Energy and angular momentum radiated for non head-on binary black hole collisions

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Abstract

We investigate the possible total radiated energy produced by a binary black hole system containing non-vanishing total angular momentum. For the scenarios considered we find that the total radiated energy does not exceed 1%. Additionally we explore the gravitational radiation field and the variation of angular momentum in the process.

After the formation of the final black hole, the model uses the Robinson-Trautman (RT) spacetimes as background. The evolution of perturbed RT geometries is carried out numerically.

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1 Introduction

The advent of powerful detectors capable of directly measuring gravitational radiation for the first time has motivated numerous investigations of systems likely to produce gravitational waves of enough intensity which are expected to be observed with these detectors. Among those systems, a prime candidate is that composed by a couple of inspiraling black holes which eventually merge into one releasing considerable amounts of energy via gravitational radiation. Clearly, because of the strong gravitational fields involved in such process, its complete description requires solving Einstein equations in their full generality which can only be done through numerical methods\textsuperscript{[1]}. Several groups are combining efforts to numerically model such system\textsuperscript{[2,3,4,5,6,7]} and although significant progress has been (and is being) made in this direction (see for instance\textsuperscript{[8,9]}), an accurate and robust implementation is still missing. In the mean time, valuable insight can be gained through approximate models of the system. These models serve both as means to gain a better understanding of the problem and also to provide information that can serve as tests for the numerical simulations, where the lack of known solutions renders the problem of accuracy assessment more involved.

Traditional models are obtained through perturbative approximations where an expansion with respect to some appropriately chosen parameter provides a reduced and manageable system of equations. Naturally, these approaches provide accurate answers only when the perturbative parameter remains small. For instance, the traditional post-Newtonian approximation (see for instance\textsuperscript{[10,11]}), can be safely used when the black holes are far enough and the relative speeds involved are much smaller than 1. However, as the holes come closer, the gravitational fields and relative speeds involved become very large and it is clear that this approximation will fail to give sensible answers. Resumation techniques\textsuperscript{[10]} and/or effective one-body expansions\textsuperscript{[11]} have been proposed to extend the regime of validity of this approach. However, even when the ambiguities proper of these options can be satisfactorily addressed, they will inevitably break in the late stages of the merger, which for the case of black holes lies in the maximum sensitivity window of earth-based detectors\textsuperscript{[2,3]}.

Another approximation, known as the close limit approximation (CLA) (see for instance\textsuperscript{[4,5]}) assumes the black holes have already merged and the perturbation parameter can be considered to be how non-Schwarzschild (or non-Kerr) the hole is. Since no-hair theorems imply the final fate of the formed black hole should be of the Kerr-type, this approach can safely be used to describe such epoch (note however, that in certain situations, the CLA has been able to produce valuable results at rather earlier times when the holes had just merged; however, it is not clear that this will hold in generic cases).

A different approximate model can be obtained by

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null tetrad ($\ell^a, m^a, \overline{m}^a, n^a$) where:
\[
g_{ab} \ell^a n^b = -g_{ab} m^a \overline{m}^b = 1
\]
with all other possible scalar products being zero; then, the metric can be expressed by
\[
g_{ab} = \ell_a n_b + m_a \overline{m}_b - m_a m_b.
\]
In terms of the coordinate system $(x^0, x^1, x^2, x^3) = (u, r, (\zeta + \xi), \frac{1}{2}(\zeta - \xi))$, where $u$ is a null coordinate and $r$ is an affine parameter along the geodesic integral lines of the vector $\ell^a$, one can express the null tetrad in terms of its components by the relations:
\[
\ell_a = (du)_a
\]
\[
\ell^a = \left( \frac{\partial}{\partial r} \right)^a
\]
\[
m^a = \xi^i \left( \frac{\partial}{\partial x^i} \right)^a
\]
\[
\overline{m}^a = \xi^i \left( \frac{\partial}{\partial \bar{x}^i} \right)^a
\]
\[
n^a = U \left( \frac{\partial}{\partial r} \right)^a + X^i \left( \frac{\partial}{\partial x^i} \right)^a
\]
with $i = 0, 2, 3$ and $\zeta = \frac{1}{2}(x^2 + i x^3)$; and where the components $\xi^i$, $U$ and $X^i$ are given by the following expressions:
\[
\xi^0 = 0,
\]
\[
\xi^2 = \frac{\xi_0^2}{r} + \lambda^2 \xi_0^2 \left( -\frac{\sigma_0}{r^2} + \frac{1}{r} \frac{\partial^2 W_0}{\partial r^2} - \frac{2}{r^2} \frac{\partial W_0}{\partial r} \right),
\]
\[
\xi^3 = \frac{\xi_0^3}{r} + \lambda^3 \xi_0^3 \left( -\frac{\sigma_0}{r^2} + \frac{1}{r} \frac{\partial^2 W_0}{\partial r^2} - \frac{2}{r^2} \frac{\partial W_0}{\partial r} \right)
\]
\[
\xi_0^2 = \sqrt{2} P_0 V, \quad \xi_0^3 = -i \xi_0^2;
\]
\[
U = r U_{00} + U_0 + \frac{U_1}{r} + \frac{U_2}{r^2} + \Delta U_3,
\]
\[
U_{00} = \frac{\dot{V}}{V},
\]
\[
U_0 = -\frac{1}{2} K V,
\]
\[
U_1 = -\frac{1}{2} \left( \bar{\Phi}_2 + \overline{\Phi}_2 \right),
\]
\[
U_2 = \frac{\lambda}{r^2} \left( \bar{\Phi}_{12} + \overline{\Phi}_{12} \right),
\]
\[
\Delta U_3 = -\frac{\lambda}{r^2} \left( \bar{\Phi}_{12}^2 W_0 + \overline{\Phi}_{12}^2 \bar{W}_0 \right);
\]
\( X^0 = 1, \)
\( X^2 = \lambda \xi^2_{[2]} \left( \frac{-\tau_0}{r^2} + \frac{2\bar{\psi}_0}{3r^3} + \frac{2}{r^2} \partial_{\nu_{\text{RT}}} \bar{W}_0 \right) - \frac{4}{r^3} \bar{\delta}_{\nu_{\text{RT}}} \overline{\bar{W}_0} \)  
\[ + \text{c.c.} \]
\[ X^3 = \lambda \xi^3_{[2]} \left( \frac{-\tau_0}{r^2} + \frac{2\bar{\psi}_1}{3r^3} + \frac{2}{r^2} \partial_{\nu_{\text{RT}}} \bar{W}_0 \right) - \frac{4}{r^3} \bar{\delta}_{\nu_{\text{RT}}} \overline{\bar{W}_0} \)  
\[ + \text{c.c.} \]

where
\[ \tau_0 = \bar{\delta}_{\nu_{\text{RT}}} \sigma_0, \]  \hspace{1cm} (13)
c.c means complex conjugate,
\[ K_V = \frac{2}{\bar{V}} \bar{\delta}_{\nu_{\text{RT}}} \bar{V} - \frac{2}{\bar{V}} \partial_{\nu_{\text{RT}}} \bar{V} \bar{V} + V^2; \]  \hspace{1cm} (14)
and where in these equations we are explicitly denoting the first order terms by introducing the first order parameter \( \lambda \) dependency, and where \( \bar{\delta}_{\nu_{\text{RT}}} \) is the edth operator [25, 26, 27], in the GHP notation [28], of the sphere with metric
\[ dS^2 = \frac{1}{r^2} \, dr^2 + \bar{d} \xi \, d\bar{\xi}, \]  \hspace{1cm} (15)
where \( P = V(u, \zeta, \bar{\zeta})P_0(\zeta, \bar{\zeta}) \), and \( P_0 \) is the value of \( P \) for the unit sphere.

The scalar \( V \) is given by
\[ V = V_{\text{RT}} + \lambda V_\lambda; \]  \hspace{1cm} (16)
where \( V_{\text{RT}} \) is the RT scalar satisfying the Robinson-Trautman equation
\[ -3M \dot{V}_{\text{RT}} = V_{\text{RT}}^4 \bar{\delta}^2 \bar{\delta} V_{\text{RT}} - V_{\text{RT}}^3 \bar{\delta}^2 V_{\text{RT}} \bar{\delta}^2 V_{\text{RT}}, \]  \hspace{1cm} (17)
\( V_\lambda \) is the linear perturbation scalar and \( \bar{\delta} \) is the edth operator of the unit sphere. One can then express \( K_V \) by
\[ K_V = K_{V_{\text{RT}}} + \lambda \, K_{V_\lambda}, \]  \hspace{1cm} (18)
where
\[ K_{V_{\text{RT}}} = \frac{2}{V_{\text{RT}}} \bar{\delta}_{V_{\text{RT}}} \bar{V}_{\text{RT}} V_{\text{RT}}, \]
\[ - \frac{2}{V_{\text{RT}}^2} \bar{\delta}_{\nu_{\text{RT}}} V_{\text{RT}} \bar{V}_{\text{RT}} V_{\text{RT}} + V_{\text{RT}}^2, \]  \hspace{1cm} (19)
and
\[ K_{V_\lambda} = \frac{2}{V_{\text{RT}}} \bar{\delta}_{V_{\text{RT}}} \bar{V}_{\text{RT}} V_\lambda - \frac{2}{V_{\text{RT}}^2} \bar{\delta}_{\nu_{\text{RT}}} V_\lambda \bar{V}_{\text{RT}} V_{\text{RT}} \]
\[ - \frac{2}{V_{\text{RT}}^2} \bar{\delta}_{\nu_{\text{RT}}} V_{\text{RT}} \bar{V}_{\text{RT}} V_\lambda + \frac{2V_\lambda}{V_{\text{RT}}} \bar{\delta}_{\nu_{\text{RT}}} V_{\text{RT}} \bar{V}_{\text{RT}} V_{\text{RT}} \]
\[ + V_\lambda V_{\text{RT}} + \frac{V_\lambda}{V_{\text{RT}}} K_{V_{\text{RT}}}. \]  \hspace{1cm} (20)

In the above equations we have \( \sigma_0 = \sigma_0(u, \zeta, \bar{\zeta}) \), \( W_0 = W_0(u, r, \zeta, \bar{\zeta}) \), \( \Psi_1^0 = \Psi_1^0(u, \zeta, \bar{\zeta}) \) and
\[ \Psi_2^0 = \Psi_2^0(u, \zeta, \bar{\zeta}), \]
\[ = \left( M_0 + \lambda \left( M_1(u, \zeta, \bar{\zeta}) + i \mu(u, \zeta, \bar{\zeta}) \right) \right); \]
\hspace{1cm} (21)
where \( \mu \) is related to \( \sigma_0 \) by
\[ \mu = \frac{1}{2i} \left( \bar{\delta}_{\nu_{\text{RT}}} \sigma_0 - \bar{\delta}_{\nu_{\text{RT}}}^2 \sigma_0 \right). \]  \hspace{1cm} (22)

In order to study the intrinsic fields at future null infinity (scri), it is convenient to consider the leading order behavior of \( W_0 \), namely
\[ W_0 = \frac{\Psi_0^0}{4r^2} + W_1; \]  \hspace{1cm} (23)
where \( \Psi_0^0 = \Psi_0^0(u, \zeta, \bar{\zeta}) \) and \( W_1 = O(1/r^2) \). It is interesting to note that the component \( \Psi_0^0 = \frac{\bar{\delta}_{\nu_{\text{RT}}}^2}{\bar{\partial}_{\nu_{\text{RT}}}} \sigma \) of the Weyl tensor is given in this case [24] by
\[ \Psi_0 = \frac{\Psi_0^0}{r^3} + \frac{\partial^4 W_1}{\partial r^4}, \]  \hspace{1cm} (24)
where the second term is of order \( O(1/r^6) \).

The remaining equations at scri are:
\[ \dot{\Psi}_0 = 3V_{\text{RT}} \frac{\bar{V}_{\text{RT}}}{V_{\text{RT}}} \dot{\Psi}_0 - \bar{\delta}_{\nu_{\text{RT}}} (M_1 + i\mu) - \bar{\delta}_{\nu_{\text{RT}}} (K_{V_{\text{RT}}}) \sigma_0, \]  \hspace{1cm} (25)
\[ \dot{\Psi}_1 = 3V_{\text{RT}} \frac{\bar{V}_{\text{RT}}}{V_{\text{RT}}} \dot{\Psi}_1 - \bar{\delta}_{\nu_{\text{RT}}} (M_1 + i\mu - \bar{\delta}_{\nu_{\text{RT}}} (K_{V_{\text{RT}}}) \sigma_0, \]  \hspace{1cm} (26)
and
\[ -6M_0 \frac{\bar{V}_{\text{RT}}}{V_{\text{RT}}} \dot{V}_{\text{RT}} + 3V_{\text{RT}} \frac{\bar{V}_{\text{RT}}}{V_{\text{RT}}} \dot{V}_{\text{RT}} - M_1 \sigma_{\text{RT}}^2 \]
\[ - 2M_1 - \frac{2}{V_{\text{RT}}} \bar{\delta}_{\nu_{\text{RT}}} \sigma_0 + 2V_{\text{RT}} \frac{\bar{V}_{\text{RT}}}{V_{\text{RT}}} \bar{\delta}_{\nu_{\text{RT}}} V_{\text{RT}} \bar{V}_{\text{RT}} \bar{\delta}_{\nu_{\text{RT}}} \sigma_0 \]
\[ - \frac{2}{V_{\text{RT}}} \bar{\delta}_{\nu_{\text{RT}}} V_{\text{RT}} \bar{V}_{\text{RT}} \bar{\delta}_{\nu_{\text{RT}}} \sigma_0 + \frac{2}{V_{\text{RT}}} \bar{\delta}_{\nu_{\text{RT}}} \sigma_0 \]
\[ - \frac{2}{V_{\text{RT}}} \bar{\delta}_{\nu_{\text{RT}}} V_{\text{RT}} V_{\text{RT}} \bar{V}_{\text{RT}} \bar{\delta}_{\nu_{\text{RT}}} \sigma_0 + \frac{2}{V_{\text{RT}}} \bar{\delta}_{\nu_{\text{RT}}} \sigma_0. \]  \hspace{1cm} (27)

The objective is to use the solutions of the previous equations to model a black hole that just formed after the non head-on collision of a previous binary system. In this model the idea is to only make use of the information of the individual masses and the total angular momentum. With all this in mind we have to chose the appropriate gauge and free functions.

In reference [23] it was discussed the gauge freedom of these spacetimes. In order not to introduce extra structure, we will chose the free functions and gauges that make
\[ M_1 = 0; \]  \hspace{1cm} (28)
where $A$ is the amplitude of the excitation, $Y_{2,0}$ is a spherical harmonic, and $(\zeta', \tilde{\zeta}')$ are the coordinates of the sphere where the pole is along the $y$ axis. Then in terms of the usual spherical harmonics with coordinates $(\zeta, \tilde{\zeta})$, one has

$$V = 1 - \frac{A}{2} \left( \sqrt{\frac{3}{2}} Y_{2,-2}(\zeta, \tilde{\zeta}) + Y_{2,0}(\zeta, \tilde{\zeta}) + \sqrt{\frac{3}{2}} Y_{2,2}(\zeta, \tilde{\zeta}) \right).$$

### 3 Fixing the parameters and the Newtonian matching

The physical system can be characterized by two stages. During the first stage two black holes are moving towards each other with some orbital angular momentum. At some moment they collide and form a single black hole; which settles down in the asymptotic future to a stationary Kerr geometry.

During the first stage of evolution we describe the gravitational radiation with the quadrupole formula, where the dynamics is worked out from the Newtonian framework. The whole motion is contained in a plane; therefore two variables are sufficient to describe the orbits. The two integrals of motion, namely energy $E$ and angular momentum $j$, allow to solve the Newtonian system.

The initial data is assumed to be given in the $(x, y)$ plane; with zero total momentum and such that the orbital angular momentum is along the positive $z$ direction.

The initial velocities can be thought to have components along the $x$ axis only. Let us call $R_0$ the initial impact parameter, at some initial relative distance $r_0$. In this way, using the relative velocity $v_0$, one has

$$E = \frac{\mu}{2} v_0^2 - \frac{Gm_1 m_2}{r_0},$$

$$j = \mu R_0 v_0;$$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass.

For this kind of motion the quadrupole formula predicts that the power of gravitational radiation is given by

$$F_Q(r) = \frac{8}{15} (Gm_1 m_2)^2 \left[ \frac{2}{\mu^4} \left( E + \frac{Gm_1 m_2}{r} \right) + \frac{11 j^2}{\mu^2 r^6} \right];$$

which generalizes the analogous equation appearing in ref. [22].

In order to continue the dynamical description after the collision of the two black holes, we need to have a merging condition in the Newtonian framework. In references [21] and [22] we have succeeded in estimating the total energy radiated in the head on black hole collision using the following criteria. When a black hole, which
mass at infinity is \( m_{i1} \), is brought to a distance \( r_{12} \) of another black hole of asymptotic mass \( m_{i2} \), its physical mass, for the stationary situation, is changed \( [3] \) to

\[
m_1 = m_{i1} + \frac{m_{i1} m_{i2}}{2 r_{12}},
\]

and similarly for the other mass.

Applying the arguments of ref. \([22]\) one concludes that the merging condition should be taken when the separation distance has the value

\[
r_c = 2 (m_1 + m_2).
\]

It is convenient to introduce the relative mass parameter \( \alpha \) and the reference mass \( m \), so that \( m_{i1} = m \) and \( m_{i2} = \alpha m \). Then the merging condition \([22]\) gives

\[
r_c = (m_1 + m_2) \left[ 1 + \sqrt{1 + \frac{2 \alpha}{(1 + \alpha)^2}} \right].
\]

Let us now consider the initial data used in ref. \([1]\) for the grazing collisions of black holes; namely: \( x_1 = 5m \), \( x_2 = -5m \), \( y_1 = m \), \( y_2 = -m \), \( v_{x1} = -0.5 \) and \( v_{x2} = 0.5 \); with \( m = 1 \) and \( \alpha = 1 \). This data corresponds to a hyperbolic Newtonian trajectory; with initial separation distance \( r = 10.198 \), and critical merging radius \( r_c = 4.449 \).

Our strategy is to follow closely the model used in \([21, 22]\); but we also want to compare our work with \([3]\). However, the initial data of \([2]\) involves two black holes with half the speed of light each; therefore in adapting this initial data to our model we must take into account relativistic effects. In references \([21]\) and \([22]\) there was no need for these concerns because the initial data was not relativistic.

Since the initial velocities are relativistic, the relative initial velocity \( v_0 \) is calculated from the expression

\[
v_0 = \frac{v_{x2} - v_{x1}}{1 - v_{x1} v_{x2}} = 0.8;
\]

where we are using geometric units in which the velocity of light and the gravitational constant have the unit value.

The energy radiated during the falling phase, calculated from \([3]\) is \( E_N = 0.00340 (m_1 + m_2) \).

In order to match the Newtonian stage with the black hole RT perturbed model we need also to set the total initial mass and initial angular momentum for the RT stage.

In previous work we have matched the Newtonian mass \( M_{New} = m_{i1} + m_{i2} \) to the initial mass \( M \); this already takes into account the field relativistic first order correction in terms of its physical masses \( m_1 \) and \( m_2 \), since, recalling equation (21) of ref. \([22]\), the initial mass would be \( m_1 + m_2 - \frac{m_1 m_2}{r_{12}} \). The system discussed in ref. \([22]\) had zero initial velocity, and therefore there was no need to take into account any other effect. Instead in our case we should take into account speed relativistic corrections.

Then, since the initial data is relativistic, the initial mass and angular momentum are calculated from

\[
M = \frac{m_{i1}}{\sqrt{1 - v_{i1}^2}} + \frac{m_{i2}}{\sqrt{1 - v_{i2}^2}} = 2.309
\]

and

\[
J = -y_1 \frac{m_{i1} v_{x1}}{\sqrt{1 - v_{x1}^2}} - y_2 \frac{m_{i2} v_{x2}}{\sqrt{1 - v_{x2}^2}} = 1.155.
\]

It should be emphasized that the relation between the initial RT stage mass and angular momentum \( M \) and \( J \) with the Newtonian mass and angular momentum \( M_{New} \) and \( j \) is just the Lorentzian factor \( \gamma = \frac{1}{\sqrt{1 - v_{i1}^2}} = 1.1547 \).

In the last section we will comment on the incidence of this factor on our results.

Let us observe that the relation between the relativistic angular momentum and total mass is \( \frac{J}{M} = 0.217 \).

Since the angular momentum is small, it can be treated as a perturbation. At the moment of the collapse, we can consider a quadrupole excitation along the \( y \) axis with amplitude \( A \), as described above. Let us take \( A = A_m + A_j \); where \( A_j \) is the contribution coming from the appearance of the angular momentum \( j \). The matching condition is given by the equation (See equations \([28]\), \([29]\) and \([30]\).)

\[
F_B = F_Q(r_c);
\]

from which one obtains in this case

\[
A_m = 0.057
\]

and

\[
A_j = 0.028.
\]

Therefore, to be explicit, the initial data for \( V_{RT} \) and \( V_\lambda \) are

\[
V_{RT} = 1 - \frac{A_m}{2} \times \left( \sqrt{\frac{3}{2}} Y_{2,-2}(\zeta, \bar{\zeta}) + Y_{2,0}(\zeta, \bar{\zeta}) + \sqrt{\frac{3}{2}} Y_{2,2}(\zeta, \bar{\zeta}) \right);
\]

and

\[
\lambda V_\lambda = -\frac{A_j}{2} \times \left( \sqrt{\frac{3}{2}} Y_{2,-2}(\zeta, \bar{\zeta}) + Y_{2,0}(\zeta, \bar{\zeta}) + \sqrt{\frac{3}{2}} Y_{2,2}(\zeta, \bar{\zeta}) \right).
\]

The constant \( M_0 \) is determined from \([33]\), and the condition that initially the mass is given by \([34]\), which sets \( M_0 = 2.302 \).
The orbital angular momentum is taken into account in the initial data for $\Psi^0_1$. It is convenient to express this initial data in terms of the auxiliary field $g$, given by

$$\Psi^0_1 = \frac{\partial g}{\partial V_{RT}} = i \frac{\partial g}{\partial V_{RT}}, \quad (55)$$

Let us note that then

$$\partial g / \partial V_{RT} \Psi^0_1 = \partial^2 g$$

so that in the stationary case one has $\partial^2 g = 0$ and $\dot{g} = 0$.

We take $g = \bar{g}$ and

$$g = g_0 Y_{1,0}(\zeta, \zeta); \quad (56)$$

where $g_0$ is related to the angular momentum in the $z$ direction by

$$g_0 = -\sqrt{12\pi J} = -4.094. \quad (57)$$

In reference [31] it was also considered a similar case of a binary system with orbital angular momentum. Their initial data was: $x_1 = 0$, $x_2 = 0$, $y_1 = m$, $y_2 = -m$, $v_{x1} = -0.8$ and $v_{x2} = 0.894$; with $m = 1.5$ and $\alpha = \frac{3}{2}$. From the Newtonian point of view this data corresponds to an elliptic motion; but with maximum and minimum radius that are smaller than the corresponding critical merging radius. For this reason, we can not compare this case with our model.

### 4 Numerical Implementation

Accurate numerical evolution of a fourth order parabolic equation, such as (17), by means of an explicit finite difference scheme is a challenge because the CFL condition requires that the time step $\Delta u$ scale as the fourth power of the spatial grid size. Nevertheless, we constructed a set of algorithms to solve these equations using second order accurate finite difference approximations (following [33]). The numerical treatment of the eth operator has been thoroughly described in [33]. This work presented a clean way to deal with derivative operators on the sphere by covering it with two coordinate patches and dealing with spin weighted quantities. Thus, it is ideally suited for our present purposes. The numerical grid on each patch is defined by $\hat{\xi}_i = q_i + ip_i$, where $q_i, p_i = -1 - 2\Delta \Lambda + (i - 1)\Delta A$ (with $\Delta A = 2(N - 5)$). The angular derivatives are discretized by centered second order finite difference approximations and information between patches is obtained through fourth order accurate interpolations. (For a detailed description of this approach see [33]).

The integration in time is based upon a third time level Adams-Bashford scheme with predictor ($\tilde{F}$) given by

$$\tilde{F}(u + \Delta u) = F(u) + \frac{\Delta u}{2} \partial_u [F(u) - F(u - \Delta u)], \quad (58)$$

and corrector

$$F(u + \Delta u) = \tilde{F}(u) + \frac{\Delta u}{2} \partial_u [\tilde{F}(u) - \hat{F}(u + \Delta u)] + O(\Delta u^3). \quad (59)$$

Where $F$ stands for $V_{RT}$ or $V_{\lambda}$ and the $\partial u$ terms are to right hand sides of equations (23).

Additionally, we implemented the iterative Crank-Nicholson algorithm [34, 35] and observed that the results obtained with both implementations agree. Since the evolution equation for $\Psi^0_1$ is linear, its numerical integration is straightforwardly done by centered second order differences at the level $(u + \Delta u/2)$.

The second order convergence of numerical solutions was confirmed in the perturbative regime using solutions of the linearized equation and second order self-convergence of the solutions was confirmed in the nonlinear regime.

### 5 Results

#### Variations of total energy

The total Bondi energy-momentum vector at any RT retarded time $u$ can be calculated from the expression

$$P^a = -\frac{1}{4\pi} \int_S P^a(\zeta, \bar{\zeta}) (\Psi_{B2}^0 + \sigma_B \hat{\sigma}_B) \, dS^2, \quad (60)$$

where $S$ is the section determined by $u = \text{constant}$,

$$P^a(\zeta, \bar{\zeta}) = \left(1, \frac{\zeta + \bar{\zeta}}{1 + \zeta}, \frac{\zeta - \bar{\zeta}}{i(1 + \zeta)}, \frac{\bar{\zeta}}{1 + \zeta}\right); \quad (61)$$

and the subscript $B$ is used to emphasize that the quantities are evaluated with respect to a Bondi frame. The mass $M$ at this section $S$ is then given by

$$M = \sqrt{P^a P_a}, \quad (62)$$

where the indices are raised and lowered by the Lorentzian flat metric $\eta_{ab}$ at $\text{scri}$. [36]

Let us note that the relations between the Bondi quantities and the intrinsic ones are

$$\Psi_{B2}^0 = \Psi_2^0 \frac{V}{V^3}, \quad (63)$$

and

$$\sigma_B = \sigma \frac{V}{V}; \quad (64)$$

therefore in our gauge one has $\sigma_B = 0$, at each RT section.

The gravitational energy radiation flux is calculated from the Bondi time derivative of the supermomentum $\Psi_{B2}^0$, namely

$$\frac{\partial \Psi}{\partial u_B} = \frac{\partial \sigma_B}{\partial u_B} \frac{\partial \sigma_B}{\partial u_B}. \quad (65)$$


If one instead considers the time change with respect to the RT time, it is convenient to have in mind that for any function $f$ one has

$$\frac{\partial f}{\partial u_B} = \frac{1}{V} \frac{\partial f}{\partial u}. \quad (66)$$

The so-called news function $\frac{\partial \sigma_B}{\partial u_B}$ can be expressed in terms of the perturbed RT fields by

$$\frac{\partial \sigma_B}{\partial u_B} = \frac{\partial^2 V}{V} + \frac{1}{V^2} \left( \dot{\sigma} - \frac{\dot{V}}{V} \sigma \right). \quad (67)$$

To calculate the total energy radiated, one could then numerically evaluate the gravitational energy radiation flux of equation (65) at different times and sum along all the elapsed time. However, it is more accurate to numerically evaluate the initial mass and subtract the final mass. This is due to the fact that the RT spacetime is known to converge asymptotically to the Schwarzschild one; more specifically, one knows that $\lim_{u \to \infty} V_{RT} = 1$; and similarly one can see that $\lim_{u \to \infty} V_{\Lambda} = 0$.

Using this procedure, and a resolution of $n = 32$ points for half a meridian of the sphere (approximately $N = 1600$ points for the whole sphere), the energy radiated $E_{RT}$ during the RT stage is found to be $E_{RT} = 0.0034 M_0$.

Then, since in units of $M_0$, the energy radiated in the first stage is $E_N = 0.0030 M_0$, the total energy radiated in the whole process is $E = 0.0064 M_0$.

**Gravitational radiation field**

The numerical calculation of the evolution of the gravitational radiation field $\Psi_4$, is shown in figure 1.

It can be seen that although the total energy radiated is rather small, the amplitude of the gravitational radiation field can be large. In other words, this model describes a noticeable burst.

This is interesting since $\Psi_4$ is precisely what gravitational wave detectors will measure.

**Variations of angular momentum**

When dealing with the notion of angular momentum one is faced with the fact that there are several inequivalent definitions of angular momentum; which are not tightly related with the notion of intrinsic angular momentum, with the exception of $B$. An appropriate definition of intrinsic angular momentum involves the selection of unique sections of future null infinity where the quantity is to be calculated. Since in our case we are taking the angular momentum as a perturbation parameter of the RT geometries, it is not essential to consider these refinements in our model. And also, since the RT spacetimes provide with a geometric unique family of sections of future null infinity, namely the sections $u = \text{constant}$, it is natural to use them to calculate the angular momentum.

Then, instead of describing the variation of the intrinsic angular momentum we describe the variation of the RT-angular momentum vector given by

$$J^k = \mathfrak{Re} \left[ \int_{S_{RT}} \frac{i}{4\pi} \bar{\ell}^k \Psi_{B1}^0 dS_B^2 \right]; \quad (68)$$

where $S_{RT}$ are the sections determined by $u = \text{constant}$, $k = 1, 2, 3$, so that $\ell^k$ are the spacelike components of $\ell^a$; and where the Bondi component $\Psi_{B1}^0$, of the Weyl tensor, is related to the RT Weyl component $\Psi_1^0$ by

$$\Psi_{B1}^0 = \frac{\Psi_1^0}{V^3}. \quad (69)$$

Figure 2 shows a very small and smooth variation of the angular momentum; which is more related to the time variation of the RT geometry than to the radiation of angular momentum; as can be seen from the nature of equation (61).
tic surface. This implies an essential difference since in RT spacetime, our initial data is given on a characteristic hypersurface of the spacetime; while, by the nature of the member that the initial data of \[2\] is given at a spacelike horizon calculations, in dynamical regimes, are only apparent horizon masses. Masses obtained from apparent energy come from comparisons of initial ADM mass and radiation of angular momentum seems to be negligible with these initial conditions. In order to consider higher values of the initial angular momentum, we would need to deal with other background geometries, as for example twisting algebraically special spacetimes. Regarding the smooth monotonic variation of it one can infer that, for this small initial angular momentum data, its behavior is driven by the exponential asymptotic behavior of the RT background geometry. There are not complicated initial variations in \(\Psi^0\), that for example do appear in \(\Psi^4\), as seen in figure 4.

When describing a concrete physical situation with these spacetimes, one is supposed to choose the gauge and fix the free functions in order to make the best representation of the system. It is somehow striking that the choice of the frame in first order has physical significance, and it is not pure gauge as one is accustomed in the studies of linearized gravity around Minkowski
spacetime.

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