Some New Results on Equivalency of Collusion-Secure Properties for Reed-Solomon Codes

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Abstract

A. Silverberg (IEEE Trans. Inform. Theory 49, 2003) proposed a question on the equivalence of identifiable parent property and traceability property for Reed-Solomon code family. Earlier studies on Silverberg’s problem motivate us to think of the stronger version of the question on equivalence of separation and traceability properties. Both, however, still remain open. In this article, we integrate all the previous works on this problem with an algebraic way, and present some new results. It is notable that the concept of subspace subcode of Reed-Solomon code, which was introduced in error-correcting code theory, provides an interesting prospect for our topic.

Keywords: Separation, Traceability, Reed-Solomon Code, Silverberg’s Problem, Subspace Subcode

1 Introduction

The growth of Internet raised the problem of illegal redistribution as a major concern in digital content industry, because copying such material is easy and no information is lost in the process. To protect digital copies, however, is a complicated task. Methods like cryptography do not resolve this problem, since the information must be decrypted at one point to be able to use it. The goal of digital fingerprinting is to discourage people from illegally redistributing their legally purchased copy. In this scenario, the distributor embeds into the digital content, using a watermark algorithm, a unique piece of information (fingerprint) for each user. If an illegal copy is found, the distributor can extract the fingerprint from it to identity the dishonest user (pirate). Because the pirate may try to damage the fingerprint before redistribution, the watermarking algorithm must ensure robustness to the distributor.

Nevertheless, the most dangerous attack against digital fingerprinting is the collusion attack introduced in [1]. The contents delivered to different users are,
since their fingerprints differ, essentially different. Two or more pirates may compare their copies and reveal the locations of part of fingerprint. With deleting or modifying those locations, pirates can generate a new copy of content in order not to be traced. This collusion attack could not only violate pirate-identifying but frame an innocent user in some cases. We are interested in designing a set of fingerprints (fingerprinting code) with which the distributor can always identity at least one colluder from a forged fingerprint with a small error probability. In particular separating code, IPP code and TA code are most important fingerprinting codes with different collusion-secure properties for generic digital data.

We will denote the $i$th component of any tuple $x$ by $x_i$ and the Hamming distance between two tuples $x, y$ by $d(x, y)$. Let $n, w, w_1$ and $w_2$ be positive integers such that $n, w, w_1 \geq 2$ and $w_1 \geq w_2$. Suppose $C$ is a code of length $n$ over $\mathbb{F}_q$.

- We define descendant set of an arbitrary nonempty subset $U$ of $C$ by
  $$\text{desc}U := \{ x \in \mathbb{F}_q^n \mid \forall i, \exists y \in U : x_i = y_i \}$$

- $C$ is a $(w_1, w_2)$-separating code provided that, if $U_1, U_2$ are disjoint subsets of $C$ such that $1 \leq |U_1| \leq w_1$ and $1 \leq |U_2| \leq w_2$, then their descendant sets are also disjoint.

- $C$ is called a $w$-identifiable parent property code (IPP code) provided that for all $x \in \mathbb{F}_q^n$, the set $\text{IPP}_w(x) := \{ U \subset C \mid x \in \text{desc}U, 1 \leq |U| \leq w \}$ is empty or $\bigcap_{U \subset \text{IPP}_w(x)} U \neq \phi$.

- $C$ is called a $w$-traceability code (TA code) provided that if $U \subset C, 1 \leq |U| \leq w$ and $x \in \text{desc}U$, there exist at least one codeword $y \in U$ such that $d(x, y) < d(x, z)$ for all $z \in C \setminus U$.

The code classes defined above are known to satisfy the following relationships.

**Proposition 1.1** (see [9]) Let $d$ be the minimum distance of a code $C$ of length $n$. Then for $C$,
$$d > n(1 - 1/w^2) \Rightarrow w\cdot TA \Rightarrow w\cdot IPP \Rightarrow (w, w)$-separating

**Proposition 1.2** (see [2]) Let $d$ be the minimum distance of a code $C$ of length $n$. If $d > n(1 - 1/(w_1w_2))$, then $C$ is a $(w_1, w_2)$-separating code.

Let $1 \leq k \leq q - 1$ be an integer. The Reed-Solomon code $\text{RS}_k(q)$ of dimension $k$ over $\mathbb{F}_q$ is defined by $\text{RS}_k(q) := \{ \text{ev}(f) \mid f \in \mathbb{F}_q[x], \deg f < k \}$, where $\text{ev} : f \in \mathbb{F}_q[x] \mapsto (f(\alpha^0), f(\alpha^1), \cdots, f(\alpha^{q-2})) \in \mathbb{F}_q^{q-1}$ and $\alpha$ is a primitive element in $\mathbb{F}_q$. It is well known that $\text{RS}_k(q)$ is a $[q - 1, k, q - k]$-linear code.
Reed-Solomon code is one of the most famous error-correcting codes and it also has an application in digital fingerprinting. A. Silverberg, et al. [8] dealt with applying list decoding method to tracing algorithms of fingerprinting codes. In their work, the collusion-secure properties of Reed-Solomon codes and other algebraic geometry codes were studied, and the following question was left as an open problem.

**Question 1** Is it the case that \( d > n - n/w^2 \) for all \( w \)-IPP Reed-Solomon codes of length \( n \) and minimum distance \( d \)?

Thus, Silverberg’s question is a problem of the equivalence of IPP and traceability for Reed-Solomon code family.

The problem was studied in [4] and [7]. In [4], they restated the separation property of Reed-Solomon codes algebraically, as a system of equations, to get the following result.

**Theorem 1.1** (see [4]) Suppose \( k - 1 \) divides \( q - 1 \). If \( \text{RS}_k(q) \) is a \((w_1, w_2)\)-separating code, then \( d > n - n/(w_1w_2) \) where \( d \) is minimum distance.

In [7], they presented the similar result as follows by establishing an additive homomorphism over finite field.

**Theorem 1.2** (see [7]) Suppose \( w^2 > q \) or \( w \) divides \( q \). If \( \text{RS}_k(q) \) is a \((w, w)\)-separating code, then \( d > n - n/w^2 \) where \( d \) is minimum distance.

As you can see, the previous works claimed the stronger fact than Silverberg’s original problem in certain cases. In this context, we naturally raise the following question, which turns out to be the main topic of this article.

**Question 2** Is it the case that \( d > n - n/(w_1w_2) \) for all \((w_1, w_2)\)-separating Reed-Solomon codes of length \( n \) and minimum distance \( d \)?

The rest of the paper is organized as follows: In Section 2, we will present a sufficient condition for non-separation of linear codes, and prove that the previous works can be derived from that condition. Some more parameter setups providing positive answer about Question 2 will be obtained in Section 3. In Section 4, the application of subspace subcodes of Reed-Solomon codes will be unveiled. We conclude the paper in Section 5 after presenting experimental results to show the extension of our work.

Throughout the remaining, \( \mathbb{F}_q \) is Galois field with order \( q = p^m \) and characteristic \( p \). Let \( r_i = \lceil \log_p w_i \rceil, i = 1, 2 \). For any polynomial \( f \) over \( \mathbb{F}_q \), let \( \text{Im}f = f(\mathbb{F}_q) \). For an arbitrary word \( x \in \mathbb{F}_q^n \), \( \text{Im}x \) is the set of all its components, i.e. \( \text{Im}x = \{x_i \mid 1 \leq i \leq n\} \). For given two sets \( E, F \subset \mathbb{F}_q \), we define \( EF := \{ab \mid a \in E, b \in F\} \) and \( E + F := \{a + b \mid a \in E, b \in F\} \). We will denote the set of all polynomials over \( \mathbb{F}_q \) of degree less than \( k \) by \( P_k \). \( n, w, w_1 \) and \( w_2 \) are positive integers satisfying \( n, w, w_1 \geq 2 \) and \( w_1 \geq w_2 \).
2 Restatement of the Previous Works

In this section we propose a sufficient condition for non-separation of linear codes, which will integrate the former results in [4] and [7]. The idea was motivated by [7], where an additive homomorphism was established such that its image set has a special property. Before presenting the major result, we will formally define such ”special property” of a set.

Let $U$ be a subset of $\mathbb{F}_q$. $U$ is called additively (multiplicatively) $(w_1, w_2)$-separable and written by $U = (E, F)_{w_1,w_2}$ provided that there exist two subsets $E, F \subset U$ with $1 \leq |E| \leq w_1$ and $1 \leq |F| \leq w_2$ such that $U \subset E+F (U \subset EF)$.

The following theorem is the main result of this section. Note that it is not just for Reed-Solomon codes, but for linear codes.

**Theorem 2.1** Let $C$ be $[n, k]_q$-linear code containing $1= (1, 1, \cdots, 1)$. Suppose there exists a codeword $c = (c_1, c_2, \cdots, c_n) \in C$ such that $|\text{Im}c| \geq 2$ and $\text{Im}c$ is $(w_1, w_2)$-separable additively or multiplicatively. Then, $C$ is not $(w_1, w_2)$-separating.

**Proof.** We will only prove when $\text{Im}c$ is additively $(w_1, w_2)$-separable, since the other case can be proven in similar way. Let $\text{Im}c = (E, F)_{w_1,w_2}$. Define $U := \{\beta \cdot 1 \mid \beta \in E \}$ and $V := \{c - \gamma \cdot 1 \mid \gamma \in F \}$. Then $U, V \subset C$ since $c, 1 \in C$ and $C$ is a linear code. Further, $U$ and $V$ are disjoint because $|\text{Im}c| \geq 2$. For all $i \in [1, n]$, there exist $\beta_i \in E$ and $\gamma_i \in F$ such that $c_i = \beta_i + \gamma_i$. If we set $x := (\beta_1, \beta_2, \cdots, \beta_n)$, it is clear that $x \in \text{desc}U \cap \text{desc}V$ which implies non-separation. \hfill $\square$

The following corollary is the Reed-Solomon code version of Theorem 2.1.

**Corollary 2.1** Let $1 \leq k \leq q-1$ be an integer. If there exists a non-constant polynomial $f$ in $P_k$ such that $\text{Im}f$ is $(w_1, w_2)$-separable additively or multiplicatively, then the code $\text{RS}_k(q)$ is not $(w_1, w_2)$-separating.

By definition, $\text{RS}_k(q) \subset \text{RS}_{k+1}(q)$, therefore the code $\text{RS}_{k+1}(q)$ is not $(w_1, w_2)$-separating if $\text{RS}_k(q)$ is not $(w_1, w_2)$-separating. Meanwhile, the inequality $d > n-n/(w_1w_2)$ is equivalent with $k-1 < (q-1)/(w_1w_2)$ for Reed-Solomon codes. Thus, it sufcies to consider the case $k = \lceil(q-1)/(w_1w_2)\rceil + 1$ when we study Question 2. In other words, if $\text{RS}_k(q)$ is not $(w_1, w_2)$-separating where $k = \lceil(q-1)/(w_1w_2)\rceil + 1$ for given $q, w_1, w_2$, Question 2 has the positive answer. (see [7])

In this context, we will reprove the previous results done on Silverberg’s open problem more simply using Corollary 2.1.
Proof of Theorem 1.1: Suppose \( d \leq n(1 - 1/(w_1w_2)) \), i.e. \( k - 1 \geq (q - 1)/(w_1w_2) \). Set \( f(x) := x^{k-1} \). Since \( k - 1 \mid q - 1 \), the polynomial \( f \) is a multiplicative homomorphism mapping \( \mathbb{F}_q^* \) to \( \mathbb{F}_q^* \). So \( \text{Im} f \) is a multiplicative subgroup of \( \mathbb{F}_q^* \) with order \( |\text{Im} f| = |\mathbb{F}_q^*/\text{Ker} f| = (q - 1)/(k - 1) \leq w_1w_2 \). For \( \mathbb{F}_q^* \) is cyclic, \( \text{Im} f \) is also cyclic, thus, it has a generator \( \gamma \). Set \( E := \{ \gamma^i w_2 \mid 0 \leq i \leq w_1 - 1 \} \) and \( F := \{ \gamma^j \mid 0 \leq j \leq w_2 - 1 \} \). Then it is easy to check that \( \text{Im} f = (E, F)w_1w_2 \), which implies non-separation by Corollary 2.1. Therefore, if \( \text{RS}_k(q) \) is \((w_1, w_2)\)-separating, then \( k - 1 < (q - 1)/(w_1w_2) \). \( \square \)

Proof of Theorem 1.2: As we mentioned above, it suffices to consider the case \( k = \lceil (q - 1)/w^2 \rceil + 1 \) only. Assume \( w^2 > q \). Then \( k = 2 \), thus \( k - 1\mid q - 1 \), which makes the condition of Theorem 1.1. Now let’s assume that \( w|q \). The polynomial \( f(x) := x^{q/w^2} - x \) is an additive homomorphism over \( \mathbb{F}_q \) and \( |\text{Im} f| = w^2 \). By finite group theory, there exist subgroups \( E, F < \text{Im} f \) with \( w \) elements, respectively, such that \( \text{Im} f = E + F \). Further, \( f \in P_k \). Thus, from Corollary 2.1 the code \( \text{RS}_k(q) \) is not \((w_1, w_2)\)-separating. \( \square \)

From the preceding proofs, we claim that the results in [4] and [7] can be integrated into a simpler scheme. We conclude this section with the following proposition that resembles Theorem 1.2 without proof.

**Proposition 2.1** Suppose \( w_1w_2 \) divides \( q \). If \( \text{RS}_k(q) \) is a \((w_1, w_2)\)-separating code, then \( d > n - n/(w_1w_2) \) where \( d \) is minimum distance.

## 3 New Parameter Setups

\( w_1, w_2 \) and \( q \) are the parameters specifying Reed-Solomon code and its separation property. The aim of this section is to propose new configurations of them that provide Question 2 with positive answer. The underlying principle is again Theorem 2.1 or Corollary 2.1.

In this section, we suppose that \( k - 1 \mid q \) where \( k = \lceil (q - 1)/(w_1w_2) \rceil + 1 \). Since \( q \) is a prime power, there exists an integer \( s \) such that \( q/(k - 1) = p^s \). One can easily check that \( p^s \) is the largest power of \( p \) which is equal or less than \( w_1w_2 \). Therefore, \( s = r_1 + r_2 \) or \( s = r_1 + r_2 + 1 \).

The main idea is to set \( f(x) := x^{k-1} - x \), prove that \( \text{Im} f \) is additively or multiplicatively \((w_1, w_2)\)-separable, and refer to Corollary 2.1. It is obvious that \( f \in P_k \) is an additive homomorphism over \( \mathbb{F}_q \) and therefore \( \text{Im} f \) is an additive group with \( p^s \) elements. Thus, the problem is to find the setups such that \( \text{Im} f \) is \((w_1, w_2)\)-separable. The first setup is \( s = r_1 + r_2 \).

**Proposition 3.1** If \( s = r_1 + r_2 \), then \( \text{Im} f \) is additively \((w_1, w_2)\)-separable.
The second setup is $[w_1/p^n] \cdot [w_2/p^r] \geq p$.

**Proposition 3.2** If $[w_1/p^n] \cdot [w_2/p^r] \geq p$, $\text{Im} f$ is additively $(w_1, w_2)$-separable.

**Proof.** If $s = r_1 + r_2$, $\text{Im} f$ is additively $(w_1, w_2)$-separating by Proposition 3.1. Assume $s = r_1 + r_2 + 1$. Then there exist three additive subgroups $E, F$ and $P$ of $\text{Im} f$ with $\text{Im} f = E + F + P$ such that $|E| = p^{r_1}, |F| = p^{r_2}$ and $|P| = p$. Moreover, $P$ is cyclic since $p$ is a prime number. Let $\alpha$ be its generator. If we set $P_1 := \{(i \cdot [w_2/p^r]) \gamma \mid 0 \leq i \leq [w_1/p^{r_1}] - 1\}$, $P_2 := \{j \gamma \mid 0 \leq j \leq [w_2/p^r] - 1\}$ and $E' = E + P_1, F' = F + P_2$, then we get $P = P_1 + P_2$ and $\text{Im} f = E' + F'$ since $[w_1/p^{r_1}] \cdot [w_2/p^r] \geq p$. Therefore, $\text{Im} f$ is additively $(w_1, w_2)$-separable. □

The results of this section can be integrated into the following theorem.

**Theorem 3.1** Suppose $k - 1$ divides $q$ with $q/(k - 1) = p^s$, and $s \leq r_1 + r_2$ or $[w_1/p^n] \cdot [w_2/p^r] \geq p$. If $\text{RS}_k(q)$ is a $(w_1, w_2)$-separating code, then $d > n - n/(w_1w_2)$ where $d$ is minimum distance.

Now we state the following lemma which will be useful for the next section.

**Lemma 3.1** The finite field $\mathbb{F}_{p^s}$ is $(w_1, w_2)$-separable if at least one of the followings hold:

- $s \leq r_1 + r_2$
- $[w_1/p^{r_1}] \cdot [w_2/p^r] \geq p$
- $w_1w_2 - w_2 \geq p^s$

**Proof.** We can prove in the first and second cases similarly with the propositions above since $\text{Im} f$ is additively isomorphism with $\mathbb{F}_{p^s}$. So we will only consider the third case. It is well known that $\mathbb{F}_{p^s}^* = \mathbb{F}_{p^s} \setminus \{0\}$ is a multiplicative cyclic group. Denote by $\alpha$ its generator. Set $E := \{\alpha^{(w_2 - 1)} \mid 0 \leq i \leq w_1 - 1\}$ and $F := \{\alpha^i \mid 0 \leq j \leq w_2 - 2\}$. Then $EF = \{\alpha^i \mid 0 \leq i \leq w_1 w_2 - w_2\}$ and $\mathbb{F}_{p^s}^* = EF$ since $w_1 w_2 - w_2 \geq p^s$. Thus, if we set $F' = F \cup \{0\}$, then $\mathbb{F}_{p^s}^* = EF'$ which implies $\mathbb{F}_{p^s} = (E, F')_{w_1, w_2}$. □
4 Application of Subspace Subcodes

In a linear code, there are some codewords all of whose components belong to a certain subset of $\mathbb{F}_q$. Collecting such codewords is a method of constructing a new code from an existing code, and it was studied in [3], [6] and [5]. Subfield subcode in [3] is a set of codewords whose components all lie in a subfield. Subgroup subcodes, or subspace subcodes were introduced in [6] and [5], where their dimensions were estimated. Let $S$ be a $v$-dimensional subspace of $\mathbb{F}_q$ where $0 \leq v \leq m$. Subspace subcode of Reed-Solomon code $C = \text{RS}_k(q)$ with $S$ is defined to be the set of codewords from $C$ whose components all lie in $S$, and is denoted by $\text{SSRS}_S(C)$. In this section, we will study application of subspace subcodes of Reed-Solomon codes to Question 2 in case $p = 2$ and $q = 2^m$. It is related to the dimensions of $\text{SSRS}_S(C)$.

$\text{SSRS}_S(C)$ is an $\mathbb{F}_2$-linear space. In [5], the explicit formula to calculate the binary dimension of $\text{SSRS}_S(C)$ denoted by $K(C, S)$ was proposed as follows :

$$K(C, S) = \sum_{j \in I_n} d_j (a_j - r_j)$$

where $I_n$ is the set consisting of the smallest integers in each modulo $n = 2^m - 1$ cyclotomic coset, $d_j$ is the cardinality of the coset containing $j$ denoted by $\Omega_j$, $e_j$ is the number of elements from $\Omega_j$ lying in the set $J = \{1, 2, \cdots, k\}$, $a_j = me_j/d_j$ and $r_j$’s are the ranks of certain $(m - v) \times a_j$ matrices called cyclotomic matrices.

As well as the explicit formula, they presented the following lower bound for the binary dimension :

$$K(C, S) \geq L(k, v) = \sum_{j \in I_n} \max\{d_j(a_j - (m - v)), 0\}$$

We will call subspace subcode $\text{SSRS}_S(C)$ trivial, provided that $K(C, S) \leq v$. Then the following lemma is immediately obtained.

**Lemma 4.1** Suppose that there exists a $v$-dimensional subspace of $\mathbb{F}_q$ denoted by $S$ such that the subspace subcode of $C = \text{RS}_k(q)$ with $S$ is non-trivial. Then, $C$ is not $(w_1, w_2)$-separating, provided that $S$ is $(w_1, w_2)$-separable.

**Proof.** There exists a codword $c \in \text{SSRS}_S(C)$ with $|\text{Im}c| \geq 2$ because of non-triviality. Therefore, by Theorem 2.1, $C$ is not $(w_1, w_2)$-separating. $\square$

By using the lower bound $L(k, v)$, we can get the more practical result about Question 2. $L(k, v)$ depends on the dimension of the parent code $k$ and the dimension of the subspace $s$, not the subspace $S$ itself. So we can restrict to $S = \mathbb{F}_{2^v}$. Suppose $\text{SSRS}_S(C)$ is trivial, then $\text{SSRS}_T(C)$ is also trivial where $T$ is a subspace of $S$. Therefore it suffices to consider the largest power $2^v$ equal or less than $w_1w_2$. 
Theorem 4.1 Let $2^v$ be the largest power equal or less than $w_1w_2$, which satisfies at least one of the following conditions hold, and suppose $L(k, v) > v$.

- $v \leq r_1 + r_2$
- $[w_1/2^{r_1}] \cdot [w_2/2^{r_2}] \geq 2$
- $w_1w_2 - w_2 \geq 2^v$

If $\text{RS}_k(q)$ is a $(w_1, w_2)$-separating code, then $d > n - n/(w_1w_2)$ where $d$ is minimum distance.

Proof. SSRS$_S(C)$ is non-trivial since $K(C, S) \geq L(k, v) > v$ where $S = \mathbb{F}_{2^v}$. Plus, $\mathbb{F}_{2^v}$ is $(w_1, w_2)$-separable by Lemma 3.1. Therefore, applying Lemma 4.1 implies the conclusion. $\square$

5 Examples and Conclusions

In this article, we presented an algebraic statement for the generalized version of Silverberg’s open problem, and exploited it to integrate the former results. Besides the previous results, we could procure some new parameter setups ensuring the equivalence of separation and traceability properties for Reed-Solomon codes. Finally using the concept of subspace subcode introduced in error-correcting code theory, we proposed a new result when the characteristic of finite field is 2.

Table-1 illustrates the contributions of our work to Silverberg’s open problem for some parameters. For each $w$ and $q$, we set $k = \lceil (q - 1)/w^2 \rceil + 1$ and check $(w, w)$-separation property of $\text{RS}_k(q)$ by the existing results. In each cell, the source of the work is written if $\text{RS}_k(q)$ is not $(w, w)$-separating. For example, "[4]" means that non-separation is proven by Theorem 1.1, and "3.1" represents that it is followed by Theorem 3.1 of our paper. The symbol "*" denotes the trivial cases $w^2 \geq q$, and "-" stands for pending cases.

**Example 1**: Let $w = 15$ and $q = 256$. Then $k = \lceil (q - 1)/w^2 \rceil + 1 = 3$. So $k - 1 \mid q$. Moreover, since $(15/2^3)^2 > 2$, the condition of Theorem 3.1 holds. Therefore $\text{RS}_k(q)$ is not $(w, w)$-separating. Now let $w = 10$ and $q = 128$. Then $(10/2^3)^2 < 2$. However, $q/(k - 1) = 64 = 2^6$, so the condition of Theorem 3.1 holds and $\text{RS}_k(q)$ is not $(w, w)$-separating.

**Example 2**: Let $w = 12$ and $q = 2048$. Then $k = 16$. Theorem 3.1 cannot be applied in this case, since $k - 1 = 15$ divides neither $q$ nor $q - 1$. The largest power of 2 equal or less than $w^2 = 144$ satisfying at least one of the conditions in Lemma 4.1 is $2^7 = 128$, for $w^2 - w = 132 > 128$. The modulo $n = 2047$
cyclotomic coset containing 1 is \( \Omega_1 = \{1, 2, 4, 8, 16, 32, 64, 128, 256, 512, 1024\} \), therefore \( d_j = |\Omega_1| = 11 \) and \( a_j = e_j = |\Omega_1 \cap J| = 5 \) where \( J = \{1, 2, \ldots, 16\} \). So \( K(C, S) \geq L(k, v) \geq \max\{d_1(a_1 - (m - v)), 0\} = 11 > 7 \), which implies \( RS_k(q) \) is not \((w_1, w_2)\)-separating by Theorem 4.1.

| \( q \) | 16 | 32 | 64 | 81 | 125 | 128 | 243 | 256 | 512 | 1024 | 2048 | 2187 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( w = 2 \) | * | * | * | * | * | * | 3.1 | 3.1 | - | - | - | - |
| \( w = 3 \) | * | * | * | 3.1 | 3.1 | - | - | - | - | - | - | - |
| \( w = 4 \) | * | * | * | * | * | * | 3.1 | 3.1 | - | - | - | - |
| \( w = 5 \) | * | 3.1 | 3.1 | - | - | - | - | - | - | - | - | - |
| \( w = 7 \) | * | * | 3.1 | 3.1 | - | - | - | - | - | - | - | - |
| \( w = 9 \) | * | * | * | * | * | * | 3.1 | 3.1 | - | - | - | - |
| \( w = 10 \) | * | * | * | 4 | 4 | 3.1 | 3.1 | - | - | - | - | - |
| \( w = 12 \) | * | * | * | * | * | 4 | 3.1 | 3.1 | 4.1 | - | - | - |
| \( w = 13 \) | * | * | * | * | * | * | 4 | 3.1 | 3.1 | 4.1 | - | - |
| \( w = 14 \) | * | * | * | * | * | * | 4 | 3.1 | 4.1 | - | - | - |
| \( w = 17 \) | * | * | * | * | * | * | * | 3.1 | 3.1 | - | - | - |
| \( w = 18 \) | * | * | * | * | * | * | 4 | 3.1 | 3.1 | - | - | - |
| \( w = 19 \) | * | * | * | * | * | * | 4 | 3.1 | - | - | - | - |
| \( w = 22 \) | * | * | * | * | * | * | 4 | 3.1 | - | - | - | - |
| \( w = 24 \) | * | * | * | * | * | * | * | 3.1 | 3.1 | - | - | - |
| \( w = 28 \) | * | * | * | * | * | * | * | 3.1 | 4.1 | 3.1 | - | - |
| \( w = 34 \) | * | * | * | * | * | * | * | 3.1 | 4.1 | - | - | - |

Table-1. Contributions to Silverberg’s Problem for Some Parameters

Thus, for a large family of Reed-Solomon codes with \( 2 \leq w \leq 40 \) and \( 16 \leq q \leq 2187 \), the separation and traceability properties are equivalent.

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