From Puiseux series to invariant algebraic curves

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Abstract

A relationship between Puiseux series satisfying an ordinary differential equation corresponding to a polynomial dynamical system and degrees of irreducible invariant algebraic curves is studied. A bound on the degrees of irreducible invariant algebraic curves for a wide class of polynomial dynamical systems is obtained. It is demonstrated that the Puiseux series near infinity can be used to find irreducible algebraic curves explicitly. As an example, all irreducible invariant algebraic curves for the famous FitzHugh–Nagumo system are obtained.

Keywords: FitzHugh–Nagumo model, invariant algebraic curves, Darboux polynomials, Puiseux series

1. Introduction

A polynomial dynamical system in \( \mathbb{C}^2 \) can be defined as

\[
x_t = P(x, y), \quad y_t = Q(x, y),
\]

where \( P(x, y), Q(x, y) \) are polynomials in the ring \( \mathbb{C}[x, y] \). An algebraic curve \( F(x, y) = 0, F(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C} \) is called an invariant algebraic curve (or a Darboux polynomial) of dynamical system (1.1) if it satisfies the following equation

\[
P(x, y)F_x + Q(x, y)F_y = \lambda(x, y)F,
\]

where \( \lambda(x, y) \in \mathbb{C}[x, y] \) is a polynomial called the cofactor of the invariant curve \( F(x, y) \).

Lemma 1.1. Let \( F(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C} \) and \( F = f_1^{n_1} \cdots f_m^{n_m} \) be its factorization in irreducible factors. Then \( F \) is an invariant algebraic curve of dynamical system (1.1) if and only if \( f_j \) is an invariant algebraic curve of dynamical system (1.1) for each \( j = 1, \ldots, m \). In addition the following relation is valid \( \lambda = n_1\lambda_1 + \cdots + n_m\lambda_m \), where \( \lambda \) is the cofactor of \( F \) and \( \lambda_j \) is the cofactor of \( f_j \).

The proof of this lemma is straightforward, see, for example, [1]. It can be observed that an invariant algebraic curve of dynamical system (1.1) is formed by solutions of the latter. A solution of dynamical system (1.1) has either empty intersection with the zero set of \( F \), or it is entirely contained in \( F = 0 \). Existence of invariant algebraic curves is a substantial
measure of integrability, for more details see, for instance, [1–4]. In view of lemma 1.1 it is an important problem to classify all irreducible invariant algebraic curves of dynamical systems.

The problem of finding a bound on degrees of irreducible invariant algebraic curves goes back to Poincaré [5]. It is still an open problem to establish an ”effective” upper bound (if any) for a generic polynomial dynamical system. Here by ”effective” upper bound we mean a bound, which allows one to find all irreducible invariant algebraic curves iterating finite amount of times the method of undetermined coefficients. For rapid methods of finding Darboux polynomials with bounded degrees see [6]. Note that there exist polynomial dynamical systems that possess irreducible invariant algebraic curves with degrees depending on the coefficients of the system. For example, if the coefficients in a cubic Liénard system with linear damping are not bounded, then the degrees of irreducible invariant algebraic curves can be arbitrary [7].

In this article our aim is to present an approach, which can be used to find an ”effective” bound for a wide class of polynomial dynamical systems.

Let us regard the variable $y$ as dependent and the variable $x$ as independent. Note that the roles can be changed. The function $y(x)$ satisfies the following first–order ordinary differential equation

$$\frac{d}{dx} y - \frac{Q(x, y)}{P(x, y)} = 0.$$  (1.3)

In what follows we suppose that the polynomials $P(x, y), Q(x, y)$ do not have non–constant common factors.

A Puiseux series in a neighborhood of the point $x = 0$ is defined as

$$y(x) = \sum_{k=k_0}^{+\infty} c_k x^{k/n}$$  (1.4)

where $k_0 \in \mathbb{Z}, n \in \mathbb{N}$.

In its turn a Puiseux series in a neighborhood of the point $x = \infty$ is defined as

$$y(x) = \sum_{k=k_0}^{+\infty} b_k x^{-k/n}$$  (1.5)

where $l_0 \in \mathbb{Z}, n \in \mathbb{N}$.

Let us formulate our main results.

**Theorem 1.1.** Suppose that there exists finite number of Puiseux series of the form (1.4) satisfying equation (1.3). Let $F(x, y)$ be an irreducible invariant algebraic curve of dynamical system (1.1). Then the degree of $F(x, y)$ with respect to $y$ does not exceed the number of distinct Puiseux series of the form (1.4) that satisfy equation (1.3).

Situation of theorem 1.1 seldom occurs. The next theorem is much more important for applications.
Theorem 1.2. Suppose that there exists finite number of Puiseux series of the form (1.5) satisfying equation (1.3). Let \( F(x, y) \) be an irreducible invariant algebraic curve of dynamical system (1.1). Then the degree of \( F(x, y) \) with respect to \( y \) does not exceed the number of distinct Puiseux series of the form (1.5) that satisfy equation (1.3).

Interestingly, the same "finiteness property" of admissible series is one of the major points in classification of meromorphic solutions of autonomous algebraic ordinary differential equations [8–11].

All the Puiseux series in neighborhoods of the points \( x = 0 \) and \( x = \infty \) satisfying equation (1.3) can be easily constructed with the help of the Newton polygon related to equation (1.3) [12, 13]. Note that the Newton diagram and the Puiseux series in a neighborhood of the point \( x = 0 \) are mainly considered in classical literature. In view of this we shall give a definition of the Newton polygon and describe an algorithm for finding the Puiseux series for both cases in detail.

Let us note that making the change of variables \( x \mapsto x - x_0, x_0 \neq 0 \) or \( y \mapsto y - y_0, y_0 \neq 0 \) one can investigate the number of admissible Puiseux series in a neighborhood of the point \( x = x_0 \) or \( y = y_0 \). A theorem similar to 1.1 can be formulated for Puiseux series centered at the point \( x = x_0 \) or \( y = y_0 \).

In this article we shall consider the following dynamical system

\[
x_t = -x^3 + ex^2 + \sigma x - y + \delta, \quad y_t = \alpha x + \beta y. \tag{1.6}
\]

The change of variables \( x \mapsto x + A, y \mapsto y + B, A = e/3, \alpha A + \beta B = 0 \) relates system (1.6) with its simplified version at \( e = 0 \). Thus without loss of generality we set \( e = 0 \). All the parameters are supposed to be from the field \( \mathbb{C} \). If \( \alpha = 0 \) and \( \beta = 0 \), then integrating the second equation, we obtain \( y(t) = C_0 \). In this case the function \( x(t) \) satisfies a simple first–order ordinary differential equation. Hence in what follows we suppose that \( \alpha \) and \( \beta \) are not simultaneously zero. Dynamical system (1.6) is the two–dimensional FitzHugh–Nagumo system [14, 15]. It is one of the most famous models describing the excitation of neural membranes and the propagation of nerve impulses along an axon. This system has been intensively studied in recent years, see [16, 17] and references therein. In article [18] first–order (with respect to \( y \)) invariant algebraic curves were derived. In this article our aim is to obtain all irreducible algebraic curves for dynamical system (1.6).

Theorem 1.3. The unique irreducible invariant algebraic curves of dynamical system (1.6) with \( e = 0 \) and \( |\alpha| + |\beta| > 0 \) are those given in table 1.
2. Proof of main results

It is known [19] that a Puiseux series of the form (1.4) that satisfy equation $F(x,y) = 0$ is convergent in a neighborhood of the point $x = 0$ (the point $x = 0$ is excluded from domain of convergence if $k_0 < 0$). Analogously, a Puiseux series of the form (1.5) that satisfy equation $F(x,y) = 0$ is convergent in a neighborhood of the point $x = \infty$ (the point $x = \infty$ is excluded from domain of convergence if $l_0 < 0$). This fact follows from the classical result if we consider the change of variables $s = x^{-1}$, which brings infinity to the origin. In other words the Puiseux series $y(s^{-1})$ satisfying the equation $G(s,y) = 0$ converges in a neighborhood of the point $s = 0$ (the point $s = 0$ is excluded from domain of convergence if $l_0 < 0$). Here $G(s,y) = s^M F(s^{-1},y) \in \mathbb{C}[s,y]$ is an algebraic curve, $M$ is the degree of $F(x,y)$ with respect to $x$. The set of all Puiseux series of the form (1.4) ((1.5)) forms a field, which we denote by $\mathbb{C}\{x\} (\mathbb{C}_\infty\{x\})$.

Let us prove the following lemma.

**Lemma 2.1.** Let $y(x)$ be a Puiseux series satisfying the equation $F(x,y) = 0$, $F_y \neq 0$ with $F(x,y)$ being an invariant algebraic curve of dynamical system (1.1). Then the series $y(x)$ satisfies equation (1.3).

**Proof.** Representing $F(x,y)$ as the product of irreducible functors $F = f_1^{n_1} \cdots f_m^{n_m}$, we see that there exists $f_j(x,y)$ such that $f_j(x,y(x)) = 0$. Differentiating this equation with respect to $x$, we get

$$f_{j,x}(x,y(x)) + y_x f_{j,y}(x,y(x)) = 0. \quad (2.1)$$

It follows from lemma 1.1 that $f_j(x,y)$ is an irreducible invariant algebraic curve of dynamical system (1.1) and satisfies the equation

$$P(x,y)f_{j,x} + Q(x,y)f_{j,y} = \lambda_j(x,y)f_j. \quad (2.2)$$

Substituting $y = y(x)$ into this equation yields

$$P(x,y(x))f_{j,x}(x,y(x)) + Q(x,y(x))f_{j,y}(x,y(x)) = 0. \quad (2.3)$$

Further, let us note that the series $y = y(x)$ cannot satisfy the equation $f_{j,y}(x,y) = 0$. Indeed, assuming the contrary we see that $f_j$ and $f_{j,y}$ intersect in an infinite number of points inside the domain of convergence of the series $y = y(x)$. It follows from the Bézout’s theorem that there exists a polynomial both dividing $f_j$ and $f_{j,y}$. Since $f_j$ is irreducible, we conclude that this divisor coincides with $f_j$. Thus we get $f_{j,y} = fh$ with $h$ being a polynomial. This relation contradicts the fact that the degree of $f_{j,y}$ is less than the degree of $f_j$ (degrees are taken with respect to $y$).

Homogeneous system of linear equations (2.1), (2.3) relating $f_{j,x}(x,y(x))$, $f_{j,y}(x,y(x))$ has non–trivial solutions. Indeed, $f_{j,y}(x,y(x)) \neq 0$ Consequently its determinant equals zero. This completes the prove. 

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Proof of theorem 1.1. The field \( \mathbb{C}\{x\} \) is algebraically closed [19]. Let \( F(x, y), F_y \neq 0 \) be an invariant algebraic curve of polynomial dynamical system (1.1). There exists uniquely determined system of elements \( y_n(x) \in \mathbb{C}\{x\} \) such that the following representation is valid [19]

\[
F(x, y) = \mu(x) \prod_{n=1}^{N} \{y - y_n(x)\}, \tag{2.4}
\]

where \( N \) is the degree of \( F(x, y) \) with respect to \( y \), \( \mu(x) \in \mathbb{C}[x] \). Moreover, if a non–constant polynomial \( g(x) \in \mathbb{C}[x] \) does not divide \( F(x, y) \), then \( F(x, y) \) has multiple factors in \( \mathbb{C}[x, y] \) if and only if the equation \( F(x, y) \) has multiple roots in \( \mathbb{C}\{x\} \) [19]. Further, it follows from lemma 2.1 that the set of elements \( y_n(x) \in \mathbb{C}\{x\} \) appearing in representation (2.4) is a subset of those satisfying equation (1.3). If the latter is finite and \( F(x, y) \) is irreducible, then \( N \) does not exceed the number of distinct Puiseux series of the form (1.4) satisfying equation (1.3). This completes the prove.

Proof of theorem 1.2. We repeat the proof of theorem 1.1 with the field \( \mathbb{C}\{x\} \) replaced by \( \mathbb{C}_\infty\{x\} \). Note that representation of \( F(x, y) \) in \( \mathbb{C}_\infty\{x\} \) reads as

\[
F(x, y) = \mu(x) \prod_{n=1}^{N} \{y - y_n(x)\}, \tag{2.5}
\]

where \( y_n(x) \in \mathbb{C}_\infty\{x\}, \mu(x) \in \mathbb{C}[x] \).

Let us prove other theorems, applicable even if equation (1.3) admits infinite number of Puiseux series.

Theorem 2.1. Let \( y_j(x) \in \mathbb{C}\{x\} \) be a Puiseux series with uniquely determined coefficients satisfying equation (1.3). Then the degree of \( y_j(x) \) in representation (2.4) of an irreducible invariant algebraic curve \( F(x, y) \) of polynomial dynamical system (1.1) is either 0, or 1.

Proof. Let \( F(x, y) \) be an irreducible invariant algebraic curve of dynamical system (1.1). The curve \( F(x, y) \) can be represented in the form (2.4). Suppose that the series \( y_j(x) \) appears in this representation at least twice. Then \( F(x, y) \) is reducible in \( \mathbb{C}\{x\} \) and consequently in \( \mathbb{C}[x, y] \) [19]. It is a contradiction. \( \square \)

Theorem 2.2. Let \( y_j(x) \in \mathbb{C}_\infty\{x\} \) be a Puiseux series with uniquely determined coefficients satisfying equation (1.3). Then the degree of \( y_j(x) \) in representation (2.5) of an irreducible invariant algebraic curve \( F(x, y) \) of dynamical system (1.1) is either 0, or 1.

Proof. We prove this theorem repeating the proof of theorem 2.1 with the field \( \mathbb{C}\{x\} \) replaced by \( \mathbb{C}_\infty\{x\} \). \( \square \)

Concluding this section let us mention that one can state the same theorems if the variables \( x \) and \( y \) change their roles or if the base of Puiseux series is the point \( x_0 \neq 0 \) and \( x_0 \neq \infty \).
3. Newton polygons and series representing solutions of ordinary differential equations

Let us consider an algebraic ordinary differential equation

\[
E \left( \frac{d^N y}{dx^N}, \ldots, \frac{dy}{dx}, y, x \right) = 0.
\]  

(3.1)

Here \( E \) is a polynomial of its arguments. Such an equation can be regarded as the sum of differential monomials of the form

\[
M = cy^{j_0} \left\{ \frac{dy}{dx} \right\}^{j_1} \ldots \left\{ \frac{d^N y}{dx^N} \right\}^{j_N}, \quad c \in \mathbb{C}.
\]  

(3.2)

Define the map \( q : M \to \mathbb{R}^2 \) by the following rules

\[
cx^{q_1}y^{q_2} \mapsto q = (q_1, q_2), \quad \frac{d^k y}{dx^k} \mapsto q = (-k, 1), \quad q(M_1M_2) = q(M_1) + q(M_2),
\]  

(3.3)

where \( c \in \mathbb{C} \) is a constant, \( M_1 \) and \( M_2 \) are differential monomials. We denote the set of all points \( p \in \mathbb{R}^2 \) corresponding to the monomials of equation (3.1) as \( S(E) \).

Definition. The convex hull of \( S(E) \) is called the Newton polygon of equation (3.1).

The boundary of the Newton polygon consists of vertices and edges. Selecting all the differential monomials of the original equation that generate the vertices and the edges of the Newton polygon, we obtain a number of sub-equations or balances. The functions solving these sub-equations produce asymptotics (at \( x \to 0 \) or \( x \to \infty \)) of solutions of equation (3.1) \([12, 13]\).

In this article we are interested in power asymptotics. Thus we substitute \( y = cx^r \), \( c \neq 0 \) into a sub-equation. If the sub-equation corresponds to an edge that is not parallel to the \( q_1 \)-axis, then \( r \in \mathbb{Q}, c \in \mathbb{C} \) are fixed. The parameter \( r \) is unique, while the parameter \( c \) may take several distinct values. Sub-equations related to edges parallel to the \( q_1 \)-axis do not have power solutions. If the sub-equation is not algebraic and corresponds to a vertex, then \( c \in \mathbb{C} \) is arbitrary (but not zero), the parameter \( r \) can be complex-valued (\( r \in \mathbb{C} \)) and there may exist a set of values for \( r \). Algebraic sub-equations related to vertices do not have non-trivial solutions.

In what follows we are working in the frames of two-dimensional Euclidean space \( \mathbb{R}^2 \). All the vectors and rays which are to appear below have the origin \((q_1, q_2) = (0, 0)\) as a starting point. By \( \psi \) we denote the angle between the external normal to an edge (external with respect to the Newton polygon) and the vector \( \vec{e}_{q_1} \). In the case of a vertex by \( \psi \) we denote the angle between the vectors \( \varepsilon(1, \Re r) \) and \( \vec{e}_{q_1} \), where \( \varepsilon = \pm 1 \) and \( r \) is the exponent in the expression \( y = cx^r \). The sign of \( \varepsilon \) is chosen in such a way that the vector \( \varepsilon(1, \Re r) \) lies in the domain bounded by the rays passing through the external normals of the edges attached to the vertex (excluding the rays themselves). If \( 0 \leq \psi < \frac{\pi}{2} \), then we obtain \( x \to \infty \) for the corresponding power asymptotics. If \( \frac{\pi}{2} < \psi \leq \pi \), then \( x \to 0 \).
It is necessary to consider both normals (for the edge) and both vectors ±(1, Re r) for the vertex whenever the Newton polygon degenerates to an edge or a vertex.

Analyzing the Newton polygon one can obtain all the power asymptotics at \( x \to 0 \) or \( x \to \infty \) \[^{[12]}\]. Further, Painlevé methods can be used to find series that satisfy equation \(^{(3.1)}\) and have the leading–order behavior \( y = cx^r \). Originally the Painlevé methods were designed for finding Laurent series, however in our case these methods are also applicable. For more details on the Painlevé methods see, for example, \[^{[2, 12, 13, 20]}\].

The equation of the form \(^{(1.3)}\) related to dynamical system \(^{(1.6)}\) with \( e = 0 \) is the following

\[
\{ -x^3 + \sigma x - y + \delta \} y_x - \{ \alpha x + \beta y \} = 0.
\]

The Newton polygon of this equation is presented in figure \(^{[1]}\).

The balances giving power asymptotics at \( x \to \infty \) and their admissible power solutions take the form

\[
(Q_2, Q_3) : \ (x^3 + y)y_x = 0, \ y(x) = -x^3; \\
Q_3 : \ x^3 y_x = 0, \ y(x) = a_0; \\
(Q_3, Q_4) : \ x^3 y_x + \alpha x = 0, \ y(x) = \frac{\alpha}{x}.
\]

In these expressions \( a_0 \neq 0 \) is an arbitrary constant. The corresponding series turn out to be Laurent series. They are the following

\[
(I) : \ y(x) = -x^3 + \left( \sigma - \frac{\beta}{3} \right) x + \delta + \ldots; \\
(II) : \ y(x) = a_0 + \frac{\alpha}{x} + \frac{\beta a_0}{2x^2} + \frac{\alpha(\sigma + \beta)}{3x^3} + \ldots; \\
(III) : \ y(x) = \frac{\alpha}{x} + \frac{\alpha(\sigma + \beta)}{3x^3} + \ldots.
\]

Series \((I)\) and \((III)\) have uniquely determined coefficients, while series \((II)\) has one arbitrary coefficient \( a_0 \). Further, series \((III)\) is a partial case of series \((II)\). Indeed, setting \( a_0 = 0 \) in \((II)\), we get \((III)\). If \( \alpha = 0 \) the edge \((Q_3, Q_4)\) disappears and we do not have series \((III)\).
Unfortunately, we are in situation with infinite number of admissible Puiseux series near the point $x = \infty$. Further, let the variables $x$ and $y$ change their roles. Now we suppose that $x = x(y)$. The equation of the form (1.3) (with $x \leftrightarrow y$) can be written as

$$\{ -x^3 + \sigma x - y + \delta \} - \{ \alpha x + \beta y \} x y = 0.$$  \hspace{1cm} (3.7)

Its Newton polygon is given in figure 2. Analyzing the Newton polygon of figure 2 we see that there exists only one balance giving power asymptotics at $y \to \infty$. This balance and its power solutions are the following

$$(Q_3, Q_4) : \quad x^3 + y = 0, \quad x^{(j)}(y) = b_0^{(j)} y^{1/3}, \quad b_0^{(j)} = \{(1)^{1/3}\}_j, \quad j = 1, 2, 3. \hspace{1cm} (3.8)$$

Here $b_0$ is one of the cubic roots of $-1$. We obtain three distinct Puiseux series. They take the form

$$x^{(j)}(y) = b_0^{(j)} y^{1/3} + \frac{1}{9} \left(b_0^{(j)}\right)^2 (\beta - 3\sigma)y^{-1/3} + \ldots, \quad j = 1, 2, 3. \hspace{1cm} (3.9)$$

Now we are in situation with finite number of admissible Puiseux series. We do not need to construct admissible Puiseux series near the origin $(x = 0$ and $y = 0)$. We shall classify irreducible invariant algebraic curves of dynamical system (1.6) in the next section.

### 4. Invariant algebraic curves of the FitzHugh–Nagumo system

Let $F(x, y)$ be an invariant algebraic curve of dynamical system (1.6) with $\epsilon = 0$. Then $F(x, y)$ satisfies the equation

$$\{ -x^3 + \sigma x - y + \delta \} F_x + \{ \alpha x + \beta y \} F_y = \lambda F. \hspace{1cm} (4.1)$$

Let us prove the following lemma.

**Lemma 4.1.** If $F(x, y)$ is an invariant algebraic curve of system (1.6), then

$$F(x, y) = \mu_0 y^N + \sum_{k=0}^{N-1} c_k(x) y^k, \quad N \in \mathbb{N} \hspace{1cm} (4.2)$$

and its cofactor $\lambda$ is $\lambda = A_2 x^2 + A_1 x + A_0$, where $A_2 = -M$ with $M$ being the degree of $F(x, y)$ with respect to $x$. 

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**Figure 2:** The Newton polygon of equation (3.7) with $\alpha \neq 0, \delta \neq 0$. 

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Proof. By direct calculations we find that there are no invariant algebraic curves that do not depend on $y$.

Let $F$ and $\lambda$ have degrees $N \in \mathbb{N}$ and $l \in \mathbb{N}_0$ with respect to $y$ accordingly. Substituting relations $F = \mu(x)y^N$, $\mu(x) \neq 0$ and $\lambda = \lambda_0(x)y^l$ with $\mu(x)$, $\lambda_0(x) \in \mathbb{C}[x]$ into equation (4.1) and balancing higher-order terms yields $l = 1$ and $\mu = -\lambda_0(x)\mu$. Since $\mu(x)$ is a polynomial, we get $\lambda_0(x) = 0$, $\mu(x) = \mu_0$, where $\mu_0$ is a constant. In addition we see that the cofactor $\lambda$ does not depend on $y$.

Now suppose that $F$ and $\lambda$ have degrees $M \in \mathbb{N}_0$ and $s \in \mathbb{N}_0$ with respect to $x$ accordingly. Taking expressions $F = \nu(y)x^M$, $\nu(y) \neq 0$ and $\lambda = A_s x^s$ with $\nu(y) \in \mathbb{C}[y]$, $A_s \in \mathbb{C}$ and arguing as above, we get $s = 2$, $A_2 = -M$. This completes the proof.

In what follows we shall assume without loss of generality that $\mu_0 = 1$.

Proof of theorem 1.3. Suppose that $F(x, y)$ is an irreducible invariant algebraic curve of FitzHugh–Nagumo dynamical system (1.6) with $e = 0$. In view of theorems 1.2, 2.2, lemma 1.1 and results of the previous section we get the following representations in the fields $\mathbb{C}_\infty \{x\}$, $\mathbb{C}_\infty \{y\}$:

$$
\mathbb{C}_\infty \{x\} : \quad F(x, y) = \left\{ y + x^3 - \left( \sigma - \frac{\beta}{3} \right) x - \delta - \ldots \right\} \prod_{j=1}^m \left( y - a_0^{(j)} - \ldots \right);
$$

(4.3)
\[ C_\infty \{ y \} : \quad F(x, y) = \nu(y) \prod_{j=1}^{\nu(y)} \left\{ x - b_{ij} y^{1/3} - \ldots \right\}^{k_j}, \quad (4.4) \]

where \( n = 0 \) or \( n = 1 \), \( m \in \mathbb{N}_0 \), \( k_j = 0 \) or \( k_j = 1 \). In addition we suppose that the last product in (4.3) is unite whenever \( m = 0 \). Using theorem 1.2 we see that the degree of \( F(x, y) \) with respect to \( x \) is at most 3. Moreover, it follows from representations (4.3), (4.4) that either \( M = 0 \), or \( M = 3 \).

First we consider the case \( M = 0 \). From expressions (4.3), (4.4), and lemma 4.1 we obtain that \( F(x, y) \) does not depend on \( x \) and \( n = 0 \), \( A_2 = 0 \). Substituting \( F(x, y) = F(y) \in \mathbb{C}[y] \) into equation (4.1) and setting to zero the coefficients at \( x^1 \) and \( x^0 \), we find \( A_1 = 0 \), \( A_0 = \beta \), \( \alpha = 0 \), and \( F(x, y) = y \).

Secondly we consider the case \( M = 3 \). As a result we obtain \( n = 1 \), \( A_2 = -3 \). Substituting \( F(x, y) = c_3(y) x^3 + c_2(y) x^2 + c_1(y) x + c_0(y) \) into equation (4.1) and setting to zero the coefficients at \( x^4 \), \( x^3 \) and \( x^2 \), we express \( c_3(y) \), \( c_1(y) \), and \( c_0(y) \) via \( c_3(y) \) and its derivatives. Note that these relations are linear in \( c_3(y) \) and its derivatives. Further, setting to zero the coefficients at \( x^1 \) and \( x^0 \) and using relations for \( c_2(y) \), \( c_1(y) \), and \( c_0(y) \), we obtain two fourth–order linear ordinary differential equations for the polynomial \( c_3(y) \). We use the method of undetermined coefficients to find their polynomial solutions of degrees 0, 1, 2.

The results are given in table 1.

Finally, we suppose that there exists a polynomial solution of degree \( K \geq 3 \). We make the substitution \( c_3(y) = a_3 y^K + a_{K-1} y^{K-1} + \ldots + a_0 \neq 0 \) and set to zero the coefficients at \( y^{K+1-j} \), \( j \geq 0 \). The algebraic equations with \( j = 0 \) give the values of \( A_1 \) and \( A_0 \):

\[
A_1 = 0, \quad A_0 = \left( K + \frac{2}{3} \right) \beta + \sigma. \quad (4.5)
\]

Solving other algebraic equations, we find the values of \( a_{K-j}, j \geq 1 \) and necessary conditions for such a polynomial solution to exist. Taking eight equations (1 \( \leq j \leq 8 \)), we see that this system is inconsistent if we require that the resulting algebraic curve is irreducible. Recall that the case \( \alpha = 0, \beta = 0 \) is simple (see section 1) and we exclude it here. This completes the proof.

5. Conclusion

In this article we have studied a relationship between invariant algebraic curves of a polynomial dynamical system and the Puiseux series satisfying an ordinary differential equation corresponding to the system. A bound on the degrees of irreducible invariant algebraic curves for a wide class of polynomial vector fields is derived. It is shown that the structure of the Puiseux series near infinity can be used to find all irreducible algebraic curves explicitly. Using this approach we have classified irreducible invariant algebraic curves of the FitzHugh–Nagumo system.

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