Deformation of Singularities via $L_\infty$-Algebras

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September 4, 2018

Abstract

This is an addendum to the paper “Deformation of $L_\infty$-Algebras” [9]. We explain in which way the deformation theory of $L_\infty$-algebras extends the deformation theory of singularities. We show that the construction of semi-universal deformations of $L_\infty$-algebras gives explicit formal semi-universal deformations of isolated singularities.

Introduction

In this paper, we apply the following general idea for the construction of moduli spaces to isolated singularities: Take the differential graded Lie algebra $L$ describing a deformation problem (for isolated singularities, this is the tangent complex) and find a minimal representative $M$ of $L$ in the class of formal $L_\infty$-algebras (see [9]). In geometric terms, $M$ is a formal DG-manifold, containing the moduli space as analytic substructure. This general concept is also sketched in [7].

We define a functor $F$ from the category of complex analytic space germs to the localization of the category of $L_\infty$-algebras by $L_\infty$-equivalence. For a singularity $X$, we take the semi-universal $L_\infty$-deformation $(V, Q^V)$ of $F(X)$ constructed in [9]. For isolated singularities, the components $V^i$ are of finite dimension. The restriction of the vectorfield $Q^V$ defines a formal map (Kuranishi-map) $V^0 \to V^1$ whose zero locus gives the formal moduli space.

1 Definitions and reminders

In the whole paper, we work over a ground field $k$ of characteristic zero.

Denote the category of formal (resp. convergent) complex analytic space germs by $\mathfrak{Anf}$ (resp. $\mathfrak{An}$). Denote the category of isomorphism classes of formal DG manifolds by $\mathsf{DG-Manf}$. We use the following superscripts to denote full subcategories of $\mathsf{DG-Manf}$:

$L$ (“local”): the subcategory of all $(M, Q^M)$ in $\mathsf{DG-Manf}$ such that $Q^M_0 = 0$;

$M$ (“minimal”): the subcategory of all $(M, Q^M)$ in $\mathsf{DG-Manf}^L$ such that $Q^M_1 = 0$;

$G$ (“$g$-finite”): the subcategory of all $(M, Q^M)$ in $\mathsf{DG-Manf}^L$ such that $H^1(M, Q^M_1)$

*Supported by: Doktorandenstipendium des Deutschen Akademischen Austauschdienstes im Rahmen des gemeinsamen Hochschulsonderprogramms III des Bundes und der Länder
is $g$-finite.

We call a morphism $f = (f_n)_{n\geq 1}$ in $\text{DG-Manf}^L$ \bf weak equivalence, if the morphism $f_1$ of DG vectorspaces is a quasi-isomorphism, i.e. if the corresponding morphism of $L_\infty$-algebras is an $L_\infty$-equivalence. Recall that by Theorem 4.4 and Lemma 4.5 of \cite{[3]}, weak equivalences define an equivalence relation in $\text{DG-Manf}^L$ and that in each equivalence class, there is a uniquely defined \bf minimal model, i.e. an object belonging to $\text{DG-Manf}^M$.

\textbf{Proposition 1.1.} We can localize the category $\text{DG-Manf}^L$ by weak equivalences ($\approx$). The quotient $\text{DG-Manf}^L/\approx$ is equivalent to the category $\text{DG-Manf}^M$ and the localization functor assigns to each object of $\text{DG-Manf}^L$ its minimal model.

\textit{Proof.} This follows directly by Corollary 2.5.7 of \cite{[7]}. \hfill $\square$

\section{The functors $F$ and $V$}

In this section we explain how to represent (formal) singularities by formal DG manifolds.

Let $C$ be the category of formal analytic algebras, $A \in \text{Ob}(C)$ and $R = (R, s)$ a \bf resolvent of $A$ over $k$, i.e. a $g$-finite free DG-algebra in $\text{gr}(C)$ such that $H^0(R, s) \cong A$ and $H^j(R, s) = 0$, for $j < 0$. For $l \geq 0$, let $I_l$ be an index set containing one index for each free algebra generator of $R$ of degree $-l$. Consider the disjoint union $I$ of all $I_l$ as graded set such that $g(i) = l$, for $i \in I_l$. Fix an ordering on $I$, subject to the condition $i < j$, if $g(i) < g(j)$.

Thus, as graded algebra, $R = k[[X^0]][X^-]$, where $X^0 = \{x_i | i \in I, g(i) = 0\}$ and $X^- = \{x_i | i \in I, g(i) \geq 1\}$ are sets of free algebra generators with $g(x_i) = -g(i)$.

Set $M := \coprod_{i \in I} ke_i$ to be the free, graded $k$-vectorspace with base $\{e_i : i \in I\}$, where $g(e_i) = g(i)$. Consider $S(M) = \coprod_{n \geq 0} M^\otimes_n$ in the usual way as graded coalgebra (see Section 1.1 of \cite{[9]}). Set

$$S(M)^* := \text{Hom}_{k-\text{graded}}(S(M), k) = \prod_{j \geq 0} \text{Hom}_{k-\text{mod}}(M^\otimes_j, k).$$

We identify products $x_{i_1} \cdots x_{i_l}$ in $R$ with the maps $M^\otimes_l \to k$, defined by $e_{i_1} \cdots e_{i_l} \mapsto 1$ and $e_{j_1} \cdots e_{j_l} \mapsto 0$ for $\{j_1, \ldots, j_l\} \neq \{i_1, \ldots, i_l\}$. Especially, we identify each constant $\lambda \in k$ with the map $k \to k$, sending 1 to $\lambda$. We have

$$R^j = \prod_{n \geq 0} \text{Hom}^j(M^\otimes_n, k)$$

and $R = \coprod_{j \geq 0} R^j$. The differential $s$ of $R$ extends naturally to $\bar{R} := \coprod_{j \geq 0} R^j$. As complexes, $R$ and $\bar{R}$ are identical, but not as graded modules. We identify $\bar{R} = S(M)^*$. Set

$$\text{Der}(R) := \coprod_{i \in \mathbb{Z}} \text{Der}^i(R, R) \quad \text{and} \quad \text{Coder}(S(M)) := \coprod_{i \in \mathbb{Z}} \text{Coder}^i(S(M), S(M)).$$
Denote \( \text{Diff}(R) \) (resp. \( \text{Codiff}(S(M)) \)) the submodule of differentials (resp. codifferentials). The following proposition explains why, for a formal DG manifold \( W \), the complex \( \text{Coder}(S(W), S(W)) \) is called tangent complex of \( W \).

**Proposition 2.1.** Take \( R \) and \( M \) as above. The natural map
\[
\text{Coder}(S(M)) \longrightarrow \text{Der}(R), \quad Q \mapsto s^Q
\]
where \( s^Q(g) = g \circ Q \), is bijective and the restriction gives rise to an isomorphism
\[
\text{Codiff}(S(M)) \longrightarrow \text{Diff}(R).
\]

**Proof.** The injectivity is clear. Surjectivity: A derivation \( s \) of degree \( j \) on \( R \) induces a differential (also denoted by \( s \)) on \( \bar{R} = S(M)^* \). We have to find a coderivation \( Q \) of degree \( j \) on \( S(M) \) such that, for \( u \in S(M)^* \), we have
\[
s^Q(u) = u \circ Q.
\]
For each \( i \in I \), set \( f_i := s(x_i) \). Then, \( f_i \) is a product \((f_i)_n \) \( n \geq 1 \) with \((f_i)_n \) \( \in \text{Hom}^{g(i)+1}(M^\otimes n, k) \). We define the coderivation \( Q \) by
\[
Q_n(m_1, \ldots, m_n) := \sum_{i \in I} (f_i)_n(m_1, \ldots, m_n) \cdot e_i,
\]
for homogeneous \( m_1, \ldots, m_n \in M \). In fact, the non-vanishing terms in the sum satisfy the condition \( g(m_1) + \ldots + g(m_n) = g(i) \), hence the sum is finite. To show that for \( u \in S(M)^* \), we have \( s(u) = u \circ Q \), it is enough to show that for all \( i \in I \), \( s(x_i) = x_i \circ Q \). But by definition, for \( m_1, \ldots, m_n \in M \), we have
\[
(x_i \circ Q)_n(m_1, \ldots, m_n) = (f_i)_n(m_1, \ldots, m_n) = (s(x_i))(m_1, \ldots, m_n).
\]
The second statement is a direct consequence of the first. \( \square \)

As consequence, the differential \( s \) on \( R \) induces a codifferential \( Q^M \) on \( S(M) \).

We consider the pair \((M, Q^M)\) as formal DG manifold in \( \text{DG-Manf}^k G \). It has the following property: The restriction of \( Q^M \) to \( M^0 \) defines a formal map \( M^0 \longrightarrow M^1 \). Its zero locus is isomorphic to \( X \).

Summarizing the above construction, to each formal space germ \( X \) with associated formal analytic algebra \( A \), we can construct a formal DG manifold \((M, Q^M)\), containing \( X \) as “subspace”. Of course, \((M, Q^M)\) depends on the choice of the resolvent \((R, s)\). But we will show that \((M, Q^M)\) is well defined up to weak equivalence, i.e. that the assignment \( X \mapsto (M, Q^M) \) defines a functor
\[
F : \text{Anf} \longrightarrow \text{DG-Manf}^k G/ \approx
\]

**Lemma 2.2.** If \( W = (W, d) \) is a DG \( k \)-vectorspace and if the dual complex \( \text{Hom}(W, k) \) is acyclic, then \( W \) is acyclic. Consequently, if \( f : V \longrightarrow W \) is a morphism of DG \( k \)-vectorspaces such that the dual complex \( f^* : W^* \longrightarrow V^* \) is a quasi-isomorphism, then \( f \) is a quasi-isomorphism.
Proof. Assume that $M$ is cyclic, i.e. there is an $n$ and an element $a \in M^n$ such that $d^n(a) = 0$ and $a \notin \text{Im} d^{n-1}$. Let $B'$ be a base of $\text{Im} d^{n-1}$. We extend $B' \cup \{a\}$ to a base $B$ of $M^n$. Let $p : M^n \to k$ be the projection on the coordinate $a$ of $B$. Then, $d^n(p) = p \circ d^{n-1} = 0$ and $p(a) = 1$, hence $p \notin \text{Im} d^n$. Contradiction! □

Lemma 2.3. Let $f : M \to M'$ be a morphism of formal DG-manifolds such that the corresponding map $S(M) \to S(M')$ is a quasi-isomorphism of complexes. Then, $f$ is a weak equivalence.

Proof. By the Decomposition Theorem for $L_\infty$-algebras (see Lemma 4.5 of [3]), we may assume that $M$ is minimal and that $f$ is strict. In this case, the homomorphism $f : S(M) \to S(M')$ of DG coalgebras is a direct sum of maps of complexes $f_1 : M \to M'$ and

$$\sum_{j \geq 2} f_1^{\otimes j} : \prod_{j \geq 2} M^{\otimes j} \to \prod_{j \geq 2} M'^{\otimes j}.$$ 

Since the sum is a quasi-isomorphism, both factors are quasi-isomorphisms. □

Corollary 2.4. Let $F : (M, Q^M) \to (M', Q^{M'})$ be a morphism of formal DG manifolds in $\text{DG-Man}^G$ and suppose that the dual map $S(M')^* \to S(M)^*$ is a quasi-isomorphism of free DG algebras, then $F$ is a weak equivalence.

Proof. This follows by Lemma 2.2 and 2.3. □

Thus, we have proved the functoriality of $F$. Next, we define a functor

$$V : \text{DG-Man}^{GM} \to \text{Anf}$$

as already mentioned above: For a minimal DG manifold $(M, Q^M)$ in $\text{DG-Man}^{MG}$, set $V(M, Q^M)$ to be the zero locus of the formal map $M^0 \to M^1$, induced by $Q^M$. It can easily be seen that the composition $V \circ F$ is the identity on $\text{Anf}$. As a consequence, we get the following theorem:

Theorem 2.5. The functor $F$ embeds $\text{Anf}$ as full subcategory into $\text{DG-Man}^{GM}$.

3 Deformations and embedded deformations

In this section we recall some classical results, showing that each deformation of a singularity is equivalent to an embedded deformation.

A morphism $G : C \to D$ of fibered groupoids over the category $\text{An}$ of complex space germs is called smooth if the following condition holds: If $\beta : b \to b'$ is a morphism in $D$ such that $G(\beta) : S \to S'$ is a closed embedding, and if $a$ is an object in $C$ such that $G(a) = b$, then there is a morphism $\alpha : a \to a'$ in $C$ such that $G(\alpha) = \beta$.

Consider a complex space germ $X$ with corresponding analytical algebra $O_X$. Suppose that $X$ is embedded in the smooth space germ $P$ with corresponding
analytic algebra $R^0$. Let $R = (R, s)$ be a g-finite, free algebra resolution of $O_X$ such that $R^0 = O_P$.

For any space germ $(S, O_S)$, set $R_S := R \otimes_C O_S$ and

$$C(S) := \{ \delta \in \text{Der}^1(R_S, R_S) \mid \delta(0) = 0 \text{ and } (s + \delta)^2 = 0 \}$$

Furthermore, let $D(S)$ be the equivalence class of deformations of $X$ with base $S$, i.e., the equivalence class of all flat morphisms $X \rightarrow S$ such that there is a cartesian diagram

\[
\begin{array}{ccc}
X & \rightarrow & X' \\
\downarrow & & \downarrow \\
\ast & \rightarrow & S
\end{array}
\] (3.1)

Then, $C$ and $D$ are fibered groupoids over $\mathfrak{An}$ and we define a morphism $G : C \rightarrow D$ as follows: For $\delta \in C(S)$, let $X$ be the space germ with $O_X = H^0(R_S, \delta + s)$ and $X \rightarrow S$ the composition of the closed embedding $X \rightarrow S \times P$ and the canonical projection $S \times P \rightarrow S$. Obviously, there is a cartesian diagram (3.1). I.e. $G(\delta) := X \rightarrow S$ is a deformation of $X$. We want to remind the proof of the well-known fact that $G$ is smooth.

Let $(A, m)$ be a local analytic algebra, $B$ a graded, g-finite free $A$-algebra and $C$ a flat DG-algebra over $A$. For $A$-modules $M$, we set $M' := M \otimes_A A/m$. The following statement is a special case of Proposition 8.20 in Chapter I of [1]:

**Proposition 3.1.** Let $\nu' \in \text{Der}_{B'}(B', B')$ be a differential and $\phi' : B' \rightarrow C'$ a surjective quasi-isomorphism of DG-algebras over $A'$. Then, there is a differential $\nu \in \text{Der}_B(B, B)$, lifting $\nu'$ and a surjective quasi-isomorphism $\phi : B \rightarrow C$ of DG-algebras, lifting $\phi'$.

**Corollary 3.2.** For all $S$ in $\mathfrak{An}$, $G(S) : C(S) \rightarrow D(S)$ is surjective.

**Proof.** For $X \rightarrow S$ in $D(S)$, we have to find a $O_S$-derivation $\delta : R_S \rightarrow R_S$ of degree 1 with $\delta(0) = 0$ such that $\delta + s$ is a differential and a surjective quasi-isomorphism $(R_S, \delta + s) \rightarrow O_X$. Since $R_S \otimes_{O_S} C = R$ and $O_X \otimes_{O_S} C = O_X$, the existence follows by Proposition 3.1.

**Corollary 3.3.** $G$ is smooth.

**Proof.** We have to show that for each $\delta \in C(S)$ and each morphism

\[
\begin{array}{ccc}
X := V(S \times P, \delta + s) & \rightarrow & X' \\
\downarrow & & \downarrow \\
S & \rightarrow & S'
\end{array}
\]

of deformations of $X$, there exist $\delta' \in C(S')$ such that $G(\delta') = X'$ and a cartesian diagram

\[
\begin{array}{ccc}
(R_S', \delta' + S) & \rightarrow & (R_S, \delta + s) \\
\downarrow & & \downarrow \\
O_{S'} & \rightarrow & O_S
\end{array}
\]
Setting $A := O_{S'}$, this follows by Proposition 3.1

In the literature (see [1], for instance), the deformation functor is defined such that a space germ $S$ maps to the quotient of $C(S)$ by the Lie group, associated to the Lie algebra $\text{Der}^0(R_S, R_S)$. In fact, $G$ factors through this quotient and the first factor is even "minimal smooth". For the construction here, we don’t need to consider this group action to get semi-universal deformations. One can say that the group action is replaced by the going - over to a minimal model.

4 A formal semi-universal deformation

In this section, we apply the new method for the construction of a formal semi-universal deformations to isolated singularities $X$. Let $(M, Q^M) := F(X)$ be the formal DG-manifold in $\text{DG-Manf}^M G$, assigned to the space germ $X$. As in Section 2 denote the resolvent of $A = O_X$, having $S(M)$ as completion, by $(R, s)$.

By Theorem 5.13 of [9], there is a semiuniversal deformation $(V, Q^V, Q)$ of $(M, Q^M)$. Recall that as graded modules $V = H[1]$, where $H$ denotes the cohomology of $\text{Coder}(S(M), S(M))$, i.e. the tangent cohomology of $X$. It is well-known that $H$ is $g$-finite.

We apply the functor $V$ to the morphism $(V \times M, Q^V + Q^M + Q) \rightarrow (V, Q^V)$ and get a morphism $Y \rightarrow Y$ in $\text{Anf}$.

**Theorem 4.1.** The morphism $Y \rightarrow Y$ is a formal semi-universal deformation of the space germ $X$.

**Proof.** Let

\[
\begin{array}{c}
\tilde{X} \\
\downarrow \\
S \\
\end{array}
\quad
\begin{array}{c}
X \\
\downarrow \\
S \\
\end{array}
\]

be any formal deformation of $X$. By Corollary 3.3 there is a morphism of the deformation $X \rightarrow S$ to an embedded deformation $\tilde{X} \rightarrow S$, where $\tilde{X}$ is such that $C_{\tilde{X}} = H^0(R_S, s + \delta)$, for a certain $\delta \in C(S)$ (see Section 3). I.e. there is a cartesian diagram

\[
\begin{array}{c}
\tilde{X} \\
\downarrow \\
S \\
\end{array}
\rightarrow
\begin{array}{c}
X \\
\downarrow \\
S \\
\end{array}
\]

Set $(B, Q^B) := F(S)$. By Proposition 2.1 to $\delta$, there corresponds a coderivation $Q_\delta$ in $\text{Coder}^{+1}(S(B \times M), S(B \times M))$, defining a deformation $(B, Q^B, Q_\delta)$ of $(M, Q^M)$. Since $(V, Q^V)$ is semi-universal, there is a morphism

\[
\begin{array}{c}
(B \times M, Q^B + Q^M + Q_\delta) \\
\downarrow \\
(B, Q^B) \\
\end{array}
\rightarrow
\begin{array}{c}
(V \times M, Q^V + Q^M + Q) \\
\downarrow \\
(V, Q^V) \\
\end{array}
\]
of deformations. Application of the functor $V$ gives a cartesian diagram

$$
\begin{array}{ccc}
\tilde{X} & \rightarrow & Y \\
\downarrow & & \downarrow \\
S & \rightarrow & Y
\end{array}
$$

which obviously respects the distinguished fiber $X \rightarrow \ast$. This shows that $\mathcal{Y} \rightarrow Y$ is versal. Since $Y$ is a formal analytic subgerm of $V^0 = H^1$, we have $\dim(TY) \leq \dim H^1$. Thus, necessarily $\mathcal{Y} \rightarrow Y$ is semi-universal (see Chapter 2.6 of [8]).

□

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