Termination of Linear Programs with Nonlinear Constraints *

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Abstract. In [16] Tiwari proved that termination of linear programs (loops with linear loop conditions and updates) over the reals is decidable through Jordan forms and eigenvectors computation. In [4] Braverman proved that it is also decidable over the integers. In this paper, we consider the termination of loops with polynomial loop conditions and linear updates over the reals and integers. First, we prove that the termination of such loops over the integers is undecidable. Second, with an assumption, we provide an algorithm to decide the termination of a class of such programs over the reals. Our method is similar to that of Tiwari in spirit but uses different techniques. Finally, we conjecture that the termination of linear programs with polynomial loop conditions over the reals is undecidable in general by reducing the problem to another decision problem related to number theory and ergodic theory, which we guess undecidable.

1 Introduction

Termination analysis is an important aspect of program verification. Guaranteed termination of program loops is necessary for many applications, especially those for which unexpected behavior can be catastrophic. For a generic loop

\[
\text{while \ (conditions) \ {updates},}
\]

it is well known that the termination problem is undecidable in general, even for a simple class of polynomial programs [3]. In [2] Blondel et al. proved that, even when all the conditions and updates are given as piecewise linear functions, the termination of the loop remains undecidable.

In [16] Tiwari proved that termination of the following programs is decidable over \( \mathbb{R} \) (the real numbers)

\[
P_0: \quad \text{while \ (BX > b) \ \{X := AX\},}
\]

where \( A \) and \( B \) are respectively \( n \times n \) and \( m \times n \) matrices, \( BX > b \) is a conjunction of linear inequalities over the state variables \( X \) and \( X := AX \)

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represents a (deterministic) linear update of each variable. Subsequently in [4] Braverman proved that the termination of $P_0$ is decidable over $\mathbb{Z}$ (the integers).

In this paper, we consider the problem of termination of the following loop:

\[ P_1 : \quad \text{while } (P(X) > 0) \quad \{ X := AX \}, \]

where $X = [x_1 \ldots x_N]^T$ is the vector of state variables of the program, $P(X) = [P_1(X) P_2(X) \ldots P_m(X)]^T$ are polynomial constraints, each $P_i(X)$ ($1 \leq i \leq m$) is a polynomial in $\mathbb{Q}[X]$ and $A$ is an $N \times N$ matrix over $\mathbb{Q}$ (the rational numbers). That is to say, we replace the linear constraints in $P_0$ with polynomial constraints and keep linear updates unchanged.

There are some well known techniques for deciding termination of some special kinds of programs. Ranking functions are most often used for this purpose. A ranking function for a loop maps the values of the loop variables to a well-founded domain; further, the values of the map decrease on each iteration. A linear ranking function is a ranking function that is a linear combination of the loop variables and constants. Recently, the synthesis of ranking functions draws increasing attention, and some heuristics concerning how to automatically generate linear ranking functions for linear programs have been proposed in [8,9,12]. In [12] Podelski et al. provided an efficient and complete synthesis method based on linear programming to construct linear ranking functions. In [5] Chen et al. proposed a method to generate non-linear ranking functions based on semi-algebraic system solving. However, existence of ranking function is only a sufficient condition on the termination of a program. It is not difficult to construct programs that terminate, but do not have ranking functions.

To solve the problem of termination of $P_1$, we do not use the technique of ranking functions. Our method is similar to that of Tiwari in spirit. Our main contributions in this paper are as follows. First, we prove that the termination of $P_1$ over $\mathbb{Z}$ is undecidable. Then it is easy to prove that, if “$>$” is replaced with “$\geq$” in $P_1$, termination of the resulted $P_1$ over $\mathbb{Z}$ is undecidable either. Second, with an assumption, we provide an algorithm to decide the termination of $P_1$ over $\mathbb{R}$. Finally, we conjecture that the termination of $P_1$ over $\mathbb{R}$ is undecidable in general by constructing a loop and reducing the problem to another decision problem related to number theory and ergodic theory, which we guess undecidable.

The rest of the paper is organized as follows. Section 2 proves the undecidability of $P_1$ and its variation over $\mathbb{Z}$. Section 3 introduces our main algorithm. The main steps of the algorithm are outlined first and some details of the steps are introduced separately in several subsections. With an assumption, we prove the correctness of our algorithm at the end of Section 3. After presenting our conjecture that the termination of $P_1$ is generally undecidable in Section 4, we conclude the paper in Section 5.

2 Undecidability of $P_1$ over $\mathbb{Z}$

Definition 1. A loop with $N$ variables is called terminating over a ring $R$ if for all the inputs $X \in R^N$, it is terminating; otherwise it is called nonterminating.
The undecidability of $P_1$ is obtained by a reduction to Hilbert’s 10th problem. Consider the following loop:

$$P_2: \quad \text{while } (x_{m+1} - f(x_1, \ldots, x_m)^2 > 0) \{ X := AX \}$$

where $X = \begin{bmatrix} x_1 & \ldots & x_{m+1} \end{bmatrix}^T$, $A = \text{diag}(1, \ldots, 1, 1/2)$ is a diagonal matrix and $f(x_1, \ldots, x_m)$ is a polynomial with integer coefficients.

**Lemma 1.** For any input $(x_1, \ldots, x_{m+1}) \in \mathbb{Z}^{m+1}$, $P_2$ terminates if and only if $f(x_1, \ldots, x_m)$ does not have integer roots.

**Proof.** ($\Rightarrow$) If $f(x_1, \ldots, x_m)$ has an integer root, say $(y_1, \ldots, y_m)$, obviously $P_1$ does not terminate with the input $Y = (y_1, \ldots, y_m, 1)$.

($\Leftarrow$) If $f(x_1, \ldots, x_m)$ has no integer roots, for any given $X \in \mathbb{Z}^{m+1}$, $-f(x_1, \ldots, x_m)^2$ is a fixed negative number. Because $(x_1, \ldots, x_m)$ will never be changed and $(1/2)^n \to 0$ as $n \to +\infty$, the loop will terminates after sufficiently large $n$ iterations.

**Theorem 1.** Termination of $P_1$ over $\mathbb{Z}$ is undecidable.

**Proof.** Because the existence of an integer root of an arbitrary Diophantine equation is undecidable, the termination of $P_1$ over $\mathbb{Z}$ is undecidable according to Lemma 1.

If “>” is substituted with “≥” in $P_1$, the loop becomes

$$P'_1: \quad \text{while } (P(X) \geq 0) \{ X := AX \}.$$ 

Analogously, we denote by $P'_2$ the loop obtained by substituting “≥” for “>” in $P_2$. It is easy to see that Lemma 1 still holds for $P'_2$. Then we get the following theorem.

**Theorem 2.** Termination of $P'_1$ over $\mathbb{Z}$ is undecidable.

3 Relatively Complete Algorithm for Termination of $P_1$ over $\mathbb{R}$

To decide whether $P_1$ terminates on a given input $X \in \mathbb{R}^N$, it is natural to consider a general expression of $A^n X$, for instance, a unified formula expressing $A^n X$ for any $n$. If one has such unified formula of $A^n X$, one can express the values of $P(X)$ after $n$ iterations. Then, for each element of $P(X)$ (each constraint), i.e., $P_i(X)$, one may try to determine whether $P_i(X) > 0$ as $n \to +\infty$ by guessing the dominant term of $P_i(X)$ and deciding the sign of the term.

That is the main idea of our algorithm which will be described formally in Subsection 3.2. At several main steps of our algorithm, a few techniques and results in number theory and ergodic theory are needed. For the sake of clarity, the details are introduced subsequently in the next subsections.

The first subsection is devoted to expressing $A^n X$ in a unified formula.
3.1 General Expression for $A^n X$ by Generating Function

In this subsection the general expression of $A^n X$ will be deduced with generating function, not Jordan form.

**Lemma 2.** Let $\alpha_1, \ldots, \alpha_d$ be a sequence of complex numbers, $d \geq 1$ and $\alpha_d \neq 0$. The following conditions on a function $f : \mathbb{N} \rightarrow \mathbb{C}$ are equivalent to each other:

i. $\sum_{n \geq 0} f(n)x^n = \frac{P(x)}{Q(x)}$, where $Q(x) = 1+\alpha_1 x+\ldots+\alpha_d x^d$ and $P(x)$ is a polynomial in $x$ of degree less than $d$.

ii. For all $n \geq 0$, $f(n+d) + \alpha_1 f(n+d-1) + \alpha_2 f(n+d-2) + \ldots + \alpha_d f(n) = 0$.

iii. For all $n \geq 0$, $f(n) = \sum_{i=1}^{k} P_i(n)\gamma^n_i$, and $Q(x) = 1+\alpha_1 x+\alpha_2 x^2+\ldots+\alpha_d x^d = \prod_{i=1}^{k} (1-\gamma_i x)^{d_i}$, where the $\gamma_i$’s are distinct, and $P_i(n)$ is a polynomial in $n$ of degree less than $d_i$.

**Corollary 1.** $A$ is a $d \times d$ square matrix with its entries in $\mathbb{Q}$. Suppose that the characteristic polynomial of $A$ is $D(x) = x^d + \alpha_1 x^{d-1} + \ldots + \alpha_d x$, where $\alpha_d \neq 0$ and $s \geq 0$. Define $f(n) = A^{n+s} X$ and let $f_j(n)$ be the $j$-th component of $f(n)$. Then for each $j$, $f_j(n)$ can be expressed as

$$f_j(n) = \sum_{i=1}^{k} p_{ji}(n)\xi^n_i,$$  

where $\xi_i$’s are all the distinct nonzero complex eigenvalues of $A$ and $p_{ji}(n)$ is a polynomial in $n$ of degree less than the multiplicity of $\xi_i$.

**Proof.** First, $A^d + \alpha_1 A^{d-1} + \ldots + \alpha_d A = 0$ because $D(x)$ is the characteristic polynomial of $A$. So, for any $n \geq 0$,

$$f(n+d-s) + \alpha_1 f(n+d-(s+1)) + \ldots + \alpha_d f(n) = A^{n+d} X + \alpha_1 A^{n+d-1} X + \ldots + \alpha_d A^{n+s} X = (A^d + \alpha_1 A^{d-1} + \ldots + \alpha_d A^s) A^n X = 0.$$  

Thus, for each $j$, $f_j(n+d-s) + \alpha_1 f_j(n+d-(s+1)) + \ldots + \alpha_d f_j(n) = 0$. By Lemma 2, $f_j(n) = \sum_{i=1}^{k} p_{ji}(n)\xi^n_i$ and $Q(x) = 1+\alpha_1 x+\ldots+\alpha_d x^{d-s} = \prod_{i=1}^{k} (1-\xi_i x)^{d_i}$ where $p_{ji}(n)$ is a polynomial in $n$ of degree less than $d_i$. It’s easy to see that $x = 0$ is not a solution of $Q(x)$ and $\sum_{i=1}^{k} d_i = d-s$. Because

$$D(x) = x^d Q\left(\frac{1}{x}\right) = x^{d-s} + \alpha_1 x^{d-s-1} + \ldots + \alpha_d x^{d-s} = x^d \prod_{i=1}^{k} (1-\xi_i x)^{d_i} = x^s \prod_{i=1}^{k} (x-\xi_i)^{d_i},$$

$\xi_i$’s are all the distinct nonzero complex eigenvalues of $A$ and $d_i$ is the multiplicity of $\xi_i$. That completes the proof.
Corollary 1. The characteristic polynomial of $A$ is

$$D(\lambda) = \lambda^n + \frac{3}{10} \lambda^4 + \frac{7}{10} \lambda^3 + \frac{11}{5} \lambda^2 + \frac{9}{10} \lambda + \frac{1}{2} = (\lambda + \frac{1}{2})(\lambda^2 - \frac{6}{5} \lambda + 1)(\lambda^2 + \lambda + 1).$$

Remark 1. According to Corollary 1, we may compute the general expression of $A^n X$ as follows. First, compute all the complex eigenvalues of $A$ and their multiplicities. Second, suppose each $f_j(n)$ of $f(n)$ is in the form of eq. (1) where the coefficients of $p_j$ are to be computed. Third, compute $f(1), \ldots, f(d)$, and obtain a set of linear equations by comparing the coefficients of the resulted $f_j(i)$ ($1 \leq i \leq d$) to those of eq. (1). Finally, by solving those linear equations, we can obtain $f_j(n)$ and $f(n)$.

Example 1. Let’s consider the following loop:

$$\text{while } (x_5 + x_1^2 + x_1 x_2 - x_2^2 - 2 x_3 x_4 - x_4^2 > 0) \{ X := AX; \}$$

where

$$A = \begin{bmatrix} 1 & -\frac{2}{5} & 0 & 0 & 0 \\ 2 & \frac{1}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -\frac{1}{2} & -1 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix}.$$

We shall show how to compute the general expression of $A^n X$ by taking use of Corollary 1. The characteristic polynomial of $A$ is

$$D(\lambda) = \lambda^n + \frac{3}{10} \lambda^4 + \frac{7}{10} \lambda^3 + \frac{11}{5} \lambda^2 + \frac{9}{10} \lambda + \frac{1}{2} = (\lambda + \frac{1}{2})(\lambda^2 - \frac{6}{5} \lambda + 1)(\lambda^2 + \lambda + 1).$$

The eigenvalues of $A$ are

$$\xi_1 = \frac{3 + 4i}{5}, \xi_2 = \frac{3 - 4i}{5}, \xi_3 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i, \xi_4 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i, \xi_5 = -\frac{1}{2}.$$

Set $f(n) = A^n X = [f_1(n) f_2(n) f_3(n) f_4(n) f_5(n)]^T$. For any $n > 0$, we have

$$f(n + 5) + \frac{3}{10} f(n + 4) + \frac{7}{10} f(n + 3) + \frac{11}{5} f(n + 2) + \frac{9}{10} f(n + 1) + \frac{1}{2} f(n) = 0.$$

Because the multiplicities of $\xi_1, \xi_2, \xi_3, \xi_4$ and $\xi_5$ are all 1, by Corollary 1 we may assume for $1 \leq j \leq 5$

$$f_j(n) = \sum_{i=1}^{5} a_{ji} x_i \xi_1^n + \sum_{i=1}^{5} b_{ji} x_i \xi_2^n + \sum_{i=1}^{5} c_{ji} x_i \xi_3^n + \sum_{i=1}^{5} d_{ji} x_i \xi_4^n + \sum_{i=1}^{5} e_{ji} x_i \xi_5^n.$$

Let $f(1), f(2), f(3), f(4)$ and $f(5)$ equal to $AX, \ldots, A^5 X$ respectively, and by solving some linear equations (see Remark 1) we can obtain

$$f_1(n) = \left( \frac{2 - i}{4} x_1 + \frac{i}{4} x_2 \right) \xi_1^n + \left( \frac{2 + i}{4} x_1 - \frac{i}{4} x_2 \right) \xi_2^n,$$

$$f_2(n) = \left( -\frac{5i}{4} x_1 + \frac{2 + i}{4} x_2 \right) \xi_1^n + \left( \frac{5i}{4} x_1 + \frac{2 - i}{4} x_2 \right) \xi_2^n,$$

$$f_3(n) = \left( \frac{1}{2} - \frac{\sqrt{3} i}{6} x_3 - \frac{2 \sqrt{3} i}{3} x_4 \right) \xi_3^n + \left( \frac{1}{2} + \frac{\sqrt{3} i}{6} x_3 + \frac{2 \sqrt{3} i}{3} x_4 \right) \xi_4^n,$$

$$f_4(n) = \left( \frac{\sqrt{3}}{6} x_3 + \left( \frac{1}{2} + \frac{\sqrt{3} i}{6} x_4 \right) \xi_3^n + \left( -\frac{\sqrt{3}}{6} x_3 + \left( \frac{1}{2} - \frac{\sqrt{3} i}{6} x_4 \right) \xi_4^n,$$

$$f_5(n) = x_5 \xi_5^n.$$
3.2 Main Algorithm

According to subsection 3.1 the general expression of $A^{n+m}X$ is a polynomial in $x_1, \ldots, x_N, n$ and $\xi_1, \ldots, \xi_q$, the nonzero complex roots of $D(x)$. If we substitute $A^{n+m}X$ for $X$ in $P(X)$ and denote the resulted $P_j(X)(1 \leq j \leq m) \in P(X)$ by $P_j(X, n)$, then $P_j(X, n)$ can be written as

$$P_j(X, n) = p_{j0}(X, n) + p_{j1}(X, n)\eta_1^n + \ldots + p_{jM}(X, n)\eta_M^n,$$

where $\eta_k (1 \leq k \leq M)$ is the product of some $\xi_j$'s.

To determine whether $P_j(X, n) > 0$ holds for all $n$. To this end, it is sufficient to know whether the dominant term (leading term) of eq. (2) is positive or not as $n \to +\infty$. In the following, we shall give a more detailed description of eq. (2) so that we can obtain the expression of the leading term of eq. (2).

Let $\eta = r_k e^{\alpha_k 2\pi i}$, where $i = \sqrt{-1}$ and $r_k$ is the modulus of $\eta$. Without loss of generality, we assume $r_1 < r_2 < \cdots < r_M$. For convenience, set $\eta_0 = r_0 = 1$. Rewrite $P_j(X, n)$ as

$$P_j(X, n) = p_{j0}(X, n)r_0^n + p_{j1}(X, n)e^{\alpha_1 2\pi i}r_1^n + \ldots + p_{jM}(X, n)e^{\alpha_M 2\pi i}r_M^n.$$

Suppose $T_j$ is the common period of all the $e^{\alpha q 2\pi i} (1 \leq q \leq M)$ where $\alpha_q$ is a rational number.

**Definition 2.** For each $j (1 \leq j \leq \sum_{i=1}^m T_i)$, if $j = \sum_{i=1}^{i-1} T_i + i$ and $1 \leq i \leq T_j$, then define

$$G_j(X, n) \triangleq P_s(X, T_sn + i - 1).$$

**Notation 1** For each $j (1 \leq j \leq \sum_{i=1}^m T_i)$, expand $G_j(X, n)$, collect the result with respect to (w.r.t.) $n^l r_k^n$, and let $C_{jkl}(X, n)$ denote the coefficient of the term $n^l r_k^n$.

Then $G_j(X, n)$ can be written as

$$C_{j10}(X, n)r_1^n + C_{j11}(X, n)nr_1^n + \ldots + C_{j1d_1}(X, n)n^{d_1}r_1^n + \ldots + C_{jM0}(X, n)r_M^n + C_{jM1}(X, n)nr_M^n + \ldots + C_{jMd_M}(X, n)n^{d_M}r_M^n,$$

where $d_l (1 \leq l \leq M)$ is the greatest degree of $n$ in $G_j(X, n)$ w.r.t. $r_l$.

It can be deduced that if $r_{l1} < r_{l2}$, the order of $n^{l1}r_1^n$ is less than the order of $n^{l2}r_1^n$ for any $l_1$ and $l_2$ when $n$ goes to infinity. Similarly, if $l_1 < l_2$, the order of $n^{l1}r_1^n$ is less than the order of $n^{l2}r_1^n$. So, it is natural to introduce an ordering on the terms $n^l r_1^n$ as follows.

**Definition 3.** We define $n^{l1}r_1^n \prec n^{l2}r_1^n$ if $r_{l1} < r_{l2}$ or $r_{l1} = r_{l2}$ and $l_1 < l_2$. A term $C_{jkl}n^l r_k^n$ in eq. (3) is said to be the leading term and $C_{jkl}$ the leading coefficient if $n^l r_k^n$ occurring in $C_{jkl}$ is the largest one under that ordering $\prec$. 

Suppose \( G_j(X, n) = P_s(X, T_s n + t) \) for some \( s \) and \( t \). For those \( e^{\alpha_s} \)'s are rational numbers, \( e^{(T_s n + t)\alpha_s} = e^{\alpha_s} \) because \( T_s \) is the common period. Because there may be some \( e^{(T_s n + t)\alpha_s} \)'s with irrational \( \alpha_s \)'s, each \( C_{jkl}(X, n) \) can be divided into three parts,

\[
C_{jkl}(X, n) = C_{jkl0}(X) + C_{jkl1}(X) + C_{jkl2}(X, n),
\]

where \( C_{jkl0}(X) \) does not contain any \( e^{(T_s n + t)\alpha_s} \), \( C_{jkl1}(X) \) contains \( e^{(T_s n + t)\alpha_s} \) with rational \( \alpha_s \) and \( C_{jkl2}(X, n) \) contains those \( e^{(T_s n + t)\alpha_s} \) with irrational \( \alpha_s \).

Further, \( C_{jkl2}(X, n) \) can be written as

\[
C_{jkl2}(X, n) = C_{jkl2}(X, \sin((nT_s + t)\alpha_{k1} 2\pi), \cos((nT_s + t)\alpha_{k1} 2\pi), \ldots, \\
\sin((nT_s + t)\alpha_{k_h} 2\pi), \cos((nT_s + t)\alpha_{k_h} 2\pi)),
\]

where \( \{\alpha_{k1} 2\pi, \ldots, \alpha_{k_h} 2\pi\} \) is a maximum rationally independent group.

**Example 2.** We continue to use the loop in Example 1 to illustrate the above concepts and notations.

Because \( |\xi_1| = |\xi_2| = 1 \), let \( \xi_1 = e^{\alpha_1 2\pi} \) and \( \xi_2 = e^{-\alpha_1 2\pi} \), where \( \alpha_1 2\pi \) is the argument of \( \xi_1 \). It’s not difficult to check that \( \alpha_1 \) is an irrational number.

For the sake of clarity, in the following we firstly reduce the expressions of \( f_1(n), f_2(n), f_3(n), f_4(n) \) and \( f_5(n) \), and then substitute them in the loop guard.

Let \( \alpha_2 = \frac{1}{7} \), then \( \xi_3 = e^{\alpha_2 2\pi}, \xi_4 = e^{-\alpha_2 2\pi} \), and \( f_1(n), f_2(n), f_3(n), f_4(n) \) and \( f_5(n) \) can be rewritten as

\[
\begin{align*}
 f_1(n) &= x_1 \cos(n\alpha_1 2\pi) + \frac{x_1 - x_2}{2} \sin(n\alpha_1 2\pi), \\
 f_2(n) &= x_2 \cos(n\alpha_1 2\pi) + \frac{5x_1 - x_2}{2} \sin(n\alpha_1 2\pi), \\
 f_3(n) &= x_3 \cos(n\alpha_2 2\pi) + \frac{\sqrt{3}x_3 + 4\sqrt{3}x_4}{3} \sin(n\alpha_2 2\pi), \\
 f_4(n) &= x_4 \cos(n\alpha_2 2\pi) - \frac{\sqrt{3}x_3 + \sqrt{3}x_4}{3} \sin(n\alpha_2 2\pi), \\
 f_5(n) &= x_5 (\frac{1}{2}).
\end{align*}
\]

Substituting \( f_1(n), f_2(n), f_3(n), f_4(n) \) and \( f_5(n) \) for \( x_1, x_2, x_3, x_4 \) and \( x_5 \) respectively in the loop guard, we get that the resulted loop guard is

\[
L_1 r_1^n + (L_{21} + L_{22} + L_{23}) r_2^n,
\]

where \( r_1 = \frac{1}{2}, r_2 = 1, \) and

\[
\begin{align*}
 L_1 &= (-1)^n x_5, \\
 L_{21} &= -\frac{x_3^2 + 2x_3 x_4 + 4x_4^2}{2} + \frac{5x_1^2 + x_2^2 - 2x_1 x_2}{4}, \\
 L_{22} &= \frac{2x_1^2 - 2x_3 x_4 - x_2^2}{2} \cos(n2\alpha_2 2\pi) - (\sqrt{3}x_3 + \sqrt{3}x_4) \sin(n2\alpha_2 2\pi), \\
 L_{23} &= \frac{x_1^2 + x_2^2 - 6x_3 x_4}{4} \cos(n2\alpha_1 2\pi) + \frac{7x_1^2 - x_2^2 - 2x_1 x_2}{4} \sin(n2\alpha_1 2\pi).
\end{align*}
\]

\[1 \text{ Later it will be proved that } C_{jkl0}(X), C_{jkl1}(X) \text{ and } C_{jkl2}(X, n) \text{ are reals for any } n.\]

\[2 \text{ “Rationally independent” will be described later.}\]

\[3 \text{ A general algorithm for checking whether an argument is a rational multiple of } \pi \text{ will be stated in detail in subsection 3.4.}\]
Since $\alpha_2 = \frac{1}{4}$, the period of $\xi_2^2 = e^{2\alpha_2 2\pi} = \cos(2\alpha_2 2\pi) + \sin(2\alpha_2 2\pi)i$ is 3. Then we compute

\[ G_1(X, n) = ((-1)^{2n} x_3) r_1^{2n} + C_{1200} r_2^{3n}, \]
\[ G_2(X, n) = ((-1)^{2n+1} x_3) r_1^{3n+1} + C_{2200} r_2^{3n+1}, \]
\[ G_3(X, n) = ((-1)^{2n+2} x_3) r_1^{3n+2} + C_{3200} r_2^{3n+2}. \]

Take $G_1(X, n)$ as an example.

\[ C_{120} = C_{1200} + C_{1201} + C_{1202}, \]
\[ C_{1200} = \frac{-x_4^2 + 2x_3 x_4 + 4x_5^2}{2} + \frac{5x_4^2 + x_2^2 - 2x_1 x_2}{4}, \]
\[ C_{1201} = \frac{2x_4^2 - 2x_3 x_4 - x_5^2}{2}, \]
\[ C_{1202} = -\frac{x_4^2 + x_2^2 - 6x_1 x_2}{4} \cos(3\alpha_2 2\pi) + \frac{7x_4^2 - x_2^2 - 2x_1 x_2}{4} \sin(3\alpha_2 2\pi). \]

**Notation 2** We denote by $C_{jkl}(X, n) > 0$ (and call $C_{jkl}(X, n)$ “positive”) if

\[ \min(C_{jkl0}(X) + C_{jkl1}(X) + C_{jkl2}(X, y_{i1}, y_{i2}, \ldots, y_{s_{k1}}, y_{s_{k2}})) > 0 \]

subject to \( \{y_{i1}^2 + y_{i2}^2 = 1, \ldots, y_{s_{k1}}^2 + y_{s_{k2}}^2 = 1\} \).

**Remark 2.** It is not difficult to see $C_{jkl}(X, n) > 0$ iff

\[ \forall (y_{i1}, y_{i2}, \ldots, y_{s_{k1}}, y_{s_{k2}}) \ ((y_{i1}^2 + y_{i2}^2 = 1) \land \cdots \land (y_{s_{k1}}^2 + y_{s_{k2}}^2 = 1) \rightarrow C_{jkl0}(X) + C_{jkl1}(X) + C_{jkl2}(X, y_{i1}, y_{i2}, \ldots, y_{s_{k1}}, y_{s_{k2}}) > 0) \]

because \( \{(y_{i1}, y_{i2}, \ldots, y_{s_{k1}}, y_{s_{k2}}): y_{i1}^2 + y_{i2}^2 = 1, i = 1, \ldots, s_k\} \) is a bounded closed set. If $>$ and $\geq$ are replaced with $\geq$ and $\geq$ respectively in the notation, we get the notation of $C_{jkl}(X, n) \geq 0$ (“nonnegative”).

Roughly speaking, for any $G_j(X, n)$, if its leading coefficient $C_{jkl}(X, n) > 0$, there exists an integer $N_1$ such that for all $n > N_1$, $G_j(X, n) > 0$. If all the leading coefficients of all the $G_j(X, n)$’s are “positive”, there exists $N'$ such that for all $n > N'$, all the $G_j(X, n)$’s are positive. Therefore, $P_j$ is nonterminating with input $X' := A^{N'} X$.

On the other hand, if $P_j$ is nonterminating, does there exist an input $X$ such that the leading coefficients of all the $G_j(X, n)$’s are “positive”? We do not know the answer yet. However, with an assumption described below, the answer is yes.

**Assumption for the main algorithm:** For any $X \in \mathbb{R}^n$ and any $C_{jkl}(X, n)$, $C_{jkl2}(X, n)$ being not identically zero implies

\[ \min(C_{jkl0}(X) + C_{jkl1}(X) + C_{jkl2}(X, y_{i1}, y_{i2}, \ldots, y_{s_{k1}}, y_{s_{k2}})) \neq 0 \]

subject to \( y_{i1}^2 + y_{i2}^2 = 1 \ (i = 1, \ldots, s_k) \).

It is not difficult to see that the assumption is equivalent to the following formula:

\[ \forall X \exists Y \ (C_{jkl2}(X, n) \equiv 0 \lor \bigwedge_{1 \leq i \leq s_k} y_{i1}^2 + y_{i2}^2 = 1 \rightarrow C_{jkl0}(X) + C_{jkl1}(X) + C_{jkl2}(X, Y) > 0) \] \]
\[ \forall X \exists Y \ (C_{jkl2}(X, n) \equiv 0 \lor \bigwedge_{1 \leq i \leq s_k} y_{i1}^2 + y_{i2}^2 = 1 \rightarrow C_{jkl0}(X) + C_{jkl1}(X) + C_{jkl2}(X, Y) < 0) . \]
Because $C_{jkl2}(X,n)$ can be written as $\sum_{i \in I} f_i(X) \sin(n\alpha, 2\pi) + f_{i2}(X) \cos(n\alpha, 2\pi)$, where $I$ is an index set, $C_{jkl2}(X,n) \equiv 0$ is equivalent to $\bigwedge_{i \in I} f_i(X) = f_{i2}(X) = 0$. Thus, the assumption can be checked with real quantifier elimination techniques.

Example 3. For those $C_{jkl}$’s in Example 2, let’s check whether they satisfy the assumption. For clarity, we present a clear proof here rather than make use of any tool for real quantifier elimination. Take $G_1(X,n)$ for example. It’s clear that $C_{1202}(X,n) \equiv 0$ if and only if $x_1 = x_2 = 0$. It is not difficult to compute

$$D = \inf_{n \geq 1} C_{1202}(X,n) = -\sqrt{\left(\frac{-x_1^2 - x_2^2 + 6x_1x_2}{4}\right)^2 + \left(\frac{7x_1^2 - x_2^2 - 2x_1x_2}{4}\right)^2}.$$  

If $x_1 = x_2 = 0$ does not hold, $\frac{5x_1^2 + x_2^2 - 2x_1x_2}{4} + D < 0$ because

$$\left(\frac{5x_1^2 + x_2^2 - 2x_1x_2}{4}\right)^2 - D^2 = \frac{-1}{16}(5x_1^2 + x_2^2 - 2x_1x_2)^2 < 0.$$  

Let $M = \min_{y_{11}^2 + y_{12}^2 = 1} (C_{1200}(X) + C_{1201}(X) + C_{1202}(X,y_{11},y_{12}))$, then

$$M = -(x_3 + x_4)^2 + \frac{5x_1^2 + x_2^2 - 2x_1x_2}{4} + D < 0$$

if $x_1 = x_2 = 0$ does not hold. Consequently, $G_1(X,n)$ satisfies the assumption. Similarly it can be proved $G_2(X,n)$ and $G_3(X,n)$ satisfy the assumption too. Thus the loop in Example 1 satisfies the assumption.

We shall show in next section how hard it is to deal with the case that the assumption does not hold. Now, we are ready to describe our main algorithm. For the sake of brevity, the algorithm is described as a nondeterministic algorithm. The basic idea is to guess a leading term for each $G_j(X,n)$ first. Then, setting its coefficient be “positive” and the coefficients of the terms with higher order be “nonnegative”, we can get a semi-algebraic system (SAS). If one of our guess is satisfiable, i.e., one of the SASs has solutions, $P_1$ is nonterminating. Otherwise, it is terminating.

Algorithm Termination

**Step 0** Compute the general expression of $A^{n+m}X$.

**Step 1** Substitute $A^{n+m}X$ for $X$ in $P(X)$, and compute all $G_j(X,n)$ (finite many, say, $j = 1, ..., L$).

**Step 2** Guess a leading term for each $G_j(X,n)$, say $C_{jkl}^nX^n_{jkl}$.

**Step 3** Construct a semi-algebraic system $S$ as follows.

$$S_j = C_{jkl}^nX^n_{jkl} \geq 0 \land \bigwedge_{(k > k_j) \lor (k = k_j, l > l_j)} C_{jkl}(X,n) \geq 0.$$

**Step 4** If one of these systems is satisfiable, return ”nonterminating”. Otherwise return ”terminating”.


Remark 3. If the assumption for the main algorithm does not hold, then Termination is incomplete. That is, if it returns “nonterminating”, the loop is nonterminating. Otherwise, it tells nothing.

Example 4. For the loop in Example 1, we have computed the $G_i(X, n)$’s in Example 2 and verified that it satisfies the assumption of the main algorithm in Example 3. We shall finish its termination decision in this example, following the steps in Termination.

By Steps 2 and 3 of Termination, we should guess leading terms and construct SASs accordingly. To be concrete, let’s take as an example one certain guess and suppose we have the following semi-algebraic system:

\[
\begin{align*}
C_{120}(X, n) &> 0, \\
C_{220}(X, n) &> 0, \\
C_{320}(X, n) &> 0.
\end{align*}
\]

According to Notation 2 and Remark 2, the above inequalities are equivalent to $(\forall y_1, y_2) y_1^2 + y_2^2 = 1 \Rightarrow$

\[
\begin{align*}
\left\{ \begin{array}{ll}
-(x_3 + x_4)^2 + \frac{5x_1^2 + x_2^2 - 2x_1 x_2}{4} & - y_1^2 + \frac{7x_1^2 - x_2^2 - 2x_1 x_2}{4} y_2 > 0, \\
-(x_4 - x_3)^2 + \frac{5x_1^2 + x_2^2 - 2x_1 x_2}{4} & - y_1^2 + \frac{7x_1^2 - x_2^2 - 2x_1 x_2}{4} y_2 > 0, \\
-(x_4 + \frac{x_4}{2})^2 + \frac{5x_1^2 + x_2^2 - 2x_1 x_2}{4} & - y_1^2 + \frac{7x_1^2 - x_2^2 - 2x_1 x_2}{4} y_2 > 0.
\end{array} \right.
\]

In Example 3, we have shown that $\frac{5x_1^2 + x_2^2 - 2x_1 x_2}{4} + D \leq 0$. Thus, the above predicate formula does not hold. In fact, none of the formulas obtained in Step 3 holds. Thus, the loop in Example 1 is terminating.

Remark 4. It is well known that real quantifier elimination is decidable from Tarski’s work [15]. Therefore, the semi-algebraic systems in Step 3 can be solved. For the tools for solving semi-algebraic systems, please be referred to [6,7,10,18].

Remark 5. There are some techniques to decrease the amount of computation of the algorithm Termination. For example, we can use Lemma 3 when guessing leading terms for each $G_j(X, n)$.

Lemma 3. Let $\xi_1, \xi_2, \ldots, \xi_m \in \mathbb{C}$ be a collection of distinct complex numbers such that $|\xi_i| = 1$ and $\xi_i \neq 1$ for all $i$. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be any complex numbers and $z_n = \alpha_1\xi_1^n + \ldots + \alpha_m\xi_m^n$. Then one of the following is true:

1. the real part $\text{Re}(z_n) = 0$ for all $n$; or
2. there is $c < 0$ such that $\text{Re}(z_n) < c$ for infinitely many $n$’s.

According to Lemma 2, if $C_{jkl} = 0$ and $C_{jkl}(X, n)$ is not identically zero w.r.t. $n$, then $C_{jkl}(X, n)$ cannot be always nonnegative. According to the former discussion

\[\{G_{i-1}\sum_{i=1}^j T_k+1(X, n), \ldots, G_{i-1}\sum_{i=1}^j T_k+T_j(X, n)\}\]

are obtained from $P_j(X, n)$. Then the leading term with the greatest order among all the leading terms of the above set should not be of the form $C_{jkl}n^{l_j}r_j^{n_l}$ where
At Step 1 of the main algorithm, we may compute \( \eta_k (1 \leq k \leq M) \) which are the products of some \( \xi_j \)'s after substituting the general expression of \( A^{n+m}X \) for \( X \) in \( P(X) \). In order to describe \( \eta_k \), we need the minimal polynomials of \( \eta_k \). In this subsection a method is presented to solve a more general problem.

In \cite{Strzebonski1989} Strzebonski gave an algorithm to compute \( \{ \alpha + \beta, \alpha - \beta, \alpha \cdot \beta, \frac{\alpha}{\beta} \} \) numerically where \( \alpha \) and \( \beta \) are given algebraic numbers. Here we want to present another more intuitive method based on symbolic computation.

In the following subsections, we shall explain the details of the main steps of Termination and prove its correctness.

### 3.3 Compute the Minimal Polynomials of \( \alpha + \beta, \alpha - \beta, \alpha \cdot \beta \) and \( \frac{\alpha}{\beta} \)

At Step 1 of the main algorithm, we may compute \( \eta_k (1 \leq k \leq M) \) which are the products of some \( \xi_j \)'s after substituting the general expression of \( A^{n+m}X \) for \( X \) in \( P(X) \). In order to describe \( \eta_k \), we need the minimal polynomials of \( \eta_k \). In this subsection a method is presented to solve a more general problem.

In \cite{Strzebonski1989} Strzebonski gave an algorithm to compute \( \{ \alpha + \beta, \alpha - \beta, \alpha \cdot \beta, \frac{\alpha}{\beta} \} \) numerically where \( \alpha \) and \( \beta \) are given algebraic numbers. Here we want to present another more intuitive method based on symbolic computation.

In the following let \( \alpha \cdot \beta \) denote one of \( \{ \alpha + \beta, \alpha - \beta, \alpha \cdot \beta, \frac{\alpha}{\beta} \} \). Without loss of generality let’s assume that the minimal polynomial of \( \alpha \) is \( f_1(x) \) whose degree is \( d_1 \) and the minimal polynomial of \( \beta \) is \( f_2(x) \) whose degree is \( d_2 \). We can bound \( \alpha \) and \( \beta \) in \( W_1 \) and \( W_2 \), respectively, where \( W_1, W_2 \) are “boxes”, by isolating the complex zeros of \( f_1(x) \) and \( f_2(x) \). Since the degree of \( \alpha \) is \( d_1 \) and the degree of \( \beta \) is \( d_2 \), the degree of \( \alpha \cdot \beta \) is at most \( d_1 \cdot d_2 \). Then there must exist \( x_1, \ldots, x_{d_1 \cdot d_2 + 1} \) in \( \mathbb{Z} \) such that

\[
x_1 + x_2 \alpha \cdot \beta + \ldots + x_{d_1 \cdot d_2 + 1} (\alpha \cdot \beta)^{d_1 \cdot d_2} = 0.
\]

Thus we can design an algorithm to enumerate all the \( (x_1, \ldots, x_{d_1 \cdot d_2 + 1}) \in \mathbb{Z}^{d_1 \cdot d_2 + 1} \) and check whether it is a solution. Since there exists one solution this algorithm must terminate and output a solution. Assume that its output is \( a_0, a_1, \ldots, a_{d_1 \cdot d_2} \) and \( f(x) = a_0 + a_1 x + \ldots + a_{d_1 \cdot d_2} x^{d_1 \cdot d_2} \). Because \( f(\alpha \cdot \beta) = 0 \), the minimal polynomial of \( \alpha \cdot \beta \) is an irreducible factor of \( f(x) \). Factor \( f(x) \) in \( \mathbb{Q} \). Without loss of generality, we assume \( f(x) = g_1(x)^{m_1} \cdot g_2(x)^{m_2} \cdot \ldots \cdot g_d(x)^{m_d} \). We can check whether \( g_j(x)(1 \leq j \leq d) \) is the minimal polynomial of \( \alpha \cdot \beta \) by solving the following semi-algebraic system (SAS):

\[
\{ g_j(x \cdot y) = 0, f_1(x) = 0, f_2(y) = 0, x \in W_1, y \in W_2 \}.
\]

If it is satisfiable, \( g_j(x) \) is the minimal polynomial of \( \alpha \cdot \beta \); otherwise it is not. Thus the minimal polynomial of \( \alpha \cdot \beta \) can be obtained.

### 3.4 Check Whether the Argument of \( \alpha \) Is a Rational Multiple of \( \pi \)

At Step 1 of the main algorithm, \( T_s \) is necessary for defining \( G_j(X, n) \). Thus for a given \( \eta_k \) we have to check whether its argument is a rational multiple of \( \pi \) and if it is, we need to know the period of \( \frac{\eta}{|\eta|} \). This subsection aims at this problem.

Suppose the minimal polynomial of \( \alpha \) is \( p(x) \) whose degree is \( d \). Without loss of generality suppose \( \alpha = re^{i\beta m_1} \). We can bound \( \alpha \) in \( W \) by isolating all the complex roots of \( p(x) \). Since the degree of \( \alpha \) is \( d \), the degree of \( \frac{\alpha}{\pi} \) must be \( d \).
Then the degree of $\alpha \cdot \bar{\alpha} = r^2$ is at most $d^2$. Thus the degree of $r$ is at most $2d^2$. The degree of $\alpha^{-1}$ is at most $2d^2$ because the degree of $\alpha^{-1}$ is the same as the degree of $\alpha$. Since the degree of $\alpha$ is $d$ the degree of $\alpha \cdot r^{-1} = e^{i2\pi}r$ is at most $2d^3$.

If $\beta$ is a rational number, $\alpha$ must be a unit root and its minimal polynomial must be a cyclotomic polynomial. As a result if $\beta$ is a rational number the minimal polynomial of $\alpha$ must be a cyclotomic polynomial whose degree is less than or equal to $2n^3$. All the cyclotomic polynomials can be computed explicitly according to the theory of cyclotomic field. Thus let $CP_j(x)$ denote the cyclotomic polynomial whose degree is $j$. Then, $\beta$ is a rational number if and only if the following is satisfiable

$$\exists r((r \neq 0) \bigwedge_{j=1}^{2d^3} (\bigvee_{i=1}^{d} CP_j(x/r) = 0) \bigwedge (p(x) = 0) \bigwedge (x \in W)).$$

Because $CP_j(x/r) = 0 \iff r^jCP_j(x/r) = 0$, the above quantifier formula is decidable. If it’s satisfiable the minimal polynomial of $\alpha$ can be computed by checking whether

$$\exists r((r \neq 0) \bigwedge (CP_j(x/r) = 0) \bigwedge (p(x) = 0) \bigwedge (x \in W))$$

is satisfiable one by one.

### 3.5 Check Rational Independence

Given a set of algebraic numbers, $\alpha_1 = e^{\beta_1 \pi i}, \ldots, \alpha_d = e^{\beta_d \pi i}$, where $\beta_1, \ldots, \beta_d$ are irrational numbers. In this subsection we present a method to check whether $\beta_1, \ldots, \beta_d$ are rationally independent.

**Definition 4.** Irrational numbers $\beta_1, \ldots, \beta_d$ are rationally independent if there does not exist rational numbers $a_1, \ldots, a_d$ such that $\sum_{j=1}^{d} a_j \beta_j \in \mathbb{Q}$.

Obviously, $\beta_1, \ldots, \beta_d$ are rationally independent if and only if $1, \beta_1, \ldots, \beta_d$ are linearly independent in $\mathbb{Q}$. It can be deduced that $\beta_1, \ldots, \beta_d$ are rationally independent if and only if $\forall (b_1, \ldots, b_d) \in \mathbb{Z}^d$, $\sum_{j=1}^{d} b_j \beta_j \notin \mathbb{Z}$.

**Lemma 4.** Let $\lambda_1, \ldots, \lambda_m$ with $m \geq 2$ be linearly dependent logarithms of algebraic numbers. Define $\alpha_j = e^{\lambda_j}(1 \leq j \leq m)$. For $1 \leq j \leq m$, let $\log A_j \geq 1$ be an upper bound for $\max\{h(\alpha_j), |\lambda_j|\}$ where $D$ is the degree of the number field $K = \mathbb{Q}(\alpha_1, \ldots, \alpha_m)$ over $\mathbb{Q}$ and $h(\alpha)$ denotes the absolute logarithmic height of $\alpha$. Then there exist rational integers $n_1, \ldots, n_m$, not all of which are zero, such that $n_1 \lambda_1 + \ldots + n_m \lambda_m = 0$ and $|n_k| < (11(m-1)D^{3m-1} \log A_1 \cdots \log A_m) \log A_k$ for $1 \leq k \leq m$.

**Remark 6.** Baker is the first one to use his transcendence arguments to establish such an estimate. However, the description here follows Lemma 7.19 in [17].
To prove the correctness of our main algorithm, we need some further results.

### 3.6 Compute the Infimum of $C_{jkl2}$

To prove the correctness of our main algorithm, we need some further results.

First, let’s introduce a lemma in ergodic theory. Let $S$ be the unit circumference. Usually any point on $S$, say $(a, b)$, is denoted as a complex number $a + bi$. Define $\pi : \mathbb{R}^m \to S^m$ as

$$
(x_1, \ldots, x_m) \to (e^{x_1 2\pi i}, \ldots, e^{x_m 2\pi i}).
$$

Define $m$-torus $T^m = \{ (e^{x_1 2\pi i}, \ldots, e^{x_m 2\pi i}) | x_j \in \mathbb{R} \}$ and $L_{\pi(\alpha)} : T^m \to T^m$ as

$$
(e^{y_1 2\pi i}, \ldots, e^{y_m 2\pi i}) \to (e^{(y_1 + \alpha_1) 2\pi i}, \ldots, e^{(y_m + \alpha_m) 2\pi i}).
$$

**Lemma 5.** [11] If $\alpha \in \mathbb{R}^m$, the translation $L_{\pi(\alpha)}(X)$ is ergodic iff for all $K \in \mathbb{Z}^m$, $(K, \alpha) \notin \mathbb{Z}$ where $(K, \alpha)$ stands for the inner product of $K$ and $\alpha$.

According to Lemma 5 we know that if $\alpha_1$ is an irrational number, the closure of $\{e^{n_1 2\pi i} \} \_{n \geq 1}$ is the unit circumference. Also, if $\alpha_1, \ldots, \alpha_m$ are rationally independent, $L_{\pi(\alpha)}(X)$ is ergodic. Thus the closure of $\{L_{\pi(\alpha)}(0) \} \_{s \geq 1}$ is $T^m$.

**Lemma 6.** If $\alpha_{k1}, \ldots, \alpha_{ks_k}$ are rationally independent and $C_{jkl2}$ is of the form

$$
C_{jkl2}(X, \sin(n_{k1} \alpha_1 2\pi), \cos(n_{k1} 2\pi), \ldots, \sin(n_{ks_k} \alpha_k 2\pi), \cos(n_{ks_k} \alpha_k 2\pi))
$$

for a fixed $X$, then

$$
\inf_{n \geq 1} \{ C_{jkl2}(X, \sin(n_{k1} \alpha_1 2\pi), \cos(n_{k1} 2\pi), \ldots, \sin(n_{ks_k} \alpha_k 2\pi), \cos(n_{ks_k} \alpha_k 2\pi)) \}
$$

is equal to $\min \{ C_{jkl2}(X, x_1, y_1, \ldots, x_{s_k}, y_{s_k}) \}$ subject to $\{ x_i^2 + y_i^2 = 1, 1 \leq i \leq s_k \}$.

**Proof.** According to Lemma 5 for any $(x_1, y_1, \ldots, x_{s_k}, y_{s_k})$ there exists a subsequence, say $(n_i)_{i \geq 1}$, such that

$$
\lim_{i \to +\infty} (\sin(n_{i1} \alpha_{k1} 2\pi), \cos(n_{i1} \alpha_{k1} 2\pi), \ldots, \sin(n_{is_k} \alpha_{ks_k} 2\pi), \cos(n_{is_k} \alpha_{ks_k} 2\pi))
$$

$$
= (x_1, y_1, \ldots, x_{s_k}, y_{s_k}).
$$
Lemma 8. Consider Lemma 7. The following lemma.

Let $\gamma_i = (n_{T,j} + j'')\alpha_{k_i} 2\pi$ (1 $\leq i \leq s_k$) and suppose that

$$C_{jk1l} = C_{jk1l}(X, \sin(\gamma_i), \cos(\gamma_i), \ldots, \sin(\gamma_{s_k}), \cos(\gamma_{s_k})).$$

Then

$$\inf_{n \geq 1} \{C_{jk1l}(X, \sin(\gamma_i), \cos(\gamma_i), \ldots, \sin(\gamma_{s_k}), \cos(\gamma_{s_k}))\}$$

is equal to $\min\{C_{jk1l}(X, x_1, y_1, \ldots, x_{s_k}, y_{s_k})\}$ subject to \{x_i^2 + y_i^2 = 1, 1 \leq i \leq s_k\}.

Proof. According to Lemma 5 and Lemma 6, it’s sufficient to prove that $T^{\alpha_k}$ is the closure of $\{(e^{\gamma_1}, \ldots, e^{\gamma_{s_k}})\}_{n \geq 1}$. Because $\{\alpha_1, \ldots, \alpha_{s_k}\}$ are rationally independent, $\{T_{j', \alpha_1}, \ldots, T_{j', \alpha_{s_k}}\}$ are rationally independent, too. Thus, $T^{\alpha_k}$ is the closure of $\{(e^{n_{T,j}' \alpha_1 2\pi}, \ldots, e^{n_{T,j}' \alpha_{s_k} 2\pi})\}_{n \geq 1}$. The result of rotating $$(e^{n_{T,j}' \alpha_1 2\pi}, \ldots, e^{n_{T,j}' \alpha_{s_k} 2\pi})$$

by $(j'' \alpha_{k_1} 2\pi, \ldots, j'' \alpha_{k_{s_k}} 2\pi)$ is $(e^{\gamma_1}, \ldots, e^{\gamma_{s_k}})$. Consequently, $T^{\alpha_k}$ is the closure of $\{(e^{\gamma_1}, \ldots, e^{\gamma_{s_k}})\}_{n \geq 1}$. That completes the proof.

3.7 Correctness

For each $j$, $P_j(X, n)$ can be written as

$$D_j10(X, n)r_1^n + D_j11(X, n)n_{r_1}r_1^n + \ldots + D_jl_{d_1}(X, n)n_{r_1}r_1^n + \ldots$$

$$D_jH0(X, n)r_2^n + D_jH1(X, n)n_{r_2}r_2^n + \ldots + D_jHd_1(X, n)n_{r_2}r_2^n.$$

The $D_{jkl}$’s in the above are real because $P_j(X, n) \in \mathbb{R}$ and those $n_{r_k}r_k$’s are of different orders. Just like $C_{jkl}$, $D_{jkl}$ can be divided into three parts,

$$D_{jkl} = D_{jkl0}(X) + D_{jkl1}(X, n) + D_{jk1l2}(X, n).$$

Because $D_{jkl0}(X)$ contains no $e^{(n_{T,j} + j'')\alpha_{k_1} 2\pi}$’s, the $e^{(n_{T,j} + j'')\alpha_{k_1} 2\pi}$’s contained in $D_{jkl1}(X, n)$ are periodic and the $e^{(n_{T,j} + j'')\alpha_{k_1} 2\pi}$’s contained in $D_{jk1l2}(X, n)$ are not, $D_{jkl0}, D_{jkl1}$ and $D_{jk1l2}$ are all real. Since $C_{jkl}(X, n)$ results from $D_{jkl}(X, n)$, we get the following lemma.

Lemma 7. For all $n \in \mathbb{N}$ and each $C_{jkl}(X, n) = C_{jkl0}(X) + C_{jkl1}(X) + C_{jk1l2}(X, n)$, $C_{jkl0}(X) \in \mathbb{R}$, $C_{jkl1}(X, n) \in \mathbb{R}$ and $C_{jk1l2}(X, n) \in \mathbb{R}$.

Lemma 8. Consider $C_{jkl}(X, n) = C_{jkl0}(X) + C_{jkl1}(X) + C_{jk1l2}(X, n)$.

1. $\inf_{n \geq 1} \{C_{jkl}(X, n)\} > 0$ if and only if

$$\min\{C_{jkl0}(X) + C_{jkl1}(X) + C_{jk1l2}(X, y_{11}, y_{12}, \ldots, y_{s_k}, y_{s_k})\} > 0$$

subject to \{y_{i1}^2 + y_{i2}^2 = 1, 1 \leq i \leq s_k\}.

2. If $I = C_{jkl0}(X) + C_{jkl1}(X) + D < 0$, then there is $c < 0$ such that $C_{jkl}(X, n) < c$ for infinitely many $n$’s, where

$$D = \min\{C_{jk1l2}(X, y_{11}, y_{12}, \ldots, y_{s_k}, y_{s_k})\}$$

subject to \{y_{i1}^2 + y_{i2}^2 = 1, 1 \leq i \leq s_k\}. 

Proof. 1. It has been proved that
\[
\inf_{n \geq 1} C_{jk}(X, n) = \min\{C_{jk}(X, y_{11}, \ldots, y_{s_k})\}
\]

subject to \(y_{11}^2 + y_{12}^2 = 1, 1 \leq t \leq s_k\). Consequently
\[
\inf_{n \geq 1} \{C_{jk}(X, n)\} = \min\{C_{jk}(X) + C_{jk}(X, n) + C_{jk}(X, y_{11}, \ldots, y_{s_k})\},
\]

subject to \(y_{11}^2 + y_{12}^2 = 1, 1 \leq t \leq s_k\).

2. Let \(Y = (y_{11}, y_{12}, \ldots, y_{s_k})\) and \(Y' = (y'_{11}, y'_{12}, \ldots, y'_{s_k})\). \(D\) can be attained because \(y_{11}^2 + y_{12}^2 = 1, 1 \leq t \leq s_k\) is a bounded closed set and \(C_{jk}(X, Y)\) is a continuous function of \(Y\). Assume that \(D = C_{jk}(X, Y')\). Since \(I < 0\) there exists a neighborhood of \(Y'\), say \(U\), such that
\[
\forall Y \in U \Rightarrow C_{jk}(X) + C_{jk}(X, Y) < I/2.
\]

Let \(c = I/2\) and \(\gamma_i = (nT_{ij} + j'' \alpha_k)2\pi\). Because of the density of
\[
\{(e^{\gamma_1}, \ldots, e^{\gamma_L})\}_{n \geq 1},
\]

there are infinitely many \(n\)'s such that \((\cos \gamma_1, \sin \gamma_1, \ldots, \cos \gamma_{s_k}, \sin \gamma_{s_k})\) lies in \(U\). Thus there are infinitely many \(n\)'s such that \(C_{jk}(X, n) < c\).

If the main algorithm, Termination, finds one solution, \(X_0\), the leading coefficient of \(G_j(X_0)\), say \(C_{jk}(X_0, n)\), satisfies \(C_{jk}(X_0, n) > 0\). According to the definition of \(\succ\) there exist \(c_j > 0\) \((j = 1, \ldots, L)\) such that \(C_{jk}(x, n) > c_j\) for all \(n\). Thus \(P_1\) is nonterminating. This means that if the algorithm outputs “nonterminating”, then \(P_1\) is nonterminating indeed.

On the other hand, if the main algorithm outputs “terminating”, then for any \(\{C_{jk}(X, n), j = 1, \ldots, L\}\) there is a subset \(V \subseteq \{1, \ldots, L\}\) such that
\[
\bigwedge_{j \in V} C_{jk}(X, n) \succ 0
\]
is not satisfiable subject to
\[
\bigwedge_{j \in V} C_{jk}(X, n) \succ 0 \land \bigwedge_{(k > k_j \lor k = k_j \land t > j)} C_{jk}(X, n) \geq 0
\]

According to the assumption for the main algorithm, we get that with the above constraints
\[
\forall j \in V, \inf_{n \geq 1} C_{jk}(X, n) \leq 0.
\]
Thus by Lemma [8] \(\forall j \in V, C_{jk}(X, n)\) is identically zero or there are infinitely many \(n\)'s and some \(c < 0\) such that \(C_{jk}(X, n) < c\). That means \(P_1\) is terminating. Therefore, we get the following theorem.

Theorem 4. Under the assumption of the main algorithm, Termination returns “terminating” if and only if \(P_1\) is terminating.
4 Conjecture

In this section, we shall discuss the general case of $P_1$ wherein our assumption for the main algorithm may not hold.

Suppose $p(X) = p(x_1, x_2, \ldots, x_m) \in \mathbb{Q}[X]$, and one of the loop conditions is $p(X)^2 > 0$. From the discussion in Section 3, we know that if we substitute $A^{n+m}X$ for $X$ in the conditions, there must be a polynomial $q$ such that the condition becomes $q(X, \sin(n\alpha_1 2\pi), \cos(n\alpha_1 2\pi), \ldots, \sin(n\alpha_m 2\pi), \cos(n\alpha_m 2\pi))^2 > 0$. Because $p$ is arbitrary, $q$ can be arbitrary. Further, $\alpha_1, \ldots, \alpha_m$ can be made rationally independent because $A$ can be arbitrary. It’s not hard to see that we can construct a program $Q$ such that it is terminating if and only if

$$S_{q, \alpha} \triangleq \{ n : q(\sin(n\alpha_1 2\pi), \cos(n\alpha_1 2\pi), \ldots, \sin(n\alpha_m 2\pi), \cos(n\alpha_m 2\pi)) = 0 \}$$

contains infinitely many elements.

For any $p(X) \in \mathbb{Q}[X]$ the decision problem “whether $\{ X : p(X) = 0 \} \cap \mathbb{Z}^{2m} = \emptyset$” is undecidable. $\mathbb{Z}^{2m}$ is a “regular” set while

$$E = \{ (\sin(n\alpha_1 2\pi), \cos(n\alpha_1 2\pi), \ldots, \sin(n\alpha_m 2\pi), \cos(n\alpha_m 2\pi)) \}_{n \geq 1}$$

is a chaotic set when $\{ \alpha_1, \ldots, \alpha_m \}$ are rationally independent according to the ergodic theory. Intuitively, deciding whether $S_{p, \alpha} = \{ X : p(X) = 0 \} \cap E = \emptyset$ is more difficult than deciding whether $\{ X : p(X) = 0 \} \cap \mathbb{Z}^{2m} = \emptyset$. So, we intuitively guess the decision problem “whether $S_{p, \alpha}$ is empty” is undecidable. Following the same idea, we guess the decision problem “whether $S_{p, \alpha}$ contains infinitely many elements” is much more difficult and thus undecidable. Thus, we make the following conjecture:

**Conjecture.** The decision problem “whether the loop $P_1$ is terminating over $\mathbb{R}$” is undecidable.

5 Conclusion

In this paper we have proved that termination of $P_1$ over $\mathbb{Z}$ is undecidable. Then we give a relatively complete algorithm, with an assumption, to determine whether $P_1$ is terminating over $\mathbb{R}$. If the assumption holds, $P_1$ is terminating iff our algorithm outputs “terminating”. If the assumption does not hold, $P_1$ is nonterminating if the algorithm outputs “nonterminating”. We demonstrate the main steps of our algorithm by an example. Finally we show how hard it is to determine the termination of $P_1$ by reducing its termination to the problem of “whether $S_{p, \alpha}$ has infinite many elements”. We conjecture the latter problem is undecidable. Thus, if our conjecture holds, the termination of $P_1$ over $\mathbb{R}$ is undecidable.

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