Congruences for Catalan and Motzkin numbers
and related sequences

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Abstract

We prove various congruences for Catalan and Motzkin numbers as well as related sequences. The common thread is that all these sequences can be expressed in terms of binomial coefficients. Our techniques are combinatorial and algebraic: group actions, induction, and Lucas’ congruence for binomial coefficients come into play. A number of our results settle conjectures of Benoit Cloitre and Reinhard Zumkeller. The Thue-Morse sequence appears in several contexts.

1 Introduction

Let \( \mathbb{N} \) denote the nonnegative integers. The divisibility of the Catalan numbers

\[
C_n = \frac{1}{n+1} \binom{2n}{n}, \quad n \in \mathbb{N},
\]

by primes and prime powers has been completely determined by Alter and Kubo
ta [4] using arithmetic techniques. In particular, the fact that \( C_n \) is odd precisely
when \( n = 2^h - 1 \) for some \( h \in \mathbb{N} \) has attracted the attention of several authors including Deutsch [9], Eğecioğlu [13], and Simion and Ullman [24] who found combinatorial explanations of this result. In the next section we will derive the theorem which gives the largest power of 2 dividing any Catalan number by using group actions. In addition to its generality, this technique has the advantage that when \( n = 2^h - 1 \) there is exactly one fixed point with all the other orbits having size divisible by 2. For other congruences which can be proven using the action of a group, see Sagan’s article [21].

By contrast, almost nothing is known about the residues of the Motzkin numbers

\[
M_n = \sum_{k \geq 0} \binom{n}{2k} C_k, \quad n \in \mathbb{N}.
\]

In fact, the only two papers dealing with this matter of which we are aware are the recent articles of Luca [19] about prime factors of \( M_n \) and of Klazar and Luca [17] about the periodicity of \( M_n \) modulo a positive integer. In section 3 we will characterize the parity of the Motzkin numbers as well as three related sequences. Surprisingly, the characterizations involve a sequence which encodes the lengths of the blocks in the Thue-Morse sequence. The block-length sequence was first studied by Allouche et. al. [2]. For more information about the Thue-Morse sequence in general, the reader is referred to the survey article of Allouche and Shallit [3].

Section 4 is devoted to congruences for the central binomial and trinomial coefficients. We are able to use these results to describe the Motzkin numbers and their relatives modulo 3. They also prove various conjectures of Benoit Cloitre [8] and Reinhard Zumkeller [25]. The Thue-Morse sequence appears again. Our main tool in this section is Lucas’ congruence for multinomial coefficients [20].

Our final section is a collection of miscellaneous results and conjectures about sequences related to binomial coefficients. These include the Apéry numbers, the central Delannoy and Eulerian numbers, Gould’s sequence, and the sequence enumerating noncrossing graphs.

## 2 Catalan numbers

If \( n, m \in \mathbb{N} \) with \( m \geq 2 \) then the order of \( n \) modulo \( m \) is

\[
\omega_m(n) = \text{largest power of } m \text{ dividing } n.
\]

If the base \( m \) expansion of \( n \) is

\[
n = n_0 + n_1 m + n_2 m^2 + \cdots
\]
then let
\[ \Delta_m(n) = \{ i : n_i = 1 \} \]
and
\[ \delta_m(n) = |\Delta_m(n)| \]
where the absolute value signs denotes cardinality. We will also use a pound sign for this purpose. If a subscript \( m \) is not used then we are assuming \( m = 2 \) and in this case \( \delta(n) \) is also the sum of the digits in the base 2 expansion of \( n \).

We wish to prove the following theorem.

**Theorem 2.1** For \( n \in \mathbb{N} \) we have
\[ \omega(C_n) = \delta(n + 1) - 1. \]

Note as an immediate corollary that \( C_n \) is odd if and only if \( n = 2^h - 1 \) for some \( h \in \mathbb{N} \). It is easy to prove this theorem from Kummer’s result about the order of a binomial coefficient [18] (or see [10, pp. 270–271]). However, we wish to give a combinatorial proof.

We will use a standard interpretation of \( C_n \) using binary trees. A binary tree \( T \) is a tree with a root \( r \) where every vertex has a left child, or a right child, or both, or neither. Note that this differs from the convention where a vertex in a binary tree must have no children or both children. It will also be convenient to consider \( T = \emptyset \) as a binary tree. With this convention, any nonempty tree can be written as \( T = (T', T'') \) where \( T' \) and \( T'' \) are the subtrees generated by the left child and by the right child of \( r \), respectively. (The subtree generated by a vertex \( v \) of \( T \) consists of \( v \) and all its descendants.) Let \( T_n \) be the set of all binary trees on \( n \) vertices. Then it is well-known that \( |T_n| = C_n \) for all \( n \in \mathbb{N} \).

The height of a vertex \( v \) is the length of the unique path from the root \( r \) to \( v \). A complete binary tree \( T_h \) has all \( 2^i \) possible vertices at height \( i \) for \( 0 \leq i \leq h \) and no other vertices. Let \( G_h \) be the group of automorphisms of \( T_h \) as a rooted tree. We will need some facts about \( G_h \).

**Lemma 2.2** We have the following

1. If \( h = 0 \) then \( G_0 = \{ e \} \) where \( e \) is the identity element, and if \( h \geq 1 \) then
   \[ G_h = Z_2 \wr G_{h-1} \]
   where \( Z_2 \) is the cyclic group of order 2 and \( \wr \) is wreath product.
2. \( |G_h| = 2^{2h-1} \).
3. If \( G_h \) acts on a set and \( O \) is an orbit of the action then \( |O| \) is a power of 2.
Proof The proof of (1) follows by noting that $T_h = (T_{h-1}, T_{h-1})$ for $h \geq 1$. Then (2) is an easy induction on $h$ using (1). Finally, (3) is a consequence of (2) and the fact that for any group action the size of an orbit always divides the order of the group.

Now $G_n$ acts on $T_n$ in the obvious way. It is this action which will permit us to calculate $\omega(C_n)$. Recall the double factorial 

$$(2d)!! = (2d - 1)(2d - 3) \cdots 3 \cdot 1.$$  

Lemma 2.3 For $n \in \mathbb{N}$, let $d = \delta(n+1) - 1$. Then given any orbit $O$ of $G_n$ acting on $T_n$ we have

$$\omega(\#O) \geq d$$

with equality for exactly $(2d)!!$ orbits.

Proof We will induct on $n$ with the result being trivial for $n = 0$. For $n \geq 1$ let $T = (T', T'') \in T_n$. We also let $n'$ and $n''$ be the number of vertices of $T'$ and $T''$ respectively, as well as setting $d' = \delta(n' + 1) - 1$ and $d'' = \delta(n'' + 1) - 1$. Clearly $n + 1 = (n' + 1) + (n'' + 1)$. It follows that

$$d \leq d' + d'' + 1$$  

(2)

with equality if and only if we have a disjoint union $\Delta(n+1) = \Delta(n'+1) \uplus \Delta(n''+1)$.

Let $O(T)$ denote the orbit of $T$. Then

$$|O(T)| = \begin{cases} |O(T')|^2 & \text{if } T' \cong T'', \\ 2|O(T')||O(T'')| & \text{otherwise.} \end{cases}$$  

(3)

Also we have, by induction, $\omega(\#O(T')) \geq d'$ and $\omega(\#O(T'')) \geq d''$.

First consider the case when $T' \cong T''$. Then $n' = n''$ and so equation (2) gives $d < 2d' + 1$. Now from (3) we obtain

$$\omega(\#O(T)) = 2\omega(\#O(T')) \geq 2d' \geq d$$

as desired for the first half of the lemma. If we actually have $\omega(\#O(T)) = d$ then this forces $2d' = d$. But since $n' = n''$ we also have $n + 1 = 2(n' + 1)$ and so $d = d'$. This can only happen if $d = d' = 0$ and consequently $n = 2^h - 1$ for some $h$. But by the third part of the previous lemma, $T_h$ is the unique tree with $2^h - 1$ vertices and $\omega(\#O(T)) = 0$. Since in this case $(2d)!! = 0!! = 1$, we have proven the present lemma when $T' \cong T''$. 
Now consider what happens when $T' \not\cong T''$. Using equations (2) and (3) as before gives

$$\omega(\#O(T)) = \omega(\#O(T')) + \omega(\#O(T'')) + 1 \geq d' + d'' + 1 \geq d$$

and again the first half of the lemma follows. When $\omega(\#O(T)) = d$ then we must have $\omega(\#O(T')) = d'$, $\omega(\#O(T'')) = d''$, and $\Delta(n+1) = \Delta(n'+1) \cup \Delta(n''+1)$. Using (3) to count orbits and induction it follows that we will be done if we can show

$$(2d)!! = \frac{1}{2} \sum_{k=1}^{d} \binom{d+1}{k} (2k-2)!!(2d-2k)!!$$

for $d \geq 1$. Rewriting this equation in hypergeometric series form we obtain the equivalent identity

$$2F_1 \left( -d - 1, -\frac{1}{2}; \frac{1}{2} - d, 1 \right) = 0$$

which is true by Vandermonde’s convolution.

We can now prove Theorem 2.1. Since the orbits of a group action partition the set acted on, we can use Lemma 2.2 (3) and Lemma 2.3 to write

$$C_n = \#T_n = (2d)!!2^d + k2^{d+1}$$

for some $k \in \mathbb{N}$. Since $(2d)!!$ is odd we can conclude $\omega(C_n) = d = \delta(n+1) - 1$ as desired.

The reader may not be happy with the last step in the proof of Lemma 2.3 since its appeal to the theory of hypergeometric series is not combinatorial. So we wish to give a bijective proof of equation (4). For this, we will interpret the double factorial in terms of binary total partitions, an object introduced and enumerated by Schröder [22]. Given a set $S$ then a binary total partition of $S$ is an unordered rooted tree $B$ satisfying the following restrictions.

1. Every vertex of $B$ has 0 or 2 children.

2. Every vertex of $B$ is labeled with a subset of $S$ in such a way that

   (a) the root is labeled with $S$ and the leaves with the 1-element subsets of $S$,
   (b) if a vertex is labeled with $A$ and its children with $A', A''$ then $A = A' \cup A''$. 

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Figure 1: A total binary partition

For example, if $S = \{1, 2, 3, 4\}$ then one possible total binary partition is displayed in Figure 1. Let $b_d$ be the number of total binary partitions on set $S$ with $|S| = d$. Then

$$b_{d+1} = (2d)!!$$

For proofs of this fact, including a combinatorial one, see the text of Stanley [26, Example 5.2.6].

It is now easy to prove (4) combinatorially. The left side counts total binary partitions $B$ of a set $S$ with $|S| = d + 1$. For the right side, note that each such $B$ can be formed uniquely by writing $S = S' \cup S''$, letting $S'$ and $S''$ label the children of the root, and then forming total binary partitions on $S'$ and $S''$ to create the rest of $B$. If $|S'| = k$ then there are $\binom{d+1}{k}$ choices for $S'$ (after which, $S''$ is uniquely determined). The factors $(2k - 2)!!$ and $(2d - 2k)!!$ count the number of ways to put total binary partitions on $S'$ and $S''$, respectively. Finally, we must sum over all possible $k$ and divide by 2 since the tree is unordered. This completes the combinatorial proof of (4).

3 Motzkin numbers and related sequences

To find the parity of $M_n$ we must first introduce a related sequence. Define $c = (c_0, c_1, c_2, \ldots) = (1, 3, 4, 5, 7, \ldots)$ inductively by $c_0 = 1$ and for $n \geq 0$

$$c_{n+1} = \begin{cases} 
  c_n + 1 & \text{if } (c_n + 1)/2 \not\in c, \\
  c_n + 2 & \text{otherwise}.
\end{cases}$$

Equivalently, $c$ is the lexicographically least sequence of positive integers such that

$$m \in c \text{ if and only if } m/2 \not\in c.$$
It follows that \( c \) contains all the positive odd integers \( m \) since in this case \( m/2 \) is not integral.

The sequence \( c \) is intimately connected with the *Thue-Morse sequence* \( t = (t_0, t_1, t_2, \ldots) = (0, 1, 1, 0, 1, 0, 1, 0, \ldots) \) which is the 0-1 sequence defined inductively by \( t_0 = 0 \) and for \( n \geq 1 \)

\[
t_n = \begin{cases} 
  t_{n/2} & \text{if } n \text{ even}, \\
  1 - t_{(n-1)/2} & \text{if } n \text{ odd}.
\end{cases}
\]

A *block* of a sequence is a maximal subsequence of consecutive, equal elements. One can show [2] that \( c_n - c_{n-1} \) is the length of the \( n \)th block of \( t \) (where we start with the 0th block and set \( c_{-1} = 0 \)).

Given a sequence \( s = (s_0, s_1, s_2, \ldots) \) and integers \( k, l \) we let

\[
ks + l = (ks_0 + l, ks_1 + l, ks_2 + l, \ldots).
\]

To simplify our notation, we will also write \( k \equiv l \pmod{m} \) as \( k \equiv_m l \) with the usual convention that if the modulus is omitted then \( m = 2 \). We can now characterize the parity of \( M_n \).

**Theorem 3.1** *The Motzkin number \( M_n \) is even if and only if either \( n \in 4c - 2 \) or \( n \in 4c - 1 \).*

**Proof** To prove this result we will need a combinatorial interpretation of \( M_n \). A *0-1-2 tree* is an ordered tree where each vertex has at most two children (but a single child is not distinguished by being either left or right). It is known that \( M_n \) is the number of 0-1-2 trees with \( n \) edges. See the articles of Donaghey [11] and Donaghey and Shapiro [12] for details. The four 0-1-2 trees with three edges are shown in Figure 2.

Now let \( S_n \) be the number of *symmetric 0-1-2 trees* which are those with \( n \) edges for which reflection in a vertical line containing the root is an automorphism of the tree. Only the first two trees in Figure 2 are symmetric. Clearly

\[
M_n \equiv S_n
\]

for all \( n \in \mathbb{N} \). Furthermore,

\[
S_{2n+1} = S_{2n}
\]

since if a symmetric 0-1-2 tree has \( 2n+1 \) edges then the root must have a single child and the subtree generated by that child must be a symmetric 0-1-2 tree with \( 2n \) edges. So to prove the theorem, it suffices to show that

\( S_{2n} \) is even if and only if \( 2n \in 4c - 2 \).
This can be restated that $S_{2n-2}$ is even iff $2n \in 4c$ which is equivalent to $n \in 2c$. So, by (6), it suffices to prove

$$S_{2n-2} \text{ is even if and only if } n \not\in c. \quad (10)$$

To prove (10), we will need a recursion involving $S_{2n-2}$. Let $T$ be a symmetric 0-1-2 tree with $2n - 2$ edges. If the root of $T$ has one child then the subtree generated by that child is a symmetric 0-1-2 tree with $2n - 3$ edges. If the root has two children then the subtree generated by one child can be any 0-1-2 tree with $n - 2$ edges as long as the subtree generated by the other is its reflection. So using (8) and (9)

$$S_{2n-2} = S_{2n-3} + M_{n-2} \equiv S_{2n-4} + S_{n-2}. \quad (11)$$

We now prove (10) by induction, where the case $n = 1$ is trivial. Suppose first that $n \not\in c$. Then by (5) we have $n - 1 \in c$ and by induction it follows that $S_{2n-4} = S_{2(n-1)-2}$ is odd. Also, since $n \not\in c$ we must have that $n$ is even. Furthermore, by (6) we have $n/2 \in c$. By induction again, $S_{n-2} = S_{2(n/2)-2}$ is odd. So $S_{2n-4} + S_{n-2}$ is even and we are done with this direction.

When $n \in c$, one can use similar reasoning to show that $S_{2n-4} + S_{n-2}$ is odd. One needs to consider the cases when $n$ is even and odd separately (and the latter case breaks into two subcases depending on whether $n - 1$ is in $c$ or not). But there are no really new ideas to the demonstration, so we omit the details.

We should note that Theorem 3.1 can also be derived from the results in [17], although it is not explicitly stated there. This theorem also permits us to determine the parity of various related sequences which we will now proceed to do.

A Motzkin path of length $n$ is a lattice path in the lattice $\mathbb{N} \times \mathbb{N}$ with steps $(1,1), (1,-1),$ and $(1,0)$ starting at $(0,0)$ and ending at $(n,0)$. It is well known...
that $M_n$ is the number of Motzkin paths of length $n$. (Note that we do not need any condition about staying above the $x$-axis since we are working in $\mathbb{N} \times \mathbb{N}$.) Define a **Motzkin prefix of length** $n$ to be a lattice path which forms the first $n$ steps of a Motzkin path of length $m \geq n$. Equivalently, a Motzkin prefix is exactly like a Motzkin path except that the endpoint is not specified. Let $P_n$, $n \geq 0$, be the number of Motzkin prefixes of length $n$. This is sequence A005773 in Sloane’s Encyclopedia [25]. The $P_n$ also count directed rooted animals with $n + 1$ vertices as proved by Gouyou-Beauchamps and Viennot [15].

**Corollary 3.2** The number $P_n$ is even if and only if $n \in 2c - 1$.

**Proof** Let $s_n$ be the number of Motzkin paths of length $n$ which are symmetric with respect to reflection in the vertical line $x = n/2$. Clearly $M_n \equiv s_n$ for all $n \geq 0$. There is also a bijection between Motzkin prefixes of length $n$ and symmetric Motzkin paths of length $2n$ gotten by concatenating the prefix with its reflection in the line $x = n$. So $P_n = s_{2n}$. Combining this with the previous congruence and Theorem 3.1 completes the proof.

Next we consider the **Riordan numbers** [25, A005043] $\gamma_n$ which count the number of ordered trees with $n$ edges where every nonleaf has at least two children. These are called **short bushes** by Bernhart [7]. If we relax the degree restriction so that the root can have any number of children then the resulting trees are called **bushes**. It is known [11, 12] that $M_n$ is the number of bushes with $n + 1$ edges. It follows that

\[ M_n = \gamma_{n+1} + \gamma_n \quad (12) \]

since every bush with $n + 1$ edges is either a short bush or has a root with one child which generates a short bush with $n$ edges.

**Corollary 3.3** The number $\gamma_n$ is even if and only if $n \in 2c - 1$.

**Proof** Given the previous corollary, it suffices to show that $\gamma_n$ and $P_n$ have the same parity. So it suffices to show that the two sequences satisfy the same recursion and boundary condition modulo 2. Now $\gamma_0 = 1 = P_0$ and we have just seen that

\[ \gamma_{n+1} \equiv \gamma_n + M_n. \]

So consider the prefixes $p$ counted by $P_{n+1}$. If $p$ goes through $(n, 0)$ then there are two possible last steps for $p$ and so such paths need not be considered modulo 2. If $p$ goes through $(n, m)$ where $m > 0$ then those $p$ ending with a $(1, 1)$ step can be paired with those ending with a $(1, -1)$ step and ignored. So we are left with prefixes going through $(n, m)$ and $(n+1, m)$ where $m > 0$. Such prefixes are
equinumerous with those ending at \((n, m)\). And since \(m > 0\), this is precisely the set of Motzkin prefixes which are not Motzkin paths. So

\[ P_{n+1} \equiv P_n - M_n \equiv P_n + M_n \]

as desired.

Finally, consider the sequence counting restricted hexagonal polyominos [25, A002212]. The reader can find the precise definition of these objects in the paper of Harary and Read [16]. We will use an equivalent definition in terms of trees which can be obtained from the polyomino version by connecting the centers of adjacent hexagons. A \textit{ternary tree} is a rooted tree where every vertex has some subset of three possible children: a left child, a middle child, or a right child. Just as with our definition of binary trees, this differs from the all or none convention for ternary trees. A \textit{hex tree} is a ternary tree where no node can have two adjacent children. (A middle child would be adjacent to either a left or a right child but left and right children are not adjacent.) Let \(H_n, n \geq 0\), be the number of hex trees having \(n\) edges.

**Corollary 3.4** The number \(H_n\) is even if and only if \(n \in 4c - 2\) or \(n \in 4c - 1\).

**Proof** In view of Theorem 3.1, it suffices to show that \(H_n\) and \(M_n\) have the same parity. Call a hex tree \textit{symmetric} if the reflection in a line containing the root leaves it invariant, and let \(h_n\) be the number of such trees with \(n\) edges. There is an obvious bijection between symmetric hex trees and symmetric 0-1-2 trees. So

\[ H_n \equiv h_n = S_n \equiv M_n \]

as desired.

4 Central binomial and trinomial coefficients

Our main tool in this section will be the following famous congruence of Lucas. If the base \(p\) expansion of \(n\) is

\[ n = n_0 + n_1p + n_2p^2 + \cdots \]

then it will be convenient to denote the sequence of digits by

\[ (n)_p = (n_0, n_1, n_2, \ldots) = (n_i). \]
Theorem 4.1 (Lucas [20]) Let $p$ be a prime and let $(n)_p = (n_i)$ and $(k)_p = (k_i)$. Then
\[
\binom{n}{k} \equiv_p \prod_i \binom{n_i}{k_i}.
\]  \hfill \blacksquare

The following corollary will be useful as well. It is also a special case of the theorem of Kummer cited in the discussion following the statement of Theorem 2.1. But this result will be sufficient for our purposes.

Corollary 4.2 Let $p$ be prime. If there is a carry when adding $k$ to $n-k$ in base $p$ then
\[
\binom{n}{k} \equiv_p 0.
\]

Proof Using the notation of the previous theorem, if there is a carry out of the $i$th place then we have $n_i < k_i$. So $(n_i/k_i) = 0$ and thus the product side of (13) is zero. \hfill \blacksquare

Most of our results in this section will have to do with congruences modulo 3 so it will be useful to have the following notation. Given $i, j$ distinct integers in $\{0, 1, 2\}$ we let
\[
T(ij) = \{n \in \mathbb{N} : (n)_3 \text{ contains only digits equal to } i \text{ or } j\}.
\]

We begin with the central binomial coefficients. Recall that $\delta_3(n)$ is the number of ones in the base three expansion of $n$. The next result settles conjectures of Benoit Cloitre and Reinhard Zumkeller [25, A074938–40].

Theorem 4.3 The central binomial coefficients satisfy
\[
\binom{2n}{n} \equiv_3 \begin{cases} (-1)^{\delta_3(n)} & \text{if } n \in T(01), \\ 0 & \text{otherwise}. \end{cases}
\]

Proof If $n$ has a 2 in its ternary expansion then there is a carry when adding $(n)_3$ to itself. So the second half of the theorem follows from the previous corollary. On the other hand, if $n \in T(01)$ then $2n \in T(02)$ and $(2n)_3$ has twos exactly where $(n)_3$ has ones. So by Lucas’ Theorem
\[
\binom{2n}{n} \equiv_3 \binom{2}{1}^{\delta_3(n)} \equiv_3 (-1)^{\delta_3(n)}
\]
giving the first half. \hfill \blacksquare
It is easy to generalize the previous theorem to arbitrary prime modulus. To state the result, we need to define
\[ \delta_{p,j}(n) = \text{number of elements of } (n)_p \text{ equal to } j \] (14)
where \( 0 \leq j < p \). Since the proof of the general case is the same as the one just given, we omit it.

**Theorem 4.4** Let \( p \) be prime and let \( S \) be the set of all \( n \in \mathbb{N} \) such that all elements of \( (n)_p \) are less than or equal to \( p/2 \). Then
\[
\binom{2n}{n} \equiv_p \begin{cases} 
\prod_j \binom{2j}{j}^{\delta_{p,j}(n)} & \text{if } n \in S, \\
0 & \text{otherwise.} 
\end{cases}
\]

It turns out that there is a connection between the central binomial coefficients modulo 3 and the Thue-Morse sequence \( t \). This may seem surprising because \( t \) is essentially a modulo 2 object. However, Theorem 4.3 will allow us to reduce questions about \( \binom{2n}{n} \mod 3 \) to questions about bit strings. We will need another one of the many definitions of \( t \) for the proof, namely
\[ t_n = \rho(\delta(n)) \] (15)
where \( \rho(k) \) is the remainder of \( k \) on division by 2. We will also need the notation that \( a \equiv_m b \) as sequences if and only if \( a_n \equiv_m b_n \) for all \( n \in \mathbb{N} \). The next result is again a conjecture of Cloitre [25, A074939].

**Theorem 4.5** We have
\[ \left\{ n : \binom{2n}{n} \equiv_3 1 \right\} \equiv_3 t. \]

**Proof** Let us call the sequence on the left of the previous congruence \( a \). Then from Theorem 4.3 we have that \( n \in a \) exactly when \( n \in T(01) \) and \( (n)_3 \) has an even number of ones. From this it follows by an easy induction that \( n = a_i \) if and only if \( (n)_3 = (n_0, n_1, n_2, \ldots) \) where \( (i)_2 = (n_1, n_2, \ldots) \) and \( n_0 \) is zero or one depending on whether \( \delta(i) \) is even or odd, respectively. So by (15) we have
\[ a_i = n \equiv_3 n_0 = \rho(\delta(i)) = t_i \]
for all \( i \geq 0 \). ■

There is an analogous conjecture of Cloitre for those central binomial coefficients with residue \(-1\) modulo 3 [25, A074938]. Since the proof is much the same as the previous one, we omit it.
Theorem 4.6 We have
\[
\left( n : \binom{2n}{n} \equiv_3 -1 \right) \equiv_3 1 - t. \quad \blacksquare
\]

We next consider the central trinomial coefficients [25, A002426]. Let \( T_n \) be the largest coefficient in the expansion of \((1 + x + x^2)^n\). It is easy [6] to express \( T_n \) in terms of trinomial coefficients
\[
T_n = \sum_{k \geq 0} \binom{n}{k, k, n-2k}
\]
where we use the convention that if any multinomial coefficient has a negative number on the bottom then the coefficient is zero. Lucas’ Theorem and its corollary generalize in the expected way to multinomial coefficients. So now we can find the residue of \( T_n \) modulo 3.

Theorem 4.7 The central trinomial coefficients satisfy
\[
T_n \equiv_3 \begin{cases} 
1 & \text{if } n \in T(01), \\
0 & \text{otherwise}.
\end{cases}
\]

Proof Modulo 3 we can restrict the sum in (16) to those \( k \) such that there is no carry in doing the triple addition \( k + k + (n - 2k) \) in base 3. So, in particular, we can restrict to \( k \in T(01) \) since if \((k)_3 = (k_i)\) contains a 2 then we will have such a carry. Furthermore, if \( k_i = 1 \) for some \( i \) then \( k + k \) has a two in the \( i \)th place, and to have no carry this forces \( n_i = 2 \).

Now let \((n - 2k)_3 = (l_i)\) and let \( S \) be the set of indices \( i \) such that \( n_i = 2 \). So we have shown that \( \Delta_3(k) \subseteq S \). Furthermore, for every \( i \not\in \Delta_3(k) \) we must have \( l_i = n_i \) since \( k \in T(01) \). So the nonzero terms in the sum correspond to subsets \( R \subseteq S \) and each such subset contributes
\[
\left( \begin{array}{c} 2 \\ 1, 1, 0 \end{array} \right)^{|R|} = 2^{|R|}.
\]

Hence, by the binomial theorem, the total contribution is
\[
\sum_{R \subseteq S} 2^{|R|} = 3^{|S|} \equiv_3 \begin{cases} 
1 & \text{if } S = \emptyset, \\
0 & \text{if } S \neq \emptyset.
\end{cases}
\]

But \( S = \emptyset \) precisely when \( n \in T(01) \), so we are done. \( \blacksquare \)

Since the \( T_n \) are related to a number of the other sequences which we have been studying, we can use the previous result to determine their behavior modulo 3. We will apply linear operations to sets the same way we do to sequences (7).
Corollary 4.8  The Motzkin numbers satisfy

\[
M_n \equiv_3 \begin{cases} 
-1 & \text{if } n \in 3T(01) - 1, \\
1 & \text{if } n \in 3T(01) \text{ or } n \in 3T(01) - 2, \\
0 & \text{otherwise}.
\end{cases}
\]

Proof Barcucci, Pinzani, and Sprugnoli [6] have shown that

\[2M_n = 3T_n + 2T_{n+1} - T_{n+2}.\] (17)

Reducing this equation modulo 3 and applying the previous theorem finishes the proof.

Corollary 4.9  The Motzkin prefix numbers satisfy

\[
P_n \equiv_3 \begin{cases} 
1 & \text{if } n \in 3T(01), \\
-1 & \text{if } n \in 3T(01) + 1 \text{ or } 3T(01) - 1, \\
0 & \text{otherwise}.
\end{cases}
\]

Proof If \(p\) is a Motzkin prefix of length \(n\) going through \((n - 1, m)\) for some \(m > 0\) then there are three ways to end the prefix and so they cancel out modulo 3. If \(p\) goes through \((n - 1, 0)\) then the first \(n - 1\) steps of \(p\) form a Motzkin path and there are two possible last steps. So \(P_n \equiv_3 2M_{n-1}\). Now apply the previous corollary to finish.

Corollary 4.10  The Riordan numbers satisfy

\[
\gamma_n \equiv_3 \begin{cases} 
1 & \text{if } n \in T(01) - 1, \\
0 & \text{otherwise}.
\end{cases}
\]

Proof Using the recursions (12) and (17) it is easy to prove inductively that \(\gamma_n \equiv_3 T_{n+1}\). Theorem 4.7 now completes the proof.

5  Miscellaneous results and conjectures

We end with various results and conjectures related to what we have done in the previous sections.
5.1 Catalan numbers to other moduli

Theorem 2.1 implies that the $k$th block of zeros in the sequence of Catalan numbers modulo 2 has length $2^k - 1$ (where we start numbering with the first block). Alter and Kubota [4] have generalized this result to arbitrary primes and prime powers. One of their main theorems is as follows.

**Theorem 5.1** (Alter and Kubota) Let $p \geq 3$ be a prime and let $q = (p + 1)/2$. The length of the $k$th block of zeros of the Catalan sequence modulo $p$ is

$$\left( p^{\omega_q(k) + \delta_{3,p} + 1} - 3 \right) / 2$$

where $\delta_{3,p}$ is the Kronecker delta.

We can improve on this theorem in several regards. First of all, when $p = 3$ we can use our results to give a complete characterization of the residue of $C_n$ and not just say when it is zero. Suppose $(n)_3 = (n_i)$. Then we let

$$T^*(01) = \{ n : n_i = 0 \text{ or } 1 \text{ for all } i \geq 1 \} \quad \text{and} \quad \delta^*_3(n) = \text{number of } n_i = 1 \text{ for } i \geq 1.$$  

**Theorem 5.2** The Catalan numbers satisfy

$$C_n \equiv_3 \begin{cases} (-1)^{\delta^*_3(n+1)} & \text{if } n \in T^*(01) - 1, \\ 0 & \text{otherwise}. \end{cases}$$

**Proof** The result is easy to verify for $n \leq 1$ so we assume $n \geq 2$. Directly from our definition of $C_n$ we have

$$C_n = \frac{4n - 2}{n + 1} C_{n-1}$$

If $n \equiv 3 \text{ 0 or 1 then } n + 1$ is invertible modulo 3 and in fact $(4n - 2)/(n + 1) \equiv 3 1$. So for $k \geq 1$ we have $C_{3k-1} \equiv_3 C_{3k} \equiv_3 C_{3k+1}$. Thus it suffices to prove the theorem for $n \equiv_3 0$. Notice that in this case $C_n \equiv_3 \binom{2n}{n}$, Furthermore $n + 1 \in T^*(01)$ if and only if $n \in T(01)$. And lastly $\delta^*_3(n + 1) = \delta_3(n)$. Applying Theorem 2.1 finishes the proof.

We should verify that we can derive the $p = 3$ block lengths in Theorem 5.1 from Theorem 5.2. First from the latter result it follows that the $k$th block must start at an integer $3a - 1$ and end at $3b - 1$ for $a, b \in \mathbb{N}$. To simplify notation, let $\omega = \omega_2(k)$. Now $(a)_3$ must contain a 2 and $(a-1)_3$ does not. It follows that $(a)_3 =
(a_0, a_1, a_2, \ldots) where a_0 = 2 and (a_1, a_2, \ldots) = (k - 1)_2. Furthermore, since b + 1 is the smallest integer larger than a whose expansion contains no twos, the first \omega + 1 elements of (b)_3 must all equal 2 and the rest must agree with the corresponding entries of (a)_3. By the same token, we must have a_1 = a_2 = \ldots = a_\omega = 1. Now one calculates the number of integers in the kth block by considering the first \omega + 1 digits of a and b to get a count of

\[ 3(a - b + 1) = 3[(3^{\omega+1} - 1) - (3^\omega + 3^{\omega-1} + \cdots + 3 + 2) + 1] = (3^{\omega+2} - 3)/2 \]

as desired. Note that not only have we been able to determine the length and starting and ending points of the block (which was also done by Alter and Kubota) but our demonstration is combinatorial as opposed to the original proof of Theorem 5.1 which is arithmetic. We had to use Lucas’ Theorem to get to this result, but that theorem also has a combinatorial demonstration using group actions [21].

When \( p \geq 5 \), the residues of \( C_n \) become more complicated, but one could use the same techniques in principle to compute them. In particular, if one is only interested in divisibility then one can derive Theorem 5.1 from Theorem 4.4 as we did for the \( p = 3 \) case above.

It is also interesting another setting where a congruence involving the Catalan numbers has arisen. Albert, Atkinson, and Klazar [1] have studied simple permutations which are those permutations of \( \{1, 2, \ldots, n\} \) mapping no nontrivial subinterval of this set onto an interval. Then the number of such simple permutations is \( 2(-1)^{n+1} - \text{Com}_n \) where \( \text{Com}_n \) is the coefficient of \( x^n \) in the compositional inverse of the formal power series \( \sum_{n \geq 1} n! x^n \) [25, A059372]. One of the results in [1] is that

\[ \text{Com}_n \equiv_3 C_{n-1}. \]

Their proof of this result uses generating functions, so it would be interesting to find a combinatorial one. Also, one would like to know the behavior of \( \text{Com}_n \) modulo other odd primes. (Albert et. al. have results for powers of two.)

The careful reader will note that we have not yet derived the residues of the hex tree numbers \( H_n \) modulo three. It is time to fill that lacuna.

**Theorem 5.3** The hex tree numbers satisfy

\[
H_n \equiv_3 \begin{cases} 
(-1)^{\delta_3(m+1)} & \text{if } n = 2m \text{ where } m \in T^*((01) - 1, \\
0 & \text{otherwise.}
\end{cases}
\]

**Proof** Suppose \( T \) is a hex tree which has a vertex with a single child. Finding the first such vertex, say in depth-first order, one can associate with \( T \) the two other hex trees which differ from \( T \) only by moving the child into the two other possible positions. So modulo 3, \( H_n \) is congruent to the number of hex tree with \( n \) edges
where every vertex has 0 or 2 children. So to be nonzero modulo 3, we must have \( n = 2m \) and the resulting trees are in bijection with binary trees on \( m \) vertices (merely remove the \( m \) leaves of the hex tree). Thus \( H_n \equiv_3 C_m \) and we are now done by Theorem 5.2.

\[ \text{■} \]

### 5.2 Motzkin numbers to other moduli

For the Motzkin numbers, nothing has been proved for moduli other than 2 and 3. However, there are some conjectures. To put them in the context of Theorem 3.1, note that the Thue-Morse block sequence \( c \) can also be described \([2]\) as the increasing sequence of all numbers of the form

\[ (2i + 1)4^j \text{ where } i, j \in \mathbb{N}. \]

The following conjecture is due in part to Tewodros Amdeberhan \([5]\).

**Conjecture 5.4 (Amdeberhan)** We have \( M_n \equiv_4 0 \) if and only if

\[ n = (4i + 1)4^{j+1} - 1 \text{ or } n = (4i + 3)4^{j+1} - 2 \text{ where } i, j \in \mathbb{N}. \]

Furthermore we never have \( M_n \equiv_8 0 \).

Amdeberhan also has a conjecture about some of the values of \( n \) for which \( M_n \) is zero modulo 5, although it is complicated.

### 5.3 Gould’s sequence

**Gould’s sequence** \([25, \text{A001316}]\) consists of the numbers \( G_n \) which count the number of odd entries in the \( n \)th row of Pascal’s triangle. More generally, we can calculate \( G_n(p) \) which is the number of entries in the \( n \)th row of Pascal’s triangle which are not zero modulo the prime \( p \). Recall the definition of \( \delta_{p,j}(n) \) in \((14)\).

**Theorem 5.5** Let \( p \) be prime. Then

\[ G_n(p) = \prod_{1 \leq j < p} (j + 1)^{\delta_{p,j}(n)}. \]

Furthermore, every entry of the \( n \)th row of Pascal’s triangle is nonzero modulo \( p \) if and only if

\[ n = qp^k - 1 \]

where \( 1 \leq q < p \) and \( k \in \mathbb{N} \). In particular

\[ G_n = 2^{\delta(n)} \]

and every entry of the \( n \)th row of Pascal’s triangle is odd if and only if \( n = 2^k - 1 \) where \( k \in \mathbb{N} \).
Proof Suppose \( \binom{n}{k} \not\equiv p 0 \) where \((n)_p = (n_i)\) and \((k)_p = (k_i)\). If \( n_i = j \) then we will not have a carry in the \( i \)th place if and only if \( 0 \leq k_i \leq j \). So there are \( j + 1 \) choices for \( k_i \) and taking the product of the number of choices for each \( i \) gives the first statement of the theorem.

Now suppose that every entry of the \( n \)th row is nonzero modulo \( p \). Since there are no carries for all \( k \), all the elements of \((n)_p\) must equal \( p - 1 \) except for possibly the last (leading) one \( n_l \). Since there can never be a carry out of \( n \)’s last place, we have the desired characterization of those \( n \) under consideration.

5.4 Sums of central binomial coefficients

The partial sums of central binomial coefficients \([25, A006134]\) also have nice congruence properties. The proof of the next result is easily obtained by using Theorem 4.3 and induction on \( n \), so we omit it. In conjunction with Theorem 4.5, it settles a conjecture of Cloitre \([25, A083096]\).

**Theorem 5.6** We have

\[
\sum_{k \geq 0} \binom{2k}{k} \equiv_3 \begin{cases} (-1)^{\delta_3(n)} & \text{if } n \in 3T(01), \\ 0 & \text{otherwise}. \end{cases}
\]

5.5 Apéry numbers and central Delannoy numbers

We can generalize our results about the central trinomial numbers as follows. Given positive integers \( r, s \) we define a sequence with the following entries

\[
a_n(r, s) = \sum_{k \geq 0} \binom{n}{k}^r \binom{n+k}{k}^s.
\]

Note that since \( r, s \) are positive, each term in this sum will have a factor of

\[
\binom{n}{k} \binom{n+k}{k} = \binom{n+k}{k, k, n-k}.
\]

Using this fact we can prove the following result. Since the demonstration is similar to that of Theorem 4.7, it is omitted. Again, this settles a conjecture of Cloitre \([8]\).

**Theorem 5.7** Let \( r, s \) be positive integers. Then

\[
a_n(r, s) \equiv_3 \begin{cases} (-1)^{\delta_3(n)} & \text{if } s \text{ is even}, \\ 1 & \text{if } s \text{ is odd and } n \in T(02), \\ 0 & \text{otherwise}. \end{cases}
\]
The central Delannoy numbers [25, A001850] are $D_n = a_n(1, 1)$. Also, the Apéry numbers [25, A005258] are $A_n = a_n(2, 1)$. So we immediately have the following corollary.

**Theorem 5.8** The central Delannoy numbers and Apéry numbers satisfy

$$D_n \equiv_3 A_n \equiv_3 \begin{cases} 1 & \text{if } n \in T(02), \\ 0 & \text{otherwise.} \end{cases}$$

### 5.6 Central Eulerian numbers

The Eulerian numbers [25, A008292] are denoted $A(n, k)$ and count the number of permutations in the symmetric group $S_n$ which have $k - 1$ descents. They can be written as

$$A(n, k) = \sum_{i=0}^{k} (-1)^i (k - i)^n \binom{n + 1}{i}.$$

Since the odd numbered rows have an odd number of elements, we define the central Eulerian numbers to be

$$E_n = A(2n - 1, n) = \sum_{i=0}^{n} (-1)^i (n - i)^{2n-1} \binom{2n}{i}.$$

We have the following congruence for these numbers.

**Theorem 5.9** The central Eulerian numbers satisfy

$$E_n \equiv_3 \begin{cases} 1 & \text{if } n \in T(01) + 1, \\ 0 & \text{otherwise.} \end{cases}$$

**Proof** Note that $k^{2n-1} = k$ for $k = 0, \pm 1$. So we have

$$E_n \equiv_3 \sum_{i=0}^{n} (-1)^i (n - i)^{2n-1} \binom{2n}{i}.$$

Applying the binomial recursion to this sum twice yields, after massive cancellation,

$$E_n \equiv_3 (-1)^{n-1} \binom{2n - 2}{n - 1}.$$

Now Theorem 4.3 will finish the proof provided $n + \delta_3(n)$ is always even. But this is easy to show by induction on $n$, so we are done.

Rows in the Eulerian triangle are symmetric, so even numbered rows have two equal elements in the middle. We will call these elements *bicentral*. Cloitre conjectured the residues of these elements modulo 3. Since the proof of this result is similar to the one just given, we will omit it.
Theorem 5.10  The bicentral Eulerian numbers satisfy

\[ A(2n, n) \equiv_3 \begin{cases} 
1 & \text{if } n \in 3T(01) + 1, \\
-1 & \text{if } n \in 3T(01) \text{ or } 3T(01) + 2, \\
0 & \text{otherwise.} 
\end{cases} \]

5.7 Noncrossing connected graphs

Noncrossing set partitions are an important object of study in combinatorics. An excellent survey of the area can be found in the article of Simion [23]. Noncrossing graphs are a generalization of noncrossing partitions which have been studied by Flajolet and Noy [14]. Consider vertices labeled 1, \ldots, n and arranged clockwise in this order around a circle. A graph on this vertex set is noncrossing if, when the edges are drawn with straight line segments between the vertices, none of the edges cross. Let \( N_n \) be the number on noncrossing connected graphs on \( n \) vertices [25, A007287]. Then it can be shown that

\[ N_n = \frac{1}{n-1} \sum_{k \geq 0} \binom{3n-3}{n+k+1} \binom{k}{n-2}. \]

We have the following conjecture about the residue of \( N_n \) modulo 3.

Conjecture 5.11  The number of noncrossing connected graphs satisfies

\[ N_n \equiv_3 \begin{cases} 
1 & \text{if } n = 3^i \text{ or } n = 2 \cdot 3^i \text{ for some } i \in \mathbb{N}, \\
-1 & \text{if } n = 3^i + 3^j \text{ for two distinct } i, j \in \mathbb{N}, \\
0 & \text{otherwise.} 
\end{cases} \]

In the first two cases, it is not hard to show that the congruence holds using Lucas’ Theorem because of the very specific form of \( (n)_3 \). However, we have been unable to prove that for all remaining \( n \) one always has \( N_n \) divisible by 3. It would be even more interesting to give a combinatorial proof of this result based on symmetries of the graphs involved.

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References

[1] M. Albert, M. Atkinson, and M. Klazar, The enumeration of simple permutations, *J. Integer Sequences* 6 (2003), Article 03.4.4, 18 pp.
[2] J.-P. Allouche, A. Arnold, J. Berstel, S. Brlek, W. Jockusch, S. Plouffe, and B. E. Sagan, A relative of the Thue-Morse sequence, *Discrete Math.* **139** (1995), 455–461.

[3] J.-P. Allouche and J. Shallit, The ubiquitous Prouhet-Thue-Morse sequence, in “Sequences and their applications, Proceedings of SETA’98,” C. Ding, T. Helleseth, and H. Niederreiter eds., Springer-Verlag, 1999, 1–16.

[4] R. Alter and K. Kubota, Prime and prime power divisibility of Catalan numbers, *J. Combin. Theory Ser. A* **15** (1973), 243–256.

[5] T. Amdeberhan, personal communication.

[6] E. Barcucci, R. Pinzani, R. Sprugnoli, The Motzkin family, *Pure Math. Appl. Ser. A* **2** No. 3–4 (1991), 249–279.

[7] F. Bernhart, Catalan, Motzkin, and Riordan numbers, *Discrete Math.* **204** (1999), 73–112.

[8] B. Cloitre, personal communication.

[9] E. Deutsch, An involution on Dyck paths and its consequences, *Discrete Math.* **204** (1999), 163–166.

[10] L. E. Dickson, “History of the Theory of Numbers, Vol. 1,” Chelsea, New York, NY, 1952.

[11] R. Donaghey, Restricted plane tree representations of four Motzkin-Catalan equations, *J. Combin. Theory, Ser. B* **22** (1977), 114–121.

[12] R. Donaghey and L. W. Shapiro, Motzkin numbers, *J. Combin. Theory, Ser. A* **23** (1977), 291–301.

[13] Ö. Eğecioğlu, The parity of the Catalan numbers via lattice paths, *Fibonacci Quart.* **21** (1983) 65–66.

[14] P. Flajolet and M. Noy, Analytic combinatorics of non-crossing configurations, *Discrete Math.* **204** (1999), 203–229.

[15] D. Gouyou-Beauchamps and G. Viennot, Equivalence of the two-dimensional directed animal problem to a one-dimensional path problem, *Adv. Appl. Math.* **9** (1988), 334–357.

[16] F. Harary and R. C. Read, The enumeration of tree-like polyhexes, *Proc. Edinburgh Math. Soc.* **17** (1970), 1–13.
[17] M. Klazar and F. Luca, On integrality and periodicity of the Motzkin numbers, preprint.

[18] E. E. Kummer Über die Ergänzungssätze zu den allgemeinen Reciprocitätsgesetzen, J. Reine Angew. Math. 44 (1852) 93–146.

[19] F. Luca, Prime factors of Motzkin numbers, preprint.

[20] E. Lucas, Sur les congruences des nombres eulériens et des coefficients différentiels des fonctions trigonométriques suivant un module premier, Bull. Soc. Math. France 6 (1877–1878), 49–54.

[21] B. E. Sagan, Congruences via Abelian groups, J. Number Theory 20 (1985), 210–237.

[22] E. Schröder, Vier combinatorische Probleme, Z. für Math. Phys. 15 (1870), 361–376.

[23] R. Simion, Noncrossing partitions, Discrete Math. 217 (2000), 367–409.

[24] R. Simion and D. Ullman, On the structure of the lattice of noncrossing partitions, Discrete Math. 98 (1991), 193–206.

[25] N. J. A. Sloane, “The On-Line Encyclopedia of Integer Sequences,” available at http://www.research.att.com/~njas/sequences/.

[26] R. P. Stanley, “Enumerative Combinatorics, Volume 2,” Cambridge University Press, Cambridge, 1999.