Invariant Linearization Criteria for Systems of Cubically Semi-Linear Second-Order Ordinary Differential Equations

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Abstract. Invariant linearization criteria of square systems of second-order quadratically semi-linear ordinary differential equations (ODEs) that can be represented as geodesic equations are extended to square systems of ODEs cubically nonlinear in the first derivatives. It is shown that there are two branches for the linearization problem via point transformations for an arbitrary system of second-order ODEs. One is when the system is at most cubic in the first derivatives. We solve this branch of the linearization problem by point transformations in the case of a square system of two second-order ODEs. Necessary and sufficient conditions for linearization by means of point transformations are given in terms of coefficient functions of the system of two second-order ODEs cubically nonlinear in the first derivatives. A consequence of our geometric approach of projection is a re-derivation of Lie’s conditions for a single second-order ODE and sheds light on more recent results on them. In particular, we show here how one can construct point transformations for reduction to the simplest linear equation by going to the higher space and just utilising the coefficients of the original ODE. We also obtain invariant criteria for the reduction of a linear square system to the simplest system. Moreover, these results contain the quadratic case as a special case. Examples are given to illustrate our results.
1. Introduction

A linearization problem involves the study of families of equations that are reducible via admissible transformations, which can be point, contact or more general, to linear equations. Lie [1] presented linearizability criteria, obtaining both algebraic and practical criteria, for a single second-order ODE to be point transformable to a linear equation via invertible changes of both the independent and dependent variables.

Lie [1] proved that necessary and sufficient conditions for a second-order ODE, \( y'' = E(x, y, y') \), to be linearizable by means of invertible point transformations are that the ODE be at most cubic in the first derivative, viz.

\[
y'' + E_3(x, y)y'^3 + E_2(x, y)y'^2 + E_1(x, y)y' + E_0(x, y) = 0 \tag{1}
\]

and the coefficients \( E_0 \) to \( E_3 \) satisfy the over-determined integrable system

\[
\begin{align*}
    b_x &= -\frac{1}{3}E_{1y} + \frac{2}{3}E_{2x} + be - E_0E_3, \\
    b_y &= E_{3x} - b^2 + bE_2 - E_1E_3 + eE_3, \\
    e_x &= E_{0y} + e^2 - eE_1 - bE_0 + E_0E_2, \\
    e_y &= \frac{2}{3}E_{1y} - \frac{1}{3}E_{2x} - be + E_0E_3, \tag{2}
\end{align*}
\]

where \( b \) and \( e \) are auxiliary variables and the suffixes \( x \) and \( y \) here and hereafter refer to partial derivatives. Since the classic work of Lie there has been continuing interest in this topic. We, inter alia, re-derive the Lie conditions (2) geometrically, by projections.

Tressé [2] also studied the linearization problem for scalar second-order ODEs. He deduced two relative invariants of the equivalence group of point transformations, the vanishing of both of which gives necessary and sufficient conditions for linearization of equation (1). These conditions are equivalent to the Lie conditions (2) (see Mahomed and Leach [3]) and can be given as the compatibility of (2) as

\[
\begin{align*}
    3(E_1E_3)_x - E_{1yy} + 2E_{2xy} - 3(E_0E_3)_y + E_2E_{1y} - 2E_2E_{2x} - 3E_{3xx} - 3E_3E_{0y} &= 0, \\
    3(E_0E_3)_x + 2E_{1xy} - 3E_{0yy} - E_{2xx} - E_1E_{2x} + 2E_1E_{1y} - 3(E_0E_2)_y + 3E_0E_{3x} &= 0. \tag{3}
\end{align*}
\]

Note that under the interchange of \( E_3 \) by \(-E_0\), \( E_2 \) by \(-E_1\) and \( x \) by \( y \), these conditions imply each other. Equations (3) provide practical criteria for linearization of equation (1) by point transformations. These conditions were also derived by the Cartan equivalence method (see Grissom et al [4]) as well as recently using a geometric argument in Ibragimov and Magri [5].
The reader is also referred to the review of various approaches in Mahomed [6]. Linearization via point and other than point transformations is of great interest and has been investigated in several works (see, e.g. [7, 8, 9, 10, 11, 12, 13, 14]).

The algebraic criteria of linearization of systems of second-order ODEs by means of point transformations have been considered in Wafo and Mahomed [11]. Practical criteria for quadratic semi-linear systems of second-order ODEs have been looked at recently as well (see Mahomed and Qadir [15]). In this paper our intention is to extend these results to cubically semi-linear square systems of second-order ODEs using geometric methods developed earlier (see Feroze et al [16]). As a by-product of our approach we re-derive the Lie conditions (2). Moreover, we present practical criteria in terms of coefficients for cubically semi-linear systems of second-order ODEs to be linearizable by point transformations. As a consequence we provide practical criteria for the class of linear second-order system of two ODEs to be reducible to the simplest system. Notwithstanding, our results subsume the linearization criteria for the quadratic case.

The outline of this paper is as follows. In the next section we present mathematical preliminaries. In section 3 we give an alternative method for obtaining the Lie conditions (2) as well as an alternative method for the construction of linearizing transformations for scalar second-order ODEs. Then in section 4 we derive practical criteria for linearization for a system of two second-order cubically semi-linear ODEs. Herein we state the relevant result for linear systems. Our theorem also contains the quadratically semi-linear equations as a corollary. In the next section we provide examples that amply illustrate our results. Finally, in section 6 we present a brief summary and conclusion.

2. Preliminaries

We first present some preliminaries. The system of geodesic equations is

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0, \quad i, j, k = 1, \ldots, n,$$

(4)

where the dot refers to total differentiation with respect to the parameter $s$ and $\Gamma^i_{jk}$ are the Christoffel symbols, which depend on $x^i$ and are given in terms of the metric tensor as

$$\Gamma^i_{jk} = \frac{1}{2} g^{im} (g_{jm,k} + g_{km,j} - g_{jk,m}).$$

(5)
The Christoffel symbols are symmetric in the lower pair of indices and have \( n^2(n+1)/2 \) coefficients. The Riemann curvature tensor is

\[
R^i_{jkl} = \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^m_{mk} \Gamma^i_{jl} - \Gamma^i_{ml} \Gamma^m_{jk},
\]

which is skew-symmetric in the lower last two indices and satisfies

\[
R^i_{jkl} + R^i_{kjl} + R^i_{ljk} = 0.
\]

A necessary and sufficient condition for a system of \( n \) second-order quadratically semi-linear ODEs for \( n \) dependent variables of the form (4) to be linearizable by point transformation and admit \( sl(n+2, \mathbb{R}) \) symmetry algebra is that the Riemann tensor vanishes (9, 17), i.e.

\[
R^i_{jkl} = 0.
\]

Practical criteria and the construction of point transformations are given in [15]. In particular, for a system of two geodesic equations (4), one has the linearization conditions (admittance of \( sl(4, \mathbb{R}) \) symmetry algebra) on the coefficients given by

\[
\begin{align*}
ay - bx + be - cd &= 0, \\
bz - cx + (ac - b^2) + (bf - ce) &= 0, \\
dz - cx - (ae - bd) - (df - e^2) &= 0, \\
(b + f)z &= (a + e)y,
\end{align*}
\]

where the Christoffel symbols are

\[
\begin{align*}
\Gamma_{11}^1 &= -a, \quad \Gamma_{12}^1 = -b, \quad \Gamma_{12}^2 = -c, \quad \Gamma_{11}^2 = -d, \\
\Gamma_{12}^2 &= -e, \quad \Gamma_{22}^2 = -f.
\end{align*}
\]

Now equation (5) together with (11) on setting \( g_{11} = p, \ g_{12} = q = g_{21} \) and \( g_{22} = r \) yield

\[
\begin{align*}
p_x &= -2(ap + dq), \\
q_x &= -bp - (a + e)q - dr, \\
r_x &= -2(bq + er), \\
p_y &= -2(bp + eq), \\
q_y &= -cp - (b + f)q - er, \\
r_y &= -2(cq + fr).
\end{align*}
\]

The construction of the linearization point transformations are found as follows (see [15]). One invokes

\[
g_{ab}(x) = \frac{\partial u^i}{\partial x^a} \frac{\partial u^j}{\partial x^b} g_{ij}(u),
\]
where $x = (x^1, \ldots, x^n)$, $u = (u^1, \ldots, u^n)$ with the requirement that $g_{ij}(u)$ be the identity matrix. For the case of two variables, we need to solve the equations

$$
\begin{align*}
  u_x^2 + v_x^2 &= p, \\
  u_x u_y + v_x v_y &= q, \\
  u_y^2 + v_y^2 &= r,
\end{align*}
$$

(13)

for which we have set $(x^1, x^2) = (x, y)$, $(u^1, u^2) = (u, v)$, $g_{11} = p$, $g_{12} = q = g_{21}$ and $g_{22} = r$ in (12).

Following Aminova and Aminov \[17\], we project the system down by one dimension and write the geodesic equations (4) as

$$
x_{aa}'' + A_{bc} x_{b} x_{c}^{a} + B_{bc} x_{b} x_{c}^{a} + C_{bc} x_{b} + D_{a} = 0, \quad a = 2, \ldots, n,
$$

(14)

where the prime now denotes differentiation with respect to the parameter $x^1$ (in \[17\] $x^n$ is used as the parameter) and the coefficients in terms of the $\Gamma_{ijk}^{a}$s are

$$
A_{bc} = -\Gamma_{bc}^{1}, \quad B_{bc} = \Gamma_{bc}^{a} - 2\delta_{(c}^{a} \Gamma_{b)1}^{1}, \quad C_{bc} = 2\Gamma_{bc}^{a} - \delta_{b}^{a} \Gamma_{i1}^{1}, \quad D_{a} = \Gamma_{i1}^{a}, \quad a, b, c = 2, \ldots, n,
$$

(15)

where we have used the notation $T_{(a,b)} = (T_{ab} + T_{ba})/2$. It is straightforward to deduce (14) and (15). Indeed, insert

$$
\dot{x}^a = \frac{dx^a}{dx^1} \dot{x}^1, \quad a = 2, \ldots, n
$$

and its derivatives

$$
\ddot{x}^a = \frac{d^2x^a}{dx^1^2} \dot{x}^1^2 + \frac{dx^a}{dx^1} \ddot{x}^1, \quad a = 2, \ldots, n
$$

into system (4). These, after cancelation of $\dot{x}^1^2$, directly yield (14) and (15). Note that in projecting down the Christoffel symbols there is degeneracy which results from the reduction of the range of the indices, so that $\Gamma_{i1}^{1}$ and $\Gamma_{i1}^{1}$ appear in the same combinations in $C_{bc}^{a}$ and $B_{bc}^{a}$, respectively. Consequently the set of coefficients $A$, $B$, $C$, $D$ have $n$ less elements than the coefficients $\Gamma_{ijk}^{a}$.

3. Re-derivation of the Lie conditions

We invoke equations (14) and (15) for $n = 2$. We also use (10) in identifying the $\Gamma_{ijk}^{a}$s with the coefficients $a$ to $f$ of the system of two geodesic equations which projects to (14). Thus we have (setting $(x^1, x^2) = (x, y)$)

$$
y'' + E_3(x, y)y'^3 + E_2(x, y)y'^2 + E_1(x, y)y' + E_0(x, y) = 0,
$$

(16)
where

\[ E_3 = A_{22} = -\Gamma^1_{22} = c, \]
\[ E_2 = B_{22}^2 = \Gamma^2_{22} - 2\Gamma^1_{12} = -f + 2b, \]
\[ E_1 = C^2 - 2\Gamma^1_{12} - \Gamma^1_{11} = -2e + a, \]
\[ E_0 = D^2 = \Gamma^2_{11} = -d. \] (17)

To re-derive the Lie conditions (2), we use the system of two geodesic equations (4) from which equation (16) arises projectively. Hence we utilize the conditions (9) which are conditions for a flat space. This requires that the coefficients \(a\) to \(f\) be in terms of the \(E_i\)s. From (17) we have

\[ a = E_1 + 2e, \]
\[ c = E_3, \]
\[ d = -E_0, \]
\[ f = 2b - E_2, \] (18)

where we have chosen \(b\) and \(e\) as yet arbitrary. These are constrained by the relations (9). We substitute (18) into (9). Equations (9) then yield

\[ E_{1y} + 2e_y - b_x + be + E_3E_0 = 0, \]
\[ b_y - E_{3x} + E_1E_3 + eE_3 + b^2 - bE_2 = 0, \]
\[ E_{0y} + e_x + eE_1 + e^2 - bE_0 + E_0E_2 = 0, \]
\[ 3b_x - 3e_y - E_{2x} - E_{1y} = 0. \] (19)

The first and last equations of (19) are easily seen to be equivalent to

\[ b_x = -\frac{1}{3}E_{1y} + \frac{2}{3}E_{2x} - be - E_0E_3, \]
\[ e_y = \frac{1}{3}E_{2x} - \frac{2}{3}E_{1y} - be - E_0E_3. \] (20)

The second and third equations of (19) as well as equations (20), on replacing \(e\) by \(-e\), are precisely the Lie conditions (2). Hence, we have provided an alternative derivation of the Lie conditions (2) by viewing the projection (16) in one higher space and looking at the flat space requirement there. If we had projected the system of two geodesic equations to a single ODE of the form (16) by using \(x^2\) instead of \(x^1\), then by interchanging \(E_3\) by \(-E_0\), \(E_2\) by \(-E_1\) and \(x^1 = x\) by \(x^2 = y\), the coefficients (17) imply the coefficients of the projected equation with independent variable \(x^2\). We state the following theorem.
Theorem 1. A necessary and sufficient condition that the scalar second-order ODE (16) has \(sl(3, \mathbb{R})\) symmetry algebra is that there is a corresponding system of two geodesic equations of the form (4) from which it is projected that admits the \(sl(4, \mathbb{R})\) symmetry algebra.

Furthermore, one can construct linearizing point transformations for (16) that satisfy (3) by resorting to the corresponding system of two geodesic equations from which (16) arises by projection. This is done by using the relations (13). This approach also results in the determination of at least one metric as a bonus. Notwithstanding, this method uses the coefficients of the equation which is linearizable and a transformation is then constructed via the relations (13). We consider two examples to illustrate this.

1. On using (18), the simple nonlinear equation
\[
y'' + y^3 - y' = 0
\] (21)
has corresponding \(a\) to \(f\) values,
\[
a = -1 + 2e, c = 1, d = 0, f = 2b.
\]
These together with the choices \(b = 0\) and \(e = 1\) satisfy the system (9). With these values of \(a\) to \(f\) we obtain from (11) particular solutions for \(p\), \(q\) and \(r\) given by
\[
p = r = \exp(2y - 2x), \quad q = -\exp(2y - 2x).
\]
Invoking (13), a linearizing point transformation to the simplest second-order ODE is
\[
u = \frac{1}{\sqrt{2}}\exp(-x + y), \quad v = \frac{1}{\sqrt{2}}\exp(-x - y),
\]
where \(u\) is the new independent variable.

2. The familiar nonlinear ODE (see, e.g. [18])
\[
y'' + 3yy' + y^3 = 0
\]
has, upon using (18),
\[
a = 3y + 2e, c = 0, d = -y^3, f = 2b.
\]
These and the choices \(b = 1/y\) and \(e = -y\) satisfy (9). A particular solution of (11) is then
\[
p = 1 + x^2 - 2xy^{-1} + y^{-2}, \quad q = (1 + x^2)y^{-2} - xy^{-3}, \quad r = y^{-4}(1 + x^2).
\]
A point transformation that linearizes the original ODE to the simplest second-order equation, after solving (13), then is
\[
u = x - y^{-1}, \quad v = \frac{1}{2}x^2 - \frac{x}{y}.
\]
where \( u \) is taken as the new independent variable. This transformation was previously obtained in [18] by mapping generators to canonical forms. As such we have presented another way of finding such transformations.

## 4. Linearization conditions for square systems

Driven by the success in obtaining the Lie conditions (2) by projection and then going back to the geodesic equations, we pursue similar conditions and practical criteria for linearization for a system of two second-order ODEs in a similar manner. Consequently, we study (14) for linearization via point transformations by resorting to a system of three geodesic equations (4). Before we do so, we need to first understand what is meant by linearization for systems of ODEs. A system of two second-order linear ODEs can possess 5, 6, 7, 8 or 15 point symmetries (see [19, 20]). The maximal symmetry algebra and hence \( \text{sl}(4, \mathbb{R}) \) is reached for the simplest system. Here we consider practical linearization criteria in terms of the coefficients for a system of two cubically semi-linear second-order ODEs of the form (14) having \( \text{sl}(4, \mathbb{R}) \) symmetry algebra. The quadratically semi-linear case was treated in [13]. Also algebraic criteria for systems of second-order ODEs have been found in [11].

We once again invoke equations (14) and (15) but now for \( n = 3 \). We therefore have

\[
\begin{align*}
\frac{d^2}{dt^2}x^2 &= A_{22}(x^2)'^3 + 2A_{23}(x^2)'^2 x^3' + A_{33} x^2' (x^3')^2 + B_{22}^2 (x^2')^2 + 2B_{23}^2 x^3' x^3' \\
&\quad + B_{33}^2 (x^3')^2 + C_{22}^2 x^2' + C_{33}^2 x^3' + D^2 = 0, \\
\frac{d^3}{dt^3}x^3 &= A_{22}(x^2)'^3 x^4' + 2A_{23} x^2' (x^3')^2 + A_{33} x^2' (x^3')^3 + B_{22}^2 (x^2')^2 + 2B_{23}^2 x^3' x^3' \\
&\quad + B_{33}^2 (x^3')^2 + C_{22}^2 x^2' + C_{33}^2 x^3' + D^3 = 0,
\end{align*}
\]

with coefficients

\[
A_{bc} = -\Gamma^1_{bc}, \quad B^a_{bc} = \Gamma^a_{bc} - 2\delta^a_{(c}\Gamma^1_{b)1}, \quad C^a_b = 2\Gamma^a_{1b} - \delta^a_b \Gamma^1_{11}, \quad D^a = \Gamma^a_{11}, \quad a, b, c = 2, 3.
\]

(23)

Here three \( \Gamma^a_{bc} \) coefficients are lost. We select \( \Gamma^1_{12}, \Gamma^2_{12} \) and \( \Gamma^3_{33} \) as arbitrary. We solve for the 15 \( \Gamma^a_{bc} \)'s of (23) in terms of the 15 coefficients \( A_{bc}, B^a_{bc}, C^a_b, D^a \) as well as \( \Gamma^1_{12}, \Gamma^2_{12} \) and \( \Gamma^3_{33} \). We only write down the \( \Gamma^a_{bc} \)'s in which the arbitrary elements appear. They are

\[
\begin{align*}
\Gamma^1_{11} &= 2\Gamma^2_{12} - C^2_2, \\
\Gamma^1_{13} &= 2\Gamma^2_{12} - C^3_{33}, \\
\Gamma^2_{22} &= 2\Gamma^2_{12} + B^2_{22},
\end{align*}
\]

8
\[
\Gamma_{23}^2 = \frac{1}{2}(\Gamma_{33}^3 + 2B_{23}^2 - B_{33}^3), \\
\Gamma_{13}^3 = \Gamma_{12}^2 + \frac{1}{2}C_{3}^3 - \frac{1}{2}C_2^2, \\
\Gamma_{23}^3 = \Gamma_{12}^1 + B_{23}^3.
\]
(24)

The others can be read-off from equations (23).

The flat space requirement for the corresponding system of three geodesic equations (4) are now imposed by means of the vanishing of the Riemann tensor, viz. (8). They are (let \((x^1, x^2, x^3) = (x, y, z)\))

\[
(\Gamma_{j2}^i)_x - (\Gamma_{j1}^i)_y + \Gamma_{m1}^i \Gamma_{j2}^m - \Gamma_{m2}^i \Gamma_{j1}^m = 0, \\
(\Gamma_{j3}^i)_x - (\Gamma_{j1}^i)_z + \Gamma_{m1}^i \Gamma_{j3}^m - \Gamma_{m3}^i \Gamma_{j1}^m = 0, \\
(\Gamma_{j3}^i)_y - (\Gamma_{j2}^i)_z + \Gamma_{m2}^i \Gamma_{j3}^m - \Gamma_{m3}^i \Gamma_{j2}^m = 0,
\]
(25)

which provide 27 conditions. Only 24 of them are linearly independent due to the identity (7). The reduction of these equations to explicit form is given in the Appendix.

These are 24 conditions (47) to (49) given in the Appendix that arise from the vanishing of the Riemann tensor as given in (25). They are the Lie-type integrability conditions for the \(\Gamma_{jk}^i\). We find that there are 7 equations in (47) to (49) which are independent of the \(\Gamma_{jk}^i\). The other 17 contain first-order partial derivatives of the \(\Gamma_{jk}^{i}\). Of these, \(\Gamma_{12,y}^2\) and \(\Gamma_{12,z}^2\) appear once each, \(\Gamma_{33,x}^{3}\) occurs three times and the rest twice each. Therefore, apart from the 7 conditions which are independent of the \(\Gamma_{jk}^i\) and given solely in terms of the coefficients of the system, there arise a further 8 conditions on the coefficients upon equating the respective \(\Gamma_{jk}^i\). Hence, we end up with 15 conditions or constraint equations on the coefficients. Now the \(\Gamma_{jk}^i\) which appear once each do not result in linearly independent equations as can easily be checked by equating them with the corresponding \(\Gamma_{jk}^i\) that were discarded. The resultant two equations that occur in this manner are linearly dependent. Thus the \(\Gamma_{12,y}^2\) and \(\Gamma_{12,z}^2\) are spurious. It is thus opportune to state the following theorem.

**Theorem 2.** A necessary and sufficient condition for the system of two cubically semi-linear ODEs

\[
y'' + A_{22}y'^3 + 2A_{23}y'^2z' + A_{33}y'z'^2 + B_{22}^2y'^2 + 2B_{23}^2y'z' + B_{33}^3z'^2 + C_{3}^{2}y' + C_{3}^{2}z' + D^2 = 0,
\]
\[
z'' + A_{22}y'^2z' + 2A_{23}y'z'^2 + A_{33}z'^3 + B_{22}^3y'^2 + 2B_{23}^3y'z' + B_{33}^3z'^2 + C_{3}^{2}y' + C_{3}^{2}z' + D^3 = 0,
\]
(26)

(where the prime denotes differentiation with respect to the independent variable \(x\) and the coefficients are in general functions of \(x, y, z\)) to be linearizable via point transformations to
the simplest system of two second-order ODEs is that its coefficients satisfy the following fifteen conditions on the coefficients functions of (20), viz.

\[
\begin{align*}
\frac{1}{2}C_{2x}^3 - D_y^3 + \frac{1}{4}C_{3x}^2C_{2x}^3 + \frac{1}{4}C_{2x}^3C_{2x}^3 - D^2B_{22}^3 - D^3B_{23}^3 &= 0, \\
B_{22x}^3 - \frac{1}{2}C_{2y}^3 - A_{22}^2D^3 + \frac{1}{2}C_{3x}^2B_{22}^3 + \frac{1}{2}C_{3x}^3B_{22}^3 - \frac{1}{2}C_{2x}^3B_{22}^3 - \frac{1}{2}B_{23}^3C_{2x}^3 &= 0, \\
B_{23x}^3 - \frac{1}{3}C_{2x}^2 - \frac{1}{6}C_{2y}^2 - \frac{1}{3}D^3A_{23} - \frac{2}{3}B_{22}^3C_{2x}^3 + \frac{2}{3}B_{23}^3C_{2x}^3 - \frac{1}{2}C_{3y}^3 &= 0, \\
\frac{1}{2}C_{2y}^2 - D_z^2 + \frac{1}{4}C_{3y}^2C_{2y}^3 + \frac{1}{4}C_{2y}^3C_{2y}^3 - B_{23}^2D^2 - B_{23}^3D^3 &= 0, \\
B_{33x}^2 - \frac{1}{2}C_{2y}^2 - D^2A_{33} + \frac{1}{2}C_{3y}^2C_{3y}^3 - \frac{1}{2}B_{23}^2C_{3y}^3 - \frac{1}{2}B_{33}^2C_{3y}^3 + \frac{1}{2}B_{23}^2C_{2y}^3 &= 0, \\
-A_{23y} + A_{22x} - A_{22}^2D^2A_{23} - A_{23}^2B_{22}^3 + A_{23}B_{22}^3 + A_{33}B_{22}^3 &= 0, \\
-A_{33y} + A_{22x} - A_{22}^2D^2 + A_{23}^2B_{22}^3 + A_{23}B_{22}^3 + A_{33}B_{22}^3 &= 0, \\
-A_{23x} + \frac{5}{3}A_{23}C_{2x}^3 + \frac{1}{3}A_{33}C_{2y}^3 - \frac{1}{3}B_{23x}^3 + B_{23}^3B_{22}^3 + \frac{1}{6}C_{2y}^3A_{23}^3 - B_{23}^3B_{22}^3 \\
-\frac{2}{3}B_{23y}^3 + \frac{1}{2}B_{23y}^3 + \frac{1}{2}B_{22x}^3 - \frac{1}{3}C_{2y}^3A_{22}^3 &= 0, \\
-A_{33x} + \frac{1}{2}C_{2y}^3A_{33}^3 + \frac{1}{2}A_{33y}C_{2y}^3 - B_{33y}^3 + B_{23}^3B_{22}^3 + B_{23}^3B_{22}^3 - B_{23}^3B_{23}^3 + B_{33}^3B_{23}^3 &= 0, \\
-\frac{2}{3}B_{22x}^3 + \frac{1}{2}C_{2y}^3 - \frac{1}{2}C_{2x}^3B_{23}^3 + D^2A_{22}^3 - \frac{2}{3}D^3A_{23}^3 - \frac{1}{3}C_{2y}^3B_{22}^3 + \frac{5}{6}B_{23}^3C_{2x}^3 \\
+ B_{23x}^3 - \frac{1}{2}C_{2x}^3B_{23}^3 + \frac{1}{2}C_{3y}^3B_{23}^3 - \frac{1}{2}C_{2y}^3B_{23}^3 &= 0, \\
-A_{22x} + \frac{1}{2}C_{2x}^2A_{22}^3 - B_{23}^3B_{33}^3 + B_{33y}^3 + B_{23}^3B_{22}^3 + B_{23}^3B_{23}^3 + B_{23}^3B_{23}^3 + \frac{1}{2}C_{2y}^3A_{22}^3 - B_{23}^3B_{22}^3 &= 0, \\
D_{y}^3 + B_{22}^3D^2 + D^3B_{23}^3 - D^3B_{23}^3 + \frac{1}{2}C_{2x}^2 - D_{z}^3 + \frac{1}{4}C_{2x}^3C_{2x}^3 - \frac{1}{4}C_{2y}^3C_{2x}^2 - B_{23}^2D^3 &= 0, \\
-2A_{23x} + \frac{4}{3}B_{33y}^3 + \frac{5}{3}A_{23}C_{2x}^3 + \frac{5}{3}A_{23}C_{2y}^3 + \frac{1}{3}C_{2y}^3A_{22}^3 - \frac{4}{3}B_{23}^3 - \frac{2}{3}C_{3y}^3A_{33}^3 + 2B_{22}^3B_{23}^3 \\
-2B_{23}^3B_{23}^3 - \frac{2}{3}B_{23y}^3 + \frac{2}{3}B_{22x}^3 &= 0, \\
B_{23x}^3 + \frac{1}{2}C_{3y}^3 - 2D^2A_{23}^3 + \frac{1}{2}C_{3y}^3B_{22}^3 + \frac{1}{2}C_{3y}^3B_{22}^3 + \frac{1}{2}C_{3y}^3B_{22}^3 - \frac{1}{2}B_{23}^3C_{2x}^3 - B_{23}^3C_{2x}^3 \\
- C_{2x}^3 - D^3A_{33}^3 &= 0, \\
-\frac{1}{2}B_{23}^3 + B_{33x}^3 + C_{3y}^3 - C_{3y}^3B_{23}^3 + C_{3y}^3B_{22}^3 + B_{23}^3C_{2x}^3 - B_{23}^3C_{2x}^3 - \frac{1}{2}C_{3y}^3 \\
-\frac{1}{2}C_{2x}^3 - 2D^3A_{33}^3 &= 0. \\
\end{align*}
\]
Corollary 1. The system of two quadratically semi-linear ODEs

\[ y'' + B_{22}^2 y'^2 + 2B_{23}^2 y' z' + B_{33}^2 z'^2 = 0, \]
\[ z'' + B_{22}^3 y'^2 + 2B_{23}^3 y' z' + B_{33}^3 z'^2 = 0, \]  

(28)

where the \( B_{ij} \)'s are functions of \( y \) and \( z \) and the dot denotes total derivative with respect to \( x \), is linearizable by point transformations to the simplest system of two equations if and only if the \( B_{ij} \)'s satisfy the four conditions on the coefficients given by

\[-B_{22}^3 B_{33}^3 + B_{23}^3 B_{32}^3 - B_{22}^3 B_{23}^3 + B_{23}^3 B_{23}^3 - B_{23}^3 B_{22}^2 = 0, \]
\[ \frac{4}{3} B_{33}^3 B_{23}^3 - \frac{4}{3} B_{23}^3 B_{32}^3 + 2B_{22}^3 B_{33}^3 - 2B_{23}^3 B_{32}^3 - \frac{2}{3} B_{23}^3 B_{22}^2 + \frac{2}{3} B_{22}^2 = 0 \]
\[-B_{23}^3 B_{32}^3 + B_{32}^3 B_{23}^3 - B_{23}^3 B_{23}^3 - \frac{2}{3} B_{23}^3 B_{32}^3 + \frac{1}{3} B_{32}^3 B_{32}^3 + \frac{2}{3} B_{22}^2 = 0, \]
\[-B_{33}^3 B_{22}^2 + B_{22}^2 B_{33}^3 - B_{22}^2 B_{23}^3 - B_{23}^3 B_{23}^3 + B_{23}^3 B_{33}^3 = 0. \]  

(29)

Remark. If one sets \( B_{22}^2 = -a, B_{23}^2 = -b, B_{33}^3 = -c, B_{22}^3 = -d, B_{23}^3 = -e \) and \( B_{33}^3 = -f \), one gets precisely the conditions (30). Hence Theorem 2 naturally contains the linearizability criteria for the quadratic case.

Corollary 2. The system of two linear (in the first derivatives) ODEs

\[ y'' + C_{2}^2 y' + C_{3}^2 z' + D^2 = 0, \]
\[ z'' + C_{2}^3 y' + C_{3}^3 z' + D^3 = 0, \]  

(30)

where the prime refers to differentiation with respect to \( x \) and the \( C_{ij} \)'s are independent of \( y \) and \( z \), is linearizable by point transformations to the simplest system of two equations if and only if the \( C_{ij} \)'s and \( D^i \)'s satisfy the three conditions on the coefficients, viz.

\[ \frac{1}{2} C_{2x}^3 + \frac{1}{4} C_{3x}^3 + \frac{1}{4} C_{2}^3 C_{2}^3 = D_{y}', \]
\[ \frac{1}{2} C_{3x}^3 + \frac{1}{4} C_{3x}^3 C_{3}^3 + \frac{1}{4} C_{2}^3 C_{2}^3 = D_{z}', \]
\[ \frac{1}{2} C_{3x}^3 - \frac{1}{2} C_{2x}^3 + \frac{1}{4} C_{3x}^3 C_{3}^3 - \frac{1}{4} C_{2}^3 C_{2}^3 = D_{z}^3 - D_{y}^2. \]  

(31)
We have provided practical criteria, necessary and sufficient conditions, for equations of the form (26) to be linearizable via point transformations to the simplest system. The question naturally arises if there are more general equations than (26) that can be linearizable to the simplest system. Indeed, there are more general systems of two second-order ODEs which can be linearized.

The most general system of \(n-1\) second-order ODEs linearizable is given by

\[
J^i_j x^{i''} + G^i_j x^{i'} x^{j'} + \Delta^i_{jkl} x^{i'} x^{j'} x^{k'} + \Lambda^i_{jkl} x^{i'} x^{k'} + \Omega^i_{jkl} x^{i'} x^{l'} + E^i = 0, \quad i = 2, \ldots, n,
\]

where the prime refers to total differentiation with respect to \(x^1\), the coefficient functions are dependent upon \(x^1, \ldots, x^n\), and are given by

\[
J^i_j = X^i_1 X^j_1 - X^i_1 X^j_1,
G^i_j = X^i_k X^j_1 - X^i_1 X^j_k,
\Delta^i_{jkl} = X^i_l X^j_1 - X^i_1 X^j_l,
\Lambda^i_{jkl} = 2X^i_1 X^j_1 - 2X^i_1 X^j_1 + X^i_1 X^j_1 - X^i_1 X^j_1,
\Omega^i_{jkl} = 2X^i_1 X^j_1 - 2X^i_1 X^j_1 + X^i_1 X^j_1 - X^i_1 X^j_1,
E^i = X^i_1 X^i_1 - X^i_1 X^i_1 - X^i_1 X^i_1, \quad i, j, k, l = 2, \ldots, n
\]

in which

\[
X^1 = X^1(x^1, \ldots, x^n), \quad X^i = X^i(x^1, \ldots, x^n), \quad i = 2, \ldots, n
\]

are invertible transformations. It is certainly not difficult to obtain (32). This is done by the substitution of (34) into the free particle system

\[
X^{i''} = 0, \quad i = 2, \ldots, n; \quad ' = \frac{d}{dX^1}.
\]

This after routine calculations yields (32) with the coefficients satisfying (33). Equation (32) is the most general system of \(n-1\) equations point transformable to the simplest system (35). Equation (32) has \(n(n-1)(n^2 + 6n - 1)/6\) coefficients.

Equation (32) can be written in normal form in terms of at most cubic first order derivatives as

\[
x^{i''} + A^i_{jkl} x^{j'} x^{k'} x^{l''} + B^i_{jkl} x^{j'} x^{k'} x^{l'} + C^i_{jkl} x^{j'} x^{l'} + D^i = 0, \quad i, j, k, l = 2, \ldots, n,
\]

provided

\[
\Delta^i_{klm} = J^i_{jkl} A^j_{klm} + G^i_{mjk} B^j_{kl},
\Lambda^i_{kl} = J^i_{jkl} B^j_{kl} + G^i_{ijk} C^j_k,
\Omega^i_{k} = J^i_{jkl} C^j_k + G^i_{kjl} D^j,
E^i = J^i_{jkl} D^j,
G^i_{pjk} A^j_{klm} = 0.
\]
The relations (37) can be obtained by solving for the second derivative in terms of the first order derivatives and inserting these into equation (32). The last equation of (37) tells us that not all the $A_{ijklm}^i$ coefficients are independent. As a matter of fact if we replace these by $A_{kl}^i$ in (36), then it turns out that this relation in (37) will now be identically satisfied. What transpires is that the quartic term disappears automatically due to $G_{ij}^i$ being skew symmetric in the lower indices and $x^p x^q$ appearing symmetrically. One also needs then to adjust the relation (37) in the latter case by

$$\Delta_{klm}^i = J_k^i A_{lm} + G_{mj}^i B_{kl}^j.$$  

(38)

The remaining equations of (37) are the same.

There are two branches of the linearization problem by point transformations for a system of $n - 1$ second-order ODEs. One is the general form (32) owing to the arbitrariness of the $\Delta_{ijkl}^i$ coefficients. The other is the form (14) in which the cubic coefficients are fewer in number. In the case of two second-order ODEs, equations (26), we have obtained explicit linearization criteria as encapsulated in Theorem 2 and their corollaries.

In the general equation (32) there are $(n - 1)n(n^2 + 6n - 1)/6$ coefficients while for (14) there are $(n - 1)n(n + 2)/2$ independent coefficients. It would be of interest to find practical criteria for the reduction of equation (32) to the simplest system via point transformations for $n = 3$. Of course it is of great interest to do this for the general system (32) for $n \geq 4$.

If one has a system of the form (32) with known coefficients which is reducible to the free particle system (35) by point transformations, then one can utilise (33) to construct a linearizing point transformation. Also, we can obtain linearizing point transformations for system (14), if it is linearizable to the simplest system (35), by invoking (33) together with (37).

In particular, one can find linearizing point transformations for the system (26) in a similar manner by solving the system (37).

Instead of using the system (33) in order to construct a linearizing point transformation there are other ways as pointed out earlier. One is to go to the higher space, once one has the coefficients at hand, and use (12) for which $g_{ij}(u)$ must be the identity matrix and where we may set $u^1$ to be the independent variable. Yet a third approach is that of mapping symmetry generators of the linearizable system, if known, to the free particle generators.
5. Examples

We present examples to illustrate our results. We have $y$ and $z$ as the dependent variables. Also the $'$ below denotes differentiation with respect to $x$. Moreover, we have included one example that does not satisfy our linearization criteria but belongs to the more general class (32) which is linearizable.

1. Consider the anisotropic oscillator system

$$
y'' + \omega_1(x)y = 0, \tag{39}
$$
$$
z'' + \omega_2(x)z = 0,
$$

The coefficients of system (39) satisfy the conditions (31) provided $\omega_1 = \omega_2$. Hence in order for the system (39) to be reducible to the free particle system one must have isotropy.

2. The simple linear system

$$
y'' + z = 0, \tag{40}
$$
$$
z'' + z = 0,
$$
do not satisfy the conditions (31). Thus this system is not transformable pointwise to the free particle system. This system does not have a Lagrangian formulation as well [21].

3. For the quadratic system

$$
y'' - y' + y'^2 = 0, \tag{41}
$$
$$
z'' - z' + z'^2 = 0,
$$

all conditions (29) are satisfied. Therefore the system (41) is reducible to the simplest system. A point transformation that does the job is

$$
u = \exp x, \ v = \exp y, \ w = \exp z, \tag{42}
$$

where $u$ is the independent variable. This can be constructed by going to the higher space as we have illustrated for the scalar ODEs in section 3.

4. Consider the cubically semi-linear system

$$
y'' + \frac{1}{x}y' + y'^2 + \left(\frac{x}{y} + \frac{x}{y^2}\right)y'^3 = 0, \tag{43}
$$
$$
z'' + \frac{1}{x}z' + z'^2 + 2y'z' + \left(\frac{x}{y} + \frac{x}{y^2}\right)y'^2z' = 0,
$$
For the system (43) all the conditions (27) hold. A linearizing point transformation to the simplest system is

\[ u = \ln xy, \quad v = \exp y, \quad w = \exp(y + z), \quad (44) \]

in which \( u \) is the independent variable.

5. Finally the system

\[
\begin{align*}
4yz^2y'^2 + 4y^2zy'z' + 2xz^2y'^3 + 8xyzy^2z' + 2xy^2yz'^2 + 2xy^2zyz'' &= y^2z^2y'' + 2xy^2zz'y'', \\
y'' + xzy'y'' + xyz'y'' - xz^2y^2y'' - xyzzy'z'y'' &= y'(zy' + y'z)(zy' + y'z + 2xy'z' + xzy''), \quad (45)
\end{align*}
\]

is not of the form given in Theorem 2. It is of the form given in (32) and is linearizable by means of the point transformation

\[ u = x \exp(yz), \quad v = xy^2z^2, \quad w = y, \quad (46) \]

where \( u \) is the independent variable.

6. Concluding remarks

Aminova and Aminov [17] had provided a procedure of projecting down 1 dimension from a system of \( n \) geodesic equations to \( n - 1 \) cubically semi-linear ODEs. Separately, we had provided [15] linearizability criteria for a square quadratically semi-linear system. These were used together to derive linearizability criteria for a single cubically semi-linear equation by projecting down from a system of 2 quadratically semi-linear equations. This provided an alternate method to prove Lie’s general result for linearizability of a single non-linear equation. It led naturally to an extension of the linearization criteria via point transformations from a scalar second-order ODE as obtained by Lie [1] to a system of two cubically semi-linear ODEs of the form (26). These provided necessary and sufficient conditions for reduction to the simplest system and hence \( sl(4, \mathbb{R}) \) symmetry algebra for equations of the form (26). Moreover, Theorem 2 provides criteria for the reduction of linear systems of two equations to the free particle system.

Lie had demonstrated [1] that only cubically semi-linear scalar equations of order two are linearizable in general. As such, it could have been hoped that the projection procedure will provide the complete solution of the linearizability problem for the system of 2 non-linear ODEs. That hope is doomed from the start as there are 5 classes of systems of 2 cubically non-linear equations that are linearizable by point transformations, having different symmetry
algebras. Moreover, the maximum symmetry algebra class of such systems of two equations is one branch of the linearization problem via point transformations as the general class is represented by \( (32) \). Why do we get a unique class in the former case and 5 in the latter? Furthermore, how many distinct classes should there be for a system of \( n \) cubically semi-linear ODEs?

We start by noting that the projection procedure and linearizability can be equally well adopted for an arbitrary system of \( n \) quadratically semi-linear second order ODEs reduced to \( n - 1 \) cubically semi-linear second order ODEs. There are two branches for the linearization problem for systems admitting the maximal algebra for \( n \geq 3 \). There is enormous computational complications that arise. As such, one would need an algebraic computational code to deal with larger systems. A code has, indeed, been prepared to construct the metric coefficients given the Christoffel symbols \([22]\). That can be extended to deal with the linearization of larger systems. Now observe that in projecting down from the system of \( n \) dependent variables to \( n - 1 \) variables, the Christoffel symbols are reduced from \( n^2(n + 1)/2 \) by \( n \), to give \( (n - 1)n(n + 2)/2 \) independent coefficients. Since we now have \( n - 1 \) equations, each with its own cubic function, there are \( (n - 1)n/2 \) cubic coefficients for the reduced system. If the number of coefficients left over after losing \( n \) equals the number of coefficients of the reduced system, we can determine one set of coefficients in terms of the other. The two expressions are obviously equal for \( n = 2 \) and the former is greater than the latter for \( n > 2 \). As such, the coefficients of the cubic system can be determined uniquely in terms of the quadratic system for \( n = 2 \), i.e. for a scalar cubically semi-linear system. For larger systems there will be infinitely many ways to write the former in terms of the latter. Hence there is a unique solution to the linearizability problem only for the scalar cubically semi-linear equation and many solutions for systems of cubically semi-linear systems!

The second question remains and has, in fact, been compounded. It is known that there are 5 and not infinitely many distinct classes. Why? The point is that all distinct ways of writing the cubic system coefficients in terms of the quadratic system coefficients will not give independent criteria as there will be transformations permissible from one definition to another. The point is to determine those that are distinct. Another way of looking at what we have done is to note that we have asked that the original system correspond to a system of geodesic equations in flat space. Then the projection gives the reduced system, which must also be of geodesics in an \( (n - 1) \)-dimensional flat space. Even if the original geodesics were curved, the projected geodesics could correspond to straight lines. For example, if the original space was a sphere and one projects along the plane containing the geodesic to a plane perpendicular to it, the resulting projected curve would be a straight line.
The minimal dimension for a system of $n$ second-order ODEs to be linearizable by point transformation is $2n + 1$. The maximum dimension of the symmetry algebra is $(n + 1)(n + 3)$ which corresponds to $sl(n + 2, \mathbb{R})$. The other submaximal symmetry algebras besides that of dimension $2n + 1$ range from $2n + 2$ to $(n + 2)^2/2$ for $n$ even and $[(n + 2)^2 + 1]/2$ for $n$ odd. Thus for $n = 2$ we have the minimum dimension to be 5 and other submaximal algebra dimensions are 6, 7 and 8. The maximum dimension for $n = 2$ is 15. For $n = 3$ the minimum dimension is 7 and the next to maximum is 13. The maximum is 24. Thus for this case there are 8 classes. Generally, for $n = 2m$, the number of classes is $2m^2 + 3$ and for $n = 2m – 1$ it is $2m^2 – 2m + 4$.

It would be important to find ways of providing the linearizability criteria for the cases of the other symmetry algebras.

**Appendix**

We take $j = 1$ in the third set of (25) as the 3 dependent equations and discard them. The involution of the first set of 9 equations of (25) gives

\[
\begin{align*}
\frac{1}{2} C_{2x}^3 - D_y^3 + \frac{1}{4} C_3 C_2^3 + \frac{1}{4} C_2^2 C_2^3 - D_2 B_{22}^3 - D_3 B_{23}^3 &= 0, \\
B_{22x}^3 - \frac{1}{2} C_{2y}^3 - A_{22} D_3^3 + \frac{1}{2} C_3 B_{22}^3 + \frac{1}{2} C_3 B_{22}^3 - \frac{1}{2} C_2 B_{22}^3 - \frac{1}{2} B_{23}^3 C_2^3 &= 0, \\
\Gamma_{12,y}^1 &= -A_{22x} - A_{22} \Gamma_{12}^0 + C_2^3 A_{22} + \Gamma_{12x}^2 + \Gamma_{12}^1 \Gamma_{12}^2 + \frac{1}{2} B_{22}^3 \Gamma_{33}^3 - \frac{1}{2} B_{22}^3 B_{33}^3 + \frac{1}{2} C_3 A_{23}, \\
\Gamma_{12,x}^2 &= D_y^3 + D_2 \Gamma_{12}^1 + D_2 \Gamma_{12}^2 - \frac{1}{4} C_2^2 C_2^3 - \Gamma_{12x}^2 + B_{22}^3 D_2^2 + D_2 B_{23}^3 - \frac{1}{2} D_{23} B_{33}^3 + \frac{1}{2} D_3 \Gamma_{33}^3, \\
\Gamma_{12,y}^1 &= -\frac{1}{2} B_{22x}^2 + \frac{2}{3} C_{2y}^2 + \Gamma_{12x}^2 + \frac{1}{4} C_2^3 \Gamma_{33}^3 - \frac{1}{4} C_2^3 B_{33}^3 \\
+ D_2 A_{22} + \frac{2}{3} D_2 B_{23}^3 - \frac{1}{6} C_2 B_{22}^3 + \frac{1}{6} B_{23}^2 C_2^3, \\
\Gamma_{12,x}^1 &= -\frac{2}{3} B_{22x}^2 + \frac{1}{3} C_{2y}^2 + \Gamma_{12x}^2 + \frac{1}{4} C_2^3 \Gamma_{33}^3 - \frac{1}{4} C_2^3 B_{33}^3 + D_2 A_{22} \\
+ \frac{1}{3} D_2 A_{23} - \frac{1}{3} C_2^3 B_{22}^3 + \frac{1}{3} B_{23}^2 C_2^3, \\
B_{23x}^3 - \frac{1}{2} B_{22x}^3 + \frac{1}{6} C_{2y}^2 - \frac{4}{3} D_{23} A_{23} - \frac{2}{3} B_{22}^3 C_2^3 + \frac{2}{3} B_{23}^2 C_2^3 - \frac{1}{2} C_{3y}^3 &= 0, \\
\Gamma_{33,y}^3 &= -2 A_{23x} + B_{33x}^3 - 2 \Gamma_{12x} A_{23} + A_{23} C_2^3 + 2 \Gamma_{12}^2 B_{23}^3 + \Gamma_{12}^1 \Gamma_{33}^3 - \Gamma_{12}^1 B_{33}^3 \\
+ \Gamma_{33}^3 B_{33}^3 - B_{33}^3 B_{33}^3 + A_{22} C_2^3 + A_{23} C_3^3 \\
\Gamma_{33,x}^3 &= -2 B_{23}^3 + B_{33x}^3 + C_{3y}^3 + 2 D_3 A_{23} - C_3^3 B_{23}^3 + C_3^2 \Gamma_{12}^1 - \Gamma_{12}^2 B_{33}^3 + C_3^2 B_{22}^3 + B_{23}^2 C_3^3
\end{align*}
\]
\[-B_{23}^2 C_2^2 - \frac{1}{2} C_3^3 B_{33}^3 + \frac{1}{2} B_{33}^3 C_2^2 + \frac{1}{2} C_3^3 \Gamma_{33}^3 - \frac{1}{2} C_2^2 \Gamma_{33}^3 + \Gamma_{33}^3 \Gamma_{12}^2. \quad (47)\]

The second set of 9 equations of (25) yields

\[
\Gamma_{12,x} = -B_{23}^3 + \frac{1}{2} C_3^3 + D^3 A_{23} - \frac{1}{2} C_3^3 B_{23}^2 + \frac{1}{4} C_2^3 B_{33}^3 + \frac{1}{4} C_3^3 \Gamma_{33}^3 \\
-\frac{1}{2} C_3^3 B_{23}^2 + \frac{1}{2} C_2^3 B_{23}^2 + \Gamma_{12}^1 \Gamma_{12}^1.
\]

\[
\Gamma_{12,y} = -A_{23}^3 - A_{23}^2 \Gamma_{12}^1 + A_{23}^1 C_2^2 + A_{12}^1 B_{23}^2 + \frac{1}{2} \Gamma_{33}^3 B_{23}^2
\]

\[
\frac{1}{4} C_3^3 \Gamma_{33}^1 - \frac{1}{2} B_{33}^3 B_{33}^3 - \frac{1}{2} B_{33}^3 \Gamma_{33}^1 + \frac{1}{2} A_{33} C_2^2,
\]

\[
\Gamma_{12,z} = -A_{23}^3 - A_{23}^2 \Gamma_{12}^1 + A_{23}^1 C_2^2 + A_{12}^1 B_{23}^2 + \frac{1}{2} \Gamma_{33}^3 B_{23}^2
\]

\[
\frac{1}{2} B_{23}^3 \Gamma_{12}^1 - \frac{1}{2} B_{33}^3 B_{33}^3 - \frac{1}{2} B_{33}^3 \Gamma_{33}^1 + \frac{1}{2} A_{33} C_2^2,
\]

\[
\Gamma_{12,x} = -\frac{1}{2} C_3^2 C_3^2 + \frac{1}{2} C_3^2 C_2^2 + \frac{1}{2} C_3^2 \Gamma_{33}^3 + D^2 A_{23}^2 + \frac{1}{2} C_3^2 B_{23}^2 + D^3 B_{33}^3 - \frac{1}{4} C_2^3 C_2^2 - \frac{1}{4} C_3^3 C_3^3 \\
+ \frac{1}{2} C_2^3 C_3^3 + \frac{1}{2} C_2^3 C_2^3 + \frac{1}{4} C_3^3 C_2^2 - B_{23}^2 D^2 - B_{33}^2 D^3 = 0,
\]

\[
B_{33}^3 - \frac{1}{2} C_3^3 - D^2 A_{33}^3 + \frac{1}{2} C_3^3 B_{33}^3 - \frac{1}{2} B_{33}^3 C_3^3 - \frac{1}{2} B_{33}^3 C_3^3 + \frac{1}{2} B_{33}^3 C_2^2 = 0,
\]

\[
\Gamma_{33}^3 = B_{33}^3 - 4 B_{23}^2 + 6 A_{23}^3 D^2 - 2 C_3^2 B_{23}^2 + 2 B_{33}^3 C_2^2 + 2 C_2^2 + C_3^2 \Gamma_{12}^1 + \frac{1}{2} C_3^3 \Gamma_{33}^3 \\
- \frac{1}{2} C_2^2 \Gamma_{33}^3 + \Gamma_{33}^3 \Gamma_{33}^1 - \frac{1}{2} B_{33}^3 C_3^3 + \frac{1}{2} C_2^2 B_{33}^3 - B_{33}^3 \Gamma_{33}^1 + 2 A_{33}^3 D^3,
\]

\[
\Gamma_{33}^3 = \frac{1}{2} C_3^3 + \frac{1}{2} C_3^2 - B_{23}^2 + 2 A_{23} \Gamma_{12}^1 + \frac{1}{2} C_3^2 \Gamma_{33}^3 + \frac{1}{2} C_3^2 \Gamma_{33}^3 \\
+ \Gamma_{12}^1 \Gamma_{33}^1 - \frac{1}{2} C_3^3 B_{33}^3 + \frac{1}{2} C_2^2 B_{33}^3 - B_{33}^3 \Gamma_{33}^1 + 2 A_{33}^3 D^3,
\]

\[
\Gamma_{33}^3 = -2 A_{33}^3 + B_{33}^3 - 2 A_{33}^2 \Gamma_{12}^1 + \frac{1}{2} A_{33} \Gamma_{33}^1 + 2 \Gamma_{12}^1 \Gamma_{33}^3 + \frac{1}{2} \Gamma_{33}^3 \Gamma_{33}^3 - \frac{1}{2} B_{33}^3 B_{33}^3 \\
+ A_{23}^3 + A_{33}^3 C_3^3.
\quad (48)\]

The last set of the 9 equations of (25) result in 6 independent conditions

\[
\Gamma_{12,y} = -B_{23}^3 + \frac{1}{2} C_3^3 A_{23} - B_{33}^3 B_{23}^2 + \frac{1}{2} B_{22}^3 B_{33}^3 + \frac{1}{2} B_{23}^3 \Gamma_{33}^3 \\
- B_{33}^3 A_{23} + \Gamma_{12}^1 \Gamma_{12}^1 - \frac{1}{2} C_3^2 A_{23} - \frac{1}{2} C_3^2 B_{23}^2 - \frac{1}{2} C_3^2 \Gamma_{33}^3 - \frac{1}{2} B_{33}^3 B_{33}^3 \\
+ \frac{1}{3} B_{23}^3 A_{33}^3 - B_{33}^3 B_{23}^2 - \frac{1}{6} C_3^2 A_{23}^3 + \frac{1}{6} C_3^2 A_{23}^3 - A_{33} \Gamma_{12}^1 + B_{23}^3 B_{33}^3 - \frac{1}{2} B_{23}^3 B_{33}^3
\]
FM thanks the HEC of Pakistan for granting him a visiting professor and NUST-CAMP for hospitality during which time this work was undertaken. AQ is grateful to the DECMA Centre and the School of Computational and Applied Mathematics for hosting him when this work was completed.

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