Acoustic black holes for relativistic fluids

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Abstract

We derive a new acoustic black hole metric from the Abelian Higgs model. In the non-relativistic limit, while the Abelian Higgs model becomes the Ginzburg-Landau model, the metric reduces to an ordinary Unruh type. We investigate the possibility of using (type I and II) superconductors as the acoustic black holes. We propose to realize experimental acoustic black holes by using spiral vortices solutions from the Navier-stokes equation in the non-relativistic classical fluids.


1 Introduction

Black hole is characterized by the event horizon, a spherical boundary out of which even the light cannot escape. In 1974, Hawking announced that black holes are not black at all, and emit thermal radiation at a temperature proportional to the horizon surface gravity. Although several decades passed, the experimental verification of Hawking radiation is still elusive.

Compared with the difficulties on the astrophysical side, analog models of general relativity is more appealing since these models are shedding light on possible experimental verifications of the evaporation of black holes. In the remarkable paper of Unruh [1], the idea of using hydrodynamical flows as analog systems to mimic a few properties of black hole physics was proposed. In this model, sound waves rather than light waves, cannot escape from the horizon and therefore it is named “acoustic (sonic) black hole”. A moving fluid with speed exceeding the local sound velocity through a spherical surface could in principle form an acoustic black hole. The event horizon is located on the boundary between subsonic and supersonic flow regions.

In general, any fluid flows could be the candidate of acoustic black holes. Many classical and quantum systems have been investigated as black hole analogues, including gravity wave [3], water [4], slow light [5,7], an optical fiber [8], and an electromagnetic waveguide [9]. But considered the detection of “Hawking radiation” of acoustic black holes, quantum fluid systems, rather than classical fluid systems are specifically preferred in that Hawking radiation is actually a quantum phenomenon. Up to now, superfluid helium II [10], atomic Bose-Einstein condensates [11], one-dimensional Fermi-degenerate noninteracting gas [12] have been proposed to created an acoustic black hole geometry in the laboratory. The possible application of acoustic black holes to quantum information theory has been suggested in [13].
The first experimental realization of acoustic black hole reported was conducted in a Bose-Einstein condensate [14]. So far it has been assumed that the system is non-relativistic, because all the atomic system is likely in such regime. However, some strongly interacting system shows some evidences for exotic behavior that the matter is based on the heavy atoms but the basic excitations has massless character [15, 16]. In fact the linear and massless dispersion relation is canonical for fermionic system. Therefore it is interesting to ask what happen to the acoustic black hole for the relativistic fluid system.

In this paper, we explore a relativistic version of acoustic black holes from the Abelian Higgs model. Another motivation comes from the feasibility of detecting Hawking temperature associated with the acoustic black hole. Since the Hawking temperature depends on the gradients of flow speed at the horizon, detecting thermal phonons radiating from the horizons is very difficult. In fact the Hawking temperature calculated from models in Bose-Einstein condensates so far is very low (∼ nano Kelvin). In contrast, the acoustic black holes created in the color superconducting phase in the dense quark matter and the pion superfluid phase at finite isospin density may provide us examples of relativistic superfluid and higher temperature acoustic black hole. The line elements of the black hole metric we obtained are found to be different from the non-relativistic version, which has been widely used for almost 30 years.

2 Acoustic black holes from Abelian Higgs model

The Abelian Higgs model is the Mexican-hat model coupled to electromagnetism with the action

\[
S = \int d^4x (\frac{-1}{4} F_{\mu\nu} F_{\mu\nu} + |(\partial_\mu - ieA_\mu)\phi|^2 + m^2|\phi|^2 - b|\phi|^4),
\]

(2.1)

where \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). The Planck units \( \hbar = c = k_B = 1 \) are used here. We will turn on the Planck constant \( \hbar \) in section 3. The corresponding equations of motion can be deduced from the action with one governing on the electromagnetic field and the other on the scalar field. We write down only the second equation of motion for the purpose of deriving the acoustic black hole geometry

\[
\Box \phi + 2ieA_\mu \nabla^\mu \phi - e^2 A_\mu A^\mu \phi + m^2 \phi - b \phi \phi = 0,
\]

(2.2)

in the \( \partial^\mu A_\mu = 0 \) gauge.

Multiplying (2.2) with \( \phi^* \), we obtain

\[
\phi^* \left( -\partial_t \partial^t \phi + \nabla_i \nabla^i \phi + 2ieA_\mu \nabla^\mu \phi - e^2 A_\mu A^\mu \phi \right) + m^2 \phi \phi^* - b \phi \phi = 0,
\]

(2.3)

where the index \( \mu = t, x_1...x_3 \) and \( i = x_1...x_3 \). We can obtain another equation by multiplying the conjugation of (2.2) with \( \phi \)

\[
\phi \left( -\partial_t^2 \phi^* + \nabla_i \nabla^i \phi^* + 2ieA_\mu \nabla^\mu \phi^* - e^2 A_\mu A^\mu \phi^* \right) + m^2 \phi \phi^* - b \phi \phi = 0.
\]

(2.4)
The combination of (2.3) and (2.4) yields two equations

\[ - \partial_t \left( \phi^* \partial_t \phi - \phi \partial_t \phi^* \right) + \nabla_i \left( \phi^* \nabla^i \phi - \phi \nabla^i \phi^* \right) + 2ieA_\mu \left( \phi^* \nabla^\mu \phi + \phi \nabla^\mu \phi^* \right) = 0. \] (2.5)

and

\[ - \partial_t \left( \phi^* \partial_t \phi + \phi \partial_t \phi^* \right) + 2|\partial_t \phi|^2 + \nabla_i \left( \phi^* \nabla^i \phi + \phi \nabla^i \phi^* \right) - 2|\partial_i \phi|^2 + 2m^2|\phi|^2 + 2ieA_\mu \left( \phi^* \nabla^\mu \phi - \phi \nabla^\mu \phi^* \right) - 2b|\phi|^4 - 2e^2A_\mu A^\mu |\phi|^2 = 0. \] (2.6)

With the assumption \( \phi = \sqrt{\rho}(\vec{x},t)e^{i\theta(\vec{x},t)} \), the above equations reduce to

\[ - \partial_t \left[ \rho (\dot{\theta} - eA_t) \right] + \nabla \cdot \left[ \rho (\nabla \theta + eA_i) \right] = 0, \] (2.7)

\[ \sqrt{\rho} \nabla \rho + (\dot{\theta} - eA_t)^2 - (\nabla \theta + eA_i)^2 + m^2 - b\rho = 0, \] (2.8)

where (2.7) is the continuity equation and (2.8) looks like an equation describing hydrodynamical fluid with an extra quantum correction term \( \sqrt{\rho} \nabla \rho \), which is the so called quantum potential. (2.7) and (2.8) are actually relativistic and thus are Lorentz invariant. This is a crucial difference from previous literature where only the Galilean invariant fluid equations were considered. Hydrodynamics is a long wavelength effective theory for physical fluids. Quantum gravity, or even very exotic field theories, when heated up to finite temperature, also behave hydrodynamically at long-wavelength and long-time scales. Now we consider perturbations around the background \( (\rho_0,\theta_0) \): \( \rho = \rho_0 + \rho_1 \) and \( \theta = \theta_0 + \theta_1 \), and rewrite (2.7) and (2.8) as

\[ - \partial_t \left[ \rho_0 \dot{\theta}_1 + (\dot{\theta}_0 - eA_t)\rho_1 \right] + \nabla \cdot \left[ \rho_0 \nabla \theta_1 + \rho_1 (\nabla \theta_0 + eA_i) \right] = 0, \] (2.9)

\[ 2(\dot{\theta}_0 - eA_t)\dot{\theta}_1 - 2(\nabla \theta_0 + eA_i) \cdot \nabla \theta_1 - b\rho_1 + D_2\rho_1 = 0, \] (2.10)

where we have defined

\[ D_2\rho_1 = -\frac{1}{2} \rho_0^{-\frac{3}{2}}(\Box \sqrt{\rho_0})\rho_1 + \frac{1}{2} \rho_0^{-\frac{5}{2}}\Box(\rho_0^{-\frac{1}{2}}\rho_1). \]

In this work, we do not consider fluctuations of the electromagnetic field \( A_\mu \). It is a background field that can affect the acoustic black hole geometry. The equations (2.9) and (2.10) corresponds to a relativistic fluids equation if \( b\rho_1 \) term and \( D_2\rho_1 \) (the quantum potential) term are absent. Actually, the \( D_2 \) term contains the second derivative of slowly varying \( \rho \), which can be negligible in the hydrodynamic region where \( k, \omega \) are small. We will turn on the quantum potential term only in next subsection. In fact it is well-known that in the Madelung representation of the condensation,

\[ \phi = \sqrt{\rho}(\vec{x},t)e^{i\theta(\vec{x},t)}, \]
the Schrödinger equation can be rewritten as a continuity equation plus an Euler equation for an irrotational and inviscid fluid when the quantum potential can be neglected. The Abelian Higgs model here shows that the same property holds for relativistic fluids.

For the simplicity of calculation, let us introduce variables $\omega_0$ and $\vec{v}_0$ with the definition

$$\omega_0 = -\dot{\theta}_0 + eA_t$$

and

$$\vec{v}_0 = \nabla \theta_0 + e\vec{A}.$$ We then obtain

$$-\partial_t \left[ \rho_0 \dot{\theta}_1 + \frac{2}{b} \left( \omega_0^2 \dot{\theta}_1 + \omega_0 \vec{v}_0 \cdot \nabla \theta_1 \right) \right] + \nabla \cdot \left[ \rho_0 \nabla \theta_1 + \frac{2}{b} \left( -\vec{v}_0 \omega_0 \dot{\theta}_1 - \vec{v}_0 \cdot \nabla \theta_1 \vec{v}_0 \right) \right] = 0.$$  (2.11)

Comparing the above equation with the massless Klein-Gordon equation

$$\frac{1}{\sqrt{-g}} \partial_\nu (\sqrt{-g} g^{\mu \nu} \partial_\mu \theta_1) = 0,$$  (2.12)

one may find

$$\sqrt{-g} g^{\mu \nu} \equiv \begin{bmatrix} -\frac{b}{2} \rho_0 - \omega_0^2 & -v_0^j \omega_0 & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
-v_0^i \omega_0 & (\frac{b}{2} \rho_0 \delta^{ij} - v_0^i v_0^j) & \cdots & \cdots & \cdots \\
\end{bmatrix}.  \tag{2.13}$$

Defining the local speed of sound

$$c_s^2 = \frac{b}{2 \omega_0^2} \rho_0,$$  (2.14)

the metric for acoustic black hole yields the form

$$g_{\mu \nu} \equiv \frac{b \rho_0}{2c_s^2} \begin{bmatrix} -(c_s^2 - v^2) & \cdots & -v^j \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
-v^i & (1 + c_s^2 - v^2) \delta^{ij} + v^i v^j & \cdots & \cdots & \cdots \\
\end{bmatrix},$$  (2.15)

where we made replacement $v^i_0 \rightarrow v^i \omega_0$ and notation $v^2 = \sum_i v^i v^i$. We now have a relativistic version of acoustic black hole which is a generalization of [11]. It is worth noting that the sound velocity $c_s$ is a function of the electromagnetic field $A_t$, which is treated as a slowly varying quantity.

In the non-relativistic limit, $c_s \ll 1$, $\vec{v} \ll 1$, and keeping only the leading order term of each line element, we can recover the metric from non-relativistic theory first obtained by Unruh [11], but with an overall factor difference

$$g_{\mu \nu} \equiv \left( \frac{\rho_0}{c_s} \right) \begin{bmatrix} -(c_s^2 - v^2) & \cdots & -v^j \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
-v^i & \delta^{ij} & \cdots & \cdots & \cdots \\
\end{bmatrix}.  \tag{2.16}$$
Equivalently, we can rewrite (2.15) as
\[
\begin{align*}
   ds^2 &= \frac{b}{2 c_s} \rho_0 \frac{1}{\sqrt{1 + c_s^2 - v^2}} \left[ -(c_s^2 - v^2) dt^2 - 2 \vec{v} \cdot d\vec{x} dt + (\vec{v} \cdot d\vec{x})^2 + (1 + c_s^2 - v^2) d\vec{x}^2 \right] \\
   &= \frac{b}{2 c_s} \rho_0 \frac{1}{\sqrt{1 + c_s^2 - v^2}} \left[ -(c_s^2 - v^2) d\tau^2 + (1 + c_s^2 - v^2) \left( \frac{v^i v^j}{c_s^2 - v^2} + \delta_{ij} \right) dx^i dx^j \right],
\end{align*}
\]
where we have defined a new time coordinate by \( d\tau = dt + \frac{\vec{v} \cdot d\vec{x}}{c_s^2 - v^2} \). In this coordinate the acoustic geometry is actually stationary.

In the case \( v_z = 0 \), but \( v_r \neq 0 \) and \( v_\phi \neq 0 \), we can simplify the metric in a Kerr-like form
\[
\begin{align*}
   ds^2 &= \frac{b}{2 c_s} \rho_0 \frac{1}{\sqrt{1 + c_s^2 - v^2}} \left[ -N^2(v_r, v_\phi) d\tau^2 + \frac{1}{N^2(v_r, v_\phi)} dr^2 + \sqrt{1 + c_s^2 - v^2} dz^2 \\
   &+ N^2(v_r, v_\phi) r^2 d\varphi^2 + \frac{(v_\phi d\tau - r d\varphi)^2}{\sqrt{1 + c_s^2 - v^2}} \right],
\end{align*}
\]
where
\[
N^2(v_r, v_\phi) = \frac{c_s^2 - v^2}{\sqrt{1 + c_s^2 - v^2}}
\]
and the coordinate transformations have been used
\[
\begin{align*}
   dt &= d\tau + \frac{v_r}{v_r^2 - c_s^2} dr, \\
   d\phi &= d\varphi + \frac{v_r v_\phi}{v_r^2 - c_s^2} dr.
\end{align*}
\]
When \( v_z \neq 0 \), the metric becomes non-spherically symmetric and non-axisymmetric. In non-relativistic limit, the geometry (2.18) reduces to a “draining bathtub” metric first appeared in [2].

At the region far from the acoustic horizon, the normal modes of the \( \theta_1 \) field are purely outgoing. The operator \( \theta_1 \) can be expanded in terms of modes in the region outside the horizon. An observer traveling with the fluid as it flows through the acoustic horizon will see that the field \( \theta_1 \) is essentially the vacuum state in analogy with the behavior of the normal modes of a scalar field in Schwarzschild coordinates by a freely falling observer [1]. The corresponding Hawking temperature at the event horizon reads
\[
T_H = \frac{\hbar}{2\pi k_B} \left. \frac{\partial (c_s - v^i)}{\partial x^i} \right|_{\text{horizon}}.
\]
The relativistic version of Hawking temperature from Abelian Higgs model is identical with the non-relativistic one.

Since there is no real gravity, the Newton constant \( G_N \) is not involved in the formulation of the acoustic “surface gravity” and Hawking temperature. One may compare the acoustic Hawking temperature with Unruh temperature in Rindler coordinates. The acoustic temperature and the Unruh temperature yield almost the same form: The acoustic temperature is proportional to the gradients of the fluid
kinetic energy (i.e. $\propto \partial_i (c_s^2 - v_i^2)|_{\text{horizon}}$) at the horizon, while the Unruh temperature ($T = \frac{\alpha_R}{2\pi}$), observed by an accelerating observer is proportional to the acceleration $a_R (= \frac{\partial v}{\partial t})$. According to Einstein equivalence principle, no experiment can distinguish the acceleration due to gravity from the inertial acceleration due to a change of velocity. Thus, the Unruh effects in Rindler coordinates and Hawking effects in astrophysical black hole spacetime are equivalent. It is surprising that the acoustic geometry provides us another way to realize the equivalence principle by the relation (Acceleration) = (Intensity of the gravitational field)=(Gradients of the kinetic energy). One point that should be made clear is that acoustic analog models of black holes could be used to mimic the kinematical but not the dynamical aspects of gravitational systems. For example, it is impossible to reproduce black hole thermodynamics using acoustic analogue models of black holes [18].

### 2.1 Dispersion relation

We have shown that by studying the linearized equations of motion of the complex scalar fields of Abelian Higgs model, the perturbations obey a Klein-Gordon equation in the curved space-time. Here we stress that since we worked in the long-wavelength and slowly varying limit, the quantum potential term (the $D_2$ term) is negligble.

In the high-momentum or eikonal regime, the contribution of the quantum potential would be important. In [17], the authors explored the dispersion relation with the contribution of the quantum potential. In our case, both the quantum potential and the relativistic effects will contribute to the dispersion relation. In this regime, the phase fluctuation $\theta_1$ and the density fluctuation $\rho_1$ can be treated as a function with a slowly varying amplitude and rapidly varying phase:

$$
\theta_1 \sim \rho_1 \sim \Re\{\exp(-i\Omega t + i\vec{k} \cdot \vec{x})\}
$$

$$
\Omega = \frac{\partial t}{\theta_1} = \frac{\partial t}{\rho_1}, \quad k_i = -\nabla_i \theta_1 = -\nabla_i \rho_1
$$

The operator $D_2$ can be approximately written as

$$
D_2 \rho_1 \equiv -\frac{1}{2} \rho_0^{-\frac{3}{2}} (\Box \sqrt{\rho_0}) \rho_1 + \frac{1}{2} \rho_0^{-\frac{1}{2}} \Box (\rho_0^{-\frac{1}{2}} \rho_1)
$$

$$
= \frac{\rho_1 (\partial_\mu \rho_0)^2}{2 \rho_0^3} - \frac{\partial_\mu \rho_0 \partial_\mu \rho_1}{2 \rho_0^3} - \frac{\rho_1 \Box \rho_0}{2 \rho_0^2} + \frac{\Box \rho_1}{2 \rho_0}
$$

$$
\approx \frac{\Box \rho_1}{2 \rho_0}
$$

$$
= -\frac{1}{2} \rho_0^{-1} (k^2 - \Omega^2) \rho_1,
$$

(2.24)
where $\mu = t, x_1 \ldots x_3$. We have similar result for $D_2 \theta_1$. Therefore, we can treat $D_2 = -\frac{1}{2} \rho_0^{-1} (k^2 - \Omega^2)$ in the eikonal regime. The Klein-Gordon equation (2.9) and (2.10) become

$$\frac{D_2 - b}{2} - \rho_0 (k_i k_j \delta_{ij} - \Omega^2) - \left( \omega_0 \Omega + v_0^i k_j \right)^2 = 0. \quad (2.25)$$

With the expression of $D_2$ above, Eq. (2.25) becomes

$$\frac{b \rho_0}{2} \left( k^2 - \Omega^2 \right) - \left( \omega_0 \Omega + v^i_0 k_i \right)^2 + \frac{1}{4} \left( k^2 - \Omega^2 \right)^2 = 0. \quad (2.26)$$

In the eikonal limit, the last term in the above equation will become dominant and we can neglect the other two terms, which gives

$$\Omega = k. \quad (2.27)$$

On the other hand, in the hydrodynamics regime, we can neglect the $D_2$ term again in the left hand of Eq.(2.25). To be definite, we choose $k^i = k \delta^{i1}$, and define $v^i = v_0^i / \omega_0$. Then we easily obtain the dispersion relation

$$\Omega = -v^1 + c_s \sqrt{1 + c_s^2 - (v^1)^2} \frac{k}{1 + c_s^2}. \quad (2.28)$$

The dispersion relation is quiet different from the result of [2]. For large wavelength and in the non-relativistic limit, it reads

$$\Omega = (c_s - v^1) \frac{1 + (c_s + v^1)/2}{2} k. \quad (2.29)$$

The sound velocity in hydrodynamic limit has a correction to the familiar value $c_s$. The main correction is simply due to the fluid motion and the next term is higher order. Since $v^1$, the generalized physical velocity in the presence of the electro-magnetic fields, depends on background electric and magnetic fields, as one can see from

$$v^1 = (\partial_x \theta_0 + e A_x) / (-\dot{\theta} + e A_t), \quad (2.30)$$

so is the sound velocity.

We stress that we assume that the background fields change very little within a wave length. If $c_s^2$ and $v^2$ can be neglected, compared with the speed of light, then the relation return to a linear dispersion relation $\Omega \approx c_s |k|$.

### 3 Non-relativistic limit: Acoustic black holes in superconductors

In the non-relativistic limit, the Abelian Higgs model reduces to the Ginzburg-Landau theory

$$i \hbar \frac{\partial}{\partial t} \psi(\vec{r}, t) = -\frac{\hbar^2}{2m^*} \vec{\nabla}^2 \psi(\vec{r}, t) + a(T) \psi(\vec{r}, t) + b(T) | \psi(\vec{r}, t) |^2 \psi(\vec{r}, t), \quad (3.1)$$
where \( \vec{D} = \nabla + \frac{2ie}{\hbar} \vec{A} \), \( m^* \) is the mass of a cooper pair, \( a(T) \) and \( b(T) \) are two parameters that depend on temperature. We have turned on the Planck constant \( \hbar \). The Ginzburg-Landau coherence length is defined by

\[
\xi(T) = \left( \frac{\hbar^2}{2m^* |a(T)|} \right)^{1/2}.
\]  

(3.2)

Assuming \( \psi(\vec{r}, t) = \sqrt{\rho(\vec{r}, t)} e^{i\theta(\vec{r}, t)} \), we have

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \tag{3.3}
\]

\[
\hbar \frac{\partial \theta}{\partial t} = \frac{\hbar^2}{2m^*} \nabla^2 \sqrt{\rho} - \frac{m^*}{2} \vec{v}^2 - a(T) - b(T) \rho. \tag{3.4}
\]

where we have used the gauge \( \nabla \cdot \vec{A} = 0 \), and the velocity is denoted by \( \vec{v} = \frac{\hbar \nabla \theta}{m^*} + \frac{2e}{m^*} \vec{A} \). The first term in the right hand of (3.4) corresponds to the quantum potential. The above two equations are completely equivalent to the hydrodynamical equations for irrotational and inviscid fluid apart from the quantum potential. In the long-wavelength approximation, the contribution coming from the linearization of the quantum potential can be neglected.

Linearizing Eqs. (3.3) and (3.4) around the background \( (\rho_0, \theta_0) \), we get

\[
\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_1 \vec{v}_0 + \rho_0 \nabla \theta_1) = 0, \tag{3.5}
\]

\[
\frac{\partial \theta_1}{\partial t} + \vec{v}_0 \cdot \nabla \theta_1 + \frac{b(T)}{m^*} \rho_1 = 0. \tag{3.6}
\]

Eliminating \( \rho_1 \), the above equations lead to

\[
\frac{\partial}{\partial t} \left( - \frac{m^*}{b(T)} \frac{\partial \theta_1}{\partial t} - \frac{m^*}{b(T)} \vec{v}_0 \cdot \nabla \theta_1 \right) + \nabla \cdot \left( \rho_0 \nabla \theta_1 - \frac{m^*}{b(T)} \vec{v}_0 \frac{\partial \theta_1}{\partial t} - \frac{m^*}{b(T)} \vec{v}_0 \vec{v}_0 \cdot \nabla \theta_1 \right) = 0, \tag{3.7}
\]

where \( \vec{v}_0 = \frac{\hbar \nabla \theta_0}{m^*} + \frac{2e}{m^*} \vec{A} \). One can extract a metric from (3.7) because it has the same form as a massless Klein-Gordon equation in curved space-time. The non-relativistic version of acoustic metric from (3.7) is read as

\[
ds^2 = \frac{\rho_0}{m^* c_s} \left\{ - (c_s^2 - \vec{v}_0^2) dt^2 - 2\vec{v}_0 \cdot d\vec{r} dt + (\vec{dr} \cdot d\vec{r}) \right\}, \tag{3.8}
\]

where \( c_s = \sqrt{\frac{b(T)}{m^*} \rho_0} \) is denoted as the “sound velocity”. Actually, one can write the sound velocity in terms of the coherent length

\[
c_s = \frac{\hbar}{\sqrt{2m^* \xi(T)}}. \tag{3.9}
\]
One should not identify the “sound velocity” defined here with the sound velocity produced by the crystal lattices. Here \(c_s\) is a function of the density \(\rho_0\) of superconducting electrons and temperature-dependent parameter \(b(T)\).

We would like to do some calculations on the necessary conditions for the formation of an acoustic event horizon in Ginzburg-Landau theory. We will show that it would be a tough task to realize acoustic black holes by using type I and type II superconductors. The definition of acoustic black hole requires \(\frac{v_0}{c_s} > 1\) in some region. From the London equation

\[
\nabla^2 \vec{B} - \frac{1}{\lambda(T)} \vec{B} = 0,
\]

and the Maxwell equation

\[
\nabla \times \vec{B} = \mu_0 \vec{j}_s,
\]

where \(\lambda(T) = \left(\frac{m_e}{\mu_0 n_s e^2}\right)^{1/2}\) and \(\vec{j}_s\) denote the penetration depth and the supercurrent, respectively. We have a finite magnetic field, say \(\vec{B} = (0, B_y(x), 0)\), outside a superconducting region. After simple computation, we find the supercurrent yields \(19\)

\[
\vec{j}_{sz} = \frac{H(T)}{\lambda(T)} \lambda(T)\]

(3.12)
The critical current density above is only valid for clean type I superconductors. On the other hand, the current density has the form

\[
\vec{j}_{sz} = e n_s v_0 s,
\]

(3.13)
where \(n_s = 2|\psi^2| = 2\rho_0\) denotes the superfluid density, and \(v_0 s\) is the velocity of the Cooper pairs. We then have

\[
\frac{v_0 s}{c_s} = \frac{\sqrt{2\kappa H(T)}}{H_{c2}} = \frac{H(T)}{H_c},
\]

(3.14)
where \(\kappa = \frac{\lambda(T)}{\xi(T)}\), and \(H_{c2} = \frac{h}{4\pi e \mu_0 \xi^2(T)} = \sqrt{2\kappa} H_c\). In the standard textbook, \(\kappa\) denotes the G-L parameter, which describes the difference between type I and type II superconductors: \(\kappa < \frac{1}{\sqrt{2}}\) is for type I, and \(\kappa > \frac{1}{\sqrt{2}}\) is for type II. \(H_{c2}\) denotes the critical magnetic field, and \(H_c\) the thermodynamic critical field. From (3.14), we can see that for type I superconductors, when \(v_0s = c_s\), the superconducting phase is broken and return to Normal states. Now we are going to solve (3.10) in the high \(\kappa\) limit, \(\lambda \gg \xi\), which is valid for the cooper-oxide superconductors. We will show that even near the normal core of the vortex, it is hard for the speed of superconducting electrons to exceed the “sound velocity”– \(c_s\). For simplicity, the vortex is assumed to be infinitely long and axially symmetric so that there are no \(z\) or angular dependence of its field distribution. We then write Eq. (3.10) in cylindrical coordinates

\[
\frac{1}{r} \partial_r (r \partial_r \vec{B}) - \frac{1}{\lambda^2} \vec{B} = 0.
\]

(3.15)
This solution is given by [19]

\[ B(r) = \frac{\phi_0}{2\pi\lambda^2} K_0(r/\lambda), \]  

where \( K_0(r/\lambda) \) is the zeroth-order modified Bessel function. The current density is derived by the Maxwell equation (3.11),

\[ j_s(r) = \frac{\phi_0}{2\pi\mu_0\lambda^2} K_1(r/\lambda) \hat{e}_\phi \]  

We are interested in the asymptotic behaviors at small radial distances \( r \ll \lambda \), where \( K_1(r/\lambda) \approx \frac{\lambda}{r} \).

The speed of Cooper pairs is then given by

\[ v_{0s} = \frac{j_s(r)}{n_s e} \]  

We then find

\[ \frac{v_{0s}}{c_s} = \frac{\sqrt{2}\xi}{r}. \]  

The formation of acoustic black holes requires \( v_{0s}/c_s > 1 \), i.e. \( r < \sqrt{2}\xi \). But at the core region of the vortex \( r < \xi \), the order parameter and the superconducting current would decay as \( r \to 0 \), since it corresponds to a normal state. (3.13) tells us that for fixed electron velocity \( v_{0s} \), the current \( j_{sz} \) decreases as \( n_s (= 2\rho_0) \) decreases. Since the “sound velocity” \( c_s \) depends on the density \( \rho_0 \), we expect that in the region \( \xi < r < \sqrt{2}\xi \), electron velocity can exceed the sound velocity. However, it would be interesting if one can verify this point experimentally.

In order to trap phonons, one need input a sink so that electrons can propagate along \( \phi \)- and \( r \)-direction simultaneously. For this purpose, we need turn to spiral flux vortices in the type II superconductors. The spiral flux vortices [20] in current-carrying type-II superconducting cylinders subjected to longitudinal magnetic fields can be used to mimic acoustic black holes. Consider a type-II superconducting cylinder of radius \( a \) containing a spiral vortex, in the presence of a longitudinal applied magnetic field \( H_\alpha \) and transport current \( I \), where \( H_\alpha \) and \( I \) are positive in the \( z \) direction. The Lorentz force imposed on the vortex were found to be radially inward. Thus the spiral vortex has a tendency to move to the center of the cylinder. One can determine the terminal speed \( v_{0s}^r \) of the contracting vortex by balancing the Lorentz force and the viscous drag force \( \eta v_{0s}^r \), where \( \eta \) is a phenomenological viscous drag coefficient per unit length of vortex [20].

Actually, the spiral vortex in the presence of a sufficiently large current density parallel to its axis, is unstable against a helical perturbation [20]. But this does not prevent the formation of acoustic black hole geometry. In the next section, we investigate the spiral vortex geometry from classical hydrodynamics in the non-relativistic frame and give the exact metric for a spiral-vortex geometry.

\[ ^1 \text{The order parameter also reaches its maximum at } r = \sqrt{2}\xi [19]. \]
3.1 Spiral-Vortex geometry

In [2], the author constructed a “draining bathtub” black hole metric by a \((2+1)\) dimensional Newtonian flow with a sink at the origin. An stable vortex solution to the Navier-Stokes equation requires each component of the fluid velocity to be non-vanishing. Such an exact solution to Navier-Stokes equation was first found by Sullivan [21] and later observed in experiments [22]. Now, we utilize the Sullivan vortex solution of Navier-Stokes equation to model a 4-dimensional acoustic black hole metric, which could be realized in experiments. The non-relativistic Navier-Stokes equation is given by

\[
\partial_t \vec{v} + \vec{v} \cdot \nabla \vec{v} = -\nabla p/\rho + \nu \nabla^2 \vec{v} + \vec{f}/\rho, \tag{3.20}
\]

where \(\vec{v}\) is the fluid velocity, \(p\) the fluid pressure, \(\rho\) the mass density, \(\nu\) the shear viscosity and \(\vec{f}\) an externally specified forcing function. The continuity equation generally accompanies the Navier-Stokes equation

\[
\partial_t \rho + \nabla \cdot (\rho \vec{v}) = 0. \tag{3.21}
\]

By defining \(\vec{v} = \nabla \psi\), the Navier-Stokes equation can be rewritten as

\[
\partial_t \psi + \frac{1}{2} (\nabla \psi)^2 + \int_0^p \frac{dp'}{\rho} - \nu \nabla^2 \psi = 0, \tag{3.22}
\]

where we have chosen vanishing external force \(\vec{f} = 0\). Now linearize these equations around a fixed background \((\rho_0, p_0, \psi_0)\) and set \(\rho = \rho_0 + \rho_1\), \(p = p_0 + p_1\), and \(\psi = \psi_0 + \psi_1\). After linearizing the equation of motion and the continuity equation, the equations for \(\rho_1\) and \(\psi_1\) yield

\[
\frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_1 \vec{v}_0 + \rho_0 \nabla \psi_1) = 0, \tag{3.23}
\]

\[
\frac{\partial \psi_1}{\partial t} + \vec{v}_0 \cdot \nabla \psi_1 + \frac{p_1}{\rho_0} - \nu \nabla^2 \psi_1 = 0. \tag{3.24}
\]

One can further assume the fluid is barotropic, which implies that \(\rho\) is a function of \(p\) only \(^2\). The barotropic assumption gives

\[
\rho_1 = \frac{\partial \rho}{\partial p} \rho_1. \tag{3.25}
\]

After rearranging (3.24), we have

\[
\rho_1 = -\frac{\partial \rho}{\partial p} \rho_0 \left( \partial_t \psi_1 + \vec{v}_0 \cdot \nabla \psi_1 - \nu \nabla^2 \psi_1 \right), \tag{3.26}
\]

\(^2\)This assumption was first used implicitly in [1] (see also [2]).
Substituting the above linearized equation of motion back into (3.23), we obtain
\[-\partial_t \left[ \frac{\partial \rho}{\partial p_0} \rho_0 \left( \partial_t \psi_1 + \vec{v}_0 \cdot \nabla \psi_1 - \nu \nabla^2 \psi_1 \right) \right]
+ \nabla \cdot \left[ \rho_0 \nabla \psi_1 - \frac{\partial \rho}{\partial p_0} \rho_0 \vec{v}_0 \left( \partial_t \psi_1 + \vec{v}_0 \cdot \nabla \psi_1 - \nu \nabla^2 \psi_1 \right) \right] = 0. \tag{3.27}

We can rewrite the above equation in a Klein-Gordon like form
\[\partial_{\mu} (\sqrt{-g} g^{\mu\nu} \partial_{\nu} \psi_1) = -\frac{\partial \rho}{\partial p_0} \rho_0 \nu (\partial_t + \vec{v}_0 \cdot \nabla) \nabla^2 \psi_1, \tag{3.28}\]
where the inverse metric is given by
\[g^{\mu\nu} \equiv \left( \frac{1}{\rho_0 c_s} \right) \begin{pmatrix} -1 & \cdots & -v^i_0 \\ \cdots & & \cdots \\ -v^i_0 & \cdots & -\left( c_s^2 \delta^{ij} - v^i_0 v^j_0 \right) \end{pmatrix}, \tag{3.29}\]
and the local speed of sound is defined by \(c_s^2 = \frac{\partial p}{\partial \rho}. \) The acoustic metric \(g_{\mu\nu}\) is given in (2.16).

What is different from before is that we have third order derivative term of \(\psi_1\) in the right hand of (3.28). The similar situation was discussed in [2] and Lorentz symmetry breaking was observed. Clearly, by writing \(\psi_1 \sim \exp(-i\omega t + i\vec{k} \cdot \vec{x})\), one see that in the eikonal approximation, the right hand of (3.28) will be dominant, while in the hydrodynamic limit, the right hand of (3.28) can be ignored. In the following, we will work in long-wavelength approximation and neglect those terms.

In the cylindrical coordinate \((r, \phi, z)\), the exact solution to Navier-Stokes equation yields [21]
\[v^r_0 = -\alpha r + \frac{2\beta \nu}{r} \left( 1 - e^{-\frac{\alpha r^2}{2\nu}} \right), \tag{3.30}\]
\[v^\phi_0 = \frac{\Gamma_0}{2\pi r} \frac{H(x)}{H(\infty)}, \tag{3.31}\]
\[v^z_0 = 2\alpha z \left[ 1 - \beta e^{-\frac{\alpha z^2}{2\nu}} \right], \tag{3.32}\]
where
\[H(x) = H\left( \frac{\alpha r^2}{2\nu} \right) = \int_0^x \exp \left( -s + \beta \int_0^s \frac{1 - e^{-\tau}}{\tau} d\tau \right) ds, \tag{3.33}\]
\(\Gamma_0\) is a constant that depends on the boundary condition, \(\nu\) denotes the shear viscosity, \(\alpha > 0\) and \(\beta > 1\) are free parameters that can be fixed in experiments. By using the coordinate transformation
\[dt = d\tau + \frac{v^r_0}{v^r_0^2 + v^z_0^2 - c_s^2} dr + \frac{v^z_0}{v^r_0^2 + v^z_0^2 - c_s^2} dz, \tag{3.34}\]
\[d\phi = d\phi + \frac{v^r_0 v^\phi_0}{v^r_0^2 + v^z_0^2 - c_s^2} \frac{dr}{r} + \frac{v^z_0 v^\phi_0}{v^r_0^2 + v^z_0^2 - c_s^2} \frac{dz}{r}. \tag{3.35}\]
We obtain a non-spherically symmetric metric by rewriting (2.16) as
\[
\begin{align*}
    ds^2 &= \frac{\rho_0}{c_s} \left\{ - (c_s^2 - v_0^2) d\tau^2 + \left( 1 - \frac{v_0^2}{v_0^2 + v_0^2 - c_s^2} \right) dr^2 + \left( 1 - \frac{v_0^2}{v_0^2 + v_0^2 - c_s^2} \right) dz^2 \\
    &\quad - \frac{2v_0^2 v_0^2}{v_0^2 + v_0^2 - c_s^2} dr dz - 2v_0^2 r d\phi d\tau + r^2 d\phi^2 \right\}.
\end{align*}
\] (3.36)

It is worth noting that when \( v_z = 0 \), we can recover the “draining bathtub” metric obtained in [2], that is to say
\[
\begin{align*}
    ds^2 &= \frac{\rho_0}{c_s} \left\{ - (c_s^2 - v_0^2) d\tau^2 + \frac{c_s^2}{c_s^2 - v_0^2} dr^2 + dz^2 - 2v_0^2 r d\phi d\tau + r^2 d\phi^2 \right\}.
\end{align*}
\] (3.37)

The form of the metric obtained in (3.36) is non-spherically symmetric and even non axi-symmetric. There exists an event horizon for the metric. The event horizon can be fixed by using the Null surface equation [24]
\[
g^\mu \nu \frac{\partial f}{\partial x^\mu} \frac{\partial f}{\partial x^\nu} = 0,
\] (3.38)
where \( f = f(\tau, r, \varphi, z) = 0 \) denotes the hypersurface. Since the radius of the event horizon would vary as a function of \( \tau \) and \( z \), we find that the event horizon is determined by
\[
\begin{align*}
    \dot{r}_H &= \left[ v_0^2 (r_H) + v_0^2 (r_H) - c_s^2 \right] \left[ v_0^2 (r_H) - r'H v_0^2 (r_H) + r'^2 v_0^2 (r_H) - c_s^2 (1 + r'^2) \right] c_s^{-2},
\end{align*}
\] (3.39)

where the dot and the prime denote the derivative with respect to \( \tau \) and \( z \), respectively.

### 4 Conclusion

In summary, a new acoustic black hole metric has been derived from the Abelian Higgs model. In the non-relativistic limit, the metric can reduce to an acoustic black hole geometry given by Unruh. Since the Abelian Higgs model describes high energy physics, our results indicate that acoustic black holes might be created in high energy physical process, such as quark matters and neutron stars. The dispersion relation was given and in the non-relativistic limit, we can recover the linear dispersion relation. We have interpreted the requirements of creating an acoustic black hole by using the language of superconductor physics. Our results have demonstrated that although it would be very difficult to mimic the acoustic black hole in type I superconductors, people would be able to to discover the acoustic black holes near the magnetic vortex in type II superconductors. We developed an acoustic black hole with spiral vortex geometry from the Navier-Stokes equation in classical Newtonian fluids.

One point to be figured out is that we have used the vortices to mimic the black hole. Actually, when a superfluid flow exceeds the Landau critical velocity, vortices can be created and excitations
with negative energy appear, which means that black hole configurations are energetically unstable since the speed of sound is also the Landau critical velocity. However, this does not mean that the supercritical flow of the superfluid vacuum is not possible. In superfluid $\text{^3He}$ the Landau velocity for vortex nucleation can be exceeded by several orders of magnitude, without vortices creation. In [23], the event horizon was proposed to be realized in superfluid $\text{^3He}$ by arranging the superfluid flow velocity to be constant but the “sound velocity” to change in space and in some region becomes less than the flow velocity of the liquid. In our above discussions, we have shown that in the ideal type II superconductors, the “sound velocity” indeed depends on the superfluid density $\rho_0$, but the superconducting (superfluid) flow velocity can be chosen to be constant by varying the current and the superfluid density to be $\rho_0$ simultaneously in (3.13). Therefore, if the “sound velocity” changes in space, superconducting (superfluid) flow velocity would exceed the “sound velocity” in some regions.

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