Solution of effective Hamiltonian of impurity hopping between two sites in a metal

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We analyze in detail all the possible fixed points of the effective Hamiltonian of a non-magnetic impurity hopping between two sites in a metal obtained by Moustakas and Fisher (MF). We find a line of non-fermi liquid fixed points which continuously interpolates between the 2-channel Kondo fixed point (2CK) and the one channel, two impurity Kondo (2IK) fixed point. There is one relevant direction with scaling dimension 1/2 and one leading irrelevant operator with dimension 3/2. There is also one marginal operator in the spin sector moving along this line. The marginal operator, combined with the leading irrelevant operator, will generate the relevant operator. For the general position on this line, the leading low temperature exponents of the specific heat, the hopping susceptibility and the electron conductivity $C_{\text{imp}}, \chi_{\text{imp}}, \sigma(T)$ are the same as those of the 2CK, but the finite size spectrum depend on the position on the line. No universal ratios can be formed from the amplitudes of the three quantities except at the 2CK point on this line where the universal ratios can be formed. At the 2IK point on this line, $\sigma(T) \sim 2\sigma_0(1 + aT^{3/2})$, no universal ratio can be formed either. The additional non-fermi liquid fixed point found by MF has the same symmetry as the 2IK, it has two relevant directions with scaling dimension 1/2, therefore also unstable. The leading low temperature behaviors are $C_{\text{imp}} \sim T, \chi_{\text{imp}} \sim \log T, \sigma(T) \sim 2\sigma_0(1 + aT^{3/2})$, no universal ratios can be formed. The system is shown to flow to a line of fermi-liquid fixed points which continuously interpolates between the non-interacting fixed point and the 2 channel spin-flavor Kondo fixed point (2CSFK) discussed by the author previously. The effect of particle-hole symmetry breaking is discussed. The effective Hamiltonian in the external magnetic field is analysed. The scaling functions for the physical measurable quantities are derived in the different regimes; their predictions for the experiments are given. Finally the implications are given for a non-magnetic impurity hopping around three sites with triangular symmetry discussed by MF.

I. INTRODUCTION

The experimental realization of overscreened multichannel Kondo model has been vigorously searched since the discovery of its non-fermi liquid behavior (NFL) by Nozièes and Blandin (NB). D. L. Cox pointed out that the NFL behaviors in heavy fermion systems like Y1-xUxPd2 may be explained by the 2 channel quadrupolar Kondo effects, but the observed electrical conductivity of such systems is linear in $T$ in contrast to $\sqrt{T}$ behavior of the 2-channel Kondo model (2CK). Vlad´ ar and Zawadowski suggested that a non-magnetic impurity tunneling between two sites in a metal can be mapped to the 2CK in which the roles of channels and spins in the original formulation are interchanged. Ralph et al. proposed that the conductance signals observed in ballistic metal point contacts may be due to the 2-channel Kondo scattering from 2-level tunneling systems, the conductance exponent 1/2 and the magnetic field dependence observed in such device is indeed in consistant with that predicted by Affleck-Ludwig's (AL) Conformal Field Theory (CFT) solution of the 2CK, however the alternative interpretation was also proposed.

Moustakas and Fisher reexamined the problem of the electron assisted tunneling of a heavy particle between two sites in a metal. In addition to bare hopping ($\Delta_0$ term in Eq. 1) and one electron assisted hopping term ($\Delta_1$ term in Eq. 2) found previously in Ref. 1, they found that an extra two electrons assisted hopping term ($\Delta_2$ term in Eq. 2) also plays an important role. Treating all these important processes carefully, they concluded that more than four channels (including spin) are needed in order to localize the impurity. In Ref. 1 they wrote down an effective Hamiltonian which includes all the important processes and employed Emery-Kivelson (EK)’s Abelian Bosonization solution of the 2CK to investigate the full phase diagram of this Hamiltonian, they found the two electron assisted hopping term plays a similar role to the bare hopping term. However, they overlooked the important fact that the canonical transformation operator $U = e^{iS^z\phi_+}$ in EK’s solution is a boundary condition changing operator, therefore their analysis of the symmetry of the fixed points and the operator contents near these fixed points are not complete. They didn’t calculate the electron conductivity which is the most important experimental measurable quantity. Furthermore the nature of the stable Fermi liquid fixed point was also left unexploited.
Affleck and Ludwig (AL) using Conformal Field Theory, pointed out that for any general quantum impurity problem, the impurity degree of freedoms completely disappear from the description of the low temperature fixed point and leave behind conformally invariant boundary conditions. CFT can also be used to classify all the possible boundary operators near any low temperature fixed points and calculate any correlation functions. For 4 pieces of bulk fermions which correspond to 8 pieces of Majorana fermions, the non-interacting theory possesses $SO(8)$ symmetry, Maldacena and Ludwig (ML) showed that finding the symmetry of the fixed points is exactly equivalent to finding the boundary conditions of the 8 Majorana fermions at the fixed points, the boundary conditions turned out to be linear in the basis which separates charge, spin and flavor. ML reduced the descriptions of the fixed points as free chiral bosons. The linear boundary conditions can also be transformed into the boundary conditions in the original fermion basis by the triality transformation Eq. (16). The boundary conditions in the original fermion basis only fall into two classes: NFL fixed points where the original fermions are scattered into spinors; fermi liquid (FL) fixed points where the original fermions only suffer just phase shifts at the boundary.

The important step in the CFT approach developed by AL is the identification of the fusion rules at various fixed points. Although the fusion rule is simple in the multichannel Kondo model, it is usually very difficult to identify in more complicated models like the one discussed in this paper.

Recently, using EK’s Abelian Bosonization approach to the 2CK, the author developed a simple and powerful method to study certain class of quantum impurity models with 4 pieces of bulk fermions. The method can identify very quickly all the possible boundary fixed points and their maximum symmetry, therefore circumvent the difficult tasks to identify the fusion rules at different fixed point or line of fixed points. It can also demonstrate the physical picture at the boundary explicitly.

In this paper, using the method developed in Ref. 13 and paying the special attention to boundary condition changing nature of $U = e^{i\beta \Phi}$, we investigate the full phase diagram of the present problem again. In Sec. II, we Abelian bosonize the effective Hamiltonian. By using the Operator Product Expansion (OPE), we get the Renormalization Group (RG) flow equations near the weak coupling line of fixed points, therefore identify the two independent crossover scales. In the following sections, we analyze all the possible fixed points of the bosonized Hamiltonian. In Sec. III, we find a line of NFL fixed points which continuously interpolates between the 2CK fixed point and the one channel two impurity Kondo (2IK) fixed point. Its symmetry is $U(1) \times O(1) \times O(5)$. This line of NFL fixed points is unstable, it has one relevant direction with scaling dimension 1/2. It also has one marginal operator in the spin sector which is responsible for this line. The OPE of this marginal operator and the leading irrelevant operator will always generate the relevant operator. For the general position on the line, although the leading exponents of the specific heat, hopping susceptibility and the electron conductivity $C_{\text{imp}}, \chi_{\text{imp}}^h, \sigma(T)$ are the same as those of the 2CK, the finite size spectrum depend on the position on the line; no universal relations can be found among the amplitudes of the three quantities. Only at the 2CK point on the line, universal ratios can be formed. However, at the 2IK point, the coefficient of $\sqrt{T}$ vanishes; we find two dimension 5/2 operators which lead to $\sigma(T) \sim 2\sigma_a(1 + T^{3/2})$, no universal ratio can be formed either. In Sec. IV, the additional NFL fixed point found by MS is shown to have the symmetry $O(7) \times O(1)$, therefore is the same fixed point as the 2IK. This fixed point is also unstable, it has two relevant directions with scaling dimension 1/2. Because the leading irrelevant operators near this NFL fixed point are first order Virasoro descendant with scaling dimension 3/2 which can be written as a total imaginary time derivative, therefore, can be dropped. The subleading irrelevant operators with dimension 2 give $C_{\text{imp}} \sim T$. However, because the ‘orbital field’ in Eq. 12 couples to a non-conserved current, $\chi_{\text{imp}}^h \sim \log T$. We also find two dimension 5/2 irrelevant operators, one of them contributes to the leading low temperature conductivity $\sigma(T) \sim 2\sigma_a(1 + T^{3/2})$. No universal ratios can be formed near this NFL fixed point. In Sec. V, we find the system flows to a stable line of FL fixed points which continuously interpolates between the non-interacting fixed point and the 2 channel spin-flavor Kondo (2CSFK) fixed point discussed by the author in Ref. 14, its symmetry is $U(1) \times O(6) \sim U(4)$. Along this line of fixed points, the electron fields of the even and the odd parity under interchanging the two sites suffer opposite continuously-changing phase shifts. We also discuss the effect of the marginal operator in the charge sector due to the P-H symmetry breaking and compare it with the magnetic operator in the spin sector. In Sec. VI, we analyse the effective Hamiltonian in the external magnetic field which break the channel symmetry. In Sec. VII, all the scaling functions for the physical measurable quantities including the real spin susceptibility are derived in different regimes. In the final section, the relevance of the results of this paper to the experiments are examined, some implications on the non-magnetic impurity hopping around three sites with triangular symmetry are also given. In Appendix A, the finite size spectrum of one complex fermion is listed. In Appendix B, the boundary conditions in the original fermion basis are derived by both Bosonization method and $\gamma$ matrix method. In Appendix C, the results on the additional NFL fixed point in Sec. IV. are rederived in a different basis.
II. BOSONIZATION OF THE EFFECTIVE HAMILTONIAN AND THE WEAK COUPLING ANALYSIS

We start from the following effective Hamiltonian for a non-magnetic impurity hopping between two sites in a metal first obtained by MF:

\[ H = H_0 + V_1 (\psi_{1\alpha}^{\dagger} \psi_{1\sigma} + \psi_{2\sigma}^{\dagger} \psi_{2\alpha}) + V_2 (\psi_{1\alpha}^{\dagger} \psi_{2\sigma} + \psi_{1\sigma}^{\dagger} \psi_{2\alpha}) + V_3 (d_1^{\dagger} d_1 - d_2^{\dagger} d_2) (\psi_{1\sigma}^{\dagger} \psi_{1\sigma} - \psi_{2\sigma}^{\dagger} \psi_{2\sigma}) + \sum_{i} \psi_{1\sigma}^{\dagger} \psi_{2\sigma} + \sum_{i} \Delta_2 2 \pi \tau_c \psi_{1\uparrow}^{\dagger} \psi_{2\uparrow} + h.c. + \cdots \] (1)

Here the two sites 1, 2 (the two real spin directions \( \uparrow, \downarrow \)) play the role of the two spin directions \( \uparrow, \downarrow \) (the two channels 1, 2) in the magnetic Kondo model. All the couplings have been made to be dimensionless. As emphasized by MF even initially \( \Delta_1, \Delta_2 \) may be negligible, they will be generated at lower energy scales, \( \cdots \) stands for irrelevant terms. In the following, we use the notation of the magnetic Kondo model and rewrite the above Hamiltonian as:

\[ H = H_0 + V_1 J_x (0) + 2 V_2 J^x (0) + 4 V_3 S^z J^z (0) + \Delta_1 (J^x (0) S^x + J^y (0) S^y) + \Delta_2 2 \pi \tau_c (S^\uparrow \psi_{1\uparrow}^{\dagger} \psi_{2\uparrow} + h.c.) + h(\int dx J^z (x) + S^z) \] (2)

where \( S^\uparrow = d_1^{\dagger} d_2, S^- = d_3^{\dagger} d_1, S^z = \frac{1}{2} (d_1^{\dagger} d_1 - d_2^{\dagger} d_2) \). We also add a ‘uniform field’ which corresponds to strain or pressure in the real experiment of Ref. Abelian-bosonizing the four bulk Dirac fermions separately:

\[ \psi_{i\alpha}(x) = \frac{P_{i\alpha}}{\sqrt{2 \pi \tau_c}} e^{-i \Phi_{i\alpha}(x)} \] (3)

Where \( \Phi_{i\alpha}(x) \) are the real chiral bosons satisfying the commutation relations

\[ [\Phi_{i\alpha}(x), \Phi_{j\beta}(y)] = i \delta_{ij} \delta_{\alpha\beta} \pi \tau_c \text{sgn}(x - y) \] (4)

The cocycle factors have been chosen as: \( P_{1+} = P_{1\downarrow} = e^{i \pi N_{1+}}, P_{2+} = P_{2\downarrow} = e^{i \pi N_{2+} + N_{1+} + N_{2\downarrow}}. \)

It is convenient to introduce the following charge, spin, flavor, spin-flavor bosons:

\[ \begin{pmatrix} \Phi_c \\ \Phi_s \\ \Phi_f \\ \Phi_{sf} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \Phi_{1+} + \Phi_{1\downarrow} + \Phi_{2+} + \Phi_{2\downarrow} \\ \Phi_{1-} + \Phi_{1\uparrow} + \Phi_{2+} - \Phi_{2\downarrow} \\ \Phi_{1\uparrow} + \Phi_{1\downarrow} - \Phi_{2+} + \Phi_{2\downarrow} \\ \Phi_{1-} - \Phi_{1\uparrow} - \Phi_{2\downarrow} + \Phi_{2\uparrow} \end{pmatrix} \] (5)

Following the four standard steps in EK solution: step 1: writing the Hamiltonian in terms of the chiral bosons Eq. step 2: making the canonical transformation \( U = \exp(-i V_1 \Phi_c (0) + i S^z \Phi_s (0)) \). step 3 shift the spin boson by \( \partial_x \Phi_s \rightarrow \partial_x \Phi_s + \frac{i \pi}{2} \) step 4: making the following reformation:

\[ S^x = \frac{\hat{a}}{\sqrt{2}} e^{i \pi N_{sf}}, \quad S^y = \frac{\hat{b}}{\sqrt{2}} e^{i \pi N_{sf}}, \quad S^z = -i \hat{a} \hat{b} = d^d - 1 \]

\[ \psi'_{sf} = \frac{1}{\sqrt{2 \pi \tau_c}} (a_{sf} - ib_{sf}) = \frac{1}{\sqrt{2 \pi \tau_c}} e^{i \pi N_{sf}} e^{-i \Phi_{sf}} \]

\[ \psi_{s,i} = \frac{1}{\sqrt{2 \pi \tau_c}} (a_{s,i} - ib_{s,i}) = \frac{1}{\sqrt{2 \pi \tau_c}} e^{i \pi d^d d^d} e^{i \pi N_{sf}} e^{-i \Phi_{s}} \] (6)

Note \( \psi_{s,i}(x) \) defined above contains the impurity operator \( e^{i \pi d^d d^d} \) in order to satisfy the anti-commutation relations with the other fermions.
The transformed Hamiltonian \( H' = UHU^{-1} \) can be written in terms of the Majorana fermions as:

\[
H' = H_0 + 2y\hat{a}_a\hat{b}_{s,i}(0)b_{sf}(0) + 2q\hat{a}_{a\hat{b}}a_s(0)b_{s,i}(0) - i\hat{a}\hat{b}q \frac{h}{v_F} + i\frac{\Delta_+}{\sqrt{2\pi\tau_c}}\hat{a}_a\hat{b}_{s,i}(0) + i\frac{\Delta_-}{\sqrt{2\pi\tau_c}}\hat{a}_b\hat{b}_{s,i}(0)
\]

where \( y = 2V_2, q = \frac{1}{2}(V_3 - \frac{\pi v_F}{2}), \Delta_\pm = \Delta_0 \pm \Delta_2 \).

As observed by MF, the above equation clearly indicate that the two electron assisted hopping term plays a similar role to the bare hopping term. From the OPE of the various operators in Eq. 7, the R. G. flow equations near the weak coupling fixed point \( q = 0 \) is

\[
\frac{d\Delta_+}{dl} = \frac{1}{2}\Delta_+ + 2y\Delta_1 \\
\frac{d\Delta_-}{dl} = \frac{1}{2}\Delta_- \\
\frac{d\Delta_1}{dl} = \frac{1}{2}\Delta_1 + 2y\Delta_+ \\
\frac{dy}{dl} = \Delta_1\Delta_+ \\
\frac{dq}{dl} = \Delta_+\Delta_-
\]

MF got the same RG flow equations (Eq.23 in Ref.9) near the weak coupling line of fixed points by using Anderson-Yuval Coulomb gas picture. Eq. 8 shows that \( \Delta_+, \Delta_-, \Delta_1 \) have the same scaling dimension 1/2 at \( q = 0 \), so are equally important.

Define \( b_{s,i}(x), b_{sf,i}(x) \) (similarly for \( \tilde{q}, \tilde{y} \))

\[
\begin{pmatrix}
\tilde{b}_{s,i}(x) \\
\tilde{b}_{sf,i}(x)
\end{pmatrix} = \frac{1}{\Delta_K} \begin{pmatrix}
\Delta_1 & \Delta_- \\ -\Delta_+ & \Delta_1
\end{pmatrix} \begin{pmatrix}
b_{s,i}(x) \\
b_{sf}(x)
\end{pmatrix}
\]

Where \( \Delta_K = \sqrt{\Delta_1^2 + \Delta_-^2} \).

Eq.8 can be rewritten as:

\[
H' = H_0 + 2\tilde{q}\tilde{a}_{a\hat{b}}a_{s,i}(0)b_{s,i}(0) + 2\tilde{y}\tilde{a}_{a\hat{b}}a_{s,i}(0)b_{s,i}(0) - i\tilde{a}\tilde{b}q \frac{h}{v_F} + i\frac{\Delta_K}{\sqrt{2\pi\tau_c}}\tilde{a}_\hat{a}\tilde{b}_{s,i}(0) \]

The R. G. flow equations which are equivalent to Eq.8 are (we set \( h = 0 \) )

\[
\frac{d\Delta_+}{dl} = \frac{1}{2}\Delta_+ + 2\tilde{y}\Delta_1 \\
\frac{d\Delta_1}{dl} = \frac{1}{2}\Delta_1 + 2\tilde{y}\Delta_+ \\
\frac{dy}{dl} = 4\tilde{q}\Delta_+ \Delta_K \\
\frac{dq}{dl} = \Delta_1\Delta_+ \]

Where the angle \( \theta \) is defined by \( \cos \theta = \frac{\Delta_+ - \Delta_1}{\Delta_K}, \sin \theta = \frac{2\Delta_1}{\Delta_K} \).

The crossover scale from the weak coupling fixed point \( q = 0 \) to a given point on the line of NFL fixed points to be discussed in section III is given by \( T_K_1 \sim D(\Delta_K)^2 \), to the additional NFL fixed point to be discussed in section IV. is given by \( T_K_2 \sim D(\Delta_K)^2 \).

As emphasized in [14], the canonical transformation \( U \) is a boundary condition changing operator, the transformed field \( \psi'_s(x) \) is related to the original field \( \psi_s(x) \) by:

\[
\begin{align*}
\psi'_s(x) &= \psi_s(x) \\
\phi'(x) &= \phi(x)
\end{align*}
\]
\[ \psi_s(x) = U^{-1} \psi_{s,i}(x) U = e^{i \pi^d} e^{i \pi^S} sgn \psi_s(x) = -isgn \psi_s(x) \]

(12)

As expected, the impurity spin \( S^z \) drops out in the prefactor of the above equation.

The above Eq. can be written out explicitly in terms of the Majorana fermions

\[
\begin{align*}
\Delta L_s^L(0) &= -b_s^L(0), & b_s^L(0) &= a_s^L(0) \\
\Delta R_s^R(0) &= b_s^R(0), & b_s^R(0) &= -a_s^R(0)
\end{align*}
\]

(13)

We find the physical picture can be more easily demonstrated in the corresponding action:

\[
S = S_0 + \frac{\gamma_1}{2} \int d\tau \hat{a} \frac{\partial \hat{a}(\tau)}{\partial \tau} - i \frac{\Delta_K}{\sqrt{2\pi} c} \int d\tau \hat{a}(\tau) \hat{b}_{sf,i}(0, \tau)
\]

\[
+ \frac{\gamma_2}{2} \int d\tau \hat{b}(\tau) \frac{\partial \hat{b}(\tau)}{\partial \tau} + i \frac{\Delta_+}{\sqrt{2\pi} c} \int d\tau \hat{b}(\tau) a_{s,i}(0, \tau)
\]

\[
+ 2\tilde{q} \int d\tau \hat{a}(\tau) \hat{b}(\tau)a_{s,i}(0, \tau) \hat{b}_{sf,i}(0, \tau) + 2\tilde{q} \int d\tau \hat{a}(\tau) \hat{b}(\tau)a_{s,i}(0, \tau) \hat{b}_{sf,i}(0, \tau)
\]

(14)

When performing the RG analysis of the action \( S \), we keep 1: \( \gamma_2 = 1, \Delta_K \) fixed, 2: \( \gamma_1 = 1, \Delta_+ \) fixed, 3: \( \Delta_K, \Delta_+ \) fixed.

We will identify all the possible fixed points or the line of fixed points and derive the R. G. flow equations near these fixed points in the following sections respectively.

III. THE LINE OF NFL FIXED POINTS

If \( \tilde{q} = \hat{y} = \Delta_+ = 0 \), this fixed point is located at \( \gamma_1 = 0, \gamma_2 = 1 \) where \( \hat{b} \) decouples, but \( \hat{a} \) loses its kinetic energy and becomes a Grassmann Lagrangian multiplier, integrating \( \hat{a} \) out leads to the boundary conditions:

\[
\tilde{b}_{sf,i}(0) = -b_{s,f,i}(0)
\]

(15)

We also have the trivial boundary conditions

\[
\Delta L_{s,i}(0) = a_{s,i}^R(0), & b_{s,i}^L(0) = \tilde{b}_{s,i}(0)
\]

(16)

By using Eqs.12, we find the boundary conditions in \( H \):

\[
\Delta L_s^L(0) = a_s^R(0), & (b_{sf}^L(0) \quad b_{sf}^R(0)) = (-\cos \theta \quad \sin \theta \quad \sin \theta \quad \cos \theta)
\]

(17)

By using Eq.13, we get the corresponding boundary conditions in \( H \):

\[
b_s^L(0) = -b_s^R(0), \quad \psi_s^L(0) = e^{2\bar{\delta}} \psi_s^R(0), \quad \theta = 2\bar{\delta}
\]

(18)

where \( \psi_s(0) = a_s(0) - ib_{sf}(0) \).

It is evident that at the fixed point, the impurity degree of freedoms totally disappear and leave behind the conformally invariant boundary conditions Eq.15. These are NFL boundary conditions. In contrast to the boundary conditions discussed previously, NFL boundary conditions are still related in the basis which separates charge, spin, and flavor, they are not in any four of the Cartan subalgebras of \( SO(8) \) group, therefore cannot be expressed in terms of the chiral bosons in Eq.15.

However, Eq.15 indicates that it is more convenient to regroup the Majorana fermions as

\[
\psi_s = a_s - ib_{sf} = e^{-i\Phi_s}, \quad \psi_{sf} = a_{sf} + ib_s = e^{-i\Phi_{sf}}
\]

(19)

The boundary condition Eq.15 can be expressed in terms of the new bosons

\[
\Phi_s^L = \Phi_{s,R} + \theta, \quad \Phi_s^L = -\Phi_{sf,R}
\]

(20)
As pointed out in Ref. 12, in the basis of Eq. 5, the physical fermion fields transform as the spinor representation of $SO(8)$, therefore in order to find the corresponding boundary conditions in the physical fermion basis, we have to find the $16 \times 16$ dimensional spinor representation of boundary conditions Eq. 18. The derivation of the boundary conditions is given in Appendix B, the results are found to be:

$$
\psi_{i\pm}^L = e^{\pm i\theta/2}S_{R,i\pm}^L, \quad S_{L,i\pm}^R = e^{\pm i\theta/2}\psi_{i\pm}^R
$$

Where the new fields are defined by:

$$
\psi_{i\pm} = \frac{1}{\sqrt{2}}(\psi_{i\uparrow} \pm \psi_{i\downarrow}), \quad S_{i\pm} = \frac{1}{\sqrt{2}}(S_{i\uparrow} \pm S_{i\downarrow})
$$

It can be checked explicitly the above boundary conditions satisfy all the symmetry requirement (namely $Z_2 \times SU_f(2) \times U_c(1)$, Time Reversal and P-H symmetry). Fermions with the even and the odd parity under $Z_2$ symmetry are scattered into the collective excitations of the corresponding parity which fit into the $S$ spinor representation of $SO(8)$, therefore the one particle S matrix and the residual electrical conductivity are the same with those of the 2CK. This is a line of NFL fixed points with $g = \sqrt{2}$ and the symmetry $O(1) \times U(1) \times O(5)$ which interpolates continuously between the 2CK fixed point and the 2IK fixed point. If $\theta = \pi$, namely $\Delta = 0$, the fixed point symmetry is enlarged to $O(3) \times O(5)$ which is the fixed point symmetry of the 2CK. If $\theta = 0$, namely $\Delta = 1$, the fixed point symmetry is enlarged to $O(1) \times O(7)$ which is the fixed point symmetry of the 2IK(b) (Fig. 1). The finite size spectrum can be obtained from the free fermion spectrum with both periodic (R) and anti-periodic (NS) boundary conditions by twisting the three Majorana fermions in the spin sector: changing the boundary condition of the Majorana fermion $b_s$ from NS sector to R sector or vice versa, twisting the complex fermion $\psi_s$ by a continuous angle $\theta = 2\delta$. The finite size spectrum of one complex fermion is derived in Appendix A. The complete finite size spectrum of this NFL fix line is listed in Table I if $0 < \delta < \frac{\pi}{2}$ and in Table II if $-\frac{\pi}{2} < \delta < 0$. The ground state energy is $E_0 = \frac{1}{16} + \frac{1}{2}(\delta)^2$ with degeneracy $d = 2$, the first excited energy is $E_1 - E_0 = \frac{3}{8} - \frac{|\delta|}{\pi}$ (if $-\frac{\pi}{4} < \delta < \frac{\pi}{2}$) or $E_1 - E_0 = \frac{1}{2} + \frac{\delta}{\pi}$ (if $-\frac{\pi}{2} < \delta < -\frac{\pi}{4}$) with $d = 4$.

If $\delta = \pm\frac{\pi}{2}(\delta = 0)$, the finite size spectrum of the 2CK (the 2IK) is recovered. The finite size spectrum of the 2CK and the 2IK are listed in Tables III and IV respectively.

The local correlation functions at this line of NFL fixed points are:

$$
\langle \hat{a}(\tau)\hat{a}(0) \rangle = \frac{1}{\tau}, \quad \langle \hat{b}_{sf,i}(0,\tau)\hat{b}_{sf,i}(0,0) \rangle = \gamma \frac{1}{\tau^3}
$$

We can also read the scaling dimension of the various fields $[\hat{b}] = 0, [\hat{a}] = 1/2, [a_{s,i}] = [\hat{b}_{s,i}] = 1/2, [\hat{b}_{sf,i}] = 3/2$. 

\[2IK(b) \quad (O(1)xO(7)) \]  \[U(1)xO(1)xO(5) \]  \[2CK \quad (O(3)xO(5)) \]

\[\theta = 0 \]  \[NFL \text{ Fixed line} \]  \[\theta = \pi \]

\[\theta = 0 \]  \[FL \text{ Fixed line} \]  \[U(1)xO(6) \]  \[\theta = \pi \]

\[Free \quad (O(8)) \]  \[U(1)xO(6) \]  \[2CSFK \quad (O(2)xO(6)) \]
FIG. 1. Phase diagram of a non magnetic impurity hopping between two sites in a metal. The line of NFL fixed points has the symmetry $O(1) \times U(1) \times O(5)$ with $g = \sqrt{2}$ which interpolates continuously between the 2CK fixed point and the 2IK fixed point. If $\theta = \pi$, the fixed point symmetry is enlarged to $O(3) \times O(5)$ which is the fixed point symmetry of the 2CK. If $\theta = 0$, the fixed point symmetry is enlarged to $O(1) \times O(7)$ which is the fixed point symmetry of the 2IK(b). This line of NFL fixed points is unstable, there is one relevant operator with dimension 1/2 which drives the system to the line of FL fixed points. There is also a marginal operator along the line. This line of FL fixed points has the symmetry $U(1) \times O(6) \sim U(4)$ with $g = 1$ which interpolates continuously between the non-interacting fixed point and the 2CSFK fixed point. If $\theta = 0$, the fixed point symmetry is enlarged to $O(2) \times O(6)$ which is the the fixed point symmetry of the 2CSFK. This line of FL fixed points is stable. There is a marginal operator along this line. The additional NFL fixed point with $g = \sqrt{2}$ has the symmetry $O(1) \times O(7)$ which is the fixed point symmetry of the 2IK(a). This additional NFL fixed point is also unstable. There are two relevant terms with scaling dimension 1/2, any linear combination of the two terms will drive the system to a given point of the line of FL fixed points. See the text for the detailed discussions on the physical properties of these fixed points or line of fixed points.

As shown in Ref. 3, at the line of fixed points, the impurity degree of freedoms completely disappear: $\tilde{h}$ decouples and $\tilde{a}$ turns into the non-interacting scaling field at the fixed point

$$\tilde{a}(\tau) \sim b_{sf}, i(0, \tau)$$

(24)

The corresponding two scaling fields in $H$ is

$$\frac{1}{\Delta_{\theta}}(-\Delta_{-}a_{s} + \Delta_{1}b_{sf})$$

(25)

Following Ref. 2, we find the impurity spin turns into

$$S_{x}(\tau) \sim -i(\tilde{b}b_{s} + \frac{1}{\Delta_{\theta}}a_{s}b_{sf}) + \cdots$$

$$S_{y}(\tau) \sim i(\tilde{b}a_{s} + \frac{1}{\Delta_{\theta}}(-\Delta_{-}a_{s} + \Delta_{1}b_{sf})b_{s}) + \cdots$$

$$S_{z}(\tau) \sim \tilde{b}\frac{1}{\Delta_{\theta}}(-\Delta_{-}a_{s} + \Delta_{1}b_{sf}) + \cdots$$

(26)

Where $\cdots$ stands for higher dimension operators and $\frac{\Delta_{+}}{\Delta_{\theta}} = \sqrt{1 - \cos \theta}$. $\frac{\Delta_{-}}{\Delta_{\theta}} = \sqrt{1 + \cos \theta}$.

The impurity spin-spin correlation function $\langle S^{a}(\tau)S^{a}(0) \rangle \sim 1/\tau$.

From Eq. 3, it is easy to see that even $y$ term itself is irrelevant near this line of fixed points, but Eq. 4 shows that it, when combined with $\Delta_{0}$ term, will generate $\Delta_{1}$, $\Delta_{2}$ terms which must be taken into account at this line of fixed points. $y$ term is "dangerous" irrelevant, similar "dangerous" irrelevant term occurred in the two channel flavor anisotropic Kondo model. This line of NFL fixed points is unstable. $\Delta_{+}$ term in Eq. 4 has scaling dimension 1/2, therefore is relevant, this term was first discovered by MF. The OPE of $a_{s, i}$ with itself will generate the dimension 2 energy momentum tensor of this Majorana fermion $a_{s, i}, (0, \tau) \frac{\partial a_{s, i}(0, \tau)}{\partial \tau}$; the OPE of this energy momentum tensor with $a_{s, i}$ will generate a first order descendant field of this primary field with dimension 3/2 $L_{-}a_{s, i}(0, \tau) = \frac{\partial a_{s, i}(0, \tau)}{\partial \tau}$ $\hat{q}$ term is the leading irrelevant operator with scaling dimension 3/2, therefore contributes to

$$C_{imp} \sim (\hat{q})^{2} T \log T$$

(27)

Where $\hat{q} = \sqrt{1 - \cos \theta}q + \sqrt{1 + \cos \theta}y$.

It is important to see there is a marginal operator $\partial \Phi_{s}^{a}(0)$ in the spin space which changes the angle $\theta$ in Eq. 20. This operator is very different from the marginal operator $\partial \Phi_{s}(0)$ in the charge sector which changes the angle $\theta_{ph}$ in Eq. 12. Combined with the leading irrelevant operator, it will always generate the dimension 1/2 relevant operator. This indicates that the existence of the line of NFL fixed points and the existence of one relevant operator are intimately connected.

However, from Eqs. 10, 23, only $q$ term contributes to the impurity susceptibility

$$\chi_{imp}^{h} \sim q^{2} \log T$$

(28)

As shown in Ref. 2, the Wilson Ration $R = 8/3$ is universal for any general spin anisotropic 2CK. However, near this line of NFL fixed points, $R$ is not universal.
The bosonized form of the $\tilde{y}$ term in $H$ is
\[
\tilde{y} = -\sqrt{\frac{1+\cos\theta}{2}} \tilde{q} + \sqrt{\frac{1-\cos\theta}{2}} y.
\]

By Eq.13, the corresponding spin-0 and spin-1 operators in $H'$ are
\[
\tilde{q} \tilde{a}(0,\tau) = \tilde{b}_{1,s,i}(0,\tau), \quad \frac{\partial \tilde{b}_{1,s,i}(0,\tau)}{\partial \tau} = \tilde{b}_{s,i}(0,\tau) \frac{\partial a_{s,i}(0,\tau)}{\partial \tau}
\]

By Eq.14, the first order correction to the single particle L-R Green functions due to the spin-0 operator can be calculated ( $x_1 > 0, x_2 < 0$)
\[
\langle \psi_{1+}(x_1,\tau_1) \psi_{1+}^\dagger(x_2,\tau_2) \rangle \sim \tilde{q} \int d\tau \langle \psi_{1+}(x_1,\tau_1) \cos \Phi_{s,i}^0(0,\tau) \partial \Phi_{s,i}^0(0,\tau) \psi_{1+}^\dagger(x_2,\tau_2) \rangle
\]
\[
= \int d\tau \langle e^{-\frac{i}{2} \Phi_{s,i}(x_1,\tau_1)} e^{\frac{i}{2} \Phi_{s,i}(x_2,\tau_2)} \rangle e^{-\frac{i}{2} \Phi_{s,i}(x_1,\tau_1) + \theta} \rangle e^{-\frac{i}{2} \Phi_{s,i}(x_2,\tau_2)} \rangle
\]
\[
\sim \tilde{q} e^{i\theta/2}(z_1 - \bar{z}_2)^{-3/2}
\]

where $z_1 = \tau_1 + ix_1$ is in the upper half plane, $\bar{z}_2 = \tau_2 + ix_2$ is in the lower half plane.

The first order correction to the single particle L-R Green functions due to the spin-1 operator can be written as a total derivative and the three point functions are the periodic function of the imaginary time.

The bosonized form of the $\hat{y}$ term in $H$ is
\[
\hat{y} = \sqrt{\frac{1-\cos\theta}{2}} \hat{q} - \sqrt{\frac{1+\cos\theta}{2}} y.
\]

The first order correction due to this dimension 5/2 operator is
\[
\langle \psi_{1+}(x_1,\tau_1) \psi_{1+}^\dagger(x_2,\tau_2) \rangle \sim \hat{y} \int d\tau \langle e^{-\frac{i}{2} \Phi_{s,i}(x_1,\tau_1)} [-\frac{1}{2} \cdot (\partial \Phi_{s,i}^0(0,\tau))^2 + \sin \theta (\cos \Phi_{s,i}^0(0,\tau) + \sin \phi)] \rangle e^{\frac{i}{2} \Phi_{s,i}(x_2,\tau_2)} \rangle
\]
\[
\sim \hat{y} e^{i\theta/2}(iC_1 + C_2 \sin \theta)(z_1 - \bar{z}_2)^{-5/2}
\]
Where $C_1, C_2$ are real numbers.

From Eq. 32, it is easy to identify another dimension 5/2 operator
\[
: (\partial \Phi_i^n(0, \tau))^2 : \cos \Phi_i^n(0, \tau)
\]
(36)

The contribution from this operator can be similarly calculated.

By using the following OPE:
\[
: e^{-\Phi(z_1)} : e^{\Phi(z_2)} := (z_1 - z_2)^{-1/4} - i \frac{1}{2} (z_1 - z_2)^{3/4} : \partial \Phi(z_2) :
\]
\[
- \frac{i}{4} (z_1 - z_2)^{7/4} : \partial^2 \Phi(z_2) : - \frac{1}{8} (z_1 - z_2)^{7/4} : (\partial \Phi(z_2))^2 : + \cdots
\]
(37)

We get the three point functions
\[
\langle e^{-\Phi(z_1)} \partial \Phi(z) e^{\Phi(z_2)} \rangle = -i/2 \frac{\zeta_2}{(z_1 - z_2)(z_1 - z_2)^{-3/4}(z - z_2)}
\]
\[
\langle e^{-\Phi(z_1)} \partial^2 \Phi(z) e^{\Phi(z_2)} \rangle = \partial_2 \frac{-i/2}{(z_1 - z_2)(z_1 - z_2)^{-3/4}(z - z_2)}
\]
(38)

By Conformal Ward identity, we can write the three point function with the energy momentum tensor
\[
\langle e^{-\Phi(z_1)} T(z) e^{\Phi(z_2)} \rangle = \frac{1/8}{(z_1 - z)^2(z_1 - z_2)^{-7/4}(z - z_2)^2}
\]
(39)

In Ref. 4, AL found that in the multi-channel Kondo model, the electron self-energy has both real and imaginary parts which are non-analytic function of the frequency $\omega$. In the presence of P-H symmetry, the imaginary part is even function of $\omega$, the real part is odd function of $\omega$, because the two parts are related by Kramers-Kronig relation. Only the part of self-energy which is both imaginary and even function of $\omega$ contributes to the electron conductivity. The factor $i$ will interchange real and imaginary part. In evaluating Eq. 33, we used the important fact that the two three point functions in Eqs. 35, 36 differ by the factor $i$.

By conformal transforming Eq. 34 to finite temperature, we get the leading term of the low temperature conductivity from channel one and parity + fermions
\[
\sigma_{1+}(T) \sim \sigma_u \frac{\pi}{2} (1 - \tilde{q} \sin \frac{\theta}{2} \sqrt{T}), \quad \sigma_u = \frac{2\pi (e \rho_T v_F)^2}{3n_i}
\]
(40)

Where $\rho_T$ is the density of states per spin per channel at the fermi energy, $n_i$ is the impurity density.

Similarly 4, we get the leading term of the low temperature conductivity from channel one and parity - fermions
\[
\sigma_{1-}(T) \sim \sigma_u \frac{\pi}{2} (1 - \tilde{q} \sin \frac{\theta}{2} \sqrt{T})
\]
(41)

Because of the global SU(2)$_f$ symmetry in the flavor sector, the same equations hold in channel 2.

Even the symmetry in the spin sector is $O(1) \times U(1)$ instead of $O(3)$ of the 2CK, the 2 channel and 2 parity fermions do make the same leading contribution to the total conductivity
\[
\sigma(T) \sim 2\sigma_u (1 - \tilde{q} \sin \frac{\theta}{2} \sqrt{T})
\]
(42)

For $\theta = \pi$, namely at the 2CK, $\tilde{q} = q$, then two universal ratios can be formed from Eqs. 27, 28, 42.

For $\theta = 0$, namely at the 2IK, $\tilde{q} = y$, the coefficient of $\sqrt{T}$ vanishes. It is evident that the 2nd order correction (actually any even order correction) to the Green function vanishes, the 3rd order correction will lead to $T^{3/2}$, but the coefficients still vanishes due to $\sim \sin \theta/2 = 0$, because odd order corrections have the same $i$ factor.

By conformal transforming Eq. 35 to finite temperature 4, we get the next-leading term of the low temperature electrical conductivity
\[
\sigma_{1+} \sim \tilde{y}(\cos \frac{\theta}{2} C_1 + \sin \frac{\theta}{2} \sin \theta C_2) T^{3/2}
\]
(43)

Putting $\theta = 0$ (then $\tilde{y} = -q$) in the above equation and adding the contribution from the operator in Eq. 36, we get the leading term at the 2IK fixed point:
\[ \sigma(T) \sim 2\sigma_u(1 + T^{3/2}) \]  

(44)

It is evident that even at the 2IK point, no universal ratios can be formed.

For general \( \theta \), the leading low temperature behaviors of the three physical measurable quantities are given by Eqs.27, 28, 42, no universal ratios can be formed.

The potential scattering term \( V_1 \) is a marginal operator which causes a phase shift in the charge sector:

\[ \Phi_{c,L} = \Phi_{c,R} + \theta_{ph} \]  

(45)

The symmetry of the fixed point is reduced to \( O(1) \times U(1) \times O(3) \times U(1) \), Eqs.40, 41 become:

\[ \sigma_{1+}(T) \sim \frac{\sigma_u}{2}(1 - \tilde{q} \sin \frac{\theta + \theta_{ph}}{2} \sqrt{T}) \]
\[ \sigma_{1-}(T) \sim \frac{\sigma_u}{2}(1 - \tilde{q} \sin \frac{\theta - \theta_{ph}}{2} \sqrt{T}) \]  

(46)

It is easy to see that in the presence of P-H symmetry breaking, in contrast to the 2CK, the different parity fermions do make different contributions to the conductivity, Eq.42 becomes

\[ \sigma(T) \sim 2\sigma_u(1 - \tilde{q} \sin \frac{\theta_{ph}}{2} \cos \frac{\theta_{ph}}{2} \sqrt{T}) \]  

(47)

Eq.43 becomes

\[ \sigma_{1+}(T) \sim \tilde{y}(\cos \theta_{ph} C_1 + \sin \theta_{ph} C_2)T^{3/2} \]
\[ \sigma_{1-}(T) \sim \tilde{y}(\cos \theta_{ph} C_1 + \sin \theta_{ph} C_2)T^{3/2} \]  

(48)

The total conductivity becomes

\[ \sim \tilde{y} \cos \frac{\theta_{ph}}{2} (\cos \frac{\theta}{2} C_1 + \sin \frac{\theta}{2} \sin \theta_{ph} C_2)T^{3/2} \]  

(49)

The total conductivity at the 2IK becomes

\[ \sigma(T) \sim 2\sigma_u(1 + \cos \frac{\theta_{ph}}{2} T^{3/2}) \]  

(50)

As shown by AL5, P-H symmetry breaking does not change the leading results of the specific heat and susceptibility.

In this section, we discussed two marginal operators, one in spin space \( \partial \Phi_{n}(0) \), another in the charge space \( \partial \Phi_{c}(0) \). Both make no contributions to the conductivity and make only subleading contributions to the thermodynamic quantities. However, \( \partial \Phi_{c}(0) \) is much more important. Combined with the leading irrelevant operator in the spin sector, it will generate the dimension 1/2 relevant operator which will always make the line of NFL fixed points unstable. Furthermore, as shown by Eqs.23, 25, even the coefficients of the leading terms of the thermodynamic quantities depend on the position on the line caused by \( \partial \Phi_{c}(0) \).

**IV. ADDITIONAL NFL FIXED POINT**

If \( \tilde{q} = \tilde{y} = \Delta_K = 0 \), this fixed point is located at \( \gamma_1 = 1, \gamma_2 = 0 \) where \( \tilde{a} \) decouples, \( \tilde{b} \) loses its kinetic energy and becomes a Grassmann Lagrangian multiplier, integrating \( \tilde{b} \) out leads to the boundary conditions :

\[ a_{s,i}^{L}(0) = -a_{s,i}^{R}(0) \]  

(51)

We also have the trivial boundary conditions

\[ b_{s,i}^{L}(0) = b_{s,i}^{R}(0), \quad b_{s,j}^{L}(0) = b_{s,j}^{R}(0) \]  

(52)

Using Eq.13, the above boundary conditions of \( H' \) correspond to the following boundary conditions of \( H \):

\[ a_{L}^{l}(0) = -a_{R}^{l}(0), \quad b_{L}^{l}(0) = b_{R}^{l}(0), \quad b_{L}^{s}(0) = b_{R}^{s}(0) \]  

(53)
The above boundary conditions can be expressed in terms of the \textit{new} chiral bosons in Eq.\ref{eq:boundary_conditions}:

$$\Phi^a_{s,L}(0) = -\Phi^a_{s,R}(0) + \pi$$ \hspace{1cm} (54)

In terms of \textit{new} physical fermions, it reads:

$$\psi_{i\pm,L}(0) = e^{\pm i \frac{\pi}{2}} S_{i\pm,R}(0)$$ \hspace{1cm} (55)

This is a NFL \textit{fixed point} with \( q = \sqrt{2} \) and the symmetry \( O(1) \times O(7) \) which is the fixed point symmetry of the 2IK(a) (Fig.\( \tilde{\text{I}} \)). At this fixed point, the original electrons scatter into the collective excitations which fit into the \( S \) spinor representation of SO(8). The finite size spectrum is listed in Table IV.

The local correlation functions at this fixed point are

$$\langle \hat{b}(\tau)\hat{b}(0) \rangle \sim \frac{1}{\tau}, \quad \langle a_{s,i}(0, \tau)a_{s,i}(0, 0) \rangle \sim \frac{\gamma^2}{\tau^3}$$ \hspace{1cm} (56)

We can also read the scaling dimensions of the various fields \([\hat{a}]=0, [\hat{b}]=1/2, [b_{sf}]=[a_{sf}]=[a_{s,i}]=3/2\).

This NFL fixed point is more unlikely to be observed by experiments, because it has \textit{two} relevant directions \( \Delta_1, \Delta_- \).

Similar to the discussions in the last section, at the 2IK fixed point, \( \hat{a} \) decouples and \( \hat{b} \) turns into the \textit{non-interacting} scaling field at the fixed point

$$\hat{b}(\tau) \sim a_{s,i}(0, \tau)$$ \hspace{1cm} (57)

The corresponding scaling field in \( H \) is

$$-b_s(0, \tau)$$ \hspace{1cm} (58)

The impurity spin turns into

$$S_x(\tau) \sim i\hat{a}a_s + \cdots$$

$$S_y(\tau) \sim i(\hat{a}b_s + a_sb_s) + \cdots$$

$$S_z(\tau) \sim i\hat{a}b_s + \cdots$$ \hspace{1cm} (59)

The impurity spin-spin correlation function \( \langle S^a(\tau)S^a(0) \rangle \sim 1/\tau \).

On the line of NFL fixed points discussed in the previous section, if \( \theta = 0 \), the fixed point symmetry is enlarged to the 2IK(b) (Fig.\( \tilde{\text{I}} \)). Although these two fixed points have the same symmetry, therefore the same finite size spectrum, but the \textit{allowed} boundary operators are very different. This additional NFL fixed point is also \textit{unstable}. \( \Delta_1, \Delta_- \) are \textit{two} relevant terms with scaling dimension 1/2, any linear combination of the two terms will drive the system to a given point on the line of FL fixed points to be discussed in the following section. They will generate two dimension 3/2 leading irrelevant operators \( L_+ b_{sf}(0, \tau), L_+ b_{s,i}(0, \tau) \) respectively\( \tilde{\text{I}} \) and two dimension 2 subleading irrelevant operators (energy-momentum tensors) \( b_{sf}(0, \tau)\frac{\partial b_{sf}}{\partial \tau}, b_{s,i}(0, \tau)\frac{\partial b_{s,i}}{\partial \tau} \). As explained in the last section, because the two leading irrelevant operators are \textit{first order Virasoro descendants}, they make no contributions. \( \gamma_2 \) term also has dimension 2. In all there are \textit{three} dimension 2 subleading irrelevant operators which contribute \( C_{imp} \sim T \).

In all, we have

$$C_{imp} \sim T$$ \hspace{1cm} (60)

From Eqs.\( \tilde{\text{I}} \), we get the susceptibility

$$\chi^h_{imp} \sim q^2 \log T$$ \hspace{1cm} (61)

By Eq.\( \tilde{\text{I}} \), the leading and subleading irrelevant operators in \( H' \) become

$$\frac{\partial b_{sf}(0, \tau)}{\partial \tau}, \quad \frac{\partial b_{s,i}(0, \tau)}{\partial \tau}, \quad b_{sf}(0, \tau)\frac{\partial b_{sf}}{\partial \tau}, \quad b_{s,i}(0, \tau)\frac{\partial b_{s,i}}{\partial \tau}$$

$$\gamma_2 \hat{b}(\tau)\frac{\partial \hat{b}(\tau)}{\partial \tau} = \gamma_2 a_s'(0, \tau)\frac{\partial a_s'(0, \tau)}{\partial \tau}$$ \hspace{1cm} (62)
By Eq. 13, the corresponding operators in $H$ are

$$\frac{\partial b_{sf}(0, \tau)}{\partial \tau}, \quad \frac{\partial a_{s}(0, \tau)}{\partial \tau}$$

They can be written in terms of the new bosons $b$: $\frac{\partial}{\partial \tau} \sin \Phi^{n}_{s}(0, \tau), \quad \frac{\partial}{\partial \tau} \cos \Phi^{n}_{s}(0, \tau)$

$$\pm \cos 2\Phi^{n}_{s}(0, \tau) \sim \langle \Phi^{n}_{s}(0, \tau) \rangle \sim \langle \Phi^{n}_{s}(0, \tau) \rangle$$

The 5th operator in Eq. 64 makes no contributions to the Green function either:

$$\langle \psi_{1+} (x_{1}, \tau_{1}) \psi_{1+} (x_{2}, \tau_{2}) \rangle \sim \int d\tau (e^{i\frac{1}{\tau} \Phi^{n}_{s}(x_{1}, \tau_{1})} \langle \cos 2\Phi^{n}_{s}(0, \tau) \rangle \sim \langle \Phi^{n}_{s}(0, \tau) \rangle \sim \langle \Phi^{n}_{s}(0, \tau) \rangle$$

It is easy to see higher order corrections due to the above operators also vanish. The $y$ and $q$ terms are two irrelevant operators with scaling dimension 5/2, they can be written in $H'$ as

$$: b(\tau) \frac{\partial b(\tau)}{\partial \tau} : \sim b(\tau) \frac{\partial b(\tau)}{\partial \tau} :$$

The bosonized forms in $H$ are

$$(\cos 2\Phi^{n}_{sf}(0, \tau) \sim \langle \Phi^{n}_{sf}(0, \tau) \rangle \sim \langle \Phi^{n}_{sf}(0, \tau) \rangle$$

$$(\cos 2\Phi^{n}_{sf}(0, \tau) \sim \langle \Phi^{n}_{sf}(0, \tau) \rangle \sim \langle \Phi^{n}_{sf}(0, \tau) \rangle$$

The first order correction due to the first operator is

$$\langle \psi_{1+} (x_{1}, \tau_{1}) \psi_{1+} (x_{2}, \tau_{2}) \rangle \sim i \int d\tau \langle e^{-\frac{1}{\tau} \Phi^{n}_{s}(x_{1}, \tau_{1})} e^{i\Phi^{n}_{s}(0, \tau)} e^{\frac{1}{\tau} \Phi^{n}_{s}(x_{2}, \tau_{2})} \rangle$$

The above integral is essentially the same as the first part of that in Eq. 55.

As explained in the preceding section, due to the factor $i$ difference than the first operator, the 2nd operator makes no contribution to the Green function. The other two dimension 5/2 operators are

$$L^{-2}b_{sf}(0, \tau) = \frac{3}{4} \frac{\partial^{2}}{\partial \tau^{2}} \sin \Phi^{n}_{s}(0, \tau), \quad L^{-2}a_{s}(0, \tau) = \frac{3}{4} \frac{\partial^{2}}{\partial \tau^{2}} \cos \Phi^{n}_{s}(0, \tau)$$

Because they can still be written as total derivatives, therefore make no contributions to the Green functions. Conformal transformation of Eq. 56 to finite temperature and scaling analysis lead to

$$\sigma(T) \sim 2\sigma_{u}(1 + T^{3/2})$$

There is no chance to form universal ratios from Eqs. 60, 61, 71.
V. THE LINE OF FL FIXED POINTS

If $\hat{q} = \hat{y} = 0$, this fixed point is located at $\gamma_1 = \gamma_2 = 0$ where both $\hat{a}$ and $\hat{b}$ lose their kinetic energies and become two Grassmann Lagrangian multipliers, integrating them out leads to the following boundary conditions:

$$a^{L,L}_{s,i}(0) = -a^{R,R}_{s,i}(0), \quad b^{L,L}_{s,i}(0) = -b^{R,R}_{s,i}(0) \quad (72)$$

We also have the trivial boundary conditions

$$\tilde{b}^{L,R}_{s,i}(0) = \tilde{b}^{R,L}_{s,i}(0) \quad (73)$$

Following the same procedures as those in the discussion of the line of NFL fixed points, the above boundary conditions correspond to the boundary conditions in $H$:

$$b^{L,R}_{s,i}(0) = \tilde{b}^{R,L}_{s,i}(0), \quad \psi^{L,R}_{s,i}(0) = e^{i\theta} \psi^{R,L}_{s,i}(0) \quad (74)$$

The above boundary condition can be expressed in terms of the new bosons in Eq. 3:

$$\Phi^{a}_{s,L} = \Phi^{a}_{s,R} + \theta \quad (75)$$

As discussed in the line of NFL fixed points, the physical fermions transform as the spinor representation of the above boundary conditions, the corresponding boundary conditions are derived in Appendix B, the result is

$$\psi^{L,R}_{1 \pm} = e^{\pm i\theta/2} \psi^{R,L}_{1 \pm} \quad (76)$$

It can be checked explicitly the above boundary conditions satisfy all the symmetry requirement. The fermion fields with the even and the odd parity under $Z_2$ suffer opposite continuously changing phase shifts $\pm e^{i\theta}$ along this line of fixed points. Depending on the sign of $\Delta_0$, the impurity will occupy either the even parity state with $S^x = \frac{1}{2}$ or the odd parity state $S^x = -\frac{1}{2}$. This simple physical picture should be expected from the starting Hamiltonian Eq. 2.

We have the trivial boundary conditions

$$\tilde{q}^{L,R}_{s,i}(0) = \tilde{q}^{R,L}_{s,i}(0) \quad (77)$$

This line of FL fixed points is stable. There is no relevant operator. There is one marginal operator in the spin sector $\partial \Phi_{s,i}^a(0)$ which changes the angle $\theta$ in Eq. 73: $\gamma_1$ and $\gamma_2$ terms which are leading irrelevant operators have dimension 2, they lead to typical fermi liquid behaviors. $\hat{q}$ term has dimension 3; in $H'$, it can be written as:

$$(\tilde{b}(\tau))^{\partial \Phi}_{s,i}^a(\tau) : b_{s,i}(0,\tau) \tilde{b}_{s,f,i}(0,\tau).$$

The $\hat{y}$ term has dimension 4: in $H'$, it can be written as:

$$(\tilde{a}(\tau))^{\partial \Phi}_{s,i}^a(\tau) : b_{s,i}(0,\tau) \tilde{a}_{s,f,i}(0,\tau).$$

This operator contents are completely consistent with our direct analysis mentioned above. This complete agreement provides a very powerful check on the method developed in Ref 5.

The local correlation functions at this fixed point are

$$\langle \tilde{a}(\tau) \hat{a}(0) \rangle \sim \frac{1}{\tau}, \quad \langle \tilde{b}_{s,f,i}(0,\tau) \hat{b}_{s,f,i}(0,0) \rangle \sim \frac{\gamma^2}{\tau^3}, \quad \langle \tilde{b}(\tau) \hat{b}(0) \rangle \sim \frac{1}{\tau}, \quad \langle a_{s,i}(0,\tau) a_{s,i}(0,0) \rangle \sim \frac{\gamma^2}{\tau^3} \quad (77)$$

From the above equation, we can also read the scaling dimension of the various fields $[\hat{a}] = [\hat{b}] = 1/2, [\tilde{a}_{s,i}] = [\tilde{b}_{s,f,i}] = 3/2$. 

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At the fixed point, $\hat{a}, \hat{b}$ turn into the \textit{non-interacting} scaling fields in $H'$

$$\hat{a}(\tau) \sim \hat{b}_{sf,i}(0, \tau) = \frac{1}{\Delta_K} (\Delta - b_s,i(0, \tau) + \Delta_1 b_{sf}(0, \tau))$$

$$\hat{b}(\tau) \sim a_{s,i}(0, \tau)$$  \hspace{1cm} (78)

The corresponding two scaling fields in $H$ are

$$\frac{1}{\Delta_K} (-\Delta - a_s + \Delta_1 b_{sf})$$

$$- b_s(0, \tau)$$  \hspace{1cm} (79)

The impurity spin turns into

$$S_x(\tau) \sim -i \frac{\Delta_1}{\Delta_K} a_s b_{sf} + \cdots$$

$$S_y(\tau) \sim i (a_s b_s + \frac{1}{\Delta_K} (-\Delta - a_s + \Delta_1 b_{sf}) b_s) + \cdots$$

$$S_z(\tau) \sim i \frac{1}{\Delta_K} (-\Delta - a_s + \Delta_1 b_{sf}) b_s + \cdots$$  \hspace{1cm} (80)

The impurity spin-spin correlation function shows typical FL behavior

$$\langle S^a(\tau) S^a(0) \rangle \sim 1/\tau^2$$  \hspace{1cm} (81)

The two leading irrelevant operator in $H$ become

$$\gamma_1 \hat{a}(\tau) \frac{\partial \hat{a}(\tau)}{\partial \tau} =$$

$$\frac{(\Delta - \Delta)^2 a_s}{\Delta_K} \frac{\partial a_s(\tau)}{\partial \tau} - 2 \frac{\Delta_1}{\Delta_K} a_s \frac{\partial b_{sf}(\tau)}{\partial \tau} + \frac{\Delta_1}{\Delta_K} b_{sf} \frac{\partial b_{sf}(\tau)}{\partial \tau}$$

$$= \cos \theta : \cos 2\Phi^s_{sf}(0, \tau) : - \frac{1}{2} : (\partial \Phi^s_{sf}(0, \tau))^2 :$$

$$- \sin \theta : \sin 2\Phi^s_{sf}(0, \tau) : - \partial^2 \Phi^s_{sf}(0, \tau) :$$

$$\gamma_2 \hat{b}(\tau) \frac{\partial \hat{b}(\tau)}{\partial \tau} \sim \gamma_2 (\cos 2\Phi^s_{sf}(0, \tau) : - \frac{1}{2} : (\partial \Phi^s_{sf}(0, \tau))^2 :)$$  \hspace{1cm} (82)

Although the first operator do depend on the angle $\theta$, its scaling dimension remains 2, therefore will not affect the exponents of any physical measurable quantities. We refer the readers to Ref.\textsuperscript{38} for the detailed similar calculations on the single particle Green function and the electron conductivity. Second order corrections to the single particle Green functions from the two leading irrelevant operators lead to $\sigma(T) \sim \sigma(0) + C(\theta)T^2$.

\section{VI. THE EFFECTS OF EXTERNAL MAGNETIC FIELD}

According to Ref.\textsuperscript{3}, the parameters in Eq.\textsuperscript{2} are

$$V_1 = \pi \rho_F V$$  \hspace{1cm} (83)

$$V_2 = \pi \rho_F V \frac{\sin k_F R}{k_F R}$$  \hspace{1cm} (84)

$$V_3 = \pi \rho_F V \sqrt{1 - \left(\frac{\sin k_F R}{k_F R}\right)^2}$$  \hspace{1cm} (85)

The external magnetic field $H$ breaks the $SU(2)$ flavor (the real spin) symmetry. It causes the energy band of spin $\uparrow$ electrons to shift downwards, that of spin $\downarrow$ to shift upwards. Channel 1 and 2 electrons have \textit{different} fermi momenta, therefore couple to the impurity with \textit{different} strength. Setting the external strain $h = 0$, the Hamiltonian is
\[ H = H_0 + i\delta v_F \int dx (\psi_{1a}^\dagger(x) \frac{\partial \psi_{1a}(x)}{\partial x} - \psi_{2a}^\dagger(x) \frac{\partial \psi_{2a}(x)}{\partial x}) + V_1 J_z(0) + + \delta V_1 J_z^f(0) + 2V_2 J_z(0) + 2\delta V_2 J_z^f(0) \\
+ 4V_3 S^x J_z^f(0) + \Delta_1 (J_z^f(0) S^x + J_z^f(0) S^y) \\
+ 4\delta V_5 S^x J_z^f(0) + \delta \Delta_1 (J_z^f(0) S^x + J_z^f(0) S^y) \\
+ \Delta_{12} S^x + \Delta_{22} 2\pi \tau_c (S^\dagger \psi_{11}^\dagger \psi_{21}^\dagger \psi_{21} + h.c.) \] (86)

Where \( J_x^n(x) = J_x^R(x) - J_x^L(x) \) and all the \( \delta \) terms are \( \sim H \).

The term \( \frac{\delta H}{\delta \theta} \int dx (\partial \Phi_f(x) \partial \Phi_f(x) + \partial \Phi_f(x) \partial \Phi_f(x)) \) does not couple to the impurity, therefore can be neglected.

It is important to observe the bare hopping term and the two electron assisted hopping term are not affected by the magnetic field. Following Ref. 38, the transformed Hamiltonian \( H' = UHU^{-1} \) is

\[ H' = H_0 + 2\gamma \hat{a}_{s,i}(0) b_{sf}(0) + 2\delta y \hat{a}_{s,i}(0) b_{s,f}(0) \\
+ 2q \hat{a}_{s,i}(0) b_{s,i}(0) + 2\delta q \hat{a}_{s,f}(0) b_{s,f}(0) \\
- i\frac{\Delta_1}{\sqrt{2\pi \tau_c}} \hat{a}_{s,f}(0) + i\frac{\delta \Delta_1}{\sqrt{2\pi \tau_c}} \hat{a}_{s,f}(0) \\
+ i\frac{\Delta_1}{\sqrt{2\pi \tau_c}} \hat{a}_{s,i}(0) + i\frac{\delta \Delta_1}{\sqrt{2\pi \tau_c}} \hat{a}_{s,i}(0) \] (87)

Performing the rotation Eq. 39, the above equation can be rewritten as

\[ H' = H_0 + 2q \hat{a}_{s,i}(0) \hat{b}_{s,i}(0) + 2\delta q \hat{a}_{s,f}(0) \hat{b}_{s,f}(0) \\
+ 2\gamma \hat{a}_{s,i}(0) \hat{b}_{s,i}(0) + 2\delta \gamma \hat{a}_{s,f}(0) \hat{b}_{s,f}(0) \\
- i\frac{\Delta_K}{\sqrt{2\pi \tau_c}} \hat{b}_{s,f}(0) + i\frac{\Delta_1}{\sqrt{2\pi \tau_c}} \hat{b}_{s,i}(0) + i\frac{\delta \Delta_1}{\sqrt{2\pi \tau_c}} \hat{b}_{s,i}(0) \] (88)

It is evident that the magnetic field \( H \) introduces, another relevant operator with scaling dimension 1/2. Under the combination of the two relevant directions in the above equation, the system flows to the line of FL fixed points with the boundary conditions

\[ \Phi^n_{s,L} = \Phi^n_{s,R} + \theta_s, \quad \Phi^n_{s,f,L} = \Phi^n_{s,f,R} + \theta_{sf} \] (89)

If \( H = 0, \theta_{sf} = 0 \), the boundary condition Eq. 39 is recovered. If \( \Delta_1 = 0, \theta_{sf} = \pi \).

There are two marginal operators along this line of FL fixed points, one \( \partial \Phi^n_{s}(0) \) is in the spin sector which changes the angle \( \theta_s \), another \( \partial \Phi^n_{s,f}(0) \) is in the spin-flavor sector which changes the angle \( \theta_{sf} \). The corresponding boundary conditions in the original fermion basis can be similarly worked out as in the last section.

VII. SCALING ANALYSIS OF THE PHYSICAL MEASURABLE QUANTITIES

In this section, following the methods developed in Ref. 40 and also considering the correction due to the leading irrelevant operators, we derive the scaling functions of the conductivity, impurity specific heat and susceptibility:

\[ A(T, \Delta_+, H, \lambda) = F(\frac{a \Delta_+}{\sqrt{T}}, \frac{bH}{\sqrt{T}}, \lambda \sqrt{T}) \] (90)

Where \( a, b \) are non-universal metric factors which depend on \( \theta, \theta_{ph} \) and the cutoff of the low energy theory, the dependence on \( \theta \) is due to the existence of the exactly marginal operator \( \partial \Phi^n_{s}(0) \) in the spin sector, the Kondo temperature is given by \( T_K \sim \lambda^{-2} \).

We confine \( T < T_K \), so \( \lambda \sqrt{T} \) is a small parameter, we expand the right hand side of Eq. 40 in terms of the leading irrelevant operator

\[ A(T, \Delta_+, H, \lambda) = f_0(\frac{a \Delta_+}{\sqrt{T}}, \frac{bH}{\sqrt{T}}) + \lambda \sqrt{T} f_1(\frac{a \Delta_+}{\sqrt{T}}, \frac{bH}{\sqrt{T}}) + (\lambda \sqrt{T})^2 f_2(\frac{a \Delta_+}{\sqrt{T}}, \frac{bH}{\sqrt{T}}) + \cdots \] (91)
For simplicity, we only consider $\Delta_+ \neq 0$ or $H \neq 0$. The general case Eq. can be discussed along the similar line of Ref.

From Eq. it is easy to observe that $\Delta_+$ term and the magnetic field $H$ term play very similar roles. In the following, we only explicitly derive the scaling function in terms of $\Delta_+$. The scaling functions in the presence of $H$ can be obtained by replacing $\Delta_+$ by $H$.

As discussed in Sec. V, depending on the sign of $\Delta_+$, the impurity is either in even parity or odd parity states, but the physical measurable quantities should not depend on if the system flows to FL1 (even parity) or FL2 (odd parity), so the above scaling function should only depend on $|\Delta_+|$.

In the following, we discuss the scaling functions of $\sigma(T) - \sigma(0), C_{\text{imp}}, \chi_{\text{imp}}, \chi_{\text{imp}}$ respectively. Here $\chi_{\text{imp}}, \chi_{\text{imp}}$ are the impurity hopping and spin susceptibility respectively.

The scaling function of the conductivity

We look at the two different limits of the $f$ functions.

Keeping $T < T_K$ fixed, let $\Delta_+ \to 0$, the system is in the Quantum Critical (QC) regime controlled by the line of NFL fixed points. We can do perturbative expansions in terms of $\Delta_+$. Symmetry and analyticity in the spinor representation of the NFL boundary condition Eq. should be fixed by the requirement of symmetry and analyticity. The perturbative expansions are

$$\sigma_i(T, \Delta_+, \lambda = 0) - \sigma(0) \sim \frac{\Delta_+}{\sqrt{T}} + \left(\frac{\Delta_+}{\sqrt{T}}\right)^3 + \cdots$$

The total conductivity $\sigma(T, \Delta_+, \lambda = 0) - \sigma(0) = 0$, therefore $f_0(x) \equiv 0$. The conductivities from the different parities have to cancel each other, otherwise, we get a non-analytic dependence at small magnetic field at finite temperature.

$$f_1(x) = 1 + x^2 + x^4 + \cdots, \quad x \ll 1$$

Substituting the above equation into Eq. we get

$$\sigma(T, \Delta_+, \lambda) - \sigma(0) = \lambda \sqrt{T} + \frac{\lambda \Delta_+^2}{\sqrt{T}} + \cdots$$

Keep $\Delta_+$ fixed, but small, let $T \to 0$, the system is in FL1 (or FL2) regime, the conductivity should reduce to the FL form

$$f_0(x) \equiv 0$$
$$f_1(x) = |x| + |x|^{-3} + \cdots, \quad x \gg 1$$
$$f_2(x) \equiv 0$$

Substituting the above equation into Eq. we have

$$\sigma(T, \Delta_+, \lambda) - \sigma(0) = \lambda |\Delta_+| + (\lambda |\Delta_+|)^3 + (\lambda |\Delta_+|^{-3} + \cdots)T^2 + \cdots$$

The above equation indicates that the coefficient of $T^2$ diverges as $\lambda |\Delta_+|^{-3}$ instead of $\Delta_+^{-4}$ as we approach to the line of NFL fixed points. This means the relevant operator $\Delta_+$ with scaling dimension $1/2$ combined with the leading irrelevant operator $\lambda$ with scaling dimension $-1/2$ near the line of NFL fixed points will turn into one of the irrelevant operators $\lambda_{FL,-2}$ with scaling dimension $-2$ near the line of FL fixed points

$$\lambda_{FL,-2} \sim \lambda |\Delta_+|^{-3}$$

First order perturbation in this operator leads to Eq.

The scaling function of the impurity specific heat

In the QC regime, the perturbative expansions give (up to possible logarithm):

$$g_0(x) = x^2 + x^4 + \cdots, \quad x \ll 1$$
$$g_1(x) \equiv 0$$
$$g_2(x) = 1 + x^2 + \cdots, \quad x \ll 1$$
Substituting the above equation into Eq. 91, we get
\[ C_{\text{imp}} = \frac{\Delta^2}{T} + \frac{\Delta^4}{T^2} + \lambda^2 T \log T + \lambda^2 \Delta^2 + \cdots \] (99)

It was known that there are accidental logarithmic violations of scaling when the number of channel is two. This violation has nothing to do with the existence of marginally irrelevant operators. Similar violation occur in itinerant magnetism.

In the FL regime, the impurity specific heat should reduce to the FL form
\[ g_0(x) = x^{-2} + \cdots, \quad x \gg 1 \]
\[ g_2(x) = c + \cdots, \quad x \gg 1 \] (100)

Substituting the above equation into Eq. 91, we get
\[ C_{\text{imp}} = T(\Delta^2 + \lambda^2 + \cdots) + \cdots \] (101)

The above equation indicates that the coefficient of \( T \) diverges as \( \Delta^2 \) as we approach to the line of NFL fixed points. This means the relevant operator \( \Delta^2 \) with scaling dimension 1/2 near the line of NFL fixed points will turn into one of the leading irrelevant operators \( \lambda_{FL,-1} \) with scaling dimension \(-1\) near the line of FL fixed points
\[ \lambda_{FL,-1} \sim \Delta^2 \] (102)

First order perturbation in this operator leads to Eq. 101. However, as shown in Eq. 96, this leading irrelevant operator make no contribution to the total conductivity, even though it makes contribution to even and odd parity conductivity separately (see Eq. 92).

*The scaling function of the impurity hopping susceptibility*

In the QC regime, the perturbative expansions give (up to possible logarithm)
\[ \chi_{\text{imp}}^h = \lambda^2 \log T + \frac{\lambda^2 \Delta^2}{T} + \cdots \] (103)

In the FL regime
\[ \chi_{\text{imp}}^h = \lambda^2 \log 1/\Delta^2 + \cdots \] (104)

The exact crossover function can be calculated along the EK line in Eq. 10. In the FL regime, the Wilson Ratio \( R = T \chi_{\text{imp}}^h / C_{\text{imp}} \sim \lambda^2 \Delta^2 \log 1/\Delta^2 \) is very small as \( \Delta \to 0 \).

*The scaling function of the impurity spin susceptibility*

In this part, we set \( \Delta = 0 \) and consider finite \( H \).

In the QC regime, the perturbative expansions give (up to possible logarithm)
\[ h_0(x) = \log aT + \cdots, \quad x \ll 1 \]
\[ h_1(x) = 0 \]
\[ h_2(x) = c_2 + x^2 + \cdots, \quad x \ll 1 \] (105)

Substituting the above equation into Eq. 101, we get
\[ \chi_{\text{imp}} = \log aT + \lambda^2 T + \lambda^2 H^2 + \cdots \] (106)

In the FL regime, it was shown in Ref. 20
\[ \chi_{\text{imp}} = \log aH + \cdots \] (107)

Actually, the whole crossover functions \( g_0(x) \), \( h_0(x) \) have been calculated along the EK line in Ref. 20.
VIII. DISCUSSIONS ON EXPERIMENTS AND CONCLUSIONS

In this paper, we brought about the rich phase diagram of a non-magnetic impurity hopping between two sites in a metal (Fig. 3). As discussed in Sec. IV, the NFL fixed point with the symmetry 2IK(a) is very unlikely to be observed, although it has very interesting behaviors $C_{\text{imp}} \sim T, \chi_{\text{imp}}^h \sim \log T$ and $\sigma(T) \sim 2\pi(1 - T^{3/2})$. The peculiar behaviors of $C_{\text{imp}}, \chi_{\text{imp}}$ are due to the ‘orbital field’ couples to a non conserved current.

Ralph et al. found that the experiment data show $\sqrt{T}$ behavior for $0.4K < T < T_{K1} \sim 4K$ and concluded the two sites system fall in the Quantum Critical (QC) regime controlled by the “2CK fixed point”. They also discussed the crossover to the FL regime in the presence of magnetic field which acts as a channel anisotropy of scaling dimension $1/\sqrt{2}$ and in the presence of asymmetry of the two sites which acts as a local magnetic field of scaling dimension $1/\sqrt{2}$. As first pointed out by MS, even the two sites are exactly symmetric, therefore, the two channel are exactly equivalent, there is another dimension 1/2 operator $\Delta_+$ which will drive the system to FL regime.

In this paper, we find the “2CK fixed point” actually is a line of NFL fixed points which interpolates continuously between the 2CK and the 2IK(a). As Gal et al. showed that under different canonical transformation than employed in this paper, the 2IK model can be mapped to the 2CK model. This paper discussed the two apparent different fixed points in a unified framework. Although P-H symmetry breaking is a relevant perturbation in the original 2IK model discussed in Refs. 13, 14, but its effect is trivial in this model. Because the two models have different global symmetry, although the fixed point is exactly the same, the allowed boundary operators are different.

We discovered a marginal operator in the spin sector which is responsible for this line of NFL fixed points. In a real system, there is always P-H symmetry breaking, therefore there is always a marginal operator in the charge sector. Eq. 17 show that the coefficient of $\sqrt{T}$ behavior depend on this breaking any way. However, the marginal operator identified in this paper is in the spin sector, it, combined with the leading irrelevant operator which contributes to the $\sqrt{T}$ behavior of the conductivity, will always generate the dimension 1/2 relevant operator. The existence of the line of NFL fixed points and the existence of the relevant operator is closely related. There is no reason why the coefficient of this relevant operator is so small that it can be neglected in the scaling analysis in the last section.

The crossover scale from the weak coupling fixed point $q = 0$ to any point on the line of NFL fixed points is given by the Kondo scale $T_{K1} \sim D(\Delta_K)^2 \sim \lambda^2$ (see Eq. 94), the crossover scale from a given point on the line of NFL fixed points to the corresponding point on the line of FL fixed points is set by $\Lambda \sim D(\Delta_K)^2$, the finite size scaling at finite temperature $T$ leads to the universal scaling function Eq. 90. Because there is no reliable way to estimate the magnitude of $\Delta_+$ which is always present. It is very hard to say if the experimental situation do fall in the QC regime controlled by any fixed point on this line.

The experiment measured the magnetic field dependence of conductance signal. Assuming $\Delta_+$ is so small that it can be neglected, scaling Eqs 96 in the field $H$ shows that in the FL regime, the conductance should depend on $\lambda |H|$ which is consistent with the experimental finding. The coefficient of $T^2$ of the conductivity should scale as $H^{-3}$. Because $T_K \sim \lambda^{-\frac{1}{3}} \sim 4K$, the lowest temperature achieved in the experiment is $T_{\min} \sim 0.1K$, if $0.1K < |H| < \lambda^{-1} \sim \sqrt{T_K} = 2K$, Eq. 101 shows that the linear $T$ coefficient of the impurity specific heat should scale as $H^{-2}$. Eq. 101 shows that the impurity susceptibility should scale as $\log |H|$. So far, there is no experiment test of these scaling relations in this range of magnetic field. It should be possible to extract $\chi_{\text{imp}}$ from experimental data, because the impurity does not carry real spin. There is no difficulty caused by the conduction electrons and the impurity having different Landé factors.

It is very difficult, but not impossible to measure $\chi_{\text{imp}}^h$ by adding pressure and strain to the system. Because $\chi_{\text{imp}}^h$ here is the hopping susceptibility. The difficulty caused by the conduction electrons and the impurity having different Landé factors is not a issue either. Unfortunately, no universal ratios can be formed among the amplitudes of the three quantities except at the 2CK point on this line of NFL fixed points, this is because (1) the strain couples to a non conserved current (2) this is a line of NFL fixed points instead of a fixed point.

The experiment also measured the conductance signal when a finite voltage $V$ is applied to the point contacts and find $eV/T$ scaling in the temperature range $0.4K < T < 4K$. It is not clear how the position on this line of NFL fixed points enter the expression of the non-equilibrium conductivity calculations. This question deserve to be addressed seriously.

Ref. 28 showed that a non-magnetic impurity hopping around 3 sites arranged around a triangular is delocalized and in the presence of asymmetry of the two sites which acts as a local magnetic field of scaling dimension $3/2$, as Gan et al. showed that under pressure and strain to the system. Because the three quantities except at the 2CK point on this line of NFL fixed points, this is because (1) the strain couples to a non conserved current (2) this is a line of NFL fixed points instead of a fixed point.

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realized at just one point on this line of NFL fixed points. It was shown in Ref. [48] that the 2CK fixed point with the symmetry SU(2) can be realized in C_{3v} or higher symmetry, because it is indeed possible for the ground state of the impurity-electrons complex to transform as the Γ_3 representation of C_{3v} group, therefore a doublet. This NFL fixed point was also shown to be stable. Similarly, Ref. [48] pointed out that the stable SU(3) NFL fixed point can be realized in the system of a non-magnetic impurity hopping around the tetrahedral or octahedral sites in a cubic crystal when the ground state is a triplet. However, as the symmetry get higher, the NFL fixed point with higher symmetry will be more unlikely to be realized by experiments, because the number of relevant processes will increase.

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\[ E_0 = \frac{1}{16} + \frac{1}{2} \left( \frac{\delta}{\pi} \right)^2 \]

\[ NS_5 \] is the state achieved by twisting NS by the angle 2δ, namely \( \psi(-l) = -e^{i2\delta} \psi(l) \). R_δ is the state achieved by twisting R by the angle 2δ, namely \( \psi(-l) = e^{i2\delta} \psi(l) \). NS+1st is the first excited state in NS sector et. al. Only when \( \frac{\pi}{4} < \delta < \frac{\pi}{2} \), the 4th row has lower energy than the 5th row. If \( \delta = 0 \), the symmetry is enlarged to \( O(1) \times O(7) \), the finite size spectrum of the 2IK fixed point is recovered. If \( \delta = -\frac{\pi}{2} \), the symmetry is enlarged to \( O(3) \times O(5) \), the finite size spectrum of the 2CK fixed point is recovered.

\[ \frac{1}{\sqrt{N}} (E - E_0) \]

\( N_{\parallel} \) is the state achieved by twisting \( N_{\parallel} \) by the angle 2δ, namely \( \psi(-l) = e^{i2\delta} \psi(l) \). \( R_{\parallel} \) is the state achieved by twisting \( R_{\parallel} \) by the angle 2δ, namely \( \psi(-l) = -e^{i2\delta} \psi(l) \). Only when \( \frac{\pi}{4} < \delta < \frac{\pi}{2} \), the 4th row has lower energy than the 5th row. If \( \delta = 0 \), the symmetry is enlarged to \( O(1) \times O(7) \), the finite size spectrum of the 2IK fixed point is recovered. If \( \delta = -\frac{\pi}{2} \), the symmetry is enlarged to \( O(3) \times O(5) \), the finite size spectrum of the 2CK fixed point is recovered.

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\[
\begin{array}{|c|c|c|c|}
\hline
O(1) & O(7) & \frac{1}{\pi \sqrt{\pi}} (E - \frac{1}{16}) & \text{Degeneracy} \\
\hline
R & NS & 0 & 2 \\
NS & R & \frac{1}{4} & 8 \\
R & NS + 1st & \frac{1}{4} & 14 \\
NS + 1st & R & \frac{1}{4} & 8 \\
R & NS + 2nd & 1 & 42 \\
R + 1st & NS & 1 & 2 \\
\hline
\end{array}
\]

TABLE IV. The finite size spectrum of the 2IK fixed point

\[
\begin{array}{|c|c|c|c|}
\hline
U(1) & O(6) & \frac{1}{\pi \sqrt{\pi}} (E - E_0) & \text{Degeneracy} \\
\hline
NS_5 & NS & 0 & 1 \\
R_1 & R & \frac{1}{4} - \frac{\delta}{\pi} & 8 \\
NS_6 & NS + 1st & \frac{1}{4} & 6 \\
R_2 + 1st & R & \frac{1}{4} + \frac{\delta}{\pi} & 16 \\
NS_8 + 1st & NS & \frac{1}{4} + \frac{\delta}{\pi} & 2 \\
\hline
\end{array}
\]

TABLE V. The finite size spectrum at the line of FL fixed points with the symmetry \(U(1) \times O(6)\) when \(0 < \delta < \frac{\pi}{2}\). \(E_0 = \frac{1}{4}(\frac{\delta}{\pi})^2\). If \(\delta = 0\), the symmetry is enlarged to \(O(8)\), the finite size spectrum of the free fermion fixed point is recovered. If \(\delta = \frac{\pi}{2}\), the symmetry is enlarged to \(O(2) \times O(6)\), the finite size spectrum of the 2CSFK is recovered.

\[
\begin{array}{|c|c|c|c|}
\hline
U(1) & O(6) & \frac{1}{\pi \sqrt{\pi}} (E - E_0) & \text{Degeneracy} \\
\hline
NS_5 & NS & 0 & 1 \\
R_5 & R & \frac{1}{4} + \frac{\delta}{\pi} & 8 \\
NS_6 & NS + 1st & \frac{1}{4} & 2 \\
NS_8 & NS + 1st & \frac{1}{4} & 6 \\
NS_8 + 2nd & NS & 2(\frac{1}{4} + \frac{\delta}{\pi}) & 1 \\
NS_8 + 1st & NS + 1st & 1 + \frac{\delta}{\pi} & 12 \\
R_8 + 1st & R & \frac{1}{4}(1 + \frac{\delta}{\pi}) & 16 \\
\hline
\end{array}
\]

TABLE VI. The finite size spectrum at the line of FL fixed points with the symmetry \(U(1) \times O(6)\) when \(-\frac{\pi}{2} < \delta < 0\). \(E_0 = \frac{1}{4}(\frac{\delta}{\pi})^2\). If \(\delta = 0\), the symmetry is enlarged to \(O(8)\), the finite size spectrum of the free fermion fixed point is recovered. If \(\delta = -\frac{\pi}{2}\), the symmetry is enlarged to \(O(2) \times O(6)\), the finite size spectrum of the 2CSFK is recovered.

\[
\begin{array}{|c|c|c|}
\hline
O(8) & \frac{1}{\pi \sqrt{\pi}} E & \text{Degeneracy} \\
\hline
NS & 0 & 1 \\
R & \frac{1}{4} & 16 \\
NS + 1st & \frac{1}{4} & 8 \\
NS + 2nd & 1 & 28 \\
NS + 3rd & \frac{1}{4} & 64 \\
R + 1st & \frac{1}{4} & 8 \\
\hline
\end{array}
\]

TABLE VII. The finite size spectrum of the free fermions with both NS and R sectors.
| $O(2)$ | $O(6)$ | $\frac{-1}{2\pi^2} (E - \frac{1}{2})$ | Degeneracy |
|--------|--------|-----------------------------------|------------|
| $R$    | $NS$   | 0                                 | 2          |
| $NS$   | $R$    |                                    | 8          |
| $R$    | $NS + 1st$ |                                  | 12         |
| $NS + 1st$ | $R$    |                                    | 16         |
| $R$    | $NS + 2nd$ |                                  | 30         |
| $R + 1st$ | $NS$   | 1                                 | 4          |

TABLE VIII. The finite size spectrum of the 2CSFK fixed point.
APPENDIX A: THE FINITE SIZE SPECTRUM OF ONE COMPLEX FERMIONS

In this appendix, we closely follow the notations of

The energy momentum tensor of one complex fermion in the complex plane is:

\[ T(z) = :\psi^*(z)\partial\psi(z) : \]  \hspace{1cm} (A1)

Where the complex fermion can be written in terms of two Majorana fermions:

\[ \psi(z) = \frac{1}{\sqrt{2}}(\psi_1(z) + i\psi_2(z)), \quad \psi^*(z) = \frac{1}{\sqrt{2}}(\psi_1(z) - i\psi_2(z)) \]  \hspace{1cm} (A2)

The mode expansions of the complex fermion are

\[ i\psi(z) = \sum_n \psi_n z^{-n-1/2}, \quad \psi_n = \frac{1}{\sqrt{2}}(\psi_1^n + i\psi_2^n) \]

\[ i\psi(z) = \sum_n \psi_n^* z^{-n-1/2}, \quad \psi_n = \frac{1}{\sqrt{2}}(\psi_1^n - i\psi_2^n) \]  \hspace{1cm} (A3)

The modes satisfy the commutation relations:

\[ \{\psi_n, \psi_m\} = \{\psi_n^*, \psi_m^*\} = 0, \quad \{\psi_n, \psi_m^*\} = \delta_{n+m,0} \]  \hspace{1cm} (A4)

Under the conformal transformation \( z = e^w \), mapping the cylinder, labeled by \( w = \tau + i\sigma \), to the plane, labeled by \( z \)

\[ \psi(w) = \sum_n \psi_n e^{-nw}, \quad \psi^*(w) = \sum_n \psi_n^* e^{-nw} \]  \hspace{1cm} (A5)

Where \( n \in Z - \theta \) in order to satisfy the boundary conditions on the cylinder

\[ \psi(w + 2\pi) = e^{i2\pi \theta} \psi(w), \quad 0 < \theta < 1 \]  \hspace{1cm} (A6)

The energy of the complex fermion on the cylinder is

\[ H = L_0 = \sum_{n \in Z} (n - \theta) :\psi_{n-\theta}^*\psi_{n+\theta} : \]  \hspace{1cm} (A7)

From the commutation relations

\[ [L_0, \psi_{n-\theta}] = -(n - \theta)\psi_{n-\theta}, \quad [L_0, \psi_{n-\theta}^*] = -(n - \theta)\psi_{n-\theta}^* \]  \hspace{1cm} (A8)

and the fact that the regularity at \( \tau = -\infty \) requires the positive modes annihilate the vacuum

\[ \psi_{n-\theta}|0\rangle_\theta = 0, \quad n - \theta > 0 \]  \hspace{1cm} (A9)

We can construct the Hilbert space

\[ \cdots \psi_{-m-\theta}^* \psi_{-n-\theta}|0\rangle_\theta \]  \hspace{1cm} (A10)

Where \( m, n \) are no negative integers.

The energy of the twisted vacuum \( |0\rangle_\theta \) is

\[ E_0 = f(\theta) = -\frac{1}{24} + \frac{1}{2}\left(\frac{\delta}{\pi}\right)^2, \quad \theta = \frac{1}{2} + \frac{\delta}{\pi} \]  \hspace{1cm} (A11)

The energies of the excited states in Eq. A10 are

\[ E = E_0 + m + \theta + n + \theta + \cdots \]  \hspace{1cm} (A12)

If \( \theta = 0 \), then the states are in Ramond (periodic) sector, because of zero modes, the ground state degeneracy is 4, the ground states are \( |0\rangle, \psi_0|0\rangle, \psi_0^*|0\rangle, \psi_0\psi_0^*|0\rangle \). If \( \theta = \frac{1}{2} \), the states are in Neveu-Schwarz (anti-periodic) sector.
APPENDIX B: THE DERIVATION OF THE BOUNDARY CONDITIONS

SO(8) has three 8-dimensional representations, one vector representation \(8_v\); two spinor representations, one \(8_s\) with positive chirality \(\Gamma = 1\), another \(8_s\) with negative chirality \(\Gamma = -1\). They can be bosonized as

\[
\psi_\mu = \begin{pmatrix} e^{-i\Phi_{1,\uparrow}} \\
 e^{-i\Phi_{2,\uparrow}} \\
 e^{-i\Phi_{2,\downarrow}} \end{pmatrix}, \quad C_\mu = \begin{pmatrix} e^{-i\Phi_c} \\
 e^{-i\Phi_s} \\
 e^{-i\Phi_f} \end{pmatrix}, \quad S_\mu = \begin{pmatrix} e^{-i\Phi_{c,s}} \\
 e^{-i\Phi_{s,f}} \\
 e^{-i\Phi_{f,s}} \end{pmatrix}
\]

(B1)

Where \(\Phi_c, \Phi_s, \Phi_f, \Phi_{sf}\) are defined by first triality transformation Eq.(3), \(\Phi_{c,s}, \Phi_{s,s}, \Phi_{f,s}, \Phi_{sf,s}\) are defined by second "triality transformations"

\[
\begin{pmatrix} \Phi_{c,s} \\
 \Phi_{s,s} \\
 \Phi_{f,s} \\
 \Phi_{sf,s} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \Phi_{1\uparrow} - \Phi_{1\downarrow} + \Phi_{2\uparrow} + \Phi_{2\downarrow} \\
 \Phi_{1\uparrow} + \Phi_{1\downarrow} + \Phi_{2\uparrow} - \Phi_{2\downarrow} \\
 \Phi_{1\uparrow} - \Phi_{1\downarrow} - \Phi_{2\uparrow} + \Phi_{2\downarrow} \\
 \Phi_{1\uparrow} + \Phi_{1\downarrow} - \Phi_{2\uparrow} - \Phi_{2\downarrow} \end{pmatrix}
\]

(B2)

Note the only difference between \(C_\mu\) and \(S_\mu\) is the change of sign of \(\Phi_{1\downarrow}\). In the basis of Eq.(3), we can rewrite Eq.(3) as

\[
C_\mu = \begin{pmatrix} e^{-i\Phi_c} \\
 e^{-i\Phi_s} \\
 e^{-i\Phi_f} \end{pmatrix}, \quad \psi_\mu = \begin{pmatrix} e^{-i\Phi_{1,\uparrow}} \\
 e^{-i\Phi_{1,\downarrow}} \\
 e^{-i\Phi_{2,\downarrow}} \end{pmatrix}, \quad S_\mu = \begin{pmatrix} e^{-i\Phi_{c,s}} \\
 e^{-i\Phi_{s,f}} \\
 e^{-i\Phi_{f,s}} \end{pmatrix}
\]

(B3)

The other two sets of bosons are expressed in terms of those in Eq.(3)

\[
\begin{pmatrix} \Phi_{1,\uparrow} \\
 \Phi_{1,\downarrow} \\
 \Phi_{2,\downarrow} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \Phi_{c} + \Phi_{s} + \Phi_{f} + \Phi_{sf} \\
 \Phi_{c} - \Phi_{s} + \Phi_{f} - \Phi_{sf} \\
 \Phi_{c} - \Phi_{s} - \Phi_{f} + \Phi_{sf} \end{pmatrix}, \quad \begin{pmatrix} \Phi_{c,s} \\
 \Phi_{s,s} \\
 \Phi_{f,s} \\
 \Phi_{sf,s} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \Phi_{c} - \Phi_{s} + \Phi_{f} + \Phi_{sf} \\
 \Phi_{c} + \Phi_{s} + \Phi_{f} - \Phi_{sf} \\
 \Phi_{c} + \Phi_{s} - \Phi_{f} - \Phi_{sf} \\
 \Phi_{c} + \Phi_{s} - \Phi_{f} + \Phi_{sf} \end{pmatrix}
\]

(B4)

It is evident that in the first basis of Eq.(3), \(\psi_\mu\) transforms as \(8_c\) and \(C_\mu\) transforms as \(8_v\), namely, \(\psi_\mu\) and \(C_\mu\) exchange roles. It is important to observe that the only difference between \(\psi_\mu\) \((\Gamma = 1)\) and \(S_\mu(\Gamma = -1)\) is the change of sign of \(\Phi_s\).

Similarly, in the basis of Eq.(B2), \(\psi_v\) transforms as \(8_s\) and \(S_v\) transforms as \(8_v\), namely, \(\psi_v\) and \(S_v\) exchange roles. The complex fermions in Eq.(4) are \(C_v\) fermions.

1. The derivation by bosonization

In Eq.(1), we bosonize new sets of complex fermions in terms of new sets of bosons \(\Phi_s^n, \Phi_{sf}^n\).

Replacing \(\Phi_{c,s}^n, \Phi_{sf}^n\) by \(\Phi_s^n, \Phi_{sf}^n\) in Eq.(3), we can construct \(\psi_v^n, S_v^n\).

Eq.(B1) and (B2) lead to

\[
\psi_{i\alpha,L}^n = e^{\pm i\theta/2} S_{i\alpha,R}^n, \quad S_{i\alpha,L}^n = e^{\pm i\theta/2} \psi_{i\alpha,R}^n
\]

(B5)

where in the exponential, we take +, if \(\alpha = \uparrow\) and −, if \(\alpha = \downarrow\).

Because the basis \(\psi_{i\alpha}\) is related to the basis \(\psi_{i\alpha}\) by a SO(8) rotation matrix in \(8_v\), therefore \(\psi_{i\alpha}^n, S_{i\alpha}^n\) are related to \(\psi_{i\alpha}, S_{i\alpha}\) by the two rotations in \(8_c, 8_s\), respectively.

We can determine the two rotations in \(8_c, 8_s\) by the following two steps.

(1) Under \(\Phi_c \rightarrow \Phi_c + \lambda, \Phi_f \rightarrow \Phi_f + \lambda, \lambda\) is an arbitrary angle.

Eq.(3) dictates

\[
\psi_{\uparrow}^f = c_{11} \psi_{\uparrow} + c_{12} \psi_{\downarrow}
\]

\[
\psi_{\downarrow}^f = c_{21} \psi_{\uparrow} + c_{22} \psi_{\downarrow}
\]

(B6)

(2) Under \(\Phi_s \rightarrow \Phi_s + \pi, \alpha_s \rightarrow -\alpha_s, b_s \rightarrow -b_s\) therefore \(\Phi_s^n \rightarrow -\Phi_s^n + \pi, \Phi_{sf}^n \rightarrow -\Phi_{sf}^n\). Under \(\Phi_{sf} \rightarrow \Phi_{sf} + \pi, \alpha_{sf} \rightarrow -\alpha_{sf}, b_{sf} \rightarrow -b_{sf}\) therefore \(\Phi_s^n \rightarrow -\Phi_s^n, \Phi_{sf}^n \rightarrow -\Phi_{sf}^n + \pi\).
Eq $\text{B8}$ dictates

$$
\psi_{i+}^n = \psi_{i+} = \frac{1}{\sqrt{2}}(\psi_{1i} + \psi_{4i})
$$

$$
\psi_{i-}^n = \psi_{i-} = \frac{1}{\sqrt{2}}(\psi_{1i} - \psi_{4i})
$$

(B7)

Remember $S$ differ from $\psi$ by the change of sign of $\Psi$, we get the similar relations for $S$ spinor

$$
S_{i+}^n = S_{i+} = \frac{1}{\sqrt{2}}(S_{1i} + S_{4i})
$$

$$
S_{i-}^n = S_{i-} = \frac{1}{\sqrt{2}}(S_{1i} - S_{4i})
$$

(B8)

Eqs B3, B7, B8 give the boundary conditions at the line of NFL fixed points Eq 21.

2. The derivation by $\gamma$ matrix

In this subsection, we follow Ref. [6] but use different conventions. $J_{kl}$ are the 28 generators of $SO(8)$ and 4 commuting generators $W_k$ form a Cartan subalgebra of $SO(8)$

$$(J_{kl})_{mp} = \delta_{km}\delta_{lp} - \delta_{kp}\delta_{lm}, \quad k,l = 1, \cdots, 8$$

$$W_k = J_{2k-1,2k}, \quad k = 1, 2, 3, 4. \quad \quad \text{B9}$$

The eight Majorana fermions in Eq 4 transform as 8. Eq 24 can be written as $e^{\theta J_{38}}$ which is a matrix in $SO(8)$ (therefore in $8_\nu$).

$\psi_{\nu}, S_{\mu}$ will transform as $16 \times 16$ spinor representation of $SO(8)$. The $16 \times 16$ matrices can be written in the block form

$$
\Gamma^i = \left(\begin{array}{cc}
0 & \gamma^i \\
\gamma^i & 0
\end{array}\right), \quad i = 1, \cdots, 8
$$

(B10)

$\Gamma^i$ satisfy $\{\Gamma^i, \Gamma^j\} = 2\delta^{ij}$ if

$$
\gamma^i \gamma^j + \gamma^j \gamma^i = 2\delta^{ij}
$$

$$
\gamma^i \gamma^j + \gamma^j \gamma^i = 2\delta^{ij}
$$

(B11)

The generators of the 16 $\times$ 16 representation are

$$
\Gamma^{ij} = \frac{1}{4}[\Gamma^i, \Gamma^j] = \frac{1}{4}\left(\begin{array}{cc}
\gamma^i \gamma^j - \gamma^j \gamma^i & 0 \\
0 & \gamma^i \gamma^j - \gamma^j \gamma^i
\end{array}\right) = \left(\begin{array}{cc}
\gamma^{ij} & 0 \\
0 & \gamma^{ij}
\end{array}\right)
$$

(B12)

Where $\gamma^{ij} = \frac{1}{2}\gamma^i \gamma^j$, $\gamma^{ij} = \frac{1}{2}\gamma^i \gamma^j$ are the generators of $8_c$ and $8_s$ respectively.

Under $\Phi_c \rightarrow \Phi_c + 2\theta$, Eq 12 transform under $e^{2\theta J_{12}}$, from Eq B4, $\psi_{\mu}, S_{\mu}$ both transform under $e^{\theta/2(J_{12} + J_{34} + J_{56} + J_{78})}$. Under $\Phi_s \rightarrow \Phi_s + \theta$, Eq 12 transform under $e^{\theta J_{34}}$, from Eq B4, $\psi_{\mu}, S_{\mu}$ transform under $e^{\theta/2(J_{12} - J_{34} + J_{56} - J_{78})}$ and $e^{-\theta/2(J_{12} - J_{34} + J_{56} - J_{78})}$ respectively. Under $\Phi_f \rightarrow \Phi_f + 2\theta$, Eq 12 transform under $e^{2\theta J_{56}}$, from Eq B4, $\psi_{\mu}, S_{\mu}$ both transform under $e^{\theta/2(J_{12} + J_{34} - J_{56} - J_{78})}$. Under $\Phi_{sf} \rightarrow \Phi_{sf} + \theta$, Eq 12 transform under $e^{\theta J_{78}}$, from Eq B4, $\psi_{\mu}, S_{\mu}$ both transform under $e^{\theta/2(J_{12} + J_{34} - J_{56} - J_{78})}$. Therefore, omitting subscript $c$, we conclude

$$
\gamma_{12} = \frac{1}{2}(J_{12} + J_{34} + J_{56} + J_{78}) = \gamma_{12, s}
$$

$$
\gamma_{34} = \frac{1}{2}(J_{12} - J_{34} + J_{56} - J_{78}) = \gamma_{34, s}
$$

$$
\gamma_{56} = \frac{1}{2}(J_{12} + J_{34} - J_{56} - J_{78}) = \gamma_{56, s}
$$

$$
\gamma_{78} = \frac{1}{2}(J_{12} - J_{34} - J_{56} + J_{78}) = \gamma_{78, s}
$$

(B13)
The 8 $\gamma_i$ matrices can be expressed as the direct products of $2 \times 2$ blocks

\[
\gamma_1 = \epsilon \times 1 \times \sigma_3, \quad \gamma_2 = \epsilon \times 1 \times \sigma_1 \\
\gamma_3 = 1 \times \sigma_3 \times \epsilon, \quad \gamma_4 = 1 \times 1 \times 1 \\
\gamma_5 = \epsilon \times \epsilon \times \epsilon, \quad \gamma_6 = \sigma_1 \times \epsilon \times 1 \\
\gamma_7 = 1 \times \sigma_1 \times \epsilon, \quad \gamma_8 = \sigma_3 \times \epsilon \times 1 
\]

(B14)

Where $\epsilon = i\sigma_2$ and $\sigma_i$ are the three Pauli matrices.

It is easy to see $\gamma_4$ is an identity matrix, the other 7 matrices are real and anti-symmetric.

It can be checked that indeed $\gamma_i^{ij} = \frac{1}{2} \gamma^i \gamma^j$, $\gamma_5^{ij} = \frac{1}{2} \gamma^i \gamma^j$ satisfy Eq. B13.

If Eq. B3 transform under $e^{iJ_{38}}$, then $\psi_\mu$ and $S_\mu$ transform under $e^{i\gamma_{38}}$ and $e^{i\gamma_{38}s}$ respectively.

Using Eq. B14, straightforward calculation show that $e^{i\gamma_{38}}$ give Eq. 76.

Under the first equation in Eq. 18, $\Phi_s \rightarrow -\Phi_s$, from Eq. B4, it is easy to see $\psi_\mu$ and $S_\mu$ transform to each other, therefore lead to Eq. 21.

**APPENDIX C: THE CALCULATIONS IN THE OLD BOSON BASIS AT THE ADDITIONAL NFL FIXED POINT**

In contrast to the boundary conditions Eq. 18 of the line of NFL fixed points, the boundary conditions Eq. 53 are in one of the four Cartan subalgebra, therefore can be expressed in terms of chiral bosons in Eq. 5:

\[
\Phi_{s,L}(0) = -\Phi_{s,R}(0) + \pi 
\]

(C1)

In terms of physical fermions, it reads:

\[
\psi_{ia,L}(0) = e^{i\pi j_\alpha} S_{ia,R}(0) 
\]

(C2)

where $j_\alpha = \pm \frac{1}{2}$.

The five operators in Eq. 63 can be written in terms of the bosons introduced in Eq. 5:

\[
\frac{\partial}{\partial \tau} \cos \Phi_{sf}(0, \tau), \quad \frac{\partial}{\partial \tau} \cos \Phi_s(0, \tau) \\
\cos 2\Phi_{sf}(0, \tau) - \frac{1}{2} : (\partial \Phi_{sf}(0, \tau))^2 : \\
- \cos 2\Phi_s(0, \tau) - \frac{1}{2} : (\partial \Phi_s(0, \tau))^2 : \\
\gamma_2(\cos 2\Phi_s(0, \tau) - \frac{1}{2} : (\partial \Phi_s(0, \tau))^2 : ) 
\]

(C3)

The four dimension 5/2 operators are

\[
(\cos 2\Phi_s(0, \tau) : -\frac{1}{2} : \partial\Phi_s(0, \tau))^2 : \cos \Phi_{sf}(0, \tau) \\
(\cos 2\Phi_s(0, \tau) : \frac{1}{2} : \partial\Phi_s(0, \tau))^2 : \cos \Phi_s(0, \tau) \\
\frac{\partial^2}{\partial \tau^2} \cos \Phi_{sf}(0, \tau), \quad \frac{\partial^2}{\partial \tau^2} \cos \Phi_s(0, \tau) 
\]

(C4)

Very similar arguments to those in Sec. IV lead to $\sigma(T) \sim 2\sigma_u(1 + T^{3/2})$.

---

1. P. Nozières and A. Blandin, J. Phys. (Paris) 41, 193 (1980)
2. D. L. Cox, Phy. Rev. Lett. 59, 1240 (1987). For a review, see D. L. Cox and M. Jarrel, J. Phys. Cond. Matt. 9825 (1996).
If two operators anticommute with each other, they will not generate any new terms.

P. Ginsbarg's lecture in fields, strings and critical phenomena.

J. Cardy's lecture in fields, strings and critical phenomena.

M. Fabrizio, A. O. Gogolin and P. Nozières, Phys. Rev. Lett. 74, 4503 (1995), Phys. Rev. B 51, 16088 (1995).

M. Fabrizio and A. O. Gogolin, Phys. Rev. B 50, 17732 (1994).

ψ

In contrast to...

Because...

We can also check the operator contents in the two special cases directly: If ∆

A. Georges and A. M. Sengupta, Phy. Rev. Lett. 74, 2808 (1995), cond-mat/9702057; I. Affleck and A. W. W. Ludwig, J. Phys. A 27, 5375 (1994).

O(1) stands for Ising sector.

M. Green, J. H. Schwarz and E. Witten, Superstring Theory, Vol.1, p282-289.

For example, it can be shown that the composite processes of one electron and two electrons hopping in the same direction as the impurity are irrelevant at all the fixed points (or line of fixed points) discussed in this paper. All the irrelevant operators consistent with the global symmetry near the fixed points (or line of fixed points) will be taken into account in evaluating the thermodynamic quantities and the correlation functions.

In contrast to the 2CK, we don’t reverse the spin here, so this Time Reversal symmetry is different than that in the 2CK.

M. Fabrizio and A. O. Gogolin, Phys. Rev. B 50, 17732 (1994).

M. Fabrizio, A. O. Gogolin and P. Nozières, Phys. Rev. Lett. 74, 4503 (1995), Phys. Rev. B 51, 16088 (1995).

J. Cardy’s lecture in fields, strings and critical phenomena.

P. Ginsbarg’s lecture in fields, strings and critical phenomena.

If two operators anticommute with each other, they will not generate any new terms.

ψ′(x) = U−1ψs(x)U = ψsf(x).

Because H = U−1HU′, the actual scaling field in H′ is ψ′s(x) instead of ψs(x). In the following, when deriving the leading or subleading irrelevant operators in H′, we will replace ψs(x) directly by ψ′s(x). Then Eq. 3 will yield the corresponding operators in H.

If keeping γ1 = γ2 = 1, we get the weak coupling RG flow Eq. 3.

See Ref. 29 for very similar discussions on the operator contents. Similar operators were also identified near the 2IK fixed point discussed in Ref. 30.

Jinwu Ye, Phys. Rev. Lett. 77, 3224 (1996).

Jinwu Ye, cond-mat/9600058

ψs, ψf decouple, therefore remains the same.

For simplicity, we omit the cocycle factors in the following expressions.

See Eq. 32 for the bosonized form of this operator in H.

I. Affleck, A. W. W. Ludwig and B. A. Jones, Phys. Rev. B 52, 9528 (1995).

We can also check the operator contents in the two special cases directly: If ∆+ = 0, namely at the 2CK fixed point, 3 ∼ 3f, the spin-0 operator is qb,i(δa), 3 ∼ 3f; If ∆− = 0, namely at the 2IK fixed point, 3 ∼ 3f, the spin-0 operator is yb,i(δa), 3 ∼ 3f.

There is a sign ambiguity due to the spinor nature of the representation, but, if θ = π, the fixed point is the 2CK, so all the 2 channel and 2 parity should make the same contribution to the conductivity, this fix the sign of Eq. 11.

When transforming to finite temperature and considering connected correlation function, we can treat energy momentum tensor T(z) as if it is primary operator, even it is only quasi-primary.

In the following, we don’t write out the other trivial factors.

Jinwu Ye, cond-mat/9600057

In NRG, either NS or R sector is chosen.

This finite size spectrum actually has g = 2. If we reduce the degeneracy in column 4 by factor 2, we get the finite size spectrum of g = 1, but still with the symmetry O(2) × O(6). See Ref. 25 for the breaking of the degeneracy from g = 2 to g = 1.

For symmetric ferromagnetic Kondo model, the impurity and the conduction electrons also asymptotically decoupled, but g = 2 in this case.

S. Sachdev and J. Ye, Phys. Rev. Lett. 69, 2411 (1992).

A. Chubukov, S. Sachdev and J. Ye, Phys. Rev. B 39, 2344 (1989).

If there exists marginal irrelevant operators, then the logarithmic violation of scalings is expected; see, J. Ye, S. Sachdev and N. Read, Phys. Rev. Lett. 70, 4011 (1993); N. Read, S. Sachdev and J. Ye, Phys. Rev. B 52, 384 (1995). If there exists marginally relevant operator, then this operator will introduce another energy scale, for example, in isotropic Kondo model, the Kondo scale is given by the marginally irrelevant operator near the weak coupling fixed point.

A. Millis, Phys. Rev. B 48, 7183 (1993).
46 I thank A. Millis for pointing this out to me.
47 N. Read and S. Sachdev, Phy. Rev. Lett. 75, 3509 (1992).
48 A. L. Moustakas and D. S. Fisher, cond-mat/9607268
49 J. Gan, Phy. Rev. Lett. 74, 2583 (1995); Phys. Rev. B 51, 8287 (1995)
50 J. V. Delft et. al., cond-mat/9702048, cond-mat/9702049.
51 We are sloppy about the discrete symmetry $Z_2$, however, the discrete symmetry may be important. For example, $O(2)$ is the semi-product of $U(1)$ and $Z_2$, their difference is clearly shown along the line of FL fixed points in Fig.1.