Analyzing the absolute stability region of implicit methods of solving ODEs

Mihail Semenov
30, Lenin Avenue, Tomsk, 634050, Russia, Tomsk Polytechnic University

Abstract
Review of implicit methods of integrating system of stiff ordinary differential equations is presented. Defines and graphically presents absolute stability region for Gears methods (backward differentiation formula) used to solve system of stiff ordinary differential equations. Recommendations for selecting the order of Gears method are given.

Keywords: Stiff systems of ordinary differential equations, implicit method, backward differentiation formulae, stability region

1. Introduction

Ordinary differential equations (ODE) are widely used for modelling real processes. Practice shows that the initial value problem (Cauchy problem) for ODE systems can be attributed to the following types: soft, stiff, ill-conditioned and rapidly oscillating. Each type is connected with specific demands to integrating methods. Stiff systems can be exemplified by problems of chemical kinetics [1, 2], nonstationary processes in complex electric circuits [3, 4], systems emerging while solving equations of heat conduction and diffusion [5], movement of celestial bodies and satellites [6], plasticity physics [7], etc.

The numerical solution of ODE systems is accompanied by problems due to the fact that at the modelling of a complex physical process speeds of local processes may vary significantly, while variables systems may be of different
orders or change within the interval of integration by orders of magnitude \[7, 8\].

Besides, in physical experiments a boundary layer may be observed which is characterized by quick changes in the object of research. Such a layer does not necessarily emerge at the beginning of the experiment, but only when some controlling parameter sharply changes or achieves a certain critical value. For the numerical solution of such problems the numerical method has to be chosen very carefully \[1, 2, 9\].

The purpose of this paper is to report on a continuing research effort aimed at the use and development of numerical methods in computer program for plasticity physics. The previous results were reported in \[8\] while using the same mathematical basis for solving stiff ODEs. The paper is organized as follows: the review of definitions stiff ODEs presented in Section 2. The approaches to finding a numerical solution of the stiff ODEs are discussed in Section 3. In Section 4, defines and graphically presents absolute stability region for backward differentiation formula and recommendations for various types of ODEs are given.

2. Stiff ordinary differential equations

Interest for stiff systems appeared at the beginning of the 20th century, initially in radio engineering (van der Pol problem, 1920 \[3, 4\]). Then there was a new wave of interest in the middle 1950s with application in studying equations of chemical kinetics, movement of celestial bodies \[6\], which contained both very slowly and very rapidly changing components. Methods like Runge-Kutta, which had been considered as highly reliable, produced mistakes in solving such problems.

One of the first attempts to give a definition of stiff systems was made by C.F. Curtiss and J.O. Hirschfelder in 1952. They proposed the following interpretation: stiff equations are equations where certain implicit methods perform better, than using classical explicit ones like Euler or Adams methods \[18\].

There are a number of interpretations of stiffness, each of which reflects certain aspects of the numerical solution (e.g. impossibility of using explicit methods of integration \[19\], presence of rapidly damped disturbances \[11, 20\], large Lipschitz constants or logarithmic norms of matrices \[21, 22\], big difference between eigenvalues of Jacobian matrix \[1, 2, 23, 24\], fullness of
Jacobian matrix [25], a priori fixed sign of the solution [26], number of transient phases [27], etc.) In some applications an important factor influencing the behavior of the numerical solution is the order of the ODE system [28] or the presence of a boundary level [24] [29], in others – only the limit behavior (gradual change) at wide intervals of integration. It is often unclear whether stiffness is attributed to a particular solution or the problem in general.

In papers [1–3, 13, 23, 24, 30] authors find it difficult to clearly define a stiff system ODE because of its complex character, so they present a working description of a stiff problem. This is a problem modelling a physical process, components of which possess incommensurable characteristic times, or a process characteristic time (reciprocal quantities of Jacobian eigenvalues) of which is much smaller than the interval of integration.

In 1970s L.F. Shampine and C.W. Gear, who had gained a wide experience of computing experiments with systems having components of the decision vector of different orders, offered their own definition of a stiff ODE system: the initial value problem for ODEs is stiff if the Jacobian \( J_{i,j} = \frac{\partial f_i}{\partial y_j}, i, j = 1, \ldots, N \) has at least one eigenvalue, for which real part is negative with high modulus, while the solution within the major part of the interval of integration changes slowly [15] [31].

In papers [23, 29] ODE system is called stiff if real components of all the eigenvalues of Jacobian are negative, i.e. \( Re(\lambda_i) < 0, i = 1, \ldots, N \) (the system is asymptotically stable) and the ratio

\[
s = \frac{\max\{|Re(\lambda_i)|, i = 1, 2, \ldots, N\}}{\min\{|Re(\lambda_i)|, i = 1, 2, \ldots, N\}}
\]

is large. The parameter \( s \) is called the stiffness ratio.

The problem of defining a stiff system is that for the stiffness ratio the boundary value where it becomes big is not given. A system of equations can be regarded as stiff if the stiffness ratio \( s \) exceeds 10, but in numerous applied problems this parameter reaches \( 10^6 \) and higher [32]. In paper [33] a concept of superstiffness is introduced, when the stiffness ratio reaches \( 10^6 \ldots 10^{12} \).

It must be pointed out that there are no simple methods for evaluating stiffness, so numerical methods working without stiffness testing are necessary.
3. Approaches to finding a numerical solution of the stiff ODEs

Let us regard a Cauchy problem for the ODEs of the first order, which can be presented as follows:

\[ Y' = F(x, Y), \quad x \in [x_0, b], \quad Y(x_0) = Y_0. \]  \hfill (1)

For numerical integrating of system (1) methods using a linear combination of the decision vector \( Y_1, Y_2, \ldots, Y_n, \ldots \) and its derivatives in some sequence of independent variable \( x_1, x_2, \ldots, x_n, \ldots \) are widely applied. Such methods are called linear methods [1, 4, 31].

Since the discovery of the stiffness in the development of numerical methods for integrating stiff systems, the following trends have appeared: the investigation of stiffness and the establishment of the theoretical apparatus stability analysis methods [3, 19], design and enhancement of the methods taking into account the specifics of the tasks [34–40] and the prospects for parallel [41, 42]. The most complete overview of the current numerical methods for solving stiff ODE systems with an extensive bibliography is presented in the papers [3, 4, 18, 29, 43, 44].

3.1. Taylor series

One of the easy ways to construct the solution to the system (1) at point, \( x_{n+1} \), if it is known at point, \( x_n \), is a method based on the expansion of solutions \( Y(x_{n+1}) \) in a Taylor series in the neighborhood of point, \( x_n \):

\[ Y(x_{n+1}) = Y(x_n) + hF(x_n, Y_n, h), \]

where

\[ F(x_n, Y_n, h) = Y'(x_n) + hY''(x_n)/2! + h^2Y'''(x_n)/3! + \ldots. \]

If this series is truncated at \( q \)-th term and replace \( Y(x_n) \) with the approximate value of \( Y_n \), thus an approximate formula is obtained:

\[ Y_{n+1} = Y_n + hF(x_n, Y_n) + hF'(x_n, Y_n)/2! + h^2F''(x_n, Y_n)/3! + \ldots + h^qF^{(q)}(x_n, Y_n)/(q+1)!). \]  \hfill (2)

If \( q = 1 \), the computational scheme of explicit Euler method [1, 2, 13, 23, 45] is obtained:

\[ Y_{n+1} = Y_n + hF(x_n, Y_n). \]
Application of the formula (2) is limited to only those tasks which can easily calculate the higher-order derivatives of the function \( F(x, Y) \) of the right side of the system (1). Though, it is usually not so.

### 3.2. Runge-Kutta methods

S. Runge (1895), K. Heun (1900) and M. Kutta (1901) put forward an approach based on constructing of the formula for \( Y_{n+1} \):

\[
Y(x_{n+1}) = Y(x_n) + h\Phi(x_n, Y_n, h),
\]

where \( h \) – integration step. The function \( \Phi(\cdot) \) is close to \( F(\cdot) \), but does not contain the derivatives from the right side of the equation. Thus, the series of explicit and implicit methods requiring \( s \)-stage calculation of the right side function at each integration step are obtained (\( s \)-stage methods).

The formulas of these methods are ideally applicable for practical calculations: they allow to change the integration step \( h \) easily. Perhaps the most famous is the formula of the 4-th order of 4-stage Runge-Kutta [4].

One of the major problems associated with the use of Runge-Kutta methods (in fact, almost of all the explicit methods) lies in choosing the size of the integration step \( h \), which provides the stability of the computational scheme [1, 2, 13, 23, 24, 45, 46]. Nevertheless, even nowadays, the explicit adaptive methods for solving stiff ODEs [38] are developed and widely used.

### 3.3. Backward differentiation formulas

These stiff tasks have made the implicit computational scheme particularly attractive and led to the development of such implicit methods which do not involve calculations based on the size of the integration step [18, 24, 31, 47–51]. The most common among them are the methods of Adams-Moulton and ”backward differentiation formulas” (more commonly known as Gear method). Having got the approximation to the solution at points, \( x_1, x_2, \ldots, x_n \), it is possible to find solutions at point, \( x_{n+1} \).

The computational schemes of the Adams-Moulton implicit methods take the following form [1, 2, 13, 23, 24, 45, 46]:

\[
Y_{n+1} = Y_n + h \sum_{i=0}^{q} \beta_i F(x_{n-i+1}, Y_{n-i+1}),
\]

where \( q \) determines the order of the method, the constants \( \beta_i, i = 0, 1, \ldots, q \) correspond to the chosen order of the method [1, 24, 31]. Moreover, the
implicit Euler’s method (the first order) and the trapezoidal method (the second order) are the special cases of the last computational scheme where $q = 0$ and $q = 1$.

The construction of the multistep methods is based on the polynomial of the degree $q$. The approximate value of the solution $Y(x)$ at the point, $x_{n+1}$ appears as a linear combination of the several approximate values of the solution and its derivative at this and the previous $q$ points. Obviously, the use of the multistep formulas requires the calculations of the $q$ units of the initial values $Y_1, Y_2, \ldots, Y_n$. The accuracy of setting these $q$ units should not be less precise than the accuracy of the formula. Thus, the polynomial can be represented as the following formula:

$$Y_{n+1} = \sum_{i=1}^{q} \alpha_i Y_{n-i+1} + h \sum_{j=0}^{q} \beta_j F(x_{n-j+1}, Y_{n-j+1}).$$

(4)

Some constants $\alpha_i$ and $\beta_i$ in equation (4) can take zero values. When $\beta_1 = \beta_2 = \ldots = \beta_q = 0$, it is possible to construct backward differentiation formulas.

4. The region of implicit methods stability

The first studies on the stability of multistep methods refer to the researches of G. Dahlquist. According to the definitions above, the stiff ODEs require the stability of numerical methods used to solve them. When getting the asymptotically stable solution of the stiff Cauchy problem, an error of the difference method should non-increase under any step, i.e. the method should be absolutely stable. Current reviews of the stability regions of multistep methods can be found in [3, 4].

The researchers [23, 45, 54] give a more clear definition to the stability method through the model first-order equation

$$y' = \lambda y, y(x_0) = y_0.$$ 

(5)

The general solution of the equation (5) takes the form of

$$y = C \cdot exp\{\lambda x\},$$

where $C$ – a constant, the solution of the equation (5) – a function $y = y_0 \cdot exp\{\lambda(x - x_0)\}$ – tends to zero, if $Re(\lambda) < 0$ and it infinitely grows in
absolute value for $Re(\lambda) > 0$, where $\lambda$ is a complex number. Further, the concept of absolute stability and the study of the absolute stability of the numerical methods are considered for the model equation (5).

Here, the stability domain of the multistep method (4) of solving the initial value problem (5) is defined as the set of points of complex numbers plane defined by the complex variable $\sigma = h\lambda$. For $\sigma = h\lambda$, this method applied to the model equation (5) is stable, i.e. it provides non-increase of an error [23, 45, 54].

To determine the region of the implicit methods stability (4), the characteristic (complex) polynomial is used:

$$
P(z) = \left(z^q - \alpha_1 z^{q-1} - \alpha_2 z^{q-2} - \ldots - \alpha_q \right) + \sigma(\beta_0 z^q + \beta_1 z^{q-1} + \beta_2 z^{q-2} + \ldots + \beta_q) = 0,$$

it may be represented in the form of [3, 4, 6, 24, 31]:

$$
\sigma(\theta) = -\frac{\sum_{k=0}^{q} \alpha_k e^{i(q-k)\theta}}{\beta_0 e^{i\theta}}, \quad (6)
$$

where $\alpha_k$, $\beta_k$ - coefficients of the method ($k=0, 1, 2, \ldots, q$; $\beta_1 = \beta_2 = \ldots = \beta_q = 0$), $q$ - the order of the method, $i$ - a unit imaginary number, $z = e^{i\theta}$ - an imaginary number, $0 \leq \theta \leq 2\pi$. To determine the region of the method absolute stability for the given value $\sigma = \sigma_0$, it is necessary to find the solution of the equation (6) relatively to $\theta$.

A set of points generated by the equation (6) corresponds to a geometrical locus of points of single radicals $\Gamma_\sigma$, for which $|z| = 1$ is true. The region of absolute stability of implicit methods (4) is considered to be an external area $\Gamma_\sigma$ since at $|\sigma| = h|\lambda| \rightarrow \infty$ implicit methods (4) are stable [3, 41, 24, 31].

To determine a geometrical locus of points described in the equation (6), a computer algebra system MathCAD is used. As a result, a geometrical locus of single radicals $\Gamma_\sigma$ is obtained (fig. 1). The region of absolute stability of methods is considered to be an external region $\Gamma_\sigma$ (shaded).

The study region of stability ($\Gamma_\sigma$ is a simple closed curve, fig. 1) shows that implicit methods (4) from 1 to 6 order inclusive are stiffly stable (first introduced by [31]). The stiffness property of the implicit method (4) from 1 to 6 order inclusive is attained at various values of a real number $\delta \leq 0$ (fig. 1, c – g marked with a dotted line). Particularly, for the methods of the first and second order it is $\delta = -0.1$; for the third order $\delta = -0.1$; for
Figure 1: Region of absolute stability for Gear methods of $q$-order: a) first (Euler method); b) second; c) third; d) forth; e) fifth; f) sixth.
Figure 2: The geometrical locus of points of single radicals \( \Gamma_\sigma \) for Gear method of the seventh order.

the fourth order \( \delta = -0.7 \); for the fifth order \( \delta = -2.4 \), for the sixth order \( \delta = -6.1 \).

For Gear’s method of the seventh order, the equation (6) will be:

\[
\sigma(\theta) = -\left( \frac{-e^{i7\theta} + \frac{980}{363}e^{i6\theta} - \frac{490}{1089}e^{i5\theta} - \frac{490}{1089}e^{i4\theta} - \frac{1225}{363}e^{i3\theta} + \frac{196}{121}e^{i2\theta} - \frac{490}{1089}e^{i\theta} + \frac{20}{363}}{140369 e^{i7\theta}} \right).
\]

Finding a solution to \( \sigma(\theta) \) relative to \( \theta \), it turns out that Gear method of the seventh order does not meet the requirements of stiff stability (fig. 2) [24, 31]. At origin of coordinates and at \( \text{Re}(\lambda) \approx -8 \), there are intersection points of \( \Gamma_\sigma \).

The results obtained for the equation (5) can be distributed on the ODEs [44]. In the case of an autonomous system of ODE of \( Y'' = AY \) type, where \( A \) is a constant matrix, it becomes possible to transact matching Jordan’s form and proceed to look for a solution to the system of ODE of \( Z' = JZ \) type, where \( J = T^{-1}AT = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \), \( \lambda_i \) – eigenvalues of matrix \( A \), \( i = 1, 2, \ldots, N \), \( Y = TZ \), \( Z = T^{-1}Y \). The matrix \( T \) is composed of eigenvectors of the matrix \( A \). Thus the initial system of ODE decomposes into \( n \) scalar equations, for which the solution can be found and the above approach to the region of stability determination applied [4, 24, 31].

If the coefficients of the system \( Y' = A(x)Y \) are not constant, the check of the eigenvalues \( A \) at each value \( x \) becomes laborious to calculate [31].

9
should be noted that operating with nonlinear systems of \( Y' = AY + G(x, y) \) type, the stability of the solution can be provided only at the origin of coordinates, moreover the stability can be broken for eigenvalues located on an imaginary axis \([19]\).

The overview of the alternative ways of defining stability regions for the implicit methods can be seen in \([3, 4, 19]\).

Implicit methods (Gear methods) can be applied for the calculation of a big category of the stiff ODEs. In this case decrease of stepsize (to the minimum possible) doesn’t always let us adapt to local solution and decrease volume of computation with required precision. Optimal strategy of using multistep methods implies availability of order autocontrol (from 1 to 6) and stepsize.

5. Conclusion

The area of the numerical methods for solving of ODEs is one of the most well-investigated topics in the mathematical literature. A number of techniques and solvers have been suggested, development, and described, but a clear definition of stiffness has not been provided, so the working definition of stiffness is still topical.

As a rule in most solvers for ordinary differential equations the explicit first order method (Euler method) or a second order method (trapezoidal method) is applied. Implicit Gear methods (backward differentiation formulas) are stiff from 1 to 6 order inclusive, so for the acceleration of the integration process of ordinary differential equations increasing order could be applied.

The results of the calculations let us define absolute stability regions for the implicit methods where changing of integration step over wide region when computational stability of the method is constant.

6. Acknowledgments

The author is grateful to Dr. S.N. Kolupaeva at Tomsk State University of Architecture and Building, Russia and Dr. J.C. Butcher at The University of Auckland, New Zealand who have been influence in the development of the ideas presented in this paper.
References

[1] D. Kahaner, C. Moler, S. Nash, Numerical Methods and Software, Prentice-Hall, Englewood Cliffs, N.J., 1989.

[2] G. Forsythe, M. Malcolm, C. Moler, Computer Methods for Mathematical Computations, Prentice-Hall, Englewood Cliffs, N.J., 1977.

[3] E. Hairer, G. Wanner, Solving Ordinary Differential Equations II: Stiff and Differential-Algebraic Problems, Springer, Berlin, 1996.

[4] J. C. Butcher, Numerical Methods for Ordinary Differential Equations, John Wiley & Sons, Ltd, Chichester, England, 2008.

[5] N. N. Kalitkin, Numerical methods for the solution of stiff equations, Matematicheskoe Modelirovanie 7 (6) (1995) 8–11.

[6] C. W. Gear, R. D. Skeel, The development of ODE methods: a symbiosis between hardware and numerical analysis, in: Proc. of the ACM Conf. on History of scientific and numeric computation.

[7] M. E. Semenov, S. N. Kolupaeva, T. A. Kovalevskaya, O. I. Daneyko, Mathematical modeling of strain hardening and evolution of the deformation defect medium in dispersion-hardened materials, NTL, Tomsk, 2007, pp. 5–41.

[8] S. N. Kolupaeva, T. A. Kovalevskaya, O. I. Daneyko, M. E. Semenov, N. A. Kulaeva, Modeling of temperature and rate dependence of the flow stress and evolution of a deformation defect medium in dispersion-hardened materials, Bulletin of the Russian Academy of Sciences: Physics 74 (11) (2010) 1527–1531.

[9] S. D. Cohen, A. C. Hindmarsh, Cvode, a stiff/nonstiff ODE solver in C, in: Conference Processing SciCADE95: Scientific Computing and Differential Equations, 1995.

[10] W. H. Enright, J. D. Pryce, Two FORTRAN Packages for Assessing Initial value Methods, Association for Computing Machinery Transaction on Mathematical Software 13 (1(3)) (1987) 1–23.

[11] K. Dekker, J. G. Verver, Stability of Runge-Kutta Methods for Stiff Nonlinear Differential Equations, North-Holland, Amsterdam, 1984.
[12] H. Rosenbrock, S. Story, Computational methods for chemist engineers, Mir, Moscow, 1968.

[13] K. Babenko (Ed.), Theory and construction of the numerical algorithm for the problems in mathematical physics, Nauka, Moscow, 1979.

[14] T. E. Hull, W. H. Enright, B. M. Fellen, A. E. Sedgwic, Comparing numerical methods for ordinary differential equations, SIAM Journal on Numerical Analysis 9 (4) (1972) 603–637.

[15] L. F. Shampine, C. W. Gear, A User’s View of Solving Stiff Ordinary Differential Equations, SIAM Review 21 (1) (1979) 1–17.

[16] F. Mazzia, F. Iavernaro, Test Set for Initial Value Problem Solvers, department of Mathematics, University of Bari, 2003. – Available at www.dm.uniba.it/~testset.

[17] L. Brugnano, F. Mazzia, D. Trigiante, Fifty years of stiffness, arXiv.org e-Print archive. 2009. – URL: http://arxiv.org/abs/0910.3780 (date of assess: 12.01.2011).

[18] C. F. Curtiss, J. O. Hirschfelder, Integration of Stiff Equations, in: Proceedings of the National Academy of Sciences of the United States of America, Vol. 38, 1952, pp. 235–243.

[19] E. Hairer, S. Nørsett, G. Wanner, Solving Ordinary Differential Equations. Nonstiff problems, Springer-Verlag, New York, 1987.

[20] U. M. Ascher, L. R. Petzold, Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations, SIAM, Philadelphia, 1998.

[21] G. Soderlind, The logarithmic norm. History and modern theory, BIT Numerical Mathematics 46 (3) (2006) 631–652.

[22] D. J. Higham, L. N. Trefethen, Stiffness of ODE, BIT Numerical Mathematics 33 (2) (1993) 286–303.

[23] V. M. Verzhbitsky, The principles of computational methods, Vysshaya shkola, Moscow, 2002.
[24] L. Chua, P.-M. Ling, Computer aided analyses of electronic circuit, algorithms and computations techniques, Prentice-Hall, Englewood Cliffs, N.J., 1975.

[25] L. A. M. Nejad, A comparison of stiff ODE solvers for astrochemical kinetics problems, Astrophys. and Space Sci. 299 (3) (2005) 1–29.

[26] E. Bertolazzi, Positive and conservative schemes for mass action kinetics, Computers Math. Applic. 32 (6) (1996) 29–43.

[27] P. Novati, An explicit one-step method for stiff problems, Computing 71 (2) (2003) 133–151.

[28] R. Weiner, B. A. Schmitt, H. Podhaisky, Rowmap a row-code with Krylov techniques for large stiff ODEs, Applied Numerical Mathematics 25 (2–3) (1997) 303–319.

[29] Yu. V. Rakitskiy, S. M. Ustinov, I. G. Chernorudski, Numerical methods for the solution of stiff systems, Nauka, Moscow, 1979.

[30] J. Ortega, G. Golub, Scientific computation and differential equations. An Introduction to Numerical Methods, Academic Press, Boston, 1992.

[31] C. W. Gear, Numerical Initial Value Problem in Ordinary Differential Equations, Prentice-Hall, Englewood Cliffs, N.J., 1971.

[32] D. Petcu, M. Dragan, Designing an ODE solving environment, Springer-Verlag, Berlin, 2000, pp. 319–338.

[33] V. N. Gridin, V. B. Mihajlov, G. A. Kuprijanov, K. V. Mihajlov, Stable numerical-analytical methods for the solution of differential-algebraic systems of equations, Matematicheskoe Modelirovanie (10) 35–50.

[34] V. A. Vshivkov, About one way of W-method construction for the stiff systems of ODEs, Computer technologies 12 (4) (2007) 42–58.

[35] A. Rahumanthan, D. Stanescu, High-order W-methods, Journal of Computational and Applied Mathematics 233 (2002) 381–395.

[36] S. S. Filippov, ABC-schemes for the stiff systems of ordinary differential equations, Doklady Academii Nauk 399 (2) (2004) 170–172.
[37] G. Hojjati, M. Y. R. Apdabili, A-EBDF: An adaptive method for numerical solution of stiff systems of ODE, Mathematics and Computers in Simulations 66 (1) (2004) 33–41.

[38] L. M. Skvortsov, Explicit two-step Runge-Kutta methods, Matematicheskoe Modelirovanie 21 (9) (2009) 54–65.

[39] W. Hundsdorfer, V. Savcenco, Analysis of multirate theta-methods for stiff ODEs, Applied Numerical Mathematics 59 (3-4) (2009) 693–706.

[40] V. Savcenco, Construction of multirate roads methods for stiff ODEs, Journal of Computational and Applied Mathematics 225 (2) (2009) 323–337.

[41] H. Podhaisky, B. A. Schmitt, R. Weiner, Design, analysis and testing of some parallel two-step W-methods for stiff systems, Applied Numerical Mathematics 42 (2002) 381-395.

[42] J. Bahi, J. Charr, R. Couturier, D. Laiymani, A parallel algorithm to solve large stiff ODE systems on grid systems, International Journal of High Performance Computational Applications 23 (2) (2009) 140–159.

[43] G. Soderlind, L. Wang, Evaluating numerical ODE/DAE methods, algorithms and software, Journal of Computational and Applied Mathematics 185 (2) (2006) 244–260.

[44] G. Hall, J. M. Watt. (Eds.), Modern numerical methods for ordinary differential equations, Clarendon Hress, Oxford, 1976.

[45] V. M. Verzhibitsky, Numerical methods (mathematical analysis and ordinary differential equations), Vysshaya shkola, Moscow, 2001.

[46] O. B. Arushanyan, S. F. Zalyetkin, Numerical solution of ordinary differential equations in Fortran, MSU Publishing, Moscow, 1990.

[47] C. W. Gear, The automatic integration of ordinary differential equations, Communications of the ACM 14 (3) (1971) 176–179.

[48] C. W. Gear, The Algorithm 407: DIFSUB for solution of ordinary differential equations [D2], Communications of the ACM 14 (3) (1971) 185–190.
[49] A. Nordsieck, On Numerical Initegration of Ordinary Differential Equations, Mathematics of Computation 16 (77) (1962) 22–49.

[50] W. L. Miranker, The Computation Theory of Stiff Differential Equations, Num. 102 in Serie - III, IAC. Institute per Le Applicazioni Del calcolo Mauro Picone, Roma, 1975.

[51] W. L. Miranker, Numerical method for stiff equations and singular perturbation problem, Vol. 5 of Mathematics and its applications, Reidel Publishing Company, Reidel, 1981.

[52] G. Dahlquist, Convergence and stability in the numerical integration of ordinary differential equations, Math. Scandinavica 4 (1956), 33–50.

[53] G. Dahlquist, A special stability problem for linear multistep methods, BIT Numerical Mathematics 3 (1) (1963) 27–43.

[54] A. A. Samarskiy, A. V. Gulin, Numerical methods, Nauka, Moscow, 1989.