Extreme gravitational lensing by supermassive black holes

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Summary. — Extreme gravitational lensing refers to the bending of photon trajectories that pass very close to supermassive black holes and that cannot be described in the conventional weak deflection limit. A complete analytical description of the whole expected phenomenology has been achieved in the recent years using the strong deflection limit. These progresses and possible directions for new investigations are reviewed in this paper at a basic level. We also discuss the requirements for future facilities aimed at detecting higher order gravitational lensing images generated by the supermassive black hole in the Galactic center.

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1. Introduction

All known astrophysical cases of bending of photon trajectories by gravitational fields can be interpreted using the conventional weak deflection paradigm, originally formulated by Einstein [1]. Yet it is clear that photons passing very close to black holes suffer very large deflections, which must be addressed in a full general relativistic context. However, integrating null geodesics in full general relativity usually leads to very complicated analytical formulae or heavy numerical codes, which typically obscure the basic physical interpretation. The Strong Deflection Limit (SDL) is an analytical tool that allows to derive simple analytical formulae describing the higher order images appearing in extreme gravitational lensing. In this work we review the main progresses achieved in the recent years in this technique and its application to the most interesting physical case: the black hole in the Galactic center (Sgr A*). All relevant formulae derived in previous works are recalled and restated here in the simplest possible form.

This work is structured as follows: § 2 explains the basic phenomenology of extreme lensing; § 3 introduces the SDL for spherically symmetric black holes; § 4 generalizes the method to spinning black holes; § 5 applies the formulae to Sgr A*, examining possible sources for extreme gravitational lensing; § 6 generalizes the method to sources very close to the black hole; § 7 contains the conclusions.
2. – Basic phenomenology

Performing an exact integration of the null geodesics equation in Schwarzschild metric, Darwin found a general expression for the deflection angle in a Schwarzschild geometry without any approximation [2]. This formula looks quite complicated, since it involves elliptic integrals. Nevertheless, taking the Weak Deflection Limit (WDL) for photon passing far enough from the black hole, it reduces to the classical Einstein’s formula for the deflection angle

\[ \alpha(u) = \frac{4GM}{c^2u} \]

where \( u \) is the impact parameter of the photon as emitted by the source, \( M \) is the mass of the black hole, \( c \) is the speed of light and \( G \) is the Newton constant.

Fig. 1 contains a comparison between the exact deflection angle and the WDL approximation, along with the SDL approximation to be introduced in the following section. The exact deflection angle and the WDL perfectly agree at impact parameters much larger than the Schwarzschild radius of the black hole. However, as the impact parameter becomes comparable to the Schwarzschild radius, the approximations underlying the derivation of the WDL formula are no longer valid. As we lower the impact parameter \( u \), the deflection grows larger and larger, until it diverges at a limiting value of the impact parameter, indicated by \( u_m \). For the Schwarzschild black hole, we have \( u_m = 3\sqrt{3}GM/c^2 \).

Very large deflections are possible for \( u \) close to \( u_m \). A deflection of order \( \pi \) means that the photon turns around the black hole and goes back to the source. The possibility of such a back-scattering was already examined by Luminet [3] and was recently revived with the name of retro-lensing [4].

A deflection of order \( 2\pi \) is experienced by photons that perform a complete loop...
around the black hole before emerging in the direction opposite to the source. Higher deflections are possible for values of \( u \) closer and closer to \( u_m \), corresponding to photons performing any number of loops around the black hole before emerging in any direction.

Photons with impact parameter exactly equal to the minimum value \( u_m \) would be injected on a circular orbit with fixed radius \( r_m \) around the black hole. The spherical surface of radius \( r_m \), containing these circular photon orbits, is also called photon sphere. In Schwarzschild metric, the radius of the photon sphere is \( r_m = 3GM/c^2 \). Photons with impact parameter \( u < u_m \) just fall inside the event horizon and are definitely lost.

The divergence of the deflection angle for \( u \to u_m \) is a genuine feature of the strong gravitational fields generated by black holes. Since it is predicted by general relativity, one can wonder whether possible observable effects of this divergence exist. Let us define the optical axis as the line joining the observer with the black hole. If we have a source lying behind the black hole, classical gravitational lensing theory states that the observer sees two images formed by photons weakly deviated by the black hole. These two images are formed by photons passing far from the black hole, which can be usually described in the WDL. Besides these two main images, the fact that there are values of \( u \) such that \( \alpha = 2\pi \) implies that there exist some photons passing very close to the black hole that perform one complete loop around the black hole and finally reach the observer. These photons form an additional pair of images (one on each side of the black hole). Furthermore, there are photons performing two loops, giving rise to one more additional pair, and so on. The divergence in the deflection angle as \( u \to u_m \) is thus responsible for the existence of two infinite sequences of higher order images appearing very close to the black hole. The angular separation from the black hole of images formed by photons with impact parameter \( u \) is simply given by \( \theta = \arcsin(u/D_{OL}) \approx u/D_{OL} \), with \( D_{OL} \) being the distance between observer and lens.

No image of distant sources can appear at angular separations less than \( \theta_m = u_m/D_{OL} \), because no deflection is possible with \( u < u_m \). The minimum angle \( \theta_m \) defines the angular radius of the so-called shadow of the black hole [2, 3]. The higher order images would appear just slightly outside the border of this shadow.

A perfect alignment of source, lens and observer on the same line would cause each pair of images to merge together and form an Einstein ring. Besides the classical WDL Einstein ring, we would have an infinite sequence of higher order Einstein rings just outside the shadow of the black hole. With respect to standard WDL gravitational lensing, extreme black hole lensing also opens the possibility of forming retro-lensing Einstein rings, i.e. ring images generated by a source exactly in front of the black hole whose photons turn around the black hole and then reach the observer.

The basic phenomenology of higher order images generated by a Schwarzschild black hole was described by Darwin in 1959 [2]. However, higher order images were practically forgotten until Virbhadra and Ellis noticed that higher order images could become observable around the supermassive black hole in the Galactic center and perhaps around central black holes of nearby galaxies [5]. After their work, higher order images have been the object of specific studies by numerous authors both numerically and analytically.

3. – Gravitational lensing in the Strong Deflection Limit: spherically symmetric black holes

The study of higher order images by numerical methods is quite complicated because it is necessary to integrate null geodesics with very high accuracy. On the other hand, Darwin himself gave an expansion of his exact deflection formula that is valid very close
to the divergence in $u_m$. It reads

\begin{equation}
\alpha(u) = -c_1 \log \left( \frac{u}{u_m} - 1 \right) + c_2 + O(u - u_m),
\end{equation}

with the coefficients $c_1 = 1$ and $c_2 = \log \left[ 216(7 - 4\sqrt{3}) \right] - \pi$ in Schwarzschild metric.

This formula approximates the divergence in $u_m$ very well, as can be seen in Fig. 1. Therefore, it provides an excellent and highly simplified starting point for the analytical treatment of higher order images. In particular, the angular position and the magnification of the higher order images assume extremely simple expressions as functions of the position of the source [2, 6]. They read

\begin{equation}
\theta_n = \theta_m \left( 1 + e^{(c_2 - \phi_s - 2n\pi)/c_1} \right)
\end{equation}

\begin{equation}
\mu_n = \frac{D_{OS}}{D_{LS}} \left( \frac{\theta_m}{c_1 \sin \phi_s} \right)^2 e^{(c_2 - \phi_s - 2n\pi)/c_1},
\end{equation}

where $D_{LS}$ is the distance from the lens to the source, $D_{OS}$ is the distance from the observer to the source, $\phi_s$ is the azimuth of the source measured in a reference frame centered on the black hole ($\phi_s = 0$ means source aligned behind the black hole), and $n = 1, 2, 3, \ldots$ identifies the order of the image.

Since the higher order images are generated in a strong gravity regime, they have the potentiality to become a test of general relativity outside the weak field approximation. In principle they can be used to distinguish general relativity from alternative theories of gravity that lead to the same expectations in weak field tests but different predictions in a strong gravity regime. However, the first step is to understand whether they are just an exceptional outcome of general relativity or they are generic to all theories of gravity.

Ref. [7] demonstrates that the deflection angle formula (2) is universal, in the sense that it is exactly the same for any spherically symmetric metric admitting a static limit (i.e. a spherical surface where $g_{tt} = 0$, with $t$ being the proper time for an observer at infinity). So, whatever the theory of gravity we consider, the black hole metric will always be characterized by a photon sphere and thus by a logarithmic divergence in the deflection angle. What changes from one metric to another is the numerical value of the coefficients $u_m$, $c_1$, $c_2$. These three coefficients are a sort of identity card of the black hole metric which is inherited by the higher order images. In principle, since the position and the luminosity of the higher order images depend on the three coefficients in the SDL formula, by the observation of the higher order images it is possible to identify the correct black hole metric and thus the correct theory of gravity.

This possibility opened by higher order images has pushed many authors [8] to calculate the SDL coefficients for several black hole metrics using the general method outlined in Ref. [7]. This method can be easily extended to the retro-lensing geometry [9, 10], taking $\phi_s = \pi$ and $n$ starting from 0 in Eqs. (3) and (4). Time delay calculations have been done in Ref. [11]. Higher orders in the expansion (2) were calculated in Ref. [12] for the Schwarzschild black hole.

It is important to note that in the intermediate region $|0.15 < \alpha < 2|$ both the WDL and the SDL approximations reproduce the exact deflection angle with an accuracy worse than 10%. The only known analytical approximations able to cover this region are based on variational methods [13].
4. – Gravitational lensing in the Strong Deflection Limit: spinning black holes

The integration of null geodesics in the Kerr black hole metric is considerably simplified by the separability of the Hamilton-Jacobi equation [14]. The geodesics equations can be put in the form of first integral equations, which can be solved in terms of elliptic functions. The explicit form of these analytical solutions is extremely complicated, but can be exploited in building fast and efficient numerical codes aimed at specific aspects of the propagation of photons in the Kerr metric (see e.g. [15, 16]). In spite of the large number of studies in this field, the phenomenology of higher order images was still poorly understood before the techniques of the SDL came into play.

There are some fundamental differences between extreme lensing by spinning black holes and spherically symmetric black holes.

Eq. (4) shows that when the source is perfectly aligned on the optical axis ($\phi_s = 0$) the magnification diverges for all higher order images (the same happens for the two classical WDL images). Within gravitational lensing theory, this divergence signals the fact that there is a unique caustic point lying on the optical axis for all images. When a source approaches this point, all images become simultaneously bright and finally degenerate into Einstein rings.

In spinning black holes, there is one separate caustic for each pair of images. All caustics are shifted from the optical axis in the sense opposite to the rotation of the black hole. The shift becomes larger and larger with the order of the images. As a consequence, if the source is close to the WDL caustic, only the WDL images will become bright while all the others stay faint. If the source is close to the first order SDL caustic, only the first order images get bright while all the others stay faint, and so on.

In addition to the shift from the optical axis, the caustics are no longer pointlike, but assume the classical shape of the 4-cusped astroid, which is typical of the breaking of spherical symmetry in gravitational lensing. When the source is inside the caustic, an additional pair of images appears.

In a series of papers devoted to spinning black holes [17, 18, 19, 20] the SDL paradigm has been gradually extended to gravitational lensing by spinning black holes. Here we report the most recent results in Ref. [20].

In order to keep all the results analytical, we can expand all formulae to second order in the black hole spin $a$, finding very accurate results up to $a = 0.1$ (in our notations, $a = 0.5$ is the extremal Kerr black hole). In this way, expressions for the position and the shape of the caustics can be explicitly derived. In a reference frame centered on the black hole with polar axis defined by the projection of the spin axis on the observer sky, the azimuthal shift from the optical axis is

$$\Delta_k = -4 \left[ \frac{k\pi}{3\sqrt{3}} + \log(2\sqrt{3} - 3) \right] a \sin i.$$  

where $i$ is the inclination of the spin axis with respect to the observer line of sight and $k$ is the caustic order, being $3, 5, 7, \ldots$ for higher order images in standard lensing configuration and $2, 4, 6, \ldots$ for retro-lensing configuration. Note that $\Delta_k$ is always negative, which means that the shift is in the clockwise sense.

The shift in the altitude angle is much smaller, being of second order in $a$:

$$\delta_k = (-1)^k \frac{1}{2} \Delta_k^2 \cot i.$$
Fig. 2. – Position and shape of the caustics in Kerr gravitational lensing. The observer is in O, the black hole is in L, with spin oriented along the vector $\mathbf{a}$. The inclination of the spin with respect to the line of sight is $i$. The azimuth of the center of the caustic in this reference frame is given by $\Delta_k$ and its altitude is $\delta_k$. The full amplitude of the astroid caustic is $2R_k$.

The semi-amplitude of the astroid caustic is

$$R_k = \frac{2}{9}(5k\pi + 8\sqrt{3} - 36)a^2 \sin^2 i. \tag{7}$$

From these equations it is evident that the shift and the extension of the caustic increase with the black hole spin, its inclination and the caustic order. The precise geometric meaning of these quantities is illustrated in Fig. 2.

The position of the images for a source close to a caustic (which is the most relevant case) can be found in two steps. First, the equation

$$\delta \vartheta_s \cos \psi + (-1)^k(\delta \phi_s - R_k \cos \psi) \sin \psi = 0, \tag{8}$$

must be solved for the angle $\psi$. Here $\delta \phi_s$ and $\delta \vartheta_s$ represent the displacement of the source from the center of the caustic in the azimuth and polar angle respectively. With reference to Fig. 2, if $\delta \phi_s > 0$ the source is on the left of the caustic; if $\delta \vartheta_s > 0$ the source is below the caustic.

For each real solution of Eq. (8) we have a distinct image. The coordinates of the images in the observer sky are then simply given by

$$\theta_{1,k} = \theta_m (1 + \epsilon_k) \cos \psi \tag{9}$$
$$\theta_{2,k} = \theta_m (1 + \epsilon_k) \sin \psi, \tag{10}$$

with $\epsilon_k = 216 (7 - 4\sqrt{3}) e^{-k\pi}$. With reference to Fig. 2, if $\theta_{1,k} > 0$ the observer sees the
image on the right of the black hole; if $\theta^{2,k} > 0$ the observer sees the image above the black hole.

The radial and tangential magnifications are respectively given by

$$\mu_r = \frac{D_{\text{OS}}}{D_{\text{LS}}} \frac{\theta_m e_k}{\theta_m \cos \psi}$$

$$\mu_t = (-1)^k \frac{D_{\text{OS}}}{D_{\text{LS}}} \frac{\theta_m \cos \psi}{(\delta \phi_s - R_k \cos^3 \psi)}$$

By these very simple formulae, we can describe the whole phenomenology of higher order images in Kerr black hole lensing, including caustic crossing, with formation and destruction of additional pairs of images.

It is interesting to note that, to lowest order in $a$, all observables in Kerr lensing are functions of $a \sin i$. In practice, only the projection of the spin on a plane orthogonal to the line of sight is accessible by gravitational lensing measurements.

Another interesting consideration regards the stability of higher order images with respect to external fields. The study of extreme lensing by a Schwarzschild black hole embedded in an external uniform gravitational field has shown that the external field causes the shift and the formation of extended astroid caustics in a way qualitatively similar to Kerr metric, but with very different quantitative aspects [21]. In particular, the caustic extension induced by realistic external fields would be much smaller than that due to an intrinsic black hole spin.

The SDL has been recently extended to the Kerr-Sen dilaton-axion spinning black hole [22].

5. – Sgr A* as an extreme gravitational lens

The radio source Sgr A* is believed to host a supermassive black hole with mass $M = 3.6 \times 10^6 M_\odot$ [23]. With a distance from the sun $D_{\text{OL}} = 8$ kpc, the shadow of the black hole would have an angular radius $\theta_m = 23$ microarcseconds. The higher order images would appear just outside the shadow of the black hole and thus demand a very high angular resolution to be detected.

Resolutions of this order can be reached with very long baseline interferometry in the radio cm-band [24]. However, at wavelengths higher than 1 mm, the scattering by electrons surrounding Sgr A* blurs the image, making any progress in the resolution useless [25]. Sub-mm imaging would probably be able to show the existence of the shadow of the black hole [26, 27].

Higher chances of seeing higher order images exist in the infrared bands, where the blur is negligible and the dust extinction still allows good observations of the Galactic Center. In addition to this, several stellar sources have been identified and accurately followed through their orbit around the black hole [23]. With these sources, we can make accurate predictions for the appearance and luminosity of secondary and higher order images [28, 10, 29]. Unfortunately, Sgr A* has its own emission in the IR, which would render the identification of higher order images definitely challenging. On the other hand, there are good chances to see a secondary image in an intermediate regime between the WDL and the SDL.

The best chances to observe higher order images come from the X-ray band, where several point sources have been detected and identified as low mass X-ray binaries [30]. In this band, the emission by Sgr A* is rather low, providing a very low noise to higher
order images [19, 20]. The chance alignment of one of these sources behind or in front of the black hole, would generate observable higher order images. In this electromagnetic band, resolutions better than 1 microarcsecond should be achievable by future X-ray interferometers in space (MAXIM, http://maxim.gsfc.nasa.gov).

6. – Sources close to black holes

All the formulae in § 3 and § 4 are strictly valid for sources very far from the black hole, where very far means that their distance is much larger than the Schwarzschild radius. Yet, we know that most black holes are surrounded by accretion disks and infalling matter, which provide very interesting sources for gravitational lensing. In order to include such sources in our formalism, we need to revise the derivation of the SDL without taking the limit $D_{LS} \gg 2GM/c^2$.

Actually, this can be done without too much trouble and the whole general method of Ref. [7] can be reformulated taking care of the source and the observer distance [31]. In particular, the azimuthal shift becomes

\[
\Delta \phi(u) = -c_1 \log \left( \frac{u/u_m - 1}{\eta_O \eta_S} \right) + c_2 + \pi, \tag{13}
\]

where $\eta_O = 1 - r_m/D_{OL}$ and $\eta_S = 1 - r_m/D_{LS}$. The coefficient $c_1$ is unchanged, whereas the general expression of $c_2$ is slightly changed (see Ref. [31] for details).

The main change from Eq. (2) to (13) is represented by the coefficients $\eta_O$ and $\eta_S$ in the argument of the logarithm. Whereas $\eta_O$ is significantly different from one only in the unrealistic situation of an observer very close to the black hole, $\eta_S$ can be less than one for sources close to the black hole and even negative for sources inside the photon sphere. This is reflected in the formula for the position of the images, which allows higher order image to form inside the shadow when the source is inside the photon sphere

\[
\theta_n = \theta_m \left( 1 + \eta_O \eta_S e^{(c_2 - \phi_s - 2n\pi)/c_1} \right). \tag{14}
\]

The value of $c_2$ in the case of the Schwarzschild black hole is explicitly

\[
c_2 = -\pi + 5 \log[6] + b_O + b_S \tag{15}
\]

\[
b_O = -2 \log \left[ 3 + \sqrt{3 + \frac{9}{D_{OL}}} \right]. \tag{16}
\]

\[
b_S = -2 \log \left[ 3 + \sqrt{3 + \frac{9}{D_{LS}}} \right], \tag{17}
\]

where the distances of observer and source to the black hole are measured in Schwarzschild radii.

The extension to sources close to the black hole can be also done for spinning black holes. Here as well everything remains qualitatively identical to the infinitely distant

\[^{(1)}\text{It is inappropriate to speak about a deflection angle if the source is not in the asymptotic region.}\]
source case. The only changes come into the expression of the shift of the caustic, which is updated to

\[ \Delta_k = -\left\{ \frac{4k\pi}{3\sqrt{3}} + 4\log\left(2\sqrt{3} - 3\right) + \log\left[\frac{(2\sqrt{D_{LS}} + \sqrt{3 + D_{LS}})^2}{9(D_{LS} - 1)}\right] \right\} a \sin i, \]

and in the expression of the semi-amplitude of the caustic

\[ R_k = \left[ \frac{2}{9}(5k\pi + 8\sqrt{3} - 36) + 2\frac{9 + 4D_{LS} - 4\sqrt{D_{LS}\sqrt{3 + D_{LS}}}}{3\sqrt{3\sqrt{D_{LS}\sqrt{3 + D_{LS}}}}} \right] a^2 \sin^2 i. \]

The position of the images can be found with the same algorithm described in § 4. The magnification, instead, is not well-defined if the source is inside the gravitational field of the black hole, because there is no reference unlensed image to compare with. The determination of surface brightness and shape of the image must take into account the gravitational and Doppler redshift and is thus better performed on specific source models.

7. – Conclusions

Extreme gravitational lensing is a very spectacular though elusive phenomenon, which demands a great effort in order to be observed. The most promising extreme lens is represented by the supermassive black hole in the Galactic center, which anyway requires resolutions of order microarcseconds for the direct observation of higher order images. Suitable known sources for gravitational lensing are giant stars in the IR band and low mass X-ray binaries in the X-ray band.

In this paper we have reviewed recent results on gravitational lensing in the Strong Deflection Limit, which is an approximation devoted to the analytical determination of the properties of higher order images. Within this framework, it has been proved that the logarithmic divergence in the deflection angle is a universal feature of all black hole metrics. There exists a general method to determine the SDL coefficients of the deflection angle for any given spherically symmetric black hole metric. This method has been applied to numerous metrics. Higher order images are formed by photons performing one or more loops around the black hole before reaching the observer. The position and the flux ratios of higher order images depend on the specific metric through the SDL coefficients and are thus able to track any possible deviations from general relativity in the strong field regime.

The SDL can be extended to spinning black holes, where several new features emerge, such as extended and shifted caustics and additional images. The position and the shape of the caustics can be expressed by simple analytical formulae and the lens equation for sources near to caustics, which represent the most physically relevant situation, can be solved.

The extension of the SDL to the case of sources very close to black holes opens the way to even more interesting investigations of extreme gravitational lensing. In fact, whereas direct observation of higher order images is very difficult and probably very far to come in the future, the contribution of higher order images to currently observed phenomena involving sources very close to black holes is already measurable at present time. Light curves of flares born in the accretion disk and spectral measurements of Iron K-lines are
heavily influenced by the presence of higher order images. With the SDL setup in its updated version, it is possible to study such phenomena, bearing in mind that the large model dependence remains the main uncertainty in all theoretical attempts to interpret the complicated physics of the supermassive black holes environment.

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