Calculating effective resistances on underlying networks of association schemes

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Recently, in the work of Jafarizadeh et al. [J. Phys. A: Math. Theor. 40, 4949 (2007); e-print arXiv:0705.2480], calculation of effective resistances on distance-regular networks was investigated, where in the first paper, the calculation was based on stratification and Stieltjes functions associated with the network, whereas in the latter one a recursive formula for effective resistances was given based on the Christoffel–Darboux identity. In this paper, evaluation of effective resistances on more general networks that are underlying networks of association schemes is considered, where by using the algebraic combinatoric structures of association schemes such as stratification and Bose–Mesner algebras, an explicit formula for effective resistances on these networks is given in terms of the parameters of the corresponding association schemes. Moreover, we show that for particular underlying networks of association schemes with diameter \(d\) such that the adjacency matrix \(A\) possesses \(d+1\) distinct eigenvalues, all of the other adjacency matrices \(A_i\), \(i \neq 0, 1\) can be written as polynomials of \(A\), i.e., \(A_i = P_i(A)\), where \(P_i\) is not necessarily of degree \(i\). Then, we use this property for these particular networks and assume that all of the conductances except for one of them, say, \(c = c_1 = 1\), are zero to give a procedure for evaluating effective resistances on these networks. The preference of this procedure is that one can evaluate effective resistances by using the structure of their Bose–Mesner algebra without any need to know the spectrum of the adjacency matrices. © 2008 American Institute of Physics.

I. INTRODUCTION

A classic problem in electric circuit theory studied by numerous authors over many years is the computation of the resistance between two nodes in a resistor network (see, e.g., Ref. 1). The effective resistance has a probabilistic interpretation based on classical random walker walking on the network. Indeed, the connection between random walks and electric networks has been recognized for some time, where one can establish a connection between the electrical concepts of current and voltage and corresponding descriptive quantities of random walks regarded as finite state Markov chains (for more details see Refs. 2 and 3). In Refs. 4 and 5, calculation of effective resistances on distance-regular networks was investigated. In this paper, we consider more general resistor networks that are underlying networks of association schemes. In fact, the theory of association schemes6 has its origin in the design of statistical experiments. We will employ the algebraic structures of the underlying networks of association schemes (such as the Bose–Mesner

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algebra) in order to calculate the effective resistances between arbitrary nodes, in terms of the parameters of the corresponding association scheme such as the diameter of the scheme, the so-called first eigenvalue matrix $P$, the valencies of the adjacency matrices, and the rank of the corresponding idempotents. As we will see, the preference of this employment is that we are able to give analytical formulas for effective resistances on these networks in terms of the known parameters of the corresponding association schemes. As it will be shown in Sec. IV, in order to calculate the effective resistances on these networks, one needs to know the spectrum of the adjacency matrices $A_i$ for $i=1, \ldots, d$. However, in the most cases the spectrum of the Bose–Mesner algebra is known (for example, in the cases of group association schemes), but the formulas for effective resistances in terms of the spectrum of the networks do not possess a closed form and evaluation of them in the most cases is not an easy task. So, we will assume that all of the conductances except for one of them, say, $c=c_1=1$, are zero. Also, we will consider particular underlying networks for which the adjacency matrix $A_1=A$ possesses $d+1$ distinct eigenvalues.

Then, we give a procedure for evaluating the effective resistances on these networks, so that one can calculate the effective resistances by using the structure of their Bose–Mesner algebra without any need to know the spectrum of the adjacency matrices.

The organization of the paper is as follows. In Sec. II, some definitions and properties related to association schemes, their underlying networks, and the effective resistance in resistor networks are reviewed. Section III is devoted to the calculation of the effective resistances on underlying resistor networks of association schemes without using the spectrum of the underlying networks. In Sec. IV, explicit formula for effective resistances is given in terms of the spectrum of underlying networks. The paper ends with a brief conclusion and the Appendix.

II. EFFECTIVE RESISTANCES ON RESISTOR NETWORKS

Let $V=\{1, 2, \ldots, N\}$ be a finite set and $R$ be a symmetry relation on $V$; then $\Gamma=(V,R)$ can be thought as a network, where the sets $V$ and $R$ consist of its vertices and edges, respectively. A general resistor network is nothing but a network $\Gamma=(V,R)$ with a resistor function $r$ (or a conductance function $\sigma$) on $R$, that is,

$$r: (i,j) \rightarrow r_{ij}(>0), \quad (i,j) \in R.$$  

For a given resistor network $\Gamma=(V,R,r)$ ($\Gamma=(V,R)$ is called the underlying network in the following), as a convention, we assume $\sigma_{ij}=1/r_{ij}$ for $(i,j) \in R$ and $\sigma_{ij}=0$ otherwise. Let $L$ be the Laplacian matrix of $\Gamma=(V,R,r)$, that is,

$$L_{ij} = \sum_{j=1, j \neq i}^{N} \sigma_{ij}, \quad L_{ii} = -\sum_{j=1, j \neq i}^{N} \sigma_{ij}, \quad i \neq j.$$  

It should be noticed that $L$ has the eigenvector $(1, 1, \ldots, 1)^t$ with eigenvalue 0. Therefore, $L$ is not invertible, so we define the pseudoinverse of $L$ as

$$L^{-1} = \sum_{i, \lambda_i \neq 0} \lambda_i^{-1} E_i,$$  

where $E_i$ is the projection operator of the eigenspace of $L$ corresponding to eigenvalue $\lambda_i$. It is well known that the effective resistances $R_{\alpha \beta}$ are given by

$$R_{\alpha \beta} = \langle \alpha | L^{-1} | \alpha \rangle + \langle \beta | L^{-1} | \beta \rangle - \langle \alpha | L^{-1} | \beta \rangle - \langle \beta | L^{-1} | \alpha \rangle,$$  

where $|\alpha\rangle$ ($\langle \alpha |$) denotes the column (row) vector with 1 in the $\alpha$ coordinate and 0 in all other coordinates. This formula may be formally derived using Kirchoff’s laws and seems to have been long known in the electrical engineering literature, with it appearing in several texts such as Ref. 8.
In the present paper we deal with special networks. These underlying networks are related to symmetric association schemes. So, we first introduce the definition and some results about symmetric association schemes.

**Definition 1:** Let $V$ be a finite (non-void) set. Presume relations $\{R_i\}_{0 \leq i \leq d}$ on $V \times V$ satisfying the following conditions:

1. $\{R_i\}_{0 \leq i \leq d}$ is a partition of $V \times V$.
2. $R_0 = \{(\alpha, \alpha) : \alpha \in V\}$.
3. $R_i = R_i^i$ for $0 \leq i \leq d$, where $R_i^i = \{(\beta, \alpha) : (\alpha, \beta) \in R_i\}$.
4. For $(\alpha, \beta) \in R_k$, the number $p_{i,j}^k = |\{\gamma \in V : (\alpha, \gamma) \in R_i \text{ and } (\gamma, \beta) \in R_j\}|$ does not depend on $(\alpha, \beta)$ but only on $i$, $j$, and $k$.

Then, $Y = (V, \{R_i\}_{0 \leq i \leq d})$ defines a symmetric association scheme of class $d$ on $V$. Further, if $p_{i,j}^k = p_{j,i}^k$ for all $i$, $j$, $k = 0, 1, \ldots, d$, then $Y$ is called commutative.

Let $Y = (V, \{R_i\}_{0 \leq i \leq d})$ be a commutative symmetric association scheme of class $d$; then the matrices $A_0, A_1, \ldots, A_d$ defined by

$$
(A_i)_{\alpha, \beta} = \begin{cases} 
1 & \text{if}(\alpha, \beta) \in R_i, \\
0 & \text{otherwise}
\end{cases}, \quad \alpha, \beta \in V
$$

(2.4)

are adjacency matrices of $Y$ and are such that

$$
A_i A_j = \sum_{k=0}^d p_{i,j}^k A_k.
$$

(2.5)

From (2.5), it is seen that the adjacency matrices $A_0, A_1, \ldots, A_d$ form a basis for a commutative algebra $A$ known as the Bose–Mesner algebra of $Y$. This algebra has a second basis $E_0, \ldots, E_d$ primitive idempotents,

$$
E_0 = \frac{1}{N} J, \quad E_i E_j = \delta_{ij} E_i, \quad \sum_{i=0}^d E_i = I,
$$

(2.6)

where $N = |V|$ and $J$ is the $N \times N$ all-1 matrix in $A$. Let $P$ and $Q$ be the matrices relating the two bases for $A$,

$$
A_j = \sum_{i=0}^d P_{i,j} E_i, \quad 0 \leq j \leq d,
$$

$$
E_j = \frac{1}{N} \sum_{i=0}^d Q_{i,j} A_i, \quad 0 \leq j \leq d.
$$

(2.7)

Then clearly

$$
PQ = QP = NI.
$$

(2.8)

It also follows that

$$
A_i E_i = P_{i,j} E_i.
$$

(2.9)

From above, it is easy to see that the $P_{i,j}$'s are the eigenvalues of $A_j$ and the columns of $E_j$ are the corresponding eigenvectors. Thus $m_i = \text{rank}(E_i)$ is the multiplicity of the eigenvalue $P_{i,j}$ of $A_j$ (provided that $P_{i,j} \neq P_{k,j}$ for $k \neq i$). We see that $m_0 = 1$, $\sum m_i = N$, and $m_i = \text{trace } E_i = N(E_i)_{ij}$ (indeed, $E_i$ has only eigenvalues 0 and 1, so rank $(E_k)$ equals the sum of the eigenvalues).
Clearly, each network \((V, R_i)\) for \(0 \leq i \leq d\) is a regular network, that is, the degree is independent of the vertex \(\alpha\), so the regular degree can be denoted by \(\kappa_i\). Without loss of generality, in the following we assume that \((V, R_1)\) is a connected network.

Now, let \(Y = (V, \{R_i\}_{0 \leq i \leq d})\) be a given association scheme; we define a special resistor network by the conductance function as
\[
\sigma_{ij} = c_{ij} (i, j) \in R_i (0 \leq i \leq d).
\]
Then the corresponding Laplacian matrix can be expressed as
\[
L = \left( \sum_{i=0}^{d} c_{i} \kappa_{i} \right) I - \sum_{i=0}^{d} c_{i} A_{i}, \tag{2.10}
\]

In the following sections, we will use the knowledge of association schemes (such as stratification associated with the underlying networks of association schemes) to give the closed form of expression for the effective resistance of this resistor network, especially for \(c = c_1 = 1, c_i = 0\) for \(i \neq 1\); we also give many explanation examples. Before that, we recall the stratification technique briefly (for more details see Ref. 4).

For a given vertex \(\alpha \in V\) (called the reference vertex), we define \(\Gamma_i(\alpha) = \{\beta \in V : (\alpha, \beta) \in R_i\}\). Then, the vertex set \(V\) can be written as the disjoint union of strata \(\Gamma_i(\alpha)\) (associate classes), i.e.,
\[
V = \bigcup_{i=0}^{d} \Gamma_i(\alpha). \tag{2.11}
\]
Moreover, for each stratum \(\Gamma_i(\alpha)\) we associate a unit vector
\[
|\phi_i\rangle = \frac{1}{\sqrt{\kappa_i}} \sum_{\alpha \in \Gamma_i(\alpha)} |\alpha\rangle. \tag{2.12}
\]

### III. Calculating Effective Resistances on Underlying Networks of Association Schemes Without Using the Spectrum of the Networks

Let us stratify the underlying networks with respect to an arbitrary chosen reference vertex \(\alpha \in V\); then all of the nodes \(\beta \in \Gamma_m(\alpha)\) possess the same effective resistance with respect to \(\alpha\) (this can be easily seen from Eqs. (2.2) and (2.3)). This allows us to write
\[
R_{\alpha \beta} = \frac{1}{\kappa_{m}} \sum_{\beta \in \Gamma_m(\alpha)} R_{\alpha \beta} = \frac{1}{\kappa_{m}} \sum_{\beta \in V} (A_m)_{\alpha \beta} R_{\alpha \beta}.
\]
where \(R_{\alpha \beta}\) denotes the effective resistances between \(\alpha\) and all of the nodes \(\beta \in \Gamma_m(\alpha)\).

In the Appendix, we show that if the adjacency matrix of a connected underlying network of association scheme with diameter \(d\) possesses \(d+1\) distinct eigenvalues, then all of the other adjacency matrices \(A_i\) for \(i = 2, \ldots, d\) can be written as polynomials of \(A\). Now, consider underlying networks for which we have \(A_m = \sum_{m=0}^{d} \kappa_{m} c_{m} A^m\) [recall that for distance-regular networks, we have \(A_m = P_m(A) = \sum_{m=0}^{d} \kappa_{m} c_{m} A^m\), where \(P_m\) is a polynomial of degree \(m\) (Ref. 4)]. Then, by using (2.3) one can obtain
\[
R_{\alpha \beta} = \frac{2}{\kappa_{m}} \sum_{m=0}^{d} c_{mn} \sum_{\beta \in V} (A^n)_{\alpha \beta} = \sum_{\beta \in V} (A^n)_{\alpha \beta} - \sum_{\beta \in V} (A^n)_{\alpha \beta}^{-1} \tag{3.1}
\]
From the fact that the effective resistances \(R_{\alpha \beta}\) are independent of the choice of the reference node \(\alpha\), one can write
\[
\sum_{\alpha \in V} R_{\alpha \beta^{(m)}} = N \cdot R_{\alpha \beta^{(m)}} = \frac{2}{N \cdot K_m} \sum_{\alpha, \beta \in V} \left( \sum_{\kappa, m = 0}^{d} c_{mn} \left[ \sum_{\alpha, \beta \in V} (A^n)_{\alpha \beta} L^{-\alpha}_{\beta} - \sum_{\alpha, \beta \in V} (A^n)_{\alpha \beta} L^{-1}_{\alpha \beta} \right] \right).
\]

(3.2)

Now, we note that
\[
\sum_{\beta \in V} (A^n)_{\alpha \beta} = \sum_{\beta \in V} A_{\alpha \beta} = K^\alpha.
\]

In the case of \(c_1 = c = 1, c_i = 0 \) for \( i \neq 1 \) (which implies that \( L = KI - A \) with \( K = K_1 \), Eq. (3.2) gives
\[
R_{\alpha \beta^{(m)}} = \frac{2}{N \cdot K_m} \sum_{\alpha, \beta \in V} \left( \sum_{\kappa, m = 0}^{d} c_{mn} \left[ K^n \text{tr}(L^{-1}) - \text{tr}(A^n L^{-1}) \right] \right) = \frac{2}{N \cdot K_m} \sum_{\alpha, \beta \in V} \left( \sum_{i=1}^{n} \kappa^{n-i} \text{tr}(A^{-i}) - n \cdot K^{n-i} \right).
\]

(3.3)

By using the equality \( A^j = K^j \) (recall that \( A^j = K^j \)), Eq. (3.3) can be rewritten as follows:
\[
R_{\alpha \beta^{(m)}} = \frac{2}{N \cdot K_m} \sum_{\alpha, \beta \in V} \left( \sum_{i=1}^{n} \kappa^{n-i} \text{tr}(A^{-i}) - n \cdot K^{n-i} \right).
\]

(3.4)

As the above formula indicates, in order to calculate \( R_{\alpha \beta^{(m)}} \), we need to evaluate \( \text{tr}(A^i) \), for all \( i = 1, 2, \ldots, d-1 \). To do this, we use the relations \( A_m = \sum_{\alpha, \beta \in V} \alpha^m \) for \( m = 1, \ldots, d \) to write \( A^j = \sum_{m=0}^{d} c_{mn} \text{tr}(A^m) \), and obtain \( \text{tr}(A^j) = n c_{\beta_0} \).

**A. Examples**

**1. Underlying network of the association scheme derived from \( Z_3 \times Z_3 \)**

In the regular representation, the elements of Abelian group \( Z_3 \times Z_3 \) are written as \( S_1 \cdot S_2 \) with \( S_1 = S \otimes I \) and \( S_2 = I \otimes S \), where \( S \) is the shift operator with period 5. i.e., \( S^2 = I_5 \). Now, we define the following adjacency matrices:

\[
A = A_1 = S_1 + S_2 + S_1 S_2 + S_1^2 + S_2^2, \quad A_2 = S_1^2 + S_2^2 + S_1 S_2 + S_1^3 + S_2^3, \quad A_3 = S_1 S_2 + S_1^2 S_2^2 + S_1 S_2^3 + S_2 S_1^2 + S_1 S_2, \quad A_4 = S_1 S_2 + S_1^2 S_2^2 + S_1 S_2^3 + S_2 S_1^2 + S_1 S_2^3.
\]

Then, one can easily see that the above adjacency matrices constitute the Bose–Mesner algebra of a symmetric association scheme. In fact, we have

\[
A^2 = 6A_0 + 2A + 2A_2 + 2A_3, \quad AA_2 = A + A_2 + 2A_3 + 2A_4, \quad AA_3 = 2A + 2A_2 + 2A_4, \quad A^4 = 2A + 2A_2 + 2A_4.
\]

(3.5)

The above relations indicate that the underlying network of the constructed association scheme is not distance regular. By using (3.5), one can evaluate the powers of \( A \) as follows:

\[
A^2 = 6A_0 + 2A + 2A_2 + 2A_3, \quad A^3 = 12A_0 + 15A + 7A_2 + 6A_3 + 6A_4, \quad A^4 = 90A_0 + 61A + 46A_2 + 56A_3 + 38A_4.
\]

(3.6)

Then, by solving (3.6) in terms of \( A_2, A_3, \) and \( A_4 \), we obtain
\[ A_2 = \frac{1}{44}(-6A^4 + 38A^3 + 54A^2 - 312A - 240A_0), \quad A_3 = \frac{1}{44}(3A^4 - 19A^3 - 5A^2 + 112A - 12A_0), \quad A_4 = \frac{1}{22}(2A^4 + 9A^3 - 29A^2 + 71A + 102A_0). \]  

That is, the coefficients \( c_{mn} \) in \( A_m = \sum_{n=0}^{4} c_{mn} A^n \) are given by

\[
\begin{align*}
  c_{11} &= 1, \quad c_{1i} = 0 \text{ for } i \neq 1, \\
  c_{20} &= \frac{66}{11}, \quad c_{21} = -\frac{78}{11}, \quad c_{22} = -\frac{27}{11}, \quad c_{23} = \frac{19}{11}, \quad c_{24} = -\frac{3}{11}, \quad c_{30} = -\frac{3}{11}, \\
  c_{31} &= \frac{28}{11}, \quad c_{32} = -\frac{5}{11}, \quad c_{33} = -\frac{19}{11}, \quad c_{34} = \frac{3}{11}, \quad c_{40} = \frac{51}{11}, \quad c_{41} = \frac{71}{11}, \quad c_{42} = -\frac{29}{11}, \quad c_{43} = \frac{9}{11}, \quad c_{44} = \frac{1}{11}, \\

&= \frac{1}{11}.
\end{align*}
\]

Then, by using (3.6) and (3.8) and substituting \( N=25 \) and \( \kappa = \kappa_2 = \kappa_3 = \kappa_4 = 6 \) in the result (3.4), we obtain the effective resistances as follows:

\[
R_{\alpha\beta(1)} = \frac{1}{75} c_{11}(25 - 1) = \frac{24}{75},
\]

\[
R_{\alpha\beta(2)} = \frac{1}{75} \{24c_{21} + 138c_{22} + 942c_{23} + 5736c_{24}\} = \frac{112}{275},
\]

\[
R_{\alpha\beta(3)} = \frac{1}{75} \{24c_{21} + 138c_{22} + 942c_{23} + 5736c_{24}\} = \frac{327}{825},
\]

\[
R_{\alpha\beta(4)} = \frac{1}{75} \{24c_{21} + 138c_{22} + 942c_{23} + 5736c_{24}\} = \frac{2942}{275}. \quad (3.9)
\]

2. Group association scheme \( S_4 \)

In group association schemes, the adjacency matrices are defined as the class sums of a group in the regular representation. For instance, in the symmetric group \( S_4 \), the conjugacy classes are given by

\[
\begin{align*}
  C_0 &= \{1\}, \quad C_1 = \{(12),(13),(14),(23),(24),(34)\}, \quad C_2 = \{(123),(132),(124),(142),(134),(143),(234),(243)\}, \\
  C_3 &= \{(12)(34),(13)(24),(14)(23)\}, \quad C_4 = \{(1324),(1234),(1243),(1423),(1432)\}. \quad (3.10)
\end{align*}
\]

Then, the adjacency matrices are defined as \( A_i = \sum_{g \in C_i} g A g^{-1} \), \( i = 0, 1, \ldots, 4 \), i.e., we have

\[
\begin{align*}
  A &= A_1 = (12) + (13) + (14) + (23) + (24) + (34), \quad A_2 = (123) + (132) + (124) + (142) + (134) + (143) + (234) + (243), \quad A_3 = (12)(34) + (13)(24) + (14)(23), \quad A_4 = (1234) + (1243) + (1324) + (1432) + (1342).
\end{align*}
\]

One can easily show that these adjacency matrices satisfy the following relations

\[
A^2 = 6A_0 + 3A_2 + 2A_3, \quad AA_2 = 4A + 4A_4, \quad AA_3 = A + 2A_4, \quad AA_4 = 4A_2 + 4A_3. \quad (3.11)
\]

By using (3.11), one can evaluate the powers of \( A \) and as in the previous example, by solving them in terms of \( A_2, A_3, \) and \( A_4 \), one can obtain the coefficients \( c_{mn} \) in \( A_m = \sum_{n=0}^{4} c_{mn} A^n \) as

\[
\begin{align*}
  c_{11} &= 1, \quad c_{1i} = 0 \text{ for } i \neq 1, \quad c_{20} = -4, \quad c_{21} = 0, \quad c_{22} = \frac{13}{12}, \quad c_{23} = 0, \quad c_{24} = -\frac{1}{48}, \quad c_{30} = 3.
\end{align*}
\]
Then, by using (3.12) and substituting \(N=24\) and \(\kappa=6\), \(\kappa_2=8\), \(\kappa_3=3\), \(\kappa_4=6\) in the result (3.4), we obtain the effective resistances as follows:

\[
R_{\alpha\beta(1)} = \frac{1}{12} c_{11}(24 - 1) = \frac{23}{72}, \quad R_{\alpha\beta(2)} = \frac{1}{96} (132 c_{22} + 6048 c_{24}) = \frac{35}{36}, \quad R_{\alpha\beta(3)} = \frac{1}{36} (132 c_{32} + 5184 c_{34}) = \frac{7}{5}, \quad R_{\alpha\beta(4)} = \frac{1}{72} (23 c_{41} + 1620 c_{43}) = \frac{145}{36}. \tag{3.13}
\]

In order to give a nontrivial example of underlying networks of association schemes, which is not distance regular, we construct association schemes with diameter 6 by decomposing the class sums of the symmetric group \(S_4\) as follows:

We define the adjacency matrices as follows:

\[
A_0 = I, \quad A = A_1 = (12) + (13) + (14), \quad A_2 = (123) + (132) + (124) + (142) + (134) + (143), \quad A_3 = (23) + (24) + (34), \quad A_4 = (1234) + (1243) + (1324) + (1342) + (1423) + (1432), \quad A_5 = (12) \times (34) + (13)(24) + (14)(23), \quad A_6 = (234) + (243). \tag{3.14}
\]

Then, one can show that the following relations are satisfied:

\[
A^2 = 3A_0 + A_2, \quad AA_2 = 2A + 2A_3 + A_4, \quad AA_3 = A_2 + A_5, \quad AA_5 = A_3 + A_4, \quad AA_4 = A_2 + 2A_5 + 3A_6, \quad AA_6 = A_4. \tag{3.15}
\]

Now, by using (3.15), one can evaluate the powers of \(A\) and by solving them in terms of \(A_2, \ldots, A_6\), one can obtain the coefficients \(c_{mn}\) in \(A_m = \sum \alpha^n c_{mn} A^n\) as

\[
c_{11} = 1, \quad c_{1i} = 0 \quad \text{for} \quad i \neq 1, \quad c_{20} = -3, \quad c_{22} = 1, \quad c_{21} = c_{23} = c_{24} = c_{25} = c_{26} = 0, \quad c_{30} = c_{32} = c_{34} = c_{36} = 0, \quad c_{31} = -\frac{23}{5}, \quad c_{33} = \frac{3}{7}, \quad c_{35} = -\frac{1}{10}, \quad c_{40} = c_{42} = c_{44} = c_{46} = 0, \quad c_{41} = \frac{19}{5}, \quad c_{43} = c_{52} = -\frac{22}{5}, \quad c_{54} = \frac{3}{7}, \quad c_{56} = -\frac{1}{10}, \quad c_{60} = -1, \quad c_{61} = c_{63} = c_{65} = 0, \quad c_{62} = \frac{68}{15}, \quad c_{64} = \frac{5}{3}, \quad c_{66} = \frac{2}{13}. \tag{3.16}
\]

Again, by using (3.16) and substituting \(N=24\) and \(\kappa=3\), \(\kappa_2=3\), \(\kappa_3=6\), \(\kappa_4=2\), \(\kappa_5=3\), \(\kappa_6=6\) in the result (3.4), we obtain the effective resistances as follows:

\[
R_{\alpha\beta(1)} = \frac{1}{36} c_{11}(24 - 1) = \frac{23}{36}, \quad R_{\alpha\beta(2)} = \frac{1}{72} c_{25}(72 - 6) = \frac{33}{36}, \quad R_{\alpha\beta(3)} = \frac{1}{36} \left( 3 c_{31}(24 - 1) + 3 c_{33} \left( \sum_{i=1}^{3} 3^{3-i} \text{tr}(A^{-i}) - 27 \right) + c_{35} \left( \sum_{i=1}^{5} 3^{3-i} \text{tr}(A^{-i}) - 405 \right) \right) = \frac{89}{90}, \quad R_{\alpha\beta(4)} = \frac{1}{36} \left( 3 c_{41}(24 - 1) + 3 c_{43} \left( \sum_{i=1}^{3} 3^{3-i} \text{tr}(A^{-i}) - 27 \right) + c_{45} \left( \sum_{i=1}^{5} 3^{3-i} \text{tr}(A^{-i}) - 405 \right) \right) = \frac{187}{180}, \quad R_{\alpha\beta(5)} = \frac{1}{36} \left( 3 c_{52}(72 - 6) + 3 c_{54} \left( \sum_{i=1}^{4} 3^{3-i} \text{tr}(A^{-i}) - 108 \right) + c_{56} \left( \sum_{i=1}^{6} 3^{3-i} \text{tr}(A^{-i}) - 1458 \right) \right) = \frac{21}{20}. \]
\[ R_{a\beta(i)} = \frac{1}{24} \left\{ c_{62}(72 - 6) + c_{66} \left( \sum_{i=1}^{4} 3^{i-1} \text{tr}(A^{i-1}) - 108 \right) + c_{66} \left( \sum_{i=1}^{6} 3^{i-1} \text{tr}(A^{i-1}) - 1458 \right) \right\} = \frac{16}{15} \]  
(3.17)

It should be noticed that, in distance-regular networks, by using the three-term recursion structure, one can obtain

\[ A^2 = AA_1 = \kappa I + a_1 A + c_2 A_2, \]
\[ A^3 = AA^2 = \kappa a_1 I + (\kappa + a_1^2 + b_1 c_2) A + (a_1 + a_2) c_2 A_2 + c_2 c_3 A_3, \]
\[ A^4 = AA^3 = I_0 I + I_1 A + I_2 A_2 + I_3 A_3 + c_2 c_3 c_4 A_4, \]
\[ A^5 = AA^4 = \kappa I_1 I + (I_0 + a_1 I_1 + b_1 I_2) A + (c_2 I_1 + a_2 I_2 + b_2 I_3) A_3 + (c_3 I_2 + a_3 I_3 + b_3 c_2 c_3) A_4 + (c_4 I_3 + a_4 c_2 c_3 c_4) A_4 + c_2 c_3 c_4 c_5 A_5, \]  
(3.18)

where

\[ I_0 := \kappa (\kappa + a_1^2 + b_1 c_2), \quad I_1 := a_1 (2 \kappa + a_1^2 + 2 b_1 c_2) + b_1 c_2 a_2, \quad I_2 := c_2 (\kappa + a_1^2 + b_1 c_2 + a_2 (a_1 + a_2) + b_2 c_3), \quad I_3 := c_2 c_3 (a_1 + a_2 + a_3). \]  
(3.19)

Therefore, by using (3.4) and (3.18), for distance-regular resistor networks with \( c_1 = c_2 = 0 \) for \( i \neq 1 \), the effective resistances \( R_{a\beta(i)} \) for \( m=1, 2, \ldots, 5 \) are obtained in terms of the intersection numbers of the network as follows:

\[ R_{a\beta(1)} = \frac{2}{\kappa} \left( \frac{N-1}{N} \right), \]
\[ R_{a\beta(2)} = \frac{2}{\kappa b_1} \left\{ b_1 + 1 - \frac{\kappa + b_1 + 1}{N} \right\}, \]
\[ R_{a\beta(3)} = \frac{2}{\kappa b_1 b_2} \left\{ b_{d-1} c_d + b_2 - \kappa + c_2 + b_1 b_2 - \frac{(\kappa + 1)(b_2 + c_2) + b_1 (\kappa + b_2)}{N} \right\}, \]
\[ R_{a\beta(4)} = \frac{2}{\kappa b_1 b_2 b_3} \left\{ -I_1 \left( 1 - \frac{2}{N} \right) - \kappa I_2 \left( 1 - \frac{2}{N} \right) - \kappa I_3 \left( \kappa + 1 - \frac{\kappa}{N} \right) + \kappa^2 \left[ 1 - \frac{4}{N} \right] + \kappa \cdot (\kappa + a_1) \right\}, \]
\[ R_{a\beta(5)} = \frac{2}{\kappa b_1 b_2 b_3 b_4} \left\{ -(I_0 + a_1 I_1 + b_1 I_2) \left( 1 - \frac{1}{N} \right) - (c_2 I_1 + a_2 I_2 + b_2 I_3) (\kappa - 2) - (c_3 I_2 + a_3 I_3 + b_3 c_2 c_3) \right. \]
\[ + b_3 c_2 c_3 c_4 \left. \right\}, \]
\[ \left( \kappa^2 \left( 1 - \frac{3}{N} \right) + \kappa \right) - (c_2 I_3 + a_4 c_2 c_3 c_4) \left( \kappa^3 \left( 1 - \frac{4}{N} \right) + \kappa (\kappa + a_1) \right) + \kappa^4 \left( 1 - \frac{5}{N} \right) + \kappa (\kappa^2 + \kappa a_1) + \kappa + a_1^2 \]
\[ + b_1 c_2. \]  
(3.20)

In Ref. 4, analytical formulas for \( R_{a\beta(i)} \), for \( i=1, 2, 3 \), on distance-regular networks have been given by using the Stieltjes function associated with the network, where the results are in agreement with (3.20). It may be noted that for distance-regular networks, the explicit formula for
effective resistances had been given by Devroye and Sbihi\textsuperscript{9} and Biggs\textsuperscript{10} by using different approaches.

It should be also noticed that by using the above result, one can evaluate the limiting value of the effective resistances in the limit of the large network size, i.e., in the limit $N \to \infty$. For instance, the effective resistances $R_{a \beta^1}$, $R_{a \beta^2}$, and $R_{a \beta^3}$ tend to $2/\kappa$, $2(b_1 + 1)/kb_1$, and $2(b_2 - c_2b_2 - \kappa)/kb_1b_2$, respectively, which are finite for the resistor networks with finite value of the valency $\kappa$.

In the following, we introduce some interesting distance-regular networks that underlie networks of association schemes derived from symmetric group $S_n$ and its nontrivial subgroups and calculate the effective resistances on these networks.

3. Association schemes derived from symmetric group $S_n$

Let $\lambda=(\lambda_1, \ldots, \lambda_m)$ be a partition of $n$, i.e., $\lambda_1 + \cdots + \lambda_m = n$. We consider the subgroup $S_m \otimes S_{n-m}$ of $S_n$ with $m \leq \lfloor n/2 \rfloor$. Then we assume the finite set $M^\lambda$ as $M^\lambda = S_m/S_m \otimes S_{n-m}$ with $|M^\lambda| = n!/m!(n-m)!$. In fact, $M^\lambda$ is the set of left cosets of $S_m \otimes S_{n-m}$ in $S_n$ which can be viewed as $(m-1)$-faces of the $(n-1)$-simplex [where, the graph $K_n$ of an $(n-1)$-simplex is the complete graph with $n$ vertices]. If we denote the vertex $i$ by the $m$-tuple $(i_1, i_2, \ldots, i_m)$, then the relations $R_k$, $k=0, 1, \ldots, m$, defined by

$$R_k = \{(i,j) : \partial(i,j) = k\}, \quad k = 0, 1, \ldots, m,$$

(3.21)

where $\partial(i,j)$ denotes the number of components of $i=(i_1, i_2, \ldots, i_m)$ which differ from the corresponding components of $j=(j_1, j_2, \ldots, j_m)$, define an association scheme on $S_n/S_m \otimes S_{n-m}$ with diameter $m+1$. Then the adjacency matrices $A_k$, $k=0, 1, \ldots, m$ are defined as

$$(A_k)_{ij} = \begin{cases} 1 & \text{if } \partial(i,j) = k, \\ 0 & \text{otherwise} \end{cases}, \quad i, j \in M^\lambda, \quad k = 0, 1, \ldots, m.$$

(3.22)

The $m+1$ primitive idempotents are defined by

$$E_{\mu} = \frac{\chi_\mu(e)}{n!} \sum_{g \in S_n} \chi_\mu(g) \rho(g),$$

(3.23)

where $e$ is the identity element, $\chi_\mu$ is the character corresponding to the irreducible submodule $S^\mu$, and $\rho$ is the representation of $S_n$ over $M^\lambda$.

One can show that the sizes of strata (valencies) are given by

$$\kappa_0 = 1, \quad \kappa_l = \binom{m}{m-l} \binom{n-m}{l}, \quad l = 1, 2, \ldots, m$$

(3.24)

[clearly we have $N=\sum_{l=0}^m \kappa_l = \binom{n}{m} = n!/m!(n-m)! = |M^\lambda|$]. If we stratify the network with respect to reference node $|\phi_0\rangle=|i_1, i_2, \ldots, i_m\rangle$, the unit vectors $|\phi_\mu\rangle$, $\mu=1, \ldots, m$ are defined as

$$|\phi_1\rangle = \frac{1}{\sqrt{\kappa_1}} \left( \sum_{i'_1 \neq i_1} \langle i'_1, i_2, \ldots, i_m \rangle + \sum_{i'_2 \neq i_2} \langle i_1, i'_2, i_3, \ldots, i_m \rangle + \cdots + \sum_{i'_m \neq i_m} \langle i_1, \ldots, i_{m-1}, i'_m \rangle \right),$$

$$|\phi_2\rangle = \frac{1}{\sqrt{\kappa_2}} \sum_{k=1}^m \sum_{i'_k \neq i_k} \langle i_1, \ldots, i_{k-1}, i'_k, i_{k+1}, \ldots, i_{m-1}, i'_m \rangle,$$

$$\vdots$$
In the following, we consider the case as in the above is a distance-regular network with intersection array as follows:

$$b_l = (m-l)(n-m-l), \quad c_l = l^2.$$  \hfill (3.26)

Then, one can obtain

$$AA_l = (m-l+1)(n-m-l+1)A_{l-1} + l(n-2l)A_l + (l+1)^2A_{l+1}.$$  \hfill (3.27)

In the following, we consider the case $m=2$, where the vertices are edges of a complete graph $K_n$ and calculate the effective resistances.

In the case of $m=2$, we have three kinds of relations as follows:

$$R_0 = \{(i,j),(i,j)\}, \quad R_1 = \{(i,j),(i,k),(i,k); j \neq k, i \neq j\}, \quad R_2 = \{(i,j),(k,l); i \neq k, j \neq l\},$$  \hfill (3.28)

for $i<j=1,2, \ldots, n; \quad k<l=1, \ldots, n$. Therefore, we have three adjacency matrices $A_0, A_1 = A,$ and $A_2$, where $A_0=I_{n(n-1)/2}$ and

$$(A)_{ij,kl} = \delta_{ik}(1-\delta_{jl}) + \delta_{jl}(1-\delta_{ik}),$$

$$(A_2)_{ij,kl} = (1-\delta_{ik})(1-\delta_{jl}), \quad i(k) < j(l) = 1,2, \ldots, n.$$  \hfill (3.29)

From (3.24) and (3.26), we have

$$\kappa_0 = 1, \quad \kappa = \kappa_1 = 2(n-2), \quad \kappa_2 = \frac{(n-2)(n-3)}{2}, \quad \{b_0,b_1;c_1,c_2\} = \{2(n-2),n-3;1,4\}.$$  \hfill (3.30)

Then, by using the recursion relations (3.27), one can write

$$A^2 = 2(n-2)I_{n(n-1)/2} + (n-2)A + 4A_2,$$

$$AA_2 = (n-3)A + 2(n-4)A_2.$$  \hfill (3.31)

Now, by using the result (3.20), the effective resistances are evaluated as follows:

$$R_{\alpha \beta^{(1)}} = \frac{n(n-1)-2}{n(n-1)(n-2)}, \quad R_{\alpha \beta^{(2)}} = \frac{n(n-1)+6}{n(n-1)(n-3)}.$$  \hfill (3.32)

IV. CALCULATING EFFECTIVE RESISTANCES ON UNDERLYING NETWORKS OF ASSOCIATION SCHEMES BY USING THE SPECTRUM OF THE NETWORKS

In the following, we use the algebraic combinatoric structures of the underlying resistor networks of association schemes in order to calculate effective resistances in (2.3) in terms of the corresponding association scheme’s parameters. By using (2.7), the Laplacian (2.1) can be written as

$$L = \sum_{k=0}^{d} \left( \sum_{i=0}^{d} c_i E_{k_i} \right) E_k.$$  \hfill (4.1)

Then, we have
\[ L^{-1} = \sum_{k=1}^{d} \frac{E_k}{\sum_{i=0}^{d} c_i(\kappa_i - P_{ik})} \]  
\( (4.2) \)

Now, for each \( \alpha \) and \( \beta \), we consider \( \alpha \) as reference vertex and \( \beta \in \Gamma_\beta(\alpha) \). Then, the diagonal entries of \( L^{-1} \) are all the same and equal to
\[
L^{-1}_{\alpha\alpha} = \sum_{k=1}^{d} \frac{\langle \alpha | E_k | \alpha \rangle}{\sum_{i=0}^{d} c_i(\kappa_i - P_{ik})} = \frac{1}{N} \sum_{k=1}^{d} \frac{m_k}{\sum_{i=0}^{d} c_i(\kappa_i - P_{ik})},
\]
where we have used the fact that \( \langle \alpha | E_k | \alpha \rangle = m_k/N \) with \( m_k = \text{rank}(E_k) \). Also, we have
\[
L^{-1}_{\beta\alpha} = \frac{1}{\sqrt{\kappa_i}} \langle \alpha | L^{-1} | \alpha \rangle = \frac{1}{\kappa_i} \langle \alpha | A_i L^{-1} | \alpha \rangle = \frac{1}{\kappa_i} \left( \sum_{k=1}^{d} A_i E_k \right) \left( \sum_{i=0}^{d} c_i(\kappa_i - P_{ik}) \right) = \frac{1}{N} \sum_{k=1}^{d} \frac{P_{mk} \langle \alpha | E_k | \alpha \rangle}{\sum_{i=0}^{d} c_i(\kappa_i - P_{ik})},
\]
(4.4)

Therefore, by using (2.3), we obtain our main result as
\[
R_{\alpha\beta} = \frac{2}{N} \sum_{k=1}^{d} \frac{m_k(\kappa_i - P_{ik})}{\sum_{i=0}^{d} c_i(\kappa_i - P_{ik})}, \quad \forall \beta \in \Gamma_\beta(\alpha).
\]
(4.5)

As the result (4.5) indicates, in order to calculate the effective resistances on underlying networks of association schemes, it suffices to know the spectrum of the adjacency matrices \( A_i \) for \( i = 1, \ldots, d \), i.e., it suffices to know \( P_{\beta\alpha} \) for \( i, l = 0, \ldots, d \). In general the sums appearing in Eq. (4.5) have not been expressed in closed form, and evaluation of them in the most cases is not an easy task. Despite these problems, the result (4.5) leads us to the following corollary:

**Corollary 1:** For the networks such that all of the conductances \( c_i \) are equal to zero except for one of them, i.e., \( c_1 = c = 1 \) and \( c_i = 0 \) for \( i \neq 1 \), we have
\[
\sum_{\alpha, \beta, \beta' = \alpha} R_{\alpha\beta} = \frac{N k_i}{2} R_{\alpha\beta} = \sum_{k=1}^{d} \frac{m_k(\kappa_i - P_{ik})}{(\kappa_i - P_{ik})} = \sum_{k=1}^{d} m_k = N - 1.
\]

In fact, this particular result is true for any \( N \)-site connected network and was long ago established by Foster\(^{11}\) and by Weinberg\(^{12}\). In Refs. 7 and 8, the sum in (4.6) has been identified as one of the sum rules for effective resistances, it also being noted that the effective resistances constitute a metric (called resistance distance) which is an invariant of the graph.

It should be noticed that, the result (4.6) is a special case of the following theorem:

**Theorem:** (Ref. 13) For any \( N \)-site underlying network of an association scheme, we have
\[
S = \frac{1}{2} \sum_{\alpha, \beta} A_{\alpha\beta} R_{\alpha\beta} = N - 1,
\]
(4.7)

where \( A_i = \sum c_i A_i \) (for the proof see Ref. 13).

**Corollary 2:** By using the result (4.7), one can obtain a linear dependence between effective resistances as follows:
\[
c_1 \kappa_1 R_{\alpha\beta(1)} + c_2 \kappa_2 R_{\alpha\beta(2)} + \cdots + c_d \kappa_d R_{\alpha\beta(d)} = \frac{2(N - 1)}{N}.
\]
(4.8)

**Proof:** For a given \( \alpha \), the conductance between \( \alpha \) and all of the nodes \( \beta \), which have the
relation \( l \) with \( \alpha (\beta \in \Gamma_\alpha) \), is \( c_l \) (the number of such nodes \( \beta \) is equal to \( \kappa_l \)). Then, by using (4.7) one can write
\[
\frac{1}{2} \sum_{\alpha, \beta} A_{\alpha \beta} R_{\alpha \beta} = \frac{1}{2} N (c_1 \kappa_1 R_{\alpha \beta 1} + c_2 \kappa_2 R_{\alpha \beta 2} + \cdots + c_d \kappa_d R_{\alpha \beta d}) = N - 1.
\]
(4.9)

Clearly, if only one of the conductances \( c_1 = c \) is nonzero, we will have
\[
R_{\alpha \beta 1} = \frac{2(N - 1)}{Nc_1 \kappa_1},
\]
(4.10)

which is the same as the result (4.6).

One should notice that for underlying networks of particular association schemes called the group association schemes, the combinatorics of characteristics of the networks such as \( m_\mu \), \( \kappa_l \), and first eigenvalue matrix \( P \) are known based on the corresponding group characteristics. In fact, for the group association schemes, the class sums \( \bar{C}_l := \sum_{g \in C_l} \) constitute the basis of the corresponding Bose-Mesner algebra and in the regular representation of the group, they are the corresponding adjacency matrices. Then (based on the group theory), for all underlying networks of (symmetric) group association schemes, one can easily evaluate
\[
m_\mu = d_\mu^2, \quad \kappa_l = \frac{|C_l|^2}{|G|} \sum_\mu \chi_\mu (g_i) \text{ with } g_i \in C_l, \quad P_{\mu l} = \frac{\kappa_l}{d_\mu} \chi_\mu (e),
\]
(4.11)

where \( \chi_\mu \) is the character of the irreducible representation \( \mu \) of the group \( G \), \( d_\mu = \chi_\mu (e) \), and \( C_l \) is the \( l \)th conjugacy class of \( G \). Then, by using the Eq. (4.5), one can obtain
\[
R_{\alpha \beta l} = \frac{2}{|G|} \sum_\mu \frac{d_\mu (d_\mu - \chi_\mu (g_i))}{\sum_i c_i \kappa_l \left( 1 - \frac{\chi_\mu (g_i)}{d_\mu} \right)}, \quad \forall \beta \in \Gamma_\alpha,
\]
(4.12)

Although the result (4.5) can be applied to all underlying networks of association schemes such as distance-regular and strongly regular networks\(^6\) and underlying networks of QD and GQD types,\(^3\) in the following we consider only special networks which we construct by using the orbits of the point groups corresponding to finite lattices, so that in the limit of the large size of the lattices, we obtain root lattices of type \( A_n \) (for more details see Ref. 14).

A. Examples

In this section, we consider some examples of the underlying networks of association schemes such as the hypercube network, the finite and infinite square lattice, and the hexagonal network.

1. Hypercube network

The hypercube of dimension \( n \) (known also as the binary Hamming scheme denoted by \( H(n, 2) \)) is a distance-regular network with \( N = 2^n \) vertices, each of which can be labeled by an \( n \)-bit binary string. Two vertices on the hypercube described by bitstrings \( \bar{x} \) and \( \bar{y} \) are connected by an edge if \( |\bar{x} - \bar{y}| = 1 \), where \( |\bar{x}| \) is the Hamming weight of \( \bar{x} \). Thus, each of the \( 2^n \) vertices on the hypercube has degree \( n \). The corresponding adjacency matrices are given by
\[
A_i = \sum_{\sigma_i} \sigma_i \otimes \sigma_i \cdots \otimes \sigma_i \otimes I_2 \cdots \otimes I_2, \quad i = 0, 1, \ldots, n,
\]
(4.13)

where the summation is taken over all possible nontrivial permutations.

It is well known that the eigenvalues \( P_\beta \) are given by
\[ P_{il} = K_i(l), \] (4.14)

where \( K_i(x) \) are the Krawtchouk polynomials defined as
\[
K_i(x) = \sum_{i=1}^{l} \binom{x}{i} \binom{n-x}{l-i} (-1)^i.
\] (4.15)

Also, we have
\[
\kappa_i = m_i = \frac{n!}{i! (n-i)!}.
\] (4.16)

Therefore, by using (4.5) we obtain effective resistance between two arbitrary nodes \( \alpha \) and \( \beta \) such that \( \beta \in \Gamma_\beta(\alpha) \) as
\[
R_{\alpha\beta} = \frac{2}{2n(n-1) \cdots (n-l+1)} \sum_{k=1}^{n} \frac{n!}{k! (n-k)!} \left( \sum_{i=1}^{L} \frac{n!}{i! (n-i)!} - K_i(l) \right)
\] (4.17)

For \( n=2 \) (square) and \( l=1 \), we have
\[
R_{\alpha\beta} = \frac{1}{4} \sum_{i=1}^{2} \frac{2(2 - K_i(1))}{i!(2-i)! (2(c_1 + c_2) - c_1 K_i(1) - c_2 K_i(2))} = \frac{1}{4} \frac{3c_1 + c_2}{c_1 + c_2},
\] (4.18)

where for \( c_2 = 0 \) and \( c = c_1 = 1 \), we obtain the simple result
\[
R_{\alpha\beta} = \frac{3}{4c} = \frac{3}{4}.
\] (4.19)

\section*{2. \( d \)-dimensional periodic networks}

Consider a \( d \)-dimensional lattice, periodic in each direction with period \( m \) and total number of \( N=m^d \) vertices. Each vertex of the lattice corresponds to a basis state \( |a> \), where \( a \) is a \( d \)-component vector with components \( a_1, \ldots, a_d \in \{0, 1, \ldots, m-1\} \). In the limit of large \( m \), this lattice tends to \( \prod_{i=1}^{d} A_i \), the Cartesian product of \( d \) root lattices \( A_i \) (for more details see Ref. 14). In the following we show that this lattice is an underlying network of an association scheme which can be derived from the finite Abelian group \( \mathbb{Z}_m^d = \mathbb{Z}_m \times \cdots \times \mathbb{Z}_m \) (\( m \geq 3 \)). Therefore, we can obtain effective resistances in terms of the scheme’s properties.

The generators of the lattice \( \prod_{i=1}^{d} A_i \) are \( S_i, \ldots, S_d \), where \( S_i = I \otimes \cdots \otimes I \otimes S_i \otimes I \otimes \cdots \otimes I \) with \( S^m = I \). Then, one can show that the point group of the lattice is \( \mathbb{Z}_m^d \times S_d \), where \( \mathbb{Z}_m^d \) is the corresponding Weyl group generated by the reflections \( S_1 \rightarrow S_1^{-1}; S_2 \rightarrow S_2^{-1}; \ldots; S_d \rightarrow S_d^{-1} \), and \( S_d \) is the symmetry group that contains all possible permutations of the simple roots \( S_1, S_2, \ldots, S_d \) (Ref. 15) (recall that this is the same as the symmetries of the corresponding Coxeter–Dynkin diagram of the lattice that consists of the \( d \) disconnected simple roots). Then, the adjacency matrix of the underlying network [which is the same as the orbit \( O(S_1) \)] is
\[
A = S_1 + \cdots + S_d + S_1^{-1} + \cdots + S_d^{-1}.
\] (4.20)

In fact, the orbits of the point group corresponding to the lattice form a partition \( P = \{ P_i, i = (i_1, \ldots, i_d) \} \) for \( \mathbb{Z}_m^d \). Then, the adjacency matrices \( A_i \) are defined as the sum of all elements of \( P_i \) in the regular representation, i.e., we define
\[ A_i = \sum_{g \in P_i} g. \]  

(4.21)

More clearly, one can see that

\[ A_{i=\{i_1, i_2, \ldots, i_d\}} = O(S_i^1S_i^2 \cdots S_i^d) = S_i^1S_i^2 \cdots S_i^d + \text{perm} + S_i^1S_i^2 \cdots S_i^d + \text{perm} + S_i^1S_i^2S_i^3 \cdots S_i^d + \text{perm} + \cdots \\
+ S_i^1S_i^2 \cdots S_i^d + \text{perm}, \]

(4.22)

where the "perm" after each term denotes all permutations of the indices 1, 2, \ldots, d in that term. From Eq. (4.22), it can be easily seen that for these networks, the spectrum of the adjacency matrices can be found easily because the adjacency matrices are simultaneously diagonalized by the Fourier matrix \( F_m \otimes \cdots \otimes F_m \). The corresponding idempotents are given by

\[ E_{i=\{i_1, i_2, \ldots, i_d\}} = E_{i_1} \otimes E_{i_2} \otimes \cdots \otimes E_{i_d} + \text{perm} + E_{-i_1} \otimes E_{i_2} \otimes \cdots \otimes E_{i_d} + \text{perm} + E_{i_1} \otimes E_{-i_2} \otimes E_{i_3} \otimes \cdots \otimes E_{i_d} + \text{perm} + \cdots + E_{i_1} \otimes E_{i_2} \otimes \cdots \otimes E_{-i_d} + \text{perm}, \]

(4.23)

where \( E_{i=\{i\}} \) with \( |i| = (1/(\sqrt[2]{m}))(1, \omega, \ldots, \omega^{m-1})^t \) for \( i = 0, 1, \ldots, m-1 \) [see Eq. (2.6)]. From (4.22), one can deduce that the eigenvalues of the adjacency matrices \( A_i \) are given by

\[ \lambda^{(i_1, \ldots, i_d)} = 2 \left[ \cos \frac{2\pi(l_1 + \cdots + l_d)}{m} + \text{perm} + \cos \frac{2\pi(-l_1 + l_2 + \cdots + l_d)}{m} + \text{perm} + \cdots \\
+ \cos \frac{2\pi(l_1 + \cdots + l_{d-1}l_{d-1} - l_d)}{m} + \text{perm} + \cos \frac{2\pi(-l_1 - l_2 + l_3 + \cdots + l_d)}{m} \\
+ \text{perm} + \cdots + \cos \frac{2\pi(-l_1 - l_2 - \cdots - l_{d-n} + l_{d-n+1} + \cdots + l_d)}{m} + \text{perm} \right], \]

\[ := [d/2]. \]

(4.24)

In the limit of large lattice size, the eigenvalues tend to

\[ \lambda^{(i_1, \ldots, i_d)}(x_1, \ldots, x_d) = 2[\cos(l_1x_1 + \cdots + l_dx_d) + \text{perm} + \cos(-l_1x_1 + l_2x_2 + \cdots + l_dx_d) + \text{perm} \\
+ \cdots + \cos(l_1x_1 + \cdots + l_{d-1}x_{d-1} - l_dx_d) + \text{perm} + \cos(-l_1x_1 - l_2x_2 + l_3x_3 \\
+ \cdots + l_dx_d) + \text{perm} + \cdots + \cos(-l_1x_1 - l_2x_2 - \cdots - l_{d-n}x_n + l_{d-n+1}x_{d-n+1} + \cdots \\
+ l_dx_d) + \text{perm}], \]

(4.25)

where \( x_t = \lim_{m \to \infty} 2\pi i_t/m \) for \( k = 1, 2, \ldots, d \). From the assumptions \( c_1 = c = 1 \) and \( c_i = 0 \) for all \( i \neq 1 \) along with the fact that \( v_t = v = d \), Eq. (4.5) implies that the effective resistances in the infinite \( d \)-dimensional lattice \( A_1 \times \cdots \times A_1 \) are obtained as

\[ R_{a\beta^{\bullet}} = \frac{1}{\kappa_1(2\pi)^d} \int_0^{2\pi} dx_1 \cdots \int_0^{2\pi} dx_d \{ \kappa_1 - 2[\cos(l_1x_1 + \cdots + l_dx_d) + \text{perm} + \cos(-l_1x_1 + l_2x_2 \\
+ \cdots + l_dx_d) + \text{perm} + \cos(-l_1x_1 + l_2x_2 + \cdots + l_{d-1}x_{d-1} - l_dx_d) + \text{perm} + \cos(-l_1x_1 - l_2x_2 + \cdots + l_dx_d) + \text{perm} + \cdots + \cos(-l_1x_1 - l_2x_2 - \cdots - l_{d-n}x_n + l_{d-n+1}x_{d-n+1} + \cdots + l_dx_d) ]^{-1}. \]

(4.26)

In the following we consider the special cases of two-dimensional (\( d=2 \)) periodic networks such that in the limit of large network size, they tend to the root lattices \( A_1 \times A_1 \) and \( A_2 \), respec-
tively. The first case is called the finite square network and the latter one is called the finite hexagonal network (although for \( n \geq 3 \), the underlying networks can be constructed similarly, but the networks do not possess so physical importance).

### B. Finite square network

For this case \((d=2)\), the point group is \((Z_2 \times Z_2) \rtimes Z_2\), which is isomorphic to the finite Heisenberg group \(H_2\). In detail, we have \(Z_2 \times Z_2 = \{e, S_1 \rightarrow S_1, S_2 \rightarrow S_2, S_1 \rightarrow S_2, S_2 \rightarrow S_1, S_1 \rightarrow S_2, S_2 \rightarrow S_1\}\) and the third cyclic group \(Z_2\) is generated by the permutation \(S_1 \leftrightarrow S_2\).

Now, we choose the ordering of elements of \(Z_m \times Z_m\) as follows:

\[
V = \{e, a, \ldots, a^{m-1}, b, ab, \ldots, b^{m-1}, ab^{m-1}, \ldots, a^{m-1}b^{m-1}\},
\]

(4.27)

where \(a^m = b^m = e\). We use the notation \((k, l)\) for the element \(a^{k}b^{l}\) of the group. Clearly, \((k, l) \times (k', l') = (k + k', l + l')\) and \((k, l)^{-1} = (-k, -l)\). Then the vertex set \(V\) of the network will be \(\{(k, l) : k, l \in \{0, 1, \ldots, m-1\}\}\). Then, the corresponding orbits are given by

\[
P_{k, k} := O((k_1, k_2)),
\]

(4.28)

where \(P_{00} = \{(0, 0)\}\) (in this case, the partition \(P\) is called homogeneous). In the regular representation of the group, for the corresponding adjacency matrices and the corresponding idempotents, we have

\[
A_{k=(k_1, k_2)} = \sum_{g \in O((k_1, k_2))} g = S_1^{k_1}S_2^{k_2} + S_1^{-k_1}S_2^{-k_2} + S_1^{k_1}S_2^{k_2} + S_1^{-k_1}S_2^{-k_2} + S_1^{k_1}S_2^{-k_1} + S_2^{k_1}S_1^{-k_1} + S_1^{k_1}S_2^{k_2} + S_2^{k_1}S_1^{-k_2},
\]

(4.29)

for \(k_1 \neq k_2\),

\[
E_{k=(k_1, k_2)} = E_{k_1} \otimes E_{k_2} + E_{-k_1} \otimes E_{k_2} + E_{k_1} \otimes E_{-k_2} + E_{-k_1} \otimes E_{-k_2} + E_{k_1} \otimes E_{k_2} + E_{-k_1} \otimes E_{k_2} + E_{k_1} \otimes E_{k_2} + E_{-k_1} \otimes E_{k_2},
\]

(4.30)

respectively, and

\[
A_{k=(k_1, k_1)} = \sum_{g \in O((k_1, k_1))} g = S_1^{k_1}S_1^{k_1} + S_1^{-k_1}S_1^{-k_1} + S_1^{k_1}S_1^{-k_1} + S_1^{-k_1}S_1^{k_1},
\]

(4.31)

\[
E_{k=(k_1, k_1)} = E_{k_1} \otimes E_{k_1} + E_{-k_1} \otimes E_{-k_1} + E_{k_1} \otimes E_{-k_1} + E_{k_1} \otimes E_{k_1} + E_{-k_1} \otimes E_{k_1} + E_{-k_1} \otimes E_{-k_1},
\]

(4.32)

Therefore, the cardinalities of the associate classes \(\Gamma(o)\) \((\kappa)\), the valencies of the adjacency matrices, and the ranks of the idempotents are given by

\[
\kappa_{0=(0, 0)} = m_{0=(0, 0)} = 1, \quad \kappa_{k=(k_1, k_2)} = m_{k=(k_1, k_2)} = 8 \text{ for } 0 \neq k_1 \neq k_2 \neq 0, \quad \kappa_{k=(k, k)} = m_{k=(k, k)} = 4.
\]

(4.33)

The eigenvalues of the adjacency matrix \(A_{k=(k_1, k_2)}\) with \(k_1 \neq k_2\) are given by

\[
\lambda_{ij}^{k=(k_1, k_2)} = 2 \left\{ \cos \frac{2 \pi (ik_1 + jk_2)}{m} + \cos \frac{2 \pi (ik_2 + jk_1)}{m} + \cos \frac{2 \pi (ik_1 - jk_1)}{m} + \cos \frac{2 \pi (ik_2 - jk_2)}{m} \right\},
\]

(4.34)

where for \(k_1 = k_2\) we have
Clearly, for a finite square lattice we have $e_{(1,0)} = c = 1$ and $e_{(i,1)} = 0$ for all $i_1 \neq 1$ and $i_2 \neq 0$. Then, by substituting (4.33) and (4.34) into (4.5), the effective resistances on the finite square lattice are given by

$$R_{\alpha \beta} = \frac{1}{m^2 k_{\alpha \beta} l_{\lambda}} \sum_{m(k_{\alpha \beta})} \left[ k_{\alpha \beta} - 2 \left( \frac{2 \pi (k_{1} j_{1} + k_{2} j_{2})}{m} + \cos \frac{2 \pi (k_{1} j_{1} + k_{2} j_{2})}{m} + \cos \frac{2 \pi (k_{1} j_{1} - k_{2} j_{2})}{m} + \cos \frac{2 \pi (k_{1} j_{1} - k_{2} j_{2})}{m} \right) \right]$$

where $R_{\alpha \beta}$ denotes the effective resistances between $\alpha$ and all the nodes $\beta \in \Gamma_{\alpha}(l_{\lambda})$. For instance for $\beta \in \Gamma_{\alpha}(l_{\lambda})$, we obtain

$$R_{\alpha \beta} = \frac{1}{m^2} \sum_{k_{1,2}} \left[ \frac{2 - \cos(2 \pi k_{1}/m) - \cos(2 \pi k_{2}/m)}{2 - \cos(2 \pi k_{1}/m) - \cos(2 \pi k_{2}/m)} \right] = \frac{R}{2} \sum_{k_{1,2}} \frac{1}{m^2} = \frac{R}{2}$$

In the limit of the large size of the finite lattice, i.e., in the limit of $m \to \infty$, we have the infinite square lattice. In this limit the eigenvalues (4.34) tend to

$$\lambda_{k_{1,2}} = 2 \left[ \cos(l_{1} x_{1} + l_{2} x_{2}) + \cos(l_{1} x_{1} + l_{2} x_{1}) + \cos(l_{1} x_{2} - l_{2} x_{1}) + \cos(l_{1} x_{1} - l_{2} x_{2}) \right]$$

where $x_{1} = \lim_{k_{1} \to \infty} 2 \pi k_{1}/m$ and $x_{2} = \lim_{k_{2} \to \infty} 2 \pi k_{2}/m$. Then, the effective resistances are calculated as follows:

$$R_{\alpha \beta} = \frac{1}{8 \pi} \int_{0}^{2 \pi} \int_{0}^{2 \pi} \frac{2 - \cos(l_{1} x_{1} + l_{2} x_{2}) - \cos(l_{1} x_{1} + l_{2} x_{1}) - \cos(l_{1} x_{2} - l_{2} x_{1}) - \cos(l_{1} x_{1} - l_{2} x_{2})}{2 - \cos x_{1} - \cos x_{2}} \times dx_{1} dx_{2}.$$ 

C. Hexagonal network

Now, we consider the finite root lattice $A_{2}$ which is called the hexagonal lattice. The point group of the lattice $A_{2}$ is $S_{3} \times Z_{2}$, where $S_{3}$ is the group of permutations of the simple roots together with the lowest root [all permutations of $S_{1}, S_{2}, \text{and } (S_{1} S_{2})^{-1}$]. With the same ordering of elements as before, the corresponding orbits are given by

$$P_{k_{1} k_{2}} := O(k_{1}, -k_{2}),$$

where $P_{00} = \{(0,0)\}$. Then, for the corresponding adjacency matrices and the corresponding idempotents, we have

$$A_{k_{1} k_{2}} = \sum_{g \in O(k_{1}, -k_{2})} S_{1}^{k_{1}} S_{2}^{k_{2}} + S_{1}^{k_{1}} S_{2}^{k_{2}} + S_{1}^{k_{1}} S_{2}^{k_{2}} + S_{1}^{k_{1}} S_{2}^{k_{2}} + S_{1}^{k_{1}} S_{2}^{k_{2}} + S_{1}^{k_{1}} S_{2}^{k_{2}} + S_{1}^{k_{1}} S_{2}^{k_{2}} + S_{1}^{k_{1}} S_{2}^{k_{2}} + S_{1}^{k_{1}} S_{2}^{k_{2}}$$

for $k_{1} \neq k_{2}$. (4.40)
\[ E_{k=(k_1,k_2)} = E_{k_1} \otimes E_{-k_2} + E_{-k_1} \otimes E_{k_2} + E_{k_1} \otimes E_{k_2} + E_{-k_1} \otimes E_{-k_2} + E_{k_1+k_2} + E_{k_2} \otimes E_{-(k_1+k_2)} + E_{k_1} \otimes E_{k_2+k_2} + E_{k_1+k_2} \otimes E_{k_2} + E_{-(k_1+k_2)} \otimes E_{k_2} + E_{k_1+k_2} \otimes E_{k_1} + E_{-(k_1+k_2)} \otimes E_{-k_2} + E_{k_1+k_2} \otimes E_{-k_2} \] 

respectively, and

\[ A_{k=(k,k)} = \sum_{g \in O((k,k))} g = S^k_{k^2} + S^k_1 S^k_2 + S^k_1 S^{2k}_2 + S^k_1 S^{4k}_2 + S^k_1 S^{6k}_2 + S^k_1 S^{8k}_2 + S^k_1 S^{10k}_2 + S^k_1 S^{12k}_2 + S^k_1 S^{14k}_2 \]

\[ E_{k=(k,k)} = E_k \otimes E_{-k} + E_k \otimes E_{2k} + E_{-k} \otimes E_{2k} + E_k \otimes E_{-k} + E_{2k} \otimes E_k + E_{-2k} \otimes E_k + E_k \otimes E_{-2k} + E_{2k} \otimes E_{-k} \]

\[ \kappa_{k(0,0)} = m_{k(0,0)} = 1, \quad \kappa_{k=(k_1,k_2)} = m_{k=(k_1,k_2)} = 12 \text{ for } 0 \neq k_1 \neq k_2 \neq 0, \text{ and } \kappa_{k=(k,k)} = m_{k=(k,k)} = 10. \]

The eigenvalues of the adjacency matrix \( A_{k=(k_1,k_2)} \) with \( k_1 \neq k_2 \) are given by

\[ \lambda_{k_1,k_2}^{(k)} = \frac{2}{m} \left\{ \cos \frac{2\pi (k_1-jk_2)}{m} + \cos \frac{2\pi (k_2-jk_1)}{m} + \cos \frac{2\pi (k_1-k_2)}{m} + \cos \frac{2\pi (k_2-k_1)}{m} \right\}, \]

where for \( k_1 = k_2 = k \) we have

\[ \lambda_{k_1,k_2}^{(k)} = \frac{2}{m} \left\{ \cos \frac{2\pi (i-j)k}{m} + \cos \frac{2\pi (i-2j)k}{m} + \cos \frac{2\pi (2i-j)k}{m} + \cos \frac{2\pi (2i+j)k}{m} \right\}. \]

Then, similar to the case of finite square lattice, one can calculate effective resistances \( R_{\alpha \beta \gamma} \).

In the limit of \( m \to \infty \), we have the infinite hexagonal lattice. In this limit the eigenvalues (4.34) tend to

\[ \lambda_{x_1,x_2}^{(\infty)} = \frac{1}{8 \pi^2} \int_0^{2\pi} \frac{3}{3 - \cos (l_1 x_1 - l_2 x_2) - \cos (l_1 x_2 - l_2 x_1)} \cdot \frac{\cos (l_1 x_1 + x_2) - \cos (l_1 x_2 + l_2 x_1) - \cos (l_1 x_1 + x_2) + l_2 x_1) \cdot \cos (l_1 x_1 + x_2)}{3 - \cos x_1 - \cos x_2 - \cos (x_1 + x_2)} dx_1 dx_2. \]
1. An example of the underlying networks of group association schemes

As it is well known, for the group \( S_n \), conjugacy classes are determined by the cycle structures of elements when they are expressed in the usual cycle notation. The useful notation for describing the cycle structure is the cycle type \([v_1, v_2, \ldots, v_n]\), which is the listing of number of cycles of each length (i.e., \( v_1 \) is the number of one cycles, \( v_2 \) is that of two cycles, and so on). Thus, the number of elements in a conjugacy class or stratum is given by

\[
|C_{[v_1, v_2, \ldots, v_n]}| = \frac{n!}{v_1! v_2! \cdots v_n!},
\]

(4.48)

On the other hand a partition \( \lambda \) of \( n \) is a sequence \((\lambda_1, \ldots, \lambda_n)\), where \( \lambda_1 \geq \cdots \geq \lambda_n \) and \( \lambda_1 + \cdots + \lambda_n = n \), where in terms of cycle types

\[
\lambda_1 = v_1 + v_2 + \cdots + v_n, \quad \lambda_2 = v_2 + v_3 + \cdots + v_n, \quad \ldots, \quad \lambda_n = v_n.
\]

(4.49)

The notation \( \lambda + n \) indicates that \( \lambda \) is a partition of \( n \). For each partition \( \lambda + n \), \( S_n \) has a corresponding conjugacy class that consists of those permutations having cycle structure described by \( \lambda \). We denote by \( C_\lambda \) the conjugacy class of \( S_n \) consisting of all permutations having cycle structure \( \lambda \). Therefore, the number of conjugacy classes of \( S_n \), namely, the diameter of its scheme is equal to the number of partitions of \( n \), which grows asymptotically as \((1/4 \pi^2)^n e^{\pi^2/12n}\).

We consider the case where the generating set consists of the set of all transpositions, i.e., \( C_1 = \langle 2, 1, 1, 1, \ldots \rangle \). The character of any transposition is known to be \( 16 \) given as

\[
\chi_\lambda(\alpha_1) = \frac{2! (n-2)! \dim(\rho_\lambda)}{n} \sum_j \left( \left( \frac{\lambda_j}{2} \right) - \left( \frac{\lambda_j'}{2} \right) \right).
\]

(4.50)

Here, \( \lambda' \) is the partition generated by transposing the Young diagram of \( \lambda \), while \( \lambda_j' \) and \( \lambda_j \) are the \( j \)th components of the partitions \( \lambda' \) and \( \lambda \), and \( \rho_\lambda \) is the irreducible representation corresponding to partition \( \lambda \). Then the eigenvalues of the adjacency matrix can be written as

\[
P_{\lambda_1} = \frac{d_{\lambda_1}}{m_\lambda} \chi_\lambda(\alpha_1) = \sum_j \left( \left( \frac{\lambda_j}{2} \right) - \left( \frac{\lambda_j'}{2} \right) \right).
\]

(4.51)

In the above calculation, we have used the following results for the characters of the \( n \)-cycles:

\[
\chi_\lambda((n)) = \begin{cases} 
(-1)^{n-k} & \text{for } \lambda = (k, 1, \ldots, 1), k \in \{1, \ldots, n\} \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
\chi_{(k, 1, \ldots, 1)}(e) = \dim(\rho_{(k, 1, \ldots, 1)}) = \left( \begin{array}{c} n-1 \\ k-1 \end{array} \right), \quad P_{\lambda_1} = \frac{1}{2} (2nk - n^2 - n).
\]

Then, one can evaluate effective resistances by using Eq. (4.36). In the following, we consider the underlying network of the group association scheme \( S_4 \) with diameter \( d=4 \), in detail. To do so, we use the conjugacy classes of \( S_4 \) given by Eq. (3.10) and the adjacency matrices \( A_1 = C_1, i = 0, 1, \ldots, 4 \), which satisfy the following Bose–Mesner algebra:

\[
A^2 = 6A_0 + 3A_2 + 2A_3, \quad AA_2 = 4A + 4A_4, \quad AA_3 = A + 2A_4, \quad AA_4 = 4A_2 + 4A_3, \quad A^2 = 8A_0 + 4A_2 + 8A_3, \quad A_2A_3 = 3A_2, \quad A_3A_4 = 4A + 4A_4, \quad A^2 = 3A_0 + 2A_3, \quad A_3A_4 = 2A + A_4, \quad A^2 = 6A_0 + 3A_2.
\]

(4.52)

By using the character table of the group \( S_4 \) and Eq. (4.11), one can obtain

\[
P_{0k} = 1, \quad k = 0, \ldots, 4, \quad P_{10} = P_{12} = P_{13} = -P_{11} = -P_{14} = 6, \quad P_{20} = P_{23} = 8, \quad P_{21} = P_{24} = 0,
\]

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\[ \begin{align*}
P_{22} &= -4, \quad P_{30} = 3, \quad P_{31} = -P_{33} = 1, \quad P_{32} = 0, \quad P_{40} = 6, \quad P_{41} = P_{43} = -P_{44} \\
&= -2, \quad P_{42} = 0.
\end{align*} \]  

(4.53)

Now, by using Eq. (4.5), we obtain

\[
R_{\alpha\beta}^{(1)} = \frac{1}{6} \left\{ \frac{1}{12c_1 + 2c_3 + 8c_4} + \frac{9}{12c_1 + 8c_2 + 4c_3 + 4c_4} \right\},
\]

\[
R_{\alpha\beta}^{(2)} = \frac{1}{36} \left\{ \frac{3}{12c_1 + 2c_3 + 8c_4} + \frac{20}{12c_2 + 3c_3 + 6c_4} - \frac{9}{4c_3 + 8c_4} + \frac{27}{12c_1 + 8c_2 + 4c_3 + 4c_4} \right\},
\]

\[
R_{\alpha\beta}^{(3)} = \frac{1}{18} \left\{ \frac{1}{12c_1 + 2c_3 + 8c_4} + \frac{20}{12c_2 + 3c_3 + 6c_4} + \frac{18}{4c_3 + 8c_4} + \frac{27}{12c_1 + 8c_2 + 4c_3 + 4c_4} \right\},
\]

\[
R_{\alpha\beta}^{(4)} = \frac{1}{48} \left\{ \frac{5}{12c_1 + 2c_3 + 8c_4} + \frac{16}{12c_2 + 3c_3 + 6c_4} + \frac{45}{4c_3 + 8c_4} + \frac{27}{12c_1 + 8c_2 + 4c_3 + 4c_4} \right\}.
\]  

(4.54)

where \( R_{\alpha\beta}^{(i)} \) denotes the effective resistance between the node \( \alpha \) and all nodes \( \beta \in \Gamma_i(\alpha) \) for \( i = 1, 2, 3, 4 \).

V. CONCLUSION

Based on the stratification of underlying networks of association schemes and their algebraic combinatoric structure such as the Bose–Mesner algebra, evaluation of effective resistances on these networks was discussed. It was shown that in these types of networks, the effective resistances between a node \( \alpha \) and all nodes \( \beta \) belonging to the same stratum as \( \alpha \) are the same. Then, by assumption that all of the conductances except for one of them are zero, a procedure for the evaluation of effective resistances on particular underlying networks for which the adjacency matrix \( A \) possesses \( d+1 \) distinct eigenvalues was given such that effective resistances can be evaluated without using the spectrum of the networks. Moreover, an explicit analytical formula for effective resistance between arbitrary nodes \( \alpha, \beta \) of an underlying resistor network of association schemes (where all of conductances are non-zero) was given in terms of the spectrum of the networks. In each case, evaluation of effective resistance on some important finite underlying networks and their corresponding infinite networks was given.

APPENDIX: POLYNOMIAL PROPERTY OF THE ADJACENCY ALGEBRA OF THE ASSOCIATION SCHEMES WITH \( d+1 \) DISTINCT EIGENVALUES

In this we show that for underlying networks of association schemes with diameter \( d \) such that the adjacency matrix \( A \) has \( d+1 \) distinct eigenvalues, all of the adjacency matrices are polynomials of \( A \), i.e., \( A_i = P_i(A) \), where \( P_i \) is not necessarily of degree \( i \). To do so, let \( A \) possess \( d+1 \) distinct eigenvalues \( P_{1k}, k=0,1, \ldots, d \). Then, by using Eq. (2.7), one can write

\[
A^i = \sum_{k=0}^{d} (P_{1k})^i E_k,
\]

or in the matrix form

\[
(1 \quad A \quad A^2 \cdots A^d)^i = V(E_0 \quad E_1 \quad E_2 \cdots E_d)^i,
\]

where \( V \) is the Vandermonde matrix.
Clearly $V$ is invertible due to the distinctness of the eigenvalues $P_{1k}$ for $k=0,1,\ldots,d$. Then, we have

$$(E_0\ E_1\ E_2\cdots E_d)^t = V^{-1}(1\ A\ A^2\cdots A^d)^t.$$}

Now, by using (2.7), we write the idempotents $E_i$ in terms of $A_j$ to obtain

$$(E_0\ E_1\ E_2\cdots E_d)^t = \frac{1}{n}Q(1\ A\ A_2\cdots A_d)^t = V^{-1}(1\ A\ A^2\cdots A^d)^t.$$}

Therefore, the adjacency matrices $A_i$, $i=0,1,\ldots,d$, can be written as polynomials of $A$, i.e., we have

$$(1\ A\ A_2\cdots A_d)^t = n(VQ)^{-1}(1\ A\ A^2\cdots A^d)^t.$$}

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