Pressure gradient errors in a covariant method of implementing the \(\sigma\)-coordinate: idealized experiments and geometric analysis

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ABSTRACT
A new approach is proposed to use the covariant scalar equations of the \(\sigma\)-coordinate (the covariant method), in which the pressure gradient force (PGF) has only one term in each horizontal momentum equation, and the PGF errors are much reduced in the computational space. In addition, the validity of reducing the PGF errors by this covariant method in the computational and physical space over steep terrain is investigated. First, the authors implement a set of idealized experiments of increasing terrain slope to compare the PGF errors of the covariant method and those of the classic method in the computational space. The results demonstrate that the PGF errors of the covariant method are consistently much-reduced, compared to those of the classic method. More importantly, the steeper the terrain, the greater the reduction in the ratio of the PGF errors via the covariant method. Next, the authors use geometric analysis to further investigate the PGF errors in the physical space, and the results illustrate that the PGF of the covariant method equals that of the classic method in the physical space; namely, the covariant method based on the non-orthogonal \(\sigma\)-coordinate cannot reduce the PGF errors in the physical space. However, an orthogonal method can reduce the PGF errors in the physical space. Finally, a set of idealized experiments are carried out to validate the results obtained by the geometric analysis. These results indicate that the covariant method may improve the simulation of variables relevant to pressure, in addition to pressure itself, near steep terrain.

1. Introduction
The pressure gradient force computational errors (PGF errors) in a terrain-following coordinate (\(\sigma\)-coordinate) can significantly affect the performance of a model, including the vorticity in the downslope of steep terrain, the blocking of cold air in the upslope of steep terrain, the potential vorticity near the tropopause over steep terrain, and so on (Smagorinsky et al. 1967; Kasahara 1974; Mahre 1984; Steppeler et al. 2003; Hoinka and Zängl 2004; Li, Chen, and Shen 2005; Hu and Wang 2007). The PGF computational form is expressed by two terms in each horizontal momentum equation in the \(\sigma\)-coordinate. Computational errors are therefore inevitable as these two terms are opposite in sign and typically of the same order near steep terrain (Haney 1991; Fortunato and Baptista 1996; Lin 1997; Ly and Jiang 1999; Berntsen 2002; Chu and Fan 2003; Shchepetkin and McWilliams 2003; Li, Chen, and Li 2012).

Much effort has been made to alleviate the PGF errors to an acceptable level (Corby, Gilchrist, and Newson 1972; Gary 1973; Zeng 1979; Qian and Zhong 1986; Blumberg and Mellor 1987; Yu 1989; Qian and Zhou 1994; Berntsen 2011; Klemp 2011; Zängl 2012), without touching this two-term PGF (the so-called classic method). Alternatively, two new methods have been proposed to create a one-term PGF to overcome the PGF errors. One is to adopt the covariant scalar equations of the \(\sigma\)-coordinate (the covariant method by Li, Wang, and Wang (2012)); and the other is to design an orthogonal terrain-following coordinate (the orthogonal method by Li et al. (2014)). Using two idealized
experiments, Li, Wang, and Wang (2012) showed that the covariant method significantly reduces the errors, compared to the classic method, in the computational space.

Many researchers have pointed out that the PGF errors of the classic method are related to terrain slope (Yan and Qian 1981; Zeng and Ren 1995; Steppeler et al. 2003; Weller and Shahrokhi 2014; Li, Li, and Wang 2016). But can the covariant method consistently reduce the PGF errors compared to the classic method as terrain slope increases? Moreover, although the calculation of a model is in the computational space, the final application of model results is in the physical space; can the covariant method reduce the PGF errors in the physical space?

In this study, we first carry out a set of sensitivity experiments of increasing terrain slope to compare the PGF errors of the classic method and those of the covariant method in the computational space. Then, we use a geometric schematic and associated idealized experiments to further investigate the PGF errors of these methods in the physical space. The results of the idealized experiments using various terrain in the computational space are presented in Section 2. The PGF errors in the physical space are compared in Section 3. Concluding remarks and a discussion are given in Section 4.

2. Idealized experiments in the computational space

Since the covariant method was shown to significantly reduce the PGF errors in the computational space, compared to the classic method, in the experiments using one kind of terrain implemented by Li, Wang, and Wang (2012), we further investigate the PGF errors of the covariant method and those of the classic method in the computational space over different kinds of terrain. We first introduce the basic parameters for all the experiments, and then compare the PGF errors of the covariant method and those of the classic method in the computational space in experiments of increasing terrain slope.

2.1. Basic parameters

For consistency, we use the same parameters as Li, Wang, and Wang (2012), except for the terrain slope. First, the definition of \( \sigma = \frac{H_T - h}{H_T - H} \) proposed by Gal-Chen and Somerville (1975) is adopted, where \( z \) represents the height, \( H_T \) is the top of the model, and \( h \) represents terrain. We use a 2D bell-shaped terrain (black curve in Figure 1), where \( H = 4 \) km is the maximum height, \( a = 5 \) km is the half width, and \( h_0 = 50 \) km is the middle point of the terrain.

Second, we use the central spatial discretization for the PGF in the horizontal and the forward scheme in the vertical for both methods. The expressions are given as follows:

Finally, we use a pressure field, as shown in Figure 1, where \( h(x) \) is defined by Equation (1), \( H \) is the maximum height of terrain, \( H_p = 300 \) km is a parameter to adjust the pressure gradient, \( p_0 = 1,015.0 \) hPa is surface pressure, and \( \lambda = 8 \) km is the typical height of the atmosphere. The domain of all the experiments is 0–100 km in the horizontal and 0–37 km in the vertical (Figure 1). The horizontal and vertical resolutions are 0.5 km and 3.7 km, respectively.

![Figure 1. The pressure field (shading) and terrain (black curve). The pressure scale (color bar on the right) is in hPa.](image)
2.2. Sensitivity experiments

Through increasing the maximum height $H$ of terrain in Equation (1) at 50-m intervals from 3 to 9 km, we carry out 121 sets of experiments (Figure 2(a)). Note that the maximum slope is almost three times the minimum in Figure 2(a).

We calculate the root-mean-square of relative errors (RMS-REs) of the PGF of the covariant method and those of the classic method (Figure 2(b)). The RMS-REs of the covariant method are consistently reduced by one order of magnitude, compared to those of the classic method. Moreover, as the terrain slope increases, the RMS-REs of the classic method significantly increase (red line in Figure 2(b) relative to black line in Figure 2(a)); however, the RMS-REs of the covariant method remain approximately the same (blue line in Figure 2(b) relative to black line in Figure 2(a)). Therefore, the steeper the terrain, the greater the reduction of the ratio of PGF errors via the covariant method.

3. Comparison of the PGF errors in the physical space

In order to compare the PGF errors of the covariant method and those of the classic method in the physical space, we first use a geometric schematic to further investigate the PGF errors in the physical space, and then carry out a set of associated idealized experiments to validate the results obtained by the geometric analysis.

The geometric schematic of PGF is shown in Figure 3. The relationship between the lines with arrow heads in Figure 3 and the variables related to PGF are all listed below:

The vertical PGF of the $z$-coordinate,

$$AF = \frac{\partial p}{\partial z};$$

(4)

The horizontal PGF of the covariant method in the computational space,

$$AB = \left( \frac{\partial p}{\partial x} \right)_{\sigma};$$

(5)
The vertical PGF of the covariant method in the computational space,

\[ AD = \frac{\partial p}{\partial \sigma} \cdot \sqrt{\left( \frac{\partial \sigma}{\partial x} \right)_z^2 + \left( \frac{\partial \sigma}{\partial z} \right)_x^2}. \]  

(6)

In addition, through the geometric relationship in Figure 3, we obtain

\[ BE = AF \cdot \tan \varphi = AD \cdot \sin \varphi = AK, \]

where \( \varphi \) is terrain slope, and

\[ \varphi = \arctan \left( \frac{\partial z}{\partial x} \right)_\sigma = \arcsin \frac{\left( \frac{\partial \sigma}{\partial x} \right)_z}{\sqrt{\left( \frac{\partial \sigma}{\partial x} \right)_z^2 + \left( \frac{\partial \sigma}{\partial z} \right)_x^2}} \]  

(8)

First, the expressions of the PGF of the covariant method and the classic method in the physical space are respectively given by

\[ \frac{\partial p}{\partial x}^\text{covan} = AB - AD \cdot \sin \varphi \]  

(9)

and

\[ \frac{\partial p}{\partial x}^\text{class} = AB - AF \cdot \tan \varphi \]  

(10)

According to Equation (7), the PGF of the covariant method expressed in Equation (9) equals the PGF of the classic method shown in Equation (10); namely, the covariant method cannot reduce the PGF errors in the physical space compared to the classic method.

Note that both the classic and the covariant method are non-orthogonal methods (Li, Wang, and Wang 2011, 2012), namely, the PGF errors in the physical space cannot be reduced by the coordinate transformation in the non-orthogonal \( \sigma \)-coordinate. But can the orthogonal method proposed by Li et al. (2014) reduce the PGF errors in the physical space?

Second, according to Figure 3, the horizontal and vertical PGFs of the orthogonal method in the computational space are

\[ AC' = \left( \frac{\partial p}{\partial x} \right)^\sigma, \]

(11)

and

\[ AG = \frac{\partial p}{\partial \sigma}, \]

(12)

respectively, where \( x' \) is the horizontal coordinate of the orthogonal terrain-following coordinate. Then, the PGF of the orthogonal method in the physical space can be expressed by

\[ \left( \frac{\partial p}{\partial x} \right)^\text{orthogonal} = AC' \cdot \cos \varphi - AG \cdot \sin \varphi = AJ - AH. \]  

(13)

Using the geometric relationship in Figure 3, we obtain

\[ AJ = AB \cdot \cos^2 \varphi = AB - AB \cdot \sin^2 \varphi, \]  

(14)

and

\[ AH = (AD - AB \cdot \sin \varphi) \sin \varphi. \]  

(15)

Substituting Equations (5) and (7) into Equations (14) and (15), we obtain

\[ \frac{AH}{AJ} = \frac{BE - \left[ \left( \frac{\partial p}{\partial x} \right)^\sigma \sin \varphi \right]}{AB - \left[ \left( \frac{\partial p}{\partial x} \right)^\sigma \sin^2 \varphi \right]}, \]  

(16)

Note that the PGF of the orthogonal method in the physical space is \( AJ - AH \) and that of the non-orthogonal method is \( AB - BE \). According to Equation (16), the PGF errors in the physical space can be reduced by the orthogonal method when the terrain slope \( \varphi \) is large enough:

1. If \( \left( \frac{\partial p}{\partial x} \right)^\sigma \sin^2 \varphi \) is large enough to make the order of \( AH \) smaller than that of \( AJ \), i.e. \( AH \) and \( AJ \) are no longer of the same order, the PGF errors in the physical space can be reduced by the orthogonal method;

2. If \( \left( \frac{\partial p}{\partial x} \right)^\sigma \sin^2 \varphi \) is large enough to make the magnitude of \( AH \) (\( AJ \)) much smaller than that of \( BE \) (\( AB \)), the orthogonal method is a comparable method to the standard stratification deduction proposed by Zeng (1979). Therefore, the PGF errors in the physical space can be reduced by the orthogonal method.

Finally, we calculate the PGF errors of the three methods, i.e. the classic method, the covariant method and the orthogonal method. Substituting Equations (4), (5), (6), (8), (11), and (12) into Equations (9), (10), and (13), and using the discretisation schemes given in Section 2.1, we can obtain the discrete expressions of the PGF of the three methods in the physical space as follows:

\[ \left( \frac{\partial p}{\partial x} \right)^\text{class} = \frac{p_{i+1,k} - p_{i-1,k}}{2\Delta x} - \frac{p_{i,k+1} - p_{i,k}}{\Delta z} \cdot \left( \frac{\partial z}{\partial x} \right)^\sigma \]  

(17)
Figure 4. REs of three methods in the computational and physical spaces. The dashed contours are for negative values. The contour interval in (a), (b), (d), and (f) is 1.0, while that in (c) and (e) is 0.1. The differences between (a) and (c) in this study and Li, Wang, and Wang (2012, Figure 6(c) and (d)) on the boundary are due to the revised boundary condition used in this study. The revised boundary condition is directly from the definition of pressure, 

\[ p = p_0 \cdot e^{-\left(\frac{-x}{\eta H} - \frac{z - H}{H h(x)}\right)} \]

to obtain the value on each boundary grid.

\[
\left(\frac{\partial p}{\partial x}\right)_{\text{physical}}^{\text{covariant}} = \frac{p_{i+1,k} - p_{i-1,k}}{2\Delta x} + \frac{p_{i,k+1} - p_{i,k}}{\Delta \sigma} \cdot \left(\frac{\partial \sigma}{\partial x}\right)_z \quad \text{(the covariant method)};
\]

\[
\left(\frac{\partial p}{\partial x}\right)_{\text{physical}}^{\text{orthogonal}} = \frac{p_{i+1,k} - p_{i-1,k}}{2\Delta x} \cdot \cos \varphi_{i,k} - \frac{p_{i,k+1} - p_{i,k}}{\Delta \sigma} \cdot \sin \varphi_{i,k} \quad \text{(the orthogonal method)}.
\]
Using Equations (17)–(19) and the parameters given in Section 2.1, we calculate the REs of the PGF of the three methods in the computational space as well as in the physical space (Figure 4). As obtained in the geometric analysis, the PGF errors of the covariant method are the same as those of the classic method in the physical space (Figure 4(b) and (d)), whereas the PGF errors of the orthogonal method are much reduced compared to those of the classic method in the physical space (Figure 4(b) and (f)). In addition, as with the covariant method, the orthogonal method can also reduce the PGF errors of the classic method in the computational space (Figure 4(a), (c), and (e)).

4. Conclusion and discussion

Through idealized experiments using increasing terrain slope in the computational space and a geometric analysis in the physical space, the present study investigates the validity of reducing the PGF errors via the covariant method proposed by Li, Wang, and Wang (2012), compared to the classic method. First, sensitivity experiments of increasing terrain slope in the computational space show that the RMS-REs of the covariant method are consistently one order of magnitude smaller than those of the classic method (Figure 2). More importantly, the steeper the terrain, the greater the reduction in the ratio of PGF errors via the covariant method, indicating that the covariant method may perform better near steep terrain.

The geometric analysis (Figure 3) and associated idealized experiments then demonstrate that, compared to the classic method, the covariant method based on the non-orthogonal $\sigma$-coordinate can reduce the PGF errors in the computational space but not in the physical space (Figure 4(a)–(d)). However, the orthogonal method proposed by Li et al. (2014) can reduce the PGF errors in the computational space as well as in the physical space (Figure 4(a) and (b), (e) and (f)).

In addition, since the covariant method cannot reduce the PGF errors in the physical space, but can significantly reduce the errors in the computational space, especially over steep terrain, the covariant method may not improve the simulation of pressure itself but could lead to improvement in the velocity (relevant to pressure, according to the momentum equations). For example, Weller and Shahrokhi (2014) used the curl-free PGF (the PGF of the covariant method is curl-free in the computational space) to obtain a better hydrostatic balance and better energy conservation.

Besides, the patterns of PGF error of the orthogonal method are different from those of the other two methods based on the non-orthogonal $\sigma$-coordinate (Figure 4(a)–(d), (e) and (f)). This is related to the difference between computational grids in the orthogonal $\sigma$-coordinate and those in the non-orthogonal $\sigma$-coordinate used in this study. Further analyses are needed to investigate the relationship between computational grids and PGF errors. Plus, the true benefits of the covariant method and the orthogonal method need to be tested using primitive equations in more idealized experiments and realistic simulations.

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