EFFECTIVE QUANTUM UNIQUE ERGODICITY FOR HECKE–MAASS NEWFORMS AND LANDAU–SIEGEL ZEROS

JESSE THORNER

Abstract. We show that Landau–Siegel zeros for Dirichlet $L$-functions do not exist or quantum unique ergodicity for $GL_2$ Hecke–Maaß newforms holds with an effective rate of convergence. This follows from a more general result: Landau–Siegel zeros of Dirichlet $L$-functions repel the zeros of all other automorphic $L$-functions from the line $\text{Re}(s) = 1$.

1. Introduction and statement of the main results

Let $(\chi_i \pmod{q_i})_{i=1}^{\infty}$ be a sequence of primitive quadratic Dirichlet characters to increasing moduli $q_i$ whose Dirichlet $L$-functions $L(s, \chi_i)$ have a greatest real zero $\beta_i = 1 - \frac{\lambda_i}{\log q_i}$. It remains an open problem to rule out the possibility that $\lim_{q_i \to \infty} \lambda_i = 0$, in which case the zeros $\beta_i$ are called Landau–Siegel zeros. Siegel proved that for all $\varepsilon > 0$, there exists an ineffective constant $c(\varepsilon) > 0$ such that $\lambda_i \geq c(\varepsilon)q_i^{-\varepsilon}$. The existence of Landau–Siegel zeros violates the generalized Riemann hypothesis (GRH), induces inequities in the distribution of primes in arithmetic progressions, and obfuscates the study of class numbers of quadratic fields [7]. However, the existence of Landau–Siegel zeros implies many desirable results, including the infinitude of twin primes [13], the existence of primes in very short intervals [9], the infinitude of very large gaps between primes [8, 12], and the existence of a large proportion of nonvanishing central values of Dirichlet $L$-functions to prime moduli [4].

Let $\Gamma = \text{SL}_2(\mathbb{Z})$, let $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ be the hyperbolic Laplacian on $\Gamma \backslash \mathbb{H}$, and let $(\varphi)$ be a basis of Hecke–Maaß newforms, orthonormal with respect to the Petersson inner product

$$\langle \Psi, \Xi \rangle = \int_{\Gamma \backslash \mathbb{H}} \Psi(x + iy)\overline{\Xi(x + iy)}d\mu,$$

which are eigenfunctions of $\Delta$ (with eigenvalues $\lambda_\varphi > \frac{1}{4}$) and all Hecke operators. Consider the probability measures $d\mu_\varphi := |\varphi(x + iy)|^2d\mu$, where $d\mu := y^{-2}dxdy$ is the usual hyperbolic measure; note that $\mu(\Gamma \backslash \mathbb{H}) = \frac{\pi}{3}$. For a test function $\Psi \in L^2(\Gamma \backslash \mathbb{H}, \mu)$, we have $\mu_\varphi(\Psi) = \langle \Psi, |\varphi|^2 \rangle$ and $\mu(\Psi) = \langle \Psi, 1 \rangle$. Lindenstrauss [21] and Soundararajan [28] proved the (arithmetic) quantum unique ergodicity conjecture of Rudnick and Sarnak [26]:

$$\lim_{\lambda_\varphi \to \infty} \mu_\varphi(\Psi) = \frac{3}{\pi} \mu(\Psi).$$

Thus, as $\lambda_\varphi \to \infty$, the $L^2$ mass of $\varphi$ equidistributes on $\Gamma \backslash \mathbb{H}$ with respect to $\mu$. No rate of convergence in terms of $\lambda_\varphi$ is known unconditionally, but an optimal rate is closely tied to GRH for a certain family of $L$-functions (see below).

We relate the rate of convergence in quantum unique ergodicity to the existence of Landau–Siegel zeros. We also prove a new bound on the set of exceptional $\varphi$ for which an effective rate of convergence is not yet unconditionally known. All implied constants are effectively computable.
Theorem 1.1. Let $\Psi \in L^2(\Gamma\backslash \mathbb{H}, \mu)$ be a test function, and let $\varphi$ denote a Hecke–Maass newform with Laplace eigenvalue $\lambda_\varphi$.

(1) If $\chi \pmod{q}$ is a primitive quadratic Dirichlet character, $A \geq 1$, and $0 < \lambda < A^{-1}$, then at least one of the following statements is true:

(a) $L(s, \chi) \neq 0$ for $s \geq 1 - \lambda/\log q$.

(b) If $\lambda_\varphi \not\in [q, q^2]$, then $\mu_\varphi(\Psi) = \frac{2}{\pi} \mu(\Psi) + O(\delta(\log \lambda_\varphi)^{1+\frac{1}{6}})$ for all except $O((\log M)^{10^{20}q})$ Hecke–Maass newforms $\varphi$ with $\lambda_\varphi \leq M$.

Soundararajan [29, p. 1477] proved a version of Theorem 1.1(2) with an exceptional set of size $\ll \varepsilon M^\varepsilon$ for all $\varepsilon > 0$, so Theorem 1.1(2) gives a noticeable improvement. (Note that the Weyl law tells us that there are $\sim \frac{1}{12}M$ Hecke–Maass newforms $\varphi$ with $\lambda_\varphi \leq M$.)

Theorem 1.1(1) shows that this exceptional set is empty if there exists a suitable sequence of primitive quadratic Dirichlet characters whose $L$-functions have Landau–Siegel zeros.

The connection between $L$-functions and quantum unique ergodicity is well-understood [18, 23, 32]. The test function $\Psi$ has an orthonormal expansion with respect to the Petersson inner product in terms of the constant function: a basis of Hecke–Maass newforms $(u_k)_{k=1}^\infty$, which are eigenfunctions of $\Delta$ and all Hecke operators; and the Eisenstein series $E(z, \frac{1}{2} + it)$ with $t \in \mathbb{R}$ [17, Chapter 15]. Write $\lambda_k = \frac{1}{4} + t_k^2 > \frac{1}{4}$ for the Laplace eigenvalue of $u_k$. By orthonormality, we have that $\langle u_k, u_k \rangle = 1$.

Let $\varphi$ be a Hecke–Maass newform with $\lambda_\varphi = \frac{1}{4} + t_\varphi^2 > \frac{1}{4}$, normalized so that $\langle \varphi, \varphi \rangle = 1$. Since $\mu_\varphi(\Psi) = \langle \Psi, |\varphi|^2 \rangle$ and $\mu(\Psi) = \langle \Psi, 1 \rangle$, the spectral theorem [17, Theorem 15.5] implies that

\[
(1.1) \quad \mu_\varphi(\Psi) - \frac{3}{\pi} \mu(\Psi) = \sum_{k=1}^\infty \langle \Psi, u_k \rangle \langle u_k, |\varphi|^2 \rangle + \frac{1}{4\pi} \int_{\mathbb{R}} \langle \Psi, E(\cdot, \frac{1}{2} + it) \rangle \langle E(\cdot, \frac{1}{2} + it), |\varphi|^2 \rangle dt.
\]

Since $\Psi$ is a fixed test function, the $\lambda_\varphi$-dependence in the rate at which $\mu_\varphi(\Psi)$ tends to $\frac{2}{\pi} \mu(\Psi)$ is completely dictated by $\lambda_\varphi$-dependence in bounds for the inner products $\langle u_k, |\varphi|^2 \rangle$ and $\langle E(\cdot, \frac{1}{2} + it), |\varphi|^2 \rangle$, where $t \in \mathbb{R}$ and $u_k$ is a fixed Hecke–Maass newform.

Watson [32, Theorem 3] proved an elegant relationship between the inner product $\langle u_k, |\varphi|^2 \rangle$ and central values of $L$-functions:

\[
|\langle u_k, |\varphi|^2 \rangle|^2 = \frac{1}{8} \frac{\Lambda(\frac{1}{2}, u_k \otimes \varphi \otimes \overline{\varphi})}{\Lambda(1, \operatorname{Ad}^2 u_k)\Lambda(1, \operatorname{Ad}^2 \varphi)^2} \ll \left| \frac{\Gamma(\frac{1+2it_\varphi+it_\delta}{2})\Gamma(\frac{1+2it_\varphi-it_\delta}{2})\Gamma(\frac{1+2it_\varphi+it_\delta}{2})^2}{\Gamma(\frac{1+2it_\delta}{2})^2\Gamma(\frac{1+2it_\delta}{2})} \right|^{1/2} L(\frac{1}{2}, u_k) L(\frac{1}{2}, \operatorname{Ad}^2 \varphi \otimes u_k) \frac{L(1, \operatorname{Ad}^2 u_k) L(1, \operatorname{Ad}^2 \varphi)^2}{L(1, \operatorname{Ad}^2 u_k) L(1, \operatorname{Ad}^2 \varphi)^2},
\]

where $\operatorname{Ad}^2$ denotes the adjoint square lift. (See Section 2 for the pertinent $L$-function notation.) An application of Stirling’s formula yields

\[
|\langle u_k, |\varphi|^2 \rangle| \ll \lambda_\varphi^{-1/4} L(1, \operatorname{Ad}^2 \varphi)^{-1} L(\frac{1}{2}, u_k \otimes \operatorname{Ad}^2 \varphi)^{1/2}.
\]

Goldfeld, Hoffstein, and Lieman [13, Appendix] proved that $L(1, \operatorname{Ad}^2 \varphi)^{-1} \ll \log \lambda_\varphi$, hence

\[
(1.2) \quad |\langle u_k, |\varphi|^2 \rangle| \ll \lambda_\varphi^{-1/4}(\log \lambda_\varphi) L(\frac{1}{2}, u_k \otimes \operatorname{Ad}^2 \varphi)^{1/2}.
\]

Their main result is $L(s, \operatorname{Ad}^2 \varphi) \neq 0$ for $s \geq 1-c/\log(\lambda_\varphi+3)$; one can check that $c = 1/500$ is permissible.
Using the unfolding method (see [6] Equation 4.2, for example), one computes

\[
|\langle E(\cdot, \frac{1}{2} + it), |\varphi|^2 \rangle| = \frac{\sqrt{\pi}}{2} \left| \frac{\Gamma(\frac{1+2it+it}{2})\Gamma(\frac{1+2it-it}{2})\Gamma(\frac{1+it}{2})^2}{\Gamma(\frac{1+it}{2}+it\lambda^2\varphi)\Gamma(\frac{1+2it}{2})} \right| \frac{\zeta(\frac{1}{2} + it)L(\frac{1}{2} + it, \text{Ad}^2\varphi)}{L(1, \text{Ad}^2\varphi)|\zeta(1+2it)|}.
\]

Stirling’s formula and standard bounds for the Riemann zeta function \( \zeta(s) \) imply that there exists an absolute and effectively computable constant \( A_1 > 0 \) such that

\[
|\langle E(\cdot, \frac{1}{2} + it), |\varphi|^2 \rangle| \ll (3 + |t|)^{A_1} \lambda^{-1/4} L(1, \text{Ad}^2\varphi)^{-1} |L(\frac{1}{2} + it, \text{Ad}^2\varphi)|.
\]

Again, we invoke the work of Goldfeld, Hoffstein, and Lieman to deduce the bound

\[
(1.3) \quad |\langle E(\cdot, \frac{1}{2} + it), |\varphi|^2 \rangle| \ll (3 + |t|)^{A_1} \lambda^{-1/4} (\log \lambda_\varphi)|L(\frac{1}{2} + it, \text{Ad}^2\varphi)|.
\]

Let \( g \) be an increasing function with \( \lim_{x \to \infty} g(x) = \infty \). Equations (1.1)-(1.3) show that if there exists an absolute constant \( A_2 > 0 \) such that

\[
(1.4) \quad L(\frac{1}{2}, u_k \otimes \text{Ad}^2\varphi) \ll_{u_k} \frac{\lambda_\varphi^{1/2}}{g(\lambda_\varphi)^{2}} \quad \text{and} \quad |L(\frac{1}{2} + it, \text{Ad}^2\varphi)| \ll (3 + |t|)^{A_2} \frac{\lambda_\varphi^{1/4}}{g(\lambda_\varphi)}
\]

for every \( t \in \mathbb{R} \) and every fixed Hecke–Maaß form \( u_k \), then

\[
(1.5) \quad \mu_\varphi(\Psi) = \frac{3}{\pi} \mu(\Psi) + O_{\Psi}(\log \lambda_\varphi).
\]

Gelbart and Jacquet [11] proved that \( L(s, \text{Ad}^2\varphi) \) is the \( L \)-function of a cuspidal automorphic representation of \( \text{GL}_3 \), and the combined work of Kim and Shahidi [19] and Ramakrishnan and Wang [25] shows that \( L(s, u_k \otimes \text{Ad}^2\varphi) \) is the \( L \)-function of a cuspidal automorphic representation of \( \text{GL}_6 \) when \( \varphi \) is not a twist of. Therefore, the rate of convergence in (1.5) is directly tied to bounds for certain \( \text{GL}_3 \) and \( \text{GL}_6 \) automorphic \( L \)-functions on the critical line \( \text{Re}(s) = \frac{1}{2} \).

We address this in a broader context. Let \( \mathbb{A} \) be the ring of adeles over \( \mathbb{Q} \), and let \( \mathfrak{F}_m \) be the set of all cuspidal automorphic representations of \( \text{GL}_m(\mathbb{A}) \) with unitary central character (which we normalize to be trivial on the diagonally embedded copy of the reals so that \( \mathfrak{F}_m \) is a discrete set). Let \( m \geq 2 \) and \( \pi \in \mathfrak{F}_m \). The analytic conductor \( C(\pi) \) of \( \pi \) (see (2.1)) is a useful measure of the spectral and arithmetic complexity of \( \pi \).

The work of Heath-Brown [13], Luo, Rudnick, and Sarnak [22]; and Müller and Speh [24] produces the convexity bound \( |L(\frac{1}{2}, \pi)| \ll_m C(\pi)^{1/4} \). In many applications, this bound barely fails to be of use, and even small improvements would yield deep results. GRH implies that \( |L(\frac{1}{2}, \pi)| \leq C(\pi)^{\theta(1)} \), which yields (1.5) with an optimal error term of \( O_{\Psi}(\lambda_\varphi^{-1/4+o(1)}) \). Subconvexity bounds of the form \( |L(\frac{1}{2}, \pi)| \ll_m C(\pi)^{1/4-\delta_m} \) for some fixed \( \delta_m > 0 \) are known for many classes of \( L \)-functions, but a nontrivial error term in (1.5) remains out of reach.

Under a weak form of the generalized Ramanujan conjecture (GRC), Soundararajan [29] proved that \( |L(\frac{1}{2}, \pi)| \ll_{m, \varepsilon} C(\pi)^{1/4} (\log C(\pi))^{-1+\varepsilon} \) for all \( \varepsilon > 0 \). By a different method, Soundararajan and the author [30] unconditionally proved

\[
(1.6) \quad |L(\frac{1}{2}, \pi)| \ll_m C(\pi)^{\frac{1}{2} (\log C(\pi))^{-1+\frac{1}{100+m}}}
\]

for all \( m \geq 1 \). We will refine the results in [30] when Landau–Siegel zeros for Dirichlet \( L \)-functions exist. In what follows, let \( \mathfrak{F}_m(Q) = \{ \pi \in \mathfrak{F}_m : C(\pi) \leq Q \} \).

**Theorem 1.2.** Let \( t \in \mathbb{R} \), and let \( m \geq 2 \) be an integer.
(1) If \( \chi \pmod{q} \) is a primitive quadratic Dirichlet character, \( A \geq 1 \), and \( 0 < \lambda < A^{-1} \), then at least one of the following statements is true:
(a) \( L(s, \chi) \neq 0 \) for \( s \geq 1 - \lambda / \log q \).
(b) If \( A \geq 1 \) and \( C(\pi) \in [q, q^A] \), then
\[
|L(\frac{1}{2} + it, \pi)| \ll_m (3 + |t|)^{O(m)}C(\pi)^{\frac{1}{3}}(\log C(\pi))^{-\frac{1}{4} \frac{1}{10^{16}m^4} (A\lambda)^{\frac{1}{4} \frac{1}{10^{16}m^4}}}
\]

(2) Let \( \delta \geq 0 \). For all except \( O_m((\log Q)^{1016 m^4 \delta}) \) of the \( \pi \in \mathfrak{F}_m(Q) \), we have
\[
|L(\frac{1}{2} + it, \pi)| \ll_m (3 + |t|)^{O(m)}C(\pi)^{\frac{1}{3}}(\log C(\pi))^{-\delta}.
\]

Remark. Brumley and Miličević [2] proved that \( \# \mathfrak{F}_m(Q) \gg_m Q^{m+1} \), so the exceptional set in Theorem [12] (2) is quite small. Theorem [12] (1) shows that this exceptional set is empty if there exists a suitable sequence of primitive quadratic characters with Landau–Siegel zeros.

Proof of Theorem [13]. We have \( C(u_k \otimes \text{Ad}^2 \varphi) \preceq_{uk} \lambda_{\varphi}^2 \) and \( C(\text{Ad}^2 \varphi) \simeq \lambda_{\varphi} \), so the theorem follows from Theorem [12] and the fact [14] implies [15].

Given \( \pi \in \mathfrak{F}_m \), define \( N_\pi(\sigma, T) = \#\{\rho = \beta + i\gamma : \beta \geq \sigma, |\gamma| \leq T, L(\rho, \pi) = 0\} \), where the zeros are counted with multiplicity. For \( L(s, \pi) \), GRH is equivalent to the assertion that \( N_\pi(\sigma, T) = 0 \) for all \( \sigma > \frac{1}{2} \). The starting point for the proof of Theorem [12] is the following inequality: If \( 0 \leq \alpha < \frac{1}{2} \), then
\[
(1.7) \quad \log |L(\frac{1}{2}, \pi)| \leq \left(\frac{1}{4} - \frac{\alpha}{10^9}\right) \log C(\pi) + \frac{\alpha}{10^7} N_\pi(1 - \alpha, 6) + 2 \log |L(\frac{1}{2}, \pi)| + O(m^2).
\]

This was proved by Soundararajan and the author [30, Theorem 1.1]. The combined work of Luo–Rudnick–Sarnak [22] and Müller–Speh [24] shows that \( 2 \log |L(\frac{1}{2}, \pi)| \ll m^2 \). Therefore, (1.7) relates bounds for \( L(\frac{1}{2}, \pi) \) to the distribution of zeros of \( L(s, \pi) \) near \( s = 1 \). Soundararajan and the author [30, Corollary 2.6] also proved the log-free zero density estimate
\[
(1.8) \quad N_\pi(\sigma, T) \ll_m (C(\pi)T)^{10^7 m^4 (1 - \sigma)}.
\]

They showed that (1.7) and (1.8) imply (1.6) (choose \( \alpha = (10^8 m^3 \log C(\pi))^{-1} \log \log C(\pi) \)).

Variations of (1.8) in which one averages over \( \pi \in \mathfrak{F}_m(Q) \) have been known for a long time assuming unproven progress toward the generalized Ramanujan conjecture (GRC) [3, 20].

The author and Zaman [31, Theorem 1.2] recently proved such an estimate unconditionally:
\[
(1.9) \quad \sum_{\pi \in \mathfrak{F}_m(Q)} N_\pi(\sigma, T) \ll_m (QT)^{10^7 m^4 (1 - \sigma)}.
\]

Theorem [12] follows from the following refinement of (1.9) arising from Landau–Siegel zeros.

**Theorem 1.3.** Let \( Q, T \geq 3 \) and \( 0 \leq \sigma \leq 1 \). Let \( m \geq 2 \). If \( \chi \pmod{q} \) is a primitive quadratic Dirichlet character having a real zero \( \beta_\chi \in (\frac{1}{2}, 1) \) and \( q \leq Q \), then
\[
\sum_{\pi \in \mathfrak{F}_m(Q)} N_\pi(\sigma, T) \ll_m \min\{1, (1 - \beta_\chi) \log(qQT)\} (qQT)^{10^7 m^4 (1 - \sigma)}.
\]

Bombieri [1, Theorem 14] proved a version of Theorem [13] for zeros of Dirichlet \( L \)-functions with \( \beta_\chi \) omitted from the count (if it exists). A classical result of Deuring and Heilbronn quantifies the extent to which \( \beta_\chi \) repels zeros of other Dirichlet \( L \)-functions from \( \text{Re}(s) = 1 \) [17, Ch. 18]. Bombieri’s result recovers this phenomenon as a corollary, along with Siegel’s ineffective upper bound on \( \beta_\chi \). When \( m \leq 4 \), Theorem [13] recovers recent work of Brumley,
the author, and Zaman [3, Theorem 1.7 when \(\pi_0\) is trivial]. When \(m \geq 5\), the proofs in [3] relied on unproven progress toward GRC, which we remove here. This opens up opportunities to study the arithmetic consequences of Landau–Siegel zeros (should they exist) in many other settings. Theorem 1.1 ultimately uses Theorem 1.3 with \(m = 3\) and \(m = 6\).

Unlike Bombieri’s setting, we can ensure that \(L(\beta, \chi \pi) \neq 0\) for all \(\pi \in \mathfrak{F}_m(Q)\), provided that \(m \geq 2\) and \(m^2(1 - \beta) \log(qQ) < c\) for a certain absolute and effectively computable constant \(c > 0\) (see Lemma 2.3 below). Combined with Theorem 1.3, this fact enables us to deduce for each \(\pi \in \mathfrak{F}_m(Q)\) a much stronger zero-free region for \(L(s, \pi)\) than unconditional methods permit, akin to the aforementioned result of Deuring and Heilbronn, when \(\beta\) is sufficiently close to 1.

Acknowledgements. I thank Kevin Ford, Peter Humphries, and the anonymous referee for their helpful comments.

2. Properties of \(L\)-functions

We recall some standard facts about \(L\)-functions arising from automorphic representations and their twists by Dirichlet characters; see [30] for a convenient summary.

2.1. Standard \(L\)-functions. Let \(\pi \in \mathfrak{F}_m\) have arithmetic conductor \(q_\pi\) with \(m \geq 2\). We express \(\pi\) is a tensor product \(\otimes_v \pi_v\) of smooth admissible representations of \(GL_m(Q_p)\) and \(GL_m(\mathbb{R})\), where \(v\) varies over the places of \(Q\). If \(v\) is nonarchimedean and corresponds with a prime \(p\), in which case we write \(\pi_p\) instead of \(\pi_v\), then there are \(m\) Satake parameters \(\alpha_{j,\pi}(p)\) from which we define

\[
L(s, \pi_p) = \prod_{j=1}^{m} \left(1 - \frac{\alpha_{j,p}(p)}{p^s}\right)^{-1} = \sum_{j=0}^{\infty} \frac{\lambda_{\pi}(p^j)}{p^{js}}.
\]

We have \(\alpha_{j,p}(p) \neq 0\) for all \(j\) whenever \(p \nmid q_\pi\), and it might be the case that \(\alpha_{j,\pi}(p) = 0\) for some \(j\) when \(p|q_\pi\). The finite part of the standard \(L\)-function \(L(s, \pi)\) associated to \(\pi\) is of the form

\[
L(s, \pi) = \prod_p L(s, \pi_p) = \sum_{n=1}^{\infty} \frac{\lambda_{\pi}(n)}{n^s}, \quad \text{Re}(s) > 1.
\]

If \(v\) is archimedean, in which case we write \(\pi_\infty\) instead of \(\pi_v\), there are \(m\) Langlands parameters \(\mu_{\pi}(j) \in \mathbb{C}\) for \(1 \leq j \leq m\) from which we define

\[
L(s, \pi_\infty) = \pi^{-\frac{ms}{2}} \prod_{j=1}^{m} \Gamma\left(s + \frac{\mu_{\pi}(j)}{2}\right).
\]

We define for \(t \in \mathbb{R}\) the analytic conductor

\[
C(\pi, t) = q_\pi \prod j=1^{m} (3 + |t + \mu_{\pi}(j)|), \quad C(\pi) = C(\pi, 0).
\]

There exists \(\theta_m \in [\frac{1}{2}, \frac{1}{2} - \frac{1}{m^2 + 1}]\) such that we have the uniform bounds

\[
|\alpha_{j,\pi}(p)| \leq p^{\theta_m}, \quad \text{Re}(\mu_{\pi}(j)) \geq -\theta_m.
\]

(This follows from [22] when \(v\) is unramified and [24] when \(v\) is ramified.) GRC asserts that one may take \(\theta_m = 0\).
The completed $L$-function $\Lambda(s, \pi) = q^{s^2/2}L(s, \pi)L(s, \pi_\infty)$ is entire of order 1. Each pole of $L(s, \pi_\infty)$ is a trivial zero of $L(s, \pi)$. Since $\Lambda(s, \pi)$ is entire of order 1, a Hadamard factorization

$$\Lambda(s, \pi) = e^{a_s+b_s} \prod_p \left(1 - \frac{s}{\rho}\right)e^{s/\rho}$$

exists, where $\rho$ varies over the nontrivial zeros of $L(s, \pi)$. These zeros satisfy $0 \leq \Re(\rho) \leq 1$.

### 2.2. Twisted $L$-functions

If $\pi \in \mathfrak{X}_m$ and $\chi \pmod q$ is a primitive Dirichlet character, then $\pi \otimes \chi \in \mathfrak{X}_m$. If $p \nmid q_\pi q$, then we have the equality of sets $\{\alpha_{j,\pi\otimes\chi}(p)\} = \{\alpha_{j,\pi}(p)\chi(p)\}$. See [30, Appendix] for a description of Lemma 2.3. By the proof of [30, Lemma 3.1], we have

This is an immediate corollary of [3, Lemma 4.2]. In the notation therein, take

Define the functions $\Lambda_\pi(n)$ and $\Lambda_{\pi\otimes\chi}(n)$ by the Dirichlet series identities

$$\sum_{n=1}^{\infty} \frac{\Lambda_\pi(n)}{n^s} = -\frac{L'}{L}(s, \pi), \quad \sum_{n=1}^{\infty} \frac{\Lambda_{\pi\otimes\chi}(n)}{n^s} = -\frac{L'}{L}(s, \pi \otimes \chi), \quad \Re(s) > 1.$$

**Lemma 2.1.** If $\pi \in \mathfrak{X}_m$, $\chi$ is a primitive quadratic character, and $\eta > 0$, then

$$\sum_{n=1}^{\infty} \frac{|\Lambda_{\pi\otimes\chi}(n)|}{n^{1+\eta}} \leq \frac{1}{\eta} + \frac{m}{2} \log \Lambda_\pi(n) + O(m^2).$$

**Proof.** This is an immediate corollary of [3, Lemma 4.2]. In the notation therein, take $\pi_0$ to be the trivial representation of $GL_1(A_Q)$, so that $m_0 = 1$ and $C(\pi_0) \asymp 1$. □

**Lemma 2.2.** If $t \in \mathbb{R}$ and $0 < \eta \leq 1$, then

$$\#\{\rho: |\rho - (1+it)| \leq \eta, \ L(\rho, \pi \otimes \chi) = 0\} \leq 5m\eta \log(q\Lambda_\pi(\pi)) + O(m^2\eta + 1)$$

and

$$\sum_{\rho} \frac{1+\eta-\beta}{|1+\eta+it-\rho|^2} \ll \frac{1}{\eta} + m \log(q\Lambda_\pi(\pi))(3 + |t|).$$

**Proof.** By the proof of [30, Lemma 3.1], we have

$$\sum_{\rho} \frac{1+\eta-\beta}{|1+\eta+it-\rho|^2} \leq \frac{1}{5\eta} \#\{\rho: |\rho - (1+it)| \leq \eta, \ L(\rho, \pi \otimes \chi) = 0\}$$

$$\leq \frac{1}{2} \log q_{\pi\otimes\chi} + \frac{1}{2} \sum_{j=1}^{m} \Re \left(\frac{1+\eta+it+\mu_{\pi\otimes\chi}(j)}{2}\right) + \sum_{n=1}^{\infty} \frac{|\Lambda_{\pi\otimes\chi}(n)|}{n^{1+\eta}}$$

The result follows from the bound $q_{\pi\otimes\chi} \leq q^m q_\pi$, Stirling’s formula, and Lemma 2.1. □

**Lemma 2.3.** Let $\chi \pmod q$ be a primitive quadratic character such that $L(s, \chi)$ has a zero $\beta_\chi \in (\frac{1}{2}, 1)$, and let $m \geq 2$. There exists an absolute and effectively computable constant $c > 0$ such that if $m^2(1-\beta_\chi)\log(qQ) < c$, then $L(\beta_\chi, \pi) \neq 0$ for all $\pi \in \mathfrak{X}_m(Q)$. □
Proof. It follows from [16, Theorem A] that for at most one \( \pi \in \mathcal{F}_m(Q) \cup \{ \chi \} \), \( L(s, \pi) \) has a zero in the interval \([1 - c(m^2 \log qQ)^{-1}, 1]\). If \( m^2(1 - \beta_\chi) \log(qQ) < c \), then \( \beta_\chi \) lies in this interval. Since \( L(\beta_\chi, \chi) = 0 \) by definition and \( \chi \notin \mathcal{F}_m(Q) \) once \( m \geq 2 \), the lemma follows. \( \square \)

3. Proof of Theorem 1.3

Let \( \pi \in \mathcal{F}_m(Q) \) with \( m \geq 2 \), and let \( \chi \pmod{q} \) be a primitive quadratic Dirichlet character. Suppose that \( L(s, \chi) \) has a real zero \( \beta_\chi \), and suppose that \( L(s, \pi) \) has a zero \( \rho_0 \) (necessarily distinct from \( \beta_\chi \) by Lemma 2.2) such that \( |\rho_0 - (1 + i\tau)| \leq \eta \), where \( \tau \in \mathbb{R}, |\tau| \leq T \), and

\[
(3.1) \quad (\log qQT)^{-1} \leq \eta \leq (200m^2)^{-1}.
\]

3.1. A zero detection criterion. Let \( k \geq 0 \) be an integer, let \( s = 1 + \eta + i\tau \), and define

\[
F(z) = L(z, \pi)L(z + 1 - \beta_\chi, \pi \otimes \chi), \quad G_k(z) = \frac{(-1)^k}{k!} \left( \frac{F'}{F} \right)^{(k)}(z).
\]

A standard calculation involving the Hadamard product as in [30, Section 4] leading up to Equation 4(2) shows that if \( k \geq 0 \) is an integer, then

\[
(3.2) \quad G_k(s) = \sum_{L(\rho, \pi) = 0} \frac{1}{(s - \rho)^{k+1}} + \sum_{L(\rho, \pi \otimes \chi) = 0} \frac{1}{(s + 1 - \beta_\chi - \rho')^{k+1}} + O\left( \frac{m^2 \eta \log(qQT)}{(200\eta)^k} \right).
\]

The hypothesized existence of \( \rho_0 \) ensures that \( |G_k(s)| \) has a large lower bound.

**Lemma 3.1.** Let \( \eta \) satisfy (3.1), and let \( \tau \in \mathbb{R} \) satisfy \(|\tau| \leq T \). Suppose that \( L(z, \pi) \) has a zero \( \rho_0 \) satisfying \(|\rho_0 - (1 + i\tau)| \leq \eta \). If \( K \geq [2000m^2 \eta \log(qQT) + O_m(1)] \) with a sufficiently large implied constant, then \( \eta^{k+1}|G_k(1 + \eta + i\tau)| \geq (2(100)^{k+1})^{-1} \) for some \( k \in [K, 2K] \).

**Proof.** By Lemma 2.2 there are at most \( 2000m^2 \eta \log(qQT) + O(1) \) zeros of \( F(z) \) such that \(|1 + \eta + i\tau - \rho| \leq 200\eta \). A result of Sós and Turán [27, Theorem] states that if \( z_1, \ldots, z_\nu \in \mathbb{C} \) and \( K \geq \nu \), then there exists an integer \( k \in [K, 2K] \) such that \(|z_1^k + \cdots + z_\nu^k| \geq (\nu|z_1|/50)^{k}\).

We apply this to (3.2) with \( K \geq [2000m^2 \eta \log(qQT) + O_m(1)] \) and \( z_1 = 1/(s - \rho_0) \), which has modulus at least \( 1/(2\eta) \). (Recall that \( s = 1 + \eta + i\tau \).) It follows that

\[
\left| \sum_{L(\rho, \pi) = 0} \frac{1}{(s - \rho)^{k+1}} + \sum_{L(\rho, \pi \otimes \chi) = 0} \frac{1}{(s + 1 - \beta_\chi - \rho')^{k+1}} \right| \geq \left( \frac{1}{50|s - \rho_0|} \right)^{k+1} \geq \frac{1}{(100\eta)^{k+1}}.
\]

We apply this to (3.2), and the lemma follows. \( \square \)

We now prove an upper bound for \( |G_k(s)| \).

**Lemma 3.2.** Let \( \eta \) satisfy (3.1), and let \( \tau \in \mathbb{R} \) satisfy \(|\tau| \leq T \). Let \( K \) be as in Lemma 3.1, and define \( N_0 = \exp(K/(300\eta)) \) and \( N_1 = \exp(40K/\eta) \). If \( k \in [K, 2K] \), then

\[
\eta^{k+1}|G_k(1 + \eta + i\tau)| \leq \eta^2 \int_{N_0}^{N_1} \left| \sum_{N_0 \leq p \leq u \atop \pi \text{ prime}} \frac{\pi(p) \log p}{p^{1+i\tau}} (1 + \chi(p)p^{\alpha - 1}) \right| \frac{du}{u} + O\left( \frac{k}{(110)^{k+1}} \right).
\]

\[\text{Use Lemma 2.2 in place of [30, (3.5)]}.\]
Proof. A direct computation shows that
\[ \eta^{k+1}|G_k(s)| = \eta \left| \sum_{n=1}^{\infty} \frac{\Lambda_\pi(n) + \Lambda_\pi \chi(n) n^{\beta \chi - 1}}{n^{1+\eta+i\tau}} (\eta \log n)^k \right|. \]

Since \(|\chi(n)| \leq 1\) and \(\beta \chi < 1\), it follows from [31, (5.6) and (5.7)] that
\[ |\eta^{k+1}G_k(s)| = \eta \left| \sum_{p \in [N_0, N_1]} \frac{\Lambda_\pi(p) + \Lambda_\pi \chi(p) p^{\beta \chi - 1}}{p^{1+\eta+i\tau}} (\eta \log p)^k \right| + O\left(\frac{m^2 \eta \log(qQT)}{(110)^k}\right). \]

Since \(N_0 > \max\{Q, q\}\), we have \(\Lambda_\pi(p) + \Lambda_\pi \chi(p) p^{\beta \chi - 1} = \lambda_\pi(p)(1 + \chi(p) p^{\beta \chi - 1}) \log p\) when \(p \in [N_0, N_1]\). The lemma now follows by partial summation and Lemma 2.4, just as in [31, Lemma 5.5].

Lemmas 3.1 and 3.2 provide a criterion for detecting zeros. Let \(K = 10^3 \eta m^4 \log(qQT) + O_m(1)\) with a sufficiently large implied constant. If \(k \in [K, 2K]\), then \(O(k(110)^{-k}) \leq \frac{1}{3} (100)^{-k-1}\). Therefore, if \(k \in [K, 2K], \ |\tau| \leq T, \ \eta\ satisfies (3.1), \) and \(L(s, \rho)\) has a zero \(\rho_0\) satisfying \(|\rho_0 - (1 + i\tau)| \leq \eta\), then it follows from Lemmas 3.1 and 3.2 that
\[
1 \leq 4(100)^{2K + 1} \eta^2 \int_{N_0}^{N_1} \left| \sum_{N_0 \leq \rho \leq u} \Lambda_\pi(p) \log p \right| p^{1+\eta+i\tau} (1 + \chi(p) p^{\beta \chi - 1}) \left| d\rho \right|. 
\]

We square both sides, apply Cauchy–Schwarz, and use Lemma 2.2 to deduce that
\[
\frac{\#\{\rho = \beta + i\gamma: \beta \geq 1 - \frac{\eta}{2}, \ |\gamma - \tau| \leq \frac{\eta}{2}\}}{m^2 \eta \log(qQT)} \ll (101)^{4K} \eta^3 \int_{N_0}^{N_1} \left| \sum_{N_0 \leq \rho \leq u} \Lambda_\pi(p) \log p \right| p^{1+\eta+i\tau} (1 + \chi(p) p^{\beta \chi - 1})^2 \left| d\rho \right|. 
\]

3.2. Proof of Theorem 1.3. In order to count detected zeros in the box \([1 - \frac{\eta}{2}, 1] \times [-T, T]\), we integrate (3.3) over \(|\tau| \leq T\) and observe that \(m^2 \eta \log(qQT) \ll K\) to obtain
\[
N_\pi(1 - \frac{\eta}{2}, T) \ll (101)^{4K} K \eta^2 \int_{N_0}^{N_1} \left| \sum_{N_0 \leq \rho \leq u} \Lambda_\pi(p) \log p \right| p^{1+\eta+i\tau} (1 + \chi(p) p^{\beta \chi - 1}) \left| d\rho \right| d\tau. 
\]

We sum (3.4) over \(\pi \in \mathfrak{S}_m(Q)\) to find that \(\sum_{\pi \in \mathfrak{S}_m(Q)} N_\pi(1 - \frac{\eta}{2}, T)\) is
\[
\ll (101)^{4K} K \eta^2 \int_{N_0}^{N_1} \left( \sum_{\pi \in \mathfrak{S}_m(Q)} \left| \sum_{N_0 \leq \rho \leq u} \Lambda_\pi(p) \log p \right| p^{1+\eta+i\tau} (1 + \chi(p) p^{\beta \chi - 1}) \right| d\tau \left| d\rho \right|. 
\]

Lemma 3.3. If \(u \in [N_0, N_1]\), then with the notation above, we have
\[
\sum_{\pi \in \mathfrak{S}_m(Q)} \left| \sum_{N_0 \leq \rho \leq u} \Lambda_\pi(p) \log p \right| p^{1+\eta+i\tau} (1 + \chi(p) p^{\beta \chi - 1}) \left| d\tau \right| \ll_m \frac{\eta}{K} \sum_{p \in [N_0, u]} \frac{(\log p)^2}{p} (1 + \chi(p) p^{\beta \chi - 1})^2. 
\]

Proof. By [10, Theorem 1], if \(a: \mathbb{Z} \to \mathbb{C}\) is an arbitrary function and \(\sum_p |a(p)| < \infty\), then
\[
\sum_{\pi \in \mathfrak{S}_m(Q)} \left| \sum_{N_0 \leq \rho \leq u} \Lambda_\pi(p) a(p) p^{-i\tau} \right|^2 d\tau \ll T^2 \int_0^\infty \left| \sum_{\pi \in \mathfrak{S}_m(Q)} \sum_{p \in [x, xe^{-i/T}]} \Lambda_\pi(p) a(p) x \right|^2 dx. 
\]
If \( Q, T, x \geq 1 \) and \( z \gg m, \varepsilon \) with a large implied constant, then by [31 (4.7)],
\[
\sum_{\pi \in \mathfrak{F}(Q)} \left| \sum_{p \in [x, x e^{1/T}]} \lambda_\pi(p)a(p) \right|^2 \ll_{m, \varepsilon} \left( \frac{x}{T \log z} + Q^{m^2 + m + \varepsilon} T^{m^2} z^{2m^2 + 2 + \varepsilon} \right) \left| \mathfrak{F}_m(Q) \right| \sum_{p \in [x, x e^{1/T}]} |a(p)|^2.
\]
Combining these bounds with \( \left| \mathfrak{F}_m(Q) \right| \ll_{m, \varepsilon} Q^{2m + \varepsilon} \) [3 Theorem A.1] and \( \varepsilon = \frac{1}{100} \), we find that
\[
\sum_{\pi \in \mathfrak{F}(Q)} \int_{-T}^T \left| \sum_{\substack{p \in [x, z]} \lambda_\pi(p)a(p)e^{-it}} \right|^2 dt \ll_m \sum_p |a(p)|^2 p \left( \frac{1}{\log z} + \frac{Q^{5m^2} T^{m^2} z^{2m^2 + 3}}{p} \right) \ll_m \frac{1}{\log z} \sum_p |a(p)|^2 p \left( 1 + \frac{Q^{5m^2} T^{m^2} z^{2m^2 + 4}}{p} \right).
\]
Choose \( y = N_0 \) and \( z = y^{1/(10m^2)} \). Given \( u \in [N_0, N_1] \), it remains to choose \( a(p) = 0 \) unless \( p \in [y, u] \), in which case \( a(p) = \frac{\log p}{p} (1 + \chi(p)p^{\beta_\pi - 1}) \).

Recall the definitions of \( K, \eta, N_0, \) and \( N_1 \). We apply Lemma 3.3 to (3.5) and find that
\[
\sum_{\pi \in \mathfrak{F}(Q)} N_\pi(1 - \frac{\eta}{2}, T) \ll (101)^4 K^3 \int_{N_0}^{N_1} \sum_{\substack{p \in [N_0, N_1]}} \frac{(\log p)^2}{p} (1 + \chi(p)p^{\beta_{\pi} - 1}) \frac{du}{u}.
\]
It follows from a lemma of Bombieri [1, Lemme C, p. 50] that
\[
\sum_{\substack{p \in [N_0, N_1]}} \frac{(\log p)^2}{p} (1 + \chi(p)p^{\beta_{\pi} - 1})^2 \ll_m K^3 \frac{1 - \beta_{\pi}}{\eta^2} \min \left\{ 1, \frac{1 - \beta_{\pi}}{\eta} \right\},
\]
so
\[
\sum_{\pi \in \mathfrak{F}(Q)} N_\pi(1 - \frac{\eta}{2}, T) \ll (101)^4 K^4 \min \{ 1, (1 - \beta_{\pi}) \log(qQT) \}.
\]
Using our choices of \( K \) and \( \eta \) and writing \( \sigma = 1 - \frac{\eta}{2} \), we conclude that
\[
\sum_{\pi \in \mathfrak{F}(Q)} N_\pi(\sigma, T) \ll_m \min \{ 1, (1 - \beta_{\pi}) \log(qQT) \} (qQT)^{10^7 m^4 (1 - \sigma)}
\]
when \( 1 - (400m^2)^{-1} \leq \sigma < 1 - (2 \log qQT)^{-1} \). If \( \sigma \geq 1 - (2 \log qQT)^{-1} \), then \( N_\pi(\sigma, T) \leq N_\pi(1 - (2 \log qQT)^{-1}, T) \). If \( \sigma < 1 - (400m^2)^{-1} \), then the result is trivial by the Riemann–von Mangoldt formula [17 Theorem 5.8] and the effective bound \( 1 - \beta_{\pi} \gg q^{-1/2} \).

4. PROOF OF THEOREM 1.2

We begin with a bound relating the size of \( L \)-functions on the line \( \text{Re}(s) = \frac{1}{2} \) to

**Lemma 4.1.** Let \( m \geq 2 \) be an integer, \( t \in \mathbb{R} \), and \( \pi \in \mathfrak{F}_m \). If \( 0 \leq \alpha < \frac{1}{2} \), then
\[
\log |L(\frac{1}{2} + it, \pi)| \leq \left( \frac{1}{4} - \frac{\alpha}{10^9} \right) \log(C(\pi)(3 + |t|)^m) + \frac{\alpha}{10^7} N_\pi(1 - \alpha, |t| + 6) + O(m^2).
\]
Proof. Recall (1.7). If \( t \in \mathbb{R} \), the bound applies to \( L(\frac{1}{2} + it, \pi) \) as well once we add \( it \) to all of the Langlands parameters \( \mu_n(j) \) for \( 1 \leq j \leq m \). Consequently, we have

\[
\begin{align*}
\log |L(\frac{1}{2} + it, \pi)| & \leq \left( \frac{1}{4} - \frac{\alpha}{10^9} \right) \log C(\pi, t) + 2 \log |L(\frac{3}{2} + it, \pi)| + O(m^2) \\
& \quad + \frac{\alpha}{10^6} \# \{ \rho = \beta + i\gamma: L(\rho, \pi) = 0, \ \beta \geq 1 - \alpha, \ |\gamma - t| \leq 6 \}. 
\end{align*}
\]

By (2.2), we have \( 2 \log |L(\frac{3}{2} + it, \pi)| \ll m^2 \). Furthermore, we have the trivial bound

\[
\# \{ \rho = \beta + i\gamma: L(\rho, \pi) = 0, \ \beta \geq 1 - \alpha, \ |\gamma - t| \leq 6 \} \ll N_\pi(1 - \alpha, |t| + 6).
\]

The lemma now follows from the bound \( C(\pi, |t| + 6) \leq C(\pi)(|t| + 6)^m \) in (2.3). \( \square \)

Let \( t \in \mathbb{R} \). By (1.3), all except \( O_m((\log Q)^{10^6m^4\delta}) \) of the \( \pi \in \mathcal{F}_m(Q) \) satisfy

\[
L(s, \pi) \neq 0 \quad \text{when} \quad |\text{Im}(s)| \leq |t| + 6, \ \text{Re}(s) \geq 1 - \frac{10^9\delta \log \log C(\pi)}{\log(C(\pi)(|t| + 6)^m)}.
\]

This and Lemma 4.1 prove Theorem 1.2(2).

For Theorem 1.2(1), let \( \chi \equiv a \pmod{q} \) be a primitive quadratic Dirichlet character whose \( L \)-function \( L(s, \chi) \) has a zero \( \beta_\chi \in (\frac{1}{2}, 1) \), and define \( \lambda = (1 - \beta_\chi) \log q \). If \( Q = C(\pi) \), \( C(\pi) \in [q, q^A] \) for some \( A \geq 1 \), and \((1 - \beta_\chi) \log(qC(\pi)T) \) is sufficiently small with respect to \( m \), then Theorem 1.3 implies that there exists a constant \( c_m > 0 \), depending at most on \( m \), such that

\[
N_\pi(\sigma, T) \leq c_m \frac{\lambda}{\log q} \log(C(\pi)T)(C(\pi)T)^{2 \times 10^7m^4(1-\sigma)}.
\]

It follows that \( \alpha N_\pi(1 - \alpha, |t| + 6) \ll m \) when

\[
\alpha = \frac{\log \left( \frac{e \log q}{100\lambda cm} \right)}{4 \times 10^7m^4 \log(C(\pi)(|t|+6)^m)}.
\]

With this choice of \( \alpha \) in Lemma 4.1 we find that

\[
\begin{align*}
\log |L(\frac{1}{2} + it, \pi)| & \leq \frac{1}{4} \log(C(\pi)(|t|+3)^m) - \frac{1}{4 \times 10^{16}m^4} \log \left( \frac{e \log q}{100\lambda cm} \right) + O_m(1) \\
& \quad + \frac{1}{4 \times 10^{16}m^4} \log \left( \frac{A\lambda}{\log C(\pi)} \right) + O_m(1).
\end{align*}
\]

The claimed result follows once we exponentiate.

References

[1] E. Bombieri. Le grand crible dans la théorie analytique des nombres. Astérisque, (18):103, 1987.
[2] F. Brumley and D. Miličević. Counting cusp forms by analytic conductor. arXiv e-prints, page arXiv:1805.00633, May 2018.
[3] F. Brumley, J. Thorner, and A. Zaman. Zeros of Rankin–Selberg \( L \)-functions at the edge of the critical strip. J. Eur. Math. Soc. Accepted for publication. Appendix B by Colin J. Bushnell and Guy Henniart.
[4] H. M. Bui, K. Pratt, and A. Zaharescu. Exceptional characters and nonvanishing of Dirichlet \( L \)-functions. Math. Ann., 380(1-2):593–642, 2021.
[5] C. J. Bushnell and G. Henniart. An upper bound on conductors for pairs. J. Number Theory, 65(2):183–196, 1997.
[6] J. Buttcane and R. Khan. On the fourth moment of Hecke-Maass forms and the random wave conjecture. Compos. Math., 153(7):1479–1511, 2017.
[7] H. Davenport. Multiplicative number theory, volume 74 of Graduate Texts in Mathematics. Springer-Verlag, New York-Berlin, second edition, 1980. Revised by Hugh L. Montgomery.
[9] J. B. Friedlander and H. Iwaniec. Exceptional characters and prime numbers in short intervals. *Selecta Math. (N.S.),* 10(1):61–69, 2004.

[12] A. Granville. Sieving intervals and Siegel zeros. *arXiv e-prints*, page arXiv:2010.01211, Oct. 2020.

[16] J. Hoffstein and D. Ramakrishnan. Siegel zeros and cusp forms. *Int. Math. Res. Not. IMRN,* (6):279–308, 1995.

[17] H. Iwaniec and E. Kowalski. *Analytic number theory,* volume 53 of *American Mathematical Society Colloquium Publications.* American Mathematical Society, Providence, RI, 2004.

[21] E. Lindenstrauss. Invariant measures and arithmetic quantum unique ergodicity. *Ann. of Math. (2),* 163(1):165–219, 2006.

[26] Z. Rudnick and P. Sarnak. The behaviour of eigenstates of arithmetic hyperbolic manifolds. *Comm. Math. Phys.,* 161(1):195–213, 1994.

[29] K. Soundararajan. Quantum unique ergodicity for SL_2(Z)\H. *Ann. of Math. (2),* 172(2):1529–1538, 2010.

[30] K. Soundararajan and J. Thorner. Weak subconvexity for central values of L-functions. *Ann. of Math. (2),* 172(2):1469–1498, 2010.

[31] J. Thorner and A. Zaman. An unconditional GL_n large sieve. *Adv. Math.,* 378:107529, 2021.

[32] T. C. Watson. *Rankin triple products and quantum chaos.* ProQuest LLC, Ann Arbor, MI, 2002. Thesis (Ph.D.)–Princeton University.

**Department of Mathematics, University of Illinois, Urbana, IL 61801, United States**

**Email address:** jesse.thorner@gmail.com