The resolvent algebra for oscillating lattice systems. Dynamics, ground and equilibrium states

Detlev Buchholz
Institut für Theoretische Physik, Universität Göttingen, 37077 Göttingen, Germany

Abstract: Within the C*-algebraic framework of the resolvent algebra for canonical quantum systems, the structure of oscillating lattice systems with bounded nearest neighbor interactions is studied in any number of dimensions. The global dynamics of such systems acts on the resolvent algebra by automorphisms and there exists a (in any regular representation) weakly dense subalgebra on which this action is pointwise norm continuous. Based on this observation, equilibrium (KMS) states as well as ground states are constructed which are shown to be regular. It is also indicated how to deal with singular interactions and non-harmonic oscillations.

Key words. resolvent algebra – dynamics – equilibrium states – local normality

1. Introduction

In this article we continue our study of the resolvent algebra with emphasis on applications. The resolvent algebra, being a novel approach to the treatment of canonical commutation relations of finite and infinite systems, was originally invented to overcome some technical difficulties in the analysis of supersymmetric models [6]. It was quickly realized that, in contrast to the familiar Weyl algebra, it provides a general C*-algebraic framework for the construction of non-trivial dynamics and of the corresponding states [7]. Moreover, it encodes distinctive properties of finite and infinite quantum systems and has an intriguing algebraic structure [5]. For a recent review of these developments, cf. [8].
It is the aim of the present article to supplement these results by a study of oscillating lattice systems with bounded nearest neighbor interaction, describing particles which are confined about their respective lattice positions by harmonic forces and which interact with each other. We shall show that for such systems the time evolution acts by automorphisms on the resolvent algebra. This fact was already established in [7] for one-dimensional lattices and we extend here these results to any number of dimensions. Moreover, we shall exhibit a subalgebra of the resolvent algebra which is weakly dense in all regular representations and on which the automorphisms act pointwise norm continuously, i.e. this subalgebra together with the respective automorphisms constitutes C*-dynamical systems.

This observation facilitates the construction of global equilibrium and ground states, a topic which has not been discussed in the W*-dynamical approach to oscillating lattice systems, proposed in [12,13] and references quoted there. There the underlying algebra is equipped with the weak topology induced by some ad hoc choice of states. Yet, choosing from the outset a weak topology on the global algebra is a quite subtle issue since different equilibrium or ground states lead in general to globally disjoint representations of the algebra [15] and hence to different topologies. The best one may hope for in the present context is that the representations of interest are quasi-equivalent on all subalgebras affiliated with finite subsets of the lattice (viz. locally normal), but this feature requires some proof.

In order to illustrate the utility of the present approach in this respect, we will construct global equilibrium (KMS) states for high temperatures as well as ground states on the above subalgebra for any of the given dynamics and show that they are all locally normal. They can then be extended to regular equilibrium and ground states on the full resolvent algebra. Results of this type seem to be relatively rare, cf. for example the books [1,4] and references quoted there. It is the primary purpose of the present article to reveal the simplifying features of our algebraic approach for the study of lattice systems; but it should be noted that the resolvent algebra can also be used for the treatment of continuous systems (field theories) [7].

The subsequent section contains definitions and two relevant technical results. In Sec. 3 we establish the existence of interacting dynamics of the resolvent algebra and exhibit the subalgebra which is stable and pointwise norm continuous under their action. Sec. 4 contains the construction of locally normal equilibrium (KMS) states for sufficiently high temperatures and Sec. 5 that of ground states. In Sec. 6 we indicate how to deal with non-harmonic dynamics and singular interactions. The article concludes with a brief summary and outlook.
2. Preliminaries

Let $\mathbb{Z}^d$ be a $d$-dimensional cubic lattice, let $\Lambda \subset \mathbb{Z}^d$ denote any of its finite subsets and, more specifically, let $\lambda \in \mathbb{Z}^d$ denote any lattice point. For the case at hand, the resolvent algebra $\mathcal{R}(\mathbb{Z}^d)$ is generated by the sums and products of the resolvents

$$(ic + \sum_j a_j P_{\lambda_j} + \sum_k b_k Q_{\lambda_k})^{-1}, \quad c \in \mathbb{R}\setminus\{0\}, \quad a_j, b_k \in \mathbb{R}^d,$$

formed by linear combinations of $d$-dimensional momentum and position operators $(P_{\lambda}, Q_{\lambda})$ associated with the lattice points $\lambda \in \mathbb{Z}^d$; the bold face product denotes scalar products. Operators at different lattice points commute, whereas at a given point they satisfy canonical commutation relations in resolvent form, cf. [7, Def. 3.1]. Similarly, one defines the resolvent algebras $\mathcal{R}(\Lambda)$ for finite subsets $\Lambda \subset \mathbb{Z}^d$ as the algebras generated by resolvents of $(P_{\lambda}, Q_{\lambda})$ with $\lambda \in \Lambda$. As a matter of fact, one proceeds from abstract C*-versions of these algebras. Yet, as has been shown in [7, Thm. 4.10], these C*-algebras are faithfully represented in any regular representation, e.g. the Fock representation. So we may assume here that we are dealing with the concretely represented resolvent algebra in some such representation.

For the construction of interacting dynamics it matters that the resolvent algebras have a non-trivial (two-sided) ideal structure [7]. We do not need to dive deeply here into this topic, cf. [5], and will only make use of the following elementary facts. Given $\lambda \in \Lambda$, one can form the operators $f(P_\lambda) g(Q_\lambda)$, where $f, g \in C_0(\mathbb{R}^d)$, the space of continuous functions which tend to 0 at infinity. The C*-algebra $\mathcal{C}(\lambda)$ generated by these operators forms an ideal in the resolvent algebra $\mathcal{R}(\lambda)$ which is isomorphic to the algebra of compact operators on some separable Hilbert space; more generally, given $\Lambda \subset \mathbb{Z}^d$, the products $\prod_{\lambda \in \Lambda} f_\lambda (P_\lambda) g_\lambda (Q_\lambda)$, where $f_\lambda, g_\lambda \in C_0(\mathbb{R}^d)$, $\lambda \in \Lambda$, form an ideal $\mathcal{C}(\Lambda) \subset \mathcal{R}(\Lambda)$ which, again, is isomorphic to the algebra of compacts [7, Thm. 5.4]. Note that the algebras $\mathcal{C}(\Lambda)$ are isomorphic to the unique C*-tensor products $\otimes_{\lambda \in \Lambda} \mathcal{C}(\lambda)$, $\Lambda \subset \mathbb{Z}^d$, since the algebras $\mathcal{C}(\Lambda)$ are postliminal (type I) and commute with each other at different points $\lambda \in \mathbb{Z}^d$.

Whereas the assignment $\Lambda \mapsto \mathcal{R}(\Lambda)$ defines a net of C*-algebras on $\mathbb{Z}^d$ so that we can proceed to its C*-inductive limit $\mathcal{R}(\mathbb{Z}^d)$, the compact algebras $\mathcal{C}(\Lambda)$ are not nested. We therefore proceed from them to the algebras $\mathcal{K}(\Lambda)$ which are formed by all compact algebras in the respective region $\Lambda$ and the unit operator.

**Definition:** Let $\Lambda \subset \mathbb{Z}^d$. The algebra $\mathcal{K}(\Lambda) \subset \mathcal{R}(\mathbb{Z}^d)$ is the unital C*-algebra formed by all algebras $\mathcal{C}(A_1)$ with $A_1 \subseteq A$, and the unit operator.
The algebras $K(A)$ are isomorphic to the unique tensor product $\otimes_{\lambda \in A} K(\lambda)$, and the assignment $A \mapsto K(A)$ defines a subnet of the resolvent algebra whose C*-inductive limit is denoted by $K(\mathbb{Z}^d)$. This subalgebra of the resolvent algebra will play a prominent role in the subsequent discussion.

Let us turn next to the definition of dynamics. We proceed from the harmonic dynamics $\alpha_{0}^{(0)}$ which, for given frequency $\varpi > 0$ and any time $t \in \mathbb{R}$, is fixed by its action on the position and momentum operators at any $\lambda \in \mathbb{Z}^d$,

$$\alpha_t^{(0)}(Q_\lambda) = \cos(t \varpi)Q_\lambda + \sin(t \varpi)/\varpi P_\lambda, \quad \alpha_t^{(0)}(P_\lambda) = \cos(t \varpi)P_\lambda - \varpi \sin(t \varpi)Q_\lambda.$$  

Note that each compact subalgebra $C(\Lambda)$, $\Lambda \subset \mathbb{Z}^d$, is stable under this action.

The full dynamics $\alpha_{s}$ is obtained from $\alpha_{0}^{(0)}$ by introducing between neighboring points in increasing subsets $\mathcal{A} \subset \mathbb{Z}^d$ some interaction. Pairs of neighboring points will be denoted by $\lambda' \sim \lambda'' \in \mathcal{A} \times \mathcal{A}$ and the interaction between any two such points is described by the (real) potential $V_{\lambda' \sim \lambda''} = V(Q_{\lambda'} - Q_{\lambda''})$, where $V \in C_0(\mathbb{R}^d)$ is kept fixed. Proceeding to the interaction picture, we consider cocycles $\gamma_{s}^{d}$ which are designed to define automorphisms of the resolvent algebra. They satisfy the cocycle equation $\alpha_{s}^{(0)}(\gamma_{t}^{d}) = \gamma_{s}^{d} \alpha_{s}^{(0)}$ for $s, t \in \mathbb{R}$ with the initial value $\partial_{t}\gamma_{t}^{(d)}|_{t=0} = i \left[ \cdot\cdot\cdot \sum_{\lambda' \sim \lambda'' \in \mathcal{A} \times \mathcal{A}} V_{\lambda' \sim \lambda''} \right]$. Here the dot stands for elements of the resolvent algebra and the square bracket denotes the commutator; cf. below for precise definitions.

We shall prove in the subsequent section that there exist solutions to these equations. With their help one can define automorphisms $\alpha_t^{d} \doteq \alpha_t^{(0)}\gamma_{-t}^{d}$, $t \in \mathbb{R}$, of the resolvent algebra, describing the dynamics in the presence of interaction between neighboring points in the sets $\mathcal{A} \subset \mathbb{Z}^d$. Moreover, these automorphisms converge pointwise in norm to automorphisms $\alpha_t$, $t \in \mathbb{R}$, of the resolvent algebra in the thermodynamic limit $\mathcal{A} \not\rightarrow \mathbb{Z}^d$. The latter automorphisms no longer leave the individual algebras $C(\Lambda)$, $\Lambda \subset \mathbb{Z}^d$, invariant, but the subalgebra $K(\mathbb{Z}^d)$ remains stable under their action, as we shall see.

For the proof of these assertions we need two technical lemmas which we supply here for later reference. We begin with a definition.

**Definition:** Let $\lambda' \sim \lambda'' \in \mathbb{Z}^d \times \mathbb{Z}^d$ and let

$$(P_{\lambda' \sim \lambda''}, Q_{\lambda' \sim \lambda''}) \doteq (P_{\lambda'} - P_{\lambda''}, Q_{\lambda'} - Q_{\lambda''}).$$

The C*-algebra generated by the resolvents of linear combinations of the components of $\left(P_{\lambda' \sim \lambda''}, Q_{\lambda' \sim \lambda''}\right)$ is denoted by $\mathcal{R}(\lambda' \sim \lambda'')$ and its subalgebra which is generated by the operators $f(P_{\lambda' \sim \lambda''})g(Q_{\lambda' \sim \lambda''})$ with $f, g \in C_{0}(\mathbb{R}^d)$ is denoted by $\mathcal{C}(\lambda' \sim \lambda'') \subset \mathcal{R}(\lambda' \sim \lambda'')$. (Note that the latter algebra is, once again, isomorphic to the algebra of compact operators on some Hilbert space; but it is not contained in $K(\mathbb{Z}^d)$.)
Lemma 2.1. Let $\lambda' \sim \lambda'' \in \mathbb{Z}^d \times \mathbb{Z}^d$ and let $V_{\lambda' \sim \lambda''}(s) = \alpha_s^{(0)}(V_{\lambda' \sim \lambda''})$, $s \in \mathbb{R}$. Then, for any $t_1, t_2 \in \mathbb{R}$, one has $\int_{t_1}^{t_2} ds V_{\lambda' \sim \lambda''}(s) \in C(\lambda' \sim \lambda'')$, where the integral is defined in the strong operator topology of the chosen regular (hence faithful) representation.

Proof. In order to simplify the notation, we put $P \doteq P_{\lambda' \sim \lambda''}$, $Q \doteq Q_{\lambda' \sim \lambda''}$ and note that $\alpha_s^{(0)}(V_{\lambda' \sim \lambda''}(Q)) = V(\cos(\pi s) Q + \sin(\pi s)/\pi P)$ is contained in $\mathcal{R}(\lambda' \sim \lambda'')$, c.f. [7, Prop. 5.2]. By von Neumann’s uniqueness theorem, the automorphic action of $\alpha_s^{(0)}$ on $\mathcal{R}(\lambda' \sim \lambda'')$ is implemented in any regular representation of this algebra by unitary operators $U_0(s)$ which depend continuously on $s \in \mathbb{R}$ in the strong operator topology. Since the potential $V$ is bounded, the above integral is well defined in this topology.

In order to see that the integral is an element of $C(\lambda' \sim \lambda'')$, we make use of the fact that the elements of $C_0(\mathbb{R}^d)$ can be approximated in norm by Schwartz test functions and assume temporarily that the potential $V$ belongs to this subspace. We then study the (distributional) kernel of the operator function $s, s' \mapsto V_{\lambda' \sim \lambda''} U_0(s - s') V_{\lambda' \sim \lambda''}$ in the Schrödinger representation. Making use of Dirac’s bra-ket notation, this kernel is given in position space by $\langle x| V_{\lambda' \sim \lambda''} U_0(s - s') V_{\lambda' \sim \lambda''}|y \rangle = V(x) \langle x| U_0(s - s')|y \rangle V(y), \quad x, y \in \mathbb{R}^d$.

The Green’s function of the harmonic oscillator, $s, s' \mapsto \langle x| U_0(s - s')|y \rangle$, is known to be continuous and bounded for fixed $s - s' \notin (\pi/\omega) \mathbb{Z}$, c.f. [3]. Hence for those values the above kernel is square integrable in $x, y$ and therefore belongs to a Hilbert Schmidt operator. Whence, the operators $\alpha_s^{(0)}(V_{\lambda' \sim \lambda''}) \alpha_{s'}^{(0)}(V_{\lambda' \sim \lambda''}) = U_0(s') V_{\lambda' \sim \lambda''} U_0(s - s') V_{\lambda' \sim \lambda''} U_0(-s)$ are compact and consequently are elements of $C(\lambda' \sim \lambda'')$ if $(s' - s) \notin (\pi/\omega) \mathbb{Z}$. Moreover, they are uniformly bounded for all $s', s \in \mathbb{R}$. 

We proceed by considering the double integral $\int_{t_1}^{t_2} ds' \int_{t_1}^{t_2} ds \alpha_{s'}^{(0)}(V_{\lambda' \sim \lambda''}) \alpha_s^{(0)}(V_{\lambda' \sim \lambda''})$. Disregarding the singular set $\{(s', s) : (s' - s) \in (\pi/\omega) \mathbb{Z}\}$ which has measure zero, the integration extends over a strong-operator continuous and bounded function which has values in the compact operators. Thus if $\Phi_n$, $n \in \mathbb{N}$, is any sequence of vectors which converges weakly to zero, one obtains by the dominated convergence theorem

$$\lim_n \| \int_{t_1}^{t_2} ds \alpha_s^{(0)}(V_{\lambda' \sim \lambda''}) \Phi_n \|^2 = \lim_n \int_{t_1}^{t_2} ds' \int_{t_1}^{t_2} ds \langle \Phi_n, \alpha_{s'}^{(0)}(V_{\lambda' \sim \lambda''}) \alpha_s^{(0)}(V_{\lambda' \sim \lambda''}) \Phi_n \rangle = 0.$$
Hence $\int_1^2 ds \alpha^{(0)}_s (V_{\lambda' \sim \lambda''})$ is compact and therefore an element of $C(\lambda' \sim \lambda'')$. This proves the assertion for potentials $V \in C_0(\mathbb{R}^d)$ which are test functions; the statement then follows from the continuity of the integral with regard to $V$ in the norm topology of $C_0(\mathbb{R}^d)$. \]

While the algebras $C(\lambda' \sim \lambda'')$ are not contained in the C*-algebra $\mathcal{K}(\mathbb{Z}^d)$, their elements induce bounded derivations of it; note that $\mathcal{K}(\mathbb{Z}^d)$ is not a simple algebra. The proof of this fact, entering into our construction of dynamics, is based on the following lemma.

**Lemma 2.2.** Let $\Lambda \subset \mathbb{Z}^d$ and let $\lambda' \sim \lambda'' \in \mathbb{Z}^d \times \mathbb{Z}^d$ be nearest neighbors. If $\lambda' \in \Lambda$, $\lambda'' \not\in \Lambda$ one has $C(\lambda' \sim \lambda'')C(\Lambda) = C(\lambda' \sim \lambda'') \subset C(\lambda' \sim \lambda'')$. Similarly, if $\lambda', \lambda'' \in \Lambda$ one has $C(\lambda' \sim \lambda'')C(\Lambda) = C(\lambda' \sim \lambda'') \subset C(\Lambda)$. (Note that $C(\lambda' \sim \lambda'') = C(\lambda' \sim \lambda')$.)

**Proof.** As to the first part of the statement, it suffices because of the tensor product structure of $C(\Lambda)$ to consider the simplest non-trivial case, where $\Lambda = \lambda'$. Now the linear space $C(\lambda' \sim \lambda'')C(\lambda')$ is generated by the operators $f(P_{\lambda' \sim \lambda''})g(Q_{\lambda' \sim \lambda''}) h(P_{\lambda'}) k(Q_{\lambda'})$, where $f, g, h, k \in C_0(\mathbb{R}^d)$. So we must show that these operators, lying in the algebra $\mathcal{R}(\lambda' \cup \lambda'')$, actually belong to its compact ideal.

This task can be accomplished by explicit computations in the Schrödinger representation of the algebra $\mathcal{R}(\lambda' \cup \lambda'')$, where one has to show that the operators are compact. Alternatively, one arrives at this conclusion by noticing that the operators lie in the principle ideal which is generated by the $4d$-fold product of resolvents involving all components of the operators $P_{\lambda' \sim \lambda''}, Q_{\lambda' \sim \lambda''}, P_{\lambda'}, Q_{\lambda'}$ in the given order, cf. [7] Prop. 3.8. This ideal, being equal to the intersection of the principle ideals formed by the individual resolvents [7] Prop. 3.8], does not depend on this order, so it coincides with the principle ideal generated by the $4d$-fold product of resolvents involving the components of the operators $P_{\lambda' \sim \lambda''}, P_{\lambda'}, Q_{\lambda' \sim \lambda''}, Q_{\lambda'}$. Since the product of the resolvents of the components of $P_{\lambda' \sim \lambda''}, P_{\lambda'}$ is of the form $f_2(P_{\lambda'}, P_{\lambda''})$ with $f_2 \in C_0(\mathbb{R}^d) \otimes C_0(\mathbb{R}^d)$, and similarly for $Q_{\lambda' \sim \lambda''}, Q_{\lambda'}$, cf. [7] Prop. 5.2], this ideal is generated by the operators $f_2(P_{\lambda'}, P_{\lambda''}) g_2(Q_{\lambda'}, Q_{\lambda''})$, where $f_2, g_2 \in C_0(\mathbb{R}^d) \otimes C_0(\mathbb{R}^d) = C_0(\mathbb{R}^{2d})$. It therefore coincides with the compact ideal $C(\lambda' \cup \lambda'')$, which consequently contains $C(\lambda' \sim \lambda'') C(\lambda')$. Similarly, one obtains $C(\lambda') C(\lambda' \sim \lambda'') \subset C(\lambda' \sim \lambda'')$, completing the proof of the first part of the statement. The second part immediately follows from the fact that $C(\lambda' \sim \lambda'') \subset \mathcal{R}(\lambda' \cup \lambda'')$ and that the subalgebra $C(\lambda' \cup \lambda'') \subset \mathcal{R}(\lambda' \cup \lambda'')$ is an ideal. \]
3. Dynamics

In this section we establish the existence of one-parameter groups of automorphisms of the resolvent algebra (dynamics) which describe nearest neighbor interactions. These results extend those obtained in [7] for lattice systems in $d = 1$ dimension. We could partially rely on results established subsequently in [12,13] and references quoted there; but we need some more detailed information and therefore present here a different, simpler argument.

As has been explained in the preceding section, we must show that for any finite $\Lambda \subset \mathbb{Z}^d$ there exist cocycles $\gamma^A_\Lambda$ which act as automorphisms on the resolvent algebra $R(\mathbb{Z}^d)$ and satisfy

$$\alpha_s^{(0)} \gamma^A_\Lambda \alpha_s^{(0)} = \gamma^A_{s+t}$$

for $s, t \in \mathbb{R}$ (3.1)

with initial condition given by the derivative (on a suitable domain)

$$\partial_t \gamma^A_{\Lambda} (\cdot) \big|_{t=0} = i \langle \cdot, V_\Lambda \rangle,$$

where $V_\Lambda = \sum_{\lambda' \sim \lambda'' \in \Lambda \times \Lambda} V_{\lambda'} \langle \lambda'', \cdot \rangle$. (3.2)

Here $\alpha^{(0)}_\Lambda$ denotes the harmonic dynamics acting at each lattice site and the dot stands for any element of the resolvent algebra. We will exhibit solutions to this problem by proceeding to its integrated version,

$$\gamma^A_t (\cdot) = \iota (\cdot) + i \int_0^t ds \left[ \gamma^A_{s} (\cdot), V_A(s) \right], \quad \text{where} \quad V_A(s) = \sum_{\lambda' \sim \lambda'' \in A \times A} V_{\lambda'} \langle \lambda'', V_A(s) \rangle.$$ (3.3)

and $\iota$ denotes the identity map. These solutions are obtained by iteration, giving the Dyson series

$$\gamma^A_t (\cdot) = \iota (\cdot) + \sum_{n=1}^{\infty} \iota^n \int_0^t ds_n \int_0^{s_n} ds_{n-1} \cdots \int_0^{s_2} ds_1 \left[ \ldots \left[ \cdot, V_A(s_1) \right], \ldots, V_A(s_n) \right].$$ (3.4)

In this form the cocycles $\gamma^A_\Lambda$ acquire some precise mathematical meaning: The functions $s \mapsto V_A(s)$ are continuous in the strong operator topology of the chosen (regular) representation, so the integrals are well defined in this topology. Moreover, since $\|V_A(s)\| = \|V_A\| < \infty$, the series is absolutely convergent in norm, so the cocycles $\gamma^A_t$ map the elements of the resolvent algebra into bounded operators. The more difficult part is the demonstration that the images of the resolvent algebra under this action are actually contained in the resolvent algebra itself and depend pointwise norm continuously on $t \in \mathbb{R}$. Since the underlying representation is faithful, one then has shown that these properties are representation independent.

To accomplish this goal we note that it suffices to establish these properties for the individual terms appearing in the Dyson series because of its convergence.
properties. Moreover, these terms can be split further into finite sums of linear maps $M_n^{X'-X''}(t)$, $t \in \mathbb{R}$, mapping the resolvent algebra $\mathcal{R}(\mathbb{Z}^d)$ into bounded operators. They are given by

$$M_n^{X'-X''}(t)(\cdot) = \int_0^t \int_0^{s_n} \int_0^{s_{n-1}} \cdots \int_0^{s_2} \int_0^{s_1} \left[ \cdots [\cdots, V_{\lambda_1' \sim \lambda_1''}(s_1), \ldots, V_{\lambda_n' \sim \lambda_n''}(s_n)] \right] \Phi \, ds_1 \cdots ds_n,$$ (3.5)

where $X' \sim X'' = \lambda_1' \sim \lambda_1'', \ldots, \lambda_n' \sim \lambda_n'' \in A \times A$ are $n$ pairs of nearest neighbors.

In the subsequent lemma we collect some pertinent properties of these maps.

**Lemma 3.1.** Let $A \subset \mathbb{Z}^d$ and let $X' \sim X'' = \lambda_1' \sim \lambda_1'', \ldots, \lambda_n' \sim \lambda_n'' \in A \times A$, $n \in \mathbb{N}$, be any collection of nearest neighbors (which need not be different from each other). The maps $M_n^{X'-X''}(t)$, $t \in \mathbb{R}$, defined in (3.5), have the properties

(i) $\|M_n^{X'-X''}(t) - M_n^{X'-X''}(s)\| \leq 2^n \|V\| |t^n - s^n|/n!$ if $s, t \geq 0$ or $s, t \leq 0$, and $M_n^{X'-X''}(0) = 0$.

(ii) $M_n^{X'-X''}(t)$ maps the resolvent algebra $\mathcal{R}(\mathbb{Z}^d)$ into itself.

(iii) Given $A_0 \subset \mathbb{Z}^d$, there is a finite $A_n \supseteq A_0$, depending on the collection of points $\lambda_1' \sim \lambda_1'', \ldots, \lambda_n' \sim \lambda_n'' \in A \times A$, such that $M_n^{X'-X''}(t)(C(A_0)) \subseteq C(A_n)$.

**Proof.** (i) Let $R \in \mathcal{R}(\mathbb{Z}^d)$ and let $\Phi$ be any vector in the underlying representation space of the resolvent algebra. Then one obtains for $t \geq s \geq 0$

$$\|M_n^{X'-X''}(t)(R) - M_n^{X'-X''}(s)(R)\| \Phi\| \leq 2^n \|\Phi\| |t^n - s^n|/n!,$$

where, in the last step, we estimated the norm of the $n$-fold commutator, making use of the fact that $\|V_{\lambda' \sim \lambda''}(s)\| = \|V\|$ does neither depend on time nor on the chosen family of neighboring points. The first part of the statement then follows for the particular choice of $t, s$ and the other cases are treated in a similar manner. The equality $M_n^{X'-X''}(0) = 0$ is immediate from relation (3.5).

(ii) The second statement is proved by induction in $n \in \mathbb{N}$, so let $R \in \mathcal{R}(\mathbb{Z}^d)$. For $n = 1$ we have $M_1^{X'-X''}(t)(R) = \int_0^t ds [R, V_{\lambda_1' \sim \lambda_1''}(s)]$; this is an element of $\mathcal{R}(\mathbb{Z}^d)$ since $\int_0^t ds V_{\lambda_1' \sim \lambda_1''}(s) \in C(\lambda_1' \sim \lambda_1'') \subset \mathcal{R}(\mathbb{Z}^d)$ according to Lemma 2.1.

For the induction step from $n$ to $n + 1$ we notice that

$$M_{n+1}^{X'-X''}(t)(R) = \int_0^t ds [M_n^{X'}(s)(R), V_{\lambda_{n+1}' \sim \lambda_{n+1}''}(s)]$$
and the function $s \mapsto M^X_n(s)(R) \in \mathcal{R}(\mathbb{Z}^d)$ is continuous in norm, as has been shown in part (i). We approximate the integral by sums of the form
\[
S_N(R) = \sum_{k=0}^N \int_{t_k}^{t_{k+1}} ds \, [M^X_n(t_k)(R), V_{\lambda^{n+1}_i - \lambda^{n+1}_j}(s)],
\]
where all distances $|t_k - t_{k+1}|$, $k = 0, \ldots, N$, tend to zero in the limit of large $N$. Applying Lemma 2.2 again, one sees that each term in these sums is an element of the resolvent algebra, hence $S_N(R) \in \mathcal{R}(\mathbb{Z}^d)$. An estimate as in part (i) yields
\[
\|M^X_{n+1} - \mathcal{M}^X_n(t)(R) - S_N(R)\| \leq 2^{n+1} \|V\|^n \sum_{k=0}^N \int_{t_k}^{t_{k+1}} ds \, |t_k^n - s^n|/n! ,
\]
showing that $M^X_{n+1} - \mathcal{M}^X_n(t)(R)$ can be approximated in norm by the sums $S_N(R)$. Since $\mathcal{R}(\mathbb{Z}^d)$ is closed in this topology, the second statement follows.

(iii) For the third statement we proceed as in part (ii) but need to have a closer look at the localization properties of the operators. We make use again of an induction argument, so let $C_0 \in \mathcal{C}(A_0)$. For $n = 1$ we get
\[
M^X_1 - \mathcal{M}^X_1(t)(C_0) = \int_0^t ds [C_0, V_{\lambda_1 - \lambda'_1}(s)] = [C_0, \int_0^t ds V_{\lambda_1 - \lambda'_1}(s)],
\]
where $\int_0^t ds V_{\lambda_1 - \lambda'_1}(s) \in \mathcal{C}(\lambda'_1 \sim \lambda''_1)$ according to Lemma 2.1. If both points $\lambda'_1, \lambda''_1 \not\in A_0$, then the commutator vanishes because of the commutativity of operators at different lattice points. If $\lambda'_1 \in A_0$, $\lambda''_1 \not\in A_0$, then we have
\[
[C_0, \int_0^t ds V_{\lambda_1 - \lambda'_1}(s)] \in \mathcal{C}(A_0) \mathcal{C}(\lambda'_1 \sim \lambda''_1) - \mathcal{C}(\lambda'_1 \sim \lambda''_1) \mathcal{C}(A_0) \subset \mathcal{C}(A_0 \cup \lambda''_1),
\]
where the second inclusion follows from Lemma 2.2. In a similar manner, one treats the case $\lambda''_1 \in A_0$, $\lambda'_1 \not\in A_0$. Finally, if $\lambda'_1, \lambda''_1 \in A_0$, then the commutator is an element of $\mathcal{C}(A_0)$ according to Lemma 2.2, completing the proof of the initial step of the induction. Next, let $M^X_{n+1} - \mathcal{M}^X_n(t)(C_0) \in \mathcal{C}(A_n)$, where $A_{n+1} \supseteq A_0$ is some finite set. For the induction step from $n$ to $n+1$ we approximate the integral $M^X_{n+1} - \mathcal{M}^X_n(t)(C_0) = \int_0^t ds [M^X_n - \mathcal{M}^X_n(s)(C_0), V_{\lambda^{n+1}_i - \lambda^{n+1}_j}(s)]$ by sums which converge in the norm topology, cf. the preceding part (ii). For the terms in these sums we have the inclusion
\[
[M^X_{n+1} - \mathcal{M}^X_n(t_k)(C_0), \int_{t_k}^{t_{k+1}} ds V_{\lambda^{n+1}_i - \lambda^{n+1}_j}(s)] \in \mathcal{C}(A_n) \mathcal{C}(\lambda^{n+1}_i \sim \lambda^{n+1}_j) - \mathcal{C}(\lambda^{n+1}_i \sim \lambda^{n+1}_j) \mathcal{C}(A_n),
\]
if $\lambda^{n+1}_i \in A_n$ or $\lambda^{n+1}_j \in A_n$; in all other cases the commutator vanishes. By another application of Lemma 2.2 we conclude that these operators are elements of $\mathcal{C}(A_{n+1})$, where $A_{n+1} \supseteq A_n$. Since the latter algebra is closed in the norm topology, this completes the proof of the lemma. □
Having control on the individual terms appearing in the Dyson series, we now need to study their behavior in the thermodynamic limit \( \Lambda \not\sim \mathbb{Z}^d \). To this end we combine them into \( n \)-th order contributions to the Dyson series, \( n \in \mathbb{N} \),

\[
D_n^A(t) = \sum_{\lambda', \lambda'' = \lambda_1' \sim \lambda_2' \ldots \sim \lambda_0'' = A \times A} M_n^{\lambda' \sim \lambda''}(t).
\]

We then have the following result for these maps.

**Lemma 3.2.** Let \( A_0 \subset \mathbb{Z}^d \) be a finite set and let \( n \in \mathbb{N} \). There exists a finite set \( A_n \supset A_0 \) such that the maps \( D_n^A(t) \), \( t \in \mathbb{R} \), defined above, satisfy

(i) the restricted maps \( D_n^A(t) | R(A_0) \) do not depend on \( A \) if \( A \supset A_0 \);

(ii) \( \| (D_n^A(t) - D_n^A(s)) | R(A_0) \| \leq 2^{d+2}n \left( L_0 \cdots (L_0 + n - 1) \| V \| \| t^n - s^n \| / n! \right) \)

for \( s, t \geq 0 \) or \( s, t \leq 0 \), where \( L_0 \) is the number of points in \( A_0 \). This bound is independent of \( A \).

**Proof.** (i) Let \( R_0 \in R(A_0) \). Because of the commutativity of operators localized at different points of the lattice and the stability of the corresponding algebras under the action of \( \alpha_0^{(0)} \), there contribute to \( M_n^{\lambda' \sim \lambda''}(R_0) \) only potentials \( V_{\lambda', \lambda''} \) which are attributed to points \( \lambda', \lambda'' \) having a lattice distance of at most \( n \) from some point in \( A_0 \). Phrased differently, \( M_n^{\lambda' \sim \lambda''}(R_0) \) is different from zero only if the corresponding points \( \lambda', \lambda'' \) all lie in the region \( A_n \times A_n \), where \( A_n \) is obtained from \( A_0 \) by surrounding every point in \( A_0 \) with a cube of side length \( 2n \). Since for points \( \lambda', \lambda'' \notin A_n \times A_n \) the corresponding terms in relation (3.6) vanish, this proves the first part of the statement.

(ii) For the proof of the second part we estimate the number \( N_n(R_0) \) of terms which contribute to \( D_n^A(t)(R_0) \), \( t \in \mathbb{R} \), in the sum (3.6) and the number of points \( L_n(R_0) \) in their respective localization regions; the norm of the individual terms has already been estimated in the preceding lemma. Once again, we rely on induction in \( n \in \mathbb{N} \). For the initial case \( n = 1 \) we note that, either, one must have \( \lambda' \in A_0 \) and there are then \( 2^d \) different possibilities for its neighboring point \( \lambda'' \), so there appear at most \( 2^dL_0 \) terms of this type in the sum. Interchanging the role of \( \lambda', \lambda'' \) one gets the same number, giving the estimate \( N_1(R_0) \leq 2^{d+1}L_0 \); moreover, the number of points \( L_1 \) in the localization region of each term increases at most by one, \( L_1 \leq L_0 + 1 \). As to the induction hypothesis, we have \( N_n(R_0) \leq 2^{d+1}nL_0 \cdots (L_0 + n - 1) \) and \( L_n \leq (L_0 + n) \). For the step from \( n \) to \( n+1 \) we note that we must add to each of the \( N_n(R_0) \) contributing configurations \( \lambda' \sim \lambda'' \) of size at most \( L_n(R_0) \) another pair \( \lambda_{n+1}' \sim \lambda''_{n+1} \) such that the resulting configurations still contribute to the corresponding sum (3.6). For a given configuration this can be done in at most \( 2^{d+1}L_n(R_0) \) different ways and since there are \( N_n \) such configurations, we get \( N_{n+1}(R_0) \leq 2^{d+1}L_n(R_0)N_n(R_0) \) and...
Lemma 3.3. Let $\Lambda \subset \mathbb{Z}^d$ and let $\gamma_t^A$, $t \in \mathbb{R}$, be the map defined by the Dyson series (3.4).

(i) The Dyson series for $\gamma_t^A$ converges absolutely in norm, uniformly on compact subsets of $t \in \mathbb{R}$. It defines automorphisms of the resolvent algebra $\mathcal{R}(\mathbb{Z}^d)$ and satisfies the cocycle relation (3.1).

(ii) The function $t \mapsto \gamma_t^A$ is absolutely continuous, uniformly on compact sets of $t \in \mathbb{R}$.

(iii) The restrictions $\gamma_t^A | \mathcal{K}(\mathbb{Z}^d)$ are automorphisms of $\mathcal{K}(\mathbb{Z}^d)$, $t \in \mathbb{R}$.

(iv) The thermodynamic limit $\gamma_t = \lim_{\Lambda \rightarrow \mathbb{Z}^d} \gamma_t^A$ exists pointwise on $\mathcal{R}(\mathbb{Z}^d)$ in the norm topology, uniformly on compact subsets of $t \in \mathbb{R}$. The limits $\gamma_t$ are automorphisms of $\mathcal{R}(\mathbb{Z}^d)$ which satisfy the cocycle relation (3.1), and $t \mapsto \gamma_t$ is pointwise norm continuous, $t \in \mathbb{R}$.

Proof. (i) According to relation (3.4) we have $\gamma_t^A = t + \sum_{n=1}^{\infty} i^n D_{-n}^A(t)$, $t \in \mathbb{R}$, where the terms $D_{-n}^A(t)$ were defined in (3.6). In the sum in (3.6) there appear at most $2^{(d+1)L^n}$ terms, where $L$ is the number of points in $\Lambda$. Hence, putting $s = 0$ in Lemma (3.1(ii)), we get

$$\|D_{-n}^A(t)\| \leq 2^{(d+1)L^n} \sup_{x' \sim x''} \|M_{-n}^{x' \sim x''}(t)\| \leq 2^{n(d+2)}L^n \|V\|/n!.$$ 

Thus the series of these norms converges uniformly for compact subsets of $t \in \mathbb{R}$. Since the individual terms in the Dyson series map the (norm-closed) resolvent algebra into itself, cf. Lemma (3.1(ii)), it follows that $\gamma_t^A(\mathcal{R}(\mathbb{Z}^d)) \subseteq \mathcal{R}(\mathbb{Z}^d)$, $t \in \mathbb{R}$. That these maps actually are automorphisms of the resolvent algebra which satisfy the cocycle relation (3.1) can be established by direct computation, based on relation (3.3). More conveniently, notice that in any regular (hence faithful) representation of $\mathcal{R}(\mathbb{Z}^d)$ the automorphisms $\gamma_t^A$ are implemented by the adjoint action of unitaries $I_t^A = e^{itH_A^{(0)}} e^{-itH_A}$, $t \in \mathbb{R}$, where

$$H_A^{(0)} = (1/2) \sum_{\lambda \in A} (P_\lambda^2 + \omega^2 Q_\lambda^2) \quad \text{and} \quad H_A = H_A^{(0)} + \sum_{\lambda' \sim \lambda'' \in A \times A} V(Q_{\lambda'} - Q_{\lambda''})$$
are selfadjoint operators. Since $\gamma^A = \text{Ad} \Gamma^A$, $t \in \mathbb{R}$, satisfies the equations (3.1) and (3.2) and can be expanded into the Dyson series (3.4), this completes the proof of the first statement.

(ii) As in the preceding step one obtains with the help of Lemma 3.1(i) the estimate for $s, t \geq 0$ or $s, t \leq 0$

$$
\|D^A_n(t) - D^A_n(s)\| \leq 2^{n(d+1)} L^n \sup_{\lambda', \lambda''} \|M^{\lambda', \lambda''}_n(t) - M^{\lambda', \lambda''}_n(s)\|
\leq 2^{n(d+2)} L^n \|V\|^n |t^n - s^n|/n!.
$$

Since the first terms in the two Dyson series cancel, this yields

$$
\|\gamma^A_t - \gamma^A_s\| \leq 2^{d+2} L \|V\| |t - s| e^{2d+2L\|V\|/t} \quad \text{for} \quad t \geq s \geq 0 \quad \text{or} \quad t \leq s \leq 0,
$$

proving absolute continuity, uniformly on compact sets of $\mathbb{R}$.

(iii) Given $A_0 \subset \mathbb{Z}^d$ and $n \in \mathbb{N}$, Lemma 3.1(iii) implies that there is some $A_n \supset A_0$ such that $M^{\lambda', \lambda''}_n(t)(\mathcal{C}(A_0)) \subset \mathcal{C}(A_n)$, $t \in \mathbb{R}$. Hence, because of their continuity properties, the terms in the Dyson series map the norm closed subalgebra $\mathcal{K}(\mathbb{Z}^d) \subset \mathcal{R}(\mathbb{Z}^d)$ into itself. Since the Dyson series converges absolutely, the assertion follows.

(iv) Let $A_0 \subset \mathbb{Z}^d$ and let $R_0 \in \mathcal{R}(A_0)$. In order to see that $\|\gamma^A_t(R_0) - \gamma^A_t(R_0)\|$ tends to zero for $A, A' \not\subset \mathbb{Z}^d$, we expand both cocycles into a Dyson series. According to Lemma 3.2(i) there exists for each $n \in \mathbb{N}$ some region $A_n \supset A_0$ such that for $A, A' \supset A_n$ the first $n$ terms in the Dyson series for $\gamma^A_t(R_0)$ and $\gamma^{A'}_t(R_0)$ coincide. Hence for such $A, A'$ one gets for sufficiently small $|t|

$$
\|\gamma^A_t(R_0) - \gamma^{A'}_t(R_0)\| \leq \sum_{k=n+1}^{\infty} \|D^A_k(t)(R_0) - D^{A'}_k(t)(R_0)\|
\leq \|R_0\| \sum_{k=n+1}^{\infty} (\|D^A_k(t)\| + \|D^{A'}_k(t)\|)
\leq 2 \|R_0\| \sum_{k=n+1}^{\infty} 2^{d+2} L_0 \cdots (L_0 + k - 1) \|V\|^k |t|^k/k!,
$$

where $L_0$ is the number of points in $A_0$ and, in the last inequality, we made use of Lemma 3.2(ii). For times $t$ satisfying $|t| 2^{d+2}\|V\| < 1$ this upper bound tends to zero in the limit of large $n$, proving the norm convergence of $\lim_{A \not\subset \mathbb{Z}^d} \gamma^A_t(R_0)$, uniformly for small times. Moreover, since the $\gamma^A_t$ are automorphisms of $\mathcal{R}(\mathbb{Z}^d)$, which is the C*-inductive limit of the algebras $\mathcal{R}(A_0)$, $A_0 \subset \mathbb{Z}^d$, this result extends to all elements of $\mathcal{R}(\mathbb{Z}^d)$. Thus the limits $\gamma^A_t = \lim_{A \not\subset \mathbb{Z}^d} \gamma^A_t$ define automorphisms of $\mathcal{R}(\mathbb{Z}^d)$ for small $t \in \mathbb{R}$. 

In order to show that the limit exists for arbitrary \( t \in \mathbb{R} \), we make use of the cocycle relation \((3.1)\) for the approximating maps, giving for any \( R \in \mathcal{R}(\mathbb{Z}^d) \) and small \( s, t \) as above
\[
\|(\alpha_s^{(0)} \gamma_t \alpha_{-s}^{-1} \gamma_s - \gamma_s^A(t))c_0(R)\| = \|(\alpha_s^{(0)} \gamma_t \alpha_{-s}^{-1} \gamma_s - \alpha_s^{(0)} \gamma_s^A(t)\alpha_{-s}^{-1} \gamma_s^A(t))c_0(R)\|
\leq \|(\gamma_t - \gamma_s^A(t))c_0(R)\| + \|(\gamma_s - \gamma_s^A(t))c_0(R)\|.
\]
Hence \( \gamma_s+t = \lim_{A \in \mathbb{Z}^d} \gamma_s^A(t) \) exists pointwise on \( \mathcal{R}(\mathbb{Z}^d) \) in norm and coincides with \( \alpha_s^{(0)} \gamma_t \alpha_{-s}^{-1} \gamma_s \) for the restricted set of \( s, t \). Repeating this procedure, one obtains convergence and the cocycle property \((3.1)\) on all of \( \mathbb{R} \).

It remains to prove the pointwise norm continuity of \( t \mapsto \gamma_t \). Because of the cocycle property of \( \gamma_t \), \( t \in \mathbb{R} \), it suffices to do this for small \( |t| \). Given \( A_0 \subset \mathbb{Z}^d \) and any \( R_0 \in \mathcal{R}(A_0) \) we get for small \( s, t \geq 0 \) or \( s, t \leq 0 \) by the same reasoning as above the estimate
\[
\|\gamma_t(R_0) - \gamma_s(R_0)\| \leq \lim_{A \in \mathbb{Z}^d} \|\gamma_t^A(R_0) - \gamma_s^A(R_0)\|
\leq \limsup_{A \in \mathbb{Z}^d} \sum_{k=1}^{\infty} \|D_k^A(t)(R_0) - D_k^A(s)(R_0)\|
\leq 2 \|R_0\| \sum_{k=1}^{\infty} 2^{(d+2)k} L_0 \cdots (L_0 + k - 1) \|V\| k^k |t^k - s^k|/k!.
\]
Since \( A_0 \) was arbitrary, the claimed continuity obtains on a norm dense sub-algebra of \( \mathcal{R}(\mathbb{Z}^d) \). But \( \gamma_t \), \( t \in \mathbb{R} \), are automorphisms, so the pointwise norm continuity is obtained on all of \( \mathcal{R}(\mathbb{Z}^d) \) by standard arguments, completing the proof of the lemma. \( \square \)

Recalling that the dynamics for finite \( A \subset \mathbb{Z}^d \) is given by \( \alpha_t^A \triangleq \alpha_t^{(0)} \gamma_{-t}^A \), where \( \gamma_t^A \) has been defined in \((3.4)\), \( t \in \mathbb{R} \), we can now state the main result of this section.

**Proposition 3.4.** Let \( V \in C_0(\mathbb{R}^d) \) be any potential and let \( A \subset \mathbb{Z}^d \). There exists a corresponding group of automorphisms \( \alpha_t^A \) of \( \mathcal{R}(\mathbb{Z}^d) \), describing harmonic oscillations of the lattice system as well as nearest neighbor interactions within the region \( A \) for the given potential.

(i) The restriction of \( \alpha_t^A \) to \( K(\mathbb{Z}^d) \subset \mathcal{R}(\mathbb{Z}^d) \) defines automorphisms of this subalgebra, and \( t \mapsto \alpha_t^A \), \( t \in \mathbb{R} \), acts pointwise norm continuously on it, i.e. \( (K(\mathbb{Z}^d), \alpha_t^A) \) is a \( C^* \)-dynamical system.

(ii) The thermodynamic limit \( \alpha_t \triangleq \lim_{A \supset \mathbb{Z}^d} \alpha_t^A \) exists pointwise on \( \mathcal{R}(\mathbb{Z}^d) \) in the norm topology, \( t \in \mathbb{R} \), and defines a group of automorphisms \( \alpha_t \) of \( \mathcal{R}(\mathbb{Z}^d) \) with nearest neighbor interactions all over \( \mathbb{Z}^d \). Its restriction \( \alpha_t \upharpoonright C(\mathbb{Z}^d) \) leaves \( C(\mathbb{Z}^d) \) invariant and acts pointwise norm continuously on this algebra, i.e. \( (K(\mathbb{Z}^d), \alpha_t) \) is a \( C^* \)-dynamical system, as well.
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(Note that the algebra $\mathcal{K}(\mathbb{Z}^d)$ is not norm-dense in $\mathcal{R}(\mathbb{Z}^d)$; but it is dense in the strong operator topology in all regular representations, cf. [10, Thm. 5.4].)

Proof. It has been shown in Lemma 3.3(i) that the maps $\gamma_t^A$ are automorphisms of $\mathcal{R}(\mathbb{Z}^d)$ which satisfy the cocycle relation (3.1), $t \in \mathbb{R}$. Since $\alpha_{\lambda}(0)$ is a group of automorphisms of $\mathcal{R}(\mathbb{Z}^d)$, this is true for $\alpha_t^A = \alpha_{\lambda t}^A$, $t \in \mathbb{R}$, as well. In fact, making use of the cocycle relation for $\gamma_t^A$, one obtains

$$
\alpha_{s+t}^A \gamma_{-s-t} = \alpha_{s+t}^A \alpha_{-t}^A \gamma_{-s-t} \gamma_{-t} = \alpha_{s+t}^A \alpha_{s}^A \gamma_{-s-t} = \alpha_{s+t}^A \gamma_{-s-t} \quad \text{for } s, t \in \mathbb{R},
$$

proving that $\alpha_t^A$ satisfies the group law.

(i) According to Lemma 3.3(iii), the restrictions of the cocycles $\gamma_t^A$ to $\mathcal{K}(\mathbb{Z}^d)$ act as automorphisms on this subalgebra; moreover the function $t \mapsto \gamma_t^A$ is absolutely continuous according to the second part of this lemma. The restriction of $\alpha_{\lambda}(0)$ to $\mathcal{K}(\mathbb{Z}^d)$ induces likewise an automorphic action on this subalgebra which is pointwise norm continuous. This follows from the fact that each algebra $\mathcal{C}(\lambda)$, $\lambda \in \mathbb{Z}^d$, is left invariant by the action of $\alpha_{\lambda}^A$ and the elements of these algebras are represented by compact operators in the (faithful) Schrödinger representation. Combining the two actions, the statement follows. (Note that the action of $\alpha_t^A$ on the full resolvent algebra is discontinuous, however.)

(ii) It was shown in Lemma 3.3(iv) that $\gamma_t = \lim_{A \to \mathbb{Z}^d} \gamma_t^A$ exists pointwise on $\mathcal{R}(\mathbb{Z}^d)$ in the norm topology, satisfies the cocycle relation (3.1), and $t \mapsto \gamma_t$ is pointwise norm continuous, $t \in \mathbb{R}$. Hence the limits $\alpha_t = \lim_{A \to \mathbb{Z}^d} \alpha_t^A \gamma_t^A$, $t \in \mathbb{R}$, exist as well in this topology and thereby define a group of automorphisms $\alpha_{\lambda}$ of the C*-algebra $\mathcal{R}(\mathbb{Z}^d)$. Since $\mathcal{K}(\mathbb{Z}^d)$ is a closed subalgebra of $\mathcal{R}(\mathbb{Z}^d)$, this strong form of convergence also implies that the restriction of $\alpha_{\lambda}$ to $\mathcal{K}(\mathbb{Z}^d)$ defines automorphisms of it. Moreover, the continuity properties of $t \mapsto \gamma_t$ and $t \mapsto \alpha_t^A$ imply that their combined action $t \mapsto \alpha_t = \alpha_t^A \gamma_t$ is pointwise norm continuous on $\mathcal{K}(\mathbb{Z}^d)$ as well, $t \in \mathbb{R}$. This completes the proof of the proposition. \(\square\)

4. Equilibrium states

In this section we turn to the analysis of states on the resolvent algebra with the aim to exhibit equilibrium (KMS) states for the dynamics constructed in the preceding section; cf. also [10] for a recent investigation of KMS states on the resolvent algebra in the one-dimensional case. In a large part of our study we will restrict attention to states on the subalgebra $\mathcal{K}(\mathbb{Z}^d)$. Only at the very end of our analysis we will extend the locally normal states to the full algebra. The reason for this approach is twofold: first, we can work in the setting of C*-dynamical systems, where we can profit from many known results; second, we
can illustrate the advantages of the present framework, where we deal with an
algebra which has a rich ideal structure, allowing to describe finite and infinite
systems at the same time. Thus, instead of changing the algebra when proceeding
to theories in “larger boxes”, we change states on our fixed algebra. This will lead
to considerable simplifications when proceeding to the thermodynamic limit.

We begin with a brief account of notation and terminology: States \( \omega \) are
positive, linear and normalized functionals on a unital \( \text{C}^* \)-algebra. By the GNS
construction, any such state yields a concrete representation (homomorphism) \( \pi \)
of the algebra into the bounded operators on some Hilbert space \( \mathcal{H} \), and a unit
vector \( \Omega \in \mathcal{H} \) such that \( \omega(\cdot) = \langle \Omega, \pi(\cdot)\Omega \rangle \), where the dot stands for any given
member of the algebra. Restricting attention to states on the algebra \( \mathcal{K}(\mathbb{Z}^d) \), we
adopt the following terminology.

**Definition:** Let \( \omega \) be a state on \( \mathcal{K}(\mathbb{Z}^d) \) and let \( A \subset \mathbb{Z}^d \) be any finite subset.

(i) \( \omega \) is **normal** at \( A \) if for every approximate identity formed by increasing
projections \( \{E_i \in \mathcal{C}(A)\}_{i \in I} \) one has \( \lim_i \omega(E_i) = 1 \). The state is said to be
**locally normal** at some infinite subset \( A \subset \mathbb{Z}^d \) if it is normal at all finite
subsets \( A \subset \mathcal{A} \). (Bold face symbols \( \mathcal{A} \) will be used for infinite subsets of \( \mathbb{Z}^d \).)

(ii) \( \omega \) is said to be singular at \( A \) if \( \omega \mathcal{C}(A) = 0 \). (Such states exist since \( \mathcal{C}(A) \)
forms an ideal of compact operators in \( \mathcal{K}(A) \) and \( \mathcal{K}(\mathbb{Z}^d) = \mathcal{K}(A) \otimes \mathcal{K}(\mathbb{Z}^d\setminus A) \).

The following technical lemma concerning such states will be used at various
points in our analysis.

**Lemma 4.1.** Let \( \omega \) be a state on \( \mathcal{K}(\mathbb{Z}^d) \) with GNS-representation \( (\pi, \mathcal{H}, \omega) \).

(i) If, for some \( A \subset \mathbb{Z}^d \), the state \( \omega \) is normal at all points \( \lambda \in A \), it is normal
at \( A \). The restriction of the representation \( \pi | \mathcal{C}(A) \) can then be extended in
the strong operator topology to a regular representation of the resolvent algebra
\( \mathcal{R}(A) \) which is quasi-equivalent to the unique Schrödinger representation of
this algebra.

(ii) If the state \( \omega \) is singular at some point \( \lambda \in \mathbb{Z}^d \), it is singular at any \( A \ni \lambda \)
and \( \pi | \mathcal{C}(A) = 0 \) for such \( A \).

**Proof.** (i) Let \( A', A'' \subset \mathbb{Z}^d \), let \( A(\lambda') \in \mathcal{K}(\lambda') \), \( \lambda' \in A' \), and let \( B(\lambda'') \in \mathcal{K}(\lambda'') \),
\( \lambda'' \in A'' \). The tensor product structure of \( \mathcal{K}(\mathbb{Z}^d) \) and the normality of \( \omega \) at \( \lambda \in A \) imply that for any approximate identity formed by increasing projections
\( \{E_i(\lambda) \in \mathcal{C}(\lambda)\}_{i \in I} \), one has

\[
\lim_i \omega(\Pi_N A(\lambda') E_i(\lambda) \Pi_N B(\lambda'')) = \lim_i \omega(\Pi_N A(\lambda') \Pi_N B(\lambda'')) A(\lambda) E_i(\lambda) B(\lambda)) = \omega(\Pi_N A(\lambda') 1 \Pi_N B(\lambda'')).
\]
The second equality obtains since $E_i(\lambda) \in \mathcal{C}(\lambda)$, $i \in I$, is an approximate identity of increasing projections and $\lim_i \omega(E_i(\lambda)) = 1$. It follows from this relation that $\lim \pi(E_i(\lambda)) = 1$ in the weak, hence also strong operator topology.

Picking approximate identities from each $\mathcal{C}(\lambda)$, $\lambda \in A$, their product defines an approximate identity of increasing projections $\{E_i(A) \in \mathcal{C}(A)\}_{i \in I}$ which consequently converges to 1 in the strong operator topology, as well. Given any other approximate identity $\{F_\kappa(A) \in \mathcal{C}(A)\}_{\kappa \in K}$ one has

$$\pi(E_i(A)) = \lim_\kappa \pi(F_\kappa(A)E_i(A)F_\kappa(A)) \leq \lim_\kappa \pi(F_\kappa(A)),$$

hence the latter approximate identity also converges to 1 in the strong operator topology. Thus $\omega$ and $\pi \upharpoonright \mathcal{C}(A)$ are normal at $A$.

Now let $\{E_i(A) \in \mathcal{C}(A)\}_{i \in I}$ be an approximate identity, i.e. $\lim_i \pi(E_i(A)) = 1$ in the strong operator topology. Clearly, $RE_i(A), E_i(A)R \in \mathcal{C}(A)$, $i \in I$, for any $R \in \mathcal{R}(A)$ since $\mathcal{C}(A) \subset \mathcal{R}(A)$ is an ideal. Moreover, the corresponding limits $\pi(R) = \lim_i \pi(RE_i(A)) = \lim_i \pi(E_i(A)R)$ exist in the strong operator topology, defining an extension of $\pi \upharpoonright \mathcal{C}(A)$ to $\mathcal{R}(A)$, denoted by the same symbol. For the proof of regularity, let $R(c) \in \mathcal{R}(A)$, $c \in \mathbb{R}\setminus\{0\}$, be any of the basic resolvents defined in relation (2.1). Then $\lim_{c \to \infty} icR(c)E_i(A) = E_i(A)$, $i \in I$, in the norm topology. Since the function $c \mapsto \pi(icR(c))$ is uniformly bounded, it follows by a three epsilon argument that

$$\lim_{c \to \infty} \pi(icR(c)) = \lim_{c \to \infty} \lim_{i \to \infty} \pi(icR(c)E_i(A)) = \lim_{i \to \infty} \pi(icR(c,f)E_i(A)) = 1$$

in the strong operator topology. The regularity of the representation then follows, cf. [21, Prop. 4.5]. The statement about quasi-equivalence is a consequence of von Neumann’s uniqueness theorem for finite regular quantum systems.

(ii) Let $\omega$ be singular at some given $\lambda \in \mathbb{Z}^d$ and let $A \ni \lambda$. Similarly as in the preceding step, one shows that for any approximate identity $\{E_i(\lambda) \in \mathcal{C}(\lambda)\}_{i \in I}$ one has $\pi(E_i(\lambda)) = 0$, $i \in I$. Hence if $A \ni \lambda$, the product $\{E_i(A) \in \mathcal{C}(A)\}_{i \in I}$ of any family of approximate identities $\{E_i(\lambda') \in \mathcal{C}(\lambda')\}_{i \in I}$, $\lambda' \in A$, satisfies $\pi(E_i(A)) = 0$, $i \in I$. Since $\pi(C) = \lim_i \pi(CE_i(A)) = 0$ for $C \in \mathcal{C}(A)$, this completes the proof of the lemma. □

We turn next to the study of the dynamics in given states and their GNS-representations. In addition to the free dynamics $\alpha_R^{(0)}$ and the interacting dynamics $\alpha_R$ we will also consider the approximating dynamics $\alpha_R^A$ for $A \subset \mathbb{Z}^d$. The following lemma is of some interest in its own right since it displays the role of singularities of states on $\mathcal{K}(\mathbb{Z}^d)$ as “impenetrable boundaries”: i.e. if a state is singular in some region, any form of interaction is turned off there in its GNS representation.
Lemma 4.2. Let $\Lambda \subset \mathbb{Z}^d$, let $\omega$ be a state on $K(\mathbb{Z}^d)$ which is normal at $\Lambda$ and singular at all points $\Lambda' \in \mathbb{Z}^d \setminus \Lambda$, and let $(\pi, \mathcal{H}, \Omega)$ be its GNS-representation. Then

(i) $\pi(K(\mathbb{Z}^d)) = \pi(K(\Lambda))$ and $\pi \upharpoonright \mathcal{C}(\Lambda') = 0$ if $\Lambda' \cap (\mathbb{Z}^d \setminus \Lambda) \neq \emptyset$

(ii) $\pi(\alpha_t(C')) = 0$, $t \in \mathbb{R}$, for any $C' \in \mathcal{C}(\Lambda')$ if $\Lambda' \cap (\mathbb{Z}^d \setminus \Lambda) \neq \emptyset$

(iii) $\pi(\alpha_t(C)) = \pi(\alpha_t^A(C))$, $t \in \mathbb{R}$, for any $C \in K(\mathbb{Z}^d)$.

Proof. (i) The first statement is an immediate consequence of the preceding lemma and the definition of the algebra $K(\mathbb{Z}^d)$, which is generated by the algebras $\mathcal{C}(\Lambda), \Lambda \subset \mathbb{Z}^d$, and the unit operator.

(ii) The crucial point is the observation that the time evolution $\alpha_\mathbb{R}$ maps the algebra $\mathcal{C}(\Lambda')$ into a norm closed subalgebra $S(\Lambda') \subset K(\mathbb{Z}^d)$, the surrounding of $\mathcal{C}(\Lambda')$, which is generated by all algebras $\mathcal{C}(\Lambda'')$ with $\Lambda'' \supseteq \Lambda'$. According to Lemma 3.1(ii) we have $\pi \upharpoonright \mathcal{C}(\Lambda'') = 0$ if $\Lambda'' \supseteq \Lambda'$, whence by continuity also $\pi \upharpoonright S(\Lambda') = 0$. The statement then follows.

Now the harmonic dynamics $\alpha_\mathbb{R}^{(i)}$ leaves all algebras $\mathcal{C}(\Lambda''), \Lambda'' \subset \mathbb{Z}^d$, invariant, so it suffices to determine the action of the cocycle $\gamma_\mathbb{R}$. Given $C' \in \mathcal{C}(\Lambda')$, we expand $\gamma_t(C')$ for small $|t|$ into a (norm-convergent) Dyson series; it is given by $\gamma_t(C') = C' + \sum_{n=1}^{\infty} t^n D_n^A(t)(C')$, where we used Lemma 3.2(i) by choosing for the $n$-th order contribution $D_n^A(t)(C')$ some sufficiently big $A'_n \supseteq \Lambda'$. Decomposing $D_n^A(t)(C')$ into the maps $M_n^{x \rightarrow x'}(t)(C')$, it follows from Lemma 3.1(iii) that $D_n^A(t)(C') \subseteq \bigcup_{n' \subseteq n, \Lambda_n \supseteq \Lambda'} \mathcal{C}(\Lambda')$. Hence, one has $\alpha_t(C') = \alpha_t^{(i)}(C') \in S(\Lambda')$ for small $|t|$ and, making use of the group law for $\alpha_\mathbb{R}$, this result extends to all $t \in \mathbb{R}$.

(iii) Let $A_0 \subset \mathbb{Z}^d$ and let $C \in \mathcal{C}(A_0)$. If $A_0 \cap (\mathbb{Z}^d \setminus A) \neq \emptyset$, the statement follows from the preceding step, so we may assume that $A_0 \subseteq A$. Similarly to the preceding step, we expand $\gamma_t(C)$ for small $|t|$ into a Dyson series, $\gamma_t(C) = C + \sum_{n=1}^{\infty} t^n D_n^A(t)(C)$, where each $A_n \supseteq A_0$ is sufficiently big. Decomposing $D_n^A(t)(C)$ into the terms $M_n^{x \rightarrow x'}(t)(C)$, it follows from Lemma 3.1(iii) that for each term one has $M_n^{x \rightarrow x'}(t)(C) \in \mathcal{C}(A_n^{x \rightarrow x'})$, where $A_n^{x \rightarrow x'} \supseteq A_0$. If $A_n^{x \rightarrow x'} \cap (\mathbb{Z}^d \setminus A) \neq \emptyset$, the corresponding terms $M_n^{x \rightarrow x'}(t)(C)$ are annihilated in the representation $\pi$ according to Lemma 4.1(ii). In other words, in the representation $\pi$ there contribute only terms $M_n^{x \rightarrow x'}(t)(C)$ to the Dyson series with $A_n^{x \rightarrow x'} \subset A$, $n \in \mathbb{N}$. The resulting series thus coincides exactly with the Dyson series for $\gamma_t^{A_0}$.

So we conclude that $\pi(\gamma_t(C)) = \pi(\gamma_t^A(C))$ for $C \in \mathcal{C}(A_0), A_0 \subset \mathbb{Z}^d$, and small $|t|$. Since the algebra $K(\mathbb{Z}^d)$ is generated by the algebras $\mathcal{C}(A_0), A_0 \subset \mathbb{Z}^d$, and the unit operator and 1 since $\pi$ is a homomorphism, this relation holds therefore for all $C \in K(\mathbb{Z}^d)$. Using the group law for the dynamics $\alpha_\mathbb{R}$ and $\alpha_\mathbb{Z}$, and the invariance of $K(\mathbb{Z}^d)$ under both dynamics, the statement follows. $\Box$
This lemma shows how one can describe finite systems on the resolvent algebra, using partially singular states. We will make use of this fact in our construction of KMS states; cf. [4] for basic definitions in this context. Given any $\Lambda \subset \mathbb{Z}^d$ and $\beta > 0$, there exists a (unique) normal KMS state $\rho_\beta^A$ on the algebra $\mathcal{K}(A)$ for the dynamics $\alpha_\beta^A \upharpoonright \mathcal{K}(A)$; note that this subalgebra is stable under the action of $\alpha_\beta^A$. The state $\rho_\beta^A$ is described in the Schrödinger representation of this finite system by the density matrix $Z^{-1}_A e^{-\beta H_A}$, where

$$H_A = (1/2) \sum_{\lambda \in \Lambda} (P_\lambda^2 + \varepsilon^2 Q_\lambda^2) + \sum_{\lambda' \sim \lambda'' \in \Lambda \times \Lambda} V_{\lambda' \sim \lambda''}(Q_{\lambda'} - Q_{\lambda''}). \quad (4.1)$$

Exploiting the tensor product structure of $\mathcal{K}(\mathbb{Z}^d)$, we extend $\rho_\beta^A$ to the full algebra. This extension is given by the product state $\omega_\beta^A \equiv \rho_\beta^A \otimes \sigma^{\mathbb{Z}^d \setminus A}$ on $\mathcal{K}(A) \otimes \mathcal{K}(\mathbb{Z}^d \setminus A)$, where $\sigma^{\mathbb{Z}^d \setminus A}$ denotes the singular state which annihilates all operators in $\mathcal{K}(\mathbb{Z}^d \setminus A)$, apart from multiples of 1. The latter state exists since $\mathcal{K}(\mathbb{Z}^d \setminus A) = \otimes_{\Lambda' \subset \mathbb{Z}^d \setminus A} \mathcal{K}(\Lambda')$ and since each algebra $\mathcal{C}(\Lambda')$ is an ideal in $\mathcal{K}(\Lambda') = \mathbb{C}1 + \mathcal{C}(\Lambda')$, $\Lambda' \subset \mathbb{Z}^d \setminus A$. The properties of the states $\omega_\beta^A$, given in the subsequent lemma, are a simple consequence of the preceding results.

**Lemma 4.3.** Let $\Lambda \subset \mathbb{Z}^d$, $\beta > 0$, and let $\omega_\beta^A$ be the state on $\mathcal{K}(\mathbb{Z}^d)$, defined above, with GNS representation $(\pi_\beta^A, \mathcal{H}_\beta^A, \Omega_\beta^A)$.

(i) $\omega_\beta^A$ is a KMS state for the full dynamics $\alpha_\beta^A$ and the given $\beta$

(ii) $\omega_\beta^A$ is normal at $\Lambda$ and singular at any point $\lambda' \in \mathbb{Z}^d \setminus A$

(iii) $\omega_\beta^A$ is a primary state, i.e. $\pi_\beta^A(\mathcal{C}(\mathbb{Z}^d))^\prime$ is a factor.

(iv) For given $\Lambda \subset \mathbb{Z}^d$ and $\beta > 0$, the state is uniquely fixed by these three properties.

**Proof.** (i) Let $C_1, C_2 \in \mathcal{K}(\mathbb{Z}^d)$ and consider the function $t \mapsto \omega_\beta^A(C_1 \alpha_t(C_2))$, $t \in \mathbb{R}$. According to Lemma 4.1(ii) this function vanishes whenever one of the operators contains a factor $C \in \mathcal{C}(A_0)$ with $A_0 \cap (\mathbb{Z}^d \setminus A) \neq \emptyset$; thus it satisfies the KMS condition. Because of the linearity of $\omega_\beta^A$ we may therefore restrict attention to the case, where $C_1, C_2 \in \mathcal{K}(A)$. It then follows from Lemma 4.2(iii) and the fact that $\mathcal{K}(A)$ is stable under the action of $\alpha_\beta^A$ that the functions $t \mapsto \omega_\beta^A(C_1 \alpha_t(C_2)) = \omega_\beta^A(C_1 \alpha_\beta^A(C_2))$ satisfy the KMS condition, bearing in mind that $\omega_\beta^A \upharpoonright \mathcal{K}(A) = \rho_\beta^A$ is by definition a KMS state for the dynamics $\alpha_\beta^A \upharpoonright \mathcal{K}(A)$.

(ii) $\omega_\beta^A \upharpoonright \mathcal{K}(A) = \rho_\beta^A$ is represented by a density matrix in the Schrödinger representation, hence normal, and its extension to $\mathcal{K}(\mathbb{Z}^d)$ is singular at all points of $\mathbb{Z}^d \setminus A$ by its very construction.

(iii) According to Lemma 4.1(i) one has $\pi_\beta^A(\mathcal{K}(\mathbb{Z}^d))^\prime = \pi_\beta^A(\mathcal{K}(A))^\prime$, and the latter algebra is a factor according to von Neumann’s uniqueness theorem.

(iv) The last statement is an immediate consequence of the uniqueness of the KMS state $\omega_\beta^A \upharpoonright \mathcal{K}(A)$ for the given dynamics $\alpha_\beta^A \upharpoonright \mathcal{K}(A)$. □
According to this lemma, there exists for every finite subset \( A \subset \mathbb{Z}^d \) and inverse temperature \( \beta > 0 \) some primary KMS state \( \omega_0^A \) for the fixed C*-dynamical system \( (\mathcal{K}(\mathbb{Z}^d), \alpha_A) \) which is normal at \( A \) and singular at all points of \( \mathbb{Z}^d \setminus A \). In order to gain control on the normality properties of these states in the thermodynamic limit, we consider KMS states for the perturbed dynamics, where the interaction between any given point \( \lambda \in A \) and its complement \( A \setminus \lambda \) is turned off. To this end we proceed from the KMS-state \( \rho_0^A \), and extend it to a state on \( \mathcal{K}(\Lambda) = \mathcal{K}(A \setminus \lambda) \otimes \mathcal{K}(A) \), putting \( \rho_0^A(\Lambda, \lambda) \equiv \rho_0^A \otimes \rho_0^\lambda \). In a regular representation of \( \mathcal{K}(\Lambda) \), where \( \rho_0^A \) is described by the density matrix \( Z_\Lambda^{-1} e^{-\beta H_\lambda} \), the state \( \rho_0^A(\Lambda, \lambda) \) is described by the density matrix \( Z_\Lambda^{-1} \otimes \lambda^\e^{-\beta H_\lambda} \), where

\[
H_{A \setminus \lambda, \lambda} = H_A - V_\lambda \quad \text{with} \quad V_\lambda = \sum_{\lambda' \in A} (V_{\lambda' \lambda} + V_{\lambda' \lambda}) .
\]

Since both states are represented by density matrices in the same normal representation of \( \mathcal{K}(\Lambda) \), we can proceed to the weak closure of this algebra. Thus, we can make use of the perturbation theory for KMS states, developed by Araki [2], where we rely on the presentation given in [9]. Applying these results, we obtain the following estimate for the norm difference between the two states.

**Lemma 4.4.** Let \( A \subset \mathbb{Z}^d \), \( \lambda \in A \), and \( \beta > 0 \). Moreover, let \( \rho_0^A, \rho_0^A(\Lambda, \lambda) \) be the two states on \( \mathcal{K}(\Lambda) \), defined above. Then

(i) \( \| \rho_0^A - \rho_0^A(\Lambda, \lambda) \| \leq 2 e^{\beta} \| V_\lambda \|/2 (e^{\beta} \| V_\lambda \|/2 - 1) \). (Note that this bound does not depend on the size of \( A \).)

(ii) The norm difference of the states does not change if one extends them to the algebra \( \mathcal{K}(\mathbb{Z}^d) = \mathcal{K}(\Lambda) \otimes \mathcal{K}(\mathbb{Z}^d \setminus A) \), putting \( \omega_0^A = \rho_0^A \otimes \sigma^{\mathbb{Z}^d \setminus A} \) and \( \omega_0^A(\Lambda, \lambda) = \rho_0^A(\Lambda, \lambda) \otimes \sigma^{\mathbb{Z}^d \setminus A} \), respectively. Here \( \sigma^{\mathbb{Z}^d \setminus A} \) denotes the singular state which satisfies \( \sigma^{\mathbb{Z}^d \setminus A} \downarrow \mathcal{C}(\lambda') = 0 \) for \( \lambda' \in \mathbb{Z}^d \setminus A \).

**Proof.** (i) Let \( (\pi, \mathcal{H}, \Omega) \) be the GNS representation of \( \mathcal{K}(\Lambda) \), induced by \( \rho_0^A \). According to [9] Thm. 5.1, there exists then a non-zero vector \( \Omega_{V_\lambda} \in \mathcal{H} \) such that

\[
\rho_0^A(\Lambda, \lambda)(\cdot) = \langle \Omega_{V_\lambda}, \cdot \Omega_{V_\lambda} \rangle/\| \Omega_{V_\lambda} \|^2 .
\]

For its norm one has \( \| \Omega_{V_\lambda} \| \leq \| e^{-\beta V_\lambda/2} \Omega \| \leq e^{\beta} \| V_\lambda \|/2 \) (Golden-Thomson inequality). Moreover, one obtains the bound \( \| \Omega_{V_\lambda} - \Omega \| \leq (e^{\beta} \| V_\lambda \|/2 - 1) \), cf. [9] Thm. 5.3]. These estimates are not optimal, but they suffice for our purposes. For they imply

\[
\| \rho_0^A - \rho_0^A(\Lambda, \lambda) \| = \| \langle \Omega, \pi(\cdot) \Omega \rangle - \langle \Omega_{V_\lambda}, \pi(\cdot) \Omega_{V_\lambda} \rangle/\| \Omega_{V_\lambda} \| \|^2 \|
\leq \| \Omega - \Omega_{V_\lambda} \| \left( 1 + \| \Omega_{V_\lambda} \| \right) + \| 1 - \Omega_{V_\lambda} \|^2
\leq 2 \| \Omega - \Omega_{V_\lambda} \| \left( 1 + \| \Omega_{V_\lambda} \| \right) ,
\]
from which the first statement follows.

(ii) The second statement is an immediate consequence of Lemma 4.2(i), completing the proof of this lemma. □

Let $\beta > 0$ be fixed and let $K_\beta$ be the weak-*-closure of the convex hull of the corresponding KMS states on $K(\mathbb{Z}^d)$ for the dynamics $\alpha_R$. Recalling that $(K(\mathbb{Z}^d), \alpha_R)$ is a C*-dynamical system, it follows from [4, Prop. 5.3.30] that all extremal points of $K_\beta$ are primary (factorial) KMS states, which are disjoint. In our proof that KMS states exist in $K_\beta$ which are normal at all points of $\mathbb{Z}^d$ we will rely on the following basic fact.

**Lemma 4.5.** Let $\omega$ be a primary state on $K(\mathbb{Z}^d)$ with factorial GNS representation $(\pi, \mathcal{H}, \Omega)$. There is a (empty, finite, or infinite) set $\Lambda \subseteq \mathbb{Z}^d$ such that $\omega$ is normal at all points $\lambda \in \Lambda$ and singular at all points $\lambda' \in \mathbb{Z}^d \setminus \Lambda$.

**Proof.** Let $\lambda \in \mathbb{Z}^d$, and let the projections $\{E_\iota(\lambda) \in C(\lambda)\}_{\iota \in \mathcal{I}}$ form an approximate identity of $C(\lambda)$. Making use of the arguments in the proof of Lemma 4.1(i), one shows that $\lim_{\iota} \pi(E_\iota(\lambda))$ lies in the center of $\pi(K(\mathbb{Z}^d))''$. Since $\pi$ is factorial it follows that $\lim_{\iota} \pi(E_\iota(\lambda)) \in \{0, 1\}$. Let $\Lambda = \{\lambda \in \mathbb{Z}^d : \lim_{\iota} \pi(E_\iota(\lambda)) = 1\}$. Then $\lim_{\iota} \pi(E_\iota(\lambda')) = 0$ for $\lambda' \in \mathbb{Z}^d \setminus \Lambda$, completing the proof. □

We are equipped now with the tools for the proof that there exist primary states in $K_\beta$ which are locally normal at all points of $\mathbb{Z}^d$. At this point we have to assume that $\beta > 0$ is sufficiently small, i.e. the temperature is sufficiently high. For we will need that the upper bound on the norm difference of states, given in Lemma 4.4, is less than 1. This is accomplished if $\beta \|V\| < 2|\ln(\sqrt{3} - 1)|$ for $\lambda \in \mathbb{Z}^d$. Note that for given potential $V$ and dimension $d$ of the lattice one has $\|V\| \leq 2^d \|V\|$.

**Lemma 4.6.** Let $V$, $\beta$ be given, where $0 < \beta \|V\| < 2^{1-d} \ln(\sqrt{3} - 1)$, and let $\Lambda_m$, $m \in \mathbb{N}$, be an increasing family of subsets, covering $\mathbb{Z}^d$. Moreover, let $K_{\beta,n} \subseteq K_\beta$ be the weak-*-closures of the convex hull of states on $K(\mathbb{Z}^d)$ given by $\{\sum_{m \geq n} p_m \omega_\beta^{A_m} : p_m \geq 0, \sum_{m=n}^{\infty} p_m = 1\}$, $n \in \mathbb{N}$. Their intersection $\bigcap_{n \in \mathbb{N}} K_{\beta,n}$ is not empty and its extremal points are primary KMS states which are locally normal on $\mathbb{Z}^d$.

**Proof.** We begin by noting that $\bigcap_{n \in \mathbb{N}} K_{\beta,n}$ is not empty since it is closed in the weak-*-topology and $1 \in K(\mathbb{Z}^d)$. Moreover, by central decomposition [4, Prop. 5.3.30] one can decompose the states in this set into primary, hence extremal KMS states for the given $\beta$. Next, let $\lambda \in \mathbb{Z}^d$ and let $n \in \mathbb{N}$ be sufficiently large such that $\lambda \in \Lambda_m$, $m \geq n$. Then, for such $m$ and corresponding primary
states $\omega_\beta^{A_m}$, one has according to Lemma 4.4(ii) and the choice of $\beta \|V\|$, bearing also in mind that by construction $\omega_\beta^{A_m \setminus \lambda, \lambda} \parallel K(\lambda) = \omega_\lambda^{\lambda} \parallel K(\lambda)$,

$$\|(\omega_\beta^{A_m} - \omega_\beta^{A_m \setminus \lambda, \lambda}) \parallel K(\lambda)\| \leq \|(\omega_\beta^{A_m} - \omega_\beta^{A_m \setminus \lambda, \lambda})\| \leq c < 1.$$ It follows that for any convex combination of the states $\omega_\beta^{A_m}$, $m \geq n$, one has

$$\| (\sum_{m \geq n} p_m \omega_\beta^{A_m} - \omega_\beta^{A_m}) \parallel C(\lambda)\| \leq \sum_{m \geq n} p_m \| (\omega_\beta^{A_m} - \omega_\beta^{A_m}) \parallel C(\lambda)\| \leq c < 1,$$

and this estimate holds for their weak-*limit points in $K_{\beta, n}$, as well.

Now let $\omega_\beta \in \bigcap_{n \in \mathbb{N}} K_{\beta, n}$ be any primary KMS state. It follows from this estimate that $\| \omega_\beta \parallel C(\lambda)\| \geq \| \omega_\beta \parallel C(\lambda)\| - \| (\omega_\beta - \omega_\lambda) \parallel C(\lambda)\| > 0$, $\lambda \in \mathbb{Z}^d$. Hence $\omega_\beta \parallel C(\lambda)$ does not vanish at any point $\lambda \in \mathbb{Z}^d$ and therefore is locally normal according to Lemma 4.5.

At this point we can finally proceed from the algebra $\mathcal{K}(\mathbb{Z}^d)$ back to the full resolvent algebra $\mathcal{R}(\mathbb{Z}^d)$, making use of Lemma 4.1(i). We thereby arrive at the main result of this section.

**Proposition 4.7.** Let $V \in C_0(\mathbb{R}^d)$ be any interaction potential between nearest neighbors on $\mathbb{Z}^d$ and let $0 < \beta \|V\| < 2^{1-d} \left| \ln (\sqrt{3} - 1) \right|$. There exist regular and primary KMS states $\omega_\beta$ on the resolvent algebra $\mathcal{R}(\mathbb{Z}^d)$ for the corresponding dynamics $\alpha_\mathcal{R}$.

**Proof.** According to Lemma 4.6 there exists some locally normal and primary KMS state $\omega_\beta$ on $\mathcal{K}(\mathbb{Z}^d)$ with GNS representation $(\pi_\beta, \mathcal{H}_\beta, \Omega_\beta)$. Since the state satisfies the KMS condition, there exists on $\mathcal{H}_\beta$ a continuous unitary representation $U_\beta(t)$ of the time translations such that $U_\beta(t) \pi_\beta(\cdot) = \pi_\beta \circ \alpha_t(\cdot) U_\beta(t)$, i.e. the representations $\pi_\beta$ and $\pi_\beta \circ \alpha_t$ are unitarily equivalent, $t \in \mathbb{R}$. Moreover, the state $\omega_\beta$ and hence $\pi_\beta$ is locally normal, so the same is true for the representations $\pi_\beta \circ \alpha_t$, $t \in \mathbb{R}$. We can therefore uniquely extend these representations to regular representations of the $C^*$-inductive limit $\mathcal{R}(\mathbb{Z}^d)$, formed by the algebras $\mathcal{R}(A), A \subset \mathbb{Z}^d$, cf. Lemma 4.1(i). Since $\mathcal{R}(\mathbb{Z}^d)$ is stable under the action of $\alpha_\mathcal{R}$, these extensions are unitarily equivalent with the same unitary intertwiners $U_\beta(t), t \in \mathbb{R}$. The statement then follows. \(\square\)

5. Ground states

Having established the existence of regular KMS states for large temperatures on the resolvent algebra, let us turn now to the opposite end of the temperature scale, i.e. the ground states. The construction of such states proceeds in exactly
the same manner as that of KMS states with one major difference: The perturbation theory underlying Lemma 4.4 is not applicable in this case, so one needs a different strategy in order to establish regularity. We will rely here on arguments, given in [2, Lem. 7.3] for one-dimensional lattice systems, which work in the present case, as well.

We proceed again from the C*-dynamical system $(\mathcal{K}(\mathbb{Z}^d), \alpha_{\mathbb{R}})$ and consider regular ground states $\rho_\Lambda^\infty$ on $\mathcal{K}(\Lambda)$ for the approximating dynamics $\alpha_{\Lambda} \upharpoonright \mathcal{K}(\Lambda)$, $\Lambda \subset \mathbb{Z}^d$, corresponding to zero temperature $\beta = \infty$. These states can be represented in the Schrödinger representation by normalized eigenvectors $\Omega_\Lambda$ for the generators of the dynamics,

$$H_\Lambda = \frac{1}{2} \sum_{\lambda \in \Lambda} (P^2_\lambda + \varpi^2 Q^2_\lambda) + \sum_{\lambda' \sim \lambda' \in \Lambda \times \Lambda} V(Q'_{\lambda} - Q'_{\lambda'}) - E_\Lambda 1,$$

where the constants $E_\Lambda$ are chosen in such a way that $H_\Lambda \Omega_\Lambda = 0$, $\Lambda \subset \mathbb{Z}^d$; note that $H_\Lambda$ is positive and has discrete spectrum. The states $\rho_\Lambda^\infty$ are extended to the full algebra $\mathcal{K}(\mathbb{Z}^d) = \mathcal{K}(\Lambda) \otimes \mathcal{K}(\mathbb{Z}^d \setminus \Lambda)$, putting $\omega_\Lambda^\infty = \rho_\Lambda^\infty \otimes \sigma_{\mathbb{Z}^d \setminus \Lambda}$, where $\sigma_{\mathbb{Z}^d \setminus \Lambda}$ is the singular state, invented in the preceding section.

Now Lemmas 4.1, 4.2 and 4.5 are applicable without changes in the present context; in Lemma 4.3 one has to put $\beta = \infty$ and, in its part (iii), to replace the term “primary state” by the more specific term “pure state”. Thus the states $\omega_\Lambda^\infty$, $\Lambda \subset \mathbb{Z}^d$, are all pure ground states for the C*-dynamical system $(\mathcal{K}(\mathbb{Z}^d), \alpha_{\mathbb{R}})$. In analogy to the preceding discussion, we consider the weak-*-closed convex (hence compact) set $K_\infty$ formed by these ground states and note that all of its extremal points are pure ground states [4, Prop. 5.3.37].

For the proof that $K_\infty$ contains locally normal states, we make use of the fact that the operators $(\mu 1 + H_\lambda)^{-1}$, $\mu > 0$, are compact in the Schrödinger representation of $\mathcal{C}(\lambda)$ and hence are elements of these algebras, $\lambda \in \mathbb{Z}^d$. The following statement replaces Lemma 4.4 in case of ground states.

**Lemma 5.1.** Let $\Lambda \subset \mathbb{Z}^d$ and $\lambda \in \Lambda$. Then

$$\omega_\infty^\lambda ( (\mu 1 + H_\lambda)^{-1} ) \geq (\mu + 2^{d+2} \|V\|)^{-1}, \quad \text{where} \quad \mu > 0.$$  

*(Note that the lower bound does not depend on the size of $\Lambda$.)

**Proof.** Proceeding to the Schrödinger representation induced by the state $\rho_\infty^\lambda$ on $\mathcal{K}(\Lambda)$, we decompose the generator of the underlying dynamics,

$$H_\Lambda = H_{\Lambda \setminus \lambda} + H_\lambda + \sum_{\lambda' \in \Lambda} (V_{\lambda' \sim \lambda} + V_{\lambda \sim \lambda'}) + (E_{\Lambda \setminus \lambda} + E_\lambda - E_A) 1.$$  

Since $H_{\Lambda \setminus \lambda}$ and $H_\lambda$ commute, they have a common normalized ground state vector $\Omega$, i.e. $H_{\Lambda \setminus \lambda} \Omega = H_\lambda \Omega = 0$. As $\| \sum_{\lambda' \in \Lambda} (V_{\lambda' \sim \lambda} + V_{\lambda \sim \lambda'}) \| \leq 2^{d+1} \|V\|$
and $H_A$ is a positive operator which has $\Omega$ in its domain, this gives

$$0 \leq \langle \Omega, H_A \Omega \rangle \leq 2^{d+1} \|V\| + (E_{A\setminus \lambda} + E_\lambda - E_A).$$

Bearing in mind that $H_{A\setminus \lambda} \geq 0$, it follows that $H_A \geq (H_{A\setminus \lambda} - 2^{d+2} \|V\|)$ and hence

$$(\nu 1 + H_A)^{-1} \leq ((\nu - 2^{d+2} \|V\|) 1 + H_{A\setminus \lambda})^{-1}$$

for $\nu > 2^{d+2} \|V\|$. Taking the expectation value of these operators in the ground state for $H_A$ and putting $\nu = (\mu + 2^{d+2} \|V\|)$, one finally obtains the lower bound $\rho_\Lambda^A((\mu 1 + H_\lambda)^{-1}) \geq (\mu + 2^{d+2} \|V\|)^{-1}$. But the chosen extension $\omega_\Lambda^A$ of $\rho_\Lambda^A$ to the algebra $\mathcal{K}(\mathbb{Z}^d)$ does not affect this lower bound, so the proof of the lemma is complete. \qed

We can proceed now exactly as in the construction of the thermodynamic limits of equilibrium states. Let $A_n \subset \mathbb{Z}^d$, $n \in \mathbb{N}$, be increasing sets, covering all of $\mathbb{Z}^d$ in the limit and let $K_{\infty,n} \subset K_{\infty}$ be the weak*-closures of the convex hulls of ground states $\omega_{\Lambda_m}^A$ with $m \geq n$. In complete analogy to Lemma 4.6, we have the following result.

**Lemma 5.2.** The intersection $\bigcap_{n \in \mathbb{N}} K_{\infty,n}$ is not empty and the extremal points of this set are pure ground states which are locally normal on $\mathbb{Z}^d$.

**Proof.** The intersection $\bigcap_{n \in \mathbb{N}} K_{\infty,n}$ is not empty because of the reasons given in the proof of Lemma 4.6 so let $\omega_\infty \in \bigcap_{n \in \mathbb{N}} K_{\infty,n}$ be an extremal (hence pure) state in this closed, convex set. Given any $\lambda \in \mathbb{Z}^d$ we choose $n \in \mathbb{N}$ sufficiently large such that $A_m \supset \lambda$, $m \geq n$. It then follows from the preceding lemma (proceeding to a suitable weak*-limit of the states for $A \not
\subset \mathbb{Z}^d$) that for $\mu > 0$ one has $\omega_\infty((\mu 1 + H_\lambda)^{-1}) \geq (\mu + 2^{d+2} \|V\|)^{-1} > 0$. Since $\omega_\infty$ is pure and thus a fortiori a primary state on $\mathcal{K}(\mathbb{Z}^d)$ and since $\lambda \in \mathbb{Z}^d$ was arbitrary, it follows from Lemma 4.5 that $\omega_\infty$ is locally normal on $\mathbb{Z}^d$. \qed

Equipped with this information, one can now establish the following proposition. Since the proof is identical to the one given for equilibrium states, we can omit it.

**Proposition 5.3.** Let $V \in C_0(\mathbb{R}^d)$ be the interaction potential between nearest neighbors on the lattice $\mathbb{Z}^d$. There exist regular and pure ground states $\omega_\infty$ for the corresponding dynamics $\alpha_{\mathbb{R}}$ on the resolvent algebra $\mathcal{R}(\mathbb{Z}^d)$.
6. More dynamics

In the preceding analysis we have considered the simplest non-trivial examples of infinite lattice systems with nearest neighbor interactions. The forces keeping the particles in a neighborhood of their respective lattice points were of harmonic nature and their interaction was described by a regular class of potentials. We have restricted attention to this case in order not to obscure the novel features of our construction of dynamics and states.

In order to indicate how one can deal with the functional analytic problems appearing in more complicated situations, such as singular interactions and non-harmonic binding forces, and to reveal also certain limitations of the present framework, we outline here three characteristic examples. Moreover, we consider only the simple case of a single particle with one degree of freedom and deal with its resolvent algebra $\mathcal{R}$ in the (faithful) Schrödinger representation.

The canonical momentum and position operators of the particle are denoted by $P, Q$ with their standard domain of essential selfadjointness $\mathcal{D} \subset L^2(\mathbb{R})$, consisting of Schwartz test functions on which $Q$ acts as a multiplication operator. The resolvent algebra $\mathcal{R}$ is then concretely given as the $C^*$-algebra which is generated by the resolvents $R(a, b, c) = (ic1 + aP + bQ)^{-1}$, $a, b \in \mathbb{R}$, $c \in \mathbb{R} \setminus \{0\}$; it contains the ideal of compact operators $C \subset \mathcal{R}$. The free Hamiltonian $H_0 = P^2$ acts on $\mathcal{D}$ and, for the sake of simplicity, we consider here only unbounded potentials $V = V(Q)$ which are defined on $\mathcal{D}$ as well and for which the resulting Hamiltonian $H = (H_0 + V)$ is essentially selfadjoint on $\mathcal{D}$.

In order to prove that for $R \in \mathcal{R}$ also all operators $e^{itH}Re^{-itH_0} - 1 \in C$, $t \in \mathbb{R}$, we make use of the fact that $e^{itH_0}Re^{-itH_0} \in \mathcal{R}$, $t \in \mathbb{R}$, since the free dynamics maps the set of resolvents of linear combinations of $P, Q$ onto itself. Thus, if one can show that for $R \in \mathcal{R}$ all differences $(e^{itH}Re^{-itH_0}e^{-itH_0} - 1)$, $t \in \mathbb{R}$, are compact operators, i.e. are elements of $C \subset \mathcal{R}$, the assertion follows. In [7] and also in the preceding analysis, this task was accomplished by proving that the operators $(e^{itH}e^{-itH_0} - 1)$, $t \in \mathbb{R}$, are compact. This strategy can work if the essential spectra of $H_0$ and $H$ coincide, but it is bound to fail otherwise. In the following we will deal with either case.

6.1. Singular potentials. Before discussing some example of a singular potential, we establish a general result about perturbations of dynamics, thereby avoiding the usage of the Dyson series which entered in the preceding analysis.

Lemma 6.1. Let $V$ be a potential, defined on the domain $\mathcal{D}$, which satisfies

\[ \int_0^t ds \| V e^{-isH_0} \Phi \|^2 \leq \| C_t \Phi \|^2 \] for $\Phi \in \mathcal{D}$, where $C_t$ are compact operators, $t \geq 0$. Then $H \doteq (H_0 + V)$ is essentially selfadjoint on $\mathcal{D}$ and $(e^{itH} e^{-itH_0} - 1) \in C$, $t \in \mathbb{R}$. 

Proof. We omit the proof of essential selfadjointness of $H$, which is based on the
fact that $V$ is relatively compact with respect to $H_0$ under the given assumption.
The second statement follows from the simple estimate for $\Phi, \Psi \in D$,

$$
|\langle \Phi, (e^{itH} e^{-itH_0} - 1)\Phi \rangle|^2 = \int_0^t ds \, |\langle \Phi, e^{isH} V e^{-isH_0} \Phi \rangle|^2
\leq \int_0^t ds' \|e^{-is' \Phi} \|^2 \int_0^t ds \|V e^{-isH_0} \Phi\|^2
$$

which implies $\|(e^{itH} e^{-itH_0} - 1)\Phi\|^2 \leq t \|C_t \Phi\|^2, \Phi \in D$. Since $C_t$ is compact, this
shows that $(e^{itH} e^{-itH_0} - 1)$ maps weakly convergent null sequences of vectors to
strongly convergent null sequences; it thus is a compact operator, i.e. an element
of $\mathcal{C}$, $t \geq 0$. Moreover, $(e^{-itH} e^{itH_0} - 1) = -e^{itH} (e^{itH} e^{-itH_0} - 1) e^{itH_0}$, so this
holds true for all $t \in \mathbb{R}$. \qed

Equipped with this technical result we can exhibit now singular potentials
which lead to dynamics of the resolvent algebra $\mathcal{R}$. We do not consider here the
most general case, but illustrate the method by a significant example.

Lemma 6.2. Let $0 < \kappa < 1/2$, $g \in \mathbb{R}$, and let $x \mapsto V(x) \doteq g \, |x|^{-\kappa}$ on $\mathbb{R}\setminus\{0\}$,
$V(0) = 0$. The corresponding Hamiltonian $H = (H_0 + V)$ is essentially selfadjoint
on $D$, and one has $(e^{itH} e^{-itH_0} - 1) \in \mathcal{C}, t \in \mathbb{R}$. Hence $\mathcal{R}$ is stable under the
adjoint action of $e^{itH}$, $t \in \mathbb{R}$.

Remark: The admissible singularities of $V$ depend on the number of space
dimensions.

Proof. It is apparent that $V$ is defined on $D$. We will show that the positive
quadratic form $\int_0^t ds \, e^{isH_0} V^2 e^{-isH_0}$ on $D \times D$ extends to a compact operator.
The statement then follows from the preceding lemma. Let $\chi$ be a smooth function
which is equal to 1 for $|x| \leq 1$ and 0 for $|x| \geq 2$. We consider approximations of the
squared potential, $n \in \mathbb{N}$,

$$
x \mapsto V_n^2(x) = g^2 (|x|^{-2\kappa} (\chi(x/n) + (1 - 2\kappa)^{-1} (x/n) \chi'(x/n))
$$

where the prime $'$ indicates the derivative. The functions $x \mapsto V_n^2(x)$ have
compact support and are absolutely integrable, hence their Fourier transforms
$p \mapsto \hat{V}_n^2(p)$ are entire analytic. Moreover, since

$$
\hat{V}_n^2(x) = \frac{d}{dx} \left( (1 - 2\kappa)^{-1} g^2 |x|^{-2\kappa} \chi(x/n) \right), \quad x \neq 0,
$$

the Fourier transforms have a zero at $p = 0$. Since $\int dx \, |x|^{-2\kappa} e^{ipx} = c |p|^{2\kappa-1}$
for $p \neq 0$ and each $\hat{V}_n^2$ is obtained from $\hat{V}_2^2$ by convolution with a test function,
it is also clear that one obtains the bound $|\hat{V}_n^2(p)| \leq c_n |p|^{2\kappa-1}$ for large $|p|$. 

After these preparations we can analyze now the kernels of the quadratic forms \( \int_0^t ds e^{itH_0} V_n^2 e^{-isH_0} \) in momentum space, which are defined in the sense of distributions. Making use of Dirac’s bra-ket notation, they are given by

\[
\langle p | \int_0^t ds e^{itH_0} V_n^2 e^{-isH_0} | q \rangle = i \sqrt{2} \langle n | p - q \rangle (1 - e^{it(p^2 - q^2)})/(p^2 - q^2).
\]

These kernels are square integrable in \( p, q \), as one sees by substituting \( k = (p - q) \), \( l = (p + q) \),

\[
\int dp \int dq |V_n^2| (p - q) (1 - e^{it(p^2 - q^2)})/(p^2 - q^2)^2
\]

\[
= 8 \int dk \int dl |V_n^2(k)|^2 \sin^2(kl/2)/k^2l^2
\]

\[
= 8 \int dk \int dl |V_n^2(k)|^2/k^2l^2 = 8 \int dk |V_n^2(k)|^2/k \int dl \sin^4(l/2)/l^2.
\]

The latter integral with respect to \( l \) exists and the integral with respect to \( k \) exists as well since the functions \( k \mapsto |V_n^2(k)|^2/k \) are continuous due to the zero of \( V_n^2(k) \) at \( k = 0 \) and the asymptotic bound \(|V_n^2(k)|^2/k| \leq c^2_n |k|^{4n-3}\).

We conclude that the forms \( \int_0^t ds e^{iH_0} V_n^2 e^{-iH_0} \) extend to operators in the Hilbert Schmidt class and hence are compact. Finally, we make use of the fact that the operators \( (V^2 - V_n^2) \) are bounded by construction with norm satisfying

\[
\| (V^2 - V_n^2) \| \leq c'n^{-2\kappa},
\]

where \( c' \) does not depend on \( n \in \mathbb{N} \). It follows that

\[
\| \int_0^t ds e^{iH_0} (V^2 - V_n^2) e^{-iH_0} \| \leq tc'n^{-2\kappa}.
\]

Hence the form \( \int_0^t ds e^{iH_0} V^2 e^{-iH_0} \) can be approximated in norm by compact operators and thus extends to a compact operator as well. The statement then follows from the preceding lemma. \( \square \)

### 6.2. Unbounded potentials

We turn now to the discussion of some non-harmonic binding force involving an unbounded potential such that \( H \) has discrete spectrum. As already mentioned, the method of proof of stability of \( \mathcal{R} \) under the corresponding dynamics, used in the preceding subsection, is then bound to fail.

**Lemma 6.3.** Let \( 0 < \kappa < 1 \), \( g, x_0 > 0 \), and let \( x \mapsto V(x) = g(x^2 + x_0^2)^{\kappa/2} \). The corresponding Hamiltonian \( H = (H_0 + V) \) is essentially selfadjoint on \( \mathcal{D} \), has a compact resolvent, and the resolvent algebra \( \mathcal{R} \) is stable under the adjoint action of \( e^{itH} \), \( t \in \mathbb{R} \).

**Proof.** Again, we do not deal here with the domain and spectral properties of \( H \) since they are well known, cf. [15 Thm. XIII.67]. For the proof of the main part of the statement we note that it suffices to establish it for the generating resolvents \( \mathcal{R} = R(a, b, c) \subset \mathcal{R} \), where \( a, b \in \mathbb{R}, c \in \mathbb{R}\backslash \{0\} \). Putting \( \Gamma(t) = e^{itH} e^{-itH_0}, t \in \mathbb{R} \),
we have in the sense of bilinear forms on $D \times D$, making use of the fundamental theorem of calculus,

$$(\Gamma(t)R\Gamma(t)^{-1} - R) = i \int_0^t ds e^{isH} [V, R(-s)] e^{-isH},$$

where $R(-s) = e^{-isH_0}Re^{isH_0} = R(a - 2s, b, c)$. Since the inverse of $R$ is linear in the operators $P, Q$ and 1, we obtain $[V, R(-s)] = -i(a - 2s) R(-s)V'R(-s)$, where $x \mapsto V'(x) = g\kappa x (x^2 + x_0^2)^{(\kappa/2 - 1)}$. So we arrive at

$$(\Gamma(t)R\Gamma(t)^{-1} - R) = \int_0^t ds (a - 2s) e^{isH} R(-s)V'R(-s) e^{-isH}.$$

But $x \mapsto V'(x) \in C_0(\mathbb{R})$, so it follows from arguments similar to those given in the proof of Lemma 2.1 that the operator function

$$s \mapsto (a - 2s) e^{isH} R(-s)V'R(-s) e^{-isH}$$

is, for almost all $s \in \mathbb{R}$, a compact operator. Since it is also bounded on compact sets of $\mathbb{R}$, its integral (defined in the strong operator topology) is a compact operator, as well. This shows that for all resolvents $R$ and $t \in \mathbb{R}$

$$(e^{-itH}Re^{itH} - e^{-itH}Re^{itH}) = e^{-itH}(\Gamma(t)R\Gamma(t)^{-1} - R)e^{itH} \in \mathcal{C}.$$

Since the set of polynomials of the resolvents is norm dense in $\mathcal{R}$, it proves that the resolvent algebra is stable under the perturbed dynamics. □

6.3. Inadmissible dynamics. We conclude this outline with the remark that the non-interacting relativistic Hamiltonian $H_m = (P^2 + m^2)^{1/2}$, $m > 0$, which is related to the boundary case $\kappa = 1$ in the preceding lemma, does not lead to an automorphism group of $\mathcal{R}$. In order to prove this, consider the resolvent of the position operator $(ic + Q)^{-1}$, $c \in \mathbb{R}\setminus\{0\}$. By an elementary computation one obtains

$$e^{itH_m}(ic + Q)^{-1}e^{-itH_m} = (ic + Q + tP(P^2 + m^2)^{-1/2})^{-1}, \quad t \in \mathbb{R}. \quad (6.1)$$

To see that the operator on the right hand side is not an element of the resolvent algebra if $t \neq 0$, we consider the unitary representation of the dilations $D(\delta)$, $\delta \in \mathbb{R}_+$, on the underlying Hilbert space whose adjoint action on the basic resolvents is given by

$$D(\delta)(ic + aP + bQ)^{-1}D(\delta)^{-1} = (ic + \delta aP + \delta^{-1}bQ)^{-1}, \quad \delta \in \mathbb{R}_+.$$

By arguments already used in the proof of [7] Thm. 4.8, one obtains the following result.
Lemma 6.4. For given \(a, b \in \mathbb{R}, c \in \mathbb{R}\setminus\{0\}\), one has in the strong operator topology

\[
\lim_{\delta \to \infty} D(\delta)(ic + aP + bQ)^{-1}D(\delta)^{-1} = \begin{cases} (ic)^{-1} & \text{if } a = 0 \\ 0 & \text{if } a \neq 0. \end{cases}
\]

Hence \(\lim_{\delta \to \infty} D(\delta)RD(\delta)^{-1} \in \mathbb{C}1\) for any \(R \in \mathcal{R}\).

Proof. By a routine computation, one obtains on the domain \(\mathcal{D}\) the equality

\[
\left(\left(ic + \delta aP + \delta^{-1}bQ\right)^{-1} - (ic + \delta aP)^{-1}\right) = \left(ic + \delta aP + \delta^{-1}bQ\right)^{-1}(-\delta^{-1}bQ)(ic + \delta aP)^{-1}
\]

\[
= (ic + \delta aP + \delta^{-1}bQ)^{-1}(ic + \delta aP)^{-1}(iab(ic + \delta aP)^{-1} - \delta^{-1}bQ).
\]

Since the resolvents are uniformly bounded for fixed \(c \neq 0\) and \((ic + \delta aP)^{-1} \to 0\) in the strong operator topology if \(\delta \to \infty\) and \(a \neq 0\), the first part of the statement follows. The second part is then a consequence of the fact that the polynomials of all resolvents form a norm dense set in \(\mathcal{R}\).

The fact that the operator on the right hand side of equation (6.1) does not belong to \(\mathcal{R}\) if \(t \neq 0\) is now a consequence of the following lemma.

Lemma 6.5. Let \(c \neq 0\). Then

\[
\lim_{\delta \to \infty} D(\delta)(ic + Q + tP(P^2 + m^2)^{-1/2})^{-1}D(\delta)^{-1} = (ic + tP/|P|)^{-1}
\]

in the strong operator topology.

Proof. The adjoint action of the dilations on the operator is given by, \(\delta \in \mathbb{R}_+\),

\[
D(\delta)(ic + Q + tP(P^2 + m^2)^{-1/2})^{-1}D(\delta)^{-1} = (ic + \delta^{-1}Q + tP(P^2 + \delta^{-2}m^2)^{-1/2})^{-1}.
\]

Now the operator \((ic + tP/|P|)^{-1}\) maps the dense set \(\mathcal{D}_0 \subset \mathcal{D}\) of test functions, vanishing in momentum space in a neighborhood of the origin, onto itself. The statement then follows from the above equality by a similar computation as in the preceding argument.

According to the preceding Lemma 6.5 the scaling limits of the operators (6.1) are not multiples of the identity, so by Lemma 6.4 they are not elements of the resolvent algebra. So we conclude that the resolvent algebra \(\mathcal{R}\) is not stable under the adjoint action of the unitaries \(U_m(t)\) for \(t \in \mathbb{R}\setminus\{0\}\). But, whereas the relativistic dynamics of particles does not leave the quantum mechanical resolvent algebra invariant, it is noteworthy that there do not appear such problems for the resolvent algebra of the corresponding relativistic field theory.
7. Conclusions

In the present article we have continued our study of the resolvent algebra, which is a C*-algebraic framework for the description of the kinematics of quantum systems, putting emphasis here on its applications.

It was already pointed out in [7] that any kinematical algebra which is capable of describing a variety of dynamics of physical interest, must have ideals. Indeed, the familiar algebra $B(H)$ of bounded operators on some Hilbert space does comply with this condition, it contains the ideal of compact operators. But, in contrast to the resolvent algebra, $B(H)$ does not contain specific information about the underlying quantum system. The ideal structure of the resolvent algebra is more complex than that of $B(H)$. Compact operators appear in disguise also in subalgebras of the resolvent algebra, i.e. they are homomorphic to compact operators, but have in general infinite multiplicity. In fact, one can extract from the nesting of these subalgebras the number of degrees of freedom of the system [5].

In the present study of infinite lattice systems we have exhibited several reasons why this ideal structure is essential. First, it greatly simplifies proofs that a given dynamics acts by automorphisms on the resolvent algebra, cf. in this context the Lemmas 6.2 and 6.3. Second, we made use of the fact that non-simple algebras, i.e. algebras having ideals, in general admit outer bounded generators describing perturbations of the dynamics, cf. Lemma 2.2. Third, the ideal structure was vital in the description of finitely localized systems on the global algebra; for there exist states which are annihilated by parts of its ideals, describing a situation, where the quantum system is confined to a box, cf. Lemma 4.3.

Making use of these features of the resolvent algebra, we were able to establish the existence of global dynamics, of global equilibrium states at high temperatures and of global ground states. All of these states are regular. Proofs to that effect were not available, to the best of our knowledge.

Our arguments were based on well known results in the context of C*-dynamical systems, the vital ingredient being the fact that we were able to show that they are applicable here. The resolvent algebra of the lattice theory does not belong to this class; yet it contains a C*-algebra formed by (in regular representations) weak-*-dense subalgebras, on which the dynamics acts point-wise norm-continuously. Exploiting this structure, we could confine our analysis to states on this subalgebra and extend the locally normal states back to regular states on the full algebra at the very end of our analysis. Thus the ideal structure entered again at this point.
The present results suggest to explore the applicability of the resolvent algebra to other problems of physical interest, such as Pauli-Fierz type models of small systems coupled to a thermal reservoir, cf. for example [17]; there we expect no major problems. More ambitiously, one can also study within this framework the asymptotic time behavior of states, where one couples two thermal reservoirs, filling half spaces, through a finite layer and ask, whether they approach a steady state in the course of time, cf. for example [10]. It seems also possible to treat along these lines models of quantum crystals, where the interaction between neighboring atoms is described by harmonic forces, cf. for example [11]; note that these forces induce automorphisms of the resolvent algebra.

Considerably more involved are infinite systems, where all particles can interact with each other. There the statistics of particles will enter, i.e. particles can no longer be treated as distinguishable, such as in the present context, where each particle is confined by binding forces to a neighborhood of its respective lattice point. As already indicated in connection with the no-go theorem concerning the relativistic dynamics, presented in Subsection 6.3, one should then no longer deal with the particle picture, but rely on the field theoretic version of the resolvent algebra [7]. We hope to return to these problems elsewhere.

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