WEAK CHAOS IN LARGE CONSERVATIVE SYSTEM – INFINITE-RANGE COUPLED STANDARD MAPS

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We study, through a new perspective, a globally coupled map system that essentially interpolates between simple discrete-time nonlinear dynamics and certain long-range many-body Hamiltonian models. In particular, we exhibit relevant similarities, namely (i) the existence of long-standing quasistationary states (QSS), and (ii) the emergence of weak chaos in the thermodynamic limit, between the present model and the Hamiltonian Mean Field model, a strong candidate for a nonextensive statistical mechanical approach.

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1. Introduction

In the last years, considerable effort has been made in order to clarify the role that nonextensive statistical mechanics plays in physics. In this context, there has been significantly growing evidence relating several, physically motivated, nonlinear dynamical systems. It has been repeatedly put forward that the statistical behaviour of a physical system descends from its microscopic dynamics. Consequently, the study of paradigmatic nonlinear dynamical systems is important in
order to describe and understand anomalies and deviations from the well known Boltzmann-Gibbs (BG) statistical mechanics. The scenario within which we are working tries to capture the most relevant features of nonextensive statistical mechanics in the complete range of dynamical systems: from extremely simple dissipative low-dimensional maps to complex conservative Hamiltonian dynamics. In the present paper we will make specific progress along these lines by focusing on a model which illustrates the deep similarities that can exist between nonlinear coupled maps and many-body Hamiltonian dynamics.

Let us first recall a paradigmatic and intensively studied many-body infinite-range coupled conservative system, namely the Hamiltonian Mean Field (HMF) model:

\[ H = K + V = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i,j=1}^{N} \left[ 1 - \cos(\theta_i - \theta_j) \right]. \]

The HMF model may be thought of as \( N \) globally coupled classical spins (inertial version of the XY ferromagnetic model). Its molecular dynamics exhibit a remarkably rich behaviour. When the initial conditions are out of equilibrium (for example, the so called waterbag initial conditions), it can present an anomalous temperature evolution (we consider \( T \equiv 2K/N \), being \( K \) the total kinetic energy). These states are characterized by a first stage (quasistationary state, QSS) whose temperature is different from that predicted by the BG theory, followed by a crossover to the expected final temperature. These QSS appear to be a consequence of the long-range coupling. They are important because their duration diverges with \( N \), thus becoming the only relevant state for a macroscopic system.

At the other end of the range of dynamical systems we may consider a dissipative, one-dimensional model, such as the logistic map (and its universality class):

\[ x_{t+1} = 1 - \mu x_t^2 \quad (t = 0, 1, 2; \quad x \in [-1, 1]; \quad \mu \in [0, 2]). \]

Because of its physical importance, the logistic map is one of the most studied low-dimensional maps. Despite its apparently simple form, it exhibits a quite complex behaviour. Important progress has recently been made which places this model as an important example of the applicability of nonextensive statistical mechanical concepts. Indeed, Baldovin and Robledo rigorously proved that, at the edge of chaos (as well as at the doubling-period and tangent bifurcations), the sensitivity to initial conditions is given by a \( q \)-exponential function. Furthermore, they proved the \( q \)-generalization of a Pesin-like identity concerning the entropy production per unit time. For the stationary state at the edge of chaos, the entropic index \( q \) can be obtained analytically. Moreover, when a small external noise is added, a two-step relaxation evolution is found, similarly to what occurs for the HMF case.

At this point, a natural question may arise. Is it possible to relate the results found for such simple maps to the anomalies found in the HMF model? Further-
more, can we treat various nonlinear dynamical systems within the nonextensive statistical mechanics theory? Many studies are presently addressing such questions.

A first step that can be done in this direction is to move closer to a Hamiltonian dynamics by considering a symplectic, conservative map. This is the case of the widely investigated Taylor-Chirikov standard map \[ \theta(t+1) = p(t+1) + \theta(t) \mod 1 \]
\[ p(t+1) = p(t) + \frac{a}{2\pi} \sin[2\pi \theta(t)] \mod 1, \]
(3)

This map may be obtained, for instance, by approximating the differential equation of a simple pendulum by a centered difference equation, and converting a second-order equation into two first-order equations.

The standard map was studied along the present lines by Baldovin et al.\[11\]. For symplectic maps, what plays a role analogous to the temperature is the variance of the angular momentum:
\[ T \equiv \sigma^2_p = \langle p^2 \rangle - \langle p \rangle^2, \]
where \( \langle \rangle \) denotes the ensemble average. Beginning with the same type of initial conditions as before (waterbag), we observe once again a two-plateaux relaxation process, suggesting a connection with the phenomena already described for the HMF model.

A step forward to capture the behaviour of the HMF system of rotors is to consider \( N \) standard maps, coupled in such way as to maintain their symplectic (hence conservative) structure. However, there are several ways to achieve this. A particular coupling in the momenta has been recently addressed\[9\] with quite interesting results such as QSS relaxation and nonergodic occupation of phase space. A different type of coupling is addressed in the next Section.

2. Symplectic coupling in the coordinates

As before, we consider \( N \) standard maps but, this time, with a global, symplectic coupling in the coordinates:
\[ \theta_i(t+1) = \theta_i(t) + p_i(t+1) \mod 1, \]
\[ p_i(t+1) = p_i(t) + \frac{a}{2\pi} \sin[2\pi \theta_i(t)] + \frac{b}{2\pi(N-1)} \sum_{j \neq i}^N \sin[2\pi(\theta_i(t) - \theta_j(t))] \mod 1. \]
(4)

This coupling arises as a natural choice. In fact, applying to the HMF model the difference procedure mentioned above for the standard map, we obtain precisely the \( a = 0 \) particular instance of model (4). This model has already been addressed in the literature\[12\] but in a quite different context, related to the study of the Lyapunov exponents in the completely chaotic regime.

We present next numerical simulations of the map system (4). We calculated the evolution of the variance of the momenta \( \sigma^2_p \). Our results show that, for waterbag initial conditions and appropriate ranges for the parameters \( a \) and \( b \), two-step relaxation processes are again found. In Fig.\[1\] we show these results for different sizes of the system. It can be seen that the crossover time \( t_c \) grows as \( t_c \sim N^{1.07 \pm 0.10} \), thus never reaching BG equilibrium when \( N \to \infty \). In other words, the \( N \to \infty \) and \( t \to \infty \) limits do not commute.
Figure 1. Temperature evolution illustrating the presence of two-step relaxation (QSS) for typical system sizes. We have used \( a = 0.05, b = 2 \), and waterbag initial conditions. Ensemble averages were done, typically over 100 realizations. Only much longer simulations could confirm, or exclude, the possibility that all curves, i.e. \( \forall N \), saturate at the equal-probability value \( 1/12 \approx 0.08 \). Inset: The crossover time \( t_c \) corresponds to the inflexion point of \( T \) versus \( \log t \).

Finally, we calculated the largest Lyapunov exponent (LLE) \( \lambda_L \) (we recall that Lyapunov exponents measure the instability of dynamical trajectories, and provide a quantitative idea of the sensitivity to the initial conditions of the system). Indeed, for the \( (a, b) \)-parameters in the range illustrated in Figs. 1 and 2, we found that the dependence of the LLE with the system size is consistent with \( \lambda_L \sim N^{-0.40 \pm 0.08} \), i.e., a clear indication of weak chaos in the thermodynamic limit.

Summarizing, we presented a conservative model consisting in \( N \) standard maps symplectically coupled through the coordinates. We have found results suggestively similar to those obtained for other nonlinear dynamical systems including the HMF model. More specifically, we found the double plateaux in the time evolution of the temperature, and a LLE which approaches zero for increasing size. We are currently studying several other quantities (e.g., correlation functions and momenta probability distribution functions), as well as the influence of \( (a, b) \) on the present ones. These results place naturally the present system within a series of nonlinear dynamical systems which starts with one-dimensional dissipative maps, follows with low-dimensional conservative maps, then many symplectically coupled maps, and ends with long-range many-body Hamiltonians. They all share important phe-
nomina, typically related, in one way or another, to weak chaos and long-standing nonergodic occupation of phase space. These features precisely constitute the scenario within which nonextensive statistical mechanics appears to be the adequate thermostatistical theory, in analogy to Boltzmann-Gibbs statistical mechanics, successfully used since more than one century for strongly chaotic and ergodic systems.

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6. By *waterbag initial conditions* we mean spins and momenta randomly taken from uniform distributions. In our simulations $q_i \in [0, 1]$ and $p_i \in [0.29, 0.31]$. See, for instance, A. Pluchino, V. Latora and A. Rapisarda, *Physica* **A338**, 60 (2004).
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