PERTURBATIONS OF CUR DECOMPOSITIONS

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ABSTRACT. The CUR decomposition is a factorization of a low-rank matrix obtained by selecting certain column and row submatrices of it. We perform a thorough investigation of what happens to such decompositions in the presence of noise. Since CUR decompositions are non-uniquely formed, we investigate several variants and give perturbation estimates for each in terms of the magnitude of the noise matrix in a broad class of norms which includes all Schatten $p$-norms. The estimates given here are qualitative and illustrate how the choice of columns and rows affects the quality of the approximation, and additionally we obtain new state-of-the-art bounds for some variants of CUR approximations.

1. Introduction

Low-rank matrix approximation has become a mainstay of applied mathematics in recent years, finding applications in signal processing [3], data compression [10], matrix completion [4], and analysis of large-scale data [34], to name but a few. Indeed, it has been observed for some time that much of the data we collect is approximately low rank (see [35] for a prolonged discussion) and thus this structure has been much exploited. One method for doing so is the CUR decomposition, which while known since at least the 1950s, has recently received much more attention following the works of Goreinov et al. [15, 14, 16], and Drineas et al. [7, 9, 20], among others [5, 28, 36] (see [1, 18] for a more detailed history of its use).

Classical low-rank matrix approximation methods arose from the Singular Value Decomposition (SVD), while more recent methods typically solve penalized optimization problems [19] or use randomized methods in some fashion [7, 9, 17] due to the lack of robustness of the SVD to noise in many applications [1], but also due to lack of interpretability of results [20].

1.1. Contributions. The main contribution of this work is to provide a thorough perturbation analysis of many different CUR approximations. The classical CUR decomposition of a low-rank matrix is to put $A = CU^\dagger R$, where $C$ and $R$ are column and row submatrices of $A$, respectively, i.e., $C = A(:, J)$ and $R = A(I, :)$. Other options, as discussed later are $A \approx CC^\dagger AR^\dagger R$, where $CC^\dagger$ and $R^\dagger R$ are orthogonal projections onto the span of the columns of $C$ and rows of $R$, respectively. We analyze what happens when we observe
The Moore–Penrose pseudoinverse of $A$ admits an easy expression given the SVD:

$$\phi(\sigma_1(A), \ldots, \sigma_k(A))$$

for some symmetric function $\phi$. The canonical examples of $\phi$ are $\phi(x, y) = \min(x, y)$ and $\phi(x, y) = \max(x, y)$.

1.2. Notations. We will use $\mathbb{K}$ to be either $\mathbb{R}$ or $\mathbb{C}$, and $[n]$ to denote $\{1, \ldots, n\}$. As column-row factorizations choose submatrices of a given matrix, if $A \in \mathbb{K}^{m \times n}$ and $I \subset [m]$, $J \subset [n]$, we let $A(I, J)$ denote the $|I| \times |J|$ submatrix of $A$ with entries $\{a_{i,j}\}_{(i,j) \in I \times J}$, and use $A(\cdot, J)$ to be the case $J = [n]$ and $A(I, \cdot)$ the case $I = [m]$.

We denote by $A = W\Sigma V^*$ (or $W_A\Sigma_AV_A^*$ if the matrix needs to be specified) the Singular Value Decomposition (SVD) of $A$, with the use of $W$ rather than the typical $U$ on account of the latter being used for the middle matrix in the CUR decomposition. The truncated SVD of order $r$ of a matrix $A$ will be denoted by $A_r = W_r\Sigma_r V_r^*$, where the columns of $W_r$ are the first $r$ left singular vectors, $\Sigma_r$ is a $r \times r$ matrix containing the largest $r$ singular values, and the columns of $V_r$ are the first $r$ right singular vectors.

Singular values are assumed to be positioned in descending order, and we label them $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_k \geq 0$, where $k = \text{rank}(A)$. If $r = k$, then $A = A_k$ and the truncated SVD $A = W_k\Sigma_k V_k^*$ is also called the compact SVD of $A$. To specify the underlying matrix, we may write $\sigma_i(A)$ for the $i$–th singular value of $A$. We will also make use of thresholding singular values of a matrix, and will denote by $[A]_\tau$ the matrix $W[\Sigma]_\tau V^*$, where $[\Sigma]_\tau(i, i) = \sigma_i(A)$ if $\sigma_i(A) \geq \tau$, and is 0 otherwise; thus the case $\tau = 0$ corresponds to the full SVD of $A$.

The Moore–Penrose pseudoinverse of $A \in \mathbb{K}^{m \times n}$ is denoted by $A^\dagger \in \mathbb{K}^{n \times m}$. Recall that this pseudoinverse is unique and satisfies the following properties: (i) $AA^\dagger A = A$, (ii) $A^\dagger AA^\dagger = A^\dagger$, and (iii) $AA^\dagger$ and $A^\dagger A$ are Hermitian. Additionally, the Moore–Penrose pseudoinverse admits an easy expression given the SVD: $A^\dagger = V_A\Sigma_A^\dagger W_A^*$, where $\Sigma_A^\dagger$ is the $n \times m$ matrix with diagonal entries $\frac{1}{\sigma_i(A)}$, $i = 1, \ldots, \text{rank}(A)$.

In our analysis we consider a general family of matrix norms as in Stewart [29]. The spectral norm is denoted by $\|A\|_2$, and is the operator norm of $A$ mapping $\mathbb{R}^n$ to $\mathbb{R}^m$ in the Euclidean norm. We consider families of submultiplicative, unitarily invariant norms $\| \cdot \| : \bigcup_{m,n=1}^\infty \mathbb{K}^{m \times n} \to \mathbb{R}$ which are normalized ($\|x\| = \|x\|_2$ for any vector $x$ considered as a matrix) and uniformly generated ($\|A\|$ can be written as $\phi(\sigma_1(A), \ldots, \sigma_k(A))$ for some symmetric function $\phi$). The canonical examples of
such families of norms are the Schatten $p$–norms ($1 \leq p \leq \infty$) given by $\|A\|_{S_p} := \|(\sigma_1(A), \ldots, \sigma_k(A))\|_{\ell_p}$. Unfortunately, while $\|\cdot\|_2$ is a thoroughly reasonable notation for the spectral norm, it is actually the Schatten $\infty$–norm. The Frobenius norm is the Schatten 2–norm but is denoted $\|\cdot\|_F$, and the Nuclear norm is the Schatten 1–norm, but is typically denoted $\|\cdot\|_*$. unless we need to specify a specific choice or norm, we will simply use the symbol $\|\cdot\|$ to denote an arbitrary submultiplicative, unitarily invariant, normalized, uniformly generated norm. Note that $\|\cdot\|_2 \leq \|\cdot\|$ for any such norm, and also that $\|AB\| \leq \|A\|_2 \|B\|$

Finally, we will use $\mathcal{N}(A)$ and $\mathcal{R}(A)$ to denote the nullspace and range of $A$, respectively.

1.3. Layout. The rest of the paper consists of a discussion of CUR decompositions in Section 2, our full perturbation analysis as well as comparison with other facets of the literature in Section 3, a discussion of rank-enforcement in CUR approximations in Section 4. We end with some brief numerical experiments in Section 5 and comments in Section 6. Supplementary proofs are given in Appendices A–C, and a table summarizing our error bounds is provided in Appendix D.

2. CUR Decompositions and Approximations

CUR approximations are low-rank approximations formed by selecting certain column and row submatrices of a given matrix, and then putting them together in some fashion. If $C$ and $R$ are such submatrices of $A$, then a CUR approximation of $A$ is a product of the form $A \approx CXR$ ($U$ will be used to denote a specific choice of $X$ shortly), where $C = A(:,J)$ for some $J \subset [n]$ and $R = A(I,:)$ for some $I \subset [m]$. For general $A$, there is a closed form for the best choice of $X$ for Frobenius norm error in the following sense.

**Proposition 2.1** ([31]). Let $A \in \mathbb{K}^{m \times n}$ and $C$ and $R$ be column and row submatrices of $A$, respectively. Then the following holds:

$$\arg\min_X \|A - CXR\|_F = C^tAR^t.$$  

The approximation $A \approx CC^tAR^tR$ corresponds to projecting $A$ onto the span of the given columns and rows, which is a natural candidate for a good approximation (though interestingly Proposition 2.1 does not hold for other norms). The quality of a CUR approximation for matrices of full rank has been considered in many works in the theoretical Computer Science literature, e.g., [2, 7, 8, 9, 20, 28, 38]. Most of these works focus on randomly sampling columns and rows to form the approximation; however, these works consider many different choices for the middle matrix $X$ in the CUR approximation. Nonetheless, there are deterministic methods of selecting columns given in [28, 36], the latter of which first uses a fast QR factorization of $A$ and subsequently implicitly forms the CUR approximation.

In the event that $A$ is actually low rank, a characterization of exact CUR decompositions was given by the authors in [18], which we restate here for the reader’s convenience.
Theorem 2.2 ([18, Theorem 5.5]). Let \( A \in \mathbb{K}^{m \times n} \), \( I \subset [m] \), and \( J \subset [n] \). Let \( C = A(:, J) \), \( R = A(I,:) \), and \( U = A(I, J) \). Then the following are equivalent:

(i) \( \text{rank}(U) = \text{rank}(A) \)
(ii) \( A = CU^\dagger R \)
(iii) \( A = CC^\dagger AR^\dagger R \)
(iv) \( A^\dagger = R^\dagger UC^\dagger \)
(v) \( \text{rank}(C) = \text{rank}(R) = \text{rank}(A) \).

Moreover, if any of the equivalent conditions above hold, then \( U^\dagger = C^\dagger AR^\dagger \).

Note that this theorem suggests at least two natural CUR approximations to a general matrix \( A \), namely \( A \approx CC^\dagger AR^\dagger R \), and \( A \approx CU^\dagger R \). We will discuss both variants and several rank truncations in the sequel.

3. Perturbation Analysis for CUR Approximations

We now turn to a perturbation analysis suggested by the CUR approximations described above. Our primary task will be to consider matrices of the form

\[
\tilde{A} = A + E,
\]

where \( A \) has low rank \( k < \min\{m, n\} \), and \( E \) is a (generally) full-rank noise matrix. We ask the question: if we choose column and row submatrices of \( \tilde{A} \), how do CUR approximations of \( \tilde{A} \) of the forms suggested by Theorem 2.2 relate to CUR decompositions of \( A \)?

To set some notation, we consider \( \tilde{C} = \tilde{A}(::, J) \), \( \tilde{R} = \tilde{A}(I,:) \), and \( \tilde{U} = \tilde{A}(I, J) \) for some index sets \( I \) and \( J \), and we write

\[
(1) \quad \tilde{C} = C + E(:, J), \quad \tilde{R} = R + E(I,:), \quad \tilde{U} = U + E(I, J),
\]

where \( C = A(:, J) \), \( R := A(I,:) \) and \( U := A(I, J) \). Thus if we choose columns and rows, \( \tilde{C} \) and \( \tilde{R} \) of \( \tilde{A} \), we seek to determine how approximation of \( \tilde{A} \) by \( \tilde{C} \) and \( \tilde{R} \) compares to the underlying approximation of the low rank matrix \( A \) by its columns and rows, \( C \) and \( R \). Theorem 2.2 suggests at least two distinct CUR approximations, namely \( \tilde{A} \approx \tilde{C}U^\dagger \tilde{R} \) and \( \tilde{A} \approx \tilde{C}C^\dagger \tilde{A}R^\dagger \tilde{R} \).

For experimentation in the sequel we will consider \( E \) to be a random matrix drawn from a certain distribution, but here we do not make any assumption on its entries. We are principally interested in the case that \( E \) is “small” in a suitable sense, and so the observed matrix \( \tilde{A} \) is really a small perturbation of the low rank matrix \( A \). To this end, most of our analysis will contain upper bounds on a CUR approximation of \( \tilde{A} \) in terms of a norm of the noise \( E \).

Note that we are interested in recovering the low-rank matrix \( A \), but the approximations suggested above are not necessarily low rank. Indeed, both approximations mentioned will typically have rank \( \min\{|I|, |J|\} \), which could be much larger than \( k \) in general. Therefore, we also consider various ways of enforcing the rank in the case that it is known or well-estimated. Unfortunately, there is no canonical way to do this as we will demonstrate in Section 4. Our perturbation estimates will analyze the following approximation errors:
• $\|A - \tilde{C}\tilde{C}^\dagger \tilde{A}\tilde{R}^\dagger \tilde{R}\|
• $\|A - \tilde{C}\tilde{U}^\dagger \tilde{R}\|
• $\|A - \tilde{C}[\tilde{U}]^\dagger \tilde{R}\|
• $\|A - \tilde{C}\tilde{U}_k^\dagger \tilde{R}\|
• $\|A - \tilde{C}_k\tilde{C}_k^\dagger \tilde{A}_k\tilde{R}_k^\dagger \tilde{R}_k\|$. 

In our discussion in Section 4, we will also discuss the approximation $A \approx (\tilde{C}\tilde{U}^\dagger \tilde{R})_k$.

For ease of notation, we will use the conventions that $E_I := E(I,:)$, $E_J := E(:,J)$, and $E_{I,J} := E(I,J)$; since $I$ and $J$ are always reserved for subsets of the rows and columns, respectively, we trust this will not cause confusion.

3.1. A Preliminary Note. Before stating our main estimations, we start with the following proposition which will be useful in estimating some of the terms that arise in the subsequent analysis.

**Proposition 3.1.** Suppose that $A, C, U, R$ are as in Theorem 2.2 such that $A = CU^\dagger R$, and suppose that $\text{rank}(A) = k$. Let $A = W_k\Sigma_k V_k^*$ be the compact SVD of $A$. Then for any unitarily invariant norm $\| \cdot \|$ on $\mathbb{K}^{m \times n}$, we have

$$\|CU^\dagger\| = \|W_{k,I}^\dagger\|, \text{ and } \|U^\dagger R\| = \|V_{k,J}^\dagger\|,$$

where $W_{k,I} := W_k(I,:)$ and $V_{k,J} := V_k(J,:)$.

**Proof.** See Appendix C. \qed

Unfortunately, it is often difficult to say much about the norms of pseudoinverses of submatrices of the compact SVD of a matrix; however, we will give some indications later of some universal bounds that can be used in certain cases.

3.2. Assumptions. To make the statement of results more simple, we will always make the following assumptions throughout the rest of this section. $\tilde{A} = A + E$ will be in $\mathbb{K}^{m \times n}$ with $\text{rank}(A) = k$, and $C, U, R, \tilde{C}, \tilde{U}, \tilde{R}, E_I, E_J, E_{I,J}$ will be as in (1) with $I \subset [m]$ and $J \subset [n]$ being the row and column index sets, respectively. We will always assume that $\text{rank}(C) = \text{rank}(U) = \text{rank}(R) = k$, and that $\| \cdot \|$ is a normalized, uniformly generated, unitarily invariant, submultiplicative norm. Given this assumption on the ranks, Proposition 3.1 is valid and will be utilized frequently.

**Remark 3.2.** For simplicity of reading, we state all bounds in the sequel for arbitrary norms satisfying the above assumptions; in particular, we use the pessimistic inequality $\|AB\| \leq \|A\|\|B\|$. But we note that at any stage, we can use the fact that $\|AB\| \leq \|A\|_2\|B\|$, which gives a better bound. In some instances, we will highlight how using the latter affects the right-hand sides of the given inequalities.

3.3. Perturbation Estimates for CUR Approximations With No Rank Enforcement. To begin, let us consider the CUR approximation suggested by the two exact decompositions of Theorem 2.2.
3.3.1. **Projection Based Approximation:** \( A \approx \tilde{C} \tilde{C}^\dagger \tilde{A} \tilde{R}^\dagger \tilde{R} \). We begin our perturbation analysis by considering the approximation suggested by Theorem 2.2(iii). Our main result is the following.

**Theorem 3.3.** The following holds:

\[
\|A - \tilde{C} \tilde{C}^\dagger \tilde{A} \tilde{R}^\dagger \tilde{R}\| \leq \|E_I\|\|AR^\dagger\| + \|E_J\|\|C^\dagger A\| + 3\|E\|.
\]

Hence,

\[
\|A - \tilde{C} \tilde{C}^\dagger \tilde{A} \tilde{R}^\dagger \tilde{R}\| \leq \|E\|((\|W_{k,I}\| + \|V_{k,I}\| + 3).
\]

Before proving this theorem, we need the following lemma.

**Lemma 3.4.** The following holds:

\[
\|A - \tilde{C} \tilde{C}^\dagger \tilde{A}\| \leq \|E_J\|\|C^\dagger A\| + \|E\|,
\]

\[
\|A - \tilde{A} \tilde{R}^\dagger \tilde{R}\| \leq \|E_I\|\|AR^\dagger\| + \|E\|.
\]

**Proof.** First, notice that

\[
\|(I - \tilde{C} \tilde{C}^\dagger)C\| = \|(I - \tilde{C} \tilde{C}^\dagger)\tilde{C} - (I - \tilde{C} \tilde{C}^\dagger)E_J\|
\leq \|(I - \tilde{C} \tilde{C}^\dagger)\tilde{C}\| + \|(I - \tilde{C} \tilde{C}^\dagger)E_J\|
\leq \|E_J\|.
\]

The final inequality arises because the first norm term is 0 by identity of the Moore–Penrose pseudoinverse and \(\|I - \tilde{C} \tilde{C}^\dagger\|_2 \leq 1\) as this is an orthogonal projection operator.

Now since \(\text{rank}(C) = \text{rank}(A) = k\), we have \(A = CC^\dagger A\); therefore,

\[
\|A - \tilde{C} \tilde{C}^\dagger \tilde{A}\| \leq \|(I - \tilde{C} \tilde{C}^\dagger)A\| + \|E\|
= \|(I - \tilde{C} \tilde{C}^\dagger)CC^\dagger A\| + \|E\|
\leq \|E_J\|\|C^\dagger A\| + \|E\|.
\]

The second inequality follows by mimicking the above argument. \(\Box\)

**Proof of Theorem 3.3.** First note that

\[
\|A - \tilde{C} \tilde{C}^\dagger \tilde{A} \tilde{R}^\dagger \tilde{R}\| \leq \|A - \tilde{C} \tilde{C}^\dagger \tilde{A}\| + \|\tilde{A} - \tilde{A} \tilde{R}^\dagger \tilde{R}\|
\]

by the triangle inequality and the fact that \(\|C^\dagger\|_2 \leq 1\). The proof is completed by first noting that the second term above satisfies \(\|A(I - \tilde{R}^\dagger \tilde{R})\| \leq \|E\| + \|A(I - \tilde{R}^\dagger \tilde{R})\|
\)

since \(I - \tilde{R}^\dagger \tilde{R}\) is a projection, and then applying the inequalities of Lemma 3.4. The second stated inequality follows directly by Proposition 3.1 and the fact that the norms of submatrices of \(E\) are at most \(\|E\|\). \(\Box\)

3.3.2. **Non-projection Based Approximation:** \( A \approx \tilde{C}[\tilde{U}]_\tau \tilde{R} \). Now we turn to considering the approximation suggested by Theorem 2.2(ii). We formulate our approximation in a slightly more general form by thresholding the singular values of \(\tilde{U}\) by a fixed parameter \(\tau \geq 0\). Of course provided \(0 \leq \tau \leq \sigma_r(\tilde{U})\) where \(\text{rank}(\tilde{U}) = r\), we have \(\tilde{C}[\tilde{U}]_\tau \tilde{R} = \tilde{C} \tilde{U}^\dagger \tilde{R}\), and hence this framework encompasses the case that no thresholding is actually done (recall the definition of \(\tilde{U}_\tau\) from Section 1.2).
This approximation scheme was studied by Osinsky et al. [25] and previously by Goreinov et al. [15, 14, 16], and we recover similar perturbation results to those in the former, but by a different proof method, which we provide in full. The reason for including our analysis is that it gives some more qualitative estimates, and additionally we get slightly better error bounds since they are in terms of submatrices of the noise $E$. Moreover, our bounds hold for arbitrary norms satisfying the conditions above (e.g., for all Schatten $p$–norms), which is a strengthening of the spectral and Frobenius norm guarantees of the aforementioned works. Additionally, our estimation techniques are amenable to performing a novel analysis of different ways of enforcing the rank in the CUR approximation, which is done in the sequel.

**Proposition 3.5.** Let $\tau \geq 0$ be fixed; then the following holds:

$$
\|A - \tilde{C}[\tilde{U}]^\dagger \tilde{R}\| \leq \|C[\tilde{U}]^\dagger \| E_I\| + \|[[\tilde{U}]^\dagger \tilde{R}]\| E_J\| + \|W_{k,j}^\dagger\| \|V_{k,j}^\dagger\| \left[2\|E_{I,j}\| + \|[\tilde{U}]_\tau - U\| + \|[[\tilde{U}]^\dagger]\| E_{I,j}\|^2 \right].
$$

**Proof.** Begin with the fact that

$$
\|A - \tilde{C}[\tilde{U}]^\dagger \tilde{R}\| \leq \|A - CU^\dagger R\| + \|CU^\dagger R - \tilde{C}[\tilde{U}]^\dagger \tilde{R}\|,
$$

and notice that the first term is 0 by the assumption on $U$. Then we have

$$
\|CU^\dagger R - \tilde{C}[\tilde{U}]^\dagger \tilde{R}\| \leq \|CU^\dagger R - C[\tilde{U}]^\dagger \tilde{R}\| + \|C[\tilde{U}]^\dagger \tilde{R} - \tilde{C}[\tilde{U}]^\dagger \tilde{R}\| \\
\leq \|CU^\dagger R - C[\tilde{U}]^\dagger \tilde{R}\| + \|C[\tilde{U}]^\dagger \tilde{R}\|_2 \|R - \tilde{R}\| + \|C - \tilde{C}\| [[\tilde{U}]^\dagger \tilde{R}]_2 \\
= \|CU^\dagger R - C[\tilde{U}]^\dagger \tilde{R}\| + \|C[\tilde{U}]^\dagger \tilde{R}\|_2 \|E_I\| + \|[[\tilde{U}]^\dagger \tilde{R}]_2\| E_J\|.
$$

To estimate the first term above, note that Lemma A.1 implies that $C = CC^\dagger C = CU^\dagger U$, and likewise $R = RR^\dagger R = UU^\dagger R$. Consequently, since $E_{I,j} = \tilde{U} - U$, the following holds:

$$
\|CU^\dagger R - C[\tilde{U}]^\dagger \tilde{R}\| = \|CU^\dagger UU^\dagger R - CU^\dagger(\tilde{U} - E_{I,j})[\tilde{U}]_\tau^\dagger (\tilde{U} - E_{I,j})U^\dagger R\| \\
\leq \|CU^\dagger (U - [\tilde{U}]_\tau)U^\dagger R\| + \|CU^\dagger [\tilde{U}]_\tau^\dagger E_{I,j}U^\dagger R\| \\
+ \|CU^\dagger E_{I,j}[\tilde{U}]_\tau^\dagger UU^\dagger R\| + \|CU^\dagger E_{I,j}[\tilde{U}]_\tau^\dagger E_{I,j}U^\dagger R\|.
$$

(2)

The first term in (2) is evidently at most $\|CU^\dagger\|_2\|UU^\dagger R\|_2\|[\tilde{U}]_\tau - U\|$, whereas the second is majorized by the same quantity on account of the fact that $\tilde{U}[\tilde{U}]_\tau^\dagger$ is a projection. Similarly, as $[\tilde{U}]_\tau^\dagger \tilde{U}$ is a projection, the third term in (2) is at most $\|CU^\dagger\|_2\|UU^\dagger R\|_2\|E_{I,j}\|$, while the final term is at most $\|CU^\dagger\|_2\|UU^\dagger R\|_2\|[\tilde{U}]_\tau^\dagger\|_2\|E_{I,j}\|^2$. Putting these observations together, and combining (2) with Proposition 3.1 yields the following:

$$
\|CU^\dagger R - C[\tilde{U}]^\dagger \tilde{R}\| \leq \|W_{k,j}^\dagger\|_2\|V_{k,j}^\dagger\|_2\|2\|E_{I,j}\| + \|[\tilde{U}]_\tau - U\| + \|[[\tilde{U}]^\dagger]\|_2\|E_{I,j}\|^2)\).
$$

Combining the estimates of (2) and (3) yield the desired conclusion. □

**Lemma 3.6.** If $\tau \geq 0$, then the following hold:
(i) $\|C[\tilde{U}]^+_r\| \leq \|[\tilde{U}]^+_r\|E_{I,J}\|W_{k,I}^+\| + \|W_{k,I}^+\|$

(ii) $\|[\tilde{U}]^+_r\tilde{R}\| \leq \|[\tilde{U}]^+_r\|E_{I,J}\|V_{k,J}^+\| + \|V_{k,J}^+\| + \|V_{k,J}^+\|$

Proof. To see (i), notice that $C = CU^+U$ by Theorem 2.2, whence applying Proposition 3.1 yields

$$\|C[\tilde{U}]^+_r\| = \|CU^+U[\tilde{U}]^+_r\|$$

$$\leq \|CU^+\|\|[U][\tilde{U}]^+_r\|_2$$

$$= \|W_{k,I}^+\|(\tilde{U} - E_{I,J})[\tilde{U}]^+_r\|_2$$

$$\leq \|W_{k,I}^+\|(\|[\tilde{U}]^+_r\|_2 + \|E_{I,J}\|\|[\tilde{U}]^+_r\|_2)$$

$$\leq \|W_{k,I}^+\|(1 + \|E_{I,J}\|\|[\tilde{U}]^+_r\|)).$$

Similarly, we have

$$\|[\tilde{U}]^+_r\tilde{R}\| \leq \|V_{k,J}^+\|(1 + \|E_{I,J}\|\|[\tilde{U}]^+_r\|).$$

Thus to prove (ii), note that

$$\|[\tilde{U}]^+_r\tilde{R}\| \leq \|[\tilde{U}]^+_r\| + \|[\tilde{U}]^+_rE_{I}\|$$

$$\leq \|V_{k,J}^+\|(1 + \|E_{I,J}\|\|[\tilde{U}]^+_r\|) + \|[\tilde{U}]^+_r\|\|E_{I}\|.$$

Theorem 3.7. Given $\tau \geq 0$, the following holds:

$$\|A - \tilde{C}[\tilde{U}]^+_r\tilde{R}\| \leq \|W_{k,I}^+\|\|E_{I}\| + \|V_{k,J}^+\|\|E_{J}\| + \|W_{k,I}^+\|\|V_{k,J}^+\|(2\|E_{I,J}\| + \|[\tilde{U}]_r - U\|)$$

$$+ \|[\tilde{U}]^+_r\|\left(\|W_{k,I}^+\|\|E_{I}\| + \|V_{k,J}^+\|\|E_{J}\| + \|W_{k,I}^+\|\|V_{k,J}^+\|\|E_{I,J}\|\right)\|E_{I,J}\| + \|E_{I}\|\|E_{I}\|.$$

Proof. Apply the conclusion of Lemma 3.6 to Proposition 3.5 and collect terms.

Corollary 3.8. Setting $\tau = 0$, we have

$$\|A - \tilde{C}[\tilde{U}]^+_r\tilde{R}\| \leq \left(\|W_{k,I}^+\| + \|V_{k,J}^+\| + 3\|W_{k,I}^+\|\|V_{k,J}^+\|\right)\|E\|$$

$$+ \|[\tilde{U}]^+_r\|\left(\|W_{k,I}^+\| + \|V_{k,J}^+\| + \|V_{k,J}^+\|\|V_{k,J}^+\| + 1\right)\|E\|^2.$$

Remark 3.9. If the tighter bound suggested in Remark 3.2 is used, then the conclusion of Theorem 3.7 becomes

$$\|A - \tilde{C}[\tilde{U}]^+_r\tilde{R}\| \leq \|W_{k,I}^+\|\|E_{I}\| + \|V_{k,J}^+\|\|E_{J}\| + \|W_{k,I}^+\|\|V_{k,J}^+\|(2\|E_{I,J}\| + \|[\tilde{U}]_r - U\|)$$

$$+ \|[\tilde{U}]^+_r\|\left(\|W_{k,I}^+\|\|E_{I}\| + \|V_{k,J}^+\|\|E_{J}\| + \|W_{k,I}^+\|\|V_{k,J}^+\|\|E_{I,J}\|\right)\|E_{I,J}\| + \|E_{I}\|\|E_{I}\|.$$

Remark 3.10. The bounds above may be simplified in a couple of ways. First, note that $\|[\tilde{U}]^+_r\|_2 \leq \tau^{-1}$. In some previous works, $\tau$ is chosen to offset the other norm terms to demonstrate the existence of nice upper bounds (e.g., taking $\tau = \|E\|$ or to be related to the inverse of the product $\|W_{k,I}^+\|\|V_{k,J}^+\|$). This is not practical; however,
we will mention in Section 3.6 how our results recover previous analyses in the literature in this direction. Additionally, if the norm on the left-hand side is the spectral norm, we may estimate \( \| [\tilde{U}]_\tau - U \|_2 \leq \tau + \| E_{I,J} \|_2 \) by adding and subtracting \( \tilde{U} \) and applying the triangle inequality.

Note that the approximation bound in Corollary 3.8 depends on \( \tilde{U} \) and hence must be considered preliminary, as this could be arbitrarily large. Without additional assumptions, not much more may be said, but in Section 3.5, we analyze how one may improve the estimates herein by choosing maximal volume submatrices, and point the reader to some existing algorithms for doing so.

3.4. Perturbation Estimates for Rank \( k \) CUR Approximations. If rank \( (A) = k \) is known in advance, we are interested in enforcing this rank in any CUR approximation of \( A \). In this section, we consider two variants of rank enforcement. Further discussion of the merits and drawbacks of both are found in Section 4.

3.4.1. Enforcing the rank on \( \tilde{U} \). If more than \( k \) columns or rows of \( \tilde{A} \) are chosen, then the rank of \( \tilde{U} \) is typically larger than \( k \). Therefore, \( \tilde{C} \tilde{U}^\dagger \tilde{R} \) is an approximation of \( A \) which has strictly larger rank. It is natural to consider then what happens if the target rank is enforced. There are many ways to enforce the rank, one of which has been utilized for some time is to do so on the matrix \( \tilde{U} \). If \( \tilde{A} \) is symmetric, positive semidefinite, then this rank-enforcement strategy is known to be deficient, but for generic matrices this is not so. For further discussion on these matters, consult Section 4. By modifying the proof of Proposition 3.5, we arrive at the following.

**Proposition 3.11.** Let \( \tilde{U}_k \) be the best rank-\( k \) approximation of \( \tilde{U} \). Then

\[
\| A - \tilde{C} \tilde{U}_k^\dagger \tilde{R} \| \leq \| W_{k,I}^\dagger \| \| E_I \| + \| V_{k,J}^\dagger \| \| E_J \| + 4 \| W_{k,I}^\dagger \| \| V_{k,J}^\dagger \| \| E_{I,J} \| \\
+ \| \tilde{U}_k^\dagger \| \left( \| W_{k,I}^\dagger \| \| E_I \| + \| V_{k,J}^\dagger \| \| E_J \| + \| W_{k,I}^\dagger \| \| V_{k,J}^\dagger \| \| E_{I,J} \| \right) \| E_{I,J} \| + \| E_I \| \| E_J \| .
\]

The presence of terms depending on \( \tilde{U} \) in the error bounds above are undesirable, so we now are tasked with estimating them. Before stating the final bound, we estimate some of the terms specifically in the following lemma.

**Lemma 3.12.** Let \( \mu \in [1,3] \) be the quantity given by Theorem B.2 (\( \mu \) depends on the norm \( \| \cdot \| \) chosen). Provided \( \sigma_k(U) > 2\mu \| E_{I,J} \| \), the following estimate holds:

\[
\| \tilde{U}_k^\dagger \| \leq \frac{\| U^\dagger \|}{1 - 2\mu \| U^\dagger \|_2 \| E_{I,J} \|}.
\]

**Proof.** Note that \( \tilde{U}_k = U + (\tilde{U}_k - U) \), and notice that \( \| U - \tilde{U}_k \| \leq \| U - \tilde{U} \| + \| \tilde{U} - \tilde{U}_k \| \), where the first term is equal to \( \| E_{I,J} \| \) by definition, and the second satisfies \( \| U - \tilde{U}_k \| \leq \| E_{I,J} \| \) by Mirsky’s Theorem. Hence \( \| U - \tilde{U}_k \| \leq 2 \| E_{I,J} \| \). Using this estimate, we see that if \( \sigma_k(U) > 2\mu \| E_{I,J} \| \geq \mu \| \tilde{U}_k - U \| \), then by Corollary B.3,

\[
\| \tilde{U}_k^\dagger \| \leq \frac{\| U^\dagger \|}{1 - \mu \| U^\dagger \|_2 \| \tilde{U}_k - U \|} \leq \frac{\| U^\dagger \|}{1 - 2\mu \| U^\dagger \|_2 \| E_{I,J} \|}.
\]
which is the desired conclusion. \hfill \qed

**Theorem 3.13.** Take the notations and assumptions of Lemma 3.12. Then,

\[
\|A - \tilde{C}\tilde{U}_k^\dagger\tilde{R}\| \leq \left(\|W_{k,I}^\dagger\|\|E_I\| + \|V_{k,J}^\dagger\|\|E_J\| + 4\|W_{k,I}^\dagger\||V_{k,J}^\dagger||E_{I,J}|\right) + \frac{\|U^\dagger\|}{1 - 2\mu\|U^\dagger\|_2\|E_{I,J}\|} \left(\|W_{k,I}^\dagger\|\|E_I\||E_{I,J}| + \|V_{k,J}^\dagger\|\|E_J\||E_{I,J}| + \|W_{k,I}^\dagger\|\|V_{k,J}^\dagger||E_{I,J}|^2 + \|E_I\||E_J|\right).
\]

**Proof.** Recalling that \(\|U - \tilde{U}_k\| \leq 2\|E_{I,J}\|\) as estimated in the proof of Lemma 3.12, the conclusion of the proof follows by combining this estimate with those of Proposition 3.11 and Lemma 3.12, and rearranging terms. \hfill \qed

**Remark 3.14.** Note that all terms in the second line in the bound of Theorem 3.13 are second order in the noise, whereas the first three terms are first order. In particular, if \(\sigma_k(U) > 4\mu\|E\|\), then

\[
\|A - \tilde{C}\tilde{U}_k^\dagger\tilde{R}\| \leq \left(1 + \frac{1}{2\mu}\right)(\|W_{k,I}^\dagger\| + \|V_{k,J}^\dagger\|) + \left(4 + \frac{1}{2\mu}\right)\|W_{k,I}^\dagger\||V_{k,J}^\dagger| + \frac{1}{2\mu}\|E\|.
\]

**Remark 3.15.** Since \(U^\dagger = W_{k,I}^\dagger\Sigma_{k,I}^\dagger V_{k,J}^\dagger\) and \(\|\Sigma_{k,I}^\dagger\| = \|A^\dagger\|\), we may replace the fractional term in Theorem 3.13 with

\[
\frac{\|W_{k,I}^\dagger\||V_{k,J}^\dagger||A^\dagger|}{1 - 2\mu\|W_{k,I}^\dagger\|_2\|V_{k,J}^\dagger\|_2\|A^\dagger\|_2\|E_{I,J}\|}
\]

thus giving a bound independent of the chosen \(U\). Indeed, this means that the error bounds in Theorem 3.13 are of the form

\[
\|A - \tilde{C}\tilde{U}_k^\dagger\tilde{R}\| \leq \|A - CU^\dagger R\| + O(\|E\|) + O(\|A^\dagger\|\|E\|^2).
\]

That is, the first order terms depend essentially only on the noise, whereas the second order terms have dependence on \(\|A^\dagger\|\). Do note that the assumptions in Theorem 3.13 do imply that \(\|E\|\|A^\dagger\| \leq C_1\) for some universal constant \(C_1\); on the other hand, it could be that this quantity is rather small, and so we leave the expression as is to denote the second order dependence on the noise matrix.

### 3.4.2. Induced from the CUR Approximation

Next let us consider what happens if we enforce the rank on both \(\tilde{C}\) and \(\tilde{R}\) and then form the projection-based approximation from these. This corresponds to finding the best \(k\)-dimensional subspace that approximates the span of the columns of \(\tilde{C}\) and projecting \(\tilde{A}\) onto this subspace. Ideally, this should well-approximate projecting \(\tilde{A}\) onto the span of the columns of \(C\) itself. To begin, we mention a straightforward lemma of Drineas and Ipsen.

**Lemma 3.16** ([6], Corollary 2.4). Let \(W_k \in \mathbb{K}^{m \times k}\) be the \(k\) dominant left singular vectors of \(B\) and let \(\tilde{W}_k \in \mathbb{K}^{m \times k}\) be the \(k\) dominant left singular vectors of \(B + E\). Then

\[
\|(I - W_kW_k^*)B\| \leq \|(I - \tilde{W}_k\tilde{W}_k^*)B\| \leq \|(I - W_kW_k^*)B\| + 2\|E\|.
\]
Lemma 3.17. The following hold:
\[
\|A - \tilde{C}_k \tilde{C}_k^\dagger \tilde{A}\| \leq 2\|E_J\|\|C^\dagger A\| + \|E\|,
\]
\[
\|A - \tilde{A}\tilde{R}_k^\dagger \tilde{R}\| \leq 2\|E_I\|\|AR^\dagger\| + \|E\|.
\]

Proof. Since rank \((C) = \text{rank} (A) = k\), we have \(A = CC^\dagger A\). If \(\tilde{C}_k = \tilde{W}_k \Sigma_k \tilde{V}_k\), then \(\tilde{C}_k \tilde{C}_k^\dagger = \tilde{W}_k \tilde{W}_k^\dagger\). A similar statement holds for \(C\). Consequently, by Lemma 3.16, we have
\[
\|C - \tilde{C}_k \tilde{C}_k^\dagger C\| \leq \|(I - C_k C_k^\dagger) C\| + 2\|C - \tilde{C}\| = 2\|E_J\|.
\]
Therefore,
\[
\|A - \tilde{C}_k \tilde{C}_k^\dagger \tilde{A}\| \leq \|A - \tilde{C}_k \tilde{C}_k^\dagger A\| + \|E\| = \|(I - \tilde{C}_k \tilde{C}_k^\dagger) CC^\dagger A\| + \|E\| \leq 2\|E_J\|\|C^\dagger A\| + \|E\|.
\]
The second stated inequality follows from the same argument \textit{mutatis mutandis}. \(\square\)

Theorem 3.18. We have
\[
\|A - \tilde{C}_k \tilde{C}_k^\dagger \tilde{A}\tilde{R}_k^\dagger \tilde{R}\| \leq 2\left(\|E_J\|\|W_{k,J}^\dagger\| + \|E_I\|\|V_{k,J}^\dagger\| + \frac{3}{2}\|E\|\right).
\]
Hence,
\[
\|A - \tilde{C}_k \tilde{C}_k^\dagger \tilde{A}\tilde{R}_k^\dagger \tilde{R}\| \leq 2\|E\|\left(\|W_{k,J}^\dagger\| + \|V_{k,J}^\dagger\| + \frac{3}{2}\right).
\]

Proof. Mimic the proof of Theorem 3.3 while applying Lemma 3.17. \(\square\)

3.5. \textbf{Refined Estimates: The Maximal Volume Case.} One drawback of the above estimates is that some of the right-hand sides maintain dependencies on the choice of the submatrix \(U\). If one assumes that maximal volume submatrices of the left and right singular values are chosen, then one can use estimates from [25] to give bounds on the corresponding spectral norms. Recall that the volume of a matrix \(B \in \mathbb{K}^{m \times n}\) is \(\prod_{i=1}^{\min\{m,n\}} \sigma_i(B)\). While finding the maximal volume submatrix of a given matrix is NP-hard, there are good approximation algorithms available, e.g. [13, 21, 24].

Proposition 3.19. Suppose that \(W_{k,I}\) and \(V_{k,J}\) are the submatrices of \(W_k\) and \(V_k\) such that \(W_{k,I}\) has maximal volume among all \(|I| \times k\) submatrices of \(W_k\) and \(V_{k,J}\) is of maximal volume among all \(|J| \times k\) submatrices of \(V_k\). Then
\[
\|W_{k,I}^\dagger\|_2 \leq \sqrt{1 + \frac{k(m - |I|)}{|I| - k + 1}}, \quad \|V_{k,J}^\dagger\|_2 \leq \sqrt{1 + \frac{k(n - |J|)}{|J| - k + 1}}.
\]
Moreover, if rank \((U) = \text{rank} (A)\), then
\[
\|U^\dagger\|_2 \leq \sqrt{1 + \frac{k(m - |I|)}{|I| - k + 1}} \sqrt{1 + \frac{k(n - |J|)}{|J| - k + 1}}\|A^\dagger\|_2.
\]
Note that (4) appears in [25], and the moreover statement follows by Proposition 2.1 and the assumption that \( \text{rank}(U) = \text{rank}(A) \). For ease of notation, since the upper bounds appearing in (4) are universal, we abbreviate the quantities there \( t(k, m, |I|) \), and \( t(k, n, |J|) \), respectively as in [25]. Regard also, that Frobenius bounds are also provided in [25], where the upper bound is \( \sqrt{k + \frac{k(m-|J|)}{|J|+1}} \).

To finish off our perturbation analysis, we will apply the conclusion of Proposition 3.19 to the bounds on rank-enforcement CUR approximations discussed in the previous section. We withhold the proofs as the following corollaries arise by simply applying (4) and (5) to previous theorems.

As a preliminary remark, we note that choosing maximal volume submatrices of the singular vectors of \( A \) automatically yields a valid CUR decomposition.

**Proposition 3.20.** Let \( A \in \mathbb{K}^{m \times n} \) have rank \( k \) and compact SVD \( A = W_k \Sigma_k V_k^* \). Suppose that \( I \subset [m] \) and \( J \subset [n] \) satisfy \( |I|, |J| \geq k \) and \( W_{k,I} \) and \( V_{k,J} \) are the maximal volume submatrices of \( W_k \) and \( V_k \), respectively. Then if \( C = A(:, J) \), \( U = A(I, J) \), and \( R = A(:, J) \), then \( A = CU^\dagger R \).

**Proof.** Notice that rank \( (W_k) = \text{rank} (V_k) = k \). There exist \( I_1, J_1 \) such that rank \( (W_{k,I_1}) = k \) and rank \( (V_{k,J_1}) = k \). Therefore, by the definition of the maximal volume submatrices, we must have rank \( (W_{k,I}) = \text{rank} (V_{k,J}) = k \); indeed, recall that the volume of \( W_{k,I} = \prod_{j=1}^{k} \sigma_j(W_{k,I}) \), and this product is 0 if \( W_{k,I} \) has rank less than \( k \) and hence cannot be of maximal volume since \( W_{k,I} \) has nonzero volume. It follows that rank \( (C) = \text{rank} (R) = k \), and hence \( A = CU^\dagger R \) by Theorem 2.2. \( \square \)

The following corollary arises from Theorems 3.13 and 3.18 and Remark 3.2; the condition on \( \sigma_k(A) \) comes from combining (5) with the condition on \( \sigma_k(U) \) in Lemma 3.12.

**Corollary 3.21.** Suppose that \( A \in \mathbb{K}^{m \times n} \) has rank \( k \) and compact SVD \( A = W_k \Sigma_k V_k^* \). Suppose also that \( W_{k,I} \) and \( V_{k,J} \) are maximal volume submatrices of \( W_k \) and \( V_k \), respectively. Then

\[
\| A - \tilde{C}_k \tilde{C}_k^\dagger \tilde{A} \tilde{R}_k^\dagger \tilde{R}_k \| \leq (2t(k, m, |I|) + 2t(k, n, |J|) + 3)\| E \|.
\]

If additionally, \( \sigma_k(A) \geq 4\mu t(k, m, |I|)t(k, n, |J|)\| E \| \), then

\[
\| A - \tilde{C}_k \tilde{U}_k^\dagger \tilde{R} \| \leq \left[ \frac{1}{2\mu} + \left( 1 + \frac{1}{2\mu} \right) \left( t(k, m, |I|) + t(k, n, |J|) \right) \right. \\
\left. + \left( 4 + \frac{1}{2\mu} \right) t(k, m, |I|)t(k, n, |J|) \right] \| E \|.
\]

**Corollary 3.22.** Suppose \( A \in \mathbb{K}^{n \times n} \) and \( |I| = |J| \). Suppose also that \( W_{k,I} \) and \( V_{k,J} \) are maximal volume submatrices as in Proposition 3.19. Abbreviate \( t := t(k, n, |I|) \). Then

\[
\| A - \tilde{C}_k \tilde{C}_k^\dagger \tilde{A} \tilde{R}_k^\dagger \tilde{R}_k \| \leq (4t + 3)\| E \|.
\]
If additionally, $\sigma_k(A) \geq 4\mu t^2 \|E\|$, then
\[
\|A - \tilde{C}\tilde{U}_k^\dagger\tilde{R}\| \leq \left( \frac{1}{2\mu} + \left( 2 + \frac{1}{\mu} \right) t + \left( 4 + \frac{1}{2\mu} \right) t^2 \right) \|E\|.
\]

3.6. Comparison with Previous Results. Osinsky and Zamarashkin [25] provides several estimates of CUR approximations in which they assume that maximal volume submatrices are chosen.

Here are some of the theorems therein stated for comparison. For simplicity of the statements, we focus on the case that $A$ is square, and exactly $k$ columns and rows are selected (i.e., $|I| = |J| = k$) and denote the factor $t$ as in Corollary 3.22. The first result we highlight is the following.

**Theorem 3.23 ([25, Theorem 2]).** There exist $I, J \subset [n]$ and $X$ such that
\[
\|A - \tilde{C}X\tilde{R}\|_2 \leq 4t\|E\|_2.
\]

This error bound is relatively good; however, the authors choose $X = [A(I, J)]_r^\dagger$ with $\tau = \frac{\|E\|}{t}$. This choice is impractical for real matrices as one does not have access to $A(I, J)$ or $\|E\|$ in practice. For comparison, Theorem 3.3 yields an upper bound of $(3 + 2t)\|E\|$. It is easily checked that when $t > 3/2$, Theorem 3.3 gives a better bound than Theorem 3.23, but nonetheless this approximation still suffers from needing to know the SVD of $A$.

The second result we highlight is the following.

**Theorem 3.24 ([25, Theorem 3]).** If $\|E\|_2 \leq \varepsilon$, and $I$ and $J$ yield the maximal volume submatrices of $W_k$ and $V_k$, respectively, then
\[
\|\tilde{A} - \tilde{C}[\tilde{U}]_2^\dagger\tilde{R}\|_2 \leq (2 + 4t + 5t^2)\varepsilon.
\]

Our estimate in Theorem 3.7 gives the same error bound as this on the right-hand side if we choose $\tau = \varepsilon$. However, our estimates obtained by directly enforcing the rank are novel and not directly considered in other works. Indeed, in certain cases, Remark 3.14 gives a much better bound than that above:

**Corollary 3.25.** If $\|E\| \leq \varepsilon$, $I$ and $J$ yield maximal volume submatrices of $W_k$ and $V_k$, respectively, and in addition $\sigma_k(U) > 4\mu\varepsilon$, then
\[
\|A - \tilde{C}\tilde{U}_k^\dagger\tilde{R}\| \leq \left( \frac{1}{2} + 3t + \frac{9}{2}t^2 \right) \varepsilon.
\]

Note that bounds of a different flavor have recently been provided by Mikhalev and Oseledets [21]. Additionally, independent, concurrent work which has an analysis of $\tilde{A} - \tilde{C}\tilde{U}_k^\dagger$ in a similar vein to that of Section 3.3.2 was done by Pan et al. [26], though their subsequent focus is more algorithmic than the present work.

4. How to enforce the rank?

Our previous analysis illustrated two natural ways to enforce the rank of CUR approximations; namely, $\tilde{C}\tilde{U}_k^\dagger\tilde{R}$ and $\tilde{C}_k\tilde{C}_k^\dagger\tilde{A}\tilde{R}_k^\dagger\tilde{R}_k$. The first has been utilized in the special case of the CUR approximation called the Nyström method [11], which is when
A is symmetric positive semi-definite (SPSD) and the same columns and rows are chosen (i.e., $A \approx CU^\dagger C^*$). It has recently been suggested by some authors that a better way to enforce the rank would be to consider $(CU^\dagger R)_k$, which means to make the CUR approximation suggested by Theorem 2.2(ii), and then take its best rank $k$ approximation [27, 33].

In particular, Pourkamali-Anaraki and Becker [27], Tropp et al. [33], and Wang et al. [37] have discussed that when approximating a SPSD matrix $K$ using the Nyström method, it is better to enforce the rank after forming the approximation rather than during the process. Specifically, Pourkamali-Anaraki and Becker [27] show that

$$\|K - (CU^\dagger C^*)_r\|_* \leq \|K - CU^\dagger_r C^*\|_*.$$  

(6)

Here $\| \cdot \|_*$ is the nuclear norm, which is the Schatten 1-norm.

However, it turns out that this can fail to be true in every Schatten $p$-norm for non-SPSD matrices. Indeed consider the modified version of Example 2 of [27], and let

$$A = \begin{bmatrix} -1 & 0 & 10 \\ 0 & 1 + \varepsilon & 0 \\ 10 & 0 & 100 \end{bmatrix},$$

and let $C$ be the first two columns of $A$, $R$ be the first two rows of $A$, and $W$ be their intersection (i.e. $U = A(1 : 2, 1 : 2)$). Clearly

$$U_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 + \varepsilon \end{bmatrix}, \quad \text{whence} \quad U_1^\dagger = \begin{bmatrix} 0 & 0 \\ 0 & 1/1+\varepsilon \end{bmatrix}.$$

We then have that

$$CU_1^\dagger R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 + \varepsilon & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

On the other hand,

$$CU^\dagger R = \begin{bmatrix} -1 & 0 & 10 \\ 0 & 1 + \varepsilon & 0 \\ 10 & 0 & 100 \end{bmatrix},$$

and

$$(CU^\dagger R)_1 = \begin{bmatrix} -1 & 0 & 10 \\ 0 & 0 & 0 \\ 10 & 0 & 100 \end{bmatrix}.$$  

Thus

$$A - CU_1^\dagger R = \begin{bmatrix} -1 & 0 & 10 \\ 0 & 0 & 0 \\ 10 & 0 & 100 \end{bmatrix},$$

but

$$A - (CU^\dagger R)_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 + \varepsilon & 0 \\ 0 & 0 & 200 \end{bmatrix}.$$
The spectrum of $A - CU^\dagger R$ is approximately $(100.9806, 1.9806, 0)$, but the spectrum of $A - (CU^\dagger R)_1$ is $(200, 1 + \varepsilon, 0)$.

For any $p \in [1, \infty]$, the Schatten $p$–norm of the first approximation is thus in the interval $[100, 103]$, whereas for every $\varepsilon \in (0, 1)$, the Schatten $p$–norm of the latter approximation lies in $[200, 202]$. In particular, the analogue of (6) does not hold for CUR approximations of non-SPSD matrices in general.

Note this is not a universal phenomenon. For matrices which are small random perturbations of SPSD matrices, the inequality (6) may be valid for certain CUR decompositions, i.e., certain choices of columns and rows.

5. Numerical Simulations

In this section, we compare the performance of various rank-enforcement methods for different structured matrices, e.g., SPSD, symmetric matrices, general random matrices, and real data matrices from the Hopkins155 motion segmentation data set [32]. In the following, we first state the set-ups of our experiments and the conclusions of the simulations follow in Section 5.1.

**Experiment 1.** First, we examine the performances of the various rank-enforcement methods for an SPSD matrix which is corrupted by noise which is SPSD (the easiest case) and noise which is symmetric but not positive semi-definite. The purpose of this basic experiment is to test how sensitive the bound of Pourkamali-Anaraki and Becker (6) is to perturbations. We first generate an SPSD matrix $A \in \mathbb{R}^{100 \times 100}$ with rank 8. $A$ is then perturbed by an SPSD or a merely symmetric random noise matrix $E$ whose entries are 0 mean and have standard deviation $\sigma = 10^{-3}$ (experiments with other noise levels are not shown, but the qualitative behavior is the same). We sample $x$ columns of $\widetilde{A}$ uniformly with replacement to form $\widetilde{C} = \widetilde{A}(\cdot, J)$, and we allow $x$ to vary from 8 to 60 (this is not the optimal sampling method for general matrices, but for random matrices uniform sampling suffices to illustrate the behavior of the different methods).

Because of the symmetry of the problem, we set $\widetilde{R} = \widetilde{C}^T$ and thus $\widetilde{U} = \widetilde{A}(J, J)$. In this and subsequent experiments, we then compute the following relative errors (the norm changes from experiment to experiment and is specified for each):

- $\|A - (\widetilde{C} \widetilde{U}^\dagger \widetilde{R})_k\|/\|A\|$  
- $\|A - \widetilde{C} \widetilde{U}^\dagger_k \widetilde{R}\|/\|A\|$  
- $\|A - \widetilde{C}_k \widetilde{C}^\dagger_k \widetilde{A} \widetilde{R}^\dagger_k \widetilde{R}_k\|/\|A\|$  
- $\|A - \widetilde{A}_k\|/\|A\|$.

In each case, $k$ is taken to be 8, the known underlying rank of $A$. The column sampling procedure is repeated 20 times so that there are 20 distinct CUR approximations of each kind for each value of $x$. Figure 1 shows the averaged errors (over the 20 choices of $\widetilde{C}$ and $\widetilde{U}$) versus the number of columns; and Figure 2 shows the results when the random noise is not SPSD.

**Experiment 2.** The first experiment showed the dependence on the SPSD structure of the matrix on the bound (6); here we turn to illustrating the CUR approximations for generic matrices $A$ which have no set structure other than being low rank. We
Figure 1. The performance of each rank-enforcement method for an SPSD matrix of rank 8 plus SPSD random noise of standard deviation $10^{-3}$; the plot shows relative error in the nuclear norm averaged over 20 trials vs. the number of columns selected.

Figure 2. The performance of each rank-enforcement method for an SPSD matrix of rank 8 plus symmetric (non-PSD) random noise of standard deviation $10^{-3}$; the plot shows relative error in the nuclear norm averaged over 20 trials vs. the number of columns selected.

generate $A \in \mathbb{R}^{100 \times 100}$ of rank 8, and perturb it by an unstructured random matrix $E$ as before. We again sample columns and rows uniformly, but this time do not choose $\tilde{R}$ and $\tilde{C}$ to be related. We allow the number of columns and rows, $x$, to vary from 8 to 60, and compute the same relative errors as in Experiment 1, except with the nuclear norm replaced by the spectral norm (we note, however, that the behavior is essentially the same for both).

Experiment 3. In this experiment, we test the performance of the rank-enforcement methods on a deterministic matrix $B$ of size $62 \times 159$, which comes from the Hopkins155 motion segmentation data set [32]. The test process is the same as in Experiment 2, and the results are shown in Figure 4.
Figure 3. The performances of each rank-enforcement method for a random matrix of rank 8 plus random noise with standard deviation $10^{-3}$ (behavior for other noise levels is similar, and so is omitted); the plot shows relative error in the spectral norm averaged over 20 trials vs. the number of columns and rows selected.

Many times in applications, a kernel matrix (which is SPSD) is formed from the data, for example as a precursor to Spectral Clustering [23]. For illustration, we test the different approximation in this case, in which from $B$ above, we generate the Gaussian kernel matrix $\tilde{A}$ of size $159 \times 159$ by setting the $i, j$-th entry of $\tilde{A}$ to be $e^{-\|B(:,i) - B(:,j)\|^2}$. Then we repeat the process in Experiment 1 by testing the rank 40 CUR approximation of $\tilde{A}$ and choosing $x$ (the number of columns) to range from 40 to 100. This value of the rank was determined empirically by analyzing the scree plot of the singular values of $\tilde{A}$.

Figure 4. The rank-7 (a) and rank-8 (b) CUR approximations of the Hopkins matrix; the plot shows relative error in the spectral norm vs. the number of columns and rows chosen.

5.1. Discussion. As seen in the experiments and figures above, the SPSD structure of matrices is crucial to the success of the rank-enforced Nyström method of [27],
Figure 5. The performance of CUR approximations on the SPSD Gaussian kernel matrix formed from the Hopkins data matrix; the target rank is set to be 40. The plot shows relative error in the spectral norm vs. the number of columns and rows chosen.

i.e., of taking $A \approx (\tilde{C}\tilde{U}^\dagger\tilde{C}^*)_k$ as opposed to $\tilde{C}\tilde{U}^\dagger\tilde{C}^*$. However, another interesting phenomenon appears, and that is that for a small oversampling of columns and rows, the new approximation introduced here of $A \approx \tilde{C}_k\tilde{C}_k^\dagger\tilde{R}_k^\dagger\tilde{R}_k$ performs better than the other rank-enforcement methods both in the SPSD and the unstructured case. We suggest the following explanation for this: this approximation corresponds to finding the best $k$–dimensional subspace which captures the span of the columns of $\tilde{C}$, and even when choosing few more than $k$ columns, this should be a good approximation to the span of the columns of $C$ itself. On the other hand, the other approximations are not projections onto a $k$–dimensional subspace in the domain and range, and thus the effect of the noise on the approximation is greater. However, as the number of columns and rows increases, the other approximations may better capture the information of $A$ by nature of better approximating the rank $k$ SVD of $\tilde{A}$; i.e., for large $k$, $\tilde{C}\tilde{U}^\dagger\tilde{R}_k \approx \tilde{A}_k$ (this is in line with the theory known from previous works, e.g., [7]; there is currently no similar theory for the projection-based approximation shown here).

6. Conclusion and Final Comments

To end, let us make some brief comments. We have provided perturbation error estimates for a variety of CUR approximation methods: estimates which hold for arbitrary matrix norms which are normalized, uniformly generated, unitarily invariant, and submultiplicative (a class which includes all Schatten $p$–norms). Our estimates qualitatively illustrate how the column and row selections affect the error, and in particular we give some more specific bounds in the case when maximal volume submatrices are chosen.

Due to the suggestion of other works on the Nyström method, we considered the effect of how the rank is enforced on CUR approximations for generic matrices, and found that, in contrast to the phenomenon observed for symmetric positive semi-definite
matrices, there is no provably better way to enforce the rank for CUR approximations of arbitrary matrices.

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Appendix A. Useful Properties of CUR Decompositions

Here we collect some useful properties of the submatrices involved in exact CUR decompositions. The proofs may be found in [18].

Lemma A.1. Suppose that $A, C, U,$ and $R$ are as in Theorem 2.2, with rank $(A) = \text{rank}(U)$. Then $\mathcal{N}(C) = \mathcal{N}(U)$, $\mathcal{N}(R^*) = \mathcal{N}(U^*)$, $\mathcal{N}(A) = \mathcal{N}(R)$, and $\mathcal{N}(A^*) = \mathcal{N}(C^*)$. Moreover,
\[ C^\dagger C = U^\dagger U, \quad RR^\dagger = UU^\dagger, \]
\[ AA^\dagger = CC^\dagger, \quad \text{and} \quad A^\dagger A = R^\dagger R. \]

Proposition A.2. Suppose that $A, C, U,$ and $R$ are as in Theorem 2.2 (but without any assumption on the rank of $U$). Then
\[ U = RA^\dagger C. \]

Proposition A.3. Suppose that $A, C, U,$ and $R$ satisfy the conditions of Theorem 2.2. Then
\[ U^\dagger = C^\dagger AR^\dagger. \]

Appendix B. Preliminaries from Matrix Perturbation Theory

Here we collect some useful facts from perturbation theory. The first is due to Weyl:

Theorem B.1. [12, Corollary 8.6.2.] If $B, E \in \mathbb{K}^{m \times n}$ and $\tilde{B} = B + E$, then for $1 \leq j \leq \min\{m,n\}$,
\[ |\sigma_j(B) - \sigma_j(\tilde{B})| \leq \sigma_1(E) = \|E\|_2. \]

Note that Theorem B.1 holds in greater generality and is due to Mirsky [22]. Therein, it was shown that for any normalized, uniformly generated, unitarily invariant norm $\|\cdot\|$,
\[ \|\text{diag}(\sigma_1(B) - \sigma_1(\tilde{B}), \ldots)\| \leq \|E\|. \]

The following Theorem of Stewart provides an estimate for how large the difference of pseudoinverses can be.

Theorem B.2. [29, Theorems 3.1–3.4] Let $\|\cdot\|$ be any normalized, uniformly generated, unitarily invariant norm on $\mathbb{K}^{m \times n}$. For any $B, E \in \mathbb{K}^{m \times n}$ with $\tilde{B} = B + E$, if rank $(\tilde{B}) = \text{rank}(B)$, then
\[ \|B^\dagger - \tilde{B}^\dagger\| \leq \mu \|\tilde{B}^\dagger\|_2 \|B^\dagger\|_2 \|E\|, \]
where $1 \leq \mu \leq 3$ is a constant depending only on the norm. If rank $(\tilde{B}) \neq \text{rank}(B)$, then
\[ \|B^\dagger - \tilde{B}^\dagger\| \leq \mu \max\{\|\tilde{B}^\dagger\|_2, \|B^\dagger\|_2\} \|E\| \text{ and } 1/\|E\|_2 \leq \|B^\dagger - \tilde{B}^\dagger\|_2. \]
The precise value of $\mu$ depends on the norm used and the relation of the rank of the matrices to their size; in particular, $\mu = 3$ for an arbitrary norm satisfying the hypotheses in Section 1.2, whereas $\mu = \sqrt{2}$ for the Frobenius norm, and $\mu = \frac{1 + \sqrt{5}}{2}$ (the Golden Ratio) for the spectral norm.

The preceding theorems yield the following immediate corollary.

**Corollary B.3.** With the assumptions of Theorem B.2, if $\tilde{B} = B + E$ and $\text{rank}(\tilde{B}) = \text{rank}(B) = k$, then

$$||B^\dagger|| - ||\tilde{B}^\dagger|| \leq \mu ||B^\dagger||_2 ||\tilde{B}^\dagger||_2 ||E||.$$  

Moreover, if $\sigma_k(B) > \mu ||E||$, then

$$\frac{||B^\dagger||}{1 + \mu ||B^\dagger||_2 ||E||} \leq ||\tilde{B}^\dagger|| \leq \frac{||B^\dagger||}{1 - \mu ||B^\dagger||_2 ||E||}.$$  

Regard that from the representation of $B^\dagger$ in terms of the SVD of $B$ mentioned in Section 1.2, we have

$$\|B^\dagger\|_2 = \frac{1}{\sigma_{\text{min}}(B)},$$  

where $\sigma_{\text{min}}(B)$ is the smallest nonzero singular value of $B$; this is sometimes how the inequalities in Corollary B.3 are written.

**Appendix C. Proof of Proposition 3.1**

First, note that by Proposition A.3 and the fact that $CC^\dagger = AA^\dagger$ (Lemma A.1), we have

$$CU^\dagger = CC^\dagger AR^\dagger = AA^\dagger AR^\dagger = AR^\dagger,$$

and likewise

$$U^\dagger R = C^\dagger A.$$

As noted in Proposition A.2, we have that

$$R = W_k(I, :)\Sigma_k V_k^{*} =: W_{k,I}\Sigma_k V_k^{*}.$$  

Consequently,

$$AR^\dagger = W_k\Sigma_k V_k^{*} (W_{k,I}\Sigma_k V_k^{*})^\dagger.$$  

To estimate the norm, let us first notice that the pseudoinverse in question turns out to satisfy

$$(W_{k,I}\Sigma_k V_k^{*})^\dagger = (V_k^{*})^\dagger \Sigma_k^{-1} W_{k,I}^\dagger.$$  

This is true on account of the fact that $W_{k,I}$ has full column rank, $V_k^{*}$ has orthonormal rows, and $\Sigma_k$ is invertible by assumption. Next, note that since $V_k^{*}$ has orthonormal rows, $(V_k^{*})^\dagger = V_k$. Putting these observations together, we have that

$$\|AR^\dagger\| = \|W_k\Sigma_k V_k^{*} (W_{k,I}\Sigma_k V_k^{*})^\dagger\|
= \|\Sigma_k V_k^{*} V_k^{*} \Sigma_k^{-1} W_{k,I}^\dagger\|
= \|\Sigma_k V_k^{*} \Sigma_k^{-1} W_{k,I}^\dagger\|
= \|W_{k,I}^\dagger\|.$$  

The second equality follows from the unitary invariance of the norm in question; to see this, write $W_k = WP$, where $W$ is the $m \times m$ orthonormal basis from the full SVD of
A, and \( P = \begin{bmatrix} I_{k \times k} & 0 \\ 0 & I_{k_0} \end{bmatrix} \); subsequently, the norm in question will be the norm of \( \begin{bmatrix} W^\dagger_k & 0 \end{bmatrix} \), which is \( \| W_k \| \). A word of caution: Equation (9) is not true if \( W_{k,I} \) is replaced by \( W_{A,I} \), the row submatrix of the full left singular vector matrix of \( A \).

By a directly analogous calculation, we have that

\[
\| C^\dagger A \| = \|(V^*_k)_k\|,
\]

whereupon the conclusion follows from the fact that \( (V^*_k)_k \) has the same norm as \( V_k \).

**Appendix D. Table of Inequalities**

| Approximation | Error Bound \((w = \| W_k^\dagger \|, v = \| V_k^\dagger \|)\) |
|---------------|-------------------------------------------------------|
| \( \bar{C} \bar{C}^\dagger \bar{A} \bar{R}^\dagger \bar{R} \) | \( (w + v + 3)\| E \| \) |
| \( \bar{C} \bar{U}^\dagger \bar{R} \) | \( (w + v + 3w)\| E \| + \| \bar{U}^\dagger \| (w + v + wv + 1)\| E \|^2 \) |
| \( \bar{C} [\bar{U}] \bar{R} \) | \( (w + v + 2w)\| E \| + wv\| [\bar{U}]_r - U \| + \| [\bar{U}]^\dagger \|_2 (w + v + wv + 1)\| E \|^2 \) |
| \( \bar{C} \bar{U}^\dagger_k \bar{R} \) | \( (w + v + 4w)\| E \| + \frac{1 - 2\mu \| \bar{U}^\dagger \|_2 \| E \| (w + v + wv + 1)\| E \|^2 \) |
| \( C_k \bar{C}^\dagger_k \bar{A} \bar{R}^\dagger_k \bar{R} \) | \( (2w + 2v + 3)\| E \| \) |

Table 1. Summary of the perturbation bounds attained in our analysis.