Faster Streaming Algorithms for Graph Spanners

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Abstract

Given an undirected graph $G = (V, E)$ on $n$ vertices, $m$ edges, and an integer $t \geq 1$, a subgraph $(V, E_S), E_S \subseteq E$ is called a $t$-spanner if for any pair of vertices $u, v \in V$, the distance between them in the subgraph is at most $t$ times the actual distance. We present streaming algorithms for computing a $t$-spanner of essentially optimal size-stretch trade offs for any undirected graph.

Our first algorithm is for the classical streaming model and works for unweighted graphs only. The algorithm performs a single pass on the stream of edges and requires $O(m)$ time to process the entire stream of edges. This drastically improves the previous best single pass streaming algorithm for computing a $t$-spanner which requires $\Theta(mn^{1.5})$ time to process the stream and computes spanner with size slightly larger than the optimal.

Our second algorithm is for StreamSort model introduced by Aggarwal et al. [2], which is the streaming model augmented with a sorting primitive. The StreamSort model has been shown to be a more powerful and still very realistic model than the streaming model for massive data sets applications. Our algorithm, which works of weighted graphs as well, performs $O(t)$ passes using $O(\log n)$ bits of working memory only.

Our both the algorithms require elementary data structures.

Keywords: streaming, spanner, approximate shortest path
1 Introduction

A spanner is a (sparse) subgraph of a given graph that preserves approximate distance between each pair of vertices. Putting in more formal words, a \( t \)-spanner of a graph \( G = (V, E) \), for any \( t \in \mathbb{N} \) is a subgraph \((V, E_S), E_S \subseteq E \) such that, for any pair of vertices, their distance in the subgraph is at most \( t \) times their distance in the original graph. The parameter \( t \) is called the stretch factor associated with the \( t \)-spanner. The concept of spanners was defined formally by Peleg and Schäffer [25] though the associated notion was used implicitly by Awerbuch [5] in the context of network synchronizers. Since then, spanner has found numerous applications in the area of distributed systems, communication networks and all pairs approximate shortest paths [5, 9, 26, 27].

Each application of spanners requires, for a specified \( t \in \mathbb{N} \), a \( t \)-spanner of smallest possible size (the number of edges). Based on the famous girth conjecture by Erdős [17], Bollobás [11], and Bondy and Simonovits [12], it follows that for any \( k \in \mathbb{N} \), there are graphs on \( n \) vertices whose \((2k-1)\)-spanner or a \( 2k \)-spanner will require \( \Omega(n^{1+1/k}) \) edges. The conjecture has been proved for \( k = 1, 2, 3 \), and \( 5 \). Note that the conjectured worst case lower bound is the same for stretch \( 2k \) and \( 2k-1 \), and by definition, a \((2k-1)\)-spanner is also a \( 2k \)-spanner. Therefore, from the perspective of an algorithmist, the aim would be to design an efficient algorithm to compute a \((2k-1)\)-spanner whose size is \( O(n^{1+1/k}) \) for any given graph.

For unweighted graphs, Halperin and Zwick [21] designed a deterministic \( O(m) \) time algorithm to compute a \((2k-1)\)-spanner of \( O(n^{1+1/k}) \) size. However, for weighted graphs, it took a series of improvements [4, 6, 14, 29, 8, 7] till an expected \( O(m) \) time algorithm for computing a \((2k-1)\)-spanner could be designed. This linear time randomized algorithm [8, 7] computes a \((2k-1)\)-spanner of size \( O(kn^{1+1/k}) \) for a given weighted graph. Recently Roditty et al. [28] derandomized this algorithm.

In this paper, we consider the problem of computing a \((2k-1)\)-spanner in streaming model and its recently extended variant StreamSort. These models capture the complexities of algorithms designed for massive data set applications more accurately, and are thus gaining ever increasing attention these days. Our algorithms for computing spanners are significantly superior to the previously existing ones, and are arguably optimal. We shall now briefly describe the streaming model, the StreamSort model, and the motivation for computing spanners in streaming environment. Then we present the (bounds of) previously existing streaming algorithms for spanners, and new results.

1.1 Streaming model

The streaming model [22] has the following two characteristics: firstly the input data can be accessed sequentially (in the form of a stream), secondly the working memory is considerably smaller than the the size of the entire input stream. So an algorithm in this model can only make a few passes over the input stream to solve the corresponding problem. The sequentiality in accessing the data and the small working memory size enforce the following restriction: during a pass, a data item once evicted from the memory can’t be brought back into the working memory. It is due to this restriction that the streaming model is more stringent than other models namely, various external memory models [1, 30], and models for competitive analysis of algorithms [23].

The features and restrictions of the streaming model have been motivated by various technological factors pertaining to massive data set applications. Due to enormity of size along with various practical and economical reasons, the input data of a massive data set application resides on secondary and tertiary storage devices. These devices are optimized for sequential access and impose substantial penalties (seek times, cache misses, pipeline stalls) for non-sequential data access. So an efficient algorithm in this model should make a small number of sequential passes over the in-
put data with a small size of working memory. The number of passes and the size of working memory are the two parameters associated with a streaming algorithm. An additional parameter is the processing time per data item. These three parameters also capture the efficiency criteria for a streaming algorithm.

This model is gaining a lot of attention currently due to emerging massive data set applications. Earlier, in this model, much attention was given to problem related to computing order statistics, outliers, histograms [3, 13, 20, 22]. Recently, much attention has been given to solving graph problems in this model, for example, approximate distances, spanners, and matching [18, 19, 24]. A typical graph problem in the streaming model involves making one or more sequential passes over the stream of edges.

Aggarwal et al. [2] introduce an extension of streaming model called StreamSort model which is more powerful and still very practical than the streaming model. An algorithm in this model performs two kinds of passes - stream pass and sort pass. The stream pass sequentially reads the input stream, processes it with its limited memory, and produces an output stream. During the pass, the output stream is written left to right, and a data item once written can’t be erased. A sort pass sorts a stream according to some well defined order and produces as an output a sorted stream. An output stream of one pass can be used as input stream for the next (stream or sort) pass. An algorithm in StreamSort model thus performs a few stream pass and a few sort passes to solve a computational problem. In a slightly simpler variant of StreamSort model, Demetrescu et al. [15] presented streaming algorithms for undirected connectivity and shortest paths problem which achieve near optimal trading off between space and the number of passes.

1.2 Computing a spanner in streaming environment and new results

Being one of the fundamental problem in its own right, computing spanners in a streaming environment is a significant problem. This problem has recently gained more relevance due to all-pairs approximate shortest path problem in streaming environment. Due to enormity of size, it is just not feasible to compute or store all-pairs distances in streaming environment for graphs appearing in massive data sets applications. So one wants to settle for approximate shortest paths to save space. A result of Thorup and Zwick [29] showed that for any data structure capable of answering \((2k-1)\)-approximate distance query would need \(\Omega(n^{1+1/k})\) space. The result obviously holds for streaming environment too. If we can compute \((2k-1)\)-spanner efficiently in streaming environment, it can be employed to solve the APASP problem in the streaming environment in the following way: For any pair of vertices, just explore the spanner to report the approximate distance. Feigenbaum et al. [19] took this approach for all-pairs approximate distances in streaming environment.

We would like to add a note that a \(k\)-pass streaming algorithm for a \((2k-1)\)-spanner of \(O(kn^{1+1/k})\) size for any weighted graph is implicit in the algorithm of [8, 7], and the processing time for each edge is also just \(O(1)\) during each pass. The working memory required has size \(O(kn^{1+1/k})\). Since each pass is a time consuming task, it is always desirable to have a single pass algorithm for computing a \((2k-1)\)-spanner. For such an algorithm, one would also aim to keep processing time per edge bounded by a constant. Feigenbaum et al. [19] made a step in this direction. As a main result in their paper, they present a single pass streaming algorithm (Theorem 2.1, [19]) for computing a \(t\)-spanner for any unweighted graph. Though they don’t mention it, their algorithm is indeed an adaptation of the algorithm of [8, 7] for streaming environment. However, the bounds their algorithm achieves are suboptimal: For any \(k \in \mathbb{N}\), their algorithm computes a \((2k+1)\)-spanner of expected size \(O(kn^{1+1/k})\) and requires expected \(\theta(k^2 n^{1/k})\) processing time per edge. Note that the size of the spanner thus computed is away from the optimal by a factor of \(\theta(n^{1/k^2})\).
In this paper, we succeed in achieving optimal bounds and size-stretch trade-offs for computing a \((2k - 1)\)-spanner in streaming environment. We achieve the following two results.

1. Given any unweighted undirected graph, and \(k \in \mathbb{N}\), a \((2k - 1)\)-spanner of expected \(O(kn^{1+1/k})\) size can be computed in classical streaming model with single pass and \(O(m)\) processing time for the entire stream (amortized constant processing time per edge).

   **Remark.** The algorithm at each stage maintains a \((2k - 1)\)-spanner of the graph seen so far. Therefore, it can also be viewed as a partial dynamic (incremental) algorithm for computing a \((2k - 1)\)-spanner of an unweighted graph with amortized \(O(1)\) time per edge insertion (the same observation, but with inferior bounds, holds for the earlier algorithm of [19]).

   If the edges appear sorted in nondecreasing order of their weights in the stream, our algorithm, without any modification at all, would work for weighted graphs as well. As a result, it requires one sort pass followed by a stream pass in the StreamSort model for computing a \((2k - 1)\)-spanner of expected \(O(kn^{1+1/k})\) size for any \(k \in \mathbb{N}\) and any weighted graph. Note that working memory has size of the order of spanner size, and though larger than \(n\), is indeed optimal for classical streaming model.

2. Given a weighted undirected graph, and \(k \in \mathbb{N}\), a \((2k - 1)\)-spanner of expected \(O(kn^{1+1/k})\) size can be computed in StreamSort model in \(O(k)\) passes total and with \(O(\log n)\) bits of working memory only. Furthermore, each Stream pass in this algorithm spends just \(O(1)\) time per edge.

We would also like to mention that the algorithms presented in our paper employ elementary data structures (link lists and arrays). The algorithms (and their analysis) presented in this paper are complete on their own.

**Remark.** Elkin and Zhang [16] address the problem of computing \((1 + \epsilon, \beta)\)-spanner in streaming environment. Their algorithm, though sheds some light on the APASP problem in streaming environment, has little practical relevance. This is because, the number of passes required, though constant, depend quite heavily on \(\epsilon, \beta\).

### 2 Preliminaries

We assume, like the previous algorithms [18, 19], that \(n\), the number of vertices is known in advance and the vertices are numbered from 1 to \(n\).

As mentioned in the introduction, our algorithm is basically a careful adaptation of the previous static linear time algorithms [8, 7, 10] in the streaming environment. The central idea of these algorithms is clustering which we define below.

**Definition 2.1** A **cluster** is a subset of vertices, and a **clustering** \(C\), is a union of disjoint clusters. Each cluster will have a unique vertex which will be called its **center**.

The uniqueness of the center of a cluster can be used to represent a clustering \(C\) as an array (of the same label \(C\)) of size \(n\) in the following way: \(C(v)\) will denote the center of the cluster containing \(v\) unless when \(v\) does not belong to any cluster, in which case \(C(v) = 0\). We shall say that a cluster \(c\) is **incident** on or **adjacent** to a vertex \(u\) if there is some vertex \(v \in c\) adjacent to \(u\). With respect to a given clustering \(C\), a vertex \(u \in V\) is said to be a **clustered** vertex if it belongs to some cluster in \(C\), and an **unclustered** vertex otherwise.
The role of clustering to achieve a small size spanner can be described intuitively as follows. Suppose we can partition the vertices into a small number of disjoint clusters, and span each of these clusters by a small set $E \subseteq E$. As a consequence of this clustering, each vertex $u \in V$ has all its neighbors grouped in various clusters. Among those edges that are incident on $u$ from same cluster, say $c$, selecting just one edge will ensure the following property. For each missing edge $(u, v)$ such that $v \in c$, there is a path connecting $u$ and $v$ using one of the selected edges and some edges from $E$, and the length of this path is at most one unit more than the diameter of the cluster containing $v$. (In order to ensure a small bound on the stretch, we need these clusters to have very small diameter). This simple idea of pruning edges lies at the core of the static algorithm of [8, 7, 10], and to materialize it they build a multilevel clustering using random sampling.

3 Algorithm for $(2k - 1)$-spanners in classical streaming model

Prior to processing the stream of edges, the algorithm constructs an initial $(k + 1)$-levels of clusterings $\{C_i|0 \leq i \leq k\}$ for the empty (without edges) graph as follows.

| Initializing the $(k + 1)$-levels of clusterings |
|--------------------------------------------------|
| Let $S_0 \leftarrow V$, $S_k = \emptyset$       |
| For $0 < i < k$,                                 |
| $S_i$ contains each element of set $S_i$        |
| independently with prob. $n^{-1/k}$            |
| For $0 \leq i \leq k$                          |
| $C_i \leftarrow \{\{v\}|v \in S_i\}$          |

We introduce two notations at this point.

$\ell(v)$ : the highest level of the clustering in which $v$ is present as a clustered vertex.

$\ell_S(v)$ : the highest level $i < k$ such that the cluster centered at $v$ is a sampled cluster in $C_i$.

Note that, in the beginning $\ell_S(v) = \ell(v)$ for all the vertices. However, as the edges are being processed, the level $\ell(v)$ of a vertex might rise.

We shall now give an overview and intuition of the algorithm. Initially, at each level $i < k$, every cluster is a singleton set. From viewpoint of clustering, the only change in a cluster during the algorithm will be that other vertices (from levels lower than the cluster) might join it. We shall always use the following convention : a cluster $c \in C_i$ is a sampled cluster if in the beginning of the algorithm, the corresponding singleton cluster was a sampled cluster. The following assertion will hold throughout.

$A$: For each $c \in C_{i+1}$, there exists a sampled cluster $c' \in C_i$ such that $c' \subseteq c$.

Now we describe the way the stream of edges is processed by the algorithm, and how the clustering evolves by upward movement of vertices. Each vertex $u \in V$ waits at its present level $\ell(u)$ for an opportunity to move to a level higher than $\ell(u)$, and the only opportunity for it to move higher is when it receives an edge incident from some sampled cluster in $C_{\ell(u)}$. We shall explain soon how this tendency of vertices to rise to higher level proves crucial to compute a sparse $(2k - 1)$-spanner. It follows from assertion $A$ that a sampled cluster $c \in C_i$ has some $c' \in C_{i+1}$ such that $c \subseteq c'$. Whenever $u$ gets such an edge, it hooks itself to the sampled cluster $c$ to join (become member of) cluster $c'$ (so $c'$ gets updated accordingly). In case, $c$ appears as sampled cluster at the next level also, the vertex $u$ will join the next level parent as well. As follows from the sampling involved in building the hierarchy of clusterings, only a very few of the clusters at any level are the
sampled clusters. So a vertex will get an opportunity to become adjacent to a sampled cluster on very few occasion, and until then, it adds edges to the spanner in a frugal manner using the smart idea of clustering, as follows. Let the vertex $v$ be member of only unsampled cluster at level $\ell(v)$. Let $c$ be the cluster at level $\ell(v)$ in which $v$ is present. In this case, the vertex $u$ just adds an edge $(u, v)$ to the spanner if $c$ was not adjacent to $u$ earlier. Vertex $u$ would keep a list storing one edge from each cluster of $C_i$ that is adjacent to it. Now, in order to determine whether the cluster $c$ was previously incident on $u$ before the edge $(u, v)$, it suffices to explore the entire list of edges incident from various clusters at level $\ell(v)$, which could be quite large. (Feigenbaum et al. [19] used this brute force search). In order to achieve amortized $O(1)$ time, we adopt a buffering approach in which we keep a buffer storing the edges at each level temporarily. The vertex $v$ will initially add the edge $(u, v)$ to its temporary buffer at level $\ell(v)$, and prune this set once there are sufficiently large number of edges using the procedure $Prune(u, i)$.

A vertex’s tendency to move to higher levels proves crucial to compute a sparse $(2k - 1)$-spanner in the following way. At lower level, there are a large number of clusters, so we can’t afford to add edges from a vertex to all these clusters. As more and more number of clusters at level $i \geq \ell(u)$ get adjacent to $u$, one of them might be a sampled cluster. Since a sampled cluster is present at higher level too (see assertion $\mathcal{A}$), getting hooked to a sampled cluster would pay $u$ in the sense that it moves to a higher level where there are fewer clusters. At level $k - 1$, there would be expected $n^{1/k}$ clusters, and once $u$ reaches this level, it can afford to add a single edge to each of its neighboring clusters.

Having given an intuitive and informal description of the algorithm above, now we shall present the algorithm and the associated data structures formally.

**Data structure** : We shall use $k$ arrays $C_i$, $i < k$ to store clustering at each level. As mentioned earlier $C_i(u)$ will store the center of the cluster in $C_i$ storing $u$. In case $u$ is not clustered at level $i$, $C_i(u)$ will store 0. Each vertex $u \in V$ keeps lists $Temp(u)$ and $\mathcal{E}(u)$. The list $\mathcal{E}(u)$ will store edges incident on $u$ from unsampled clusters at level $\ell(u)$, and $Temp(u)$ will act as a buffer for these edges which we shall purge once the number of edges in $Temp(u)$ exceeds the number of edges in $\mathcal{E}(u)$.

### Processing an edge $(u, v)$ from the stream

1. **Assigning the edge to the endpoint at lower level**
   - If $\ell(u) > \ell(v)$, then swap $(u, v)$.
   - $i \leftarrow \ell(u)$, $x \leftarrow C_i(v)$, $h \leftarrow \ell_S(x)$,

2. **Processing the edge**
   - **If** $h > i$
     - For $j = i + 1$ to $h$, do
       - $C_j(u) \leftarrow x$
     - $\ell(u) \leftarrow h$
     - $\mathcal{E}_S \leftarrow Temp(u) \cup \mathcal{E}(u) \cup \{(u, v)\}$
     - $Temp(u) \leftarrow \emptyset$, $\mathcal{E}(v) \leftarrow \emptyset$
   - **Else**
     - $Temp(u) \leftarrow Temp(u) \cup \{(u, v)\}$
     - **If** $|Temp(u)| \geq |\mathcal{E}(u)|$, then $Prune(u, i)$.

The **If** condition in step 2 checks whether there is any sampled cluster containing $v$ at level $\ell(u)$ or higher, and if so, the vertex $u$ joins a cluster. Otherwise, the clustering remains unchanged. It is
easy to observe that the assertion $A$ will hold after every edge is processed.

**The procedure Prune($u, i$)**: The procedure uses a boolean array $A[1..n]$ as a scratch space. The array $A$ is initialized to 1. First it scans the list $\mathcal{E}(u)$ and sets to 1 entries in $A$ corresponding to clusters in $C_i$ neighboring to $u$. It then scans the edges in the list $\mathit{Temp}(u)$, and eliminates an edge if the corresponding cluster was already incident, otherwise it adds it to $\mathcal{E}(u)$. Afterwords, we scan the updated list $\mathcal{E}(u)$ once to undo the changes made in array $A$ so that $A$ is initialized back to its start stage (all entries set to 0).

**Observation 3.1** For each vertex $u \in V$, $|\mathit{Temp}(u)| \leq |\mathcal{E}(u)|$ except before the invocation of Prune($u$) when $|\mathit{Temp}(u)|$ exceeds $|\mathcal{E}(u)|$ by one.

### 3.1 Analyzing the running time

It takes $O(1)$ time for processing an edge except when it invokes Prune(). Let us analyze the total time spent in a single call of Prune($u, i$). It follows from the description of the procedure that the total time required by Prune($u$) is of the order of $|\mathcal{E}_i(u)| + |\mathit{Temp}_i(u)|$, which by Observation 3.1 is $O(|\mathit{Temp}_i(u)|)$. So it suffices to charge $O(1)$ cost to each edge of $\mathit{Temp}_i(u)$ to account for the time spent in a call of Prune($u, i$). Note that an edge is processed only once by Prune($u, i$) while being a member of $\mathit{Temp}_i(v)$. This is because, after Prune($u, i$) procedure, either the edge gets discarded forever or it becomes a member of $\mathcal{E}_i(u)$. Hence it suffices to charge $O(1)$ cost to each edge in order to account for the total computational cost charged to all calls of Prune() during the algorithm. Hence total time spent in required for processing the stream of edges is $O(m)$.

Let $\mathcal{E}^+$ be the set $\cup_{i<k, u \in V}(\mathit{Temp}_i(u) \cup \mathcal{E}_i(u)) \cup \mathcal{E}$ at any stage of the algorithm.

In the following section, we shall prove that : the set $\mathcal{E}^+$ at any given moment is a $(2k - 1)$-spanner for the set of edges appeared in the stream till that moment, and its expected size $O(kn^{1+1/k})$. This way, the algorithm can also be viewed as an incremental algorithm for computing a $(2k - 1)$-spanner.

### 4 The stretch and the size of the spanner computed by the algorithm

#### 4.1 Analysis of the stretch of the spanner

First we state an important Lemma.
Lemma 4.1  Let \( c \) be any cluster in \( C_i \). Each vertex \( v \in c \) is connected to its center through at most \( i \) edges from \( E \).

Proof: The proof is based on induction on \( i \) and the number of edges of the stream seen so far. Let \( x \) be the center of the cluster \( c \). If \( c \) is a singleton cluster, there is nothing to prove, so assuming otherwise, let \( u \neq x \) be a vertex which belongs to \( c \). Now observe the process by which \( u \) joined the cluster \( c \). The vertex \( u \) became member of \( c \) only in the situation where an edge \((u, v)\) appeared in the stream with vertex \( v \) being a member of some sampled cluster \( c' \) in \( C_{i-1} \). The assertion \( A \) implies that, \( c' \) is a subset of \( c \) and so has \( x \) as its center. Now applying inductive assertion, there is a path \( \subseteq E \) between \( v \) and \( x \) with length \( i - 1 \). This path concatenated with the edge \((u, v)\) (also in \( E \)), is a path \( \subseteq E \) between \( u \) and \( x \) of length at most \( i \).

The streaming algorithm processes each edge of the stream and discards a dispensable edge only through the procedure \( Prune() \). In order to prove that \( E^+ \) is a \((2k - 1)\)-spanner, we shall show that for each edge \((u, v)\) discarded by the algorithm, there is a path in \( E^+ \) of length at most \((2k - 1)\) that connects \( u \) and \( v \). Without loss of generality, assume that the edge \((u, v)\) got discarded during \( Prune(u, i) \). Now the edge \((u, v)\) could be discarded only if we had already selected some other edge \((u, w)\) in \( E_i(u) \) incident from the same cluster in \( C_i \) to which \( v \) belongs. Lemma 4.1 implies that the center of each cluster in \( C_i \) is connected to its members through a path in \( E \) with length at most \( i \). Hence \( u \) and \( w \), being the members of the same cluster, are connected by a path in \( E \) with length at most \( 2i \). This path concatenated with the edge \((u, w) \in E(u)\), is a path in \( E^+ \) between \( u \) and \( v \) with length at most \( 2i + 1 \), which is at most \( 2k - 1 \) since \( i < k \) always. Hence we can conclude that \( E^+ \) at any moment is a \((2k - 1)\)-spanner for the the set of edges appeared in the stream till that moment.

4.2 Analyzing the size of the spanner

In the algorithm, a vertex \( u \) contributes edges to \( E \) only when its level \( \ell(u) \) increases. So \( |E| \leq n(k - 1) \). Let us count the expected number of edges in \( E_i(u) \) and \( Temp_i(u) \). It follows from Observation 3.1 that the number of edges in \( Temp_i(u) \) is at most \( |E_i(u)| + 1 \). So it suffices to bound the number of edges in \( E_i(u) \).

First we would like to make an observation. When an edge \((u, v)\) appears in the stream with \( \ell(u) \leq \ell(v) \) and let vertex \( v \) does not belong to a sampled cluster at any level from \( \ell(u) \) onwards. This edges makes \( u \) adjacent to the cluster containing \( v \) at level \( \ell(v) \). Note from the algorithm that although the vertex \( v \) is clustered from every level \( \ell(u) \) to \( \ell(v) \), it is only the cluster at level \( \ell(v) \) which gets adjacent to \( u \) by edge \((u, v)\). So the sets \( \{E_i(u)\} \) are disjoint always. It also follows from the procedure \( Prune() \) that \( E_i(u) \) stores one edge per cluster at level \( i \) that gets adjacent to \( u \).

We shall give a bound on the expected size of \( |E_i(u)| \). For any arbitrary but fixed stream of edges, let \((c_1, c_2, \ldots)\) be the clusters at level \( i \) arranged in the chronological order of their getting incident on to \( u \). When a cluster from \( C_i \) gets adjacent to \( u \) and the cluster is a sampled cluster, the vertex \( u \) will hook onto that cluster and move to the next level. It follows from the algorithm that from this time onwards, \( u \) won’t add any edge to \( Temp_i(u) \) or \( E_i(u) \). So an edge incident from \( c_j \) will be selected in \( E_i(u) \) if none of \( c_1, \ldots, c_{j-1} \) were a sampled cluster. From the sampling of clusters done in the beginning of the algorithm, it follows that each cluster at level \( i \) is a sampled cluster independently with probability \( p = n^{-1/k} \). So an edge incident from \( c_j \) on \( u \) will be added to
\( \mathcal{E}_i(u) \) with probability \((1 - n^{-1/k})^j\). Hence the expected number of edges in \( \mathcal{E}_i(u) \) is
\[
\sum_{1 \leq j} (1 - n^{-1/k})^{j-1} \leq n^{1/k}
\]

Since there are \( n \) vertices, it follows that the expected size of the spanner computed by the streaming algorithm will be \( O(kn^{1+1/k}) \). Note that it could be that vertex \( u \) moves to level higher than \( i \) even when it gets adjacent to some sampled cluster at some level \( > i \). But that would only decrease the number of edges contributed as analyzed above.

**Theorem 4.1** Given any \( k \in \mathbb{N} \), a \((2k-1)\)-spanner of expected size \( O(\min(m, kn^{1+1/k})) \) for an unweighted graph can be computed in streaming model in one pass with amortized constant processing time per edge. The working memory required is \( O(kn^{1+1/k}) \).

Now we shall show that the algorithm for classical streaming model described above will work for weighted graphs as well if the edges appear in the increasing order of edge weights.

We shall employ the following observation which follows from the procedure \( \text{Prune}() \).

**Observation 4.1** Consider any vertex \( u, c \in C_i \), \( i < k \) and the period during which \( \ell(u) \leq i \). Among all the edges in the stream that get incident on \( u \) from \( c \) in this period, the edge that appears first in the stream is surely present in the spanner.

**Proof:** Let \((u, v), v \in c \) be the first edge incident on \( u \) from \( c \) during the period \( \ell(u) \leq i \). It will be added to \( \text{Temp}_i(u) \) initially like any other edge. When \( \text{Prune}(u, i) \) is invoked near future, and the edge \((u, v)\) is processed, it is clear that \( A[C_i(v)] = 0 \) since by definition there was no edge prior to \((u, v)\) which is incident on \( u \) from \( c \). Hence \((u, v)\) gets added to \( \mathcal{E}_i(u) \) and subsequently to the spanner. \( \square \)

Along similar lines, we can infer the following observation.

**Observation 4.2** Consider any cluster \( c \in C_i \), and let \( v \) be a vertex present in \( c \). For the period \( \ell(v) = i \), let \( E_v \) be the edges that gets incident on \( v \) from vertices lying at level \( \leq i \). All the edges lying on the path from \( v \) to the center of \( c \) appeared before any edge in the set \( E_v \).

Let the edges in the stream appear in the non decreasing order of their weights. Let our single pass algorithm (designed for unweighted graph) processes this stream ignoring the edge weights. We shall show that the spanner computed will also be a \((2k-1)\)-spanner of the original graph with weighted edges. Let \((u, w)\) be an edge discarded by the algorithm. and let us suppose it got discarded during \( \text{Prune}(u, i) \), for some \( i < k \). Let \( w \in c \in C_i \), it follows from Observation 4.1 that there is some edge, say \((u, v), v \in c \) that appeared before \((u, w)\) in the stream and got added to the spanner. From the arguments used in the proof of Lemma 4.1, it follows that \( v \) and \( w \) were connected by a path of at most \( 2i \) edges from set \( E \). All these edges and the edge \((u, v)\) form a path in the spanner of length at most \( 2i + 1 \). Using Observation 4.1 and 4.2, it also follows that all these edges appeared before the edges \((u, w)\) in the stream. Hence each of them is at most as heavy as \((u, w)\) since the edges appeared in the stream in the nondecreasing order of their weights. So there is a path between \( u \) and \( w \) in the spanner consisting of at most \( 2i + 1 \) edges each one being at most as heavy as \((u, w)\). Hence the spanner is indeed a \((2k-1)\)-spanner.

Thus we can conclude that our single pass streaming algorithm originally designed for unweighted graphs will also compute a \((2k-1)\)-spanner for weighted graph provided the edges appear in nondecreasing order of their weights. So an algorithm for computing a \((2k-1)\)-spanner in StreamSort model would be as follows.
1. First run a sort pass on the input stream \( I \) which will produce an output stream \( O \) where edges appear in the nondecreasing order of their weights.

2. Execute our single pass algorithm of earlier section (originally designed for unweighted graphs) on the stream \( O \) ignoring the weights.

**Theorem 4.2** Given any \( k \in \mathbb{N} \), a \((2k - 1)\)-spanner of expected size \( O(\min(m, kn^{1+1/k})) \) for weighted graph can be computed in StreamSort model with one sort pass followed by one stream pass and it requires amortized constant processing time per edge during the stream pass and the working memory required is \( O(kn^{1+1/k}) \).

In the following section we shall describe an algorithm for computing \((2k - 1)\)-spanner in StreamSort model which will require \( O(\log n) \) bits of working memory and perform \( O(k) \) passes only.

5 Algorithm for \((2k - 1)\)-spanners in StreamSort model

We shall now present an algorithm for computing a \((2k - 1)\)-spanner in StreamSort model. The algorithm works for weighted graphs as well and will require just \( O(\log n) \) bits of working memory and \( O(k) \) alternating passes of Streaming and Sorting.

The algorithm can be viewed as a streaming version of the static RAM algorithm for computing \((2k - 1)\)-spanner given by [8]. We provide a brief overview of the algorithm below. The algorithm executes \( k \) iterations. Each iteration begins with a partially built spanner \( E_S \), a subset of edges \( E' \) for which decision of including them into spanner has yet to be made, a subset \( V' \subset V \) such that end point of each edge in \( E' \) is present in \( V' \). In addition, \( i \)th iteration begins with a clustering \( C_{i-1} \) which partitions \( V' \) into disjoint clusters such that each edge in \( E' \) is an inter-cluster edge. The clustering \( C_{i-1} \) has the following crucial property.

P : For each edge \((u, v) \in E'\), there is a path from \( u \) to the center of its cluster in \( C_{i-1} \) with \( i - 1 \) edges each of weight not more than that of \((u, v)\). The first iteration begins with \( V' = V, E' = E, E_S = \emptyset, C_0 = \{\{v\} | v \in V\} \).

Execution of \( i \)th iteration selects each cluster from \( C_{i-1} \) independently with probability \( n^{-1/k} \). This sampling forms the basis of defining the clustering for \( i \)th iteration. Namely, \( C_i \) consists of the clusters sampled in \( i \)th iteration with every vertex not belonging to any sampled cluster joining its nearest neighboring sampled cluster (if any). In addition to it, processing of each vertex in \( V' \) contributes some edges to spanner and discards a few in the \( i \)th iteration. We shall describe the exact description of the \( i \)th iteration and its execution in StreamSort model soon. But before that, we need to preprocess the initial stream of edges, and introduce a few key ideas which lead to execution of \( i \)th iteration in StreamSort model in \( O(1) \) passes.

5.1 Augmenting the initial edge stream, and two sorting primitives

Our algorithm will receive just a stream of edges. In order to execute our algorithm, we will associate some more fields with each edge and vertex. We do so as a preprocessing phase of the algorithm. **Preprocessing of initial edge stream** : We preprocess the initial stream of edges to produce another stream such that for an edge between \( u, v \in V \) in the stream, we introduce two edges denoted as \((u, v)\) and \((v, u)\) in the output stream. We shall use \((u, v)\) to denote the edge associated with vertex \( u \) and we shall use \((v, u)\) to denote the edge associated with vertex \( v \).

In addition, we augment the data structure of each edge \((u, v)\) with the following additional fields.
• *lcenter* and *rcenter* storing the center of cluster to which \( u \) and \( v \) belong in present clustering. Since the initial clustering is \( \{ \{ x \} | x \in V \} \), \( C(u) \leftarrow u \) and \( C(v) \leftarrow v \).

• *spanner-edge* : which is set to 1 if \((u, v)\) is selected as spanner, and set to -1 if it has not to be added to spanner, and to 0 if no such decision has been made. So initially, this field is set to 0 for each edge.

• *sampled-edge* : which is set to 1 if either of \( u \) or \( v \) belong to a sampled cluster during an iteration.

For each vertex \( u \in V' \), we store the following additional variables.

• \( C(u) \) : the center of the cluster in present clustering containing \( u \). Initially \( C(u) \leftarrow u \).

• \( sampled(u) \) : a boolean variable which is true during an iteration if \( u \) belongs to sampled cluster.

• \( N(u) \) : the weight of the edge incident on \( u \) from nearest neighboring sampled cluster.

Main idea is to show that for processing various steps of an iteration, we need to sort the edges and vertices in a suitable total order such that each task of \( i \)th iteration can be executed by performing a few *Sort* passes and a few *Stream* passes. We shall first introduce two total orders on the set of edges.

1. \( \preceq_0 \)
   An edge \((x, y)\) precede \((p, q)\) in \( \preceq_0 \) if
   \[
   \min(x, y) < \min(p, q) \text{ or } \min(x, y) = \min(p, q) \text{ and } \max(x, y) < \max(p, q).
   \]

2. \( \preceq_{(C, C')} \)
   Given two clustering \( C, C' \) on a set of vertices \( V' \), we define an order \( \preceq_{(C, C')} \) on the set of vertices \( V' \) and edges \( E' \) as follows.
   - a vertex \( u \) would precede vertex \( v \) in the total order \( \preceq_{(C, C')} \) if
     \[
     C(u) < C(v) \quad \text{or} \quad C(u) = C(v) \text{ and } C'(u) < C'(v).
     \]
     We break the tie, that is, \( C(u) = C(v) \) and \( C'(u) = C'(v) \) by comparing the labels \( u \) and \( v \).
   - an edge \((u, v)\) would precede another edge \((x, y)\) in the order \( \preceq_{(C, C')} \) if
     \[
     C(u) < C(x) \quad \text{or} \quad C(u) = C(x) \text{ and } C'(v) < C'(y).
     \]
     We break the tie, that is, \( C(u) = C(x) \) and \( C'(v) = C'(y) \), by resorting to lexicographic comparison of \((u, v)\) and \((x, y)\).
   - a vertex \( u \) precede an edge \((x, y)\) in the order \( \preceq_{(C, C')} \) if \( C(u) \leq C(x) \).

**Lemma 5.1** Suppose we want to arrange all the edges so that if there is an edge between two vertices \( u \) and \( v \), then its two occurrences \((u, v)\) and \((v, u)\) occur together. This goal can be achieved by a sorting according to the order \( \preceq_0 \).

We now state the following Lemma which would highlight the importance of arranging edges according to the order \( \preceq_{(C, C')} \).
Lemma 5.2 If the list of edges $E'$ is arranged according to the order $\preceq_{(c, c')}$, then for any two clusters $c \in C, c' \in C'$,
(i) the set of edges $\{(u, v)\mid u \in c\}$, i.e. the edges emanating from the cluster $c$ appear as a sub-list, say $L_c$.
(ii) the set of edges $E'(c, c')$ appear as a sub-list within the sub-list $L_c$.

Corollary 5.1 If either $C \in C'$ is the clustering $\{\{u\}\mid u \in V\}$, then in the total order $\preceq_{(c, c')}$, all edges incident on a vertex $u$ appear together as a sub-list and immediately succeed the vertex $u$.

5.2 Algorithm for $(2k - 1)$-spanner in StreamSort model

Algorithm:
As mentioned earlier, the algorithm will execute $k - 1$ iterations. The $i$th iteration will begin with a tuple $(V', E', E_S, C_{i-1})$, where $E_S$ is a partially built spanner, $E' \subseteq E$ consists of those edges for which decision of selecting into spanner (or discarding) has not been made yet. Moreover, each endpoint of an edge in $E'$ is present in $V'$ and the clustering $C_{i-1}$ partitions $V'$ into disjoint clusters such that each edge in $E'$ is an inter cluster edge and the property $P_{i-1}$ is satisfied:

Our algorithm does not do any processing on the edges of $E_S$ and basically processes only $E'$ and $V'$ in the stream. The various fields of the data structures associated with $E'$ and $V'$ store the following information in the beginning of $i$th iteration – the fields $lcenter$ and $recenter$ of each edge $(u, v) \in E'$ store $C_{i-1}(u)$ and $C_{i-1}(v)$ respectively. The $sampled-edge$ field of each edge is reset, and $sampled$ field of each vertex is also reset. $N(v)$ of each vertex stores $\infty$.

We now present the four basic tasks of the $i$th iteration for computing a $(2k - 1)$-spanner and their execution in StreamSort model as follows.

1. Forming a sample of clusters:
   Sample each cluster from $C_{i-1}$ independently with probability $n^{-1/k}$. However, if $i = k - 1$, then sample no cluster.

   Execution in StreamSort model: Perform a sorting pass on the stream of vertices $V'$ and edges $E'$ according to the order $\preceq_{(C_{i-1}, C_0)}$. Consequently, the vertices (and their edges) belonging to same cluster in $C_{i-1}$ appear together in the stream. We make a Stream pass on this stream and do the following. We pick each cluster independently with probability $n^{-1/k}$ and set the field $sampled$ of the vertices of the sampled clusters accordingly, and also set the field $sampled-edge$ of each edge emanating from them.

2. Finding nearest neighboring sampled clusters for vertices:
   For each vertex not belonging to any sampled cluster, if it is adjacent to one or more sampled cluster, compute the least weighted edge incident from the nearest sampled cluster; let $N(v)$ stores the weight of the edge.

   Execution in StreamSort model: We sort the stream according to $\preceq_0$ so that for an edge between $u$ and $v$, the two occurrences $(u, v)$ and $(v, u)$ appear together. We make a Stream pass and if $sampled-edge(u, v)$ is set to 1, then we set $sampled-edge(v, u)$ to 1 as well. After this, we sort the stream according to the order $\preceq_{(C_0, C_0)}$. As a result, we can observe the following. All edges incident on a vertex $v$ appear contiguously in the stream. We process each
vertex \( v \in V \) in this stream as follows. If \( v \) is not sampled, then we select the least weighted \textit{sampled-edge} incident on it. If \((v, x)\) is such an edge then we set \( C(v) \leftarrow \text{recenter}(v, x) \) (so \( v \) gets assigned to the cluster containing \( x \) in \( C_i \)), set \textit{spanner-edge}(v, x) to 1 and let \( N(v) \) store weight of the edge \((v, x)\). However, in case, \( v \) is not adjacent to any marked edge, we set \( N(v) \) to \( \infty \).

3. **Adding edges to the spanner**
   Each vertex \( v \) not belonging to any sampled cluster does the following: For each cluster \( c \in C_{i-1} \), incident on \( v \) in the clustering with edge of weight less than that of \( N(v) \), we select the least weight edge from \( E'(v, c) \) and mark it as a spanner edge.

Execution in StreamSort model: We perform a Sort pass on the stream according to the order \( \preceq (C_0, C_{i-1}) \) so that all the edges incident on a vertex from same cluster in the clustering \( C_{i-1} \) are contiguous and a vertex precedes immediately all the edges incident on it. We process a vertex \( v \) in the stream as follows. For each cluster incident on \( v \) with edge of weight less than \( N(v) \), we mark least weight edge incident on \( v \) from that cluster as spanner-edge and mark others as non-spanner edge.

We make a Sort pass over the stream of edges so that both the occurrences of an edge are together and then delete both of them if any of them has spanner-edge field set to -1.

4. **Defining the clustering \( C_i \)**
   Keep only those vertices which belong to sampled cluster or were adjacent to sampled cluster.

Execution in StreamSort: We make a Stream pass and delete all those vertices \( v \) for which \( \text{sampled}(v) = 0 \) and \( N(v) = \infty \). If a vertex \( u \) belonged to a sampled cluster, then it continues to belong to same cluster. If it did not belong then unless it is deleted, it was adjacent to some sampled cluster and \( \text{calC}(u) \) was set to the center of new cluster in the second step. This defines a clustering \( C_i \) for all the vertices among \( V' \) which survived \( i \)th iteration. We need to set the \( \text{lecenter} \) and \( \text{recenter} \) of each edge now according to the new clustering \( C_i \). We do so as follows. We make a Sort pass on the edges \( E' \) and vertices \( V' \) according to the order \( \preceq (C_0, C_i) \). Consequently all edges incident on a vertex \( v \) will appear together. We assign \( \text{lecenter}(v, w) \) of each edge to \( C(v) \) and reset \( \text{recenter}(v, w) \). We make a Sort pass according to \( \preceq_0 \) so that both the occurrences of an edge appear together. We then perform a Stream pass and for each pair of edges \((u, v)\) and \((v, u)\) that appear consecutive now, we set \( \text{recenter}(u, v) \leftarrow \text{lecenter}(v, u) \) and \( \text{recenter}(v, u) \leftarrow \text{lecenter}(u, v) \).

It is obvious that each step of \( i \)th iteration is executed in StreamSort model using a constant number of Stream passes and Sort passes. Since the algorithm is a streaming version on the static RAM algorithm, its correctness follows from the correctness of the latter. However, for sake of completeness, we shall now provide an overview of the correctness of the algorithm.

A simple inductive argument can be given to show that \( P_i \) holds at the end of \( i \)th iteration. And on this basis, it follows that for any edge \((u, v)\) that we delete from \( E' \), there is a path in the spanner \( E_S \) with at most \((2i - 1)\)-edges joining \( u \) and \( v \). So at the end of the algorithm, the set \( E_S \) will indeed be a \((2k - 1)\)-spanner. Also note that the number of clusters incident on a vertex with weight less than nearest neighboring sampled cluster in \( C_{i-1} \) is a geometric random variable with mean \( n^{1/k} \). Hence expected number of spanner edges contributed by a vertex in an iteration is \( O(n^{1/k}) \). Since there are \( k - 1 \) iterations, the expected size of the spanner computed by the algorithm is \( O(kn^{1+1/k}) \).
Theorem 5.1 Given any \( k \in \mathbb{N} \), a \((2k - 1)\)-spanner of expected size \( O(\min(m, kn^{1+1/k})) \) for weighted graph can be computed in StreamSort model with \( O(\log n) \) bits of working memory and \( O(k) \) sort passes and stream passes. Furthermore, it requires constant processing time per edge during each stream pass.

6 Conclusion and open problems

We presented single pass algorithm for computing a \((2k - 1)\)-spanner of expected \( O(kn^{1+1/k}) \) size with \( O(m) \) processing time for the entire stream (amortized constant processing time per edge). We also showed that in the StreamSort model, the algorithm can be extended for weighted graph as well and would require one sort pass followed by a stream pass. However, the working memory in both these algorithm is of the order of size of spanner, which though optimal for classical streaming model, is very large. We then provide an algorithm for computing spanner in StreamSort model with \( O(\log n) \) working memory and \( O(k) \) passes. It can be seen that these two algorithm achieve optimal or near optimal performance in all aspects - number of passes, amortized processing time per edge, working memory size in both models. One aspect which is not truly optimal is the expected size of the spanner which is away from the conjectured lower bound by a factor of \( k \) at most. An important open question is: Can we get rid of multiplicative factor \( k \) from the the size \( O(kn^{1+1/k}) \) of \((2k - 1)\)-spanner computed in streaming model? Note that this factor is present in case of the static randomized algorithm as well. So either a more careful and involved analysis of randomized algorithm would be required or some fundamentally new approach should be pursued to answer this question.

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