A SMOOTH COUNTEREXAMPLE TO THE
HAMILTONIAN SEIFERT CONJECTURE IN $\mathbb{R}^6$

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Abstract. A smooth counterexample to the Hamiltonian Seifert conjecture in $\mathbb{R}^6$ is found. Namely, we construct a smooth proper function $H : \mathbb{R}^{2n} \to \mathbb{R}$, $2n \geq 6$, such that the level set $\{H = 1\}$ is non-singular and has no periodic orbits for the Hamiltonian flow of $H$ with respect to the standard symplectic structure. The function $H$ can be taken to be $C^0$-close and isotopic to a positive-definite quadratic form so that $\{H = 1\}$ is isotopic to an ellipsoid. This is a refinement of previously known constructions giving such functions only for $2n \geq 8$. The proof is based on a new version of a symplectic embedding theorem applied to the horocycle flow.

1. Introduction

Hamiltonian systems tend to have periodic orbits whenever the energy level is non-singular and compact. For example, Hofer, Zehnder, and Struwe (see [HZ1], [St], and [HZ2]) have shown that periodic orbits must exist on almost all energy levels of a proper smooth function on the standard symplectic $\mathbb{R}^{2n}$. This implies Viterbo’s theorem, [Vi], proving Weinstein’s conjecture, [We]: a compact smooth hypersurface of contact type in the standard $\mathbb{R}^{2n}$ carries at least one closed characteristic. (The reader interested in a detailed discussion should consult [HZ2].)

The first examples of smooth compact hypersurfaces in $\mathbb{R}^{2n}$ without closed characteristics were found only recently. As a consequence, one obtains a smooth proper function on $\mathbb{R}^{2n}$ which has at least one regular level set without periodic orbits of its Hamiltonian flow. These examples, constructed in [Gi1] and, independently, by M. Herman [Her3] required the ambient space to be at least eight-dimensional, i.e., $2n \geq 8$, in the smooth case. A $C^{3-\epsilon}$-smooth hypersurface in $\mathbb{R}^{6}$ was found by M. Herman [Her3]. In the present paper we extend the results of [Gi1] to $2n = 6$ by constructing such examples in dimension six (Section 2).
To carry out this extension, we show that, as a consequence of a general symplectic embedding theorem proved below, the horocycle flow can be “symplectically” embedded into $\mathbb{R}^5$. (See Section 3.) This is the only part of the argument of [Gi1] missing in the six-dimensional case. The paper is concluded by a list of all Hamiltonian flows known to the author to have no periodic orbits (Section 4).

Although formally independent, this paper can be considered as a follow-up of [Gi1]. In particular, we refer to some of the proofs, not only the results, given therein. When possible and appropriate the original notation is kept.

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2. Main Theorems

Let $i: Q \hookrightarrow V$ be an embedded smooth compact hypersurface without boundary in a $2n$-dimensional symplectic manifold $(V, \sigma)$. Recall that a characteristic of a two-form $\eta$ of rank $(2n - 2)$ on a $(2n - 1)$-dimensional manifold is an integral curve of the field of directions formed by the null-spaces $\ker \eta$. Our main result is the following

**Theorem 1.** Assume that $2n \geq 6$ and $i^* \sigma$ has only a finite number of closed characteristics. Then there exists a $C^\infty$-smooth embedding $i': Q \to V$, which is $C^0$-close and isotopic to $i$, such that $i'^* \sigma$ has no closed characteristics.

**Remark 2.** For $2n \geq 8$, Theorem 1 as well as other results stated in this section are known. See [Gi1] and [Her3]. This paper improves the dimensional constraint by two. As mentioned above, for $2n = 6$, an example of such a $C^3-\epsilon$-smooth embedding $i'$ was found by M. Herman [Her3].

As is the case for $2n \geq 8$, the embedding $i'$ can be chosen to coincide with $i$ outside a finite number of small balls each “centered” on a closed characteristic of $i^* \sigma$.

An irrational ellipsoid $Q$ in the standard symplectic vector space $\mathbb{R}^{2n} = V$ corresponds to a collection of $n$ harmonic oscillators (a quadratic Hamiltonian) whose frequencies are linearly independent over $\mathbb{Q}$. Thus $Q$ has exactly $n$ periodic orbits. Applying Theorem 1, we obtain

**Corollary 3.** For $2n \geq 6$, there exists a $C^\infty$-smooth embedding $S^{2n-1} \to \mathbb{R}^{2n}$ such that the pull back of $\sigma_{2n}$ to $S^{2n-1}$ has no closed characteristics.
Corollary 4. For $2n \geq 6$, there exists a $C^\infty$-smooth function $h: \mathbb{R}^{2n} \to \mathbb{R}$, $C^0$-close and isotopic (with a compact support) to a positive definite quadratic form, such that the Hamiltonian flow of $h$ has no closed trajectories on the level set $\{h = 1\}$.

A similar result involving no symplectic embeddings, but only two-forms on $Q$, is used in the proof of Theorem 1. Let $\dim Q = 2n - 1$ and let $\eta$ be a closed maximally non-degenerate (i.e., of rank $(2n - 2)$) two-form on $Q$.

Theorem 5. Assume that $2n - 1 \geq 5$ and that $\eta$ has a finite number of closed characteristics. Then there exists closed a maximally non-degenerate 2-form $\eta'$ on $Q$ (homotopic to $\eta$) without closed characteristics.

Recall that the forms $\eta$ and $\eta'$ are said to be homotopic if there exists a family $\eta_\tau$, $\tau \in [0, 1]$, of closed maximally non-degenerate forms in the same cohomology class connecting $\eta = \eta_0$ with $\eta' = \eta_1$.

Remark 6. Theorem 5 extends to the real analytic case: one can make the form $\eta'$ real analytic, provided that $Q$ and $\eta$ are real analytic. The argument is the same as that used in the construction of a real analytic version of Wilson’s, $[W]$, or Kuperberg’s, $[KuK]$, flow. (See $[Gh]$.)

3. Proofs

As is pointed out in $[Gi1]$ (Remark 3.5), to prove Theorems 1 and 3 it is sufficient to find a “symplectic” embedding of the horocycle flow into $\mathbb{R}^5$. (We will elaborate on this later.) The existence of such an embedding is a consequence of the following general construction.

Let $N$ and $W$ be manifolds of equal dimensions. The manifold $N$ is assumed to be compact, perhaps with boundary, while $W$ may be open but must be a manifold without boundary. Let $\sigma$ be a symplectic form on $W$. Abusing notation, also denote by $\sigma$ the pull-back of $\sigma$ to $W \times \mathbb{R}$ under the natural projection $W \times \mathbb{R} \to W$. To avoid confusion, we will sometimes indicate the domain of a form by a subscript, e.g., $\sigma_W$ or $\sigma_{W \times \mathbb{R}}$.

Theorem 7. Let $\omega_t$, $t \in [0, 1]$, be a family of symplectic forms on $N$ in a fixed cohomology class: $[\omega_t] = \text{const}$. Assume that there is an embedding $j_0: N \to W \times \mathbb{R}$ such that $j_0^* \sigma = \omega_0$. Then there exists an embedding $j_1: N \to W \times \mathbb{R}$, isotopic to $j_0$, with $j_1^* \sigma = \omega_1$.

Remark 8. Since $\sigma_W$ is symplectic, the composition of $j_1$ with the projection to $W$ is necessarily an immersion. When $\partial N = \emptyset$, Theorem
follows immediately from Moser’s theorem [Mo]. Here, however, we are more interested in the case where \( \partial N \neq \emptyset \) and Moser’s theorem does not apply.

**Proof.** Assume first that \( N \) is a manifold with boundary.

Following Gromov, [Gr3] (p. 336), let us approximate the family \( \omega_t \) by a family which piecewise linearly interpolates a finite sequence of symplectic forms \( \omega(k), k = 0, 1, \ldots, q, \) such that \( \omega(k) = \omega(k-1) + df_k \wedge dg_k, \) where \( f_k \) and \( g_k \) are smooth functions on \( N. \) We assume, of course, that the new and the original families have the same endpoints: \( \omega_0 = \omega(0) \) and \( \omega_1 = \omega(q). \) The existence of this approximation is easy to verify.

Arguing inductively, we will find an embedding \( i_k \) with \( i_k^* \sigma = \omega(k) \) using the existence of \( i_{k-1} \) with \( i_{k-1}^* \sigma = \omega(k-1). \) The base of induction is given by \( i_0 = j_0. \) After \( i_k \) is constructed, it will remain to set \( j_1 = i_q. \)

Thus assume that we have \( i_{k-1}. \) For the sake of simplicity, we identify \( N \) with \( i_{k-1}(N) \) and denote \( f_k \) and \( g_k \) by \( f \) and \( g. \) The construction of \( i_k \) will be carried out in two steps: first, find a symplectic embedding \( \tilde{i} : N \rightarrow W \times \mathbb{R}^2 \) and then push it back into \( W \times \mathbb{R}. \)

The first step. Denote by \((t, y)\) the standard linear coordinates on \( \mathbb{R}^2. \) Let us think of \( W \times \mathbb{R} \) as given by the equation \( y = 0 \) in \( W \times \mathbb{R}^2 \) so that the form \( \sigma_{W \times \mathbb{R}} \) is the restriction of \( \sigma_W + dt \wedge dy. \) However, to construct \( \tilde{i} \), it is more convenient to use a different system of coordinates on a neighborhood of \( N. \)

First, let us extend \( N \) a little bit beyond its boundary so that the resulting open manifold \( N_1 \) containing \( N \) is still embedded into \( W \times \mathbb{R} \) transversely to the direction of the \( t \)-axis. As a consequence, the restriction of \( \sigma_W \) to \( N_1 \) is a symplectic form, which we denote for the sake of simplicity by \( \sigma. \) Note that the restriction of \( \sigma \) to \( N \) is just \( \omega(k-1), \) the same as \( \sigma|_{W \times \mathbb{R}}|_{N}. \) In what follows we will need to take \( N_1 \) so that the closure \( \bar{N}_1 \) is a smooth embedded manifold with boundary and that \( \bar{N}_1 \) is still transversal to the \( t \)-axis. Since \( W \times \mathbb{R} \) is embedded into \( W \times \mathbb{R}^2, \) so are \( N_1 \) and \( \bar{N}_1. \)

Let \( I = (-\epsilon, \epsilon) \) and \( J = (-\delta, \delta) \) and let \( \tilde{t} \) and \( \tilde{y} \) be the natural coordinates on these intervals.

**Lemma 9.** For sufficiently small \( \epsilon > 0 \) and \( \delta > 0, \) there exists an embedding \( \psi : N_1 \times I \times J \rightarrow W \times \mathbb{R}^2 \) such that

\[
\psi^*(\sigma_W + dt \wedge dy) = \sigma + d\tilde{t} \wedge d\tilde{y}
\]

and, in addition, \( \psi|_{N_1} \) is the original embedding \( N_1 \hookrightarrow W \times \mathbb{R} \) and \( \psi(N_1 \times I \times 0) \subset W \times \mathbb{R}. \)
The existence of $\psi$ is a consequence of the symplectic neighborhood theorem applied to a small neighborhood of $\hat{N}_1$ in $W \times \mathbb{R}$ considered as a submanifold in $W \times \mathbb{R}^2$. For the sake of completeness we give a direct proof.

*Proof* of the lemma. To find $\psi$, let us first extend $N_1$ beyond its boundary in the same way as $N_1$ extends $N$. In particular, the closure of the resulting manifold $N_2$ is assumed to be transversal to the $t$-axis direction. Let, as before, $\sigma = \sigma_{W \times \mathbb{R}^1|N_2}$.

A neighborhood $U$ of $N_2$ in $W \times \mathbb{R}^2$ is diffeomorphic to $N_2 \times D^2$. The diffeomorphism from $N_2 \times D^2$ to $U$ can be chosen to be fiber preserving: each fiber $q \times D^2$, $q \in N_2$, goes to a disk in some plane $p \times \mathbb{R}^2$, $p \in W$, where $p$ depends on $q$. To be more precise, let us pick $U$ so that its intersection with every plane $p \times \mathbb{R}^2$ is a disjoint union of a finite number of two-dimensional disks. By means of the linear structure in $\mathbb{R}^2$ we identify each of these disks with a small disk centered at the origin. One can choose $U$ such that different fibers give rise to the same disk for all points of $N_2$. In this way, a neighborhood $U$ of $N_2$ in $W \times \mathbb{R}^2$ turns into the direct product $N_2 \times D^2$.

Let $\bar{t}$ and $\bar{y}$ be the restrictions of $t$ and $y$ to the disk $D^2$ centered at the origin. We extend $\bar{t}$ and $\bar{y}$ to $U$ using the direct product decomposition $U = N_2 \times D^2$ and denote by $t$ and $y$ the restrictions of $\bar{t}$ and $\bar{y}$, as functions on $W \times \mathbb{R}^2$, to $U$. Then $y = \bar{y}$ and $t = \bar{t} + h$, where $h$ is a function on $N_2$. The restriction of $\sigma + dt \wedge dy$ to $U$ is therefore equal to $\sigma + d\bar{t} \wedge d\bar{y} + dh \wedge d\bar{y}$.

Denote by $\xi_h$ the Hamiltonian vector field of $h$ on $N_2$, i.e., $i_{\xi_h} \sigma = -dh$ and set $v = -\bar{y} \xi_h$ on $U = N_2 \times D^2$ by using the direct product structure. A straightforward calculation shows that the local flow $\psi^\tau$ of $v$ on $U$ is such that

$$(\psi^\tau)^* (\sigma + d\bar{t} \wedge d\bar{y} + \tau dh \wedge d\bar{y}) = \sigma + d\bar{t} \wedge d\bar{y}$$

whenever $\psi^\tau$ is defined. Since $v$ vanishes on the section $\bar{y} = 0$ of $N_2 \times D^2$, the time-one flow $\psi$ is defined as a mapping of a small neighborhood of $\hat{N}_1$ to $U$. Thus choosing $\epsilon > 0$ and $\delta > 0$ sufficiently small, we get $\psi: N_1 \times I \times J \to U$ with

$$\psi^* (\sigma_W + dt \wedge dy) = \psi^* (\sigma + d\bar{t} \wedge d\bar{y} + dh \wedge d\bar{y}) = \sigma + d\bar{t} \wedge d\bar{y}$$

and such that $\psi = id$ on $N_1 \times I \times 0$. Also, $\psi$ preserves $\bar{t}$ and $\bar{y}$ because $L_v \bar{t} = L_v \bar{y} = 0$. This completes the proof of the lemma. \qed

Denote by $F: \mathbb{R}^2 \to I \times J$ an arbitrary symplectic immersion. Without lost of generality we may assume that $F$ sends the origin to the
origin. (To construct $F$ we may, for example, first squeeze symplectically $\mathbb{R}^2$ into a narrow infinite strip and then roll it up onto an annulus lying in $I \times J$.) Let us extend the functions $f$ and $g$ on $N$ to some smooth compactly supported functions on $N_1$, which we denote by $\tilde{f}$ and $g$ again. As in the proof of Lemma (B') on p. 336 in [Gr3], define the embedding $\tilde{j}: N_1 \to N_1 \times I \times J$ as the graph of $F \circ (f,g): N_1 \to I \times J$. Clearly, $\tilde{j}$ has compact support and $\tilde{j}^*(\sigma + dt \wedge dy) = \sigma + df \wedge dg = \omega_{(k)}$. The symplectic embedding $\tilde{i}: N \to W \times \mathbb{R}^2$ such that $\tilde{i}^*(\sigma_W + dt \wedge dy) = \omega_{(k)}$ is just $\psi \circ \tilde{j}$ restricted to $N$.

The second step is to modify $\tilde{i}$ to make it fit into $W \times \mathbb{R}$. We will alter $\tilde{j}$ so that to transform it into an embedding $j$ with the image in $N_1 \times I \times 0$ and the same pull-back form. Then it will remain to set $i = \psi \circ j$.

Denote by $\pi$ the projection $N_1 \times I \times J \to N_1 \times I$ along $J$. We claim that the restriction of $\pi$ to the image $\tilde{j}(N_1)$ is an embedding. This is an immediate consequence of the fact that $\tilde{j}(N_1)$ is a graph of some smooth function $N_1 \to I \times J$.

Thus $\tilde{j}(N_1)$ is a graph of a smooth function $\pi(\tilde{j}(N_1)) \to J$ with compact support. Let us extend this function to a smooth compactly supported function $H: N_1 \times I \to J$. Clearly, $\tilde{j}(N_1)$ lies on the graph $\Gamma$ of $H$.

Finally, we claim that $\Gamma$ with the restriction of $\sigma + dt \wedge dy$ is “symplectomorphic” to $(N_1 \times I, \sigma)$. The symplectomorphism is given by the straightening of the characteristics on $\Gamma$. This is possible because the $\tilde{t}$-component of a characteristic is nonzero and, on the complement to a compact set, the characteristics are just the straight lines parallel to $I$.

To give a more rigorous definition of the diffeomorphism, let us think of $H$ as a time-dependent Hamiltonian on $N_1$ with $I$ representing the time axis. Denote its local flow in the extended phase-space $N_1 \times I$ by $\phi^t$. (To be more precise about the notation, $\phi^t$ sends a point $(p, \tilde{t}') \in N_1 \times I$ to the point $(\phi^t(p, \tilde{t}'), \tilde{t} + \tilde{t}')$. The characteristics of $\Gamma$ project under $\pi$ to the integral curves of $\phi^t$. (See, e.g., [A1].) Since $supp H$ is compact in $N_1 \times I$, there exists $\tilde{t}_0 > -\epsilon$ such that $supp H$ is entirely contained in the region $\{\tilde{t} > \tilde{t}_0\}$. Now we define the desired symplectomorphism $\phi: \Gamma \to N_1 \times I$ by sending a point $(p, \tilde{t}, H(p, \tilde{t})) \in \Gamma$ to the point $(\phi^{\tilde{t}_0 - \tilde{t}}(p, \tilde{t}), \tilde{t}) \in N_1 \times I$ when $\tilde{t} > \tilde{t}_0$ and setting $\phi = id$ in the region $\tilde{t}_0 \geq \tilde{t} > -\epsilon$. By restricting the composition $\phi \circ \tilde{j}$ to $N$, we obtain an embedding $j: N \to N_1 \times I$ such that $j^* \sigma = j^*(\sigma + dt \wedge dy) = \omega_{(k)}$. 
Finally, $i_k = \psi \circ j$ satisfies the desired conditions: $i_k(N) \subset W \times \mathbb{R}$ and $i_k^* \sigma_W = \omega(k)$.

When $\partial N = \emptyset$, the above argument goes through for $N_2 = N_1 = N$ because the flow of $v$ exists for all times $\tau \in \mathbb{R}$. Alternatively, in this case Theorem 7 follows immediately from Moser’s theorem as is pointed out in Remark 8.

\[ \square \]

Before applying Theorem 7, let us recall the definition of the horocycle flow.

Let $\Sigma$ be a closed surface with a hyperbolic metric, i.e., a metric with constant negative curvature $K = -1$. Denote by $M = ST^*\Sigma$ the unit cotangent bundle of $\Sigma$ and by $N$ a small closed tubular neighborhood of $M$ in $T^*\Sigma$. Let $\lambda$ be the restriction to $M$ (or $N$) of the canonical Liouville one-form “$p\,dq$” and by $\theta$ the connection form on the principle circle bundle $M \to \Sigma$ (taken with the negative sign). We also denote by $\theta$ its pull-back to $N$ under the radial projection $N \to M$. It is easy to see that $d\theta$ is the pull-back of the metric area form on $\Sigma$. For any $t \in [0,1]$, the sum $\omega_t = d\lambda + t d\theta$ is a symplectic form on $N$.

In what follows, the pair $(M,\omega_1|_M)$ is referred to as the “horocycle flow”, for its characteristics form the flow lines of the true horocycle flow. (See [Gi2] and [Gi3] for details.) Our goal is to show that $(M,\omega_1|_M)$ can be embedded “symplectically” to $(\mathbb{R}^5,\sigma)$ where $\sigma$ is the pull-back to $\mathbb{R}^5$ of the standard symplectic form on $\mathbb{R}^4$. (By a symplectic embedding we mean the one that pulls back $\sigma$ to $\omega_1$ and which is nowhere tangent to the characteristics of $\sigma$ in $\mathbb{R}^5$.) This will follow from (but is actually equivalent to) the existence of a symplectic embedding of $(N,\omega_0)$ to $(\mathbb{R}^5,\sigma)$.

By Theorem 4 it is sufficient to find a symplectic embedding of $(N,\omega_0)$, i.e., for the standard symplectic form on $N$. To construct such an embedding, let us start with a Lagrangian immersion $\Sigma \to \mathbb{R}^4$ (see [Gr1] and [Gr3]), which we may assume to have only simple double points. Extend this immersion to a symplectic immersion of a small neighborhood $V$ of $\Sigma$ in $T^*\Sigma$. The Lagrangian immersion $\Sigma \to \mathbb{R}^4$ can be lifted to an embedding $\Sigma \to \mathbb{R}^5$ so that $\sigma$ pulls back to the zero form. As a consequence, the symplectic embedding $V \to \mathbb{R}^4$ can be lifted to a symplectic embedding $(V,d\lambda) \to (\mathbb{R}^5,\sigma)$. Applying a dilation in $\mathbb{R}^4$ and in $T^*\Sigma$, we can make $V$ arbitrarily large. In particular, we can take $V$ such that $N \subset V$ which results into an embedding $(N,\omega_0 = d\lambda) \to (\mathbb{R}^5,\sigma)$. Thus we have proved
Corollary 10. There exists an embedding \( j: N \to \mathbb{R}^5 \) such that \( j^* \sigma = \omega_1 \). There exists an embedding \( M \to \mathbb{R}^5 \) which is nowhere tangent to the characteristics of \( \sigma \) in \( \mathbb{R}^5 \) and such that the pull-back of \( \sigma \) is \( \omega_1 \).

Remark 11. The existence of a symplectic embedding of \( (N, \omega_t) \) into \( \mathbb{R}^6 \) is an immediate consequence of Gromov’s theorem on symplectic embeddings ([Gr3], pp. 335–336). Theorem 7 refines some of the assertions of Gromov’s theorem by reducing the dimension of the ambient space by one.

Remark 12. It is not very clear how to show that \( (M, \omega_1|_M) \) admits no symplectic embeddings into \( \mathbb{R}^4 \). Assume that it does. Then since \( M \) is a closed hypersurface in \( \mathbb{R}^4 \), the complement \( \mathbb{R}^4 \setminus M \) has two connected components: bounded, \( U_b \), and unbounded, \( U_u \). Let \( V_b \) and \( V_u \), respectively, be the bounded and unbounded components of \( T^* \Sigma \setminus M \) equipped with \( \omega_1 \). By a symplectic neighborhood theorem, one can attach either \( V_b \) to \( U_u \) (and \( V_u \) to \( U_b \)) or \( V_b \) to \( U_b \) (and \( V_u \) to \( U_u \)) with a symplectic result. In the former case, the resulting manifold \( V_b \cup_M U_u \) would be symplectomorphic to \( \mathbb{R}^4 \) at infinity, and so homeomorphic and (even symplectomorphic) to the standard \( \mathbb{R}^4 \) by a theorem of Gromov and McDuff, [Gr2] and [McD]. As a consequence, the symplectic form on it must be exact. This is impossible because \( \Sigma \subset V_b \) and \( \int_{\Sigma} \omega_1 > 0 \). However, the second way of attachment is not so easy to rule out.

The proofs of Theorems 1 and 5 repeat word-for-word the proofs of their higher-dimensional counterparts (Theorems 2.1 and 2.5) given in [Gi1]. In fact, the proof of Theorem 5 becomes even simpler than the proof of Theorem 2.5 because now \( M \) has codimension one in \( N \) and therefore one does not have the \( D^2k \)-component (\( z \)-coordinates). The condition (8), which the embedding \( M \to \mathbb{R}^5 \) must satisfy, is an immediate consequence of the fact that the embedding is nowhere tangent to the characteristics or equivalently that we have a symplectic embedding \( N \to \mathbb{R}^5 \). Finally, the argument used in [Gi1] to prove Theorem 2.1 literally applies to Theorem 1.

4. A list of Hamiltonian flows without periodic orbits

In this section we list all examples (known to the author) of smooth Hamiltonian systems on symplectic manifolds \((V, \omega)\) with a compact regular energy level having no periodic orbits. This list divides in two, depending on whether or not the cohomology class of the symplectic form (near the level) is exact.
Case 1: The form $\omega$ is not required to be exact. In this case, one can take the torus $V = T^{2n}$, $2n \geq 4$, with an irrational translation-invariant symplectic structure $\omega$. Then choose a Hamiltonian $H$ on $V$ so that the level $\{H = 1\}$ is the union of two standard embedded tori $T^{2n-1} \subset T^{2n}$. Since $\omega$ is irrational, the characteristics of $\omega|_{T^{2n-1}}$ form an quasiperiodic flow on $T^{2n-1}$. Such a flow obviously has no periodic orbits. In fact, for a suitable choice of $H$, none of the levels $\{H = c\}$ with $c \in (0.5, 1.5)$ carries a periodic orbit. This example is due to Zehnder [Ze]. As shown by M. Herman, the flow in question exhibits remarkable stability properties [Her1], [Her2].

Case 2: The form $\omega$ is exact near the energy level. In this case we should distinguish whether $\dim V = 4$ or $\dim V \geq 6$.

When $\dim V = 4$, the only known example is the horocycle flow described as a Hamiltonian system in Section 3. (Here $V = T^*\Sigma$, where $\Sigma$ is a compact surface with a metric of constant negative curvature $-1$, $\omega = \omega_1$ is the twisted symplectic form, and $H$ is the standard metric Hamiltonian.) It is an old result of Hedlund that the horocycle flow has no periodic orbits on $M = \{H = 1\}$ [He]. The horocycle flow naturally arises as a Hamiltonian system for the motion of a charge in a magnetic field on the surface $\Sigma$. This is the only known such system without periodic orbits. (See [Gi2] and [Gi3] for a detailed discussion.)

Observe that $N$ is $G \setminus \text{PSL}(2, \mathbb{R}) \times (1-\epsilon, 1+\epsilon)$ with $G = \pi_1(\Sigma)$. Then $H$ becomes the projection to the second component. The flow we just described is the Hamiltonian flow of $H$ with respect to some symplectic form. Instead of $G = \pi_1(\Sigma)$ we can take any discrete subgroup such that the quotient is compact and smooth.

An example of a $C^1$-smooth divergence-free vector field on $S^3$ having no periodic orbits is due G. Kuperberg, [KuG], as well as the construction of $C^\infty$-smooth volume preserving flows with a finite number of periodic orbits on closed three-manifolds. These examples are more along the lines of our Theorem 5 rather than genuine Hamiltonian flows.

Finally, when $\dim V \geq 6$, we have the flows on $\mathbb{R}^{2n}$ obtained in the present paper and [Gi1] by means of inserting a symplectic plug to kill periodic orbits on a given energy level. When $2n \geq 8$, there is an alternative construction of such a plug due to M. Herman [Her3] as the symplectization of Wilson’s plug, [Wi]. These methods allow one to use instead of an ellipsoid any compact hypersurface in $\mathbb{R}^{2n}$ (with a finite number of closed characteristics) as the initial manifold $Q$. For instance, the methods apply to non simply connected hypersurfaces.
found by Laudenbach [La]. Finally, one can combine these examples by employing, when possible, one of the above manifolds as the core of the plug to find new flows without closed orbits or with a finite number of them. (See Remark 3.6 (ii) of [Gi1].) For instance, taking $S^1$ as the core, Cieliebak [Ci] constructed embeddings $S^{2n-1} \subset \mathbb{R}^{2n}$, $2n \geq 4$, with a very interesting location and geometry of closed characteristics.

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THE HAMILTONIAN SEIFERT CONJECTURE IN $\mathbb{R}^6$

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