A Bayesian Nonparametric IRT Model

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Abstract: This paper introduces a flexible Bayesian nonparametric Item Response Theory (IRT) model, which applies to dichotomous or polytomous item responses, and which can apply to either unidimensional or multidimensional scaling. This is an infinite-mixture IRT model, with person ability and item difficulty parameters, and with a random intercept parameter that is assigned a mixing distribution, with mixing weights a probit function of other person and item parameters. As a result of its flexibility, the Bayesian nonparametric IRT model can provide outlier-robust estimation of the person ability parameters and the item difficulty parameters in the posterior distribution. The estimation of the posterior distribution of the model is undertaken by standard Markov chain Monte Carlo (MCMC) methods based on slice sampling. This mixture IRT model is illustrated through the analysis of real data obtained from a teacher preparation questionnaire, consisting of polytomous items, and consisting of other covariates that describe the examinees (teachers). For these data, the model obtains zero outliers and an R-squared of one. The paper concludes with a short discussion of how to apply the IRT model for the analysis of item response data, using menu-driven software that was developed by the author.

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1 Introduction

Given a set of data, consisting of person’s individual responses to items of a test, an item response theory (IRT) model aims to infer each person’s ability on the test, and to infer the test item parameters. In typical applications of an IRT model, each item response is categorized into one of two or more categories. For example, each item response may be scored as either correct (1) or incorrect (0). From this perspective, a categorical regression model, which includes person ability parameters and item difficulty parameters, provides an interpretable approach to inferring from item response data. One basic example is the Rasch (1960) model. This model can be characterized as a logistic regression model, having the dichotomous item score as the dependent variable. The predictors (covariates) of this model include $N$ person indicator (0,1) variables, corresponding to regression coefficients that define the person ability parameters; and include $I$ item indicator (0,-1) variables, corresponding to coefficients that define the item difficulty parameters.

In many item response data sets, there are observable and unobservable covariates that influence the item responses, in addition to the person and item factors. If the additional covariates are not fully accounted for in the given IRT model, then the estimates of person ability and item difficulty parameters can become noticeably biased. Such biases can be (at least) partially-alleviated by including the other, observable covariates into the IRT (regression) model, as control variables. However, for most data collection protocols, it is not possible to collect data on all the covariates that help determine the item responses (e.g., due to time, financial, or ethical constraints). Then, the unobserved covariates, which influence the item responses, can bias the estimates of the ability and item parameters in an
IRT model that does not account for these covariates.

A flexible mixture IRT model can provide robust estimates of person ability parameters and item difficulty parameters, by accounting for any additional unobserved latent covariates that influence the item responses. Modeling flexibility can be maximized through the use of a Bayesian nonparametric (BNP) modeling approach.

In this chapter we present a BNP approach to infinite-mixture IRT modeling, based on the general BNP regression model introduced by Karabatsos and Walker (2012). We then illustrate this model, called the BNP-IRT model, through the analysis of real item response data. The analysis was conducted using a menu-driven (point-and-click) software, developed by the author (Karabatsos 2014a, 2014b).

In the next section, we give a brief overview of the concepts of mixture IRT modeling, and BNP infinite-mixture modeling. Then in Section 3, we introduce our basic, BNP-IRT model. This is a regression model consisting of person ability and item difficulty parameters, constructed via the appropriate specification of person and item indicator predictor variables, as mentioned above. While the basic model assumes dichotomous item scores and unidimensional person ability, our model can be easily extended to handle polytomous responses (with item response categories not necessarily ordered), extra person-level and/or item-level covariates, and/or multidimensional person ability parameters. In Section 4, we describe the Markov chain Monte Carlo (MCMC) methods that can be used to estimate the posterior distribution of the model parameters. (This is a highly technical section which can be skipped when reading this chapter). In Section 5, we describe methods for evaluating the fit of our BNP-IRT model. Section 6 provides an empirical illustration of the BNP-IRT model through the analysis of polytomous response data. The data were obtained from an
administration of a questionnaire that was designed to measure teacher preparation. Section 7 ends with a brief overview of how to use the menu-driven software to perform data analysis using the BNP-IRT model. That section also includes a brief discussion of how to extend the BNP-IRT model for cognitive IRT.

The remained of this chapter makes use of the following notational conventions. Let \( \mathbf{U} = (U_1, \ldots, U_i, \ldots, U_I)^\top \) denote a random vector for the scores on a test with \( I \) items. A realized value of the item response vector is denoted by \( \mathbf{u} = (u_1, \ldots, u_i, \ldots, u_I)^\top \). We assume that each item \( i = 1, \ldots, I \) has \( m_i + 1 \) possible discrete-valued scores, indexed by \( u = 0, 1, \ldots, m_i \).

We use lower cases to denote a probability mass function (pmf) of a value \( u \) discrete random variable (or vector, \( \mathbf{u} \)) or a probability density function (pdf) of a value \( u \) of a continuous random variable (or \( \mathbf{u} \)), such as \( f(u) \) or \( f(\mathbf{u}) \), respectively. The given pmf (or pdf) \( f(u) \) corresponds to a cumulative distribution function (cdf), denoted by upper case \( F(u) \), which gives the probability that the random variable \( U \) does not exceed \( u \). \( F(u) \) is sometimes more simply referred to as the distribution function. Thus, for example, \( N(\mu, \sigma^2) \), \( U(0, b) \), \( \text{IG}(a, b) \) and \( \text{Be}(a, b) \) (or cdfs \( N(\cdot | \mu, \sigma^2) \), \( U(\cdot | 0, b) \), \( \text{IG}(\cdot | a, b) \) and \( \text{Be}(\cdot | a, b) \), respectively), denote the univariate normal, uniform, inverse-gamma, and beta distribution functions, respectively. They correspond to pdfs \( n(\cdot | \mu, \sigma^2) \), \( u(\cdot | 0, b) \), \( \text{ig}(\cdot | a, b) \), \( \text{be}(\cdot | a, b) \), with mean and variance parameters \((\mu, \sigma^2)\), minimum and maximum parameters \((0, b)\), shape and rate parameters \((a, b)\), and shape parameters \((a, b)\), respectively. Also, if \( \mathbf{\beta} \) is a realized value of a \( K \)-dimensional random vector, then \( N(\mathbf{\beta} | \mathbf{0}, \mathbf{V}) \) denotes the cdf of the multivariate \((K\text{-variate})\) normal distribution with mean vector of zeros \( \mathbf{0} \) and \( K \times K \) variance-covariance matrix \( \mathbf{V} \), distribution function \( n(\mathbf{0}, \mathbf{V}) \), and corresponding to pdf \( n(\mathbf{\beta} | \mathbf{0}, \mathbf{V}) \). The pmf or
pdf of $u$ given values of one or more variables $x$ is written as $f(u \mid x)$ (with corresponding cdf $F(u \mid x)$); given a vector of parameter values $\zeta$ is written as $f(u \mid \zeta)$ (with corresponding cdf $F(u \mid \zeta)$), and conditionally on variables and given parameters is written as $f(u \mid x; \zeta)$ (with corresponding cdf $F(u \mid x; \zeta)$). Also, $\sim$ means ”distributed as”, $\sim_{ind}$ means ”independently distributed,” and $\sim_{iid}$ means ”independently and identically distributed.” For example, $U \sim F, U \sim_{iid} F(u), U \sim F(u \mid x; \zeta), U \sim_{iid} F(\zeta), \beta \sim N(0, \Sigma), \sigma^2 \sim IG(a, b)$. The preceding notation may replace $U$ by $U$, replace $F$ by $f$, replace $N$ by $n$, and/or replace $IG$ by $ig$.

2 Mixture IRT and Bayesian Nonparametrics

For any given vector of item response data $u = (u_1, \ldots, u_i, \ldots, u_I)\top$, a discrete-mixture IRT model admits the general form

$$f_{G_x}(u \mid x) = \int f(u \mid x; \beta, \Psi(x))dG_x(\Psi) = \sum_{j=1}^{J} f(u \mid x; \beta, \Psi_j(x))\omega_j(x).$$

conditionally on any given value of a vector of any covariates $x$. In this expression, $f(u \mid x; \beta, \Psi(x))$ is the kernel of the mixture, and $G_x$ is a mixture distribution that may (or may not) depend on the same covariates.

Also, as show in (1), this pmf is based on a mixture of $J$ pmfs $f(u \mid x; \beta, \Psi_j(x)), j = 1, \ldots, J$. Here, $\beta$ is a vector of (any available) fixed parameters that are not subject to the mixture, the $\Psi_j(x), j = 1, \ldots, J$, are random parameters that are subject to the mixture that may be covariate dependent, and $J$ is the number of mixture components. In addition,
\( \omega_j(x), j = 1, \ldots, J, \) are mixture weights that sum to one for every given covariate value \( x \in X \). The mixture model (1) is called a discrete (continuous) mixture model if \( G_x \) is discrete (continuous); it is called a finite (infinite) mixture model if \( J \) is finite (infinite).

A simple example is given by the finite mixture Rasch model for dichotomous item scores (Rost, 1990, 1991; von Davier & Rost, vol. 1, chap. 23), which assumes that

\[
f(u | x; \beta, \Psi_j(x)) = \prod_{i=1}^{J} \frac{\exp(\theta_j - \beta_{ij})^{u_i}}{1 + \exp(\theta_j - \beta_{ij})}, \tag{2}
\]

with a finite number of \( J \) components and mixture weights that are not covariate-dependent (i.e., \( \omega_j(x) = \omega_j, j = 1, \ldots, J < \infty \)). The ordinary Rasch (1960) model for dichotomous item scores is the special case of the model defined by (1) and (2) for \( J = 1 \).

An infinite-mixture model is given by (1) for \( J = \infty \). A general BNP infinite-mixture IRT model assumes that the mixture distribution has the general form

\[
G_x(\cdot) = \sum_{j=1}^{\infty} \omega_j(x) \delta_{\Psi_j(x)}(\cdot), \tag{3}
\]

where \( \delta_{\Psi}(\cdot) \) denotes a degenerate distribution with support \( \Psi \). This Bayesian model is completed by the specification of a prior distribution on \( \{\Psi_j(x)\}_{j=1,2,\ldots}, \{\omega_j(x)\}_{j=1,2,\ldots}, \) and \( \beta \) with large supports.

A common example is a Dirichlet process mixed IRT model, which assumes that the mixing distribution is not covariate-dependent (i.e., \( G(x) = G(\cdot) \)), along with a random mixing distribution \( G(\cdot) \) constructed as

\[
G(\cdot) = \sum_{j=1}^{\infty} \omega_j \delta_{\Psi_j(\cdot)} \text{where } \omega_j = v_j \prod_{k=1}^{j-1} (1 - v_k)
\]

for random draws \( v_j \sim iid \text{Be}(1, \alpha) \) and \( \Psi_j \sim iid G_0 \), for \( j = 1, 2, \ldots \). Here, \( G \) is a Dirichlet
process (DP), denoted $G \sim \text{DP}(\alpha, G_0)$, with baseline parameter $G_0$ and precision parameter $\alpha$ (Sethuraman, 1994). The DP$(\alpha, G_0)$ has mean (expectation) $\mathbb{E}[G(\cdot)] = G_0(\cdot)$ and variance $\mathbb{V}[G(\cdot)] = G_0(\cdot)\{1 - G_0(\cdot)\}/(\alpha + 1)$ (Ferguson, 1973).

An important generalization of the DP prior includes the Pitman-Yor (Poisson-Dirichlet) prior (Ishwaran & James, 2001), which assumes that $\nu_j \sim \text{iid Be}(\alpha_{1j}, \alpha_{2j})$, for $j = 1, 2, \ldots$, for some $\alpha_{1j} = 1 - \alpha_1$ and $\alpha_{2j} = \alpha_2 + j\alpha_1$ with $0 \leq \alpha_1 < 1$ and $\alpha_2 > -\alpha_1$. The special case defined by $\alpha_1 = 0$ and $\alpha_2 = \alpha$ results in the DP$(\alpha G_0)$.

Another important generalization of the DP is given by the Dependent Dirichlet process (DDP) (MacEachern, 1999, 2000, 2001), which provides a model for the covariate-dependent random distribution, denoted $G_x$. The DDP model assumes that $G_x \sim \text{DP}(\alpha_x, G_{0x})$, marginally for each $x$. Specifically, the DDP defines a covariate-dependent random distribution $G_x$ of the form given in equation (3), and incorporates this dependence either through covariate-dependent atoms $\Psi_j(x)$, a covariate-dependent baseline $G_{0x}$, and/or covariate-dependent stick-breaking weights of the form $\omega_j(x) = \nu_j(x) \prod_{k=1}^{j-1}(1 - \nu_k(x))$, for $j = 1, 2, \ldots$. For example, the ANOVA-linear DDP (De Iorio et al. 2004), denoted $G_x \sim \text{ANOVA-DDP}(\alpha, G_0, x)$, constructs a dependent random distribution $G_x(\cdot) = \sum_{j=1}^{\infty} \omega_j \delta_{x^\top \beta_j}(\cdot)$, via covariate-dependent atoms $x^\top \beta_j$, along with $\beta \sim G$ and $G \sim \text{DP}(\alpha, G_0(\beta))$.

Many examples of DP-mixture and DDP mixture IRT models can be found in the literature (Qin, 1998; Duncan & MacEachern, 2008; Miyazaki & Hoshino, 2009; Farina et al. 2009; San Martin et al. 2011; San Martin, et al., 2011; Karabatsos & Walker, 2012).
3 Presentation of the Model

The BNP-IRT model is a special case of the Bayesian nonparametric regression (infinite-mixture) model introduced by Karabatsos and Walker (2012). These authors demonstrated that the model tended to have better predictive performance relative to DP-mixed and DDP mixed regression models. As will be shown, the BNP-IRT model is suitable for dichotomous or polytomous item responses.

First, we present the basic BNP-IRT model for dichotomous item responses. Let $D = \{(u_{pi}, x_{pi})_{i=1}^{I}\}_{p=1}^{P}$ denote a set of item-response data, including dichotomous responses $u_{pi} \in \{0, 1\}$. Also, $x_{pi}$ denotes a covariate vector that describes person $p = 1, ..., P$ and item $i = 1, \ldots, I$.

The basic BNP-IRT model is defined as

$$f(D \mid X; \zeta) = \prod_{p=1}^{P} \prod_{i=1}^{I} f(u_{pi} \mid x_{pi}; \zeta)$$

(4a)

$$f(u_{pi} \mid x_{pi}; \zeta) = P(U_{pi} = 1 \mid x_{pi}; \zeta)^{u_{pi}}[1 - P(U_{pi} = 1 \mid x_{pi}; \zeta)]^{1-u_{pi}}$$

(4b)

$$\Pr(U = 1 \mid x; \zeta) = 1 - F^*(0 \mid x; \zeta) = \int_{0}^{\infty} f(u_{pi}^* \mid x_{pi}; \zeta) du^*$$

(4c)

$$= \int_{0}^{\infty} \sum_{j=-\infty}^{\infty} n(u^* \mid \mu_j + x_{pi}^{\top} \beta, \sigma^2) \omega_j(x_{pi} \mid \beta_\omega, \sigma_\omega) du^*$$

(4d)

$$\omega_j(x \mid \beta_\omega, \sigma_\omega) = \Phi \left( \frac{j - x^{\top} \beta_\omega}{\sigma_\omega} \right) - \Phi \left( \frac{j - 1 - x^{\top} \beta_\omega}{\sigma_\omega} \right)$$

(4e)

$$(\mu_j, \sigma^2_\mu) \sim N(\mu_j \mid 0, \sigma^2_\mu U(\sigma_\mu \mid 0, b_{\sigma_\mu})$$

(4f)

$$(\beta, \beta_\omega) \sim N(\beta \mid 0, \sigma^2 v_{\text{diag}}(\infty, J_{NI}) N(\beta_\omega \mid 0, \sigma^2_\omega v_\omega I_{NI+1})$$

(4g)

$$(\sigma^2, \sigma^2_\omega) \sim IG(\sigma^2 \mid a_0/2, a_0/2) IG(\sigma^2_\omega \mid a_\omega/2, a_\omega/2).$$

(4h)
Under the model, the data likelihood is given by equations (4a)-(4e) given parameters $\zeta = (\mu, \sigma_\mu, \beta, \beta_\omega, \sigma^2, \sigma_\omega)$ with $\mu = (\mu_j)_{j=-\infty}^\infty$. By default, the model assumes that $x_{pi}$ is a binary indicator vector with $NI + 1$ rows, having constant (1) in the first entry, a “1” in entry $p + 1$ to indicate person $p$, and “−1” in entry $i + (p + 1)$ to indicate item $i$. Specifically, each vector $x_{pi}$ is defined by

$$x_{pi} = (1, 1(p = 1), \ldots, 1(p = N), -1(i = 1), \ldots, -1(i = I))^T,$$

where $1(\cdot)$ denotes the indicator (0,1) function. Then, in terms of the coefficient vector $\beta = (\beta_0, \beta_1, \ldots, \beta_{PI})$, each coefficient $\beta_{p+1} = \theta_p$ represents the ability of person $p = 1, \ldots, P$. Likewise, each coefficient $\beta_{i+p+1}$ represents the difficulty of item $i = 1, \ldots, I$. The covariate-dependent mixture weights $\omega_j(x)$ in (4e) are specified by a cumulative ordered probits regression, based on the choice of a standard normal cdf for $\Phi(\cdot)$ with latent mean $x^T \beta_\omega$ and variance $\sigma^2_\omega$, for the ”ordinal categories” $j = 0, \pm 1, \pm 2, \ldots$, where coefficient vector $\beta_\omega$ contains additional person parameters and item parameters.

As shown in (4f)-(4h), the Bayesian model parameters $\zeta$ have joint prior density

$$\pi(\zeta) = \prod_{j=-\infty}^{\infty} \text{n}(\mu_j \mid 0, \sigma^2_\mu)\text{n}(\sigma_\mu \mid 0, b_\sigma)\text{n}(\beta \mid 0, \sigma^2 \text{diag}(\infty, vJ^T_{NI}))$$

$$\times \text{n}(\beta_\omega \mid 0, \sigma^2_\omega v_\omega I_{NI+1}) \text{ig}(\sigma^2 \mid a_0/2, a_0/2) \text{ig}(\sigma^2_\omega \mid a_\omega/2, a_\omega/2),$$

where $J^T_{NI}$ denotes the vector of $NI$ ones, and $I_{NI+1}$ is the identity matrix of dimension $NI + 1$. As shown in (5), the full specification of their prior density relies on the choice of the parameters $(b_\sigma, v, a_0, v_\omega, a_\omega)$. In Section 6, where we illustrate the BNP-IRT model.
through the analysis of a real item response data set, we suggest some useful default choices for these prior parameters.

As shown by the model equations in (4a)-(4d), the item response function $\Pr(U = 1 \mid x; \zeta)$ is modeled by a covariate($x$)-dependent location mixture of normal distributions for the latent variables $u_{pi}^*$. The random locations $\mu_j$ of this mixture corresponds to mixture weights $\omega_j(x)$, $j = 0, \pm 1, \pm 2, \ldots$. Conditionally on a covariate vector, $x_{pi}$ and model parameters, the latent mean and variance of the mixture can be written as:

$$\mathbb{E}[U_{pi}^* \mid x_{pi}; \beta, \beta_\omega, \sigma^2, \sigma_\omega] = \mu_{pi}^* = \sum_{j=-\infty}^{\infty} (\mu_j + x_{pi}^T \beta) \omega_j(x_{pi}; \beta_\omega, \sigma_\omega),$$

$$\mathbb{V}[U_{pi}^* \mid x_{pi}; \beta, \beta_\omega, \sigma^2, \sigma_\omega] = \sum_{j=-\infty}^{\infty} \{[(\mu_j + x_{pi}^T \beta) - \mu_{pi}^*]^2 + \sigma_\omega^2\} \omega_j(x_{pi}; \beta_\omega, \sigma_\omega),$$

respectively (Marron & Wand, 1992).

The BNP-IRT model can be viewed as an extension of the DP-mixed binary logistic generalized linear model (Mukhopadhyay & Gelfand, 1997). In terms of the responses $u$, the extension can be written as

$$f(u \mid x) = \sum_{j=1}^{\infty} \frac{\exp(\mu_j + x^T \beta)^u}{1 + \exp(\mu_j + x^T \beta)} \omega_j,$$

$$\omega_j = v_j \prod_{k=1}^{j-1} (1 - v_k)$$

$$v_j \sim \text{Be}(1, \alpha), \quad j = 1, 2, \ldots$$

$$\mu_j \sim \text{N}(0, \sigma_\mu^2), \quad j = 1, 2, \ldots$$

$$\beta \sim \text{N}(0, \Sigma_\beta).$$
This model thus defines a mixture of logistic cdfs for the inverse link function, with weights \( \omega_j \) that are not covariate-dependent. In contrast, as shown in (4c)–(4d), the BNP-IRT model in (4) is based on a mixture of normal cdfs for the inverse link function. The BNP-IRT model is more flexible than the DP model, because the former uses covariate-dependent mixture weights, as shown in (4e).

In other words, if \( \mu_j = 0 \) for all \( j \), then the BNP-IRT model reduces to the Rasch IRT model with "normal-ogive" response functions; all items are assumed to have common slope (discrimination) parameter that is proportional to \( 1/\sigma \). Nonzero values of \( \mu_j \), along with the covariate-dependent mixture weights \( \omega_j(x; \beta_\omega, \sigma_\omega) \), for \( j = 0, \pm 1, \pm 2, \ldots \), allows the BNP-IRT model to shift the location of each response function across persons and items. Value of \( \mu_j > 0 \) (\( \mu_j < 0 \)) shifts the response function to the left (right). The BNP-IRT model allows for this shifting in a flexible manner, accounting for any outlying responses (relative to a normal-ogive Rasch model). This feature enables inferences of person and item parameters from the BNP-IRT that model are robust against such outliers.

According to Bayes’ theorem, a set of data \( D \) updates of the prior probability density \( \pi(\zeta) \) in (5) leads to posterior probability density

\[
\pi(\zeta \mid D) = \frac{f(D \mid X; \zeta)\pi(\zeta)}{\int f(D \mid X; \zeta)\pi(\zeta)\,d\zeta}.
\]

Also, conditionally on \( (x_{pi}, D) \), the posterior predictive pmf and the posterior expectation
(E) and variance (V) of the item response $U_{pi}$ are given by

$$f(u_{pi} | x_{pi}, D) = \int f(u_{pi} | x_{pi}, \zeta) \pi(\zeta | D) d\zeta,$$

(6)

$$E[U_{pi} | x_{pi}, D] = f(U_{pi} = 1 | x_{pi}, D) = f(1 | x_{pi}, D),$$

(7)

$$V[U_{pi} | x_{pi}, D] = f(1 | x_{pi}, D)[1 - f(1 | x_{pi}, D)],$$

(8)

respectively.

It is straightforward to extend the BNP-IRT regression model to other types of response data by making appropriate choices of covariate vector $x$ (corresponding to coefficients $\beta, \beta_{\omega}$). Such extensions are described as follows:

1. Suppose that for each item $i = 1, \ldots, I$ the responses are each scored in more than two categories, say $m_i + 1$ nominal or ordinal categories denoted as $u' = 0, 1, \ldots, m_i$, with $u' = 0$ the reference category. Then the model can be extended to handle such polytomous item responses using the Begg and Gray (1984) method. Specifically, the model would assume the response to be defined by $u_{pi} = 1(u'_{pi} > 0)$ each covariate vector $x_{pi}$ to be defined by a binary indicator vector:

$$x_{pi} = (1, 1(p = 1), \ldots, 1(p = N), 1(i = 1)1(u'_{pi} = 1), \ldots, 1(i = I)1(u'_{pi} = 1), \ldots, 1(i = 1)1(u'_{pi} = m_i), \ldots, 1(i = I)1(u'_{pi} = m_i))^T.$$

Then in terms of coefficient vector $\beta = (\beta_0, \beta_1, \ldots, \beta_{1+p+m*1})$, coefficient $\beta_{1+p} = \theta_p$, $p = 1, \ldots, P$, represents the latent ability of person $p$ and the coefficient $\beta_{1+p+(u-1)I+i}$ represents the latent difficulty of item $i = 1, \ldots, I$ and category $u = 1, \ldots, m^*$, where
\[ m^* = \max_i m_i. \]

2. If the data has additional covariates \((x_1, \ldots, x_q)\) which describe either the persons (e.g., socioeconomic status), test items (e.g., item type), or type of response (e.g., response time), associated with each person \(p\) and item \(i\), then these covariates can be added as the last \(q\) elements to each of the covariate vectors \(x_{pi}\), such that \(x_{pi} = (\ldots, x_{1i}, \ldots, x_{qi})^\top, p = 1, \ldots, P\) and \(i = 1, \ldots, I\). Then, specific elements of coefficient vector \(\beta\), namely the elements \(\beta_k, k = \dim(\beta) - q + 1, \ldots, \dim(\beta)\), would represent the associations of the \(q\) covariates with the responses.

3. Similarly, suppose that given test consists of measuring one or more of \(D \leq I\) measurement dimensions. Then we can extend the model to represent such multidimensional items, by including \(D\) binary \((0,1)\) covariates into the covariate vectors \(x_{pi}\), \(p = 1, \ldots, P\) and \(i = 1, \ldots, I\), such that the first set of elements of \(x_{pi}\) defined by

\[
x_{pi} = (1, 1(p = 1)1(d_i = 1), \ldots, 1(p = N)1(d_i = 1), \ldots, 1(p = 1)1(d_i = D), \ldots, 1(p = N)1(d_i = D), \ldots)^\top,
\]

where \(d_i \in \{1, \ldots, D\}\), denotes the measurement dimension of item \(i\). Then specific elements of the coefficient vector \(\beta\), namely the elements \(\beta_k\), for \(k = 2, \ldots, ND + 1\), indicate each person’s ability on dimension \(d = 1, \ldots, D\).
4 Parameter Estimation

By using latent-variable Gibbs sampling methods for Bayesian infinite-mixture models (Kalli et al. 2011), it is possible to conduct exact MCMC sampling from the posterior distribution of the BNP-IRT model parameters. More specifically, introducing latent variables $(u_{pi}, z_{pi} \in \mathbb{Z}, u_{pi}^* \in \mathbb{R})_{N \times I}$ and a fixed decreasing function such as $\xi_l = \exp(-l)$, the conditional likelihood of the BNP-IRT model can be written as

$$\prod_{p=1}^{P} \prod_{i=1}^{I} \prod_{j} \mathbf{1}(0 < u_{pi}^* < \xi_{z_{pi}}) \xi_{z_{pi}}^{-1} \mathcal{N}(u_{pi}^* | \mu_{z_{pi}} + x_{pi}^T \beta, \sigma^2) \omega_{z_{pi}}(x_{pi}^T \beta_{\omega}, \sigma_{\omega}).$$

(9)

For each $(p, i)$, after marginalizing over the latent variables in (9) we obtain the original model likelihood $f(u_{pi} | x_{pi}; \zeta)$ in (4a). Importantly, conditionally on the latent variables, the infinite-dimensional BNP-IRT model can be treated as a finite-dimensional model, which then makes the task of MCMC sampling feasible (of course, even a computer cannot handle an infinite number of parameters). Given all variables, save the latent variables $(z_{i})_{i=1}^{n}$, the choice of each $z_{i}$ has finite maximum value $\pm N_{max}$, where $N_{max} = \max_{\pi} \max_{i} \max_{j} \{\max_{j} \mathbb{I}(u_{pi} < \xi_{j}) \; | \; j \}$.

Then standard MCMC methods can be used to sample the full conditional posterior distributions of each latent variable and model parameter repeatedly for a sufficiently large number of times, $S$. If the prior $\pi(\zeta)$ is proper (Robert & Casella, 2004, sect. 10.4.3), then, for $S \rightarrow \infty$, this sampling process constructs a discrete-time Harris ergodic Markov chain

$$\{(u_{pi}^{(s)}), (z_{pi}^{(s)}), (z_{pi}^{*^{(s)}}), \zeta^{(s)} = ((u_{pi}), (z_{pi}), \mu, \sigma_{\mu}, \beta, \sigma^2, \beta_{\omega}, \sigma_{\omega})^{(s)}\}_{s=1}^{S}.$$
which, upon after marginalizing out all the latent variables \((w^{(s)}_pi, (z^{(s)}_pi, (z^{* (s)}_pi), has the posterior distribution \(\Pi(\zeta | D_n)\) as its stationary distribution (for definitions, see Meyn & Tweedie, 1993; Nummelin, 1984; Roberts & Rosenthal, 2004). (The next paragraph provides more details about the latent variables, \(z^{* (s)}_pi\).

The full conditional posterior distribution are as follows: the one of \(w^{(s)}_pi\) is \(u(w^{(s)}_pi | 0, \xi | z^{(s)}_pi)\); \(w^{* (s)}_pi\) has a truncated normal distribution; the one of \(z^{(s)}_pi\) is a multinomial distribution independently for \(p = 1, \ldots, P\) and \(i = 1, \ldots, I\); the full conditional distribution of \(\mu_j\) is a normal distribution (sampled using a Metropolis-Hastings algorithm), independently for \(j = -N_{max}, \ldots, N_{max}\); \(\sigma_\mu\) can be sampled using a slice sampling algorithm involving a stepping-out procedure (Neal, 2003); the one \(\beta\) is multivariate normal distribution; and the full conditional posterior distribution of \(\sigma^2_\omega\) is inverse-gamma. Also, upon sampling of truncated normal latent variables \(z^{* (s)}_pi\) that have full conditional densities proportional to \(n(z^{* (s)}_pi | \mathbf{x}_i^T \beta_\omega, \sigma_\omega)1(z^{* (s)}_pi - 1 < z^{* (s)}_pi < z^{* (s)}_pi)\), independently for \(p = 1, \ldots, P\) and \(i = 1, \ldots, I\), the full conditional posterior distribution of \(\beta_\omega\) is multivariate normal distribution and the one of \(\sigma^2_\omega\) is inverse-gamma distribution. For further details of the MCMC algorithm, see Karabatsos and Walker (2012).

In practice, obviously only a MCMC chain based on a finite number \(S\) can be generated. The convergence of finite MCMC chains to samples from posterior distributions can be assessed using the following two procedures (Geyer, 2011): (i) viewing univariate trace plots of the model parameters to evaluate MCMC mixing (Robert & Casella, 2004); and (ii) conducting a batch-mean (or subsampling) analysis of the finite chain, which would provide 95% Monte Carlo Confidence intervals (MCCIs) of all the posterior mean and quantile estimates of the model parameters (Flegal & Jones, 2011). Convergence can be confirmed both
by trace plots that look stable and "hairy" and 95% MCCIs that, for all practical purposes, are sufficiently small. If convergence is not attained for the current choice of S samples of a MCMC chain, additional MCMC samples should be generated until convergence is obtained.

5 Model Fit

The fit of the BNP-IRT model to a set of item response data, D, can be assessed on the basis of its posterior predictive pmf, defined in (6).

More specifically, the fit to a given response $u_{pi}$ can be assessed by its standardized response residual

$$r_{pi} = \frac{u_{pi} - \mathbb{E}[U|X, D]}{\sqrt{\mathbb{V}_n[U|X, D]}}.$$  

Response $u_{pi}$ can be judged to be an outlier when $|r_{pi}|$ is greater than exceeds two or three.

A global measure of the predictive fit of a regression model, indexed by $m \in \{1, \ldots, M\}$, is provided by the mean-squared predictive error criterion

$$D(m) = \sum_{p=1}^{P} \sum_{i=1}^{I} (u_{pi} - \mathbb{E}[U_{pi} | x_{pi}, D])^2 + \sum_{p=1}^{P} \sum_{i=1}^{I} \mathbb{V}_n[U_{pi} | x_{pi}, \overline{m}].$$

(Laud & Ibrahim, 1995; Gelfand & Ghosh, 1998). The first term of $D(m)$ measures the goodness-of-fit ($\text{Gof}(m)$) of the model to the data, while its second term is a penalty for model complexity. Among a set of $m = 1, \ldots, M$ that is compared, the model with the highest predictive accuracy for the data set $D$ is identified as the one with the smallest value of $D(m)$.

The proportion of variance explained by the regression model is given by the $R$-squared
\( R^2 \) statistic

\[
R^2 = 1 - \frac{\sum_{p=1}^{P} \sum_{i=1}^{I} \{u_{pi} - \mathbb{E}[U_{pi} | x_{pi}, D]\}^2}{\sum_{p=1}^{P} \sum_{i=1}^{I} \{u_{pi} - \overline{u}\}^2},
\]

where \( \overline{u} = \frac{1}{PI} \sum_{p=1}^{P} \sum_{i=1}^{I} u_{pi}. \)

The standardized residuals \( r_{pi} \), the \( D(m) \) criterion, and \( R^2 \) can each be estimated as a simple by-product of an MCMC algorithm.

6 Empirical Example

Using the BNP-IRT model, we analyzed a set of polytomous response data obtained from the 2006 Progress in International Reading Literacy Study. A total of \( N = 244 \) fourth-grade U.S. teachers rated their own teaching preparation level in a ten-item questionnaire \( (I = 10) \). Each item was scored on a scale ranging from zero to two.

For this questionnaire, the latent person ability was assumed to represent the level of teaching preparation. The ten items addressed the following areas: education level (named CERTIFICATE), English LANGUAGE, LITERATURE, teaching reading (PEDAGOGY), PSYCHOLOGY, REMEDIAL reading, THEORY of reading, children’s language development (LANGDEV), special education (SPED), and second language (SECLANG) learning. The CERTIFICATE item was scored on a scale of 0 = bachelor’s, 1 = master’s, 2 = doctoral, while the other 9 questionnaire items were each scored on a scale consisting of 0 = not at all, 1 = overview or introduction to topic, and 2 = area of emphasis. Each of the ten items described a type of training for literacy teachers, as prescribed by the National Research Council (2010).

We considered three additional covariates for the BNP-IRT model, namely AGE level
(scored in nine ordinal categories), FEMALE status, and Miss:FEMALE, an indicator (0,1) of missing value for FEMALE status. Overall, 2,419 of the total possible 2,440 item were observed. Three of the 244 teachers had missing values for FEMALE, which were imputed using information from the observed values of all the variables mentioned above.

Given that each of the 10 items item was scored on a polytomous scale (3 categories), and that we were interested in additional covariates over and beyond the person-indicator and item-indicator covariates, we analyzed the data using the BNP-IRT model, using extensions #1 and #2 of the basic BNP-IRT model in Section 3 above. Also, the parameters of the prior pdf (5) of the model were chosen as \((b_{\sigma\mu}, v, a_0, v_w, a_w) = (1, 10, 1000, 1, .01)\).

To estimate the posterior distribution of the BNP-IRT model parameters, we ran the MCMC sampling algorithm in Section 4 for 62,000 iterations. We used 12,000 MCMC samples for posterior inference, retaining every fifth sample beyond the first 2,000 iterations (burn-in) to obtain (pseudo-) independence between them. Trace plots for the univariate parameters displayed adequate mixing (i.e., exploration of the posterior distribution), and a batch-mean (subsampling) analysis of the 12,000 MCMC samples revealed 95% Monte Carlo Confidence intervals of the posterior mean and quantile estimates (reported below) that typically had half-widths less than .2. If desired, smaller half-widths could have been obtained by generating additional MCMC samples.

For the BNP-IRT model, the standardized response residuals ranged from \(-.21\) to \(.20\), meaning that the model had no outliers (i.e., all the absolute standardized residuals were well below two). Globally, the model fit analyses yielded criterion value \(D(m) = 2.76\) (with \(\text{Gof}(m) = .03\) and Penalty \(P(m) = 2.73\)) for the 2,419 responses in the data set. Also, the BNP-IRT model attained an R-squared of one.
The estimated posterior means of the person ability parameters were found to be distributed with mean .00, standard deviation .46, minimum −.66, and maximum 3.68 for the 244 persons. Figure 1 presents a box plot of the marginal posterior distributions (full range, interquartile range, median), for all the remaining parameters, including the item-difficulty parameters and the slope coefficients of the covariates AGE, FEMALE, and Miss:FEMALE. Parameter labels such as CERTIFICATE(1) and CERTIFICATE(2) refer to the difficulty of the CERTIFICATE item, with respect its rating categories 1 and 2, respectively. The most difficult item was REMEDIAL(2) (with posterior median difficulty of .27), and the easiest item was SECLANG(1) (posterior median difficulty −1.81). Also, the covariates AGE and FEMALE were each found to have a significant positive association with the rating response, since they had coefficients with 75% posterior intervals that excluded zero (this type of interpretation of significance was justified by Li & Lin, 2010). The box plot also presents the marginal posterior distributions for all the item and covariate parameters in $\beta$, the mixture weights, and the variance parameters $\sigma_\mu$, $\sigma^2$, and $\sigma^2_\omega$.

7 Discussion

In this chapter, we proposed and illustrated a practical and yet flexible BNP-IRT model, which can provide robust estimates of person ability and item difficulty parameters. We demonstrated the suitability of the model through the analysis of real polytomous item
response data. The model showed excellent predictive performance for the data, with no item response outliers.

For the BNP-IRT model, a user-friendly and menu-driven software, entitled: "Bayesian Regression: Nonparametric and Parametric Models" is freely downloadable from the authors website (Karabatsos, 2014a,b). The BNP-IRT model can be easily specified by clicking the menu options "Specify New Model" and "Binary infinite homoscedastic probits regression model." Afterwards, the response variable, covariates, and prior parameters can be selected by the user. Then, to run for data analysis, the user can click the "Run Posterior Analysis" button to start the MCMC sampling algorithm in Section 4 for a chosen number of iterations. Upon completion of the MCMC run, the software automatically opens a text output file containing the results, which includes summaries of the posterior distribution of the model obtained from the MCMC samples. The software also allows the user to check for MCMC convergence through menu options that can be clicked to construct trace plots or run a batch- mean analyses that produces 95% Monte Carlo confidence intervals of the posterior estimates of the model parameters. Other menu options allow the user to construct plots (e.g., box plots) and text with the estimated marginal posterior distributions of the model parameters or residual plots and text reports the fit of the BNP-IRT model in greater detail.

Currently, the software provides a choice of 59 statistical models, including a large number of BNP regression models. The choice allows the user to specify DP-mixture (or more generally, stick-breaking-mixture) IRT models, with the mixing done either on the intercept parameter or the entire vector of regression coefficient parameters.

An interesting extension of the BNP-IRT model would involve specifying the kernel of the mixture by a cognitive model. For example, one may consider the multinomial processing
tree (MPT) model (e.g., Batchelder & Riefer, 1999) with parameters that describe the latent processes underlying the responses. Such an extension would provide a flexible, infinite-mixture of cognitive models that allows cognitive parameters to vary flexibly as a function of (infinitely-many) covariate-dependent mixture weights.

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Figure Caption

Figure 1. For the BNP-IRT model, a box plot of the marginal posterior distributions of the item, covariate, and prior parameters. For each of these model parameters, the box plot presents the range, interquartile range, and median.
