Fast quantum noise in Landau-Zener transition

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(Dated: September 16, 2018)

We show by direct calculation starting from a microscopic model that the two-state system with time-dependent energy levels in the presence of fast quantum noise obeys the master equation. The solution of master equation is found analytically and analyzed in a broad range of parameters. The fast transverse noise affects the transition probability during much longer time (the accumulation time) than the longitudinal one. The action of the fast longitudinal noise is restricted by the shorter Landau-Zener time, the same as in the regular Landau-Zener process. The large ratio of time scales allows solving the Landau-Zener problem with longitudinal noise only, then solving the same problem with the transverse noise only and matching the two solutions. The correlation of the longitudinal and transverse noise renormalizes the Landau-Zener transition matrix element and can strongly enhance the survival probability, whereas the transverse noise always reduces it. Both longitudinal and transverse noise reduce the coherence. The decoherence time is inverse proportional to the noise intensity. If the noise is fast, its intensity at which the multi-quantum processes become essential corresponds to a deeply adiabatic regime. We briefly discuss possible applications of the general theory to the spin relaxation of molecular magnets.

PACS numbers: 03.65.-w;33.80.Be;32.80.-t;

I. INTRODUCTION

Landau-Zener (LZ) theory deals with a two-state system, whose levels vary in time by sweeping of some physical parameter and would cross at some moment. Transitions between the two states are induced by the transition matrix element $\Delta$, which in static or adiabatic case causes the level repulsion (avoided crossing according to the Wigner-Neumann theorem). The LZ theory is one of basic dynamic problems of quantum mechanics. It has a broad range of applications from the collision theory and quantum chemistry to disordered solids and molecular magnets.

The LZ transitions in noisy environment recently attracted new attention in connection with the problem of the qubit decoherence. New experimental realization of qubits rose again the question to what extent it is possible to minimize the decoherence simultaneously maintaining sufficiently strong coupling to the external signal.

The problem of the LZ transition in noisy environment has relatively long history. One of the first considerations based on ideas of stochastic trajectories belongs to Kasumok. In the pioneering work Kadanov calculated the transition amplitude in the presence of a fast transverse Gaussian classical noise with a specific (exponential) two-time correlation function. This solution was simplified and extended to general shape of correlation function by one of the authors (VP) and N. Sinitsyn. In the same work a situation in which the transitions are produced by noise as well as by regular Hamiltonian was considered. Pokrovsky and Scheidl calculated the two-time correlation function of the transition probabilities for the LZ system subject to a fast classical transverse noise. Longitudinal noise was considered by Kayanuma, who proved that strong fast longitudinal noise does not change the LZ transition probability. Gefen et al. and Ao and Rammer considered more wide range of parameters and found the situations in which the noise changes the transition probability. In the work a rather detailed analysis of different limiting cases of temperature, coupling to the phonon bath, its spectral width and sweeping rate was presented. There occurred a controversy between the works and. Generally, there is no complete agreement between different authors on what happens in the adiabatic regime (very slow sweeping) in the presence of the longitudinal noise. Motivated by this disagreement Kayanuma and Nakamura performed a comprehensive analytical and numerical study of the LZ transition in the presence of longitudinal noise. In particular they obtained a formula for the case of strong decoherence which is valid in both low-temperature and high-temperature limits. In all these works the quantum nature of the longitudinal noise was taken in account.

Despite of significant progress a complete theory of the LZ transition in noisy environment still does not exist. Theoretical works considered either quantum longitudinal noise with transitions originated from the regular transition matrix element or the classical transverse noise. Quite recently Wubs et al. have found a beautiful exact formula for the transition probability of the 2-state system interacting with the phonon bath at zero temperature. The noise had both longitudinal and transverse components. Their correlation and quantum nature were substantial. No limitations to the noise strength and spectral width were assumed. However, the limitation of zero temperature (phonon bath is in the ground state) does not allow to extend these results to more realistic situations.

The purpose of this article is to present a theoretical description of the LZ system subject to a fast quantum
noise which has both transverse and longitudinal components. It is not yet complete theory, since it does not cover slow and intermediate noise, but in its range of applicability it allows to understand clearly all relevant physical regimes and phenomena. We will show that, due to the fastness of the noise, the LZ transition in the presence of the longitudinal noise and the transitions due to the transverse noise are separated in time, whereas the correlation between the transverse and longitudinal noise leads to a renormalization of the regular transition matrix element in the LZ Hamiltonian. For a moderately strong transverse noise we derive master equations governing the population of the two states and study their solution. If the transverse noise is strong and also fast, the 2-state system falls into adiabatic regime. The population of levels comes to the equilibrium with the spin bath if the bath is in the state of thermal equilibrium. We argue that a very strong noise is classical and adiabatic. In this situation, as it was shown in Eq.(1), the populations of the two levels are equal.

The plan of the article is as follows. In section 2 we introduce the Hamiltonian and characterize the noise. In section 3 we present simple heuristic arguments resulting in master equations. In section 4 we derive the master equations starting from microscopic Hamiltonian for the case of the transverse noise only and zero LZ transition matrix element. In section 5 we derive the renormalization of the regular transition matrix element due to correlation of longitudinal and transverse noise. In section 6 we analyze the influence of the longitudinal noise. In section 7 find the solution of master equation and study it. In section 8 we match the solution of the master equation with the solution of the LZ problem without transverse noise. Section 9 contains discussion and conclusions. Here we compare our theory with that by Wubs et al.16 We briefly analyze possible applications of our theory to molecular magnets.

II. STATEMENT OF THE PROBLEM

We consider a 2-state system interacting with a noisy environment. The latter is a large system (bath) with a stationary density matrix. We neglect the influence of the LZ transitions onto the state of the bath. For a definiteness we will speak about the phonon bath, though it can include other Boson excitations like spin waves, excitons, photons. Then the total Hamiltonian of the system can be represented as follows:

$$H = H_2 + H_b + H_{int}$$

The term $H_2$ in equation (1) represents the two-state system:

$$H_2 = \frac{\Omega(t)}{2} \sigma_z + \Delta \sigma_x,$$

(2)

where $\sigma_x$ and $\sigma_z$ are Pauli matrices and $\Omega(t)$ is the time-dependent frequency or the energy difference between the so-called diabatic levels turning into zero at $t = 0$; $\Delta$ is the regular transition matrix element. If $t$ would be not time but some parameter of the Hamiltonian, then non-zero $\Delta$ provides repulsion of the adiabatic levels (the Wigner-Neumann theorem on avoided levels crossing). For brevity we will call further the regular transition matrix element $\Delta$ the LZ gap. Usually the linear approximation for the frequency $\Omega(t) = \Omega t$ proposed by Landau and Zener is acceptable, but sometimes it is necessary to go beyond this approximation. Namely, in real experiment the sweeping of $\Omega(t)$ stops at some finite value, which can be not large in the frequency scale of the problem. Therefore we will keep notation $\Omega(t)$ throughout this article. The term $H_b$ in equation (1) is the phonon bath Hamiltonian:

$$H_b = \sum_q \omega_q b_q^\dagger b_q,$$

(3)

where $b_q$ and $b_q^\dagger$ are the operators of the phonon annihilation and creation; $\omega_q$ are the phonon frequencies and $q$ are their momenta. The interaction Hamiltonian reads:

$$H_{int} = u_{\parallel} \sigma_z + u_{\perp} \sigma_x$$

(4)

The Hermitian operators $u_{\parallel}$ and $u_{\perp}$ responsible for the longitudinal and transverse noise depend linearly on the phonon operators. Each of them is a sum of two Hermitian conjugated operators containing either phonon annihilation or creation operators only:

$$u_{\alpha} = \eta_{\alpha} + \eta_{\alpha}^*, \; \eta_{\alpha} = \frac{1}{\sqrt{V}} \sum_q g_{\alpha}(q) b_q; \; \alpha = \parallel, \perp$$

(5)

where $g_{\alpha}(q)$ are complex coupling amplitudes and $V$ is the volume of the system supporting phonons. Quantum character of the noise manifests itself in non-commutativity of operators $\eta_{\alpha}$ and $\eta_{\alpha}^*$. The problem consists in calculation of transition and surviving probabilities for the 2-state system at a fixed noise density matrix. In the absence of the noise, the transitions amplitudes constitute the LZ transition matrix belonging to the SU(2) group and depending on the dimensionless LZ parameter $\gamma_{LZ} = \Delta^2/\Omega$:

$$T_{LZ} = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$$

(6)

$$\alpha = e^{-\pi \gamma_{LZ}} ; \; \beta = \frac{\sqrt{2\pi} \exp \left(-\frac{\pi \gamma_{LZ}}{2} + i\pi \right)}{\sqrt{\gamma_{LZ} + 1} \left(-i\gamma_{LZ} \right)}$$

If $\gamma_{LZ}$ is small, the system with the probability close to 1 remains in initial diabatic state; if $\gamma_{LZ}$ is large the system with probability close to 1 proceeds along the adiabatic state, i.e. changes the initial diabatic state to the alternative one. A characteristic time necessary for the LZ transition is $\tau_{LZ} = \max \left(\Delta/\Omega, \Omega^{-1/2} \right)$. 

The noise correlation time is completely described by the noise correlation functions:

\[
\langle \eta_\alpha (t) \eta_\beta (t') \rangle = \frac{1}{V} \sum_q g_\alpha (q) g_\beta^\ast (q) (n_q + 1) e^{i\omega_q(t-t')}
\]

(7)

\[
\langle \eta_\alpha^\dagger (t) \eta_\beta (t') \rangle = \frac{1}{V} \sum_q g_\alpha (q) g_\beta^\ast (q) n_q e^{i\omega_q(t-t')}
\]

(8)

Here \( n_q = \langle b_q^\dagger b_q \rangle \) are the average phonon occupation numbers and \( \langle ... \rangle \) means averaging over the phonon bath ensemble, not necessarily in thermal equilibrium. The Fourier-components of the correlation functions read:

\[
\langle \eta_\alpha \eta_\beta^\dagger \rangle_\omega = \frac{2\pi}{V} \sum_q g_\alpha (q) g_\beta^\ast (q) (n_q + 1) \delta (\omega - \omega_q)
\]

(9)

\[
\langle \eta_\alpha^\dagger \eta_\beta \rangle_\omega = \frac{2\pi}{V} \sum_q g_\alpha (q) g_\beta^\ast (q) n_q \delta (\omega + \omega_q)
\]

(10)

Note that one of the two correlators contains only positive, whereas the second one contains only negative frequencies. If the noise is in equilibrium at temperature \( T \), the Fourier-transforms of correlation functions obey a simple relation (\( \omega > 0 \)):

\[
\frac{\langle \eta_\alpha \eta_\beta^\dagger \rangle_\omega}{\langle \eta_\alpha^\dagger \eta_\beta \rangle_{-\omega}} = e^{\frac{\pi}{2}}
\]

(11)

Let us denote \( \omega_g \) the range of frequencies in which the coupling coefficients \( g_\alpha (q) \) do not vanish. If the occupation numbers \( n_q \) are of the same order of magnitude for all states within this region of frequencies, then \( \omega_g \) determines the spectral width of the noise \( \Delta \omega \). In some cases, for example at low temperature \( T \ll \omega_g \), there appears a second, smaller scale of frequency (\( T \)). The noise correlation time is \( \tau_n = 1/\Delta \omega \). It is different for the two correlation functions \( \langle \eta_\alpha (t) \eta_\beta^\dagger (t') \rangle \) and \( \langle \eta_\alpha^\dagger (t) \eta_\beta (t') \rangle \) at low temperature and it is equal to \( \omega_g^{-1} \) for both at high temperature. By definition the noise is fast if \( \tau_n \ll \min \left( \Omega^{-1/2}, \Delta^{-1} \right) \) or \( \Delta \omega \gg \max \left( \Omega^{1/2}, \Delta \right) \).

Besides its spectral characteristics the noise is characterized by its strength. The most natural measure of the noise strength is the average square of its amplitude:

\[
\langle u_\alpha^2 (t) \rangle = \langle \eta_\alpha (t) \eta_\alpha^\dagger (t) \rangle + \langle \eta_\alpha^\dagger (t) \eta_\alpha (t) \rangle = \frac{1}{V} \sum_q |g_\alpha (q)|^2 (2n_q + 1)
\]

(12)

The noise is weak if \( \langle u_\alpha^2 (t) \rangle \ll \hat{\Omega} \). Weak noise can be accounted as a small perturbation to the LZ result. We call the noise moderately strong if it obeys the inequality: \( \langle u_\alpha^2 (t) \rangle \ll \tau_n^{-2} \). Though for moderately strong noise the perturbation theory is generally invalid at accumulation time scale determined in the next section, we will show that it works during sufficiently small intervals of time still longer than \( \tau_n \). Most of our results relate to the moderately strong noise. The noise is called strong if \( \langle u_\alpha^2 (t) \rangle \geq \tau_n^{-2} \). During another time scale, which we call the relaxation time \( \tau_{\tau_n} = \left( \langle u_\alpha^2 (t) \rangle \tau_n \right)^{-1} \) (see \ref{5}, where it is called phase relaxation time) the probability of finding the system in state 1 or 2 changes significantly. When the noise is strong, the relaxation time becomes less than the noise correlation time \( \tau_n \). It is convenient to introduce dimensionless coupling function \( \lambda_{\alpha q} = |g_\alpha (q)| / (\omega_g a^2) \), where \( a \) is the lattice constant. In condensed matter systems the values \( \lambda_{\alpha q} \) never become large. Migdall\ref{20} argued that large coupling constants would lead to the lattice instability and reconstruction. Though his arguments related directly to the electron-phonon coupling, his idea is very general. If \( \lambda_{\alpha q} \) are not large, a large value of \( \langle u_\alpha^2 (t) \rangle \tau_n^2 \) can be reached only if phonon occupation numbers \( n_q \) become large. In the equilibrium case it means that the temperature must be large. Very strong noise is classical irrespectively of its specific density matrix. An analogue of the LZ parameter for the noise reads: \( \gamma_n = \langle u_\alpha^2 (t) \rangle / \hat{\Omega} \). If it is small, the noise brings only a small perturbation to the LZ picture; if it is large, the occupation numbers of the 2-state system follow adiabatically the instantaneous value of frequency. We will return to this point later.

III. HEURISTIC APPROACH

We start with an auxiliary problem specifying \( \Delta = 0 \) and \( \eta_\parallel = 0 \), so that all transitions are only due to the transverse noise. Since \( \tau_n \ll \hat{\Omega}^{-1/2} \), the instantaneous frequency of the 2-state system does not change during correlation time of the noise and it can be considered as a constant. Thus, it is possible to calculate the instantaneous rate of transition probability using the standard quantum mechanical technique for transitions between stationary energy levels.

![FIG. 1: Feynman graph for a 3-phonon process. Thin solid lines correspond to the state 1; thick solid lines correspond to the state 2; dashed blue lines correspond to phonons.](image-url)
which must be complemented by conservation law. In the framework of the considered model the next after
single-phonon is three-phonon transition shown in Fig. 1. Its contribution to the transition probability reads:
\[
p_{1\rightarrow2}^{(3)} (t) = 2\pi \left\langle \eta_\perp \eta_\perp \right\rangle_{\Omega (t)} \tag{13}
\]
In the framework of the considered model the next after single-phonon is three-phonon transition shown in Fig. 1. Its contribution to the transition probability reads:

\[
p_{1\rightarrow2}^{(3)} (t) = 2\pi \times \left\langle \int \frac{d\omega_1 d\omega_2 \eta_\omega_1 \eta_\omega_2 \eta_\omega_1 (t) - \omega_1 - \omega_2}{\omega_1 \omega_2} \right. \times \text{herm. conj.} \right\rangle \tag{14}
\]

The 3-phonon contribution can be neglected if the noise is moderately strong. In the same approximation it is possible to neglect the correction to the transition frequency due to the interaction with the phonon bath. Thus, the occurrence numbers of the diabatic states \( N_{1,2} \) at negative time obey following master equation:
\[
\dot{N}_1 = 2\pi \left( -\left\langle \eta_\perp \eta_\perp \right\rangle_{\Omega (t)} N_1 + \left\langle \eta_\perp \eta_\perp \right\rangle_{\Omega (t)} N_2 \right),
\]
which must be complemented by conservation law \( N_1 + N_2 = 1 \). For positive time equation [15] must be modified as follows:
\[
\dot{N}_1 = 2\pi \left( -\left\langle \eta_\perp \eta_\perp \right\rangle_{\Omega (t)} N_1 + \left\langle \eta_\perp \eta_\perp \right\rangle_{\Omega (t)} N_2 \right),
\]
The noise produces transitions as long as its spectral width exceeds the instantaneous frequency \( |\Omega (t)| \). The accumulation time estimated from this requirement is \( \tau_{\text{acc}} = \left( \frac{\Omega}{\tau_n} \right)^{-1} \). Since the noise is assumed to be fast the accumulation time \( \tau_{\text{acc}} \) is much longer than the noise correlation time \( \tau_n \). In real experiment the sweeping of frequency may stop or saturate before the accumulation time is reached. The master equations enable one calculating the occupation numbers at any time rather than asymptotically at \( t \to \infty \). The accumulation time is also much longer than the LZ time \( \tau_{\text{LZ}} \). Therefore, it is possible to neglect the action of the noise during the LZ time interval \( \tau_{\text{LZ}} \) and neglect the LZ gap \( \Delta \) beyond this time interval. It means that the action of the fast transverse noise and of the LZ gap are separated in time as it was earlier shown for classical noise.\( ^{11} \) The solution of the LZ problem without transverse noise and the noise transition problem with zero LZ gap \( \Delta \) should be matched at some intermediate time. It will be done in Section VIII.

The action of the fast longitudinal noise is very different from that of the transverse one. The longitudinal noise does not produce transitions in the absence of the LZ gap. Therefore, its action is effectively limited to the LZ time interval. In the next section we demonstrate that the fast longitudinal noise must be sufficiently strong to produce a substantial change in the LZ transition probability. Namely it must satisfy an inequality \( \langle \eta_\perp \eta_\perp \rangle \simeq \Omega/|\Delta\tau_n| \gg \Omega \). An analogous criterion for the transverse noise is much more liberal: \( \langle \eta_\perp \eta_\perp \rangle \simeq \Omega \). For a comprehensive analysis of the longitudinal noise action we refer the reader to the cited articles.\( ^{12,13,14} \) Beyond the LZ time interval the classical longitudinal noise modulates the transverse noise by a factor \( \exp \left( -i \int_0^t u_\parallel \mathrm{d}\tau \right) \).

Correlation functions \( \left\langle \eta_\perp (t) \eta_\perp (t') \right\rangle \) must be substituted by \( \left\langle \eta_\perp (t) \eta_\perp (t') \exp \left( -i \int_0^t u_\parallel \mathrm{d}\tau \right) \right\rangle \). Neglecting the correlation between longitudinal and transverse noise and employing the Gaussian statistics of the noise, one can express the latter correlator as follows:
\[
\left\langle \eta_\perp (t) \eta_\perp (t') \exp \left( -i \int_0^t u_\parallel \mathrm{d}\tau \right) \right\rangle = \left\langle \eta_\perp (t) \eta_\perp (t') \right\rangle \times \exp \left[ -\frac{1}{2} \int_{t'}^t \int_{t'}^t dt_1 dt_2 \left\langle u_\parallel (t_1) u_\parallel (t_2) \right\rangle \right] \tag{17}
\]

The transverse noise correlator decays rapidly when the modulus of difference \( |t - t'| \) exceeds \( \tau_n \). Therefore, the value in the exponent in the right-hand side of equation [17] can be estimated as \( \left\langle u_\parallel^2 \right\rangle \tau_n^2 \ll 1 \). This estimates shows that the longitudinal noise can be neglected beyond the LZ time interval. In the next section we consider this question in more details.

IV. DERIVATION OF MASTER EQUATIONS.

![FIG. 2: An example of a term in the perturbation theory. Points correspond to vertexes \( V_I (t_j) \).](image)

Our goal is to find the dependence of the occupation numbers \( N_{\alpha} \) (\( \alpha = 1, 2 \)) on time. The same problem can be formulated as calculation of the average value of projectors \( P_\alpha = |\alpha \rangle \langle \alpha | = \frac{1}{2} (1 \pm \sigma_z) \). We consider the case \( \Delta = 0, u_\parallel = 0 \). The calculation will be performed in the interaction representation with the diagonal time-dependent unperturbed Hamiltonian \( H_0 = -\frac{\Omega(t)}{2} \sigma_z + H_b = -\frac{\Omega(t)}{2} (|1 \rangle \langle 1 | - |2 \rangle \langle 2 |) + H_b \) and the interaction Hamiltonian \( V = u_\perp \sigma_\alpha = u_\perp |1 \rangle \langle 2 | + |2 \rangle \langle 1 | \). Being transformed to the interaction representation, the interaction Hamiltonian depends on time as follows:
\[
V_I (t) = u_\perp (t) \left( |1 \rangle \langle 2 | + \text{e}^{-i \int_0^t \Omega (\tau) d\tau} [2 \langle 1 | \text{e}^{i \int_0^t \Omega (\tau) d\tau} \right), \tag{18}
\]
where
\[ u_\perp(t) = e^{iH_0(t-t_0)} u_\perp(t_0) e^{-iH_0(t-t_0)} \]  
\[ (19) \]

The time-dependent occupation numbers can be expressed in terms of the evolution operator \( U_I(t, t_0) \) in the interaction representation:
\[ N_\alpha(t) = \text{Tr} \left[ \rho_0 U_I^{-1}(t, t_0) P_\alpha U_I(t, t_0) \right] \]  
\[ (20) \]

where \( \rho_0 \) is the initial density matrix which is the direct product of two independent density matrices \( \rho_0 = \rho_2 \rho_0 \), where the first factor is the density matrix of the 2-state system and the second one is the same for the bath. For calculation or these averages we employ a simplified version of the Keldysh-Schwinger technique\textsuperscript{21,22} used already for a similar purpose in\textsuperscript{10,11,12}. Each of the two evolution operators is presented as a series of time ordered integrals. A general term of such an expansion contains a product of two multiple time integrals. With each time variable \( t_k \) a vertex \( V_I(t_k) \) is associated. The product of vertices is ordered chronologically in \( U_I(t, t_0) \) and antichronologically in \( U_I^{-1}(t, t_0) \). All operators of \( V_I(t_k) \) belonging to \( U_I^{-1}(t, t_0) \) are located on the left ("later") than corresponding operators belonging to \( U_I(t, t_0) \). A particular contribution is graphically depicted in Fig. 2 after the averaging over phonon bath is performed. It consists of two lines both starting at \( t_0 \) and ending at \( t \). The upper line symbolizes \( U_I(t, t_0) \) and the lower one symbolizes \( U_I^{-1}(t, t_0) \). Vertices on these lines correspond to the operators \( V_I(t_k) \). Each vertex contains one phonon operator \( u(t_k) \) and changes the state of the 2-state system. The presence of the projection operator \( P_\alpha \) in equation (20) implies that the state closest to the final time \( t \) on both lines must be \( |\alpha\rangle \). Applying the Wick’s rule for phonons, one should form all possible pairing of (phonon) noise lines\textsuperscript{23}. However, in the case of fast moderately strong noise the main contribution to the occupation numbers comes from the graphs without overlapping or overlapping of the phonon lines. An example of a graph without overlapping is shown in Fig. 3. In comparison with elementary graphs without phonon line overlapping or crossing (Fig. 4a, b) the contributions of the graphs containing overlapping or crossing (Fig. 4c, d) have additional small factors of the order of \( \langle u_\perp^4 \rangle \tau_n^2 \) and can be neglected if the transverse noise is moderately strong. Indeed the time interval between the ends of each phonon lines is about \( \tau_n \).

FIG. 3: A typical graph without phonon line crossings dominantly contributing to the survival and transition probability.

To derive the master equation we consider a set of graphs differing the occupation numbers \( N_\alpha(t+\Delta t) \) from those for \( N_\alpha(t) \) with \( \Delta t \) satisfying the following strong inequality \( \tau_n \ll \Delta t \ll \langle \langle u_\perp^4 \rangle \rangle^{-1} \). First, there are graphs with phonon lines connecting the interval \( (t, t+\Delta t) \) with the interval \( (t_0, t) \). Their contribution can be neglected since it is relatively proportional to a small ratio \( \tau_n/\Delta t \). The contribution of \( k \) non-overlapping or intersecting noise lines inside the interval \( (t, t+\Delta t) \) is proportional to \( \langle \langle u_\perp^4 \rangle \rangle \tau_n \Delta t \rangle^k \ll 1 \). Therefore, the dominant contribution to the set comes from graphs containing exactly zero or one line inside the interval \( (t, t+\Delta t) \). Note that it is a coarse-grain description: the master equation is invalid at time scale \( \tau_n \) and shorter. The graphical equation connecting \( N_\alpha(t+\Delta t) \) with \( N_\alpha(t) \) is shown in Fig. 5.

FIG. 5: Graphic equation connecting \( N_\alpha(t+\Delta t) \) and \( N_\alpha(t) \)

To find their analytical expression, we calculate the contribution of the 3 elementary subgraphs for \( \alpha = 1 \) shown in Fig. 6.

FIG. 6: Three elementary graphs with one phonon line in the interval \( t, t+\Delta t \).

They read:
\[ \Gamma_1 \approx \int_0^{\Delta t} dt \int_{t}^{t+\Delta t} dt' \left\langle u_\perp(t_1) u_\perp(t_2) \right\rangle e^{i \int_{t_1}^{t_2} i\Omega(\tau) d\tau} \]
\[ \approx 2\pi \left\langle u_\perp u_\perp \right\rangle \Omega(t) \Delta t \]  
\[ (21) \]
In this calculation we used the fastness of the noise \( \tau_n \ll \Delta t, \tau_n \ll \sqrt{\Omega} \) to substitute the integral in the exponent by \( \Omega(t) (t_1 - t_2) \) and to extend the integration over the difference \( t_1 - t_2 \) to infinite limits. The graph 6a connects \( N_1 (t + \Delta t) \) to \( N_2 (t) \), two others graphs connect \( N_1 (t + \Delta t) \) to \( N_1 (t) \). Different signs in the contributions \( (21) \) and \( (22, 23) \) are associated with the fact that the vertex at the upper line contains a factor \(-i\), whereas it acquires the factor \(+i\) at the lower line.

Collecting all contributions together, we arrive at the Master equations \( (24, 25) \). We rewrite them in a unified form:

\[
\frac{dN_1}{dt} = 2\pi \left[ N_2 \left[ \theta (\Omega) \langle \eta \eta^\dagger \rangle_\Omega + \theta (-\Omega) \langle \eta^\dagger \eta \rangle_\Omega \right] - N_1 \left[ \theta (-\Omega) \langle \eta \eta^\dagger \rangle_{-\Omega} + \theta (\Omega) \langle \eta^\dagger \eta \rangle_{-\Omega} \right] \right]_{\Omega=\Omega(t)} \tag{24}
\]

where \( \theta(x) \) is the step function equal to 1 at positive and 0 at negative arguments. The master equation looks simpler when the variable \( s_z = \frac{N_1-N_2}{2} = N_1 - \frac{1}{2} \) is used:

\[
\frac{ds_z}{dt} = 2\pi \left[ -s_z \times \left[ \langle \eta \eta^\dagger \rangle_{\|\Omega} + \langle \eta^\dagger \eta \rangle_{-\|\Omega} \right] - \frac{1}{2} \text{sign} \Omega \times \left[ \langle \eta \eta^\dagger \rangle_{\|\Omega} - \langle \eta^\dagger \eta \rangle_{-\|\Omega} \right] \right]_{\Omega=\Omega(t)} \tag{25}
\]

The Fermi Golden Rule or the first Born approximation at a fixed moment of time can be applied since the frequency variation during the noise correlation is small and the perturbation caused by the noise in the corresponding stationary problem is weak. The perturbation theory is valid if \( |u_\perp| \ll \Omega(t) \). Within the accumulation time interval the instantaneous frequency is of the same order of magnitude as the spectral width of the noise: \( \Omega(t) \sim \tau_n^{-1} \). The same inequality ensures that the frequency exceeds the width of the levels and the change of frequency due to the interaction with the noise.

Now the question about the influence of the longitudinal noise onto the master equation is in order. First we demonstrate that correlations of the type \( \langle u_{\|} (t) u_{\|} (t') \rangle \) do not change the master equation. Indeed, let us consider the influence of the longitudinal noise onto the difference between \( N_{\alpha} (t + \Delta t) \) and \( N_{\alpha} (t) \). In analogy with the case of the transverse noise, the contribution of one-longitudinal phonon line inside the interval \( (t, t+\Delta t) \) must be taken in account. This contribution does not depend on preceding evolution of the system. Therefore, it is the same as it would be in the absence of the transverse noise. Since the longitudinal noise does not produce transitions in the absence of the transverse noise, the total contribution of 3 graphs of Fig. 6 for longitudinal noise is zero.

V. RENORMALIZATION OF THE LZ GAP

![FIG. 7: Graphs containing mixed noise correlator (dash-dot line) and responsible for the LZ gap renormalization.](image)

The problem of mixed correlations between transverse and longitudinal noise is more subtle. The line of mixed correlation starts at one state, to say 1, and ends at another one (2) as shown in Fig. 7. The self-energy part associated with the graphs of the Fig. 7 reads:
\[ \Pi_{mix} = |1\rangle \langle 2| e^{-i \int_{t_0}^{t_1} \Omega(t) d\tau} \times \left[ \int_{-\infty}^{0} \langle u_{\perp} (t') u_{\perp} (0) \rangle dt' - \int_{0}^{\infty} \langle u_{\perp} (t') u_{\parallel} (0) \rangle dt' \right] + \text{herm. conj.} \] (26)

This operator has the same form as the operator \( \Delta \sigma_x \) in the interaction representation. Thus, at a time scale much longer than \( \tau_n \), the mixed correlation renormalizes the LZ gap to a value

\[ \tilde{\Delta} = \Delta + i \int_{0}^{\infty} \langle [u_{\perp} (t'), u_{\perp} (0)] \rangle dt' \] (27)

For the transformation of the integrals in equation (26) into integral in equation (27) we employed the time-translation invariance: \( \langle u_{\parallel} (-t) u_{\perp} (0) \rangle = \langle u_{\parallel} (0) u_{\perp} (t) \rangle \). Thus, the statement that one can neglect the action of the transverse noise within the LZ time interval is not completely correct: it is legitimate to neglect transverse-transverse correlations, but the mixed correlations can significantly change the LZ gap up to turning it into zero and changing its sign.

The commutator entering equation (27) does not depend on the phonon occupation numbers, i.e. on temperature. It is instructive to express the renormalization of the LZ gap in terms of the phonon model (Section II equations (5) (7) (8)):

\[ \tilde{\Delta} - \Delta = -\frac{1}{V} \sum_{q} g_{\parallel}(q) g_{\perp}(q) \] (28)

\[ \left\langle \prod_{j} \exp \left[ \pm 2i \int_{t_0}^{t_1} u_{\parallel} (\tau_j) \ d\tau_j \right] \right\rangle = \exp[-2^{2k-1} \int_{t_0}^{t_1} d\tau_1 \cdots \int_{t_0}^{t_2} d\tau_{2k} \sum_{c} (\pm \langle u_{\parallel} (j_1) u_{\parallel} (j_2) \rangle \langle u_{\parallel} (j_3) u_{\parallel} (j_4) \rangle + \cdots)] \] (30)

where summation is performed over all possible divisions of arguments \( \tau_j \) into pairs. Each correlator vanishes if the modulus of corresponding time difference \( |\tau - \tau'| \) exceeds \( \tau_n \). The substantial range of integration over remaining variable \( (\tau + \tau')/2 \) is about \( \tau_{LZ} = \Delta/\Omega \). Therefore the order of magnitude of the number obtained in the exponent (30) after integration is \( \sim \left( \langle u_{\parallel}^2 \rangle \tau_n \tau_{LZ} \right)^k \). This value must be of the order or larger than 1 to ensure a significant change of the transition probability by the longitudinal noise. This requirement is equivalent to the inequality \( \langle u_{\parallel}^2 \rangle \gtrsim \Omega/ (\Delta \tau_n) \gg \Omega \). For the transverse noise the analogues criterion is much softer: \( \langle u_{\perp}^2 \rangle \gtrsim \Omega \).

VI. LONGITUDINAL NOISE

Next we consider the action of the purely longitudinal noise within the LZ time interval. For this problem the transverse noise will be ignored. If the noise is classical, the proper diagonal Hamiltonian is \( H_0 = \left( -\frac{\Omega(t)}{2} + u_{\parallel} \right) \sigma_z + H_b \), whereas the non-diagonal part is \( V = \Delta \sigma_x \). In the interaction representation the non-diagonal part acquires the following form:

\[ V_I (t) = \Delta \left( |1\rangle \langle 2| e^{i \int_{t_0}^{t_1} (\Omega - 2u_{\parallel}) d\tau} + \text{herm. conj.} \right) \] (29)

Calculation of the transition probability is very similar to that considered above (see equation (20) and Fig. 2), but the vertexes correspond to \( V_I (t) \) given by (29) (we will call them \( \Delta \)-vertexes) and instead of connecting pairs of noise amplitudes it is necessary to calculate average of a product \( \prod_{j} \exp \left[ \pm 2i \int_{t_0}^{t_1} u_{\parallel} (\tau_j) \ d\tau_j \right] \). Number of the signs – in the exponent is equal to the number of signs +. Therefore, the dependence on the initial moment of time \( t_0 \) vanishes. The Gaussian statistics allows to calculate the average of the product:

The quantum longitudinal noise does not commute with itself at different moments of time. Therefore, it must be included into the non-diagonal Hamiltonian. Then, besides of \( \Delta \)-vertexes one should consider the noise vertexes. We obtain the same estimate considering the contribution of an elementary graph with one noise line. However, for the fast longitudinal noise it is possible to find not only an estimate, but equations for the transition amplitudes. First we prove that the contribution of the irreducible graphs\( \dagger \), which do not connect different branches of the Keldysh contour, is zero. Indeed the remote ends of such graphs are separated by the time interval of the order of \( \tau_n \ll \tau_{LZ} \). Therefore it is possible to integrate over the time difference at a fixed "center of
time" or "slow time" as we did in the case of the transverse noise. In contrast to the latter case, for longitudinal noise the integral does not depend on slow time since the longitudinal noise vertex in the interaction representation does not contain time-dependent phase factor (it connects identical states of the 2-state system). By the same reason it is the same for states 1 and 2. Therefore in a slow time scale much longer than \( \tau_n \) (but much less than \( \tau_{LZ} \)) the contribution of such graphs is proportional to unit operator for the 2-state system and can be completely ignored. The irreducible noise graphs connecting different branches of Keldysh contour form a 4-pole vertex (see Fig. 8), which by the same reason does not depend on slow time and connects identical states at each branch of the Keldysh contour. We denote this vertex \( \Gamma \).

\[ \text{FIG. 8: Graphs containing the longitudinal noise only. Triangles correspond to the LZ gap } \Delta. \]

a). graphs with one noise loop. In the slow time scale they are equivalent to addition of a constant energy.

b). graphs with a line connecting Keldysh branches.

c). general graphic equation for \( P \). See explanation in the text.

Since the number of \( \Delta \)–vertices between vertexes \( \Gamma \) is arbitrary we need to extend the number of amplitudes in consideration. Namely, we define transition amplitudes \( P_{\alpha \beta, \alpha_0 \beta_0} (t, t_0) \) as the average value of the operator \( |\alpha\rangle \langle \beta| \) at the moment \( t \) if the density matrix for 2-state system at the moment \( t_0 \) was \( |\alpha_0\rangle \langle \beta_0| \). In the accepted approximation these amplitudes obey a system of linear integral equations:

\[
P_{\alpha \beta, \alpha_0 \beta_0} (t, t_0) = P_{\alpha \beta, \alpha_0 \beta_0}^{(0)} (t, t_0) - \Gamma \int_{t_0}^{t} P_{\alpha \beta, \alpha_0 \beta_0}^{(0)} (t', t_0) P_{\alpha' \beta', \alpha_0 \beta_0} (t', t_0) dt'
\]

(31)

Here \( P_{\alpha \beta, \alpha_0 \beta_0}^{(0)} (t, t_0) \) denotes the transition amplitude in the absence of noise. The validity of equations (31) is limited by moderately strong noise. Otherwise the amplitudes \( P_{\alpha \beta, \alpha_0 \beta_0} (t, t_0) \) vary significantly at a time scale \( \tau_n \). In this case more complicated equations with a nonlocal in time kernel \( \Gamma \) and amplitudes depending on 4 time arguments must be used.

VII. SOLUTION OF THE MASTER EQUATION AND NOISE DIAGNOSTIC

The master equation (25) allows an explicit solution:

\[
s_z (t) = s_z (t_0) \exp \left( - \int_{t_0}^{t} f (\tau) d\tau \right) + \int_{t_0}^{t} g (t') \exp \left( - \int_{t'}^{t} f (\tau) d\tau \right) dt',
\]

(32)

where \( f (t) = 2\pi \left( \langle n n \rangle_{|t|} + \langle n^\dagger n^\dagger \rangle_{-|t|} \right) \) and \( g (t) = -\pi \text{sign} \Omega (t) \left( \langle n n \rangle_{|t|} - \langle n^\dagger n^\dagger \rangle_{-|t|} \right) \). For classical noise \( g (t) = 0 \), and equation (32) reproduces the result obtained in the reference (23). At zero temperature \( \langle n n \rangle_{|t|} \) = 0, and \( g (t) = -\frac{1}{2} \pi \text{sign} \Omega (t) f (t) \). In these two cases the measurement of occupation numbers or \( s_z (t) \) gives direct information on spectral power of noise.

In classical case we find 4 \( \pi \left( \langle n n \rangle^{\dagger}_{|t|} - \langle n^\dagger n^\dagger \rangle_{-|t|} \right) = \frac{1}{4} \ln |s_z + \text{sign} \Omega (t)| \). In general case it is possible to find both spectral functions \( f (t) \) and \( g (t) \) by performing two series of measurements with different initial states. Thus, the 2-state system is an ideal noise analyzer.

Next we consider regimes of very fast and very slow (adiabatic) frequency sweeping. In the regime of fast sweeping \( \langle n^\dagger n \rangle \ll \Omega \) the perturbation theory for equation (32) is valid. Indeed the integral \( \int_{t_0}^{t} f (\tau) d\tau \) can be rewritten in terms of spectral power as follows:

\[
\int_{t_0}^{t} f (\tau) d\tau = 2\pi \Omega^{-1} \int_{\Omega (t_0)}^{\Omega (t)} \left( \langle n n \rangle_{|\omega|} + \langle n^\dagger n^\dagger \rangle_{-|\omega|} \right) d\omega
\]

(33)

The integral in the r.-h. side of equation (33) reaches its maximum value, equal to the average square fluctuation \( \langle n n \rangle^2 \) at \( \Omega (t_0) = -\infty \) and \( \Omega (t) = +\infty \). If the condition of fast sweeping is satisfied, the exponent in equation (32) can be expanded into a series over small noise parameter \( \gamma_n = \langle n^\dagger n + n n \rangle / \Omega \). The variation \( \Delta s_z (t) = s_z (t) - s_z (t_0) \) is small at any time. In the leading approximation it reads:

\[
\Delta s_z (t) = -2\pi \Omega^{-1} \int_{\Omega (t_0)}^{\Omega (t)} \left( \langle n n \rangle_{|\omega|} (s_z (t_0) + \text{sign} \omega) \right) d\omega
\]

(34)

\[+ \langle n^\dagger n^\dagger \rangle_{-|\omega|} (s_z (t_0) - \text{sign} \omega) \right) d\omega
\]

In the opposite regime of slow (adiabatic) sweeping the noise parameter \( \gamma_n \) is large. In this case the exponents in equation (32) vary very rapidly allowing asymptotic calculation of \( s_z \). However, in adiabatic regime it is simpler to start directly with the Master equation (25). Neglecting the time derivative in it, we find the adiabatic
solution:
\[
    s_z(t) = \frac{g(t)}{f(t)} = -\frac{\text{sign}(t)}{2} \left( \langle \eta \eta \rangle_{\Omega(t)} - \langle \eta \eta \rangle_{-\Omega(t)} \right).
\]

If the photon bath is in equilibrium with temperature \( T \), equation (35) implies \( s_z(t) = -\tanh \frac{\Omega(t)}{2T} \). As it could be expected, at slow sweeping the two-state system adiabatically accepts the equilibrium population with the temperature of the bath. This conclusion shows that in the case of the quantum noise one must be more careful with the asymptotic behavior of the time-dependent frequency than in genuine LZ problem or even in the analogous problems with the classical noise. In the latter problems the linear approximation for \( \Omega(t) = \hat{\Omega} t \) was satisfactory. However, this approximation may be invalid for the quantum noise if the sweeping stops before the frequency \( \Omega(t) \) reaches the spectral width of the noise. In the opposite case the value \( s_z(t) \) saturates after \( t = \tau_{\text{acc}} \). In classical adiabatic case \( s_z(t) \) becomes zero after a short time \( \tau_{\text{tr}} = \left( \frac{\langle \eta \eta \rangle}{2} \right)^{-1} \). A similar time scale for the longitudinal noise was introduced by Kayanuma and Nakamura [18].

At the edge of the adiabatic regime \( \gamma_n \sim 1 \) the fast noise is still moderately strong, i.e. \( \sqrt{\langle \eta \eta \rangle} \ll \tau_n^{-1} \). It means that, when the noise becomes strong \( \sqrt{\langle \eta \eta \rangle} \gg \tau_n^{-1} \), the system is already in deeply adiabatic regime. If the phonon bath is in equilibrium, the 2-state system also is in equilibrium with the noise. This equilibrium state is established in a time-independent, but strongly nonlinear system. The interaction of the two-level system characterized by the time-independent frequency \( \Omega \) with the strong noise renormalizes the frequency and creates a finite width for each level. The situation become simpler in the limit of very strong noise \( \sqrt{\langle \eta \eta \rangle} \gg \tau_n^{-1} \gg \Omega \). In this case the initial energy difference \( \Omega \) between levels can be neglected. The two states become equivalent and their occupation numbers are equal \((1/2)\), i.e. \( s_z = 0 \). The same result can be obtained from the fact that, as we already argued, the very strong noise must be classical. Equation (35) can be considered as an interpolation between weak and very strong noise. Therefore, it gives a reasonable description of intermediate regime.

VIII. TRANSITIONS IN THE PRESENCE OF THE LZ GAP AND NOISE

As we demonstrated earlier, for the fast moderately strong noise, the effective time of the LZ transition due to the regular LZ gap \( \tau_{\text{LZ}} = \Delta/\hat{\Omega} \) is much less than the accumulation time \( \tau_{\text{acc}} = \left( \hat{\Omega} \tau_n \right)^{-1} \). Therefore it is possible to ignore the transverse noise within the LZ time interval \( |t| \lesssim \tau_{\text{LZ}} \) and to ignore the LZ gap \( \Delta \) beyond this interval. In this section we match the LZ solution modified by longitudinal noise inside the LZ interval with the solution of the problem with the transverse noise and \( \Delta = 0 \) (see Section VII) beyond this interval. For this purpose we choose a time scale \( t_1 \) such that \( \tau_{\text{LZ}} \ll t_1 \ll \tau_{\text{acc}} \) and first consider the solution (32) of the noise problem with \( \Delta = 0 \) at \( t = -t_1 \). For simplification we accept \( t_0 = -\infty \). Since \( t_1 \ll \tau_{\text{acc}} \) it can be replaced by 0 in the solution (32) with high precision \( \sim t_1/\tau_{\text{acc}} \). Thus, at the left edge of the interval \((-t_1, t_1)\) we find:

\[
    s_z(-t_1) \simeq s_z(-\infty) = \exp \left( -\int_{-\infty}^{0} f(\tau) d\tau \right) s_z(-\infty) + \int_{-\infty}^{0} g(t') \exp \left( -\int_{t'}^{0} f(\tau) d\tau \right) dt'.
\]

This value can be treated as an initial condition \( s_z(-\infty) \) at \( t = -\infty \) for the LZ problem with the longitudinal noise. If the solution of this problem is known, the value \( s_z(\infty) \) at \( t = +\infty \) can be calculated. The information necessary to make this calculation effective is the knowledge of two numbers if there is no coherence in the initial system. The density matrix \( \rho(\infty) \) at \( t = +\infty \) is obviously a linear function of the initial density matrix \( \rho(-\infty) \). There exists a linear 4\times4 matrix \( \Lambda \) performing this transformation:

\[
    \rho_{\alpha\beta}^{(\infty)} = \Lambda_{\alpha\beta,\mu\nu} \rho_{\mu\nu}^{(-\infty)}
\]

The requirement that \( \text{Tr}\rho^{(+)} = 1 \) if \( \text{Tr}\rho^{(-)} = 1 \) implies the following equation: \( \lambda_{\alpha\beta,\mu\nu} = \delta_{\mu\nu} \). If \( \rho_{\text{LZ}}^{(-)} = \rho_{21}^{(-)} = 0 \), the equation (37) results in following relationship between \( s_z^{(+)} \) and \( s_z^{(-)} \):

\[
    s_z^{(+)} = (A_1 + A_2) s_z^{(-)} + (A_1 - A_2),
\]

where we introduced abbreviations \( A_1 \) and \( A_2 \) for \( A_{11,11} \) and \( A_{22,22} \), respectively. If the longitudinal noise is absent or sufficiently weak, the LZ values for \( \Lambda_n \) are:

\[
    A_1 = A_2 = \exp \left( -2\pi \gamma_n \tau_{\text{LZ}} \right) - \frac{1}{2}
\]

If \( \langle u_{n}^{2} \rangle \ll \hat{\Omega}/(\Delta \tau_n) \), the longitudinal noise is weak enough to neglect the longitudinal-longitudinal correlations. Still the correlation of the longitudinal and transverse noise can significantly change the effective LZ gap (see equation (27)).

The value \( s_z^{(+)} \) from equation (39) serves in turn as initial condition at \( t = +0 \) for the Master equation. Its solution (39) at \( t = +\infty \) leads to the final result:
\[ s_z(+\infty) = (\Lambda_1 + \Lambda_2)e^{-2\pi \gamma_L s_z(-\infty)} + (\Lambda_1 - \Lambda_2)e^{-\pi \gamma_L \pi \Omega} \]

\[
\int_0^\infty \left( \langle \eta^\dagger \eta \rangle_\Omega - \langle \eta^\dagger \eta \rangle_{-\Omega} \right) \exp \left[ -\frac{2\pi}{\Omega} \int_0^\infty \left( \langle \eta^\dagger \eta \rangle_\omega + \langle \eta^\dagger \eta \rangle_{-\omega} \right) d\omega \right] \left[ (\Lambda_1 + \Lambda_2)e^{-\frac{4\pi}{\Omega} \int_0^\infty \langle \eta^\dagger \eta \rangle_\omega + \langle \eta^\dagger \eta \rangle_{-\omega} d\omega} - 1 \right] d\Omega
\]

Here \( \gamma_L = \langle u^2 \rangle / \Omega \). We remind that the occupation numbers are related to \( s_z \) as \( N_{1,2} = 1/2 \pm s_z \). Below we write the survival probability for the case when the longitudinal-longitudinal correlations can be neglected.

\[ P_{1 \rightarrow 1} = \frac{1}{2} \left[ 1 + e^{-2\pi \gamma_L \left( 2e^{-2\pi \gamma_L} - 1 \right)} \right] + \frac{\pi}{\Omega} \]

\[
\int_0^\infty \left( \langle \eta^\dagger \eta \rangle_\Omega - \langle \eta^\dagger \eta \rangle_{-\Omega} \right) \exp \left[ -\frac{2\pi}{\Omega} \int_0^\infty \left( \langle \eta^\dagger \eta \rangle_\omega + \langle \eta^\dagger \eta \rangle_{-\omega} \right) d\omega \right] \left[ (2e^{-2\gamma_L} - 1)e^{-\frac{4\pi}{\Omega} \int_0^\infty \langle \eta^\dagger \eta \rangle_\omega + \langle \eta^\dagger \eta \rangle_{-\omega} d\omega} - 1 \right] d\Omega
\]

This result with precision of notations coincides with the exact result by Wubs et al., equations (6-8), obtained without any limitations to the strength of noise and ratios of characteristic time scales. Surprisingly the multi-phonon processes as well as the longitudinal-longitudinal noise correlations do not contribute at all to the survival and transition probabilities even at very high noise intensity. At zero temperature such a high noise level can be reached only by enhancement of the coupling amplitudes. Though large coupling amplitudes are physically

\[ \langle u^2 \rangle / \Omega \ll (\Delta \tau_n)^{-1} \]. In this case \( s_z(-\infty) = 1/2 \), the values \( \Lambda_{1,2} \) are determined by equation (63) and from equation (40) we find:

\[ P_{1 \rightarrow 1} = \exp \left( -2\pi \left( \gamma_L + \gamma_L \right) \right) = \exp \left[ -\frac{2\pi}{\Omega} \left( \Delta - \frac{1}{V} \sum_q g_\parallel (q) g_\parallel (q) \right) \right] + \left( \eta_\parallel (0) \eta_\parallel (0) \right) \]

\[ IX. \ DISCUSSION \ AND \ CONCLUSIONS \]

In the case of weak transverse noise or very fast sweeping \( \gamma_L \ll 1 \) equations (40, 41) turn into the result (63) and the LZ survival probability, respectively. In the opposite case of the strong transverse noise or slow sweeping \( \gamma_L \gg 1 \) the occupation numbers accept their stationary values at fixed instantaneous frequency independently on the value of LZ parameter \( \gamma_L \). The classical noise corresponds to large phonon occupation numbers \( n_\parallel \). In this case the operators \( \eta \) and \( \eta^\dagger \) commute; all terms containing commutators \( \langle \eta^\dagger \eta \rangle_\Omega - \langle \eta^\dagger \eta \rangle_{-\Omega} \) can be neglected. Then theory reproduces the result for classical fast noise. It is instructive to compare equation (41) with the exact survival probability for \( T = 0 \) obtained in the recent article by Wubs et al.. At zero temperature the average value \( \langle \eta^\dagger \eta \rangle \) as well as its Fourier transform turns into zero. This fact allows to calculate the integrals in equation (41). More physically visible way of obtaining the same result is to keep in mind that there is no live phonon at \( T = 0 \) and only spontaneous emission of phonons is possible. Therefore, if initially only the lower state was populated, the phonon cannot be emitted before the levels crossing. This consideration immediately gives \( s_z(-\infty) = s_z(+\infty) = 1/2 \) and \( s_z(+) = \frac{1}{2} \left( 2e^{-2\gamma_L} - 1 \right) \). Employing general equation (62), we find the value \( s_z(+\infty) = \frac{1}{2} \left[ 2 \exp \left( -2\pi \left( \gamma_L + \gamma_L \right) \right) - 1 \right] \) and the survival probability:

\[ P_{1 \rightarrow 1} = \exp \left( -2\pi \left( \gamma_L + \gamma_L \right) \right) = \exp \left[ -\frac{2\pi}{\Omega} \left( \Delta - \frac{1}{V} \sum_q g_\parallel (q) g_\parallel (q) \right) \right] + \left( \eta_\parallel (0) \eta_\parallel (0) \right) \]
implausible, as a mathematical model they are absolutely legitimate. The fact that these high-intensity processes do not play role in the transitions supports our speculations about possible extension of the master equation beyond their range of validity at least as a reasonable interpolation procedure.

Our theory is relevant to molecular magnets, first of all because the condition of the noise fastness is perfectly satisfied in the experiment. Indeed, the highest magnetic field rate used in the experiments with Fe₈ and Mn₁₂ was $10^3$ Gs/s. This rate corresponds to $\dot{\Omega} = 10^{10}s^{-2}$. The lowest temperature used in the cited measurements was about 0.05K. The dimensionless ratio of the value $\sqrt{\dot{\Omega}}$ to the smaller of the noise spectral widths is $\hbar \sqrt{\dot{\Omega}}/T \sim 10^{-5}$. The estimates of the noise intensity is not so simple. The transverse noise is very weak for transitions with a large change of the spin projection, for example, from +10 to -10, since the standard coupling of the deformations to spin can change the spin projection only by ±1, ±2. Therefore, the transverse noise for such a transition appears only in the 10-th order of the perturbation caused by acoustic phonons. However, for transitions with the small change of the spin projection the transverse noise is of the same order of the perturbation caused by acoustic phonons. Thus, $|\mathbf{q}|^2$, which is proportional to $q$ at small wave vectors. Thus, $\sqrt{\langle u_1^2 \rangle} \sim E_a (E_a/E_D)^4 \ll E_a = \hbar \omega_q$. This inequality shows that the condition of moderately strong noise is well satisfied, whereas the value $\langle u_1^2 \rangle / \hbar^2 \dot{\Omega}$ is large. This fact together with the experimental fact that the hysteresis curve significantly narrows at $T \sim 2 – 3K$ shows that the noise level is overestimated by this simple formula and more accurate theory is necessary. However, there is no doubt, that the thermal noise becomes important at a temperature of few Kelvin and that the noise is fast and quantum. Though the longitudinal noise does not produce transitions between diabatic levels, it is effective for transitions between adiabatic levels.

In conclusion, we derived the master equations for moderately strong fast quantum transverse noise. We obtained its solution at any moment of time. We demonstrated that the action of the regular LZ transition matrix element and the longitudinal-longitudinal noise correlation is limited by the LZ time scale, whereas the action of the transverse noise is accumulated during much longer accumulation time. We showed that the mixed longitudinal-transverse noise correlation leads to the renormalization of the LZ gap $\Delta$. In the limiting case of adiabatic transverse noise the 2-state system adiabatically follows its stationary state at an instantaneous value of frequency independently on the value of LZ parameter. The separation of time scales allows to derive exact transition probability with the LZ gap, longitudinal and transverse noise taken in account simultaneously. The transition probabilities depend explicitly on the noise commutator reflecting the quantum nature of the noise. It plays an important role especially in the adiabatic regime. In the extreme quantum regime at zero temperature our result coincides with exact result by Wubs et al. We argued that the strong-noise effects such as multiphonon processes and change of frequency appear only in the adiabatic regime for the fast noise and do not change substantially the transition probability.

Acknowledgement 1. Our thanks due to Drs. N. Sinitsyn, Y. Gefen and A. Abanov for useful discussions and B. Dobrescu for participation in the initial stage of this work and a valuable remark. This work was supported by the DOE under the grant DE-FG02-06ER46278 and partly by the NSF under the grant DMR-0321573. VP acknowledges the hospitality and support of the Max Planck Institut für Komplexe Systeme, Dresden, during the Workshop on Mesoscopics, August-September 2006.

1 L.D. Landau, Phys. Z. Sowietunion, 2, 46 (1932).
2 C. Zener, Proc. R. Soc. A137, 696 (1932).
3 E.C.G. Stuckelberg, Helv. Phys. Acta 5, 369 (1932).
4 A.J. Legget, S. Chakravarty, A.T. Dorsey, M.P.A. Fisher, A. Garg, and W. Zwerger, Rev. Mod. Phys. 59,1 (1987).
5 U. Weiss, Quantum Dissipative Systems, World Scientific, Singapore, 1998.
6 A. Izmalkov, M. Grajcar, E. Ilichev, N. Oukhanski, Th. Wagner, H.-G. Meyer, W. Krech, M. H. S. Amin, Alec Brink van den Maassen, A. M. Zagoskin, Europhys. Lett. 65, 844 (2004); I. Chiorescu, P. Bertet, K. Samba, Y. Nakamura, C.J.P.M. Harmans, and J.E. Mooij, Nature 431, 159 (2004).
7 M. Kusunoki, Phys. Rev. B 20, 2512 (1979).
8 Y. Kayanuma, J. Phys. Soc. Jpn. 54, 2037 (1985)
9 The noise is called transverse if it produces transitions between the two states in the absence of regular transition matrix element.
10 Y. Kayanuma, J. Phys. Soc. Jpn. 53, 108 (1984)
11 V.L. Pokrovsky and N.A. Sinitsyn, Phys. Rev. B 67, 144303 (2003).
12 V.L. Pokrovsky and S. Scheidt, Phys. Rev. B 70, 014416 (2004).
13 Y. Gefen, E. Ben-Jacob, and A.O. Caldeira, Phys. Rev B 36, 2770 (1987)
14 P. Ao and J. Rammer, Phys. Rev. B 43, 5397 (1991)
15 Y. Kayanuma and H. Nakamura, Phys. Rev. B 57, 13099 (1998)
The result for survival probability obtained\textsuperscript{16} is a consequence of the theorem proposed by Brandobler and Elser\textsuperscript{18} as a hypothesis and proved rigorously by Dobrescu and Sinitsyn\textsuperscript{19}. Wubs et al. have found the same method independently, but a little later. Important steps toward the proof of the Brandobler-Elser hypothesis were made by A.V. Shytov (Phys. Rev. B \textbf{71}, 085301 (2005)), N.A. Sinitsyn (J. Phys. A \textbf{37}, 10691 (2004)) and M.V. Volkov and V.N. Ostrovsky (J. Phys. B, \textbf{37}, 4069 (2004), Ib. \textbf{38}, 907 (2005)).

\textsuperscript{16} S. Brandobler and V. Elser, J. Phys. A: Math. Gen. \textbf{26}, 1211 (1993).
\textsuperscript{18} S. Brandobler and V. Elser, J. Phys. A: Math. Gen. \textbf{26}, 1211 (1993).
\textsuperscript{19} B. Dobrescu and N.A. Sinitsyn, J. Phys. B: At. Mol. Phys., \textbf{39}, 1253 (2006).