Roots of a characteristic equation with complex coefficients associated with differential-difference equations

Rafał Kapica, Radosław Zawiski

Abstract

We analyse placement of roots of a characteristic exponential polynomial with complex coefficients associated with a first order differential-difference equation. We provide necessary and sufficient conditions for all the roots to be in the complex open left half-plane assuring stability of the differential-difference equation. The conditions are expressed explicitly in terms of complex coefficients of the characteristic exponential polynomial, what makes them easy to use in applications.

Keywords: first order differential-difference equation with complex coefficients, stability of differential-difference equation, characteristic exponential polynomial of differential-difference equation, retarded differential-difference equation

2020 Subject Classification: 30C15, 34K06, 34K20, 34K41

1 Introduction

Consider a differential-difference equation

\[ x'(t) = Ax(t) + Bx(t - \tau) + N(x(t), x(t - \tau)), \]

where \( x : [t_0, \infty) \to \mathbb{R}^n \), \( A, B \) are constant \( n \times n \) matrices and \( N : \mathbb{R}^n \times \mathbb{R}^n \) is a nonlinear function satisfying

\[ \lim_{\|x\| + \|y\| \to 0} \frac{N(x, y)}{\|x\| + \|y\|} = 0 \]

with the Euclidean norm \( \| \cdot \| \). We are interested in an asymptotic stability of (1), as given by the following

Definition 1. A solution \( u \) of (1) is asymptotically stable on \([t_0, \infty)\) if \( u \) is defined for every \( t \geq t_0 \) and if for every \( t_1 \geq t_0 \) there exists a \( \delta > 0 \) such that every solution \( x \) of (1) which satisfies

\[ \max_{t_1 \leq t \leq t_1 + \tau} \{ \|x(t) - u(t)\| + \|x'(t) - u'(t)\| \} \leq \delta \]

also satisfies

\[ \lim_{t \to \infty} \|x(t) - u(t)\| = 0. \]

With the above it can be shown that the null solution \( x(t) \equiv 0 \) of (1) is asymptotically stable if all roots \( s \) of the characteristic equation

\[ \det [sI - A - B e^{-s\tau}] = 0 \]

have a negative real part i.e. \( \text{Re}(s) < 0 \). Under appropriate conditions, for example if \( A \) and \( B \) commute - see or - the problem of stability of (1) is equivalent to finding conditions...
on the coefficients of a so-called exponential polynomial $P(s) := s - \lambda - \gamma e^{-\sigma \tau}$, which will guarantee that every root $s$ of

$\quad s - \lambda - \gamma e^{-\sigma \tau} = 0 \tag{3}$

is such that $\text{Re}(s) < 0$. The coefficients $\lambda$ and $\gamma$ in [3] are eigenvalues of $A$ and $B$, respectively. For the rest of this paper by mentioning stability we refer to a situation when all the roots of $[3]$ have negative real parts.

All of the above sketches a general formulation of the problem of stability of first order differential-difference equations. The literature contains two intertwined approaches to this problem - one based on analysis of some form of [1] in time domain and one based on analysis of [3]. In the latter approach the case with $\lambda, \gamma \in \mathbb{C}$ is well understood - see [4], where the author obtained necessary and sufficient conditions for stability of $s - a - c e^{-s} = 0$ with $a, c \in \mathbb{R}$. In a more general form $A(s) + B(s) e^{-\sigma \tau} = 0$ this problem is widely studied in [2, Chapter 13]. For a through exposition of other methods of analysis of the real $\lambda$ and $\gamma$ case see [11] and references therein.

The case $\lambda, \gamma \in \mathbb{C}$ is less analysed. In particular, some sufficiency results can be found in [11], where the author presents a numerical analysis of

$\quad \lambda x(t) + \gamma x(t - \tau), \quad \tag{4}$

and stipulates asymptotic stability for every $\tau > 0$ if $-\text{Re}\lambda > |\gamma|$. The authors of [3] provide, based on algorithmic criteria, some sufficiency result for specific values of complex $\lambda$ and $\gamma$, proving also the result in [11] for some cases. Authors of [16] built on [3] and provide additional sufficient conditions for stability. In one of the latest works [6] the author manages to find the necessary and sufficient conditions for the zeros of [3] to be in the left complex half-plane, but these are expressed in a highly non-trivial fashion. The argument in [6] is based on an analysis of the Lambert W function, what complicates applications of obtained conditions.

Our approach in this paper is motivated by [7], where the author was able to state necessary and sufficient conditions for stability of [3] for $\lambda, \gamma \in \mathbb{C}$ and $\tau = 1$. Results of [7] are, however, based on specific analysis of roots of [3] which is uneasy to trace for different values of $\tau$. This may explain why, although it precedes many of the works mentioned above, [7] did not receive sufficient recognition. This approach, nevertheless, combined with analysis of root placement depending on $\tau$, as shown in [11] Chapter 5.3.2 or [8, Proposition 6.2.3], allowed us to obtain necessary and sufficient conditions for stability of [3] based explicitly on a relation between $\lambda, \gamma \in \mathbb{C}$ and $\tau > 0$. The conditions do not require to calculate any specific roots of a transcendental equation associated with [3], as is in the case of [7], and thus provide a generalisation of the latter.

2 Preliminaries

The following observation can be found in [3] or [7].

**Lemma 2.** Let $a, b, c, d, \tau \in \mathbb{R}$ and let $\{s_0\}$ be the set of roots of

$\quad s - (a + ib) - (c + id) e^{-\sigma \tau} = 0 \tag{5}$

and $\{z_0\}$ be the set of roots of

$\quad z - a - e^{-ib\tau}(c + id) e^{-\sigma \tau} = 0. \tag{6}$

Then $\text{Re}(s_0) < 0$ for all $s_0$ if and only if $\text{Re}(z_0) < 0$ for all $z_0$.

**Proof.** Let $z_0$ be a root of [6]. Then $s_0 = z_0 + ib$ is a root of [5]. Conversely, let $s_0$ be a root of [5]. Then $z_0 = s_0 - ib$ is a root of [6]. As $\text{Re}(s_0) = \text{Re}(z_0)$ the result follows.

We will also use the following result concerning parameter $\beta \in \mathbb{R}$ and real functions

$\quad L, R : [0, \infty) \to \mathbb{R}, \quad L(r) := \frac{r}{r^2 + 1}, \quad R(r) := \arctan(r) + \beta. \tag{7}$

2
Lemma 3. Let $\beta \in \mathbb{R}$ and put

$$A = \{r \in [0, \infty) : L(r) \leq R(r)\},$$

where real functions $L$ and $R$ are given by (7). Then:

(i) $A = [0, \infty)$ if and only if $\beta \geq 0$,
(ii) $A = [r_0, \infty)$ with $r_0 > 0$ if and only if $\beta \in (-\frac{\pi}{2}, 0)$, wherein the correspondence $(0, \infty) \ni r_0 \leftrightarrow \beta \in (-\frac{\pi}{2}, 0)$ is one-to-one,
(iii) set $A$ is empty if and only if $\beta \leq -\frac{\pi}{2}$.

Proof. Let us consider function $\varphi : [0, \infty) \to \mathbb{R}$ given by $\varphi = R - L$. Clearly, we have

$$\varphi'(r) = \frac{2r^2}{(r^2 + 1)^2} > 0, \quad \varphi(0) = \beta, \quad \varphi(r) \to \beta + \frac{\pi}{2} \quad \text{as} \quad r \to \infty.$$

In particular $\varphi$ is strictly increasing and $\varphi([0, \infty)) = [\beta, \beta + \frac{\pi}{2})$.

If $\beta \geq 0$, then $\varphi(r) \geq 0$ for $r \in [0, \infty)$, i.e. $A = [0, \infty)$. On the other hand if $L(0) \leq R(0)$, then $\beta \geq 0$. This gives assertion (i).

Suppose $\beta \in (-\frac{\pi}{2}, 0)$. Hence $\varphi(0) < 0$ and $\varphi(r) > 0$ for large enough $r > 0$. Then there exists a unique $r_0 > 0$ such that $\varphi(r_0) = 0$. This shows that $A = [r_0, \infty)$. If now $A = [r_0, \infty)$ for some $r_0 > 0$, then $\beta < 0$ by (i). To finish the proof it is enough to notice that for $\beta \leq -\frac{\pi}{2}$ we have $\varphi(r) < 0$ for $r \in [0, \infty)$, i.e. $A = \emptyset$.

Corollary 4. Equation $L(r) = R(r)$ has exactly one solution if and only if $\beta \in (-\frac{\pi}{2}, 0)$.

3 Main results

By Lemma 2 we restrict our attention to (6). Taking $\eta = u + iv = e^{-ib\tau}(c + id)$ the conditions for stability of (3) are given on an $(u, iv)$-complex plane in terms of regions that depend on $a$ and $\tau$.

Remark 5. We take the principal argument of $\lambda$ to be $\text{Arg}\, \lambda \in (-\pi, \pi]$.

Let $\mathbb{D}_r \subset \mathbb{C}$ be an open disc centred at 0 with radius $r > 0$. We shall require the following subset of the complex plane, depending on $\tau > 0$ and $a \in (-\infty, \frac{1}{\tau})$, namely:

- for $a < 0$:
  $$\Lambda_{\tau,a} := \{\eta \in \mathbb{C} \setminus \mathbb{D}_a : \text{Re}\, \eta + a < 0, |\eta| < |\eta_{\pi}|, \quad |\text{Arg}\, \eta| > \tau \sqrt{|\eta|^2 - a^2} + \text{arctan}\left(-\frac{1}{a} \sqrt{|\eta|^2 - a^2}\right)\} \cup \mathbb{D}_a,$$
  where $\eta_{\pi}$ is such that
  $$\sqrt{|\eta_{\pi}|^2 - a^2} + \text{arctan}\left(-\frac{1}{a} \sqrt{|\eta_{\pi}|^2 - a^2}\right) = \pi;$$

- for $a = 0$:
  $$\Lambda_{\tau,a} := \{\eta \in \mathbb{C} \setminus \{0\} : \text{Re}\, \eta < 0, |\eta| < \frac{\pi}{2\tau}, |\text{Arg}\, \eta| > \tau |\eta| + \frac{\pi}{2}\};$$

- for $0 < a \leq \frac{1}{\tau}$:
  $$\Lambda_{\tau,a} := \{\eta \in \mathbb{C} : \text{Re}\, \eta + a < 0, |\eta| < |\eta_{\pi}|, \quad |\text{Arg}\, \eta| > \tau \sqrt{|\eta|^2 - a^2} + \text{arctan}\left(-\frac{1}{a} \sqrt{|\eta|^2 - a^2}\right) + \pi\},$$
  where $\eta_{\pi}$ is such that $|\eta_{\pi}| > a$ and
  $$\sqrt{|\eta_{\pi}|^2 - a^2} + \text{arctan}\left(-\frac{1}{a} \sqrt{|\eta_{\pi}|^2 - a^2}\right) = 0.$$
Figure 1: Outer boundaries of the $\Lambda_{\tau,a}$, defined in (8) with $\eta = u + iv$, for $a = -1.5$ and different values of $\tau$: dotted for $\tau = 0.5$, dash-dotted for $\tau = 1$, dashed for $\tau = 2$. The solid line shows a circle with radius $|a| = 1.5$.

Figure 2: Outer boundaries of the $\Lambda_{\tau,a}$, defined in (9) with $\eta = u + iv$, for $a = 0$ and different values of $\tau$: dotted for $\tau = 0.5$, dash-dotted for $\tau = 1$, dashed for $\tau = 2$. 
The zeros of (3) are in the left half plane $\mathbb{C}^-$, according to Lemma 2, if and only if the roots of (6) belong to $\mathbb{C}^-$. Thus for $\lambda = a + ib, \gamma = c + id$ and $\eta := e^{-ibr}\gamma$ we have the following

**Theorem 6.** Let $\tau > 0$, $a \leq \frac{1}{2}$ and let $\Lambda_{\tau,a}$ be the closure of $\Lambda_{\tau,a}$ given by (8), (9) or (10), depending on $a$. Then every solution of the equation

$$s - a - \eta e^{-s\tau} = 0$$

(11)

belongs to $\mathbb{C} \setminus \mathbb{C}_+$ if and only if $\eta \in \Lambda_{\tau,a}$.

**Proof.** For any $\tau > 0$ and $a \leq 0$ there is $0 \in \Lambda_{\tau,a}$ and taking $\eta = 0$ the statement of the proposition obviously holds true, while for $0 < a \leq \frac{1}{2}$ we have $0 \notin \Lambda_{\tau,a}$. Thus for the remainder of the proof assume that $\eta \neq 0$.

1. Let $\eta = u + iv$ and let $s = x + iy$. Simple trigonometric identities show that every solution of (11) must satisfy simultaneously the pair of equations

$$x = a + y \tan(y\tau + \alpha)$$

$$y \sec(y\tau + \alpha) \sin(\alpha) e^{a\tau + y\tau \tan(y\tau + \alpha)} = \frac{u}{v},$$

where $\alpha := \arctan\left(\frac{u}{v}\right)$. Thus the problem is equivalent to finding the conditions for which $x \leq 0$.

Equations (12) show that infinitely many solutions of (11) move continuously with $\tau \in (0, \delta)$ for some $\delta > 0$. In the limit as $\tau \to 0$ there is

$$x = a + u, \quad y = v,$$

and the solutions start in $\mathbb{C} \setminus \mathbb{C}_+$ i.e. with $x \leq 0$ if and only if $a + u \leq 0$.

2. Let us establish when at least one of the solutions crosses the imaginary axis for the first time as $\tau$ increases from zero upwards. At the crossing of the imaginary axis there is $s = i\omega$ for some $\omega \in \mathbb{R}$. In view of (11) we can treat $s$ as an implicit function of $\tau$.

Figure 3: Outer boundaries of the $\Lambda_{\tau,a}$, defined in (10) with $\eta = u + iv$, for $a = 0.25$ and different values of $\tau$: dotted for $\tau = 0.5$, dash-dotted for $\tau = 1$, dashed for $\tau = 2$. Figures 1, 2 and 3 show $\Lambda_{\tau,a}$ for fixed values of $a$ and varying $\tau$, while Figure 4 shows $\Lambda_{\tau,a}$ for fixed $\tau$ and varying $a$. 
Figure 4: Outer boundaries of $\Lambda_{\tau,a}$, defined in (8)–(10) with $\eta = u + iv$, for $\tau = 1$ and different values of $a$: solid for $a = -1.5$, dashed for $a = 0$ and dotted for $a = 0.25$.

and check the direction in which zeros of it cross the imaginary axis by analysing the $\text{sgn Re} \frac{ds}{d\tau}$ if $s = i\omega$. By calculating the implicit function derivative we have

$$\frac{ds}{d\tau} = -\frac{s^2 - as}{1 - a\tau + s\tau}.$$  

As $\text{sgn Re } z = \text{sgn Re } z^{-1}$ we have if $s = i\omega$ that

$$\text{sgn Re } \frac{ds}{d\tau} = \text{sgn } \frac{1}{\omega^2 + a^2} > 0$$

and the zeros cross from the left to the right half-plane. As the sign of the above does not depend on $\tau$, the direction of the crossing remains the same for every value of $\tau$. Thus with $\eta = u + iv$ a necessary condition for the solutions of (11) to be in $\mathbb{C} \setminus \mathbb{C}_+$ is

$$a + u \leq 0.$$  

(13)

3. Fix $\tau = \tau_0$, where $\tau_0 > 0$ is the delay for which the first crossing happens i.e. $\tau_0$ is the smallest $\tau$ for which crossing happens at all. Taking the complex conjugate of (11) at the crossing, i.e. with $s = i\omega$ for some $\omega \in \mathbb{R}$, we obtain

$$-i\omega - a - \bar{\eta}e^{i\omega\tau_0} = 0.$$  

(14)

Using (11) for $s = i\omega$ and (14) to eliminate the exponential part we obtain $\omega^2 = |\eta|^2 - a^2$. From here we see that for every $\eta = u + iv$ satisfying

$$|\eta| < |a|$$

(15)

i.e. $\eta \in D_{|a|}$, the crossing does not exist, regardless of $\tau$, and all the solutions of (11) are in $\mathbb{C} \setminus \mathbb{C}_+$.

4. Let us focus on the other case and so assume additionally that $|\eta| \geq |a|$. Then the crossing takes place at $s = \pm i\sqrt{|\eta|^2 - a^2}$. Choosing to work further with $s = i\sqrt{|\eta|^2 - a^2}$ and substituting it into (11) at the crossing we get

$$\eta = -ae^{i\sqrt{|\eta|^2 - a^2\tau_0}} + \sqrt{|\eta|^2 - a^2}e^{i\frac{\pi}{2} + \sqrt{|\eta|^2 - a^2\tau_0}}$$

$$= e^{i\sqrt{|\eta|^2 - a^2\tau_0}} \left(-a + \sqrt{|\eta|^2 - a^2}e^{i\frac{\pi}{2}}\right),$$  

(16)
For $s = -i \sqrt{\eta^2 - a^2}$ the equation corresponding to (16) is

$$\eta = e^{-i \sqrt{\eta^2 - a^2} \tau_0} \left(-a + \sqrt{\eta^2 - a^2} e^{-i \frac{\varphi}{2}}\right).$$  \hspace{1cm} (17)$$

Let $\eta$ be a solution of (16). Then $\overline{\eta}$ is a solution of (17), and these solutions are obviously symmetric about the real axis.

5. It will be more convenient to use different notation that the one in (16) or (17). Define $\gamma_+: |a|, \infty) \to \mathbb{C}$ and $\gamma_- : |a|, \infty) \to \mathbb{C}$ as the right side of (16) and (17), respectively, i.e.

$$\gamma_+(w) := e^{i \sqrt{w^2 - a^2} \tau_0} \left(-a + \sqrt{w^2 - a^2} e^{i \frac{\varphi}{2}}\right),$$  \hspace{1cm} (18)$$

$$\gamma_-(w) := e^{-i \sqrt{w^2 - a^2} \tau_0} \left(-a + \sqrt{w^2 - a^2} e^{-i \frac{\varphi}{2}}\right).$$  \hspace{1cm} (19)$$

Let $\Gamma_+ := \gamma_+(|a|, \infty))$ be the image of (18) and $\Gamma_- := \gamma_-(|a|, \infty))$ be the image of (19). We easily see that $\gamma_+(w) = \gamma_-(w)$ for every $w \geq |a|$ and so $\Gamma_+$ is symmetric to $\Gamma_-$ about the real axis.

Up to this moment all considerations were done regardless of the sign of parameter $a$. In the reminder of the proof we will consider cases $a < 0$, $a = 0$, $a \in (0, \frac{1}{\tau_0}]$ and $a > \frac{1}{\tau_0}$ separately.

6. Fix now $a < 0$ and let a function describing a continuous argument increment of (18) be given by $\Delta \gamma_+ : |a|, \infty) \to [0, \infty)$,

$$\Delta \gamma_+(w) = \sqrt{w^2 - a^2} \tau_0 + \arctan \left(-\frac{1}{a} \sqrt{w^2 - a^2}\right).$$  \hspace{1cm} (20)$$

We easily see that it is a non-negative, strictly increasing function. We also see that $\Delta \gamma_- : |a|, \infty) \to (-\infty, 0]$ and $\Delta \gamma_-(w) = -\Delta \gamma_+(w)$ for every $w \in |a|, \infty)$.

Looking at (18) note that the first component has modulus 1 and introduces counter-clockwise rotation, while the second component is always in the first quadrant, with a positive real part equal to $-a$, and its modulus is strictly increasing and tends to infinity as $w \to \infty$. Thus $\Gamma_+$ is a curve that is a counter-clockwise outward spiral that begins in $-a \in \mathbb{C}$. An exemplary pair of $\Gamma_+$ and $\Gamma_-$ curves is shown in Fig. 5.

7. Let a set $\{\eta_{2k-1}\pi\}_{k \in \mathbb{N}}$ be such that the argument increment along $\Gamma_+$ as $w$ changes from $|a|$ to $|\eta_{(2k-1)}\pi|$ is equal to $(2k - 1)\pi$, that is

$$\Delta \gamma_+(|\eta_{(2k-1)}\pi|) = (2k - 1)\pi.$$  \hspace{1cm} (21)$$

Let us also denote upper and lower open half-planes by

$$\Pi_+ := \{s \in \mathbb{C} : \text{Im}(s) > 0\} = \{s \in \mathbb{C} : \text{Arg} s \in (0, \pi)\}$$

$$\Pi_- := \{s \in \mathbb{C} : \text{Im}(s) < 0\} = \{s \in \mathbb{C} : \text{Arg} s \in (-\pi, 0)\}.$$ 

Due to constraint (13) we take into account only these parts of $\Gamma_+$ (or $\Gamma_-$) that lie to the left of $u = -a$ line, as depicted in Fig. 5. Let us now focus on the closure of the first part of $\Gamma_+$ that lies in $\Pi_+$, i.e. $\gamma_+([|a|, |\eta_\pi|])$. By (20) and (21) for every $w \in [|a|, |\eta_\pi|]$ we have $\Delta \gamma_+(w) \in [0, \pi]$. For the case of the part of $\Gamma_-$ equal to $\gamma_-([|a|, |\eta_\pi|])$ the argument expression gives $\Delta \gamma_-(w) = -\Delta \gamma_+(w)$. Putting both cases together and returning to the notation of (16) and (17), our equation of interest becomes

$$|\text{Arg} \eta| = \sqrt{|\eta|^2 - a^2} \tau_0 + \arctan \left(-\frac{1}{a} \sqrt{|\eta|^2 - a^2}\right), \quad |\eta| \leq |\eta_\pi|,$$  \hspace{1cm} (22)$$

where $\eta_\pi$ is such that

$$\Delta \gamma_+(|\eta_\pi|) = \sqrt{|\eta_\pi|^2 - a^2} \tau_0 + \arctan \left(-\frac{1}{a} \sqrt{|\eta_\pi|^2 - a^2}\right) = \pi.$$
Figure 5: (a): Curves $\Gamma_+$ (solid line) and $\Gamma_-$ (dash-dotted line) drawn for $\tau_0 = 1$ and $a = -1.5$ with $|\eta| = w \in (|a|, 10)$. The constraint related to $a$ and expressed by (13) is marked with a dotted line. The crossings of the real negative semi-axis by $\Gamma_+$ (and $\Gamma_-$) are at $\eta_r$ and $\eta_{3\pi}$. The crossings of $u = -a$ by $\Gamma_+$, as $w$ increases, are at $\eta_1$, $\eta_2$ and $\eta_3$; (b): enlargement of the central part of (a) with crossings of dotted line. The crossings of the real negative semi-axis by $|\eta|$ with $\Gamma_0$.

8. The set of all $\eta \in \mathbb{C}$ that satisfy (22) is the boundary of the $\Lambda_{\eta_0, a}$ region, shown in Fig. 1 for different values of $\tau_0$. To show that for every $\eta$ inside this boundary the roots of (11) are in $\mathbb{C} \setminus \mathbb{C}_+$ consider the following. For every $\eta$ in the half-plane $\{u + iv \in \mathbb{C} : a + u < 0\}$ simple geometric considerations show that there exists exactly one $\eta_0$ fulfilling (22) and such that $\text{Arg} \eta = \text{Arg} \eta_0$. Conversely, let us fix $\eta_0$ fulfilling (22) and consider a function $\tau = \tau(|\eta|)$ defined on a ray from the origin and passing through $\eta_0$. More precisely, define

$$D_{\eta_0} := \{\eta = u + iv \in \mathbb{C} : |\eta| > |a| \text{ and } a + u < 0 \text{ and } \text{Arg} \eta = \text{Arg} \eta_0\}$$

and let $D'_{\eta_0} := \{t \geq 0 : t = |\eta|, \eta \in D_{\eta_0}\}$. Now reformulate the equality in (22) to express $\tau$ as a function $\tau : D'_{\eta_0} \to (0, \infty)$,

$$\tau(t) = \frac{\arctan \left(\frac{1}{a} \sqrt{t^2 - a^2}\right) + |\text{Arg} \eta_0|}{\sqrt{t^2 - a^2}}. \quad (23)$$

This is a well-defined positive continuous function. Really, for positivity note that for $u \leq 0$ there is $|\text{Arg} \eta_0| \geq \frac{\pi}{2}$, while for $u \in (0, -a)$ consider the following trigonometric identity

$$\arctan \left(\frac{1}{a} \sqrt{t^2 - a^2}\right) + |\text{Arg} \eta_0| = \arctan \left(\frac{1}{a} \sqrt{t^2 - a^2}\right) + \arctan \left(\frac{v}{u}\right)$$

$$= \arctan \left(\frac{\frac{u}{a} \sqrt{t^2 - a^2} + |v|}{u - \frac{1}{a} \sqrt{t^2 - a^2}|v|}\right)$$

and the estimation $\frac{u}{a} \sqrt{t^2 - a^2} > -\sqrt{u^2 + v^2 - a^2} > -|v|$. The derivative of (23) is given by

$$\frac{d\tau}{dt}(t) = \frac{t}{t^2 - a^2} \left(\frac{a}{t^2 - \tau(t)}\right). \quad (24)$$
As $a < 0$ we have $\frac{d\tau}{dt} < 0$ for every $t \in D_{\tau_0}^\prime$ and $\tau$ is a decreasing function. Thus for every $\eta \in D_{\tau_0}$ such that $|\eta| \leq |\eta_0|$ we have $\tau(|\eta|) \geq \tau(|\eta_0|) = \tau_0$, that is

$$|\text{Arg} \eta| \geq \sqrt{|\eta|^2 - a^2}\tau_0 - \arctan \left( \frac{1}{a} \sqrt{|\eta|^2 - a^2} \right), \quad |\eta| \leq |\eta_\pi|. \quad (25)$$

As the above is true for every $\eta_0$ fulfilling (22), condition (25) is true for every $\eta \in \Lambda_{\tau_0,a} \setminus \{ \eta \in \mathbb{C} : |\eta| \leq |a| \}$. Stated otherwise, for a given $\eta' \in \Lambda_{\tau_0,a} \setminus \{ \eta \in \mathbb{C} : |\eta| \leq |a| \}$ the time $\tau'$ for which the first crossing happens, as stated in point 3. In fact, although (16) still cannot be the smallest delay for which the first crossing happens, as stated in point 3. In fact, although (16) still describes (11) with a root corresponding to $|a|$, as shown in Fig. 1.

Results of the previous point show that the only parts of $\Gamma_+$ and $\Gamma_-$ that we need to consider are the ones already discussed i.e. $\gamma_+(|[a],|\eta_\pi|)) \cup \gamma_-(|[a],|\eta_\pi|)).$

Really, let $\eta_k, k = 1, 2, \ldots$ be consecutive points where $\Gamma_+$ crosses the constraint line $u = -a$, as depicted in Fig. 5. Then for every $\eta_0 \in \gamma_+(|[a],|\eta_\pi|)) \cup \gamma_-(|[a],|\eta_\pi|))$

there exists

$$\eta_k \in \gamma_+(|[a],|\eta_\pi|) \cup \gamma_-(|[a],|\eta_\pi|)), \quad k \in \mathbb{N}$$

such that

$$\text{Arg} \eta_0 = \text{Arg} \eta_k \quad \text{and} \quad |\eta_0| < |\eta_k|.$$\nThe result of point 8 now gives a contradiction as $\tau_0$ cannot be the smallest delay for which $\tau'$ comes now directly from (16). The analysis of points 5–9 simplifies greatly resulting in a necessity condition of the form

$$|\text{Arg} \eta| \geq |\eta|\tau_0 + \frac{\pi}{2}, \quad |\eta| < \frac{\pi}{2\tau_0}. \quad (27)$$

12. Assume now $0 < a$ and $\tau_0$ is as in point 3. Equations (18) and (19) have the same form. The difference now is that the second product term in (18) is constantly in the second quadrant, with a negative real part $-a$ and imaginary part tending to $+\infty$ as $w \to \infty$.

This changes e.g. the behaviour of the continuous argument increment function $\Delta \gamma_+$, as it is in general no longer strictly increasing. In fact for $0 < a$ we have $\Delta \gamma_+ : [a, \infty) \to (0, \infty)$,

$$\Delta \gamma_+(w) = \sqrt{w^2 - a^2}\tau_0 + \arctan \left( -\frac{1}{a} \sqrt{w^2 - a^2} \right) + \pi \quad (28)$$

and $\Delta \gamma_- : [a, \infty) \to [0, \infty), \Delta \gamma_-(w) = -\Delta \gamma_+(w)$ for every $w \geq a$. As (28) is a differentiable function its derivative is

$$\frac{d\Delta \gamma_+}{dw}(w) = \frac{w}{\sqrt{w^2 - a^2}} \left( \tau_0 - \frac{a}{w^2} \right). \quad (29)$$
13. Fix $0 < a < \frac{1}{\tau_0}$, where $\tau_0$ is as in point 3. Similarly as in points 7 and 8 we focus initially on a part of $\Gamma_+$ given by $\gamma_+([a, \eta_2])$, as indicated in Fig. 6. Take $\eta_1$ that fulfills \eqref{eq:gamma_1} and with $|\eta_1| = w_1 < w_m$. For such $\eta_1$ we have

$$\Delta \gamma_+(w_m) < \arg \eta_1 = \Delta \gamma_+(w_1) \leq \pi.$$ 

Define a ray from the origin and passing through $\eta_1$ by

$$D_{\eta_1} := \{ \eta = u + iv \in \mathbb{C} : |\eta| > |a| \text{ and } a + u < 0 \text{ and } \arg \eta = \Delta \gamma_+(w_1) \}$$

and let $D_{\eta_1}^* := \{ t \geq 0 : t = |\eta|, \eta \in D_{\eta_1} \}$. To express $\tau$ as a function on this ray, i.e., $\tau : D_{\eta_1}^* \to (0, \infty)$ we now reformulate \eqref{eq:tau_2} to obtain

$$\tau(t) := \frac{\arctan \left( \frac{t}{a \sqrt{t^2 - a^2}} \right) + \Delta \gamma_+(w_1)}{\sqrt{t^2 - a^2}}, \quad \text{for } t \geq 0, \eta \in D_{\eta_1}.$$ 

where $\Delta \gamma_+(w_1) = \arg \eta_1$. Note also that as $w_1 < w_m$ there exists $\eta_2$, with $|\eta_2| = w_2$, such that $\eta_2 \in D_{\eta_1} \cap \gamma_+([a, \eta_2])$ and $w_m < w_2 \leq |\eta_2|$. The derivative of \eqref{eq:tau_3} is again expressed by \eqref{eq:tau_4'}, namely

$$\frac{d\tau}{dt}(t) = \frac{t}{t^2 - a^2} \left( \frac{a}{t^2} - \tau(t) \right),$$
but, unlike in point 8, this derivative is in general not negative due to \( a > 0 \). In fact, at the intersections \( \{ \eta_1, \eta_2 \} = D_{\eta_1} \cap \gamma_+([a, \eta_a]) \) we find
\[
\frac{d\tau}{dt}(w_1) = \frac{w_1}{w_1^2 - a^2} \left( \frac{a}{w_1^2} - \tau(w_1) \right) = \frac{w_1}{w_1^2 - a^2} \left( \frac{a}{w_1^2} - \tau_0 \right)
\]
\[
= \frac{1}{\sqrt{w_1^2 - a^2}} \left( \frac{d\Delta \gamma_+(w_1)}{d\tau} \right) > 0,
\]
where the last inequality comes from (30); similarly
\[
\frac{d\tau}{dt}(w_2) = \frac{1}{\sqrt{w_2^2 - a^2}} \left( -\frac{d\Delta \gamma_+(w_2)}{d\tau} \right) < 0.
\]
We see that \( \tau \) is an increasing function in a neighbourhood of \( t_1 = w_1 \) and a decreasing one in a neighbourhood of \( t_2 = w_2 \) i.e. at the boundaries of the \( \Lambda_{\eta_1, a} \) region shown in Fig. 6. If we show that \( \tau \) has only one extreme value - a local maximum - inside \( \Lambda_{\eta_1, a} \), that is for some \( t \in (w_1, w_2) \), then with a reasoning of point 8 we will show that for every \( \eta \) inside \( \Lambda_{\eta_1, a} \) region the roots of (33) are in \( \mathbb{C} \setminus \mathbb{C}_1 \).
We are interested in the number of solutions of \( \frac{d\tau}{dt}(t) = 0 \), what is equivalent to the number of solutions of
\[
a = \frac{\arctan \left( \frac{1}{a} \sqrt{t^2 - a^2} \right) + \beta}{\sqrt{t^2 - a^2}}.
\]
where \( \beta = \Delta \gamma_+(w_1) - \pi \). Define \( r := \frac{1}{a} \sqrt{t^2 - a^2} \). Then \( r > 0 \) is a bijective image of \( t > a \) and (34) can be rearranged to
\[
\frac{r}{r^2 + 1} = \arctan(r) + \beta.
\]
As \( \frac{\pi}{2} < \Delta \gamma_+(w_m) < \arctan \eta_1 \leq \pi \) we have \( \beta \in (-\frac{\pi}{2}, 0] \) and by Corollary 4 we infer that there is only one local extremum i.e. local maximum of \( \tau \) for \( t \in (w_1, w_2) \). What follows, for every \( \eta \in D_{\eta_1} \), \( w_1 \leq |\eta| \leq w_2 \) we have \( \tau(|\eta|) \geq \tau_0 \) i.e.
\[
\arctan \left( \frac{1}{a} \sqrt{|\eta|^2 - a^2} \right) = \arctan \left( \frac{1}{a} \sqrt{|\eta|^2 - a^2} \right) + \pi.
\]
Thus by the definition of \( D_{\eta_1} \) and symmetry about the real axis we obtain that for every \( \eta \) with \( |\eta| \leq |\eta| \) such that
\[
|\arctan \eta| \geq \sqrt{|\eta|^2 - a^2} \tau_0 - \arctan \left( \frac{1}{a} \sqrt{|\eta|^2 - a^2} \right) + \pi
\]
the time \( \tau \) for this \( \eta \) to be such that the first root of (11) reaches the imaginary axis is bigger than or equal to \( \tau_0 \). Argument similar to the one in point 8 shows that if \( |\arctan \eta| \geq \Delta \gamma_+(w_m) \) then the only region we need to consider is the one given by (36). Thus we distinguish a ray
\[
D_{\eta_m} = \{ \eta = u + iv \in \mathbb{C} : |\eta| > |a|, a + u < 0, \quad \arg \eta = \Delta \gamma_+(w_m) \}
\]
together with a delay time function based on it, namely \( \tau_m : D_{\eta_m} \to (0, \infty) \),
\[
\tau_m(t) = \frac{\arctan \left( \frac{1}{a} \sqrt{t^2 - a^2} \right) + \Delta \gamma_+(w_m) - \pi}{\sqrt{t^2 - a^2}},
\]
where \( \Delta \gamma_+(w_m) = \arg \eta_m \), see Fig. 6. The above analysis shows that for \( \tau_m \) we have \( \tau_m(t) \leq \tau_0 \) for every \( t \in D_{\eta_m} \), where the equality holds only for \( t = w_m \).
14. Take now, without loss of generality due to symmetry, $\eta \in \mathbb{C} \cap \Pi_+$ such that $\Re \eta < -a$ and $\frac{\pi}{2} < \text{Arg } \eta < \Delta_{\gamma_+}(w_m) = \text{Arg } \eta_m$. We claim that for every such $\eta$ there is $\tau(|\eta|) < \tau_0$, where $\tau$ is defined on a ray containing $\eta$. Really, let us fix $\eta$ as above and assume otherwise i.e. $\tau(|\eta|) \geq \tau_0$. Then there exists $\eta_0$ that fulfills (10), $\text{Arg } \eta_0 = \text{Arg } \eta$ and $\Delta_{\gamma_+}(w_0) = \text{Arg } \eta_0 + 2\pi$, where $w_0 = |\eta_0|$ (see Fig. [3]). As $\eta \in D_{\eta_0}$ we have $\tau : D_{\eta_0} \rightarrow (0, \infty)$ defined as in (31) but on the ray $D_{\eta_0}$, and such that for $t = |\eta|$ it takes the value

$$\tau(t) = \frac{\arctan \left( \frac{1}{a} \sqrt{t^2 - a^2} \right) + \text{Arg } \eta_0 + \pi}{\sqrt{t^2 - a^2}},$$

(38)

where we used a fact that $\Delta_{\gamma_+}(w_0) = \text{Arg } \eta_0 + 2\pi$. Note that for a fixed $t$ the above is a continuous function of $\text{Arg } \eta_0 \in \left(\frac{\pi}{2}, \pi\right)$. Let us take a sequence $\{\eta_0^k\}_{k \in \mathbb{N}}$ such that $\eta_0^k$ fulfills (10), $|\eta_0^k| < |\eta_0^{k+1}|$ for every $k \in \mathbb{N}$ and $\eta_0^k \rightarrow \eta_m^*$ as $k \rightarrow \infty$, where $\text{Arg } \eta_m^* = \text{Arg } \eta_m$. Geometry of the problem shows that for every $t \in \mathbb{N}$ we have

$$\text{Arg } \eta_0^k < \text{Arg } \eta_0^{k+1} < \text{Arg } \eta_m^* \quad \text{and} \quad D_{\eta_0^k} \subset D_{\eta_0^{k+1}} \subset D_{\eta_m^*}.$$

For the fixed $t$ from (38) consider a continuous, strictly increasing function $\tau_t : [\text{Arg } \eta_0, \pi] \rightarrow (0, \infty)$,

$$\tau_t(\text{Arg } \xi) = \frac{\arctan \left( \frac{1}{a} \sqrt{t^2 - a^2} \right) + \text{Arg } \xi + \pi}{\sqrt{t^2 - a^2}}.$$

Our hypothesis now gives

$$\tau_0 \leq \tau(t) < \lim_{k \rightarrow \infty} \tau_t(\text{Arg } \eta_0^k) = \tau_t(\text{Arg } \eta_m^*) = \tau_m(t) \leq \tau_0,$$

where we used strict monotonicity and continuity of $\tau_t$, continuity of $\gamma_+$, definition of $D_{\eta_m}$ and boundedness of $\tau_m$ given by (57). The above contradiction proves our claim. Thus with $0 < a \leq \frac{1}{\tau_0}$ for the roots of (11) to be in $\mathbb{C} \setminus \mathbb{C}_+$ the region given by (36) is the only allowable one for $\eta$.

15. Fix $a > \frac{1}{\tau_0}$, where $\tau_0$ is as in point 3. By (30) and a comment directly below it the continuous argument increment function $\Delta_{\gamma_+}$ given by (28) is now strictly increasing with range $\Delta_{\gamma_+}([a, \infty)) = [\pi, \infty)$. The minimal value of $\Delta_{\gamma_+}(w) = \pi$ for $w = |a|$ and point 14 shows that if the roots of (11) are in $\mathbb{C} \setminus \mathbb{C}_+$ then $\eta = -a$; there is no such $\eta$ that the roots of (11) are in $\mathbb{C}_-$.

16. For sufficiency let $\tau_0 > 0$ be given and $a \leq \frac{1}{\tau_0}$. The behaviour of the roots described in points 1 and 2 does not change. Every $\eta \in \Lambda_{\gamma_0, a}$, where $\Lambda_{\gamma_0, a}$ is defined accordingly to $a$, is either inside $D_{|a|}$ or satisfies (23), (27) or (36). Following backwards the reasoning in points 5–13 we reach the boundary condition (22), (26) or equality in (36), for which the roots of (11) are on the imaginary axis, what happens exactly when $\eta$ is at the boundary of $\Lambda_{\gamma_0, a}$.

Corollary 7. Let $\tau > 0$ and let $\lambda, \gamma, \eta \in \mathbb{C}$ such that $\lambda = a + ib$ with $a \leq \frac{1}{\tau}$, $b \in \mathbb{R}$. Then

(i) every solution of the equation $s - a - \eta e^{-st} = 0$ belongs to $\mathbb{C}_-$ if and only if $\eta \in \Lambda_{\tau, a};$

(ii) every solution of

$$s - \lambda - \gamma e^{-st} = 0$$

(39)

and its version with conjugate coefficients

$$s - \bar{\lambda} - \bar{\gamma} e^{-st} = 0$$

(40)

belongs to $\mathbb{C}_-$ if and only if $\gamma e^{-ib\tau} \in \Lambda_{\tau, a}$.

Proof. Part (i) follows from the continuity of (23) or (31). Part (ii) follows from (i) and Lemma 2 by defining $\eta = \gamma e^{-ib\tau}$ for the case of (39), while for the case of (40) by the real-axis symmetry of $\Lambda_{\tau, a}$ we have $\eta \in \Lambda_{\tau, a}$ if and only if $\bar{\eta} = \bar{\gamma} e^{ib\tau} \in \Lambda_{\tau, a}$. □
4 Examples

In the following examples we illustrate how the necessary and sufficient conditions of Theorem 6 can be compared with and improve known literature results. Note initially that the stability condition discussed in [1] and later proved in [3], that is $- \Re \lambda > |\gamma|$, follows immediately from (15). Note also that as Corollary 7 concerns the placement of roots of the characteristic equation (3), it gives also a necessary and sufficient condition for stability of (4). With that in mind we give the following examples.

4.1 Example 1

Consider a differential-difference equation

$$x'(t) = i20x(t) + \gamma x(t - 0.1),$$

(41)

where $\Re \gamma > 0$. Equation (41) is a special case of (4), for which necessary and sufficient conditions of stability were found in [3]. By Corollary 7 equation (41) is stable if and only if $\gamma e^{-i2} \in \Lambda_{0.1,0}$, where $\Lambda_{0.1,0}$ is given by (9). We thus obtain that (41) is stable if and only if $\gamma < 20 - 5 \pi$, what is equivalent to the condition given by [3, Theorem 3.1].

4.2 Example 2

Consider a differential-difference equation

$$x'(t) = \left(\frac{1}{4} + i\frac{1}{4}\right) x(t) - \left(\frac{-1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) x(t - 1),$$

(42)

for which the corresponding characteristic equation takes the form

$$s - \left(\frac{1}{4} + i\frac{1}{4}\right) - \left(-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right) e^{-s} = 0.$$

(43)

By Corollary 7 and (10) (or, in fact, by investigating Fig. 4) we see that (43) is stable.

4.3 Example 3

In [7] the author considers a system of type (1), where - due to the approach method - the results can be stated only for a fixed delay $\tau = 1$. By using a there-described transformations the exemplary system takes the form

$$x'(t) = Bx(t - 1) + N(x(t), x(t - 1)),$$

(44)

where $N$ satisfies (2) and

$$B = \left(\begin{array}{cc} -1 & \frac{1}{\sqrt{8}} \\ -1 & -1 \end{array}\right).$$

As $A = 0$ and thus $\lambda = 0$, we are interested only in eigenvalues of $B$, which are $-1 \pm i\frac{1}{\sqrt{8}}$. The author concludes that the system is stable.

With conditions (9) we can improve results in [7] by finding a maximal delay $\tau$ for which (44) remains stable. Let $\eta = -1 + i\frac{1}{\sqrt{8}}$. Then $|\eta| = \frac{3\sqrt{2}}{4}$ and $\text{Arg } \eta = \pi - \arctan \frac{1}{\sqrt{8}}$ and by (9) we obtain that (44) is stable if and only if

$$0 < \tau < \frac{1}{|\eta|} \left(\text{Arg } \eta - \frac{\pi}{2}\right) = \frac{2\sqrt{2}}{3} \left(\frac{\pi}{2} - \arctan \frac{1}{\sqrt{8}}\right).$$

Note that we do not need to consider $\bar{\eta}$ due to the symmetry of $\Lambda_{\tau,0}$ about the real axis.
5 Acknowledgements

The authors would like to thank Prof. Yuriy Tomilov for mentioning to them reference [7].

The research of Rafał Kapica was supported by the Faculty of Applied Mathematics AGH UST statutory tasks within subsidy of Ministry of Education and Science.

The work of Radosław Zawiski was performed when he was a visiting researcher at the Centre for Mathematical Sciences of the Lund University, hosted by Sandra Pott, and supported by the Polish National Agency for Academic Exchange (NAWA) within the Bekker programme under the agreement PPN/BEK/2020/1/00226/U/00001/A/00001.

References

[1] V. K. Barwell, *Special stability problems for functional differential equations*, BIT 15 (1975), 130–135.
[2] R. Bellman and K. L. Cooke, *Differential-Difference Equations*, Mathematics in Science and Engineering, vol. 6, Academic Press, London, 1963.
[3] B. Cahlon and D. Schmidt, *On stability of a first-order complex delay differential equation*, Nonlinear Analysis: Real World Applications 3 (2002), 413–429.
[4] N. D. Hayes, *Roots of the transcendental equation associated with a certain difference-differential equation*, Journal of the London Mathematical Society 25 (1950), 226–232.
[5] T. S. Motzkin and O. Taussky, *Pairs of matrices with property L*, Transactions of the American Mathematical Society 73 (1952), 108–114.
[6] J. Nishiguchi, *On parameter dependence of exponential stability of equilibrium solutions in differential equations with a single constant delay*, Discrete and Continuous Dynamical Systems 36 (2016), 5657–5679.
[7] V. W. Noonburg, *Roots of a transcendental equation associated with a system of differential-difference equations*, SIAM Journal of Applied Mathematics 17 (1969), 198–205.
[8] J. R. Partington, *Linear Operators and Linear Systems: An Analytical Approach to Control Theory*, London Mathematical Society Student Texts, vol. 60, Cambridge University Press, Cambridge, UK, 2004.
[9] G. Stépán, *Retarded dynamical systems: Stability and characteristic functions*, Longman Scientific and Technical, Harlow, 1989.
[10] J. Wei and C. Zhang, *Stability analysis in a first-order complex differential equations with delay*, Nonlinear Analysis 59 (2004), 657–671.
[11] M. Wim and N. Silviu-Iulian, *Stability, Control, and Computation for Time-Delay Systems*, SIAM, Philadelphia, 2014.