MULTIPOLE GRAVITATIONAL LENSING AND HIGH-ORDER PERTURBATIONS ON THE QUADRUPOLE LENS

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ABSTRACT

An arbitrary surface mass density of the gravitational lens can be decomposed into multipole components. We simulate the ray tracing for the multipolar mass distribution of the generalized Singular Isothermal Sphere model based on deflection angles, which are analytically calculated. The magnification patterns in the source plane are then derived from an inverse shooting technique. As has been found, the caustics of odd mode lenses are composed of two overlapping layers for some lens models. When a point source traverses this kind of overlapping caustics, the image numbers change by ±4, rather than ±2. There are two kinds of caustic images. One is the critical curve and the other is the transition locus. It is found that the image number of the fold is exactly the average value of image numbers on two sides of the fold, while the image number of the cusp is equal to the smaller one. We also focus on the magnification patterns of the quadrupole (m = 2) lenses under the perturbations of m = 3, 4, and 5 mode components and found that one, two, and three butterfly or swallowtail singularities can be produced, respectively. With the increasing intensity of the high-order perturbations, the singularities grow up to bring sixfold image regions. If these perturbations are large enough to let two or three of the butterflies or swallowtails make contact, then eightfold or tenfold image regions can be produced as well. The possible astronomical applications are discussed.

Key words: gravitational lensing: strong – methods: analytical – methods: numerical

1. INTRODUCTION

Strong gravitational lensing is a gravitational light deflection phenomenon that produces easily visible distorted images such as Einstein rings, arcs, and multiple images. The strong lensing allows us to probe the mass distribution of galaxies, groups and clusters, as well as invisible dark substructures (Dalal & Kochanek 2002; Rusin et al. 2003; Yoo et al. 2006; Orban de Xivry & Marshall 2009). Moreover, the magnification effect of gravitational lensing also provides chances to study galaxies, black holes, and active nuclei that are too small or too faint to be resolved or detected with current instruments (Treu 2010).

One of the most distinct qualities of gravitational lensing is the image multiplicities of distant quasars or galaxies lensed by foreground galaxies or galaxy clusters. For non-singular lenses, it is well known that the total image number is odd (Burke 1981; Schneider et al. 1992). Because most of the dark matter halos are triaxial ellipsoids (Kassiola & Kovner 1993; Evans et al. 2000; Jing & Suto 2002), their planar projections should correspond to the elliptic lenses (Kassiola & Kovner 1993). If a source lies within the central caustic of a typical elliptic lens (or quadrupole lens), then there will be five images produced. However, the image located near the lens center is usually highly demagnified and faint (Rusin & Ma 2001; Winn et al. 2004), resulting in an even image number with many observed quasars. Keeton et al. (2000) and Evans & Witt (2001) have demonstrated that sextuple or octuple images of lensed quasars are likely to exist, brought about by the elliptic lens being perturbed by external shears or the lens isophote deviating from pure ellipses, such as boxiness or diskiness. B1359+154 is the first example of galaxy-scale gravitational lensing in which six images are observed on the same background quasar. The density configuration of the lens, which was proposed to include three primary lens galaxies (Myers et al. 1999; Rusin et al. 2001) is complex.

The critical curves and caustics are crucial for qualitatively understanding strong gravitational lensing. In general, the critical curves are closed curves in the lens plane (image plane) where the Jacobian matrices vanish and the magnifications are infinite. They divide the lens plane into image regions of positive and negative parities. The caustics are the corresponding curves obtained by mapping the critical curves into a source plane via the lens equation (Schneider et al. 1992, 2006). Accordingly, the critical curves are the images of the caustics. In fact, there are also other images of the caustics (Blandford & Narayan 1986; Suyu & Blandford 2006) and they are the so-called transition loci (Finch et al. 2002), which have received less attention so far. Nevertheless, they are important for us to understand the spatial distribution of multiple images of strong lensing, which is one of the main topics of this paper. The caustics of a typical elliptic lens can be divided into radial and tangential caustics. The tangential caustic commonly comprises cusp points and fold lines, where the cusps are the generic singularities. In addition to the cusp, there are some other high-order singularities, such as swallowtails and butterflies. Detailed information about the singularity theories in strong gravitational lensing can be obtained elsewhere (e.g., Petters 1993; Aazami & Petters 2009; Orban de Xivry & Marshall 2009). The caustics divide the source plane into regions with different image multiplicities. When a source traverses a single caustic, either fold or cusp, the total number of images changes by ±2. However, in a case where
the caustics are composed of two overlapping layers, the situation will be different, as presented later in the main text of this paper.

In practice, decomposing the projected mass distribution or planar potential into multipole modes is often used as a method to study gravitational lensing (e.g., Trotter et al. 2000; Evans & Witt 2003; Kochanek & Dalal 2004; Bernstein & Nakajima 2009). In addition to the monopole \((m = 0)\), the quadrupole \((m = 2)\) is usually the most important component and has been investigated intensively (Kovner 1987; Kormann et al. 1994; Witt & Mao 2000; Finch et al. 2002). The components with an order higher than 2 are not quite dominant in the expansions of real lenses and are usually more complex than the quadrupole. Nevertheless, their perturbation may have significant impacts on image properties and should not be totally neglected. For this purpose, by using analytical and numerical methods, we focus on the properties of these high-order components and then study their perturbations on the \(m = 2\) lenses with different intensities and phase differences. Although it is not clear if there are such strong high-order perturbations in reality, it is still worth investigating theoretical predictions of multiple image lenses.

The paper is arranged as follows. In Section 2, we investigate the properties of the \(m = 2, 3, 4, \) and \(5\) mode gravitational lenses. Different image multiplicities and overlapping caustics are investigated. In Section 3, the images of the caustics including critical curves and transition loci are studied. Then, we discuss how the image number changes when a point source traverses the overlapping caustics. In Section 4, we focus on the magnification patterns of the quadrupole \((m = 2)\) lenses perturbed by the \(m = 3, 4, \) and \(5\) mode components. Then, we interpret how the swallowtail and butterfly singularities appear and evolve under perturbations with different intensities and phase differences. Finally, in Section 5, the conclusions of the paper and a discussion are given.

2. PROPERTIES OF MULTIPOLe LENSING

2.1. Multipole Decomposing and Lens Equations

Under the condition that the size of the lens object is much smaller than the distance from the lens to the observer and source, the lens can generally be assumed to be a thin mass plane. A very useful way to describe lensing mass distribution is multipole expansion. Through this method, an arbitrary mass surface density or convergence \(\kappa(\theta)\) can be decomposed into multipole components (Huterer et al. 2005; Schneider et al. 2006) with the following function:

\[
\kappa(\theta) = \kappa_0(\theta) + \sum_{m=1}^{\infty} \kappa_m(\theta) \cos m[\phi - \phi_m(\theta)].
\]

Here, \((\theta, \phi)\) are the polar coordinates and \(\phi_m(\theta)\) is the phase for the mode \(m\) at radius \(\theta\).

\[
\kappa_0(\theta) = \frac{1}{2\pi} \int_0^{2\pi} \kappa(\theta) d\phi
\]

is the monopole \((m = 0)\) component of the surface mass density.

To simplify the multipole lensing problem, we apply the familiar Singular Isothermal Sphere (SIS) lens model as the monopole \((m = 0)\) distribution. Note that this spherical mass distribution yields flat rotation curves (Schneider et al. 1992; Maoz & Rix 1993). Then, one multipole component with a radial profile similar to the SIS is added to the \(m = 0\) mode. As a first step, for simplicity, the phase of the multipole \(\phi_m(\theta)\) is set to 0. Therefore, the multipolar mass surface density of the generalized SIS model is given by

\[
k(\theta) = \frac{\theta_E}{2\theta}(1 + k_m \cos m\phi),
\]

where the \(\theta_E\) is the Einstein radius of the SIS lens model. Hence, \(k_0(\theta) = \theta_E/2\theta\) represents the profile of the \(m = 0\) mode component, and \(k_m(\theta) = k_m0(\theta) (0 \leq k_m \leq 1)\) expresses the radial profiles of the multipolar components.

Before ray-tracing simulations with the surface density field \(\kappa(\theta)\), one must know the deflection (or lensing) potential and the deflection angle as a function of the position in the lens plane. The deflection potential depends on the two-dimensional Poisson equation \(\nabla^2 \psi(\theta) = 2k(\theta)\). By solving this differential equation in a polar coordinate, we can derive the deflection potential

\[
\psi(\theta) = \theta_E \theta \left(1 - \frac{k_m}{m^2-1} \cos m\phi\right), \quad m \neq 1,
\]

\[
\psi(\theta) = \theta_E \theta \left(1 + \frac{k_1}{2} \ln \theta \cos \phi\right), \quad m = 1.
\]

Hence, the scaled deflection angle \(\alpha = \nabla \psi(\theta)\) is the first derivation of the deflection potential. It can be written as radial and tangential components:

\[
\alpha_{\text{rad}} = \theta_E - \frac{\theta_E k_m}{m^2-1} \cos m\phi, \quad m \neq 1,
\]

\[
\alpha_{\text{tan}} = \frac{m\theta_E k_m}{m^2-1} \sin m\phi, \quad m \neq 1,
\]

\[
\alpha_{\text{rad}} = \theta_E + 2 \frac{\theta_E k_1}{2} \cos \phi + \frac{\theta_E k_1}{2} \ln \theta \cos \phi, \quad m = 1,
\]

\[
\alpha_{\text{tan}} = -\frac{\theta_E k_1}{2} \ln \theta \sin \phi, \quad m = 1.
\]

Based on the deflection potential \(\psi\), we found the shear \(\gamma = \kappa\) for the modes \(m \neq 1\) (using Equation (13) in the paper Bernstein & Nakajima 2009). Consequently, the magnification in the lens plane is

\[
\mu = \frac{1}{(1-k)^2 - \gamma^2} = \frac{1}{1-2k}.
\]

and therefore the critical curves where \(\mu\) is infinite is given by

\[
\theta = \frac{\theta_E + \theta_E k_m \cos m\phi}{\theta_E}.
\]

Alternatively, the deflection potential \(\psi\), the deflection angle \(\alpha\), and the shear \(\gamma\) can be calculated numerically by the fast Fourier transform method via \(\kappa(\theta)\) (refer to Bartelmann 2010 for more details). Solving these equations in the Fourier space imposes periodic boundary conditions. The lens grids could be set in the center of a larger grid, while the densities outside the inner grid are padded with zeros. The larger grid must be at least twice as large as the inner grid, and the larger the better (Li et al. 2005; Amara et al. 2006). In this work, the ray-tracing simulation was calculated with an inner grid of \(1024 \times 1024\), which corresponds to a resolution of about 0.004 \(\theta_E\).
Finally, the lens equation $\beta = \theta - \alpha$, which describes the transformation between the lens plane ($\theta$, $\phi$) and the source plane ($\beta$, $\varphi$), can also be written in the polar coordinate as

$$\beta^2 = (\theta - \alpha_{\text{cen}})^2 + \alpha_{\text{tan}}^2,$$

$$\tan(\phi - \varphi) = \frac{\alpha_{\text{cen}}}{\theta - \alpha_{\text{cen}}}.$$

\[ (12) \quad (13) \]

### 2.2. Image Multiplicities of the Multipole Lenses

We simulate the multipole lenses of modes $m = 2$, 3, 4, and 5, corresponding to the quadrupole, sextupole, octupole, and decagonal pole modes, respectively. The monopole lens ($m = 0$) is a circular symmetric lens, and this simple lens model has been well known to us. A dipole moment ($m = 1$) corresponds to making the lens lopsided with more mass on one side of the center than the other, or equivalently, a shift in the center of the mass. The dipole moment vanishes if the coordinate origin is shifted to this point (Dominik 1999; Schneider et al. 2006; Jog & Combes 2009). Furthermore, the expressions of the deflection angle for $m = 1$ are not similar in appearance to the other modes, and it is divergent when $\theta$ tends to infinity in Equations (8) and (9). Therefore, only the $m = 2$, 3, 4, and 5 modes of lens components were taken into account throughout the paper.

Magnification patterns can exhibit caustic profiles in the source plane well and be computed by the inverse ray-shooting technique (Kayser et al. 1986; Schneider & Weiss 1987; Mediavilla et al. 2006). The main idea is shooting a bundle of light rays from the observer to each grid of the lens, and then the light deflects to the source plane according to deflection angle $\alpha$. The place in the source plane that is irradiated by more light rays corresponds to a larger area of the lens plane, and it means higher magnification levels there. Accordingly, these patterns present the tangential caustics of the multipole lenses clearly. Because a singular isothermal lens does not have a formal radial critical curve (Evans & Wilkinson 1998; Huterer et al. 2005), the radial caustic (or pseudo-caustic in this situation) associated with the center singular in the lens plane will not be discussed. After deriving the magnification patterns, we set them as source maps so that their images can be obtained through ray-tracing simulations (i.e., mapping these magnification patterns to the lens plane). The magnification patterns and their images are shown on the left and middle columns of Figure 1. The right column displays the magnification maps in the lens plane, which are calculated through Equation (10).

The magnification patterns of the $m = 2$, 3, 4, and 5 modes are shown in the left column from top to bottom, respectively, where $k_m$ is set to 0.3 in each case. The numbers for the magnification patterns denote different multifold image regions, and the numbers in the middle column denote the different image regions with reference to the left column. Due to the extraordinary complexities brought about by numerous intersecting folds in the caustics of the $m = 4$ and 5 mode lenses, their corresponding multifold image regions are not denoted in Figures 1(h) and (k). Since the middle column shows the images of the magnification patterns, they can well present the critical curves and transition loci, which are the images of the caustics (Finch et al. 2002). The magnification maps on the right column can only show the critical curves by high magnification red regions, not including the information about transition loci.

In Figure 1(a), one can see the familiar astroid caustic, inside of which is the 4-image region (hereafter, we ignore the demagnified image in the center of the lens), and outside of which is the 2-image region (hereafter, the 1-image region outside of the pseudo-caustic is not considered). The cusps can be divided into major cusps and minor cusps. A source near the major cusp can be mapped into three close images around the tangential critical curve on the same side of the source with respect to the lens center, while a source on the minor cusp will be mapped into three close images on the opposite side of the source (Xu et al. 2009). In Figure 1(a), the left and right cusps are the major cusps, while the top and bottom ones are the minor cusps. The areas near the major cusps are slightly redder than those of the minor cusps, which means that the former have a higher level of magnification. Figure 1(g) shows the tangential caustic composed of eight cusps. The four cusps in the horizontal and vertical directions are the major cusps, while the four cusps in the ±45° directions are the minor cusps.

As the numbers denote in Figure 1(g), it can yield eight, six, four, and two images in different regions from the inner area to outside of the magnification pattern. On the other hand, as shown in Figures 1(d) and (j), for the magnification patterns of the $m = 3$ and $m = 5$ mode components, the tangential caustics only show three and five cusps, respectively, instead of six and ten. In these two cases, the differences of the image numbers between two sides of the caustics are 4.

In fact, the cusp numbers for $m = 3$ and 5 are reduced by half because the major and minor cusps overlap with each other. If we put Equations (6)–(7) and Equation (11) into Equations (12)–(13), then the expression for the caustics in the form of parametric equations can been derived as

$$\beta = \frac{m\theta E k_m}{m^2 - 1} \sqrt{m^2 \cos^2 m\phi + \sin^2 m\phi},$$

$$\psi = \phi - \arctan \frac{m \theta E k_m}{m} - (\pi).$$

Because the arc tangent function has a phase uncertainty for the shift of $\pi$, the $\pi$ in the bracket is only needed when $\theta < \alpha_{\text{cen}}$ in Equation (13), i.e., $c_0 \cos m\phi < 0$ in Equations (6) and (11). Coupled with Equation (15), one can find that $\phi(\phi + \pi) = \phi(\phi + \pi) + m\pi$ if $m$ is even and $\phi(\phi + \pi) = \pi/2$ if $m$ is odd. It is also easily obtained that $\beta(\phi) = \beta(\phi + \pi)$ from Equation (14). Thus, for the odd mode lens, the two points $\theta(\phi)$ and $\theta(\phi + \pi)$ on the critical curves correspond to the same point on the caustics, which means that these caustics overlap in two layers. Furthermore, the major (minor) cusps correspond to the points $c_0 \cos m\phi = \pm 1$ on the critical curves, so the distances from both major and minor cusps to the center of the source plane are exactly

$$\beta_{\text{cusp}} = \frac{m^2 \theta E k_m}{m^2 - 1}.$$

Therefore, in the left column of Figure 1, the major and the minor cusps show the same distances to the source center.

Figure 2 is similar to Figure 1, except that we set $k_m = 0.6$, which is twice the value used in Figure 1. From top to bottom, the left column lays out the magnification patterns for modes $m = 2$, 3, 4, and 5, respectively, while their images are shown in the middle column. There are white squares in each center of the magnification patterns, located in 4-, 6-, 8-, and 10-image regions in turn. The red squares are located in the 2-, 4-, and 6-image regions in turn. As can be seen in the middle column, there are 4, 6, 8, and 10 white images, and 2, 4, and 6 red images from top to bottom, respectively.
The white images lie approximately on a circle; in other words, they appear like incomplete Einstein rings. Additionally, the white areas outside the critical curves are located slightly farther from the lens centers than those inside. The red square in Figure 2(g) is located near the minor cusp, so that its three close images are on the opposite side of the source. The red square in Figure 2(j) is located near the overlapping cusps (a major and a minor), so there are three close images on the same side of the source and the other three on the opposite side. Since the multipole intensity relative to $m = 0$ used in Figure 1 ($k_m = 0.3$) is smaller than that used in Figure 2 ($k_m = 0.6$), the critical curves of the former are rounder than those of the latter. If the intensities of the multipoles meet the condition of $k_m = 0$, then the critical curves will turn into Einstein rings. In addition, because the $\beta_{\text{cusp}}$ in Figure 2 are twice as large as those in Figure 1 (they are not drawn according to proportion in the two figures), the areas or the probabilities that yield more than two images of a single source for the cases in Figure 2 would be much larger than those in Figure 1.

3. THE IMAGES OF CAUSTICS

Caustics are the divisions of different image multiplicities in the source plane and can be mapped into the lens plane to obtain their images. The critical curve is the most familiar one, but there are other types as introduced in the first section. Finch et al. (2002) have studied the images of the caustics for the $m = 2$ lens by the analytical method. We extend this further by using numerical methods. Figure 3 shows the caustics of the $m = 2$ and 3 mode components as well as their images. Due to the many intersecting folds in the caustics of the $m = 4$ and 5 mode lenses, we will not study them intensively. Investigations of caustic images can be divided into two parts, one for the folds and the other for the cusps.

As shown in Figures 3(a) and (b), each fold of the $m = 2$ lens has three images. The images of different folds are distinguished by four different colors. Except for the solid thick critical curve, there is a dashed/dotted curve outside/inside of the critical curve. They are outer and inner 2–4 (two-image region to
Figure 2. Similar to Figure 1, but for $k_m = 0.6$ in contrast. The white and red areas on the middle column are the images of the white and red squares on the left column.

four-image region, similar hereafter) transition loci, which can also be found in Figures 1(b) and 2(b). In Figures 3(c) and (d), one can find that each fold in the $m = 3$ mode lens has four images. As a result of the overlapping caustics, there are two critical curves, as indicated by the solid thick lines in Figure 3(d). The one indicated by a dashed curve is the outer 2–6 transition locus, and the one indicated by a dotted curve is the inner 2–6 transition locus. They can also be found in Figures 1(e) and 2(e). Therefore, we can take the folds of the $m = 2$ lens as three-image regions, which divide the lens plane into two-image and four-image regions, and the folds of the $m = 3$ lens as four-image regions, which divide the lens plane into two-image and six-image regions. We found that the image number of the fold is the average value of the two sides of the fold. Among these images, the number of critical curves is the layer number of the fold, and the rest of the images of the fold are the transition loci in the lens plane.

It is intriguing when a source traverses such an overlapped fold. For example, in Figure 2(j), there are red and white squares in the 6 image and 10 image regions, respectively, and they have 6 and 10 images as shown in Figure 2(k) correspondingly. The fold between the two squares has eight images, including two critical curves and six transition loci. If this red square traverses the overlapping fold into the 10-image region, the left three images will move to the left, traversing the three 6–10 transition loci into the 10-image regions as the right three images move to the left into the 10-image regions simultaneously. Two new images will appear on the critical curves denoted by two white arrows, and then each of them splits into two images, resulting in four new images once the source is well within the 10-image region. At last, the total images number is 10. The process is similar to reverse traversing, but with images being eliminated in the two critical curves.

The cusps of the caustics in Figure 3 can be distinguished by the crossover points of different color folds. As shown in Figure 3(b), each cusp of the $m = 2$ lens has two images, one of which is the tangent point of the critical curve and a transition locus, and another of which is the cusp of the other transition locus. Each cusp of the $m = 3$ lens also has two images. As shown in Figure 3(d), one image is the tangent point of the
critical curve and the outer transition locus, i.e., the image of the major cusp, located on the same side as the source. The other image is the tangent point of the critical curve and the inner transition locus, i.e., the image of the minor cusp, located on the opposite side of the source. Therefore, we can regard the cusps of the modes $m = 2$ and $m = 3$ as 2-image regions. We found that the image number of the cusp is equal to the smaller of the image numbers on either side. Similar to the folds, there are also two kinds of images. One is the tangent point of the critical curve and transition locus, and their image number is the layer number of the cusp. The outer and inner tangent points relative to the critical curve correspond to the major and minor cusps, respectively. Another kind of image is the cusps of the transition loci.

In Figure 2(j), the cusp on the right side of the red square has two images, because it must be consistent with the image number of the 2-image region. If the red square in the 6-image region moves to the right to traverse the overlapping cusp into the 2-image region, as implied in Figure 2(k), then the left three images will move to the right and merge into one image on the tangent point of the critical curve and transition locus, while the right three images will also merge into one image on the right tangent point simultaneously. Then, the two merged images will move into the 2-image regions, respectively. Finally, there will be only two images in total. The process is similar for reverse traversing, but with each of the two images splitting into three images.

4. QUADRUPOLE LENSES PERTURBED BY HIGH-ORDER MODES

In the expansion of projected real galaxies or simulated dark matter halos, besides the $m = 0$ component, the $m = 2$ component is usually the most important one. Although the intensities of the $m = 3, 4,$ and $5$ mode components hardly exceed the $m = 2$ mode, their influences on strong gravitational lensing are still significant and can be directly reflected in the perturbations on the caustic of the $m = 2$ lens. For these higher-order modes, a non-zero $\phi_m$ must be included since circular symmetry is broken by the $m = 2$ mode. In reality, the intensities of the high-order perturbations on the $m = 2$ lens are all functions of the radius. However, to simplify the perturbation problem, we assume that $\phi_m$ do not depend on the radius. In these cases, the matter distribution formula of $m = 2$ plus the high-order mode ($m = 3, 4,$ and $5$) is given simply by

$$\kappa(\theta) = \frac{\theta_E}{2\alpha} \left[ 1 + k_2 \cos 2\phi + \frac{k_m}{m^2 - 1} \cos m(\phi - \phi_m) \right],$$

and the deflection potential is given by

$$\psi(\theta) = \frac{\theta_E \theta}{2\alpha \theta_E k_m} \left[ 1 - \frac{k_2}{2^2 - 1} \cos 2\phi - \frac{k_m}{m^2 - 1} \cos m(\phi - \phi_m) \right].$$

Hence, the scaled deflection angle is also derived as:

$$\alpha_{rad} = \theta_E - \frac{\theta_E k_2}{2^2 - 1} \cos 2\phi - \frac{\theta_E k_m}{m^2 - 1} \cos m(\phi - \phi_m),$$

$$\alpha_{tan} = \frac{2\theta_E k_2}{2^2 - 1} \sin 2\phi + \frac{m \theta_E k_m}{m^2 - 1} \sin m(\phi - \phi_m),$$

and the critical curve is

$$\theta = \theta_E + \theta_E k_2 \cos 2\phi + \theta_E k_m \cos m(\phi - \phi_m).$$

We can put Equations (19)–(21) into Equations (12)–(13) to get the caustics, but it is not easy because the expression is very cumbersome. However, it is very convenient to derive the magnification patterns by the inverse shooting technique through the deflection angle and lens equation. The results are shown in Figures 4–6 for the modes $m = 3, 4,$ and $5$, respectively. We then discuss the three kinds of perturbations, respectively, based on the magnification patterns.

The $m = 3$ component in the gravitational lens sometimes plays a very important role through its perturbation of the $m = 2$ mode (e.g., Irwin & Shmakova 2006). We examine the perturbations of the $m = 3$ mode with different relative intensities $k_3$ and phases $\phi_3$. In Figure 4, the three rows from top to bottom correspond to the anticlockwise rotations of the $m = 3$ component for the phases $\phi_3$ of $0, \pi/4,$ and $\pi/2$, respectively. The sum of $k_3$ and $k_3$ is kept constant at 0.5, while $k_3$ increases from 0.0 to 0.5 from left to right. As shown on the magnification patterns of the top row in Figure 4, the left major cusp begins to shrink under a small perturbation, and a butterfly singularity appears. Then, the top and bottom two minor cusps become tilted to the left. The butterfly structure becomes more obvious with stronger perturbations, and the triangular pattern from the butterfly moves to the primary caustic and overlaps with it eventually. It also confirms that the $m = 3$ caustic is composed of two overlapping layers in a triangular shape. The middle row of Figure 4 shows the patterns of the $m = 3$ mode perturbations rotated $\pi/4$ anticlockwise. Instead of a butterfly catastrophe, a swallowtail catastrophe appears on the fold. With a stronger $m = 3$ component, the triangle that evolved from the swallowtail also moves and finally overlaps with the primary caustic. The bottom row of Figure 4 shows the results with a phase of $\pi/2$. In this case, the changing of patterns is similar to the top row, but with a butterfly evolved from a minor cusp. The $m = 3$ perturbation can produce six images via a butterfly or swallowtail catastrophe. In general, the area of the sixfold image region grows with an increasing perturbation intensity.
Figure 4. Magnification patterns of \( m = 2 \) lenses perturbed by the \( m = 3 \) components. From top to bottom, the perturbations phases are 0, \( \pi/4 \), and \( \pi/2 \), respectively, and from left to right \( k_3 \) is set from 0.0 to 0.5, respectively, with the constraint of \( k_2 + k_3 = 0.5 \).

Figure 5. Same as Figure 4, except perturbed by the \( m = 4 \) components, and the perturbations corresponding to the phases 0, \( \pi/8 \), and \( \pi/4 \), respectively.

The \( m = 4 \) component is relatively more universal in strong lenses, such as the elliptical galaxy models with boxy or disky isophotes (Naab et al. 1999; Evans & Witt 2001). Therefore, we also study their perturbations on \( m = 2 \) mode components with different intensities \( k_4 \) and phases \( \phi_4 \). Given that the \( m = 4 \) mode does not change after rotating \( \pi/2 \), the three rows from top to bottom present the results with a phase shift of 0, \( \pi/8 \), and \( \pi/4 \), respectively. As before, the sum of \( k_2 \) and \( k_4 \) is kept constant at 0.5, while \( k_4 \) increases from 0.0 to 0.5 from left to right. As shown on the magnification patterns of the top row in Figure 5, the butterflies appear at the top and bottom minor cusps. The two butterflies become more obvious and begin to make contact with each other with increasing perturbation. At last, the magnification pattern gradually evolves to have eight distinct cusps. The middle row of Figure 5 shows the case with a phase of \( \pi/8 \). There are no butterflies, but swallowtails. With the increasing of the \( m = 4 \) component, the magnification pattern also evolves into the pure \( m = 4 \) caustic. The bottom row shows the results of the \( m = 4 \) perturbation rotated \( \pi/4 \) anticlockwise. They are similar to the top row of Figure 5, but the two butterflies
Figure 6. Same as Figure 4, except perturbed by the $m = 5$ components.

The $m = 5$ mode component is rarely investigated in strong gravitational lensing. Nevertheless, we study it in a similar way to the $m = 3$ and 4 modes. The three rows from top to bottom show results with anticlockwise rotations of $0$, $\pi/4$, and $\pi/2$, respectively. The components of $k_2$ and $k_3$ were still set as before. As shown in the top row of Figure 6, a butterfly appears at the left major cusp, and two swallowtails appear on the right folds. The butterfly and the swallowtails become more and more obvious with increasing perturbation. Then, the three structures begin to make contact with each other. At last, the magnification pattern gradually evolves to the pattern of $m = 5$. The middle row in Figure 6 shows that there are three swallowtails but no butterfly. With increasing perturbation, the magnification pattern also evolves to the pure $m = 5$ mode caustic at the end. The bottom row shows a result similar to that of the top row, but the butterfly evolved from a minor cusp. The perturbation of the $m = 5$ mode can also produce sixfold image regions by the butterflies or swallowtails. If the perturbation is large enough to let two of the butterflies or swallowtails make contact, then it can produce eightfold image regions. Similarly, if three of the butterflies or swallowtails make contact, then tenfold image regions are produced. In general, the areas of the sixfold, eightfold, or tenfold image regions grow with increasing perturbation intensity. However, in this ideal case, the eightfold image region disappears in the end.

Butterflies and swallowtails are high-order singularities on the caustics. The swallowtail structure evolves on a fold, and it can increase two cusps. The butterfly structure evolved from a cusp and it has three cusps, so it also corresponds to increasing two cusps (Orban de Xivry & Marshall 2009). The numbers of the cusps for modes $m = 2$, 3, 4, and 5 are 4, 6, 8, and 10, respectively (see Figures 1 and 2, and the overlapped cusp has been counted as 2). Thus, for the $m = 2$ mode lens perturbed by $m = 3$, 4, and 5 modes, the number of cusps mutates from 4 to 6, 8, and 10, respectively. Therefore, 1, 2, and 3 swallowtail or butterfly singularities should be yielded under such perturbations, as has been shown in Figures 4–6. Additionally, since the numbers of the major and the minor cusps are equal in each mode of the lenses (see Figures 1 and 2), each pair of cusps increased by a butterfly or a swallowtail must be one major and the other minor. The number of butterflies or swallowtails as well as their positions are determined by the phase differences between the high-order modes and the $m = 2$ mode.

5. CONCLUSIONS AND DISCUSSION

The aim of this paper is to examine the properties of the pure multipoles of strong gravitational lenses and high-order perturbations on the $m = 2$ lens. At first, we assume a simplified multipole surface distribution of mass $\kappa$ and then calculate the deflection potential $\psi$ and deflection angle $\alpha$ using analytical and numerical methods. The ray-tracing simulations based on the two methods give consistent results, but there is no doubt that using the analytical deflection angle should be more accurate and time-saving. Through the deflection angle and lens equation, the magnification patterns of the source plane are derived by the inverse shooting technique. These patterns clearly display the...
tangential caustics in the source plane. Therefore, the tangential caustics of these multipole lenses and their corresponding images, including transition loci in the lens plane, can be studied in great detail. It is found that the lenses of the \( m = 2, 3, 4, \) and 5 modes, with singular points in the lens center, can produce 4, 6, 8, and 10 images of a single source at most, respectively.

The quadrupole mass distribution could produce four-image regions, and the generalized quadrupole includes the \( m = 2 \) lens, the elliptic lens, and the binary lens. The sextupole distribution could produce six-image regions, and similarly, the generalized sextupole includes the \( m = 3 \) lens and the triangular distribution of three galaxies. The lenses objects of B1608+656 are two galaxies, which provide a quadrupole component and generate four images of a single source (Surpi & Blandford 2003; Suyu & Blandford 2006). The quasar B1359+154 at \( z_f = 3.235 \) has six images, and they are yielded by three lens galaxies at \( z_f \approx 1 \) (Rusin et al. 2001). This example is not the same type as the \( m = 3 \) lens in this work. However, it gives strong support to our sextupole lens model. In Figure 1, we find that even for the same \( k_{\text{int}} \), the multifold image region with the largest image number decreases with increasing \( m \). Thus, the regions with an image number larger than 10 could be produced similarly, but the possibilities become lower and lower. Although these multipole lens models are very simple, they provide new opportunities for studying the image multiplicities with an image number larger than four, which could hardly be acquainted with general quadrupole lenses.

Through the generalized SIS lens model, we studied the overlapped caustics and their images. The expressions of magnification \( \mu \) for the multipole lenses in the lens plane are calculated. In addition, the critical curves and the caustics are all derived analytically. The caustics of odd mode lenses consist of two overlapping layers as shown in the simulations, and these overlapping phenomena are also theoretically proven. If a point source traverses this kind of caustic, then the image number changes by \( \pm 4 \), rather than \( \pm 2 \). The overlapping caustics could not be detached by the odd mode perturbations, but can be detached easily by even modes. Since the possible internal or external perturbations are very complex, the overlapping caustics are indubitably not universal in real cases of gravitational lensing.

The images of the caustic include not only the critical curves but also the transition loci (Finch et al. 2002). The critical curves divide the lens plane into regions of positive and negative parities, while the transition loci divide the lens plane into regions of different image multiplicities. In general, the tangential caustic commonly comprises cusp points and fold lines. The image number of the fold is the average of the numbers on both sides of the fold, among which the number of critical curves is the layer number of the fold. The rest images are the transition loci in the lens plane. The image number of the cusp is equal to the smaller of the image numbers on either side of the cusp. Generally speaking, there are also two ways for the cusp images to appear. One is the tangent point of the critical curve and transition locus, and the number is the layer number of the cusp. Another category is the cusp of the transition locus. The tangent point of the critical curve and transition locus can also be interpreted to be two images but in contact with each other. This is the reason that the image number of the cusp is smaller than that of the fold. The outer tangent point relative to the critical curve corresponds to the major cusp, while the inner tangent point corresponds to the minor cusp.

For the planar projections of the dark matter halos, besides the \( m = 0 \) component, the \( m = 2 \) mode is usually the largest one. In reality, there is almost no lens dominated by the \( m = 3, 4, \) and 5 modes, while their effects can be embodied in the perturbations on the \( m = 2 \) lens. In this paper, we study the magnification patterns of the mode \( m = 2 \) with perturbations by the \( m = 3, 4, \) and 5 components. One, two, and three swallowtail or butterfly singularities are yielded under these high-order perturbations. Additionally, each pair of increased cusps by a butterfly or a swallowtail must be one major and the other minor. The number for the butterflies or the swallowtails as well as their positions is determined by the phase differences between the high-order modes and the \( m = 2 \) mode. With an increase in high-order perturbations, the butterfly or swallowtail singularities grow up to bring sixfold image regions. If these perturbations are large enough to let two or three of the butterflies or swallowtails make contact, then eightfold or tenfold image regions will appear. In general, the stronger the high-order perturbations, the larger the areas of the multifold image regions. These results are consistent with and also complements of the sextuplet and octuplet image theories derived by Keeton et al. (2000) and Evans & Witt (2001).

Studying the multipole lenses could also help us to understand the number of giant arcs produced by sources located near the caustics. Because high-order mode lenses can produce caustics with longer circumferences and more cusps, the probability of generating giant arcs is higher than that of low-order lenses. Furthermore, even a small high-order perturbation on the quadrupole lens could also increase the length of caustics and the number of cusps, by bringing butterfly or swallowtail structures. Therefore, we conclude that dark matter halos, which are more complex and have stronger multipole components, should have a higher probability of generating giant arcs than the simple quadrupole lens. Indeed, the quadrupole lens can also yield more arcs than the monopole lens (i.e., circular lens) for the same reason, which has been illustrated by Meneghetti et al. (2007), who found that elliptical lenses with realistic density profiles produce a number of arcs larger by a factor of 10 than circular lenses with the same mass. Substructures in dark matter halos could play vital roles in generating high-order singularities and increasing the probability of producing giant arcs (Li et al. 2006; Meneghetti et al. 2007). These effects could also be interpreted to be brought about by the multipole components introduced by large substructures. Additionally, real lenses should possibly include external shears produced by neighbor galaxies or merging clusters. External shears on the lens system can stretch the shape of the caustics and can also add swallowtail or butterfly catastrophes (Chang & Refsdal 1984; Schneider & Weiss 1986; Keeton et al. 2000). Therefore, the external shear should also be able to increase the probability of generating arcs.

The cases discussed in this work are all simplified exercises. In these models, the strengths of the multipoles fall off with the radius as the SIS lens and phase dependence on the radius are neglected. Actually, in the real situation, the multipole components with many other radial profiles and phase dependence could also generate high-order singularities or multifold image regions. Nevertheless, they may not be resolved analytically and may not provide the situation of overlapping caustics. Realistic gravitational lenses can be decomposed into many multipole components, which have complex radial profiles and phase dependence on the radius. Thus, more complex and accurate modeling should take into account these realistic conditions in future investigations. The magnification patterns and the images of them for any arbitrary dark matter halos can be
derived from ray-tracing simulations. Therefore, some useful information such as multifold image regions could be presented in the images of magnification patterns, so it may provide a powerful tool in the theoretic research of the multiplicities of strong gravitational lensing.

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