A 2-DIMENSIONAL COMPLEX KLEINIAN GROUP WITH INFINITE LINES IN THE LIMIT SET LYING IN GENERAL POSITION

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Abstract. In this article we present an example of a discrete group \( \Sigma \subset PSL(3, \mathbb{R}) \) whose action on \( \mathbb{P}^2_\mathbb{C} \) does not have invariant projective subspaces, is not conjugated to complex hyperbolic group and its limit set in the sense of Kulkarni on \( \mathbb{P}^2_\mathbb{R} \) has infinite lines in general position.

Introduction

In \cite{[6]} the author has constructed an example of 3-manifold which admits an exotic projective structure, as part of this construction the author, by means of the use of the Pappus’s theorem, has build up a discrete group \( \Sigma \) of \( PSL(3, \mathbb{R}) \) acting on \( \mathbb{P}^2_\mathbb{R} \) which has a fractal curve as a unique closed minimal set. The main purpose of this note is to show that considering the action of \( \Sigma \) on \( \mathbb{P}^2_\mathbb{C} \) one gets and example of a discrete group whose discontinuity region in the Kulkarni sense is non empty, even in case that the respective region on \( \mathbb{P}^2_\mathbb{R} \) might be empty, its limit set in the sense of Kulkarni has infinite lines in general position and is not conjugated either to a group which has invariant projective spaces or a complex hyperbolic group, which where the only known examples. More precisely we show:

**Theorem 0.1.** There is a discrete group \( \Sigma \subset PSL(3, \mathbb{R}) \) acting on \( \mathbb{P}^2_\mathbb{C} \) with the following properties:

(i) The group \( \Sigma \) is complex Kleinian;

(ii) There is a fractal curve \( C \subset \mathbb{P}^2_\mathbb{R} \) which is the minimal close set for the action of \( \Sigma \);

(iii) The discontinuity region and the discontinuity region in the Kulkarni sense agree;

(iv) There is a fractal curve \( T \subset Gr(\mathbb{P}^2_\mathbb{C}) \) such that:

\[
\Lambda(\Sigma) = \bigcup T;
\]

(v) The number of lines on \( \Lambda(\Sigma) \) lying on general position is infinite;

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The group $\Sigma_C$ does not have invariant lines;

The group $\Sigma_C$ is not complex hyperbolic, i.e. is not conjugated to a group which leaves the unitary complex ball invariant.

This article is organized as follows: in section 1 we introduce some terms and notations which will be used along the text and finally in section 2 we present the proof of theorem 0.1.

1. Preliminaries and Notations

1.1. The Kulkarni’s Limit Set.

**Definition 1.1.** Given a group $G$ acting on a topological space $X$, then $X$ is said to be a discontinuity region for $X$ if for every pair of compact subsets $C$ and $D$ of , the cardinality of the set \{\{g \in G : g(C) \cap D \neq \emptyset\}\}, is finite.

R. Kulkarni introduced in [4] a concept of limit set that has the important property of assuring that its complement is a region of discontinuity. Let us proceed to define it. Let $L_0(G)$ be the closure of the set of points in $X$ with infinite isotropy group. Let $L_1(G)$ be the closure of the set of cluster points of orbits of points in $X \setminus L_0(G)$ Finally, let $L_2(G)$ be the closure of the set of cluster points of $GK$, where $K$ runs over all the compact subsets of $X \setminus (L_0(G) \cup L_1(G))$. We have:

**Definition 1.2** (See [4]). Let $X$ be a topological space and $G$ be a group of homeomorphisms of $X$. The Kulkarni limit set of $G$ in $X$ is the set:

$\Lambda(G) := L_0(G) \cup L_1(G) \cup L_2(G)$.

The Kulkarni region of discontinuity of $G$ is

$\Omega(G) := X \setminus \Lambda(G)$.

A group is said to be Kleinian if $\Omega(G)$ is non empty.

An easy argument shows that $\Omega(X)$ is a discontinuity region for $G$.

1.2. Projective geometry. We recall that given a a vector space $V$ over a field $K$ is to be defined as

$PV = (V \setminus \{0\})/ \approx$

where $\approx$ denotes the equivalence relation given by $x \approx y$ if and only if $x = cy$ for some nonzero scalar $c$. If $[\cdot] : V \setminus \{0\} \rightarrow PV$ represents the quotient map, then a nonempty set $H \subset PV$ is said to be a line projective if there is a $K$-linear subspace $\tilde{H} \subset V$ of dimension 2 such that $[\tilde{H}] = H$. It is well known that, given $p, q \in PV$ there is a unique line passing through $p$ and $q$. Also set $Gr(PV)$ to be the space of lines, it can be seen that $Gr(PV) = PV$ if $K$ is either $\mathbb{C}$ or $\mathbb{R}$ and $V$ has dimension 3.

The group of projective automorphisms is:

$PSL(V) := GL(V)/ \approx$
where \( \approx \) denotes the equivalence relation given by \( x \approx y \) if and only if \( x = cy \) for some nonzero scalar \( c \). The elements in \( PSL(V) \) are called projective transformations, they act in a natural way on \( PV \) and it is verified that take lines into lines. Also given \( \bar{M} : V \rightarrow V \) a nonzero linear transformation which is not necessarily invertible. Let \( \text{Ker}(M) \) be its kernel and let \( \text{Ker}(M) \) denote its projectivisation, the seudo-projective transformation induced by \( \bar{M} \) is the map \( M : PV \setminus \text{Ker}(V) \rightarrow PV \ M([v]) = [M(v)] \).

2. Proof of the Main Theorem

**Proposition 2.1.** In [6], by means of the Pappus’s Theorem, the author has constructed a discrete group \( \Sigma \subset PSL(3, \mathbb{R}) \) with the following properties.

(i) Algebraically \( \Sigma \) is the modular group \( \mathbb{Z}_2 \ast \mathbb{Z}_3 \).

(ii) There are fractal curves \( \alpha_1 : \mathbb{S}^1 \rightarrow \mathbb{P}_\mathbb{R}^2, \alpha_2 : \mathbb{S}^1 \rightarrow \text{Gr}(\mathbb{P}_\mathbb{R}^2) \) such that \( \alpha_2(\mathbb{S}^1) \) is the unique transverse linefield to \( \alpha_1(\mathbb{S}^1) \), \( \alpha_1(\mathbb{S}^1) \) is the section to \( \alpha_2(\mathbb{S}^1) \)

\[
\Sigma \alpha_1(\mathbb{S}^1) = \alpha_1(\mathbb{S}^1), \quad \Gamma^* \alpha_2(\mathbb{S}^1) = \alpha_2(\mathbb{S}^1) \quad \text{and} \quad \alpha_2(p) \cap \alpha_1(\mathbb{S}^1) = \{ \alpha_1(p) \}, \text{ for each } p \in \mathbb{S}^1.
\]

(iii) There are transformations \( \gamma \in \Sigma \) (resp. \( \Sigma^* \)) with an attracting and a repelling fixed point. From now on, we will call such transformations Loxodromic and we will denote the respective attracting and repelling fixed points by \( \gamma_+ \) and \( \gamma_- \), (resp. \( \gamma_+^* \) and \( \gamma_-^* \)).

(iv) There are loxodromic elements \( \gamma, \tau \in \Sigma \) such that:

\[
\{ \gamma_+^*, \gamma_-^* \} \cap \{ \tau_+, \tau_- \} = \emptyset;
\]

(v) The set of repelling and attracting loxodromic fixed points are dense in \( \alpha_1(\mathbb{S}^1) \) and \( \alpha_2(\mathbb{S}^1) \) respectively, that is:

\[
\alpha_1(\mathbb{S}^1) = \{ \gamma_+, \gamma_- : \gamma \in \Sigma \text{ is loxodromic} \};
\]

\[
\alpha_2(\mathbb{S}^1) = \{ \gamma_+^*, \gamma_-^* : \gamma \in \Sigma^* \text{ is loxodromic} \}.
\]

(vi) Let \( (\gamma_m) \subset \Sigma \) be a sequence of distinct elements. Thus for any \( \epsilon > 0 \) there is some \( n \in \mathbb{N} \) and points \( p,q \in \mathbb{S}^1 \) such that

\[
\gamma_n(\mathbb{P}_\mathbb{R}^2 \setminus B_\epsilon(\alpha_2(q))) \subset B_\epsilon(\alpha_1(p))
\]

where \( B_\epsilon(x) \) denotes open ball with radius \( \epsilon \), center \( \alpha_i(x) \) and with respect the spherical metric \( \rho \).

**Lemma 2.2.** Let \( \gamma \in \Sigma \) be a loxodromic element, then there are \( s, t \in \mathbb{S}^1 \) distinct elements such that \( \alpha_1(s) = \gamma_+, \alpha_1(t) = \gamma_- \), \( \text{Fix}(\gamma) = \{ \alpha_1(s), \alpha_1(t), \gamma_c \} \), where \( \gamma_c \) is the unique element in \( \alpha_2(s) \cap \alpha_2(t) \), which is also a saddle point.

**Proof.** Let \( \gamma \in \Sigma \) be a loxodromic element. Thus \( \gamma_+, \gamma_- \in \alpha_1(\mathbb{S}^1) \). In consequence there are \( x,x_2 \in \mathbb{S}^1 \) such that \( \alpha_1(x_1) = \gamma_+ \) and \( \alpha_1(x_2) = \alpha_- \). Thus

\[
\gamma(\alpha_2(x_i)) \cap \alpha_1(\mathbb{S}^1) = \{ \alpha_1(x_i) \}.
\]
Since $\alpha_2$ is $\Sigma^*$-invariant we conclude $\gamma(\alpha_2(x_i)) = \alpha_2(x_i)$. In consequence $\alpha_2(x_1) \cap \alpha_2(x_2)$ is fixed by $\gamma$. Therefore the fixed set of $\gamma$ consist on exactly 3 points, each of one is contained in $\mathbb{P}^2_{\mathbb{C}}$. \hfill \Box

A similar argument yields

**Corollary 2.3.** Let $\gamma \in \Sigma$ be a loxodromic element, then there are $s, t \in S^1$ distinct elements such that $\alpha_2(s) = \gamma^+_s$, $\alpha_2(t) = \gamma^+_t$, $Fix(\gamma) = \{ \alpha_2(s), \alpha_2(t), \gamma^+_s \}$, where $\gamma^+_s$ is the unique element in $Gr(\mathbb{P}^2_R)$ which contains $\alpha_1(s)$ and $\alpha_1(t)$, which is also a saddle point.

**Lemma 2.4.** The group $\Sigma$ does not have invariant real lines.

**Proof.** On the contrary let us assume that there is $\ell \in Gr(\mathbb{P}^2_R)$ which is $\Sigma$-invariant. Now since $\alpha_1(S^1)$ is a fractal curve we conclude that there is $\gamma \in \Sigma$ a loxodromic element such that $\gamma_+ \not\in \ell$. Now, let $s, t \in S^1$ be distinct elements such that $\alpha_1(s) = \gamma_+$, $\alpha_1(t) = -\gamma_-$, $\alpha_2(s) = \gamma^+_t$ and $\alpha_2(t) = \gamma^+_s$. Since $\ell$ is invariant under the action of $\gamma$, Corollary 2.3 yields $\ell = \alpha_2(t)$.

Claim 1. If $\tau \in \Sigma$ is a loxodromic element, thus $\alpha_1(t)$ is a fixed point of $\tau$ which is not saddle. Now, let $\tau \in \Sigma$ be a loxodromic element. Then by Lemma 2.2 there are $s_1, t_1 \in S^1$ distinct elements such that $\alpha_1(s_1) = \tau_+$, $\alpha_2(t_1) = \tau_-$, $\alpha_2(s_1) = \gamma^+_t$ and $\alpha_2(t_1) = \gamma^+_s$. Since $\alpha_2(t_1)$ is $\tau$-invariant and $\alpha_2(S^1)$ is the unique transverse linefield to $\alpha_2(S^1)$, Corollary 2.3 yields that either $\alpha_2(t) = \alpha_2(t_1)$ or $\alpha_2(t) = \alpha_2(s_1)$. In consequence $\alpha_1(t)$ is a fixed point of $\tau$ which is not saddle.

From the previous claim it follows $\alpha_1(t) \in \cap \{ \tau_+, \tau_- : \tau \in \Sigma \text{ is loxodromic} \}$, which contradicts part (iv) of Proposition 2.1. \hfill \Box

Dualizing the previous argument one gets

**Lemma 2.5.** The group $\Sigma$ does not have fixed points on $\mathbb{P}^2_R$.

**Lemma 2.6.** Let $M \subset \mathbb{P}^2_R$ be a closed, non empty set such that $\text{Stab}(M, \Sigma)$ is a subgroup of finite index of $\Sigma$, then $\alpha_1(S^1) \subset M$.

**Proof.** Let us assume that there is $\gamma \in \Sigma$ a loxodromic element such that $\gamma_+ \not\in M$. Thus by Lemmas 2.4 and 2.5 it follows that $M \setminus Fix(\gamma)$ has at least one element, say $y$. Since $\text{Stab}(M, \Sigma)$ has finite index, there is $k \in \mathbb{N}$ such that $\gamma^k \in \text{Stab}(M, \Sigma)$. Thus $(\gamma^{mk}(y)) \subset M$ and $\gamma^m(y) \underset{m \to \infty}{\longrightarrow} \gamma_+$, which is a contradiction. \hfill \Box

**Lemma 2.7.** Let $(T_m) \subset \Sigma$ be a sequence of distinct elements and $T \in QP(3, \mathbb{R})$ be a seudoprojective transformation such that $T_m \underset{m \to \infty}{\longrightarrow} T$, then there are $\tau, s \in S^1$ such that $\text{Im}(T) = \alpha_1(r)$ and $\text{Ker}(T) = \alpha_2(s)$.

**Proof.** For each $m \in \mathbb{N}$, define $\epsilon_m = \frac{\text{diam}_y(\mathbb{P}^2_{\mathbb{R}})}{8^m}$. 


Thus by part 2 of Proposition 2.1 there is a subsequence of \((T_m)\), still denoted \((T_m)\), sequences \((r_m), (s_m) \in S^1\) such that \(r_m \xrightarrow{m \to \infty} r\), \(s_m \xrightarrow{m \to \infty} s\) and
\[
T_m(\mathbb{P}^2_{\mathbb{R}} \setminus B_{t_m}(\alpha_2(r_m))) \subset B_{t_m}(\alpha_1(s_m)).
\]

Let \(K \subset \mathbb{P}^2_{\mathbb{R}} \setminus \alpha_2(r)\) be a non empty compact set. Since \(\alpha_2(r_m) \xrightarrow{m \to \infty} \alpha_2(r)\), it follows that there is \(m_0 \in \mathbb{N}\) such that:
\[
B_{t_m}(\alpha_2(r_m)) \subset B_d(\alpha_2(r))\text{ for all } m \geq m_0,
\]
where \(d = 4^{-1} \rho(\alpha_2(r), K)\). Thus
\[
T_m(K) \subset B_{t_m}(\alpha_1(s_m))\text{ for all } m \geq m_0.
\]

Now let \(\epsilon > 0\), since \(\alpha_1(s_m) \xrightarrow{m \to \infty} \alpha_1(s)\), it follows that there is \(n_0 \in \mathbb{N}\) such that:
\[
B_{t_m}(\alpha_1(s_m)) \subset B_\epsilon(\alpha_1(s))\text{ for all } m \geq n_0.
\]

In consequence
\[
T_m(K) \subset B_\epsilon(\alpha_1(s))\text{ for all } m \geq \text{max}\{m_0, n_0\},
\]
which concludes the proof. \(\square\)

For each \(\ell \in Gr(\mathbb{P}^2_{\mathbb{R}})\), let us denote by \(\langle \ell \rangle\) the unique complex line which contains \(\ell\). Also let us denote the action of \(\Sigma\) on \(\mathbb{P}^2_{\mathbb{C}}\) by \(\Sigma_C\).

**Proposition 2.8.** The group \(\Sigma_C\) does not have fixed points.

**Proof.** On the contrary, let us assume that there is a point \(p \in \mathbb{P}^2_{\mathbb{C}}\) which is fixed by \(\Sigma_C\). Now, let \(\gamma \in \Sigma_C\) be a loxodromic element. Thus my Lemma 2.2 it follows that \(p \in \text{Fix}(\gamma) \subset \mathbb{P}^2_{\mathbb{R}}\), which contradicts lemma 2.5. \(\square\)

**Proposition 2.9.** The group \(\Sigma_C\) is not affine.

**Proof.** On the contrary, let us assume that there is a line \(\ell \in Gr(\mathbb{P}^2_{\mathbb{C}})\) which is \(\Sigma^*\)-invariant. Now an easy calculation shows that \(\ell_1 = \ell \cap \mathbb{P}^2_{\mathbb{R}}\) is either a point or a a real line which is \(\Sigma\)-invariant. Which contradicts Lemmas 2.5 and 2.4. \(\square\)

**Proposition 2.10.** The group \(\Sigma_C\) is not complex hyperbolic.

**Proof.** On the other hand, let us assume that \(\Sigma_C\) is conjugated to a complex hyperbolic group. Thus there is a smooth manifold \(M\) diffeomorphic to the 3-dimensional sphere \(S^3\) which is \(\Sigma_C\)-invariant. Now let \(\gamma \in \Sigma_C\) be a loxodromic element and \(y \in M \setminus \text{Fix}(\gamma)\), thus \(\gamma^m(y) \xrightarrow{m \to \infty} \gamma^+\). In consequence, \(\tilde{M} = M \cap \mathbb{P}^2_{\mathbb{R}}\) is a non empty, compact, \(\Sigma\)-invariant, smooth submanifold of \(\mathbb{P}^2_{\mathbb{R}}\). Let \(n_0 = \text{max}\{\text{dim}_\mathbb{R}(N) : N \subset M\}\) be a connected component. Thus \(n_0\) is finite since \(M\) is compact. Now let \(N_0\) be a connected component of \(M\) such that \(\text{dim}_\mathbb{R}(N_0) = n_0\). Since \(M\) is compact it follows that \(\text{Stab}(N_0, \Sigma_C)\) is a subgroup of \(\Sigma_C\) with finite index. Thus Lemma 2.6 yields \(\alpha_1(S^1) \subset N_0\). Since \(N_0\) is smooth, compact and connected we conclude \(n_0 = 2\). Thus \(N_0 = \mathbb{P}^2_{\mathbb{R}}\), which is a contradiction since \(\mathbb{P}^2_{\mathbb{R}}\) cannot be immersed in \(S^3\). \(\square\)
Recall that given a family of continuous functions $A = \{ f_\alpha : X \to Y \}_{\alpha \in I}$, where $X$ and $Y$ are topological spaces. The equicontinuity region of $A$ is to be defined as the sets of points $x \in X$ which has an open neighborhood such that each sequence $(f_m) \subset A$ has a subsequence which is convergent on $U$ with respect the compact-open topology.

**Corollary 2.11.** The equicontinuity set of $\Sigma C$ is non empty open set and is the union of all the lines induced by $\alpha_2(S^1)$, that is:

$$Eq(\Sigma C) = \mathbb{P}^2_C \setminus \bigcup_{x \in S^1} \langle \alpha_2(x) \rangle.$$

**Proof.** Let $(T_m) \subset \Sigma C$ be a sequence of distinct elements. Thus, see [2], there is a transformation $\tilde{T} \in M(3, \mathbb{R})$, a subsequence of $(T_m)$, still denoted $(\tilde{T}_m)$, and a sequence $\tilde{T}_m \in SL(3, \mathbb{R})$ such that $\tilde{T}_m \xrightarrow{m \to \infty} \tilde{T}$ as $\mathbb{C}$-linear transformations and with respect the compact-open topology. Now by considering $\tilde{T}$ and $(\tilde{T}_m)$ as $\mathbb{R}$-linear transformations and applying Lemma 2.7 we deduce $[Ker(\tilde{T}) \setminus \{0\}] = \alpha_2(r)$ and $[Im(\tilde{T}) \setminus \{0\}] = \{\alpha_1(s)\}$. Thus considering $\tilde{T}$ as $\mathbb{C}$-linear transformation one gets $Ker(T) = \langle \alpha_2(r) \rangle$ and $Im(T) = \{\alpha_1(s)\}$. Now the description of $Eq(\Sigma C)$ follows from part (v) of Proposition 2.1.

Finally, let $z \in S^1$, $y \in \mathbb{P}^2_R \setminus \alpha_2(z)$, $\ell \in Gr(\mathbb{P}^2_R)$ such that $y, \alpha_1(z)$ and $x \in \langle \ell \rangle \setminus \mathbb{P}^2_R$. Clearly $x \in Eq(\Sigma C)$. \qed

**Proposition 2.12.** The number of lines on $\Lambda_{Kul}(\Sigma C)$ lying in general position is infinite.

**Proof.** Since $\Sigma C$ does not have invariant lines on fixed point, it can be shown, see [1], that the number of lines in general position lying on $\Lambda(\Sigma C)$ is either 3 or infinite. On the other hand, in [1] it is showed a group which have exactly 3 lines on its limit set in the Kulkarni sense, should satisfy have a normal commutative subgroup $N$ with finite index. Which is not possible for $\Sigma C$ since it is, algebraically, the modular group. \qed

Now it follows easily

**Proposition 2.13.** The group $\Sigma C$ is a Kleinian group such that the equicontinuity region agrees with the discontinuity region in the Kulkarni’s sense, that is:

$$\Omega_{Kul}(\Sigma C) = \mathbb{P}^2_C \setminus \bigcup_{x \in S^1} \langle \alpha_2(x) \rangle.$$

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