MOD-2 EQUIVALENCE OF THE $K$-THEORETIC EULER AND SIGNATURE CLASSES

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This note proves that, as $K$-theory elements, the symbol classes of the de Rham operator and the signature operator on a closed manifold of even dimension are congruent mod 2. An equivariant generalization is given pertaining to the equivariant Euler characteristic and the multi-signature.

1. Introduction

It is well-known that the Euler characteristic $\chi(M)$ and the signature $\text{Sign}(M)$ of a closed oriented manifold $M$ of dimension $4n$ are two integers of the same parity. This fact is an easy consequence of Poincaré duality and we briefly recall its proof. Let $\beta_k = \dim H_k(M; \mathbb{R})$. By Poincaré duality, $\beta_k = \beta_{4n-k}$ and there is a non-degenerate bilinear form (the “intersection form”) on $H_{2n}(M; \mathbb{R})$. Let $\beta_{2n}^+$ (respectively $\beta_{2n}^-$) be the dimension of a maximal subspace of $H_{2n}(M; \mathbb{R})$ on which the form is positive definite (respectively negative definite). By definition, $\text{Sign}(M) = \beta_{2n}^+ - \beta_{2n}^-$. The following mod-2 congruence relation follows:

$$
\chi(M) = \sum_{k=0}^{4n} (-1)^k \beta_k \equiv \beta_{2n}^+ + \beta_{2n}^- + \sum_{k=0}^{2n-1} 2\beta_k \equiv \beta_{2n}^+ - \beta_{2n}^- = \text{Sign}(M).
$$

We prove an analogous result on the level of the symbols (as $K$-theory classes) of the de Rham and signature operators on even-dimensional manifolds; we also give an equivariant generalization.

We first recall the construction of the two operators (cf. [2]).

Let $M$ be a closed oriented smooth manifold of dimension $2n$. Equip $M$ with a Riemannian metric. (The symbols, as $K$-theory classes, of the two operators will be independent of the choice of Riemannian metric and hence are smooth invariants. This is due to the fact that, for any metrics $g_0$ and $g_1$ and for $t \in [0, 1], tg_0 + (1-t)g_1$ is a Riemannian metric.) Let

$$
\Omega^* = \Gamma(\Lambda^*(T^*M \otimes \mathbb{C})),
$$

the space of smooth sections of the exterior algebra bundle $\Lambda^* = \Lambda^*(T^*_cM)$ associated with the complexified cotangent bundle $T^*_cM = T^*M \otimes \mathbb{C}$; i.e., $\Omega^*$ is the space of complex differential forms. Let

$$
D = d + \delta : \Omega^* \to \Omega^*
$$

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where $d$ is the exterior derivative and $\delta$ its adjoint with respect to the Hermitian product on $\Omega^*$ induced by the Riemannian metric on $M$. The de Rham operator $D^0$ is defined to be the restriction of $D$ to the subspace $\Omega^{\text{even}}$ of even-degree forms and thus takes value in the subspace $\Omega^{\text{odd}}$ of odd-degree forms:

$$D^0 = D|_{\Omega^\text{even}} : \Omega^\text{even} = \Gamma(\Lambda^\text{even}) \to \Omega^\text{odd} = \Gamma(\Lambda^\text{odd}).$$

The bundle map Hodge star $\ast : \Lambda^k \to \Lambda^{2n-k}$ is defined on each fiber $(\Lambda^k)_x$ by

$$\alpha \wedge \ast \beta = \langle \alpha, \beta \rangle \cdot \text{vol}(M)_x \text{ for } \alpha, \beta \in (\Lambda^k)_x$$

where $\text{vol}(M) \in \Omega^{2n}$ is the volume form. The Hodge star has the property that $\ast(\ast \alpha) = (-1)^k \alpha$ for $\alpha \in \Lambda^k$.

Define $\tau : \Lambda^k \to \Lambda^{2n-k}$ by letting

$$\tau = i^{n+k(k-1)} \ast.$$ 

It is easy to verify that $\tau$ is an involution on $\Lambda^*$. Then $\tau$ decomposes $\Lambda^*$ into $\Lambda^+ \oplus \Lambda^-$, the $+1$ and $-1$ eigenbundles. The map $\tau$ induces an involution on $\Omega^*$ (which we also call $\tau$) and decomposes $\Omega^*$ into $\Omega^+ \oplus \Omega^-$, where $\Omega^\pm$ are the $(\pm 1)$-eigenspaces. Note that $\Omega^\pm = \Gamma(\Lambda^\pm)$ and that $D$ interchanges $\Omega^+$ and $\Omega^-$ (since $D\tau = -\tau D$). The signature operator $D^+$ is defined to be the restriction of $D$ to $\Omega^+$:

$$D^+ = D|_{\Omega^+} : \Omega^+ = \Gamma(\Lambda^+) \to \Omega^- = \Gamma(\Lambda^-).$$

We now recall the $K$-theoretic symbol class associated with an elliptic differential operator. Suppose $E_1$ and $E_2$ are two complex vector bundles over $M$. Let $\pi : T^*_C M \to M$ be the bundle projection. Associated with a differential operator $P : \Gamma(E_1) \to \Gamma(E_2)$, there is the (leading) symbol of $P$,

$$\sigma(P) : \pi^* E_1 \to \pi^* E_2.$$ 

$\sigma(P)$ is a bundle homomorphism. If, over the complement of the zero section of $T^*_C M$, $\sigma(P)$ is a bundle isomorphism, $P$ is then said to be elliptic. The symbol $\sigma(P)$ of an elliptic operator $P$ determines a class $[\sigma(P)]$, the $K$-theoretic symbol class of $P$, in

$$K(T^*_C M) = \tilde{K}(D(T^*_C M)/S(T^*_C M)), $$

where $D(T^*_C M)$ and $S(T^*_C M)$ denote the closed unit disc bundle and the unit sphere bundle associated with $T^*_C M$. We will review in more detail the transition from $\sigma(P)$ to $[\sigma(P)]$ in §2.
It is a standard fact that $D^0$ and $D^+$ are elliptic. We call $[\sigma(D^0)]$ the $K$-theoretic Euler class of $M$ and $[\sigma(D^+)]$ the $K$-theoretic signature class of $M$. It is their relationship in the abelian group $K(T^*_\mathcal{C}M)$ that our Main Theorem pertains to.

**Main Theorem.** If $\dim M$ is even, then $[\sigma(D^0)] \equiv [\sigma(D^+)] \mod 2K(T^*_\mathcal{C}M)$.

(Here, $2K(T^*_\mathcal{C}M)$ means $\{\alpha + \alpha : \alpha \in K(T^*_\mathcal{C}M)\}$.)

The Main Theorem implies, as a corollary, the mod-2 congruence between $\chi(M)$ and $\text{Sign}(M)$. To see this, consider the index homomorphism $\text{Index} : K(T^*_\mathcal{C}M) \to \mathbb{Z}$ and recall that $\text{Index}([\sigma(D^0)]) = \chi(M)$ and $\text{Index}([\sigma(D^+)]) = \text{Sign}(M)$.

2. Preliminaries on $K$-theory

We review and establish some $K$-theoretic results.

Let $(X, A)$ be a pair of connected compact Hausdorff spaces. Let $E_1$ and $E_2$ be two complex vector bundles over $X$ with a bundle isomorphism $\sigma : E_1|_A \to E_2|_A$. The triple $(E_1, E_2; \sigma)$ determines a class $[E_1, E_2; \sigma]$ in $\tilde{K}(X/A)$. We first recall its construction.

Let $X_i = X \times \{i\}$ and $A_i = A \times \{i\}$, $i = 1, 2$; let $Y = X_1 \cup_g X_2$, with $g : (a, 1) \mapsto (a, 2)$ for $a \in A$. (For notational convenience, we regard $E_i$ as a bundle not only over $X$ but also over $X_1$ and $X_2$.) We first construct bundles $E_{i,j}$’s over $Y$. To produce $E_{i,j}$, we glue $E_i|_{X_1}$ and $E_j|_{X_2}$ via $\varepsilon_{i,j} : E_i|_{A_1} \to E_j|_{A_2}$ where $\varepsilon_{1,2} = \sigma$, $\varepsilon_{2,1} = \sigma^{-1}$, and $\varepsilon_{i,i} = \text{Id}_{E_i|_A}$.

Consider $E_{1,2} - E_{2,2} \in \tilde{K}(Y)$. Evidently, its restriction to $X_2$ is $0 \in \tilde{K}(X_2)$. As $X_2$ is a retract of $Y$ and $X/A \cong Y/X_2$, the long exact sequence of $K$-theory for the pair $(Y, X_2)$ yields the following short exact sequence:

$$0 \to \tilde{K}(X/A) \to \tilde{K}(Y) \to \tilde{K}(X_2) \to 0.$$  

Hence, $E_{1,2} - E_{2,2}$ is the image of a unique element in $\tilde{K}(X/A)$, which we name $[E_1, E_2; \sigma]$. It can be shown that this element is invariant under variation of $\sigma$ within its own homotopy class of bundle isomorphisms; see $\mathbb{I}$.

**Lemma 2.1.** In the above notation, we have the following identities in $\tilde{K}(X/A)$:

1. $[E_1, E_2; \sigma] + [E_1', E_2'; \sigma'] = [E_1 \oplus E_1', E_2 \oplus E_2'; \sigma \oplus \sigma'].$
2. $[E_1, E_2; \sigma] + [E_2, E_3; \rho] = [E_1, E_3; \rho \circ \sigma].$
3. $[E_2, E_1; \sigma^{-1}] = -[E_1, E_2; \sigma].$
Proof: Part 1 is a direct consequence of the definition.
For Part 2, note that $\sigma \oplus \rho : E_1|_A \oplus E_2|_A \to E_2|_A \oplus E_3|_A$ is homotopic through bundle isomorphisms to

$$\varphi = \begin{pmatrix} 0 & -\text{Id} \\ \rho \circ \sigma & 0 \end{pmatrix} : E_1|_A \oplus E_2|_A \to E_2|_A \oplus E_3|_A$$

via

$$t \mapsto \begin{pmatrix} \text{Id} & 0 \\ t\rho & \text{Id} \end{pmatrix} \begin{pmatrix} \text{Id} & 0 \\ 0 & \text{Id} \end{pmatrix} \begin{pmatrix} \text{Id} & 0 \\ t\rho & \text{Id} \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & \rho \end{pmatrix}, \quad t \in [0,1].$$

Thus

$$[E_1, E_2; \sigma] + [E_2, E_3; \rho] = [E_1 \oplus E_2, E_2 \oplus E_3; \sigma \oplus \rho] \quad \text{(by Part 1)}$$

$$= [E_1 \oplus E_2, E_2 \oplus E_3; \varphi] \quad \text{(by } \varphi \simeq (\sigma \oplus \rho))$$

$$= [E_1 \oplus E_2, E_3 \oplus E_2; (\rho \circ \sigma) \oplus (-\text{Id})] \quad \text{(by the formula for } \varphi)$$

$$= [E_1, E_3; \rho \circ \sigma] + [E_2, E_2; (-\text{Id})] \quad \text{(by Part 1)}$$

$$= [E_1, E_3; \rho \circ \sigma] + [E_2, E_2; \text{Id}]$$

where the last equality follows from the fact that $\text{Id}$ and $-\text{Id}$ are homotopic through isomorphisms of complex vector bundles. It is trivial that $[E_2, E_2; \text{Id}] = 0$ (which can be checked in $\tilde{K}(Y)$). Part 2 then follows.

Part 3 follows directly from Part 2. $\square$

Given an isomorphism of complex vector bundles $\sigma : E_1 \to E_2$, let $\sigma^* : E_2^* \to E_1^*$ denote the dual isomorphism; if $E_1$ and $E_2$ are equipped with Hermitian metrics, let $\hat{\sigma} : E_2 \to E_1$ denote the adjoint bundle map. (Given Hermitian vector spaces $V$ and $W$ and a complex linear map $g : V \to W$, the adjoint $\hat{g} : W \to V$ is defined by $\langle gv, w \rangle = \langle v, \hat{g}w \rangle$ for $v \in V$ and $w \in W$. The adjoint of a bundle map can be fiberwise defined.) Since any complex linear map $g : V \to W$ remains complex linear when viewed as a map $g : \overline{V} \to \overline{W}$ where $\overline{V}$ and $\overline{W}$ are the conjugates $V$ and $W$, we may view $\hat{g}$ as $\hat{g} : \overline{W} \to \overline{V}$. In the same way, we may view $\hat{\sigma}$ as $\hat{\sigma} : E_2^* \to E_1^*$.

Lemma 2.2. Suppose that $E_1$ and $E_2$ are complex vector bundles equipped with Hermitian metrics and that $\sigma : E_1|_A \to E_2|_A$ is an isomorphism of complex vector bundles. Then, in $\tilde{K}(X/A)$, we have:

1. $[\overline{E_2}, \overline{E_1}; \hat{\sigma}] = [E_2^*, E_1^*; \sigma^*].$
2. $[E_1, E_2; \sigma]^* = [E_1^*, E_2^*; (\sigma^*)^{-1}].$
3. $[E_2, E_1; \hat{\sigma}] = -[E_1, E_2; \sigma].$
Proof: For a complex linear map $g : V \rightarrow W$ between finite-dimensional Hermitian vector spaces, we have the following commutative diagram

$$
\begin{array}{ccc}
W & \xrightarrow{\bar{g}} & V \\
\downarrow & & \downarrow \\
W^* & \xrightarrow{g^*} & V^*
\end{array}
$$

where the vertical maps are the complex linear isomorphisms $w \mapsto \langle \cdot, w \rangle : W \rightarrow \mathbb{R} \in W^*$ and $v \mapsto \langle \cdot, v \rangle : V \rightarrow \mathbb{R} \in V^*$.

Part 1 then follows from the corresponding diagram of isomorphisms of bundles over $A$.

For Part 2, recall that the bundle $E_{i,j}$ over $Y$ is given by gluing the bundle $E_i$ over $X_1$ with $E_j$ over $X_2$ via a bundle isomorphism $\varepsilon_{i,j} : E_i|_{A_1} \rightarrow E_j|_{A_2}$. It follows that the bundle $E_{i,j}^*$ is formed by gluing $E_i^*$ with $E_j^*$ via $\varepsilon_{i,j}^* : E_i^*|_{A_2} \rightarrow E_j^*|_{A_2}$, or equivalently via $(\varepsilon_{i,j}^*)^{-1} : E_i^*|_{A_1} \rightarrow E_j^*|_{A_2}$. Part 2 readily follows.

Part 3 follows from a string of equalities:

$$
[E_2, E_1; \bar{\sigma}] = [(E_2)^*, (E_1)^*; \sigma^*] \quad \text{(by Part 1)}
$$

$$
= \left[ (E_2)^*, (E_1)^*; ((\sigma^{-1})^*)^{-1} \right] 
$$

$$
= \left[ E_2, E_1; \sigma^{-1} \right]^* \quad \text{(by Part 2)}
$$

$$
= - \left[ E_1, E_2; \sigma \right]^* \quad \text{(by Part 3 of Lemma 2.1)}
$$

$$
= - \left[ [E_1, E_2; \sigma] \right]^*
$$

where the last equality follows from the bundle equivalence between the conjugate of a complex vector bundle and its dual. \(\square\)

3. Proof of the Main Theorem

We now collect the preliminary results to prove our Main Theorem.

Proof of the Main Theorem: Let $M$ be an even-dimensional closed smooth manifold.

Because $\dim M$ is even, $\Lambda^{\text{even}}$ and $\Lambda^{\text{odd}}$ are both invariant under the involution $\tau$. Thus, $\tau$ decomposes $\Lambda^{\text{even}}$ and $\Lambda^{\text{odd}}$ into $(\pm 1)$-eigenbundles, resulting in the following decomposition of $\Lambda^*$:

$$
\Lambda^* = \Lambda^{\text{even}} \oplus \Lambda^{\text{odd}}
$$

$$
= (\Lambda^{\text{even},+} \oplus \Lambda^{\text{even},-}) \oplus (\Lambda^{\text{odd},-} \oplus \Lambda^{\text{odd},+}).
$$
Likewise, $\Omega^{\text{even}}$ and $\Omega^{\text{odd}}$ can be decomposed into $(\pm 1)$-eigenspaces of $\tau$, resulting in the following decomposition of $\Omega^*$:

$$
\Omega^* = \Omega^{\text{even}} \oplus \Omega^{\text{odd}}
$$

$$
= (\Omega^{\text{even},+} \oplus \Omega^{\text{even},-}) \oplus (\Omega^{\text{odd},-} \oplus \Omega^{\text{odd},+}).
$$

Then, $D^0$ can be diagonalized as follows:

$$
D^0 = D^{0,+} \oplus D^{0,-} : \Omega^{\text{even},+} \oplus \Omega^{\text{even},-} \rightarrow \Omega^{\text{odd},-} \oplus \Omega^{\text{odd},+}.
$$

Regrouping the summands in (1), we have

$$
\Omega^* = (\Omega^{\text{even},+} \oplus \Omega^{\text{odd},+}) \oplus (\Omega^{\text{odd},-} \oplus \Omega^{\text{even},-}).
$$

Clearly,

$$
(\Omega^{\text{even},+} \oplus \Omega^{\text{odd},+}) \subset \Omega^+ \quad \text{and} \quad (\Omega^{\text{odd},-} \oplus \Omega^{\text{even},-}) \subset \Omega^-.
$$

Since $\Omega^* = \Omega^+ \oplus \Omega^-$, the decomposition (2) implies

$$
\Omega^+ = \Omega^{\text{even},+} \oplus \Omega^{\text{odd},+} \quad \text{and} \quad \Omega^- = \Omega^{\text{odd},-} \oplus \Omega^{\text{even},-}.
$$

Then, $D^+$ can be diagonalized as follows:

$$
D^+ = D^{+,0} \oplus D^{+,1} : \Omega^{\text{even},+} \oplus \Omega^{\text{odd},+} \rightarrow \Omega^{\text{odd},-} \oplus \Omega^{\text{even},-}.
$$

Since $\Omega^{\text{even/odd},\pm} = \Gamma(\Lambda^{\text{even/odd},\pm})$, we have

$$
\sigma(D^0) = \sigma(D^{0,+}) \oplus \sigma(D^{0,-}) : \pi^* \Lambda^{\text{even,+}} \oplus \pi^* \Lambda^{\text{even,-}} \rightarrow \pi^* \Lambda^{\text{odd,-}} \oplus \pi^* \Lambda^{\text{odd,+}},
$$

and

$$
\sigma(D^+) = \sigma(D^{+,0}) + \sigma(D^{+,1}) : \pi^* \Lambda^{\text{even,+}} \oplus \pi^* \Lambda^{\text{odd,+}} \rightarrow \pi^* \Lambda^{\text{odd,-}} \oplus \pi^* \Lambda^{\text{even,-}}.
$$

By Part 1 of Lemma 2.1,

$$
[\sigma(D^0)] = [\sigma(D^{0,+})] + [\sigma(D^{0,-})] \quad \text{and} \quad [\sigma(D^+)] = [\sigma(D^{+,0})] + [\sigma(D^{+,1})].
$$

It is a matter of definition that

$$
D^{0,+} = D^{+,0} \quad \text{and} \quad D^{0,-} = \hat{D}^{+,1}.
$$

Thus,

$$
\sigma(D^{0,+}) = \sigma(D^{+,0}) \quad \text{and} \quad \sigma(D^{0,-}) = \sigma(\hat{D}^{+,1}) = \sigma(D^{+,1}).
$$

Therefore

$$
[\sigma(D^{0,+})] = [\sigma(D^{+,0})]
$$

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and, by Part 3 of Lemma 2.2,
\[
[\sigma(D^0, -)] = [\sigma(D^{+0})] = -[\sigma(D^{+1})].
\]

In summary,
\[
[\sigma(D^0)] + [\sigma(D^+)] = [\sigma(D^{0,+})] + [\sigma(D^{0,-})] + [\sigma(D^{+0})] + [\sigma(D^{+1})] = 2[\sigma(D^{0, +})].
\]

\[\square\]

4. Discussion

By the $K$-theoretic Thom isomorphism theorem for complex vector bundles, $K(T^*_CM)$ is a free $K(M)$-module of rank 1 generated by the $K$-theory Thom class. In this way, the Main Theorem admits interpretation in $K(M)$. The $K$-theory Thom class of $T^*_CM \to M$ is simply the element
\[
[\pi^*\Lambda^{even}, \pi^*\Lambda^{odd}; \mu] \in K(T^*_CM)
\]
with
\[
\mu(v, \omega) = v \wedge \omega - \iota_v \omega \quad \text{for } v \in (T^*_CM)_x \text{ and } \omega \in (\Lambda^{even})_x
\]
where $\iota_v : \Lambda^k \to \Lambda^{k-1}$ is the interior multiplication by $v$. This is precisely the $K$-theoretic Euler class, i.e., the symbol class of the de Rham operator $[\sigma(D^0)] \in K(T^*_CM)$, which corresponds (via Thom isomorphism) to $1 \in K(M)$. Thus, letting $S \in K(M)$ denote the element corresponding (via the Thom isomorphism) to $[\sigma(D^+)] \in K(T^*_CM)$ and translating the Main Theorem into a statement in $K(M)$, we have that $S = 1 + 2x$ for some $x \in K(M)$.

When $M$ admits an orientation-preserving smooth action by a compact Lie group $G$, the Main Theorem admits an equivariant generalization. (This is due to a suggestion by Sylvain Cappell.) Equip $M$ with a $G$-invariant metric, i.e., a metric with respect to which $G$ acts isometrically. Then, $[\sigma(D^0)]$ and $[\sigma(D^+)]$ can both be interpreted as $G$-equivariant $K$-theory classes, i.e., elements of $K_G(T^*_CM)$, and the proof for the Main Theorem can be adapted to show the following.

**Theorem.** Suppose that dim $M$ is even and that a compact Lie group $G$ acts smoothly on $M$ preserving orientation. Then $[\sigma(D^0)] \equiv [\sigma(D^+)] \mod 2K_G(T^*_CM)$.

When $G$ is finite, we may apply $G$-Index : $K_G(T^*_CM) \to R(G)$ to deduce from this result the mod-2 equivalence of the equivariant Euler characteristic and multi-signature (as virtual complex representations of $G$).
In closing, we mention a few related works. One is [4], which shows that the $K$-homology class of the de Rham operator is trivial; another is [3], which discusses the equivariant $KO$-theoretic Euler characteristic. A more recent work, [5], shows among other results that the mod-8 reduction of the $K$-homology class of the signature operator is an oriented homotopy invariant.

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