Charge-spin response and collective excitations in Weyl semimetals

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Weyl semimetals are characterized by unconventional electromagnetic response. We present analytical expressions for all components of the frequency- and wave-vector-dependent charge-spin linear-response tensor of Weyl fermions. The spin-momentum locking of the Weyl Hamiltonian leads to a coupling between charge and longitudinal spin fluctuations, while transverse spin fluctuations remain decoupled from the charge. Based on the response tensor, we investigate the low-energy collective excitations of interacting Weyl fermions. For a local Hubbard interaction, the charge-spin coupling leads to a dramatic change of the zero-sound dispersion: its velocity becomes independent of the interaction strength and the chemical potential and is given solely by the Fermi velocity. In the presence of long-range Coulomb interactions, the coupling transforms the plasmon modes into spin plasmons. For a model with two Weyl nodes, the collective modes are strongly affected by the presence of parallel static electric and magnetic fields, due to the chiral anomaly. In particular, their dispersion goes through a singularity as the chemical potential of one of the Weyl cones is tuned through the Weyl node. We discuss possible experiments that could provide smoking-gun evidence for Weyl physics.

I. INTRODUCTION

Within the large field of topological condensed matter physics, the study of Weyl semimetals (WSMs) [1–6] has received a strong boost by the discovery of several candidate materials [7–23]. In WSMs, non-degenerate bands touch at points in momentum space, called “Weyl nodes.” This requires time-reversal or spatial inversion symmetry to be broken since otherwise all bands would be spin degenerate. In magnetic WSMs such as the candidates Mn$_3$(Ge, Sn) [13, 17, 19, 20, 24, 25] and Co$_3$Sn$_2$S$_2$ [21, 22, 26] as well as (Gd, Nd)PtBi in a magnetic field [16, 18, 23], time-reversal symmetry (TRS) is broken. On the other hand, time-reversal-symmetric WSMs with broken inversion symmetry have been realized as well [7–12, 14, 15].

In WSMs, Weyl nodes act as sources and sinks of Berry curvature (of, say, the lower band), which is analogous to a magnetic field in momentum space. The corresponding magnetic monopole charge, or “chirality,” is given by the flux of the Berry curvature over a Fermi surface enclosing the node and is quantized to an integer value. Since the net monopole charge summed over all Weyl nodes in the Brillouin zone must be zero, Weyl nodes always appear in pairs of opposite chirality [27, 29]. Near each node, the band dispersion is linear and the effective Hamiltonian has the form of the well-known Weyl Hamiltonian [30, 31]. The corresponding massless low-energy quasiparticles, the Weyl fermions, exhibit a chiral or Adler-Bell-Jackiw anomaly [32–34], which means that, in the presence of parallel electric and magnetic fields, the number of fermions close to Weyl nodes of opposite chirality is not conserved separately. WSMs show a plethora of exotic optical and transport properties, some, but not all, of which are caused by the chiral anomaly. Examples are an anomalous Hall effect in WSMs with broken time-reversal symmetry [35]: a chiral magnetic effect, i.e., a dynamical current parallel to the magnetic field in WSMs with broken inversion symmetry [5, 36, 37]; and a negative magnetoresistance for the magnetic field parallel to the electric field [38]. Moreover, the chiral anomaly leads to the appearance of a term proportional to $E \cdot B$ in the electromagnetic action, with a non-uniform prefactor [39]. This term implies a magnetoelectric effect—an applied electric field generates a magnetization, while an applied magnetic field generates an electric polarization [2, 5, 6]. The magnetoelectric effect is encoded in the coupled charge-spin linear-response tensor, the frequency and wave-vector dependence of which has not been calculated completely so far.

In this paper, we investigate the response of Weyl fermions to time- and space-dependent electric and magnetic perturbations by calculating the composite charge-spin linear-response tensor in Sec. II. We analyze all its components for a single Weyl node and discuss the consequences for WSMs with a pair of Weyl nodes. In Sec. III we study the collective excitations of the Weyl fermions within the random phase approximation, exploring in particular the impact of the coupling between density and spin excitations. The chiral anomaly significantly affects the excitation modes, which can be probed by optical pump-probe experiments. Finally, we summarize our results in Sec. IV.

II. LINEAR RESPONSE

We start from an effective Hamiltonian describing the low-energy physics of Weyl fermions in the vicinity of a Weyl node residing at $Q$ in the Brillouin zone,

$$\mathcal{H} = \chi v_F k \cdot \sigma - \mu \chi,$$

(1)
where \( \mathbf{k} \) is the momentum relative to the Weyl node, \( \sigma \) is the vector of Pauli matrices representing the electron spin, \( \chi = \pm 1 \) denotes the chirality describing the relative orientation of spin and momentum, and \( \mu_\alpha \) is the chirality-dependent chemical potential. We set \( \hbar = 1 \) and assume isotropic Fermi velocity \( v_F \). However, our results are qualitatively insensitive to the anisotropy.

The first-order response to electromagnetic perturbations is described by a \( 4 \times 4 \) linear-response tensor, the components of which are determined by correlation functions,

\[
\Pi_{\alpha\alpha'}(q, i\omega_n) = \frac{1}{N} \int_0^\beta d\tau e^{i\omega_n\tau} (T_\tau \alpha(q, \tau)\alpha'(-q, 0)), \tag{2}
\]

where \( \alpha, \alpha' \in \{\rho, \sigma\} \) refer to the Fourier-transformed density and spin operators defined as \( \rho(q) = \sum_{\mathbf{k}\sigma} c_{\mathbf{k}+\mathbf{q}, \sigma} c_{\mathbf{k}, \sigma} \) and \( \sigma(q) = \sum_{\mathbf{k}\zeta\zeta'} c_{\mathbf{k}+\mathbf{q}, \zeta} c_{\mathbf{k}, \zeta'} \), respectively, in terms of the fermionic annihilation (creation) operators \( c_{\mathbf{k}\sigma} (c_{\mathbf{k}\sigma}^\dagger) \). \( \beta = 1/k_BT \) is the inverse temperature, \( i\omega_n \) are Matsubara frequencies, and \( T_\tau \) is the time-ordering directive in imaginary time. All response functions can be written as sums of interband (denoted by superscript \(-\)) and intraband (superscript \(+\)) contributions: \( \Pi_{\alpha\alpha'} = \Pi_{\alpha\alpha'}^+ + \Pi_{\alpha\alpha'}^- \). The retarded response functions are obtained by analytic continuation \( i\omega_n \to \omega + i\delta \).

### A. Separate charge and spin responses

The charge and the diagonal spin response functions have been calculated before. The retarded charge response functions \( \Pi_{\rho\rho}(q, \omega) \) were obtained by Lv and Zhang [40], see also Ref. [41]. At zero temperature they are given by

\[
\Pi_{\rho\rho}^\mp(q, \omega) = \frac{1}{2N} \sum_{\mathbf{k}} \left( \frac{1 \mp k' \cdot \mathbf{k}}{k' k} \right) \times \left[ \frac{1}{v_F(k \pm k')} - \omega + i\delta \right] \tag{3}
\]

where \( k' = k + q, k = |\mathbf{k}| \) etc. For completeness, the explicit form of the charge response tensor \( \Pi_{\rho\rho}(q, \omega) \) is presented in Appendix A. It only depends on the magnitude of the wave vector \( q \), as expected for the isotropic Hamiltonian [1].

The spin response tensors contain diagonal and off-diagonal components in spin space. The diagonal terms

\[
\Pi_{\sigma\sigma}(q, \omega) = \frac{1}{2N} \sum_{\mathbf{k}} \left( \frac{1 \pm k' k_m + k' k_n - k' k_l}{k' k} \right) \times \left[ \frac{1}{v_F(k \pm k')} - \omega + i\delta \right], \tag{4}
\]

where \( l, m, n \) refer to three orthogonal coordinate axes with \( \varepsilon_{lmn} = +1 \), have recently been evaluated by Thakur et al. [42]. On the other hand, the off-diagonal components read

\[
\Pi_{\sigma\sigma'}(q, \omega) = \frac{1}{2N} \sum_{\mathbf{k}} \left[ \frac{-ik' k_m + k' k_n - i(k'_l - k'_m) k_k}{v_F(k \pm k')} - i(k'_c \pm k'_m) \right]. \tag{5}
\]

Evidently, the spin response depends on both the magnitude and the direction of \( q \). We consider \( q \) along the positive \( z \)-axis, without loss of generality. The diagonal terms have long-dwavelength \( (\Pi_{\sigma\sigma}(q, \omega)) \) and transverse \( (\Pi_{\sigma\sigma'}(q, \omega) = \Pi_{\sigma\sigma'}^x(q, \omega)) \) components, which differ in the orientation between the wave vector and the spin. The imaginary parts of the longitudinal and transverse diagonal spin responses are related to the charge response by multiplicative factors [42]. For the real parts, the situation is more complicated since they contain the unphysical cutoff-dependent term \( \Lambda^2/6\pi^2 v_F \), where \( \Lambda \) is the ultraviolet cutoff for the momentum sum [42, 43]. This term arises from the presence of the infinite sea of negative-energy states for the effective Weyl Hamiltonian. The spin response function should be regularized by taking the \( \mu = 0 \) ground state as the reference system [42, 43, 46], which corresponds to subtracting the \( \Lambda^2 \) term from the spin response tensor. Details of the calculation are given in Appendix B where we also reproduce the explicit results for the diagonal components [42].

Among the off-diagonal terms, only the transverse components \( \Pi_{\sigma\sigma'}^x(q, \omega) \) differ from zero. Hence, the transverse and longitudinal components of the spin response decouple, which can be attributed to the rotational invariance of the model about the wave vector \( q \). By rewriting the momentum sum in Eq. (5) as an integral and performing the angular part, we obtain

\[
\Pi_{\sigma\sigma'}^x(q, \omega) = \frac{i}{16\pi^2 q^2} \int dk \int dk' \left( k \mp k' \right) \times \left[ q^2 - (k \pm k')^2 \right] \frac{1}{v_F(k \pm k') - \omega + i\delta} \tag{6}
\]

where \( \omega \to -\omega \). The integrals can be evaluated explicitly. The results for general doping, which to our knowledge have not been reported previously, are given in Appendix B. In the undoped case, the response function becomes simple and purely imaginary:

\[
\Pi_{\sigma\sigma'}^m(q, \omega) = i \frac{q \omega}{24\pi^2 v_F^2}. \tag{7}
\]

The usually considered off-diagonal response function \( \Pi_{\sigma\sigma'}^- \) with \( \sigma_+ = \sigma_+ \pm i\sigma_y \) is related to the calculated components by

\[
\Pi_{\sigma\sigma'}^- (q, \omega) = 2i \Pi_{\sigma\sigma'}^x (q, \omega) - 2 \Pi_{\sigma\sigma'}^m (q, \omega), \tag{8}
\]

where symmetries have been used. Hence, the real part of \( \Pi_{\sigma\sigma'}^- \) is related to the imaginary part of \( \Pi_{\sigma\sigma'}^x \).
Re \( \Pi \) Weyl cone: wave-vector and frequency dependence of (a) \( \rho_x \rho_y \) contribution can be decomposed into two parts, an extrinsic one which is nonzero already in the undoped case and an intrinsic one that only emerges upon doping. For the intrinsic case, the Fermi level lies at the Weyl node, and only interband transitions from the completely filled valance band to the empty conduction band contribute. The imaginary and real parts of the intrinsic charge-spin response are given by

\[
\text{Im} \Pi_{\rho_{x} \rho_{y}}(q \hat{z}, \omega) = \frac{q \omega}{24 \pi \nu_F^2} \ln \left| \frac{4 \nu_F^2 \Lambda^2}{\nu_F^2 \omega^2 - \omega^2} \right|,
\]

\[
\text{Re} \Pi_{\rho_{x} \rho_{y}}(q \hat{z}, \omega) = \frac{q \omega}{24 \pi \nu_F^2} \theta(\omega - \nu_F q),
\]

where \( \theta(x) \) is the Heaviside step function. In the presence of doping, the charge-spin response depends only
on the magnitude of the chemical potential $\mu$, due to the particle-hole symmetry of the Weyl Hamiltonian. For electron doping, $\mu = v_F k_F > 0$, the imaginary and real parts of the extrinsic contribution read

\[
\text{Im} \Pi_{\rho\sigma z}^{ex}(q, \omega) = \frac{\omega}{8\pi q v_F} \left[ \theta(v_F q - \omega) \left[ \alpha(q, \omega) - \alpha(q, -\omega) \right] \theta(2\mu - v_F q - \omega) + \alpha(q, \omega) \theta(2\mu - v_F q + \omega) \theta(v_F q + \omega - 2\mu) \right] + \theta(\omega - v_F q) \left[ -\alpha(-q, -\omega) \theta(2\mu + v_F q - \omega) \theta(v_F q + \omega - 2\mu) \frac{q^2}{3} \theta(2\mu - v_F q - \omega) - \alpha(q, -\omega) \beta(q, -\omega) - \alpha(-q, -\omega) \beta(-q, -\omega) \right],
\]

(12)

\[
\text{Re} \Pi_{\rho\sigma z}^{ex}(q, \omega) = \frac{\omega}{8\pi^2 q v_F^3} \left[ 8\mu^2 - \alpha(q, \omega) \beta(q, \omega) - \alpha(-q, \omega) \beta(-q, \omega) - \alpha(q, -\omega) \beta(q, -\omega) - \alpha(-q, -\omega) \beta(-q, -\omega) \right],
\]

(13)

where we have defined

\[
\alpha(q, \omega) = \frac{1}{12v_F q} \left( (2\mu + \omega)^3 - 3v_F^2 q^2 (2\mu + \omega) + 2v_F^3 q^3 \right)
\]

and

\[
\beta(q, \omega) = \ln \left| \frac{2\mu + \omega - v_F q}{v_F q - \omega} \right|.
\]

(14)

(15)

The full response function in the doped case is of course $\Pi_{\rho\sigma z} = \Pi_{\rho\sigma z}^n + \Pi_{\rho\sigma z}^{ex}$. This response function satisfies

\[
\Pi_{\rho\sigma z}(q, \omega) = \frac{\omega}{v_F q} \Pi_{\rho\rho}(q, \omega),
\]

(16)

which relates it to the charge response.

The wave-vector and frequency dependence of the longitudinal charge-spin response function $\Pi_{\rho\sigma z}$ of a single Weyl cone with $\chi = +1$ is presented in Fig. 2 for the undoped and doped cases. In Fig. 3 we plot the same data for cuts at fixed wave numbers. For the undoped case, when the valence band is completely occupied and the conduction band is empty, the imaginary part shows a step at $\omega = v_F q$, indicating the onset of interband particle-hole excitations (the finite slope of the step arises from the small imaginary part $\delta = 0.005$ used for the calculations). The real part depends linearly on $q$ and $\omega$ for small wave number or frequency. By dint of the Kramers-Kronig relations, the step in the imaginary part implies a logarithmic divergence (rounded by $\delta > 0$) at $\omega = v_F q$ in the real part. Upon doping, a nonzero imaginary part appears for $v_F q - 2\mu < \omega < v_F q$, resulting from intraband particle-hole excitations, while interband transitions are Pauli blocked for $v_F q < \omega < 2\mu - v_F q$. Consequently, the step in the imaginary part and the peak in the real part become inverted and the real part can even change sign. Note that this does not imply a thermodynamic instability since the response of the system to external perturbations is governed by the full response tensor $\Pi_{\alpha\alpha'}$; an instability would be signaled by a negative real part of an eigenvalue.

The $4 \times 4$ response tensor $\Pi_{\alpha\alpha'}$ is related to the $6 \times 6$
electromagnetic susceptibility by \[^{51}\]

\[
\chi_{ij}^{ee} = \frac{\partial P_i}{\partial E_j} = -\frac{e^2}{q_i q_j} \pi_{\rho\mu}, \tag{17}
\]

\[
\chi_{ij}^{mm} = \frac{\partial M_i}{\partial B_j} = \left(\frac{g_B \mu_B}{2}\right)^2 \pi_{\sigma_i \sigma_j}, \tag{18}
\]

\[
\chi_{ij}^{em} = \frac{\partial P_i}{\partial B_j} = i \frac{e g_B \mu_B}{2} \frac{1}{q_i} \pi_{\rho \sigma_j}, \tag{19}
\]

\[
\chi_{ij}^{me} = \frac{\partial M_i}{\partial E_j} = i \frac{e g_B \mu_B}{2} \frac{1}{q_j} \pi_{\sigma_i \rho}, \tag{20}
\]

where we have taken the electron charge to be \(-e\) and its magnetic moment to be \(-g\mu_B/2\). The charge-spin response tensor thus implies a magnetization response to an electric field and a polarization response to an electric field, as expected from the topological \(E \cdot B\) term in the electromagnetic action \[^{39}\]. Moreover, the fact that only the longitudinal charge-spin response is nonzero is consistent with the longitudinal nature of this term.

The charge-spin response originates from the coupling of spin and momentum. Consequently, it changes sign for a Weyl cone with opposite chirality, \(\chi = -1\), in Eq. \(^1\). Hence, the coupled response vanishes for a WSM with a pair of Weyl cones described by Eq. \(^1\) with \(\chi = \pm 1\), as long as the chirality-dependent chemical potentials \(\mu_\chi\) are equal. One could thus expect that WSMs with two Weyl nodes at the same energy cannot show charge-spin coupling in equilibrium. However, this is not true—a Weyl cone with opposite chirality can only be generated by changing the overall sign of the Weyl Hamiltonian but also by flipping the sign of the dispersion in only one direction. This is exemplified by the Hamiltonian \[^{48, 52, 53}\]

\[
H = -\left[m(2 - \cos k_x - \cos k_y) + 2t(\cos k_z - \cos k_0)\right] \sigma_z - 2t \sin k_x \sigma_x - 2t \sin k_y \sigma_y, \tag{21}
\]

which has a pair of Weyl nodes with opposite chirality at \((0, 0, \pm k_0)\). The corresponding Fermi-velocity tensors are diag\((-2t, -2t, \pm 2t \sin k_0\)). The Weyl points are thus anisotropic.

The numerical calculation of the separate charge and spin responses for this model yields qualitatively the same behavior as discussed in Sec. \(\Pi A\) (not shown). However, for the coupled charge-spin response, the broken rotational symmetry plays a key role. The longitudinal response \(\Pi_{\rho\sigma}\) for \(q = qz\) vanishes due to opposite contributions from the two nodes. But \(\Pi_{\rho\sigma}\) for \(q = q\hat{x}\) and \(\Pi_{\rho\sigma}\) for \(q = q\hat{y}\) contain equal contributions from the two nodes and hence yield similar behavior as shown in Fig. \[^3\] For general orientation of \(q\), the coupled charge-spin response is only nonzero for the spin direction given by the projection of \(q\) into the \(xy\) plane. If \(q\) is rotated towards the \(z\)-axis, the absolute value of the response function decreases smoothly with the polar angle \(\theta_q\) of \(q\). The anisotropy has important implications for the collective excitations, as discussed in the following section.

III. COLLECTIVE EXCITATIONS

Having investigated the linear charge-spin response of free Weyl fermions, we now turn to the collective excitations of a “Weyl liquid” in the presence of interactions. The interacting response functions are calculated within the random phase approximation (RPA). Since the transverse charge-spin response vanishes, the \(4 \times 4\) RPA response tensor decomposes into two \(2 \times 2\) blocks describing (i) the coupled charge and longitudinal spin response and (ii) the transverse spin response. Our interest is in the former part. We use a charge-spin basis, in which the coupled response takes the form

\[
\tilde{\Pi} = \begin{pmatrix}
\Pi_{\rho\rho} & \Pi_{\rho\sigma_z} \\
\Pi_{\sigma_z\rho} & \Pi_{\sigma_z\sigma_z}
\end{pmatrix}. \tag{22}
\]

We first consider the consequences of an on-site interaction and then of long-range Coulomb repulsion. These two cases are understood as the extreme limits of the screened Coulomb interaction. Details of the calculations are relegated to Appendix \[^D\].

A. On-site Hubbard interaction

For an ordinary Fermi liquid with on-site Hubbard repulsion, the collective excitations in the low-temperature, collisionless regime consist of gapless zero-sound modes. We here consider a WSM with Hubbard repulsion of strength \(U > 0\) in the collisionless regime, taking \(T =\)
In the presence of charge-spin coupling, the zero sound mixes with the longitudinal spin waves and the dispersion changes dramatically: it becomes lightlike, propagating with the Fermi velocity, as shown by the blue line in Fig. 4. The dispersion of this coupled zero-sound mode is critical in the sense of having the minimum possible velocity for undamped excitations. Its velocity does not depend on the interaction strength or on the chemical potential, although the linewidth does. The collective modes can be clearly seen in the imaginary part of the RPA response function, Im Tr $\hat{\Pi}_{RPA}(q, \omega)$, which is plotted in Fig. 5. Besides the zero-sound branch, which is strongly modified by charge-spin coupling, we observe a gapped branch in Fig. 5(a), which stems from pure spin fluctuations. The charge-spin coupling leads to its mixing with charge modes in Fig. 5(b) but the dispersion remains qualitatively unchanged. The linearly dispersing branch at low frequencies in Fig. 5(a) also consists of spin fluctuations. They are resonant with the intraband continuum and thus damped. In Fig. 5(b), this branch is completely removed by the mixing with the charge modes.

What happens in a WSM with pairs of Weyl nodes of opposite chirality? For the lattice model of a TRS-breaking WSM given by Eq. (21), we find that the longitudinal charge-spin response for $q$ along the $z$-direction changes sign for the two nodes while this does not happen for $q$ in the $xy$ plane. Consequently, the charge-spin response $\Pi_{\sigma z}(q^2, \omega)$ vanishes and the zero-sound mode along the $z$-direction is described only by the charge response. In contrast, the zero sound in the $xy$ plane is described by the collective excitations of density and spin. Evidently, the collective modes are highly anisotropic—in the long-wavelength limit, the velocity in the $z$-direction depends on microscopic parameters such as the interaction strength and the charge density, whereas in the $xy$ plane it is given by the Fermi velocity. For general directions of $q$, the charge-spin coupling is continuously
Weyl cones, \( \mu > 0 \). If the charge-spin response has opposite sign for the two Weyl nodes, the zero sound is determined only by charge density fluctuations. As \( \nu \) becomes nonzero by application of parallel static \( E \) and \( B \) fields, \( \mu^+ \) shifts upward while \( \mu^- \) shifts downward. Consequently, the zero sound obtains contributions from spin fluctuations and is redshifted (dashed green line in Fig. [6]). At \( \nu = 1 \), \( \mu^- \) crosses a Weyl node, which can be understood as a Lifshitz transition. At this transition, spin fluctuations have maximal contribution to the zero-sound wave and it becomes lightlike with velocity \( v_F \). Beyond \( \nu = 1 \), when \( \mu^- \) lies in the valance band, the frequency increases again. On the other hand, if the charge-spin response has the same sign for the two nodes, the zero sound is not affected by \( \nu \) and remains lightlike (solid purple line).

For the lattice model of Eq. (21), the zero sound along the \( z \)-direction is redshifted when the \( E \) and \( B \) fields are applied along the \( z \)-direction, while the sound in the \( xy \) plane is not affected. At \( \nu = 1 \), sound modes in all directions become degenerate and lightlike. Beyond \( \nu = 1 \), the zero sound along \( z \) is blueshifted again. Therefore, the zero sound parallel to \( E \) and \( B \) can be taken as a signature of the chiral anomaly, with a dip to an isotropic frequency indicating the Lifshitz transition point.

B. Long-range Coulomb interaction

Now we consider Weyl fermions with a long-range Coulomb interaction. The RPA response is described by

\[
\Pi_{\text{RPA}}(\mathbf{q}, \omega) = \Pi(q, \omega) \left[ 1 + \frac{V(q)}{2} (1 + \tau_z) \hat{\Pi}(q, \omega) \right]^{-1}
\]

in the charge-spin basis. Here, \( V(q) = 4\pi e^2/\kappa q^2 \) is the Fourier-transformed Coulomb interaction, with \( \kappa \) being the dielectric constant. The difference in the interaction vertex compared to Eq. (23) arises from the fact that the Coulomb interaction only acts in the charge channel in the RPA, whereas the Hubbard interaction can be decomposed into charge and spin channels. Using that \( \Pi_{pp, \sigma_\sigma} - \Pi_{p\sigma_z, \sigma_z} = 0 \), the response function can be rewritten as

\[
\Pi_{\text{RPA}}(\mathbf{q}, \omega) = \frac{1}{1 + V(q) \Pi_{pp}(\mathbf{q}, \omega)} \hat{\Pi}(\mathbf{q}, \omega).
\]

Thus, similarly to the ordinary Fermi liquid and to the two-dimensional Dirac liquid, the zero sound morphs into plasmonic modes in the presence of long-range Coulomb interaction. Interestingly, unlike for a local interaction, all components of the charge-spin response tensor are uniformly enhanced at the RPA level and only the charge response governs this enhancement. The plasmon dispersion is thus given by the zeros of the RPA dielectric function

\[
\epsilon_{\text{RPA}}(\mathbf{q}, \omega) = 1 + V(q) \Pi_{pp}(\mathbf{q}, \omega),
\]
which also only depends on the charge response. The dispersion is thus the same as when charge-spin coupling is ignored, as in Ref. [41]. The dispersion can be obtained in the long-wavelength limit, keeping only the leading order in \( q \) and is given by [41]

\[
\omega_\rho(q) = \omega_0 \left( 1 - \frac{v_F^2 q^2}{8 \mu^2} \left[ 1 + \frac{\nu_0^2}{\nu_0^2 (1 - \nu_0^2/2)^2} \right] \right),
\]

(30)

where \( \nu_0 = \omega_0/2\mu \) and

\[
\omega_0 = \mu \sqrt{\frac{2 \alpha_\kappa}{3\pi \kappa^* (\omega_0)}}
\]

(31)

is the plasmon frequency at \( q \to 0 \). It is determined by \( \alpha_\kappa = e^2/\kappa v_F \) and the frequency-dependent effective background dielectric function

\[
\kappa^*(\omega) = 1 + \frac{\alpha_\kappa}{6\pi} \ln \left| \frac{4v_F^2 \Lambda^2}{4\mu^2 - \omega^2} \right|.
\]

(32)

The plasmons are manifested as sharp peaks in the electron energy-loss function

\[
\mathcal{E}_{\text{loss}}(q, \omega) = -\text{Im} \frac{1}{\epsilon_{\text{RPA}}(q, \omega)},
\]

(33)

which can be probed for example by electron energy-loss spectroscopy. Figure 7 shows \( \mathcal{E}_{\text{loss}} \) in the \((q, \omega)\) plane for a single Weyl cone. The plasmon dispersion calculated from Eq. (29) is shown as the red line, which agrees well with the position of the peak in \( \mathcal{E}_{\text{loss}} \). The nonzero loss below the line \( \omega = v_F q \) stems from intraband particle-hole excitations. We see that the plasmon dispersion is gapped in the Weyl liquid, similarly to the ordinary Fermi liquid.

The fundamental difference between the collective modes in the Weyl liquid and in a normal Fermi liquid is that plasmons in the former carry spin, and hence can be called spin plasmons. This is a signature of spin-orbital locking. A schematic representation of the spin plasmon is shown in Fig. 8. A density fluctuation is accompanied by a 90° out-of-phase longitudinal spin fluctuation. This phase shift and the amplitudes \( \Delta \rho \) and \( \Delta \sigma_z \) of the charge and spin fluctuations, respectively, can be calculated from the eigenvectors of the response tensor. They are plotted in the inset in Fig. 7. We find that the charge amplitude is linear in \( q \) in the long-wavelength limit. Therefore, the spin fluctuation associated with spin plasmon becomes dominant for long wavelengths, and for \( q \to 0 \) the spin plasmon becomes a pure spin excitation.

Spin plasmons have also been proposed for the two-dimensional helical liquid formed by the surface states of a topological insulator [47]. The main differences to our case are that (i) the spin plasmon of the topological insulator is a surface plasmon, whereas we are considering bulk excitations and (ii) the spin plasmon of the helical surface liquid carries transverse spin fluctuations, whereas the WSM spin plasmon is associated with longitudinal spin fluctuations.

For WSMs with pairs of opposite-chirality nodes, the
charge-spin coupling has remarkable consequences for the plasmon excitations. When charge-spin coupling has opposite sign for the two nodes with opposite chirality, the propagating plasmons do not carry any spin polarization for $E \cdot B = 0$. For $E \cdot B \neq 0$, the two Weyl cones have different populations, which results in the generation of a spin polarization by density fluctuations. The effect is shown in Fig. 9 as a function of charge pumping, denoted by $\nu$, see Eq. (26). When charge pumping is turned on by parallel electric and magnetic fields, the spin-plasmon frequency starts to decrease slightly, reaches a minimum at the Lifshitz transition point $\nu = 1$, and then increases again [21]. On the other hand, the spin amplitude associated with the spin plasmon initially increases with $\nu$. At $\nu = 1$, the chemical potential is at the $\chi = -1$ Weyl node and the charge-spin coupling reaches its maximum, resulting in the maximum amplitude of the spin-wave component. Beyond $\nu = 1$, one of the nodes becomes hole doped and the spin-wave amplitude decreases again. In case of the lattice model in Eq. (21), plasmons propagating along the $z$-direction exhibit similar behavior when parallel $E$ and $B$ fields are applied along $z$. However, plasmons propagating in the $xy$ plane always have spin-wave component whose amplitude is insensitive to $\nu$. Therefore, the spin wave associated with the collective mode can be used as a fingerprint of the position of the Weyl node.

It should be possible to detect the coupled spin-charge collective modes using optical pump-probe spectroscopy. In the following, we call the excitations “spin plasmons” but the discussion remains valid in the limit of short-range interactions. The basic idea is to generate a propagating longitudinal spin wave and to detect the density wave coupled to it. We consider the TRS-breaking WSM as before, which has two Weyl nodes displaced along the $z$-direction. An intense circularly polarized light pulse propagating along the $x$-direction is incident on a $y-z$ surface of the sample. The pulse generates a spin polarization, which is localized at the surface, with a typical length scale given by the optical penetration depth. This spin-polarization pattern can be decomposed into plane waves $e^{iqx}$. These pure spin waves dress with charge fluctuations to form spin plasmons, which then propagate through the bulk of the sample with group velocity $d\omega_{pl}/dq$. The charge fluctuation associated with the spin plasmons could be detected at the opposite surface for example by means of the energy-gain (anti-Stokes) lines in inelastic light scattering. The dispersion $\omega_{pl}(q)$ leads to a characteristic temporal distribution of the charge fluctuations at the probe surface, which could be accessed by varying the delay between pump and probe pulses.

On the other hand, if the pump probe is incident on an $xy$ surface, a spin polarization is also generated but does not couple to the plasmon, which has $\mathbf{q} = q\hat{z}$, and hence we predict a null result at the opposite surface. However, if parallel electric and magnetic fields are applied along $z$, charge-spin mixing is activated and a nonzero charge fluctuation at the opposite surface is expected. It becomes strongest when the chemical potential for one chirality reaches the Weyl node.

Our scheme is quite different from the one proposed by Raghu et al. [17] for the helical surface states of a topological insulator. There, a transverse spin wave with given wave vector is coherently excited by the incident light. This approach does not work in our case since the spin wave needs to be longitudinal to couple to the plasmon.

IV. SUMMARY AND CONCLUSIONS

To summarize, we have investigated the electric and magnetic response of Weyl fermions by calculating all terms of the $4 \times 4$ charge-spin linear-response tensor for Weyl nodes with arbitrary doping. We have found that the charge and the longitudinal (i.e., parallel to the vector) spin responses are strongly coupled because of the spin-momentum locking in the Weyl Hamiltonian. The transverse spin response does not couple to the charge and longitudinal spin response to linear order.

With the full response tensor in hand, we have examined the collective excitations of Weyl fermions in the presence of electron-electron interaction. For a local, Hubbard-type interaction, we obtain zero sound with a constant velocity given by the Fermi velocity $v_F$ of the Weyl fermions, independently of the interaction strength $U$ or the chemical potential $\mu$. This is a dramatic change compared to the result when charge-spin coupling is ignored—in that case the dispersion is linear only in the long-wavelength limit and the velocity strongly depends on $U$ and $\mu$. The charge-spin coupling also causes the zero sound modes to have mixed charge and longitudinal spin character, and so gives rise to the possibility to “hear” the spin fluctuations, as proposed earlier for surface states of topological insulators [17].

On the other hand, for a long-range Coulomb interaction, we have found a propagating spin plasmon, which also consists of charge and longitudinal spin fluctuations. While the character of the collective plasma modes is thus strongly affected by charge-spin coupling, their dispersion is still determined by the zeros of the dielectric function, which only depends on the charge response.

Using a tight-binding model for a WSM that breaks TRS and has two Weyl nodes, we show that the collective modes are strongly influenced by the chiral anomaly. For a local interaction, the zero sound is typically strongly anisotropic but becomes isotropic exactly when the chemical potential of one of the Weyl cones is tuned to the Weyl point by parallel static electric and magnetic fields. On the other hand, for the Coulomb interaction, we have proposed a pump-probe experiment involving the spin plasmons that gives a null result in the absence of parallel electric and magnetic fields and yields the largest signal again when the chemical potential of one of the Weyl cones is tuned to the Weyl point. These signatures thus have the potential to lead to smoking-gun experi-
ments for the chiral anomaly and the presence of Weyl nodes. As another possible direction for future research, the tunability of the spin plasmons by means of the chiral anomaly suggests the combination of plasmonics and spintronics in Weyl systems.

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Appendix A: Charge response

In this appendix, we reproduce the explicit form of the charge response tensor \[^{40,41}\] for completeness. The charge response can be written as the sum of a contribution from the undoped system ("intrinsic") and a contribution due to doping ("extrinsic"),

\[ \Pi_{pp}(q,\omega) = \Pi_{pp}^{in}(q,\omega) + \Pi_{pp}^{ex}(q,\omega). \]  

(A1)

The imaginary and real parts of the intrinsic contribution are given by

\[ \text{Im} \Pi_{pp}^{in}(q,\omega) = \frac{q^2}{24\pi v_F} \theta(\omega - v_F q), \]  

(A2)

\[ \text{Re} \Pi_{pp}^{in}(q,\omega) = \frac{q^2}{24\pi^2 v_F} \ln \left| \frac{4v_F^2\Lambda^2}{v_F^2 q^2 - \omega^2} \right|, \]

respectively, where \( \theta(x) \) is the Heaviside step function. For electron doping, \( \mu = v_F k_F > 0 \), the imaginary and real parts of the extrinsic contribution read

\[ \text{Im} \Pi_{pp}^{ex}(q,\omega) = \frac{1}{8\pi^2 v_F} \left[ \theta(v_F q - \omega) [\alpha(q,\omega) - \alpha(q,-\omega)] \theta(2\mu - v_F q - \omega) + \alpha(q,\omega) \theta(2\mu - v_F q + \omega - 2\mu) \right. \]

\[ + \left. \theta(\omega - v_F q) [\alpha(-q,-\omega) \theta(2\mu + v_F q - \omega) \theta(v_F q + \omega - 2\mu) - \frac{q^2}{3} \theta(2\mu - v_F q - \omega)] \right], \]

(A4)

\[ \text{Re} \Pi_{pp}^{ex}(q,\omega) = \frac{1}{8\pi^2 v_F} \left[ \frac{8\mu^2}{3v_F^2} - \alpha(q,\omega) \beta(q,\omega) - \alpha(-q,\omega) \beta(-q,\omega) - \alpha(q,-\omega) \beta(q,-\omega) - \alpha(-q,-\omega) \beta(-q,-\omega) \right], \]

(A5)

where we have defined

\[ \alpha(q,\omega) = \frac{1}{12v_F^2 q} [(2\mu + \omega)^3 - 3v_F^2 q^2 (2\mu + \omega) + 2v_F^3 q^3] \]

(A6)

and

\[ \beta(q,\omega) = \ln \left| \frac{2\mu + \omega - v_F q}{v_F q - \omega} \right|. \]

(A7)

Appendix B: Spin response

Here, we discuss the calculation of the 3 \( \times \) 3 spin response tensor. The components can be written as

\[ \Pi_{\sigma_1\sigma_m}(q, i\omega_n) = -\frac{1}{N} \sum_k \sum_{\lambda,\lambda'} \langle \phi_{\lambda'}(k + q) | \sigma_1 | \phi_{\lambda}(k) \rangle \langle \phi_{\lambda}(k) | \sigma_m | \phi_{\lambda'}(k + q) \rangle \left( n_{\lambda'}^F(k) - n_{\lambda'}^F(k + q) \right) / (i\omega_n + \epsilon_{\lambda}(k) - \epsilon_{\lambda'}(k + q)), \]

(B1)

where \( \lambda, \lambda' = \pm \) refer to the two bands with energies \( \epsilon_{\pm} = \pm v_F |k| \) and eigenvectors (periodic parts of Bloch states) given by

\[ |\phi_{ \pm } (k) \rangle = \left( \begin{array}{c} \cos \frac{\theta_k}{2} e^{-i\varphi_k/2} \\ \sin \frac{\theta_k}{2} e^{i\varphi_k/2} \end{array} \right), \]

\[ |\phi_{ - } (k) \rangle = \left( \begin{array}{c} \sin \frac{\theta_k}{2} e^{-i\varphi_k/2} \\ -\cos \frac{\theta_k}{2} e^{i\varphi_k/2} \end{array} \right), \]

(B2)
where $\theta_k$ and $\varphi_k$ are the polar and azimuthal angle of $k$, respectively. $n_k^\lambda$ is the Fermi-Dirac distribution function for the band $\lambda$, which becomes a step function for the zero-temperature limit considered here.

The response tensor has diagonal $(i = j)$ and off-diagonal $(i \neq j)$ terms. The interband and intraband contributions to diagonal terms can be simplified to [42]

$$
\Pi_{\sigma_\sigma}^\tau(q, \omega) = \frac{1}{2N} \sum_k \left( 1 + \frac{k_m'k_m + k_n'k_n - k_l'k_l}{k'k} \right) \left[ \frac{1}{v_F(k \pm k') - \omega - i\delta} + \frac{1}{v_F(k \pm k') + \omega + i\delta} \right],
$$

where again $k' = k + q$ and $l, m, n$ refer to three orthogonal coordinate axes with $\varepsilon_{lmn} = +1$.

Taking $q = q\hat{z}$, the diagonal components can be categorized into longitudinal and transverse responses. After rewriting the momentum sum as an integral and performing the angular part, the longitudinal part becomes

$$
\Pi_{\sigma_\sigma}^\tau(q\hat{z}, \omega) = \frac{1}{16\pi^2q^3} \int dk \int dk' (k \pm k')^2 \left[ \frac{1}{v_F(k \pm k') - \omega - i\delta} + (\omega \rightarrow -\omega) \right],
$$

while the transverse response is described by

$$
\Pi_{\sigma_\sigma}^\tau(q\hat{z}, \omega) = \Pi_{\sigma_\sigma}^\tau(q\hat{z}, \omega)
$$

Here, the limits of the integrals over $k$ and $k'$ depend on the chemical potential in such a way that only transitions from occupied to empty states are included. The integrals also depend on an ultraviolet cutoff, which is necessary since the Hamiltonian in Eq. (1) describes an infinite sea of negative-energy states [46, 63].

The diagonal spin responses for a single Weyl cone were calculated by Thakur et al. [42]. The real parts contain a term $\Lambda^2/6\pi^2 v_F$, which depends on the ultraviolet cutoff $\Lambda$ [42, 43, 45]. The spin response function is regularized by taking the $\mu = 0$ ground state as the reference [42, 43, 46], which amounts to subtracting the $\Lambda^2$ term.

The result for the intrinsic (undoped) contributions can be expressed in terms of the charge response as

$$
\Pi_{\sigma_\sigma}^{in}(q\hat{z}, \omega) = \frac{\omega^2}{v_Fq^2} \Pi_{\rho\rho}^{in}(q\hat{z}, \omega),
$$

$$
\Pi_{\sigma_\sigma}^{in}(q\hat{z}, \omega) = \frac{\omega^2 - v_Fq^2}{v_Fq^2} \Pi_{\rho\rho}^{in}(q\hat{z}, \omega).
$$

However, for the extrinsic (doping) part, the relationship between the spin and charge response is only valid for the longitudinal response function,

$$
\Pi_{\sigma_\sigma}^{ex}(q\hat{z}, \omega) = \frac{\omega^2 - v_Fq^2}{v_Fq^2} \Pi_{\rho\rho}^{ex}(q\hat{z}, \omega),
$$

whereas the transverse spin response does not satisfy a simple relation to the charge response [42]. The imaginary part of the extrinsic transverse response function reads [42]

$$
\text{Im}\, \Pi_{\sigma_\sigma}^{ex}(q\hat{z}, \omega) = \frac{\omega^2 - v_Fq^2}{32\pi^2 v_F^4 q^3} \left[ \theta(v_Fq - \omega) \left| \gamma(q, \omega) - \gamma(q, -\omega) \right| \theta(2\mu - v_Fq - \omega)
+ \gamma(q, \omega) \theta(2\mu - v_Fq + \omega) \theta(v_Fq + \omega - 2\mu)
+ \theta(\omega - v_Fq) \left( \gamma(-q, -\omega) \theta(2\mu + v_Fq - \omega) \theta(v_Fq + \omega - 2\mu) + \frac{4q^2}{3} \theta(2\mu - v_Fq - \omega) \right) \right],
$$

where the function $\gamma(q, \omega)$ is defined as

$$
\gamma(q, \omega) = 2q \alpha(q, \omega) + q^2 (2\mu - v_Fq + \omega),
$$

with $\alpha(q, \omega)$ is defined in Eq. A6. The real part is given by [42]

$$
\text{Re}\, \Pi_{\sigma_\sigma}^{ex}(q\hat{z}, \omega) = -\frac{\omega^2 - v_Fq^2}{2v_F^2 q^2} \Pi_{\rho\rho}^{ex}(q\hat{z}, \omega) - \frac{\omega^2 - v_Fq^2}{16\pi^2 v_F^4 q^3} \sum_{n=\pm1} P \int_{k_q}^{k_{k-\eta}} dk P \int_{k_{k+\eta}}^{k_{k+q}} dk' \left[ \frac{v_F}{v_Fk' + \eta v_Fk + \omega} + (\omega \rightarrow -\omega) \right]
- \frac{1}{16\pi^2 v_F^4 q^3} \sum_{n=\pm1} \int_{k_q}^{k_{k-\eta}} dk \int_{k_{k+q}}^{k_{k+q}} dk' (k' - \eta k) \left[ (k' + \eta k)^2 + q^2 \right].
$$
We have re-evaluated the integrals since Eq. (D4) in Ref. [42] contains an ambiguous factor 0/0 for $c = 0$ and is incorrect if one naively cancels $c$ before setting $c = 0$. Our result reads

$$
\text{Re} \Pi_{x\sigma \sigma'}^\text{ex}(q \hat{z}, \omega) = -\frac{\omega^2 - v_F^2 q^2}{2v_F^2 q^2} \Pi_{x\rho\rho}^{\text{ex}}(q \hat{z}, \omega) - \frac{\mu^2}{4\pi^2 v_F^2} - \frac{\omega^2 - v_F^2 q^2}{32\pi^2 v_F^4} \left( \theta(v_F q - \mu) \left[ \xi(q, \omega) \beta(-q, \omega) + \xi(q, -\omega) \beta(-q, -\omega) - \xi(-q, \omega) \beta(q, \omega) - \xi(-q, -\omega) \beta(q, -\omega) \right] \\
+ \theta(\mu - v_F q) \left[ (2\mu + \omega) \ln \frac{\xi(q, \omega)}{\xi(-q, -\omega)} + (2\mu - \omega) \ln \frac{\xi(q, -\omega)}{\xi(-q, \omega)} - 2\omega \ln \frac{v_F q + \omega}{v_F q - \omega} \right] \\
+ v_F q \left[ \zeta(q, \omega) + \zeta(-q, \omega) + \zeta(q, -\omega) + \zeta(-q, -\omega) \right] \right),
$$

(B12)

where $\xi(q, \omega) = 2\mu + v_F q + \omega$ and

$$
\zeta(q, \omega) = \ln \left| \frac{2\mu + v_F q + \omega}{v_F q + \omega} \right|.
$$

(B13)

This also simplifies the expressions since the last term in Eq. (B11) is found to reduce to the momentum- and frequency-independent term $-\mu^2/4\pi^2 v_F^2$.

Note that the full longitudinal response function, $\Pi_{\sigma,\sigma} = (\omega^2/v_F^2 q^2) \Pi_{\rho\rho}$, has the same sign as the charge response function [42]. This is different from the linear response of the helical surface states of a topological insulator, where the transverse spin response is related to the charge response and the relative factor is $-\omega^2/v_F^2 q^2$ [47].

The inter- and intraband contributions to the off-diagonal spin responses can be obtained from

$$
\Pi_{\sigma\tau,\tau m}(q, \omega) = \frac{1}{2N} \sum_k \left[ \pm \frac{k_t k_m + k'_t k_m}{kk'} + i \left( \frac{k'_t}{k'} \pm \frac{k_m}{k} \right) \right] \frac{1}{v_F(k \pm k') - \omega - i\delta} \\
+ \left[ \pm \frac{k_t k_m + k'_t k_m}{kk'} - i \left( \frac{k'_t}{k'} \pm \frac{k_m}{k} \right) \right] \frac{1}{v_F(k \pm k') + \omega + i\delta}.
$$

(B14)

It is evident that only the transverse part of the off-diagonal spin response contributes. For $q = q \hat{z}$, we have $\Pi_{\sigma\tau,\tau} = \Pi_{\sigma\sigma} = 0$ and after performing the angular integrals, Eq. (B14) simplifies to

$$
\Pi_{\sigma\tau,\tau}(q \hat{z}, \omega) = \frac{i e^2 v_F^2}{16\pi^2 q^2} \int dk \int dk' \left[ q^2 - (k \pm k')^2 \right] \frac{1}{v_F(k \pm k') - \omega - i\delta} - (\omega \rightarrow -\omega).
$$

(B15)

The integration limits again depend on the chemical potential in such a way that only transitions from occupied to empty states are included. The response can be decomposed into intrinsic and extrinsic parts. Due to spin-momentum locking, it is proportional to the off-diagonal current-current correlation function, which in turn is related to the Hall conductivity [42, 47], giving

$$
\sigma_{xy}(q, \omega) = \frac{i e^2 v_F^2}{\omega} \Pi_{\sigma\tau,\tau}(q, \omega).
$$

(B16)

As pointed out by Burkov and Balents [49], an overall constant in the Hall conductivity cannot be determined within a continuum model. This problem carries over to the off-diagonal transverse spin response. In the evaluation of Eq. (B14) or Eq. (B15), it appears as an ambiguity of how to regularize the cutoff-dependent term. Specifically, for a lattice model, the sum over $k$ in Eq. (B14) depends on the location of the Weyl node in momentum space. In the static and uniform limit, Burkov and Balents [49] determine this offset from the known limit of vanishing anomalous Hall effect in a trivial insulator. This leads to the Hall conductance being proportional to the separation of nodes [39, 44], [48, 49]. The generalization of this approach to the frequency- and wave-vector-dependent response is difficult since it would involve the regularization of a cutoff-dependent function of $q$ and $\omega$. Since the transverse spin (or Hall) response is not central for this paper, we do not attempt this here. Accordingly disregarding the cutoff-dependent term, the intrinsic part is found to be purely imaginary,

$$
\Pi_{\sigma\tau,\tau}^\text{in}(q \hat{z}, \omega) = i \frac{q \omega}{24\pi^2 v_F^2}.
$$

(B17)
For the extrinsic contribution, a lengthy derivation yields

\[
\text{Im} \Pi_{\sigma_x, \sigma_y}^{\text{ex}}(q\mathbf{z}, \omega) = \frac{1}{16\pi^2 v_F^2 q^2} \left[ \frac{\omega^2 - v_F^2 q^2}{4} \left( \xi(-q, \omega)\xi(q, \omega) \ln \left| \frac{\xi(-q, \omega)}{\xi(q, \omega)} \right| + \xi(q, -\omega)\xi(-q, -\omega) \ln \left| \frac{\xi(q, -\omega)}{\xi(-q, -\omega)} \right| \right) + \left[ \xi(q, \omega)\xi(-q, \omega) + \xi(q, -\omega)\xi(-q, -\omega) \right] \ln \left| \frac{v_F q + \omega}{v_F q - \omega} \right| + 4v_F q \omega (\mu^2 - v_F^2 q^2) \right],
\]

\[
\text{Re} \Pi_{\sigma_x, \sigma_y}^{\text{ex}}(q\mathbf{z}, \omega) = \frac{v_F^2 q^2 - \omega^2}{64\pi^2 v_F^2 q^2} \left[ \xi(q, \omega)\xi(-q, \omega)\theta(v_F q - \omega)\theta(2\mu - v_F q + \omega) - \xi(q, -\omega)\xi(-q, -\omega)\theta(\omega - v_F q)\theta(2\mu + v_F q - \omega)\theta(v_F q + \omega - 2\mu) \right],
\]

where \(\xi(q, \omega) = 2\mu + v_F q + \omega\).

**Appendix C: Coupled charge-spin response**

In this appendix, we discuss the main steps of the calculation of the charge-spin response for a single Weyl cone with \(\chi = +1\). The charge-spin response function can be calculated from the density-spin correlations as

\[
\Pi_{\rho_{\sigma}}(q, i\omega_n) = -\frac{1}{N} \sum_k \sum_{\lambda, \lambda'} \langle \phi_{\lambda'}(k + q) | \phi_{\lambda'}(k) \rangle \langle \phi_{\lambda}(k) | \sigma_i | \phi_{\lambda}(k + q) \rangle \frac{n^F_\lambda(k) - n_\lambda^F(k + q)}{i\omega_n + \epsilon_\lambda(k) - \epsilon_{\lambda'}(k + q)},
\]

see Appendix C for definitions of symbols. At zero temperature, the interband (superscript \(-\)) and intraband (superscript \(+\)) contributions simplify to

\[
\Pi_{\rho_{\sigma}}^{\pm}(q, \omega) = \frac{1}{2N} \sum_k \left[ \left( \frac{k_i}{k^2} \mp \frac{k_i'}{k'^2} \pm i \frac{k_m k_{n'} - k_{m'} k_n}{kk'} \right) \frac{1}{v_F(k \pm k')} - \omega - i\delta \right]
\]

\[
+ \left( \frac{k_i}{k} \pm \frac{k_i'}{k'} \mp i \frac{k_m k_{n'} - k_{m'} k_n}{kk'} \right) \frac{1}{v_F(k \pm k') + \omega + i\delta}. \tag{C2}
\]

Further calculation shows that the transverse components of the charge-spin response, \(\Pi_{\rho_{\sigma}}^{\pm}(q, \omega)\) for \(\hat{\mathbf{I}} \perp \mathbf{q}\), vanish. We can thus write the vector of the response functions \(\Pi_{\rho_{\sigma}}^{\pm}\) as

\[
\Pi_{\rho_{\sigma}}^{\pm}(q, \omega) = \Pi_{\rho_{\sigma}}^{\pm}(q, \omega) \hat{\mathbf{q}} = \Pi_{\rho_{\sigma}}^{\pm}(q\mathbf{z}, \omega) \hat{\mathbf{q}},
\]

where \(\hat{\mathbf{q}} \equiv \mathbf{q}/q\) is the unit vector in the direction of \(\mathbf{q}\) and \(\sigma_q \equiv \sigma \cdot \mathbf{q}\).

Taking \(\mathbf{q} = q\mathbf{z}\), without loss of generality, the non-vanishing response functions can be written in terms of the polar angles of wave vectors as

\[
\Pi_{\rho_{\sigma}}^{\pm}(q\mathbf{z}, \omega) = \frac{1}{2N} \sum_k \langle \cos \theta_{k'} \mp \cos \theta_k \rangle \left[ \frac{1}{v_F(k \pm k')} - \omega - i\delta - (\omega \rightarrow -\omega) \right], \tag{C4}
\]

Due to particle-hole symmetry, the charge-spin response is identical for \(\mu \rightarrow -\mu\), and we now consider an electron-doped Weyl node, i.e., \(\mu > 0\). The response can be decomposed into a sum of intrinsic (undoped) and extrinsic (doping) parts, \(\Pi_{\rho_{\sigma}} = \Pi_{\rho_{\sigma}}^{\text{in}} + \Pi_{\rho_{\sigma}}^{\text{ex}}\). The intrinsic part only contains interband contributions, whereas the extrinsic part can be divided into inter- and intraband contributions. We first evaluate the intrinsic part, which is given by

\[
\Pi_{\rho_{\sigma}}^{\text{in}}(q\mathbf{z}, \omega) = \frac{1}{16\pi^2 q^2} \int_0^\Lambda d\mathbf{k} \int_{|k-q|}^{k+q} dk' \frac{1}{v_F(k + k') - \omega - i\delta} - (\omega \rightarrow -\omega), \tag{C5}
\]

where we have made the limits of integration explicit. \(\Lambda\) is the ultraviolet cutoff for the momentum integral. Using the Sokhatskii-Plemelj formula

\[
\frac{1}{x \pm i\epsilon} = \mathcal{P} \frac{1}{x} \mp i\pi\delta(x) \tag{C6}
\]
and taking $\omega > 0$, we write

$$\text{Im} \Pi_{\rho\sigma}^{\text{in}}(q, \omega) = \frac{1}{16\pi^2 v_F^2} \int_0^\Lambda dk \int_{[k-q]}^{k+q} dk' (k + k') \frac{1}{[q^2 - (k - k')^2]} \delta(\tilde{\omega} - k - k'),$$

(C7)

$$\text{Re} \Pi_{\rho\sigma}^{\text{in}}(q, \omega) = \frac{1}{16\pi^2 q^2 v_F^2} \mathcal{P} \int_0^\Lambda dk \int_{[k-q]}^{k+q} dk' (k + k') \frac{1}{[q^2 - (k - k')^2]} \left( \frac{1}{k + k' - \tilde{\omega}} - \frac{1}{k + k' + \tilde{\omega}} \right),$$

(C8)

with $\tilde{\omega} = \omega/v_F$. The imaginary part is easy to calculate,

$$\text{Im} \Pi_{\rho\sigma}^{\text{in}}(q, \omega) = \frac{q \omega}{24\pi^2 v_F^2} \theta(\omega - v_F q).$$

(C9)

The real and imaginary parts are related by causality. However, since the imaginary part diverges for $\omega \to \infty$, the standard Kramers-Kronig relation is not applicable. A generalized Kramers-Kronig relation can be used, though. The $n$-th order generalized Kramers-Kronig relation for a response function $\chi(\omega)$ that diverges as $\omega^{n-1}$ for $\omega \to \infty$ is given by [42, 64]

$$\frac{\text{Re} \chi(\omega)}{\omega^n} = \frac{1}{\omega} \lim_{\zeta \to 0} \frac{\text{Re} \chi(\zeta)}{\zeta^n} + \frac{1}{\pi} \mathcal{P} \int_{-\infty}^\infty d\zeta \frac{\text{Im} \chi(\zeta)}{\zeta^n (\zeta - \omega)}.$$  

(C10)

Here, the imaginary part of the charge-spin response diverges linearly with $\omega$, thus we employ the second-order Kramers-Kronig relation, which yields

$$\text{Re} \Pi_{\rho\sigma}^{\text{in}}(q, \omega) = \omega \lim_{\zeta \to 0} \frac{\text{Re} \Pi_{\rho\sigma}^{\text{in}}(q, \zeta)}{\zeta} + \frac{\omega^2}{\pi} \mathcal{P} \int_{-\infty}^\infty d\zeta \frac{\text{Im} \Pi_{\rho\sigma}^{\text{in}}(q, \zeta)}{\zeta^2 (\zeta - \omega)}.$$  

(C11)

The first term on the right-hand side evaluates to

$$\frac{q \omega}{24\pi^2 v_F^2} \left( \ln \frac{4\Lambda^2 - q^2}{q^2} - \frac{1}{2} \left[ \frac{(2\Lambda)^3}{q} \ln \frac{2\Lambda + q}{2\Lambda - q} - 2(2\Lambda)^2 - 6q\Lambda \ln \frac{2\Lambda + q}{2\Lambda - q} + \frac{16}{3} q^2 \right] \right).$$

(C12)

Using the identities

$$\lim_{x \to \infty} \left( \frac{x^3}{y} \ln \frac{x + y}{x - y} - 2x^2 \right) = \frac{2y^2}{3},$$

(C13)

$$\lim_{x \to \infty} x \ln \frac{x + y}{x - y} = 2y$$

(C14)

in the limit of large cutoff $\Lambda$, the terms in the angular bracket vanish, and the expression simplifies to

$$\frac{q \omega}{24\pi^2 v_F^2} \ln \frac{4\Lambda^2}{q^2}.$$  

(C15)

The second term on the right-hand side of Eq. (C11) can be rewritten as

$$\frac{2\omega^2}{\pi} \int_0^\infty d\zeta \frac{\omega}{\zeta^2 (\zeta^2 - \omega^2)} \text{Im} \Pi_{\rho\sigma}^{\text{in}}(q, \zeta) = \frac{q \omega}{24\pi^2 v_F^2} \ln \left| \frac{v_F^2 q^2}{v_F^2 q^2 - \omega^2} \right|.$$  

(C16)

Therefore, the real part of the intrinsic charge-spin response is given by

$$\text{Re} \Pi_{\rho\sigma}^{\text{in}}(q, \omega) = \frac{q \omega}{24\pi^2 v_F^2} \ln \left| \frac{4v_F^2 \Lambda^2}{v_F^2 q^2 - \omega^2} \right|.$$  

(C17)

Direct evaluation of the integral in Eq. (C8) gives the same result.

The extrinsic contribution for electron doping can be calculated in a similar fashion. The imaginary and real part
are given by

\[ \text{Im} \Pi_{\rho\rho}^\text{ex}(q\mathbf{z},\omega) = \frac{\omega}{8\pi^2 v_F^2} \left( \theta(v_F q - \omega) \theta(2\mu - v_F q - \omega) [\alpha(q,\omega) - \alpha(q, -\omega)] + \theta(v_F q - \omega) \theta(2\mu - v_F q + \omega) \theta(v_F q + \omega - 2\mu)\alpha(q,\omega) \right. \]
\[ \left. + \theta(\omega - v_F q) \theta(2\mu + v_F q - \omega) \theta(v_F q + \omega - 2\mu) [-\alpha(-q, -\omega)] + \theta(\omega - v_F q) \theta(2\mu - v_F q - \omega) \left( -\frac{q^2}{3} \right) \right). \]

(C18)

\[ \text{Re} \Pi_{\rho\rho}^\text{ex}(q\mathbf{z},\omega) = \frac{\omega}{8\pi^2 v_F^2 q} \left[ \frac{8\mu^2}{3v_F^2} - \alpha(q,\omega)\beta(q,\omega) - \alpha(-q,\omega)\beta(-q,\omega) - \alpha(q,-\omega)\beta(q,-\omega) - \alpha(-q,-\omega)\beta(-q,-\omega) \right], \]

(C19)

where the functions \( \alpha \) and \( \beta \) are defined in Eqs. \([A6]\) and \([A7]\), respectively. The step functions represent boundaries in the \((q,\omega)\) plane for interband and intraband particle-hole excitations.

### Appendix D: Dispersion of collective modes

Here, we calculate the dispersion relation of the coupled collective modes. The real part of the charge response function \( \Pi_{\rho\rho} \) has intrinsic and extrinsic parts given in Eqs. \([A3]\) and \([A5]\), respectively. In the long-wavelength limit with \( v_F q \ll \omega \ll 2\mu \), the real part of \( \Pi_{\rho\rho} \) to order \( q^4 \) is

\[ \Pi_{\rho\rho}(q,\omega) \approx \frac{q^2}{24\pi^2 v_F} \left[ \ln \left( \frac{4v_F^2 \Lambda^2}{4\mu^2 - \omega^2} \right) \frac{4\mu^2}{\omega^2} \left( 1 - \frac{v_F^2 q^2}{4\mu^2} [1 + \mathcal{K}(\omega/2\mu)] \right) \right], \]

(D1)

with

\[ \mathcal{K}(u) = \frac{u^2 - 3/5}{u^2(1-u^2)^2}. \]

(D2)

For a local Hubbard interaction \( U \), the dispersion of zero sound is determined by \( 1 + U \text{Re} \Pi_{\rho\rho}(q,\omega_{zs}(q)) = 0 \) so that

\[ \omega_{zs}(q) \approx \sqrt{\frac{U}{24\pi^2 v_F}} \sqrt{\frac{2\mu q}{1 + U v_F^2 q^2 \ln \left( \frac{4v_F^2 \Lambda^2}{4\mu^2 - \omega^2} \right) \left( 1 - \frac{v_F^2 q^2}{8\mu^2} [1 + \mathcal{K}(\omega/2\mu)] \right)}}. \]

(D3)

In the long-wavelength limit, the dispersion becomes

\[ \omega_{zs}(q) \approx \sqrt{\frac{U}{24\pi^2 v_F}} 2\mu q. \]

(D4)

Strictly speaking, Eq. \([D3]\) should be solved self-consistently to obtain the dispersion. However, since the frequency satisfies \( \omega \ll 2\mu \), we can ignore \( \omega^2 \) in the denominator of the logarithmic term and also take the long-wavelength form \( \omega_0 \) for calculating \( \mathcal{K} \).

For the Coulomb interaction \( V(q) = 4\pi e^2/\kappa q^2 \), the dispersion of the spin plasmon is determined by \( 1 + V(q) \text{Re} \Pi_{\rho\rho}(q,\omega) = 0 \), which in the long-wavelength limit gives

\[ \kappa^*(\omega) - \frac{4\mu^2 \alpha_c}{6\pi \omega^2} \left( 1 - \frac{v_F^2 q^2}{4\mu^2} [1 + \mathcal{K}(\omega/2\mu)] \right) \approx 0, \]

(D5)

with the effective background dielectric function

\[ \kappa^*(\omega) = 1 + \frac{\alpha_c}{6\pi} \ln \left| \frac{4v_F^2 \Lambda^2}{4\mu^2 - \omega^2} \right|. \]

(D6)

Hence, the dispersion reads

\[ \omega_{pl}(q) \approx \mu \sqrt{\frac{2\alpha_c}{3\pi \kappa^*(\omega)}} \left( 1 - \frac{v_F^2 q^2}{8\mu^2} [1 + \mathcal{K}(\omega/2\mu)] \right). \]

(D7)
Again, Eq. (D7) should in principle be solved self-consistently but we can use the $q = 0$ plasma frequency $\omega_0 = \mu \sqrt{2\alpha_\kappa/3\pi\kappa^*(0)}$ for calculating $\mathcal{K}$ and $\kappa^*(\omega)$ with reasonable accuracy.
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