Symplectic double for moduli spaces of $G$-local systems on surfaces

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To the memory of Andrei Zelevinsky

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Abstract

A decorated surface $S$ is a topological oriented surface with punctures and holes, equipped
with a finite set of special points on the boundaries of holes, considered modulo isotopy. Each
hole boundary has at least one special point.

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Let G be a split semi-simple algebraic group over \( \mathbb{Q} \). We introduce a moduli space \( \mathcal{D}_{G,S} \), and define a collection of special rational coordinate systems on it.

The moduli space \( \mathcal{D}_{G,S} \) is the symplectic double of the Poisson moduli space \( \mathcal{X}_{G,S} \) of framed G-local systems on S. Its dimension is \( \dim \mathcal{D}_{G,S} = 2 \dim \mathcal{X}_{G,S} \). Its symplectic form is upgraded to a \( K_2 \)-symplectic structure for which the special coordinates are \( K_2 \)-Darboux coordinates.

1 Introduction

Dual pairs of moduli spaces related to G-local systems. We defined in [FG1] a pair of moduli spaces \( \mathcal{A}_{G,S} \) and \( \mathcal{X}_{G,S} \) closely related to the moduli space of G-local systems on S. We usually consider the \( \mathcal{X} \)-moduli space for the adjoint group G, and the \( \mathcal{A} \)-moduli space for its universal cover \( \tilde{G} \), using the notation \( (\mathcal{A}_{\tilde{G},S}, \mathcal{X}_{G,S}) \) for the dual pair.

Each of the moduli spaces is equipped with a positive atlas, equivariant under the action of the mapping class group \( \Gamma_S \) of S. This allows to define their manifolds of positive real points. The space \( \mathcal{X}_{\text{PGL}_2} (\mathbb{R}_{>0}) \) is identified with the modified Teichmüller space, and \( \mathcal{A}_{\text{SL}_2} (\mathbb{R}_{>0}) \) is Penner’s decorated Teichmüller space of S. Furthermore, their points with values in any semifield are well defined. The real tropical points of each of the spaces are identified with the appropriate modifications of Thurston’s measured laminations. For a group G of higher rank we get dual pairs of higher Teichmüller and lamination spaces.

The moduli space \( \mathcal{X}_{G,S} \) is equipped with a \( \Gamma_S \)-equivariant Poisson structure.

The moduli space \( \mathcal{A}_{G,S} \) is equipped with a \( \Gamma_S \)-equivariant class in \( K_2 \). Notice that a \( K_2 \)-class on a space \( \mathcal{A} \) gives rise to a closed 2-form on \( \mathcal{A} \).

There is a map \( p : \mathcal{A}_{\tilde{G},S} \to \mathcal{X}_{G,S} \) respecting these structures.

A variety of properties of these two moduli spaces is best explained by the fact that they admit \( \Gamma_S \)-equivariant cluster structures of two different types.

Cluster \( K_2 \)-varieties, which we called before cluster \( \mathcal{A} \)-varieties, are geometric incarnations of cluster algebras of Fomin-Zelevinsky [FZI].

Cluster Poisson varieties, which we call before cluster \( \mathcal{X} \)-varieties are dual geometric objects [FG2].

A dual pair \( (\mathcal{A}, \mathcal{X}) \) of cluster varieties can be assigned to any quiver. One has a Langlands type involution \( \mathcal{A} \to \mathcal{A}^\vee \), \( \mathcal{X} \to \mathcal{X}^\vee \) on cluster varieties. There is a deep duality between cluster varieties \( \mathcal{A}^\vee \) and \( \mathcal{X} \).

The space \( \mathcal{A}_{\tilde{G},S} \) (respectively \( \mathcal{X}_{G,S} \)) has a structure of \( \Gamma_S \)-equivariant cluster \( K_2 \)- (respectively cluster Poisson) variety. This means that there is a \( \Gamma_S \)-equivariant collection of rational coordinate systems of specific cluster nature on each of the spaces. Dual pairs of moduli spaces \( (\mathcal{A}_{\tilde{G},S}, \mathcal{X}_{G,S}) \) provide interesting examples of cluster varieties arising in geometry.

The Langlands dual pair of moduli spaces is given by the pair \( (\mathcal{A}_{G^L,S}, \mathcal{X}_{G,S}) \), where \( G^L \) is the Langlands dual group to G.

Cluster symplectic double \( \mathcal{D} \). In [FG3] we defined a symplectic double of a cluster Poisson variety \( \mathcal{X} \), called the cluster symplectic variety \( \mathcal{D} \). Its symplectic form is a \( K_2 \)-symplectic form.

So one assigns now to a quiver a triple of cluster varieties \( (\mathcal{A}, \mathcal{X}, \mathcal{D}) \):
• A cluster $K_2$-variety $A$;
• A cluster Poisson variety $X$;
• A cluster $K_2$-symplectic variety $D$ – the symplectic double of the cluster Poisson variety.

The cluster symplectic variety $D$ appears during the construction of quantization of the cluster Poisson variety $X$. The quantised algebra of regular functions $\mathcal{O}_q(D)$ on $D$ was realized in loc. cit. as the algebra of $q$-difference operators on the cluster $K_2$-variety $A$.

Let us elaborate on this: although we do not use it anywhere later on in the paper, this shows how the quantum symplectic double appears, motivating the main goals of this paper. The reader might skip this, and jump to discussion of the moduli space $D_G, S$ on the next page.

One can think about the cluster symplectic variety $D$ as of a cluster analog of the cotangent bundle to $A$. This analogy can be seen as follows.

The cotangent bundle $T^*M$ of a manifold $M$ is the quasiclassical limit of the algebra of differential operators on $M$. We define differential operators on $M$ locally. Namely, a manifold $M$ is given by a collection of coordinate domains $U_i \subset \mathbb{R}^n$ and gluing maps $\varphi_{ij}: U_i \to U_j$.

We define a global polynomial differential operator $D$ on $M$ as a collection of operators $\{D_i\}$ on the spaces of functions $C^\infty(U_i)$ generated by infinitesimal translations and multiplication by linear functions in $\mathbb{R}^n$, such that the linear maps $\varphi_{ij}^*: C^\infty(U_j) \to C^\infty(U_i)$ induced by the gluing maps $\varphi_{ij}$ intertwine them: $\varphi_{ij}^*(D_jf) = D_i \varphi_{ij}^*(f)$.

Similarly to this, a cluster variety $A$ is glued from split algebraic tori $T_i = (\mathbb{C}^*)^n$ by positive birational maps $\varphi_{ij}: T_i \to T_j$. On each cluster coordinate torus $T_i = (\mathbb{C}^*)^n$ we consider a non-commutative algebra of operators on functions generated by the following ones: the operators on functions induced by the shift by $q$ of one of the coordinates, $z_i \mapsto -qz_i$, and the operators of multiplication by the coordinates $z_i$. It is called the algebra of $q$-difference operators on $T_i$, and denoted by $D_q(T_i)$.

The most non-trivial step is that instead of the linear maps $\varphi_{ij}^*$ induced by the gluing maps $\varphi_{ij}$ we use the intertwiners. Namely, we consider the Hilbert space $H_i := L^2(\mathbb{R}^n, \omega)$ associated to the real positive part $(\mathbb{R}^*_+)^n \subset (\mathbb{C}^*_+)^n$ of the coordinate torus $T_i$, identify it with $\mathbb{R}^n$ by the logarithm map, and equip it with the Lebesgue measure $\omega = d\log x_1 \ldots d\log x_n$, where $x_i = \log z_i$. Then we define, by using the (non-compact) quantum dilogarithm function, a unitary operator, called the intertwiner

$$I_{ij}: H_j \longrightarrow H_i.$$

The crucial point is that the unitary intertwiner $I_{ij}$ induces a unique birational map of the fields of fractions of the algebras of $q$-difference operators on the cluster tori,

$$\psi_{ij}^*: \text{Frac}(D_q(T_j)) \longrightarrow \text{Frac}(D_q(T_i)),$$  \hspace{1cm} (1)

which has the “intertwining property”:

$$I_{ij}(Ff) = \psi_{ij}(F)(f), \quad \forall f \in H_j, \quad \forall F \in \text{Frac}(D_q(T_j)).$$  \hspace{1cm} (2)
This is similar to the isomorphism of the ring of polynomial differential operators in $\mathbb{R}^n$ induced by the Fourier transform in $\mathbb{R}^n$. The intertwiner was defined in the first arXive version of [FG2], and elaborated in [FG3]. The key property (2) requires clarification since it involves fractions of q-difference operators, and uses the Schwarz space in $\mathcal{H}_i$ rather then the Hilbert space to state the intertwining property correctly.

Summarising, the analog of the algebra of global differential operators is the algebra $\mathcal{O}_q(D)$ of $q$-difference operators on the cluster $K_2$-variety $\mathcal{A}$. It consists of $q$-difference operators $D_i \in D_q(T_i)$ on each cluster torus, related by transformations (1): $\psi_{ij}^*(D_j) = D_i$. The $q \to 1$ limit of the algebra $\mathcal{O}_q(D)$ is the algebra of functions on the cluster symplectic double $\mathcal{D}$.

The cluster symplectic double is glued from split algebraic tori $T_i$ by cluster symplectic $D$-transformations, which are the $q \to 1$ limits of the non-commutative birational transformations (1). It turns out that the formulas for them coincide with the formulas of Fomin-Zelevinsky defining mutations in cluster algebras with principal coefficients [FZIV]. It is remarkable that, although our approaches and motivations were different, we arrived, independently, to the same formulas. This connection deserves to be better understood.

**Symplectic double moduli space $\mathcal{D}_{G,S}$.** It is natural to ask whether there is a $\Gamma_S$-equivariant moduli space $\mathcal{D}_{G,S}$ related to the dual pair of moduli spaces $(\mathcal{A}_{G,S}, \mathcal{X}_{G,S})$ the same way as the cluster symplectic variety $\mathcal{D}$ is related to the dual pair $(\mathcal{A}, \mathcal{X})$. In Section 2 we introduce such a symplectic moduli space.

The story goes as follows. Let $S_D$ be the topological double of $S$. It is a topological surface obtained by gluing the decorated surface $S$ with its “mirror” $S^\circ$, given by the same surface with the opposite orientation, along the corresponding boundary components. The marked points on $S$ match under the gluing with the ones on $S^\circ$, and give rise to punctures on the double. So $S_D$ is a topological surface with punctures, equipped with an orientation reversing involution $\sigma$ flipping $S$ and $S^\circ$. The moduli space $\mathcal{D}_{G,S}$ is a relative of (a finite cover of) the moduli space of $G$-local systems on the double $S_D$.

The main result of this paper is a construction of a cluster symplectic double atlas on the space $\mathcal{D}_{G,S}$. It is a $\Gamma_S$-equivariant collection of special rational coordinate systems on the space $\mathcal{D}_{G,S}$; different coordinate systems are related by cluster symplectic $\mathcal{D}$-transformations for the cluster symplectic double of the space $\mathcal{X}_{G,S}$. This is a new construction even for $SL_2$.

In the case when $S$ is a compact surface without boundary we have

$$\mathcal{D}_{G,S} = \text{Loc}_{G,S} \times \text{Loc}_{G,S^\circ}.$$  

Here $\text{Loc}_{G,S}$ is the moduli space of $G$-local systems on $S$. It is already symplectic. This is the only case when there are no coordinates on $\mathcal{D}_{G,S}$ or $\text{Loc}_{G,S}$, and we have nothing new to say.

In general the definition of the moduli space $\mathcal{D}_{G,S}$ is somewhat subtle: it contains a closed subvariety whose points do not parametrise any kind of local systems on $S_D$.

**Key features of the symplectic moduli space $\mathcal{D}_{G,S}$.** There is a Poisson map

$$\pi : \mathcal{D}_{G,S} \longrightarrow \mathcal{X}_{G,S} \times \mathcal{X}_{G,S^\circ},$$  

(3)
and an involution \( i \) of \( D_{G,S} \) interchanging the two projections in (3).

There is an embedding \( j : \mathcal{X}_{G,S} \hookrightarrow D_{G,S} \). Its image is Lagrangian. The following diagram, where \( \Delta_X \) is the diagonal in \( \mathcal{X}_{G,S} \times \mathcal{X}_{G,S'} \), is commutative:

\[
\mathcal{X}_{G,S} \xleftarrow{j} D_{G,S} \quad \Downarrow \pi
\]
\[
\Delta_X \xleftarrow{} \mathcal{X}_{G,S} \times \mathcal{X}_{G,S'}
\]

The pair \( (\mathcal{X}_{G,S}, D_{G,S}) \) with the maps \( i, j, \pi \) has a symplectic groupoid structure [W] related to the Poisson space \( \mathcal{X}_{G,S} \), where \( D_{G,S} \) is the space of morphisms, and \( \mathcal{X}_{G,S} \) is the space of objects.

There is a map respecting the closed 2-forms

\[
\varphi : A_{G,S} \times A_{G,S'} \longrightarrow D_{G,S}.
\]

The symplectic double \( D_{G,S} \) sits in a commutative \( \Gamma_S \)-equivariant diagram:

\[
A_{G,S} \times A_{G,S'} \quad \xrightarrow{\varphi} \quad D_{G,S} \quad \xleftarrow{\pi}
\]

\[
\mathcal{X}_{G,S} \times \mathcal{X}_{G,S'}
\]

The very existence of a \( \Gamma_S \)-equivariant positive atlas on the space \( D_{G,S} \) implies that there is a \( \Gamma_S \)-equivariant symplectic space \( D_{G,S}(\mathbb{R}_{>0}) \) of its positive points. It is isomorphic to \( \mathbb{R}^{-2\chi(S)\dim G} \).

The cluster \( D \)-coordinates allow to produce a *-algebra \( O_q(D_{G,S}) \) - a non-commutative \( q \)-deformation of the algebra \( O(D_{G,S}) \) of regular functions on the moduli space \( D_{G,S} \).

There is a canonical class in \( K_2 \) of the moduli space \( D_{G,S} \) providing the symplectic form. It gives rise to a canonical geometric quantisation line bundle with connection \( (\mathcal{L}, \nabla) \) on the moduli space \( D_{G,S} \), whose curvature is the symplectic form.

**\( D \)-laminations.** Dylan Allegretti [A] in his Yale Thesis defined integral \( D \)-laminations on decorated surfaces. Denote by \( D_L(S; \mathbb{Z}) \) the set of integral \( D \)-laminations on \( S \). D. Allegretti proved that given an ideal triangulation \( T \) of \( S \), there is a natural isomorphism of sets

\[
a_T : D_L(S; \mathbb{Z}) \xrightarrow{\sim} (\mathbb{Z} \times \mathbb{Z})^{\{\text{edges of } T\}}.
\]

Given a flip \( T \rightarrow T' \) of ideal triangulations, there is a piecewise linear map, obtained by the tropicalisation of the cluster symplectic double transformations (given by formulas (43)-(44)):

\[
\mu_{T \rightarrow T'} : (\mathbb{Z} \times \mathbb{Z})^{\{\text{edges of } T\}} \longrightarrow (\mathbb{Z} \times \mathbb{Z})^{\{\text{edges of } T'\}}.
\]
Allegretti proved that it intertwines the isomorphisms $a_T$ and $a_{T'}$, making the following diagram commutative:

$$
\begin{array}{ccc}
\{(\mathbb{Z} \times \mathbb{Z})\text{ edges of } T\} & \overset{(Z \times Z)}{\mapsto} & \{(\mathbb{Z} \times \mathbb{Z})\text{ edges of } T'\} \\
D_L(S; \mathbb{Z}) & \overset{\mu_{T \rightarrow T'}}{\downarrow} & D_{PGL_2, S}(\mathbb{Z}). \\
D_L(S; \mathbb{Z}) & \overset{\mu_{T \rightarrow T'}}{\downarrow} & D_{PGL_2, S}(\mathbb{Z}). \\
\end{array}
$$

This just means that there is a $\Gamma_S$-equivariant isomorphism of sets

$$A_{\mathbb{Z}} : D_L(S; \mathbb{Z}) \sim \rightarrow D_{PGL_2, S}(\mathbb{Z}).$$

Furthermore, D. Allegretti defined measured $D$-laminations on $S$, and proved that there is a similar isomorphism between the set $D_L(S; \mathbb{R})$ of all measured $D$-laminations on $S$ and the set $D_{PGL_2, S}(\mathbb{R})$ of the real tropical points of $D_{PGL_2, S}$:

$$A_{\mathbb{R}} : D_L(S; \mathbb{R}) \sim \rightarrow D_{PGL_2, S}(\mathbb{R}).$$

**Organization of the paper.** We define the moduli space $D_{G, S}$ and establish its first properties in Section 2. We introduce the coordinates in the $SL_2$ case in Section 3. We made an effort to make the paper accessible to geometers. So a geometrically inclined reader can skip Section 2, and go to Section 3 where we emphasize connection with the geometry and Teichmuller theory. We start in Section 3.1 from the case when $S$ is a disc with $m$ points on the boundary. We consider in Section 3.2 the case of a surface with holes without marked points (and in fact a more general set-up of a surface with a simple lamination). The general case is a mixture of these two. In Section 3.4 we consider the moduli space $D_{G, S}$ for $SL_2$.

In Section 4 we define the coordinates for any group $G$.

In Section 6 we recall the definition of the quantum double and its classical counterpart, the symplectic double, borrowing from [FG3]. We start from the quantum story since it is the simplest way to get the formulas in the classical setting. To simplify the exposition, we restrict in Section 6 to the simply-laced case. The general case is discussed in *loc. cit.*.

**Remark.** The first draft of this paper originally appeared as the last Section of the arXive preprint arXiv:math/0702397, later published in [FG3] without that Section.

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2 The symplectic double moduli spaces

In Section 2.1 we introduce a moduli space $D_{G,S}^*$. To define the special coordinates it is sufficient to deal with it. However this space is only an approximation to the right moduli space $D_{G,S}$. One of the reasons is that we should have a natural map $\varphi$, see (5). Its cluster analog is a key feature of the cluster symplectic double. However, as we see for $G = SL_2$ in Section 3, there is no such a map even for the spaces of positive points:

$$A_{G,S}(\mathbb{R}_{>0}) \times A_{G,S^0}(\mathbb{R}_{>0}) \not\rightarrow D_{G,S}^*(\mathbb{R}_{>0}).$$

Assuming that $S = S$ has no marked points, we define in Section 2.2 a moduli space $D_{G,S}$ equipped with a map (5). Theorem 2.4 provides a birational isomorphism $D_{G,S}^* \sim \rightarrow D_{G,S}$.

2.1 The moduli space $D_{G,S}^*$

Flag variety. The flag variety $B$ for $G$ parametrises all Borel subgroups in $G$. It is isomorphic to $G/B$ where $B$ is a Borel subgroup of $G$. A $G$-local system $\mathcal{L}$ on a space gives rise to the associated local system of flag varieties $\mathcal{L}_B := \mathcal{L} \times_G B$.

Decorated flags. Let $U$ be a maximal unipotent subgroup of $G$. The decorated flag variety $A_G$, also known as the principal affine space for $G$, is isomorphic to $G/U$. A $G$-local system $\mathcal{L}$ gives rise to the associated decorated flag local system $\mathcal{L}_A$:

$$\mathcal{L}_A := \mathcal{L} \times_G A_G \cong \mathcal{L}/U.$$ 

There is a canonical projection $\mathcal{L}_A \rightarrow \mathcal{L}_B$.

The configuration space of $n$ decorated flags is defined by

$$Conf_n(A_G) := G \backslash A^n_G.$$ 

The Cartan group $H$ of $G$ acts on $A_G$ from the right. So the group $H^n$ acts on $Conf_n(A_G)$.

When the group $G$ is simply-connected, a positive structure on the space $Conf_n(A_G)$ was defined in Section 8 of [FG1]. The space $Conf_n(A_G)$ has a structure of the cluster $A$-variety. The case when $G = SL_m$ is described in Section 10 of loc. cit.

Twisted $G$-local systems. Denote by $s_G$ a central element in $G$ given by the image of $-e$ under a principal embedding $SL_2 \rightarrow G$. For example, if $G = SL_m$, then $s_G = (-1)^m e$.

Let $T^*_S$ be the punctured at the zero section tangent bundle of $S$. A twisted $G$-local system on $S$ is a $G$-local system on $T^*_S$ with the monodromy $s_G$ along a simple loop around the origin in a tangent space to a point of $S$. 

7
The moduli space $\mathcal{D}^{\ast}_{G,S}$. Let $S$ be an arbitrary decorated surface with $k > 0$ holes $h_i$. We glue it to its mirror $S^\circ$, matching the corresponding pairs of marked points. We get a new topological surface $S_D$, equipped with a set of the glued marked points. Deleting them we arrive at the topological double $S_D$ of $S$:

$$S_D := S_D' - \{\text{glued marked points}\}.$$ 

We call the deleted points punctures. The surface $S_D$ carries an unbounded simple lamination $\gamma$, called the neck lamination, consisting of boundary components of $S$ and $S^\circ$ glued together, minus the punctures. It is a union of circles and open segments, whose endpoints contain all punctures of $S_D$.

Let $\sigma : S_D \rightarrow S_D$ be the involution interchanging the two halves of the double. Let $C\tilde{G}$ be the center of the group $\tilde{G}$. Let us consider a subgroup $\Delta\tilde{G} \subset \text{Hom}(H_1(S_D, \mathbb{Z}), C\tilde{G})$ of all maps $f \in \text{Hom}(H_1(S_D, \mathbb{Z}), C\tilde{G})$ invariant under the action of the involution $\sigma$. The group $\text{Hom}(H_1(S_D, \mathbb{Z}), C\tilde{G})$, and hence its subgroup $\Delta\tilde{G}$, act on the twisted $\tilde{G}$-local systems on $S_D$.

**Definition 2.1.** The moduli space $\mathcal{D}^*_{G,S}$ parametrises pairs $(L, \beta)$ where $L$ is a twisted $\tilde{G}$-local system on $S_D$ with unipotent monodromies around the punctures, and a framing $\beta$ given by:

- a collection of flat sections of the associated flag local system $L_B$: near each of the punctures, and over each loop of the neck lamination $\gamma$.

The moduli space $\mathcal{D}^*_{G,S}$ is the quotient of $\mathcal{D}^*_{G,S}'$ by the action of the group $\Delta\tilde{G}$:

$$\mathcal{D}^*_{G,S} := \mathcal{D}^*_{G,S}' / \Delta\tilde{G}.$$ 

2.2 The moduli space $\mathcal{D}_{G,S}$ and its first properties

Below we assume that the decorated surface $S$ has no marked points on the boundary. So it is just an oriented surface $S$ with boundary.

Recall that $\tilde{G}$ denotes a simply-connected split algebraic group over $\mathbb{Q}$, and $G$ is its adjoint group. Let us recall the definition of the moduli space $\mathcal{A}_{G,S}$ and $\mathcal{X}_{G,S}$.

**Definition 2.2 ([FG1]).** i) A decoration on a twisted $\tilde{G}$-local system $L$ on $S$ is a locally constant section $\alpha_L$ of the restriction $L_A|_{\partial S}$ of the decorated flag bundle $L_A$ to the boundary $\partial S$ of $S$. The moduli space $\mathcal{A}_{G,S}$ parametrises decorated twisted $G$-local systems on $S$.

ii) The moduli space $\mathcal{X}_{G,S}$ parametrises $G$-local systems $L$ on $S$ equipped with a framing – a flat section of the associated flag local system $L_B$ over the boundary.

There is a canonical map $p : \mathcal{A}_{G,S} \rightarrow \mathcal{X}_{G,S}$, obtained by forgetting the decoration and pushing a twisted $\tilde{G}$-local system to a $G$-local system on $S$.

A framing $\beta$ on a $G$-local system $L$ on $S$ is the same thing as a $H^\mathbb{C}_G$-local subsystem, $k = \pi_0(\gamma)$, given by the preimage of $\beta$ under the map $L_A|_{\partial S} \rightarrow L_B|_{\partial S}$:

$$F_\beta \subset L_A|_{\partial S}.$$ 

(8)
Definition 2.3. i) The moduli space $\mathcal{D}_{G,S}'$ parametrises gluing data $(L_\pm, \beta_\pm, \alpha)$, where:

1. $(L_\pm, \beta_\pm)$ are twisted framed $\tilde{G}$-local systems on $S$ and $S^o$ with isomorphic restrictions to the boundaries.

2. A $H^k_G$-equivariant map of local subsystems (8) assigned to the framings $\beta_+$ and $\beta_-:
\alpha : \mathcal{F}_{\beta_+} \to \mathcal{F}_{\beta_-}$.

ii) The moduli space $\mathcal{D}_{G,S}$ is the quotient of $\mathcal{D}_{G,S}'$ by the action of the group $\Delta \tilde{G}$.

Properties of the moduli space $\mathcal{D}_{G,S}$. Here are some properties of the moduli space $\mathcal{D}_{G,S}$ matching similar properties of the cluster symplectic double.

i) A point $p \in \mathcal{X}_{\tilde{G},S}$ determines its mirror image, $p^o \in \mathcal{X}_{\tilde{G},S^o}$. Equipping the pair $(p, p^o)$ with the tautological gluing data, we arrive at a point of $\mathcal{D}_{G,S}'$. So we get an embedding $j' : \mathcal{X}_{\tilde{G},S} \hookrightarrow \mathcal{D}_{G,S}'$. By the very definition,

$$\mathcal{X}_{G,S} = \mathcal{X}_{G,S}/\text{Hom}(\pi_1(S), C_G), \quad \mathcal{D}_{G,S} = \mathcal{D}_{G,S}'/\text{Hom}(\pi_1(S), C_G).$$

Therefore the embedding $j'$ provides an embedding

$$j : \mathcal{X}_{G,S} \hookrightarrow \mathcal{D}_{G,S}.$$

Next, there is a natural restriction map

$$\pi : \mathcal{D}_{G,S} \longrightarrow \mathcal{X}_{G,S} \times \mathcal{X}_{G,S^o}.$$

So we arrive at a commutative diagram similar to (4):

$$\begin{array}{ccc}
\mathcal{X}_{G,S} & \xrightarrow{j} & \mathcal{D}_{G,S} \\
\downarrow & & \downarrow \pi \\
\Delta \mathcal{X}_{G,S} & \hookrightarrow & \mathcal{X}_{G,S} \times \mathcal{X}_{G,S}^{\text{op}}
\end{array}$$

ii) There are canonical maps

$$\varphi : \mathcal{A}_{G,S} \times \mathcal{A}_{G,S^o} \longrightarrow \mathcal{D}_{G,S}' \longrightarrow \mathcal{D}_{G,S}.$$

Namely, a pair

$$(L_+, \alpha_+) \in \mathcal{A}_{G,S}, \quad (L_-, \alpha_-) \in \mathcal{A}_{G,S^o}$$

produces a gluing data $(L_\pm, \beta_\pm, \alpha) \in \mathcal{D}_{G,S}'$, where the framings $(\beta_+, \beta_-)$ are the images of the decorations $(\alpha_+, \alpha_-)$, and we set $\alpha(\alpha_+) := \alpha_-$. The second map is the canonical projection.
Forgetting the framing $\beta$, we get a projection to the moduli space of twisted $\tilde{G}$-local system on $S$:

$$\text{pr} : D_{G,S} \rightarrow \mathcal{L}_{\tilde{G},S}. \quad (9)$$

It is a Galois cover over the generic point with the Galois group $W^k$, where $W$ is the Weyl group of $G$, and $k = \pi_0(\gamma)$. The space $\mathcal{L}_{\tilde{G},S}$ is symplectic. It provides a $\Gamma_S$-invariant symplectic structure on an open part of $D_{G,S}$.

Our next goal is the following

**Theorem 2.4.** There is a canonical embedding, which is a birational isomorphism:

$$D^*_{G,S} \hookrightarrow D_{G,S}.$$ 

To define it, we need a digration on cutting and gluing of some moduli spaces.

### 2.2.1 The moduli space $X_{G,S;\gamma}$

Let us introduce a moduli space related to the pair $(S; \gamma)$, where $\gamma$ is a simple lamination on a surface $S$.

**Definition 2.5.** The moduli space $X_{G,S;\gamma}$ parametrises pairs $(\mathcal{L}, \beta)$ where $\mathcal{L}$ is a twisted framed $G$-local system on $S$ and a framing $\beta$ is a flat section of the flag bundle $\mathcal{L}_B$ over the curve $\gamma$ and the boundary $\partial S$. The pair $(\mathcal{L}, \beta)$ is called a twisted framed $G$-local system on $(S; \gamma)$.

**Example.** When $\gamma$ is empty we recover the moduli space $X_{G,S}$.

Let $S - \gamma$ be the surface obtained by cutting $S$ along the curve $\gamma$. Restricting a framed $G$-local system on $(S; \gamma)$ to $S - \gamma$ we get the restriction map

$$\text{Res} : X_{G,S;\gamma} \rightarrow X_{G,S - \gamma}. \quad (10)$$

It is not a map onto: the monodromies around the matching boundary components $\gamma_{\pm,i}$ of the surface $S - \gamma$ must coincide.

**Definition 2.6.** $X^\text{red}_{G,S - \gamma}$ the subspace of $X_{G,S - \gamma}$ determined by the condition that the monodromies around the loops $\gamma_{\pm,i}$ coincide for every $i$.

The following result is an algebraic-geometric version of the cutting and gluing techniques developed in Section 7 of [FG1], in particular Theorem 7.6 there. Observe that in loc. cit. we established cutting and gluing properties of the Teichmüller space $X_{G,S}(\mathbb{R}_{>0})$, while their algebraic-geometric analog requires consideration of the moduli space $X_{G,S;\gamma}$.

**Theorem 2.7.** Let us assume that the center of $G$ is trivial. Then the restriction map

$$\text{Res} : X_{G,S;\gamma} \rightarrow X^\text{red}_{G,S - \gamma} \quad (11)$$

is a fibration over the generic point of $X^\text{red}_{G,S - \gamma}$ with the structure group $H^\pi_{G,\gamma}$. The space $X_{G,S;\gamma}$ is rational.
Proof. Follows the proof of Theorem 7.6 in *loc. cit.*

The restriction map (10) is a Poisson map, which admits the following alternative description. The monodromies along the connected components of $\gamma$ provide a map

$$\mu_\gamma : \mathcal{X}_{G,S,\gamma} \to H^k, \quad k = \pi_0(\gamma).$$

The lifts of characters of the torus $H^k$ commute under the Poisson bracket on $\mathcal{X}_{G,S,\gamma}$. They provide a Hamiltonian action of the group $H^k$ on $\mathcal{X}_{G,S,\gamma}$. The corresponding Hamiltonian reduction map is the map (10).

**Proof of Theorem 2.4.** By assigning to a twisted framed $\tilde{G}$-local system on $S_D$ its restrictions to $S$ and $S^o$ we get an injective map

$$\mathcal{D}^*_G,S \hookrightarrow \mathcal{D}'_{G,S}.$$  \hspace{1cm} (12)

Taking the quotients by the action of the subgroup $\Delta_{\tilde{G}}$, we get an injective map $\mathcal{D}^*_G,S \hookrightarrow \mathcal{D}'_{G,S}$.

One defines a version $\mathcal{D}^\sharp_{G,S}$ of the moduli space $\mathcal{D}'_{G,S}$ by replacing in Definition 2.3 the simply-connected group $\tilde{G}$ by any split semi-simple group $G$. Then one sees the following:

1) Recall that $G'$ denotes the adjoint group of $\tilde{G}$. Then one has

$$\mathcal{D}^\sharp_{G,S} \overset{\text{Def}}{=} \mathcal{D}'_{G,S}, \quad \mathcal{D}^\sharp_{G',S} \overset{\text{Th. 2.7}}{\sim} \mathcal{X}_{G',S,\gamma}.$$  \hspace{1cm} (13)

Here the second map is a birational isomorphism by Theorem 2.7.

2) The space $\mathcal{D}^\sharp_{G',S}$ is the quotient of the space $\mathcal{D}'_{G,S}$ by the subgroup

$$\Delta^\sharp_{G} \subset \text{Hom}(\pi_1(S), C_{\tilde{G}}) \times \text{Hom}(\pi_1(S^o), C_{\tilde{G}})$$

consisting of all pairs of maps $(f, f^o)$ which agree on the subgroup $\pi_1(\partial S) \subset \pi_1(S)$. Evidently,

$$\text{Hom}(\pi_1(S_D), C_{\tilde{G}}) = \Delta^\sharp_{G}.$$  \hspace{1cm} (13)

3) The space $\mathcal{X}_{G',S,\gamma}$ is naturally birationally isomorphic to the quotient $\mathcal{D}_{G,S}^*/\text{Hom}(\pi_1(S_D), C_{\tilde{G}})$. Using isomorphism (13), the map (12) provides a map

$$\mathcal{D}_{G,S}^*/\text{Hom}(\pi_1(S_D), C_{\tilde{G}}) \hookrightarrow \mathcal{D}'_{G,S}/\Delta^\sharp_{G}.$$  \hspace{1cm} (13)

Using 1) and 2) it is interpreted as a map: $\mathcal{X}_{G',S,\gamma} \to \mathcal{D}^\sharp_{G',S} \sim \mathcal{X}_{G',S,\gamma}$. Since it is tautologically a birational isomorphism, the map (12) is also a birational isomorphism.
3 Cluster coordinates on the symplectic double for $SL_2$

3.1 The symplectic double of the space of positive configurations of points on $\mathbb{RP}^1$

A collection of $m$ distinct points $(p_1, ..., p_m)$ on an oriented circle is *positive* if its order is compatible with the orientation of the circle. Denote by $\text{Conf}_m^+(\mathbb{RP}^1)$ the moduli space of positive configurations of $m$ points on the circle modulo the action of the group $PSL_2(\mathbb{R})$. We denote by $\text{Conf}_m^-(\mathbb{RP}^1)$ a similar space of the *negative* configurations of points on the circle—a configuration is negative if reversing its order we get a positive configuration.

The moduli space $\text{Conf}_m^+(\mathbb{RP}^1)$ parametrises ideal geodesic $m$-gons. Indeed, we identify the oriented $\mathbb{RP}^1$ with the boundary of the oriented hyperbolic disc, and assign to a configuration of points $(p_1, ..., p_m)$ on $\mathbb{RP}^1$ the ideal geodesic $m$-gon with vertices at these points.

Pick horocycles $h_1, ..., h_m$ at the vertices $p_1, ..., p_m$ of an ideal geodesic $m$-gon. For any two horocycles $h_i, h_j$ there is a number $l(h_i, h_j)$—the distance between them along the geodesic connecting $p_i$ and $p_j$. Namely, if the discs bounded by the horocycles are disjoint, it is the length of the geodesic segment between them. Otherwise it is its negative.

When $m = 2n$ is even, there is a map, which we call the *Casimir map*:

$$C : \text{Conf}_{2n}^+(\mathbb{RP}^1) \longrightarrow \mathbb{R}.$$ (14)

Namely, set

$$C(p_1, ..., p_{2n}) := l(h_1, h_2) - l(h_2, h_3) + \ldots + l(h_{2n-1}, h_{2n}) - l(h_{2n}, h_1).$$

It does not depend on the choices of the horocycles. Alternatively, for every vertex $p_i$ there is a map sending a geodesic $p_{i-1}p_i$ to the geodesic $p_{i+1}p_i$. It assigns to every point $c \in p_{i-1}p_i$ the intersection of the horocycle passing through $c$ and centered at $p_i$ with the geodesic $p_{i+1}p_i$. The composition of these maps is a map of the geodesic $p_1p_2$ to itself preserving the length. It preserves the orientation if $m$ is even and reverses it if $m$ is odd. So when $m$ is even, it is a translation by the number (14). If $m$ is odd, it has a unique stable point, providing a preferred collection of horocycles.

Denote by $\text{Conf}_m^B(\mathbb{CP}^1)$ the moduli space of pairs $\{(z_1, ..., z_m), \alpha\}$, considered modulo the action of $PGL_2(\mathbb{C})$, where $(z_1, ..., z_m)$ is a configuration of $m$ distinct points on $\mathbb{CP}^1$, and $\alpha$ is an isotopy class of a simple oriented loop passing through the points so that their order is compatible with the loop orientation.

A point of $\text{Conf}_m^B(\mathbb{CP}^1)$ can be thought of as a sphere with a complete hyperbolic structure with $m$ cusps $(p_1, ..., p_m)$, plus a simple geodesic polygon connecting them. Cutting along the geodesic polygon isotopic to $\alpha$ we get two ideal geodesic $m$-gons. Their vertices provide a positive configuration of points on $\mathbb{RP}^1$, and a negative one. So we get a map

$$\text{Cut}_m : \text{Conf}_m^B(\mathbb{CP}^1) \longrightarrow \text{Conf}_m^+(\mathbb{RP}^1) \times \text{Conf}_m^-(\mathbb{RP}^1).$$

**Theorem 3.1.** When $m$ is odd, the map $\text{Cut}_m$ is an isomorphism.

When $m$ is even, it is a principal $\mathbb{R}$-fibration over the subspace given by pairs of configurations of points with the same values of the Casimirs.
**Proof.** Consider two geodesic polygons

\[ P = (p_1, \ldots, p_m) \quad \text{and} \quad P^\circ = (p_1^\circ, \ldots, p_m^\circ). \]

Choose horocycles \((h_1, \ldots, h_m)\) at the vertices of \( P \). Choose a horocycle \( h_1^\circ \) at the vertex \( p_1^\circ \) of \( P^\circ \). Then there is a unique horocycle \( h_2^\circ \) at the vertex \( p_2^\circ \) such that \( l(h_1^\circ, h_2^\circ) = l(h_1, h_2) \). Similarly there is a unique horocycle \( h_3^\circ \) at the vertex \( p_3^\circ \) such that \( l(h_2^\circ, h_3^\circ) = l(h_2, h_3) \). And so on, till we get to the original point \( p_1^\circ \). Here it is an *a priori* non-trivial condition that the horocycle constructed by using the horocycle \( h_m^\circ \) coincides with the horocycle \( h_1^\circ \). So, for given horocycles \((h_1, \ldots, h_m)\), we need horocycles \((h_1^\circ, \ldots, h_m^\circ)\) satisfying a system of linear equations

\[
l(h_1, h_2) = l(h_1^\circ, h_2^\circ), \quad l(h_2, h_3) = l(h_2^\circ, h_3^\circ), \quad \ldots, \quad l(h_m, h_1) = l(h_m^\circ, h_1^\circ).\]

When \( m \) is odd, it has a unique solution. When \( m \) is even, the solution exists only when the values of the Casimirs are equal, and is parametrised by one parameter: there is an action of the group \( \mathbb{R} \) on the solutions given by \( h_i^\circ \mapsto h_i^\circ + (-1)^i c \).

Finally, there is a unique way to glue the ideal polygon \( P \) with the horocycles \((h_1, \ldots, h_m)\) and the ideal polygon \( P^\circ \) with the horocycles \((h_1^\circ, \ldots, h_m^\circ)\), matching the horocycles, getting a hyperbolic surface with \( m \) punctures. The result does not depend on the choice of horocycles \((h_1, \ldots, h_m)\), as well as the horocycle \( h_1^\circ \). In particular, we constructed an inverse map

\[
\text{Glue}_{2n+1} : \text{Conf}^+_m(\mathbb{R}P^1) \times \text{Conf}^{-}_m(\mathbb{R}P^1) \sim \rightarrow \text{Conf}^{\sharp}_m(\mathbb{C}P^1).
\]

Gluing polygons give a punctured surface (a surface with a complete metric) if and only if around each vertex on the glued surface there exists a horocycle.

![Figure 1: Gluing two ideal geodesic pentagons into a hyperbolic sphere with 5 cusps.](image)

### 3.1.1 Coordinates on the space \( \text{Conf}^{\sharp}_m(\mathbb{C}P^1) \)

Let \(((z_1, \ldots, z_m), \alpha)\) be a point of \( \text{Conf}^{\sharp}_m(\mathbb{C}P^1) \). Let \( P_m \) be a convex \( m \)-gon whose vertices are parametrised by the points \((z_1, \ldots, z_m)\) so that the order of the vertices is compatible with a
cyclic order of the points on the loop $\alpha$. Let $T$ be a triangulation of $P_m$. Let us assign to $T$ a coordinate system $\{x_E, b_E\}$ on the space $\text{Conf}_m^2(\mathbb{C}\mathbb{P}^1)$, where $\{E\}$ are the edges of $T$. The set of edges of $T$ does not include the sides of the polygon.

We think of a configuration of complex points $(z_1, ..., z_m)$ as of a hyperbolic structure on an oriented sphere $S^2$ with cusps $(p_1, ..., p_m)$. Pick horocycles $(h_1, ..., h_m)$ at the cusps. Let $G_\alpha$ be the ideal geodesic polygon with vertices at the cusps which is isotopic to $\alpha$. Cutting $S^2$ along the geodesic polygon $G_\alpha$, we get two ideal geodesic polygons $P$ and $P^o$.

Consider a geodesic triangulation of $P$ realising the triangulation $T$. An edge $E$ determines an ideal geodesic quadrilateral $(p_a, p_b, p_c, p_d)$ of this triangulation with the diagonal $E = p_a p_c$. Let $l(h_i, h_j)$ (respectively $l^o(h_i, h_j)$) be the distance between the horocycles $h_i, h_j$ along the geodesic connecting $p_i$ and $p_j$ inside of $P$ (respectively $P^o$).

**Definition 3.2.** The coordinates $x_E$ and $b_E$ assigned to the edge $E$ are given by

$$x_E := l(h_a, h_b) - l(h_b, h_c) + l(h_c, h_d) - l(h_d, h_a), \quad b_E := l^o(h_i, h_j) - l(h_i, h_j).$$

Evidently they are independent of the choice of the horocycles $h_i$. The coordinates $\{x_E\}$ are the standard coordinates on the space of ideal geodesic polygons $P$.

The exponents $X_E := \exp(x_E)$ and $B_E := \exp(b_E)$ form a coordinate system $\{X_E^T, B_E^T\}$ assigned to a triangulation $T$. We show below that the coordinate systems corresponding to triangulations $T$ of the $m$-gon provide a positive real atlas on the space $\text{Conf}_m^2(\mathbb{C}\mathbb{P}^1)$.

### 3.1.2 Geometric interpretation of the cluster symplectic double of type $A_m$

The definition of cluster symplectic double us recalled in the Appendix.

In particular, a root system of type $A_m$ gives rise to a cluster symplectic double which we denote by $\mathcal{D}_{A_m}$. Let $\mathcal{D}_{A_m}^+$ be the space of its real positive points. The space $\mathcal{D}_{A_m}$ is equipped with a cluster atlas $\{X_E^T, B_E^T\}$ whose coordinate systems are parametrised by the triangulations $T$ of a convex $(m + 3)$-gon.

**Theorem 3.3.** There is a unique isomorphism

$$\text{Conf}_{m+3}^2(\mathbb{C}\mathbb{P}^1) \xrightarrow{\sim} \mathcal{D}_{A_m}^+$$

sending the atlas $\{X_E^T, B_E^T\}$ on $\text{Conf}_{m+3}^2(\mathbb{C}\mathbb{P}^1)$ to the cluster atlas on the symplectic double $\mathcal{D}_{A_m}^+$.

**Proof.** Consider an ideal geodesic quadrilateral with vertices $p_1, p_2, p_3, p_4$. Denote by $(B_{ij}, X_{ij})$ the pair of coordinates assigned to the edge $p_ip_j$. Let us calculate how the coordinate $B_{13}$ changes under the flip at the edge $E_{13} = p_1p_3$, see Fig. 2.

We will use shorthands $l_{ij} = l(h_i, h_j)$, $l^o_{ij} = l^o(h_i, h_j)$. Recall the Plücker relation

$$\exp(l_{13})\exp(l_{24}) = \exp(l_{12})\exp(l_{34}) + \exp(l_{14})\exp(l_{23}).$$
There is a similar relation for the \( \exp(l_{ij}^o) \). Then the flipped \( B \)-coordinate \( B_{24} \) equals

\[
B_{24} := \frac{\exp(l_{24}^o)}{\exp(l_{24})} = \frac{\left(\exp(l_{12}^o + l_{34}^o) + \exp(l_{14}^o + l_{23}^o)\right)\exp(l_{13}^o)}{\left(\exp(l_{12} + l_{34}) + \exp(l_{14} + l_{23})\right)\exp(l_{13})} = \frac{B_{12}B_{34} + X_{13}B_{14}B_{23}}{(1 + X_{13})B_{13}}.
\]

This agrees with the mutation formula for the \( B \)-coordinates. The mutation formulas for the \( X \)-coordinates are the standard ones. The theorem is proved.

Figure 2: A flip at the edge \( E_{13} \).

The symplectic structure \([FG3]\). The symplectic structure on the space \( \text{Conf}_m^+ (\mathbb{C}P^1) \) in the coordinate system \( \{b_E, x_E\} \) related to any ideal triangulation \( T \) of the \( m \)-gon \( P \) is given by

\[
-\frac{1}{2} \sum_{E,F} \varepsilon_{EF} db_E \wedge db_F - \sum_E db_E \wedge dx_E.
\]

The corresponding Poisson bracket is given by

\[
\{x_E, x_F\} = \varepsilon_{EF} x_E x_F, \quad \{x_E, b_F\} = \delta_{EF} \quad \{b_E, b_F\} = 0.
\]

The space \( \text{Conf}_m^+ (\mathbb{R}P^1) \) is a Poisson space with the Poisson structure given in the coordinate system \( \{x_E\} \) related to any ideal triangulation \( T \) by the formulas

\[
\{x_E, x_F\} = \varepsilon_{EF} x_E x_F.
\]

It is non-degenerate if \( m \) is odd, providing a symplectic structure on \( \text{Conf}_m^+ (\mathbb{R}P^1) \). When \( m \) is even, the Casimir function \( C \) generates the center of the Poisson structure.

The cutting map \( \text{Cut}_m \) is a Poisson map. Notice that \( \varepsilon_{EF}^o = -\varepsilon_{EF} \).

3.1.3 Theorem 3.1 and the Bers double uniformization theorem.

The moduli space \( \text{Conf}_m^+ (\mathbb{R}P^1) \) is the Teichmüller space parametrising complex structures on the disc \( \hat{D}_m \) with \( m \) marked points on the boundary. The moduli space \( \text{Conf}_m^+ (\mathbb{C}P^1) \) should be viewed as a baby version of the space of quasifuchsian groups for the disc \( \hat{D}_m \), with Theorem 3.1 being the analog of the Bers double uniformization theorem.
Let $S$ be a closed hyperbolic surface. Let $\partial_\infty \pi_1(S)$ be the boundary at infinity of the fundamental group of $S$. It is a cyclic $\pi_1(S)$-set, homeomorphic to an oriented circle. A map $\partial_\infty \pi_1(S) \to \mathbb{RP}^1$ is positive if it preserves the cyclic order. The Teichmüller space $\mathcal{T}(S)$ is identified ([FG1, Lemma 1.1]) with the set of $\pi_1(S)$-equivariant positive maps
\[
\varphi : \partial_\infty \pi_1(S) \to \mathbb{RP}^1 \text{ modulo the action of } \text{PSL}_2(\mathbb{R}).
\]

**Definition 3.4.** A representation $\rho : \pi_1(S) \to \text{PSL}_2(\mathbb{C})$ is quasifuchsian if the induced map $\psi_\rho : \partial_\infty \pi_1(S) \to \mathbb{CP}^1$ is a homeomorphism on its image, i.e. its limit set is a Jordan curve.

A quasifuchsian representation $\rho$ is uniquely described by the $\pi_1(S)$-equivariant map $\psi_\rho$.

The condition that $\rho$ is quasifuchsian is equivalent to the following condition on the limit set $C_\rho := \psi_\rho(\partial_\infty \pi_1(S))$ of a representation $\rho$: the convex core, defined as the convex hull of $C_\rho$ modulo the action of $\pi_1(S)$, is compact, i.e. its projection to $\mathcal{H}^3/\rho(\pi_1(S))$ is compact.

Denote by $Q(S)$ the space of quasifuchsian representations of $\pi_1(S)$ modulo the conjugation. Then cutting $\mathbb{CP}^1$ along the limit set $C_\rho$ we get the Bers map
\[
\beta : Q(S) \to \mathcal{T}_S \times \mathcal{T}_{S^0}.
\]

Indeed, let $\mathbb{CP}^1 - C_\rho = D \cup D^0$. Then $\pi_1(S)$ acts discretely on $D$ and $D^0$, providing Riemann surfaces $D/\rho(\pi_1(S))$ and $D^0/\rho(\pi_1(S))$ homeomorphic to $S$. The disc $D^0$ and the second surface have the opposite orientation. Here is the Bers double uniformization theorem.

**Theorem 3.5.** The Bers map is an isomorphism.

Unlike the Bers theorem, Theorem 3.1 has a simple constructive proof. Here is an approach for a new proof of the Bers double uniformization theorem, as a limit of Theorem 3.1.

Pick a finite subset $C_{2n+1} \subset \partial_\infty \pi_1(S)$. Then $\psi_\rho(C_{2n+1})$ is a configuration of $2n + 1$ points on $\mathbb{CP}^1$, and the limit curve $C_\rho$ provides a loop $\alpha$. So we get a point of $\text{Conf}^+_2\mathbb{CP}^1$. The Bers map is described by two $\pi_1(S)$-equivariant positive maps
\[
\varphi, \varphi^0 : \partial_\infty \pi_1(S) \to \mathbb{RP}^1.
\]

Their restriction to $\psi_\rho(C_{2n+1})$ should converge to the cutting map: as the subset $C_{2n+1}$ approximates $\partial_\infty \pi_1(S)$, the vertical arrows in the diagram below should approximate isomorphisms
\[
\begin{array}{ccc}
Q(S) & \xrightarrow{\beta} & \mathcal{T}_S \times \mathcal{T}_{S^0} \\
\downarrow & & \downarrow \\
\text{Conf}^+_2\mathbb{CP}^1 & \to & \text{Conf}^+_2\mathbb{RP}^1 \times \text{Conf}^+_2\mathbb{RP}^1
\end{array}
\]

**Conjecture 3.6.** When the subsets $C_{2n+1}$ approximate $\partial_\infty \pi_1(S)$, the maps $\psi_{C_{2n+1}}$ converge to a limit
\[
\psi : \partial_\infty \pi_1(S) \to \mathbb{CP}^1,
\]
providing a quasifuchsian representation of $\pi_1(S)$.  

3.2 Teichmüller space for a closed surface with a simple lamination

Let $\Sigma$ be a closed oriented hyperbolic surface. A simple lamination on $\Sigma$ is a finite collection \( \{\gamma_i\} \) of simple non-trivial disjoint nonisotopic loops on $\Sigma$ modulo isotopy. So $\gamma := \cup_i \gamma_i$ is a curve without self-intersections.

Let us introduce a moduli space $X_{\Sigma,\gamma}$ assigned to a simple lamination $\gamma$ on $\Sigma$. It will differ from $X_{\Sigma}$ by including some nodal surfaces and some discrete data. It has a stratification parametrised by collections of components of $\gamma$. The open stratum parametrises complex structures on $\Sigma$ plus a choice of an orientation for every loop of $\gamma$. Let us define the stratum assigned to a collection of loops $\{\gamma_1, \ldots, \gamma_k\}$. Let us pinch these loops to the nodes $p_1, \ldots, p_k$, getting a singular surface $\Sigma_{p_1,\ldots,p_k}$ with a simple lamination $\gamma_{p_1,\ldots,p_k}$ given by the image of $\gamma - \{\gamma_1, \ldots, \gamma_k\}$.

We say that a horocycle $c'$ at a node $p$ is obtained from a horocycle $c$ at $p$ by a shift by $l \in \mathbb{R}$ if both are on the same side of $p$, and the distance from $c$ to $c'$ in the off $p$ direction is $l$.

**Definition 3.7.** The stratum $X_{S,\gamma p_1,\ldots,p_k}^+$ parametrises complex structures on $\Sigma_{p_1,\ldots,p_k}$ plus the following gluing data:

- An orientation for every loop $\gamma_i$ of the simple lamination $\gamma_{p_1,\ldots,p_k}$ on $\Sigma_{p_1,\ldots,p_k}$.
- For every node $p_i$, a pair of horocycles $(c_{-i}, c_{+i})$ centered at the node $p_i$, located at the different sides to the node, and defined up to a shift by the same number, see Fig. 3:

\[
(c_{-i}, c_{+i}) \sim (c_{-i} + a, c_{+i} + a).
\]

![Figure 3: A codimension one stratum for a genus three surface with a two-loop lamination.](image)

Cut the surface $\Sigma$ with a hyperbolic metric along the geodesic isotopic to a loop $\gamma_i$. We get a surface $\Sigma_i$ with geodesic boundary. The points of the Teichmüller space for $\Sigma$ are obtained from the ones for $\Sigma_i$ via a gluing procedure introducing one real parameter – the Dehn twist along the loop $\gamma_i$.

Here is a standard definition of the Dehn twist parameters. Take a universal cover $\tilde{\Sigma}_i$ of $\Sigma_i$. It is obtained by cutting out from the hyperbolic plane $\mathcal{H}$ geodesic half discs bounded by the preimages of the boundary geodesic loops $\gamma_{\pm,i}$ on $\Sigma_i$. Choose a pair of boundary geodesics $g_{\pm}$ on $\tilde{\Sigma}_i$ projecting to $\gamma_{\pm,i}$. The geodesics $g_{\pm}$ are oriented, so that their orientations agree

---

1We show in Section 2.2 that it is the set of $\mathbb{R}_{>0}$-points of a moduli space $X_{\text{PGL}_2, S, \gamma}$ defined there.
with the orientations of the boundary components $\gamma_{\pm,i}$ induced by the surface orientation. The Dehn twist parameter assigned to $\gamma_i$ parametrises orientation reversing isometries $f : g_+ \to g_-$:

$$\{\text{Dehn twists for } \gamma_i\} \sim \{\text{maps } f : g_+ \to g_- \text{ such that } f(x + c) = f(x) - c\}. \quad (16)$$

It is convenient for us to modify slightly this definition. Observe that a choice of the orientation of the loop $\gamma_i$ which enters in the definition of the stratum $\mathcal{X}^+_{\Sigma, \gamma_1, p_1, \ldots, p_k}$ provides a simultaneous choice of ends of the geodesics $g_\pm$. It provides therefore orientations of these geodesics, directed out of the chosen ends. The group $\mathbb{R}$ acts by translations of the geodesics so that a shift by a positive number moves a point according to the orientation. The Dehn twists assigned to $\gamma_i$ are parametrised by orientation preserving isometries $f : g_+ \to g_-:

$$\{\text{Dehn twists for } \gamma_i\} \sim \{\text{maps } f : g_+ \to g_- \text{ such that } f(x + c) = f(x) + c\}. \quad (17)$$

Let us glue the strata into a space $\mathcal{X}^+_{\Sigma, \gamma}$ so that pinching $\gamma_i$ to the node $p_i$ we get in the limit the corresponding stratum, such that the following condition holds:

- The gluing transforms the Dehn twist action of $\mathbb{R}$ into the action of $\mathbb{R}$ provided by shifting the horocycle $c_i$ by $l \in \mathbb{R}$.

The data (17) is the same as a choice of a pair of horocycles centered at the chosen ends of the geodesics $g_+$ and $g_-$, defined up to their shifts by the same number, see Fig 4. Indeed, given two such horocycles $c_+$ and $c_-$ there is a unique orientation preserving isometry $g_+ \to g_-$ identifying $c_+ \cap g_+$ and $c_- \cap g_-$. Pinching the geodesic isotopic to the loop $\gamma_i$ on $\Sigma$ we shrink

$$\text{Figure 4: A pair of geodesics with a pair of horocycles centered at the chosen ends.}$$

the geodesics $g_+$ and $g_-$ to cusps, but keep a pair of horocycles centered at the cusps, defined up to a common shift. Thus in the limit we get a point of the corresponding stratum.

The group $\mathbb{R}$ acts on (17) by $(t_a f)(x) := f(x + a)$, $a \in \mathbb{R}$. There is an action of the group $\mathbb{R}^k$ on the space $\mathcal{X}^+_{\Sigma, \gamma}$: an element $a \in \mathbb{R}$ in the factor assigned to a loop $\gamma_i$ acts shifting by $a$ the Dehn parameter if $\gamma_i$ was not shrank to a node, and by shifting the horocycle $c_{+,i}$ by $a$ otherwise. It makes $\mathcal{X}^+_{\Sigma, \gamma}$ into a principal $\mathbb{R}^k$-fibration.

The stratum assigned to $\Sigma_{p_1, \ldots, p_k}$ is fibered over the stratum of the Weil-Peterson completion of the classical Teichmüller space of $\Sigma$ assigned to $\{\gamma_1, \ldots, \gamma_k\}$. The latter stratum is of real codimension $2k$, while the former is of real codimension $k$. The Weil-Peterson stratum is the
quotient of our stratum by the action of $\mathbb{R}^k$. Our strata lie inside of the space $X_{\Sigma;\gamma}^+$, while the Weil-Peterson strata lie on the boundary of the Teichmüller space.

There is a natural action of the group $(\mathbb{Z}/2\mathbb{Z})^n$ on the space $X_{\Sigma;\gamma}^+$, where $n$ is the number of connected components of the lamination $\gamma$. It preserves the stratification, and acts on the stratum $X_{\Sigma;\gamma; p_1,...,p_k}^+$ via the quotient $(\mathbb{Z}/2\mathbb{Z})^{n-k}$, by changing the orientations of the $n-k$ loops of $\gamma$ which were not shrank to the nodes. The quotient $X_{\Sigma;\gamma}^+/(\mathbb{Z}/2\mathbb{Z})^n$ is a manifold with corners of depth $\leq n$, obtained by completion of the classical Teichmüller space of $\Sigma$.

### 3.3 The modified Teichmüller space for the double.

Let $S$ be an oriented hyperbolic surface with $n > 0$ holes $h_i$. Denote by $S^o$ the same surface with the opposite orientation. The double $S_D$ of $S$ is defined by gluing the surfaces $S$ and $S^o$ along the corresponding parts of the boundaries, see Fig. 5. It is an oriented surface without holes. It carries a simple lamination $\gamma$ obtained by gluing the boundaries of $S$ and $S^o$. Cutting the double $S_D$ along $\gamma$ we recover $S$ and $S^o$.

**Definition 3.8.** The moduli space $D^+_S$ is the space $X^+_{S_D;\gamma}$ for the lamination $\gamma$.

**Coordinates on $D^+_S$.** Denote by $h_1,...,h_n$ the holes on $S$, and by $\partial h_1,...,\partial h_n$ the corresponding boundary components of $S$. Shrink the holes $h_1,...,h_n$ on $S$ to punctures $p_1,...,p_n$, getting a surface $S'$. An ideal triangulation of $S'$ is a triangulation of $S'$ with vertices at the punctures.

**Theorem 3.9.** Given an ideal triangulation of $S'$, the space $D^+_S$ has a coordinate system which identifies it with $\mathbb{R}^{-6\chi(S)}$.

![Figure 5: Left: gluing surfaces $S$ and $S^o$; Right: shrinking holes on $S$, getting a surface $S'$.](image)

**Proof.** Take the universal cover $\tilde{S}$ of $S$, and the universal cover $\tilde{S^o}$ of $S^o$. Take an ideal edge $E$ connecting punctures $p_1$ and $p_2$ on $S'$, see Fig. 5. Let $E^o$ be its mirror image on $S^o$.

Choose a pair of geodesics $g_1, g_2$ on $\tilde{S}$ projecting to the boundary geodesics corresponding to the holes $h_1,h_2$. The orientation of the loop $\gamma_i$ determines an end $e_i$ of the geodesic $g_i$. There is a geodesic $g_E$ on $S$ realizing the edge $E$ which spirals around the holes $h_1,h_2$ towards the ends $e_1,e_2$. Consider the geodesic $\tilde{E}$ on $\tilde{S}$ projecting to the geodesic $g_E$. It has the ends at $e_1,e_2$. 


Figure 6: A geometric description of the coordinate $B_E$.

A choice of the geodesics $g_1$ and $E$ determines uniquely the geodesic $\tilde{E}$. Then the geodesic $g_2$ is determined uniquely by $\tilde{E}$. Take a similar data $g_1^\circ, g_2^\circ$ and $\tilde{E}^\circ$ on $\tilde{S}^\circ$ assigned to $E^\circ$. The gluing data contains a pair of horocycles $(c_1, c_1^\circ)$ centered at the ends $e_1, e_1^\circ$ of the geodesics $g_1, g_1^\circ$, defined up to a common shift, and a similar pair of horocycles $(c_2, c_2^\circ)$.

**Definition 3.10.** The coordinate $b_E$ assigned to the edge $E$ is the difference of lengths of geodesics $\tilde{E}$ and $\tilde{E}^\circ$, measured using pairs of horocycles $(c_1, c_2)$ and $(c_1^\circ, c_2^\circ)$, see Fig. 6:

$$b_E := l_{\tilde{E}}(c_1, c_2) - l_{\tilde{E}^\circ}(c_1^\circ, c_2^\circ).$$

Here $l_{\tilde{E}}(c_1, c_2)$ is the distance between $\tilde{E} \cap c_1$ and $\tilde{E} \cap c_2$. Clearly $b_E$ does not depend on shift of pairs of horocycles $(c_1, c_1^\circ)$ and $(c_2, c_2^\circ)$. It is a new coordinate assigned to the edge $E$.

The edge $E$ determines an ideal quadrilateral $(E_{12}, E_{23}, E_{34}, E_{41})$ with the diagonal $E = E_{13}$, see Fig. 2.

**Definition 3.11.** The coordinate $x_E$ assigned to the edge $E$ is given by

$$x_E := l_{E_{12}}(c_1, c_2) - l_{E_{23}}(c_2, c_3) + l_{E_{34}}(c_3, c_4) - l_{E_{41}}(c_4, c_1).$$

The coordinate $x_E$ is independent of the choice of horocycles $c_i$. It is the standard shear coordinate assigned to the edge $E$.

### 3.4 The moduli space $\mathcal{D}_S^*$

Let $S$ be a surface with holes. We introduce a moduli space $\mathcal{D}_S$, and identify the set of its positive points with the space $\mathcal{D}_S^+$.

We start with a moduli space $\mathcal{D}_S^*$ which is an open part of the moduli space $\mathcal{D}_S$. Its advantage is that it can be defined as a moduli space of local systems on $S$.

Recall the canonical involution $\sigma : S_D \to S_D$ and the subgroup of the $\sigma$-invariant maps

$$\Delta_{SL_2} \subset \text{Hom}(H_1(S_D, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}).$$

(18)

Realising $\mathbb{Z}/2\mathbb{Z}$ as the center of the group $SL_2$, we make the group $\text{Hom}(H_1(S_D, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z})$ act on the space of twisted $SL_2$-local systems on $S_D$. 

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Definition 3.12. The moduli space $\mathcal{D}^\ast_S$ parametrises the orbits of the subgroup $\Delta_{SL_2}$ on the moduli space of twisted framed $SL_2$-local systems on $S_D$.

The moduli space $\mathcal{X}_S := \mathcal{X}_{PGL_2 \ast S}$ parametrises $PGL_2$-local systems on $S$ with a framing, i.e. a choice of an eigenspace for the monodromy around every boundary component of $S$.

Let us define an atlas on the space $\mathcal{D}^\ast_S$ whose coordinate systems are parametrized by ideal triangulations $T$ of $S'$. Given such a $T$, we define a rational coordinate system $(B_E, X_E)$, where $E$ runs through the edges of $T$. This atlas on $\mathcal{D}^\ast_S$ has a structure of the cluster symplectic double of the Poisson moduli space $\mathcal{X}_S$.

Given a pair of vectors $v_1, v_2$ in a two-dimensional symplectic vector space, denote by $\Delta(v_1, v_2)$ the area of the parallelogram spanned by these vectors. Take an ideal quadrilateral with vertices parametrised by a set $\{1, 2, 3, 4\}$. Pick a non-zero vector in the fiber of the two dimensional vector bundle associated with $L$ over each of the vertices of the quadrilateral which projects to the eigenline defining the framing at that vertex. We get a pair of configurations of four non-zero vectors in a two dimensional vector space:

$$(l_1, l_2, l_3, l_4) \quad \text{and} \quad (l'^o_1, l'^o_2, l'^o_3, l'^o_4),$$

well defined up to an action of the group $(\mathbb{C}^*)^4$, where an element $\lambda_i$ from the $i$-th factor $\mathbb{C}^*$ multiplies each of the vectors $l_i$ and $l'^o_i$ by $\lambda_i$. The vectors $l_i$ and $l'^o_i$ are assigned to the vertex $i$ of the quadrilateral. The $B$- and $X$-coordinates assigned to the edge $(1, 3)$ are

$$B_{13} := \frac{\Delta(l'^o_1, l'^o_3)}{\Delta(l_1, l_3)}, \quad X_{13} := \frac{\Delta(l_1, l_4)\Delta(l_2, l_3)}{\Delta(l_1, l_2)\Delta(l_3, l_4)}.$$

Multiplying both $l_i$ and $l'^o_i$ by $\lambda_i$ we do not change $B_{13}$. Clearly one has

$$X'^o_{13} = \frac{\Delta(l'^o_1, l'^o_3)\Delta(l'^o_3, l'^o_4)}{\Delta(l'^o_1, l'^o_4)\Delta(l'^o_2, l'^o_3)} = X^{-1}_{13} B_{12} B_{34} B_{14} B_{23}.$$  \hspace{1cm} (19)

Theorem 3.13. (i) The rational functions $(B_E, X_E)$ assigned to an ideal triangulation $T$ of $S'$ provide a coordinate system on the moduli space $\mathcal{D}^\ast_S$.

(ii) The atlas given by these coordinate systems is the cluster atlas for the double of $\mathcal{X}_S$.

(iii) There is a canonical isomorphism $\mathcal{D}^\ast_S(\mathbb{R}_{>0}) = \mathcal{D}^\ast_S^{-1}$.  

Proof. (i) The functions $B_E$ do not change under the action of the subgroup $(18)$ on the moduli space of framed twisted $SL_2$-local systems on $S_D$. Indeed, acting by this subgroup we alter $\Delta(l'^o_1, l'^o_3)$ and $\Delta(l_1, l_3)$ by the same sign. So the functions $B_E$ live on the space of orbits of the subgroup $(18)$. The same is evidently true for the functions $X_E$.

Observe that if we did not take the quotient by the action of the subgroup $(18)$, the functions $(B_E, X_E)$ would not have the chance to be coordinates.

The claim that they are coordinates has the same proof as the proof of the general Theorem 4.4, so we skip it here.

(ii) We claim that our $(B, X)$-coordinates behave under a flip just like the ones on the symplectic double $\mathcal{X}_S$. The $X$-coordinates are the same as for $\mathcal{X}_S$. Let us calculate how the coordinate $B_{13}$ changes under the flip at the edge $(1, 3)$, see Fig. 2. Recall the Plücker relation

$$\Delta(l_1, l_3)\Delta(l_2, l_4) = \Delta(l_1, l_2)\Delta(l_3, l_4) + \Delta(l_1, l_4)\Delta(l_2, l_3),$$
and a similar relation for the configuration \((l_0^0, l_0^1, l_0^2, l_0^3, l_0^4)\). The flipped \(B\)-coordinate \(B_{24}\) equals
\[
B_{24} := \frac{\Delta(l_0^0, l_0^4)}{\Delta(l_2, l_4)} = \frac{\left(\Delta(l_1^0, l_1^4)\Delta(l_3, l_4) + \Delta(l_0^0, l_0^4)\Delta(l_2, l_0^3)\right)\Delta(l_1, l_3)}{\left(\Delta(l_1, l_2)\Delta(l_3, l_4) + \Delta(l_1, l_4)\Delta(l_2, l_3)\right)\Delta(l_0^0, l_0^3)} = \frac{B_{12}B_{34} + X_{13}B_{14}B_{23}}{1 + X_{13}B_{13}}.
\]
This agrees with the mutation formula (44) for the \(B\)-coordinates from Section 6.

(iii) The space \(\mathcal{X}_S^+ \coloneqq \mathcal{X}_S(\mathbb{R}_{>0})\) is identified with the modified Teichmüller spaces parametrising complex structures on \(S\) plus eigenvalues of the monodromies around the boundary components ([FG1]). The canonical projection
\[
\pi : \mathcal{D}_{S_D}^+(\mathbb{R}_{>0}) \longrightarrow \mathcal{X}_S^+ \times \mathcal{X}_S^+.
\]
is a principal fibration with the fiber \(\mathbb{R}^k\). Its image is a linear subspace in the logarithmic coordinates given by the condition that the monodromies around the holes \(h_i\) and \(h_i^o\) coincide. On the other hand, cutting the double \(S_D\) along \(\gamma\) we get a projection
\[
\pi : \mathcal{D}_{S_D}^+ \longrightarrow \mathcal{X}_S^+ \times \mathcal{X}_S^+
\]
with the same image, which is also a principal \(\mathbb{R}^k\)-fibration. So to construct an isomorphism \(\mathcal{D}_{S_D}^+ \to \mathcal{D}_{S_D}^+(\mathbb{R}_{>0})\) it is sufficient to define a map of principal \(\mathbb{R}^k\)-bundles (21) \(\to\) (20) over the same base.

The open stratum \(\mathcal{X}_{S_D;\gamma,\emptyset}^+\) parametrizes pairs (a complex structures on \(S_D\), a choice of an orientation for each loop \(\gamma_i\)). Translating into the language of positive local systems ([FG1], Section 11), \(\mathcal{X}_{S_D;\gamma,\emptyset}^+\) parametrises pairs \((\mathcal{L}, \beta)\), where \(\mathcal{L}\) is a positive \(PGL_2(\mathbb{R})\)-local system on \(S_D\) (i.e. \(X_E > 0\) for all coordinates \(X_E\) of a coordinate system on \(\mathcal{X}_S^+\)), and \(\beta\) encodes choice of an eigenspace of the monodromy of \(\mathcal{L}\) for each loop \(\gamma_i\). Let us define an open \(\mathbb{R}^k\)-equivariant embedding
\[
j : \mathcal{X}_{S_D;\gamma,\emptyset}^+ \hookrightarrow \mathcal{D}_S^+(\mathbb{R}_{>0}).
\]
Cutting \(S_D\) along \(\gamma\) and restricting the pair \((\mathcal{L}, \beta)\) to the obtained surface we get framed \(PGL_2(\mathbb{R})\)-local systems on \(S\) and \(S^o\). Since they arose from points of the Teichmüller space, they are positive. Their monodromies around the loops \(\partial h_i\) and \(\partial h_i^o\) coincide, and conjugate to a diagonal matrix different from the identity, with positive diagonal entries. Next, the group \(\mathbb{R}_+^\ast \cong \mathbb{R}\) acts on the gluing data \(\alpha\) in Definition 3.12 restricted to \(\partial h_i\) by multiplying it by \(\lambda_i \in \mathbb{R}_+^\ast\), as well as on the Dehn twist parameters for \(\partial h_i\). There is an \(\mathbb{R}\)-equivariant bijection
\[
\{\text{Dehn twist parameters for } \partial h_i, \text{ an orientation of } \partial h_i\} \coloneqq \{\text{Gluing data for } \partial h_i \text{ in Def. 3.12}\}
\]
Moreover, by the very definition, \(B_E = \exp(b_E), X_E = \exp(x_E)\). Thus \(B_E > 0\). We get the embedding \(j\). It extends to an \(\mathbb{R}^k\)-equivariant embedding of \(\mathcal{D}_S^+ \hookrightarrow \mathcal{D}_S^+(\mathbb{R}_{>0})\). Since both spaces are principal \(\mathbb{R}^k\)-fibrations over the same base, we are done. \(\square\)
4 Special coordinates on the symplectic double for general $G$

4.1 Main construction

Let us construct a positive atlas on the moduli space $\mathcal{D}^{*}_{G,S}$, whose coordinate systems are parametrized by the same set as the ones on $\mathcal{X}_{G,S}$, and have the properties of the cluster symplectic double atlas. Choose an ideal triangulation $T$ of $S'$.

The $X$-coordinates on $\mathcal{D}^{*}_{G,S}$. Given a triangulation $T$, they are the inverse images of the $X$-coordinates on $\mathcal{X}_{G,S}$ for the projection $\mathcal{D}^{*}_{G,S} \to \mathcal{X}_{G,S}$ given by the restriction from $S_D$ to $S$.

The $B$-coordinates on $\mathcal{D}^{*}_{G,S}$. Choose a triangle $t$ of the triangulation $T$. Our goal is to produce a pair of points in $\text{Conf}_3(A_{\tilde{G}})$ assigned to the triangle $t$ on $S'$ and a point of $\mathcal{D}^{*}_{G,S}$, well defined up to a diagonal action of the group $H^3_{\tilde{G}}$, that is a point of

$$\left(\text{Conf}_3(A_{\tilde{G}}) \times \text{Conf}_3(A_{\tilde{G}})\right)/H^3_{\tilde{G}}.$$  \hspace{1cm} (22)

Denote by $p_1, p_2, p_3$ the vertices of the triangle $t$. They are either punctures or marked points on $S'$. Pick a path $E$ connecting $p_i$ and $p_j$ on $S'$.

A) Let us consider first the case when both $p_i$ and $p_j$ are punctures. Denote by $h_i$ and $h_j$ the corresponding holes on $S$. Choose a point $x_s$ on the boundary loop $\gamma_s$ of the hole $h_s$. Take a path $E^+ \subset S \subset S_D$ connecting points $x_i$ and $x_j$, which shrinks to a path isotopic to $E$ as we shrink the holes $h_i$ and $h_j$ to the punctures $p_i$ and $p_j$. The isotopy class of $E^+$ considered up to winding around the loops $\gamma_i$ and $\gamma_j$ is uniquely defined. Let $E^-$ be the mirror of $E^+$ under the involution of $S_D$ interchanging $S$ and $S'$.

Pick a triangle $t^+ \subset S \subset S_D$ with vertices at $x_i$'s which shrinks to $t$. Let $t^- \subset S^o \subset S_D$ be its mirror. Points of $\mathcal{D}^{*}_{G,S}$ are orbits of the group (7) acting on the following data:

1. A twisted $\tilde{G}$-local system $\mathcal{L}$ on $S_D$ and
2. A flat section $\beta_{\gamma_i}$ of the flag bundle $\mathcal{L}_B$ over each loop $\gamma_i$.

Pick an decorated flag $A_i$ at the fiber of $\mathcal{L}_A$ over the point $x_i$ projecting to the restriction of $\beta_{\gamma_i}$ to $x_i$. Since the triangle $t^+$ is contractible, the decorated flags at its vertices provide a configuration of three decorated flags. The same for $t^-$. We get two triples:

$$(a_1, a_2, a_3) \in \text{Conf}_3(A_{\tilde{G}}) \quad \text{and} \quad (a_1^o, a_2^o, a_3^o) \in \text{Conf}_3(A_{\tilde{G}}).$$  \hspace{1cm} (23)

Lemma 4.1. The triples (23) are well defined up to the diagonal action of the group $H^3_{\tilde{G}}$, producing a point in (22).
Proof. Follows immediately from the two observations:

(i) The decorated flag $A_i$ is well defined up to the action of the group $\tilde{H}_G$.

(ii) Altering the triangle $t^+$ by rotating the point $x_i$ around the loop $\gamma_i$, we alter the decorated flag $A_i$ by the monodromy around the loop. So $a_i$ and $a_i^o$ are multiplied by the same element of $\tilde{H}_G$.

B) The case when one or two of the endpoints $p_i, p_j$ of $E$ are marked points is treated similarly, and is in fact simpler.

According to Section 8 of [FG1], there is a set $I_t$ parametrising the $A$-coordinates on the configuration space $\text{Conf}_3(A_{\tilde{G}})$. So each $i \in I_t$ provides two numbers: $A_i(a_1, a_2, a_3)$ and $A_i(a_1^o, a_2^o, a_3^o)$. The coordinate $B_{t,i}$ related to the triangle $t$ is defined as their ratio:

$$B_{t,i} := \frac{A_i(a_1^o, a_2^o, a_3^o)}{A_i(a_1, a_2, a_3)}, \quad i \in I_t. \quad (24)$$

Lemma 4.2. Ratio (24) does not depend on the choices in the construction of triples (23).

Proof. The only fact we need is the following property of the $A$-coordinates on $\text{Conf}_3(A_{\tilde{G}})$:

Lemma 4.3. Each $i \in I_t$ determines a character $\chi_i$ of the group $H^3_{\tilde{G}}$ such that one has

$$A_i(h_1a_1, h_2a_2, h_3a_3) = \chi_i(h_1, h_2, h_3)A_i(a_1, a_2, a_3), \quad i \in I_t, \quad \forall (h_1, h_2, h_3) \in H^3_{\tilde{G}}.$$

Proof. Follows from the definition of positive atlas on $\text{Conf}_3(A_{\tilde{G}})$ in Section 8 of [FG1].

Lemma 4.2 follows immediately from Lemmas 4.3 and 4.1.

Denote by $D_{G,S}$ the cluster symplectic double of the cluster variety $X_{G,S}$. We distinguish it from the moduli space $D^*_G$. It is easy to check that

$$\dim D_{G,S} = \dim D^*_{G,S} = 2\dim X_{G,S}. \quad (25)$$

Theorem 4.4. There is a canonical rational surjective at the generic point map of spaces

$$D^*_{G,S} \longrightarrow D_{G,S}. \quad (26)$$

The proof of Theorem 4.4 will show that the map (26) is a finite cover at the generic point.

Conjecture 4.5. The map (26) is a birational isomorphism.

Proof. The claim of Theorem 4.4 is equivalent to the following:

1. The rational functions $(X^T_i, B^T_i)$ on the moduli space $D^*_{G,S}$ assigned to an ideal triangulation $T$ of $S'$ are independent;

2. The functions $(X^T_i, B^T_i)$ for different ideal triangulations $T$ are related by cluster transformations for the symplectic double of the cluster $X$-variety $X_{G,S}$.

We start from the proof of the Claim 2.
Proof of Claim 2. Let us show that the \((B, X)\)-coordinates on \(D_{G,S}^*\) for different ideal triangulations \(T\) are related by cluster double transformations.

A flip \(T \rightarrow T'\) at an edge \(E\) of \(T\) is decomposed into a composition of mutations as in Section 10 of [FG1]. We need to show that this sequence of mutations transforms the \((B, X)\)-coordinates assigned to \(T\) to the ones for \(T'\).

The double \(X\)-coordinates are just the usual cluster \(X\)-coordinates on \(X_{G,S}\), so the claim follows from the corresponding claim for the \(X\)-coordinates proved in loc. cit.

Our \(B\)-coordinates on \(D_{G,S}^*\) are defined as ratios of appropriate \(A\)-coordinates. Precisely, consider an ideal quadrilateral with vertices parametrised by the set \(\{1, 2, 3, 4\}\), so that the \(E\) is the diagonal \((1, 3)\), see Fig. 2. We assigned to it a pair of configurations of decorated flags \((A_1, A_2, A_3, A_4)\) and \((A_1^0, A_2^0, A_3^0, A_4^0)\).

The \(A\)-coordinates we use are cluster \(A\)-coordinates (loc. cit.).

To prove that our \(B\)-coordinates on \(D_{G,S}^*\) behave under the mutations just as the cluster \(B\)-coordinates, consider the diagram

\[
\text{Conf}_4(A) \times \text{Conf}_4(A) \xrightarrow{\varphi} \left( \text{Conf}_4(A) \times \text{Conf}_4(A) \right) / H^4_{G} \xrightarrow{\pi} \text{Conf}_4(B) \times \text{Conf}_4(B).
\]

The \(B\)-coordinates live on the middle space (by the same argument as in Lemma 4.1). Since \(\varphi\) is surjective, to check that they transform as the cluster \(B\)-coordinates it is sufficient to do it for the lifted coordinates \(\varphi^*B_i\). To prove the latter we employ the fact that the map \(\varphi\) commutes with mutations: the computation checking this was carried out in the proof of the part ii) of Theorem 3.13.

Proof of Claim 1. Given a triangulation \(T\), the rational functions \((X^T_i, B^T_i)\) provide a rational map \(\psi_q : D_{G,S} \rightarrow D_q\), where \(q\) is the seed assigned to the triangulation \(T\), and \(D_q\) is the corresponding seed torus. There is a diagram

\[
\begin{array}{ccc}
D_{G,S} & \xrightarrow{\pi} & \Delta_{X_{G,S}} \subset X_{G,S} \times X_{G,S^o} \\
\psi_q \downarrow & \sim & \\
D_q & \xrightarrow{\pi_q} & \Delta_{X_q} \subset X_q \times X_{q^o}
\end{array}
\]  

(27)

Here \(\Delta_{X_{G,S}}\) and \(\Delta_{X_q}\) are the diagonals in the corresponding products, defined as the invariants of the involution \(\sigma\). In particular, \(\Delta_{X_{G,S}} = X_{G,S}^{\text{red}}\).

The map \(\pi\) is the restriction map.

The right vertical map is a birational isomorphism given by the rational map to the cluster seed torus for \(X_{G,S} \times X_{G,S^o}\).  

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The square is commutative by construction: this is evident for the projection to $X_{G,S}$, and follows from the definition of the $B$-coordinates for the projection to $X_{G,S'}$. Indeed, the functions $X^i$ on $D_{G,S}$ defined by cluster formulas (40) coincide with the $X_i$-functions for the mirror triangulation $T^o$ on $S^o \subset S_D$.

By Theorem 2.7 the map $\pi$ at the generic point is a principal fibration with the structure group $H^G_k$, where $k$ is the number of holes without marked points on $S$.

The map $\psi_q$ at the generic point is a map of fibrations. The map $\psi_q$ transforms faithfully the action of the torus $H^G_k$. Thus, thanks to (25), the map $\psi_q$ is surjective at the generic point.

4.2 Functions $B(\alpha)$ on the moduli space $X_{G,S;\gamma}$

We start with a generalisation of the moduli space introduced in Section 2.2.1. Let $S$ be a decorated surface, and $\gamma'$ a simple lamination on it, given by a collection of simple non-intersecting loops. The punctured boundary of $S$ is the boundary of $S$ minus the marked points:

$$\partial^* S := \partial S - \{\text{marked points}\}.$$  

Let us set

$$\gamma := \gamma' \cup \partial^* S.$$  

**Definition 4.6.** The moduli space $X_{G,S;\gamma}$ parametrises pairs $(L, \beta)$ where $L$ is a $G$-local system on $S$, and $\beta$ is a framing, given by a flat section of the associated flag local system $L_B$ on $\gamma$.

Take an ordered collection of points $z_1, ..., z_k$ on $\gamma$. For each consecutive pair of points $z_i, z_{i+1}$ consider an arbitrary path $\alpha_{i,i+1}$ on $S$ which does not intersect $\gamma$ and connects the points $z_i$ and $z_{i+1}$. It is oriented from $z_i$ towards $z_{i+1}$. Travelling along these paths we get a loop

$$\alpha(z_1, ..., z_k) = \alpha_{1,2} \circ \alpha_{2,3} \circ \ldots \circ \alpha_{k,1}.$$  

The loop can have selfintersections. We consider it up to isotopies such that:

- the paths $\alpha_{i,i+1}$ end on $\gamma$, and their interior parts do not intersect $\gamma$.

Let $(L, \beta)$ be a framed $G$-local system on $(S; \gamma)$. Just as in the definition of the $B$-coordinates, pick a decorated flag $A_{z_i}$ in the fiber of the decorated flag local system $L_A$ at the point $z_i$ which projects to the flag $B_{z_i}$ in the fiber of $L_B$ over $z_i$ provided by the framing $\beta$. Transporting the decorated flags $A_{z_i}$ and $A_{z_{i+1}}$ along the arc $\alpha_{i,i+1}$ into the same point of the arc, we get a configuration of two decorated flags, denoted by $(A_{z_i}, A_{z_{i+1}})$.  

Recall the $H$-invariant, given by the birational isomorphism

$$h : \text{Conf}_2(A_G) \xrightarrow{\sim} H.$$  

It has the following property:

$$h(tA_1, A_2) = th(A_1, A_2), \quad h(A_1, tA_2) = w_0(t)h(A_1, A_2), \quad t \in H.$$  

(28)
We apply the $H$-invariant map to the configuration $(A_{z_i}, A_{z_{i+1}})_\alpha$, getting

$$h_\alpha(A_{z_i}, A_{z_{i+1}}) := h((A_{z_i}, A_{z_{i+1}})_\alpha) \in H.$$ 

Consider an alternating product

$$B(\alpha) := \frac{h_\alpha(A_{z_1}, A_{z_2}) h_\alpha(A_{z_3}, A_{z_4}) \cdots}{w_0 h_\alpha(A_{z_2}, A_{z_3}) w_0 h_\alpha(A_{z_4}, A_{z_5}) \cdots w_0 h_\alpha(A_{z_k}, A_{z_1})}.$$  

(29)

Thanks to (28), rescaling the flag $A_{z_i} \mapsto t A_{z_i}$ we do not change the $B(\alpha)$. So we get a rational function $B(\alpha)$ on the space $X_{G,S;\gamma}$, which assigns to a framed $G$-local system $(L, \beta)$ the value of the invariant $B(\alpha)$.

The cyclic shift of the points $s : (z_1, \ldots, z_k) \mapsto (z_2, \ldots, z_k, z_1)$ changes the $B(\alpha)$ as follows:

$$B(s(\alpha)) = w_0 B(\alpha)^{-1}.$$ 

Examples. 1. If our surface is the double $S_D$, and $\alpha_E$ is a loop on $S_D$ obtained by doubling an ideal edge $E$ on the original surface, then $B(\alpha)$ is just the $B$-coordinate function $B_E$.

2. When $G = SL_2$ and $\alpha$ is a 4-gon on $S$ with vertices at the $\gamma$, the function $B(\alpha)$ is a generalisation of the $X$-coordinate. To get the latter we restrict to a contractable quadrilateral. Then $B(\alpha)$ is the cross-ratio of the configuration of four points on $P^1$ provided by the framing at the vertices of the quadrilateral.

It would be interesting to calculate the function $B(\alpha)$ in a cluster double coordinate system related to an ideal triangulation of the half $S \subset S_D$.

For $G = SL_2$ the functions $B(\alpha)$ is studied in the Yale Thesis of Dylan Allegretti [A], who discovered their close relationship to the $F$-polynomials of Fomin-Zelevinsky [FZIV].

5 A complex analog of Fenchel-Nielsen coordinates

5.1 Construction of coordinates for $G = SL_2$

Let $S$ be a closed surface, i.e. a surface without boundary and punctures. Let $g$ be the genus of $S$. We assume that $g > 1$. Consider a collection $3g - 3$ simple non-intersecting loops on $S$ which determine a pair of pants decomposition of $S$: cutting $S$ along these loops we get $2g - 2$ pair of pants. The gluing pattern is described by a trivalent graph $\Gamma$: its vertices $v$ correspond to pairs of pants denoted $P_v$, and its edges $E$ correspond to the loops, denoted $\alpha_E$. So such a graph $\Gamma$ has Betti number $g$; it has $2g - 2$ vertices and $3g - 3$ edges. Denote by $V_\Gamma$ and $E_\Gamma$ the sets of the vertices and edges of the graph $\Gamma$.

The homology classes $[\alpha_E]$ of the loops generate a Lagrangian sublattice $L_\alpha$ of $H_1(S, \mathbb{Z})$. Indeed, each pair of pants gives a relation, and there is a single relation between these relations. So its rank is $(3g - 3) - (2g - 2) + 1 = g$. Denote by $\mathbb{Z}[X]$ the free abelian group generated by a set $X$. We arrive at an isomorphism of abelian groups

$$\text{Coker} \left( \mathbb{Z}[V_\Gamma] \longrightarrow \mathbb{Z}[E_\Gamma] \right) = L_\alpha \subset H_1(S, \mathbb{Z}).$$  

(30)
Let us define a dual collection of loops \( \{ \beta_E \} \). Let us choose once forever an orientation of the graph \( \Gamma \). Let \( v_E^+ \) and \( v_E^- \) be the vertices of the edge \( E \), so that \( E \) is oriented from \( v_E^+ \) to \( v_E^- \). One can have \( v_E^+ = v_E^- \), in which case \( E \) is a loop.

The pairs of pants \( \mathcal{P}_{v_E^+} \) and \( \mathcal{P}_{v_E^-} \) contain \( \alpha_E \). They coincide if \( E \) is a loop. Denote by \( \beta_E^+ \) a half loop on \( \mathcal{P}_{v_E^+} \) shown on the top left of Fig 7. It intersects the loop \( \alpha_E \) at two points. Denote by \( \beta_E^- \) a similar half loop on \( \mathcal{P}_{v_E^-} \) intersecting \( \alpha_E \) at the same two points. The orientation of the half loops does not play any role. Set \( \beta_E := \beta_E^+ \cup \beta_E^- \). See Fig. 7 and 8.

Figure 7: \( E \) is not a loop. Gluing two pairs of pants along a green boundary loop we get four holed sphere. The complimentary red loop \( \beta_E \) is obtained by gluing two red half loops.

Figure 8: \( E \) is a loop. Gluing green boundary loops on a pair of pants we get a torus with a hole \( T_E^0 \). On the left: a half loop on a pair of pants ending on the left boundary circle is glued into a half loop \( \beta_E^+ \) on \( T_E^0 \). On the right: a similar construction of a half loop \( \beta_E^- \). The loop \( \beta_E \) on \( T_E^0 \) is the union of \( \beta_E^+ \cup \beta_E^- \).

So we get two collections of loops: \( \{ \alpha_E \} \) and \( \{ \beta_E \} \).

Let \( \alpha \) be a lamination on \( S \) given by the union of the loops \( \alpha_E \):

\[
\alpha = \bigcup_{E \in \mathcal{E}_1} \alpha_E.
\]
Then the coarse moduli space $\text{Loc}_{SL_2,S;\alpha}$ from Section 2 parametrizes pairs $(\mathcal{L}, \varphi)$ where $\mathcal{L}$ is a twisted $SL_2$-local systems on $S$, and $\varphi$ is a framing of $\mathcal{L}$ over the lamination $\alpha$, which amounts to a choice of an eigenline of the monodromy of $\mathcal{L}$ along each of the loops $\alpha_E$.

Forgetting the framing, we get a $2^{3g-3}:1$ cover of the coarse moduli space $\text{Loc}_{SL_2,S}$ of twisted $SL_2$-local systems on $S$: 

$$\pi_\alpha : \text{Loc}_{SL_2,S;\alpha} \rightarrow \text{Loc}_{SL_2,S}.$$ 

**Complex analogs of Fenchel-Nielsen coordinates for $SL_2$.** Given two complimentary sets of loops $\{\alpha_E, \beta_E\}$, let us define a collection of rational functions $\{M_E, B_E\}$ on the space $\text{Loc}_{SL_2,S;\alpha}$, parametrized by the edges $E$ of the graph $\Gamma$.

A) Take a loop $\alpha_E$. Our choice of an orientation of the edge $E$ provides an orientation of the loop $\alpha_E$ such that the pair of pants $P_{\alpha_E}$ is on the left.

Then, given a twisted framed $SL_2$-local system $(\mathcal{L}, \varphi)$ on $S$, the monodromy along the loop $\alpha_E$ preserves the one dimensional subspace determined by the framing. The eigenvalue $\mu_E$ of the monodromy in this subspace provides a function 

$$M_E : \text{Loc}_{SL_2,S;\alpha} \rightarrow \mathbb{C}^*, \ (\mathcal{L}, \varphi) \mapsto \mu_E.$$ 

B) Take a loop $\beta_E$. Let us define a rational function 

$$B_E : \text{Loc}_{SL_2,S;\alpha} \rightarrow \mathbb{C}^*.$$ 

The loops $\beta_E$ and $\alpha_E$ intersect at two points $x, y$. So $\beta_E - \{x \cup y\}$ is a union of two arcs: 

$$\beta_E - \{x \cup y\} = \beta_E^+ \cup \beta_E^-.$$ 

Take a non-zero vector $v_x$ at the eigenline $L_x \subset \mathcal{L}_x$ at the point $x$ of the monodromy of $\mathcal{L}$ along the loop $\alpha_E$. It is the eigenline providing a framing over $\alpha_E$. Take a similar vector $v_y \in L_y \subset \mathcal{L}_y$ over the point $y$. Moving the vectors $v_x, v_y$ along the arc $\beta_E^+$ to the same point, we define a number 

$$\Delta_{\beta_E^+}(v_x,v_y) \in \mathbb{C}^*.$$ 

We use the fact that $\mathcal{L}$ is a twisted $SL_2$-local system on $S$: otherwise number (32) is well defined only up to a sign. Similarly, using the arc $\beta_E^-$ we get a number $\Delta_{\beta_E^-}(v_x,v_y) \in \mathbb{C}^*$. Set 

$$B_E := \frac{\Delta_{\beta_E^+}(v_x,v_y)}{\Delta_{\beta_E^-}(v_x,v_y)}.$$ 

Evidently the ratio $B_E$ does not depend on the choice of the non-zero vectors $v_x$ and $v_y$. So we get a rational function (31).

Changing the orientation of an edge $E$ results in inversion of both $M_E$ and $B_E$.

The functions $\{M_E, B_E\}$ do not define a rational coordinate system on the space $\text{Loc}_{SL_2,S;\alpha}$ for the following reason. We are going to show that there is a canonical non-trivial action of a group $\text{Hom}(L_\alpha, \mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^g$ on $\text{Loc}_{SL_2,S;\alpha}$ which preserves the functions $\{M_E, B_E\}$. 

29
5.2 Complex analogs of Fenchel-Nielsen coordinates for arbitrary $G$.

Recall that orientations of the edges $E$ provide orientations of the loops $\alpha_E$. Using these orientations, the semi-simple parts of the monodromies along the loops $\alpha_E$ provide a map

$$\{M_E\} : \text{Loc}_{G,S,\alpha} \to H^\mathcal{E}_E \cong H^{3g-3}.$$ 

Let $\text{Conf}_2^\mathcal{E}(\mathcal{A}) \subset \text{Conf}_2(\mathcal{A})$ be the subspace parametrization pairs of decorated flags in generic position. There is a canonical isomorphism

$$\Delta : \text{Conf}_2^\mathcal{E}(\mathcal{A}) \sim \to H.$$ 

Using this, and generalizing the set up of formula (33) from $SL_2$ to arbitrary group $G$ by replacing the vectors $v_x, v_y$ there by arbitrary decorated flags $A_x, A_y$ at the points $x, y$ which lift the framings over these points, the loop $\beta_E$ provides an $H$-invariant:

$$B_E := \frac{\Delta_+^{\mathcal{E}}(A_x, A_y)}{\Delta_-^{\mathcal{E}}(A_x, A_y)} \in H.$$ 

So we get a map

$$\{B_E\} : \text{Loc}_{G,S,\alpha} \to H^\mathcal{E}_E \cong H^{3g-3}.$$ 

Changing the orientation of an edge $E$ results in the inversion $h \mapsto h^{-1}$ of both $M_E$ and $B_E$.

The space $\mathcal{L}_{G,S,\alpha}$ on which the coordinates live. Let us formulate first our results. Let $G$ be any split semi-simple algebraic group.

Recall the Lagrangian sublattice $L_{\alpha} \subset H_1(S, \mathbb{Z})$ generated by the loops $\{\alpha_E\}$.

Consider the following finite abelian group:

$$\text{Hom}(L_{\alpha}, \text{Cent}(G)) \cong \text{Cent}(G)^9.$$  \hspace{1cm} (34)

**Proposition 5.1.** The group $\text{Hom}(L_{\alpha}, \text{Cent}(G))$ acts effectively at the generic point of the coarse moduli space $\text{Loc}_{G,S,\alpha}$.

**Definition 5.2.** The space $\mathcal{L}_{G,S,\alpha}$ is the quotient of $\text{Loc}_{G,S,\alpha}$ by the action of the group (34).

So it is related to the original moduli space $\text{Loc}_{G,S}$ via the following diagram, where $c_G$ is the order of the center of $G$, and the numbers at the vertical arrows are their degrees:

$$\xymatrix{ \text{Loc}_{G,S,\alpha} & \mathcal{L}_{G,S,\alpha} \\
\text{Loc}_{G,S} \ar[u]^{c_G^{3g-3}} & \ar[l]^{c_G^g} }$$ 

Notice that the space of $G$-local systems on $S$ in general is not a rational variety, i.e. it is not birationally isomorphic to a projective space. So it can not have a rational coordinate system since the latter, by definition, provides a birational isomorphism with a projective space.
Theorem 5.3. a) The functions \( \{M_E, B_E\} \) define a rational coordinate system on \( \mathcal{L}_{SL_2, S; \alpha} \).

b) The space \( \mathcal{L}_{G, S; \alpha} \) is rational. The functions \( \{M_E, B_E\} \) are a part of a rational coordinate system on \( \mathcal{L}_{G, S; \alpha} \).

Below we prove simultaneously Proposition 5.1 and Theorem 5.3.

Proof. Denote by \( \mathcal{M}_{G, \mathcal{P}_v} \) the coarse moduli space of twisted G-local systems on a pair of pants \( \mathcal{P}_v \) equipped with framings at the three boundary loops. We denote by \( \mathcal{L}_v \) a point of \( \mathcal{M}_{G, \mathcal{P}_v} \). Consider the subspace

\[
\mathcal{M}_{G, \Gamma} \subset \prod_{v \in \Gamma} \mathcal{M}_{G, \mathcal{P}_v}
\]

defined by the condition that for each loop \( \alpha_E \) the monodromies of the local systems \( \mathcal{L}_{v_E}^+ \) and \( \mathcal{L}_{v_E}^- \) around \( \alpha_E \) coincide. There is a surjective restriction map

\[
\text{Res} : \text{Loc}_{G, S; \alpha} \longrightarrow \mathcal{M}_{G, \Gamma}.
\]

The automorphism group of a generic framed G-local system on a space with non-abelian fundamental group is the center \( \text{Cent}(G) \) of the group \( G \).

There is a canonical map

\[
\text{Cent}(G)^{\mathcal{V}_\Gamma} \longrightarrow \text{Cent}(G)^{\mathcal{E}_\Gamma}.
\]

It assigns to a collection of central element \( \{c_v\} \) at the vertices \( v \) a collection of central elements \( \{c_E\} \) at the oriented edges \( E \) where \( c_E := c_{s(E)}/c_{t(E)} \), where \( s(E) \) is the source of the arrow \( E \), and \( t(E) \) is its target. Then, since \( \text{Cent}(G) \subset H \), one has

\[
\text{Im}\left(\text{Cent}(G)^{\mathcal{V}_\Gamma} \longrightarrow \text{Cent}(G)^{\mathcal{E}_\Gamma}\right) \subset H^{\mathcal{E}_\Gamma}.
\]

Lemma 5.4. The group

\[
H^{\mathcal{E}_\Gamma}/\text{Im}\left(\text{Cent}(G)^{\mathcal{V}_\Gamma} \longrightarrow \text{Cent}(G)^{\mathcal{E}_\Gamma}\right)
\]

acts simply transitively on the fiber of the map \( \text{Res} \) over a generic point of \( \mathcal{M}_{G, \Gamma} \).

Proof. Given a collection of framed twisted G-local systems \( \{\mathcal{L}_v\} \) on pairs of pants \( \mathcal{P}_v \) whose monodromies around all loops \( \alpha_E \) coincide, and given any collection of isomorphisms

\[
\{i_E\} \in \prod_{E \in \mathcal{E}_\Gamma} \text{Isom}(\mathcal{L}_{s(E)}|_{\alpha_E} \rightarrow \mathcal{L}_{t(E)}|_{\alpha_E}),
\]

one can glue a twisted G-local system \( \mathcal{L} \) on \( S \) with a framing on the \( \alpha \). So a gluing data \( (\{\mathcal{L}_v\}, \{i_E\}) \) determines uniquely such an \( \mathcal{L} \) which restricts to the collection \( \{\mathcal{L}_v\} \). Let us find out when two gluing data \( (\{\mathcal{L}_v\}, \{i_E\}) \) and \( (\{\mathcal{L}_v'\}, \{i'_E\}) \) determine isomorphic \( \mathcal{L} \)'s on \( S \).

The automorphism group of a generic twisted framed G-local system on a circle is the Cartan group \( H \) of \( G \) - the centralizer of a generic element of \( G \). Therefore for a generic G-local system, the group \( H^{\mathcal{E}_\Gamma} \) acts simply transitively on the space of gluing isomorphisms (36). The group \( \text{Cent}(G) \) acts by automorpsims of \( \mathcal{L}_v \) for each vertex \( v \) of \( \Gamma \). So the group
Cent(G)\textsuperscript{Fr} acts by automorphisms of the collection \(\{L_v\}\). It does not change the isomorphism classes of the \(L_v\)'s, but does change the collection of isomorphisms (36). Evidently the group Cent(G)\textsuperscript{Fr} acts on isomorphisms (36) via its image in Cent(G)\textsuperscript{Fr}. For generic \(L_v\) one has

\[
\text{Aut}(L_v) = \text{Cent}(G).
\]

So the isomorphism classes of the glued \(L\)'s on \(S\) are the orbits of the group (35).

Now we can finish the proof of Proposition 5.1. Indeed, it is clear from (30) that one has

\[
\text{Cent}(G)\textsuperscript{Fr}/\text{Im}(\text{Cent}(G)\textsuperscript{Fr} \rightarrow \text{Cent}(G)\textsuperscript{Fr}) = \text{Hom}(L_\alpha, \text{Cent}(G)).
\]

Let us prove now Theorem 5.3.

a) Recall the restriction map \(\text{Res} : \mathcal{L}_{G,S;\alpha} \rightarrow \mathcal{M}_{G,\Gamma}\).

If \(G = SL_2\), then \(\mathcal{M}_{G,P_v} = \mathbb{G}_m^3\), with the monodromies around the three boundary loops providing the isomorphism. The fibers over the generic points are rational by Lemma 5.4. So the total space \(\mathcal{L}_{SL_2,S;\alpha}\) is rational. Moreover, the restriction map for \(SL_2\) boils down to the monodromies of the twisted framed \(SL_2\)-local systems over the loops of the lamination \(\alpha\):

\[
\text{Res} = M_\alpha : \text{Loc}_{SL_2,S;\alpha} \rightarrow (\mathbb{G}_m)^{3g-3}.
\]

Rescaling a component \(i_E\) of the gluing data by \(\lambda\) rescales \(B_E\) by \(\lambda^2\), and leaves untouched the other \(B\)-coordinates. Therefore given a generic fiber \(M_\alpha^{-1}(x)\), the functions \(\{B_E\}\) provide its isomorphism with a torus

\[
\{B_E\} : M_\alpha^{-1}(x) \rightarrow (\mathbb{G}_m/\pm 1)^{3g-3}.
\]

Definition 5.2 of the space \(\mathcal{L}_{SL_2,S;\alpha}\) kills the action of the group \((\pm 1)^{3g-3}\) on the isomorphisms \(i_E\). So the functions \((M_E, B_E)\) separate generic points, providing a birational isomorphism

\[
(M_E, B_E) : \text{Loc}_{SL_2,S;\alpha} \sim \rightarrow (\mathbb{G}_m)^{6g-6}.
\]

b) Given a pair of pants \(\mathcal{P}\), the moduli space \(\mathcal{M}_{G,\mathcal{P}}\) of framed twisted G-local systems on \(\mathcal{P}\) has a positive structure. In the case when Cent(G) is trivial this was proved in [FG1]. The monodromy around the three boundary loops provides a positive map

\[
\mu_\mathcal{P} : \mathcal{M}_{G,\mathcal{P}} \rightarrow H^3.
\]

Its fiber \(\mathcal{M}_{G,\mathcal{P}}^{\text{un}}\) over the unit element \(\mathcal{M}_{G,\mathcal{P}}^{\text{un}}\) parametrizes the subspace of unipotent framed G-local systems. It is a positive space. In particular it is rational. One can split non-canonically the map \(\mu_\mathcal{P}\), getting a positive projection \(\nu_\mathcal{P} : \mathcal{M}_{G,\mathcal{P}} \rightarrow \mathcal{M}_{G,\mathcal{P}}^{\text{un}}\) and therefore a positive birational isomorphism

\[
(\nu_\mathcal{P}, \mu_\mathcal{P}) : \mathcal{M}_{G,\mathcal{P}} \rightarrow \mathcal{M}_{G,\mathcal{P}}^{\text{un}} \times H^3.
\]

Therefore there is a birational isomorphism

\[
(\nu_\mathcal{P}, \mu_\mathcal{P}) : \text{Loc}_{G,\mathcal{P};\alpha} \rightarrow \prod_{v \in \mathcal{V}_\mathcal{P}} \mathcal{M}_{G,P_v}^{\text{un}} \times (H \times H)^{\text{Fr}}. \tag{37}
\]

Then the arguments are just as in the \(SL_2\) case. \(\square\)
6 Appendix: The quantum cluster symplectic double

Definition 6.1. A quiver is a datum

\[ \mathbf{q} = (\Lambda, \{e_i\}, (\cdot, \cdot)). \]

Here \( \Lambda \) is a lattice, \( \{e_i\} \) is its basis, and \((\cdot, \cdot)\) a skew-symmetric \( \mathbb{Z} \)-valued bilinear form on \( \Lambda \).

A mutation of a quiver \( \mathbf{q} \) in the direction of a basis vector \( e_k \) is a new quiver

\[ \tilde{\mathbf{q}} = \left(\tilde{\Lambda}, \{\tilde{e}_i\}, (\cdot, \cdot)\right). \]

It has the same lattice and form as the original quiver \( \mathbf{q} \), and a new basis \( \{\tilde{e}_i\} \) defined by

\[ \tilde{e}_i := \begin{cases} e_i + (e_i, e_k) e_k & \text{if } i \neq k, \\ -e_k & \text{if } i = k. \end{cases} \]

Consider the double \( \Lambda_D \) of the lattice \( \Lambda \):

\[ \Lambda_D := \Lambda \oplus \Lambda^\vee, \quad \Lambda^\vee := \text{Hom}(\Lambda, \mathbb{Z}). \]

It gives rise to a split algebraic torus

\[ T_\Lambda := \text{Hom}(\Lambda_D, \mathbb{C}^*). \]

The basis \( \{e_i\} \) of \( \Lambda \) provides the dual basis \( \{e_i^\vee\} \) of \( \Lambda^\vee \). So a quiver \( \mathbf{q} \) provides a basis

\[ \{e_i, e_i^\vee\} \]

of \( \Lambda_D \). (39)

The basis (39) of the lattice \( \Lambda_D \) gives rise to the coordinates \( \{X_i, B_j\} \) of the torus \( T_\Lambda \).

The lattice \( \Lambda_D \) with the form \((\cdot, \cdot)_D\) gives rise to the quantum torus algebra \( T \). Precisely, the basis (39) gives rise to a set of the "quantum coordinates" \( (X_i, B_j) \) – generators of the quantum torus algebra \( T \) – satisfying the relations

\[ B_i B_j = B_j B_i, \quad q^{-1} X_i B_i = q B_i X_i, \quad B_i X_j = X_j B_i, \quad i \neq j, \quad q^{-(e_i, e_j)} X_i X_j = q^{-(e_j, e_i)} X_j X_i. \]

Denote by \( \mathbb{T} \) the (non-commutative) fraction field of \( T \). Recall the quantum dilogarithm power series, although known as the quantum exponential:

\[ \Psi_q(x) = \prod_{k=1}^{\infty} (1 + q^{2k-1} x)^{-1}. \]

Let us introduce the following notation

\[ B^+_{k} := \prod_{i \mid (e_k, e_i) > 0} B^{(e_k, e_i)}_i, \quad B^-_{k} := \prod_{i \mid (e_k, e_i) < 0} B^{-(e_k, e_i)}_i. \]

\[ X_i^o := X_i \prod_{j \in I} B^{(e_i, e_j)}_j. \]

Notice that all variables which appear in the definition of \( X_i^o \) commute.

Theorem-Definition 6.2. The conjugation by \( \Psi_q(X_k)/\Psi_q(X_k^o) \) provides an automorphism

\[ \mu^{*}_{e_k} : \mathbb{T} \longrightarrow \mathbb{T}. \]

Notice that, although \( \Psi_q(X_k) \) is a power series, we get a birational automorphism.
Mutations on the classical level. Let us apply the quantum automorphism \((41)\) to the generators \(\{\tilde{X}_i, \tilde{B}_j\}\) assigned to the mutated basis \((\tilde{e}_i, \tilde{e}_j^\vee)\), express the result via the generators \(\{X_i, B_j\}\) assigned finally to the original basis \((e_i, e_j^\vee)\), and set \(q = 1\). Then we calculate the obtained birational transformation of the torus \(T_\Lambda\):

\[\mu_{e_k}^* : \mathbb{Q}(T_\Lambda) \longrightarrow \mathbb{Q}(T_\Lambda).\]  

(42)

**Theorem 6.3.** The action of the birational automorphism \((42)\) on the coordinates \(\{\tilde{X}_i, \tilde{B}_j\}\), expressed in terms of the coordinates \(\{X_i, B_j\}\) is given by

\[
\begin{align*}
\mu_{e_k}^* : \tilde{X}_i &\mapsto \begin{cases} 
X_k^{-1} & \text{if } i = k \\
X_i(1 + X_k^{-\text{sgn}(\varepsilon_{ik})} - \varepsilon_{ik}) & \text{if } i \neq k.
\end{cases} \\
\mu_{e_k}^* : \tilde{B}_k &\mapsto \frac{B_k^- + X_k B_k^+}{B_k(1 + X_k)}, \\
\tilde{B}_j &\mapsto B_j \text{ if } j \neq k.
\end{align*}
\]  

(43)

There is a symplectic form on the torus \(T_\Lambda\), given in coordinate \((X_i, B_j)\) by

\[
\Omega_q = -\frac{1}{2} \sum_{i,j} (e_i, e_j) \cdot d \log B_i \wedge d \log B_j - \sum_i d \log B_i \wedge d \log X_i.
\]  

(45)

The Poisson structure provided by the symplectic form is given in coordinates \(\{X_i, B_j\}\) by

\[
\{B_i, B_j\} = 0, \quad \{X_i, B_j\} = \delta_{ij} X_i B_j, \quad \{X_i, X_j\} = \varepsilon_{ij} X_i X_j.
\]  

(46)

The symplectic form is obtained by applying the \(d \log \wedge d \log\) map to a class

\[
W_q = -\frac{1}{2} \sum_{i,j} (e_i, e_j) \cdot B_i \wedge B_j - \sum_i B_i \wedge X_i \in \Lambda^2 \mathbb{Q}(T_\Lambda)^*.
\]  

(47)

**Theorem 6.4.** i) Given a mutation \(q \to \tilde{q}\) in the direction \(e_k\), one has

\[
\mu_{e_k}^* W_{\tilde{q}} - W_q = (1 + X_k^\circ) \wedge X_k^\circ - (1 + X_k) \wedge X_k.
\]

ii) The mutations preserve the symplectic, and hence the Poisson structure.

The second claim follows immediately from the first. The first is Proposition 2.14 in [FG3]. The first claim implies that there is a canonical class in \(K_2(T_\Lambda)\) preserved by the mutations. So it gives rise to a canonical line bundle with connection on the symplectic double. Notice that the part ii) follows immediately from Theorem-Definition 6.2. Indeed, the classical limit of an automorphism of a quantum torus algebra preserves the corresponding Poisson structure.
The classical cluster symplectic double. Now we are ready to define the cluster symplectic double variety. The construction follows the definition of cluster Poisson and $K_2$-varieties given in [FG2]. Starting with a quiver $q$, we assign to it the split algebraic torus $\mathcal{T}_q$ with the cluster symplectic double coordinates $(B_i, X_i)$. Then we mutate the quiver $q$ in the directions of all basis vectors, getting new split algebraic tori, and continue this process indefinitely. We glue each pair of split tori $\mathcal{T}_q$ and $\mathcal{T}_{q'}$ related by a quiver mutation according to the mutation formula (43)-(44). Finally, given two quivers $q$ and $q'$ related by a sequence of mutations, such that there is an isomorphism of quivers $i : q \rightarrow q'$ which induces the same isomorphism of tori $i_\mathcal{T} : \mathcal{T}_q \rightarrow \mathcal{T}_{q'}$, as the sequence of cluster mutations relating $q$ and $q'$, we identify the tori $\mathcal{T}_q$ and $\mathcal{T}_{q'}$ according the isomorphism $i_\mathcal{T}$. This way we get a possibly non-separable prescheme, which by abuse of terminology is called a cluster symplectic double variety.

By talking about a cluster symplectic double variety structure on an actual space $\mathcal{D}$ we mean that $\mathcal{D}$ has a collection of rational coordinate systems, assigned to cluster mutations of a quiver $q$ as explained above, and related by the compositions of cluster symplectic double transformations (43)-(44). Precisely, given a space $\mathcal{D}$ we have to provide the following:

- A collection of quivers, usually infinite, such that any two of them are related by a sequence of quiver mutations inside of a given collection.
- A construction assigning to each of the quivers a cluster symplectic double rational coordinate system on $\mathcal{D}$.
- A proof that the coordinates systems assigned to any pair of the quivers related by a quiver mutation are related by the cluster symplectic double transformations (43)-(44).

This is precisely what we do in the paper: define a moduli space $\mathcal{D}_{G,S}$ assigned to a pair $(G, S)$; consider the collection of quivers introduced in [FG1] for the pair $(PGL_m, S)$; construct the cluster symplectic double rational coordinate systems on the moduli space $\mathcal{D}_{PGL_m, S}$, and prove that they are related by the cluster symplectic double transformations (43)-(44).

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