Fermionic T-duality and Morita Equivalence

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ABSTRACT

In this paper we investigate the relationship between the so-called fermionic T-duality and the Morita equivalence of noncommutative supertori. We first get an action satisfying the BRST invariance under nonvanishing constant R-R and NS-NS backgrounds in the hybrid formalism. We investigate the effect of bosonic T-duality transformation together with fermionic T-duality transformation in this background and look for the resultant symmetry of transformations. We find that the duality transformations correspond to Morita equivalence of noncommutative supertori. In particular, we obtain the symmetry group $SO(2, 2, \mathbb{V}_0^0)$ in two dimensions, where $\mathbb{V}_0^0$ denotes Grassmann even number whose body part belongs to $\mathbb{Z}$.
1 Introduction

One typical symmetry of string theory is T-duality. It originates from the symmetry of string worldsheet action under a shift of string coordinate and relates string theories on generically different target space-time backgrounds. At the level of string spectrum, T-duality exchanges the momentum and the winding mode. Recently, besides this usual T-duality which is now referred as the bosonic T-duality, a different kind of T-duality has been proposed in the process of understanding the dual superconformal symmetry displayed by planar scattering amplitudes of super Yang-Mills theory from the viewpoint of superstring theory \cite{1, 2}. Its origin is the symmetry of tree level superstring theory under a shift of fermionic coordinate rather than bosonic one, and thus it is dubbed fermionic T-duality. Similar to the bosonic T-duality, the fermionic T-duality maps superstring theory on one supersymmetric background to that on another supersymmetric background.

As a new kind of duality, one may be curious about the fermionic T-duality itself. Indeed, there have been works exploring properties and various aspects of the fermionic T-duality as follows. It has been shown that some basic symmetry structures of pure spinor string theory are preserved under the fermionic T-duality transformation up to one-loop level \cite{3}. The self-duality of the supercoset sigma model description of superstring has been investigated in \cite{4}. In the context of supergravity, the problem of complexification of supergravity fields after the fermionic T-duality has been considered in \cite{5}. Supersymmetric generalization of duality, superduality, which may connect to the fermionic T-duality has been also given \cite{6}.

As alluded to above, the fermionic T-duality is quite similar to the bosonic one from the worldsheet viewpoint. Actually, this is also the case in the canonical formulation of the fermionic T-duality \cite{7}, where the duality transformation is formally represented as the exchange of momentum and winding. If we now recall the well established fact that the bosonic T-duality is related to the Morita equivalence of noncommutative tori \cite{8, 9}, such similarity opens up the possibility of uncovering the mathematical structure of the fermionic T-duality via Morita equivalence. We note that in the bosonic noncommutative torus case the metric is flat and the background NS-NS B-field is constant. Incidentally, the NS-NS B-field is related with noncommutativity of space and with the T-duality \cite{10, 11}, while
the graviphoton flux is related with nonanticommutativity of space \[12,13\] and with the fermionic T-duality \[1\]. In this paper, we will investigate underlying mathematical structure of the fermionic T-duality with a flat metric, a constant B-field, and a constant self-dual graviphoton flux.

Now, let us give a brief historical description of the bosonic T-duality in connection with noncommutativity. Connes, Douglas, and Schwarz \[10\] first showed that two dimensional noncommutative tori can emerge from toroidal compactifications of M(atrix) theory with nonvanishing NS-NS field backgrounds. Then, Rieffel and Schwarz \[8\] showed that the actions of the group $SO(n, n, \mathbb{Z})$ on an antisymmetric $n \times n$ matrix which represents noncommutativity parameters for an $n$-dimensional noncommutative torus yield Morita equivalent tori. A subsequent work by Schwarz \[9\] showed that compactifications on Morita equivalent tori are physically equivalent, corresponding to T-duality in string theory. This bosonic T-duality usually connects different NS-NS field backgrounds. However note that, as was shown in the Green-Schwarz formalism \[14\] as well as in the pure spinor formalism \[15\], even the bosonic T-duality transformations can connect different R-R field backgrounds. Therefore a decisive factor for the bosonic or fermionic T-duality is not the background fields but the symmetry transformations via which dual theories are connected: in the bosonic T-duality case, it is a shift symmetry along bosonic coordinates, and in the fermionic T-duality case, it is a shift symmetry along fermionic coordinates.

Since the bosonic T-duality is related with the Morita equivalence of noncommutative tori representing different NS-NS backgrounds, we wonder whether the fermionic T-duality relating different R-R field backgrounds \[1\] is related with the Morita equivalence of noncommutative supertori \[16\]. For this purpose, we investigate the fermionic T-duality transformations under the presence of both R-R and NS-NS background fields. Then we compare the obtained symmetry of the above T-duality transformations with the symmetry between noncommutative supertori \[17\] related by Morita equivalence. In a flat geometry with a constant B-field, and a constant self-dual graviphoton flux, we show that the fermionic T-duality corresponds to the Morita equivalence of noncommutative supertori in two and four

\[1\] We would like to note that the bosonic T-duality changes the form degree of the R-R fields in the usual case while the fermionic T-duality shifts the value of the R-R fields.
This paper is organized as follows. In section 2, we investigate the fermionic T-duality transformations in the presence of NS-NS and R-R background fields. In section 3, we look into the relationship between this duality symmetry and the Morita equivalence of noncommutative supertori. We conclude with discussion in section 4.

2 Bosonic and Fermionic T-duality in NS-NS and R-R Backgrounds

We start with type II superstring compactified on Calabi-Yau three-fold, where the background of the constant NS-NS B-field $B_{\mu\nu}$ and the constant self-dual graviphoton field strength $F^{\alpha\beta}$ are turned on in four-dimensional spacetime. We consider the case such that the closed string metric $g_{\mu\nu}$ and the dilaton $\phi$ are also constant. Then we neglect the dilaton coupling $\sim \int d^2z \phi R^{(2)}$. The worldsheet action can be explicitly written down using the hybrid formalism \[18\] (or pure spinor formalism in four dimensions \[19\]) as

$$S = \frac{1}{2\pi\alpha'} \int d^2z \left[ \frac{1}{2} (g_{\mu\nu} + 2\pi\alpha' B_{\mu\nu}) \partial X^\mu \partial X^\nu + \tilde{p}_\alpha \partial \tilde{\theta}^\alpha + \tilde{p}_\dot{\alpha} \partial \tilde{\bar{\theta}}^{\dot{\alpha}} + \tilde{q}_\alpha \tilde{\partial} \tilde{\theta}^\alpha + \tilde{\bar{q}}_{\dot{\alpha}} \tilde{\partial} \tilde{\bar{\theta}}^{\dot{\alpha}} + 2\pi\alpha' F^{\alpha\beta} q_\alpha \tilde{q}_{\dot{\beta}} \right] + S_C, \tag{1}$$

where $\mu = 0, \ldots, 3$ and $\alpha, \dot{\alpha} = 1, 2$. $p$ denotes the conjugate momentum of fermionic coordinate $\theta$. We use the tilde to express the worldsheet chirality, while the bar is used to express the chirality in four dimensions. $S_C$ consists of the action for the chiral bosons \[18\] (or the action for the pure spinor in the pure spinor formalism \[19\]) and the action for the compactified direction. $q_\alpha, \tilde{q}_\alpha$ are the chiral supercharges as worldsheet variables, defined by

$$q_\alpha = -\bar{p}_\alpha - i\sigma^{\mu}_{\alpha\dot{\alpha}} \tilde{\theta}^\alpha \partial X^\mu + \frac{1}{2} \tilde{\partial} \partial \theta_\alpha - \frac{3}{2} \tilde{\partial} (\theta_\alpha \bar{\theta}),$$

$$\tilde{q}_\alpha = -\bar{\tilde{p}}_\alpha - i\sigma^{\mu}_{\alpha\dot{\alpha}} \tilde{\bar{\theta}}^{\dot{\alpha}} \partial X^\mu + \frac{1}{2} \tilde{\partial} \partial \tilde{\theta}_\alpha - \frac{3}{2} \tilde{\partial} (\tilde{\theta}_\alpha \tilde{\bar{\theta}}). \tag{2}$$

Note that in the case of $B_{\mu\nu} = 0$, the action (1) is reduced to the one discussed in \[20, 12, 13\].

We introduce the chiral coordinate $Y^\mu$ as

$$Y^\mu = X^\mu + \partial \theta^\alpha \sigma^{\mu}_{\alpha\dot{\alpha}} \tilde{\theta}^\alpha + \partial \bar{\theta}^{\dot{\alpha}} \sigma^{\mu}_{\alpha\dot{\alpha}} \tilde{\bar{\theta}}^{\dot{\alpha}}. \tag{3}$$
The worldsheet action (1) can be rewritten in terms of $Y^\mu$ as

$$S = \frac{1}{2\pi\alpha'} \int d^2 z \left[ \frac{1}{2} (g_{\mu\nu} + 2\pi\alpha' B_{\mu\nu}) \partial Y^\mu \partial Y^\nu - q_\alpha \tilde{\partial} \theta^\alpha + \tilde{d}_\alpha \tilde{\partial} \tilde{\theta}^\alpha - \tilde{d}_\alpha \tilde{\partial} \tilde{\theta}^\alpha + \tilde{d}_\alpha \tilde{\partial} \tilde{\theta}^\alpha + 2\pi\alpha' F^{\alpha\beta} q_\alpha \tilde{q}_\beta \right] + \text{(surface term)},$$

where $\tilde{d}_\alpha$, $\tilde{\partial} \tilde{\theta}^\alpha$ are the anti-chiral supercovariant derivatives as worldsheet variables, defined by

$$\tilde{d}_\alpha = \tilde{p}_\alpha - i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \partial X_\mu - \theta \tilde{\partial} \tilde{\theta}_\alpha + \frac{1}{2} \tilde{\theta}_\alpha \partial (\theta \tilde{\theta}),$$

$$\tilde{\partial} \tilde{\theta}^\alpha = \tilde{\partial} \tilde{\theta}^\alpha - i\tilde{\theta}^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \tilde{\partial} X_\mu - \tilde{\theta} \tilde{\partial} \tilde{\theta}_\alpha + \frac{1}{2} \tilde{\theta}_\alpha \tilde{\partial} (\tilde{\theta} \tilde{\theta}).$$

In the action (4), $q_\alpha$, $\tilde{q}_\alpha$ and $\tilde{d}_\alpha$, $\tilde{\partial} \tilde{\theta}^\alpha$ are regarded as the conjugate momenta of $\theta$'s and are independent of $Y^\mu$. Then the action is now quadratic in terms of these variables and there is no backreaction. This consistency can be also explained if we assume that the graviphoton field strength $F^{\alpha\beta}$ comes from the R-R five-form field strength in ten-dimensional type IIB superstring. This assumption is consistent with the fact that supersymmetric Yang-Mills theory on non(anti)commutative $\mathcal{N} = 1$ superspace, in which non(anti)commutativity is induced by $F^{\alpha\beta}$, is reproduced by the coupling between D-branes and the R-R five-form field strength [21]. This coupling can be determined through the calculation of the worldsheet disk amplitudes. Under the assumption, it is shown that the backgrounds satisfy the equation of motion. On the other hand, other R-R backgrounds induce the different kind of deformation, such as $\Omega$-background deformation from R-R three-form field strength [22].

We consider the bosonic and fermionic T-duality on this system. Here we restrict ourselves to the case that the worldsheet does not have any nontrivial cycle because the fermionic T-duality may not be a symmetry for the worldsheet with nonzero genus as discussed in [1]. Since in the worldsheet action (1) the bosonic part (first line) and the fermionic part (second line) are decoupled, we can apply the Buscher procedure [23] to two parts separately. For bosonic part, we decompose $Y^\mu = (\hat{Y}^i, \tilde{Y}^l)$ and we apply the duality transformation to $\hat{Y}^i$. We also decompose the NS-NS background $E_{\mu\nu} = \frac{1}{2\pi\alpha'} g_{\mu\nu} + B_{\mu\nu}$ as

$$E_{\mu\nu} = \begin{pmatrix} E_a & E_b \\ E_c & E_d \end{pmatrix}.$$
Since $E_{\mu\nu}$ is constant, we have a shift isometry $\hat{Y}^i \rightarrow \hat{Y}^i + y^i$. We rewrite the bosonic part of the worldsheet action (4) by introducing the gauge fields $A$ and $\tilde{A}$ with the constraint $\partial A - \partial \tilde{A} = 0$:

$$S_B = \frac{1}{2} \int d^2z \left[ A^i E_a \tilde{A} + A^i E_b \tilde{\partial Y} + \partial \tilde{Y}^t E_c \tilde{A} + \partial \tilde{Y}^t E_d \tilde{\partial Y} + \hat{Y}^t (\partial A - \partial \tilde{A}) \right]. \quad (7)$$

By integrating out $\hat{Y}'$, we have $A = \partial \tilde{Y}$ and $\tilde{A} = \tilde{\partial \tilde{Y}}$ for some $\tilde{Y}$ and the above action comes back to the original form. Instead, by reshuffling the terms, one can show that

$$S_B = \frac{1}{2} \int d^2z \left[ (A^i + (\partial \tilde{Y} E_c + \partial Y^\mu E_a^{-1}) E_a^t (\tilde{A} + E_a^{-1} (E_b \tilde{\partial Y} - \tilde{Y}'))
\right.$$ 

$$+ \partial \tilde{Y}^t E_a^{-1} \tilde{\partial Y}' - \partial \tilde{Y}^t E_a^{-1} E_b \tilde{\partial Y} + \partial \tilde{Y}^t E_c E_a^{-1} \tilde{\partial Y}' + \partial \tilde{Y}^t (E_d - E_c E_a^{-1} E_b) \tilde{\partial Y} \right], \quad (8)$$

where the first line in (8) is integrated out and it generates the shift of dilaton $\phi$ as $\phi \rightarrow \phi - \frac{1}{2} \log \det E_a$. Then the dilaton is again constant. From the second line one can read off the transformation of $E_{\mu\nu}$ as

$$E_{\mu\nu} \rightarrow E'_{\mu\nu} = \begin{pmatrix} E_a^{-1} & -E_a^{-1} E_b \\ E_c E_a^{-1} & E_d - E_c E_a^{-1} E_b \end{pmatrix}. \quad (9)$$

Note that the T-duality transformation which we examined above is slightly different from the conventional one since we take the transformation for the chiral coordinate $Y^\mu$, while we usually consider the transformation for $X^\mu$. The transformation of $E_{\mu\nu}$ is the same as the conventional one but $F^{\alpha\beta}$ is unchanged in our case because of the decoupling between the bosonic part and the fermionic part in the action (4).

For fermionic part, we apply the fermionic T-duality transformation given in (1). Although we have the graviphoton background $F^{\alpha\beta}$, it turns out that one can perform the transformation in the same way as in (1) because the term containing $F^{\alpha\beta}$ depends only on $q_\alpha$ and $\tilde{q}_\alpha$. Since the action (4) does not contain the square of the derivative, we add the following surface term to the action (4)

$$S_b = \frac{1}{(2\pi \alpha')^2} \int d^2z (f^{-1})_{\alpha\beta} (\partial \theta^\alpha \tilde{\partial} \tilde{\theta}^\beta - \tilde{\partial} \tilde{\theta}^\alpha \partial \theta^\beta), \quad (10)$$

In conventional T-duality in terms of $X^\mu$, the rank of R-R fields is changed due to the flipping of the spinor chirality.
where $f^{\alpha \beta} = f^{\beta \alpha}$ is constant. Since the background preserves chiral supersymmetry which can be regarded as a shift isometry in the fermionic direction $\theta^\alpha \to \theta^\alpha + \rho^\alpha$, $\tilde{\theta}^\alpha \to \tilde{\theta}^\alpha + \tilde{\rho}^\alpha$, then one can dualize $\theta^\alpha$ and $\tilde{\theta}^\alpha$ by introducing the fermionic gauge fields $(A^\alpha, \tilde{A}^\alpha)$ and $(\hat{A}^\alpha, \hat{\tilde{A}}^\alpha)$ with the constraints $\partial A^\alpha - \partial \tilde{A}^\alpha = \partial \hat{A}^\alpha - \partial \hat{\tilde{A}}^\alpha = 0$ as

$$S_F = \frac{1}{2\pi \alpha'} \int d^2z \left[ -q_\alpha \tilde{A}^\alpha - \tilde{q}_\alpha \hat{A}^\alpha + (2\pi \alpha')^{-1}(f^{-1})_{\alpha \beta}(A^\alpha \hat{\tilde{A}}^\beta - \hat{A}^\alpha \tilde{A}^\beta) + 2\pi \alpha' F^{\alpha \beta} q_\alpha \tilde{q}_\beta + \chi_\alpha (\partial A^\alpha - \partial \tilde{A}^\alpha) + \hat{\chi}_\alpha (\partial \hat{A}^\alpha - \partial \hat{\tilde{A}}^\alpha) + \tilde{\partial}_\alpha \tilde{\theta}^\alpha + \tilde{\bar{\partial}}_\alpha \tilde{\bar{\theta}}^\alpha \right], \quad (11)$$

where $\chi_\alpha$ and $\hat{\chi}_\alpha$ are the fermionic Lagrange multipliers for the above constraints. By integrating out the fermionic gauge fields, the action (11) becomes

$$S_F = \frac{1}{2\pi \alpha'} \int d^2z \left[ -2\pi \alpha' f^{\alpha \beta} \left( \partial \tilde{\chi}_\alpha \tilde{\chi}_\beta - \tilde{\partial} \tilde{\chi}_\alpha \partial \chi_\beta \right) + 2\pi \alpha' \tilde{q}_\alpha f^{\alpha \beta} \partial \chi_\beta + 2\pi \alpha' q_\alpha f^{\alpha \beta} \tilde{\partial} \tilde{\chi}_\beta + 2\pi \alpha' (F^{\alpha \beta} + f^{\alpha \beta}) q_\alpha \tilde{q}_\beta + \tilde{d}_\alpha \tilde{\partial} \tilde{\theta}^\alpha + \tilde{\bar{d}}_\alpha \tilde{\bar{\partial}} \tilde{\bar{\theta}}^\alpha \right]. \quad (12)$$

The first term is the surface term and we drop it out. The dual fermionic coordinates are identified as

$$\theta'^\alpha = -2\pi \alpha' f^{\alpha \beta} \chi_\beta, \quad \tilde{\theta}'^\alpha = -2\pi \alpha' f^{\alpha \beta} \hat{\chi}_\beta. \quad (13)$$

$F^{\alpha \beta}$ is just shifted by the constant $f^{\alpha \beta}$ as

$$F^{\alpha \beta} \to F'^{\alpha \beta} = F^{\alpha \beta} + f^{\alpha \beta}. \quad (14)$$

We also have the constant shift of the dilaton from the integration of the fermionic gauge fields.

In the next section, we will show that the bosonic and fermionic T-duality transformations (9), (14) and other symmetries are combined into the supersymmetric version of Morita equivalence. In the case of $F^{\alpha \beta} = 0$, when we put the D-branes filling the four-dimensional spacetime, the backgrounds $B_{\mu \nu}$ induces noncommutativity on the D-branes [24, 11]. Under the limit

$$g_{\mu \nu} \sim (\alpha')^2, \quad B_{\mu \nu} : \text{finite for } \alpha' \to 0, \quad (15)$$

the non(anti)commutativity among the coordinates are

$$[X^\mu, X^\nu] = \Theta^{\mu \nu} \quad \text{or} \quad [Y^\mu, Y^\nu] = \Theta^{\mu \nu}. \quad (16)$$
Here the noncommutativity parameters $\Theta^{\mu\nu}$ ($\Theta^t = -\Theta$) and the open string metric $G_{\mu\nu}$ are respectively obtained as \[24, 11\]

$$\Theta^{\mu\nu} = (B^{-1})^{\mu\nu}, \quad G_{\mu\nu} = -(2\pi\alpha')^2(Bg^{-1}B)_{\mu\nu}. \quad (17)$$

Then the duality transformations of the parameters $\Theta^{\mu\nu}$ can be obtained from (9) as

$$\Theta^{\mu\nu} = \begin{pmatrix} \Theta_a & \Theta_b \\ \Theta_c & \Theta_d \end{pmatrix} \rightarrow \Theta'^{\mu\nu} = \begin{pmatrix} \Theta_a^{-1} & -\Theta_a^{-1}\Theta_b \\ \Theta_c\Theta_a^{-1} & \Theta_d - \Theta_c\Theta_a^{-1}\Theta_b \end{pmatrix} \quad (18)$$

The above and other symmetries are combined into $SO(n, n, \mathbb{Z})$ group which gives the Morita equivalence of noncommutative tori \[8, 9\].

When $F^{\alpha\beta}$ is also turned on, the fermionic coordinates become nonanticommutative \[20, 12, 13\] as

$$[Y^\mu, Y^\nu] = \Theta^{\mu\nu}. \quad \{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta}, \quad \text{(others)} = 0, \quad (19)$$

where the nonanticommutativity parameter $C^{\alpha\beta}$ is defined by

$$C^{\alpha\beta} = (2\pi\alpha')^2 F^{\alpha\beta}. \quad (20)$$

Then if we assume that we can apply the result of the duality transformation \[14\], $C^{\alpha\beta}$ transforms as

$$C^{\alpha\beta} \rightarrow C'^{\alpha\beta} = C^{\alpha\beta} + e^{\alpha\beta}, \quad (21)$$

where $e^{\alpha\beta} = (2\pi\alpha')^2 f^{\alpha\beta}$. Here we note that under the fermionic T-duality transformation adding surface term \[13\] and removing the first term from \[12\] change the boundary condition of the D-branes (see appendix). However, if we start from the full ten-dimensional pure spinor formalism, we do not need to add or remove the surface term since the square of the derivative of $\theta$ is already present, and thus the problem with the boundary condition does not occur. It turns out that in this case the R-R background is shifted by the square of the constant Killing spinor \[1\]. So we use the transformation rule \[21\] even under the presence of the D-branes.

\[3\]In our case, the Killing spinor is constant in ten-dimensional pure spinor formalism. Then the shift is also constant.
3 Relation to Morita Equivalence

Before considering the relationship between the fermionic T-duality and Morita equivalence, we first briefly review the Morita equivalence of noncommutative super tori \[16\]. An \( n \)-dimensional noncommutative torus \((A^n_\theta)\) is an associative algebra with involution having unitary generators \( U_1, \ldots, U_n \) obeying the relations

\[
U_i U_j = e^{2\pi i \theta_{ij}} U_j U_i, \quad i, j = 1, \ldots, n, \tag{22}
\]

where the noncommutativity parameters \( \theta_{ij} \) form a real \( n \times n \) anti-symmetric matrix \( \Theta \).

The endomorphism algebra of the module of noncommutative torus is Morita equivalent to the given noncommutative torus. When \( U_i \)'s belong to the given noncommutative torus and \( Z_i \)'s belong to its endomorphism algebra, then \( Z_i \)'s commute with the \( U_i \)'s, i.e.,

\[
U_i Z_j = Z_j U_i \quad \text{where} \quad Z_i Z_j = e^{2\pi i \theta_{ij}} Z_j Z_i, \quad i, j = 1, \ldots, n. \tag{23}
\]

Let \( D \) be a lattice in \( G = M \times \hat{M} \), where \( M = \mathbb{R}^p \times \mathbb{Z}^q \) with \( 2p + q = n \) and \( \hat{M} \) is its dual. The embedding map \( \Phi \) under which \( D \) is the image of \( \mathbb{Z}^n \) determines a projective module \( E \) on which the algebra of noncommutative torus acts. In the Heisenberg representation the operators \( U_i \) can be defined by

\[
U_{(m,\hat{s})} f(r) = e^{2\pi i <r, \hat{s}>} f(r + m), \quad m, r \in M, \, \hat{s} \in \hat{M}, \, f \in E, \tag{24}
\]

where we denoted \( U_i := U_{\vec{e}_i} \) with \( \vec{e}_i := (m, \hat{s}) \), and \( <r, \hat{s}> \) is usual inner product between \( M \) and \( \hat{M} \). In this representation, we get

\[
U_{(m,\hat{s})} U_{(n,\hat{t})} = e^{2\pi i (<m,\hat{t}> - <n,\hat{s}>)} U_{(n,\hat{t})} U_{(m,\hat{s})}. \tag{25}
\]

Denoting the basis as \( \vec{e}_i := (m, \hat{s}) \), \( \vec{e}_j := (n, \hat{t}) \) and the embedding map as \( \Phi = (\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n) \), then \( \theta_{ij} \) can be expressed as

\[
\theta_{ij} = \vec{e}_i \cdot J_0 \vec{e}_j \quad \text{where} \quad J_0 = \begin{pmatrix} 0 & I_p \\ -I_p & 0 \end{pmatrix}, \tag{26}
\]

\[\text{Here we assume } n \text{ is even and } n = 2p.\]
and thus \[ \Theta = \Phi^t J_0 \Phi. \] (27)

Therefore one can see that the condition for Morita equivalent dual torus \[ \Phi \] can be written as \[ \Phi^t J_0 \Phi' = K, \] (28)

where \( \Phi \) is the embedding map of a given torus and \( \Phi' \) is the embedding map of the dual torus, and \( K \) is an \( n \times n \) matrix whose elements belong to \( \mathbb{Z} \).

In the supersymmetric case, one can put the relation (28) into the following form \[ \tilde{\Phi}^t \tilde{J}_0 \tilde{\Phi}' = B^t J_0 B' + F^t \hat{J}_0 F' = \tilde{K}, \] where \( \tilde{J}_0 = \begin{pmatrix} 0 & I_r \\ I_r & 0 \end{pmatrix} \). (29)

Here \( \tilde{\Phi} := \begin{pmatrix} B \\ F \end{pmatrix} \) and \( \tilde{\Phi}' := \begin{pmatrix} B' \\ F' \end{pmatrix} \) are the embedding maps of the given supertorus and the dual supertorus respectively, the elements of the matrix \( \tilde{K} \) belong to \( \mathbb{Z} \), and \( r \) depends on the number of supersymmetry generators \[ \Phi \text{ and } \Phi' \text{ in Eq. (28) can be regarded as different sets of basis vectors which yield commuting generators } U_i \text{'s and } Z_i \text{'s. Thus the inner product between the two bases which have the same rank should be nonsingular. Therefore together with the condition for a dual torus, } K \text{ should be a nonsingular } n \times n \text{ integer matrix. The same holds for } \tilde{\Phi} \text{ and } \tilde{\Phi}' \text{ with } \tilde{K}. \]
where we used $J_0^{-1} = -J_0$. This shows that the duality condition \((2.21)\) does not restrict the soul part $F'$ of the dual map $\tilde{\Phi}'$ in the defining relation \((3.22)\) for the noncommutativity parameters $\tilde{\Theta}'$ of the dual torus. Therefore, as it was stressed in \([10]\), when two $\tilde{\Theta}$'s have the same body parts and only differ over the soul parts, then the two corresponding tori are Morita equivalent.

At this point, we want to consider the result obtained by the T-dual transformations in the previous section. There the noncommutativity parameters were given by \((1.13)\). In the present notation $\tilde{\Theta}^{\mu\nu}, C^{\alpha\beta}$ together give $\tilde{\Theta} = \Theta \oplus C$. Namely, $\Theta$ in the previous section corresponds to $\Theta_B$ here, and $C$ corresponds to $\Theta_F$. If we only consider the T-dual transformation of $\hat{Y}^i$ coordinates in the previous section, then we end up with $\Theta \rightarrow \Theta^{-1}$ as it was shown in \((1.18)\). This agrees with \((3.22)\), which is what we get when both $F$ and $F'$ vanish, up to an allowed $GL(n, \mathbb{Z})$ transformation of $\tilde{K}$ as we mentioned above. For the fermionic part, $C$ changes by a shift in \((2.1)\), which agrees with our above statement that the two tori with the same body parts are Morita equivalent.\(^6\) Therefore the T-dual, both bosonic and fermionic, transformations correspond to the Morita equivalence of noncommutative supertori represented by $\Theta$'s and $C$'s.

In order to analyze the symmetry structure of the above Morita equivalence we first consider the following linear fractional transformation, an action of an element of $SO(n, n, \mathcal{V}_Z^0)$ where $\mathcal{V}_Z^0$ denotes Grassmann even number whose body part belongs to $\mathbb{Z}$.

$$g \tilde{\Theta} := (A \tilde{\Theta} + B)(C \tilde{\Theta} + D)^{-1} \quad \text{with} \quad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SO(n, n, \mathcal{V}_Z^0). \quad (33)$$

Now we consider a case when $g$ becomes $\sigma_n = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$ with $n$ denoting the dimension of a torus:

$$\sigma_n \tilde{\Theta} = \tilde{\Theta}^{-1} = (\Theta_B + \Theta_F)^{-1} = \tilde{\Theta}_b^{-1} \sum_{m=0}^{\infty} (-\tilde{\Theta}_s \tilde{\Theta}_b^{-1})^m \quad (34)$$

where $\tilde{\Theta}_b$ and $\tilde{\Theta}_s$ are the body and soul parts of $\tilde{\Theta}$, respectively. In order to understand this relation we now express $\tilde{\Theta}'$ directly using the dual embedding map $\tilde{\Phi}'$. From the relation

\(^6\) This conforms with the result in section 2, Eq.\((1.14)\), that $F^{\alpha\beta}$ can be shifted by an arbitrary constant $f^{\alpha\beta}$. Such shift does not affect the physics at the string tree-level \([1]\).
we have
\[ \tilde{\Phi}' = (\tilde{\Phi}' \tilde{J}_0) - \tilde{K}, \quad (35) \]
and thus
\[ \tilde{\Theta}' = \tilde{\Phi}' \tilde{J}_0 \tilde{\Phi}' = \tilde{K}(\tilde{\Phi}' \tilde{J}_0 \tilde{\Phi})^{-1} \tilde{K}, \quad (36) \]
Since
\[ \tilde{J}_0 = \begin{pmatrix} J_0 & 0 \\ 0 & \tilde{J}_0 \end{pmatrix} = \begin{pmatrix} -J_0 & 0 \\ 0 & \tilde{J}_0 \end{pmatrix}, \]
we can write \( \tilde{\Phi}' \tilde{J}_0 \tilde{\Phi}' = -\Theta_B + \Theta_F \). Therefore (36) can be written as
\[ \tilde{\Theta}' = \tilde{K}(\tilde{\Theta}_B + \Theta_F)^{-1} \tilde{K} = -\tilde{K} \tilde{\Theta}_b^{-1} \sum_{m=0}^{\infty} (\hat{\Theta}_s \hat{\Theta}_b^{-1})^m \tilde{K}, \quad (37) \]
where \( \hat{\Theta}_b \) is the body part of \( \Theta_B \) and \( \hat{\Theta}_s \) is the soul part of \(-\Theta_B + \Theta_F\). Note that the body part of \( \Theta_B \) and that of \( \hat{\Theta} \) are the same. Since \( \hat{\Theta}_b^{-1} \) in (37) and \( \tilde{\Theta}_b^{-1} \) in (34) are the same, \( \tilde{\Theta}^{-1} \) is just differ from \( \tilde{\Theta} \) in the soul part up to the action of \( \tilde{K} \in GL(n, \mathbb{Z}) \). This shows that \( \sigma_n \) generates a Morita equivalent torus.

In general, one can apply the bosonic T-dual transformations partially such as to the coordinates \( \hat{Y}^i \), \( 1 \leq i \leq 2p \) with \( 2p < n \) as in (6). We assume now \( n = 2p + q \). Then the noncommutativity parameters \( \Theta \) is given and transformed by (18), where \( \Theta_a \) is \( 2p \times 2p \) and \( \Theta_d \) is \( q \times q \). In this case, we consider \( \sigma_{2p} \in SO(n, n, V^0_L) \) given by
\[ \sigma_{2p} = \begin{pmatrix} 0 & 0 & I_{2p} & 0 \\ 0 & I_q & 0 & 0 \\ I_{2p} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_q \end{pmatrix}. \]
One can easily check that the action of \( \sigma_{2p} \) defined by (33) yields \( \Theta' \) in (18); \( \sigma_{2p} \Theta = \Theta' \). Still we have to show that this transformed \( \Theta' \) corresponds to a Morita equivalent torus. As it was shown in (8) (see also [16]), this transformed \( \Theta' \) can be obtained from a dual embedding
map $\Phi'$ satisfying (28); 

$$
\Phi' = \begin{pmatrix}
J_0(T_a^t)^{-1} & -J_0(T_a^t)^{-1}T_b^t \\
0 & I_q \\
0 & T_c^2
\end{pmatrix},
$$

(38)

where $J_0$ is the $2p \times 2p$ matrix defined before, and $T_a$ is $2p \times 2p$, $T_b$ is $q \times 2p$, $T_c$ is $q \times q$ such that $T_a^tJ_0T_a := -\Theta_a$, $T_b := \Theta_b$, and $\Theta_d := T_c^t - T_c$. Namely one can check that $(\Phi')^tJ\Phi' = \Theta$ where 

$$
J = \begin{pmatrix}
(J_0)_{2p} & 0 & 0 \\
0 & 0 & I_q \\
0 & -I_q & 0
\end{pmatrix},
$$

and $\Phi^tJ\Phi = -\Theta$ when the original embedding map $\Phi$ is given by 

$$
\Phi = \begin{pmatrix}
T_a & 0 \\
0 & I_q \\
T_b & T_c
\end{pmatrix}.
$$

(39)

Therefore $\Theta' = \sigma_{2p}\Theta$ and $\Theta$ are Morita equivalent. Since the noncommutativity parameters with the same body parts yield Morita equivalent tori, we can say that the general T-dual transformations given by (18) and (21) yield Morita equivalent noncommutative supertori.

The fact that the same body parts up to elements in $V^0$ yield equivalent tori dictates us another symmetry action of the following element of $SO(n, n, V_0^0)$ 

$$
\nu(\tilde{N}) = \begin{pmatrix}
I_n & \tilde{N} \\
0 & I_n
\end{pmatrix},
$$

where $\tilde{N}$ is an antisymmetric $n \times n$ matrix whose entries are in $V_0^0$. The action of $\nu(\tilde{N})$ is given by 

$$
\nu(\tilde{N})\tilde{\Theta} = \tilde{\Theta} + \tilde{N}.
$$

(40)

Finally, we consider the “rotation” $\rho(\tilde{R}) \in SO(n, n, V_0^0)$ given by 

$$
\rho(\tilde{R}) = \begin{pmatrix}
\tilde{R}^t & 0 \\
0 & \tilde{R}^{-1}
\end{pmatrix},
$$

where $\tilde{R} \in GL(n, V_0^0)$. It was shown in [16] that the action of the above element to a basis $\{\tilde{E}_i\} (i = 1, 2, \cdots, n)$ for a given torus with $\tilde{\Theta}$ yields an isomorphic torus with $\tilde{\Theta}' = \tilde{R}\tilde{\Theta}\tilde{R}^t$. 

13
Thus we have shown that the three elements of $SO(n, n, \mathcal{V}_0^Z)$, which are $\sigma_n$ (or $\sigma_{2p}$ with $2p \leq n$), $\rho(\tilde{R})$, and $\nu(\tilde{N})$, yield Morita equivalent noncommutative $n$-supertori. Therefore, the bosonic and fermionic T-duality transformations that we considered in section 2 correspond to the Morita equivalence of noncommutative supertori related by the above symmetry transformations. In the $n = 2$ case, the above three elements generate the group $SO(2, 2, \mathcal{V}_2^0)$.

4 Conclusion

In this paper, we show that under the bosonic and fermionic T-duality transformations the relation between the corresponding dual background fields can be dictated by the Morita equivalence of noncommutative supertori that can be constructed with the dual background fields. Especially, when we restrict ourselves to the duality transformations along two coordinate directions only, then we obtain the symmetry group $SO(2, 2, \mathcal{V}_2^0)$ which is the symmetry group of the Morita equivalence of noncommutative supertori in two dimensions. However the problem of the boundary condition for D-branes still remains. We have used the hybrid formalism in four dimensions since it is easier to analyze than the full ten-dimensional pure spinor formalism. The above problem does not appear in the ten-dimensional formalism.

We have discussed the extended T-duality in tree level, i.e. in the worldsheet without genus. In the case of bosonic T-duality, it is the exact symmetry for all order of the genus expansion when the dualizing coordinate $X^i$ is compact. The holonomy of the auxiliary gauge field $\int_C A^i dz + \bar{A}^i d\bar{z}$ gives the winding number, where $C$ is the nontrivial cycle on the worldsheet. For fermionic T-duality, in order to extend it to all genus, one needs the non-periodic fermionic variable satisfying $\theta \rightarrow \theta + \xi_C$ when $\theta$ goes around the cycle $C$ as discussed in [1].

If the superstring background becomes nonconstant, then the symmetry structure of the bosonic and the fermionic T-duality might be different from what we have considered in this paper. It would be interesting to extend to a more general case with extended symmetry of T-duality as mentioned or the supergroup duality considered in [6]. A supersymmetric extension of non-abelian T-duality [25] which is recently extended to R-R background [26]
would be also interesting.

Appendix: Boundary Condition for D-branes

In this appendix, we discuss the boundary condition of the worldsheet action \([11]\) when spacetime-filling D-branes are present. In particular we examine how the surface term \([10]\) changes the boundary condition. Since the surface term \([10]\) consists of the fermionic fields only, we focus on the boundary condition of the fermionic fields. The relevant part of the action is

\[
\frac{1}{2\pi\alpha'} \int d^2z \left( -q_\alpha \partial \theta^\alpha - \tilde{q}_\alpha \partial \tilde{\theta}^\alpha + 2\pi\alpha' F^{\alpha\beta} q_\alpha \tilde{q}_\beta \right).
\]

As it is studied in \([20, 12, 13]\), the fields \(q_\alpha\) and \(\tilde{q}_\alpha\) can be integrated out using their equation of motion

\[
\tilde{\partial} \theta^\alpha = 2\pi\alpha' F^{\alpha\beta} \tilde{q}_\beta, \quad \partial \tilde{\theta}^\alpha = -2\pi\alpha' F^{\alpha\beta} q_\beta,
\]

which leads to the following action

\[
S_{\text{eff}} = \frac{1}{(2\pi\alpha')^2} \int d^2z (F^{-1})_{\alpha\beta} \partial \tilde{\theta}^\alpha \tilde{\theta}^\beta.
\]

The equation of motion for \(\theta^\alpha\) and \(\tilde{\theta}^\alpha\) becomes \(\partial \tilde{\partial} \theta^\alpha = \partial \tilde{\partial} \tilde{\theta}^\alpha = 0\). Then the solution of \(\theta^\alpha\) and \(\tilde{\theta}^\alpha\) are written as the sum of the holomorphic part and the anti-holomorphic part as

\[
\theta^\alpha(z, \tilde{z}) = \theta^\alpha_L(z) + \theta^\alpha_R(z), \quad \tilde{\theta}^\alpha(z, \tilde{z}) = \tilde{\theta}^\alpha_L(z) + \tilde{\theta}^\alpha_R(z).
\]

When we consider the case of open string attached on the D-brane, the surface term in the equation of motion is

\[
(F^{-1})_{\alpha\beta} \left( \partial \tilde{\partial} \theta^\alpha \delta \tilde{\partial}^\beta + \partial \tilde{\partial} \tilde{\theta}^\alpha \delta \theta^\beta \right) \bigg|_{z=\tilde{z}} = 0.
\]

The above is satisfied by choosing the boundary condition as

\[
\theta^\alpha(z = \tilde{z}) = \tilde{\theta}^\alpha(z = \tilde{z}), \quad \tilde{\partial} \theta^\alpha(z = \tilde{z}) = -\tilde{\partial} \tilde{\theta}^\alpha(z = \tilde{z}),
\]

\footnote{The boundary condition for bosonic fields is the same as the case of usual noncommutativity \([11]\) except that \(X^\mu\) is replaced by \(Y^\mu\).}
where the second condition comes from $q_\alpha(z = \tilde{z}) = \tilde{q}_\alpha(z = \tilde{z})$ to preserve the half of supersymmetry. Under the boundary condition (46), the two point correlators are given in [20, 12, 13]. Here we focus the one of them as

$$\langle \theta_L^\alpha(z)\tilde{\theta}_R^\beta(\tilde{w}) \rangle = -\frac{(2\pi\alpha')^2 F^{\alpha\beta}}{\pi i} \log(z - \tilde{w}).$$  \hspace{1cm} (47)

The contribution to the boundary condition from the surface term (11) is

$$\delta S_b = \frac{1}{(2\pi\alpha')^2} \int dx \left( f^{-1} \right)_{\alpha\beta} \left( \partial \tilde{\theta}^\alpha \delta \theta^\beta + \tilde{\partial} \theta^\alpha \tilde{\delta} \tilde{\theta}^\beta + \partial \theta^\alpha \tilde{\delta} \tilde{\theta}^\beta + \tilde{\partial} \tilde{\theta}^\alpha \delta \theta^\beta \right) \bigg|_{z = \tilde{z} = x}.$$  \hspace{1cm} (48)

The first and second terms have the same form with (45) and then cancel each other because of the boundary condition (46). So if we can have

$$\partial \theta^\alpha(z = \tilde{z}) = -\tilde{\partial} \tilde{\theta}^\alpha(z = \tilde{z}),$$  \hspace{1cm} (49)

then the third and fourth terms also cancel each other, the surface term (11) does not change the boundary condition (46). On the other hand if (49) holds, two point correlator (47) must vanish. This leads to contradiction. However, as we mentioned in section 2, if we start from the full ten-dimensional pure spinor formalism, we do not need to add or remove the surface term since the square of the derivative of the fermionic coordinate $\theta$ is already present, and thus the problem with the boundary condition disappears.

**Acknowledgments**

The authors thank KIAS for hospitality during the time that this work was done. This work was supported by the National Research Foundation (NRF) of Korea grants funded by the Korean government (MEST) [NRF-2009-0075129 (E. C.-Y.), NRF-2009-0084601 (H. N.), and NRF-2008-331-C00071 (H. S.)].

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