On Convergence of Partial Derivatives of Multidimensional Convolution Operators

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Abstract
In this paper, we prove some results on convergence properties of higher order partial derivatives of multidimensional convolution-type singular integral operators being applied to the class of functions which are integrable in the sense of Lebesgue.

Keywords: Fatou-type convergence; Approximation of partial derivatives; Integral operators.

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1. Introduction

To reach the beginning of the singular integral theory, which is an immense field of application, it is necessary to go up to well-known Fourier integrals. If Fourier integrals are considered in general frame, the following integral operators are obtained:

\[ L_w(f; x) = \int_A f(t) K_w(t - x) \, dt, \quad x \in A, \quad w \in \Lambda, \tag{1.1} \]

where \( \Lambda \) is an index set consisting of \( w \) which are real numbers (parameters) and \( w_0 \) is an accumulation point of indicated index set, or it equals infinity, \( A \) is a desired subset of the set of all real numbers \( R \) and \( K_w \) is a kernel with some assumptions on it. The operators of type (1.1) were widely studied in [4]. Also, several results and discussions related to multidimensional analogues of the operators of type (1.1) can be found in [18, 19]. Also, for further reading, we refer the reader to [1, 9, 15, 20–23] and references therein. In particular, Fatou-type convergence of the operators (see [5]) is studied, for example, in [9, 16, 20, 21].

Singular integrals, in fact, have been studied in many different ways, but in this article we focus on the convergence properties of their derivatives. In the year 1962, Taberski [20] proved a theorem concerning Fatou-type convergence of higher order derivatives of singular integrals of type (1.1) whose kernels were supposed to be \( 2\pi \)–periodic. In the proof of this theorem, Taberski [20] used an auxiliary function, whose higher order derivatives coincide with derivatives of the same order of original function. For this function, he used an asymptotic formula for De la Vallée Poussin’s singular integrals, which is given by Matsuoka [13], which can be seen as an alternative
version of usual Taylor series in regard to trigonometric functions. In the same year, Žornickaja [24] studied the similar problem under the concept of almost everywhere convergence by taking the set Λ as the set of all natural numbers denoted by N. Žornickaja [24] also gave some kernel examples which does not fit the presented theorem’s hypotheses. Later on, Gadžiev [7] proved a theorem concerning approximation by first order derivatives of the operators of type (1.1) under the existence of right and left derivatives of the integrable functions at indicated points by assuming that A is an arbitrary bounded interval in R. This theorem may be categorized as Fatou-type convergence theorem. In the proof of this theorem, Gadžiev [7] used first order Taylor polynomial as an auxiliary function (see also [6, 8]). We remark here that, in the works above, the kernel functions satisfy standard approximate identity properties. Then, Karsli and Ibikli [10] proved some theorems concerning approximation of higher order derivatives of functions in more general function spaces. In this respect, Karsli [11, 12] studied similar problem in the framework of linear and nonlinear integral operators using desired order Taylor polynomial in the proving stage, respectively. In fact, Taylor series are very important for the proofs of approximation theory. For different usages, we refer the reader to [1–3].

Let \( R^n = R \times R \times \cdots \times R \) denote usual finite dimensional Euclidean space with elements, such as \( t := (t_1, \ldots, t_n) \) and \( x := (x_1, \ldots, x_n) \). Further, let \( \Lambda \) be an index set consisting of real numbers (parameters) \( w \) and \( w_0 \) be an accumulation point of indicated index set or \( w_0 = \infty \), separately. Under the conditions assigned to the function \( K_w : R^n \rightarrow R_0^+ \), we obtained some approximation properties of higher order partial derivatives of the operators given by

\[
(G_w f)(x) = \int_D f(t) K_w(t - x) \, dt, \quad x \in D^o
\]

with respect to desired component \( x_j \), where \( D \) denotes a closed box in \( R^n \), \( D^o \) stands for its interior and \( j \) is an arbitrary number between \( j = 1, 2, \ldots, n \). This study is a continuation of the study [22] and contains some results concerning multidimensional analogues of the results proved in [10, 11, 20]. Following similar steps used in [7, 10, 11, 20] with some additional considerations, we state and prove main theorems of this study.

### 2. Main Results

#### Case 1: Domain of integration is bounded

Let us consider the following operators

\[
(G_w f)(x) = \int_D f(t) K_w(t - x) \, dt.
\]

The explicit form of these operators can be written as follows:

\[
(G_w f)(x_1, \ldots, x_n) = \prod_{A_1}^{B_1} \cdots \prod_{A_n}^{B_n} f(t_1, \ldots, t_n) K_w(t_1 - x_1, \ldots, t_n - x_n) \, dt_n \cdots dt_1
\]

with \( x \in D^o \) and \( t \in D \), where \( D = [A_1, B_1] \times \cdots \times [A_n, B_n] \) is a closed box in \( R^n \) and \( D^o = ]A_1, B_1[ \times \cdots \times ]A_n, B_n[ \) is an open box. Here, \( A_j \) and \( B_j \) with \( A_j \neq B_j \) are certain real numbers for every fixed \( j = 1, 2, \ldots, n \).

Let \( L_1(D) \) denote the space of all functions \( f \) which are integrable in the sense of Lebesgue on \( D \) with respect to usual Lebesgue measure \( dt \). Any function in this space satisfies the property such that \( \|f\|_{L_1(D)} := \int_D |f(t)| \, dt < \infty \).

Here, the kernel \( K_w(t) \) satisfies the following conditions:

1. \( K_w : R^n \rightarrow R_0^+ \) is a measurable function on its domain for every fixed \( w \in \Lambda \).

2. \( \lim_{(x,w) \rightarrow (x_0,w_0)} \int_D K_w(t - x) \, dt = 1 \), where \( w \in \Lambda \), \( x \in D^o \) and \( x_0 \) is an accumulation point of \( D^o \).

3. \( \|K_w(\cdot - x)\|_{L_1(D)} \) is uniformly bounded on \( \Lambda \) for all \( x \in D \) by a constant \( M \).

Here, \( |.| \) denotes usual Euclidean distance.
Theorem 2.1. Assume that $K_w(t)$ and $\frac{\partial^m}{\partial t_j^m} K_w(t)$ are continuous functions with respect to $t$ on $\mathbb{R}^n$ for every fixed $w \in \Lambda$, $v = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Suppose that the following conditions

$$\lim_{w \to w_0} \sup_{|\xi| \geq \varepsilon} \left| \frac{\partial^v}{\partial t_j^v} K_w(t) \right| = 0, \forall \xi > 0$$ \hfill (2.1)

hold for every fixed $j = 1, 2, \ldots, n$ and $v = 0, 1, \ldots, m$, together with conditions (1)-(3). If $f \in L_1(D)$ possesses at $a := (a_1, \ldots, a_n) \in D^n$ finite $m$-th order partial derivative with respect to $j$-th variable $f_j^{(m)}(a)$ and there exists a neighborhood $[a_1 - \eta, a_1 + \eta] \times \cdots \times [a_n - \eta, a_n + \eta] \subset D$ with $\eta > 0$ of $a$ on which the functions $f_j^{(m)}$ and $f$ are continuous, then

$$\lim_{(x, w) \to (a, w_0)} \frac{\partial^m}{\partial x_j^m} (G_w f) (x) = f_j^{(m)}(a)$$

on any set $S$ consisting of $(x, w)$ on which the functions expressed as

$$\sup_{w \in \Lambda} \int_{a_1 - x_1 - \zeta}^{a_1 - x_1 + \zeta} \cdots \int_{a_n - x_n - \zeta}^{a_n - x_n + \zeta} |t_j|^m \left| \frac{\partial^m}{\partial t_j^m} K_w(t_1, \ldots, t_n) \right| dt_n \cdots dt_1 \leq C_v$$ \hfill (2.2)

and

$$\sup_{w \in \Lambda} \int_{a_1 - x_1 - \zeta}^{a_1 - x_1 + \zeta} \cdots \int_{a_n - x_n - \zeta}^{a_n - x_n + \zeta} |t_j|^{m-v} \left| \frac{\partial^m}{\partial t_j^m} K_w(t_1, \ldots, t_n) \right| dt_n \cdots dt_1 \leq C_v$$ \hfill (2.3)

are bounded, where $j = 1, 2, \ldots, n$ and $v = 1, 2, \ldots, m$. Here, $C_v$ are positive constants for every fixed positive real number $\zeta$ that makes the value of integral finite as required.

Proof. We define the function $g(t)$ by

$$g(t) := g_{t_j},$$

where

$$g_{t_j} := f(a) + (t_j - a_j) \left. \frac{\partial f(t)}{\partial t_j} \right|_{t=a} + \cdots + \frac{(t_j - a_j)^m}{m!} \left. \frac{\partial^m f(t)}{\partial t_j^m} \right|_{t=a}$$

such that $\left. \frac{\partial^k g(t)}{\partial t_j^k} \right|_{t=a} = f_j^{(k)}(a), k = 0, \ldots, m$ for $a \in D^o$. By linearity of the operators, we can write

$$(G_w f)(x) = (G_w (f + g - g))(x) = (G_w g)(x) + (G_w (f - g))(x).$$

Differentiating both sides of the following equation up to order $m$ with respect to $x_j$ and writing the definition of $g$ in $(G_w g)(x)$ such that

$$(G_w g)(x) = \int_D g(t) K_w(t - x) dt,$$

one easily obtains

$$\frac{\partial^m}{\partial x_j^m} (G_w g)(x) = \frac{\partial^m}{\partial x_j^m} \int_D g(t) K_w(t - x) dt = (-1)^m \int_D g(t) \frac{\partial^m}{\partial t_j^m} K_w(t - x) dt.$$
In view of Fubini’s Theorem (see [14]) and \( m \) times application of integration by parts with respect to \( t_j \), we have the following equality:

\[
(-1)^m \left\{ \int_{[x,w]} \left\{ \sum_{k=0}^{m-1} (-1)^k g_{t_j}^{(k)} \frac{\partial^{m-k} I_m(t-x)}{\partial t_j^{m-k}} K \right\} dt \right\}
\]

\[
= (-1)^m \left\{ \int_{[x,w]} \left\{ \sum_{k=0}^{m-1} (-1)^k g_{t_j}^{(k)} \frac{\partial^{m-k} I_m(t-x)}{\partial t_j^{m-k}} K \right\} dt \right\}
\]

\[
+ \int_{D} g_{t_j}^{(m)} K(t-x)dt
\]

\[
= : J_1(x, w) + J_2(x, w),
\]

where \( 1 \leq i, j \leq n \). Performing some analysis on \( J_1(x, w) \), we deduce that it tends to zero as \( (x, w) \to (a, w_0) \) as a straightforward consequence of (2.1). Obviously, there holds for \( J_2(x, w) \)

\[
\lim_{(x, w) \to (a, w_0)} \int_{D} K(t-x) \frac{\partial^{m} f}{\partial t_j^{m}} g(t) dt
\]

\[
= \lim_{(x, w) \to (a, w_0)} \frac{\partial^{m} f}{\partial t_j^{m}} \bigg|_{t=a} \int_{D} K(t-x) dt.
\]

Hence, by condition (2), the desired result follows, that is,

\[
\lim_{(x, w) \to (a, w_0)} \frac{\partial^{m} (G_w g)}{\partial x_j^{m}} (x) = \frac{\partial^{m} f}{\partial t_j^{m}} \bigg|_{t=a} = f_{j}^{(m)}(a).
\]

To complete the proof, we will show that

\[
\lim_{(x, w) \to (a, w_0)} \frac{\partial^{m} (G_w (f - g))}{\partial x_j^{m}} (x) = 0.
\]

By the hypotheses, the functions \( \frac{\partial^{m} f}{\partial t_j^{m}}(t) \) for \( m = 1, \ldots, n \) and \( f(t) \) are continuous at \( a \in D^n \). Therefore, according to \( \varepsilon - \delta \) criterion of continuity for all \( \varepsilon > 0 \) there exists a number \( \delta > 0 \) such that the following relations hold there (see also [20]):

i. \( |f(t) - f(a)| < \varepsilon \) provided that \( t \in D^1 \), where \( D^1 = |a_1 - \delta, a_1 + \delta| \times \cdots \times |a_n - \delta, a_n + \delta| \).

ii. \( |f(a_1, \ldots, t_j, \ldots, a_n) - f(a_1, \ldots, a_j, \ldots, a_n)| < \varepsilon \) provided that \( |t_j - a_j| < \delta \).

iii. \( \left| \frac{f(a_1, \ldots, t_j, \ldots, a_n) - g_{t_j}}{(t_j - a_j)^m} \right| < \varepsilon \) provided that \( |t_j - a_j| < \delta \), since

\[
\lim_{t_j \to a_j} \frac{f(a_1, \ldots, t_j, \ldots, a_n) - g_{t_j}}{(t_j - a_j)^m} = 0,
\]

where \( a \in D^n \) and

\[
g_{t_j} = f(a) + (t_j - a_j) \frac{\partial f(t)}{\partial t_j} \bigg|_{t=a} + \ldots + (t_j - a_j)^m \frac{\partial^{m} f(t)}{\partial t_j^{m}} \bigg|_{t=a}.
\]

Here, \( \delta > 0 \) represents the smallest number for which the relations written above are provided simultaneously. In the light of above observations, for a sufficiently small \( \delta > 0 \), we obtain the following equality:
\[
\frac{\partial^m}{\partial x_j^m} (G_w(f - g)) (x) = (-1)^m \int_D [f(t) - g(t)] \frac{\partial^m}{\partial w_j^m} K_w(t - x) dt \\
= (-1)^m \left\{ \int_{D_1} + \int_{D_1 \setminus D_2} \right\} [f(t) - g(t)] \frac{\partial^m}{\partial w_j^m} K_w(t) dt \\
= k(x, w) + l(x, w).
\]

We first consider \(k(x, w)\) such that

\[
k(x, w) = (-1)^m \int_{D_1} [f(t) \pm f(a_1, \ldots, t_j, \ldots, a_n) \pm f(a) - g(t)] \frac{\partial^m}{\partial w_j^m} K_w(t - x) dt \\
= (-1)^m \int_{D_1} [-f(a_1, \ldots, t_j, \ldots, a_n) + f(a)] \frac{\partial^m}{\partial w_j^m} K_w(t - x) dt \\
+ (-1)^m \int_{D_1} [f(a_1, \ldots, t_j, \ldots, a_n) - g(t)] \frac{\partial^m}{\partial w_j^m} K_w(t - x) dt \\
+ (-1)^m \int_{D_1} [f(t) - f(a)] \frac{\partial^m}{\partial w_j^m} K_w(t - x) dt \\
= k_1(x, w) + k_2(x, w) + k_3(x, w).
\]

Let \(|x_i - a_i| < \frac{\delta}{2}\), where \(i = 1, 2, \ldots, n\). Using relations (i) and (ii), we obtain the following inequality for \(|k_1(x, w)| + |k_3(x, w)|\)

\[
(|k_1(x, w)| + |k_3(x, w)|) \leq 2 \varepsilon \int_{D^2} \left| \frac{\partial^m}{\partial w_j^m} K_w(t) \right| dt,
\]

where \(D^2 = [a_1 - x_1 - \delta, a_1 - x_1 + \delta[ \times \cdots \times ] a_n - x_n - \delta, a_n - x_n + \delta] \). By the hypothesis (2.3), we can write with peace of mind that

\[
\lim_{(x, w) \to (a, w_0)} \left( |k_1(x, w)| + |k_3(x, w)| \right) = 0
\]
on \(S\). If we deal with \(k_2(x, w)\) using relation (iii), we obtain

\[
|k_2(x, w)| = \left| (-1)^m \int_{D_1} \frac{f(a_1, \ldots, t_j, \ldots, a_n) - g_t_i}{(t_j - a_j)^m} (t_j - a_j)^m \frac{\partial^m}{\partial w_j^m} K_w(t - x) dt \right| \\
\leq \varepsilon \int_{D^2} \left| \frac{\partial^m}{\partial w_j^m} K_w(t) \right| dt \\
= \varepsilon \int_{D^2} \left| \left( (t_j + x_j - a_j)^m - t_j^m + t_j^m \right) \frac{\partial^m}{\partial w_j^m} K_w(t) \right| dt \\
\leq \varepsilon \int_{D^2} \left| (t_j + x_j - a_j)^m - t_j^m \right| \left| \frac{\partial^m}{\partial w_j^m} K_w(t) \right| dt \\
+ \varepsilon \int_{D^2} \left| t_j^m \right| \left| \frac{\partial^m}{\partial w_j^m} K_w(t) \right| dt \\
= \varepsilon k_{21} (x, w) + \varepsilon k_{22} (x, w).
\]
It is easy to see that $k_{22}(x, w)$ is bounded on $S$ by condition (2.2). Using well-known formula (see [17]) given by
\[ z_s^1 - z_s^2 = (z_1 - z_2)(z_1^{s-1} + z_1^{s-2}z_2 + \ldots + z_2^{s-1}), \quad z_1, z_2 \in R, \]
where $s$ is a natural number with $s \neq 0$, for $k_{21}(x, w)$, we obtain
\[
k_{21}(x, w) = \int_{D^2} |(t_j + x_j - a_j)^m - t_j^m| \left| \frac{\partial^m}{\partial t_j^m} K_w(t) \right| dt
\]
\[
\leq |x_j - a_j| \int_{D^2} |(t_j + x_j - a_j)^{m-1}| \left| \frac{\partial^m}{\partial t_j^m} K_w(t) \right| dt
\]
\[
+ |x_j - a_j| \int_{D^2} |(t_j + x_j - a_j)^{m-2}| |t_j| \left| \frac{\partial^m}{\partial t_j^m} K_w(t) \right| dt
\]
\[
+ \ldots +
\]
\[
+ |x_j - a_j| \int_{D^2} |t_j|^{m-1} \left| \frac{\partial^m}{\partial t_j^m} K_w(t) \right| dt.
\]
If we apply same operations to each integral above, we see that $k_{21}(x, w)$ is bounded above by a sum consisting of the following expressions:
\[
|x_j - a_j|^v \int_{D^2} |t_j|^{m-v} \left| \frac{\partial^m}{\partial t_j^m} K_w(t) \right| dt, \quad v = 1, 2, \ldots, m.
\]
By the hypothesis (2.4), $k_{21}(x, w)$ is bounded on $S$. Hence
\[
\lim_{(x, w) \to (a, w_0)} k_2(x, w) = 0
\]
on $S$. Since $f - g$ is Lebesgue integrable on $D$ and
\[
\|l(x, w)\| \leq \left| \int_{D \setminus D_1} [f(t) - g(t)] \frac{\partial^m}{\partial t_j^m} K_w(t-x) dt \right|
\]
\[
\leq \sup_{2 \leq |w|} \left| \frac{\partial^m}{\partial t_j^m} K_w(u) \right| \int_{D} |f(t) - g(t)| dt,
\]
we see that
\[
\lim_{(x, w) \to (a, w_0)} l(x, w) = 0.
\]
From above observations, we deduce that
\[
\lim_{(x, w) \to (a, w_0)} \frac{\partial^m}{\partial x_j^m} (G_w(f - g))(x) = 0.
\]
Therefore, the claim follows, that is,
\[
\lim_{(x, w) \to (a, w_0)} \frac{\partial^m}{\partial x_j^m} (G_wf)(x) = f^{(m)}(a).
\]
Thus, the proof is completed.
Case 2: Domain of integration is $R^n$

We consider the following operators

$$ (E_w f)(x) = \int_{R^n} f(t) K_w(t - x) dt. $$

The explicit form of these operators can be written as follows:

$$ (E_w f)(x_1, \ldots, x_n) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(t_1, \ldots, t_n) K_w(t_1 - x_1, \ldots, t_n - x_n) dt_1 \cdots dt_n, $$

with $x \in R^n$.

Let $L_1(R^n)$ denote the space of all functions $f$ which are integrable in the sense of Lebesgue on $R^n$ with respect to usual Lebesgue measure $dt$. Any function in this space satisfies the property such that $\|f\|_{L_1(R^n)} := \int_{R^n} |f(t)| dt < \infty$.

Here, the kernel $K_w(t)$ satisfies the following conditions:

1. $K_w : R^n \to R^+$ is a measurable function on its domain for every fixed $w \in \Lambda$.
2. $\lim_{w \to w_0} \int_{R^n} K_w(t) dt = 1$, where $w \in \Lambda$.
3. $\|K_w(\cdot)\|_{L_1(R^n)}$ is uniformly bounded on $\Lambda$ by a constant $K$.
4. $\lim_{w \to w_0} \int_{|t| \geq \xi} K_w(t) dt = 0$, $\forall \xi > 0$.

Here, $\| \cdot \|$ denotes usual Euclidean distance.

Now, we state and prove the last theorem.

**Theorem 2.2.** Assume that $K_w(t)$ and $\frac{\partial^m}{\partial t_j^m} K_w(t)$ are continuous functions with respect to $t$ on $R^n$ for every fixed $w \in \Lambda$, $v = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$. Suppose that the following conditions

$$ \lim_{w \to w_0} \sup_{|t| \geq \xi} \left| \frac{\partial^v}{\partial t_j^v} K_w(t) \right| = 0, \forall \xi > 0 $$

$$ \sup_{w \in \Lambda} \int_{R^n} |t_j|^m \left| \frac{\partial^m}{\partial t_j^m} K_w(t) \right| dt < \infty \quad (2.5) $$

hold for every fixed $j = 1, 2, \ldots, n$ and $v = 0, 1, \ldots, m$, together with conditions (1)-(4). If $f \in L_1(R^n)$ possesses at $a := (a_1, \ldots, a_n) \in R^n$ finite $m - th$ order partial derivative with respect to $j - th$ variable $f^{(m)}_j(a)$ and there exists a neighborhood $|a_1 - \eta, a_1 + \eta| \times \cdots \times |a_n - \eta, a_n + \eta|$ with $\eta > 0$ of $a$ on which the functions $f^{(m)}_j$ and $f$ are continuous, then

$$ \lim_{(x, w) \to (a, w_0)} \frac{\partial^m}{\partial x_j^m} (E_w f)(x) = f^{(m)}_j(a), $$

on any set $S$ consisting of $(x, w)$ on which the functions expressed as

$$ \sup_{w \in \Lambda} \int_{R^n} \left| \frac{\partial^m}{\partial t_j^m} K_w(t) \right| dt \quad (2.6) $$

and

$$ |x_j - a_j|^v \int_{R^n} |t_j|^m v \left| \frac{\partial^m}{\partial t_j^m} K_w(t) \right| dt \leq T_v \quad (2.7) $$

are bounded, where $j = 1, 2, \ldots, n$ and $v = 1, 2, \ldots, m$. Here, $T_v$ are certain positive constants.
Proof. By the hypotheses, the functions \( \frac{\partial^m}{\partial t^m} f(t) \) for \( m = 1, ..., n \) and \( f(t) \) are continuous at \( a \in \mathbb{R}^n \). Therefore, for all \( \varepsilon > 0 \) there exists an appropriate number \( \delta > 0 \) such that the following relations hold there:

i. \( |f(t) - f(a)| < \varepsilon \) provided that \( t \in D_1 \), where \( D_1 = ]a_1 - \delta, a_1 + \delta[ \times \cdots \times ]a_n - \delta, a_n + \delta[ \).

ii. \( |f(a_1, ..., t_j, ..., a_n) - f(a_1, ..., a_j, ..., a_n)| < \varepsilon \) provided that \( |t_j - a_j| < \delta \).

Further, for above \( \delta > 0 \), we define an auxiliary function \( h^1 \) as follows:

\[
  h^1(t) = \begin{cases} 
    h(t), & t \in D_1, \\
    0, & t \in \mathbb{R}^n \setminus D_1 
  \end{cases}
\]

where \( h(t) \) is defined by \( h(t) = h_{t_j} \), where

\[
  h_{t_j} := f(a) + (t_j - a_j) \frac{\partial f(t)}{\partial t_j} \bigg|_{t=a} + \ldots + \frac{(t_j - a_j)^m}{m!} \frac{\partial^m f(t)}{\partial t_j^m} \bigg|_{t=a}
\]

such that \( \frac{\partial^k h}{\partial t_j^k} \bigg|_{t=a} = f_j^{(k)}(a), k = 0, ..., m \).

We continue further with the following observation:

\[
  \left| \frac{f(a_1, ..., t_j, ..., a_n) - h_{t_j}}{(t_j - a_j)^m} \right| < \varepsilon \quad \text{provided that } |t_j - a_j| < \delta, \text{ since }
  \lim_{t_j \to a_j} \frac{f(a_1, ..., t_j, ..., a_n) - h_{t_j}}{(t_j - a_j)^m} = 0.
\]

Let \( |x_j - a_j| < \delta/2 \), where \( j = 1, 2, ..., n \). By linearity of the operators, we can write

\[
  (E_w f)(x) = (E_w(f + h^1 - h^1))(x) = (E_w h^1)(x) + (E_w(f - h^1))(x).
\]

Differentiating both sides of the following equation up to order \( m \) with respect to \( x_j \) such that one easily obtains

\[
  \frac{\partial^m}{\partial x_j^m} (E_w h^1)(x) = \frac{\partial^m}{\partial x_j^m} \int_{\mathbb{R}^n} h^1(t) K_w(t-x) dt
  = (-1)^m \int_{\mathbb{R}^n} h^1(t) \frac{\partial^m}{\partial t_j^m} K_w(t-x) dt.
\]

In view of Fubini’s Theorem and \( m \) times application of integration by parts with respect to \( t_j \), we have the following equality:

\[
  \lim_{(x, w) \to (a, w_0)} \int_{D_1} K_w(t-x) \frac{\partial^m}{\partial t_j^m} h(t) dt
  = \lim_{(x, w) \to (a, w_0)} \frac{\partial^m}{\partial t_j^m} \int_{\mathbb{R}^n} K_w(t-x) dt
  - \lim_{(x, w) \to (a, w_0)} \frac{\partial^m}{\partial t_j^m} f(t) \bigg|_{t=a} \int_{\mathbb{R}^n \setminus D_1} K_w(t-x) dt.
\]

Hence, by conditions (2) and (4), the desired result is obtained, that is,

\[
  \lim_{(x, w) \to (a, w_0)} \frac{\partial^m}{\partial x_j^m} (E_w h^1)(x)
  = \lim_{(x, w) \to (a, w_0)} \frac{\partial^m}{\partial t_j^m} f(t) \bigg|_{t=a} = f_j^{(m)}(a).
\]

To complete the proof, we will show that

\[
  \lim_{(x, w) \to (a, w_0)} \frac{\partial^m}{\partial x_j^m} (E_w(f - h^1))(x) = 0.
\]

In the light of observations given in the beginning of the proof, for \( \delta > 0 \), we obtain the following equality:
\[
\frac{\partial^m}{\partial x_j^m} (E_w(f - h^j)) (x) = (-1)^m \int_{D_1} [f(t) - h(t)] \frac{\partial^m}{\partial t_j^m} K_w(t - x) dt
\]
\[
= (-1)^m \left\{ \int_{D_1} + \int_{D_1 \setminus D_1} \right\} [f(t) - h(t)] \frac{\partial^m}{\partial t_j^m} K_w(t - x) dt
\]
\[
= : k(x, w) + l(x, w).
\]

We first consider \( k(x, w) \) such that

\[
k(x, w) = (-1)^m \int_{D_1} [f(t) \pm f(a_1, ..., t_j, ..., a_n) \pm f(a) - h(t)] \frac{\partial^m}{\partial t_j^m} K_w(t - x) dt
\]
\[
= (-1)^m \int_{D_1} [-f(a_1, ..., t_j, ..., a_n) + f(a)] \frac{\partial^m}{\partial t_j^m} K_w(t - x) dt
\]
\[
+(-1)^m \int_{D_1} [f(a_1, ..., t_j, ..., a_n) - h(t)] \frac{\partial^m}{\partial t_j^m} K_w(t - x) dt
\]
\[
+(-1)^m \int_{D_1} [f(t) - f(a)] \frac{\partial^m}{\partial t_j^m} K_w(t - x) dt
\]
\[
= : k_1(x, w) + k_2(x, w) + k_3(x, w).
\]

Let \(|x_j - a_j| < \frac{\delta}{2^n}\), where \( j = 1, 2, ..., n \). Using relations (i) and (ii), we obtain the following inequality for \(|k_1(x, w)| + |k_3(x, w)|\)

\[
(|k_1(x, w)| + |k_3(x, w)|) \leq 2\varepsilon \int_{R^n} \left| \frac{\partial^m}{\partial t_j^m} K_w(t) \right| dt.
\]

By (2.6), we can write

\[
\lim_{(x, w) \to (a, w_0)} (|k_1(x, w)| + |k_3(x, w)|) = 0.
\]

If we deal with \( k_2(x, w) \) using relation (iii), we obtain

\[
|k_2(x, w)| = \left| (-1)^m \int_{D_1} \frac{(f(a_1, ..., t_j, ..., a_n) - h_{a_j})}{(t_j - a_j)^m} (t_j - a_j)^m \frac{\partial^m}{\partial t_j^m} K_w(t - x) dt \right|
\]
\[
\leq \varepsilon \int_{R^n} \left| (t_j + x_j - a_j)^m \frac{\partial^m}{\partial t_j^m} K_w(t) \right| dt
\]
\[
= \varepsilon \int_{R^n} \left| ((t_j + x_j - a_j)^m - t_j^m) \frac{\partial^m}{\partial t_j^m} K_w(t) \right| dt
\]
\[
\leq \varepsilon \int_{R^n} \left| (t_j + x_j - a_j)^m - t_j^m \right| \left| \frac{\partial^m}{\partial t_j^m} K_w(t) \right| dt
\]
\[
+ \varepsilon \int_{R^n} \left| t_j^m \right| \left| \frac{\partial^m}{\partial t_j^m} K_w(t) \right| dt
\]
\[
= : \varepsilon k_{21}(x, w) + \varepsilon k_{22}(x, w),
\]
where

\[ k_{21}(x, w) = \int_{\mathbb{R}^n} |(t_j + x_j - a_j)^m - t_j^m| \left\| \frac{\partial^m}{\partial t_j^m} K_w(t) \right\| dt \]

and

\[ k_{22}(x, w) = \int_{\mathbb{R}^n} |t_j|^m \left\| \frac{\partial^m}{\partial t_j^m} K_w(t) \right\| dt. \]

It is easy to see that \( k_{22}(x, w) \) is bounded on \( S \) by condition (2.5). Using well-known formula given by

\[ z_1^s - z_2^s = (z_1 - z_2)(z_1^{s-1} + z_1^{s-2}z_2 + \ldots + z_2^{s-1}), \ z_1, z_2 \in \mathbb{R}, \]

where \( s \) is a natural number with \( s \neq 0 \), for \( k_{21}(x, w) \), we obtain

\[
k_{21}(x, w) \leq |x_j - a_j| \int_{\mathbb{R}^n} |(t_j + x_j - a_j)^{m-1}| \left\| \frac{\partial^m}{\partial t_j^m} K_w(t) \right\| dt
\]

+ \[ |x_j - a_j| \int_{\mathbb{R}^n} |t_j| |t_j|^{m-2} \left| \frac{\partial^m}{\partial t_j^m} K_w(t) \right| dt
\]

+ \[ |x_j - a_j| \int_{\mathbb{R}^n} |t_j|^{m-1} \left| \frac{\partial^m}{\partial t_j^m} K_w(t) \right| dt. \]

If we apply same operations to each integral, we see that \( k_{21}(x, w) \) is bounded above by a sum consisting of the following expressions:

\[ |x_j - a_j|^v \int_{\mathbb{R}^n} |t_j|^{m-v} \left\| \frac{\partial^m}{\partial t_j^m} K_w(t) \right\| dt, \ v = 1, 2, \ldots, m. \]

By the hypothesis (2.7), \( k_{21}(x, w) \) is bounded on \( S \). Hence

\[
\lim_{(x, w) \to (a, w_0)} k_2(x, w) = 0
\]

on \( S \). Since \( f \) is Lebesgue integrable on \( \mathbb{R}^n \) and

\[
|l(x, w)| \leq \left| \int_{\mathbb{R}^n \setminus D_1} f(t) \frac{\partial^m}{\partial t_j^m} K_w(t - x) dt \right|
\]

\[
\leq \sup_{\frac{\epsilon}{2} \leq |u|} \left| \frac{\partial^m}{\partial t_j^m} K_w(u) \right| \int_{\mathbb{R}^n} |f(t)| dt,
\]

we see that

\[
\lim_{(x, w) \to (a, w_0)} l(x, w) = 0.
\]
From above observations, we deduce that
\[
\lim_{(x,w)\to(a,w_0)} \frac{\partial^m}{\partial x^m_j} (E_w(f - h^1))(x) = 0.
\]
Therefore, the claim follows, that is,
\[
\lim_{(x,w)\to(a,w_0)} \frac{\partial^m}{\partial x^m_j} (E_w f)(x) = f_j^{(m)}(a).
\]
Thus, the proof is completed.

3. Application

Example 3.1. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by \( f(x,y) = e^{-(x^2+y^2)} \) and
\[
(E_w f)(x,y) = \frac{1}{4\pi w} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t,s) e^{-\frac{(x-t)^2+(y-s)^2}{4w}} ds dt,
\]
where the used kernel is of Gauss-Weierstrass type. This version is obtained by applying change of variables method to well-known version. Here, \( \Lambda = (0,1) \) and \( w_0 = 0 \). Some properties of above operators can be found in [15]. Figure 1 demonstrates the convergence of \( \frac{\partial^2 (E_w f)(x,y)}{\partial x^2} \) to \( \frac{\partial^2 f(x,y)}{\partial x^2} \) (dark blue) for \( w = 0.5 \) (green) and \( w = 0.2 \) (red) on \([-5, 5] \times [-5, 5]\). Figure 1 is generated by using a computer algebra system (CAS) Mathematica.

Figure 1. Demonstration

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