Counting Maximum Matchings in Planar Graphs Is Hard

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Abstract

Here we prove that counting maximum matchings in planar, bipartite graphs is #P-complete. This is somewhat surprising in the light that the number of perfect matchings in planar graphs can be computed in polynomial time. We also prove that counting non-necessarily perfect matchings in planar graphs is already #P-complete if the problem is restricted to bipartite graphs. So far hardness was proved only for general, non-necessarily bipartite graphs.

1 Introduction

The complexity class #P contains the counting version of problems in NP. A problem is #P-complete if it is in #P and also #P-hard, that is, any problem in #P is polynomial reducible to it. In his seminal paper, Valiant defined the #P-complete complexity class and proved that counting perfect matchings in bipartite graphs is #P-complete [5]. This result showed that a counting problem might be hard even if its decision version is in P.

On the other hand, counting perfect matchings in planar graphs can be done in polynomial time by the Fisher-Kasteleyn-Temperley algorithm [3, 4]. Using a polynomial interpolation, Valiant proved that counting non-necessarily perfect matchings (short: matchings) in bipartite graphs is also #P-complete [6]. Jerrum used this result to prove that counting matchings in planar graphs is also #P-complete [1, 2]. He reduced the problem of counting matchings in bipartite graphs to the problem of counting matchings in planar graphs. The proof uses a crossing gadget which is non-bipartite.

In this paper, we modify Jerrum’s crossing gadget to be a bipartite, planar graph. In this way, we can prove that counting matchings in planar graphs is already #P-complete when restricted to planar, bipartite graphs. Then we reduce the problem of counting matchings in planar, bipartite graphs to
the problem of counting maximum matchings in planar, bipartite graphs to prove \#P-completeness of counting maximum matchings in planar, bipartite graphs.

2 Preliminaries

For any graph $G$, $M(G)$ denotes the number of matchings in $G$ and $MM(G)$ denotes the number of maximum matchings of $G$.

We will denote the computational problem of counting matchings in a graph by \#Match. Similarly, \#MAXIMUMMATCH denotes the computational problem of counting maximum matchings in a graph. When the problem is restricted to special classes of graphs, it is emphasized by prefixes; Pl- will denote planar, B- will denote bipartite. For example, \#Pl-B-MATCH denotes the computational problem of counting matchings in planar, bipartite graphs.

**Definition 1.** Let $G = (V, E)$ be a graph, and let $w : V \to W$ be a weight function mapping complex numbers to the vertices of $G$, where $W \subseteq \mathbb{C}$. Then the matching polynomial of $G$ is defined as

$$MP(G) := \sum_{M \in M(G) \setminus \{\emptyset\}} \prod_{v \notin M} w(v),$$

(1)

where $M(G)$ is the set of non-necessarily perfect matchings of $G$, and a vertex $v$ is not in a matching $M$ if none of the edges of $M$ is incident to $v$. The empty product is defined as 1, that is, each perfect matching also contributes to the matching polynomial with a value 1. The set $W$ is given by fixed formulae using the basic arithmetic operations and roots with integer degrees.

We will denote the computational problem of computing the matching polynomial of a planar bipartite graph by \#Pl-B-W-Match. The size of $W \setminus \{1\}$ equals $k$ will be emphasized by denoting the problem by \#Pl-B-W(k)-Match.

Observe that $MP(G)$ equals $M(G)$ when each weight is 1.

3 Counting matchings in planar, bipartite graphs is hard

We prove this hardness results very similar to the proof of Jerrum. The only difference between our proof and Jerrum’s proof is in the sub-gadgets building the crossing gadget and in the number of different weights used in the matching polynomial.

**Theorem 3.1.** Counting matchings in planar, bipartite graphs is \#P-complete.

**Proof.** We first prove that \#B-Match $\leq$ \#Pl-B-W(6)-Match. Then with 6 polynomial interpolations, we prove that \#Pl-B-W(6)-Match $\leq$ \#Pl-B-Match.

To prove \#B-Match $\leq$ \#Pl-B-W(6)-Match, we create a planar, bipartite crossing gadget. By replacing all crossing edges of a bipartite graph $G$, we construct a vertex-weighted, planar, bipartite graph $G'$, such
that \( MP(G') = C^{k} M(G) \), where \( C \) is

\[
\left( \frac{\sqrt{\frac{11+9\sqrt{3}}{4}}}{4} \right)^{2} \left( 1 - \sqrt{3}i \right) \cdot \frac{\sqrt{105 - 77}}{2(\sqrt{105} + 3)} \approx -15.24493 + 42.854005i
\]

and \( k \) is the number of crossings in \( G \). Clearly, the first \( n \) digits of \( C \) can be computed in \( O(poly(n)) \) time.

Similar to Jerrum, the crossing gadget is built from two sub-gadgets, \( \Delta_1 \) and \( \Delta_2 \). Both sub-gadgets have the topology and parametrized vertex weights shown in Figure 1. We will call the edges incident to \( x, y \) and \( z \) external edges.

It is easy to see that the matching polynomial of the gadget is

\[
b^3c + 3b^2 + (x + y + z)(b(4 + 3ab + 2bc + ab^2c)) + 
+ (xy + xz + yz)(3 + 8ab + 3bc + 3a^2b^2 + 4ab^2c + a^2b^3c) + 
+ xyz(c(2 + 9ab + 6a^2b^2 + a^3b^3) + 3(2a + 4a^2b + a^3b^2)).
\]

We are looking for parameter values \( a, b, \) and \( c \) such that the matching polynomial of \( \Delta_1 \) be

\[
C_1(1 + xy + xz + yz)
\]

and the matching polynomial of \( \Delta_2 \) be

\[
C_2(1 + xyz),
\]

for some complex, non-zero numbers \( C_1 \) and \( C_2 \). That is, we want the matching polynomial of subgadget \( \Delta_1 \) to contribute to the matching polynomial when all or exactly one of its external edges are in a matching. Similarly, we want the matching polynomial of \( \Delta_2 \) to contribute to the matching polynomial when all or none of the external edges are in a matching.

Observing that \( b = 0 \) cannot yield a solution, we have to solve the equation system for \( \Delta_2 \):

\[
c(2 + 9ab + 6a^2b^2 + a^3b^3) + 3(2a + 4a^2b + a^3b^2) \quad b^3c + 3b^2 \quad 4 + 3ab + 2bc + ab^2c \quad 0 \quad 3 + 8ab + 3bc + 3a^2b^2 + 4ab^2c + a^2b^3c \quad 0
\]

Using Equation 7, we can express \( c \) as

\[
c = \frac{-4 - 3ab}{2b + ab^2}
\]

to get to the reduced system

\[
-8 - 30ab - 21a^2b^2 - 4a^3b^3 \quad 2b^3 \quad -6b - 6ab^2 - 2a^2b^3 = 0
\]
A possible solution is

\[
a = \frac{-3 + \sqrt{3}i}{\sqrt{2}(11 + 9\sqrt{3}i)} \approx -0.684979 + 0.767825i, \quad b = \sqrt[3]{\frac{11 + 9i\sqrt{3}}{4}} \approx 1.598509 + 0.527535i \quad (12)
\]

from which \(c\) and \(c_2\) can be also expressed:

\[
c = \frac{-2 - \sqrt{3}i}{\sqrt[3]{\frac{11 + 9\sqrt{3}}{4}}} \approx -1.063699 + 0.921191i
\]

and

\[
C_2 = \left(\frac{\sqrt[3]{\frac{11 + 9\sqrt{3}}{4}}}{4}\right)^2 (1 - \sqrt{3}i) \approx 1.299527 - 0.564309i.
\]

The weights for \(\Delta_1\) can be computed from the following equation system:

\[
\begin{align*}
3 + 8ab + 3bc + 3a^2b^2 + 4ab^2c + a^2b^3c &= b^3c + 3b^2c \quad (13) \\
4 + 3ab + 2bc + ab^2c &= 0 \quad (14) \\
c(2 + 9ab + 6a^2b^2 + a^3b^3) + 3(2a + 4a^2b + a^3b^2) &= 0 \quad (15)
\end{align*}
\]
We can again express $c$ from the second equation, and thus we get the following reduced system:

$$
-6b - 6ab^2 - 2a^2b^3 = 2b^3 \quad (16)
$$

$$
-8 - 30ab - 21a^2b^2 - 4a^3b^3 = 0 \quad (17)
$$

We know that $\sqrt[3]{105-13 \over 8}$ is a solution of the equation

$$
4x^3 + 21x^2 + 30x + 8 = 0. \quad (18)
$$

Then setting $ab = \sqrt[3]{105-13 \over 8}$, that is, $a = \sqrt[3]{105-13 \over 8}$, substituting it in equation 16 we arrive to the following equation:

$$
-6b - 6 \sqrt[3]{105-13 \over 8} b - 2 \left( \frac{\sqrt[3]{105-13 \over 8}}{8} \right)^2 b = 2b^3 \quad (19)
$$

has to be solved for $b$, that is,

$$
b = \pm \frac{1}{4}i \sqrt{\frac{77 - \sqrt{105}}{2}}. \quad (20)
$$

Then a possible solution is

$$
a = -i \frac{\sqrt[3]{105-13 \over 2(77 - \sqrt{105})}}, \quad b = \frac{1}{4}i \sqrt{\frac{77 - \sqrt{105}}{2}},
$$

from which $c$ and $c_1$ can also be expressed:

$$
c = -2i \frac{71 - 3 \sqrt{105}}{\sqrt{2(679 + 29 \sqrt{105})}} \quad (22)
$$

and

$$
C_1 = \frac{\sqrt[3]{105-77}}{2(\sqrt{105 + 3})},
$$

Still following Jerrum, we can pack three $\Delta_1$ and three $\Delta_2$ sub-gadgets to arrive to our planar, bipartite crossing gadget, see Figure 2. Let $G$ be an arbitrary bipartite graph. For any planar drawing of $G$, replace each crossing edges $(x, y)$ and $(z, w)$ with the crossing gadget. Let the so-obtained graph be $G_1$. For each vertex in $G_1$ which is not in the crossing gadget, the weight is set to 1. Recall that $\Delta_1$ contribute to the matching polynomial only when all or one of the ’$a_1$’ vertices are incident to external edges in the matching. Also observe that $\Delta_2$ contributes to the matching polynomial only if all or none of the ’$a_2$’ vertices are incident to external edges in the matching. Furthermore, any contribution of $\Delta_1$ is $C_1$, and any contribution of $\Delta_2$ is $C_2$. Therefore, the matching polynomial of the gadget in Figure 2 is

$$
C_1^3 C_2^3 (1 + xy + zw + xyzw). \quad (24)
$$

It is also easy to see that the crossing gadget is a bipartite graph with $x$ and $w$ being in one of the vertex
Figure 2: The planar, bipartite crossing gadget. Vertex weights \( a_1, b_1 \) and \( c_1 \) are the weights of sub-gadget \( \Delta_1 \). Vertex weights \( a_2, b_2 \) and \( c_2 \) are the weights of sub-gadget \( \Delta_2 \). See text for details.

Thus, we get that the matching polynomial of \( G_1 \) is the number of matchings in \( G \) times \( (C_1C_2)^{3k} \), where \( k \) is the number of crossings in the drawing of \( G \).

Next, we prove that \( \#\text{Pl-B-W}(6)\text{-Match} \leq \#\text{Pl-B-Match} \). We do it by showing that for any \( k > 0 \), \( \#\text{Pl-B-W}(k+1)\text{-Match} \leq \#\text{Pl-B-W}(k)\text{-Match} \). Let \( G = (V,E) \) be an arbitrary planar, bipartite graph, and let \( w : V \to W \) be a weight function with \( |W \setminus \{1\}| = k + 1 \). Let \( x \in W \) be any of the weights but 1, let \( X \subseteq V \) be the subset of vertices with weight \( x \), and let \( m = |X| \). Then we construct \( m + 1 \) graphs, \( G_1, G_2, \ldots, G_{m+1} \). In the graph \( G_i = (V_i, E_i) \), \( i \) new edges are attached to each vertex in \( X \). Then the weights of the vertices \( V \setminus X \) have the weights as they had in \( G \), the vertices in \( X \) and the further vertices incident to the new edges have weight 1. Clearly, for the weight function \( v_i : V_i \to W_i \) it holds that \( |W_i \setminus \{1\}| = k \).
It is easy to see that the matching polynomial of $G$ is $f(x)$, where

$$f(y) := \sum_{j=0}^{m} a_j y^j$$

is the matching polynomial of a graph $G_y$, which has the same topology than the graph $G$, and almost all the same weights except that each weight $x$ is replaced with the indeterminate $y$.

It is also easy to see that the matching polynomial of $G_i$ is

$$\sum_{j=0}^{m} a_j (i+1)^j.$$  

Indeed, the matching polynomial of $G_i$ restricted to those matchings $M$ in which $j$ vertices in $X$ are not incident to edges in $M \cap E$ is $a_j (i+1)^j$. Indeed, for each vertex in $X$ which are not incident to any edge in $X \cap E$, there are $i$ ways to be incident to an edge in $E_i \setminus E$, and there is a possible way not to be covered by the matching.

That is, if we could compute the matching polynomial for each $G_i$, then we could evaluate $f$ at $m+1$ different points. In this way, we could compute the coefficients $a_j$ with polynomial interpolation, and thus, we also could compute $MP(G) = f(x)$.

Applying this polynomial reduction 6 times we arrive to

$$\#\text{Pl-B-W}(6)-\text{Match} \leq \#\text{Pl-B-W}(0)-\text{Match} = \#\text{Pl-B-Match}.$$  

It is worth explaining why this chain of reductions is indeed a polynomial reduction when the computations involve complex numbers with irrational coefficients. We would like to compute the matching polynomial of $G_1$ which can be written in the form

$$MP(G_1) = \sum_{i_1=0}^{3k} \sum_{i_2=0}^{3k} \sum_{i_3=0}^{9k} \sum_{i_4=0}^{9k} \sum_{i_5=0}^{9k} \sum_{i_6=0}^{3k} \lambda_{i_1, i_2, i_3, i_4, i_5, i_6} a_1^{i_1} b_1^{i_2} c_1^{i_3} a_2^{i_4} b_2^{i_5} c_2^{i_6},$$

where $k$ is the number of crossings of $G$, and $a_1, b_1, c_1, a_2, b_2$ and $c_2$ are the six complex weights computed above. Observe that all coefficients $\lambda_{i_1, i_2, i_3, i_4, i_5, i_6}$ are integers, and these integers are computed during the polynomial interpolation. That is, the computation during the polynomial interpolations remains in the domain of integer numbers, furthermore these integers are surely smaller than the maximum of the possible number of matchings in a graph, that is, the number of digits necessary to express the numbers appearing during the polynomial interpolation are at most polynomial with the size of the problem. Then we have to compute

$$M(G) = \frac{1}{C^{3k}} MP(G_1) = \frac{1}{C^{5k}} \sum_{i_1=0}^{9k} \sum_{i_2=0}^{9k} \sum_{i_3=0}^{3k} \sum_{i_4=0}^{9k} \sum_{i_5=0}^{9k} \sum_{i_6=0}^{3k} \lambda_{i_1, i_2, i_3, i_4, i_5, i_6} a_1^{i_1} b_1^{i_2} c_1^{i_3} a_2^{i_4} b_2^{i_5} c_2^{i_6}.$$  

Observe that there are polynomial number of terms to add in equation (28) and the sum of these terms is
an integer when divided by $C^{3k}$. Furthermore, each monom as well as $C$ comes from a field extension with a fixed (constant) dimension, so the right hand side can be formally – and thus, explicitly – computed in polynomial time.

4 Counting the maximum matchings in planar, bipartite graphs is hard

Theorem 4.1. Counting the maximum matchings in planar, bipartite graphs is $\#P$-complete.

Proof. We prove it by proving the polynomial reduction $\#\text{Pl-B-Match} \leq \#\text{Pl-B-MaximumMatch}$. Let $G = (U, V, E)$ be an arbitrary bipartite, planar graph with $|U| \leq |V|$. We construct a graph $G' = (U, V', E')$, that we obtain from $G$ by attaching an edge to each vertex in $U$. Clearly, any maximum matching in $G'$ has size $|U|$. We claim that there is a bijection between the matchings of $G$ and the maximum matchings of $G'$. Indeed, from a maximum matching $M'$ of $G'$, we can remove the edges of $M'$ which are in $E' \setminus E$, thus we get a matching of $G$. For the other direction, any matching $M$ of $G$ can be extended to a maximum matching of $G'$ by covering the vertices of $U$ not in the matching $M$ by the edges in $E' \setminus E$. If these two transformations are denoted by $\varphi$ and $\varphi'$, it is easy to see that

$$\varphi' (\varphi(M)) = M,$$

and

$$\varphi (\varphi'(M')) = M'$$

for any matching $M$ of $G$ and any maximum matching $M'$ of $G'$. Therefore,

$$M(G) = MM(G'),$$

which completes the proof.

5 Discussion

In this paper, we showed that counting maximum matchings in bipartite, planar graphs is $\#P$-complete via proving that counting matchings in planar graphs is already $\#P$-complete when the problem is restricted to planar, bipartite graphs. For this, we had to construct a planar, bipartite crossing gadget which has 6 different complex weights. It remains an open problem if similar gadget exists with less number of and/or only integer weights.

A similar problem to the one discussed in this paper is the computational problem of counting maximal matchings in graphs. A maximal matching $M$ of $G$ is a matching such that no matching $M'$ of $G$ exists such that $M \subset M'$. To our best knowledge, the computational complexity of counting maximal matchings
is unknown even for simple graphs, though our results suggests the conjecture that it might be hard even restricting the problem to planar, bipartite graphs.

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