Cardinal Characteristics of Models of Set Theory

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Abstract

We continue our investigation from [38] of Shelah’s interpretability orders \( \preceq^*_\kappa \) as well as the new orders \( \preceq^*_\times \kappa \). In particular, we give streamlined proofs of the existence of minimal unstable, unsimple and nonlow theories in these orders, and we give a similar analysis of the hypergraph examples \( T_{n,k} \) of Hrushovski [7]. Using the technology of [37], we prove that if \( B \) is a complete Boolean algebra with the \( \lambda \)-c.c., then no nonprincipal ultrafilter on \( U^\lambda \) -saturates any unsimple theory.

1 Introduction

In [34], Shelah introduced the interpretability orders \( \preceq^*_\kappa \) as abstractions of Keisler’s order \( \preceq \). In [38], we cast these orders in terms of cardinal characteristics of nonstandard models of \( ZFC^- \). Furthermore, we introduced new interpretability orders \( \preceq^*_\times \kappa \), which refine \( \preceq^*_\kappa \) and which smooth out many technical difficulties. In this paper, we demonstrate the power of these notions in several applications.

To begin, we review the setup of [38].

Definition 1.1. \( ZFC^- \) is \( ZFC \) without powerset, and with replacement strengthened to collection, and with choice strengthened to the well-ordering principle; this is as in [5].

Say that \( \hat{V} \models ZFC^- \) is an \( \omega \)-model, or is \( \omega \)-standard, if every natural number of \( \hat{V} \) is standard (i.e. has finitely many \( \hat{\in} \)-predecessors).

\( V \) will denote a transitive model of \( ZFC^- \). \( \hat{V} \) will denote an \( \omega \)-nonstandard model of \( ZFC^- \). Frequently \( \hat{V} \) will come from an embedding \( j : V \preceq \hat{V} \), where \( V \) is transitive. Whenever \( \hat{V} \models ZFC^- \), we will identify \( HF \) (the hereditarily finite sets) with its copy in \( \hat{V} \). Other elements of \( \hat{V} \) will usually be decorated with a hat, for instance we write \( \hat{\omega} \) rather than \( (\omega)^V \). Given \( X \subseteq \hat{V} \), we say that \( X \) is an internal subset of \( \hat{V} \) if there is some \( \hat{X} \in \hat{V} \)

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such that $X = \{\hat{y} \in \hat{V} : \hat{y} \in \hat{X}\}$. In this case, we usually identify $X$ with $\hat{X}$ and will write that $X \in \hat{V}$.

Suppose $V \models ZFC^-$ is transitive, and $j : V \preceq \hat{V}$. Say that $X \subseteq \hat{V}$ is pseudofinite (with respect to $\hat{V}$) if there is some $\hat{X} \in \hat{V}$ finite in the sense of $\hat{V}$, with $X \subseteq \hat{X}$. So if $\hat{X} \in \hat{V}$, then $\hat{X}$ is pseudofinite if and only if it is finite in the sense of $\hat{V}$.

Suppose $M \in V$ is an $L$-structure (it follows that $L \in V$). Note that $j(M)$ is a $j(L)$-structure, where possibly some of the symbols of $j(L)$ are nonstandard; let $j_{\text{std}}(M)$ be the “reduct” to $L$. Say that $j_{\text{std}}(M)$ is $\kappa$-pseudosaturated if for every pseudofinite $A \subseteq j_{\text{std}}(M)$ with $|A| < \kappa$, and for every $n < \omega$: every type $p(\pi) \in S^n(A)$ is realized in $j_{\text{std}}(M)$. (It is enough to take $n = 1$.)

**Convention.** We operate entirely in $ZFC$; thus everything is a set, including formulas. Whenever $T$ is a complete countable theory, we suppose $T$ comes equipped with an an injection from the symbols of $T$ into $\omega$. In particular, whenever $T \in V \models ZFC^-$, $T$ is countable in $V$.

**Remark 1.2.** As an exercise in terminology (Lemma 3.10 of [38]), note that when $\kappa > \aleph_0$, we have that $j_{\text{std}}(M)$ is $\kappa$-pseudosaturated if and only if every pseudofinite partial type over $j_{\text{std}}(M)$ of cardinality less than $\kappa$ is realized in $j_{\text{std}}(M)$. To parse this: a partial type over $j_{\text{std}}(M)$ is pseudofinite if it is contained in some set $\hat{X} \in \hat{V}$ which is finite in the sense of $\hat{V}$; we can suppose by the Separation schema that $\hat{X}$ is a finite set of $j(L)$-formulas over $j(M)$, but we cannot arrange for all the formulas in $\hat{X}$ to be standard (they may involve nonstandard symbols, and may be of nonstandard length), and the crux of the issue is whether we can arrange that $\hat{V} \models \"X is consistent\"$.

The reader may take the following as the definition of $\preceq_{\lambda, \kappa}^*$, for our purposes; this is Lemma 3.6 of [38]. Following [20], we consider every structure to be $1$-saturated.

**Lemma 1.3.** Suppose $\lambda$ is an infinite cardinal, $\kappa$ is an infinite cardinal or $1$, and $T_0, T_1$ are complete countable theories. Then the following are equivalent:

(A) $T_0 \preceq_{\lambda, \kappa}^* T_1$;

(B) There is some countable transitive $V \models ZFC^-$ with $T_0, T_1 \in V$, and some $M_i \models T_i$ both in $V$, such that whenever $j : V \preceq \hat{V}$, if $\hat{V}$ is $\kappa$-saturated and $\omega$-nonstandard and if $j_{\text{std}}(M_i)$ is $\lambda^+$-saturated, then $j_{\text{std}}(M_0)$ is $\lambda^+$-saturated.

We say that $T_0 \preceq_{\kappa}^* T_1$ if $T_0 \preceq_{\lambda, \kappa}^* T_1$ for all $\lambda$. Note that we follow the original indexing system of Shelah [34], and so $\preceq_{\aleph_0}$ refines Keisler’s order $\succeq$.

It is a serious annoyance that the choice of $M_i \in V$ matters, and indeed is often delicate. The following remedy is the conjunction of Theorem 3.13 ($\kappa > \aleph_0$) and Corollary 7.7 ($\kappa = \aleph_0$) from [38].

**Theorem 1.4.** Suppose $V \models ZFC^-$ is transitive and $j : V \preceq \hat{V}$ is $\omega$-nonstandard, and $T \in V$ is a complete countable theory, and $M_0, M_1 \in V$ are two models of $T$. Suppose $\kappa$ is an infinite cardinal. Then $j_{\text{std}}(M_0)$ is $\kappa$-pseudosaturated if and only if $j_{\text{std}}(M_1)$ is.
This gives a new way of viewing Keisler’s fundamental theorem on saturation of ultrapowers, since if \( U \) is an ultrafilter on \( \mathcal{P}(\lambda) \), and if we let \( j : V \preceq \hat{V} := V^\lambda/U \) be the Loš embedding, then for every \( M \in V, M^\lambda/U \cong j_{\text{std}}(M) \). Further, if \( U \) is \( \lambda \)-regular then every subset of \( V^\lambda/U \) of size at most \( \lambda \) is pseudofinite, and hence \( j_{\text{std}}(M) \) is \( \lambda^+ \)-saturated if and only if it is \( \lambda^+ \)-pseudosaturated.

Theorem 1.4 suggests the following tweak to the interpretability orders:

**Definition 1.5.** Suppose \( V \models ZFC^- \) is transitive, \( j : V \preceq \hat{V} \) is \( \omega \)-nonstandard, and suppose \( T \) is a complete countable theory with \( T \in V \). Suppose \( \kappa \) is an infinite cardinal. Then say that \( \hat{V} \) \( \kappa \)-pseudosaturates \( T \) if for some or every \( M \models T \) with \( M \in V \), \( j_{\text{std}}(M) \) is \( \kappa \)-pseudosaturated.

Suppose \( \lambda \) is infinite and \( \kappa \) is infinite or 1. Then say that \( T_0 \preceq_{\lambda^+}^\times T_1 \) if there is some countable transitive \( V \models ZFC^- \) containing \( T_0, T_1 \) such that whenever \( j : V \preceq \hat{V} \), if \( \hat{V} \) is \( \kappa \)-saturated and \( \omega \)-nonstandard, and if \( \hat{V} \) \( \lambda^+ \)-pseudosaturates \( T_1 \), then also it \( \lambda^+ \)-pseudosaturates \( T_0 \). Say that \( T_0 \preceq_{\kappa}^\times T_1 \) if \( T_0 \preceq_{\lambda^+}^\times T_1 \) for all infinite \( \lambda \).

Theorem 10.6 of [38] states that \( \preceq_{\mathcal{K}}^{\times} \subseteq \preceq_{\mathcal{K}}^{\times} \), in other words: to prove \( T_0 \preceq_{\kappa}^{\times} T_1 \) it is enough to show \( T_0 \preceq_{\kappa}^{\times} T_1 \). In practice, this turns out to be much cleaner. In any case, we suspect that \( \preceq_{\mathcal{K}}^{\times} = \preceq_{\mathcal{K}}^{\times} \).

In Section 3, we lift Malliaris’s theorem that Keisler’s order is local [14] to the context of \( \preceq_{\mathcal{K}}^{\times} \), and introduce the notion of patterns. Patterns have been studied under different notation by Shelah [35] and then Malliaris [15], although Malliaris was the first to connect them to Keisler’s order.

In Sections 4, 5 and 6 we use this machinery to give streamlined proofs of the existence of minimal unstable, unsimple and nonlow theories in \( \preceq_{\mathcal{K}}^{\times} \) (which are thus minimal in all of the other orders \( \preceq_{\mathcal{K}}^{\times}, \preceq_{\mathcal{K}}^{\times} \) and \( \preceq \) as well). In particular, we prove the following, where \( T_{rg} \) is the theory of the random graph, and \( T_{nlow} \) is the supersimple nonlow theory introduced by Casanovas and Kim [3], and where \( T_{rf} \) is the theory of the random binary function [1].

**Theorem 1.6.** Suppose \( V \models ZFC^- \) is transitive, and \( j : V \preceq \hat{V} \) is \( \omega \)-nonstandard, and \( \lambda \) is an infinite cardinal, and \( T \in V \) is a complete countable theory. Then:

(A) If \( \hat{V} \) \( \lambda^+ \)-pseudosaturates \( T \) and \( T \) is unstable, then for all disjoint, pseudofinite \( X_0, X_1 \subseteq \hat{V} \) with \( |X_i| \leq \lambda \), there exist disjoint \( \hat{X}_0, \hat{X}_1 \in \hat{V} \) with \( X_0 \subseteq \hat{X}_0 \) and \( X_1 \subseteq \hat{X}_1 \). If \( T = T_{rg} \) then the converse holds.

(B) If \( \hat{V} \) \( \lambda^+ \)-pseudosaturates \( T \) and \( T \) is unsimple, then every pseudofinite partial function from \( \hat{V} \) to \( \hat{V} \) of cardinality at most \( \lambda \) can be extended to some internal partial function \( \hat{f} \) from \( \hat{V} \) to \( \hat{V} \). If \( T = T_{rf} \) then the converse holds.

(C) If \( \hat{V} \) \( \lambda^+ \)-pseudosaturates \( T \) and \( T \) is nonlow, then the conclusion in (A) holds, and further: for every \( X \subseteq \hat{V} \) with \( |X| \leq \lambda \), and for every \( \hat{n} < \hat{\omega} \) nonstandard, there exists

\[1\] We will define low in Section 2 but for now, note—there are two definitions of “low” in circulation. We follow the original definition of Buechler [1], so in particular low implies simple.
some $\hat{X} \in \hat{V}$ such that $\hat{V} \models "|\hat{X}| = \hat{n}"$ and such that $X \subseteq \hat{X}$. If $T = T_{nlow}$ then the converse holds.

We can cast this theorem more systematically as follows.

**Definition 1.7.** Suppose $V \models ZFC^-$ is transitive, and $T \in V$ is a complete countable theory, and $j : V \preceq \hat{V}$ is $\omega$-nonstandard. Then let $\lambda_V(T)$ be the least infinite cardinal such that $\hat{V}$ does not $\lambda_V(T)^+\text{-pseudosaturate } T$; possibly $\lambda_V(T) = \infty$ (if $\hat{V} \lambda$-pseudosaturates $T$ for all $\lambda$).

In other words, each complete countable theory $T$ induces a cardinal characteristic of models of $ZFC^-$; we are interested in determining which of these cardinal characteristics can be separated. When $\lambda_V(T) > \aleph_0$, then it follows from the definition that $\hat{V}$ does $\lambda_V(T)^+\text{-pseudosaturate } T$; when $\lambda_V(T) = \aleph_0$, then $\hat{V}$ may or may not $\aleph_0\text{-pseudosaturate } T$, but the situation is understood (see Corollary 7.7 of [38]).

**Question.** Suppose $T$ is a complete countable theory. What is the function $\hat{V} \mapsto \lambda_V(T)$?

To rephrase, Theorem 1.6 determines $\lambda_V(T_{rg})$, $\lambda_V(T_{rf})$ and $\lambda_V(T_{nlow})$, and shows that these are the maximal possible values for unstable, unsimple and nonlow theories, respectively.

In Section 7, we give a similar treatment of the hypergraph examples $T_{n,k}$. Namely, for $n > k \geq 2$, let $T_{n,k}$ be the random $k$-ary $n$-clique free hypergraph. These are due to Hrushovski [7]; Malliaris and Shelah use them in [21] to prove that Keisler’s order has infinitely many classes.

In Section 8, we recall the setup of full Boolean-valued models from [37], and recast our results in terms of ultrafilters. In particular, for every ultrafilter $U$ on a complete Boolean algebra $\mathcal{B}$, and for every complete countable theory $T$, we define what it means for $U$ to $\lambda^+\text{-saturate } T$. Keisler’s order can be framed as follows: $T_0 \preceq \lambda T_1$ if and only if for every complete Boolean algebra $\mathcal{B}$ with the $\lambda^+\text{-c.c.}$ and for every ultrafilter $U$ on $\mathcal{B}$, if $U$ $\lambda^+\text{-saturates } T_1$, then $U$ $\lambda^+\text{-saturates } T_0$; and $T_0 \preceq T_1$ if and only if $T \preceq \lambda T_1$ for all $\lambda$.

In Section 9, we tie off several strands of non-saturation arguments. To give the reader context, we quote the following theorems. (B) is due to Malliaris and Shelah [26]. I prove (C) and (D) in [39], and I prove (A) in [37]. As notation, if $\mathcal{B}$ is a complete Boolean algebra, then $\mathcal{B}$ has the $\kappa\text{-c.c.}$ (chain condition) if $\mathcal{B}$ has an antichain of size $\lambda$, and $c.c.(\mathcal{B})$ is the least $\kappa$ for which this fails, i.e. the least cardinality $\kappa$ for which $\mathcal{B}$ has no antichain of size $\kappa$.

**Theorem 1.8.** The following are all true.

(A) Suppose $\lambda$ is an infinite cardinal, and $\mathcal{B}$ is a complete Boolean algebra with an antichain of size $\lambda$ (i.e. failing the $\lambda\text{-c.c.}$). Then there is a $\lambda^+\text{-good ultrafilter } U$ on $\mathcal{B}$ (i.e. an ultrafilter $U$ which $\lambda^+\text{-saturates every complete countable theory } T$).

(B) If there is a supercompact cardinal $\lambda$, then there is a complete Boolean algebra $\mathcal{B}$ with the $\lambda\text{-c.c.}$ and an ultrafilter $U$ on $\mathcal{B}$, such that $U$ $\lambda^+\text{-saturates exactly the simple theories.
(C) There is some complete Boolean algebra $\mathcal{B}$ and some $\aleph_1$-incomplete ultrafilter $\mathcal{U}$ on $\mathcal{B}$, such that $\mathcal{U}$ c.c.$(\mathcal{B})^+$-saturates exactly the low theories.

(D) Suppose $\mathcal{B}$ is a complete Boolean algebra, and $\mathcal{U}$ is an $\aleph_1$-incomplete ultrafilter on $\mathcal{B}$. Then $\mathcal{U}$ does not c.c.$(\mathcal{B})^+$-saturate any nonlow theory.

We complete the picture with the following. Here, we say that the ultrafilter $\mathcal{U}$ on $\mathcal{B}$ is principal if $\bigwedge \mathcal{U}$ is nonzero (this coincides with the usual definition if $\mathcal{B} = \mathcal{P}(\lambda)$.)

**Theorem 1.9.** Suppose $\mathcal{B}$ is a complete Boolean algebra and $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathcal{B}$. Then $\mathcal{U}$ does not c.c.$(\mathcal{B})^+$-saturate any unsimple theory.

We also give a short proof of (D) above, making use of our new machinery.

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## 2 Preliminaries

We collect together some facts and terminology we will need.

### 2.1 Model-Theoretic Preliminaries

**Definition 2.1.** Suppose $T$ is a complete countable theory and $\varphi(\overline{x}, \overline{y})$ is a formula; for convenience, we write it as $\varphi(x, y)$. Then:

- $\varphi(x, y)$ has the independence property (IP) if there are $(b_n : n < \omega)$ such that for all disjoint $u, v \subseteq \omega$, $\{\varphi(x, b_n : n \in u) : n \in v\}$ is consistent. Otherwise, $\varphi(x, y)$ has NIP.

- $\varphi(x, y)$ has the strict order property of the second kind ($SOP_2$) if there are $(b_s : s \in \omega^{<\omega})$, such that for each $\eta \in \omega^\omega$, $(\varphi(x, b_{\eta|n}) : n < \omega)$ is consistent, but whenever $s, t \in \omega^{<\omega}$ are incomparable, $\varphi(x, b_s) \land \varphi(x, b_t)$ is inconsistent.

- $\varphi(x, y)$ has the tree property of the second kind ($TP_2$) if there are $(b_{n,m} : n, m < \omega)$ such that for all $n < \omega$ and for all $m < m' < \omega$, $\exists \overline{x}(\varphi(x, b_{n,m}) \land \varphi(x, b_{n,m'}))$ is inconsistent, but such that for all $\eta \in \omega^\omega$, $\{\varphi(x, b_{\eta|n})\}$ is consistent. Otherwise $\varphi(x, y)$ has $NTP_2$.

- $\varphi(x, y)$ has the finite dividing property if for every $k$ there is some indiscernible sequence $(b_n : n < \omega)$ over the empty set such that $\{\varphi(x, b_n) : n < \omega\}$ is $k$-consistent but not consistent.

**Remark 2.2.** The tree property of the first kind $TP_1$ is equivalent to the strict order property of the second kind $SOP_2$; we pick the term $SOP_2$ to use. See [10] for a comparison.
We recall that in simple theories, forking (equivalently dividing) is a well-behaved independence relation.

**Definition 2.3.** Suppose $T$ is a complete countable theory. Then $T$ is low if $T$ is simple and does not have the finite dividing property.

**Remark 2.4.** There are multiple definitions of low in use. Our definition is equivalent to the original definition of Buechler [1], and is also how the low is defined in [9], for instance. In other places in the literature, the hypothesis that $T$ is simple is dropped, e.g. as in [22].

We recall two dichotomy theorems of Shelah. The following is Theorem 0.2 of [33]:

**Theorem 2.5.** $T$ is unsimple if and only if either $T$ has TP$_2$ or else $T$ has SOP$_2$.

The following is also well-known:

**Theorem 2.6.** $T$ is unstable if and only if either $T$ has IP or SOP$_2$.

**Proof.** Theorem II.4.7 of [32] states that $T$ is unstable if and only if either $T$ has IP or else $T$ has SOP$_2$. But if $T$ has SOP$_2$ then $T$ has SOP, and if $T$ has SOP then $T$ is unstable, so the theorem follows. □

### 2.2 Pseudosaturation

We recall some facts from [38].

The following is a key cardinal characteristic of models of set theory; in the context of ultrapower embeddings, the definition is due to Malliaris and Shelah [25].

**Definition 2.7.** Suppose $(L, <)$ is a linear order with proper initial segment $\omega$. If $\kappa, \theta$ are infinite regular cardinals, then a $(\kappa, \theta)$-pre-cut in $L$ is a pair of sequences $(\bar{a}, \bar{b}) = (a_\alpha : \alpha < \kappa)$, $(b_\beta : \beta < \theta)$ from $L$, such that for all $\alpha < \alpha', \beta < \beta'$, we have $a_\alpha < a_{\alpha'} < b_{\beta'} < b_\beta$. $(\bar{a}, \bar{b})$ is a cut if there is no $c \in L$ with $a_\alpha < c < b_\beta$ for all $\alpha, \beta$. Let the cut spectrum of $(L, <)$ be $\mathcal{C}(L, <) := \{(\kappa, \theta) : L \text{ admits a } (\kappa, \theta) \text{ cut}\}$. Define $\text{cut}(L, <) = \min\{\kappa + \theta : (\kappa, \theta) \in \mathcal{C}(L, <)\}$.

Suppose $\hat{V} \models ZFC^-$ is nonstandard. Define $\mathcal{C}_\varphi = \mathcal{C}(\hat{\omega}, \hat{\zeta})$, and define $\text{p}_\varphi = \text{cut}(\hat{\omega}, \hat{\zeta})$.

$p_\varphi$ is the smallest cardinal characteristic of models of set theory that is relevant for pseudosaturation. The following two theorems are mild generalizations of results of Malliaris and Shelah [25, 19] to the context of $\preceq_{<\lambda}$. The first is Theorem 5.3 in [38], the second is the conjunction of Corollary 5.4 and Theorem 5.7.

**Theorem 2.8.** Suppose $\hat{V} \models ZFC^-$ is $\omega$-nonstandard and $p_\varphi \geq \aleph_1$. Suppose $p(x) = \{\varphi_\alpha(x, \hat{a}_\alpha) : \alpha < \lambda\}$ is a type over $\hat{V}$ of cardinality $\lambda < p_\varphi$, and suppose $\{\hat{a}_\alpha : \alpha < \lambda\}$ is pseudofinite. Then $p(x)$ is realized in $\hat{V}$, provided either of the following conditions are met.

(A) There is some $n < \omega$ such that each $\varphi_\alpha(x, a_\alpha)$ is $\Sigma_n$.

(B) Every countable subset of $\hat{V}$ is pseudofinite.

**Theorem 2.9.** Suppose $V \models ZFC^-$ is transitive, $T \in V$ is a complete countable theory, and $j : V \preceq \hat{V}$ is $\omega$-nonstandard. Then $\lambda_{\varphi}(T) \geq p_\varphi$. If $T$ has SOP$_2$ then equality is attained.
2.3 Full Boolean-Valued Models

We recall the setup of [37]. This won’t be used until Section 8 and the reader may wish to defer reading this subsection until then.

As a convention, if $X$ is a set and $\mathcal{L}$ is a language, then $\mathcal{L}(X)$ is the set of formulas of $\mathcal{L}$ with parameters taken from $X$.

Suppose $\mathcal{B}$ is a complete Boolean algebra. A $\mathcal{B}$-valued structure is a pair $(M, \| \cdot \|_M)$ where:

1. $M$ is a set;
2. $\varphi \mapsto \| \varphi \|_M$ is a map from $\mathcal{L}(M)$ to $\mathcal{B}$;
3. If $\varphi$ is a logically valid sentence then $\| \varphi \|_M = 1$;
4. For every formula $\varphi \in \mathcal{L}(M)$, we have that $\| \neg \varphi \|_M = \neg \| \varphi \|_M$;
5. For all $\varphi, \psi$, we have that $\| \varphi \land \psi \|_M = \| \varphi \|_M \land \| \psi \|_M$;
6. For every formula $\varphi(x)$ with parameters from $M$, $\| \exists x \varphi(x) \|_M = \bigvee_{a \in M} \| \varphi(a) \|_M$;
7. For all $a, b \in M$ distinct, $\| a = b \|_M < 1$.

We are only interested in the case when $M$ is full, i.e. when in fact $\| \exists x \varphi(x, \alpha) \|_M = \max_{a \in M} \| \varphi(a, \alpha) \|_M$. If $T$ is a theory, then we write $M \models^B T$, and say that $M$ is a full $\mathcal{B}$-valued model of $T$, if $\| \varphi \|_M = 1$ for all $\varphi \in T$.

For example, (ordinary) $\mathcal{L}$-structures are the same as full $\{0, 1\}$-valued $\mathcal{L}$-structures, which can thus be viewed as full $\mathcal{B}$-valued structures for any $\mathcal{B}$. Also, if $M$ is an $\mathcal{L}$-structure and $\lambda$ is a cardinal, then $M^\lambda$ is a $\mathcal{P}(\lambda)$-valued $\mathcal{L}$-structure; moreover, we have the canonical elementary embedding $i : M \subseteq M^\lambda$, given by the diagonal map. We call this the pre-Loś embedding. More generally, for any complete Boolean algebra $\mathcal{B}$ we can define the $\mathcal{B}$-valued structure $M^\mathcal{B}$.

If $M$ is a full $\mathcal{B}$-valued model of $T$ and $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$, then we can form the specialization $M/\mathcal{U} \models T$, which comes equipped with a canonical surjection $[\cdot]_\mathcal{U} : M \to M/\mathcal{U}$, satisfying that for all $\varphi(\alpha) \in \mathcal{L}(M)$, $M/\mathcal{U} \models \varphi([\alpha]_\mathcal{U})$ if and only if $\| \varphi(\alpha) \|_M \in \mathcal{U}$. This generalizes the ultrapower construction $M^\lambda/\mathcal{U}$; note that the Loś embedding of $M$ into $M^\lambda/\mathcal{U}$ is the composition of the pre-Loś embedding with $[\cdot]_\mathcal{U}$.

In [37], we prove the following compactness theorem for full Boolean-valued models:

**Theorem 2.10.** Suppose $\mathcal{B}$ is a complete Boolean algebra, $X$ is a set, $\Gamma \subseteq \mathcal{L}(X)$, and $F_0, F_1 : \Gamma \to \mathcal{B}$ with $F_0(\varphi(\alpha)) \leq F_1(\varphi(\alpha))$ for all $\varphi(\alpha) \in \Gamma$. Then the following are equivalent:

(A) There is some full $\mathcal{B}$-valued structure $M$ and some map $\tau : X \to M$, such that for all $\varphi(\alpha) \in \Gamma$, $F_0(\varphi(\alpha)) \leq \| \varphi(\tau(\alpha)) \|_M \leq F_1(\varphi(\alpha))$;
(B) For every finite $\Gamma_0 \subseteq \Gamma$ and for every $c \in B_+$, there is some $\{0,1\}$-valued $L$-structure $M$ and some map $\tau : X \to M$, such that for every $\varphi(\bar{a}) \in \Gamma$, if $c \leq F_0(\varphi(\bar{a}))$ then $M \models \varphi(\tau(\bar{a}))$, and if $c \leq -F_1(\varphi(\bar{a}))$ then $M \models -\varphi(\tau(\bar{a}))$.

Here is a first application: given $B$-valued models $M \subseteq N$, say that $M \preceq N$ if $\| \cdot \|_M \subseteq \| \cdot \|_N$. Say that $N$ is $\lambda^+$-saturated if for every $M_0 \preceq N$ with $|M_0| \leq \lambda$ and for every $M_1 \models M_0$ with $|M_1| \leq \lambda$, there is some elementary embedding $f : M_1 \subseteq N$ extending the inclusion from $M_0$ into $N$. Then in [37], we show that for every $B$-valued structure $M$ and for every $\lambda$, there is an elementary extension $N \models M$ such that $N$ is full and moreover $\lambda^+$-saturated.

Suppose $T$ is a complete countable theory, and $U$ is an ultrafilter on the complete Boolean algebra $B$. We observe in [37] that if there is some $\lambda^+$-saturated $M \models T$ with $M/U \lambda^+$-saturated, then for every $\lambda^+$-saturated $M \models T$, $M/U$ is $\lambda^+$-saturated. We define that $U$ $\lambda^+$-saturates $T$ in this case.

Finally, in [37] we give the following convenient characterization of Keisler’s order:

**Theorem 2.11.** Suppose $T_0, T_1$ are theories. Then $T_0 \preceq T_1$ if and only if for every $\lambda$, for every complete Boolean algebra $B$ with the $\lambda^+$-c.c., and for every ultrafilter $U$ on $B$, if $U$ $\lambda^+$-saturates $T_1$, then $U$ $\lambda^+$-saturates $T_0$.

**Convention.** $V$ will denote a full $B$-valued model of $ZFC^-$ for some complete Boolean algebra $B$, often associated with an elementary embedding $i : V \preceq V$ for some transitive $V$ (for example, if $V = V^B$ is the Boolean ultrapower, then $i$ would be the pre-Łoś embedding).

**Definition 2.12.** Suppose $B$ is a complete Boolean algebra, $V \models ZFC^-$ is transitive, and $i : V \preceq V \models^B ZFC^-$. Given $X \subseteq V$, let $i_{std}(X) = \{ a \in V : \|a\in i(X)\|_B = 1 \}$. Suppose $M \subseteq V$ is a structure in the countable language $\mathcal{L}$, with domain $\text{dom}(M)$. Then let $i_{std}(M)$ be the full $B$-valued $\mathcal{L}$-structure defined as follows. Its domain is $i_{std}(\text{dom}(M))$. Given a formula $\varphi(a_i : i < n) \in \mathcal{L}(i_{std}(\text{dom}(M)))$, let $\|\varphi(a_i : i < n)\|_{i_{std}(M)} = \|i(M) \models \varphi(a_i : i < n)\|_V$. (Note that in practice, we usually denote $M$ and $\text{dom}(M)$ by the same symbol $M$.)

The following corollaries are proven in [38].

**Corollary 2.13.** Suppose $B$ is a complete Boolean algebra, $U$ is an ultrafilter on $B$, $\lambda$ is a cardinal, and $T$ is a complete countable theory. Then the following are equivalent:

(A) $U$ $\lambda^+$-saturates $T$.

(B) For some or every transitive $V \models ZFC^-$, and for some or every $i : V \preceq V$ with $V$ $\lambda^+$-saturated, and for some or every $M \models T$ with $M \subseteq V$, $i_{std}(M)/U$ is $\lambda^+$-saturated.

**Corollary 2.14.** Suppose $B$ is a complete Boolean algebra, $U$ is an ultrafilter on $B$, $\lambda$ is a cardinal, and $T$ is a complete countable theory. Then the following are equivalent:

(A) $U$ $\lambda^+$-saturates $T$. 

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(B) For some or every transitive $V \models ZFC^-$ with $T \in V$, and for some or every $i : V \preceq V$ with $V$ $\lambda^+$-saturated, and for some or every $M \models T$ with $M \in V$, $i_{\text{std}}(M)/U$ is $\lambda^+$-saturated.

(C) For some or every transitive $V \models ZFC^-$, for some or every $i : V \preceq V$ with $V$ $\lambda^+$-saturated, $V/U$ $\lambda^+$-pseudosaturates $T$.

2.4 Boolean Ultrapowers

We will also need Boolean ultrapowers (starting in Section 8). These are implicit in the work of Scott and Solovay [31], and made explicit by Vopenka [40]. We follow the notation of Mansfield [27] and Hamkins and Sebald [6].

Definition 2.15. Suppose $M \models T$. Let the set of all partitions of $B$ by $M$, denoted $M^B$, be the set of all functions $a : M \to B$, such that for all $a, b \in M$, $a(b) = 0$, and such that $\bigvee_{a \in M} a(a) = 1$. Given $\varphi(a_i : i < n) \in L(M^B)$, put $\|\varphi(a_i : i < n)\|_B = \bigvee_{M \models \varphi(a_i : i < n)} \bigwedge_{i < n} a_i(a_i)$. (One must check that this does not change if we add dummy parameters to $\varphi$, but this is straightforward.)

Let $i : M \to M^B$ be the embedding sending $a \in M$ to the function $i(a) : M \to B$ which takes the value 1 on $a$, and 0 elsewhere. We call this the pre-Łoś embedding.

The following theorem is the conjunction of Corollary 1.2 and Theorem 1.4 of [27] (in the special case of a relational language).

Theorem 2.16. Suppose $M$ is a $\{0, 1\}$-valued structure and $B$ is a complete Boolean algebra (so $M$ is also a full $B$-valued structure). Then $M^B$ is a full $B$-valued $L$-structure, and $i : M \preceq M^B$.

3 Patterns

In this section, we introduce the notion of patterns, which will give a method of computing $\lambda_T^\hat{V}$ when $\hat{V}$ is $\aleph_1$-saturated.

We first recall a theorem due to Malliaris [14], that says that ultrapowers are saturated if and only if they are locally saturated. We phrase it in the terminology of $\leq_{\aleph_1}^{\hat{V}}$; Malliaris’s proof translates to this context verbatim.

Theorem 3.1. Suppose $V \models ZFC^-$ is transitive, $j : V \preceq \hat{V}$ is $\aleph_1$-saturated, and $T \in V$ is a complete countable theory. Suppose $M \models T$ with $M \in V$, and $\lambda$ is a cardinal. Then the following are equivalent.

(A) $j_{\text{std}}(M)$ is $\lambda^+$-pseudosaturated;

(B) For every formula $\varphi(x, \overline{y})$ and for every positive, pseudofinite $\varphi$-type $p(x)$ over $M$ of cardinality at most $\lambda$, $p(x)$ is realized in $M$. 

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The following is interchangeable with the terminology of characteristic sequences, arrays and diagrams of Malliaris [15], although we find patterns to be more convenient.

**Definition 3.2.** If $I$ is an index set, then a pattern on $I$ is some $\Delta \subseteq [I]^{<\aleph_0}$ which is closed under subsets. A $\Delta$-clique is a subset $X \subseteq I$ with $[X]^{<\aleph_0} \subseteq \Delta$. If $\Delta$ is a pattern on $I$ and $\Delta'$ is a pattern on $J$, then say that $\Delta'$ is an instance of $\Delta$ if for all $s \in [J]^{<\aleph_0}$ there is some map $f : s \to I$ such that for all $t \subseteq s$, $t \in \Delta'$ if and only if $f[t] \in \Delta$. Say that two patterns $\Delta, \Delta'$ are equivalent if they are instances of each other.

Note that every pattern is equivalent to one on $\omega$.

**Definition 3.3.** Suppose $V \models ZFC^-$ is transitive, $j : V \preceq \hat{V}$, and suppose $\Delta \in V$ is a pattern on $I$ (so $I \in V$). Then let $\lambda^V_\Delta(\Delta)$ be the least $\lambda$ such that there is some pseudofinite $X \subseteq j(I)$ with $|X| \leq \lambda$, such that $[X]^{<\aleph_0} \subseteq j(\Delta)$, and such that there is no $X \in j(\Delta)$ with $X \subseteq \hat{X}$. If there is no such $\lambda$ then let $\lambda^V_\Delta(\Delta) = \infty$.

For any pattern $\Delta$ on $\omega$, and for any $j : V \preceq \hat{V}$, we have that $\lambda^V_\Delta(\Delta) \geq p_\Delta$ by Theorem 5.3(A) of [38].

**Remark 3.4.** Technically we should refer to the pair $(I, \Delta)$ in the definition of equivalence, but $I$ will always be clear from context.

Also we should refer to $\lambda^V_\Delta(I, \Delta)$ but $j$ will be clear from context.

**Lemma 3.5.** Suppose $V \models ZFC^-$ is transitive, and $j : V \preceq \hat{V}$. Suppose $\Delta$ is a pattern on $I$ and $\Delta'$ is a pattern on $I'$, with $\Delta, I, \Delta', I' \in V$. If $\Delta$ is an instance of $\Delta'$, then $\lambda^V_\Delta(\Delta) \geq \lambda^V_\Delta(\Delta')$. Hence, if $\Delta$ and $\Delta'$ are equivalent, then $\lambda^V_\Delta(\Delta) = \lambda^V_\Delta(\Delta')$.

**Proof.** It suffices to prove the first part. Note that $V \models (\Delta \text{ is an instance of } \Delta')$, since this is a finitary condition.

Suppose $X \subseteq j(I)$ is pseudofinite with $|X| < \lambda^V_\Delta(\Delta')$; choose $X \subseteq \hat{X}$ with $X \in V$ finite in the sense of $\hat{V}$; by replacing $\hat{X}$ with $\hat{X} \cap j(I)$ we can suppose $\hat{X} \subseteq j(I)$. Since $j(\Delta)$ is an instance of $j(\Delta')$ in $\hat{V}$, we can find some $\hat{X}' \subseteq j(I')$ finite in the sense of $\hat{V}$ and some map $f : \hat{X} \to \hat{X}'$ in $\hat{V}$, such that for all $\hat{s} \subseteq \hat{x}$ in $\hat{V}$, $\hat{s} \in j(\Delta)$ if and only if $f[\hat{s}] \in j(\Delta')$. Define $X' = f[\hat{X}]$ (as computed in $V$). Then by definition of $\lambda^V_\Delta(\Delta')$ we can find some $\hat{s}' \in j(\Delta')$ with $X' \subseteq \hat{s}'$. Let $\hat{s} = f^{-1}(\hat{s}')$; then $\hat{s} \in j(\Delta)$ and $X \subseteq \hat{s}$, as desired. \hfill $\Box$

We now connect this with pseudosaturation.

**Definition 3.6.** Suppose $T$ is a complete countable theory, $\varphi(\overline{x}, \overline{y})$ is a formula of $T$ and $\Delta$ is a pattern on $I$. Then say that $\varphi(\overline{x}, \overline{y})$ admits $\Delta$ if we can choose $(\overline{a}_i) : i \in I)$ from $\mathcal{C}[\overline{y}]$ (where $\mathcal{C}$ is the monster model of $T$, or just any $[I]^+\text{-universal model}$), such that for every $s \in [I]^{<\aleph_0}$, $\exists \overline{x} \bigwedge_{i \in s} \varphi(\overline{x}, \overline{a}_i)$ is consistent if and only if $s \in \Delta$ (whether or not $\varphi$ admits $\Delta$ depends on $T$; if there is confusion, we will say that $(T, \varphi)$ admits $\Delta$). Say that $T$ admits $\Delta$ if some formula of $T$ does.

Suppose $T$ is a complete countable theory, $M \models T$ and $\varphi(\overline{x}, \overline{y})$ is a formula of $T$. Then we can form a pattern $\Delta_{M, \varphi} := \{s \in [M][\overline{y}]^{<\aleph_0} : M \models \exists \overline{x} \bigwedge_{i \in s} \varphi(\overline{x}, \overline{a}_i)\}$. Then for all patterns
\( \Delta, \varphi \) admits \( \Delta \) if and only if \( \Delta \) is an instance of \( \Delta_{M, \varphi} \). Hence for all \( M, N \models T, \Delta_{M, \varphi} \) and \( \Delta_{N, \varphi} \) are equivalent. Write \( \Delta_{\varphi} = \Delta_{M, \varphi} \), for some arbitrary choice of \( M \models T \). We will only refer to \( \Delta_{\varphi} \) in contexts where we just need its equivalence class.

If \( T \) is a complete countable theory in \( V \) and \( \varphi(\bar{x}, \bar{y}) \) is a formula of \( T \), then let \( \lambda^T_V(T, \varphi) = \lambda^T_V(\Delta_{\varphi}) \). By Lemma 3.5 this only depends on the equivalence class of \( \Delta_{\varphi} \), and so is well-defined.

For each \( n \), let \( \lambda^{loc,n}_V(T) \) be the minimum over all formulas \( \varphi(\bar{x}, \bar{y}) \) of \( T \) with \( |\bar{x}| \leq n \) of \( \lambda^T_V(T, \varphi) \); so this is a descending sequence of cardinals (which necessarily stabilizes). Let \( \lambda^{loc}_V(T) = \min_n \lambda^{loc,n}_V(T) \).

**Theorem 3.7.** Suppose \( V \models ZFC^- \) is transitive and \( j : V \preceq \hat{V} \) is \( \omega \)-nonstandard and \( T \in V \) is a complete countable theory. Then \( \lambda^T_V(T) = \lambda^{loc}_V(T, \varphi) \). If \( \hat{V} \) is \( \aleph_1 \)-saturated, then \( \lambda^T_V(T) = \lambda^{loc}_V(T) = \lambda^{loc,1}_V(T) \).

**Proof.** This is basically Theorem 3.1 restated. The point is the following: suppose \( M \models T \) with \( M \in V \), and let \( \varphi(\bar{x}, \bar{y}) \) be a formula of \( T \). Let \( \Delta = \Delta_{M, \varphi} = \{ s \in [M]^{|\bar{y}|} : M \models \exists \bar{x} \in s \varphi(\bar{x}, \bar{y}) \} \) and let \( \hat{\Delta} = j(\Delta) \). Then positive \( \varphi \)-types \( p(\bar{x}) \) over \( j_{std}(M) \) correspond exactly to subsets \( X \) of \( j_{std}(M)[|\bar{y}|] \) with \( [X]^{<\aleph_0} \subseteq \hat{\Delta} \), and, assuming \( p(\bar{x}) \) is pseudofinite, \( p(\bar{x}) \) is realized in \( j_{std}(M) \) if and only if there is some \( \hat{X} \in \hat{\Delta} \) with \( X \subseteq \hat{X} \). Moreover, it suffices to consider types in a single variable \( x \).

Malliaris proved the following corollary as Lemma 5.14 of [13] for Keisler’s order \( \preceq \), under the terminology of characteristic sequences:

**Corollary 3.8.** Suppose \( T_0, T_1 \) are complete countable theories. Suppose that for every pattern \( \Delta \), if \( T_0 \) admits \( \Delta \) then \( T_1 \) admits \( \Delta \). Then \( T_0 \preceq_{\aleph_1} T_1 \), and hence \( T_0 \preceq_{\aleph_1} T_1 \) and \( T_0 \preceq T_1 \).

Although this is not the line of investigation of the current work, this theorem suggests a natural ordering on theories (first proposed by Shelah [35] under notation similar to Malliaris’s), namely: put \( T_0 \preceq T_1 \) if and only if for every pattern \( \Delta \), if \( T_0 \) admits \( \Delta \) then \( T_1 \) (it is enough to consider just formulas \( \varphi(x, \bar{y}) \) where \( x \) is a single variable; this may be preferable). \( \preceq \) detects many more properties than do the interpretability orders; for instance, it follows from our work that if \( T_0 \) has \( IP \) and \( T_1 \) is \( NIP \), then \( T_0 \not\preceq T_1 \), and so \( DLO \) is not maximal in \( \preceq \). Shelah defines the “straightly maximal” theories to be the maximal class of \( \preceq \); one simple example is \( Th(P(\omega), \preceq) \).

### 4 A Minimal Unstable Theory

Let \( T_{rg} \) be the theory of the random graph. Malliaris proved in [16] that \( T_{rg} \) is a \( \preceq \)-minimal unstable theory. Malliaris and Shelah prove in [20] that \( T_{rg} \) is a \( \preceq^1_1 \)-minimal unstable theory, although that proof has some additional complications. In fact, Malliaris’s proof goes through to show that \( T_{rg} \) is a \( \preceq^1_1 \)-minimal theory; the argument is even simplified.

The following essentially describes an \( (\omega, 2) \)-array as in [15].
**Definition 4.1.** Let $\Delta(IP)$ be the pattern on $\omega \times 2$, defined to be the set of all $s \in [\omega \times 2]^{<\aleph_0}$ such that for all $n < \omega$, $\{(n,0),(n,1)\} \not\subseteq s$, i.e. $s$ is a partial function from $\omega$ to 2. (Think of $(n,0)$ as being $\neg(n,1)$.)

And the following is Claim 3.7 from [15].

**Lemma 4.2.** Suppose $T$ is a complete countable theory and $\varphi(\overline{x}, \overline{y})$ is a formula of $T$. Define $\theta(\overline{x}, \overline{y}_0, \overline{y}_1) = \varphi(\overline{x}, \overline{y}_0) \land \neg \varphi(\overline{x}, \overline{y}_1)$. Then $\theta(\overline{x}, \overline{y}_0, \overline{y}_1)$ admits $\Delta(IP)$ if and only if $\varphi(\overline{x}, \overline{y}_1)$ has the independence property, which is the case if and only if $\varphi(\overline{x}, \overline{y})$ has the independence property.

Thus $T$ has the independence property if and only if it admits $\Delta(IP)$.

The following lemma is helpful in understanding the invariant $\lambda_\varphi(\Delta(IP))$. It is a translation of remarks in [16] into the context of models of $ZFC^-$, for instance see the discussion in Example 2 in Section 3.2.

**Lemma 4.3.** Suppose $V \models ZFC^-$ is transitive, and $j : V \preceq \hat{V}$. Then $\lambda_\varphi(\Delta(IP))$ is the least $\lambda$ such that for some $\hat{n} < \hat{\omega}$, there are disjoint, pseudofinite $X_0, X_1 \subseteq \hat{V}$ each of size at most $\lambda$, such that there are no disjoint $X_0, X_1 \subseteq \hat{X}_i$.

**Proof.** Let $\lambda$ be the least cardinal as in the statement of the lemma (possibly $\infty$).

We first show that $\lambda_\varphi(\Delta(IP)) \leq \lambda$. So suppose $X \subseteq \hat{n} \times 2$ is given, with $|X| < \lambda$ and with $\{X\}^{<\aleph_0} \subseteq j(\Delta(IP))$; i.e. $X$ is a partial function from $\hat{n}$ to 2. For each $i < 2$, define $X_i = \{\hat{m} < \hat{n} : (\hat{m}, i) \in X\}$; by hypothesis, there exist disjoint $\hat{X}_i : i < 2$ in $\hat{V}$ with each $\hat{X}_i \subseteq \hat{X}_i$; by replacing $\hat{X}_i$ by $\hat{X}_i \cap \hat{n}$, we can suppose $\hat{X}_i \subseteq \hat{n}$. Then $X \subseteq \hat{X}_0 \times \{0\} \cup \hat{X}_1 \times \{1\} \in j(\Delta(IP))$.

Conversely, we show that $\lambda \leq \lambda_\varphi(\Delta(IP))$. So suppose we are given disjoint, pseudofinite $X_0, X_1 \subseteq \hat{V}$ each of cardinality less than $\lambda$. Choose $\hat{Y} \in \hat{V}$, finite in the sense of $\hat{V}$, with $X_0, X_1 \subseteq \hat{Y}$, and choose a bijection $\hat{f} : \hat{Y} \to |\hat{Y}|$ in $\hat{V}$. In $V$, define $X = \hat{f}[X_0] \times \{0\} \cup \hat{f}[X_1] \times \{1\}$. Note that $|X| \leq \lambda$ and $\{X\}^{<\aleph_0} \subseteq j(\Delta(IP))$ and $X$ is pseudofinite. By hypothesis, there exists $\hat{X} \in j(\Delta(IP))$ with $X \subseteq \hat{X}$. Let $\hat{X}_i = \hat{f}^{-1}[\{\hat{n} : (\hat{n}, i) \in \hat{X}\}]$ for each $i < 2$, then $\hat{X}_0, \hat{X}_1 \in \hat{V}$ are disjoint and each $\hat{X}_i \subseteq \hat{X}_i$. 

Thus:

**Theorem 4.4.** Suppose $V \models ZFC^-$ is transitive, and $j : V \preceq \hat{V}$ is $\omega$-nonstandard, and $\lambda$ is an infinite cardinal. Then the following are equivalent:

(A) $\hat{V}$ $\lambda^+$-pseudosaturates $T_{rg}$;

(B) $\hat{V}$ $\lambda^+$-pseudosaturates some unstable theory;

(C) $\lambda < \lambda_\varphi(\Delta(IP))$. 

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Proof. (A) implies (B) is trivial.
(B) implies (C): suppose $T \in V$ is unstable such that $\hat{V} \lambda^+$-pseudosaturates $T$, i.e. $\lambda < \lambda_{\hat{V}}(T)$. Now $T$ either has $SOP_2$ or else $IP$, by Theorem 2.6. If $T$ has $SOP_2$ then $\lambda_{\hat{V}}(T) \leq p_{\hat{V}} \leq \lambda_{\hat{V}}(\Delta(IP))$. If on the other hand $T$ has $IP$, then $\hat{T}$ admits $\Delta(IP)$ so we get $\lambda_{\hat{V}}(T) \leq \lambda_{\hat{V}}(\Delta(IP))$ in any case.

(C) implies (A): let $M|\mathcal{V} = T_{rg}$ with $M \in V$, and let $p(x)$ be a pseudofinite partial type over $\mathcal{A} \subseteq \mathcal{V}$, where $A$ is pseudofinite. Let $X_0 = \{ a \in A : R(x,a) \in p(x) \}$ (defined in $\mathcal{V}$) and let $X_1 = \{ a \in A : \neg R(x,a) \in p(x) \}$, and conclude by Lemma 4.3.

We immediately get the following corollary.

**Corollary 4.5.** $T_{rg}$ is a $\leq_1^\omega$-minimal unstable theory. That is, if $T$ is unstable then $T_{rg} \leq_1^\omega T$. Thus, this also holds for $\leq_\kappa^\omega$ and $\leq_\kappa^*$ for every $\kappa$, and also for $\leq$.

### 5 A Minimal Unsimple Theory

In [16], Malliaris proved the existence of a $\leq$-minimal $TP_2$ theory (namely, $T_{feq}$). In view of Theorem 2.5 and Theorem 2.9 (due to Malliaris and Shelah [25] for Keisler’s order), $T_{feq}$ must also be a $\leq$-minimal unsimple theory. More recently in [20], Malliaris and Shelah prove that $T_{feq}$ is a $\leq_1^\omega$-minimal unsimple theory.

We perform the routine translations into the language of $\leq_1^\omega$. However, we prefer to use the following as our flagship unsimple theory:

**Definition 5.1.** Let $T_{rf}$ be the theory of the random binary function. That is, $T_{rf}$ is the model completion of the empty theory in the language containing a single binary function symbol $F$.

$T_{rf}$ is shown to be $NSOP_1$ and to admit quantifier elimination in [11]. In particular, $T_{rf}$ is $NSOP_2$. Further, $T_{rf}$ is $TP_2$ via the formula $f(x,y_0) = y_1$.

We now proceed as in the previous section. The following definition is equivalent to the notion of $(\omega,\omega,1)$-arrays from [15].

**Definition 5.2.** Let $\Delta(TP_2)$ be the pattern on $\omega \times \omega$ consisting of all $s \in [\omega \times \omega]^{<\aleph_0}$ satisfying: for all $n < \omega$, $|s \cap \{n\} \times \omega| \leq 1$, i.e. $s$ is a partial function from $\omega$ to $\omega$.

The following is then trivial. (This is Claim 3.8 of [15], although our choice of definition of $TP_2$ absorbs all of the work.)

**Lemma 5.3.** Suppose $\varphi(x,y)$ is a formula of $T$. Then $\varphi(x,y)$ has $TP_2$ if and only if $\varphi(x,y)$ admits $\Delta(TP_2)$. Thus $T$ has $TP_2$ if and only if $T$ admits $\Delta(TP_2)$.

The following is essentially Theorem 6.9 of [16]. To explain the terminology: we are viewing a partial function $f : \hat{V} \to \hat{V}$ as a subset of $\hat{V}^2 \subseteq \hat{V}$, so the statement that $f$ is pseudofinite just means that the domain and range of $f$ are pseudofinite.
Lemma 5.4. Suppose \( V \models ZFC^- \) is transitive, and \( j : V \preceq \hat{V} \) is \( \omega \)-nonstandard, and \( \lambda \) is a cardinal. Then \( \lambda_{\hat{V}}(\Delta(TP_2)) \) is the least \( \lambda \) such that there is a pseudofinite partial function \( f : \hat{V} \to \hat{V} \) which cannot be extended to an internal partial function \( \hat{f} \) from \( \hat{V} \) to \( \hat{V} \).

Proof. Let \( \lambda \) be the least such cardinal as in the statement of the lemma.

First we show that \( \lambda_{\hat{V}}(\Delta(TP_2)) \leq \lambda \). Given some pseudofinite \( f \subseteq \hat{\omega} \times \hat{\omega} \) with \( [f]^{<\aleph_0} \subseteq j(\Delta(TP_2)) \), and of cardinality less than \( \lambda \), note that \( f \) is a partial function from \( \hat{V} \) to \( \hat{V} \), and so we can find some internal partial function \( \hat{f} \) from \( \hat{V} \) to \( \hat{V} \) extending \( f \). Choose \( \hat{n} < \hat{\omega} \) large enough so that \( f \subseteq \hat{n} \times \hat{n} \); let \( \hat{X} = \{ \hat{m} < \hat{n} : \hat{f}(\hat{m}) \) is defined and \( < \hat{n} \} \). Then \( \hat{f} \mid_{\hat{X}} \in j(\Delta(TP_2)) \).

Conversely, we show that \( \lambda \leq \lambda_{\hat{V}}(\Delta(TP_2)) \). Suppose \( f \) is a pseudofinite partial function from \( \hat{V} \) to \( \hat{V} \) of cardinality less than \( \lambda_{\hat{V}}(\Delta(TP_2)) \). We can find some \( \hat{X} \in \hat{V} \), finite in the sense of \( \hat{V} \), such that \( f \) is a partial function from \( \hat{X} \) to \( \hat{X} \). By relabeling, we can suppose \( \hat{X} = \hat{n} \times \hat{n} \). Then \( [f]^{<\aleph_0} \subseteq j(\Delta(TP_2)) \), and thus we can find \( \hat{f} \in j(\Delta(TP_2)) \) with \( f \subseteq \hat{f} \). Then \( \hat{f} \) is as desired.

Putting it all together:

Theorem 5.5. Suppose \( V \models ZFC^- \) is transitive, and \( j : V \preceq \hat{V} \) is \( \omega \)-nonstandard, and \( \lambda \) is a cardinal. Then the following are equivalent:

(A) \( \hat{V} \) \( \lambda^+ \)-pseudosaturates \( T_{rf} \);

(B) \( \hat{V} \) \( \lambda^+ \)-pseudosaturates some unsimple theory;

(C) \( \lambda < \lambda_{\hat{V}}(\Delta(TP_2)) \).

Proof. (A) implies (B) is trivial.

(B) implies (C): suppose \( T \in V \) is unsimple and \( \hat{V} \) \( \lambda^+ \)-pseudosaturates \( T \), i.e. \( \lambda < \lambda_{\hat{V}}(T) \). By Theorem 2.5, \( T \) either has SOP_2 or else TP_2; if \( T \) has SOP_2 then \( \lambda_{\hat{V}}(T) = p_{\hat{V}} \leq \lambda_{\hat{V}}(\Delta(TP_2)) \). If on the other hand \( T \) has TP_2, then \( T \) admits \( \Delta(TP_2) \) so we get \( \lambda_{\hat{V}}(T) \leq \lambda_{\hat{V}}(\Delta(TP_2)) \) in any case.

(C) implies (A): Let \( F : \omega^2 \to \omega \) be such that \( (\omega, F) \models T_{rf} \), and write \( \hat{F} = j(F) \) (we also use \( F \) to denote the symbol in the language). Let \( p(x) \) be a pseudofinite partial type over \( (\hat{\omega}, \hat{F}) \), say \( p(x) \) is over \( X \subseteq \hat{n} \) with \( |X| \leq \lambda \). We need to show \( p(x) \) is realized in \( (\hat{\omega}, \hat{F}) \). We can suppose \( p(x) \) is nonalgebraic.

Let \( \kappa = |\hat{n}| \) as computed in \( \mathbb{V} \); i.e. \( \kappa \) is the cardinality of \( \{ \hat{m} \in \hat{V} : \hat{m} < \hat{n} \} \) in \( \mathbb{V} \). I claim that \( \kappa > \lambda \). Suppose towards a contradiction that \( \kappa \leq \lambda \). In \( \mathbb{V} \), choose a bijection \( f : (\hat{n} - 1) \to \hat{n} \). By Lemma 5.4, we can find some internal partial function \( \hat{f} \) extending \( f \). But then \( \hat{f} \mid_{\hat{n} - 1} = f \), and \( \hat{f} \) cannot be internal, contradiction.

Thus we can find some \( Y \subseteq \hat{n} \) such that \( X \subseteq Y \) and \( |Y \setminus X| = \lambda \). Extend \( p(x) \) to a complete type \( q(x) \) over \( Y \) such that \( q(x) \) is induced by some function \( f : (Y \cup \{x\})^2 \to Y \cup \{x\} \). More precisely, \( q(x) \) is nonalgebraic, and for every \( a, b \in Y \cup \{x\} \), if we write \( c = f(a, b) \), then \( q(x) \models F(a, b) = c \). Note that \( q(x) \) is logically implied by \( \{ F(a, b) = c \} : a, b \in Y \cup \{x\}, c = f(a, b) \}. 

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Since $\lambda < \lambda_\nu(\Delta(T P_2))$, we can find some function $\hat{f} : (\hat{n} \cup \{x\})^2 \to \hat{n} \cup \{x\}$ extending $f$. Thus we can find $a_\ast \geq \hat{n}$ such that $\hat{F} \upharpoonright (\hat{n} \cup \{a_\ast\})$ is given by $\hat{f}$; then $a_\ast$ realizes $q(x)$.

We immediately get the following corollary.

**Corollary 5.6.** $T_{rf}$ is a $\preceq_1^\kappa$-minimal unsimple theory. That is, if $T$ is unsimple then $T_{rf} \preceq_1^\kappa T$. Thus, this also holds for $\preceq_\kappa^\kappa$ and $\preceq_\kappa^\kappa$ for every $\kappa$, and also for $\preceq_\kappa$.

### 6 A Minimal Nonlow Theory

In this section, we proceed similarly to Sections 4 and 5 to show that there is a minimal nonlow theory in $\preceq_1^\omega$, namely $T_{nlow}$. In [39] I proved that a different theory, $T_{Cas}$ (introduced by Casanovas in [2]) is a minimal nonlow theory in Keisler’s order. The arguments there show that $T_{Cas}$ is in fact a $\preceq_\aleph_1^\omega$-minimal nonlow theory; however, Malliaris and Shelah show in [18] that supersimplicity is a dividing line in $\preceq_1^\ast$, and hence $T_{Cas}$ is not a $\preceq_1^\ast$-minimal nonlow theory (seeing as it is not supersimple). $T_{nlow}$ is the first example of a supersimple nonlow theory [3], due to Casanovas and Kim.

We now describe $T_{nlow}$. The language $L_{nlow}$ is $(R, E, P, Q, U_n, P_n, Q_n^i, F_n : 1 \leq n < \omega, i < 2)$, where $P, Q, U_n, P_n, Q_n^i$ are each unary relation symbols, and $R, E$ are binary relation symbols, and $F$ is an $n$-ary relation symbol. (Our notation differs from [3]: our $P$ is their $Q^0$, our $Q$ is their $Q^1$, our $U_n$ is their $P_n$, our $P_n$ is their $Q_n^0$, and our $Q_n^i$ is their $Q_n^{i+1}$.)

$T_{nlow}$ is the model completion of the following axioms:

1. The universe is the disjoint union of $P$ and $Q$, both infinite;
2. $E$ is an equivalence relation on the universe;
3. $R \subseteq P \times Q$ and $R \subseteq E$;
4. Each $U_n$ is an equivalence class of $E$;
5. Each $P_n = U_n \cap P$;
6. $U_n \cap Q$ is the disjoint union of $Q_n^0$ and $Q_n^1$;
7. For each $a \in P_n$, the set $u_a := \{b \in Q_n^0 : R(a, b)\}$ has exactly $n$ elements; moreover $a \mapsto u_a$ is a bijection from $P_n$ to $[Q_n^0]^n$;
8. $F_n$ induces the bijection (also denoted $F_n$) from $[Q_n^0]^n$ to $P_n$ given by $F_n(u_a) = a$. More formally, for all $(b_0, \ldots, b_{n-1})$, if some $b_i \not\in Q_n^0$ or else there are $i < j$ with $b_i = b_j$, then $F_n(b_0, \ldots, b_{n-1}) = b_0$. Otherwise, $F_n(b_0, \ldots, b_{n-1}) \in P_n$, and $u_{F_n(b_0, \ldots, b_{n-1})} = \{b_0, \ldots, b_{n-1}\}$.
As notation, we will let $U_\omega$ be the complement of $\bigcup_{1 \leq n < \omega} U_n$, and we will let $P_\omega = P \setminus \bigcup_{1 \leq n < \omega} P_n$, and we will let $Q_\omega = Q \setminus \bigcup_{1 \leq n < \omega, i < 2} Q_n^i$. These are type-definable sets, and so we can view them also as partial types.

In [3] it is shown that $T_{n\text{low}}$ is well-defined. Moreover, $X$ is algebraically closed if and only if $X$ is closed under each $F_n$, and for all $a \in P_n$, $u_a \subseteq X$. Further, $X$ is supersimple, with forking relation given by: $A \downarrow_C B$ if and only if $\text{acl}(AC) \cap \text{acl}(BC) \subseteq \text{acl}(C)$, and further, if $a \in A \setminus \bigcup_n P_n$ and $a$ is not $E$-related to any element of $C$, then $a$ is not $E$-related to any element of $B$. (This is equivalent to $\text{acl}(AC) \cap \text{acl}(BC) \subseteq \text{acl}(C)$ once we eliminate the imaginaries $[a/E]$.) Finally, the formula $R(x, y)$ visibly witnesses that $T_{n\text{low}}$ has the finite dividing property, and hence is nonlow.

$T_{n\text{low}}$ does not have quantifier elimination (note that the axioms above are not universal). However, whenever $A$ is algebraically closed, then complete types over $A$ are determined by their quantifier-free part. The following lemma follows from the proof of Proposition 4.2 in [2]:

**Lemma 6.1.** Let $M \models T_{n\text{low}}$ and let $C \subseteq M$ be algebraically closed. Write $C = A \cup B$ where $A = C \cap P^M$ and $B = C \cap Q^M$. Write $A_n = A \cap P_n$ for each $1 \leq n < \omega$, and write $B_n^i = B \cap Q_n^i$ for each $1 \leq n < \omega$ and $i < 2$.

(I) Suppose $1 \leq n < \omega$. Let $X \subseteq [B_n^0]^{n-1} \times B_n^1$ be given. Let $p_X^I(x)$ be the partial type over $C$ which asserts:

- $Q_n^0(x)$;
- $\neg R(a, x)$ for each $a \in A_n$
- For every $v \in [B_n^0]^{n-1}$ and for every $b \in B_n^1$, $R(F_n(v \cup \{x\}), b)$ if and only if $(v, b) \in X$.

Then $p_X^I(x)$ generates a complete type over $C$ that does not fork over $\emptyset$. Moreover, all nonalgebraic complete types over $C$ extending $Q_n^0(x)$ are of this form.

(II) Suppose $1 \leq n < \omega$. let $X \subseteq A$ be given. Let $p_X^{II}(x)$ be the partial type over $C$ that asserts:

- $Q_n^1(x)$;
- $x \neq b$ for each $b \in B_n^1$;
- For each $a \in A_n$, $R(a, x)$ holds if and only if $a \in X$.

Then $p_X^{II}(x)$ generates a complete type over $C$ that does not fork over $\emptyset$. Moreover, all nonalgebraic complete types over $C$ extending $\{Q(x)\} \cup Q_n^1(x)$ are of this form.

(III) Suppose $c \in C \cap U_\omega$ and $X \subseteq A \cap [c]_E$. Then let $p_{X,c}^{III}(x)$ be the partial type over $C$ which asserts:

- $x \neq b$ for each $b \in B \cap [c]_E$;
- $x E c$;
- $Q(x)$;
- For every $a \in A \cap [c]_E$, $R(a, x)$ holds if and only if $a \in X$.

Then $p_{X, c}^{III}(x)$ generates a complete type over $C$ that does not fork over $\emptyset$. Moreover, all nonalgebraic complete types over $C$ extending $Q_\omega(x) \land x E c$ for some $c \in C$ are of this form.

(IV) Suppose $c \in C \cap U_\omega$ and $X \subseteq B \cap [c]_E$. Then let $p_{X, c}^{IV}(x)$ be the partial type over $C$ which asserts:
- $x \neq a$ for each $a \in A \cap [c]_E$;
- $x E c$;
- $P(x)$;
- For every $b \in B \cap [c]_E$, $R(x, b)$ holds if and only if $b \in X$.

Then $p_{X, c}^{IV}(x)$ generates a complete type over $C$ that does not fork over $\emptyset$. Moreover, all nonalgebraic complete types over $C$ extending $P_\omega(x) \land x E c$ for some $c \in C$ are of this form.

(V) There are unique types over $C$ extending $P_\omega(x)$ and $Q_\omega(x)$ respectively, which additionally assert that $x$ is not $E$-related to any element of $C$. Further, for each $n < \omega$ and for each nonalgebraic type $p(x)$ over $C$ extending $P_n(x)$, there is a type $q(\bar{x}) \in S^n(C)$ extending $\bigwedge_{i < n} Q_0^n(x_i)$, such that $p(x)$ is realized in $M$ if and only if $q(\bar{x})$ is realized in $M$; further $q(\bar{x})$ can be chosen independently of the choice of $M \supseteq C$. Namely, for some or any realization of $p(x)$ in the monster model, let $\bar{b}$ be an enumeration of $u_a$ and let $q(\bar{x}) = tp(\bar{b})$; this works because $a$ and $\bar{b}$ are interdefinable.

We now introduce the relevant patterns.

**Definition 6.2.** Given $I \subseteq \omega \setminus \{0\}$ infinite, let $\Delta_I(FDP)$ be the pattern on $I \times \omega$ defined by: $s \in \Delta_I(FDP)$ if $s \in \{m\} \times [\omega]^{<m}$ for some $m \in I$. Let $\Delta_I'(FDP)$ be the pattern on $I \times \omega$, defined to be all $s$ with each $|s \cap \{m\} \times \omega| \leq m$.

Write $\Delta(FDP) = \Delta_{\omega \setminus \{0\}}(FDP)$.

Easily, $\lambda_{\forall}(\Delta(TP_2)) \leq \lambda_{\forall}(\Delta(FDP))$. Also, $T_{nlow}$ admits $\Delta(FDP)$.

The following is straightforward.

**Lemma 6.3.** Suppose $\varphi(\bar{x}, \bar{y})$ is a formula of $T$. Then $\varphi(\bar{x}, \bar{y})$ has the finite dividing property if and only if for some infinite $I \subseteq \omega \setminus \{0\}$, $\varphi(\bar{x}, \bar{y})$ admits some $\Delta$ with $\Delta_I(FDP) \subseteq \Delta \subseteq \Delta_I'(FDP)$. Hence $T$ has the finite dividing property if and only if $T$ admits some such $\Delta$.  

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Suppose $\varphi(\overline{x}, \overline{y})$ admits some such $\Delta$, via \((\overline{a}_{m,n} : (m,n) \in I \times \omega)\). Then by compactness and Ramsey’s theorem, for each $m \in I$ we get an indiscernible sequence \((\overline{b}_{m,n} : n < \omega)\) such that \(\{\varphi(\overline{x}, \overline{b}_{m,n}) : n < \omega\}\) is $m$-consistent but $m + 1$-inconsistent. Hence $\varphi(\overline{x}, \overline{y})$ has the finite dividing property.

Conversely, suppose $\varphi(\overline{x}, \overline{y})$ has the finite dividing property; choose $I \subseteq \omega \setminus \{0\}$ infinite, and indiscernible sequences \((\overline{b}_{m,n} : n < \omega) : m \in I\)$, so that each \(\{\varphi(\overline{x}, \overline{b}_{m,n}) : m \in I\}\) is $m$-consistent but $m + 1$-inconsistent. Let $\Delta$ be the set of all $s \in [\omega \times \omega]^{<\aleph_0}$ such that \(\{\varphi(\overline{x}, \overline{b}_{m,n}) : (m,n) \in s\}\) is consistent. Clearly, $\Delta_I(FDP) \subseteq \Delta \subseteq \Delta^*_I(FDP)$. \(\square\)

The following allows us to compare the various $\lambda^*_I(\Delta)$'s from Lemma 6.3 in particular $\lambda^*_I(\Delta(FDP))$ is maximal among them. As convenient notation, if $\hat{V} \models ZFC^-$ and $\hat{n} < \hat{\omega}$, then let $[\hat{V}]^{\leq \hat{n}}$ denote $\{\hat{u} \in \hat{V} : \hat{V} \models |\hat{u}| \leq \hat{n}\}$.

**Lemma 6.4.** Suppose $V$ is a transitive model of $ZFC^-$, and $j : V \preceq \hat{V}$ is $\omega$-nonstandard. Suppose $I \subseteq \omega \setminus \{0\}$ is infinite, and $\Delta \in V$ is such that $\Delta_I(FDP) \subseteq \Delta \subseteq \Delta^*_I(FDP)$. If $\lambda < \lambda^*_I(\Delta)$, then for every $\hat{m}_* < \hat{\omega}$ with $\hat{m}_*$ nonstandard, and for every pseudofinite $X \subseteq \hat{V}$ of cardinality at most $\lambda$, there is $\hat{X} \in [\hat{V}]^{\leq \hat{m}_*}$ with $X \subseteq \hat{X}$. If $\Delta = \Delta_I(FDP)$, then the converse is true as well.

Hence $\lambda^*_I(\Delta) \leq \lambda^*_I(\Delta_I(FDP)) = \lambda^*_I(\Delta(FDP))$.

**Proof.** Suppose first $\lambda < \lambda^*_I(\Delta)$, and $\hat{m}_*$, $X$ are as above. By relabeling, we can suppose $X \subseteq \hat{n}_*$ for some $\hat{n}_* < \hat{\omega}$. By decreasing $\hat{m}_*$, we can suppose $\hat{m}_* \in j(I)$ while keeping $\hat{m}_*$ nonstandard. Write $Y = \{(\hat{m}_*, \hat{n}) : \hat{n} \in X\}$. Since $\hat{m}_*$ is nonstandard we have $[Y]^{<\aleph_0} \subseteq j(\Delta)$, thus we can find $\hat{Y} \in j(\Delta)$ with $Y \subseteq \hat{Y}$. Let $\hat{X} = \{\hat{n} < \hat{n}_* : (\hat{m}_*, \hat{n}) \in \hat{Y}\}$; then $\hat{X} \in [\hat{n}_*]^{\leq \hat{m}_*}$ with $X \subseteq \hat{X}$.

Next, suppose $\Delta = \Delta_I(FDP)$; let $Y \subseteq \hat{n}_* \times \hat{n}_*$ be of size at most $\lambda$ with $[Y]^{<\aleph_0} \subseteq j(\Delta_I(FDP))$. Then $Y \subseteq \{\hat{m}_*\} \times \hat{n}_*$ for some $\hat{m}_* < \hat{n}_*$ with $\hat{m}_* \in j(I)$, so let $X = \{\hat{n} < \hat{n}_* : (\hat{m}_*, \hat{n}) \in Y\}$. By hypothesis we can find $\hat{X} \supseteq X$ with $\hat{X} \in [\hat{V}]^{\leq \hat{m}_*}$. Then $\hat{Y} := \{\hat{m}_*\} \times (\hat{X} \cap \hat{n}_*)$ is as desired. \(\square\)

We can now wrap up the proof that $T_{\text{nonlow}}$ is a minimal nonlow theory. (B) implies (C) is due to Malliaris [13] in the context of regular ultrafilters on $\mathcal{P}(\lambda)$.

**Theorem 6.5.** Suppose $V \models ZFC^-$ is transitive, $j : V \preceq \hat{V}$ is $\omega$-nonstandard, and $\lambda$ is given. Then the following are equivalent:

(A) $\hat{V} \lambda^+\text{-pseudosaturates } T_{\text{nonlow}}$.

(B) $\hat{V} \lambda^+\text{-pseudosaturates some nonlow theory.}$

(C) $\lambda < \lambda^*_I(\Delta(IP))$ and $\lambda < \lambda^*_I(\Delta(FDP))$.

**Proof.** (A) implies (B) is trivial.

(B) implies (C): suppose $T \in V$ is nonlow; (B) is equivalent to $\lambda < \lambda^*_I(T)$. Now $T$ is unstable, so $\lambda^*_I(T) \leq \lambda^*_I(\Delta(IP))$. If $T$ is unsimple, then $\lambda^*_I(T) \leq \lambda^*_I(\Delta(TP_2)) \leq$
\(\lambda_T(\Delta(FDP))\), and if \(T\) has the finite dividing property then \(\lambda_T(T) \leq \lambda_T(\Delta(FDP))\) by Lemma 6.4. Hence \(\lambda_T(T) \leq \lambda_T(\Delta(FDP))\) in any case, and (C) holds.

(C) implies (A): suppose \(\lambda < \lambda_V(\Delta(IP))\) and \(\lambda < \lambda_V(\Delta(FDP))\), and let \(M \models T_{nlow}\) have universe \(\omega\) (say), with \(M \in V\). Write \(M = (\omega, R, B, E, P, Q, U_n, P_n, Q_n, F_n : 1 \leq n < \omega, i < 2)\) and write \(j(M) = (\hat{\omega}, R, E, \hat{P}, \hat{Q}, \hat{U}_n, \hat{Q}_n, \hat{F}_n : 1 \leq \hat{n} < \hat{\omega})\) (so \(j_{\text{std}}(M) = (\hat{\omega}, R, E, \hat{P}, \hat{Q}, \hat{U}_n, P_n; \hat{Q}_n, \hat{F}_n : 1 \leq \hat{n} < \hat{\omega})\)). In terms of our previous notation, we write, for instance, \(U_\omega = U \cup \bigcup_{n<\omega} \hat{U}_n\), a type-definable subset of \(j_{\text{std}}(M)\) which is not definable in \(\hat{V}\), and we write \(\hat{U}_\omega = \hat{U} \cup \bigcup_{n<\omega} \hat{U}_n\), a type-definable subset of \(j(M)\) in the sense of \(\hat{V}\) which is not type-definable in \(j_{\text{std}}(M)\).

We show that \(j_{\text{std}}(M)\) is \(\lambda^+\)-pseudosaturated.

So let \(p(x)\) be a pseudofinite partial type over \(j_{\text{std}}(M)\) of cardinality at most \(\lambda\). We first of all claim that we can suppose \(p(x)\) is a type over an algebraically closed set. Indeed, choose \(\hat{n}_0 < \hat{\omega}\) such that \(p(x)\) is over \(\hat{n}_0\). Since algebraic closures of finite sets in \(T_{nlow}\) are finite, we have that the algebraic closure of \(\hat{n}_0\) in \(j(M)\), as computed in \(\hat{V}\), is pseudofinite as desired. Let \(C = \text{acl}_{j_{\text{std}}(M)}(\hat{n}_0);\) then \(|C| \leq \lambda\), and further \(C \subseteq \text{acl}_{j(M)}(\hat{n}_0)\) is pseudofinite. Choose \(\hat{n}_* < \hat{\omega}\) with \(C \subseteq \hat{n}_*\). (If algebraic closures of finite sets were infinite, we would need to use overflow arguments instead.) Write \(A = C \cap \hat{P}\) and write \(B = C \cap \hat{Q}\). For each \(1 \leq \hat{n} < \hat{\omega}\), let \(A_{\hat{n}} = A \cap \hat{P}_{\hat{n}};\) for each \(1 \leq \hat{n} < \hat{\omega}\) and for each \(i < 2\), let \(B_{\hat{n}} = B \cap \hat{Q}_{\hat{n}}\).

We can suppose \(p(x)\) is a complete nonalgebraic type over \(C\). We can also suppose \(p(x)\) is of one of the forms (I) through (IV) of Lemma 6.1.

Suppose first \(p(x)\) is of form (I), (say), there is \(1 \leq n < \omega\) and \(X \subseteq [B_n]_{n-1} \times B_n^1\) such that \(p(x) = p_X^I(x)\). Since \(\lambda < \lambda_V(\Delta(IP))\), we can find disjoint \(Y_0, Y_1 \subseteq [\hat{Q}_n]_{n-1} \times Q_n^1\) with \(X \subseteq Y_0\) and with \([B_n]_{n-1} \times B_n^1 \setminus X \subseteq Y_1\). Let \(\hat{p}(x) \in \hat{V}\) be the partial type asserting that \(Q_n^0(x)\), and \(\neg R(a, x)\) for each \(a \in P_n \cap \hat{n}_*\), and for every \((v, b) \in Y_0, R(F_n(v \cup \{x\}, b))\) holds, and for every \((v, b) \in Y_1, R(F_n(v \cup \{x\}, b))\) fails. Easily, \(\hat{V}\) believes \(\hat{p}(x)\) is a consistent finite type, and hence \(\hat{p}(x)\) is realized. But \(p(x) \subseteq \hat{p}(x)\), so \(p(x)\) is realized as desired.

If \(p(x)\) is of form (II) or of form (III), or of the form \(p_{V,c}^X\) for some \(c \in \hat{U}_\omega\), then a similar argument works.

The crucial case is when \(p(x)\) is of the form \(p(x) = p_{V,c}^X(x)\) for some \(c \in C \cap \hat{U}_{\hat{m}}\), with \(\hat{m} < \hat{\omega}\) nonstandard. Write \(X_i = X \cap \hat{Q}_{i\hat{m}}^0\) for each \(i < 2\). Since \(\lambda < \lambda_V(\Delta(FDP))\) we can find \(\hat{X}_0 \subseteq [\hat{Q}_{\hat{m}}]_{\hat{m}-1} \times X_0 \subseteq \hat{X}_0\). Also, we can find disjoint \(\hat{Y}_0, \hat{Y}_1 \subseteq \hat{Q}_{\hat{m}}^1\) with \(X_1 \subseteq \hat{Y}_0\) and \(B_{\hat{m}}^1 \setminus X_1 \subseteq \hat{Y}_1\). Let \(\hat{p}(x) \in \hat{V}\) be the partial type asserting that \(x Ec\) and \(R(x, a)\) for each \(a \in X_0 \cup \hat{Y}_0\) and \(\neg R(x, a)\) for each \(a \in \hat{Y}_1\). Easily, \(\hat{V}\) believes \(\hat{p}(x)\) is a consistent finite type, and hence \(\hat{p}(x)\) is realized. But \(p(x) \subseteq \hat{p}(x)\), so \(p(x)\) is realized as desired.

We immediately get the following corollary.

**Corollary 6.6.** \(T_{nlow}\) is a minimal nonlow theory in \(\preceq_1^\chi\). Thus, this also holds for \(\preceq_\kappa^\chi\) and \(\preceq_\kappa^*\) for every \(\kappa\), and also for \(\preceq\).
7 Hypergraphs Omitting Cliques

In this section, we analyze the major class of examples of simple theories with interesting amalgamation properties.

**Definition 7.1.** For each $2 \leq k < n < \omega$, let $T_{n,k}$ be the theory of the random $k$-ary, $n$-clique free hypergraph.

These were introduced by Hrushovski [7], who proved that each $T_{n,2}$ is unsimple, in fact it has $SOP_2$ and so is maximal in Keisler’s order. We shall mainly be interested in the case of $T_{n,k}$ for $k \geq 3$; these are simple, with forking given by equality. In [21], Malliaris and Shelah prove that for all $k < k' - 1$, $T_{k+1,k} \not\subseteq T_{k+1,k'}$ (note that they subtract one from the indices). In [36], we show that this holds for all $k < k'$.

The following are the relevant patterns:

**Definition 7.2.** Suppose $S \subseteq [I]^k$ for some $k$, and suppose $n > k$. Then let $\Delta_{n,k}(S)$ be the pattern on $[I]^{k-1}$, consisting of all $s \subseteq [I]^{k-1}$ such that there is no $v \in [I]^{n-1} \setminus [v]^{k-1}$ such that $s \subseteq v$.

Clearly, then, if $S \subseteq [I]^k$ is $n$-clique free, then $(T_{n,k}, R(x,\overline{y}))$ admits $\Delta_{n,k}(S)$.

**Definition 7.3.** For each $k \geq 2$ and for each $n > k$, let $S_k$ be a random $k$-ary graph on $\omega$, and let $\Delta_n = \Delta_{n,k}(R_k)$. (Up to equivalence, this does not depend on the choice of $S_k$.)

Note that for every index set $I$ and for every $R \subseteq [I]^k$ and for every $n > k$, $\Delta_n, k(R)$ is an instance of $\Delta_{n,k}$. Also note that admitting $\Delta_{n,2}$ implies $SOP_3$.

It is not immediate that $T_{n,k}$ admits $\Delta_{n,k}$, since we did not require $S_k$ to be triangle-free. Towards this, the following fact will be helpful.

**Definition 7.4.** Suppose $\Delta$ is a pattern on $I$. For each $n < \omega$, let $\Delta^n$ be the pattern on $[I]^{\leq n}$ consisting of all $s \subseteq [I]^{\leq n}$ such that $\bigcup s \in \Delta$.

**Theorem 7.5.** Suppose $\Delta$ is a pattern on $I$ and $1 \leq n < \omega$.

1. If $T$ is a complete countable theory, then $T$ admits $\Delta$ if and only if $T$ admits $\Delta^n$.

2. If $V \models ZFC^-$ is transitive with $\Delta, I \subseteq V$ and if $j : V \subseteq \tilde{V}$ with $\tilde{V}$ not $\omega$-standard, then $\lambda_\tilde{V}(\Delta) = \lambda_\tilde{V}(\Delta^n)$.

3. If $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$, then $\lambda_\mathcal{U}(\Delta) = \lambda_\mathcal{U}(\Delta^n)$.

**Proof.** (1): Note that $\Delta$ is an instance of $\Delta^n$ (using $[I]^1 \subseteq [I]^{\leq n}$), so it suffices to show that if $T$ admits $\Delta$ then $T$ admits $\Delta^n$. Suppose $\varphi(x,y)$ admits $\Delta$ (really $x,y$ could be tuples). Let $\overline{y} = (y_i : i < n)$ and let $\psi(x,\overline{y}) = \bigwedge_{i<n} \varphi(x,y_i)$. Easily then $\psi(x,\overline{y})$ admits $\Delta^n$.

(2): Since $\Delta$ is an instance of $\Delta^n$, by Lemma 3.3 it suffices to show that $\lambda_\Delta(\Delta) \leq \lambda_\Delta(\Delta^n)$. Write $(I, \hat{\Delta}, \hat{\Delta}^n) = j(I, \Delta, \Delta^n)$, and suppose $X \subseteq \hat{\Delta}^n$ is pseudofinite and of cardinality less than $\lambda_\Delta(\Delta)$. Write $Y = \bigcup X$. Then $Y \subseteq \hat{\Delta}$ is pseudofinite and of cardinality less than $\lambda_\Delta(\Delta)$, so there is some $\hat{s} \in \hat{\Delta}$ with $Y \subseteq \hat{s}$. Then $\hat{s}^{\leq n} \in \hat{\Delta}^n$ satisfies $X \subseteq \hat{s}^{\leq n}$, as desired.

(3): follows from (2) and Theorem 8.5. 

\[\square\]
We then obtain the following:

**Lemma 7.6.** Suppose \( k \geq 2 \) and \( n > k \). Let \( S \) be any \( k \)-ary hypergraph on \( I \); then there is \( i_\ast < \omega \) and a \( k \)-ary, \( n \)-clique free hypergraph \( S' \) on \( I' \) such that \( \Delta_{n,k}(S) \) is an instance of \( \Delta_{n,k}(S')^{i_\ast} \).

**Proof.** Write \( I' = \omega \times (n-1) \). Let \( S' \) be the \( k \)-ary graph on \( I' \) consisting of the set of all order-preserving injections from \( v \) to \( n-1 \), for \( v \in S \). Since \( S' \) is \( n \)-clique free, it suffices to show that \( \Delta_{n,k}(S) \) is an instance of \( \Delta_{n,k}(S')^{i_\ast} \), for \( i_\ast := (\binom{n-1}{k-1}) \). In fact, we will embed all of \( \Delta_{n,k}(S) \) into \( \Delta_{n,k}(S')^{i_\ast} \) at once.

Given \( v \in [\omega]^{k-1} \), define \( F(v) \subseteq [\omega \times (n-1)]^{k-1} \) to be the set of all order-preserving injections from \( v \) to \( n-1 \). Easily, then, for every \( s \subseteq [I]^{k-1} \) finite, \( s \in \Delta_{n,k}(S) \) if and only if \( F[s] \in \Delta_{n,k}(S') \).

Thus each \( T_{n,k} \) admits \( \Delta_{n,k} \). Actually, more is true:

**Theorem 7.7.** Suppose \( 2 \leq n \leq n' \). Then \( T_{n',k} \) admits \( \Delta_{n,k} \).

**Proof.** By Lemma 7.6 \( T_{n',k} \) admits \( \Delta_{n',k} \), so by Theorem 7.5 it suffices to note that \( \Delta_{n,k} \) is an instance of \( \Delta_{n',k}^{i_\ast} \) for \( i_\ast \) large enough. For this it suffices to show that if \( S \subseteq [I]^k \), then there is some \( S' \subseteq [I']^k \) such that \( \Delta_{n,k}(S) \) is an instance of \( \Delta_{n',k}(S')^{i_\ast} \). Write \( I' = I \cup u_\ast \), where \( u_\ast \) is a new \( n'-n \)-element set. Put \( S' = S \cup \{v' \in [I']^k : v' \cap u_\ast \neq \emptyset \} \). Write \( i_\ast = (\binom{k-1+(n'-n)}{k-1}) \). We claim this works.

Indeed, suppose \( v \in [I]^{k-1} \); then define \( F(v) \subseteq [I']^{k-1} \) via \( F(v) = [v \cup u_\ast]^{k-1} \). Easily, for all \( s \subseteq [[I]^{k-1}]^{<\omega} \), \( s \in \Delta_{n,k}(S) \) if and only if \( F[s] \in \Delta_{n',k}(S')^{i_\ast} \), so this works.

We now aim to prove that \( T_{n,k} \) is the \( \leq \)-minimal theory admitting \( \Delta_{n,k} \). As a preliminary case, we have to show that if \( T \) admits \( \Delta_{n,k} \), then \( T \) is unstable (this should be clear but we spell out the details in our formalism).

**Lemma 7.8.** Suppose \( n > k \geq 2 \). Then \( \Delta(IP) \) is an instance of \( (\Delta_{n,k})^{i_\ast} \) for some \( i_\ast \).

**Proof.** By Theorem 7.7 it suffices to consider the case \( n = k + 1 \); in this case we will be able to set \( i_\ast = k-1 \).

Let \( u_\ast \) be a \( k-2 \)-element set.

Let \( S \) be the \( k \)-ary graph on \( \omega \times 2 \cup u_\ast \) consisting of all \( w \in [(\omega \times 2) \cup u_\ast]^k \) of the form \( u_\ast \cup \{(n,0),(n,1)\} \), for some \( n < \omega \). Then \( \Delta_{k+1,k}(S) \) is an instance of \( \Delta_{k+1,k} \), so it suffices to show that \( \Delta(IP) \) is an instance of \( \Delta_{k+1,k}(S)^{k-1} \).

Given \( (j,i) \in \omega \times 2 \), define \( F(j,i) \) to be the set of all \( v \in [u_\ast \cup \{(j,0),(j,1)\}]^{k-1} \) other than \( u_\ast \cup \{(j,i)\} \). Then clearly, for any \( s \subseteq [\omega \times 2] \) finite, \( s \in \Delta(IP) \) if and only if \( F[s] \in \Delta(S)^{k-1} \).

We thus obtain the following easily.
Theorem 7.7. Suppose $2 \leq k < \omega$, suppose $V \models ZFC^-$ is transitive, and suppose $j : V \subseteq \hat{V}$. Then $\hat{V}$ $\lambda^+$-pseudosaturates $T_{n,k}$ if and only if $\lambda < \lambda_{\hat{V}}(\Delta_{n,k})$. In particular, $T_{n,k}$ is a $\leq^1_{\mathfrak{i}^+}$-minimal theory admitting $\Delta_{n,k}$ (so this is true for the other interpretability orders as well).

Proof. By Theorem 7.7 if $\hat{V}$ $\lambda^+$-pseudosaturates $T_{n,k}$ then $\lambda < \lambda_{\hat{V}}(\Delta_{n,k})$.

So suppose $\lambda < \lambda_{\hat{V}}(\Delta_{n,k})$. Let $M \models T_{n,k}$ have universe $\omega$, and let $p(x)$ be a pseudofinite partial type over $j(\omega)(M) = (\omega, R)$. (We also use $R$ for the symbol in the language.)

Let $X_0 = \{ \hat{a} \in \hat{n} : R(x, \hat{a}) \in p(x) \}$ and let $X_1 = \{ \hat{a} \in \hat{n} : \neg R(x, \hat{a}) \in p(x) \}$. Since $[X_0]^{<\omega} \subseteq \hat{j}(\Delta_{n,k}(R))$ we can find $X'_0 \subseteq \hat{j}(\Delta_{n,k}(R))$ with $X_0 \subseteq X'_0$. By Lemma 7.8, we can find disjoint $X_0, X_1 \subseteq \hat{n}$ with $X_0 \subseteq X_0'$ and $X_1 \subseteq X_1'$; we can suppose $X_0 = X_0'$ (by replacing them with $X_0 \cap X_0'$).

Let $q(x) \in \hat{V}$ be the pseudofinite partial type, defined in $\hat{V}$ via: $q(x) = \{ R(x, \hat{a}) : \hat{a} \in X_0 \} \cup \{ \neg R(x, \hat{a}) : \hat{a} \in X_1 \}$. Clearly $p(x) \subseteq q(x)$ and $\hat{V}$ believes $q(x)$ is a consistent finite type, so $q(x)$ must be realized in $j(M)$ and we are done. \qed

Corollary 7.10. For all $n' \geq n \geq k$, $T_{n,k} \leq_{\mathfrak{i}^+} T_{n',k}$.

Before moving on, we show that if $T$ admits $\Delta_{n,k}$ then it does so in a particularly nice way. Some definitions are in order:

Definition 7.11. Suppose $n > k \geq 2$. Let $L_{n,k}^-$ be the language $\{ \langle, S \rangle \}$, where $S$ is $k$-ary, and let $K_{n,k}^-$ be the class of all finite $L$-structures $(M, \langle^M, S^M \rangle)$ where $\langle^M$ is a linear order and where $S^M \subseteq [M]^k$ (i.e. is irreflexive and symmetric). Let $L_{n,k} = L_{n,k}^- \cup \{ Q \}$, where $Q$ is $k - 1$-ary, and let $K_{n,k}$ be the class of all finite substructures $(M, \langle^M, S^M, Q^M \rangle)$, where $\langle^M$ is a linear order, and $S^M \subseteq [M]^k$, and $Q^M \subseteq [M]^{k-1}$, and there is no $w \in [M]^{n-1}$ with $[w]^k \subseteq S$ and $[w]^{k-1} \subseteq Q$.

Clearly, $K_{n,k}$ and $K_{n,k}^-$ are Fraisse classes, and hence we can let $\tilde{T}_{n,k}$ and $\tilde{T}_{n,k}^-$ be the limit theories.

In the following theorem, given structures $M, N$, we let $(N)_M$ be the set of all substructures of $N$ which are isomorphic to $M$.

Theorem 7.12. $K_{n,k}$ and $K_{n,k}^-$ are both Ramsey classes—that is, letting $K$ be either of these classes, then whenever $A \subseteq B \in K$, and whenever $n < \omega$, there is some $C \in K$ such that whenever $c : \binom{C}{n} \to n$ is a coloring, there is some $B' \in \binom{B}{n}$ such that $c$ is constant on $\binom{B'}{n}$.

Proof. This is a special instance of the Nešetřil-Rödl theorem 28. \qed

We can apply this to our situation in the following manner:
Theorem 7.13. Suppose \( n > k \geq 2 \). \( T \) is a complete countable formula, and suppose \( \varphi(x, y) \) is a formula of \( T \) (where possibly \( x, y \) are tuples) which admits \( \Delta_{n,k} \). Let \( \mathcal{C} \) be the monster model of \( T_{n,k} \).

Then for any \( M \in \mathbf{K}_{n,k} \) we can find a sequence \( (b_u : u \in [M]^{k-1}) \) from \( \mathcal{C} \), and some \( a \in \mathcal{C} \), such that, writing \( M^- := M \upharpoonright_{\mathcal{L}_{n,k}} \in \mathbf{K}^-_{n,k} \):

- For every \( w \in [M]^{<\aleph_0} \), \( tp_{\mathcal{C}}(b_u : u \in [w]^{k-1}) \) depends only on \( tp_{M^-}(w) \) (for this to make sense, we are using that \( w \) has a canonical enumeration from \( <^M \), and hence also \( (b_u : u \in [w]^{k-1}) \) can be enumerated unambiguously by fixing an ordering of \( |w|^{k-1} \);

- For every \( w \in [M]^{<\aleph_0} \), \( tp_{\mathcal{C}}(a/(b_u : u \in [w]^{k-1})) \) depends only on \( tp_{M}(w) \);

- For every \( u \in Q^M \), \( \mathcal{C} \models \varphi(a, b_u) \);

- For every \( s \in [[M^{k-1}]^{<\aleph_0}] \), \( \{ \varphi(x, b_u : u \in s) \} \) is consistent if and only if there is no \( w \in [M]^{n-1} \) such that \( |w|^k \subseteq S^M \) and \( |w|^{k-1} \subseteq s \).

Proof. This follows easily from Theorem 7.12 and compactness.

We deduce two interesting corollaries from this. The following corollary is likely not optimal:

Corollary 7.14. Suppose \( k, k' \geq 2 \). Then \( T_{k'+1,k'} \) admits \( \Delta_{k+1,k} \) if and only if \( k = k' \).

More generally, suppose \( n > k > 2 \) and \( n' > k' \geq 2 \), and either \( \binom{n-1}{k-1} < \binom{k-1}{k-1} \) or else \( \binom{n-1}{k-1} < k' \). Then \( T_{n',k'} \) does not admit \( \Delta_{n,k} \).

Proof. Note that the first statement follows, since \( T_{n,k} \) always admits \( \Delta_{n,k} \) (by Lemmas 7.5 and 7.6), and when \( k = 2 \) and \( k' > 2 \) then \( T_{k'+1,k'} \) has \( NSOP_3 \) and hence cannot admit \( T_{k+1,k} \).

Suppose towards a contradiction that \( n > k > 2 \) and \( n' > k' \geq 2 \), and either \( \binom{n-1}{k-1} < \binom{k-1}{k-1} \) or else \( \binom{n-1}{k-1} < k' \), and yet \( T_{n',k'} \) admits \( \Delta_{n,k} \).

Let \( \varphi(\bar{x}, \bar{y}) \) be a formula of \( T_{n',k'} \) admitting \( \Delta_{n,k} \). Let \( (\mathcal{C}, R^\mathcal{C}) \) be the monster model of \( T_{n,k} \) and let \( M = \mathcal{C}^-_{n,k} \), write \( M^- = M \upharpoonright_{\mathcal{L}_{n,k}} = \mathcal{C}^-_{n,k} \). Then we can find \( \bar{b}_u : u \in [M]^{k-1} \) from \( \mathcal{C}^\mathcal{C} \) and \( \bar{a} \in \mathcal{C}^\mathcal{C} \) as in Theorem 7.12.

Let \( q(\bar{y}) = tp(\bar{b}_u) \) for some or any \( u \in [M]^{k-1} \). Let \( p(\bar{x}, \bar{y}) = tp(\bar{a}, \bar{b}_u) \) for some or any \( u \in Q^M \), so \( p(\bar{x}, \bar{y}) \) extends \( q(\bar{y}) \). Choose \( w \in S^M \). Let \( r(\bar{x}, \bar{b}_u : u \in [w]^{k-1}) := \bigcup \{ p(\bar{x}, \bar{b}_u) : u \in [w]^{k-1} \} \).

Note that \( r(\bar{x}) \) is inconsistent, since \( w \in S^M \) and \( \varphi(\bar{x}, \bar{b}_u) \in p(\bar{x}, \bar{b}_u) \) for each \( u \in [w]^{k-1} \). Nonetheless, I claim that for every proper \( X \subseteq [w]^{k-1} \), \( \bigcup \{ p(\bar{x}, \bar{b}_u) : u \in X \} \) is consistent; suppose towards a contradiction this failed. Choose \( \{ v(u) : u \in X \} \) from \( Q^M \), such that \( tp_{M^-}(v(u) : u \in X) = tp_{M^-}(u : u \in X) \). Then \( \bigcup \{ p(\bar{x}, \bar{b}_{v(u)}) : u \in X \} \) is inconsistent; but \( \bar{a} \) realizes it.

In particular, for all \( u_0, u_1 \in [w]^{k-1} \), \( p(\bar{x}, \bar{b}_{u_0}) \cup p(\bar{x}, \bar{b}_{u_1}) \) is consistent (using \( k > 2 \)). Hence the only way for \( r(\bar{x}) \) to be inconsistent is for it to create an \( n' \)-clique. In other words, we
must be able to find \( \overline{y} = (y_i : i < i_*) \) from \( \overline{x} \) and \( r = \{a_j : j < j_*\} \in [\mathcal{C}]^{j_*} \), such that 
\[ i_* + j_* = n' + 1, \text{ and such that } r \text{ is an } R^e\text{-clique}, \text{ and for every } t \in [r]^{< k'}, \text{ there is some } u(t) \in [w]^{k-1} \text{ such that } t \subseteq u(t) \text{ and such that } p(\overline{x}, \overline{b}_{u(t)}) \models \text{"} \overline{y} \cup t \text{ is an } R\text{-clique} \text{"}. \]

Note that if \( |r| < k' \) then this implies \( p(\overline{x}, u(r)) \) is inconsistent, a contradiction. So \( |r| \geq k' \).

Suppose first \((n' - 1)/k'_{k-1} < (n-1)/k_{k-1}\). Let \( X = \{u(t) : t \in [r]^{k'-1}\} \). Then \( X \subseteq [w]^{k-1} \) and yet 
\[ \bigcup \{p(\overline{x}, b_u) : u \in X\} \text{ is inconsistent, contradiction.} \]

Finally, suppose \((n-1)/k_{k-1} < k'\). For each \( u \in [w]^{k-1} \), choose \( j_u < j_* \) such that \( a_{j_u} \not\in b_u \) (possible as otherwise \( p(\overline{x}, b_u) \) would be inconsistent). But then \( \{a_{j_u} : u \in [w]^{k-1}\} \) cannot be covered by any \( \overline{b}_u \); since \( \{a_{j_u} : u \in [w]^{k-1}\} \) has cardinality at most \( k' - 1 \), this is a contradiction.

\[ \square \]

Also, the following corollary will be helpful in [36]:

**Corollary 7.15.** Suppose \( T \) is a countable simple theory and \( \varphi(x, y) \) is a formula of \( T \) which admits \( \Delta_{n,k} \) (so \( n > k \geq 3 \), since \( T \) is simple). Let \( \mathcal{C} \) be the monster model of \( T \). Then for every index set \( I \) and for every \( S \subseteq [I]^k \), we can find some countable \( N \preceq \mathcal{C} \) and some \( (b_u : u \in [I]^{k-1}) \) from \( \mathcal{C} \), such that for all \( s \in [[I]^{k-1}]^{< \aleph_0} \):

- If there is no \( w \in [I]^{n-1} \) such that \([w]^k \subseteq S \) and \([w]^{k-1} \subseteq s \), then \( \{\varphi(x, b_u) : u \in [w]^{k-1}\} \) does not fork over \( N \);
- Otherwise, \( \{\varphi(x, b_u) : u \in [w]^{k-1}\} \) is inconsistent.

**Proof.** Let \( M \models \hat{T}_{n,k} \) be \([I]^+\)-saturated and let \( M^- = M \upharpoonright_{\hat{T}_{n,k}} \). Let \( \mathcal{C} \) be the monster model of \( T_{n,k} \). Choose \( (b_u : u \in [M]^{k-1}) \) and \( a \) from \( \mathcal{C} \) as in Theorem 7.12.

Choose \( M_0 \preceq M \) countable such that \( tp(a/b_{u_0} : u \in [M]^{k-1}) \) does not fork over \( (b_u : u \in [M_0]^{k-1}) \), and let \( N \preceq \mathcal{C} \) be a countable elementary substructure containing \( (b_u : u \in [M_0]^{k-1}) \). By the saturation hypothesis on \( M \), there is an embedding of \( (I, S) \) into \( (M\setminus M_0, S \cap [M\setminus M_0]^k) \). Hence, it suffices to show that for all \( s \in [[M\setminus M_0]^{k-1}]^{< \aleph_0} \), if there is no \( w \in [M\setminus M_0]^{n-1} \) such that \([w]^k \subseteq S^M \) and \([w]^{k-1} \subseteq s \), then \( \{\varphi(x, b_u) : u \in [w]^{k-1}\} \) does not fork over \( N \). Suppose \( w \) is given as such. Write \( v = \bigcup s \). Since \( M \) is \( \aleph_1 \)-saturated, we can find some \( v' \in [M\setminus M_0]^{< \aleph_0} \) such that \( tp_{M^-}(v/M_0) = tp_{M^-}(v'/M_0) \) and such that, if \( f : v \to v' \) is the unique order-preserving bijection, then for all \( u \in s \), \( f[u] \in Q^M \). Then \( \{\varphi(x, b_{f[u]}) : u \in s\} \) does not fork over \( N \), since this is a subset of \( tp(a/b_{u} : u \in [M]^{k-1}) \). Hence \( \{\varphi(x, b_u) : u \in s\} \) does not fork over \( N \). \[ \square \]

### 8 The Analysis in Terms of Ultrafilters

We phrase what we have done so far in terms of ultrafilters.
Definition 8.1. Given an index set $I$ and a complete Boolean algebra $B$, an $I$-distribution in $B$ is a function $A : [I]^{<\omega} \to B_+$ (i.e. the positive elements of $B$), such that $A(\emptyset) = 1$, and $s \subseteq t$ implies $A(s) \geq A(t)$. If $D$ is a filter on $B$, we say that $A$ is in $D$ if $\text{im}(A) \subseteq D$.

If $A, B$ are $I$-distributions in $B$, then say that $B$ refines $A$ if $B(s) \leq A(s)$ for all $s \in [I]^{<\omega}$. Say that $A$ is multiplicative if for all $s \in [I]^{<\omega}$, $A(s) = \bigwedge_{t \subseteq s} A(\{i\})$.

Suppose $I, J$ are index sets, $B$ is a complete Boolean algebra, $A$ is a $J$-distribution and $\Delta$ is a pattern on $I$. Then say that $A$ is a $(J, \Delta)$-distribution if for every $s \in [J]^{<\omega}$ and for every $c \in B_+$ such that $c$ decides $A_t$ for all $t \subseteq s$, there is some $f : s \to I$ such that for all $t \subseteq s$, $c \leq A_{f[t]}$ if and only if $f[t] \in \Delta$. If $T$ is a complete countable theory and $\varphi(\bar{\pi}, \bar{\eta})$ is a formula of $T$, then say that $A$ is a $(T, \varphi)$-Lo"{s} map if $A$ is a $(J, \Delta, \varphi)$-distribution.

Suppose $U$ is an ultrafilter on the complete Boolean algebra $B$, and $\Delta$ is a pattern on $I$. Then let $\lambda_U(\Delta)$ be the least $\lambda$ such that there is some $(\lambda, \Delta)$-distribution in $U$ with no multiplicative refinement in $U$. If $\varphi(\bar{\pi}, \bar{\eta})$ is a formula of $T$ then let $\lambda_U(T, \varphi) = \lambda_U(\Delta, \varphi)$. In other words, $\lambda_U(T, \varphi)$ is the least $\lambda$ such that there is some $(\lambda, T, \varphi)$-Lo"{s} map $A$ in $U$ with no multiplicative refinement in $U$.

Let $\lambda_U(T)$ be the least infinite cardinal $\lambda$ such that $U$ does not $\lambda^+$-saturates $T$. Then always $\lambda_U(T) \geq \aleph_1$, and possibly $\lambda_U(T) = \infty$. Further, $U$ always $\lambda_U(T)$-saturates $T$.

Remark 8.2. In [37], we defined the notion of a $(\lambda, T, \varphi)$-Lo"{s} map for sequences of formulas $(\varphi_\alpha(\bar{\pi}, \bar{\eta}_\alpha) : \alpha < \lambda)$, where the $\bar{\eta}_\alpha$’s are all disjoint from each other and from $\bar{\pi}$. The above definition corresponds to the special case where each $\varphi_\alpha(\bar{\pi}, \bar{\eta}_\alpha) = \varphi(\bar{\pi}, \bar{\eta}_\alpha)$ for some fixed formula $\varphi(\bar{\pi}, \bar{\eta})$. We also defined that a $A$ is a $(\lambda, T)$-Lo"{s} map if it is a $(\lambda, T, \varphi)$-Lo"{s} map for some $\varphi$.

We have the following important but straightforward consequences of Theorem 2.10.

Lemma 8.3. Suppose $U$ is an ultrafilter on the complete Boolean algebra $B$, suppose $V \models ZFC^-$ is transitive, suppose $i : V \not\preceq V$ is $\lambda^+$-saturated, and suppose $\Delta \in V$ is a pattern on $I$. Finally, suppose $A$ is a $J$-distribution in $B$. Then $A$ is a $(J, \Delta)$-distribution if and only if there are $(a_j : j \in J)$ from $\text{std}(I)$, such that for all $s \in [J]^{<\omega}$, $\|\{a_j : j \in s\} \in i(\Delta)\|_V = A(s)$. The above definition corresponds to the special case where each $\varphi_\alpha(\bar{\pi}, \bar{\eta}_\alpha) = \varphi(\bar{\pi}, \bar{\eta}_\alpha)$ for some fixed formula $\varphi(\bar{\pi}, \bar{\eta})$. We also defined that a $A$ is a $(\lambda, T, \varphi)$-Lo"{s} map if it is a $(\lambda, T, \varphi)$-Lo"{s} map for some $\varphi$.

Proof. First, suppose $A$ is a $(J, \Delta)$-distribution. Choose $V_0 \preceq V$ with $|V_0| = \lambda$, such that $i(I, \Delta) \in V_0$.

We claim that Theorem 2.10 implies there is $V_1 \supseteq V_0$ and $a_j \in V_1$ for each $j \in J$, such that each $\|a_j \in i(I)\|_{V_1} = 1$, and such that for all $s \in [J]^{<\omega}$, $\|\{a_j : j \in s\} \in i(\Delta)\|_{V_1} = A(s)$. Indeed, let $\Gamma = L(V_0) \cup \{a_j \in i(I) : j \in J\} \cup \{\{a_j : j \in s\} \in i(\Delta) : s \in [J]^{<\omega}\}$. We define $F_0 = F_1 = F : \Gamma \to B$ via $F|_{L(V_0)} = \| \cdot \|_{V_0}$, and $F(\{a_j \in i(I)\}) = 1$, and $F(\{a_j : j \in s\} \in i(\Delta)) = A(s)$. Then clearly Theorem 2.10 applies and gives $V_1$ as desired.

Since $V$ is $\lambda^+$-saturated, we can in fact choose such $(a_j : j \in J)$ from $V$.

Conversely, suppose $(a_j : j \in J)$ are given. Suppose $s \in [J]^{<\omega}$ and $c \in B_+$ are given, such that $c$ decides $A_t$ for every $t \subseteq s$. Let $\Delta_0 = \{t \subseteq s : c \leq A_t\}$, a pattern on $s$. We need to find $\{i_j : j \in s\}$ such that for all $t \subseteq s$, $\{i_j : j \in t\} \in \Delta$ if and only if $t \in \Delta_0$. Suppose towards a contradiction this were impossible. Then by elementarity of $V \preceq V$, we would
have \(||\text{there exists } (b_j : j \in J) \text{ from } i(I) \text{ such that for all } t \subseteq s, \{b_j : j \in t\} \in i(\Delta) \text{ if and only if } t \in \Delta_0 \vdash V = 0. \text{ But } 0 < c \leq ||(a_j : j \in J) \text{ is a sequence from } i(I) \text{ and for all } t \subseteq s, \{a_j : j \in t\} \in i(\Delta) \text{ if and only if } t \in \Delta_0 \vdash V, \text{ contradiction.} \square"

**Lemma 8.4.** Suppose \( \mathcal{U} \) is an ultrafilter on the complete Boolean algebra \( \mathcal{B} \), suppose \( V \models ZFC^- \) is transitive, suppose \( i : V \preceq V \) is \( \lambda^+ \)-saturated, and suppose \( \Delta \in V \) is a pattern on \( I \). Suppose \( A \) is a \((J, \Delta)\)-distribution in \( \mathcal{B} \) as witnessed by \( (a_j : j \in J) \), i.e. each \( a_j \in i_{\text{std}}(I) \), and for all \( s \in [J]^{<\aleph_0}, \{a_j : j \in s\} \in i(\Delta) \| V = A(s) \). Finally, suppose \( B \) is a \( J \)-distribution in \( \mathcal{B} \). Then \( B \) is a multiplicative refinement of \( A \) if and only if there is some \( Y \in i_{\text{std}}(\Delta) \) such that for all \( s \in [J]^{<\aleph_0}, \{a_j : j \in s\} \subseteq Y \| V = B(s) \).

**Proof.** First, suppose \( B \) is a multiplicative refinement of \( A \). Choose \( V_0 \preceq V \) with \( |V_0| \leq \lambda \), such that \( i(I), i(\Delta) \) and each \( a_j \in V_0 \).

We claim that Theorem 2.10 implies there is \( V_1 \supseteq V_0 \) and \( Y \in V_1 \) such that \( |Y| \in i(\Delta) \| V = 1 \), and for all \( s \in [J]^{<\aleph_0}, \{a_j : j \in s\} \subseteq Y \| V = B(s) \). Indeed, let \( \Gamma = \mathcal{L}(V_0) \cup \{ \text{"} Y \in i(\Delta) \text{"} \} \cup \{ \{a_j : j \in s\} \subseteq Y : s \in [J]^{<\aleph_0} \} \). We define \( F_0 = F_1 = F : \Gamma \rightarrow \mathcal{B} \) via \( F \big|_{\mathcal{L}(V_0)} = \| : \{a_j : j \in s\} \} \subseteq Y \| V = 1, \) and \( F(\{a_j : j \in s\} \subseteq Y) = B(s) \). Then clearly Theorem 2.10 applies and gives \( V_1 \) as desired.

Since \( V \) is \( \lambda^+ \)-saturated, we can in fact choose such \( Y \in V \).

Conversely, suppose \( Y \) is given. Trivially, \( B \) is multiplicative. Furthermore, since each \( B(s) = \{a_j : j \in s\} \subseteq Y \| V \) and \( |Y| \in i(\Delta) \| V = 1 \), we have that \( B(s) \leq \{a_j : j \in s\} \in i(\Delta) \| V = A(s) \), so \( B \) refines \( A \). \( \square \)

As a first example of how these lemmas are used, we have the following. It strengthens a theorem from [37], which states that \( \mathcal{U} \lambda^+ \)-saturates \( T \) if and only if every \((\lambda, T)\)-\( \text{Lo"{s}} \) map in \( \mathcal{U} \) has a multiplicative refinement in \( \mathcal{U} \).

**Theorem 8.5.** Suppose \( \mathcal{U} \) is an ultrafilter on the complete Boolean algebra \( \mathcal{B} \), suppose \( V \models ZFC^- \) is transitive, suppose \( i : V \preceq V \) is \( \lambda^+ \)-saturated, and suppose \( \Delta \in V \) is a pattern on \( I \). Then \( \lambda < \lambda_\mathcal{U}(\Delta) \) if and only if \( \lambda < \lambda_{\mathcal{V}/\mathcal{U}}(\Delta) \). Thus, for all complete countable theories \( T, \lambda_\mathcal{U}(T) \) is the minimum of all \( \lambda_\mathcal{U}(T, \varphi(\overline{x}, \overline{y})) \), for \( \varphi(\overline{x}, \overline{y}) \) a formula of \( T \); this is the same as the minimum of all \( \lambda_\mathcal{U}(T, \varphi(x, \overline{y})) \) for all formulas \( \varphi(x, \overline{y}) \) of \( T \) with \( x \) a single variable.

**Proof.** Write \( \hat{V} = V/\mathcal{U} \) and let \( j : V \preceq \hat{V} \) be the composition \( \lfloor \cdot \rfloor_\mathcal{U} \circ i \). We know that \( \mathcal{U} \lambda^+ \)-saturates \( T \) if and only if \( \hat{V} \lambda^+ \)-pseudosaturates \( T \) if and only if \( \lambda < \lambda_{\hat{V}}(T) \), and since \( \hat{V} \) is \( \aleph_1 \)-saturated, we know by Theorem 3.7 that \( \lambda_{\hat{V}}(T) = \lambda_{\hat{V}}^{\text{loc}}(T) = \lambda_{\hat{V}}^{\text{loc}, 1}(T) \). Thus it suffices to show the first claim. So suppose \( \Delta \in V \) is a pattern on \( I \).

Suppose first \( \lambda \geq \lambda_\mathcal{U}(\Delta) \). Then we can find a \((\lambda, \Delta)\)-distribution in \( \mathcal{U} \) with no multiplicative refinement in \( \mathcal{U} \); call it \( A \). By Lemma 8.3, we can choose \( (a_\alpha : \alpha < \lambda) \) from \( i_{\text{std}}(I) \), such that for all \( s \in [\lambda]^{<\aleph_0}, \{a_\alpha : \alpha \in s\} \in i(\Delta) \| V = A(s) \). Write \( \hat{a}_\alpha = [a_\alpha]_\mathcal{U} \) and write \( X = \{\hat{a}_\alpha : \alpha < \lambda\} \subseteq j(I) \). Since \( V \) is \( \lambda^+ \)-saturated, an easy application of Theorem 2.10 shows that \( X \) is pseudofinite (this is Lemma 4.8 of [38]). It suffices to show there is no \( Y \in i(\Delta) \) with \( \hat{X} \subseteq \hat{Y} \); suppose towards a contradiction there was some such \( Y \). Write \( \hat{Y} = \ldots \)
Choose \( Y \in V \) such that \( \| Y' \in i(\Delta) \) then \( Y = Y' \), and otherwise \( Y = \emptyset \| V = 1; \) then \( [Y]_\mu = \hat{Y} \) and further \( \| Y \in i(\Delta) \| V = 1 \). Define \( B(s) = \| \{ a_\alpha : \alpha \in s \} \subseteq Y \| V; \) by Lemma \[8.3\] this is a multiplicative refinement of \( A \) in \( U \), contradicting the choice of \( A \). Thus no such \( Y \) exists, and this witnesses \( \lambda \geq \lambda^*_V(\Delta) \).

Conversely, if \( \lambda \geq \lambda^*_V(\Delta) \), then we can find some pseudofinite \( X \subseteq j(I) \) of size at most \( \lambda \) with \( [X]_{<\aleph_0} \subseteq j(\Delta) \), such that there is no \( \hat{Y} \in j(\Delta) \) with \( X \subseteq \hat{Y} \). Enumerate \( X = \{ [a_\alpha]_\mu : \alpha < \lambda \}; \) by a similar trick as above, we can arrange that each \( \| a_\alpha \in i(I) \| V = 1 \). For each \( s \in [\lambda]_{<\aleph_0} \) define \( A(s) = \| \{ a_\alpha : \alpha \in s \} \in i(\Delta) \| V. \) Then \( A \) is an \( (I, \Delta) \)-distribution in \( U \), by Lemma \[8.3\] Suppose towards a contradiction that \( B \) were a multiplicative refinement of \( A \) in \( U \). By Lemma \[8.4\], we can choose \( Y \in i_{std}(\Delta) \) such that for all \( s \in [\lambda]_{<\aleph_0} \), \( \| \{ a_\alpha : \alpha \in s \} \in Y \| V = B(s); \) then \( \hat{Y} = [Y]_\mu \) contradicts our choice of \( X \).

We now give characterizations of \( \lambda_\mu(\Delta(IP)) \), \( \lambda_\mu(\Delta(TP_2)) \) and \( \lambda_\mu(\Delta(FDP)) \).

**Lemma 8.6.** Suppose \( B \) is a complete Boolean algebra and \( U \) is an ultrafilter on \( B \). Write \( I = \lambda \times 2 \). Then \( \lambda_\mu(\Delta(IP)) \) is the least \( \lambda \) such that there is a \( I \)-distribution \( A \) in \( U \) of the following form, with no multiplicative refinement in \( U \). Namely, for some \( V \models ZFC^- \) and for some \( (a_\alpha, i : (\alpha, i) \in I) \) from \( V \), we have that each \( A(s) = \bigwedge \{ \| a_\alpha, 0 \neq a_\beta, 1 \| V : (\alpha, 0), (\beta, 1) \in s \} \).

**Proof.** Easily, any such \( A \) is a \( (\lambda \times 2, \Delta(IP)) \)-distribution. Conversely, suppose \( A \) is a given \( (\lambda, \Delta(IP)) \)-distribution in \( U \). Choose some transitive \( V \models ZFC^- \), choose \( i : V \subseteq V \) with \( V \lambda^+ \)-saturated, and choose \( (x_\alpha : \alpha < \lambda) \) a pseudofinite sequence from \( i_{std}(\omega \times \omega) \) such that for all \( s \in [\lambda]_{<\aleph_0} \), \( \| \{ x_\alpha : \alpha \in s \} \in i(\Delta(IP)) \| V = A(s); \) this is possible by Lemma \[8.3\]

For each \( \alpha < \lambda \), choose \( m_\alpha \) such that \( \| x_\alpha \in \{ m_\alpha \} \times 2 \| V = 1 \), and choose \( k_\alpha \) such that \( \| x_\alpha \in \omega \times \{ k_\alpha \} \| V = 1 \); this is possible by fullness of \( V \) (note \( k_\alpha \) is determined by the pair \( (\| k_\alpha = 0 \| V, \| k_\alpha = 1 \| V) \). For each \( \alpha < \lambda \), there is a unique \( f(\alpha) < 2 \) such that \( \| k_\alpha = f(\alpha) \| V \in U \). For each \( s \in [\lambda]_{<\aleph_0} \), let \( A'(s) = A(s) \wedge \bigwedge_{\alpha \in s} \| k_\alpha = f(\alpha) \| V; \) then this is a conservative refinement of \( A \) in \( U \). Thus \( A' \) has a multiplicative refinement in \( U \) if and only if \( A \) does.

For each \( i < 2 \), let \( \{ a_\alpha, i : \alpha < \lambda \} \) list \( \{ m_\beta : f(\beta) = i \} \), with repetitions if necessary. Let \( A'' \) be the \( \lambda \times 2 \)-distribution defined from \( (a_\alpha, i : \alpha < \lambda, i < 2) \) as in the statement of the lemma. Note that \( A'' \) is in \( U \), since whenever \( f(\beta) \neq f(\beta') \), we have \( \| m_\beta \neq m_{\beta'} \| V \in U \). Thus \( A'' \) has a multiplicative refinement in \( U \), which easily gives a multiplicative refinement of \( A' \).

**Corollary 8.7.** Suppose \( U \) is an ultrafilter on the complete Boolean algebra \( B \). Then the following are equivalent:

(A) \( U \lambda^+ \)-saturates \( T_{rg}; \)

(B) \( U \lambda^+ \)-saturates some unstable theory;

(C) \( \lambda < \lambda_\mu(\Delta(IP)) \).
It is a major open problem in the subject, see e.g. Problem (1) in the list of open problems in [20], to determine the Keisler class of the random graph model-theoretically. Examples of theories in this class are rather sparse; for instance, one can show that the theory of an algebraically closed field of with a generic automorphism. ACFA is incomplete; one must specify the characteristic, and also the isomorphism type of the automorphism restricted to the algebraic closure of the emptyset.

**Question.** Suppose $T$ is a completion of ACFA. Is $T$ equivalent to the random graph in $\leq_1^\omega$, or at least in $\leq_2$?

(A) $\leq (C)$ of the following lemma is Lemma 6.8 of [16], in the special case where $U$ is an ultrafilter on $\mathcal{P}(\lambda)$. The other inequalities are implicit in Section 6 there, although they are formulated differently. In [17], Malliaris defines $\mathcal{U}$ to be $\lambda^+$-good for equality if $\lambda$ is less than the value in (C).

**Lemma 8.8.** Suppose $B$ is a complete Boolean algebra and $\mathcal{U}$ is an ultrafilter on $B$. Then the following cardinals are equal:

(A) $\lambda_U(\Delta(TP_2))$;

(B) The least $\lambda$ such that there are $V \models B ZFC^-$ and $(a_\alpha : \alpha < \lambda)$ from $V$, such that there is no multiplicative $\lambda$-distribution $B$ in $\mathcal{U}$ such that each $B(\{\alpha, \beta\})$ decides $\|a_\alpha = a_\beta\|_V$ (necessarily as dictated by $\mathcal{U}$);

(C) The least $\lambda$ such that there are $V \models B ZFC^-$ and $(a_\alpha : \alpha < \lambda)$ from $V$, such that $[a_\alpha]_U \neq [a_\beta]_U$ for all $\alpha \neq \beta$, and such that there is no multiplicative $\lambda$-distribution $B$ in $\mathcal{U}$ with $B(\{\alpha, \beta\}) \leq \|a_\alpha \neq a_\beta\|_V$ for all $\alpha \neq \beta$.

**Proof.** Let $\lambda_A, \lambda_B, \lambda_C$ be the cardinals defined in items (A), (B), (C).

$\lambda_C \leq \lambda_B$: suppose $\lambda < \lambda_C$, we show $\lambda < \lambda_B$. Let $V, (a_\alpha : \alpha < \lambda)$ be given. Let $E$ be the equivalence relation on $\lambda$ defined via: $E(\alpha, \beta)$ if $[a_\alpha]_U = [a_\beta]_U$ (i.e. $\|a_\alpha = a_\beta\|_V \in U$). Let $I \subseteq \lambda$ be a choice of representative for $\lambda/E$, i.e., such that each $\alpha < \lambda$ is $E$-related to exactly one $\beta \in I$. Let $f : \lambda \to I$ be the map witnessing this, so for all $\alpha < \lambda$ and for all $\beta \in I$, $[a_\alpha]_U = [a_\beta]_U$ if and only if $\beta = f(\alpha)$. Since $\lambda < \lambda_C$, we can find a multiplicative $I$-distribution $B_0$ in $\mathcal{U}$ with each $B_0(\alpha, \beta) \leq \|a_\alpha \neq a_\beta\|_V$. Define $B$, a $\lambda$-distribution in $\mathcal{U}$, via $B(s) = B_0(f(s)) \land \bigwedge_{\alpha \in s} \|a_\alpha = a_{f(\alpha)}\|_V$. Then $B$ is clearly witnesses $\lambda < \lambda_B$.

$\lambda_B \leq \lambda_A$: suppose $\lambda < \lambda_B$, we show $\lambda < \lambda_A$. Suppose $A$ is a given $(\lambda, \Delta(TP_2))$-distribution in $\mathcal{U}$. Choose some transitive $V \models ZFC^-$, and let $i : V \leq V$ with $\mathcal{V} \lambda^+$-saturated, and choose $(x_\alpha : \alpha < \lambda)$ a sequence from $i_{\text{std}}(\omega \times \omega)$ such that for all $s \in [\lambda]^{<\omega}$, $\|\{x_\alpha : \alpha \in s\} \in i(\Delta(TP_2))\|_V = A(s)$; this is possible by Lemma 8.3.

For each $\alpha < \lambda$, choose $n_\alpha, m_\alpha$ such that $\|x_\alpha = (n_\alpha, m_\alpha)\|_V = 1$ (possible by fullness of $V$). By two applications of $\lambda < \lambda_B$, we can find a multiplicative distribution $B$ in $\mathcal{U}$ such that for all $\alpha < \beta < \lambda$, $B(\{\alpha, \beta\})$ decides $\|n_\alpha = n_\beta\|_V$ and decides $\|m_\alpha = m_\beta\|_V$, from which it follows that $B$ is a multiplicative refinement of $A$. 
\[\lambda_A \leq \lambda_C: \text{ suppose } \lambda < \lambda_A, \text{ we show } \lambda < \lambda_C. \text{ So suppose } V, (a_\alpha : \alpha < \lambda) \text{ are given.}
\]
Define a \(\lambda\)-distribution \(A\) in \(U\) via \(A(s) = \bigwedge_{\alpha < \beta \leq s} ||a_\alpha \neq a_\beta||_V\). Easily this is a \((\lambda, \Delta(TP_2))\)-distribution, and thus it has multiplicative refinement \(B\) in \(U\). \(B\) is as desired. \(\square\)

**Corollary 8.9.** Suppose \(U\) is an ultrafilter on the complete Boolean algebra \(B\). Then the following are equivalent:

(A) \(U\) \(\lambda^+\)-saturates \(T_{rf}\);

(B) \(U\) \(\lambda^+\)-saturates some unsimple theory;

(C) \(\lambda < \lambda_U(\Delta(TP_2))\).

The corresponding invariants for \(\lambda_U(\Delta(FDP))\) have been studied under various guises. \(\lambda\)-OK was first defined by Kunen [12], and \(\lambda\)-flexibility was first defined by Malliaris in [13]. Previously these definitions were made only in the case of \(B = P(\lambda)\).

**Definition 8.10.** The ultrafilter \(U\) on the complete Boolean algebra \(B\) is \(\lambda\)-OK if whenever \(A\) is a \(\lambda\)-distribution in \(U\) such that for all \(s, t \in [\lambda]^n\), \(A(s) = A(t)\), we have that \(A\) has a multiplicative refinement in \(U\). (Note in this case that \(A\) is determined by the descending sequence \(\langle A(n) : n < \omega \rangle\).) \(U\) is \(\lambda\)-flexible if \(U\) is \(\aleph_1\)-incomplete and, for every \(U\)-nonstandard \(n \in (\omega, <)^B/U\), there is some multiplicative \(\lambda\)-distribution \(B\) in \(U\), such that for all \(s \in [\lambda]^n\), \(B(s) \leq ||n \geq n||_{(\omega, <)^B}\).

So clearly, if \(U\) is \(\aleph_1\)-complete, then \(U\) is \(\lambda\)-OK for all \(\lambda\).

We first remark that this definition of \(\lambda\)-flexibility coincides with the one given by Malliaris. We recall some definitions from [37] and (independently) [30, 29]. Suppose \(B\) is a complete Boolean algebra, and \((a_i : i \in I)\) is a sequence from \(B\). Then say that \((a_i : i \in I)\) is \(I\)-regular if it has the finite intersection property, and infinite intersections are 0, and the set of all \(c \in B_+\) which decide \(a_i\) for every \(i \in I\) is dense in \(B_+\); this coincides with the usual definition when \(B\) is \(\lambda^+\)-distribution. The \(\lambda\)-distribution \(B\) is \(\lambda\)-regular if \((B(\{\alpha\}) : \alpha < \lambda)\) is \(\lambda\)-regular, or equivalently \((B(s) : s \in [\lambda]^{<\aleph_0})\) is \([\lambda]^{<\aleph_0}\)-regular. The filter \(D\) is \(\lambda\)-regular if it contains a \(\lambda\)-regular family.

The following is motivated by Mansfield’s argument in Theorem 4.1 of [27].

**Theorem 8.11.** Suppose \(U\) is an ultrafilter on \(B\), and \([n]_U \in (\omega, <)^B/U\) is nonstandard, and \(B\) is a multiplicative \(\lambda\)-distribution in \(U\). Then the following are equivalent:

(A) For every \(s \in [\lambda]^n\), \(B(s) \leq ||n \geq n||_{(\omega, <)^B}\).

(B) \(B\) is \(\lambda\)-regular, and for every \(c \in B\), if \(c\) decides \(B(s)\) for each \(s \in [\lambda]^{<\aleph_0}\) (or equivalently, \(c\) decides \(B(\{\alpha\})\) for each \(\alpha < \lambda\)) and if we write \(n := |\{\alpha < \lambda : c \leq B(\{\alpha\})\}|\), then \(c \leq ||n \geq m||_{(\omega, <)^B}\).

In particular, if \(U\) is \(\lambda\)-flexible, then \(U\) is \(\lambda\)-regular.
Proof. (A) implies (B): We show that for every $c \in B$ nonzero, there is $c' \leq c$ nonzero such that $c'$ decides each $B(\{\alpha\})$, and there is $m_* < \omega$ such that $c' \leq \|m = m_*\|_{(\omega, <)\beta}$ and $|\{\alpha < \lambda : c \leq B(\{\alpha\})\}| \leq m_*$. This clearly suffices.

So let $c \in B$ be nonzero. Choose $c_0 < c$ nonzero, such that there is some $m_0 < \omega$ with $c_0 \leq \|m = m_*\|_{(\omega, <)\beta}$. Try to find, by induction on $m \leq m_* + 1$, a descending sequence $(c_m : m \leq m_* + 1)$ such that for each $m$, $|\{\alpha < \lambda : c_m \leq B(\{\alpha\})\}| \geq m$. There must be some stage $m < m_* + 1$ at which we cannot continue, since for any $s \in [\lambda]^{m_* + 1}$, $c_0 \land B(s) = 0$ (since $B(s) \leq \|n > m_*\|_{(\omega, <)\beta}$). So we get some $m < m_* + 1$ such that if we set $s = \{\alpha < \lambda : c_m \leq B(\{\alpha\})\}$, then for all $\alpha \not\in s$, $c_m \leq \neg B(\{\alpha\})$. So $c_m$ is desired.

(B) implies (A): suppose $s \in [\lambda]^n$. Then for any $c \leq B(s)$ nonzero such that $c$ decides each $B(\{\alpha\})$, we have that $c \leq \|n \geq n\|_{(\omega, <)\beta}$. Since the set of all such $c$ is dense below $B(s)$, we must have that $B(s) \leq \|n \geq n\|_{(\omega, <)\beta}$. \qed

We have the following theorem connecting all of these notions. It is a translation of Observation 9.9 of [24] into our context. Parente [30] 29 has independently proven that if $U$ is $\lambda$-$OK$ and is $\aleph_1$-incomplete, then $U$ is $\lambda$-regular.

**Theorem 8.12.** Suppose $U$ is an ultrafilter on the complete Boolean algebra $B$. Then $\lambda_U(\Delta(FDP))$ is the least $\lambda$ such that $U$ is not $\lambda$-$OK$. Additionally, if $U$ is $\aleph_1$-incomplete, then this is the least $\lambda$ such that $U$ is not $\lambda$-flexible.

**Proof.** Choose some transitive $V \models ZFC^{-}$, and some $i : V \preceq V$ with $V \lambda^{+}$-saturated. Write $V = V/U$ and let $j : V \preceq V$ be the usual embedding.

Suppose first that $\lambda < \lambda_U(\Delta(FDP))$, and $A$ is a $\lambda$-distribution with $A(s) = A(t)$ for all $|s| = |t|$. Then it is easy to see that $A$ is a $(\lambda, \Delta(FDP))$-distribution, so by hypothesis $A$ has a multiplicative refinement in $U$; thus $U$ is $\lambda$-$OK$. Conversely, suppose $U$ is $\lambda$-$OK$; we show $\lambda < \lambda_U(\Delta(FDP))$, using the characterization of Lemma 6.4. This suffices by Theorem 8.5.

So suppose $m_*, n_\iota \in i(\omega)$ and $\{n_\alpha : \alpha < \lambda\} \subseteq i(\omega)$ are given with $[m_\iota]_U$ nonstandard and $[m_\iota]_U < [n_\iota]_U$, and each $[n_\alpha]_U < [n_*]_U$. We can suppose each $\|m_* < n_*\|_V = \|n_* < n_*\|_V = 1$. Define a $\lambda$-distribution $A$ in $U$, via $A(s) = \|m_\iota \geq n\|_V$ for each $s \in [\lambda]^n$. Since $U$ is $\lambda$-$OK$, we can find a multiplicative refinement $B$ of $A$ in $U$. By a similar argument to Lemma 8.4, we can find $X \in [n_*]^{<m_*}$ (i.e. $X \in V$ and $\|X \in [n_*]^{<m_*}\|_V = 1$) so that for all $\alpha < \lambda$, $\|n_\alpha \in X\|_V = B(\{\alpha\})$. Then $X := [X]_U$ is as desired.

Suppose next that $U$ is $\lambda$-flexible; we show that $U$ is $\lambda$-$OK$. So suppose $A$ is a distribution in $U$ such that for all $s, t \in [\lambda]^n$, $A(s) = A(t)$. If $a := \bigwedge_n A(n) \in U$ then obviously the constant distribution with value $a$ is a refinement in $U$. Otherwise, we can suppose $\bigwedge_n A(n) = 0$ (by intersecting each $A(s)$ with $\neg a$). Define $m \in (\omega, <)^B$ via $m(n) = A(n) \land \neg A(n + 1)$. The fact that $m \in (\omega, <)^B$ follows form $\bigwedge_n A(n) = 0$ and $A(0) = A(\emptyset) = 1$. $m$ is $U$-nonstandard since each $\|m \geq n\|_{(\omega, <)\beta} = A(n) \in U$. Thus we can find a multiplicative distribution $B$ in $U$, such that for all $s \in [\lambda]^n$, $B(s) \leq A(s) = \|m \geq n\|_V$.

Finally, suppose $U$ is $\lambda$-$OK$ and $\aleph_1$-incomplete, and let $[m]_U$ be a $U$-nonstandard element of $(\omega, <)^B/U$. Define $A(s) = \|m \geq n\|_{(\omega, <)\beta}$ for each $s \in [\lambda]^n$ and let $B$ be a multiplicative refinement of $A$ in $U$. Then for all $s \in [\lambda]^n$, $B(s) \leq \|m \geq n\|_V = A(s)$. \qed

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Remark 8.13. It follows that if $\mathcal{U}$ is $\lambda$-OK, then $\mu_\mathcal{U} > \lambda$, and so $\mathcal{U}$ $\lambda^+$-saturates every stable theory. The special case where $\mathcal{B} = \mathcal{P}(\lambda)$ and $\mathcal{U}$ is $\aleph_1$-incomplete (i.e. $\lambda$-flexible) was proved similarly by Malliaris and Shelah in [23]. Hence: if $\mathcal{U}$ is $\aleph_1$-complete (and thus $\lambda$-OK for all $\lambda$) and if $T$ is stable, then $\lambda_\mathcal{U}(T) = \infty$, i.e. $\mathcal{U}$ $\lambda^+$-saturates $T$ for every $\lambda$.

Corollary 8.14. Suppose $\mathcal{U}$ is an ultrafilter on $\mathcal{B}$ and $\lambda$ is given. Then the following are equivalent:

(A) $\mathcal{U}$ $\lambda^+$-saturates $T_{\text{nlow}}$;

(B) $\mathcal{U}$ $\lambda^+$-saturates some nonlow theory;

(C) $\mathcal{U}$ $\lambda^+$-saturates $T_{rg}$ and $\mathcal{U}$ is $\lambda$-OK.

9 The Chain Condition and Saturation

Malliaris and Shelah prove some special cases of the following in [26], but their arguments do not generalize.

Theorem 9.1. Suppose $\mathcal{B}$ is a complete Boolean algebra; write $\lambda = c.c.(\mathcal{B})$. Suppose $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathcal{B}$. Then $\mathcal{U}$ does not $\lambda^+$-saturate any nonsimple theory. In fact, we can find a $(\lambda, \Delta(TP_2))$-distribution $\mathbf{A}$ in $\mathcal{U}$, such that if $\mathcal{B}$ is a complete subalgebra of $\mathcal{B}_*$ where $\mathcal{B}_*$ has the $\lambda$-c.c., then $\mathbf{A}$ has no multiplicative refinement in $\mathcal{B}_*$.

Proof. It suffices to show the second claim. Note that $\lambda > \aleph_0$, as otherwise $\mathcal{B}$ would be finite, and so would not admit any nonprincipal ultrafilters. Thus $\lambda$ is regular; this is a theorem due to Erdös and Tarski [1], or see Theorem 7.15 of Jech [8].

Let $\sigma$ be the completeness of $\mathcal{U}$, i.e. the least cardinal such that there is a descending sequence $(a_\alpha : \alpha < \sigma)$ from $\mathcal{U}$ with $\bigcap_{\alpha < \sigma} a_\alpha = 0$. It is not hard to see that $\sigma < \lambda$, and moreover there is an antichain $\mathcal{C}$ of $\mathcal{B}$ of size $\sigma$ such that for every $X \in [\mathcal{C}]^{<\sigma}$, $\bigvee X \notin \mathcal{U}$. Enumerate $\mathcal{C} = (c_\gamma : \gamma < \sigma)$.

Let $S \subseteq \lambda$ be the set of all $\alpha < \lambda$ with $\text{cof}(\alpha) = \sigma$, so $S$ is stationary in $\lambda$. For each $\alpha \in S$, let $L_\alpha : \sigma \to \alpha$ be a cofinal, increasing map, and let $\delta_\alpha \in (\lambda, <)^\mathcal{B}$ be the element such that for all $\gamma < \sigma$, $\|\delta_\alpha - L_\alpha(\gamma)\|_{(\lambda, <)^\mathcal{B}} = c_\gamma$. This determines $\delta_\alpha$, since $\mathcal{C}$ is a maximal antichain. In particular, we have that $\|\delta_\alpha - \alpha\|_{(\lambda, <)^\mathcal{B}} = 1$, and for all $\beta < \alpha$, $\|\delta_\alpha - \beta\|_{(\lambda, <)^\mathcal{B}} \in \mathcal{U}$. In particular, for all $\alpha < \beta$ both in $S$, $\|\delta_\alpha - \delta_\beta\|_{(\lambda, <)^\mathcal{B}} \in \mathcal{U}$.

For each $s \in [\lambda]^{<\aleph_0}$, put $\mathbf{A}(s) = \bigwedge_{\alpha \neq \beta \in s} \|\delta_\alpha \neq \delta_\beta\|_{(\lambda, <)^\mathcal{B}}$; so $\mathbf{A}$ is a $(\lambda, \Delta(TP_2))$-distribution in $\mathcal{U}$. Suppose $\mathcal{B}$ is a complete subalgebra of $\mathcal{B}_*$, where $\mathcal{B}_*$ has the $\lambda$-c.c. We show that $\mathbf{A}$ has no multiplicative refinement in $\mathcal{B}_*$, i.e. there is no multiplicative $\lambda$-distribution $\mathbf{B}$ in $\mathcal{B}_*$ such that for all $\alpha < \beta$, $\|\delta_\alpha - \delta_\beta\|_{(\lambda, <)^\mathcal{B}} \
ot\in \mathcal{U}$.

Suppose there were. For each $\alpha < \lambda$ there is some $f(\alpha) < \sigma$ with $\mathbf{B}(\{\alpha\}) \wedge c_{f(\alpha)}$ nonzero, i.e. with $\mathbf{B}(\{\alpha\}) \wedge \|\delta_\alpha - L_\alpha(f(\alpha))\|_{(\lambda, <)^\mathcal{B}/\mathcal{U}}$ nonzero. Write $g(\alpha) = L_\alpha(f(\alpha)) < \alpha$. By Fodor’s Lemma (using that $\lambda$ is regular), we can find a stationary set $S' \subseteq S$ on which $g$ is constant, say with value $\gamma$. Since $\mathcal{B}_*$ has the $\lambda$-c.c., $\mathbf{B}(\{\alpha\}) \wedge \|\delta_\alpha = \gamma\|_{(\lambda, <)^\mathcal{B}/\mathcal{U} : \alpha \in S'}$ is not an
antichain, so we can choose \( \alpha < \beta \) both in \( S' \) such that \( \textbf{B}(\{ \alpha \}) \wedge \textbf{B}(\{ \beta \}) \wedge \| \delta_\beta = \gamma \|_{(\lambda, <)^{\mathcal{S}/\mathcal{U}}} \) is nonzero. But \( \textbf{B}(\{ \alpha, \beta \}) \leq \| \delta_\alpha \neq \delta_\beta \|_{(\lambda, <)^{\mathcal{S}/\mathcal{U}}} \), a contradiction.

I proved the following (minus the “in fact” clause) in [39]; that result in turn built off of some special cases proven by Malliaris and Shelah in [22].

**Theorem 9.2.** Suppose \( \mathcal{B} \) is a complete Boolean algebra; write \( \lambda = \text{c.c.}(\mathcal{B}) \). Suppose \( \mathcal{U} \) is an \( \aleph_1 \)-incomplete ultrafilter on \( \mathcal{B} \). Then \( \mathcal{U} \) does not \( \lambda^+ \)-saturate any nonlow theory. In fact, there is a \( (\lambda, \Delta(FDP)) \)-distribution \( \textbf{A} \) in \( \mathcal{U} \) such that if \( \mathcal{B} \) is a complete subalgebra of \( \mathcal{B}_* \) and \( \mathcal{B}_* \) has the \( \lambda \)-c.c., then \( \mathcal{A} \) has no multiplicative refinement in \( \mathcal{B}_* \).

**Proof.** It suffices to show the second claim. Note that \( \lambda > \aleph_0 \), as otherwise \( \mathcal{B} \) would be finite, and so would not admit any nonprincipal ultrafilters.

Let \( \mathcal{U} \) be an \( \aleph_1 \)-incomplete ultrafilter on \( \mathcal{B} \); then we can choose a descending sequence \( (c_n : n < \omega) \) from \( \mathcal{U} \) such that \( c_0 = 1 \) and \( \bigwedge_n c_n = 0 \). Let \( \textbf{A} \) be the distribution in \( \mathcal{U} \), defined by \( \textbf{A}(s) = c_{|s|} \). Then \( \textbf{A} \) is a \( (\lambda, \Delta(FDP)) \)-distribution. Suppose \( \mathcal{B}_* \) has the \( \lambda \)-c.c. and \( \mathcal{B} \) is a complete subalgebra of \( \mathcal{B}_* \). If \( \textbf{A} \) has a multiplicative refinement \( \textbf{B} \) in \( \mathcal{U} \), then by Theorem 8.11 \( \textbf{B} \) would be a \( \lambda \)-regular distribution. But we note in [37] that no complete Boolean algebra with the \( \lambda \)-c.c. admits a \( \lambda \)-regular family.

It will be convenient for us in [36] if we phrase our results in terms of \( (\lambda, T) \)-Los maps instead of \( (\lambda, \Delta) \)-distributions. We recall from Remark 8.2 that if \( T \) admits \( \Delta \) and \( \textbf{A} \) is a \( (\lambda, \Delta) \)-distribution, then \( \textbf{A} \) is a \( (\lambda, T) \)-Los map.

**Corollary 9.3.** In Theorem 9.1 and 9.2 the distribution \( \textbf{A} \) is a \( (\lambda, T_{rf}) \)-Los map or a \( (\lambda, T_{nlow}) \)-Los map.

**Proof.** Since \( T_{rf} \) admits \( \Delta(TP_2) \), every \( (\lambda, \Delta(TP_2)) \)-distribution is a \( (\lambda, T_{rf}) \)-Los map; and since \( T_{nlow} \) admits \( \Delta(FDP) \), every \( (\lambda, \Delta(FDP)) \)-distribution is a \( (\lambda, T_{nlow}) \)-Los map.

**Remark 9.4.** It follows from results in [38] that if \( T \) is a complete countable theory with the NFCP, then any ultrafilter \( \mathcal{U} \) satisfies \( \lambda_\mathcal{U}(T) = \infty \), but if \( T \) has FCP and \( \mathcal{U} \) is \( \aleph_1 \)-incomplete, then \( \lambda_\mathcal{U}(T) \leq |\mathcal{B}| \) (see Remark 9.9 and Corollary 9.12). Remark 8.13 of the present work says that if \( \mathcal{U} \) is \( \aleph_1 \)-complete and if \( T \) is stable, then \( \lambda_\mathcal{U}(T) = \infty \), and Theorem 1.9 says that if \( \mathcal{U} \) is nonprincipal and \( T \) is unstable, then \( \lambda_\mathcal{U}(T) \leq \text{c.c.}(\mathcal{B}) \). Malliaris and Shelah show in [22] that if \( \mathcal{U} \) is an ultrafilter on \( \mathcal{P}(\lambda) \) with \( |\lambda^{\mathcal{U}}| = 2^\lambda \) and if \( T \) is unstable, then \( \lambda_\mathcal{U}(T) \leq 2^\lambda \); in particular, this holds whenever \( \lambda \) is inaccessible and \( \mathcal{U} \) is a uniform, \( \aleph_1 \)-complete ultrafilter on \( \mathcal{P}(\lambda) \).

But the following is open:

**Conjecture.** Suppose \( \mathcal{B} \) is a complete Boolean algebra and \( \mathcal{U} \) is a nonprincipal ultrafilter on \( \mathcal{B} \), and suppose \( T \) is unstable. Then \( \lambda_\mathcal{U}(T) \leq |\mathcal{B}| \).
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