Research Article

Note on a Class of Subsets of AG(3, q) with Intersection Numbers 1, q and n with respect to the Planes

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We give a new and correct proof of a result of O. Ferri and S. Ferri (1995) on $q^2$-caps of AG(3, q) in this paper; moreover we prove that sets of AG(3, q) of type $(1, q, q+1)$ with respect to the planes of AG(3, q) have size at most $q^2$ with equality if and only if $K$ is a cap.

1. Introduction

Let $G$ denote either a finite projective space or a finite affine space, and let $\{m_0, \ldots, m_s\}$ be a set of nonnegative integers with $m_0 < m_1 < \cdots < m_s$. A subset $\mathcal{K}$ of $G$ is of class $[m_0, \ldots, m_s]$ with respect to the subspaces of dimension $d$ of $G$ if any subspaces intersect $\mathcal{K}$ either in $m_0, \ldots, m_{s-1}$ or $m_s$ points, and $\mathcal{K}$ is of type $(m_0, \ldots, m_s)$ with respect to the subspaces of dimension $d$ if for every integer $m_i, i \in \{0, \ldots, s\}$ there exists a $d$-subspace $H$ meeting $\mathcal{K}$ in exactly $m_i$ points. The numbers $m_i$ are called intersection numbers of $\mathcal{K}$. As usual, by a $k$-set we mean a set of size $k$.

In the literature one can find many papers devoted to the study of $k$-sets of given type, not only in affine and projective geometries (cf., e.g., [1–15]), and most of these results are characterizations of classical geometric objects. Recently, characterizations of Hermitian varieties and quadrics of PG(r, q) as $k$-sets with given intersection numbers with respect to more than one family of subspaces (e.g., with respect to planes and solids) have been considered [16, 17].

A cap of an affine or projective space of dimension $\geq 3$ is a subset of points no three of which are collinear.

In 1995, O. Ferri and S. Ferri [18] gave a characterization of $q^2$-caps of AG(3, q) in terms of sets with three given intersection numbers with respect to the planes. Their result reads as follows.

Theorem 1 (see [18]). Let $K$ be a subset of AG(3, q) with $|K| = q^2$ points and of type $(1, q, n)$. Then, $n = q + 1$ and $K$ is a cap of AG(3, q).

Unfortunately, a step of the proof of that theorem is not correct; in fact it contains a counting argument which does not give the contradiction they want (see [18] page 71 line +7). However, the statement of the result is true as we are going to prove in Lemma 4.

In this paper we will prove the following slight extension of the O. Ferri and S. Ferri result.

Theorem 2. Let $\mathcal{K}$ be a set of AG(3, q) of size $k \geq q^2$ and with three intersection numbers $1, q$, and $n$. Then $n \geq q + 1$, and $n = q + 1$ if and only if $k = q^2$. Moreover, if $k = q^2$ the set $\mathcal{K}$ is a cap.

Thus, it follows that the sets of type $(1, q, q+1)$ of AG(3, q) have size at most $q^2$ and that equality holds if and only if they are caps.

2. Proof of Theorem 1

In this section, first, we briefly recall the basic equations for a $k$-set of AG(3, q) with three intersection numbers, and then we will assume that $k = q^2$ and we will give the proof of Theorem 1.
2.1. The Basic Equations for $k$-Sets with Intersection Numbers $l$, $q$, and $n$. Let $t_i$, $(i = 1, q, n)$, denote the number of planes intersecting $K$ in exactly $i$-points (such numbers are called characters of $\mathcal{K}$).

Double counting gives

$$t_1 + t_q + t_n = q^3 + q^2 + q,$$

$$t_1 + qt_q + nt_n = k \left( q^2 + q + 1 \right),$$

$$q(q - 1)t_1 + n(n - 1)t_n = k(k - 1)(q + 1).$$

From (1) it follows that

$$t_1 = \left( q^2 n \left( q^2 + q + 1 \right) - k \left( q^2 + q + 1 \right) \right),$$

$$\times (n + q - 1) + k(k - 1)(q + 1),$$

$$\times \left( (n - 1)(q - 1) \right)^{-1},$$

$$t_q = \frac{n \left( q^2 + q + 1 \right) (k - q) - k(k - 1)(q + 1)}{(q - 1)(n - q)},$$

$$t_n = \frac{k(k - 1)(q + 1) - (k - q) q \left( q^2 + q + 1 \right)}{(n - 1)(n - q)}.$$  

From (1) if follows that

$$t_1 = \left( q^2 n \left( q^2 + q + 1 \right) - k \left( q^2 + q + 1 \right) \right)\times (n + q - 1) + k(k - 1)(q + 1)\times \left( (n - 1)(q - 1) \right)^{-1},$$

$$t_q = \frac{n \left( q^2 + q + 1 \right) (k - q) - k(k - 1)(q + 1)}{(q - 1)(n - q)},$$

$$t_n = \frac{k(k - 1)(q + 1) - (k - q) q \left( q^2 + q + 1 \right)}{(n - 1)(n - q)}.$$  

The proof of Theorem 1 follows from Lemmas 3 and 4.

Let us end with the following easy consequence of Theorem 2.

**Corollary 5.** A subset $\mathcal{K}$ of points of $AG(3, q)$ of type $\{1, q, q + 1\}$ has size $k \leq q^2$, and $k = q^2$ if and only if $\mathcal{K}$ is a cap of $AG(3, q)$.

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