THE GENERALIZED LINEAR SAMPLING AND
FACTORIZATION METHODS ONLY DEPENDS ON THE SIGN OF CONTRAST ON THE BOUNDARY

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Abstract. We extend the applicability of the Generalized Linear Sampling Method (GLSM) [2] and the Factorization Method (FM)[16] to the case of inhomogeneities where the contrast changes sign. Both methods give an exact characterization of the target shapes in terms of the farfield operator (at a fixed frequency) using the coercivity property of a special solution operator. We prove this property assuming that the contrast has a fixed sign in a neighborhood of the inhomogeneities boundary. We treat both isotropic and anisotropic scatterers with possibly different supports for the isotropic and anisotropic parts. We finally validate the methods through some numerical tests in two dimensions.

1. Introduction. We extend the applicability of the Generalized Linear Sampling Method (GLSM) [2] and the Factorization Method (FM)[16] to the case of inhomogeneities with sign changing contrast. More specifically we shall prove that these methods are applicable under the assumption that the contrast sign is kept constant only in a neighborhood of the inhomogeneities boundary. This type of assumption is compatible with the one used for proving discreteness of the Transmission Eigenfrequencies (frequencies for which the homogeneous Interior Transmission Problem has non trivial solutions [6]) which is another additional ingredient for these sampling methods. Under these assumptions one obtains in particular an exact characterization of the target shapes in terms of the range of the farfield operator (at a fixed frequency). One of the key ingredient to prove this exact characterization is based on a factorization of the farfield operator. This factorization involves three operators that should satisfy some specific properties. The main result of this paper is to prove this coercivity property in the case of sign changing contrast both for isotropic and anisotropic scatters with possibly different supports for the isotropic and anisotropic parts.

This work is indeed closely related to the study of ITP for sign changing contrasts [6, 17, 9, 4, 8, 15]. However, proving the above mentioned coercivity property requires to also deal with bounds on the solution outside the inhomogeneity. The main idea of our proof is to introduce an artificial contrast in order to isolate the terms in the region where the contrast changes sign. Using classical regularity results for pde we then are able to demonstrate that these terms are only related to
compact perturbations of a case with constant sign contrast. This is then enough to prove the shape characterization using FM [16, 14] and GLSM [2] with the additional assumption that ITP is well posed. Let us quote that a similar result for FM and isotropic scatterers has been obtained independently by [10] using different techniques.

Our methodology could be applied to other type of perturbations (such as sound-soft or sound-hard inclusions) inside the inhomogeneities to demonstrate coercivity property in those cases [18].

The article is organized as follows. In Section 2 we introduce the scalar wave equation for orthotropic media and demonstrate that the farfield operator can be factorized in a similar way as for the isotropic case (although with some additional technicalities with respect to [5]). The obtained factorization does not require any correlation between the supports of the isotropic parameters and the anisotropic ones (which then may be different). In Section 3 we demonstrate the coercivity of the middle operator (denoted $T$) which is shown to hold true if the contrasts have fixed (and compatible) sign in a neighborhood of the boundary of $D$. In Section 4 we conclude that the Generalized Linear Sampling Method or the Factorization Method could be applied in these cases. Finally in Section 5, we give some numerical illustrations.

2. Model problem. The model problem we are interested in is the scattering of scalar waves by an orthoptic medium in the frequency domain. For a wave number $k > 0$, the total field solves the following scalar wave equation:

$$\div(A \nabla u) + k^2 nu = 0 \text{ in } \mathbb{R}^d$$

with $d = 2$ or $3$ and with $n \in L^\infty(\mathbb{R}^d)$ denoting the refractive index such that the support of $n - 1$ is included into $\overline{D_n}$ with $D_n$ a bounded domain with Lipschitz boundary and connected complement and such that $\Im(n) \geq 0$. We assume that $A$ is at least in $L^\infty(\mathbb{R}^d)^{d \times d}$ (we denote by $Id$ the identity matrix) and that the support of $A - Id$ is included into $\overline{D_A}$ with $D_A$ a bounded domain with Lipschitz boundary and connected complement and such that $\Im(A \zeta \cdot \bar{\zeta}) \leq 0$ and $\Re(A \zeta \cdot \bar{\zeta}) \geq c|\zeta|^2$ for $\zeta \in \mathbb{C}^d$ and for some positive constant $c$. We introduce a domain $D$ such that $D_n \cup D_A \subset D$ with $D$ a bounded domain with Lipschitz boundary and connected complement. In the following we will assume that the simply connected components of $D$ will either have boundary where $A$ is not equal to $Id$ or $n$ is not equal to one if $A$ is identically equal to $Id$. Therefore we will have $D_n \cup D_A = D$.

We introduce the unit sphere $S^{d-1} := \{x \in \mathbb{R}^d \text{ s.t. } |x| = 1\}$ we are interested in the cases where the total field is generated by plane waves, $u'(\theta, x) := e^{ik|x|\theta}$ with $x \in \mathbb{R}^d$ and $\theta \in S^{d-1}$ and we denote by $u^s$ the scattered field defined by

$$u^s(\theta, \cdot) = u(\theta, \cdot) - u^s(\theta, \cdot) \text{ in } \mathbb{R}^d,$$

which is assumed to be satisfying the Sommerfeld radiation condition,

$$\lim_{r \to \infty} \int_{|x| = r} \left| \frac{\partial u^s}{\partial r} - iku^s \right|^2 \, ds = 0.$$  

Our data for the inverse problem will be formed by noisy measurements of so-called farfield pattern $u^\infty(\theta, \hat{x})$ defined by

$$u^s(\theta, x) = \frac{e^{ik|x|}}{|x|(d-1)/2} \left( u^\infty(\theta, \hat{x}) + O(1/|x|) \right)$$
as $|x| \to \infty$ for all $(\theta, \hat{x}) \in \mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$. The goal is to be able to reconstruct $D$ from these measurements (without knowing $n$ and $A$). From those measurements we introduce the farfield operator $F : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$, defined by

$$Fg(\hat{x}) := \int_{\mathbb{S}^{d-1}} u^\infty(\theta, \hat{x})g(\theta)ds(\theta), \ \hat{x} \in \mathbb{S}^{d-1}$$

Let us define for $\psi \in \{ f \in L^2(D) \ s.t \ f|_{D_A} \in H^1(D_A) \}$, the unique function $w \in H^1_{\text{loc}}(\mathbb{R}^d)$ satisfying

$$\begin{cases}
\text{div}(A\nabla w) + nk^2w = -k^2(n-1)\psi - \text{div}((A - Id)\nabla \psi) \text{ in } \mathbb{R}^d, \\
\lim_{r \to \infty} \int_{|x|=r} |\partial w/\partial r - ikw|^2 \ ds = 0.
\end{cases}$$

By linearity of the forward scattering problem, $Fg$ is nothing but the farfield pattern of $w$ solution of (1) with $\psi = v_\psi$ in $D$, where

$$v_\psi(x) := \int_{\mathbb{S}^{d-1}} e^{ik\hat{x} \cdot \theta}g(\theta)ds(\theta), \ g \in L^2(\mathbb{S}^{d-1}), \ x \in \mathbb{R}^d.$$ 

We introduce $X(D) = \{(f, h) \in L^2(D) \times L^2(D_A) \ s.t \ h = \nabla f \text{ in } D_A\}$, we identify $X$ and its adjoint. Finally we consider the norm on $X(D)$ defined by

$$\|(f, h)\|_X = \|f\|_{L^2(D)} + \|h\|_{L^2(D_A)} = \|f\|_{H^1(D_A)} + \|f\|_{L^2(D_A)}$$

Now consider the (compact) operator $H : L^2(\mathbb{S}^{d-1}) \to X$ defined by

$$Hg := \left[ \begin{array}{c}
v_g|_{D_n} \\
\nabla v_g|_{D_A}
\end{array} \right],$$

and the (compact) operator $G : \overline{R(H)} \subset X \to L^2(\mathbb{S}^{d-1})$ defined by

$$G \left[ \begin{array}{c}
\psi \\
\nabla \psi
\end{array} \right] := w^\infty|_{\mathbb{S}^{d-1}}$$

where $w^\infty$ is the farfield of $w \in H^1_{\text{loc}}(\mathbb{R}^d)$ solution of (1) and where $\overline{R(H)}$ denotes the closure of the range of $H$ in $X$. Then clearly

$$F = GH$$

One can also decompose $G$ to get the second factorization of the farfield operator. More precisely, for the case under consideration, since the farfield pattern of $w$ has the following expression [5]:

$$w^\infty(\hat{x}) = -\int_{D_n} e^{-ik\hat{x} \cdot y} \frac{1-n}{2}(1-n)k^2(\psi(y) + w(y))dy - \int_{D_A} (A - Id)\nabla \varphi e^{-ik\hat{x} \cdot y} \cdot \nabla (\psi(y) + w(y))dy,$$

one simply has $G = H^*T\psi$, where $H^* : X \to L^2(\mathbb{S}^{d-1})$ is the adjoint of $H$ given by

$$H^* \left[ \begin{array}{c}
\varphi \\
\nabla \varphi
\end{array} \right] (\hat{x}) :=$$

$$\int_{D_n} e^{-ik\hat{x} \cdot y} \varphi(y)dy + \int_{D_A} (A - Id)\nabla \varphi e^{-ik\hat{x} \cdot y} \cdot \nabla \varphi dy, \ \varphi \in X, \ \hat{x} \in \mathbb{S}^{d-1},$$

and $T : X \to X$ is defined by

$$T \left[ \begin{array}{c}
\psi \\
\nabla \psi
\end{array} \right] := \left[ \begin{array}{c}
-k^2(1-n)(\psi + w) \\
-(A - Id) \cdot (\nabla \psi(y) + \nabla w(y))
\end{array} \right].$$
with \( w \in H^1_{\text{loc}}(\mathbb{R}^d) \) being the solution of (1)(with \( \psi_1 = \psi \) and \( \psi_2 = \nabla \psi \)). Finally we get

\[
F = H^*TH,
\]

**Remark 1.** We remark that \( T \) is independent of the type of incident waves (either plane waves, point sources ...). We presented our results for plane waves but the properties of \( T \) presented in this paper remain true for other type of incident waves and measurements such as the one considered in [3].

3. Key properties of the factorization operators. In the following we will review important properties of the operators involved in the factorization (4) and (6). First we will state the classical properties of \( H \) and \( G \), in particular a range characterization of the obstacle \( D \) which is at the heart of both the GLSM and the FM. Then we will study the coercivity of \( T \) for sign changing contrasts.

3.1. Range characterization. First we assume that our obstacle \( D \) will be composed of several disjoint simply connected components. Those components will either have \( A = \text{Id} \) and \( n \neq 1 \) or \( A \neq \text{Id} \) in a neighborhood of their boundaries. The characterization of the obstacle \( D \) in terms of the range of \( G \) is based on the solvability of the interior transmission problem (for given regular boundary values \( f \) and \( h \)):

\[
\begin{align*}
\text{div}(A\nabla u) + k^2 nu &= 0 \quad \text{in } D, \\
\Delta v + k^2 v &= 0 \quad \text{in } D, \\
(u - v) &= f \quad \text{on } \partial D, \\
\frac{\partial}{\partial \nu}(u - v) &= h \quad \text{on } \partial D,
\end{align*}
\]

where \((u,v) \in \mathcal{Y}(D)\) and \( \mathcal{Y}(D) \) is a space of solutions that will be specified later. We will assume that the following hypothesis holds true.

**Hypothesis 1.** We assume that \( k^2 \in \mathbb{R}_+ \) is such that problem (7) has a unique solution for all regular (to be specified later) functions \( f \) and \( h \).

This hypothesis and the interior transmission problem stated above are incomplete in the sense that we did not specify \( \mathcal{Y}(D) \). This space actually depends on the properties of \( A \) and \( n \). For example if we assume that \( D_n \subset D_A = D \), (7) can be studied for \((u,v) \in \mathcal{Y}(D) = H^1(D) \times H^1(D) \). In this case we know from [4] that hypothesis 1 is for instance true if \( A - \text{Id} \) and \( n - 1 \) have the same sign and do not change sign in a neighborhood of \( \partial D \). The case where \( D_A = \emptyset \) should be considered for \((u,v) \in L^2(D) \times L^2(D) \) such that \( u - v \in H^2(D) \). In this case Hypothesis 1 is true if \( n - 1 \) does not change sign in a neighborhood of \( \partial D \). The case where \( n = 1 \) in a neighborhood of \( \partial D \) has been less studied in the literature and the only case where we know that hypothesis 1 is true is when \( A - \text{Id} \) does not change sign in all \( D \) and \( n = 1 \) in \( D \). Finally when \( A = \text{Id} \) in a neighborhood of \( \partial D \), but not in all \( D \), and \( n - 1 \) does not change sign in a neighborhood of \( \partial D \), there is no clearly stated result in the literature about this case. Let us mention however that surface integral method applied to (7) (as proposed in [9]) would be an appropriate tool to study this case.

**Lemma 3.1.** If hypothesis 1 holds true, we have that \( z \) is inside \( D \) if and only if \( \phi_z(\hat{x}) := e^{-ik\hat{x} \cdot z} \in \mathcal{R}(G) \).

We also give the following lemmas from [5]:
Lemma 3.2. $G$ is compact and if hypothesis 1 holds true, its range is dense in $L^2(S^{d-1})$.

Lemma 3.3. The compact operator $H$ has a dense range in the space $\{(f,h) \in L^2(D) \times L^2(D_A) \text{ s.t. } h = \nabla f \text{ in } D_A \text{ and } \Delta f + k^2 f = 0 \text{ in } D\} \subset X(D)$.

3.2. Coercivity of the middle operator $T$. Our hypothesis on $D$ implies that we can split the simply connected component into two categories. The first one is such that $A-Id$ does not equal zero on a neighborhood of the boundary and the second one is such that $A-Id = 0$ and $n-1$ does not change sign in a neighborhood of the boundary. We will give a coercivity result for each of those two configurations and then merge them into a combined condition on $n$ and $A$ under which we have the coercivity of $T$ defined in (5).

For both cases we will need the following equality.

Lemma 3.4. We have the following identity, for $\psi = (\psi_1, \psi_2) \in X(D)$ and $T$ defined in (5):

$$\Im\langle T\psi, \psi \rangle = k \int_{S^{d-1}} |w|^2 \text{d}S - \int_{D_A} \Im(A)(\nabla w + \nabla \psi) \cdot (\nabla(w) + \nabla\psi) + \kappa^2 \int_{D_n} \Im(\nabla w)^2.$$

Proof. We recall that for any $\psi \in X(D)$ there exists a unique $w \in H^1_{loc}(\mathbb{R}^d)$ that solves (1). The definition of $T$, (5) gives:

$$\langle T\psi, \psi \rangle = -\int_{D_A} (A-Id)(\nabla\psi + \nabla w) \cdot (\nabla\psi + \nabla w) - k^2 \int_{D_n} (1-n)|\psi + w|^2 + \int_{D_A} (A-Id)(\nabla\psi + \nabla w) \cdot (\nabla w) + k^2 \int_{D_n} (1-n)(\psi + w)^2 w.$$

Using (1) and integrating by parts over a ball $B_R$ such that $D \subset B_R$ we have:

$$-\int_{B_R} \nabla w \cdot \nabla w - k^2 w \Delta w + \int_{B_R} \partial_{\overline{w}} \frac{\partial w}{\partial r} \overline{w} = \int_{D_A} (A-Id)(\nabla\psi + \nabla w) \cdot (\nabla w) + k^2 \int_{D_n} (1-n)(w + \psi)^2 w.$$

Substituting in (8), taking the imaginary part and letting $R$ to $+\infty$ prove the lemma. \hfill $\Box$

3.2.1. The case where $\overline{D_A} \subsetneq D_n$. We first consider the case where $\overline{D_A} \subsetneq D_n$ which can be seen as an extension of the case $D_A = \emptyset$. The Herglotz wave operator reduces to $Hg = [v_{y}|_{D}, \overline{v_y}|_{D_A}]$.

Hypothesis 2. There exist $\alpha \geq 0, c > 0$ such that either $\Re(n-1) + \alpha \Im(n) \geq c$ or $\Re(1-n) + \alpha \Im(n) \geq c$ in a neighborhood of $\partial D$.

Theorem 3.5. If $\overline{D_A} \subsetneq D_n = D$ and hypothesis 2 and 1 holds true , then there exists $\mu > 0$ such that the operator $T$ defined in (5) verifies

$$|\langle T\psi, \psi \rangle| \geq \mu \|\psi\|^2_{X(D)},$$

for all $\psi \in \mathcal{R}(H)$.

Proof. We will proceed by a contradiction argument, therefore we assume:

$$\|\psi_\ell\|_{X(D)} = 1 \Delta \psi_\ell + k^2 \psi_\ell = 0 \text{ in } D \quad \text{and } |\langle T\psi_\ell, \psi_\ell \rangle| \to 0 \text{ as } \ell \to \infty.$$

Up to changing the initial sequence, one can assume that $\psi_\ell$ weakly converges to $\psi$ in $L^2(D)$. One easily see that $\psi$ satisfies

$$\Delta \psi + k^2 \psi = 0 \text{ in } D.$$
We denoted \( w_\ell \in H^2_{loc}(\mathbb{R}^d) \) the solution of
\[
\begin{aligned}
\text{div}(A\nabla w_\ell) + nk^2 w_\ell &= -k^2(n-1)\psi_\ell - \text{div}((A - Id)\nabla \psi_\ell) \quad \text{in } \mathbb{R}^d, \\
\lim_{r \to \infty} \int_{|x|=r} |\frac{\partial w_\ell}{\partial r} - ikw_\ell|^2 \, ds &= 0.
\end{aligned}
\]
(9)

where the last term on the right hand side goes to zero (because of the strong convergence results), and using the assumption on \( n_0 \) we conclude that \( \lim_{\ell \to 0} \| (T\psi_\ell, \psi_\ell) \| \geq k^2 c/2 > 0 \), which is a contradiction.

\[ \square \]

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3.2.2. The case \( D_n \subset D_A \). Without loss of generality we will consider the case where \( D_n = \emptyset \) and \( D = D_A \) rather than the case \( D_n \subset D_A = D \) in order to lighten the notation. This is possible because thanks to compact embedding from \( H^1 \) to \( L^2 \), terms that come from contrast in \( n \) will go to zero in the proof similarly to the previous section. Therefore in the following \( D_A = D \) and \( \| \|_{X(D)} \) will be equal to the \( \| H^1(D) \| \).

**Hypothesis 3.** \( A \) is \( C^1 \) in a neighborhood \( V \) of \( \partial D \) and if either of both conditions apply:

- there exists \( \alpha \geq 0 \), \( c > 0 \) such that \( \Re (A - Id) - \alpha \Im (A) \geq c \) in \( V \)
- \( \Re (A) \) is positive definite and there exists \( \alpha \geq 0 \), \( c > 0 \) and \( 0 < \eta \leq 1 \) such that
  \[
  \Re (Id - A) X \cdot X + (1 - \eta) \Re (A) Y \cdot Y - \alpha \Im (A) (X + Y) \cdot (X + Y) \geq c X \cdot X
  \]

in \( V \) for all \( X, Y \in \mathbb{C}^d \).

**Theorem 3.6.** If \( D_n = \emptyset \), \( D = D_A \) and hypothesis 3 holds true, there exists \( \mu > 0 \) such that the operator \( T \) defined in (5) verifies

\[
|⟨T\psi, \psi⟩| \geq \mu \|\psi\|_{X(D)}^2
\]

for all \( \psi \in \overline{\mathcal{R}(H)} \).

**Proof.** We introduce \( A_0 \) such that \( A_0 = A \) inside \( V \) a neighborhood of \( \partial D \) and \( A_0 \) verifies hypothesis 3 in all \( D \). Since we suppose that \( A \) is \( C^1 \) inside \( V \) we can choose \( A_0 \) to be \( C^1 \) inside all \( D \). We also introduce \( \Omega = \text{supp}(A - A_0) \), by construction \( \Omega \subset D \).

We will proceed by a contradiction argument, therefore we assume:

\[
\|\psi_\ell\|_{X(D)} = 1 \quad \text{and} \quad |⟨T\psi_\ell, \psi_\ell⟩| \to 0 \quad \text{as} \quad \ell \to \infty
\]

and that \( \psi_\ell \) weakly converges in \( H^1(D) \) to \( \psi \) that satisfies

\[
\Delta \psi + k^2 \psi = 0 \quad \text{in} \quad D.
\]

The solution \( w_\ell \) satisfying (1) with \( v = \psi_\ell \) weakly converges in \( H^1(D) \) to \( w \in H^1(\mathbb{R}^d) \) satisfying (1) with \( v = \psi \).

Lemma 3.4 implies that \( w^\infty \to 0 \) in \( L^2(\mathbb{S}^{d-1}) \) and therefore \( w^\infty = 0 \). The Rellich theorem and unique continuation theorem imply that \( w = 0 \) outside \( D \). Thus we have that \( u = \psi + w \) and \( v = \psi \) solve the interior transmission eigenvalue problem (7). Hypothesis 1 implies that that \( \psi = w = 0 \). Let us introduce the intermediate (scattered) field \( u_{0,\ell}^s \) that solves:

\[
\begin{aligned}
\text{div}(A_0 \nabla u_{0,\ell}^s) + k^2 u_{0,\ell}^s &= -\text{div}((A_0 - Id) \nabla \psi_\ell) \quad \text{in} \quad \mathbb{R}^d, \\
\lim_{r \to \infty} \int_{|x|=r} \left| \frac{\partial u_{0,\ell}^s}{\partial r} - i k u_{0,\ell}^s \right|^2 \, ds &= 0.
\end{aligned}
\]

We denote by \( u_{0,\ell} = u_{0,\ell}^s + \psi_\ell \) the total field. We also introduce \( u_\ell^s \) that solves:

\[
\begin{aligned}
\text{div}(A \nabla u_\ell^s) + k^2 u_\ell^s &= -\text{div}((A - A_0) \nabla u_{0,\ell}) \quad \text{in} \quad \mathbb{R}^d, \\
\lim_{r \to \infty} \int_{|x|=r} \left| \frac{\partial u_\ell^s}{\partial r} - i k u_\ell^s \right|^2 \, ds &= 0.
\end{aligned}
\]
We have
\[ |(T \psi_\ell, \psi_\ell)| = \int_D (A - I) \nabla (u_\ell^0 + u_{0,\ell}) \nabla \psi_\ell dx = \int_D (A_0 - I) \nabla u_{0,\ell} \nabla \psi_\ell dx + \int_D (A - I) \nabla u_\ell \nabla \psi_\ell dx \]
\[ + \int_\Omega (A - A_0) \nabla u_{0,\ell} \nabla \psi_\ell dx \]
\[ \geq \int_D (A - I) \nabla u_{0,\ell} \nabla \psi_\ell dx - \int_D ((A - I) \nabla u_\ell \nabla \psi_\ell dx \]
\[ + \int_\Omega (A - A_0) \nabla u_{0,\ell} \nabla \psi_\ell dx \]
\[ \geq \int_D (A - I) \nabla u_{0,\ell} \nabla \psi_\ell dx - \int_D (A - I) \nabla u_\ell \nabla \psi_\ell dx \]
\[ - \int_\Omega (A - A_0) \nabla u_{0,\ell} \nabla \psi_\ell dx \] 
(14)

Since \( u_{0,\ell} \in H^1(D) \) satisfies \( \text{div}(A_0 \nabla u_{0,\ell}) + k^2 u_{0,\ell} = 0 \) in \( D \), we infer by interior elliptic regularity that \( u_{0,\ell} \in H^2(\Omega) \) (from [12] and the fact that \( A_0 \) is \( C^1 \)). Due to compact embeddings from \( H^2 \) to \( H^1 \), we deduce that \( u_{0,\ell} \) converges strongly to zero in \( H^1(\Omega) \). For the same reasons we deduce that \( \psi_\ell \) converges strongly to zero in \( H^1(\Omega) \).

By continuity of the forward scattering problem verified by \( u_\ell^0 \) and the strong convergence of \( u_{0,\ell} \) in \( H^1(\Omega) \), we deduce that \( u_\ell^0 \) strongly converges to zero in \( H^1_{loc}(\mathbb{R}^d) \). We therefore deduce that for \( \ell \) large enough (14) becomes:

\[ |(T \psi_\ell, \psi_\ell)| \geq \frac{1}{2} \int_D (A - I) \nabla u_{0,\ell} \nabla \psi_\ell dx \] 
(15)

To treat \( |(T \psi_\ell, \psi_\ell)| = \int_D (A_0 - I) \nabla u_{0,\ell} \nabla \psi_\ell dx \) we need to consider two cases depending on the compatibility of the sign of \( A_0 - I \) and \( I d \) (as in [7]). First we consider the case when there exist \( \alpha \geq 0 \) and \( c > 0 \) such that \( \Re(A_0 - I) - \alpha \Im(A_0) \geq c > 0 \). Since \( u_{0,\ell}^0 \) solves (12) we deduce that:

\[ (T_0 \psi_\ell, \psi_\ell) = - \int_D (A_0 - I) \nabla u_{0,\ell} \nabla \psi_\ell + |u_{0,\ell}|^2 \]
\[ - \int_{\mathbb{R}^d} |\nabla u_{0,\ell}|^2 + \int_{\mathbb{R}^d} |u_{0,\ell}|^2 \]
\[ + \int_{\mathbb{R}^d} |u_{0,\ell}|^2 + ik \int_{\mathbb{R}^d} |u_{0,\ell}|^2 \] 
(16)

The weak convergence of \( u_{0,\ell} \) in \( H^1_{loc}(\mathbb{R}^d) \) and \( u_{0,\ell} \) in \( H^1(D) \) imply the strong convergence in \( L_2^2(\mathbb{R}^d) \) and \( L_2^2(D) \) respectively. Therefore the last three terms in the equality above go to zero. Moreover (15) implies that \( |(T_0 \psi_\ell, \psi_\ell)| \) go to zero. Therefore the first term in (16) goes also to zero and the hypothesis on \( A_0 \) implies that

\[ \| - \int_D (A_0 - I) \nabla u_{0,\ell} \nabla \psi_\ell + |u_{0,\ell}|^2 \| \geq c/2 \| u_{0,\ell} \|_{H^1(D)} \]

Therefore \( \| u_{0,\ell} \|_{H^1(D)} \) goes to zero as well as \( \| u_{0,\ell} \|_{H^1_{loc}(\mathbb{R}^d)} \). This implies that \( \| \psi_\ell \|_{H^1(D)} \to 0 \) which is a contradiction.

Then we consider the case when \( \Re(A_0) \) is positive definite. We cannot use (16) since the terms involving \( u_{0,\ell} \) and \( u_{0,\ell}^0 \) do not have the same sign. From the definition of \( T_0 \) we have:

\[ (T_0 \psi_\ell, \psi_\ell) = - \int_D (A_0 - I) \nabla \psi_\ell \nabla \psi_\ell - \int_D (A_0 - I) \nabla u_{0,\ell} \nabla \psi_\ell. \]

Using equation (1) verified by \( u_{0,\ell}^0 \) we have:

\[ (T_0 \psi_\ell, \psi_\ell) = - \int_D (A_0 - I) \nabla \psi_\ell \nabla \psi_\ell + \int_{\mathbb{R}^d} A_0 \nabla u_{0,\ell} \nabla \psi_\ell \]
\[ - 2i \int_D S(A_0) \nabla u_{0,\ell} \nabla \psi_\ell - ik \int_{\mathbb{R}^d} |u_{0,\ell}|^2 - \int_{\mathbb{R}^d} k^2 u_{0,\ell} u_{0,\ell} \]
The last two terms go to zero because of regularity and compact embedding. The real part of the remaining term is
\[
\int_D (\text{Id} - \Re(A_0)) \nabla \psi_x \nabla \overline{\psi}_x + \int_D \Re(A_0) \nabla u_{0,\ell}^* \nabla \overline{u}_{0,\ell}^*
\]
\[
- i \int_D \Im(A_0) \nabla u_{0,\ell}^* \nabla (\overline{\psi}_x + \overline{u}_{0,\ell}^*) - \Im(A_0) \nabla \overline{u}_{0,\ell}^* \nabla (\psi_x + u_{0,\ell}^*)
\]
and the imaginary part is
\[
- \int_D \Im(A_0) \nabla (\psi_x + u_{0,\ell}^*) \nabla (\overline{\psi}_x + \overline{u}_{0,\ell}^*),
\]
both will go to zero. Those two terms can be combined through a positive parameter \(\lambda\) in order to form the following quantity:
\[
\int_D (\text{Id} - \Re(A_0)) \nabla \psi_x \nabla \overline{\psi}_x + \int_D \Re(A_0) \nabla u_{0,\ell}^* \nabla \overline{u}_{0,\ell}^*
\]
\[
- \lambda \int_D \Im(A_0) \nabla (\psi_x + u_{0,\ell}^*) \nabla (\overline{\psi}_x + \overline{u}_{0,\ell}^*)
\]
\[
- i \int_D \Im(A_0) \nabla u_{0,\ell}^* \nabla (\overline{\psi}_x + \overline{u}_{0,\ell}^*) - \Im(A_0) \nabla \overline{u}_{0,\ell}^* \nabla (\psi_x + u_{0,\ell}^*)
\]
\[
+ \int_D \Re(A_0) \nabla u_{0,\ell}^* \nabla \overline{u}_{0,\ell}^*.
\]

If we denote the quantity under the integral over \(D\) in this identity by \(M(\nabla \psi_x, \nabla u_{0,\ell}^*)\). We observe that :
\[
M(X,Y) = (I - \Re(A_0))X \cdot \overline{X} + (1 - \eta)\Re(A_0)Y \cdot \overline{Y} - \lambda \Im(A_0)(X + Y) \cdot (\overline{X} + \overline{Y})
\]
\[
+ (\eta \Re(A_0))^{1/2}X + i(\eta \Re(A_0))^{-1/2} \Im(A_0)(X + Y)^2
\]
\[
- (\eta \Re(A_0))^{-1/2} \Im(A_0)(X + Y)^2.
\]

Our assumption on \(A_0\) implies that for
\[
\lambda > \alpha + \sup_{x \in D} \|\Im(A_0(x))\| / (\eta \|\Re(A_0(x))\|)
\]
we have
\[
(T_0 \psi_x, \psi_x) \geq c/2 \|\psi_x\|^2_{H^1(D)}.
\]
This implies that \(\|\psi_x\|^2\) goes to zero which is a contradiction. \(\square\)

**Remark 2.** One can weaken the regularity assumption on \(A\) in \(V\) (e.g. example piecewise \(C^1\)) as long as one obtain an interior regularity property (e.g. \(u_{0,\ell} \in H^s(\Omega)\) where \(s > 1\)) which implies strong convergence through compact embeddings [13].

3.2.3. A final coercivity result. We introduce \(D = \bigcup_i D_i^V \cup \bigcup_i D_i^A\) where the \(D_i\) are simply connected disjoint components. We assume that \(A - \text{Id}\) is not zero in the neighborhood of the boundary \(D_i^A\) and \(A - \text{Id}\) equals zero in the neighborhood of the boundary \(D_i^V\). With those notation and the result of Theorems 3.6 and 3.5 we can give the final result under Hypothesis 1 in the case of many disjoint scatters.

**Theorem 3.7.** Assume \(A\) has \(C^1\) regularity in \(D_i^A \cap V\) and that either conditions apply :

- there exist \(c > 0\) and \(\alpha > 0\) such that either \(\Re(A - \text{Id}) - \alpha \Im(A) \geq c > 0\) in \(\bigcup_i D_i^A \cap V\) and \(\Re(1 - n) + \alpha \Im(n) \geq c > 0\) in \(\bigcup_i D_i^V \cap V\)
- \(\Re(A)\) is positive definite in \(\bigcup_i D_i^A \cap V\) and there exists \(\alpha \geq 0\), \(c > 0\) and \(0 < \eta \leq 1\) such that \(\Re(n - 1) + \alpha \Im(n) \geq c > 0\) in \(\bigcup_i D_i^V \cap V\) and \(\Re(\text{Id} - A)X \cdot \overline{X} + (1 - \eta)\Re(A)Y \cdot \overline{Y} - \alpha \Im(A)(X + Y) \cdot (\overline{X} + \overline{Y}) \geq cX \cdot \overline{X}\) in \(\bigcup_i D_i^A \cap V\) for all \(X, Y \in \mathbb{C}^d\).
We have that $T$ defined by (5) verifies:

$$|\langle T\psi, \psi \rangle| \geq \mu \|\psi\|_X^2$$

where $\psi \in \overline{\mathcal{R}(H)}$.

**Proof.** We set $D_1 = \bigcup_i D'_i$ and $D_2 = \bigcup_i D''_i$. In this case we have that

$$\langle T\psi, \psi \rangle = \langle T\psi|_{D_1}, \psi|_{D_1} \rangle + \langle T\psi|_{D_2}, \psi|_{D_2} \rangle$$

By the linearity of the forward scattering problem, if we introduce the two total fields associated to the two incidents waves $\psi_1 = \psi|_{D_1}$ in $D_1$ and 0 in $D_2$ and $\psi_2 = \psi|_{D_2}$ in $D_2$ and 0 in $D_1$, denoted $u_1 = u_1^a + \psi_1$ and $u_2 = u_2^a + \psi_2$. Then we have:

$$\langle T\psi, \psi \rangle = \langle T_1\psi_1, \psi_1 \rangle_{D_1} + \langle T_2\psi_2, \psi_2 \rangle_{D_2} - \int_{D_1} (A - Id) \nabla u_2^a \cdot \nabla \psi_1$$

$$- \int_{D_1} k^2(1 - n)u_2^a \psi_1 - \int_{D_2} k^2(1 - n)u_1^a \psi_2 + (A - Id) \nabla u_1^a \cdot \nabla \psi_2$$

where $T_1$ and $T_2$ are the operators corresponding to $D_1$ and $D_2$ respectively. We clearly see that the last two terms go to zero (using a compactness argument). Therefore if $T_1$ and $T_2$ have the same sign, we obtain that $T$ is coercive. The sign of $T_1$ and $T_2$ are given in the proofs of Theorems 3.6 and 3.5 respectively, which allows us to conclude.

**Remark 3.** We concentrate on sign changing contrast but we believe that both the results and the methods of the proofs could be straightforwardly extended to inclusion of any kind (sound-soft, sound-hard, Robin condition,...) strictly included inside the penetrable obstacle.

**Remark 4.** We concentrate on the wave equation but the FM and GLSM might be applied to Electrical Impedance Tomography (EIT) [11]. Despite the similarity between sampling methods for EIT and inverse scattering our proofs do not extend straightforwardly to the EIT equation. As far as we know the case where the contrast changes sign strictly inside is still an open question for EIT.

4. Application to the GLSM and factorization methods.

4.1. Application to the GLSM. We recall that the farfield pattern of the green function $\Phi_z$ is given by,

$$\phi_z(\hat{x}) := e^{-ik\hat{x} \cdot z}$$

and that that lemma 3.1 give a range characterization of $D$. In order to use this range characterization the GLSM framework introduce the cost functional $J_\alpha$, defined for $g \in L^2(\mathbb{R}^{d-1})$ by

$$J_\alpha(\phi_z, g) = \alpha |\langle Fg, g \rangle| + \alpha^{1-\gamma}|\langle Fg - \phi_z, g \rangle| + \|Fg - \phi_z\|^2$$

where $\gamma \in [0, 1]$. We also introduce

$$j_\alpha(\phi_z) = \inf_{g \in L^2(\mathbb{R}^{d-1})} J_\alpha(\phi_z, g).$$

From the results of [2], [1] and [3] (partly based on lemmas 3.3, 3.2 and 3.1), we obtain the following characterization of $D$. 

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Theorem 4.1. Assume that Hypothesis 1 and the hypothesis of theorem 3.7 hold true. For \( z \in \mathbb{R}^d \) let us introduce \( g^\circ \alpha \) such that \( J_\alpha(\phi_z, g^\circ \alpha) \leq J_\alpha(\phi_z) + p(\alpha) \) with \( p(\alpha) = O(\alpha) \).

Then \( z \in D \) if and only if \( \limsup_{\alpha \to 0} |(Fg^\circ \alpha, g^\circ \alpha)| < \infty \). Moreover, we have that the sequence of Herglotz wave functions associated with \( g^\circ \alpha \) converges strongly to the solution \( v \) of (7) with \( (f, g) = (\Phi_z, \frac{\partial \Phi_z}{\partial \nu}) \) as \( \alpha \) goes to zero.

For the noisy case, consider \( F^\delta : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1}) \) a compact operator such that:

\[
\|F^\delta - F\| \leq \delta.
\]

Then consider for \( \alpha > 0 \) and \( \phi \in L^2(\mathbb{S}^{d-1}) \) the functional \( J_\alpha^\delta(\phi, \cdot) : L^2(\mathbb{S}^{d-1}) \to \mathbb{R}, \)

\[
J_\alpha^\delta(\phi, g) = \alpha |(F^\delta g, g)| + \alpha^{1-\gamma} \|g\|^2 + \alpha^{1-\gamma} |(F^\delta g - \phi_z, g)| + \|F^\delta g - \phi_z\|^2,
\]

where \( \gamma \in [0, 1] \). We obtain the following asymptotic characterization of \( D \).

Theorem 4.2. Assume that the hypothesis of the previous theorem hold true. For \( z \in \mathbb{R}^d \) let us denote by \( g^{\circ,\alpha,\delta} \) the minimizer of \( J_\alpha^\delta(\phi_z, \cdot) \) over \( L^2(\mathbb{S}^{d-1}) \).

Then \( z \in D \) if and only if \( \limsup_{\alpha \to 0} \limsup_{\delta \to 0} |(F^\delta g^{\circ,\alpha,\delta}, g^{\circ,\alpha,\delta})| + \alpha^{-\gamma} \|g^{\circ,\alpha,\delta}\|^2 < \infty \).

Moreover, there exists \( \delta_0(\alpha) \) such that for all \( \delta(\alpha) \leq \delta_0(\alpha) \), \( Hg^{\circ,\alpha,\delta(\alpha)} \) converges strongly to the solution \( v \) of (7) with \( (f, g) = (\Phi_z, \frac{\partial \Phi_z}{\partial \nu}) \) as \( \alpha \) goes to zero.

4.2. Application to the factorization method. From [16] we have the following theorem for the factorization method:

Theorem 4.3. For \( F = H^*TH \), assume that:

- \( H^* \) is compact with dense range.
- \( \Im(T) \) is compact and non negative on the range of \( H \).
- \( |\Re(T)| \) is one to one or \( \Im(T) \) is strictly positive on \( \Re(H) \).
- \( |\Re(T)| = C + K \), where \( K \) is compact and \( C \) is a self adjoint coercive operator.

Then the range of the operators \( F_\#^{1/2} \) and \( H^* \) coincide, where \( F_\# = |\Re(F)| + \Im(F) \).

The first three assumptions are direct consequences of lemmas 3.3, 3.2 and 3.1. For the last assumption the application of the Factorization method is more restrictive than the GLSM as it relies on the fact that the real part of \( T \) have to be of the form “coercive + compact”. In section 3 we have proven that \( T \) is actually of the form \( T_0 + K \), where \( K \) is compact and \( T_0 \) extend assumptions on the contrast in a neighborhood of the boundary \( \partial D \) to all \( D \). Therefore for the factorization method to work we need to find a set of assumptions on the contrasts inside all \( D \) that ensure that \( |\Re(T_0)| \) is of the form “coercive+compact”. Such hypothesis for \( D_A \) can be found in [7] (Theorem 4.8) and in [16] for \( D_n \). Those results allow us to state the following theorem,

Theorem 4.4. Assume that Hypothesis 1 holds true and \( A \) has \( C^1 \) regularity in \( D_A \cap V \) and that either conditions apply:

- there exist \( c > 0 \) and \( \alpha > 0 \) such that either \( \Re(A - Id) \geq c > 0 \) in \( \bigcup_i D_A^i \cap V \) and \( \Re((1 - n) \geq c > 0 \) in \( \bigcup_j D_A^j \cap V \)
- there exists \( \alpha \geq 0, c > 0 \) and \( 0 < \eta \leq 1 \) such that \( \Re((n - 1) + \alpha \Re((1 - n) \geq c > 0 \) in \( \bigcup_i D_A^i \cap V \) and \( (1 - \Re(A) - \alpha \Im(A)) \) is positive definite and \( \Re(A) - \frac{1}{\alpha} \Im(A) \geq 0 \) for some \( \alpha > 0 \) in \( \bigcup_i D_A^i \cap V \).

Then \( z \) is inside \( D \) if and only if \( \phi_z \in \Re(F_\#^{1/2}) \).
5. **Numerical experiments.** We restrict ourselves to the two dimensional isotropic case (\(A \text{ is the identity matrix} \)) and will introduce the algorithms for the discrete version of the GLSM and FM. We identify \(S^1 \) with the interval \([0, 2\pi] \). In order to collect the data of the inverse problem we solve numerically (1) for \(N \) incident fields \(u_i(\frac{2\pi j}{N}, \cdot), j \in 0, ..., N - 1 \) using a finite element solver. The discrete version of \(F \) is the matrix \(\mathbf{F} := (u_{\infty}(\frac{2\pi j}{N}, \frac{2\pi k}{N}))_{0 \leq j, k \leq N} \). We add some noise to the data to build a noisy farfield matrix \(\mathbf{F}^\delta \) where \((\mathbf{F}^\delta)_{j,k} = F_{j,k}(1 + \sigma N_{j,k}) \) for \(\sigma > 0 \) and \(N_{j,k} \) an uniform complex random variable in \([-1, 1]^2 \). Similarly we consider the discrete version of the green function \(\Phi_z(j) = \phi_z(\frac{2\pi j}{N}) \) for \(j \in 0, ..., N \).

We apply both the factorization method and the GLSM to kite shape obstacle where with \(n = 0.2 \) except within a disk stricly inside the kite where \(n = 2 \). We choose \(N = 100 \) and a wavelength \(\lambda = \frac{2\pi}{k} = 0.5 \). We fix the regularization parameter \(\alpha \) as explained in [2] for the GLSM and using the Morozov discrepancy principle for the factorization method. Figure 1 shows that there is no significant change in the ability of the methods to reconstruct the inclusion when the contrast changes sign or not. The axes of the figure are measured in \(\lambda \).

![Figure 1](image_url)

**Figure 1.** First line : Factorization method (left) and GLSM (right) without sign changing contrast. Second line : Factorization method (left) and GLSM (right) with sign changing contrast.
REFERENCES

[1] L. Audibert, A. Girard and H. Haddar, Identifying defects in an unknown background using differential measurements, *Inverse Problems and Imaging*, 9 (2015), 625–643.

[2] L. Audibert and H. Haddar, A generalized formulation of the linear sampling method with exact characterization of targets in terms of farfield measurements, *Inverse Problems*, 30 (2014), 035011, 20pp.

[3] L. Audibert and H. Haddar, The generalized linear sampling method for limited aperture measurements, *SIAM J. Imaging Sci.*, 10 (2017), 845–870.

[4] A.-S. Bonnet-Ben Dhia, L. Chesnel and H. Haddar, On the use of T-coercivity to study the interior transmission eigenvalue problem, *C. R. Math. Acad. Sci. Paris*, 349 (2011), 647–651.

[5] F. Cakoni and D. Colton, *Qualitative Methods in Inverse Scattering Theory*, Interaction of Mechanics and Mathematics. Springer-Verlag, Berlin, 2006. An introduction.

[6] F. Cakoni, D. Colton and H. Haddar, *Inverse Scattering Theory and Transmission Eigenvalues*, volume 88 of CBMS Series, SIAM publications, 2016.

[7] F. Cakoni and I. Harris, The factorization method for a defective region in an anisotropic material, *Inverse Problems*, 31 (2015), 025002, 22pp.

[8] H. Haddar, F. Cakoni and S. Meng, Boundary integral equations for the transmission eigenvalue problem for axwell equations, *J. Integral Equations Appl.*, 27 (2015), 375–406.

[9] A. Cossonnière and H. Haddar, Surface integral formulation of the interior transmission problem, *J. Integral Equations Appl.*, 25 (2013), 341–376.

[10] L. Evgeny and L. Armin, Monotonicity in inverse medium scattering,

[11] B. Gebauer, The factorization method for real elliptic problems, *Zeitschrift für Analysis und ihre Anwendungen*, 25 (2006), 81–102.

[12] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[13] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, volume 24 of Monographs and Studies in Mathematics, Pitman (Advanced Publishing Program), Boston, MA, 1985.

[14] A. Kirsch, The factorization method for a class of inverse elliptic problems, *Mathematische Nachrichten*, 278 (2005), 258–277.

[15] A. Kirsch, A note on Sylvester’s proof of discreteness of interior transmission eigenvalues, *Comptes Rendus Mathématique*, 354 (2016), 377–382.

[16] A. Kirsch and N. Grinberg, *The Factorization Method for Inverse Problems*, volume 36 of Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, Oxford, 2008.

[17] J. Sylvester, Discreteness of transmission eigenvalues via upper triangular compact operators, *SIAM J. Math. Anal.*, 44 (2012), 341–354.

[18] J. Yang, B. Zhang and H. Zhang, The factorization method for reconstructing a penetrable obstacle with unknown buried objects, *SIAM Journal of Applied Mathematics*, 73 (2013), 617–635.

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