DETRENDED FLUCTUATION ANALYSIS FOR CONTINUOUS REAL VARIABLE FUNCTIONS

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Abstract. Based on the well-known Detrended Fluctuation Analysis (DFA) for time series, in this work we describe a DFA for continuous real variable functions. Under certain conditions, DFA accurately predicts the long-term auto-correlation of the time series, depending on the value of certain scaling parameter. We show that for continuous functions, the proposed continuous DFA also exhibits fractal properties and approximates a power law with scaling exponent one.

Keywords—Detrended fluctuation analysis, self-similar, power law.

Introduction

Time series analysis is a useful study area as it provides means to understand the dynamics of data collected from diverse research areas such as business, economics, medicine, volcanology, among many others [10, 3, 12]. One of the important aspects in a time series is to determine whether the data has autocorrelation. This property is related to the concept of long-memory [8], i.e it helps to determine if there exists a relation between the data in the past and the data in the future. One way to quantify this relation is through the computation of an exponent called the Hurst exponent [17, 18]. In this sense, the Detrended Fluctuation Analysis (DFA) may be regarded as a powerful method to detect self-similar patterns in non-stationary time series. It works by transforming a time series into a new time series which approximates a power law [16]; depending on the scaling exponent, one may conclude that the original time series was autocorrelated or not. This method is widely used today, for example, in [1, 11, 5, 3, 14, 15, 13], just to mention a few. Roughly speaking, DFA is a modified root mean square analysis so it seemed plausible to adapt its discrete context to the continuous setting. The ensuing natural question, which we answer in the affirmative in this paper, was whether a continuous version of the DFA would also manifest a power law. It is worth mentioning that the classical DFA model presents some problems in mitigating non-stationaries [2].

In this work we propose a DFA that is applied to continuous real functions. As in the classical DFA for time series, it will consist of two steps. In the first step we define the function called the integral process associated to a integrable function, and establish that this is a sum of two self similar fractals functions, where we regard a fractal function in the sense of Hutchinson [6, 7]. The second step consists in removing the trend and define a detrended function $F$, and prove that this function $F$ is approximately a power law.

We divide this work in two sections. In the Section 1, we review the concept of self similar fractal functions and provide some examples of such functions. In
Section 2 we present the main results of the paper, namely, we give the DFA version for continuous functions and prove that it approximates a power law with scaling exponent one.

The first author gratefully acknowledges support from CONACyT grant 1002291.

1. Self Similar Fractal Function

In this section we will introduce the definition of self similar fractal function, and give some examples. Since in the classical DFA the integrated process, a time series, is a self similar process, the definition of self similar fractal real variable function is important, as we would expect the integrated process for continuous functions to have self similar or fractal properties. The definition given here was introduced by Hutchinson [6, 7]. There are, in the literature, other definitions for self similar functions; however, the one presented here seems to give a more natural generalization of the concept of self similar sets, its formulation makes it easy to give examples and, as it will be shown here, preserves some properties that have the geometrical fractals, for example translation invariance.

**Definition 1.1.** Let $N \geq 2$. A scaling law $S$ is defined to be an $N$-tuple $(S_1, \ldots, S_N)$ of Lipschitz maps $S_i : \mathbb{R}^n \to \mathbb{R}^n$. We denote the Lipschitz constants by $\text{Lip}(S_i)$.

**Definition 1.2.** Let $I = I_1 \sqcup \cdots \sqcup I_N$ be a partition of an interval $I$ into $N$ disjoint subintervals. Given maps $g_i : I_i \to \mathbb{R}^n$ for $i = 1, \ldots, N$, define the function $\bigcup_{i=1}^N g_i : I \to \mathbb{R}^n$ by

$$(\bigcup_{i=1}^N g_i)(x) = g_j(x), \quad x \in I_j.$$ 

**Definition 1.3.** Let $f : I \to \mathbb{R}^n$ be a function where $I \subset \mathbb{R}$ is a closed bounded interval. Let $I = I_1 \sqcup \cdots \sqcup I_N$ be a partition of $I$ into disjoint subintervals and let $\phi_i : I \to I_i$ be an onto and increasing Lipschitz map for each $i \in \{1, \ldots, N\}$. Given a scaling law $S = (S_1, \ldots, S_N)$, define $Sf : I \to \mathbb{R}^n$ by

$$Sf = \bigcup_{i=1}^N S_i \circ f \circ \phi_i^{-1}.$$ 

We say that $f$ satisfies the scaling law $S$, or that $f$ is a **self similar fractal function** if

$$f = Sf.$$ 

Note that in Definition 1.3 the use of Lipschitz maps may be seen as the analogous of verifying scaling properties in $f$, and by the disjoint union we can interpret that we are joining the pieces after the scaling, so this emulates the behavior of known fractals. Consider the following important example.

**Example 1.4.** The function $f : [a, b] \to \mathbb{R}$ given by $f(x) = cx$, where $c \in \mathbb{R}$ is a constant, is a self similar fractal function.

**Proof.** Consider the following disjoint partition of the interval $[a, b]$ divided by $= [a, \frac{a+b}{2}], \left(\frac{a+b}{2}, b\right]$ and let

$$\phi_1 : [a, b] \to \left[ a, \frac{a+b}{2} \right], \quad \phi_1(x) = \frac{x}{2} + \frac{a}{2}$$ 

and

$$\phi_2 : \left(\frac{a+b}{2}, b\right] \to \left(\frac{a+b}{2}, b\right], \quad \phi_2(x) = \frac{x}{2} + \frac{a}{2}.$$ 

...
\[ \phi_2 : [a, b] \to \left[ \frac{a + b}{2}, b \right], \quad \phi_2(x) = \frac{x}{2} + \frac{b}{2}. \]

It is easy to see that \( \phi_1 \) and \( \phi_2 \) are Lipschitz maps because they are differentiable, and derivative is positive.

Consider the scaling law \((S_1, S_2) = \left( \frac{x}{2} + \frac{a}{2c}, \frac{x}{2} + \frac{b}{2c} \right)\), then a straightforward computation shows that

\[ Sf = \left( \bigoplus_{i=2}^2 S_i \circ f \circ \phi_i^{-1} \right) = cx. \]

and hence (1) follows. Then \( f \) is a self similar fractal function. \( \Box \)

The following lemma is relevant to establish the translations invariance of self similar fractal function.

**Lemma 1.5.** Let \( I \) be a bounded and closed interval in \( \mathbb{R} \) and let \( S : I \to \mathbb{R} \) be a Lipschitz map. Given a constant \( c \in \mathbb{R} \), consider the set \( Y = \{ x + c : x \in I \} \). If we define \( \bar{S} : Y \to \mathbb{R} \) in \( Y \) by \( \bar{S}(x + c) = S(x) + c \), then \( \bar{S} \) is a Lipschitz map.

**Proof.** Let \( y_1, y_2 \in Y \) i.e \( y_1 = x_1 + c \) and \( y_2 = x_2 + c \) then

\[ |\bar{S}(y_1) - \bar{S}(y_2)| = |S(x_2) - S(x_1)| \leq \text{Lip}(S)|x_1 - x_2| = \text{Lip}(S)|y_1 - y_2|. \]

Then, \( \bar{S} \) is a Lipschitz map, as wanted. \( \Box \)

**Theorem 1.6.** Let \( I \) be a closed and bounded interval and let \( c \in \mathbb{R} \). If \( f : I \to \mathbb{R} \) is a self similar fractal function, then \( f + c \) is also a self similar fractal function.

**Proof.** By hypothesis \( f \) admits the following representation

\[ f = \bigoplus_{i=1}^N S \circ f \circ \phi_i^{-1}, \]

for some scaling law \((S_1, \ldots, S_N)\) and some family of Lipschitz maps \( \phi_i \)'s.

For every \( i \in \{1, 2, \ldots, N\} \) consider the family of sets \( Y_i = \{ f \circ \phi_i^{-1}(x) + c : x \in I_i \} \), and let us define a function \( S_i \) in these sets, by \( S_i(f \circ \phi_i^{-1}(x) + c) = S_i(f \circ \phi_i^{-1}(x)) + c \), which is a Lipschitz by Lemma 1.5 for every \( i \in \{1, 2 \ldots, N\} \). Then:

\[ \bigoplus_{i=1}^N S \circ (f + c) \circ \phi_i^{-1}(x) = \bigoplus_{i=1}^N S(f \circ \phi_i^{-1} + c) \]

\[ = \bigoplus_{i=1}^N (S(f \circ \phi_i^{-1}) + c) \]

\[ = \left( \bigoplus_{i=1}^N S(f \circ \phi_i^{-1}) \right) + c \]

\[ = f + c. \]

Hence the function \( f + c \) is a self similar fractal function, with scaling law \((S_1, S_2, \ldots, S_N)\). \( \Box \)
2. CLASSICAL AND CONTINUOUS DFA MODEL

The classical DFA may be regarded as a procedure to transform a time series into a new time series which approximates a power law. It consists on two steps. This section contains the main contribution of the paper, namely, we present the corresponding two steps which will define a DFA for continuous functions and prove that it approximates a power law.

2.1. First step: integrated function. The first step of the classical DFA is the following, consider a time series of size \( M \in \mathbb{N} \), \( x(i), i = 1, \ldots, M \), then define the integrated time series given by \( y(i) = \sum_{j=1}^{i} (x(j) - \bar{x}) \), where \( \bar{x} = \frac{1}{M} \sum_{i=1}^{M} x(i) \) is the average value of the time series. This new time series is a self similar process.

**Definition 2.1.** Let \( M \) be a positive real number. Suppose that \( x : [0, M] \to \mathbb{R} \) is an integrable function. We define the integrated function of \( x(t) \) to be the function \( y : [0, M] \to \mathbb{R} \) given by the formula:

\[
y(t) := \int_{0}^{t} (x(s) - \bar{x}) \, ds,
\]

where \( \bar{x} := \frac{1}{M} \int_{0}^{M} x(s) \, ds \).

There is a correspondence between time series \( a(i) \) of size \( M \in \mathbb{N} \), and simple functions \( x : [0, M] \to \mathbb{R} \) given by \( x(t) = \sum_{i=1}^{M} a(i) \chi_{[i-1,i]}(t) \). Note that \( \bar{x} = \bar{x} \). Furthermore, if \( b(i) \) is the integrated time series of \( a(i) \) and \( y(t) \) is the integrated function of \( x(t) \), then \( y(t) = \sum_{i=1}^{M} b(i) \chi_{[i-1,i]}(t) \). Hence the definition of integrated function generalizes the notion of the integrated series.

One of the features in the DFA method is that the integrated time series is a self similar process. We wonder if the integrated function is a self similar fractal function as in Definition 1.3. We were unable to prove it. However, as we show in the next theorem, it is the sum of two such functions.

**Theorem 2.2.** Let \( x : [0, M] \to \mathbb{R} \) be an integrable function with \( M > 0 \).

\( \text{(1)} \) If there exists \( \delta > 0 \) such that \( x(s) \geq \delta \) for all \( s \), then \( z : [0, M] \to \mathbb{R} \) defined by

\[
z(t) = \int_{0}^{t} x(s) \, ds,
\]

is a self similar fractal function.

\( \text{(2)} \) The integrated function of \( x(t) \) is a sum of two self similar fractal functions.

**Proof.** To prove part (1), consider the disjoint partition of \( [0, M] = [0, \frac{M}{2}] \cup [\frac{M}{2}, M] \) and the increasing Lipschitz maps \( \phi_1, \phi_2 \) gives by:

\[
\phi_1 : [0, M] \to \left[ 0, \frac{M}{2} \right], \quad \phi_1(t) = \frac{t}{2},
\]

and
\[ \phi_2 : [0, M] \to \left[ \frac{M}{2}, M \right], \quad \phi_2(t) = \frac{t}{2} + \frac{M}{2}. \]

with inverse maps:
\[ \phi_1^{-1}(t) = 2t, \]
\[ \phi_2^{-1}(t) = 2t - M. \]

Define the map \( S_1 \) on the image of \( z \circ \phi_1^{-1} \) by:
\[ S_1(z(\phi_1^{-1}(t))) = S_1 \left( \int_0^{2t} x(s) \, ds \right) := \int_t^0 x(s) \, ds, \]

define the map \( S_2 \) on the image of \( z \circ \phi_2^{-1} \) by:
\[ S_2(z(\phi_2^{-1}(t))) = S_2 \left( \int_0^{2t-M} x(s) \, ds \right) := \int_t^0 x(s) \, ds. \]

If we show that \( S_1 \) and \( S_2 \) are Lipschitz maps on the images of \( z \circ \phi_1^{-1} \) and \( z \circ \phi_2^{-1} \) respectively, then by Kirzbraun Theorem \([4]\), we can extend these maps to Lipschitz maps on \( \mathbb{R} \).

By continuity of \( x(s) \) in the compact set \([0, M]\), there exist \( K \) and \( k \) given by:
\[ K = \sup \left\{ x(s) : s \in \left[ 0, \frac{M}{2} \right] \right\} \]
and
\[ k = \inf \left\{ x(2s) : s \in \left[ 0, \frac{M}{2} \right] \right\}. \]

For \( t, \tau \in [0, \frac{M}{2}] \) with \( t < \tau \) we have the following:
\[ (3) \quad \int_t^\tau x(s) \, ds \leq \int_t^\tau Kds = (\tau - t)K, \]
and
\[ (4) \quad k(\tau - t) = \int_t^\tau kds \leq \int_t^\tau x(2s) \, ds, \]

note that by hypothesis \( x(s) \geq \delta > 0 \) so we have that \( K > 0 \) y \( k > 0 \). Then we may choose \( R > 0 \) such that: \( K \leq 2Rk \). Hence, using \((3)\) and \((4)\) we have that:
\[ (5) \quad \int_t^\tau x(s) \, ds \leq 2R \int_t^\tau x(2s) \, ds. \]

And using a variable change \( \tau = \frac{\tau}{2} \), to the integral on the right in \((5)\) we obtain
\[ \int_t^\tau x(s) \, ds \leq R \int_{2t}^{2\tau} x(s) \, ds. \]

Since by hypothesis \( x(s) \geq \delta > 0 \), the integrals in the last inequality are non negatives, so we obtain
\[ (6) \quad \left| \int_t^\tau x(s) \, ds \right| \leq R \left| \int_{2t}^{2\tau} x(s) \, ds \right|. \]
Then, if we add a zero to the integral on the left hand side of (6) we obtain
\[ \left| \int_t^\tau x(s) \, ds \right| = \left| \int_0^t x(s) \, ds - \left( \int_0^t x(s) \, ds + \int_t^\tau x(s) \, ds \right) \right| = \left| \int_0^t x(s) \, ds - \int_0^\tau x(s) \, ds \right|, \]
and if we add a zero to the integral on right hand side of (6)
\[ \left| \int_{2t}^{2\tau} x(s) \, ds \right| = \left| \int_0^{2t} x(s) \, ds - \left( \int_0^{2t} x(s) \, ds + \int_{2t}^{2\tau} x(s) \, ds \right) \right| = \left| \int_0^{2t} x(s) \, ds - \int_{2t}^{2\tau} x(s) \, ds \right|. \]
With this (6) becomes:
\[ \left| \int_0^t x(s) \, ds - \int_0^\tau x(s) \, ds \right| \leq R \left| \int_0^{2t} x(s) \, ds - \int_{2t}^{2\tau} x(s) \, ds \right|, \]
thus
\[ |S_1(z(\phi_1^{-1}(t))) - S_1(z(\phi_1^{-1}(\tau)))| \leq R|z(\phi_1^{-1}(t)) - z(\phi_1^{-1}(\tau))|. \]
This prove that $S_1$ is a Lipschitz map on the image of $z \circ \phi_1^{-1}$. The proof that $S_2$ is a Lipschitz map on the image of $z \circ \phi_2^{-1}$ is analogous, so we omit it.

Then we have that
\[ \int_0^t x(s) \, ds = \sum_{i=1}^2 S_i \circ z \circ \phi_i^{-1}, \]
i.e. $z(t)$ is a self similar fractal function.

Now to prove part (2), consider $\bar{x} = \frac{1}{M} \int_0^M x(s) ds$ and let $\delta > 0$. Then there exists $c > 0$ such that $x(t) - \bar{x} + c \geq \delta > 0$. Define $x_1(t) = x(t) - \bar{x} + c$. From the previous part a) we have that, $z_1(t) = \int_0^t x_1(s) \, ds$ is a self-similar fractal function, in other words
\[ z_1(t) = \int_0^t x_1(s) \, ds = \int_0^t (x(s) - \bar{x} + c) \, ds = y(t) + ct, \]
where $y(t)$ is the integrated function of $x(s)$. On the other hand, by Example 1.3 the function $ct$ is a self similar fractal function. Hence
\[ y(t) = z_1(t) - ct. \]
is a sum of a two self similar fractals functions.

The following corollary establish a sufficient condition for a function to be self similar fractal.

**Corollary 2.3.** Let $x : [0, M] \to \mathbb{R}$ continuous and differentiable function, and suppose that there exists $\delta > 0$ such that $\frac{dx}{dt} \geq \delta$, then $x(t)$ is a self similar fractal function.
Proof. Since by the hypothesis \( \frac{dx}{ds} \geq \delta > 0 \), it is possible to apply Theorem 2.2 then:

\[ z(t) = \int_0^t \frac{dx}{ds} ds = x(t) - x(0), \]

is a self similar fractal function, and then \( x(t) \) is a self similar fractal function by Theorem 1.6. □

2.2. Second step: detrended function. The second step in the classical DFA model consists in removing the trend from the integrated time series \( y(i) \) of a given time series \( x(i) \), \( i = 1, \ldots, N \). To achieve this, we restrict the integrated time series \( y(i) \) on subintervals of size \( n \), with \( 1 < n < N \). With the data in each window of size \( n \), the line of least squares is calculated. The \( y \)-coordinate value of this line is denoted by \( y_n(i) \). The process of removing the trend from the integrated time series \( y(i) \) is performed by subtracting the value of \( y_n(i) \) in each window. For each \( n \), the characteristic length is obtained for the fluctuations of the integrated and trendless time series:

\[ F(n) = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (y(i) - y_n(i))^2}. \]

For the continuous case, in this second step, we will require the use of continuous functions on compact intervals. As a consequence, we are able to define the integrated functions described in subsection 2.1 and apply Theorem 2.2.

We now propose one way of removing the trend from the integrated function \( y(t) \) of a given continuous function \( x: [0, M] \to \mathbb{R} \), where \( M \) is a positive number, as in Definition 2.1. One may try removing the trend by means of a least squares approximation, that is, by restricting the integrated function \( y(t) \) on subintervals of size \( n \), with \( 1 < n < M \), and on each window of size \( n \), approximate the graph of \( y(t) \) by means of a linear approximation and proceed to remove the trend from \( y(t) \) by subtracting the value of the \( y \)-coordinate of the corresponding line. However, unlike the discrete case, this approach will not lead us to obtain a power law, even when an approximation by polynomials replaces the linear approximation. For this reason, we consider a different way to remove the trend, by means of a methodology analogous to the technique of differentiation in the context of time series. Let \( 0 < m < M \). Restrict the integrated function \( y(t) \) given in Definition 2.1 to the interval \([m, M]\) and let \( 0 < r < m \). Consider the difference

\[ (7) \quad y(t) - y(t - r) = \int_0^t (x(s) - \bar{x}) ds - \int_0^{t-r} (x(s) - \bar{x}) ds = \int_{t-r}^t (x(s) - \bar{x}) ds. \]

The expression in (7) may be regarded as the process of removing the trend of \( y(t) \), taking into account the immediate past, instead of the tendency established by windows. Then as in the methodology introduced by Peng and his collaborators, [14, 13], consider the square root of the average of the squared values obtained in (7) as a function of \( r \). This gives the detrended function \( F: [0, m] \to \mathbb{R} \) defined by

\[ (8) \quad F(r) := \sqrt{\frac{1}{(M - m)} \int_m^M (y(t) - y(t - r))^2 dt}. \]
The following lemma will be useful in the sequel.

**Lemma 2.4.** The detrended function $\mathcal{F}$ defined in (8) is continuous at $r = 0$.

**Proof.** Since the square root function is continuous at zero, it will suffice to show that the function $r \mapsto \left(\frac{1}{M-m}\right) \int_{m}^{M} (y(t) - y(t-r))^2 \, dt$ is continuous at $r = 0$. Let $\epsilon > 0$. Since $x(t) - \bar{x}$ is a continuous function, there exists $M > 0$ such that $|x(t) - \bar{x}| < M$. Let $\delta = \sqrt{\frac{\epsilon}{M}}$. Then, if $0 < r < \delta$, using the equality in (2.1), it follows that

$$
|y(t) - y(t-r)| \leq \int_{t-r}^{t} |x(s) - \bar{x}| \, ds < M\delta = \sqrt{\epsilon}.
$$

and hence

$$
\left| \frac{1}{(M-m)} \int_{m}^{M} (y(t) - y(t-r))^2 \, dt \right| < \epsilon,
$$

as wanted. \hfill \Box

We are now ready to state and prove our main result.

**Theorem 2.5.** Let $x : [0, M] \to \mathbb{R}$ a continuous function and let $y : [0, M] \to \mathbb{R}$ be the integrated function of $x(t)$. Fix $m$ such that $0 < m < M$ and consider the detrended function $\mathcal{F} : [0, m] \to \mathbb{R}$ given by:

$$
\mathcal{F}(r) = \sqrt{\frac{1}{(M-m)} \int_{m}^{M} (y(t) - y(t-r))^2 \, dt}
$$

Then $\mathcal{F}(r)$ approximates a power law. More precisely, for all $\epsilon > 0$, there exists $0 < \delta < m$ such that if $0 < r < \delta$, then

- in case $\bar{x} \neq 0$, we have
  $$
  |\mathcal{F}(r) - \bar{x}r| < \epsilon,
  $$
- in case $\bar{x} = 0$, we have
  $$
  |\mathcal{F}(r) - r| < \epsilon.
  $$

**Proof.** Since by Lemma 2.4 the detrended function $\mathcal{F}$ is continuous at $r = 0$, so are the function $r \mapsto \mathcal{F}(r) - \bar{x}r$ and $r \mapsto \mathcal{F}(r) - r$. The result now follows by the definition of continuity at $r = 0$. \hfill \Box

As in the traditional DFA model, we obtain a power law, but in this case we do not obtain a complete spectrum of values for the exponent of the power law [14]. In the classical DFA, the scaling exponent close to one indicated the existence of long-range correlations, while the scaling exponent equal one corresponds to the so-called $1/f$ noise [9]. In our case, the fact that we obtain the scaling exponent equal one may be due to the fact that a function may be regarded as a deterministic object. Finally, we point out that, by Example 1.4 and Theorem 2.5, the detrended function $\mathcal{F}$ is approximately a self similar fractal function.
References

[1] B. Blasius. Covid-19 cases fit power-law distribution during initial phase of pandemic. *Chaos*, 30(9), 2020.

[2] R. M Bryce and K. B Sprague. Revisiting detrended fluctuation analysis. *Scientific Reports*, 2(315), 2012.

[3] J-Y. Chiang, J-W. Huang, L-Y. Lin, C-H. Chang, F-Y. Chu, Y-H. Lin, C-K. Wu, J-K. Lee, J-J. Hwang, J-L. Lin, and F-T. Chiang. Detrended fluctuation analysis of heart rate dynamics is an important prognostic factor in patients with end-stage renal disease receiving peritoneal dialysis. *PLOS ONE*, 10.1371/journal.pone.0147282, 2016.

[4] H. Federer. *Geometric Measure Theory*. Springer-Verlag Berlin, Heidelberg, 1969.

[5] R. Hardstone, S-S Poil, G. S., R. Jansen, V. V. Nikulin, H. D. Mansvelder, and K. Linkenkaer-Hansen. Detrended fluctuation analysis: a scale-free view on neuronal oscillations. *Frontiers in Psychology*, 2012.

[6] J. E. Hutchinson. Fractals and self similarity. *Indiana Univ. Math. J.*, 30(5), 1981.

[7] J. E. Hutchinson and L. Rüschendorf. Self similar fractals and self similar random fractals. In Bandt C., Graf S., and Zahle M., editors, *Fractal Geometry and Stochastics II*, chapter 3, pages 109–123. Birkhauser, Basel, 2000.

[8] S. Ghosh R. Kulik J. Beran, Y. Feng. *Long-Memory Processes, Probabilistic Properties and Statistical Methods*. Springer-Verlag Berlin, Heidelberg, 2013.

[9] W. Li and D. Holste. Universal 1/f noise, crossovers of scaling exponents, and chromosome-specific patterns of guanine-cytosine content in DNA sequences of the human genome. *Phys. Rev. E*, 71:0419410–19, 2005.

[10] M.C. Mariani, P. K. Asante, M. A Masum Bhuiyan, M. P. Beccar-Varela, S. Jaroszewicz, and O. K. Tweneboah. Long-range correlations and characterization of financial and volcanic time series. *Mathematics*, 8, 2020.

[11] L.F. Mártont, S.T. Brassai, L. Bakóa, and L. Losonczi. Detrended fluctuation analysis of EEG signals. *Procedia Technology*, 12:125–132, 2014.

[12] S. S. Pal and S. Kar. Time series forecasting for stock market prediction through data discretization by fuzzistics and rule generation by rough set theory. *Mathematics and Computers in Simulation*, 8, 2019.

[13] C-K. Peng, S.V. Buldyrev, S. Havlin, H.E. Stanley M. Simons, and A.L. Goldberger. Mosaic organization of DNA nucleotides. *Phys. Rev. E*, 49(2), 1994.

[14] C-K. Peng, S. Havlin, H. E. Stanley, and A. L. Goldberger. Quantification of scaling exponents and crossover phenomena in nonstationary heartbeat time series. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 5(82), 1995.

[15] E.E Rodríguez, E. Hernández-Lemus, B.A. Itzá-Ortiz, and I. Jiménez. Multichannel detrended fluctuation analysis reveals synchronized patterns of spontaneous spinal activity in anesthetized cats. *PLoS ONE*, 6(10), 2011.

[16] M. R. Schroeder. *Fractals, Chaos, Power Laws: Minutes from an Infinite Paradise*. Dover Books on Physics, 1991.

[17] M.A. Sánchez Granero, J.E. Trinidad Segovia, and J. García Pérez. Some comments on Hurst exponent and the long memory processes on capital markets. *Physica A Statistical Mechanics and its Applications*, 387(22), 2008.

[18] N. Wynn Watkins and C. Franzke. A brief history of long memory: Hurst, Mandelbrot and the road to ARFIMA, 1951-1980. *Entropy*, 19(9), 2017.

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