Non-geodesic Motion in General Relativity and Thermodynamics

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Abstract

In a previous article a relationship was established between the linearized metrics of General Relativity associated with geodesics and the Dirac Equation of quantum mechanics. In this paper the extension of that result to arbitrary curves is investigated. In the case of scalar fields, a relationship between mass and temperature is also worked out.

KEY WORDS: non-geodesic motion, quantum wave equations, thermodynamics.

1 Introduction

Before laying out the formalism proper, we need to clarify notation. Throughout the paper, \((\mathcal{M}, g)\) will denote a connected four dimensional Hausdorff manifold, with metric \(g\) of signature -2. At every point \(p\) on the space-time manifold \(\mathcal{M}\) we erect a local tetrad \(e_0(p), e_1(p), e_2(p), e_3(p)\) such that a point \(x\) has coordinates \(x = (x^0, x^1, x^2, x^3) = x^a e_a\) in this tetrad coordinate system, while the spinor \(\psi\) can be written as \(\psi = \psi^i e_i(p)\), where \(\psi^i\) represent the coordinates of the spinor with respect to the tetrad at \(p\). Also at \(p\) we can establish a tangent vector space \(T_p(\mathcal{M})\), with basis \(\{\partial_0, \partial_1, \partial_2, \partial_3\}\)
and a dual 1-form space, denoted by $T_p^*$ with basis \{\(dx_0, dx_1, dx_2, dx_3\)\} at \(p\), defined by

\[
dx^\mu \partial_\nu \equiv \partial_\nu x^\mu = \delta^\mu_\nu.
\]

We refer to the basis \{\(dx^0, dx^1, dx^2, dx^3\)\} as “the basis of one forms dual to the basis \{\(\partial_0, \partial_1, \partial_2, \partial_3\)\} of vectors at \(p\).”

With notation clarified, we note that in a previous paper [3], the Dirac equation associated with quantum mechanics was directly related to motion along a geodesic. The linkage was accomplished in a natural way by associating a generalized Dirac equation with those operators which are duals of differential one-forms, obtained by linearizing the metrics of General Relativity (expressed locally as a Minkowski metric). Specifically if

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta_{ab} dx^a dx^b
\]

where \(a\) and \(b\) refer to local tetrad coordinates and \(\eta\) to a rigid Minkowski metric of signature -2, then associated with this metric and the vector \(ds\) is the scalar \(\bar{ds}\) and a matrix \(\bar{ds} \equiv \gamma_a dx^a\) respectively, where \(\{\gamma_a, \gamma_b\} = 2\eta_{ab}\).

In addition, the \(\bar{ds}\) matrix can be considered as the dual of the expression \(\bar{s} \equiv \gamma^a \frac{\partial}{\partial x^a}\) which in turn enabled us to define a generalized Dirac equation

\[
\bar{s}\psi \equiv \gamma^a \frac{\partial \psi}{\partial x^a} = \frac{\partial \psi}{\partial s},
\]

associated with the motion of a particle along a geodesic.

This contrasts with the usual method of changing classical (including special relativity) Hamiltonians into quantum wave equations. For example, the Dirac equation was first obtained by substituting the momentum operator for the four-momentum term in the linearized relativistic Hamiltonian. Similarly, Erwin Schrödinger used the “purely formal procedure”[1] of replacing \(\frac{\partial W}{\partial t}\) in the Hamilton-Jacobi equation with \(\pm \frac{\hbar}{2\pi} \frac{\partial}{\partial t}\) to obtain his wave equation. In both cases, the transition to quantum mechanics relied upon additional formal assumptions associated with Hilbert Space theory, and the final form of the wave equation depended only indirectly upon the underlying geometry of Minkowski space.
2 Non-geodesic Motion

The generalized Dirac equation defined above relies on the definition of the four-momentum in special relativity and upon the fact that $\tilde{\partial}_s$ and $d\tilde{s}$ are parallel along geodesics, and consequently their product is an exact differential $\frac{d\psi}{ds}d\tilde{s}$. In contrast, when accelerations are introduced we will find that in general

$$\frac{d\tilde{s}}{ds}\tilde{\partial}_s\psi = \frac{1}{2}\left\{\frac{d\tilde{s}}{ds},\tilde{\partial}_s\psi\right\} + \frac{1}{2}\left[\frac{d\tilde{s}}{ds},\tilde{\partial}_s\psi\right]$$

$$= \frac{d\psi}{ds} + \frac{d\tilde{s}}{ds} \wedge \tilde{\partial}_s\psi,$$ \hfill (4)

and that it is the dot product of $d\tilde{s}$ and its dual $\tilde{\partial}_s$ that conserve the form of the exact differential. In addition, no one seems to have noticed that this term can also be directly related to the Hamilton-Jacobi characteristic function \cite{2} associated with a natural motion, which we now formulate as a Lemma and corollary.

**Lemma 1** Let $F(q,t)$ be a function and $\psi(F) = \exp(kF)$ where $k$ is constant then $p^a = \eta^{ab}\frac{\partial F}{\partial q^b}$ iff $k\gamma^a\psi = \eta^{ab}\frac{\partial \psi}{\partial q^b}$.

**Proof:** Trivial. It is sufficient to substitute.

The following corollary immediately follows:

**Corollary 1** If $k = 1$ and $F = S = \int \eta^{ab}p_adq_b = \int Hdt - pdq$ is the Hamilton-Jacobi function then

$$\gamma^a\frac{\partial \psi}{\partial x^a} = \gamma^a p_a\psi.$$ \hfill (6)

Indeed, the Hamilton-Jacobi function can be directly related to the metric expressed locally it tetrad coordinates as follows:

$$S = \int m\frac{ds}{dt}ds = \int mdt - pdq, \quad \text{where} \quad m = m_0\frac{dt}{d\tau},$$ \hfill (7)
and chosen units whereby \( c = 1 \) for the velocity of light.

Equation (6) represents the most general form of a “wave-equation” with respect to a tetrad coordinate system associated with a particle moving along a curve with tangent vector \( \left( \frac{du}{ds}, -\frac{dx^1}{ds}, -\frac{dx^2}{ds}, -\frac{dx^3}{ds} \right) \). In the case of motion along a geodesic, there exists an eigenvector \( \psi \) such that \( \gamma^a p_a \psi = \frac{\partial \psi}{\partial s} = m\psi \) and equation (6) reduces to the Dirac equation

\[
\gamma^a \frac{\partial \psi}{\partial x^a} = m\psi.
\]  

(8)

It is also worth noting that if we take \( k = i = \sqrt{-1} \) that we can also derive the Dirac equation by considering mass to be a gauge term.

In the case of equation (7), the Hamilton-Jacobi function can be re-written in covariant form in a general coordinate system as follows:

\[
dS = g^{\mu \nu} p_\mu dx_\nu,
\]  

(9)

with the corresponding wave equation

\[
\gamma^\mu \frac{\partial \psi}{\partial x^\mu} = \gamma^\mu p_\mu \psi
\]  

(10)

associated with the action along a curve, provided \( 2g^{\mu \nu} = \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu \). Also, the Hamilton-Jacobi function has the advantage that it encapsulates information both about the metric \( d\tilde{s} \) and its dual \( \frac{\partial \psi}{\partial x^\mu} \) in that

\[
d\psi = \frac{\partial \psi}{\partial x_\mu} dx^\mu = g^{\mu \nu} \frac{\partial \psi}{\partial x^\mu} dx^\nu.
\]  

(11)

Moreover from equation (9), we see that the first expression on the right hand side of equation (4) has ten field terms associated with the dot product of the tangent vector with the gradient of the wave function. As a dot product it is the component of \( \tilde{\partial}_s \) along \( \tilde{ds} \) which essentially defines the wave-equation and allows us to clarify the meaning of equation (4) and generalize equation (8) to

\[
\gamma^a \frac{\partial \psi}{\partial x^a} = m\lambda(s)\psi,
\]  

(12)
where \( \lambda(s) = \cos(\theta(s)) \) is a directed cosine which varies along the curve. Note also that when \( \lambda(s) = 1 \) then \( \tilde{ds} \) and its dual spinor \( \tilde{\partial}_s \) are strictly parallel and we get the usual Dirac equation.

Finally, the second term is equivalent to a cross product \( \tilde{d}s \wedge \frac{\partial}{\partial s} \tilde{\psi} \) of the same two terms and plays a role similar to \( E \wedge B \) in Electrodynamics. In fact, it is worth noting the similarity between equation (4) and the famous equation for Lorentz Force on a charge of size \( e \) moving in an electric field \( \vec{E} \):

\[
\vec{F} = e[\vec{E} + (\vec{v} \times \vec{B})].
\] (13)

3 Non-geodesic Motion associated with the Hamiltonian

In the previous section we related the Hamilton-Jacobi characteristic function directly to the general form of the wave equation of quantum mechanics. At the same time because of the equivalence principle it was noted that the general form of the wave equation is determined only locally and not globally, especially when we consider motion along a non-geodesic. Indeed, the existence of non-geodesics suggests that other factors other than gravity are at work. The dynamics in such cases is usually analyzed in terms of test particles. We will continue then for the purpose of this article to use a somewhat “classical” approach to quantum mechanics, in that we will continue to associate a wave \( \psi \) and a generalized Dirac equation with a particle moving along a curve. Moreover, from a mathematical perspective \( \psi \) can be an \( L^p \) function. However, for the purpose of quantum mechanics, we will take \( \psi \in L^2 \) or \( \psi \in L^2 \otimes H \) where \( H \) is a Hilbert Space.

3.1 The Physics Interpretation

Although the wave function can be given a precise mathematical meaning both as an \( L^2 \) function and in terms of probability of the state of the system, from a physics perspective things are more nuanced. The word “state” can be assigned multiple interpretations depending on the physics of the system and on the question been asked. Indeed, in Lemma 1 no restrictions were
put on the wave function, other than the fact that it could be written as $\psi = e^{kF}$. And as it turns out this is a rather weak condition in that any eigenvector solution to a first order differential equation must involve the exponential. The state, therefore, may refer to position, momentum, force, temperature, potential, electric and magnetic fields etc. In its most general form, we can write

$$\gamma_a \frac{\partial \psi}{\partial x^a} = \phi$$

where $\phi$ would be defined by the physics of the problem. For example, Maxwell’s equations in Minkowski space can be written in spinor form as

$$i\alpha^a \frac{\partial \psi}{\partial x^a} = -4\pi \phi,$$

where $\phi_0 = \rho$ is charge density, and $\phi_a = j_a$, $a \in \{1, 2, 3\}$ is a current density. Also in this case, $\phi_0 = 0$ and $\phi_a = H_a - iE_a$, where $H_a$ and $E_a$ are the magnetic and electric fields respectively [5].

For the purpose of this article, we will confine ourselves to working with eigenvector equations, which means we are essentially looking for the invariant states of the system. For example, if $A$ is an operator such that $A\psi = \pm \psi$ then $\psi$ can be interpreted as either an axis of rotation or as the axis of reflection which remain invariant under the operation. These axes correspond to the stable states or the states of equilibrium of the system associated with the operator. One of the challenges then for physics is to find the equilibrium conditions associated with the relevant operators (such as the Hamiltonian or Spin operators), solve their eigenvector equations and then interpret their results. From a methodological perspective, it should be noted that when physical states are not in equilibrium or invariant then they are more difficult to access. This can be seen in the uncertainty principle, where both the position operator $x$ and the momentum operators $p_x$ do not have the same eigenfunctions. Consequently, the physical system cannot be in both invariant states simultaneously and therefore both cannot be measured at the same time.

With these observations in mind, we now reconsider Lemma 1 from the perspective of the Hamiltonian function. Indeed taking our cue from Hamilton’s equations, the canonical equations of motion expressed in a local
tetrad coordinate system are given by
\[ \frac{dx^a}{d\tau} = -g^{ab} \frac{\partial K}{\partial p^b}, \]
\[ \frac{dp^a}{d\tau} = g^{ab} \frac{\partial K}{\partial x^b}, \] (16)
where \( K \equiv H \frac{\partial^2}{\partial \tau^2} \) and \( H = mc^2 \) can be identified with the Hamiltonian as it appears in the Hamiltonian-Jacobi function of Equation (7). It now follows from Lemma 1 that the covariant form of the wave equation associated with the Hamiltonian and the dual of the metric can be written as
\[ \gamma^\mu \frac{\partial}{\partial x^\mu} \psi' = \gamma^\mu Dp_\mu \psi', \] (17)
where \( \psi' = \psi' (K) \) and \( Dp_\mu = \dot{p}_\mu + \Gamma^\mu_{\nu\eta} \dot{p}_\nu p_\eta \). We also note that both \( \psi \) and \( \psi' \) are not in general simultaneous eigenvectors of \( p^\mu \).

For the remainder of this article, we will restrict ourselves to working with a scalar field in Minkowski space, and in so doing avoid problems arising from the connection. We will also drop the prime on \( \psi' \) and write \( \psi \). In the case of a unit rest mass, Equation (17) then reduces to
\[ \gamma^a \frac{\partial}{\partial x^a} \psi (s) = k \gamma_a \left( \frac{d^2 x^a}{ds^2} \right) \psi (s). \] (18)
Note that this is equivalent to introducing a gravitational potential of the form \( \dot{x}^a = \Gamma^a_{bc} \frac{ds}{dx^b} \frac{ds}{dx^c} \). Indeed, the weak field approximation is a special case of equation (18) and reduces to
\[ \nabla \psi (s) = k \gamma^a \left( \frac{d^2 x^a}{ds^2} \right) \psi (s), \] (19)
with the understanding that \( \tilde{t} = 0 \).

In the case of a particle of mass \( m (s) \) we can rewrite equation (19) in the form
\[ m^2 \gamma^a \frac{\partial}{\partial x^a} \psi (s) = km^2 \gamma_a \left( \frac{d^2 x^a}{ds^2} \right) \psi (s), \] (20)
which in Minkowski space is invariant under Lorentz transformations and covariant under a change of curve parameter as expressed in the following lemma:
Lemma 2 Let $\tau$ and $s$ be two parameters of a curve $\sigma \in (\mathcal{M}, g)$ such that \[ \frac{ds}{m(s)} = \frac{d\tau}{M(\tau)} \] along the curve then

\[ m^2 \gamma^a \frac{\partial \psi(s)}{\partial x^a} = k m^2 \gamma_a \frac{d^2 x^a}{ds^2} \psi(s) \]

is covariant under a change of parameter.

Proof: Using \( \frac{ds}{m(s)} = \frac{d\tau}{M(\tau)} \), direct substitution gives:

\[ m^2 \gamma^a \frac{\partial \psi}{\partial x^a} = k m^2 \gamma_a \frac{d^2 x^a}{ds^2} \psi \]

and the result follows.

Finally note that using Special Relativity, we can identify the parameters $m$ and $M$ with mass. Specifically, if $s$ is the proper time and $\tau = t$ the local time then \[ \frac{dt}{ds} = \gamma = \frac{M}{m} \] where $m$ is the rest mass and $M = \gamma m$ is the mass in the moving frame.

3.2 Scalar Fields

It is clear that from a mathematical perspective there are many possible solutions to equation (18) depending on the initial conditions. For example, \( \ddot{x}^a = -k x^a \) describes simple harmonic motion. Moreover, under a parameter change of the form \( \frac{ds}{m(s)} = \frac{d\tau}{M(\tau)} \) (see Lemma), \( p^a = M \frac{dx^a}{d\tau} = M \frac{dx^a}{ds} \frac{ds}{d\tau} = m \frac{dx^a}{ds} \) remains invariant and

\[ ds^2 = \frac{m^2}{M^2} d\tau^2. \] (21)

To avoid confusion, let us consider two different parameterizations for $\sigma$ such that $M = m \frac{ds}{d\tau}$ and $\frac{ds}{d\tau}$ is a variable. Note that it is possible for $M$ to be a variable along the curve if $m$ is a constant and for $m$ to be a variable if $M$ is constant along $\sigma$. Equivalently, it is possible to have two different curves $\sigma_1$ and $\sigma_2$ parameterized by $s$ and $\tau$ respectively such that $m$ is constant along $\sigma_1$ and varies along $\sigma_2$ and vice versa for $M$. For example, in the case
of particle motion along a curve $\sigma$, parameterized by a parameter $\tau$, where $\tau$ denotes the proper time along another curve such that $\dot{x}^a = \frac{dx^a}{d\tau} = \frac{\sigma^a}{M}$, it follows that

$$\int_{\sigma} \dot{x}^a dx^a = \int_{\sigma} \dot{\psi} \dot{x}^a d\tau = \int_{\sigma} \frac{p^a}{M} \frac{p_a}{M} d\tau = \int_{\sigma} \left( \frac{E^2 - p^2}{m^2} \right) d\tau = \int_{\sigma} \frac{m^2}{M^2} d\tau \quad (22)$$

where $m^2 = E^2 - p^2$ is a dynamical variable along the curve $\sigma$. In the case that both curves coincide and $ds = d\tau$ then $M = m$. Moreover, if we consider one of the curves to be a geodesic, then $\frac{ds}{m(a)} = \frac{d\tau}{M}$ are exact differentials, since $M$ is a constant along the geodesic. This suggests a possible relationship to entropy and temperature, which we will make more explicit in the next section.

Finally, we note that if we choose $\tau$ to be the world time of Minkowski space then the approach above is equivalent to the Stueckelberg approach and equation (21) is identical with equation (14) described by Horowitz in his paper *On the Definition and Evolution of States in Relativistic Classical and Quantum Mechanics*.

### 3.3 Relationship of Mass to Temperature

We now solve (18) for a gas of $n$ independent particles. In other words, we are taking the simplest of all models and considering the wave-function associated with each particle to be a scalar field. Specifically

$$m^2 \gamma_a \frac{\partial \psi}{\partial x^a} = -kM^2 \gamma_a \ddot{x}^a \psi, \quad (23)$$

implies that

$$m^2 \gamma_a \frac{\partial \tau}{\partial x^a} \frac{\partial \psi}{\partial \tau} = -kM^2 \gamma_a \ddot{x}^a \psi. \quad (24)$$

Taking the inner product of both sides with $\gamma_a \dot{x}^a = \gamma_a \frac{dx^a}{d\tau}$ gives

$$m^2 \frac{\partial \psi}{\partial \tau} = -kM^2 \dot{x}^a \ddot{x}^a \psi. \quad (25)$$
Solving for $\psi$ gives

$$\psi = e^{-k \frac{E}{T}}$$  \hspace{1cm} (26)$$

such that $\frac{E}{T} = \int M^2 \dot{x}_a \ddot{x}_a d\tau$. In particular, if we consider two different parameterizations along a geodesic (in terms of world time and proper time) then both $M$ and $m$ are constant and in this case we can write

$$\psi = c \exp \left( -\frac{k M^2}{2m^2} (-\dot{t}^2 + \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2) \right).$$ \hspace{1cm} (27)$$

Moreover for a system (gas) of $n$ independent (no interactions between them) identical particles with unit mass (in the $s$ frame) the wave function denoted by $\psi(x_1^n, \ldots, x_n^n)$ is given by

$$\psi = c \exp \left[ -k \left( \frac{M}{m} \right) \left( \frac{M}{2m} \right) \sum_n (-\dot{t}^2 + \dot{x}_1^n + \dot{x}_2^n + \dot{x}_3^n) \right],$$ \hspace{1cm} (28)$$

which at any given $t$ coincides with a Bose-Einstein distribution for free particles with $\psi_t = c \exp \left( -\frac{M}{kT} \sum_n (\dot{x}_1^n + \dot{x}_2^n + \dot{x}_3^n) \right)$ where $T = (k k_B M/m)^{-1}$ plays the role of temperature and $k_B$ is Boltzmann’s constant. Also note that the above distribution could also be considered to represent a Boltzmann’s distribution if the gas were composed of $n$ distinguishable particles. The difference between the two cases would be in the normalizing constants.

**Remarks:**

(1): In the above interpretation we have separated out the variables in the wave function by writing $\psi(t, x) = \psi(t) \psi_t(x)$. We can consider $\psi(t, x)$ to be Lorentz invariant but not $\psi_t(x)$. However, this does not detract from the theory. Rather it indicates the key role of the observer when it comes to a local interpretation of the physics phenomena.

(2) If we let $m$ to be the mass along the curve $\sigma(s)$ and $M$ be the mass defined with respect to the time parameter along a local geodesic then we can define the temperature $T = k \frac{M}{m}$ and note that if the particle has positive acceleration then the temperature is rising, if the temperature is decreasing then the particle has negative acceleration and if there is no acceleration then the temperature is constant. It also would mean that as a massive
particle approaches the velocity of light “c”, its temperature would become infinite.

(3) If one associates absolute zero with the absence of all motion with respect to the “world time” in the Stueckelberg frame of motion then at absolute zero all interactions between matter, including that of the gravitational field, would have to cease. From the viewpoint of General Relativity this would mean that there is no mass at absolute zero, and in this sense the above equation is consistent.

4 Conclusion

The article set out to explore the relationship between particle motion and wave equations within the framework of General Relativity focusing primarily on non-geodesic motion. As noted in a previous article these wave equations can be identified with the wave equations of quantum mechanics if the proper boundary conditions are imposed. In the process, we established for scalar fields a relationship between mass and temperature.

In addition, this approach seems to be comparable to the work of Stueckelberg and Horowitz [4] on the evolution of states in relativistic dynamics. Indeed, if $ds$ is considered to be an independent variable along a curve then $ds^2 = \frac{m^2}{M^2} d\tau^2$, with $M$ and $m$ having the units of mass as in equation (21), and with $\frac{m}{M}$ being associated with an increase of temperature per unit mass. It follows that $ds = \frac{m}{M} d\tau$ is an exact differential along the curve and can be associated with entropy.

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