Elementary SU(2) Operations to Manipulate the Entropy in Large 2-Level Quantum Systems

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Abstract. SU(2) operations are recurrently required in quantum information because of their simplicity. The SU(2) formalism states that quantum systems can be controlled easily, step by step, by means of such operations if convenient vector bases (normally composed by entangled states) are used as grammar. In this work, we explore the entanglement gained under such operations for the classes GHZ and W, as two well known representatives of maximal entangled states for larger systems.

1. Introduction
In Quantum Information, it is worth recommendable to decompose the evolution of a large number of qubits in terms of SU(2) operations. Such fact is due to the well-known results about the universality of two-qubit processing and the existence of universal sets of operations on such systems [1]. Several decompositions for the processing between two qubits have been proposed as cosine-sine decomposition [2] and other convenient unitary factorizations [3–8]. As instance, high-tech commercial appliances as D-Wave and IBM-Q both use qubits in the form of two-level systems (superconducting circuits or ions with an interconnected architecture). Particularly, SU(2) decomposition [9] exploits the possibility to express the quantum evolution for more complex systems composed by several two-level subsystems on more natural bases according to their dynamics. The quest for such decomposition is then be able to reach any arbitrary state (entangled or not) for the complex system from the simplest two-level representations (or vice versa). Here, the entanglement and their full comprehension become central.

The aim of this article is to set some ideas about the connectivity among arbitrary states under the analysis of the SU(2) decomposition for the dynamics (introduced first for bipartite systems [10] and then generalized in [9]). Second section resumes and formalizes the SU(2) decomposition. Third section reviews some findings for the SU(2) reduction for systems including Ising-Heisenberg and Dzyaloshinskii-Moriya interactions. Fourth section reviews some universal basic operations able to be settled there together with examples connecting separable states with strong-entangled states. Fifth section discusses the entanglement relations in such exemplary connections. Last section sets the conclusions.

2. SU(2) decomposition formalism
The SU(2) decomposition is achievable for any system allowing an even number of non-degenerate energy eigenstates \(|b_j\rangle \in \mathcal{H}^{2N}|j = 1, \ldots, 2N\rangle\), with \(\{E_i\}\) the corresponding
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eigenvalues. Then, one can set \( N \) pairs of new orthogonal states via rotations (constructed from selected pairs of the energy eigenstates \([9]\) \( \{|\alpha_i\rangle\} \), and \( N \) pairs of labels \( \{j(i), k(i)\}, i = 1, 2, ..., N \), with \( k(i) = j(i) + 1 \in \{2, 4, ..., 2N\} \), such that

\[
|\alpha_j(i)\rangle = A_i^+ |b_{2i-1}\rangle - B_i e^{-i\phi} |b_{2i}\rangle, \quad |\alpha_k(i)\rangle = B_i^* |b_{2i-1}\rangle + A_i e^{-i\phi} |b_{2i}\rangle,
\]

where \( |A_i|^2 + |B_i|^2 = 1 \). Each pair forms one of the orthogonal subspaces conforming the entire Hilbert space of the system:

\[
\mathcal{H}_i^2 = \text{span}(\{|b_{2i-1}\rangle, |b_{2i}\rangle\}) = \text{span}(\{|\alpha_j(i)\rangle, |\alpha_k(i)\rangle\}) \to \mathcal{H}_i^{2N} = \bigoplus_{i=1}^N \mathcal{H}_i^2.
\]

Such decomposition permits to express the Hamiltonian as a \( 2 \times 2 \)-blocks matrix. The structure is then inherited to the evolution matrix (here, rows and columns have been rearranged to form blocks with consecutive entries),

\[
H = \bigoplus_{i=1}^N S_{Hi} = \left( \begin{array}{ccc}
S_{H1} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & S_{HN}
\end{array} \right) \quad \to \quad U = \bigoplus_{i=1}^{2N} S_{Ui} = \left( \begin{array}{ccc}
S_{U1} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & S_{UN}
\end{array} \right),
\]

where each pair of blocks \( S_{Hi}, S_{Ui} \) obeys the Schrödinger equation

\[
S_{Hi}S_{Ui} = i\hbar \frac{\partial S_{Ui}}{\partial t}.
\]

Therefore, the dynamics is decomposed into the dynamics of \( N \) artificial two-level systems the evolution of which may be visualized on \( N \) Bloch spheres; each one generated by a pair \( \alpha_j(i), \alpha_k(i) \) \([9]\). In addition, using \( A_i = r_{A_i} e^{i\gamma_{A_i}}, B_i = r_{B_i} e^{i\gamma_{B_i}}, \Delta_1^i = \frac{1}{\sqrt{2}}(E_{2i} \pm E_{2i-1}) \), and \( \Gamma_i = \gamma_{A_i} - \gamma_{B_i} \), the blocks can be expressed as

\[
S_{Hi} = \Delta_1^i I_i - 2r_{A_i} r_{B_i} \Delta_1^- \cos \Gamma_i X_i + 2r_{A_i} r_{B_i} \Delta_1^- \sin \Gamma_i Y_i - (r_{A_i}^2 - r_{B_i}^2) \Delta_1^- Z_i
\|
\equiv \Delta_1^i I_i + S_{Hi}^0,
\]

\[
S_{Ui} = e^{-i\Delta_1^i t} \left( \cos \Delta_1^- t I_i + 2ir_{A_i} r_{B_i} \cos \Gamma_i \sin \Delta_1^- t X_i - 2ir_{A_i} r_{B_i} \sin \Gamma_i \sin \Delta_1^- t Y_i \right)
\]

\[+i(r_{A_i}^2 - r_{B_i}^2) \sin \Delta_1^- t Z_i \right) \equiv e^{-i\Delta_1^i t} S_{Ui}^0,
\]

where \( \{I_i, X_i, Y_i, Z_i\} \) is the extended Pauli basis of \( U(2) \) (including the identity) on each \( \mathcal{H}_i^2 \) space using \( \{\alpha_j(i), \alpha_k(i)\} \) as vector basis. Last expressions show that \( S_{Hi}^0, S_{Ui}^0 \) themselves satisfy (4), denoting \( S_{Ui} \in U(1) \times SU(2), S_{Ui}^0 \in SU(2) \).

3. Formalism applied to Ising-Heisenberg and Dzyaloshinskii-Moriya interactions

Consider a chain of \( 2d \) qubits arranged in \( d \) correspondent pairs (each one formed by qubits \( i \) and \( i + d, i = 1, 2, ..., d \) weakly interacting through Ising-Heisenberg (IH) or Dzyaloshinskii-Moriya (DM) interactions together with some local operations involving local magnetic fields on certain qubits. In \([9]\), it has been proved such system posseses a \( SU(2) \) decomposition with \( N = 2^{2d-1} \) on the generalized Bell states (GBS) basis, defined as:

\[
|\Psi_{x_i}\rangle = \bigotimes_{s=1}^d |\Psi_{i_s}\rangle
\]
\( i_s = I^d_{I^d} \), the \( s \)-digit of number \( I \in \{0, 1, \ldots, 4^d - 1\} \) written in base-4 with \( d \) digits, and \( \{\Psi_{2i+j}\} = |\beta_{i,j}\rangle \) are the Bell states for the \( s \)-pair of qubits with \( s = 1, 2, \ldots, d \). In the following, we will use \( |\Psi_{I_s}\rangle \) indistinctly to \( |\Psi_I\rangle \) \( (I \) in base-10). All concrete interactions in the chain could include all possible spatial directions \( (j = 1, 2, 3) \) of 2-local III interactions between the qubits of each correspondent pair \( k' = 1, 2, \ldots, d \) \( (H^k_{ND}) \) together with:

- **Type I** (2-local interaction): it admits one 1–local interaction on the qubits of pair \( k' \) using a magnetic field in the direction \( j' \) \( (H^j_{ND}) \).
- **Type IIa** (4-local interaction): it admits one 2–local IH interaction between the qubits of two pairs \( k', k'' \) in the direction \( j' \) \( (H^k_{ND}) \).
- **Type IIb** (4-local interaction): it admits one 2–local DM interaction between the qubits of two pairs \( k', k'' \) in the direction \( j' \) and parity \( p \) \( (H^k_{ND}) \).
- **Type III** (2-local interaction): it admits a 2–local DM interaction between the qubits of pair \( k' \) in the direction \( j' \) with all possible parities \( (H^k_{ND}) \).

Here, \( D \) and \( ND \) subscripts indicate Hamiltonians generating diagonal and non-diagonal entries respectively. Direction \( j' \) in DM means the interaction terms include spins in the directions \( j'' \) and \( j''' \) of each qubits in the two groups being considered (non-correspondent or correspondent respectively), being \( (j', j'', j''') \) a permutation from \( (1, 2, 3) \) with parity \( p \). The locality indicated in each interaction considers the previous entanglement between the qubits in each pair \( [9] \). In addition, it has been proved that each interaction generate respectively only 2, 8, 8 and 2 different and independently blocks through the whole evolution matrix \( U(t) \) depending on the exchanges between the scripts in base-4 of those basis elements \( |\Psi_{I_s}\rangle \) being related in each block:

(a) \((0,j') \leftrightarrow (j',0), (j'',j''') \leftrightarrow (j''',j'')\) for Types I and III (involving only one digit in \( I_s^d \)).

And

b) Similarly, the eight different combinations for Types IIa and IIb (involving two digits in \( I_s^d \)) \([9]\).

### 4. Universal elementary \( SU(2) \) operations to reproduce large entangled states

Operations previous presented develop certain basic entanglement in the chain. While Type I and III are able to modify the entanglement in the selected correspondent pair \( k' \), instead Type IIa and IIb interactions are 4–local, entangling the pairs \( k', k'' \). We will consider such latter operations mixing a couple of basis elements \( |\psi_I\rangle \) and \( |\psi_J\rangle \) both related through the rules settled in the previous section into \( |\phi_{IJ}\rangle = \cos(\theta/2) |\Psi_I\rangle + e^{i\phi} \sin(\theta/2) |\Psi_J\rangle \). In addition, we use the generalized bipartite concurrence for pure states \( C^2(Tr^S(\rho_{IJ})) = 2(1 - Tr^S(\rho_{IJ}^2)) \) \([11]\) (running from 0 for separable states to 2 \((m-1)/m\) for maximally entangled ones, where \( m = \min(m_1, m_2) \), being \( m_1, m_2 \) the Hilbert space dimensions of each subsystem. The partial trace is taken over all parts except \( s \in S \) with \( \rho_{IJ} = |\phi_{IJ}\rangle \langle \phi_{IJ}| \). Thus, using \( \phi' = \phi + \phi'_{ij} - \phi'_{jk} \) if \( \phi_i = \delta_{i,2}\pi \), and \( i_k = I^d_{S^d} \) (recall that \( j, k, \) and \( k' \) are parameters related with the 2 and 4–local operations), one has \([9]\):

- **Type I** and **III** generates the entanglement \( C^2(Tr^S(\rho_{IJ})) = 1 - \sin^2 \theta (\cos \phi'\delta_{i\nu,j\nu} + (-1)^i\delta_{i\nu,j\nu}) \) on \( S = \{s\} \) with \( s \in \{k', k'' + d\} \)

- **Type IIa** and **IIb** generates the entanglement \( C^2(Tr^S(\rho_{IJ})) = \frac{3}{2} - \frac{1}{2} \sin^2 \theta (\cos^2 \phi'\delta_{i\nu,j\nu} + \sin^2 \phi'(1 - \delta_{i\nu,j\nu} \delta_{j\nu,i\nu})) \) on \( S = \{k', k''\} \)
We bear out the maximum entanglement is 1 and 3/2 respectively.

5. Behavior of entanglement entropy for large entangled characteristic states

In particular, it is shown that specific groups of transformations generate recursively the states \(|GHZ\rangle^{2d}\) and \(|W\rangle^{2d}\) for arbitrary \(d\) (indicated by the subscript) departing from the single Bell states \(|\beta_0\rangle^{2d}\) and \(|\beta_1\rangle^{2d}\) respectively:

\[
|GHZ\rangle^{2d} \quad T_{GHZ} \quad |GHZ\rangle^{2(d+1)}
\]

\[
|W\rangle^{2d} \quad T_{W} \quad |W\rangle^{2(d+1)}
\]

They can be written in the GBS as [9]:

\[
|GHZ\rangle^{2d} = \frac{1}{\sqrt{2}} \sum_{i=0}^{d} \sum_{j=1}^{d} \prod_{k=1}^{d} \left( |\Psi_0\rangle_j + (-1)^i |\Psi_3\rangle_j \right),
\]

\[
|W\rangle^{2d} = \frac{1}{\sqrt{2}^d} \prod_{i=1}^{d} \prod_{j=1}^{d} \left( \delta_{i,j} - 1 \right) \prod_{i=1}^{d} \left( |\Psi_0\rangle_j + |\Psi_3\rangle_j \right) \otimes |\Psi_1\rangle_i
\]

the technical details for \(T_{GHZ}\) and \(T_{W}\) are given in [9] but Figure 1 depicts synthetically such processes. Additional local operations achievable with Type I and III Hamiltonians let to transform \(|GHZ\rangle^{2d}\) and \(|W\rangle^{2d}\) into any states in the \(GHZ\)-class and \(W\)-class [12].

Despite \(\lim_{n \to \infty} S_{n}(|\psi\rangle) = S_{\text{VNE}}(|\psi\rangle)\), the Von-Neumann entropy, in this work we use \(\alpha = 2\). For a \(N\)-qubit state, \(S_{\alpha}^{RUI}(|\psi\rangle) \leq \ln(2^N - N(d-1)/2)\) and \(S_{\alpha}^{RUI}(|\psi\rangle) = 0\) only if \(|\psi\rangle\) is separable [13].
Figure 1. Recursive process based on $SU(2)$ 2– and 4–local operations to generate large entangled states, a) $|GHZ\rangle^{2d}$, and b) $|W\rangle^{2d}$.

Figure 1 depicts the recursive process detailed in [9] based on the set of Hamiltonians depicted there. The initially 2–separable state being integrated is $|\Psi_0\rangle_{d+1}$ (the subscript depicts the pair being added; note this state is achievable with Type I or Type III interactions from more basic separable resources). It is notable that few operations are required to obtain $|GHZ\rangle^{2(d+1)}$ compared with the series of detailed operations to get $|W\rangle^{2(d+1)}$ [9]. In addition, operations to get $|GHZ\rangle^{2(d+1)}$ involves centrally only one 4–local operation involving two pairs to become entangled, while $|W\rangle^{2(d+1)}$ requires $d$ of them to make entangling operations through the entire original pairs [9]. Those processes show why the entanglement in $|W\rangle^{2d}$ states is stronger than the $|GHZ\rangle^{2d}$ one under qubit losses due to the number of operations among the original qubits with the newest qubit being added.

In the following, to quantify the entanglement of states, we use the Rényi-Ingarden-Urbanik entropy [13]:

$$S_{\alpha}^{RIU}(|\psi\rangle) = \min_{U_{loc}} S_{\alpha}(U_{loc}|\psi\rangle)$$

being $U_{loc}$ any tensor product of local operations on each qubit and $S_{\alpha}(|\psi\rangle)$ the Rényi entropy of $|\psi\rangle = \sum_{i=1}^{2^N} \alpha_i |i\rangle$ (with $p_i = |\alpha_i|^2$):

$$S_{\alpha}(|\psi\rangle) = \frac{1}{1-\alpha} \ln \sum_{i=1}^{2^N} p_i^\alpha$$

Despite $\lim_{\alpha \to 1} S_{\alpha}(|\psi\rangle) = S_{VN}(|\psi\rangle)$, the Von-Neumann entropy, in this work we use $\alpha = 2$. For a $N$–qubit state, $S_{\alpha}^{RIU}(|\psi\rangle) \leq \ln(2^N - Nd(d-1)/2)$ and $S_{\alpha}^{RIU}(|\psi\rangle) = 0$ only if $|\psi\rangle$ is separable [13]. Thus, the $S_{\alpha}^{RIU}$ entropy in the steps of the processes depicted in the Figure 1 (without intermediate steps) can be quantified. For $|GHZ\rangle$ states, $S_2^{RIU} = \ln(2)$ for all $d \geq 1$. While, for $|W\rangle$ states, $S_2^{RIU} = \ln(2d)$ for all $d \geq 1$ (that is clear because the forms of such states are well-known in the computational basis). Despite, some intermediate steps could reach states with a larger entropy (particularly for $|GHZ\rangle$ states due to the 4–local intermediate operations). Clearly, due to definition (16), all states in the $GHZ$–class and $W$–class have respectively the same entropy that $|GHZ\rangle^{2d}$ and $|W\rangle^{2d}$ (for each $d$).
The last aspect can be analysed because there is a relation between $|GHZ\rangle^4$ and $|W\rangle^4$ \cite{9} in terms of the $SU(2)$ operations. Such process is depicted in Figure 2, where a sequence of $SU(2)$ operations carries out a separable state $|0000\rangle$ onto $|W\rangle^4$, crossing by $|GHZ\rangle^4$.

Figure 2. Process based on $SU(2)$ 2– and 4–local operations transiting from $|GHZ\rangle^4$ to $|W\rangle^4$.

Figure 3 resumes the behavior of $S_{2RIU}^{\alpha}$ in the last two cases. For the process depicted in the Figure 1a, the orange dashed line shows the constant $S_{2RIU}^{\alpha}$ entropy for the recursive construction of $|GHZ\rangle^{2d}$ states ($d \geq 1$), despite intermediate steps (not shown) increase temporarily the entropy in order to integrate new quantum resources. While, the green dashed line shows the increase of $S_{2RIU}^{\alpha}$ (in agreement with Figure 2b) in the recursive construction of larger $|W\rangle^{2d}$ states ($d \geq 1$).

Figure 3. Behavior of $S_{2RIU}^{\alpha}$ in the processes of construction of $|W\rangle^4$ departing from a separable state and crossing by $|GHZ\rangle^4$ (solid line, showing the intermediate steps) and the recursive construction of $|GHZ\rangle^{2d}$ and $|GHZ\rangle^{2d}$ (dashed line).
For the case depicted in the Figure 2 (showing the intermediate steps), the change of the entropy in the construction of $|W\rangle^4$ state is shown in Figure 3 as a unique and sequential process (orange and green solid line) departing from the separable $|0000\rangle$ state and crossing by $|GHZ\rangle^4$. Note in the intermediate steps how entropy raises momentarily while $|GHZ\rangle^4$ is reached. After, the entropy raises again until it reaches the $|W\rangle^4$ state entropy.

6. Conclusions

Entanglement quantification is still an unclear physical and mathematical measure about which we known few. It should comprise the complexity of quantum states in terms of their quantum correlations which involves each possible subsystem of them.

Due to the $SU(2)$ decomposition proposes a basic settlement for the control and understanding of larger quantum systems in terms of simplest ones: two-level systems, it automatically provides natural simpler $SU(2)$ operations on a convenient grammar of states in agreement with their physical nature and their possible entangling interactions. Thus, such interactions occurring by pairs (as well as we know in the universe) can be analysed in a programmed way to understand the entanglement departing from the tiniest quantum correlations among the most basic individual subsystems until increasingly to reach strong quantum correlations on the entire composed system.

For the case involving Ising-Heisenberg and Dzyaloshinskii-Moriya interactions, the grammar of $SU(2)$ decomposition is well known in the form of generalized Bell states. Then, on it, basic entangling operations can be defined to then produce extended entanglement. Thus, it has been addressed to shown how such operations can manage the raising of the entropy by combining 2 and 4−local operations.

In the current work, we revisited the construction of the representative states $|GHZ\rangle$ and $|W\rangle$ for larger systems in terms of their entropy approached with the $S_{RIU}^2$ entropy measure. The recursive construction developed through the basic $SU(2)$ operations exhibits how the measurement of entanglement is related with the basic quantum correlations being introduced to reach each state.

Despite $GHZ$−class and $W$−class (of dimension $2d$) contains lots of quantum states sharing their entropy with $|GHZ\rangle^{2d}$ and $|W\rangle^{2d}$ respectively (all of them obtained from those remarkable states through 1−local operations), the problem of raising (or reducing) the entropy and manipulate general states with $SU(2)$ operations departing from separable states is still open. In such terms, the analysis presented shows a concrete behavior of entanglement.

While $|W\rangle^{2d}$ states promise to be a starting point for the preparation of other general states, it contrasts with the constant $S_{RIU}^2$ entropy of $|GHZ\rangle^{2d}$, which exhibits the limitation of our entanglement measures to show clearly the full complexity of entanglement by noting how the inclusion of new entangled resources into $|GHZ\rangle^{2d}$ does not increase the value of the measure. In any case, those examples illustrate how their construction requires the fine manipulation of them, raising and lowering such restricted measures only as a partial aspect of the entanglement preparation.

References
[1] Boykin P O, Mor T, Pulver M, Roychowdhury V and Vatan F 1999 On universal and fault tolerant quantum computing Preprint quant-ph/9906054
[2] M"ott"onen M, Vartiaienen J J, Bergholm V and Salomaa M M 2004 Phys. Rev. Lett. 93 130502
[3] Knill E 1995 Approximation by Quantum Circuits Preprint quant-ph/9508006v1
[4] Vatan F and Williams C 2004 Phys. Rev. A 69 032315
[5] Vidal G and Dawson C 2004 Phys. Rev. A 69 010301.
[6] Urias J 2010 J. Math. Phys. 51 072204
[7] Li C K, Jones R and Yin X 2013 Int. J. Quantum Inf. 11 1350015
[8] Delgado F 2017 Quantum Inf. Comput. 17 0721
[9] Delgado F 2018 *Entropy* 20 610
[10] Delgado F 2015 *Int. J. Quantum Inf.* 13 1550055
[11] Uhlmann A 2016 *Sci. China Phys. Mech. Astron.* 59 630301
[12] Wang A 2004 Generalized GHZ-class and W-class concurrence and entanglement vectors of the multipartite systems consisting of qubits *Preprint* quant-ph/0406114v3
[13] Enríquez M, Puchała Z and Życzkowski K 2015 *Entropy* 17 5063