Multiple Fourier series and lattice point problems

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Abstract

For the multiple Fourier series of the periodization of some radial functions on $\mathbb{R}^d$, we investigate the behavior of the spherical partial sum. We show the Gibbs-Wilbraham phenomenon, the Pinsky phenomenon and the third phenomenon for the multiple Fourier series, involving the convergence properties of them. The third phenomenon is closely related to the lattice point problems, which is a classical theme of the analytic number theory. We also prove that, for the case of two or three dimension, the convergence problem on the Fourier series is equivalent to the lattice point problems in a sense. In particular, the convergence problem at the origin in two dimension is equivalent to Hardy’s conjecture on Gauss’s circle problem.

1 Introduction

It is well known as the Gibbs-Wilbraham phenomenon that, for the Fourier series of piecewise continuous functions, in the neighborhood of each jump, the partial sums overshoot the jump by approx 9% of the jump. This phenomenon can be seen not only in one dimension but also in higher dimensions (see for example [4, 24, 42]).

In one dimension, it is also well known as the localization property that, if the function is zero on an interval, then the Fourier series converges to zero there. However, in higher dimensions this property is no longer valid. In 1993, Pinsky, Stanton and Trapa [36] showed that, for the Fourier series of the indicator function of a $d$-dimensional ball with $d \geq 3$, the spherical partial sum diverges at the center of the ball. That is, the Fourier series diverges at a smooth point - even a point of
local constancy - of the function, resulting from global rather than local properties. This phenomenon is called the Pinsky phenomenon.

In 2010, the third phenomenon was discovered in [18, 22]. Namely, for the Fourier series of the indicator function of a $d$-dimensional ball with $d \geq 5$, the spherical partial sum diverges at all rational points, while it converges almost everywhere. This third phenomenon was proved by using results on the lattice point problems.

The study of lattice point problems is a classical theme of analytic number theory which is concerned with the number of integer points. It has a long history and deep accumulations since G. Voronoï, G. H. Hardy, E. Landau, J. G. Van der Corput, V. Jarník and A. Walfisz, see [5, 15, 16]. For example, in $\mathbb{R}^2 = \{(x_1, x_2) : x_1$ and $x_2$ are real numbers}, by $A(s)$ we denote the number of lattice points, where are points with integral co-ordinates, inside the circle

$$x_1^2 + x_2^2 = s,$$

see Figure 1. Then $A(s)$ is the same as the area of the polygon in Figure 2 since the polygon is the union of the unit squares whose centers are the lattice points inside the circle. Let $P(s) = A(s) - \pi s$, where $\pi s$ is the area of this circle. Gauss showed that

$$P(s) = O(s^{1/2}) \text{ as } s \to \infty.$$  

In the above $O$ is Landau’s symbol, that is, $f(s) = O(g(s))$ as $s \to \infty$ means that

$$\limsup_{s \to \infty} \frac{|f(s)|}{g(s)} < \infty$$  

for the positive valued function $g$. Similarly, $f(s) =
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\[ \text{o}(g(s)) \text{ as } s \to \infty \text{ means that } \lim_{s \to \infty} \frac{f(s)}{g(s)} = 0. \] In 1915 Hardy \[8\] proved that

\[ P(s) \neq O(s^\theta) \text{ if } \theta \leq 1/4. \]

In fact P(s) \(\neq o(s^{1/4} \log^{1/4} s)\). The best bound on \(\theta\) for \(P(s) = O(s^\theta)\) is a very famous open problem known as Gauss’s circle problem. While Hardy’s conjecture(\[9\]) is \(P(s) = O(s^{1/4+\varepsilon})\) for any \(\varepsilon > 0\), the most sharp result up to now is \(P(s) = O(s^{131/416}(\log s)^{18637/8320})\) by M. Huxley \[12\] in 2003, where 131/416 = 0.3149\ldots.

(Recently, Bourgain and Watt \[1\] gave \(\theta = 517/1648 = 0.31371\ldots\) in arXiv, 2017.)

Note that \(A(s)\) is a special case of

\[
\sum_{m_1^2 + \cdots + m_d^2 \leq s} \exp \left( 2\pi i \sum_{k=1}^{d} m_k x_k \right),
\]

where \(m_1, \ldots, m_d\) are integers and \(x_1, \ldots, x_d\) are real numbers, that is, \(A(s)\) is the case \((x_1, \ldots, x_d) = (0, \ldots, 0)\) of (1.1) and \(d = 2\). The sum (1.1) is related to the Fourier series. Especially the research on the sum (1.1) by Czechoslovakian mathematician B. Novák (1938–2003) is very important for the study of the convergence problem of multiple Fourier series.

Recently, Taylor \[43, 44\] found that the Pinsky phenomenon arises even in two-dimension. He treated the radial function

\[
U(x) = \begin{cases} 
\frac{1}{\sqrt{a^2 - |x|^2}}, & |x| < a, \\
0, & |x| \geq a, \end{cases} \quad x \in \mathbb{R}^2, \quad a > 0,
\]

Then \(U(x)\) is the fundamental solution to the wave equation on \(\mathbb{R} \times \mathbb{T}^2\), evaluated at \(t = a\). Our aim in this paper which is motivated by Taylor \[43, 44\] is to study the Gibbs-Wilbraham phenomenon, the Pinsky phenomenon and the third phenomenon on the Fourier series of

\[
U_{\beta,a}(x) = \begin{cases} 
(a^2 - |x|^2)^{\beta}, & |x| < a, \\
0, & |x| \geq a, \end{cases} \quad x \in \mathbb{R}^d, \quad a > 0, \quad \beta > -1,
\]

for \(\beta > -1\), \(a > 0\) and dimension \(d\), involving the convergence properties of them. If \(\beta = 0\), then \(U_{\beta,a}(x)\) is the same as the indicator function of the ball centered at the origin and of radius \(a\).
By $\mathbb{R}^d$, $\mathbb{Z}^d$ and $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ we denote the $d$-dimensional Euclidean space, integer lattice and torus, respectively. In this paper, however, we always identify $\mathbb{T}^d$ with $(-1/2, 1/2]^d$, that is, $x \in \mathbb{T}^d$ means $x \in (-1/2, 1/2]^d$ and $\mathbb{T}^d \subset \mathbb{R}^d$. Let $\mathbb{Q}$ be the set of all rational numbers, and let $\mathbb{Q}^d = \{(x_1, \cdots, x_d) : x_1, \cdots, x_d \in \mathbb{Q}\}$.

For an integrable function $F(x)$ on $\mathbb{R}^d$, its Fourier transform $\hat{F}(\xi)$ and its Fourier spherical partial integral $\sigma_\lambda(F)(x)$ of order $\lambda \geq 0$ are defined by

$$\hat{F}(\xi) = \int_{\mathbb{R}^d} F(x)e^{-2\pi i \xi x} \, dx, \quad \xi = (\xi_1, \cdots, \xi_d) \in \mathbb{R}^d, \quad (1.3)$$

$$\sigma_\lambda(F)(x) = \int_{|\xi| < \lambda} \hat{F}(\xi)e^{2\pi i \xi x} \, d\xi, \quad |\xi| = \sqrt{\sum_{k=1}^d \xi_k^2}, \quad x \in \mathbb{R}^d, \quad (1.4)$$

respectively, where $\xi x$ is the inner product $\sum_{k=1}^d \xi_k x_k$. Also, for an integrable function $f(x)$ on $\mathbb{T}^d$, its Fourier coefficients $\hat{f}(m)$ and its Fourier spherical partial sum $S_\lambda(f)(x)$ of order $\lambda \geq 0$ are defined by

$$\hat{f}(m) = \int_{\mathbb{T}^d} f(x)e^{-2\pi i m x} \, dx, \quad m = (m_1, \cdots, m_d) \in \mathbb{Z}^d, \quad (1.5)$$

$$S_\lambda(f)(x) = \sum_{|m| < \lambda} \hat{f}(m)e^{2\pi i m x}, \quad |m| = \sqrt{\sum_{k=1}^d m_k^2}, \quad x \in \mathbb{T}^d, \quad (1.6)$$

respectively.

For an integrable function $F(x)$ on $\mathbb{R}^d$, we consider its periodization

$$f(x) = \sum_{m \in \mathbb{Z}^d} F(x + m), \quad x \in \mathbb{T}^d, \quad (1.7)$$

Note that in (1.7) the series converges with respect to the $L^1$-norm on $\mathbb{T}^d$ and then $f$ is an integrable function on $\mathbb{T}^d$. Then it is known as the Poisson summation formula that the equation

$$\hat{f}(m) = \hat{F}(m), \quad m \in \mathbb{Z}^d \quad (1.8)$$

holds, see for example [40, Theorem 2.4 (page 251)]. The left hand side of (1.8) is defined by (1.5) and the right hand side of (1.8) is defined by (1.3) with $\xi = m$.

In particular, we denote by $u_{\beta,a}(x)$ the periodization of $U_{\beta,a}(x)$. That is,

$$u_{\beta,a}(x) = \sum_{m \in \mathbb{Z}^d} U_{\beta,a}(x + m), \quad x \in \mathbb{T}^d. \quad (1.9)$$
If $\beta = 0$, then $u_{0,a}$ is the periodization of the indicator function of the ball centered at the origin and of radius $a$. See Figure 3 and also Figure 4. Let

$$ u_{0,a}(x) = \sum_{m \in \mathbb{Z}^d} U_{0,a}(x + m), \quad x \in \mathbb{T}^d, $$

$$ U_{0,a}(x) = \begin{cases} 
1, & |x| < 1, \\
1/2, & |x| = 1, \\
0, & |x| > 1,
\end{cases} \quad x \in \mathbb{R}^d. $$

If $d = 2$, then

$$ S_\lambda(u_{0,a})(x) = \pi a^2 + a \sum_{0 < |m| < \lambda} \frac{J_1(2\pi a|m|)}{|m|} e^{2\pi i mx} $$

and

$$ \lim_{\lambda \to \infty} S_\lambda(u_{0,a})(0) = u_{0,a}(0), $$

see Remark 6.1. More precisely, we have

$$ \lim_{\lambda \to \infty} \left( \pi a^2 + a \sum_{0 < |m| < \lambda} \frac{J_1(2\pi a|m|)}{|m|} \right) = \sum_{|m| < a} 1 + \frac{1}{2} \sum_{|m| = a} 1. \quad (1.10) $$

The equation (1.10) is well known as Hardy’s identity [8]. See Corollary 6.3 for $\lim S_\lambda(u_{0,a})(x), \ x \in \mathbb{T}^2.$
We first show an identity (Theorem 2.1) for the periodization of any integrable radial function with compact support. Then it turns out that the difference between the Fourier partial sum and the Fourier partial integral is closely related to lattice point problems. Therefore, to study the convergence problem on the Fourier series of \( u_{\beta,a}(x) \) we must also investigate the behavior of \( \sigma_\lambda(U_{\beta,a})(x) \) as \( \lambda \to \infty \) and lattice point problems.

In the case \( \beta = 0 \), the convergence problem on the Fourier series of the function \( u_{0,a} \) has been studied in detail by [18, 19, 20, 21, 22, 23, 24, 36, 34, 35]. Some of results in these papers were proved by using Novák’s results in [28, 29, 30, 32]. His results were very useful and sufficient for the affirmative results of the convergence problem on the Fourier series of the function \( u_{0,a} \).

On the other hand, in the case \( -1 < \beta < 0 \), Novák’s results on the sum (1.1) are not sufficient to study the convergence problem on the Fourier series of \( u_{\beta,a} \). Our results on the convergence of the Fourier series of \( u_{\beta,a} \) are obtained by using the best estimates up to now on lattice point problems. Therefore, if the lattice point problems will be improved in the future, then our results can be also improved. Actually, the convergence problem on the Fourier series and the lattice point problems are equivalent in a sense as we will show in Section 7. In particular, the convergence problem at the origin in two dimension is equivalent to Hardy’s conjecture on Gauss’s circle problem, see Remark 7.1.

To state our main results, for \( a > 0 \), let

\[
E_a = \{ x \in \mathbb{T}^d : x \neq 0 \text{ and } |x - m| \neq a \text{ for all } m \in \mathbb{Z}^d \}, \tag{1.11}
\]
\[
G_a = \{ x \in \mathbb{T}^d : x \neq 0 \text{ and } |x - m| = a \text{ for some } m \in \mathbb{Z}^d \}. \tag{1.12}
\]

Then \( \mathbb{T}^d = \{0\} \cup G_a \cup E_a \). See Figures 4. If \( 0 < a < 1/2 \), then

\[
E_a = \{ x \in \mathbb{T}^d : x \neq 0 \text{ and } |x| \neq a \},
\]
\[
G_a = \{ x \in \mathbb{T}^d : |x| = a \},
\]
because \( x \in \mathbb{T}^d \) means \( x \in (-1/2, 1/2]^d \) and then \( \{ x \in \mathbb{T}^d : |x - m| = a \} \) is empty for \( m \neq 0 \). For \( a > 0 \), let also

\[
r_d(a : x) = \sum_{m \in \mathbb{Z}^d, |x-m|=a} 1, \quad x \in \mathbb{T}^d. \tag{1.13}
\]
Then \( r_d(a : x) = 0 \) for \( x \in E_a \). If \( 0 < a < 1/2 \), then \( r_d(a : 0) = 0 \) and \( r_d(a : x) = 1 \) for \( x \in G_a \).

![Figure 4: \( E_a \) and \( G_a \) for \( a = 1/4 \) and \( a = 3/4 \)](image_url)

In the following, Theorem 1.1 and Corollary 1.2 deal with the behavior of \( S_\lambda(u_{\beta,a}) \) at \( x = 0 \) including the Pinsky phenomenon, Theorem 1.3 deals with the pointwise behaviors including the third phenomenon, Theorem 1.4 deals with the Gibbs-Wilbraham phenomenon near \( G_a \). Theorem 1.5 deals with the almost everywhere convergence.

Our first result is on the behavior of \( S_\lambda(u_{\beta,a})(0) \) which includes the Pinsky phenomenon. Let \( \Gamma(s) \) be the Gamma function. For \( \beta > -1 \), \( a > 0 \) and the dimension \( d \), let

\[
P_{\beta,a}^{[d]} = \frac{\Gamma(\beta + 1)}{\Gamma(d/2)} a^{(d-3)/2 + \beta} \pi^{(d-4)/2 - \beta}
\]

and

\[
L_{\beta,a} = \frac{\Gamma(\beta + 1)}{2} \left( \frac{a}{\pi} \right)^{\beta} \left( \frac{\sin \frac{\beta \pi}{2}}{\frac{\beta \pi}{2}} \right),
\]

where \( \left( \sin \frac{\beta \pi}{2} \right) / \frac{\beta \pi}{2} \) is regarded as 1 if \( \beta = 0 \), that is, \( L_{0,a} = 1/2 \).

**Theorem 1.1.** Let \( \beta > -1 \) and \( a > 0 \).
(i) If \( d \geq 1 \) and \( \beta > \frac{d-3}{2} \), then

\[
S_\lambda(u_{\beta,a})(0) = u_{\beta,a}(0) + r_d(a : 0) (L_{\beta,a} + o(1)) \lambda^{-\beta} + O(\lambda^{\frac{d-3}{2}-\beta})
\]
as \( \lambda \to \infty \). (1.16)

(ii) If \( d \geq 2 \) and \(-1 < \beta \leq \frac{d-3}{2} \), then \( S_\lambda(u_{\beta,a}) \) reveals the Pinsky phenomenon. More precisely,

\[
S_\lambda(u_{\beta,a})(0) = u_{\beta,a}(0) + \left(\sigma_\lambda(U_{\beta,a})(0) - U_{\beta,a}(0)\right)
+ r_d(a : 0) (L_{\beta,a} + o(1)) \lambda^{-\beta} + o(\lambda^{\frac{d-3}{2}-\beta}) \quad \text{as} \quad \lambda \to \infty,
\]

where

\[
\sigma_\lambda(U_{\beta,a})(0) - U_{\beta,a}(0) = -P_{\beta,a}^{[d]} \cos \left(2\pi a \lambda - \frac{d - 1 + 2\beta}{4} \pi\right) \lambda^{\frac{d-3}{2}} + O(\lambda^{\frac{d-5}{2}}).
\]

For example, even if \( d = 2 \), we can see the Pinsky phenomena on the graphs of \( S_\lambda(u_{\beta,a}) \) with \( \beta = \frac{-1}{2} \) and \( a = \frac{1}{4} \) for \( \lambda \in \mathbb{N} \), as Taylor pointed out in [43, 44], see Figure 6. In this case \( P_{\beta,a}^{[d]} = 4 \).

If \( d = 3 \), \( \beta = 0 \) and \( a = \frac{1}{4} \), then \( P_{\beta,a}^{[d]} = \frac{2}{\pi} \), see Figure 7. If \( d = 4 \), \( \beta = \frac{1}{2} \) and \( a = \frac{1}{4} \), then \( P_{\beta,a}^{[d]} = \frac{1}{8} \), see Figure 8.

Next, we state the pointwise behaviors of \( S_\lambda(u_{\beta,a}) \) on \( \mathbb{T}_d \setminus \{0\} \) which includes the third phenomenon. We define the number \( c(d) \) for the dimension \( d \) as the following:

\[
c(d) = d - \frac{2d}{d+1} - \frac{d+1}{2} + \frac{2}{d+1} = \frac{d(d-4)-1}{2(d+1)}, \quad (1.17)
\]
Figure 5: $u_{\beta,a}(x_1, x_2) \ (d = 2, \ \beta = -1/2, \ a = 1/4)$.

Figure 6: $S_\lambda(u_{\beta,a})(x_1, x_2) \ (d = 2, \ \beta = -1/2, \ a = 1/4)$ for $\lambda = 9, 10, 11, 12$. 
Figure 7: $S_\lambda(u_{\beta,a})(x,0,0)$ ($d = 3$, $\beta = 0$, $a = 1/4$) for $\lambda = 46, 47, 48, 49$.

Figure 8: $S_\lambda(u_{\beta,a})(x,0,0)$ ($d = 4$, $\beta = 1/2$, $a = 1/4$) for $\lambda = 43, 44, 45, 46$. 
that is,
\[ c(1) = -1, \quad c(2) = -5/6, \quad c(3) = -1/2, \quad c(4) = -1/10. \]

**Theorem 1.3.** Let \( \beta > -1 \) and \( a > 0 \).

(i) If \( 1 \leq d \leq 4 \), then,

(a) for \( \beta > -1 \),
\[
\lim_{\lambda \to \infty} \frac{S_\lambda(u_{\beta,a})(x) - u_{\beta,a}(x)}{\lambda^{-\beta}} = r_d(a : x) L_{\beta,a} \quad \text{for all } x \in G_a.
\]

(b) for \( \beta > c(d) \),
\[
\lim_{\lambda \to \infty} S_\lambda(u_{\beta,a})(x) = u_{\beta,a}(x) \quad \text{uniformly on any compact set in } E_a.
\]

(ii) If \( d \geq 5 \), then
\[
\lim_{\lambda \to \infty} \frac{1}{\lambda^{d/2}} \left| \frac{S_\lambda(u_{\beta,a})(x) - u_{\beta,a}(x)}{\lambda^{-\beta}} - r_d(a : x) L_{\beta,a} \right| = 0
\]
for all \( x \in (E_a \cup G_a) \setminus \mathbb{Q}^d \),

and
\[
0 < \limsup_{\lambda \to \infty} \frac{1}{\lambda^{d/2}} \left| \frac{S_\lambda(u_{\beta,a})(x) - u_{\beta,a}(x)}{\lambda^{-\beta}} - r_d(a : x) L_{\beta,a} \right| < \infty
\]
for all \( x \in (E_a \cup G_a) \cap \mathbb{Q}^d \).

Namely, if \( (d-5)/2 - \beta \geq 0 \), then \( S_\lambda(u_{\beta,a}) \) reveals the third phenomenon.

Note that \( r_d(a : x) = 0 \) for \( x \in E_a \) in the above. For example, we can see the Gibbs-Wilbraham phenomena, the Pinsky phenomena and the third phenomena on the graphs of \( S_\lambda(u_{\beta,a}) \) with \( d = 5, \beta = -1/2 \) and \( a = 1/4 \) for \( \lambda \in \mathbb{N} \), see Figures 10 and 11.

**Remark 1.1.** In the case \( 2 \leq d \leq 4 \) and \(-1 < \beta \leq c(d)\), the pointwise convergence of \( S_\lambda(u_{\beta,a}) \) on \( E_a \) is an open problem, which is closely related to the lattice point problems. Especially, if \( d = 2 \) and \( x = 0 \), then the pointwise convergence of \( S_\lambda(u_{\beta,a})(0) \) is equivalent to Hardy’s conjecture on Gauss’s circle problem (see Section 7). If \( d = 4 \) and \(-1 < \beta < -1/2\), then we get a partial result on the divergence of \( S_\lambda(u_{\beta,a}) \) on \( T^d \cap \mathbb{Q}^d \) (33).
Figure 9: $u_{\beta,a}(x,0,0,0,0) \ (d = 5, \beta = -1/2, a = 1/4)$.

Figure 10: $S_\lambda(u_{\beta,a})(x,0,0,0,0) \ (d = 5, \beta = -1/2, a = 1/4)$ for $\lambda = 100, \ldots, 103$.

Figure 11: $S_\lambda(u_{\beta,a})(x,0,0,0,0) \ (d = 5, \beta = -1/2, a = 1/4)$ for $\lambda = 100, \ldots, 103$: Expansion of the part $0.2 \leq x \leq 0.5$ in Figure 10.
Next, we state the Gibbs-Wilbraham phenomenon near $G_a$ for $1 \leq d \leq 4$.

**Theorem 1.4.** Let $1 \leq d \leq 4$, $c(d) < \beta \leq 0$ and $0 < a < 1/2$. Then, on any neighborhood of the set $G_a$, $S_\lambda(u_{\beta,a})$ reveals a phenomenon like the Gibbs-Wilbraham phenomenon. More precisely, the following holds: For each $x_0 \in G_a$, let $\{x_\lambda^\pm\} \subset T^d$ be the sequences which satisfy $\lim_{\lambda \to \infty} x_\lambda^\pm = x_0$ and $|x_\lambda| = a \mp (2 \pm \beta)/(4\lambda)$. Then

$$
\lim_{\lambda \to \infty} S_\lambda(u_{\beta,a})(x_\lambda^\pm) - u_{\beta,a}(x_\lambda) = C_\beta,a^\pm,
$$

where

$$
C_\beta,a^\pm = \mp \frac{\Gamma(\beta+1)a^2(2 \pm \beta)^\beta}{\pi 2^\beta} \int_\pi^\infty \frac{\sin s}{(s \pm \beta \pi)^{\beta+1}} \, ds.
$$

(1.18)

If $d = 3$, $\beta = 0$ and $a = 1/4$, then we can see both the Gibbs-Wilbraham phenomena and the Pinsky phenomena on the graphs in Figure 7. If $d = 4$, $\beta = 1/2$ and $a = 1/4$, then we can see the Pinsky phenomena but not see the Gibbs-Wilbraham phenomena on the graphs in Figure 8.

**Remark 1.2.** (i) The constant $G_\beta,a^+$ is positive and $G_\beta,a^-$ is negative. Especially

$$
G_{0,a}^+ = \mp \frac{1}{\pi} \int_\pi^\infty \frac{\sin s}{s} \, ds = \mp \frac{1}{2} \pm \frac{1}{\pi} \int_0^\pi \frac{\sin s}{s} \, ds = \pm 0.08949 \cdots.
$$

(ii) In the case $2 \leq d \leq 4$ and $-1 < \beta \leq -1/2$, Theorem 1.4 is also an open problem by the same reason as Remark 1.1.

Finally, we state the almost everywhere convergence of $S_\lambda(u_{\beta,a})$ for $d \geq 4$.

**Theorem 1.5.** Let $d \geq 4$, $\beta > -1/2$ and $a > 0$. Then

$$
\lim_{\lambda \to \infty} S_\lambda(u_{\beta,a})(x) = u_{\beta,a}(x), \quad a.e. \ x \in T^d.
$$

**Remark 1.3.** (i) From Theorems 1.3 and 1.5 we get Theorem in [18] as a corollary, which is the case $\beta = 0$.

(ii) In the case $d \geq 4$ and $-1 < \beta \leq -1/2$, the almost everywhere convergence of $S_\lambda(u_{\beta,a})$ is an open problem.

We first prove in Section 2 a fundamental identity for the periodization $f$ of any integrable radial function $F$ on $\mathbb{R}^d$ with compact support. This identity gives the
relation among $S_\lambda(f)(x)$, $\sigma_\lambda(F)(x)$ and the term related to lattice point problems. In Section 3 we collect the results with respect to Bessel functions. In Section 4 we investigate the behavior of $\sigma_\lambda(U_{\beta,a})(x)$. We show the uniform convergence of $\sigma_\lambda(U_{\beta,a})(x)$ on any compact subset of $E_a$, the Pinsky phenomenon at the origin, and, the Gibbs-Wilbraham phenomenon near the spherical surface. In Section 5 we prove lemmas related to the lattice point problems. Then, in Section 6, using the results in Sections 2-5 we prove the main results, in which it turns out that the third phenomenon is closely related to the lattice point problems. Finally, in Section 7 we give the relation between the convergence of the spherical partial sum and lattice point problems.

2 Fundamental identity

Let $J_\nu$ be the Bessel function of order $\nu$. If $\nu > -1$, then

$$\frac{J_\nu(s)}{s^\nu} \to \frac{1}{2^\nu \Gamma(\nu + 1)} \quad \text{as} \quad s \to 0.$$  

For this reason, in this paper we always regard

$$\frac{J_\nu(s)}{s^\nu} = \frac{1}{2^\nu \Gamma(\nu + 1)} \quad \text{for} \quad s = 0. \quad (2.1)$$

For $\nu > -1$, let

$$\Lambda_\nu(t : s) = \frac{J_\nu(2\pi t \sqrt{s})}{s^\frac{\nu}{2}}, \quad t \geq 0, \quad s > 0. \quad (2.2)$$

By (2.1) we regard

$$\Lambda_\nu(t : 0) = \frac{(\pi t)^\nu}{\Gamma(\nu + 1)}, \quad t \geq 0.$$  

For any $\alpha > -1$ we define $D_\alpha(s : x)$, $\mathcal{D}_\alpha(s : x)$ and $\Delta_\alpha(s : x)$ as the following:

$$D_\alpha(s : x) = \begin{cases} \frac{1}{\Gamma(\alpha + 1)} \sum_{|m|^2 < s} (s - |m|^2)^\alpha e^{2\pi imx}, & s > 0, \\ 0, & s = 0, \end{cases} \quad x \in \mathbb{R}^d,$$

$$\mathcal{D}_\alpha(s : x) = \begin{cases} \frac{1}{\Gamma(\alpha + 1)} \int_{|\xi|^2 < s} (s - |\xi|^2)^\alpha e^{2\pi i x \xi} d\xi, & s > 0, \\ 0, & s = 0, \end{cases} \quad x \in \mathbb{R}^d,$$
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\[ \Delta_\alpha(s : x) = D_\alpha(s : x) - D_\alpha(s : x), \quad s \geq 0, \ x \in \mathbb{R}^d. \] \hspace{1cm} (2.3)

We consider a function \( \phi \) on \([0, \infty)\) and the periodization \( f_\phi \) of the function \( F_\phi(x) = \phi(|x|), \ x \in \mathbb{R}^d \), that is,

\[ f_\phi(x) = \sum_{m \in \mathbb{Z}^d} F_\phi(x + m) = \sum_{m \in \mathbb{Z}^d} \phi(|x + m|), \quad x \in \mathbb{T}^d. \] \hspace{1cm} (2.4)

If the radial function \( F_\phi \) is integrable on \( \mathbb{R}^d \), then the Fourier transform \( \hat{F}_\phi \) is also radial and has the form

\[ \hat{F}_\phi(\xi) = \int_{\mathbb{R}^d} \phi(|x|) e^{-2\pi i x \xi} dx = 2\pi \int_{0}^{\infty} \phi(t) \frac{J_{\frac{d}{2} - 1}(|2\pi t|\xi|)}{(t|\xi|)^{\frac{d}{2} - 1}} t^{d-1} dt, \] \hspace{1cm} (2.5)

that is,

\[ \hat{F}_\phi(\xi) = 2\pi \int_{0}^{\infty} \phi(t) \Lambda_{\frac{d}{2} - 1}(t : |\xi|^2) t^{\frac{d}{2}} dt. \] \hspace{1cm} (2.6)

For the equation (2.5), see [40, Theorem 3.3, page 155] if \( d \geq 2 \). If \( d = 1 \), then (2.5) is also valid by the equation \( J_{-\frac{1}{2}}(s) = \sqrt{2/(\pi s)} \cos s \). Let

\[ A_\phi(s) = 2\pi \int_{0}^{\infty} \phi(t) \Lambda_{\frac{d}{2} - 1}(t : s) t^{\frac{d}{2}} dt, \quad s \geq 0. \] \hspace{1cm} (2.7)

Then

\[ \hat{F}_\phi(\xi) = A_\phi(|\xi|^2). \] \hspace{1cm} (2.8)

Moreover, if the support of \( \phi \) is compact, then \( A_\phi \) is infinitely differentiable and

\[ A_\phi^{(j)}(s) = 2\pi (-\pi)^j \int_{0}^{\infty} \phi(t) \Lambda_{\frac{d}{2} - 1 + j}(t : s) t^{\frac{d}{2} + j} dt, \quad s \geq 0, \quad j = 1, 2, \cdots , \]

since

\[ \frac{\partial}{\partial s} \Lambda_\mu(t : s) = (-\pi t) \Lambda_{\mu + 1}(t : s), \] \hspace{1cm} (2.9)

by the Bessel recurrence formula (see (2.15) below).

Now, we define \( d_\sharp \) as the smallest integer that is greater than \((d - 1)/2\), that is,

\[ d_\sharp = \left\lfloor \frac{d - 1}{2} \right\rfloor + 1, \] \hspace{1cm} (2.10)

where \( \lfloor t \rfloor \) is the integer part of \( t \geq 0 \). In this section we prove the following:
Theorem 2.1. Let \( \phi \) be a function on \([0, \infty)\) with compact support. Suppose that \( F_\phi \) is integrable on \( \mathbb{R}^d \). Let \( f_\phi \) be the periodization of the function \( F_\phi \). Then

\[
S_\lambda(f_\phi)(x) = \sigma_\lambda(F_\phi)(x) + \sum_{j=0}^{d_\lambda} (-1)^j \Delta_j(\lambda^2 : x) A^{(j)}(\lambda^2)
+ (-1)^{d_\lambda+1} \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \int_0^{\lambda^2} D_{d_\lambda}(s : x - m) A_{d_\lambda+1}(s) ds,
\]

for all \( x \in \mathbb{T}^d \).

As \( \phi \) in Theorem 2.1 take

\[
\phi_{\beta,a}(t) = \begin{cases} (a^2 - t^2)^{\beta}, & 0 \leq t < a, \\ 0, & t \geq a, \end{cases}
\]

with \( \beta > -1 \) and \( a > 0 \). Then \( U_{\beta,a}(x) = \phi_{\beta,a}(|x|) \), \( x \in \mathbb{R}^d \), and the function \( u_{\beta,a} \) is the periodization of \( U_{\beta,a}(x) \) defined by (1.9). In this case we denote \( A_\phi \) by \( A_{\beta,a} \). Then \( A_{\beta,a} \) can be calculated explicitly by using (2.5), (2.8) and (2.19) below. Its derivatives \( A^{(j)}_{\beta,a} \) are also calculated by (2.9). That is,

\[
\begin{align*}
A_{\beta,a}(s) &= \Gamma(\beta + 1) \pi^\beta a^{\frac{d}{2}+\beta} \Lambda_{\frac{d}{2}+\beta}(a : s), \\
A^{(j)}_{\beta,a}(s) &= (-1)^j \Gamma(\beta + 1) \pi^{\beta-j} a^{\frac{d}{2}+\beta+j} \Lambda_{\frac{d}{2}+\beta+j}(a : s).
\end{align*}
\]

Then we have the following corollary:

Corollary 2.2. Let \( \beta > -1 \) and \( a > 0 \). Then

\[
S_\lambda(u_{\beta,a})(x) = \sigma_\lambda(U_{\beta,a})(x) + \sum_{j=0}^{d_\lambda} (-1)^j \Delta_j(\lambda^2 : x) A^{(j)}_{\beta,a}(\lambda^2)
+ (-1)^{d_\lambda+1} \sum_{m \in \mathbb{Z}^d \setminus \{0\}} \int_0^{\lambda^2} D_{d_\lambda}(s : x - m) A_{d_\lambda+1}(s) ds,
\]

for all \( x \in \mathbb{T}^d \).

To prove Theorem 2.1, first we state the properties of the Bessel functions, \( D_\alpha \) and \( \mathcal{D}_\alpha \). The following are the recurrence formulas for Bessel functions.

\[
\frac{d}{ds} \left( s^\nu J_\nu(s) \right) = s^\nu J_{\nu-1}(s), \quad \frac{d}{ds} \left( \frac{J_\nu(s)}{s^\nu} \right) = - \frac{J_{\nu+1}(s)}{s^\nu}.
\]
The Bessel functions have also the following asymptotic behavior.

\[
J_\nu(s) = \frac{s^\nu}{2^\nu \Gamma(\nu + 1)} + O(s^{\nu+1}) \quad \text{as} \quad s \to 0, \quad (2.16)
\]

\[
J_\nu(s) = \sqrt{\frac{2}{\pi s}} \cos \left( s - \frac{2\nu + 1}{4} \pi \right) + O(s^{-3/2}) \quad \text{as} \quad s \to \infty. \quad (2.17)
\]

If \( \alpha > -1 \), then

\[
\int_0^t D_\alpha(s : x) \, ds = D_{\alpha+1}(t : x), \quad t \geq 0, \quad x \in \mathbb{R}^d, \quad (2.18)
\]

and

\[
D_\alpha(s : x) = \frac{1}{\Gamma(\alpha + 1)} \int_{|\xi|^2 < s} (s - |\xi|^2)^\alpha e^{2\pi i \xi x} \, d\xi = \frac{s^{\frac{d}{2}+\alpha} J_{\frac{d}{2}+\alpha}(2\pi \sqrt{s}|x|)}{\pi^\alpha (\sqrt{s}|x|)^{\frac{d}{2}+\alpha}}. \quad (2.19)
\]

If \( \alpha > (d - 1)/2 \), for example, \( \alpha = d_* \), then

\[
D_\alpha(s : x) = \sum_{m \in \mathbb{Z}^d} D_\alpha(s : x - m) = \frac{s^{\frac{d}{2}+\alpha}}{\pi^\alpha} \sum_{m \in \mathbb{Z}^d} \frac{J_{\frac{d}{2}+\alpha}(2\pi \sqrt{s}|x - m|)}{(|\sqrt{s}|x - m|)^{\frac{d}{2}+\alpha}}, \quad (2.20)
\]

where the sum in (2.20) converges absolutely. In the above the equation (2.18) follows from the definition. For the equalities (2.19) and (2.20), see (4.2) and (4.3) in [24], respectively. By (2.15) and (2.19) we have

\[
\begin{cases}
\frac{\partial}{\partial s} D_\alpha(s : x) = D_{\alpha-1}(s : x), \quad &\alpha > 0, \\
\frac{\partial}{\partial s} D_0(s : x) = \pi \left( \frac{\sqrt{s}}{|x|} \right)^{\frac{d}{2}+1} J_{\frac{d}{2}+1}(2\pi |x| \sqrt{s}), \quad &\alpha = 0.
\end{cases} \quad (2.21)
\]

Proof of Theorem 2.1. Combining (1.8) with (2.8) we have

\[
\hat{f}_\phi(m) = \hat{F}_\phi(m) = A_\phi(|m|^2), \quad m \in \mathbb{Z}^d.
\]
Then

\[
S_\lambda(f_\phi)(x) - D_0(\lambda^2 : x)A_\phi(\lambda^2) \\
= \sum_{|m| < \lambda} \hat{f}_\phi(m)e^{2\pi imx} - \left( \sum_{|m| < \lambda} e^{2\pi imx} \right) A_\phi(\lambda^2) \\
= \sum_{|m| < \lambda} \left( A_\phi(|m|^2) - A_\phi(\lambda^2) \right) e^{2\pi imx} \\
= \sum_{|m| < \lambda} \left( - \int_{|m|^2}^{\lambda^2} A_\phi^{(1)}(s) \, ds \right) e^{2\pi imx} \\
= - \int_{0}^{\lambda^2} \left( \sum_{|m|^2 < s} e^{2\pi imx} \right) A_\phi^{(1)}(s) \, ds \\
= - \int_{0}^{\lambda^2} D_0(s : x)A_\phi^{(1)}(s) \, ds.
\]

Using (2.18) and integration by parts, we have

\[
- \int_{0}^{\lambda^2} D_0(s : x)A_\phi^{(1)}(s) \, ds \\
= - \left[ D_1(s : x)A_\phi^{(1)}(s) \right]_{0}^{\lambda^2} + \int_{0}^{\lambda^2} D_1(s : x)A_\phi^{(2)}(s) \, ds \\
= -D_1(\lambda^2 : x)A_\phi^{(1)}(\lambda^2) + \left[ D_2(s : x)A_\phi^{(2)}(s) \right]_{0}^{\lambda^2} - \int_{0}^{\lambda^2} D_2(s : x)A_\phi^{(3)}(s) \, ds \\
= \cdots \\
= \sum_{j=1}^{d_1} (-1)^j D_j(\lambda^2 : x)A_\phi^{(j)}(\lambda^2) + (-1)^{d_1+1} \int_{0}^{\lambda^2} D_{d_1}(s : x)A_\phi^{(d_1+1)}(s) \, ds.
\]

Therefore, by (2.20) we have

\[
S_\lambda(f_\phi)(x) \\
= \sum_{j=0}^{d_1} (-1)^j D_j(\lambda^2 : x)A_\phi^{(j)}(\lambda^2) + (-1)^{d_1+1} \sum_{m \in \mathbb{Z}^d} \int_{0}^{\lambda^2} D_{d_1}(s : x - m)A_\phi^{(d_1+1)}(s) \, ds.
\]
Using (2.21) and integration by parts again, we have

\[\int_0^{\lambda^2} D_{d_t}(s : x) A^{(d_t+1)}_\phi(s) \, ds = \int_0^{\lambda^2} D_{d_t-1}(s : x) A^{(d_t)}_\phi(s) \, ds = \ldots.\]

\[= \sum_{j=0}^{d_t} (-1)^j D_{d_t-j}(\lambda^2 : x) A^{(d_t-j)}_\phi(\lambda^2) + \ldots + (-1)^{d_t+1} \int_0^{\lambda^2} \frac{\sqrt{s}}{|x|} \frac{\lambda^{d_t-1}}{2^{d_t-1}} J_{d_t-1}(2\pi|x|s) A_\phi(s) \, ds = \ldots\]

\[= \sum_{j=0}^{d_t} (-1)^j D_{d_t-j}(\lambda^2 : x) A^{(j)}_\phi(\lambda^2) + \ldots + (-1)^{d_t+1} \int_0^{\lambda^2} \frac{J_{d_t-1}(2\pi|x|u)}{(|x|u)^{d_t-1}} u^{d_t-1} \, du.\]

Here we note that, by (2.8) and (2.5),

\[\sigma_\lambda(F_\phi)(x) = \int_{|\xi|<\lambda} \hat{F}_\phi(\xi) e^{2\pi i x \xi} \, d\xi = \int_{|\xi|<\lambda} A_\phi(|\xi|^2) e^{2\pi i x \xi} \, d\xi = 2\pi \int_0^{\lambda^2} \frac{J_{d_t-1}(2\pi|x|u)}{(|x|u)^{d_t-1}} u^{d_t-1} \, du.\]

Therefore

\[S_\lambda(f_\phi)(x) = \sigma_\lambda(F_\phi)(x) + \sum_{j=0}^{d_t} (-1)^j \Delta_j(\lambda^2 : x) A^{(j)}_\phi(\lambda^2) + \ldots + (-1)^{d_t+1} \sum_{m \in \mathbb{Z}^d, m \neq 0} \int_0^{\lambda^2} D_{d_t}(s : x - m) A^{(d_t+1)}_\phi(s) \, ds.\]

3 Further properties of Bessel functions

In this section we collect properties of Bessel functions which we use in the following sections. Adding to (2.17), we first recall the following asymptotic behavior of the
Bessel functions:

\[ J_\nu(s) = \sqrt{\frac{2}{\pi s}} \left( \cos \left( s - \frac{2\nu + 1}{4} \pi \right) - \frac{4\nu^2 - 1}{8s} \sin \left( s - \frac{2\nu + 1}{4} \pi \right) \right) + O(s^{-5/2}) \]

as \( s \to \infty \). (3.1)

**Lemma 3.1** (Gradshteyn and Ryzhik [6], page 676, 6.561, 14). Let \( \nu + 1 > -\mu > -1/2 \). Then

\[
\int_0^\infty x^\mu J_\nu(x) \, dx = \frac{2\mu \Gamma((1 + \nu + \mu)/2)}{\Gamma((1 + \nu - \mu)/2)}.
\]

Take \( \nu = \frac{1}{2} + \beta \) and \( \mu = -\frac{1}{2} - \beta \) in Lemma 3.1 and we have the following corollary.

**Corollary 3.2.** Let \( \beta > -1 \). Then

\[
\int_0^\infty \frac{J_{\frac{1}{2} + \beta}(s)}{s^{\frac{1}{2} + \beta}} \, ds = \frac{1}{2^\beta \Gamma(1 + \beta)} \sqrt{\frac{\pi}{2}}.
\]

**Lemma 3.3.** Let \( \beta > -1 \). Then

\[
\int_u^\infty \frac{J_{\frac{1}{2} + \beta}(s)}{s^{\frac{1}{2} + \beta}} \, ds = \sqrt{\frac{2}{\pi}} u^{-\beta - 1} \cos \left( u - \frac{\beta}{2} \pi \right) + O(u^{-2-\beta}) \quad \text{as} \quad u \to \infty.
\]

**Proof.** Use (3.1) and two equations

\[
\int_u^\infty \frac{\cos(s - \theta)}{s^{1+\beta}} \, ds = u^{-\beta-1} \cos \left( u - \theta + \frac{\pi}{2} \right) + O(u^{-2-\beta}),
\]

\[
\int_u^\infty \frac{\sin(s - \theta)}{s^{2+\beta}} \, ds = O(u^{-2-\beta}),
\]

and we have the conclusion. \( \square \)

**Lemma 3.4** (Gradshteyn and Ryzhik [6], page 683, 6.575, 1). Let \( \nu + 1 > \mu > -1 \), \( A > 0 \) and \( B > 0 \). Then

\[
\int_0^\infty J_{\nu+1}(As) J_\mu(Bs) s^{\nu-\mu} \, ds = \begin{cases} 
0 & (A < B) \\
\frac{(A^2 - B^2)^{\nu-\mu} B^{\mu}}{2^{\nu-\mu} A^{\nu+1} \Gamma(\nu - \mu + 1)} & (A > B).
\end{cases}
\]
Take
\[ \nu + 1 = \mu + \beta + 1, \ A = 1 \text{ and } B = \frac{t}{a}, \]
in Lemma 3.4 and we have the following corollary.

**Corollary 3.5.** Let \( \mu > -1, \ a > 0, \ \beta > -1 \) and \( t > 0 \). Then
\[
2^\beta \Gamma(\beta + 1) a^{2\beta} \int_0^\infty \frac{J_\mu(\frac{t}{a}s)J_{\mu+\beta+1}(s)}{(\frac{t}{a})^\mu s^\beta} ds = \begin{cases} 
0 & (t > a) \\
(a^2 - t^2)^\beta & (0 < t < a) 
\end{cases},
\]

**Lemma 3.6.** Let \( \nu > -1, \ \mu > -1, \ A > 0 \) and \( B > 0 \). Then
\[
 J_\nu(Au)J_\mu(Bu) = \frac{1}{\pi\sqrt{ABu}} \left( \cos \left( (A - B)u - \frac{\nu - \mu - 1}{2\pi} \right) \right. \\
&+ \cos \left( (A + B)u - \frac{\nu + \mu + 1}{2\pi} \right) \\
&+ \left. \frac{1}{\sqrt{AB}} \left( \frac{1}{A} + \frac{1}{B} \right) O(u^{-2}) \right) \text{ as } u \to \infty.
\]

**Proof.** Use \[2.17\] and \( 2 \cos \theta \cos \phi = \cos(\theta + \phi) + \cos(\theta - \phi) \), and we have the conclusion. \qed

Let
\[ \text{sign}(r) = \begin{cases} 
1, & r > 0, \\
0, & r = 0, \\
-1, & r < 0.
\end{cases} \]

**Lemma 3.7.** For \( \beta > -1 \), let
\[
\psi_\beta(\lambda, r) = |r|^\beta \int_{2\pi |r| \lambda}^\infty \frac{1}{s^{\beta+1}} \cos \left( s - \text{sign}(r) \frac{\beta + 1}{2\pi} \right) ds, \quad \lambda > 0, \ r \in \mathbb{R} \setminus \{0\}. \quad (3.2)
\]
Then \( \psi_\beta \) has the following properties:

(i) For each \( r \in \mathbb{R} \setminus \{0\} \),
\[
\psi_\beta(\lambda, r) = |r|^{-1} O(\lambda^{-\beta-1}) \text{ as } \lambda \to \infty. \quad (3.3)
\]

(ii) If \( \beta > 0 \), then
\[
\psi_\beta(\lambda, r) = O(\lambda^{-\beta}) \text{ as } \lambda \to \infty \quad (3.4)
\]
uniformly with respect to \( r \in \mathbb{R} \setminus \{0\} \).
(iii) If $-1 < \beta \leq 0$, then, for each $\lambda > 0$, $-\psi_\beta(\lambda, r)$ has values
\[
\left(\frac{2 + \beta}{4\lambda}\right)^\beta \int_\pi^\infty \frac{-\sin s}{(s + \frac{\beta}{2} \pi)^{\beta+1}} ds \quad \text{at} \quad r = \frac{2 + \beta}{4\lambda},
\]
and
\[
\left(\frac{2 - \beta}{4\lambda}\right)^\beta \int_\pi^\infty \frac{\sin s}{(s - \frac{\beta}{2} \pi)^{\beta+1}} ds \quad \text{at} \quad r = -\frac{2 - \beta}{4\lambda}.
\]

Proof. (i) Using the equation
\[
\int_v^\infty \cos(s - \theta) s^{\beta+1} ds = O(v^{-\beta-1}), \tag{3.5}
\]
we have $\psi_\beta(\lambda, r) = |r|^\beta O(|r|\lambda^{-\beta-1}) = |r|^{-1} O(\lambda^{-\beta-1})$.

(ii) Changing of variables, we have
\[
\psi_\beta(\lambda, r) = \frac{1}{(2\pi\lambda)^\beta} \int_1^\infty \frac{\cos(2\pi|\lambda u - \text{sign}(r)\frac{\beta+1}{2} \pi)}{u^{\beta+1}} du,
\]
which shows $\psi_\beta(\lambda, r) = O(\lambda^{-\beta})$ uniformly with respect to $r$, since $\beta > 0$.

(iii) If $r > 0$, then
\[
-\psi_\beta(\lambda, r) = r^\beta \int_{2\pi r \lambda}^\infty \frac{-\sin\left(s - \frac{\beta}{2} \pi\right)}{s^{\beta+1}} ds = r^\beta \int_{2\pi r \lambda - \frac{\beta}{2} \pi}^\infty \frac{-\sin s}{(s + \frac{\beta}{2} \pi)^{\beta+1}} ds,
\]
and, if $r < 0$, then
\[
-\psi_\beta(\lambda, r) = (-r)^\beta \int_{2\pi (-r) \lambda}^\infty \frac{\sin\left(s + \frac{\beta}{2} \pi\right)}{s^{\beta+1}} ds = (-r)^\beta \int_{2\pi (-r) \lambda + \frac{\beta}{2} \pi}^\infty \frac{\sin s}{(s - \frac{\beta}{2} \pi)^{\beta+1}} ds.
\]
Then we have the conclusion.

Lemma 3.8. Let $\mu > -1$, $\beta > -1$ and $a > 0$. If $0 < t < a$ or $t > a$, then
\[
\int_{2\pi a \lambda}^\infty J_\mu\left(\frac{t}{a} u\right) J_{\mu + \beta + 1}(u) du = \frac{1}{\pi a^\beta} \left(\frac{a}{t}\right)^{\mu+\frac{1}{2}} \psi_\beta(\lambda, a - t) + \frac{1}{a^{\beta+1}} \left(\frac{a}{t}\right)^{\mu+\frac{1}{2}} \left(1 + \frac{a}{t}\right) O(\lambda^{-\beta-1}) \quad \text{as} \quad \lambda \to \infty,
\]
where the term $O(\lambda^{-\beta-1})$ is uniform with respect to $t$. 
Proof. By Lemma 3.6 we have
\[
J_{\mu} \left( \frac{t}{a} u \right) J_{\mu+\beta+1}(u) = \frac{1}{\pi u} \sqrt{\frac{a}{t}} \left( \cos \left( \frac{|a-t|}{a} u - \text{sign}(a-t) \frac{1+\beta}{2} \pi \right) + \cos \left( \frac{a+t}{a} u - \frac{2\mu+\beta+1}{2} \pi \right) \right) + \sqrt{\frac{a}{t}} \left( 1 + \frac{a}{t} \right) O(u^{-2}).
\]

Let
\[
I_1 = \frac{1}{\pi} \left( \frac{a}{t} \right)^{\mu+\frac{1}{2}} \int_{2\pi a \lambda}^{\infty} \cos \left( \frac{|a-t|}{a} u - \text{sign}(a-t) \frac{\beta+1}{2} \pi \right) u^{\beta+1} \, du,
\]
\[
I_2 = \frac{1}{\pi} \left( \frac{a}{t} \right)^{\mu+\frac{1}{2}} \int_{2\pi a \lambda}^{\infty} \cos \left( \frac{a+t}{a} u - \frac{2\mu+\beta+1}{2} \pi \right) u^{\beta+1} \, du.
\]

Then
\[
\int_{2\pi a \lambda}^{\infty} J_{\mu} \left( \frac{t}{a} u \right) J_{\mu+\beta+1}(u) \frac{1}{\left( \frac{a}{t} \right)^{\mu} u^\beta} \, du = I_1 + I_2 + \left( \frac{a}{t} \right)^{\mu+\frac{1}{2}} \left( 1 + \frac{a}{t} \right) a^{-\beta-1} O(\lambda^{-\beta-1}).
\]

Changing of variables and using (3.5), we have
\[
I_1 = \frac{1}{\pi} \left( \frac{a}{t} \right)^{\mu+\frac{1}{2}} \left( \frac{|a-t|}{a} \right)^{\beta} \int_{2\pi |a-t| \lambda}^{\infty} \cos \left( s - \text{sign}(a-t) \frac{\beta+1}{2} \pi \right) s^{\beta+1} \, ds
\]
\[
= \frac{1}{\pi a^{\beta}} \left( \frac{a}{t} \right)^{\mu+\frac{1}{2}} \psi_\beta(\lambda, a-t)
\]
and
\[
I_2 = \frac{1}{\pi} \left( \frac{a}{t} \right)^{\mu+\frac{1}{2}} \left( \frac{a+t}{a} \right)^{\beta} \int_{2\pi (a+t) \lambda}^{\infty} \cos \left( s - \frac{2\mu+\beta+1}{2} \pi \right) s^{\beta+1} \, ds
\]
\[
= \left( \frac{a}{t} \right)^{\mu+\frac{1}{2}} \left( \frac{a+t}{a} \right)^{\beta} (a+t)^{-\beta-1} O(\lambda^{-\beta-1})
\]
\[
= \frac{1}{\pi a^{\beta+1}} \left( \frac{a}{t} \right)^{\mu+\frac{1}{2}} \left( \frac{a}{a+t} \right) O(\lambda^{-\beta-1}).
\]

Then we have the conclusion. \qed

Lemma 3.9 (Gradshteyn and Ryzhik [6], page 683, 6.574, 2). Let \( \nu+\mu+1 > \kappa > 0 \) and \( b > 0 \). Then
\[
\int_{0}^{\infty} J_{\nu}(b s) J_{\mu}(b s) \frac{s^\kappa}{s^\kappa} \, ds = \frac{b^{\kappa-1} \Gamma(\kappa) \Gamma(\frac{\nu+\nu+\kappa+1}{2})}{2^\nu \Gamma(\frac{\mu-\nu-\kappa+1}{2}) \Gamma(\frac{-\nu+\nu+\kappa+1}{2}) \Gamma(\frac{\mu+\nu+1}{2})}.\]
Especially, for \( \mu > -1 \) and \( \beta > 0 \),
\[
\int_0^\infty \frac{J_{\mu+\beta+1}(s)J_{\mu}(s)}{s^\beta} \, ds = 0.
\]

**Lemma 3.10** (Gradshteyn and Ryzhik [6], page 660, 6.512, 3). Let \( \nu > 0 \). Then
\[
\int_0^\infty J_\nu(s)J_{\nu-1}(s) \, ds = \frac{1}{2}.
\]

**Lemma 3.11.** Let \( \mu > -1 \) and \( \beta \geq 0 \). Then
\[
\int_u^\infty \frac{J_{\mu+\beta+1}(s)J_{\mu}(s)}{s^\beta} \, ds = \begin{cases} 
-\frac{1}{\pi \beta} \sin \left( \frac{\beta \pi}{2} \right) u^{-\beta} + O(u^{-1-\beta}), & \text{if } \beta > 0, \\
O(u^{-1}), & \text{if } \beta = 0.
\end{cases}
\]
as \( u \to \infty \).

**Proof.** By Lemma 3.6 we have
\[
J_{\mu+\beta+1}(s)J_\mu(s) = \frac{1}{\pi s} \left( \cos \left( \frac{\beta + 1}{2} \pi \right) + \cos \left( 2s - \frac{2\mu + 2 + \beta}{2} \pi \right) \right) + O(s^{-2}). \tag{3.6}
\]

Using
\[
\frac{1}{\pi} \int_u^\infty \frac{1}{s^{\beta+1}} \cos \left( \frac{\beta + 1}{2} \pi \right) \, ds = \begin{cases} 
-\frac{1}{\pi \beta} \sin \left( \frac{\beta \pi}{2} \right) u^{-\beta}, & \beta > 0, \\
0, & \beta = 0,
\end{cases}
\]
and (3.5), we have the conclusion. \( \square \)

**Corollary 3.12.** Let \( \mu > -1 \) and \( \beta > -1 \). Then
\[
2^\beta \Gamma(\beta + 1) a^{2\beta} \int_0^{2\pi a \lambda} \frac{J_{\mu+\beta+1}(s)J_{\mu}(s)}{s^\beta} \, ds
= L_{\beta,a} \lambda^{-\beta} + \begin{cases} 
O(\lambda^{-\beta-1}), & \beta \geq 0, \\
O(1), & -1 < \beta < 0,
\end{cases} \quad \text{as } \lambda \to \infty. \tag{3.7}
\]

In the above \( L_{\beta,a} \) is defined by (1.15), that is,
\[
L_{\beta,a} = \Gamma(\beta + 1) \left( \frac{a}{\pi} \right)^\beta \left( \frac{\sin \frac{\beta \pi}{2}}{\frac{\beta \pi}{2}} \right),
\]
where \( \left( \sin \frac{\beta \pi}{2} \right) / \frac{\beta \pi}{2} \) is regarded as 1 if \( \beta = 0 \).
**Proof. Case 1.** Let \( \beta \geq 0 \). By Lemmas 3.9 and 3.10 we have
\[
\int_0^\infty \frac{J_{\mu+\beta+1}(s)J_\mu(s)}{s^\beta} \, ds = \begin{cases} 
0, & \text{if } \beta > 0, \\
\frac{1}{2}, & \text{if } \beta = 0.
\end{cases}
\]
Then
\[
\int_0^{2\pi\alpha} \frac{J_{\mu+\beta+1}(s)J_\mu(s)}{s^\beta} \, ds = -\int_{2\pi\alpha}^\infty \frac{J_{\mu+\beta+1}(s)J_\mu(s)}{s^\beta} \, ds + \begin{cases} 
0, & \text{if } \beta > 0, \\
\frac{1}{2}, & \text{if } \beta = 0.
\end{cases}
\]
By Lemma 3.11 we have (3.7).

**Case 2.** Let \( -1 < \beta < 0 \). By (3.6) we have
\[
\int_0^{2\pi\alpha} \frac{J_{\mu+\beta+1}(s)J_\mu(s)}{s^\beta} \, ds = \frac{1}{\pi} \int_1^{2\pi\alpha} \left\{ \cos \left( \frac{\beta + 1}{2} \pi \right) + \cos \left( 2s - \frac{2\nu + 2 + \beta}{2} \pi \right) \right\} \, ds + O(\lambda^{-\beta-1})
\]
\[
= \frac{1}{\pi} \left( -\sin \left( \frac{\beta \pi}{2} \right) \right) \frac{(2\pi\alpha)^{-\beta}}{-\beta} + O(1).
\]
Since
\[
\int_0^1 \frac{J_{\mu+\beta+1}(s)J_\mu(s)}{s^\beta} \, ds
\]
is a constant, we have the conclusion. \( \square \)

The following lemma is an extension of [24, Lemma 3.3] which is for the case \( \beta = 0 \).

**Lemma 3.13.** Let \( \nu > 0, \lambda > 0, \omega \geq 0 \) and \( \beta > -1 \). Then
\[
\int_0^\lambda \frac{J_{\nu+\beta+1}(s)J_\nu(\omega s)}{\omega^\nu s^\beta} \, ds = \int_0^\lambda \frac{J_{\nu+\beta}(s)J_{\nu-1}(\omega s)}{\omega^{\nu-1}s^\beta} \, ds - \frac{J_{\nu+\beta}(\lambda)J_\nu(\omega \lambda)}{\omega^\nu \lambda^\beta}. \tag{3.8}
\]
In the above, if \( \omega = 0 \), then by (2.1) we regard (3.8) as
\[
\int_0^\lambda \frac{J_{\nu+\beta+1}(s)}{s^\beta} \frac{s^\nu}{2^\nu \Gamma(\nu + 1)} \, ds = \int_0^\lambda \frac{J_{\nu+\beta}(s)}{s^\beta} \frac{s^{\nu-1}}{2^\nu \Gamma(\nu)} \, ds - \frac{J_{\nu+\beta}(\lambda)}{\lambda^\beta} \frac{\lambda^\nu}{2^\nu \Gamma(\nu + 1)}.
\]
Proof. **Case 1.** Let $\omega > 0$. Then, using (2.15) and integration by parts, we have

\[
\int_0^\lambda J_{\nu + \beta}(s)J_{\nu - 1}(\omega s)\omega^{\nu + 1} ds
\]

\[
= \int_0^\lambda J_{\nu + \beta}(s)\omega^{\nu + 1} ds
\]

\[
= \int_0^\lambda \frac{J_{\nu + \beta}(s)}{s^{\nu + \beta}}(\omega s)^\nu J_{\nu - 1}(\omega s)\omega ds
\]

\[
= \left[ \frac{J_{\nu + \beta}(s)}{s^{\nu + \beta}}(\omega s)^\nu J_{\nu}(\omega s) \right]_0^\lambda + \int_0^\lambda \frac{s^{\nu - 1}}{s^{\nu + \beta}}(\omega s)^\nu J_{\nu}(\omega s) ds
\]

\[
= \frac{J_{\nu + \beta}(\lambda)}{\lambda^{\nu + 1}} \omega^{\nu + 1} + \int_0^\lambda \frac{s^{\nu - 1}}{s^{\nu + \beta}}(\omega s)^\nu J_{\nu}(\omega s) ds.
\]

**Case 2.** Let $\omega = 0$. Then,

\[
\int_0^\lambda \frac{s^{\nu - 1}}{s^{\nu + \beta}} \frac{2\nu - 1}{\Gamma(\nu)} ds
\]

\[
= \int_0^\lambda \frac{s^{\nu - 1}}{s^{\nu + \beta}} \frac{2\nu - 1}{\Gamma(\nu)} ds
\]

\[
= \left[ \frac{J_{\nu + \beta}(s)}{s^{\nu + \beta}}(\omega s)^\nu J_{\nu}(\omega s) \right]_0^\lambda + \int_0^\lambda \frac{s^{\nu - 1}}{s^{\nu + \beta}} \frac{2\nu - 1}{\Gamma(\nu)} ds
\]

\[
= \frac{J_{\nu + \beta}(\lambda)}{\lambda^{\nu + 1}} \omega^{\nu + 1} + \int_0^\lambda \frac{s^{\nu - 1}}{s^{\nu + \beta}} \frac{2\nu - 1}{\Gamma(\nu)} ds.
\]

Therefore we have the conclusion. 

Using Lemma 3.13 several times, we have the following.

**Corollary 3.14.** Let $d \geq 3$, $\beta > -1$, $a > 0$ and $t \geq 0$, and let $d_\sharp$ be as in (2.10).

(i) If $t > 0$, then

\[
\int_0^{2\pi a} \frac{J_{\frac{d}{2} + \beta}(s)J_{\frac{d}{2} - 1}(\frac{z}{a})}{(\frac{z}{a})^\frac{d}{2} - 1 s^{d_\sharp}} ds
\]

\[
= \int_0^{2\pi a} \frac{J_{\frac{d}{2} + \beta - d_\sharp + 1}(s)J_{\frac{d}{2} - d_\sharp}(\frac{z}{a})}{(\frac{z}{a})^\frac{d}{2} - 1 s^{d_\sharp}} ds - \sum_{\ell=1}^{d_\sharp - 1} \frac{J_{\frac{d}{2} + \beta - \ell}(2\pi a\lambda)J_{\frac{d}{2} - \ell}(2\pi t\lambda)}{(\frac{z}{a})^\frac{d}{2} - \ell (2\pi a\lambda)^{\ell}}.
\]
where the first term of the right hand side in the above equation is equal to

\[
\int_0^{2\pi\lambda} J_{\beta+\frac{1}{2}}(s)J_{-\frac{1}{2}}\left(\frac{s}{2}\right) ds, \quad \text{if } d \text{ is odd},
\]

\[
\int_0^{2\pi\lambda} J_{\beta+1}(s)J_0\left(\frac{s}{2}\right) ds, \quad \text{if } d \text{ is even}.
\]

(ii) If \( t = 0 \), then

\[
\int_0^{2\pi\lambda} J_{\frac{d+\beta}{2}}(s) s^{\frac{d-1}{2}} ds
\]

\[
= \int_0^{2\pi\lambda} J_{\frac{d+\beta-d_1+1}{2}}(s) s^{\frac{d-1}{2}} ds - \sum_{\ell=1}^{d_1-1} \frac{(\pi\lambda)^{\frac{d-\beta+1}{2}}}{\Gamma\left(\frac{d}{2} - \ell + 1\right)(2\pi\lambda)^{\beta}},
\]

where the first term of the right hand side in the above equation is equal to

\[
\begin{cases}
\frac{\sqrt{2\pi}}{\pi} \int_0^{2\pi\lambda} J_{\beta+\frac{1}{2}}(s) s^{\beta+\frac{1}{2}} ds, & \text{if } d \text{ is odd}, \\
\int_0^{2\pi\lambda} J_{\beta+1}(s) s^{\beta} ds, & \text{if } d \text{ is even}.
\end{cases}
\]

4 Fourier inversion for the function \( U_{\beta,a}(x) \)

Recall that \( U_{\beta,a}(x) = \phi_{\beta,a}(|x|) \) with

\[
\phi_{\beta,a}(t) = \begin{cases}
(a^2 - t^2)^{\beta}, & 0 \leq t < a, \\
0, & t \geq a.
\end{cases}
\]

Then \( \phi_{\beta,a} \) and \( U_{\beta,a} \) have the following properties:

(i) \( \phi_{\beta,a} \) is continuous and bounded variation if and only if \( \beta > 0 \);

(ii) \( \phi_{0,a} \) is discontinuous and bounded variation and \( U_{0,a} \) is the indicator function of the ball in \( \mathbb{R}^d \) centered at the origin with the radius \( a \);

(iii) \( \phi_{\beta,a} \) is not bounded variation if and only if \( \beta < 0 \);

(iv) \( U_{\beta,a} \) is integrable on \( \mathbb{R}^d \) if and only if \( \beta > -1 \) for all dimensions \( d \).
In this section, for $\beta > -1$, we investigate the behavior of the Fourier spherical partial integral $\sigma_\lambda(U_{\beta,a})(x)$ as $\lambda \to \infty$.

For $a > 0$, let

$$\tilde{E}_a = \{ x \in \mathbb{R}^d : x \neq 0 \text{ and } |x| \neq a \},$$

$$\tilde{G}_a = \{ x \in \mathbb{R}^d : |x| = a \}.$$  \hspace{1cm} (4.1)

Then $\mathbb{R}^d = \{0\} \cup \tilde{E}_a \cup \tilde{G}_a$.

**Theorem 4.1.** Let $d \geq 1$, $a > 0$ and $\beta > -1$. Then

$$\sigma_\lambda(U_{\beta,a})(x) = 2^\beta \Gamma(\beta + 1)a^{2\beta} \int_0^{2\pi a \lambda} \frac{J_{\beta-1}(\frac{|x|}{a}s)J_{\beta+\beta}(s)}{\left(\frac{|x|}{a}\right)^{2-1}s^{\beta}} ds,$$  \hspace{1cm} (4.3)

for all $x \in \mathbb{R}^d$ and $\lambda > 0$. Moreover, $\sigma_\lambda(U_{\beta,a})$ has the following properties:

(i) At $x = 0$,

$$\sigma_\lambda(U_{\beta,a})(0) = U_{\beta,a}(0) - P_{\beta,a}^{[d]} \cos \left(2\pi a \lambda - \frac{d - 1 + 2\beta}{4}\pi\right) \lambda^\frac{d-3}{2} - \beta$$

$$+ O(\lambda^\frac{d-3}{2} - \beta) \text{ as } \lambda \to \infty,$$  \hspace{1cm} (4.4)

where $P_{\beta,a}^{[d]}$ is the constant defined by (1.14). Consequently,

(a) if $\beta > (d-3)/2$, then

$$\sigma_\lambda(U_{\beta,a})(0) = U_{\beta,a}(0) + O(\lambda^\frac{d-3}{2} - \beta) \text{ as } \lambda \to \infty,$$

(b) if $-1 < \beta \leq (d-3)/2$, then $\sigma_\lambda(U_{\beta,a})(x)$ reveals the Pinsky phenomenon, that is,

$$\liminf_{\lambda \to \infty} \frac{\sigma_\lambda(U_{\beta,a})(0) - U_{\beta,a}(0)}{\lambda^{(d-3)/2 - \beta}} = -P_{\beta,a}^{[d]},$$

$$\limsup_{\lambda \to \infty} \frac{\sigma_\lambda(U_{\beta,a})(0) - U_{\beta,a}(0)}{\lambda^{(d-3)/2 - \beta}} = P_{\beta,a}^{[d]}.$$

(ii) For $x \in \tilde{G}_a$,

$$\lim_{\lambda \to \infty} \frac{\sigma_\lambda(U_{\beta,a})(x)}{\lambda^{-\beta}} = L_{\beta,a},$$
where $L_{\beta,a}$ is the constant defined by (1.15), and consequently,

$$
\lim_{\lambda \to \infty} \sigma_\lambda(U_{\beta,a})(x) = \begin{cases} 
0, & \beta > 0, \\
\frac{1}{2}, & \beta = 0, \\
\infty, & -1 < \beta < 0. 
\end{cases}
$$

(iii) For $x \in \tilde{E}_a$,

$$
\sigma_\lambda(U_{\beta,a})(x) = U_{\beta,a}(x) + O(\lambda^{-\beta-1}) \quad \text{as} \quad \lambda \to \infty,
$$

where the last term $O(\lambda^{-\beta-1})$ is uniform on any compact subset of $\tilde{E}_a$.

(iv) If $\beta > 0$, then

$$
\lim_{\lambda \to \infty} \sigma_\lambda(U_{\beta,a})(x) = U_{\beta,a}(x) \quad \text{for} \quad x \in \tilde{E}_a \cup \tilde{G}_a
$$

where the convergence is uniform on any compact subset of $\tilde{E}_a \cup \tilde{G}_a$ and $\sigma_\lambda(U_{\beta,a})$ does not reveal the Gibbs-Wilbraham phenomenon.

(v) If $-1 < \beta \leq 0$, then

$$
\lim_{\lambda \to \infty} \sigma_\lambda(U_{\beta,a})(x) = U_{\beta,a}(x) \quad \text{for} \quad x \in \tilde{E}_a,
$$

where the convergence is uniform on any compact subset of $\tilde{E}_a$, and $\sigma_\lambda(U_{\beta,a})$ reveals a phenomenon like the Gibbs-Wilbraham phenomenon near $\tilde{G}_a$. More precisely, the following holds: For each $x_0 \in \tilde{G}_a$, let $\{x_\lambda^\pm\}$ be the sequences which satisfy $\lim_{\lambda \to \infty} x_\lambda^\pm = x_0$ and $|x_\lambda^\pm| = a \mp (2 \pm \beta)/(4\lambda)$. Then

$$
\lim_{\lambda \to \infty} \frac{\sigma_\lambda(U_{\beta,a})(x_\lambda^\pm) - U_{\beta,a}(x_\lambda^\pm)}{\lambda^{-\beta}} = G_{\beta,a}^\pm,
$$

and

$$
\lim_{\lambda \to \infty} \frac{\sigma_\lambda(U_{\beta,a})(x_\lambda^-) - U_{\beta,a}(x_\lambda^-)}{\lambda^{-\beta}} = G_{\beta,a}^-,
$$

where $G_{\beta,a}^\pm$ are the constants defined by (1.18).

Remark 4.1. The function $U_{\beta,a}$ is piecewise smooth in the sense of Pinsky [34] if and only if $\beta$ is a nonnegative integer. In this case the result in Theorem 4.1 (i) is contained in [34, Theorem 1a].
In the following we first prove (4.3) in Subsection 4.1. Then, using (4.3), we prove (i)–(v) of Theorem 4.1 in Subsections 4.2–4.6, respectively. Let

\[ U_{\beta,a,\lambda}(t) = 2^\beta \Gamma(\beta + 1)a^{2\beta} \int_0 \frac{J_{\frac{d}{2}-1} \left(\frac{t}{a}s\right) J_{\frac{d}{2}+\beta} \left(s \right)}{\left(\frac{\pi}{a}\right)^{\frac{d}{2}-1} s^{\beta}} ds. \]  

Then (4.3) means that

\[ \sigma_{\lambda}(U_{\beta,a})(x) = U_{\beta,a,\lambda}(|x|). \]  

### 4.1 Proof of (4.3)

By (2.19) the Fourier transform of \( U_{\beta,a}(x) \) is expressed by the following:

\[ \hat{U}_{\beta,a}(\xi) = \Gamma(\beta + 1)D_{\beta}(a^2 : \xi) = \Gamma(\beta + 1)\frac{a^{d+2\beta}}{\pi^\beta} \frac{J_{\frac{d}{2}+\beta}(2\pi|a|\xi)}{(a|\xi|)^{\frac{d}{2}+\beta}}. \]  

Since \( \hat{U}_{\beta,a}(\xi) \) is a radial function, using (2.5), we have

\[ \sigma_{\lambda}(U_{\beta,a})(x) = \int_{|\xi| < \lambda} \hat{U}_{\beta,a}(\xi)e^{2\pi i \xi \cdot x} d\xi \]

\[ = 2\pi \Gamma(\beta + 1)\frac{a^{d+2\beta}}{\pi^\beta} \int_0^\lambda \frac{J_{\frac{d}{2}-1}(2\pi|x|) J_{\frac{d}{2}+\beta}(2\pi as)}{(as)^{\frac{d}{2}+\beta}} s^{d-1} ds \]

\[ = 2^\beta \Gamma(\beta + 1)a^{2\beta} \int_0^{2\pi a \lambda} \frac{J_{\frac{d}{2}-1} \left(\frac{|x|}{a}s\right) J_{\frac{d}{2}+\beta} \left(s \right)}{\left(\frac{|x|}{a}\right)^{\frac{d}{2}-1} s^{\beta}} ds. \]

This shows (4.3).

### 4.2 Proof of Theorem 4.1 (i)

Since \( \sigma(U_{\beta,a})(0) = U_{\beta,a,\lambda}(0) \) and \( U_{\beta,a}(0) = \phi_{\beta,a}(0) = a^{2\beta} \), it is enough to prove that

\[ U_{\beta,a,\lambda}(0) = a^{2\beta} - P_{\beta,a} \cos \left(2\pi a \lambda - \frac{d - 1 + 2\beta}{4} \right) \lambda^{-\frac{d-\beta}{4}} + O(\lambda^{-\frac{d-\beta}{4}}). \]  

To do this we show the following two lemmas. The first lemma is for the cases \( d = 1, 2 \). For the cases \( d \geq 3 \) we have the conclusion by using both lemmas.

#### Lemma 4.2

Let \( \beta > -1, a > 0 \) and \( \lambda > 0 \). Then

\[ U_{\beta,a,\lambda}(0) = a^{2\beta} - P_{\beta,a} \cos \left(2\pi a \lambda - \frac{\beta}{2} \right) \lambda^{-\beta-1} + O(\lambda^{-\beta-2}) \]  

as \( \lambda \to \infty \).

\[ U_{\beta,a,\lambda}(0) = a^{2\beta} - P_{\beta,a} \cos \left(2\pi a \lambda - \frac{2\beta + 1}{4} \right) \lambda^{-\beta-\frac{1}{2}} + O(\lambda^{-\beta-\frac{3}{2}}) \]  

as \( \lambda \to \infty \).
Lemma 4.3. Let \( d \geq 3 \), \( \beta > -1 \), \( a > 0 \) and \( \lambda > 0 \), and let
\[
\mathcal{P}^{[d]}_{\beta,a,\lambda}(t) = -\frac{\Gamma(\beta + 1) a^\beta}{(\pi \lambda)^\beta} \sum_{\ell=1}^{d-1} J_{\frac{d-\ell}{2}}(2\pi a \lambda) J_{\frac{d-\ell}{2}}(2\pi t \lambda) \left( \frac{\ell}{a} \right)^{d-\ell}, \quad t \geq 0, \tag{4.9}
\]
where \( d_+ \) is defined by (2.10). Then
\[
U^{[d]}_{\beta,a,\lambda}(t) = \begin{cases} 
U^{[1]}_{\beta,a,\lambda}(t) + \mathcal{P}^{[d]}_{\beta,a,\lambda}(t), & \text{if } d \text{ is odd}, \\
U^{[2]}_{\beta,a,\lambda}(t) + \mathcal{P}^{[d]}_{\beta,a,\lambda}(t), & \text{if } d \text{ is even}, 
\end{cases} \quad t \geq 0. \tag{4.10}
\]
Moreover, if \( t \neq 0 \), then \( \mathcal{P}^{[d]}_{\beta,a,\lambda}(t) = O(\lambda^{-\beta - 1}) \) as \( \lambda \to \infty \). If \( t = 0 \), then
\[
\mathcal{P}^{[d]}_{\beta,a,\lambda}(0) = -a^{2\beta} \cos \left( \frac{\pi a \lambda - \frac{\beta}{2}}{\lambda} \right) + O(\lambda^{-\beta - 2}) \quad \text{as } \lambda \to \infty. \tag{4.11}
\]

Remark 4.2. (i) In Lemma 4.2 if \( \beta = -1/2 \), then
\[
U^{[2]}_{-1/2,a,\lambda}(0) = \frac{1 - \cos (2\pi a \lambda)}{a} + O(\lambda^{-1}) \quad \text{as } \lambda \to \infty. \tag{4.12}
\]
since \( P^{[2]}_{-1/2,a} = a^{-1} \). This shows the amplitude of Pinsky phenomenon which was found by Taylor [44].
(ii) Recently, Grafakos and Teschl [7] showed some related results with Lemma 4.3.

Proof of Lemma 4.2. Let \( d = 1 \). By (4.5) with (2.1) and Corollary 3.2 we have
\[
U^{[1]}_{\beta,a,\lambda}(0) = 2^\beta \Gamma(\beta + 1) a^{2\beta} \sqrt{\frac{\pi}{2}} \int_0^{2\pi a \lambda} \frac{J_{\frac{1}{2} + \beta}(s)}{s^{1/2 + \beta}} ds
\]
and
\[
a^{2\beta} = 2^\beta \Gamma(\beta + 1) a^{2\beta} \sqrt{\frac{\pi}{2}} \int_0^{\infty} \frac{J_{\frac{1}{2} + \beta}(s)}{s^{1/2 + \beta}} ds,
\]
respectively. Then by Lemma 3.3 we have
\[
U^{[1]}_{\beta,a,\lambda}(0) - a^{2\beta} = -2^\beta \Gamma(\beta + 1) a^{2\beta} \sqrt{\frac{\pi}{2}} \int_0^{\infty} \frac{J_{\frac{1}{2} + \beta}(s)}{s^{1/2 + \beta}} ds
\]
\[
= -P^{[1]}_{\beta,a} \cos \left( \frac{\pi a \lambda - \frac{\beta}{2}}{\lambda} \right) \lambda^{-\beta - 1} + O(\lambda^{-\beta - 2}).
\]
Let $d = 2$. By (4.5) with (2.1) we have

$$U_{\beta,a,\lambda}^{[2]}(0) = 2^\beta \Gamma(\beta + 1) a^{2\beta} \int_0^{2\pi a \lambda} \frac{J_{1+\beta}(s)}{s^{\beta}} ds$$

$$= 2^\beta \Gamma(\beta + 1) a^{2\beta} \left[ \frac{J_{\beta}(s)}{s^{\beta}} \right]_0$$

$$= a^{2\beta} - a^{2\beta} \Gamma(\beta + 1) \frac{J_{\beta}(2\pi a \lambda)}{(\pi a \lambda)^{\beta}}$$

$$= a^{2\beta} - P_{\beta,a}^{[2]} \cos \left( 2\pi a \lambda - \frac{2\beta + 1}{4}\pi \right) \lambda^{-\beta - 1/2} + O(\lambda^{-\beta - 3/2}).$$

Then the proof is complete. \(\square\)

**Proof of Lemma 4.3.** The equation (4.10) follows from Collorary 3.14 and the definition of $U_{\beta,a,\lambda}^{[d]}(t)$ (see (4.5)). If $t \neq 0$, then from (2.17) it follows that

$$J_{\frac{d}{2}-\ell+\beta}(2\pi a \lambda) J_{\frac{d}{2}-\ell}(2\pi t \lambda) = O(\lambda^{-1}), \quad \ell = 1, \ldots, d - 1.$$

Then we have $P_{\beta,a,\lambda}^{[d]}(t) = O(\lambda^{-\beta - 1})$. If $t = 0$, then by (2.1) we regard

$$\frac{J_{\frac{d}{2}-\ell+\beta}(2\pi a \lambda) J_{\frac{d}{2}-\ell}(2\pi t \lambda)}{(\frac{\pi}{\lambda})^{\frac{d}{2}-\ell}} = \frac{(\pi a \lambda)^{\frac{d}{2}-\ell}}{\Gamma(\frac{d}{2} - \ell + 1)} J_{\frac{d}{2}-\ell+\beta}(2\pi a \lambda), \quad \ell = 1, \ldots, d - 1.$$

Then, using (2.17), we have

$$P_{\beta,a,\lambda}^{[d]}(0)$$

$$= - \frac{\Gamma(\beta + 1) a^{\beta} \sum_{\ell=1}^{d - 1} \frac{(\pi a \lambda)^{\frac{d}{2}-\ell}}{\Gamma(\frac{d}{2} - \ell + 1)} J_{\frac{d}{2}-\ell+\beta}(2\pi a \lambda)}{(\pi a \lambda)^{\beta}}$$

$$= - \frac{\Gamma(\beta + 1) a^{\beta} (\pi a \lambda)^{\frac{d}{2}-1}}{(\pi a \lambda)^{\beta}} J_{\frac{d}{2}-1+\beta}(2\pi a \lambda) + O(\lambda^{\frac{d-\beta}{2}})$$

$$= \left( - \frac{\Gamma(\beta + 1)}{\Gamma(\frac{d}{2})} a^{\frac{d-\beta}{2}+\beta} \lambda^{\frac{d-\beta}{2}-\beta} \cos \left( 2\pi a \lambda - \frac{d - 1 + 2\beta}{4}\pi \right) \right) + O(\lambda^{\frac{d-\beta}{2}}).$$

This is the conclusion. \(\square\)
4.3 Proof of Theorem 4.1 (ii)
For \( x \in \tilde{G}_a \), by (4.5) and (4.6) we have
\[
\sigma_\lambda(U_{\beta,a})(x) = U_{\beta,a,\lambda}^{[d]}(a) = 2^\beta \Gamma(\beta + 1) a^{2\beta} \int_0^{2\pi a \lambda} \frac{J_{\frac{d}{2}+\beta}(s)J_{\frac{d}{2}-1}(s)}{s^\beta} \, ds.
\]
Then, using Corollary 3.12 with \( \mu = \frac{d}{2} - 1 \), we have
\[
U_{\beta,a,\lambda}^{[d]}(a) = L_{\beta,a} \lambda^{-\beta} + \begin{cases} O(\lambda^{-\beta-1}), & \beta \geq 0, \\ O(1), & -1 < \beta < 0, \end{cases} \quad \text{as } \lambda \to \infty,
\]
which shows the conclusion.

4.4 Proof of Theorem 4.1 (iii)
Let \( x \in \tilde{E}_a \) and \( |x| = t \). Then \( 0 < t < a \) or \( t > a \). In this case, by (4.5), (4.6) and Corollary 3.5, we have
\[
\sigma_\lambda(U_{\beta,a})(x) - U_{\beta,a}(x) = U_{\beta,a,\lambda}^{[d]}(t) - \phi_{\beta,a}(t)
\]
\[
= -2^\beta \Gamma(\beta + 1) a^{2\beta} \int_0^{2\pi a \lambda} \frac{J_{\frac{d}{2}-1}(\frac{t}{a}s)J_{\frac{d}{2}+\beta}(s)}{(\frac{t}{a})^{\frac{d}{2}-1} s^\beta} \, ds.
\]
Using Lemma 3.8 with \( \mu = \frac{d}{2} - 1 \), we have
\[
U_{\beta,a,\lambda}^{[d]}(t) - \phi_{\beta,a}(t)
\]
\[
= -2^\beta \Gamma(\beta + 1) a^{2\beta} \frac{\psi_{\beta}(\lambda, a-t) + \left(\frac{a}{t}\right)^{\frac{d}{2}+1} \psi_{\beta}(\lambda, a-t)}{\pi} O(\lambda^{-\beta-1}).
\]
Since \( \psi_{\beta}(\lambda, a-t) = |a-t|^{-1}O(\lambda^{-\beta-1}) \) by Lemma 3.8 (i),
\[
U_{\beta,a,\lambda}^{[d]}(t) - \phi_{\beta,a}(t) = O(\lambda^{-\beta-1})
\]
uniformly on any closed interval in \((0, a) \cup (a, \infty)\). This shows the conclusion.

4.5 Proof of Theorem 4.1 (iv)
Let \( \beta > 0 \). By (4.13) we have
\[
U_{\beta,a,\lambda}^{[d]}(a) - \phi_{\beta,a}(a) = U_{\beta,a,\lambda}^{[d]}(a) = L_{\beta,a} \lambda^{-\beta} + O(\lambda^{-\beta-1}).
\]
If \(0 < t < a\) or \(t > a\), then by (4.14) and Lemma 3.7 (ii) we have
\[
U_{\beta,a,\lambda}^{[d]}(t) - \phi_{\beta,a}(t) = \left(\frac{a}{t}\right)^{d-1} O(\lambda^{-\beta}) + \left(\frac{a}{t}\right)^{d-1} \left(1 + \frac{a}{t}\right) O(\lambda^{-\beta-1}),
\]
where the terms \(O(\lambda^{-\beta})\) and \(O(\lambda^{-\beta-1})\) are uniform with respect to \(t\). Hence, \(U_{\beta,a,\lambda}^{[d]}(t)\) converges to \(\phi_{\beta,a}(t)\) uniformly on any closed interval in \((0, \infty)\). That is, \(\sigma_\lambda(U_{\beta,a})(x)\) converges to \(U_{\beta,a}(x)\) uniformly on any compact subset of \(\tilde{E}_a \cup \tilde{G}_a = \mathbb{R}^d \setminus \{0\}\), and consequently, it does not reveal any phenomenon like the Gibbs-Wilbraham phenomenon.

### 4.6 Proof of Theorem 4.1 (v)

Let \(-1 < \beta \leq 0\). The first assertion is already shown by (iii). Next we show the Gibbs-Wilbraham phenomenon. Here, we recall that
\[
G_{\beta,a}^\pm = \pm \frac{\Gamma(\beta + 1) a^\beta (2 \pm \beta)^{\beta}}{\pi 2^\beta} \int_{\pi}^{\infty} \frac{\sin s}{(s \pm \frac{\beta}{2} \pi)^{\beta+1}} ds.
\]
By (4.14) we have
\[
\sigma_\lambda(U_{\beta,a})(x_\lambda^+) - U_{\beta,a}(x_\lambda^+) = U_{\beta,a,\lambda}^{[d]}(|x_\lambda^+|) - \phi_{\beta,a}(|x_\lambda^+|)
= - \frac{2^\beta \Gamma(\beta + 1) a^\beta}{\pi} \left(\frac{a}{|x_\lambda^+|}\right)^{d-1} \psi_\beta(\lambda, a - |x_\lambda^+|) + \left(\frac{a}{|x_\lambda^+|}\right)^{d-1} \left(1 + \frac{a}{|x_\lambda^+|}\right) O(\lambda^{-\beta-1}).
\]
Since \(a - |x_\lambda^+| = \frac{2 + \beta}{4 \lambda}\), using Lemma 3.7 (iii), we have
\[
-\psi_\beta(\lambda, a - |x_\lambda^+|) = \left(\frac{2 + \beta}{4 \lambda}\right)^\beta \int_{\pi}^{\infty} \frac{-\sin s}{(s \pm \frac{\beta}{2} \pi)^{\beta+1}} ds.
\]
Observing
\[
\left(\frac{a}{|x_\lambda^+|}\right)^{d-1} = \left(1 - \frac{2 + \beta}{4 \lambda a}\right)^{-\frac{d-1}{2}} = 1 + O(\lambda^{-1}),
\]
we conclude that
\[
\frac{\sigma_\lambda(U_{\beta,a})(x_\lambda^+) - U_{\beta,a}(x_\lambda^+)}{\lambda^{-\beta}}
= - \frac{2^\beta \Gamma(\beta + 1) a^\beta}{\pi} \left(\frac{2 + \beta}{4}\right)^\beta \int_{\pi}^{\infty} \frac{\sin s}{(s \pm \frac{\beta}{2} \pi)^{\beta+1}} ds \left(1 + O(\lambda^{-1})\right) + O(\lambda^{-1})
\rightarrow G_{\beta,a}^\pm\text{ as } \lambda \rightarrow \infty.
In a similar way we also have
\[ \frac{\sigma_\lambda(U_{\beta,a}(x_\lambda^\ast) - U_{\beta,a}(x_\lambda))}{\lambda^{-\beta}} \to G_{\beta,a}^- \quad \text{as} \quad \lambda \to \infty. \]

5 Results related to lattice point problems

The terms \( \Delta_j(s : x) \), \( j = 0, 1, \cdots \), are closely related to lattice point problems which have been studied by Landau, Jarník, Szegő, Novák and others, see \[5, 13, 15, 16, 26, 27, 46\]. Recall that
\[ \Delta_\alpha(s : x) = D_\alpha(s : x) - D_\alpha(s : x), \quad \alpha > -1, \ s \geq 0, \ x \in \mathbb{R}^d, \]
where
\[
D_\alpha(s : x) = \begin{cases} 
\frac{1}{\Gamma(\alpha + 1)} \sum_{|m|^2 < s} (s - |m|^2)^\alpha e^{2\pi i m x}, & s > 0, \ x \in \mathbb{R}^d, \\
0, & s = 0,
\end{cases}
\]
\[
D_\alpha(s : x) = \begin{cases} 
\frac{1}{\Gamma(\alpha + 1)} \int_{|\xi|^2 < s} (s - |\xi|^2)^\alpha e^{2\pi i \xi x} d\xi, & s > 0, \ x \in \mathbb{R}^d, \\
0, & s = 0,
\end{cases}
\]

In this section we consider the behavior of \( \Delta_\alpha(s : x) \) as \( s \to \infty \). In Stein \[39\], the estimation of \( \Delta_{d-1}(s : x) \) was treated. We shall prove the following four lemmas: In the following \( f(s) = \Omega(g(s)) \) means \( f(s) \neq o(g(s)) \).

Lemma 5.1. Let \( d \geq 1 \). Then, as \( s \to \infty \),
\[ \Delta_\alpha(s : x) = \begin{cases} 
O(s^{\frac{d}{2} - \frac{d}{2} + \varepsilon}), & \text{if } \alpha = 0, \\
O(s^{\frac{d}{2} - \frac{d}{2} + \varepsilon}) \text{ for every } \varepsilon > 0, & \text{if } 0 < \alpha \leq \frac{d-1}{2}, \\
O(s^{\frac{d}{2} - \frac{d}{2} + \varepsilon}), & \text{if } \alpha > \frac{d-1}{2},
\end{cases} \] (5.1)
uniformly with respected to \( x \in \mathbb{T}^d \).

Lemma 5.2. Let \( d \geq 5 \). If \( 0 \leq \alpha < (d - 4)/2 \), then, as \( s \to \infty \),
\[ \Delta_\alpha(s : x) = \begin{cases} 
O(s^{\frac{d}{2} - 1}), \ \Omega(s^{\frac{d}{2} - 1}) & \text{for } x \in \mathbb{T}^d \cap \mathbb{Q}^d, \\
o(s^{\frac{d}{2} - 1}) & \text{for } x \in \mathbb{T}^d \setminus \mathbb{Q}^d.
\end{cases} \] (5.2)

If \( (d - 4)/2 \leq \alpha \leq (d - 1)/2 \), then, for every \( \varepsilon > 0 \), as \( s \to \infty \),
\[ \Delta_\alpha(s : x) = \begin{cases} 
O(s^{\frac{d}{2} - \frac{d-1}{2} + \varepsilon}), & \text{for } x \in \mathbb{T}^d \cap \mathbb{Q}^d, \\
o(s^{\frac{d}{2} - \frac{d-1}{2} + \varepsilon}) & \text{for } x \in \mathbb{T}^d \setminus \mathbb{Q}^d.
\end{cases} \] (5.3)
The following lemma gives more precise information on the estimate (5.2) for $x \in \mathbb{T}^d \cap \mathbb{Q}^d$.

**Lemma 5.3.** Let $d \geq 5$, $\alpha = 0$ and $\beta > -1$. For $k \in \mathbb{N} = \{1, 2, \ldots \}$, let

$$
\ell_k = \ell_k(d, \beta) = \frac{1}{2a} \left( k + \frac{d + 2\beta + 1}{4} - \frac{1}{4} \right),
$$

$$
m_k = m_k(d, \beta) = \frac{1}{2a} \left( k + \frac{d + 2\beta + 1}{4} + \frac{1}{4} \right),
$$

Then, for large $k \in \mathbb{N}$, there exists $s_k \in [\ell_k, m_k]$ such that

$$
|\Delta_0(s_k : x)| \geq C(s_k) s_k^{d-1}, \quad x \in \mathbb{T}^d \cap \mathbb{Q}^d,
$$

where $C(x)$ is a positive constant dependent on $x$, but independent of $s_k$ for large $k$.

**Lemma 5.4.** Let $d \geq 4$ and $0 \leq \alpha \leq (d-1)/2$. Then, for every $\varepsilon > 0$, as $s \to \infty$,

$$
\Delta_\alpha(s : x) = O(s^{d/2 + \frac{d-2}{2} \alpha + \varepsilon}) \quad \text{for a.e. } x \in \mathbb{T}^d.
$$

To prove above four lemmas we state known results (Theorems 5.5–5.9): See also Jarník [14] for Theorem 5.5. For $\alpha \geq 0$, let

$$
P_\alpha(s : x) = D_\alpha(s : x) - \frac{\pi^{d/2}s^{d/2 + \alpha}}{\Gamma(d/2 + \alpha + 1)} \delta(x), \quad x \in \mathbb{R}^n, \quad s \geq 0,
$$

where $\delta(x)$ is the indicator function of $\mathbb{Z}^d$.

**Theorem 5.5 (Landau [25, 27]).** Let $d \geq 2$. Then, for $x \in \mathbb{R}^d$,

$$
P_\alpha(s : x) = \begin{cases} 
O(s^{\frac{d}{2} + \alpha - \frac{d-1}{2}}), & \text{if } 0 \leq \alpha < \frac{d-1}{2}, \\
O(s^{\frac{d-1}{2} \log s}), & \text{if } \alpha = \frac{d-1}{2}, \\
O(s^{\frac{d+1}{2} + \alpha}), & \text{if } \alpha > \frac{d-1}{2}. 
\end{cases}
$$

**Theorem 5.6 (Novák [32]).** Let $d \geq 5$, and let $0 \leq \alpha < (d-4)/2$. Then

$$
P_\alpha(s : x) = \begin{cases} 
O(s^{\frac{d}{2} - 1}), & \text{if } x \in \mathbb{Q}^d, \\
o(s^{\frac{d}{2} - 1}), & \text{if } x \notin \mathbb{Q}^d, \\
O(s^{\frac{d}{2} + \frac{d}{2} \log \tau} s), & \text{for a.e. } x \in \mathbb{R}^d,
\end{cases}
$$

where $\tau = 3d$ if $\alpha = 0$ and $\tau = 3d - 1$ if $\alpha > 0$. 
Theorem 5.7 (Novák [30]). Let \( d \geq 3 \). Then, for all \( x \in \mathbb{Q}^d \), there exists a positive constant \( K_d(x) \) such that

\[
\int_0^s |P_0(t : x)|^2 \, dt = \begin{cases} 
K_d(x)s^2 \log s + O(s^2 \log \frac{1}{2} s), & \text{if } d = 3, \\
K_d(x)s^3 + O(s^{5/2} \log s), & \text{if } d = 4, \\
K_d(x)s^4 + O(s^3 \log^2 s), & \text{if } d = 5, \\
K_d(x)s^{d-1} + O(s^{d-2}), & \text{if } d \geq 6.
\end{cases}
\]

(5.11)

Remark 5.1. In Theorem 5.7 the positive constant \( K_d(x) \) is given explicitly for each \( x \in \mathbb{Q}^d \), see [22]. For example, if \( d \geq 4 \) and \( x = 0 \), then

\[
K_d(0) = \frac{\pi^d (2^d + 8) \zeta(d - 2)}{12(d - 1)(2^d - 1)\zeta(d)\Gamma^2(d/2)},
\]

where \( \zeta \) is the Riemann’s zeta function, see [46].

Theorem 5.8 (Kuratsubo [17]). Let \( d \geq 2 \). Then, for every \( \tau > \frac{3}{2} \),

\[
P_0(s : x) = O(s^{\frac{d}{4}} \log^\tau s) \quad \text{for a.e. } x.
\]

Remark 5.2. Theorems 5.5–5.8 valid for \( \Delta_\alpha(s : x) \) instead of \( P_\alpha(s : x) \), if \( x \in \mathbb{T}^d \). Actually,

\[
\Delta_\alpha(s : x) - P_\alpha(s : x) = \begin{cases} 
0, & \text{if } x = 0, \\
-\mathcal{D}_\alpha(s : x) = O(s^{\frac{d-1}{4} + \frac{\alpha}{2}}), & \text{if } x \in \mathbb{T}^d \setminus \{0\},
\end{cases}
\]

since (see (2.19) and (2.17))

\[
\mathcal{D}_\alpha(s, x) = \begin{cases} 
\frac{\pi^d s^{\frac{d}{2} + \alpha}}{\Gamma\left(\frac{d}{2} + \alpha + 1\right)}, & x = 0, \\
\frac{s^{\frac{d}{2} + \alpha} J_{\frac{d}{2} + \alpha}(2\pi \sqrt{s}|x|)}{\left(\sqrt{s|x|}\right)^{\frac{d}{2} + \alpha}} = O(s^{\frac{d-1}{4} + \frac{\alpha}{2}}), & x \neq 0.
\end{cases}
\]

Therefore,

\[
|\Delta_\alpha(s : x)| = |P_\alpha(s : x)| + O(s^{\frac{d-1}{4} + \frac{\alpha}{2}}), \quad \text{if } x \in \mathbb{T}^d.
\]

Remark 5.3. Let \( d = 1 \). Then

\[
\Delta_\alpha(s : x) = \begin{cases} 
O(1), & \text{if } \alpha = 0, \\
O(s^{\frac{\alpha}{2}}), & \text{if } \alpha > 0,
\end{cases}
\]

(5.12)
uniformly with respect to $x \in T$. Actually, for all $s > 0$, choosing $N \in \mathbb{N}$ such that $N < \sqrt{s} \leq N + 1$, we have by an elementary calculation

$$
\Delta_0(s : x) = \frac{\sin(2\pi(N + \frac{1}{2})x)}{\sin \pi x} - \frac{\sin(2\pi\sqrt{s}x)}{\pi x} = O(1).
$$

For $\alpha > 0$, from (2.19) and (2.20) it follows that

$$
\Delta_\alpha(s : x) = \frac{s^{\frac{1}{2}+\alpha}}{\pi^\alpha} \sum_{m \in \mathbb{Z}, m \neq 0} J_{\frac{1}{2}+\alpha}(2\pi\sqrt{s}|x - m|) \left(\sqrt{s}|x - m|\right)^{\frac{1}{2}+\alpha}.
$$

By (2.17) we have

$$
|J_{\frac{1}{2}+\alpha}(2\pi\sqrt{s}|x - m|)| \leq \frac{C}{(\sqrt{s}|x - m|)^{1+\alpha}},
$$

for some positive constant $C$. Since $|x - m| \geq 1/2$ for $x \in T$ and $m \neq 0$, the sum converges absolutely and $\Delta_\alpha(s : x) = O(s^{\frac{\alpha}{2}})$.

The following is the Riesz’ convexity theorem. (see [40, page 285] and [3, page 13]).

**Theorem 5.9.** Let $0 \leq \alpha_0 < \alpha_1 < \infty$. For $x \in \mathbb{R}^d$, let $V_0(s : x)$ and $V_1(s : x)$ be two positive nondecreasing functions with respect to $s > 0$. Assume that

$$
|\Delta_{\alpha_i}(s : x)| \leq V_i(s : x), \quad i = 0, 1.
$$

Then, for $0 \leq \theta \leq 1$,

$$
|\Delta_{(1-\theta)\alpha_0 + \theta \alpha_1}(s : x)| \leq CV_0(s : x)^{1-\theta}V_1(s : x)\theta,
$$

where $C$ is a positive constant dependent on $\alpha_1$, $\alpha_0$ and $\theta$, and independent of $s$ and $x$.

**Proof of Lemma 5.1.** The case $d = 1$ has been already proven in Remark 5.3. Then we consider the case $d \geq 2$. In general, for a function $f : [0, \infty) \rightarrow \mathbb{R}$, the difference of $f$ with $h > 0$ is defined by

$$
\delta_h f(s) = \delta_h^1 f(s) = f(s + h) - f(s),
$$
and

$$
\delta_h^{k+1} f(s) = \delta_h^k f(s + h) - \delta_h^k f(s), \quad k = 1, 2, \cdots.
$$

Then

$$
\delta_h^k f(s) = \int_s^{s+h} ds_1 \cdots \int_{s_{k-1}}^{s_{k-1}+h} f^{(k)}(s_k) ds_k.
$$

Let $k = d_\#$ as in (2.10) and use this relation for $f(s) = \Delta_k(s : x)$. Then, from (2.18) and (2.21) we see that

$$
\delta_h^k \Delta_k(s : x) - h^k \Delta_0(s : x)
= \int_s^{s+h} ds_1 \cdots \int_{s_{k-1}}^{s_{k-1}+h} \left( \Delta_0(s_k : x) - \Delta_0(s : x) \right) ds_k
= \int_s^{s+h} ds_1 \cdots \int_{s_{k-1}}^{s_{k-1}+h} \left( D_0(s_k : x) - D_0(s : x) \right) ds_k
= \int_s^{s+h} ds_1 \cdots \int_{s_{k-1}}^{s_{k-1}+h} \left( \sum_{s \leq |m|^2 < s_k} e^{2\pi im \cdot x} - \int_{s \leq |\xi|^2 < s_k} e^{2\pi i \xi \cdot x} \right) ds_k.
$$

Using $s_k \leq s + kh$ inside the integration, we have

$$
\left| \sum_{s \leq |m|^2 < s_k} e^{2\pi im \cdot x} - \int_{s \leq |\xi|^2 < s_k} e^{2\pi i \xi \cdot x} \right|
\leq \sum_{s \leq |m|^2 < s_k} 1 + \int_{s \leq |\xi|^2 < s_k} \frac{1}{1}
\leq \sum_{s \leq |m|^2 < s + kh} 1 + \int_{s \leq |\xi|^2 < s + kh} \frac{1}{1}
= D_0(s + kh : 0) - D_0(s : 0) + v_d \left( (s + kh)^2 - s^2 \right)
\leq |\Delta_0(s + kh : 0)| + |\Delta_0(s : 0)| + 2v_d \left( (s + kh)^2 - s^2 \right)
\leq |\Delta_0(s + kh : 0)| + |\Delta_0(s : 0)| + dkhv_d(s + kh)^{\frac{d}{2} - 1},
$$
where $v_d$ is the volume of the $d$-dimensional unit ball. Hence
\[
\left| \delta_h^k \Delta_k(s : x) - h^k \Delta_0(s : x) \right| \\
\leq \int_s^{s+h} ds_1 \cdots \int_{s_{k-1}+h}^{s_k+h} \left| \sum_{s \leq |m|^2 < s_k} e^{2\pi i m} - \sum_{s \leq |\xi|^2 < s_k} e^{2\pi i \xi} \right| ds_k \\
\leq h^k \left( |\Delta_0(s + kh : 0)| + |\Delta_0(s : 0)| + dkhv_d(s + kh)^{\frac{d}{2} - 1} \right).
\]
By Theorem 5.5 with $\alpha = 0$ and $x = 0$, we have
\[
\left| \delta_h^k \Delta_k(s : x) - h^k \Delta_0(s : x) \right| \leq Ch^k \left( s^\frac{d}{2} - \pi t^2 + hs^{\frac{d}{2} - 1} \right), \quad \text{if} \ kh \leq s,
\]
where $C$ is a positive constant independent of $h$, $s$ and $x$.

Next we estimate $\delta_h^k \Delta_k(s : x)$. By (2.19) and (2.20) we have
\[
\left| \delta_h^k \Delta_k(s : x) \right| = \left| \delta_h^k \left( \sum_{m \in \mathbb{Z}^d, m \neq 0} \frac{s^{\frac{1}{2} + \frac{d}{4} + k} J_{\frac{d}{4} + k}(2\pi \sqrt{m} |x - m|)}{\pi^k} \right) \right| \\
= \sum_{m \in \mathbb{Z}^d, m \neq 0} \delta_h^k \left( s^{\frac{1}{2} + \frac{d}{4} + k} J_{\frac{d}{4} + k}(2\pi \sqrt{m} |x - m|) \right) \frac{1}{\pi^k |x - m|^{\frac{d}{2} + k}}.
\]
Here, using a well known inequality on the Bessel function (see the equation (37) in [25, page 472]),
\[
\left| \delta_h^k \left( s^{\frac{1}{2} + \frac{d}{4} + k} J_{\frac{d}{4} + k}(2\pi \sqrt{st}) \right) \right| \leq C s^{\frac{d-1}{2}} t^\frac{d}{4} (\min(s, h^2t))^{\frac{d}{2}},
\]
where $C$ is a positive constant independent of $s$, $t$ and $h$, we have
\[
\left| \delta_h^k \Delta_k(s : x) \right| \leq C \sum_{m \in \mathbb{Z}^d, m \neq 0} s^{\frac{d-1}{2}} (\min(s, h^2 |x - m|^2))^{\frac{d}{2}} \frac{1}{|x - m|^{\frac{d}{2} + k}} \\
= \left( \sum_{|x - m|^2 \leq s/h^2, m \neq 0} \frac{h^k s^{\frac{d-1}{2}}}{|x - m|^{\frac{d}{2} + k}} + \sum_{|x - m|^2 > s/h^2} \frac{s^{\frac{d-1}{2}} + h^k s^{\frac{d-1}{2}} + k}{|x - m|^{\frac{d}{2} + k}} \right) \\
\leq C \left( h^k s^{\frac{d-1}{2}} \left( \frac{s}{h^2} \right)^{\frac{d-1}{4}} + s^{\frac{d-1}{4} + \frac{1}{2}} \left( \frac{s}{h^2} \right)^{\frac{d-1}{4} + \frac{1}{2}} \right) = Ch^k \left( \frac{s}{h} \right)^{\frac{d-1}{2}}.
\]
That is
\[
\left| \delta_h^k \Delta_k(s : x) \right| \leq Ch^k \left( \frac{s}{h} \right)^{\frac{d-1}{2}}.
\]

(5.14)
Therefore, combining (5.13) and (5.14), we have
\[
|\Delta_0(s : x)| \leq h^{-k} \left( |\delta_h^k \Delta_k(s : x) - k^k \Delta_0(s : x)| + |\delta_h^k \Delta_k(s : x)| \right)
\leq C \left( s^\frac{d}{2} - \frac{s^d}{\pi^{d/2}} + h s^{\frac{d}{2} - 1} \right) + C \left( \frac{s}{h} \right)^{\frac{d+1}{2}}, \quad \text{if } kh \leq s.
\]
Then, setting \( h = (s/h)^{\frac{d+1}{2}} \), that is, \( h = s^{\frac{1}{d+1}} \), we have that
\[
|\Delta_0(s : x)| \leq C s^{\frac{d+1}{2} - \frac{d}{4}}, \quad \text{if } s \geq k^{d+1}.
\]
where \( C \) is a positive constant independent of \( x \in \mathbb{T}^d \) and \( s \geq k^{d+1} \).

If \( \alpha > (d - 1)/2 \), then we have by (2.19) and (2.20) that
\[
\Delta_{\alpha}(s : x) = s^{\frac{d+\alpha}{\pi^{\alpha}}} \sum_{m \in \mathbb{Z}^d, m \neq 0} J_{\frac{d+\alpha}{2}}(2\pi \sqrt{s}|x - m|) \left( \sqrt{s}|x - m| \right)^{\frac{d+\alpha}{2}}.
\]
Moreover, since \(|x - m| \geq 1/2 \) for \( x \in \mathbb{T}^d \) and \( m \neq 0 \), the sum converges absolutely and
\[
|\Delta_{\alpha}(s : x)| \leq C s^{\frac{d+\alpha}{4} + \frac{\alpha}{2}},
\]
where \( C \) is a positive constant independent of \( x \in \mathbb{T}^d \) and \( s > 0 \). Therefore, we have
\[
\Delta_{\alpha}(s : x) = \begin{cases} O(s^{\frac{d}{4} + \frac{\alpha}{2}}), & \text{if } \alpha = 0, \\ O(s^{\frac{d}{4} + \frac{\alpha}{2} + \varepsilon}), & \text{if } \alpha > \frac{d-1}{2}, \end{cases}
\]
uniformly with respect to \( x \in \mathbb{T}^d \).

Applying Theorem 5.9 as
\[
\alpha_0 = 0, \quad \alpha_1 = \frac{d - 1}{2} \quad \text{and} \quad \alpha = \theta \alpha_1,
\]
we have
\[
\Delta_{\alpha}(s : x) = O(s^{\frac{d}{4} + \frac{\alpha}{2} + \varepsilon}) \quad \text{for every } \varepsilon > 0, \text{ if } 0 < \alpha \leq \frac{d-1}{2},
\]
since
\[
(1 - \theta) \left( \frac{d}{2} - \frac{d}{d+1} \right) + \theta \left( \frac{d-1}{4} + \frac{\alpha_1}{2} \right) = \frac{d}{2} - \frac{d}{d+1} + \frac{\alpha}{d+1}.
\]
Then the proof is complete. \( \square \)
Remark 5.4. We have the following comparison between Lemma 5.1 and Landau’s estimate (5.9):
\[
\frac{d}{2} - \frac{d}{d + 1} + \alpha \frac{d}{d + 1} < \frac{d}{2} + \alpha - \frac{d}{d + 1 - 2\alpha}, \quad \text{if } 0 < \alpha < \frac{d - 1}{2}.
\]

Proof of Lemma 5.2. If \(0 \leq \alpha < (d - 4)/2\), then, by Theorem 5.6 and Remark 5.2 we have (5.2) immediately. Next we show (5.3). By Theorems 5.5, 5.6 and Remark 5.2 we have
\[
\Delta_\alpha(s : x) = \begin{cases} O(s^{\frac{d}{2} - 1}) & \text{for } x \in \mathbb{T}^d \cap \mathbb{Q}^d, \\ o(s^{\frac{d}{2} - 1}) & \text{for } x \in \mathbb{T}^d \setminus \mathbb{Q}^d, \end{cases} \quad \text{if } \alpha < \frac{d - 4}{2},
\]
and
\[
\Delta_\alpha(s : x) = O(s^{\frac{d - 1}{2} \log s}) \quad \text{for } x \in \mathbb{T}^d, \quad \text{if } \alpha = \frac{d - 1}{2}.
\]

Applying Theorem 5.9 as \(\alpha_0 = \frac{d - 4}{2}, \alpha_1 = \frac{d - 1}{2}\) and \(\alpha = (1 - \theta)\alpha_0 + \theta\alpha_1\), we have
\[
\Delta_\alpha(s : x) = O(s^{\frac{d - 1 + \alpha + \epsilon}{2}}) \quad \text{for every } \epsilon > 0, \text{ if } \frac{d - 4}{2} \leq \alpha \leq \frac{d - 1}{2},
\]

since
\[
(1 - \theta) \left( \frac{d}{2} - 1 \right) + \theta \left( \frac{d - 1}{2} \right) = \frac{d - 1 + \alpha}{3}.
\]

Proof of Lemma 5.3. By Theorem 5.7 and Remark 5.2 we have
\[
\int_0^s |\Delta_0(t : x)|^2 \, dt = K_d(x)s^{d - 1} + O(s^{d - 2} \log^\tau s),
\]
where \(\tau = 2\) if \(d = 5\), \(\tau = 0\) if \(d \geq 6\). Using
\[
m_k^2 - \ell_k^2 = \frac{1}{(2a)^2} \left( k + \frac{d + 2\beta + 1}{4} \right),
\]
\[
m_k^{2(d - 1)} - \ell_k^{2(d - 1)} = \frac{d - 1}{(2a)^2(d - 1)} k^{2d - 3} + O(k^{2d - 4}),
\]
we have
\[
\frac{1}{m_k^2 - \ell_k^2} \int_0^{m_k^2} |\Delta_0(t : x)|^2 \, dt = \frac{1}{m_k^2 - \ell_k^2} \left( \int_0^{m_k^2} |\Delta_0(t : x)|^2 \, dt - \int_0^{\ell_k^2} |\Delta_0(t : x)|^2 \, dt \right) = K_d(x) \frac{m_k^{2(d - 1)} - \ell_k^{2(d - 1)}}{m_k^2 - \ell_k^2} + O(k^{2(d - 2) - 1} \log^\tau k)
\]
\[
= \frac{d - 1}{(2a)^2(d - 2)} K_d(x) k^{2d - 4} + O(k^{2(d - 2) - 1} \log^\tau k).
\]
Hence we can take $\tilde{K}_d(x)$ such that
$$\frac{1}{m_k^2 - \ell_k^2} \int_{\ell_k^2}^{m_k^2} |\Delta_0(t : x)|^2 dt \geq \tilde{K}_d(x)^2 m_k^{2d - 4} \quad \text{for large } k.$$ 

Therefore, for large $k$, there exists $s_k \in [\ell_k^2, m_k^2]$ such that
$$|\Delta_0(s_k : x)| \geq \tilde{K}_d(x) m_k^{d - 2} \geq \tilde{K}_d(x) s_k^{\frac{d}{2} - 1}.$$ 

Proof of Lemma 5.4. By Theorems 5.5, 5.8 and Remark 5.2 we have
$$\Delta_0(s, x) = \begin{cases} O(s^d \log^\tau s), & \text{if } \alpha = 0, \\ O(s^{\frac{d-1}{2}} \log s), & \text{if } \alpha = \frac{d-1}{2}, \end{cases} \quad \text{for a.e. } x \in \mathbb{T}^d.$$ 

Applying Theorem 5.9 as
$$\alpha_0 = 0, \alpha_1 = \frac{d-1}{2} \quad \text{and} \quad \alpha = \theta \alpha_1,$$
we have
$$\Delta_0(s, x) = O(s^{\frac{d}{2} + \frac{d-2}{d(d-1)} \alpha + \varepsilon}) \quad \text{for every } \varepsilon > 0, \text{ if } 0 < \alpha < \frac{d-1}{2},$$
since
$$(1 - \theta) \frac{d}{4} + \theta \frac{d-1}{2} = \frac{d}{4} + \frac{d-2}{2(d-1)} \alpha.$$ 

Remark 5.5. We have the following comparison between Lemma 5.1 and Lemma 5.4:
$$\frac{d}{4} + \frac{d-2}{2(d-1)} \alpha < \frac{d}{2} - \frac{d}{d+1} + \frac{\alpha}{d+1}, \quad \text{if } d \geq 4 \text{ and } 0 \leq \alpha < \frac{d-1}{2}.$$ 

6 Proof of the main theorems

In this section we prove the pointwise convergence of the Fourier series of the function $u_{\beta, \alpha}(x)$ described in the main theorems (Theorems 1.1–1.5). First we state a generalized Hardy’s identity (Theorem 6.1) and three lemmas (Lemmas 6.4–6.6). Next, using the generalized Hardy’s identity and three lemmas, we will prove the main theorems in Subsection 6.2. The proofs of Theorem 6.1 and Lemmas 6.4–6.6 are in Subsections 6.3, 6.4, 6.5 and 6.6 respectively.
6.1 Generalized Hardy’s identity and three lemmas

Recall that $u_{\beta,a}(x)$ is the periodization of $U_{\beta,a}(x) = \phi_{\beta,a}(|x|)$ with

$$\phi_{\beta,a}(t) = \begin{cases} (a^2 - t^2)^\beta, & 0 \leq t < a, \\ 0, & t \geq a. \end{cases}$$

That is,

$$u_{\beta,a}(x) = \sum_{m \in \mathbb{Z}^d} U_{\beta,a}(x + m) = \sum_{|x + m| < a} (a^2 - |x + m|^2)^\beta, \quad x \in \mathbb{T}^d. \quad (6.1)$$

Let $\Delta_j, A^{(j)}_{\beta,a}, d_\sharp$ and $\psi_\beta$ be as in (2.3), (2.13), (2.10) and (3.2), respectively, and let

$$K_{\beta,a}(s : x) = \sum_{j=0}^{d_\sharp} (-1)^j \Delta_j(s : x) A^{(j)}_{\beta,a}(s), \quad (6.2)$$

and

$$G_{\beta,a}(\lambda, x) = \frac{\Gamma(\beta + 1)(2a)^\beta}{\pi} \sum_{m \in \mathbb{Z}^d \setminus \{0\}, |x-m| \neq a} \left( \frac{a}{|x-m|} \right)^{\frac{d+1}{2} + d_\sharp} \psi_\beta(\lambda, |a - |x-m||). \quad (6.3)$$

First we show the convergence of the infinite sum in (6.3). If $x \in \mathbb{T}^d$ and $m \in \mathbb{Z}^d \setminus \{0\}$, then $|x - m| > 1/2$ and then

$$\sup_{x \in \mathbb{T}^d} \left\{ \sum_{m \in \mathbb{Z}^d \setminus \{0\}, |x-m| \neq a} \frac{1}{|x-m|^{\frac{d+1}{2} + d_\sharp}} \right\} \leq C \left\{ \sum_{m \in \mathbb{Z}^d \setminus \{0\}, |x-m| \neq a} \frac{1}{|x-m|^{\frac{d+1}{2} + d_\sharp}} \right\} M_a(x) \lambda^{-\beta - 1} \quad (6.4)$$

since $\frac{d+1}{2} + d_\sharp > d$. By Lemma 3.7 (i) we have that

$$\psi_\beta(\lambda, |a - |x-m||) = |a - |x-m||^{-1} O(\lambda^{-\beta - 1}) \quad \text{as} \quad \lambda \to \infty.$$
Hence,

\[ G_{\beta,a}(\lambda, x) = M_a(x)O(\lambda^{-\beta - 1}) \quad \text{as} \quad \lambda \to \infty. \]  

Next, recall that

\[ r_d(a : x) = \sum_{m \in \mathbb{Z}^d, |x - m| = a} 1, \quad x \in \mathbb{T}^d, \]

and let

\[ \tilde{r}_d(a : x) = \sum_{m \in \mathbb{Z}^d \setminus \{0\}, |x - m| = a} 1, \quad x \in \mathbb{T}^d. \]  

Then

\[ \tilde{r}_d(a : 0) = r_d(a : 0) \quad \text{and} \quad \tilde{r}_d(a : x) = r_d(a : x) = 0 \quad \text{for} \quad x \in E_a. \]

**Theorem 6.1** (Generalized Hardy’s identity). Let \( d \geq 1, \beta > -1 \) and \( a > 0 \). Then

\[ S_{\lambda}(u_{\beta,a})(x) = u_{\beta,a}(x) + (\sigma_{\lambda}(U_{\beta,a})(x) - U_{\beta,a}(x)) \]

\[ + \left( G_{\beta,a}(\lambda : x) + \tilde{r}_d(a : x) \left( L_{\beta,a} + o(1) \right) \lambda^{-\beta} \right) \]

\[ + K_{\beta,a}(\lambda^2 : x) + O(\lambda^{-\beta - 1}) \quad \text{as} \quad \lambda \to \infty \]  

for all \( x \in \mathbb{T}^d \).

If \( 0 < a < 1/2 \), then \( \tilde{r}_d(a : x) = 0 \) and \( u_{\beta,a}(x) = U_{\beta,a}(x) \) in \( x \in \mathbb{T}^d \). Combining these and \((6.6)\), we have the following corollary of Theorem 6.1.

**Corollary 6.2.** Let \( d \geq 1, \beta > -1 \) and \( 0 < a < 1/2 \). Then

\[ S_{\lambda}(u_{\beta,a})(x) = \sigma_{\lambda}(U_{\beta,a})(x) + K_{\beta,a}(\lambda^2 : x) + O(\lambda^{-\beta - 1}) \quad \text{as} \quad \lambda \to \infty \]  

for all \( x \in \mathbb{T}^d \).

If \( d = 2, \beta = 0 \) and \( a > 0 \), then \( U_{0,a}(x) \) is the indicator function of the ball \( \{ x \in \mathbb{R}^2 : |x| < a \} \). In this case, from Theorem 4.1 (i) (a), (ii) and (iii) it follows that

\[ \lim_{\lambda \to \infty} \left( \sigma_{\lambda}(U_{0,a})(x) - U_{0,a}(x) \right) = \begin{cases} L_{0,a} = \frac{1}{2}, & |x| = a, \\ 0, & |x| \neq a, \end{cases} \; x \in \mathbb{R}^2. \]

Then we conclude that

\[ \lim_{\lambda \to \infty} \left( \sigma_{\lambda}(U_{0,a})(x) - U_{0,a}(x) \right) + \tilde{r}_2(a : x)L_{0,a} = \frac{1}{2} r_2(a : x) = \frac{1}{2} \sum_{|x+m| = a} 1, \; x \in \mathbb{T}^2. \]
By (6.6) and Lemma 6.4 below we also have

\[ G_{0,a}(\lambda : x) + K_{0,a}(\lambda^2 : x) \to 0 \quad \text{as} \quad \lambda \to \infty, \quad x \in T^2. \]

Since \( u_{0,a}(x) = \sum_{|x+m|<a} 1 \) (see (6.1)), we have the following corollary.

**Corollary 6.3** ([18, Theorem] and [2, page 446]). Let \( d = 2, \beta = 0 \) and \( a > 0 \). Then

\[
\lim_{\lambda \to \infty} S_\lambda(u_{0,a})(x) = \sum_{|x+m|<a} 1 + \frac{1}{2} \sum_{|x+m|=a} 1,
\]

for all \( x \in T^2 \).

**Remark 6.1.** Let \( d = 2, \beta = 0 \) and \( a > 0 \). By (1.8) and (4.7) together with (2.1) we have \( \hat{u}_{0,a}(0) = \pi a^2 \) and \( \hat{u}_{0,a}(m) = a J_1(2\pi a|m|)/|m| \) for \( m \neq 0 \). Then

\[
S_\lambda(u_{0,a})(x) = \pi a^2 + \sum_{0<|m|<\lambda} \frac{J_1(2\pi a|m|)}{|m|} e^{2\pi imx}.
\]

In particular, if \( x = 0 \), then Corollary 6.3 shows Hardy’s identity that

\[
\lim_{\lambda \to \infty} \left( \pi a^2 + \sum_{0<|m|<\lambda} \frac{J_1(2\pi a|m|)}{|m|} \right) = \sum_{|m|<a} 1 + \frac{1}{2} \sum_{|m|=a} 1, \tag{6.11}
\]

which was studied by Voronoï [15] (1905), Hardy [8] (1915) and Hardy and Landau [10] (1924). Therefore, we shall call the identity (6.9) the generalized Hardy’s identity.

**Remark 6.2** (The ”serendipitous phenomenon” by M. Taylor). The observation of Taylor [33 44] can be stated as follows under our notation: For \( d = 2, \beta = -1/2 \) and \( 0 < a < 1/2 \), Corollary 6.2 implies that

\[
S_\lambda(u_{-\frac{1}{2},a})(x) = \sigma_\lambda(U_{-\frac{1}{2},a})(x) + \frac{\Delta_0(\lambda^2 : x)}{\lambda} \sin(2\pi a\lambda) + O(\lambda^{-\frac{1}{2}}). \tag{6.12}
\]

He found a certain choppiness of the graphs of \( S_\lambda(u_{-\frac{1}{2},a})(x) \) outside the disk \( \{|x| < a\} \). This choppiness is absent for the analogous partial Fourier inversion \( \sigma_\lambda(U_{-\frac{1}{2},a})(x) \), since the graphs of \( \sigma_\lambda(U_{-\frac{1}{2},a})(x) \) are rotational symmetry. He also found a further surprise that, for certain discrete values of \( \lambda \), this choppiness magically clears up,
and $S_\lambda(u_{-\frac{1}{2},a})(x)$ behaves about as nicely on the torus as does $\sigma_\lambda(U_{-\frac{1}{2},a})(x)$ on the Euclidean space. Now, from (6.12) we know that the set of these discrete values of $\lambda$ is

$$\{\lambda > 0 : \sin(2\pi a \lambda) = 0\}.$$  

For example, if $a = 3/(4\pi)$, then $\lambda = (2\pi/3)n$ for $n = 1, 2, \ldots$, which are correspondent with Figure 7A $(n = 6)$ and Figure 7F $(n = 7)$ in [43]. For the coefficient of $\sin(2\pi a \lambda)$, we can calculate by Lemma 5.1 as

$$\Delta_0(\lambda^2 : x) = O(\lambda^{-\frac{1}{3}}).$$

In the rest of this subsection we state three lemmas. Recall that

$$c(d) = d - \frac{2d}{d + 1} - \frac{d + 1}{2} = \frac{d - 5}{2} + \frac{2}{d + 1} = \frac{d(d - 4) - 1}{2(d + 1)} = \frac{d - 3}{2} - \frac{d - 1}{d + 1}.$$

**Lemma 6.4.** Let $d \geq 1$, $\beta > -1$ and $a > 0$. Then

$$K_{\beta,a}(\lambda^2 : x) = O(\lambda^{c(d) - \beta}) \text{ uniformly on } \mathbb{T}^d.$$  

Consequently, if $1 \leq d \leq 4$ and $\beta > c(d)$, then

$$\lim_{\lambda \to \infty} K_{\beta,a}(\lambda^2 : x) = 0 \text{ uniformly on } \mathbb{T}^d,$$

and, if $d \geq 2$, then

$$\lim_{\lambda \to \infty} \frac{|K_{\beta,a}(\lambda^2 : x)|}{\lambda^{\frac{d-3}{2} - \beta}} = 0 \text{ uniformly on } \mathbb{T}^d.$$

**Lemma 6.5.** Let $d \geq 5$, $\beta > -1$ and $a > 0$. Then

$$\lim_{\lambda \to \infty} \frac{|K_{\beta,a}(\lambda^2 : x)|}{\lambda^{\frac{d-3}{2} - \beta}} = 0, \quad \text{if } x \in \mathbb{T}^d \setminus \mathbb{Q}^d,$$

$$0 < \limsup_{\lambda \to \infty} \frac{|K_{\beta,a}(\lambda^2 : x)|}{\lambda^{\frac{d-5}{2} - \beta}} < \infty, \quad \text{if } x \in \mathbb{T}^d \cap \mathbb{Q}^d.$$  

**Lemma 6.6.** Let $d \geq 4$, $\beta > -1/2$ and $a > 0$. Then

$$\lim_{\lambda \to \infty} K_{\beta,a}(\lambda^2 : x) = 0, \quad a.e. x \in \mathbb{T}^d.$$

### 6.2 Proof of Theorems 1.1–1.5

In this section, using Theorem 6.1 and Lemmas 6.4, 6.6, we prove the main theorems.
6.2.1 Proof of Theorem 1.1

For all $\beta > -1$, from (6.6) and Lemma 6.4 it follows that

$$G_{\beta,a}(\lambda : 0) = O(\lambda^{-\beta-1}) \quad \text{and} \quad K_{\beta,a}(\lambda^2 : 0) = O(\lambda^{\frac{d-\beta}{2}}),$$

respectively. Since $\tilde{r}_d(a,0) = r_d(a,0)$ as in (6.8), by Theorem 6.1 we have

$$S_{\lambda}(u_{\beta,a})(0) = u_{\beta,a}(0) + (\sigma_\lambda(U_{\beta,a})(0) - U_{\beta,a}(0)) + r_d(a : 0)(L_{\beta,a} + o(1))\lambda^{-\beta} + O(\lambda^{\frac{d-\beta}{2}}), \quad (6.13)$$

If $\beta > (d-3)/2$, then, using Theorem 4.1 (i) (a), we have

$$\sigma_\lambda(U_{\beta,a})(0) - U_{\beta,a}(0) = O(\lambda^{\frac{d-\beta}{2}}).$$

Combining this with (6.13), we have

$$S_{\lambda}(u_{\beta,a})(0) = u_{\beta,a}(0) + r_d(a : 0)\lambda^{-\beta} + O(\lambda^{\frac{d-\beta}{2}}),$$

which shows (i).

If $d \geq 2$ and $-1 < \beta \leq (d-3)/2$, then, using (4.4) in Theorem 4.1 (i), we have

$$\sigma_\lambda(U_{\beta,a})(0) - U_{\beta,a}(0) = -P^{[d]}_{\beta,a}\cos\left(2\pi a\lambda - \frac{d-1+2\beta}{4}\pi\right)\lambda^{\frac{d-3}{2}} + O(\lambda^{\frac{d-5}{2}}),$$

which shows (ii).

6.2.2 Proof of Theorem 1.3

By Theorem 4.1 (ii) and (iii) we have

$$\lim_{\lambda \to \infty} \frac{\sigma_\lambda(U_{\beta,a})(x) - U_{\beta,a}(x)}{\lambda^{-\beta}} = \begin{cases} L_{\beta,a}, & x \in \tilde{G}_a, \\ 0, & x \in \tilde{E}_a, \end{cases}$$

which shows

$$\lim_{\lambda \to \infty} \frac{\sigma_\lambda(U_{\beta,a})(x) - U_{\beta,a}(x)}{\lambda^{-\beta}} + \tilde{r}_d(a : x)L_{\beta,a} = r_d(a : x)L_{\beta,a}, \quad x \in E_a \cup G_a. \quad (6.14)$$

Proof of (i) (a) Let $1 \leq d \leq 4$. By (6.6) and Lemma 6.4 we have

$$\frac{G_{\beta,a}(\lambda : x)}{\lambda^{-\beta}} = O(\lambda^{-1}), \quad \frac{K_{\beta,a}(\lambda^2 : x)}{\lambda^{-\beta}} = O(\lambda^{\frac{d-3}{2}} + \frac{2}{\pi\lambda}) = o(1).$$
By Theorem 6.1 and (6.14) we have

\[
S_\lambda(u_{\beta,a})(x) - u_{\beta,a}(x) \\
= \frac{\sigma_\lambda(U_{\beta,a})(x) - U_{\beta,a}(x)}{\lambda^{-\beta}} + \frac{G_{\beta,a}(\lambda : x)}{\lambda^{-\beta}} + r_d(a : x)\left(L_{\beta,a} + o(1)\right) \\
+ \frac{K_{\beta,a}(\lambda^2 : x)}{\lambda^{-\beta}} + O(\lambda^{-1}) \\
\to r_d(a : x)L_{\beta,a} \quad \text{as} \quad \lambda \to \infty,
\]

which shows the conclusion.

**Proof of (i) (b)** Let \(\beta > c(d) = \frac{d-5}{2} + \frac{2}{d+1}\). Then by Lemma 6.4 we have

\[
K_{\beta,a}(\lambda^2 : x) = O(\lambda^{\frac{d-5}{2} + \frac{2}{d+1} - \beta}) = o(1) \quad \text{uniformly on} \quad \mathbb{T}^d.
\]

If \(x \in E_a\), then \(\hat{r}_d(a : x) = 0\) as (6.8). By Theorem 4.1 (iii) and (6.6) we have

\[
\lim_{\lambda \to \infty} \left(\sigma_\lambda(U_{\beta,a})(x) - U_{\beta,a}(x)\right) = 0, \quad \lim_{\lambda \to \infty} G_{\beta,a}(\lambda : x) = 0
\]

uniformly on any compact set in \(E_a\). Hence, by Theorem 6.1 we have

\[
S_\lambda(u_{\beta,a})(x) - u_{\beta,a}(x) \\
= (\sigma_\lambda(U_{\beta,a})(x) - U_{\beta,a}(x)) + G_{\beta,a}(\lambda : x) + K_{\beta,a}(\lambda^2 : x) + O(\lambda^{-\beta-1}) \\
\to 0 \quad \text{as} \quad \lambda \to \infty
\]

uniformly on any compact set in \(E_a\), which shows the conclusion.

**Proof of (ii)** Let \(d \geq 5\) and \(x \in E_a \cup G_a\). By Theorem 6.1 we have

\[
\frac{1}{\lambda^{\frac{d-5}{2}}} \left(\frac{S_\lambda(u_{\beta,a})(x) - u_{\beta,a}(x)}{\lambda^{-\beta}} - r_d(a : x)L_{\beta,a}\right) \\
= \frac{1}{\lambda^{\frac{d-5}{2}}} \left(\frac{\sigma_\lambda(U_{\beta,a})(x) - U_{\beta,a}(x)}{\lambda^{-\beta}} + r_d(a : x)\left(L_{\beta,a} + o(1)\right) - r_d(a : x)L_{\beta,a}\right) \\
+ \frac{G_{\beta,a}(\lambda : x)}{\lambda^{\frac{d-5}{2} - \beta}} + \frac{K_{\beta,a}(\lambda^2 : x)}{\lambda^{\frac{d-5}{2} - \beta}} + O(\lambda^{-\frac{d-5}{2} - 1}).
\]

By (6.14), (6.6) and Lemma 6.5 we have

\[
\lim_{\lambda \to \infty} \frac{1}{\lambda^{\frac{d-5}{2}}} \left(\frac{S_\lambda(u_{\beta,a})(x) - u_{\beta,a}(x)}{\lambda^{-\beta}} - r_d(a : x)L_{\beta,a}\right) = \lim_{\lambda \to \infty} \frac{K_{\beta,a}(\lambda^2 : x)}{\lambda^{\frac{d-5}{2} - \beta}} = 0
\]
for all $x \in (E_a \cup G_a) \setminus \mathbb{Q}^d$, and

$$0 < \limsup_{\lambda \to \infty} \frac{1}{\lambda^\frac{d-2}{2}} \left| \frac{S_\lambda(u_{\beta,a})(x) - u_{\beta,a}(x)}{\lambda^{-\beta}} - r_d(a : x)L_{\beta,a} \right| = \limsup_{\lambda \to \infty} \frac{|K_{\beta,a}(\lambda^2 : x)|}{\lambda^\frac{d-2}{2} - \beta} < \infty$$

for all $x \in (E_a \cup G_a) \cap \mathbb{Q}^d$. The proof is complete.

6.2.3 Proof of Theorem 1.4

Let $1 \leq d \leq 4$, $c(\beta) < \beta \leq 0$ and $0 < a < 1/2$. Then by Lemma 6.4 we have

$$K_{\beta,a}(\lambda^2 : x^\pm_{\lambda}) = \sigma(1).$$

If $m \neq 0$, then $|m - x_0| \geq 1 - a$, since $|x_0| = a$. Then, for large $\lambda$, we have

$$|a - x^\pm_{\lambda} - m| \geq |m - x_0| - |x^\pm_{\lambda} - x_0| - a \geq 1 - 2a - |x^\pm_{\lambda} - x_0| \geq \frac{1 - 2a}{2} > 0.$$

Hence, by (6.6) we have

$$G_{\beta,a}(\lambda : x^\pm_{\lambda}) = O(\lambda^{-1}).$$

Since $x^\pm_{\lambda} \in E_a$, $\hat{r}_d(a : x^\pm_{\lambda}) = 0$ as in (6.8). By Theorem 4.1 (iii) and (6.6) we have

$$\frac{S_\lambda(u_{\beta,a})(x^\pm_{\lambda}) - u_{\beta,a}(x^\pm_{\lambda})}{\lambda^{-\beta}} = \left( \frac{\sigma_\lambda(U_{\beta,a})(x^\pm_{\lambda}) - U_{\beta,a}(x^\pm_{\lambda})}{\lambda^{-\beta}} \right) + \frac{G_{\beta,a}(\lambda : x^\pm_{\lambda})}{\lambda^{-\beta}} + \frac{K_{\beta,a}(\lambda^2 : x^\pm_{\lambda})}{\lambda^{-\beta}} + O(\lambda^{-1})
\rightarrow G_{\beta,a}^\pm \text{ as } \lambda \to \infty,$$

which is the conclusion.

6.2.4 Proofs of Theorem 1.5

Let $d \geq 4$, $\beta > -1/2$ and $a > 0$. Then by lemma 6.6 we have

$$\lim_{\lambda \to \infty} K_{\beta,a}(\lambda^2 : x) = 0, \text{ a.e. } x \in \mathbb{T}^d.$$

Let $x \in E_a$. Then $\hat{r}_d(a : x) = 0$ as in (6.8). By Theorem 4.1 (iii) and (6.6) we have

$$\sigma_\lambda(U_{\beta,a})(x) - U_{\beta,a}(x) = O(\lambda^{-\beta - 1}) \text{ and } G_{\beta,a}(\lambda : x) = O(\lambda^{-\beta - 1}).$$
Then by Theorem 6.1 we have, for a.e. $x \in E_a$,

\[
S_\lambda(u_{\beta,a})(x) - u_{\beta,a}(x) = (\sigma_\lambda(U_{\beta,a})(x) - U_{\beta,a}(x)) + G_{\beta,a}(\lambda : x) + K_{\beta,a}(\lambda^2 : x) + O(\lambda^{-\beta-1})
\]

\[
\rightarrow 0 \quad \text{as} \quad \lambda \rightarrow \infty,
\]

which shows the conclusion, since the measure of $T^d \setminus E_a$ is zero.

### 6.3 Proof of Theorem 6.1

By Corollary 2.2 and (6.2) we have that, for all $x \in T^d$,

\[
S_\lambda(u_{\beta,a})(x) = \sigma_\lambda(U_{\beta,a})(x) + K_{\beta,a}(\lambda^2 : x)
\]

\[
+ (-1)^{d_1+1} \sum_{m \in 2^d \setminus \{0\}} \int_0^{\lambda^2} \mathcal{D}_{d_1}(s : x - m)A_{\beta,a}^{(d_1+1)}(s) \, ds.
\]

Using (2.19) and (2.13) we have

\[
\mathcal{D}_{d_1}(s : x - m)A_{\beta,a}^{(d_1+1)}(s) = \frac{\pi^{d_1} J_{d_1+1/2}((2\pi \sqrt{s})^1 |x - m|)}{\pi^{d_1} (\sqrt{s} |x - m|)^{d_1+1/2}} (-1)^{d_1+1} \frac{\Gamma(\beta + 1)}{\pi^\beta (d_1+1)} \frac{J_{\beta}((2\pi \sqrt{s})^1 |x - m|)}{\pi^\beta} \frac{J_{\beta+d_1+1}((2\pi a \sqrt{s})^1)}{\pi a}.
\]

Then, letting $u = 2\pi a \sqrt{s}$, we have

\[
(-1)^{d_1+1} \int_0^{\lambda^2} \mathcal{D}_{d_1}(s : x - m)A_{\beta,a}^{(d_1+1)}(s) \, ds = 2\pi \Gamma(\beta + 1) a^{2\beta} \int_0^{2\pi a \lambda} J_{\beta}((x-m)^1/a) J_{\beta+d_1+1}((x-m)^1/a) \frac{u^\beta}{(x-m)^1/a} \, du.
\]

If $|x - m| \neq a$, then Corollary 3.5 shows that

\[
2\pi \Gamma(\beta + 1) a^{2\beta} \int_0^{\infty} \frac{J_{\beta}((x-m)^1/a) J_{\beta+d_1+1}(u)}{(x-m)^1/a} \frac{u^\beta}{(x-m)^1/a} \, du = U_{\beta,a}(x - m) = \begin{cases} 
0, & \text{if } a < |x - m|, \\
(a^2 - |x - m|^2)^\beta, & \text{if } 0 < |x - m| < a.
\end{cases}
\]

(6.16)
If \( m \neq 0 \) and \(|x - m| \neq a\), then Lemma 3.8 shows that

\[
2^\beta \Gamma(\beta + 1) a^{2\beta} \int_{2\pi a \lambda}^\infty J_{\frac{d}{2} + d_j} \left( \frac{|x - m|}{a} u \right) \frac{J_{\frac{d}{2} + \beta + d_j + 1}(u)}{\left( \frac{|x - m|}{a} \right)^{\frac{d}{2} + d_j} u^\beta} \, du
\]

\[
= 2^\beta \Gamma(\beta + 1) a^{\beta} \left( \frac{a}{|x - m|} \right)^{\frac{d}{2} + d_j} \psi(\lambda, a - |x - m|) + \frac{1}{|x - m|^{\frac{d}{2} + d_j + 1}} O(\lambda^{-\beta - 1}),
\]

(6.17)

since \(|x - m| \geq 1/2\) for \( m \neq 0 \) and \( x \in \mathbb{T}^d\). Combining (6.15), (6.16) and (6.17), and observing (6.3)–(6.5), we have

\[
(-1)^{d_j + 1} \sum_{m \in \mathbb{Z}^d \setminus \{0\}, \, |x - m| \neq a} \int_0^{\lambda^2} D_{d_j}(s : x - m) A_{\beta, a}^{(d_j + 1)}(s) \, ds
\]

\[
= \left( \sum_{m \in \mathbb{Z}^d \setminus \{0\}, \, |x - m| \neq a} U_{\beta, a}(x - m) \right) - G_{\beta, a}(\lambda : x) + O(\lambda^{-\beta - 1}).
\]

\[
= u_{\beta, a}(x) - U_{\beta, a}(x) - G_{\beta, a}(\lambda : x) + O(\lambda^{-\beta - 1}).
\]

On the other hand, if \( m \neq 0 \) and \(|x - m| = a\), then, using (6.15) and Corollary 3.12, we have

\[
(-1)^{d_j + 1} \int_0^{\lambda^2} D_{d_j}(s : x - m) A_{\beta, a}^{(d_j + 1)}(s) \, ds
\]

\[
= 2^\beta \Gamma(\beta + 1) a^{2\beta} \int_{2\pi a \lambda}^\infty J_{\frac{d}{2} + d_j} \left( \frac{a}{|x - m|} u \right) \frac{J_{\frac{d}{2} + \beta + d_j + 1}(u)}{u^\beta} \, du
\]

\[
= L_{\beta, a} \lambda^{-\beta} + \begin{cases} O(\lambda^{-\beta - 1}), & \beta \geq 0, \\ O(1), & -1 < \beta < 0 \end{cases}
\]

\[
= (L_{\beta, a} + o(1)) \lambda^{-\beta}.
\]

Therefore, we have the conclusion.

### 6.4 Proof of Lemma 6.4

To estimate

\[
K_{\beta, a}(\lambda^2 : x) = \sum_{j=0}^{d_j} (-1)^j \Delta_j(\lambda^2 : x) A_{\beta, a}^{(j)}(\lambda^2),
\]
we combine the estimates of $\Delta_j(\lambda^2 : x)$ and $A_{\beta,a}(\lambda^2)$. Firstly, by (2.17) we see that

$$A_{\beta,a}(s) = (-1)^j \frac{\Gamma(\beta + 1)}{\pi^{\beta-j}} a_{\beta+j} J_{\frac{1}{2}+\beta+j} \left(2\pi a \sqrt{s}\right) \cdot \frac{s^{\frac{1}{2}+\beta+j}}{s^{\frac{1}{2}+\beta+j}}.$$  

Comparing (6.20), (6.21) and (6.22), we have the conclusion.

That is, for some positive constant $C$,

$$|A_{\beta,a}(\lambda^2)| \leq C\lambda^{-\left(\frac{d}{2}+\beta+j+\frac{1}{2}\right)}, \quad \lambda \geq 1,$$  

(6.19)

For the terms $\Delta_j(s : x)$, we use Lemma 5.1, that is,

$$\Delta_j(\lambda^2 : x) = \begin{cases} O(\lambda^{d-2d\pi^{-1}}), & \text{if } j = 0, \\ O(\lambda^{d-2d\pi^{-1}+2j+\varepsilon}) & \text{for every } \varepsilon > 0, \\ O(\lambda^{d_{1j}+j}), & \text{if } j > \frac{d-1}{2}, \end{cases}$$

If $j = 0$, then

$$|\Delta_0(\lambda^2 : x)A_{\beta,a}(\lambda^2)| \leq C\lambda^{d-2d\pi^{-1}} \lambda^{-\left(\frac{d}{2}+\beta+\frac{1}{2}\right)} = C\lambda^{\frac{d\pi}{2}+\frac{2\pi}{(d+1)}-\beta}. \quad (6.20)$$

If $0 < j < d_1$, then $0 < j < (d-1)/2$ and then, for any small $\varepsilon > 0$,

$$|\Delta_j(\lambda^2 : x)A_{\beta,a}(\lambda^2)| \leq C\lambda^{d-2d\pi^{-1}+2\pi^{-1}+\varepsilon} \lambda^{-\left(\frac{d}{2}+\beta+j+\frac{1}{2}\right)} = C\lambda^{\frac{d\pi}{2}+\frac{2\pi}{(d+1)}-\beta+\varepsilon}. \quad (6.21)$$

If $j = d_1$, then $j > (d-1)/2$ and then

$$|\Delta_{d_1}(\lambda^2 : x)A_{\beta,a}(\lambda^2)| \leq C\lambda^{d_{1d_1}+d_{1}+\frac{d}{2}+\beta+j+\frac{1}{2}} = C\lambda^{-1}. \quad (6.22)$$

Comparing (6.20), (6.21) and (6.22), we have the conclusion.

### 6.5 Proof of Lemma 6.5

By Lemma 5.2, we have that, as $\lambda \rightarrow \infty$,

$$\Delta_j(\lambda^2 : x) = \begin{cases} O(\lambda^{d-2}) & \text{for } x \in \mathbb{T}^d \cap \mathbb{Q}^d \text{ and if } 0 \leq j < (d-4)/2, \\ o(\lambda^{d-2}) & \text{for } x \in \mathbb{T}^d \setminus \mathbb{Q}^d \text{ and if } 0 \leq j < (d-4)/2, \\ O(\lambda^{2(d-1)\pi^{-1}+\varepsilon}) & \text{for } x \in \mathbb{T}^d \text{ and if } (d-4)/2 \leq j \leq (d-1)/2. \end{cases}$$
Combining this and (6.19), we have the following estimates: If $0 \leq j < (d - 4)/2$ and $x \in T^d \setminus Q^d$, then, for some decreasing function $\varphi(\lambda)$ which satisfies $\varphi(\lambda) \to 0$ as $\lambda \to \infty$,

$$|\Delta_j(\lambda^2 : x)A_{\beta,a}^{(j)}(\lambda^2)| \leq C\lambda^{d-2}\varphi(\lambda)\lambda^{-\left(\frac{d}{2} + j + \frac{1}{2}\right)} = C\lambda^{\frac{d-4}{2} - \beta - j}\varphi(\lambda).$$

If $0 \leq j < (d - 4)/2$ and $x \in T^d \cap Q^d$, then, for some decreasing function $\varphi(\lambda)$ which satisfies $\varphi(\lambda) \to 0$ as $\lambda \to \infty$,

$$|\Delta_j(\lambda^2 : x)A_{\beta,a}^{(j)}(\lambda^2)| \leq C\lambda^{d-2}\lambda^{-\left(\frac{d}{2} + j + \frac{1}{2}\right)} = C\lambda^{\frac{d-4}{2} - \beta - j}.\]$$

If $(d - 4)/2 \leq j \leq (d - 1)/2$, then, for small $\varepsilon > 0$,

$$|\Delta_j(\lambda^2 : x)A_{\beta,a}^{(j)}(\lambda^2)| \leq C\lambda^{\frac{2(d-1)}{d-4} + \varepsilon}\lambda^{-\left(\frac{d}{2} + j + \frac{1}{2}\right)} = C\lambda^{\frac{d-7}{2} - \beta - \frac{1}{2} + \varepsilon} \leq C\lambda^{-\frac{1}{2} - \beta + \varepsilon}.$$

If $j = d^*$, then we have (6.22). Comparing these estimates, we have

$$|K_{\beta,a}(\lambda^2 : x)| \leq \begin{cases} C\lambda^{\frac{d-4}{2} - \beta}\varphi(\lambda) & \text{for } x \in T^d \setminus Q^d, \\ C\lambda^{\frac{d-4}{2} - \beta} & \text{for } x \in T^d \cap Q^d, \end{cases}$$

which shows

$$\lim_{\lambda \to \infty} \frac{|K_{\beta,a}(\lambda^2 : x)|}{\lambda^{\frac{d-4}{2} - \beta}} = 0, \quad \text{if } x \in T^d \setminus Q^d,$$

$$\limsup_{\lambda \to \infty} \frac{|K_{\beta,a}(\lambda^2 : x)|}{\lambda^{\frac{d-4}{2} - \beta}} < \infty, \quad \text{if } x \in T^d \cap Q^d.$$

Next we shall prove

$$0 < \limsup_{\lambda \to \infty} \frac{|K_{\beta,a}(\lambda^2 : x)|}{\lambda^{\frac{d-4}{2} - \beta}}, \quad \text{if } x \in T^d \cap Q^d.$$

If $j = 0$ and $x \in T^d \cap Q^d$, then, using Lemma 5.3, we have

$$|\Delta_0(\lambda^2_k : x)| \geq C(x)\lambda_k^{d-2}, \quad \lambda_k \in [\ell_k, m_k], \quad k \in \mathbb{N}.$$  

On the other hand, by (6.18),

$$A_{\beta,a}(\lambda^2) = \frac{\Gamma(\beta + 1) a^{\frac{d}{2} + \beta - \frac{1}{2}}}{\pi^{\beta + 1}} \frac{\cos\left(2\pi a\lambda - \frac{d + 2\beta + 1}{4}\right)}{\lambda^{\frac{d}{2} + \beta + \frac{1}{2}}} + O(\lambda^{-\left(\frac{d}{2} + \beta + \frac{3}{2}\right)}) \quad \text{as } \lambda \to \infty.$$
If $\lambda \in [\ell_k, m_k]$, then
\[
k\pi - \frac{1}{4}\pi \leq 2\pi a\lambda - \frac{d + 2\beta + 1}{4}\pi \leq k\pi + \frac{1}{4}\pi,\]
that is,
\[
\left| \cos \left( 2\pi a\lambda - \frac{d + 2\beta + 1}{4}\pi \right) \right| \geq \frac{1}{\sqrt{2}}.
\]
Therefore, if $x \in T^d \cap \mathbb{Q}^d$, then, for $\lambda_k \in [\ell_k, m_k]$,
\[
|\varDelta_0(\lambda_k^2 : x)A_{\beta,a}(\lambda_k^2)| \geq C_{\beta,a}C(x)\lambda_k^{d-2}\lambda_k^{-(\frac{d}{2} + \beta + \frac{1}{2})} = C_{\beta,a}C(x)\lambda_k^{\frac{d-5}{2} - \beta},
\]
that is,
\[
0 < \limsup_{\lambda \to \infty} \frac{|\varDelta_0(\lambda^2 : x)A_{\beta,a}(\lambda^2)|}{\lambda^{\frac{d-5}{2} - \beta}}.
\]
The proof is complete.

### 6.6 Proof of Lemma 6.6

If $j = d$, then we have (6.22). If $0 \leq j \leq (d - 1)/2$, then, using Lemma 5.4 and (6.19), we have, for small $\epsilon > 0$,
\[
|\varDelta_j(\lambda^2 : x)A_{\beta,a}(\lambda^2)| \leq C\lambda^{\frac{d}{2} + \frac{d^2}{2}j + \epsilon}\lambda^{-(\frac{d}{2} + \beta + j + \frac{1}{2})} = C\lambda^{\frac{d}{2} - \beta - \frac{1}{2} + \epsilon} \leq C\lambda^{\frac{d}{2} - \beta + \epsilon},
\]
\[
\lambda \geq 1, \text{ a.e. } x \in T^d.
\]
This shows the conclusion.

### 7 A relation between multiple Fourier series and lattice point problems

Let $d \geq 1$, $\beta > -1$ and $0 < a < 1/2$. Then Corollary 6.2 has shown that
\[
S_\lambda(u_{\beta,a})(x) = \sigma_\lambda(U_{\beta,a})(x) + K_{\beta,a}(\lambda^2 : x) + O(\lambda^{-\beta-1}) \quad \text{as } \lambda \to \infty
\]
for all $x \in T^d$. The behavior of $\sigma_\lambda(U_{\beta,a})(x)$ has been clarified by Theorem 4.1. Therefore, to clarify the behavior of $S_\lambda(u_{\beta,a})(x)$ we need to investigate the term
\[
K_{\beta,a}(\lambda^2 : x) = \sum_{j=0}^{d/2} (-1)^j \varDelta_j(\lambda^2 : x)A_{\beta,a}^{(j)}(\lambda^2).
\]
Since the estimates for the terms $A_{\beta,a}^{(j)}(s)$ are gotten as $6.19$, the convergence problem depends only on the terms $\Delta_j(s : x)$, which are connected with the lattice point problem. Especially, the estimate for $\Delta_0(s : x)$ is very important and difficult.

First we state the following theorem, which will be proven later:

**Theorem 7.1.** Let $d = 2$ or $3$, $0 < a < 1/2$ and $x_0 \in \mathbb{T}^d$. Then,

$$S_\lambda(u_\beta,a)(x_0) - \sigma_\lambda(U_\beta,a)(x_0) = o(1) \text{ as } \lambda \to \infty \text{ for all } \beta > -1,$$

if and only if

$$\Delta_0(s : x_0) = O(s^{\frac{d}{2} + \varepsilon}) \text{ as } s \to \infty \text{ for all } \varepsilon > 0. \quad (7.1)$$

For $d = 1$ the convergence property of $S_\lambda(u_\beta,a)(x)$ and $\sigma_\lambda(U_\beta,a)(x)$ are well known and the estimate $(7.1)$ holds. Actually, we have $\Delta_0(s : x) = O(1)$ for all $x \in \mathbb{T}$, see Remark 5.3. However, for $d = 2$ and $d = 3$, the estimate $(7.1)$ is an open problem, see Remarks 7.1 and 7.2, respectively. For $d \geq 4$, see Remarks 7.3 and 7.4.

**Remark 7.1.** If $d = 2$, then the following fact is well known (see [30, Hauptsatz 3] and Remark 5.2):

$$C_2(x) t^\frac{1}{2} \left( \frac{1}{t} \int_0^t |\Delta_0(s : x)|^2 \, ds \right)^{\frac{1}{2}} < D_2(x) t^\frac{1}{4} \text{ for all } x \in \mathbb{T}^2, \quad (7.2)$$

where $C_2(x)$ and $D_2(x)$ are positive constants depending on $x \in \mathbb{T}^2$. Therefore it is natural to conjecture that $\Delta_0(s : x) = O(s^{\frac{1}{2} + \varepsilon})$ for all $x \in \mathbb{T}^2$. However,

"$\Delta_0(s : 0) = O(s^{\frac{1}{4} + \varepsilon})$ for all $\varepsilon > 0$"

is an open problem as famous Hardy’s conjecture on Gauss’s circle problem, since

$$\Delta_0(s : 0) = D_0(s : 0) - D_0(s : 0) = \sum_{|m|^2 < s} 1 - \int_{|\xi|^2 < s} d\xi,$$

which is the difference between the number of lattice points inside the circle and the area of the circle. Up to now the best result on this problem is $\Delta_0(s : 0) = O(s^{131/416}(\log s)^{18637/8320})$ by M. N. Huxley [12] in 2003. (Recently, Bourgain and Watt [1] gave $\theta = 517/1648 = 0.31371\ldots$ in arXiv, 2017.) By Theorem 7.1, it turns out that Hardy’s conjecture on Gauss’s circle problem is equivalent to

$$S_\lambda(u_\beta,a)(0) - \sigma_\lambda(U_\beta,a)(0) = o(1) \text{ as } \lambda \to \infty \text{ for all } \beta > -1.$$
Remark 7.2. If \( d = 3 \) and for all \( x \in \mathbb{T}^3 \) then (see [30, Hauptsatz 3] and Remark 5.2)
\[
C_3(x) t^{\frac{3}{2}} < \left( \frac{1}{t} \int_0^t |\Delta_0(s : x)|^2 \, ds \right)^{\frac{1}{2}} < D_3(x) t^{\frac{3}{2}} \log^\frac{1}{2} t \quad \text{for all } x \in \mathbb{T}^3,
\]
(7.3)
where \( C_3(x) \) and \( D_3(x) \) are positive constants depending on \( x \in \mathbb{T}^3 \). Therefore it is natural to conjecture that \( \Delta_0(s : x) = O(s^{\frac{3}{2} + \varepsilon}) \) for all \( x \in \mathbb{T}^3 \). However, this problem is also very hard. Up to now the best result on this problem is \( \Delta_0(s : 0) = O(s^{\frac{21}{32} + \varepsilon}) \) by D. R. Heath-Brown [11] in 1999.

For \( d \geq 4 \) we have the following theorem. The proof will be given later:

**Theorem 7.2.** Let \( d \geq 4 \), \( \beta > -1 \) and \( 0 < a < 1/2 \). For \( x_0 \in \mathbb{T}^d \), if (7.1) holds, then
\[
S_\lambda(u_{\beta,a})(x_0) - \sigma_\lambda(U_{\beta,a})(x_0) = o(1) \quad \text{as } \lambda \to \infty.
\]

Remark 7.3. If \( d \geq 4 \) and \( x \in \mathbb{T}^d \cap \mathbb{Q}^d \), then (see [30, Hauptsatz 1] and Remark 5.2)
\[
\left( \frac{1}{t} \int_0^t |\Delta_0(s : x)|^2 \, ds \right)^{\frac{1}{2}} = M_d(x) t^{\frac{d-1}{2}} + O(t^{\frac{d-3}{2} + \varepsilon}),
\]
where \( M_d(x) \) is a positive constant depending on \( x \). Therefore, the estimate (7.1) fails at \( x \in \mathbb{T}^d \cap \mathbb{Q}^d \) for \( d \geq 4 \).

By Propositions 7.1 and 7.2 we have the following corollary immediately:

**Corollary 7.3.** Let \( d \geq 2 \), \( \beta > -1 \) and \( 0 < a < 1/2 \). Assume that, for all \( \varepsilon > 0 \),
\[
\Delta_0(s : x) = O(s^{\frac{d-1}{4} + \varepsilon}) \quad \text{as } s \to \infty \quad \text{for a.e. } x \in \mathbb{T}^d.
\]
(7.4)
Then
\[
S_\lambda(u_{\beta,a})(x) - \sigma_\lambda(U_{\beta,a})(x) = o(1) \quad \text{as } \lambda \to \infty \quad \text{for a.e. } x \in \mathbb{T}^d.
\]

Consequently,
\[
\lim_{\lambda \to \infty} S_\lambda(u_{\beta,a})(x) = u_{\beta,a}(x) \quad \text{for a.e. } x \in \mathbb{T}^d.
\]

**Remark 7.4.** If \( d = 2 \) or \( d = 3 \), then (7.4) is conjectured naturally by (7.2) and (7.3). If \( d \geq 4 \), then the following is known (see [31, Theorem 5] and Remark 5.2)
\[
\left( \frac{1}{t} \int_0^t |\Delta_0(s : x)|^2 \, ds \right)^{1/2} = O(t^{\frac{d-1}{4}}) \quad \text{for a.e. } x.
\]
Therefore (7.4) is conjectured naturally for \( d \geq 4 \) also.
Proof of Theorem 7.1. First observe that Theorem 6.1 implies that
\[
S_{\lambda}(u_{\beta,a})(x_0) = \sigma_{\lambda}(U_{\beta,a})(x_0) + o(1) \quad \text{as} \quad \lambda \to \infty,
\]
if and only if
\[
K_{\beta,a}(\lambda^2 : x_0) = \sum_{j=0}^{d_\varepsilon} (-1)^j \Delta_j(\lambda^2 : x_0) A^{(j)}(\lambda^2) = o(1).
\tag{7.5}
\]
If \( j = d_\varepsilon \), then by (6.22) we have
\[
\Delta_{d_\varepsilon}(\lambda^2 : x_0) A^{(d_\varepsilon)}(\lambda^2) = O(\lambda^{-1-\beta}).
\]
Note that \( d_\varepsilon = 1 \) if \( d = 2 \) and \( d_\varepsilon = 2 \) if \( d = 3 \). For the case \( d = 3 \) and \( j = 1 \), by (6.21), we have, for small \( \varepsilon_0 > 0 \),
\[
\Delta_1(\lambda^2 : x_0) A^{(1)}(\lambda^2) = O(\lambda^{-1-\beta+\varepsilon_0}).
\]
Hence
\[
K_{\beta,a}(\lambda^2 : x_0) = \begin{cases} 
\Delta_0(\lambda^2 : x_0) A^{(d_\varepsilon)}(\lambda^2) + O(\lambda^{-1-\beta}), & d = 2, \\
\Delta_0(\lambda^2 : x_0) A^{(d_\varepsilon)}(\lambda^2) + O(\lambda^{-1-\beta+\varepsilon_0}), & d = 3.
\end{cases}
\]
Since we can take \( \varepsilon_0 > 0 \) as \( -1 - \beta + \varepsilon_0 < 0 \) in the case \( d = 3 \), we see that (7.5) is equivalent to
\[
\Delta_0(\lambda^2 : x_0) A^{(1)}(\lambda^2) = o(1). 
\tag{7.6}
\]
(i) Assume that the estimate (7.1) holds. Then, for all \( \beta > -1 \), we can take \( \varepsilon > 0 \) as \( -1 - \beta + \varepsilon < 0 \) and
\[
|\Delta_0(\lambda^2 : x_0) A^{(d_\varepsilon)}(\lambda^2)| \leq C\lambda^{-\frac{d_\varepsilon - 1}{4} + \varepsilon} \lambda^{-\left(\frac{d_\varepsilon}{2} + \beta + \frac{1}{2}\right)} = C\lambda^{-\beta+\varepsilon},
\]
where we use (7.1) and (6.19). This shows (7.6).
(ii) Conversely, assume that the estimate (7.6) holds for all \( \beta > -1 \). Then by (6.18) we have that
\[
\frac{\Delta_0(\lambda^2 : x)}{\lambda^{\frac{d}{2} + \beta + \frac{1}{2}}} \cos \left( 2\pi a\lambda - \frac{d + 2\beta + 1}{4} \right) = o(1)
\]
for all \( \beta > -1 \). Now, for all \( \varepsilon > 0 \), take \( \beta(1) \) and \( \beta(2) \) such that \( -1 < \beta(1) < -1 + \varepsilon = \beta(2) < \beta(1) + 1 \). Then
\[
\frac{\Delta_0(\lambda^2 : x)}{\lambda^{\frac{d}{2} + \beta(i) + \frac{1}{2}}} \cos \left( 2\pi a\lambda - \frac{d + 2\beta(i) + 1}{4} \right) = o(1) \quad (i = 1, 2). \tag{7.7}
\]
Let $\theta_0 = (\beta(2) - \beta(1))\pi/4$. Then $0 < \theta_0 < \pi/4$ and 
\[
\min_{\theta \in \mathbb{R}} (\max\{|\cos \theta|, |\cos(\theta - 2\theta_0)|\}) = \min_{-\pi \leq \theta \leq \pi} (\max\{|\sin \theta|, |\sin(\theta - 2\theta_0)|\}) \geq \sin \theta_0.
\]
Therefore,
\[
\min_{\lambda > 0} \left(\max\left\{\left|\cos\left(2\pi a\lambda - \frac{d + 2\beta(i) + 1}{4}\pi\right)\right| : i = 1, 2\right\}\right) \geq \sin \theta_0.
\]
Combining this and (7.7), and observing that $\beta(1) < \beta(2)$, we conclude that 
\[
\Delta_0(\lambda^2 : x_0) = \frac{\Delta_0(\lambda^2 : x_0)}{\lambda^{\frac{d-\beta(2)}{2}+\frac{1}{2}}} = o(1).
\]
This shows the conclusion.

Proof of Theorem 7.2. From Theorem 6.1 it is enough to prove that 
\[
K_{\beta,a}(\lambda^2 : x_0) = \sum_{j=0}^{d_2} (-1)^j \Delta_j(\lambda^2 : x_0) A_{\beta,a}^{(j)}(\lambda^2) = o(1).
\]
By the assumption and Lemma 5.1 for all $\varepsilon > 0$, 
\[
\Delta_\alpha(s : x_0) = \begin{cases} 
O(s^{\frac{d-1}{4}+\varepsilon}), & \text{if } \alpha = 0, \\
O(s^{\frac{d-1}{2}+\varepsilon}), & \text{if } \alpha = \frac{d-1}{2}.
\end{cases}
\]
Applying Theorem 5.9 as 
\[
\alpha_0 = 0, \quad \alpha_1 = \frac{d-1}{2} \quad \text{and} \quad \alpha = \theta\alpha_1,
\]
we have 
\[
\Delta_\alpha(s : x_0) = O(s^{\frac{d-1}{4}+\frac{\varepsilon}{2}+\varepsilon}), \quad \text{if } 0 \leq \alpha \leq \frac{d-1}{2},
\]
since 
\[
(1 - \theta)\frac{d-1}{4} + \theta\frac{d-1}{2} + \alpha = \frac{d-1}{4} + \frac{\alpha}{2}.
\]
Take $\varepsilon > 0$ as $-1 - \beta + \varepsilon < 0$. Then we have by (6.19) and (6.22) 
\[
|K_{\beta,a}(\lambda^2 : x_0)| \leq \sum_{j=0}^{d_2-1} |\Delta_j(\lambda^2 : x_0) A_{\beta,a}^{(j)}(\lambda^2)| + O(\lambda^{-1-\beta})
\]
\[
\leq C \sum_{j=0}^{d_2-1} \lambda^{\frac{d-1}{2}+j+\varepsilon} \lambda^{-\left(\frac{d}{2}+\beta+j+\frac{1}{2}\right)} + O(\lambda^{-1-\beta})
\]
\[
= O(\lambda^{-1-\beta+\varepsilon})
\]
\[
= o(1).
\]
The proof is complete.
References

[1] J. Bourgain and N. Watt, Mean square of zeta function, circle problem and divisor problem revisited, arXiv:1709.04340v1.

[2] L. Brandolini and L. Colzani, Localization and convergence of eigenfunction expansions, J. Fourier Anal. Appl. 5 (1999), no. 5, 431–447.

[3] K. Chandrasekharan and S. Minakushisundaram, Typical means, Oxford Univ. Press, 1952.

[4] L. Colzani and M. Vignati The Gibbs Phenomenon for Multiple Fourier Integrals, J. Approximation theory. 80 (1995), 119-131.

[5] F. Fricker, Einführung in die Gitterpunktlehre, Birhäuser, 1982

[6] I. S. Gradshteyn and I. M. Ryzhik, Tables of integrals, series, and products (Seventh edition), Academic Press, 2007.

[7] L. Grafakos and G. Teschl, On Fourier Transforms of Radial Functions and Distributions, J. Fourier Anal. Appl. 19 (2013), 167–179.

[8] G. H. Hardy, On the expression of a number as the sum of two squares, Quart. J. Math., 46 (1915), 263–283.

[9] G. H. Hardy, The average order of the arithmetical functions $P(x)$ and $\Delta(x)$, Proc. London Math. Soc. 15 (1917), 192–213.

[10] G. H. Hardy and E. Landau, The lattice points of a circle, Proc. Roy. Soc. London Ser. A, 105 (1924), 244–258.

[11] D. R. Heath-Brown, Lattice points in the sphere, Number theory in progress, Vol. 2 (Zakopane-Kościelisko, 1997), 883-892, de Gruyter, Berlin, 1999.

[12] M. N. Huxley, Exponential sums and lattice points. III. Proc. London Math. Soc., (3) 87 (2003), no. 3, 591–609.
[13] A. Ivic, E. Krätzel, M. Kühleitner, and W. G. Nowak, Lattice points in large regions and related arithmetic functions: recent developments in a very classic topic, Elemenare und analytische Zahlentheorie, Franz Steiner Verlag Stuttgart, 2006, 89-128.

[14] V. Jarník, Bemerkungen zu Landauschen Methoden in der Gitterpunktlehre, Number Theory and Analysis, (A collection of papers in honor of Edmund Landau (1877-1938), Edited by Turan, Plenum, New York, 1969, 139–159.

[15] E. Krätzel, Lattice points, Kluwer Academic Publication, 1988.

[16] E. Krätzel, Analytische Funktion in der Zahlentheorie, B. G. Teubner Stuttgart-Leipzig-Weisbaden, 2000.

[17] S. Kuratsubo, On a theorem of B. Novák on lattice-point problem, Science Reports Hirosaki Univ., 29(1982), 111-113.

[18] S. Kuratsubo, On pointwise convergence of Fourier series of indicator function of $N$ dimensional ball, Sci. Report Hirosaki Univ., 43 (1996), 199–208.

[19] S. Kuratsubo, On pointwise convergence of Fourier series of radial function in several variables, Proc. Amer. Math. Soc., 127 (1999), 2987–2994.

[20] S. Kuratsubo, Analysis of the Fourier inversion formula of the indicator function of a $d$ dimensional ball and an extension of the Voronoï-Hardy’s identity, In: Proceedings of the 4th International Conference on Analytic Number Theory and Spatial Tesselations, Drahomanov National Pedagogical Univ. (Kyiv Ukraine), 2008, 22–28.

[21] S. Kuratsubo, On an extension of the Voronoï-Hardy identity and multiple Fourier series, RIMS Kokyuroku Bessatsu, B14, 2009, 35–51.

[22] S. Kuratsubo, On pointwise convergence of Fourier series of the indicator function of $d$ dimensional ball, J. Fourier Anal. Appl., 16 (2010), 52–59.

[23] S. Kuratsubo, E. Nakai and K. Ootsubo, On the Pinsky phenomenon of Fourier series of the indicator function in several variables, Mem. Osaka-Kyoiku Univ. Ser. III Nat. Sci. Appl. Sci., 55 (2006), 1–20.
[24] S. Kuratsubo, E. Nakai and K. Ootsubo, Generalized Hardy identity and relations to Gibbs-Wilbraham and Pinsky phenomena, J. Funct. Anal., 259 (2010), 315–342.

[25] E. Landau, Zur analytischen Zahlentheorie der definiten quadratischen Formen über die Gitterpunkte in einem mehrdimensionalen Ellipsoid, Sitzungsber. der Königlich Preussischen Akad. Wiss., 31 (1915), 458-476.

[26] E. Landau, Vorlesungen über Zahlentheorie II, Hirzel, Leipzig, 1927; Chelsea, New York, 1969.

[27] E. Landau, Ausgewählte Abhandlungen zur Gitterpunktlehre, Herausgegeben von Arnold Walfisz, VEB Deutscher Verlag der Wissenschaften, Berlin 1962.

[28] B. Novák, Über Gitterpunkte mit Gewichten in mehrdimensionalen Ellipsoiden, Czechoslovak Math. J., 17 (1967), 609–623.

[29] B. Novák, Verallgemeinerung eines Petersonschen Satzes und Gitterpunkte mit Gewichten, Acta Arith., 13 (1967/1968), 423–454.

[30] B. Novák, Mittelwertsätze der Gitterpunktlehre, chechoslovak Math. J., 19 (1969), 154–180.

[31] B. Novák, Mean value theorems in the theory of lattice points with weight II, Comm. Math. Univ. Carolinae, 11(1970), 53–81.

[32] B. Novák, Über Gitterpunkte in mehrdimensionalen Ellipsoiden, Czechoslovak Math. J., 22 (1972), 495–507.

[33] K. Ootsubo, S. Fujima, S. Kuratsubo and E. Nakai, Kuratsubo phenomenon of the Fourier series of some radial functions in 4 dimension, in preparation.

[34] M. Pinsky, Pointwise Fourier inversion and related eigenfunction expansions, Comm. Pure Appl. Math., 47 (1994), 653–681.

[35] M. Pinsky, Introduction to Fourier Analysis and Wavelets, Brooks/Cole, CA, 2002.
[36] M. Pinsky, N. Stanton and P. Trapa, Fourier series of radial function in several variables, J. Funct. Anal. 116 (1993), 111–132.

[37] M. Pinsky and M. Taylor, Pointwise Fourier inversion: a Wave Equation Approach, J. Fourier Anal. Appl., 3 (1997), 647–703.

[38] E. M. Stein, Localization and summability of multiple Fourier series, Acta Math., 100 (1958), 93–147.

[39] E. M. Stein, On certain exponential sums arising in multiple Fourier series, Annals of Math., 73(1961), 87-109.

[40] E. M. Stein and G. Weiss, Introduction to Fourier analysis on Euclidean spaces, Princeton Univ. Press, 1971.

[41] M. Taylor, Pointwise Fourier inversion on tori and other compact manifolds, J. Fourier Anal. Appl., 5 (1999), 449–463.

[42] M. Taylor, The Gibbs phenomenon, the Pinsky phenomenon, and variants for eigenfunction expansions, Comm. Partial Differential Equations, 27 (2002), 565–605.

[43] M. Taylor, Double Fourier series with simple singularities–A graphical case study, preprint. [http://www.unc.edu/math/Faculty/met/fourier.html](http://www.unc.edu/math/Faculty/met/fourier.html)

[44] M. Taylor, Serendipitous Fourier inversion, preprint. [http://www.unc.edu/math/Faculty/met/SERENE.pdf](http://www.unc.edu/math/Faculty/met/SERENE.pdf)

[45] G. Voronoï, Sur le développement, à l’aide des fonctions cylindriques, des sommes doubles \( \sum f(pm^2 + 2qmn + rn^2) \), où \( pm^2 + 2qmn + rn^2 \) est une forme positive à coefficients entiers, Verhandlungen des dritten Int. Math. -Kongresses in Heidelberg, 241–245, 1905.

[46] A. Walfisz, Gitterpunkte in mehrdimensionalen Kugeln, Warszawa, 1957.

[47] A. Zygmund, Trigonometric Series. 2nd ed., Cambridge, UK: Cambridge University Press, 1959.
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