Boundedness in a quasilinear fully parabolic Keller-Segel system of higher dimension with logistic source

Cibing Yang  Xinru Cao  Zhaoxin Jiang  Sining Zheng†
School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, P.R. China

March 10, 2015

Abstract. This paper deals with the higher dimension quasilinear parabolic-parabolic Keller-Segel system involving a source term of logistic type $u_t = \nabla \cdot (\phi(u)\nabla u) - \chi \nabla \cdot (u \nabla v) + g(u)$, $\tau v_t = \Delta v - v + u$ in $\Omega \times (0,T)$, subject to nonnegative initial data and homogeneous Neumann boundary condition, where $\Omega$ is smooth and bounded domain in $\mathbb{R}^n$, $n \geq 2$, $\phi$ and $g$ are smooth and positive functions satisfying $k s^p \leq \phi$ when $s \geq s_0 > 1$, $g(s) \leq as - \mu s^2$ for $s > 0$ with $g(0) \geq 0$ and constants $a \geq 0$, $\tau, \chi, \mu > 0$.

It was known that the model without the logistic source admits both bounded and unbounded solutions, identified via the critical exponent $\frac{2}{n}$. On the other hand, the model is just a critical case with the balance of logistic damping and aggregation effects, for which the property of solutions should be determined by the coefficients involved. In the present paper it is proved that there is $\theta_0 > 0$ such that the problem admits global bounded classical solutions, regardless of the size of initial data and diffusion whenever $\frac{k}{\mu} < \theta_0$. This shows the substantial effect of the logistic source to the behavior of solutions.

Keywords: Boundedness; Keller-Segel system; Chemotaxis; Global existence; Logistic source.

Mathematics Subjection Classification: 92C17; 35K55; 35B35; 35B40.

*Supported by the National Natural Science Foundation of China (11171048)
†Corresponding author. E-mail: 1145250006@qq.com (C. Yang), caoxinru@gmail.com (X. Cao), jxzdut@163.com (Z. Jiang), snzheng@dlut.edu.cn (S. Zheng)
1 Introduction

In this paper, we consider the higher dimension quasilinear parabolic-parabolic Keller-Segel system with logistic source

\[
\begin{aligned}
\begin{cases}
u_t &= \nabla \cdot (\phi(u)\nabla u) - \nabla \cdot (\psi(v)\nabla v) + g(u), & (x,t) \in \Omega \times (0,T), \\
\tau v_t &= \Delta v - v + u, & (x,t) \in \Omega \times (0,T), \\
\frac{\partial u}{\partial n} &= \frac{\partial v}{\partial n} = 0, & (x,t) \in \partial \Omega \times (0,T), \\
u(x,0) &= u_0(x), & v(x,0) = v_0(x), & x \in \Omega,
\end{cases}
\end{aligned}
\]

(1.1)

where \(\Omega \subset \mathbb{R}^n (n \geq 2)\) is a bounded domain with smooth boundary, \(\tau > 0\). Functions \(\phi, \psi \in C^2([0,\infty))\) satisfy

\[
\begin{align}
\psi(s) &= \chi s \\
\phi(s) &= \begin{cases} 
0, & s \leq 0, \\
k s^p, & s \geq s_0,
\end{cases}
\end{align}
\]

(1.2)

(1.3)

with \(\chi > 0, k > 0, p \in \mathbb{R}, s_0 > 1, \) and \(g \in C^\infty([0,\infty))\) fulfills

\[
g(s) \leq a s - \mu s^2, \quad s > 0
\]

(1.4)

with \(g(0) \geq 0\), and constants \(a \geq 0, \mu > 0\). Here, \(u\) and \(v\) represent the density of cells and the concentration of chemical signal respectively. The classical Keller-Segel system can be obtained by setting \(\phi \equiv \psi \equiv 1\) and \(g \equiv 0\) in (1.1), which models the mechanism of chemotaxis, and has been extensively studied since 1970, we refer to [9, 13, 23, 25] and the reference therein.

Eq. (1.1) with \(g(u) \equiv 0\) is a type of refined models pursued by Hillen and Painter [8], with the bacterial cells having a positive size, the so-called volume-filling effect. Beyond this, more general functions \(\phi\) and \(\psi\) are involved to denote the diffusivity and chemotatic sensitivity, respectively [5, 10, 22]. When \(\phi(s) \sim s^p\) and \(\psi(s) \sim s^q\) for large \(s\), a critical exponent \(\frac{2}{n}\) on the interplay of \(\phi\) and \(\psi\) has been found to identify boundedness and unboundedness. Namely, if \(q - p < \frac{2}{n}\), then all solutions are global and uniformly bounded [24] [18]; however, if \(q - p > \frac{2}{n}\), unbounded solutions do exist [22], even finite-time blow-up may occur under some additional conditions \(n \geq 3\) and \(q \geq 1\) [14, 18].

Apart from the aforementioned system, a source of logistic type is included in (1.1) to describe the spontaneous growth of cells. The effect of preventing ultimate growth has been widely studied [16, 17, 20, 19]. In the related classical semilinear chemotaxis systems, that is when \(\phi(u) \equiv 1\) and \(\psi(u) = \chi u\) with \(\chi > 0\), such proliferation mechanisms are known to prevent chemotactic collapse: In [19], for instance, it was proved that if \(\tau = 0\) and \(\mu > \frac{(n-2)\chi}{n}\), solutions of the parabolic-elliptic system are global and remain bounded. The same conclusion is true for the fully parabolic system with \(\tau > 0\) if either \(n \leq 2\), \(\mu > 0\) [16], or \(n \geq 3\) and \(\mu > \mu_0\) with some constant \(\mu_0(\chi) > 0\) [20]. This is in sharp contrast to the possibility of blow-up which is known to occur in such systems when \(g \equiv 0\) and \(n \geq 2\) [16, 13, 15, 25].

In this context, we intend to study (1.1) with \(\tau > 0\) under the conditions (1.2)–(1.4). It is our purpose to investigate the interaction among the triple of nonlinear diffusion, aggregation and the logistic
absorption. By taking \( u^{\gamma-1} \) as the test function to the first equation, and then substituting the second equation, the standard \( L^\gamma \) estimate argument yields

\[
\frac{1}{\gamma} \frac{d}{dt} \int_\Omega u^\gamma \leq - (\gamma - 1) \int_\Omega u^{\gamma+p-2} |\nabla u|^2 + \frac{\gamma - 1}{\gamma + q - 1} \int_\Omega u^{\gamma+q} - \frac{\gamma - 1}{\gamma + q - 1} \int_\Omega u^{\gamma+q-1} \Delta v
+ a \int_\Omega u^\gamma - \mu \int_\Omega u^{\gamma+1}. \tag{1.5}
\]

Comparing the terms \( \frac{\gamma - 1}{\gamma + q - 1} \int_\Omega u^{\gamma+q} \) and \(-\mu \int_\Omega u^{\gamma+1}\), it is easy to find that \( q = 1 \) is critical. It has been proved that when \( q < 1 \), the logistic dampening rules out the occurrence of blow-up regardless of diffusion [2]. And when \( q > 1 \), the strong diffusion with \( q - p < \frac{2}{n} \) ensures global boundedness by [18], without the help of the logistic damping. The critical case \( q = 1 \) is more involved: from (1.5) we may expect that under the balance of logistic damping and aggregation effects, the coefficients would determine weather the solution is bounded. In [3], it has been proved that when \( q = 1 \) with \( \tau = 0 \), the solutions are bounded if \( \mu > (1 - \frac{2}{m(1-p)+}) \chi \) for the parabolic-elliptic case. This makes an agreement with the above expectation. The main result of the present paper is the following theorem for the fully parabolic Keller-Segel system.

**Theorem 1.** Suppose that \( \Omega \subset \mathbb{R}^n \) \((n \geq 2)\) is a bounded domain with smooth boundary, and \( \chi, \mu, \tau > 0 \). Assume that \( \psi(u) = \chi u, \phi \) and \( g \) satisfy (1.2)-(1.3), \( g \) fulfills (1.4), \( u_0 \in C^0(\bar{\Omega}) \) and \( v_0 \in W^{1,r}(\Omega) \) (with some \( r > n \)) both are nonnegative. Then there is \( \theta_0 > 0 \) such that if \( \frac{\chi}{\mu} < \theta_0 \), for any nonnegative \( u_0 \in C^0(\bar{\Omega}) \) and \( v_0 \in W^{1,r}(\Omega) \) with \( r > n \), Eq. (1.1) uniquely admits a classical solution \((u,v)\) such that \( u \in C^0(\bar{\Omega} \times [0, \infty)) \) \& \( C^2(\bar{\Omega} \times (0, \infty)) \) \& \( L^\infty_{loc}([0,T_{max}); W^{1,r}(\Omega)) \). Moreover, \((u,v)\) is bounded in \( \Omega \times (0, \infty) \).

**Remark 1.** We underline that the above result is independent of the value of \( p \) in (1.3), and thus extends the analogue result for the semilinear case [20]. Moreover, due to the technique used here, the convexity of \( \Omega \) (required in [20]) is unnecessary in our theorem.

Unlike using the trace embedding technique to estimate the boundary integral in [11, 12], our approach strongly relies on the Maximal Sobolev Regularity.

The paper is arranged as follows. In section 2, we deal with the local existence and the extensibility of classical solution to (1.1) as well as a variation of Maximal Sobolev Regularity. Section 3 will be devoted to prove Theorem 1.

## 2 Preliminaries

The local solvability to (1.1) for sufficiently smooth initial data can be addressed by methods involving standard parabolic regularity theory in a suitable fixed point framework. In fact, one can thereby also derive a sufficient condition for extensibility of a given local-in-time solution. Details of the proof can be founded in [2].

**Lemma 2.1.** Suppose \( \Omega \subset \mathbb{R}^n \) \((n \geq 3)\) is a bounded domain with smooth boundary, \( \phi \) and \( \psi \) satisfy (1.2)-(1.3), \( g \) fulfills (1.4), \( u_0 \in C^0(\bar{\Omega}) \) and \( v_0 \in W^{1,r}(\Omega) \) (with some \( r > n \)) both are nonnegative. Then
there exists \((u, v) \in (C^0(\Omega \times [0, T_{\text{max}}]) \cap C^{2,1}_t(\Omega \times (0, T_{\text{max}})))^2\) with \(T_{\text{max}} \in (0, \infty)\) classically solving (1.1) in \(\Omega \times (0, T_{\text{max}})\). Moreover, if \(T_{\text{max}} < \infty\), then
\[
\limsup_{t \nearrow T_{\text{max}}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty.
\] (2.1)

Note that in \((1.3)\), the term \(-\frac{\gamma - 1}{\gamma + q - 1} \int_{\Omega} u^{\gamma + q - 1} \Delta v\) is unsigned. We thus intend to estimate its absolute value adequately. For this purpose, we will make use of the following property referred to as a variation of Maximal Sobolev Regularity, which will play an important role in the proof of our main result. The following Lemma is not the original version of a corresponding statement in [7, Theorem 3.1], but by means of a simple transformation by including an exponential weight function as in [7].

**Lemma 2.2.** Let \(r \in (1, \infty), \tau > 0\). Consider the following evolution equation
\[
\begin{aligned}
&\tau v_t = \Delta v - v + u, \quad (x, t) \in \Omega \times (0, T), \\
&\frac{\partial v}{\partial \nu} = 0, \quad (x, t) \in \partial \Omega \times (0, T), \\
&v(x, 0) = v_0(x), \quad x \in \Omega.
\end{aligned}
\] (2.2)

For each \(v_0 \in W^{2, r}(\Omega) (r > n)\) with \(\frac{\partial v}{\partial \nu} = 0\) on \(\partial \Omega\) and any \(u \in L^r((0, T); L^r(\Omega))\), there exists a unique solution
\[
v \in W^{1, r}((0, T); L^r(\Omega)) \cap L^r((0, T); W^{2, r}(\Omega)).
\]

Moreover, there exists \(C_r > 0\), such that if \(s_0 \in [0, T), v(\cdot, s_0) \in W^{2, r}(\Omega)(r > n)\) with \(\frac{\partial v(\cdot, s_0)}{\partial \nu} = 0\), then
\[
\int_{s_0}^{T} \int_{\Omega} e^{\frac{s}{\tau}} |\Delta v|^r \leq C_r \int_{s_0}^{T} \int_{\Omega} e^{\frac{s}{\tau}} u^r + C_r \tau e^{\frac{s}{\tau}} \left( \|v(\cdot, s_0)\|_{L^r(\Omega)} + \|\Delta v(\cdot, s_0)\|_{L^r(\Omega)} \right),
\] (2.3)

**Proof.** Let \(w(x, s) = e^s v(x, \tau s)\). We derive that \(w\) satisfies
\[
\begin{aligned}
w_s(x, s) &= \Delta w(x, s) + e^s u(x, \tau s), \quad (x, s) \in \Omega \times (0, T), \\
\frac{\partial w}{\partial \nu} &= 0, \quad (x, s) \in \partial \Omega \times (0, T), \\
w(x, 0) &= v_0(x), \quad x \in \Omega.
\end{aligned}
\] (2.4)

Applying the Maximal Sobolev Regularity ([7, Theorem 3.1]) to \(w\), we obtain that
\[
\int_{0}^{T} \int_{\Omega} |\Delta w(x, s)|^r \leq C_r \int_{0}^{T} \int_{\Omega} e^{s} u(x, \tau s)|^r + C_r \|v_0\|_{L^r(\Omega)} + C_r \|\Delta v_0\|_{L^r(\Omega)}.
\] (2.5)

Substituting \(v\) into the above inequality and changing the variables imply
\[
\int_{0}^{T} \int_{\Omega} e^{\frac{s}{\tau}} |\Delta v|^r \leq C_r \int_{0}^{T} \int_{\Omega} e^{\frac{s}{\tau}} u^r + C_r \tau \|v_0\|_{L^r(\Omega)} + C_r \tau \|\Delta v_0\|_{L^r(\Omega)}.
\]

Consequently, for any \(s_0 > 0\), replacing \(v(t)\) by \(v(t + s_0)\), we prove (2.3). \(\square\)
3 Proof of Theorem \[1\]

In this section, we are going to prove our main result. Since the regularity obtained in (2.5) requires that the initial data satisfy homogeneous Neumann boundary conditions, we will perform a small time shift and use any positive time as the “initial time” to guarantee that the respective boundary condition is satisfied naturally.

Specifically, given any \(s_0 \in (0,T_{\text{max}})\) such that \(s_0 \leq 1\), from the regularity principle asserted by Lemma 2.1, we know that \((u(\cdot,s_0),v(\cdot,s_0)) \in C^2(\bar{\Omega})\) with \(\frac{\partial v(\cdot,s_0)}{\partial n} = 0\) on \(\partial \Omega\), so that in particular we can pick \(M > 0\) such that

\[
\sup_{0 \leq \theta \leq s_0} \|u(\cdot,\theta)\|_{L^\infty(\Omega)} \leq M, \quad \sup_{0 \leq \theta \leq s_0} \|v(\cdot,\theta)\|_{L^\infty(\Omega)} \leq M, \quad \text{and} \quad \|\Delta v(\cdot,s_0)\|_{L^\infty(\Omega)} \leq M. \tag{3.1}
\]

Now we proceed to derive an a priori estimate which will constitute the main part of the work.

**Lemma 3.1.** Suppose \(\Omega \subset \mathbb{R}^n, n \geq 3\), is a bounded domain with smooth boundary, \(\tau > 0\) and \(\chi \in \mathbb{R}\). For any \(\gamma > 1, \eta > 0\), there exist \(\mu, \tau, \eta > 0\) and \(C = C(\gamma, |\Omega|, \mu, \chi, \eta, u_0, v_0) > 0\) such that if \(\mu > \mu_{\gamma, \eta}\), then

\[
\|u(\cdot,t)\|_{L^\gamma(\Omega)} \leq C
\]

for all \(t \in (s_0, T_{\text{max}})\).

**Proof.** We fix \(s_0 \in (0,T)\) such that \(s_0 \leq 1\). For arbitrary \(\gamma > 1\), take \(u^{\gamma-1}\) as a test function for the first equation in (1.1) and integrate by part to obtain

\[
\frac{1}{\gamma} \frac{d}{dt} \int_{\Omega} u^\gamma = -(\gamma - 1) \int_{\Omega} u^{\gamma-2} \phi(u) \nabla u \cdot \nabla v + \chi(\gamma - 1) \int_{\Omega} u^{\gamma-1} \nabla u \cdot \nabla v + a \int_{\Omega} u^\gamma - \mu \int_{\Omega} u^{\gamma+1} \\
\leq \chi \frac{\gamma - 1}{\gamma} \int_{\Omega} u^{\gamma} \nabla v + a \int_{\Omega} u^\gamma - \mu \int_{\Omega} u^{\gamma+1} \\
= -\chi \frac{\gamma - 1}{\gamma} \int_{\Omega} u^\gamma \Delta v + a \int_{\Omega} u^\gamma - \mu \int_{\Omega} u^{\gamma+1} \\
= -\frac{\gamma + 1}{\tau \gamma} \int_{\Omega} u^{\gamma} - \chi \frac{\gamma - 1}{\gamma} \int_{\Omega} u^\gamma \Delta v + \left( a + \frac{\gamma + 1}{\tau \gamma} \right) \int_{\Omega} u^\gamma - \mu \int_{\Omega} u^{\gamma+1} \tag{3.2}
\]

for all \(t \in (s_0, T_{\text{max}})\). Here by Young’s inequality, for any \(\varepsilon > 0\), there exists \(c_1 > 0\) such that

\[
\left( a + \frac{\gamma + 1}{\tau \gamma} \right) \int_{\Omega} u^{\gamma} \leq \varepsilon \int_{\Omega} u^{\gamma+1} + c_1(a, \varepsilon, \gamma)|\Omega|, \tag{3.3}
\]

where \(c_1(a, \varepsilon, \gamma) = \frac{1}{\gamma}(1 + \frac{\varepsilon}{\gamma})^{-(\gamma+1)}\varepsilon^{-\gamma}(a + \frac{\gamma + 1}{\tau \gamma})^{\gamma+1}\). Young’s inequality also implies that

\[
-\chi \frac{\gamma - 1}{\gamma} \int_{\Omega} u^\gamma \Delta v \leq \chi \int_{\Omega} u^\gamma |\Delta v| \\
\leq \eta \int_{\Omega} u^{\gamma+1} + c_2 \eta^{-\gamma} \chi^{\gamma+1} \int_{\Omega} |\Delta v|^{\gamma+1} \tag{3.4}
\]

with \(c_2 = \sup_{\gamma > 1} \frac{\gamma}{\gamma}(1 + \frac{\varepsilon}{\gamma})^{-(\gamma+1)} < \infty\). By substituting (3.3) and (3.4) into (3.2), we find that

\[
\frac{d}{dt} \left( \frac{1}{\gamma} \int_{\Omega} u^\gamma \right) \leq -\frac{\gamma + 1}{\tau} \frac{1}{\gamma} \int_{\Omega} u^\gamma - (\mu - \varepsilon - \eta) \int_{\Omega} u^{\gamma+1} + c_2 \eta^{-\gamma} \chi^{\gamma+1} \int_{\Omega} |\Delta v|^{\gamma+1}
\]
\[ + c_1(a, \varepsilon, \gamma)|\Omega| \] (3.5)

holds for all \( t \in (s_0, T_{\max}) \). Applying the variation-of-constants formula to the above inequality shows that

\[
\frac{1}{\gamma} \int_{\Omega} u^\gamma(\cdot, t) \leq e^{-\frac{2\gamma}{3}\frac{s}{t}} \left( e^{-\frac{\gamma}{3}(t-s)} \int_{\Omega} u^\gamma(\cdot, s_0) - (\mu - \varepsilon - \eta) \int_{s_0}^{t} e^{-\frac{\gamma}{3}(t-s)} \int_{\Omega} u^{\gamma+1} + c_2 \right)
\]

\[
+ c_2 \eta^{-\gamma} \int_{s_0}^{t} e^{-\frac{\gamma}{3}(t-s)} \int_{\Omega} |\Delta v|^{\gamma+1} + c_3 |\Omega| \int_{s_0}^{t} e^{-\frac{\gamma}{3}(t-s)}
\]

\[
\leq - (\mu - \varepsilon - \eta) e^{-\frac{\gamma}{3}(t-s)} \int_{s_0}^{t} e^{\frac{\gamma}{3}(t-s)} u^{\gamma+1}
\]

\[
+ c_2 \eta^{-\gamma} \int_{s_0}^{t} e^{-\frac{\gamma}{3}(t-s)} \int_{\Omega} |\Delta v|^{\gamma+1} + c_3(a, \varepsilon, \gamma, |\Omega|, s_0)
\] (3.6)

for all \( t \in (s_0, T_{\max}) \), where

\[ c_3(a, \varepsilon, \gamma, |\Omega|) = c_1 |\Omega| \int_{s_0}^{t} e^{-\frac{\gamma}{3}(t-s)} + \frac{1}{\gamma} \int_{\Omega} u^\gamma(\cdot, s_0) \]

is independent of \( t \). Next, we apply Lemma [22] to see that there is \( C_\gamma > 0 \) such that

\[
c_2 \eta^{-\gamma} \int_{s_0}^{t} e^{-\frac{\gamma}{3}(t-s)} \int_{\Omega} |\Delta v|^{\gamma+1}
\]

\[
\leq c_2 \eta^{-\gamma} \int_{s_0}^{t} e^{-\frac{\gamma}{3}(t-s)} \int_{\Omega} |\Delta v|^{\gamma+1} + C_\gamma \tau e^{\frac{\gamma}{3}(t-s)} |v(\cdot, s_0)|^{\gamma+1}\]

\[
\int_{W^{2,\gamma+1}(\Omega)}
\] (3.7)

Inserting (3.7) into (3.6) with some rearrangement, we finally arrive at

\[
\frac{1}{\gamma} \int_{\Omega} u^\gamma(\cdot, t) \leq - (\mu - \varepsilon - \eta - c_2 C_\gamma \eta^{-\gamma} \int_{\Omega} v(\cdot, s_0)|^{\gamma+1}) e^{-\frac{\gamma}{3}(t-s)} \int_{\Omega} e^{\frac{\gamma}{3}(t-s)} u^{\gamma+1} + c_3
\]

\[
+ c_2 C_\gamma \tau \eta^{-\gamma} \int_{s_0}^{t} e^{-\frac{\gamma}{3}(t-s)} |v(\cdot, s_0)|^{\gamma+1}\]

\[
\int_{W^{2,\gamma+1}(\Omega)}
\] (3.8)

for all \( t \in (s_0, T_{\max}) \). Let \( \mu_{\gamma, \eta} = \eta + c_2 C_\gamma \eta^{-\gamma} e^{-\gamma s_0} \), we can choose \( \varepsilon \in (0, \mu - \mu_{\gamma, \eta}) \) such that

\[ \mu - \varepsilon - \eta - c_2 C_\gamma \eta^{-\gamma} e^{-\gamma s_0} \geq 0. \]

It is entailed that

\[
\frac{1}{\gamma} \int_{\Omega} u^\gamma(\cdot, t) \leq c_4
\] (3.9)

for all \( t \in (s_0, T_{\max}) \) with \( c_4 = c_3 + c_2 C_\gamma \tau \eta^{-\gamma} e^{-\gamma s_0} \). This completes the proof by the above inequality together with (3.11).

Next, we invoke the well established Moser iteration to get boundedness of \((u, v)\).

Proof of Theorem [7] By Morse’s iteration (Lemma A.1 in [18]), we claim that there is \( \gamma_0(n, p) > n > 0 \), determined via (A.8)–(A.10) in Lemma A.1 of [18], such that if

\[
\|u(\cdot, t)\|_{L^\gamma(\Omega)} < \infty
\] (3.10)
for all $\gamma \geq \gamma_0$ and all $t \in (s_0, T_{\text{max}})$, then there exists $C_1 > 0$ such that
\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_1
\] (3.11)
for all $t \in (s_0, T_{\text{max}})$. Actually, (3.10) implies that $\nabla v$ is bounded. It is easy to check that all assumptions of Lemma A.1 are fulfilled.

Let $\theta_0$ satisfy
\[
\inf_{\eta > 0} \mu_{\eta, \gamma_0} = \inf_{\eta > 0} (\eta + c_2 C_{\gamma_0+1}\eta^{-\gamma_0}\lambda^{\gamma_0+1}) = \frac{1}{\theta_0^\lambda}.
\]
We see that $\lambda < \theta_0$ implies $\mu > \mu_{\eta, \gamma_0}$ with some $\eta > 0$. We know By Lemma A.1 that (3.10) holds, and hence (3.11) is true. Combining with (3.11), we get that $u$ is bounded in $(0, T_{\text{max}})$. The boundedness of $v$ can be obtained by the standard parabolic regularity. Finally, Lemma 2.1 yields that $(u, v)$ is global by contradiction.

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