Ghostpeakons and characteristic curves for the Camassa–Holm, Degasperis–Procesi and Novikov equations

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Abstract

We derive explicit formulas for the characteristic curves associated with the multi-peakon solutions of the Camassa–Holm, Degasperis–Procesi and Novikov equations. Such a curve traces the path of a fluid particle whose instantaneous velocity equals the elevation of the wave at that point (or the square of the elevation, in the Novikov case). The peakons themselves follow characteristic curves, and the remaining characteristic curves can be viewed as paths of “ghostpeakons” with zero amplitude; hence, they can be obtained as solutions of the ODEs governing the dynamics of multipeakon solutions. However, the previously known solution formulas for multipeakons only cover the case when all amplitudes are nonzero, since they are based upon inverse spectral methods unable to detect the ghostpeakons. We show how to overcome this problem by taking a suitable limit in terms of spectral data, in order to force a selected peakon amplitude to become zero. Similar techniques also make it possible to compute arbitrary (non-interlacing) peakon solutions of the two-component Geng–Xue equation, starting from the known formulas for interlacing solutions; a special case is included here for illustration. In addition, we also use direct integration to compute the characteristic curves for the solution of the Degasperis–Procesi equation where a shockpeakon forms at a peakon–antipeakon collision.

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1 Introduction

In this article, we are concerned with characteristic curves associated to solutions of the Camassa–Holm equation [7]

\[ m_t + m_x u + 2mu_x = 0, \quad m = u - u_{xx}, \]  

(1.1)

the Degasperis–Procesi equation [24, 23]

\[ m_t + m_x u + 3mu_x = 0, \quad m = u - u_{xx}, \]  

(1.2)

and the Novikov equation [35, 52]

\[ m_t + m_x u^2 + 3muu_x = 0, \quad m = u - u_{xx}. \]  

(1.3)

These equations are nonlinear PDEs for the function \( u(x, t) \). Here they are formulated using the auxiliary quantity \( m(x, t) = u(x, t) - u_{xx}(x, t) \), but they can easily be written in terms of
\( u \) alone, for example \( u_t - u_{xxx} + 3u u_x - uu_{xxx} - 2u_x u_{xx} = 0 \) instead of (1.1). All three are integrable systems in the sense of having Lax pairs, multisoliton solutions, and infinitely many conservation laws.

**Definition 1.1.** If \( u(x, t) \) is a solution of the Camassa–Holm equation (1.1) or the Degasperis–Procesi equation (1.2), the associated **characteristic curves** (or **characteristics**) \( x = \xi(t) \) are the solutions curves of the ODE

\[
\dot{\xi}(t) = u(\xi(t), t),
\]

where the dot denotes time derivative, \( \dot{\xi} = d\xi/dt \). For Novikov’s equation (1.3), the characteristic curves associated with a solution \( u(x, t) \) are the solution curves of the ODE

\[
\dot{\xi}(t) = u(\xi(t), t)^2,
\]

which has \( u^2 \) instead of \( u \) on the right-hand side.

Our principal new result is the derivation of explicit formulas for the characteristics associated with the so-called **multipeakon** solutions of these equations – weak solutions in the form of a superposition of peaked solitons:

\[
u(x, t) = \sum_{i=1}^{N} m_i(t) e^{-|x-x_i(t)|}.
\]

We achieve this by viewing the characteristics as the trajectories of zero-amplitude “ghost-peakons”, and using a limiting procedure to obtain the formulas for the ghostpeakons from the previously known formulas for ordinary peakons with nonzero amplitude. This approach avoids the difficulties of trying to find the characteristics by direct integration of the ODEs (1.4) and (1.5).

Starting in Section 1.2 we shall explain in detail how this works and why it is of interest, but first we briefly outline the main results.

### 1.1 Overview of the results

For all three PDEs mentioned above, the \( N \)-peakon solutions (1.6) are known completely explicitly; there are formulas, containing only elementary functions, for the time-dependence of the positions \( x_k(t) \) and amplitudes \( m_k(t) \) such that the ansatz (1.6) satisfies the PDE. The exact formulas will be given in Section 2 for the general case, and in Example 1.2 below for a particular case (the Camassa–Holm three-peakon solution). For now, it suffices to know that the positions, which we assume to be numbered in increasing order \( x_1 < x_2 < \cdots < x_N \), are given by formulas of the structure

\[
x_k(t) = A \ln \frac{f_k(t)}{g_k(t)}, \quad 1 \leq k \leq N,
\]

where \( A = 1 \) for the Camassa–Holm and Degasperis–Procesi equations and \( A = 1/2 \) for Novikov’s equation. The details of the functions \( f_k(t) \) and \( g_k(t) \) depend on which equation we are looking at, but in all three cases they are rational expressions in the quantities

\[
b_i(t) = b_i(0) e^{t/\lambda_i}, \quad 1 \leq i \leq N,
\]
Figure 1. Graph of a pure three-peakon solution \( u(x, t) = \sum_{k=1}^{3} m_k(t) e^{-|x-x_k(t)|} \) of the Camassa–Holm equation, “pure” meaning that the amplitudes \( m_k(t) \) are all positive. The solution is given by exact formulas for \( x_k(t) \) and \( m_k(t) \), as explained in Example 1.2. The graph is plotted here in a somewhat novel way (see Section 1.6) using the ghostpeakon formulas derived later in this article, with the mesh consisting of lines \( t = \text{constant} \) together with characteristic curves (lifted to the surface, of course). The dimensions of the box are \( |x| \leq 12, |t| \leq 12 \) and \( -1 \leq u \leq 5/2 \).

where the constant parameters \( \{\lambda_i, b_i(0)\}_{i=1}^{N} \) are uniquely determined by the initial values \( \{x_k(0), m_k(0)\}_{k=1}^{N} \) (or vice versa).

What we are going to prove is that for each \( k = 1, \ldots, N-1 \), the family of characteristic curves \( x = \xi(t) \) filling out the region in the \((x, t)\) plane between the peakon trajectories

\[
x = x_k(t) = A \ln \frac{f_k(t)}{g_k(t)} \quad \text{and} \quad x = x_{k+1}(t) = A \ln \frac{f_{k+1}(t)}{g_{k+1}(t)}
\]

is described by the explicit formula

\[
x = \xi(t) = A \ln \frac{f_k(t) + \theta f_{k+1}(t)}{g_k(t) + \theta g_{k+1}(t)}, \quad (1.8)
\]

where \( \theta \in (0, \infty) \) is a parameter in one-to-one correspondence with the initial value \( \xi(0) \in (x_k(0), x_{k+1}(0)) \). We will also give similar formulas for the families of characteristic curves in the exterior regions \( x < x_1(t) \) and \( x > x_N(t) \).

Example 1.2 (CH three-peakon characteristics). Figure 1 shows the graph of a three-peakon solution \( u(x, t) \) of the Camassa–Holm equation, i.e., a solution of the form (1.6) with \( N = 3 \). It is a “pure peakon solution”, meaning that the amplitudes \( m_1(t), m_2(t) \) and \( m_3(t) \) are all
Figure 2. Spacetime plot of the peakon trajectories \( x = x_k(t) \) for the Camassa–Holm pure three-peakon solution shown in Figure 1. All curves are computed from exact formulas as described in Example 1.2. Since this is a pure peakon solution, \( x_1(t) < x_2(t) < x_3(t) \) holds for all \( t \in \mathbb{R} \).
Figure 3. A selection of characteristic curves $x = \xi(t)$ for the Camassa–Holm pure three-peakon solution shown in Figure 1 and Figure 2. All curves are computed from exact formulas as described in Example 1.2.
which, in turn, depend on the constant parameters $\lambda$ in a number of exponentials. From the solution formulas above, we see that each time-dependent (positive) quantity is given by the explicit formulas (2.3) derived by Beals, Sattinger and Szmigielski [1, 2]. In the case $N = 3$, these formulas reduce to

$$
\begin{align*}
  x_1 &= \ln \frac{\Delta^0}{\Delta^2}, \\
  x_2 &= \ln \frac{\Delta^0}{\Delta_1}, \\
  x_3 &= \ln \frac{\Delta^0}{\Delta_0} = \ln \Delta^0,
\end{align*}$$

(1.9a)

$$
\begin{align*}
  m_1 &= \frac{\Delta^0 \Delta^2}{\Delta^1 \Delta^2}, \\
  m_2 &= \frac{\Delta^0 \Delta^2}{\Delta^1 \Delta_1}, \\
  m_3 &= \frac{\Delta^0 \Delta^2}{\Delta^1 \Delta_0},
\end{align*}$$

(1.9b)

in terms of certain expressions $\Delta^0_k$, $\Delta^1_k$ and $\Delta^2_k$ defined by

$$
\begin{align*}
  \Delta^0 &= 1, \\
  \Delta^1 &= \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3, \\
  \Delta^2 &= (\lambda_1 \lambda_2)^2 b_1 b_2 \\
  &+ (\lambda_1 \lambda_3)^2 b_1 b_3 \\
  &+ (\lambda_2 \lambda_3)^2 b_2 b_3, \\
  \Delta^3 &= (\lambda_1 \lambda_2 \lambda_3)^2 (\lambda_1 - \lambda_2)^2 (\lambda_1 - \lambda_3)^2 (\lambda_2 - \lambda_3)^2 b_1 b_2 b_3,
\end{align*}$$

(1.10)

which, in turn, depend on the constant parameters $\lambda_1$, $\lambda_2$, $\lambda_3$ (real, nonzero, distinct) and on the time-dependent (positive) quantities

$$
b_k = b_k(t) = b_k(0) e^{t/\lambda_k}, \quad 1 \leq k \leq 3.
$$

(1.11)

Untangling these definitions, we see that $x_3(t)$ is given by the explicit formula

$$
x_3(t) = \ln \Delta^0_1(t) \\
= \ln \left(b_1(t) + b_2(t) + b_3(t) \right) \\
= \ln \left(b_1(0) e^{t/\lambda_1} + b_2(0) e^{t/\lambda_2} + b_3(0) e^{t/\lambda_3} \right),
$$

(1.12)

and similarly for the other variables, but with more involved expressions:

$$
x_2(t) = \ln \frac{(\lambda_1 - \lambda_2)^2 b_1(t) b_2(t) + (\lambda_1 - \lambda_3)^2 b_1(t) b_3(t) + (\lambda_2 - \lambda_3)^2 b_2(t) b_3(t)}{\lambda_1^2 b_1(t) + \lambda_2^2 b_2(t) + \lambda_3^2 b_3(t)},
$$

(1.13)

and so on.

The parameter values used in Figure 1 and Figure 2 are (cf. Example 2.4)

$$
\lambda_1 = \frac{2}{5}, \quad \lambda_2 = 1, \quad \lambda_3 = 3, \quad b_1(0) = \frac{25}{13}, \quad b_2(0) = \frac{1}{e}, \quad b_3(0) = \frac{1}{13}.
$$

(1.14)

From the solution formulas above, we see that each $x_k(t)$ is the logarithm of a rational function in a number of exponentials $e^{\alpha t}$ with different growth rates $\alpha$. As $t \to \pm \infty$, when a single
exponential term dominates in each $\Delta_k^\alpha$, the peakons asymptotically travel in straight lines with constant velocities $1/\lambda_1 = 5/2$, $1/\lambda_2 = 1$ and $1/\lambda_3 = 1/3$, and those numbers are also the limiting values of the amplitudes $m_k(t)$ as $t \to \pm \infty$.

The peakon solution formulas and the curves in Figure 2 are well-known, but Figure 3 contains something new, namely a selection of characteristic curves given by the ghostpeakon formulas from our Theorem 3.4 and Corollary 3.5. (Actually, these curves were seen as mesh lines already in Figure 1; more about this in Section 1.6.) The characteristics $x = \xi(t)$ in the leftmost region $x < x_1(t)$ are given by the formula

$$
\xi(t) = \ln \frac{\Delta_1^0 + \theta \Delta_3^0}{\Delta_2^2 + \theta \Delta_2^0},
$$

(1.15a)

those in the region $x_1(t) < x < x_2(t)$ are

$$
\xi(t) = \ln \frac{\Delta_3^0 + \theta \Delta_2^0}{\Delta_2^2 + \theta \Delta_2^1},
$$

(1.15b)

for $x_2(t) < x < x_3(t)$ we have

$$
\xi(t) = \ln \frac{\Delta_2^0 + \theta \Delta_1^0}{\Delta_1^2 + \theta \Delta_1^0},
$$

(1.15c)

and in the rightmost region $x_3(t) < x$ the formula is

$$
\xi(t) = \ln \frac{\Delta_1^0 + \theta \Delta_1^0}{\Delta_0^2 + \theta \Delta_0^2},
$$

(1.15d)

where in each case $\theta$ is a positive parameter. As $\theta$ runs through the values $0 < \theta < \infty$, the corresponding characteristic curves sweep out their respective regions, and in the limit as $\theta \to 0^+$ or $\theta \to \infty$, the formula for $\xi(t)$ reduces to the appropriate $x_k(t)$ (or to $-\infty$ or $+\infty$, in the exterior regions $x < x_1$ and $x > x_3$).

In Figure 3 (and Figure 1) the values of $\theta$ were taken in geometric progressions with ratio $e$, in order for the curves to be approximately evenly spaced.

Further examples, involving also the more complicated case of peakon–antipeakon interactions, will be given in later sections.

1.2 Background: Peakons

Now let us go back and start from the beginning.

The Camassa–Holm equation (1.1) was proposed as a model for shallow water waves by Camassa and Holm [7], with the sought function $u(x, t)$ being a horizontal velocity component. They discovered that the equation admits peaked soliton solutions, *peakons* for short. The $N$-peakon solution has the remarkably simple form

$$
u(x, t) = \sum_{i=1}^{N} m_i(t) e^{-|x-x_i(t)|},
$$

(1.16)
a linear combination of $e^{-|x|}$-shaped waves, whose positions $x_k(t)$ and amplitudes $m_k(t)$ have to satisfy a certain system of $2N$ coupled ODEs in order for $u(x, t)$ to be a weak solution of the PDE (1.1). These ODEs are the canonical equations generated by the Hamiltonian

$$H(x_1, \ldots, x_N, m_1, \ldots, m_N) = \frac{1}{2} \sum_{i,j=1}^{N} m_i m_j e^{-|x_i - x_j|}$$

(1.17)

with the variables $x_k$ as positions and the variables $m_k$ as momenta, namely

$$\dot{x}_k = \frac{\partial H}{\partial m_k} = \sum_{i=1}^{N} m_i e^{-|x_k - x_i|},$$

$$\dot{m}_k = -\frac{\partial H}{\partial x_k} = \sum_{i=1}^{N} m_k m_i \text{sgn}(x_k - x_i) e^{-|x_k - x_i|}.$$  

(1.18)

Here we use the convention $\text{sgn}(0) = 0$, and it is assumed that all $x_k$ are distinct.

The Degasperis–Procesi equation (1.2) and the Novikov equation (1.3) were found not via physical considerations but by purely mathematical means, using integrability tests to single out integrable systems of a form similar to the Camassa–Holm equation [24, 23, 35, 52]. There are many interesting similarities and differences among these three PDEs. They all have peakon solutions of the form (1.16). The Lax pair for (1.1) involves a second-order ODE in the $x$ direction, while those for (1.2) and (1.3) are of order three. And any weak solution of (1.1) or (1.3) is by necessity continuous, whereas (1.2) also admits discontinuous weak solutions, in particular “shockpeakons” [43] (see Example 4.5 and Example 4.6).

For the Degasperis–Procesi equation, the ODEs which must be satisfied in order for (1.16) to be a solution of the PDE are

$$\dot{x}_k = \sum_{i=1}^{N} m_i e^{-|x_k - x_i|},$$

$$\dot{m}_k = 2 \sum_{i=1}^{N} m_k m_i \text{sgn}(x_k - x_i) e^{-|x_k - x_i|},$$

(1.19)

which differ from the Camassa–Holm peakon ODEs (1.18) only by the factor of 2 in the equations for $\dot{m}_k$. And in the Novikov case, the system of peakon ODEs is

$$\dot{x}_k = \left( \sum_{i=1}^{N} m_i e^{-|x_k - x_i|} \right)^2,$$

$$\dot{m}_k = \left( \sum_{i=1}^{N} m_i e^{-|x_k - x_i|} \right) \left( \sum_{i=1}^{N} m_k m_i \text{sgn}(x_k - x_i) e^{-|x_k - x_i|} \right),$$

(1.20)

which looks like (1.18) except that $\dot{x}_k$ and $\dot{m}_k$ are both multiplied by the factor $\sum_{i=1}^{N} m_i e^{-|x_k - x_i|}$. Note that this factor depends on $k$, so it is not just a matter of multiplying the whole vector field by a single function. That is, Novikov peakons are genuinely different, and cannot be obtained from Camassa–Holm peakons just by a time-reparametrization.
The ODEs take a simpler form if we define the shorthand notation
\[ u(x_k) = \sum_{i=1}^{N} m_i e^{-|x_k - x_i|} \]  
(1.21)
and
\[ u_x(x_k) = -\sum_{i=1}^{N} m_i \text{sgn}(x_i - x_k) e^{-|x_k - x_i|}. \]  
(1.22)
Here the left-hand sides are just abbreviations for the functions on phase space given by the right-hand sides, but the notation is supposed to convey the idea that (1.21) is what one gets by substituting \( x = x_k \) into the peakon ansatz (1.6),
\[ u(x_k(t), t) = \sum_{i=1}^{N} m_i(t) e^{-|x_k(t) - x_i(t)|}, \]
and similarly (1.22) is obtained by letting \( x = x_k \) in the derivative \( u_x(x, t) \) of the peakon ansatz, or more precisely (since \( u \) is not differentiable at \( x = x_k \)) in the average of the left and right derivatives,
\[ \frac{1}{2} \left( u_x(x_k(t)^-, t) + u_x(x_k(t)^+, t) \right) = -\sum_{i=1}^{N} m_i(t) \text{sgn}(x_i(t) - x_k(t)) e^{-|x_k(t) - x_i(t)|}. \]
Using this shorthand notation (1.21) and (1.22), the Camassa–Holm peakon ODEs (1.18) take the form
\[ \dot{x}_k = u(x_k), \quad \dot{m}_k = -m_k u_x(x_k), \]  
(1.23)
the Degasperis–Procesi peakons ODEs (1.19) can be written as
\[ \dot{x}_k = u(x_k), \quad \dot{m}_k = -2m_k u_x(x_k), \]  
(1.24)
and the Novikov peakons ODEs (1.20) become
\[ \dot{x}_k = u(x_k)^2, \quad \dot{m}_k = -m_k u(x_k) u_x(x_k). \]  
(1.25)
These complicated systems of ODEs are in fact finite-dimensional integrable systems, whose solutions can be found explicitly in terms of elementary functions. Direct integration is only feasible if \( N = 1 \) or \( N = 2 \), but the solutions for arbitrary \( N \) have been found with the help of inverse spectral techniques (supposing all amplitudes \( m_k \) are nonzero; see Section 1.3). For the Camassa–Holm peakon ODEs (1.23) this was done by Beals, Sattinger and Szmigielski [1, 2]. The general solution of the Degasperis–Procesi peakon ODEs (1.24) is due to Lundmark and Szmigielski [44, 45], and the Novikov peakon ODEs (1.25) were solved by Hone, Lundmark and Szmigielski [34]. We will state the solution formulas in Section 2.
If we assume that the positions \( x_k \) are numbered in increasing order
\[ x_1 < x_2 < \cdots < x_N \]
at time \( t = 0 \) (say), then by continuity this ordering must be preserved by the unique solution of the initial value problem for the ODEs, at least for some time interval \( -T < t < T \). In fact, for all three PDEs it turns out that in the “pure peakon case” where all the amplitudes \( m_k \) are positive, the solution exists for all \( t \in \mathbb{R} \), with the ordering and the positivity preserved. Qualitatively speaking, these pure peakon solutions look rather similar for all three equations; they exhibit the familiar exchange-of-velocity and phase shift phenomena seen in multisoliton solutions of other integrable PDEs (cf. Example 1.2 above). The same holds for pure antipeakon solutions, where all amplitudes are negative.

For mixed “peakon–antipeakon” solutions, where some amplitudes are positive and some are negative, the situation is much more complicated. In general, collisions will occur where \( x_k(t) = x_{k+1}(t) \) for some \( k \), and at the instant of such a collision the amplitudes \( m_k \) and \( m_{k+1} \) blow up, one to \( +\infty \) and the other to \( -\infty \). The question of how the solution \( u(x, t) \) of the PDE may be continued past a collision is very subtle, and our three equations behave differently in this regard, so we will postpone the discussion of this to the separate sections about each equation.

1.3 Ghostpeakons

In the discussion of multipeakon solutions (1.16) above, we assumed that all amplitudes \( m_k \) are nonzero. In fact, if \( m_k(0) = 0 \) then \( m_k(t) \) will be identically zero, since \( m_k \) is a factor in the right-hand side of the equation for \( \dot{m}_k \) in the peakon ODEs (1.23), (1.24), (1.25). And in that case, the corresponding term \( m_k(t) e^{-|x - x_k(t)|} \) in the sum (1.16) would simply be absent, not entering in any way into the solution \( u(x, t) \) of the PDE. So from that point of view it is very natural to ignore the case when some \( m_k \) is zero.

However, in the system of ODEs which determines the dynamics of the peakons, the equation for \( x_k \) is still nontrivial even if \( m_k = 0 \), and it describes the position of a zero-amplitude “ghostpeakon” which is driven by the other peakons but does not have any influence on them.

Definition 1.3. A ghostpeakon is a peakon whose amplitude \( m_k(t) \) is identically zero. More precisely, a solution \( \{x_i(t), m_i(t)\}_{i=1}^N \) of the \( N \)-peakon ODEs (1.23), (1.24) or (1.25) is said to have a ghostpeakon at site \( k \) if \( m_k(t) = 0 \) for all \( t \). The corresponding function \( x_k(t) \) will be referred to as the position of the ghostpeakon, and its trajectory is the curve \( x = x_k(t) \) in the \((x, t)\) plane.

The explicit peakon solution formulas referred to above (to be described in detail in Section 2) actually only cover the case when all amplitudes \( m_k \) are nonzero. The reason for this is that their derivation involves changing to new variables defined by the spectrum of a vibrating string made up of weightless thread connecting \( N \) point masses whose locations and weights are in one-to-one correspondence with the positions and amplitudes of the peakons. (For the Camassa–Holm equation, it is just an ordinary string described by the standard linear second-order wave equation, while the Degasperis–Procesi and Novikov cases involve a “cubic string” described by a third-order equation instead.) A peakon of amplitude zero corresponds to a point mass of weight zero, which leaves no trace whatsoever in the spectrum of the string.

Thus, ghostpeakons are not directly accessible through these inverse spectral methods. Nevertheless, in this paper we will show how one can extract the ghostpeakon solutions from
the solution formulas for ordinary peakons via a relatively simple (but still not entirely trivial) limiting procedure.

Our motivation for studying this problem is at least fourfold, as we will describe next.

1.4 Motivation I: The peakon ODEs as an integrable system

To begin with, if we view the ODEs for peakons as a finite-dimensional integrable system in its own right, we should be able to integrate this system no matter what the initial data are. It is a somewhat peculiar situation to know what the solution formulas are in the case when all amplitudes \( m_k \) are nonzero, but not in the apparently simpler case when some of them vanish identically!

We may mention right away that it is enough if we can find the solution formulas for the “\( N+1 \) case” with \( N \) ordinary peakons and just one ghostpeakon, as we do in Theorem 3.4. Then the general “\( N+K \) case” with \( N \) peakons and \( K \) ghostpeakons is also solved immediately. This is because ghostpeakons do not affect other peakons; in particular, they do not affect other ghostpeakons, so if there are several of them, each one simply follows a one-ghostpeakon trajectory.

1.5 Motivation II: Characteristic curves for multipeakon solutions

As we stated in Definition 1.1, given a solution \( u(x, t) \) of the Camassa–Holm equation (defined in some time interval \( I \) containing \( t = 0 \), say), one defines the associated characteristic curves \( x = \xi(t) \) as the solutions of the ODE

\[
\dot{\xi}(t) = u(\xi(t), t), \quad \xi(0) = \xi_0. \tag{1.26}
\]

One thinks of each characteristic curve as describing the path of a “fluid particle” which starts out at the position \( x = \xi_0 \) at time \( t = 0 \), and at each instant travels with a velocity which equals the wave’s elevation \( u \) at the particle’s current location. The initial value \( \xi_0 \) serves as a label which identifies a particular characteristic curve (or fluid particle), and as \( \xi_0 \) varies over \( \mathbb{R} \) we get a family of characteristic curves filling the strip \( (x, t) \in \mathbb{R} \times I \) (or possibly only a subset thereof, if the ODEs suffer some kind of breakdown).

Now recall from the Camassa–Holm peakon ODEs (1.23) that the equation for the position of the \( k \)th peakon is \( \dot{x}_k = u(x_k) \), shorthand for

\[
\dot{x}_k(t) = u(x_k(t), t). \tag{1.27}
\]

This shows that the trajectories of the peakons, \( x = x_k(t) \), are particular characteristic curves associated to a multipeakon solution

\[
u(x, t) = \sum_{k=1}^{N} m_k(t)e^{-|x-x_k(t)|}.
\]

In particular, if there is a ghostpeakon at position \( k \), then it follows a characteristic curve for the solution \( u(x, t) \), a solution which actually only contains the contributions from the ordinary
(non-ghost) peaks. Turning this around, we can think of the situation as follows: if we start with a solution containing only ordinary peaks and are interested in finding a characteristic curve which is not a peakon trajectory, say \( x = \xi(t) \) with \( \xi(0) = \xi_0 \) distinct from all \( x_k(0) \), we can imagine a ghostpeakon being added to the system with position \( \xi_0 \) at time \( t = 0 \), and obtain the characteristic curve as the trajectory of that ghostpeakon. Thus, finding exact solution formulas for ghostpeakons will tell us explicitly what the characteristic curves \( x = \xi(t) \) are in the case of multipeakon solutions. This is of interest in the study of peakon–antipeakon collisions (see below), and it might also provide an idea of what to expect when studying other kinds of solutions, where such explicit formulas are not available.

Of course, the same things apply to the Degasperis–Procesi and Novikov equations, except that the characteristic curves for the Novikov equation are defined using the square of the elevation instead of just the elevation:

\[
\dot{\xi}(t) = u(\xi(t), t)^2. \tag{1.28}
\]

Characteristic curves play a most important role in the general study of the Camassa–Holm equation and its relatives, as we will try to illustrate by a very brief review. Consider first the initial-value problem for the Camassa–Holm equation on the real line. (Periodic solutions are also much studied, but we will leave them aside here.) There are various works \([20, 40, 53, 50, \text{and many others}]\) which prove existence and uniqueness of a solution in a suitable function space, at least on some time interval \( 0 \leq t < T \). The limitation \( t < T \) is unavoidable in general, since for certain initial data \( u_0(x) = u(x,0) \) it may happen that the solution leaves the nice function space after some finite time \( T \); typically the solution \( u \) remains continuous but its derivative \( u_x \) blows up. This idea of “wave breaking in finite time” is present already in the original Camassa–Holm paper \([7]\), where they consider the slope \( u_x \) at an inflection point of \( u \), to the right of the maximum of \( u \), and sketch a proof showing that this slope must tend to \(-\infty\) in finite time. A fleshed-out argument was given in a later paper \([8]\). Another type of condition implying finite-time blowup involves assuming a sign change from positive to negative in \( m_0(x) = m(x,0) \), where \( m = u - u_{xx} \). In this case, one follows a characteristic curve emanating from a point where \( m_0 \) changes sign, and aims to show that \( u_x \to -\infty \) along that curve after finite time. Many arguments of this kind follow the approach described in detail by Constantin \([19]\). See also (for example) McKean \([48, 49]\), Jiang, Ni and Zhou \([37, 64]\) and Brandolese \([3]\). We also mention some similar studies for the Degasperis–Procesi equation \([63, 41, 42, 25]\) and the Novikov equation \([51, 36, 12]\).

If the solution blows up after finite time, the question arises whether it is possible to continue it past the singularity. It turns out that the answer is yes, but the continuation is not unique, and additional conditions must be imposed in order to single out some particular continuation of interest. Global weak solutions were first studied for the Camassa–Holm equation by Constantin and Escher \([21]\), and have since been investigated in great detail \([22, 60, 61, 5, 6, 29, 30, 32, 33, 27, 4]\). The idea is to remove the singularity by changing to new variables, whose very definition involve characteristic curves. This means that even the “simple” task of verifying that a proposed solution \( u(x, t) \) really satisfies the reformulated equation will require some knowledge of the characteristics associated with that solution. To appreciate how complicated this might be even for \( N \)-peakon solutions with small \( N \), see
in particular the works by Holden, Raynaud and Grunert concerning peakon–antipeakon solutions [29, 32, 27].

In the theory of global weak solution of the Camassa–Holm equation, a fundamental distinction is that between conservative solutions and dissipative solutions. Conservative solutions preserve the $H^1$ energy

$$E(t) = \int_\mathbb{R} (u^2 + u_x^2) \, dx$$

for almost all $t$; for example, at a peakon–antipeakon collision $E(t)$ momentarily drops to a lower value as the peakon and antipeakon merge into a single peakon, but then it immediately returns to its previous value again as the peakon and antipeakon reappear. Dissipative solutions are characterized by the condition that $E(t)$ is nonincreasing, so once the energy drops to a lower value it can’t go back up, which means that the merged peakons must stay together, with some energy having been lost to dissipation at the collision. See Example 3.7 and Example 3.8 for illustrations. As an intermediate case, introduced by Grunert and Holden [27], one may also consider $\alpha$-dissipative solutions, where only the fraction $\alpha \in [0, 1]$ of the energy concentrated at the collision is lost; thus, $\alpha = 0$ is the conservative case and $\alpha = 1$ is the fully dissipative case.

The corresponding theory is rather different for the Degasperis–Procesi equation, which admits much weaker solutions than the Camassa–Holm and Novikov equations; the wave profile $u$ doesn’t even have to be continuous [16, 17, 18, 43].

Concerning Novikov’s equation, Chen, Chen and Liu [11] have recently studied existence and uniqueness of global conservative weak solutions, again with a characteristics-based approach. Interestingly, Novikov’s equation always conserves the $H^1$ energy $E(t)$ automatically, and it is the conservation of the quartic energy-like quantity

$$F(t) = \int_\mathbb{R} (u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4) \, dx$$

which singles out the conservative weak solutions. There are related works by other authors as well [39, 56, 57, 58, 59, 62].

Finally, we mention that characteristic curves also play a prominent role in many numerical schemes for solving Camassa–Holm-type equations [9, 10, 28, 31, 13, 14, 15].

1.6 Motivation III: Simpler plotting of peakon solutions

In addition to the theoretical interest in knowing as much as possible about the characteristic curves, there is also the bonus that it makes it easier to produce good three-dimensional plots of peakon solutions. Ordinarily, if plotting the graph $u = u(x, t)$ directly from the formula (1.16),

$$u(x, t) = \sum_{k=1}^N m_k(t) e^{-|x-x_k(t)|},$$

with $x_k(t)$ and $m_k(t)$ given by the explicit solution formulas, there is firstly the nuisance that one might need to manually provide the software with hints about where the function is not differentiable, in order to prevent the peaks from being smoothed out or otherwise inaccurately represented. More seriously, when plotting peakon–antipeakon solutions there are problems with cancellation in the summation (1.16) and with $|u_x|$ being very large near collisions. If we instead use the explicit formulas for the characteristic
Figure 4. Graph of a conservative peakon–antipeakon solution \( u(x,t) = m_1(t) e^{-|x-x_1(t)|} + m_2(t) e^{-|x-x_2(t)|} \) of the Camassa–Holm equation, plotted as explained in Section 1.6. Thus, the mesh consists of lines \( t = \) constant, which are lifted to the surface so that they illustrate the peakon wave profile \( u(x) = \sum m_k e^{-|x-x_k|} \) at a given time, together with characteristic curves (also lifted to the surface) given by the ghostpeakon formulas from Corollary 3.5. The parameters in the solution formulas are given by (2.9) with \( c_1 = 2 \) and \( c_2 = 1 \), so that the collision takes place at \((x,t) = (0,0)\) and the asymptotic velocity is 2 for the peakon and \(-1\) for the antipeakon. The wave profile at the instant of collision consists of just a single peakon: \( u(x,0) = e^{-|x|} \). The dimensions of the box are \(|x| \leq 8, |t| \leq 7\) and \(-1 \leq u \leq 2\).
1.7 Motivation IV: Handling arbitrary peakon configurations in the two-component Geng–Xue equation

The lessons learned from ghostpeakons are extremely useful in the study of ordinary (non-ghost) peakon solutions of the Geng–Xue equation [26], a coupled two-component generalization of the Novikov equation:

\[
\begin{align*}
    m_t + (m_x u + 3m u_x) v &= 0, & m &= u - u_{xx}, \\
    n_t + (n_x v + 3n v_x) u &= 0, & n &= v - v_{xx}.
\end{align*}
\]  

(1.29)

In this system, a peakon in one component \(u(x, t)\) is not allowed to occupy the same position as a peakon in the other component \(v(x, t)\), and therefore there are many inequivalent peakon configurations, depending on the order in which the peakons in \(u\) and \(v\) occur relative to each other.

The solution formulas for the interlacing configuration, where the peakons alternate (one peakon in \(u\), then one in \(v\), then one in \(u\) again, then one in \(v\), and so on) have been derived by inverse spectral methods [46, 47]. However, it is not obvious how to apply these techniques in non-interlacing cases, since then the two Lax pairs for the Geng–Xue equation do not seem to provide sufficiently many constants of motion to make it possible to integrate the peakon ODEs (see Example 6.1). And regardless of whether it is possible or not, the procedure is very intricate, so one would like to avoid having to do it all over again for each and every possible peakon configuration.

Instead, our idea is to derive the solution formulas for an arbitrary configuration by starting from an interlacing solution (with a larger number of peakons) and driving selected amplitudes to zero by taking suitable limits, thus turning some of the peakons into ghostpeakons, in such a way that the remaining ordinary peakons occur in the desired configuration. The details are quite technical and will be described in a separate article, but as a preview we show in Section 6 what the results look like for a case with relatively few peakons (two in \(u\), followed by two in \(v\)).

1.8 Outline of the article

The structure of our paper is straightforward. Section 2 recalls the explicit formulas for the general solution of the \(N\)-peakon ODEs (1.18), (1.19) and (1.20) when all the amplitudes \(m_k\) are nonzero. This includes explaining all the notation needed for the rest of the article. The ghostpeakon formulas for Camassa–Holm peakons with arbitrary \(N\) are then derived and exemplified in Section 3, and the corresponding results for Degasperis–Procesi and Novikov ghostpeakons are given in Section 4 and Section 5, respectively. In addition, Section 4 also contains a couple of examples where we compute the characteristics for the Degasperis–Procesi one-shockpeakon solution by direct integration. Finally, as a sample of the ghostpeakon-inspired general results about non-interlacing Geng–Xue peakons to be published elsewhere, Section 6 gives the explicit formulas for the \(2 + 2\) “maximally non-interlacing” peakon solution.
2 Review: Formulas for $N$-peakon solutions with nonzero amplitudes

In this section we collect the known formulas for the $N$-peakon solutions of the Camassa–Holm, Degasperis–Procesi and Novikov equations, where it is understood that all peakons have nonzero amplitude $m_k(t)$. The notation defined here will also be used to state our new formulas for ghostpeakon solutions in later sections, and the $N$-peakon solution formulas will be needed for our proofs.

2.1 Camassa–Holm peakons

First we formulate the explicit formulas for Camassa–Holm $N$-peakon solutions. The solutions for $N = 1$ and $N = 2$ were computed already in the original Camassa–Holm paper [7]. For $N \geq 3$ direct integration seems to be virtually impossible, but the solution for arbitrary $N$ was found with the help of inverse spectral methods by Beals, Sattinger and Szmigielski [1, 2]. Note that they use a different normalization of the Camassa–Holm equation, so their formulas differ by various factors of 2 from the ones that we quote here, and they also use the opposite sign convention for the parameters $\lambda_k$.

First some notation. For any integers $a$ and $k$, and for a fixed $N \geq 1$, let

$$
\Delta^a_k = \Delta^a_k(N) = \begin{cases}
\Delta(\lambda_i^a, \ldots, \lambda_i^a) \dfrac{\lambda^a_i \cdot \prod_{r=1}^{k} b_i^a}{\prod_{r=1}^{k} \lambda_i^a} \quad & \text{if } 1 \leq k \leq N, \\
1 & \text{if } k = 0, \\
0 & \text{otherwise},
\end{cases}
$$

(2.1)

where $\Delta$ on the right-hand side denotes the Vandermonde determinant:

$$
\Delta(x_1, x_2, \ldots, x_k) = \prod_{1 \leq i < j \leq k} (x_j - x_i).
$$

The superscript $a$ in $\Delta^a_k$ is just a label, whereas in $\lambda_i^a$ it denotes the $a$th power of the constant $\lambda_i$.

In terms of these quantities $\Delta^a_k$, the Beals–Sattinger–Szmigielski formulas for the general solution $(x_k(t), m_k(t))_{k=1}^N$ of the Camassa–Holm $N$-peakon ODEs (1.18), with the constraint that $m_k \neq 0$ for all $k$, are

$$
x_{N+1-k}(t) = \ln \dfrac{\Delta^0_k}{\Delta^2_{k-1}}, \quad m_{N+1-k}(t) = \dfrac{\Delta^0_k \Delta^2_{k-1}}{\Delta^1_k \Delta^1_{k-1}}, \quad k = 1, \ldots, N,
$$

(2.3)

where $\Delta^a_k$ depends on $t$ via

$$
b_k = b_k(t) = b_k(0) e^{t/\lambda_k}.
$$

(2.4)

We will often let this time dependence be understood, and write just $\Delta^a_k$ and $b_k$ instead of $\Delta^a_k(t)$ and $b_k(t)$.

The parameters $\lambda_k$ are the eigenvalues of a certain symmetric $N \times N$-matrix, hence they are real, and they are also known to be nonzero and distinct. The number of positive (negative)
eigenvalues $\lambda_k$ equals the number of positive (negative) amplitudes $m_k$. The parameters $b_k(0)$ are always positive; they appear as the residues of the so-called modified Weyl function [2].

**Remark 2.1.** The set of parameters

\[
\{\lambda_k, b_k(0)\}_{k=1}^N,
\]

which is referred to as the spectral data (eigenvalues $\lambda_k$ and residues $b_k$), is uniquely determined by the set of initial data

\[
\{x_i(0), m_i(0)\}_{i=1}^N
\]

up to ordering. The solution formulas (2.3) are given by symmetric functions which are invariant under relabeling $(\lambda_k, b_k) \rightarrow (\lambda_{\sigma(k)}, b_{\sigma(k)})$ for any permutation $\sigma \in S_N$. So we may prescribe an ordering if we like, for example

\[
\lambda_1 < \cdots < \lambda_N.
\]

Since the eigenvalues $\lambda_k$ are determined by a polynomial equation of degree $N$, there is no explicit formula for this correspondence in terms of radicals (except for small $N$), but the inverse map is explicitly provided by (2.3) (with $t = 0$).

**Example 2.2** (CH one-peakon solution). For $N = 1$, the solution formulas (2.3) reduce to

\[
x_1 = \ln \frac{\Delta_1^0}{\Delta_0^1}, \quad m_1 = \frac{\Delta_1^0 \Delta_2^1}{\Delta_1^1 \Delta_2^0} = \frac{b_1 \cdot 1}{\lambda_1 b_1 \cdot 1}.
\]

Taking the time dependence $b_1 = b_1(t) = b_1(0) e^{t/\lambda_1}$ into account, we see that the one-peakon solution $u = m_1 e^{-|x-x_1|}$ is just a travelling wave with constant velocity $\dot{x}_1 = 1/\lambda_1$ and constant amplitude $m_1 = 1/\lambda_1 \neq 0$:

\[
x_1(t) = t \frac{\lambda_1}{\lambda_1} + \ln b_1(0), \quad m_1(t) = \frac{1}{\lambda_1}.
\]

(This of course also follows immediately from direct integration of the Camassa–Holm peakon ODEs (1.18), which for $N = 1$ are just $\dot{x}_1 = m_1$, $\dot{m}_1 = 0$.)

**Example 2.3** (CH two-peakon solution). Letting $N = 2$ in (2.3) we get the two-peakon solution, $u = m_1 e^{-|x-x_1|} + m_2 e^{-|x-x_2|}$, with

\[
x_1 = \ln \frac{\Delta_1^0}{\Delta_2^0}, \quad x_2 = \ln \frac{\Delta_2^0}{\Delta_0^0}, \quad m_1 = \frac{\Delta_1^0 \Delta_2^1}{\Delta_1^1 \Delta_2^0}, \quad m_2 = \frac{\Delta_0^0 \Delta_2^1}{\Delta_0^1 \Delta_2^0},
\]

where

\[
\Delta_0^a = 1, \quad \Delta_1^a = \lambda_1^a b_1 + \lambda_2^a b_2, \quad \Delta_2^a = (\lambda_1 - \lambda_2)^2 \lambda_1^a \lambda_2^a b_1 b_2.
\]
Written out explicitly, the formulas are

\[
x_1(t) = \ln \left( \frac{(\lambda_1 - \lambda_2)^2 b_1 b_2}{\lambda_1 b_1 + \lambda_2 b_2} \right), \quad m_1(t) = \frac{\lambda_1^2 b_1 + \lambda_2^2 b_2}{\lambda_1 \lambda_2 (\lambda_1 b_1 + \lambda_2 b_2)}, \quad (2.6)
\]

\[
x_2(t) = \ln (b_1 + b_2), \quad m_2(t) = \frac{b_1 + b_2}{\lambda_1 b_1 + \lambda_2 b_2},
\]

with the time dependence \( b_k = b_k(t) = b_k(0) e^{t/\lambda_k} \).

The Camassa–Holm equation is invariant with respect to translations \( x \rightarrow x - x_0, t \rightarrow t - t_0 \), and we can use this freedom to reduce the number of parameters by two in the solution formulas. It turns out the eigenvalues \( \lambda_k \) are unaffected by such translations, while the residues \( b_k \) are the only essential parameters, and we can make \( b_1(0) \) and \( b_2(0) \) take any positive values by a suitable translation. A particularly useful choice in the pure peakon case, say with \( 0 < \lambda_1 < \lambda_2 \), is to take

\[
c_1 = \frac{1}{\lambda_1} > c_2 = \frac{1}{\lambda_2} > 0, \quad b_1(0) = \frac{c_1}{c_1 - c_2}, \quad b_2(0) = \frac{c_2}{c_1 - c_2}.
\]

Then the two-peakon solution takes the following form, which is symmetric with respect to the reversal \( (x, t) \rightarrow (-x, -t) \) since \( x_1(t) = -x_2(-t) \) and \( m_1(t) = m_2(-t) \):

\[
x_1(t) = -\ln \left( \frac{c_1 e^{-c_1 t} + c_2 e^{-c_2 t}}{c_1 - c_2} \right), \quad m_1(t) = \frac{c_1 e^{-c_1 t} + c_2 e^{-c_2 t}}{e^{-c_1 t} + e^{-c_2 t}},
\]

\[
x_2(t) = \ln \left( \frac{c_1 e^{c_1 t} + c_2 e^{c_2 t}}{c_1 - c_2} \right), \quad m_2(t) = \frac{c_1 e^{c_1 t} + c_2 e^{c_2 t}}{e^{c_1 t} + e^{c_2 t}}. \quad (2.8)
\]

For the peakon–antipeakon case, say with \( \lambda_1 > 0 > \lambda_2 \), we can take

\[
c_1 = \frac{1}{\lambda_1} > 0, \quad c_2 = \frac{-1}{\lambda_2} > 0, \quad b_1(0) = \frac{c_1}{c_1 + c_2}, \quad b_2(0) = \frac{c_2}{c_1 + c_2}
\]

\[ (2.9) \]

to get

\[
x_1(t) = -\ln \left( \frac{c_1 e^{-c_1 t} + c_2 e^{c_2 t}}{c_1 + c_2} \right), \quad m_1(t) = \frac{c_1 e^{-c_1 t} + c_2 e^{c_2 t}}{e^{-c_1 t} - e^{c_2 t}},
\]

\[
x_2(t) = \ln \left( \frac{c_1 e^{c_1 t} + c_2 e^{-c_2 t}}{c_1 + c_2} \right), \quad m_2(t) = \frac{c_1 e^{c_1 t} + c_2 e^{-c_2 t}}{e^{c_1 t} - e^{-c_2 t}}. \quad (2.10)
\]

This will make the peakon–antipeakon collision take place at the origin \((x, t) = (0, 0)\). Even though \( m_1(t) \) and \( m_2(t) \) are undefined at \( t = 0 \), the function \( u(x, t) \) given for \( t \neq 0 \) by \( u(x, t) = m_1 e^{-|x-x_1|} + m_2 e^{-|x-x_2|} \) together with (2.10), can be extended continuously by setting

\[
u(x, 0) = (c_1 - c_2) e^{-|x|}. \quad (2.11)
\]

This provides the \textit{conservative} continuation past the collision (see Section 1.5 and Figure 4). If we instead let \( u(x, t) = (c_1 - c_2) e^{-|x-(c_1-c_2)t|} \) for \( t \geq 0 \), then we get the \textit{dissipative} solution; if \( c_1 \neq c_2 \), the peakon and the antipeakon merge into a single peakon or antipeakon which continues on its own after the collision, whereas if \( c_1 = c_2 \) they annihilate completely, so that \( u(x, t) = 0 \) for all \( t \geq 0 \). The \textit{a-dissipative} solution \( [27] \) with \( 0 < \alpha < 1 \) is obtained by defining
\[ u(x, t) \] for \( t > 0 \) using (2.10) with \( c_1 \) and \( c_2 \) replaced with new constants \( d_1 \) and \( d_2 \) which are determined from the equations

\[
\begin{align*}
d_1 - d_2 &= c_1 - c_2, \\
d_1^2 + d_2^2 &= c_1^2 + c_2^2 - 2c_1c_2\alpha, \quad (2.12)
\end{align*}
\]

reflecting the conservation of momentum and the loss of a fraction \( \alpha \) of the energy concentrated at the collision. (For the Camassa–Holm equation, the momentum \( \int_R u \, dx = \sum_{i=1}^N m_i \) is always conserved.)

**Example 2.4 (CH three-peakon solution).** Letting \( N = 3 \) in (2.3) gives the three-peakon solution formulas, already written out in Example 1.2. In this case, for given eigenvalues \( \lambda_1, \lambda_2, \lambda_3 \), the translations \( x \to x - x_0, t \to t - t_0 \) can be used to make two out of the three parameters \( b_1(0), b_2(0), b_3(0) \) take any values that we like, but the third one remains as an essential parameter. That is, unlike the two-peakon case, where (up to translation) there is a unique solution for each pair of eigenvalues, there is a one-parameter family of inequivalent three-peakon solutions for each triple of eigenvalues. By taking

\[
\lambda_k = \frac{1}{c_k} \quad \text{with} \quad c_1 > c_2 > c_3, \quad c_1 \neq 0, \quad c_2 \neq 0, \quad c_3 \neq 0,
\]

\[
\begin{align*}
b_1(0) &= \frac{c_1^2}{(c_1 - c_2)(c_1 - c_3)}, \\
b_2(0) &= \frac{c_2^2 e^K}{(c_1 - c_2)(c_2 - c_3)}, \\
b_3(0) &= \frac{c_3^2}{(c_1 - c_3)(c_2 - c_3)}, \quad (2.13)
\end{align*}
\]

we obtain the three-peakon solution in a form which depends on the asymptotic velocities \( c_k \) and one more essential parameter \( K \in \mathbb{R} \), and which has the property that the asymptotes for the curves \( x = x_1(t) \) and \( x = x_3(t) \) lie symmetrically with respect to the origin, whereas the asymptotes for the curve \( x = x_2(t) \) are symmetrically placed with respect to the point \( (x, t) = (K, 0) \). Changing the sign of \( K \) corresponds to the reversal \( (x, t) \to (\pm x, \mp t) \), so \( K = 0 \) gives a three-peakon solution which is symmetric with respect to this reversal. The parameter values (1.14) in Example 1.2 were obtained by taking \( c_1 = 5/2, c_2 = 1, c_3 = 1/3 \) and \( K = -1 \) in (2.13).

### 2.2 Degasperis–Procesi peakons

Next, we present the \( N \)-peakon solution formulas for the Degasperis–Procesi equation. Readers who are only interested in the Camassa–Holm case can proceed directly to Section 3 about Camassa–Holm ghostpeakons.

For \( N = 1 \) and \( N = 2 \), the solution of the Degasperis–Procesi \( N \)-peakon ODEs (1.19) was found using direct integration by Degasperis, Holm and Hone [23], and the general solution for arbitrary \( N \) was derived by Lundmark and Szmigielski [45] using inverse spectral methods. To avoid certain complications, we consider first the pure peakon case where all amplitudes \( m_k \)
are positive. Then the solution is given in terms of spectral data \((\lambda_k, b_k)_{k=1}^N\) where the eigenvalues \(\lambda_k\) are positive and distinct, and the residues \(b_k\) are positive and have the time dependence \(b_k(t) = b_k(0) e^{t/\lambda_k}\), just like for Camassa–Holm peakons. The solution formulas are

\[
\begin{align*}
x_{N+1-k}(t) &= \ln \frac{U_k^0}{U_{k-1}^1}, \\
m_{N+1-k}(t) &= \frac{(U_k^0 U_{k-1}^1)^2}{W_k W_{k-1}}, \tag{2.14}
\end{align*}
\]

for \(k = 1, \ldots, N\), where

\[
U_k^a = \begin{cases}
\sum_{1 \leq i_1 < \cdots < i_k \leq N} \left( \prod_{r=1}^k \lambda_{i_r}^a b_{i_r} \right) \frac{\Delta(\lambda_{i_1}, \ldots, \lambda_{i_k})^2}{\Gamma(\lambda_{i_1}, \ldots, \lambda_{i_k})}, & 1 \leq k \leq N, \\
1, & k = 0, \\
0, & \text{otherwise},
\end{cases} \tag{2.15}
\]

and

\[
W_k = \begin{vmatrix}
U_k^0 & U_{k-1}^1 \\
U_{k+1}^0 & U_k^1
\end{vmatrix}, \tag{2.16}
\]

with \(\Gamma\) similar to the Vandermonde determinant \(\Delta\) (see (2.2)) but with plus instead of minus:

\[
\Gamma(x_1, x_2, \ldots, x_k) = \prod_{1 \leq i < j \leq k} (x_i + x_j). \tag{2.17}
\]

The superscript \(a\) in \(U_k^a\) is just a label, whereas in \(\lambda_i^a\) it denotes the \(a\)th power of the constant \(\lambda_i\).

**Example 2.5** (DP two-peakon solution). For convenience, let us write

\[
U_k = U_k^0, \quad V_k = U_k^1, \tag{2.18}
\]

which is the original notation used by Lundmark and Szmigielski [45]. Then the Degasperis–Procesi two-peakon solution is

\[
\begin{align*}
x_1(t) &= \ln \frac{U_2}{V_1} = \ln \frac{(\lambda_1 - \lambda_2)^2 b_1 b_2}{\lambda_1 b_1 + \lambda_2 b_2}, \\
x_2(t) &= \ln \frac{U_1}{V_0} = \ln (b_1 + b_2), \\
m_1(t) &= \frac{(U_2 V_1)^2}{W_2 W_1} = \frac{(\lambda_1 b_1 + \lambda_2 b_2)^2}{\lambda_1 \lambda_2 \left( \lambda_1 b_1^2 + \lambda_2 b_2^2 + \frac{4 \lambda_1 \lambda_2}{\lambda_1 + \lambda_2} b_1 b_2 \right)}, \tag{2.19} \\
m_2(t) &= \frac{(U_1 V_0)^2}{W_1 W_0} = \frac{(b_1 + b_2)^2}{\lambda_1 b_1^2 + \lambda_2 b_2^2 + \frac{4 \lambda_1 \lambda_2}{\lambda_1 + \lambda_2} b_1 b_2}.
\end{align*}
\]

where \(b_k = b_k(t) = b_k(0) e^{t/\lambda_k}\).
Just as for the Camassa–Holm equation (Example 2.3), the eigenvalues are the only essential parameters in the two-peakon solution; changing \( b_1(0) \) and \( b_2(0) \) only amounts to a translation in the \((x, t)\) plane. With
\[
c_1 = \frac{1}{\lambda_1} > c_2 = \frac{1}{\lambda_2} > 0, \quad b_1(0) = \frac{\sqrt{c_1(c_1 + c_2)}}{c_1 - c_2}, \quad b_2(0) = \frac{\sqrt{c_2(c_1 + c_2)}}{c_1 - c_2},
\]
the pure two-peakon solution takes the symmetric form
\[
x_2(t) = -x_1(-t) = \frac{1}{2} \ln \left( \frac{c_1 + c_2}{(c_1 - c_2)^2} + \ln \left( \sqrt{c_1} e^{c_1 t} + \sqrt{c_2} e^{c_2 t} \right) \right),
\]
\[
m_2(t) = m_1(-t) = \frac{\left( \sqrt{c_1} e^{c_1 t} + \sqrt{c_2} e^{c_2 t} \right)^2}{e^{2c_1 t} + e^{2c_2 t} + \frac{4}{c_1 + c_2} e^{(c_1 + c_2) t}}.
\]

**Example 2.6 (DP three-peakon solution).** Similarly, for \( N = 3 \) the solution becomes
\[
x_1(t) = \ln \frac{U_3}{V_2}, \quad m_1(t) = \frac{(U_3 V_2)^2}{W_3 W_2} = \frac{(V_2)^2}{\lambda_1 \lambda_2 \lambda_3 W_2},
\]
\[
x_2(t) = \ln \frac{U_2}{V_1}, \quad m_2(t) = \frac{(U_2 V_1)^2}{W_2 W_1},
\]
\[
x_3(t) = \ln \frac{U_1}{V_0} = \ln U_1, \quad m_3(t) = \frac{(U_1 V_0)^2}{W_1 W_0} = \frac{(U_1)^2}{W_1},
\]
where
\[
U_{-1} = V_{-1} = 0, \quad U_0 = V_0 = 1, \quad U_1 = b_1 + b_2 + b_3, \quad V_1 = \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3,
\]
\[
U_2 = \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1 + \lambda_2} b_1 b_2 + \frac{(\lambda_1 - \lambda_3)^2}{\lambda_1 + \lambda_3} b_1 b_3 + \frac{(\lambda_2 - \lambda_3)^2}{\lambda_2 + \lambda_3} b_2 b_3,
\]
\[
V_2 = \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1 + \lambda_2} \lambda_1 \lambda_2 b_1 b_2 + \frac{(\lambda_1 - \lambda_3)^2}{\lambda_1 + \lambda_3} \lambda_1 \lambda_3 b_1 b_3 + \frac{(\lambda_2 - \lambda_3)^2}{\lambda_2 + \lambda_3} \lambda_2 \lambda_3 b_2 b_3,
\]
\[
U_3 = \frac{(\lambda_1 - \lambda_2)^2 (\lambda_1 - \lambda_3)^2 (\lambda_2 - \lambda_3)^2}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)} b_1 b_2 b_3,
\]
\[
V_3 = \frac{(\lambda_1 - \lambda_2)^2 (\lambda_1 - \lambda_3)^2 (\lambda_2 - \lambda_3)^2}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)} \lambda_1 \lambda_2 \lambda_3 b_1 b_2 b_3,
\]
\[
U_4 = V_4 = 0,
\]
and consequently
\[
\begin{align*}
W_0 &= 1, \\
W_1 &= U_1 V_1 - U_2 V_0 \\
&= \lambda_1 b_1^2 + \lambda_2 b_2^2 + \lambda_3 b_3^2 + \frac{4\lambda_1\lambda_2}{\lambda_1 + \lambda_2} b_1 b_2 + \frac{4\lambda_1\lambda_3}{\lambda_1 + \lambda_3} b_1 b_3 + \frac{4\lambda_2\lambda_3}{\lambda_2 + \lambda_3} b_2 b_3, \\
W_2 &= U_2 V_2 - U_3 V_1 \\
&= \frac{(\lambda_1 - \lambda_2)^4}{(\lambda_1 + \lambda_2)^2}\lambda_1\lambda_2 (b_1 b_2)^2 + \frac{(\lambda_1 - \lambda_3)^4}{(\lambda_1 + \lambda_3)^2}\lambda_1\lambda_3 (b_1 b_3)^2 + \frac{(\lambda_2 - \lambda_3)^4}{(\lambda_2 + \lambda_3)^2}\lambda_2\lambda_3 (b_2 b_3)^2 \\
&\quad + \frac{4\lambda_1\lambda_2\lambda_3 b_1 b_2 b_3}{(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)} \left( (\lambda_1 - \lambda_2)^2 (\lambda_1 - \lambda_3)^2 b_1 + (\lambda_2 - \lambda_1)^2 (\lambda_2 - \lambda_3)^2 b_2 + (\lambda_3 - \lambda_1)^2 (\lambda_3 - \lambda_2)^2 b_3 \right), \\
W_3 &= U_3 V_3 = \lambda_1\lambda_2\lambda_3 (U_3)^2.
\end{align*}
\]

A symmetric way of writing the Degasperis–Procesi pure three-peakon solution, analogous to what we saw in Example 2.4 for the Camassa–Holm equation, is obtained by taking

\[
\begin{align*}
\lambda_k &= \frac{1}{c_k} \quad \text{with} \quad c_1 > c_2 > c_3 > 0, \\
b_1(0) &= \frac{c_1 \sqrt{(c_1 + c_2)(c_1 + c_3)}}{(c_1 - c_2)(c_1 - c_3)}, \\
b_2(0) &= \frac{c_2 \sqrt{(c_1 + c_2)(c_2 + c_3)}}{(c_1 - c_2)(c_2 - c_3)} e^K, \\
b_3(0) &= \frac{c_3 \sqrt{(c_1 + c_3)(c_2 + c_3)}}{(c_1 - c_3)(c_2 - c_3)}.
\end{align*}
\]

(2.23)

In this form, the solution depends on the asymptotic velocities \(c_1, c_2, c_3\), together with one more essential parameter \(K \in \mathbb{R}\), where \(K = 0\) gives a solution symmetric under the reversal \((x, t) \rightarrow (-x, -t)\).

**Remark 2.7.** The solution formulas (2.14) also work for pure antipeakon solutions, with all amplitudes negative, the only difference being that all \(\lambda_k\) are negative in this case. The situation for mixed peakon–antipeakon solutions is considerably more complicated. To begin with, the peakon solution formulas only provide the solution up until the first peakon–antipeakon collision, since at that time the PDE solution \(u(x, t)\) develops a jump discontinuity, so to continue past the collision one has to go outside the peakon world and consider *shockpeakons* [43]. (Cf. Example 4.5 and Example 4.6 below.) Secondly, the eigenvalues \(\lambda_k\) (which in the Degasperis–Procesi case are eigenvalues of a non-symmetric matrix) need not be real and simple anymore. If the eigenvalues are complex and simple, and \(\lambda_i + \lambda_j \neq 0\) for all \(i\) and \(j\), then the formulas work without modification; the eigenvalues occur in complex-conjugate pairs, and whenever \(\lambda_j = \lambda_i\), then also \(b_j = \overline{b_i}\), which will make all \(U_k^a\) real-valued, and the formulas provide a solution of the peakon ODEs in any time interval where the positions \(x_k(t)\) given by the formulas satisfy the ordering assumption \(x_1(t) < \cdots < x_N(t)\). If there are eigenvalues of multiplicity
greater than one, then the formulas must be modified, taking into account that the partial fraction decomposition of the Weyl function will involve coefficients $b_k^{(i)}$ whose time dependence is given by a polynomial in $t$ times the exponential $e^{t/\lambda_i}$; see Szmigielski and Zhou [54, 55]. There is also the possibility of resonant cases, where $\lambda_i + \lambda_j = 0$ for one or more pairs $(i, j)$. Such cases can be handled using a limiting procedure, but as far as we are aware the resulting formulas have not been published, with the exception of the symmetric two-peakon solution with $\lambda_1 + \lambda_2 = 0$, which is easily computed by direct integration (see Example 4.5 below).

### 2.3 Novikov peakons

As far as pure peakon solutions are concerned, Novikov’s equation is fairly similar to the Degasperis–Procesi equation. The $N$-peakon solutions are governed by the ODEs (1.20), and the general solution for positive amplitudes $m_k$ was derived by Hone, Lundmark and Szmigielski [34]. It is once again given in terms of spectral data $(\lambda_k, b_k)^N_{k=1}$ where the eigenvalues $\lambda_k$ are positive and distinct, and the residues $b_k$ are positive and have time dependence $b_k(t) = b_k(0) e^{t/\lambda_k}$.

Recall the notation $U_k^{(i)}$ and $W_k$ from (2.15) and (2.16), and also let

$$Z_k = \begin{vmatrix} U_k^{-1} & U_k^{(i)} \\ U_k^{-1} & U_k \end{vmatrix} = \begin{vmatrix} T_k & U_{k-1} \\ T_{k+1} & U_k \end{vmatrix},$$

where $T_k = U_k^{-1}$ and $U_k = U_k^{(i)}$ is the notation used by Hone, Lundmark and Szmigielski [34]. In terms of these quantities, the solution formulas are

$$x_{N+1-k}(t) = \frac{1}{2} \ln \frac{Z_k}{W_{k-1}}, \quad m_{N+1-k}(t) = \frac{\sqrt{Z_k W_{k-1}}}{U_k U_{k-1}},$$

for $k = 1, \ldots, N$. The fact that $W_k$ and $Z_k$ are positive follows in the pure peakon case from an explicit combinatorial formula due to Lundmark and Szmigielski which expresses them as sums of positive quantities [45, Lemma 2.20]. A different argument, which also works in the mixed peakon–antipeakon case, is given by Kardell and Lundmark [38].

**Example 2.8** (Novikov two-peakon solution). The Novikov two-peakon solution is

$$x_1(t) = \frac{1}{2} \ln \frac{Z_2}{W_1} = \frac{1}{2} \ln \frac{(\lambda_1 - \lambda_2)^4 \lambda_1 \lambda_2}{(\lambda_1 + \lambda_2)^2 \lambda_1 \lambda_2} \frac{b_1 b_2}{b_1 b_2},$$

$$x_2(t) = \frac{1}{2} \ln \frac{Z_1}{W_0} = \frac{1}{2} \ln \left( \frac{b_1^2}{\lambda_1} + \frac{b_2^2}{\lambda_2} + \frac{4}{\lambda_1 + \lambda_2} b_1 b_2 \right),$$

$$m_1(t) = \frac{\sqrt{Z_2 W_1}}{U_2 U_1} = \sqrt{\frac{(\lambda_1 - \lambda_2)^4 \lambda_1 \lambda_2}{(\lambda_1 + \lambda_2)^2 \lambda_1 \lambda_2} \left( \lambda_1 b_1^2 + \lambda_2 b_2^2 + \frac{4 \lambda_1 \lambda_2}{\lambda_1 + \lambda_2} b_1 b_2 \right)} \left( \lambda_1 b_1^2 + \lambda_2 b_2^2 + \frac{4 \lambda_1 \lambda_2}{\lambda_1 + \lambda_2} b_1 b_2 \right)^{1/2},$$

$$m_2(t) = \frac{\sqrt{Z_1 W_0}}{U_1 U_0} = \frac{\sqrt{\frac{(\lambda_1 - \lambda_2)^4 \lambda_1 \lambda_2}{(\lambda_1 + \lambda_2)^2 \lambda_1 \lambda_2} \left( \lambda_1 b_1^2 + \lambda_2 b_2^2 + \frac{4 \lambda_1 \lambda_2}{\lambda_1 + \lambda_2} b_1 b_2 \right)}}{b_1 + b_2},$$

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where the expression for \( m_1 \) can be simplified to

\[
m_1(t) = \left( \frac{\lambda_1 b_1^2 + \lambda_2 b_2^2 + \frac{4\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} b_1 b_2}{\sqrt{\lambda_1 \lambda_2} (b_1 + b_2)} \right)^{1/2}
\]

in the pure peakon case, since then all spectral data are positive.

Pure antipeakon solutions are obtained from pure peakon solutions simply by keeping all \( x_k(t) \) and changing the signs of all \( m_k(t) \), which in terms of spectral data is accomplished by keeping all \( \lambda_k \) and changing the signs of all \( b_k(t) \). Note that both peakons and antipeakons move to the right, since \( \dot{x}_k = u(x_k)^2 \geq 0 \).

Mixed peakon–antipeakon solutions have been studied in detail by Kardell and Lundmark [38], and it turns out that the behaviour of the Novikov equation differs from that of the Camassa–Holm and Degasperis–Procesi equations. The eigenvalues \( \lambda_k \) may be complex, but as long as they are simple and have positive real part (which is the generic case), the solutions will still be described by the same formulas (2.25) as in the pure peakon case. Despite everything moving to the right, there will be peakon–antipeakon collisions where a faster soliton catches up with a slower one. At the collision, the corresponding amplitudes blow up to \( \pm \infty \), but the wave profile \( u(x, t) \), which is defined by the formulas for \( x_k(t) \) and \( m_k(t) \) for all \( t \) except the instants of collision (which are isolated), extends to a continuous function defined for all \( t \in \mathbb{R} \), providing a global conservative weak solution; cf. the discussion at the end of Section 1.5, and see Example 5.5 for an illustration. For some initial conditions, the eigenvalues have multiplicity greater than one, but the solution formulas in those non-generic cases are also known (although we will not state them here, since that would require quite a lot of additional notation). For complex eigenvalues, the peakon–antipeakon solutions exhibit periodic or quasi-periodic behaviour, with peakons colliding, separating, and colliding again, infinitely many times. The eigenvalues always lie in the right half of the complex plane, \( \text{Re} \lambda_k \geq 0 \). However, for \( N \geq 3 \) there may be conjugate pairs lying on the imaginary axis, leading to resonances where some \( \lambda_i + \lambda_j = 0 \); that non-generic case is also covered by modified solution formulas which are known (but not described here).

This concludes our review of the known solution formulas for peakons (with nonzero amplitude), and we now turn to our new results about ghostpeakons and characteristic curves.

### 3 Camassa–Holm ghostpeakons

In this section, we will state and prove the explicit formulas for Camassa–Holm multipeakon solutions where one or several amplitudes \( m_k(t) \) are identically zero. Then the corresponding positions \( x_k(t) \) of those ghostpeakons give characteristic curves for the multipeakon solution \( u(x, t) = \sum m_i e^{-|x-x_i|} \) formed by the nonzero-amplitude peakons. As explained already in Section 1.4, it is enough to find the solution for the case with \( N \) ordinary peakons and just one ghostpeakon.

**Remark 3.1.** Before we go into the details, let us say a few words about other possible approaches, to give a better understanding of the philosophy behind our proof. The obvious first
thing to try is direct integration: if we seek the \((N+1)\)-peakon solution where one particular amplitude \(m_k\) is zero and the other amplitudes are nonzero, then the \(N\)-peakon solution formulas (2.3) give us the functions \((x_i(t), m_i(t))_{i \neq k}\) explicitly, and we can plug those expressions into \(\dot{x}_k = u(x_k)\) to get a single remaining equation for the ghostpeakon’s position \(x_k(t)\). This works well for \(k = 1\) or \(k = N+1\), since then we get an easily solved ODE where the variables \(x_k\) and \(t\) are separated, but this approach runs into trouble for \(2 \leq k \leq N\). Let us illustrate this for the case with \(N = 2\) peakons and one ghostpeakon.

**Example 3.2** (CH two-peakon characteristics by direct integration). From (1.18), the Camassa–Holm three-peakon ODEs are

\[
\begin{align*}
\dot{x}_1 &= m_1 + m_2 E_{12} + m_3 E_{13}, \\
\dot{x}_2 &= m_1 E_{12} + m_2 + m_3 E_{23}, \\
\dot{x}_3 &= m_1 E_{13} + m_2 E_{23} + m_3, \\
\dot{m}_1 &= m_1(-m_2 E_{12} - m_3 E_{13}), \\
\dot{m}_2 &= m_2(m_1 E_{12} - m_3 E_{23}), \\
\dot{m}_3 &= m_3(m_1 E_{13} + m_2 E_{23}),
\end{align*}
\]

(3.1)

where we assume \(x_1 < x_2 < x_3\) as always, and use the abbreviations \(E_{12} = e^{x_1-x_2}, E_{13} = e^{x_1-x_3}\) and \(E_{23} = e^{x_2-x_3}\).

If we seek the solution where \(m_3\) is identically zero, the equations for \(x_1(t), x_2(t), m_1(t), m_2(t)\) reduce to the Camassa–Holm two-peakon ODEs

\[
\begin{align*}
\dot{x}_1 &= m_1 + m_2 E_{12}, \\
\dot{x}_2 &= m_1 E_{12} + m_2, \\
\dot{m}_1 &= -m_1 m_2 E_{12}, \\
\dot{m}_2 &= m_2 m_1 E_{12},
\end{align*}
\]

(3.2)

so we obtain those functions immediately from the two-peakon solution formulas (2.6):

\[
\begin{align*}
x_1(t) &= \ln \left( \frac{(1 - \lambda_2^2) b_1 b_2}{\lambda_1^2 b_1 + \lambda_2^2 b_2} \right), \\
\dot{x}_2(t) &= \ln(b_1 + b_2), \\
m_1(t) &= \frac{\lambda_2^2 b_1 + \lambda_2^2 b_2}{\lambda_1 \lambda_2 (\lambda_1 b_1 + \lambda_2 b_2)}, \\
m_2(t) &= \frac{b_1 b_2}{\lambda_1 b_1 + \lambda_2 b_2},
\end{align*}
\]

where \(b_k = b_k(t) = b_k(0) e^{t/\lambda_k}\). The remaining equation for \(x_3(t)\) becomes

\[
\dot{x}_3 = m_1 E_{13} + m_2 E_{23} = (m_1 e^{x_1} + m_2 e^{x_2}) e^{-x_3} = \left( \frac{b_1}{\lambda_1} + \frac{b_2}{\lambda_2} \right) e^{-x_3},
\]

so that

\[
\frac{d}{dt} e^{x_3(t)} = e^{x_3(t)} \dot{x}_3(t) = \frac{b_1(t)}{\lambda_1} + \frac{b_2(t)}{\lambda_2} = \frac{b_1(0)}{\lambda_1} e^{t/\lambda_1} + \frac{b_2(0)}{\lambda_2} e^{t/\lambda_2},
\]

which is easily integrated to give the desired ghostpeakon formula:

\[
x_3(t) = \ln \left( b_1(t) + b_2(t) + \theta \right) = \ln \left( b_1(0) e^{t/\lambda_1} + b_2(0) e^{t/\lambda_2} + \theta \right).
\]

(3.3)
Comparison with the formula \( x_2(t) = \ln(b_1(t) + b_2(t)) \) shows that the constant of integration \( \theta \) must be positive, in order for \( x_2 < x_3 \) to hold. When \( \theta \) runs through all positive values, equation (3.4) thus gives the family of characteristic curves \( x = \xi(t) \) for the two-peakon solution in the outer right region \( x > x_2(t) \) in the \((x, t)\)-plane (cf. the discussion in Section 1.5):

\[
\xi(t) = \ln(\Delta_2^0 + \theta) = \ln(b_1 + b_2 + \theta), \quad \theta > 0. \tag{3.4}
\]

The case with \( m_1 \) identically zero is just as easy: the two-peakon solution formulas (2.6) give \((x_2(t), x_3(t), m_2(t), m_3(t))\), and \( x_1(t) \) can then be found by direct integration of the separable ODE

\[
\dot{x}_1 = m_2 E_{12} + m_3 E_{13} = e^{x_1}(m_2 e^{-x_2} + m_3 e^{-x_3}).
\]

Renaming \((x_1, x_2, x_3, m_2, m_3)\) to \((\xi, x_1, x_2, m_1, m_2)\) then gives the characteristic curves \( x = \xi(t) \) for the two-peakon solution in the outer left region \( x < x_1(t) \); in our notation they take the form

\[
\xi(t) = \ln(\Delta_3^0 + \theta \Delta_1^0) = \ln\left(\frac{0 + \theta (\lambda_1 - \lambda_2)^2 b_1 b_2}{(\lambda_1 - \lambda_2)^2 \lambda_1^2 \lambda_2^2 b_1 b_2 + \theta (\lambda_1^2 b_1^2 + \lambda_2^2 b_2^2)}\right), \quad \theta > 0. \tag{3.5}
\]

(Compare this with the formula \( x_1(t) = \ln\left(\frac{(\lambda_1 - \lambda_2)^2 b_1 b_2}{\lambda_1^2 b_1^2 + \lambda_2^2 b_2^2}\right)\), which is obtained in the limit as \( \theta \to \infty \).

To highlight the similarity to (3.3) or (3.4), equation (3.5) can be written as

\[
\xi(t) = -\ln(\hat{b}_1(0) e^{-t/\lambda_1} + \hat{b}_2(0) e^{-t/\lambda_2} + \hat{\theta}), \quad \hat{\theta} > 0,
\]

if we let

\[
\hat{b}_1(0) = \frac{\lambda_2^2}{(\lambda_1 - \lambda_2)^2 b_1(0)}, \quad \hat{b}_2(0) = \frac{\lambda_1^2}{(\lambda_1 - \lambda_2)^2 b_2(0)}, \quad \hat{\theta} = \frac{\lambda_1^2 \lambda_2^2}{\theta}.
\]

So the “outer” cases were easy, but in the “middle” case with \( m_2 \) identically zero, we are left with the equation

\[
\dot{x}_2 = m_1 E_{12} + m_3 E_{23} = m_1(t) e^{x_1(t)} e^{-x_2} + m_3(t) e^{-x_3(t)} e^{x_2}, \tag{3.6}
\]

where the functions \( x_1(t), x_3(t), m_1(t), m_3(t) \) are explicitly given by the two-peakon solution formulas (2.6) after relabeling, with \( x_3 \) and \( m_3 \) taking the places of \( x_2 \) and \( m_2 \). Since we have \( e^{-x_2} \) in one of the terms on the right-hand side and \( e^{x_2} \) in the other term, the variables \( x_2 \) and \( t \) do not separate, and it seems like we are stuck. In spite of this, Grunert and Holden [27, Sect. 4] managed to find a way of integrating equation (3.6) via a sequence of rather ingenious substitutions. In our notation, the resulting formula for the characteristics in the region between the two peaks is

\[
\xi(t) = \ln(\Delta_2^0 + \theta \Delta_1^0) = \ln\left(\frac{(\lambda_1 - \lambda_2)^2 b_1 b_2 + \theta (b_1 + b_2)}{\lambda_1^2 b_1 + \lambda_2^2 b_2 + \theta}\right), \quad \theta > 0.
\]

(Note that this reduces to \( x_1(t) = \ln\left(\frac{(\lambda_1 - \lambda_2)^2 b_1 b_2}{\lambda_1^2 b_1^2 + \lambda_2^2 b_2^2}\right) \) and \( x_2(t) = \ln(b_1 + b_2) \) as \( \theta \to 0 \) or \( \theta \to \infty \), respectively.) However, there is no obvious way of generalizing their approach to an arbitrary number of peakons, so already the characteristic curves for the three-peakon solution that we showed in Example 1.2 seem to be out of reach for the method of direct integration.
Remark 3.3. Another idea is to fix all initial data \( x_i(0) \) and \( m_i(0) \) except for one amplitude \( m_k(0) = \varepsilon \neq 0 \) that we allow to vary. Then all the spectral data \( \{\lambda_i, b_i(0)\}_{i=1}^{N+1} \) in the \((N+1)\)-peakon solution formulas will be functions of \( \varepsilon \). Taking the limit as \( \varepsilon \to 0 \), these solution formulas must reduce to the usual (already known) \( N \)-peakon solution formulas for the functions \( x_i(t) \) and \( m_i(t) \) with \( i \neq k \), plus the trivial formula \( m_k(t) = 0 \) and one additional (previously unknown) formula which gives the ghostpeakon’s position \( x_k(t) \). However, this is easily doable only when \( N = 1 \), the trivial case with one peakon and one ghostpeakon, since in this case the eigenvalues \( \lambda_1 \) and \( \lambda_2 \) are the roots of a quadratic equation with coefficients depending on the initial data (including \( \varepsilon \)), so we can write \( \lambda_1(\varepsilon) \) and \( \lambda_2(\varepsilon) \) explicitly with formulas involving nothing worse than square roots. For \( N = 2 \) and \( N = 3 \), we get a cubic and a quartic equation for the eigenvalues, so the formulas become horrible, and for \( N \geq 4 \) we get a quintic, so there are no formulas at all in terms of radicals. In principle, one does not need the exact expressions for \( \lambda_k \) and \( b_k(0) \) as functions of \( \varepsilon \), only their asymptotic behaviour as \( \varepsilon \to 0 \), and this should be possible to figure out using perturbation techniques, but that doesn’t seem to be worth the trouble, given that we will present a much simpler way below.

Our idea is instead to reparametrize the spectral data in the \((N+1)\)-peakon solution formulas; keep the parameters \( \{\lambda_i, b_i(0)\}_{i=1}^N \) but replace the parameters \( \lambda_{N+1} \) and \( b_{N+1}(0) \) by two new parameters \( \varepsilon \neq 0 \) and \( \theta > 0 \) defined by

\[
\lambda_{N+1} = \frac{1}{\varepsilon}, \quad b_{N+1}(0) = \varepsilon^{2p} \theta
\]

for some suitably chosen integer \( p \). If we let \( \varepsilon \) vary and keep all other parameters fixed, all the functions \( x_i(t; \varepsilon) \) and \( m_i(t; \varepsilon) \) will depend on \( \varepsilon \), and it turns out that each one of them will have a removable singularity at \( \varepsilon = 0 \). Moreover, the value at \( \varepsilon = 0 \) will be \( m_k(t; 0) = 0 \) for exactly one index \( k \) (which depends on the chosen exponent \( p \)), and the limiting formula for \( x_k(t; 0) \) then gives us the position of a ghostpeakon at that site. The other \( x_i(t; 0) \) and \( m_i(t; 0) \) reduce to the \( N \)-peakon solution with parameters \( \{\lambda_i, b_i(0)\}_{i=1}^N \). We emphasize here that the limiting procedure is performed in the space of spectral variables.

Here, finally, is our theorem about the solution with \( N \) ordinary peakons together with one ghostpeakon at position \( N + 1 - p \) for some \( 0 \leq p \leq N \). Recall the notation \( \Delta^0_k = \Delta^0_k(N) \) from (2.1).

Theorem 3.4. Fix some \( p \) with \( 0 \leq p \leq N \). The solution of the Camassa–Holm \((N+1)\)-peakon ODEs (1.18) with \( x_1 < \cdots < x_{N+1} \) and all amplitudes \( m_k(t) \) nonzero except for \( m_{N+1-p}(t) = 0 \) is as follows: the position of the ghostpeakon is given by

\[
x_{N+1-p}(t) = \ln \frac{\Delta^0_{p+1} + \theta \Delta^0_p}{\Delta^0_p + \theta \Delta^0_{p-1}}, \quad 0 < \theta < \infty,
\]

while the other peakons are given by the \( N \)-peakon solution formulas (2.3) up to relabeling.
Here \( \theta \in (0, \infty) \) is a constant in one-to-one correspondence with the ghostpeakon’s initial position \( x_{N+1-p}(0) \), while the quantities \( \{ \lambda_k, b_k \}_{k=1}^{N} \) appearing in the expressions \( \Delta_k^a = \Delta_k^a(N) \) have the usual time dependence \( \lambda_k = \text{constant}, \ b_k(t) = b_k(0) e^{t/\lambda_k} \).

**Proof.** The formulas for the \((N+1)\)-peakon solution are given by (2.3) with \( N+1 \) instead of \( N \):

\[
\begin{align*}
x_{N+1-k}(t) &= \begin{cases} 
\ln \frac{\Delta^0_{k+1}}{\Delta_k^1}, & 0 \leq k < p, \\
\frac{\Delta^0_k}{\Delta^2_{k-1}}, & p < k \leq N,
\end{cases} \\
m_{N+1-k}(t) &= \begin{cases} 
\frac{\Delta^0_{k+1} \Delta^2_k}{\Delta^1_{k+1} \Delta^1_k}, & 0 \leq k < p, \\
\frac{\Delta^0_k \Delta^2_{k-1}}{\Delta^1_k \Delta^1_{k-1}}, & p < k \leq N.
\end{cases}
\end{align*}
\]

(3.8)

Thus, \( \lambda_{N+1} = 1/\varepsilon \) and

\[
b_{N+1} = b_{N+1}(t) = b_{N+1}(0) e^{t/\lambda_{N+1}} = \varepsilon^{2p} \theta e^{t} = \varepsilon^{2p} \Theta,
\]

where \( \Theta = \Theta(t) = \theta e^{t} \).

When we perform these substitutions in the definition (2.1) of \( \tilde{\Delta}_k^a = \Delta_k^a(N+1) \), split the sum...
according to whether \( i_k \leq N \) or \( i_k = N + 1 \), and write \( \Delta^q_k = \Delta^q_k(N) \), we obtain (for \( 1 \leq k \leq N + 1 \))

\[
\Delta^q_k = \sum_{1 \leq i_1 < \cdots < i_k \leq N+1} \left( \prod_{r=1}^{k} \lambda_i^a b_{i_r} \right) \Delta(\lambda_{i_1}, \ldots, \lambda_{i_k})^2
\]

\[
= \sum_{1 \leq i_1 < \cdots < i_k \leq N} \left( \prod_{r=1}^{k} \lambda_i^a b_{i_r} \right) \Delta(\lambda_{i_1}, \ldots, \lambda_{i_k})^2
\]

\[
+ \sum_{1 \leq i_1 < \cdots < i_{k-1} \leq N} \left( \prod_{r=1}^{k-1} \lambda_i^a b_{i_r} \right) \Delta(\lambda_{i_1}, \ldots, \lambda_{i_{k-1}})^2 \lambda^a_{N+1} b_{N+1} \prod_{s=1}^{k-1} (\lambda_{i_s} - \lambda_{N+1})^2
\]

\[
= \Delta^q_k + \sum_{1 \leq i_1 < \cdots < i_{k-1} \leq N} \left( \prod_{r=1}^{k-1} \lambda_i^a b_{i_r} \right) \Delta(\lambda_{i_1}, \ldots, \lambda_{i_{k-1}})^2 \lambda^a_{N+1} \prod_{s=1}^{k-1} (\epsilon \lambda_{i_s} - 1)^2
\]

\[
= \Delta^q_k + \sum_{1 \leq i_1 < \cdots < i_{k-1} \leq N} \left( \prod_{r=1}^{k-1} \lambda_i^a b_{i_r} \right) \Delta(\lambda_{i_1}, \ldots, \lambda_{i_{k-1}})^2 \Theta \frac{\epsilon^{2(p-a)} (1 + \Theta(\epsilon))}{(\epsilon^2)^{k-1}}
\]

Consequently, expressed in terms of the new parameters (and in particular considered as being functions of \( \epsilon \)) the positions and amplitudes in the \( N + 1 \)-peakon solution take the form

\[
x_{N+1-k}(t; \epsilon) = \ln \frac{\Delta^0_{k+1}}{\Delta^2_k} = \ln \frac{\Delta^0_{k+1} + \Delta^2_k \Theta \epsilon^{2(p-k)} (1 + \Theta(\epsilon))}{\Delta^2_k + \Delta^2_{k-1} \Theta \epsilon^{2(p-k)} (1 + \Theta(\epsilon))}
\]

\[
= \begin{cases} 
\ln \frac{\Delta^0_{k+1} + \Theta(\epsilon)}{\Delta^2_k + \Theta(\epsilon)}, & k < p, \\
\ln \frac{\Delta^0_{p+1} + \Delta^2_p \Theta + \Theta(\epsilon)}{\Delta^2_p + \Delta^2_{p-1} \Theta + \Theta(\epsilon)}, & k = p, \\
\ln \frac{\Delta^0_k \Theta + \Theta(\epsilon)}{\Delta^2_{k-1} \Theta + \Theta(\epsilon)}, & k > p,
\end{cases}
\]

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and
\[ m_{N+1-k}(t; \varepsilon) = \frac{\Delta_{k+1}^0 \Delta_k^2}{\Delta_{k+1}^1 \Delta_k^1} \]
\[ = \frac{\Delta_{k+1}^0 + \Delta_k^0 \Theta \varepsilon^{2(p-k)} \{1 + \Theta (\varepsilon)\}}{\Delta_{k+1}^1 + \Delta_k^1 \Theta \varepsilon^{2(p-k)-1} \{1 + \Theta (\varepsilon)\}} \times \]
\[ \frac{\Delta_k^2 + \Delta_{k-1}^2 \Theta \varepsilon^{2(p-k)} \{1 + \Theta (\varepsilon)\}}{\Delta_k^1 + \Delta_{k-1}^1 \Theta \varepsilon^{2(p-k)-1} \{1 + \Theta (\varepsilon)\}} \]
\[ = \begin{cases} \\
\frac{\Delta_{k+1}^0 \Delta_k^2 + \Theta (\varepsilon)}{\Delta_{k+1}^1 \Delta_k^1 + \Theta (\varepsilon)}, & k < p, \\
\varepsilon \frac{\Delta_{p+1}^0 + \Delta_p^0 \Theta + \Theta (\varepsilon)}{(\Delta_{p+1}^0 \Theta + \Theta (\varepsilon))(\Delta_p^0 + \Theta (\varepsilon))}, & k = p, \\
\frac{\Theta^2 \Delta_{k-1}^0 \Delta_k^2 + \Theta (\varepsilon)}{\Theta^2 \Delta_{k-1}^1 \Delta_k^1 + \Theta (\varepsilon)}, & k > p.
\end{cases} \]

In the limit \( \varepsilon \to 0 \), these expressions reduce to those given in (3.7) and (3.8). Note in particular that \( m_{N+1-p}(t; 0) = 0 \), i.e., we really kill the peakon at that position, as claimed.

The expressions which remain after taking the limit \( \varepsilon \to 0 \) still satisfy the peakon ODEs. Indeed, this is a purely differential-algebraic question. Because of the ordering property \( x_1 < \cdots < x_{N+1} \), we can remove the absolute value signs in the ODEs, and then the satisfaction of the ODEs by the proposed solution formulas boils down to certain meromorphic functions of \( t \) and of the parameters being identically zero. After the reparametrization, these meromorphic functions will have removable singularities at \( \varepsilon = 0 \), so if they are zero for all \( \varepsilon \neq 0 \), they will remain zero also for \( \varepsilon = 0 \).

If we rename
\[ (x_1, x_2, \ldots, x_{N+1-p}, \ldots, x_N, x_{N+1}) \]
to
\[ (x_1, x_2, \ldots, x_{\text{ghost}}, \ldots, x_{N-1}, x_N) \]
and similarly for \( m_k \), we see that Theorem 3.4 tells us how to add a ghostpeakon to a given nonzero-amplitude \( N \)-peakon solution. (Or as many ghostpeakons as we like, since they are independent of each other.) Which formula to use for describing the ghostpeakon’s trajectory \( x = x_{\text{ghost}}(t) \) depends on which pair of peakons we want it to lie between. The collection of all possible such ghostpeakon trajectories, together with the peakon trajectories \( x = x_i(t) \) themselves, constitutes the family of characteristic curves for the \( N \)-peakon solution \( u(x, t) \), i.e., the curves \( x = \xi(t) \) such that \( \dot{\xi}(t) = u(\xi(t), t) \). So we can rephrase the theorem as the following corollary.

**Corollary 3.5.** For the Camassa–Holm \( N \)-peakon solution given by (2.3), namely
\[ x_{N+1-k}(t) = \ln \frac{\Delta_k^0}{\Delta_{k-1}^2}, \quad m_{N+1-k}(t) = \frac{\Delta_k^0 \Delta_{k-1}^2}{\Delta_k^1 \Delta_{k-1}^1}, \quad 1 \leq k \leq N, \]
Figure 5. Graph of a conservative Camassa–Holm solution $u(x, t)$ with two peakons (positive amplitude) and one antipeakon (negative amplitude), computed from exact formulas as described in Example 3.7. The dimensions of the box are $|x| \leq 8$, $|t| \leq 8$ and $-2 \leq u \leq 3$.

The characteristic curves $x = \zeta(t)$ in the $k$th interval from the right, i.e.,

$$x_{N-k}(t) < \zeta(t) < x_{N+1-k}(t)$$

(where $0 \leq k \leq N$, $x_0 = -\infty$, $x_{N+1} = +\infty$), are given by

$$\zeta(t) = \ln \frac{\Delta^0_{k+1} + \theta \Delta^0_k}{\Delta^2_k + \theta \Delta^2_{k-1}}, \quad \theta > 0.$$  \hspace{1cm} (3.10)

Remark 3.6. Note that the function $\zeta(t)$ given by (3.10) ranges over all values between $x_N^{-k}(t) = \ln \frac{\Delta^0_{k+1}}{\Delta^2_k}$ as the parameter $\theta$ ranges over all positive numbers.

Example 3.7 (Conservative CH peakon solution). In Example 1.2 we plotted a pure three-peakon solution given by the solution formulas (2.3), together with some of its characteristic curves obtained from Corollary 3.5. From the same formulas, but using the parameter values

$$\lambda_1 = \frac{1}{3}, \quad \lambda_2 = 1, \quad \lambda_3 = -\frac{1}{2}, \quad b_1(0) = \frac{9}{10}, \quad b_2(0) = \frac{e^2}{6}, \quad b_3(0) = \frac{4}{15}$$ \hspace{1cm} (3.11)
Figure 6. Spacetime plot of the peakon trajectories $x = x_k(t)$ for the $N = 3$ conservative peakon–antipeakon Camassa–Holm solution shown in Figure 5. The dashed lines indicate the times $t = t'$ and $t = t''$ of the two peakon–antipeakon collisions that occur; for all other $t$ the strict ordering $x_1(t) < x_2(t) < x_3(t)$ holds. The solution starts out with $m_1$ and $m_2$ positive and $m_3$ negative, so that the antipeakon is on the far right for $t < t'$. At the first collision, where $x_2(t') = x_3(t')$, the corresponding amplitude factors $m_2$ and $m_3$ blow up to $\pm \infty$ and change their signs so that it is the middle peakon that plays the role of the antipeakon for $t' < t < t''$ ($m_1$ positive, $m_2$ negative, $m_3$ positive). Similarly, $m_1$ and $m_2$ switch signs at the second collision where $x_1(t'') = x_2(t'')$, and the solution ends up with the antipeakon on the far left for $t > t''$ ($m_1$ negative, $m_2$ and $m_3$ positive).
Figure 7. A selection of characteristic curves for the $N = 3$ conservative peakon–antipeakon Camassa–Holm solution shown in Figure 5 and Figure 6. All curves are computed from exact formulas as described in Example 3.7. The blue curve is a typical characteristic curve in the region $x_1 < x < x_2$, and the red curve is a typical characteristic curve in the region $x_2 < x < x_3$. 
instead of (1.14), we get a mixed peakon–antipeakon solution instead of a pure peakon solution. The asymptotic velocities (and amplitudes) as $t \to \pm \infty$ are

$$c_1 = \frac{1}{\lambda_1} = 3, \quad c_2 = \frac{1}{\lambda_2} = 1, \quad c_3 = \frac{1}{\lambda_3} = -2.$$  

Since $c_1 > c_2 > 0 > c_3$, so there are two peakons and one antipeakon. (The values for $b_k(0)$ in (3.11) were obtained by taking these numbers $c_k$ together with $K = 2$ in the formulas (2.13).)

The wave profile $u(x, t)$ is illustrated in Figure 5, while the peakon trajectories $x = x_k(t)$ are plotted in Figure 6. A selection of characteristic curves $x = \xi(t)$ are plotted in Figure 7; they are also given by the same formulas as in Example 1.2 but with the new parameter values (3.11).

In contrast to the pure peakon case, the solution formulas for $m_k(t)$ are not globally defined; there is an instant

$$t = t' \approx -0.758404$$  

(easily determined numerically) when the quantity

$$\Delta_1^1(t) = \lambda_1 b_1(t) + \lambda_2 b_2(t) + \lambda_3 b_3(t) = \frac{3e^{3t}}{10} + \frac{e^{2t} + 2e^{-2t}}{15}$$

becomes zero, which causes $m_2$ and $m_3$ to blow up, since $\Delta_1^1$ occurs in the denominator of the formulas for $m_2$ and $m_3$ in (1.9b). Similarly, there is another instant

$$t = t'' \approx 0.182763$$

when the expression

$$\Delta_2^2(t) = \lambda_1 \lambda_2 (\lambda_1 - \lambda_2)^2 b_1(t) b_2(t) + \lambda_1 \lambda_3 (\lambda_1 - \lambda_3)^2 b_1(t) b_3(t) + \lambda_2 \lambda_3 (\lambda_2 - \lambda_3)^2 b_2(t) b_3(t)$$

$$= \frac{e^{2+4t}}{45} - \frac{e^t}{36} - \frac{e^{2-t}}{20}$$

occuring in the denominator of $m_1$ and $m_2$ becomes zero. These times $t'$ and $t''$, when two amplitudes $m_k$ blow up (one to $+\infty$ and the other to $-\infty$), are also the instants of collisions between the corresponding peakon and antipeakon:

$$x_1(t') < x_2(t') = x_3(t'), \quad x_1(t'') = x_2(t'') < x_3(t'').$$  

(3.14)

For all other $t \in \mathbb{R}$, the strict ordering $x_1(t) < x_2(t) < x_3(t)$ holds.

From the point of the peakon ODEs, the solution $\{x_k(t), m_k(t)\}_{k=1}^3$ given by the explicit formulas for $t \in \mathbb{R} \setminus \{t', t''\}$ is obtained by analytic continuation in the complex $t$ plane around simple poles at $t'$ and $t''$.

More importantly, from the point of view of the PDE, there is cancellation which causes the function

$$u(x, t) = \sum_{k=1}^3 m_k(t) e^{-|x-x_k(t)|}, \quad t \in \mathbb{R} \setminus \{t', t''\},$$

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with \((x_k(t), m_k(t))\) given by the explicit formulas, to have finite limits as \(t \to t'\) and as \(t \to t''\). More specifically, \(\Delta_1^1\) cancels in the sum

\[
m_2 + m_3 = \frac{\Delta_0^0 \Delta_1^2}{\Delta_1^1 \Delta_1^1} + \frac{\Delta_0^0 \Delta_0^2}{\Delta_1^1},
\]

so that this expression remains finite at \(t = t'\), and \(\Delta_2^1\) cancels in the sum

\[
m_1 + m_2 = \frac{\Delta_0^0 \Delta_2^2}{\Delta_1^1 \Delta_2^1} + \frac{\Delta_0^0 \Delta_1^2}{\Delta_2^1 \Delta_1^1}
= \frac{\lambda_1^2 (\lambda_2 + \lambda_3) b_1 + \lambda_2^2 (\lambda_1 + \lambda_3) b_2 + \lambda_3^2 (\lambda_1 + \lambda_2) b_3}{\lambda_1 \lambda_2 \lambda_3 \Delta_1^1},
\]

so that this expression is finite at \(t = t''\). Extending \(u\) continuously by defining

\[
u(x, t') = \lim_{t \to t'} u(x, t) = m_1(t') e^{-|x-x_1(t')|} + (m_2 + m_3)(t') e^{-|x-x_2(t')|} (3.15)
\]

and

\[
u(x, t'') = \lim_{t \to t''} u(x, t) = (m_1 + m_2)(t'') e^{-|x-x_1(t'')|} + m_3(t'') e^{-|x-x_3(t'')|},
\]

we get a global weak solution \(u(x, t)\) of the Camassa–Holm equation, which has been continued past the singularities at \(t'\) and \(t''\), where the derivative \(u\) blows up, although \(u\) itself remains bounded. This is the so-called \textit{conservative} multipeakon solution; see \textsection 1.5 for references.

\textbf{Example 3.8} (Dissipative CH peakon solution). The \textit{dissipative} multipeakon solution is another global weak solution, where \(u(x, t)\) is continued past singularities in a different way, as illustrated in Figure 8. (Again, see \textsection 1.5 for references.) In the scenario of the previous example, the conservative and the dissipative solutions agree for \(t \leq t'\), up until the time of the first collision. The wave profile \(u(x, t')\) defined by (3.15) has the two-peakon form

\[
u(x, t') = \tilde{m}_1(t') e^{-|x-\tilde{x}_1(t')|} + \tilde{m}_2(t') e^{-|x-\tilde{x}_2(t')|} (3.17)
\]

with

\[
\tilde{x}_1(t') = x_1(t') \approx -2.1819596,
\tilde{m}_1(t') = m_1(t') \approx 3.0170075,
\tilde{x}_2(t') = x_2(t') \approx 0.6337836,
\tilde{m}_2(t') = m_2(t') \approx -1.0170075,
\]

(3.18)

where the values are obtained from exact formulas, except that \(t'\) has to be found by solving the equation \(\Delta_1^1(t) = 0\) numerically. In contrast to the conservative solution above, which again contains three peakons immediately after the collision, the dissipative solution continues as a
Figure 8. Graph of a dissipative Camassa–Holm solution $u(x, t)$. The solution starts out identical to the conservative solution in Figure 5, with two peakons and one antipeakon, but here they merge at collisions so that only one peakon remains in the end. See Example 3.7 for details. The dimensions of the box are $|x| \leq 8, |t| \leq 6$ and $-2 \leq u \leq 3$. 
Figure 9. Spacetime plot of the peakon trajectories \( x = x_k(t) \) for the dissipative Camassa–Holm solution shown in Figure 8. The solution starts out with two peakons on the left (at \( x_1 \) and \( x_2 \)) and one antipeakon on the right (at \( x_3 \)). At the first collision, when \( x_2(t') = x_3(t') \), the colliding peakon and antipeakon merge into a single antipeakon, which then collides with the leftmost peakon at time \( t = t'' \), after which the solution continues as a single peakon. For \( t \leq t' \), the solution is given by the three-peakon solution formulas, and for \( t' < t \leq t'' \) it is given by the two-peakon solution formulas, with new spectral data computed as described in Example 3.8. And for \( t'' < t \), the solution is just a one-peakon solution – a travelling wave where the wave profile \( u(x, t'') \) is translated with constant speed.
Figure 10. A selection of characteristic curves (plotted from the exact formulas) for the $N = 3$ dissipative peakon–antipeakon Camassa–Holm solution shown in Figure 8 and Figure 9. See Example 3.8.
two-peakon solution, given by the usual explicit formulas (2.6), but with new parameters \( \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{b}_1(t') \) and \( \tilde{b}_2(t') \) determined from the values (3.18) via the general relations

\[
\frac{1}{\tilde{\lambda}_1} + \frac{1}{\tilde{\lambda}_2} = m_1 + m_2, \quad \frac{1}{\tilde{\lambda}_1 \tilde{\lambda}_2} = m_1 m_2 (1 - e^{x_1 - x_2})
\]  

(3.19)

and

\[
b_1 + b_2 = e^{x_2}, \quad \frac{b_1}{\tilde{\lambda}_1} + \frac{b_2}{\tilde{\lambda}_2} = m_1 e^{x_1} + m_2 e^{x_2}.
\]  

(3.20)

(These formulas originate in the study of the forward spectral problem; we will not go into the details here. It is by solving these equations for \( x_1, x_2, m_1, m_2 \) that the Camassa–Holm two-peakon solution formulas (2.6) are obtained.) This gives

\[
\tilde{\lambda}_1 \approx 0.3365925, \quad \tilde{b}_1(t') = \tilde{b}_1(0) e^{t' / \tilde{\lambda}_1} \approx 0.06432834,
\]

\[
\tilde{\lambda}_2 \approx -1.02991776, \quad \tilde{b}_2(t') = \tilde{b}_2(0) e^{t' / \tilde{\lambda}_2} \approx 1.82039986.
\]  

(3.21)

The remaining peakon and the antipeakon then collide when

\[
\tilde{\Delta}_1(t) = \tilde{\lambda}_1 \tilde{b}_1(0) e^{t' / \tilde{\lambda}_1} + \tilde{\lambda}_2 \tilde{b}_2(0) e^{t' / \tilde{\lambda}_2}
\]

vanishes, namely at

\[
t = t''' \approx 0.373326, \quad x_1(t'''') = x_2(t'''') \approx 0.9013432,
\]  

(3.22)

and merge into a single peakon of amplitude 2:

\[
u(x, t) = 2 e^{-|x - x_1(t''') - 2t|} \quad \text{for } t \geq t'''.
\]  

(3.23)

(The amplitude here is exactly 2, which is due to the momentum \( \int_{\mathbb{R}} u \, dx = \sum m_i \) being conserved even for dissipative solutions. Initially, the momentum is \( m_1 + m_2 + m_3 = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 3 + 1 + (-2) = 2 \), and so it stays equal to 2.) The peakon trajectories for the dissipative solution are plotted in Figure 9, while Figure 10 shows a selection of characteristic curves.

### 4 Degasperis–Procesi ghostpeakons

We now leave the Camassa–Holm equation, and move on to the Degasperis–Procesi peakon ODEs (1.19). In addition to deriving ghostpeakons formulas analogous to those for the Camassa–Holm equation in Section 3, we also use direct integration to compute the characteristics for the one-shockpeakon solution formed at a peakon–antipeakon collision; see Example 4.5 for the symmetric case and Example 4.6 for the asymmetric case.

For simplicity, we will formulate Theorem 4.1 and Corollary 4.2 for pure peakon solutions, but to some extent they remain valid also for mixed peakon–antipeakon solutions; see Remark 4.3.
Theorem 4.1. Fix some $p$ with $0 \leq p \leq N$. The solution of the Degasperis–Procesi $(N+1)$-peakon ODEs (1.19) with $x_1 < \cdots < x_{N+1}$ and all amplitudes $m_k(t)$ positive except for $m_{N+1-p}(t) = 0$ is as follows: the position of the ghostpeakon is given by

$$x_{N+1-p}(t) = \ln \frac{U_p^0 + \theta U_p^1}{U_p^1 + \theta U_p^{p-1}}, \quad 0 < \theta < \infty,$$

(4.1)

while the other peakons are given by the general solution formulas (2.14), up to renumbering:

$$x_{N+1-k}(t) = \begin{cases} 
\ln \frac{U_{k+1}^0}{U_k^1}, & 0 \leq k < p, \\
\ln \frac{U_k^0}{U_{k-1}^1}, & p < k \leq N,
\end{cases}$$

(4.2)

$$m_{N+1-k}(t) = \begin{cases} 
(\frac{U_{k+1}^0 U_k^1}{W_{k+1} W_k})^2, & 1 \leq k < p, \\
(\frac{U_k^0 U_{k-1}^1}{W_k W_{k-1}})^2, & p < k \leq N.
\end{cases}$$

Here $\theta \in (0, \infty)$ is a constant in one-to-one correspondence with the ghostpeakon’s initial position $x_{N+1-p}(0)$, while the quantities $|\lambda_k, b_k|_{k=1}^{N-1}$ appearing in the expressions $U_k^a$ and $W_k$ have the usual time dependence $\lambda_k = \text{constant}, b_k(t) = b_k(0) e^{t/\lambda_k}$.

Proof. The (pure) $(N+1)$-peakon solution of the Degasperis–Procesi equation is given by (2.14) with $N+1$ instead of $N$:

$$x_{N+1-k} = \ln \frac{\bar{U}_{k+1}^0}{\bar{U}_k^1}, \quad m_{N+1-k} = \frac{(\bar{U}_{k+1}^0 \bar{U}_k^1)^2}{\bar{W}_{k+1} \bar{W}_k}, \quad 0 \leq k \leq N$$

where $\bar{U}_k^a = U_k^a(N+1)$ and $\bar{W}_k = W_k(N+1)$.

Replace the parameters $\lambda_{N+1}$ and $b_{N+1}(0)$ with

$$\varepsilon = \frac{1}{\lambda_{N+1}}, \quad \theta = \lambda_{N+1}^p b_{N+1}(0),$$

(4.3)

and let

$$\Theta = \Theta(t) = \theta e^{\varepsilon t} = \lambda_{N+1}^p b_{N+1}(t).$$

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Then
\[
\tilde{U}_k^a = \sum_{1 \leq i_1 < \cdots < i_k \leq N+1} \left( \prod_{r=1}^{k} \lambda_{i_r}^a b_{i_r} \right) \frac{\Delta(\lambda_{i_1}, \ldots, \lambda_{i_k})^2}{\Gamma(\lambda_{i_1}, \ldots, \lambda_{i_k})} \\
= \sum_{1 \leq i_1 < \cdots < i_k \leq N} \left( \prod_{r=1}^{k} \lambda_{i_r}^a b_{i_r} \right) \frac{\Delta(\lambda_{i_1}, \ldots, \lambda_{i_k})^2}{\Gamma(\lambda_{i_1}, \ldots, \lambda_{i_k})} \\
+ \sum_{1 \leq i_1 < \cdots < i_{k-1} \leq N} \left( \prod_{r=1}^{k-1} \lambda_{i_r}^a b_{i_r} \right) \frac{\Delta(\lambda_{i_1}, \ldots, \lambda_{i_{k-1}}) \lambda_{N+1}^b b_{N+1} \prod_{i=1}^{k-1} (\lambda_{i} - \lambda_{N+1})^2}{\Gamma(\lambda_{i_1}, \ldots, \lambda_{i_{k-1}}, \lambda_{N+1})} \\
= U_k^a + \sum_{1 \leq i_1 < \cdots < i_{k-1} \leq N} \left( \prod_{r=1}^{k-1} \lambda_{i_r}^a b_{i_r} \right) \frac{\Delta(\lambda_{i_1}, \ldots, \lambda_{i_{k-1}})^2}{\Gamma(\lambda_{i_1}, \ldots, \lambda_{i_{k-1}})} \epsilon^{a-\Theta} \epsilon^{p} \prod_{i=1}^{k-1} (\epsilon \lambda_{i} - 1)^2 \\
= U_k^a + \sum_{1 \leq i_1 < \cdots < i_{k-1} \leq N} \left( \prod_{r=1}^{k-1} \lambda_{i_r}^a b_{i_r} \right) \frac{\Delta(\lambda_{i_1}, \ldots, \lambda_{i_{k-1}})^2}{\Gamma(\lambda_{i_1}, \ldots, \lambda_{i_{k-1}})} \Theta \epsilon^{p-a} \epsilon^{k-1} (1 + \Theta (\epsilon)) \\
= U_k^a + U_k^a \Theta \epsilon^{p-k-a+1} (1 + \Theta (\epsilon)) \quad \text{(as } \epsilon \to 0),
\]
and hence
\[
\tilde{W}_k = \begin{bmatrix} U_k^0 + U_k^1 \Theta \epsilon^{p-k+1} (1 + \Theta (\epsilon)) & U_k^1 + U_k^2 \Theta \epsilon^{p-k+1} (1 + \Theta (\epsilon)) \\
U_k^0 \Theta \epsilon^{p-k} (1 + \Theta (\epsilon)) & U_k^1 \Theta \epsilon^{p-k} (1 + \Theta (\epsilon)) \end{bmatrix} \\
= \begin{bmatrix} \theta_{k-1} \Theta \epsilon^{p-k+1} (1 + \Theta (\epsilon)) + W_{k-1} \Theta^2 \epsilon^{2(p-k)+1} (1 + \Theta (\epsilon)) \\
W_k + \Theta (\epsilon), & k \leq p, \\
W_{k-1} \Theta^2 + \Theta (\epsilon) / \epsilon^{2(k-p)+1}, & k > p,
\end{bmatrix}
\]
so that the peakon solution formulas take the form
\[
\frac{x_{N+1-k} = \ln \frac{\tilde{U}_{k+1}^0}{\tilde{U}_k^0}}{\ln \frac{\tilde{U}_{k+1}^1}{\tilde{U}_k^1}} = \ln \frac{U_{k+1}^0 + U_k^0 \Theta \epsilon^{p-k} (1 + \Theta (\epsilon))}{U_k^0 + U_k^1 \Theta \epsilon^{p-k} (1 + \Theta (\epsilon))} \\
= \begin{cases} 
\ln \frac{U_{k+1}^0}{U_k^0 + \Theta (\epsilon)}, & k < p, \\
\ln \frac{U_{k+1}^0 + U_k^0 \Theta + \Theta (\epsilon)}{U_k^0 + U_k^0 \Theta + \Theta (\epsilon)}, & k = p, \\
\ln \frac{U_k^0 + U_{k+1}^1 \Theta + \Theta (\epsilon)}{U_k^0 \Theta + \Theta (\epsilon)}, & k > p,
\end{cases}
\]
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and
\[ m_{N+1-k} = \frac{(U_{k+1} \Theta^1_k)^2}{W_{k+1} W_k} \]
\[ = \frac{[U_{k+1} + \epsilon U_k \Theta^p - \epsilon (1 + \Theta (\epsilon))]^2 \left( U_k^1 + U_k^1 \Theta \epsilon - \epsilon (1 + \Theta (\epsilon)) \right)^2}{W_{k+1} W_k} \]
\[ = \begin{cases} \frac{[U_{k+1} + \epsilon (1 + \Theta (\epsilon))]^2 \left( U_k^1 + \epsilon (1 + \Theta (\epsilon)) \right)^2}{W_{k+1} + \Theta (\epsilon) \left[ W_k + \Theta (\epsilon) \right]}, & k < p, \\ \frac{\epsilon^2 \left( U_{p+1} + \epsilon U_p \Theta + \Theta (\epsilon) \right)^2 \left( U_p^1 + U_p^1 \Theta + \Theta (\epsilon) \right)^2}{W_p \Theta^2 + \Theta (\epsilon) \left[ W_{p-1} + \Theta (\epsilon) \right]}, & k = p, \\ \frac{U_k \Theta + \Theta (\epsilon) \left[ W_{k-1} + \Theta (\epsilon) \right]}{W_k \Theta^2 + \Theta (\epsilon) \left[ W_{k-1} + \Theta (\epsilon) \right]}, & k > p. \end{cases} \]

In the limit \( \epsilon \to 0 \), we see that \( m_{N+1-p} \to 0 \), while the other expressions reduce to those given in (4.1) and (4.2).

In the same way as for Corollary 3.5, we obtain the following result just by relabeling.

**Corollary 4.2.** Write \( U_k = U_k^0 \) and \( V_k = U_k^1 \). For the Degasperis–Procesi pure \( N \)-peakon solution given by (2.14), namely
\[ x_{N+1-k}(t) = \ln \frac{U_k}{V_{k-1}}, \quad m_{N+1-k}(t) = \frac{(U_k V_{k-1})^2}{W_k W_{k-1}}, \quad 1 \leq k \leq N, \]
the characteristic curves \( x = \zeta(t) \) in the \( k \)th interval from the right,
\[ x_{N-k}(t) < \zeta(t) < x_{N+1-k}(t) \]
(where \( 0 \leq k \leq N, x_0 = -\infty, x_{N+1} = +\infty \)), are given by
\[ \zeta(t) = \ln \frac{U_{k+1} + \theta U_k}{V_k + \theta V_{k-1}}, \quad \theta > 0. \] (4.4)

**Remark 4.3.** Although we have formulated Theorem 4.1 and Corollary 4.2 for pure peakon solutions, they are also valid (in the appropriate time interval) for mixed peakon–antipeakon solutions, in the generic case where all eigenvalues are simple and there are no resonances \( \lambda_i + \lambda_j = 0 \); see Remark 2.7. The eigenvalues and residues may even be complex (appearing in complex-conjugate pairs). In the proof, given a “target configuration” of \( N \) eigenvalues and \( N \) residues for which we want to obtain the ghostpeakon formulas, we add a simple positive eigenvalue \( \lambda_{N+1} = 1/\epsilon > 0 \) and a corresponding real nonzero residue \( b_{N+1} \). However, it may happen that this residue \( b_{N+1} \) must be negative in order for \( x_1 < \cdots < x_{N+1} \) to hold, and according to (4.3) this means that also \( \theta \) must be negative. Such cases, where we need to use a negative ghost parameter \( \theta \), can be seen in Example 4.6 below.
Example 4.4 (DP three-peakon characteristics). The trajectories in the pure three-peakon solution are given by

\[ x_1(t) = \ln \frac{U_3}{V_2}, \quad x_2(t) = \ln \frac{U_2}{V_1}, \quad x_3(t) = \ln \frac{U_1}{V_0} = \ln U_1, \]

where the expressions \( U_k \) and \( V_k \) were written out in detail in Example 2.6. By Corollary 4.2, the characteristic curves in the intervals outside and between the peakons are

\[ \xi(t) = \ln \frac{U_4 + \theta U_2}{V_3 + \theta V_2} = \ln \frac{\theta U_3}{V_5 + \theta V_2} \quad \text{in } (-\infty, x_1(t)), \]

\[ \xi(t) = \ln \frac{U_3 + \theta U_2}{V_2 + \theta V_1} \quad \text{in } (x_1(t), x_2(t)), \]

\[ \xi(t) = \ln \frac{U_2 + \theta U_1}{V_1 + \theta V_0} = \ln \frac{U_2 + \theta U_1}{V_1 + \theta} \quad \text{in } (x_2(t), x_3(t)), \]

\[ \xi(t) = \ln \frac{U_1 + \theta U_0}{V_0 + \theta V_{-1}} \quad \ln(U_1 + \theta) \quad \text{in } (x_3(t), \infty), \]  

where \( 0 < \theta < \infty \) in all cases. The curves look fairly similar to those for the Camassa–Holm case shown in Figure 3.

The following two examples illustrate what the characteristic curves look like for a peakon–antipeakon collision in the Degasperis–Procesi equation. Here our theorem only applies before the collision, since the solution continues after the collision in the form of a so-called shockpeakon [43, Theorem 3.3]. This symmetric situation occurs when the eigenvalues satisfy \( \lambda_{1,2} = \pm \lambda \), so that \( \lambda_1 + \lambda_2 = 0 \). This is a resonant case where the usual formulas (2.19) do not apply (since they contain \( \lambda_1 + \lambda_2 \) in the denominators), but the solution can easily be found by direct integration instead. The results below are illustrated in Figure 11 for \( \lambda = 1 \).

By translations along the \( x \) and \( t \) axes, we can arrange for the collision to occur at \( (x, t) = (0, 0) \); then the solution depends on the single parameter \( \lambda > 0 \). Before the collision, for \( t < 0 \), we have \( x_1(t) < 0 < x_2(t) \) and \( m_1(t) > 0 > m_2(t) \), and the solution is

\[ x_1(t) = -x_2(t) = \frac{t}{\lambda}, \quad m_1(t) = -m_2(t) = \frac{1}{\lambda(1 - e^{2t/\lambda})} \quad (t < 0). \]  

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Figure 11. Peakon trajectories and a selection of other characteristic curves for a symmetric peakon–antipeakon solution ($\lambda_1 = 1$, $\lambda_2 = -1$) of the Degasperis–Procesi equation; see Example 4.5. The peakon and the antipeakon travel with constant speed and collide head-on at the origin, forming a shockpeakon which remains stationary at $x = 0$ (dashed line): $u(x, t) = -\sgn(x) e^{-|x| \lambda + \lambda t}$ for $t \geq 0$. All characteristic curves have a finite lifespan; the curve through the point $(x, t) = (\zeta_0, 0)$ reaches the shock at time $t = \exp(e^{\zeta_0 \lambda} - 1) - 1$. 
Thus,

\[ u(x, t) = m_1(t) e^{-|x-x_1(t)|} + m_2(t) e^{-|x-x_2(t)|} = \frac{e^{-|x-t/\lambda|} - e^{-|x+t/\lambda|}}{\lambda(1-e^{2t/\lambda})} \]

\[ = \begin{cases} \frac{e^{x-t/\lambda} - e^{x+t/\lambda}}{\lambda(1-e^{2t/\lambda})} = \frac{e^{x-t/\lambda}}{\lambda}, & x < t/\lambda, \\ \frac{e^{-x+t/\lambda} - e^{-x-t/\lambda}}{\lambda(1-e^{2t/\lambda})} = \frac{\sinh x}{\lambda \sinh(t/\lambda)}, & t/\lambda \leq x \leq -t/\lambda, \\ \frac{e^{-x+t/\lambda} - e^{-x-t/\lambda}}{\lambda(1-e^{2t/\lambda})} = \frac{-e^{-x-t/\lambda}}{\lambda}, & -t/\lambda < x, \end{cases} \quad (4.7) \]

In particular, \( u(x_1(t), t) = -u(x_2(t), t) = 1/\lambda \) for all \( t < 0 \), in agreement with the fact that the peakons travel with the constant speed \( 1/\lambda \).

We can also easily find the characteristic curves by direct integration in this case. The characteristics \( x = \xi(t) \) to the left of \( x_1 \) are given by

\[ \dot{\xi} = \frac{e^{\xi-t/\lambda}}{\lambda} \iff \int \frac{d\xi}{e^{\xi-t/\lambda}} = \int \frac{dt}{\lambda \sinh(t/\lambda)} \iff \xi(t) = -\ln(\theta + e^{-t/\lambda}), \quad \theta = e^{\xi(0)} - 1 > 0, \quad (4.8) \]

and similarly to the right of \( x_2 \),

\[ \xi(t) = \ln(\theta + e^{-t/\lambda}), \quad \theta = e^{\xi(0)} - 1 > 0. \quad (4.9) \]

In between, we have

\[ \dot{\xi} = \frac{\sinh \xi}{\lambda \sinh(t/\lambda)} \iff \int \frac{d\xi}{\sinh \xi} = \int \frac{dt}{\lambda \sinh(t/\lambda)} \iff \ln \tanh \frac{\xi}{2} = \ln \tanh \frac{t}{2\lambda} + C \]

\[ \iff \xi(t) = -2 \text{artanh}(\tanh \frac{\theta}{2} \tanh \frac{t}{2\lambda}), \quad (4.10) \]

with \( \theta = \lim_{t \to -\infty} \xi(t) \in \mathbb{R} \).

The limiting wave profile at the collision is

\[ u(x, 0) = -\frac{\text{sgn}(x) e^{-|x|}}{\lambda}, \quad (4.11) \]

which has a jump of size \(-2/\lambda\) at \( x = 0 \). The continuation of the solution is

\[ u(x, t) = -\frac{\text{sgn}(x) e^{-|x|}}{\lambda + t}, \quad t \geq 0, \quad (4.12) \]

a shockpeakon which just sits at \( x = 0 \), with the size of the jump decaying to zero as \( t \to \infty \).

The characteristic curves for \( t \geq 0 \) to the left of the shock \( (x < 0) \) are given by

\[ \dot{\xi} = \frac{e^{-\xi}}{\lambda + t} \iff \xi(t) = -\ln\left(\theta - \ln\left(1 + \frac{t}{\lambda}\right)\right), \quad \theta = e^{-\xi(0)} > 1, \quad (4.13) \]
and similarly to the right of the shock ($x > 0$) we have
\[
\zeta(t) = \ln \left( \theta - \ln \left( 1 + \frac{4}{\lambda} \right) \right), \quad \theta = e^{\zeta(0)}> 1. \quad (4.14)
\]
From this it follows that each characteristic curve has a finite lifespan; it ceases to exist when it collides with the shock, i.e., when $\zeta(t)$ becomes zero, which happens when
\[
\frac{t}{\lambda} = e^{\theta - 1} - 1 = \exp(e^{\zeta(0)} - 1) - 1. \quad (4.15)
\]

**Example 4.6 (DP asymmetric collision).** Next, we look at the characteristics for the asymmetric peakon–antipeakon collision [43, Theorem 3.5] which occurs when $\lambda_1 + \lambda_2 \neq 0$. Let us assume that the peakon at $x_1$ is stronger than the antipeakon at $x_2$, i.e., $m_1(t) > -m_2(t) > 0$. Then the shockpeakon which forms at the collision will move to the right. This case occurs when $b_1(0)$ and $b_2(0)$ are positive and $0 < \lambda_1 < -\lambda_2$, so that
\[
\kappa := \sqrt{-\lambda_2/\lambda_1} > 1. \quad (4.16)
\]
The results below are illustrated in Figure 12 for $\lambda_1 = 1/4$ and $\lambda_2 = -1/2$.

The solution is given by the same formulas (2.19) as in the pure peakon case, but only up until the time of collision, which is
\[
t_0 = \frac{1}{\lambda_1 - \lambda_2} \log \left( \frac{\kappa^2 - \kappa b_2(0)}{\kappa^2 + 1} b_1(0) \right) > 0, \quad (4.17)
\]
By translation, we can arrange for the collision to occur at $(x, t) = (0, 0)$. This is accomplished by taking
\[
b_1(0) = \frac{\kappa^2 - \kappa}{\kappa^2 + 1}, \quad b_2(0) = \frac{\kappa + 1}{\kappa^2 + 1}. \quad (4.18)
\]
Indeed, to get the collision at $t = 0$, we must take $\frac{b_1(0)}{b_2(0)} = \frac{\kappa^2 - \kappa}{\kappa^2 + 1}$, and $x_2(0) = 0$ if and only if $b_1(0) + b_2(0) = 1$. With these parameter values, the solution before the collision (i.e., for $t < 0$) is given by
\[
x_1(t) = \ln \frac{(\lambda_1 - \lambda_2)^2 b_1 b_2}{\lambda_1 b_1 + \lambda_2 b_2} = -\ln \frac{(\kappa^2 + \kappa) e^{-t/\lambda_1} - (\kappa - 1) e^{-t/\lambda_2}}{\kappa^2 + 1},
\]
\[
x_2(t) = \ln (b_1 + b_2) = \ln \frac{(\kappa^2 - \kappa) e^{t/\lambda_1} + (\kappa + 1) e^{t/\lambda_2}}{\kappa^2 + 1},
\]
\[
m_1(t) = \frac{(\lambda_1 b_1 + \lambda_2 b_2)^2}{\lambda_1 \lambda_2 \left( \lambda_1 b_1^2 + \lambda_2 b_2^2 + \frac{4\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} b_1 b_2 \right)}
\]
\[
= \frac{1}{\lambda_1} \cdot \frac{((\kappa - 1) e^{t/\lambda_1} - (\kappa + 1) e^{t/\lambda_2})^2}{\kappa^2 ((\kappa - 1) e^{t/\lambda_1} + (\kappa + 1) e^{t/\lambda_2}) (e^{t/\lambda_2} - e^{t/\lambda_1})} > 0, \quad (4.19)
\]
\[
m_2(t) = \frac{(b_1 + b_2)^2}{\lambda_1 b_1^2 + \lambda_2 b_2^2 + \frac{4\lambda_1 \lambda_2}{\lambda_1 + \lambda_2} b_1 b_2}
\]
\[
= \frac{1}{\lambda_2} \cdot \frac{((\kappa - 2) - (\kappa + 1) e^{t/\lambda_2})^2}{((\kappa - 1) e^{t/\lambda_1} + (\kappa + 1) e^{t/\lambda_2}) (e^{t/\lambda_2} - e^{t/\lambda_1})} < 0.
\]
Figure 12. Peakon trajectories and a selection of other characteristic curves for an asymmetric peakon–antipeakon solution ($\lambda_1 = 1/4$, $\lambda_2 = −1/2$) of the Degasperis–Procesi equation; see Example 4.6. The limiting wave profile at the collision is a shockpeakon given by $u(x, 0) = (v + \frac{1}{\sigma})e^{\sigma x}$ for $x < 0$ and $u(x, 0) = (v - \frac{1}{\sigma})e^{-\sigma x}$ for $x > 0$, where $v = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} = 2$ and $\sigma = \sqrt{-\lambda_1\lambda_2} = \frac{1}{2\sqrt{2}}$. In particular, $u(0^\pm, 0) = v \mp \frac{1}{\sigma}$, so there is a jump of size $-\frac{2}{\sigma}$ at the origin. The shockpeakon travels with constant velocity $v$ (dashed line), and the jump at time $t > 0$ is $-\frac{2v}{\sigma t}$, which decays to zero as $t \to \infty$. The characteristic curve (dotted line) through the point $(x, t) = (\Xi, 0)$, where $\Xi = -\ln(1 - e^{\nu\sigma}Ei(-\nu\sigma)) \approx -0.55842$, approaches the shockpeakon trajectory $x = vt$ as $t \to \infty$. Characteristics to the left of that curve approach lines $x$ = constant as $t \to \infty$, and characteristics to the right of it hit the shock after finite time. All characteristics to the right of the shock turn from left to right at the same instant, namely $t = \frac{1}{\nu} - \sigma \approx 0.146$ (except, of course, those that have already hit the shock and ceased to exist).
Only the terms containing \( e^{t/\lambda_2} \) contribute asymptotically as \( t \to -\infty \), since \( e^{t/\lambda_1} \to 0 \) and \( e^{t/\lambda_2} \to \infty \). More precisely, \( e^{t/\lambda_1} = e^{\delta t} e^{t/\lambda_2} \), where \( \delta = \frac{1}{\lambda_1} - \frac{1}{\lambda_2} > 0 \), which implies that

\[
\begin{align*}
x_1(t) &= \frac{t}{\lambda_1} + \ln \frac{\kappa^2 + 1}{\kappa^2 + \kappa} + \mathcal{O}(e^{\delta t}), \\
x_2(t) &= \frac{t}{\lambda_2} + \ln \frac{\kappa + 1}{\kappa^2 + \kappa} + \mathcal{O}(e^{\delta t}), \\
m_1(t) &= \frac{1}{\lambda_1} + \mathcal{O}(e^{\delta t}), \\
m_2(t) &= \frac{1}{\lambda_2} + \mathcal{O}(e^{\delta t}),
\end{align*}
\]

as \( t \to -\infty \). To see the behaviour at the collision, we just use the Maclaurin expansions of the exponentials to get

\[
\begin{align*}
x_1(t) &= \frac{\kappa^2 + \kappa - 1}{\kappa^2 \lambda_1} t + \mathcal{O}(t^2), \\
x_2(t) &= \frac{\kappa^2 - \kappa - 1}{\kappa^2 \lambda_1} t + \mathcal{O}(t^2), \\
m_1(t) &= -\frac{1}{2t} + \frac{1}{2\kappa^2 \lambda_1} + \mathcal{O}(t), \\
m_2(t) &= \frac{1}{2t} + \frac{\kappa^2 - \kappa - 1}{2\kappa^2 \lambda_1} + \mathcal{O}(t),
\end{align*}
\]

as \( t \to 0^− \). It follows that

\[
\begin{align*}
u(x_1(t), t) = m_1(t) + m_2(t) e^{x_1(t) - x_2(t)} &= \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\sqrt{-\lambda_1 \lambda_2}} + \mathcal{O}(t), \\
u(x_2(t), t) = m_1(t) e^{x_1(t) - x_2(t)} + m_2(t) &= \frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\sqrt{-\lambda_1 \lambda_2}} + \mathcal{O}(t),
\end{align*}
\]

as \( t \to 0^− \). The limiting wave profile at the collision is thus

\[
u(x, 0) = \left( \frac{\nu - \text{sgn}(x)}{\sigma} \right) e^{-|x|},
\]

where

\[
\nu = \frac{1}{\lambda_1} + \frac{1}{\lambda_2} > 0, \quad \sigma = \sqrt{-\lambda_1 \lambda_2} > 0.
\]

It has a jump of size \(-2/\sigma\) at \( x = 0 \), and continues as a shockpeakon moving to the right with constant velocity \( \nu \) and shock strength decaying to zero as \( t \to \infty \):

\[
u(x, t) = \left( \frac{\nu - \text{sgn}(x)}{\sigma + t} \right) e^{-|x-\nu t|}, \quad t \geq 0.
\]

The characteristic curves before the collision are given by our general ghostpeakon formula (4.1), except that we must let the parameter \( \theta \) be negative in some cases; cf. Remark 4.3.
To the left of $x_1$, we have
\[
\xi_{\text{left}}(t) = \ln \frac{\theta \frac{(\lambda_1 - \lambda_2)^2}{\lambda_1 + \lambda_2} b_1 b_2}{\frac{(\lambda_1 - \lambda_2)^2}{\lambda_1 + \lambda_2} \lambda_1 \lambda_2 b_1 b_2 + \theta (\lambda_1 b_1 + \lambda_2 b_2)}
= -\ln \left( \frac{(\kappa^2 + \kappa) e^{-t/\lambda_1} - (\kappa - 1) e^{-t/\lambda_2}}{\kappa^2 + 1} + \frac{\lambda_1 \lambda_2}{\theta} \right),
\]
where $-\infty < \theta < 0$, with the asymptotics
\[
\xi_{\text{left}}(t) = \frac{t}{\lambda_1} + \ln \frac{\kappa^2 + 1}{\kappa^2 + \kappa} + \Theta \left( e^{t/\lambda_1} \right), \quad t \to -\infty,
\]
and
\[
\xi_{\text{left}}(t) = -\ln \left( 1 + \frac{\lambda_1 \lambda_2}{\theta} \right) + \Theta(t), \quad t \to 0^-.
\]

The characteristics to the right of $x_2$ are
\[
\xi_{\text{right}}(t) = \ln (b_1 + b_2 + \theta)
= \ln \left( \frac{(\kappa^2 - \kappa) e^{t/\lambda_1} + (\kappa + 1) e^{t/\lambda_2}}{\kappa^2 + 1} + \theta \right)
\]
where $0 < \theta < \infty$, with the asymptotics
\[
\xi_{\text{right}}(t) = \frac{t}{\lambda_2} + \ln \frac{\kappa + 1}{\kappa^2 + 1} + \Theta \left( e^{-t/\lambda_2} \right), \quad t \to -\infty,
\]
and
\[
\xi_{\text{right}}(t) = \ln (1 + \theta) + \Theta(t), \quad t \to 0^-.
\]

And between $x_1$ and $x_2$, the characteristics are
\[
\xi_{\text{mid}}(t) = \ln \frac{\theta (\lambda_1 - \lambda_2)^2 b_1 b_2}{\lambda_1 b_1 + \lambda_2 b_2 + \theta (b_1 + b_2)}
= \ln \left( \frac{\theta (\kappa^2 + \kappa) e^{t/\lambda_1} - \theta (\kappa^2 + \kappa) e^{t/\lambda_2}}{(\kappa^2 - \kappa) e^{t/\alpha_1} + (\kappa + 1) e^{t/\lambda_2} + \theta} \right)
\]
where $-\infty < \theta < 0$, with the asymptotics
\[
\xi_{\text{mid}}(t) = \ln \frac{-\theta (\kappa + 1)}{\lambda_1 (\kappa^2 + \kappa^2)} + \Theta \left( e^{-t/\lambda_2} \right), \quad t \to -\infty,
\]
and
\[
\xi_{\text{mid}}(t) = \ln \frac{(\kappa^2 + \kappa - 1) + \frac{\theta}{\lambda_1} (\kappa^2 - \kappa - 1)}{\kappa^2 \lambda_1 \left( 1 + \frac{\theta}{\lambda_1} \right)} t + \Theta \left( t^2 \right), \quad t \to 0^-.
\]
The rate of steepening of the slope \( u_x \) along this middle family of characteristic curves turns out to be

\[
\begin{align*}
  u_x(\xi_{\text{mid}}(t), t) &= -m_1(t) e^{x_1(t)-\xi_{\text{mid}}(t)} + m_2(t) e^{x_2(t)-\xi_{\text{mid}}(t)} \\
  &= \frac{1}{t} + \mathcal{O}(t), \quad t \to 0^-,
\end{align*}
\]

(4.35)

for every \( \theta < 0 \). We may note that this agrees with the blowup rate for solutions with initial data in \( H^s(\mathbb{R}) \) for \( s > 3/2 \) given by Escher, Liu and Yin [25, Theorem 3.1], even though peakons are not smooth enough to be in those function spaces.

Let us also investigate the characteristic curves of the shockpeakon solution (4.25), which provides the continuation of the solution past the collision. Here we use direct integration again. The ODE for the characteristics starting to the left of the shock \( (\xi(0) < 0) \) is

\[
  \dot{\xi} = \left( \nu + \frac{1}{\sigma + t} \right) e^{-\nu t},
\]

which gives

\[
  e^{-\xi(t)} - e^{-\xi(0)} = -\int_0^t \left( \nu + \frac{1}{\sigma + \tau} \right) e^{-\nu \tau} d\tau
  = -\int_0^{\nu t} \left( 1 + \frac{1}{\nu \sigma + s} \right) e^{-s} ds
  = -e^{\nu \sigma} \int_{\nu \sigma}^{\nu \sigma + \nu t} \left( 1 + \frac{1}{r} \right) e^{-r} dr
  = e^{\nu \sigma} \left( F(\nu \sigma + \nu t) - F(\nu \sigma) \right),
\]

where \( F \) is a decreasing function determined by

\[
  F'(r) = -\left( 1 + \frac{1}{r} \right) e^{-r}, \quad r > 0.
\]

If we normalize by requiring that \( F(r) \to 0 \) as \( r \to \infty \), we can write

\[
  F(r) = e^{-r} - \text{Ei}(-r),
\]

(4.36)

where \( \text{Ei} \) is the standard exponential integral available in many software packages,

\[
  \text{Ei}(z) = -\int_{-\infty}^{z} \frac{e^{-\zeta}}{\zeta} d\zeta.
\]

(4.37)

The function \( \text{Ei}(z) \) has the following properties for real \( z \): it is negative and decreasing for \( z < 0 \), with

\[
  \lim_{z \to -\infty} \text{Ei}(z) = 0, \quad \lim_{z \to 0^-} \text{Ei}(z) = -\infty,
\]

and it is increasing for \( z > 0 \), with

\[
  \lim_{z \to 0^+} \text{Ei}(z) = -\infty, \quad \text{Ei}(z) = e^z \left( z^{-1} + \mathcal{O}(z^{-2}) \right) \quad \text{as } z \to \infty.
\]
There is a unique positive zero $z_0 \approx 0.372507$.

Since $\text{Ei}(-r) < 0$ for $r > 0$, we have

\[ F(r) > e^{-r}. \] (4.38)

Thus, the characteristics to the left of the shock are

\[ x = \xi(t) = -\ln\left(e^{-\xi(0)} + e^{\nu \sigma}\left(F(v \sigma + vt) - F(v \sigma)\right)\right). \] (4.39)

This is an increasing function of $\xi(0)$, so different characteristic curves never intersect. Note that the expression in brackets,

\[ e^{-\xi(0)} + e^{\nu \sigma}\left(F(v \sigma + vt) - F(v \sigma)\right), \]

is a decreasing function (which agrees with $\xi(t)$ being increasing), and it tends to 

\[ e^{-\xi(0)} - e^{\nu \sigma} F(v \sigma) \]

as $t \to \infty$. Since $e^{\nu \sigma} F(v \sigma) > 1$ by (4.38), there is exactly one initial value

\[ \xi(0) = -\ln\left(e^{\nu \sigma} F(v \sigma)\right) = -\ln\left(1 - e^{\nu \sigma} \text{Ei}(-v \sigma)\right) =: \Xi < 0 \] (4.40)

for which this limit equals zero. This particular characteristic curve,

\[ x = \xi(t) = -\ln\left(e^{\nu \sigma} F(v \sigma + vt)\right), \] (4.41)

therefore tends to $\infty$ as $t \to \infty$, and by (4.38) it satisfies

\[ \dot{\xi}(t) < -\ln\left(e^{\nu \sigma} e^{-v \sigma \Xi}\right) = v t, \]

so that it always stays to the left of the shock (which travels along the curve $x = vt$). In fact, asymptotically it catches up with the shock:

\[ \xi(t) = -\ln\left(e^{\nu \sigma} \left(e^{-(v \sigma + vt)} - \text{Ei}(-(v \sigma + vt))\right)\right) \]

\[ = -\ln\left(e^{-vt} - e^{\nu \sigma} \text{Ei}(-v \sigma - vt)\right) \]

\[ = vt + o(1), \quad \text{as } t \to \infty. \]

The characteristic curves to the left of this special curve, i.e., those with $\xi(0) < \Xi$, asymptotically slow down to a halt,

\[ \lim_{t \to \infty} \xi(t) = -\ln\left(e^{-\xi(0)} - e^{-\Xi}\right), \]

while the curves with $\Xi < \xi(0) < 0$ have a finite lifespan; they cease to exist when they cross the shock curve $x = vt$, which they must do, since the expression for $\xi(t)$ tends to $\infty$ in finite time.

Similarly, the characteristics starting to the right of the shock ($\xi(0) > 0$) satisfy

\[ \dot{\xi} = \left(v - \frac{1}{\sigma + t}\right) e^{v t - \xi}, \]
so

\[
e^{\xi(t)} - e^{\xi(0)} = \int_0^t \left( v - \frac{1}{\sigma + \tau} \right) e^{\nu \tau} \, d\tau
\]

\[
= \int_0^\nu t \left( 1 - \frac{1}{\nu \sigma + s} \right) e^s \, ds
\]

\[
= e^{-\nu \sigma} \int_{\nu \sigma}^{\nu \sigma + \nu t} \left( 1 - \frac{1}{r} \right) e^r \, dr
\]

\[
= e^{-\nu \sigma} \left( G(\nu \sigma + \nu t) - G(\nu \sigma) \right),
\]

(4.42)

where \( G(z) \) is uniquely determined up to an additive constant by

\[
G'(r) = \left( 1 - \frac{1}{r} \right) e^r, \quad r > 0.
\]

For definiteness, let us take

\[
G(r) = e^r - \text{Ei}(r) \quad (= F(-r)).
\]

(4.43)

Thus, the characteristic curves to the right of the shock are

\[
x = \xi(t) = \ln \left( e^{\xi(0)} + e^{-\nu \sigma} \left( G(\nu \sigma + \nu t) - G(\nu \sigma) \right) \right).
\]

(4.44)

This is an increasing function of \( \xi(0) \), so different characteristic curves never intersect. Note that \( G(r) \) is decreasing for \( 0 < r \leq 1 \) and increasing for \( r \geq 1 \). This means that \( \xi(t) \) is increasing for all \( t \geq 0 \) if \( \sigma \nu \geq 1 \), and initially decreasing and then increasing if \( \sigma \nu < 1 \). This is as expected, since \( v < 1/\sigma \) means that the right part of the shock peakon formed at the collision dips down to \( u < 0 \) (recall that \( u(x^+,0) = v + \frac{1}{2} \)), making the characteristics to the right of the shock go left until the shock strength has decayed enough for \( u \) to be positive everywhere. Thus, all these characteristics turn from left to right at the same instant, namely when \( \inf_{x \in \mathbb{R}} u(x,t) = v - \frac{1}{\sigma + t} \) becomes zero so that \( u \) is identically zero to the right of the shock, i.e., when \( t = \frac{1}{v} - \sigma \); this can be seen in Figure 12.

These curves all have a finite lifespan, since the shock catches up with each one of them sooner or later. Indeed,

\[
vt < \xi(t)
\]

\[
\iff e^{\nu t} < e^{\xi(0)} + e^{-\nu \sigma} \left( G(\nu \sigma + \nu t) - G(\nu \sigma) \right)
\]

\[
\iff e^{\nu \sigma + \nu t} - G(\nu \sigma + \nu t) < e^{\nu \sigma + \xi(0) - G(\nu \sigma)}
\]

\[
\iff \text{Ei}(\nu \sigma + \nu t) < e^{\nu \sigma + \xi(0) - G(\nu \sigma)}.
\]

Here, the right-hand side is a constant greater than \( e^{\nu \sigma + 0} - G(\nu \sigma) = \text{Ei}(\nu \sigma) \), and the left-hand side increases from \( \text{Ei}(\nu \sigma) \) to \( \infty \) as \( t \) goes from 0 to \( \infty \), so there is a unique value of \( t \) (depending on \( \xi(0) \), and of course also on \( \nu \sigma \)) for which the shock reaches the characteristic curve.
5 Novikov ghostpeakons

The derivation of the ghostpeakon formulas for Novikov’s equation is rather similar to what we have seen for the Camassa–Holm and Degasperis–Procesi equations in Theorem 3.4 and Theorem 4.1 – especially the latter, as far as notation is concerned. As in the DP case, we formulate the theorems for pure peakon solutions first, and comment on their validity for mixed peakon–antipeakon solutions in Remark 5.3.

**Theorem 5.1.** Fix some $p$ with $0 \leq p \leq N$. The solution of the Novikov $(N+1)$-peakon ODEs (1.19) with $x_1(0) < \cdots < x_N(0)$ and all amplitudes $m_k(t)$ positive except for $m_{N+1-p}(t) = 0$ is as follows: the position of the ghostpeakon is given by

$$x_{N+1-p}(t) = \frac{1}{2} \ln \frac{Z_{p+1} + \theta Z_p}{W_p + \theta W_{p-1}}, \quad 0 < \theta < \infty,$$

(5.1)

while the other peakons are given by the general solution formulas (2.25) up to renumbering:

$$x_{N+1-k}(t) = \begin{cases} \frac{1}{2} \ln \frac{Z_{k+1}}{W_k}, & 0 \leq k < p, \\ \frac{1}{2} \ln \frac{Z_k}{W_{k-1}}, & p < k \leq N, \end{cases}$$

(5.2)

$$m_{N+1-k}(t) = \begin{cases} \sqrt{Z_{k+1}W_k}, & 0 \leq k < p, \\ \sqrt{Z_kW_{k-1}}U_kU_{k-1}, & p < k \leq N. \end{cases}$$

**Proof.** Consider the reparametrization

$$\epsilon = \frac{1}{\sqrt{\lambda_{N+1}}}, \quad \theta = \lambda_{N+1}^{2p-1} b_{N+1}(0)^2,$$

(5.3)

which (for $\epsilon > 0$) is equivalent to

$$\lambda_{N+1} = \frac{1}{\epsilon^2}, \quad b_{N+1}(0) = \epsilon^{2p-1} \sqrt{\theta}.$$

(5.4)

The same computation as in the proof of Theorem 4.1, but with $\epsilon^2$, $p - \frac{1}{2}$ and $\sqrt{\theta}$ instead of $\epsilon$, $p$ and $\theta$, gives

$$\tilde{U}_k = U_k^a + U_{k-1}^a \sqrt{\Theta} \epsilon^{2(p-k-a)+1} (1 + O(\epsilon)) \quad (\text{as } \epsilon \to 0),$$

(5.5)

Thus, the general solution formulas (2.25) are valid for the Novikov $(N+1)$-peakon ODEs (1.19) with $0 \leq p \leq N$.
where $\Theta = \Theta(t) = \lambda_{N+1}^{2p-1} b_{N+1}(t)^2 = \theta e^{2\epsilon t}$, and thus

$$W_k = \begin{vmatrix} U^{0}_{k+1} + U^{0}_{k-1} \sqrt{\Theta} e^{2(p-k)+1} & 1 + \Theta(\epsilon) \\ U^{1}_{k+1} + U^{1}_{k-2} \sqrt{\Theta} e^{2(p-k)+1} & U^{1}_{k-1} + U^{1}_{k-2} \sqrt{\Theta} e^{2(p-k)+1} + \Theta(\epsilon) \end{vmatrix}$$

$$= W_k + \begin{vmatrix} U^{0}_{k-1} - U^{1}_{k-2} \sqrt{\Theta} e^{2(p-k)+1} & 1 + \Theta(\epsilon) \\ U^{0}_{k+1} - U^{1}_{k-2} \sqrt{\Theta} e^{2(p-k)+1} & U^{1}_{k-1} - U^{1}_{k-2} \sqrt{\Theta} e^{2(p-k)+1} + \Theta(\epsilon) \end{vmatrix}$$

$$= \begin{cases} W_k + \Theta(\epsilon), & k < p, \\
W_k - \Theta(\epsilon) - \Theta(\epsilon), & k = p, \\
W_k - \Theta(\epsilon) - \Theta(\epsilon), & k > p. \end{cases}$$

Similarly,

$$\tilde{Z}_k = \begin{vmatrix} U^{0}_{k-1} - U^{1}_{k-1} \sqrt{\Theta} e^{2(p-k)+3} & 1 + \Theta(\epsilon) \\ U^{0}_{k+1} + U^{0}_{k-1} \sqrt{\Theta} e^{2(p-k)+1} + \Theta(\epsilon) \\ U^{0}_{k+1} + U^{1}_{k-1} \sqrt{\Theta} e^{2(p-k)+1} + \Theta(\epsilon) \end{vmatrix}$$

$$= \begin{cases} Z_k + \Theta(\epsilon), & k < p + 1, \\
Z_{p+1} + Z_{p-1} + \Theta(\epsilon), & k = p + 1, \\
Z_{k-1} - \Theta(\epsilon) - \Theta(\epsilon), & k > p + 1. \end{cases}$$

Thus, the peakon solution formulas reduce to

$$x_{N+1-k}(t) = \frac{1}{2} \ln \frac{\tilde{Z}_{k+1}}{W_k} = \begin{cases} \frac{1}{2} \ln \frac{Z_{k+1} + \Theta(\epsilon)}{W_k + \Theta(\epsilon)}, & k < p, \\
\frac{1}{2} \ln \frac{Z_{p+1} + Z_{p-1} + \Theta(\epsilon)}{W_k + W_{k-1} + \Theta(\epsilon)}, & k = p, \\
\frac{1}{2} \ln \frac{Z_{k-1} + \Theta(\epsilon)}{W_k + \Theta(\epsilon)}, & k > p. \end{cases}$$
and
\[
m_{N+1-k}(t) = \sqrt{Z_{k+1}W_k} \frac{U_{k+1}U_k}{U_{k+1}U_k} \begin{cases} \frac{\sqrt{(Z_{k+1} + \theta(\epsilon))(W_k + \theta(\epsilon))}}{U_{k+1} + \theta(\epsilon)} U_{k+1} \theta(\epsilon), & k < p, \\ \varepsilon \sqrt{(Z_{p+1} + Z_p \sqrt{\theta(\epsilon)})(W_p + W_{p-1} \sqrt{\theta(\epsilon)})} \frac{(U_p \sqrt{\theta(\epsilon)})(U_p + \theta(\epsilon))}{U_{k} \sqrt{\theta(\epsilon)}U_{k-1} \sqrt{\theta(\epsilon)}}, & k = p, \\ \frac{\sqrt{(Z_k \theta + \theta(\epsilon))(W_{k-1} \theta + \theta(\epsilon))}}{(U_k \sqrt{\theta(\epsilon)})(U_k-1 \sqrt{\theta(\epsilon)})}, & k > p. \end{cases}
\]

It follows that \( m_{N+1-p} \to 0 \) as \( \varepsilon \to 0 \), while the other expressions tend to those given in (5.1) and (5.2).

**Corollary 5.2.** For the Novikov pure \( N \)-peakon solution given by (2.25),

\[
x_{N+1-k}(t) = \frac{1}{2} \ln \frac{Z_k}{W_{k-1}}, \quad m_{N+1-k}(t) = \sqrt{Z_kW_{k-1}} \frac{U_k}{U_{k-1}}, \quad 1 \leq k \leq N,
\]

the characteristic curves \( x = \xi(t) \) in the \( k \)th interval from the right,

\[
x_{N-k}(t) < \xi(t) < x_{N+1-k}(t), \quad 0 \leq k \leq N,
\]

are given by

\[
\xi(t) = \frac{1}{2} \ln \frac{Z_{k+1} + \theta Z_k}{W_{k+1} + \theta W_k}, \quad \theta > 0.
\]

**Remark 5.3.** Pure antipeakon solutions are obtained from pure peakon solutions by changing \( u \) to \(-u\), which in terms of spectral data is effected by keeping the eigenvalues \( \lambda_k \) and changing the sign of every \( b_k \). In the proof of Theorem 5.1, we see from (5.4) that we can change the sign of \( b_{N+1} \) by letting \( \varepsilon \to 0^- \) instead of \( 0^+ \), i.e., we take \( \varepsilon = -1/\sqrt{\lambda_{N+1}} \) in (5.3). Everything else is unchanged, including the positivity of \( \theta \), so the ghostpeakon formulas are valid without change for pure antipeakon solutions. Similarly it is seen that they are valid for (conservative) peakon–antipeakon solutions, in the generic case where the (possibly complex) eigenvalues are simple and have positive real part; such solutions are still given by the usual formulas (2.25). For every such solution \( u \) there is a corresponding solution \(-u\) obtained by changing the signs of all \( b_k \), and the two are handled by taking \( \varepsilon = \pm 1/\sqrt{\lambda_{N+1}} \). Unlike the DP case, there is never any need to use negative \( \theta \).

Novikov’s equation also admits a great variety of non-generic peakon–antipeakon solutions, where some eigenvalues may have multiplicity greater than one and/or lie on the imaginary axis. Those solutions are described by separate formulas (see Kardell and Lundmark [38]), where the quantities \( U_k^\theta \) (and consequently \( W_k \) etc.) are modified via limiting procedures that commute with the limit \( \varepsilon \to 0 \) considered here. It follows that the ghostpeakon formulas are valid also for solutions of this kind, provided only that all \( W_k \) and \( Z_k \) are replaced with their suitably modified counterparts.

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The Novikov N-peakon solution

Theorem 5.4. As already explained in Section 1.6, we can use $u$ information about the sign of $u$ cases, but we need another formula for $u$ to make parametric plots of multipeakon solutions, where $\xi = u(\xi)^2$ gives us no information about the sign of $u(\xi)$.

**Theorem 5.4.** The Novikov N-peakon solution (2.25) satisfies

$$u(x_{N+1-k}) = \frac{Y_k}{\sqrt{Z_k W_{k-1}}}, \quad (5.6)$$

where

$$Y_k = \begin{vmatrix} T_k & V_{k-2} \\ T_{k+1} & V_{k-1} \end{vmatrix} = \begin{vmatrix} U_{k-1}^1 & U_{k-2}^1 \\ U_{k+1}^1 & U_{k-1}^1 \end{vmatrix}. \quad (5.7)$$

Moreover, if $x = \xi(t)$ is the characteristic curve (5.5), then

$$u(\xi) = \frac{Y_{k+1} + \theta Y_k}{\sqrt{(Z_{k+1} + \theta Z_k)(W_k + \theta W_{k-1})}}. \quad (5.8)$$

**Proof.** Equation (5.6) is proved by direct computation (with empty sums interpreted as zero):

$$u(x_{N+1-k}) = \sum_{i=1}^{N} m_{N+1-i} e^{-|x_{N+1-k} - x_{N+1-i}|}$$

$$= \sum_{1 \leq i < k} m_{N+1-i} \frac{e^{x_{N+1-k}}}{e^{x_{N+1-i}}} + m_{N+1-k} + \sum_{k \leq i \leq N} m_{N+1-i} \frac{e^{x_{N+1-k}}}{e^{x_{N+1-i}}}$$

$$= \sum_{1 \leq i < k} \sqrt{Z_i W_{i-1}} \frac{W_{i-1}}{U_i U_{i-1}} + \sqrt{Z_k W_{k-1}} \frac{W_{k-1}}{U_k U_{k-1}} + \sum_{k \leq i \leq N} \sqrt{Z_i W_{i-1}} \frac{W_{i-1}}{U_i U_{i-1}}$$

$$= \sqrt{Z_k W_{k-1}} \sum_{1 \leq i < k} \left( \frac{V_{i-1}}{U_i} - \frac{V_{i-2}}{U_{i-1}} \right) + \sqrt{Z_k W_{k-1}} \frac{W_{k-1}}{U_k U_{k-1}} + \sqrt{Z_k} \sum_{k \leq i \leq N} \left( \frac{T_i}{U_{i-1}} - \frac{T_{i+1}}{U_i} \right)$$

$$= \sqrt{Z_k W_{k-1}} \sum_{1 \leq i < k} \left( \frac{V_{i-2}}{U_{i-1}} + \frac{Z_k W_{k-1}}{U_k U_{k-1}} \right) + \sqrt{Z_k} \sum_{k \leq i \leq N} \left( \frac{W_{k-1}}{U_k} \right)$$

$$= \frac{Z_k U_k V_{k-2} + Z_k W_{k-1} + W_{k-1} U_{k-1} T_{k+1}}{\sqrt{Z_k W_{k-1}} U_k U_{k-1}}$$

(5.6)
where the numerator is
\[ Z_k(U_k V_{k-2} + W_{k-1}) + W_{k-1} U_{k-1} T_{k+1} \]
\[ = Z_k U_{k-1} V_{k-1} + W_{k-1} U_{k-1} T_{k+1} \]
\[ = U_{k-1}(Z_k V_{k-1} + W_{k-1} T_{k+1}) \]
\[ = U_{k-1}(T_k U_k - T_{k+1} U_{k-1}) V_{k-1} + (U_{k-1} V_{k-1} - U_k V_k) T_{k+1} \]
\[ = U_{k-1} U_k (T_k V_{k-1} - T_{k+1} V_{k-2}) \]
\[ = U_{k-1} U_k Y_k, \]
as desired.

Regarding (5.8) we proceed as in the proof of Theorem 5.1, i.e., we consider the \((N + 1)\)-peakon solution and make again the same substitutions as in that proof, so that the peakon at \(x_{N+1-p}\) will turn into a ghostpeakon as \(\epsilon \to 0\). For the \((N + 1)\)-peakon solution, we have just showed that
\[ u(x_{N+1-p}) = u(x_{(N+1)-1-(p+1)}) = \frac{\tilde{Y}_{p+1}}{\sqrt{Z_{p+1} W_p}}, \]
where we know from the previous proof that
\[ Z_{p+1} = Z_{p+1} + Z_p \Theta + \Theta (\epsilon), \quad W_p = W_p + W_{p-1} \Theta + \Theta (\epsilon), \]
and in the same way we derive
\[ \tilde{Y}_{p+1} = \frac{\tilde{Y}_{p+1}}{\tilde{Y}_{p+2}} - \frac{\tilde{Y}_p}{\tilde{Y}_{p+1} - \tilde{Y}_p} \]
\[ = \frac{U_{p+1} + U_{p-1} \sqrt{\Theta} (1 + \Theta (\epsilon))}{U_p + U_{p-1} \sqrt{\Theta} (1 + \Theta (\epsilon))} \]
\[ = Y_{p+1} + \Theta Y_p + \Theta (\epsilon). \]

Letting \(\epsilon \to 0\) and relabelling as in Corollary 5.2, we thus obtain (5.8).

\[ \Box \]

**Example 5.5.** As an illustration, consider the solution of the \(N = 5\) Novikov peakon ODEs which is given by the explicit formulas (2.25) with the complex spectral data
\[ \lambda_1 = \lambda_2 = \frac{1}{2} + \frac{i}{2}, \quad \lambda_3 = \lambda_4 = \frac{1}{2} + i, \quad \lambda_5 = 1, \quad b_k(0) = 1 \quad (1 \leq k \leq 5). \]  
(5.9)
This is a peakon-antipeakon solution, where \(m_k(t)\) and \(m_{k+1}(t)\) blow up at those instants \(t = t_c\) when a collision \(x_k(t_c) = x_{k+1}(t_c)\) occurs. As always, \(u(x, t) = \sum_{i=1}^{N} m_i(t) e^{-|x-x_i(t)|}\) extends by continuity to a globally defined conservative weak solution, which is given by the formulas in Theorem 5.4 without any singularities, since the troublesome factor \(U_{k-1} U_k\) was cancelled in the proof. The graph \(u = u(x, t)\) is plotted from those formulas in Figure 13, while the peakon trajectories \(x = x_k(t)\) are shown in Figure 14 and on a larger domain in Figure 15.
Figure 13. A conservative peakon–antipeakon solution of Novikov’s equation, with $N = 5$ and spectral data (5.9), as described in Example 5.5. A peakon with velocity 1 interacts with a breather-like cluster of four peakons/antipeakons which velocity 1/2. The peakon trajectories $x = x_k(t)$ are shown in Figure 14. The dimensions of the box are $|x| \leq 12, |t| \leq 12, |u| \leq 1.2$.

For $1 \leq k \leq 4$, the reciprocal eigenvalues $1/\lambda_k$ all have real part 1/2, which means that as $t \to \pm\infty$ one will see a cluster of four peakons/antipeakons travelling together, with the positions $x_k(t)$ displaying oscillations superimposed on an overall drift with velocity 1/2. The peakons and antipeakons in the cluster interact in a breather-like manner, with infinitely many collisions taking place, in this case (asymptotically periodically since $\text{Im}(1/\lambda_1) = \frac{1}{2}$ and $\text{Im}(1/\lambda_3) = 1$ are commensurable, but quasi-periodic oscillations are also possible.

At collisions, the peakon and the antipeakon approach each other very closely; for the Novikov equation, the leading term in $x_{k+1}(t) - x_k(t)$ as $t \to t_c$ is a positive constant times $(t - t_c)^j$, where $j = 1$ (fourth power) is the generic case, although “higher-order collisions” with $j > 1$ are also possible. In contrast, for Camassa–Holm collisions the leading term is always a positive constant times the second power $(t - t_c)^2$, as shown by Beals, Sattinger and Szmigielski [2].

Since $1/\lambda_5 = 1$, there will also be a lone peakon with asymptotic velocity 1. As seen in Figure 15, the periodic pattern of the 4-peakon cluster as $t \to +\infty$ (after interaction with the single peakon) does not look the same as when $t \to -\infty$ (before the interaction), so the peakon scattering process involves more than simply a phase shift in this case. We refer to Kardell and Lundmark [38] for details.
Figure 14. Peakon trajectories $x = x_k(t)$ for the $N=5$ conservative peakon–antipeakon Novikov solution shown in Figure 13. The curve $x = x_2(t)$ is shown in blue and $x = x_4(t)$ in red. The strict ordering $x_1 < x_2 < x_3 < x_4 < x_5$ holds for almost all $t$, the exceptions being the isolated (but infinitely many) instants $t_c$ when a peakon–antipeakon collision occurs: $x_k(t_c) = x_{k+1}(t_c)$. Near a collision, the curves stay very close together, and therefore they appear to overlap in the figure; in fact $x_{k+1}(t) - x_k(t)$ is approximately a constant times $(t - t_c)^2$. See Example 5.5.
**Figure 15.** The same as Figure 14, but on a larger domain. Note that the four-peakon cluster which emerges after the interaction with the single peakon is not merely a shifted version of the cluster seen before the interaction – the pattern of oscillation within the cluster has also changed.
6 Preview: Non-interlacing Geng–Xue peakons

As mentioned in Section 1.7, ghostpeakon-inspired methods can be used to derive the general peakon solution of the two-component Geng–Xue equation (1.29). This will be explained fully in a separate paper. Here we only give an example to illustrate what the results look like in a relatively small case.

Example 6.1. Consider the configuration with two peakons in \( u(x, t) \) at \( x_1 \) and \( x_2 \), followed by two peakons in \( v(x, t) \) at \( x_3 \) and \( x_4 \), where \( x_1 < x_2 < x_3 < x_4 \):

\[
\begin{align*}
  u(x, t) &= m_1(t) e^{-|x-x_1(t)|} + m_2(t) e^{-|x-x_2(t)|}, \\
  v(x, t) &= n_3(t) e^{-|x-x_3(t)|} + n_4(t) e^{-|x-x_4(t)|}.
\end{align*}
\]  

(6.1)

The governing ODEs for the 8 unknowns \( x_1, x_2, x_3, x_4, m_1, m_2, n_3, n_4 \) are

\[
\begin{align*}
  \dot{x}_k &= u(x_k) v(x_k), \\
  \dot{m}_k &= m_k \left( u(x_k) v_x(x_k) - 2u_x(x_k) v(x_k) \right), \\
  \dot{n}_k &= n_k \left( u_x(x_k) v(x_k) - 2u(x_k) v_x(x_k) \right),
\end{align*}
\]  

(6.2)

which is shorthand for

\[
\begin{align*}
  \dot{x}_1 &= (m_1 + m_2 E_{12})(n_3 E_{13} + n_4 E_{14}), \\
  \dot{x}_2 &= (m_1 E_{12} + m_2)(n_3 E_{23} + n_4 E_{24}), \\
  \dot{x}_3 &= (m_1 E_{13} + m_2 E_{23})(n_3 + n_4 E_{34}), \\
  \dot{x}_4 &= (m_1 E_{14} + m_2 E_{24})(n_3 E_{34} + n_4), \\
  \dot{m}_1 &= m_1 \left( (m_1 + m_2 E_{12})(n_3 E_{13} + n_4 E_{14}) - 2m_2 E_{12}(n_3 E_{13} + n_4 E_{14}) \right) \\
  &= m_1(m_1 - m_2 E_{12})(n_3 E_{13} + n_4 E_{14}), \\
  \dot{m}_2 &= m_2 \left( (m_1 E_{12} + m_2)(n_3 E_{23} + n_4 E_{24}) - 2(-m_1 E_{12})(n_3 E_{23} + n_4 E_{24}) \right) \\
  &= m_1(3m_1 E_{12} + m_2)(n_3 E_{23} + n_4 E_{24}), \\
  \dot{n}_3 &= n_3 \left( (-m_1 E_{13} - m_2 E_{23})(n_3 + n_4 E_{34}) - 2(m_1 E_{13} + m_2 E_{23})n_4 E_{34} \right) \\
  &= n_3(m_1 E_{13} + m_2 E_{23})(-n_3 - 3n_4 E_{34}), \\
  \dot{n}_4 &= n_4 \left( (-m_1 E_{14} - m_2 E_{24})(n_3 E_{34} + n_4) - 2(m_1 E_{14} + m_2 E_{24})(-n_3 E_{34}) \right) \\
  &= n_4(m_1 E_{14} + m_2 E_{24})(n_3 E_{34} - n_4),
\end{align*}
\]  

(6.3)

where \( E_{ij} = e^{-|x_i-x_j|} = e^{x_i-x_j} \) for \( i < j \).

Here we are explicitly seeking non-ghost solutions of (6.3), i.e., \( m_1, m_2, n_3, n_4 \) are supposed to be nonzero; otherwise we would consider a different peakon configuration from the outset (with fewer terms in the multipeakon ansatz). For simplicity, let us also assume that these amplitudes are positive, so that we get a globally defined solution and don't have to deal with the subtle question of peakon–antipeakon collisions, where in general shockpeakons will form [47].

The solution formulas will be given in (6.13) below, but let us first see what we would obtain if we were to apply the inverse spectral setup involving two different Lax pairs for the
Geng–Xue equation [46]. We do not go into any details, but this produces two time-invariant polynomials

\[ A(\lambda) = 1 - 2\lambda (m_1 e^{x_1} + m_2 e^{x_2})(n_3 e^{-x_3} + n_4 e^{-x_4}), \quad \bar{A}(\lambda) = 1, \]  

(6.4)

and two auxiliary polynomials

\[ B(\lambda) = m_1 e^{x_1} + m_2 e^{x_2}, \quad \bar{B}(\lambda) = n_3 e^{x_3} + n_4 e^{x_4}, \]  

(6.5)

which are such that the Weyl functions

\[ W(\lambda) = -\frac{B(\lambda)}{A(\lambda)} = \frac{a_1}{\lambda - \lambda_1}, \quad \bar{W}(\lambda) = -\frac{\bar{B}(\lambda)}{\bar{A}(\lambda)} = -b_\infty \]  

(6.6)

have a simple time evolution given by

\[ \dot{\lambda}_1 = 0, \quad \dot{a}_1 = \frac{a_1}{\lambda_1}, \quad \dot{b}_\infty = 0. \]  

(6.7)

We thus get two constants of motion

\[ \lambda_1 = \frac{1}{2(m_1 e^{x_1} + m_2 e^{x_2})(n_3 e^{-x_3} + n_4 e^{-x_4})}, \quad b_\infty = n_3 e^{x_3} + n_4 e^{x_4}, \]  

(6.8)

plus the fact that the quantity

\[ a_1 = \frac{1}{2(n_3 e^{-x_3} + n_4 e^{-x_4})} \]  

(6.9)

evolves as

\[ a_1(t) = a_1(0) e^{t/\lambda_1}. \]  

(6.10)

There is also another constant of motion

\[ b_\infty^* = m_1 e^{-x_1} + m_2 e^{-x_2} \]  

(6.11)

coming from an adjoint spectral problem (or in this case directly from the symmetry of the setup).

Notice that the expressions for \( \dot{x}_2 \) and \( \dot{x}_3 \) in (6.3) are in fact equal, and moreover can be expressed in terms of the constant of motion \( \lambda_1 \):

\[ \dot{x}_2 = \dot{x}_3 = (m_1 e^{x_1} + m_2 e^{x_2})(n_3 e^{-x_3} + n_4 e^{-x_4}) = \frac{1}{2\lambda_1}. \]  

This allows us to integrate those equations:

\[ x_2(t) = \frac{t}{2\lambda_1} + x_2(0), \quad x_3(t) = \frac{t}{2\lambda_1} + x_3(0). \]  

(6.12)

However, at this point there is no obvious additional information which would allow us to continue and compute the time dependence of all our 8 unknowns individually, so without some further bright ideas we are stuck.
In the interlacing case (say with $K + K$ peakons), where the peakons in $u$ and $v$ occur alternatingly, the polynomials $A, \bar{A}, B, \bar{B}$ have higher degrees than what we are seeing here, and then the Weyl functions define eigenvalues

$$\{\lambda_i\}_{i=1}^K, \quad \{\mu_j\}_{j=1}^{K-1}$$

which are constants of motion, and residues

$$\{a_i\}_{i=1}^K, \quad \{b_j\}_{j=1}^{K-1}$$

which evolve according to $\dot{a}_i = a_i/\lambda_i$ and $\dot{b}_j = b_j/\mu_j$. Together with the constants of motion $b_\infty$ and $b_\infty^*$, we therefore get $4K$ spectral variables, which turn out to determine the $4K$ peakon variables uniquely.

But whenever there is non-interlacing (i.e., if two or more peakons in $u$ appear next to each other, or similarly for $v$) then the polynomial degrees drop, and the Lax pairs do not provide us with enough constants of motion for the ODEs to be explicitly solvable; it’s just that we don’t seem to get them from the Lax pairs. (At least not directly, by applying the setup from the interlacing case.)

To obtain the general pure peakon solution of (6.3), we use instead the strategy described in Section 1.7: we start from the known formulas for the $3 + 3$ interlacing solution [46, 47], with $m_1, n_2, m_3, n_4, m_5, n_6$ positive, and use methods similar to those explained earlier in this paper to deduce the solution in the limiting case when $m_5 \to 0$ and $n_2 \to 0$. We can then discard the variables with indices 2 and 5 (which now refer to ghostpeakons), and relabel the remaining indices $(1, 3, 4, 6)$ to $(1, 2, 3, 4)$. This leaves the following formulas:

$$x_4 = \frac{1}{2} \ln \left( \frac{2(C_2 + C_3 + K_2)a_1 + 2K_2C_4}{a_1 + C_4} \right),$$

$$x_3 = \frac{1}{2} \ln \left( \frac{2(C_2 + C_3)a_1}{a_1 + C_4} \right),$$

$$x_2 = \frac{1}{2} \ln \left( \frac{2C_2a_1}{C_1 + K_1(a_1 + C_1)} \right),$$

$$x_1 = \frac{1}{2} \ln \left( \frac{2C_1a_1}{C_1 + K_1(a_1 + C_1)} \right),$$

$$n_4 = \frac{1}{2} e^{x_4} \cdot \frac{1}{a_1 + C_4},$$

$$n_3 = \frac{1}{2} e^{x_3} \cdot \frac{C_4}{a_1(a_1 + C_4)},$$

$$m_2 = \frac{1}{2} e^{x_2} \cdot \frac{a_1}{C_2 \lambda_1 (a_1 + C_1)},$$

$$m_1 = \frac{1}{2} e^{x_1} \cdot \frac{C_1 + K_1(a_1 + C_1)}{\lambda_1 C_2 (a_1 + C_1)},$$

(6.13)

where the time dependence comes from $a_1 = a_1(t) = a_1(0) e^{t/\lambda_1}$ (cf. (6.10)), and where the eight constants $a_1(0), \lambda_1, C_1, C_2, C_3, C_4, K_1, K_2$ are positive parameters. Comparison to (6.8) and (6.11) shows that

$$b_\infty = n_3 e^{x_4} + n_4 e^{x_4} = C_2 + C_3 + K_2, \quad b_\infty^* = m_1 e^{-x_4} + m_2 e^{-x_2} = \frac{1 + K_1}{\lambda_1 C_2},$$

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so $K_1$ and $K_2$ are related to the previously known constants of motion $b_{\infty}$ and $b_{\infty}^*$, but we also have the following four constants of motion which do not seem to come from the Lax pairs:

$$\frac{C_1}{\lambda_1} = \frac{1}{m_1 e^{x_1} - \frac{1}{a_1}} = \frac{m_1 e^{x_1} (m_1 e^{x_1} + m_2 e^{x_2})}{m_2 e^{x_2}},$$

$$2C_2 \lambda_1 = \frac{1}{2} e^{2x_2} \frac{2 \lambda_1}{a_1} = \frac{e^{2x_2}}{m_1 e^{x_1} + m_2 e^{x_2}},$$

$$C_2 + C_3 = \frac{1}{a_1} = \frac{n_3 e^{-x_3} + n_4 e^{-x_4}}{e^{-2x_3}},$$

$$2C_4 = 2 \left( \frac{1}{2n_4 e^{-x_4} - a_1} \right) = \frac{n_3 e^{-x_3}}{n_4 e^{-x_4} (n_3 e^{-x_3} + n_4 e^{-x_4})}.$$ (6.14)

Concerning the dynamics of this “maximally non-interlacing” $2 + 2$ peakon configuration, note in particular that peakons number 2 and 3 travel in parallel straight lines, with the constant velocity $(2\lambda_1)^{-1}$, as we saw in (6.12). With the solution written in the form (6.13), this becomes

$$x_2(t) = \frac{1}{2} \ln \left( 2 C_2 a_1(0) e^{t/\lambda_1} \right) = \frac{t}{2 \lambda_1} + \frac{\ln 2 + \ln a_1(0) + \ln C_2}{2},$$

$$x_3(t) = \frac{1}{2} \ln \left( 2 (C_2 + C_3) a_1(0) e^{t/\lambda_1} \right) = \frac{t}{2 \lambda_1} + \frac{\ln 2 + \ln a_1(0) + \ln (C_2 + C_3)}{2}. \quad (6.15)$$

For the positions of the other peakons, we immediately obtain the following asymptotics, by retaining the dominant terms in (6.13):

$$x_1(t) = \frac{1}{2} \ln \frac{2C_1 C_2}{K_1} + o(1),$$

$$x_4(t) = \frac{t}{2 \lambda_1} + \frac{\ln 2 + \ln a_1(0) + \ln (C_2 + C_3 + K_2)}{2} + o(1),$$

as $t \to \infty$, and

$$x_1(t) = \frac{t}{2 \lambda_1} + \frac{\ln 2 + \ln a_1(0) + \ln C_2 - \ln (1 + K_1)}{2} + o(1),$$

$$x_4(t) = \frac{1}{2} \ln \left( 2 K_2 C_4 \right) + o(1),$$

as $t \to -\infty$. Thus, as $t \to \infty$, peakon number 4 asymptotically travels in a straight line parallel to the path of peakons number 2 and 3, while peakon number 1 slows to a stop, and vice versa as $t \to -\infty$.

Note also that, in contrast to the other peakon equations treated in this paper, the amplitudes $m_1(t)$, $m_2(t)$, $n_3(t)$, $n_4(t)$ need not tend to constants as $t \to \pm \infty$, except that we always have

$$\lim_{t \to \infty} m_1(t) = \frac{1}{\lambda_1} \sqrt{\frac{C_1 K_1}{2C_2}}, \quad \lim_{t \to -\infty} n_4(t) = \sqrt{\frac{K_2}{2C_3}}.$$  

Instead, the amplitudes will typically grow or decay exponentially, although some of them may tend to constant values in borderline cases (for particular parameter values).
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