SOME FREE ENTROPY DIMENSION INEQUALITIES FOR SUBFACTORS

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ABSTRACT. Suppose \( N \subset M \) is an inclusion of \( II_1 \)-factors of finite index. If \( N \) can be generated by a finite set of elements, then there exist finite generating sets \( X \) for \( N \) and \( Y \) for \( M \) such that \( \delta_0(X) \geq \delta_0(Y) \), where \( \delta_0 \) denotes Voiculescu’s microstates (modified) free entropy dimension. Moreover, given \( \epsilon > 0 \) one has

\[
\delta_0(F) \geq \delta_0(G) \geq ([M : N]^{-2} - \epsilon) \cdot (\delta_0(F) - 1) + 1 - \epsilon
\]

for certain generating sets \( F \) for \( N \) and \( G \) for \( M \).

Connections between free probability and subfactor theory have existed for at least a decade. In [7] Radulescu studied finite index subfactors of the free group factors. Stefan applied microstate techniques to study finite index subfactors of the interpolated free group factors. He showed in [9] that finite index subfactors of free group factors have certain irreducibility properties like primality (see also [6] where Ozawa uses \( C^* \) techniques to generalize these primality results). More recently Shlyakhtenko and Ueda have used free probability techniques to construct irreducible subfactors of \( \mathcal{L}(F^{\infty}) \) ([8]).

We are interested in the problem of computing the microstates free entropy dimension of generating sets of a finite inclusion of subfactors. Suppose \( N \subset M \) is a finite inclusion of \( II_1 \)-factors and \( X \) and \( Y \) are finite sets of generators for \( N \) and \( M \), respectively. One would expect the following:

\[
\delta_0(X) = [M : N](\delta_0(Y) - 1) + 1.
\]

This formula is a straightforward analogue of Shreier’s theorem on the free groups: if \( H \subset G \) is an inclusion of groups of finite index and \( G \) is free group on \( n \) generators, then \( H \) is a free group on \( [G : H](n - 1) + 1 \) generators.

In this remark we find generating sets \( X \) for \( N \) and \( Y \) for \( M \) such that \( \delta_0(X) \geq \delta_0(Y) \). This inequality is clearly weaker than the equality previously stated since \( [M : N] \geq 1 \). The argument makes use of the basic construction in [3] and the hyperfinite inequality in [5]. Using this monotonicity result, duality in the basic construction, and a computation of the free entropy dimension for amplifications, it follows that for \( \epsilon > 0 \) there exist generating sets \( X \) for \( N \) and \( Y \) for \( M \) such that

\[
\delta_0(X) \geq \delta_0(Y) \geq ([M : N]^{-2} - \epsilon) \cdot (\delta_0(X) - 1) + 1 - \epsilon.
\]

The \( \epsilon \)'s can be dropped provided that \( [M : N] \in \mathbb{Q} \). As a corollary it follows that if \( L(F_r) \) is the interpolated free group with index \( r \) and \( M \) is a \( II_1 \)-factor such that for some \( \epsilon > 0 \) and any finite set of generators \( X \) for \( M \) \( \delta_0(X) < r - \epsilon \), then \( L(F_r) \) cannot contain \( M \) as a subfactor of finite index.

Section 1 is a brief list of preliminaries and assumptions. Section 2 will prove the monotonicity result. Section 3 deals with the free entropy dimension of amplifications and Section 4 will present the final inequality. Section 5 consists of applications of the preceding material.

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1. Preliminaries and Assumptions

Throughout suppose $M$ is a $II_1$-factor with tracial state $\varphi$ and $N \subset M$ is a subfactor. We will always assume that $M$ embeds into the ultraproduct of the hyperfinite $II_1$-factor (this guarantees the existence of microstates for any set of generators for $M$). For any $n \in \mathbb{N}$, $\cdot_2$ denotes the norm on $(M_k(\mathbb{C}))^n$ given by $\|(x_1, \ldots, x_n)\|_2 = (\sum_{j=1}^n \|x_j\|^2)^{1/2}$ where $\|x_j\|$ is the tracial state on the $k \times k$ complex matrices. We will maintain the notation introduced in [5] and [10]. If $F = \{a_1, \ldots, a_n\}$ is a finite set of elements in $M$, we abbreviate $\Gamma_R(a_1, \ldots, a_n; m, k, \gamma)$ by $\Gamma_R(F; m, k, \gamma)$ and in a similar way we write the associated microstate sets and quantities $\delta_0(F), \mathbb{P}_e(F)$, and $\mathbb{K}_e(F)$. We point out that $F$ does not necessarily consist of selfadjoint elements and thus, the microstate spaces are tacitly assumed to consist of matricial approximants in $*$-moments. This will cause no problems, since all these concepts make sense with non-selfadjoint elements (as has been observed by many before), and moreover, clearly coincide with the selfadjoint quantities. Finally, if $G = \{b_1, \ldots, b_p\}$ is another finite set of elements in $M$, then we denote by $\Gamma_R(F \cup G; m, k, \gamma)$ the set $\Gamma_R(a_1, \ldots, a_n, b_1, \ldots, b_p; m, k, \gamma)$ and write all the associated microstate quantities $\delta_0(F \cup G), \mathbb{P}_e(F \cup G)$, and $\mathbb{K}_e(F \cup G)$ with respect to $\Gamma_R(F \cup G; m, k, \gamma)$. Finally, $\Gamma(F; m, k, \gamma)$ will denote the set of all $k \times k$ $*$-microstates (no restrictions on the operator norms) with degree of approximation $(m, \gamma)$.

2. A Monotonic Property of $\delta_0$

Recall the basic construction and some of its properties introduced in [3]. On $L^2(M)$ consider the orthogonal projection $e_N$ onto $L^2(N) \subset L^2(M)$. Denote by $M_1$ the von Neumann subalgebra of $B(L^2(M))$ generated by $M$ and $e_N$. $M_1$ is called the basic construction for $N \subset M$. $M_1$ turns out to be a type $II$-factor and when $[M : N] < \infty$, then $M_1$ is a type $II_1$-factor.

By [3] any inclusion of $II_1$-factors of finite index arise from the basic construction. By this I mean that there exists a $II_1$-subfactor $N_1$ of $N$ of finite index such that the basic construction $P$ for $N_1 \subset N$ is isomorphic to $M$ in such a way that the two copies of $N$ (one in $P$ and one in $M$) are canonically identified. The crucial point is the fact that the projection $f$ onto $L^2(N_1) \subset N$ together with a set of generators for $N$ is a generating set for $M$ (identified with $P$) in such a way that this set of generators for $M$ has an element (namely, $f$) which commutes with any element of $N_1$.

With these preliminaries out of the way we come to the main point of this section which more or less says that if $N$ can be finitely generated, then there exist finite generating sets $Y$ for $N$ and $Z$ for $M$ such that $\delta_0(Y) \geq \delta_0(Z)$. In order to deal with some technical issues in Section 4, I will prove below a bloated version of this simple statement.

**Lemma 2.1.** If $X$ is a finite set of generators for $N$, then there exist finite sets of selfadjoints $F$ and $G$ such each generates a copy of the hyperfinite $II_1$-factor, $F'' \subset G''$, $X \cup F$ generates $N$, $X \cup G$ generates $M$, $f_0 \in G$, and

$$\delta_0(X \cup F) \geq \delta_0(X \cup G).$$

**Proof.** Suppose $\mathcal{R}$ is a copy of the hyperfinite $II_1$-factor in $N_1$ and pick a finite set of selfadjoint generators $F$ for $\mathcal{R}$. $f$ commutes with $\mathcal{R}$. Hence by the computations in [4] and the hyperfinite inequality for $\delta_0$ in [5]

$$\delta_0(X \cup F \cup f) \leq \delta_0(X \cup F) + \delta_0(F \cup \{f\}) - \delta_0(F) = \delta_0(X \cup F) + 1 - 1 = \delta_0(X \cup F).$$

There exists a von Neumann algebra $\mathcal{R}_1 \subset M$ isomorphic to the hyperfinite $II_1$-factor such that $F \cup \{f\} \subset \mathcal{R}_1$. Pick a finite set of selfadjoint generators $G$ for $\mathcal{R}_1$ with $f \in G$. $F'' \subset G''$. It is clear that $X \cup G$ generates $M$. By [11] and the hyperfinite inequality for $\delta_0$
\[
\delta_0(X \cup G) \leq \delta_0(X \cup G \cup F \cup \{f\}) \\
\leq \delta_0(X \cup F \cup \{f\}) + \delta_0(G \cup F \cup \{f\}) - \delta_0(F \cup \{f\}) \\
\leq \delta_0(X \cup F \cup \{f\}) + 1 - 1 \\
\leq \delta_0(X \cup F \cup \{f\}).
\]

Thus, \(\delta_0(X \cup G) \leq \delta_0(X \cup F \cup \{f\}) \leq \delta_0(X \cup F)\). \qed

3. Amplifications and Compressions

In this section we will compute formulae or bounds relating the free entropy dimension of a set of generators for a \(II_1\)-factor and corresponding sets of generators for an amplification or cutdown of the factor. We start with the rational case and then use this to derive weaker results for the general case.

3.1. The Rational Case. Throughout this section denote by \(x_1, \ldots, x_n\) a finite set of elements in a tracial von Neumann algebra \(A\) and by \(\langle e_{ij} \rangle_{1 \leq i, j \leq p}\) the canonical system of matrix units for \(M_p(\mathbb{C})\). Set \(X = \{x_1 \otimes I, \ldots, x_n \otimes I\} \subset A \otimes M_p(\mathbb{C})\) and \(Y = \langle I \otimes e_{ij} \rangle_{1 \leq i, j \leq p} \subset A \otimes M_p(\mathbb{C})\). Denote by \(\psi\) the canonical tracial state on \(A = A \otimes M_p(\mathbb{C})\). Observe that \(\delta_0(X) = \delta_0(x_1, \ldots, x_n)\) and \(\delta_0(Y) = \delta_0(\langle e_{ij} \rangle_{1 \leq i, j \leq p})\). Our main goal in this section is to show:

Lemma 3.1. \(\delta_0(X \cup Y) = 1 - \frac{1}{p^2} + \frac{1}{p^2} \cdot \delta_0(x_1, \ldots, x_n)\).

Proof. Given \(k \in \mathbb{N}\) denote by \(\pi_k : M_p(\mathbb{C}) \to M_k(\mathbb{C})\) the map which sends an arbitrary \(x \in M_p(\mathbb{C})\) to the \(k \times k\) matrix obtained by taking each entry of \(x_{ij}\) and stretching it into the \([\frac{k}{p}] \times [\frac{k}{p}]\) matrix with \(x_{ij}\) repeated along the diagonals and 0’s elsewhere. It is clear that \(\pi_k\) is a \(*\)-homomorphism. Set \(\xi_k = \langle \pi_k(e_{ij}) \rangle_{1 \leq i, j \leq p}\). Given \(m \in \mathbb{N}\) and \(\gamma > 0\), \(\xi_k \in \Gamma(Y ; m, k, \gamma)\) for \(k\) large enough.

Recall the the relative microstate spaces defined in [5]: \(\Xi(X ; m, k, \gamma)\), \(K_r(\Xi(X))\), and \(P_r(\Xi(X))\) relative to the sequence \(\langle \xi_k \rangle_{k=1}^\infty\) given in the preceding paragraph. By [4] and [5]

\[
\delta_0(X \cup Y) = 1 - \frac{1}{p^2} + \limsup_{\epsilon \to 0} \frac{\log \|X(\Xi(X))\|}{\log \epsilon} = 1 - \frac{1}{p^2} + \limsup_{\epsilon \to 0} \frac{\log \|K_r(\Xi(X))\|}{\log \epsilon}.
\]

Hence, it suffices to show that \(\limsup_{\epsilon \to 0} \frac{\log \|X(\Xi(X))\|}{\log \epsilon} \leq \frac{1}{p^2} \cdot \delta_0(X)\) and \(\limsup_{\epsilon \to 0} \frac{\log \|K_r(\Xi(X))\|}{\log \epsilon} \geq \frac{1}{p^2} \cdot \delta_0(X)\).

Concerning the first claim given \(t > 0\) there exists an \(\epsilon_0 > 0\) such that for \(\epsilon_0 > \epsilon > 0\), \(\mathbb{K}_t(X) < (\delta_0(x_1, \ldots, x_n) + t) \|\epsilon\|\). Find \(m > 10\) and \(\gamma \leq (p + 1)^{-3} \|\epsilon\|\) such that for \(k\) sufficiently large there exists an \(\epsilon\)-net \(\langle z_r^{(1)}, \ldots, z_r^{(n)} \rangle_{r \in \Omega(k)}\) for \(\Gamma(x_1, \ldots, x_n ; m, k, \gamma)\) with \(k^{-2} \cdot \log (\|\Omega(k)\|) \leq |\log \epsilon| \cdot (\delta_0(x_1, \ldots, x_n) + 2t)\). Denote by \(b_r^{(l)}\) the \(k \times k\) matrix which has \(z_r^{(l)}\) repeated along the diagonal \(p^3\) times and 0 in all other entries. I claim that \(\langle (b_r^{(l)}, \ldots, b_r^{(n)}) \rangle_{r \in \Omega([\frac{k}{p}])}\) is an \(4p^3\epsilon\)-cover for \(\Xi(X ; m, k, 2\gamma)\).

Towards this end suppose \(a = (a_1, \ldots, a_n) \in \Xi(X ; m, k, \gamma)\). For any \(*\)-monomial \(f\) in \(n\) noncommuting variables

\[
\psi(f(x_1 \otimes e_{ii}, \ldots, x_n \otimes e_{ii})) = \frac{1}{p} \cdot \psi((f(x_1 \otimes I, \ldots, x_n \otimes I)).
\]

Thus,

\[
(1) \quad p \cdot (\pi_k(e_{11})a_1 \pi_k(e_{11}), \ldots, \pi_k(e_{11})a_n \pi_k(e_{11})) \in \Gamma(x_1, \ldots, x_n ; m, k, 2\gamma).
\]

For any \(1 \leq i \leq p\) and \(1 \leq l \leq n\), \((I \otimes e_{ij})(x_l \otimes I)(I \otimes e_{ji}) = x_l \otimes e_{ii}\) so that
Finally, for any $1 \leq l \leq n$, $x_l - \sum_{i=1}^{p} x_i (I \otimes e_{ii}) = 0$ so that

\begin{equation}
|a_l - \sum_{i=1}^{p} a_i \pi_k(e_{ii})|_2 < \epsilon.
\end{equation}

For each $1 \leq l \leq n$ denote by $\eta_l$ the $\left[ \frac{k}{p} \right] \times \left[ \frac{k}{p} \right]$ submatrix of $\pi_k(e_{11})a_i\pi_k(e_{11})$, i.e., the matrix formed from the nonzero entries of $\pi_k(e_{11})a_i\pi_k(e_{11})$. It is clear from (1) that $\eta = (\eta_1, \ldots, \eta_n) \in \Gamma(x_1, \ldots, x_n; m, [\frac{k}{p}], \gamma)$. There exists some $r \in \Omega([\frac{k}{p}])$ such that $|\zeta_{rk} - \eta|_2 < \epsilon$. Using (1) and (2) we get

\begin{equation}
|a_l - b_{rk}^{(l)}|_2 \leq \left| \sum_{i=1}^{p} a_i \pi_k(e_{ii}) - b_{rk}^{(l)} \right|_2 + \epsilon = \left| \sum_{i=1}^{p} \left( a_i \pi_k(e_{ii}) - b_{rk}^{(l)} \pi_k(e_{ii}) \right) \right|_2 + \epsilon
\end{equation}

\begin{align*}
&\leq \sum_{i=1}^{p} \left| \pi_k(e_{11})a_i \pi_k(e_{11}) - \pi_k(e_{11})b_{rk}^{(l)} \pi_k(e_{11}) \right|_2 + (p + 1)\epsilon \\
&\leq \sum_{i=1}^{p} \left| \pi_k(e_{11})a_i \pi_k(e_{11}) - \pi_k(e_{11})b_{rk}^{(l)} \pi_k(e_{11}) \right|_2 + 2(p + 1)\epsilon \\
&\leq \sum_{i=1}^{p} p \cdot |\eta_l - e_{ik}|_2 + 3(p + 1) \\
&\leq 4p^2 \epsilon.
\end{align*}

It follows that $|a - (b_{rk}^{(1)}, \ldots, b_{rk}^{(n)})|_2 < 4p^2 \epsilon$ as desired. Thus,

\begin{equation}
\mathbb{K}_{4p^2 \epsilon}(\Xi(X)) \leq \mathbb{K}_{4p^2 \epsilon}(\Xi(X; m, 2\gamma)) \leq \limsup_{k \to \infty} k^{-2} \cdot \log(\#\Omega([\frac{k}{p}])) \leq \frac{|\log \epsilon|}{p^2} \cdot \left( \delta_0(x_1, \ldots, x_n) + 2t \right).
\end{equation}

Hence, for $\epsilon$ sufficiently small,

\begin{equation}
\frac{\mathbb{K}_\epsilon(\Xi(X))}{|\log \epsilon|} \leq \frac{1}{p^2} \cdot \delta_0(x_1, \ldots, x_n) + 2t,
\end{equation}

and $t > 0$ being arbitrary we have that $\limsup_{\epsilon \to 0} \frac{\mathbb{K}_\epsilon(\Xi(X))}{|\log \epsilon|} \leq \frac{1}{p^2} \cdot \delta_0(x_1, \ldots, x_n)$. This completes the upper bound claim.

For the lower bound claim suppose $t > 0$. For any $\epsilon_0 > 0$ there exists an $\epsilon > 0$ such that $\epsilon_0 > \epsilon$ and $\mathbb{P}_\epsilon(X) \geq |\log \epsilon| \cdot (\delta_0(x_1, \ldots, x_n) + t)$. Suppose now that $m \in \mathbb{N}$ and $\gamma > 0$ are given. For $k > \frac{p}{\gamma}$ sufficiently large there exists an $\epsilon$-separated subset $(\langle y_{sk}^{(1)}, \ldots, y_{sk}^{(n)} \rangle)_{s \in \Pi_k}$ of $\Gamma(X; m, [\frac{k}{p}], \gamma)$ satisfying $k^{-2} \cdot \log(\#\Pi_k) \geq \frac{1}{p^2} \cdot |\log \epsilon| \cdot \delta_0(x_1, \ldots, x_n)$. For each such $k$ and $1 \leq l \leq n$ denote by $b_{sk}^{(l)}$ the $k \times k$ matrix obtained by repeating $y_{sk}^{(l)}$ repeated $p$ times along the diagonal and 0’s elsewhere. Now if $1 \leq j \leq m$, $1 \leq i_1, \ldots, i_j \leq n$, and $v_1, \ldots, v_{j+1} \in \xi_k \cup \{I_k\}$ where $I_k$ is the $k \times k$ identity matrix, then

\begin{align*}
\text{tr}_k(v_1 b_{sk}^{(i_1)} v_2 b_{sk}^{(i_2)} \cdots v_k b_{sk}^{(i_j)} v_{k+1}) &= \text{tr}_k(b_{sk}^{(i_1)} \cdots b_{sk}^{(i_j)} v_1 \cdots v_{k+1}) \\
&= p \cdot \text{tr}_{\left[ \frac{k}{p} \right]}(b_{sk}^{(i_1)} \cdots b_{sk}^{(i_j)}) \cdot \text{tr}_k(v_1 \cdots v_{k+1})
\end{align*}
By Corollary 3.2 there exists a finite set of generators $M$. Proof. $M$ is a finite set of generators for $\langle X; m, k, p\gamma\rangle$. This being true for any finitely many elements, so can be computed in the von Neumann algebra $eAe$. Lemma 3.4. Recall that for each $\alpha \in (0, \infty)$ there exists a type $II_1$-factor $M_\alpha$. Suppose $\alpha \in \mathbb{Q}_{>0}$. If $M$ can be finitely generated then there exist two finite sets of generators $F$ for $M$ and $G$ for $M_\alpha$, such that

$$\delta_0(G) = 1 - \frac{1}{\alpha^2} + \frac{1}{\alpha^2} \cdot \delta_0(F).$$

Proof. There exist natural numbers $m$ and $n$ such that $\alpha = \frac{m}{n}$. Because $M$ can be generated by finitely many elements, so can $M_{\frac{n}{m}}$. Fix such a set of generators $X$ for $M_{\frac{n}{m}}$ canonically identified in $M_1 \otimes M_m(\mathbb{C})$ and denote by $E$ the obvious system of matrix units for $M_m(\mathbb{C})$ canonically identified in $M_{\frac{n}{m}} \otimes M_m(\mathbb{C})$. Set $G = X \cup E$ and observe that the von Neumann algebra generated by $G$ is $M_1 \otimes M_m(\mathbb{C}) = M_\alpha$. From Lemma 3.1 we have that

$$\delta_0(G) = 1 - \frac{1}{m^2} + \frac{1}{m^2} \cdot \delta_0(X).$$

By Corollary 3.2 there exists a finite set of generators $F$ for $M$ such that $\delta_0(X) = n^2(\delta_0(F) - 1) + 1$. Substituting this into the above equation gives the result. □

3.2. The General Case. Suppose again that $\infty > \alpha > 0$. As always, $M$ is a $II_1$-factor.

Lemma 3.4. Suppose $M$ can be generated by a finite number of elements and $0 < \alpha < \infty$. For $\epsilon > 0$ there exist finite sets of generators $F$ and $G$ for $M$ and $M_\alpha$, respectively, such that

$$\delta_0(G) < 1 - \frac{1}{\alpha^2} + \left(\frac{1}{\alpha^2} + \epsilon\right) \cdot \delta_0(F) + \epsilon.$$
Proof. $M_n = e(M \otimes B(H))e$ for some finite dimensional Hilbert space and $e$ is a projection in $M \otimes B(H)$ with trace $\alpha$. Suppose $n \in \mathbb{N}$ is a multiple of $\dim H$. For some $j \in \mathbb{N}$, $\dim H \cdot \frac{1}{n} \leq \alpha \leq \dim H \cdot \frac{i+1}{n}$. Choose a system of matrix units in $\langle e_{pq} \rangle_{1 \leq p,q \leq n}$ in $M \otimes B(H)$. It can be arranged so that $e_{11} + \cdots + e_{jj} \leq e$ and that the algebra the matrix units form is $A \otimes B(H)$ where $A$ is a type I finite factor in $M$. Set $f_n = e - (e_{11} + \cdots + e_{jj})$. Because $M$ can be generated by finitely many elements, $e_{11} Me_{11}$ can also. Fix a finite set of generators $X_n$ for $e_{11} Me_{11}$. In this same way we fix such a finite set of generators $Y$ for $M_n$. Set $E_n = \langle e_{pq} \rangle_{1 \leq p,q \leq j}$. Now consider the finite set $G_n = X_n \cup E \cup Y f_n \cup f_n Y \cup \{f_n\}$

where $Y f_n = \{y f : y \in Y\}$ and $f_n Y = \{f_n y : y \in Y\}$. $G_n$ is a set of generators for $M_n$.

I will bound $\delta_0(G_n)$ by appealing again to the relative (or conditional) microstate space result in [5]. Although it was phrased in terms of selfadjoint microstate spaces, it holds true in the non-selfadjoint case as well. We condition with respect to microstates for $f_n$. For each $k$ denote by $p_k$ the diagonal $k \times k$ matrix with 1’s on the last $\lceil \varphi(f_n) \rceil k$ entries and 0’s everywhere else. Observe that for any $m \in \mathbb{N}$ and $\gamma > 0$, for sufficiently large $k$, $p_k \in \Gamma(f_n; m, k, \gamma)$. By [5]

$$
\delta_0(G_n) = \delta_0(f_n) + \limsup_{\epsilon \to 0} \frac{K_n(\Xi(X_n \cup \langle e_{pq} \rangle_{1 \leq p,q \leq j} \cup Y f_n \cup f_n Y))}{|\log \epsilon|}
$$

where the covering quantity is taken relative to $\langle p_k \rangle_{k=1}^\infty$.

Suppose $t > 0$. By Corollary 3.2 $\delta_0(X_n \cup E_n) = 1 - \frac{1}{j^2} + \frac{1}{j^2} \cdot \delta_0(X_n)$. Here $\delta_0(X)$ is computed in the von Neumann algebra $e_{11} Me_{11}$ and $\delta_0(X_n \cup E_n)$ is computed in the von Neumann algebra $e_{11} Me_{11} \otimes M_n(\mathbb{C})$. Thus, for $\epsilon > 0$ sufficiently small there exist $m \in \mathbb{N}$ and $\gamma > 0$ (dependent upon $\epsilon$) and an $\epsilon$-cover of $\Gamma(X_n \cup E_n; m, k, 2\gamma)$ such that for $k$ sufficiently large the logarithm of the cardinality of the cover is greater than

$$
|\log \epsilon| \cdot k^2 \cdot \left(1 - \frac{1}{j^2} + \frac{1}{j^2} \cdot \delta_0(X_n) + t\right).
$$

We can also choose $m$ and $\gamma$ so that if $(a, b) \in \Xi(Y f_n \cup f_n Y; m, k, \gamma)$ (relative to the sequence $\langle p_k \rangle_{k=1}^\infty$), then $|a p_k - a|, |p_k b - b| < \epsilon$, where $a p$ and $p b$ are the tuples obtained by multiplying each entry from the left, respectively, by $p$. Finally, there exists $m_1 > m$ and $\gamma_1 < \gamma$ such that if $a \in \Xi(X_n \cup E_n; m_1, k, \gamma_1)$ (microstate space relative to $\langle p_k \rangle_{k=1}^\infty$), then $p_k^{-1} a p_k$ (the $(\#X_n + j^2)$-tuple obtained by compressing each entry by $p_k$) is in $\Xi(X_n \cup E; m, k, 2\gamma)$ and $|p_k^m a^m p_k - a| < \epsilon$.

Now for each $k$ find an $\epsilon$-cover $\langle \xi_{rk} \rangle_{r \in \Omega_k}$ of $\Gamma(X_n \cup E_n; m, k - [\varphi(f)] k, \gamma)$. For each such $r$ and $k$ denote by $a_{rk}$ the $(\#X_n + j^2)$-tuple whose $r$th entry is the $k \times k$ matrix which the $i$th component of $\xi_{rk}$ as a submatrix in the left corner and 0’s elsewhere. For each $k$ find an $\epsilon$-net $\langle v_{sk} \rangle_{s \in \Omega_k}$ in the $|\cdot|_2$-ball of $\langle M_n(\mathbb{C}) \rangle_{\#Y p_k \otimes p_k M_n(\mathbb{C})}^{\#Y p_k \otimes p_k (M_n(\mathbb{C}))}$ of radius $K = \left\| (Y f_n, Y) \right\|_2 + 1$. $\langle M_n(\mathbb{C}) \rangle_{\#Y p_k \otimes p_k M_n(\mathbb{C})}^{\#Y p_k \otimes p_k (M_n(\mathbb{C}))}$ is $\Xi(Y f_n \cup f_n Y)$. The standard volume comparison test allows us to assume that $\# \Pi_k < \left(\frac{K^2}{\epsilon^2}\right)^{4k^2 \varphi(f)}$.

I claim that $\langle (a_{rk}, v_{sk}) \rangle_{(r,s) \in \Omega_k \times \Pi_k}$ is a $4\epsilon$-cover for $\Xi(X_n \cup E \cup Y f_n \cup Y f_n; m_1, k, \gamma_1)$. To see this suppose $(x, y, z) \in \Xi(X_n \cup E_n \cup Y f_n \cup f_n Y; m, k, \gamma)$ where $x$ is a $(\#X_n + j^2)$-tuple (corresponding to $X_n \cup E$), $y$ is a $\#Y_n$-tuple corresponding to $Y f_n$, and $z$ is $\#Y_n$-tuple corresponding to $f_n Y$. By the choice of $m_1$ and $\gamma_1$, $|x - p_k \otimes p_k x^p_k| < \epsilon$ and $p_k \otimes p_k x^p_k \in \Xi(X_n \cup E_n; m, k, 2\gamma)$. It follows from the previous paragraph that there exists an $r_0 \in \Omega_k$ such that $|a_{rk} - p_k \otimes p_k x^p_k| < \epsilon$, whence $|a_{rk} - x| < 2\epsilon$. Again by the choice of $m_1$ and $\gamma_1$, $|y p_k - y|, |p_k z - z| < \epsilon$. There exists an $s_0 \in \Pi_k$ satisfying $|\langle y p_k, p_k z \rangle - v_{sk}| < \epsilon$ whence $|\langle y, z \rangle - v_{sk}| < 3\epsilon$. Thus, $|\langle x, y, z \rangle - (a_{rk}, v_{sk})| < 4\epsilon$. From the preceding paragraph we now have:
\[ K_{5\epsilon} (\Xi (X_n \cup E_n \cup Y f_n \cup f_n Y)) \leq K_{5\epsilon} (\Xi (X_n \cup E_n \cup Y f_n \cup f_n Y; m_1, \gamma_1)) \]
\[ \leq \limsup_{k \to \infty} k^{-2} \cdot \log (\# \Omega_k \cdot \# \Pi_k) \]
\[ \leq \limsup_{k \to \infty} k^{-2} \cdot \log (\# \Omega_k) + \limsup_{k \to \infty} k^{-2} \cdot \log (\# \Pi_k) \]
\[ \leq (1 - \varphi(f))^2 \cdot |\log \epsilon| \cdot \left( 1 - \frac{1}{j^2} + \frac{1}{j^2} \cdot \delta_0(X) + t \right) + |\log \epsilon| \cdot 4 \varphi(f_n) \]

This being true for all \( \epsilon \) sufficiently small and \( t > 0 \) being arbitrary (4) yields
\[ \delta_0(G_n) \leq \delta_0(f_n) + (1 - \varphi(f_n))^2 \cdot \left( 1 - \frac{1}{j^2} + \frac{1}{j^2} \cdot \delta_0(X_n) \right) + \varphi(f_n). \]

By Corollary 3.2 we can find a finite set of generators \( F_n \) for \( M \) such that
\[ \delta_0(X_n) = \left( \frac{n}{\dim H} \right)^2 (\delta_0(F_n) - 1) + 1. \]

Substituting this above gives
\[ \delta_0(G_n) \leq \delta_0(f_n) + 1 + \frac{n^2}{(j \cdot \dim H)^2} \cdot (\delta_0(F_n) - 1) + \varphi(f_n) \]

As \( n \to \infty \), \( \varphi(f_n) \to 0 \), \( \delta_0(f_n) \to 0 \), and \( \frac{n^2}{(j \cdot \dim H)^2} \to \frac{1}{\alpha^2} \). This yields the desired result. \( \square \)

**Corollary 3.5.** If \( M \) can be generated by finitely many elements and \( 0 < \alpha < \infty \), then for \( \epsilon > 0 \) there exist finite sets of generators \( F \) and \( G \) for \( M \) and \( M_n \), respectively such that
\[ \delta_0(F) < 1 - \alpha^2 + (\alpha + \epsilon)^2 \cdot \delta_0(G) + \epsilon. \]

## 4. Weak Bounds for an Inclusion of II_1-Factors of Finite Index

In this section I will use the results of the previous two sections to derive the main inequality concerning \( \delta_0 \) and finite index subfactors. \( N \subset M \) is an inclusion of II_1-subfactors with \( [M : N] < \infty \). Moreover, assume \( N \) can be generated by finitely many elements. First for the rational case:

**Lemma 4.1.** Suppose \([M : N] \in \mathbb{Q}\). Then there exist finite generating sets \( X \) for \( N \) and \( Y \) for \( M \) such that
\[ \delta_0(X) \geq \delta_0(Y) \geq [M : N]^{-2} \cdot (\delta_0(X) - 1) + 1. \]

**Proof.** Write \([M : N] = \frac{m}{n}\) where \( m, n \in \mathbb{N}\). Find a copy \( \mathcal{R} \) of the hyperfinite II_1-factor in \( N_1 \) and a system of matrix units \( E = (e_{ij})_{1 \leq i, j \leq n} \) in \( \mathcal{R} \). Choose a finite set of selfadjoint generators \( D \) for \( \mathcal{R} \). Because \( N \) can be finitely generated so can \( e_{11}N e_{11} \). Choose a finite set of generators \( Z \) for \( e_{11}N e_{11} \). Set \( X = Z \cup E \cup D \). \( X \) clearly generates \( N \). Lemma 2.3 implies
\[ \delta_0(X) \geq \delta_0(X \cup G) \]
for some finite set of selfadjoint elements \( G \) satisfying \( f_0 \in G \), \( \mathcal{R} \subset G'' \simeq \mathcal{R} \), and \( (X \cup G)'' = M \).

Denote by \( e_N \in B(L^2(M)) \) the projection onto \( L^2(N) \subset L^2(M) \). Lemma 2.3 applied to the inclusion \( M \subset M_1 \) and the generating set \( X \cup G \) for \( M \) yields a finite set of selfadjoint elements \( W \) such that \( \mathcal{R} \subset W'' \simeq R \), \( e_N \in W \), and \( \delta_0(X \cup G) \geq \delta_0(X \cup G \cup W) \).
Because $e_N \in N'$, $e_1e_N$ is a projection of trace $\frac{1}{m}$ in the hyperfinite $II_1$-factor $W''$. Find a system of matrix units $(f_{ij})_{1 \leq i,j \leq m}$ in $W''$ such that $f_{11} = e_1e_N$. Clearly

$$f_{11}M_1f_{11} = e_1e_NM_1e_Ne_1 = e_1N_1e_N \simeq e_1Ne_1.$$

It follows from Corollary 3.2 that $H = e_NZe_N \cup (f_{ij})_{1 \leq i,j \leq m}$ is a generating set for $M_1$ such that $\delta_0(Z) = m^2(\delta_0(H) - 1) + 1$ where $\delta_0(Z)$ is computed in the von Neumann algebra $e_1Ne_1$. By Corollary 4.2 of [5], algebraic invariance of $\delta_0$ ([11]), and Corollary 3.2

$$\delta_0(X \cup G \cup W) = \delta_0(Z \cup E \cup G \cup W)$$

$$= \delta_0(Z \cup E \cup G \cup W \cup (f_{ij})_{1 \leq i,j \leq m})$$

$$= \delta_0(Z \cup Ze_N \cup E \cup G \cup W \cup (f_{ij})_{1 \leq i,j \leq m})$$

$$\geq \delta_0(e_NZe_N \cup (f_{ij})_{1 \leq i,j \leq m})$$

$$= \frac{1}{m^2} \cdot \delta_0(Z) - \frac{1}{m^2} + 1$$

$$= [M : N]^{-2} \cdot (\delta_0(X) - 1) + 1.$$

Set $Y = X \cup G$. $Y$ generates $M$ and by the first and second paragraphs and the computations above

$$\delta_0(X) \geq \delta_0(Y) = \delta_0(X \cup G) \geq \delta_0(X \cup G \cup W) \geq [M : N]^{-2} \cdot (\delta_0(X) - 1) + 1.$$

We can drop the condition that $[M : N] \in \mathbb{Q}$ in the preceding lemma at the cost of obtaining an approximate version of the resultant inequality:

**Lemma 4.2.** Suppose $[M : N] < \infty$. For each $n \in \mathbb{N}$ there exist finite generating sets $X_n$ for $N$ and $Y_n$ for $M$ and sequences of real numbers $\langle \alpha_n \rangle_{n=1}^{\infty}$ and $\langle \beta_n \rangle_{n=1}^{\infty}$ with $\alpha_n \to [M : N]^{-1}$ and $\beta_n \to 0$ such that

$$\delta_0(X_n) \geq \delta_0(Y_n) \geq \alpha_n^2 \cdot (\delta_0(X_n) - 1) + 1 - \beta_n.$$

**Proof.** Write $[M : N] = \alpha$ and assume $\epsilon > 0$ is given. Suppose $\alpha < n \in \mathbb{N}$. Find the largest $m \in \mathbb{N}$ satisfying $\frac{m}{n} \leq \alpha^{-1} < \frac{m+1}{n}$ and set $\alpha_n = \frac{m}{n}$. Consider $\mathcal{R} \subset N_1$ in Section 2 and fix a finite set of generators $F$ for $\mathcal{R}$ ($F$ is independent of $n$). Find a system of matrix units $E_n = \langle e_{ij} \rangle_{1 \leq i,j \leq m}$ in $\mathcal{R}$ with $\varphi(e_{11}) = \frac{n}{m}$ (the matrix algebra generated by $E_n$ will not contain the identity of $N$). Because $N$ can be finitely generated so can $e_1Ne_1$. Set $f_n = I - e_1 + \cdots + e_{mm}$. Choose a finite set of generators $Y$ for $N$ and $Z_n$ for $e_1Ne_1 \subset N$ and arrange it so that $F \subset Y$ and $e_1Fe_1 \subset Z_n$. Define $X_n = Z_n \cup E_n \cup f_nY \cup Yf_n \cup \{f_n\}$. If $n$ is large enough, then the proof of Lemma 3.4 implies that

$$\delta_0(X_n) \leq 1 - \frac{1}{m^2} + \frac{1}{m^2} \cdot \delta_0(Z_n) + \epsilon.$$

Whence, $\delta_0(Z_n) \geq m^2 \cdot \delta_0(X_n) - m^2 + 1 - m^2\epsilon$.

Lemma 2.3 implies

$$\delta_0(X_n) \geq \delta_0(X_n \cup G_n)$$

for some finite set of selfadjoint elements $G_n$ satisfying $f \in G_n$, $\mathcal{R} \subset G_n' \simeq \mathcal{R}$, and $(X \cup G_n)'' = M$. Here $f$ is the projection onto $L^2(N_1) \subset L^2(N)$ where $N_1 \subset N$ is as in the beginning of section 2.
Denote by $e_N \in B(L^2(M))$ the projection onto $L^2(N) \subset L^2(M)$. Lemma 2.3 applied to the inclusion $M \subset M_1$ and the generating set $X \cup G_n$ for $M$ yields a finite set of selfadjoint elements $W_n$ such that $\mathcal{R} \subset W_n'' \cong R, e_N \in W$, and $\delta_0(X_n \cup G_n) \geq \delta_0(X_n \cup G_n \cup W_n)$.

Because $e_N \in N'$, $e_{11}e_N$ is a projection of trace $\frac{1}{n}$ in the hyperfinite $II_1$-factor $W_n''$. Find a system of matrix units $(f_{ij})_{1 \leq i,j \leq n}$ in $W_n''$ such that $f_{11} = e_{11}e_N$. Clearly

$$f_{11}M_1f_{11} = e_{11}e_NM_1e_{11} = e_{11}Ne_{11}e_{11} \simeq e_{11}Ne_{11}.$$

It follows from Corollary 3.2 that $H_n = e_NZ_n e_N \cup (f_{ij})_{1 \leq i,j \leq n}$ is a finite set of generators for $M_1$ such that $\delta_0(Z_n) = n^2(\delta_0(H_n) - 1) + 1$ where $\delta_0(Z_n)$ is computed in the von Neumann algebra $e_{11}Ne_{11}$.

By Corollary 4.2 of [5], algebraic invariance of $\delta_0 ([11])$, and Corollary 3.2

$$\delta_0(X_n \cup G_n \cup W_n) = \delta_0(X_n \cup G_n \cup W_n \cup (f_{ij})_{1 \leq i,j \leq n})$$

$$\geq \delta_0(e_NZ_n e_N \cup (f_{ij})_{1 \leq i,j \leq n})$$

$$= \frac{1}{n^2} \cdot \delta_0(Z_n) - \frac{1}{n^2} + 1$$

$$\geq \alpha_n^2 \cdot (\delta_0(X_n) - 1) + 1 - \alpha_n^2 \cdot \epsilon$$

Set $Y_n = X_n \cup G_n$. $Y_n$ generates $M$ and by what preceded

$$\delta_0(Y_n) \geq \delta_0(X_n \cup G_n \cup W_n) \geq \alpha_n^2 \cdot (\delta_0(X_n) - 1) + 1 - \alpha_n^2 \cdot \epsilon.$$

Since $\epsilon > 0$ was arbitrary we are done.

\[ \square \]

5. Applications

We conclude with a few simple applications of the preceding section. Suppose $L(F_r)$ is the interpolated free group factor with index $r > 1$ ([11], [7]).

**Corollary 5.1.** If $L(F_r)$ is a finite index subfactor of $M$, then there exists a finite generating set $Y$ for $M$ such that $\delta_0(Y) \geq [M : L(F_r)]^{-2} \cdot (r - 1) + 1$.

**Proof.** Set $N = L(F_r)$ and recall from the proof of the Lemma 4.2 where $Z_n$ is chosen to be a set of generators for $eNe$ where $e \in N$ is a projection of trace $[M : N]/n$. By results of Radulescu and Dykema ([1], [7]), $eL(F_r)e \cong L(F_s)$ where $s = \frac{n^2}{[M : N]} \cdot (r - 1) + 1$ and we may choose $Z_n$ so that $\delta_0(Z_n) \geq \frac{n^2}{[M : N]} \cdot (r - 1) + 1$. Constructing the generating set $Y = Y_n$ for $M$ with respect to this $Z_n$ we have again by the proof of Lemma 4.2 that

$$\delta_0(Y) \geq \frac{1}{n^2} \cdot \delta_0(Z_n) - \frac{1}{n^2} + 1 \geq [M : L(F_r)]^{-2} \cdot (r - 1) + 1.$$

\[ \square \]

Using the basic construction the same result can be obtained concerning finite index subfactors of $L(F_r)$. First for a preliminary lemma:

**Lemma 5.2.** Suppose there exists a $t > 0$ such that for any finite set of generators $F$ for $M$, $\delta_0(F) \leq t$. If $\alpha > 0$, then for any finite set of generators $G$ for $M_\alpha$, $\delta_0(G) \leq 1 - \alpha^{-2} + t\alpha^{-2}$.
Proof. First consider the case where $\alpha = p^{-1}$, $p \in \mathbb{N}$. If $X$ is finite set of generators for $M_{p^{-1}}$ then consider the set of generators $Y$ for $M_{p^{-1}} \otimes M_p(\mathbb{C}) = M$ consisting of elements of the form $I \otimes e_{ij}$, $1 \leq i, j \leq p$ where the $e_{ij}$ are the canonical system of matrix units for $M_p(\mathbb{C})$ and elements of the form $x \otimes e_{11}$ where $x \in X$. It follows from Corollary 3.2 and the hypothesis that $1 - p^{-2} + p^{-2} \cdot \delta_0(X) = \delta_0(Y) < t$.

Now suppose $\alpha$ is arbitrary, $X$ is a set of generators for $M_\alpha$, and let $\epsilon > 0$. Suppose $n \in \mathbb{N}$ and find a system of matrix units $E = \langle e_{ij} \rangle_{1 \leq i, j \leq m}$ such that $\varphi(e_{11}) = (\alpha n)^{-1}$ and $m$ is the largest integer such that $m \cdot (\alpha n)^{-1} \leq 1$. Notice that this system of matrix units will not contain the identity. By [11], $\delta_0(X) \leq \delta_0(X \cup E)$. Set $f = I - (e_{11} + \cdots + e_{nn})$. Denote by $Y$ the set consisting of all elements of the form $e_{11}xx_1$ where $x \in X$. $Y$ is a generating set for $e_{11}M_{e_{11}} \simeq M_{\alpha^{-1}}$. $X \cup E$ and $Y \cup E \cup fX \cup Xf$ generate the same unital $\ast$-algebra whence by [11] their free entropy dimensions are equal. Set $Z = Y \cup E \cup fX \cup Xf \cup \{f\}$. Now by the proof of Lemma 3.4 if $n$ is chosen large enough, then

$$
\delta_0(X) \leq \delta_0(X \cup E) \leq \delta_0(Y \cup E \cup fX \cup Xf) \leq \delta_0(Z) \\
\leq 1 - \frac{1}{m^2} \cdot \frac{1}{m^2} \cdot \delta_0(Y) + \epsilon \\
\leq 1 - \frac{1}{m^2} + \frac{n^2}{m^2} \cdot (t - 1) + \frac{1}{m^2} + \epsilon.
$$

As $n \to \infty$, $m \to \infty$ and $\frac{n^2}{m^2} \to \alpha^{-1}$. Thus, $\delta_0(X) \leq 1 + \alpha^{-2}(t - 1) + \epsilon$. $\epsilon > 0$ being arbitrary it follows that $\delta_0(X) \leq 1 - \alpha^{-2} + \epsilon$.

Corollary 5.3. If $N$ satisfies the property that for some $\epsilon > 0$ and any finite set of generators $X$ for $N$, $\delta_0(X) < r - \epsilon$, then $L(F_r)$ cannot contain $N$ as a subfactor of finite index.

Proof. This follows from duality. More precisely suppose $L(F_r)$ contains $N$ as a subfactor of finite index. Perform the basic construction to arrive at the inclusion $N \subset L(F_r) \subset M$. By [3] $[M : L(F_r)] = [L(F_r) : N] = \alpha < \infty$. On the other hand $M \simeq N_\alpha$. For any finite set of generators $X$ for $N_\alpha$ we have by hypothesis and the previous lemma, $\delta_0(X) \leq 1 - \alpha^{-2} + (r - \epsilon) \cdot \alpha^{-2} < \alpha^{-2}(r - 1) + 1$. By Corollary 5.1 $L(F_r)$ cannot be a finite index subfactor of $M$ of index $\alpha$. This is absurd.

Using [2] we get:

Corollary 5.4. If $M$ can be generated by a sequence of Haar unitaries $\langle u_j \rangle_{j=1}^\infty$ with the property that $u_{j+1}u_j u_{j+1}^* \in \{u_1, \ldots, u_j\}''$ for each $j$, then $M$ cannot contain $L(F_r)$ as a finite index subfactor and $L(F_r)$ cannot contain $M$ as a finite index subfactor.

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