Free hyperplane arrangements associated to labeled rooted trees

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Abstract
To each labeled rooted tree is associated a hyperplane arrangement, which is free with exponents given by the depths of the vertices of this tree. The intersection lattices of these arrangements are described through posets of forests. These posets are used to define coalgebras, whose dual algebras are shown to have a simple presentation by generators and relations.

0 Introduction
This article is centered on labeled rooted trees. This kind of tree is very classical in combinatorics, since the enumerative results of Cayley [1]. More recently, it has surfaced, with more algebraic structure, in the study of pre-Lie algebras [2] and in relation with a Hopf algebra in renormalization [3].

Here, we deal with other algebraic and geometric aspects of rooted trees. It is not yet clear if these new aspects are related to the previous ones.

The starting point is the definition of a hyperplane arrangement for each individual rooted tree. One can consider a rooted tree as a poset, and from this viewpoint directly follow the equations of the arrangement. Apart from being quite simple, these arrangements have the remarkable property of being free, in the sense of Saito [4]. More precisely, the arrangement associated to a rooted tree is free with exponents given by the depths of the vertices of this tree.

After the first version of this article was completed, I learned from R. Stanley that this result is a consequence of the theory of graphical arrangements [5] §3, Th. 3.3 and supersolvable lattices [6] Prop. 2.8, Ex. 4.6. A graphical arrangement is known to be free if and only if the corresponding graph is chordal. As the comparability graphs for rooted trees are indeed chordal, one can recover in this way Theorem [7]. My proof of freeness avoids using these general theories and gives explicit information on logarithmic vector fields and differential forms.

Besides, it turns out that the intersection lattices of these hyperplane arrangements admit a neat combinatorial description in terms of forests of labeled rooted trees. This leads to the definition of a partial order on the set of forests labeled by a finite set, which contains intersection lattices as intervals.

Next, inspired by similarity with the study of binary leaf-labeled trees made in [8], the posets of forests are used to define coalgebras on forests. The dual algebras of these coalgebras are then shown to have a simple presentation by generators and relations.
Thanks to David Bessis for helpful discussions on the topology of the complement.

1 The category of labeled rooted trees

Let $I$ be a finite set. A tree on $I$ is a connected, simply connected graph on the vertex set $I$ endowed with a distinguished vertex called the root.

It will be convenient to use the following equivalent definition. A tree on $I$ is given by a partial order relation on the finite set $I$ with a unique minimal element (the root) and such that the interval between the root and any element of $I$ is a chain. The order relation corresponding to a tree $T$ is denoted by $\leq_T$.

The depth of a vertex $i$ is defined to be the number of edges in the maximal chain from the root to $i$ and is denoted by $\text{Depth}(i)$, see Fig. 1.

By convention, trees are drawn with their root at the bottom and edges are oriented towards the root, in the decreasing direction for the poset structure.

The valence of a vertex is the number of its incoming edges.

A linear tree is a tree whose vertices have valence at most 1.

The category $\text{Tree}$ is defined as follows. Its objects are the trees on $I$ for all finite sets $I$. Morphisms from a tree on $I$ to a tree on $J$ are the maps from $I$ to $J$ which are morphisms of posets.

2 The hyperplane arrangement of a rooted tree

Let $I$ be a finite set. Consider the vector space $\mathbb{C}^I$ with coordinates $x_i$. Let $T$ be a tree on $I$. The equations

$$x_i = x_j \quad \text{if} \quad i \leq_T j$$

(1)

define a central hyperplane arrangement in $\mathbb{C}^I$ denoted by $H_T$.

Note that the complement of the union of these hyperplanes is an algebraic variety defined over $\mathbb{Z}$ which depends functorially from the tree in the category $\text{Tree}$ of rooted labeled trees defined above.

Let $Q_T$ be the product of all the equations $x_i - x_j$ for $i \leq_T j$. It is called the defining form of the arrangement $H_T$.

Here is the first main result of this article.

Theorem 1 The arrangement $H_T$ is free. Its exponents are the depths of the vertices of $T$. 

Figure 1: A labeled rooted tree and the depths of its vertices.
Remark that the arrangement $H_T$ for $T$ a linear tree on $I$ is simply the braid arrangement given by the reflection hyperplanes of the symmetric group of $I$. Therefore the classical freeness of this braid arrangement is recovered as a special case of Theorem 1.

The proof of Theorem 1 uses the Saito criterion of freeness [10, Th. 1.8 (ii)] involving logarithmic vector fields and is the object of the next section.

### 3 Logarithmic vector fields

First one can associate a vector field $\theta_i$ to each vertex $i$ of $T$. Define the vector field $\theta_i$ on $\mathbb{C}^I$ by

$$\theta_i = \sum_{j \geq T_i} \prod_{k < T_i} (x_k - x_j) \partial_j.$$  \hspace{1cm} (2)

The degree of $\theta_i$ is one less than the depth of $i$ in $T$.

This definition can be restated as

$$\theta_i(x_j) = \begin{cases} \prod_{k < T_i}(x_k - x_j) & \text{if } i \leq_T j, \\ 0 & \text{otherwise}. \end{cases}$$ \hspace{1cm} (3)

Let $\text{Der}(H_T)$ be the space of logarithmic vector fields for the arrangement $H_T$, i.e. the space of polynomial vector fields $\theta$ on $\mathbb{C}^I$ such that $\alpha$ divides $\theta(\alpha)$ for all hyperplanes $\alpha$ of $H_T$.

**Proposition 1** The vector fields $\theta_i$ belong to $\text{Der}(H_T)$.

**Proof.** Let $j$ and $k$ be distinct vertices of $T$ with $j \leq_T k$.

Assume first that neither $i \leq_T j$ nor $i \leq_T k$. Then $\theta_i(x_j - x_k)$ is zero.

Assume then that $i \not\leq_T j$ but $i \leq_T k$. As $T$ is a tree, these conditions together with $j \leq_T k$ implies that $j <_T i$. It follows that $\theta_i(x_j - x_k) = -\prod_{\ell < T_i}(x_\ell - x_k)$ is divisible by $x_j - x_k$.

Assume now that $i \leq_T j$, then also $i \not\leq_T k$ by transitivity. Then $\theta_i(x_j - x_k) = \prod_{\ell < T_i}(x_\ell - x_j) - \prod_{\ell < T_i}(x_\ell - x_k)$ is divisible by $x_j - x_k$, because this expression vanishes when $x_j = x_k$.

Therefore $\theta_i(x_j - x_k)$ is always divisible by $x_j - x_k$. \hfill \blacksquare

One can now prove Theorem 1.

**Proof.** By Formula (3), the matrix $\Theta = [\theta_i(x_j)]$ is triangular for the poset structure on $I$ given by $\leq_T$. The determinant of the matrix $\Theta$ is therefore

$$\prod_{(i,j): k < T_i} (x_k - x_i),$$ \hspace{1cm} (4)

which is the defining form $Q_T$ of the arrangement $H_T$. Hence the Saito criterion applies and the arrangement $H_T$ is free. The exponents are by definition one more than the degrees of the vector fields $\theta_i$, which are one less than the depths of the vertices of $T$.

Freeness of the arrangement $H_T$ means that $\text{Der}(H_T)$ is a free module over the ring of polynomials on $\mathbb{C}^I$. The proof via the Saito criterion implies further that the set of vector fields $\{\theta_i\}_{i \in I}$ is a basis of $\text{Der}(H_T)$.
4 Logarithmic differential forms

Define a differential 1-form \( \omega_i \) for each vertex \( i \) of \( T \) by

\[
\omega_i = \sum_{j \leq T_i} \left( \prod_{\substack{k \leq T_i \atop k \neq j}} \frac{1}{x_k - x_j} \right) dx_j.
\] (5)

Recall the definition of the space of logarithmic differential forms for an hyperplane arrangement [10]. A differential form \( \omega \) with coefficients in the field of rational functions on \( \mathbb{C}^I \) is called a logarithmic differential form for \( H_T \) if \( Q_T \omega \) and \( Q_T d\omega \) are polynomial forms on \( \mathbb{C}^I \).

**Proposition 2** The forms \( \omega_i \) are logarithmic 1-forms for \( H_T \).

**Proof.** It is clear that \( Q_T \omega_i \) is polynomial. On the other hand, one has

\[
-d\omega_i = \sum_{j \leq T_i} \sum_{k \leq T_i \atop k \neq j} \left( \prod_{\substack{l \leq T_i \atop l \neq k}} \frac{1}{x_l - x_j} \right) \frac{1}{(x_k - x_j)^2} dx_k \wedge dx_j.
\] (6)

Consider now the coefficient of \( dx_k \wedge dx_j \) in \( d\omega_i \). It is sufficient to prove that it has only a simple pole at \( x_k = x_j \). This follows from the vanishing of

\[
\left( \prod_{\substack{l \leq T_i \atop \neq k}} \frac{1}{x_l - x_j} \right) - \left( \prod_{\substack{l \leq T_i \atop \neq j}} \frac{1}{x_l - x_k} \right)
\] (7)

when \( x_k = x_j \).

As the arrangement \( H_T \) is free, it is known that the space of logarithmic 1-forms is a free module over the the ring of polynomials on \( \mathbb{C}^I \). Furthermore this module is in duality with the module of logarithmic vector fields by restriction of the duality between vector fields and 1-forms [10].

More precisely, one has the following proposition in the case of \( H_T \).

**Proposition 3** The set of logarithmic 1-forms \( (\omega_i)_{i \in I} \) is the dual basis to the basis \( (\theta_i)_{i \in I} \).

**Proof.** One has

\[
\langle \omega_i, \theta_{i'} \rangle = \sum_{i' \leq T_j \leq T_i} \left( \prod_{\substack{l \leq T_i \atop \neq j}} \frac{1}{x_l - x_j} \right) \left( \prod_{j' \leq T_{i'}} x_{j'} - x_j \right) = \sum_{i' \leq T_j \leq T_i} \left( \prod_{\substack{k \leq T_i \atop k \neq j}} \frac{1}{x_k - x_j} \right).
\] (8)

The proposition now follows from Lemma [11] below.
Lemma 1 The sum
\[
\sum_{i' \leq T} \left( \prod_{k \not\geq j, k \neq i'} \frac{1}{x_k - x_j} \right) \tag{9}
\]
equals 1 if \(i' = i\) and vanishes otherwise.

Proof. This sum is clearly equal to 1 when \(i = i'\) and 0 when \(i' > T\). Assume now that \(i' < T\). Then the sum is homogeneous of negative degree. As poles are at most of order one, it is sufficient to prove that all residues vanish. The residue at \(x_{j_0} = x_{k_0}\) is given by the value of
\[
\left( \prod_{i' \leq T, k \not\geq j_0, k \neq k_0} \frac{1}{x_k - x_{j_0}} \right) - \left( \prod_{i' \leq T, k \not\geq j_0, k \neq k_0} \frac{1}{x_k - x_{k_0}} \right) \tag{10}
\]
at \(x_{j_0} = x_{k_0}\), which is zero.

5 Topology of the complement

Let \(M_T\) denote the complexified complement of the union of all hyperplanes of \(H_T\). Using the general theory of supersolvable arrangements \([11, 14]\) and the results of \([6]\) on graphical arrangements, the complement \(M_T\) can be precisely described.

One can map a tree \(T\) to the comparability graph \(\Gamma_T\) for the relation \(\leq_T\). In this way, a rooted tree is seen as a special case of a graph on \(I\). Then \(H_T\) is the graphical arrangement corresponding to \(\Gamma_T\). It is also easy to see that the graph \(\Gamma_T\) is chordal (see \([6]\) for the definition).

Recall from \([6, \S 3]\) the notions of simplicial vertex and vertex elimination ordering of a graph. For the graph \(\Gamma_T\), one has

Lemma 2 The leaves of \(T\) (maximal vertices for \(\leq_T\)) are simplicial vertices. Any total order extending the partial order \(\leq_T\) on \(I\) gives a vertex elimination ordering.

Such an ordering will be called a leaf-removal ordering.

As \(\Gamma_T\) is chordal, Theorem 3.3 of \([6]\) implies that \(H_T\) is a supersolvable arrangement and that each leaf-removal ordering gives a modular chain in \(L_T\). By results of Terao \([14]\), each modular chain in a supersolvable arrangement gives rise to a description of the complement as an iterated fibration of punctured \(C\).

Hence the complement \(M_T\) is an iterated fibration of the spaces \(C \setminus S_i\) for \(i \in I\), where \(S_i\) is a finite set of distinct points in \(C\) of cardinality the depth of \(i\) in \(T\). The order of the fibration tower is given by the chosen leaf-removal ordering of \(I\). From this, one can deduce the following proposition.

Proposition 4 The space \(M_T\) is a \(K(\pi, 1)\) space and its fundamental group is an iterated extension of free groups on \(\text{Depth}(i)\) generators for \(i \in I\), in the order given by any leaf-removal ordering of \(I\).
6 Cohomology of the complement

One can recall the results of Orlik and Solomon [7] on the cohomology of the complexified complement.

Proposition 5 The cohomology of the complex complement of \( H_T \) is generated by the differential forms \( d(\log(x_i - x_j)) \) for \( i \leq_T j \).

In the same way, the similar results of Gelfand and Varchenko [15] on the filtered algebra of locally constant integral functions on the real complement of a real hyperplane arrangement can be applied.

It would be interesting to find the relations for these algebras, i.e. to describe the dependent sets of hyperplanes. As a first step in this direction, one can give a generating set of relations between the linear forms \( \alpha_{i,j} = x_i - x_j \) for \( i \leq_T j \).

Lemma 3 Any linear relation between the linear forms \( \alpha_{i,j} \) for \( i \leq_T j \) is a linear combination of relations

\[
\alpha_{i,j} + \alpha_{j,k} - \alpha_{i,k} = 0,
\]

with \( i \leq_T j \leq_T k \).

Proof. Let \( \ell \) be the root of \( T \). Consider an arbitrary linear relation

\[
\sum_{i \leq_T j} \lambda_{i,j} \alpha_{i,j} = 0.
\]

This relation can be rewritten, using all relations

\[
\alpha_{i,j} = \alpha_{\ell,j} - \alpha_{\ell,i},
\]

as a relation of the following kind:

\[
\sum_{i \neq \ell} \mu_i \alpha_{\ell,i} = 0.
\]

Now the coefficients \( \mu_i \) vanish because the forms \( \alpha_{\ell,i} \) are obviously linearly independent. Note that among relations [11] only relations [13] involving the root are really used.

7 The lattice of a labeled rooted tree

Let \( L_T \) be the intersection lattice of the arrangement \( H_T \), i.e. the lattice of intersections of the hyperplanes of \( H_T \) for the reverse inclusion order.

7.1 Characteristic polynomial

The characteristic polynomial of \( H_T \) can be deduced from a theorem of Terao on free arrangements (Main Theorem of [13], see also [9, Th. 5.1]). Recall that the characteristic polynomial of the hyperplane arrangement \( H_T \) is

\[
\chi_T(y) = \sum_{a \in L_T} \mu(a) y^{\dim a},
\]

where \( \mu \) is the Möbius function of the lattice \( L_T \).
Proposition 6  The characteristic polynomial of the arrangement $H_T$ is
\[
\chi_T(y) = \prod_{i \in I} (y - \text{Depth}(i)),
\]  
where \text{Depth}(i) is the depth of $i$ in $T$.

The number of chambers (connected components) in the real complement is related by a theorem of Zaslavsky [16] to the value at $y = -1$ of the characteristic polynomial.

Proposition 7  The number of chambers in the real arrangement $H_T$ is given by
\[
\prod_{i \in I} (\text{Depth}(i) + 1).
\]  

7.2 Partial order \subseteq on the set of forests

Here is defined a partial order on the set of forests on $I$, denoted by $\subseteq$. This poset will be shown in the next paragraphs to contain (as intervals) all lattices $L_T$ for trees $T$ on $I$. A similar (but different) partial order on forests has appeared in [8] and Exercise 5.29 of [12].

Let $I$ be a finite set. A forest on $I$ is a simply connected graph on the vertex set $I$ where each connected component has a distinguished vertex called its root. Therefore a forest on $I$ is a partition of $I$ together with a rooted tree on each part. The partition of $I$ underlying a forest $F$ is denoted by $\pi(F)$. The set of forests on $I$ is denoted by $\text{for}(I)$. Vertices of a forest which are not roots are called nodes. Let $N(F)$ be the set of nodes of a forest $F$.

One can see a forest on $I$ as a structure of poset on $I$. The partial order is the ascendance relation, i.e. a vertex $i$ is lower than $j$ if they belong to the same tree in $F$ and $i$ is on the path from $j$ to the root of this tree. This order relation is denoted by $\leq_F$. This is an extension of the definition of a tree as a poset in §1.

Let us define the partial order $\subseteq$ on $\text{for}(I)$ as follows. Let $F$ and $F'$ be two forests on $I$. Then set $F' \subseteq F$ if
- the partition $\pi(F')$ is finer than $\pi(F)$,
- each tree of $F'$ is induced by $F$ as a poset.

This clearly defines a partial order relation.

Lemma 4  If $F' \subseteq F$, then $N(F')$ is contained in $N(F)$.

Proof. A root in $F$ is a minimal element for $\leq_F$ so it is also a minimal element for $\leq_{F'}$ i.e. a root in $F'$.

Lemma 5  If $F' \subseteq F$ and $N(F') = N(F)$, then $F = F'$.

Proof. If $F$ and $F'$ have the same nodes, they have the same roots, hence the same number of connected components. Therefore their partitions are the same, hence $F = F'$.

The poset $(\text{for}(I), \subseteq)$ has a unique minimal element, which is the forest made uniquely of roots and will be denoted by $\hat{0}$.
7.3 Intervals for \( \subseteq \)

Let \( T \) be a fixed tree on \( I \). This section gives a simple description of the interval between \( \hat{0} \) and \( T \) in the poset \( (\text{for}(I), \subseteq) \).

**Lemma 6** A forest \( F \subseteq T \) is uniquely determined by its partition \( \pi(F) \).

**Proof.** Given \( \pi(F) \), one can reconstruct \( F \) by inducing the partial order \( \leq_T \) on the parts of \( \pi(F) \).

**Lemma 7** A partition \( \pi \) can be written \( \pi(F) \) for a forest \( F \subseteq T \) if and only if each part of \( \pi \) has a unique minimal element for the partial order induced by \( \leq_T \).

**Proof.** If \( \pi = \pi(F) \), then each part of \( \pi \) is a tree, so has a unique minimal element. Conversely, if a part \( c \) of \( \pi \) has a unique minimal element, then in fact it is a tree, because a poset induced by a tree is either a tree or a forest.

**Lemma 8** Let \( F \subseteq T \) and \( F' \subseteq T \). Then \( F \subseteq F' \) if and only if \( \pi(F) \) is finer than \( \pi(F') \).

**Proof.** By definition of the partial order \( \subseteq \), if \( F \subseteq F' \) then \( \pi(F) \) is finer than \( \pi(F') \). Conversely, assume that \( \pi(F) \) is finer than \( \pi(F') \). As \( F \) and \( F' \) are recovered from their partitions by inducing \( \leq_T \), \( F \) is in fact induced from \( F' \), so \( F \subseteq F' \).

7.4 Lattices \( L_T \) and forests

In this section, one obtains a description of the lattice \( L_T \) in terms of forests.

**Proposition 8** The elements of \( L_T \) are in bijection with the forests \( F \) on \( I \) which satisfy \( F \subseteq T \).

**Proof.** Let \( a \) be an element of \( L_T \), i.e. the intersection of some hyperplanes \( \alpha_{i,j} \) with \( i \leq_T j \).

Define a relation \( \leq_a \) on \( I \) by setting \( i \leq_a j \) if \( \alpha_{i,j} \) vanishes on \( a \). Remark that \( i \leq_a j \) implies \( i \leq_T j \). That \( \leq_a \) is a partial order follows from relations \( [11] \). Let \( \pi_a \) be the partition of \( I \) given by the connected components of \( \leq_a \).

Now consider a part \( c \) of \( \pi_a \). Let us prove that \( c \) has a unique minimal element. Let \( i \) and \( j \) be two minimal elements of \( c \) and assume additionally that there exists \( k \) in \( c \) such that \( i \leq_a k \) and \( j \leq_a k \). One has either \( i \leq_T j \) or \( j \leq_T i \). Then relations \( [11] \) imply that \( i \leq_a j \) or \( j \leq_a i \), so that in fact \( i = j \). Now any two minimal elements \( i \) and \( j \) in \( c \) can be connected by an alternating chain

\[
i \leq_a k_0 \geq_a i_0 \leq_a k_1 \geq_a i_1 \leq_a \cdots \leq_a k_N \geq_a j,
\]

where one can assume without restriction that \( i_0, \ldots, i_{N-1} \) are minimal elements. Then a repeated application of the preceding argument implies that \( i = i_0 = i_1 = \cdots = i_{N-1} = j \). Therefore each part \( c \) of \( \pi_a \) has a unique minimal element for \( \leq_a \).

The partial order \( \leq_a \) on each part \( c \) is induced by \( \leq_T \). Indeed let \( i, j \) be two elements of \( c \). As said before, if \( i \leq_a j \), then \( i \leq_T j \). Conversely, assume that
\( i \leq_T j \). Let \( k \) be the minimal element of \( c \) for \( \leq_a \). Then \( k \leq_a i \) and \( k \leq_a j \), so relations \( \text{[10]} \) implies that \( i \leq_a j \).

Therefore to each element \( a \) of \( L_T \) is associated a forest \( F_a \subseteq T \), which is defined as a poset by \( i \leq_{F_a} j \) if and only if \( \alpha_{i,j} \) vanishes on \( a \).

Conversely, one can map each forest \( F \subseteq T \) to the intersection \( a_F \) of the hyperplanes \( \alpha_{i,j} \) for \( i \leq_F j \). This set of linear forms is closed with respect to the relations \( \text{[11]} \). Therefore the linear forms of \( H_T \) vanishing on \( a_F \) are exactly the \( \alpha_{i,j} \) for \( i \leq_F j \).

Furthermore, it is easy to see that the element of \( L_T \) associated in this way to the forest \( F_a \) is exactly \( a \).

This gives the sought-for bijection.

**Theorem 2** The interval between the minimal forest \( \hat{0} \) and a tree \( T \) is isomorphic to the lattice \( L_T \), i.e. the reverse inclusion order in \( L_T \) is mapped by the bijection to the relation \( \subseteq \).

**Proof.** Let \( a \) and \( b \) in \( L_T \) and let \( F_a \) and \( F_b \) be the corresponding forests.

That \( a \) is smaller than \( b \) in \( L_T \) means that each equation \( \alpha_{i,j} = 0 \) satisfied in \( a \) is also satisfied in \( b \). This imply that the partition \( \pi(F_a) \) is finer than the partition \( \pi(F_b) \).

Conversely, if the partition \( \pi(F_a) \) is finer than the partition \( \pi(F_b) \), then each tree of \( a \) is induced by a tree of \( b \) as a poset hence each relation satisfied in \( a \) is also satisfied in \( b \).

For an example of lattice \( L_T \), see Fig. 2.

This result shows that there is some similarity between the poset \( (\text{for}(I), \subseteq) \) on rooted vertex-labeled forests and the poset on forests of leaf-labeled binary trees introduced in [2] and further studied in [3], notably because the characteristic polynomials of intervals have a nice factorization in both cases. This will motivate the construction of a coalgebra in §8.

**Proposition 9** The dimension of a element \( a \in L_T \) is mapped by the bijection to the number of nodes of the associated forest \( F_a \).

**Proof.** It is known that \( L_T \) is a ranked lattice. All maximal chains have the same length and contain one element in each dimension. On the other hand, consider a maximal chain in the interval \([\hat{0}, T]\) in \( \text{for}(I) \). As the set of nodes must grow at each step of the chain by Lemma 5, it must grow by one element only. From this follows the lemma.

**Lemma 9** The interval between \( \hat{0} \) and a forest \( F \) is a lattice.

**Proof.** There exists a tree \( T \) such that \( F \subseteq T \). To build one such \( T \), choose one root \( r \) of \( F \) and glue the other roots of \( F \) to some vertices of the tree with root \( r \). Therefore the interval \([\hat{0}, F]\) is an interval in the lattice \([\hat{0}, T]\).

**Lemma 10** Let \( T \) be a tree, \( F_1 \) and \( F_2 \) be forests in \( L_T \) and \( F \) be the supremum of \( F_1 \) and \( F_2 \). Assume that \( N(F_1) \cap N(F_2) = \emptyset \). Then \( N(F) = N(F_1) \cup N(F_2) \).
Proof. Let $N_1 = N(F_1)$ and $N_2 = N(F_2)$ for short. Let $R$ be the complement of $N_1 \sqcup N_2$ in $I$. By Lemma 1, one has an inclusion $N_1 \sqcup N_2 \subseteq N(F)$.

As the lattice $L_T$ is graded by the number of nodes, it is enough to prove that there exists a forest $F$ greater than $F_1$ and $F_2$ with nodes $N_1 \sqcup N_2$.

Consider the partition $\pi$ of $I$ which is the sup of the partitions $\pi_1$ and $\pi_2$ associated to $F_1$ and $F_2$ respectively.

Let $c$ be an arbitrary part of $\pi$ endowed with the partial order $\leq$ induced from $\leq_T$. Then any minimal element of $c$ is in $R$. Assume on the contrary that it is in $N_1$ for example. Then it cannot be minimal in $c$, for it is already not minimal in the part of $\pi_1$ in which it is contained, because it is a node there. The same is true for $N_2$ by symmetry.

Now take $r \in R$. As $I$ is finite, the part $c$ of $\pi$ containing $r$ can be built by iterated closures of $\{r\}$ with respect to the partitions $\pi_1$ and $\pi_2$. Closing $\{r\}$ with respect to $\pi_1$ can only add elements of $N_1$. Then closing with respect to $\pi_2$ can only add elements of $N_2$. This goes on in the same way, adding in an alternating way elements of $N_1$ and $N_2$ until the full part $c$ of $\pi$ containing $r$ is reached. This implies that each part of $\pi$ contains an unique element of $R$, which is its minimum.

Therefore $\pi$ defines a forest $F$ in $L_T$, which is greater than $F_1$ and $F_2$ by construction and has $R$ as its set of roots. So $N(F) = N_1 \sqcup N_2$ and the proof is done.
7.5 Cardinality

Here is computed the cardinality polynomial of the graded lattice \( L_T \), which counts elements of \( L_T \) according to their rank. More precisely, a recursion is found for a refinement of the cardinality polynomial.

Define the refined generating polynomial

\[
C_T(y, z) = \sum_{F \in L_T} y^{\text{crk}(F)} z^{\text{Stump}(F)},
\]

where \( \text{Stump}(F) \) is one less than the cardinal of the part of \( \pi(F) \) containing the root of \( T \) and \( \text{crk}(F) \) is the corank of \( F \) (i.e. one less than the number of roots). The value at \( z = 1 \) of \( C(y, z) \) is the cardinality polynomial.

Let \( T \) be a tree on \( I \). Let \( o(T) \) be the tree on \( I \sqcup \{j\} \) obtained from \( T \) by grafting the root of \( T \) on a new root \( j \).

**Proposition 10** One has

\[
C_{o(T)}(y, z) = zC_T(y, z) + yC_T(y, 1 + z).
\]

**Proof.** Let \( i \in I \) be the root of \( T \). Let \( F \) be a forest in \( L_{o(T)} \).

Assume first that \( F \) does not have a tree with root \( i \). The set of such forests in \( L_{o(T)} \) is in bijection with the set of forests in \( L_T \) in the following way. Necessarily \( i \) belongs to a tree of \( F \) with root \( j \), and \( i \) is the only vertex related by an edge to \( j \). By removing \( j \), one can therefore define a forest \( F' \) with a tree of root \( i \). This forest \( F' \) is in \( L_T \). Conversely, by grafting back \( j \) under the root \( i \) in any forest \( F' \) in \( L_T \), one gets an element \( F \) of \( L_{o(T)} \). This gives the sought-for bijection.

Through this bijection, the number of roots is unchanged between \( F \) and \( F' \).

The number of nodes in the component of \( j \) in \( F \) is one more than the number of nodes in the component of \( i \) in \( F' \). This gives the term \( zC_T(y, z) \).

Assume now that \( F \) does have a tree with root \( i \). The set of such forests in \( L_{o(T)} \) is in bijection with the set of pairs \( (F', S) \) where \( F' \) is a forest in \( L_T \) and \( S \) is a subset of the set of nodes in the component of \( i \) in \( F' \). Consider the partition of \( I \) defined by gathering the parts containing \( i \) and \( j \) in the partition of \( I \sqcup \{j\} \) defined by \( F \), then removing \( j \). Taking the order induced by \( \leq_T \) on each part of this partition define a forest \( F' \) in \( L_T \). The forest \( F' \) differs from \( F \) only by the replacement of the trees \( t_i \) with root \( i \) and \( t_j \) with root \( j \) in \( F \) by a tree \( t'_i \) of root \( i \) in \( F' \). The set \( S \) is defined as the set of nodes in the component of \( j \) in \( F \). Conversely, one can recover \( F \) from the data of \( F' \) and \( S \). It is enough to recover the trees \( t_i \) and \( t_j \). The tree \( t_i \) of \( F \) is induced by the tree \( t'_i \) on the set of vertices not in \( S \). The tree \( t_j \) can be defined by replacing \( i \) by \( j \) in the tree induced on \( \{i\} \sqcup S \) by the tree \( t'_j \).

The number of roots in \( F \) is one more than the number of roots in \( F' \). The number of nodes in the component of \( j \) in \( F \) is the cardinal of \( S \). This gives the term \( yC_T(y, 1 + z) \).

On the other hand, the lattice \( L_T \) is isomorphic to the product of the lattices \( L_{T_k} \) where the \( T_k \) are the trees with root of valence one obtained by separately grafting back the different subtrees of the root. For example, Fig. 3 depicts this decomposition for the rooted tree of Fig. 3.
Proposition 11  One has

\[ C_T(y, z) = \prod_k C_{T_k}(y, z). \]  \hfill (21)

Proof. The bijection between \( L_T \) and \( \prod_k L_{T_k} \) can be described as follows. An element of \( L_T \) is a forest \( F \) with a unique tree containing the root of \( T \). One can decompose this tree into trees contained in \( T_k \) as in Fig. 3. Each other tree of \( F \) is contained in some unique \( T_k \). Collecting trees according to the \( T_k \) in which they are contained, one gets a collection of forests in \( L_{T_k} \), which is the image of \( F \) in \( \prod_k L_{T_k} \). The statement of the proposition follows easily. \hfill \blacksquare

Together, Prop. 10 and Prop. 11 give a recursive procedure for computing the refined cardinality polynomial.

8 Coalgebras on forests

In this section is defined a coalgebra, using the poset for \( (I) \).

Let \( F \) be a forest on \( I \) and \( N \) be a subset of \( N(F) \). Define

\[ \gamma(F, N) = \sum_{\substack{F' \subseteq F \\ N(F') = N}} F'. \]  \hfill (22)

As this sum is without multiplicity, \( \gamma(F, N) \) can also be seen as a set.

From now on, modules, algebras and coalgebras are in the category of \( \mathbb{Z} \)-graded abelian group with Koszul sign rules for the tensor product.

An orientation \( o \) of a forest \( F \) is a maximal exterior product of the set \( N(F) \cup \{ R_F \} \), where \( R_F \) is an auxiliary element. An oriented forest is a tensor product \( o \otimes F \), where \( o \) is an orientation of the forest \( F \).

Let \( F(I) \) be the free \( \mathbb{Z} \)-module on the set of oriented forests on \( I \) modulo the relations \( (-o) \otimes F = -(o \otimes F) \). This module is graded by \( \deg(o \otimes F) = \#N(F) \).

A map \( \Delta \) from \( F(I) \) to \( F(I) \otimes F(I) \) is defined as follows:

\[ \Delta(o \otimes F) = \sum_{N(F) = N_1 \cup N_2} (o_1 \otimes \gamma(F, N_1)) \otimes (o_2 \otimes \gamma(F, N_2)), \]  \hfill (23)

where the orientations satisfy \( o = o_1 \land r \land o_2 \) modulo \( R = R_1 \land r \land R_2 \).
Lemma 11 Let $F$ be a forest in $L_T$. Let $N_1$ be a subset of $N(F)$ and fix a partition $N_1 = N_1' \uplus N_1''$. Then (not taking care of orientations), one has

$$\sum_{F_1 \in \gamma(F, N_1)} \gamma(F_1, N_1') \otimes \gamma(F_1, N_1'') = \gamma(F, N_1') \otimes \gamma(F, N_1'').$$

(24)

Proof. It is sufficient to give a bijection between the set of triples $(F_1', F_1'', F_1)$ satisfying

$$F_1' \in \gamma(F, N_1'), \quad F_1'' \in \gamma(F, N_1''),$$

and the set of pairs $(F_1', F_1'')$ satisfying

$$F_1' \in \gamma(F, N_1'), \quad F_1'' \in \gamma(F, N_1'').$$

In one direction, a triple is mapped to a pair by forgetting $F_1$.

In the other direction, let $F_1'$ and $F_1''$ be a pair as above. By Lemma 11, the supremum $F_1$ of $F_1'$ and $F_1''$ (well-defined by Lemma 9) satisfies the conditions $F_1' \subseteq F_1$, $F_1'' \subseteq F_1$, and $N(F) = N_1$.

Proposition 12 Formula (23) endows $F(I)$ with the structure of a cocommutative coassociative counital coalgebra with coproduct $\Delta$.

Proof. Cocommutativity is clear from Formula (23). Coassociativity is deduced from the following formula for the double coproduct, which is a consequence of Lemma 11 (details are left to the reader)

$$\sum_{N_F = N_1 \uplus N_2 \uplus N_3} (o_1 \otimes \gamma(F, N_1)) \otimes (o_2 \otimes \gamma(F, N_2)) \otimes (o_3 \otimes \gamma(F, N_3)),$$

(30)

where $o = o_1 \land r \land o_2 \land s \land o_3$ modulo $R = R_1 \land r \land R_2 \land s \land R_3$. The counit is the projection to the degree-zero component, which is one-dimensional.

Lemma 12 Let $F$ be a forest on $I$ and $F_1 \otimes F_2$ be a term in the coproduct $\Delta(F)$. Then the set of nodes of $F_1$ and $F_2$ are disjoint in $I$.

Proof. Obvious from Formula (23).

Lemma 13 Let $F$ be a forest on $I$ and $F_1 \otimes F_2 \otimes \cdots \otimes F_k$ be a term in the iterated coproduct of $F$. Assume that one has $i \leq_{F_1} j$ for some $\ell$. Then one has $i \leq_F j$. In words, each ascendance relation in a term of a coproduct of $F$ is also satisfied in $F$.

Proof. By definition of the partial order $\subseteq$, each tree in each $F_1$ is induced by $F$ as a poset.
9 Algebras on forests

9.1 Partial order on forests

Consider the following partial order $\preceq$ on the set of forests on $I$. One sets $F \preceq F'$ if the identity of $I$ is a morphism of posets from $(I, \leq_F)$ to $(I, \leq_{F'})$. Note that in order to check if $F \preceq F'$, it is enough to check the relations $i \leq_{F'} j$ for all edges $i \leftarrow j$ of $F$.

Observe that this partial order is completely different from the partial order $\subseteq$ introduced in §7.2. For example, there are distinct forests with the same nodes which are comparable for $\preceq$ (compare to Lemma 5).

9.2 Generators and relations

Recall that the ambient category is that of $\mathbb{Z}$-graded abelian groups with Koszul sign rules. So there are appropriate signs to be inserted whenever two elements are exchanged.

In this category, consider the commutative associative unital algebra $M(I)$ defined by generators $\Omega_{i,j}$ of degree 1 for $i \neq j$ in $I$ and relations

$$\Omega_{i,k} \Omega_{j,k} = 0,$$

(31)

for $i, j, k$ pairwise different in $I$, and

$$\Omega_{i_0,i_1} \Omega_{i_1,i_2} \cdots \Omega_{i_n,i_0} = 0,$$

(32)

for $n \geq 1$ and pairwise different $i_0, \ldots, i_n$ in $I$.

**Proposition 13** The algebra $M(I)$ is spanned by monomials $m_F$ indexed by the set for $(I)$ of forests of rooted trees on $I$.

**Proof.** One can represent (up to sign) a monomial in the generators as an oriented graph on $I$ by drawing an oriented edge $i \leftarrow j$ for the generator $\Omega_{i,j}$.

The relation (31) says exactly that graphs containing two edges going out of the same vertex vanish. Therefore any cycle in a non-vanishing graph is oriented.

The relations (32) say exactly that graphs containing an oriented cycle vanish.

It follows that the algebra $M(I)$ is spanned by monomials indexed by oriented simple graphs on $I$ with no cycle and no divergence of arrows, i.e. rooted forests on $I$. For each forest $F$, choose (well defined up to sign) a monomial $m_F$.

In fact, this set of monomials $(m_F)_F$ form a basis of $M(I)$. This property can be seen directly or will follow from Theorem 3.

9.3 Isomorphism

Consider now the dual algebra $F^*(I)$ of the coalgebra $F(I)$, for the pairing $(,) : F^*(I) \otimes F(I) \rightarrow \mathbb{Z}$. Let $(F^*)_F$ be the basis of $F^*(I)$ dual to the basis $F$ of $F(I)$ for this pairing.

Define elements $F^*_{i,j}$ of degree 1 in $F^*(I)$ by

$$(F^*_{i,j}, F_{k,\ell}) = \delta_{i,k} \delta_{j,\ell},$$

(33)
where $F_{k,\ell}$ is the unique forest of degree 1 on $I$ with only one node, labeled $\ell$ and attached to a root labeled $k$. The orientation of $F_{k,\ell}$ is prescribed by $\ell \wedge R$.

**Theorem 3** The algebra $F^*(I)$ is isomorphic to the algebra $M(I)$.

First, one has to define a map from $M(I)$ to $F^*(I)$.

**Proposition 14** There is a map $\rho$ from $M(I)$ to $F^*(I)$ satisfying

$$\rho(\Omega_{i,j}) = F_{i,j}^*.$$  \hfill (34)

**Proof.** One has to show that the elements $F_{i,j}^*$ of $F^*(I)$ satisfy the relations \[\text{and } \text{ and } \text{ and } \text{ and } \text{ and } \](31) and \[\text{and } \text{ and } \text{ and } \text{ and } \text{ and } \text{ and } \](32).

To check the relation (31), it is sufficient to prove that the term $F_{i,k} \otimes F_{j,k}$ does not appear in the coproduct $\Delta(F)$ of any forest $F$ on $I$. This follows from Lemma 13 because $k$ is a node of both $F_{i,k}$ and $F_{j,k}$.

To check the relation (32), it is sufficient to prove that the term $F_{i_0,i_1} \otimes F_{i_1,i_2} \otimes \cdots \otimes F_{i_{n-1},i_n}$ does not appear in the iterated coproduct of any forest $F$ on $I$. Assume that it does appear in the iterated coproduct of $F$. Then by Lemma 13 one has

$$i_0 \leq_F i_1 \leq_F i_2 \leq_F \cdots \leq_F i_{n-1} \leq_F i_n,$$

which would mean that there is a cycle in $F$, which is absurd.

Therefore the map $\rho$ is well-defined.

The crucial part of the proof of Theorem 3 is the following Lemma.

**Lemma 14** Let $F$ be a forest on $I$. The image by $\rho$ of the monomial $m_F$ is the sum (with coefficients $\pm 1$) of $G^*$ over the set of all forests $G$ which are $\geq_F$ than $F$ and have the same nodes as $F$.

**Proof.** Write $m_F$ as a product $\Omega_{i_{0,j_0}} \cdots \Omega_{i_{n,j_n}}$ over the set of edges of $F$, in some order. Then $\rho(m_F)$ is $F_{i_0,j_0}^* \cdots F_{i_{n,j_n}}^*$.

Let $G$ be a forest on $I$. The coefficient of $G^*$ in $F_{i_0,j_0}^* \cdots F_{i_{n,j_n}}^*$ is (up to sign) the coefficient of $F_{i_0,j_0} \otimes \cdots \otimes F_{i_{n,j_n}}$ in the iterated coproduct of $G$.

Assume that the coefficient of $G^*$ in $\rho(m_F)$ is non-zero. Then the coefficient of $F_{i_0,j_0} \otimes \cdots \otimes F_{i_{n,j_n}}$ in the iterated coproduct of $G$ is non-zero. Lemma 13 implies, for each edge $i \leftrightarrow j$ of $F$, that $i \leq_G j$. Hence by definition of $\leq$, one has $F \leq_G G$.

By homogeneity of the product, one also has that $G$ and $F$ have the same number of nodes and the same number of roots. As $F \leq_G G$, each root of $F$ is also a root of $G$, so $F$ and $G$ have the same roots and nodes.

Now consider any forest $G$ with the same nodes and roots as $F$ and satisfying $F \leq_G G$. For such a $G$, the tensor $F_{i_0,j_0} \otimes \cdots \otimes F_{i_{n,j_n}}$ does appear with coefficient $\pm 1$ in the iterated coproduct of $G$, in the term of the sum of $N(G) = \{j_0, j_1, \ldots, j_n\}$ corresponding to the partition of $N(G)$ into singletons. Indeed, for any $k$, $F \leq_G G$ implies $i_k \leq_G j_k$, which in turn implies that $F_{i_k,j_k}$ belongs to the set $\gamma(G, \{j_k\})$.

The proof of Theorem 3 can now be completed.

**Proof.** By Lemma 13, the image by $\rho$ of the set $(m_F)_F$ which spans $M(I)$ is a basis of $F^*(I)$. As the rank of $F^*(I)$ is the cardinal of $\text{for}(I)$, one deduces that $(m_F)_F$ is in fact a basis of $M(I)$. So $\rho$ is an isomorphism.
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