1. Abstract

In the framework of the study of the $Sp(n)$-orbits in the real Grassmannian $G^k(4n)$ of $k$-dimensional non oriented subspaces of a real $4n$-dimensional vector space $V$, here we consider the case of the isoclinic subspaces whose set we indicate with $IC$. Endowed $V$ with an Hermitian quaternionic structure $(Q,\langle,\rangle)$, a subspace $U$ is isoclinic if for any compatible complex structure $A \in Q$ the principal angles of the pair $(U, AU)$ are all the same, say $\theta^A$. We will show that, fixed an admissible hypercomplex basis $(I, J, K)$, to any such subspace $U$ we can associate two set of invariants, namely a triple $(\xi, \chi, \eta)$ and a pair $(\Gamma, \Delta)$ where $\Gamma$ itself is a function of $(\xi, \chi, \eta)$. We prove that the angles of isoclinicity $(\theta^I, \theta^J, \theta^K)$ together with $(\xi, \chi, \eta, \Delta)$ determine its $Sp(n)$-orbit. In particular if $\dim U = 8k + 2$ or $\dim U = 8k + 6$ with $k \geq 0$ the last set reduce to the pair $(\xi = \pm 1, \chi = \pm 1)$.

2. Summary

In [18] we determined the set of invariants characterizing the $Sp(n)$-orbits in the Grassmannian $G^k(4n)$ of $k$-dimensional non oriented real subspaces of a $4n$-dimensional real vector space $V$. There we endowed $V$ with a quaternionic structure $Q$, an Hermitian metric $\langle,\rangle$ and denoted by $S(Q)$ the sphere of compatible complex structures of $Q$. Given a $k$-dimensional subspace $U \subseteq V$, for any $A \in S(Q)$ we denoted by $\omega^A = \langle X, AY \rangle$, $X, Y \in U$ the skew-symmetric $A$-Kähler form restricted to $U$. Such form assumes a standard form represented w.r.t. some orthonormal basis $\{X_i\}$ by the skew-symmetric matrix

$$(\omega^A_{ij}) = \langle X_i, AX_j \rangle \equiv \begin{cases} \geq 0 & \text{if } i \text{ is odd and } j = i + 1, \\ 0 & \text{otherwise}, \end{cases}$$

for $i \leq j \leq k$. We call $\{X_i\}$ a standard basis of $\omega^A|U$ and, for $i$ odd, $i < k$, the linear span $L(X_i, X_{i+1})$ a standard 2-plane. By the skew symmetry of $\omega^A$, the standard bases are never unique (even if $\dim U = 1$ they are defined up to sign). Moreover, by the assumption that the quantities $\omega^A_{ij}$ are non negative, for $i$ odd and $j = i + 1 \leq k$ one has that the entries $\omega^A_{ij}$ are the cosines of the principal angles $\theta_{ij} \in [0, \pi/2]$ of the pair of 2-planes $\langle L(X_i, X_{i+1}), AL(X_i, X_{i+1}) \rangle$. Moreover we call $\omega^A$-standard subspaces the uniquely determined subspaces $U_i \subseteq U$ associated to the same principal angle.

Chosen an admissible (hypercomplex) basis $(I, J, K)$ of $Q$, we denoted by $B(U)$ the set of triples of standard bases of the skew-symmetric forms $\omega^A_{IJ}|U, \omega^A_{IJ}|U, \omega^A_{IK}|U$. Necessary and sufficient conditions for a pair of subspaces $U, W$ to belong to the same $Sp(n)$-orbit are stated in the

**Theorem 2.1.** [18] Let $(I, J, K)$ be an admissible basis of $Q$. The non oriented subspaces $U^m$ and $W^m$ of $V$ are in the same $Sp(n)$-orbit iff

1. they share the same $I, J, K$ principal angles i.e.

$$\theta^I(U) = \theta^I(W), \quad \theta^J(U) = \theta^J(W), \quad \theta^K(U) = \theta^K(W)$$

for one and hence any admissible basis $(I, J, K)$ or equivalently the singular values of the projectors $Pr^{IU} : U \to IU$, $Pr^{JI} : JU \to IU$, $Pr^{KU} : KU \to IU$ of $Pr^{IW} : W \to IW$, $Pr^{JIW} : JU \to IW$, $Pr^{KIW} : KU \to IW$ for one and hence any admissible basis $(I, J, K)$.

2. there exist three (orthonormal) standard bases $\{(X_i), \{Y_i\}, \{Z_i\}\} \in B(U)$ and $\{(X'_i), \{Y'_i\}, \{Z'_i\}\} \in B(W)$ whose relative position is the same or equivalently

$$C_{IJ} = C'_{IJ}, \quad C_{IK} = C'_{IK},$$

where $C_{IJ} = \langle X_i, Y_j \rangle$, $C'_{IJ} = \langle X'_i, Y'_j \rangle$, $C_{IK} = \langle X_i, Z_j \rangle$, $C'_{IK} = \langle X'_i, Z'_j \rangle$. 

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The determination of the principal angles between a pair of subspaces $S, T$ is a well-known problem solved by the singular value decomposition of the orthogonal projector of $S$ onto $T$. The problem to determine the $Sp(n)$-orbits in $G^\mathbb{R}(k, 4n)$ of $k$-dimensional real subspaces for $k = 1, \ldots, 4n$, turns then into the one of determining the existence of a triple of bases belonging to $B(S)$ and $B(T)$, satisfying the condition 2) of the Theorem (2.1). This is not an easy problem to solve because, as already said, for any $A \in S(Q)$, the standard bases of $A^\mathbb{R}$ are never unique. An iterative procedure to determine a canonical form of the matrices $C_{IJ}, C_{IK}$ of a subspace $U$ (that we denoted canonical matrices) is given in [18].

The idea is to refer $C_{IJ}, C_{IK}$ to some canonical bases $\{(X_1), (Y_1), (Z_1)\} \in B(U)$ built up through the determination of the mutual principal angles and associated principal vectors of the uniquely defined standard subspaces $U_i^i, U_j^j, U_k^k$ of the forms $\omega_i|U, \omega_j^j|U, \omega_k^k|U$ respectively.

In [18] we showed in fact that a necessary condition for a pair of subspaces $U, W$ to belong to the same $Sp(n)$-orbit is that the principal angles between the $\omega_i^j, \omega_j^j, \omega_k^k$ standard subspaces $(U_i^i, U_j^j)$ as well as $(U_i^i, U_k^k)$ and $(U_j^j, U_k^k)$ are the same, where the indices $i, j, k$ range through all standard subspaces.

But such procedure requires that at each step there exists at least one pair of standard subspaces (whose size reduce at the end of the current iteration) with one or more principal angles of multiplicity equal to one. In most of the cases this procedure leads to the construction of the canonical matrices associated to $U$ and consequently to the full set of invariants (for a fixed admissible basis) characterizing its $Sp(n)$-orbit. But there are some degenerate cases (see [18]) where the aforementioned procedure can not be applied tout court. Among these, there are the isoclinic subspaces $i.e.$ subspaces $U$ characterized by the fact that, for all $A \in S(Q)$, all principal angles between $U$ and $AU$ equal $\theta^U$ which is the angle of isoclinicity. They do not exhaust all the degenerate cases (see [18]) but surely they have an important role and this article deals with them.

The purpose of this article is to determine the $Sp(n)$-orbit in the real Grassmannian $G^\mathbb{R}(k, 4n)$ of $k$-dimensional isoclinic subspaces and we will end up with a theorem giving the full set of invariants characterizing their orbits. We denote by $IC^{2m}$ the set of $2m$-dimensional isoclinic subspaces. We consider only even dimensions since the set $IC^{2m+1}$ is formed only by the real Hermitian product (r.h.p.) subspaces $U$ (see [19]) characterized by the fact that $U \perp AU, \forall A \in S(Q)$. From Theorem (2.1) one has that all and only such isoclinic subspaces share the same orbit in $G^\mathbb{R}(2m, 1, 4n)$.

First we prove that, fixed an admissible basis $(I, J, K)$, for any unitary $X_1 \in U$, and being $X_2, Y_2, Z_2$ the triple of unitary vectors orthogonal to $X_1$ such that the $2$-planes $L(X_1, X_2), L(X_1, Y_2), L(X_1, Z_2)$ are standard $2$-planes of $\omega_i^j|U, \omega_j^j|U, \omega_k^k|U$ respectively, the cosines $\xi = <X_2, Y_2>, \chi = <X_2, Z_2>, \eta = <Y_2, Z_2>$ do not depend on $X_1$.

Furthermore, we prove that, fixed an admissible basis $(I, J, K)$, any $U \in IC^{2m}, m > 1$ is characterized by a pair of invariants $(\Gamma, \Delta)$ which, together with $(\xi, \chi, \eta)$, determine the canonical form of the matrices $C_{IJ}, C_{IK}$. Being $\Gamma$ a function of $(\xi, \chi, \eta)$, according to the Theorem (2.1), one has that $(\xi, \chi, \eta, \Delta)$ and the triple $(\theta^I, \theta^J, \theta^K)$ of the angles of isoclinicity determine the $Sp(n)$-orbit of $U$.

We say that a pair of isoclinic subspaces $U, W, \text{even of different dimensions}$, have the same angles of isoclinicity when, for all compatible complex structure $A \in S(Q)$, the angles of isoclinicity of the pairs $(U, AU)$ and $(W, AW)$ are the same. We also say that $U$ is a $2$-planes decomposable subspace if it admits an orthogonal decomposition into isoclinic $2$-planes with same angles of isoclinicity of $U$. Furthermore we say that $U$ is orthogonal if at least one among $(\theta^I, \theta^J, \theta^K)$ equals $\pi/2$. The main result of this paper is given in the Theorem (6.20) that here we report:

**Theorem 2.2.** Let $U \in IC^{2m}$. Let fix an admissible basis $(I, J, K)$ and denote by $(\theta^I, \theta^J, \theta^K)$ the angles of isoclinicity of the pairs $(U, IU), (U, JU), (U, KU)$ respectively. For $k \geq 0$:

- If $2m = 8k + 2$ or $2m = 8k + 6$, $U$ is $2$-planes decomposable i.e. is orthogonal sum of $U_i \in IC^2$ with same angle of isoclinicity of $U$. In this case $(\Gamma, \Delta) = (1, 0)$ and the pair $(\xi, \chi, \eta) = (\pm 1, \pm 1)$ determine the matrices $C_{IJ}, C_{IK}$. The $Sp(n)$-orbit is then determined by the angles $(\theta^I, \theta^J, \theta^K)$ and by the pair $(\xi, \chi, \eta)$.
- If $2m = 8k + 4$, then $U$ is orthogonal sum of $U_i \in IC^4$ with same angle of isoclinicity of $U$ and characterized by the same pair $(\Gamma, \Delta)$. In this case $\Gamma^2 + \Delta^2 = 1$ and the canonical matrices are determined by $(\xi, \chi, \eta, \Delta)$. In particular this case always occurs if $U$ is orthogonal in which case $(\Gamma, \Delta) = (1, 0)$. The $Sp(n)$-orbit is then characterized by $(\theta^I, \theta^J, \theta^K)$ and $(\xi, \chi, \eta, \Delta)$. In particular, if $\xi = \pm 1$ and $\chi = \pm 1$ we are in the first case.
- If $2m = 8k$ then $U$ is orthogonal sum of $U_i \in IC^8$ with same angle of isoclinicity of $U$. The canonical matrices are determined by $(\xi, \chi, \eta, \Delta)$ where $\Gamma^2 + \Delta^2 \leq 1$ and the $Sp(n)$-orbit by $(\theta^I, \theta^J, \theta^K)$ and $(\xi, \chi, \eta, \Delta)$. If in particular $\Gamma^2 + \Delta^2 = 1$ we are in the previous case and if furthermore $\xi = \pm 1$ and $\chi = \pm 1$ we are in the first case.

3. Preliminaries

In this paragraph we recall some definitions regarding the structures, angles and groups we need in this paper. For a wider treatment one can refer to [17]. Let $V$ be a $4n$-dimensional real vector space. We endow $V$ with an Hermitian quaternionic structure and an (H-valued)-Hermitian product (see [17]) whose definitions we recall in the sequel.
Definition 3.1.  
(1) A triple \( \mathcal{H} = \{J_1, J_2, J_3\} \) of anticommuting complex structures on \( V \) with \( J_1 J_2 = J_3 \) is called a hypercomplex structure on \( V \).

(2) The 3-dimensional subalgebra
\[
Q = \text{span}_\mathbb{R}(\mathcal{H}) = \mathbb{R}J_1 + \mathbb{R}J_2 + \mathbb{R}J_3 \cong \mathfrak{sp}_1
\]
of the Lie algebra \( \text{End}(V) \) is called a quaternionic structure on \( V \).

Note that two hypercomplex structures \( \mathcal{H} = \{J_1, J_2, J_3\} \) and \( \mathcal{H}' = \{J'_1, J'_2, J'_3\} \) generate the same quaternionic structure \( Q \) iff they are related by a rotation, i.e.
\[
J'_\alpha = \sum \beta A_\beta^\alpha J_\beta, \quad (\alpha = 1, 2, 3)
\]
with \((A_\beta^\alpha) \in SO(3)\). A hypercomplex structure generating \( Q \) is called an admissible (hypercomplex) basis of \( Q \). We denote by \( S(Q) \) the 2-sphere of complex structures in \( Q \) i.e. \( S(Q) = \{aJ_1 + bJ_2 + cJ_3, \ a, b, c \in \mathbb{R}, \ a^2 + b^2 + c^2 = 1\} \).

Definition 3.2. An Euclidean scalar product \( \langle, \rangle \) in \( V \) is called Hermitian with respect to a hypercomplex structure \( \mathcal{H} = (J_\alpha) \) (resp. the quaternionic structure \( Q = \text{span}_\mathbb{R}(\mathcal{H}) \)) if and only if, for any \( X, Y \in V \),
\[
\langle J_\alpha X, J_\alpha Y \rangle = \langle X, Y \rangle, \quad (\alpha = 1, 2, 3)
\]
(respectively
\[
\langle JX, JY \rangle = \langle X, Y \rangle, \quad (\forall J \in S(Q))\).

Definition 3.3. A hypercomplex structure \( \mathcal{H} \) (resp. quaternionic structure \( Q \)) together with an Hermitian scalar product \( \langle, \rangle \) is called a Hermitian hypercomplex (resp. Hermitian quaternionic) structure on \( V \) and the triple \((V^4n, \mathcal{H}, \langle, \rangle)\) (resp. \((V^4n, Q, \langle, \rangle)\)) is an Hermitian hypercomplex (resp. quaternionic) vector space.

For an introduction and a survey of some results on Hermitian hypercomplex and Hermitian quaternionic structures one can refer among others to \([17, 2, 13]\).

The group \( Sp(1) \) is the group under multiplication of unitary quaternions. It is a Lie group whose Lie algebra \( \mathfrak{sp}_1 = Im \mathbb{H} \cong \mathbb{Q} \). For any quaternion \( q \in Sp(1) \), let consider the unitary homothety in the \( \mathbb{H} \)-module \( V \).
\[
q : X \mapsto Xq, \quad X \in V.
\]

To these transformations belong for instance the automorphisms \( I = R_{-i}, J = R_{-j}, K = R_{-k} \) given by the right multiplications by \(-i, -j, -k\) being \((1, i, j, k)\) a basis of \( \mathbb{H} \) satisfying the multiplication table obtainable from the conditions
\[
i^2 = j^2 = k^2 = -1; \ ij = -ji = k.
\]

Proposition 3.4. \([4]\) We denote by \( B \) the set of such bases. The unitary homotheties are rotations of \( V^{4n} \) that leave invariant any characteristic line. \(^1\)

Moreover for any \( X \in V \) the angle \( \overline{X}, \overline{X}q \) does not depend on \( X \) and it is
\[
\cos \overline{X}, \overline{X}q = Re(q).
\]

Restricting to the action of \( Sp(1) \) determines then an inclusion
\[
\lambda : Sp(1) \hookrightarrow SO(4n).
\]

We define \( Sp(n) \) to be the subgroup of \( SO(4n) \) commuting with \( \lambda(Sp(1)) \) i.e. \( Sp(n) \) is the centralizer of \( \lambda Sp(1) \) in \( SO(4n) \). Then the group \( Sp(n) \) is the group of automorphisms of an Hermitian hypercomplex vector space.

As an \( \mathbb{H} \)-module, on a quaternionic Hermitian vector space \((V^{4n}, \{I, J, K\}_\mathbb{H}, \langle, \rangle)\), once identified the hypercomplex basis with \((R_{-i}, R_{-j}, R_{-k})\) for some basis \((i, j, k)\) of \( Im(\mathbb{H}) \) with \((1, i, j, k) \in B \), we define the \((\mathbb{H} \text{-valued})\)-Hermitian product \((\cdot) \) by:
\[
(\cdot) : V \times V \to \mathbb{H}
\]
\[
(X, Y) \mapsto X \cdot Y = \langle X, Y \rangle + \langle X, IY \rangle i + \langle X, JY \rangle j + \langle X, KY \rangle k.
\]

The Hermitian product in \([2]\) is definite positive. Observe that if the pair \((X, Y)\) is an orthonormal oriented basis of the 2-plane \( U = L(X, Y) \), for any \( A \in S(Q) \), \( \omega^A(X, Y) \in [-1, 1] \) is the cosine of the A-Kähler angle \( \Theta^U \) (see definition at the end of this section).

The Hermitian product in \( \mathbb{H}^n \) is canonical \([3, 17]\) i.e. it does not depend on the particular basis \((1, i, j, k) \in B \). Observe that, when considering a different admissible basis \((I', J', K') = (I, J, K)C, C \in SO(3) \), the new quantities

\(^1\)The following definition appeared in \([3]\). We regard \( V \cong \mathbb{R}^{4n} \) as a right module over the skew-field \( \mathbb{F} \) of quaternions by identifying \( \mathbb{R}^{4n} \) with \( \mathbb{H}^n \) and by letting \( \mathbb{F} \) act by right multiplication. The subspaces \( U^{4h} \subset \mathbb{H}^n \) of real dimension \( 4h \) real image of the subspaces of \( \mathbb{H}^n \) of quaternionic dimension \( h \) are called characteristic subspaces. A characteristic subspace of dimension 4 (resp. 8, 12, \ldots) is called characteristic line, (resp. plane, 3-plane, \ldots). According to the definition given in \([17]\) a characteristic line is a 4-dimensional quaternionic subspace.
<X, Y'>, <X, Y', Y' > in \( \mathbb{B} \) are now the component of \( X \cdot Y \) w.r.t. the basis \((i', j', k') = (i, j, k)C\) of \( lm(\mathbb{H}) \). If instead one considers a fixed basis \( B \in \mathcal{B} \), the Hermitian product \( \mathbb{B} \) is defined up to an inner automorphisms of \( \mathbb{H} \).

We recall the definition of the principal angles between a pair of subspaces of a real vector space \( V \) (see [6,11] among others).

**Definition 3.5.** Let \( A, B \subseteq V \) be subspaces, \( \dim k = \dim(A) \leq \dim(B) = l \geq 1 \). The principal angles \( \theta_i \in [0, \pi/2] \) between the subspaces \( A \) and \( B \) are recursively defined for \( i = 1, \ldots, k \) by

\[
\cos \theta_i = \frac{\langle a_i, b_i \rangle}{\|a_i\|\|b_i\|} = \max\{\frac{\langle a, b \rangle}{\|a\|\|b\|} : a \perp a_m, b \perp b_m, m = 1, 2, \ldots, i - 1\}
\]

where the \( a_j \in A, b_j \in B \). The pairs \( (a_i, b_i) \), \( i = 1, \ldots, k \) are called related principal vectors.

In words, the procedure is to find the unit vector \( a_1 \in A \) and the unit vector \( b_1 \in B \) which minimize the angle between them and call this angle \( \theta_1 \). Then consider the orthogonal complement in \( A \) to \( a_1 \) and the orthogonal complement in \( B \) to \( b_1 \) and iterate. The principal angles \( \theta_1, \ldots, \theta_k \) between the pair of subspaces \( A, B \) are some of the critical values of the angular function

\[ \phi_{A,B} = A \times B \to \mathbb{R} \]

associating with each pair of non-zero vectors \( a \in A, b \in B \) the angle between them. Moreover the principal angles are the diagonal entries of the orthogonal projector \( P^A : B \to A \) stated in the theorem of Afriat ([8, 11]):

**Theorem 3.6.** ([8], [11]) In any pair of subspaces \( A^k \) and \( B^l \) there exist orthonormal bases \( \{u_i\}_{i=1}^k \) and \( \{v_j\}_{j=1}^l \) such that \( \langle u_i, v_j \rangle \geq 0 \) and \( \langle v_i \times v_j \rangle = 0 \) if \( i \neq j \).

**Proof.** It is a direct consequence of the following

**Lemma 3.7.** Given finite dimensional subspaces \( A, B \), let \( a_1, b_1 \) attain

\[
\max\{\langle a, b \rangle, \quad a \in A, b \in B, \quad \|a\| = 1, \|b\| = 1\}
\]

(i.e. the pair \( (a_1, b_1) \) are the first principal vectors). Then

1. for some \( \alpha \geq 0 \),
\[ P^B a_1 = \alpha b_1, \quad P^A b_1 \]
2. \( a_1 \perp (b_1 \cap B) \) and \( b_1 \perp (a_1 \cap A) \) which leads the diagonal form of the matrix of the projector \( P^B \) (and \( P^A \)).

To see that 1) holds, note that \( P^B a_1 = \alpha b \) where \( a, b \) minimize \( \|a_1 - ab\|^2 \) for \( b \in B, \|b\| = 1 \) and \( a \) a scalar. Thus to minimize \( \|a_1 - ab\|^2 = \alpha^2 - 2\alpha < a_1, b > + 1 \) we must maximize \( < a_1, b > \). Moreover \( \langle a_1, b_1 \rangle \) is the cosine of the first principal angle.

For 2), let \( A_1 = a_1 \cap A \) (resp. \( B_1 = b_1 \cap B \)). If \( a \in A_1 \), then \( a \perp b_1 \) since \( < a_1, b_1 > = < P^A a_1, b_1 > = < a, b_1 \rangle = < a, a_1 \rangle = 0 \). Likewise if \( b \in B_1 \) then \( b \perp a_1 \). We proceed letting \( a_2 \) and \( b_2 \) attain

\[
\max\{\langle a, b \rangle, \quad a \in A_1, b \in B_1, \quad \|a\| = 1, \|b\| = 1\}
\]

and continue till we have exhausted \( A \) and \( B \).

From the proof of (3.6) we have that the principal angles between a pair of subspaces \( A, B \) of \( V \) can also defined as the singular value of the orthogonal projector \( P^A \) (or equivalently \( P^B \)).

Let recall the definition and some properties of isoclinic subspaces.

**Definition 3.8.** A pair of non oriented subspaces \( A \) and \( B \) of same dimension are said to be isoclinic and the angle \( \phi \in [0, \frac{\pi}{2}] \) is said to be angle of isoclinicity between them if either of the following conditions hold:

1. the angle between any non-zero vector of one of the subspaces and the other subspace is equal to \( \phi \);
2. \( GG^* = \cos^2 \phi I \) for the matrix \( G := < a_i, b_j > \) of the orthogonal projector \( P^B_\phi : B \to A \) with respect to any orthonormal basis \( \{a_i\} \) of \( A \) and \( \{b_j\} \) of \( B \);
3. all principal angles between \( A \) and \( B \) equal \( \phi \).

**Definition 3.9.** We denote by \( \mathcal{I}^{2m} \) the set of \( 2m \)-dimensional subspaces of \( V \) such that, for any \( A \in S(Q) \), the pair \( (U, A') \) is isoclinic. When we do not need to specify the dimension we just use the notation \( \mathcal{I} \) and we call them simply isoclinic subspaces.

The fact that we consider only even dimensions subspaces follows from the

**Proposition 3.10.** Let \( U \) be an odd dimension isoclinic subspace. Then \( U \) is a real hermitian product subspace (r.h.p.s.) i.e. for one and hence any admissible basis \((I, J, K)\) the pairs \((U, IU), (U, JU), (U, KU)\) are strictly orthogonal. Then \( \mathcal{I}^{2m+1} \) is the set of all and only the real Hermitian product \((2m + 1)\)-dimensional subspaces. All r.h.p. subspaces share the same orbit.
Proof. The first statement is obvious since, for any \( A \in S(\mathbb{Q}) \), by the skew-symmetry of \( \omega^A \) one (and then all) principal angle is necessarily equal to \( \pi/2 \). For the last statement observe that any orthonormal basis \( B \) is a standard basis of \( \omega^I, \omega^J, \omega^K \) restricted to \( U \) which implies that w.r.t. \( B \) one has \( C_{12} = C_{1K} = Id \) and the conclusion follows from the aforementioned Theorem (2.1).

Therefore, this paper deals with isoclinic subspaces of even dimension. Throughout the article we will fix an admissible basis \((I, J, K)\) and, given \( U \in \mathbb{R}^{2m} \), we denote by \( \theta^I, \theta^J, \theta^K \) the respective angles of isoclinicity. If the pair \((U, IU)\) (resp. \((U, JU)\), resp. \((U, KU)\)) is strictly orthogonal (i.e. if all principal angles are \( \pi/2 \)) we say that \( U \) is \( I \)-orthogonal (resp. \( J \)-orthogonal, resp. \( K \)-orthogonal) and in general we speak of single orthogonality (or 1-orthogonality). When two (resp. three) of the above pair are strictly orthogonal we speak of double (resp. triple)-orthogonality. By saying that \( U \) is orthogonal (without specifying the complex structures) we mean that at least one among \( \theta^I, \theta^J, \theta^K \) equals \( \pi/2 \).

Fixed an admissible basis \( \mathcal{H} \) of \( \mathbb{Q} \), throughout this paper we will define some functions \( f : V \times V \times \ldots \times V \to \mathbb{R} \). If \( A \) is one of them and it is constant on its domain, will say that \( A \) is an invariant of \( U \). If furthermore the invariant \( A \) does not depend on the chosen oriented basis \( \mathcal{H} \), we will say that \( A \) is an intrinsic property of \( U \).

Given a pair of non oriented subspaces \((U, W)\) we denote by \( \bar{U}, \bar{W} \) the Euclidean angle \( \phi \in [0, \pi/2] \) they form. We recall (14) that if \( \theta_1, \ldots, \theta_p \) are the principal angles between \( U \) and \( W \) one has

\[
\cos \phi = \cos \theta_1 \cdot \cos \theta_2 \cdots \cos \theta_p.
\]

Finally we recall the notion of \( \text{K"ahler angle} \) which is defined in a real vector space \( V \) endowed with a complex structure \( I \).

**Definition 3.11.** Let \((V^{2n}, I)\) be a real vector space endowed with a complex structure \( I \). For any pairs of non parallel vectors \( X, Y \in V \) their \( \text{K"ahler angle} \) is given by

\[
\Theta^I = \arccos \frac{< X, JY >}{|X||Y| \sin XY} = \arccos \frac{< X, JY >}{\text{mis}(X \wedge Y)}.
\]

Then \( 0 \leq \Theta^I \leq \pi \). It is straightforward to check that the \( \text{K"ahler angle} \) is an intrinsic property of the oriented 2-plane \( U = L(X, Y) \). For this reason one speaks of the \( \text{K"ahler angle} \) of an oriented 2-plane.

The cosine of the \( \text{K"ahler angle} \) of the pair of 2-planes with opposite orientation \( U \) and \( \bar{U} = L(Y, X) \) have opposite sign i.e. \( \cos \Theta^I(U) = -\cos \Theta^I(\bar{U}) \), then, if one disregards the orientation of the 2-plane \( U \), we can consider the absolute value of the right hand side of equation (3) restricting the \( \text{K"ahler angle} \) to the interval \([0, \pi/2]\). In this case the \( \text{K"ahler angle} \) of the 2-plane \( U \) coincides with one of the two identical principal angles, say \( \theta^I(U) \), between the pairs of 2-plane \( U \) and \( IU \) (same as the pair \((\bar{U}, \bar{IU})\)) which are always isoclinic as one can immediately verify, then

\[
\cos \theta^I(U) = |\cos \Theta^I(U)|
\]

and one has

\[
\cos(\bar{U}, IU) = \cos(\bar{U}, \bar{IU}) = \frac{< X, JY >^2}{\text{mis}^2(X \wedge Y)} = \cos^2 \theta^I(U) = \cos^2 \Theta^I(U).
\]

The \( \text{K"ahler angle} \) measures the deviation of a 2-plane from holomorphicity. For instance the \( \text{K"ahler angle} \) of a totally real plane \( U \) (i.e. a plane such that \( U \perp IU \)) is \( \Theta^I(U) = \pi/2 \) and the one of an oriented complex plane \( U \) (i.e. \( U = IU \)) is \( \Theta^I(U) \in \{0, \pi\} \).

Generalizing the notion of \( \text{K"ahler angle} \), in an Hermitian quaternionic vector space \((V^{4n}, Q, <, >)\) we will speak of the \( A \)-\( \text{K"ahler angle} \) of an oriented 2-plane \( U \) with \( A \in S(\mathbb{Q}) \). We will denote it by \( \Theta^A(U) \).

4. 2-DIMENSIONAL SUBSPACES

The simplest example of an even dimension isoclinic subspace \( U \subseteq V \) is a 2-plane. By the skew-symmetry of the \( A \)-\( \text{K"ahler form} \) for any \( A \in S(\mathbb{Q}) \), any 2-dimensional subspace of \( U \) is isoclinic with \( AU \). Therefore as a set one has that \( G_{2}(2, 4n) = \mathbb{C}^{2} \). The study of the orbits of the oriented 2-planes in the real Grassmannian under the action of the groups \( Sp(n) \) and \( Sp(n) \cdot Sp(1) \) is carried out in [17]. There we introduced the imaginary measure and the characteristic deviation of an oriented 2-plane \( U \) proving that such invariants characterize its orbits under the action of the groups \( Sp(n) \) and \( Sp(n) \cdot Sp(1) \). Namely, let \( L, M \) be an oriented basis of \( U \). The purely imaginary quaternion

\[
\mathcal{I}M(U) = \frac{\mathfrak{I}(L \cdot M)}{\text{mis}(L \wedge M)},
\]

is an intrinsic property of an oriented 2-plane \( U \subseteq (V^{4n}, Q, <, >) \) i.e. it does not depend neither on the chosen oriented generators \( L, M \) nor on the admissible basis \( \mathcal{H} \) of \( \mathbb{Q} \). Moreover \( Sp(n) \) preserves \( \mathcal{I}M(U) \). In particular, if the pair \( L, M \) is an orthonormal oriented basis of \( U \), then \( \mathcal{I}M(U) = L \cdot M \). We called it imaginary measure of the oriented 2-plane \( U \). Disregarding the orientation of \( U \) and being \((L, M)\) some orthonormal basis, it is \( \mathcal{I}M(U) = \{ \pm L \cdot M \} \) i.e. is the set made of a pair of conjugated pure imaginary quaternions. We proved that
The imaginary measure $IM(U)$ represents the full system of invariants for the $Sp(n)$-orbits in the real Grassmannian of 2-planes $G_2(2,4n)$ as well as in $G_2^*(2,4n)$ (the Grassmannian of the oriented 2-planes) i.e. a pair of 2-planes $(U, W)$ of $(V^{4n}, Q, <, >)$ are in the same $Sp(n)$-orbit iff $IM(U) = IM(W)$.

Let consider a triple of standard bases $(X_1, X_2), (X_1, Y_2), (X_1, Z_2)$ with a common leading vector $X_1$ of the non oriented 2-plane $U$. By definition one has that $cos \theta^I = \langle X_1, I X_2 \rangle$, $cos \theta^J = \langle X_1, J Y_2 \rangle$, $cos \theta^K = \langle X_1, K Z_2 \rangle$ are non negative, and computed $\xi = \langle X_2, X_2 \rangle, \chi = \langle X_2, Z_2 \rangle, \eta = \langle Y_2, Z_2 \rangle$, where $(\xi, \chi, \eta) \in \{-1, 1\}$ one has that the matrices $C_{IJ}$ and $C_{IK}$ w.r.t. the standard bases $(X_1, X_2), (X_1, Y_2)(X_1, Z_2)$ are given by

$$C_{IJ} : \begin{pmatrix} 1 & 0 & \xi \\ 0 & 1 & 0 \end{pmatrix} \quad C_{IK} : \begin{pmatrix} 1 & 0 & \chi \\ 0 & 1 & 0 \end{pmatrix}$$

It is straightforward to verify that the pair $(\xi, \chi)$ is an invariant of $U$. Therefore, according to Theorem (2.1), together with the triple $(\theta^I, \theta^J, \theta^K)$, such pair determines the $Sp(n)$-orbits of the (non oriented) 2-plane $U$.

This is accordance with the Theorem (4.1). In fact, If $U$ has a triple orthogonality then clearly $IM(U) = 0$. In this case any orthonormal basis is a standard basis of $\omega^I, \omega^J, \omega^K$ which implies $\xi = \chi = 1$. Else suppose, without lack of generality, that $cos \theta^I \neq 0$ and that let $(X_1, X_2)$ be an $\omega^I$-standard basis. Then

$$IM(U) = X_1 \cdot X_2 = \pm (cos \theta^I + \xi \cos \theta^J + \chi \cos \theta^K).$$

Given a pair of 2-planes $U, W$ with $IM(U) = IM(W)$, according to Theorem (4.1), they are in the same orbit. Since they share the same pair $(\xi, \eta)$ and the same triple $(\theta^I, \theta^J, \theta^K)$, they are in the same $Sp(n)$-orbit also according to Theorem (2.1). Viceversa if they share the same pair $(\xi, \eta)$ and the same triple $(\theta^I, \theta^J, \theta^K)$ which implies that they belong to the same $Sp(n)$-orbit according to Theorem (2.1) then clearly $IM(U) = IM(W)$.

5. 4-DIMENSIONAL ISOCLINIC SPACES

5.1. Invariants of a 4 dimensional isoclinic subspace. Let consider a 4-dimensional subspace $U$ such that the pair $(U, AU), A \in S(Q)$ is isoclinic with angle $\theta^A$. From 2) of the Definition (3.8), for any (principal) vector $X$ in $U$, one has $Pr_{AU} \circ Pr_U^{AU}(X) = \cos^2 \theta^A X$. Any pair of orthonormal basis $(X_1, X_2, X_3, X_4)$ of $U$ and $(AX_1, AX_2, AX_3, AX_4)$ of $AU$ are made of principal vectors of $U$ and $AU$ respectively. In general the pairs $(X_1, AX_i)$ are not related principal vectors unless $U \perp AU$.

Then, w.r.t. the pair of aforementioned bases, for the skew-symmetric matrix of the projector $Pr_{AU}^U : U \rightarrow AU$ (or equivalently for the matrix of the form $\omega^A$ w.r.t. the basis $(X_1, X_2, X_3, X_4)$ of $U$) the condition $(\omega^A)^2 \circ \omega^A = \cos^2 \theta^A 1d$ leads to the following

**Proposition 5.1.** Let $A \in S(Q)$. The pair $(U, AU)$ of 4-dimensional subspaces is isoclinic iff the matrix of $\omega^A$ w.r.t. the orthonormal basis $(X_1, X_2, X_3, X_4)$ has the form

$$\omega^A : \begin{pmatrix} 0 & a & b & c \\ -a & 0 & \pm c & \mp b \\ -b & \mp c & 0 & \pm a \\ -c & \pm b & \mp a & 0 \end{pmatrix}.$$  \hspace{1cm} (6)

It is a matrix with orthogonal rows and columns (see [23]) whose square norms evidently equal the square cosine of the angle of isoclinicity $\theta^A$ between the pair $(U, AU)$ i.e.

$$\cos \theta^A = \sqrt{a^2 + b^2 + c^2}.$$  

Moreover, recalling that for any orthonormal basis $(X_1, X_2, X_3, X_4)$ $a, b, c$ represent respectively the cosines of the A-Kähler angles $\cos \Theta^A(U_{12}), \cos \Theta^A(U_{13}), \cos \Theta^A(U_{14})$ where $U_{ij} = L(X_1, X_j), j = 2, 3, 4$ and from [41] one has

$$\cos^2 \theta^A = \cos^2 \Theta^A(U_{12}) + \cos^2 \Theta^A(U_{13}) + \cos^2 \Theta^A(U_{14}) = \cos^2(U_{12}, \bar{U}_{12}) + \cos^2(U_{13}, \bar{U}_{13}) + \cos^2(U_{14}, \bar{U}_{14}).$$

Observe that, from the above definition, given a pair of vectors $X, Y$ and being the pair of 4-dimensional quaternionic subspaces $Q X = L(X, I X, J X, K X)$ and $Q Y$ always isoclinic as it can be easily verified, the Hermitian and the characteristic angle of any pair of vectors $X, Y$ (see [17], [3]) equal respectively the angle of isoclinicity of the pair $(Q X, Q Y)$ and the (Euclidean) angle between the same pair of subspaces.

The quaternionic characteristic angle $\varphi$ between a pair of vectors $L, M$ of a quaternionic vector space $V^{4n}$ is given by

$$\cos \varphi = \frac{|[L \cdot M]|^2}{m^{4n} L \cdot M} = \frac{\langle L, M \rangle^2 + \langle L, \bar{M} \rangle^2 + \langle L, KM \rangle^2 + \langle L, \bar{K}M \rangle^2 + \langle L, \bar{K} \bar{M} \rangle^2 + \langle L, K M \rangle^2}{< L, L > < M, M >}.$$  \hspace{1cm} (7)

The **Hermitian angle** between the same pair of vectors of $V$ is defined as

$$\cos \psi = \frac{|[L \cdot M]|}{|L||M|} = \frac{\langle L, M \rangle^2 + \langle L, \bar{M} \rangle^2 + \langle L, KM \rangle^2 + \langle L, \bar{K}M \rangle^2 + \langle L, \bar{K} \bar{M} \rangle^2 + \langle L, K M \rangle^2}{\sqrt{< L, L > < M, M >}}.$$  \hspace{1cm} (8)

Both angles do not depend on the admissible basis of $Q$. Therefore the Hermitian angle $\psi$ between a pair of vectors $L, M$ is just the angle between such pair computed by using the Hermitian product, whereas the characteristic angle $\varphi$ is the angle between the 4-dimensional characteristic lines they span over $\mathbb{R}$. It is $\cos \varphi = \cos^4 \psi$. 

\footnote{The quaternionic characteristic angle $\varphi$ between a pair of vectors $L, M$ of a quaternionic vector space $V^{4n}$ is given by $\cos \varphi = \frac{|[L \cdot M]|^2}{m^{4n} L \cdot M} = \frac{\langle L, M \rangle^2 + \langle L, \bar{M} \rangle^2 + \langle L, KM \rangle^2 + \langle L, \bar{K}M \rangle^2 + \langle L, \bar{K} \bar{M} \rangle^2 + \langle L, K M \rangle^2}{< L, L > < M, M >}$. 

The **Hermitian angle** between the same pair of vectors of $V$ is defined as $\cos \psi = \frac{|[L \cdot M]|}{|L||M|} = \frac{\langle L, M \rangle^2 + \langle L, \bar{M} \rangle^2 + \langle L, KM \rangle^2 + \langle L, \bar{K}M \rangle^2 + \langle L, \bar{K} \bar{M} \rangle^2 + \langle L, K M \rangle^2}{\sqrt{< L, L > < M, M >}}$. 

Both angles do not depend on the admissible basis of $Q$. Therefore the Hermitian angle $\psi$ between a pair of vectors $L, M$ is just the angle between such pair computed by using the Hermitian product, whereas the characteristic angle $\varphi$ is the angle between the 4-dimensional characteristic lines they span over $\mathbb{R}$. It is $\cos \varphi = \cos^4 \psi$.}
Consider now a subspace $U$ such that the pairs $(U, IU), (U, JU), (U, KU)$ are isoclinic where $(I, J, K)$ is some admissible basis. To this set belong for instance all totally complex, quaternionic and real Hermitian product (r.h.p.) subspaces (see [19]). In particular in this paragraph we consider a 4-dimensional subspace $U$. We can add to the list above all 4-dimensional complex subspaces (see [19]). Let fix an orthonormal basis $B = (X_1, X_2, X_3, X_4)$ of $U$. Keeping the same notations used in the Proposition (5.1), we denote by $(a, b, c), (a', b', c'), (a, b, c)$ the entries of the first row of the matrices representing respectively $\omega^I, \omega^J, \omega^K$ w.r.t. the basis $B$ whose form is given in (9).

**Proposition 5.2.** Let $U$ be a 4 dimensional subspace and $(I, J, K)$ an admissible basis. Suppose the pairs $(U, IU), (U, JU), (U, KU)$ are isoclinic and $\theta^I, \theta^J, \theta^K$ the respective angles of isoclinicity. Then for any $A = \alpha_1 I + \alpha_2 J + \alpha_3 K \in S(Q)$ the pair $(U, AU)$ is isoclinic and therefore $U \in \mathcal{I}^4$. The angle of isoclinicity $\theta^A$ between the pair $(U, AU)$ is given by

$$\cos^2 \theta^A = \frac{1}{4} \text{Tr}[(\alpha_1 \omega^I + \alpha_2 \omega^J + \alpha_3 \omega^K)^2] = -\frac{1}{4} \text{Tr}[\omega^A]^2$$

**Proof.** The first statement follows from the Proposition (5.1) since $\omega^A = \alpha_1 \omega^I + \alpha_2 \omega^J + \alpha_3 \omega^K$ has clearly the form given in (5.1). The angle of isoclinicity and considering for instance the vector $X_1$ of $B$, one has $\cos^2 \theta^I = \frac{1}{4} \text{Tr}[(\alpha_1 \omega^I + \alpha_2 \omega^J + \alpha_3 \omega^K)^2] + 2\alpha_1 \alpha_2(a + b + c) = \alpha_1 \alpha_2$. It is $(a + b + c) = -\frac{1}{4} \text{Tr}(\omega^I \cdot \omega^J)$ (resp. $(a + b + c) = -\frac{1}{4} \text{Tr}(\omega^I \cdot \omega^K)$). Moreover $\text{Tr}[\omega^A]^2 = -4 \cos^2 \theta^I, \text{Tr}[\omega^A]^2 = -4 \cos^2 \theta^J,$ $\text{Tr}[\omega^A]^2 = -4 \cos^2 \theta^K$ which leads to the expression (9). From the similarity invariance of the trace we have the

**Corollary 5.3.** Fixed an admissible basis $(I, J, K)$, the quantities $(a + b + c), (a + b + c), (a + b + c)$ are invariant of $U$.

**Corollary 5.4.** Let $U \in \mathcal{I}^4$. Fixed an orthonormal basis $B$, for any $A \in S(Q)$ the matrix of $\omega^A$ is given in (9) with the same choice of sign. In particular if $B$ is a standard basis of some compatible complex structure, the common choice of sign is the upper one.

**Proof.** Fixed an orthonormal basis, the matrices of the forms $\omega^I, \omega^J, \omega^K$ have necessarily the same choice of sign being, from (10), the quantities $(a + b + c), (a + b + c), (a + b + c)$ independent from the row. Consequently, for any $A \in S(Q)$ and a fixed orthonormal basis, the matrix of $\omega^A = \alpha_1 \omega^I + \alpha_2 \omega^J + \alpha_3 \omega^K$, given in (9), has the same choice of sign. The second statement is straightforward.

Any unitary vector $X_1 \in U$ is simultaneously a principal vector of the pairs $(U, IU), (U, JU), (U, KU)$. Suppose that $U$ is not an orthogonal subspace (w.r.t. $(I, J, K)$) and let

$$X_2 = \frac{1}{\cos \theta^I} I^{-1} P_{RU} X_1, \quad Y_2 = \frac{1}{\cos \theta^J} J^{-1} P_{RU} X_1, \quad Z_2 = \frac{1}{\cos \theta^K} K^{-1} P_{RU} X_1$$

be the unitary vectors such that the $(X_1, X_2), (X_1, Y_2), (X_1, Z_2)$ are (orthonormal) standard bases of the 2-planes $U^I, U^J, U^K$ of $\omega^I, \omega^J, \omega^K$ they generate. The quantities $<X_1, IX_2>, <X_1, JY_2>, <X_1, KZ_2>$ are the (non negative) cosines of the principal angles of the pairs $(U^I, U^I), (U^J, U^J), (U^K, U^K)$ or equivalently the absolute value of the cosine of the $I, J, K$-Kähler angles of the 2-planes $U^I, U^J, U^K$ respectively. Let $(X_1, X_2, X_3, X_4)$ (resp. $(X_1, Y_2, Y_3, Y_4)$, resp. $(X_1, Z_2, Z_3, Z_4)$) be a standard form of the bases $\omega^I$ (resp. $\omega^J$, resp. $\omega^K$) with the common leading vector $X_1$.

One has that

$$(a + b + c) = \cos \theta^I <X_1, JX_2>, \quad (a + b + c) = \cos \theta^J <X_1, KX_2>, \quad (a + b + c) = \cos \theta^K <X_1, ZX_2>$$

$$<X_1, JY_2>, <X_1, KZ_2>$$

Let moreover denote by

$$\xi = <X_2, Y_2>, \quad \chi = <X_2, Z_2>, \quad \eta = <Y_2, Z_2>$$

where $\xi, \chi, \eta \in [-1, 1]$. In case $U$ is an orthogonal subspace we

**Claim 5.5.** Let $X_1 \in U$ unitary. In case $\cos \theta^I = 0$ (resp. $\cos \theta^J = 0$, resp. $\cos \theta^K = 0$), the pair $(Y_2, Z_2)$ (resp. $(X_2, Z_2)$, resp. $(X_2, Y_2)$) is given in (11). Any vector of $IU$ (resp. $JU$, resp. $KU$) can be consider a principal vector related to $X_1$. We assume $X_2 = Y_2$ (resp. $Y_2 = X_2$, resp. $Z_2 = X_2$) then $\xi = 1$ (resp. $\chi = 0$, resp. $\eta = 0$).

In case $\cos \theta^I = \cos \theta^K = 0$ (resp. $\cos \theta^J = \cos \theta^K = 0$, resp. $\cos \theta^I = \cos \theta^J = 0$) then we assume $X_2 = Y_2 = Z_2$ with $X_2$ (resp. $Y_2$, resp. $Z_2$) given in (11) and then $\xi = \chi = \eta = 1$.

Finally in case all three cosines equal zero, $U$ is a r.h.p. subspace. As in previous point we can assume $X_2 = Y_2 = Z_2$ with $X_2$ any unitary vector orthogonal to $X_1$ and then $\xi = \chi = \eta = 1$. 


Proposition 5.6. The cosines \( \xi = \langle X_2, Y_2 \rangle, \chi = \langle X_2, Z_2 \rangle, \eta = \langle Y_2, Z_2 \rangle \) are invariants of \( U \).

Proof. In case \( U \) is orthogonal we consider the assumptions of the Claim \((5.3)\). One has

\[
\begin{align*}
(aa' + bb' + cc') &= \langle X_2, Y_2 \rangle > \cos \theta^I \cos \theta^J, \\
(a\bar{a} + b\bar{b} + c\bar{c}) &= \langle X_2, Z_2 \rangle > \cos \theta^I \cos \theta^K, \\
(a'\bar{a} + b'\bar{b} + c'\bar{c}) &= \langle Y_2, Z_2 \rangle > \cos \theta^I \cos \theta^K.
\end{align*}
\]

The first equality is obtained by substituting in the first of \((12)\) \( X_2 = \langle X_2, Y_1 \rangle + \langle X_2, Y_2 \rangle + \langle X_2, Y_3 \rangle + \langle X_2, Y_4 \rangle = Y_4 \). The independence from \( X_1 \) follows from Corollary \((5.3)\). The other equalities follow in the same way.

Clearly if \( U \) is 2-planes decomposable then \( \xi, \chi, \eta \) all assume value \( \pm 1 \) with \( \eta = \xi \cdot \chi \). It follows that in case of double or triple orthogonality \( U \) is a 2-planes decomposable subspace.

Applying the last result, from equation \((10)\) we have the following equivalent expression for the angle of isoclinicity \((13)\)

\[
cos^2 \theta^I = \alpha_1^2 \cos^2 \theta^I + \alpha_2^2 \cos^2 \theta^J + \alpha_3^2 \cos^2 \theta^K + 2\xi \alpha_1 \alpha_2 \cos \theta^I \cos \theta^J + 2\chi \alpha_1 \alpha_3 \cos \theta^I \cos \theta^K + 2\eta \alpha_2 \alpha_3 \cos \theta^J \cos \theta^K.
\]

Remark 5.7. Let \( U \in \mathbb{IC}^{2m} \), \( m \leq 2 \). The cosines \( \xi, \chi, \eta \) are not an intrinsic properties of \( U \) i.e. they depend on the chosen admissible basis.

Proof. In case of a 2-plane \( U \), let consider the admissible basis \((I, J, K)\) and let \( X_2, Y_2, Z_2 \) be the usual triple and suppose that \( \xi = \langle X_2, Y_2 \rangle > \chi = \langle X_2, Z_2 \rangle > 1 = \eta = \xi = \langle Y_2, Z_2 \rangle > 1 \).

Let now consider the admissible basis \((-I, -J, K)\) (admissible bases are related by a rotation then they have the same orientation). The new vectors \( X_2 \mapsto -X_2, Y_2 \mapsto -Y_2, Z_2 \mapsto Z_2 \), then \( \xi = 1, \chi = -1, \eta = -1 \).

An example for the dimension 4 is given by an \( I \)-complex 4 dimensional subspace with quaternionic Kähler angle \( \cos \theta \) (i.e. \( \theta \) is the angle of isoclinicity of the pairs \((U,JU = KU)\) (see \((19)\)). In this case w.r.t. an adapted basis \((I, J, K)\) in \((19)\) we proved that \( X_2, Y_2, Z_2 \) are mutually orthogonal. Then \( \xi = \chi = \eta = 0 \). If \( (J', J', K') \) is another admissible basis with \( J' = \alpha_1 I + \alpha_2 J + \alpha_3 K, J' = \beta_1 I + \beta_2 J + \beta_3 K, K' = \gamma_1 I + \gamma_2 J + \gamma_3 K \) one has that

\[
\xi' = \alpha_1 \beta_1 (1 - \cos \theta), \quad \chi' = \alpha_1 \gamma_1 (1 - \cos \theta), \quad \eta' = \beta_1 \gamma_1 (1 - \cos \theta)
\]

which account for the dependence of \((\xi', \chi', \eta')\) on \((\alpha_1, \beta_1, \gamma_1)\).

We finally introduce an intrinsic property of the elements of \( \mathbb{IC}^4 \). Later we will see that the statement below is valid for all subspaces \( U \in \mathbb{IC} \) regardless their dimension.

Proposition 5.8. Let \((I, J, K)\) be an admissible hypercomplex basis and \( U \in \mathbb{IC}^4 \) with angles of isoclinicity equal respectively to \((\theta^I, \theta^J, \theta^K)\). Then \( S = \cos \theta^I + \cos \theta^J + \cos \theta^K \) is an intrinsic property of \( U \) not depending on the admissible basis.

Proof. It follows directly from \((13)\) recalling that any change of admissible basis is represented by a matrix belonging to the special orthogonal group.

5.2. Canonical matrices \( C_{IJ}, C_{IK} \) of \( U \in \mathbb{IC}^4 \). It is well known that, if \( U \) and \( W \) are a pair of \( m \)-dimensional subspaces of the \( n \)-dimensional space \( V \) and \( U^\perp \) and \( W^\perp \) the respective orthogonal complements in \( V \), the non zero principal angles of the pair \((U, W)\) are the same as those between \((U^\perp, W^\perp)\). Furthermore, the non zero principal angles between \( U \) and \( W \) are the same as those between \((U^\perp, W^\perp)\). We recall some well known properties of the principal angles of a pair of subspaces.

Proposition 5.9. \((24)\) Let \( U \) and \( W \) be a pair of respectively \( p \) and \( q \) dimensional subspaces with \( p \geq q \) of the \( n \)-dimensional space \( V \) and \( U^\perp \) and \( W^\perp \) the respective orthogonal complement in \( V \). Denote by \( \Theta^I(U, W) \) (resp. \( \Theta^J(U, W) \)) the vector of the \( q \) principal angles between \( U \) and \( W \) arranged in non decreasing (resp. non increasing) order. One has the following properties for the principal angles of the pair \((U, W)\) and of their orthogonal complements \((U^\perp, W^\perp)\):

1. \( \{\Theta^J(U, W), 0, \ldots, 0\} = \{\Theta^J(U^\perp, W^\perp), 0, \ldots, 0\} \),
2. \( \{\Theta^I(U, W), 0, \ldots, 0\} = \{\Theta^I(U^\perp, W^\perp), 0, \ldots, 0\} \),
3. \( \{\pi/2, \ldots, \pi/2, \Theta^J(U, W)\} = \{\pi/2 - \Theta^J(U, W^\perp), 0, \ldots, 0\} \),

where \( l = \max(\dim U - \dim W, 0) \) and extra 0s at the end may need to added on either side to match the sizes.

In particular if we consider a 4 dimensional subspace \( U \), from the Proposition \((5.9)\), it follows that, given a pair of orthogonal decompositions into 2-planes of \( U \) i.e. \( U = U_1 \oplus U_2 = W_1 \oplus W_2 \), there exists orthonormal bases \((X_1, X_2)\)
of $U_1$, $(X_3, X_4)$ of $U_2$, $(Y_1, Y_2)$ of $W_1$ and $(Y_3, Y_4)$ of $W_2$ w.r.t. which the orthogonal Gram matrix $G = \langle X_i, Y_j \rangle$ assumes the CS (Cosine-Sine) form:

$$G = \begin{pmatrix}
\cos \alpha_1 & 0 & \sin \alpha_1 & 0 \\
0 & \cos \alpha_2 & 0 & \sin \alpha_2 \\
-\sin \alpha_1 & 0 & \cos \alpha_1 & 0 \\
0 & -\sin \alpha_2 & 0 & \cos \alpha_2
\end{pmatrix}$$

with non-negative entries in the upper triangular part.

Such bases are the bases of principal vectors of the pair $(U_1, W_1)$ and $(U_2, W_2)$. If $\alpha_1 \neq \alpha_2$, the principal vectors are defined up to sign. The related pairs are then defined up to contemporary change of sign. Observe that the pairs $(X_1, Y_3)$ and $(X_2, Y_4)$ are pairs of related principal vectors of the subspaces $(U_1, W_2)$ whereas $(-X_1, Y_1)$ and $(-X_4, Y_2)$ are pairs of related principal vectors of $(U_1, W_1)$. In case $\alpha_1 = \alpha_2$, the bases of related principal vectors are defined up to a common orthogonal transformation in $U_1, U_2, W_1, W_2$.

In the following we determine the canonical bases of any $U \in \mathbb{C}^4$ and associated canonical matrices $C_{IJ}$ and $C_{IK}$ according to the definition given in [18]. We recall that the canonical bases are triples $\{X_i\}, \{Y_i\}, \{Z_i\}$ of $\omega^I, \omega^J, \omega^K$-standard bases respectively built up using some determined procedure. In general they are not unique (each triple may depend for instance on an arbitrary choice of some vector to start the procedure); nevertheless they are canonical in the sense that the relative Gram matrices $C_{IJ} = \langle X_i, Y_j \rangle$, $C_{IK} = \langle X_i, Z_k \rangle$, $C_{JK} = \langle Y_j, Z_k \rangle$ are independent from the chosen triple. According to the Theorem (2.1) the knowledge of $C_{IJ}$ and $C_{IK}$ is necessary to determine the $Sp(n)$-orbit of $U$. In this degenerate case however we cannot apply tout court the procedure described in [18].

In fact such iterative procedure needs at each step the uniqueness (up to sign) of the first (smallest) principal angle of a pair of standard subspaces of $\omega^I$ and $\omega^J$ and of $\omega^J$ and $\omega^K$. In our case we have only one standard subspace for any $A \in S(Q)$ which is $U$ itself and, because of the isolincity, the choice of the first pair of related principal vectors necessary to start the procedure is not unique. Here we will show that to overtake this problem we need to add one extra condition to such procedure which will allow us to determine the canonical bases and associated pair of canonical matrices $C_{IJ}, C_{IK}$.

In the following if $U$ is orthogonal sum of a pair of 2-planes both isolinic with their $I, J, K$-images with angles $\theta^I, \theta^J, \theta^K$ respectively we say that $U$ is a 2-planes decomposable subspace. In a 2-plane decomposable subspace the values of $\xi, \chi, \eta$ are all clearly equal to ±1.

Let $X_3$ be the unique vector such that $\langle X_3, IX_4 \rangle = \cos \theta^I$ i.e. $L(X_3, X_4)$ is an $\omega^I$-standard 2-plane and analogously $Y_4 = -\frac{1}{\cos \theta^J}X_4$ the unique vector such that $\langle Y_3, JY_4 \rangle = \cos \theta^J$. Clearly the vectors $X_3$ and $Y_3$ belong to $U$.

Analogously we consider $L(X_4, Z_3)$ and the vectors $X_4, Z_3$ of such 2-plane such that $(X_4, X_4)$ is a pair of orthonormal bases consistently oriented with the pair $(X_2, Z_2)$ (then $< X_2, Y_2 > > 0$). Again, let $\tilde{X}_3 = \frac{-1}{\cos \theta^J}X_3$ be the unique vector such that $\langle \tilde{X}_3, I\tilde{X}_4 \rangle = \cos \theta^I$ i.e. $L(\tilde{X}_3, \tilde{X}_4)$ is an $\omega^I$ standard 2-plane and $\tilde{Z}_3 = \frac{-1}{\cos \theta^K}Z_3$ the unique vector such that $< Z_3, KZ_4 >= \cos \theta^K$ i.e. $L(Z_3, Z_4)$ is an $\omega^K$ standard 2-plane. The vectors $\tilde{X}_3$ and $\tilde{Z}_3$ belong to $U$. Proceeding in the same way considering the oriented 2-plane $L(Y_2, Z_2)$ we determine the pair $(\tilde{Y}_4, \tilde{Z}_4)$ and consequently $(\tilde{Y}_3, \tilde{Z}_3)$. Namely one has

- the pair $(X_4, Y_4) \in L(X_2, Z_2)$ where $X_4 = Y_4 \cdot \frac{X_2}{\sqrt{1-\mu}}$, $Y_4 = -\frac{X_3 + \mu Y_3}{\sqrt{1-\mu}}$;
- the pair $(\tilde{X}_4, Z_4) \in L(X_2, Z_2)$ where $\tilde{X}_4 = \frac{X_2 - \mu X_3}{\sqrt{1-\mu^2}}$, $\tilde{Z}_4 = -\frac{X_3 + \mu Y_3}{\sqrt{1-\mu^2}}$;
- the pair $(Y_4, \tilde{Z}_4) \in L(Y_2, Z_2)$ where $\tilde{Y}_4 = \frac{Z_2 - \mu Y_3}{\sqrt{1-\mu^2}}$, $\tilde{Z}_4 = \frac{Z_3 + \mu Z_4}{\sqrt{1-\mu^2}}$.

**Proposition 5.10.**
If both $\cos \theta^I \neq 0$ and $\cos \theta^J \neq 0$ one has $X_3 = \frac{-1}{\cos \theta^J}X_3 = \frac{-1}{\cos \theta^I}X_3 = Y_3$,
if both $\cos \theta^I \neq 0$ and $\cos \theta^K \neq 0$ one has $X_3 = \frac{-1}{\cos \theta^K}X_3 = \frac{-1}{\cos \theta^J}X_3 = Z_3$,
if both $\cos \theta^J \neq 0$ and $\cos \theta^K \neq 0$ one has $X_3 = \frac{-1}{\cos \theta^K}X_3 = \frac{-1}{\cos \theta^J}X_3 = Z_3$.

**Proof.** We prove that $X_3 = Y_3$ and $(X_1, X_2, X_3, X_4)$ and $(X_1, Y_2, X_3, Y_4)$ are a pair of standard bases of $\omega^I$ and $\omega^J$ restricted to $U = L(X_1, X_2, X_3, X_4) = L(X_1, Y_2, X_3, Y_4)$ since the other proofs are similar. From the Proposition (5.5) we know that fixed any unitary vector $\tilde{X} \in U$, the cosine of the angle between the unitary vectors $\frac{-1}{\cos \theta^J}X_3$ and $\frac{-1}{\cos \theta^K}X_3$ is always the same and equals $\xi$. In particular considering the vector $X_4$, if $\tilde{Y}_3 = -\frac{1}{\cos \theta^I}X_3$ and...
\[
X_3 = -\frac{I^{-1}Pr^{IU}X_4}{\cos \theta^I}, \quad \text{one has } \langle X_3, \tilde{Y}_3 \rangle = \xi. \text{ Denoting by }
\]
\[
X_3 = -\frac{I^{-1}Pr^{IU}X_4}{\cos \theta^I}, \quad \tilde{Y}_3 = -\frac{J^{-1}Pr^{IU}X_4}{\cos \theta^J}, \quad Y_3 = -\frac{J^{-1}Pr^{IU}Y_4}{\cos \theta^J},
\]
one has
\[
\tilde{Y}_3 = -\frac{1}{\cos \theta^J}J^{-1}Pr^{IU}(\langle X_4, Y_2 \rangle > Y_2 + \langle X_4, Y_4 \rangle > Y_4) = -\frac{1}{\cos \theta^J}(\langle X_4, Y_2 \rangle + \langle -\cos \theta^JX_1 \rangle) + \langle X_4, Y_4 \rangle + \langle -\cos \theta^JY_3 \rangle =
\]
\[
= \langle X_4, Y_2 \rangle > X_1 + \langle X_4, Y_4 \rangle > Y_3.
\]
Then
\[
\xi = \langle X_3, \tilde{Y}_3 \rangle = \langle X_4, Y_2 \rangle > X_1, Y_3 >
\]
Since the orthonormal bases \((X_2, X_4)\) and \((Y_2, Y_4)\) of \(L(X_2, Y_2)\) are consistently oriented by hypothesis one has \(\langle X_4, Y_4 \rangle = \langle X_2, Y_2 \rangle = \xi\) which implies that \(\langle X_3, Y_3 \rangle = 1\ i.e.
\]
\[
X_3 = Y_3.
\]
The other equalities of the Proposition are proved in the same way.

We define the following standard bases of \(\omega^I\) and \(\omega^I\) respectively.

**Definition 5.11.** Let \(U \in \mathbb{I}C^4\), \((I, J, K)\) be an admissible basis and \(\theta^I, \theta^J, \theta^K\) the respective angles of isoclinicity. In case none among \(\xi, \chi, \eta\) is equal to \(\pm 1\) (in particular if \(U\) is neither orthogonal nor 2-planes decomposable), for any unitary \(X_1 \in U\), that we call leading vector, we define the following standard bases of \(\omega^I\) and \(\omega^K\) respectively:
\[
\{X_i\} = \{X_1, X_2\} = \frac{I^{-1}Pr^{IU}X_3}{\cos \theta^I}, \quad X_3 = -\frac{I^{-1}Pr^{IU}X_4}{\cos \theta^I}, \quad Y_3 = \frac{Y_2 - \chi X_2}{\sqrt{1 - \chi^2}},
\]
\[
\{Y_i\} = \{X_1, Y_2\} = \frac{J^{-1}Pr^{IU}X_4}{\cos \theta^J}, \quad Y_3 = -\frac{J^{-1}Pr^{IU}Y_4}{\cos \theta^J}, \quad Y_4 = \frac{Y_3 + \chi Y_2}{\sqrt{1 - \chi^2}}.
\]
the \(\omega^I\) and \(\omega^I\)-chains of \(U\) centered on \(X_1\), and the following standard bases of \(\omega^I\) and \(\omega^K\) respectively:
\[
\{\tilde{X}_i\} = \{X_1, X_2\} = \frac{I^{-1}Pr^{IU}X_3}{\cos \theta^I}, \quad \tilde{X}_3 = -\frac{I^{-1}Pr^{IU}\tilde{X}_4}{\cos \theta^I}, \quad \tilde{X}_4 = \frac{Z_2 - \chi Z_2}{\sqrt{1 - \chi^2}},
\]
\[
\{\tilde{Z}_i\} = \{X_1, Z_2\} = \frac{K^{-1}Pr^{KU}X_3}{\cos \theta^K}, \quad \tilde{Z}_3 = -\frac{K^{-1}Pr^{KU}Z_4}{\cos \theta^K}, \quad \tilde{Z}_4 = \frac{Z_2 + \chi Z_2}{\sqrt{1 - \chi^2}}.
\]
the \(\omega^I\) and \(\omega^K\)-chains of \(U\) centered on \(X_1\) and the following standard bases of \(\omega^I\) and \(\omega^K\) respectively:
\[
\{\tilde{Y}_i\} = \{X_1, Y_2\} = \frac{J^{-1}Pr^{IU}X_4}{\cos \theta^J}, \quad \tilde{Y}_3 = -\frac{J^{-1}Pr^{IU}\tilde{Y}_4}{\cos \theta^J}, \quad \tilde{Y}_4 = \frac{Z_2 - \chi Z_2}{\sqrt{1 - \chi^2}},
\]
\[
\{\tilde{Z}_i\} = \{X_1, Z_2\} = \frac{K^{-1}Pr^{KU}X_4}{\cos \theta^K}, \quad \tilde{Z}_3 = -\frac{K^{-1}Pr^{KU}Z_4}{\cos \theta^K}, \quad \tilde{Z}_4 = \frac{Z_2 + \chi Z_2}{\sqrt{1 - \chi^2}}.
\]
the \(\omega^I\) and \(\omega^K\)-chains of \(U\) centered on \(X_1\). We denote by \(\Sigma(X_1)\) the set of the six chains with leading vector \(X_1\).

Clearly \(\Sigma(X_1)\) is uniquely determined by the leading vector \(X_1\).

Let consider now the case that at least one among \((\xi, \chi, \eta)\) equals \(\pm 1\) (this is always the case if \(U\) is orthogonal (claim (5.3) or 2-plane-decomposable). Let denote by \(U_2^I = L(X_1, X_2), U_2^J = L(X_1, Y_2), U_2^K = L(X_1, Z_2)\) the \(\omega^I, \omega^J, \omega^K\)-standard 2-plane respectively determined by \(X_1\) and by \((U_1^{I}), (U_1^{J}), (U_1^{K})\) their orthogonal complements in \(U\).

Consider first the case that only one among \((\xi, \chi, \eta)\) equals \(\pm 1\). We define \(\Sigma(X_3)\) in case \(\xi = \pm 1\); the cases \(\chi = \pm 1\) and \(\eta = \pm 1\) can be treated similarly. If \(\xi = \pm 1\) one has \(U_2^I = L(X_1, X_2) = L(X_1, Y_2) = U_1^{I}\) and clearly \((U_1^{I})^{\perp} = (U_1^{J})^{\perp}\). Being \(X_3 = Y_3 \in (U_1^{I})^{\perp} \cap (U_1^{J})^{\perp}\), in this case any vector \(X_3 \in (U_1^{I})^{\perp}\) is a principal vector of the pair \((U_1^{I})^{\perp}, (U_1^{J})^{\perp}\). We assume \(X_3 = \tilde{X}_3 = \tilde{Y}_3\). Such unitary vector is determined up to sign. Moreover we assume \(Z_3 = \tilde{Z}_3\). It follows that \(Z_4 = \tilde{Z}_4\). To solve the ambiguity of sign we choose \(X_4\) in order that the pair \((X_2, Z_3)\) and \((X_2, X_4)\) are consistently oriented.

Mutatis mutandis, in case \(\chi = \pm 1\) or \(\eta = \pm 1\) we will have always \(X_3 = \tilde{X}_3 = \tilde{Z}_3 = \tilde{Y}_3\) defined up to sign, then we can give the following.

**Definition 5.12.** Let \(\{X_i\} = \{X_1, X_2, X_3, X_4\}\). If one among \((\xi, \chi, \eta)\) equals \(\pm 1\), in particular if \(U\) has a single orthogonality, we define the following chains:
\[
\xi = \pm 1: \quad \{X_i\} = \{X_i\}, \quad \{Y_i\} = \{X_1, \xi X_2, X_3, \xi X_4\} = \{\tilde{Y}_i\}, \quad \{Z_i\} = \{X_1, Z_2, X_3, Z_4\} = \{\tilde{Z}_i\};
\]
\[
\chi = \pm 1: \quad \{X_i\} = \{X_i\}, \quad \{Y_i\} = \{X_1, Y_2, X_3, Y_4\} = \{\tilde{Y}_i\}, \quad \{Z_i\} = \{X_1, Y_2, X_3, Y_4\} = \{\tilde{Z}_i\};
\]
\[
\eta = \pm 1: \quad \{X_i\} = \{X_i\}, \quad \{Y_i\} = \{X_1, Y_2, X_3, Y_4\} = \{\tilde{Y}_i\}, \quad \{Z_i\} = \{X_1, \eta Y_2, X_3, \eta Y_4\} = \{\tilde{Z}_i\}.
\]
Similarly to the choice made when \(\xi = \pm 1\), in case \(\chi = \pm 1\) or \(\eta = \pm 1\), to solve the ambiguity of sign and have a unique \(\Sigma(X_1)\), we choose \(X_4\) in order that the pair \((X_2, Y_2)\) and \((X_2, X_4)\) are consistently oriented.

Finally, since \(\eta = \xi \cdot \chi\), in case a pair among \((\xi, \chi, \eta)\) are equal to \(\pm 1\) then all of them are equal to \(\pm 1\) (namely either \(\eta = \xi = \chi = 1\) or two of them are equal to -1 and the other to 1), we have the following
**Definition 5.13.** In case $U$ is a 2-planes decomposable subspace i.e. $\xi, \chi, \eta$ are all equal to $\pm 1$ we define the following chains:

\[ \{X_i\} = \{\hat{X}_i\}, \{Y_i\} = \{X_1, \xi X_2, X_3, \xi X_4\} = \{\hat{Y}_i\}, \{Z_i\} = \{X_1, \chi X_2, X_3, \chi X_4\} = \{\hat{Z}_i\} \]

In particular if $U$ has a double or triple orthogonal, one has $\{X_i\} = \{\hat{X}_i\} = \{Y_i\} = \{\hat{Y}_i\} = \{Z_i\} = \{\hat{Z}_i\}$.

**Proposition 5.14.** The function $\Sigma : X_1 \mapsto \Sigma(X_1)$ is defined only if $U$ is not a 2-planes decomposable subspaces.

**Proof.** In case $U$ is not a 2-planes decomposable subspace the vector $X_3(= Y_3) = \hat{X}_3(= Z_3) = \hat{Y}_3(= \hat{Z}_3)$ is uniquely defined (after solving the ambiguity in sign as stated beforehand). In case $U$ is a 2-plane decomposable subspace any unitary vector in $(U^1_i)^{\perp}$ can be chosen as $X_3$ leading to a different set of chains. In fact, for any $X_i \in U$, although the decomposition of $U$ into standard 2-planes is independent from the chosen $X_3$, the $\omega^1, \omega^2, \omega^3$-standard basis in $(U^1_i)^{\perp} = (U^j_i)^{\perp} = (U^{k_i})^{\perp}$ undergo a rotation. \[\square\]

In all orthogonal cases at least one among $\xi, \chi, \eta$ equals 1. Observe that in case of double orthogonality we consider any standard basis centered on $X_i$ relative to the only non null cosine, whereas, in case of triple orthogonality of $U$, $\{X_i\} = \{Y_i\} = \{Z_i\} = \{\hat{X}_i\} = \{\hat{Y}_i\} = \{\hat{Z}_i\}$ is any orthonormal basis of the r.h.p. 2-planes decomposable subspace $U$ centered on $X_1$.

**Proposition 5.15.** Let $U \in \mathbb{C}^4$ be not a 2-planes decomposable subspace. The application mapping $\Sigma(X) \mapsto \Sigma(Y)$, where $X, Y \in U$ are a pair of leading vectors, belongs to the real representation of $Sp(n)$. Moreover $Sp(n)$ is transitive on $\Sigma$.

**Proof.** Let $U \in \mathbb{C}^4$ be not a 2-planes decomposable subspace and consider $\Sigma(X_1) = \{(X_1), \{Y_1\}, \{\hat{X}_1\}, \{\hat{Y}_1\}, \{\hat{Z}_1\}\}$. Let $C$ be a linear transformation on $U$. The images $\{CX_1, CY_1, \{C\hat{X}_1\}, \{C\hat{Y}_1\}, \{C\hat{Z}_1\}\}$ are standard bases iff $C$ is an orthogonal map and commutes with $I, J, K$ i.e. iff $C \in Sp(n)$. We prove that they are chains of $U$ centered on $CX_1 = \hat{X}_1$. In fact let $\Sigma(X_1) = \{(X_1), \{Y_1\}, \{\hat{X}_1\}, \{\hat{Y}_1\}, \{\hat{Z}_1\}\}$. The subspaces $L(CX_1, CY_2)$ (resp. $L(CX_1, CY_2), L(CX_1, CZ_2)$) are standard 2-plane and therefore $CX_2 = \hat{X}_2$, (resp. $CY_2 = \hat{Y}_2, CZ_2 = \hat{Z}_2$). As an orthogonal transformation, $C$ preserves angles and then $CX_4 = \hat{X}_4$ (resp. $CY_4 = \hat{Y}_4, CZ_4 = \hat{Z}_4$). Consequently $CX_4 = \hat{X}_4$ (resp. $CX_3 = \hat{X}_3, CZ_3 = \hat{Z}_3$) since $L(CX_3, CX_4) = L(\hat{X}_3, \hat{X}_4)$ (resp. $L(CY_3, CY_4) = L(\hat{Y}_3, \hat{Y}_4)$). The bases $(X_3, X_4)$, $(\hat{X}_3, \hat{X}_4)$ being a pair of standard 2-planes with non trivial intersection. The fact that the action is transitive follows from the transitivity of $Sp(n)$ on unitary vectors. \[\square\]

**Corollary 5.16.** Let $X_1 \in U$. For any pair of chains belonging to $\Sigma(X_1)$ the relative Gram matrix is an invariant of $U$. In particular the angles $\hat{X}_2, \hat{Y}_2, \hat{X}_2, \hat{Y}_2, \hat{Z}_2, \hat{Y}_2, \hat{Z}_2$ are invariants.

The last statement follows from the invariance of $<X_1, Y_2>, <\hat{X}_4, Z_2>, <\hat{Y}_4, Z_2>$.

W.r.t. the chains $\{X_i\}, \{Y_i\}$ and $\{\hat{X}_i\}, \{\hat{Y}_i\}$, the Gram matrices $C_{IJ} = <X_i, Y_j>$ and $C'_{IK} = <\hat{X}_i, Z_j>$ assume the form

\[
C_{IJ} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \xi & 0 & -\sqrt{1 - \xi^2} \\
0 & 1 & 0 & \xi \\
0 & \sqrt{1 - \xi^2} & 0 & \xi
\end{pmatrix}, \quad C'_{IK} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \chi & 0 & -\sqrt{1 - \chi^2} \\
0 & 1 & 0 & \chi \\
0 & \sqrt{1 - \chi^2} & 0 & \chi
\end{pmatrix}.
\]

Observe that such forms differs from the classical CS decomposition given in [16]. Even if the pairs $(X_2, Y_2)$ and $(X_2, Z_2)$ are related principal vectors (i.e. if $\xi$ and $\chi$ are both negative), the pair $(X_2, Y_4)$ (resp. $(X_2, Z_4)$) consists of related principal vector only if $\xi = 1$ (resp. $\chi = 1$).

Clearly $L(X_3, X_4) = L(\hat{X}_3, \hat{X}_4)$. The bases $(X_3, X_4)$ and $(\hat{X}_3, \hat{X}_4)$, being $\omega^1$-standard bases, are consistently oriented. Let

\[
C : \begin{pmatrix}
< X_3, \hat{X}_3 > \\
< X_4, \hat{X}_4 >
\end{pmatrix} = \begin{pmatrix}\Gamma & -\Delta \\
\Delta & \Gamma\end{pmatrix}
\]

the orthogonal matrix of the change of basis. The orthogonal matrices $C_{IJ} = <X_i, Y_j>$ and $C_{IK} = <X_i, Z_j>$ of the relative position of the basis $\{X_i\} = \{X_1, X_2, X_3, X_4\}$, $\{Y_i\} = \{X_1, Y_2, X_3, Y_4\}$ and $\{Z_i\} = \{X_1, Z_2, X_3, Z_4\}$ are given by

\[
C_{IJ} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \xi & 0 & -\sqrt{1 - \xi^2} \\
0 & 1 & 0 & \xi \\
0 & \sqrt{1 - \xi^2} & 0 & \xi
\end{pmatrix}, \quad C_{IK} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \chi & 0 & -\sqrt{1 - \chi^2} \\
0 & -\Delta \sqrt{1 - \chi^2} & \Gamma & -\chi \Delta \\
0 & \Delta \sqrt{1 - \chi^2} & \Gamma & \chi \Delta
\end{pmatrix}.
\]
To determine $\Gamma = \langle X_3, \tilde{X}_3 \rangle$, being $\langle Y_2, Z_2 \rangle = \langle Y_2, X_2 \rangle = \langle Z_2, X_2 \rangle > + \langle Y_2, X_4 \rangle = \langle Z_2, X_4 \rangle$, we get $\eta = \xi + \sqrt{1 - \xi^2} \sqrt{1 - \chi^2} \Gamma$. From the above expression, in case neither $\xi$ nor $\chi$ equal 1, we get:

$$
\Gamma = \frac{\eta - \xi}{\sqrt{1 - \xi^2} \sqrt{1 - \chi^2}} \in [-1, 1].
$$

**Proposition 5.17.** If none among $\xi, \chi, \eta$ is equal to $\pm 1$ the value of $\Gamma \in [-1, 1]$ is given in (17). If instead at least one among $\xi, \chi, \eta$ is equal to $\pm 1$ then $\Gamma = 1$. In particular this happens if $U$ is orthogonal or is a 2-planes decomposable subspace. In all cases, the pair $(\Gamma, \Delta)$ is an invariant of $U$.

**Proof.** In case none among $\xi, \chi, \eta$ is equal to $\pm 1$, the invariance of the pair $(\Gamma, \Delta)$ follows from Corollary (5.16). In case at least one among $\xi, \chi, \eta$ is equal to $\pm 1$, by construction in all chains given in the Definitions (5.12) and (5.13) it is $X_3 = \tilde{X}_3$. We remark that if $\eta = -1$ and none among $\xi, \chi$ equals $\pm 1$, the (17) is perfectly defined giving $\Gamma = -1$. The difference in sign follows from the different sign of $X_4$ in the two constructions. \qed

From Corollary (5.16), which states in particular the invariance of the triple $(\xi, \chi, \eta)$ and of the pair $(\Gamma, \Delta)$, one has the

**Proposition 5.18.** The matrices $C_{IJ}$ and $C_{IK}$ given in (16) w.r.t. the chains $\{X_i\}, \{Y_i\}$ and $\{X_i\}, \{Z_i\}$ centered on a common leading vector are invariant of $U$.

We underline the following fact.

**Remark 5.19.** The pair $(\Gamma, \Delta)$ in not an intrinsic property of $U$.

As an example to show the dependance of $(\Gamma, \Delta)$ from the admissible basis, we consider the $I$-complex 4-dimensional subspace with quaternionic Kähler angle $\cos \theta$ of Remark (5.7). In that case, if $\alpha = 0$ it is $\xi = \chi = 0$ and

$$
\Gamma = \eta = \beta_1 \gamma_1 (1 - \cos \theta)
$$

which clearly depends on $\beta_1$ and $\gamma_1$.

Since if $U$ is orthogonal one has $(\Gamma, \Delta) = (1, 0)$, we can state the following

**Corollary 5.20.** In case one among $\xi, \chi, \eta$ is equal to $\pm 1$, the matrices $C_{IJ}$ and $C_{IK}$ are given in (16) i.e. $C_{IK} = C'_{IK}$. In particular

1. If $U$ is $I$ or $J$-orthogonal one has $C_{IJ} = \text{Id}$.
2. If $U$ is $K$-orthogonal one has $C_{IK} = \text{Id}$.
3. In case of double or triple orthogonality one has $C_{IJ} = C_{IK} = \text{Id}$.

Following the definition given in (16), we give the

**Definition 5.21.** Let $U \in \mathbb{C}^4$. Fixed an admissible basis $(I, J, K)$, for any leading vector $X_1$, we call the chains $\{X_i\}, \{Y_i\}, \{Z_i\}$ (resp. the matrices $C_{IJ}$ and $C_{IK}$) determined above canonical bases (resp. canonical matrices) of the subspace $U \in \mathbb{C}^4$.

Clearly for any leading vector we have a different set of canonical bases. As explained beforehand, we denote them "canonical" since, by the invariance of $(\xi, \chi, \eta, \Delta)$, the matrices $C_{IJ}$ and $C_{IK}$ are invariants of $U \in \mathbb{C}^4$ having to the unique forms given in (16) regardless the leading vector $X_1$. We summarize the results obtained in the following

**Proposition 5.22.** Fixed an admissible basis $(I, J, K)$, to any $U \in \mathbb{C}^4$ we can associate the orthogonal canonical matrices $C_{IJ}$ and $C_{IK}$ given in (16) representing the mutual position of the canonical (standard) bases $\{X_i\}, \{Y_i\}, \{Z_i\}$ of $\omega^I, \omega^J, \omega^K$. Such matrices depend on the triple of invariants $(\xi, \chi, \eta)$ and on the sign of $\Delta = \langle X_4, \tilde{X}_3 \rangle > \pm \sqrt{1 - \Gamma^2}$ where $\Gamma = \Gamma(\xi, \chi, \eta)$ is given in (17) if none among $\xi, \chi, \eta$ is equal to $\pm 1$ else $\Gamma = 1$. The second case happens in particular if $U$ is orthogonal or a 2-planes decomposable subspace.

Then according to Theorem (2.1) we state the

**Theorem 5.23.** The invariants $(\xi, \chi, \eta, \Delta)$ together with the angles $(\theta^I, \theta^J, \theta^K)$ determine the orbit of any $U \in \mathbb{C}^4$. In particular if $U$ is orthogonal or 2-planes decomposable (in which case $(\Gamma, \Delta) = (1, 0)$) the first set reduces to the pair $(\xi, \chi)$.

In particular, from the Definition (5.12) and the Corollary (5.20) one has

**Proposition 5.24.**

1. In case $U$ is $I$-orthogonal (resp. $J$-orthogonal, resp. $K$-orthogonal), the $\text{Sp}(n)$-orbit is characterized by the triple $(\chi, \theta^I, \theta^K)$ (resp. $(\chi, \theta^J, \theta^K)$, resp. $(\xi, \theta^I, \theta^J)$).
2. In case of double orthogonality of $U$, the $\text{Sp}(n)$-orbit depends on the only non null cosine.
3. All subspaces with a triple orthogonality (i.e. all r.h.p. subspaces) share the same $\text{Sp}(n)$-orbit.
We complete the analysis of 4-dimensional isoclinic subspaces, by giving some examples of orthogonal and non orthogonal cases.

Let consider an $I$-complex 4-dimensional subspace $U$. In [19] we proved that $AU$ is the same for any $A \in I^{\perp} \cap S(Q)$ and we called $I^{\perp}$-Kähler angle and denoted by $\theta^{\perp}$ the angle of isoclinicity of the pair $(U, AU)$. Then w.r.t. an adapted basis $(I, J, K)$ one has $(\cos \theta^{I}, \cos \theta^{J}, \cos \theta^{K}) = (1, \theta^{I^{\perp}}, \theta^{J^{\perp}})$. This is a typical elements of the set $\mathcal{IC}^4$. In particular in the first example we consider the non orthogonal case i.e. the case when the $I^{\perp}$-Kähler angle of $U$ is not $\pi/2$. The results obtained thereof apply also to a 4-dimensional quaternionic subspace which is a particular case of a non totally $I$-complex subspace characterized by $(\cos \theta^{I} = \cos \theta^{J} = \cos \theta^{K} = 1)$. W.r.t. any admissible basis in the quaternionic case $Pr^{\text{Ad}}X_1 = X_1$, $\forall A \in S(Q)$, $\forall X \in U$ then $X_2 = -IX_1$, $Y_2 = -JX_1$, $Z_2 = -KX_1$ which implies $\xi = \chi = \eta = 0$.

More generally, in [19] we proved that the triple $(X_2, Y_2, Z_2)$ of a 4-dimensional $I$-complex subspace $U$ is orthonormal iff the admissible basis is an adapted one. Then, w.r.t. any adapted basis $(I, J, K)$, in both $I$-complex and quaternionic case, $\xi = \chi = \eta = 0$ which implies $\Gamma = <X_3, \tilde{X}_3> = 0$. To compute $\Delta$, we apply the formulas given in section 5.2. One has

$$X_4 = Y_2, \ Y_4 = -X_2, \ \tilde{X}_4 = Z_2, \ Z_4 = -X_2.$$  

Furthermore

$$\tilde{X}_3 = -I^{-1}Pr^{JU}X_4 = IX_4 = IZ_2.$$  

Therefore the set $(\xi, \chi, \eta, \Gamma, \Delta) = (0, 0, 0, 0, -1)$ is an invariant (resp. an intrinsic property) of an $I$-complex subspace (resp. quaternionic subspace). The chains $\{X_1\}, \{Y_1\}, \{\tilde{X}_1\}, \{Z_1\}$ of an $I$-complex subspace with leading vector $X_1 \in U$ w.r.t. the adapted basis $(I, J, K)$ are:

\begin{align*}
\{X_1\} &= \{X_1, X_2, X_3, X_4\} = \{X_1, -IX_1, Z_2, -IZ_2\} = \{X_1, X_2, Z_2, Y_2\} \\
\{Y_1\} &= \{X_1, Y_2, X_3, Y_4\} = \{X_1, -IZ_2, Z_2, IX_1\} = \{X_1, Y_2, Z_2, -X_2\} \\
\{\tilde{X}_1\} &= \{X_1, X_2, \tilde{X}_3, \tilde{X}_4\} = \{X_1, -IX_1, IZ_2, Z_2\} = \{X_1, X_2, -Y_2, Z_2\} \\
\{Z_1\} &= \{X_1, Z_2, X_3, Z_4\} = \{X_1, Z_2, IZ_2, IX_1\} = \{X_1, Z_2, -Y_2, -X_2\}
\end{align*}

In particular for a quaternionic subspace it is $Y_2 = -JX_1$ and $Z_2 = -KX_1$ then

\begin{align*}
\{X_1\} &= \{X_1, -IX_1, -KX_1, -JX_1\} \\
\{Y_1\} &= \{X_1, -JX_1, -KX_1, IX_1\} \\
\{\tilde{X}_1\} &= \{X, -IX_1, JX_1, -KX_1\} \\
\{Z_1\} &= \{X_1, -KX_1, JX_1, IX_1\}
\end{align*}

W.r.t. the canonical bases $\{X_1\}, \{Y_1\}, \{\tilde{X}_1\}$ the canonical matrices [10] of an $I$-complex 4-dimensional subspace are

\begin{align*}
C_{IJ} = C_{IK} &= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}, \\
C_{IK} &= \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0
\end{pmatrix}.
\end{align*}

We give the following Proposition whose result is in accordance with what stated [19].

**Proposition 5.25.** All 4-dimensional quaternionic subspaces are in the same $Sp(n)$-orbit, whereas the $I^{\perp}$-Kähler angle $\theta^{\perp} = \theta^{J}$ determines the $Sp(n)$-orbit of an $I$-complex subspace where $(I, J, K)$ is an adapted basis.

We give a pair of examples for the orthogonal case where we recall $(\Gamma, \Delta) = (1, 0)$: we consider 4-dimensional totally complex subspaces and 4-dimensional real Hermitian product subspaces.

In the first case $(\cos \theta^{I} = 1, \cos \theta^{J} = \cos \theta^{K} = \theta^{I^{\perp}} = 0)$ then the 4-dimensional totally complex subspaces represent a case of double orthogonality. According to point 2 of the Proposition 5.24, $\xi = \chi = \eta = 1$, $C_{IJ} = C_{IK} = Id$ and the $Sp(n)$-orbit is determined by the only non null cosine of the angle of isoclinicity which equals 1 for all totally complex subspaces. Therefore

**Proposition 5.26.** Let $I \in S(Q)$. All 4-dimensional totally $I$-complex subspaces share the same $Sp(n)$-orbit.

This is accordance with the result that appear in [19]. We give an additional prove of this fact. In [18] we proved that a pair of subspaces $U, W$ are in the same orbit if there exist a basis in $U$ and one in $W$ w.r.t. which the matrices
$H$ of the Hermitian products are the same. W.r.t. a standard basis $B = \{X_1, X_2, X_3, X_4\}$ of $\omega^1|_U$, computing the Hermitian product given in (2), one has

$$H|_U = \begin{pmatrix}
1 & i & 0 & 0 \\
-i & 1 & 0 & 0 \\
0 & 0 & 1 & i \\
0 & 0 & -i & 1
\end{pmatrix}$$

which is clearly independent from $B$.

The case of a 4-dimensional real Hermitian product subspaces (r.h.p.s.), is the only case of triple orthogonality being $(\cos \theta^I, \cos \theta^J, \cos \theta^K) = (0, 0, 0)$. Applying the results stated in point (3) of the Proposition 5.24, the canonical matrices $C_{IJ}$ and $C_{IK}$ of all r.h.p. subspaces w.r.t. any admissible basis $(I, J, K)$ and any orthonormal (standard and canonical) basis are the identity matrix. We have then the confirmation that

**Proposition 5.27.** All 4-dimensional r.h.p. subspaces share the same $Sp(n)$-orbit.

6. Isoclinic subspaces of dimension $2m > 4$

The purpose of this article is to determine the $Sp(n)$-orbit of an isoclinic subspace $U \in \mathbb{C}^{2m}$ in the real Grassmannian $G^2(2m, 4n)$. After having studied the 2 and 4-dimensional cases, in the following we consider $m > 2$. We prove that, similar to the result achieved in dimension 4, to any $U \in \mathbb{C}^{2m}$, $m > 2$ we can associate two sets of invariants: a triple $(\xi, \chi, \eta)$ and the pair $(\Gamma, \Delta)$ which determine the canonical form of the matrices $C_{IJ}, C_{IK}$. In fact, since $\Gamma$ is a function of $(\xi, \chi, \eta)$, according to Theorem (2.3) appeared in [18], the vector $(\xi, \chi, \eta, \Delta)$, and the triple $(\theta^I, \theta^J, \theta^K)$ of the angles of isoclinicity w.r.t. a fixed admissible basis $(I, J, K)$ determine the $Sp(n)$-orbit of $U$.

6.1. The triple of invariants $(\xi, \chi, \eta)$. Let $U \in \mathbb{C}^{2m}$ and $(\theta^I, \theta^J, \theta^K)$ be the angles of isoclinicity w.r.t. the admissible basis $(I, J, K)$. Let $X_1 \in U$ be a generic unitary vector. If $U$ is not orthogonal the triple $(X_2, Y_2, Z_2)$ is given in (11) otherwise it is obtainable from Claim (5.5). Generalizing the result of the Proposition 5.24, one has the following

**Proposition 6.1.** Let $U \in \mathbb{C}^{2m}$. For any vector $X_1 \in U$ and fixed an admissible basis $(I, J, K)$ the triple $(\xi = < X_2, Y_2 >, \chi = < X_2, Z_2 >, \eta = < Y_2, Z_2 >)$ is an invariant of $U$. Such triple is not an intrinsic property of $U$.

**Proof.** In case $U$ is orthogonal we first consider the assumptions of the Claim (5.5). Let $A = \alpha_1 I + \alpha_2 J + \alpha_3 K$, $A \in S(\mathbb{Q})$. Calculating as in the proof of the position (5.2) regarding the 4-dimensional case, and considering for example the first row of the matrices of $\omega^I, \omega^J, \omega^K$ w.r.t. some orthonormal basis of $U$ whose entries we denote by $a_i, a_i', a_i^\prime$, $i = 1, \ldots, 2m$ respectively, one has

$$\cos^2 \theta_A = \alpha^2_1 \cos^2 \theta^I + \alpha^2_2 \cos^2 \theta^J + \alpha^2_3 \cos^2 \theta^K + 2\alpha_1 \alpha_2 (\sum_{i=1}^{2m} a_i a_i') + 2\alpha_1 \alpha_3 (\sum_{i=1}^{2m} a_i a_i') + 2\alpha_2 \alpha_3 (\sum_{i=1}^{2m} a_i a_i').$$

By the hypothesis of isoclinicity, for any $A \in \mathbb{Q}$, the quantities $\sum_{i=1}^{2m} a_i a_i', \sum_{i=1}^{2m} a_i a_i', \sum_{i=1}^{2m} a_i a_i'$ do not depend on $X_1$. As in the 4-dimensional case in fact we can assume that one among $\alpha_1, \alpha_2, \alpha_3$ is zero and the result follows.

Considering in particular a triple of standard bases $\{X_1\}, \{Y_i\}, \{Z_i\}$ centered on a common vector $X_1 = Y_1 = Z_1$ it is straightforward to verify that

$$\sum_{i=1}^{2m} a_i a_i' = \cos \theta^I < X_1, JX_2 >= \cos \theta^I < X_1, JY_2 >= < X_2, Y_2 > \cos \theta^I \cos \theta^J,$$

$$\sum_{i=1}^{2m} a_i a_i' = \cos \theta^I < X_1, JX_2 >= \cos \theta^I < X_1, JZ_2 >= < X_2, Z_2 > \cos \theta^I \cos \theta^K,$$

$$\sum_{i=1}^{2m} a_i a_i' = \cos \theta^I < X_1, JX_2 >= \cos \theta^I < X_1, JZ_2 >= < Y_2, Z_2 > \cos \theta^I \cos \theta^K.$$

For instance for the first equality

$$\cos \theta^I < X_1, JX_2 >= \cos \theta^I < Y_1, J \sum_{i=1}^{2m} < X_2, Y_i > Y_i > >= \cos \theta^I \cos \theta^J < X_2, Y_2 >$$

and the stated invariance of the triple $(\xi, \chi, \eta)$ from the leading vector is proved.

As an example of the dependence of $(\xi, \chi, \eta)$ on the admissible basis we can consider a $2m$-dimensional $I$-complex subspace. In [11] we showed that it admits a decomposition into 4-dimensional $I$-complex subspaces. Such addends are mutually Hermitian orthogonal i.e. they belong to orthogonal quaternionic subspaces. Then the conclusion follows from Remark 5.7.

We will show afterwards in the Proposition 6.29 that, like in the 4-dimensional case, the content of the Proposition 6.1 is part of a more general result.
6.2. The associated subspaces of type $U^{IJ}, U^{IK}, U^{JK}$ of $U \in \mathcal{I}^{2m}$.
Let $U \in \mathcal{I}^{2m}$ and $(\theta^I, \theta^J, \theta^K)$ be the angles of isoclinicity w.r.t. the admissible basis $(I, J, K)$.

**Definition 6.2.** We say that a subspace $W \subset U \in \mathcal{I}^{2m}$ is of type $U^{IJ}$ (resp. of type $U^{IK}$, resp. of type $U^{JK}$) if the pair $(W, JW)$ (resp. $(W, JW)$ and $(W, KW)$, resp. $(W, JW)$ and $(W, KW)$) are isoclinic with angles $(\theta^I, \theta^J, \theta^K)$ (resp. $(\theta^I, \theta^K)$, resp. $(\theta^J, \theta^K)$).

When dealing with the 4-dimensional case, to any leading vector $X_1 \in U$ we associated a set of six chains. According to the Proposition 5.13, only if $U$ is not a 2-planes decomposable subspace we defined the vector $\Sigma(X_1)$ whose entries are exactly the six chains given in the Definition 5.11 if none among $\xi, \chi, \eta$ is equal to $\pm 1$ or in Definition 5.12 if only one among $\xi, \chi, \eta$ is equal to $\pm 1$. In these cases in fact the correspondence $X_1 \mapsto \Sigma(X_1)$ is one to one. Moreover we recall that in the second case, which happens in particular is $U$ has a single orthogonality, one has $\Gamma = 1$ (see the Proposition 5.17).

Applying the same procedure described in subsection 5.2, fixed a unitary vector $X_1 \in U$, we build the chains given in the Definitions 5.11 and 5.12 in case $U$ is not 2-planes decomposable and the ones defined in the Definition 5.13 otherwise. The result of the Proposition 5.10 is clearly still valid i.e.

$$X_3 = Y_3, \quad \tilde{X}_3 = \tilde{Z}_3, \quad \tilde{Y}_3 = \tilde{Z}_3$$

In particular if at least one among $\xi, \chi, \eta$ equals $\pm 1$, one has that $X_3 = Y_3 = \tilde{X}_3 = \tilde{Y}_3 = \tilde{Z}_3$ which implies that $(\Gamma, \Delta) = (1, 0)$.

Denoting by $\{X_1\}_{R}, \{Y_1\}_{R}, \{\tilde{X}_1\}_{R}, \{\tilde{Z}_1\}_{R}$ the linear span of the chains built on a common leading vector, one has that, if $\dim U = 4$, $\{X_1\}_{R} = \{Y_1\}_{R} = \{\tilde{X}_1\}_{R} = \{\tilde{Z}_1\}_{R} = U$.

This is not in general the case when $2m > 4$. By construction however $\{X_1\}_{R} = \{Y_1\}_{R}$ and then, according to the Definition 6.2, such subspace is of type $U^{IJ}$. We will denote it by $U^{IJ}(X_1)$. Analogously by construction $\{\tilde{X}_1\}_{R} = \{\tilde{Z}_1\}_{R}$ is of type $U^{IK}$ and $\{Y_1\}_{R} = \{\tilde{Z}_1\}_{R}$ is of type $U^{JK}$. We will denote them by $U^{IK}(X_1)$ and $U^{JK}(X_1)$ respectively.

**Proposition 6.3.** If at least one among $(\xi, \chi, \eta)$ is equal to $\pm 1$, in particular if $U$ is orthogonal, one has $U^{IJ}(X_1) = U^{IK}(X_1) = U^{JK}(X_1) \in \mathcal{I}^4$ with same angles of isoclinicity of $U$. Furthermore if only one among $(\xi, \chi, \eta)$ is equal to $\pm 1$, in particular if $U$ has a single orthogonality, such subspace is unique. This is not the case if $U$ is 2-planes decomposable.

**Proof.** If at least one among $(\xi, \chi, \eta)$ is equal to $\pm 1$, from the Definition 5.12 and 5.13, one has $\{X_1\} = \{\tilde{X}_1\}$ and the first statement follows. In case $U$ is not 2-planes decomposable (in particular if $U$ has a single orthogonality) the unicity follows from the unicity of $\Sigma(X_1)$. Otherwise (in particular if $U$ has a double or triple orthogonality) let $X_3, X_3 \in (U^I)^2 \cap U$ and denote by $U^I_2 = L(X_3, X_4), U^J_2 = L(X_3, X_4)$ the $\omega^I, \omega^J$-standard 2-planes containing $X_3$ and $\tilde{X}_3$ respectively. One has that $U^I_1 \oplus U^I_2$ and $U^J_1 \oplus U^J_2$ are a pair of different 4-dimensional isoclinic subspace unless $X_3 \in L(X_3, X_4)$.

**Definition 6.4.** Let $U \in \mathcal{I}^{2m}$. We call associated subspaces of $U$ of type $U^{IJ}$ and $U^{IK}$ (resp. of type $U^{IJ}$ and $U^{JK}$, resp. of type $U^{IK}$ and $U^{JK}$) a pair of 4-dimensional subspaces of type $U^{IJ}$ and $U^{IK}$ (resp. of type $U^{IJ}$ and $U^{JK}$, resp. of type $U^{IK}$ and $U^{JK}$) generated by the $\omega^J$ and $\omega^J$ (resp. $\omega^I$ and $\omega^K$, resp. $\omega^J$ and $\omega^K$) chains centered on a common vector.

Clearly if one among $\xi, \chi, \eta$ equals $\pm 1$ and $X_1 \in U$, any pair among $U^{IJ}(X_1), U^{IK}(X_1), U^{JK}(X_1)$ are associated subspaces. In the following we will need only associated subspaces of type $U^{IJ}$ and $U^{IK}$. Anyway all properties given in the next proposition for subspaces of one kind (for example $U^{IJ}$) or for a pair of subspaces of different kind (for instance $U^{IJ}$ and $U^{IK}$) can be extended to all 3 kinds or to all three pairs.

**Proposition 6.5.** Let $U \in \mathcal{I}^{2m}$.

1. Let $U \in \mathcal{I}^{2m}$ be not 2-planes decomposable. Let $X \in U$ and consider $U^{IJ}(X)$. Then for any $X_1 \in U^{IJ}(X)$ one has $U^{IJ}(X) = U^{IJ}(X_1)$.

   **Proof.** If none among $\xi, \chi, \eta$ equals $\pm 1$, for any $X_1 \in U^{IJ}(X)$ the standard 2-planes $L(X_1, X_2 = \frac{1}{\cos \theta^I} X_1)$ and $L(X_1, Y_2 = \frac{1}{\cos \theta^J} X_2)$ are both in $U^{IJ}(X)$. Then also the pair $(X_4, Y_4)$ belonging to $L(X_2, Y_2)$ is in $U^{IJ}(X)$ and consequently $X_3 = Y_3$ is in $U^{IJ}(X)$. Mutatis mutandis, the same proof can be used for $U^{IK}(X)$ or $U^{JK}(X)$. In case only one among $\xi, \chi, \eta$ equals $\pm 1$, suppose without lack of generality that $\chi = \pm 1$ and consider the uniquely determined $U^{IJ}(X_1)$. Applying the proof above, one has that $U^{IJ}(X_1) = U^{IJ}(X)$. By the Proposition 6.2 one has $U^{IJ}(X_1) = U^{IK}(X_1) = U^{JK}(X_1)$ and we can extend such property to $U^{IK}(X)$ and $U^{JK}(X)$.

2. The orthogonal sum of subspaces of $U$ of type $U^{IJ}$ is a subspace of type $U^{IJ}$. Furthermore, let $W_1 \subset U^{2m}$ be a subspace of type $U^{IJ}$. The subspaces $W_1 \cap U$ is of type $U^{IJ}$. 
Proof. Straightforward. □

(3) Let $U \in I\mathcal{C}^{2m}$ be not 2-planes decomposable. If a pair of 4-dimensional subspaces $U_1, U_2$ of $U$ of type $U^{1J}$ has no trivial intersection then $U_1 = U_2$. As a consequence for any $U \supset X \notin U_1$ one has $U_1 \cap U^{1J}(X) = \{0\}$. In particular if $X \in U_1^\perp$ all vectors of the $\omega^J$ and $\omega^J$ chains centered on $X$ are in $U_2^\perp$.

Proof. The first statement is straightforward from point [1]. For the second, let $X_1 \in U_1^\perp$. Without lack of generality we can suppose that both $\cos \Theta$ and $\cos \Theta'$ are not zero. Then $X_2 = \frac{1}{\cos \Theta} X \in U_2^\perp$ since, from previous point, $U_2^\perp$ is of type $U^{1J}$ and for the same reason also $Y_2 = \frac{1}{\cos \Theta'} X \in U_1^\perp$. Then the 2-plane $L(X_2, Y_2) \subset U_2^\perp$ which implies that the pair $(X_4, Y_4)$ is in $U_1^\perp$ as well as $X_3 = Y_3$. Then $L(X_1, X_2) \supset L(X_3, X_4) \subset U_1^\perp$.

(4) Let $U^{1J}(X_1) = L(X_1, X_2, X_3, X_4)$ and $U^{1K}(X_1) = L(X_1, X_2, \tilde{X}_3, \tilde{X}_4)$ be a pair of associated subspaces. Then either $U^{1J}(X_1) \cap U^{1K}(X_1) = L(X_1, X_2)$ or $U^{1J}(X_1) = U^{1K}(X_1)$. In the last case $U^{1J}(X_1) = U^{1K}(X_1) \in I\mathcal{C}^4$ (with same angles of isoclinicity of $U$).

Proof. In case at least one among $\xi, \chi, \eta$ equals $\pm 1$, from the Proposition [5,6], one has that $U^{1J}(X_1) = U^{1K}(X_1) = U^{1K}(X_1)$. Otherwise the 4-dimensional subspaces $U^{1J}(X_1)$ and $U^{1K}(X_1)$ intersect by construction in $L(X_1, X_2)$ since $L(X_1, X_2)$ is a standard 2-plane of $\omega^J$ in both of them. Suppose $S \in U^{1J}(X_1) \cap U^{1K}(X_1)$ with $S \notin L(X_1, X_2)$. Then $S = S_1 + S_2$ with $S_1 \in L(X_1, X_2)$ and $S_2 \in L(X_1, X_2)^\perp = L(X_3, X_4)$ as a vector of $U^{1J}(X_1)$ and $S = T_1 + T_2$ with $T_1 \in L(X_1, X_2)$ and $T_2 \in L(X_1, X_2)^\perp = L(X_3, X_4)$ as a vector of $U^{1K}(X_1)$. Being $S_1 = T_1$, by the uniqueness of the orthogonal decomposition, it follows $S_2 = T_2$. Then $L(X_3, \tilde{X}_3) \cap L(X_3, \tilde{X}_4) \neq \{0\}$, which implies that $L(X_3, \tilde{X}_4) = L(X_3, X_4)$ i.e. $U^{1J}(X_1) = U^{1K}(X_1)$.

(5) Let $X_1 \in U$ unitary and $U^{1J}(X_1)$ and $U^{1K}(X_1)$ be a pair of associated 4-dimensional subspaces. Then either $U^{1J}(X_1) = U^{1K}(X_1)$ i.e. $U^{1J}(X_1) \in I\mathcal{C}^4$ with same angles of isoclinicity of $U$ or there exists a $1:1$ correspondence between standard 2-planes of $\omega^J|_{U^{1J}(X_1)}$ and associated subspaces of type $U^{1K}$.

Proof. From previous point either $U^{1J}(X_1) \cap U^{1K}(X_1) = L(X_1, X_2)$ or $U^{1J}(X_1) = U^{1K}(X_1)$ in which case $U^{1J}(X_1) \in I\mathcal{C}^4$ (with same angles of isoclinicity of $U$). Suppose $U^{1J}(X_1) \notin I\mathcal{C}^4$. Let $X_1 \in U^{1J}(X_1)$, $\tilde{X}_1 \notin L(X_1, X_2)$. Clearly $U^{1K}(X_1) \neq U^{1J}(X_1) = U^{1J}(X_1)$ by the assumption that $U^{1J}(X_1) \notin I\mathcal{C}^4$. We prove that $U^{1K}(X_1) \neq U^{1K}(X_1)$. In fact if $U^{1K}(X_1) = U^{1K}(X_1)$ one has that $\tilde{X}_1 \in U^{1K}(X_1)$ and consequently by previous point $U^{1K}(X_1) = U^{1J}(X_1)$ contradiction.

Let $X_1 \in U$ unitary and $U^{1J}(X_1)$ and $U^{1K}(X_1)$ be the associated subspaces. It is straightforward to verify that the Proposition [6,16] and the Proposition [6,18] can be applied also to the 4-dimensional subspaces of type $U^{1J}, U^{1K}, U^{1K}$ leading to the invariance of the Gram matrices $g_{1J} = < X_1, Y_1 >$ and $g_{1K} = < X_1, Z_1 >$ w.r.t. the chains $\{X_1\}, \{Y_1\}, \{Z_1\}$ centered on $X_1$. The form of such matrices is still given in [19]. In this case the subspaces $L(X_3, X_4)$ and $L(\tilde{X}_3, \tilde{X}_4)$ are in general different as well as the pairs of associated subspaces $U^{1J}(X_1)$ and $U^{1K}(X_1)$. Therefore $g_{1K}$ is not in general an orthogonal matrices (differently from $g_{1J}$). Recalling that from the Proposition [6,17] one has that $\Gamma = < X_3, \tilde{X}_3 >$ is given in [17] in case none among $\xi, \chi, \eta$ equals $\pm 1$ and $\Gamma = 1$ otherwise we state the following

**Proposition 6.6.** The Gram matrix $(L(X_3, X_4) \times L(\tilde{X}_3, \tilde{X}_4))$ is given by

\[
\begin{pmatrix}
< X_3, \tilde{X}_3 > & < X_3, \tilde{X}_4 > \\
< X_4, \tilde{X}_3 > & < X_4, \tilde{X}_4 > 
\end{pmatrix}
\begin{pmatrix}
\Gamma & -\Delta \\
\Delta & \Gamma 
\end{pmatrix}
\]

Therefore the 2-planes $L(X_3, X_4)$ and $L(\tilde{X}_3, \tilde{X}_4)$ are isoclinic with the cosine of the angle of isoclinicity $\theta$ given by $\cos \theta = \sqrt{\Gamma^2 + \Delta^2}$.

Proof. We prove that $< X_3, \tilde{X}_4 > = -< X_3, \tilde{X}_4 >$. In fact

\[
< X_3, \tilde{X}_4 > = < I X_3, I \tilde{X}_4 > = -< \frac{P^{1U} X_4}{\cos \Theta}, I \hat{X}_4 > = -< \frac{P^{1U} X_4}{\cos \Theta} > =< \frac{1}{\cos \Theta} X_4, \hat{X}_4 > = < \hat{X}_4, \tilde{X}_4 > = < \hat{X}_4, \tilde{X}_4 > > = -< X_3, \tilde{X}_4 >
\]

Let now compute

\[
< X_4, \hat{X}_4 > = < Y_2 - \xi X_2, Z_2 - \chi X_2 > = < Y_2 - \xi X_2, Z_2 - \chi X_2 > = \eta - \xi \chi - \xi \chi + \chi \chi = \eta - \chi \xi = \frac{\eta - \chi \xi}{\sqrt{1 - \xi^2} \sqrt{1 - \chi^2}}
\]

□
It is easy to check that $\Delta = <X_4, \bar{X}_3>$ is given by the following equivalent expressions where $\epsilon = \frac{1}{\cos \theta^t \sqrt{1 - \xi^2} \sqrt{1 - \chi^2}}$:

$$\Delta = <X_4, \bar{X}_3> = \frac{1}{\cos \theta^t} \frac{<X_4, I\bar{X}_3>-<X_3, I\bar{X}_4>-\epsilon <Y_2, I\bar{X}_4>-\epsilon <Y_2, IZ_2>}{\sqrt{1 - \chi^2}} = \frac{1}{\cos \theta} <P^I U X_1, P^K U X_3> = \frac{1}{\cos \theta} <P^I U \tilde{X}_4, P^I U X_3> = -\epsilon <Y_2, IZ_2>.$$ 

The Proposition (5.15) and the Proposition (5.18) lead to the following strong characterization of the subspaces of type $U^{IJ}$.

**Proposition 6.7.** Let $U \in IG_{2m}$. All 4-dimensional subspaces of $U$ of type $U^{IJ}$ have the same value of the pair $(\Gamma, \Delta)$ (w.r.t. the associated subspaces of type $U^{IK}$).

In the Proposition (5.5) we stated that given a unitary vector $X_1 \in U$ and determined the associated pair $U^{IJ}(X_1), U^{IK}(X_1)$, for any vector $\bar{X} \in U^{IJ}(X_1)$ one has that $U^{IJ}(\bar{X}) = U^{IJ}(X_1)$ (point (1)) whereas, unless $\bar{X} \notin L(X_1, X_2)$, it is $U^{IK}(\bar{X}) \neq U^{IK}(X_1)$ (point (3)). Nevertheless, from the Proposition 6.7 it follows the interesting

**Corollary 6.8.** Let $U^{IJ}(X_1) \subset U$ be a 4-dimensional subspace of type $U^{IJ}$. Suppose that $U^{IJ}(X_1) \neq U^{IK}(X_1)$ and denote by $U^{IK}$ the different subspaces associated to $U^{IJ}(X_1)$ determined by different unitary vectors of $U^{IJ}(X_1)$. The cosines of the principal angles between the pairs of associated subspaces ($U^{IJ}(X_1), U^{IK}$) are always given by $(1, 1, \sqrt{\Gamma^2 + \Delta^2}, \sqrt{\Gamma^2 + \Delta^2})$.

Applying the Proposition 5.15 we extend the result of the Proposition 5.22 to state the following

**Proposition 6.9.** Let $\{X_i\},\{Y_i\},\{Z_i\}$ be the triple of $\omega^i, \omega^j, \omega^K$-chains centered on $X_1 \in U$. The Gram matrices $g_{IJ} = <X_i, Y_j>$ and $g_{IK} = <X_i, Z_j>$ are invariants of $U$.

We recall that the form of these matrices is given in (16) and that in general $g_{IK}$, differently from $g_{IJ}$, is not an orthogonal matrix.

Let $U \in IG_{2m}$ and let $U^{IJ}(X_1) = L(X_1, X_2, X_3, X_4)$ and $U^{IK}(X_1) = \{\bar{X}_i\} = (X_1, X_2, \bar{X}_3, \bar{X}_4)$ be the pair of associated subspaces centered on $X_1$.

**Proposition 6.10.** Chosen $X_1 \in U$, the 4-dimensional subspace $U^{IJ}(X_1)$ is isoclinic with $KU^{IJ}(X_1)$. Then for any leading vector $X_1$ all subspaces $U^{IJ}(X_1) \in IG^4$ with angles of isoclinicity $(\theta^i, \theta^j, \gamma)$ where

$$\cos^2 \gamma = <X_1, KX_2>^2 + <X_1, KX_3>^2 + <X_1, KX_4>^2 = \cos \theta^K \sqrt{1 - \chi^2},$$

all 4-dimensional subspaces of $U$ of type $U^{IJ}$ share the same matrix of $\omega^K$ w.r.t. the generating chain.

**Proof.** If some among $\xi, \chi, \eta$ equals $\pm1$ from the Proposition (6.3), $U^{IJ}(X_1) = U^{IK}(X_1)$ is a 4-dimensional isoclinic subspace with same angles $(\theta^i, \theta^j, \gamma)$ of $U$. Consider then the case that none among $\xi, \chi, \eta$ equals $\pm1$ (in particular the case that $U$ is not orthogonal). From Corollary 6.8 we have that the cosines of the principal angles of the pair $(U^{IJ}(X_1), U^{IK}(X_1))$ are given by $(1, 1, \sqrt{\Gamma^2 + \Delta^2}, \sqrt{\Gamma^2 + \Delta^2})$ with $\gamma$ given in (17) and

$$\Delta = <X_4, \bar{X}_3> = <X_4, \bar{X}_3> = <X_3, KX_4>.$$ 

This is another expression for $\Delta$ besides the ones given in (10). We have then that the cosine $<X_1, KX_3>$ is an invariant of $U$.

Let consider a new leading vector $\bar{X}_1 \in U^{IJ}(X_1)$ which in the general case will be $\bar{X}_1 = aX_1 + bX_2 + cX_3 + dX_4$, $a^2 + b^2 + c^2 + d^2 = 1$. One has

$$\bar{X}_2 = \frac{1}{\cos \theta^j} X_2 = (aX_2 - bX_1 + cX_4 - dX_3);$$

$$\bar{Y}_2 = \frac{1}{\cos \theta^j} Y_2 = (aY_2 + b\sqrt{1 - \xi^2} Y_3 + cY_4 + d\sqrt{1 - \xi^2} X_3).$$

Let compute $\frac{1}{\cos \theta^j} X_2$ and $\frac{1}{\cos \theta^j} Y_2$. In order that $L(X_2, X_4)$ and $L(Y_2, Y_4)$ have the same orientation it is:

$$X_2 = \xi Y_2 - \sqrt{1 - \xi^2} Y_4,$$

$$X_4 = \sqrt{1 - \xi^2} Y_2 + \xi Y_4.$$ 

Then computing

$$\frac{1}{\cos \theta^j} X_2 = -\xi X_1 + \sqrt{1 - \xi^2} X_3,$$

$$\frac{1}{\cos \theta^j} X_4 = -\sqrt{1 - \xi^2} X_1 - \xi X_3.$$ 

Then

$$\bar{Y}_2 = aY_2 + b(-\xi X_1 + \sqrt{1 - \xi^2} X_4) + cY_4 + d(-\sqrt{1 - \xi^2} X_1 - \xi X_3) =$$

and being $Y_2 = \xi X_2 + \sqrt{1 - \xi^2} X_4$ and $Y_4 = -\sqrt{1 - \xi^2} X_2 + \xi X_4$ one has

$$\bar{Y}_2 = (-b\xi + d\sqrt{1 - \xi^2}) X_1 + (a\xi + b\sqrt{1 - \xi^2}) X_2 + (b\sqrt{1 - \xi^2} - d\xi) X_3 + (a\sqrt{1 - \xi^2} + c\xi) X_4.$$
One can verify that $|\bar{Y}_2| = 1$ and that $< \bar{X}_2, \bar{Y}_2 > = \xi$. Being $\bar{X}_4 = \frac{X_2 - \xi X_1}{\sqrt{1 - \xi^2}}$, one has
$$\bar{X}_4 = -dX_1 - cX_2 + bX_3 + aX_4$$
and
$$\bar{X}_3 = -\frac{I^{-1}PU \bar{X}_4}{\cos \theta^I} = -cX_1 + dX_2 + aX_3 - bX_4.$$ 

Through simple calculations one has
$$< \bar{X}_1, K\bar{X}_3 > = (ad + bc)[< X_1, KX_2 > - < X_3, KX_4 >] + (ab - cd)[< X_2, KX_3 > - < X_1, KX_4 >] = +\cos (a^2 + \xi^2) < X_1, KX_3 > - b^2 < X_2, KX_4 > + ab < X_2, KX_3 > - < X_1, KX_4 >.$$ 

This is the value of $< \bar{X}_1, K\bar{X}_3 > = < X_1, KX_3 >$ for any chain centered on a unitary vector $\bar{X}_1 \in U^{1J}(X_1)$. In particular such expression must be valid in case $\bar{X}_1 \in L(X_1, X_2)$ i.e. in case $c = d = 0$. In such case, it is
$$< \bar{X}_1, K\bar{X}_3 > = +a^2 < X_1, KX_3 > - b^2 < X_2, KX_4 > + ab < X_2, KX_3 > - < X_1, KX_4 >$$

which implies
$$< X_1, KX_3 > = < X_1, KX_3 >, \quad < X_1, KX_3 > = < X_2, KX_4 >.$$ 

The matrix of $\omega^K_{|U}$, w.r.t. the $\omega^I$-chain $(X_1, X_2, X_3, X_4)$ has then the form (6) and the conclusion follows from the Proposition (5.1). Namely, computing, one has
$$\omega^K_{|U} = \begin{pmatrix} 0 & \chi \cos \theta^K & \Delta \cos \theta^K \sqrt{1 - \chi^2} & -\chi \cos \theta^K \\ -\chi \cos \theta^K \sqrt{1 - \chi^2} & 0 & -\Delta \cos \theta^K \sqrt{1 - \chi^2} \sqrt{1 - \chi^2} & 0 \\ -\Delta \cos \theta^K \sqrt{1 - \chi^2} & -\Delta \cos \theta^K \sqrt{1 - \chi^2} & 0 & -\chi \cos \theta^K \\ -\chi \cos \theta^K \sqrt{1 - \chi^2} & 0 & -\chi \cos \theta^K \sqrt{1 - \chi^2} & 0 \end{pmatrix}.$$ 

which ends the proof. \hfill \Box

Observe that, if $U$ is orthogonal, we can affirm beforehand that we can assume $\Gamma = 1$ (and $\Delta = 0$). In this case we have a confirmation that $\gamma = \cos \theta^K$ and $U \in \mathcal{IC}^4$ with angles $(\theta^I, \theta^J, \theta^K)$. The same conclusion is valid also if $\Delta = -1$ which happens for instance in case of 4-dimensional quaternionic and I-complex subspaces (see examples in previous section) and in general if $\Gamma^2 + \Delta^2 = 1$ i.e. if $U^{1J}(X_1)$ is of type $U^{1K}$ as well. In fact one has the

**Corollary 6.11.** Let $U$ be a 4-dimensional subspace of type $U^{1J}$. Then $\gamma = \cos \theta^K$ iff $\Gamma^2 + \Delta^2 = 1$. In this case, for any $X \in U$, $U^{1K}(X) = U^{1K}(X) = U$. 

**Proof.** The solutions of $(\Gamma^2 + \Delta^2 + \xi^2(1 - \Gamma^2 - \Delta^2)) = 1$ are $\Gamma^2 + \Delta^2 = 1$ and $\chi = \pm 1$. But in the second case, from the Proposition (5.17), it is $\Gamma = 1$. The second statement follows from the property (1) of the Proposition (5.1) and from the hypothesis that $X_3 \in L(X_3, X_4)$. \hfill \Box

**6.3. Orthogonal decomposition of $U \in \mathcal{IC}^{2m}$ ($m \geq 4$) into isoclinic addends.** In this section we show that any $U \in \mathcal{IC}^{2m}$ admits an orthogonal decomposition into isoclinic addends $U_1$. Although such decomposition is in general not unique, the addends have all the same dimension and, for any $A \in S(Q)$, the angle of isoclinicity of all pair $(U_i, AU_i)$ is the same as the one of the pair $(U, AU)$. Fixed an admissible basis $(I, J, K)$ and being $(\theta^I, \theta^J, \theta^K)$ the respective angles of isoclinicity, the addends are characterized by the same values of a pair of invariants $(\Gamma, \Delta)$. It will turn out that the angles $(\theta^I, \theta^J, \theta^K)$ together with $(\xi, \chi, \eta, \Delta)$ determines the $Sp(\eta)$-orbit of $U$. We first prove the following

**Theorem 6.12.** Let $U \in \mathcal{IC}^{2m}$, $m > 4$ and, for any $A \in S(Q)$, denote by $\theta^A$ the angle of isoclinicity of the pair $(U, AU)$. For any unitary $X_1 \in U$ there exists an 8-dimensional subspace $U^8 \in \mathcal{IC}^8$ containing $X_1$ such that, for any $A \in S(Q)$, the pair $(U^8, AU^8)$ has angle $\theta^K$. The subspace $U^8$ is unique unless it is furtherly decomposable into isoclinic subspaces with same angle $\theta^K$ for any $A \in S(Q)$. 

**Proof.** In case $U$ is a non orthogonal 2-planes decomposable subspace any $\omega^J$-standard 2-plane is isoclinic with angles $(\theta^I, \theta^J, \theta^K)$. If $U$ is double orthogonal (resp. triple orthogonal), any standard 2-plane associated to the non null cosine (resp. any 2-plane) is a standard 2-plane for $\omega^I, \omega^J, \omega^K$. Then $U^8$ is obtained by summing up four mutually orthogonal such standard 2-planes (the first containing $X_1$). Clearly $U^8$ is not uniquely determined. Observe that the orthogonal sum of a pair of such 2-planes is a 4-dimensional isoclinic subspace with angles $(\theta^I, \theta^J, \theta^K)$ characterized then by $\Gamma = 1$ and $\Delta = 0$.

Consider now the case that $U$ is not a 2-planes decomposable subspace. Let $U^{1J}(X_1) = L(X_1, X_2, X_3, X_4)$ and $U^{1K}(X_1) = L(X_1, X_2, X_3, X_4)$ be the pair of (unique) associated subspaces centered on $X_1$. Let $\Gamma$ and $\Delta$ as in the Proposition (5.1) and expression (17). From the Proposition (6.7) the pair $(\Gamma, \Delta)$ is an invariant of all 4-dimensional subspaces of type $U^{1J}$ of $U$. 

In the sequel, for simplicity we will not indicate the leading vector and denote by \((U^{1J})^\perp\) (resp. \((U^{1K})^\perp\)) the subspace \((U^{1J}(X_1))\perp \cap U\) (resp. the subspace \((U^{1K}(X_1))\perp \cap U\)). From the Proposition \([6.3]\), the subspace \((U^{1J})^\perp\) (resp. \((U^{1K})^\perp\)) is of type \(U^{1J}\) (resp. of type \(U^{1K}\)).

The subspaces \((U^{1J})^\perp\) and \((U^{1K})^\perp\) are in general different unless \(\Gamma^2 + \Delta^2 = 1\).

As stated in Corollary \([6.11]\), in case \(\Gamma^2 + \Delta^2 = 1\) the subspaces \((U^{1J})^\perp = U^{1K} \in \mathcal{L}^4\) with angles \(\theta^I, \theta^J, \theta^K\) w.r.t. \((I, J, K)\). In this case, considering any unitary vector \(X_0 \in (U^{1J})\perp\), built the \(\omega^J\)-chains \(\{X\}\), \(\{\tilde{X}\}\) centered on \(X_0\) and being \(U^{1J}(X_0)\) and \(U^{1K}(X_0)\) the associated subspaces of type \(U^{1J}\) and \(U^{1K}\), one has that \(U^{1J}(X_0) = U^{1K}(X_0) \in \mathcal{L}^4\) with angles \(\theta^I, \theta^J, \theta^K\) w.r.t. \((I, J, K)\). Furthermore from point \((2)\) of the Proposition \([6.5]\) one has that \(U^{1J}(X_1) \perp U^{1J}(X_3)\).

The 8-dimensional subspace \(U_8 = U^{1J}(X_1) \perp U^{1J}(X_3) = U^{1K}(X_1) \perp U^{1K}(X_3)\) is isoclinic with \(I^{U_8}, J^{U_8}, K^{U_8}\) with angles \((\theta^I, \theta^J, \theta^K)\). Because of the arbitrariness in the choice of \(X_0\), also in this case \(U^8\) is not uniquely defined.

Let then consider the case that \(\Gamma^2 + \Delta^2 \neq 1\). From the Proposition \([6.3]\) the cosines of the pair of non null principal angles of the pair \((U^{1J})^\perp, (U^{1K})^\perp\) are both equal to \(\sqrt{\Gamma^2 + \Delta^2}\). Let \(U_1 \subset (U^{1J})^\perp\) and \(U_2 \subset (U^{1K})^\perp\) be the pair of isoclinic 2-planes generated by the related principal vectors associated to such non null principal angles. One has

\[
(U^{1J})^\perp = U_1 \perp W; \quad (U^{1K})^\perp = U_2 \perp W
\]

being \(W\) the \((2m - 6)\)-dimension intersection \((U^{1J})^\perp \cap (U^{1K})^\perp\).

**Lemma 6.13.** The subspaces \(U_1\) and \(U_2\) are \(\omega^J\)-standard.

**Proof.** Both \((U^{1J})^\perp\) and \((U^{1K})^\perp\) are isoclinic with \(I(U^{1J})^\perp\) and \(I(U^{1K})^\perp\) with angle \(\theta^I\). Then for any \(X_1 \in (U^{1J})^\perp\) the vector \(X_2 = \frac{I^{\perp U^I} X_1}{\cos \theta^I X_1}\) belongs to \((U^{1J})^\perp\) and analogously for any \(\tilde{X}_1 \in (U^{1K})^\perp\) the vector \(\tilde{X}_2 = \frac{I^{\perp U^I} \tilde{X}_1}{\cos \theta^I \tilde{X}_1}\) is in \((U^{1K})^\perp\). Consequently any \(X_1 \in (U^{1J})^\perp \cap (U^{1K})^\perp = W\) has the corresponding in \(W\) which implies that both \(U_1\) and \(U_2\) are \(\omega^J\)-standard 2-planes.

It follows that

\[
P_{\perp U} U_1 = IU_1, \quad P_{\perp U} U_2 = IU_2
\]

and \(P_{\perp U} W = IW\).

Consider a unitary vector \(X_7 \in U_1\) and the \(\omega^J\)-chain \((X_7, X_8, X_5, X_6)\) generating the subspace \(U^{1J}(X_7) = L(X_7, X_8) \perp L(X_5, X_6)\). From previous Lemma \(U_1 = L(X_7, X_8)\). From the Proposition \([6.5]\) point \(4)\), \(U^{1J}(X_7) \subset U^{1J}(X_1)\). The \(\omega^J\)-standard 2-plane \(L(X_5, X_6) \subset W\). In fact, from previous Lemma, \(U^{1J}(X_7) = L(X_7, X_8) \perp L(X_5, X_6) \subset U^{1J}(X_1)\). \(W\) and, being \(L(X_5, X_6) \perp L(X_7, X_8)\), the conclusion follows.

Let consider now the vector \(X_3\) as new leading vector. From the Proposition \([6.5]\), one has that \(U^{1J}(X_3) = U^{1J}(X_7)\).

In fact the \(\omega^J\)-chain is now \((X_5, X_6, -X_7, -X_8)\) and \(U^{1J}(X_3) = L(X_5, X_6) \perp L(X_7, X_8)\). Let \(U^{1K}(X_5) = L(X_5, X_6) \perp L(-X_7, -X_8)\) the associated subspace of type \(\omega^J\) where the generators form the \(\omega^J\)-chain centered on \(X_5\). From point \(4\) of the Proposition \([6.5]\), the chain \(U^{1K}(X_5) \subset (U^{1K})^\perp = U_2 \perp W\).

**Proposition 6.14.** One has

\[
L(\tilde{X}_7, \tilde{X}_8) = U_2
\]

**Proof.** By assumption, the orthogonal projection of \(L(X_7, X_8)\) onto \((U^{1K})^\perp = U_2 \perp W\) is onto \(U_2\) with the cosine of the principal angles equal to \(\sqrt{\Gamma^2 + \Delta^2}\). From the Proposition \([6.7]\) as 2-planes of the associates chains \((X_5, X_6, -X_7, -X_8)\) and \((X_5, X_6, -X_7, -X_8)\) the subspaces \(L(X_7, X_8)\) and \(L(\tilde{X}_7, \tilde{X}_8)\) are isoclinic with the cosine of the principal angles equal to \(\sqrt{\Gamma^2 + \Delta^2}\). Then \(L(\tilde{X}_7, \tilde{X}_8) = U_2\).

Therefore \(L(X_3, X_1, X_7, X_9) = L(\tilde{X}_3, \tilde{X}_1, \tilde{X}_7, \tilde{X}_9)\) and the 8-dimensional subspace

\[
U_8 = L(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8) = L(X_1, X_2, \tilde{X}_3, \tilde{X}_4, X_5, X_6, \tilde{X}_7, \tilde{X}_8)
\]

is orthogonal sum of a pair of uniquely defined subspaces of type \(U^{1J}\) that is \(U_8^J = U^{1J}(X_1) \perp U^{1J}(X_3)\) and orthogonal sum of the pair of the (uniquely defined) relative associated subspaces of type \(U^{1K}\) that is \(U_8^K = U^{1K}(X_1) \perp U^{1K}(X_3)\). This implies that \(U_8^J\) is isoclinic with \(I^{U_8^J}, J^{U_8^J}, K^{U_8^J}\) with angles \((\theta^I, \theta^J, \theta^K)\). By the uniqueness of the adds it follows the uniqueness of \(U_8^J\). To conclude the proof we need the following

**Lemma 6.15.** Let \(U \in \mathcal{L}^{2m}\), \((I, J, K)\) be some admissible basis and \((\theta^I, \theta^J, \theta^K)\) be the angles of isoclinicity of the pairs \((U, IU), (U, JU), (U, KU)\) respectively. Any k-dimensional subspace \(W \subset U\) isoclinic with \(IW, JW, KW\) with angles equal respectively to \((\theta^I, \theta^J, \theta^K)\) belongs to \(\mathcal{L}^k\) and, for any \(A \in S(Q)\), the angle of isoclinicity of the pair \((W, AW)\) equals the one of the pair \((U, AU)\).
Proof. Let \( A = \alpha I + \beta J + \gamma K \) be a compatible complex structure and let \( \theta^A \) be the angle of isoclinicity of the pair \((U, AU)\). Consider the subspace \( AW = \{ AX = \alpha IX + \beta JX + \gamma KX, X \in W \} \). If \( \theta^A = \pi/2 \) the pair \((W, AW)\) is isoclinic with angle \( \pi/2 \). Suppose then that \( \theta^A \neq \pi/2 \). We prove that \( Pr^{AU} W = AW \) implies that the pair \((W, AW)\) is isoclinic with angle of isoclinicity equal to \( \theta^A \). Projecting the generic vector \( AW \ni \tilde{Y} = \alpha I \tilde{X} + \beta J \tilde{X} + \gamma K \tilde{X} \) with \( \tilde{X} \in W \) onto \( U \) one has
\[
Pr^U \tilde{Y} = Pr^U (\alpha I \tilde{X} + \beta J \tilde{X} + \gamma K \tilde{X}) = \alpha Pr^U I \tilde{X} + \beta Pr^U J \tilde{X} + \gamma Pr^U K \tilde{X}.
\]
By hypothesis the pairs \((W, IW), (W, JW), (W, KW)\) are isoclinic with angles \( \theta^J, \theta^K, \theta^K \) respectively which implies that \( Pr^{IU}W = IW \) and \( Pr^{IJ}W = W \). Therefore \( Pr^{IU} \tilde{Y} \in W \). This implies that the pair \((W, AW)\) is isoclinic with the angle of isoclinicity equal to \( \theta^A \). □

Lemma 6.16. For any \( X \in U_1^g \) the associated subspaces \( U^{IJ}(X) \) and \( U^{IK}(X) \) are both in \( U_1^8 \).

Proof. For any \( X_1 \in U_1^g \) one has that \( X_2, Y_2, Z_2 \) completing the standard 2-planes of \( \omega^J, \omega^J, \omega^K \) respectively are in \( U_1^8 \) (by assumption in case of triple orthogonality) and so are the 2-plane \( L(X_2, Y_2), L(X_2, Z_2), L(Y_2, Z_2) \) and consequently \( X_4, Y_4, Z_4 \) and the whole chains \( \{X_1\}, \{Y_1\}, \{Z_1\} \). □

Let now consider the 4-dimensional \( \omega^J \)-standard subspace \( L(X_3, X_4, X_7, X_8) = L(\tilde{X}_3, \tilde{X}_4, \tilde{X}_7, \tilde{X}_8) \). The Gram matrix w.r.t. the bases \((X_3, X_4, -X_7, -X_8)\) and \((\tilde{X}_3, \tilde{X}_4, -\tilde{X}_7, -\tilde{X}_8)\) has necessarily the form
\[
A = \begin{pmatrix}
0 & a & -\Delta & \Delta \\
\Delta & \Delta & -a & b \\
-a & b & \Delta & -\Delta \\
b & -a & \Delta & \Delta
\end{pmatrix}.
\]
From the Proposition (point \( \square \)) we know that, by changing leading vector with \( X \in L(X_5, X_6) \), one has that \( U^{IJ}(X) = U^{IJ}(X_3) \) and \( U^{IK}(X) = U^{IK}(X_3) \). We then consider a new leading vector \( X_5 \in L(X_5, X_6) \) and let \((\tilde{X}_5, \tilde{X}_6, \tilde{X}_7, \tilde{X}_8)\) be the new chains determined in order that the pair \((\tilde{X}_5, \tilde{X}_7)\) are related principal vectors of the pair \((L(X_3, X_4), L(X_7, X_8))\). The change of basis in \( L(X_5, X_6) \) to diagonalize the minors \( A_{12,34} \) and \( A_{34,41} \) is given by the orthogonal transformation \( A_{5,6} = C(X_5, X_6) \) represented by the matrix
\[
C : \frac{1}{\sqrt{a^2 + b^2}} \begin{pmatrix}
a & -b \\
b & a
\end{pmatrix}
\]
which induce the same transformation onto \( L(X_7, X_8) \) and \( L(\tilde{X}_7, \tilde{X}_8) \).

The Gram matrix w.r.t. the bases \((X_3, X_4, \tilde{X}_7, \tilde{X}_8)\) and \((\tilde{X}_3, \tilde{X}_4, \tilde{X}_7, \tilde{X}_8)\) is the following
\[
\begin{pmatrix}
\Gamma & -\Delta & 0 & 0 \\
-\Delta & \Gamma & 0 & 0 \\
0 & 0 & \Gamma & -\Delta \\
0 & 0 & -\Delta & \Gamma
\end{pmatrix}.
\]
By construction the vector \( \tilde{X}_7 \) is the principal vector associated to \( X_3 \) of the pair of isoclinic subspaces \((L(X_3, X_4), L(\tilde{X}_7, \tilde{X}_8))\) and in fact \( \cos X_3, \tilde{X}_7 \geq 0 \) whereas, being \( \cos X_3, \tilde{X}_8 \leq 0 \) the pair \((X_3, \tilde{X}_8)\) is not formed by related principal vectors.

Let \((X_5, X_6, \tilde{X}_7, \tilde{X}_8)\) and \((\tilde{X}_5, \tilde{X}_6, \tilde{X}_7, \tilde{X}_8)\) the \( \omega^J \) and \( \omega^J \) chains of \( U^{IJ}(X_3) \) and \((\tilde{X}_5, \tilde{X}_6, \tilde{X}_7, \tilde{X}_8)\) the \( \omega^J \) and \( \omega^K \) chains of \( U^{IK}(X_3) \).

W.r.t. the bases \((X_1, X_2, X_3, \tilde{X}_4, \tilde{X}_6, \tilde{X}_7, \tilde{X}_8)\) and \((X_1, Z_2, \tilde{Z}_3, \tilde{X}_4, \tilde{Z}_6, \tilde{X}_7, \tilde{X}_8)\), and denoting by \( \Sigma = \sqrt{1 - \Gamma^2 - \Delta^2} \) the matrix \( C_{IK} \) is given by
\[
C_{IK} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\Sigma & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]
W.r.t. the $\omega^J$-standard basis $(X_1, X_2, X_3, X_4, \bar{X}_5, \bar{X}_6, \bar{X}_7, \bar{X}_8)$ and the $\omega^J$-standard basis $(\bar{X}_1, Y_2, X_3 = Y_3, Y_4, X_5, X_6, X_7, Y_8)$ the matrix $C_{IJ}$ is given by

$$C_{IJ} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \xi & 0 & -\sqrt{1-\xi^2} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{1-\xi^2} & 0 & \xi & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{1-\xi^2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \xi \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

(22)

The aforementioned $\omega^J$ (resp. $\omega^K$)-standard bases are union of a pair of $\omega^J$ (resp. $\omega^K$)-chains of $U_\theta^8$ generating a pair of orthogonal subspaces of type $U^{IJ}$ (resp. $U^{IK}$). The independence of $(\xi, \chi, \eta, \Gamma, \Delta)$ from the leading vector, accounts for the independence of both matrices from the leading vector $X_1 \in U_\theta^8$. We call canonical bases of $U_\theta^8$ the set of such bases determined by the leading vector $X_1 \in U_\theta^8$ and canonical matrices $C_{IJ}$ and $C_{IK}$ the matrices given in (22) and (21) respectively.

We can continue this construction considering an unitary vector $\tilde{X}_1 \in (U_\theta^8)_\perp$ which will determine a second 8-dimensional subspace $U_\theta^2$ orthogonal to the first and so on. Then we can state the following

**Proposition 6.17.** Any $U \in \mathcal{IC}_{8k}$ admits an orthogonal decomposition

$$U = U_\theta^1 \oplus U_\theta^2 \oplus U_\theta^3 \oplus \ldots \oplus U_\theta^k,$$

where all 8-dimensional addends are isoclinic i.e. $U_i \in \mathcal{IC}_8$, $i = 1, \ldots, k$ with same angles for all pairs $(U_i, AU_i)$ as $(U, AU)$.

**Proposition 6.18.** To any $U \in \mathcal{IC}_{8k}$ we can associate the values $\Gamma$ and $\Delta$ of all 4-dimensional subspaces of type $U_\theta^{IJ}$ belonging to $U$.

We can then state the

**Corollary 6.19.** Let $U \in \mathcal{IC}_{8k}$. For the canonical matrices one has $C_{IJ} = \bigoplus_{i=1}^k (C_{IJ})$, and

$$C_{IK} = \bigoplus_{i=1}^k (C_{IK})$$

where the blocks are given in (22) and in (21) respectively. Such matrices depend neither on the particular decomposition of $U$ into isoclinic 8-dimensional subspaces nor on the canonical bases chosen for each addend.

So we proved that if $\dim U = 8k$, $U$ admits an orthogonal decomposition into 8-dimensional isoclinic subspaces $U_i$ whose angles of isoclinicity are $\theta^J, \theta^J, \theta^K$. According to the Theorem (2.1) and Corollary (6.19) the orbit is determined by the triple $(\theta^J, \theta^J, \theta^K)$, by the triple $(\xi, \chi, \eta)$ and by $\Delta$ (recall that $\Gamma$ is a function of $(\xi, \chi, \eta)$).

If $\dim U = 8k + 2$ or $\dim U = 8k + 6$ necessarily $\xi = \pm 1, \chi = \pm 1, \eta = \xi \cdot \chi$ and $U$ decomposes into orthogonal sum of 2 dimensional isoclinic addends $U_i$. For any $A \in S(Q)$ the angle of isoclinicity of the pair of standard 2-planes $(U_i, AU_i)$ is the same as the one of the pair $(U, AU)$. From the Proposition (5.17) one has $\Gamma = 1$. The orbit in this case is determined by the triple $(\theta^J, \theta^J, \theta^K)$ and by the pair $(\xi, \chi)$.

Finally if $\dim U = 8k + 4$, then $U$ is direct orthogonal sum of isoclinic 4-dimensional subspaces $U_i$ in which case $\Gamma^2 + \Delta^2 = 1$. For any $A \in S(Q)$ the angle of isoclinicity of the pair $(U_i, AU_i)$ is the same as the one of the pair $(U, AU)$. The orbit is determined by the triple $(\theta^J, \theta^J, \theta^K)$, by the triple $(\xi, \chi, \eta)$ and by the sign of $\Delta = \pm \sqrt{1-\Gamma^2}$ with $\Gamma = \Gamma(\xi, \chi, \eta)$. Recalling that the canonical matrices are given in (22) and (21), we can now state the final result.

**Theorem 6.20.** Let $U \in \mathcal{IC}_{2m}$. Let fix an admissible basis $(I, J, K)$ and denote by $(\theta^J, \theta^J, \theta^K)$ the angles of isoclinicity of the pairs $(U, IU), (U, JU), (U, KU)$ respectively. For $k \geq 0$:

- If $2m = 8k + 2$ or $2m = 8k + 6$, $U$ is 2-planes decomposable i.e. is orthogonal sum of $U_i \in \mathcal{IC}_2$ with same angle of isoclinicity of $U$. In this case $(\Gamma, \Delta) = (1, 0)$ and the pair $(\xi, \chi) = (\pm 1, \pm 1)$ determine the matrices $C_{IJ}, C_{IK}$. The $Sp(n)$-orbit is then determined by the angles $(\theta^J, \theta^J, \theta^K)$ and by the pair $(\xi, \chi)$.

- If $2m = 8k + 4$, $U$ is orthogonal sum of $U_i \in \mathcal{IC}_4$ with same angle of isoclinicity of $U$ and characterized by the same pair $(\Gamma, \Delta)$. In this case $\Gamma^2 + \Delta^2 = 1$ and the canonical matrices are determined by $(\xi, \chi, \Gamma, \Delta)$. In particular this case occurs if $U$ is orthogonal in which case $(\Gamma, \Delta) = (1, 0)$. The $Sp(n)$-orbit is then characterized by $(\theta^J, \theta^J, \theta^K)$ and $(\xi, \chi, \eta, \Delta)$. In particular, if $\xi = \pm 1$ and $\chi = \pm 1$ we are in the first case.

- If $2m = 8k$ then $U$ is orthogonal sum of $U_i \in \mathcal{IC}_8$ with same angle of isoclinicity of $U$. The canonical matrices are determined by $(\xi, \chi, \Gamma, \Delta)$ where $\Gamma^2 + \Delta^2 \leq 1$ and the $Sp(n)$-orbit by $(\theta^J, \theta^J, \theta^K)$ and $(\xi, \chi, \eta, \Delta)$. If in particular $\Gamma^2 + \Delta^2 = 1$ we are in the previous case and if furthermore $\xi = \pm 1$ and $\chi = \pm 1$ we are in the first case.
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