MINIMAL POLYNOMIALS OF SIMPLE HIGHEST WEIGHT MODULES OVER CLASSICAL LIE ALGEBRAS

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Abstract. We completely determine the minimal polynomial of an arbitrary simple highest weight module $L(\lambda)$ over a complex classical Lie algebra $g \subseteq \mathfrak{gl}_N$ relative to its defining module $\pi = \mathbb{C}^N$. These results are applied to ordering on primitive ideals and algebraic properties of Howe duality correspondence.

1. Introduction

Let $g$ be a complex reductive Lie algebra and $\pi$ be a finite-dimensional $g$-module. The minimal polynomial of a $g$-module $V$ is a polynomial $q_{\pi,V}$ in one variable which records certain matrix identities involving generators of the Lie algebra $g$ acting on $V$. Minimal polynomials were explicitly introduced by Oshima in [13], but their theory goes back to the work of Bracken and Green, O’Brien, Cant and Carey, and Gould on characteristic identities, see [2] and the references therein. We consider minimal polynomials in the setting of a classical Lie algebra $g \subseteq \mathfrak{gl}_N$ with the defining module $\pi = \mathbb{C}^N$ and completely determine the minimal polynomial $q_{\pi,V}$ for an arbitrary simple highest weight module $V = L(\lambda)$. Here $g = \mathfrak{gl}_N, \mathfrak{o}_N$, or $\mathfrak{sp}_N$ (in the symplectic case, $N$ is even). This generalizes many earlier results: an explicit formula for $q_{\pi,V}$ when $V = M(\lambda)$ is a Verma module, [2], and Oshima and Oda’s description of the minimal polynomials of $V = M_p(\lambda)$, the generalized Verma module of scalar type (under some technical restrictions), [11].

To state our results in full would require more notation, but they are easy to describe in a qualitative form. The roots of the minimal polynomial of $L(\lambda)$ are expressed in terms of the shuffle decomposition, a combinatorial object associated to the highest weight $\lambda$. In the general linear case, it is described via a simplified version of the Robinson–Schensted–Knuth algorithm. The symplectic and orthogonal cases are analogous, but somewhat more involved. The defining module plays a very distinguished role in invariant theory and representation theory of a classical Lie group $G$: it is in many ways the smallest and the simplest module, and it is very useful in constructing and studying finite-dimensional $G$-modules. Our results may be viewed as a manifestation of this principle in the context of minimal polynomials and (infinite-dimensional) highest weight modules over the Lie algebra $g$.

There are several important reasons to study minimal polynomials: they are related to explicit description of primitive ideals in $U(g)$, noncommutative Capelli identities, the effect of Howe duality on ideals, [10] and Section 6, the ordering on the primitive ideals and the multiplicity of the adjoint representation in the primitive quotients of $U(g)$, Section 7. It is easy to see that for a fixed $\pi$, the minimal polynomial of a module $V$ depends only on its annihilator ideal $I = \text{Ann}_{U(g)}V$ in

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the universal enveloping algebra \( U(\mathfrak{g}) \). Hence it makes sense to speak about the minimal polynomial \( q_I \) of an ideal. Moreover, if \( I \subset J \) is an inclusion of ideals then the polynomial \( q_I \) divides \( q_J \). By Duflo’s theorem, the annihilator of a simple highest weight module \( L(\lambda) \), but for a singular highest weight \( \lambda \), the minimal polynomial can be a stronger invariant.

Another motivation for studying the minimal polynomial comes from the theory of the reductive dual correspondence (or Howe duality). The relation between the centers of the universal enveloping algebras of two Lie algebras forming a reductive dual pair is fairly well understood through the theory of Capelli identities. It has important applications to harmonic analysis on reductive Lie groups. In we have studied a more general situation: the effect of Howe duality on a general, not necessarily centrally generated, ideal. Here we prove a simple relation between the minimal polynomials of the modules \( V \) and \( V' \) that correspond to each other under an algebraic version of the Howe duality.

This paper is organized as follows. In Section 2 we collect some mostly known facts about the Harish-Chandra projection map in the relative setting. The main result there, Proposition 2.5, is a criterion to determine whether a given \( ad(\mathfrak{g}) \)-invariant subspace \( T \subset U(\mathfrak{g}) \) annihilates \( L(\lambda) \). In Section 3 we define the minimal polynomial, Definition 3.8, and explain our approach to computing it through the use of formal resolvents, Proposition 3.9. The next two sections contain our main results, first in the general linear case, Theorem 4.15 and then in the symplectic and orthogonal cases, Theorem 5.10. A key role in our analysis is played by a certain relative Harish-Chandra projection map that allows us to relate minimal polynomials for modules over \( \mathfrak{gl}_n \) and \( \mathfrak{gl}_{n-1} \) and apply induction. For an irreducible reductive dual pair of Lie algebras \((\mathfrak{g}, \mathfrak{g}')\), we establish in Section 6 a relation between the minimal polynomials of modules over \( \mathfrak{g} \) and \( \mathfrak{g}' \) in an algebraic Howe duality with each other. Section 7 is devoted to applications to primitive ideals.

2. Harish-Chandra projection

In this paper, the ground field is \( \mathbb{C} \), although many results extend to the case of an arbitrary field \( \mathbb{K} \) of characteristic zero. The case of the field \( \mathbb{C}(t) \) of rational functions in several variables seems especially promising for applications to the study of prime ideals in enveloping algebras. Such extensions hold under assumption that Lie algebras are split over \( \mathbb{K} \), with little change in the proofs.

Let \( \mathfrak{g} \) be a split reductive Lie algebra, \( \mathfrak{h} \) a Cartan subalgebra, \( \mathfrak{R} \subset \mathfrak{h}^* \) be the root system of \((\mathfrak{g}, \mathfrak{h})\) and \( \mathfrak{g}_\alpha \) be the root subspace corresponding to \( \alpha \in \mathfrak{R} \). Choose a system of simple roots \( \mathfrak{B} \subset \mathfrak{R} \) and denote by \( \mathfrak{R}^+ \) and \( \mathfrak{R}^- \) the corresponding subsets of \( \mathfrak{R} \) consisting of positive and negative roots. For any subset \( S \subset \mathfrak{B} \) we let \( \mathfrak{R}_S = \mathbb{Z}S \cap \mathfrak{R} \). Then \( \mathfrak{R}_S \) is a root system and we consider the following subspaces of \( \mathfrak{g} \):

\[
\mathfrak{n}^+_S = \bigoplus_{\alpha \in \mathfrak{R}^+ \setminus \mathfrak{R}_S} \mathfrak{g}_\alpha, \quad \mathfrak{n}^-_S = \bigoplus_{\alpha \in \mathfrak{R}^- \setminus \mathfrak{R}_S} \mathfrak{g}_\alpha, \quad \mathfrak{m}_S = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{R}_S} \mathfrak{g}_\alpha.
\]
It is well known that \( p_S = m_S \oplus n_S^+ \) is a standard parabolic of \( g \) and all standard parabolics arise in this way. In particular, for \( S = \emptyset \) we get \( m_S = h, p_S = b \) is the Borel subalgebra, and \( n_S^+ = n \) is its nilradical.

Denote by \( U(g) \) the universal enveloping algebra of \( g \). The Lie algebra \( g \) has a triangular decomposition \( g = n_S^+ \oplus m_S \oplus n_S^- \) and the Poincare–Birkhoff–Witt (PBW) theorem gives rise to a canonical isomorphism

\[
U(g) \simeq (n_S^- U(g) + U(g)n_S^+) \oplus U(m_S).
\]

Let \( Z \) be the center of the Levi factor \( m_S \). Then \( Z \) is the subspace of \( h \) which is the annihilator of \( S \subset h^* \). It is clear that the adjoint action of \( Z \) on \( U(g) \) is semisimple, and the subspace of invariants under this action is a subalgebra \( U(g)^Z \subset U(g) \) spanned by the \( ad(h) \)-weight vectors of \( U(g) \) whose weight belongs to \( Z \).

**Definition 2.2.** The relative Harish-Chandra projection map \( pr_{g/m} : U(g) \to U(m) \) is the projection onto the second summand in the decomposition (2.1).

The map \( pr_{g/h} \), which we call the (absolute) Harish-Chandra projection map, is well known in representation theory. Its restriction to the center \( Z(g) \) of the universal enveloping algebra \( U(g) \) is an injective algebra homomorphism from \( Z(g) \) to \( U(h) \) whose image consists of \( W \)-invariants in \( U(h) \) with respect to the shifted action of the Weyl group \( W \). Composing \( pr_{g/h} \) with the algebra automorphism of \( U(h) \simeq P(h^*) \) induced by the \( \rho \)-shift, one obtains the Harish-Chandra homomorphism, in fact, an isomorphism between \( Z(g) \) and the algebra \( P(h^*)^W \) of \( W \)-invariant polynomial functions on \( h^* \). The Harish-Chandra homomorphism arises from the action of \( Z(g) \) on highest weight modules as follows. Let \( M \) be a highest weight module over \( g \) with highest weight vector \( v_\lambda \). Thus \( M \) is generated by \( v_\lambda \) as a \( U(g) \)-module, \( n v_\lambda = 0 \), and \( hv_\lambda = \lambda(h)v_\lambda \) for \( h \in h \). Denote by \( M_\lambda \) the linear span of all weight vectors in \( M \) of weight lower than \( \lambda \), then \( M = C v_\lambda \oplus M_\lambda \). Let \( v^* \) be the linear functional on \( M \) which vanishes on \( M_\lambda \) and such that \( \langle v^*, v \rangle = 1 \). Then for any element \( u \in U(g) \), we have an equality

\[
\langle v^*, u v_\lambda \rangle = \langle v^*, pr_{g/h}(u) v_\lambda \rangle.
\]

The associative algebra \( U(h) \) is canonically identified with the commutative algebra of polynomial functions on \( h^* \), and we denote by \( ev_\lambda \) the homomorphism from \( U(h) \) to \( \mathbb{C} \) which is the evaluation at \( \lambda \in h^* \). The identity (2.3) says that for any \( u \in U(g) \), the coefficient of \( v_\lambda \) in the expansion of \( uv_\lambda \) with respect to a basis of \( M \) consisting of weight vectors is \( ev_\lambda(pr_{g/h}(u)) \).

A few elementary properties of the relative projection map are collected in the following proposition (cf. [7], 5.12).

**Proposition 2.4.** Fix a subset \( S \subset B \) and let \( m = m_S \).

1. The map \( pr_{g/m} \) is a homomorphism of \( (U(m), U(m)) \)-bimodules. Its restriction to the subalgebra \( U(g)^Z \subset U(g) \) is an algebra homomorphism.

2. If \( u \in n_S^- U(g) \) or \( u \in U(g)n_S^- \) then \( pr_{g/m}(u) = 0 \).

3. If \( u \in U(g) \) is an \( ad(h) \)-eigenvector of weight \( \mu \) and \( \mu \neq ZS \) then \( pr_{g/m}(u) = 0 \).

**Proof.** Since the adjoint action of \( m_S \) on \( U(g) \) stabilizes \( n_S^-, n_S^+ \), and \( m_S \), we see that both summands in the decomposition (2.1) are invariant under the left and
right multiplication by $m_S$, hence they are $(U(m_S), U(m_S))$ subbimodules of $U(g)$. Assertion (1) follows.

Assertion (2) is immediate from the definition. Since both summands in the decomposition (2.1) are also $ad(h)$-invariant, $pr_{g/m}$ is an $ad(h)$-equivariant map. The weights of $ad(h)$ on $U(m_S)$ are precisely $ZS$, which implies assertion (3). (When $S = \emptyset$, we let $ZS = \{0\}$.)

For any $\lambda \in h^*$ there is a unique up to an isomorphism simple highest weight module $L(\lambda)$ with highest weight $\lambda$. The next proposition gives a powerful criterion for determining whether an $ad(g)$-invariant subspace of $U(g)$ annihilates $L(\lambda)$.

**Proposition 2.5.** Let $P \subseteq U(g)$ be an $ad(g)$-invariant subspace, $P_0$ the zero weight space of $P$, and $\lambda \in h^*$ a highest weight. Then the following conditions are equivalent:

1. $P$ annihilates $L(\lambda)$;
2. $pr_{g/h}(p)$ vanishes at $\lambda$ for all $p \in P$;
3. $pr_{g/h}(p)$ vanishes at $\lambda$ for all $p \in P_0$.

**Proof.** Clearly, (2) implies (3). We will prove that (3) implies (1) and (1) implies (2). Suppose that $P$ annihilates $L(\lambda)$, then it follows in particular that $Pv_\lambda = 0$. From (2.1) $pr_{g/h}(P)v_\lambda = 0$, therefore (2) holds.

The $ad(g)$-invariance of $P$ implies that for any subspace $a \subseteq g, aP = Pa + P$. By the Poincare–Birkhoff–Witt theorem, $U(g) = U(n^-)U(h)$, so the two-sided ideal $\langle P \rangle = U(g)PU(g)$ generated by the subspace $P$ is equal to $U(n^-)PU(h)$. Applying the elements of this ideal to the highest weight vector $v_\lambda$, we find that $\langle P \rangle L(\lambda) = U(g)PU(g)v_\lambda = U(n^-)PU(h)v_\lambda = U(n^-)Pv_\lambda$.

Now let us decompose $P$ as $P_+ \oplus P_0 \oplus P_-$, where $P_+$ (respectively, $P_-$) is spanned by the positive (respectively, negative) $ad(h)$-weight vectors in $P$. The action of $P_+$ annihilates $v_\lambda$, so $Pv_\lambda = pr_{g/h}(P_0)v_\lambda + L(\lambda)_c$. Suppose that condition (3) holds, so that $pr_{g/h}(P_0)$ acts by 0 on $v_\lambda$, then the submodule $\langle P \rangle L(\lambda)$ of $L(\lambda)$ is contained in $L(\lambda)_c$. Therefore, it is a proper submodule of a simple module, implying that $\langle P \rangle$ annihilates $L(\lambda)$.

The second condition of the proposition says that $\lambda$ is in Joseph’s “characteristic variety” of $P$. In a different language, the weight $\lambda$ appears in the Jacquet functor of the Harish-Chandra bimodule $U(g)/\langle P \rangle$ with respect to the nilradical $(n^-, n^+) \subset (g, g)$, cf [8, 18].

Another important property of Harish-Chandra projections is the composition formula.

**Proposition 2.6.** Let $S \subseteq B$ and suppose that the root system $RS$ admits a basis (i.e. a system of simple roots) $BS \subseteq B$. If $S' \subseteq S$ then we may construct a Levi subalgebra $m' = m_S$, of the reductive Lie algebra $m$ as above and define the Harish-Chandra projection $pr_{m/m'} : U(m) \rightarrow U(m')$. Moreover, $m'$ is also the Levi subalgebra of the Lie algebra $g$ itself associated with the subset $S' \subseteq B$, so that the map $pr_{g/m'} : U(g) \rightarrow U(m')$ is defined. Then the following composition formula holds:

$$pr_{m/m'} \circ pr_{g/m} = pr_{g/m'}.$$  

**Proof.** This follows immediately from the definitions. □
3. Minimal Polynomials and Formal Resolvents

We begin with a very general notion of minimal polynomial for the elements of an associative, but possibly noncommutative, algebra.

**Definition 3.1.** Let $R$ be an associative algebra over $C$. For any element $a \in R$, the non-zero monic polynomial $q \in C[u]$ of minimal degree such that $q(a) = 0$ is called the minimal polynomial of the element $a$.

It is clear that all polynomials $f \in C[u]$ annihilating a fixed element $a \in R$ form a (possibly zero) ideal. Since the ring of polynomials in one variable over a field is a principal ideal domain, we conclude that $f(a) = 0$ holds for a polynomial $f \in C[u]$ if and only if $f$ is divisible by the minimal polynomial of $a$. In particular, if a minimal polynomial exists, it is unique.

This notion of minimal polynomial, which can be defined for associative algebras over any field, simultaneously generalizes minimal polynomials in field theory and minimal polynomials of matrices.

**Definition 3.2.** Let $a$ be an element of an associative algebra $R$. We call the formal power series

$$T_a(u) = (u - a)^{-1} = u^{-1} \sum_{k=0}^{\infty} a^k u^{-k} \in R[[u^{-1}]]$$

the formal resolvent of $a$.

There is a well known characterization of the minimal polynomial in terms of the formal resolvent.

**Proposition 3.4.** Let $a \in R$ and $p \in C[u]$ be a polynomial in one variable. Then $p(a) = 0$ if and only if $p(u)T_a(u)$ is a polynomial in $u$ (a priori, it is only a Laurent formal power series in $u^{-1}$). Moreover, both these conditions are equivalent to the vanishing of the coefficient of $u^{-1}$ in the expansion of $p(u)T_a(u)$.

**Proof.** It is well known (and easy to check) that $(p(u)p(v))(u-v)^{-1}$ is a polynomial in $u$ and $v$. Therefore, $p(u)T_a(u) = p(u)(u-a)^{-1} = p(a)(u-a)^{-1} + (a$ polynomial in $u)$. From the expansion of $(u-a)^{-1}$ in powers of $u^{-1}$ we find that the coefficient of $u^{-k-1}$ in $p(u)T_a(u)$ is $p(a)a^k$. This proves the proposition. □

**Corollary 3.5.** An element $a \in A$ admits a minimal polynomial if and only if its formal resolvent $T_a(u)$ is a rational function of $u$ with denominator in $C[u]$.

We are particularly interested in the case where $R$ is the algebra of $N \times N$ matrices over another algebra $A$.

**Corollary 3.6.** Suppose that $a$ is an $N \times N$ matrix over $A$. Then the minimal polynomial of $a$, viewed as an element of $R = M_N(A)$, is the monic least common denominator in $C[u]$ of the entries of the matrix $T_a(u)$.

**Remark 3.7.** We may interpret the proposition as saying that the multiset of the roots of the minimal polynomial of $a$ coincides with the multiset of the poles of the formal resolvent of $a$, where both the roots and the poles are considered with their multiplicities. This formulation has important applications to functional calculus in Banach algebras.
The following construction is due to Oshima in [13], cf. [2][11]. Fix a Lie algebra $\mathfrak{g}$, an $N$-dimensional $\mathfrak{g}$-module $\pi$ with a basis $\{v_1, \ldots, v_N\}$, and a $\mathfrak{g}$-equivariant linear map $p : \text{End } \pi \to U(\mathfrak{g})$. Then the elements $p(E_{ij}), 1 \leq i, j \leq N$ (where $E_{ij}$ are the ordinary matrix units) can be arranged into an $N \times N$ matrix $F_\pi \in M_N(U(\mathfrak{g}))$. Let $V$ be a $\mathfrak{g}$-module and denote by $F_{\pi,V}$ the image of the matrix $F_\pi$ in the associative algebra $M_N(\text{End } V)$ induced by the natural homomorphism $U(\mathfrak{g}) \to \text{End } V$.

**Definition 3.8.** With the assumptions above, the minimal polynomial $q_{\pi,V}$ is the minimal polynomial of the matrix $F_{\pi,V}$.

Gould proved in [2] that if $\mathfrak{g}$ is a semisimple Lie algebra then there exists “universal” polynomial with coefficients in $Z(\mathfrak{g})$ annihilating $F_{\pi,V}$. It follows that if a module $V$ has an infinitesimal character then it admits a minimal polynomial in the sense of Definition 3.1 i.e. with coefficients in $C$.

Our approach to computing the minimal polynomial of a simple highest weight module $L(\lambda)$ is based on the combination of the criterion of Proposition 3.9, which allows one to determine whether the $ad$-invariant subspace of $U(\mathfrak{g})$ spanned by the entries of the matrix $f(F_\pi)$ annihilates $L(\lambda)$ by considering the Harish-Chandra projection, and of the relation between the minimal polynomial and the poles of the formal resolvent of a matrix given in Proposition 3.4. Let us summarize these remarks in a proposition.

**Proposition 3.9.** Fix a weight $\lambda \in \mathfrak{h}^*$. Denote by $pr : U(\mathfrak{g}) \to \mathbb{C}$ the composition of the Harish-Chandra projection map $pr_{\mathfrak{g}/\mathfrak{h}}$ and the evaluation at $\lambda$. Then the roots of the minimal polynomial of $L(\lambda)$ coincide with the poles of the Laurent formal power series $pr_{\mathfrak{g}/\mathfrak{h}}(u) \in \mathbb{C}((u^{-1}))$, preserving the multiplicities. Moreover, a polynomial $f(u)$ is divisible by the minimal polynomial of $L(\lambda)$ if and only if the coefficient of $u^{-1}$ in the formal power series expansion $f(u)pr_{\mathfrak{g}/\mathfrak{h}}(u) \in \mathbb{C}((u^{-1}))$ is 0. In this case, the expansion reduces to a polynomial in $u$.

We now have in our possession a convenient technique of computing minimal polynomials of simple highest weight modules. However, an important obstacle remains: the Harish-Chandra projection map $pr_{\mathfrak{g}/\mathfrak{h}}$ is notoriously difficult to compute explicitly. This reflects upon the complexity of the structure of the primitive spectrum. At this juncture, we specialize to $\pi = \mathbb{C}^N$, the defining module for $\mathfrak{g}$, for a symplectic or orthogonal Lie algebra $\mathfrak{g}$ and the contragredient $\mathbb{C}^N^*$ for a general linear algebra $\mathfrak{g} = \mathfrak{gl}_N$. In these cases we are able to employ the relative Harish-Chandra projections arising from the canonical chain of reductive Lie subalgebras of $\mathfrak{g}$ of decreasing rank (as in the Gelfand–Tsetlin construction and its analogues for symplectic and orthogonal Lie algebras) and compute the absolute projection in an inductive fashion using the composition formula (2.7).

**Remark 3.10.** The arguments just given let one hope that similar computations can be carried out for any weight multiplicity-free module $\pi$. Such modules exist for semisimple Lie algebras of all types except $F_4$ and $E_8$. For the case of a general linear algebra, we have successfully applied this technique to determination of the quantized elementary divisors, see Section 6 and [15].
4. Minimal polynomials: general linear case

Let \( \mathfrak{g} = \mathfrak{gl}_n \) be the Lie algebra of \( n \times n \) matrices. The algebra \( \mathfrak{g} \) acts on its defining module \( \mathbb{C}^n \), an \( n \)-dimensional complex vector space with a basis \( \{v_1, \ldots, v_n\} \). A basis of \( \mathfrak{gl}_n \) is given by the matrix units \( E_{ij} \), \( 1 \leq i, j \leq n \). They satisfy the following commutation relation:

\[
\{E_{ij}, E_{kl}\} = \delta_{jk} E_{il} - \delta_{il} E_{kj}.
\]

We arrange the generators \( \{E_{ij}\} \) into a single \( n \times n \) matrix \( E \in M_n(U(\mathfrak{gl}_n)) \):

\[
E = \begin{bmatrix}
E_{11} & E_{12} & \cdots & E_{1n} \\
E_{21} & E_{22} & \cdots & E_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
E_{n1} & E_{n2} & \cdots & E_{nn}
\end{bmatrix} \in M_n(U(\mathfrak{gl}_n)).
\]

Choose a triangular decomposition \( \mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^- \), where \( \mathfrak{n}^+ \) is spanned by \( \{E_{ij} : i < j\} \), \( \mathfrak{n}^- \) is spanned by \( \{E_{ij} : i > j\} \) and \( \mathfrak{h} \) is spanned by \( \{E_{ii}\} \). The elements \( H_1 = E_{11}, \ldots, H_n = E_{nn} \) form a basis of \( \mathfrak{h} \), and we identify the weights \( \lambda \in \mathfrak{h}^* \) with the \( n \)-tuples \( (\lambda_1, \ldots, \lambda_n) \) of their values on the elements of this basis:

\[
\lambda_i = \lambda(H_i) = \lambda(F_{ii}).
\]

We also set

\[
\rho = (n-1, \ldots, 0), \quad \lambda = \lambda + \rho = (\lambda_1 + \rho_1, \ldots, \lambda_n + \rho_n), \quad \text{where} \quad l_i = \lambda_i + n - i.
\]

Denote by \( \mathcal{T}(u) \) the formal resolvent of \( E \) (Definition 3.2), viewed as an element of the algebra of \( n \times n \) matrices over \( U(\mathfrak{gl}_n) \). Then

\[
\mathcal{T}(u) = (uI_n - E)^{-1} = u^{-1} + \sum_{k \geq 1} E^k u^{-k-1} \in M_n(U(\mathfrak{gl}_n))[u^{-1}].
\]

**Proposition 4.6.** The entries of the matrix \( \mathcal{T}(u) \) transform under the adjoint action of \( \mathfrak{gl}_n \) in the same way as the entries of the matrix \( E \). More explicitly,

\[
\{E_{ij}, \mathcal{T}_{kl}(u)\} = \delta_{jk} T_{il}(u) - \delta_{il} T_{kj}(u).
\]

**Proof.** This follows from the standard commutator identity

\[
[x, (X^{-1})_{kl}] = - \sum_{p,q} (X^{-1})_{kp} [x, X_{pq}](X^{-1})_{ql},
\]

where \( X \) is an invertible \( n \times n \) matrix with entries in an associative algebra, \( x \) is an element of the same algebra, and the indices \( l, m, p, q \) are between 1 and \( n \). We take \( x = E_{ij}, X = uI_n - E \), and use the commutation relations (4.1). A more conceptual proof can be given using the adjoint action of the group \( GL_n \) on the linear span of the matrix entries of the \( n \times n \) matrix \( uI_n - E \) and of its inverse \( \mathcal{T}(u) \). \( \square \)

Our goal is to determine the minimal polynomial of an arbitrary simple highest weight \( \mathfrak{gl}_n \)-module following the strategy outlined in the previous section. We begin by computing some relative Harish-Chandra projections with respect to the maximal proper standard parabolic subalgebra with the Levi factor \( \mathfrak{gl}_{n-1} \oplus \mathfrak{gl}_1 \).

Let \( S \) be the complement of the last simple root in the root system \( B \) of \( \mathfrak{gl}_n \) (in the usual enumeration). The parabolic subalgebra \( \mathfrak{p}_S = \mathfrak{n}^+_S \oplus \mathfrak{m}_S \) is the \( (n - 1) \)st standard maximal parabolic of \( \mathfrak{gl}_n \) and consists of the matrices stabilizing the codimension one subspace of \( \mathbb{C}^n \) spanned by the first \( n - 1 \) standard basis vectors. We have a triangular decomposition \( \mathfrak{g} = \mathfrak{n}^+_S \oplus \mathfrak{m}_S \oplus \mathfrak{n}^-_S \), where the Levi factor
\( m_S = \mathfrak{gl}_{n-1} \oplus \mathfrak{gl}_1 \) is spanned by \( E_{ij} \) (1 \( \leq i, j \leq n-1 \)) together with \( E_{nn} \), the upper nilradical \( n_S^+ \) is spanned by \( E_{i,n} \) (1 \( \leq i \leq n-1 \)) and the lower nilradical \( n_S^- \) is spanned by \( E_{n,j} \) (1 \( \leq j \leq n-1 \)).

Denote by \( E' \) the upper left corner \((n-1) \times (n-1)\) submatrix of \( E \) formed by the elements \( E_{ij} \) with 1 \( \leq i, j \leq n-1 \) and by \( T'(u) \) its formal resolvent, cf. (1.5).

Let us write \( \lambda = (\lambda', \lambda_n) \), so that \( \lambda'_p = \lambda_p \) for 1 \( \leq p \leq n-1 \).

Suppose that a weight \( \lambda \in \mathfrak{h}^* \) has been fixed. Let \( pr : U(\mathfrak{gl}_n) \to \mathbb{C} \) be the composition of the absolute Harish-Chandra projection \( pr_{/\mathfrak{h}} \) and the evaluation homomorphism \( ev_\lambda : U(\mathfrak{h}) \to \mathbb{C}, H_i \mapsto \lambda_i \). Let the map \( pr' : U(\mathfrak{gl}_{n-1}) \to \mathbb{C} \) have a similar meaning with \( \mathfrak{gl}_{n-1} \) in place of \( \mathfrak{gl}_n \). Consider the “relative” map \( pr_n : U(\mathfrak{gl}_n) \to U(\mathfrak{gl}_{n-1}) \), which is the composition of the relative Harish-Chandra projection \( pr_{\mathfrak{gl}/m_S} : U(\mathfrak{gl}_n) \to U(\mathfrak{gl}_{n-1}) \otimes U(\mathfrak{gl}_1) \) and the partial evaluation homomorphism \( H_n \mapsto \lambda_n \). By Proposition (2.6) these maps are related by the composition formula \( pr = pr' \circ pr_n \).

**Proposition 4.9.** The following formulas describe the effect of the relative projection map \( pr_n \) on the matrix series \( T(u) \):

\[
pr_n T_{nn}(u) = \frac{1}{u - \lambda_n}, \quad pr_n T_{ij}(u) = (1 - \frac{1}{u - \lambda_n}) T'_{ij}(u-1), 1 \leq i, j \leq n-1.
\]

**Proof.** Since the matrices \( T(u) \) and \( uI_n - E \) are mutually inverse, we have:

\[
\sum_{k=1}^{n} T_{ik}(u)(u\delta_{kj} - E_{kj}) = \delta_{ij}.
\]

First, let \( i = j = n \). Splitting the sum into two parts, corresponding to whether \( 1 \leq k \leq n-1 \) or \( k = n \) and applying \( pr_n \) to both sides, we get:

\[
- \sum_{p=1}^{n-1} pr_n(T_{ip}(u)E_{pn}) + pr_n(T_{nn}(u)(u - E_{nn})) = 1.
\]

Note that since \( E_{pn} \in n_S^+ \), all terms in the first sum in the left hand side vanish, and taking into account that \( E_{nn} \in m_S \) and \( pr_n E_{nn} = \lambda_n \), we obtain \( pr_n T_{nn}(u)(u - \lambda_n) = 1 \), proving the first formula.

Now suppose that \( 1 \leq i, j \leq n-1 \). Splitting the sum into two parts as before and applying \( pr_n \), we get:

\[
\sum_{p=1}^{n-1} pr_n(T_{ip}(u)(u\delta_{pj} - E_{pj})) - pr_n(T_{in}(u)E_{nj}) = \delta_{ij}.
\]

Using the commutation relations (4.7), we may replace \(- T_{in}(u)E_{nj} \) with \(- E_{nj} T_{in}(u) + [E_{nj}, T_{in}(u)] \) = \(- E_{nj} T_{in}(u) + \delta_{ij} T_{nn}(u) - T_{ij}(u) \). Since \( E_{nj} \in n_S \), \( pr_n(E_{nj} T_{in}(u)) = 0 \) and after routine transformations, we obtain:

\[
\sum_{p=1}^{n-1} pr_n(T_{ip}(u)((u - 1)\delta_{pj} - E_{pj}) = \delta_{ij}(1 - pr_n T_{nn}(u)).
\]

Substituting the explicit form of \( pr_n T_{nn}(u) \) just proved, this may be restated in the matrix form as follows:

\[
((u - 1)I_{n-1} - E')(pr_n T(u))' = (1 - \frac{1}{u - \lambda_n})I_{n-1}.
\]
Multiplying both sides by $T'(u - 1)$ on the left, we obtain the desired formula for $pr T_{ij}(u), 1 \leq i, j \leq n - 1$.

In order to formulate our main result for the general linear Lie algebra, we need to introduce certain combinatorial objects.

**Definition 4.12.** A **falling sequence** is an arithmetic progression with the step $-1$.

Next we construct a canonical decomposition of an arbitrary sequence $l$ into a disjoint union (or shuffle) of falling subsequences, called the **shuffle decomposition**. (We shall refer to these subsequences as the **parts** of the shuffle decomposition.) This decomposition is best described by an algorithm. For an empty sequence the shuffle decomposition is empty. Suppose that the shuffle decomposition for a sequence $l' = (l_1, \ldots, l_{n-1})$ has already been constructed. Let $S$ be its longest part ending in $l_n + 1$. Then the shuffle decomposition of $l = (l_1, \ldots, l_n)$ is obtained by appending $l_n$ to the end of $S$, and leaving the other parts intact. If no such (non-empty) $S$ exists, we create a new part consisting of a single term $l_n$. It is clear that all parts constructed by this algorithm are going to be falling sequences, and the decomposition has a certain minimality property.

**Example 4.13.** Let $l = (3, 3, 2, 4, 1, 3, 2, 2, 1)$. Reading the sequence $l$ from left to right, we easily obtain the shuffle decomposition:

$$
\{3\} \to \{3\} \sqcup \{3\} \to \{3, 2\} \sqcup \{3\} \to \{3, 2\} \sqcup \{3\} \sqcup \{4\} \to \{3, 2, 1\} \sqcup \{3\} \sqcup \{4\} \to \{3, 2, 1\} \sqcup \{3, 2\} \sqcup \{4, 3, 2\} \to \{3, 2, 1\} \sqcup \{3, 2\} \sqcup \{3, 2, 1, 4, 3, 2\}.
$$

Note that on the last step both $\{4, 3, 2\}$ and $\{3, 2\}$ end in 2, but we append 1 to the former subsequence because it is the longer of the two.

**Remark 4.14.** The shuffle decomposition of $l$ can also be constructed by reading its components from right to left.

The following theorem completely determines the minimal polynomial of a simple highest weight $\mathfrak{gl}_n$-module.

**Theorem 4.15.** Let $L(\lambda)$ be the simple highest weight $\mathfrak{gl}_n$-module with highest weight $\lambda, l = \lambda + \rho$, and the multiset $A$ consist of the last (i.e. the smallest) terms of the parts of the shuffle decomposition of the $n$-element sequence $l$. Then $A$ is the multiset of roots of the minimal polynomial of $L(\lambda)$.

**Proof.** By Proposition 3.9 we need to relate the locations and the maximum orders of the poles of $pr T_{ij}(u), 1 \leq i, j \leq n$ to the shuffle decomposition of the $n$-element sequence $l = \lambda + \rho$. This is done by induction in $n$ using the formulas (4.10) for the relative map $pr_n$ and the composition formula $pr = pr' \circ pr_n$.

From the first formula (4.10) we see that the minimal polynomial of $L(\lambda)$ must be divisible by $u - \lambda_n$. Let $f(u) = (u - \lambda_n)g(u)$. Suppose that $1 \leq i, j \leq n - 1$, then from the second formula (4.10) we get:

$$
(4.16) \quad f(u)pr T_{ij}(u) = (u - \lambda_n)g(u)pr' (pr_n T_{ij}(u)) = (u - 1 - \lambda_n)g(u)pr' T_{ij}'(u - 1).
$$

By the inductive assumption, the poles of $pr' T_{ij}'(u)$ are the last terms of the parts of the shuffle decomposition of $\lambda' + \rho' = (\lambda_1 + n - 2, \ldots, \lambda_{n-1})$, thus the poles of $pr' T_{ij}'(u - 1)$ are the last terms of the parts of the shuffle decomposition of $l' = (\lambda_1 + n - 1, \ldots, \lambda_{n-1} + 1)$. There are two cases to consider: either one of these...
parts, which we call $S$, ends in $\lambda_n + 1$, or none of them does. In the former case, we append $\lambda_n$ to the end of $S$, changing the location of the corresponding pole from $\lambda_n + 1$ to $\lambda_n$. In the latter case, we create a new part consisting of $\lambda_n$. In either case, we see from (4.10) and the algorithm for constructing the shuffle decomposition that as we pass from $l'$ to $l$, the relation between the shuffle decomposition and the poles of the formal resolvent remains unaffected. □

Remark 4.17. One may ask whether the other elements of the shuffle decomposition of $l = \lambda + \rho$ admit a similar interpretation. The answer is affirmative and is related to the notion of quantized elementary divisors of the module $L(\lambda)$, see [14].

5. Minimal Polynomials: Symplectic/Orthogonal Case

Following [10], we consider split realizations of orthogonal Lie algebras and introduce notation that simultaneously incorporates the cases of symplectic and orthogonal Lie subalgebras of the general linear Lie algebra $\mathfrak{gl}_N$. Many of the formulas below appear most naturally in the context of the twisted Yangians, introduced by Olshanski in [13] and further studied in [10], but we will not pursue the connection in this paper.

Let $N = 2n$ (even case) or $N = 2n + 1$ (odd case). We consider a complex vector space $C^N$ with a basis $\{v_i : -n \leq i \leq n\}$, where $v_0$ is absent in the even case. This space is endowed with a non-degenerate bilinear form $(\cdot, \cdot)_\pm$, where the subscript + (respectively, −) indicates that the form is symmetric (respectively, skew-symmetric):

$$\langle v_i, v_j \rangle_+ = \delta_{i,-j} \quad \text{(orthogonal)}, \quad \langle v_i, v_j \rangle_- = \text{sgn } i \delta_{i,-j} \quad \text{(symplectic)}.$$

Let $\theta_{ij} = 1$ in the orthogonal case, and $\theta_{ij} = \text{sgn } i \text{ sgn } j$ in the symplectic case. The classical Lie algebra $\mathfrak{g}$ is the subalgebra of $\mathfrak{gl}_N$ consisting of $N \times N$ matrices skew-symmetric with respect to the form and is spanned by the following elements:

$$F_{ij} = E_{ij} - \theta_{ij} E_{-j,-i},$$

where $E_{ij}$ are the usual matrix units. The generators (5.2) satisfy the following commutation relations:

$$[F_{ij}, F_{kl}] = \delta_{kj} F_{il} - \delta_{il} F_{kj} - \theta_{k,-j} \delta_{i,-k} F_{-j,l} + \theta_{i,-i} \delta_{i,-l} F_{k,-i}.$$ (5.3)

Arrange $\{F_{ij}\}$ into an $N \times N$ matrix $F \in M_N(U(\mathfrak{g}))$, cf (122). Choose a triangular decomposition $\mathfrak{g} = n^+ \oplus \mathfrak{h} \oplus n^-$, where $n^+$ is spanned by $\{F_{ij} : i < j\}$, $n^-$ is spanned by $\{F_{ij} : i > j\}$ and $\mathfrak{h}$ is spanned by $\{F_{ii}\}$. We have $F_{ii} = -F_{-i,-i}$, the elements $H_1 = F_{-n,-n}, \ldots, H_n = F_{-1,-1}$ form a basis of $\mathfrak{h}$, and we will identify the weights $\lambda \in \mathfrak{h}^*$ with the $n$-tuples $(\lambda_1, \ldots, \lambda_n)$ of their values on the elements of this basis:

$$\lambda_i = \lambda(H_i) = \lambda(F_{i-n-1,i-n-1}).$$

Further, we set $\epsilon = 0$ for $\mathfrak{g}$ even orthogonal, $\epsilon = \frac{1}{2}$ for $\mathfrak{g}$ odd orthogonal, and $\epsilon = 1$ for $\mathfrak{g}$ symplectic. The half sum of the positive roots $\rho$ and the $\rho$-shifted highest weight $l = \lambda + \rho$ are then given by the $n$-tuples

$$\rho = (\epsilon + n - 1, \ldots, \epsilon), \quad l = \lambda + \rho = (\lambda_1 + \rho_1, \ldots, \lambda_n + \rho_n),$$

where $l_i = \lambda_i + n - i + \epsilon$.

Denote by $T(u)$ the formal resolvent of the $N \times N$ matrix $F$ over $U(\mathfrak{g})$. 

\[ T(u) = \frac{1}{u - \lambda - \rho + \epsilon} = \sum_{n=0}^{\infty} T^{(n)}(u), \]
Proposition 5.6. The entries of the matrix $T(u)$ transform under the adjoint action of $g$ in the same way as the entries of the matrix $F$. More explicitly,

$$[F_{ij}, T_{kl}(u)] = \delta_{kj} T_{il}(u) - \delta_{il} T_{kj}(u) - \theta_{k,-j} \delta_{i,-l} T_{-j,l}(u) + \theta_{i,-l} \delta_{-j,k} T_{k,-i}(u).$$

(5.7)

**Proof.** This is proved in the same way as Proposition 4.6. \qed

Let $S$ be the complement of the first simple root in the root system $B$ of $g$ (in standard enumeration). The parabolic subalgebra $p_S = n^+_S \oplus m_S$ is the first standard maximal parabolic of $g$ and consists of the matrices in $g \subset gl_N$ that stabilize the one-dimensional isotropic subspace of $\mathbb{C}^N$ spanned by the first standard basis vector $v_{-n}$. We have a triangular decomposition $g = n^+_S \oplus m_S \oplus n^-_S$, where the Levi factor $m_S = g' \oplus gl_1$ is spanned by $F_{ij}$ ($i, j \neq \pm n$) together with $F_{nn} = -F_{-n,-n}$, the upper nilradical $n^+_S$ is spanned by $F_{-n,j}$ ($j \neq \pm n$) and the lower nilradical $n^-_S$ is spanned by $F_{i,n}$ ($i \neq \pm n$). The Lie algebra $g'$ has the same Dynkin type as $g$ (i.e. even orthogonal, odd orthogonal, or symplectic) but its rank is one smaller than the rank of $g$.

Denote by $F'$ the $(N - 2) \times (N - 2)$ submatrix of $F$ formed by the elements $F_{ij}$ with $i, j \neq \pm n$ and by $T'(u) \in M_{n-1}(U(g'))[[u^{-1}]]$ its formal resolvent. Let $tr' T(u)$ be the sum of the $N - 2$ diagonal entries of the $N \times N$ matrix $T(u)$ in rows with indices different from $\pm n$. Let us write $\lambda = (\lambda_1, \lambda')$, so that $\lambda_p = \lambda_p$ for $2 \leq p \leq n$.

As in the general linear case, the maps $pr : U(g) \to \mathbb{C}, pr' : U(g') \to \mathbb{C}$ denote the compositions of the absolute Harish-Chandra projections with the evaluation at $\lambda$ and $\lambda'$. The relative map $pr_n : U(g) \to U(g')$ denotes the composition of the relative Harish-Chandra projection $pr_{g/m_S} : U(g_{\lambda_n}) \to U(g') \otimes U(gl_1)$ and the evaluation homomorphism $H_n \to \lambda_n$. By Proposition 2.6 these maps are related by the composition formula $pr = pr' \circ pr_n$.

**Proposition 5.8.** Let $v = u - \rho_l = u - (n - 1 + \epsilon)$ and $l = \lambda + \rho$. The following formulas describe the effect of the relative projection map $pr_n$ on the matrix series $T(u)$:

$$pr_n T_{nn}(u) = \frac{1}{v + l_1},$$

$$pr_n T_{ij}(u) = \frac{1}{v + l_1} T'_{ij}(u - 1), i, j \neq \pm n,$$

$$pr_n T_{-n, -n}(u) = \frac{1 - \text{tr}' T(u)}{v - l_1} \quad \text{(orthogonal case)};$$

$$pr_n T_{-n, -n}(u) = \frac{1 - 2(v + l_1)^{-1} - \text{tr}' T(u)}{v - l_1} \quad \text{(symplectic case)}.$$ (5.9)

Moreover, in the symplectic and even orthogonal case we have:

$$pr \text{tr} T(u) = \frac{v - 2\epsilon}{v - 1/2} \left(1 - \prod_{i=1}^{n} \frac{(v - 1)^2 - l_i^2}{v^2 - l_i^2}\right),$$

while in the odd orthogonal case

$$pr \text{tr} T(u) = \frac{v}{v - 1/2} - \frac{v - 1}{v - 1/2} \left(1 - \prod_{i=1}^{n} \frac{(v - 1)^2 - l_i^2}{v^2 - l_i^2}\right).$$

(5.10)

(5.11)

**Remark 5.12.** When both sides of (5.10) and (5.11) are expanded in negative powers of $u$, one recovers the Perelomov-Popov formulas that describe the action of the Gelfand invariants $\text{tr} F' \in Z(g)$ in the simple highest weight module $L(\lambda)$. \[19\]
Proof. The proof of the first two formulas is similar to the proof of the corresponding result (Proposition 4.9) in the general linear case. Therefore, some steps in the proof will only be sketched. Since the matrices $T(u)$ and $uI_N - F$ are mutually inverse, we have:

$$
(5.13) \quad \sum_k T_{ik}(u)(u\delta_{kj} - F_{kj}) = \delta_{ij}.
$$

First, let $i = j = n$. Splitting the sum into parts and proceeding as before, we find that $pr_n(T_{nn}(u)(u - F_{nn})) = 1$, therefore, $pr_nT_{nn}(u)(u + \lambda_n) = 1$, and since $u + \lambda_1 = v + l_1$, we have established the first formula. Now suppose that $i, j \neq \pm n$. Splitting the sum in the left hand side of (5.13) into three parts, according to whether $k \neq \pm n$, $k = n$ or $k = -n$ and applying $pr_n$ to both sides, we get:

$$
\sum pr_n T_{ip}(u)(u\delta_{pj} - F_{pj}) - pr_n(T_{i,-n}(u) F_{-nj}) - pr_n(T_{in}(u) F_{nj}) = \delta_{ij}.
$$

Here $\sum$ indicates the sum over the index $p \neq \pm n$. Note that since $F_{-nj} \in \mathfrak{n}_\mathcal{S}^+$, the middle term in the left hand side is zero. Using the commutation relations (5.7), we may replace $-T_{in}(u)F_{nj}$ with $-F_{nj} T_{in}(u) + [F_{nj}, T_{in}(u)] = -F_{nj} T_{in}(u) + \delta_{ij} T_{nn}(u) - T_{ij}(u)$. Since $F_{nj} \in \mathfrak{n}_\mathcal{S}$, $pr_n(F_{nj} T_{in}(u)) = 0$ and after routine transformations, we obtain:

$$
\sum pr_n T_{ip}(u)((u - 1)\delta_{pj}) = \delta_{ij}(1 - pr_n T_{nn}(u)).
$$

Proceeding in the same way as in the proof of Proposition 4.9 we establish the formula for $pr_n T_{ij}(u), i, j \neq \pm n$. \[\square\]

An important distinction of the symplectic/orthogonal case that we are considering now from the general linear case treated earlier is that $\mathrm{End} \pi = \mathrm{End} \mathbb{C}^N$ decomposes into the direct sum $\Lambda^2 \mathbb{C}^N \oplus S^2 \mathbb{C}^N$ of $\mathfrak{g}$-submodules.

**Definition 5.14.** Fix an infinitesimal character of $\mathfrak{g}$-modules. A polynomial $p \in \mathbb{C}[t]$ is called even (respectively, odd) modulo center if the identity

$$
p(F)_{ij} = \pm \theta_i \theta_j p(F_{-j,-i}) \mod \mathrm{Ann} M(\lambda)
$$

holds for all $i, j$ with the plus (respectively, the minus) sign.

**Proposition 5.15.** Suppose that $V$ is a module with infinitesimal character. Then the minimal polynomial of $V$ is either even or odd.

In analogy with the general linear case, we shall prove that the roots of the minimal polynomial of a $\mathfrak{g}$-module $L(\lambda)$ are obtained from $\lambda$ via a combinatorial algorithm. If $l$ is an $n$-element sequence $(l_1, \ldots, l_n)$, let $l' = (l_2, \ldots, l_n)$ and $l^* = (-l_{n-1}, \ldots, -l_1)$. The mirror image of the $k$th term of the $2n$-element sequence $l \cup l^*$ is the $(2n + 1 - k)$th term of the sequence, and a similar convention applies to subsequences. The shuffle decomposition of $l \cup l^*$ is constructed inductively and represents this $2n$-element sequence as a disjoint union of a number of pairs of falling subsequences (referred to as the parts of the decomposition), with the additional property that the parts in each pair are the mirror images of each other. For an empty $l$, the decomposition is empty. Suppose that the shuffle decomposition of $l' \cup l^*$ has already been constructed. Let $S$ be its longest part which begins with $l_1 - 1$, and $S^*$ the mirror image of $S$. Then appending $l_1$ to the beginning of $S$ and $-l_1$ to the end of $S^*$ and leaving the other parts intact, we get the shuffle decomposition of $l \cup l^*$. If no such (non-empty) $S$ existed, we create two new
one-term parts, \( \{ l_1 \} \) and \( \{- l_1 \} \) instead. The shuffle decomposition is called *odd* if one of the parts is a falling subsequence of \( l \) ending in \( \epsilon \) (or equivalently, if one of the parts is a falling subsequence of \( l^* \) starting with \(-\epsilon \)). Otherwise, the shuffle decomposition is *even*.

**Theorem 5.16.** Let \( \lambda \) be a weight for a classical Lie algebra \( \mathfrak{g} \) and \( l = \lambda + \rho \). Denote by \( \hat{A} \) the set of the first terms of the sequences comprising the shuffle decomposition of \( l \cup l^* \). Let \( A = \hat{A} \) if the shuffle decomposition is *even*, and \( A = \hat{A} \\setminus \{ \epsilon \} \) if it is *odd*. Then the roots of the minimal polynomial of the simple highest weight \( \mathfrak{g} \)-module \( L(\lambda) \) are \( \{ n - 1 + \epsilon - \alpha : \alpha \in A \} \).

**Proof.** The proof proceeds along the same lines as the proof of Theorem 4.13. However, instead of dealing with the poles of \( pr \mathcal{T}_{-n,-n}(u) \) directly, we need to keep track of whether the minimal polynomial \( f \) of \( L(\lambda) \) is even or odd modulo center, Definition 5.14. This allows us to restrict attention to the poles of \( pr \mathcal{T}_{nn}(u) \) and \( pr \mathcal{T}_{ij}(u), i, j \neq \pm n \). \( \square \)

### 6. Behavior of minimal polynomials under transfer

In [16], Section 5, we considered an algebraic version of the Howe duality in the context of modules over Lie algebras forming a reductive dual pair. We first review its definition and then investigate the effect this duality has on the minimal polynomials. The main result of this section is a noncommutative generalization of a well known property of matrices over a commutative field: suppose that \( A \) and \( B \) are two rectangular matrices of the same size, so that \( AB \) and \( A'B \) are square matrices, then the minimal polynomials \( p_{AB}(\lambda) \) and \( p_{A'B}(\lambda) \) are either the same or differ by a factor of \( \lambda \).

**Definition 6.1.** Let \( (\mathfrak{g}, \mathfrak{g}') \) is a reductive dual pair of Lie algebras and \( \mathcal{A} \) be a fixed non-zero module over the corresponding Weyl algebra \( \mathcal{W} \). Then a \( \mathfrak{g} \)-module \( V \) and a \( \mathfrak{g}' \)-module \( V' \) are in the algebraic Howe duality with each other if there exists an \( \text{Ad}(G) \)-invariant and \( \text{Ad}(G') \)-invariant subspace \( \mathcal{A}_0 \) of \( \mathcal{A} \) such that the quotient \( \mathcal{A}/\mathcal{A}_0 \) is isomorphic to \( V \otimes V' \) as a \( \mathfrak{g} \)-module and a \( \mathfrak{g}' \)-module.

For an irreducible reductive dual pair of general linear Lie algebras \( (\mathfrak{gl}_n, \mathfrak{gl}_k) \), the Weyl algebra \( \mathcal{W} \) may be identified with the algebra of polynomial coefficient differential operators on \( n \times k \) matrices, corresponding to the realization \( W = M_{k,n} \oplus M_{n,k} \) for the symplectic vector space. We denote by \( E \) the \( n \times n \) matrix \( (1.2) \) for \( \mathfrak{gl}_n \) and by \( E' \) the analogous \( k \times k \) matrix for \( \mathfrak{gl}_k \). Let \( X \) be the \( k \times n \) matrix with entries \( x_{ai} \), the coordinate functions on \( M_{k,n} \) and \( D \) be the \( k \times n \) matrix with entries \( \partial_{bj} \), the corresponding partial derivatives. The *unnormalized embeddings* \( L \) and \( R \) of \( \mathfrak{gl}_k \) and \( \mathfrak{gl}_n \) into \( \mathcal{W} \) are given by the following explicit formulas:

\[
(6.2) \quad L(E') = XD', \quad R(E) = X'D.
\]

The action of \( \mathfrak{gl}_k \) on the polynomial functions on \( M_{k,n} \) via \( L \) arises from the action of the group \( GL_k \) on the \( k \times n \) matrices by the left matrix multiplication, and similarly for \( \mathfrak{gl}_n \), \( R \), and the right matrix multiplication. In the language of the classical invariant theory, the differential operators \( R(E_{ij}) \) are the *polarization operators* with respect to the \( GL_k \)-action and likewise for the operators \( L(E'_{ij}) \) and the \( GL_n \)-action. Any module over the Weyl algebra acquires a structure of a \( \mathfrak{gl}_n \)-module and \( \mathfrak{gl}_k \)-module by pulling back the \( \mathcal{W} \)-action via the maps \( R \) and \( L \).
Theorem 6.3. Let \((\mathfrak{g}, \mathfrak{g}') = (\mathfrak{gl}_n, \mathfrak{gl}_k)\) and the modules \(V\) and \(V'\) over \(\mathfrak{gl}_n\) and \(\mathfrak{gl}_k\) are in an unnormalized algebraic Howe duality with each other. Then the following divisibility properties hold for their minimal polynomials \(q = q_V\) and \(q' = q_{V'}:\)

\[
q(u) \mid uq'(u + k - n), \quad q'(u) \mid uq(u - k + n).
\]

The statement of the theorem is symmetric with respect to exchanging \(n\) and \(k\), and its conclusion given does not depend on the particular module over the Weyl algebra used to define algebraic Howe duality.

Proof. The first part of the proof takes place in the Weyl algebra. Consider the matrices over \(W\) obtained by applying \(R\) and \(L\) entrwise to the matrices \(E\) and \(E'\). By an abuse of notation, we will also denote them \(E\) and \(E'\). Similar conventions apply to the matrix resolvents \(T(u)\) and \(T'(u)\) of \(E\) and \(E'\).

Proposition 6.5. For any \(r \geq 0\), the following identity holds:

\[
\sum_l (E)_l x_{al} = \sum_b (E' + (n - k)I_k)_{ab} x_{bi}.
\]

Proof. We apply induction in \(r\). For \(r = 0\) both sides are equal to \(x_{ai}\). For \(r = 1\), we compute

\[
\sum_l E_{il} x_{al} = \sum_{b,l} x_{bi} \partial_{bl} x_{al} = \sum_{b,l} (x_{al} \partial_{bl} x_{bi} + \delta_{ab} x_{bi} - \delta_{il} x_{al}) = (n - k) x_{ai} + \sum_b E'_{ab} x_{bi} = \sum_b (E' + (n - k)I_k)_{ab} x_{bi}.
\]

If the identity is true for \(r \geq 1\) then

\[
\sum_l (E')_l^{r+1} x_{al} = \sum_{l,m} E_{im} (E')_{ml} x_{al} = \sum_{c,m} E_{im} (E' + (n - k)I_k)_{ac}^{r} x_{cm} = \sum_{c,m} (E' + (n - k)I_k)_{ac}^{r} E_{im} x_{cm} = \sum_c (E' + (n - k)I_k)_{ac}^{r} (E' + (n - k)I_k)_{cb} x_{bi} = (E' + (n - k)I_k)_{ab}^{r+1} x_{bi}.
\]

Since each entry of the matrix \(E\) commutes with each entry of the matrix \(E'\), the exchange of the \(E\)-factor and \(E'\)-factor in the second line is justified.

Multiplying the \(r\)th identity \([6.0]\) by \(u^{-1-r}\) and summing over \(r \geq 0\), we obtain the identity

\[
\sum_l T_{il} (u) x_{al} = \sum_b T'_{ab} (u + k - n) x_{bi}, \quad T(u)X^l = (T'(u + k - n)X)^l.
\]

Now multiplying both sides by \(\partial_{aj}\) and summing over \(a\), we see that

\[
\sum_l T_{il} (u) E_{ij} = \sum_{a,l} T_{il} (u) x_{ai} \partial_{aj} = \sum_{a,b} T'_{ab} (u + k - n) x_{bi} \partial_{aj}.
\]

Adding \(\sum_l T_{il} (u) (uI_n - E)_{ij} = (I_n)_{ij}\) to both sides, we finally arrive at the identity

\[
u T_{ij} (u) = (I_n)_{ij} + \sum_{a,b} T'_{ab} (u + k - n) x_{bi} \partial_{aj}, \quad u T(u) = I_n + (T'(u + k - n)X)^l D.
\]

Now let us consider the quotient \(A/A_0\) of a module over the Weyl algebra that realizes the algebraic Howe duality between \(V\) and \(V'\). Since \(q'\) is the minimal
polynomial of the \( \mathfrak{gl}_k \)-module \( V' \), each matrix entry of \( q'(u + k - n)T'(u + k - n) \) acts on this quotient by a polynomial in \( u \). It follows that each matrix entry of \( uq'(u + n - k)T(u) \) acts on the \( \mathfrak{gl}_n \)-module \( V \) by a polynomial in \( u \), establishing that \( uq'(u + n - k) \) is divisible by the minimal polynomial \( q(u) \) of the \( \mathfrak{gl}_n \)-module \( V \). □

7. Applications to primitive ideals

**Theorem 7.1.** Suppose that a simple highest weight module \( L(\lambda) \) over \( \mathfrak{gl}_n \) has regular infinitesimal character. Then the minimal polynomial of \( L(\lambda) \) is uniquely determined by the tau invariant of \( L(\lambda) \). Moreover, for simple highest weight modules in the infinitesimal character of the trivial representation, the minimal polynomial and the \( \tau \)-invariant of \( L(\lambda) \) completely determine each other. Inclusions of primitive ideals correspond to divisibility of their minimal polynomials and inclusions of their \( \tau \) invariants.

**Theorem 7.2.** Let \( \pi \) be the adjoint representation of \( \mathfrak{sl}_n \) extended to \( \mathfrak{gl}_n \) trivially on the center. Then the multiplicity of \( \pi \) in the primitive quotient \( U(\mathfrak{gl}_n)/\text{Ann } L(\lambda) \) is one less than the degree of the minimal polynomial of \( L(\lambda) \).

8. Concluding remarks

It should be pointed out that in analogy with the classification of the conjugacy classes of matrices (or of the adjoint orbits in \( \mathfrak{g} \)), the minimal polynomial is only the first member in a larger family of invariants of modules and ideals. Therefore, it is insufficient for classification of the primitive ideals of \( U(\mathfrak{g}) \). In the case of the general Lie algebra \( \mathfrak{gl}_n \), we introduced in [15] a family of invariants \( q_k \), \( 1 \leq k \leq n \) generalizing the minimal polynomial. Their definition is inspired by Oshima’s construction of explicit generators for the induced ideals of \( U(\mathfrak{gl}_n) \). These invariants may be viewed as quantizations of the Fitting invariants of \( n \times n \) matrices and analogously to the theory of elementary divisors of matrices, they separate completely prime primitive ideals of \( U(\mathfrak{gl}_n) \). We hope that similar generalizations are possible also for the symplectic and orthogonal Lie algebras.

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