Abstract. A famous conjecture of Erdős and Sós states that every graph with average degree more than $k - 1$ contains all trees with $k$ edges as subgraphs. Motivated by this problem, we propose two variants of this conjecture by imposing bounds on the minimum and the maximum degree of the host graph.

As a corollary of our main result we get that the Erdős-Sós conjecture holds approximately, if the size of the embedded tree is linear in the size of the graph, and the maximum degree of the tree is sublinear.

Key words. Erdős-Sós conjecture, embedding of trees, extremal combinatorics

1. Introduction. Typical problems in extremal graph theory ask, how many edges in a graph force it to contain a given subgraph. A classical example of a result in this area is Turán’s Theorem, which determines the average degree that guarantees the containment of the complete graph $K_r$. A more complex example is the Erdős-Stone Theorem [7], which essentially determines the average degree condition guaranteeing that the host graph contains a fixed non-bipartite graph. On the other hand, for a general bipartite graph the problem is wide open. If the embedded graph is a tree, the celebrated conjecture of Erdős and Sós asserts that an average degree greater than $k - 1$ forces a copy of any tree of order $k + 1$.

Conjecture 1.1 (The Erdős-Sós conjecture). Every graph $G$ with $\overline{\text{deg}}(G) > k - 1$ contains any tree on $k + 1$ vertices.

Here $\overline{\text{deg}}(G)$ means the average degree of $G$; similarly, we denote the minimum and the maximum degree of $G$ by $\delta(G)$ and $\Delta(G)$, respectively.

There are many partial results concerning the conjecture. It has been verified for some special families of host graphs [5, 19, 4, 21, 6], special families of trees embedded [17, 10, 9], or when the size of the host graph is only slightly larger than the size of the tree [22, 20, 12]. Finally, a solution of this conjecture for large $k$, based on an extension of the regularity lemma, has been announced in the early 1990’s by Ajtai, Komlős, Simonovits, and Szemerédi. This result will be published as a sequence of three papers [1, 3, 2].

Observe that the conjecture is optimal, since a graph with average degree at most $k - 1$ may have only $k$ vertices. Also observe that if we replace the condition on the average degree by a stronger condition $\delta(G) > k - 1$, we can embed any tree on $k + 1$ vertices in $G$ in a greedy manner. Note that each graph with average degree $\overline{\text{deg}}(G) \geq 2k$ contains a subgraph with $\delta(G) \geq k$. Such a subgraph can be found by repeatedly deleting vertices of degree smaller than $k$ from $G$.

After one verifies that the Erdős-Sós conjecture is true for both trees of diameter at most three, and for paths (this was done already by Erdős and Gallai in 1959 [8]) one can observe that such trees can be embedded even in the case when the host graph contains a vertex of degree at least $k$ and its minimum degree is at least $k/2$. This is trivial for trees of small diameter, while for the case of paths this follows from the mentioned proof of Erdős and Gallai.

*The author was supported by the Czech Science Foundation, grant number GJ16-07822Y.

†Faculty of Mathematics and Physics, Charles University & The Czech Academy of Sciences, Institute of Computer Science, Pod Vodárenskou věží 2, 182 07 Prague, Czech Republic. With institutional support RVO:67985807. (vaclavrozhon@gmail.com).
While this local condition on the minimum and maximum degree of $G$, indeed, suffices both for these special cases, it already fails for trees of diameter four, as is demonstrated by the following example from [13]. Let $T$ be a tree consisting of a vertex connected to centres of three stars on $k/3$ vertices and let $G$ be a graph consisting of a vertex complete to either two cliques of size $k/2$, or $K_{k/2,k/2}$. Then $\Delta(G) \geq k$ and $\delta(G) \geq k/2$, but $T$ is not contained in $G$ (see Figure 1.1). This example shows that it would be naïve to try to prove the Erdős-Sós conjecture in the most general setting using only the local consequence of the bound on the average degree on the maximum and minimum degree of $G$. We will actually show in Section 2.1 that trees of diameter at most three and paths are special cases; with high probability, a random tree on $k+1$ vertices cannot be embedded in the host graph with two cliques from Figure 1.1.

Despite this fact, we devote this paper to this local approach to the Erdős-Sós conjecture, showing for example, that it can be used to prove an approximate version of the Erdős-Sós conjecture for trees such that their size is linear in the size of the host graph, while their maximum degree is sublinear. The idea to use only conditions on the minimum and maximum degree comes from the paper of Havet, Reed, Stein, and Wood [13]. We mention their results in the next section.

One advantage of the local approach is that we will be able to formulate a skew version of our main result, i.e., if we know that the size of one colour class of the tree is at most $rk$, the conditions that we impose on the host graph can be weakened. This is in the spirit of a paper of Klímašová, Piguet, and Rozhoň [16], who apply the same idea to the similar Loebl-Komlós-Sós conjecture.

We are now ready to state the main result of this paper, Theorem 2.8. Roughly speaking, it states that one can embed a tree with $k$ vertices and skew $r$ in every large enough host graph with nontrivial number of vertices of degree roughly $k$ and with minimum degree roughly $rk$. We have to further assume that the degree of the tree is sublinear.

**Theorem 2.8.** For any $r, \eta > 0$ there exists $n_0$ and $\gamma > 0$ such that the following holds. Let $G$ be a graph of order $n > n_0$ and $T$ a tree of order $k$ with two colour classes $T_1, T_2$ such that $|T_1| \leq rk$ and $\Delta(T_2) \leq \gamma k$. If $\delta(G) \geq rk + \eta n$, and at least $\eta n$ vertices of $G$ have degree at least $k + \eta n$, then $G$ contains $T$.

Note that the result is interesting only if $k > \eta n/2$, otherwise we may embed $T$ in $G$ in a greedy manner. Hence we interpret this result as one for trees of size linear in the size of the host graph; for such a class of trees this result is nontrivial.

As we will see in Section 2.4, the assumptions that the maximum degree of $T$
is sublinear and that the host graph contains nontrivial proportion of high degree vertices cannot be dropped. The error term $\eta n$ also cannot be completely dropped.

A simple corollary of Theorem 2.8 is that the Erdős-Sós conjecture holds approximately (with error term linear in $n$) for trees with sublinear maximum degree.

**Theorem 2.10.** For any $\eta > 0$ there exists $n_0$ and $\gamma > 0$ such that for every $n > n_0$, any graph of order $n$ with average degree $\overline{\Delta}(G) \geq k + \eta n$ contains every tree on $k$ vertices with maximum degree $\Delta(T) \leq \gamma k$.

The theorem is again trivial if $k \leq \eta n / 2$ and the only interesting case is thus when the size of the tree is linear in the size of the host graph. Although this theorem is only a special case of the announced result of Ajtai, Komlós, Simonovits, and Szemerédi [1, 3, 2], we still believe that it is of interest, since its proof is relatively straightforward.

2. The local approach. Motivated by the fact that the two conditions $\delta(G) \geq k/2$ and $\Delta(G) \geq k$ do not suffice to ensure that $G$ contains every tree on $k + 1$ vertices, we ask the following natural questions.

1. Which trees can be embedded in any host graph satisfying $\Delta(G) \geq k$ and $\delta(G) \geq k/2$?
2. What is the smallest constant $c_1$ such that every graph with $\Delta(G) \geq k$ and $\delta(G) \geq c_1 k$ allows embedding of any tree with $k + 1$ vertices? Strictly speaking, the smallest constant may not exist. On the other hand, setting $c_1 = 1$ clearly suffices.
3. Is there a minimal $c_2$ such that every graph with $\Delta(G) \geq c_2 k$ and $\delta(G) \geq k/2$ allows embedding of any tree with $k + 1$ vertices?
4. What is the minimal number of vertices of degree at least $k$ that a graph $G$ with $\delta(G) \geq k/2$ has to contain, so that it then allows embedding of any $T$ on $k + 1$ vertices?

We now discuss these questions. The second question was considered in the paper of Havet, Reed, Stein, and Wood [13], and we only state their results.

2.1. Restricting the class of embedded trees. We observe that the example graph with two cliques from Figure 1.1 actually provides a large class of trees on $k + 1$ vertices that cannot be embedded in this graph.

**Proposition 2.1 (Wagner, personal communication).** For even $k$ it holds that the probability that a random unlabelled tree of size $k + 1$ can be embedded in the graph $G$ consisting of a vertex complete to two cliques of size $k/2$ is in $O\left(k^{-3/2}\cdot B^k\right)$.

**Proof.** We at first classify trees on $k + 1$ vertices that can be embedded in $G$. A vertex $u \in T$ is a centroid, if after removing it from $T$ we obtain a family of trees such that each tree is of size at most $k/2$. Since the size of the graph is the same as the size of the tree that we embed, only a centroid of $T$ can be embedded in the vertex of $G$ complete to all other vertices. Since $k + 1$ is odd, the centroid of the tree is unique. Hence, $T$ can be embedded if and only if the subtrees created after removing its centroid can be partitioned into two classes such that the number of vertices in each class is $k/2$. We call such trees balanced.

Let $r_k$ be the number of unlabelled rooted trees with $k$ vertices. A formula of Otter (see e.g. page 481 of [11]) states that $r_k = \Theta(k^{3/2} \cdot B^k)$ for some positive constant $B$. Similarly, the number of unlabelled unrooted trees $s_k$ is in $\Theta(k^{5/2} \cdot B^k)$ for the same constant $B$ (again page 481 of [11]).

Note that the number of balanced trees of order $k + 1$ is at most $r_k^{2k/2+1}$, since each
such tree can be decomposed into two rooted trees with $k/2 + 1$ vertices each. Hence the number of balanced trees is in $O(k^{-3}B_k)$. Comparing this with the sequence $s_k$, we conclude that the probability that a random unlabelled tree is balanced goes to 0 at a rate of at least $k^{-1/2}$.

2.2. Greater minimum degree. The second question was considered by Havet, Reed, Stein, and Wood in [13]. They conjectured the following:

**Conjecture 2.2** (Conjecture 1.1 in [13]). If $G$ is a graph such that $\delta(G) \geq \lfloor 2k/3 \rfloor$ and $\Delta(G) \geq k$, then $G$ contains any tree on $k + 1$ vertices.

As one can see from the example in Figure 1.1, this is tight. As an evidence for their conjecture, they prove its two following weakened variants. The first variant relaxes the condition on the maximum degree:

**Theorem 2.3** (Theorem 1.2 in [13]). There is a function $g$ such that any graph $G$ with $\delta(G) \geq \lfloor 2k/3 \rfloor$ and $\Delta(G) \geq g(k)$ contains any tree on $k + 1$ vertices.

The second weakening on the other hand shows that the constant $c_1$ from the second question is strictly smaller than 1.

**Theorem 2.4** (Theorem 1.3 in [13]). There is a constant $\varepsilon > 0$ such that if $G$ is a graph with $\delta(G) \geq (1 - \varepsilon)k$ and $\Delta(G) \geq k$, then $G$ contains any tree on $k + 1$ vertices.

2.3. Greater maximum degree. The third question seems to be similar to the previous one. The example in Figure 1.1 shows that we have to take $\Delta(G) \geq 4k/3$. We conjecture that this is tight:

**Conjecture 2.5.** (Klimošová, Piquet, Rozhoň) If $G$ is a graph such that $\delta(G) \geq k/2$ and $\Delta(G) \geq 4k/3$, then $G$ contains any tree on $k + 1$ vertices.

If true, this conjecture would imply that the constant $2/3$ from Theorem 2.3 can be improved to $1/2$. We were able to verify the weakening of Conjecture 2.5 with $\Delta(G) \geq 4k/3$ replaced by $\Delta(G) \geq g(k)$ for some function $g$ for trees of diameter at most four.

**Theorem 2.6** ([15]). If $G$ is a graph with $\delta(G) \geq k/2$ and $\Delta(G) \geq 2k^7$, then $G$ contains any tree on $k + 1$ vertices of diameter at most four.

Note that the Erdős-Sós conjecture was also verified for trees of diameter four in [17], but these two results are incomparable.

2.4. Many high degree vertices. Finally we consider the question of how many vertices of degree $k$ a graph $G$ with $\delta(G) \geq k/2$ has to have so as to contain all trees on $k + 1$ vertices. We propose the following conjecture.

**Conjecture 2.7.** (Klimošová, Piquet, Rozhoň) Every graph $G$ on $n$ vertices with $\delta(G) \geq k/2$ and at least $\frac{n}{2\sqrt{k}}$ vertices of degree at least $k$ contains every tree of order $k + 1$.

Note that the fraction $\frac{1}{2\sqrt{k}}$ cannot be substantially improved due to the following example in the spirit of the example from Figure 1.1.

Let $k$ be an odd square and $T$ be a tree of order $k + 1$ consisting of a vertex connected to centres of $\sqrt{k}$ stars on $\sqrt{k}$ vertices. Let $G$ be a graph consisting of two disjoint cliques of order $\frac{k-1}{2}$ and $\frac{k+1}{2}$, and an independent set of $\frac{\sqrt{k}}{2}$ vertices complete to both cliques. A simple calculation shows that the proportion of high
degree vertices of $G$ is
\[ \frac{\sqrt{k-1}}{2} < \frac{1}{2\sqrt{k}}. \]

Note that for any $c < 1$ the left hand side is larger than $\frac{c}{2\sqrt{k}}$ for sufficiently large $k$. One can check that $G$ does not contain $T$. This shows that the expression $\frac{c}{2\sqrt{k}}$ in the theorem cannot be strengthened to $\frac{c}{2\sqrt{k}}$ for any $c < 1$.

We prove a weakened variant of Conjecture 2.7. Specifically, we show that it is asymptotically true if the number of high degree vertices of $G$ as well as the size of the tree $T$ is linear in the size of $G$ and, moreover, the maximum degree of $T$ is sublinear. As we have already mentioned, we state a finer version of this result for skewed trees. Specifically, if we know that the skew of the tree $T$ is at most $r$, then $G$ contains $T$ even if its minimum degree is roughly $rk$.

**Theorem 2.8.** For any $r, \eta > 0$ there exist $n_0$ and $\gamma > 0$ such that the following holds. Let $G$ be a graph of order $n > n_0$ and $T$ a tree of order $k$ with two colour classes $T_1, T_2$ such that $|T_1| \leq rk$ and $\Delta(T_2) \leq \gamma k$. If $\delta(G) \geq rk + \eta n$, and at least $\eta n$ vertices of $G$ have degree at least $k + \eta n$, then $G$ contains $T$.

We postpone the proof of Theorem 2.8 to the last section of this chapter. As a special case for $r = 1/2$, we get the following weakening of Conjecture 2.7.

**Corollary 2.9.** For any $\eta > 0$ there exist $n_0$ and $\gamma > 0$ such that the following holds. Let $G$ be a graph of order $n > n_0$ and $T$ a tree of order $k$ such that $\Delta(T) \leq \gamma k$. If $\delta(G) \geq k/2 + \eta n$, and at least $\eta n$ vertices of $G$ have degree at least $k + \eta n$, then $G$ contains $T$.

Finally, Corollary 2.9 yields an approximate version of the Erdős-Sós conjecture for trees with sublinear degree.

**Theorem 2.10.** For any $\eta > 0$ there exist $n_0$ and $\gamma > 0$ such that for every $n > n_0$, any graph of order $n$ with average degree $\overline{\deg}(G) \geq k + \eta n$ contains every tree on $k$ vertices with maximum degree $\Delta(T) \leq \gamma k$.

**Proof.** Let $\eta' = \eta/2$ and let $G$ be a graph on $n \geq n_0 = \frac{n_0 \cdot 2 \cdot 0.9(\eta')}{\eta}$ vertices (here $n_0 \cdot 2 \cdot 0.9(\eta')$ means the output of Corollary 2.9 with input $\eta'$). Suppose that $k \geq \eta n/2$.

We choose a subgraph $G' \subseteq G$ such that $\overline{\deg}(G') \geq k + \eta n$ and $\delta(G') \geq k/2 + \eta n/2$. Hence, we know that the size of $G'$ is at least $k + \eta n \geq \eta n \geq n_0 \cdot 2 \cdot 0.9(\eta')$.

We claim that at least $\eta' |G'|$ vertices of $G'$ have degree at least $k + \eta' n$ and hence we may apply Corollary 2.9. Otherwise, most of the vertices of $G'$ have degree less than $k + \eta' n$ and we may compute that
\[ \overline{\deg}(G') \leq \eta' \cdot n + (1 - \eta') \cdot (k + \eta' n) < \eta' n + (k + \eta' n) = k + 2\eta' n = k + \eta n, \]

a contradiction.

Let us state one more remark regarding Theorem 2.8. Although the result of Ajtai, Komlós, Simonovits, and Szemerédi [1, 3, 2] implies that the condition on the maximum degree $\Delta(T)$ in Theorem 2.10 is only an imperfection, it cannot be omitted in the statement of Theorem 2.8. We show that the theorem is false if we omit this condition.

Specifically, we show that for all $0 < r < 1/3$ there exists $\eta > 0$ such that the following is true. Let $G$ be a graph on $n$ vertices consisting of two disjoint copies of complete bipartite graphs with colour classes of sizes $rk + \eta n$ and $k/2 + \eta n$. Moreover,
\[ k = 2 + \eta n \]

Fig. 2.1. Example showing that the condition on bounded degree is needed in the statement of Theorem 2.8.

\( \eta n \) additional vertices are complete to both larger colour classes of the two bipartite graphs (see Figure 2.1). Let \( T \) be a tree on \( k \) vertices consisting of a vertex \( x \) complete to centres of \( rk \) stars of sizes \( \lfloor \frac{1}{r} \rfloor \) and \( \lceil \frac{1}{r} \rceil \). The smaller colour class of \( T \) has size \( rk \).

Note that for fixed \( r \) the maximum degree of this smaller colour class of \( T \) is constant, though it is not true for the larger colour class, hence Theorem 2.8 does not apply. We claim that the tree \( T \) is not contained in \( G \) if we choose \( \eta \) sufficiently small. Suppose that there is an embedding of \( T \) in \( G \). Since \( G \) is bipartite with one colour class of size at most \( 2rk + 3\eta n < (1-r)k \) if \( k \) is big enough and \( \eta \) sufficiently small, the vertex \( x \) must be embedded in the larger colour class. Out of \( (1-r)k - 1 \) leaves at least \( (1-r)k - 1 - \eta n \cdot \lceil \frac{1}{r} \rceil > k/2 + \eta n \) have to be embedded in the same colour class as \( x \), a contradiction.

Theorem 2.8 is thus an example of an asymptotic result that does not seem to have a natural exact strengthening. On the other hand, we believe that the assumption on the sublinear maximum degree in Corollary 2.9 can be dropped.

3. The regularity method. In this section we state several preparatory results that will be later used for the proof of Theorem 2.8. In the first subsection we introduce the notion of a regular pair and state the regularity lemma. In the following subsection we show, how one can split a tree in a controllable number of smaller trees. Finally, in the last section we propose two embedding lemmas. The first will enable us to embed a special subset of vertices of a tree in the host graph, while the second will enable us to embed a small tree in a regular pair.

3.1. The regularity lemma. We say that \((X, Y)\) is an \( \varepsilon \)-regular pair, if for every \( X' \subseteq X \) and \( Y' \subseteq Y \), \(|X'| \geq \varepsilon |X|\) and \(|Y'| \geq \varepsilon |Y|\) it holds that \(|d(X',Y') - d(X,Y)| \leq \varepsilon \).

We say that a partition \( \{V_0, V_1, \ldots, V_m\} \) of \( V(G) \) is an \( \varepsilon \)-regular partition, if \(|V_0| \leq \varepsilon |V(G)|\), and all but at most \( \varepsilon m^2 \) pairs \((V_i, V_j), 1 \leq i < j \leq m\), are \( \varepsilon \)-regular. Each set of the partition is called cluster. We call the cluster \( V_0 \) the garbage set. We call a regular partition equitable if \(|V_i| = |V_j|\) for every \( 1 \leq i < j \leq m \).

Lemma 3.1 (Szemerédi’s regularity lemma). For every \( \varepsilon > 0 \) there is \( n_0 \) and \( M \) such that every graph of size at least \( n_0 \) admits an \( \varepsilon \)-regular equitable partition \( \{V_0, \ldots, V_m\} \) with \( 1/\varepsilon \leq m \leq M \).

Given an \( \varepsilon \)-regular pair \((X, Y)\), we call a vertex \( x \in X \) typical with respect to a set \( Y' \subseteq Y \) if \( \deg(x,Y') \geq (d(X,Y) - \varepsilon)|Y'| \). Note that from the definition of regularity it follows that all but at most \( \varepsilon |X| \) vertices of \( X \) are typical with respect to any subset of \( Y \) of size at least \( \varepsilon |Y| \).
3.2. Partitioning trees. Here we state a crucial lemma from [14] that allows us to partition the tree in controllable number of small subtrees that we also call microtrees. These trees are neighbouring with a set of vertices of bounded size consisting of vertices that we call seeds. Moreover, we need to work separately with seeds from different colour classes of $T$. In the following definition, the set $W_A \cup W_B$ is the set of seeds of $T$ and the set $D_A \cup D_B$ is the set of its microtrees.

**Definition 3.2.** [14, Definition 3.3] Let $T$ be a tree on $k+1$ vertices. An $\ell$-fine partition of $T$ is a quadruple $(W_A, W_B, D_A, D_B)$, where $W_A, W_B \subseteq V(T)$ and $D_A$ and $D_B$ are families of subtrees of $T$ such that

1. the three sets $W_A, W_B$ and $\{V(K)\}_{K \in D_A \cup D_B}$ partition $V(T)$ (in particular, the trees in $K \in D_A \cup D_B$ are pairwise vertex disjoint),
2. $\max\{|W_A|, |W_B|\} \leq 336k/\ell$,
3. for $w_1, w_2 \in W_A \cup W_B$ their distance is odd if and only if one of them lies in $W_A$ and the other one in $W_B$,
4. $|K| \leq \ell$ for every tree $K \in D_A \cup D_B$,
5. for each $K \in D_A$ we have $N_T(V(K)) \setminus V(K) \subseteq W_A$. Similarly for each $K \in D_B$ we have $N_T(V(K)) \setminus V(K) \subseteq W_B$,
6. $|N(V(K)) \cap (W_A \cup W_B)| \leq 2$ for each $K \in D_A \cup D_B$,
7. if $N(V(K)) \cap (W_A \cup W_B)$ contains two vertices $z_1, z_2$ for some $K \in D_A \cup D_B$, then $\text{dist}_T(z_1, z_2) \geq 6$.

We did not list all properties of $\ell$-fine partition from [14], only those we need.

**Lemma 3.3.** [14, Lemma 3.5] Let $T$ be a tree on $k+1$ vertices and let $\ell \in \mathbb{N}, \ell < k$. Then $T$ has an $\ell$-fine partition.

In the subsequent applications we are always working with $\ell = \beta k$ for some small $\beta > 0$.

Since we work with trees with sublinear degree, we may further constrain the $\ell$-fine partition in such a way that all of its seeds are only from one colour class of $T$. We call this simpler structure one-sided $\ell$-fine partition.

**Definition 3.4.** Let $T \in T_{k+1}$ be a tree and $T_1, T_2$ its colour classes. Let $\Delta = \max_{v \in T_2} \deg(v)$. A one-sided $\ell$-fine partition of $T$ is a pair $(W, D)$, where $W \subseteq V(T_1)$ and $D$ is a family of subtrees of $T$ such that

1. the two sets $W$ and $\{V(K)\}_{K \in D}$ partition $V(T)$,
2. $|W| \leq 336k(1 + \Delta)/\ell$,
3. $|K| \leq \ell$ for every tree $K \in D$,
4. For each $K \in D$ we have $N_T(V(K)) \setminus V(K) \subseteq W$,
5. We can split $D$ into two subfamilies, $D = D' \cup D''$, in such a way that all trees from $D'$ have at most two neighbours $z_1, z_2 \in W$ such that $\text{dist}_T(z_1, z_2) \geq 4$, while all of at most $336k/\ell$ trees from $D''$ are singletons with at most $\Delta$ neighbours in $W$.

**Lemma 3.5.** Let $T \in T_{k+1}$ and let $\ell \in \mathbb{N}, \ell < k$. Then $T$ has a one-sided $\ell$-fine partition.

**Proof.** Let $(W_A, W_B, D_A, D_B)$ be an $\ell$-fine partition of $T$. Suppose that $W_B \subseteq T_2$. Let $W = W_A \cup N(W_B)$ and define $D$ as the set of trees of the forest $T \setminus W$. The conditions (1), (2), and (4) are clearly satisfied. Each vertex from $W$ is now a singleton tree in $D$. Define $D''$ as the family of these singleton trees and set $D' = D \setminus D''$. Each tree in $D''$ clearly satisfies the conditions (3) and (5). Each tree from $D'$ is either a tree from $D_A$, or a subtree of a tree from $D_B$, all such trees satisfy the condition.
Finally recall that for each tree from $D_A \cup D_B$ with two neighbours $z_1$ and $z_2$ in $W_A \cup W_B$ we have $dist_T(z_1, z_2) \geq 6$. Thus, all trees from $D_A$ satisfy the condition (5). Each tree from $D_B$ with two neighbours $z_1, z_2 \in W_B$ was split into one tree with two neighbours in $W$, such that their distance in $T$ is at least 4, and maybe several other trees with only one neighbour in $W$. All such trees also satisfy (5).

### 3.3. Embedding in regular pairs

In this section we present two embedding lemmas. The first will be used to embed the seeds of a one-sided partition, together with the set $D''$, in vertices of two neighbouring clusters.

**Proposition 3.6.** For any $d, \beta, \varepsilon > 0$, $\varepsilon \leq d^2/100$ there exist $k_0$ and $\gamma > 0$ such that the following holds.

Let $T$ be a tree of order $k \geq k_0$ and $T_2$ one of its colour classes such that $\Delta(T_2) \leq \gamma k$. Moreover, let $(W, D) = D' \cup D''$ be its one-sided $\beta k$-fine partition. Let $v_1$ and $v_2$ be two clusters of vertices forming an $\varepsilon$-regular pair of density at least $d$. Suppose that $|v_1| = |v_2| \geq k/M_{L3.1}(\varepsilon)$, where $M_{L3.1}(\varepsilon)$ is the output of the regularity lemma (Lemma 3.1) with an input $\varepsilon$. Let $U \subseteq v_1$, $|U| \leq 2\sqrt{\varepsilon}|v_1|$. Then there is an injective mapping $\varphi$ of $W \cup (\bigcup D'')$ that embeds vertices of $W$ in $v_1 \setminus U$ and vertices of $\bigcup D''$ in $v_2$.

**Proof.** Choose $\gamma, k_0 > 0$ such that

$$\gamma = \frac{\beta d}{2000M_{L3.1}(\varepsilon)},$$

$$k_0 = \frac{10}{\gamma}.$$

Note that in this case we have

$$\left| \bigcup D'' \right| \leq |W| \leq \frac{336(1 + \gamma k)}{\beta} \leq \frac{500\gamma k}{\beta}$$

(definition of $\gamma$)

$$= \frac{500\beta dk}{\beta} \cdot 2000M_{L3.1}(\varepsilon) = \frac{dk}{4M_{L3.1}(\varepsilon)} \leq \frac{d}{4}|v_1|.$$

Take an arbitrary vertex $r \notin \bigcup D''$ of $T$ and root the tree at $r$. Order all vertices of $W \cup (\bigcup D'')$ according to an order, in which they are visited by a depth-first search starting at $r$. Let $U' = v_1 \cup v_2$ be the set of vertices of $v_1$ not typical to $v_2$ together with vertices of $v_2$ not typical to $v_1$. We will provide an algorithm that gradually defines a partial embedding $\varphi$ of the vertices of $W \cup (\bigcup D'')$ such that $\varphi(W) \subseteq v_1 \setminus (U \cup U')$ and $\varphi((\bigcup D'')) \subseteq v_2 \setminus U'$.

We iterate over the sequence $x_1, x_2, x_3, \ldots$ of vertices from $W \cup (\bigcup D'')$, where the vertices are ordered by the depth-first search. In the $i$-th step we deal with the vertex $x = x_i$. At first we deal with the case $x \in W$.

Suppose that $y \in \bigcup D''$ is the already embedded parent of $x$ (if $y \notin \bigcup D''$, our task is simpler). We want to embed $x$ in an arbitrary neighbour of $y$ in $v_1 \setminus (U \cup \varphi(W) \cup U')$. To do so, it suffices to verify that $N(y) \setminus (U \cup \varphi(W) \cup U')$ is nonempty. This can be done with the help of the fact that $\varphi(y)$ is typical to $v_1$ and together with our bound on $|W|$: 8
\[ |N(y) \setminus (U \cup \varphi(W) \cup U')| \geq |v_1|((d - \varepsilon) - 2\sqrt{\varepsilon} - \frac{d}{4} - \varepsilon) > 0. \]

Similarly, suppose that \( x \in \bigcup D'' \). From the definition of \( D'' \) we know that its parent \( y \) is certainly in \( W \) and \( \varphi(y) \) is typical to \( v_2 \). Now we similarly verify that

\[ |N(y) \setminus (\varphi(\bigcup D'') \cup U')| \geq |v_2|((d - \varepsilon) - \frac{d}{4} - \varepsilon) > 0. \]

Next, we state a similar proposition that enables us to embed small trees from a fine partition of \( T \) in the regular pairs of the host graph. The proposition is a variation on a folklore result and is similar to e.g.

**Proposition 3.7.** For all \( 1 \geq d, \varepsilon > 0 \) such that \( \varepsilon < d^2/100 \) there exists \( \beta > 0 \) such that the following holds.

Let \( v_1, u, v \) be three clusters of vertices such that \( v_1 u \) and \( uv \) are \( \varepsilon \)-regular pairs of density at least \( d \). Let \( v_1, v_2 \) be two (not necessarily distinct) vertices of \( v_1 \). Suppose that \( |v_1| = |u| = |v| \geq k/M_{L3.1}(\varepsilon) \). Let \( K \) be a tree of order at most \( \beta k \) and let \( x_1, x_2 \) be its two vertices from the same colour class of \( K \) such that if \( v_1 \neq v_2 \), then \( x_1 \neq x_2 \).

Let \( U \) be a subset of vertices of \( u \cup v \) such that \( |u \setminus U| \geq 4\sqrt{\varepsilon}|u| \) and \( |v \setminus U| \geq 4\sqrt{\varepsilon}|v| \).

Moreover, suppose that either

1. the vertices \( v_1, v_2 \) are typical to \( u \) and \( \deg(v_1, u) - |U \cap u| \geq 4\sqrt{\varepsilon}|u| \),
2. or we have \( |N(v_i) \setminus (u \setminus U)| \geq 3\varepsilon|u| \) for \( i = 1, 2 \).

Then there is an injective mapping \( \varphi \) of \( K \) in \( u \cup v \) such that \( \varphi(K) \cap U = \emptyset \).

Moreover, \( \varphi(x_1) \) is a neighbour of \( v_1 \) and \( \varphi(x_2) \) is a neighbour of \( v_2 \).

**Proof.** We show the proof for the harder case when \( v_1 \neq v_2 \). Choose

\[ \beta = \varepsilon/M_{L3.1}(\varepsilon). \]

From this we get

\[ |v_1| \geq \frac{k}{M_{L3.1}(\varepsilon)} = \beta \cdot \frac{M_{L3.1}(\varepsilon)}{\varepsilon} \cdot \frac{k}{M_{L3.1}(\varepsilon)} = \frac{\beta k}{\varepsilon}. \]

Note that \( u \setminus U \) contains at least \( 3\varepsilon|u| \) vertices, and similarly for \( v \). Hence there are at most \( \varepsilon|u| \) vertices in \( u \) that are not typical to \( v \setminus U \), and similarly for \( v \). We will use only typical vertices for embedding, so let \( U' \) denote the set \( U \) together with vertices not typical to \( u \setminus U \) or \( v \setminus U \), respectively. Observe that for each such vertex \( u \in u \) we have

\[ |N(u) \cap (v \setminus U')| \geq (d - \varepsilon)|v \setminus U| - \varepsilon|v| \]

\[ \geq (d - \varepsilon)4\sqrt{\varepsilon}|v| - \varepsilon|v| \]

\[ \geq \sqrt{\varepsilon} \cdot 4\sqrt{\varepsilon}|v| - \varepsilon|v| \geq 2\varepsilon|v| \]

\[ \geq \varepsilon|v| + \beta k \geq \varepsilon|v| + |K|, \]

and similar holds for any \( u \in v \). This means that during embedding we may always find a neighbour of \( u \) in \( v \setminus U' \) that was not yet used for embedding. The same applies for both vertices \( v_1, v_2 \). In the case (1) the vertices \( v_1, v_2 \) are typical to \( u \) and hence
we have
\[ |N(v_i) \cap (u \setminus U')| \geq (d(v_1, u) - \varepsilon)|u| - |U' \cap u| \]
\[ \geq \deg(v_1, u) - |U \cap u| - 2\varepsilon|u| \geq 4\sqrt{\varepsilon}|u| - 2\varepsilon|u| \geq 2\varepsilon|u| \]
\[ |u| \geq \beta k / \varepsilon \]
\[ \geq \varepsilon|u| + \beta k \geq \varepsilon|u| + |K|, \]

while in the case (2) we have
\[ |N(v_i) \cap (u \setminus U')| \geq |N(v_i) \cap (u \setminus U)| - \varepsilon|u| \]
\[ \geq 2\varepsilon|u| \geq |u| \geq \beta k / \varepsilon \]
\[ \geq \varepsilon|u| + \beta k \geq \varepsilon|u| + |K|. \]

We start by embedding the path \( t_1 = x_1, t_2, \ldots, t_\ell = x_2 \) connecting \( x_1 \) with \( x_2 \) in \( K \). Embed \( x_1 \) in an arbitrary vertex of \( u \setminus U' \). For \( i \) going from 2 to \( \ell - 2 \) we always map \( t_i \) to a neighbour of \( v(t_{i-1}) \) not lying in \( U' \). Now we observe that both \( N(v_2) \cap (u \setminus U') \) and \( N(t_{\ell-2}) \cap (v \setminus U') \) have sizes at least \( \varepsilon|v_1| \), thus there is an edge connecting those two neighbourhoods. Map \( t_{\ell-1} \) and \( t_\ell \) in the two endpoints of the edge. The rest of the tree can be then embedded in the greedy manner.

4. Proof of Theorem 2.8. In this section we prove Theorem 2.8. We split the proof into three parts. At first we preprocess the host graph by applying the regularity lemma and we partition the tree by applying Lemma 3.5. In the second part we find a suitable matching structure in the host graph. In the last part we embed the tree in the host graph.

Preprocessing the host graph and the tree. Fix \( \eta, r \). Suppose that \( \eta < 1 \). Choose \( d, \varepsilon, \beta, n_0 \) such that

\[ d = \frac{(\eta r)^2}{1000}, \]
\[ \varepsilon = \frac{(\eta rd)^{10}}{10^{15}}, \]
\[ \beta = \min \left( \beta_{P3.7}(d, \varepsilon), \frac{\eta d}{10^8 \cdot M_{L3.1}(\varepsilon)} \right), \]
\[ \gamma = \gamma_{P3.6}(d, \varepsilon, \beta), \]
\[ n_0 = \max \left( n_{0, L3.1}(\varepsilon), \frac{2}{\eta} k_{0, P3.6}(d, \varepsilon, \beta) \right). \]

Let \( G \) be a fixed graph on \( n \geq n_0 \) vertices with at least \( \eta n \) vertices of degree 
\( k + \eta m \) and with \( \delta(G) \geq rk + \eta m \). Suppose that \( k \geq \eta m / 2 \), otherwise we embed the tree \( T \) greedily. We apply the regularity lemma (Lemma 3.1) on \( G \) with \( \varepsilon_{L3.1} = \varepsilon \) and obtain an \( \varepsilon \)-regular equitable partition \( V_0, V_1, \ldots, V_m \) with \( 1 / \varepsilon \leq m \leq M_{L3.1}(\varepsilon) \) clusters. Each cluster has average degree at least \( rk + \eta m \).

Erase all edges within sets \( V_i \) of the partition, between irregular pairs, and between pairs of density lower than \( d \). We have erased at most \( m \cdot \left( \frac{n/m}{2} \right) \leq \frac{n^2}{m} \leq \varepsilon n^2 \) edges within the sets \( V_i \), at most \( \varepsilon m^2 \cdot (n/m)^2 = \varepsilon n^2 \) edges in irregular pairs, and at most \( \left( \frac{m}{2} \right) \cdot d \cdot (n/m)^2 \leq d \cdot n^2 \) edges in pairs of low density. Erase the garbage set \( V_0 \) and all of at most \( \varepsilon n \cdot n \) incident edges. Note that we have erased at most \( (3 \varepsilon + d)n^2 \) edges. We abuse the notation and still call the resulting graph \( G \).
Note that the quantity $\sum_{1 \leq i \leq m} |V_i| \cdot \deg(V_i)$ dropped down by at most $(6\varepsilon + 2d)n^2$. Thus there are at most $\sqrt{6\varepsilon + 2d} \cdot m$ clusters such that their average degree dropped down by more than $\sqrt{6\varepsilon + 2d} \cdot n$. Delete all such clusters and incident edges. We again call the resulting graph $G$. The average degree of each cluster of $G$ that was not deleted at first dropped by at most $\sqrt{6\varepsilon + 2d} \cdot n$. Then we erased at most $\sqrt{6\varepsilon + 2d} \cdot m$ clusters, so now it is at least $rk + \eta m - 2 \cdot \sqrt{6\varepsilon + 2d} \cdot n > rk + \eta m/2$. Moreover, $G$ contains at least $(\eta - \varepsilon - \sqrt{6\varepsilon + 2d})n \geq \eta m/2$ vertices of degree at least $k + \eta m - 2 \cdot \sqrt{6\varepsilon + 2d} \cdot n \geq k + \eta m/2$. Hence, there exists a cluster, without loss of generality it is $V_1$, such that the proportion of vertices of degree at least $k + \eta m/2$ in that cluster is at least $\eta/2 \geq \varepsilon$. If we denote by $L$ this set of high degree vertices of $V_1$, then we have $\deg(V_1, V_i) \geq \deg(L, V_i) - \varepsilon |V_i|$ from regularity of each pair $(V_1, V_i)$. This yields that $\deg(V_1) \geq \deg(L) - \varepsilon n \geq k + \eta m/3$. Moreover, if it is the case that $\deg(V_1) > 2k$, we erase several regular pairs from neighbouring $V_1$ so as to achieve $\deg(V_1) \leq 2k$. After deletion the average degree of each cluster is still at least $k + \eta m/2
ot\leq k + \eta m/3$.

The cluster graph $G$ of $G$ is a graph such that its vertex set are the clusters of $G$ and there is an edge between two vertices of $V$ if and only if there is a regular pair of density at least $d$ between the corresponding two clusters in $G$. The weight of each edge $uv$ is the average degree of $u$ in $v$. We use boldface font to denote the vertices and sets of vertices of $G$. The vertex set of $G$ is denoted $v_1, \ldots, v_m$, where each $v_i$ corresponds to the cluster $V_i$ of $G$.

After preprocessing the host graph we turn our attention to the tree $T$. Let $T_1, T_2$ be its colour classes such that $|T_1| \leq rk$ and $\Delta(T_2) \leq \gamma k$. We apply Lemma 3.5 with parameter $\ell_3.5 = \beta k$ and obtain its one-sided $\beta k$-fine partition $(W, D), D = D' \cup D''$ such that $|\tilde{W}| \leq 336(1 + \gamma k)/\beta$ and $|\tilde{D'}| \leq 336/\beta$. Moreover, for each $K \in D'$ we have $|K| \leq \beta k$ and for each $K \in D''$ we have $|K| = 1$. Also note that $W \subseteq T_1$.

Structure of the host graph. We now find a suitable structure in the cluster graph $G$ that will be used for the embedding of $T$. It suffices to look at the cluster $v_1$, that will serve for the embedding of the seeds of $T$, and its neighbourhood.

Let $M$ a maximal matching in $N(v_1)$. We will denote by $M$ both the graph and its underlying vertex set. Suppose that $uv \in M$. Note that from the condition on maximality we get that there cannot be two vertices $x \neq y \in N(v_1) \setminus M$ such that both $xu$ and $yv$ are edges of $G$. Thus there are two possibilities for each edge $uv$: either only one of its endpoints have neighbours in $N(v_1) \setminus M$, or both of its endpoints have just one neighbour in $N(v_1) \setminus M$. We can get rid of the second special case as follows. For each vertex in $N(v_1) \setminus M$ we either delete it if it is a common neighbour of at least $\eta m/40$ matching pairs, or we delete all edges in at most $2 \cdot \eta m/40$ regular pairs connecting the vertex with these matching pairs. In this way we delete at most $40/\eta$ clusters and the degree of all remaining clusters of $G$ drops down by at most $\eta m/20 \cdot |v_1| + 40/\eta \cdot |v_1| \leq (\eta/20 + 40\varepsilon/\eta) \cdot n \leq \eta m/10$. We abuse the notation and still call the resulting graph $G$. The degree of $v_1$ is at least $k + \eta m/3 - \eta m/10 \geq k + \eta m/5$ and the average degree of every cluster is similarly at least $rk + \eta m/5$. The matching $M$ is still maximal in $N(v_1)$. Moreover, we can split $M$ into two colour classes, $M = M_1 \cup M_2$, in such a way that only clusters from $M_2$ have neighbours in $N(v_1) \setminus M$. Let $O_1 = N(v_1) \setminus M$. Note that it is an independent set. Define $O_2 = N(O_1) \setminus \{v_1\} \cup M$. Note that $N(v_1) \cap O_2 = \emptyset$. All these sets are shown in Figure 4.1.

Embedding. We split the last part further into three subparts. At first we give an overview of the method that we use for the construction of the mapping $\varphi$. Then we formulate several preparatory technical claims. In the last part we propose the
embedding algorithm.

**Overview.** We gradually construct an injective mapping $\varphi$ from $T$ to $G$. In each step $\varphi$ denotes the partial embedding that we already constructed. The idea behind the embedding process is very straightforward – we will try to embed microtrees of $\mathcal{D}$ inside the regular pairs in $M$ and 'through' the vertices of $O$. We will, however, have to overcome several technical difficulties.

One of the standard approaches of embedding trees, pursued, e.g., in [16], is to start by embedding the seeds of $T$ in vertices of two clusters (one for each colour class) such that the neighbourhood of these special clusters is sufficiently rich. Moreover, we embed the seeds in such vertices that are typical to almost all neighbouring clusters. We then split the microtrees in $T$ into several subsets and embed each subset of microtrees in some part of the neighbourhood of the special clusters. Here we take a different approach. We start in the same way by embedding the seeds $W$ of $T$ in a high degree cluster of $G$ that we call $v_1$. We then propose an algorithm that iterates over clusters in the neighbourhood of $v_1$, each time finding two clusters that can be used for embedding of a microtree.

There are two main technical difficulties that we have to overcome. Recall that each seed is embedded in a vertex that is typical to almost all clusters. This means that when we choose a pair of clusters that will be used for embedding, we have to find a microtree that has not yet been embedded such that its adjacent seeds are embedded in vertices typical to the first cluster from the pair. We can ensure that there will be such microtree, unless the number of vertices that remain to be embedded, is very small, specifically $\sqrt{\varepsilon k}$. To ensure that we can embed the whole tree $T$, we at first allocate a small fraction of vertices $F \subseteq \bigcup (M \cup O)$ that we do not use for the embedding during the main embedding procedure. When only at most $\sqrt{\varepsilon k}$ vertices
remain to be embedded, we finally embed this small proportion of trees in the set $F$.

The second technical problem is that we cannot ensure that all the microtrees have the same skew. This complicate the main embedding procedure that would have been simpler in the case of microtrees with uniform skew. During the embedding procedure we behave against intuition and sometimes redefine an embedding of some microtrees.

**Preparations.** Note that there may be at most $\sqrt{\varepsilon}|v_1|$ vertices of $v_1$ that are not typical to more than $\sqrt{\varepsilon}m$ clusters. Indeed, otherwise there would be at least $\varepsilon m|v_1|$ pairs of a cluster and a vertex not typical to it, which in turn implies existence of a cluster such that more than $\varepsilon|v_1|$ vertices are not typical to it, a contradiction with the $\varepsilon$-regularity. For each cluster $v \in M_1 \cup O_1$ fix its arbitrary subset $F_v$ of size $\lfloor \eta_{rd}|v|/300 \rfloor$. By the same reasoning there are at most $\sqrt{\varepsilon}|v_1|$ vertices that are not typical to more than $\sqrt{\varepsilon}m$ clusters.

We invoke Proposition 3.6 with parameters $d_{p, 3.6} = d, \beta_{p, 3.6} = \beta, \varepsilon_{p, 3.6} = \varepsilon$. We also choose $v_2, p_{3.6} = v_2$ to be any cluster from the neighbourhood of $v_1, p_{3.6} = v_1$.

Finally, we define the set $U_{p, 3.6}$ to be the set of at most $2\sqrt{\varepsilon}|v_1|$ vertices not typical to more than $\sqrt{\varepsilon}m$ neighbouring clusters $v_i$, or their subsets $F_{v_i}$. Note that due to our initial choice of constants all the conditions from the statement of the proposition are satisfied. Hence we embed the vertices of $W$ in $v_1$, while the vertices of $\bigcup D''$ will be embedded in $v_2$. Moreover, each vertex from $W$ is embedded in a vertex typical to all but at most $\sqrt{\varepsilon}m$ clusters $v_i$ and their fixed subsets $F_{v_i}$ of size $\lfloor \eta_{rd}|v|/300 \rfloor$.

Note that each microtree $K \in D'$ has at most two neighbours in $W$. We call a cluster $u \neq v_1$ nice with respect to $K \in D'$, if the at most two neighbours of $K$ are embedded in vertices typical to $u$. Note that each vertex from $W$ was mapped to a vertex that is typical to all but at most $\sqrt{\varepsilon}m$ clusters, thus for each tree $K$ there are at most $2\sqrt{\varepsilon}m$ clusters that are not nice to $K$. We will now, yet again, employ a simple doublecounting argument. This time we doublecount pairs consisting of microtrees from $D'$ and clusters that are not nice to them; each such connection is weighted by the size of the tree. We get that there are at most $2\sqrt{\varepsilon}m$ clusters such that if we take all trees such that the cluster is not nice to them, then the union of all such trees contains more than $\sqrt{\varepsilon}k$ vertices. Delete all such clusters and if they are from $M$, delete also their neighbours in $M$. We also delete the cluster $v_2$. Observe that the average degree of each cluster is still at least

$$rk + \eta n/10 - 4\sqrt{\varepsilon}m|v_1| - |v_2|$$

$$\begin{align*}
\text{if } m \geq \sqrt{\varepsilon}n \\
\text{if } m < \sqrt{\varepsilon}n
\end{align*}$$

Similarly, the degree of $v_1$ is still at least $\deg(v_1) \geq k + \eta n/20$. We still call the new graph $G$. We also know for each $u \in N(v_1)$ that the number of vertices in microtrees such that $u$ is not nice to them is at most $\sqrt{\varepsilon}k$.

Now we will define a small set $F \subseteq \bigcup (M \cup O)$ that will be used at the end for embedding of several leftover microtrees with at most $\sqrt{\varepsilon}k$ vertices.

**Claim 4.1.** There is a set $F \subseteq \bigcup (M \cup O)$ satisfying $|F| \leq \eta_{rd}\deg(v_1)/100$, $F_u \subseteq F \cap u$ for any $u \in M_1 \cup O_1$ and $|F \cap u| = |F \cap v|$ for any $uv \in M$. Moreover, if we extend our partial mapping $\varphi$ of $W \cup \bigcup D''$ in such a way that the extended mapping satisfies $\varphi(T) \cap F = \emptyset$ and $\varphi$ is defined on the whole $T$ except of some $D \subseteq D$ with $|\bigcup D| \leq \sqrt{\varepsilon}k$, then we can injectively extend $\varphi$ to the whole tree $T$. 

13
Proof. We define $F$ as follows. For each $u \in M_1 \cup O_1$ we add $F_u$ to $F$. Then for each set $F_u$ we find a set of the same size in some neighbour $v \neq v_1$ of $u$ and also add this set to $F$. We call this set $G_u$ and find it as follows. For $uv \in M$ we take $G_u = F_v$. For $u \in O_1$ we find its neighbouring cluster in $O_2 \cup M_2$ with at least $\lceil \eta rd(u)/300 \rceil$ vertices that were not yet added to $F$ and we set $G_u$ to be this set (we explain later, why we always find a suitable neighbouring cluster). In the case when $F_u \in O_1$, but $G_u \in M_2$, it is no longer true that $|F \cap u'| = |F \cap v'|$ for some matching edge $u'v' \in M$. We again establish the condition by adding $\lceil \eta rd(u')/300 \rceil$ vertices from $u'$ to $F$. The construction implies that

$$|F| \leq 3 \cdot \sum_{u \in M_1 \cup O_1} \lceil \eta rd(u)/300 \rceil \leq \eta rd(v_1)/100.$$  

Now we explain, why each cluster $u \in O_1$ has a neighbour in $M_2 \cup O_2$ with at least $\lceil \eta rd(u)/300 \rceil$ vertices that are not yet in $F$. Since we know that

$$\deg(u \cup (M_2 \cup O_2)) \geq rk$$

there is certainly a cluster $v \in M_2 \cup O_2$ such that $\deg(u, v) > \deg(u, F \cap v)$, hence $\deg(u, v \setminus F) = \deg(u, v) - \deg(u, v \cap F) > \deg(u, v) / 2 \geq d(v)/2$, meaning that there is a subset of at least $d(v)/2 > \lceil \eta rd(v)/300 \rceil$ vertices in $v$ that can be used to define $G_u$.

Now we show how to embed any $D$ of small size in $F$. We define the embedding $\varphi$ of all trees $K \in D$ in a step-by-step manner. Suppose that $u \in M_1 \cup O_1$ and $G_u \subseteq v$. If the at most two neighbours $z_1, z_2$ of $K \in W$ are embedded in two vertices of $v_1$ that are typical to set $F_u$ and, moreover, $|\varphi(T) \cap F_u| \leq \frac{\varepsilon}{2} |F_u|$ and $|\varphi(T) \cap G_u| \leq \frac{\varepsilon}{2} |G_u|$, we can compute that for $i = 1, 2$ we have

$$|F_u \setminus \varphi(T)| \geq \left(1 - \frac{d}{2}\right) |F_u| \geq 4\sqrt{\varepsilon} |u|$$

and similarly $|G_u \setminus \varphi(T)| \geq 4\sqrt{\varepsilon} |v|$. We also have

$$|N(v_i) \cap (F_u \setminus \varphi(T))| \geq |N(v_i) \cap F_u| - |\varphi(T) \cap F_u|$$

where $v_i$ is typical to $F_u$

$$\geq (d - \varepsilon) |F_u| - |\varphi(T) \cap F_u| \geq \frac{d}{3} |F_u| \geq 3\varepsilon |u|.$$  

Hence we can use Proposition 3.7 case (2) with parameters $U_{p3.7} = \bar{F} \cup \varphi(T)$, where $\bar{F}$ means the complement of $F$ in our graph, $d_{p3.7} = d$, $\varepsilon_{p3.7} = \varepsilon$, $\beta_{p3.7} = \beta$, $v_{1,p3.7} = v_1, u_{p3.7} = u, v_{p3.7} = v, K_{p3.7} = K, v_{1,p3.7} = \varphi(z_1), v_{2,p3.7} = \varphi(z_2)$. The proposition then allows us to embed $K$.

Now it suffices to show that for any $K$ we always find a suitable $u$ such that $\varphi(z_1), \varphi(z_2)$ are typical to $F_u$ and both $F_u$ and $G_u$ do not contain many embedded vertices of $T$. Recall that vertices $\varphi(z_1), \varphi(z_2)$ are typical to all but at most $\sqrt{\varepsilon} m$ sets $F_u$. If we cannot use for embedding any other set $F_u$ from remaining clusters of $M_1 \cup O_1$, it means that we have embedded at least $\frac{d}{2} \cdot \lceil \eta rd(v_1)/300 \rceil$ vertices to this set $F_u$, or we have embedded at least the same number of vertices in the appropriate
set $G_u$. This means that the number of vertices we have embedded is at least

\[
\left( |M_1 \cup O_1| - 2\sqrt{\varepsilon m} \right) \cdot \left( \frac{d}{2} \cdot |\varepsilon d| v_1^2 / 300 \right) \geq \left( \frac{|M \cup O_1|}{2} - 2\sqrt{\varepsilon m} \right) \cdot \frac{d^2 \eta}{700} |v_1|
\]

\[
\geq \left( \frac{\deg(v_1)}{2} - 2\sqrt{\varepsilon m} |v_1| \right) \cdot \sqrt[3]{\varepsilon}
\]

\[
|v_1| \leq n \Rightarrow \left( \frac{k}{2} - 2\sqrt{\varepsilon n} \right) \cdot \sqrt[3]{\varepsilon}
\]

\[
n \geq \left( \frac{1}{2} - \frac{4\sqrt{\varepsilon}}{\eta} \right) \cdot \sqrt[3]{\varepsilon} k
\]

\[
> \sqrt[3]{\varepsilon} k,
\]

a contradiction.

**Embedding algorithm.** So far we have embedded the set $W$ in vertices of $v_1$ that are typical to almost all clusters in the neighbourhood of $v_1$. We also embedded the small set $D'$. We invoke Claim 4.1 to get a small set $F$. Now we will gradually embed microtrees from $D$ in $\bigcup (M \cup O) \setminus F$, until the number of vertices of microtrees that were not embedded yet is at most $\sqrt[3]{\varepsilon} k$. Then we embed the remaining parts of $T$ in $F$ using Claim 4.1. We will use the following notation for the sake of brevity.

**Definition 4.2.** We say that a cluster $u$ is full, if

\[
|u \cap (\varphi(T) \cup F) | \geq |u| - 4\sqrt{\varepsilon} |u|.
\]

We say that a cluster $u \in N(v_1)$ is saturated, if

\[
|u \cap (\varphi(T) \cup F) | \geq \deg(v_1, u) - 4\sqrt{\varepsilon} |u|.
\]

We say that a matching edge $uv \in M$ is saturated, if

\[
| (u \cup v) \cap (\varphi(T) \cup F) | \geq \deg(v_1, (u \cup v)) - 8\sqrt{\varepsilon} |u| - 3k.
\]

Note that every full cluster is also saturated. The intuition behind these definitions will be clear from the statements of the following claims.

**Claim 4.3.** If $u \in N(v_1)$ is not saturated and $v \in N(u) \setminus \{v_1\}$ is not full, then, unless $|\text{dom}(\varphi)| \geq k - \sqrt[3]{\varepsilon} k$, we may injectively extend $\varphi$ to some $K \in D$ that was not yet embedded in such a way that $\varphi(K \cap D_1) \subseteq u$, $\varphi(K \cap D_2) \subseteq v$, and $\varphi(K) \cap F = \emptyset$.

**Proof.** We have ensured that all trees of $D$ such that $u$ is not nice to them have at most $\sqrt[3]{\varepsilon} k$ vertices. Hence there is a yet nonembedded tree $K \in D$ such that its at most two neighbours $t_1, t_2$ in $W$ are embedded in vertices of $v_1$ that are typical to $u$. We may now apply Proposition 3.7 (1) with $d_{p, 3.7} := d, \varepsilon_{p, 3.7} = \varepsilon, \beta_{p, 3.7} = \beta, v_{1, p, 3.7} = v_1, u_{p, 3.7} = u, v_{p, 3.7} = v, K_{p, 3.7} = K, v_{i, p, 3.7} = \varphi(t_i), x_{i, p, 3.7} = N(t_i) \cap K, U_{p, 3.7} = \varphi(T) \cup F$. The proposition then allows us to extend injectively $\varphi$ to $K$.

**Claim 4.4.** Let $\varphi$ be a partial embedding of $T$ in $G$.

1. There exists either an unsaturated vertex of $O_1$ or an unsaturated edge of $M$.
2. Suppose that $\varphi(D_1) \cap (M_2) = \emptyset$ and let $u \in O_1$. There exists a vertex in $N(u) \setminus \{v_1\}$ that is not full.
Proof. 1. Suppose that for each edge $uv \in M$ we have

$$| (u \cup v) \cap (\varphi(T) \cup F) | \geq \deg(v_1, u \cup v) - 8\sqrt{\varepsilon} |u| - \beta k$$

$$\geq \deg(v_1, u \cup v) \left(1 - \frac{8\sqrt{\varepsilon} |v_1| + \beta k}{2d|v_1|}\right)$$

$$|v_1| \geq n/M_{L,1}(\varepsilon), k \leq n$$

$$\geq \deg(v_1, u \cup v) \left(1 - \frac{4\sqrt{\varepsilon}}{d} - \frac{\beta n}{2dn/M_{L,1}(\varepsilon)}\right)$$

$$k \leq d, \beta \leq d/M_{L,1}(\varepsilon)$$

$$\geq \deg(v_1, u \cup v)(1 - \eta/100)$$

and suppose that for each $u \in O_1$ we have (after similar calculation)

$$|u \cap (\varphi(T) \cup F)| \geq \deg(v_1, u)(1 - \eta/100).$$

Hence we have

$$|\bigcup(M \cup O_1) \cap (\varphi(T) \cup F)| \geq \deg(v_1)(1 - \eta/100)$$

$$= \eta \deg(v_1)/100 + \deg(v_1)(1 - \eta/50)$$

$$\geq \eta \deg(v_1)/100 + (k + \eta k/20)(1 - \eta/50)$$

$$> |F| + k,$$

a contradiction.

2. Similarly as in the previous case we can compute that we have embedded at least $|v|(1 - \eta/100) \geq \deg(u, v)(1 - \eta/100)$ vertices into each full cluster $v$.

Since we know that $\deg(v_1) \leq 2k$ and $\deg(u) \geq rk + \eta k/20$, we thus have

$$|\bigcup(M_2 \cup O_2) \cap (\varphi(T \cap D_2) \cup F)| \geq \deg(u)(1 - \eta/100)$$

$$\geq \eta \deg(u)/50 + \deg(u)(1 - \eta/30)$$

$$\geq \eta \deg(v_1)/100 + (rk + \eta k/20)(1 - \eta/30)$$

$$> |F| + rk$$

$$\geq |F| + |D_2|,$$

a contradiction. \hfill \square

We can now finish the proof of Theorem 2.8.

Proof. We will gradually embed microtrees from $D'$ in $\bigcup(M \cup O)$ in a specified manner using Claim 4.3, until $|\text{dom}(\varphi)| \geq k - \sqrt{\varepsilon}k$, or all edges of $M$ and all vertices of $O$ are saturated — from Claim 4.4 (1) we know that the latter actually cannot be true. When $|\text{dom}(\varphi)| \geq k - \sqrt{\varepsilon}k$, we finish by applying Claim 4.1 on our set $F$. We split the embedding procedure into three phases:

1. Phase 1 — saturating the matching edges of $M$. In the first phase we embed gradually the microtrees of $D'$ in the edges of $M$ in such a way that for each $K \in D'$ we have $\varphi(K \cap D_1) \subseteq M_1$. We run the process of applying Claim 4.3 for each edge $uv$ until either $u \in M_1$ is saturated, $v \in M_2$ is full, or $|\text{dom}(\varphi)| \geq k - \sqrt{\varepsilon}k$.

2. Phase 2 — saturating the clusters in $O$. We repeatedly pick a cluster $v \in O_1$ and then embed trees from $D'$ in it by repeatedly applying Claim 4.3 in such a way that for each embedded $K$ we have $\varphi(K \cap D_1) \subseteq O_1$ and $\varphi(K \cap D_2) \subseteq O_2$. 

3. Phase 3 — saturating the clusters in $O$ in the remaining edges of $M$. We repeat step 2 on the remaining edges of $M$. 


We finish by invoking Claim 4.4 (2) the cluster \( v \) has always a neighbour that is not full and can be, thus, used for embedding. Hence we can apply this procedure until all clusters from \( O_1 \) are saturated, or \( |\text{dom}(\varphi)| \geq k - \sqrt[4]{\varepsilon}k \).

3. \textbf{Phase 3 – finalising the matching \( M \).} All clusters in \( O_1 \) are now saturated. Our goal is now to show how to saturate the remaining edges of \( M \). This may not be possible with current \( \varphi \) as it is defined right now, since it could have for example happened that after the first phase we completely filled one cluster from a matching pair, while the other cluster remained almost empty. We solve this problem by potentially redefining the embedding of several microtrees that were embedded in \( M_1 \cup M_2 \) in Phase 1.

Note that for each edge \( uv \in M, u \in M_1 \), it is true that either \( u \) is saturated, or \( v \) is full at the end of Phase 1. We deal with the first case in part (a). In the latter case we did not embed anything in \( v \) in Phase 2. We redefine embedding of all trees that were embedded in \( uv \) and saturate this edge in part (c).

(a) If \( u \) is saturated, we repeatedly embed trees in \( uv \) in such a way that for each \( K \in D' \) we have \( \varphi(K \cap D_1) \subseteq v \). We do this until either \( u \) is full, or \( v \) is saturated. In the latter case the whole edge is saturated. We deal with the first case in (b).

(b) Suppose that \( u \) is full, but \( v \) is not saturated. Note that Claim 4.1 ensures that \( |F \cap u| = |F \cap v| \). Hence it must be the case that \( |\varphi(T) \cap u| \geq |\varphi(T) \cap v| \). Moreover, in Phase 2 we did not embed trees in \( u \). This means that there exists a tree \( K \in D' \) that was embedded in the matching edge \( uv \) in such a way that \( |\varphi(K) \cap u| \geq |\varphi(K) \cap v| \). As long as it is true that \( |\varphi(T) \cap u| \geq |\varphi(T) \cap v| \), we find any tree \( K \) with this property and we redefine its embedding. When this procedure ends, we have \( \left| \varphi(T) \cap u \right| - \left| \varphi(T) \cap v \right| \leq \beta k \). We call this inequality a balancing condition.

(c) Finally it suffices to show how to saturate an edge \( uv \) fulfilling the balancing condition (note that if \( \varphi(T) \cap uv = \emptyset \), then the matching edge certainly fulfills the condition). We again embed the microtrees in \( uv \) one after another. Unless one of the clusters is saturated, we choose to embed \( K \in D' \) in such a way that the colour class of \( K \) with less vertices is embedded in the cluster such that more of its vertices were already used for embedding of \( T \). In this way we ensure that the balancing condition still holds.

After one cluster, say \( u \), becomes saturated, we continue by embedding only in such a way that for each \( K \in D' \) we have \( \varphi(K \cap D_1) \subseteq v \). We do this until either \( v \) becomes saturated, or \( u \) is full. In the first case the whole edge \( uv \) is clearly saturated. In the other case note that we have \( \left| \left( \varphi(T) \cup F \right) \cap u \right| \geq |u| - 4\sqrt[4]{\varepsilon} |u| \geq \deg(v_1,u) - 4\sqrt[4]{\varepsilon} |v| \) and \( \left| \left( \varphi(T) \cup F \right) \cap v \right| \geq \left| \left( \varphi(T) \cup F \right) \cap u \right| - \beta k \geq \deg(v_1,u) - 4\sqrt[4]{\varepsilon} |u| - \beta k \) due to our balancing condition. Hence the matching edge is saturated.

We described an algorithm that terminates when \( |\text{dom}(\varphi)| \geq k - \sqrt[4]{\varepsilon}k \), or all edges of \( M \) and all vertices of \( O \) are saturated. But the latter cannot happen due to Claim 4.4 (1). We finish by invoking Claim 4.1.

\( \blacksquare \)

5. \textbf{Conclusion.} This paper, as well as some similar results are part of the author’s Bachelor’s thesis [18].

I would like to thank Stephan Wagner for providing the proof of Proposition
2.1. My great thanks go to Diana Piguet and Tereza Klimošová for many helpful discussions and comments regarding the paper.

REFERENCES

[1] M. Ajtai, J. Komlós, M. Simonovits, and E. Szemerédi, On the approximative solution of the Erdős-Sós conjecture on trees. In preparation.
[2] M. Ajtai, J. Komlós, M. Simonovits, and E. Szemerédi, The solution of the Erdős-Sós conjecture for large trees. In preparation.
[3] M. Ajtai, J. Komlós, M. Simonovits, and E. Szemerédi, Some elementary lemmas on the Erdős-Sós conjecture on trees. In preparation.
[4] S. Balasubramanian and E. Dobson, Constructing trees in graphs with no $k^2,s$, Journal of Graph Theory, 56 (2007), pp. 301–310, https://doi.org/10.1002/jgt.20261, http://dx.doi.org/10.1002/jgt.20261.
[5] S. Brandt and E. Dobson, The Erdős-Sós conjecture for graphs of girth 5, Discrete Mathematics, 150 (1996), pp. 411 – 414, https://doi.org/https://doi.org/10.1016/0012-365X(95)00207-D, http://www.sciencedirect.com/science/article/pii/0012365X9500207D.
[6] E. Dobson, Constructing trees in graphs whose complement has no $K^2,s$, Combinatorics, Probability and Computing, 11 (2002), p. 343–347, https://doi.org/10.1017/S0963548302005102.
[7] P. Erdős and A. H. Stone, On the structure of linear graphs, Bulletin of the American Mathematical Society, 52 (1946), pp. 1087–1091.
[8] P. Erdős and T. Gallai, On maximal paths and circuits of graphs, Acta Mathematica Hungarica, 10 (1959), pp. 337–356, https://doi.org/10.1007/BF02024498.
[9] G. Fan, The Erdős-Sós conjecture for spiders of large size, Discrete Mathematics, 313 (2013), pp. 2513 – 2517, https://doi.org/https://doi.org/10.1016/j.disc.2013.07.021, http://www.sciencedirect.com/science/article/pii/S0012365X13003324.
[10] G. Fan and L. Sun, The Erdős–Sós conjecture for spiders, Discrete Mathematics, 307 (2007), pp. 3055 – 3062, https://doi.org/https://doi.org/10.1016/j.disc.2007.03.018, http://www.sciencedirect.com/science/article/pii/S0012365X07001070.
[11] P. Flajolet and R. Sedgewick, Analytic combinatorics, (2010).
[12] A. Görlich and A. Zak, On Erdős-Sós conjecture for trees of large size, Electr. J. Comb., 23 (2016), p. P1.52.
[13] F. Havet, B. Reed, M. Stein, and D. R. Wood, A Variant of the Erdős-Sós Conjecture, ArXiv e-prints, (2016), https://arxiv.org/abs/1606.09343.
[14] J. Hladký, J. Komlós, D. Piguet, M. Simonovits, M. Stein, and E. Szemerédi, The approximate Loebl–Komlós–Sós Conjecture IV: Embedding techniques and the proof of the main result, SIAM J. Discrete Math., 31 (2017), pp. 1072–1148.
[15] T. Klimošová, D. Piguet, and V. Rozhoš, Embedding trees of small diameter. In preparation.
[16] T. Klimošová, D. Piguet, and V. Rozhoš, A version of the Loebl-Komlós-Sóos conjecture for skewed trees, ArXiv e-prints, (2018), https://arxiv.org/abs/1802.00679.
[17] A. McLennan, The Erdős-Sós conjecture for trees of diameter four, Journal of Graph Theory, 49 (2005), pp. 291–301, https://doi.org/10.1002/jgt.20083, http://dx.doi.org/10.1002/jgt.20083.
[18] V. Rozhoš, Sufficient conditions for embedding trees, 2018. Bachelor’s Thesis.
[19] J.-F. Saclé and M. Woźniak, The Erdős-Sós conjecture for graphs without $C_4$; J. Comb. Theory Ser. B, 70 (1997), pp. 367–372, https://doi.org/10.1006/jctb.1997.1758, http://dx.doi.org/10.1006/jctb.1997.1758.
[20] G. Tiner, On the Erdős-Sós conjecture for graphs on $n = k+3$ vertices, Ars Comb., 95 (2010).
[21] M. Wang, G.-r. Li, and A.-d. Lui, A result of Erdős-Sós conjecture, Ars Combinatoria, 55 (2000).
[22] M. Woźniak, On the Erdős-Sós conjecture, Journal of Graph Theory, 21 (1996), pp. 229–234, https://doi.org/10.1002/(SICI)1097-0118(199602)21:2<229::AID-JGT13>3.0.CO;2-E, http://dx.doi.org/10.1002/(SICI)1097-0118(199602)21:2<229::AID-JGT13>3.0.CO;2-E.