Spheres are rare

VINCENT RIVASSEAU\textsuperscript{(a)}

Laboratoire de Physique Théorique, CNRS UMR 8627, Université Paris-Sud - 91405 Orsay, France, EU and Perimeter Institute - Waterloo, Canada

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Abstract – We prove that triangulations of homology spheres in any dimension grow much slower than general triangulations. Our bound states in particular that the number of triangulations of homology spheres in 3 dimensions grows at most like the power 1/3 of the number of general triangulations.

Introduction. – The “Gromov question” [1] asks whether in dimensions higher than 2 the number of triangulations of the sphere grows exponentially in the number of glued simplices, as happens in dimension 2, for which explicit formulas are known [2–4]. It has not been answered until now [5–7]. It is usually formulated for triangulations that are homeomorphic to a sphere. But we do not know counterexamples showing that such an exponential bound could not hold also more generally for homology spheres, although we are conscious that homotopy constraints are much stronger than homology constraints.

Understanding general triangulations is important in the quantum gravity [8–10] context. Early rank-$d$ tensor models were introduced to classify these triangulations [11]. Group field theory (GFT) [12–14] can be seen as a development of these models. Their Feynman amplitudes are also the spin foams of loop quantum gravity [9], allowing to introduce the powerful quantum field theory methods in the latter approach. However, spin foams are 2-simplicial complexes with no well-defined $d$-dimensional homology. The search for improved GFTs led to the introduction of colored group field theory [15–17] which has such a well-defined $d$-homology, and in which coloring acts as a book-keeping device. The corresponding Feynman graphs have edges of $D+1$ different colors meeting at vertices of degree $D+1$. Hence they are dual to colored triangulations by $D+1$ simplices, which can glue together only hyperfaces of the same color. Colored GFTs were soon found to admit a new kind of $1/N$ expansion [18–20], which for tensors of rank three and more differs from the standard $1/N$ expansion of random matrix models. This expansion is indexed by an integer, called the degree. This degree is not a topological invariant but a sum over genera of jackets. Jackets are ribbon graphs embedded in the tensor graphs, which provide global Hegaard decompositions of the triangulation [21]. The $1/N$ expansion allows to compute an associated critical behavior [22].

Coloring also led to the related development of a general theory of unsymmetrized random tensor models, in which interactions are required to obey a tensor invariance principle, and colors simply allow to track the position of the indices of the tensor [23,24]. This formalism, in turn, led to the definition and study of renormalizable quantum field theories of the tensorial type [25–28], in which the propagator softly breaks the tensor invariance in the infrared. These theories were then shown to display the remarkable property of asymptotic freedom [29,30]. Renormalizable GFTs which include the “Boulatov-type” gauge invariance of spin foam models have also been recently defined and studied [27,31,32], and they also display asymptotic freedom [33].

In this letter we perform a small step towards applying this new circle of ideas to the Gromov question. We prove a rather obvious result that we nevertheless could not find in the existing literature, namely that spherical triangulations are rare among all triangulations in any dimension. More precisely, we give in the second section a necessary condition for a colored triangulation $\Gamma$ to have a trivial homology. It states that the rank of the incidence matrix of edges and faces for the dual graph $G$ of the

\textsuperscript{(a)}E-mail: rivass@th.u-psud.fr
restricts oneself to bicolored lines of the colored triangulation. Since the initial triangulation is made of the vertices of a homology spheres made of \(n\) tetrahedra grow as \(n^{d+1}\) (up to \(K^n\) factors). Hence in dimension 3 spherical triangulations cannot grow faster than the cubic root of general triangulations.

An outlook of the connection with the tensor program for quantum gravity is provided in the last section.

Spherical triangulations. – To any ordinary triangulation is associated a unique colored triangulation, namely its barycentric subdivision. If the initial triangulation is made of \(n\) simplices of dimension \(d\) (i.e., with \(d+1\) summits), the barycentric subdivision is made of \(a \cdot n\) colored simplices, where \(a\) only depends on the dimension \(d\). For instance, in \(d = 3\), a tetrahedron is decomposed by barycentric subdivision (see fig. 1) into 24 colored tetrahedra, hence \(a = 24\). This map is injective since the initial triangulation is made of the vertices of a single color, together with lines that correspond to given bicolored lines of the colored triangulation.

Hence in order to study the Gromov question, one can restrict oneself to colored triangulations. A bound in \(K^n\) for spherical colored triangulations in \(d = 3\) would translate into a bound in \(K^{24n}\) for spherical ordinary triangulations and so on.

Colored triangulations [34,35] triangulate pseudo-manifolds [16], and in contrast with the usual definition of spin foams in loop quantum gravity, they have a well-defined \(d\)-dimensional homology. They are in one-to-one correspondence with dual edge-colored graphs. Bipartite graphs correspond to triangulations of orientable pseudo-manifolds. Since they are associated to simple field theories [15] we expect they are the most natural objects for quantum gravity.

Therefore we consider from now on the category of (vacuum) connected bipartite edge-colored graphs with \(d+1\) colors and uniform coordination \(d+1\) (one edge of every color) at each vertex. The Gromov question can be rephrased as whether the number of such graphs with \(n\) vertices dual to spherical colored triangulations is bounded by \(K^n\).

We call \(V\), \(E\) and \(F\) the set of vertices, edges and faces of the graph \(G\). Faces are simply defined as the two-colored connected components of the graph, hence come in \(d(d+1)/2\) different types, the number of pairs of colors. We also put \(|V| = n\). \(n\) is the order of the graph and is even since the graph is bipartite. Also the graph being bipartite is naturally directed (i.e., there is a canonical orientation of each edge). We write \(T\) for a generic spanning tree of \(G\) and \(G/T\) for the contracted graph with one vertex also called the rosette associated to \(G\) and \(T\). The nullity of \(G\) (number of loops, namely number of edges in any rosette) is \(|L| = |E| - |V| + 1 = 1 + n(d-1)/2\). Jackets are ribbon graphs passing through all vertices of the graph. There are \((d/2)!\) such jackets in dimension \(d\) [19]. Any jacket of genus \(g\) provides a Heegaard decomposition of the triangulated pseudo-manifold. In dimension 3 it gives a decomposition into two handle-bodies bounded by a common genus \(g\) surface [21].

The edge-face incidence matrix \(\varepsilon_{ef}\) describes the incidence relation between edges and faces. Let us orient each face arbitrarily: \(\varepsilon_{ef}\) is then \(+1\), \(-1\) or 0 depending on whether the face goes through the edge in the direct sense, opposite sense or does not go through \(e\) (see footnote 1).

Group field theory [12–14] can be used to write connections on \(G\) with a structure group \(\mathcal{G}\). To each edge of a group field theory graph \(G\) is associated a generator \(h_e \in \mathcal{G}\) representing parallel transport along \(e\). The curvatures of the connection are the family of group elements \(\prod_{f\in F} h_{e(f)}\), for all faces \(f \in F\), where the product is taken in the right order of the face.

The generators \(h_e\) for the edges \(e\) in any given spanning tree \(\mathcal{T}\) of \(G\) are irrelevant for the computation of \(\pi_1(G)\), as they can be fixed to 1 through the usual \(G|V|\) gauge

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{(Color online) Refining and coloring a triangulation using barycentric subdivision.}
\end{figure}
invariance on connections. Indeed setting \( h_e = 1 \ \forall e \in \mathcal{T} \) is equivalent to consider the retract \( G/\mathcal{T} \) of \( G \) with a single vertex, hence the rosette associated to \( G \) and \( \mathcal{T} \) (see footnote \(^2\)).

The fundamental group \( \pi_1(G) \) of \( G \) admits then a presentation with one such generator \( h_e \) per edge of \( G/\mathcal{T} \) and the relations

\[
\prod_{e \in \mathcal{F}} h_e^{\epsilon_{ef}} = 1, \quad \forall f \in \mathcal{F}. \tag{1}
\]

Hence the space of flat connections (for which curvature is 1 for all faces) is the representation variety of \( \pi_1(G) \) into \( G \) [36].

In particular \( G \) is simply connected if and only if the set of flat connections is just a point hence if the set of equations (1) has \( h_e = 1, \ \forall e \in G/\mathcal{T} \), as its unique solution.

The homology of \( G \) is even simpler, as it corresponds to the case of a commutative group \( G \). \( G \) has trivial first-homology (i.e., \( H_1(G) = 0 \)) if and only if the set of commutative equations

\[
\sum_{e \in \mathcal{L} = G/\mathcal{T}} \epsilon_{ef} h_e = 0, \quad \forall f \in \mathcal{F} \tag{2}
\]

has \( h_e = 0 \) as unique solution.

By (commutative) gauge invariance the rank \( r_G \) of the matrix \( \epsilon_{ef} \) is equal to the rank of the reduced matrix \( \epsilon_{ef}^\mathcal{T} \) where the edges \( e \) run over the reduced set of edges of \( G/\mathcal{T} \). We have certainly

\[
r_G \leq \inf(|\mathcal{L}|, |\mathcal{F}|). \tag{3}
\]

**Lemma 1.** The edge-colored graph \( G \) has trivial first-homology if and only if the rank of \( \epsilon_{ef}^\mathcal{T} \) is maximal, i.e., equal to \( |\mathcal{L}| = 1 + (d - 1)n/2 \). This reads:

\[
r_G = 1 + (d - 1)n/2. \tag{4}
\]

**Proof.** The rank \( r_G \) cannot be larger than \( |\mathcal{L}| \) by (3). If it is strictly smaller, it would mean that the linear map from \( \mathbb{R}^L \) to \( \mathbb{R}^F \) represented by the matrix \( \epsilon_{ef} \) would have a non-trivial homology space \( H_1(G) \) would have non-trivial solutions.

Remark that the rank condition (4) implies that \( |\mathcal{F}| \) must be at least \( 1 + (d - 1)n/2 \).

In dimension \( d \) we have \( (d!)^2 \) jackets and for each of them the relation

\[
2 - 2g(J) = n - |E| + F_J \implies g(J) = 1 + (d - 1)n/4 - F_J/2. \tag{5}
\]

\(^2\)After this fixing the gauge transformations are reduced to a single global conjugation of all remaining \( h_e \) by \( G \).

Since each face belongs to exactly \((d - 1)! \) jackets, we have \( \sum_J F_J = (d - 1)! |\mathcal{F}| \) and the degree of the graph is

\[
\omega(G) = \sum_J g(J) =
\]

\[
\times (d - 1)! |\mathcal{F}|^2 + 2d - 1 + d(d - 1)n/8 - |\mathcal{F}|/2. \tag{6}
\]

It means that for a graph \( G \) with \( H_1(G) = 0 \), the degree obeys the bound

\[
0 \leq \omega(G) \leq (d - 1)! \times |\mathcal{F}|/2 + (d - 1)(d - 2)n/8. \tag{7}
\]

In dimension \( d \), duality exchanges the \( k \)-th and \((d - k)\)-th Betti numbers.

**Lemma 2.** In a graph \( G \) with \( H_1(G) = 0 \), hence dual to a triangulation \( \Gamma \) such that \( H_2(\Gamma) = 0 \), there exists at least one jacket \( J_0 \) whose genus is bounded by

\[
g(J_0) \leq \frac{d - 1}{d} \left[ 1 + \frac{(d - 2)n}{4} \right]. \tag{8}
\]

**Proof.** We just divide the bound (7) by the number \((d!)^2/2 \) of the jackets.

Since spheres have trivial homologies hence zero Betti numbers between 1 and \( d - 1 \), it follows that graphs dual to spherical triangulations all obey Lemma 2. Of course triangulations of true (homotopy) spheres could be much rarer but we will not investigate this question here.

**Low genus bounds.** – In the previous section we proved that colored graphs dual to spherical triangulations must have at least one jacket of relatively low genus. Let us now exploit this condition to bound the number of such graphs. They are Feynman graphs and occur with their correct weights in the perturbative expansion of a random tensor theory with action

\[
e^{\lambda_{T_0} \cdots T_d + \lambda\bar{T}_0 \cdots \bar{T}_d - \sum_{i=0}^d \bar{T}_i T_i}. \tag{9}
\]

where \( T \) and \( \bar{T} \) are tensors whose indices are contracted according to the pattern of the complete graph on \( d + 1 \) vertices. This is detailed at length in [15,17].

Consider now a particular jacket \( J_0 \). Suppress all strata not in that jacket reduces the tensorial action to a matrix action of the type

\[
e^{\lambda \text{Tr} M_0 \cdots M_d + \lambda \text{Tr} M_0 \cdots M_d - \sum_{i=0}^d \text{Tr} M_i M_i^\dagger}. \tag{10}
\]

We know from Euler’s formula that the corresponding ribbon Feynman graphs with \( n \) vertices have genus \( g \) bounded by \( g \leq g_{\text{max}}(n) = I \left( \frac{2 + n(d - 1)}{4} \right) \), where \( I \) means “integer part”. For \( d = 3 \) this means that the genus of a bipartite ribbon graph of the \( \varphi^4 \) type is bounded by \( n/2 \), where \( n \) is the (even) order of the graph.

Let \( T_{d,g,n} \) be the number of ribbon graphs of order \( n \) and genus \( g \) corresponding to the action (10). Our main bound is

**Lemma 3.** There exists a constant \( K_d \) such that

\[
|T_{d,g,n}| \leq K_d n^{2g}. \tag{11}
\]
Proof. Because of the bipartite character of action (10), \( n = 2p \) is even. We want to count the number of Wick contractions matching 4p fields and 4p anti-fields on \( p \) vertices and \( p \) anti-vertices giving rise to a ribbon graph of genus \( g \). The edges of such a graph can always be decomposed into a spanning tree \( T \) of \( n - 1 \) edges, a dual tree \( \bar{T} \) in the dual graph made of \(|F| - 1 = n + 1 - 2g\) edges, and a set of 2g “crossing edges” \( CE \) (see footnote 3). Paying an overall factor \( 2^{2n} \) we can preselect as \((A, \bar{A})\), \((B, \bar{B})\) and \((C, \bar{C})\) the fields and anti-fields which Wick-contract, respectively, into \( T \), \( \bar{T} \) and \( CE \). Building the Wick contractions between the \((n - 1)\) fields of \( A \) and the \( n - 1 \) anti-fields of \( \bar{A} \) to form the labeled tree \( T \) certainly costs at most \((n - 1)!\), the total number of such contractions. Contracting the tree \( T \) to a single vertex we obtain a cyclic ordering of the remaining \( 2(n + 1) \) fields and anti-fields of \( B \cup \bar{B} \cup C \cup \bar{C} \). Let us delete for the moment on the cycle the 2g fields of \( C \) and the 2g anti-fields of \( \bar{C} \). Building the dual tree \( \bar{T} \) out of contractions of the \( n + 1 - 2g \) fields of \( B \) and the \( n + 1 - 2g \) anti-fields of \( \bar{B} \) must create a new face per edge, hence the number of corresponding Wick contractions is bounded by the number of non-crossing matchings between \( B \) and \( \bar{B} \) on the cycle. We know that the total number of such non-crossing matchings between 2p objects is the Catalan number \( C_p \leq (p^2)^p \), hence we obtain a bound \( 4^{n + 1} \) for the Wick contractions of the fields of \( B \) with the anti-fields of \( \bar{B} \). Finally the number of contractions joining the 2g fields of \( C \) to the 2g anti-fields of \( \bar{C} \) to create the edges of \( CE \) can be bounded by any possible way, hence by \((2g)!\). Using the standard vertex symmetry factor \([p]^!\)^2 of Feynman graphs coming from expanding the exponential action in (10) (and since \( g \leq g_{\text{max}}(n) = I(\frac{2 + n(d - 1)}{4}) \)), we easily conclude that building \( T \), \( \bar{T} \) and \( CE \) costs at most \( K^d n^{2d} \) Wick contractions, and we get (11) with \( K_d = 3^d K \).

Notice that we did not try at all to optimize \( K_d \) (in particular in the proof of the Lemma above we did not try to use the colors which give further constraints, as they would not improve on the factor \( n^2g \)). Remark also that the upper bound (11) of Lemma 3 does not contradict the well-established apparently larger asymptotic behavior [37],

\[
T_{g,n} \lesssim n^{-\infty} c_g \cdot n^{5/2(g+1)} 12^n,
\]

for fixed \( g \) and large \( n \) which is, e.g., used to define double scaling in matrix models. Indeed this asymptotic behavior cannot be maintained when \( g \) grows with \( n \), as we know that \( g \leq n/2 \).

Let us neglect fixed powers of \( n \) since they can be absorbed into new \( K^n \) factors. In dimension \( d \) general connected graphs at order \( n \) grow as \( K^n n^{(d-1)/2} \), as expected for a \( \varphi^{d+1} \) interaction. The number of graphs satisfying Lemma 2 on the other hand is bounded by \( (d!/2 \text{ to choose the jacket } J_0) \) times the number of ribbon graphs with genus \( g \) \( \leq \frac{d-1}{2} \left[ 1 + \left( \frac{d-2}{4} \right)^2 \right] \) in that \( J_0 \) jacket. By Lemma 3 it is therefore bounded by \((K')^n \cdot n^{2g(J_0)} \), hence by \((K'')^n \cdot n^{(d-1)/2 - n/2} \).

Putting together these results we obtain:

**Theorem 1.** There exist constants \( K \) and \( K' \) such that the number \( ST_n \) of spherical triangulations with \( n \) simplices is bounded by

\[
ST_n \leq K^n n^{(d-1)(d-2)/4 (d-1)}
\]

and such that the number of general triangulations \( T_n \) obeys

\[
T_n \geq (K')^n n^{2g(J_0)}.
\]

Since in any dimension \( (d-1)(d-2)/4 \) \( < d - 1/2 \), we have

\[
\lim_{n \to \infty} ST_n/T_n = 0,
\]

which means that triangulations of spheres are always rare among general triangulations.

In dimension 3 we get that triangulations of homology spheres grow at most as \( (n!)^{1/3} \) whether general triangulations grow at least as \( n^d \). Hence spherical triangulations cannot grow faster than a cubic root of general triangulations.

**Outlook.** We would like to have a microscopic theory of quantum gravity to sum over all spaces irrespectively of their topology and to generate a macroscopic spacetime such as the one we observe (large and of trivial topology). Since spheres are rare, this cannot be done without some non-trivial ponderation factor to favor them. Random tensor models have precisely such a factor; they ponder triangulations not by 1 but by \( \lambda |V|^N |F| \) where \( \lambda \) plays the role of the cosmological constant, \( N \) is the size of the tensor and \( |F| \) is a discretization of the Einstein-Hilbert action. Indeed for piecewise flat equilateral triangulations, curvature is concentrated on the \((d-2)\)-dimensional simplices, hence is associated to the faces of the dual graphs. In group field theories this connection to the Einstein-Hilbert action extends to non-equilateral triangulations through the addition of the group connections and holonomy conditions.

The 1/\( N \) expansion of random tensor models has melons (i.e., very particular “stacked” triangulations of the spheres) as their leading graphs. It can therefore lead from a perturbative phase around “no space at all” to the condensation of a primitive kind of space-time, namely the continuous random tree (CRT) [38], called branched polymer by physicists. Indeed melons, equipped with the simplest metric, namely the graph distance, are branched polymers [39]. This melonic CRT phase

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3There are usually many different such decompositions but we just choose arbitrarily one of them for each graph.

4The related renormalizable tensorial field theories [25–27,31,32] also favor these special spherical “melonic” triangulations which dominate the renormalization group flow in the deep ultraviolet regime.
(of Hausdorff dimension 2) however cannot be the end of the tensor story. Indeed submelonic triangulations include, e.g., at least all graphs planar in a fixed jacket. Such graphs are exactly well-labeled trees [4]. The labels create shortcuts on the CRT, leading to the very different 2D Brownian sphere phase (of Hausdorff dimension 4 [40])

We hope that investigation of all submelonic contributions in higher rank random tensors will uncover similar but more complicated shortcuts on the melonic CRT. Ideally it could then lead through a sequence of phase transitions to an effective space-time similar to the one we observe, in which, e.g., topological, spectral and Hausdorff dimensions all appear equal to 4.

However, even if this is the case, many mysteries would remain. Let us briefly discuss one of them. The two-dimensional phase transition from planar graphs to Brownian spheres or the higher-dimensional phase transition from melons to the continuous random tree occurs for the unstable sign of the coupling constant \( \lambda \), where all graphs add up with the same sign. But we want to (Borel)-sum all triangulations (not only planar or melonic ones). Random tensors models can do that using the loop vertex expansion [23,41,42], but only for the other sign of the coupling constant, in which amplitudes alternate with their order. It is also for this stable sign of the coupling constant that renormalizable tensor group field theories [25–27,31,32] have been proved asymptotically free [29], meaning the coupling constant \( \lambda \) in such theories grows naturally and should unavoidably reach some critical value and generate a phase transition.

It is tempting but difficult to integrate all these insights into a single coherent picture, ideally that of a renormalizable tensor group field theory whose renormalization group trajectory would lead through a sequence of phase transitions from no space at all to the 4D space we know of, equipped with general relativity as an effective theory. Indeed a major difficulty —underlined, e.g., by Ambjørn [43]— is this incoherence of sign: Borel summability and asymptotic freedom require one sign of the coupling, when the CRT and Brownian sphere phase transitions occur for the other sign.⁵ We propose constructive field theory, and more specifically the loop vertex expansion (LVE) [42], as a very promising tool to connect both phenomena through a rigorous analytic continuation. Indeed the LVE has been recently shown especially adapted to the study of random tensors of rank 3 or more, since the leading tree graphs of the LVE are exactly the melons which dominate the tensorial \( 1/N \) expansion [44].

Answering the Gromov question is not a prerequisite for the tensor track program [45,46], which proposes to ultimately (Borel)-sum over all triangulations anyway.

However, it is unclear whether submelonic corrections and the precise geometrogenetic phase transitions they could generate can be investigated in detail if we remain unable to answer the relatively simple and natural Gromov question. A positive answer would reinforce the analogy between planar graphs and random matrices on one side and spherical triangulations and random tensors on the other. It would lend some weight to the hope that, e.g., the equally weighted measure on spherical triangulations in dimensions 3 and 4 could converge (in the Gromov-Hausdorff sense) to a new kind of compact random space generalizing the Brownian two-dimensional sphere [47–49] to higher dimensions. Also it would open the possibility that double or multiple scalings beyond the melonic graphs could be found within the spherical triangulations; in that case we would expect the result to be stable, in contrast to what happens in matrix models.

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⁵They also realize a nice concrete toy model for the holographic principle, since all information about the faces of the triangulation is captured by the labels on the tree which has a single face as its boundary.

⁶This difficulty also is the main reason for which double scaling in matrix models is unstable.
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