THE CATENARY IN SPACE FORMS

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Abstract. In this paper, the notion of the catenary curve in the sphere and in the hyperbolic plane is introduced. In both spaces, a catenary is defined as the shape of a hanging chain when its weight is measured with respect to a given geodesic of the space. Several characterizations of the catenary are established in terms of the curvature of the curve and of the angle that its unit normal makes with a vector field of the ambient space. Furthermore, in the hyperbolic plane, we extend the concept of catenary substituting the geodesics by a horocycle or the hyperbolic distance by the horocycle distance.

1. Introduction and objectives

In this paper we will study the concept of catenary in the sphere and in the hyperbolic plane. Our purpose is twofold. First, we want to extend to these space forms the hanging chain problem and find its solution. Secondly, we want to give several characterizations of the catenary in terms of its curvature. We first recall the derivation of the catenary in Euclidean plane. The hanging chain problem in Euclidean plane is formulated as follows: find the shape that a hanging inextensible chain adopts under its own weight when suspended from its endpoints in a uniform gravitational field. The catenary is the solution of the problem, and the history behind the derivation of the solution of the hanging chain problem goes back to the works of Galileo, Hooke, Bernouilli, Leibniz and Huygens among others; see [4,12] for an account of the history of the catenary. Let \( \mathbb{R}^2 \) denote the Euclidean plane where \((x,y)\) stand for the Cartesian coordinates and \(\langle \cdot, \cdot \rangle\) is the Euclidean metric. The hanging chain problem is equivalent to finding a curve \( y: [a,b] \to \mathbb{R}^2, \ y = y(x), \) that minimizes the potential energy

\[
(1) \quad \int_a^b \sigma gy(x)\sqrt{1 + y'(x)^2} \, dx.
\]

Here \( g \) is the gravitational constant and \( \sigma \) is the density per unit length along the chain. The height is measured with respect to the straight-line line of equation \( y = 0. \) In particular, it is assumed in [1] that \( y(x) > 0 \) for all \( x \in [a,b]. \) Simplifying
in (1) the constant $\sigma_g$ by 1, the hanging chain problem consists in determining a function $y(x)$ such that

\begin{equation}
\mathcal{E}[y] = \int_a^b y\sqrt{1+y'^2} \, dx
\end{equation}

is minimum among all curves with prescribed endpoints $y(a)$ and $y(b)$. Consequently, the solution $y(x)$ is a critical point of the energy $\mathcal{E}$. Using standard variational arguments, the Euler-Lagrange equation associated to (2) is

\begin{equation}
y'' \left(1 + y'^2\right)^{3/2} = \frac{1}{y\sqrt{1+y'^2}}.
\end{equation}

The solution of (3) is the catenary $y(x) = \frac{1}{a} \cosh(ax+b)$, $a, b \in \mathbb{R}$, $a > 0$. Note that the left-hand side of equation (3) is the curvature $\kappa_e$ of the plane curve $y = y(x)$. The right-hand side of (3) has the following geometric interpretation. Since the unit normal vector $\mathbf{n}$ of the curve $y = y(x)$ is $\mathbf{n} = (-y', 1)/\sqrt{1+y'^2}$, then $\langle \mathbf{n}, \partial_y \rangle = 1/\sqrt{1+y'^2}$ and equation (3) can be expressed as

\begin{equation}
\kappa_e = \frac{\langle \mathbf{n}, \partial_y \rangle}{y}.
\end{equation}

The vector field $\partial_y$ just indicates the opposite direction of the gravity at each point of $\mathbb{R}^2$. Equation (4) shows that the hanging chain problem is equivalent to a coordinate-free prescribed curvature problem involving the unit normal vector of the curve, a vector field of $\mathbb{R}^2$ and the distance to the straight-line $y = 0$. In fact, equation (4) forms part of the one-parameter family of prescribed curvature equations

\begin{equation}
\kappa_e = \alpha \frac{\langle \mathbf{n}, \partial_y \rangle}{y},
\end{equation}

where $\alpha \in \mathbb{R}$ is a real parameter: see [17, pp. 25] and [22]. Equation (5) characterizes the critical points of the energy $\int_a^b y^\alpha \sqrt{1+y'^2} \, dx$ and the solutions are called $\alpha$-catenaries. If $\alpha = 0$, the critical points of this energy are simply the geodesics of $\mathbb{R}^2$.

The catenary is also related to the shape of a rotational surface of minimum area as we now explain. Consider the Euclidean space $\mathbb{R}^3$ with Cartesian coordinates $(x, y, z)$. Let $y = y(x)$ be a curve of $\mathbb{R}^3$ contained in the $xy$-coordinate plane. Let $S$ be the surface of revolution obtained by rotating this curve about the $x$-axis. The area of $S$ is $A(S) = 2\pi \int_a^b y \sqrt{1+y'^2} \, dx$. Therefore, a surface of revolution with minimum area is a critical point of energy functional $A$. Since $A$ is a multiple of $\mathcal{E}$, their critical points coincide. Thus, the generating curve $y(x)$ of a surface revolution with minimum area is the catenary. The surface of revolution obtained by rotating a catenary is called a catenoid.

The catenary is a classical curve studied in geometry and of interest to the present day. For other properties of the catenary, see [10, 11, 21, 23, 24].
The above presentation of the catenary will be the guide to propose the hanging chain problem in the unit sphere $S^2$ and in the hyperbolic plane $\mathbb{H}^2$. The objectives of this paper can be structured in the following three targets:

(T1) State the analogous hanging chain problem in $S^2$ and in $\mathbb{H}^2$ and find the corresponding Euler-Lagrange equation. The problem is determining which reference line is used to measure the weight of the curve.

(T2) Obtain an analogous formulation of the prescribed curvature equation (4) in terms of a vector field of the ambient space which represents the direction of 'gravity'.

(T3) Rotate the catenary in $S^3$ and $\mathbb{H}^3$ and determine any unique properties of the mean curvature of the resulting surface.

A parameter $\alpha$ will be introduced in the expression of the energy by replacing the distance $d$ to the reference line by a power $d^\alpha$. The critical points will be called $\alpha$-catenaries, or simply catenaries if $\alpha = 1$. Catenaries in the space form $S^2$ will be discussed in Section 2 where we consider the weight in the hanging chain problem measured with respect to a geodesic of $S^2$ and a plane of $\mathbb{R}^3$. In hyperbolic plane $\mathbb{H}^2$, the reference lines will be geodesics as well as horocycles, giving a greater richness to the hanging chain problem. This work is carried out in Section 3.

2. The catenary in the sphere

In this section we will consider the hanging chain problem in the unit sphere $S^2$. We first state the problem, and then give different characterizations of its critical points.

2.1. Spherical catenaries: definition. Consider the unit sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$. Motivated by the hanging chain problem, the reference line to measure the weight is a geodesic of $S^2$. Let us fix the great circle $P = \{(x, y, z) \in S^2 : z = 0\}$. For computational arguments, it is better to consider local coordinates in $S^2$. Let $\Psi$ be the standard parametrization of $S^2$ given by

\[ \Psi(u, v) = (\cos u \cos v, \cos u \sin v, \sin u), \]

where $u \in [-\pi/2, \pi/2]$ and $v \in \mathbb{R}$. The distance $d$ of a point $(x, y, z) = \Psi(u, v)$ to the geodesic $P$ is $d = |\arcsin(z)| = |u|$. We will consider curves of $S^2$ that do not intersect $P$. The geodesic $P$ separates $S^2$ in two domains, namely, the half-spheres $S^2_+ = \{(x, y, z) \in S^2 : z > 0\}$ and $S^2_- = \{(x, y, z) \in S^2 : z < 0\}$. Without loss of generality, all curves will be contained in the upper half-sphere $S^2_+$. Let $\gamma: [a, b] \to S^2_+$ be a regular curve and we calculate its weight measured with respect to $P$. Let us write $\gamma(t) = \Psi(u(t), v(t))$, $t \in [a, b]$, with the condition $u(t) \in (0, \pi/2]$ because $\gamma(t) \in S^2_+$. The arc-length element is $\sqrt{u'^2 + v'^2(\cos u)^2} dt$. Consequently,
the energy to minimize is
\begin{equation}
E_S[\gamma] = \int_a^b d^a \sqrt{u'^2 + v'^2 (\cos u)^2} \, dt,
\end{equation}
where \( \alpha \in \mathbb{R} \) is a parameter.

**Definition 2.1.** A critical point of \( E_S \) is called a spherical \( \alpha \)-catenary.

Since spherical \( \alpha \)-catenaries will appear as solutions of a prescribed curvature problem, it is necessary to have a suitable expression for the curvature of a curve in \( S^2 \).

Here the curvature is understood to be the geodesic curvature \( \kappa_s \) of \( \gamma \) in \( S^2 \). The sign of \( \kappa_s \) depends on the orientation of \( S^2 \). We will assume that this orientation is the unit normal vector \( N \) of \( S^2 \) is \( N(p) = -p, p \in S^2 \). In such a case,
\begin{equation}
\kappa_s = \frac{\langle \gamma'', \gamma' \times N(\gamma) \rangle}{|\gamma'|^3} = \frac{\langle \gamma, \gamma' \times \gamma'' \rangle}{|\gamma'|^3}
= \frac{1}{|\gamma'|^3} \left( v'(2u'^2 \sin u + v'^2 (\cos u)^2 \sin u + u'' \cos u) - u'v'' \cos u \right).
\end{equation}

Concerning objective T1, we have the following result.

**Theorem 2.2.** Let \( \gamma(t) = \Psi(u(t), v(t)) \) be a regular curve in \( S^2_+ \). Then \( \gamma \) is a spherical \( \alpha \)-catenary with respect to the geodesic \( P \) if and only if its curvature \( \kappa_s \) satisfies
\begin{equation}
\kappa_s = \frac{\alpha v' \cos u}{|u'||\gamma'|}.
\end{equation}

**Proof.** We calculate the Euler-Lagrange equation associated to the energy \([7]\). The Lagrangian of \( E_S \) is \( J[u, v] = u^a \sqrt{u'^2 + v'^2 (\cos u)^2} \). Since a similar computations will be done later in a similar context (see Proposition ), assume that the Lagrangian is of type
\begin{equation}
J[u, v] = f(u) \sqrt{u'^2 + v'^2 (\cos u)^2} = f(u)|\gamma'|.
\end{equation}

A curve \( \gamma \) is a critical point of \( E_S \) if and only if \( \gamma \) satisfies the next two equations,
\begin{equation}
\frac{\partial J}{\partial u} - \frac{d}{dt} \left( \frac{\partial J}{\partial u'} \right) = 0, \quad \frac{\partial J}{\partial v} - \frac{d}{dt} \left( \frac{\partial J}{\partial v'} \right) = 0.
\end{equation}

After some computations, equations \([11]\) are, respectively,
\begin{equation}
\frac{v' \cos u}{|\gamma'|} (f'v' \cos u - f \sin u) - f \frac{d}{dt} \left( \frac{u'}{|\gamma'|} \right) = 0,
\end{equation}
\begin{equation}
\frac{f'v' \cos u}{|\gamma'|} + f \frac{d}{dt} \left( \frac{v' (\cos u)^2}{|\gamma'|} \right) = 0.
\end{equation}
The above two equations can be written in terms of \( \kappa_s \). For this, we obtain from (8),

\[
\frac{d}{dt} \left( \frac{u'}{\gamma'} \right) = \frac{v' \cos u}{|\gamma'|^3} \left( v' \left( u'^2 \sin u + u'' \cos u \right) - u' v'' \cos u \right)
\]

(13)

\[
= v' \cos u \left( \kappa_s - \frac{v' \sin u}{|\gamma'|} \right),
\]

and

\[
\frac{d}{dt} \left( \frac{v'(\cos u)^2}{|\gamma'|} \right) = -\frac{u' \cos u}{|\gamma'|^3} \left( v' \left( u'^2 \sin u + u'' \cos u \right) - u' v'' \cos u \right)
\]

(14)

\[
= -u' \kappa_s \cos u.
\]

From (13) and (14), the Euler-Lagrange equations (12) are

\[
u' \cos u \left( \frac{f'v' \cos u}{f|\gamma'|} - \kappa_s \right) = 0,
\]

(15)

\[
v' \cos u \left( \frac{f'v' \cos u}{f|\gamma'|} - \kappa_s \right) = 0.
\]

Since \( \cos u \neq 0 \) and \( u'^2 + v'^2 \neq 0 \) because \( \gamma \) is a regular curve, we deduce from (15)

\[
f'v' \cos u - \kappa_s = 0.
\]

(16)

In the particular case that \( f(u) = u^\alpha \), then (16) is just (9). \( \square \)

Equation (9) is second order, but a first integration is possible because the Lagrangian \( J[u, v] \) does not depend on the function \( v \). Indeed, there exists a constant \( c > 0 \) such that

\[
\frac{\partial J}{\partial v'} = u^\alpha \frac{v'(\cos u)^2}{\sqrt{u'^2 + v'^2(\cos u)^2}} = c.
\]

(17)

Let us write \( \gamma \) in a non-parametric way \( u = u(v) \), or equivalently, \( \gamma(v) = \Psi(u(v), v) \). Then (17) rewrites as

\[
u^\alpha \frac{(\cos u)^2}{\sqrt{u'^2 + (\cos u)^2}} = c.
\]

(18)

Hence, an expression for \( u = u(v) \) is deduced, obtaining

\[
\int^u \frac{du}{\cos u \sqrt{u'^2 + (\cos u)^2} - c^2} = \frac{v}{c}.
\]

(19)

This integral yields immediately the following corollary.

**Corollary 2.3.** Let \( \gamma(v) = \Psi(u(v), v) \) be a regular curve in \( S^2_+ \). Then \( \gamma \) is a spherical \( \alpha \)-catenary with respect to the geodesic \( P \) if and only if \( u = u(v) \) satisfies (19).
Going back, equation (9) can be viewed as a prescribed equation of a curve in the sphere. However, the curve $\gamma$ is the image of a plane curve $\beta(t) = (u(t), v(t))$ under the parametrization (6) by letting $\gamma(t) = \Psi(\beta(t))$. Thus, equation (9) can be reformulated in terms of the curvature $\kappa_\beta$ of $\beta$. With this purpose, let us assume that $\beta$ is parametrized by $\beta(v) = (u(v), v)$. Then $\kappa_\beta = u''/(1 + u'^2)^{3/2}$. The value of $u'$ can be obtained directly from (18). For the computation of $u''$, let us eliminate $u'$ from equation (18). Then, we differentiate with respect to $v$, obtaining $u''$. For spherical catenaries ($\alpha = 1$), a straightforward computation yields

$$\kappa_\beta = \frac{c \cos u}{((\cos u)^2(u^2(\cos u)^2 - c^2) + c^2)^{3/2}}.$$  

Some spherical catenaries are illustrated in Figure 1. These plots have been made in Mathematica software (26). We explain briefly the method to draw them. In order to avoid the discussion whether $\beta$ is a graph on the $u$-axis or on the $v$-axis, it is better to assume that $\beta$ is parametrized by arc-length. Then $u'(t)^2 + v'(t)^2 = 1$ and according to (20), the functions $u(t)$ and $v(t)$ satisfy the ODE system,

$$\begin{cases}
u'(t) = \cos \theta(t) \\
u'(t) = \sin \theta(t) \\
(21) \quad \theta'(t) = \kappa_\beta = \frac{c \cos u}{((\cos u)^2(u^2(\cos u)^2 - c^2) + c^2)^{3/2}}.
\end{cases}$$

Recall that the variation of the angle function $\theta(t)$ measures the curvature of $\beta$. Given initial conditions $u(0) = u_0$, $v(0) = v_0$ and $\theta(0) = \theta_0$, Mathematica solves numerically the ODE (21) and graphically represents the solution $t \mapsto \Psi(u(t), v(t))$.

**Figure 1.** Spherical catenaries in $S^2_+$ (top view) for different initial conditions in the ODE system (21). The initial conditions are $u(0) = 0.8$, $v(0) = 0$ and $\theta(0) = \pi/2$ and different values for the parameter $c$: $c = 0.2$ (left), $c = 0.4$ (middle) and $c = 0.5$ (right).

**Remark 2.4.** In the classical hanging chain problem, gravity is measured with respect to a straight-line. In Euclidean 3-space $\mathbb{R}^3$, a hanging chain is subjected to
the gravitational force centered at the center of the earth. The resulting solution was called the “true catenary” in [13]. If this problem is restricted to curves supported on \( \mathbb{S}^2 \), then the energy functional is simply \( \int_a^b |\gamma'(t)| \, dt \) because \( |\gamma(t)| = 1 \). Thus the solutions of the problem are the geodesics of \( \mathbb{S}^2 \).

2.2. Spherical catenaries: characterizations. In this subsection, we give a geometric interpretation of (9) answering to the objective T2. Let \( P_N \) and \( P_S \) denote the north and south poles of \( \mathbb{S}^2 \), respectively. With the parametrization \( \Psi = \Psi(u, v) \), the meridians \( v = \text{const.} \) of \( \mathbb{S}^2 \) are the geodesics orthogonal to the reference line \( P \).

Consider the unit vector field \( X \in \mathfrak{X}(\mathbb{S}^2 \setminus \{P_N, P_S\}) \) whose integral curves are the meridians of \( \mathbb{S}^2 \). The vector field \( X \) can be expressed in terms of the canonical vector fields \( \{\partial_x, \partial_y, \partial_z\} \) of \( \mathfrak{X}(\mathbb{R}^3) \) and the parametrization (6),

\[
X(\Psi(u, v)) = -\sin u \cos v \partial_x - \sin u \sin v \partial_y + \cos u \partial_z.
\]

Theorem 2.5. Let \( \gamma \) be a regular curve in \( \mathbb{S}^2_+ \). Then \( \gamma \) is a spherical \( \alpha \)-catenary with respect to the geodesic \( P \) if and only if its geodesic curvature \( \kappa_s \) satisfies

\[
\kappa_s = \alpha \left\langle \mathbf{n}, X(\gamma) \right\rangle \frac{1}{d},
\]

where \( \mathbf{n} \) is the unit normal vector of the curve \( \gamma \) (as a tangent vector on \( \mathbb{S}^2 \) orthogonal to \( \gamma' \)), and \( d \) is the distance to \( P \).

Proof. Since the unit normal vector to \( \mathbb{S}^2 \) along \( \gamma \) is \( \mathbf{N} = -\gamma \), the vector \( \mathbf{n} \) is

\[
\mathbf{n} = \frac{\gamma' \times \mathbf{N}(\gamma)}{|\gamma'|} = \frac{\gamma \times \gamma'}{|\gamma'|}.
\]

From the definition of \( X \), \( \langle \mathbf{n}, X(\gamma) \rangle = v' \cos u |\gamma'|. \) Equation (23) follows from (9). Recall that the function \( u \) is just the distance \( d \). \( \square \)

Since \( X \) indicates the direction of the geodesic orthogonal to \( P \), \( X \) plays the same role as the vector field \( \partial_y \) plays in the formula (5). It therefore follows that equation (23) is the version of (5) for \( \alpha \)-catenaries in the space form \( \mathbb{S}^2 \).

The last part of this section addresses the third objective T3. A rotational surface in the 3-dimensional sphere \( \mathbb{S}^3 \) will be constructed by using a spherical catenary \( \gamma \) as its generating curve and being the geodesic \( P \) its rotation axis. For this, let \( \mathbb{S}^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \} \) and let \( \mathbb{S}^2 \hookrightarrow \mathbb{S}^3 \) be the embedding defined by \( (x_1, x_2, x_3) \mapsto (x_1, x_2, x_3, 0) \). This identifies \( \mathbb{S}^2 \) with \( \mathbb{S}^2 \times \{0\} \subset \mathbb{S}^3 \). Let \( \gamma: I \to \mathbb{S}^2 \subset \mathbb{S}^3 \) be a curve contained in \( \mathbb{S}^2_+ \). Denote by \( S_\gamma \) the surface of revolution in \( \mathbb{S}^3 \) obtained by rotating \( \gamma \) with respect to \( P \subset \mathbb{S}^2 \times \{0\} \).
whose rotation axis is $P$ is $\mathcal{G} = \{ \mathcal{R}_s : s \in \mathbb{R} \}$, where

\[
\mathcal{R}_s = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & \cos s & -\sin s \\
0 & 0 & \sin s & \cos s
\end{pmatrix}.
\]

In order to simplify the computations, we assume that $\gamma$ is parametrized by $\gamma(t) = \Psi(u(t), t)$. Then the parametrization of $S_\gamma$ is

\[
\Phi(t, s) = \mathcal{R}_s \cdot \gamma(t) = (\cos (u(t)) \cos t, \cos (u(t)) \sin t, \sin (u(t)) \cos s, \sin (u(t)) \sin s),
\]

where $t \in I \subset \mathbb{R}$ and $s \in \mathbb{R}$. The expression of the mean curvature of $S_\gamma$ computed with the aid of the parametrization (25) is

\[
H = \frac{E h_{22} - 2 F h_{12} + G h_{11}}{2(EG - F^2)},
\]

where, as usually, $\{E, F, G\}$ and $\{h_{11}, h_{12}, h_{22}\}$ are the coefficients of the first and second fundamental form of $S_\gamma$ for the parametrization (25):

- $E = \langle \Phi_t, \Phi_t \rangle$,
- $F = \langle \Phi_t, \Phi_s \rangle$,
- $G = \langle \Phi_s, \Phi_s \rangle$,
- $h_{11} = \langle N, \Phi_{tt} \rangle$,
- $h_{12} = \langle N, \Phi_{ts} \rangle$,
- $h_{22} = \langle N, \Phi_{ss} \rangle$.

Here $N$ is the unit normal vector field on $S_\gamma$. Note that $N(\Phi(t, s))$ is not only orthogonal to $\Phi_t(t, s)$ and $\Phi_s(t, s)$, but also to $\Phi(t, s)$ since $N$ is a tangent vector of $S^3$. After a straightforward computation, we obtain

\[
H = \frac{\cos u(\cos u + \cos 3u) + (3 \cos 2u - 1)u'^2 - 2u'' \sin u \cos u}{2 \sin u(u'^2 + (\cos u)^2)^{3/2}}.
\]

From (8), the geodesic curvature $\kappa_s$ of $\gamma$ is

\[
\kappa_s = \frac{1}{|\gamma'|} \left(2u'^2 \sin u + (\cos u)^2 \sin u + u'' \cos u\right).
\]

From (28), we can eliminate $u''$ by writing it in terms of $\kappa_s$. Replacing in (27), the expression of the mean curvature $H$ of $S_\gamma$ is rewritten as

\[
H = \frac{(\cos u)^2 - \sin u |\gamma'| \kappa_s}{\sin u |\gamma'|}.
\]

Thus, $H = 0$ if and only if $\sin u |\gamma'| \kappa_s = (\cos u)^2$. However, from (9), $\gamma$ is a spherical $\alpha$-catenary if $|\gamma'| \kappa_s = \alpha \cos u$. Note that the term $\sin u$ in the numerator of (29) is just the Euclidean distance of the point $\Psi(u, v)$ to the plane $\Pi$ of equation $z = 0$. This suggests to consider the distance to this plane $\Pi$. Since the distance is measured with respect to an object of $\mathbb{R}^3$, the problem is of extrinsic nature and the only condition required to the curve $\gamma$ is that it must be contained within $S^3_\perp$.

More generally, given a surface $S$ of $\mathbb{R}^3$, the extrinsic hanging chain problem consists in finding the shape of a hanging chain supported on $S$ where the weight is measured
with respect to a reference plane of $\mathbb{R}^3$. If, as usually, the $z$-axis indicates the
direction of the gravity, the reference plane can be assumed to be the plane $\Pi$ of
equation $z = 0$. The extrinsic hanging chain problem was studied in the XIX century
by Bobillier ([6]), although it has not yet received much interest in the literature.
See also [1 Ch. VII] and [18], and more recently, [16]. If $S$ is the sphere $S^2$, a
solution of this problem will be called an extrinsic spherical catenary (Bobillier used
the expression “spherical chaˆınette” for this curve). In this paper, we recall this
problem and its solution and, in addition, we give credit the work of Bobillier, an
almost forgotten French mathematician ([19]). Introducing a real parameter $\alpha \in \mathbb{R}$,
the energy of the hanging chain is

$$
E^x_S[\gamma] = \int_a^b (\sin u)^\alpha \sqrt{u''^2 + v''^2(\cos u)^2} \, dt.
$$

**Proposition 2.6.** Let $\gamma$ be a regular curve in $S^2_+$. Then $\gamma$ is an extrinsic spherical
$\alpha$-catenary with respect to the plane $\Pi$ if and only if its geodesic curvature $\kappa_s$ satisfies

$$
\kappa_s = \frac{\alpha v'(\cos u)^2}{\sin u|\gamma'|},
$$
or equivalently, if

$$
\kappa_s = \frac{\alpha \langle n, \partial_z \rangle}{\sin u}.
$$

**Proof.** The Euler-Lagrange equations for (30) follow directly from (16), being now
$f(u) = (\sin u)^\alpha$. □

Equation (32) is analogous to (5) because the term $\sin u$ in the denominator of (32)
is the height with respect to $\Pi$ and the vector field $\partial_z$ is parallel to the direction of
the gravity with respect to $\Pi$.

Finally, we answer the question of when the mean curvature of the rotational surface
$S_\gamma$ is identically zero (objective T3).

**Corollary 2.7.** Let $\gamma$ be a regular curve in $S^2_+ \times \{0\}$. Then $S_\gamma$ is minimal if and
only if $\gamma$ is an extrinsic spherical catenary with respect to the plane $\Pi$.

**Proof.** Without loss of generality, let $\gamma(t) = \Psi(u(t), t)$. From (29), the mean
curvature $H$ vanishes if and only if $(\cos u)^2 = \sin u|\gamma'|\kappa_s$ and this identity is just (31)
when $\alpha = 1$. □

This result in $S^3$ is analogous to the relation between the catenoid of $\mathbb{R}^3$ and the
catenary curve. Rotational surfaces in $S^3$ with zero mean curvature are known (see
for example [3, 25]). Among these surfaces, the Clifford torus is the most famous
example because it is the only minimal embedded torus in $S^3$ ([8]). The Clifford
torus corresponds to the case $\kappa_s \equiv 1$ and $u(t) = \pi/4$ in (31).
3. THE CATENARY IN THE HYPERBOLIC PLANE

In this section, the hanging chain problem in the hyperbolic plane \( \mathbb{H}^2 \) is investigated. The model for \( \mathbb{H}^2 \) will be the upper half-plane \( (\mathbb{R}_+^2, \langle \cdot, \cdot \rangle) \), where \( \mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 : y > 0\} \) and the metric is \( \langle \cdot, \cdot \rangle = \frac{dx^2 + dy^2}{y^2} \).

The hanging chain problem in hyperbolic plane is richer than in Euclidean plane because there are various ways of measuring the weight of a curve. Besides taking the (hyperbolic) distance to a given geodesic, there is also the possibility to consider the horocycle distance to this geodesic. Moreover, horocycles can also serve as reference lines to measure the weight of the curve. Horocycles have some analogies with the straight-lines of \( \mathbb{R}^2 \) and provide a new kind of geometry in \( \mathbb{H}^2 \) called the horospherical geometry ([20]). This section is divided in three subsections according this variety of choices:

1. Hyperbolic catenary: the reference line is a geodesic and the height is the hyperbolic distance.
2. Hyperbolic horo-catenary: the reference line is a geodesic and the height is the horocycle distance.
3. Horo-catenary: the reference line is a horocycle and the height is the hyperbolic distance.

Let \( \gamma : [a, b] \to \mathbb{H}^2 \) be a regular curve parametrized by \( \gamma(t) = (u(t), v(t)) \). The energy to minimize in all these situations is of type

\[
[\gamma] \mapsto \int_a^b \omega(u, v) \frac{\sqrt{u'^2 + v'^2}}{v} \, dt
\]

where \( \omega = \omega(u, v) \) is a function on the variables \( u \) and \( v \). Here \( \sqrt{u'^2 + v'^2}/v \, dt \) is the arc-length element of \( \mathbb{H}^2 \). Thus this energy can be interpreted as the length of \( \gamma \) in the conformal metric \( \tilde{g} = \frac{v^2}{r}(du^2 + dv^2) \) being its critical points geodesics in the conformal space \( (\mathbb{R}_+^2, \tilde{g}) \).

3.1. Hyperbolic catenary. The first case to investigate follows the same motivation as in Euclidean plane. The reference line is a geodesic \( L \) of \( \mathbb{H}^2 \) which will be fixed. The weight of the curve is measured with the hyperbolic distance to \( L \). Without loss of generality, we assume that the geodesic is the vertical line \( L = \{(0, y) : y > 0\} \). The hyperbolic distance \( d \) of a point \( (x, y) \in \mathbb{H}^2 \) to \( L \) is

\[
d = \log \frac{x + r}{y}, \quad r = \sqrt{x^2 + y^2}.
\]

Given a curve \( \gamma(t) = (u(t), v(t)) \), define the energy

\[
E_H[\gamma] = \int_a^b d^a \frac{\sqrt{u'^2 + v'^2}}{v} \, dt, \quad d = \log \frac{u + r}{v},
\]
where \( r(t) = \sqrt{u(t)^2 + v(t)^2} \) and \( \alpha \in \mathbb{R} \) is a parameter. Let us observe that \( \mathcal{E}_H \) is a particular case of (33) by choosing \( \omega(u, v) = d^\alpha \). Implicitly, it is assumed that \( d \neq 0 \), that is, \( u \neq 0 \). Equivalently, the curve \( \gamma \) is contained in one of the domains \( \mathbb{H}^2_+ = \{ (x, y) \in \mathbb{H}^2 : x > 0 \} \) or \( \mathbb{H}^2_- = \{ (x, y) \in \mathbb{H}^2 : x < 0 \} \). Since they are mapped into each other by means of the isometry \( (x, y) \mapsto (-x, y) \), it will be assumed that \( \mathcal{E}_H \) acts on the class of all curves \( \gamma \) contained in \( \mathbb{H}^2_+ \).

**Definition 3.1.** A critical point of \( \mathcal{E}_H \) is called an **hyperbolic \( \alpha \)-catenary**.

As in \( \mathbb{R}^2 \) and \( \mathbb{S}^2 \), hyperbolic catenaries will be characterized in terms of their curvature \( \kappa_h \) understood as the curvature as curve of the hyperbolic plane \( \mathbb{H}^2 \).

**Theorem 3.2.** Let \( \gamma(t) = (u(t), v(t)) \) be a regular curve in \( \mathbb{H}^2_+ \). Then \( \gamma \) is an hyperbolic \( \alpha \)-catenary with respect to the geodesic \( L \) if and only if its curvature \( \kappa_h \) satisfies

\[
\kappa_h = -\frac{\alpha uu' + vv'}{d \sqrt{u'^2 + v'^2}}.
\]

**Proof.** The Lagrangian of \( \mathcal{E}_H \) is \( J[u, v] = d^\alpha \sqrt{u'^2 + v'^2} \). The Euler-Lagrange equations of \( \mathcal{E}_H \) are calculated by using (11). A computation shows that the two equations of (11) are, respectively,

\[
\begin{align*}
v' \left( \alpha \frac{uu' + vv'}{rd} + u' + v \frac{u'v'' - v'u''}{u'^2 + v'^2} \right) &= 0, \\
u' \left( \alpha \frac{uu' + vv'}{rd} + u' + v \frac{u'v'' - v'u''}{u'^2 + v'^2} \right) &= 0.
\end{align*}
\]

Since \( \gamma \) is a regular curve, both equations (36) lead to

\[
\alpha \frac{uu' + vv'}{rd} + u' + v \frac{u'v'' - v'u''}{u'^2 + v'^2} = 0.
\]

To facilitate the calculations, it is better to reformulate (37) in terms of the Euclidean curvature \( \kappa_e \) of \( \gamma \) considering \( \mathbb{R}^2_+ \) endowed with the Euclidean metric, and later, to relate \( \kappa_e \) with \( \kappa_h \). Let \( m(t) = u'(t)^2 + v'(t)^2 \). The Euclidean curvature \( \kappa_e \) of \( \gamma(t) = (u(t), v(t)) \) is

\[
\kappa_e = \frac{u'v'' - v'u''}{m^{3/2}},
\]

hence equation (37) can be written as

\[
\kappa_e = \frac{1}{v \sqrt{m}} \left( \alpha \frac{uu' + vv'}{rd} + u' \right).
\]

On the other hand, the curvature \( \kappa_h \) of \( \gamma \) is related to \( \kappa_e \) because the hyperbolic metric is conformal to the Euclidean one: see [5, Chapter 1]. This relation is
(40) \[ \kappa_h = v \kappa_e + \frac{u'}{\sqrt{m}}. \]
The result concludes from (39) and (40).

Concerning T2, the next step consists into writing equation (35) in a similar manner as the formula (5) involving the curvature \( \kappa_h \) and a vector field of \( H_2 \). Consider the unit vector field \( Y \in \mathcal{X}(H^2) \) given by

\[ Y(x, y) = y \left( \frac{y}{\sqrt{x^2 + y^2}} \partial_x - \frac{x}{\sqrt{x^2 + y^2}} \partial_y \right). \]

The integral curves of \( Y \) are the geodesics orthogonal to \( L \) which measure the distance of a point of \( H^2 \) from \( L \).

**Theorem 3.3.** A regular curve \( \gamma \) in \( H^2_+ \) is a hyperbolic \( \alpha \)-catenary with respect to the geodesic \( L \) if and only if its curvature \( \kappa_h \) satisfies

(41) \[ \kappa_h = \alpha \frac{\langle n, Y \rangle}{d}. \]

**Proof.** Since \( \gamma'(t) = (u'(t), v'(t)) \), the unit normal vector of \( \gamma \) as a curve of \( H^2 \) is

(42) \[ n(t) = v \left( -\frac{v'}{r \sqrt{m}} \right). \]

Thus

\[ \langle n, Y(\gamma) \rangle = -\frac{uu' + vv'}{r \sqrt{m}}. \]

Using this identity in the expression for \( \kappa_h \) in (35) gives (41).

As a consequence, Theorem 3.3 is the version in \( H^2 \) of the statement (5) for Euclidean catenaries.

### 3.2. Hyperbolic horo-catenaries

Consider a modified version of the above hanging chain problem replacing the hyperbolic distance by the horocycle distance to \( L \). The **horocycle distance** to the geodesic \( L \) is defined as the distance of a point \( (x, y) \in H^2 \) to \( L \) measured by the horocycle passing through \( (x, y) \) and orthogonal to \( L \). The expression of the horocycle distance of \( (x, y) \) to \( L \) is \( |x|/y \). We now find the solution of this problem. Let \( \gamma(t) = (u(t), v(t)), t \in [a, b], \) be a regular curve contained in \( H^2_+ \). The energy that measures the weight of \( \gamma \) with the horocycle distance is

(43) \[ E_{hor}^{H}[\gamma] = \int_a^b d_{hor} \sqrt{u'^2 + v'^2} dt, \quad d_{hor} = \frac{u}{v}. \]

**Definition 3.4.** A critical point of \( E_{hor}^{H} \) is called an \( \alpha \)-hyperbolic horo-catenary.
We give an answer to the objective T1.

**Theorem 3.5.** Let $\gamma(t) = (u(t), v(t))$ be a regular curve in $\mathbb{H}^2_+$. Then $\gamma$ is an $\alpha$-hyperbolic horo-catenary with respect to the geodesic $L$ if and only if its curvature $\kappa_h$ satisfies

$$\kappa_h = -\alpha \frac{uu' + vv'}{d_{\text{hor}} v \sqrt{m}}.$$  \hfill (44)

**Proof.** We calculate the Euler-Lagrange equations associated to $E_{\text{hor}}$. The Lagrangian of (43) is

$$J[u,v] = d_{\text{hor}} \sqrt{u'^2 + v'^2}.$$  

Computing (11) for this $J$, we obtain

$$u' \left( \alpha (uu' + vv') + d_{\text{hor}} u' v + d_{\text{hor}} v^2 \frac{u'' u - u' v'}{u'^2 + v'^2} \right) = 0,$$$$

$$v' \left( \alpha (uu' + vv') + d_{\text{hor}} u' v + d_{\text{hor}} v^2 \frac{u'' v - u' v'}{u'^2 + v'^2} \right) = 0.$$  \hfill (45)

Since $u'$ and $v'$ cannot simultaneously vanish by regularity of $\gamma$, we deduce from (45) that the Euler-Lagrange equation is

$$\alpha (uu' + vv') + d_{\text{hor}} u' v + d_{\text{hor}} v^2 \frac{u'' u - u' v'}{u'^2 + v'^2} = 0.$$  \hfill (46)

By virtue of the expression (38) for the Euclidean curvature $\kappa_e$, equation (46) can be rewritten as

$$v \kappa_e = - \left( \alpha \frac{uu' + vv'}{d_{\text{hor}} v \sqrt{m}} + \frac{u'}{\sqrt{m}} \right).$$  \hfill (47)

Finally, the identity (44) is consequence of (40) and (47). \hfill $\square$

In order to answer to T2 for $\alpha$-hyperbolic horo-catenaries, define the vector field $W \in \mathfrak{X}(\mathbb{H}^2)$ by

$$W(x,y) = y \partial_x - x \partial_y.$$  

This vector field is proportional to $Y$ at each point of $\mathbb{H}^2$.

**Theorem 3.6.** Let $\gamma$ be a regular curve in $\mathbb{H}^2$. Then $\gamma$ is a $\alpha$-hyperbolic horo-catenary with respect to the geodesic $L$ if and only if its curvature $\kappa_h$ satisfies

$$\kappa_h = \alpha \frac{\langle n, W \rangle}{d_{\text{hor}}}.$$  \hfill (48)

**Proof.** Using the expression (42) for $n$, we have

$$\langle n, W(\gamma) \rangle = - \frac{uu' + vv'}{v \sqrt{m}}.$$
This identity and \((47)\) give \((48)\). □

To conclude this subsection we investigate problem T3 for this type of catenaries. We embed the hyperbolic plane \(\mathbb{H}^2\) into the 3-dimensional hyperbolic space \(\mathbb{H}^3 = (\mathbb{H}^3, \frac{1}{2}(dx_1^2 + dx_2^2 + dx_3^2))\) via the embedding \((x, y) \in \mathbb{H}^2 \mapsto (x, 0, y) \in \mathbb{H}^3\). With this identification, the geodesic \(L \subset \mathbb{H}^2\) is the \(x_3\)-axis in \(\mathbb{H}^3\). Let \(S_\gamma\) denote the surface of revolution obtained by rotating \(\gamma(t) = (u(t), 0, v(t))\) with respect to the \(x_3\)-axis. In the upper half-space model of \(\mathbb{H}^3\), the rotations that leave pointwise fixed the \(x_3\)-axis coincides with the Euclidean rotations of \(\mathbb{R}^3\) with the same axis. A parametrization of \(S_\gamma\) is \((t, s) \mapsto (u(t) \cos s, u(t) \sin s, v(t)), t \in [a, b], s \in \mathbb{R}\).

**Theorem 3.7.** Let \(\gamma\) be a regular curve in \(\mathbb{H}^2\). Then \(\gamma\) is a hyperbolic horo-catenary with respect to the geodesic \(L\) if and only if the surface \(S_\gamma\) is minimal in \(\mathbb{H}^3\).

**Proof.** In the upper half-space model of \(\mathbb{H}^3\), the mean curvature \(H\) of a surface \(S\) can be computed with the aid of the (Euclidean) mean curvature \(H_e\) of \(S\) when \(S\) is viewed as a submanifold of the Euclidean space \(\mathbb{R}^3\). There is a relation between \(H\) and \(H_e\), namely,

\[ H(p) = x_3H_e(p) + N_3(p), \]

where \(p = (x_1, x_2, x_3)\) and \(N = (N_1, N_2, N_3)\) is the Euclidean unit normal vector of \(S\) ([5, Chapter 1]). In the particular case of the rotational surface \(S_\gamma\), the value of \(H_e\) is

\[ H_e = \frac{\kappa_e}{2} + \frac{v'}{2u\sqrt{m}}, \]

and the expression of \(N\) is

\[ N = \frac{1}{\sqrt{m}}(-v' \cos s, -v' \sin s, u'). \]

Thus \(N_3 = u/\sqrt{m}\), and using \((50)\), the mean curvature \(H\) given in \((49)\) satisfies

\[ 2H = v\kappa_e + \frac{vv'}{u\sqrt{m}} + 2\frac{u'}{\sqrt{m}}. \]

Then \(H = 0\) if and only if

\[ v\kappa_e + \frac{vv'}{u\sqrt{m}} + 2\frac{u'}{\sqrt{m}} = 0. \]

But this identity \((51)\) is just equation \((47)\) for \(\alpha = 1\) because \(d_{\text{hor}} = u/v\). This proves the result. □

Theorem 3.7 shows a geometric interpretation of the generating curves of minimal rotational surfaces of \(\mathbb{H}^3\) proving that these curves are the solutions of a hanging chain problem in \(\mathbb{H}^2\).
3.3. Horo-catenaries. We investigate the hanging chain problem replacing the geodesic $L$ by a horocycle $H$ and measuring the weight of the curve $\gamma$ using the hyperbolic distance to $H$. Without loss of generality we can assume that $H$ is the horocycle $H = \{(t, 1) : t \in \mathbb{R}\}$. If $(x, y) \in \mathbb{H}^2$, the hyperbolic distance $d_b$ from $(x, y)$ to $H$ is the length throughout the geodesic orthogonal to $H$ passing through $(x, y)$. This distance is $d_b = \log(y)$. Note that this distance coincides with the Busemann function in the horospherical geometry when the ideal point is $\infty$ (9).

The horocycle $H$ divides $\mathbb{H}^2$ in two non-isometric domains, namely, $\mathbb{H}^2(+) = \{(x, y) \in \mathbb{H}^2 : y > 1\}$ and $\mathbb{H}^2(-) = \{(x, y) \in \mathbb{H}^2 : y < 1\}$. From now on, we will assume that all curves are contained in $\mathbb{H}^2(+)$. Let $\gamma : [a, b] \to \mathbb{H}^2$ be a curve contained in $\mathbb{H}^2(+)$. Define the energy

$$E_{\text{hor}}[\gamma] = \int_a^b (d_b)^{\alpha} \frac{\sqrt{u'^2 + v'^2}}{v} dt, \quad d_b = \log v,$$

where $\gamma(t) = (u(t), v(t))$ and $\alpha \in \mathbb{R}$ is a parameter.

**Definition 3.8.** A critical point of $E_{\text{hor}}$ is called an $\alpha$-horo-catenary.

We characterize the $\alpha$-horo-catenaries in terms of their curvature $\kappa_h$.

**Theorem 3.9.** Let $\gamma(t) = (u(t), v(t))$ be a regular curve contained in $\mathbb{H}^2(+)$. Then $\gamma$ is an $\alpha$-horo-catenary with respect to the horocycle $H$ if and only if its curvature $\kappa_h$ satisfies

$$\kappa_h = \alpha \frac{u'}{d_b \sqrt{m}}.$$

**Proof.** The Lagrangian of (52) is

$$J[u, v] = (d_b)^{\alpha} \frac{\sqrt{u'^2 + v'^2}}{v}.$$

The computation of the Euler-Lagrange equations (11) gives, respectively,

$$v' \left( \frac{(\alpha - d_b)u'}{\sqrt{m}} - v d_b \frac{u''}{m^{3/2}} \right) = 0,$$

$$u' \left( \frac{(\alpha - d_b)u'}{\sqrt{m}} - v d_b \frac{u''}{m^{3/2}} \right) = 0.$$

Since $\gamma$ is regular, the Euler-Lagrange equations (55) are equivalent to

$$\frac{\alpha - d_b}{\sqrt{m}} u' - v d_b \frac{u''}{m^{3/2}} = 0.$$
In terms of the Euclidean curvature $\kappa_e$ of $\gamma$, equation \ref{eq:56} can be expressed as

\begin{equation}
\kappa_e = \frac{\alpha - db}{vdb\sqrt{m}} v'.
\end{equation}

Then the result is now a consequence of \ref{eq:40} and \ref{eq:57}. 

As $J$ does not depend on $u$, it is deduced a first integration of the Euler-Lagrange equation \ref{eq:56}.

**Corollary 3.10.** Let $\gamma$ be a regular curve contained in $\mathbb{H}^2(+)$. Then $\gamma$ is an $\alpha$-horo-catenary with respect to the horocycle $\mathcal{H}$ if and only if $\gamma$ can be locally expressed as

\begin{equation}
\gamma(v) = \left( \int_v^\infty \frac{ct}{\sqrt{(\log t)^{2\alpha} - c^2t^2}} dt, v \right),
\end{equation}

where $c$ is a constant of integration.

**Proof.** From the expression of $J$ in \ref{eq:54}, we deduce that there exists a constant $c$ such that $\frac{\partial J}{\partial u'} = c$. This identity is

\begin{equation}
\frac{u'(\log v)^\alpha}{v\sqrt{u'^2 + v'^2}} = c.
\end{equation}

Suppose that $\gamma$ writes locally as $\gamma(v) = (u(v), v)$. Then \ref{eq:59} is

\begin{equation}
\frac{u'(\log v)^\alpha}{v\sqrt{1 + u'^2}} = c.
\end{equation}

Hence it immediately follows \ref{eq:58}.

Concerning objective T2, we will express \ref{eq:53} in terms of a vector field $V$ of $\mathbb{H}^2$. The weight of $\gamma$ is measured using the geodesics orthogonal to $\mathcal{H}$. In the upper half-plane model of $\mathbb{H}^2$, these geodesics are the vertical straight-lines. The unitary vector field $V \in \mathfrak{X}(\mathbb{H}^2)$ whose integral curves are these geodesics is $V(x, y) = y \partial_y$.

**Theorem 3.11.** Let $\gamma$ be a regular curve contained in $\mathbb{H}^2(+)$. Then $\gamma$ is an $\alpha$-horo-catenary with respect to the horocycle $\mathcal{H}$ if and only if its curvature $\kappa_h$ satisfies

\begin{equation}
\kappa_h = \alpha \frac{\langle n, V \rangle}{db}.
\end{equation}

**Proof.** The proof is immediate from \ref{eq:53}, the expression \ref{eq:42} for $n$ and the definition of the vector field $V$. 

\begin{flushright}
□
\end{flushright}
Equation (60) is the version of the formula (5) in the context of $\alpha$-horo-catenaries. We close this section with some pictures of catenaries (case $\alpha = 1$). See Figure 2. These figures have been plotted using Mathematica again and we explain the process to obtain. We have the expressions (39), (47) and (57) for the Euclidean curvature $\kappa_e$. The curve $\gamma(t) = (u(t), v(t))$ is assumed to be parametrized by the Euclidean arc-length. Consequently, writing $\gamma'(t) = (\cos \theta(t), \sin \theta(t))$ for some function $\theta = \theta(t)$, it follows that the functions $u(t), v(t)$ and $\theta(t)$ satisfy the ODE system

$$\begin{cases}
u'(t) &= \cos \theta(t) \\
v'(t) &= \sin \theta(t) \\
\theta'(t) &= \kappa_e(t).\end{cases}$$

Finally, and distinguishing the three types of catenaries of $\mathbb{H}^2$, the system (61) have been numerically solved with Mathematica once initial conditions have been prescribed,

$$u(0) = u_0, \quad v(0) = v_0, \quad \theta(0) = \theta_0.$$

Figure 2. Catenaries in hyperbolic plane considering the upper half-plane model: hyperbolic catenary (left), hyperbolic horo-catenary (middle), horo-catenary (right). These curves are solutions of (61) and the initial conditions (62) are: $u_0 = 1, \; v_0 = 3$ and $\theta_0 = \pi/2$ (left and middle); $u_0 = 0, \; v_0 = 1.8$ and $\theta_0 = 0$ (right).

The horo-catenary in Figure 2 is a graph on the $x$-axis. This is not only a particular example, but it holds in general.

**Proposition 3.12.** Let $\gamma$ be a regular horo-catenary with respect to the horocycle $\mathcal{H}$. Then $\gamma$ is a vertical line or $\gamma$ is an entire bounded graph on the $x$-axis.
Proof. Let \( \gamma(t) = (u(t), v(t)), \ t \in I \subset \mathbb{R} \), be a parametrization by Euclidean arc-length, where \( I \) is the maximal domain. Then the ODE system (61) is

\[
\begin{cases}
    u'(t) = \cos \theta(t) \\
v'(t) = \sin \theta(t) \\
    \theta'(t) = \frac{1 - \log v}{v \log v} u'.
\end{cases}
\]

We distinguish two cases.

(1) Suppose there exists \( t_0 \) such that \( u'(t_0) = 0 \). Then the solution of (63) is \( u(t) = u(t_0), \ v(t) = v(t_0) + t - t_0 \) and \( \theta(t) = \pi/2 \). This proves that \( \gamma \) is a vertical straight-line.

(2) Suppose \( u'(t) \neq 0 \) for all \( t \). This implies that \( \gamma \) is a graph on some interval \( J = (a, b) \) of the \( x \)-axis. Reparametrizing \( \gamma \), we can write \( \gamma(x) = (x, v(x)) \), \( x \in J \) and we will prove that \( J = \mathbb{R} \). Equation (59) becomes

\[
\log v / \sqrt{1 + v'^2} = c.
\]

Since \( c \neq 0 \) and \( \gamma \) is contained in \( \mathbb{H}^2(+) \), we have \( \log(v) > 0 \). From (64) we deduce

\[
0 < c = \frac{\log v}{v \sqrt{1 + v'^2}} \leq \frac{\log v}{v}.
\]

The function \( t \mapsto \log t / t \) is bounded in \( (1, \infty) \) with the property \( \lim_{t \to \infty} \log t / t = \lim_{t \to 1} \log t / t = 0 \). Thus, we conclude from (65) that \( v' \) is a bounded function. Moreover, there exist two constants \( m_1, m_2 \in \mathbb{R} \) such that \( 1 < m_1 < m_2 \) and \( m_1 \leq v(x) \leq m_2 \) for all \( x \in (a, b) \). The fact that \( v' \) is bounded proves finally that \( (a, b) = \mathbb{R} \).

\[\square\]

4. Conclusions and outlook

The catenary is the solution of the hanging chain problem in \( \mathbb{R}^2 \) and this makes the catenary also attractive to other fields of science, engineering, architecture and arts. However, the hanging chain problem has not been posed in spaces other than Euclidean one. Among these spaces, it is clear that the sphere \( \mathbb{S}^2 \) and the hyperbolic plane \( \mathbb{H}^2 \) are the natural choices to extend the hanging chain problem because they are the models of the elliptic and the hyperbolic geometry, respectively. The purpose of this paper has been to solve this problem in \( \mathbb{S}^2 \) and in \( \mathbb{H}^2 \) obtaining a natural concept of catenary in both space forms.

Catenaries defined in \( \mathbb{S}^2 \) and \( \mathbb{H}^2 \) share the same properties of the Euclidean catenary. For example, we obtain a characterization in terms of the curvature of the curve, or
a coordinate-free characterization of the curvature depending on the angle between the unit normal of the curve and a vector field of the ambient space.

There are a number of problems in which this article could be expanded. For example, a question concerns to investigate the existence of closed spherical catenaries. In view of the pictures of Figure 1 it seems plausible that such catenaries do exist (see also [3]). Besides the closed catenaries, surely there are other catenaries that never closes and they are turning around the north pole of $S^2$. The problem that arises is that the curvature function is periodic, but this does not ensure that the catenary closes: see a discussion of this problem in [2].

As in the sphere, it would be interesting to classify the catenaries in hyperbolic plane. Proposition 3.12 is just a sample button, but the work to be done goes beyond that. According to Figure 2 several questions are reasonable to ask. For example, (i) when does a catenary intersect the ideal boundary of $\mathbb{H}^2$? and in such a case, whether the intersection is orthogonal; (ii) is every horo-catenary periodic? (iii) which are the properties of those horo-catenaries contained in $\mathbb{H}^2(\_)$?

Another extension of the paper would be to consider the shape of a hanging surface in $S^3$ and $\mathbb{H}^3$. In Euclidean space, the analogue of the catenary in the two-dimensional case is called a singular minimal surface ([15]). The extension is straightforward using the characterization [4]. So, it suffices by replacing the curvature of the catenary $\kappa_e$ by the mean curvature $H$ of the surface and the unit normal $n$ of the curve by the unit normal vector field $N$ to the surface. For example, in the space form $S^3$, the shape of a hanging surface with respect to $S^2 \times \{0\}$ is characterized by the equation $H = \langle N, X \rangle / d$, where $d$ is the distance to $S^2 \times \{0\}$ and $X \in \mathfrak{X}(S^3)$ is the unit vector field tangent to the meridians of $S^3$ which are orthogonal to $S^2 \times \{0\}$.

Finally, it could be interesting to obtain some geometric properties of the rotational surfaces in $S^3$ and in $\mathbb{H}^3$ constructed by catenaries (in its different possibilities). Although, initially the hanging chain problem has no relation to the problem for rotational surfaces with minimum area, in some cases we have proved a connection between both problems (Corollary 2.7 and Theorem 3.7). In the rest of types of catenaries, it seems to be interesting to investigate geometric properties of the rotational surfaces of $S^3$ and $\mathbb{H}^3$ whose generating curves are catenaries of $S^2$ and $\mathbb{H}^2$.

**Declaration of competing interest**

The author declares that he has no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
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REFERENCES

[1] P. Appell, Traité de Mécanique Rationnelle, Tome I. Gauthier-Villars, 6th. Ed. Paris, 1941.
[2] J. Arroyo, O.J. Garay, J.J. Mencía, When is a periodic function the curvature of a closed plane curve? Amer. Math. Monthly 115 (2008), 405–414.
[3] J. Arroyo, O.J. Garay, A. Pámpano, Delaunay surfaces in $S^3(\rho)$. Filomat 33 (2019), 1191–1200.
[4] F. Behroozi, A fresh look at the catenary. Eur. J. Phys. 35 (2014), 055007.
[5] A. L. Besse, Einstein Manifolds. Springer-Verlag, Berlin,1987.
[6] E. Bobillier. Statique. De l’équilibre de la chaintette sur une surface courbe. Ann. Math. Pures Appl. [Ann. Gergonne] 20 (1829/30), 153–175.
[7] R. Böhm, S. Hildebrandt, E. Taush, The two-dimensional analogue of the catenary. Pacific J. Math. 88 (1980), 247–278.
[8] S. Brendle, Embedded minimal tori in $S^3$ and the Lawson conjecture. Acta Math. 211 (2013), 177–190.
[9] H. Busemann, The Geometry of Geodesics. Academic Press Inc., New York, 1955.
[10] V. Coll, M. Harrison, Two generalizations of a property of the catenary. Amer. Math. Monthly, 121 (2014), 109–119.
[11] V. Coll, J. Dodd, A characteristic averaging property of the catenary. Amer. Math. Monthly, 123 (2016), 683–688.
[12] G. Conti, R. Paolettini, A. Trotta, The catenary in history and applications. Science & Philosophy 5 (2017), 69–94.
[13] G. Colombo, L. Mari, M. Rigoli, Remarks on mean curvature flow solitons in warped products. Discrete Contin. Dyn. Syst. Ser. S 13 (2020), 1957–1991.
[14] J. Denzler, A. Hinz, Catenaria vera – the true catenary. Expo. Math. 17 (1999) 117–142.
[15] U. Dierkes, G. Huisken, The $n$-dimensional analogue of the catenary: existence and nonexistence. Pacific J. Math. 141 (1990), 47–54.
[16] R. Ferréol, Catenary of a surface. https://mathcurve.com/courbes3d.gb/chainette/chainette.shtml 2018.
[17] M. Giaquinta, S. Hildebrandt, Calculus of Variations I, Springer-Verlag Berlin Heidelberg, New York, Second Edition, 2004.
[18] C. Gudermann, De curvis catenariis sphaericis dissertatio analytico-geometrica. J. Reine Angew. Math. 33 (1846), 189–225.
[19] C. Haubrichs dos Santos, Étienne Bobillier (1798-1840): parcours mathématique, enseignant et professionnel. Université de Lorraine, 2015. Thesis (Ph.D.)
[20] S. Izumiya, Horospherical geometry in the hyperbolic space. Noncommutativity and singularities, Advanced Studies in Pure Mathematics, vol. 55, Mathematical Society of Japan, Tokyo, 2009, pp. 31–49.
[21] D.S. Kim, Y.H. Kim, S. Park, Center of gravity and a characterization of parabolas. Kyungpook Math. J. 55 (2015), 473–484.
[22] F. Kuczmaszski, J. Kuczmaszki, Hanging around in non-uniform fields. Amer. Math. Monthly 122 (2015), 941–957.
[23] G.V. McIlvaine, A new first-principles approach for the catenary. Expo. Math. 37 (2019), 333–346.
[24] G.V. McIlvaine, A new first-principles approach for the catenary. Expo. Math. 38 (2020), 377–390.
[25] J.B. Ripoll, Uniqueness of minimal rotational surfaces in $S^3$. Amer. J. Math. 111 (1989), 537–547.
[26] Wolfram Research, Inc., Mathematica, Version 13.0.0, Champaign, IL (2021).

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