Dark energy as generalised superfluid excitations

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In this paper we present a generic form of the cosmological Equation of State derived from the formalism of Statistical Mechanics, assuming that a cosmic acceleration scenario can be interpreted as a system of quasi–particles. By considering a generalised superfluidity approach, in which the energy spectrum can be modulated via a parameter $s$, a negative equation of state arises as excitations associated with an exotic superfluid system. We show that for $s \sim 0.293$, $w \simeq -1$, which can be related to the standard equation of state for the $\Lambda$CDM model.

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Introduction: From Statistical Mechanics to standard FLRW cosmology.– The observed dynamics of our universe is often described in terms of the FLRW metric. Even though, the adoption of this convenient geometry in Einstein’s equations does not give information about the specific form for the Equation of State (EoS) of the effective cosmological fluid, $P = w \rho P$. Here, the total energy density in the universe is denoted by $\rho$ and the pressure by $P$. The standard concordance model, $\Lambda$CDM, fits in this scenario where the fluid can be identified into matter (including baryonic and cold dark matter) $\rho_m$, radiation $\rho_r$, and a not so well understood dark energy $\rho_{\Lambda}$. Towards this direction, many attempts have been done starting from the simple relation above described by a barotropic EoS (notice that here we call $w$ as the EoS parameter) to some complex ones with an explicit relation between $P$ and $\rho$. All of them to achieve the current cosmic acceleration. As it is standard, to obtain this behaviour we require an energy density with significant negative pressure at late times. This means that the evolution ratio between the pressure and energy density is negative, i.e. $w(z) = P/\rho < 0$. All reasonable fitting dark energy models available in the literature are in agreement at this point [1,5].

The evolution Friedmann equation for a spatially flat universe

$\left(\frac{H(z)}{H_0}\right)^2 \propto (\rho_m + \rho_{\Lambda}) \left[\Omega_m (1+z)^3 + \Omega_{\Lambda} f(z)\right],$ \hspace{0.5cm} (1)

where $H_0$ is the Hubble parameter and the index $m$ denotes the matter components related to radiation, barionic and cold dark matter. From this equation it is possible to compute the EoS, $w(z)$. This can be done when the current value for the dark energy density is written as $\rho_{0(\Lambda)} = \rho_{\Lambda}(z)f^{-1}(z)$, with $f(z) = \exp\left[3\int_0^z \frac{1+w(z)}{1+z} \, dz\right]$.

By modelling $w(z)$, we can give directly an entire evolution description of Eq. (1), as e.g., in the case of quintessence models $w = \text{constant}$, the solution for $f(z)$ is $f(z) = (1 + z)^{\frac{3(1+w)}{3-w}}$. If we consider the case of the Cosmological Constant $\Lambda (w = -1)$ we obtain $f = 1$. Other cases explore a dark energy density $\rho_{\Lambda}$ with varying and non-varying $w(z)$, just to cite a few.

Notice that for a given functional form of $f(z)$, the contribution of the dark energy density to $H(z)$, goes to more negative values of $w(z)$. This is an impact in the evolution of dark energy on the dynamical age of the universe. As we mentioned, to get a dark energy model with late-time negative pressure we can think in two frameworks: on one hand, a quintessence model which shows a wide application in tracker the slow roll condition of scalar fields$^1$ and demands a constant EoS [1], a references therein. In numerical terms, according to Planck 2018 [12], the dark energy EoS parameter for a flat universe is $w = -1.006 \pm 0.045$, which is consistent with $\Lambda$. On the other hand, for kinessence models, the EoS is a function of $z$ and several dark energy models with different parameterisations of $w(z)$ has been discussed in the literature [13].

According to the latter ideas, several types of fluid are used for cosmological viable scenarios, as:

- Non–relativistic matter: An ordinary non-relativistic matter EoS as $w = 0$ (i.e cold dust), correspond to a diluted fluid with $\rho \propto a^{-3} = V_3^{-1}$, where $V_3$ is the volume in 3 space–like dimensions.

- Ultra–relativistic matter: An ultra-relativistic matter EoS given by $w = 1/3$ (e.g. radiation, or matter in the very early universe), correspond to a fluid as $\rho \propto a^{-4}$.

In an expanding universe, the energy density decreases faster than the volume expansion, because radiation has

\footnote{An associated particle can be identified with a boson with zero spin.}

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a momentum and, by the de Broglie hypothesis a wavelength is redshifted.

- Cosmic accelerated inflation: This case can be characterised by a dark energy EoS. In the simplest scenario, this correspond to the EoS of Λ with \( w = -1 \). The solution of the conservation equation for the scale factor is not valid and \( a \propto e^{Ht} \). In general, the expansion of the universe is accelerating for any EoS \( w < -1/3 \).

- Hypothetical fluids: In this case, a hypothetical phantom energy is considered to be with a EoS \( w < -1 \), and we will have a Big Rip. Using current data, it is impossible to distinguish between phantom \( w < -1 \) and non-phantom \( w \geq -1 \).

We should mention that the first two cases can be derived from first principles, since that can be deduced from quantum and statistical mechanics. The dark energy and phantom–like fluids cannot be derived as the latter. From the standard model conception, it seems that in our universe exist only two types of particles: bosons and fermions. The nature of this difference relies basically on some intrinsic property, the so–called spin. As a consequence, the statistics, i.e., how these particles occupy a single quantum energy state, depends on their quantum nature. Furthermore, it is well known that the system under consideration displays a significant departure from their classical behaviour at certain critical temperature, depending on whether particles are the constituents of the system (bosons or fermions). In other words, the quantum behaviour of the corresponding particles must be taken into account. Bosons obey Bose–Einstein statistics and fermions obey Fermi–Dirac statistics. Summarising, the corresponding properties of both systems are quite different when quantum properties are taken into account, the bosons do not fulfil the Pauli’s exclusion principle, while fermions do.

Thus, according to the concepts described above, it can be demonstrated that the occupation number in the Grand Canonical Ensemble can be expressed as

\[
\langle n(\epsilon) \rangle _\pm = \frac{1}{\exp \beta (\epsilon - \mu) \pm 1},
\]

where \((+)^\) stands for fermions and \((-)\) for bosons. As usual, \( \beta = 1/\kappa T \), being \( \kappa \) the Boltzmann’s constant and \( T \) the temperature. Additionally, \( \epsilon_p \) is the corresponding single–particle energy spectrum and \( \mu \) is the chemical potential. Thus, we have to know, through the dynamics, what is the specific functional form of the energy spectrum for a single particle to analyse the occupation number Eq. (2), which contains relevant thermodynamical information of the system. At this point, it is important to mention that the corresponding EoS contains, in fact, the aforementioned relevant thermodynamical information, if one knows the dynamical properties of the system, i.e., if these properties can be deduced from the experiment or a microscopic model of matter. In other words, the analytical form of the EoS depends on the substance under consideration, and its deduction is dependent on the experiment and/or a microscopic model of matter.

Let us calculate the EoS for some thermodynamical systems, by using all the information given above. For this goal, let us appeal to Ref. [15], to exemplify the method. We assume that the semiclassical single–particle energy spectrum for a given system can be expressed as \( \epsilon = \alpha p^s \), where \( p \) is the corresponding momentum, \( s \) is some positive real number and \( \alpha \) is a constant with an adequate dimensions. The system under consideration is ideal, i.e., interactions among its constituents are neglected. To generalise, our system can be situated in \( n \) space–like dimensions. Under these circumstances it is straightforward to show that the EoS is given by

\[
P = \frac{s}{n} \left( \frac{U}{V_n} \right),
\]

where, \( P \) is the pressure, \( U \) is the internal energy, and \( V_n \) is the hyper volume under consideration. Thus, we can define the energy density for any space–like dimension as \( \rho_e = \left( U/V_n \right) \).

By using the general relation Eq. (3), we are able to recover the standard results in 3 space–like dimensions \( (V_3) \):

- Non–relativistic system: EoS of the form \( P = \frac{2}{3} (U/V_3) = \frac{2}{3} \rho_e \) corresponding to a single–particle energy spectrum \( \epsilon = \frac{p^2}{2m} \).

- Ultra–relativistic system: the EoS is this case is of the form \( P = \frac{1}{3} (U/V_3) = \frac{1}{3} \rho_e \), which corresponds to the single–particle energy spectrum \( \epsilon = cp \), with \( c \) the speed of light.

Let us remark that the properties of the system, and in particular the EoS, strongly depends on the functional form of the single–particle energy spectrum. Due to the functional form of the single particle energy spectrum \( \epsilon \sim p^s \), we can obtain an infinite set of EoS in 3 space–like dimensions, depending on the value of the parameter \( s \). However, the last assertion can lead to exotic single–particle energy spectra which can be related, in principle, to the kinetic energy of each particle.

Finally, notice that in the case of radiation the EoS is given by \( P = \frac{1}{3} (U/V_3) = \frac{1}{3} \rho_e \). Nevertheless, in this case, we have to make some additional assumptions because for these systems, the total number of particles is not conserved, and consequently, the corresponding chemical potential must be set equal to zero. Moreover, let us remark that the EoS for
radiation (together with the corresponding EoS for dust \( P = 0 \)) can be obtained also from first principles without deeper conceptual complications.

Finally, we focus on the following scenarios for different values of the parameter \( s \). For instance, notice that in 3 space–like dimensions when \( s = 3 \), the relation between the pressure and the energy density is such that \( P = (U/V)_3 = \rho_c \), which is a consequence of a semiclassical single–particle energy spectrum of the form \( \epsilon = \alpha p^3 \). Notice that an EoS of the form \( P = -(U/V)_3 = -\rho_c \) (in the case of dark energy) implies a single–particle energy spectrum \( \epsilon = \alpha p^3 \) by using Eq. (3). The last assumption has no physical meaning, since a dispersion relation \( \sim p^{-3} \), leads to a negative number of micro–states due to the above analysis is only valid for real positive values of \( s \) and it is unable to reproduce the standard EoS for dark energy.

**Superfluid excitations as dark energy EoS.** In this line of thought, we propose that a generalised Landau’s roton spectrum for dark energy in 3 space–like dimensions can be written as

\[
\epsilon_p = \Delta + \frac{(p - p_0)^s}{\alpha},
\]

for a real positive number \( s \). This energy spectrum is a generalisation of the spectrum associated with superfluidity behaviour of liquid Helium II when \( s = 2 \), or the so–called Landau’s roton spectrum [15] [16]. In Eq. (4), \( \Delta \) is some gap energy. The term \( p_0 \), can be interpreted as the momentum near to the extremal of the energy spectrum Eq. (4), i.e., has the behaviour of a quasi–particle. Excitations with momenta close to \( p_0 \) are referred to as rotons in the usual non–relativistic case, i.e., for \( s = 2 \), when the chemical potential \( \mu \) is set to be zero.

We compute the relation between the pressure \( P \) and the internal energy \( U \) associated with Eq. (4), in other words, the corresponding EoS. Consequently, we obtain by\(^4\)

\[
\frac{PV}{\kappa T} = \frac{4\pi V}{h^3} \int_0^\infty e^{-\left[\Delta + \frac{(p - p_0)^s}{\alpha} / \kappa T\right]} p^2 dp \simeq \frac{\kappa T}{V}, \tag{5}
\]

where \( h \) is the Planck’s constant and \( \frac{\kappa T}{V} \) is the equilibrium number of quasi–particles that we call for our purposes generalised rotons.

To solve the integral Eq. (5) we take, as was mentioned above \( s > 0 \), together with the assumption that for low temperatures, i.e., for temperatures of interest, the minimal value of the occupation number \( n(\epsilon_p) \) is at most \( \exp(-\Delta/\kappa T) \) which is larger than unity. Thus, we can express the occupation number Eq. (2) for bosons as \( n(\epsilon_p) \approx \exp(-\beta \epsilon_p) \). With the change of variables \( p = p_0 + (\alpha \kappa T)^{1/s} x \), we get from Eq. (5)

\[
\frac{PV}{\kappa T} = \frac{4\pi V}{h^3} \int_0^\infty e^{-\left[\Delta + \frac{(p - p_0)^s}{\alpha} / \kappa T\right]} p^2 dp \simeq \frac{\kappa T}{V}, \tag{6}
\]

\[
\frac{P \rho_E}{\kappa T} = \frac{4\pi V}{h^3} \left[ (\alpha \kappa T)^{1/s} \right] e^{-\Delta/\kappa T} \left[ \int_0^\infty e^{-\left(1 + \frac{(\alpha \kappa T)^{1/s}}{p_0} x + \frac{(\alpha \kappa T)^{2/s}}{p_0^2} x^2 \right)} dx \right],
\]

without losing generality [15] we are able to obtain

\[
\frac{PV}{\kappa T} \left[ (\alpha \kappa T)^{1/s} \int_0^\infty e^{-\Delta/\kappa T f(s)} \right] \frac{\kappa T}{V}, \tag{7}
\]

where

\[
f(s) = 3 - \Gamma\left(\frac{1}{s}\right) - \frac{(\alpha \kappa T)^{1/s}}{p_0} \Gamma\left(\frac{2}{s}\right) \left[ (\alpha \kappa T)^{2/s} \int_0^\infty \frac{1}{x^3} dx \right], \tag{8}
\]

and \( \Gamma(\mu) \) is the Gamma function. To achieve the corresponding internal energy \( U \), we employ the useful thermodynamic functions \( A = -PV \) (Helmholtz free energy) and \( S = -\left(\frac{\partial A}{\partial T}\right)_V \) (entropy), with \( U = A + TS \).

After straightforward calculations, we obtain for the energy density \( \rho_E \)

\[
\frac{U}{V} \equiv \rho_E = \frac{4\pi V}{h^3} \left[ (\alpha \kappa T)^{1/s} \right] e^{-\Delta/(\kappa T)} \left[ 3(\Delta s + \kappa T) - (\alpha \kappa T)^{1/s} \left[ 2(\Delta s + s \kappa T)\Gamma\left(\frac{1}{s}\right) - (\alpha \kappa T)^{1/s} \left[ (\alpha \kappa T)^{2/s} \int_0^\infty \frac{1}{x^3} dx \right] \right] \right], \tag{9}
\]

Multiplying and dividing by \( (s \kappa T f(s)) \) we get

\[
P = s \rho_E \left( \frac{f(s)}{g(s)} \right), \tag{10}
\]

where \( g(s) \) is given by

\[
g(s) = \frac{1}{\kappa T} \left[ 3(\Delta s + \kappa T) - (\alpha \kappa T)^{1/s} \left[ 2(\Delta s + s \kappa T)\Gamma\left(\frac{1}{s}\right) - (\alpha \kappa T)^{1/s} \left[ (\alpha \kappa T)^{2/s} \int_0^\infty \frac{1}{x^3} dx \right] \right] \right]. \tag{11}
\]

Notice that for even values of the parameter \( s \) the term \( s \left( \frac{f(s)}{g(s)} \right) \) is always positive. The interesting scenarios for the EoS Eq. (10) are given for odd values of the parameter \( s \), together with values between \( 0 < s < 1 \), in which the right hand term of Eq. (10) becomes negative. This term seems to be relevant in the search for negative contributions on the energy density of the EoS Eq. (10) near to \( p_0 \), that we interpreted as dark energy.

\(^4\) From this point forward, the volume \( V \) is assumed to be in 3 space–like dimensions.
We define our EoS Eq. (10) in terms of \( w \) and \( s \) as

\[
    w(s) = \frac{P}{\rho E} = s \left( \frac{f(s)}{g(s)} \right),
\]

with units of \([T]\) = Kelvin, \([\Delta]\) = Joules, \(\alpha\) = seconds/meter, \([p_0]\) = kilograms meters/seconds and, \(\kappa = 1.3 \times 10^{-23}\) Joules/Kelvin.

In Figure (1) we plot Eq. (12) in function of the parameter \( s \) at low temperatures. We have three relevant regions which depict the behaviour of the system depending on the value of \( s \). Specifically, when \( s \) is large enough we have that \( w(s) \to 0 \), i.e., the system behaves as dust. We obtain that for values \( s \gtrsim 0.293 \) the system behaves as a normal fluid, in the sense that the energy density is positive. The interesting scenario seems to be placed for values \( s \lesssim 0.293 \) in which the energy density becomes negative. For \( s \lesssim 0.293 \) the system behaves as phantom fluid with \( w(s) < -1 \). For \( s \sim 0.293 \), the value of \( w(s) \) approaches to \(-1\) in a range of low temperatures according to our approximation. Using the facts described above, we can rewrite Eq. (4) with \( s \sim 0.293 \) as \( \epsilon_p \sim (p - p_0)^{0.293} \). Therefore, in our approach the excitations (or quasi–particles) that we call generalised rotons, close enough to \( p_0 \) of the corresponding energy spectrum Eq. (4), are able to mimic dark energy when \( s \sim 0.293 \).

Discussion.– In this paper, we present an EoS described by a generic Landau’s roton spectrum in 3 space–like dimensions, which can reproduce dark energy characteristics in certain laboratory conditions. To achieve this condition, we define a free parameter \( s \) that modulates the \((p - p_0)\) term that appears in Eq. (4), which value can be used to obtain direct information of the corresponding EoS for the superfluid. We obtain that the value \( s \sim 0.293 \), reproduces \( w(s) \sim -1 \). This result leads to a scenario that mimics a \( \Lambda \)CDM EoS that can be eventually extended (and perhaps tested) with superfluidity in the laboratory. These type of fractional powers of \( s \), are not an unusual topic in modern physics, e.g., they are related to properties of fractional derivatives and consequently, with fractional kinetics [17,20]. Even more, they can be used in several physical applications, see for instance [19,22] and references therein. For instance, in Ref. [17] it was analysed a fractional Schrödinger–type wave equation in 1–dimension for a free particle. In this case, the corresponding energy spec-
tum is given by $\epsilon_k = \eta \sqrt{\hbar \gamma}$, where $\gamma$ takes real positive values and $\eta$ is defined as $\eta \equiv \frac{1}{2} \left( m^2 \gamma - 1 + 2 \gamma - 1 \right)^{-1}$. It is noteworthy to mention that this result is in agreement with our approach when $s = 2\gamma$. Finally, it is remarkable that our proposal gives insights that can eventually be applied at cosmological scales to understand the nature of dark energy in the universe.

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