QUADRATIC FIELDS, ARTIN-SCHREIER EXTENSIONS, AND BELL NUMBERS

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Abstract. In this article, we prove a modulo \( p \) congruence which connects the class number of the quadratic field \( \mathbb{Q}(\sqrt{-1}^{p-1/2}p) \) and the trace of a certain monomial in a root \( \theta \) of the Artin-Schreier polynomial \( \theta^p - \theta - 1 \) over the field \( \mathbb{F}_p \) of \( p \) elements. This formula has a flavor of Dirichlet’s class number formula which connects the class number and the \( L \)-value. The proof of our formula is based on several formulae satisfied by the Bell number, where the latter is defined as the number of partitions of \( \{1, 2, ..., n\} \) and a purely combinatorial object. Among such formulae, we prove a generalization of the so called “trace formula” due to Barsky and Benzaghou which describes the special values of the Bell polynomials modulo \( p \) by the trace mentioned above.

1. Introduction

Since the 19th century, the study of the class numbers of number fields has been one of the major subjects in number theory. They are important in themselves, have many applications to Diophantine problems like Fermat’s Last Theorem, and are related with many other mathematical objects (see e.g. [6, 11, 13, 16, 19, 22, 33, 39–41]). As far as the author knows, however, no direct relation is known between the class numbers of the quadratic extensions of the field \( \mathbb{Q} \) of rational numbers and the Artin-Schreier extension \( \mathbb{F}_{p^p} \) of the field \( \mathbb{F}_p \) of \( p \) elements. In this article, we prove such a formula.

Let \( p > 3 \) be a prime and set \( p^* := (-1)^{(p-1)/2}p \). Let \((\cdot/p)\) be the quadratic residue symbol modulo \( p \), \( \text{Tr} : \mathbb{F}_{p^p} \rightarrow \mathbb{F}_p \) be the trace map, and take \( \theta \in \mathbb{F}_{p^p} \) such that \( \theta^p = \theta + 1 \). Let \( h(p^*) \) be the class number of \( \mathbb{Q}(\sqrt{p^*}) \), and \( \epsilon_p = t_p + u_p\sqrt{p} > 1 \) be the fundamental unit of \( \mathbb{Q}(\sqrt{p}) \subset \mathbb{R} \) with \( t_p, u_p \in 2^{-1}\mathbb{Z} \). Finally, set \( \tau_p(a) := \sum_{j=1}^{p-1} j p^j - 1 \). Then, our formula is stated as follows:

Theorem 1.1. The following congruence holds for every \( a \in \mathbb{Z} \) such that \( p \nmid a \);

\[
\left( -\frac{2a}{p} \right)_2 \text{Tr}(\theta^{\tau_p(a)}) \equiv \left( \frac{p - 1}{2} \right)! \equiv \begin{cases} 
\left(-1\right)^{\frac{h(p)+1}{2}} t_p \mod p & \text{if } p \equiv 1 \mod 4, \\
\left(-1\right)^{\frac{h(p)-1}{2}} \mod p & \text{if } p \equiv -1 \mod 4.
\end{cases}
\]

Since \( \tau_p(a) + 1 \equiv (1 - p^{-a})^{-1} \mod (p^p - 1)/(p - 1) \) (cf. Lemma 3.7) is associated with the Riemann zeta function \( \zeta(s) := \prod_p (1 - p^{-s})^{-1} \), it is not so strange to compare Theorem 1.1
with the following consequence of the analytic class number formula (cf. [21, §49 and §51]):

\[ e^{\frac{1}{4} \sqrt{p} \operatorname{Res}_{s=1} \zeta_{\mathbb{Q}(\sqrt{p})}(s)} = \begin{cases} h(p) & \text{if } p \equiv 1 \text{ mod } 4, \\ \epsilon_p & \text{if } p \equiv -1 \text{ mod } 4 \text{ (and } p > 3), \end{cases} \]

where \( \zeta_{\mathbb{Q}(\sqrt{p})}(s) \) is the Dedekind zeta function of \( \mathbb{Q}(\sqrt{p}) \) and \( i \) is the imaginary unit.

In addition, what is interesting is that our proof of Theorem 1.1 is based on some properties of the Bell number \( b_n \), which is defined as the number of partitions of \( \{1, 2, \ldots, n\} \) and a purely combinatorial object. The Bell number \( b_n \) has a natural subdivision by the Stirling numbers \( S(n, j) \) of the second kind, which are the numbers \( S(n, j) \) of the partitions of \( \{1, 2, \ldots, n\} \) into non-empty \( j \) subsets for non-negative integers \( j \). Since \( S(n, j) = 0 \) for \( j > n \), the series

\[ b_n(x) := \sum_{j=0}^{\infty} S(n, j)x^j, \]

defines a polynomial in \( x \) satisfying \( b_n(1) = b_n \), which we call the Bell polynomial. For known properties of these classical sequences, we refer the reader to [1–5, 8, 12, 14, 15, 17, 24, 25, 30, 37, 38] and references therein.

A key ingredient of our proof of Theorem 1.1 is the following congruence, which is a generalization of the “trace formula” due to Barsky and Benzaghou \([3, \text{Théorème } 2]\).

**Theorem 1.2** (Theorem 3.1). Let \( a, m \in \mathbb{Z}_{\geq 0} \) such that \( p \nmid a \). Then, it holds that

\[ a^m b_m(a^{-1}) \equiv -\operatorname{Tr}(\theta^{\tau_p(a)}) \operatorname{Tr}(\theta^{m-1-\tau_p(a)}) \mod p. \]

Here, we should emphasize that it is classically known that \( b_m(x) \) satisfies a linear recursion

\[ b_{m+p}(x) \equiv b_{m+1}(x) + xp b_m(x) \mod p \]
(see Example 2.5), and hence it is not surprising that the sequence \( (b_m(a^{-1}) \mod p)_m \) is described by some traces of polynomials in \( \theta \). Theorem 1.2 states, however, that this sequence can be described by the trace of the monomial \( \theta^{m-1-\tau_p(a)} \) in \( \theta \) (up to a subtle arithmetic constant in Theorem 1.1), which is what is surprising. Moreover, Theorem 1.2 (see also Lemma 4.3) tells us some consecutive zeros as follows:

(1)

\[ b_{\tau_p(a)+1}(a^{-1}) \equiv \cdots \equiv b_{\tau_p(a)+p-1}(a^{-1}) \equiv b_{\tau_p(a)+p+1}(a^{-1}) \equiv \cdots \equiv b_{\tau_p(a)+2p-1}(a^{-1}) \equiv 0 \mod p. \]

In fact, eq. (1), which was established by Radoux [35] for \( a = 1 \) and stated by Junod [24, p. 78] in general, is a key ingredient of the proof of Theorem 1.1 (see §4).

**Remark 1.3.** In fact, the statement of Theorem 1.2 itself is essentially equivalent to Junod’s “theorem” [24, Theorem 5]. However, his proof contains several errors including an imprecise intermediate result [24, Theorem 3]. In this article, we correct these errors.

In §2, we introduce the weighted Bell polynomial (cf. [9,10]) as a generalization of the Bell polynomial and recall its basic properties. In §3, we prove Theorem 1.2 in a generalized form for weighted Bell polynomials. The idea of its proof is the same as that in [24]. However, we shall clarify the crucial ideas. In §4, we first reduce Theorem 1.1 to the Hankel determinant formula in Theorem 4.1. After that, we prove Theorem 4.1 along Radoux’s proof [38] for \( a = 1 \).

\[ ^1 \text{In this view, it is more natural to call this formula the “monomial trace formula.”} \]
2. Weighted Bell polynomials and the generating series

Among several generalizations of $S(n,j)$, Carlitz \[9, 10\] introduced the weighted Stirling “number” $R(n,j;\lambda)$ of the second kind, which is a polynomial in $\lambda$ defined by

$$R(n,j;\lambda) := \sum_{m \geq 0} \binom{n}{m} S(m,j) \lambda^{n-m} \in \mathbb{Z}[\lambda].$$

Here, $\binom{n}{m}$ is the binomial coefficient. The classical Stirling number $S(n,j)$ is recovered as the constant term $R(n,j;0)$. Since $R(n,j;\lambda)$ is characterized by the initial condition

$$\sum_{n \geq 0} R(n,0;\lambda)z^n = \sum_{n \geq 0} \lambda^n z^n = \frac{1}{1-\lambda z},$$

and the recursive condition

$$R(n,j;\lambda) = (\lambda + j)R(n,1,j,\lambda) + R(n-1,j-1,\lambda),$$

its ordinary generating series is given by

$$F_j(\lambda,z) := \sum_{n \geq 0} R(n,j;\lambda)z^n = \frac{1}{1-\lambda z} \prod_{i=1}^{j} \frac{z}{1-(\lambda + i)z}.$$ 

We can identify $R(n,j;\lambda)$ with the coefficients of the asymptotic expansion of the Tate twist of the absolute zeta function $\zeta_{P^j/F_1}(z - \lambda)$ of the $j$-dimensional projective space \[27, 28\] because

$$z^{-1}F_j(\lambda,z^{-1}) = \prod_{i=0}^{j} \frac{1}{z - \lambda - i} = \zeta_{P^j/F_1}(z - \lambda).$$

Let $b_n(x,\lambda)$ be a polynomial in $x$ and $\lambda$ defined by

$$b_n(x,\lambda) := \sum_{j \geq 0} R(n,j;\lambda)x^j,$$

which is denoted by $r_n(t)$ in \[23\]. We call $b_n(x,\lambda)$ the \textit{weighted Bell polynomial}. For related sequences, see e.g. \[9, \S 7\] and \[7, 31, 32\]. Since $R(n,j;0) = S(n,j)$, we see that $b_n(x,0) = b_n(x)$. Moreover, since

$$b_n(x,\lambda) = \sum_{j \geq 0} \sum_{m \geq 0} \binom{n}{m} S(m,j) \lambda^{n-m} x^j = \sum_{m \geq 0} \binom{n}{m} b_m(x) \lambda^{n-m},$$

the exponential generating series of $b_n(x,\lambda)$ is given by

$$E(x,\lambda,z) := \sum_{n \geq 0} b_n(x,\lambda) \frac{z^n}{n!} = e^{\lambda z} \sum_{n \geq 0} b_n(x) \frac{x^n}{n!} = e^{x(e^z-1)+\lambda z} \in \mathbb{Q}[\lambda][[x, z]]$$

(cf. \[31, Theorem 3.1\]). On the other hand, eq. (2) leads us to the following representation of the ordinary generating series of $b_n(x,\lambda)$.

**Theorem 2.1.** Let $F(x,\lambda,z) := \sum_{n \geq 0} b_n(x,\lambda)z^n$. Then, we have

$$F(x,\lambda,z) = \frac{1}{1-\lambda z} \sum_{n \geq 0} \prod_{j=1}^{n} \frac{x^j}{1-(\lambda + j)z}.$$ 

Although its proof is obvious, Theorem 2.1 leads us to several significant applications. For example, we can generalize \[31, Theorem 3.2\] in a simpler manner.
Corollary 2.2. The following identity holds in \( \mathbb{Z}[\lambda][[x, z]] \):

\[
F(x, \lambda, z) = e^{-x} \sum_{n \geq 0} \frac{x^n}{(1 - (\lambda + n)z)n!}.
\]

Proof. It is sufficient to prove that

\[
(1 - \lambda z)F(x, \lambda, z) = e^{-x} \sum_{n \geq 0} \left( \frac{1 - \lambda z}{1 - (\lambda + n)z} \cdot \frac{x^n}{n!} \right).
\]

Moreover, it is sufficient to compare the coefficients of \( x^n/n! \) for \( n \geq 1 \) as elements of \( \mathbb{Z}[\lambda][[z]] \), which boils down to checking the following partial fractional decomposition

\[
\prod_{j=1}^{n} \frac{jz}{1 - (\lambda + j)z} = \sum_{j=1}^{n} (-1)^{n-j} \binom{n}{j} \frac{jz}{1 - (\lambda + j)z} = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \frac{1 - \lambda z}{1 - (\lambda + j)z}.
\]

Here, the second equality is a consequence of \( \sum_{j \geq 0} (-1)^{n-j} \binom{n}{j} = 0 \). On the other hand, the first equality holds because if we divide the both sides by \( z \), then we obtain the same rational functions of \( z \)

\[
\sum_{j=1}^{n} \frac{\rho_j(\lambda)}{z - (\lambda + j)^{-1}}
\]

which is characterized by simple poles at \( z = (\lambda + j)^{-1} (1 \leq j \leq n) \) with residues

\[
\rho_j(\lambda) := -\frac{j}{\lambda + j} \prod_{1 \leq i \leq n, i \neq j} \frac{\lambda - i}{\lambda + j} = - \frac{n!}{(\lambda + j)^2} \prod_{1 \leq i \leq n, i \neq j} \frac{1}{\lambda - i} = (-1)^{n-j-1} \binom{n}{j} \frac{j}{(\lambda + j)^2}
\]

and a zero at \( z = \infty \). This completes the proof. \( \square \)

Remark 2.3. In [31, Theorem 3.2], Mezö stated Corollary 2.2 for \( \lambda = r \in \mathbb{Z}_{\geq 0} \) in terms of the hypergeometric function \( _1F_1 \) as follows:

\[
F(x, r, z) = \frac{e^{-x}}{1 - rz} _1F_1 \left( \begin{array}{c} r - z^{-1} \\ r + 1 - z^{-1} \end{array} \bigg| x \right) = \frac{e^{-x}}{1 - rz} \sum_{n \geq 0} \binom{n-1}{j} \frac{r - z^{-1} + j}{r - z^{-1} + j + 1} \frac{x^n}{n!}.
\]

An advantage of \( F(x, \lambda, z) \) over \( E(x, \lambda, z) \) is that its reduction modulo an arbitrary integer is well-defined, and it has several applications to arithmetic properties of \( b_n(x, \lambda) \) modulo integers. For instance, we can generalize [3, Theorem 1] and [24, Theorem 1] following the same method as the original proofs in [3, 24].

Corollary 2.4. Let \( m \in \mathbb{Z}_{\geq 1} \). Then, the following congruence holds in \( (\mathbb{Z}/m\mathbb{Z})[[x, \lambda]][[z]] \):

\[
F(x, \lambda, z) \equiv \frac{\sum_{k=0}^{m-1} (xz)^k \prod_{j=k+1}^{m-1} (1 - (\lambda + j)z)}{\prod_{j=0}^{m-1} (1 - (\lambda + j)z) - (xz)^m} \mod m\mathbb{Z}.
\]

In particular, for every prime \( p \), every positive integer \( n \), and every integer \( r \), the following congruence holds in \( (\mathbb{Z}_p/(np/2)\mathbb{Z}_p)[[x]][[z]] \):

\[
F(x, r, z) \equiv \frac{\sum_{k=0}^{np-1} (xz)^{np-k} \prod_{j=1}^{k} (1 - (r - j)z)}{(1 - z^{p-1})^n - (xz)^{np}} \mod \frac{np}{2} \mathbb{Z}_p.
\]
Proof. First, note that by substituting $qm + k$ for $n$ in Theorem 2.1, we can decompose the single summation over $n \geq 0$ to a double summation as follows:

$$F(x, \lambda, z) = \frac{1}{1 - \lambda z} \sum_{q \geq 0} \sum_{k=0}^{m-1} \prod_{j=1}^{k} \frac{xz}{1 - (\lambda + j)z} \prod_{j=1}^{k} \frac{xz}{1 - (\lambda + qm + j)z}.$$ 

Therefore, we obtain Corollary 2.4 as follows:

$$F(x, \lambda, z) \equiv \frac{1}{1 - \lambda z} \sum_{k=0}^{m-1} \prod_{j=1}^{k} \frac{xz}{1 - (\lambda + j)z} \sum_{q \geq 0} \left( \prod_{j=0}^{m-1} \frac{xz}{1 - (\lambda + j)z} \right)^q \mod m$$

$$\equiv \sum_{k=0}^{m-1} (xz)^k \prod_{j=1}^{k} (1 - (\lambda + j)z)^{-1} \left( 1 - \prod_{j=0}^{m-1} \frac{xz}{1 - (\lambda + j)z} \right)^{-1} \mod m$$

$$\equiv \sum_{k=0}^{m-1} (xz)^k \prod_{j=k+1}^{m-1} (1 - (\lambda + j)z) - (xz)^m \mod m.$$ 

For eq. (4), let $n = n_0 p^\nu$ with $\nu \in \mathbb{Z}$ and $n_0 \in \mathbb{Z} \setminus p\mathbb{Z}$. Then, [18, Lemma 1.3] implies that

$$\prod_{j=0}^{np-1} (1 - (r + j)z) \equiv \left( \prod_{j=0}^{p^{\nu+1}-1} (1 - jz) \right)^{n_0} \equiv (1 - z^{p-1})^{p^\nu} \mod \frac{1}{2} p^{p+1} \mathbb{Z}_p.$$ 

On the other hand, the following congruence holds:

$$\sum_{k=0}^{np-1} (xz)^k \prod_{j=k+1}^{np-1} (1 - (r + j)z) = \sum_{k=0}^{np-1} (xz)^k \prod_{j=1}^{np-k} (1 - (r + np - j)z)$$

$$\equiv \sum_{k=0}^{np-1} (xz)^{np-k} \prod_{j=1}^{k} (1 - (r - j)z) \mod np,$$ 

By combining them with Corollary 2.4, we obtain eq. (4). \qed

**Example 2.5.** Suppose that $p$ is odd, $n = 1$, and $r = 0$. Then, eq. (4) implies that

$$(1 - z^{p-1} - (xz)^p) \sum_{n \geq 0} b_n(x) z^n \equiv \sum_{k=0}^{p-1} (xz)^k \prod_{j=k+1}^{p-1} (1 - jz) \mod p.$$ 

By comparing the coefficients of $z^{n+p}$, we obtain the congruence [36, (5)] mentioned in the previous section:

$$b_{n+p}(x) - b_{n+1}(x) - x^p b_n(x) \equiv 0 \mod p.$$ 

In fact, in a similar manner (with some auxiliary congruences), we can prove that

$$b_{n+p}(x, \lambda) - b_{n+1}(x, \lambda) - (\lambda^p - \lambda + x^p) b_n(x, \lambda) \equiv 0 \mod p.$$ 

In the case of $\lambda \in \mathbb{Z}$, the above congruence has already appeared in the proof of [23, Theorem 3.1] and was rediscovered in [32, Theorem 4]. In fact, both eq. (5) and eq. (6) hold also for $p = 2$, which we can check more directly (cf. [42], [43, Lemma 4]).
Let \( p \) be an arbitrary prime. In this section, we prove the following “trace formula” for weighted Bell polynomials \( b_n(x, \lambda) \) specialized at \( x = a \) for an integer \( a \), which is a generalization of [3, Théorème 2].

**Theorem 3.1.** Let \( a, m \in \mathbb{Z}_{\geq 0} \) such that \( p \nmid a \) and \( \text{Tr} : \mathbb{F}_{p^m}[\lambda] \rightarrow \mathbb{F}_p[\lambda] \) be the \( \mathbb{F}_p \)-linear extension of \( \text{Tr} : \mathbb{F}_p \rightarrow \mathbb{F}_p \). Then, the following congruence holds in \( \mathbb{F}_p[\lambda] \):

\[
a^m b_m(a^{-1}, \lambda) \equiv - \text{Tr}(\theta^{2/(\lambda)}) \text{Tr} \left( \frac{(a \lambda + \theta)^m}{\theta^{1+2/(\lambda)}} \right) \mod p.
\]

**Remark 3.2.** Before the proof of Theorem 3.1, recall that

\[
b_m(x, \lambda) = \sum_{j=0}^{m} \binom{m}{j} b_j(x) \lambda^{m-j}
\]

(cf. eq. (3)). Therefore, it is sufficient to prove Theorem 3.1 for \( \lambda = 0 \). In this case, the statement itself has already appeared in Junod’s paper [24, Theorem 5]. However, his proof is imprecise. Moreover, he stated an intermediate result [24, Theorem 3], which is incorrect. In this section, we correct these errors and give self-contained proofs. The ideas of our proofs are the same as [24], but we shall clarify the crucial ideas.

**Remark 3.3** \((p = 2)\). The proof of Theorem 3.1 for \( p = 2 \) is quite easy: In this case, we may assume that \( a = 1 \). Since \( \tau_2(1) = 1 \), \( b_{n+2} \equiv b_{n+1} + b_n \mod 2 \), and \( \theta^3 = 1 \), it is sufficient to prove the formula for \( m = 0, 1, 2 \), which boils down to checking that

\[
b_0(1) = 1, \quad b_1(1) = 1, \quad b_2(1) = 2,
\]

and

\[
\text{Tr}(\theta^{-2}) \equiv 1 \mod 2, \quad \text{Tr}(\theta^{-1}) \equiv 1 \mod 2, \quad \text{Tr}(\theta^0) \equiv 2 \mod 2.
\]

In what follows, we fix an odd prime \( p \). Let \( n \in \mathbb{Z}_{\geq 1} \). Define two polynomials in \( \mathbb{Z}[x, z] \) by

\[
g_{x, n}(z) := (1 - z^{p-1})^n - (xz)^{np} \quad \text{and} \quad g_{x, n}^*(z) := z^{np} g_{x, n}(z^{-1}) = (z^p - z)^n - x^{np}.
\]

In what follows, \( \overline{\mathbb{F}} \) denotes a fixed algebraic closure of each field \( F \). Let \( \mathcal{O} \) be the ring of integers in \( \overline{\mathbb{Q}}_p \) and \( \zeta_n \in \mathcal{O}^\times \) be a primitive \( n \)-th root of unity. Note that \( g_{x, n}^*(z) \in \mathcal{O}[[z]] \) if and only if \( x \in \mathcal{O}^\times \).

**Lemma 3.4.** \(2\) For every \( x \in \mathcal{O}^\times \), \( g_{x, n}^*(z) \) has distinct \( np \) roots in \( \mathcal{O} \).

**Proof.** Since \( g_{x, n}^*(z) = \prod_{m=1}^{n} g_{x, m, 1}^*(z) \) and any distinct two factors \( g_{x, m, 1}^*(z) \) \((1 \leq m \leq n)\) have no common roots, it is sufficient to prove that each factor \( g_{x, m, 1}^*(z) \) has \( p \) distinct roots in \( \mathcal{O}^\times \), i.e., \( g_{x, m, 1}^*(z) \) is separable. Indeed, the derivative of \( g_{x, m, 1}^*(z) \)

\[
\frac{\partial}{\partial z} g_{x, m, 1}^*(z) = p z^{p-1} - 1
\]

has no common roots with \( g_{x, n}^*(z) \) itself since \( x \in \mathcal{O}^\times \). \( \square \)

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\(2\)In the proof of [24, Theorem 3], Junod used [24, Lemma], which states that if one takes \( \theta \in \overline{\mathbb{Q}}_p \) so that \( \theta^p = \theta + 1 \) and \( a \in \mathbb{Z} \) so that \( p \nmid a \), then the polynomial \( g_{a, n}^*(z) \) has distinct \( np \) roots in the ring \( \mathbb{Z}_p[\theta]/npZ_p[\theta] \). This statement is incorrect when \( p \nmid n \) because we need to extend the ring from \( \mathbb{Z}_p[\theta]/npZ_p[\theta] \cong \mathbb{F}_{p^m}(\zeta_n) \) to \( \mathbb{F}_{p^m} \). Moreover, his proof is based on the claim that \( a \theta + m \) \((0 \leq m \leq np - 1)\) form the distinct \( np \) roots of \( g_{a, n}^*(z) \) in \( \mathbb{F}_{p^m} \), which is wrong whenever \( n > 1 \). In order to remedy this error, we replace [24, Lemma] by Lemma 3.4.

\(3\)In view of the proof, it is sufficient to assume that \( x^{p^{(p-1)}} \not\equiv p(1 - p^{(p-1)}) \mod (\zeta_n) \) (e.g. \( x \in \overline{\mathbb{Q}}_p \)).
The following formula, which is a corrected and generalized version of [24, Theorem 3], gives a representation of $b_m(x)$ for $x \in O^\times$ as a trace-like sum over the roots of the polynomial $g_{x,n}(z)$. It is a key ingredient in the proof of Theorem 1.2 and originate from [3, Lemme 4], where the latter was established decades after the studies [2, 35–37] in 1970’.

**Proposition 3.5.** Let $m \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{\geq 1}$. Then, for every $x \in O^\times$,

$$-\frac{x^{p-1}}{n} \sum_{l=0}^{n-1} \frac{\zeta_l^m}{\Theta^{p-1} \sum_{\Theta \in \mathbb{Q}_p} \prod_{j=1}^{k} \left( \frac{\Theta + j}{x} \right) \in \mathbb{Z}_p[x]}$$

and it congruent to $b_m(x)$ in $\mathbb{Z}_p[x]/np\mathbb{Z}_p[x]$. In particular, for every $(p-1)$-th root of unity $\xi \in \mathbb{Z}_p^\times$, the following congruence holds in $\mathbb{Z}_p/np\mathbb{Z}_p$:

$$\xi^m b_m(\xi^{-1}) \equiv -\frac{1}{n} \sum_{\Theta \in \mathbb{Q}_p} \frac{\Theta^m}{\Theta^{p-1} \sum_{\Theta \in \mathbb{Q}_p} \prod_{j=1}^{k} \left( \frac{\Theta + j}{\xi} \right) \mod np\mathbb{Z}_p}.$$

**Proof.** Define a rational function in $\mathbb{Q}_p(z)$ by

$$F_{x,n}(z) := \frac{1}{g_{x,n}(z)} \sum_{k=0}^{np-1} (xz)^{np-k-1} \prod_{j=1}^{k} (1 + jz).$$

Then, Lemma 3.4 implies that $F_{x,n}(z)$ has a partial fractional decomposition as follows

$$F_{x,n}(z) = \sum_{\eta \in \mathbb{Q}_p} \frac{\rho_{x,n}(\eta)}{z - \eta},$$

where $\rho_{x,n}(\eta) := (z - \eta)F_{x,n}(z)|_{z=\eta}$ (cf. the proof of Corollary 2.2). In particular, we obtain

$$\frac{1}{m!} \frac{\partial^m}{\partial z^m} F_{x,n}(z) \bigg|_{z=0} = -\sum_{\eta \in \mathbb{Q}_p} \frac{\rho_{x,n}(\eta)}{\eta^{m+1}}.$$

Moreover, for every root $\eta$ of $g_{x,n}(z)$, we have

$$\frac{\partial}{\partial z} g_{x,n}(\eta) = -n(p-1)(1-\eta^{p-1})^{n-1}\eta^{p-2} - npx(\eta)^{np-1} = -nx(\eta)^{np-1}\eta^{p-1} - p,$$

and hence

$$\rho_{x,n}(\eta) = \frac{1}{\frac{\partial}{\partial z} g_{x,n}(\eta)} \sum_{k=0}^{np-1} (\eta)^{np-k-1} \prod_{j=1}^{k} (1 + j\eta) = -\frac{\eta^{p-1} - 1}{nx(\eta^{p-1} - p)} \sum_{k=0}^{np-1} \prod_{j=1}^{k} \left( \frac{1}{\eta} + j \right).$$
By combining the above calculations and rewriting \( \eta^{-1} \) to \( \Theta \), we obtain the following equality:

\[
\frac{1}{m!} \frac{\partial^m}{\partial z^m} F_{x,n}(z) \bigg|_{z=0} = \frac{1}{n!} \sum_{\eta \in \mathbb{Q}_p, \ g_{x,n}(\eta) = 0} \frac{\eta^{p-1} - 1}{\eta^{m+1}(\eta^{p-1} - p)} \sum_{k=0}^{n-1} \left( \frac{1}{x \eta} + \frac{j}{x} \right)
\]

\[
= \frac{1}{n!} \sum_{\Theta \in \mathbb{Q}_p, \ g_{x,n}(\Theta) = 0} \frac{\Theta^m (\Theta - \Theta_1)}{1 - p \Theta^{p-1}} \sum_{k=0}^{n-1} \frac{\Theta^{-1} x^k}{\Theta - \Theta_1} \sum_{j=1}^k \left( \frac{\Theta + j}{x} \right).
\]

Since \( x \in \mathcal{O}^x \), the leftmost side belongs to \( \mathbb{Z}_p[x] \), which implies the first statement. On the other hand, Corollary 2.4 implies that

\[
b_m(x) \equiv \frac{1}{m!} \frac{\partial^m}{\partial z^m} F_{x,n}(z) \bigg|_{z=0} \mod np\mathbb{Z}_p[x].
\]

Therefore, we obtain the second statement. 4 For the last statement, it is sufficient to note that, \( \theta^p - \theta = \zeta^p_n \) if and only if \( (\xi^{-1} \theta)^p - (\xi^{-1} \theta) = \zeta^p_n (\xi^{-1})^p \).

**Remark 3.6.** The above proof relies on Corollary 2.4, which lifts the ordinary generating series \( F(x, 0, z) \in (\mathbb{Z}[x]/np\mathbb{Z}[x])[\langle z \rangle] \) of \( (b_m(x) \mod np\mathbb{Z}[x])_m \) to a rational function \( F_{x,n}(z) \in \mathbb{Q}_p(x, z) \). Such a rational approximation goes back to Barsky’s pioneering work [2]. Note that \( F(x, 0, z) \) itself is highly transcendental because even a specialization \( F(1, 0, z) = e^{-1}(1 - z)^{-1} F_1(1 - z^{-1}, 1) \) satisfies no algebraic differential equations over \( \mathbb{C}[\langle z \rangle] \) [26, Theorem 3.5].

In the proof of Theorem 3.1, we shall use the following lemma, which is a part of [24, Proposition 4]. The author believes that the following proof is more natural than [24] involving more technical calculations. Additionally, the author expects that our proof would lead us to a deeper understanding of Theorem 3.1 because it relates \( \tau_p(a) := \sum_{j=1}^{p-1} j p^{a-1} \) more directly to the cyclotomic polynomial \( (x^p - 1)/(x - 1) \) than [24].

**Lemma 3.7.** Suppose that \( a \not\equiv 0 \mod p \). Then, it holds that

\[
\tau_p(a) (p^a - 1) \equiv 1 \mod \frac{p^p - 1}{p - 1}.
\]

In particular, if \( \eta \in \mathbb{F}_{p^p} \) has norm 1, then \( \eta^{\tau_p(a)} \) is the unique \( (p^a - 1) \)-th root of \( \eta \) in \( \mathbb{F}_{p^p} \).

**Proof.** By calculating the derivative of \( x^{ap} - 1 \in \mathbb{Z}[x] \) in two ways, we obtain

\[
\left( \frac{x^{ap} - 1}{x^a - 1} \right)' (x^a - 1) + \frac{x^{ap} - 1}{x^a - 1} \cdot ax^{a-1} = apx^{ap-1}.
\]

---

4In this step, Junod made a crucial error to substitute the congruence

\[
\frac{\partial}{\partial z} g_{x,n}(z) = -nx(xn)^{np-1} \eta^{p-1} - p \eta^{p-1} \equiv -nx(xn)^{np-1} \eta^{p-1} \mod np\mathbb{Z}[x]
\]

to \( \rho_{x,n}(\eta) = (\frac{\partial}{\partial z} g_{x,n}(\eta))^{-1} \sum_{k=0}^{p-1}(xn)^{np-k} \eta^{p-1} \prod_{j=1}^{k} (1 + j \eta) \). Since \( (\partial/\partial z) g_{x,n}(z) \in \mathcal{O}^x \) only if \( p \nmid n \), the above substitution cannot be justified.
Since \( a \not\equiv 0 \mod p \), there exists a polynomial \( \psi(x) \in \mathbb{Z}[x] \) such that \( (x^{ap} - 1)/(x^a - 1) = \psi(x)(x^p - 1)/(x - 1) \), and hence

\[
\left( \frac{x^{ap} - 1}{x^a - 1} \right)' \bigg|_{x=p} (p^a - 1) + ap^{a-1} \psi(p) \cdot \frac{p^p - 1}{p - 1} = a \cdot (p^a)^a.
\]

By taking modulo \( (p^p - 1)/(p - 1) \), we obtain the desired congruence. \(\square\)

**Proof of Theorem 3.1.** As noted in Remarks 3.2 and 3.3, we may assume that \( \lambda = 0 \) and \( p \geq 3 \). By applying Proposition 3.5 for \( n = 1 \) and Lemma 3.7, we obtain a congruence in \( \mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{F}_p \)

\[
a^mb_m(a^{-1}) \equiv - \sum_{\theta \in \mathbb{F}_p, \, g_1^*(\theta) = 0} \theta^m \sum_{k=0}^{p-1} \prod_{j=1}^{k} (\theta + ja) \equiv - \sum_{\theta \in \mathbb{F}_p, \, \theta^p = \theta + 1} \theta^m \sum_{k=0}^{p-1} \prod_{j=0}^{k} \theta^{p^k} \equiv - \sum_{\theta \in \mathbb{F}_p, \, \theta^p = \theta + 1} \theta^{m-\tau_p(a)} \sum_{k=0}^{p-1} \bigl( \theta^{p^k} \bigr)^{\tau_p(a)} \mod p.
\]

Since \( a \not\equiv 0 \mod p \), \( \theta^{p^k} (0 \leq k \leq p-1) \) form the roots of the polynomial \( g_1^*(z) = z^p - z - 1 \), and hence \( \sum_{k=0}^{p-1} (\theta^{p^k})^{\tau_p(a)} = \text{Tr}(\theta^{\tau_p(a)}) \in \mathbb{F}_p \). This completes the proof. \(\square\)

4. PROOF OF THEOREM 1.1

In this section, we complete the proof of Theorem 1.1. Let \( p \) be an odd prime. Then, since \( \tau_p(a) \equiv (p-1)/2 \mod p - 1 \) and \( \text{Tr}(\theta^{-1}) = -1 \), Theorem 3.1 for \( m = \tau_p(a) \) implies that

\[
\left( \frac{a}{p} \right)_2 \text{Tr}(\theta^{\tau_p(a)}) \equiv b_{\tau_p(a)}(a^{-1}) \mod p,
\]

where \( (a/p)_2 \) denotes the quadratic residue symbol modulo \( p \). On the other hand, by [11,33], we have

\[
\left( \frac{p-1}{2} \right)! \equiv \begin{cases} 
(-1)^{-\frac{b(p+1)}{2}}t_p \mod p & \text{if } p \equiv 1 \mod 4, \\
(-1)^{-\frac{b(p+1)}{2}} \mod p & \text{if } p \equiv -1 \mod 4 \text{ and } p > 3.
\end{cases}
\]

Therefore, Theorem 1.1 follows from the following congruence.

**Theorem 4.1.** Let \( p \) be an odd prime and \( a \in \mathbb{Z}_{\geq 1} \) such that \( p \nmid a \). Then, it holds that

\[
b_{\tau_p(a)}(a^{-1}) \equiv \left( \frac{-2}{p} \right)_2 \left( \frac{p-1}{2} \right)! \mod p.
\]

For \( a = 1 \), the above congruence was already obtained by Radoux in [35, p. 881] and [38, (13)], whose proof based on the following Hankel determinant formula.

**Lemma 4.2** ([38, (12)], [15, Theorem 1], see also [1]). Let \( n \in \mathbb{Z}_{\geq 1} \), and set \( n \times n \) matrix \( B_n(x) := (b_{i+j}(x))_{0 \leq i,j \leq n-1} \). Then, we have

\[
\det B_n(x) = \left( \prod_{j=0}^{n-1} j! \right) x^{\frac{n(n-1)}{2}}.
\]
In particular, if \( n = p \) is an odd prime and \( a \in \mathbb{Z} \) not divisible by \( p \), then it holds that
\[
\det \mathbb{B}_p(a^{-1}) \equiv \left( \frac{2a}{p} \right) \left( \frac{p-1}{2} \right) \mod p.
\]

We also use the following lemma, whose proof is given in the appendix.

**Lemma 4.3** (cf. [3, Lemme 3]). Let \( p \) be an odd prime, and \( n \in \mathbb{Z} \). Then, \( \text{Tr}(\theta^n) \) coincides with \(-1\) times the coefficient of \( \theta^{p-1} \) in the \( \mathbb{F}_p \)-linear representation of \( \theta^n \) for the basis consisting of \( \theta^0 = 1, \theta, \ldots, \theta^{p-1} \). In particular, we have
\[
\text{Tr}(\theta^0) = \text{Tr}(\theta^1) = \cdots = \text{Tr}(\theta^{p-2}) = \text{Tr}(\theta^p) = \text{Tr}(\theta^{p+1}) = \cdots = \text{Tr}(\theta^{2p-3}) = 0
\]
and
\[
\text{Tr}(\theta^{-1}) = \text{Tr}(\theta^{p-1}) = -1.
\]

Now, we can prove Theorem 4.1 following Radoux’s idea.

**Proof of Theorem 4.1.** Set \( \mathbb{B}_p^{(n)}(x) := (b_{n+i+j}(x))_{0 \leq i, j < p-1} \). Then, Example 2.5 shows that
\[
\det \mathbb{B}_p^{(n+1)}(x) = \det \begin{pmatrix}
 b_{n+1}(x) & b_{n+2}(x) & \cdots & b_{n+p}(x) - b_{n+1}(x) \\
 b_{n+2}(x) & b_{n+3}(x) & \cdots & b_{n+p+1}(x) - b_{n+2}(x) \\
 \vdots & \vdots & \ddots & \vdots \\
 b_{n+p}(x) & b_{n+p+1}(x) & \cdots & b_{n+2p-1}(x) - b_{n+p}(x)
\end{pmatrix}
\equiv x^p \cdot (-1)^{p-1} \det \mathbb{B}_p^{(n)}(x) \equiv \cdots \equiv x^m \det \mathbb{B}_p^{(0)}(x) \mod p.
\]

Since \( \tau_p(a) \equiv (p-1)/2 \mod p - 1 \) and \( a^{\tau_p(a)} \equiv (a/p)_2 \mod p \), Lemma 4.2 implies that
\[
\det \mathbb{B}_p^{(\tau_p(a))}(a^{-1}) \equiv a^{-\tau_p(a)} \det \mathbb{B}_p^{(0)}(a^{-1}) \equiv \left( \frac{2}{p} \right) \left( \frac{p-1}{2} \right) \mod p.
\]

On the other hand, by Theorem 3.1 and Lemma 4.3, we have
\[
b_{\tau_p(a)+1}(a^{-1}) \equiv \cdots \equiv b_{\tau_p(a)+p-1}(a^{-1}) \equiv b_{\tau_p(a)+p+1}(a^{-1}) \equiv \cdots \equiv b_{\tau_p(a)+2p-1}(a^{-1}) \equiv 0 \mod p
\]
(cf. [25, 35] for \( a = 1 \)), and hence
\[
\det \mathbb{B}_p^{(\tau_p(a))}(a^{-1}) \equiv \det \begin{pmatrix}
 b_{\tau_p(a)}(a^{-1}) & 0 & \cdots & 0 \\
 0 & 0 & \cdots & 0 \\
 0 & 0 & \cdots & a^{-\tau_p(a)}b_{\tau_p(a)}(a^{-1}) \\
 \vdots & \vdots & \ddots & \vdots \\
 0 & a^{-\tau_p(a)}b_{\tau_p(a)}(a^{-1}) & \cdots & 0
\end{pmatrix}
\equiv (-1)^{\frac{p-1}{2}} b_{\tau_p(a)}(a^{-1}) \mod p.
\]

By comparing the above two formulae, we obtain the desired congruence. \(\square\)

**Remark 4.4.** In [24, Theorem 5], Junod stated his trace formula in the form
\[
a^mb_m(a^{-1}) \equiv -a^{\tau_p(a)}b_{\tau_p(a)}(a^{-1}) \text{Tr}(\theta^{m-\tau_p(a)}) \mod p.
\]

It follows from Theorem 3.1 and the fact that \( (a/p)_2 \text{Tr}(\theta^{\tau_p(a)}) \) is independent of \( a \) (but has deep arithmetic nature as stated in Theorem 1.1). However, Junod’s proof of the latter fact depends on the assumption that \( (a^{-1} \theta)^p \equiv a^{-1} \theta + 1 \mod p \), which is wrong unless \( a \equiv 1 \mod p \). Now, we can fill the gap by combining eq. (7) and Theorem 4.1.
Remark 4.5. The right hand side of Theorem 4.1, or equivalently $b_{p(a)}(a^{-1}) \mod p$, can be described by the special value $\Gamma_p(1/2)$ of Morita’s $p$-adic $\Gamma$-function [34], and hence by a quadratic Gauss sum [20, Theorem 1.7]. In fact, these quantities are strongly related to the Artin-Schreier curve defined by $x^p - x = y^2$ [29, §§15–16]. Hence, it should be an interesting problem to give geometric interpretations of more general $b_m(a^{-1}) \mod p$.

APPENDIX A. PROOF OF LEMMA 4.3

Set $a_{n,0}, \ldots, a_{n,p-1} \in \mathbb{F}_p$ so that $\theta^n = \sum_{i=0}^{p-1} a_{n,i} \theta^i$. Then, for every $m \in \mathbb{Z}_{\geq 0}$, we have

$$\theta^{np^m} = \left( \sum_{j=0}^{p-1} a_{n,j} \theta^j \right)^{p^m} = \sum_{j=0}^{p-1} a_{n,j} (\theta + m)^j = \sum_{i=0}^{p-1} \left( \sum_{j=i}^{p-1} a_{n,j} m^{j-i} \right) \theta^i.$$  

On the other hand, since $\theta^0 = 1, \theta, \ldots, \theta^{p-1}$ are linearly independent over $\mathbb{F}_p$, we see that

$$\text{Tr}(\theta^n) = \sum_{m=0}^{p-1} \theta^{np^m} = \sum_{m=0}^{p-1} \sum_{j=0}^{p-1} \left( \sum_{i=0}^{m} a_{n,j} m^{j-i} \right) \equiv \sum_{j=0}^{p-1} a_{n,j} \sum_{m=0}^{p-1} m^j \mod p.$$ 

Therefore, the claimed identity follows from the following congruence:

$$S_j(p) := \sum_{m=0}^{p-1} m^j \equiv \begin{cases} -1 \mod p & \text{if } j > 0 \text{ and } j \equiv 0 \mod p - 1 \\ 0 \mod p & \text{otherwise} \end{cases}.$$ 

Since the case of $j \equiv 0 \mod p - 1$ is obvious, we may assume that $1 \leq j \leq p - 2$. By summing up

$$\sum_{m=0}^{j} \binom{j+1}{m} k^m = (k+1)^{j+1} - k^{j+1} (k \in \mathbb{Z}_{\geq 0})$$

with respect to $0 \leq k \leq p - 1$, we have

$$\sum_{m=0}^{j} \binom{j+1}{m} S_m(p) = p^{j+1} \equiv 0 \mod p.$$ 

By induction on $j$, we obtain the desired congruence.

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