Complexity characterization in a probabilistic approach to dynamical systems through information geometry and inductive inference

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Abstract

Information geometric techniques and inductive inference methods hold great promise for solving computational problems of interest in classical and quantum physics, especially with regard to complexity characterization of dynamical systems in terms of their probabilistic description on curved statistical manifolds. In this paper, we investigate the possibility of describing the macroscopic behavior of complex systems in terms of the underlying statistical structure of their microscopic degrees of freedom by the use of statistical inductive inference and information geometry. We review the maximum relative entropy formalism and the theoretical structure of the information geometrodynamical approach to chaos on statistical manifolds $\mathcal{M}_S$. Special focus is devoted to a description of the roles played by the sectional curvature $\kappa_{\mathcal{M}_S}$, the Jacobi field intensity $J_{\mathcal{M}_S}$ and the information geometrodynamical entropy $S_{\mathcal{M}_S}$. These quantities serve as powerful information-geometric complexity measures of information-constrained dynamics associated with arbitrary chaotic and regular systems defined on $\mathcal{M}_S$. Finally, the application of such information-geometric techniques to several theoretical models is presented.

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1. Introduction

It is commonly accepted that one of the major goals of physics is modeling and predicting natural phenomena using relevant information about the system of interest. Taking this statement seriously, it is reasonable to expect that the laws of physics should reflect the methods for manipulating information. Indeed, the less controversial opposite point of view may be considered where the laws of physics are used to manipulate information. This is exactly the point of view adopted in quantum information science where information is manipulated using the laws of quantum mechanics [1]. In this paper, we wish to explore an alternative viewpoint: perhaps the laws of physics are nothing but rules of...
inference [2]. In this view, the laws of physics are not laws of nature but merely the rules we follow when processing the information that happens to be relevant to the physical problem under consideration. When the information available is sufficient to make unequivocal, unique assessments of truth, we speak of making deductions: on the basis of this or that information we deduce that a certain proposition is true. In the cases where we do not have statements that lead to unequivocal conclusions, we speak of using inductive reasoning, and the system for this reasoning is probability theory [3]. The word ‘induction’ refers to the process of using limited information about a few special cases to draw conclusions about more general situations. Following this line of reasoning, interesting probabilistic approaches describing complex dynamical systems have already been investigated [4–6].

The study of the relationship between entropy and the complexity [7] of the trajectories of a dynamical system has always been an active field of research [8–10]. Chaotic behavior is a particular case of complex behavior and it will be the subject of the present work. In this paper, we make use of the so-called entropic dynamics (ED) [11]. ED is a theoretical framework that arises from the combination of inductive inference (maximum entropy (ME) methods [12–16]) and information geometry (IG) [17]. The most intriguing question being pursued in ED stems from the possibility of deriving dynamics from purely entropic arguments. This is clearly valuable in circumstances where microscopic dynamics may be too far removed from the phenomena of interest, such as in complex biological or ecological systems, or where it may be unknown or perhaps even non-existent, as in economics. It has already been shown that entropic arguments do account for a substantial part of the formalism of quantum mechanics, a theory that is presumably fundamental [18]. Perhaps the fundamental theories of physics are not so fundamental; they may just be consistent, objective ways of manipulating information. Following this line of thought, we extend the applicability of information-geometric techniques and inductive inference methods to computational problems of interest in classical and quantum physics. In particular, we focus on the complexity characterization of dynamical systems in terms of their probabilistic description of curved statistical manifolds and identify relevant measures of chaoticity associated with such an information geodynamical approach to chaos (IGAC) [4–6, 19–22].

The structure of this paper is as follows. In section 2, we review the maximum relative entropy formalism. In section 3, we present an introduction to the main features of the IGAC. In section 4, we introduce the information-geometric indicators of chaos for our theoretical model: sectional \( (K_{M}) \) and Ricci \( (R_{M}) \) curvatures, Jacobi field intensity \( (J_{M}) \), IGE \( (S_{M}) \) and information-geometric complexity \( (C_{M}) \). In section 5, we present five applications of our techniques to the study of the dynamical complexity of suitable statistical models. Firstly, we characterize the chaotic behavior of an ED Gaussian model describing an arbitrary system of \( J \) degrees of freedom in the absence of correlations. Secondly, we discuss the asymptotic temporal behavior of the information-geometric complexity of the maximum probability trajectories of a 2\( l \)-dimensional Gaussian statistical manifold in the presence of correlations between the macrovariables labeling the macrostates of the system [23]. Thirdly, an information-geometric analogue of the Zurek–Paz quantum chaos criterion of linear entropy growth [6, 20] for a system of uncoupled, anisotropic (random frequency) and inverted harmonic oscillators (IHOs) is examined. Fourth example, we present the information-geometric characterization of regular and chaotic quantum energy level statistics [4, 5]. Our fifth and final example applies the IGAC to model and describe the scattering-induced quantum entanglement of two micro-correlated, spinless, structureless, non-relativistic particles, where each particle is represented by a minimum uncertainty wave packet [24]. Final remarks are presented in section 6.

2. On the maximum relative entropy formalism

In 1957, Jaynes [25] showed that maximizing statistical mechanic entropy for the purpose of revealing how gas molecules were distributed was equivalent to maximizing Shannon’s information entropy [26] with statistical mechanical information. In traditional statistical mechanics, Boltzmann applied relevant information regarding the gas molecules in a closed system at equilibrium, whereas Gibbs used information relevant to an open system at equilibrium. However, as Jaynes noted, both solutions can be shown to be special cases of the ME principle. It is important to emphasize that this method of maximizing entropy is true for assigning probabilities regardless of the information specifics [16]. As a consequence, the fact that Boltzmann and Gibbs used information relevant at equilibrium does not, in general, define entropy as only applying at equilibrium. Either of them could have instead included information that was time dependent as well. It is important to observe that Boltzmann included many assumptions into his solutions, such as assuming that all microstates were independent and had equal probability of occurring. The result of this was the interpretation that his probability assignments were for individual microstates. Gibbs, on the other hand, made no such assumptions. Therefore, his probability assignments described the system as a whole. This makes the Gibbs’ method more general. In fact, it has been shown that Gibbs’ entropy, not Boltzmann’s, is numerically equivalent to thermodynamic entropy [27].

Jaynes’ use of the method of ME for assigning probabilities is commonly known as MaxEnt [27]. This method has evolved to a more general method, the method of maximum (relative) entropy (MrE) [13–16], that has the advantage of not only assigning probabilities but updating them (we use the term ‘updating’ because as we gain valuable information, we ‘update’ to new probabilities) when new information is given in the form of constraints on the family of allowed posteriors. This is similar in function to Bayes theorem, but using macroscopic information (expectation values) as opposed to microscopic information (data). We point out that one of the drawbacks of the MaxEnt method was the inability to include data. Although we do not include information in the form of data in this paper, this problem
has been solved by showing that the MrE method can be used when either data, constraint information or both are present [28]. For our purposes here, we will proceed to describe the simple case of how MrE is used to update from one probability to another given macroscopic constraint information only. Our first concern when using the MrE method to update from a prior to a posterior distribution is to define the space in which the search for the posterior will be conducted. Normally, when using the MrE method, the updating is based on three pieces of information: prior information (the prior), the known or assumed relationship between the sets of microvariables and macrovariables (the model) and constraints on what the new model or posterior is allowed to be. Some good general examples of this are located in [29, 30]. However, for our purposes in the present work we follow the simpler approach where our space of inquiry is limited to one dimension of one or several quantities, \( X \in \mathcal{X} \).

We intend to maximize an appropriate entropy form,

\[
S[P, P_{\text{old}}] = -\int dX \, P(X) \ln \frac{P(X)}{P_{\text{old}}(X)},
\]

subject to the appropriate constraints where \( P(X) = P(x_1, x_2, \ldots, x_l) \), \( l \) is the dimensionality of the microspace \( \mathcal{X} \) and \( P_{\text{old}}(X) \) contains our prior information. We impose the usual normalization constraint,

\[
\int dX \, P(X) = 1,
\]

and include additional information about \( X \) in the form of a constraint on the expected value of some function \( f(X) \),

\[
\int dX \, P(X) f(X) = \langle f(X) \rangle = F.
\]

We proceed by maximizing (1) subject to constraints (2) and (3). The purpose of maximizing the logarithmic relative entropy \( S[P, P_{\text{old}}] \) is to determine the value of \( P(X) \) that is closest to \( P_{\text{old}}(X) \) given the normalization and information constraints. Using the Lagrange multipliers formalism, we set the variation of \( S[P, P_{\text{old}}] \) with respect to \( P \) to be equal to zero, to obtain

\[
\delta \left[ S[P, P_{\text{old}}] + \alpha \left( \int dX P(X) - 1 \right) + \beta \left( \int dX P(X) f(X) - F \right) \right] = 0.
\]

The quantities \( \alpha \) and \( \beta \) are Lagrange multipliers whose actual values are determined by the value of the constraints themselves. Substituting (1) into (4), we obtain (after some algebra)

\[
-\int dX \left\{ \ln \frac{P(X)}{P_{\text{old}}(X)} + 1 - \alpha - \beta f(X) \right\} \delta P(X) = 0, \quad \forall \delta P.
\]

Therefore, the terms inside the curly brackets in (4) must sum to zero, yielding

\[
P_{\text{new}}(X) = P_{\text{old}}(X)e^{\left[\alpha - \beta f(X)\right]}. \tag{6}
\]

In order to determine the Lagrange multipliers \( \alpha \) and \( \beta \), we substitute our solution (6) into the constraint equations (2) and (3), respectively. Substituting (6) into (2), we obtain

\[
\int dX \, P_{\text{old}}(X) e^{\beta f(X)} = e^{(1-\alpha)}. \tag{7}
\]

Then substituting (7) into (6) yields

\[
P_{\text{new}}(X) = P_{\text{old}}(X) \frac{e^{\beta f(X)}}{Z}.	ag{8}
\]

where

\[
Z = e^{(1-\alpha)} = \int dX e^{\beta f(X)} P_{\text{old}}(X). \tag{9}
\]

The Lagrange multiplier \( \beta \) is determined by first substituting (8) into (3),

\[
\int dX \left[ P_{\text{old}}(X) \frac{e^{\beta f(X)}}{Z} \right] f(X) = F. \tag{10}
\]

Now, \( \beta \) can be determined by rewriting (10) as

\[
\frac{\partial \ln Z}{\partial \beta} = F. \tag{11}
\]

Finally, the selected posterior \( P_{\text{new}}(X) \) has the expected canonical form

\[
P_{\text{new}}(X) = P_{\text{old}}(X) \frac{e^{\beta f(X)}}{Z}. \tag{12}
\]

When the expectation (3) is the first moment, the canonical form is clearly seen. However, if the second moment is used a normal distribution is produced, as in our example below. In conclusion, the ME method can be described as a general method to update from a prior distribution \( q(x) \) to a posterior distribution \( p(x) \) when new information becomes available. It is perhaps worth emphasizing that in this approach, entropy is a tool for reasoning that requires no interpretation in terms of heat, multiplicities, disorder, uncertainty or amount of information [16].

In what follows, we will use this inference methodology together with information-geometric methods to characterize the temporal complexity (chaoticity, dynamical stochasticity) of suitable statistical models of physical relevance.

3. On the information geometry of chaos

IGAC is an application of ED to complex systems of arbitrary nature. It is the information-geometric analogue of conventional geometrodynamical approaches [5, 19, 31–33] where the classical configuration space \( \Gamma_E \) is replaced by a statistical manifold \( \mathcal{M}_S \) (\( \mathcal{M}_S \) is defined below in (16)) with the additional possibility of considering chaotic dynamics arising from non-conformally flat metrics (the Jacobi metric is always conformally flat, instead). It is an information-geometric extension of the Jacobi geometrodynamics (the geometricization of a Hamiltonian system by transforming it into a geodesic flow [34]). The reformulation of dynamics in terms of a geodesic problem allows for the application of a wide range of well-known geometrical techniques to the investigation of the solution
space and properties of the equations of motion. The power of the Jacobi reformulation lies in the fact that all dynamical information is collected into a single geometric object (namely, the manifold on which geodesic flow is induced) in which all the available manifest symmetries of the system are retained. For example, the integrability of a system is connected with the existence of Killing vectors and tensors on this manifold. The sensitive dependence of trajectories on initial conditions, which is a key ingredient of chaos, can be investigated from the equation of geodesic deviation. In the Riemannian [31] and the Finslerian [32] (a Finsler metric is obtained from a Riemannian metric by relaxing the requirement that the metric be quadratic on each tangent space) geometrodynamical approach to chaos in classical Hamiltonian systems, an active field of research concerns the possibility of finding a rigorous relation among the sectional curvature, the Lyapunov exponents, and the Kolmogorov–Sinai (KS) dynamical entropy (i.e. the sum of positive Lyapunov exponents) [35]. Using information-geometric methods, we have investigated in some detail the aforementioned open research problem [4–6, 19–22].

IGAC arises as a theoretical framework to study chaos in informational geodesic flows describing physical, biological or chemical systems [4–6, 33]. The main goal of an ED model is that of inferring ‘macroscopic predictions’ in the absence of detailed knowledge of the microscopic nature of the arbitrary complex systems being considered. More explicitly, by ‘macroscopic prediction’ we mean knowledge of the statistical parameters (expectation values) of the probability distribution function that best reflects what is known about the system. This is an important conceptual point. The probability distribution reflects the system in general, not the microstates. We then select the relevant information about the system. In other words, we have to select the macrospace of the system. It is worth mentioning that the coexistence of macroscopic and microscopic dynamics for a given physical (biological, chemical) system from a dynamical and statistical point of view has always been a very important subject of investigation [36]. For example, in certain fluid systems showing the Rayleigh–Benard convection [37], the macroscopic chaotic behavior (macroscopic chaos) is a manifestation of the underlying molecular interaction of a very large collection of molecules (microscopic or molecular chaos). Moreover, macroscopic chaos arising from an underlying microscopic chaotic molecular behavior has been observed in chemical reactions. In order to study the underlying dynamics in rate equations of chemical reactions, a mesoscopic description has been adopted which is given by a set of transition probabilities among chemicals. In such a description, the underlying dynamics of macroscopic motion is that of stochastic processes and the evolution of the probability distribution associated with each chemical is investigated [33].

In what follows, we schematically outline the main features underlying the construction of an arbitrary form of ED. First, the microstates of the system under investigation must be defined. For the sake of simplicity, we assume the system is characterized by an \( l \)-dimensional microspace \( X \) with microstates \( X \equiv \{x_1, \ldots, x_l\} \). The quantities \( x_k \) with \( k = 1, 2, \ldots, l \) denote the degrees of freedom of the microstates of the system. We assume that each degree of freedom \( x_k \) is subject to \( n \)-information constraints. The \( n \)-information constraints may (although not necessarily) correspond to the moments of \( P(X|\Theta) \). For example, if we consider a system in which the relevant information constraints are given by the first two moments of \( P(X|\Theta) \), namely the expectation values \( \mu_k \) and variances \( \sigma_k \),

\[
\langle x_k \rangle = \mu_k \quad \text{and} \quad \langle (x_k - \langle x_k \rangle)^2 \rangle^{1/2} = \sigma_k,
\]

then the resulting posterior probability distribution function is a Gaussian distribution \( p_k(x_k|\mu, \sigma) \). For general purposes, it is convenient to define an \( nl \)-dimensional macroscopic vector \( \Theta \) with statistical coordinates \( \theta_k^{(m)} \) (where \( k = 1, 2, \ldots, l \) labels the microstates and \( m = 1, 2, \ldots, n \) enumerates the information constraints). The set \( \Theta \) defines the \( nl \)-dimensional space of macrostates of the system, namely the parameter space \( \Theta_0 \). In the \( l \)-dimensional Gaussian case, the vector \( \Theta \) is \( nl \)-dimensional and has coordinates \( (\mu_1, \mu_2, \ldots, \mu_l; \sigma_1, \ldots, \sigma_l) \).

At this point, we make a working hypothesis that the microstates are statistically independent. Then, in addition to information constraints, each distribution \( p_k(x_k|\theta_k^{(m)}) \) of each degree of freedom \( x_k \) must satisfy the usual normalization conditions

\[
\int dx_k p_k(x_k|\theta_k^{(m)}) = 1.
\]

Once the microstates have been defined and the relevant (linear or nonlinear) information constraints selected, we are left with a set of \( l \)-dimensional vector probability distributions

\[
P(X|\Theta) = \prod_{k=1}^{l} p_k(x_k|\theta_k^{(m)})
\]

(assuming statistical independence, identically distributed microvariables and assuming as an additional working hypothesis that the prior probability distribution is uniform) encoding the relevant available information about the system.

The statistical manifold \( M_S \),

\[
M_S = \left\{ P(X|\Theta) = \prod_{k=1}^{l} p_k(x_k|\theta_k^{(m)}) \right\},
\]

is defined as the set of probabilities \( \{ P(X|\Theta) \} \) described above with \( X \in \mathbb{R}^l, \Theta \in \Theta_0 \), where \( \Theta_0 \) is comprised of the \( n \)-parameter sub-spaces corresponding to each of the \( n \)-information constraint according to \( \Theta_0 = \bigotimes_{m=1}^{n} T^{(m)}(\theta_k^{(m)}) \). Each point of the geodesic on an \( nl \)-dimensional statistical manifold \( M_S \) represents a macrostate \( \Theta \) parameterized by the macroscopic dynamical variables \( \{\theta_k^{(m)}\} \) defining the macrostate of the system. Furthermore, each macrostate is in one-to-one correspondence with the probability distribution \( P(X|\Theta) \) representing the maximally probable description of the system being considered. Thus, the set of macrostates forms the parameter space \( \Theta_0 \), while the set of probability distributions forms the statistical manifold \( M_S \). Referring to our Gaussian example above, \( \Theta_0 \) is the direct product of the parameter sub-spaces \( \Theta_p \) (corresponding to the first moment \( n = 1 \) of the Gaussian, i.e. the expectation \( \mu \)) and \( \Theta_\sigma \) (corresponding to the second moment \( n = 2 \) of the Gaussian, \( \sigma \)).
i.e. the variance $\sigma$), where $I_{\mu} = (-\infty, +\infty)_\mu$ and $I_\sigma = (0, +\infty)_\sigma$.

A measure of distinguishability among macrostates is obtained by assigning a probability distribution $P(X|\Theta) \equiv \mathcal{M}_S$ to each macrostate $\Theta$. Assignment of a probability distribution to each state endows $\mathcal{M}_S$ with a metric structure.

Specifically, the Fisher–Rao information metric $g_{ab}(\Theta)$ [17],

\[ g_{ab}(\Theta) = \int dX P(X|\Theta) \partial_a \ln P(X|\Theta) \partial_b \ln P(X|\Theta); \]

\[ a, b = 1, \ldots, nl \] and $\partial_a = \frac{\partial}{\partial \varphi^a}$, \quad (17)

defines a measure of distinguishability among macrostates on $\mathcal{M}_S$.

It is known from IG [17] that there is a one-to-one relation between elements of the statistical manifold $\mathcal{M}_S$ and the parameter space $I_{\Theta_0}$. Specifically, the statistical manifold $\mathcal{M}_S$ is homeomorphic to the parameter space $I_{\Theta_0}$. This implies the existence of a continuous, bijective map $h_{\mathcal{M}_S, I_{\Theta_0}}$,

\[ h_{\mathcal{M}_S, I_{\Theta_0}} : \mathcal{M}_S \ni P(X|\Theta) \to \Theta \in I_{\Theta_0}, \quad (18) \]

where $h_{\mathcal{M}_S, I_{\Theta_0}}^{-1}(\Theta) = P(X|\Theta)$. The inverse image $h_{\mathcal{M}_S, I_{\Theta_0}}^{-1}$ is the so-called homeomorphism map. It is worth pointing out that the possible chaotic behavior of the set of macrostates $\Theta$ is strictly related to the selected relevant information about the set of microstates $X$ of the system. In other words, the assumed Gaussian characterization of the degrees of freedom $\{x_j\}$ of each macrostate of the system has deep consequences for the macroscopic behavior of the system itself. More generally, within our theoretical construct, ‘the macroscopic behavior of an arbitrary complex system is a consequence of the underlying statistical structure of the macroscopic degrees of freedom of the system being considered’.

It should be noted that coupled constraints would lead to a ‘generalized’ product rule in (15) and to a metric tensor (17) with non-trivial off-diagonal elements (covariance terms). In the presence of correlated degrees of freedom $\{x_j\}$, the ‘generalized’ product rule becomes

\[ P_{tot}(x_1, \ldots, x_n) = \prod_{j=1}^n P_j(x_j), \quad (19) \]

where

\[ P_{tot}'(x_1, \ldots, x_n) \equiv P_n(x_n|x_1, \ldots, x_{n-1}) \]

\[ P_{n-1}(x_{n-1}|x_1, \ldots, x_{n-2}) \cdots P_2(x_2|x_1) P_1(x_1) \quad (20) \]

For instance, correlations in the degrees of freedom may be introduced in terms of the following information-constraints:

\[ x_j = f_j(x_1, \ldots, x_{j-1}), \quad \forall j = 2, \ldots, n. \quad (21) \]

In such a case, we obtain

\[ P_{tot}'(x_1, \ldots, x_n) = \delta(x_n - f_n(x_1, \ldots, x_{n-1})) \delta \times (x_{n-1} - f_{n-1}(x_1, \ldots, x_{n-2})) \cdots \delta(x_2 - f_2(x_1)) P_1(x_1), \quad (22) \]

where the $j$th probability distribution $P_j(x_j)$ is given by

\[ P_j(x_j) = \int \cdots \int dx_1 \cdots dx_{j-1} dx_{j+1} \cdots dx_n P_{tot}'(x_1, \ldots, x_n). \quad (23) \]

Correlations between the microscopic degrees of freedom of the system $\{x_j\}$ (microcorrelations) are conventionally introduced by means of the correlation coefficients $r_{ij}^{(micro)}$,

\[ r_{ij}^{(micro)} = r(x_i, x_j) \equiv \frac{\langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle}{\sigma_i \sigma_j}; \quad (24) \]

with $r_{ij}^{(micro)} \in (-1, 1)$ and $i, j = 1, \ldots, n$. For the $2n$-dimensional Gaussian statistical model in the presence of microcorrelations, the system is described by the following probability distribution $P(X|\Theta)$:

\[ P(X|\Theta) = \frac{1}{\gamma_0 2 \pi^{n/2} C(\Theta)^{1/2}} \exp \left[ -\frac{1}{2} (X - M)^T C^{-1}(\Theta) (X - M) \right] \]

\[ \neq \int \cdots \int \frac{1}{(2\pi \sigma_j^2)^{n/2}} \exp \left[ -\frac{(x_j - \mu_j)^2}{2\sigma_j^2} \right] \quad (25) \]

where $X = (x_1, \ldots, x_n)$, $M = (\mu_1, \ldots, \mu_n)$ and $C(\Theta)$ is the $(2n \times 2n)$-dimensional (non-singular) covariance matrix.

Once $\mathcal{M}_S$ and $I_{\Theta_0}$ are defined, the ED formalism provides the tools to explore dynamics driven on $\mathcal{M}_S$ by entropic arguments. Specifically, given a known initial macrostate $\Theta^{(initial)}$ (probability distribution) and that the system evolves to a final known macrostate $\Theta^{(final)}$, the possible trajectories of the system are examined in the ED approach using ME methods. We emphasize that ED can be derived from a standard principle of least action (Maupertuis–Euler–Lagrange–Jacobi type) [11, 38].

The main differences are that the dynamics being considered here, namely entropic dynamics, is defined on a space of probability distributions $\mathcal{M}_S$, not on an ordinary linear space $V$, and the standard coordinates $q_a$ of the system are replaced by statistical macrovariables $\varphi^a$. The geodesic equations for the macrovariables of the Gaussian ED model are given by nonlinear second order coupled ordinary differential equations

\[ \frac{d^2 \varphi^a}{dt^2} + \Gamma^a_{bc} \frac{d \varphi^b}{dt} \frac{d \varphi^c}{dt} = 0. \quad (26) \]

The geodesic equations in (26) describe reversible dynamics whose solution is the trajectory between an initial $\Theta^{(initial)}$ and a final macrostate $\Theta^{(final)}$. The trajectory can be traversed equally well in both directions. A geodesic on a curved statistical manifold $\mathcal{M}_S$ represents the maximum probability path a complex dynamical system explores in its evolution between initial and final macrostates $\Theta^{(initial)}$ and $\Theta^{(final)}$, respectively.

We point out that this path is obtained by the use of a principle of probable inference, namely the maximum relative entropy method. Our theoretical formalism allows us to analyze important physics problems through statistical inference and information-geometric techniques.
4. Information geometric measures of temporal complexity

In this section, mainly following one of our previous works [22], we introduce the relevant indicators of chaoticity (temporal complexity, dynamical stochasticity; in general there is no one-to-one relation between chaos and complexity; chaos may imply complexity but necessarily vice versa) within our theoretical formalism. Specifically, once the Fisher–Rao information metric is given, we apply standard methods of Riemannian differential geometry to study the information-geometric structure of the manifold $\mathcal{M}_S$ underlying the ED. The connection coefficients defined in the standard manner as

$$\Gamma_{abc}^d$$

underlying the ED. The connection coefficients defined in the standard manner as $\Gamma_{abc}^d$ Ricci tensor $\mathcal{R}_{abc}$, Riemannian curvature tensor $\mathcal{R}_{abcd}$, sectional curvatures $K_{\mathcal{M}_S}$ [39], scalar curvature $K_{\mathcal{M}_S}$, Weyl anisotropy tensor $\mathcal{W}_{abcd}$ (the anisotropy of the manifold underlying the system dynamics plays a significant role in the mechanism of instability), the Jacobi vector field intensity $\mathcal{J}_{\mathcal{M}_S}$, the information-geometrodynamical entropy (IGE) $S_{\mathcal{M}_S}$ can be calculated in the standard manner [21, 31, 32].

In order to characterize the chaotic behavior of complex entropic dynamical systems, we are primarily concerned with the signs of the scalar and sectional curvatures $K_{\mathcal{M}_S}$ of $\mathcal{M}_S$, the asymptotic behavior of Jacobi fields $\mathcal{J}_a$ on $\mathcal{M}_S$, the existence a non-vanishing Weyl anisotropy tensor $\mathcal{W}_{abcd}$ and the asymptotic behavior of the IGE $S_{\mathcal{M}_S}$. It is crucial to observe that true chaos is identified by the occurrence of two features [32]: (i) strong dependence on initial conditions and exponential divergence of the Jacobi vector field intensity, i.e. stretching of dynamical trajectories; (ii) compactness of the configuration space manifold, i.e. folding of dynamical trajectories.

4.1. Sectional and scalar curvatures

Once the Fisher–Rao information metric $g_{ab}$ is given, we use standard differential geometry methods applied to the space of probability distributions to characterize the geometric properties of $\mathcal{M}_S$. Recall that the Ricci scalar curvature $\mathcal{R}$ is given by

$$\mathcal{R} = g^{ab} \mathcal{R}_{ab},$$

(27)

where $g^{ab} g_{bc} = \delta^a_c$ so that $g^{ab} = (g_{ab})^{-1}$. The Ricci tensor $\mathcal{R}_{ab}$ is given by

$$\mathcal{R}_{ab} = \partial_a \Gamma^c_{bc} - \partial_b \Gamma^c_{ac} + \Gamma^e_{ac} \Gamma^d_{bd} - \Gamma^d_{ad} \Gamma^e_{bd}.$$  

(28)

The Christoffel symbols $\Gamma^e_{ab}$ appearing in the Ricci tensor are defined in the standard manner as

$$\Gamma^e_{ab} = \frac{1}{2} g^{ec} (\partial_a g_{bd} + \partial_b g_{ad} - \partial_d g_{ab}).$$

(29)

It can be shown that the Ricci scalar curvature can be written as

$$\mathcal{R}_{\mathcal{M}_S} = \mathcal{R}_{abcd} g^{ac} g^{bd} = \mathcal{R}_a = \sum_{i \neq j} \mathcal{K}(e_i, e_j).$$

(30)

The scalar curvature is the sum of all sectional curvatures $\mathcal{K}(e_i, e_j)$ of planes spanned by pairs of orthonormal basis elements $\{e_i = \partial_{\gamma_i(p)}\}$ of the tangent space $T_p \mathcal{M}_S$ with $p \in \mathcal{M}_S$ [39].

$$\mathcal{K}(u, v) = \frac{\mathcal{R}_{abcd} u^a v^b w^d v^d}{(g_{ac} g_{bd} - g_{ad} g_{bc}) u^a v^b u^c v^d};$$

$$u \rightarrow h^i, v \rightarrow h^j \text{ with } i \neq j,$$

(31)

where $\langle e_a, h^b \rangle = \delta_a^b$. Note that the sectional curvatures completely determine the curvature tensor.

The negativity of the Ricci scalar $\mathcal{R}_{\mathcal{M}_S}$ implies the existence of expanding directions in the configuration space manifold $\mathcal{M}_S$. Indeed, since $\mathcal{R}_{\mathcal{M}_S}$ is the sum of all sectional curvatures of planes spanned by pairs of orthonormal basis elements $\{e_a = \partial_{\gamma_a}\}$, the negativity of the Ricci scalar is only a sufficient (not necessary) condition for local instability of geodesic flow. For this reason, the negativity of the scalar provides a strong criterion of local instability. Scenarios may arise where negative sectional curvatures are present, but the positive ones could prevail in the sum so that the Ricci scalar is non-negative despite the instability in the flow in those directions. Consequently, the signs of $K_{\mathcal{M}_S}$ are of crucial significance for the proper characterization of chaos.

4.2. Killing vectors

Yet another useful way of understanding the anisotropy of the $\mathcal{M}_S$ is the following. It is known that in $N$ dimensions, there are at most $\frac{N(N+1)}{2}$ independent Killing vectors (directions of symmetry of the manifold). Since $\mathcal{M}_S$ is not a pseudosphere, the information metric tensor does not admit the maximum number of Killing vectors $K_a$ defined as

$$\mathcal{L}_K g_{ab} = \mathcal{D}_a K_b + \mathcal{D}_b K_a = 0,$$

(32)

where $\mathcal{D}_a$, defined as

$$\mathcal{D}_a K_b = \partial_a K_b - \Gamma^c_{ba} K_c,$$

(33)

is the covariant derivative operator with respect to the connection $\Gamma$ defined in (29). The Lie derivative $\mathcal{L}_K (g_{ab})$ of the tensor field $g_{ab}$ along a given direction $K$ measures the intrinsic variation of the field along that direction (i.e. the metric tensor is Lie transported along the Killing vector) [40].

Locally, a maximally symmetric space of Euclidean signature is a plane, a sphere or a hyperboloid, depending on the sign of $\mathcal{R}$. In our case, none of these scenarios occur. As will be seen in what follows, this fact has a significant impact on the integration of the geodesic deviation equation on $\mathcal{M}_S$. At this juncture, we emphasize that it is known that the anisotropy of the manifold underlying system dynamics plays a crucial role in the mechanism of instability. In particular, fluctuating sectional curvatures require also that the manifold be anisotropic. However, the connection between curvature variations along geodesics and anisotropy is far from clear [22]. Krylov was the first to emphasize [41] the use of $\mathcal{R} < 0$ as an instability criterion in the context of an $N$-body system (a gas) interacting via van der Waals forces, with the ultimate hope to understand the relaxation process in a gas. However, Krylov neglected the problem of compactness of the configuration space manifold which is important for making inferences about exponential mixing of geodesic
flows [42]. Compactness [43, 44] is required in order to discard trivial exponential growths due to the unboundedness of the ‘volume’ available to the dynamical system. In other words, the folding needs to have a dynamics that is actually able to mix the trajectories, making it practically impossible, after a finite interval of time, to discriminate between trajectories that were very near each other at the initial time.

When the space is not compact, even in the presence of strong dependence on initial conditions, it could be possible in some instances (although not always) to distinguish between different trajectories originating within a small distance and then evolving subject to exponential instability. As a final remark, we note that since homeomorphisms preserve compactness and since $M_0$ and $I_0$ are homeomorphic, it is sufficient to show that the parameter space $I_0$ is compact in order to ensure that the corresponding submanifold of $M_0$ is itself compact.

4.3. Jacobi fields

A powerful mathematical tool to investigate the stability or instability of a geodesic flow is the Jacobi–Levi-Civita equation (JLC equation) for geodesic spread [31]. The JLC equation covariantly describes how nearby geodesics locally scatter, and relates the stability or instability of a geodesic flow with curvature properties of the ambient manifold. For the sake of clarity, consider the behavior of a family of neighboring geodesics $\{\theta^a_{M_0}(\tau; \xi)\}_{\xi \in \mathbb{R}^{n_0 \times N}}$ on a statistical manifold $M_0$ with $\dim_\mathbb{R} M_0 = N$. The geodesics $\theta^a_{M_0}(\tau; \xi)$ are solutions of equation (26). The relative geodesic spread on a non-maximally symmetric curved manifold as $M_0$ is characterized by the JLC equation [39, 45]

$$\frac{D^2 J^a}{D\tau^2} + R_{\mu \nu \alpha b} \frac{\partial \theta^b}{\partial \tau} J^\mu \frac{\partial \theta^\nu}{\partial \tau} J^\alpha = 0,$$

(34)

where the covariant derivative $\frac{D^2 J^a}{D\tau^2}$ in (34) is defined as [46]

$$\frac{D^2 J^a}{D\tau^2} = \frac{d^2 J^a}{d\tau^2} + 2\Gamma_{bc}^a \frac{d\theta^b}{d\tau} \frac{d\theta^c}{d\tau} J^d + \Gamma_{bc}^d \frac{d\theta^a}{d\tau} \frac{d\theta^b}{d\tau} J^d + \Gamma_{bc}^d \frac{d\theta^d}{d\tau} \frac{d\theta^a}{d\tau} J^b + \Gamma_{bc}^d \frac{d\theta^d}{d\tau} \frac{d\theta^b}{d\tau} J^c,$$

(35)

and the Jacobi vector field $J^a$ is given by [47]

$$J^a = \delta \xi \delta^a = \left. \frac{\partial \theta^a}{\partial \xi^b} \right|_{\tau} \delta \xi^b.$$

(36)

Equation (34) forms a system of $N$ coupled ordinary differential equations linear in the components of the deviation vector field (36) but nonlinear in derivatives of the metric (17). It describes the linearized geodesic flow: the linearization ignores the relative velocity of the geodesics. When the geodesics are neighboring but their relative velocity is arbitrary, the corresponding geodesic deviation equation is the so-called generalized Jacobi equation [48]. The nonlinearity is due to the existence of velocity-dependent terms in the system. Neighboring geodesics accelerate relative to each other with a rate directly measured by the curvature tensor $R_{abcd}$. The non-trivial integration of (34) leads to the following expression for the Jacobi vector field intensity:

$$J_{M_0} \equiv \| J^a \| \overset{\text{def}}{=} (g_{ab}J^a J^b)^{1/2}.$$

(37)

For the application of the asymptotic temporal behavior of $J_{M_0}(\tau)$ as a reliable indicator of chaoticity, see our previous works [5, 19, 20]. The geodesic spread on $M_0$ is described by means of an exponentially divergent Jacobi vector field intensity $J_{M_0}$, a classical feature of chaos. In our approach, inspired by the work presented in [49], the quantity $\lambda_{M_0}$, defined as

$$\lambda_{M_0} \overset{\text{def}}{=} \lim_{\tau \to \infty} \left[ \frac{1}{\tau} \ln \left( \frac{\| J_{M_0}(\tau) \|^2 + \| J_{M_0}(\tau) \|^2}{\| J_{M_0}(0) \|^2 + \| J_{M_0}(0) \|^2} \right) \right],$$

(38)

characterizes the exponential growth rate of average statistical volumes (see (55)) in $M_0$. This suggests that $\lambda_{M_0}$ may play the same role as the standard Lyapunov exponent [50]. Lyapunov exponents are asymptotic quantities since they are defined in the limit as time approaches infinity.

4.4. The Weyl projective curvature tensor

The Weyl projective curvature tensor [51] (or the anisotropy tensor) $\mathcal{W}_{abcd}$ is defined as

$$\mathcal{W}_{abcd} = R_{abcd} - \frac{R_{M_0}}{N(N-1)} (g_{bd} g_{ac} - g_{bc} g_{ad}),$$

(39)

where $N = n l$ is the dimension of the curved manifold. In (39), the quantity $R_{abcd}$ is the Riemann curvature tensor defined in the usual manner by

$$R_{abcd} = \partial_a \Gamma^a_{bc} - \partial_b \Gamma^a_{ac} + \Gamma^a_{bd} \Gamma^d_{ac} - \Gamma^a_{cd} \Gamma^d_{ac}.$$

(40)

Considerations regarding the negativity of the Ricci curvature as a strong criterion of dynamical instability and the necessity of compactness of $M_0$ in ‘true’ chaotic dynamical systems require further investigation.

4.5. Information geometric complexity

Once the distances among probability distributions have been assigned, a natural next step is to obtain measures for extended regions in the space of distributions. Consider an $N$-dimensional volume of the statistical manifold $M_0$ of distributions $P(X(\theta))$ labelled by parameters $\theta^a$ with $a = 1, \ldots, N$. The parameters $\theta^a$ are coordinates for the point $P$ and in these coordinates it may not be obvious how to write an expression for a volume element $dV_{M_0}$. However, within a sufficiently small region (volume element) any curved space looks flat. Curved spaces are 'locally flat'. The idea then is rather simple: within that very small region, we should use Cartesian coordinates and the metric takes a very simple form, namely the identity matrix $\delta_{ab}$. In locally Cartesian coordinates $x^a$ the volume element is simply given by the product

$$dV_{M_0} = dx^1 dx^2 \cdots dx^N,$$

(41)

which, in terms of the old coordinates, is

$$dV_{M_0} = \left. \frac{\partial x}{\partial \theta} \right|_{\theta^a} d\theta^1 d\theta^2 \cdots d\theta^N = \left. \frac{\partial x}{\partial \theta} \right|_{\theta^a} d^N \theta.$$

(42)

The problem at hand is the calculation of the Jacobian $\left. \frac{\partial x}{\partial \theta} \right|_{\theta^a}$ of the transformation that takes the metric $g_{ab}$ into its Euclidean form $\delta_{ab}$.
Let the new coordinates be defined by \( \chi^{\prime} = \mathbb{E}^N (\theta^1, \ldots, \theta^N) \). A small change \( d\Theta \) corresponds to a small change \( d\chi \),
\[
d\chi^\prime = X^\prime_a d\theta^a, \quad \text{where} \quad X^\prime_a \equiv \frac{\partial \chi^\prime}{\partial \theta^a}.
\]
and the Jacobian is given by the determinant of the matrix \( X^\prime_a \),
\[
\left| \frac{\partial \chi}{\partial \Theta} \right| = \left| \det \left( X^\prime_a \right) \right|.
\]
The distance between two neighboring points is the same whether we compute it in terms of the old or the new coordinates,
\[
dl^2 = g_{ab} d\theta^a d\theta^b = \delta_{ab} d\chi^a d\chi^b.
\]
Therefore, the relation between the old and the new metric is
\[
g_{ab} = \delta_{ab} X^\prime_a X^\prime_b.
\]
Taking the determinant of (46), we obtain
\[
g \equiv \det \left( g_{ab} \right) = \left[ \det \left( X^\prime_a \right) \right]^2
\]
and therefore
\[
\left| \det \left( X^\prime_a \right) \right| = \sqrt{g}.
\]
Finally, we have succeeded in expressing the volume element totally in terms of the coordinates \( \Theta \) and the known metric \( g_{ab} (\Theta) \),
\[
dV_{M_\Theta} = \sqrt{g} dV = \sqrt{g} d^N \Theta.
\]
The volume of any extended region on the manifold is given by
\[
V_{M_\Theta} = \int dV_{M_\Theta} = \int \sqrt{g} d^N \Theta.
\]
Observe that \( \sqrt{g} d^N \Theta \) is a scalar quantity and is therefore invariant under general coordinate transformations \( \Theta \rightarrow \Theta^\prime \), preserving orientation. The square root of the metric tensor transforms according to
\[
\sqrt{g (\Theta^\prime)} = \frac{\partial \Theta^\prime}{\partial \Theta} \sqrt{g (\Theta)}.
\]
and the flat infinitesimal volume element \( d^N \Theta \) transforms as
\[
d^N \Theta \rightarrow \left| \frac{\partial \Theta}{\partial \Theta^\prime} \right| d^N \Theta^\prime.
\]
Thus, from (51) and (52) we obtain
\[
\sqrt{g (\Theta)} d^N \Theta \rightarrow \sqrt{g (\Theta^\prime)} d^N \Theta^\prime.
\]
Equation (53) implies that the infinitesimal statistical volume element is invariant under general coordinate transformations that preserve orientation, that is with positive Jacobian.

The volume of an extended region \( \Delta V_{M_\Theta} (\tau; \tilde{\varsigma}) \) of \( \mathcal{M}_S \) is defined by
\[
\Delta V_{M_\Theta} (\tau; \tilde{\varsigma}) \equiv V_{M_\Theta} (\tau; \tilde{\varsigma}) - V_{M_\Theta} (0; \tilde{\varsigma}) \equiv \int_{\Theta(0; \tilde{\varsigma})}^{\Theta(\tau; \tilde{\varsigma})} \sqrt{g} d^N \Theta,
\]
where \( \Theta(\tau; \tilde{\varsigma}) \) are solutions of the geodesic equations (26) and \( \tilde{\varsigma} = (\varsigma^1, \ldots, \varsigma^N) \) is the quantity parameterizing the family of geodesics \( \{ \Theta^a (\tau; \tilde{\varsigma}) \}_{\tau \in \mathbb{R}^{+}} \). The quantity that encodes relevant information about the stability of neighboring volume elements is the average volume (information-geometric complexity) \( C_{M_\Theta} (\tau; \tilde{\varsigma}) \) defined as [19]
\[
C_{M_\Theta} (\tau; \tilde{\varsigma}) \equiv \left( \Delta V_{M_\Theta} (\tau; \tilde{\varsigma}) \right)_{\tau \rightarrow \infty} = \frac{1}{\tau} \int_0^\tau \Delta V_{M_\Theta} \left( \tau^\prime; \tilde{\varsigma} \right) d\tau^\prime.
\]
We will term as \( C_{M_\Theta} (\tau; \tilde{\varsigma}) \) the information-geometric complexity of the maximally probable trajectories \( \Theta (\tau; \tilde{\varsigma}) \). The IGC \( C_{M_\Theta} (\tau; \tilde{\varsigma}) \) in (55) represents the temporal average of the \( N \)-fold integral over maximum probability trajectories (geodesics) and serves as a measure of the number of the accessible macrostates in the configuration (statistical) manifold \( \mathcal{M}_S \) after a finite temporal increment \( \tau \). In other words, \( C_{M_\Theta} (\tau; \tilde{\varsigma}) \) can be interpreted as the temporal evolution of the system’s uncertainty volume \( C_{M_\Theta} (0; \tilde{\varsigma}) \). For instance, \( C_{M_\Theta} (0; \tilde{\varsigma}) \) may be a spherical volume of initial points whose center is a given point on an attractor and whose surface consists of configuration points from nearby trajectories. An attractor is a subset of the manifold \( \mathcal{M}_S \) toward which almost all sufficiently close trajectories converge asymptotically, covering it densely as time goes on. Strange attractors are called chaotic attractors. Chaotic attractors have at least one finite positive Lyapunov exponent [52]. As the center of \( C_{M_\Theta} (0; \tilde{\varsigma}) \) and its surface points evolve in time, the spherical volume becomes an ellipsoid with principal axes in the directions of contraction and expansion. The average rates of expansion and contraction along the principal axes are the Lyapunov exponents [53].

### 4.6. Information geometrodynamical entropy

Finally, the asymptotic regime of diffusive evolution describing the possible exponential increase of average volume elements on \( \mathcal{M}_S \) provides another useful indicator of dynamical chaoticity. The exponential instability characteristic of chaos forces the system to rapidly explore large areas (volumes) of \( \mathcal{M}_S \). We remark that the exponential instability does not necessarily imply that trajectories explore large areas (volumes) of \( \mathcal{M}_S \). The folding mechanism is responsible for a dense exploration of areas (volumes) that may be small in principle. It is interesting to note that this asymptotic behavior appears also in the conventional description of quantum chaos where the von Neumann entropy increases linearly at a rate determined by the Lyapunov exponents. The linear increase of entropy as a quantum chaos criterion was introduced by Zurek and Paz [54]. In our information-geometric approach a relevant quantity that may be useful in studying the degree of instability characterizing ED models is the IGE defined as [4–6, 19–22]
\[
\mathcal{S}_{M_\Theta} (\tau; \tilde{\varsigma}) = \mathcal{S}_{M_\Theta} [\Theta (\tau; \tilde{\varsigma}), \Theta (0; \tilde{\varsigma})]
\]
\[
= \lim_{\tau \rightarrow \infty} \left\{ \frac{1}{\tau} \int_0^\tau d\tau^\prime \left[ \int_{\Theta(0; \tilde{\varsigma})}^{\Theta(\tau; \tilde{\varsigma})} \sqrt{g} d^N \Theta \right] \right\},
\]
where \( g = |\text{det}(g_{ab})| \). More synthetically,

\[
S_{M_\lambda}(\tau; \vec{z}) \overset{\text{def}}{=} \lim_{\tau \to \infty} \ln C_{M_\lambda}(\tau; \vec{z})
\]

with \( C_{M_\lambda}(\tau; \vec{z}) \) being the information-geometric complexity defined in (55). The IGE is intended to capture the temporal complexity (chaoticity) of ED models on curved statistical manifolds \( M_\lambda \) by considering the asymptotic temporal behaviors of the average statistical volumes occupied by the evolving macrovariables labelling points on \( M_\lambda \).

5. Applications

In this section, we present five applications of the IGAC. Firstly, we study the chaotic behavior of an ED Gaussian model describing an arbitrary system of \( l \) uncorrelated degrees of freedom and show that the hyperbolicity of the non-maximally symmetric 2\( l \)-dimensional statistical manifold \( M_\lambda \) underlying such an ED Gaussian model leads to linear IGE growth and to exponential divergence of the Jacobi vector field intensity [22]. Secondly, we study the asymptotic behavior of the dynamical complexity of the maximum probability trajectories on Gaussian statistical manifolds in the presence of correlation-like terms between the macrovariables labeling the macrostates of the system under investigation. In the presence of correlation-like terms, we observe a power-law decay of the information-geometric complexity at a rate determined by the correlation coefficient [23]. We also present an information-geometric analogue of the Zurek–Paz quantum chaos criterion of linear entropy growth [54]. This analogy is presented by studying the information geometrodynamics of an ensemble of macroscopic, random frequency, IHOs [6, 20]. Next, we apply the IGAC to study the ED on curved statistical manifolds induced by classical probability distributions in common use in the study of regular and chaotic quantum energy level statistics. In doing so, we suggest an information-geometric characterization of regular and chaotic quantum energy level statistics [4, 5]. Finally, we apply the IGAC to characterize the quantum entanglement produced by a head-on collision between two Gaussian wave packets interacting via a scattering process [24, 55].

5.1. Gaussian statistical model in the absence of correlations

As a first example, we apply the IGAC to study the dynamics of a system with \( l \) degrees of freedom, each one described by two pieces of relevant information, its mean expected value and its variance (Gaussian statistical macrostates). The line element \( ds^2 = g_{ab}(\Theta) d\Theta^a d\Theta^b \ (a, b = 1, \ldots, 2l) \) on \( M_\lambda \) is defined by [22]

\[
ds^2 = \sum_{k=1}^{2l} \left( \frac{1}{\sigma_k^2} d\mu_k^2 + \frac{2}{\sigma_k^2} d\sigma_k^2 \right).
\]

This leads us to consider an ED model on a non-maximally symmetric 2\( l \)-dimensional statistical manifold \( M_\lambda \). The manifold \( M_\lambda \) possesses a constant negative Ricci curvature that is proportional to the number of degrees of freedom of the system, \( R_{M_\lambda} = -l \). The system explores statistical volume elements on \( M_\lambda \) at an exponential rate, while the IGE \( S_{M_\lambda} \) increases linearly in time (statistical evolution parameter) and is proportional to the number of degrees of freedom of the system, \( S_{M_\lambda} \sim l \). The parameter \( \lambda \) characterizes the family of probability distributions on \( M_\lambda \). For the case being considered here, \( \lambda \) does indeed play the role of the standard Lyapunov exponent [50].

Recall that the finite Lyapunov exponent in the direction \( v \in \mathbb{R}^{2l} \) of a trajectory \( \Theta(\tau, \Theta_0) \) satisfying the differential equation \( \dot{\Theta} = A(\tau) \Theta \) with \( \Theta(0, \Theta_0) = \Theta_0 \) is defined as [56]

\[
\lambda(v) \overset{\text{def}}{=} \lim_{\tau \to \infty} \ln \left[ \frac{\sqrt{\langle \dot{\Theta}^T \Theta \rangle}}{\sqrt{\langle \dot{\Theta} \dot{\Theta} \rangle}} \right].
\]

The brackets \( \langle \cdot , \cdot \rangle \) in (59) denote the standard scalar product in \( \mathbb{R}^{2l} \) and \( \dot{\Theta} = \dot{\Theta}^T \Theta \Theta_0 \) is the asymptotically regular fundamental matrix of the differential equation \( \dot{\Theta} = A(\tau) \Theta \) [57]. For instance, in the 2\( l \)-dimensional Gaussian statistical model considered here, the set of differential equations to consider is

\[
\frac{d\Theta(\tau)}{d\tau} = A(\tau) \Theta(\tau) = 0,
\]

where \( \Theta(\tau) \) is the \( 2l \)-dimensional vector \( \Theta(\tau) \equiv (\mu_1(\tau), \ldots, \mu_l(\tau); \sigma_1(\tau), \ldots, \sigma_l(\tau)) \) whose components are solutions of geodesic equations describing the evolution of the macrostates of the system [22]. In the asymptotic limit, the \( 2l \times 2l \) matrix \( A(\tau) \) can be approximated by a diagonal matrix with constant coefficients, \( A(\tau) \approx \text{diag}(0, \lambda_1, 0, \lambda_2, \ldots, 0, \lambda_l) \). A straightforward calculation leads to an asymptotically regular \( 2l \times 2l \) fundamental matrix

\[
\dot{\Theta}(\tau) \approx \text{diag}(c_1 \tau, c_2 \exp(\lambda_1 \tau), c_3 \tau, c_4 \exp(\lambda_2 \tau), \ldots, c_l \exp(\lambda_l \tau)),
\]

with \( c_i \in \mathbb{R}, \forall i = 1, \ldots, l \). Therefore, equation (59) leads to the following interesting result:

\[
\lambda_{\text{max}}(v) = \max_{v} \{\lambda_1, \ldots, \lambda_l\}, \quad \forall v \in \mathbb{R}^{2l}.
\]

Thus, the quantities \( \lambda_k \) with \( k = 1, \ldots, l \) are indeed Lyapunov exponents. In this case, for the sake of simplicity, we have assumed that \( \lambda_i = \lambda, \forall i, j = 1, \ldots, l \). The asymptotic linear IGE growth of \( S_{M_\lambda}(\tau) \) may be considered the information-geometric analogue of the von Neumann entropy growth introduced by Zurek–Paz, a quantum feature of chaos.

The geodesics on \( M_\lambda \) are hyperbolic trajectories. Using the JLC equation for geodesic spread, we show that the Jacobi vector field intensity \( J_{M_\lambda} \) diverges exponentially and is proportional to the number of degrees of freedom of the system, \( J_{M_\lambda} \sim l \exp(\lambda \tau) \). The exponential divergence of the Jacobi vector field intensity \( J_{M_\lambda} \) is a classical feature of chaos. Therefore, we conclude that [22]

\[
R_{M_\lambda} = -l, \quad J_{M_\lambda} \sim l \exp(\lambda \tau), \quad S_{M_\lambda} \sim l \lambda \tau.
\]

By virtue of (63), we observe that \( R_{M_\lambda}, S_{M_\lambda} \) and \( J_{M_\lambda} \) are proportional to the number of Gaussian-distributed
microstates of the system. This proportionality, even though proven in a very special case, suggest that there may be a substantial link among these information-geometric indicators of chaoticity.

5.2. Gaussian statistical model in the presence of macro-correlations

As a second example, we apply the IGAC to study the information constrained dynamics of a system with \( l \) degrees of freedom, each one described by two ‘correlated’ pieces of relevant information, its mean expected value and its variance (Gaussian statistical macrostates). The line element \( ds^2 = g_{ab}(\theta) d\theta^a d\theta^b \) (\( a, b = 1, \ldots, 2l \)) on the 2l-dimensional Gaussian statistical manifold \( \mathcal{M}_S \) in the presence of non-trivial off-diagonal terms is given by

\[
ds^2_{\mathcal{M}_S} = \sum_{j=1}^{2l} \left( \frac{1}{\sigma_j^2} d\mu_j^2 + \frac{2r_j}{\sigma_j^2} d\mu_j d\sigma_j + \frac{2}{\sigma_j^2} d\sigma_j^2 \right).
\]

We consider positive coefficients \( r_j \in (0, 1), \forall j = 1, \ldots, l \). From (64), it can be shown that the Ricci scalar curvature \( \mathcal{R}_{\mathcal{M}_S} (r_1, \ldots, r_l) \) of such a 2l-dimensional manifold is given by

\[
\mathcal{R}_{\mathcal{M}_S} (r_1, \ldots, r_l) = -2^l \sum_{k=1}^{2l} (2 - r_k^2)^{-3}.
\]

Note that in the limit of vanishing correlation strengths \( \{r_k\} \), \( \mathcal{R}_{\mathcal{M}_S} = -l \) as shown in [22]. Applying the IGAC formalism, we are able to compute the asymptotic temporal behavior of the dynamical complexity of geometric trajectories for the correlated 2l-dimensional Gaussian statistical model. The technical details that will be omitted in what follows can be found in [23]. It turns out that [23]

\[
S_{\mathcal{M}_S} (\tau; \{\lambda_k\}, \{r_k\}) \sim -\frac{l}{2} \sum_{k=1}^{2l} \text{ln} \left[ \Lambda_1 (r_k) + \frac{\Lambda_2 (r_k, \lambda_k)}{\tau} \right],
\]

where

\[
\Lambda_1 (r_k) \overset{\text{def}}{=} \frac{2r_k \sqrt{1 - r_k^2}}{1 + \sqrt{1 + 4r_k^2}},
\]

\[
\Lambda_2 (r_k, \lambda_k) \overset{\text{def}}{=} \sqrt{\frac{(1 + 4r_k^2)(2 - r_k^2)}{r_k^2 \lambda_k}} \ln \left[ \frac{\Sigma (r_k, \lambda_k, \alpha_\pm)}{\lambda_k} \right],
\]

\[
\alpha_\pm (r_k) \overset{\text{def}}{=} \frac{3 \pm \sqrt{1 + 4r_k^2}}{2}.
\]

The quantity \( \Sigma (r_k, \lambda_k, \alpha_\pm) \) is a strictly positive function of its arguments. For \( r_k = r, \forall k \) and \( s = 1, \ldots, l \), the IGEC \( S_{\mathcal{M}_S} (\tau; l, \lambda, r) \) becomes

\[
S_{\mathcal{M}_S} (\tau; l, \lambda, r) \sim -\frac{l}{2} \sum_{k=1}^{2l} \text{ln} \left[ \Lambda_1 (r) + \frac{\Lambda_2 (r, \lambda)}{\tau} \right].
\]

In this case, it is clear that the IGEC presents a power-law decay, where the power is related to the cardinality \( l \) of the microscopic degrees of freedom characterized by correlated pieces of macroscopic information. Furthermore, the IGEC reaches a saturation value quantified by the set \( \{r_k\} \) (the correlation strengths). It appears that macro-correlations lead to the emergence of an asymptotic information-geometric compression of the explored statistical macrostates on the configuration manifold of the model in its evolution between the initial and final macrostates \( \Theta^{\text{initial}} (0) \) and \( \Theta^{\text{final}} (\tau) \), respectively.

5.3. Ensemble of random frequency macroscopic inverted harmonic oscillators

For the third example, we employ ED in conjunction with ‘Newtonian entropic dynamics’ (NED) [58]. In NED, we explore the possibility of using well-established principles of inference to derive Newtonian dynamics from relevant prior information codified into an appropriate statistical manifold. The basic assumption is that there is an irreducible uncertainty in the location of particles so that the position of a particle is defined by a probability distribution. The corresponding configuration space is a statistical manifold \( \mathcal{M}_S \) the geometry of which is defined by the Fisher–Rao information metric. The trajectory follows from a principle of inference, namely the method of maximum relative entropy. There is no need for additional ‘physical’ postulates such as an action principle or equation of motion, nor for the concept of mass, momentum or phase space, not even the notion of time. The resulting ‘entropic’ dynamics reproduces Newton’s mechanics for any number of particles interacting among themselves and with external fields. Both the mass of the particles and their interactions are explained as a consequence of the underlying statistical manifold.

In our special application, we consider a manifold with a line element \( ds^2 = g_{ab}(\theta) d\theta^a d\theta^b \) (\( a, b = 1, \ldots, l \)) given by [6, 20]

\[
ds^2 = [1 - \Phi (\theta)] \delta_{ab} (\theta) d\theta^a d\theta^b, \quad \Phi (\theta) = \sum_{k=1}^{l} u_k (\theta_k),
\]

where

\[
u_k (\theta_k) = -\frac{1}{2} \omega_k^2 \theta_k^2, \quad \theta_k = \theta_k (s).
\]

The geodesic equations for the macrovariables \( \theta_k (s) \), are strongly nonlinear and their integration is not trivial. However, upon a suitable change of the affine parameter \( s \) used in the geodesic equations, we may simplify the differential equations for the macroscopic variables parameterizing points on the manifold \( \mathcal{M}_S \overset{\text{def}}{=} \mathcal{M}_S^{\text{IHO}} \) with the metric tensor \( g_{ab} \). Recalling that the notion of chaos is observer dependent and upon changing the affine parameter \( s \) to \( \tau \) in such a way that \( ds^2 = 2(1 - \Phi)^2 ds^2 \), we obtain new geodesic equations describing a set of macroscopic IHOs.

In this example, the IGEC \( S_{\mathcal{M}_S^{\text{IHO}}} (\tau; \omega_1, \ldots, \omega_l) \) reads

\[
S_{\mathcal{M}_S^{\text{IHO}}} (\tau; \omega_1, \ldots, \omega_l) = \lim_{\tau \to \infty} C_{\mathcal{M}_S^{\text{IHO}}} (\tau; \omega_1, \ldots, \omega_l),
\]

where

\[
C_{\mathcal{M}_S^{\text{IHO}}} (\tau; \omega_1, \ldots, \omega_l) = \frac{1}{\tau} \int_0^\tau \Delta V_{\mathcal{M}_S^{\text{IHO}}} (\tau'; \omega_1, \ldots, \omega_l) d\tau'.
\]
and

\[ \Delta V_{\text{rho}}(\tau'; \omega_1, \ldots, \omega_l) = \int_{[0, \infty)} d\tilde{\vartheta}' \left( 1 + \frac{1}{2} \sum_{j=1}^{l} \omega_j^2 \tilde{\vartheta}'_j^2 \right)^{1/2}. \]  

(72)

Substituting (71) and (72) into (70), we obtain the general expression for \( S_{\text{rho}}(\tau; \omega_1, \ldots, \omega_l) \),

\[ S_{\text{rho}}(\tau; \omega_1, \ldots, \omega_l) \equiv \lim_{\tau \to \infty} \ln \left\{ \frac{1}{\tau} \int_{0}^{\tau} \left[ \int_{[0, \infty)} d\tilde{\vartheta}' \right] \right\}. \]

(73)

To evaluate (73) we observe that \( \Delta V_{\text{rho}}(\tau') \) in (72) can be written as

\[ \Delta V_{\text{rho}}(\tau'; \omega_1, \ldots, \omega_l) \approx \frac{1}{l} 2^{l/2} \left( \prod_{i=1}^{l} \omega_i \right) \left[ \sum_{j=1}^{l} \omega_j^2 \tilde{\vartheta}'_j^2 \right]^{1/2}. \]

(74)

Since the \( t \)-Newtonian equations of motion for each IHO are given by

\[ \frac{d^2 \vartheta_j}{dt^2} - \omega_j^2 \vartheta_j = 0, \quad \forall \ j = 1, \ldots, l, \]  

(75)

the asymptotic behavior of such macrovariables on manifold \( M_{\text{IHO}} \) is given by

\[ \vartheta_j(\tau) \approx \omega_j e^{\omega_j \tau}, \quad \vartheta_j \in \mathbb{R}, \quad \forall \ j = 1, \ldots, l. \]  

(76)

We therefore obtain

\[ \Delta V_{\text{rho}}(\tau; \omega_1, \ldots, \omega_l) \approx \frac{1}{l} 2^{l/2} \left( \prod_{i=1}^{l} \omega_i \right) \left[ \sum_{j=1}^{l} \omega_j^2 e^{2 \omega_j \tau} \omega_j^2 \right]^{1/2}. \]

(77)

Upon averaging (77), we find that

\[ C_{\text{rho}}(\tau; \omega_1, \ldots, \omega_l) \approx \frac{1}{l} \int_{0}^{\tau} \frac{1}{2} \left[ \sum_{j=1}^{l} \omega_j^2 e^{2 \omega_j \tau} \omega_j^2 \right] \]  

(78)

where \( \Omega = \sum_{i=1}^{l} \omega_i \). As a working hypothesis, we assume \( \Sigma_i = \Sigma_j \equiv \Sigma \), \( \forall \ i, j = 1, \ldots, l \). Furthermore, assume that \( n \to \infty \) so that \( \omega \to \infty \) that the spectrum of frequencies becomes continuum, and as an additional working hypothesis, assume that this spectrum is linearly distributed (Ohmic frequency spectrum),

\[ \rho_{\text{Ohmic}}(\omega) = \frac{2}{\Omega_{\text{cut-off}}} \omega \quad \text{with} \quad \int_{0}^{\Omega_{\text{cut-off}}} \rho_{\text{Ohmic}}(\omega) d\omega = 1, \]

\[ \Omega_{\text{cut-off}} = \xi \Omega, \quad \xi \in \mathbb{R}. \]  

(79)

Thus, we obtain

\[ C_{\text{rho}}(\tau; \omega_1, \ldots, \omega_l) \approx \frac{1}{l} \left( \frac{\xi^2 \Omega^2}{2} \right)^{1/2} \exp \left( \frac{\xi \Omega \tau}{\tau} \right). \]

(80)

Finally, substituting (80) into (70) yields [4, 5]

\[ S_{\text{rho}}(\tau; \omega_1, \ldots, \omega_l) \approx \Omega \tau, \quad \Omega = \sum_{i=1}^{l} \omega_i. \]  

(81)

Equation (81) displays the asymptotic, linear IGE growth of the generalized \( l \)-set of IHOs and extends the result of Zurek–Paz to an arbitrary set of anisotropic IHOs [54] in a classical information-geometric setting. In order to ensure the compactification of the parameter space of the system (and therefore \( M_S \) itself), it is possible to choose a Gaussian distributed frequency spectrum for the IHOs. With this choice of frequency spectrum, the folding mechanism required for true chaos is restored in a statistical (averaging over \( \omega \) and \( \tau \)) sense. This example may be considered as the information geometric analogue of the Zurek–Paz model used to investigate the implications of decoherence for quantum chaos. In their work, Zurek and Paz considered a chaotic system, a single unstable harmonic oscillator characterized by a potential \( V(x) = -\frac{\Omega^2}{2} x^2 \) (\( \Omega \) is the Lyapunov exponent), coupled to an external environment. In the reversible classical limit [59], the von Neumann entropy of such a system increases linearly at a rate determined by the Lyapunov exponent,

\[ S_{\text{quantum}}(\tau) \approx \Omega \tau, \]

(82)

with \( \Omega \) playing the role of the Lyapunov exponent.

5.4. Regular and chaotic quantum spin chains

In our fourth example, we apply the IGAC to study the ED on curved statistical manifolds induced by classical probability distributions commonly used in the study of regular and chaotic quantum energy level statistics. In doing so, we suggest an information-geometric characterization of a special class of regular and chaotic quantum energy level statistics.

Recall that the theory of quantum chaos (quantum mechanics of systems whose classical dynamics are chaotic) is not primarily related to few-body physics. Indeed, in real physical systems such as many-electron atoms and heavy nuclei, the origin of complex behavior is the very strong interaction among many particles. To deal with such systems, a famous statistical approach has been developed which is based on random matrix theory (RMT). The main idea of this approach is to neglect the detailed description of the motion and to treat these systems statistically, bearing in mind that the interaction among particles is so complex and strong that generic properties are expected to emerge. Once again, this is exactly the philosophy underlying the ED approach to complex dynamics. It is known that the asymptotic behavior of computational costs and entanglement entropies of integrable and chaotic Ising spin chains are very different [60]. Prosen considered the question of time efficiency in implementing an up-to-date version of the t-DMRG (time-dependent density-matrix renormalization group) for a family of Ising spin-1/2 chains in an arbitrarily
oriented magnetic field which undergoes a transition from an integrable (transverse Ising) to a non-integrable chaotic regime as the magnetic field is varied. An integrable (regular) Ising chain in a general homogeneous, transverse magnetic field is defined through the Hamiltonian $H_{\text{regular}}(0, 2)$, where

$$H(\hbar_x, \hbar_y) = \sum_{j=0}^{n-2} \sigma_j^x \sigma_{j+1}^x + \sum_{j=0}^{n-1} (h^x \sigma_j^y + h^y \sigma_j^y). \quad (83)$$

In this case, the computational cost presents a polynomial growth in time, $D[j]_{\text{regular}}(t) \propto t$, while the entanglement entropy is characterized by a logarithmic growth,

$$S_{\text{regular}}(0, 2) = c_{\text{von Neumann}}^{(0,2)} \propto \ln t + c'. \quad (84)$$

The constant $c$ depends exclusively on the value of the fixed transverse magnetic field intensity $B_1$, while $c'$ depends on $B_2$ and on the choice of the initial local operators of the finite index used to calculate the operator space entanglement entropy. In contrast, a quantum chaotic Ising chain in a general homogeneous tilted magnetic field is defined through the Hamiltonian $H_{\text{chaotic}}(1, 1)$, where $H$ is given in (83). In this case, the computational cost presents an exponential growth in time, $D[j]_{\text{chaotic}}(t) \propto e^{K_q \tau}$, while the entanglement entropy is characterized by linear growth,

$$S_{\text{chaotic}}(1, 1) = c_{\text{von Neumann}}^{(1,1)} \propto K_q \tau. \quad (85)$$

The quantity $K_q$ is a constant, is asymptotically independent of the number of indexes of the initial local operators used to calculate the operator space entropy, depends only on the Hamiltonian evolution and not on the details of the initial state observable or error measures and can be interpreted as a kind of quantum dynamical entropy.

It is well known that the quantum description of chaos is characterized by a radical change in the statistics of quantum energy levels [61]. The transition to chaos in the classical limit of quantum systems is associated with a drastic change in the statistics of the nearest-neighbor spacings of quantum energy levels. In the regular regime the distribution agrees with the Poisson statistics, while in the chaotic regime the Wigner–Dyson distribution works very well. Uncorrelated energy levels are characteristic of quantum systems corresponding to a classically regular motion, while a level repulsion (a suppression of small energy level spacing) is typical of systems that are classically chaotic. A standard quantum example is provided by the study of energy level statistics of a hydrogen atom in a strong magnetic field. It is known that level spacing distribution (LSD) is a standard indicator of quantum chaos [62]. It displays characteristic level repulsion for strongly non-integrable quantum systems, whereas for integrable systems there is no repulsion due to the existence of conservation laws and quantum numbers. In [60], the authors calculate the LSD of the spectra of $H_{\text{regular}}(0, 2)$ and $H_{\text{chaotic}}(1, 1)$. They found that for $H_{\text{regular}}(0, 2)$, the nearest neighbor LSD is described by a Poisson distribution. For $H_{\text{chaotic}}(1, 1)$, they found that the nearest neighbor LSD is described by a Wigner–Dyson distribution. Therefore, they conclude that $H_{\text{regular}}(0, 2)$ and $H_{\text{chaotic}}(1, 1)$ indeed represent generic regular and quantum chaotic systems, respectively.

We encode the relevant information about the spin chain in a suitable composite-probability distribution taking into account the quantum spin chain and the configuration of the external magnetic field in which they are immersed.

In the ME method, the selection of relevant variables is made on the basis of intuition guided by experiment; it is essentially a matter of trial and error. The variables should include those that can be controlled or experimentally observed, but there are cases where others must also be considered. Our objective here is to choose the relevant microvariables of the system and select the relevant information concerning each one of them. In the integrable case, the Hamiltonian $H_{\text{regular}}(0, 2)$ describes an antiferromagnetic Ising chain immersed in a transverse, homogeneous magnetic field $B_{\text{transverse}} = B_1 B_2$, where the LSD of its spectrum is given by the Poisson distribution

$$p_{\text{Poisson}}(x_A | \mu_A) = \frac{1}{\mu_A} \exp \left( -\frac{x_A}{\mu_A} \right). \quad (86)$$

The microvariable $x_A$ represents the spacing of the energy levels, whereas the macrovariable $\mu_A$ is the average spacing. The chain is immersed in a transverse magnetic field, which has just one component $B_1$ in the Hamiltonian $H_{\text{regular}}(0, 2)$. Observe that the exponential distribution is identified by information theory as the ME distribution if only one piece of information (the expectation value) is known. Thus, we translate this piece of information in the IGAC formalism by coupling the probability (86) to an exponential bath $p_B^{(\text{exponential})}(x_B | \mu_B)$ given by

$$p_B^{(\text{exponential})}(x_B | \mu_B) = \frac{1}{\mu_B} \exp \left( -\frac{x_B}{\mu_B} \right), \quad (87)$$

where the microvariable $x_B$ is the intensity of the magnetic field and the macrovariable $\mu_B$ is the average intensity. More correctly, $x_B$ should be the energy arising from the interaction of the transverse magnetic field with the spin-1/2 particle magnetic moment, $x_B = -\mu_B \cdot \vec{B} = -\mu_B \cos \psi$ where $\psi$ is the tilt angle. For the sake of simplicity, let us set $\mu_B = 1$. Then in the transverse case $\psi = 0$ and therefore $x_B = B = B_L$. This is our best guess and we justify it by noting that the magnetic field intensity is indeed a relevant quantity in this experiment (see equation (84)). Its components are varied during the transition from integrable to chaotic regimes. In the regular regime, we say the magnetic field intensity is set to a well-defined value $\langle x_B \rangle = \mu_B$. Finally, the chosen composite probability distribution $p^{(\text{integrable})}(x_A, x_B | \mu_A, \mu_B)$ encoding relevant information about the system is given by

$$p^{(\text{integrable})}(x_A, x_B | \mu_A, \mu_B) = \frac{1}{\mu_A \mu_B} \exp \left[ -\frac{x_A}{\mu_A} - \frac{x_B}{\mu_B} \right]. \quad (88)$$

Again, we point out that our probability (88) is our best guess and, of course, must be consistent with numerical simulations and experimental data in order to have some merit. We point out that equation (88) is not fully justified from a theoretical point of view, a situation that occurs due to the lack of a systematic way of selecting the relevant microvariables of the system (and to choose the appropriate information
about such microvariables). Let us denote by $\mathcal{M}^{\text{integrable}}_S$ the two-dimensional curved statistical manifold underlying our information geometrodynamics. The line element $d\mathbf{s}^2$ on $\mathcal{M}^{\text{integrable}}_S$ is given by

$$d\mathbf{s}^2_{\text{integrable}} = d\mathbf{s}^2_{\text{Poisson}} + d\mathbf{s}^2_{\text{Exponential}} = \frac{1}{\mu_A^2} d\mathbf{m}_A^2 + \frac{1}{\mu_B^2} d\mathbf{m}_B^2.$$  

(89)

Applying the IGAC to the line element in (89) leads us to conclude polynomial growth in $C_s$ information geometrodynamics. The line element $d\mathbf{s}^2_{\text{integrable}}$ on $\mathcal{M}^{\text{integrable}}_S$ is given by

$$d\mathbf{s}^2_{\text{integrable}} = d\mathbf{s}^2_{\text{Poisson}} + d\mathbf{s}^2_{\text{Exponential}} = \frac{1}{\mu_A^2} d\mathbf{m}_A^2 + \frac{1}{\mu_B^2} d\mathbf{m}_B^2.$$  

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(89)

The quantity $C_{IG}$ is a constant proportional to the number of exponential probability distributions in the composite distribution used to calculate the IGE; $c_{IG}$ is a constant that depends on the values assumed by the statistical macrovariables $\mu_A$ and $\mu_B$. Equations (90) may be interpreted as the information-geometric analogue of the computational complexity $D^{(\text{regular})}_s(\tau)$ and the entanglement entropy $S^{(\text{regular})}_s(0, 2)$ defined in standard quantum information theory, respectively. We may not state that they are the same since we are not fully justifying, from a theoretical standpoint, our choice of the composite probability (88).

In the chaotic case, the Hamiltonian $H_{\text{chaotic}}(1, 1)$ describes an antiferromagnetic Ising chain immersed in a tilted, homogeneous magnetic field $B_{\text{tilted}} = B_\perp \hat{B}_\perp + B_\parallel \hat{B}_\parallel$, with the LSD of its spectrum given by the Poisson distribution $p_{\text{Wigner–Dyson}}(x'_A|\mu'_A)$.

$$p_{\text{Wigner–Dyson}}(x'_A|\mu'_A) = \frac{\pi x'_A}{2\mu'_A} \exp \left( -\frac{\pi x'_A^2}{4\mu'_A^2} \right).$$  

(91)

where the microvariable $x'_A$ represents the spacing of the energy levels and the macrovariable $\mu'_A$ is the average spacing. The chain is immersed in the tilted magnetic vector field, which has two components $B_\perp$ and $B_\parallel$ in the Hamiltonian $H_{\text{chaotic}}(1, 1)$. The Gaussian distribution is identified by information theory as the ME distribution if only the expectation value and the variance are known. We translate this information in the IGAC formalism by coupling the probability (91) to a Gaussian $p_B^{(\text{Gaussian})}(x'_B|\mu'_B, \sigma'_B)$

$$p_B^{(\text{Gaussian})}(x'_B|\mu'_B, \sigma'_B) = \frac{1}{\sqrt{2\pi \sigma'_B^2}} \exp \left( -\frac{(x'_B - \mu'_B)^2}{2\sigma'_B^2} \right).$$  

(92)

where the microvariable $x'_B$ is the intensity of the magnetic field, the macrovariable $\mu'_B$ is the average intensity of the magnetic energy arising from the interaction of the tilted magnetic field with the spin-$1/2$ particle magnetic moment and $\sigma'_B$ is its covariance: during the transition from the integrable to the chaotic regime, the magnetic field intensity is being varied (experimentally). It is being tilted and its two components $(B_\perp$ and $B_\parallel$) are being varied as well. Our best guess based on the experimental mechanism that drives the transitions between the two regimes is that the magnetic field intensity (actually the microvariable $\mu B \cos \varphi$) is Gaussian-distributed (two macrovariables) during this change. In the chaotic regime, we say the magnetic field intensity is set to a well-defined value $(x'_B = \mu'_B$) with covariance $\sigma'_B = \sqrt{(x'_B - x'_B^2)^2}$. Thus, the chosen composite probability distribution $p^{(\text{chaotic})}(x'_A, x'_B|\mu_A, \mu_B, \sigma'_B)$ encoding relevant information about the system is given by

$$p^{(\text{chaotic})}(x'_A, x'_B|\mu_A, \mu_B, \sigma'_B) = \frac{\pi}{2\mu'_A^2} x'_A \exp \left( -\frac{\pi x'_A^2}{4\mu'_A^2} \right) \int \exp \left[ -\left( \frac{\pi x'_A^2}{4\mu'_A^2} + \frac{(x'_B - \mu'_B)^2}{2\sigma'_B^2} \right) \right].$$  

(93)

Let us denote by $\mathcal{M}^{(\text{chaotic})}_S$ the three-dimensional curved statistical manifold underlying our ED model. The corresponding line element $d\mathbf{s}^2_{\text{chaotic}}$ on $\mathcal{M}^{(\text{chaotic})}_S$ is given by

$$d\mathbf{s}^2_{\text{chaotic}} = d\mathbf{s}^2_{\text{Wigner–Dyson}} + d\mathbf{s}^2_{\text{Gaussian}} = \frac{4}{\mu'_A} d\mathbf{m}_A^2 + \frac{1}{\sigma'_B^2} d\mathbf{m}_B^2 + \frac{2}{\sigma'_B^2} d\mathbf{m}_B^2.$$  

(94)

Applying the IGAC machinery to the line element in (94), we obtain exponential growth for $\mathcal{M}^{(\text{chaotic})}_S$.

$$c^{(\text{chaotic})}_{IG}(\tau) \propto C_{IG} \exp(K_{IG} \tau), \quad S^{(\text{chaotic})}_{IG}(\tau) \propto K_{IG} \tau.$$  

(95)

The constant $C_{IG}$ encodes information about the initial conditions of the statistical macrovariables parameterizing elements of $\mathcal{M}^{(\text{chaotic})}_S$. The constant $K_{IG}$, given by

$$K_{IG} \propto \frac{dS^{(\text{chaotic})}_{IG}(\tau)}{d\tau} \propto \lim_{\tau \to \infty} \frac{1}{\tau} \ln \left[ \frac{\mathcal{J}_{M_s}(\tau)}{\mathcal{J}_{M_s}(0)} \right] = \lambda_{M_s},$$  

(96)

is the model parameter of the chaotic system and depends on the temporal evolution of the statistical macrovariables. It plays the role of the standard Lyapunov exponent of a trajectory and is, in principle, an experimentally observable quantity. The quantity $\mathcal{J}_{M_s}(\tau)$ is the Jacobi field intensity and $\lambda_{M_s}$ may be considered the information-geometric analogue of the leading Lyapunov exponent in conventional Hamiltonian systems. Given an explicit expression for $K_{IG}$ in terms of the observables $\mu'_A, \mu'_B$ and $\sigma'_B$, a clear understanding of the relation between the IGGE (or $K_{IG}$) and the entanglement entropy (or $C_{IG}$) becomes the key point that deserves further study. Equations (95) are the information-geometric analogue of the computational complexity $D^{(\text{chaotic})}_s(\tau)$ and the entanglement entropy $S^{(\text{chaotic})}_S(1, 1)$, defined in standard quantum information theory, respectively. This result requires a deeper analysis in order to be fully understood. One of the major limitations of our findings is the lack of a detailed account of the comparison of theory with experiment. This point will be among our primary concerns in future works. Some considerations may, however, be carried out at the present stage. The experimental observables in our theoretical models are the statistical macrovariables characterizing the composite probability distributions. In the integrable case,
where the coupling between a Poisson and an exponential distribution is considered, $\mu_A$ and $\mu_B$ are the experimental observables. In the chaotic case, the coupling between a Wigner–Dyson and a Gaussian distribution is considered, $\mu_A^*, \mu_B^*$ and $\sigma_B^*$ play the role of the experimental observables. We believe that one way of testing our theory may be to determine a numerical estimate of the leading Lyapunov exponent $\lambda_{\text{max}}$ or the Lyapunov spectrum for the Hamiltonian systems under investigation directly from experimental data (measurement of a time series) and compare it to our theoretical estimate for $\lambda_{\text{max}}$ [63]. However, we are aware that it may be rather difficult to evaluate Lyapunov exponents numerically. Otherwise, knowing that the mean values of the positive Lyapunov exponents are related to the KS dynamical entropy, we suggest to measure the KS entropy $K$ directly from a time signal associated with a suitable combination of our experimental observables and compare it to our indirect theoretical estimate for $K_{\text{IG}}$ from the asymptotic behaviors of our statistical macrovariables [64]. We are aware that the basis of our discussion is rather qualitative. However, we hope that with additional study, especially in clarifying the relation between the IGE and the entanglement entropy, our theoretical information geometric characterization will find experimental support in future. For these reasons, the statement that our findings may be relevant to experiments verifying the existence of chaoticity and related dynamical properties on a macroscopic level in energy level statistics in chaotic and regular quantum spin chains is purely a conjecture at this stage.

### 5.5. Quantum entangled wave packets

As our final example, we apply the IGAC to characterize the quantum entanglement produced by a head-on collision between two identical (but distinguishable) spinless, structureless, non-relativistic particles of mass $m$, each represented by minimum uncertainty Gaussian wave packets interacting via a scattering process [55]. Before colliding, the two particles are in the form of disentangled Gaussian wave packets, each characterized by a width $\sigma_0$ in momentum space. The initial distance between the two particles is $R_0$ and their average initial momenta—setting the Planck constant $\hbar$ to be equal to $1$—are $\mp k_0$, respectively. After some straightforward algebra [55], it can be shown that the initial (pre-collisional) two-particle square wave amplitude in momentum space is given by

$$
P^{(\text{QM})}_{\text{pre}}(k_1, k_2 | k_0, \sigma_0) = \frac{1}{2\pi \sigma_0^2} \exp \left[ -\frac{(k_1 - k_0)^2 + (k_2 + k_0)^2}{2\sigma_0^2} \right].$$

(97)

where $\sigma_0 = \frac{\sigma_0}{\hbar}$, $\pm k_0 = \{k_1/2\}_0 = \{p_{1/2}\}_0/h = \pm \frac{m \pi}{\hbar}$ with $(p_{1/2})_0 = p_0$, $(p_{3/2})_0 = -p_0$, $\sigma_0$ is defined as in (13) and we have made use of the center of mass and relative coordinates whose conjugate momenta are defined as $K \equiv k_1 + k_2 \in (-\infty, +\infty)$ and $k \equiv \frac{1}{2} (k_1 - k_2) \in (-\infty, +\infty)$ with $k_{1/2} = \frac{\pi}{\sqrt{2}} \in (-\infty, +\infty)$.

Similarly, following [55], and after some tedious algebra, one finds that the final (long time limit, post-collisional) two-particle square wave amplitude in momentum space is given by

$$
P^{(\text{QM})}_{\text{post}}(k_1, k_2 | k_0, \sigma_0; r_{\text{QM}}) \approx \exp \left\{ -\frac{1}{2(1 - r_{\text{QM}}^2)} \left[ \frac{(k_1 - k_0)^2 + (k_2 + k_0)^2}{\sigma_0^2} - 2r_{\text{QM}} (k_1 - k_0)(k_2 + k_0) \right] \right\}.

(98)

with

$$
r_{\text{QM}} = \sqrt{8(2k_0^2 + \sigma_B^2)}R_0a_0 \ll 1,

(99)

where the parameter $a_0$ has the dimension of length and is defined as the s-wave scattering length [65]. It is evident from (99) that $r_{\text{QM}}$ is non-zero and positive. Thus, $f(k_0) = \frac{e^{\text{th}_0}}{k_0} \sin \frac{\theta(k_0)}{k_0} \approx \frac{\theta(k_0)}{k_0} + O(\theta^2) \tilde{k}_L \ll 1 - a_0$, where $\theta(k_0) \equiv \theta(k_0) \approx -\frac{2\sqrt{2}k_0\sigma_B}{3\hbar} \approx -k_0a_0$ denotes the s-wave scattering phase shift, $\mu$ is the reduced mass $\mu = m/2$, $f(k)$ is the scattering amplitude and $L$ is the range of the scattering potential $V$ given by

$$
V(x) = \begin{cases} V, & 0 \leq x \leq L, \\ 0, & x > L. \end{cases}

(100)

where $V$ denotes the height (for $V > 0$; repulsive potential) or depth (for $V < 0$; attractive potential) of the potential. The quantity $\theta(k)$ is the s-wave scattering phase shift considered around $k = k_0$ (i.e., assuming that our wave packet is well localized around $k = k_0$) and in the limit of low-energy scattering, i.e. $\theta(k) \ll 1$.

We conjecture that the quantum entanglement produced by a head-on collision between two Gaussian wave packets is a macroscopic manifestation emerging from specific underlying microscopic statistical structures. Specifically, we propose that $P^{(\text{IG})}_{\text{pre}}(k_1, k_2 | k_0, \sigma_0)$ can be interpreted as a limiting case (initial time limit) arising from a Gaussian probability distribution $P^{(\text{IG})}_{\text{pre}}(x, y | \mu_x, \mu_y; \sigma)$,

$$
P^{(\text{IG})}_{\text{pre}}(x, y | \mu_x, \mu_y; \sigma) \equiv \frac{1}{2\pi\sigma^2} \times \exp \left[ -\frac{(x - \mu_x)^2 + (y - \mu_y)^2}{2\sigma^2} \right].

(101)

Upon setting $x \rightarrow k_1, \ y \rightarrow k_2, \ (x) = \mu_x \rightarrow \mu_{k_1} \equiv +k_0, \ (y) = \mu_y \rightarrow \mu_{k_2} \equiv -k_0$ and $\sigma \rightarrow \sigma_{k_0}$, we obtain

$$
P^{(\text{IG})}_{\text{pre}}(x, y | \mu_x, \mu_y; \sigma) \rightarrow P^{(\text{IG})}_{\text{pre}}(k_1, k_2 | \mu_{k_1}, \mu_{k_2}; \sigma):

$$
P^{(\text{IG})}_{\text{pre}}(k_1, k_2 | \mu_{k_1}, \mu_{k_2}; \sigma) \equiv \frac{1}{2\pi\sigma^2} \times \exp \left[ -\frac{(k_1 - \mu_{k_1})^2 + (k_2 - \mu_{k_2})^2}{2\sigma^2} \right],

(102)

enabling the identification

$$
P^{(\text{QM})}_{\text{pre}}(k_1, k_2 | k_0, \sigma_0) = P^{(\text{IG})}_{\text{pre}}(k_1, k_2 | k_0, \sigma_0).

(103)
The variances \( \sigma \), and \( \sigma_0 \), in the random variables \( x \) and \( y \), respectively, are given by the standard definition (13). We remark that, in general, \( \sigma \neq \sigma_0 \). In the present example, however, it is sufficient to consider \( \sigma = \sigma_0 = \sigma \).

We propose that \( P_{QM}^{(k_1, k_2|k_0, \sigma_0; r_{QM})} \) can be viewed as a limiting case (final or long time limit) arising from a Gaussian probability distribution \( P_{QM}^{(G)}(x, y|\mu_x, \mu_y; \sigma, r) \),

\[
P_{QM}^{(G)}(x, y|\mu_x, \mu_y; \sigma, r) = \exp \left\{ -\frac{1}{2(1-r^2)} \left[ \frac{(x-\mu_x)^2}{\sigma^2} - 2r \frac{(x-\mu_x)(y-\mu_y)}{\sigma^2} + \frac{(y-\mu_y)^2}{\sigma^2} \right] \right\} 2\pi\sigma^2 \sqrt{1-r^2}.
\]

(104)

where the micro-correlation coefficient \( r \) is defined as in (24). In the present example, the micro-correlation coefficient \( r \) is considered to have compact support over the line segment \([0, 1]\); that is, \( r \in [0, 1] \). Upon setting \( \mu_{k_1} \rightarrow \pm k_0, \mu_{k_2} \rightarrow -k_0 \) and \( \sigma \rightarrow \sigma_{k_0} \), we obtain

\[
P_{QM}^{(G)}(k_1, k_2|k_0, \sigma_0; r) = \exp \left\{ -\frac{1}{2(1-r^2)} \left[ \frac{k_1-k_0 \sigma_0}{\sigma_0} - 2r \frac{(k_1-k_0)(k_2+k_0)}{\sigma_0} + \frac{k_2+k_0 \sigma_0}{\sigma_0} \right] \right\} 2\pi\sigma_{k_0}^2 \sqrt{1-r^2}.
\]

(105)

In this case, when both the weak correlation \( (r \ll 1) \) and the weak scattering conditions \( (|\theta(k_0)| \ll 1) \) are satisfied, we obtain excellent overlapping between (98) and (105), so that

\[
P_{QM}^{(G)}(k_1, k_2|k_0, \sigma_0; r_{QM}) \simeq P_{QM}^{(G)}(k_0, \sigma_0; r)
\]

for \( r \ll 1, r_{QM} \ll 1 \) and \( |\theta(k_0)| \ll 1 \),

(106)

assuming that \( k_0, \sigma_0, r \) and \( r_{QM} \) are fixed numerical constants and letting \( k_{1,2} \) assume values in the neighborhood of \( k_0 \).

At this stage our conjecture is only mathematically sustained by the formal identities (103) and (106). To render our conjecture physically relevant, recall that s-wave scattering can also be described in terms of a scattering potential \( V(x) \) and the scattering phase shift \( \theta(k) \). Integrating the radial part of the Schrödinger equation with this potential for the scattered wave and imposing the matching condition at \( x = L \) for its solution and its first derivative leads to

\[
k_{in} \cot(k_{in}L) = k_{out} \cot(k_{out}L + \theta),
\]

(107)

with

\[
k_{in} = \frac{\sqrt{2} \mu (E - V)}{h}, \quad 0 < x < L,
\]

(108)

\[
k_{out} = \frac{\sqrt{2} \mu E}{h}, \quad x > L.
\]

(109)

The quantities \( \mu \) and \( E \) are the reduced mass and kinetic energy of the two-particle system in the relative coordinates, respectively; \( k_{in} \) and \( k_{out} \) represent the conjugate-coordinate wave vectors inside and outside the potential region,

8 From online lecture notes on Nuclear Physics II by Dr Michael Hunt at Nottingham University.

respectively. Equation (107) indicates that the scattering potential \( V(x) \) shifts the phase of the scattered wave at points beyond the scattering region.

Our information geometric modeling may be briefly described in the following way. The pre-collisional scenario is characterized by the information geometric dynamics on the curved statistical manifold \( M_{QM}^{(uncorr)} \) of uncorrelated Gaussian probability distributions \( P_{QM}^{(G)}(k_1, k_2|\mu_{k_1}, \mu_{k_2}, \sigma) \) given in (102). The geodesic trajectories on \( M_{QM}^{(uncorr)} \) for the non-correlated Gaussian system is given by

\[
\langle p_{1b}(\tau) \rangle = \mu_1(\tau; 0) = -\sqrt{p_0^2 + 2\sigma_0^2} \tanh(A_0\tau),
\]

(110)

\[
\langle p_{2b}(\tau) \rangle = \mu_2(\tau; 0) = \sqrt{p_0^2 + 2\sigma_0^2} \tanh(A_0\tau),
\]

(111)

\[
\langle \sigma_b(\tau) \rangle = \sigma(\tau; 0) = \sqrt{\frac{1}{2} p_0^2 + \sigma_0^2} \frac{1}{\cosh(A_0\tau)}.
\]

(112)

The post-collisional scenario is characterized by the information geometric dynamics on the curved statistical manifold \( M_{QM}^{(corr)} \) of correlated Gaussian probability distributions \( P_{QM}^{(G)}(k_1, k_2|\mu_{k_1}, \mu_{k_2}, \sigma) \) given in (105). The geodesic trajectories on \( M_{QM}^{(corr)} \) for the non-correlated Gaussian system is given by

\[
\langle p_{1a}(\tau) \rangle = \mu_1(\tau; r) = -\sqrt{1 - r} \left( p_0^2 + 2\sigma_0^2 \right) \tanh(A_0\tau),
\]

(113)

\[
\langle p_{2a}(\tau) \rangle = \mu_2(\tau; r) = \sqrt{1 - r} \left( p_0^2 + 2\sigma_0^2 \right) \tanh(A_0\tau),
\]

(114)

\[
\langle \sigma_a(\tau) \rangle = \sigma(\tau; r) = \sqrt{\frac{1}{2} p_0^2 + \sigma_0^2} \frac{1}{\cosh(A_0\tau)}.
\]

(115)

where the subscript ‘\( a \)’ denotes the initial state, the subscripts ‘\( 1 \)’ and ‘\( 2 \)’ denote particle 1 and particle 2, respectively; subscripts ‘\( b \)’ and ‘\( a \)’ denote ‘before’ and ‘after’ collision, respectively and

\[
A_0 \equiv \frac{1}{t_0} \sqrt{\frac{2p_0}{\sigma_0}} \quad \text{and} \quad \frac{\sigma}{p_0} \ll 1.
\]

\[
\frac{\sigma}{p_0} \ll 1 \quad \text{and} \quad \frac{1}{t_0} \left( \frac{\sqrt{2} \sigma_0}{\sigma_0} + \frac{1}{2} \frac{(\sigma_0)}{p_0} \right)^2
\]

\[
- \frac{3}{8} \left( \frac{\sigma_0}{p_0} \right)^4 + O \left( \left( \frac{\sigma_0}{p_0} \right)^6 \right).
\]

(116)

The two sets of geodesic curves comprised of \{\( p_{1b}(\tau) \), \( p_{2b}(\tau) \), \( \sigma_b(\tau) \)\} (for the non-correlated model) and \{\( p_{1a}(\tau) \), \( p_{2a}(\tau) \), \( \sigma_a(\tau) \)\} (for the correlated model) are joined at the junction \( \tau = 0 \); \( \tau \ll 0 \) (before collision) for the non-correlated model and \( \tau \gg 0 \) (after collision) for the correlated model. We recognize that the momenta \( \langle p_{1b}(\tau) \rangle \) and \( \langle p_{1a}(\tau) \rangle \) asymptotically converge to \( \sqrt{p_0^2 + 2\sigma_0^2} \) and \( -\sqrt{(1 - r)} \left( p_0^2 + 2\sigma_0^2 \right) \) toward \( \tau = -\infty \) and \( \tau = +\infty \), respectively (the same is true for \( -\langle p_{2b}(\tau) \rangle \) and \( -\langle p_{2a}(\tau) \rangle \)), while \( \langle \sigma_b(\tau) \rangle \) and \( \langle \sigma_a(\tau) \rangle \) are identical and vanishingly
small toward $\tau = \pm \infty$. Furthermore, we observe that there is continuity between $\langle p_{1/2\nu}(\tau) \rangle$ and $\langle p_{1/2\nu}(\tau) \rangle$ and between $\langle \sigma_{\nu}(\tau) \rangle$ and $\langle \sigma_{\nu}(\tau) \rangle$ at the junction $\tau = 0$.

A question that now arises is how to determine the scattering phase shift in view of the fact that our statistical model is correlated after collision. Initially, we need to examine how correlations affect the momentum geodesic curve $\langle p_{1/2}(\tau) \rangle$. For this purpose we define the momentum-difference curve $\langle p(\tau) \rangle \equiv \frac{1}{2} \{ \langle p_2(\tau) \rangle - \langle p_1(\tau) \rangle \}$.

Comparison of the following two equations,

\[
\langle p(\tau; 0) \rangle = \frac{1}{2} \{ \langle p_{2b}(\tau) \rangle - \langle p_{1b}(\tau) \rangle \} = \sqrt{p_0^2 + 2\sigma_0^2 \tanh(A_0\tau)},
\]

\[
\langle p(\tau; r) \rangle = \frac{1}{2} \{ \langle p_{2a}(\tau) \rangle - \langle p_{1a}(\tau) \rangle \} = \sqrt{(1 - r) \left( p_0^2 + 2\sigma_0^2 \right) \tanh(A_0\tau)},
\]

which follow from (110), (111), (113) and (114), indicates that at any arbitrary time $\tau \geq 0$

\[
\langle p(\tau; 0) \rangle \geq \langle p(\tau; r) \rangle,
\]

while both (117) and (118) share the functional argument $A_0\tau$. Condition (119) implies that the correlation causes a reduction in the momentum for any $\tau \geq 0$ (relative to the non-correlated case). This situation is analogous to the change in momentum caused by a repulsive scattering potential (see (108) and (109)). It is then reasonable to assume that there exists some connection between the scattering potential and the correlation. Provided that this connection is established, one should be able to determine the scattering phase shift in terms of the correlation via equations (107)–(109). In this way, one can ultimately establish a connection between quantum entanglement and the statistical micro-correlation.

Recall that before collision (at the affine time $-\tau_0$) particles 1 and 2 are separated by a linear distance $R_0$. Each particle has momenta $p_0$ and $-p_0$, respectively, and the same momentum spread $\sigma_0$. Then from (110)–(112), we have

\[
p_0 = \langle p_{1b}(\tau_0) \rangle = -\langle p_{2b}(\tau_0) \rangle = \sqrt{p_0^2 + 2\sigma_0^2 \tanh(A_0\tau_0)},
\]

\[
\sigma_0 = \langle \sigma_{1b}(\tau_0) \rangle = \sqrt{\frac{1}{2} p_0^2 + \sigma_0^2 \frac{1}{\cosh(A_0\tau_0)}},
\]

For arbitrary $\tau \geq 0$ after collision, the system of particles 1 and 2, which initially carried momenta $p_0$ and $-p_0$, respectively, at $\tau = -\tau_0$ before collision, now carries the relative conjugate-momentum $\langle p(\tau; r) \rangle$ given by (118) due to the correlation. With non-vanishing micro-correlation the wave packets experience the effect of a repulsive potential; the magnitude of the wave vectors (or momenta) decreases relative to the corresponding non-correlated value. One may rewrite (109)–(109) as

\[
k_r \cot(k_r L) = k_0 \cot(k_0 L + \theta_0),
\]

with

\[
k_r = \frac{\sqrt{2\mu (\mathcal{E} - V)}}{\hbar}, \quad 0 < x < L,
\]

where $k_r$ and $k_0$ represent the wave vectors with and without the correlation, respectively. The connection between the correlation and the scattering potential can be established by combining (123) and (124). From (119), one finds that the correlation renders

\[
k_0 \rightarrow k_r \equiv \sqrt{1 - r} k_0.
\]

Then using (123)–(125), we determine the scattering potential,

\[
V = r \mathcal{E} = \frac{h^2 k_r^2}{2\mu} = \frac{p_0^2}{2\mu}.
\]

Equation (126) clearly establishes a connection between the correlation coefficient and the scattering potential: the correlation coefficient is the ratio of the scattering potential to the initial relative kinetic energy of the system. From (126) it is evident that our interaction potential is repulsive, i.e. $V > 0$, since we consider non-negative micro-correlations, $r \in [0, 1]$.

With the potential determined, one can determine the scattering phase shift $\theta_0$ (for low-energy $s$-wave scattering) by combining equations (122)–(124) and (126), the result being

\[
\tan \theta_0 \frac{k_0 L - p_0 L / \hbar}{\approx} \frac{c}{3} \theta_0 = -\frac{r (k_0 L)^3}{3}.
\]

By means of (126) and (127) we can express the scattering phase shift in terms of the scattering potential

\[
\theta_0 \approx \frac{2 \mu V k_0 L^3}{3 \hbar^2} = -\frac{2 \mu V p_0 L^3}{3 \hbar^3},
\]

which is in agreement with [66]. This is the first significant finding that allows us to state that our conjecture is also physically motivated.

As the scattering potential has been determined, so the scattering amplitude too can be determined. To this end, we write

\[
f(k_0) = \frac{e^{ik_0} \sin \theta_0}{k_0} \approx \frac{\theta_0}{\theta_0} = -\alpha_s
\]

for low-energy $s$-wave scattering, $k_0 L = p_0 L / \hbar < 1$. Thus, we finally obtain the scattering cross-section:

\[
\Sigma = 4\pi |f(k_0)|^2 \approx 4\pi r^2 k_0 L^5 = 9 = \frac{16\pi \mu^2 V^2 L^5}{9 \hbar^4} \approx 4\pi \alpha_s^2.
\]

In order to properly analyze entanglement, the entanglement entropy obtained from the long time limit post-collisional wave function is required. In most cases, however, this must be performed numerically. Thus, to approach the problem analytically and simultaneously gain insights into the problem, it is convenient to make use of the linearized version of the entropy of the system, i.e. of the purity of the system [55]. The purity function is defined as

\[
\rho_A \equiv \text{Tr}_B(\rho_{AB}),
\]

where $\rho_A \equiv \text{Tr}_B(\rho_{AB})$ is the reduced density matrix of particle $A$ and $\rho_{AB}$ is the two-particle density matrix associated with the post-collisional two-particle wave
function. For pure two-particle states, the smaller the value of $\mathcal{P}$ the higher the entanglement. That is, the loss of purity provides an indicator of the degree of entanglement. Hence, a disentangled product state corresponds to $\mathcal{P} = 1$. Under the assumption that the two particles are well separated both initially (before collision) and finally (after collision), and further assuming that the colliding Gaussian wave packets are very narrow in momentum space ($\sigma_k \ll 1$ such that the phase shift can be treated as a constant $\theta(k_0)$), it follows that the purity of the post-collisional two-particle wave function is approximately given by [55]

$$
\mathcal{P} = 1 - 4 \left(2k_0^2 + \sigma_k^2\right) R_0 \alpha_s + \mathcal{O} \left(\alpha_s^2\right).
$$

(132)

Employing the scattering cross section $\Sigma = 4\pi \left| f (k_0) \right|^2 \approx \frac{4\pi^2 k_0^2 L}{9} = \frac{16\pi^2 \sigma_k^2 L}{9b} \approx 4\pi \alpha_s^2$, we may express the purity in an alternative manner, namely,

$$
\mathcal{P} = 1 - \frac{2 \left(2k_0^2 + \sigma_k^2\right) R_0 \sqrt{\Sigma}}{\sqrt{\pi}} + \mathcal{O} \left(\Sigma\right).
$$

(133)

Equations (132) and (133) above demonstrate how the entanglement can be measured from the loss of purity by the use of the scattering length or cross section. By combining (132) and the square of (129), we find that the purity

$$
\mathcal{P} \approx 1 - \frac{4k_0^2 \left(2k_0^2 + \sigma_k^2\right) R_0 L^3}{3} = 1 - \frac{8\mu V \left(2k_0^2 + \sigma_k^2\right) R_0 L^3}{3\hbar^2}.
$$

(134)

Equation (134) implies that the purity $\mathcal{P}$ can be expressed in terms of physical quantities such as the scattering potential height $V$ and the initial quantities $k_0$, $\sigma_0$ and $R_0$ via (130) and (126). This is the second significant finding obtained within our hybrid approach (quantum dynamical results combined with information geometric modeling techniques) that allows us to explain how the interaction potential height $V$ and the incident particle energies $E$ control the strength of the entanglement. The role played by $r$ in the quantities $\mathcal{P}$ and $V$ suggests that the physical information about quantum scattering—and therefore about quantum entanglement—is encoded in the statistical correlation coefficient, specifically in the covariance term $\text{Cov}(k_1, k_2) \equiv \langle \delta k_1 \delta k_2 \rangle = \langle \delta k_1 \rangle \langle \delta k_2 \rangle$ appearing in the definition of $r$.

The correlation coefficient $r$ can now be expressed in terms of the physical quantities such as the scattering potential, the scattering cross section and the purity. Solving equations (126), (130) and (134) for $r$, we obtain

$$
r = \frac{V}{E} = \frac{2\mu V}{\hbar^2 k_0^2} = \frac{2\mu V}{p_0^2},
$$

(135)

$$
\approx \frac{3\sqrt{\Sigma}}{2\sqrt{\pi} k_0^3 L},
$$

(136)

$$
\approx \frac{3 \left(1 - \mathcal{P}\right)}{4k_0^2 \left(2k_0^2 + \sigma_k^2\right) R_0 L^3}.
$$

(137)

In view of (99), (130) and (136), one obtains the following relation:

$$
V = \frac{4\hbar^2 k_0^2 \left(2k_0^2 + \sigma_k^2\right) R_0}{3\mu},
$$

(138)

which indicates that the uniform scattering potential density is solely determined by the initial conditions of the given system.

From (117)–(119) it is observed that for the micro-correlated Gaussian system considered here, more time is required to attain the same momentum value compared with the non-correlated Gaussian system. For example, in order to attain the same value as the initial momentum $p_0$, the non-correlated system and the micro-correlated system would require time intervals $\tau_0$ and $\tau_*$, respectively, where

$$
p_0 = \sqrt{p_0^2 + 2\sigma_0^2 \tanh \left(A_0 \tau_0\right)},
$$

(139)

$$
p_0 = \sqrt{(1 - r) \left(p_0^2 + 2\sigma_0^2\right) \tanh \left(A_0 \tau_*\right)}.
$$

(140)

Combining (139) and (140), we obtain

$$
\tanh \left(A_0 \tau_*\right) = \left(1 - r\right)^{-1/2} \tanh \left(A_0 \tau_0\right).
$$

(141)

Rewriting and expanding both sides of (141), we have

$$
1 - 2e^{-2A_0 \tau_*} + O \left(e^{-4A_0 \tau_*}\right) = \left(1 - r\right)^{-1/2} \left[1 - 2e^{-2A_0 \tau_0} + O \left(e^{-4A_0 \tau_0}\right)\right].
$$

(142)

Rounding (142) off and arranging terms, we obtain

$$
e^{-2A_0 \left(\tau_* - \tau_0\right)} \approx \left(1 - r\right)^{-1/2} \left[\frac{1}{2} \left(1 - r\right)^{-1/2} - 1\right] e^{2A_0 \tau_0},
$$

(143)

The first term on the right-hand side of (143) can be approximated to 1 since $(1 - r)^{-1/2} = 1 + \frac{1}{2} r + O \left(r^2\right)$ and $r \ll 1$. However, $r$ in the second term should not be disregarded in the same manner because $[(1 - r)^{-1/2} - 1] e^{2A_0 \tau_0} = \frac{1}{2} r + O \left(r^2\right)$ $e^{2A_0 \tau_0}$ is not negligible. Therefore, we may rewrite (143) as

$$
e^{-2A_0 \Delta} \approx 1 - \left[\left(1 - r\right)^{-1/2} - 1\right] \cdot \eta_\Delta,
$$

(144)

where $\Delta \equiv \tau_* - \tau_0$ represents a new quantity that we term ‘prolongation’, and

$$
\eta_\Delta \equiv \frac{1}{2} e^{2A_0 \tau_0} \left(\frac{p_0}{\sigma_0}\right)^2 \exp \left[\left(\frac{\sigma_0}{p_0}\right)^2 - \frac{3}{4} \left(\frac{\sigma_0}{p_0}\right)^4\right] + O \left(\left(\frac{\sigma_0}{p_0}\right)^6\right)
$$

(145)

for $\frac{p_0}{\sigma_0} \ll 1$ due to (2). The quantities $\tau_*$ and $\tau_0$ are the temporal intervals required for a particle to reach the same value of momentum $k_0$ from $0$ in the post-collisional scenario, in the presence and in the absence of correlations $r$, respectively. From (144) we find that

$$
\Delta \left(k_0, \sigma_0, r\right) \propto \ln \left[1 - \left(1 - r\right)^{-1/2} - 1\right] \cdot \eta_\Delta\right].
$$

(146)

Here, we can find the upper bound value of $r$ by means of (146) and (145),

$$
r < \frac{2}{\eta_\Delta}.
$$

(147)

The prolongation serves to quantify the time required by a micro-correlated system—relative to a corresponding non-correlated one—to attain the same momentum value.
in the mechanism of instability. In particular, fluctuating manifold underlying system dynamics plays a crucial role respectively. The fact that \( M \) is isotropic. It is known that the anisotropy of the curvatures are given by  
\[ K_{\mu} = -\frac{1}{2} K_{\mu i}, \quad K_{\mu j} = -\frac{1}{2} K_{-\mu j}, \quad K_{\alpha} = -\frac{1}{2} K_{-\alpha}. \]  
(149)

The Ricci scalar \( R_{M^\text{corr}} \) and Weyl projective \( W_{abcd} \) curvatures are given by

\[ R_{M^\text{corr}} = -\frac{1}{2} = R_{M^\text{uncorr}} \quad \text{and} \quad W_{abcd} = 0, \]  
(150)

respectively. The fact that \( W_{abcd} = 0 \) implies that the manifold \( M^\text{corr} \) is isotropic. It is known that the anisotropy of the manifold underlying system dynamics plays a crucial role in the mechanism of instability. In particular, fluctuating sectional curvatures require also that the manifold be anisotropic.

It can be shown [24] that the JLC equation (34) on \( M^\text{corr} \) reduces to

\[ \frac{D^2 J_{M^\text{corr}}}{D\tau^2} + Q J_{M^\text{corr}} = 0, \]  
(151)

where

\[ Q = \frac{\kappa}{\kappa - \omega^2} = -A^2 < 0 \quad \text{and} \quad J_{M^\text{corr}} \]  
is defined in (37). Since \( Q < 0 \), unstable solutions of equation (151) assume the form

\[ J_{M^\text{corr}} (\tau) = \frac{1}{\sqrt{-Q}} \omega (0) \sinh (\sqrt{-Q} \tau), \]  
(152)

where

\[ \omega (0) = \frac{dJ_{M^\text{corr}}}{d\tau} \bigg|_{\tau = 0}. \]

Recalling the definition of the hyperbolic sine function \( \sinh x = \frac{1}{2} \left( e^x - e^{-x} \right) \), it is clear that the geodesic deviation on \( M^\text{corr} \) is described by means of an exponentially divergent Jacobi vector field intensity \( J_{M^\text{corr}} \), a classical feature of chaos. In order to evaluate (38), we use (152) to find that

\[ |J_{M^\text{corr}} (\tau)|^2 = \frac{\omega^2 (0)}{Q} \sinh^2 (\sqrt{-Q} \tau), \]

and

\[ \left| \frac{dJ_{M^\text{corr}}}{d\tau} \right|^2 = \omega^2 (0) \cosh^2 (\sqrt{-Q} \tau). \]

Thus, for the case being considered, the Lyapunov exponents \( \lambda_{M^\text{corr}} = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} \ln |\lambda_{M^\text{corr}}| = 2\sqrt{-Q} \). Therefore, it follows that

\[ \lambda_{M^\text{corr}} > 2 \sqrt{-Q} = 2A_0 > 0. \]  
(153)

From (153) we observe the following points: the classical chaoticity does not depend on the statistical correlation, i.e. \( \lambda_{M^\text{corr}} = \lambda_{M^\text{uncorr}} \equiv \lambda_m = 2A_0 \), and the Lyapunov exponents can be determined solely from the initial conditions (see equation (116)).

Yet another finding uncovers an interesting qualitative connection between quantum entanglement quantified by the purity \( P \) in (134) and the information geometric complexity (IGC) of motion on the uncorrelated and correlated curved statistical manifolds \( M^\text{uncorr} \) and \( M^\text{corr} \), respectively. The information-geometric complexity as defined in [67] represents the volume of the effective parametric space explored by the system in its evolution between the chosen initial and final macrostates. The volume itself is in general given in terms of a multidimensional integral over the geodesic paths connecting the initial and final macrostates. For additional details, see [67]. Here, omitting technical details and following the works [24, 68, 69], one finds that

\[ C^\text{corr} (\tau; r) = \frac{8}{\lambda_m} \left[ \frac{3}{4} - \frac{1}{4} \frac{\sinh (\lambda_m \tau)}{\tau} \right], \]  
(154)

where \( C^\text{corr} \) denotes the IGC on \( M^\text{corr} \). Similarly, \( C^\text{uncorr} \equiv C^\text{IG} \) (\( r \rightarrow 0 \)) represents the IGC on \( M^\text{uncorr} \). As an aside we point out that (154) confirms that an increase in the correlational structure of the dynamical equations for the statistical variables labelling a macrostate of a system implies a reduction in the complexity of the geodesic paths on the underlying curved statistical manifolds [23, 68]. In other words, making macroscopic predictions in the presence of correlations is easier than in their absence.

The technical details that will be omitted in what follows can be found in [23]. By direct computation, the IGE is found
to be
\[ S_{M_{G}^{\text{corr.}}} (\tau; r) = \frac{\lambda_{M_{G}}}{2} \ln \left( \frac{1-r}{1+r} \right). \]  

(155)

For non-correlated Gaussian statistical models the IGE is given by \( S_{M_{G}^{\text{uncorr.}}} (\tau; 0) = S_{M_{G}^{\text{corr.}}} (\tau; r \to 0) \). Observe that in contrast to the macro-correlated case, the IGE for the micro-correlated case presents linear growth in the affine temporal parameter \( \tau \). Both the IGC and the IGE decrease in presence of micro-correlations. In particular, the IGE decreases by the factor \( \sqrt{\frac{1+r}{1-r}} < 1 \) for \( r > 0 \), whereas the IGE decreases by \( \frac{1}{2} \ln \left( \frac{1+r}{1-r} \right) < 0 \) for \( r > 0 \). With the quantities \( C_{\text{IG}}^{(\text{corr.})} (\tau; r) \) and \( S_{M_{G}^{\text{corr.}}} (\tau; r) \) in hand, we make the following interesting observations. From (154) we find that
\[ r = \frac{\Delta C^{2}}{C_{\text{total}}^{2}}, \]  

(156)

where
\[ \Delta C^{2} = C_{\text{IG}}^{(\text{uncorr.})} - C_{\text{IG}}^{(\text{corr.})} \]  

(157)

and
\[ C_{\text{total}}^{2} = C_{\text{IG}}^{(\text{uncorr.})} + C_{\text{IG}}^{(\text{corr.})}. \]  

(158)

Combining (134) and (154), it follows that
\[ \mathcal{P} \approx 1 - \eta_{c} \cdot \frac{\Delta C^{2}}{C_{\text{total}}^{2}}. \]  

(159)

where the dimensionless coefficient \( \eta_{c} \) reads
\[ \eta_{c} = \frac{4}{3} k_{0}^{2} \left( 2k_{0}^{2} + \sigma_{0}^{2} \right) R_{0} L_{3}. \]  

(160)

From (159) it is evident that the scattering-induced quantum entanglement and the information-geometric complexity of motion are connected. In particular, when purity approaches unity (entanglement-free scenario), the difference between the correlated and uncorrelated information geometric complexities approaches zero.

6. Conclusions

In this paper, we have introduced a theoretical construct that allows us to describe the macroscopic behavior of complex systems in terms of the underlying statistical structure of their microscopic degrees of freedom through statistical inference and information geometry methods. We reviewed the MrE formalism and the theoretical structure of the IGAC on curved statistical manifolds \( M_{G} \). Special focus was devoted to a description of the roles played by the sectional curvature \( K_{M_{G}} \), the Jacobi field intensity \( J_{M_{G}} \), and the IGE \( S_{M_{G}} \) as information geometric indicators of chaoticity (complexity). Four applications of these information-geometric techniques combined with ME methods were presented.

Firstly, we studied the chaotic behavior of a Gaussian statistical model describing an arbitrary system of \( l \) uncorrelated degrees of freedom and found that the hyperbolicity of the non-maximally symmetric 2\( l \)-dimensional statistical manifold \( M_{G} \) underlying such a Gaussian model leads to linear IGE growth and to exponential divergence of the Jacobi vector field intensity [22]. Secondly, we studied the asymptotic behavior of the dynamical complexity of the maximum probability trajectories on Gaussian statistical manifolds in the presence of correlation-like terms between macrovariables labeling the macrostates of the system under investigation. In the presence of correlation-like terms, we observed a power-law decay of the information geometric complexity at a rate determined by the correlation coefficient. We also presented an information-geometric analogue of the Zurek–Paz quantum chaos criterion of linear entropy growth. This analogy was motivated by studying the information geometrodynamics of an ensemble of random frequency macroscopic IHOS. The IGAC was also employed to study the ED on curved statistical manifolds induced by classical probability distributions commonly used in the study of regular and chaotic quantum energy level statistics. In doing so, we suggest an information-geometric characterization of regular and chaotic quantum energy level statistics.

Finally, the IGAC was used to describe the scattering-induced quantum entanglement between two Gaussian wave packets. The IGAC was used to analyze our specific two-variable micro-correlated Gaussian statistical model. The manifolds \( M_{G}^{\text{corr.}} \) and \( M_{G}^{\text{uncorr.}} \) were used to model the quantum entanglement induced by head-on scattering (in the s-wave approximation) of two spinless, structureless, non-relativistic particles, each represented by minimum uncertainty wave packets. Equation (128) allowed us to connect the entanglement strength—quantified in terms of purity—to the scattering potential and incident particle energies (135 and 134)). It was also found that it is possible to relate the statistical entanglement duration \( \Delta \) to the scattering potential \( V(x) \) and incident particle energies \( E \) (135 and 146). Recall that the prolongation \( \Delta \) was defined as the time required for the observed momentum difference between a correlated and a corresponding non-correlated system to vanish. The prolongation encodes information about how long it would take an entangled system to overcome the momentum gap generated by the scattering phase shift. The entangled system only attains the full value of momentum (i.e. the momentum value as seen in the corresponding non-correlated system) when the scattering phase shift vanishes. For this reason, the prolongation represents the temporal duration over which the entanglement is active.

The micro-correlation coefficient \( r \), a quantity that parameterizes the entanglement of microscopic degrees of freedom of the system, can be understood as the ratio of the potential to kinetic energy of the system. When \( r \neq 0 \) the wave packets experience the effect of a repulsive potential; the magnitude of the wave vectors (momenta) decreases relative to their corresponding non-correlated value. The upper bound value of \( r \) depends on \( p_{0} \) and \( \sigma_{0} \) in such a manner that \( r \) increases as \( p_{0} \) decreases. This result constitutes a significant, explicit connection between micro-correlations (the correlation coefficient \( r \)) and physical observables (the macrovariable \( p_{0} \)). For \( r \) values close to its upper bound, the prolongation \( \Delta \) becomes infinitely large. On the other hand, with \( r \) vanishing (i.e. no micro-correlation) \( \Delta \) is identically zero. With \( r \) fixed, however, the prolongation \( \Delta \) depends on \( p_{0} \) and \( \sigma_{0} \). Thus, the prolongation \( \Delta \) can
be controlled by the initial conditions $p_0$ and $\sigma_0$ as well as $r$. Maximal prolongation occurs when $r$ is greatest and the ratio $\sigma_0/p_0$ is smallest. For small initial $r$ and $p_0$, $\Delta$ would be correspondingly small, suggesting that for such scenarios quantum entanglement is transient. Furthermore, a quantitative relation between quantum entanglement (purity) and the IGC (159) was uncovered.

The complexity of geodesic paths on $M_S^{\text{(corr.)}}$ and $M_S^{\text{(uncorr.)}}$ was characterized through the asymptotic computation of the IGE and the Lyapunov exponents on each manifold. The Lyapunov exponents in both cases were found to be the same positive definite constant, $\lambda_{M_S^{\text{(corr.)}}} = \lambda_{M_S^{\text{(uncorr.)}}} \equiv \lambda_{M_S} = 2A_0 > 0$. The IGE $S_{M_S^{\text{(corr.)}}} (\tau; r)$ in the presence of micro-correlations assumes a smaller initial value relative to the non-correlated case $S_{M_S^{\text{(uncorr.)}}} (\tau; 0)$, while the growth characteristics of both correlated and non-correlated IGEs were found to be similar. Specifically, the larger the micro-correlation (i.e. the closer $r$ is to 1) the lower the initial value of the IGE. Thus, the stronger the initial micro-correlation, the larger the gap between $S_{M_S^{\text{(corr.)}}} (\tau = 0; r)$ and $S_{M_S^{\text{(uncorr.)}}} (\tau = 0; 0)$. This implies that $S_{M_S^{\text{(corr.)}}} (\tau; r) < S_{M_S^{\text{(uncorr.)}}} (\tau; 0)$. When micro-correlations vanish (i.e. when $r = 0$), we obtain the expected result $S_{M_S^{\text{(corr.)}}} (\tau; 0) = S_{M_S^{\text{(uncorr.)}}} (\tau; 0)$. The appearance of micro-correlation terms in the elements in the Fisher–Rao information metric leads to the compression of $S_{\text{IG}}^{\text{(corr.)}} (\tau; r)$ by the fraction $\sqrt{1-r}$ and thus to a reduction of the complexity of the path leading from $\Theta_{\text{initial}}$ to $\Theta_{\text{final}}$.

We emphasize that at this stage of development, IGAC remains an ambitious unifying information-geometric theoretical construct for the study of chaotic dynamics with several unsolved problems. However, based on our findings, we believe that it provides an interesting, innovative and potentially powerful way of studying and understanding the very important and challenging problems of classical and quantum chaos through statistical inference and information-geometric techniques. To clarify the relationship between our IGE and conventional measures of complexity, for instance the topological entropy (the supremum of the KS metric entropy), further investigation is required. Finally, we believe that our IGE may play an important role in both classical and quantum information science, but at the moment this remains a conjecture [4, 6].

One final remark. In this study, we have used information-geometric techniques and inductive inference methods to tackle some computational problems of interest in classical and quantum physics. Specifically, we have provided an information geometric characterization of the complexity of dynamical systems in terms of their probabilistic description on curved statistical manifolds. The maximally probable trajectories of the system arise through implementation of a principle of inference, the ME method, and the constructed indicators of complexity of such trajectories are defined in terms of asymptotic temporal averages. This leads to the following consideration. Most standard characterizations of chaoticity are based on the Boltzmann ergodic hypothesis, which relies on a ‘frequency’ interpretation of probabilities where concepts such as coarse graining (or randomization) appear. Such artificial concepts are not needed in our information-geometric characterization based on ME methods [16] and they may even lead to incorrect results [27, 70]. Does this mean that ergodic theory is unnecessary for our purposes [71]? Certainly, ergodic theory is explicitly used in the characterization of chaos in terms of the KS dynamical entropy and Lyapunov exponents [72]. It appears that it is implicitly used in the formulation of the IGAC and the IGE by way of focusing on the above-mentioned asymptotic temporal averages. Thus it remains to be seen whether or not the IGAC and the IGE are ‘special cases’ of a broader picture.

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