A mode parabolic equations method with the resonant mode interaction

Trofimov M.Yu.\textsuperscript{1}, Kozitskiy S.B.\textsuperscript{1}, Zakharenko A.D.\textsuperscript{1,2}

\textsuperscript{1}Il'ichev Pacific Oceanological Institute, 43 Baltiiskaya St., Vladivostok, 690041, Russia

\textsuperscript{2}Far Eastern Federal University, 8 Sukhanova str., Vladivostok, 690950, Russia

\textit{e-mail: trofimov@poi.dvo.ru, skozi@poi.dvo.ru, zakharenko@poi.dvo.ru}

Abstract

A mode parabolic equation method for resonantly interacted modes was developed. The flow of acoustic energy is conserved for the derived equations with an accuracy adequate to the used approximation. The testing calculations were done for ASA wedge benchmark and proved excellent agreement with COUPLE program.

1 Introduction

Adiabatic acoustic equations appeared as a convenient tool for solving three-dimensional problems of ocean acoustics since the work of Collins \textsuperscript{[1]} and in refined version since the work of Trofimov \textsuperscript{[2]}. Further this method was extended to interactive modes \textsuperscript{[3]}. Needless to say, that all this goes back to Burridge and Weinberg \textsuperscript{[4]}. Approach to interacting modes was known outside the parabolic equation scope \textsuperscript{[5, 6, 7]}.

Here the method of adiabatic mode parabolic equation is extended to the case of resonantly interacting modes, more concretely the considered interaction arises when the wavenumbers of modes are close to each other. The most intensive interaction of such a type is observed when the mode of discrete spectrum transforms into the mode of continuous spectrum, during the propagation, or vice versa. Such transmutations of modes are common for shallow-water acoustics. We derive a system of mode parabolic equations which describes this situation. It can be easily solved numerically by the Crank-Nicholson implicit difference scheme in combination with the Gauss-Seidel iteration method. The application of the corresponding computer code to the ASA wedge benchmark problem gives the excellent results.

An additional advantage of our method is that the systematic use of multiscale expansions gives to the applied approach strictness and completeness.
2 Basic equations and boundary conditions

We consider the propagation of time-harmonic sound in the three-dimensional waveguide \( \Omega = \{(x, y, z)|0 \leq x \leq \infty, -\infty \leq y \leq \infty, -H \leq z \leq 0 \} \) (z-axis is directed upward), described by the acoustic Helmholtz equation

\[
(\gamma P)_x + (\gamma P)_y + (\gamma P)_z + \gamma \kappa^2 P = 0 ,
\]

(2.1)

where \( \gamma = 1/\rho \), \( \rho = \rho(x, y, z) \) is the density, \( \kappa(x, y, z) \) is the wavenumber.

We assume the appropriate radiation conditions at infinity in \( x, y \) plane, the pressure-release boundary condition at \( z = 0 \)

\[
P = 0 \quad \text{at} \quad z = 0 ,
\]

(2.2)

and rigid boundary condition \( \partial u/\partial z = 0 \) at \( z = -H \). At \( x = 0 \) we impose the Dirichlet boundary condition

\[
P = g(z, y) \quad \text{at} \quad x = 0 ,
\]

(2.3)

modelling the sound source located outside \( \Omega \). The parameters of medium may be discontinuous at the nonintersecting smooth interfaces \( z = h_1(x, y), \ldots, h_m(x, y) \), where the usual interface conditions

\[
\gamma_+ \left( \frac{\partial P}{\partial z} - h_x \frac{\partial P}{\partial x} - h_y \frac{\partial P}{\partial y} \right)_+ = \gamma_- \left( \frac{\partial P}{\partial z} - h_x \frac{\partial P}{\partial x} - h_y \frac{\partial P}{\partial y} \right)_- ,
\]

(2.4)

are imposed. Hereafter we use the denotations \( f(z_0, x, y)_+ = \lim_{z \searrow z_0} f(z, x, y) \) and \( f(z_0, x, y)_- = \lim_{z \nearrow z_0} f(z, x, y) \).

As will be seen below, it is sufficient to consider the case \( m = 1 \), so we set \( m = 1 \) and denote \( h_1 \) by \( h \).

Assuming that \( x \)-axis is the preferred direction of propagation, we introduce a small parameter \( \epsilon \) (the ratio of the typical wavelength to the typical size of medium inhomogeneities), the slow variables \( X = \epsilon x \) and \( Y = \epsilon^{1/2} y \) (the so called “parabolic scaling”) and postulate the following expansions for the parameters \( \kappa^2, \gamma \) and \( h \):

\[
\kappa^2 = \kappa_0^2(X, z) + \epsilon \nu(X, Y, z) ,
\gamma = \gamma_0(X, z) + \epsilon \gamma_1(X, Y, z) ,
\]

(2.5)

\[
h = h_0(X) + \epsilon h_1(X, Y) .
\]

To model the attenuation effects we admit \( \nu \) to be complex. Namely, we take \( \text{Im} \nu = 2\eta \beta n_0 \), where \( \eta = (40 \pi \log_{10} \epsilon)^{-1} \) and \( \beta \) is the attenuation in decibels per wavelength. This implies that \( \text{Im} \nu \geq 0 \).
At first we consider a solution to the Helmholtz equation (2.1) in the form of the WKB-ansatz

$$P = (u_0(X,Y,z) + \epsilon u_1(X,Y,z) + \ldots) \exp \left( \frac{i}{\epsilon} \theta(X,Y,z) \right). \quad (2.6)$$

Introducing this ansatz into equation (2.1), boundary condition (2.2) and interface conditions (2.4), all rewritten in the slow variables, we obtain the sequence of the boundary value problems at each order of $\epsilon$.

From the equations at $O(\epsilon^{-2})$ and $O(\epsilon^{-1})$ we can conclude that $\theta$ is independent of $z$ and $Y$. Using this information and the Taylor expansion, we can formulate the interface conditions at $h_0$ which are equivalent to interface conditions (2.4) up to $O(\epsilon^2)$:

$$\begin{align*}
(u_0 + \epsilon h_1 u_0)_+ &= (\text{the same terms})_-, \quad (2.7) \\
((\gamma_0 + \epsilon h_1 \gamma_0 + \epsilon \gamma_1 \times (u_{0z} + \epsilon h_1 u_{0zz} + \epsilon u_{1z} - \epsilon i k h_0 u_0))_+ &= (\text{the same terms})_-.
\end{align*} \quad (2.8)$$

### 3 The problem at $O(\epsilon^0)$

At $O(\epsilon^0)$ we obtain

$$(\gamma_0 u_{0z})_z + \gamma_0 n_0^2 - \gamma_0 (\theta_X)^2 u_0 = 0, \quad (3.1)$$

with the interface conditions of the order $\epsilon^0$

$$u_{0+} = u_{0-}, \quad \left( \gamma_0 \frac{\partial u_0}{\partial z} \right)_+ = \left( \gamma_0 \frac{\partial u_0}{\partial z} \right)_- \quad \text{at} \quad z = h_0, \quad (3.2)$$

and boundary conditions $u = 0$ at $z = 0$ and $\partial u / \partial x$ at $z = -H$. We seek a solution to problem (3.1), (3.2) in the form

$$u_0 = A(X,Y)\phi(X,z). \quad (3.3)$$

From eqs. (3.1) and (3.2) we obtain the following spectral problem for $\phi$ with the spectral parameter $k^2 = (\theta_X)^2$

$$\begin{align*}
(\gamma_0 \phi_z)_z + \gamma_0 n_0^2 \phi - \gamma_0 k^2 \phi &= 0, \\
\phi(0) &= 0,
\end{align*} \quad (3.4)$$

$$\begin{align*}
\frac{\partial \phi}{\partial z} &= 0 \quad \text{at} \quad z = -H, \\
\phi_+ &= \phi_-,
\end{align*} \quad \text{at} \quad z = h_0.$$
This spectral problem, considering in the Hilbert space \( L_2, \gamma_0 \) with the scalar product
\[
(\phi, \psi) = \int_{-H}^{0} \gamma_0 \phi \psi \, dz,
\]
has countably many solutions \((k_j^2, \phi_j)\), \( j = 1, 2, \ldots \) where the eigenfunction can be chosen as real functions. The eigenvalues \( k_j^2 \) are real and have \(-\infty\) as a single accumulation point \([9]\).

Let \( u_0 = A_j(X,Y)\phi_j(X,z) \) where \( \phi_j \) is a normalized eigenfunction with the corresponding eigenvalue \( k_j^2 > 0 \) and \( A_j \) is an amplitude function to be determined at the next order of \( \epsilon \). The normalizing condition is
\[
(\phi, \phi) = \int_{-H}^{0} \gamma_0 \phi^2 \, dz,
\]

\section{4 The derivatives of eigenfunctions and wavenumbers with respect to \( X \)}

Before considering the problem at \( O(\epsilon^1) \) we should consider the problem of calculation the derivatives of eigenfunctions and wavenumbers with respect to \( X \).

Differentiating spectral problem (3.4) with respect to \( X \) we obtain the boundary value problem for \( \phi_jX \)
\[
(\gamma_0 \phi_jX)_z + \gamma_0 n_0^2 \phi_jX - \gamma_0 k_j^2 \phi_jX = - (\gamma_0 X \phi_jz)_z - (\gamma_0 n_0^2 X \phi_j + 2k_jXk_j\gamma_0 \phi_j + \gamma_0 X k_j^2 \phi_j),
\]
\[
\phi_jX(0) = 0, \quad \phi_jX_z(-H) = 0,
\]
with interface conditions at \( z = h_0 \)
\[
\phi_jX_+ - \phi_jX_- = - h_0X (\phi_jz_+ - \phi_jz_-),
\]
\[
\gamma_0+ \phi_jX z_+ - \gamma_0- \phi_jX z_- = - (\gamma_0 X \phi_jz_+ - \gamma_0 X \phi_jz_-) - h_0X \left( ((\gamma_0 \phi_jz)_z)_+ - ((\gamma_0 \phi_jz)_z)_- \right).
\]

We seek a solution to problem (4.1), (4.2) in the form
\[
\phi_jX = \sum_{l=0}^{\infty} C_{j,l} \phi_l,
\]
where
\[
C_{j,l} = \int_{-H}^{0} \gamma_0 \phi_jX \phi_l \, dz.
\]
Multiplying (4.1) by $\phi_l$ and then integrating resulting equation from $-H$ to 0 by parts twice with the use of interface conditions (4.2), we obtain

$$
(k_l^2 - k_j^2) C_{jl} = \int_{-H}^{0} \gamma_0 x \phi_j \phi_l dz - \int_{-H}^{0} \left( \gamma_0 u_0 \right)_X \phi_j \phi_l dz + 2k_j k_j \delta_{jl} + k_j^2 \int_{-H}^{0} \gamma_0 x \phi_j \phi_l dz + \left\{ h_0 \times \sum_{k=0}^{N} \left[ \left( \frac{1}{\gamma_0} \right) + \left( \frac{1}{\gamma_0} \right)_- \right] - h_0 \times \phi_j \phi_l \left[ \left( \gamma_0 (k_j^2 - n_0^2) \right)_+ - \left( \gamma_0 (k_j^2 - n_0^2) \right)_- \right] \right\} \bigg|_{z=h_0},
$$

(4.5)

where $\delta_{jl}$ is the Kronecker delta. The coefficients $C_{jl}$ can be found from this equation when $l \neq j$ and at $l = j$ we have the formula for $k_j X$. Differentiating normalizing condition (3.6) we obtain

$$
\left( \int_{-H}^{0} \gamma_0 \phi_j^2 dz \right)_X \bigg|_{X=h_0} = \left( \int_{-H}^{0} \gamma_0 \phi_j^2 dz \right)_X + \left( \int_{-H}^{0} \gamma_0 \phi_j^2 dz \right)_X = \int_{-H}^{0} \gamma_0 \phi_j^2 dz + 2 \int_{-H}^{0} \gamma_0 \phi_j \phi_j dz + h_0 \times \phi_j^2 \left[ \gamma_0 - \gamma_0 \right]_{z=h_0} = 0,
$$

(4.6)

which gives the equation for $C_{jj}$:

$$
2C_{jj} = - \int_{-H}^{0} \gamma_0 \phi_j^2 dz + h_0 \times \phi_j^2 \left[ \gamma_0 - \gamma_0 \right]_{z=h_0}.
$$

(4.7)

5 The problem at $O(\epsilon^1)$

Let $\{\theta_j | j = M, \ldots, N\}$ be a set of phases. We seek a solution to the Helmholtz equation (2.1) in the form

$$
P = \sum_{j=M}^{N} (u_0^{(j)}(X, Y, z) + \epsilon u_1^{(j)}(X, Y, z) + \ldots) \exp \left( \frac{i}{\epsilon} \theta_j \right).
$$

(5.1)

At $O(\epsilon^1)$ we obtain

$$
\sum_{j=M}^{N} \left( \left( \gamma_0 u_{1z}^{(j)} \right)_z + \gamma_0 n_0^2 u_1^{(j)} - \gamma_0 k_j^2 u_1^{(j)} \right) \exp \left( \frac{i}{\epsilon} \theta_j \right)
$$

$$
= \sum_{j=M}^{N} \left( -i \gamma_0 k_j u_0^{(j)} - 2i \gamma_0 k_j u_0^{(j)} - i \gamma_0 k_j u_0^{(j)} + \gamma_1 k_j^2 u_0^{(j)} - \gamma_0 u_0^{(j)} \nu \gamma_0 u_0^{(j)} \right) \exp \left( \frac{i}{\epsilon} \theta_j \right),
$$

(5.2)
with the boundary conditions \( u_1^{(j)} = 0 \) at \( z = 0 \), \( \partial u_1^{(j)}/\partial z = 0 \) at \( z = -H \), and the interface conditions at \( z = h_0(X,Y) \) for each \( j = M, \ldots, N \):

\[
\begin{align*}
&u_1^{(j)}(z) - u_1^{(j)} = h_1(u_{0z}^{(j)} - u_{0z}^{(j)}) , \\
&\gamma_0+u_1^{(j)}(z) - \gamma_0 u_1^{(j)} = h_1 \left( \left( \gamma_0 u_{0z}^{(j)} \right)_+ - \left( \gamma_0 u_{0z}^{(j)} \right)_- \right) \\
&\quad + \gamma_1-u_1^{(j)}(z) - \gamma_1+u_1^{(j)} - ik_j h_0 x u_0^{(j)} (\gamma_0 - \gamma_0) \tag{5.3}
\end{align*}
\]

We seek a solution to problem (5.2), (5.3) in the form

\[
u_1^{(j)} = \sum_{l=0}^{\infty} B_{jl}(X,Y) \phi_l(z,X), \tag{5.4}
\]

and introduce coefficients \( E_{jl} \) at \( j \neq l \) by the equality

\[
A_j E_{jl} = B_{jl} = \int_{-H}^{0} \gamma_0 \nu_1^{(j)} \phi_l \, dz. \tag{5.5}
\]

Multiplying (5.2) by \( \phi_l \) and then integrating resulting equation from \(-H\) to 0 by parts twice with the use of interface conditions (5.3), we obtain

\[
\sum_{j=M}^{N} \left( A_j \cdot \left\{ \frac{h_1}{\gamma_0} \left[ \left( \frac{\gamma_1}{\gamma_0} \right)_+ - \left( \frac{\gamma_1}{\gamma_0} \right)_- \right] - ik_j h_0 X \phi_j \phi_l [\gamma_0+ - \gamma_0-] \\
- h_1 \gamma_0^2 \phi_j \phi_l \left[ \left( \frac{1}{\gamma_0} \right)_+ - \left( \frac{1}{\gamma_0} \right)_- \right] \right\}_{z=h_0} + A_j (k_j^2 - k_j^2) E_{jl} \right) \exp \left( \frac{i}{\epsilon} \theta_j \right)
\]

\[
= \sum_{j=M}^{N} \left( -ik_j A_j \int_{-H}^{0} \gamma_0 X \phi_j \phi_l \, dz - 2ik_j A_j \int_{-H}^{0} \gamma_0 X \phi_l \, dz \right.
\]

\[
- 2ik_j A_j X \int_{-H}^{0} \gamma_0 \phi_j \phi_l \, dz - ik_j X A_j \int_{-H}^{0} \gamma_0 \phi_j \phi_l \, dz \\
+ k_j^2 A_j \int_{-H}^{0} \gamma_1 \phi_j \phi_l \, dz - A_j Y \int_{-H}^{0} \gamma_0 \phi_j \phi_l \, dz \\
- A_j \int_{-H}^{0} \left( \gamma_1 \phi_j \phi_l \right)_+ \, dz - A_j \int_{-H}^{0} \gamma_1 n_0^2 \phi_j \phi_l \, dz \\
\left. - A_j \int_{-H}^{0} \nu \phi_j \phi_l \, dz \right) \exp \left( \frac{i}{\epsilon} \theta_j \right) \tag{5.6}
\]
The terms $A_j(k_j^2 - k_l^2)E_{jl}$ in these expressions can be omitted because of the resonant condition $|k_l - k_j| < \epsilon$.

As

$$-ik_jA_j \int_{-H}^{0} \gamma_0 \phi_j \phi_l dz - 2ik_jA_j \int_{-H}^{0} \gamma_0 \phi_j \phi_l dz$$

$$= ik_jA_j(C_{lj} - C_{jl}) - ik_jA_j h_0 \phi_j \phi_l [\gamma_0^+ - \gamma_0^-]_{z=h_0} ,$$

and

$$-A_j \int_{-H}^{0} (\gamma_1 \phi_{jl})_z \phi_l dz = A_j \int_{-H}^{0} \gamma_1 \phi_{jl} \phi_l dz$$

$$+ A_j \gamma_0 \phi_{jl} \phi_l \left[ \left( \frac{\gamma_0^+}{\gamma_0^-} \right) - \left( \frac{\gamma_0^-}{\gamma_0^-} \right) \right] ,$$

we obtain, after some algebra,

$$\sum_{j=M}^{N} \left( A_j \cdot \left\{ h_1 \phi_l \left[ \left( \gamma_0 \phi_{jl} \right)_+ - \left( \gamma_0 \phi_{jl} \right)_- \right] - h_1 \gamma_0^2 \phi_{jl} \phi_l \right\} \right) \exp \left( \frac{i}{\epsilon} \theta_j \right)$$

$$= \sum_{j=M}^{N} \left( ik_jA_j (C_{lj} - C_{jl}) - 2ik_jA_jX \delta_{jl} - ik_jX A_j \delta_{jl} - A_jYY \delta_{jl} \right)$$

$$+ k^2_jA_j \int_{-H}^{0} \gamma_1 \phi_j \phi_l dz + A_j \int_{-H}^{0} \gamma_1 \phi_{jl} \phi_l dz - A_j \int_{-H}^{0} \gamma_1 \phi_{jl} \phi_l dz - A_j \int_{-H}^{0} \nu \gamma_0 \phi_{jl} \phi_l dz$$

$$-A_j \int_{-H}^{0} \nu \gamma_0 \phi_{jl} \phi_l dz \right) \exp \left( \frac{i}{\epsilon} \theta_j \right) .$$

**Proposition 5.1.** The solvability condition for the problem at $O(\epsilon^1)$ is a system of parabolic wave equations for $l = M, \ldots, N$

$$2ik_lA_{l,X} + ik_lX A_l + A_{l,Y} + \alpha_{l} A_l + \sum_{j=M,j\neq l}^{N} \alpha_{lj} A_j \exp(\theta_{lj}) = 0 ,$$
where $\alpha_{lj}$ is given by the following formula
\begin{equation}
\alpha_{lj} = \int_{-\infty}^{0} \gamma_0 \phi_j \phi_l \, dz + \int_{-\infty}^{0} \gamma_1 \left( n_0^2 - k_j^2 \right) \phi_j \phi_l \, dz - \int_{-\infty}^{0} \gamma_1 \phi_j \phi_{lz} \, dz \\
- i k_j \left( C_{lj} - C_{jl} \right) + \left\{ h_1 \phi_l \left[ ((\gamma_0 \phi_{jz})_z)_+ - ((\gamma_0 \phi_{jz})_z)_- \right] - \right. \\
- h_1 \gamma_0^2 \phi_{jz} \phi_{lz} \left[ \left( \frac{1}{\gamma_0} \right)_+ - \left( \frac{1}{\gamma_0} \right)_- \right] \right\} \bigg|_{z=h_0}.
\end{equation}

and
\begin{equation}
\theta_{lj} = i(\theta_l - \theta_j) \quad (5.10)
\end{equation}

Using spectral problem (3.4) the interface terms in (5.9) can be rewritten also as
\begin{equation}
\left\{ h_1 \phi_l \left[ k_j^2 \left( \gamma_0^+ - \gamma_0^- \right) - \left( n_0^2 \gamma_0 \right)_+ + \left( n_0^2 \gamma_0 \right)_- \right] \right. \\
- h_1 \gamma_0^2 \phi_{jz} \phi_{lz} \left[ \left( \frac{1}{\gamma_0} \right)_+ - \left( \frac{1}{\gamma_0} \right)_- \right] \right\} \bigg|_{z=h_0}.
\end{equation}

We shall refer to the quantities and variables $X, Y, \nu, \theta_j$, and $A_j$ as the asymptotic ones. Considerations of initial-boundary value problems in a (partially) bounded domain require the use of physical quantities and variables, which will be $x, y, \tilde{\nu} = \epsilon \nu$, $\tilde{A}_j(x, y) = A_j(\epsilon x, \sqrt{\epsilon} y) = A_j(X, Y)$, and
\begin{equation}
\tilde{\theta}_j = \int X^{1 \epsilon \theta_j, x} \, dX = \int x k_j \, dx.
\end{equation}

It can be easily verified that equations (5.8) in physical variables has the same form
\begin{equation}
2 ik_l \tilde{A}_{l,x} + ik_l \tilde{A}_j + \tilde{A}_{l,yy} + \tilde{\alpha}_{ll} \tilde{A}_l + \sum_{j=M, j \neq l}^{N} \tilde{\alpha}_{lj} \tilde{A}_j \exp(\tilde{\theta}_{lj}) = 0, \quad (5.11)
\end{equation}

where $\tilde{\alpha}_{lj}$ are expressed by the same formulas as $\alpha_{lj}$ with $\nu$ replaced by $\tilde{\nu}$, $\tilde{\theta}_{lj} = (\theta_l - \theta_j)$.

6 Initial-boundary value problems for mode parabolic equation

For eq. (5.11) we shall consider the initial-boundary value problem in domain of the form \{(x, y) | 0 \leq x < \infty, Y_1 \leq y \leq Y_2\} with the initial condition
\begin{equation}
\tilde{A}_j(0, y) = g_j(y) = (g, \phi_j) = \int_{-H}^{0} \gamma_0 g(z, y) \phi_j(z) \, dz, \quad (6.1)
\end{equation}
interface conditions (6.4) and transparent boundary conditions at \( y = Y_1 \) and \( y = Y_2 \).

### 6.1 Vertical interfaces and boundaries and corresponding interface and boundary conditions

We consider vertical interfaces along smooth curves of the form \( \{(x, y) | y = \mathcal{I}(x)\} \). Such an interface is formed mostly by the jump of topography \( h_1 \) at \( y = \mathcal{I}(x) \). The usual interface conditions for eq. (2.1) are

\[
\mathcal{P}|_{y=\mathcal{I}(x)+0} = \mathcal{P}|_{y=\mathcal{I}(x)-0} \quad \gamma \frac{\partial \mathcal{P}}{\partial n} \bigg|_{y=\mathcal{I}(x)+0} = \gamma \frac{\partial \mathcal{P}}{\partial n} \bigg|_{y=\mathcal{I}(x)+0},
\]

where \( \partial / \partial n \) denotes the normal derivative. Assuming that \( \mathcal{P} = \exp(\bar{\theta}_j) \bar{A}_j \phi_j \), we have

\[
\gamma \frac{\partial \mathcal{P}}{\partial n} = \gamma \mathcal{P}_y - \gamma \mathcal{I}_x \mathcal{P}_x = \gamma_0 \exp(\bar{\theta}) [A_{j,y} - i\mathcal{I}_x k_j A_j] \phi_j + O(\epsilon).
\]

Multiplying eq. (6.3) by \( \phi_j \) and integrating with respect to \( z \), we have

\[
\int_{-H}^0 \gamma \frac{\partial \mathcal{P}}{\partial n} \, dz = \exp(\bar{\theta}) [A_{j,y} - i\mathcal{I}_x k_j A_j] + O(\epsilon),
\]

The interface conditions at \( y = \mathcal{I}(x) \) modulo \( \epsilon \) now become

\[
A_j_L = A_j_R, \tag{6.4}
\]

\[
(A_{j,y} - i\mathcal{I}_x k_j A_j)_L = (A_{j,y} - i\mathcal{I}_x k_j A_j)_R,
\]

where we use the denotations \( f(x_0, y_0)_R = \lim_{y_0 \downarrow y_0} f(x_0, y) \) and \( f(x_0, y_0)_L = \lim_{y_0 \uparrow y_0} f(x_0, y) \), \((x_0, y_0)\) is the interface point. As \( k_j \) is assumed to be continuous through the interface, then finally the interface conditions take the form

\[
A_j_L = A_j_R, \tag{6.5}
\]

\[
(A_{j,y})_L = (A_{j,y})_R.
\]

The analogous considerations give the following boundary conditions at the boundary \( y = \mathcal{B}(x) \):

\[
A_j|_{y=\mathcal{B}(x)} = 0
\]
at the soft boundary and

\[
(A_{j,y} - i\mathcal{B}_x k_j A_j)|_{y=\mathcal{B}(x)} = 0 \tag{6.6}
\]
at the rigid boundary.
7 Energy flux conservation for parabolic equations (5.11)

The time averaged acoustic energy flux through the plane \( x = x_0 \) is defined as

\[
J(x_0) = \frac{1}{2\omega} \text{Im} \int_{-\infty}^{\infty} I(x_0, y) \, dy,
\]

where

\[
I(x_0, y) = \int_{-H}^{0} \gamma P_x(x_0, y, z) P^*(x_0, y, z) \, dz,
\]

\( P^* \) is the complex conjugate of \( P \).

As is well known, if \( P \) is a solution of the Helmholtz equation (2.1) then the corresponding energy flux is conserved, that is

\[
\frac{dJ}{dx} = 0.
\]

Proposition 7.1. Assume that \( \text{Im} \tilde{\nu} = 0 \). Let \( \{ A_j | j = M, \ldots, N \} \) be a solution to equations (5.11) with interface conditions (6.5) and boundary condition (6.6). Then for

\[
P = \sum_{j=M}^{N} A_j \exp(i\theta_j) = \sum_{j=M}^{N} \bar{A}_j \exp\left(i\frac{\epsilon}{\epsilon} \theta_j\right)
\]

\[
\frac{dJ}{dx} = O(\epsilon^2)
\]

Proof. First calculate the derivative of the flux with respect to \( x \) for the anzats used:

\[
2\omega \frac{dJ}{dx} = \frac{d}{dx} \int_{-\infty}^{\infty} \left[ \sum_{l=M}^{N} k_l |A_l|^2 
\right. \\
+ \epsilon \sum_{l=M}^{N} \sum_{j=M}^{N} \text{Im} \left( C_{lj} A_l A_j^* \exp \left( i \frac{\epsilon}{\epsilon} (\theta_l - \theta_j) \right) \right) \\
+ \epsilon \sum_{l=M}^{N} \text{Im}(A_l X A_l^*) \right] \, dy
\]

\[
= \int_{-\infty}^{\infty} \left[ \epsilon \sum_{l=M}^{N} \sum_{j=M}^{N} (k_l - k_j) C_{lj} \text{Re} \left( A_j A_l^* \exp \left( i \frac{\epsilon}{\epsilon} (\theta_j - \theta_l) \right) \right) \\
+ \epsilon \sum_{l=M}^{N} (k_l |A_l|^2) x \right] \, dy + O(\epsilon^2).
\]

Consider now the sum on \( l \) of the equations (5.8) multiplied by \( A_l^* \) subtracted conjugate equation multiplied by \( A_l \) and integrate the result on \( y \) from minus
infinity to infinity:
\[
\sum_{l=M}^{N} \int_{-\infty}^{\infty} \left[ \left( 2ik_{l}A_{l,XX} + ik_{l,XX}A_{l} + A_{l,YY} + \sum_{j=M}^{N} \alpha_{lj}A_{j} \exp(\theta_{lj}) \right) A_{l}^* - \right.
\]
\[
\left. \left( -2ik_{l}A_{l,RR}^* - ik_{l,RR}A_{l}^* + A_{l,YY} + \sum_{j=M}^{N} \alpha_{lj}^*A_{j}^* \exp(\theta_{lj}^*) \right) A_{l} \right] dy = 0.
\]

Further the terms with the second derivative with respect to \(Y\) vanish due to boundary conditions. After some transformation we have:
\[
\sum_{l=M}^{N} \sum_{j=M}^{N} \int_{-\infty}^{\infty} (\alpha_{lj}A_{j} \exp(\theta_{lj})A_{l}^* - \alpha_{lj}^*A_{j} \exp(\theta_{lj}^*)A_{l}) dy
\]
\[
+ \sum_{l=M}^{N} 2i \int_{-\infty}^{\infty} (k_{l}|A_{l}|^2)_{X} dy = 0,
\]
then substitute for \(\alpha_{lj}\) its expression (5.9)
\[
\sum_{l=M}^{N} \sum_{j=M}^{N} \int_{-\infty}^{\infty} \left( -ik_{j}(C_{lj} - C_{jl})A_{j} \exp\left(\frac{i}{\epsilon}(\theta_{j} - \theta_{l})\right)A_{l}^* + \right.
\]
\[
\left. -ik_{j}(C_{lj} - C_{jl})A_{j}^* \exp\left(\frac{i}{\epsilon}(\theta_{l} - \theta_{j})\right)A_{l} \right) dy + \sum_{l=M}^{N} 2i \int_{-\infty}^{\infty} (k_{l}|A_{l}|^2)_{X} dy = 0,
\]
and collect terms
\[
\sum_{l=M}^{N} \sum_{j=M}^{N} \int_{-\infty}^{\infty} \left( -ik_{j}(C_{lj} - C_{jl})2 \text{Re}(A_{j} \exp\left(\frac{i}{\epsilon}(\theta_{j} - \theta_{l})\right))A_{l}^* \right) dy
\]
\[
+ \sum_{l=M}^{N} 2i \int_{-\infty}^{\infty} (k_{l}|A_{l}|^2)_{X} dy = 0,
\]
write double sums separately for terms with \(C_{lj}\) and \(C_{jl}\)
\[
\sum_{l=M}^{N} \sum_{j=M}^{N} \int_{-\infty}^{\infty} \left( -ik_{j}C_{lj}2 \text{Re}(A_{j} \exp\left(\frac{i}{\epsilon}(\theta_{j} - \theta_{l})\right))A_{l}^* \right) dy
\]
\[
+ \sum_{l=M}^{N} \sum_{j=M}^{N} \int_{-\infty}^{\infty} \left( ik_{j}C_{jl}2 \text{Re}(A_{j} \exp\left(\frac{i}{\epsilon}(\theta_{j} - \theta_{l})\right))A_{l}^* \right) dy
\]
\[
+ \sum_{l=M}^{N} 2i \int_{-\infty}^{\infty} (k_{l}|A_{l}|^2)_{X} dy = 0,
\]
Figure 1: ASA wedge benchmark geometry. Harmonic point source with frequency 25 Hz is located at depth 100 m. Sound speed in the water layer is 1.5 km/s, in the bottom is 1.7 km/s. Density of the water is 1000 kg/m³, of the bottom is 1500 kg/m³. Attenuation in the water is absent, in the bottom is 0.5 dB/λ till the depth of 1 km, then attenuation linearly increases up to 2.5 dB/λ at depth 1.5 km.

exchange indexes \( l \) and \( j \) in the second double sum and finally get

\[
\sum_{l=M}^{N} \sum_{j=M}^{N} \int_{-\infty}^{\infty} \left( i(k_l - k_j)C_{lj}2 \text{Re}(A_j \exp(i\frac{y}{\epsilon}(\theta_j - \theta_l))A_l^*) \right) dy \\
+ \sum_{l=M}^{N} 2i \int_{-\infty}^{\infty} (k_l|A_l|^2) X \ dy = 0 ,
\]

The last equation coincides modulo 2i with the \( O(\epsilon) \)-part of (7.1).

8 Conclusion

In this article a mode parabolic equation method for resonantly interacted modes was developed. The proposed method is an essential extension of the early proposed method of adiabatic mode parabolic equation because it can
Figure 2: Mode waveforms at distance $X=2$ km and wavenumbers $k_j(X)$ for the ASA wedge.
Figure 3: Transmission loss for the ASA wedge, receiver depth=30 m. Comparison with COUPLE program (Meansquare error=0.9 dB) and Adiabatic MPE.

serve all possible problems of shallow water acoustics. The flow of acoustic energy is conserved for the derived equations with an accuracy adequate to the used approximation. The proposed method was tested. The testing calculations were done for ASA wedge benchmark and proved excellent agreement with COUPLE program.

References

[1] Collins M. D. The adiabatic mode parabolic equation // J. Acoust. Soc. Amer. 1993. V. 94, N. 4. P. 2269-2278.

[2] Trofimov M.Yu. Narrow-angle parabolic equations of adiabatic single-mode propagation in horizontally inhomogeneous shallow sea // Acoust. Phys. 1999, V. 45., P. 575-580.

[3] Abawi, A. T., Kuperman, W. A., Collins, M. D. The coupled mode parabolic equation // J. Acoust. Soc. Amer. 1997. V. 102, N. 1. P. 233-238.
Figure 4: Transmission loss for the ASA wedge, receiver depth=150 m. Comparison with COUPLE program (Meansqu. error=1.3 dB) and Adiabatic MPE.

[4] Burridge, R. & Weinberg, H. Horizontal rays and vertical modes. In Wave propagation and underwater acoustics, ed. by J.R.Keller and I.S.Papadakis, Lecture Notes in Physics, Vol. 70. Springer-Verlag, NewYork, 1977.

[5] F. B. Jensen, W. A. Kuperman, M. B. Porter and H. Schmidt // Computational Ocean Acoustics (AIP Press, New York, 1994).

[6] M. B. Porter and E. L. Reiss, A numerical method for bottom interacting ocean acoustic normal modes // J. Acoust. Soc. Am. 77, 1760–1767 (1985).

[7] Abawi, A. T., An energy-conserving one-way coupled mode propagation model. // The Journal of the Acoustical Society of America vol. 111 issue 1 January 2002. p. 160-167

[8] A. H. Nayfeh, Perturbation methods. John Wiley and Sons, New York, London, Sydney, Toronto, 1973.
Figure 5: Transmission loss, XZ plain. Top: interacted modes, bottom: adiabatic modes

Figure 6: Transmission loss for the ASA wedge, plain XY, receiver depth=30 m. Across slope propagation. Adiabatic modes. Total depth is 600 m, wedge bottom depth is 200 m. Attenuation in the bottom is 0.5 dB/λ till the depth of 500 m, then attenuation linearly increases up to 5.5 dB/λ at depth 600 m. The other parameters are the same as in classical ASA wedge benchmark.
Figure 7: Transmission loss for the ASA wedge, receiver depth=30 m. Across slope propagation. Interacted modes vs. adiabatic modes.

[9] Naimark M. A. Linear differential operators. Elementary theory of linear differential operators: with additional material by the author, Part I. F. Ungar Pub. Co., New York, 1967. 144 p.