Generalized Transportation Cost Spaces

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Abstract. The paper is devoted to the geometry of transportation cost spaces and their generalizations introduced by Melleray et al. (Fundam Math 199(2):177–194, 2008). Transportation cost spaces are also known as Arens–Eells, Lipschitz-free, or Wasserstein 1 spaces. In this work, the existence of metric spaces with the following properties is proved: (1) uniformly discrete metric spaces such that transportation cost spaces on them do not contain isometric copies of $\ell_1$, this result answers a question raised by Cúth and Johanis (Proc Am Math Soc 145(8):3409–3421, 2017); (2) locally finite metric spaces which admit isometric embeddings only into Banach spaces containing isometric copies of $\ell_1$; (3) metric spaces for which the double-point norm is not a norm. In addition, it is proved that the double-point norm spaces corresponding to trees are close to $\ell^d_{\infty}$ of the corresponding dimension, and that for all finite metric spaces $M$, except a very special class, the infimum of all seminorms for which the embedding of $M$ into the corresponding seminormed space is isometric, is not a seminorm.

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1. Introduction

1.1. Definitions

Let $(M,d)$ be a metric space. Consider a real-valued finitely supported function $f$ on $M$ with a zero sum, that is,

$$\sum_{v \in M} f(v) = 0. \quad (1)$$

A natural and important interpretation of such a function is the following: $f(v) > 0$ means that $f(v)$ units of a certain product are produced or stored at point $v$; $f(v) < 0$ means that $-f(v)$ units of the same product
are needed at \( v \). The number of units can be any real number. With this in mind, \( f \) may be regarded as a transportation problem. For this reason, we denote the vector space of all real-valued functions finitely supported on \( M \) with a zero sum by \( \text{TP}(M) \), where \( \text{TP} \) stands for transportation problems.

For a metric space \( M \) with the base point, which is a distinguished point usually denoted by \( O \), there is a canonical embedding of \( M \) into \( \text{TP}(M) \) given by the formula:

\[
v \mapsto 1_v - 1_O,
\]

where \( 1_u(x) \) for \( u \in M \) is the indicator function defined as:

\[
1_u(x) = \begin{cases} 
1 & \text{if } x = u, \\
0 & \text{if } x \neq u.
\end{cases}
\]

The goal of this work is to study different norms on the vector space \( \text{TP}(M) \) for which this embedding is an isometric embedding.

One of the most commonly used norms on \( \text{TP}(M) \) satisfying this condition is that related to the transportation cost and defined in the following way.

A transportation plan is a plan of the following type: we intend to deliver

- \( a_1 \) units of the product from \( x_1 \) to \( y_1 \),
- \( a_2 \) units of the product from \( x_2 \) to \( y_2 \),
-
- \( a_n \) units of the product from \( x_n \) to \( y_n \),

where \( a_1, \ldots, a_n \) are nonnegative real numbers, and \( x_1, \ldots, x_n, y_1, \ldots, y_n \) are elements of \( M \), which do not have to be distinct.

This transportation plan is said to solve the transportation problem \( f \) if

\[
f = a_1(1_{x_1} - 1_{y_1}) + a_2(1_{x_2} - 1_{y_2}) + \cdots + a_n(1_{x_n} - 1_{y_n}).
\]

The cost of transportation plan (3) is defined as \( \sum_{i=1}^n a_i d(x_i, y_i) \). We introduce the transportation cost norm (or just transportation cost) \( \|f\|_{\text{TC}} \) of a transportation problem \( f \) as the minimal cost of transportation plans solving \( f \). It is easy to see that the minimum is attained—we consider finitely supported functions—and that \( \|\cdot\|_{\text{TC}} \) is a norm (see [47, Proposition 3.16]). We introduce the transportation cost space \( \text{TC}(M) \) on \( M \) as the completion of \( \text{TP}(M) \) with respect to the norm \( \|\cdot\|_{\text{TC}} \). The introduced above notions are very natural and were introduced independently and not-so-independently by many different people, whence a variety of names. There are also very important and actively studied notions, which are somewhat different from the introduced above, but are closely related to them. A survey of these definitions, relations between them, and some historical notes are provided in Sect. 1.6.

Our main reasons for choosing the term transportation cost space are:

1. This term will make it immediately clear to as many people as possible what is the topic of this paper;
2. This terminology helps us to develop a suitable language and to build the right intuition for working with the norm
∥·∥_TC; (3) It reflects the history of the subject and the initial motivation for introducing these notions.

1.2. Preliminaries

Theorem 1.1 ([21]). A plan

\[ f = a_1(x_1 - y_1) + a_2(x_2 - y_2) + \cdots + a_n(x_n - y_n) \]  

with \( a_i > 0, i = 1, \ldots, n \), is optimal; that is, it has the minimal cost if and only if there exists a 1-Lipschitz real-valued function \( l \) on \( M \) such that

\[ l(x_i) - l(y_i) = d(x_i, y_i) \]

for all pairs \( x_i, y_i \).

Denote by \( L \) the set of all 1-Lipschitz functions on \( M \). The following is an immediate corollary of Theorem 1.1.

Corollary 1.2.

\[ \|f\|_{TC} = \sup_{l \in L} \left| \sum_{v \in M} l(v)f(v) \right|. \]  

Corollary 1.3. Embedding (2) is isometric if we endow \( TP(M) \) with the norm \( \|\cdot\|_{TC} \).

It is not difficult to see that if \( L \) in the right-hand side of (5) is replaced with a subset \( K \) in the set of all 1-Lipschitz functions, one obtains a seminorm on \( TP(M) \):

\[ \|f\|_K = \sup_{l \in K} |l(f)|, \quad \text{where} \quad l(f) = \sum_{v \in M} l(v)f(v). \]  

It is clear that embedding (2) is isometric as an embedding from \( M \) into the seminormed space \( (TP(M), \|\cdot\|_K) \) if and only if for every two points \( u, v \in M \) and any \( \varepsilon > 0 \), there exists \( l \in K \) such that \( |l(u) - l(v)| \geq d(u, v) - \varepsilon \). In this paper, our focus is mainly on sets \( K \) satisfying the conditions:

A. All functions in \( K \) are 1-Lipschitz, that is \( K \subseteq L \).

B. For every \( u, v \in M \), there is a function \( l \in K \) satisfying the condition \( |l(u) - l(v)| = d(u, v) \).

Obviously, for \( K \) satisfying A and B, mapping (2) is an isometric embedding of \( M \) into \( (TP(M), \|\cdot\|_K) \).

The study of the seminormed spaces \( (TP(M), \|\cdot\|_K) \) for \( K \) satisfying the conditions A and B and different from \( L \) was initiated in [29], and related results were obtained in [48, 49].

Now, we give two simple examples of function sets satisfying conditions A and B.

1. The set of all distance functions \( l_v(\cdot) := d(v, \cdot), v \in M \). This set will be denoted by \( F \) because it was first used in the theory of metric embeddings by Fréchet [11].
2. The set of all functions of the form
\[ \phi_{u,v} = \frac{d(v,\cdot) - d(u,\cdot)}{2}, \quad u, v \in M. \] (7)

We denote this set by \( \mathcal{DP} \) (double-point). The set was introduced in [29, Section 1.2.2].

In cases where \( \| \cdot \|_K \) is a norm, we denote the completion of the normed space \( (\text{TP}(M), \| \cdot \|_K) \) by \( \text{TP}_K(M) \). Also, in the cases where \( K = \mathcal{L}, \mathcal{DP}, \) or \( \mathcal{F} \), we use \( \text{TP}_{\mathcal{L}}(M), \text{TP}_{\mathcal{DP}}(M), \) or \( \text{TP}_{\mathcal{F}}(M) \), respectively. Observe that \( \text{TP}_{\mathcal{L}}(M) = \text{TC}(M) \). The same notation will be used in cases where \( \| \cdot \|_K \) is a seminorm. In such cases, \( \text{TP}_K(M) \) denotes the completion of the quotient of \( \text{TP}(M) \) over \( \ker \| \cdot \|_K \) with respect to the norm induced by \( \| \cdot \|_K \) on this quotient.

1.3. Statement of Results

Section 2 is devoted to analysis of the comment of Melleray, Petrov, and Vershik [29, Comment 3, p. 185] which can be, by Proposition 1.9, restated as: In contrast to the existence of the maximal norm of the form \( \| \cdot \|_K \) with \( K \) satisfying \( A \) and \( B \), there is no minimal norm of the form \( \| \cdot \|_K \); moreover, it can happen that for a given norm of this form, the infimum of the norms which are less than a given norm, is a seminorm, but not a norm.

We show in Corollary 2.5, that for finite metric spaces, with exception of a small class, the infimum of seminorms of the form \( \| \cdot \|_K \) with \( K \) satisfying \( A \) and \( B \) is not a seminorm.

As it was discovered over the last decade, one of the differences between spaces \( \text{TC}(M) \) and other spaces of the form \( \text{TP}_K(M) \), is a substantial “presence” of \( \ell_1 \)-subspaces in \( \text{TC}(M) \). Compare the results in [5–9,13] and Proposition 1.9, which shows that there does not have to be any such presence in \( \text{TP}_K(M) \) for general \( K \). Section 3 is devoted to \( \ell_1 \)-subspaces in \( \text{TC}(M) \).

The main result of this section, Theorem 3.1, gives a negative answer to the following question [7, Question 2, p. 3410]: Let \( M \) be an infinite uniformly discrete metric space. Does \( \text{TC}(M) \) contain a subspace isometric to \( \ell_1 \)? Recall that a metric space \( M \) is called uniformly discrete if there exists a constant \( \delta > 0 \) such that
\[ \forall u, v \in X \ (u \neq v) \Rightarrow d_X(u,v) \geq \delta. \]

In Sect. 4, it is shown that there exist a class of metric spaces for which the double-point norm introduced in [29] is not a norm—it has a nontrivial kernel.

In Sect. 5, it is proved that the space \( \text{TP}_{\mathcal{DP}}(M) \) for a tree \( M \) is close to \( \ell_1^n \) of the corresponding dimension, and thus is quite different from the transportation cost space on a tree.

Recall that a metric space is called locally finite if all of its balls of finite radius have finite cardinality. In Sect. 6, a class of locally finite metric spaces \( M \) satisfying the following condition is found: all Banach spaces containing \( M \) isometrically contain linear isometric copies of \( \ell_1 \). In particular, this is true for spaces of the form \( \text{TP}_K(M) \). This result is also motivated by the following problem considered in [16,32,33]:
Problem 1.4. For what Banach spaces $X$ do there exist locally finite metric spaces $M$ such that each finite subset of $M$ embeds isometrically into $X$, but $M$ does not embed isometrically into $X$?

Our result reveals a new class of Banach spaces for which the phenomenon described in Problem 1.4 occurs.

1.4. Some Interesting Directions in the Theory of $\text{TP}_K(M)$ Spaces

We refer to [28] for basic theory of cotype of Banach spaces, and to [35] for relevance of this theory for metric embeddings as well as for an additional background needed for reading this section.

In our opinion, one of the most interesting and challenging directions in the study of the spaces $\text{TP}_K(M)$ is related to the following well-known facts: (a) Finite metric spaces admit bilipschitz embeddings with distortions arbitrary close to 1 into every Banach space with trivial cotype; (b) Locally finite metric spaces admit bilipschitz embeddings into every Banach space with trivial cotype, whose distortions are bounded by an absolute constant, see [2,34] and [35, Chapters 1 and 2]; in [32], it was shown that this constant does not exceed $4 + \varepsilon$ for every $\varepsilon > 0$.

To determine whether isometric embeddings of $M$ into $\text{TP}_K(M)$ are of a different nature, it is crucial to develop tools needed to advance the following direction of research.

Direction 1.5. Characterize metric spaces $M$ for which we can find a set $K$ of functions on $M$ satisfying conditions $A$ and $B$, and such that $\text{TP}_K(M)$ has nontrivial cotype.

The discussion presented in [35, Section 11.1] suggests the important relevant problem:

Problem 1.6. Can one find a sequence $\{G_n\}_{n=1}^{\infty}$ of expanders and the corresponding sets $K_n$ of Lipschitz functions satisfying conditions $A$ and $B$ such that the direct sum $(\bigoplus_{n=1}^{\infty} \text{TP}_{K_n}(G_n))^2$ has nontrivial cotype?

It should be mentioned that, by virtue of Proposition 1.9, each Banach space containing $M$ isometrically also contains a subspace isometric to $\text{TP}_K(M)$ for suitably chosen $K$ satisfying $A$ and $B$. This shows the significance of the following general direction in the study of $\text{TP}_K(M)$:

Direction 1.7. Given a metric space $M$, find sets $K$ of functions on $M$ satisfying conditions $A$ and $B$ for which one can describe the Banach-space-theoretical structure of $\text{TP}_K(M)$.

Another direction which we regard as fruitful is:

Direction 1.8. Find metric spaces $M$ for which the linear structure of Banach spaces $\text{TP}_K(M)$ satisfies certain geometric conditions for every choice of $K$ satisfying the conditions $A$ and $B$.

Proposition 1.9 reveals that Direction 1.8 is similar to the following: Find metric spaces for which the linear structure of Banach spaces admitting an isometric embedding of $M$ satisfies certain geometric conditions.
A few results of this type are already available: Godefroy and Kalton [14] proved that if $M$ is a separable Banach space, then any Banach space containing isometric copy of $M$ contains a linearly isometric copy of $M$. Dutrieux and Lancien [10] introduced the notion of a representing subset and found several interesting examples of such sets. The definition of representing subsets is provided in Sect. 6, where the existence of locally finite metric spaces representing $\ell_1$ is proved.

1.5. Isometric Embeddings into Banach Spaces and Spaces $TP_K(M)$

The goal of this section is to show that Banach spaces of the form $TP_K(M)$ are present in all Banach spaces containing isometric copies of $M$.

**Proposition 1.9.** If a metric space $M$ admits an isometric embedding into a Banach space $X$, then there exists a set $K$ of functions on $M$ satisfying A and B such that $TP_K(M)$ is linearly isometric to a subspace of $X$.

**Remark 1.10.** Since $\| \cdot \|_K$ can have a nontrivial kernel, a linear isometry $E : TP_M(M) \to X$ is understood as a linear map satisfying $\|Ex\|_X = \|x\|_K$, although this map can have a kernel.

**Proof.** The isometric image of $M$ in $X$ may be shifted so that one of the elements of $M$ coincides with 0. With this in mind, let us identify elements of $M$ and their isometric images in $X$.

Denote by $X^*$ the dual space of $X$ and by $S(X^*)$ its unit sphere. It is clear that the restrictions of elements of $S(X^*)$ to $M$ are 1-Lipschitz functions. Denote by $S$ this set of restrictions. Let us show that the space $TP_S(M)$ admits a linear isometric embedding into $X$ given by $1_v - 1_0 \mapsto v$, where $v \in M$ is identified with its image in $X$.

It suffices to establish that, for any finite collections $\{a_i\} \subset \mathbb{R}$ and $\{v_i\} \subset M$, the equality

$$
\left\| \sum_i a_i (1_{v_i} - 1_0) \right\|_S = \left\| \sum_i a_i v_i \right\|_X
$$

holds. Setting $f = \sum_i a_i (1_{v_i} - 1_0) \in TP(M)$, one arrives at:

$$
\left\| \sum_i a_i v_i \right\|_X = \sup_{x^* \in S(X^*)} \left| \sum_i a_i x^*(v_i) \right| = \sup_{x^* \in S(X^*)} \left| \sum_i a_i (x^*(v_i) - x^*(0)) \right|
$$

$$
= \sup_{l \in S} \left| \sum_{v \in M} l(v) f(v) \right| = \left\| \sum_i a_i (1_{v_i} - 1_0) \right\|_S.
$$

\[\square\]
Remark 1.11. In particular, $TP_K(M)$ can be strictly convex, which makes it different from $TP_L(M)$, see [7, Proposition 2]. It should be mentioned that some metric spaces, e.g., unweighted graphs which are neither complete graphs nor paths [36, Observation 5.1] do not admit isometric embeddings into strictly convex Banach spaces.

1.6. Historical and Terminological Remarks

We are aware of three directions of research for which it was natural to introduce notions which either coincide or are closely related to the notions of the transportation cost and the transportation cost space. These are:

1. Study of algebraically “free” topological (or metric) structures which contain a given topological (or metric) structure as a substructure.
2. Developing the notion of a distance between two probability distributions on a metric space.
3. Studying the notion of a transport of one finite positive measure into another.

Some of the works representing direction (1) are: Markov [26,27], Shimrat [40], Arens–Eells [1], Michael [30], Kadets [15], Pestov [37], Weaver [46] and Godefroy–Kalton [14].

Arens and Eells [1] introduced, for a metric space $M$, the linear space which we denote $TP(M)$ and the norm on it, which we denote $∥·∥_{TC}$. Their goal for introducing these notions was to prove the following result: “Every metric space can be isometrically embedded as a closed subset of a normed linear space”. In this connection, they derived a version of Theorem 1.1, more precisely, they proved that the dual of the normed space $(TP(M), ∥·∥_{TC})$ is the space of Lipschitz functions vanishing at a base point. Arens and Eells [1] did not consider the completion of this normed space because for the completion, the stated above result is false.

The completion of the space constructed by Arens and Eells was considered by Kadets [15], Pestov [37], Weaver [46], and Godefroy–Kalton [14] (see also [30]). Weaver [46, Definition 2.2.1] defined, what he named Arens–Eells space, as the completion of $(TP(M), ∥·∥_{TC})$. Kadets, Pestov, and Godefroy–Kalton defined an equivalent object in the dual way. Namely, they considered the space $Lip_0(M)$ of all Lipschitz functions on the space $M$ which vanish at a base point $O$. This is a Banach space with respect to the norm defined as the Lipschitz constant. Kadets, Pestov, and Godefroy–Kalton consider the closed subspace of $(Lip_0(M))^*$ spanned by the point evaluation functionals on $Lip_0(M)$. We denote the point evaluation functional corresponding to point $x$ by $δ(x)$ and exclude $δ(O)$ from consideration because it is a zero functional.

Observation 1.12. The norm of a finite linear combination $\sum_{x \in A} a_x δ(x)$ in the dual space $(Lip_0(M))^*$ is the same as the transportation cost of the transportation problem $\sum_{x \in A} a_x (1_x - 1_O)$.

This observation follows immediately from the fact that $Lip_0(M)$ is the dual of $(TP(M), ∥·∥_{TC})$. Thus, the spaces studied by Kadets, Pestov, and Godefroy–Kalton are all isometric to the completion of $(TP(M), ∥·∥_{TC})$. Kadets [15] denoted this space $\tilde{X}$ and did not give any name to it, Pestov
called it the *free Banach space*, and Godefroy–Kalton \[14\] called it the *Lipschitz-free Banach space*. Thus, all these names correspond to spaces which are canonically (in the sense of Observation 1.12) isometric to the space $TC(M)$.

Apparently, it is impossible to list all works corresponding to the directions (2) and (3). An ample bibliography, a wide range of contributors, and relevant discussions are presented by Villani \[45, pp. 106–111\]. We mention only the names of Kantorovich \[17,18\], Kantorovich–Gavurin \[21\], and Kantorovich–Rubinstein \[22,23\] for direction (3) and Vasershtein \[43\] (currently spelled as Wasserstein) for direction (2).

Kantorovich and Gavurin \[21\] introduced $TP(M)$, $\|\cdot\|_{TC}$, observed Theorem 1.1, and developed an approach to finding transportation plans of minimum cost.

A nice source for learning the basic definitions and results of directions corresponding to (2) and (3) above is \[44, Chapters 1 and 7\]. Another interesting source is \[20, §4 in Chapter VIII\] (see also English translation in \[19\]). Note that in \[44\] the discussion is mostly limited to probability measures; while in \[19,20\], the discussion is limited to compact metric spaces. As it is pointed out in \[44\], a passage from arbitrary finite positive Borel measures to probability measures can be achieved by normalization. As for compactness, all of the main results can be generalized to the setting of general complete separable metric space (see \[44\]), such spaces are also called *Polish spaces*. There are some obstacles which are to be overcome for some of more general spaces, see Remark 1.14.

Let us present basic notions of the theory developed in \[19,20,44\]. For a Polish space $(M,d)$, let $\mathcal{B}(M)$ denote the set of all finite positive Borel measures $\mu$ on $X$ satisfying

$$\int_M d(x,x_0) d\mu(x) < \infty,$$

for some (hence all) $x_0 \in M$. A *coupling* of a pair of finite positive Borel measures $(\mu, \nu)$ with the same total mass on $M$ is a Borel measure $\pi$ on $M \times M$ such that $\mu(A) = \pi(A \times M)$ and $\nu(A) = \pi(M \times A)$ for every Borel measurable $A \subset M$. The set of couplings of $(\mu, \nu)$ is denoted $\Pi(\mu, \nu)$. The quantity

$$T_1(\mu, \nu) := \inf_{\pi \in \Pi(\mu, \nu)} \left( \int_{M \times M} d(x,y) d\pi(x,y) \right),$$

is called the *minimal translocation work* between $\mu, \nu \in \mathcal{B}(X)$ in \[17\] and the *optimal transportation cost* in \[44, p. 3\]. Kantorovich and Rubinstein \[22,23\] (see also \[44, Section 7.1\]) proved that the set $\mathcal{M}$ of all differences $\mu - \nu$ of measures satisfying the conditions above forms a normed space if we endow it with the norm

$$\|\mu - \nu\|_{KR} = T_1(\mu, \nu),$$

and that the dual of the space $(\mathcal{M}, \|\cdot\|_{KR})$ is the space $\text{Lip}_0(M)$ with its usual norm. Observe that in the case where $\mu$ and $\nu$ are atomic measures
with finitely many atoms, the difference $\mu - \nu$ can be regarded as an element of $\text{TP}(M)$, and we have $\|\mu - \nu\|_{\text{KR}} = \|\mu - \nu\|_{\text{TC}}$.

The normed space $(\mathcal{M}, \| \cdot \|_{\text{KR}})$ is not complete if the space $M$ is not uniformly discrete. In fact, if there is a sequence of pairs $x_i, y_i \in M$ such that all elements of the set $\{x_i, y_i\}$ are distinct and $d(x_i, y_i) \leq 2^{-i}$, then the sequence of measures $\{\sum_{i=1}^{n} (\delta(x_i) - \delta(y_i))\}_{n=1}^{\infty}$, where $\delta(x)$ is the unit atomic measure supported on $\{x\}$, converges in the norm described above, but not to a difference of two finite measures. This example is well known, see [46, Proposition 2.3.2].

The relation between the normed spaces $(\mathcal{M}, \| \cdot \|_{\text{KR}})$ and the transportation cost spaces (as we define them in Sect. 1.1) is described in the following result (see Weaver [46, Section 2.3] or [47, Section 3.3] for the case where the metric space $M$ is compact), which shows that the completion of $(\mathcal{M}, \| \cdot \|_{\text{KR}})$ coincides with $\text{TC}(M)$. We believe that this result is known to experts. However, since a suitable reference has not been found, its proof is presented below.

**Theorem 1.13.** If $(M, d)$ is a Polish metric space, the space $(\text{TP}(M), \| \cdot \|_{\text{TC}})$ is dense in $(\mathcal{M}, \| \cdot \|_{\text{KR}})$. Hence $\text{TC}(M)$ can be regarded as the completion of $(\mathcal{M}, \| \cdot \|_{\text{KR}})$.

**Proof.** Fix the base point $O$. Since $\text{TP}(M) \subset \mathcal{M}$ and for both spaces the dual space is $\text{Lip}_0(M)$, it suffices to show that for each pair $\mu, \nu$ of finite Borel measures on $M$ satisfying (8) and having the same total masses, and for each $\varepsilon > 0$, there exists $f \in \text{TP}(M)$ such that, for every $1$-Lipschitz function $l$ on $M$ satisfying $l(O) = 0$, there holds:

$$|l(\mu - \nu) - l(f)| < \varepsilon.$$  \hfill (10)

Denote by $B(O, R)$ the closed ball in $M$ of radius $R$ centered at $O$. Using condition (8) for both $\mu$ and $\nu$, we conclude that there exists $R \in (0, \infty)$ satisfying

$$\left| \int_{M \setminus B(O, R)} l(v) d\mu(v) \right| \leq \int_{M \setminus B(O, R)} d(O,v) d\mu(v) < \frac{\varepsilon}{6}$$

and

$$\left| \int_{M \setminus B(O, R)} l(v) d\nu(v) \right| \leq \int_{M \setminus B(O, R)} d(O,v) d\nu(v) < \frac{\varepsilon}{6}.$$

By Ulam’s theorem (see [3, Theorem 1.4] and [12, Theorem 16.3.1]), there exists a compact set $K \subset B(O, R)$ such that

$$\mu(B(O, R) \setminus K) < \frac{\varepsilon}{6R} \quad \text{and} \quad \nu(B(O, R) \setminus K) < \frac{\varepsilon}{6R},$$

whence both

$$\left| \int_{B(O, R) \setminus K} l(v) d\mu(v) \right| < \frac{\varepsilon}{6}$$
and
\[ \left| \int_{B(O,R) \setminus K} l(v) d\nu(v) \right| < \frac{\varepsilon}{6}. \]

Next, we split \( K \) into a finite number \( \{K_n\}_{n \in T} \) of pairwise disjoint Borel subsets of diameter \( \varepsilon/(3(\mu(M) + \nu(M))) \) each. Define a function \( f \) on \( M \) as follows: In each of the sets \( K_n \), we pick a point \( t_n \) and let \( f(t_n) = \mu(K_n) - \nu(K_n) \) for \( n \in T \). If \( f \) is extended as 0 to the rest of \( M \), the obtained function is not necessarily in TP(\( M \)). To balance this, the 0-extension is modified at point \( O \) by taking \( f(O) = -\sum_{n \in T} f(t_n) \), thus implying \( f \in \text{TP}(M) \). If \( t_n \neq O \), the following inequality holds:

\[ \left| \int_{K_n} l(v) d(\mu - \nu)(v) - l(t_n) f(t_n) \right| = \left| \int_{K_n} (l(v) - l(t_n))(\mu - \nu)(v) \right| \leq \int_{K_n} |l(v) - l(t_n)| d\mu(v) + \int_{K_n} |l(v) - l(t_n)| d\nu(v) \leq \frac{\varepsilon}{3} \cdot \frac{\mu(K_n) + \nu(K_n)}{\mu(K) + \nu(K)}, \]

where, in the last inequality, the fact that \( l \) is 1-Lipschitz and the assumption on the diameter of \( K_n \) are used.

Since \( l(O) = 0 \), one has:

\[ |l(\mu - \nu) - l(f)| \leq \sum_{n \in T} \left| \int_{K_n} l(v)d(\mu - \nu)(v) - l(t_n) f(t_n) \right| + \left| \int_{B(O,R) \setminus K} l(v) d\mu(v) \right| + \left| \int_{B(O,R)} l(v) d\nu(v) \right| + \left| \int_{M \setminus B(O,R)} l(v) d\mu(v) \right| + \left| \int_{M \setminus B(O,R)} l(v) d\nu(v) \right| \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \varepsilon. \]

\( \square \)

**Remark 1.14.** We do not know how to prove an analogue of Theorem 1.13 in the case of general metric spaces. See [3, p. 234–235] in this connection. Nonseparability will not be an obstacle if we consider uniformly discrete spaces, because in such spaces finite Borel measures have countable support. The general metric case is beyond the subject of this work. For a systematic exposition of generalizations of results of Kantorovich and Rubinstein we refer the reader to [38].

Finally, it is worth mentioning that in Computer Science, the transportation cost is often called *Earth Mover’s Distance*. This name was introduced by Rubner–Tomasi–Guibas [39] in their work on computer vision. They knew the notion of transportation cost and their goal was to generalize it to the cases where total demand can be less than the total supply. This more general case is not considered in the present paper as such generalized transportation problems do not form a vector space.

Several authors published their opinions on the most suitable choice of the name for the spaces which we call *transportation cost spaces*. Vershik [41,42] provided an argument in favor of the name *Kantorovich space*. Villani [45, pp. 106–107] decided in favor of Wasserstein, and Weaver [47, p. 125] defended the name *Arens–Eells space*. It is interesting to mention that Villani
decided in favor of Wasserstein only because this term is more popular on the Internet than the others. In view of the information presented above, the argument of Vershik towards the Kantorovich space is the most convincing. However, we decided not to follow Vershik’s suggestion because the term Kantorovich space has already become a standard term for another object in Functional Analysis, see, for example, [24]. On the other hand, the relevance of the term transportation cost is mentioned even by the authors who are on the side of other terms, see [31, p. 762], [44, Introduction] (cost of transfer plan in [47, Section 3.3]).

2. On the Smallest Seminorm of the Type $\| \cdot \|_K$

Obviously, for every metric space $M$, the norm $\| \cdot \|_L$ is the largest seminorm of the type $\| \cdot \|_K$ for $K$ satisfying conditions A and B. In contrast, the class of finite metric spaces $M$ for which $\text{TP}(M)$ possesses the smallest seminorm of the type $\| \cdot \|_K$ for $K$ satisfying conditions A and B is rather narrow; and the goal of this section is to present its complete description. We start with a very simple existence result.

**Proposition 2.1.** Let $M$ be a finite subset of $\mathbb{R}$ with the induced metric. Then there exists the smallest seminorm of the form $\| \cdot \|_K$ on $\text{TP}(M)$.

**Proof.** In fact, let $M = \{x_1, \ldots, x_n\} \subset \mathbb{R}$ with $x_1 < x_2 < \cdots < x_n$. By the condition B, the set $K$ should contain a 1-Lipschitz function $l$ such that $|l(x_1) - l(x_n)| = |x_1 - x_n|$. It is clear that, for the same function $l$, we have $|l(x_i) - l(x_j)| = |x_i - x_j|$, and also that any two such functions can by obtained from each other by adding a constant and multiplying by $\pm 1$. Thus, any 1-element set $K$ containing such function leads to the minimal semi-norm of the described type. $\Box$

In the sequel, the following generalization of the notion of a linear triple ([4, p. 56]) will be used.

**Definition 2.2.** A collection $r = \{r_i\}_{i=1}^n$, $n \geq 3$, of points in a metric space $(M, d)$ is called a linear tuple if the sequence $\{d(r_i, r_1)\}_{i=1}^n$ is strictly increasing and if, for $1 \leq i < j < k \leq n$, the equality below holds:

$$d(r_i, r_k) = d(r_i, r_j) + d(r_j, r_k). \quad (11)$$

A linear triple is a linear tuple with $n = 3$.

Our next goal is to describe the condition which, for finite metric spaces $M$, is equivalent to existence of the smallest seminorm of the form $\| \cdot \|_K$ on $\text{TP}(M)$.

In the following, we assume that a metric space $M$ contains at least two points.

**Definition 2.3.** We say that a metric space $M$ satisfies the min-condition if it contains a finite set of pairs $\{u_i, v_i\}$ ($i \in \mathcal{I}$) having the following two properties:
I. For each $i \in I$, every point $x$ in $M$, different from $u_i$ and $v_i$, is such that $u_i, x, v_i$ is a linear triple.

II. For each pair $\{x, y\}$ of distinct points in $M$, there is $i \in I$ such that exactly one of the following four conditions holds:

1. The pairs $\{x, y\}$ and $\{u_i, v_i\}$ coincide.
2. Exactly one of the points $\{x, y\}$ coincides with one of the points $\{u_i, v_i\}$, and the remaining point, denote it $z$, is such that $u_i, z, v_i$ is a linear triple.
3. $u_i, x, y, v_i$ is a linear tuple.
4. $u_i, y, x, v_i$ is a linear tuple.

If pairs $\{u_i, v_i\}$ and $\{x, y\}$ in $M$ satisfy one of the conditions 1–4 in II we say that $\{x, y\}$ is on a geodesic between $u_i$ and $v_i$.

To exemplify this definition, notice that a finite subset $M$ of $\mathbb{R}$ satisfies the min-condition and the corresponding set of pairs consists of one pair $\{u_1, v_1\}$, where $u_1$ is the minimal element of $M$ and $v_1$ is the maximal element of $M$.

More interesting examples of metric spaces satisfying the min-condition are even cycles—in terminology of Graph theory—with their graph distances. More general examples are weighted even cycles provided that the weights are symmetric in the following sense: if we label vertices by $x_1, \ldots, x_{2n}$ in the cyclic order, the weight of the edge joining $x_k$ and $x_{k+1}$ is the same as the weight of the edge joining $x_{n+k}$ and $x_{n+k+1}$, where the addition is mod $(2n)$. The corresponding set of pairs is the set $\{x_i, x_{i+n}\}, i = 1, \ldots, n$.

The main result in this section is:

**Theorem 2.4.** A finite metric space $M$ satisfies the min-condition if and only if there exists the smallest seminorm of the form $\|\cdot\|_K$ on $\text{TP}(M)$ with $K$ satisfying conditions $A$ and $B$.

**Proof.** “Only if”: For each $i$, consider a 1-Lipschitz function $l_i$ such that $d(u_i, v_i) = |l_i(u_i) - l_i(v_i)|$. Condition I implies that the function $l_i$ is uniquely determined up to addition of a constant and multiplication by $-1$. Therefore, each set $K$ satisfying both $A$ and $B$ should contain at least one representative from each set $\mathbb{L}_i$ of functions obtained from $l_i$ by adding all possible real constants and multiplying by $\pm 1$. Let $\mathcal{R}$ be a collection of such representatives, we select one representative in each $\mathbb{L}_i$. Evidently, $\|\cdot\|_K \geq \|\cdot\|_R$. On the other hand, condition II implies that any such $\mathcal{R}$ satisfies condition B. Thus, the seminorm corresponding to the set $\mathcal{R}$ is the smallest seminorm of the desired type.

“If”: Let us construct a set which can be called a minimal set of pairs as follows. Starting with the set comprising all pairs of distinct points in $M$, we remove those pairs which are on geodesics between other pairs. This procedure results in a set of pairs $\{(u_i, v_i)\}$ satisfying II. If it satisfies I, then $M$ satisfies the min-condition.

It is clear that to complete the proof, it suffices to show that if the obtained set of pairs does not satisfy I, then for any set $K$ satisfying $A$ and
\( \mathbf{B} \) on \( M \), there is another set \( \tilde{K} \) satisfying \( \mathbf{A} \) and \( \mathbf{B} \) on \( M \) and a function \( f \in \text{TP}(M) \) such that \( \|f\|_{\tilde{K}} < \|f\|_{K} \).

It can be noticed that \( K \) is equivalent—in the sense that induces the same seminorm \( \| \cdot \|_K \)—to some set of 1-Lipschitz functions containing 1-Lipschitz functions \( l_i \) satisfying \( l_i(u_i) = 0 \), \( l_i(v_i) = d(u_i, v_i) \). It may be assumed without loss of generality that the pair \((u_1, v_1)\) does not satisfy \( \mathbf{I} \), implying that there is \( w \in M \) such that

\[
d(u_1, v_1) < d(u_1, w) + d(w, v_1). \tag{12}
\]

In addition, the next two inequalities hold because otherwise the pair \((u_1, v_1)\) should be deleted from the minimal collection of pairs:

\[
d(u_1, w) < d(u_1, v_1) + d(v_1, w) \tag{13}
\]

\[
d(v_1, w) < d(u_1, v_1) + d(u_1, w) \tag{14}
\]

The value of \( l_1(w) \) is in the interval \([d(u_1, v_1) - d(w, v_1), d(u_1, w)]\). On the other hand, inequality (12) implies that this interval does not reduce to one point, and consequently at least one of the following inequalities holds:

\[
l_1(w) < d(u_1, w), \quad l_1(w) > d(u_1, v_1) - d(w, v_1). \tag{15}
\]

Despite the asymmetry between the different conditions in (15) caused by different roles of \( u_1 \) and \( v_1 \) in the definition of \( l_1 \), one can check that the cases in (15) can be considered in a similar way. Thence, it suffices only to consider the case \( l_1(w) < d(u_1, w) \). In this case, consider the function \( f \in \text{TP}(M) \) given by

\[
f = \tau(1_{v_1} - 1_{u_1}) + (1_{v_1} - 1_w),
\]

where \( \tau > 0 \) will be selected later. Let \( \tilde{K} \) be given as the set of all functions of the form \( l_z := d(z, \cdot) \), where \( z \) is any of the endpoints of pairs \( \{(u_i, v_i)\} \) except \( v_1 \). The choice of \( \{(u_i, v_i)\} \) implies that \( \tilde{K} \) satisfies condition \( \mathbf{B} \). It is clear that \( \tilde{K} \) also satisfies condition \( \mathbf{A} \). It remains to show that there exists \( f \in \text{TP}(M) \) such that \( \|f\|_{K} > \|f\|_{\tilde{K}} \).

Observe that

\[
\|f\|_{K} \geq |l_1(f)| = |(\tau + 1)d(u_1, v_1) - l_1(w)|.
\]

We assume that \( \tau > 0 \) is large enough to ensure that

\[
|l_1(f)| = (\tau + 1)d(u_1, v_1) - l_1(w) > (\tau + 1)d(u_1, v_1) - d(u_1, w) > 0,
\]

whence \( l_1(f) > l_{u_1}(f) > 0 \). To complete the proof of \( \|f\|_{K} > \|f\|_{\tilde{K}} \), it has to be shown that for a suitably chosen \( \tau > 0 \), one has \( l_1(f) > |l_z(f)| \) for all \( z \) of the described type. Indeed,

\[
l_z(f) = \tau(d(z, v_1) - d(z, u_1)) + (d(z, v_1) - d(z, w)),
\]

and, therefore

\[
|l_z(f)| \leq \tau|d(z, v_1) - d(z, u_1)| + d(v_1, w).
\]

To achieve the desired goal, observe that

\[
|d(z, v_1) - d(z, u_1)| < d(u_1, v_1),
\]
for every \( z \neq u_1 \), because otherwise \( \{u_1, v_1\} \) would be on a geodesic of either between \( u_1 \) and \( z \), or between \( v_1 \) and \( z \). In any of the cases, we get a contradiction with the fact that the pair \( \{u_1, v_1\} \) belongs to the minimal set of pairs. Thus, for a sufficiently large \( \tau > 0 \), the inequality
\[
(\tau + 1)d(u_1, v_1) - d(u_1, w) > \tau|d(z, v_1) - d(z, u_1)| + d(v_1, w),
\]
holds for all \( z \neq u_1 \) belonging to the described set, and we are done. \( \square \)

**Corollary 2.5.** Let \( M \) be a finite metric space. If \( M \) does not satisfy the min-condition, then the infimum of all seminorms \( \| \cdot \|_K \) on \( TP(M) \) over \( K \) satisfying the conditions \( A \) and \( B \) is not a seminorm.

**Proof.** Assume the contrary. Let \( \| \cdot \|_{\inf} \) be the seminorm on \( TP(M) \), which is an infimum of all seminorms of the form \( \| \cdot \|_K \), with \( K \) satisfying conditions \( A \) and \( B \). Let \( X \) be the quotient of the seminormed space \( (TP(M), \| \cdot \|_{\inf}) \) by the kernel of \( \| \cdot \|_{\inf} \) and \( O \) denote the base point in \( M \). For each element \( v \in M \), denote by \( \tilde{v} \) the image of \( 1_v - 1_O \) in \( X \). It follows from the description of conditions \( A \) and \( B \)—see the paragraph below the description—that the map \( v \mapsto \tilde{v} \) is an isometry of \( M \) into \( X \). It is also clear that \( \tilde{O} = 0 \).

By Proposition 1.9, there exists a seminorm of the form \( \| \cdot \|_{\mathcal{N}} \) on \( TP(M) \) with \( \mathcal{N} \) satisfying \( A \) and \( B \) and such that
\[
\left\| \sum_i a_i(1_{v_i} - 1_O) \right\|_{\mathcal{N}} = \left\| \sum_i a_i\tilde{v}_i \right\|_X
\]
for any \( \sum_i a_i(1_{v_i} - 1_O) \in TP(M) \). On the other hand, by the construction
\[
\left\| \sum_i a_i\tilde{v}_i \right\|_X = \inf_K \left\| \sum_i a_i(1_{v_i} - 1_O) \right\|_K.
\]

Since \( M \) does not satisfy the min-condition, by Theorem 2.4, \( \| f \|_{\inf} \) is strictly less than \( \| f \|_{\mathcal{N}} \) for some functions \( f \in TP(M) \), which is a contradiction because any function in \( TP(M) \) can be written in the form \( \sum_i a_i(1_{v_i} - 1_O) \). \( \square \)

**Remark 2.6.** For small metric spaces, the result of Corollary 2.5 admits a simple direct proof. Consider, for example, an equilateral set \( M = \{a, b, c\} \) with all distances equal to 1. Let \( \mathcal{K}_1 = \{d(x, a), d(x, b)\} \) and \( \mathcal{K}_2 = \{d(x, b), d(x, c)\} \). It is clear that both sets satisfy the conditions \( A \) and \( B \). It is easy to check that
\[
\|21_a - 1_b - 1_c\|_{\mathcal{K}_2} = 1 \quad \text{and} \quad \|1_a + 1_b - 21_c\|_{\mathcal{K}_1} = 1.
\]

Therefore
\[
\|21_a - 1_b - 1_c\|_{\inf} \leq 1 \quad \text{and} \quad \|1_a + 1_b - 21_c\|_{\inf} \leq 1.
\]

On the other hand,
\[
\|(21_a - 1_b - 1_c) + (1_a + 1_b - 21_c)\|_{\inf} = \|3(1_a - 1_c)\|_{\inf}.
\]

If \( \| \cdot \|_{\inf} \) would be a seminorm, this would lead to a contradiction because \( \|3(1_a - 1_c)\|_{\mathcal{K}} = 3 \) for each \( \mathcal{K} \) satisfying \( A \) and \( B \).
3. On $\ell_1$-Subspaces in $\text{TP}_C(M)$

The next statement brings out one of the main outcomes of this paper.

**Theorem 3.1.** There exists an infinite uniformly discrete metric space $M$ such that $\text{TC}(M)$ does not contain an isometric copy of $\ell_1$.

**Proof.** Consider a metric space whose vertex set is $\mathbb{N}$, while its metric is quite different from the standard. It is a close-to-equilateral metric defined as follows. Let $h : \mathbb{N} \to (1, 2)$ be a strictly increasing function and the metric $d$ be given by

$$
d(i, j) = \begin{cases} h(\min\{i, j\}) & \text{if } i \neq j \\
0 & \text{if } i = j. \end{cases}
$$

(16)

It is clear that $d$ is a metric for any choice of $h$. This metric space is a generalization of the space suggested in [7, Remark 10, Example 2].

In this case, one can find a very handy description of the space $\text{TC}(M)$ and the norm on it, which turns out to be equivalent to the $\ell_1$-norm $\|f\|_1$, which is well defined since $f$ is a real-valued function on $\mathbb{N}$. In fact, let $f \in \text{TP}(M)$, then the amount of the available product is equal to $\|f\|_1/2$. Since each unit of product is to be moved to a distance which is between 1 and 2 (see (16)), we get that the cost of an optimal transportation plan is between $\|f\|_1/2$ and $\|f\|_1$. Observe also that $\text{TP}(M)$ contains all finitely supported sequences contained in the kernel of the functional $(1, \ldots, 1, \ldots) \in \ell_\infty$. Therefore, the space $\text{TC}(M)$ consists of all sequences of the intersection $\ell_1 \cap \ker(1, \ldots, 1, \ldots)$, and its norm satisfies $\|f\|_1/2 \leq \|f\|_{\text{TC}} \leq \|f\|_1$.

For the sequel, it will be convenient to introduce the notion of a (generalized) transportation plan for $f \in \text{TC}(M)$ as a representation of $f$ by means of a convergent series of the form:

$$
f = \sum_{i=1}^{\infty} a_i (1_{x_i} - 1_{y_i})
$$

(17)

with $a_i > 0$ and

$$
\sum_{i=1}^{\infty} a_i < \infty.
$$

(18)

We would like to emphasize that condition (18) is different from the condition in the standard description of the completion, which in our case can be described as the set of sums of all series of the form

$$
\sum_{k=1}^{\infty} \sum_{i=s_k+1}^{s_{k+1}} a_i (1_{x_i} - 1_{y_i})
$$

for some $0 = s_1 < s_2 < \cdots < s_k < \cdots$, $\{x_i\}$, $\{y_i\}$, and $\{a_i\}$ with

$$
\sum_{k=1}^{\infty} \left\| \sum_{i=s_k+1}^{s_{k+1}} a_i (1_{x_i} - 1_{y_i}) \right\|_{\text{TC}} < \infty.
$$

(19)
The reason for which we use (18) instead of the standard condition (19) is: the conditions are equivalent for the spaces which we consider. In fact, because all distances in $M$ are between 1 and 2, we have

$$\sum_{i=s_k+1}^{s_k+1} a_i \leq \left\| \sum_{i=s_k+1}^{s_k+1} a_i (1_{x_i} - 1_{y_i}) \right\|_{TC} \leq 2 \sum_{i=s_k+1}^{s_k+1} a_i,$$

in the case where the transportation plan given by $\sum_{i=s_k+1}^{s_k+1} a_i (1_{x_i} - 1_{y_i})$ is optimal.

The cost of the transportation plan (17) is defined as $\sum_{i=1}^{\infty} a_i d(x_i, y_i)$. Observe that since $d(x_i, y_i) \leq 2$ for all $x_i$ and $y_i$, this cost is always finite if $\sum_{i=1}^{\infty} a_i < \infty$. The definition of a completion also implies that $\|f\|_{TC}$ is the infimum of costs of generalized transportation plans for $f$ for every $f \in TC(M)$.

It turns out that in this metric space $M$, for each $f \in TC(M)$, there exists a minimum-cost generalized transportation plan, which can be described in the following way. Denote by $f_i$ the value of $f \in TC(M)$ at $i$. Let $m \in \mathbb{N}$ be such that $\sum_{i=1}^{m} |f_i| \geq \|f\|_{1}/2$ and $\sum_{i=1}^{m-1} |f_i| < \|f\|_{1}/2$. We call $m$ the median of the support of $f$. We represent $f$ as a sum of “beginning” and “end”, namely

$$b = \sum_{i=1}^{m-1} f_i 1_i + \text{sign} f_m \left( \frac{\|f\|_{1}}{2} - \sum_{i=1}^{m-1} |f_i| \right) 1_m,$$

and

$$c = \sum_{i=m+1}^{\infty} f_i 1_i + \text{sign} f_m \left( \frac{\|f\|_{1}}{2} - \sum_{i=m+1}^{\infty} |f_i| \right) 1_m.$$

Observe that

$$b + e = f \quad \text{and} \quad \|b\|_1 = \|c\|_1 = \|f\|_{1}/2. \quad (20)$$

Lemma 3.2. A generalized transportation plan $f = \sum_{i=1}^{\infty} a_i (1_{x_i} - 1_{y_i})$ with $a_i > 0$ is optimal if the following conditions are satisfied:

1. All $x_i, y_i$ are in the support of $f$.
2. The signs of the function $a_i (1_{x_i} - 1_{y_i})$ at $x_i$ and $y_i$ are the same as the signs of $f$ restricted to $x_i$ and $y_i$.
3. Out of each pair $x_i, y_i$, one point is in the support of $b$ and the other is in the support of $e$.

Proof. It can be readily seen that (20) implies the existence of such plans. Also, it is easy to see that the cost of each of them equals:

$$\sum_{i=1}^{m-1} |f_i| h(i) + \left( \|f\|_{1}/2 - \sum_{i=1}^{m-1} |f_i| \right) h(m). \quad (21)$$

It remains to show that the cost of a generalized transportation plan cannot be less than (21). We prove this in three steps labeled as (i)–(iii) according to items (1)–(3) above, respectively.
(i) We show that if a generalized transportation plan

\[ \sum_{i=1}^{\infty} \alpha_i (1_{x_i} - 1_{y_i}), \]  

(22)

with \( \alpha_i > 0 \) does not satisfy (1), it can be modified in such a way that its cost decreases and ultimately one obtains a plan satisfying (1).

In fact, if \( t \not\in \text{supp} f \), then the series of coefficients of \( 1_t \) in (22) adds to 0, so the part of (22) containing \( 1_t \) can be written as

\[ \sum_{k=1}^{\infty} \alpha_{n_k} (1_t - 1_{y_{n_k}}) + \sum_{j=1}^{\infty} \alpha_{m_j} (1_{x_{m_j}} - 1_t), \]  

(23)

where \( \{n_k\}_{k=1}^{\infty} \) and \( \{m_j\}_{j=1}^{\infty} \) are two disjoint—possibly finite—subsets in \( \mathbb{N} \) and \( \sum_{k=1}^{\infty} \alpha_{n_k} = \sum_{j=1}^{\infty} \alpha_{m_j} = \alpha \).

It can be noticed that the sum (23) admits the representation:

\[ \sum_{j,k=1}^{\infty} \nu_{j,k} (1_{x_{m_j}} - 1_{y_{n_k}}), \]  

(24)

where \( \nu_{j,k} \geq 0 \) and \( \sum_{j,k=1}^{\infty} \nu_{j,k} = \alpha \). From here, one derives that

\[ \sum_{k=1}^{\infty} \alpha_{n_k} d(t, y_{n_k}) + \sum_{j=1}^{\infty} \alpha_{m_j} d(x_{m_j}, t) > \sum_{j,k=1}^{\infty} \nu_{j,k} d(x_{m_j}, y_{n_k}). \]

This is because all distances are in the open interval (1, 2).

The procedure may be repeated for all points \( t \) violating (1) and in the limit we get a generalized transportation plan satisfying (1) whose cost does not exceed the cost of the original plan. Therefore, it may be assumed that (22) satisfies condition (1) of Lemma 3.2.

(ii) Suppose that the plan (22) does not satisfy (2). Let \( q \) be a point at which the condition is not satisfied, implying that \( f_q \neq 0 \) and that \( 1_q \) is present in three nonzero sums:

\[ \sum_{k=1}^{\infty} \alpha_{n_k} (1_q - 1_{y_{n_k}}) + \sum_{j=1}^{\infty} \alpha_{m_j} (1_{x_{m_j}} - 1_q) + \sum_{l=1}^{\infty} \alpha_{s_l} \text{sign} f_q (1_q - 1_{z_{s_l}}), \]  

(25)

where \( \{n_k\} \) and \( \{m_j\} \) are disjoint, \( \sum_{k=1}^{\infty} \alpha_{n_k} = \sum_{j=1}^{\infty} \alpha_{m_j}, \sum_{l=1}^{\infty} \alpha_{s_l} = |f_q|, \)

\( z_{s_l} = y_{s_l} \) if \( \text{sign} f_q = 1 \), while \( z_{s_l} = x_{s_l} \) if \( \text{sign} f_q = -1 \).

Now we modify the sum \( \sum_{k=1}^{\infty} \alpha_{n_k} (1_q - 1_{y_{n_k}}) + \sum_{j=1}^{\infty} \alpha_{m_j} (1_{x_{m_j}} - 1_q) \) in the transportation plan exactly in the same way as in (i), and get a cheaper plan, where the condition (2) is satisfied for \( q \), and no violators of conditions (1) or (2) are added.

Repeating the procedure for all points \( q \) violating (2), in the limit, we reach a generalized transportation plan satisfying (2) and (1), whose cost does not exceed the cost of the original plan. Hence, one may assume that (22) satisfies (1) and (2) of Lemma 3.2.

(iii) Assume the contrary to (3). Let \( \sum_{i \in A} \alpha_i (1_{x_i} - 1_{y_i}) \) be the sum of all terms of the generalized transportation plan in which both \( x_i \) and \( y_i \)
are $\leq m$ and $\sum_{i \in B} \alpha_i (1_{x_i} - 1_{y_i})$ be the sum of all terms of the generalized transportation plan in which both $x_i$ and $y_i$ are $\geq m$.

Let us show that conditions (20) imply that

$$\sum_{i \in A} \alpha_i = \sum_{i \in B} \alpha_i. \quad (26)$$

In fact, the sum $\sum_{i \in \mathbb{N} \setminus (A \cup B)} \alpha_i (1_{x_i} - 1_{y_i})$ contributes equally to the $\ell_1$-norm of both $b$ and $e$. Therefore, by (20), the remaining contributions should be also equal yielding (26) due to the fact that the transportation plan satisfies (1) and (2).

Condition (26) implies that we can redesign the generalized transportation plan (22) in such a way that the product is moved from $x_i$, $i \in A$, to $y_i$, $i \in B$, and from $x_i$, $i \in B$ to $y_i$, $i \in A$. As a result we get a cheaper generalized transportation plan satisfying conditions (1)–(3) of Lemma 3.2. \hfill \square

Now, suppose that there is a subspace of $\text{TC}(M)$ isometric to $\ell_1$, and let vectors $\{x_n\}_{n=1}^{\infty} \subset \text{TC}(M)$ be isometrically equivalent to the unit vector basis of $\ell_1$.

Lemma 3.3. The vectors $\{x_n\}$ have disjoint supports.

Proof. Suppose that $i$ is in the support of both $x_n$ and $x_p$. We may assume that the signs of $x_n(i)$ and $x_p(i)$ are different, changing $x_n$ to $-x_n$, if needed. Assume $x_n(i) < 0$ and $x_p(i) > 0$.

It suffices to show that $\|x_n + x_p\|_{\text{TC}} < \|x_n\|_{\text{TC}} + \|x_p\|_{\text{TC}}$, as it leads to a contradiction.

To achieve this, it is enough to establish that the sum of minimum-cost generalized transportation plans for $x_n$ and $x_p$ is not a minimum-cost transportation plan for $x_n + x_p$ since it can be improved. This can be seen as follows: since $x_n(i) < 0$ and $x_p(i) > 0$, in the sum of minimum-cost generalized transportation plans for $x_n$ and $x_p$, we deliver $-x_n(i)$ units to $i$ and move $x_p(i)$ units from $i$. It is clear that we can move $\min \{-x_n(i), x_p(i)\}$ units directly, and since all triangle inequalities in $M$ are strict, this will decrease the cost of the plan. \hfill \square

Let $m_1$ be the median of the support of $x_1 = \{x_{1,i}\}_{i=1}^{\infty}$. For $j \in \mathbb{N}$, denote by $[1,j]$ the interval $\{1, \ldots, j\}$ of integers. Observe that it is impossible that $\text{supp} x_1 \subset [1,m_1]$. Indeed, this would imply that $|x_{1,m_1}| > \|x_1\|_1/2$, which cannot happen for a function with zero sum. Hence, there are elements in $\text{supp} x_1$ which are larger than $m_1$. Let $k$ be the least such element.

Since $\{x_n\}$ are disjointly supported, there exists $x_p$ such that all elements of the support of $x_p$ are larger than $k$. We assert that in this case $\|x_1 + x_p\|_{\text{TC}} < \|x_1\|_{\text{TC}} + \|x_p\|_{\text{TC}}$, getting a contradiction with the assumption that $\{x_n\}$ is isometrically equivalent to the unit vector basis of $\ell_1$. Denote by $m_p$ the median of the support of $x_p$ and by $m_+$ the median of the support of $x_1 + x_p$.

Let us analyze the relations between the optimal transportation plans for $x_1$, $x_p$, and $x_1 + x_p$. Since $x_1$ and $x_p$ are disjointly supported, in the
optimal transportation plan for $x_1 + x_p$ we have to move $\|x_1\|_{1/2} + \|x_p\|_{1/2}$ units of product.

Those $\|x_1\|_{1/2}$ units of product in $x_1$, which were located/needed in the lower half of support of $x_1$, both in the plan for $x_1$ and in the plan for $x_1 + x_p$, will be moved to/from locations corresponding to larger elements of $\mathbb{N}$, more precisely, to some locations corresponding to the upper half of support of $x_1$ and some locations corresponding to the upper half of support of $x_1 + x_p$, respectively. Because the distance $d(i, j)$, $i \neq j$, depends only on $\min\{i, j\}$, the cost of these relocations in both cases will be $\|x_1\|_{TC}$.

After that, in the optimal transportation plan for $x_1 + x_p$ we need to pick the “next” $\|x_p\|_{1/2}$ units of product of $x_1 + x_p$ located between $m_1$ and $m_1^+$ (possibly inclusive) and move them from/to for distances $h(i)$ corresponding to their locations.

Observe that we do almost the same in the optimal transportation plan for $x_p$, but there, of course, we pick only units corresponding to $x_p$.

Since both $m_1$ and $k$ are less than any element of $\text{supp}x_p$, in the first case some of the locations corresponding to these $\|x_p\|_{1/2}$ units of product of $x_1 + x_p$ will be strictly smaller than the locations for lower $\|x_p\|_{1/2}$ units of $x_p$.

Since $h(i)$ is a strictly increasing function, this implies that the cost of relocation of these $\|x_p\|_{1/2}$ units in the optimal plan for $x_1 + x_p$ is strictly smaller than $\|x_p\|_{TC}$.

4. On the Kernel of the Seminorm $\| \cdot \|_{\mathcal{DP}}$

In [29], the seminorm $\| \cdot \|_{\mathcal{DP}}$ is called the double-point norm. However, it appears that, for some metric spaces $M$, the seminorm $\| \cdot \|_{\mathcal{DP}}$ is not a norm.

Observation 4.1. The seminorm $\| \cdot \|_{\mathcal{DP}}$ is not a norm if and only if there exists a nonzero function $f \in \text{TP}(M)$ such that

$$\sum_{x \in M} f(x)d(v, x) \text{ does not depend on } v. \quad (27)$$

Proof. In fact, $\|f\|_{\mathcal{DP}}$ is the supremum over $u$ and $v$ of

$$\left| \sum_{x \in M} \frac{d(v, x) - d(u, x)}{2} f(x) \right| = \frac{1}{2} \left| \sum_{x \in M} f(x)d(v, x) - \sum_{x \in M} f(x)d(u, x) \right|. \quad \Box$$

Examples of such metric spaces $M$ are provided below.

Example 4.2. Let $M$ be a 4-cycle and

$$f = 1_{x_1} - 1_{x_2} + 1_{x_3} - 1_{x_4}, \quad (28)$$

where $x_1, x_2, x_3, x_4$ are vertices of the cycle in the cyclic order. Then, condition (27) is satisfied and hence $\|f\|_{\mathcal{DP}} = 0$ and $\| \cdot \|_{\mathcal{DP}}$ is not a norm.
Example 4.3. The preceding example can be generalized to a sufficient condition for existence of $f \neq 0$ with $\|f\|_{D^P} = 0$. The condition is the existence in $M$ of a 4-tuple $x_1, x_2, x_3, x_4$, such that
\begin{align*}
    d(x_1, x_2) &= d(x_2, x_3) = d(x_3, x_4) = d(x_4, x_1), \\
    d(x_1, x_3) &= d(x_2, x_4) = 2d(x_1, x_2),
\end{align*}
and, in addition, for each $x \in M$, we have:
\begin{align*}
    d(x, x_1) + d(x, x_3) &= d(x, x_2) + d(x, x_4).
\end{align*}
In this case, function (28) also satisfies $\|f\|_{D^P} = 0$.

Example 4.4. Another class of metric spaces for which $\|\cdot\|_{D^P}$ is not norm can be constructed in the following way.

Given $m \in \mathbb{N}$, we construct a metric space $M$ of cardinality $2^m$ as a union of two disjoint sets, $A$ and $B$ satisfying $|A| = |B| = m$. The metric on $M$ is defined as
\begin{align*}
    d(x, y) = \begin{cases}
        0 & \text{if } x = y \\
        a & \text{if } x, y \in A, x \neq y \\
        a & \text{if } x, y \in B, x \neq y \\
        c & \text{if } x \in A, y \in B.
    \end{cases}
\end{align*}
For $d(x, y)$ to be a metric it is necessary and sufficient that
\begin{equation}
    0 < a \leq 2c. \tag{29}
\end{equation}
Let $f = 1_A - 1_B \in \text{TP}(M)$, where for a subset $U \subseteq M$, its indicator is:
\begin{align*}
    1_U(x) = \begin{cases}
        1 & \text{if } x \in U \\
        0 & \text{if } x \notin U.
    \end{cases}
\end{align*}
Then, with a suitable choice $m, a$, and $c$, for every $v \in M$,
\begin{equation}
    \sum_{x \in M} f(x)d(v, x) = 0, \tag{30}
\end{equation}
and the conclusion follows from Observation 4.1. For $v \in A$ or $v \in B$, equation (30) becomes
\begin{equation}
    (m - 1)a - mc = 0. \tag{31}
\end{equation}
Meanwhile, Eq. (31) can be written as
\begin{equation}
    a = \left(\frac{m}{m - 1}\right)c. \tag{32}
\end{equation}
Consequently, if $c > 0$ is arbitrary, $m$ is any integer satisfying $m \geq 2$, and $a$ is given by (32), condition (29) is satisfied. Therefore, with such selection of the parameters, (30) holds.
5. The Spaces \( TPDP(T) \) when \( T \) is a Finite Tree

It is well known that the space \( TP_L(T) \) for a finite tree \( T \) is isometric to \( \ell^d_1 \) of the corresponding dimension, see [13] and [9, Proposition 2.1]. In this section we show that \( TPDP(T) \) is quite different, and that it changes in the “undesirable” direction, as we are mostly interested, see Sect. 1.4, in moving “away from \( \ell^\infty_n \)”.

**Proposition 5.1.** If \( T \) is a finite tree, possibly weighted, then

\[
d_{BM}(TPDP(T), \ell^{\lvert E(T) \rvert}_\infty) \leq 4.
\]

**Proof.** Assume that \( T \) is a rooted tree and, for each its edge \( e \), consider the function \( f_e := 1_w - 1_z \in TP(T) \), where \( w \) and \( z \) are the ends of \( e \), and \( w \) is the one closer to the root. Then, every \( f \in TP(M) \) can be written in the form:

\[
f = \sum_{e \in E(T)} a_e f_e. \tag{34}
\]

The norm on \( TPDP(T) \) can be calculated in the following way: Consider a path \( P \) in \( T \) with ends \( u \) and \( v \). Suppose \( P \) is directed in such a way that first \( P \) goes towards the root, denote this first part by \( P_1 \), and then goes away from the root, denote the second part by \( P_2 \). Set:

\[
P(f) = \sum_{e \in P_1} d_e a_e - \sum_{e \in P_2} d_e a_e,
\]

where \( d_e \) is the weight (length) of the edge \( e \). Then

\[
\|f\|_{DP} = \max_{P} |P(f)|. \tag{35}
\]

To justify (35), observe that

\[
\|f\|_{DP} = \max_{u,v \in M} \left| \sum_{x \in M} \frac{d(v,x) - d(u,x)}{2} f(x) \right|.
\]

Let

\[
g(x) = \frac{d(v,x) - d(u,x)}{2},
\]

and let \( w \) and \( z \) be the ends of an edge \( e \). Then,

\[
g(w) - g(z) = \begin{cases} 0 & \text{if } e \text{ is not in } P, \\ d_e & \text{if } e \text{ is in } P \text{ and } w \text{ is closer to } u. \end{cases}
\]

Combining this formula with (34), one derives (35).

By (35), the norm \( \|f\|_{DP} \), up to a factor of 2, is equivalent to

\[
\max\{|P(f)| : P \text{ is a descending path in } T\},
\]

and up to a factor of 4, the norm \( \|f\|_{DP} \) is equivalent to the next one:

\[
\max\{|P(f)| : P \text{ is a path in } T \text{ with root being one of its ends}\}.
\]

Let us assign to each edge \( g \) the real number \( s_g \) defined as the sum of numbers \( d_e a_e \) over all edges connecting \( g \) with the root, including \( g \). Clearly,
this defines a bijective linear map $D$ from $TP(T)$ to the space of real-valued functions on the edge set $E(T)$. The discussion above implies that
\[
\frac{1}{4} \|f\|_{DP} \leq \|Df\|_{\ell_\infty(E(T))} \leq \|f\|_{DP},
\]
and in this way proves (33).

\[\square\]

6. Locally Finite Representing Subsets in Banach Spaces

To begin with, let us recollect the following definition given in [10].

Definition 6.1. A subset $K$ of a separable Banach space $X$ is said to be a representing subset of $X$ if every Banach space containing an isometric copy of $K$ contains an isometric copy of $X$.

In this connection, the following problem arises.

Problem 6.2. Characterize Banach spaces for which there exist locally finite representing sets.

This problem is motivated by the applications which we mention at the end of this section. Below, we solve this problem for $\ell_1$ by proving the following analogue of [10, Proposition 4.3].

Proposition 6.3. There exist locally finite metric spaces representing $\ell_1$.

Proof. Let $\{e_i\}_{i=1}^\infty$ be the unit vector basis of $\ell_1$. Let $M$ be the subset of $\ell_1$ consisting of all vectors of the form $\sum_{i \in A} 2^i e_i$, where $A$ is a finite subset of $\mathbb{N}$; we assume that $\sum_{i \in \emptyset} 2^i e_i = 0$. We endow this subset with the $\ell_1$-metric and consider it as a metric space. Obviously, this is a locally finite metric space.

Suppose that $T$ is an isometric embedding of the metric space $M$ into a Banach space $X$. Without loss of generality assume that $T(0) = 0$. Let $f_i = 2^{-i} T(2^i e_i)$. It is easy to see that these vectors should have norm 1. Our goal is to show that they are isometrically equivalent to the unit vector basis of $\ell_1$. To achieve this goal it suffices to prove that, for each finite collection $\Theta = \{\theta_i\}_{i=1}^n$ with $\theta_i = \pm 1$, there exists a normalized linear functional $F_{\Theta} \in X^*$ such that $F_{\Theta}(f_i) = \theta_i$ for $i = 1, \ldots, n$. Let $A_+ = \{i \in \{1, \ldots, n\} : \theta_i = 1\}$ and $A_- = \{i \in \{1, \ldots, n\} : \theta_i = -1\}$. Then $x_+ = \sum_{i \in A_+} 2^i e_i$ and $x_- = \sum_{i \in A_-} 2^i e_i$ are in $M$, whence we have:

\[
\|x_+\|_1 = \|x_+\|_X = \sum_{i \in A_+} 2^i, \quad \|x_-\|_1 = \|x_-\|_X = \sum_{i \in A_-} 2^i, \quad (36)
\]
and

\[
\|x_+ - x_-\|_1 = \|T(x_+) - T(x_-)\|_X = \sum_{i=1}^n 2^i.
\]

Thus, there exists $F \in X^*$, $\|F\| = 1$, such that $F(T(x_+)) - F(T(x_-)) = \sum_{i=1}^n 2^i$. By (36), this implies $F(T(x_+)) = \sum_{i \in A_+} 2^i$ and $F(T(x_-)) = -\sum_{i \in A_-} 2^i$. 

Next, let us verify that

\[ F(f_j) = \begin{cases} 
1 & \text{if } j \in A_+ \\
-1 & \text{if } j \in A_- 
\end{cases} \]

and, therefore, \( F \) is the desired functional \( F_\Theta \).

Consider \( j \in A_+ \) (the case where \( j \in A_- \) is similar). Observe that \( 2^j e_j \) is on a geodesic joining 0 and \( x_+ \) in the sense that

\[ \|2^j e_j\|_1 + \|x_+ - 2^j e_j\|_1 = \|x_+\|_1. \]

Since \( T \) is an isometry and \( T(0) = 0 \), we get that

\[ \|T(2^j e_j)\|_X + \|T(x_+) - T(2^j e_j)\|_X = \|T(x_+)\|_X. \]

Thus, \( F(T(2^j e_j)) = 2^j \) and \( F(f_j) = 1. \) \qed

**Corollary 6.4.** If a Banach space \( X \) contains \( \ell_1^n \) isometrically for each \( n \in \mathbb{N} \), but does not contain \( \ell_1 \) isometrically, then there exists a locally finite metric space \( M \) such that \( X \) contains isometrically each finite subset of \( M \), but does not contain an isometric copy of \( M \).

**Proof.** Let \( M \) be the locally finite subset of \( \ell_1 \) constructed in the proof of Proposition 6.3. It is clear that each finite subset of \( M \) is isometric to a subset of \( \ell_1^n \) for sufficiently large \( n \). Thus, the Banach space \( X \) contains isometrically every finite subset of \( M \). On the other hand, by Proposition 6.3, \( X \) does not contain an isometric copy of \( M \). \qed

**Examples** of spaces satisfying the conditions of Corollary 6.4: \( c_0 \), \( c(\alpha) \), where \( \alpha \) is a countable ordinal, direct sums \((\oplus_{n=1}^\infty \ell_1^n)_p, (\oplus_{n=1}^\infty \ell_\infty^n)_p \) for \( 1 < p < \infty \). The necessary definitions can be found in [25].

The spaces \( c(\alpha) \), where \( \alpha \) is a countable ordinal, and \((\oplus_{n=1}^\infty \ell_1^n)_p \) for \( 1 < p < \infty \), are new examples of Banach spaces, for which there exists a locally finite metric space \( M \) such that \( X \) contains isometrically each finite subset of \( M \), but does not contain \( M \) isometrically. Previously known examples are available in [16, Theorem 2.9], [32,33].

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