is associated to the symmetric random walk), it is somewhat reasonable to have a competition between the growth speed of \( \ell(N) \) and the strength of the function \( H \). On the other hand, under the hypothesis \( \ell = \ell(N) \geq e^{CN} \) for some constant \( c \), the high density limit holds for any perturbation \( H \in C^{1,2} \).

2.3. Large deviations. We state in the sequel a large deviations principle associated to the law of large numbers of Theorem 2.1. Denote by \( C(\mathbb{T}) \) the Banach space of continuous functions \( H : \mathbb{T} \rightarrow \mathbb{R} \) under the supremum norm \( \| \cdot \|_\infty \). Denote by \( C^{1,2} = C^{1,2}([0, T] \times \mathbb{T}) \) the set of functions \( H : [0, T] \times \mathbb{T} \rightarrow \mathbb{R} \) such that \( H \) is \( C^1 \) in time and \( C^2 \) in space. Let \( \mathcal{D}_{C(\mathbb{T})} = \mathcal{D}( [0, T], C(\mathbb{T}) ) \) be the Skorohod space of càdlàg trajectories taking values on \( C(\mathbb{T}) \).

Define the functional \( J_H : \mathcal{D}_{C(\mathbb{T})} \rightarrow \mathbb{R} \) by

\[
J_H(u) = \int_0^T \left[ H(t, x)u(t, x) - H(0, x)u(0, x) \right] dx + \int_0^T \int_0^t \left[ -u(s, x)(\partial_s H(s, x) + \Delta H(s, x) + (\nabla H(s, x))^2) \right. \\
+ \left. b(u(s, x))(1 - e^{H(s, x)}) + d(u(s, x))(1 - e^{-H(s, x)}) \right] ds \quad (2.7)
\]

Recalling that \( \gamma : \mathbb{T} \rightarrow \mathbb{R}_+ \) is the non-negative \( C^4 \) function which appears in Theorem 2.1 and Theorem 2.2, let \( I : \mathcal{D}_{C(\mathbb{T})} \rightarrow [0, +\infty) \) be given by

\[
I(u) = \begin{cases} 
\sup_{H \in C^{1,2}} J_H(u), & \text{if } u(0, \cdot) = \gamma(\cdot), \\
+\infty, & \text{otherwise}.
\end{cases}
\]

Definition 2.5. Denote by \( \mathcal{D}_{\alpha} \subseteq \mathcal{D}_{C(\mathbb{T})} \) the set of all profiles \( \psi : [0, T] \times \mathbb{T} \rightarrow \mathbb{R} \) satisfying:

- \( \psi(0, \cdot) = \gamma(\cdot) \),
- \( \psi \in C^{2,3} \),
- \( \psi \geq \varepsilon \) for some \( \varepsilon > 0 \),
- there exists a function \( H \in C^{1,2} \), with \( \| \partial_s H \|_\infty \leq \pi \sqrt{\alpha} \), such that \( \psi \) is the solution of (2.4).

We are in position now to state the main result of this paper. Let \( P_N \) be the probability measure on the set \( \mathcal{D}_{C(\mathbb{T})} \) induced by the stochastic process \( X^N(t) \) defined by (2.1) and (2.2).

Theorem 2.6. Under the hypothesis of Theorem 2.1, additionally assume that \( X^N(0, \cdot) \) is a deterministic profile for each \( N \in \mathbb{N} \). Let \( \ell = \ell(N) = N^\alpha \) for some fixed \( \alpha > 0 \). Then:

1) For every closed set \( C \subseteq \mathcal{D}_{C(\mathbb{T})} \),

\[
\limsup_{N \to \infty} \frac{1}{\ell_N} \log P_N(C) \leq -\inf_{u \in \mathcal{C}} I(u).
\]

2) For every open set \( \mathcal{O} \subseteq \mathcal{D}_{C(\mathbb{T})} \),

\[
\liminf_{N \to \infty} \frac{1}{\ell_N} \log P_N(\mathcal{O}) \geq -\inf_{u \in \mathcal{O} \cap \mathcal{D}_{\alpha}} I(u).
\]

We note that the assumption that the initial conditions are deterministic prevents the occurrence of large deviations from the initial profile, also known as static large deviations. Our main interest here are the dynamical large deviations, that is, the large deviations coming from the dynamics. Moreover, the lower bound holds only over sets intersects with \( \mathcal{D}_{\alpha} \), which has no explicit representation. On the other hand, in the case \( \ell = \ell(N) \) grows at least exponentially together with some technical assumptions, we were able to describe the full picture of large deviations.
Theorem 2.7. Assume the hypothesis of Theorem 2.1 and additionally assume that $X^N(0, \cdot)$ are deterministic profiles for each $N \in \mathbb{N}$, that $b, d$ are concave functions and $\gamma(\cdot)$ is a positive constant profile. Let $\ell = \ell(N) \geq cN$ for some constant $c > 0$. Then:
1) For every closed set $C \subseteq \mathcal{D}_C(T)$,
\[ \limsup_{N \to \infty} \frac{1}{\ell N} \log P_N(C) \leq - \inf_{u \in C} I(u). \]
2) For every open set $O \subseteq \mathcal{D}_C(T)$,
\[ \liminf_{N \to \infty} \frac{1}{\ell N} \log P_N(O) \geq - \inf_{u \in O} I(u). \]

Remark 2.8. The above hypothesis that $b, d$ are concave functions has been assumed in some related works as [6, 13, 17]. On the other hand, the assumption that the initial profile $\gamma(\cdot)$ is a constant is somewhat an ad hoc assumption.

3. HIGH DENSITY LIMIT FOR THE PERTURBED PROCESS

3.1. Semi-discrete scheme. The proof of Theorem 2.2 is done in two steps. First, we prove that the solution $\psi$ of the initial value problem (2.4) is close to the solution of some suitable spatial discretization $\psi^N$. Then, we prove that the (deterministic) solution of that spatial discretization $\psi^N$ is close to the random density of particles defined by $X^N(t)$. This subsection deals with the convergence of the just mentioned spatial discretization. Since the time variable is kept continuous, we call such discrete approximation of a semidiscrete approximation. For short, denote $x_k = k/N$, $\psi_k = \psi(t, x_k)$, $H_k = H(t, x_k)$, $\partial^2_x H_k = (\partial^2_x H)(t, x_k)$, and by $S^N_{\pm 1}$ denote the shifts of $\pm N^{-1}$. That is,
\[ S^N_1 f(s, \frac{k}{N}) = f(s, \frac{k+1}{N}), \quad S^N_{-1} f(s, \frac{k}{N}) = f(s, \frac{k-1}{N}). \]

We define the semidiscrete approximation $\psi^N(t) = (\psi^N_1(t), \ldots, \psi^N_T(t))$ of the initial value problem (2.4) as the solution of the following system of ODE’s:
\[
\begin{cases}
\frac{d}{dt} \psi^N_k &= N^2 \left( \psi^N_{k+1} - 2 \psi^N_k + \psi^N_{k-1} \right) - \partial_x H_k \cdot N \left( \psi^N_{k+1} - \psi^N_{k-1} \right) - \partial^2_x H_k \cdot \frac{1}{2} \left( S^N_1 + S^N_{-1} + 2 \right) \psi^N_k \\
&\quad + e^{H_k} b(\psi^N_k) - e^{-H_k} d(\psi^N_k), \quad k \in \mathbb{T}_N, \\
\psi^N(0) &= \gamma \left( \frac{k}{N} \right), \quad k \in \mathbb{T}_N. 
\end{cases}
\] (3.1)

At a first glance, one may think that this semidiscrete scheme is not a correct one in order to approximate (2.4). Noting that the difference $N \left( \psi^N_{k+1} - \psi^N_{k-1} \right)$ on the above should heuristically approximate twice the derivative $\partial_x \psi$ together with the equality $-2 \partial_x \left( \psi \partial_x H \right) = -2 \partial_x \psi \partial_x H + 2 \psi \partial^2_x H$ should dismiss any doubt.

Denote by $\| \cdot \|_L$ the Lipschitz constant of a given function.

Proposition 3.1. Let be $\psi$ be the solution of initial value problem (2.4) and let $\psi^N$ be the solution of semidiscrete approximation (3.1). Then, for $N \geq \| \partial_x H \|_\infty + 1$,
\[ \sup_{t \in [0,T]} \max_{k \in \mathbb{N}} \left| \psi^N_k - \psi_k \right| \leq \frac{\exp \left\{ (3C_\ast + 1)T \right\}}{N}. \]
where
\[ C_\ast = \max \left\{ \| e^H \|_\infty \cdot \| b \|_L, \| e^{-H} \|_\infty \cdot \| d \|_L, \| \partial_x H \|_\infty, \| \partial^2_x H \|_\infty, \| \partial_x \psi \|_\infty \right\}. \] (3.2)

To prove the result above we will need the next auxiliary lemma about the following system of ordinary differential equations on the time interval $[0, T]$:
\[
\begin{cases}
\frac{d}{dt} \varphi_k &= N^2 \left( \varphi_{k+1} - 2 \varphi_k + \varphi_{k-1} \right) - N \left( \varphi_{k+1} - \varphi_{k-1} \right) \partial_x H_k \\
&\quad + C_\ast \left( \varphi_{k+1} + |\varphi_k| + \varphi_{k-1} + N^{-1} \right), \\
\varphi_k(0) &= 0, \quad k \in \mathbb{T}_N.
\end{cases}
\] (3.3)
We say that $\overline{\varphi} = (\overline{\varphi}_1, \ldots, \overline{\varphi}_n)$ is a supersolution of (3.3) if

$$
\begin{align*}
\frac{d}{dt} \overline{\varphi}_k & \geq N^2 (\overline{\varphi}_{k+1} - 2\overline{\varphi}_k + \overline{\varphi}_{k-1}) - N(\overline{\varphi}_{k+1} - \overline{\varphi}_{k-1}) \partial_x H_k \\
& \quad + C_\ast (\overline{\varphi}_{k+1} + |\varphi_k| + \varphi_{k-1} + N^{-1}), \\
\overline{\varphi}_k(0) & \geq 0, \ k \in T_N,
\end{align*}
$$

(3.4)

and we say that $\underline{\varphi} = (\underline{\varphi}_1, \ldots, \underline{\varphi}_n)$ is a subsolution of (3.3) if

$$
\begin{align*}
\frac{d}{dt} \underline{\varphi}_k & \leq N^2 (\underline{\varphi}_{k+1} - 2\underline{\varphi}_k + \underline{\varphi}_{k-1}) - N(\underline{\varphi}_{k+1} - \underline{\varphi}_{k-1}) \partial_x H_k \\
& \quad + C_\ast (\underline{\varphi}_{k+1} + |\varphi_k| + \varphi_{k-1} + N^{-1}), \\
\underline{\varphi}_k(0) & \leq 0, \ k \in T_N,
\end{align*}
$$

where the function $\psi$ is the solution of (2.4).

**Lemma 3.2 (Principle of sub and supersolutions).** Let $\overline{\varphi} , \underline{\varphi} , \varphi$ be a supersolution, a subsolution and a solution of (3.3), respectively. Then, for $N \geq N_0 = \|\partial_x H\|_{\infty} + 1$,

$$
\overline{\varphi}_k(t) \geq \varphi_k(t) \geq \underline{\varphi}_k(t),
$$

(3.5)

for any $k \in T_N$ and any $t \in [0, T]$.

**Proof.** We will prove only that $\overline{\varphi} \geq \varphi$, being the second inequality analogous.

We claim that it is enough to prove that, assuming strict inequalities in (3.4), it would imply $\overline{\varphi} > \varphi$. In fact, assume that $\overline{\varphi}$ is a supersolution, that is, it satisfies (3.4) and define $\zeta(t) = \overline{\varphi}(t) + \varepsilon t$. Hence,

$$
\begin{align*}
\frac{d}{dt} \zeta & = \frac{d}{dt} \overline{\varphi} + \varepsilon \geq N^2 (\overline{\varphi}_{k+1} - 2\overline{\varphi}_k + \overline{\varphi}_{k-1}) - N(\overline{\varphi}_{k+1} - \overline{\varphi}_{k-1}) \partial_x H_k \\
& \quad + C_\ast (\overline{\varphi}_{k+1} + |\varphi_k| + \varphi_{k-1} + N^{-1}) + \varepsilon \\
& \geq N^2 (\zeta_{k+1} - 2\zeta_k + \zeta_{k-1}) - N(\zeta_{k+1} - \zeta_{k-1}) \partial_x H_k \\
& \quad + C_\ast (\zeta_{k+1} + |\zeta_k| + \zeta_{k-1} + N^{-1}) - 3C_\ast t \varepsilon + \varepsilon.
\end{align*}
$$

Therefore, $\zeta$ is a (strict) supersolution once $-3C_\ast t \varepsilon + \varepsilon > 0$ or, equivalently, if $t < 1/(3C_\ast)$. Partitioning the time interval $[0, T]$ into a finite number of intervals of length strictly smaller than $1/(3C_\ast)$ allows us to conclude that $\zeta$ is a strict supersolution in the time interval $[0, T]$. Hence $\zeta > \varphi$ and since $\varepsilon > 0$ is arbitrary, we get $\overline{\varphi} \geq \varphi$. This concludes the proof of the claim.

In view of the previous claim, assume now that $\overline{\varphi}$ is a strictly supersolution (that is, satisfies (3.4) with strict inequalities). Let us prove now that it implies the first (strict) inequality in (3.5).

Suppose by contradiction that there is a first time $t_\ast > 0$ and a site $k \in T_N$ such that:

- $\overline{\varphi}_k(t_\ast) = \varphi_k(t_\ast)$.
- For any $t < t_\ast$ and any $j \in \mathbb{T}_N$, $\overline{\varphi}_j(t) > \varphi_j(t)$.

Note that the last item above implies $\overline{\varphi}_j(t^*) \geq \varphi_j(t^*)$ for $j \neq k$. We thus have

$$
0 \geq \frac{d}{dt} \overline{\varphi}_k(t^*) - \frac{d}{dt} \varphi_k(t^*) \\
> N^2 (\overline{\varphi}_{k+1}(t^*) - \varphi_{k+1}(t^*) + \overline{\varphi}_{k-1}(t^*) - \varphi_{k-1}(t^*)) \\
- N (\overline{\varphi}_{k+1}(t^*) - \varphi_{k+1}(t^*) - \overline{\varphi}_{k-1}(t^*) + \varphi_{k-1}(t^*)) \partial_x H_k \\
+ C_\ast (\overline{\varphi}_{k+1} - \varphi_{k+1} + \overline{\varphi}_{k-1} - \varphi_{k-1}) \\
\geq (N^2 - N \partial_x H_k) (\overline{\varphi}_{k+1}(t^*) - \varphi_{k+1}(t^*) + \overline{\varphi}_{k-1}(t^*) - \varphi_{k-1}(t^*)).
$$

(3.6)

Note that (3.6) is greater than zero for $N \geq \|\partial_x H\|_{\infty} + 1$, leading to a contradiction and concluding the proof. 

$\square$
Proof of Proposition 3.1. Our goal is to estimate \( |\psi^N(t, x_k) - \psi(t, x_k)| \). To do this, let us define the error function
\[
e_k = e_k(t) \overset{\text{def}}{=} \psi_k^N - \psi_k.
\]
(3.7)

Note that \( e_k(0) = 0 \). To not overload notation, the dependence on time will often be dropped. Using a Taylor expansion, for any \( k \in \mathbb{T}_N \) there exist \( c_k \in (x_k, x_{k+1}) \) and \( \bar{c}_k \in (x_{k-1}, x_k) \) such that
\[
\psi_{k+1} = \psi_k + \frac{\partial_x \psi_k}{N} + \frac{\partial_x^2 \psi_k}{2!N^2} + \frac{\partial_x^3 \psi_k}{3!N^3} + \frac{\partial_x^4 \psi(t, c_k)}{4!N^4},
\]
\[
\psi_{k-1} = \psi_k - \frac{\partial_x \psi_k}{N} + \frac{\partial_x^2 \psi_k}{2!N^2} - \frac{\partial_x^3 \psi_k}{3!N^3} + \frac{\partial_x^4 \psi(t, \bar{c}_k)}{4!N^4}.
\]

Adding the equations above we have that
\[
\psi_{k+1} + \psi_{k-1} = 2\psi_k + \frac{\partial_x^2 \psi(t, \bar{d}_k)}{N^2} + \frac{a_k}{N^4},
\]
(3.8)

where \( a_k = \frac{1}{4!}(\partial_x^4 \psi(t, c_k) + \partial_x^4 \psi(t, \bar{c}_k)) \). Since \( \psi \) is the solution of the PDE (2.4),
\[
\partial_x^2 \psi_k = \partial_t \psi_k + 2\xi \left( \psi_k \partial_x H_k \right) - e^{H_k} b(\psi_k) + e^{-H_k} d(\psi_k),
\]
and replacing this into (3.8) gives us
\[
N^2(\psi_{k+1} - 2\psi_k + \psi_{k-1}) - \frac{a_k}{N^2} = \partial_t \psi_k + 2\xi \left( \psi_k \partial_x H_k + \psi_k \partial_x^2 H_k \right) - e^{H_k} b(\psi_k) + e^{-H_k} d(\psi_k).
\]
(3.9)

Observe that above we still have a first order derivative of \( \psi \), which we want to write in terms of \( \psi_{k+1} \) and \( \psi_{k-1} \). In order to do so, we apply again a Taylor expansion, telling us that, for \( k \in \mathbb{T}_N \), there exist \( d_k \in (x_k, x_{k+1}) \) and \( \bar{d}_k \in (x_{k-1}, x_k) \) such that
\[
\psi_{k+1} = \psi_k + \frac{\partial_x \psi_k}{N} + \frac{\partial_x^2 \psi(t, d_k)}{2!N^2} \quad \text{and} \quad \psi_{k-1} = \psi_k - \frac{\partial_x \psi_k}{N} + \frac{\partial_x^2 \psi(t, \bar{d}_k)}{2!N^2}.
\]

Subtracting the equations above we have that
\[
\psi_{k+1} - \psi_{k-1} = \frac{2}{N} \partial_x \psi_k + \frac{\pi_k}{N^2},
\]
where \( \pi_k = \frac{1}{2}(\partial_x^2 \psi(t, d_k) - \partial_x^2 \psi(t, \bar{d}_k)) \). Replacing this into (3.9), we get
\[
\partial_t \psi_k = N^2(\psi_{k+1} - 2\psi_k + \psi_{k-1}) - N(\psi_{k+1} + \psi_{k-1})\partial_x H_k - 2\psi_k \partial_x^2 H_k + e^{H_k} b(\psi_k) - e^{-H_k} d(\psi_k) - \frac{\pi_k \partial_x H_k}{N} - \frac{a_k}{N^2}.
\]

Recall the definition (3.7). Since \( \psi^N \) is the solution of (3.1), we obtain that
\[
\frac{d}{dt} e_k = N^2(e_{k+1} - 2e_k + e_{k-1}) - N(e_{k+1} - e_{k-1})\partial_x H_k - \left[ \frac{1}{2}(S_1^N + S_1^N + 2) \psi^N \right] \partial_x^2 H_k + e^{H_k} (b(\psi^N_k) - b(\psi_k)) - e^{-H_k} (d(\psi^N_k) - d(\psi_k)) + \frac{\pi_k \partial_x H_k}{N} + \frac{a_k}{N^2}.
\]

Since
\[
\left| 2\psi_k - \frac{1}{2}(S_1^N + S_1^N + 2) \psi_k \right| \leq \frac{||\partial_x \psi||_\infty}{N},
\]
then
\[
\frac{d}{dt} e_k \leq N^2(e_{k+1} - 2e_k + e_{k-1}) - N(e_{k+1} - e_{k-1})\partial_x H_k - \left[ \frac{1}{2}(S_1^N + S_1^N + 2) e_k \right] \partial_x^2 H_k + e^{H_k} (b(\psi^N_k) - b(\psi_k)) - e^{-H_k} (d(\psi^N_k) - d(\psi_k)) + \frac{\pi_k \partial_x H_k}{N} + \frac{a_k}{N^2} + \frac{||\partial_x \psi||_\infty}{N} \partial_x^2 H_k.
\]

Recalling (3.2), we get that
\[
\frac{d}{dt} e_k \leq N^2(e_{k+1} - 2e_k + e_{k-1}) - N(e_{k+1} - e_{k-1})\partial_x H_k + C_e(e_{k+1} + |e_k| + e_{k-1} + N^{-1}).
\]
We have therefore proved that \((e_1, \ldots, e_N)\) is a subsolution for (3.3). Consider now \(z_k(t) = \exp(\lambda C_s t)/N\), where \(\lambda > 0\). Noting that \(z_k(t)\) does not depend on the spatial variable, a simple calculation permits to check that it is a supersolution of (3.3) provided
\[
\lambda > 3 + \frac{1}{C_*}.
\]
Fix henceforth some \(\lambda\) satisfying the condition above. By the Lemma 3.2 we have that
\[
e_k(t) \leq \frac{\exp(\lambda C_* t)}{N} \leq \frac{\exp(\lambda C_* T)}{N}, \quad \forall k \in \mathbb{T}_N, \forall t \in [0, T].
\]
Repeating the previous arguments to \(-e_k(t)\), we can analogously obtain that
\[
-e_k(t) \leq \frac{\exp(\lambda C_* t)}{N} \leq \frac{\exp(\lambda C_* T)}{N}, \quad \forall k \in \mathbb{T}_N, \forall t \in [0, T].
\]
Thus we conclude that \(|e_k(t)| \leq \frac{\exp(\lambda C_* T)}{N}\) for \(k \in \mathbb{T}_N\) and \(t \in [0, T]\), which implies
\[
\sup_{t \in [0, T]} \max_{k \in \mathbb{T}_N} |\psi^N_k - \psi_k| \leq \frac{\exp(\lambda C_* T)}{N},
\]
finishing the proof. \(\square\)

3.2. Dynkin Martingale. Denote
\[
\begin{align*}
\Delta_N f(k) &= N^2 \left[ f\left(\frac{k+1}{N}\right) + f\left(\frac{k-1}{N}\right) - 2f\left(\frac{k}{N}\right) \right] \quad \text{and} \quad (3.10) \\
\tilde{\nabla}_N f(k) &= \frac{N}{2} \left[ f\left(\frac{k+1}{N}\right) - f\left(\frac{k-1}{N}\right) \right]. \quad (3.11)
\end{align*}
\]
Note that (3.10) is the discrete Laplacian while (3.11) is not the usual discrete derivative but it also approximates the continuous derivative in the case \(f\) is smooth. Recall the Markov process defined in Subsection 2.2. It can be also defined through its infinitesimal generator \(L_N\), which acts on functions \(f : \Omega_N \to \mathbb{R}\) as
\[
L_N f(\eta) = \sum_{k \in \mathbb{T}_N} N^2 \eta_k \exp \{ H_{k+1} - H_k \} \left[ f(\eta^{k,k+1}) - f(\eta) \right] + \sum_{k \in \mathbb{T}_N} N^2 \eta_k \exp \{ H_{k-1} - H_k \} \left[ f(\eta^{k,k-1}) - f(\eta) \right] + \sum_{k \in \mathbb{T}_N} \ell b(\ell^{-1} \eta_k) \exp \{ H_k \} \left[ f(\eta^{k,+}) - f(\eta) \right] + \sum_{k \in \mathbb{T}_N} \ell d(\ell^{-1} \eta_k) \exp \{ -H_k \} \left[ f(\eta^{k,-}) - f(\eta) \right],
\]
where
\[
\eta^{k,k+1}_j = \begin{cases} 
\eta_j, & \text{if } j \neq k, k \pm 1 \\
\eta_j - 1, & \text{if } j = k \text{ and } \eta_k \geq 1 \\
\eta_j + 1, & \text{if } j = k \pm 1 \text{ and } \eta_k \geq 1 \\
\eta_k, & \text{if } j = k \text{ and } \eta_k = 0 \\
\eta_k, & \text{if } j = k \pm 1 \text{ and } \eta_k = 0
\end{cases}
\]
and
\[
\eta^{k,+}_j = \begin{cases} 
\eta_j, & \text{if } j \neq k \\
\eta_j + 1, & \text{if } j = k
\end{cases} \quad \text{and} \quad \eta^{k,-}_j = \begin{cases} 
\eta_j, & \text{if } j \neq k \\
\eta_j - 1, & \text{if } j = k \text{ and } \eta_k \geq 1 \\
\eta_k, & \text{if } j = k \text{ and } \eta_k = 0
\end{cases}.
\]
It is a well-known fact that the process
\[
M_f(t) = f(\eta(t)) - f(\eta(0)) - \int_0^t L_N f(\eta(s)) ds
\]
is a martingale with respect to the natural filtration, which is the so-called Dynkin martingale, see [14, Appendix] for instance. Fix some $k \in \mathbb{T}_N$. Picking up the particular $f(\eta) = \eta_k$ gives us that

$$
M_k(t) = \eta_k(t) - \eta_k(0) - \int_0^t \left[ -N^2 \eta_k(s) \left( \exp \{ H_{k+1} - H_k \} + \exp \{ H_{k-1} - H_k \} \right) + N^2 \eta_{k+1}(s) \exp \{ H_k - H_{k+1} \} + N^2 \eta_{k-1}(s) \exp \{ H_k - H_{k-1} \} + \ell b(\ell^{-1} \eta_k) \exp \{ H_k \} - \ell d(\ell^{-1} \eta_k) \exp \{ -H_k \} \right] ds
$$

is a martingale. Since $H$ has a finite Lipschitz constant, a Taylor expansion gives us that

$$
\exp \{ H_{k+1} - H_k \} = 1 + H_{k+1} - H_k + \frac{(H_{k+1} - H_k)^2}{2!} + \text{err} \left( \frac{H_{k+1} - H_k}{2!}, s \right),
$$

where the error term $\text{err} \left( \frac{H_{k+1} - H_k}{2!}, s \right)$ is $O(N^{-3})$, uniformly on $k \in \mathbb{T}_N$. This allows us to rewrite the above martingale as

$$
M_k(t) = \eta_k(t) - \eta_k(0) - \int_0^t \left[ N^2 \left( \eta_{k+1}(s) + \eta_{k-1}(s) - 2 \eta_k(s) \right) - \eta_k(s) N^2 \left( H_{k+1} + H_{k-1} - 2H_k \right) + \eta_{k+1}(s) N^2 \left( H_k - H_{k+1} \right) + \eta_{k-1}(s) N^2 \left( H_k - H_{k-1} \right) + \ell b(\ell^{-1} \eta_k) \exp \{ H_k \} - \ell d(\ell^{-1} \eta_k) \exp \{ -H_k \} + A_k(s) \right] ds,
$$

where

$$
A_k(s) = N^2 \left[ \frac{1}{2} (H_{k+1} - H_k)^2 \eta_{k+1}(s) + \frac{1}{2} (H_{k-1} - H_k)^2 \eta_{k-1}(s) - \frac{1}{2} (H_{k+1} - H_k)^2 \eta_k(s) - \frac{1}{2} (H_{k-1} - H_k)^2 \eta_k(s) + \text{err} \left( \frac{H_{k+1} - H_k}{2!}, s \right) \eta_{k+1}(s) + \text{err} \left( \frac{H_{k+1} - H_k}{2!}, s \right) \eta_{k-1}(s) - \text{err} \left( \frac{H_{k+1} - H_k}{2!}, s \right) \eta_k(s) \right].
$$

Using by Taylor that $H_{k+1} - H_k = \pm \frac{1}{N} \partial_z H_k + \frac{1}{2N^2} \partial_{xx}^2 H_k + O(N^{-3})$, (3.10) and (3.11) we can rewrite the martingale $M_k(t)$ as

$$
M_k(t) = \eta_k(t) - \eta_k(0) - \int_0^t \left[ \Delta_N \eta_k(s) - \eta_k(s) \Delta_N H_k - 2 \bar{\nabla}_N \eta_k(s) \partial_x H_k - \frac{1}{2} (\eta_{k+1} + \eta_{k-1}) \partial_{xx}^2 H_k + \ell b(\ell^{-1} \eta_k) \exp \{ H_k \} - \ell d(\ell^{-1} \eta_k) \exp \{ -H_k \} + A_k + O(N^{-1}) \eta_{k+1}(s) + O(N^{-1}) \eta_{k-1}(s) \right] ds.
$$

Dividing the equation above by $\ell$ and using that the discrete Laplacian approximates the continuous Laplacian, it yields that

$$
Z^N(t, \frac{s}{N}) = X^N(t, \frac{s}{N}) - X^N(0, \frac{s}{N}) - \int_0^t \left[ \Delta_N X^N(s, \frac{k}{N}) - 2 \bar{\nabla}_N X^N(s, \frac{k}{N}) \partial_x H_k - \frac{1}{2} \left( X^N(s, \frac{k+1}{N}) + X^N(s, \frac{k-1}{N}) + 2X^N(s, \frac{k}{N}) \right) \partial_{xx}^2 H_k + b(X^N(s, \frac{k}{N})) \exp \{ H_k \} - d(X^N(s, \frac{k}{N})) \exp \{ -H_k \} + B_k(s) \right] ds
$$

is a martingale for each $k \in \mathbb{T}_N$, now in a suitable form to our future purposes, where

$$
B_k(s) = \frac{1}{2N^2} (\partial_x H_k)^2 \Delta_N X^N(s, \frac{k}{N}) + O(N^{-1}) X^N(s, \frac{k+1}{N}) + O(N^{-1}) X^N(s, \frac{k}{N}) + O(N^{-1}) X^N(s, \frac{k-1}{N})
$$

is a term which will not contribute in the limit as $N$ goes to infinity, as we shall see later.

It is a convenient moment to argue why the Entropy Method (see [14]) is not followed in this work. Because we pursue an almost sure limit in the supremum norm, in order
to approach the problem via the Entropy Method, it would be necessary to compare some Dynkin martingale with the solution of the initial value problem \((2.4)\) in a extremely fast way. However, since the solution of \((2.4)\) does not even appear in the Dynkin martingale, we cannot foresee a clear approach to do that. The Relative Entropy Method seems to be inappropriate as well: in general, the model here defined possess no invariant measure since the total quantity of particles explodes as times goes to infinity.

3.3. Duhamel’s Principle. In this subsection we provide a version of Duhamel’s Principle for the martingales in \((3.12)\), which will be necessary in the proof of Theorem 2.2.

The Duhamel’s Principle is a general, wide applicable idea, which goes as follows. Let \(X(t)\) be the time trajectory of some dynamics, and assume that the dynamics is given by the superposition of two dynamics, let us say \(D_1\) and \(D_2\), where \(D_1\) is a linear dynamics. Then \(X(t)\) can be written as the sum of \(X(0)\) evolved by \(D_1\) with the time integral from zero to \(t\) of the evolution by \(D_1\) from a given time \(s\) up to \(t\) of the infinitesimal contribution of \(D_2\) on \(X(s)\).

Next we provide a general statement from which we will get the Duhamel’s Principle for the martingales in \((3.18)\). Let \(T_N(t) = e^{t\Delta N}\) the semigroup on \(C(\mathbb{R}^{T_N})\) generated by the discrete Laplacian \(\Delta N\).

**Proposition 3.3.** Let \(\mathcal{X} : [0,T] \rightarrow \mathbb{R}^{T_N}\) be a constant by parts and continuous from the right function and let \(Z : [0,T] \rightarrow \mathbb{R}^{T_N}\) be a continuous from the right function related to \(\mathcal{X}\) by

\[
\mathcal{X}(t) = \mathcal{X}(0) + \int_0^t \Delta N \mathcal{X}(s) ds + \int_0^t \mathcal{F}(s, \mathcal{X}(s)) ds + Z(t)
\]

where \(\mathcal{F} : [0,T] \times \mathbb{R}^{T_N} \rightarrow \mathbb{R}^{T_N}\) is a continuous function. Then

\[
\mathcal{X}(t) = T_N(t)\mathcal{X}(0) + \int_0^t T_N(t-s)\mathcal{F}(s, \mathcal{X}(s)) ds + \int_0^t T_N(t-s) dZ(s).
\]

Before proving the proposition above, let us make a break to explain the meaning of the last integral in the right hand side of \((3.14)\) and provide an integration by parts formula for it. Its meaning is given by:

\[
\int_0^t T_N(t-s) dZ(s) \overset{\text{def}}{=} \lim_{\|\mathcal{P}\| \rightarrow 0} \sum_{i=1}^n T_N(t-s_i) [Z(s_i) - Z(s_{i-1})],
\]

where \(0 = s_0 < \cdots < s_n = t\) corresponds to a partition \(\mathcal{P}\) of the interval \([0,t]\) and \(\|\mathcal{P}\|\) is its mesh. Expanding the right side of the above equation, we get

\[
\sum_{i=1}^n T_N(t-s_i) [Z(s_i) - Z(s_{i-1})] = \sum_{i=1}^n T_N(t-s_i)Z(s_i) - \sum_{i=0}^{n-1} T_N(t-s_{i+1})Z(s_i)
\]

\[
= \sum_{i=1}^{n-1} [T_N(t-s_i) - T_N(t-s_{i+1})]Z(s_i) + Z(t).
\]

Now dividing and multiplying each parcel in last sum above by \((s_{i+1} - s_i)\) and then taking the limit as \(\|\mathcal{P}\| \rightarrow 0\), we deduce that

\[
\int_0^t T_N(t-s) dZ(s) = \int_0^t \frac{d}{dt}T_N(t-s)Z(s) ds + Z(t) - Z(0).
\]

Due to \(T_N(t) = e^{t\Delta N}\), we obtain that

\[
\int_0^t T_N(t-s) dZ(s) = \int_0^t \Delta N T_N(t-s) Z(s) ds + Z(t) - Z(0),
\]

which is the desired integration by parts formula.
Proof of Proposition 3.3. In what follows, the subindex \( k \) denotes the \( k \)-th entry of the respective vector function. Let \( \mu_k \) be the signed measure on \([0, T]\) given by the Lebesgue measure plus deltas of Dirac on the jumps of \( \lambda_k \), where each delta is multiplied by the corresponding size jump of \( \lambda_k \). From (3.13), we get a relation between Radon-Nikodym derivatives given by

\[
\frac{dX_k}{d\mu_k}(t) = \left[ \Delta_N X_k(t) + F_k(t, X(t)) \right] \mathbb{1}_{\{X_k(t) = x_k(t^-)\}} + \frac{dZ_k}{d\mu_k}(t),
\]

\( \mu_k \)-almost everywhere. By the integration by parts formula described in (3.15), we only need to show that

\[
X(t) = T_N(t)X(0) + \int_0^t T_N(t-s)F(s, X(s)) ds + \int_0^t \Delta_N T_N(t-s)Z(s) ds + Z(t)
\]

(3.17)
since \( Z(0) = 0 \). Denote by \( G(t) \) the expression on the right hand side of equation above. Since \( G(0) = X(0) \), in order to show the equality (3.17) it is sufficient to check that

\[
\frac{dG_k}{d\mu_k}(t) = \left[ \Delta_N G_k(t) + F_k(t, X(t)) \right] \mathbb{1}_{\{X_k(t) = x_k(t^-)\}} + \frac{dZ_k}{d\mu_k}(t),
\]

for \( k = 1, \ldots, N \), which is an elementary calculation, as we see below:

\[
\begin{align*}
\frac{dG_k}{d\mu_k} &= \frac{d}{d\mu_k} \left[ T_N(t)X(0) + \int_0^t T_N(t-s)F(s, X(s)) ds + \int_0^t \Delta_N T_N(t-s)Z(s) ds + Z(t) \right]_k \\
&= \left[ \frac{\partial}{\partial t} T_N(t)X(0) + T_N(t)F(t, X(t)) + \int_0^t \frac{\partial}{\partial t} T_N(t-s)F(s, X(s)) ds \\
&\quad + \Delta_N T_N(t)Z(t) + \int_0^t \frac{\partial}{\partial t} \Delta_N T_N(t-s)Z(s) ds \right]_k \mathbb{1}_{\{X_k(t) = x_k(t^-)\}} + \frac{dZ_k}{d\mu_k},
\end{align*}
\]

concluding the proof. \[ \square \]

We are going to deal now with a Duhamel’s Principle for the martingales in (3.12). To not overload notation, the spatial variable \( k \) will be omitted in the sequel. Keeping this in mind, (3.12) can be shortly written as

\[
X^N(t) = X^N(0) + \int_0^t \left[ \Delta_N X^N(s) - 2\nabla_N X^N(s) \partial_x H(s) - \frac{1}{2} \left( S_1^N + S_2^N + 2 \right) X^N(s) \partial_{xx} H(s) \\
+ b(X^N(s)) \exp \{ H(s) \} - d(X^N(s)) \exp \{ - H(s) \} + B(s) \right] ds + Z^N(t).
\]

(3.18)

Below, when we say that a stochastic process evolving on \( \mathbb{R}^{TN} \) is a martingale, we mean that each one of its \( N \) coordinates are martingales. Below we state a Duhamel’s Principle for \( X^N(t) \).

Corollary 3.4. Let \( Z^N(t) \) be the martingale defined by (3.18). Then

\[
X^N(t) = T_N(t)X^N(0) + \int_0^t T_N(t-s) \left[ -2\nabla_N X^N(s) \partial_x H(s) \\
- \frac{1}{2} \left( S_1^N + S_2^N + 2 \right) X^N(s) \partial_{xx} H(s) + b(X^N(s)) \exp \{ H(s) \} \\
- d(X^N(s)) \exp \{ - H(s) \} + B(s) \right] ds + \int_0^t T_N(t-s) dZ^N(s).
\]

(3.19)
Proof. This is an immediate consequence of Proposition 3.3 by taking
\[
\mathcal{F}(s, X^N(s)) = -2N X^N(s) \partial_x H(s) - \frac{1}{2} \left( S_1^N + S_1^N + 2 \right) X^N(s) \partial^2_{xx} H(s) + b(X^N(s)) \exp \{ H(s) \} - d(X^N(s)) \exp \{ -H(s) \} + B(s).
\]

Next, we present a Duhamel’s Principle for the solution \( \psi^N(t) \) of the ODE system (3.1).

**Corollary 3.5.** The solution \( \psi^N(t) \) of (3.1) satisfies
\[
\psi^N_k(t) = T_N(t) \psi^N_k(0) + \int_0^t T_N(t - s) \left[ -\frac{1}{2} \left( S_1^N + S_1^N + 2 \right) \psi^N_k(s) \partial^2_{xx} H_k(s) \\
- 2N \psi^N_k(s) \partial_x H_k(s) + b(\psi^N_k(s)) \exp \{ H_k(s) \} - d(\psi^N_k(s)) \exp \{ -H_k(s) \} \right] ds
\]
for \( k = 1, \ldots, N \).

**Proof.** It is also a direct consequence of Proposition 3.3, considering in this case \( Z \equiv 0 \).

### 3.4. Proof of the high density limit

In this section we prove the Theorem 2.2. Before going through details, let us explain the involved ideas. Noting the resemblance of (3.19) and (3.20), we would like to have that
\[
\sup_{t \in [0, T]} \| Y^N(t) \|_\infty \to 0 \quad \text{a.s.,}
\]
where
\[
Y^N(t) = \int_0^t T_N(t - s) dZ^N(s)
\]
is the only (random) term which differs (3.19) from (3.20). Since the solution \( \psi^N(t) \) of the semi-discrete scheme converges to the solution of the corresponding PDE (see Section 3.1, Gronwall inequality would finish the job, assuring that the \( X^N(t) \) converges to the solution of the PDE (2.4). However, (3.21) is not true, or at least, it is not clear to us how to argue that. The reason of this is the following: an essential ingredient to prove that a process as \( Y^N \) goes to zero is that the corresponding martingale \( Z^N(t) \) is bounded, which is not actually true in our case.

To overcome the aforementioned obstacle, we will mixture ideas from the original strategy of [5] with the approach of [10]. Instead of working with \( X^N(t) \), we will deal with a stopped process \( \overline{X}^N(t) \) close to \( X^N(t) \). Fixing \( \varepsilon_0 > 0 \), consider the stopping time
\[
\tau = \inf \left\{ t : \| X^N(t) - \psi^N(t) \|_\infty > \varepsilon_0 \right\}
\]
and define
\[
\overline{X}^N(t) = \begin{cases} X^N(t), & \text{if } t \leq \tau, \\
W^N(t), & \text{if } t > \tau,
\end{cases}
\]
where \( W^N(t) = (W^N_1(t), \ldots, W^N_N(t)) \) is defined as the solution of
\[
\begin{align*}
\frac{d}{dt} W^N_k = & \ N^2 \left( W^N_{k+1} - 2W^N_k + W^N_{k-1} \right) - N \left( W^N_{k+1} - W^N_{k-1} \right) \partial_x H_k \\
& - \frac{1}{2} \left( S_1^N + S_1^N + 2 \right) W^N_k \partial^2_{xx} H_k + e^{H_k} b(W^N_k) - e^{-H_k} d(W^N_k), \quad k \in T_N \text{ and } t > \tau,
\end{align*}
\]
where \( \overline{X}^N(t) \) is stochastic process that evolves deterministically once the original process \( X^N(t) \) gets \( \varepsilon_0 \)-away of the solution of the corresponding system of ODE’s and it is equal to \( X^N(t) \) before that time. Moreover, the deterministic evolution follows the dynamics...
of the system of ODE's, having \( X^N(\tau) \) as initial condition at time \( t = \tau \). The reason we can work with \( \overline{X}^N(t) \) instead of \( X^N(t) \) is that
\[
\lim_{N \to \infty} \sup_{t \in [0,T]} \| \overline{X}^N(t) - \psi^N(t) \|_{\infty} = 0 \quad \text{a.s.} \tag{3.22}
\]
implies
\[
\lim_{N \to \infty} \sup_{t \in [0,T]} \| X^N(t) - \psi^N(t) \|_{\infty} = 0 \quad \text{a.s.}
\]
as can be readily checked. Therefore, our goal from now on is to prove (3.22). Denote \( \overline{X}_k^N(\cdot) = \overline{X}^N(\cdot, k/N) \). The main features of \( \overline{X}^N(t) \) are the following. First, its version of Duhamel's Principle is given by
\[
\overline{X}_k^N(t) = T_N(t) \overline{X}_k^N(0) + \int_0^t T_N(t - s) \left[ -\frac{1}{2} \left( S^N_1 + S^N_2 + 2 \right) \overline{X}_k^N(s) \partial^2_{xx} H_k(s) \right. \\
- 2 \nabla N \overline{X}_k^N(s) \partial_k H_k(s) + b \left( \overline{X}_k^N(s) \right) \exp \{ H_k(s) \} - d \left( \overline{X}_k^N(s) \right) \exp \{ - H_k(s) \} \Big] ds + \overline{Y}_k^N(t)
\]
where
\[
\overline{Y}^N(t) = \int_0^t T_N(t - s) d\overline{Z}^N(s \wedge \tau),
\]
and \( \overline{Z}^N \) is the martingale obtained through (3.18) replacing \( X^N \) by \( \overline{X}^N \). The proof of (3.23) above is also a consequence of Proposition 3.3 and its proof is omitted. Second, but not less important, is the fact that there exists some \( C > 0 \) such that
\[
\sup_{t \in [0,T]} \| \overline{X}^N(t) \|_{\infty} \leq C \tag{3.24}
\]
for all large enough \( N \in \mathbb{N} \). The inequality above can be argued as follows. Since the solution \( \psi \) of the PDE (2.4) is smooth and defined on a compact domain, it is bounded. Proposition 3.1 tells us that \( \psi^N \) converges uniformly to \( \psi \), hence \( \psi^N \) is bounded as well by some constant \( c_1 > 0 \). By the definition of the stopping time \( \tau \), the process \( \overline{X}^N(t) \) is bounded by \( c_1 + \varepsilon_0 \) for any time \( t < \tau \). After time \( \tau \), the process runs deterministically under the same dynamics of \( \psi^N \), but with the random initial condition given by \( \overline{X}^N(\tau) \) at time \( \tau \). Since \( \| \overline{X}^N(\tau) \|_{\infty} \leq c_1 + \varepsilon_0 + \frac{1}{2} \), an argument on super-solutions (similar to that one presented in the Section 3.1) gives that \( \overline{X}^N(t) \) is also bounded for some constant for all times \( t > \tau \).

To obtain the necessary martingales, we provide a general statement in the next proposition. Despite this is a well-known result, we could not find any reference in the literature in a suitable form. For this reason, we include it here for sake of completeness.

**Proposition 3.6.** Let \( (X_t)_{t \geq 0} \) be a continuous time Markov chain taking values on the countable set \( \Omega \). Denote by \( \lambda : \Omega \times \Omega \to \mathbb{R}_+ \) the rates of jump, assume that \( \lambda(x,x) = 0 \) for all \( x \in \Omega \) and
\[
\sup_{x \in \Omega} \left\{ \sum_{y \in \Omega} \lambda(x,y) \right\} < \infty.
\]
This continuous time Markov chain can described as follows. When at the state \( x \in \Omega \), the next state is chosen according to the minimum of a family of independent exponentials of parameter \( \lambda(x,z) \), where \( z \in \Omega, z \neq x \). If the minimum of such exponentials is attained at the exponential of parameter \( \lambda(x,y) \), the process remains at \( x \) during a period of time equals to the value of this exponential and then jumps to \( y \). Denote by \( N_t(x,y) \) the number of times the process has made the transition from \( x \) to \( y \) in the time interval \( [0,t] \). Then
\[
\mathcal{M}_t = N_t(x,y) - \lambda(x,y) \int_0^t \mathbf{1}_{[X_s = x]} ds
\]
is a martingale with respect to the natural filtration.
Proof. Denote by \( \mu \) the initial distribution and by \( \mathcal{F}_t \) the natural filtration, i.e., the \( \sigma \)-algebra generated by the process until time \( t \geq 0 \). Let \( 0 \leq u < t \),

\[
\mathbb{E}_\mu \left[ N_t(x, y) - \lambda(x, y) \int_0^t 1_{[X_s = x]} ds \bigg| \mathcal{F}_u \right] = N_u(x, y) - \lambda(x, y) \int_0^u 1_{[X_s = x]} ds \\
+ \mathbb{E}_\mu \left[ N_t(x, y) - N_u(x, y) - \lambda(x, y) \int_u^t 1_{[X_s = x]} ds \bigg| \mathcal{F}_u \right].
\]

By the Markov Property, in order to show is null the second parcel in the r.h.s. of the equation above, it is sufficient to proof that

\[
\mathbb{E}_z \left[ N_t(x, y) - \lambda(x, y) \int_0^t 1_{[X_s = x]} ds \right] = 0
\]

for any \( z \in \Omega \) and any \( t \geq 0 \). Let \( 0 = t_0 < t_1 < \cdots < t_n = t \) be a partition of the interval \([0, t]\). Expression (3.25) can be rewritten as

\[
\sum_{i=0}^{n-1} \mathbb{E}_z \left[ N_{t_{i+1}}(x, y) - N_{t_i}(x, y) + \lambda(x, y) \int_{t_i}^{t_{i+1}} 1_{[X_s = x]} ds \right].
\]

Since the probability of two or more jumps in an interval of length \( h \) is \( O(h^2) \), it is enough to show that

\[
\mathbb{E}_z \left[ N_{t_{i+1}}(x, y) - N_{t_i}(x, y) - \lambda(x, y) \int_{t_i}^{t_{i+1}} 1_{[X_s = x]} ds \right] = O((t_{i+1} - t_i)^2).
\]

By the Markov Property, it is enough to assure that \( \mathbb{E}_z | N_h(x, y) - \lambda(x, y) h | = O(h^2) \). On his hand, this is a consequence of the definition of \( N_h(x, y) \).

Denote \( \delta f(t) = f(t) - f(t^-) \). As an application of the Proposition 3.6 in our model, we have:

**Lemma 3.7.** For any \( k = 0, 1, \ldots, N - 1 \), the following processes are martingales with respect to the natural filtration:

\[
\mathcal{M}_t^{N,1} = \ell \left[ X_k^N(t) - X_k^N(0) \right] - \int_0^t \ell N^2 \left[ X_{k-1}^N(s) e^{H_k - H_{k-1}} - 2 X_k^N(s) e^{H_k+1 - H_k} + X_{k+1}^N(s) e^{H_{k+2} - H_{k+1}} ds \right] ds
\]

(3.26)

\[
\mathcal{M}_t^{N,2} = \ell^2 \sum_{s \leq t} (\delta X_k^N(s))^2 - \int_0^t \ell N^2 \left[ X_{k-1}^N(s) e^{H_k - H_{k-1}} + X_k^N(s) e^{H_k+1 - H_k} + X_{k+1}^N(s) e^{H_{k+2} - H_{k+1}} ds \right] ds.
\]

(3.27)

\[
\mathcal{M}_t^{N,3} = - \ell^2 \sum_{s \leq t} \delta X_k^N(s) \delta X_{k+1}^N(s) - \int_0^t \ell N^2 \left[ X_k^N(s) e^{H_{k+1} - H_k} + X_{k+1}^N(s) e^{H_{k+2} - H_{k+1}} \right] ds.
\]

(3.28)

**Proof.** As we shall see below, each of the expressions (3.26), (3.27), and (3.28) are the number of times some kind of transitions has been made minus the integral in time of the corresponding rates. In (3.26), the parcel

\[
\ell \left[ X_k^N(t) - X_k^N(0) \right]
\]

of that expression counts how many times in \([0, t]\) the Markov process \((\eta_k)_{k \geq 0}\) has made a transition \( \eta_k = j \) to \( \eta_k = j + 1 \) for some \( j \in \mathbb{N} \), minus how many times the process has made a transition \( \eta_k = j + 1 \) to \( \eta_k = j \), normalized by the parameter \( \ell \).

In (3.27), the parcel

\[
\ell^2 \sum_{s \leq t} (\delta X_k^N(s))^2
\]
of that expression counts how many times in \([0, t]\) the process has made a transition \(\eta_k = j\) to \(\eta_k = j \pm 1\) for some \(j \in \mathbb{N}\).

In (3.28), the parcel

\[-\ell^2 \sum_{s \leq t} \delta \mathbf{X}_k^N(s) \delta \mathbf{X}_{k+1}^N(s)\]

of that expression counts how many times in \([0, t]\) particles have jumped between the sites \(k\) and \(k + 1\). Since the integral parts in (3.26), (3.27) and (3.28) are the integrals in time of the respective rates, recalling Proposition 3.6 finishes the proof. \(\square\)

Together with (3.23) and (3.24), the next lemma will be also an ingredient in the proof of (3.22).

**Lemma 3.8.** Recall the constant \(C > 0\) as in (3.24). Then, there exists some \(a = a(C, T) > 0\) such that, for any \(\varepsilon > 0\),

\[
P \left( \exp \left( -4T \text{sup}_{[0, T]} \| \mathbf{Y}^N(t) \|_{\infty} > \varepsilon \right) \right) \leq 4N^3 \exp(-a\varepsilon^2\ell). \]

The proof of Lemma 3.8 is similar to the of proof of Lemma 4.10 in [5]. Before proving it, we need the following Lemma 3.9 and recall two results of [5]. Denote

\[\nabla_N^+ f(k) = N \left[ f \left( \frac{k+1}{N} \right) - f \left( \frac{k}{N} \right) \right] \text{ and } \nabla_N^- f(k) = N \left[ f \left( \frac{k}{N} \right) - f \left( \frac{k+1}{N} \right) \right].\]

Let \(\langle \cdot, \cdot \rangle\) be the inner product in \(\mathbb{R}^{T_N}\) defined by

\[
\langle f, g \rangle = \frac{1}{N} \sum_{k \in T_N} f(k)g(k). \tag{3.29}
\]

**Lemma 3.9.** The process

\[
\sum_{s \leq t} \left( \delta \langle \mathbf{Z}^N(t), \varphi \rangle \right)^2 - \langle N\ell \rangle^{-1} \int_0^t \left( \langle \mathbf{X}^N(s)e^{-\nabla_N^+ H/N}, (\nabla_N^+ \varphi)^2 + (\nabla_N^- \varphi)^2 \rangle \right) ds
\]

\[-(N\ell)^{-1} \int_0^t \left( b(\mathbf{X}^N(s))e^H + d(\mathbf{X}^N(s))e^{-H}, \varphi^2 \right) ds\]

is a mean zero martingale with respect to the natural filtration.

**Proof.** First, note that the process \(\mathbf{X}^N\) and \(\mathbf{Z}^N\) have the same jumps of discontinuity. Thus, given \(\varphi \in S^N\), we have that

\[
\sum_{s \leq t} \left( \delta \langle \mathbf{Z}^N(t), \varphi \rangle \right)^2 = \sum_{s \leq t} \frac{1}{N^2} \left( \sum_{k=0}^{N-1} \varphi_k \delta X_k^N(s) \right)^2
\]

\[= \sum_{s \leq t} \frac{1}{N^2} \sum_{k=0}^{N-1} \varphi_k^2 (\delta X_k^N(s))^2 + \sum_{s \leq t} \frac{2}{N^2} \sum_{k=0}^{N-1} \varphi_k \varphi_{k+1} \delta X_k^N(s) \delta X_{k+1}^N(s), \]

so, by (3.27) and (3.28), the process below is a martingale:

\[
\sum_{s \leq t} \left( \delta \langle \mathbf{Z}^N(t), \varphi \rangle \right)^2 - \sum_{k=0}^{N-1} \int_0^t \frac{\varphi_k^2}{\ell} \left( \mathbf{X}_{k-1}^N(s)e^{H_k-H_{k-1}} + 2\mathbf{X}_k^N(s)e^{H_{k+1}-H_k} + \mathbf{X}_{k+1}^N(s)e^{H_{k+2}-H_{k+1}} \right)
\]

\[+ \frac{\varphi_k^2}{N^2\ell} \left( b(\mathbf{X}_k^N(s))e^{H_k} + d(\mathbf{X}_k^N(s))e^{-H_k} \right) ds\]

\[+ \sum_{k=0}^{N-1} \int_0^t \frac{2\varphi_k \varphi_{k+1}}{\ell} \left( \mathbf{X}_k^N(s)e^{H_{k+1}-H_k} + \mathbf{X}_{k+1}^N(s)e^{H_{k+2}-H_{k+1}} \right) ds. \tag{3.30}
\]
Observe that
\[ \sum_{k=0}^{N-1} \frac{\varphi_k^2}{N} \left( b(X_k^N(s))e^{H_k} + d(X_k^N(s))e^{-H_k} \right) = \left( b(X_N^N(s))e^H + d(X_N^N(s))e^{-H} \right)^2, \] (3.31)
and
\[ \sum_{k=0}^{N-1} \left[ \varphi_k^2 (X_{k-1}^N(s))e^{H_k-H_{k-1}} + 2X_k^N(s)e^{H_{k+1}-H_k} + X_{k+1}^N(s)e^{H_{k+2}-H_{k+1}} \right] = N^{-1} \left\langle X^N(s)e^{\nabla^H/N}, (\nabla^{+}N\varphi)^2 + (\nabla^{-}N\varphi)^2 \right\rangle. \] (3.32)
Thus, applying (3.31) and (3.32) in (3.30), we conclude that
\[ \sum_{s \leq t} \left( \delta(t)\right)^2 - (N\ell)^{-1} \int_0^t \left\langle X^N(s)e^{\nabla^H/N}, (\nabla^{+}N\varphi)^2 + (\nabla^{-}N\varphi)^2 \right\rangle ds \]
\[ - (N\ell)^{-1} \int_0^t \left( b(X_N^N(s))e^H + d(X_N^N(s))e^{-H}, \varphi^2 \right) ds \]
is a mean zero martingale. \qed

Lemma 3.10 (Lemma 4.3 in [5]). Let \( f = N1_{k/N,(k+1)/N} \). Then,
\[ \left\langle (\nabla^H Nf)^2 + (\nabla^{-}Nf)^2 + (T_N f)^2, 1 \right\rangle \leq h_N(t), \]
where \( \int_0^t h_N(s)ds \leq CN + t \).

Lemma 3.11 (Lemma 4.4 in [5]). Let \( m(t) \) be a bounded martingale of finite variation defined on \([t_0, t_1] \) with \( m(t_0) = 0 \) and satisfying:
\[ \begin{align*}
\text{i) } & \text{ is a right-continuous with left limits,} \\
\text{ii) } & |\delta m(t)| \leq 1 \text{ for } t_0 \leq t \leq t_1, \\
\text{iii) } & \sum_{t_0 \leq s \leq t} (\delta m(s))^2 - \int_{t_0}^t g(s)ds \text{ is a mean 0 martingale with } 0 \leq g(s) \leq h(s), \text{ where } h(s) \\
& \text{ is a bounded deterministic function and } g(s) \text{ is adapted to the natural filtration.}
\end{align*} \]

Then
\[ \mathbb{E} \exp \left( m(t_1) \right) \leq \exp \left( \frac{3}{2} \int_{t_0}^{t_1} h(s)ds \right). \]

Proof of the Lemma 3.8. Fix \( \bar{T} \in (0, T], k \in \mathbb{T}_N \) and consider \( f = N1_{k/N,(k+1)/N} \). Define
\[ m(t) = \left\langle \int_0^t T_N(\bar{T} - s)dZ_N(s), f \right\rangle, \text{ para todo } 0 \leq t \leq \bar{T}. \]
which satisfies \( m(\bar{T}) = Z^N(\bar{T}, k/N) \). Since \( Z^N \) is a (vector) martingale, then \( \int_0^t T_N(\bar{T} - s)dZ_N(s) \) is a zero mean (vector) martingale, hence \( m(t) \) is a zero mean martingale on \( 0 \leq t \leq \bar{T} \) as well. By the integration by parts formula (3.15), the discontinuity jumps of \( m(t) \) are the same discontinuity jumps of \( (Z^N(t), T_N(\bar{T} - t)) \). Therefore, by the Lemma 3.9,
\[ \sum_{s \leq t} (\delta m(s))^2 - (N\ell)^{-1} \int_0^t \left\langle X^N(s)e^{\nabla^H/N}, (\nabla^{+}N\bar{T} - s)f)^2 + (\nabla^{-}N\bar{T} - s)f)^2 \right\rangle ds \]
\[ - (N\ell)^{-1} \int_0^t \left( b(X_N^N(s))e^H + d(X_N^N(s))e^{-H}, (T_N(\bar{T} - s)f)^2 \right) ds \]
is a mean 0 martingale. For \( \theta \in [0, 1] \), consider \( \theta \ell m(t) \) instead of \( m(t) \). Rewrite the martingale above as
\[ (\theta\ell)^2 \sum_{s \leq t} (\delta m(s))^2 - (\theta\ell)^2 \int_0^t g(s)ds. \]
Recall the constant $C > 0$ given in (3.24). Since $X^N(s) e^{\frac{H}{2N}}$ and $b(X^N(s)) e^H + d(X^N(s)) e^{-H}$ are bounded in modulus by a constant $\pi(C)$ and recalling the Lemma 3.10, we have that

$$(\theta \ell)^2 g(s) \leq \pi(C) \theta^2 \ell N^{-1} h_N(t).$$

So, by the Lemma 3.11,

$$\mathbb{E}[\exp(\theta \ell m(t))] \leq \exp\left(\frac{3}{2} \pi(C) \theta^2 \ell N^{-1} \int_0^t h_N(s) ds\right) \leq \exp\left(\pi(C) \theta^2 (1 + t N^{-1})\right). \tag{3.33}$$

Fix $\varepsilon > 0$. By Chebychev's inequality we obtain that

$$\mathbb{P}\left[\mathcal{Y}^N(\bar{t}, k/N) > \varepsilon\right] \leq \mathbb{E}[\exp(\theta \ell \mathcal{Y}^N(\bar{t}, k/N))] \exp(-\theta \ell \varepsilon) = \mathbb{E}[\exp(\theta \ell m(\bar{t}))] \exp(-\theta \ell \varepsilon).$$

Since $\bar{t} \leq T$, we may assume that $\bar{t}/N \leq 1$. Then by (3.33)

$$\mathbb{P}\left[\mathcal{Y}^N(\bar{t}, k/N) > \varepsilon\right] \leq \exp(\theta \ell (\pi(C) \theta - \varepsilon)) \leq \exp(-\varepsilon^2 a(C)),$$

where $a(C)$ is a function of $\pi(C)$, $\varepsilon$ and $\theta$. Arguing analogously with $\mathbb{P}[\mathcal{Y}^N(\bar{t}, k/N) < -\varepsilon]$, we can conclude that, for $0 < \bar{t} < T$ and $k \in \mathbb{T}_N$,

$$\mathbb{P}\left[\|\mathcal{Y}^N(\bar{t}, k/N)\|_\infty > \varepsilon\right] \leq 2 \exp(-\varepsilon^2 a(C)),$$

and taking the supremum over $k \in \mathbb{T}_N$, it yields

$$\mathbb{P}\left[\|\mathcal{Y}^N(\bar{t}, \cdot)\|_\infty > \varepsilon\right] \leq 2N \exp(-\varepsilon^2 a(C)). \tag{3.34}$$

By the integration by parts formula (3.15) and Fubini's Theorem, we deduce that

$$\int_0^t \Delta_N \mathcal{Y}^N(s) ds = \mathcal{Y}^N(t) - \mathcal{Y}^N(0).$$

Then, for $n T N^{-2} \leq t \leq (n + 1) T N^{-2}$ with $n = 0, \ldots, N^2 - 1$,

$$\int_{n T N^{-2}}^{(n + 1) T N^{-2}} \Delta_N \mathcal{Y}^N(s) ds = \mathcal{Y}^N(t) - \mathcal{Y}^N(n T N^{-2}) - \mathcal{Y}^N(n T N^{-2}) + \mathcal{Y}^N((n + 1) T N^{-2}).$$

So, taking the supremum norm and recalling the definition of the discrete Laplacian,

$$\|\mathcal{Y}^N(t)\|_\infty \leq \|\mathcal{Y}^N(n T N^{-2})\|_\infty + 4 N^2 \int_{n T N^{-2}}^{(n + 1) T N^{-2}} \|\mathcal{Y}^N(s)\|_\infty ds + \|\mathcal{Y}^N(t) - \mathcal{Y}^N(n T N^{-2})\|_\infty.$$

Using Gronwall’s inequality and taking the supremum on the time we get that

$$\sup_{[n T N^{-2}, (n + 1) T N^{-2}]} \|\mathcal{Y}^N(t)\|_\infty \leq \left(\|\mathcal{Y}^N(n T N^{-2})\|_\infty + \sup_{[n T N^{-2}, (n + 1) T N^{-2}]} \|\mathcal{Z}^N(t) - \mathcal{Z}^N(n T N^{-2})\|_\infty\right) e^{\delta T}. \tag{3.35}$$

Observe that $\delta(\mathcal{Z}^N(t) - \mathcal{Z}^N(n T N^{-2})) = \delta \mathcal{Z}^N(t) = \delta \mathcal{X}^N(t)$. Then, by Lemma 3.7, for $k$ fixed and $\theta \in [0, 1],$

$$\left((\theta \ell)^2 \sum_{n T N^{-2} \leq s \leq t} (\delta(\mathcal{Z}^N(t) - \mathcal{Z}^N(n T N^{-2})))^2 - \theta^2 \ell \int_{n T N^{-2}}^{t} N^2 [X^N_{k-1}(s) e^{H_k - H_{k-1}} + 2 X^N_k(s) e^{H_{k+1} - H_k} + X^N_{k+1}(s) e^{H_{k+2} - H_{k+1}} + [b(X^N_k(s)) e^{H_k} + d(X^N_k(s)) e^{-H_k}] ds, $$

is a mean zero martingale for $n T N^{-2} \leq t \leq (n + 1) T N^{-2}$. Again recalling the constant $C$ as in (3.24), we rewrite the martingale above as

$$\left((\theta \ell)^2 \sum_{n T N^{-2} \leq s \leq t} (\delta(\mathcal{Z}^N(t) - \mathcal{Z}^N(n T N^{-2})))^2 - \theta^2 \ell \int_{n T N^{-2}}^{t} N^2 g(s) ds. .$$
And by Lemma 3.11, we have that
\[ E \left[ \exp \left( \theta \ell (Z^N((n+1)TN^{-2}) - Z^N(nTN^{-2})) \right) \right] \leq \exp\left( \pi(C) \theta^2 \ell T \right). \]

Fix \( \varepsilon > 0 \). Applying Doob’s inequality, we obtain that
\[
P \left[ \sup_{[nTN^{-2}, (n+1)TN^{-2}]} (Z^N(t) - Z^N(nTN^{-2})) > \varepsilon \right]
= \mathbb{P} \left[ \sup_{[nTN^{-2}, (n+1)T^{-2}]} \exp \left( \theta \ell (Z^N(t) - Z^N(nTN^{-2})) \right) > \exp(\theta \ell \varepsilon) \right]
\leq E \exp \left( \theta \ell (Z^N(t) - Z^N(nTN^{-2})) \right) \exp(-\theta \ell \varepsilon)
\leq \exp\left( \pi(C) \theta^2 \ell T - \theta \ell \varepsilon \right) = \exp \left( -a(C, T) \ell \varepsilon^2 \right)
\]

By analogous arguments to the above ones, we also get the bound
\[
P \left[ \sup_{[nTN^{-2}, (n+1)TN^{-2}]} (Z^N(t) - Z^N(nTN^{-2})) < -\varepsilon \right] \leq \exp\left( -a(C, T) \ell \varepsilon^2 \right).
\]

Taking the supremum norm, we have that
\[
P \left[ \sup_{[nTN^{-2}, (n+1)TN^{-2}]} \|Z^N(t) - Z^N(nTN^{-2})\|_\infty > \varepsilon \right] \leq 2N \exp\left( -a(C, T) \ell \varepsilon^2 \right). \tag{3.36}
\]

Therefore, by (3.35)
\[
P \left[ e^{-4T} \sup_{[nTN^{-2}, (n+1)TN^{-2}]} \|Y^N(t)\|_\infty > \varepsilon \right]
\leq \mathbb{P} \left[ \|Y^N(nTN^{-2})\|_\infty > \varepsilon \right] + P \sup_{[nTN^{-2}, (n+1)TN^{-2}]} \|Z^N(t) - Z^N(nTN^{-2})\|_\infty > \varepsilon,
\]

and by (3.34) and (3.36)
\[
P \left[ e^{-4T} \sup_{[nTN^{-2}, (n+1)TN^{-2}]} \|Y^N(t)\|_\infty > \varepsilon \right] \leq 4N \exp\left( -a(C, T) \ell \varepsilon^2 \right).
\]

Since
\[
P \left[ e^{-4T} \sup_{[0,T]} \|Y^N(t)\|_\infty > \varepsilon \right] \leq \sum_{n=0}^{N-1} \mathbb{P} \left[ e^{-4T} \sup_{[nTN^{-2}, (n+1)TN^{-2}]} \|Y^N(t)\|_\infty > \varepsilon \right],
\]

hence
\[
P \left[ e^{-4T} \sup_{[0,T]} \|Y^N(t)\|_\infty > \varepsilon \right] \leq 4N^3 \exp\left( -a(C, T) \ell \varepsilon^2 \right),
\]

concluding the proof.

\section*{Corollary 3.12. Let \( Y^N(t) = \int_0^T \tau X(t)(t-s)dZ^N(s) \) and assume \( \frac{N^4\|\partial_x H\|_\infty^2 / \pi^2 \log N}{\ell} \to 0 \) as \( N \to \infty \). Then
\( \frac{N^4\|\partial_x H\|_\infty^2 / \pi^2 \sup_{[0,T]} \|Y^N(t)\|_\infty}{\ell} \to 0 \) a.s.}

\section*{Proof. By the Lemma 3.8,}
\[
P \left[ e^{-4T} \sup_{[0,T]} \|Y^N(t)\|_\infty > \varepsilon \right] \leq 4N^3 \exp\left(-a\varepsilon^2 \ell \right),
\]

therefore
\[
P \left[ e^{-4TN^4\|\partial_x H\|_\infty^2 / \pi^2} \sup_{[0,T]} \|Y^N(t)\|_\infty > \varepsilon \right] \leq 4N^3 \exp\left( \frac{-a\varepsilon^2 \ell}{N^4\|\partial_x H\|_\infty^2 / \pi^2} \right).
\]
By hypothesis \( c \log(N) N^4 \| \partial_x H \|^2 / \pi^2 < \ell \), for any \( c \) constant and \( N \) large enough. Then
\[
\sum_{N=1}^{\infty} 4N^3 \exp \left( \frac{-a^2 \ell}{N^4 \| \partial_x H \|^2 / \pi^2} \right) < \sum_{N=1}^{\infty} \frac{1}{N^{1+\delta}} < \infty.
\]
So we have that
\[
\sum_{N=1}^{\infty} \mathbb{P} \left[ e^{-4T N^4 \| \partial_x H \|^2 / \pi} \sup_{[0,T]} \| \bar{Y}^N(t) \|_{\infty} > \varepsilon \right] < \infty
\]
and Borel-Cantelli Lemma leads us to
\[
N^4 \| \partial_x H \|^2 / \pi \sup_{[0,T]} \| \bar{Y}^N(t) \|_{\infty} \to 0 \text{ a.s.}
\]

An orthonormal basis of to the vector space \( \mathbb{R}^T \mathbb{N} \) with respect to the inner product (3.29) composed by eigenvectors of the discrete Laplacian is now required.

For \( m \) even, with \( 2 \leq m \leq N - 1 \), define
\[
\varphi_{m,N}(k) = \sqrt{2} \cos(\pi mkN^{-1}) \quad \text{and} \quad \phi_{m,N}(k) = \sqrt{2} \sin(\pi mkN^{-1}).
\]
Let \( \varphi_{0,N} \equiv 1 \) and, only in the case \( N \) is even, define also \( \varphi_{N,N}(k) = \cos(\pi k) \). These functions \( \varphi_{m,N} \) and \( \phi_{m,N} \) are eigenvectors of \( \Delta_N \) associated to the eigenvalue
\[
-\beta_{m,N} \overset{\text{def}}{=} -2N^2 \left( 1 - \cos(\pi mN^{-1}) \right).
\]
An orthonormal basis of eigenvectors is then given by
\[
\{ \varphi_{0,N} \} \cup \{ \varphi_{2,N}, \varphi_{2,N}, \ldots, \varphi_{N-2,N}, \varphi_{N-2,N} \} \quad \text{if } N \text{ is odd}
\]
and
\[
\{ \varphi_{0,N} \} \cup \{ \varphi_{2,N}, \varphi_{2,N}, \ldots, \varphi_{N-2,N}, \phi_{N-2,N} \} \cup \{ \varphi_{N,N} \} \quad \text{if } N \text{ is even}.
\]
Additionally let us define \( \phi_{0,N} = \phi_{N,N} \equiv 0 \). Provided by this orthonormal basis of eigenvectors, we can write the semigroup associated to the discrete Laplacian in the following concise form. If \( N \) is odd, given \( g \in \mathbb{R}^T \mathbb{N} \),
\[
T_N(t)g = \sum_{m \in \{0, \ldots, N-1\}}^{m \text{ is even}} e^{-\beta_{m,N}t} \left( \langle g, \varphi_{m,N} \rangle \varphi_{m,N} + \langle g, \phi_{m,N} \rangle \phi_{m,N} \right)
\]
and, if \( N \) is even,
\[
T_N(t)g = \sum_{m \in \{0, \ldots, N\}}^{m \text{ is even}} e^{-\beta_{m,N}t} \left( \langle g, \varphi_{m,N} \rangle \varphi_{m,N} + \langle g, \phi_{m,N} \rangle \phi_{m,N} \right).
\]
To make notation short, we will simply write
\[
T_N(t)g = \sum_{m} e^{-\beta_{m,N}t} \left( \langle g, \varphi_{m,N} \rangle \varphi_{m,N} + \langle g, \phi_{m,N} \rangle \phi_{m,N} \right) \quad (3.37)
\]
being implicitly understood the set over the sum above is taken. We are now in position to prove the high density limit for the perturbed process.

**Proof of Theorem 2.2.** Our goal is to show that \( \sup_{[0,T]} \| \bar{X}^N(t) - \psi(t) \|_{\infty} \) converges almost surely to zero. In view of Proposition 3.1, it is enough to show that \( \sup_{[0,T]} \| \bar{X}^N(t) - \psi^N(t) \|_{\infty} \)
converges almost surely to zero. Denote \( e^N(t) := \overline{X}^N(t) - \psi^N(t) \). Using the Duhamel's Principle (3.19) for \( X^N \) and the Duhamel's Principle (3.20) for \( \psi^N \), we get that

\[
\|e^N(t)\|_\infty \leq \|T_N(t)e^N(0)\|_\infty + \left\| \int_0^t T_N(t-s)d\overline{Z}^N(s) \right\|_\infty \\
+ \left\| \int_0^t T_N(t-s) \left[ -2\overline{\nabla}_N e^N(s)\partial_z H(s) - \frac{1}{2}(S^N_1 + S^N_{-1} + 2)e^N(s)\partial^2_z H(s) \\
+ e^H(s) \left( b(\overline{X}^N(s) - b(\psi^N_k(s))) - e^{-H(s)} \left( d(\overline{X}^N(s)) - d(\psi^N_k(s)) \right) + B(s) \right) \right] ds \right\|_\infty .
\]

Note that \( \frac{1}{2}\|\langle S^N_1 + S^N_{-1} + 2 \rangle e^N \|_\infty \leq 2\|e^N\|_\infty \) and, as \( T_N \) is contraction, we also have that \( \|T_N(t)e^N(0)\|_\infty \leq \|e^N(0)\|_\infty \). Let

\[
\mathcal{C} \overset{\text{def}}{=} \max \left\{ \|e^H\|_\infty \cdot \|b\|_L , \|e^{-H}\|_\infty \cdot \|d\|_L \right\},
\]

where \( \|b\|_L \) and \( \|d\|_L \) are the Lipschitz constants of functions \( b \) and \( d \), respectively. Then

\[
\|e^N(t)\|_\infty \leq \|e^N(0)\|_\infty + \|\overline{Y}^N(t)\|_\infty + \left\| \int_0^t 2T_N(t-s)\overline{\nabla}_N e^N(s)\partial_z H(s)ds \right\|_\infty \\
+ \left\| \int_0^t 2\|e^N(s)\|_\infty \|\partial^2_z H(s)\|_\infty ds \right\|_\infty + \left\| \int_0^t 2\mathcal{C}\|e^N(s)\|_\infty ds \right\|_\infty + \left\| \int_0^t \|B(s)\|_\infty ds \right\|_\infty . \quad (3.38)
\]

We will deal first with the third term on the right hand side of the above inequality. Using that

\[
\overline{\nabla}_N[e^N(s)\partial_z H(s)] = \overline{\nabla}_N e^N(s)\partial_z H(s) + e^N(s)\overline{\nabla}_N\partial_z H(s),
\]

we obtain

\[
\left\| \int_0^t 2T_N(t-s)\overline{\nabla}_N e^N(s)\partial_z H(s)ds \right\|_\infty \leq \left\| \int_0^t 2T_N(t-s)\overline{\nabla}_N e^N(s)\partial_z H(s)ds \right\|_\infty \\
+ \left\| \int_0^t 2T_N(t-s)\overline{\nabla}_N e^N(s)\partial_z H(s)ds \right\|_\infty .
\]

Then, since \( T_N(t) \) commutes with \( \overline{\nabla}_N \) and \( T_N(t) \) is a contraction semigroup,

\[
\left\| \int_0^t 2T_N(t-s)\overline{\nabla}_N e^N(s)\partial_z H(s)ds \right\|_\infty \leq 2 \int_0^t \left\| \overline{\nabla}_N T_N(t-s) e^N(s)\partial_z H(s) \right\|_\infty ds + \int_0^t \left\| \overline{\nabla}_N \partial_z H(s) \right\|_\infty \|e^N(s)\|_\infty ds . \quad (3.39)
\]

By the expression (3.37) for the heat semigroup, we then have that

\[
\overline{\nabla}_N T_N(t-s) e^N(s)\partial_z H(s) = \overline{\nabla}_N \sum_m e^{-\beta_m N(t-s)} \langle e^N(s)\partial_z H(s), \varphi_{m,N} \rangle \varphi_{m,N} + \langle e^N(s)\partial_z H(s), \phi_{m,N} \rangle \phi_{m,N} \\
= \sum_m e^{-\beta_m N(t-s)} \langle e^N(s)\partial_z H(s), \varphi_{m,N} \rangle \overline{\nabla}_N \varphi_{m,N} + \langle e^N(s)\partial_z H(s), \phi_{m,N} \rangle \overline{\nabla}_N \phi_{m,N} .
\]

By the definition of \( \varphi_{m,N} \) e \( \phi_{m,N} \) there exists a constant \( c \) such that

\[
\left\| \overline{\nabla}_N \varphi_{m,N} - (-\pi m \phi_{m,N}) \right\| \leq \frac{c}{N} \quad \text{e} \quad \left\| \overline{\nabla}_N \phi_{m,N} - \pi m \varphi_{m,N} \right\| \leq \frac{c}{N} .
\]
Taylor expansion, one can deduce that

Applying the Cauchy-Schwarz inequality and the definition of facts we then get that

Applying this fact to \((N + 4)\), giving us that

\[
\sum_m \exp[\frac{-2N^2(1 - \cos(\pi mN^{-1}))(t - s)}{2N^2} + O(N^{-3})] \leq N.
\]

By a Taylor expansion, one can deduce that \(1 - \cos(\pi mN^{-1}) \geq \frac{\pi^2 m^2}{2N^2} + O(N^{-3})\) and using these two facts we then get that

\[
2 \int_0^t \|\nabla T_N(t - s)[e^N(s)\partial_x H(s)]\|_\infty ds \leq 4c \int_0^t \|\partial_x H(s)\|_\infty \|e^N(s)\|_\infty ds
\]

\[
+ 4 \int_0^t \sum_m \exp \left[-2N^2 \left(\frac{\pi^2 m^2}{2N^2} + O(N^{-3})\right)(t - s)\right] m \|\partial_x H(s)\|_\infty \|e^N(s)\|_\infty ds.
\]

Applying the Cauchy-Schwarz inequality and the definition of \(\beta_{m,N}\),

\[
2 \int_0^t \|\nabla T_N(t - s)[e^N(s)\partial_x H(s)]\|_\infty ds \leq \frac{4c}{N} \int_0^t \sum_m \exp[-2N^2(1 - \cos(\pi mN^{-1}))(t - s)] \|\partial_x H(s)\|_\infty \|e^N(s)\|_\infty ds
\]

\[
+ 4 \int_0^t \sum_m \exp[-2N^2(1 - \cos(\pi mN^{-1}))(t - s)] m \|\partial_x H(s)\|_\infty \|e^N(s)\|_\infty ds.
\]

Applying this fact to \((3.39)\) we infer that

\[
\left\|\int_0^t T_N(t - s)\nabla e^N(s)\partial_x H(s)ds\right\|_\infty \leq \int_0^t (4c\|\partial_x H(s)\|_\infty + \|\nabla \partial_x H(s)\|_\infty) \|e^N(s)\|_\infty ds
\]

\[
+ 4\pi \int_0^t \sum_m \exp \left[-(\pi^2 m^2 + O(N^{-1}))(t - s)\right] m \|\partial_x H(s)\|_\infty \|e^N(s)\|_\infty ds.
\]

We apply now the inequality above on \((3.38)\), giving us that

\[
\|e^N(t)\|_\infty \leq \|e^N(0)\|_\infty + \|\nabla^N(t)\|_\infty + \int_0^t \left(2\|\partial_x^2 H(s)\|_\infty + 2C\right) \|e^N(s)\|_\infty ds
\]

\[
+ \int_0^t \|B(s)\|_\infty + \int_0^t \left(4c\|\partial_x H(s)\|_\infty + \|\nabla \partial_x H(s)\|_\infty\right) \|e^N(s)\|_\infty ds
\]

\[
+ 4\pi \int_0^t \sum_m \exp \left[-(\pi^2 m^2 + O(N^{-1}))(t - s)\right] m \|\partial_x H(s)\|_\infty \|e^N(s)\|_\infty ds.
\]

By Gronwall’s inequality, we get that

\[
\|e^N(t)\|_\infty \leq \left(\|e^N(0)\|_\infty + \|\nabla^N(t)\|_\infty + \int_0^t \|B(s)\|_\infty ds\right) \exp\left\{\int_0^t 2\|\partial_x^2 H(s)\|_\infty + 2C\right.
\]

\[
+ 4c\|\partial_x H(s)\|_\infty + \|\nabla \partial_x H(s)\|_\infty + 4\pi \sum_m \exp \left[-(\pi^2 m^2 + O(N^{-1}))(t - s)\right] m \|\partial_x H(s)\|_\infty ds \right\}.
\]
Since
\[\int_0^t 4\pi \sum_m \exp \left[ - \left( \pi^2 m^2 + O(N^{-1}) \right) (t-s) \right] m \| \partial_x H(s) \|_\infty ds \]
\[\leq 4\| \partial_x H \|_\infty \sum_m \frac{1 - \exp \left[ - \left( \pi^2 m^2 + O(N^{-1}) \right) t \right]}{\pi m} \leq 4\| \partial_x H \|_\infty \sum_m \frac{1}{m} \leq 4\| \partial_x H \|_\infty \log N ,\]
then
\[\| e^N(t) \|_\infty \leq \left( \| e^N(0) \|_\infty + \| \overline{X}^N(t) \|_\infty + \int_0^t \| B(s) \|_\infty ds \right) \times \exp \left\{ \int_0^t 2\| \partial_x^2 H(s) \|_\infty + C + 4e\| \partial_x H(s) \|_\infty + \| \nabla \partial_x H(s) \|_\infty ds \right\} N^4\| \partial_x H \|_\infty / \pi \].

Taking
\[C \triangleq \exp \left\{ \int_0^t 2\| \partial_x^2 H(s) \|_\infty + C + 4e\| \partial_x H(s) \|_\infty + \| \nabla \partial_x H(s) \|_\infty ds \right\} ,\]
we conclude that
\[\| e^N(t) \|_\infty \leq \left( \| e^N(0) \|_\infty + \| \overline{X}^N(t) \|_\infty + \int_0^t \| B(s) \|_\infty ds \right) C N^4\| \partial_x H \|_\infty / \pi \].

Moreover, we observe that
\[\| e^N(0) \|_\infty = \| \overline{X}^N(0) - \psi(0) \|_\infty \leq \frac{\eta_x(0)}{\ell} - \psi(0,x) = \frac{1}{\ell} \left[ \| \ell \psi(0,x) - \ell \psi(0) \|_\infty \right] \leq \frac{1}{\ell},\]
thus \(\| e^N(0) \|_\infty C N^4\| \partial_x H \|_\infty / \pi \to 0\) as \(N \to \infty\) due to the assumption (2.6). Now recalling Lemma 3.12 one can conclude the proof. \(\square\)

4. Large Deviations

4.1. Radon-Nikodym derivative. An important ingredient in the proof of large deviations consists in obtaining a law of large numbers for a class of perturbed processes. To find the rate function we need to calculate the Radon-Nikodym derivative \(\frac{d\mathbb{P}_N}{d\mathbb{P}_N^H} \) where \(\mathbb{P}_N^H\) and \(\mathbb{P}_N\) are measures induced by processes considering \(H \equiv 0\) and a general \(H \in C^{1,2}\), respectively. This is the content of the next proposition.

**Proposition 4.1** (An expression for the Radon-Nikodym derivative). Considering the model described above, the Radon-Nikodym derivative restricted to \(F_t = \sigma(X_s : 0 \leq s \leq t)\) is given by

\[\frac{d\mathbb{P}_N}{d\mathbb{P}_N^H} \bigg|_{F_t} = \exp \left\{ -\ell N \left[ \int_0^t \frac{1}{N} \sum_{k=0}^{N-1} \left[ b(X^N_k(s))(1 - e^{H_k}) + d(X^N_k(s))(1 - e^{-H_k}) - X^N_k(s) \left( \Delta_N H_k + \frac{1}{2} \left( \nabla^2_N H_k \right)^2 + \left( \nabla_N H_k \right)^2 + O(1/N) \right) \right] ds \right. \right. \]
\[\left. - \left. X^N_k(s) \left( \Delta_N H_k + \frac{1}{2} \left( \nabla^2_N H_k \right)^2 + \left( \nabla_N H_k \right)^2 + O(1/N) \right) \right] \bigg|_{F_t} \}
\[+ \frac{1}{N} \sum_{k=0}^{N-1} \left( H_k(t)X^N_k(t) - H_k(0)X^N_k(0) - \int_0^t X^N_k(s) \partial_x H_k ds \right) \bigg|_{F_t} \}
\[\right. \bigg\}.
\]

In particular, we can write
\[\frac{d\mathbb{P}_N}{d\mathbb{P}_N^H} \bigg|_{F_t} = \exp \left\{ -\ell N \left[ J_H(X^N) + O(1/N) \right] \right\},\]
where
\[
J_H(u) = \int_0^t \int_\Omega \left[ b(u(s,y))(1 - e^{H(s,y)}) + d(u(s,y))(1 - e^{-H(s,y)})
- u(s,y) \left( \Delta H(s,y) + (\nabla H(s,y))^2 \right) \right] dy\,ds
+ \int_\Omega \left[ H(t,y)u(t,y) - H(0,y)u(0,y) - \int_0^t u(s,y)\partial_s H(s,y)\,ds \right] dy.
\]

Now we are in position to prove the Proposition 4.1 which is the basis for deriving the rate function of large deviations. To do so, we need the following general result which can be found in [14, Appendix 1, page 320].

**Proposition 4.2.** Let \( P \) and \( \mathcal{P} \) be the probability measures corresponding to two continuous time Markov chains on some countable space \( E \), with bounded waiting times \( \lambda \) and \( \lambda \), respectively, and with transition probabilities \( p \) and \( \mathcal{P} \), respectively. Assume that \( p \) and \( \mathcal{P} \) vanish at the diagonal, that is, \( p(x,x) = \mathcal{P}(x,x) = 0 \) for all \( x \in E \). Assume that \( P \) is absolutely continuous with respect to \( \mathcal{P} \). Then, the Radon-Nikodym derivative of \( P \) with respect to \( \mathcal{P} \) restricted to \( \mathcal{F}_t = \sigma(X(s) : 0 \leq s \leq t) \) is given by
\[
\frac{dP}{d\mathcal{P}}(X) = \exp \left\{ - \int_0^t \lambda(X(s)) - \lambda(X(s))ds + \sum_{s \leq t} \log \left( \frac{\lambda(X(s)p(X(s-),X(s)))}{\lambda(X(s)\mathcal{P}(X(s-),X(s)))} \right) \right\},
\]
where \( X \) denotes a pure jump càdlàg time trajectory on \( E \).

In the case of our work, \( P = \mathbb{P} \) and \( \mathcal{P} = \mathbb{P}^H \). The probabilities \( \mathbb{P} \) and \( \mathbb{P}^H \) are associated to trajectories \( \eta(t) \) of course. However, recalling the definition (2.1), we will often write \( X(t, \lambda) \) instead of \( \eta(t) \), which makes notation shorter and enlightens ideas. Furthermore, recall the notation \( H_k = H(t, \frac{\lambda}{N}) = H(t, \frac{\lambda}{N}) \), where this last equality holds since \( H \) is assumed to be smooth and write for simplicity \( X(t, \cdot) = X_N(t, \cdot) \).

For fixed \( N \), long but elementary calculations give us that
\[
\lambda(X_N(t)) = \sum_{k=0}^{N-1} \ell \left[ b(X_k^N(t)) + d(X_k^N(t)) + 2N^2X_k^N(t) \right],
\]
\[
\bar{\lambda}(X_N(t)) = \sum_{k=0}^{N-1} \ell \left[ b(X_k^N(t))e^{H_k} + d(X_k^N(t))e^{-H_k} + N^2X_k^N(t)e^{H_k}e^{-H_{k+1}} + e^{H_{k+1}} \right],
\]
\[
p(X_N(s-), X_N(s)) = \begin{cases} \ell b(X_k^N(s-))/\lambda(X_N(s-)), & \text{if } \eta_k(s) = \eta_k(s-) + 1; \\
\ell d(X_k^N(s-))/\lambda(X_N(s-)), & \text{if } \eta_k(s) = \eta_k(s-) - 1; \\
N^2\ell X_k^N(s+)/\lambda(X_N(s-)), & \text{if } \eta_k(s) = \eta_k(s-) - 1 \\
& \text{and } \eta_{k+1}(s) = \eta_{k+1}(s-) + 1; \\
N^2\ell X_k^N(s+)/\lambda(X_N(s-)), & \text{if } \eta_k(s) = \eta_k(s-) - 1 \\
& \text{and } \eta_{k-1}(s) = \eta_{k-1}(s-) + 1; \end{cases}
\]
and
\[
\bar{p}(X_N(s-), X_N(s)) = \begin{cases} \ell b(X_k^N(s-))e^{H_k}/\bar{\lambda}(X_N(s-)), & \text{if } \eta_k(s) = \eta_k(s-) + 1; \\
\ell d(X_k^N(s-))e^{-H_k}/\bar{\lambda}(X_N(s-)), & \text{if } \eta_k(s) = \eta_k(s-) - 1; \\
N^2\ell X_k^N(s-)e^{H_{k+1}-H_k}/\bar{\lambda}(X_N(s-)), & \text{if } \eta_k(s) = \eta_k(s-) - 1 \\
& \text{and } \eta_{k+1}(s) = \eta_{k+1}(s-) + 1; \\
N^2\ell X_k^N(s-)e^{H_{k-1}-H_k}/\bar{\lambda}(X_N(s-)), & \text{if } \eta_k(s) = \eta_k(s-) - 1 \\
& \text{and } \eta_{k-1}(s) = \eta_{k-1}(s-) + 1. \end{cases}
\]
Proof of Proposition 4.1. Given a path \( \eta(t) \), define the sets of times

\[
B^k_t = \{ s \leq t : \eta_k(s) = \eta_k(s-) + 1 \},
\]

\[
D^k_t = \{ s \leq t : \eta_k(s) = \eta_k(s-) - 1 \},
\]

\[
J^{k,k+1}_t = \{ s \leq t : \eta_k(s) = \eta_k(s-) - 1 \text{ and } \eta_{k+1}(s) = \eta_{k+1}(s-) + 1 \},
\]

\[
J^{k,k-1}_t = \{ s \leq t : \eta_k(s) = \eta_k(s-) - 1 \text{ and } \eta_{k-1}(s) = \eta_{k-1}(s-) + 1 \}.
\]

Note that \( B^k_t \) represents the set of times at which some particle is created at the site \( k \) and we have similar interpretations for \( D^k_t \), \( J^{k,k+1}_t \) and \( J^{k,k-1}_t \). Invoking Proposition 4.2, the expressions (4.2), (4.3), (4.4) and the sets defined above, we deduce that

\[
\frac{\text{d}P_N}{\text{d}P_N|_{F_t}} = \exp \left\{ - \int_0^t \sum_{k=0}^{N-1} \ell \left[ b(X^N_k(s))(1 - e^{H_k}) + d(X^N_k(s))(1 - e^{-H_k}) \right. \right.
\]

\[
\left. + N^2 X^N_k(s) \left( 2 - e^{H_{k+1} - H_k} - e^{H_{k-1} - H_k} \right) \right] ds
\]

\[
+ \int_0^t \sum_{s \in B^k_t} (-H_k) + \sum_{s \in D^k_t} H_k + \sum_{s \in J^{k,k+1}_t} (H_k - H_{k+1}) + \sum_{s \in J^{k,k-1}_t} (H_k - H_{k-1}) \right\}.
\]

Since \( H \) is smooth, by a Taylor expansion on the exponential function,

\[
2 - e^{H_{k+1} - H_k} - e^{H_{k-1} - H_k} = -H_{k+1} + H_k - \frac{1}{2!}(H_{k+1} - H_k)^2 - H_{k-1} + H_k - \frac{1}{2!}(H_{k-1} - H_k)^2 + O(1/N^3),
\]

hence

\[
N^2 X^N_k(s) \left( 2 - e^{H_{k+1} - H_k} - e^{H_{k-1} - H_k} \right) = -X^N_k(s) \left( \Delta_N H_k + \frac{1}{2} \left( (\nabla^+_N H_k)^2 + (\nabla^-_N H_k)^2 \right) + O(1/N) \right).
\]

Moreover,

\[
\sum_{s \in B^k_t} (-H_k) + \sum_{s \in D^k_t} H_k + \sum_{s \in J^{k,k+1}_t} (H_k - H_{k+1}) + \sum_{s \in J^{k,k-1}_t} (H_k - H_{k-1})
\]

\[
= \int_0^t (-H_k) dB^k_t + \int_0^t H_k dD^k_t + \int_0^t (H_k - H_{k+1}) dJ^{k,k+1}_t + \int_0^t (H_k - H_{k-1}) dJ^{k,k-1}_t
\]

\[
= - \int_0^t H_k (dB^k_t - dD^k_t - dJ^{k,k+1}_t + dJ^{k-1,k}_t - dJ^{k,k-1}_t + dJ^{k,k+1}_t)
\]

\[= - \int_0^t H_k d\eta_k(t).
\]

Therefore,

\[
\frac{\text{d}P_N}{\text{d}P_N|_{F_t}} = \exp \left\{ - \ell N \left[ \int_0^t \frac{1}{N} \sum_{k=0}^{N-1} \left[ b(X^N_k(s))(1 - e^{H_k}) + d(X^N_k(s))(1 - e^{-H_k}) \right. \right. \right.
\]

\[
\left. \left. - X^N_k(s) \left( \Delta_N H_k + \frac{1}{2} \left( (\nabla^+_N H_k)^2 + (\nabla^-_N H_k)^2 \right) + O(1/N) \right) \right] ds + \frac{1}{\ell N} \sum_{k=0}^{N-1} \int_0^t H_k d\eta_k(t) \right\}.
\]

Applying the integration by parts formula for Stieltjes measures (see for instance [8, Exercise 6.4, page 470]) and the relation (2.1), we are lead to

\[
\frac{1}{\ell N} \int_0^t H_k d\eta_k(t) = \frac{1}{\ell N} \left[ H_k(t)\eta_k(t) - H_k(0)\eta_k(0) - \int_0^t \eta_k(s) d\eta_k H_k ds \right]
\]

\[
= \frac{1}{N} \left[ H_k(t)X^N_k(t) - H_k(0)X^N_k(0) - \int_0^t X^N_k(s) d\eta_k H_k ds \right].
\]
Therefore,
\[
\frac{dP_N}{dP_N^H}\Big|_{F_i} = \exp \left\{ -\ell N \left[ \int_0^t \frac{1}{N} \sum_{k=0}^{N-1} \left[ b(X_k^N(s)) (1 - e^{H_k}) + d(X_k^N(s)) (1 - e^{-H_k}) - X_k^N(s) \left( \Delta_N H_k + \frac{1}{2} \left( (\nabla^2_N H_k)^2 + (\nabla_N \Delta H_k) \right) + O(1/N) \right) \right] ds \\
+ \frac{1}{N} \sum_{k=0}^{N-1} \left( H_k(t) X_k^N(t) - H_k(0) X_k^N(0) - \int_0^t X_k^N(s) \partial_s H_k ds \right) \right] \right\}
\]
where
\[
J_H(u) = \int_0^t \int_T \left[ b(u(s, y)) (1 - e^{H(s, y)}) + d(u(s, y)) (1 - e^{-H(s, y)}) - u(s, y) \left( \Delta H(s, y) + (\nabla H(s, y))^2 \right) \right] dy ds \\
+ \int_T \left[ H(t, y) u(t, y) - H(0, y) u(0, y) - \int_0^t u(s, y) \partial_s H(s, y) ds \right],
\]
finishing the proof. \(\square\)

4.2. Large deviations upper bound. With the aid of the Theorem 4.1, we will get the upper bound for the large deviations. Recall that \(P_N, E_N\) denote the probability and expectation, respectively, on trajectories of the particle system, while \(P_N^H, E_N^H\) denote probability and expectation induced by the density of particles \(X^N\), respectively. Furthermore, the super-index \(H\) on \(P_N^H, E_N^H, P_N, E_N\) have analogous meaning, but considering instead the perturbed process defined on Subsection 2.2. Let \(\mathcal{O} \subseteq \mathcal{P}(T)\) be an open set. Then
\[
P_N[\mathcal{O}] = P_N[X^N \in \mathcal{O}] = E_N[1_{\{X^N \in \mathcal{O}\}}] = E_N\left[ \frac{dP_N}{dP_N^H} \frac{dP_N^H}{dP_N} 1_{\{X^N \in \mathcal{O}\}} \right]
= E_N\left[ e^{-\ell N J_H(X^N)} e^{\ell N J_H(X^N)} 1_{\{X^N \in \mathcal{O}\}} \right] \leq \sup_{x \in \mathcal{O}} e^{-\ell N J_H(x)} E_N\left[ e^{\ell N J_H(X^N)} 1_{\{X^N \in \mathcal{O}\}} \right]
\leq \sup_{x \in \mathcal{O}} e^{-\ell N J_H(x)},
\]
Therefore,
\[
\limsup_{N \to \infty} \frac{1}{\ell N} \log P_N[\mathcal{O}] \leq -\inf_{x \in \mathcal{O}} J_H(x).
\]
Optimizing over the set of perturbations, we then get
\[
\limsup_{N \to \infty} \frac{1}{\ell N} \log P_N[\mathcal{O}] \leq -\sup_{H} \inf_{x \in \mathcal{O}} J_H(x).
\] (4.5)
To pass to compact sets, we will apply the classical Minimax Lemma. To be used in the sequel, we recall that
\[
\limsup_{n \to \infty} \frac{1}{a_n} \log(b_n + c_n) = \max \left\{ \limsup_{n \to \infty} \frac{1}{a_n} \log b_n, \limsup_{n \to \infty} \frac{1}{a_n} \log c_n \right\}
\] (4.6)
for any sequence of real numbers such that \(a_n \to \infty\) and \(b_n, c_n > 0\).

**Proposition 4.3** (Minimax Lemma). Let \(\mathcal{K} \subseteq S\) compact, where \((S, d)\) is a Polish space. Given \(-J_H\) a family of upper semi-continuous functions, it holds that
\[
\inf_{\mathcal{O}_1, \ldots, \mathcal{O}_M} \max_{1 \leq j \leq M} \inf_{\mathcal{O}_j} \sup_{H} -J_H(x) \leq \sup_{x \in \mathcal{K}} \inf_{H} -J_H(x),
\] (4.7)
where first infimum is taken over all finite open coverings \(\mathcal{O}_1, \ldots, \mathcal{O}_M\) of \(\mathcal{K}\).
For a proof of above, see [14, page 363]. Let now \( \mathcal{K} \) be a compact set of \( \mathcal{D}([0,T], C(\mathbb{T})) \). Taking \( \{O_1, \ldots, O_M\} \) a finite open covering of \( \mathcal{K} \), then

\[
\limsup_{N \to \infty} \frac{1}{tN} \log P_N[\mathcal{K}] \leq \limsup_{N \to \infty} \frac{1}{tN} \log \left( P_N[O_1] + \cdots + P_N[O_M] \right)
\]

(4.6)

\[
\leq \max_{1 \leq j \leq M} \left\{ \limsup_{N \to \infty} \frac{1}{tN} \log P_N[O_j] \right\}
\]

(4.5)

\[
\leq \max_{1 \leq j \leq M} \left\{ \inf_{x \in O_j} \limsup_{N \to \infty} D(x) \right\}
\]

(4.7)

\[
\leq - \inf_{x \in \mathcal{K}} D(x),
\]

which furnishes the upper bound for compact sets. The next proposition is the usual key to pass to closed sets. Denote by \( \{P_n\}_{n \in \mathbb{N}} \) a general sequence of probability measures on some metric space \( \Omega \). It is a consequence of (4.6) the following standard result:

**Proposition 4.4.** A sequence of measures \( \{P_n\}_{n \in \mathbb{N}} \) on \( \Omega \) is said to be exponentially tight if, for any \( b < \infty \), there exists a compact set \( \mathcal{K}_b \subseteq \Omega \) such that

\[
\limsup_{n \to \infty} \frac{1}{a_n} \log P_n[\mathcal{K}_b^c] \leq -b,
\]

(4.8)

where \( a_n \) is constant depending on \( n \). Suppose that \( \{P_n\}_{n \in \mathbb{N}} \) is exponentially tight and we have the large deviations upper bound for compact sets, that is, for each compact set \( \mathcal{K} \subseteq \Omega \), it holds that

\[
\limsup_{n \to \infty} \frac{1}{a_n} \log P_n[\mathcal{K}] \leq - \inf_{x \in \mathcal{K}} I(x).
\]

(4.9)

Then, for any closed \( C \subseteq \Omega \),

\[
\limsup_{n \to \infty} \frac{1}{a_n} \log P_n[C] \leq - \inf_{x \in C} I(x).
\]

**Proof.** Note that, given the subsets \( C \) and \( \mathcal{K}_b \) of \( \Omega \), with closed \( C \) and compact \( \mathcal{K}_b \), we have that

\[
\frac{1}{a_n} \log P_n[C] \leq \frac{1}{a_n} \log \left( P_n[C \cap \mathcal{K}_b] + P_n[K_b^c] \right).
\]

Taking the upper limit and using (4.6) in the above equation, we have that

\[
\limsup_{n \to \infty} \frac{1}{a_n} \log P_n[C] \leq \max \left\{ \limsup_{n \to \infty} \frac{1}{a_n} \log P_n[C \cap \mathcal{K}_b], \limsup_{n \to \infty} \frac{1}{a_n} P_n[K_b^c] \right\}.
\]

Since \( \mathcal{K}_b \cap C \) is compact, by (4.9) and (4.8),

\[
\limsup_{n \to \infty} \frac{1}{a_n} \log P_n[C] \leq \max \left\{ - \inf_{x \in C \cap \mathcal{K}_b} I(x), -b \right\} \leq \max \left\{ - \inf_{x \in C} I(x), -b \right\}.
\]

Taking \( b \to \infty \), we conclude the proof. \( \square \)

In view of above, in order to prove the large deviations upper bound, it remains to assure exponential tightness for the sequence of probability measures \( P_N \) on \( \mathcal{D} \) induced by the random element \( X^N \) and the probability \( P_N \). Denote by \( \| \cdot \|_1 \) the \( L^1 \)-norm on \( \mathbb{T} \) with respect to the Lebesgue measure.

**Proposition 4.5.** Let \( C \in \mathbb{R} \) be such that \( C - \|X^N(0)\|_1 > T \|b\|_\infty \). Then,

\[
\frac{1}{tN} \log P_N \left[ \sup_{t \in [0,T]} \|X^N(t)\|_1 > C \right] \leq -I(C - \|X^N(0)\|_1),
\]

(4.10)

for any \( N \in \mathbb{N} \), where \( I(x) = x \log \left( \frac{x}{\|b\|_\infty} \right) - x + \|b\|_\infty \).
which implies that, almost surely,

\[ W^N(t) \geq \sum_{k \in \mathbb{T}_N} \eta_k(t), \quad \forall t \in [0, T], \]

which implies that, almost surely,

\[ \frac{1}{\ell N} W^N(t) \geq \frac{1}{\ell N} \sum_{k \in \mathbb{T}_N} \eta_k(t) = \|X^N(t)\|_1, \quad \forall t \in [0, T]. \tag{4.11} \]

Proposition 4.6. For every continuous function \( H : [0, +\infty) \times \mathbb{T} \to \mathbb{R} \) and \( \varepsilon > 0 \),

\[ \lim_{\delta \downarrow 0} \limsup_{N \to \infty} \frac{1}{\ell N} \log \mathbb{P}_N \left[ \sup_{[t-s] < \delta} \left| \langle X^N(t), H(t) \rangle - \langle X^N(s), H(s) \rangle \right| > \varepsilon \right] = -\infty. \tag{4.13} \]

Proof. Partitioning the time interval \([0, T]\) in intervals of size at most \( \delta \) and applying the triangular inequality together with (4.6), one can see that it is enough to assure that

\[ \lim_{\delta \downarrow 0} \limsup_{N \to \infty} \frac{1}{\ell N} \log \mathbb{P}_N \left[ \sup_{k \delta \leq t \leq (k+1)\delta} \left| \langle X^N(t), H(t) \rangle - \langle X^N(k\delta), H(k\delta) \rangle \right| > \varepsilon \right] = -\infty \tag{4.14} \]

in order to have (4.13). Therefore, our goal from now on is to prove (4.14) for fixed \( K \in \{1, \ldots, [T/\delta]\} \). Since \(|x| = \max\{x, -x\}\) and using (4.6), it is enough to show that

\[ \lim_{\delta \downarrow 0} \limsup_{N \to \infty} \frac{1}{\ell N} \log \mathbb{P}_N \left[ \sup_{K\delta \leq t \leq (K+1)\delta} \left( \langle X^N(t), H(t) \rangle - \langle X^N(K\delta), H(K\delta) \rangle \right) > \varepsilon \right] = -\infty \tag{4.15} \]

and

\[ \lim_{\delta \downarrow 0} \limsup_{N \to \infty} \frac{1}{\ell N} \log \mathbb{P}_N \left[ \sup_{K\delta \leq t \leq (K+1)\delta} \left( \langle X^N(t), H(t) \rangle - \langle X^N(K\delta), H(K\delta) \rangle \right) < -\varepsilon \right] = -\infty. \tag{4.16} \]
We will only prove (4.15) whereas the argument for (4.16) is similar. Analogously to (4.1), we may find
\[
A^N_n(t) = \int_0^t \frac{1}{N} \sum_{k=0}^{N-1} \left[ b(X^N_k(s))(1-e^{aH_k}) + d(X^N_k(s))(1-e^{-aH_k}) - X^N_k(s)\left( a\Delta_N H_k + \frac{a^2}{2} \left( (\nabla_X H_k)^2 + (\nabla_N H_k)^2 \right) + O(1/N) \right) \right] ds
\]
\[
+ \frac{a}{N} \sum_{k=0}^{N-1} \left( H_k(t)X^N_k(t) - H_k(K\delta)X^N_k(K\delta) - \int_{K\delta}^t X^N_k(s)\partial_s H_k ds \right)
\]
such that \( \exp \left\{ -\ell N A^N_n \right\} \) is a mean-one martingale. Define \( R^N_n \) by the equality
\[
R^N_n(t) = A^N_n(t) - \frac{a}{N} \sum_{k=0}^{N-1} \left( H_k(t)X^N_k(t) - H_k(K\delta)X^N_k(K\delta) \right)
\]
\[
= A^N_n(t) - a \left[ \langle X^N(t), H(t) \rangle - \langle X^N(K\delta), H(K\delta) \rangle \right].
\]
Then,
\[
P_N \left[ \sup_{K\delta \leq t \leq (K+1)\delta} \left( \langle X^N(t), H(t) \rangle - \langle X^N(K\delta), H(K\delta) \rangle \right) > \varepsilon \right] = P_N \left[ \sup_{K\delta \leq t \leq (K+1)\delta} (A^N_n(t) - R^N_n(t)) > a\varepsilon \right] = P_N \left[ \sup_{K\delta \leq t \leq (K+1)\delta} e^{\ell N(A^N_n(t) - R^N_n(t))} > e^{a\varepsilon} \right].
\]
Define the event
\[
E = \left[ \sup_{t \in [0,T]} \| X^N(t) \|_1 \leq C \right].
\]
Restrict to \( E \), it is straightforward to check that \( |R^N_n| \leq m(H,b,d)C\delta \), where \( m(H,b,d) \) is a constant depending only on \( H \), on its first and second derivatives and on the Lipschitz constant of \( b \) and \( d \). Note that the factor \( \delta \) appears since the integral in time is taken over the interval \( [K\delta,t] \). Hence, partitioning into \( E \) and \( E^c \), we have that
\[
P_N \left[ \sup_{K\delta \leq t \leq (K+1)\delta} e^{\ell N(A^N_n(t) - R^N_n(t))} > e^{a\varepsilon} \right]
\]
\[
\leq P_N \left[ \sup_{K\delta \leq t \leq (K+1)\delta} e^{\ell N A^N_n(t)} > e^{\ell N(a\varepsilon - m(H,b,d)C\delta)} \right] + P_N \left[ E^c \right].
\]
By Doob's inequality, the right hand side of above is bounded from above by
\[
\mathbb{E}_N \left[ e^{\ell N A^N_n(t)} \right] / e^{\ell N(a\varepsilon - m(H,b,d)C\delta)} + P_N \left[ E^c \right] = \exp \left\{ -\ell N(a\varepsilon - m(H,b,d)C\delta) \right\} + P_N \left[ E^c \right].
\]
Applying the logarithm function in (4.17), dividing it by \( \ell N \), taking the \( \limsup_N \), and recalling (4.6) give us that
\[
\limsup_{N \to \infty} \frac{1}{\ell N} \log P_N \left[ \sup_{K\delta \leq t \leq (K+1)\delta} e^{\ell N(A^N_n(t) - R^N_n(t))} > e^{a\varepsilon} \right]
\]
\[
\leq \max \left\{ -\left( a\varepsilon - m(H,b,d)C\delta \right), \limsup_{N \to \infty} \frac{1}{\ell N} \log P_N \left[ E^c \right] \right\}.
\]
Applying Proposition 4.5, we can bound the expression above by
\[
\max \left\{ -\alpha \varepsilon + m(H, b, d)C\delta, \lim_{N \to \infty} -I(C - \|X^N(0)\|_1) \right\}
\]
\[
= \max \left\{ -\alpha \varepsilon + m(H, b, d)C\delta, -I(C - \|\psi(0)\|_1) \right\}.
\]
Since \(\lim_{x \to \infty} I(x) = \infty\), we are allowed to first choose \(C\) large, then \(\delta\) small, and then finally \(a\) large, leading us to conclude that
\[
\limsup_{N \to \infty} \frac{1}{\ell N} \log P_N \left[ \sup_{K\delta \leq t \leq (K+1)\delta} e^{\ell N(A^N(t) - R^N(t))} > e^{a\varepsilon N} \right] = -\infty,
\]
finishing the proof. \(\square\)

**Proposition 4.7.** The sequence of measures \(\{P_N\}_{N \in \mathbb{N}}\) on \(\mathcal{D}_{C(T)}\) is exponentially tight.

**Proof.** Using the (4.13), we obtain the sequence of compact sets satisfying (4.8). Define the following sets:
\[
L_c = \left\{ u \in \mathcal{D}_{C(T)} : \|u_0\|_\infty \leq c \right\},
\]
\[
C_{\delta,1/n} = \left\{ u \in \mathcal{D}_{C(T)} : \sup_{|t-s| < \delta} \|u_t - u_s\|_\infty \leq 1/n \right\},
\]
\[
A = \left( \cap_{n=1}^\infty C_{\delta,1/n} \right) \cap L_c.
\]
By the Arzelá-Ascoli Theorem, the set \(A\) is pre-compact, hence \(\overline{A}\) is compact. Taking \(\{H_j\}_{j \in \mathbb{N}}\) a dense set in \(C(\mathbb{T})\), let us define
\[
C_{\delta,1/n}^{H_j} = \left\{ u \in \mathcal{D}_{C(T)} : \sup_{|t-s| < \delta} \left| \int u_t(x)H_j(t,x)dx - \int u_s(x)H_j(s,x)dx \right| \leq 1/n \right\}
\]
and
\[
B_\delta = L_c \cap \left( \cap_{j,n=1}^\infty C_{\delta,1/n}^{H_j} \right).
\]
Our goal is to prove that \(\overline{B_\delta}\) is compact, so it suffices to verify that \(B_\delta \subseteq A\). Let \(u \in \left( \cap_{n=1}^\infty C_{\delta,1/n}^{C_{\delta,1/n}} \right)\), then there exists \(n_0 \in \mathbb{N}\) such that \(u \in C_{\delta,1/n_0}\), that is, there exists \(|t-s| < \delta\) such that \(\|u_t - u_s\|_\infty > 1/n\). Since \(\{H_j\}_{j \in \mathbb{N}}\) is dense, there exists \(H_{j_0}\) with \(\int u_t(x)H_{j_0}(t,x)dx - \int u_s(x)H_{j_0}(s,x)dx > 1/n\), hence \(u \in (C_{\delta,1/n_0}^{H_{j_0}})\). Finally we show (4.8). Note that
\[
\limsup_{N \to \infty} \frac{1}{\ell N} \log P_N \left[ B_\delta \right]
\]
\[
= \limsup_{N \to \infty} \frac{1}{\ell N} \log P_N \left[ L_c \cap \left( \cap_{j,n=1}^\infty C_{\delta,1/n}^{H_j} \right) \right]
\]
\[
\leq \limsup_{N \to \infty} \frac{1}{\ell N} \log \left[ P_N \left[ L_c \right] + \sum_{j,n=1}^\infty P_N \left[ C_{\delta,1/n}^{H_j} \right] \right]
\]
\[
\leq \max \left\{ \limsup_{N \to \infty} \frac{1}{\ell N} \log P_N \left[ L_c \right], \limsup_{N \to \infty} \frac{1}{\ell N} \log \left[ \sum_{j,n=1}^\infty P_N \left[ C_{\delta,1/n}^{H_j} \right] \right] \right\},
\]
where in second inequality we have used (4.6). Since
\[
\limsup_{N \to \infty} \frac{1}{\ell N} \log P_N \left[ \|X^N(0)\|_\infty > c \right] = -\infty,
\]
then
\[
\limsup_{N \to \infty} \frac{1}{\ell N} \log P_N \left[ B_\delta \right] \leq \limsup_{N \to \infty} \frac{1}{\ell N} \log \left[ \sum_{j,n=1}^\infty P_N \left[ C_{\delta,1/n}^{H_j} \right] \right].
\] (4.18)
By (4.13), there exists $\delta_0$ such that
\[
\limsup_{N \to \infty} \frac{1}{\ell N} \log P_N \left( \left( C_H \right)^{\ell} \right) \leq -\frac{-b_0}{\varepsilon},
\]
and there exists $N_0$ such that for all $N > N_0$,
\[
\frac{1}{\ell N} \log P_N \left( \left( C_H \right)^{\ell} \right) \leq -\frac{-b_0}{\varepsilon}.
\]
Therefore,
\[
\sum_{j,n=1}^{\infty} P_N \left( \left( C_H \right)^{\ell} \right) \leq \sum_{j,n=1}^{\infty} \exp\{ -b_0 \ell N \} = \frac{e^{-b_0 \ell N}}{1 - e^{-b_0 \ell N}} \leq 2e^{-b_0 \ell N}.
\]
Then, coming back to (4.18),
\[
\limsup_{N \to \infty} \frac{1}{\ell N} \log P_N \left( \left( B \right)^{\ell} \right) < \limsup_{N \to \infty} \frac{1}{\ell N} \log \left( 2e^{-b_0 \ell N} \right) = -b_0.
\]
Now, taking $b = b_0$ we obtain the exponential tightness (4.8) hence finishing the proof. \[\square\]

Therefore, with the Lemma 4.4 and Proposition 4.7 at hand we have concluded the proof of the upper bound for large deviations.

4.3. **Large deviations lower bound in the power law case.** Next, we obtain a non-variational formulation of the rate functional $I$ for profiles $\psi$ which are solutions of the partial differential equation corresponding to the perturbed process associated to some perturbation $H$.

**Proposition 4.8.** Given $H \in C^{1,2}$, let $\psi = \psi^H$ be the unique solution of (2.4). Then,
\[
I(\psi) \overset{\text{def}}{=} \sup_G J_G(\psi) = J_H(\psi) = \int_0^t \int_T \left[ (\partial_x H)^2 \psi + b(\psi) \Gamma(H) + d(\psi) \Gamma(-H) \right] dx \, ds, \tag{4.19}
\]
where $\Gamma(y) = 1 - e^y + ye^y$, $y \in \mathbb{R}$.

**Proof.** Multiplying the PDE (2.4) by a test function $G \in C^{1,2}$ and integrating in space and time, we get that
\[
\int_0^t \int_T G \partial_t \psi \, dx \, ds = \int_0^t \int_T G^2 \partial_{xx} \psi - 2G \partial_x \psi \partial_x H + G[e^H b(\psi) - e^-H d(\psi)] \, dx \, ds.
\]
Using integration by parts and that
\[
Ge^H b(\psi) = b(\psi)\overline{\Gamma}(G, H) - b(\psi)(1 - e^G),
\]
\[
-Ge^{-H} d(\psi) = d(\psi)\overline{\Gamma}(-G, -H) - d(\psi)(1 - e^{-G}),
\]
where $\overline{\Gamma}(x, y) = 1 - e^x + xe^y$, we infer that
\[
\int_T \left[ G(t, x)\psi(t, x) - G(0, x)\psi(0, x) - \int_0^t \psi(s, x)\partial_t G(s, x) \, ds \right] dx = \int_0^t \int_T \partial_{xx} G(s, x)\psi(s, x) \, dx \, ds
\]
+ \int_0^t \int_T 2\psi(s, x)\partial_x G(s, x)\partial_x H(s, x) \, dx \, ds + \int_0^t \int_T b(\psi(s, x))\overline{\Gamma}(G(s, x), H(s, x))
- b(\psi(s, x))(1 - e^{G(s,x)}) + d(\psi(s, x))\overline{\Gamma}(-G(s, x), -H(s, x)) - d(\psi(s, x))(1 - e^{-G(s,x)}) \, dx \, ds,
\]
Recall the definition of $J_H$ in (2.7). The equality above allows us to deduce that
\[
J_G(\psi) = \int_0^t \int_T \left[ -\psi(\partial_x G)^2 + 2\psi\partial_x G\partial_x H + b(\psi)\overline{\Gamma}(G, H) + d(\psi)\overline{\Gamma}(-G, -H) \right] dx \, ds.
\]
Finally, noting that $2\partial_x G\partial_x H = -(\partial_x G - \partial_x H)^2 + (\partial_x G)^2 + (\partial_x H)^2$, we arrive at
\[
J_G(\psi) = \int_0^t \int_T \left[ -(\partial_x G - \partial_x H)^2 \psi + (\partial_x H)^2 \psi + b(\psi)\overline{\Gamma}(G, H) + d(\psi)\overline{\Gamma}(-G, -H) \right] dx \, ds.
\]
Fix \( y \in \mathbb{R} \). Since the function \( x \mapsto \Gamma(x, y) \) assumes its maximum at \( x = y \) and \(- (\partial_x G - \partial_x H)^2\) assumes its maximum at \( G = H \), we conclude that \( I(\psi) = \sup_G j_G(\psi) = j_H(\psi) \). Since \( \Gamma(y) = \Gamma(y, y) \), we obtain (4.19).

Solutions of (2.4) for some \( H \) provides the special representation above for the rate function. It is thus natural to find the set of profiles \( \psi \) for which we may find a perturbation \( H \) fulfilling the requirements in order to permit the high density limit (towards \( \psi \)).

**Proposition 4.9.** Let \( \psi \in C^{2,3} \) such that \( \psi \geq \varepsilon \) for some \( \varepsilon > 0 \). Then, there exists a unique solution \( H \in C^{1,2} \) of the elliptic equation

\[
\partial_{xx}^2 H + \frac{\partial_x \psi}{\psi} \partial_x H = \frac{\partial^2_x \psi - \partial_t \psi}{2\psi} + e^H b(\psi) - e^{-H} d(\psi) .
\]

**Proof.** For each fixed time \( t \in [0, T] \), equation (4.20) is a non-linear second order ordinary differential equation on the interval \([0, 1]\). As an ODE in \([0, 1]\) any of its solutions can be written as the sum of a particular solution of (4.20) plus some solution of the homogeneous part

\[
\partial_{xx}^2 H + \frac{\partial_x \psi}{\psi} \partial_x H = 0 .
\]

Solving (4.21) and then properly choosing constants allows to find a particular solution of (4.20) such that \( H(0) = H(1) \), \( \partial_x H(0) = \partial_x H(1) \) and \( \partial_{xx}^2 H(0) = \partial_{xx}^2 H(1) \), that is, such a solution \( H \) belongs to \( C^{1,2} \). Details are omitted here.

By Proposition 4.8, a profile which is a solution of (2.4) for some \( H \) provides a special representation for the rate function. This together with Proposition 4.9 are the motivation for the definition of the set \( \mathcal{P}_{\text{pert}}^\alpha \) given in Definition 2.5.

Due to the Proposition 4.9 and Remark 2.4, given \( \psi \in \mathcal{P}_{\text{pert}}^\alpha \), we can find \( H = H(\psi) \in C^{1,2} \) such that the assumptions of Theorem 2.2 are satisfied. In words, the perturbed process (under the perturbation \( H \)) has a high density limit, and the limiting profile is the aforementioned \( \psi \). We are now in position to prove the lower bound for trajectories in \( \mathcal{P}_{\text{pert}}^\alpha \). Before, we need to gather some ingredients, which are given by the next four lemmas.

**Lemma 4.10.** Let \( C \in \mathbb{R} \) be such that \( C - \|X^N(0)\|_1 > T\|b + H\|_\infty \). Then,

\[
\frac{1}{EN} \log \mathbb{P}_N \left[ \sup_{t \in [0, T]} \|X^N(t)\|_1 > C \right] \leq - I(C - \|X^N(0)\|_1) ,
\]

for any \( N \in \mathbb{N} \), where \( I(x) = x \log \left( \frac{x}{\|b + H\|_\infty} \right) - x + \|b + H\|_\infty \).

**Proof.** Note that the probability above is the one associated to the perturbed process. The proof of the inequality (4.22) is exactly the same as that one of Proposition 4.5 once we replace \( \|b\|_\infty \) by \( \|b + H\|_\infty \).

**Lemma 4.11.** The expectation \( \mathbb{E}_N \left[ \frac{1}{EN} \log \frac{d\mathbb{P}_N}{d\mathbb{P}_N^\psi} \right]^2 \) is uniformly bounded on \( N \in \mathbb{N} \).

**Proof.** By Proposition 4.1, it not difficult to see that

\[
\left| \frac{1}{EN} \log \frac{d\mathbb{P}_N}{d\mathbb{P}_N^\psi} \right| \leq f(X^N) \overset{\text{def}}{=} \tilde{c} \int_0^t \left( \|X^N(t)\| + \|X^N(0)\| + \int_0^t \|X^N(s)\| \, ds \right) \, dx
\]

for some \( \tilde{c} = c(H) > 0 \). Observe that

\[
f(X^N) \leq \tilde{c} \cdot (2 + t) \sup_{t \in [0, T]} \|X^N(t)\|_1 .
\]

As a consequence of Lemma 4.10,

\[
\frac{1}{EN} \log \mathbb{P}_N \left[ \frac{f(X^N)}{c(2 + t)} > C \right] \leq \frac{1}{EN} \log \mathbb{P}_N \left[ \sup_{t \in [0, T]} \|X^N(t)\|_1 > C \right] \leq - I(C - \|X^N(0)\|_1) ,
\]
for any $N \in \mathbb{N}$, where $C$ and $I$ above are the same as in the statement of Lemma 4.10. Replacing $C$ by $\sqrt{k}/c(2+t)$, where $k \in \mathbb{N}$ is large enough, we infer that
\[
\mathbb{P}_N^H \left[ f(X^N) > \sqrt{k} \right] \leq \exp \left\{ -\ell N I \left( \frac{\sqrt{k}}{c(2+t)} - \|X^N(0)\|_1 \right) \right\},
\]
thus
\[
\mathbb{P}_N^H \left[ f(X^N)^2 > k \right] \leq \exp \left\{ -\ell N I \left( \frac{\sqrt{k}}{c(2+t)} - \|X^N(0)\|_1 \right) \right\}
\]
\[
\leq \exp \left\{ -I \left( \frac{\sqrt{k}}{c(2+t)} - \|X^N(0)\|_1 \right) \right\},
\]
for all $k \geq k_0$ with $k_0 \in \mathbb{N}$. Keep in mind that the choice of $k_0$ does not depend on $\ell$ neither $N$, see the statement of Lemma 4.10. Since $I(x) = x \log \left( \frac{e^x}{\|be^H\|_\infty} \right) - x + \|be^H\|_\infty$, some simple analysis permits to deduce that
\[
\sum_{k \geq k_0} \mathbb{P}_N^H \left[ f(X^N)^2 > k \right] \leq c_1 < \infty,
\]
for some suitably large $k_0 \in \mathbb{N}$. This allows to finish the proof. \qed

Recall the definition of $\mathcal{D}_\text{pert}^\alpha$ given in Definition 2.5.

**Lemma 4.12.** Let $\psi \in \mathcal{D}_\text{pert}^\alpha$, $\mathcal{O}$ be an open set of $\mathcal{D}(\mathbb{T})$ such that $\psi \in \mathcal{O}$ and $H \in C^{1,2}$ the solution of (4.20). Then
\[
\lim_{N \to \infty} \mathbb{E}_N^H \left[ \mathbf{1}_{\{X^N \in \mathcal{O}\}} \frac{1}{\ell N} \log \frac{d\mathbb{P}_N^H}{d\mathbb{P}_N^{\psi}} \right] = 0. \tag{4.23}
\]

**Proof.** By the Lemma (4.11) and the Cauchy-Schwarz inequality,
\[
\mathbb{E}_N^H \left[ \mathbf{1}_{\{X^N \in \mathcal{O}\}} \frac{1}{\ell N} \log \frac{d\mathbb{P}_N^H}{d\mathbb{P}_N^{\psi}} \right] \leq \sqrt{\mathbb{E}_N^H \left[ \left\{ X^N \in \mathcal{O}\right\} \right]} \sqrt{\mathbb{E}_N^H \left[ \left( \frac{1}{\ell N} \log \frac{d\mathbb{P}_N^H}{d\mathbb{P}_N^{\psi}} \right)^2 \right]},
\]
which proves (4.23) due to the Theorem 2.2, concluding the proof. \qed

We make now the classical connection between the rate function and the entropy between the process of reference and the perturbed process.

**Lemma 4.13.** Let
\[
H \left( \mathbb{P}_N^H | \mathbb{P}_N \right) \overset{\text{def}}{=} \mathbb{E}_N^H \left[ \log \frac{d\mathbb{P}_N^H}{d\mathbb{P}_N} \right]
\]
be the relative entropy of $\mathbb{P}_N^H$ with respect to $\mathbb{P}_N$. Then,
\[
\lim_{N \to \infty} \frac{1}{\ell N} H \left( \mathbb{P}_N^H | \mathbb{P}_N \right) = I(\psi),
\]
where $\psi$ is the (unique) solution of (2.4).

**Proof.** Note that
\[
\frac{1}{\ell N} H \left( \mathbb{P}_N^H | \mathbb{P}_N \right) = \frac{1}{\ell N} \mathbb{E}_N^H \left[ \log \frac{d\mathbb{P}_N^H}{d\mathbb{P}_N} \right] = -\frac{1}{\ell N} \mathbb{E}_N^H \left[ \log \frac{d\mathbb{P}_N}{d\mathbb{P}_N^H} \right].
\]
Recalling the expression 4.1 for the Radon-Nikodym derivative, we get that
\[
\frac{1}{\ell N} H \left( \mathbb{P}_N^H | \mathbb{P}_N \right) = \mathbb{E}_N^H \left[ J_H(X^N) + O(1/N) \right].
\]
By Lemma 4.11, $\{ J_H(X^N) \}$ is a uniformly integrable sequence (with respect to $\mathbb{P}_N^H$). Since $J_H : \mathcal{D}(\mathbb{T}) \to \mathbb{R}$ is a continuous function and $\mathbb{P}_N^H$ converges weakly to a delta of Dirac at $\psi$, we conclude that
\[
\lim_{N \to \infty} \frac{1}{\ell N} H \left( \mathbb{P}_N^H | \mathbb{P}_N \right) = J_H(\psi) = I(\psi),
\]
by Proposition 4.8, which finishes the proof. \qed
We are in position to finally prove the Proposition 2.8.

Proof of lower bound for profiles in $\mathcal{D}_\text{pert}^\alpha$. Fix an open set $\mathcal{O}$. Given $\psi \in \mathcal{O} \cap \mathcal{D}_\text{pert}^\alpha$, there exists $H \in C^{1,2}$ such that $\psi$ is solution of (2.4) and $\|\partial_x H\|_\infty < \pi \sqrt{\alpha}$. Denote by $\mathbb{P}_N^{H,\mathcal{O}}$ the probability on the space $\mathcal{D}_\alpha$ given by

$$
\mathbb{P}_N^{H,\mathcal{O}}[A] \overset{\text{def}}{=} \frac{\mathbb{P}_N^{H}[\{x^N \in \mathcal{O}\}]}{\mathbb{P}_N^{H}[x^N \in \mathcal{O}]} ,
$$

for any $A$ measurable subset of $\mathcal{D}_\alpha$. Under this definition,

$$
\frac{1}{\ell N} \log \mathbb{P}_N[\mathcal{O}] = \frac{1}{\ell N} \log \mathbb{P}_N[\{x^N \in \mathcal{O}\}]
= \frac{1}{\ell N} \log \mathbb{E}_N \left[ \mathbb{1}_{\{x^N \in \mathcal{O}\}} \frac{d\mathbb{P}_N}{d\mathbb{P}_N^H} \right]
= \frac{1}{\ell N} \log \mathbb{E}_N^H \left[ \mathbb{1}_{\{x^N \in \mathcal{O}\}} \frac{d\mathbb{P}_N}{d\mathbb{P}_N^H} \right]
= \frac{1}{\ell N} \log \mathbb{E}_N^{H,\mathcal{O}} \left[ \frac{d\mathbb{P}_N}{d\mathbb{P}_N^H} \right] + \frac{1}{\ell N} \log \mathbb{P}_N^{H}[\{x^N \in \mathcal{O}\}] .
$$

(4.24)

Since $\mathcal{O}$ is an open set and $\psi \in \mathcal{O}$, by the Theorem 2.2 and the Portmanteau Theorem,

$$
\liminf_{N \to \infty} \mathbb{P}_N^{H}[\{x^N \in \mathcal{O}\}] \geq 1 ,
$$

hence the second parcel on (4.24) converges to zero as $N \to \infty$. Since the logarithm is a concave function, by Jensen inequality the first parcel in (4.24) is bounded from below by

$$
\mathbb{E}_N^{H,\mathcal{O}} \left[ \frac{1}{\ell N} \log \frac{d\mathbb{P}_N}{d\mathbb{P}_N^H} \right] = \mathbb{E}_N^{H} \left[ \mathbb{1}_{\{x^N \in \mathcal{O}\}} \frac{1}{\ell N} \log \frac{d\mathbb{P}_N}{d\mathbb{P}_N^H} \right] .
$$

(4.25)

Adding and subtracting terms, we can rewrite (4.25) as

$$
\frac{1}{\ell N} \mathbb{P}_N^{H}[\{x^N \in \mathcal{O}\}]
\left\{ - \frac{1}{\ell N} H(\mathbb{P}_N^H) - \mathbb{E}_N^{H} \left[ \mathbb{1}_{\{x^N \in \mathcal{O}\}} \frac{1}{\ell N} \log \frac{d\mathbb{P}_N}{d\mathbb{P}_N^H} \right] \right\} .
$$

(4.26)

Again by the Theorem 2.2 and the Portmanteau Theorem, we have that $\mathbb{P}_N^{H}[\{x^N \in \mathcal{O}\}]$ goes to one as $N$ increases to infinity. By Lemma 4.12 the second term inside braces in (4.26) vanishes as $N \to \infty$. Thus

$$
\liminf_{N \to \infty} \frac{1}{\ell N} \log \mathbb{P}_N[\mathcal{O}] \geq \lim_{N \to \infty} \frac{1}{\ell N} H(\mathbb{P}_N^H) = -I(\psi) ,
$$

where the last equality has been assured in Lemma 4.13. Optimizing the inequality above over $\psi \in \mathcal{D}_\text{pert}^\alpha$ leads us to (2.8) hence concluding the proof.

\[\square\]

4.4. Large deviations lower bound in the exponential case. In this section we will assume that $\ell(N) = e^{cN}$ and $\gamma$ is a constant profile in order to obtain a full large deviations principle. The scheme of proof here follows the same ideas of [13] and it is included here for sake of completeness.

Definition 4.14. Denote by $\mathcal{D}_\text{pert}^\infty \subseteq \mathcal{D}_{C(T)}$ the set of all profiles $\psi : [0, T] \times T \to \mathbb{R}$ satisfying:

- $\psi(0, \cdot) = \gamma(\cdot) \equiv \gamma$,
- $\psi \in C^{2,3}$,
- $\psi \geq \varepsilon$ for some $\varepsilon > 0$.

Repeating *ipsis litteris* the arguments of the previous subsection, under the hypothesis that $\ell(N) = e^{cN}$ we get that, given an open set $\mathcal{O} \subset \mathcal{D}_{\{[0, T], C(T)\}}$, for any $\psi \in \mathcal{D}_\text{pert}^\infty \cap \mathcal{O}$, we have that

$$
\liminf_{N \to \infty} \frac{1}{\ell N} \log \mathbb{P}_N[\mathcal{O}] \geq -I(\psi) .
$$
In what follows, we will say that a sequence \( \rho_n \in \mathcal{D}([0,T], C(\mathbb{T})) \) approximates \( \rho_0 \in \mathcal{D}([0,T], C(\mathbb{T})) \) if \( \rho_n \) converges to \( \rho_0 \) in the topology of \( \mathcal{D}([0,T], C(\mathbb{T})) \) and

\[
\lim_{n \to \infty} I(\rho_n) = I(\rho_0) .
\]

(4.27)

To conclude the proof of the lower bound large deviations it only remains to prove that any profile \( \rho_0 \in \mathcal{D}([0,T], C(\mathbb{T})) \) such that \( I(\rho_0) < \infty \) can be approximated by a sequence \( \rho_n \in \mathcal{D}_{\operatorname{pert}}^\infty \). In the usual terminology, we have to assure that the set \( \mathcal{D}_{\operatorname{pert}}^\infty \) is \( I \)-dense. In plain words, (4.27) together with the \( I \)-density of \( \mathcal{D}_{\operatorname{pert}}^\infty \) imply the lower bound in the Theorem 2.7.

Let us start by splitting the functional \( J_H \) into the \( H \)-dependent part, denoted by \( J^1_H \), and the part which does depend on \( H \), denoted by \( J^2 \). That is:

\[
J^1_H(\rho) = \int_T \left[ H(t,x)\rho(t,x) - H(0,x)\rho(0,x) \right] dx
\]

\[
+ \int_0^t \int_T \left[ - \rho(s,x) \left( \partial_s H(s,x) + \Delta H(s,x) + (\nabla H(s,x))^2 \right) \right. \\
- \left. b(\rho(s,x))e^{H(s,x)} - d(\rho(s,x))e^{-H(s,x)} \right] dx ds,
\]

and

\[
J^2(\rho) = \int_0^t \int_T b(\rho(s,x)) + d(\rho(s,x)) dx ds .
\]

Hence we define \( I^1(\rho) = \sup_{H \in C^{1,1}} J^1_H(\rho) \) if \( u(\cdot,0) = \gamma(\cdot) \), and \( I^1(\rho) = \infty \) otherwise, which gives us that

\[
I(\rho) = I^1(\rho) + J^2(\rho) .
\]

**Proposition 4.15.** The functional \( I^1 : \mathcal{D}([0,T], C(\mathbb{T})) \to \mathbb{R}_+ \cup \{+\infty\} \) is convex.

**Proof.** The functions \( b \) and \( d \) are assumed to be concave, thus \( J^1_H \) is a convex functional, see (4.28). Since the supremum of convex functions is a convex function, then \( I^1 \) is a convex function. \( \Box \)

**Proposition 4.16.** The rate function \( I : \mathcal{D}([0,T], C(\mathbb{T})) \to \mathbb{R}_+ \cup \{+\infty\} \) is a lower semi-continuous (l.s.c.) function, that is,

\[
\liminf_{\rho \to \rho_0} I(\rho) \geq I(\rho_0)
\]

for any \( \rho_0 \in \mathcal{D}([0,T], C(\mathbb{T})) \). Moreover, \( I^1 : \mathcal{D}([0,T], C(\mathbb{T})) \to \mathbb{R}_+ \cup \{+\infty\} \) is also lower semi-continuous and \( J \) is continuous.

**Proof.** We start by noting that \( J^1_H, J^2 : \mathcal{D}([0,T], C(\mathbb{T})) \to \mathbb{R} \) are continuous functionals in the Skorohod topology (see [3]) hence they are l.s.c. Since the supremum of l.s.c. functions is a l.s.c. function, we deduce that \( I^1 \) is l.s.c. And since the sum of l.s.c. functions is a l.s.c. function, we infer that \( I : \mathcal{D}([0,T], C(\mathbb{T})) \to \mathbb{R}_+ \cup \{+\infty\} \) is also a l.s.c. function. \( \Box \)

The next proposition tell us that time discontinuous space-time profiles play no role in the large deviations behavior.

**Proposition 4.17.** If \( \rho \in \mathcal{D}([0,T], C(\mathbb{T})) \) and \( \rho \notin C([0,T] \times \mathbb{T}) \), then \( I(\rho) = +\infty \).

**Proof.** We claim first that, if \( f : [0,T] \to \mathbb{R} \) is discontinuous at \( a \in [0,T] \) and has side limits at \( a \), and \( F,G : \mathbb{R} \to \mathbb{R} \) are continuous functions, then

\[
\sup_{H \in C([0,T])} \left\{ \int_0^T f(s) \partial_s H(s) ds - \int_0^T F(f(s))G(H(s)) ds \right\} = +\infty .
\]

(4.29)

In fact, let \( H_n : [0,T] \to \mathbb{R} \) such that \( H_n \) has support in the interval \( [a - 1/n^2, a + 1/n^2] \), \( H_n \in C^\infty([0,T]) \), \( H_n(a) = n \) and \( 0 \leq H_n \leq n \), that is, \( H_n \) is close to a delta of Dirac times the constant \( 1/n \) in the sense of Schwartz distributions.
Since the $L^1$-norm of $H_n$ is of order $1/n$, it is easy to check that
$$
\int_0^T F(f(s))G(H_n(s)) \, ds
$$
converges as $n \to \infty$. On the other hand, it is easy to check that the integral
$$
\int_0^T f(s)\partial_s H_n(s) \, ds
$$
is of order $n[f(a^+) - f(a^-)]$. These two facts imply (4.29), proving the claim. The statement
of the proposition is a then straightforward adaptation of the claim above, and details are
omitted here. □

**Proposition 4.18.** The set of profiles $\rho \in C([0, T] \times \mathbb{T})$ such that $\rho(0, \cdot) \equiv \gamma$ and $\rho \geq \varepsilon > 0$ for some $\varepsilon = \varepsilon(\rho) > 0$ is $I$-dense.

**Proof.** If $\rho_0 \in \mathcal{D}([0, T], C(\mathbb{T}))$ is such that $I(\rho_0) < \infty$, then $\rho_0(0, \cdot) \equiv \gamma$ and we known by Proposition 4.17 that $\rho_0 \in C([0, T] \times \mathbb{T})$. Let $\rho_n = \frac{\gamma}{n} + (1 - \frac{1}{n})\rho_0$, which converges to $\rho_0$ as $n \to \infty$. Since $I$ is l.s.c., then
$$
\liminf_{n \to \infty} I(\rho_n) \geq I(\rho_0).
$$
Since $J^2$ is continuous, then
$$
\lim_{n \to \infty} J^2(\rho_n) = J^2(\rho_0).
$$
And since $I^1$ is convex, then
$$
\limsup_{n \to \infty} I(\rho_n) \leq \limsup_{n \to \infty} \frac{1}{n} I(\gamma) + \limsup_{n \to \infty} (1 - \frac{1}{n})I(\rho_0) = I(\rho_0).
$$
Therefore, $\lim_{n \to \infty} I(\rho_n) = I(\rho_0)$. □

**Proposition 4.19.** The set of profiles $\rho \in C^{\infty,0}([0, T] \times \mathbb{T})$ such that $\rho(0, \cdot) \equiv \gamma$ and $\rho \geq \varepsilon > 0$ for some $\varepsilon = \varepsilon(\rho) > 0$ is $I$-dense.

**Proof.** By the Proposition 4.18, it is enough to prove the $I$-density of the set above on the set of profiles $\rho \in C([0, T] \times \mathbb{T})$ such that $\rho(0, \cdot) \equiv \gamma$ and $\rho \geq \varepsilon > 0$ for some $\varepsilon = \varepsilon(\rho) > 0$. Let $\Psi_{\delta} : \mathbb{T} \to \mathbb{R}$ be an approximation of identity, that is, $\int_{\mathbb{T}} \Psi_{\delta}(x)dx = 1$, $\Psi_{\delta} \geq 0$, $\text{supp}(\Psi_{\delta}) \subset (-\delta, \delta)$, $\Psi_{\delta}$ is symmetric around zero and $\Psi_{\delta} \in C^{\infty}(\mathbb{T})$. Denote by $(\Psi_{\delta} * \rho)(t, x)$ the spatial convolution of $\Psi_{\delta}$ with $\rho \in C([0, T], C^{\infty}(\mathbb{T}))$ and note that $(\Psi_{\delta} * \rho)(0, x) \equiv \gamma$.

It is simple to check that $\Psi_{\delta} * \rho$ converges to $\rho$ as $\delta \searrow 0$. Thus, by the Proposition 4.16,
$$
\lim_{\delta \to 0} J(\Psi_{\delta} * \rho) = J(\rho), \quad (4.30)
$$
and
$$
\liminf_{\delta \to 0} I^1(\Psi_{\delta} * \rho) \geq I^1(\rho).
$$
On the other hand, since $I^1$ is convex and (spatially) translation invariant, we get that
$$
I^1(\Psi_{\delta} * \rho) \leq \int_{\mathbb{T}} I^1(T_x \rho) \Psi_{\delta}(x) \, du = \int_{\mathbb{T}} I^1(\rho) \Psi_{\delta}(x) \, dx = I^1(\rho),
$$
where $T_x$ denotes the rotation of $x$ on the torus $\mathbb{T}$. Thus $\limsup_{\delta \to 0} I^1(\Psi_{\delta} * \rho) \leq I^1(\rho)$, which leads us to
$$
\lim_{\delta \to 0} I^1(\Psi_{\delta} * \rho) = I^1(\rho). \quad (4.31)
$$
Putting together (4.30) and (4.31) concludes the proof. □

**Proposition 4.20.** The set of profiles $\rho \in C^{\infty,\infty}([0, T] \times \mathbb{T})$ such that $\rho(0, \cdot) \equiv \gamma$ and $\rho \geq \varepsilon > 0$ for some $\varepsilon = \varepsilon(\rho) > 0$ is $I$-dense.
Proof. By the Proposition 4.19, it is enough to assure the $I$-density on the set of profiles $\rho \in C^{0,0}([0,T] \times \mathbb{T})$ such that $\rho(0,\cdot) \equiv \gamma$ and $\rho \geq \varepsilon > 0$ for some $\varepsilon = \varepsilon(\rho) > 0$. Let henceforth be $\rho$ with these properties and such that $I(\rho) < \infty$.

Let $\Psi_{1/n} \in C^\infty(\mathbb{R})$ be a time-approximation of identity such that $\Psi_{1/n}$ has support in $(-1/n,0)$ and is non-negative with integral one. We define now a suitable kind of time translation. Set, for $t \in [0,T]$,

$$
\sigma_t \rho(s,x) = \begin{cases} 
\rho(s+t,x) & \text{for } 0 \leq s \leq T-t, \\
\rho(T,x) & \text{for } T-t \leq s \leq T,
\end{cases}
$$

and set, for $t \in [-T,0]$,

$$
\sigma_t \rho(s,x) = \begin{cases} 
\rho(s+t,x) & \text{for } -t \leq s \leq T, \\
\rho(0,x) & \text{for } 0 \leq s \leq -t.
\end{cases}
$$

For $n \in \mathbb{N}$ such that $1/n < T/2$, let

$$
\rho_n(t,x) = \int_{-T}^{T} \Psi_{1/n}(s) \sigma_s \rho(t,x) ds.
$$

The importance of choosing the support of $\Psi_{1/n}$ on $(-1/n,0)$ is that $\rho_n(0,x) \equiv \gamma$. It is easy to check that $\rho_n$ converges to $\rho$ hence $J(\rho_n)$ converges to $J(\rho)$ as $n \to \infty$. By the convexity of $I^\gamma$ and an adaptation of [13, Prop. 3.1], we get that $I^\gamma(\rho_n) \leq I^\gamma(\rho) + \frac{c}{n}$, where $c = c(\rho)$ is a constant. This inequality and the lower semi-continuity of $I^\gamma$ implies that $\lim_{n \to \infty} I(\rho_n) = I(\rho)$, concluding the proof.

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