Symmetric multiplicative formality of the Kontsevich operad

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Abstract

In his famous paper entitled “Operads and motives in deformation quantization”, Maxim Kontsevich constructed (in order to prove the formality of the little $d$-disks operad) a topological operad, which is called in the literature the Kontsevich operad, and which is denoted $K_d$ in this paper. This operad has a nice structure: it is a multiplicative symmetric operad, that is, it comes with a morphism from the symmetric associative operad. There are many results in the literature concerning the formality of $K_d$. It is well known (by Kontsevich) that $K_d$ is formal over reals as a symmetric operad. It is also well known (independently by Syunji Moriya and the author) that $K_d$ is formal as a multiplicative nonsymmetric operad. In this paper, we prove that the Kontsevich operad is formal over reals as a multiplicative symmetric operad, when $d \geq 3$.

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1 Introduction

The object we look at in this paper is the Kontsevich operad that we denote by $K_d := \{K_d(n)\}_{n \geq 0}$ (the letter “d” stands for the dimension of the ambient space). Recall that $K_d(0)$ and $K_d(1)$ are the one point spaces, and for $n \geq 2$, $K_d(n)$ is the Kontsevich compactification of the configuration space of $n$ points in $\mathbb{R}^d$. For more details about the Kontsevich operad, we refer the reader to the paper [7] Section 4 of Sinha. This latter author shows [6] that the operad $K_d$ is weakly equivalent, as a topological operad, to the little $d$-disks operad. He also proves [7] that $K_d$ is a multiplicative symmetric operad. Recall that an operad $O$ is said to be a multiplicative symmetric operad if there is a morphism $\Sigma[As] \to O$ in the category of symmetric operads from the symmetric associative operad to $O$. Here $\Sigma[As](n) = \Sigma_n$, the symmetric group on $n$ letters. Since this paper is devoted to the study of the formality of $K_d$, we need the following definition.

**Definition 1.1.** Let $O$ be a topological operad. We say that $O$ is formal over a field $\mathbb{K}$ if there exists a zigzag

$$C_*(O; \mathbb{K}) \leftarrow \cdots \rightarrow H_*(O; \mathbb{K})$$

(1.1)

of quasi-isomorphisms between the singular chain of $O$ and its homology.
• If \( O \) is equipped with a symmetric structure, and if all morphisms in (1.1) are of symmetric operads, we say that \( O \) is formal as a symmetric operad.

• If \( O \) is a multiplicative nonsymmetric operad (that is, there is a morphism \( \{\ast\}_{n \geq 0} = \mathcal{A}s \longrightarrow O \) from the associative operad to \( O \)), and if (1.1) holds in the category of multiplicative nonsymmetric operads, we say that \( O \) is formal as a multiplicative operad.

• If \( O \) is a multiplicative symmetric operad, and if (1.1) holds in the category of multiplicative symmetric operads, we say that \( O \) is formal as a multiplicative symmetric operad.

There are many formality theorems concerning \( \mathcal{K}_d \) in the literature, but we are only interested in the following two known results. The first one is due to Kontsevich [2], and it says that the operad \( \mathcal{K}_d \) is formal over reals as a symmetric operad. The second one, independently due to Syunji Moriya [5] and the author [8], states that \( \mathcal{K}_d \) is formal over reals as a multiplicative operad. One natural question arises: are these two results immediately imply that \( \mathcal{K}_d \) is formal as a multiplicative symmetric operad? As we will see in this text, the answer to this question is not immediate, and is given by the following theorem, which is the main result of this paper.

**Theorem 1.2.** For \( d \geq 3 \), the Kontsevich operad \( \mathcal{K}_d \) is formal over reals as a multiplicative symmetric operad.

As we said before, the multiplicative formality of \( \mathcal{K}_d \) was independently discovered by Moriya [5] and by the author [8]. These two authors used this formality to explicitly compute the natural Gerstenhaber algebra structure on the homology of the space of long knots. We suspect that Theorem 1.2 could be used, in one way or another, to understand the natural Swiss-Cheese algebra structure on the pair formed by the homology of the space of long knots and the homology of the space of long links.

**Outline of the paper** In Section 2 we prove a crucial Lemma 2.4. The idea of the proof is to first use Lemma 2.1 and next use a “symmetric version” of [8, Lemma 2.7]. This crucial lemma will be the main ingredient in proving Theorem 1.2.

In Section 3 we prove Theorem 1.2 by applying Lemma 2.4 to the specific zigzag between \( C_\ast(\mathcal{K}_d; \mathbb{R}) \) and its homology \( H_\ast(\mathcal{K}_d; \mathbb{R}) \).

## 2 Equivalences of multiplicative symmetric operads

The goal here is to prove Lemma 2.4, which is the main result of this section. This lemma will be the key ingredient in the proof of Theorem 1.2 which will be done in Section 3. Throughout the section we reserve the letter \( \mathcal{C} := (\mathcal{C}, \otimes, 1) \) for the base category for operads. It is a symmetric monoidal model category that is cofibrantly generated. Since the proof of Lemma 2.4 is in some sense the “symmetric version” of [8, Lemma 2.7], we begin by recalling the classical adjunction between the categories \( \mathcal{O}P_{ns}(\mathcal{C}) \) and \( \mathcal{O}P_{s}(\mathcal{C}) \) of nonsymmetric operads and symmetric operads respectively. Next we prove an intermediate Lemma 2.1. We end the section with the proof of Lemma 2.4.

We start by recalling the adjunction (2.1) below. To do that, we recall the definition of a symmetric operad and of a nonsymmetric operad in \( \mathcal{C} \). First of all, a symmetric sequence in \( \mathcal{C} \) consists of a sequence \( \{X_n\}_{n \geq 0} \) in which each \( X_n \) is an object in \( \mathcal{C} \) equipped with an action of the symmetric group \( \Sigma_n \). By action we mean a morphism \( \alpha_{X_n} : \Sigma_n \otimes X_n \longrightarrow X_n \) such that the obvious diagrams
Proof. The result comes from the fact that the following two squares commute. Here $\Sigma_n \otimes X_n$ is the coproduct $\Sigma_n \otimes X_n = \bigsqcup_{\sigma \in \Sigma_n} X_n$, “$\circ$” is the composition in $\Sigma_n$, $i: \{e_{\Sigma_n}\} \hookrightarrow \Sigma_n$ is the inclusion of the unit, and $\Sigma_n \otimes (\Sigma_n \otimes X_n)$ is naturally isomorphic to $(\Sigma_n \times \Sigma_n) \otimes X_n$. A symmetric operad in $C$ consists of a symmetric sequence $O = \{O(n)\}_{n \geq 0}$ endowed with a unit $1 \to O(1)$, and a collection of morphisms

$$O(k) \otimes O(i_1) \otimes \cdots \otimes O(i_k) \to O(i_1 + \cdots + i_k)$$

that satisfy natural equivariance properties, unit and associative axioms (May’s axioms, see [4, Definition 1.1]). If we forget the symmetric structure, we obtain the so called nonsymmetric operad. Symmetric operads (respectively nonsymmetric operads) form a category that we denote by $OP_s(C)$ or just by $OP_s$ (respectively by $OP_{ns}(C)$ or just by $OP_{ns}$). These two categories are related by an adjunction

$$\Sigma[-]: OP_{ns}(C) \rightleftarrows OP_s(C): U,$$

where $U$ is the forgetful functor, and $\Sigma[-]$ is defined as $\Sigma[P](n) = \Sigma_n \otimes P(n)$. Therefore, there is a bijection $\Psi: \text{Hom}(P, U(Q)) \cong \text{Hom}(\Sigma[P], Q)$. Recall that this bijection sends a morphism $f: P \to U(Q)$ to $\Psi_f: \Sigma[P] \to Q$ defined as the composition

$$\Sigma[P] \xrightarrow{\Sigma[f]} \Sigma[U(Q)] \xrightarrow{\eta_Q} Q,$$

where $\eta = \{\eta_Q: \Sigma[U(Q)] \to Q\}_{Q \in OP_s}$ is the counit of the adjunction (2.1). That is, $\Psi_f = \eta_Q \circ \Sigma[f]$. Here the counit is defined as follows. For all $n \geq 0$, the morphism $(\eta_Q)_n = \alpha_{Q(n)}: \Sigma_n \otimes U(Q)(n) \to Q(n)$ comes from the action of $\Sigma_n$ on $Q(n)$.

**Lemma 2.1.** Let $g: Q \to Q'$ be a morphism in $OP_s(C)$. Let

$$\begin{array}{ccc}
\mathcal{U}(Q) & \xrightarrow{U(g)} & \mathcal{U}(Q') \\
\downarrow f & & \downarrow f' \\
P & \xrightarrow{h} & P'
\end{array}$$

be a commutative square in the category $OP_{ns}(C)$. Then the diagram

$$\begin{array}{ccc}
Q & \xrightarrow{g} & Q' \\
\downarrow \Psi_f & & \downarrow \Psi_{f'} \\
\Sigma[P] & \xrightarrow{\Sigma[h]} & \Sigma[P']
\end{array}$$

is a commutative square in $OP_s(C)$.

**Proof.** The result comes from the fact that the following two squares

$$\begin{array}{ccc}
\Sigma[U(Q)] & \xrightarrow{\Sigma[U(g)]} & \Sigma[U(Q')] \\
\downarrow \Sigma[f] & & \downarrow \Sigma[f'] \\
\Sigma[P] & \xrightarrow{\Sigma[h]} & \Sigma[P']
\end{array} \quad \begin{array}{ccc}
Q & \xrightarrow{g} & Q' \\
\downarrow \eta_Q & & \downarrow \eta_{Q'} \\
\Sigma[U(Q)] & \xrightarrow{\Sigma[U(g)]} & \Sigma[U(Q')] \\
\downarrow \Sigma[h] & & \downarrow \Sigma[h'] \\
\Sigma[P] & \xrightarrow{\Sigma[h]} & \Sigma[P']
\end{array}$$
commute. The first square commutes because it is the image of the commutative square \([2.2] \) under the functor \( \Psi \), and the second one commutes because \( g_n : Q(n) \rightarrow Q'(n) \) is \( \Sigma_n \)-equivariant (since, by hypothesis, the morphism \( g \) is of symmetric operads) for all \( n \).

In order to state and prove the most important result (Lemma 2.4 below) of this section, we recall some notations and some classical facts. The first notation is that of a symmetric operad \( A_s = \{ A_s(n) \}_{n \geq 0} \) in \( C \), which is a nonsymmetric operad. It is defined by \( A_s(n) = 1 \) for all \( n \). Recall that \( 1 \) is the unit for the tensor product in \( C \).

**Definition 2.2.** A multiplicative symmetric operad is a symmetric operad \( Q \) together with a morphism \( \Sigma[A_s] \rightarrow Q \) in the category of symmetric operads. Similarly, a multiplicative operad is a nonsymmetric operad \( O \) together with a morphism \( A_s \rightarrow O \) in \( OP_{ns} \).

The second notation is \( C^N \), which represents the category of sequences \( X = \{ X(n) \} \) of objects in \( C \). This category is endowed with the cofibrantly generated model structure in which weak equivalences, fibrations and cofibrations are all levelwise. The third and the last notation we will need is \( \Sigma C^N \), which denotes the category of symmetric sequences in \( C \). This latter category is also endowed with a model structure (also cofibrantly generated) induced by the adjunction \( \text{Sym} : C^N \rightleftarrows \Sigma C^N : U \), where \( U \) is of course the forgetful functor, and \( \text{Sym} \) is the functor defined as \( \text{Sym}(P)(n) = \Sigma_n \otimes P(n) \). Recall that a morphism \( f : P \rightarrow Q \) in \( \Sigma C^N \) is a weak equivalence (respectively a fibration) if \( U(f) \) is a weak equivalence in \( C^N \) (respectively \( U(f) \) is a fibration in \( C^N \)).

**Remark 2.3.** The following items are easy.

(i) If \( X \) is a cofibrant object in \( C^N \), then so is \( \text{Sym}(X) \). More generally, the functor \( \text{Sym} \) preserves cofibrations (this easily follows from adjunction relations).

(ii) The associative operad \( A_s \) is cofibrant as an object in \( C^N \). Indeed, the unit \( 1 \) in \( C \) is cofibrant since it is one of the axioms in the definition of a symmetric monoidal model category.

(iii) From (i) and (ii), it follows that the symmetric operad \( \Sigma[A_s] \) is cofibrant as an object in \( \Sigma C^N \).

Before stating Lemma 2.4 we mention that the model structure (or more precisely the semimodel structure) we consider on \( OP_s(C) \) (respectively on \( OP_{ns}(C) \)) is the one from [1] Theorem 12.2.A] (respectively the one from the nonsymmetric version of [1] Theorem 12.2.A]).

**Lemma 2.4.** Let \( g : R \rightarrow Q \) and \( g' : R \rightarrow Q' \) be two morphisms in \( OP_s(C) \), and let

\[
\begin{array}{ccc}
U(Q) & \xrightarrow{\sim} & U(R) \\
\downarrow f & & \downarrow \eta \\
\sim A_s & \xrightarrow{h} & \sim A_s \\
\end{array}
\]

be a commutative diagram in \( OP_{ns}(C) \). Assume that \( A \) is cofibrant as an object in \( C^N \). Then there is a commutative diagram

\[
\begin{array}{ccc}
\Sigma[A_s] & \xrightarrow{\sim} & \Sigma[A_s] \\
\uparrow \Psi_f & & \uparrow \Psi_{f'} \\
\Sigma[A_s] & \xrightarrow{\sim} & \Sigma[A_s] \\
\end{array}
\]
in the category $\mathcal{O}P_s(\mathcal{C})$ of symmetric operads.

**Proof.** By applying Lemma 2.1 to the diagram (2.3), we obtain the following commutative diagram in $\mathcal{O}P_s(\mathcal{C})$

$$
\begin{array}{c}
\begin{array}{ccc}
Q & \sim & R \\
\uparrow \psi_f & & \uparrow \psi_g \\
\Sigma[A_s] & \sim & \Sigma[A]
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
R & \sim & Q' \\
\uparrow \psi_f' & & \uparrow \psi_g' \\
\Sigma[A_s] & \sim & \Sigma[A_s]
\end{array}
\end{array}
\] (2.4)

The rest of the proof is exactly the same way as that of [8, Lemma 2.7], but for symmetric operads. To be more precise, one has the following commutative diagram (in the category $\mathcal{O}P_s(\mathcal{C})$) constructed as follows.

$$
\begin{array}{c}
\begin{array}{ccc}
Q & \sim & R \\
\uparrow \psi_f & & \uparrow \psi_g \\
\Sigma[A_s] & \sim & \Sigma[A]
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
R & \sim & Q' \\
\uparrow \psi_f' & & \uparrow \psi_g' \\
\Sigma[A_s] & \sim & \Sigma[A_s]
\end{array}
\end{array}
\] (2.5)

- The morphisms $\eta_1$ and $\eta_2$ comes from the factorization axiom applied to $\Psi_f$. One can apply this axiom here because $\Sigma[A]$ is cofibrant as an object in $\Sigma \mathcal{C}^N$. Indeed, by hypothesis, the operad $\mathcal{A}$ is cofibrant as an object in $\mathcal{C}^N$, which implies (by Remark 2.3 (i)) that $\Sigma[A]$ is cofibrant in $\Sigma \mathcal{C}^N$.

- The object $\tilde{R}$ is the pushout of the diagram formed by morphisms $\eta_1$ and $\Sigma[h]$.

- The morphism $h_1$ is a weak equivalence because of the following. Recall first that the axiom of relative properness for symmetric operads [11 Theorem 12.2.B] says that if $P_1$ and $P_2$ are two objects in $\mathcal{O}P_s(\mathcal{C})$ that are cofibrant as objects in $\Sigma \mathcal{C}^N$, then the pushout of a weak equivalence along a cofibration

$$
P_1 \sim \rightarrow P_3 \\
\uparrow \sim \downarrow \vdots \\
P_2 \sim \rightarrow P_4
$$

-gives a weak equivalence $P_3 \sim \rightarrow P_4$. Since the operad $\Sigma[A]$ is cofibrant as an object in $\Sigma \mathcal{C}^N$, and since $\Sigma[A_s]$ is also cofibrant as an object in $\Sigma \mathcal{C}^N$ (see Remark 2.3 (iii)), it follows by the axiom of relative properness that $h_1$ is a weak equivalence.

- The morphism $h_3: \tilde{R} \sim \rightarrow Q'$ comes from the universal property of the pushout. Moreover, it is a weak equivalence because of the two out of three axiom since $g' \eta_2 = h_3 h_1$ and $g', \eta_2$ and $h_1$ are weak equivalences.
One has also a commutative diagram

\[
\begin{array}{ccc}
\Sigma[A] & \xrightarrow{\eta_1} & \mathcal{R} \\
\Sigma[k] & \sim & \mathcal{R} \\
\Sigma[A_s] & \xrightarrow{h_2} & \mathcal{R} \\
\end{array}
\]

(2.6)

in which the morphism \( h_4 : \tilde{\mathcal{R}} \to Q \) comes from the universal property of pushout. This latter morphism is a weak equivalence because of the two out of three axiom since \( g\eta_2 \) and \( h_1 \) are weak equivalences.

Now from diagrams (2.5) and (2.6), we obtain

\[
\begin{array}{ccc}
Q & \xleftarrow{\sim} & \tilde{\mathcal{R}} & \xrightarrow{\sim} & Q' \\
\Psi_f & & \sim & & \sim \\
\Sigma[A_s] & \xleftarrow{\sim} & \Sigma[A_s] & \xleftarrow{\sim} & \Sigma[A_s],
\end{array}
\]

which is a commutative diagram in \( OP_\ast(C) \). This ends the proof.

\[\square\]

3 Formality of the Kontsevich operad as a multiplicative symmetric operad

The goal of this short section is to prove Theorem 1.2 announced in the introduction. The base category \( C \) for operads is taken to be \( \text{Ch}_\mathbb{R} \), the category of nonnegatively chain complexes over the ground field \( \mathbb{R} \). This latter category is equipped with its standard model structure. That is, weak equivalences (quasi-isomorphisms) and fibrations (epimorphisms) are all level wise.

Before starting the proof of Theorem 1.2, we recall the specific zigzag

\[
\begin{array}{ccc}
C_\ast(K_d) & \xleftarrow{\sim} & C_\ast(F_d) & \xrightarrow{\sim} & D_d' & \xleftarrow{\sim} & H_\ast(K_d) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
C_\ast(K_1) & \xleftarrow{\sim} & C_\ast(F_1) & \xrightarrow{\sim} & D_1' & \xleftarrow{\sim} & H_\ast(K_1).
\end{array}
\]

(3.1)

between the singular chain \( C_\ast(K_d) \) and the homology \( H_\ast(K_d) \) of the Kontsevich operad. This zigzag holds in the category \( OP_{ns}(\text{Ch}_\mathbb{R}) \) of nonsymmetric operads in \( \text{Ch}_\mathbb{R} \), and was explicitly built by Lambrechts and Volić in [3] (see [3, Section 9] and [3, Section 8] for the construction of \( C_\ast(F_d) \xrightarrow{\sim} D_d' \) and \( H_\ast(K_d) \xrightarrow{\sim} D_d' \) respectively). They built (3.1) in order to develop the Kontsevich’s proof [2] for the formality of the little \( d \)-disks operad.

The idea of the following proof is the same as that of [8, Theorem 1.3], except that here we work with symmetric operads.
Proof of Theorem 1.2. We start with the following two observations. The first one, which is well known, says that all the operads appearing in the first row of (3.1) are symmetric. The second observation (which is also well known) is the fact that all the operads in the second row (except $C_*(F_1)$) are the associative operad $A_s$ in chain complexes. From these two observations, if we denote $C_*(F_1)$ by $A$, one can rewrite (3.1) as

\[
\begin{array}{c}
\mathcal{U}(C_*(K_d)) \sim \mathcal{U}(C_*(F_d)) \sim \mathcal{U}(D'_{\mathcal{L}}) \sim \mathcal{U}(H_*(K_d)) \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
A_s \quad \sim \quad A \quad \sim \quad A_s \quad \sim \quad A_s,
\end{array}
\]

where $\mathcal{U}(\cdot)$ is the forgetful functor from (2.1). Since it is clear that $A = C_*(F_1)$ is cofibrant as an object in $Ch_N^{kR}$, and since the two morphisms from $A$ to $A_s$ are the same, the desired result follows from Lemma 2.4.

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