Relationship between symmetry protected topological phases and boundary conformal field theories via the entanglement spectrum

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Abstract
Quantum phase transitions out of a symmetry-protected topological (SPT) phase in $(1+1)$ dimensions into an adjacent, topologically distinct SPT phase protected by the same symmetry or a trivial gapped phase, are typically described by a conformal field theory (CFT). At the same time, the low-lying entanglement spectrum of a gapped phase close to such a quantum critical point is known (Cho et al arXiv:1603.04016), very generally, to be universal and described by (gapless) boundary conformal field theory. Using this connection we show that symmetry properties of the boundary conditions in boundary CFT can be used to characterize the symmetry-protected degeneracies of the entanglement spectrum, a hallmark of non-trivial symmetry-protected topological phases. Specifically, we show that the relevant boundary CFT is the orbifold of the quantum critical point with respect to the symmetry group defining the SPT, and that the boundary states of this orbifold carry a quantum anomaly that determines the topological class of the SPT. We illustrate this connection using various characteristic examples such as the time-reversal breaking ‘Kitaev chain’ superconductor (symmetry class D), the Haldane
phase, and the $\mathbb{Z}_8$ classification of interacting topological superconductors in symmetry class BDI in $(1 + 1)$ dimensions.

Keywords: symmetry-protected topological phases, entanglement spectrum, boundary conformal field theory

(Some figures may appear in colour only in the online journal)

1. Introduction

The recent progress in our understanding of phases of matter has revealed that there are plenty of phases that go beyond Landau’s symmetry breaking paradigm [1]. Having various quantum disordered phases, which are not characterized by spontaneous symmetry breaking, we can ask if all these phases are (topologically) equivalent to each other or not. At least for gapped phases of matter, which are our focus in this paper, the (partial) answer to this question is known. There are at least three broad classes of quantum disordered phases: (i) topologically trivial phases, (ii) phases with intrinsic topological order, including symmetry-enriched topological phases, the fractional quantum Hall effect, and (iii) symmetry protected topological (SPT) phases, including electronic topological band insulators. The literature on these classes of phases of matter is by now too vast to exhaustively list here, but see for example [1–8] for reviews, and [9–13] and [14–31], for recent studies on symmetry enriched topological phases and SPT phases, respectively.

In this paper, we will establish a link between $(1 + 1)d$ SPT phases which are gapped, and boundary conformal field theories (CFTs) which are gapless. In particular, we will associate specific types of boundary conditions in boundary CFTs (BCFTs) to an SPT phase.

A motivation to connect gapped SPT phases to CFT (or BCFT), which describes gapless critical points or critical phases, comes naturally from the following observation. By definition, distinct SPT phases cannot be adiabatically deformed into each other while preserving the symmetries that define the SPT phase, without going through a quantum critical point at which the gap closes; In other words, distinct SPT phases are separated from each other by a quantum critical point which is typically a CFT. Thus in the phase diagram, a given SPT phase is typically in proximity of a CFT. (The SPT belongs to the ‘theory space’ of quantum field theories that can be reached from the CFT by applying perturbations relevant in the renormalization group sense.) One may then wonder to which extent a given CFT describing such a quantum critical point knows about SPT phases which are located just in its immediate neighborhood. Since an arbitrarily small gap is enough to define a topological phase, the question which relevant operator (‘massive deformation’) of a given CFT gives rise to a specific topological or trivial phase in its vicinity can be deduced solely from data contained in the CFT.

In this paper, we associate a particular BCFT with a given $(1 + 1)$-dimensional SPT phase by using a number of different arguments. One of our main arguments, which we believe to be the most fundamental and universal, uses the entanglement spectrum. The entanglement spectrum has generally been proven to be a useful tool to study SPT phases [32, 33]. In particular, it has been previously claimed and proven, using matrix product states (MPSs), that the entanglement spectrum of the ground state of a $(1 + 1)$-dimensional SPT phase is degenerate, and that the degeneracy of the entanglement spectrum is protected by the symmetries which define the SPT phase (‘symmetry protected degeneracy’). In this paper, by establishing a connection to BCFTs, we will develop an analytical understanding of the entanglement spectrum of SPT
phases near their proximate quantum critical points, which are described by a CFT. We can then use the knowledge of the corresponding BCFTs to study SPT phases.

In another argument, we try to detect non-trivial properties of a given SPT phase by first attaching an ideal ‘lead’ (a gapless quantum field theory) to the SPT phase (see figure 2). We then ‘shoot’ quasiparticles (e.g. electrons) from the ‘lead’ into the SPT phase and measure their scattering off from the SPT phase to learn something about the SPT phase. Such an approach has been applied to non-interacting fermionic SPT phases in all dimensions [34] and has proven to be quite powerful. E.g. from the properties of the scattering matrix, one can obtain the 10-fold classification of topological insulators and superconductors [14, 15, 35]. In the present paper, we generalize this approach to (1 + 1)-dimensional SPT phases with interactions, by using BCFTs.

As an application and illustration of our framework, we will discuss archetypical topological states in one spatial dimension, such as the time-reversal breaking topological superconductor in symmetry class D (a fermionic SPT phase) [36], and the spin-1 Haldane chain (a bosonic SPT phase). We will also apply our framework to topological superconductors in symmetry class BDI in (1 + 1) dimensions. For this system, Fidkowski and Kitaev [37, 38] found a reduction of the non-interacting classification of topological insulators/superconductors. While at the non-interacting level, (1 + 1)-dimensional topological superconductors in symmetry class BDI are classified by an integer topological invariant, Fidkowski and Kitaev found that, with interaction, the $\mathbb{Z}$ classification reduces to the smaller $\mathbb{Z}_8$ classification. It would be quite interesting to understand in further detail how the non-interacting classification reduces to this smaller classification in the presence of interactions. By linking SPT phases to BCFTs, we deepen our understanding of this phenomenon.

For loosely related works, see, for example, [39], and [40]. (The latter work studied the role of boundaries in the entanglement spectrum in (1 + 1)-dimensional (gapless) CFTs, as opposed to the gapped (1 + 1)-dimensional SPT phases discussed in the present work.) In [41], the relationship between gapped phases in (1 + 1) dimensions and boundary states in boundary CFTs was discussed in the context of the (continuous) MERA tensor network representation of quantum ground states and their holographic duality.

The rest of the paper is organized as follows.

In section 2, we provide various setups allowing us to make a connection between SPT phases and BCFTs. In particular, we consider gapless CFTs which are in contact with gapped SPT phases, and we discuss the entanglement spectrum of gapped (1 + 1)-dimensional SPT phases close to a quantum critical point. (See also [42].)

In section 3 we discuss the problem of identifying a suitable boundary state of the CFT for a given SPT phase. This is also related to the question as to how we describe, in the language of CFT, the symmetry-protected degeneracy of the entanglement spectrum, a hallmark of (1 + 1)-dimensional SPT phases. In order to achieve this goal, we will propose to use boundary states of an orbifold CFT, which is obtained from the original CFT by orbifolding it by the symmetry group defining the SPT.

This methodology is demonstrated in the two simplest examples of (1 + 1)-dimensional SPT phases, namely, the Kitaev chain (section 4), and the spin-1 Haldane chain (section 5). We will also discuss the $\mathbb{Z}_8$ classification of Fidkowski–Kitaev in the class BDI Majorana chain in section 6 from this perspective. The symmetry group of this system involves time-reversal which, due to its anti-unitarity, needs to be treated somewhat differently from unitary on-site symmetries. We conclude in section 7.
2. Boundary conformal field theories (BCFTs) and symmetry protected topological phases (SPTs)

In this section, we give an overview of a set of arguments which support the advocated relation between BCFTs and SPTs: (A) The Jackiw–Rebbi domain wall; (B) the scattering from SPT phases; and (C) the entanglement spectrum.

2.1. Interface between trivial and topological phases

A principle that underlies all non-interacting topological phases of Fermions can be well illustrated by the Jackiw–Rebbi domain wall (and its analog in different dimensions and in different symmetry classes); let us consider a massive Majorana fermion system described by the action

\[ S = S_s + S_l \]

\[ S_s = \frac{1}{4\pi} \int d\tau d\mu \left[ \psi_L \mathbf{\partial} \psi_L + \psi_R \mathbf{\partial} \psi_R \right], \]

\[ S_l = -im \int d\tau d\mu \psi_L \psi_R, \]  (1)

where \( \psi_L (\psi_R) \) is a left-moving (right-moving) real fermion field, and \( v \) is the Fermi velocity. Depending on the sign of the mass \( m \), the gapped phase is a topologically trivial/non-trivial phase (topological/ordinary superconductor) in symmetry class D in the ‘ten-fold way’ classification of topological insulators and superconductors [14, 15, 35] (At the face of it, it may appear that there is no symmetry that is required to protect a system of Fermions in symmetry class D. However, to be more precise, in order for this topological phase to be stable, fermion parity needs to be preserved. We will come back later to the role of symmetries protecting the SPT phase.) Which sign of the mass realizes a topological phase depends on the ultraviolet (UV) physics which is not encoded in the low-energy action displayed in equation (1). However, when we make a domain wall in the mass, there is an isolated zero energy Majorana mode which is insensitive to UV (short distance) physics.

One can also consider a coupling constant which is space-dependent, \( m \rightarrow m(x) \). In particular, one can consider a mass profile where \( m(x) \) has alternate signs for \( x > 0 \) and \( x < 0 \) as

\[ m(x) \rightarrow \begin{cases} +|m|, & x \rightarrow +\infty \\ -|m|, & x \rightarrow -\infty. \end{cases} \]  (2)

The mass profile \( m(x) \) crosses zero somewhere in between, say at \( x = 0 \). This geometry realizes an interface between topologically trivial and topologically non-trivial gapped phases. This Jackiw–Rebbi domain wall traps a Majorana fermion, which is the hallmark of a topological phase.

One of the purposes of this paper is to extend this Jackiw–Rebbi phenomenon to interacting settings (see below). In fact, we will claim that BCFT is a natural language to discuss interacting Jackiw–Rebbi phenomena. In topological phases, details of the profile \( m(x) \) do not matter, and we can adiabatically deform it and make the interface between the topological and non-topological phases as smooth as possible. If we do so, the transient region, where \( m(x) = 0 \), can be made very long (compared to the UV cut-off length scale of the theory), and it then looks like a segment of a CFT. In passing, note that because of topology, the number of stable boundary modes (zero modes) should not change even if we make the transient region as long as possible. (See figure 1.)
The above non-interacting setting can be generalized to more generic, interacting SPT phases and quantum critical points. A given gapped phase in \((1 + 1)\) dimensions can be obtained as a ‘massive deformation’ of a CFT,

\[ S_* \rightarrow S_* - \lambda \int dt dx \mathcal{O}(t, x) \tag{3} \]

where \(\mathcal{O}(x)\) is a relevant operator, and \(\lambda \in \mathbb{R}\) is the coupling constant (see also \([42]\)). To consider an interface separating trivial and topological phases, one can also in this case consider a coupling constant which is space-dependent, \(\lambda \rightarrow \lambda(x)\). (In discussing an interface in free-fermion systems, this prescription of creating an interface essentially exhausts all possible interesting cases. As a working hypothesis, we assume this prescription is generic enough even for interacting fermion systems.) In particular, one can consider a profile where \(\lambda\) has alternate sign for \(x > 0\) and \(x < 0\) as

\[ \lambda(x) \rightarrow \begin{cases} +|\lambda|, & x \rightarrow +\infty \\ -|\lambda|, & x \rightarrow -\infty. \end{cases} \tag{4} \]

Figure 1. Deformation of the domain wall. (a) A domain wall with the size \(a\), which is of the lattice scale. SPT phases will localize a zero mode at the domain wall. (b) The domain wall can smoothly be deformed to a bigger spatial region. In this manipulation of the domain wall, the topological zero mode cannot be removed. (c) When we push the domain wall to \(L \gg a\), we effectively find a critical mode localized at the length scale \(L\). Even in this limit, the topological zero mode will be superposed with the critical mode whose level spacing will be determined by the non-topological scale \(\sim 1/L\). This picture suggests that the boundary zero mode of the SPT phases can be thought as the critical mode localized at the UV scale ‘\(a\)’ as mentioned in the main text.
$\lambda(x)$ crosses zero in somewhere in between, say at $x = 0$. This geometry realizes an interface between topologically trivial and topologically non-trivial gapped phases. As before, in topological phases details of the profile $\lambda(x)$ do not matter, and we can make the interface between the topological and non-topological phases as smooth as possible. If we do so, the transient region can be made very long (relative to the UV cut-off), and it then looks like a long region of a CFT described by the action $S$. In passing, note that because of topology, the number of stable boundary modes (zero modes) should not change even if we make the transient region as long as possible.

As before, this is precisely the setting of BCFT. The CFT realized in the critical region near $x = 0$ can be viewed as terminated by two (different) gapped phases on the left ($x \to -\infty$) and on the right ($x \to +\infty$). At low energies, the interface between the CFT and any of the adjacent gapped phases is expected to renormalize into a conformal invariant (boundary) fixed point of the CFT, and in this infrared (IR) limit, the two gapped phases simply look like a two conformally invariant boundary conditions of the CFT. In other words, this suggests that there is a correspondence between gapped topological phases in $(1 + 1)$ dimensions and conformal invariant boundary conditions or boundary states in BCFT. This point will be further elaborated in the following.

### 2.2. Scattering from topological phases

The second argument relating SPTs and BCFTs is motivated by the scattering matrix formulation of topological invariants of free fermion SPTs. In [34], properties of the scattering matrix that describes scattering of free fermion states (in the 'ideal lead') off a given topological phase were discussed. As an example, let us consider a quasi-1d system and use the following construction: we connect two $2N$-channel wires (in the Majorana basis) to the two sides of a quasi-1d scattering region, which is a gapped phase (see figure 2). We are after the topological properties of the gapped region. This situation can be modeled by the following single-particle Hamiltonian

$$\mathcal{H} = -i \frac{d}{dx} \sigma_3 \otimes I_N + V(x),$$

where $V(x)$ is a potential. For our purpose, we may choose $V(x) = m \sigma_2 \otimes I_N$ inside the gapped region, whereas $V(x) = 0$ in the lead. The single-particle Hamiltonian satisfies the particle-hole constraint, $\mathcal{H}^* = -\mathcal{H}$, and belongs to symmetry class D. Let us consider an asymptotic state with energy $\varepsilon = k$ entering the scattering region $[0, L]$ of length $L$ located to the right of $x = 0$ with amplitudes $\chi_{in}$.
and a scattered state emerging from the disorder potentials with the same energy $\varepsilon$ and the amplitudes $\chi_{\text{out}I}$

\[
\Psi_{\text{out}}(x) = \begin{cases} 
\chi_{\text{out}I} e^{ik(x-L)} n_+, & L < x, \\
\chi_{\text{out}I} e^{-ik} n_-, & x < 0,
\end{cases}
\]

where the (column) vectors $n_{\pm}$ are given\(^6\) by $n_+ := (0, ..., 1, ..., 0)^T$, $n_- := (0, ..., 0; 0, ..., 1, ..., 0)^T$. Note that $e^{ik} n_+$ and $e^{-ik} n_-$ are for $k > 0$, a right-moving and a left-moving wave function, respectively, since the eigenvalue of the momentum operator $-id/dx$ is positive (negative).

The $2N \times 2N$ scattering matrix relates incoming and outgoing amplitudes in the two regions I and II as

\[
\begin{pmatrix} 
\chi_{\text{in}I} \\
\chi_{\text{in}II}
\end{pmatrix} = S \begin{pmatrix} 
\chi_{\text{in}I} \\
\chi_{\text{in}II}
\end{pmatrix}, \quad S(\varepsilon) = \begin{pmatrix} 
(r(\varepsilon)) & (t'(\varepsilon)) \\
(t(\varepsilon)) & (r'(\varepsilon))
\end{pmatrix},
\]

where $r(r')$ and $t(t')$ are $N \times N$ matrices representing the reflection part and transmission part of the scattering matrix. Here we use the (standard) convention that $r$ and $t$ describe the reflection and transmission coefficients of the incoming states from left hand side ($\chi_{\text{in}I}$), while $r'$ and $t'$ describe the reflection and transmission coefficients of the incoming states from right hand side ($\chi_{\text{in}II}$)—compare figure 2.

Topological properties of the scatterer are fully encoded in, and can be read off from the $S$-matrix as follows. Since the scatterer (i.e. a gapped $(1+1)d$ phase) is gapped, if $L$ is large enough, (almost) all\(^7\) incoming electrons eventually get reflected back from the scatterer. We can thus focus on the reflection part, $r$, of the $S$-matrix. Depending on the underlying symmetry of the problem, the reflection matrix is subject to a set of constraints. For symmetry class D, for example, the set of all reflection matrices (at $\varepsilon = 0$), denoted by $\mathcal{R}$, is disconnected, $\pi_0(\mathcal{R}) = \mathbb{Z}_2$, which corresponds to the $\mathbb{Z}_2$ classification of class D in $(1+1)$ dimensions. These two sectors are distinguished by the $\mathbb{Z}_2$-valued topological index,

\[
\text{sgn } \det r(\varepsilon = 0) = \pm 1.
\]

Here, when $\text{sgn } \det r = 1$ the gapped system attached to the lead is trivial. On the other hand, when $\text{sgn } \det r = -1$ the gapped system attached to the lead is non-trivial. In this way, the topological character of the bulk is fully encoded in the scattering matrix\(^8\).

If we further impose, for example, time-reversal symmetry which squares $+1$, the relevant symmetry class is class BDI of the Altland–Zirnbauer classification [14, 43]. The topological classification at the level of non-interacting fermions is given in terms of a integer-valued topological invariant (‘the winding number’). In the scattering matrix approach, the integer topological invariant is given by the number of negative eigenvalues of the reflection matrix $r$. (In passing, we note that this non-interacting topological invariant fails to capture the reduction

\(^{6}\)The superscript $^T$ denotes the transpose.

\(^{7}\)All, when $L = \infty$.

\(^{8}\)Similarly, the transmission can detect the quantum phase transition separating trivial topological phases.
of the non-interacting classification from \( \mathbb{Z} \) to \( \mathbb{Z}_8 \) found by Fidkowski and Kitaev \cite{37, 38} in the presence of interactions.

The construction described above is precisely the typical setting of BCFTs. The gapless ideal lead that we use to detect topological properties of the SPT phase is a special case of a CFT, which has its boundary condition set by the SPT. In BCFTs the boundary can be probed by correlations of bulk fields. For example, BCFT computes the \( \text{(right–left)} \) fermion two-point function in the presence of a boundary, which is given by \cite{44–47}

\[
\langle \psi^a_L(z) \psi^b_R(\bar{z}) \rangle = \frac{r_{ab}}{z - \bar{z}}.
\]

Here, the physical spacetime consists of the upper half complex plane, and the boundary is located on the real axis, \( z = \bar{z} \). In the absence of interactions, the amplitude \( r \) of this function contains the information about the single-particle scattering matrix (the only existing scattering matrix in the absence of interactions). When the topological winding number is zero, the single particle Green function in the CFT is given by \( \text{sgn} \det r = 1 \). On the other hand, when the topological winding number is non-zero, \( \text{sgn} \det r = -1 \).

The above consideration shows that a CFT can be used as an external ‘probe’ to look into possible topological bulk states, although the framework presented so far has been limited to non-interacting fermion systems. However, BCFTs in general are not limited to non-interacting systems and are expected to give us a framework to study interacting \((1+1)d\) SPT phases in general. The reason why our consideration so far is limited to free-fermion systems is the fact that we focused on the single-particle \( S \)-matrix, or the single-particle fermion Green’s function \( \langle \psi^a_L(z) \psi^b_R(\bar{z}) \rangle \). For non-interacting problems, unitarity restricts \( |r| = 1 \). In the presence of interactions, however, even if they only act on the boundary, it is not difficult to find examples of boundary conditions where \( |r| < 1 \), and in particular, we have examples where \( r = 0 \). This is known for example in the context of the two-channel Kondo and related models \cite{47}. In these interacting systems, unitary of the \( S \)-matrix can be violated within the single-particle sector (while unitary in the full many-particle Hilbert space is of course preserved). BCFTs are not limited to the description of the single-particle fermion Green’s function \( \langle \psi^a_L(z) \psi^b_R(\bar{z}) \rangle \), but give us the description of the full (‘many-body’, or ‘Fock-’) Hilbert space in the presence of interactions, even if the interactions are only operative within the gapped region (i.e. only at the boundary of the ideal lead (CFT)). In section 6, we will show that our approach based on BCFTs indeed yields the \( \mathbb{Z}_8 \) classification of Fidkowski–Kitaev in the presence of interactions. In that section, the CFT (‘in the lead’) is taken to consist of non-interacting Fermions, while all interactions occur solely on the boundary. In the language of the entanglement spectrum, to be discussed in the following section 2.3, this corresponds physically to a situation of a quantum phase transition out of the interacting SPT phase into a trivial phase, described by non-interacting massless Fermions. Since, as will be described in the next section, one of the boundaries of the BCFT describing the entanglement spectrum corresponds to an interface of the CFT (in the present case a non-interacting theory) with the fully interacting SPT phase, all interactions are incorporated into that boundary condition.

### 2.3. The entanglement spectrum

The last argument in this section is based on the entanglement Hamiltonian and the entanglement spectrum of SPT phases. This is the most general and most fundamental of the arguments we are giving. As we will see, it will ‘automatically’ choose for us a gapless CFT, and a suitable BCFT. As we will now explain, the entanglement Hamiltonian of the SPT plays the role of the ‘expanded domain wall’ or the ‘lead’ of the previous two sections.
Vital tools for the study of one-dimensional (and other) gapped phases are the entanglement entropy and the entanglement spectrum. In gapped \((1 + 1)\) d phases which are adjacent to a CFT with central charge \(c\), i.e. in which the correlation length \(\xi\) is much larger than the microscopic length \(a\) (‘scaling limit’), it is well known that the entanglement entropy behaves as

\[ S_A \simeq \left(\frac{c}{6}\right) \ln(\xi/a) + \text{constant}. \]  

A topological phase, being gapped, can be tuned to have a minimally (‘infinitely’) short correlation length, \(\xi = a\). For such a representative of the topological phase only the constant term in (11) remains. Because it is a topological phase, it is not possible to make the entanglement entropy vanish completely, while preserving the symmetry which protects the SPT phase under consideration. This non-vanishing constant part (the part which is not controlled by the correlation length) is a key to classifying gapped phases in one spatial dimension. Of course, the constant term in (11) is sensitive to the a redefinition of the short distance scale \(a\) (cutoff). Nevertheless, the presence of a constant term which cannot be made to vanish in a topological phase is related to the presence of the degeneracy of the entanglement spectrum of this phase. This connection is spelled out in Equation (28) below, where the appearance of a non-vanishing constant term in the entanglement entropy is directly related to the appearance of degeneracies in the entanglement spectrum. This then attributes a universal significance to the non-vanishing constant term.

Indeed, it was recently shown in [42] that the entire low-lying entanglement spectrum of a gapped phase close a quantum critical point, such as the SPT under consideration, is universal and described, very generally, by the CFT describing the quantum critical point itself, but on a finite interval of length \(\ell = \ln(\xi/a)\) with suitable boundary conditions. In short, the entanglement spectrum is described by a boundary conformal field theory. In particular, the boundary condition at one end of the finite interval is determined by the specific gapped phase in the vicinity of the quantum critical point; different boundary conditions correspond in general to different gapped phases adjacent to the same quantum critical point. In other words, there is a mapping between gapped phases in the vicinity of the quantum critical point, and boundary conditions on the CFT on a finite interval which describes the entanglement Hamiltonian. In fact, this materializes in complete generality the connection, discussed above, between the gapped \((1 + 1)\) dimensional SPT phase and the gapless boundary CFT (see figure 3).

From this point of view, the constant part of the entanglement entropy in (11) should come from the fact that we have to specify particular boundary conditions on the CFT, in order for the resulting boundary CFT to represent the entanglement Hamiltonian of the SPT. I.e. the constant part of the entanglement entropy is related to the boundary states in a given CFT.

Roughly speaking, the classification problem of 1d gapped phase is thus related to the classification of the boundary states in CFT. Since we take the \(\xi \to a\) limit, the constant part of the entropy will turn out to be given by an overlap between two boundary states. This is somewhat reminiscent of the Affleck–Ludwig boundary entropy [48], except that here we have to consider an opposite limit (for details see below).

Specifically, making a connection with the setting discussed in the previous section, i.e. with the scattering off of SPT phases, the entanglement Hamiltonian (somewhat surprisingly) realizes precisely the same setting, but in a completely general context. Note that while in the previous setting, there may be an ambiguity as to our choice of ideal leads. In the entanglement Hamiltonian, on the other hand, the lead (i.e. the CFT) is ‘automatically’ chosen.

As a side remark, a connection between gapped topological phases and critical systems can be also made by following the construction in [49] of the so-called bulk entanglement...
In [49], it was shown that for an SPT, by using a bipartition of position space into regions A and B with the property that the interface between A and B grows with the volume (in 1D with length), a gapless entanglement Hamiltonian can emerge. Concretely, using a MPS construction of a gapped SPT phase (e.g. the Haldane phase), and rearranging tensors in the MPS in a staggered way gives rise to a transfer matrix of a critical system (the six-vertex model), describing a spin-1/2 Heisenberg spin chain. We can think of this construction as generating an entanglement Hamiltonian which sits at the quantum phase transition out of the Haldane phase into the dimerized phase of the spin-1 chain driven by staggering. This quantum phase transition is in the universality class of the unstaggered spin-1/2 Heisenberg chain, describing the CFT of the (gapless) entanglement Hamiltonian.

**Figure 3.** Entanglement Hamiltonian of SPT phase (from Cho et al arXiv:1603.04016).

(i) The ground state of the $(1+1)$ dimensional SPT phase of correlation length $\xi$, in the vicinity of a quantum critical point on the infinite space $-\infty < x < +\infty$, bipartitioned into region A (positive $x$) and region B (negative $x$). (ii) The entanglement Hamiltonian (defined on a space with coordinate $u$ which is different from $x$) is that of the CFT describing the quantum critical point, but confined to a finite interval of length $\ell = \ln(\xi/a)$. On the right hand side of the interval is an interface of the CFT with the gapped SPT, providing one boundary condition. On the other side of the interval the CFT simply ends, providing another boundary conditions (‘free boundary condition’). (iii) The situation depicted in (ii) is a generalization of the ‘expanded Jackiw–Rebbi domain wall’ depicted in figure 1(c), to the case of a completely general interacting SPT.

spectrum. In [49], it was shown that for an SPT, by using a bipartition of position space into regions A and B with the property that the interface between A and B grows with the volume (in 1D with length), a gapless entanglement Hamiltonian can emerge. Concretely, using a MPS construction of a gapped SPT phase (e.g. the Haldane phase), and rearranging tensors in the MPS in a staggered way gives rise to a transfer matrix of a critical system (the six-vertex model), describing a spin-1/2 Heisenberg spin chain. We can think of this construction as generating an entanglement Hamiltonian which sits at the quantum phase transition out of the Haldane phase into the dimerized phase of the spin-1 chain driven by staggering. This quantum phase transition is in the universality class of the unstaggered spin-1/2 Heisenberg chain, describing the CFT of the (gapless) entanglement Hamiltonian.
3. Symmetry-protected degeneracy in entanglement spectrum: general considerations

Matrix product states (MPSs) provide a convenient framework to discuss entanglement and in particular the entanglement spectrum of gapped phases in \((1 + 1)\) dimensions. In particular, the symmetry protected degeneracy of the entanglement spectra of SPT phases—a hallmark of SPT phases—can be understood from the MPS perspective \([32, 50]\). In this section, we will understand the symmetry protected degeneracy of the entanglement spectra of SPT phases from the perspective of continuous field theories, in particular orbifold BCFTs, in order to provide an alternative point of view.

Specifically, in this section we consider the BCFT describing the entanglement Hamiltonian, as discussed in the preceding section 2.3: the entanglement spectrum is that of the CFT describing the quantum phase transition itself, but on an interval of length \(\ell = \ln(\xi/a)\) with two boundary conditions A and B at the two ends. In this section we focus on the requirements on these boundary conditions, which arise from the fact that we are describing the entanglement spectrum of a SPT which is protected by a symmetry group \(G\).

3.1. Quick review of boundary conformal field theory (BCFT)

From the discussion in the previous section, we associate a particular BCFT (or a particular boundary condition, and boundary state of a CFT) with a given SPT phase; different SPT phases in proximity of the same quantum critical point are described by different boundary conditions on the same CFT. (More precisely, the correspondence between a BCFT and a SPT phase is not one-to-one, but rather several different BCFTs may correspond to a given SPT phase, because there may be several different quantum critical points through which one can exit a given SPT phase into other, neighboring phases. By staying close to a particular quantum critical point, we pick a particular BCFT-description of a given SPT phase.)

Let us first briefly review the general framework of BCFTs. Consider a CFT on a finite spatial interval with boundary conditions specified by \(A\) and \(B\) at the two boundaries. In BCFT, we may compute the partition function by using one of the following two alternative pictures: \([51]\) the so-called open string picture (sometimes also called loop channel picture), and the so-called closed string picture (sometimes also called tree channel picture). First, in the open string picture, the partition function (at inverse temperature \(\beta\)) is written as a trace of the Hamiltonian \(\hat{H}_{\text{open}}^{AB}\) of the finite interval of length \(\ell\) with boundary conditions \(A\) and \(B\) at the two end points (boundaries) of the interval:

\[
Z_{AB} = \operatorname{Tr}_{\mathcal{H}_{AB}} e^{-\beta \hat{H}_{\text{open}}^{AB}} = \operatorname{Tr}_{\mathcal{H}_{AB}} q^{L_0}.
\]

(12)

Here \(\mathcal{H}_{AB}\) denotes the Hilbert space of quantum states on the interval. In the second line, the partition function is rewritten, by using the ‘folding procedure’, so that say only holomorphic (left-moving) degrees of freedom appear\(^9\); here \(\hat{H}_L\) is the Hamiltonian defined purely in the holomorphic sector where it can be expressed in terms of the (holomorphic) Virasoro generator \(L_0\) and the central charge \(c\) as \(\hat{H}_L = L_0 - c/24\). All terms in the partition function in equation (12) are powers of \(q\).

\(^9\)Owing to its assumed conformal invariance, at each boundary right movers can be viewed as left movers, analytically continued into the lower half complex plane.
\[
q = e^{-\pi \beta / \ell}
\]
related to the length \((\approx \ell)\) of the system (2\(\ell\) for the left-movers after ‘folding’) and the inverse temperature \(\beta\).

The structure of the ‘open string’ Hilbert space \(\mathcal{H}_{AB}\) depends on the choice of the boundary conditions \(A\) and \(B\). In particular, the Hilbert space \(\mathcal{H}_{AB}\) can be decomposed into different irreducible representations \(\phi_a\) of the Virasoro (or more generally, a larger ‘chiral’) algebra of the CFT, which is supported on a vector space \([\phi_a]\),

\[
\mathcal{H}_{AB} = \bigoplus_a n_{AB}^a [\phi_a].
\]

The non-negative integers \(n_{AB}^a\) represent the multiplicity with which the irreducible representation \(\phi_a\) occurs. Hence, the partition function can be written in the form

\[
Z_{AB} = \sum_a n_{AB}^a \chi_a(q),
\]

where \(\chi_a(q)\) is the partition function associated with the representation \(\phi_a\) (and is usually called its ‘character’). The partition function can also be computed, alternatively, by exchanging the roles of the space and imaginary (Euclidean) time coordinates. In the resulting closed string picture, the partition function can be written in terms of boundary states \(|A\rangle\) and \(|B\rangle\) as

\[
Z_{AB} = \langle A | e^{-\ell \hat{H}_{\text{closed}}} | B \rangle = \langle A | \tilde{q}^{1/2} (\hat{H}_L + \hat{H}_R) | B \rangle,
\]

where \(\hat{H}_{\text{closed}}\) is the Hamiltonian of the CFT on a space with periodic boundary conditions (a circle) of circumference \(\beta\), acting on the corresponding Hilbert space \(\mathcal{H}_{\text{closed}}\), which is contained in the tensor product of holomorphic (left-moving) and anti-holomorphic (right-moving) degrees of freedom. The boundary states can be expanded in terms of so-called Ishibashi states as

\[
|A\rangle = \sum_a A_a |a\rangle, \quad \langle A| = \sum_a \tilde{A}_a \langle a|.
\]

The Ishibashi states are special states in the closed string Hilbert space \(\mathcal{H}_{\text{closed}}\) in which the holomorphic (left-moving) and the anti-holomorphic (right-moving) basis states of the Hilbert space are maximally entangled\(^{10}\). This leads to the second line of equation (16) above, where \(\hat{H}_L + \hat{H}_R = L_0 + \hat{L}_0 - c/12\) is the Hamiltonian of the CFT with periodic boundary conditions (on the unit circle) and

\[
\tilde{q} = e^{-4\pi \beta},
\]

implying

\[
\langle (a| \tilde{q}^{1/2} (L_0 + \hat{L}_0 - c/12) |b\rangle = \delta_{ab} \chi_a (\tilde{q} = (e^{-2\pi})^2).
\]

Then, the partition function can be written as

\[
Z_{AB} = \sum_a \tilde{A}_a B_a \chi_a (\tilde{q}).
\]

\(^{10}\) See [52] for a discussion from the entanglement perspective.
The two representations, equation (12) and equation (20) of the same partition function are related by a modular transformation of the space-(imaginary)time torus: By using the modular $S$-matrix,

$$\chi_a(q) = \sum_b S^b_a \chi_b(\tilde{q}),$$

(21)

one sees that the integer coefficients $n_{ab}^a$ in equation (15), and the expansion coefficients $A_a, B_a$ in equation (20), are related via

$$\sum_a n_{AB}^a S^b_a = \tilde{A}_b B_a, \quad \text{and} \quad n_{AB}^a = \sum_a \tilde{A}_a B_a S^b_a.$$ 

(22)

In the limit $\tilde{q} \to 0$ ($q \to 1$), we have the Affleck–Ludwig boundary entropy: [48]

$$\log Z_{AB} = \log \left( \sum_{a,b} n_{AB}^a S^b_a \chi_b(\tilde{q}) \right) \sim \log \left( \sum_a n_{AB}^a S^b_a \right) + \log \chi_0(\tilde{q}),$$

$$\sim \log \tilde{A}_0 + \log B_0 + \log \chi_0(\tilde{q})$$

$$\to \log \tilde{A}_0 + \log B_0.$$ 

(23)

(In the second line, only the dominant sector in the considered limit $\tilde{q} \to 0$—the vacuum sector $b = 0$, which was assumed to be unique—survives. In the last line, use was made of $\lim_{\tilde{q} \to 0} \chi_0(\tilde{q}) = 1$.)

### 3.2. Quick review of orbifold CFTs

A natural and general framework to discuss the action of discrete symmetries in CFTs is the so-called orbifold CFT [53, 54]. In order to discuss BCFT in the context SPT phases, we need to discuss the notion of the orbifold in BCFT [54]. First, before discussing the orbifold of a CFT with boundaries (i.e. of BCFT), we give a very brief overview of orbifold CFTs in the bulk (i.e. on a space with periodic boundary conditions—in the absence of boundaries) [55, 56]. Orbifold CFTs can be obtained from a parent CFT by modding out (‘gauging’) a discrete symmetry group $G$. The partition function of an orbifold CFT on a torus is known to have the following structure,

$$Z = \frac{1}{|G|} \sum_{g,h \in G} \varepsilon(g|h) Z(g,h),$$

(24)

where $Z(g,h)$ denotes the partition function in the sector twisted by group elements $g$ and $h$ in the (imaginary) time and space directions, respectively (see below). Here $[g,h] = ghg^{-1}h^{-1}$ denotes the commutator in the group. The sector-dependent phases, $\varepsilon(g|h)$, are called discrete torsion, and will be defined in detail below [57]. In each sector, the (bulk) partition function is given by

$$Z(g,h) = \text{Tr}_{H_h} \left[ \hat{q}^{L_h} \hat{q}^{R_h} \right] = \sum_{(j,j')} \chi_0^g(\tilde{q}) \chi_0^g(\tilde{q}).$$

(25)
Here, $H_h$ is the Hilbert space of the sector twisted\footnote{Both right- and left-moving factors of the bulk Hilbert space are twisted by the same group element $h$.} by $h$. Each twisted-sector Hilbert space $H_h$ is decomposed into irreducible representations (denoted by $(j)$ and $(\bar{j})$) of the left- and right-moving Virasoro (or possibly of some larger chiral \cite{53}) algebra, and we introduced the corresponding chiral blocks (‘characters’)

$$
\chi_{h,(j)}^{g}(q) = \text{Tr}_{H_{h,(j)}} \left[ \hat{g} \hat{h}_j \right].
$$

(26)

Here $\hat{g}$ is a representation of the group element $g \in \mathcal{G}$ on the Hilbert space $H_{h,(j)}$. (Note that for each group element $h$, the sum over group elements $g$ commuting with $h$ (i.e., $[g, h] = e$) in the total partition function, equation (24), projects onto $N_h$-invariant states where

$$
N_h = \{ g \in \mathcal{G} | [g, h] = e \}
$$

(27)

is the ‘normalizer’ of $h$.)

3.3. Symmetry-protected degeneracy

After the above review of general BCFT, and of bulk orbifold CFTs, we will now discuss the symmetry-protected degeneracy of the entanglement spectrum of an SPT phase. Let us start by observing that the multiplicity coefficients $n_{AB}^a$ appearing in equation (15) for $Z_{AB}$ are closely related to the symmetry-protected degeneracy: All states in the representation $a$ of the Virasoro (or larger chiral) algebra are at least $n_{AB}^a$-fold degenerate. In particular, the ground state in each representation $a$ appearing in $Z_{AB}$ is $n_{AB}^a$-fold degenerate, which can be seen by taking the limit $q \to 0$ ($\tilde{q} \to 1$). In this limit, the partition function behaves as $Z_{AB} \sim \sum_a n_{AB}^a q^{-c/24 + h_0}$ where $h_0$ is the lowest energy state in a given Virasoro representation $a$, which we assume non-degenerate. Observe that the multiplicity $n_{AB}^a$ can be extracted by taking the limit $q \to 0$, which is opposite to the limit $\tilde{q} \to 1$ taken in the Affleck–Ludwig boundary entropy. As in the boundary entropy, one can express the degeneracy $n_{AB}^a$ in terms of the data constituting the boundary states as follows by taking the limit $q \to 0$ (using the second equation in equation (22)) :

$$
\log Z_{AB} = \log \left( \sum_a n_{AB}^a \chi_a(q) \right)
\sim \log \left( n_{AB}^0 \chi_0(q) \right)
\sim \log(n_{AB}^0)
$$

(28)

Thus, as far as the identity representation ‘0’ appears in $Z_{AB}$, $n_{AB}^0 = \sum_a \hat{A}_a B_{aB} S_{aB}$ yields the degeneracy.

While the multiplicities $n_{AB}^a$ are to be closely related to the symmetry-protected degeneracy, in the above discussion we have not mentioned symmetry at all. In the following, we will be interested in the situation where the multiplicity $n_{AB}^a$ results from a discrete symmetry of the BCFT, and if so, we are interested in relating it to the property of the boundary conditions set by SPT phase. In other words, the multiplicity (degeneracy) could be simply an accidental one. On the other hand, if the SPT phase of interest is topologically non-trivial, we expect $n_{AB}^a > 1$ is enforced by symmetry. We want to be able to understand the multiplicity (degeneracy) as arising from the symmetry that protects the SPT Phase.
3.4. Projective representation in the open string channel

Coming back to the open string picture, equation (12), the partition function can be written as a chiral block (as discussed above):

$$Z_{AB} = \text{Tr}_{\mathcal{H}_{AB}} \left[ e^{-\beta \hat{H}_{\text{open}}^{AB}} \right] = \text{Tr}_{\mathcal{H}_{AB}} q^{\hat{p}_L}$$

(i.e. we used the ‘folding procedure’ to write the partition function purely in terms of the chiral (left-moving) sector of the theory.) The trace here is taken with respect to the Hilbert space $\mathcal{H}_{AB}$, which is determined by boundary conditions $A$ and $B$. As in a typical set-up of orbifold CFTs [53], we assume a decomposition of the Hilbert space of the form

$$\mathcal{H}_{AB} = \bigoplus_a r_a \otimes [\phi_a],$$

(30)

where $r_a$ and $[\phi_a]$ denote an irreducible representation of the finite group $G$ and of the Virasoro (chiral) algebra, respectively. Then, the partition function can be written as

$$Z_{AB} = \sum_a \rho_a(1) \chi_a(q),$$

(31)

where $\rho_a(g)$ is the group character of the irreducible representation $r_a$ evaluated on the group element $g \in G$. In this description, the degeneracy factor from equation (15) appears in the form

$$\rho_a(1) = \dim r_a = n^a_{AB},$$

(32)

and is attributed to the invariance of the Hamiltonian under the symmetry group $G$ and to the appearance of representations of $G$ of dimension larger than one in the spectrum.

We now consider a slight generalization of the partition function $Z_{AB}$ written in equation (29) above, namely the ‘open-string orbifold partition function’ defined by

$$Z_{AB}^\text{orb} = |G|^{-1} \sum_{g \in G} \text{Tr}_{\mathcal{H}_{AB}} \left[ \hat{g} e^{-\beta \hat{H}_{\text{open}}^{AB}} \right].$$

(33)

Here, we denote by $\hat{g}$ the representation of the group element $g$ on the Hilbert space $\mathcal{H}_{AB}$. With the decomposition (30), $\hat{g}$ can be decomposed accordingly into irreducible components as

$$\hat{g} = \bigoplus_a D_a(g),$$

(34)

where $D_a(g)$ is the representation matrix of $g$ in the irreducible representation $r_a$. If we think of the finite interval of length $\ell$ on which the BCFT resides from the point of view of the expanded domain wall picture of section 2.1, we see that for small size $\ell$ the gapless BCFT region reduces to the local domain wall at which we expect to see the appearance of a projective representation of the symmetry group $G$ defining the SPT phase. Therefore, we expect to see a projective representation of the symmetry group on the Hilbert space $\mathcal{H}_{AB}$, since this just describes the expanded version of the domain wall (section 2.1). Therefore we will be interested in the possible appearance of projective representations of the group $G$, for which the representation matrices $D_a$ will in general satisfy the composition law

$$D_a(g)D_a(h) = \omega(g|h)D_a(gh),$$

(35)

where $g, h \in G$, and where $\omega(g|h)$ is a two-cocycle, i.e. and element of the 2nd cohomology group $H^2(G, U(1))$. Note that in the direct sum decomposition in (30), all representations $r_a$
should have the same two-cocycle. (In general, one can take a direct product of two representations having different two-cocycles, but not a direct sum thereof.)

Using the decomposition (30), the orbifold partition function in equation (33) can be expressed in terms of the ‘twisted partition functions’

$$Z_{AB}^g = \text{Tr}_{\hat{H}_{AB}} \left[ g e^{-\beta H_{AB}} \right] = \sum_a \rho_a(g) \chi_a(q),$$

(36)

where $\rho_a(g) = \text{tr} D_a(g)$ defines the character of a representation $D_a$ in the usual manner. The twisted partition function $Z_{AB}^g$ thus extracts the characters of the representations of $G$. This twisted partition function may then be used to identify the representation $\hat{g}$ appearing in the untwisted partition function $Z_{AB}$ since knowing its character for all $g \in G$ helps us identify the nature of the associated representation.

To be more precise: we are interested in knowing whether the representation $\hat{g}$ in equation (34) is projective or not. On the other hand, as will be explained in the next section, knowing only the values of character of a representation for all $g \in G$, one cannot determine whether the representation is projective. This is only possible once we know the two-cocycle. Therefore, coming back to the context of SPT phases: in order to diagnose whether a projective representation occurs in the spectrum or not, i.e. in order to diagnose whether the boundary states $A$ and $B$ correspond to topologically distinct gapped phases, we propose a diagnostic that we call the symmetry-enforced vanishing of the partition function, to be discussed in the next section.

3.5. Symmetry-enforced vanishing of the partition function

To illustrate the notion of the symmetry-enforced vanishing of the partition function, which we will define momentarily in a more precise fashion, we note the following properties of projective representations of a discrete group $G$. First of all, it is well known\(^\text{12}\) that the character of a non-projective irreducible representation of a finite group always vanishes on at least one group element unless the representation is one-dimensional. We will refer to this as an accidental vanishing of the character. On the other hand, the character of a projective representation, irreducible or not, is forced to vanish in the following sense [59]: for a given projective representation with two-cocycle $\omega(g|h)$, we define

$$\varepsilon(g|h) = \omega(g|h) \omega(h|g)^{-1}, \quad \text{when } [g, h] = e.$$  \hspace{1cm} (37)

The character $\rho$ of the projective representation evaluated on the group element $h$ vanishes, $\rho(h) = 0$, if a group element $g \in N_h$ exists such that $\varepsilon(g|h) \neq 1$. To see this, we note that $\rho(ghg^{-1})$ can be written as,

$$\rho(ghg^{-1}) = \omega(gh|g^{-1})^{-1} \omega(g|h)^{-1} \omega(g^{-1}|g) \omega(h|e) \rho(h).$$  \hspace{1cm} (38)

In particular, when $g$ and $h$ commute, $\rho(h) = \varepsilon(h|g) \rho(h)$ and hence $\rho(h)$ must vanish\(^\text{13}\) when $\varepsilon(g|h) \neq 1$\(^\text{14}\). Thus in short, while the vanishing of its character alone does not allow us to determine whether the representation is projective or non-projective, if the vanishing is

\(^{12}\) See e.g. [58].

\(^{13}\) Clearly, $\varepsilon(g|h) \neq 1$ also implies $\rho(g) = 0$ by the same argument.

\(^{14}\) We note that the representation $\rho$ is not projective if and only if $\varepsilon(g|h) = 1$ for all commuting group elements $g$ and $h$. Moreover, as already mentioned in the paragraph above equation (37), the dimension of a representation must be greater than one if its character vanishes on at least one of the group elements.
enforced, in the above sense, this gives us a strong indication that the corresponding representation is projective.

Similar to the above statement at the level of the group character $\rho(h)$, we will argue below that the orbifold partition function allows us to infer whether non-trivial two-cocycles of the representations are included in the partition sum, i.e. whether the representations are projective. In particular, we introduce the notion of the symmetry-enforced vanishing of the partition function. Consider the case where the twisted partition function vanishes,

$$Z_{AB}^h = \text{Tr} \hat{H}_{\text{tot}} \left[ \hat{h} e^{-\beta H_{\text{orb}}^\text{open}} \right] = 0, \quad (39)$$

and where this vanishing of $Z_{AB}^h$ is enforced by symmetry. (Note that in view of equation (36) this vanishing implies that $\rho_a(h) = 0$ for all irreducible representations $a$ occurring in equation (30), since the conformal characters $\chi_a(q)$ are linearly independent. Then, since the character of a one-dimensional representation does not vanish, this implies in view of equations (31) and (32) the appearance of multiplicities $n_{AB} > 1$ for all irreps in the spectrum of $H_{\text{orb}}^\text{open}$. In order to formulate the precise meaning of the symmetry enforced vanishing, it is convenient to go to the closed string picture, in which $Z_{AB}^h$ can be expressed in terms of boundary states

$$Z_{AB}^h = h \langle A | e^{-\frac{i}{\beta} H_{\text{tot}}} | B \rangle_h = h \langle A | \hat{g}^{\frac{1}{2}} (\hat{h}_L + \hat{h}_R) | B \rangle_h, \quad (40)$$

which generalizes equation (16). Here $|A\rangle_h$ is a boundary state in the sector twisted by the group element $h \in G$. Thus, the vanishing of $Z_{AB}^h$ means

$$h \langle A | e^{-\frac{i}{\beta} H_{\text{tot}}} | B \rangle_h = 0. \quad (41)$$

Suppose now the boundary state in the sector twisted by $h$ is not invariant under the symmetry operation $g$, but picks up an ‘anomalous phase’\textsuperscript{16} factor $\epsilon_B(g|h)$:

$$\hat{g} | B \rangle_h = \epsilon_B(g|h) | B \rangle_h, \quad \text{when} \quad g \in N_h. \quad (42)$$

Then, since $\hat{g}$ is a symmetry of the Hamiltonian, we have

$$h \langle A | \hat{g}^{\frac{1}{2}} (\hat{h}_L + \hat{h}_R) | B \rangle_h = h \langle A | \hat{g}^{\frac{1}{2}} (\hat{h}_L + \hat{h}_R) \hat{g} | B \rangle_h, \quad (43)$$

from which it follows that

$$\epsilon_A(g|h) h \langle A | \hat{g}^{\frac{1}{2}} (\hat{h}_L + \hat{h}_R) | B \rangle_h = \epsilon_B(g|h) h \langle A | \hat{g}^{\frac{1}{2}} (\hat{h}_L + \hat{h}_R) | B \rangle_h. \quad (44)$$

Thus, unless $\epsilon_A(g|h) = \epsilon_B(g|h)$, the twisted partition function must vanish. When the partition function vanishes due to the anomalous phases $\epsilon_A(g|h)$ and $\epsilon_B(g|h)$, we call this situation a symmetry-enforced vanishing of the (twisted) partition function. Note that equation (44) reads, in view of equation (36),

$$\rho_a(h) = \frac{\epsilon_B(g|h)}{\epsilon_A(g|h)} \rho_a(h), \quad g \in N_h \quad (45)$$

\textsuperscript{15} Here $\hat{g}$ denotes the representation of the group element $g$ on the Hilbert space of the bulk CFT, of which the boundary state $|B\rangle$ and its twisted variant $|B\rangle_a$ are elements.

\textsuperscript{16} Note that the so-defined phase factor $\epsilon_B(g|h)$ is an object entirely different from the phase $\epsilon(g|h)$—no subscript $B$—defined in equation (37).
for all irreducible representations $a$ appearing in equation (30). Therefore, we argue that when this happens, the gapped phase which determines the corresponding boundary condition, and which hence determines the boundary state, is a non-trivial SPT phase.

Equation (42) is the central claim of this section. In appendix B, we will show that the anomalous phase in equation (42) is also present in the ground states of $G$-equivariant $(1 + 1)d$ topological quantum field theories. (See, e.g. (B.2).) Roughly speaking, $G$-equivariant topological quantum field theories are topological quantum field theories in the presence of a background gauge field with gauge group $G$. Such topological quantum field theories are expected to describe the universal properties (i.e. those appearing in the zero-correlation length limit) of $(1 + 1)d$ SPT phases. We have thus established a link between $(1 + 1)d$ SPT phases and BCFT.

In the following sections, we will demonstrate that such a symmetry-enforced vanishing of the twisted partition function occurs indeed in various characteristic examples of SPT phases: the time-reversal breaking Kitaev (superconducting) chain in symmetry class D, the Haldane phase, and the time-reversal invariant Majorana chain in symmetry class BDI. Observe that the partition function may vanish accidentally even when there is no anomalous phase. This should be distinguished from the vanishing of the partition function which is enforced by symmetry. In general, we do not expect a vanishing of the partition function which is not enforced is a consequence of the topological features of an SPT phase.

The assumption we made in equation (42) deserves more discussion, since there are in principle more generic possibilities for the action of the symmetry on boundary states, besides the one listed in equation (42). When boundary conditions (boundary states) break symmetries, we expect the symmetry operation $\hat{g}$ will in general map one boundary state into another. On the other hand, for boundary states that arise from $(1 + 1)$ dimensional SPT phases, we do not expect that they break the symmetry defining the SPT phase. Hence, one may expect that the symmetry operation leaves boundary states invariant (up to a phase), as in equation (42). (This point will be further illustrated in the next section). However, in principle, there is a logical possibility that $\hat{g}$ maps a boundary state into another boundary state. I.e. there could in principle exist a multiplet of boundary states that are mapped on to each other by $\hat{g}$. While we do not have a formal proof, in all examples of SPTs we looked at, a given boundary state is a singlet under the symmetry defining the SPT phase, as in equation (42). However, for other more complicated examples, there may be a multiplet of boundary states.

Let us contrast this with a slightly different context. In [60], $(1 + 1)d$ CFTs which appear at boundaries (edges) of $(2 + 1)d$ SPT phases are considered. These $(1 + 1)d$ edge theories of $(2 + 1)d$ SPT phases are expected to be ‘ingappable’ once the symmetry defining the SPTs are strictly enforced (also on the $(1 + 1)d$ edge theory). In [60], possible boundary conditions (boundary states) in the $(1 + 1)d$ edge theories are investigated for various examples of $(2 + 1)d$ SPT phases. It is found that there exist no conformally invariant boundary conditions on these $(1 + 1)d$ CFTs that preserve the symmetries of the underlying $(2 + 1)d$ SPT phase. In other words, all conformally invariant boundary conditions of these $(1 + 1)d$ CFTs are not invariant under the symmetry. This is expected since as boundaries of the $(2 + 1)d$ SPTs, these $(1 + 1)d$ CFT cannot themselves have boundaries, and since this statement refers to such theories that respect the symmetry of the SPT phase. Indeed, viewed from the perspective of the present paper, if a $(1 + 1)d$ CFT appearing at the boundary of the $(2 + 1)d$ SPT was gapable while preserving all the symmetries of the SPT, then (following section 2.3 or [42]) the entanglement spectrum of that gapped $(1 + 1)d$ theory at the boundary would be a BCFT with boundary conditions preserving the symmetries of the SPT. Thus, the absence of boundary conditions on the $(1 + 1)d$ CFT which preserve the symmetries of the SPT implies that this CFT is ingappable.
3.6. Boundary conditions and anomalous phases

In order to provide more intuition about the symmetry-enforced vanishing of the partition function, let us now show that the anomalous phase (42) can be interpreted as a quantum anomaly. As we discussed in the previous sections, we associate a BCFT with a SPT phase; the SPT serves as a boundary condition on a given CFT. By rotating (Euclidean) spacetime by $\pi/2$, namely $(x, \tau) = (-\tilde{\tau}, \tilde{x}) =: (\sigma_1, \sigma_2)$, we then introduce boundary states located at an ‘initial’ imaginary time in the rotated coordinates, $\tilde{\tau} = \sigma_1 = 0$, in the form

$$\begin{aligned}
\hat{\Phi}(\sigma_2) - U\hat{\Phi}(\sigma_2)\end{aligned}$$

| $B\rangle_h = 0, \quad (\text{at } \sigma_1 = 0) \quad (46)$$

which encode the boundary condition located at $x = 0$ in the unrotated coordinates. Here, $\hat{\Phi}(\sigma_2)$ denotes a (column) vector of quantum field operators representing fundamental degrees of freedom of the CFT under consideration, $U$ is a matrix acting on the column vector $\hat{\Phi}$ of fields, and $|\cdots\rangle_h$ represents a state in the $h$-twisted sector. By definition, states in the $h$-twisted sector obey

$$\begin{aligned}
\hat{\Phi}(\sigma_2 + \beta) - \hat{h}\hat{\Phi}(\sigma_2)\hat{h}^{-1}\end{aligned}$$

| $\cdots\rangle_h = 0. \quad (47)$$

Note that for a given boundary state, there may not be a simple description in terms of a fundamental field $\hat{\Phi}$, as that given in equation (46). However, when such description is available, we can develop an intuitive picture as follows.

Let the symmetry $g$ act on fundamental fields $\hat{\Phi}$ as

$$\begin{aligned}
\hat{g}\hat{\Phi}(\sigma_2)\hat{g}^{-1} = U_g\hat{\Phi}(\sigma_2),
\quad (48)$$

where $U_g$ is a matrix acting on the components of the (column) vector $\hat{\Phi}$. Let us now act with $g$ on the boundary condition,

$$\begin{aligned}
\hat{\Phi}(\sigma_2) - U\hat{\Phi}(\sigma_2)\end{aligned}$$

| $B\rangle_h = 0
\Rightarrow \hat{g}\left[\hat{\Phi}(\sigma_2) - U\hat{\Phi}(\sigma_2)\right]\hat{g}^{-1}\hat{g}|B\rangle_h = 0
\Rightarrow \left[U_g\hat{\Phi}(\sigma_2) - UU_g\hat{\Phi}(\sigma_2)\right]\hat{g}|B\rangle_h = 0
\Rightarrow \left[\hat{\Phi}(\sigma_2) - U^{-1}_gUU_g\hat{\Phi}(\sigma_2)\right]\hat{g}|B\rangle_h = 0. \quad (49)$$

By definition, our problem preserves the symmetry $g$, and hence we should have $U^{-1}_gUU_g = U$. If the boundary condition is invariant, then we may expect that so is the boundary state, $\hat{g}|B\rangle_h = |B\rangle_h$. However this expected invariance may be broken quantum mechanically: the boundary state may not be invariant, but may acquire a phase, $\varepsilon_B(g|h)$, under the action of the symmetry, as defined in equation (42). The phase $\varepsilon_B(g|h)$ can then be considered as a kind of quantum anomaly. While the boundary condition is invariant under the symmetry, the corresponding quantum mechanical state may not be. This anomaly signals the non-trivial topological properties of the corresponding ‘bulk’ SPT phase.

4. The Kitaev chain (Class D)

In this section, we apply the discussion from the preceding section to a simple fermionic SPT phase in $(1+1)d$, the Kitaev chain. The Kitaev chain is a fermionic SPT phase protected by fermion number parity conservation (symmetry class D).
In the continuum limit the Kitaev chain is described by the action (1), or equivalently in terms of the Hamiltonian

\[ H = H_0 + H_1, \]

\[ H_0 = \int_0^\ell dx \left[ \psi_L^+ (+i\partial_x) \psi_L + \psi_R(-i\partial_x) \psi_R \right], \]

\[ H_1 = \int_0^\ell dx i m \psi_L \psi_R, \]  

where (anti-)periodic boundary conditions on the Majorana fermions are imposed, \( \psi_L(x + l) = \pm \psi_L(x) \), \( \psi_R(x + l) = \pm \psi_R(x) \). The fermi velocity \( v \) was set to unity for simplicity. The real fermion fields \( \psi_L, \psi_R \) obey the canonical anticommutation relations

\[ \{ \psi_L(x), \psi_L(x') \} = 2\pi \sum_{n \in \mathbb{Z}} \delta(x - x' + \ell n), \]

\[ \{ \psi_R(x), \psi_R(x') \} = 2\pi \sum_{n \in \mathbb{Z}} \delta(x - x' + \ell n). \]

The fermionic Hamiltonian (50) preserves fermion number parity, \([H, \hat{g}_f] = 0\), where

\[ \hat{g}_f = (-1)^F, \quad F = \frac{1}{2\pi} \int_0^\ell dx i \psi_L \psi_R. \]  

Fermion parity \( \hat{g}_f \) is the only symmetry of the Hamiltonian that we consider in this section (which is a member of symmetry class D). I.e. the symmetry group protecting the SPT in equation (50) (and its \( N_f \)-flavor generalization discussed below) is \( G = Z^F_2 \) where the superscript \( F \) stands for fermion parity.

We also consider the generalization to \( N_f \) flavors of real (Majorana) fermions described by the Hamiltonian

\[ H = \sum_{a=1}^{N_f} \int_0^\ell dx \left[ \psi'^{a+}_L(+)i\partial_x) \psi'^{a-}_L + \psi'^{a+}_R(-i\partial_x) \psi'^{a-}_R \right]. \]  

The fermion fields obey the canonical anticommutation relations

\[ \{ \psi'^{a+}_L(x), \psi'^{b+}_L(x') \} = 2\pi \delta^{ab} \sum_{m \in \mathbb{Z}} \delta(x - x' + \ell m), \]

\[ \{ \psi'^{a+}_R(x), \psi'^{b+}_R(x') \} = 2\pi \delta^{ab} \sum_{m \in \mathbb{Z}} \delta(x - x' + \ell m). \]

The Hamiltonian of the fermionic theory with \( N_f \) flavors in equation (53) commutes with the total fermion number parity operator, given by

\[ \hat{g}_f = (-1)^F, \quad F = \sum_{a=1}^{N_f} F_a, \]  

where \( F_a \) is the total fermion number operator for the \( a \)th flavor,

\[ F_a = \frac{1}{2\pi} \int_0^\ell dx i \psi'^{a+}_L \psi'^{a-}_R. \]
The Hamiltonian (50) realizes two gapped phases separated by a quantum phase transition at $m = 0$. The two gapped phases can be topologically distinguished by a $\mathbb{Z}_2$ topological invariant. (Which sign of the mass term realizes the topological or trivial phase cannot be distinguished from the above continuum model, but the relative topological charge of the two gapped phases is well-defined.) It is well known that the entanglement spectrum of the topologically non-trivial phase is at least two-fold degenerate, while that of the trivial phase does not support any degeneracy [32, 42, 61].

Following the discussion in section 2.3, the low-lying entanglement spectrum is described in the scaling limit by the spectrum of an appropriate BCFT, i.e. an appropriate CFT with the boundary conditions specified by the topological properties of the gapped SPT phase. The spectrum of the BCFT is described (upon folding) by a chiral CFT defined on a circle of length $2 \times \ell$.

$$H = \int_0^{2\ell} dx \psi_L \partial_x \psi_L,$$

(57)

where the fermion field obeys either antiperiodic (‘NS’) or periodic (‘R’) boundary conditions,

$$\psi_L(x + 2\ell) = -\psi_L(x), \quad \text{or} \quad \psi_L(x + 2\ell) = +\psi_L(x).$$

(58)

These two boundary conditions (which describe two different BCFTs) correspond to the trivial and topological states of the Kitaev chain (1), as we will review momentarily.

Corresponding to these two boundary conditions, we consider the two partition functions

$$\tilde{Z}_{AA}(q) = \text{Tr}_A q^H, \quad \tilde{Z}_{PA}(q) = \text{Tr}_P q^H.$$  

(59)

Here, $\tilde{Z}_{AB}$ denotes the chiral partition function with spatial and temporal periodicity conditions labeled by $A$ and $B$, respectively; and $P$ ($A$) stand for periodic (antiperiodic) boundary conditions. In equation (59) the temporal direction is always anti-periodic (which is well known to follow in general from the Fermion path integral). In addition to these partition functions we consider, following our discussion in section 3, the sector twisted by the only non-trivial group element of the symmetry group $G = \mathbb{Z}_F^2$, the fermion number parity operator $\hat{g}_f = (-1)^F$. (Recall that fermion number parity is the only symmetry of the Hamiltonian in symmetry class D, which we consider in this section.) Following section 3.5 we are thus lead to consider, in addition to equation (59), the partition functions

$$\tilde{Z}_{AP}(q) = \text{Tr}_A (-1)^F q^H, \quad \tilde{Z}_{PP}(q) = \text{Tr}_P (-1)^F q^H.$$  

(60)

As is well-known, $\tilde{Z}_{PP}$ actually vanishes, $\tilde{Z}_{PP} = 0$. This is due to the fermion zero mode. (This should be distinguished from the zero mode that causes the symmetry protected degeneracy in the entanglement spectrum we are after.) As we will now explain, the vanishing $\tilde{Z}_{PP} = 0$ is precisely an example of a symmetry-enforced vanishing of the partition function discussed in general terms in section 3, the symmetry being Fermion number parity. As we will explain in the following, when this partition function is described within the boundary state formalism, the corresponding boundary state will pick up an anomalous phase of the kind defined in equation (42).

---

17 Note that $\tilde{Z}_{AB}$ is technically an object different from $Z_{AB}$ in equation (29), since in the latter the subscripts denote boundary conditions on a non-chiral CFT defined on a finite interval, whereas in the former the subscripts denote periodicity conditions on the chiral (say, only left-moving) fermion degrees of freedom (even there is of course a connection, which is recalled below). For that reason the former partition function is distinguished from the latter by a different symbol.
4.1. Boundary states

To discuss the symmetry-enforced vanishing of the partition function of the current theory (class D) from the CFT point of view, we consider the free fermion CFT that results from setting $m = 0$ in the Hamiltonian (1). Consider this (gapless) free-fermion CFT on the interval $x_1 \leq x \leq x_2$. At the two boundaries $x = x_1$ and $x = x_2$ of this interval, let us consider the following boundary conditions on the fermion field\(^ {18} \)

$$\psi_L(x_1) = \eta_1 \psi_R(x_1), \quad \psi_L(x_2) = -\eta_2 \psi_R(x_2), \quad (61)$$

where $\eta_1, \eta_2 = \pm 1$. In terms of the scattering matrix language discussed in section 2.2, these boundary conditions correspond to the reflection coefficients (matrices)

$$r = \eta_2, \quad r' = \eta_1. \quad (62)$$

(Compare equation (8).) The topological invariant computed from these reflection coefficients (matrices) is given by

$$\text{sgn det } r = \eta_2, \quad \text{sgn det } r = \eta_1. \quad (63)$$

When $\eta_1 = \eta_2$ we obtain (upon employing the ‘folding procedure’) from equation (61) a system of chiral (say, left-moving) fermions on an interval of length $2\ell$ with anti-periodic (‘NS’) boundary conditions. As is well known, this spectrum has no degeneracies. On the other hand, when $\eta_1 = -\eta_2$, the resulting system of chiral (say, left-moving) fermions on an interval of length $2\ell$ has periodic (‘R’) boundary conditions, which has a two-fold degeneracy (as is also well known). The choice $\eta_1 = -\eta_2$ for the pair of boundary conditions corresponds to a domain wall in the mass term—see figure 1. Thus the condition $\eta_1 = -\eta_2$ localizes a non-trivial zero mode in the gapless region $x_1 \leq x \leq x_2$ by the Jackiw–Rebbi mechanism, as discussed in section 2.2.

Let us now construct the boundary states corresponding to the boundary conditions in equation (61). Consider e.g. the boundary condition at $x = x_1$, given by

$$\psi_L(\tau, x_1) = \eta_1 \psi_R(\tau, x_1), \quad (0 \leq \tau \leq \beta), \quad (64)$$

where the fermions $\psi_L$ and $\psi_R$ possess their (natural) anti-periodic boundary conditions in imaginary time $\tau$. We now make the rotation by $\pi/2$ of (Euclidean) spacetime discussed in the paragraph surrounding equation 46, namely $(x, \tau) = (-\bar{x}, \bar{\tau})$. Since the fermion fields $\psi_L$ and $\psi_R$ are holomorphic (anti-holomorphic) functions of conformal weight (scaling dimension) $1/2$, they transform under the $\pi/2$ rotation $(\tau + i\beta) = (-\bar{\tau} + i\bar{\beta}) = (\bar{\tau} + i\bar{\beta})$ as

18 The difference in sign for the boundary conditions at $x = x_1$ and $x = x_2$ arises from the fact that both $\psi_L(z)$ and $\psi_R(z)$ transform as spinors under rotations in Euclidean two-dimensional spacetime, where $z = \tau + ix$. Here, $\bar{z} = \tau - ix$ (or, equivalently, from the fact that they have conformal weight (‘scaling dimension’) $h = 1/2$). That this leads to the signs displayed in equation (61) can easily be seen as follows. Consider first the situation where $x_1 = 0$ and $x_2 \to +\infty$; there is hence only one boundary, namely the one at $x = x_1 \to 0$. The spacetime is then the upper half complex plane, $\text{Im } \bar{z} \geq 0$, and the boundary is located on the real axis, $\text{Im } \bar{z} = 0$, which is a ‘lower boundary’. Compare this with the situation where $x_2 = 0$ and $x_1 \to -\infty$; there is now also only one boundary, namely the one at $x_2 = 0$. The spacetime is then the lower half complex plane, $\text{Im } \bar{z} \leq 0$, and the boundary is again located on the real axis, which is however now an ‘upper boundary’.—Now, the two situations of an ‘upper boundary’ and of a ‘lower boundary’ are related to each other by reflection about the real axis, $z \to -\bar{z}$, and $\bar{z} \to -z$. Let us impose on the ‘lower boundary’ the condition $\psi_L(x_1 = 0) = \eta_1 \psi_R(x_1 = 0)$, or equivalently $\psi_L(z) = \eta_1 \psi_R(z)$ when $z = \bar{z}$ (hence $\text{Im } z = 0$). Then the same boundary condition would read at an ‘upper boundary’ $e^{i\pi/2} \psi_L(-z) = \eta_1 e^{-i\pi/2} \psi_R(-z)$, implying $\psi_L(-z) = -\eta_1 \psi_R(-z)$ when $z = \bar{z}$ (thus $\text{Im } z = 0$), and therefore $\psi_L(x_1 = 0) = -\eta_1 \psi_R(x_1 = 0)$. (Here we used $\psi_L(e^{i\pi/2}z) = (1/e^{i\pi/2})\psi_L(z)$, and $\psi_R(e^{i\pi/2}z) = (1/e^{-i\pi/2})\psi_R(z)$.) For this reason, the same boundary condition appears with the opposite sign of $\eta_1$ at the ‘upper boundary’ as compared to the ‘lower boundary’. 

22
\[ \psi_L(\tau + i\chi) = e^{i\pi/4} \psi_L(\tilde{\tau} + i\chi), \]
\[ \psi_R(\tau - i\chi) = e^{-i\pi/4} \psi_R(\tilde{\tau} - i\chi). \]

This implies that the boundary condition (64) reads in the rotated coordinates
\[ \psi_L(\bar{\tau}_1, \bar{x}) = (-i)\eta_L \psi_R(\bar{\tau}_1, \bar{x}), \quad (0 \leq \bar{x} \leq \beta). \]

The boundary states \([B(\eta)]\) represent an operator statement of the boundary condition (66) on the closed string Hilbert space,
\[ [\psi_L(\bar{\tau}, \bar{x}) - i(-\eta)\psi_R(\bar{\tau}, \bar{x})] [B(-\eta)] = 0, \]

with anti-periodicity in \(0 \leq \bar{x} \leq \beta\), which is inherited from the anti-periodicity in \(\tau\). For simplicity we now set \(\bar{\tau} = x_1 = 0\), and omit writing the \(\bar{\tau}\) coordinate. The boundary state describing the boundary condition at \(x = x_2\) satisfies the same equation with \(\eta \rightarrow \eta_1\) (not \(\eta \rightarrow -\eta_2\); see the footnote immediately above equation (61)).

Following the discussion in section 3 we must now twist the boundary states defined in equation (67) by a group element of the symmetry group of the SPT phase. As mentioned above, in the current case of symmetry class D, there is only one non-trivial group element, which is the fermion parity operator \(\hat{g}_f\) defined in equation (55). Because the fermion parity operators changes the periodicity on both, the left- and the right-moving fermions in equation (67) from anti-periodic to periodic, the twisted boundary state \([B(\eta)]_{\hat{g}_f}\) satisfies the equation
\[ [\psi_L(\bar{x}) - i\eta \psi_R(\bar{x})] [B(\eta)]_{\hat{g}_f} = 0, \]

with periodicity in \(0 \leq \bar{x} \leq \beta\). Upon Fourier transforming the Hermitian (Majorana) fermion operators,
\[ \psi_L(\bar{x}) = \frac{2\pi}{\beta} \sum_s e^{-i2\pi s\bar{x}/\beta} \psi_{sL}, \quad \psi_{sL}^\dagger = \psi_{-sL} \]
\[ \psi_R(\bar{x}) = \frac{2\pi}{\beta} \sum_s e^{i2\pi s\bar{x}/\beta} \psi_{sR}, \quad \psi_{sR}^\dagger = \psi_{-sR} \]

where the mode-index \(s \in \mathbb{Z} + \frac{1}{2} (s \in \mathbb{Z})\) for anti-periodic (periodic) boundary conditions in \(\bar{x}\), equation (68) reads
\[ [\psi_{sL} - i\eta \psi_{sR}] [B(\eta)]_{\hat{g}_f} = 0, \quad (s \in \mathbb{Z}). \]

This determines the boundary state to be of the form
\[ [B(\eta)]_{\hat{g}_f} = \exp \left\{ \frac{i}{\eta} \sum_{s=1}^{+\infty} [\psi_{sR}^\dagger \psi_{sL}^\dagger] \right\} [B(\eta)]_0. \]

where the ‘zero-mode contribution’ (from \(s = 0\)), \([B(\eta)]_0\), is determined by
\[ [\psi_{0L} - i\eta \psi_{0R}] [B, \eta]_0 = 0. \]

The zero modes satisfy \((\psi_{0L})^2 = (\psi_{0R})^2 = 1\). The zero-mode contribution to the boundary state can be constructed by considering the following fermion creation and annihilation operators (we immediately discuss here the general case of \(N_f\) Majorana flavors, \(a = 1, ..., N_f\)),
\[ f_a^\dagger = \frac{1}{2} (\psi_{0L}^a + i\psi_{0R}^a), \quad f_a = \frac{1}{2} (\psi_{0L}^a - i\psi_{0R}^a). \]
where $|0\rangle$ denotes the Fock vacuum of the $f_a$-fermions. In view of equations (70), (69), the boundary state $|B, \eta = +\rangle_0$ is then nothing but the Fock vacuum $|0\rangle$ itself,

$$|B, \eta = +\rangle_0 = e^{i\phi \pi} |0\rangle,$$

(74)

On the other hand, the boundary state $|B, \eta = -\rangle_0$ can be constructed as

$$|B, \eta = -\rangle_0 = e^{-i\phi \pi} \prod_{a=1}^{N_f} f_a^\dagger |0\rangle.$$

(75)

In the above representation of $|B, \eta\rangle_0$, the ambiguous phases $\phi, \pi$ are not fixed by the boundary condition. These phases will not affect our analysis in this section, and hence will be set to zero henceforth. We will come back to the issue of a suitable choice of the phase at the end of this section, and also in section 6, in which a proper choice of the phase is more crucial. (We however comment that one common convention for the phase is $|B, \eta = +\rangle_0 = e^{-i\phi \pi} |0\rangle$, $|B, \eta = -\rangle_0 = e^{i\phi \pi} f_a^\dagger |0\rangle$. (These equations are written, for simplicity, for the case of a single flavor $N_f = 1$.) One motivation for this phase convention is that the following relations $\psi\psi, |B, \eta\rangle_0 = \psi^{-i\phi \pi} |B, -\eta\rangle_0$; $\psi\psi |B, \eta\rangle_0 = \psi^{i\phi \pi} |B, -\eta\rangle_0$ bear resemblance to the operator product expansions of the Ising CFT, $\psi_L \sigma \sim \psi^{i\phi \pi} \mu, \psi_L \sigma \sim \psi^{-i\phi \pi} \sigma, \psi_L \mu \sim \psi^{i\phi \pi} \sigma$, where $\sigma$ and $\mu$ are the Ising spin operator and the disorder operator, respectively. (See e.g. appendix E of [62], or [63]).)

Following the general discussion of section 3.5, in particular, equation (42), we now ask about the properties of the boundary states under the action of the fermion number parity $\hat{g}_f$, the only element of the symmetry group of the present SPT. Its explicit form within the zero mode sector of the closed string Hilbert space

$$\hat{g}_f = (i\psi_{0L}^1 \psi_{0R}^1) (i\psi_{0L}^2 \psi_{0R}^2) \cdots (i\psi_{0L}^{N_f} \psi_{0R}^{N_f}),$$

(76)

when there are the $N_f$ flavors of Majorana fermions. This implies that the fermion number parity operator acting on the boundary states gives

$$\hat{g}_f |B, \pm\rangle_0 = (\pm 1)^{N_f} |B, \pm\rangle_0.$$ 

(77)

Therefore, when $N_f$ is even, there is no anomaly neither for $|B, +\rangle_0$ nor for $|B, -\rangle_0$. On the other hand, when $N_f$ is odd one would conclude that $|B, -\rangle_0$ is anomalous while $|B, +\rangle_0$ is not. This is consistent with the $\mathbb{Z}_2$ classification of $(1+1)d$ topological superconductors in symmetry class D.

Upon closer inspection however, equation (77) would look strange since the two states $|B, \pm\rangle_0$ should be treated on the equal footing. In fact, it should be noted that there is a phase ambiguity in defining the boundary states and the fermion number parity operator. In the above analysis, we implicitly made a particular choice where the fermion number parity of the ground state $|0\rangle$ is $+1$. In principle, one could assign a different fermion number parity eigenvalue, e.g. by modifying the definition of the fermion number parity operator, $\hat{g}_f \rightarrow -\hat{g}_f$. Alternatively, instead of using $f_a^\dagger, f_a$, one could define $c_a := f_a^\dagger$ and $c'_a := f_a$, which leads to $|B, -\rangle_0 = |0\rangle$ and $|B, +\rangle_0 = \prod_a c'_a |0\rangle$. In this convention, one would then be led to claim $\hat{g}_f |B, -\rangle_0 = |B, -\rangle_0$ while $\hat{g}_f |B, +\rangle_0 = (-1)^{N_f} |B, +\rangle_0$. Thus, there is some ambiguity when deducing the fermion number parity eigenvalue. Such an ambiguity of the fermion number parity eigenvalue of the ground state, however, does not affect our conclusion, since, independent of the phase choice, when $N_f$ is odd, we cannot make both $|B, +\rangle_0$ and $|B, -\rangle_0$ anomaly-free. In conclusion, our analysis of the anomalous phase of the boundary state (as defined

\[19\] The explicit form in terms of non-zero modes follows immediately from equations (55), (56), (69).
in equation (42) of section 3.5) leads to the (known) result that there is a $\mathbb{Z}_2$ classification for $(1 + 1)d$ SPT phases in symmetry class D.

5. The Haldane phase and the compact boson theory

The Haldane phase of the SU(2) spin-1 quantum spin chain is historically the first and the canonical example of a one-dimensional symmetry-protected topological phase. The Haldane phase was shown to be a stable symmetry-protected topological phase if one of the following discrete symmetries is imposed [33]:

(i) **TRS** Time-reversal acts on a spin-1 operator as

\[
T : \quad \hat{T} S \hat{T}^{-1} = -S, \quad \hat{T} \tilde{S} \hat{T}^{-1} = -\tilde{S}.
\]  

Note that $T^2 = +1$.

(ii) The dihedral group of $\pi$-rotations about $x, y$ and $z$ axes ($D_2$) Consider a $\pi$-rotation around a particular vector in spin space: e.g. $\pi$-rotation around $z$-axis is

\[
R_z^\pi : \quad S' \to -S', \quad S' \to -S' \quad S' \to +S'.
\]  

Take any two of $R_x^\pi, R_y^\pi$. The third transformation is given by the product of other two. So, this is $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry.

(iii) **Link inversion** This can be realized as (site inversion) + (translation). The one-site translation is given by $S_j \to S_{j+1}$, while the site parity transformation is $S_j \to S_{-j}$. If combined, the link inversion $L$ acts on the spin operator at site $j$ as $L : S_j \to S_{j+1} \to S_{-j-1}$.

In the following, we will focus on the protection of the Haldane phase by $\mathbb{Z}_2 \times \mathbb{Z}_2$ (dihedral) symmetry.

5.1. Field theory descriptions of the Haldane phase

The Haldane phase is known to be adjacent to at least three CFTs the compactified free boson $U(1)$ ($c = 1$), the $SU(2)_2$ Wess–Zumino–Witten theory ($c = 3/2$), and the $SU(3)_1$ Wess–Zumino–Witten theory ($c = 2$). In this section, we will focus on the $c = 1$ CFT, and discuss its neighboring gapped phase; the Haldane phase (non-trivial SPT phase) and the so-called large $D$-phase.

We start from the free boson theory on a spatial ring of circumference $\ell$ defined by the partition function $Z = \int \mathcal{D}[\phi] \exp(iS)$ with the action

\[
S = \frac{1}{4\pi\alpha'} \int dt \int_0^\ell dx \left[ \frac{1}{v} (\partial_t \phi)^2 - v (\partial_x \phi)^2 \right],
\]  

where the spacetime coordinate of the edge theory is denoted by $(t, x)$, $v$ is the velocity, $\alpha'$ is the coupling constant, and the $\phi$-field is compactified as

\[
\phi \sim \phi + 2\pi R,
\]  

where $R$ is the radius of the compactification.
with the compactification radius $R$. The canonical commutation relation is
\[ [\phi(x,t), \partial_t \phi(x',t)] = i2\pi \alpha'/\ell \sum_{n\in\mathbb{Z}} \delta(x - x' - n\ell). \]  
(82)

We use the chiral decomposition of the boson field, and introduce the dual field $\theta$ as
\[ \phi = \varphi_L + \varphi_R, \quad \theta = \varphi_L - \varphi_R. \]  
(83)

The mode expansion of the chiral boson fields is given by $(x^\pm = vt \pm x)$
\[ \varphi_L(x^+) = x_L + \pi \alpha' p_L x^+ / \ell + i \sqrt{\alpha'/2} \sum_{n\in\mathbb{Z}} \delta_R e^{-i \omega_n x^+ / \ell}, \]  
(84)
\[ \varphi_R(x^-) = x_R + \pi \alpha' p_R x^- / \ell + i \sqrt{\alpha'/2} \sum_{n\in\mathbb{Z}} \delta_R e^{i \omega_n x^- / \ell}, \]
where $[\alpha_m, \alpha_{-n}] = [\tilde{\alpha}_m, \tilde{\alpha}_{-n}] = m \delta_{mn}$ and $[x_L, p_L] = [x_R, p_R] = i$. The compactification condition on the boson fields implies that the allowed momentum eigenvalues are given by
\[ p = \frac{1}{2} (p_L + p_R) = k \frac{R}{\alpha'}, \quad \tilde{p} = \frac{1}{2} (p_L - p_R) = \frac{R}{\alpha'} w, \]  
(85)
\[ p_L = k \frac{R}{\alpha'} w, \quad p_R = k \frac{R}{\alpha'} w, \]
where $k$ and $w$ are integers. In terms of these momentum eigenvalues, the compactification conditions on the boson fields are
\[ \varphi_L(x + \ell) = \varphi_L(x) + \pi \alpha' p_L, \]
\[ \varphi_R(x + \ell) = \varphi_R(x) - \pi \alpha' p_R, \]
\[ \phi(x + \ell) - \phi(x) = \pi \alpha' (p_L - p_R) = 2\pi R w, \]  
(86)
\[ \theta(x + \ell) - \theta(x) = \pi \alpha' (p_L + p_R) = 2\pi \alpha'/R. \]

The Hilbert space is constructed as a tensor product of the bosonic oscillator Fock spaces, each of which is generated by pairs of creation and annihilation operators $\{\alpha_m, \alpha_{-m}\}_{m>0}$ and $\{\tilde{\alpha}_m, \tilde{\alpha}_{-m}\}_{m>0}$, and the zero mode sector associated with $x_{L,R}$ and $p_{L,R}$. We will denote states in the zero mode sector by specifying their momentum eigenvalues as
\[ |p, \tilde{p}\rangle = |k,R,R w/\alpha'|, \quad k, w \in \mathbb{Z}, \]  
(87)
or more simply as $|k, w\rangle$. Alternatively, the Fourier transformation of the momentum eigenkets defines the 'position' eigenkets, which we denote by
\[ |\phi_0, \theta_0\rangle \quad 0 < \phi_0 \leq 2\pi R, \quad 0 < \theta_0 \leq 2\pi \alpha'/R. \]  
(88)

The two bases are related by
\[ |p, \tilde{p}\rangle = \int_0^{2\pi} d\phi_0 \int_0^{2\pi \alpha'/R} d\theta_0 e^{-i\phi_0 \theta_0} |\phi_0, \theta_0\rangle. \]  
(89)

The single-component compactified boson theory is invariant under various symmetry operations. First of all, in the free boson theory, when there is no perturbation, there are two conserved $U(1)$ charges, one for each left- and right-moving sector. Corresponding to
these conserved quantities, the free boson theory is invariant under the following $U(1) \times U(1)$ symmetry
\[
U_{\delta \phi, \delta \theta} : \phi \to \phi + \delta \phi, \quad \theta \to \theta + \delta \theta,
\]
\[
\phi_L \to \phi_L + \delta \phi_L, \quad \phi_R \to \phi_R + \delta \phi_R.
\]
where $\delta \phi_L = \frac{\delta \phi + \delta \theta}{2}$ and $\delta \phi_R = \frac{\delta \phi - \delta \theta}{2}$. In terms of the conserved charges, the generators of the $U(1) \times U(1)$ transformations are given by
\[
\hat{U}_{\delta \phi, \delta \theta}^L = e^{i \delta \phi_{N_L}/(\alpha' \pi)} = e^{i \delta \phi_{\phi L}},
\]
\[
\hat{U}_{\delta \phi, \delta \theta}^R = e^{i \delta \phi_{N_R}/(\alpha' \pi)} = e^{i \delta \phi_{\phi R}},
\]
\[
\hat{U}_{\delta \phi, \delta \theta} = \hat{U}_{\delta \phi, \delta \theta}^L \hat{U}_{\delta \phi, \delta \theta}^R = e^{i(\delta \phi + \delta \theta)}.
\]
Note that $\hat{U}_{\delta \phi, \delta \theta}$ acts on the momentum eigenkets as
\[
\hat{U}_{\delta \phi, \delta \theta} | p, \bar{p} \rangle = e^{i(\delta \phi + \delta \theta)} | p, \bar{p} \rangle.
\]
Another important symmetry in our discussion of the Haldane phase is particle-hole symmetry. Particle-hole symmetry or charge conjugation ($C$-symmetry) is unitary and acts on the bosonic fields as
\[
\hat{C} : \phi \to -\phi + n_c \pi R, \quad \theta \to -\theta + \frac{m_c \pi \alpha'}{R},
\]
where $(n_c, m_c) \in \{0, 1\}$. From these transformation laws of the boson fields, we read off the action of $C$-symmetry on the position basis as
\[
\hat{C} | \phi_0, \theta_0 \rangle = e^{i \delta} | -\phi_0 + n_c \pi R, -\theta_0 + m_c \pi \alpha'/R \rangle,
\]
where $e^{i \delta}$ is an unknown phase factor. In order to have the relation $\hat{C} | p, \bar{p} \rangle \propto | -p, -\bar{p} \rangle$, expected from the commutation relation between $\hat{C}$ and $p, \bar{p}$, the phase $\delta$ has to be a constant (independent of $\phi_0$ and $\theta_0$). The action of $C$-symmetry on the momentum eigenstates is given by
\[
\hat{C} | p, \bar{p} \rangle = e^{i \delta} e^{-i m_c \pi R - i \bar{p} n_c \pi \alpha'/ R} | -p, -\bar{p} \rangle = e^{i \delta} e^{-i m_c \pi R - i \bar{p} n_c \pi \alpha'/ R} | -p, -\bar{p} \rangle,
\]
where $p = k/R$ and $\bar{p} = w R/\alpha'$. Since $\delta$ is constant, the phase ambiguity is fixed once we specify the action of $\hat{C}$ on a reference state, e.g. $| p, \bar{p} \rangle = | 0, 0 \rangle$. In our analysis presented below, the reference state and its charge conjugation parity eigenvalue $e^{i \delta}$ plays an important role.

Following [64, 65] (see also [66]), we now adopt the convention
\[
\alpha' = \frac{1}{2}, \quad R = \frac{1}{2}, \quad \frac{R}{\sqrt{\alpha'}} = \frac{1}{\sqrt{2}}, \quad v = 1.
\]
This set of the parameters realizes the free fermion point in the moduli space of $c = 1$ CFTs. (In our conventions the SU(2) point (the self-dual radius) is realized when $R/\sqrt{\alpha'} = 1$.)
In this convention, the Haldane phase can be described by the sine-Gordon model:

\[
H = \int dx \left\{ \frac{1}{2\pi} \left[ (\partial_x \theta)^2 + (\partial_x \phi)^2 \right] - \lambda \cos(2\phi) \right\},
\]

where the bosonic fields are compactified as \( \phi \sim \phi + \pi \) and \( \theta \sim \theta + 2\pi \). The field \( \theta \) is introduced to represent the canonical conjugate variable of the \( \phi \), and the fields \( (\phi, \theta) \) are related to (slow-modes of) the microscopic spin \( (S_x(x), S_y(x), S_z(x)) \) by

\[
S_x(x) + i S_y(x) \sim e^{i \theta(x)}, \quad S_z(x) \sim \partial_x \phi.
\]

Considering the Haldane model as a model of the (hard-core) boson, we can also relate the fields \( (\phi, \theta) \) to (slow-modes of) the microscopic boson \( b(x) \) and its density fluctuation \( \delta \rho(x) \)

\[
b(x) \sim \sqrt{\bar{\rho}} e^{i \theta(x)}, \quad \delta \rho(x) = \rho(x) - \bar{\rho} \sim \frac{1}{\pi} \partial_x \phi.
\]

where \( \bar{\rho} \) is the average density of the boson.

Under \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) symmetry (the \( \pi \) rotations of spins around \( S_x, S_y, \) and \( S_z \)-axis), the phase variables are transformed as

\[
R^x_\pi : \phi \to -\phi, \quad \theta \to -\theta \\
R^y_\pi : \phi \to -\phi, \quad \theta \to -\theta + \pi \\
R^z_\pi : \phi \to \phi, \quad \theta \to \theta + \pi.
\]

These transformation can be generated by combining the charge conjugation \( \hat{C} \) and the \( U(1) \) phase rotation \( U_{\delta \theta = \pi} \). (On the other hand, time-reversal acts on the phase fields as \( T : \phi \to -\phi, \theta \to \theta + \pi \)).

The cosine term \(-\lambda \cos(2\phi)\) in equation (97) is allowed by the symmetry (though it is not the only perturbation allowed by the symmetry). The theory (97) describes the phase transition between the trivial Mott insulator and symmetry protected topological insulator, i.e. the Haldane insulator [64]. The transition is triggered by changing sign of the coefficient \( \lambda \) of the cosine term in the effective theory equation (97).

5.2. The entanglement spectrum of the Haldane phase

Following our general considerations, we now discuss the BCFT description of the entanglement spectrum of the Haldane phase. Setting \( \lambda = 0 \) in equation (97) the relevant CFT is the single-component compactified free boson. To identify the relevant boundary conditions, let us first consider the two gapped phases realized in the Hamiltonian (97), by taking \( \lambda \to \infty \) and \( \lambda \to -\infty \). In the both phases, the cosine term strongly pins the \( \phi \) field to its minima: For \( \lambda \to \infty \), \( \phi \) is pinned at 0 mod \( \pi \). On the other hand, for \( \lambda \to -\infty \), \( \phi \) is pinned at \( \frac{\pi}{2} \) mod \( \pi \).

Let us next consider the domain wall between the two phases by changing \( \lambda \) as a function of \( x \). The domain wall is realized by the following configuration of \( \lambda(x) \):

\[
\lambda(x) = \begin{cases} 
-\Lambda & \text{for } x < 0, \\
0 & \text{for } x \in [0, \ell], \\
+\Lambda & \text{for } x > \ell.
\end{cases}
\]

(101)
We will take the limit \( \Lambda \to +\infty \) so that the theory of \( x < 0 \) or \( x > \ell \) is in its ground states of the cosine term of equation (97) with the corresponding sign of the coefficient \( \Lambda \). (The conventional domain wall picture can be then realized by taking \( \ell \to 0^\pm \).) Hence we effectively consider a critical boson theory which is spatially sandwiched by the two topologically distinct insulator phases. We thus consider the boundary condition:

\[
\begin{align*}
\phi(x = 0) &= \frac{\pi}{2} \mod \pi, \\
\phi(x = \ell) &= 0 \mod \pi.
\end{align*}
\tag{102}
\]

Before calculating the spectrum of the BCFT, and hence the entanglement spectrum, let us discuss the presence of the domain wall mode from somewhat complementary point of view. We expect that there should be a zero mode, i.e. a solitonic operator, in the critical regime, which is identified with the topological boundary modes of the Haldane chain. We would like to identify this soliton operator in the language of the CFT. For this, we need to look carefully into the boundary condition imposed on the boson field \( \phi \). From the boundary condition, we find that

\[
\int_0^\ell dx \frac{\partial \phi}{\partial x} = \int_0^\ell dx \delta \rho(x) = n \pm \frac{1}{2}, \quad n \in \mathbb{Z}.
\tag{103}
\]

Hence the soliton object we consider is created by

\[
z_{\uparrow} \sim e^{\frac{\phi}{\pi}}, \quad \text{and} \quad z_{\downarrow} \sim e^{-\frac{\phi}{\pi}}.
\tag{104}
\]

Let us emphasize that the fields \( z_{\sigma}, \sigma \in \{\uparrow, \downarrow\} \) are the creation operators of the half charge of the fundamental boson and thus are the fractional degrees of freedom of the original boson. Furthermore, it is straightforward to check that they satisfy the projective symmetry group representation. Furthermore, it is now straightforward to check that the configuration

\[
\lambda(x) = \begin{cases} 
+\Lambda & \text{for } x < 0, \\
0 & \text{for } x \in [0, \ell], \\
+\Lambda & \text{for } x > \ell,
\end{cases}
\tag{105}
\]

has no non-trivial degenerate zero mode realizing the projective symmetry representation for both the limits \( \Lambda \to \pm \infty \).

Dependent on the sign of \( \Lambda \), there are two drastically different behaviors of the spectrum in terms of topological degeneracy, which is the focus of our interest. Hence we discuss the two cases separately. In general, the mode expansion of the boson field \( \phi(t, x) \) is the following \cite{63, 67}:

\[
\phi(t, x) = \frac{\Delta \phi - p \pi}{\ell} x + \sum_{n \in \mathbb{Z}, n \neq 0} \alpha_n \frac{e^{-2\pi i nt}}{n} \sin \left( \frac{2\pi n \ell}{\ell} x \right).
\tag{106}
\]

in which \( p \in \mathbb{Z} \) determines the winding of the bosons, \( \Delta \phi = \phi(x = \ell) - \phi(x = 0) \mod \pi \) to be determined by the boundary conditions, and \( \alpha_n \) is the harmonic oscillator satisfying \( [\alpha_n, \alpha_m] = m \delta_{n+m,0} \). The entanglement Hamiltonian in terms of the mode decomposition can be written as: \cite{63, 67}

\[
H = \frac{2}{\ell} \left[ \left( \frac{\Delta \phi}{\pi} - p \right)^2 + \sum_{n > 0} \alpha_{-n} \alpha_n \right]. \quad p \in \mathbb{Z}.
\tag{107}
\]
When $\Delta \phi = 0$, the lowest state of the tower for $p = 0$ is non-degenerate and so are all states in the tower of states. Thus the entanglement spectrum is trivial. The lowest states of the tower $p = 0$ and $p = 1$ are degenerate, and all states in the spectrum are at least doubly-degenerate. Furthermore, by state-operator correspondence, the two lowest states corresponds to the spinor (104), which transform projectively under symmetry. Thus the degeneracy in the entanglement spectrum is protected by symmetry as exactly the same way as the physical boundary zero modes.

5.3. Boundary states

Let us now use the boundary states to show (again) the symmetry-protected degeneracy. We will also derive the anomalous phase of the boundary state in the twisted sector.

The boundary state with $\phi(0) = \phi_0$ can be explicitly constructed as

$$|D(\phi_0)\rangle = \sqrt{\frac{1}{R\sqrt{2}}} \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \alpha_{-n} \tilde{\alpha}_{-n}\right) \sum_{p=k/R, k \in \mathbb{Z}} e^{-ip\phi_0/R} |p,0\rangle. \quad (108)$$

This state is invariant under $U_{\delta \theta}$ and $C$. The partition function can be computed from the boundary state as

$$\langle D(\phi_0) | \hat{q}^{1/2} (H_L + H_R) | D(\phi_0') \rangle = \frac{1}{\eta(\tilde{q})} \sum_{m \in \mathbb{Z}} q^{(m+1/2)^2}. \quad (109)$$

This spectrum shows that all states are at least doubly degenerate.

5.3.1. $\mathbb{R}_\pi^2$-twisted sector. Following our general discussion, we now consider boundary states in twisted sectors. In particular, we will confirm the symmetry-enforced vanishing of the partition function, by computing the anomalous phase of boundary states that may be picked up under the action of symmetry. Let us first now consider the twist by $R_\pi$

$$\theta(x + \ell) = \theta(x) + 2\pi \alpha' k/R + \pi \alpha'/R, \quad (110)$$

where $k$ is an integer. With this twist, the allowed momentum is now

$$p = \frac{1}{2} (p_L + p_R) = \frac{1}{R} (k + 1/2), \quad (111)$$

as one can see from the mode expansion of the boson fields. The boundary state with $\phi(0) = \phi_0$ in the presence of the twist is

$$|D(\phi_0)\rangle_{R_\pi} = \sqrt{\frac{1}{R\sqrt{2}}} \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \alpha_{-n} \tilde{\alpha}_{-n}\right) \sum_{p=(k+1/2)/R, k \in \mathbb{Z}} e^{-ip\phi_0/R} |p,0\rangle. \quad (112)$$
When $\phi_0 = \pi R$, the symmetry $\hat{C}$ acts on the boundary state as

$$
\hat{C} \sum_{p=\frac{1}{2}(k+\frac{1}{2}), k \in \mathbb{Z}} e^{-ip\phi_0/R} |p, 0\rangle
= \sum_{p=\frac{1}{2}(k+\frac{1}{2}), k \in \mathbb{Z}} e^{-ip\phi_0/R} - |p, 0\rangle
= \sum_{p=\frac{1}{2}(k+\frac{1}{2}), k \in \mathbb{Z}} e^{ip\phi_0/R} |p, 0\rangle.
$$

(113)

Since $e^{ip\phi_0/R} = e^{i(k+1/2)\pi} = (-1)e^{-i(k+1/2)\pi} = (-1)e^{-ip\phi_0/R}$, we conclude that the boundary state picks up a minus sign under the action of $\hat{C}$:

$$
\hat{C}|D(\pi R)\rangle_{R^x_\pi} = -|D(\pi R)\rangle_{R^x_\pi}.
$$

(114)

We thus conclude the corresponding partition function is forced to zero due to symmetry.

5.3.2. $R^x_\pi$ twisted sector. Let us now consider orbifolding by $R^x_\pi$:

$$
\phi(x + \ell) = -\phi(x) + 2\pi Rn,
$$

$$
\theta(x + \ell) = -\theta(x) + 2\pi \frac{\alpha'}{R} m,
$$

(115)

where $n, m$ are some integers. This twist sets the momentum to be zero, $p = \tilde{p} = 0$, and the mode expansion compatible with the twist is given by

$$
\phi(x) = x_L + x_R + \cdots
$$

(116)

where $\cdots$ represents oscillator modes. In the twisted sector the zero mode $x_L + x_R$ can only take its fixed point value 0 or $\pi R$. Thus, there are two independent states in the zero mode sector, $|0\rangle_{R^x_\pi}$ and $|\pi R\rangle_{R^x_\pi}$. (See, for example, [25, 68] for the Dirichlet boundary states can then be constructed from the states as

$$
|B(\phi_0)\rangle_{R^x_\pi} \propto \exp(\text{oscillator part})|\phi_0\rangle_{R^x_\pi},
$$

(117)

where $\phi_0 = 0$ or $\phi_0 = \pi R$.

In addition to the Dirichlet boundary states in the twisted sector, the orbifold theory twisted by $R^x_\pi$ allows the boundary states in the untwisted sector. They are simply given by a suitable linear combination of the boundary states which are invariant under $R^x_\pi$. These boundary states represent boundary conditions in which the boson field is pinned at a certain value $\phi_0$, where $\phi_0$ can be arbitrary. When $\phi_0$ is at the fixed points of the symmetry, $\phi_0 = 0$ or $\phi_0 = \pi R$, these untwisted boundary states by themselves fail to satisfy the Cardy condition. It is then necessary to consider the boundary states in the twisted sector considered above.

The two zero mode states $|\phi_0\rangle_{R^x_\pi}$, and hence the two boundary states $|B(\phi_0)\rangle_{R^x_\pi}$, are orthogonal to each other, and hence the partition function

$$
_{R^x_\pi}\langle B(0)|\hat{q}^{\text{mod}}|B(\pi R)\rangle_{R^x_\pi} = 0
$$

(118)

vanishes. To see if this is symmetry enforced, we need to consider the action of symmetry on these boundary states, say, $R^x_\pi$. I.e. $R^x_\pi|B(\phi_0)\rangle_{R^x_\pi}$. To this end, let us first consider Neumann boundary states in the twisted sector. They are given by

$$
|N(\theta_0)\rangle_{R^x_\pi} = e^{(\text{osc. part})} \frac{1}{\sqrt{2}} (|0\rangle_{R^x_\pi} \pm |\pi R\rangle_{R^x_\pi}),
$$

(119)
where $\theta_0 = 0$ or $\theta_i = \pi \alpha'/R$. Since $R^c_{\pi}$ shifts $\theta$ by $\pi$, we expect that $R^c_{\pi}$ exchanges $|N(0)\rangle_{R^c_{\pi}}$ and $|N(\pi \alpha'/R)\rangle_{R^c_{\pi}}$. That is,

\begin{align}
R^c_{\pi}|D(0)\rangle_{R^c_{\pi}} &= |D(0)\rangle_{R^c_{\pi}}, \\
R^c_{\pi}|D(\pi \alpha'/R)\rangle_{R^c_{\pi}} &= -|D(\pi \alpha'/R)\rangle_{R^c_{\pi}},
\end{align}

up to a possible common over all phase. The anomalous minus sign picked up by $|D(\pi \alpha'/R)\rangle_{R^c_{\pi}}$ under $R^c_{\pi}$ shows the vanishing of the partition function is enforced by the symmetry of the Haldane phase.

The discrete torsion phase $\varepsilon (g|h) \in H^2(G, U(1))$, here computed for the case of a non-trivial SPT phase protected by the symmetry group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, is consistent with the known classification $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$. It is also interesting to note that the way we probe the topological properties of the SPT phase here, namely, by twisting one $\mathbb{Z}_2$ (threading flux) and measuring the induced charge of the other $\mathbb{Z}_2$, is in perfect harmony with the Kunneth formula for evaluating the second group cohomology of $G = \mathbb{Z}_2 \times \mathbb{Z}_2$. [69]

6. (1 + 1) d topological superconductors in symmetry class BDI

In this section, we consider topological superconductors in symmetry class BDI, and the $\mathbb{Z}_8$ classification of Fidkowski and Kitaev [37, 70]. Following our general framework, we will use BCFT to detect the $\mathbb{Z}_8$ classification. Our analysis in terms of boundary states gives an alternative perspective of the $\mathbb{Z}_8$ classification of Fidkowski and Kitaev in terms of quantum anomalies of boundary states of CFT.

We emphasize that, in our analysis below, we will use boundary states in free fermion CFTs to detect the $\mathbb{Z}_8$ classification in class BDI, which arises from the reduction of the $\mathbb{Z}$ classification in the presence of interactions. While all calculations will be done here entirely within the context of free fermion manipulations, nevertheless, it should be noted that (i) boundary states are constructed in the many-body Hilbert space (the Fock space). Moreover, (ii) anomalous phases that boundary states may acquire upon the action of symmetry operations are expected to ‘survive’ or to ‘be protected’, even in the presence of interactions (see section 3.6 for related discussion), in analogy to various kinds of quantum anomalies in quantum field theories.

We also note that technically, the following discussion has much resemblance to the analysis of quantum anomalies at the boundary of (2 + 1)-dimensional topological crystalline superconductors [20], for which the classification is $\mathbb{Z}_8$ [71]. In [72], a quantum anomaly of the corresponding (1 + 1)-d edge theory was identified to diagnose the $\mathbb{Z}_8$ classification by using cross-cap states in CFTs. The boundary states discussed in this section of the present paper, when restricted to the zero-mode sector of the closed-string Hilbert space, are identical to those appearing in the cross-cap states that arose in the analysis of the (1 + 1)-d edges, which are obtained by ‘gauging’ a mirror (or ‘reflection’, or ‘parity’) symmetry. In fact, this is consistent with the fact that the classification of non-interacting (2 + 1)-dimensional topological insulators and topological superconductors with mirror symmetry is identical to that of non-interacting (1 + 1)-dimensional topological insulators and superconductors without mirror symmetry [73]. Our analysis presented in the present work, based on a quantum anomaly of the (0 + 1)d boundaries of (1 + 1)-d gapped SPT phases, implies that the classification for (2 + 1)-dimensional topological insulators and superconductors with mirror symmetry is the same as that of (1 + 1)-dimensional topological insulators and superconductors without mirror symmetry.

[20] In the present context, the term ‘topological crystalline insulator or superconductor’ refers to a topological phase that is protected by a ‘mirror’, ‘reflection’, or ‘parity’ symmetry.
mirror symmetry even in the presence of the interactions, where a certain $\mathbb{Z}$ classification, such as that occurring in symmetry class BDI, is reduced to a $\mathbb{Z}_8$ classification. In this section, we will particularly be interested in the BDI class.

Consider the CFT consisting of $N_f$ flavors of non-interacting non-chiral (i.e. right and left moving) real (Majorana) fermions described by the Hamiltonian (53). In addition to fermion number parity conservation, we impose on the system time-reversal symmetry

\[
\hat{T}\psi^R_{\ell}(t,x)\hat{T}^{-1} = \psi^R_{\ell}(t,x),
\]

\[
\hat{T}\psi^L_{\ell}(t,x)\hat{T}^{-1} = \sigma \psi^L_{\ell}(t,x),
\]

\[
\hat{T}^2 = \sigma F, \quad \hat{T}\hat{t}\hat{T}^{-1} = -i,
\]

where $\sigma = \pm 1$, and $a = 1,\ldots,N_f$. The fermion parity operator $(-1)^F$ was defined in equation (55). The case of interest for us is $\sigma = +1$, relevant for symmetry class BDI.

To discuss the action of time-reversal on boundary states of the corresponding free fermion CFT, we will now implement the $\pi/2$ rotation of Euclidean (i.e. imaginary time) spacetime, discussed already in the paragraphs surrounding equations (46) and (65). Since this rotation involves imaginary (Euclidean) time, we first need to reformulate the condition of time reversal invariance, equation (121), as a condition involving imaginary time. This can be understood, e.g. by using the fact that the non-interacting Majorana fermion theory in (53) satisfies the CPT Theorem (since this is a theory of Lorentz invariant Dirac/Majorana fermions).

In the present case of Majorana fermions, charge-conjugation $C$ acts trivially, and therefore the condition of time-reversal invariance in equation (121) is satisfied if and only if the following condition of ‘parity’, or equivalently ‘spatial reflection’ $\mathcal{R}$ symmetry is satisfied

\[
\mathcal{R}\psi^R_{\ell}(t,x)\mathcal{R}^{-1} = \psi^R_{\ell}(t,\ell - x),
\]

\[
\mathcal{R}\psi^L_{\ell}(t,x)\mathcal{R}^{-1} = (-\sigma) \psi^L_{\ell}(t,\ell - x),
\]

\[
\mathcal{R}^2 = (-\sigma)^F, \quad \mathcal{R}\hat{t}\mathcal{R}^{-1} = i \text{ (unitary)},
\]

where we considered the situation where the fermions are defined on a spatial circle of circumference $\ell$ with coordinate $x$. It can be verified\(^\text{21}\) (e.g. by checking that this forbids the same mass terms) that one needs to change the sign $\sigma \rightarrow (-\sigma)$ as indicated, when going from equation (121) to equation (122).

Note that since $\mathcal{R}$ in equation (122) acts only on the spatial coordinate $x$, the same equation holds true when real time $t$ is replace by imaginary (Euclidean) time $\tau$, i.e. $t \rightarrow \tau$, in that equation. The imaginary (Euclidean) time version of time reversal from equation (121) is then the same equation as equation (122), after the rotation by $\pi/2$ of (Euclidean) spacetime, $(x,\tau) = (\tilde{x},\tilde{\tau})$, which was already discussed in the paragraph surrounding equation (46), is implemented. Denoting the Euclidean-time version of time-reversal by $\hat{P}$ (standing for ‘parity’), the imaginary time version of time reversal symmetry reads

\[
\hat{P}\psi^R_{\ell}(\tilde{\tau},\tilde{x})\hat{P}^{-1} = \psi^R_{\ell}(\tilde{\tau},\beta - \tilde{x}),
\]

\[
\hat{P}\psi^L_{\ell}(\tilde{\tau},\tilde{x})\hat{P}^{-1} = (-\sigma) \psi^L_{\ell}(\tilde{\tau},\beta - \tilde{x}),
\]

\[
\hat{P}^2 = (-\sigma)^F, \quad \hat{P}\hat{t}\hat{P}^{-1} = i \text{ (unitary)}.
\]

As a brief check, note that the simple free fermion boundary conditions in equation (62) characterized by numbers $\eta = \pm 1$ are invariant under the time reversal transformation defined in

\(^{21}\) See e.g. [72].
equation (121) when $\sigma = +1$ (the sign relevant for class BDI) as expected. Equivalently, the boundary state $|B(\eta)\rangle$ defined in equation (67), can be written in the form of equation (46) with

$$\hat{\Phi}(\tilde{x}) = \begin{pmatrix} \psi_L(\tilde{x}) \\ \psi_R(\tilde{x}) \end{pmatrix} \quad \text{and} \quad U = i\eta \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}. \quad (124)$$

Following the steps in equation (49) we find that this boundary condition preserves the symmetry $\hat{P}$ (i.e. time reversal), except that we still need to discuss the action of $\hat{P}$ on the boundary state itself, i.e. $\hat{P}|B\rangle$. This will be done in detail below.

We will now implement the analysis of section 3, specifically section 3.5. The discrete symmetry group of the current problem is $\mathbb{Z}_F \times \mathbb{Z}_T$ generated by fermion parity $\hat{g}_f = (-1)^f$ (as in section 4) and time reversal symmetry, for which we use the formulation in terms of $\hat{P}$, as in equation (123). These two symmetry operations commute. Following section 3.5, we choose to twist the boundary state (i.e. we twist in imaginary time $\tau = \tilde{x}$) by the fermion parity operator $\hat{g}_f$ in the same way as was done in section 4.

It follows from equation (123) that the fermion zero modes, equation (70), satisfy

$$\hat{P}_0 \psi^a_0 \hat{P}_0^{-1} = \psi^a_R, \quad \hat{P}_0 \psi^a_R \hat{P}_0^{-1} = -\psi^a_L, \quad (s \in \mathbb{Z} + \frac{1}{2}, \text{ or } s \in \mathbb{Z}). \quad (125)$$

This implies that $\hat{P}$, when acting on the boundary state twisted by fermion number parity, $|B(\eta)\rangle_{\hat{g}_f}$ in equation (71), commutes with the exponential in that equation when $\sigma = +1$ (relevant for class BDI); therefore, we only need to discuss the action of $\hat{P}$ on the ‘zero-mode contribution’ $|B(\eta)\rangle_0$ defined in equation (71). Let us denote by $\hat{P}_0$ the projection of the operator $\hat{P}$ on the zero-mode sector. It can be verified by direct calculation that an explicit expression for $\hat{P}_0$ satisfying

$$\hat{P}_0 \psi^a_0 \hat{P}_0^{-1} = \psi^a_R, \quad \hat{P}_0 \psi^a_R \hat{P}_0^{-1} = -\psi^a_L, \quad (s \in \mathbb{Z} + \frac{1}{2}, \text{ or } s \in \mathbb{Z}). \quad (126)$$

as required by equation (125) is given by

$$\hat{P}_0 = e^{i\delta} \prod_{a=1}^{N_f} \frac{1}{\sqrt{2}} \left( 1 - \psi^a_0 \psi^a_R \right), \quad (127)$$

where $e^{i\delta}$ is a so-far unknown phase factor which will be discussed in more detail shortly. Moreover, one also verifies that

$$\hat{P}_0^2 = e^{2i\delta} (i)^{N_f} \hat{g}_f, \quad \text{and} \quad \hat{g}_f \hat{P} = \hat{P}_0 \hat{g}_f \quad (128)$$

on the zero mode sector.

Let us now calculate the action of $\hat{P}_0$ on the zero-mode contribution $|B(\eta)\rangle_0$ of the boundary state (equation (71)). By using the representation in terms of the $f$-fermions [defined in equation (73)], $\psi_0 \psi_R = i(2f^2 - 1)$, $\hat{P}_0$ can be written as

$$\hat{P}_0 = e^{i\delta} \prod_{a} \frac{1}{\sqrt{2}} \left[ 1 - i(2n_a - 1) \right], \quad (129)$$
where $n_a = f_a^f f_a^{\dagger}$. Then, the action of $\hat{P}_0$ on the zero-mode part of the boundary states is given by

$$
\hat{P}_0 |B, + \rangle_0 = e^{\delta} \prod_a \frac{1}{\sqrt{2}} [1 - i(2n_a - 1)] |0_f\rangle
$$

$$
= e^{\delta} \prod_a \frac{1}{\sqrt{2}} [1 + i] |0_f\rangle = e^{+i\pi N_f} e^{i\delta} |B, + \rangle_0.
$$

$$
\hat{P}_0 |B, - \rangle_0 = e^{\delta} \prod_a \frac{1}{\sqrt{2}} [1 - i] |B, - \rangle_0 = e^{-i\pi N_f} e^{i\delta} |B, - \rangle_0.
$$

(130)

The relative phase between $\hat{P}_0 |B, + \rangle_0$ and $\hat{P}_0 |B, - \rangle_0$, the key object of our discussion in section 3.5, is $e^{+i\pi N_f/2}$, which is independent of the choice of $e^{i\delta}$ (the choice of the action of $\hat{P}_0$ on the reference state $|0_f\rangle$, the Fock vacuum of the $f$-fermions), and vanishes when $N_f = 4 \times$ integer. In other words, one cannot make both boundary states anomaly-free unless $N_f = 4 \times$ integer. One then immediately concludes, following section 3.5, that the classification is at least $\mathbb{Z}_4$.

To give a slightly different perspective, let us now adopt the point of view that the zero-mode parts (see equation (71)) of the (conformal) boundary states correspond to ground states of SPT phases. Then, the expectation value

$$
0 \langle B \pm | \hat{P}_0 |B \pm \rangle_0 = e^{\pm i\pi N_f} e^{i\delta}
$$

implied by equation (130) can be interpreted, in the space-time path-integral picture, as the partition function $Z_{\text{Klein}}$ of the corresponding topological quantum field theory on a space-time which has the topology of the Klein bottle. (More precisely, the corresponding quantum field theory is a $G$-equivariant spin topological quantum field theory. [74–76]) In the corresponding topological quantum field theory, the partition function on the Klein bottle is $Z_{\text{Klein}} = 1$ for the topologically trivial phase, whereas it is $Z_{\text{Klein}} = e^{\pm i\pi N_f/2}$ for the topologically non-trivial phase. (The Klein bottle partition function does not detect the $\mathbb{Z}_8$ classification, but only the subgroup, $\mathbb{Z}_4$.) Thus, the comparison with the topological quantum field theory suggests the choice $e^{i\delta} = e^{\pm i\pi N_f/4}$. Observe also that the ‘unwanted’ phase $e^{2i\delta}(i)^{N_f}$ in equation (128) can be removed by choosing $0 \langle B \pm | \hat{P}_0 |B \pm \rangle_0 = e^{-i\pi N_f/4}$.

(Otherwise, equation (128) would suggest that, within the zero mode sector, the symmetry is realized projectively.)

In short, the $\mathbb{Z}_4$ anomalous phase (130) derived from BCFT is consistent with the corresponding topological quantum field theory. To detect the $\mathbb{Z}_8$ classification, we need to consider the partition function on the real projective plane $\mathbb{R}P^2$. This can be achieved by considering the expectation value of a partial reflection, [74–76] as opposed to the full reflection considered in equation (131). We should note that our general method outlined in section 3.5 was based on the set of anomalous phases $\xi_B(g|h)$ defined in equation (42) where $g$ and $h$ are two commuting group elements. As stated in the paragraph between equations (124) and (125), the relevant symmetry group for symmetry class BDI that we are considering here is $\mathbb{Z}_2^T \times \mathbb{Z}_2^T$, generated by fermion parity $\hat{g}_f = (-1)^F$ and $\hat{P}$ (time-reversal). In order to arrive at the results displayed in equations (130) and (131), that led to the classification which was at least $\mathbb{Z}_4$, we did not make use of the information from the anomalous phases $\xi_B(g|h)$ for all commuting group elements $g$ and $h$, in particular not for all combinations involving the orientation-reversing symmetry $\hat{P}$ (time-reversal symmetry), since we have not computed them. Instead, we gave in the paragraph below equation (131) a slightly different perspective of the effect of the presence of an orientation-reversing symmetry. The reason why we did not compute directly the anomalous phase $\xi_B(g|h)$ for all combinations of group elements $g$ and $h$, in particular
all combinations involving the orientation-reversing symmetry, was purely technical, and we plan on coming back to this direct computation in future work. We expect that this will allow us to reproduce the complete $\mathbb{Z}_8$ classification, within a suitable setup.—At the same time we should also note that, on the other hand, for a SPT phase protected by a *unitary on-site symmetry group* $G$ (i.e. one that does not include orientation-reversing symmetries), the complete set of *anomalous phases* $\varepsilon_B(g|h)$ (‘discrete torsion phases’) from equation (42) is known to completely characterize and classify $(1 + 1)d$ SPT phases protected by $G$.

7. Discussion

In this paper, we have given a description of the entanglement spectrum of SPT phases in $(1 + 1)$ dimensions which are in vicinity of a quantum critical point described by a CFT, in terms of a boundary CFT (BCFT) associated with that describing the quantum critical point. We also introduced a diagnostic tool, the symmetry-enforced vanishing of the twisted partition function, which allows us to identify the presence of a non-trivial cocycle, and of a projective representation of the symmetry group defining the SPT phase in the entanglement spectrum, and to identify the topological class of the SPT phases. From the perspective of CFTs, our formalism allows us to identify SPT phases that can be proximate to a given CFT. Hence, it gives us the structure of the phase diagram (the ‘theory space’) around the CFT. As yet another perspective, our formalism can be thought of as a proper generalization of the Jackiw–Rebbi soliton from non-interacting fermion systems to generic SPT phases.

While we have made a connection between boundary states in CFTs on one hand, and SPT phases on the other, it should be emphasized, again, that the correspondence is not one-to-one: Many different BCFTs can correspond to a given SPT phase. For example, for a Haldane system, one can attach as an ideal lead the $c = 1$ compactified boson, the $SU(2)_2$, or the $SU(3)_1$ CFTs. All these CFTs are proximate to the Haldane phase, in the sense discussed in this paper, i.e. they describe, respectively, three possible (conformal) quantum critical points through which one can exit the Haldane phase into other (typically non-topological) phases. Therefore, while one should be able to use boundary states of these CFTs to diagnose the topological properties of the Haldane phase, when it comes to classifying SPT phases, it is not optimal to use BCFTs, because several different BCFTs (the three mentioned above and also others) can be used to describe the same SPT phase (here the Haldane phase). In this sense, our BCFT approach is complementary to other approaches, such as e.g. the Matrix Product state [MPS] approach. In fact, the complete classification of boundary states in CFTs so far has not been achieved, while (Bosonic) SPT phases in $(1 + 1)d$ are completely classified by the 2nd cohomology group $H^2(G, U(1))$. In other words, boundary states in CFTs or BCFTs seem to have ‘too much information’. (I.e. the set of all boundary states seem to be much bigger than the set of all possible SPT phases in $(1 + 1)d$.)

Another issue which may be related to this is the difference between symmetry-protected degeneracy in the entanglement spectrum and the boundary entropy. The former degeneracy should exist at all length scales, while the latter should emerge only in the long-wave length limit, and looks like a much more non-trivial property than the symmetry-protected degeneracy. Since when describing $(1 + 1)$-dimensional SPTs we are interested in the properties of BCFTs which are in fact independent of the length scale, there may be an efficient way to extract this topological information out of the BCFTs. One can speculate that such information can be extracted by a procedure such as the ‘topological twist’, when applicable. Such a procedure essentially turns (B)CFTs into topological field theories, and hence the resulting (‘topologically twisted’) theory will only contain information that is independent of the length
scale. In other words, such a hypothetical procedure should ‘remove’ the unnecessary information from the CFT, so that only topologically relevant information remains. Along this line of thought, we note that the complete classification of boundary states in $(1 + 1)d$ topological quantum field theories (TQFTs) is actually much better understood than that in BCFTs, and has been studied, e.g. in the work of Moore and Segal [77]. (See appendix B for some details.) In short, we proposed that there is a close connection between SPT phases and boundary states in $(1 + 1)d$ TQFTs—see also the discussion in the last paragraph of section 6.

Finally, we end by mentioning that the appearance of a defect CFT which appears as an interface between two different $(1 + 1)d$ CFT connected via a gapped SPT region provides an interesting generalization of the set up discussed in this paper. However, we leave this topics for future work.

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Note added. After the key results of the work reported here were obtained, a preprint, [78], which discusses the properties of CFTs that may appear as a continuous quantum phase transition between two different SPT phases sharing the same symmetry. Our analysis concerning the entanglement spectrum of SPTs, which is described by BCFTs (as opposed to bulk CFTs), shares some similarities with [78].

Appendix A. Fractional branes and discrete torsions

In this appendix we would like to make a few historical remarks. In the context of D-branes in string theory, it was observed that open string states in the open string channel (i.e. states in BCFTs in the open string channel) may form a projective representation of an orbifold group $G$, as in equation (35). [54–56, 59] (Typically, this statement is phrased in terms of spacetime quantum fields living on D-branes.)

Boundary states that are relevant to SPT phases are those that are invariant (up to an anomalous phase) under the action of the symmetry defining the SPTs. In the terminology of orbifold CFTs and D-branes, they are D-branes that are localized at orbifold fixed points. Such D-branes are called fractional branes. Fractional branes which may exist in a theory with discrete torsion have been also discussed. (Here, the theory means string theory which includes both open and closed string, and describes interactions among them.)

In the context of fractional branes, it was argued that a two-cocycle that appears in the action of orbifold group $G$ in the open string channel is directly related to the discrete torsion that appears in the closed string Hilbert space. As the notation suggests, if $\omega(g|h)$ is the two-cocycle that appears in the open string picture, it was argued that $\varepsilon(g|h)$ in equation (37) is nothing but the discrete torsion that appears in the closed string Hilbert space.
The argument goes as follows. (Here, we follow [79].) We aim to establish the more direct connection between the discrete torsion phase $\varepsilon(g|h)$ and the projective representations that appear as boundary degrees of freedom (Chan–Paton factors in the string theory terminology). Let us consider a vertex operator $V$ in the $g$-twisted sector $H_g$, $V \in H_g$. We consider an amplitude

$$\langle \text{tr} (D(g)\psi) V \rangle$$

(A.1)

where $\psi$ describes the boundary degrees of freedom (the Chan–Paton factor). The expectation value here is taken in the CFT defined on the disc (the disc amplitude). Let us now take another group element $h$ in the symmetry group, which satisfies $[h, g] = e$. We assume the amplitude (A.1) be invariant under the action of $h$. To see the consequence of this invariance, we first note that, under the action of $h$, the vertex operator $V$ and the Chan–Paton wave function transform as

$$h \cdot V = \varepsilon(h|g)V, \quad h \cdot \psi = D(h)\psi D(h^{-1}).$$

(A.2)

Hence, the disc amplitude is transformed to

$$\langle \text{tr} (D(g)D(h)\psi D(h^{-1}))\varepsilon(g|h)V \rangle.$$  

(A.3)

Demanding the invariance of the disc amplitude, we conclude

$$D(h^{-1})D(g)D(h)\varepsilon(g|h) = D(g).$$

(A.4)

This is nothing but equation (37).

On general grounds, this intimate connection between discrete torsion (in the closed string picture) and projective representations (in the open string picture) has a close connection to SPT phases, which are classified by projective representations (the second group cohomology). In our paper, we have identified analogs of discrete torsion phases within BCFTs and in terms of boundary states to make this connection. In string theory, however, it seems uncommon to assign different discrete torsions to different (fractional branes localized at) orbifold fixed points—if one fixes a discrete torsion once for all for the theory, one needs to use the same discrete torsion all times. In the physics of SPT phases, however, we consider different discrete torsions for different boundary states at different orbifolds fixed points.

In identifying $\varepsilon(g|h)$ in equation (37) as a discrete torsion, the symmetry-enforced vanishing of the projective characters (see our discussion around (38)) plays an important role. It was argued that the relation between discrete torsions and two-cocycles can be inferred by factorizing the cylinder amplitude between two fractional branes in the closed string channel. Since the factorization in the closed string channel will be achieved by constructing boundary states for the D-branes with discrete torsion, this consistency check amounts to verifying that these boundary states are well-projected, and to checking that from $H_g$ only states invariant under the $N_g$ projection (equation (27)) contribute to the amplitude. This can be checked by using the symmetry-protected vanishing of the projective characters, $\rho(h) = 0$, when there is a $g$ which commutes with $h$ and $\varepsilon(g|h) \neq 1$.

**Appendix B. Classification of boundary conditions of (1 + 1)d $G$-equivariant open and closed oriented TFT**

In this appendix, we give a very brief review of the classification of (1 + 1)d $G$-equivariant oriented open and closed topological field theories [TFTs] by Moore and Segal [77], including the classification of boundary conditions on the open TFTs. $G$-equivariant refers to the
The fact that there is an action of the symmetry group $G$. For the purpose of comparing with SPT phases protected by the symmetry group $G$ discussed in this paper, we assume that the Hilbert space associated with a closed string, the ‘closed sector Hilbert space’, consists of a unique state (the ‘ground state’ of the closed string). [This corresponds to that assumption that the algebra $H_g = 1$ associated with the untwisted sector ($g = 1$ is the identity) of the closed string is a simple algebra.] Generalizations to symmetry broken phases and symmetry fractionalization [80, 81], or combinations thereof, which turn out to correspond to semisimple algebras, are discussed in [77].

Before moving to the details, we shall note that Moore–Segal’s $G$-equivariant open and closed TFTs apply only to a single closed SPT chain, and to its counterpart with open boundary conditions (an open chain). In other words, they cannot be applied to junction systems consisting of a SPT phase and a trivial phase discussed in the main text.

We will describe the following facts in order:

(i) The quantum state of a closed string with background flux $g \in G$, where $G$ is the symmetry group, is described by an element $|\Psi\rangle_g \in \mathcal{H}_g$ in a one-dimensional Hilbert space $\mathcal{H}_g$. We refer to $|\Psi\rangle_g$ as the ground state with background flux $g$. (Compare figure B1(a).) The Hilbert spaces $\mathcal{H}_g$ where $g \in G$, belong to the ‘closed string sector’.

A multiplication is defined by the following rule

$$\mathcal{H}_g \otimes \mathcal{H}_h \rightarrow \mathcal{H}_{gh}, \quad |\Psi\rangle_g |\Psi\rangle_h = \omega(g|h) |\Psi\rangle_{gh},$$

(B.1)

where $\omega(g|h) \in \mathbb{Z}^2(G, U(1))$ is a 2-group cocycle. (Compare figure B1(b).)

(ii) The action of a group element $g \in G$ on the closed sector is defined as

$$\hat{g} : \mathcal{H}_h \rightarrow \mathcal{H}_{gh^{-1}}, \quad \hat{g} |\Psi\rangle_h = \frac{\omega(g|h)}{\omega(g|gh^{-1}g)} |\Psi\rangle_{gh^{-1}}.$$  

(B.2)

(iii) The Hilbert space $\mathcal{H}_{AA}$ of the open TFT with the same boundary conditions ‘A’ for both ends is a simple algebra, i.e. a matrix algebra. Specifically, one associates with $\mathcal{H}_{AA}$ a single vector space $V_A$ so that $\mathcal{H}_{AA} \cong \text{End}(V_A)$, the vector space of all linear maps from $V_A$ into itself.

Figure B1. (a) The Hilbert space $\mathcal{H}_g$ associated with the SPT phase with $g$-twisted boundary condition. The black dot is a point on $S^1$ at which the $G$-bundle is trivialized. (b) The ‘fusion’ process of two closed strings. (c) $G$-symmetry action. In (b) and (c), the dashed lines with symmetry elements $1, g \in G$ represent holonomies along the lines.
(iv) Specifying a boundary condition ‘A’ on the TFT amounts to specifying a projective representation $D_A$ on the vector space $V_A$, whose cocycle is $\omega^*$, the complex conjugate of the cocycle $\omega$ introduced in item (i) above. One often refers to this as the ‘$\omega^*$-projective representation’ $(V_A, D_A)$, $D_A : G \rightarrow \text{Aut}(V_A)$.

(v) The $G$-equivariant boundary state $|A\rangle_g \in \mathcal{H}_g$ characterizing the boundary condition $A$ is written as

$$|A\rangle_g = \rho_A(g) |\Psi\rangle_g,$$

where $\rho_A(g) = \text{Tr}_{V_A} D_A(g)$ is the character of $\omega^*$-projective representation $D_A$ defined in item (iv) above.

The properties (i) and (ii) listed above are properties of the closed TFT sector. On the other hand, properties (iii), (iv) and (v) are additional properties of open TFTs. (There are important relationships between open and closed TFT. The so-called ‘equivariant Cardy condition’, figure B5, briefly reviewed below, turns out to be essential for describing their relationship.)

In the following two sections we will review in some more detail the properties of closed as well as of open $G$-equivariant TFTs.

### B.1. G-equivariant closed TFT

A $G$-equivariant TFT is a functor $Z$ from the geometric category of manifolds with the structure of background $G$ gauge fields to the algebraic category of complex vector spaces. In $(1+1)$ dimensions, an object of the geometric category is a pointed circle $(S^1, pt, g)$ with twisted boundary condition by $g \in G$ at $pt \in S^1$ as shown in figure B1(a). We have a collection of Hilbert spaces $\mathcal{H}$ consisting Hilbert spaces $\mathcal{H}_g$ of twisted boundary condition by $g$. A morphism is cobordism with a background $G$ gauge field together with a trivialization at points on boundary circles. For example, the fusion process of two closed strings (figure B1(b)) induces $\mathcal{H}$ to have a $G$-graded algebraic structure $\mathcal{H}_g \otimes \mathcal{H}_h \rightarrow \mathcal{H}_{gh}$. The cylinder with the $g$-holonomy (figure B1(c)) induces the $g$-symmetry action $\mathcal{H}, \hat{g} : \mathcal{H}_h \rightarrow \mathcal{H}_{gh^{-1}}$. As mentioned above, we assume the ground state is unique, hence $\mathcal{H}_g \cong \mathbb{C}$ is generated by the ground $|\Psi\rangle_g$ in the presence of the $g$-twisted boundary condition.
One can show that the algebra $\mathcal{H}$ is associative $(\phi_1 \phi_2) \phi_3 = \phi_1 (\phi_2 \phi_3)$, $\phi_i \in \mathcal{H}$, from figure B2, which implies the multiplication of the algebra is characterized by a 2-group cocycle $\omega(g|h) \in Z^2(G, U(1))$ as equation (B.1). The $G$-twisted commutativity represented in figure B3 means

$$|\Psi\rangle_g |\Psi\rangle_h = (\hat{g} |\Psi\rangle_h) |\Psi\rangle_g,$$  

which leads to equation (B.2).

For group elements $g, h \in G$ with $[g, h] = 0$, the factor in the rhs of equation (B.2) is invariant under the 1-group coboundary $\omega(g|h) \mapsto \omega(g|h)e^{i\theta(g)}e^{i\theta(h)}e^{-i\theta(gh)}$. Thus, we can define an invariant determined by the 2nd group cohomology $H^2(G, U(1))$ as
An algebra $R$ is said to be a Frobenius algebra when $R$ is (1) finite dimension, (2) associative, (3) unital (existence of unit $1_R$), and (4) has non-degenerate form $Q(x, y)$ satisfying the Frobenius condition $Q(xy, z) = Q(x, yz)$.

\[ \varepsilon(g|h) = \frac{\omega(g|h)}{\omega(h|g)}, \quad ([g, h] = 0), \]  
\[ (B.6) \]

which is the discrete torsion phase.

**B.2. G-equivariant open and closed TFT**

Next, we extend the closed TFT to include open strings. The new object is the oriented interval $I_{AB} = [0, 1]$ with boundary conditions $A, B$ as shown in figure B4(a). We denote the Hilbert space associated with the interval $I_{AB}$ by $\mathcal{H}_{AB}$. The fusion process (figure B4(b)) induces a multiplication $\mathcal{H}_{AB} \otimes \mathcal{H}_{BC} \to \mathcal{H}_{AC}$. In particular, $\mathcal{H}_{AA}$, the Hilbert space with the same boundary conditions $A$ for both ends, is a Frobenius algebra\(^{22}\). The rectangle with $g$-holonomy (figure B4(c)) induces the $g$-action $\hat{g} : \mathcal{H}_{AB} \to \mathcal{H}_{AB}$.

\(^{22}\) An algebra $R$ is said to be a Frobenius algebra when $R$ is (1) finite dimension, (2) associative, (3) unital (existence of unit $1_R$), and (4) has non-degenerate form $Q(s, y)$ satisfying the Frobenius condition $Q(s, yz) = Q(s, y)$. 

Figure B5. The $G$-equivariant Cardy condition.

Figure B6. The definition of boundary state. (a) A correlation function on upper half plane with boundary condition $A$. (b) A correlation function with insertion of the boundary state $|A\rangle$. 

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In open and closed TFTs, the open sectors are not independent from the closed sectors and vice versa: Open and closed sectors are connected by the closed to open map (figure B4(d)) \( t_g : \mathcal{H}_g \rightarrow \mathcal{H}_{AA} \), and the open to closed map (figure B4(e)) \( \psi^A : \mathcal{H}_{AA} \rightarrow \mathcal{H}_g \). We have several algebraic relations from the open and closed geometric category [77]. Among them, the \( G \)-equivariant Cardy condition provides a strong constraint, which states that the double twist diagram in the open string sector factorizes in the closed sector (figure B5), and is written as

\[
\pi^A_{g,B} = t_{g,B} \circ \psi^A, \quad \pi^A_{g,B}(\psi) = \sum_{\mu} \psi_{\mu} \hat{g}_{\mu}(\psi_{\mu}). \tag{B.7}
\]

Here, \( \{ \psi_{\mu} \} \) is the basis of \( \mathcal{H}_{AB} \) and \( \{ \psi_{\mu}^* \} \) is that basis dual of \( \mathcal{H}_{BA} \) which satisfies \( \text{Tr}_{\mathcal{H}_{BA}}(\psi_{\mu}^* \psi_{\nu}) = \delta_{\mu\nu} \).

The \( G \)-equivariant boundary state \( |A\rangle_g \in \mathcal{H}_g \) in the \( g \)-twisted sector associated with the boundary condition \( A \) is defined by the image of the open to closed map from \( 1_A \) (\( 1_A \) is the ‘disc state’ in the open sector),

\[
|A\rangle_g := \psi^A(1_A). \tag{B.8}
\]

This is because the defining property of the boundary states is the equivalence between correlation functions on the upper half plane with the boundary condition \( A \) (figure B6(a)) and correlation functions in the closed string sector with insertion of the boundary state \( |A\rangle_g \) (figure B6(b)).

We shall make a clarification as to what boundary conditions \( \{A, B, \ldots\} \) are, in the cases where the closed string sector Hilbert space consists of unique ground state.

First, we show (iii) according to [82]. Since \( \mathcal{H}_1 \) is 1-dimensional, the boundary state \( |A\rangle \) of the untwisted sector is proportional to the ground state as \( |A\rangle = \alpha |\Psi\rangle \). Taking the closed to open map, form the Cardy condition (B.7) we have \( \sum_{\mu} \psi_{\mu} \hat{g}_{\mu} = \alpha 1_A \). Here \( \chi = \sum_{\mu} \psi_{\mu} \hat{g}_{\mu} \) is known as the ‘Euler element’ and this equation means \( \chi \) is invertible, which follows that \( \mathcal{H}_{AA} \) is semisimple. Then, we can assume \( \mathcal{H}_{AA} \) is a direct sum of matrix algebras \( \mathcal{H}_{AA} \cong \bigoplus \text{Mat}(\mathbb{C}^{d}) \).

Moreover, we can show \( \mathcal{H}_{AA} \) is simple: Suppose \( \mathcal{H}_{AA} \) consists of two simple algebras. Then, a general element in \( \mathcal{H}_{AA} \) forms \( X_1 \oplus X_2 \in \text{Mat}(\mathbb{C}^{d_1}) \oplus \text{Mat}(\mathbb{C}^{d_2}) \), and the basis of \( \mathcal{H}_{AA} \) is given by

\[
\{( |i\rangle \langle j|, 0), (0, |k\rangle \langle l|) \}_{ij=1,\ldots,d_1, k,l=1,\ldots,d_2} \tag{B.9}
\]

We have

\[
\pi^A_{1_A}(X_1 \oplus X_2) = \text{Tr}(X_1) 1_{d_1 \times d_1} \oplus \text{Tr}(X_2) 1_{d_2 \times d_2} \tag{B.10}
\]

since \( \sum_{ij} |i\rangle \langle j| \otimes \sum_{kl} |k\rangle \langle l| = \text{Tr}(X_1) 1_{d_1 \times d_1} \) and the same for \( X_2 \). Equation (B.10) means \( \pi^A_{1_A}(\mathcal{H}_{AA}) \) is 2-dimensional center of \( \mathcal{H}_{AA} \), which gives rise to a contradiction: The factorization property states

\[
t_{1_A} \circ \psi^A(X_1 \oplus X_2) \sim t_{1_A}(|\Psi\rangle) \sim 1_{d_1 \times d_1} \oplus 1_{d_2 \times d_2} \tag{B.11}
\]

which means \( \pi^A_{1_A}(\mathcal{H}_{AA}) \) is 1-dimensional. Thus, \( \mathcal{H}_{AA} \) should consist of a single matrix algebra.

A proof of (iv) is illustrated as follows [82]: one can show that the symmetry action \( \hat{g} : \mathcal{H}_{AA} \rightarrow \mathcal{H}_{AA} \) is an automorphism \( \hat{g}(\psi_1 \psi_2) = \hat{g}(\psi_1) \hat{g}(\psi_2) \) from the open geometric category. Any automorphisms of the matrix algebra are written as

\[
\hat{g}(X) = D_A(g) X [D_A(g)]^{-1}, \quad X \in \mathcal{H}_{AA}, \tag{B.12}
\]

for some \( \omega_A \)-projective representation \( V_A \). Then, the \( G \)-twisted centrally condition [77] \( t_g(A)(|\phi\rangle \langle \psi|) \) reads \( \{ t_g(A)(|\Psi\rangle_g) D_A(g) \} X = X \{ t_g(A)(|\Psi\rangle_g) \} \)
$D_A(g)$ for all $X \in \mathcal{H}_A$, which implies $D_A(g) \sim [g_A(|\Psi_i\rangle)]^{-1}$ up to a constant factor. Moreover, the closed to open map satisfies $\tau_{g,A}(|\Psi_i\rangle)_h A(|\Psi_i\rangle)_h = \tau_{g,A}(|\Psi_i\rangle)_g A(|\Psi_i\rangle)_g$ from the open and closed geometric category \cite{77}, which implies $g \mapsto [\tau_{g,A}(|\Psi_i\rangle)]^{-1}$ is a $\omega^*$-projective representation, thus $\omega_A \sim \omega^*$.

Finally, we prove the formula of the boundary state (B.3). Since $\mathcal{H}_g$ is 1-dimensional, we can assume $|A\rangle_A = \pi^A(1_A) \sim \beta |\Psi_i\rangle$ with a constant $\beta$. From the Cardy condition, $\tau_{g,A} \circ \pi^A(1_A) = \pi^A(1_A) = \text{tr}(D_A(g))[D_A(g)]^{-1}$, which implies $\beta \sim \text{Tr}(D_A(g))$. The constant factor is fixed by $g \mapsto <A|A\rangle_g = \text{Tr}_{\mathcal{H}_A}(g) = \text{tr}(D_A(g))\text{tr}(D_A(g))$.

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