Multivariate Zipper Fractal Functions

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ABSTRACT
A novel approach to zipper fractal interpolation theory for functions of several variables is presented. Multivariate zipper fractal functions are constructed and then perturbed through free choices of base functions, scaling functions, and a binary matrix called signature to obtain their zipper $\alpha$-fractal versions. In particular, we propose a multivariate Bernstein zipper fractal function and study its coordinate-wise monotonicity which depends on the values of signature. We derive bounds for the graph of a multivariate zipper fractal function by imposing conditions on the scaling factors and the Hölder exponent of the associated germ function and base function. The box dimension result for multivariate Bernstein zipper fractal function is derived. Finally, we study some constrained approximation properties for multivariate zipper Bernstein fractal functions.

1. Introduction

Interpolation is a basic and fundamental subject in numerical analysis and approximation theory for the continuous representation of discrete data. A standard way to obtain a bivariate interpolation from univariate interpolation functions is by using a tensor product if the underlying two variables are considered separately. This procedure is also adapted to multivariate interpolation when data from a multivariate function are prescribed on a Cartesian product of grid points. There are numerous ways to approximate multivariate functions by using, for instance, multivariate polynomials [1–3], splines [4–6], tensor products splines [7], local methods, global methods, blending-function methods [8], and Hermite Interpolation Formulas [9]. All these methods may have advantages and disadvantages depending on the nature of the data and the application. When data is generated from a very irregular multivariate function, the above methods are not ideal to provide a deep understanding of the true multivariate features. This paper proposes a new approach to describe non-linear patterns associated with a multivariate data generating function by means of zipper multivariate fractal interpolation functions (FIFs).
Fractal surfaces continue to draw attention to scientists and engineers due to their useful applications in various areas such as medical sciences, surface physics, chemistry, bio-engineering, metallurgy, computer science, electrical engineering, and earth science. Fractal surfaces have been found to be good approximations of natural surfaces in these areas because of their special properties, such as self-similarity, visualization at different scales, and a non-integral fractal dimension. The construction of fractal surfaces using iterated function systems (IFSs) with co-planar boundary were first introduced by Massopust in [10] with different scaling factors. The construction of fractal surfaces with arbitrary boundary values but equal scaling factors was taken up by Geronimo and Hardin in [11]. Hardin and Massopust investigated more general fractal functions defined on complexes of simplices \( D \subseteq \mathbb{R}^n \) into \( \mathbb{R}^m \) in [12]. Bouboulis and Dalla [13, 14] constructed fractal surfaces using IFSs over grids or rectangular domains. Using tensor product of cardinal spline, Chand and Navascue\'s proposed bicubic fractal surfaces [15]. The theory of fractal surfaces has been investigated along various directions, for instance, [16–20]. The shape preserving fractal surfaces are developed recently using blending functions and univariate fractal functions, see for instance [21–23]. Aseev [24] introduced the construction of fractals by using the idea of a zipper, where the entire graph can be mapped to two consecutive nodes in two different ways. Subsequently, the theory of multi-zipper was investigated by Tetenov et. al [25]. Introducing such a binary array called the signature of a zipper, the class of affine zipper FIFs was introduced recently into the literature by Chand et al. [26]. Further, the calculus of zipper FIFs and cubic zipper FIFs are studied by Reddy [27] and the approximation by smooth zipper fractal functions is investigated in [28].

In this paper, we introduce the concept of multivariate zipper fractal interpolation functions (ZFIFs) to interpolate and approximate multivariate data or a multivariate function by using a suitable binary matrix called zipper \( \epsilon \). These multivariate zipper fractal functions are more general than the existing classical and fractal approximants. Based on the existence of ZFIFs, we construct a novel class of multivariate Bernstein zipper \( \alpha \)-fractal functions using Bernstein polynomials \( B_{n_1,\ldots,n_m}f \) [29, 30] as base functions in its IFS for a given \( f \in C \left( \prod_{k=1}^{m} I_k \right) \), where the \( I_k \) are compact intervals in \( \mathbb{R} \). Multivariate Bernstein zipper \( \alpha \)-fractal functions \( f_{n_1,n_2,\ldots,n_m}^{\alpha,\epsilon} \) converge uniformly to \( f \) as \( n_i \to \infty \) for all \( i \), without having to alter the scaling functions. We prove that the multivariate Bernstein polynomial \( B_{n_1,\ldots,n_m}f \) is Lipschitz if \( f \) is. Employing the Hölder exponents of the germ function, the base function, and the scaling factors, we derive bounds for box-dimension of the graph of a multivariate zipper \( \alpha \)-fractal function. Our results are more general than several existing results in univariate and multivariate cases [31–33].

This paper is organized as follows. Section 2 introduces the basics of univariate zipper fractal functions including its construction. Section 3 is concerned
with the constructive existence of multivariate zipper fractal interpolation on a
given multivariate data set by means of a binary signature matrix. In addition,
a multivariate (germ) function is fractalized using a zipper setting to present
its fractal version through a suitable base function. When the base function is
taken to be a multivariate Bernstein function we obtain multivariate Bernstein
zipper \( \alpha \)-fractal functions. These together with some of their approximation-
theoretic properties are introduced in Section 4. Coordinate-wise monotonicity
approximation is studied in Section 5. In the sequence of this, we derive bounds
for the box dimension of the graph of a multivariate zipper \( \alpha \)-fractal function
in Section 6. Similar bounds are obtained for multivariate zipper Bernstein
\( \alpha \)-fractal functions under the assumption that \( B_n f \) is Hölderian given that \( f \)
is. Placing restrictions on the scaling factors, it is shown in Section 7 that
multivariate Bernstein zipper \( \alpha \)-fractal functions preserve the non-negativity of
multivariate germ functions, and satisfy one-sided approximation results.

2. Basics of zipper fractal functions

In this section, we discuss the basics of IFSs and zippers and present the con-
struction of zipper fractal functions. More details can be found in [24, 26, 34].
In the following, for an \( m \in \mathbb{N} \), we denote by \( \mathbb{N}_m := \{1, 2, \ldots, m\} \) the initial
segment of \( \mathbb{N} \) of length \( m \).

**Definition 2.1.** Let \( 1 < N \in \mathbb{N} \) and let \( w_i : X \to X, i \in \mathbb{N}_{N-1} \), be non-surjective
maps on a complete metric space \( (X, d) \). Then, the system \( \tilde{I} := \{X; w_i, i \in \mathbb{N}_{N-1}\} \)
is called an IFS with vertices \( \{k_1, k_2, \ldots, k_N\} \subset X \) provided that
\[
w_i(k_1) = k_i \quad \text{and} \quad w_i(k_N) = k_{i+1}.
\]
The points \( k_1 \) and \( k_N \) are called the initial and final point of the IFS, respectively.

**Definition 2.2.** For a binary vector \( \epsilon := (\epsilon_1, \epsilon_2, \ldots, \epsilon_{N-1}) \in \{0, 1\}^{N-1} \) called
signature, let \( w_i : X \to X, i \in \mathbb{N}_{N-1} \), be non-surjective maps on a complete
metric space \( (X, d) \) such that \( w_i \) satisfies
\[
w_i(k_1) = k_{i+\epsilon_i} \quad \text{and} \quad w_i(k_N) = k_{i+1-\epsilon_i}
\]
for a given set \( \{k_1, k_2, \ldots, k_N\} \subset X \).

Then, the system \( \tilde{I} = \{X; w_i, i \in \mathbb{N}_{N-1}\} \) is called a zipper with vertices \( \{k_1, k_2, \ldots, k_N\} \). Any non-empty compact set \( A \subset X \) satisfying the self-referential equation
\[
A = \bigcup_{i=1}^{N-1} w_i(A),
\]
is called the attractor or zipper fractal corresponding to the zipper \( \tilde{I} \).
Clearly, an IFS is a particular case of a zipper when the signature satisfies $\epsilon_i = 0$, for all $i \in \mathbb{N}_{N-1}$.

Next, we will review the construction of zipper FIFs (ZFIFs) from a suitable zipper which is constructed from a given set of interpolation data.

Suppose $2 < N \in \mathbb{N}$. Let a set of interpolation points $\{(x_i, y_i) \in I \times \mathbb{R} : i \in \mathbb{N}_N\}$ be given where $x_1 < x_2 < \cdots < x_N$ is a partition of the interval $I := [x_1, x_N]$ and $y_i \in [c, d] \subset \mathbb{R}, \forall i \in \mathbb{N}_N$. Let us set $I_i := [x_i, x_{i+1}]$ and $D_i := I \times [c, d]$. Define maps $w_i : D_i = [x_i, x_{i+1}] \ni \epsilon \mapsto \epsilon = \{\epsilon_i\}_{i=1}^N$ be given where

$$\epsilon = \{\epsilon_i\} = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_{N-1}\}.$$

Note that if $u_i^\epsilon(x) := a_i x + b_i$ and $\epsilon_i = 1$, then the horizontal scaling factors $a_i$ can be negative.

Define $v_i^\epsilon : D_i = [x_i, x_{i+1}] \ni y \mapsto y_i + q_i$, where $\alpha_i$ and $q_i$ are continuous functions on $I$ such that $\|\alpha_i\|_\infty < 1$, and

$$v_i^\epsilon(x_1, y_1) = y_i + \epsilon, \quad v_i^\epsilon(x_N, y_N) = y_i + (1 - \epsilon), \quad i \in \mathbb{N}_{N-1}. \quad (2.2)$$

Here $v_i^\epsilon$ either contracts or flips the graph of $f$ over $I$ to $I_i$. Using these maps, we define maps $w_i : D_i = [x_i, x_{i+1}] \ni (x, y) \mapsto (u_i^\epsilon(x), v_i^\epsilon(x, y))$, $\forall (x, y) \in D_i$.

The zipper IFS for the construction of ZFIFs is then given by

$$\tilde{T} := \{D_i, w_i^\epsilon, i \in \mathbb{N}_{N-1}\}$$

with vertices $\{v_i = (x_i, y_i)\}_{i=1}^N$ and signature $\epsilon = \{\epsilon_1, \epsilon_2, \ldots, \epsilon_{N-1}\}$. For more details, please consult [26].

**Theorem 2.1.** The above zipper $\tilde{T} = \{D_i, w_i^\epsilon, i \in \mathbb{N}_{N-1}\}$ enjoys the following properties.

(i) There exists a unique non-empty compact set $G \subset K$ such that

$$G = \bigcup_{i=1}^{N-1} w_i^\epsilon(G).$$

(ii) $G$ is the graph of a continuous function $f^\epsilon : I \rightarrow \mathbb{R}$ which interpolates the data $\{(x_i, y_i) : i \in \mathbb{N}_N\}$, i.e., $G = \{(x, f^\epsilon(x) : x \in I\}$ and, for $i \in \mathbb{N}_N$, $f^\epsilon(x_i) = y_i$.

The above theorem shows the existence of a zipper interpolation function whose graph is the attractor of an associated zipper IFS.

To obtain a recursive formula for the ZFIF $f^\epsilon$, we proceed as follows. Let $\epsilon \in \{0, 1\}^{N-1}$ be fixed, and let

$$\tilde{C}(I) := \{g \in C(I) : g(x_1) = y_1, g(x_N) = y_N\}.$$
Then, \( \widetilde{C}(I) \) is a closed \textit{metric} subspace of \( C(I) \) and complete with respect to the metric \( d \) induced by the sup-norm.

Now define a Read-Bajraktarević operator \( T : \widetilde{C}(I) \rightarrow \widetilde{C}(I) \) by

\[
(Tg)(x) := \sum_{i=1}^{N-1} v_i^\epsilon ((u_i^\epsilon)^{-1}(x), g \circ (u_i^\epsilon)^{-1}(x)) \chi_{u_i^\epsilon(I)}(x), \quad x \in I.
\]

Clearly, as \( \|\alpha_i\|_{\infty} < 1 \), \( T \) is contraction on \( (\widetilde{C}(I), d) \). By the Banach fixed point theorem, \( T \) has a unique fixed point \( f^\epsilon \) which obeys the self-referential equation

\[
f^\epsilon = \sum_{i=1}^{N-1} v_i^\epsilon ((u_i^\epsilon)^{-1}, f^\epsilon \circ (u_i^\epsilon)^{-1}) \chi_{u_i^\epsilon(I)}.
\]

We call this interpolating function \( f^\epsilon \) a zipper fractal interpolation function (ZFIF) corresponding to the given data \{\( (x_i, y_i) : i \in \mathbb{N}_N \)\} and the signature \( \epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{N-1}) \in \{0,1\}^{N-1} \) for a fixed scaling function vector \( \alpha := (\alpha_1, \alpha_2, \ldots, \alpha_{N-1}) \).

For a prescribed function \( f \in C(I) \), if we choose

\[
q_i(x) := f(u_i(x)) - \alpha_i(x)b(x),
\]

for \( i \in \mathbb{N}_{N-1} \), and \( y_i = f(x_i) \), for \( i \in \mathbb{N}_N \), where \( b \) is called a base function satisfying \( f(x_1) = b(x_1) \) and \( f(x_N) = b(x_N) \), then the corresponding ZFIF \( f_i^\epsilon \) is called a zipper \( \alpha \)-fractal function. The concept of such zipper fractal functions will be extended to the multivariate setting in the next section.

3. Multivariate zipper fractal functions

In the first part of this section, we show the existence of multivariate ZFIFs in a deterministic way with constant scaling functions. This concept is then used to perturb any multivariate function \( f \) to construct its fractal analogue by using a suitable base function in the second part.

3.1. Multivariate zipper fractal interpolation

For \( m \in \mathbb{N} \), we adopt the following notation.

\[
\mathbb{N}_{m,0} := \{0, 1, \ldots, m\}, \quad \partial \mathbb{N}_{m,0} := \{0, m\}, \quad \text{int} \ \mathbb{N}_{m,0} := \{1, \ldots, m - 1\}.
\]

Finite tuples of elements from \( \mathbb{N}_m \) are denoted by expressions like \( j := (j_1, \ldots, j_m) \). Furthermore, let

\[
\epsilon := (\epsilon^1, \ldots, \epsilon^m) \in \prod_{k=1}^{m} \{0,1\}^{\mathbb{N}_k},
\]

\[
\mathcal{I} := \prod_{k=1}^{m} I_k, \quad I_k := [a_k, b_k] \subset \mathbb{R}, \quad a_k < b_k, \quad k \in \mathbb{N}_m.
\]
where $\prod$ denotes the Cartesian product of sets.

Let $2 \leq m \in \mathbb{N}$ and let $C(I)$ denote the Banach space of continuous functions $f : I \to \mathbb{R}$ equipped with the sup-norm. For each $k \in \mathbb{N}_m$, define a partition of $I_k$ by

$$a_k =: x_{k,0} < \cdots < x_{k,N_k} := b_k.$$  

Consider the set of interpolation data points

$$\Delta := \begin{cases} (x_{1,j_1}, \ldots, x_{m,j_m}, y_j) \in I \times \mathbb{R} : j \in \prod_{k=1}^m \mathbb{N}_{k,0} \end{cases}.$$  

Since $\{a_k = x_{k,0}, \ldots, x_{k,N_k} = b_k\}$ is the partition of $I_k$, denote the $j_k$th sub-interval of $I_k$ by $I_{k,j_k} = [x_{k,j_k-1}, x_{k,j_k}]$, $j_k \in \mathbb{N}_{N_k}$. For every $j_k \in \mathbb{N}_{N_k}$, consider an affine map $u_{k,j_k}^k : I_k \to I_{k,j_k}$ satisfying

$$|u_{k,j_k}^k(x) - u_{k,j_k}^k(x')| \leq \alpha_{k,j_k}|x - x'|, \quad \forall x, x' \in I_k,$$  

(3.1) where $0 \leq \alpha_{k,j_k} < 1$, and

$$\begin{cases} u_{k,j_k}^k(x_{k,0}) = x_{k,j_k-1} + \epsilon_{j_k}^k \quad \text{and} \quad u_{k,j_k}^k(x_{k,N_k}) = x_{k,j_k} - \epsilon_{j_k}^k, \quad \text{if $j_k$ is odd}, \\ u_{k,j_k}^k(x_{k,0}) = x_{k,j_k} - \epsilon_{j_k}^k \quad \text{and} \quad u_{k,j_k}^k(x_{k,N_k}) = x_{k,j_k-1} + \epsilon_{j_k}^k, \quad \text{if $j_k$ is even}. \\ \end{cases}$$  

(3.2)

From (3.2), it is easy to check that

$$(u_{k,j_k}^k)^{-1}(x_{k,j_k}) = (u_{k,j_k+1}^k)^{-1}(x_{k,j_k}), \quad \forall j_k \in \mathbb{N}_{N_k,0}.$$  

(3.3)

For each $k \in \mathbb{N}_m$, define a map $\tau_k : \mathbb{N}_{N_k} \times \{0, N_k\} \to \mathbb{Z}$ by

$$\begin{cases} \tau_k(j,0) := j - 1 + \epsilon_j^k \\ \tau_k(j,N_k) := j - \epsilon_j^k \\ \tau_k(j,0) := j - \epsilon_j^k \\ \tau_k(j,N_k) := j - 1 + \epsilon_j^k \\ \end{cases}$$  

(3.4)

Using (3.4), we can rewrite (3.2) as

$$u_{k,j_k}^k(x_{k,i_k}) = x_{k,\tau_k(j_k,i_k)}, \quad \forall j_k \in \mathbb{N}_{N_k}, \quad i_k \in \partial \mathbb{N}_{N_k}, \quad k \in \mathbb{N}_m.$$  

(3.5)

Let $\mathcal{K} := I \times \mathbb{R}$. For each $j \in \prod_{k=1}^m \mathbb{N}_{N_k}$, define a continuous function $\psi_j^k : \mathcal{K} \to \mathbb{R}$ satisfying the following conditions:

$$\psi_j^k(x_1, i_1, \ldots, x_m, i_m, y_1, \ldots, y_m) = y_{\tau_1(j_1, i_1), \ldots, \tau_m(j_m, i_m)}, \quad \forall i \in \prod_{k=1}^m \partial \mathbb{N}_{N_k,0}$$  

(3.6)

and

$$|\psi_j^k(x_1, \ldots, x_m, y) - \psi_j^k(x_1, \ldots, x_m, y')| \leq \gamma_j|y - y'|,$$  

(3.7)

for all $(x_1, \ldots, x_m) \in I$ and $y, y' \in \mathbb{R}$, where $0 \leq \gamma_j < 1$.

Next, for any $j \in \prod_{k=1}^m \mathbb{N}_{N_k}$, we define $W_j^e : \mathcal{K} \to \mathcal{K}$ by

$$W_j^e(x_1, \ldots, x_m, y) := (u_1^{j_1}(x_1), \ldots, u_m^{j_m}(x_m), \psi_j(x_1, \ldots, x_m, y)).$$  

(3.8)
The system

\[ I^\epsilon = \left\{ K, W_j^\epsilon : j \in \prod_{k=1}^m N_k \right\}. \]  

(3.9)

is called multi-zipper IFS with vertices \( \Delta = \{(x_{k,j_1}, \ldots, x_{k,m}, y_j) : j \in \prod_{k=1}^m N_k \} \) and signature \( \epsilon \). Let us consider

\[ G = \left\{ g \in C(\mathcal{I}) : g(x_{1,j_1}, \ldots, x_{m,j_m}) = y_j, \forall j \in \prod_{k=1}^m \partial \mathbb{N}_{N_k,0} \right\} \]

endowed with the uniform metric

\[ \rho(f, g) = \max \left\{ |f(x_1, \ldots, x_m) - g(x_1, \ldots, x_m)| : (x_1, \ldots, x_m) \in \prod_{k=1}^m I_k \right\} \]

for \( f, g \in G \). Then \((G, \rho)\) is complete metric space.

Define a Read-Bajraktarović operator \( T^\epsilon : G \to G \) on \((G, \rho)[16]\) by

\[ T^\epsilon g(x) := \sum_{j \in \prod_{k=1}^m N_k} \nu^\epsilon_j((u_{1,j_1}^1)^{-1}(x_1), \ldots, (u_{m,j_m}^m)^{-1}(x_m)), \]

\[ g(((u_{1,j_1}^1)^{-1}(x_1), \ldots, (u_{m,j_m}^m)^{-1}(x_m)) \chi_{u_j^\epsilon}(\mathcal{I})(x), \]  

(3.10)

for all \( x := (x_1, \ldots, x_m) \in \mathcal{I} \).

One observes that \( T^\epsilon g \) is not continuous for all \( \epsilon \). In order to achieve continuity, we restrict the signature \( \epsilon \) to \( \epsilon_{j_k}^k = \epsilon_{j_{k+1}}^k \), for each \( j_k \in \mathbb{N}_{N_k-1} \), where \( \epsilon_{j_k}^k \) denotes the \( j_k \)th component of the binary column vector \( \epsilon^k \).

**Theorem 3.1.** Let \( \Delta := \{(x_{1,j_1}, \ldots, x_{m,j_m}, y_j) : j \in \prod_{k=1}^m N_{k,0} \} \) be a set of multivariate interpolating data points and \( \epsilon = (\epsilon^1, \ldots, \epsilon^m) \in \prod_{k=1}^m \{0,1\}^{N_k} \) be a signature for the IFS \( I^\epsilon = \left\{ K, W_j^\epsilon : j \in \prod_{k=1}^m N_k \right\} \) as defined in (3.9). Assume that for all \( j_k \in \text{int} \ N_{N_k,0}, \ 1 \leq k \leq m, \)

\[ (u_{k,j_k}^k)^{-1}(x_{k,j_k}) = (u_{k,j_{k+1}}^k)^{-1}(x_{k,j_k}) =: x_k^*, \]

(3.11)

\[ \nu_{j_1,\ldots,j_k}^\epsilon(x_1, \ldots, x_{k-1}, x_k^*, x_{k+1}, \ldots, x_m, y) \]

\[ = \nu_{j_1,\ldots,j_{k+1}}^\epsilon(x_1, \ldots, x_{k-1}, x^*_k, x_{k+1}, \ldots, x_m, y), \]

where \( (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m) \in \prod_{i=1,i\neq k}^m I_i, \ y \in \mathbb{R} \). Then, there exists a continuous function \( f^\epsilon : \mathcal{I} \to \mathbb{R} \) such that
(i) $f^e$ interpolates the given multivariate data set $\Delta$, that is,

$$ f^e(x_{1,j_1}, \ldots, x_{m,j_m}) = y_j, \quad \forall j \in \prod_{k=1}^{m} \mathbb{N}_{N_k,0}. $$

(ii) $G := \{ (x, f^e(x)) : x \in \mathcal{I} \}$ is the graph of the Zipper fractal function $f^e$ and satisfies

$$ G = \bigcup_{j \in \prod_{k=1}^{m} \mathbb{N}_k} W_j^e(G). $$

**Proof** The proof of this theorem is similar to the bivariate case as explained in [17], but for the reader’s convenience, we give a short explanation of it.

It follows from (3.10) that $Tg$ is continuous on $\prod_{k=1}^{m} I_{k,j_k}$. To prove that $Tg$ is continuous on the $m$-dimensional hyperrectangle $\prod_{k=1}^{m} I_k$, it is sufficient to show that $Tg$ is well-defined on the hyperrectangle $\prod_{k=1}^{m} I_{k,j_k}$.

**Claim:** $T^e$ is well-defined.

Assume $j_k \in \text{int} \ \mathbb{N}_{N_k,0}$, $1 \leq k \leq m$, and $X := (x_1, \ldots, x_k, \ldots, x_m) \in \mathcal{I}$, with $x_k = x_{k,j_k}$. Then there are following two cases:

**Case (i):** Assume $x_{k,j_k}$ is an element of $I_{k,j_k}$. Then, by (3.11), we have

$$ T^e f(X) = v_{j_1, \ldots, j_k, \ldots, j_m}((u_{1,j_1}^{e^{1}})^{-1}(x_1), \ldots, (u_{k,j_k}^{e^{k}})^{-1}(x_k), \ldots, (u_{m,j_m}^{e^{m}})^{-1}(x_m)), $$

**Case (ii):** Consider $x_{k,j_k}$ as an element of $I_{k,j_k+1}$. Then, by (3.11), we have

$$ T^e f(X) = v_{j_1, \ldots, j_{k+1}, \ldots, j_m}((u_{1,j_1}^{e^{1}})^{-1}(x_1), \ldots, (u_{k,j_k+1}^{e^{k}})^{-1}(x_k), \ldots, (u_{m,j_m}^{e^{m}})^{-1}(x_m)). $$

Similarly, we can check the other possible cases. Hence, $Tg$ is well-defined on the boundary of $\prod_{k=1}^{m} I_{k,j_k}$ and therefore continuous on $\mathcal{I}$.

Let $i := (i_1, \ldots, i_m) \in \prod_{k=1}^{m} \mathbb{N}_{N_k,0}$. Choose $I := (l_1, \ldots, l_m) \in \prod_{k=1}^{m} \partial \mathbb{N}_{N_k,0}$, $j \in \prod_{k=1}^{m} \mathbb{N}_k$ such that $i = (\tau_1(j_1, l_1), \ldots, \tau_m(j_m, l_m))$. According to the definition
of $\tau_k$, we have $(u_{k,j_k}^k)^{-1}(x_{k,i_k}) = x_{k,i_k}$, for all $k \in \mathbb{N}_m$. Using (3.6) and (3.10), we obtain

$$T^e g(x_{1,i_1}, \ldots, x_{m,i_m}) = v^e_j((u_{1,j_1}^1)^{-1}(x_{1,i_1}), \ldots, (u_{m,j_m}^m)^{-1}(x_{m,i_m})),
$$

$$f((u_{1,j_1}^1)^{-1}(x_{1,i_1}), \ldots, (u_{m,j_m}^m)^{-1}(x_{m,i_m})))
$$

$$= v^e_j(x_{1,i_1}, \ldots, x_{m,i_m}, f(x_{1,i_1}, \ldots, x_{m,i_m}))
$$

$$= v^e_j(x_{1,i_1}, \ldots, x_{m,i_m}, y_j) = y_{\tau_1(j_1,i_1) \ldots \tau_m(j_m,i_m)} = y_1.
$$

Therefore, $T^e f \in \mathcal{G}$ and this shows that $T^e$ is a map from $\mathcal{G}$ to $\mathcal{G}$.

Now, let $f, g \in \mathcal{G}$, $X := (x_1, \ldots, x_m) \in \prod_{k=1}^m I_{k,j_k}$ and

$$\|y\|_\infty := \max \left\{ \gamma_j : j \in \prod_{k=1}^m \mathbb{N}_{N_k} \right\}.
$$

Using (3.7) and (3.10), we establish the contractivity of $T$ as follows:

$$|(T^e f - T^e g)(X)| =
$$

$$\left| v^e_j((u_{1,j_1}^1)^{-1}(x_{1,i_1}), \ldots, (u_{m,j_m}^m)^{-1}(x_{m}), f((u_{1,j_1}^1)^{-1}(x_{1}), (u_{m,j_m}^m)^{-1}(x_{m})))
$$

$$- v^e_j((u_{1,j_1}^1)^{-1}(x_{1}), \ldots, (u_{m,j_m}^m)^{-1}(x_{m}), g((u_{1,j_1}^1)^{-1}(x_{1}), \ldots, (u_{m,j_m}^m)^{-1}(x_{m}))) \right|
$$

$$\leq \gamma_j \left| f((u_{1,j_1}^1)^{-1}(x_{1}), \ldots, (u_{m,j_m}^m)^{-1}(x_{m})) - g((u_{1,j_1}^1)^{-1}(x_{1}), \ldots, (u_{m,j_m}^m)^{-1}(x_{m})) \right|
$$

$$\leq \|y\|_\infty \|f - g\|_\infty.
$$

As $X \in \prod_{k=1}^m I_{k,j_k}$ was arbitrary,

$$\|T^e f - T^e g\|_\infty \leq \|y\|_\infty \|f - g\|_\infty.
$$

Using the Banach fixed point theorem, we conclude that $T^e$ has a unique fixed point $f^e$ in the complete metric spaces $\mathcal{G}$, i.e. $T^e f^e = f^e$. Equivalently,

$$f^e(x_1, \ldots, x_m) = \sum_{j \in \prod_{k=1}^m \mathbb{N}_{N_k}} v^e_j((u_{1,j_1}^1)^{-1}(x_{1,i_1}), \ldots, (u_{m,j_m}^m)^{-1}(x_{m}),
$$

$$f^e((u_{1,j_1}^1)^{-1}(x_{1}), \ldots, (u_{m,j_m}^m)^{-1}(x_{m})) x_{u^e_j(\mathcal{I})}(x_1, \ldots, x_m),
$$

(3.12)

for all $(x_1, \ldots, x_m) \in \mathcal{I}$.

Let us assume that for $X = (x_1, \ldots, x_m)$,

$$(u^e_j)^{-1}(X) = ((u_{1,j_1}^1)^{-1}(x_1), \ldots, (u_{m,j_m}^m)^{-1}(x_m))
$$

and

$$u^e_j(X) = (u_{1,j_1}^1(x_1), \ldots, u_{m,j_m}^m(x_m)).$$
Then, the self-referential equation associated with the multizipper FIF is given by
\[
    f^\epsilon(X) = \sum_{j \in \prod_{k=1}^{m} \mathbb{N}_{N_k}} \left( (u_j^\epsilon)^{-1}(X), f^\epsilon((u_j^\epsilon)^{-1}(X))) X_{u_j^\epsilon}(I)(X), \quad \forall X \in \mathcal{I}. \tag{3.13}
\]
The above equation can be rewritten as
\[
    f^\epsilon(u_j^\epsilon(X)) = v_j^\epsilon(X, f^\epsilon(X)), \quad \forall X \in u_j(I). \tag{3.14}
\]
This unique fixed point \( f^\epsilon \) interpolates the data points \( \Delta \). For the graph of \( f^\epsilon \), \( G = \{(X, f^\epsilon) : X \in \mathcal{I}\} \), we obtain by (3.8) and (3.14),
\[
\bigcup_{j \in \prod_{k=1}^{m} \mathbb{N}_{N_k}} W_j^\epsilon(G) = \bigcup_{j \in \prod_{k=1}^{m} \mathbb{N}_{N_k}} \{W_j^\epsilon(X, f^\epsilon(X)) : X \in \mathcal{I} \}
\]
\[
= \bigcup_{j \in \prod_{k=1}^{m} \mathbb{N}_{N_k}} \{(u_j^\epsilon(X), v_j^\epsilon(X, f^\epsilon(X))) : X \in \mathcal{I} \}
\]
\[
= \bigcup_{j \in \prod_{k=1}^{m} \mathbb{N}_{N_k}} \{(u_j^\epsilon(X), f^\epsilon(u_j^\epsilon(X))) : X \in \mathcal{I} \}
\]
\[
= \{(X, f^\epsilon(X)) : X \in \mathcal{I} \} = G.
\]
The unique fixed point \( f^\epsilon \) of \( T^\epsilon \) is called a multivariate zipper FIF corresponding to the IFS (3.9).

**Remark 3.1.** Note that in the construction of a multivariate zipper FIF, we have to assign \( \epsilon^k \) is either a zero or one column matrix for \( k = 1, 2, \ldots, m \). Then, we can obtain \( 2^m \)-multivariate FIFs by zipper methodology for the same set of scalings. When all \( \epsilon = 0 \), then the multivariate zipper fractal function reduces to a simple multivariate fractal function [33].

### 3.2. Multivariate zipper \( \alpha \)-fractal functions

For a given multivariate function \( f \in C(\mathcal{I}) \), consider a grid
\[
\Delta := \left\{ (x_{1,j_1}, \ldots, x_{m,j_m}) \in \mathcal{I} : j \in \prod_{k=1}^{m} \mathbb{N}_{N_k,0} \right\},
\]
on its domain where \( a_k := x_{k,0} < \cdots < x_{k,N_k} := b_k \) for each \( k \in \mathbb{N}_m \). First, we construct a continuous function \( b : \mathcal{I} \to \mathbb{R} \) satisfying the conditions
\[
b(x_{1,j_1}, \ldots, x_{m,j_m}) = f(x_{1,j_1}, \ldots, x_{m,j_m}), \quad \forall j \in \prod_{k=1}^{m} \partial \mathbb{N}_{N_k,0}, \tag{3.15}
\]
For $k \in \mathbb{N}_m$, we define affine maps $u_{k,jk}^e : I_k \to I_{k,jk}$ by

$$u_{k,jk}^e(x) := a_{k,jk}(x) + b_{k,jk}, \quad j_k \in \mathbb{N}_{N_k},$$

where $a_{k,jk}$ and $b_{k,jk}$ are chosen so that each map $u_{k,jk}^e$ satisfies (3.1) and (3.2).

For $j \in \prod_{k=1}^m \mathbb{N}_{N_k}$, further define continuous variable scaling functions

$$\alpha_j : \mathcal{I} \to \mathbb{R}$$

satisfying

(i) $\|\alpha_j\|_\infty < 1$,

(ii) for all $j_k \in \text{int } \mathbb{N}_{N_k,0}$ and $(u_{k,jk}^e)^{-1}(x_{k,jk}) = (u_{k,jk+1}^e)^{-1}(x_{k,jk}) = x_k^*$, \((x_1, \ldots, x_m) \in \mathcal{I},

$$\alpha_{j_1 \ldots j_{k-1} j_k \ldots j_m}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_m, y) = \alpha_{j_1 \ldots j_{k+1} j_k \ldots j_m}(x_1, \ldots, x_{k+1}, x_k, \ldots, x_m) \in \prod_{i=1, i \neq k}^m I_i, y \in \mathbb{R}.

Further, define $v_j^e : \mathcal{I} \to \mathbb{R}$ by

$$v_j^e(X, y) := f\left(u_{1,j_1}^e(x_1), \ldots, u_{m,j_m}^e(x_m)\right) + \alpha_j(X)(y - b(X)).$$

Then, for all $j \in \prod_{k=1}^m \mathbb{N}_{N_k}, 1 = (l_1, \ldots, l_m) \in \prod_{k=1}^m \partial \mathbb{N}_{N_k}$, we get

$$v_j^e(x_{1,l_1}, \ldots, x_{m,l_m}, f(x_{1,l_1}, \ldots, x_{m,l_m})) = f(u_{1,j_1}^e(x_{1,l_1}), \ldots, u_{m,j_m}^e(x_{m,l_m}))

= f(x_{1,\tau_1(l_1)}, \ldots, x_{m,\tau_m(l_m)}))

= y_{\tau_1(l_1) \ldots \tau_m(l_m)}(j_1, l_1, \ldots, l_m).

In other words, $v_j^e$ satisfies (3.6).

Now, suppose that $j_k \in \text{int } \mathbb{N}_{N_k,0}, 1 \leq k \leq m$, and that

$$x_k^* := (u_{k,j_k}^e)^{-1}(x_{k,j_k}) = (u_{k,j_k+1}^e)^{-1}(x_{k,j_k}).$$

For any $y \in \mathbb{R},$

$$v_{j_1,\ldots,j_{k-1} j_k j_{k+1}, \ldots, j_m}^e(x_1, \ldots, x_{k-1}, x_k^*, x_{k+1}, \ldots, x_m, y)

= f(u_{1,j_1}^e(x_1), \ldots, u_{k-1,j_{k-1} j_k j_{k+1}, \ldots, j_m}^e(x_{k-1}), x_{k,j_k}^*, u_{k+1,j_{k+1} j_{k+2}, \ldots, j_m}^e(x_{k+1}), \ldots, u_{m,j_m}^e(x_m))

+ \alpha_{j_1,\ldots,j_{k-1} j_k j_{k+1}, \ldots, j_m}(x_1, \ldots, x_{k-1}, x_k^*, x_{k+1}, \ldots, x_m)(y - b(x_1, \ldots, x_m))

= f(u_{1,j_1}^e(x_1), \ldots, u_{k-1,j_{k-1} j_k j_{k+1}, \ldots, j_m}^e(x_{k-1}), x_{k,j_k}^*, u_{k+1,j_{k+1} j_{k+2}, \ldots, j_m}^e(x_{k+1}), \ldots, u_{m,j_m}^e(x_m))

+ \alpha_{j_1,\ldots,j_{k-1} j_k j_{k+1}, \ldots, j_m}(x_1, \ldots, x_{k-1}, x_k^*, x_{k+1}, \ldots, x_m)(y - b(x_1, \ldots, x_m))

= v_{j_1,\ldots,j_{k-1} j_k j_{k+1}, \ldots, j_m}^e(x_1, \ldots, x_{k-1}, x_k^*, x_{k+1}, \ldots, x_m, y).$$
Therefore, \( v_j^e \) satisfies (3.11), (3.6), and (3.7), for all \( j \in \prod_{k=1}^{m} \mathbb{N}_{N_k} \). Theorem 3.11 now implies that the IFS

\[
I^e = \left\{ \mathcal{K}, W_j^e : j \in \prod_{k=1}^{m} \mathbb{N}_{N_k} \right\}
\]

(3.19)
defined in (3.9), where the maps \( u_{k,jk}^e \) and \( v_j^e \) are now defined as in (3.16) and (3.18), determines a fractal function referred to as a multivariate zipper \( \alpha \)-fractal function and denoted by \( f_{\alpha,b}^e \).

The fractal function \( f_{\alpha,b}^e \) is the fixed point of the RB operator \( T^e : \mathcal{G} \to \mathcal{G} \) given by

\[
T^e g(X) = f(X) + \sum_{j \in \prod_{k=1}^{m} \mathbb{N}_{N_k}} \alpha_j((u_j^e)^{-1}(X))(f - b)((u_j^e)^{-1}(X))\chi_{u_j^e(I)}(X), \forall X \in \mathcal{I}.
\]

(3.20)

The fixed point \( f_{\alpha,b}^e \) satisfies the self-referential equation

\[
f_{\alpha,b}^e(X) = f(X) + \sum_{j \in \prod_{k=1}^{m} \mathbb{N}_{N_k}} \alpha_j((u_j^e)^{-1}(X))(f_{\alpha,b}^e - b)((u_j^e)^{-1}(X))\chi_{u_j^e(I)}(X),
\]

\( \forall X \in \mathcal{I} \).

or, equivalently,

\[
f_{\alpha,b}^e(u_j^e(X)) = f(u_j^e(X)) + \alpha_j(X)(f_{\alpha,b}^e(X) - b(X)),
\]

(3.21)

for all \( X \in \prod_{k=1}^{m} I_{k,jk}, j \in \prod_{k=1}^{m} \mathbb{N}_{N_k} \).

We can easily establish the following inequality from (3.21):

\[
\|f_{\alpha,b}^e - f\|_{\infty} \leq \frac{\|\alpha\|_{\infty}}{1 - \|\alpha\|_{\infty}}\|f - b\|_{\infty},
\]

(3.22)
where \( \|\alpha\|_{\infty} := \max \left\{ \|\alpha_j\|_{\infty} : j \in \prod_{k=1}^{m} \mathbb{N}_{N_k} \right\} \). From (3.22), we observe that \( \|f_{\alpha,b}^e - f\|_{\infty} \to 0 \) as \( \|\alpha\|_{\infty} \to 0 \).

4. Multivariate Bernstein zipper fractal function

To obtain convergence of a multivariate \( \alpha \)-fractal function \( f_{\alpha,b}^e \) to \( f \) without altering the scaling function \( \alpha \), we take as base functions \( b \) multivariate
Bernstein polynomials $B_n f(X)$ [29, 30] of $f$. The $n = (n_1, \ldots, n_m)$-th Bernstein polynomial for $f \in C(I)$ is given by

$$B_n f(X) = \sum_{k_1=0}^{n_1} \cdots \sum_{k_m=0}^{n_m} f \left( x_{1,0} + (x_{1,N_1} - x_{1,0}) \frac{k_1}{n_1}, \ldots, x_{m,0} + (x_{m,N_m} - x_{m,0}) \frac{k_m}{n_m} \right) \prod_{r=1}^{m} b_{kr,n_r}(x_r), \quad (4.1)$$

where

$$b_{kr,n_r}(x_r) = \binom{n_r}{k_r} \frac{(x_r - x_{r,0})^{kr} (x_{r,N_r} - x_r)^{n_r - kr}}{(x_{r,N_r} - x_{r,0})^{n_r}}, \quad 0 \leq k_r \leq n_r,$$

for $r = 1, \ldots, m$ and $n_1, \ldots, n_m \in \mathbb{N}$.

If we use as base function $b(X) := B_n f(X)$ in (3.18), then the IFS (3.19) becomes

$$I^\varepsilon_n = \left\{ K, W^\varepsilon_j : j \in \prod_{k=1}^{m} \mathbb{N}_{N_k} \right\}, \quad (4.2)$$

where

$$W^\varepsilon_j(X,y) := \left( u^\varepsilon_{1,j_1}(x_1), \ldots, u^\varepsilon_{m,j_m}(x_m), v^\varepsilon_j(X,y) \right),$$

and

$$v^\varepsilon_j(X,y) := f \left( u^\varepsilon_{1,j_1}(x_1), \ldots, u^\varepsilon_{m,j_m}(x_m) \right) + \alpha_j(X)(y - B_n f(X)),$$

for all $j \in \prod_{k=1}^{m} \mathbb{N}_{N_k}$. This IFS determines a multivariate zipper $\alpha$-fractal function

$$f^{\alpha,\varepsilon}_{\Delta, B_n} := f^{\alpha,\varepsilon}_{\Delta, n} := f^{\alpha,\varepsilon}_{n}$$

(we use these three notations interchangeably) referred to as a multivariate Bernstein zipper $\alpha$-fractal function corresponding to the continuous function $f : I \rightarrow \mathbb{R}$. It satisfies the self-referential equation

$$f^{\alpha,\varepsilon}_{\Delta, B_n} \circ u^\varepsilon_j = f \circ u^\varepsilon_j + \alpha_j (f^{\alpha,\varepsilon}_{\Delta, n} - B_n f), \quad \text{on } X \text{ and for all } j \in \prod_{k=1}^{m} \mathbb{N}_{N_k}. \quad (4.3)$$

**Definition 4.1.** Define an operator $F^{\alpha,\varepsilon}_{\Delta, B_n} : C(I) \rightarrow C(I)$ by

$$F^{\alpha,\varepsilon}_{\Delta, B_n}(f) := f^{\alpha,\varepsilon}_{\Delta, B_n} = f^{\alpha,\varepsilon}_{\Delta, n},$$

where $\Delta$ is the set of data points, $B_n$ a multivariate Bernstein operator and $\alpha$ is scaling function. We call this operator a multivariate Bernstein zipper $\alpha$-fractal operator.
Theorem 4.1. The multivariate Bernstein zipper \( \alpha \)-fractal operator

\[ \mathcal{F}^{\alpha,\epsilon}_{\Delta,B_n} : C(I) \to C(I) \]

is linear and bounded.

Proof The proof of this theorem is the same as in the univariate case for the \( \alpha \)-fractal operator in [15].

Now we give an upper bound of the uniform error between \( f \) and its Bernstein fractal perturbation function \( f^{\alpha,\epsilon}_{\Delta,B_n} \). For the multivariate Bernstein operator \( B_n \) [35, p. 115**], we have the following error estimation:

\[
|B_nf(X) - f(X)| \leq C\omega_1(f; \frac{1}{\sqrt{n_1}}, \frac{1}{\sqrt{n_2}}, \ldots, \frac{1}{\sqrt{n_m}}) \quad \text{for all } X \in I, \tag{4.4}
\]

where \( C > 0 \) is independent of \( f, X, n \) and

\[
\omega_1(f; \delta_1, \ldots, \delta_m) = \sup \{|f(X) - f(Y)|; |x_i - y_i| \leq \delta_j, j = 1, 2, \ldots, m\}.
\]

Theorem 4.2. Let \( f \in C(I) \). Then the multivariate Bernstein zipper \( \alpha \)-fractal function \( f^{\alpha,\epsilon}_{\Delta,B_n} \) satisfies the following error estimates:

\[
\|f^{\alpha,\epsilon}_{\Delta,B_n} - f\|_{\infty} \leq \frac{\|\alpha\|_{\infty}}{1 - \|\alpha\|_{\infty}} C\omega_1 \left(f; \frac{1}{\sqrt{n_1}}, \frac{1}{\sqrt{n_2}}, \ldots, \frac{1}{\sqrt{n_m}}\right). \tag{4.5}
\]

Furthermore, \( f^{\alpha,\epsilon}_{\Delta,B_n} \) converges uniformly to \( f \) as \( n_i \to \infty \), for all \( 1 \leq i \leq m \).

Proof From (4.3), we get

\[
\|f^{\alpha,\epsilon}_{\Delta,B_n} - f\|_{\infty} \leq \|\alpha\|_{\infty} \|f^{\alpha,\epsilon}_{\Delta,B_n} - B_nf\|_{\infty} \leq \|\alpha\|_{\infty} \|f^{\alpha,\epsilon}_{\Delta,B_n} - f\|_{\infty} + \|\alpha\|_{\infty} \|f - B_nf\|_{\infty}.
\]

Hence,

\[
\|f^{\alpha,\epsilon}_{\Delta,B_n} - f\|_{\infty} \leq \frac{\|\alpha\|_{\infty}}{1 - \|\alpha\|_{\infty}} \|f - B_nf\|_{\infty}. \tag{4.6}
\]

By (4.4) and (4.6), we have the required error estimate (4.5). Both Ref. [29] and (4.5), we know that \( \|f - B_nf\|_{\infty} \to 0 \) as \( n_i \to \infty \), for all \( 1 \leq i \leq m \). Employing this result in (4.6), we obtain \( \|f^{\alpha,\epsilon}_{\Delta,B_n} - f\|_{\infty} \to 0 \), as \( n_i \to \infty \) for all \( 1 \leq i \leq m \). Therefore, \( f^{\alpha,\epsilon}_{\Delta,B_n} \) converges uniformly to \( f \) as \( n_i \to \infty \) for all \( 1 \leq i \leq m \).

Example 4.1. In this example, we provide an illustration of Theorem 4.2. Let \( f(X) := \sin(\frac{\pi}{2}xy) \) in \( I := I_1 \times I_2 \) where \( I_1 = I_2 := [0, 1] \), \( \alpha_{j_1,j_2} = 0.5 \) for all \( (j_1,j_2) \in \mathbb{N}_3 \times \mathbb{N}_3 \). Consider the grid on \( \mathbb{R}^2 \) given by

\[
\Delta := \{(x_i, y_j) \in \mathbb{R}^2 : x_i, y_j \in \{0, \frac{1}{3}, \frac{2}{3}, 1\}\}.
\]

The original bivariate function \( f(x) = \sin(\frac{\pi}{2}xy) \) is constructed in Figure 1(a). For the interpolation data of \( f \) on \( \Delta \), we have constructed fractal functions in
Figure 1. Multivariate Bernstein zipper $\alpha$-fractal functions.

Figure 1(b)–(e) corresponding to different values of the signature. Figure 1(f) is the plot of $f_{\Delta_{B_{20,20}}}^{\alpha,\epsilon}$ with binary signature matrix 1. One can observe from Figure 1(d) and (f) that $f_{\Delta_{B_{20,20}}}^{\alpha,\epsilon}$ provides a better approximation for $f \in C(I)$ than that by $f_{\Delta_{B_{3,3}}}^{\alpha,\epsilon}$.
5. Coordinate-wise monotonic multivariate Bernstein zipper \(\alpha\)-fractal functions

Multivariate monotonic interpolation functions play an important role in, for instance, empirical option pricing models \([36]\) in finance, design of aggregation operators in multi-criteria decision-making and fuzzy logic \([37]\), dose-response curves and surfaces in biochemistry and pharmacology. Some work on monotonic surface approximation can be found in \([22, 38, 39]\). In this section, we develop coordinate-wise monotonic ZFIFs without using differentiability of the multivariate ZFIFs on rectangular grids. Note that the choice of scaling function depends on the sign of the signature as described in the following:

**Theorem 5.1.** Let \(f \in C(I)\) be nonzero and increasing with respect to the variable \(x_i\). Let

\[
g^\epsilon_j(X) := f(u^\epsilon_j(X)), \quad \gamma^\epsilon_j := \min_{X \in I} \frac{\partial g^\epsilon_j(X)}{\partial x_l}, \quad \Gamma^\epsilon_j := \max_{X \in I} \frac{\partial g^\epsilon_j(X)}{\partial x_l}, \quad \Gamma_n := \max_{X \in I} \frac{\partial B_n f}{\partial x_l}(X).
\]

Then, \(f^\alpha,\epsilon(X)\) is increasing with respect to the variable \(x_l\) if the partial derivative \(f_{x_l}\) exists and the scaling functions \(\alpha_j\) given in (3.17) satisfy the following conditions:

\[
(i) \quad 0 \leq \alpha_j(X) \leq \frac{\gamma^\epsilon_j}{\Gamma_n}, \quad \text{if} \quad \epsilon_{jl} = 0 \quad \text{and} \quad j_l \text{ odd, or,} \quad \epsilon_{jl} = 1 \quad \text{and} \quad j_l \text{ even};
\]

\[
(ii) \quad \frac{\Gamma^\epsilon_j}{\Gamma_n} \leq \alpha_j(X) \leq 0, \quad \text{if} \quad \epsilon_{jl} = 0 \quad \text{and} \quad j_l \text{ even, or,} \quad \epsilon_{jl} = 1 \quad \text{and} \quad j_l \text{ odd.}
\]

for \(X \in I, j \in \prod_{k=1}^m N_k:\)

**Proof** Let \(X' := (x_1, \ldots, x_{j_l}, \ldots, x_m), X'' := (x_1, \ldots, x_l, \ldots, x_m) \in \prod_{k=1}^m I_k\) where \(x_{j_l} < x_l''\) and \(f(X'') \geq f(X')\). Then,

\[
f^\alpha,\epsilon_{\Delta;n}(u^\epsilon_j(X'')) - f^\alpha,\epsilon_{\Delta;n}(u^\epsilon_j(X')) = f(u^\epsilon_j(X'')) - f(u^\epsilon_j(X')) + \alpha_j(X)(f^\alpha,\epsilon_{\Delta;n}(X'')) - f^\alpha,\epsilon_{\Delta;n}(X'))(B_n f(X'')) - B_n f(X')).
\]

As \(B_n f\) is increasing with respect to the variable \(x_l\) \([29]\), \(\Gamma_n > 0\). Now there are two cases:

**Case (i):** \(\epsilon_{jl} = 0\) and \(j_l\) odd, or, \(\epsilon_{jl} = 1\) and \(j_l\) even, i.e., \(u^\epsilon_j\) is increasing.

In this case, \(f(u^\epsilon_j(X))\) is increasing with respect to the variable \(x_l\), which implies that \(\gamma^\epsilon_j\) is nonnegative. Using the mean value theorem for several
variables applied to \( f(u^ε_{j_1,...,j_m}(X'')) - f(u^ε_{j_1,...,j_m}(X')) \) and \((B_n f(X'') - B_n f(X'))\), yields
\[
\begin{align*}
 f^\alpha_{\Delta; n}(u^ε_{j_1,...,j_m}(X'')) - f^\alpha_{\Delta; n}(u^ε_{j_1,...,j_m}(X')) \\
 &\geq \gamma^ε_{j_1,...,j_m}(x''_l - x'_l) - \alpha^ε_{j_1,...,j_m}(X) \Gamma_n(x''_l - x'_l) + \alpha^ε_{j_1,...,j_m}(X) \\
 &\quad \cdot (f^\alpha_{\Delta; n}(X'') - f^\alpha_{\Delta; n}(X')) \\
 &= (\gamma^ε_{j_1,...,j_m} - \alpha^ε_{j_1,...,j_m}(X) \Gamma_n)(x''_l - x'_l) \\
 &\quad + \alpha^ε_{j_1,...,j_m}(X)(f^\alpha_{\Delta; n}(X'') - f^\alpha_{\Delta; n}(X')).
\end{align*}
\]
If \( \alpha^ε_{j_1,...,j_m}(X) \geq 0 \), then we need \( \gamma^ε_{j_1,...,j_m} - \alpha^ε_{j_1,...,j_m}(X) \Gamma_n(x''_l - x'_l) \geq 0 \) which yields the first condition in (5.1).

**Case (ii):** \( \epsilon^l_{j_i} = 0 \) and \( j_i \) even, or, \( \epsilon^l_{j_i} = 1 \) and \( j_i \) odd, i.e., \( u^l_{i,j_i} \) is decreasing.

In this case, \( f(u^ε_{j_1,...,j_m}(X)) \) is decreasing with respect to the variable \( x_l \), which ensures that \( \gamma^ε_{j_1,...,j_m}(X) \) is non-positive.

If \( \alpha^ε_{j_1,...,j_m}(X) \leq 0 \), then an application of the mean value theorem for several variables applied to \( f(u^ε_{j_1,...,j_m}(X'')) - f(u^ε_{j_1,...,j_m}(X')) \) and \((B_n f(X'') - B_n f(X'))\), yields
\[
\begin{align*}
 f^\alpha_{\Delta; n}(u^ε_{j_1,...,j_m}(X'')) - f^\alpha_{\Delta; n}(u^ε_{j_1,...,j_m}(X')) \\
 &\leq \Gamma^ε_{j_1,...,j_m}(x''_l - x'_l) - \alpha^ε_{j_1,...,j_m}(X) \Gamma_n(x''_l - x'_l) \\
 &\quad + \alpha^ε_{j_1,...,j_m}(X)(f^\alpha_{\Delta; n}(X'') - f^\alpha_{\Delta; n}(X')) \\
 &= (\Gamma^ε_{j_1,...,j_m} - \alpha^ε_{j_1,...,j_m}(X) \Gamma_n)(x''_l - x'_l) \\
 &\quad + \alpha^ε_{j_1,...,j_m}(X)(f^\alpha_{\Delta; n}(X'') - f^\alpha_{\Delta; n}(X')).
\end{align*}
\]
Thus, \( \Gamma^ε_{j_1,...,j_m} - \alpha^ε_{j_1,...,j_m}(X) \Gamma_n(x''_l - x'_l) \leq 0 \) if the second inequality in (5.1) is true.

Since fractal interpolation is an iterative process, it ensures that \( f^\alpha_{\Delta; n} \) is increasing with respect to the variable \( x_l \).

**Remark 5.1.** Using similar arguments, we can construct coordinate-wise monotonically decreasing multivariate Bernstein zipper \( \alpha \)-fractal functions \( f^\alpha_{\Delta; n}(X) \) for coordinate-wise monotonically decreasing functions \( f \in C(\mathcal{I}) \).

**Example 5.1.** Let \( f(X) := \exp(xy) \) in \( \mathcal{I} := I_1 \times I_2 \), where \( I_1 = I_2 := [0, 1] \). Note that the function \( f \) is monotonically decreasing. Consider a grid on \( \mathbb{R}^2 \) as
\[
\Delta := \{(x_i, y_j) \in \mathbb{R}^2 : x_i, y_j \in \{0, 1/2, 1\}\}.
\]
For Bernstein zipper fractal functions, we fix \( n = (1, 1) \) and \( \epsilon = (0, 1) \). When scaling functions are chosen arbitrarily as
\[
\alpha_{11}(X) := 0.90, \alpha_{22}(X) := -0.19 \cos(xy), \alpha_{12}(X) := 0.15 \sin(xy), \alpha_{21}(X) := 0.90,
\]
then \( \alpha_l \)'s are not satisfying conditions of Theorem 5.1. In this case, multivariate ZFIF \( f^\alpha_{\Delta; n} \) is depicted in Figure 2(a), which is non-monotonic in \( x \)-direction.
When $\alpha_j$’s are not chosen as in Theorem 5.1.

Figure 2. Coordinate-wise monotonic and non-monotonic multivariate Bernstein ZFIFs.

Table 1. Restrictions for computing scaling functions $\alpha_j(X)$.

| $j = (i,j)$ | $\gamma^j$ | $\Gamma^j_n$ | Range of $\alpha_j(X)$ |
|------------|------------|-------------|----------------------|
| (1, 1)     | 0.00       | 0.32        | [0]                  |
| (1, 2)     | 0.25       | 0.82        | [0, 0.15]            |
| (2, 1)     | -0.41      | 0.00        | [0]                  |
| (2, 2)     | -1.36      | -0.32       | [-0.19, 0]           |

Now, we choose scaling functions according to Theorem 5.1. For this, we have to take any scaling functions according to the restrictions prescribed in Table 1. In particular, we have taken

$$
\alpha_{11}(X) := 0.00, \alpha_{22}(X) := -0.19 \cos(xy), \alpha_{12}(X) := 0.15 \sin(xy), \alpha_{21}(X) := 0.00
$$

to construct a multivariate ZFIF in Figure 2(b) which is monotonic in $x$-direction. It illustrates the importance of Theorem 5.1 for coordinate-wise monotonic multivariate ZFIFs.

6. Box dimension of multivariate ZFIF

In this section, we derive bounds for the box dimension of the graph of a multivariate ZFIF. Furthermore, we show that a multivariate Bernstein polynomial $B_n f$ is Hölderian with exponent $\beta$ provided that $f$ is Hölderian with exponent $\beta$. This will be used to obtain estimates for the box dimension of graphs of multivariate zipper Bernstein fractal functions.

Definition 6.1. Let $A \in \mathbb{R}_0^+$ and $0 < \beta \leq 1$. Then, $\text{Lip}_A \beta$ is defined as the set of all functions $f : \mathcal{K} \subset \mathbb{R}^m \to \mathbb{R}$ satisfying

$$
|f(X_2) - f(X_1)| \leq A \|X_2 - X_1\|^\beta, \quad \forall X_1, X_2 \in \mathcal{K}.
$$

Such functions are also called uniformly Hölderian with exponent $\beta$. 
In the next theorem, we provide estimates for the fractal dimension of the graph of a multizipper \( F_{\alpha} \). For this purpose, we use uniform partitions of \( I_k = [0, 1], k \in \mathbb{N}_m \). Based on the structure of the IFSs (3.19) and (4.2), we choose \( u_{k,jk}^k : I_k \to I_{k,jk} \) as

\[
\begin{aligned}
   &u_{k,jk}^k(x_k) := \\
   &\begin{cases} \\
      1 - 2\epsilon_{jk}^k x_k + \frac{j_k - 1 + \epsilon_{jk}^k}{N}, & \text{if } j_k \text{ is odd;} \\
      -1 + 2\epsilon_{jk}^k x_k + \frac{j_k - \epsilon_{jk}^k}{N}, & \text{if } j_k \text{ is even.}
   \end{cases}
\end{aligned}
\]

**Definition 6.2.** [40] Let \( A \) be a non-empty bounded subset of \( \mathbb{R}^n \). Suppose \( \Lambda(\delta) \) denote the smallest number of \( m \)-dimensional cube of side \( \delta \) that can cover \( A \). The lower and upper box-counting dimensions of \( A \) are defined as

\[
\dim_B(A) = \lim_{\delta \to 0} \frac{\log(N_\delta(A))}{-\log(\delta)},
\]

\[
\overline{\dim}_B(A) = \lim_{\delta \to 0} \frac{\log(N_\delta(A))}{-\log(\delta)},
\]

respectively. If these are equal and finite, we refer to the common value as the box-counting dimension or box dimension of \( A \):

\[
\dim_B(A) = \lim_{\delta \to 0} \frac{\log(N_\delta(A))}{-\log(\delta)}.
\]

Suppose that the IFS (3.19) generates a multizipper \( F_{\alpha}^{(a,e)} \). Then, we have the following result.

**Theorem 6.1.** Let \( f, b \in C(I) \) with Hölder exponents \( \xi_1, \xi_2 \in (0, 1) \). Let \( G \) be the graph of the fractal function \( f_n^{(a,e)} \) associated with the IFS (3.19). Suppose that the interpolation points are not contained in an hyperplane of \( \mathbb{R}^{m+1} \). Let \( \gamma := \min\{\xi_1, \xi_2\} \) and let

\[
\gamma := \sum_{j_1=1}^{N} \sum_{j_1=2}^{N} \cdots \sum_{j_k=1}^{N} \|\alpha_j\|_\infty,
\]

Then, we have the following bounds for the box dimension of \( G \) based on the magnitude of \( \gamma \):

(i) If \( \gamma \leq 1 \), then \( m \leq \dim_B(G) \leq m + 1 - \xi \);

(ii) If \( \gamma > 1 \) and \( N^{(\xi - m)} \gamma \leq 1 \), then

\[
m \leq \dim_B(G) \leq m + 1 - \xi + \frac{\log(\gamma)}{\log(N)};
\]

(iii) If \( \gamma > 1 \) and \( N^{(\xi - m)} \gamma > 1 \), then

\[
m \leq \dim_B(G) \leq 1 + \frac{\log(\gamma)}{\log(N)}.
\]
Proof. Our aim is to calculate the box dimension of the graph of the fractal function $f_{n}^{(\alpha, \epsilon)}$. For this, we consider a cover $\Lambda(r)$ of $G$ whose elements are $m$-cubes with sides of length $\frac{1}{N^r}$ and of the form

$$\left[\frac{p_1}{N^r}, \frac{p_1}{N^r}\right] \times \left[\frac{p_2}{N^r}, \frac{p_2}{N^r}\right] \times \ldots \times \left[\frac{p_m}{N^r}, \frac{p_m}{N^r}\right],$$

for $p_i = 1, 2, \ldots, N^r$, $r \in \mathbb{N}_0$, $i \in \mathbb{N}_m$, and $c \in \mathbb{R}$. Suppose $\mathcal{N}(r)$ is the number of such cubes necessary to cover the graph $G$. Let $\mathcal{N}_0(r)$ be the smallest number of arbitrary $(m+1)$-cubes of size

$$\frac{1}{N^r} \times \frac{1}{N^r} \times \cdots \times \frac{1}{N^r}((m+1) \text{-times}).$$

required to cover $G$. Hence, $\mathcal{N}_0(r) \leq \mathcal{N}(r)$.

Each arbitrary $(m+1)$-dimensional cube can be covered by at most $2^m (m+1)$-cube of the form (6.2). Thus, $\mathcal{N}(r) \leq 2^m \mathcal{N}_0(r)$ and, therefore,

$$\mathcal{N}_0(r) \leq \mathcal{N}(r) \leq 2^m \mathcal{N}_0(r).$$

Hence, we can use covers of the form (6.2) to compute the box dimension of the graph $G$ of $f_{n}^{(\alpha, \epsilon)}$.

Denote by $\Lambda(r, p_1, p_2, \ldots, p_m)$ the collection of $(m+1)$-cubes in

$$\left[\frac{p_1}{N^r}, \frac{p_1}{N^r}\right] \times \left[\frac{p_2}{N^r}, \frac{p_2}{N^r}\right] \times \ldots \times \left[\frac{p_m}{N^r}, \frac{p_m}{N^r}\right]$$

for $p_i = 1, 2, \ldots, N^r$ of the form (6.2) consisting of $\mathcal{N}(r, p_1, p_2, \ldots, p_m)$ $(m+1)$-dimensional cubes. One observes that

$$\mathcal{N}(r) = \sum_{p_1=1}^{N^r} \sum_{p_2=1}^{N^r} \cdots \sum_{p_m=1}^{N^r} \mathcal{N}(r, p_1, p_2, \ldots, p_m).$$

Now we analyze the image of the of the sub-interval $\left[\frac{p_i-1}{N^r}, \frac{p_i}{N^r}\right]$ under the maps $u_{k,jk}^{\epsilon}$, defined in (6.1).

$$u_{k,jk}^{\epsilon} \left[\frac{p_i - 1}{N^r}, \frac{p_i}{N^r}\right] = \left[\frac{\epsilon_{k,jk}(p_i, jk) - 1}{N^{r+1}}, \frac{\epsilon_{k,jk}(p_i, jk)}{N^{r+1}}\right],$$

where

$$\epsilon_{k,jk}(p_i, jk) = \begin{cases} p_i + (jk - 1)N^r; & \text{for } \epsilon_{k,jk} = 0, jk \text{ is odd}, \text{or}, \epsilon_{k,jk} = 1, jk \text{ is even}, \\ -p_i + jkN^r + 1; & \text{for } \epsilon_{k,jk} = 1, jk \text{ is odd}, \text{or}, \epsilon_{k,jk} = 0, jk \text{ is even}. \end{cases}$$
For $j \in \prod_{k=1}^{m} \mathbb{N}_k$, the image of $\Lambda(r, p_1, p_2, \ldots, p_m)$ under the map $W_f^j$ is contained in

$$\left[\left(\frac{k_1^1 (p_1, j_1) - 1}{N^{r+1}}, \frac{k_1^1 (p_1, j_1)}{N^{r+1}}\right) \times \left(\frac{k_2^1 (p_2, j_2) - 1}{N^{r+1}}, \frac{k_2^1 (p_2, j_2)}{N^{r+1}}\right) \times \ldots \times \left(\frac{k_m^1 (p_m, j_m) - 1}{N^{r+1}}, \frac{k_m^1 (p_m, j_m)}{N^{r+1}}\right)\right] \times \mathbb{R},$$

Therefore, we obtain

$$\mathcal{N}(r + 1, p_1, p_2, \ldots, p_m) = \sum_{j_1=1}^{N} \sum_{j_2=1}^{N} \ldots \sum_{j_m=1}^{N} \sum_{p_1, p_2, \ldots, p_m=1}^{N} \mathcal{N}(r, k_1^1 (p_1, j_1), k_2^1 (p_2, j_2), \ldots, k_m^1 (p_m, j_m)).$$

(6.5)

As $f$ and $b$ are uniform Hölderian on $I$ with exponents $\xi_1, \xi_2 \in (0, 1]$, we obtain the following estimates for $X = (x_1, x_2, \ldots, x_m), X' = (x'_1, x'_2, \ldots, x'_m) \in \prod_{k=1}^{m} \left[\frac{p_k - 1}{N^{r+1}}, \frac{p_k}{N^{r+1}}\right]$:

$$|f(u_j(X)) - f(u_j(X'))| \leq \frac{A_1}{N^{\xi_1 (r+1)}},$$

$$|b(X) - b(X')| \leq \frac{A_2}{N^{\xi_2 r}}.$$

(6.6)

Thus, the maximum height of $\nu_j(\Lambda(r, p_1, p_2, \ldots, p_m))$ is bounded above by

$$\|\alpha_j\|_\infty \mathcal{N}(r, p_1, p_2, \ldots, p_m) \leq \frac{A_1}{N^{\xi_1 (r+1)}} + \frac{A_2 \|\alpha_j\|_\infty}{N^{\xi_2 r}}.$$ 

Now,

$$\mathcal{N}(r, k_1^1 (p_1, j_1), k_2^1 (p_2, j_2), \ldots, k_m^1 (p_m, j_m)) \leq \left(\frac{\|\alpha_j\|_\infty \mathcal{N}(r, p_1, p_2, \ldots, p_m)}{N^r} + \frac{A_1}{N^{\xi_1 (r+1)}} + \frac{A_2 \|\alpha_j\|_\infty}{N^{\xi_2 r}}\right) N^{r+1} + 2$$

$$= \|\alpha_j\|_\infty \mathcal{N}(r, p_1, p_2, \ldots, p_m)N + A_1 N^{(1-\xi_1)(r+1)}$$

$$+ A_2 \|\alpha_j\|_\infty N^{(1-\xi_2) r+1} + 2.$$

This produces an estimate of the form

$$\sum_{j_1=1}^{N} \sum_{j_2=1}^{N} \ldots \sum_{j_m=1}^{N} \mathcal{N}(r, k_1^1 (p_1, j_1), k_2^1 (p_2, j_2), \ldots, k_m^1 (p_m, j_m)) \leq$$

$$\mathcal{N}(r, p_1, p_2, \ldots, p_m)N^r + A_1 N^{(1-\xi_1)(r+1)+m} + A_2 N^{(1-\xi_2) r+1} + 2N^m.$$  

(6.7)
Substituting the above estimate into (6.5), we obtain

\[
\mathcal{N}(r + 1) = \sum_{p_1, p_2, \ldots, p_m = 1}^{N'} \sum_{j_1 = 1}^{N_1} \sum_{j_2 = 1}^{N_2} \cdots \sum_{j_m = 1}^{N_m} \mathcal{N}(r, k_1^j(p_1, j_1), k_2^j(p_2, j_2), \ldots, k_m^j(p_m, j_m))
\]

\[\leq \sum_{p_1, p_2, \ldots, p_m = 1}^{N'} \mathcal{N}(r, p_1, p_2, \ldots, p_m) N \gamma + A_1 N^{(1 - \xi)(r + 1) + m} + 2N^m \]  

\[= \mathcal{N}(r) N \gamma + A_1 N^{(1 - \xi)(r + 1) + m + 2N^m} \]

\[\leq \mathcal{N}(r) N \gamma + A_1 N^{(r + 1)(m + 1 - \xi)} + A_2 N^{(r + 1)(m + 1 - \xi) + 2N^m} \]

Continuing this process yields

\[
\mathcal{N}(r) \leq \mathcal{N}(0) N^r \gamma^r + (1 + N^{(\xi - m) \gamma}) N^{r(\xi - m)(r - 1)} \mathcal{N}(m + 1 - \xi) \mathcal{C},
\]  

(6.9)

Thus, we have the following three cases:

**Case (i):** \( \gamma \leq 1 \).

As \( N \geq 2 \) and \( \xi \in (0, 1), N^{k(\xi - m)} \leq 1 \), for \( k \geq 1 \). By (6.9), we have

\[
\mathcal{N}(r) \leq \mathcal{N}(0) N^r \gamma^r + rN^{r(\xi - m) \gamma} + 2N^{(\xi - m) \gamma^2} + \cdots + N^{r(\xi - m)(r - 1)} \mathcal{N}(m + 1 - \xi) \mathcal{C},
\]

(6.10)

Hence,

\[
\dim_B(G) = \lim_{r \to \infty} \frac{\log(\mathcal{N}(r))}{\log(N^r)} \leq \lim_{r \to \infty} \frac{\log(C_1 rN^{r(\xi - m)})}{\log(N^r)} = m + 1 - \xi.
\]  

(6.11)

By the continuity of the fractal function, we have that \( \dim_B(G) \geq m \), and using (6.11), we obtain

\[m \leq \dim_B(G) \leq m + 1 - \xi.\]

**Case (ii):** \( \gamma > 1 \) and \( N^{(\xi - m) \gamma} \leq 1 \).
By (6.9), we have
\[ N(r) \leq N(0) N^r \gamma^r + r N^{r(m+1-\xi)} C \]
\[ \leq C_2 r N^{r(m+1-\xi)} , \quad \text{where} \quad C_2 = C + N(0). \] (6.12)

Hence
\[ \dim_B(G) \leq \lim_{r \to \infty} \frac{\log(C_2 r N^{r(m+1-\xi)\gamma^r})}{\log(N^r)} \]
\[ = m + 1 - \xi + \frac{\log(\gamma)}{\log(N)}. \] (6.13)

Hence, using the above inequality, we obtain
\[ m \leq \dim_B(G) \leq m + 1 - \xi + \frac{\log(\gamma)}{\log(N)}. \]

**Case (iii):** \( \gamma > 1 \) and \( N(\xi-m) \gamma > 1. \)

Again by (6.9)
\[ N(r) \leq N(0) N^r \gamma^r + \left[ \frac{N^{r(\xi-m)} \gamma^r - 1}{N^{(\xi-m)} \gamma - 1} \right] N^{r(m+1-\xi)} C \]
\[ \leq N(0) N^r \gamma^r + \left[ \frac{N^{r(\xi-m)} \gamma^r}{N^{(\xi-m)} \gamma} - 1 \right] N^{r(m+1-\xi)} C \] (6.14)
\[ \leq N(0) N^r \gamma^r + \frac{N^r \gamma^r}{N^{(\xi-m)} \gamma} - 1. \]

Using (6.14) and arguments similar to those above, we obtain
\[ \dim_B(G) \leq 1 + \frac{\log(\gamma)}{\log(N)}. \]

**Remark 6.1.** (a) The cases (i)–(iii) in Theorem 6.1 imply that the box dimension estimates of ZFIFs are independent of the signature matrix \( \epsilon. \)
(b) In Box-dimension estimate (i) of above theorem, if we take \( \xi = 1 \) for \( \gamma \leq 1, \) then \( \dim_B(G) \) is exactly equal to \( m. \)
(c) For \( \gamma > 1 \) and \( \xi = 1, \) it is not easy to compute a lower bound for the box-dimension when \( m > 1 \) due to the fact \( N(r) \) is bigger than a negative number, which is obviously true.

It is known that for a univariate function \( f \in \text{Lip}_A \beta, \) the corresponding univariate Bernstein function \( B_n f \in \text{Lip}_A \beta [41]. \) We need a similar result for the computation of the box dimension of the graph of a multivariate Bernstein zipper fractal function. We know that \( B_n f \) is a multivariate polynomial, and hence it is Lipschitz continuous. Then, it is also Hölderian with exponent \( \beta \in (0,1]. \) But Bernstein’s multivariate polynomial \( B_n f \) preserves both the exponent and the Lipschitz constant of \( f, \) and we give this interesting proof in the following:
Proposition 6.1. If a multivariate function \( f \) defined on \([0, 1]^m\) is in \( \text{Lip}_A \beta \) then the corresponding multivariate Bernstein polynomial \( B_n f \) is also an element of \( \text{Lip}_A \beta \).

Proof We know that
\[
B_n f (X) = \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} \cdots \sum_{j_m=0}^{n_m} \prod_{p=1}^{m} b_{j_p, n_p} (x_p) f \left( \frac{k_1}{n_1}, \frac{k_2}{n_2}, \ldots, \frac{k_m}{n_m} \right), \tag{6.15}
\]
where \( b_{j_p, n_p} (x_p) := \binom{n_p}{k_p} x_p^{k_p} (1 - x_p)^{n_p - k_p} \), \( p \in \mathbb{N}_m \), and \( X := (x_1, x_2, \ldots, x_m) \). Let
\[
A_{k_p, l_p}^{n_p} (x_p, y_p) := \frac{n_p!}{k_p! l_p! (n_p - k_p - l_p)!} x_p^{k_p} (y_p - x_p)^{l_p} (1 - y_p)^{n_p - k_p - l_p} \tag{6.16}
\]
for \( p \in \mathbb{N}_m \) and \( x_p, y_p \in I \).

The following results is valid for \( n \in \mathbb{N} \):
\[
\sum_{k=0}^{n} \binom{n}{k} j^k (1 - y)^{n-k} f \left( \frac{j}{n} \right) = \sum_{k=0}^{n} \sum_{k=0}^{n-k} \frac{n!}{k!(n-k)!} x^k (y-x)^{l} (1-y)^{n-k-l} f \left( \frac{k}{n} \right). \tag{6.17}
\]
Let \( X := (x_1, x_2, \ldots, x_m) \) and \( Y := (y_1, y_2, \ldots, y_m) \). Then, there are \( 2^m \) possible arrangements in the corresponding arguments of \( X \) and \( Y \). Take as one possible case \( x_1 \leq y_1, x_2 \geq y_2, x_3 \geq y_3, \ldots, x_m \geq y_m \). Equation (6.17) implies
\[
B_n f (X) = \sum_{j_1=0}^{n_1} \cdots \sum_{j_m=0}^{n_m} \prod_{p=1}^{m} b_{j_p, n_p} (x_p) \sum_{j_1=0}^{n_1} b_{j_1, n_1} (x_1) f \left( \frac{j_1}{n_1}, \frac{j_2}{n_2}, \ldots, \frac{j_m}{n_m} \right)
\]
\[
= \sum_{j_2=0}^{n_2} \cdots \sum_{j_m=0}^{n_m} \prod_{p=1}^{m} b_{j_p, n_p} (x_p) \sum_{j_1=0}^{n_1} \sum_{l_1=0}^{n_1-k_1} A_{j_1, l_1}^{n_1} (y_1, x_1) \cdot f \left( \frac{k_1}{n_1}, \frac{j_2}{n_2}, \ldots, \frac{j_m}{n_m} \right)
\]
\[
= \sum_{k_1=0}^{n_1} \sum_{l_1=0}^{n_1-k_1} \sum_{j_3=0}^{n_3} \cdots \sum_{j_m=0}^{n_m} \prod_{p=1}^{m} b_{j_p, n_p} (x_p) A_{j_1, l_1}^{n_1} (y_1, x_1) \cdot \sum_{j_2=0}^{n_2} b_{j_2, n_2} (x_2) f \left( \frac{k_1}{n_1}, \frac{j_2}{n_2}, \ldots, \frac{j_m}{n_m} \right)
\]
\[
= \sum_{k_1=0}^{n_1} \sum_{l_1=0}^{n_1-k_1} \sum_{j_3=0}^{n_3} \cdots \sum_{j_m=0}^{n_m} \prod_{p=1}^{m} b_{j_p, n_p} (x_p) A_{j_1, l_1}^{n_1} (y_1, x_1) \cdot \sum_{k_2=0}^{n_2} \sum_{l_2=0}^{n_2-k_2} A_{j_2, l_2}^{n_2} (x_2, y_2) f \left( \frac{k_1}{n_1}, \frac{k_2 + l_2}{n_2}, \ldots, \frac{j_m}{n_m} \right),
\]
\[ B_{n_1f}(X) = \sum_{k_1=0}^{n_1} \sum_{l_1=0}^{n_1-k_1} \sum_{k_2=0}^{n_2} \sum_{l_2=0}^{n_2-k_2} \sum_{n_3=0}^{n_3} \cdots \sum_{n_m=0}^{n_m} \prod_{p=1, p \neq 1}^m b_{j_1, l_1}(x_1) A_{j_2, l_2}(x_2, y_2) f \left( \frac{k_1}{n_1}, \frac{k_2 + l_2}{n_2}, \ldots, \frac{j_m}{n_m} \right). \]

By (6.18) and (6.19), we can write

\[ \sum_{k_1=0}^{n_1} \sum_{l_1=0}^{n_1-k_1} \sum_{k_2=0}^{n_2} \sum_{l_2=0}^{n_2-k_2} \sum_{n_3=0}^{n_3} \cdots \sum_{n_m=0}^{n_m} \prod_{p=1, p \neq 1}^m A_{j_1, l_1}(x_1, x_1) A_{j_2, l_2}(x_2, y_2) f \left( \frac{k_1}{n_1}, \frac{k_2 + l_2}{n_2}, \ldots, \frac{k_m + l_m}{n_m} \right). \]

Similarly, we get

\[ \sum_{k_1=0}^{n_1} \sum_{l_1=0}^{n_1-k_1} \sum_{k_2=0}^{n_2} \sum_{l_2=0}^{n_2-k_2} \sum_{n_3=0}^{n_3} \cdots \sum_{n_m=0}^{n_m} \prod_{p=1, p \neq 1}^m A_{j_1, l_1}(x_1, x_1) A_{j_2, l_2}(x_2, y_2) f \left( \frac{k_1}{n_1}, \frac{k_2 + l_2}{n_2}, \ldots, \frac{k_m + l_m}{n_m} \right). \]

By (6.18) and (6.19), we can write

\[ |B_{n_1f}(X) - B_{n_1f}(Y)| \leq \sum_{k_1=0}^{n_1} \sum_{l_1=0}^{n_1-k_1} \sum_{k_2=0}^{n_2} \sum_{l_2=0}^{n_2-k_2} \sum_{n_3=0}^{n_3} \cdots \sum_{n_m=0}^{n_m} \prod_{p=1, p \neq 1}^m A_{j_1, l_1}(y_1, x_1) A_{j_2, l_2}(x_2, y_2) f \left( \frac{k_1}{n_1}, \frac{k_2 + l_2}{n_2}, \ldots, \frac{k_m + l_m}{n_m} \right) \]

\[ - f \left( \frac{k_1 + l_1}{n_1}, \frac{k_2}{n_2}, \ldots, \frac{k_m}{n_m} \right). \]

Suppose \( \max \left\{ \frac{k_p}{n_p} : p \in \mathbb{N}_m \right\} = \frac{k_a}{n_a} \) for some \( a \in \mathbb{N}_m \). Then, (6.20) becomes

\[ |B_{n_1f}(X) - B_{n_1f}(Y)| \leq \sum_{k_1=0}^{n_1} \sum_{l_1=0}^{n_1-k_1} \sum_{k_2=0}^{n_2} \sum_{l_2=0}^{n_2-k_2} \sum_{n_3=0}^{n_3} \cdots \sum_{n_m=0}^{n_m} \prod_{p=1, p \neq 1}^m A_{j_1, l_1}(y_1, x_1) A_{j_2, l_2}(x_2, y_2) f \left( \frac{k_a}{n_a} \right)^\beta. \]
\[
\begin{align*}
&= A \sum_{k_1=0}^{n_1} \sum_{l_1=0}^{n_1-k_1} A_{j_1,l_1}^{n_1} (y_1, x_1) \sum_{k_2=0}^{n_2} \sum_{l_2=0}^{n_2-k_2} A_{j_2,l_2}^{n_2} (x_2, y_2) \\
&\cdots \sum_{k_a=0}^{n_a} \sum_{l_a=0}^{n_a-k_a} A_{j_a,l_a}^{n_a} (x_a, y_a) \left( \frac{k_a}{n_a} \right)^\beta \cdots \sum_{k_m=0}^{n_m} \sum_{l_m=0}^{n_m-k_m} A_{j_m,l_m}^{n_m} (x_m, y_m) \\
&= A \sum_{l_1=0}^{n_1} b_{l_1,n_1} (x_1) \cdots \sum_{l_a=0}^{n_a} b_{l_a,n_a} (x_a) \left( \frac{l_a}{n_a} \right)^\beta \cdots \sum_{l_m=0}^{n_m} b_{l_m,n_m} (x_m) \\
&= A \sum_{l_a=0}^{n_a} b_{l_a,n_a} (x_a) \left( \frac{l_a}{n_a} \right)^\beta = AB_{n_a} (x^\beta, (y_a - x_a)) \leq A (y_a - x_a)^\beta \\
&= A \|X - Y\|^\beta. \quad (6.21)
\end{align*}
\]

Similarly, in all other cases, we get the same inequality. Therefore,
\[
B_n f \in \text{Lip}_A^\beta.
\]

**Corollary 6.1.** Let \( f \in C(\mathcal{I}) \) with Hölder exponent \( \xi \in (0, 1] \). Let \( G \) be the graph of the multivariate Bernstein fractal function \( f_n^{(\alpha, \epsilon)} \) associated with the IFS (4.2). Suppose that the interpolation points are not contained in an \( m \)-dimensional hyperplane of \( \mathbb{R}^{m+1} \). Let

\[
\gamma := \sum_{j_1=1}^N \sum_{j_2=2}^N \cdots \sum_{j_k=1}^N \|\alpha_j\|_\infty.
\]

Then, the box dimension of \( G \) satisfies the estimates (i), (ii), (iii) of Theorem 6.1.

**Proof** As \( f \) is Hölderian with exponent \( \xi \), Lemma 6.1 ensures that \( B_n f \) is Hölderian with the same exponent. Thus, the results in Theorem 6.1 are also valid for the box dimension of the graphs of multivariate Bernstein fractal functions. \[\square\]

### 7. Constrained multivariate Bernstein zipper \( \alpha \)-fractal approximation

In this section, we study constrained approximation by multivariate Bernstein zipper \( \alpha \)-fractal functions. The choice of scaling function is fixed in [33], whereas we have taken a general choice scaling functions.

**Theorem 7.1.** Let \( f \in C(\mathcal{I}) \) and suppose \( f(X) \geq 0 \) for all \( X \in \mathcal{I} \). Consider the set

\[
\Delta := \left\{ (x_{1,j_1}, \ldots, x_{m,j_m}) : j \in \prod_{k=1}^m \mathbb{N}_{N_k,0} \right\}
\]

where \( a_k := x_{k,0} < \cdots < x_{k,N_k} =: b_k \) for each \( k \in \mathbb{N}_m \), \( I_k := [a_k, b_k] \), and \( \alpha : \mathcal{I} \to \mathbb{R} \) is a continuous scaling function. Then, the sequence \( \{I_n^\epsilon\} \) of IFSs (4.2)
determines a sequence \( \{ f_{\Delta;n}^{\alpha,\epsilon} \} \) of positive multivariate Bernstein zipper \( \alpha \)-fractal functions that converges uniformly to \( f \) if the scaling functions \( \alpha_j(X) \) are chosen as in (3.17) and according to
\[
\max \left\{ \frac{-\phi^\epsilon(f;j)}{C_n - \phi_n}, \frac{C_n - \Phi^\epsilon(f;j)}{\Phi_n} \right\} \leq \alpha_j(X) \leq \min \left\{ \frac{\phi^\epsilon(f;j)}{\Phi_n}, \frac{C_n - \Phi^\epsilon(f;j)}{C_n - \phi_n} \right\},
\]
(7.1)
for \( j \in \prod_{k=1}^m \mathbb{N}_{N_k} \), where
\[
\phi^\epsilon(f;j) := \min_{X \in I} f(u_j^\epsilon(X)), \quad \Phi^\epsilon(f;j) := \max_{X \in I} f(u_j^\epsilon(X)),
\]
\[
\phi_n := \min_{X \in I} B_nf(X), \quad \Phi_n := \max_{X \in I} B_nf(X),
\]
and \( C_n \) is a positive real number strictly greater than both \( \phi_n \) and \( \|f\|_\infty \).

**Proof** By Theorem 4.2, there exists a sequence \( \{ f_{\Delta;n}^{\alpha,\epsilon} \} \), for \( n_k \in \mathbb{N} \), of multivariate Bernstein zipper \( \alpha \)-fractal functions that converges to \( f \) for any given non-negative function \( f \in C(I) \). By [29], \( B_n \) is a positive linear operator and thus \( B_nf(X) \geq 0 \), for all \( X \in I \), which implies the positivity of \( \Phi_n \).

Let \( q_{n,j}^\epsilon(X) := f(u_j^\epsilon(X)) - \alpha_j(X)B_nf(X) \). By (4.3), we obtain
\[
f_{\Delta;n}^{\alpha,\epsilon}(u_j^\epsilon(X)) = f(u_j^\epsilon(X)) + \alpha_j(X)(f_{n}^{\alpha,\epsilon}(X) - B_nf(X))
\]
\[
= \nu_{n,j}^\epsilon(X,f_{\Delta;n}^{\alpha,\epsilon}(X)).
\]
(7.2)
As \( \nu_{n,j}^\epsilon(X,y) \in [0,C_n] \), \( j \in \prod_{k=1}^m \mathbb{N}_{N_k} \), for all \( (X,y) \in I \times [0,C_n] \) this implies
\[
f_{\Delta;n}^{\alpha,\epsilon}(u_j^\epsilon(X)) \in [0,C_n], \quad \forall X \in I.
\]
Therefore, in order to prove that \( f_{\Delta;n}^{\alpha}(X) \in [0,C_n] \), for all \( X \in I \), it suffices to show that \( \nu_{n,j}^\epsilon(X,y) \in [0,C_n] \), for all \( (X,y) \in I \times [0,C_n] \). Now, one can prove the results by using univariate case arguments as proposed in Theremom 5 of [28].

In the above theorem, we have seen that for every continuous function \( f : I \rightarrow \mathbb{R} \) with \( f \geq 0 \) on \( I \), there exists a sequence of positive multivariate Bernstein zipper \( \alpha \)-fractal functions which converges to \( f \) in the sup-norm. The next result considers the case when the difference of two functions in \( C(I) \) is positive.

**Theorem 7.2.** Let \( f, g \in C(I) \) and \( f \geq g \) on \( I \). For all \( n \in \mathbb{N}^m \) let \( f_{\Delta;n}^{\alpha,\epsilon} \) be multivariate Bernstein \( \alpha \)-fractal functions associated with the IFS \( I_n^\epsilon \), where
\[
\Delta := \left\{ (x_{1,j_1}, \ldots, x_{m,j_m}) : j \in \prod_{k=1}^m \mathbb{N}_{N_k,0} \right\}
\]
such that \( a_k := x_{k,0} < \cdots < x_{k,N_k} := b_k \) for \( k \in \mathbb{N}_m \), \( I_k := [a_k, b_k] \) and \( \alpha_j \) taken as in (3.17).

Then, the sequence \( \{f^\epsilon_n\} \) of IFSs determines a sequence of multivariate Bernstein zipper \( \alpha \)-fractal functions \( \{f_{\Delta,n}^{\alpha,\epsilon}\} \) such that \( f_{\Delta,n}^{\alpha,\epsilon} \geq g \) on \( \mathcal{I} \) and which converges uniformly to \( f \) if the continuous scaling functions \( \alpha_j(X) \) are chosen as in (3.17) and satisfy

\[
0 \leq \alpha_j(X) \leq \min \left\{ \frac{\phi^\epsilon(f - g, j)}{\Phi_n(f) - \phi(g)}, 1 \right\}, \tag{7.3}
\]

where \( \phi^\epsilon(f - g, j) := \min_{X \in \mathcal{I}} B_n f(X) \) and \( \phi(g) := \min_{X \in \mathcal{I}} g(X) \).

**Proof** By (4.3), we can rewrite the functional equation of \( f_{\Delta,n}^{\alpha,\epsilon} \) as follows.

\[
f_{\Delta,n}^{\alpha,\epsilon}(X) = f(X) + \sum_{j \in m, \prod_{k=1}^m N_k} \alpha_j((u_j^\epsilon)^{-1}(X))(f_{\Delta,n}^{\alpha,\epsilon}((u_j^\epsilon)^{-1}(X))
- B_n((u_j^\epsilon)^{-1}(X)))\chi_{u_j^\epsilon}(I)(X), \quad X \in \mathcal{I}. \tag{7.4}
\]

This functional equation is a rule to get the values of \( f_{\Delta,n}^{\alpha,\epsilon} \) at \((N^{r+2} + 1)^m \) distinct points in \( \mathcal{I} \) in \((r+1)\)-th iteration using the value of \( f_{\Delta,n}^{\alpha,\epsilon} \) at \((N^{r+1} + 1)^m \) points in \( \mathcal{I} \) at the \( r \)-th iteration.

Let us begin the iteration process with the nodal points \( X_i, i \in \mathbb{N} \). We establish that the \( p \)-th iterated image of \( X \) satisfies \( f_{\Delta,n}^{\alpha,\epsilon}(X) \geq g(X) \). For the 0-th iteration, we have

\[
f_{\Delta,n}^{\alpha,\epsilon}(X) \geq g(X),
\]

since \( f_{\Delta,n}^{\alpha,\epsilon} \) interpolates \( f \) at the nodes and \( f(X) \geq g(X) \).

Now, suppose that \( f_{\Delta,n}^{\alpha,\epsilon} \geq g \). We show that

\[
f_{\Delta,n}^{\alpha,\epsilon}(u_j^\epsilon(X)) \geq g((u_j^\epsilon(X)), \quad \forall X \in \mathcal{I}, \ \forall j \in \prod_{k=1}^m N_k.
\]

From the fixed point equation (7.4), this is equivalent to proving that

\[
f(u_j^\epsilon(X)) + \alpha_j(X)f_{\Delta,n}^{\alpha,\epsilon}(X) - \alpha_j(X)B_n f(X) - g(u_j^\epsilon(X)) \geq 0. \tag{7.5}
\]

Choosing \( \alpha_j(X) \) as nonnegative and using the \( p \)-th iterated image yields

\[
f(u_j^\epsilon(X)) + \alpha_j(X)g(X) - \alpha_j(X)B_n f(X) - g(u_j^\epsilon(X)) \geq 0.
\]

For the validity of the above inequality, it suffices to choose \( \alpha_j \) so that

\[
0 \leq \alpha_j(X) \leq \min \left\{ \frac{\phi^\epsilon(f - g, j)}{\Phi_n(f) - \phi(g)}, 1 \right\}. \tag{7.6}
\]

If \( \alpha_j, j \in \prod_{k=1}^m N_k \) satisfies (7.3), then \( f_{\Delta,n}^{\alpha,\epsilon} \geq g \) on a dense subset of \( \mathcal{I} \). By a density and continuity argument, \( f_{\Delta,n}^{\alpha,\epsilon}(X) \geq g(X) \) for all \( X \in \mathcal{I} \). \( \square \)
**Corollary 7.1.** Let \( f, g \in C(I) \) and \( f \geq g \) on \( I \). Consider the partition

\[
\Delta := \left\{ (x_{1,j_1}, \ldots, x_{m,j_m}) : j \in \prod_{k=1}^{m} \mathbb{N}_{N_k}, 0 \right\}
\]

with \( a_k := x_{k,0} < \cdots < x_{k,N_k} =: b_k \), for each \( k \in \mathbb{N}_m \), \( I_k := [a_k, b_k] \), and a continuous scaling function \( \alpha_j : I \to \mathbb{R} \).

Then, there exist sequences \( \{f_{\Delta,n}^{\alpha,j}\} \) and \( \{g_{\Delta,n}^{\alpha,j}\} \) of multivariate Bernstein zipper \( \alpha \)-fractal function converging to \( f \) and \( g \), respectively, with \( f_{\Delta,n}^{\alpha,j} \geq g_{\Delta,n}^{\alpha,j} \) on \( I \), if the scaling functions satisfy (3.17) as well as the following estimate:

\[
0 \leq \alpha_j(X) \leq \min \left\{ \frac{\phi^\epsilon(f - g, j)}{\Phi_n(f - g)}, 1 \right\}, \quad j \in \prod_{k=1}^{m} \mathbb{N}_{N_k}, \quad \text{(7.7)}
\]

where

\[
\phi^\epsilon(f - g, j) := \min_{X \in I} (f - g)(u^\epsilon_j(X))
\]

and

\[
\Phi_n(f - g) := \max_{X \in I} B_n(f - g)(X).
\]

**Proof** We obtain the result by taking \( f \) as \( f - g \) and \( g = 0 \) in Theorem 7.2. \( \square \)

In the following theorem, we construct a sequence of increasing multivariate Bernstein zipper FIFs and a one-sided approximation by a convex continuous function defined in an \( m \)-dimensional hyperrectangle. For this theorem, we adopt the following notation: For \( n = (n_1, \ldots, n_m) \in \prod_{k=1}^{m} I_k \), let \( n + 1 := (n_1 + 1, \ldots, n_m + 1) \).

**Theorem 7.3.** Let \( f \in C(I) \) be convex and suppose \( \alpha_j \) are non-negative scaling functions as in (3.17). Then, for \( n_i \in \mathbb{N}_k, i \in \mathbb{N}_m \),

\[
f_{\Delta,n}^{\alpha,j}(X) \leq f_{\Delta,n+1}^{\alpha,j}(X), \quad \text{for all } X \in I. \quad \text{(7.8)}
\]

Moreover, for \( n_i \in \mathbb{N}_k, i \in \mathbb{N}_m \),

\[
f_{\Delta,n}^{\alpha,j}(X) \leq f(X), \quad \text{for all } X \in I. \quad \text{(7.9)}
\]

**Proof** By (4.3), we have self-referential equations for \( f_{\Delta,n}^{\alpha,j} \) and \( f_{\Delta,n+1}^{\alpha,j} \), \( j \in \prod_{k=1}^{m} \mathbb{N}_{N_k}, X \in I \), of the form

\[
\begin{align*}
f_{\Delta,n}^{\alpha,j}(u^\epsilon_j(X)) &= f(u^\epsilon_j(X)) + \alpha_j(X) \cdot (f_{\Delta,n}^{\alpha,j}(X) - B_n f(X)), \\
f_{\Delta,n+1}^{\alpha,j}(u^\epsilon_j(X)) &= f(u^\epsilon_j(X)) + \alpha_j(X)(f_{\Delta,n+1}^{\alpha,j}(X) - B_n+1 f(X))
\end{align*} \quad \text{(7.10)}
\]
From (7.10), we obtain
\[
\begin{align*}
    f_{\Delta;n+1}^{\alpha,\epsilon}(u_j^e(X)) - f_{\Delta;n}^{\alpha,\epsilon}(u_j^e(X)) &= \alpha_j(X)(f_{\Delta;n+1}^{\alpha,\epsilon}(X) - f_{\Delta;n}^{\alpha,\epsilon}(X)) \\
    &+ \alpha_j(X)(B_n f - B_{n+1} f)(X).
\end{align*}
\]

Reference [29][Theorem 5]** implies that \((B_n f - B_{n+1} f)(X) \geq 0\) and the above equation thus takes the form
\[
\begin{align*}
    f_{\Delta;n+1}^{\alpha,\epsilon}(u_j^e(X)) - f_{\Delta;n}^{\alpha,\epsilon}(u_j^e(X)) &\leq \alpha_j(X)(f_{\Delta;n+1}^{\alpha,\epsilon}(X) - f_{\Delta;n}^{\alpha,\epsilon}(X)).
\end{align*}
\]

As the construction of fractal function is an iterative process, we infer from the above equation that
\[
\begin{align*}
    f_{\Delta;n+1}^{\alpha,\epsilon}(X) &\geq f_{\Delta;n}^{\alpha,\epsilon}(X), \text{ for all } X \in \mathcal{I}.
\end{align*}
\]

As \(f_{\Delta;n}^{\alpha,\epsilon}\) converges uniformly to \(f\), (7.8) implies (7.9).

8. Conclusions

In this work, we have introduced multivariate zipper fractal interpolation prescribed on multivariate data given on a Cartesian grid by a binary signature matrix. Multivariate zipper \(\alpha\)-fractal functions are constructed and its approximation properties studied. We have derived bounds for box-dimension of the graph of a multivariate zipper \(\alpha\)-fractal function based on the scaling factors and the Hölder exponents of a given function and base function. It was found that our methodology provides the same bound for \(2^m\) multivariate fractal functions for a fixed scaling function. Using multivariate Bernstein function as the base function, we have studied some shape-preserving aspects of multivariate Bernstein zipper \(\alpha\)-fractal functions. It is found that the scaling function of monotonicity preserving properties of multivariate zipper fractal function depends on the values of signature. Finally, an one-sided approximation to a convex continuous multivariate function by the increasing sequence of multivariate Bernstein zipper fractal functions is proven.

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