Exponentially-fitted Fourth-Order Taylor’s Algorithm for Solving First-Order ODEs

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Abstract. In this paper, exponentially-fitted variants of the classical fourth-order Taylor’s algorithm suitable for solving first-order ordinary differential equations are constructed. The methodology is based on the six-step flow chart proposed in [9]. The absolute stability properties of the new variants are presented. Implementation of the new schemes on some test problems showed that the new methods compared favourably with other well-known fourth-order methods.

1. Introduction
This work considers the first-order IVP

\[ u' = f(t, u), \quad t \in [t_0, t_n], \quad u(t_0) = u_0 \]  

(1)

is considered. One of the earliest known methods for solving (1) is the Taylor algorithm. The Taylor algorithm of order \( p \) requires that higher derivatives of the problem be obtained up to order \( p \) and it can integrate (1) up to machine accuracy if its solution is a linear combination of members of the set

\[ \{1, t, t^2, \ldots, t^p\} \]  

(2)

In fact, several classical methods ([4, 11, 1, 19, 20, 21]) for solving (1) are derived based on the Taylor algorithm. However, conventional Taylor methods are less accurate when applied to problems which solutions are not strictly polynomials. This barrier is overcome by fitting the conventional methods exponentially. Exponentially fitted methods are usually enhanced variants of the underlying classical methods designed to efficiently approach problems whose solutions consist of exponential terms. These enhanced methods are usually obtained by replacing some higher order monomials in (2) with exponentials or trigonometric (see [9, 18, 15]). One of the earliest known work involving the use of exponential fitting for differential equations is that of Liniger and Willoughby [13]. The authors in [13] constructed integration formulae containing free parameters chosen such that a given function \( \exp(q) \), \( q \in \mathbb{R} \), exactly satisfies the proposed method. Using a linear 2-step formula, Jackson and Kenue proposed an A-stable fourth-order exponentially-fitted formulae. Following [10], the author in [5] proposed an exponentially-fitted single-step second-derivative. In this direction, specific Runge–Kutta methods have been constructed by different authors [6, 2, 7, 3, 16, 17]. Recently, exponentially-fitted variants of the classical Simpson’s method were proposed by the authors in [22].
In this work, exponentially-fitted variants of the fourth-order Taylor algorithm for solving (1) using the proposed six-step flow chart in [9] is presented. Numerical implementations of the resulting methods are also presented.

2. The Six-Step Flow Procedure

In this section, the six-step flow chart proposed in [9] is briefly reviewed.

**Step I** Write down the linear difference operator \( \mathcal{L}[h,a] \) related to the scheme under consideration. Vector \( a \) is defined as the list of coefficients for which we need to find expressions.

**Step II** Determine the maximum value of \( M \) such that the algebraic system

\[
\{ \mathcal{L}_m^*(a) = 0 | m = 0, \cdots, M - 1 \}, \quad \text{with} \quad L_m^*(a) := h^{-m} \mathcal{L}[h,a]t^m|_{t=0}
\]

can be solved.

**Step III** Construct

\[
E_0^+(\pm \omega_h,a) := \exp (\mp \omega_h t) \mathcal{L}[h,a] \exp (\pm \omega_h t)
\]

where \( \omega_h := h\omega \) and we build

\[
G^+(\Omega_h,a) := \frac{1}{2} [E_0^+ (\omega_h,a) + E_0^+ (-\omega_h,a)] , \quad \text{and}
\]

\[
G^-(\Omega_h,a) := \frac{1}{2\omega_h} [E_0^+ (\omega_h,a) - E_0^+ (-\omega_h,a)] , \quad \text{with} \quad \Omega_h := \omega_h^2.
\]

**Step IV** A reference set of the form

\[
\{1,t,\cdots,t^K,\exp (\pm \omega t),t \exp (\pm \omega t),\cdots,t^P \exp (\pm \omega t)\} \quad (3)
\]

with \( M \) elements are chosen such that \( K + 2P = M - 3 \).

**Step V** Solve the algebraic system

\[
\begin{cases}
L_k^*(a) , \quad k = 0, \cdots, K \\
G^{\pm (p)}(\Omega_h,a) , \quad p = 0, \cdots, P . \quad \text{for} \quad a.
\end{cases}
\]

**Step VI** Investigate the error expression associated with the exponentially fitted method.

3. Construction of Method

The classical fourth-order Taylor algorithm for solving (1) is given by

\[
u_{j+1} = u_j + hu'_j + \frac{1}{2} h^2 u''_j + \frac{1}{6} h^3 u^{(3)}_j + \frac{1}{24} h^4 u^{(4)}_j . \quad (4)
\]

To construct the exponentially-fitted variants of (4), we proceed by rewriting (4) in a more general way as

\[
u_{j+1} = \delta_0 u_j + \gamma_1 h u'_j + \gamma_2 h^2 u''_j + \gamma_3 h^3 u^{(3)}_j + \gamma_4 h^4 u^{(4)}_j . \quad (5)
\]

The linear difference operator \( \mathcal{L}[h,a] \), associated with (5) is obtained as

\[
\mathcal{L}[h,a]u(t) = u(t+h) - \delta_0 u(t) - \gamma_1 h u'(t) - \gamma_2 h^2 u''(t) - \gamma_3 h^3 u^{(3)}(t) - \gamma_4 h^4 u^{(4)}(t)
\]
where \( a := (\delta_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4) \). The resulting system upon the application of step II of the flow chart is compatible with \( M = 5 \) and the solution given as

\[
\delta_0 = 1, \quad \gamma_1 = 1, \quad \gamma_2 = \frac{1}{2}, \quad \gamma_3 = \frac{1}{6}, \quad \gamma_4 = \frac{1}{24}.
\] (6)

The values obtained in (6) are the coefficients of the conventional fourth–order Taylor’s method (4).

To construct the exponentially–fitted variants of (4), the \( G \) functions are obtained as

\[
G^+(Z, a) = -\delta_0 - \gamma_4 Z^2 - \gamma_2 Z + \cosh \left( \sqrt{Z} \right)
\] (7)

\[
G^-(Z, a) = -\gamma_1 - \gamma_3 Z + \frac{\sinh \left( \sqrt{Z} \right)}{\sqrt{Z}}
\] (8)

where \( Z = z^2, z = \omega h = \omega_h \) and \( \omega = \text{frequency} \).

Step IV of the flow chart requires that the set 3 be considered. In this work, since \( M = 5 \) there are only three possibilities as listed below:

- \( K = 4, P = -1 \),
- \( K = 2, P = 0 \),
- \( K = 0, P = 1 \)

The coefficients in each of the cases above are given respectively as:

**S1 : \((K=4,P=-1)\)** The coefficients is given by (6).

**S2 : \((K=2,P=0)\)**

\[
\begin{align*}
\delta_0 &= 1, \\
\gamma_1 &= 1, \\
\gamma_2 &= \frac{1}{2}, \\
\gamma_3 &= \frac{\sinh(z)-z}{2z^4} = \frac{1}{6} + \frac{z^2}{120} + \frac{z^4}{5040} + \frac{z^6}{362880} + \frac{z^8}{39916800} + O(z^9), \\
\gamma_4 &= -\frac{1}{24} + \frac{z^2}{5040} + \frac{z^4}{40320} + \frac{z^6}{362880} + \frac{z^8}{479001600} + O(z^9)
\end{align*}
\] (9)

**S3 : \((K=0,P=1)\)**

\[
\begin{align*}
\delta_0 &= 1, \\
\gamma_1 &= \frac{3 \sinh(z) - \cosh(z)}{2z^4} = 1 - \frac{z^2}{120} - \frac{z^4}{5040} - \frac{z^6}{30240} + \frac{z^8}{1209600} + O(z^9), \\
\gamma_2 &= -\frac{2z^2}{2z^4} = \frac{1}{2} - \frac{z^2}{720} - \frac{z^4}{20160} - \frac{z^6}{1209600} + O(z^9), \\
\gamma_3 &= -\frac{z \sinh(z) - z \cosh(z)}{2z^4} = \frac{1}{6} + \frac{z^2}{60} + \frac{z^4}{1680} + \frac{z^6}{5040} + \frac{z^8}{181440} - \frac{z^{10}}{6652800} + O(z^9), \\
\gamma_4 &= \frac{z \sinh(z) - 2z \cosh(z) + 2}{2z^4} = \frac{1}{24} + \frac{z^2}{360} + \frac{z^4}{13440} + \frac{z^6}{5040} + \frac{z^8}{181440} + \frac{z^{10}}{6652800} + O(z^9)
\end{align*}
\] (10)

It can be seen that the exponentially–fitted variants reduce to (4) as \( z \) approaches zero.

**4. Error Analysis :: Local Truncation Error (lte)**

Following [9], the leading term of the local truncation error (lte) for an exponentially fitted method with respect to 3 takes the form (see [9])

\[
lte^{EF}(t) = (-1)^{P+1}h^M \frac{L^*_{K+1}(a(Z))}{(K+1)!Z^{P+1}} D^{K+1}(D^2 - \omega^2)^{P+1}u(t).
\] (11)

The \( lte \) for the methods presented in this work are respectively given as:
5. Existence and Uniqueness of Solution

The existence of a unique solution of (1) is guaranteed by the following theorem

**Theorem 5.1** Let \( f(t, u) \) be defined and continuous for all points \((t, u)\) in the region \( D \) defined by \( a \leq t \leq b, -\infty < u < \infty \), and \((t, u^*)\) are both in \( D \),

\[
| f(t, u) - f(t, u^*) | \leq L |u - u^*|
\]

then if \( \eta_0 \) is any given number, there exists a unique solution \( u(t) \) to the initial value problem (1), where \( u(t) \) is continuous and differentiable for all \((t, u) \in D\). Lambert [11].

6. Absolute Stability Analysis

**Definition 6.1** The region of absolute stability is a region of the \( v - \theta \) plane, throughout which \(|R(v, \theta)| < 1\). Any closed curve defined by \(|R(v, \theta)| = 1\) is a stability boundary. Also, any interval \((\alpha, \beta)\) of the real line is said to be the interval of stability if the method is stable for all \( v \in (\alpha, \beta) \).

The absolute stability of the constructed method is established using the standard linear test problem \( u' = ku \). Applying the constructed method (5) to this test problem gives:

\[
u_{j+1} = \left( \delta_0 + \gamma_1 hk + \gamma_2 h^2 k^2 + \gamma_3 h^3 k^3 + \gamma_4 h^4 k^4 \right) u_j\]

Setting \( v = hk \) in the above expression, the stability function of the constructed method is obtained as

\[
R_4(v; \omega_h) = \frac{u_{j+1}}{u_j} = \delta_0 + \gamma_1 v + \gamma_2 v^2 + \gamma_3 v^3 + \gamma_4 v^4
\]

For each of the constructed methods, the region of stability is presented in Figure(6) and Figure(6).
7. Numerical Results

In this section, the constructed exponentially–fitted methods are implemented on two test examples. The results obtained are compared with those of the conventional fourth–order Taylor’s method and fourth–order four–stage Runge–Kutta method. From Figure 3 and Figure 4, we see that our newly constructed methods perform better as expected compared to the method of Huta [8] for reasonable values of the steplength (h). Also considering the various variants of our constructed method, it is seen that the variant with \((K,P) = (4,0)\) performs best amongst the others, this is due to the nature of the solutions of the problems.

7.1. Test Problem 1

The IVP \( u' - u = t, \quad u(0) = 1 \) with the exact solution \( u(t) = 2e^t - t - 1 \), where \((\omega = 1)\) is considered here. The test problem is solved with different values of steplength \(h\) and the maximum absolute value for each steplength are presented in Figure( 3). As seen in Figure(3), the exponentially fitted variant \( S2:(2,0) \) solves the problem exactly up to machine accuracy (since the exact solution is a linear combination of elements of the fitting space of the variant \( S2:(2,0) \)). The variant behaved like a fourth order method. The classical fourth-order Taylor and Runge–Kutta methods are less accurate compared with the exponentially fitted variants.
S2:(2,0) and S3:(0,1).

Figure 3. Plot of Maximum absolute errors for Problem 1, $h = 2^{-k}, k = 2(1)10$

7.2. Problem 2
The second test problem considered is the IVP: $u' = \alpha u + e^{\alpha t}, \quad u(-1) = -e^{-\alpha}$ with the analytical solution $u(t) = t e^{\alpha t}$. Using $\alpha = \omega = 1$, the problem is solved with varied steplength $h$ and the maximum absolute error for each of the steplength are presented in Figure(4). As seen from Figure(4) the classical fourth-order Taylor and Runge–Kutta methods are less accurate, the exponentially fitted variant S2:(2,0) behaved like a fourth order method. Since the solution is in the fitting space of S3:(0,1), the variant S3:(0,1) solves the problem exactly up to machine’s accuracy (since the exact solution is a linear combination of elements of the fitting space of the variant S3:(0,1)).

Figure 4. Plot of Maximum absolute errors for Problem 2, $h = 2^{-k}, k = 2(1)10$
8. Conclusion
Exponentially-fitted variants of the conventional fourth-order Taylor method for first–order ordinary differential equations was constructed. The constructed methods are self–starting and of algebraic order four. The stability property of the proposed methods are also analysed. Results obtained from the test problems showed that the exponentially–fitted methods gave more accurate results compare with their conventional counterparts.

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