TRIPLY-CHARMED HEXAQUARK STATES WITH THE QCD SUM RULES

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Abstract

In this article, we construct the charmed-diquark-charmed-diquark-charmed-diquark type current to study the axialvector triply-charmed hexaquark state with the QCD sum rules in details. In calculations, we take the energy scale formula $\mu = \sqrt{M_H^2 - (3M_c^2)}$ to choose the pertinent energy scale of the QCD spectral density so as to enhance the pole contribution and improve the convergent behavior of the operator product expansion. If the spin-breaking effects are small for the triply-charmed hexaquark states, the ground state hexaquark states with $J^P = 0^+, 1^+$, and $2^+$ have the masses about 5.8 GeV and narrow widths.

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1 Introduction

A number of charmonium-like and bottomonium-like states were observed after the observation of the $X(3872)$, the most elusive meson up to now, by the Belle collaboration [1]. It is very difficult to find rooms to accommodate those exotic $X$, $Y$ and $Z$ states in the $q\bar{q}$ meson spectrum comfortably even for the charge-neutral mesons, such as the $Y(4260)$, $Y(4360)$, $Y(4660)$, etc. The charged charmonium-like states and bottomonium-like states are very good candidates for the hidden-charm and hidden-bottom tetraquark states or molecular states [2, 3, 4, 5, 6, 7, 8, 9, 10]. The QCD sum rules play an important role in diagnosing the nature of those new charmonium-like states [8, 11, 12, 13, 14, 15].

Now let us discuss how to construct the interpolating currents to study the tetraquark states in the QCD sum rules. The scattering amplitude for one-gluon exchange is proportional to

$$\left(\frac{\lambda^a}{2}\right)_{ij} \left(\frac{\lambda^a}{2}\right)_{kl} = -\frac{N_c + 1}{4N_c} t^A_{ik} t^A_{lj} + \frac{N_c - 1}{4N_c} t^S_{ik} t^S_{lj},$$

(1)

where

$$t^A_{ik} t^A_{lj} = \delta_{ij} \delta_{kl} - \delta_{il} \delta_{kj} = \epsilon_{mik} \epsilon_{mj},$$
$$t^S_{ik} t^S_{lj} = \delta_{ij} \delta_{kl} + \delta_{il} \delta_{kj},$$

(2)

the $\lambda^a$ is the Gell-Mann matrix, the $i$, $j$, $k$, and $l$ are color indexes, the $N_c$ is the color number. The negative sign in front of the $t^A_{ik} t^A_{lj}$ represents the interaction is attractive and favors forming the diquark correlations in color antitriplet, the positive sign in front of the $t^S_{ik} t^S_{lj}$ represents the interaction is repulsive and disfavors forming the diquark correlations in color sextet.

The diquark operators $\epsilon^{ijk} q_i^c C T q_k'$ in color antitriplet have five structures, where $CT = C\gamma_5$, $C$, $C\gamma_\mu\gamma_5$, $C\gamma_\mu$ and $C\sigma_{\mu\nu}$ correspond to the scalar, pseudoscalar, vector, axialvector and tensor diquarks, respectively. The QCD sum rules indicate that the favored quark-quark configurations are the scalar and axialvector diquark states, the axialvector diquark states have slightly larger masses than the corresponding scalar diquark states [14, 15], in the case of the heavy-light diquark states, they have almost degenerated masses [17]. In the QCD sum rules, we usually choose the scalar and axialvector diquark operators to construct the tetraquark current operators to interpolate the diquark-antidiquark type tetraquark states with the lowest masses. For example, we study the $Z_c(3900)$ with the $C\gamma_5 \otimes \gamma_\mu C - C\gamma_\mu \otimes \gamma_5 C$ type tetraquark current [12]. The masses and decay

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widths of the diquark-antidiquark type tetraquark states have been studied extensively with the QCD sum rules [8 11 12 13 14 15].

In previous works, we studied the energy scale dependence of the QCD sum rules for the exotic $X$, $Y$, and $Z$ states, which are very good candidates for the hidden-charm and hidden-bottom tetraquark states and molecular states, for the first time, and suggested a formula,

$$\mu = \sqrt{M^2_{\pi^+\pi^-} - (2M_Q)^2},$$

with the effective heavy quark mass $M_Q$ to choose the best energy scales of the QCD spectral densities [12 13 14 16]. In calculations, we observe that the energy scalar formula can enhance the pole contributions remarkably and improve the convergent behaviors of the operator product expansion remarkably also.

In 2017, the LHCB collaboration observed the doubly charmed baryon state $\Xi_{cc}^{++}$ in the $\Lambda_c^+ K^-\pi^+\pi^-$ mass spectrum [19]. The observation of the $\Xi_{cc}^{++}$ provides valuable experimental information on the strong correlation between the two charm quarks, which maybe shed light on the spectroscopy of the doubly-charmed baryon states, tetraquark states, pentaquark states, and hexaquark states. For the heavy-quark-heavy-quark systems $QQ$, only the axialvector diquark operators $\varepsilon^{ijk}Q_j^TC\gamma_\mu Q_k$ and tensor diquark operators $\varepsilon^{ijk}Q_j^TC\sigma_{\mu\nu}Q_k$ can exist due to the Fermi-Dirac statistics, we usually take the axialvector diquark operators $\varepsilon^{ijk}Q_j^TC\gamma_\mu Q_k$ as the basic constituents to construct the six-quark currents. In Ref. [24], we extend our previous works to study the scalar-diquark-scalarscalar-diquark, $\varepsilon^{aij}u_i^TQ_j^TC\gamma_5d_j - \varepsilon^{bkl}u_k^TC\gamma_5c_l - \varepsilon^{cmn}d_m^TQ_n^TC\gamma_5c_n$ type hexaquark states $uuddcc$ with the QCD sum rules in details. In Ref. [25], we construct the color-singlet-color-singlet type currents to study the scalar and axialvector (triply-charmed) $\Xi_{cc}, \Sigma_c$ dibaryon states with QCD sum rules in details. In this article, we extend our previous works to study the charmed-diquark-charmed-diquark-charmed-diquark type hexaquark states with the QCD sum rules.

The article is arranged as follows: we derive the QCD sum rules for the mass and pole residue of the axialvector triply-charmed hexaquark state in Sect.2; in Sect.3, we present the numerical results and discussions; and Sect.4 is reserved for our conclusion.

## 2 QCD sum rules for the triply-charmed hexaquark states

We choose the scalar ($S$) and axialvector ($A$) charmed diquark operators as the basic constituents to construct the triply-charmed diquark currents $J(x)$, $J_\mu(x)$ and $J_{\mu\nu}(x)$,

\[
\begin{align*}
J(x) &= \varepsilon^{abc}\varepsilon^{aij}\varepsilon^{bkl}\varepsilon^{cmn}u_i^T(x)C\gamma_\alpha c_j(x)d_k^T(x)C\gamma_\alpha c_l(x)u_m^T(x)C\gamma_5c_n(x), \\
J_\mu(x) &= \varepsilon^{abc}\varepsilon^{aij}\varepsilon^{bkl}\varepsilon^{cmn}u_i^T(x)C\gamma_\alpha c_j(x)d_k^T(x)C\gamma_5c_l(x)u_m^T(x)C\gamma_\mu c_n(x), \\
J_{\mu\nu}(x) &= \varepsilon^{abc}\varepsilon^{aij}\varepsilon^{bkl}\varepsilon^{cmn}u_i^T(x)C\gamma_\mu c_j(x)d_k^T(x)C\gamma_\nu c_l(x)u_m^T(x)C\gamma_5c_n(x) + (\mu \leftrightarrow \nu),
\end{align*}
\]

where the $a, b, c, \cdots$ are color indexes, the $C$ is the charge conjugation matrix. Those diquark-diquark type currents couple potentially to the hexaquark states ($H$) with the spin-parity $J^P = 0^+, 1^+$ and $2^+$, respectively,

\[
\begin{align*}
\langle 0|J(0)|H_0(p)\rangle &= \lambda_{H_0}, \\
\langle 0|J_\mu(0)|H_1(p)\rangle &= \lambda_{H_1} \varepsilon_\mu, \\
\langle 0|J_{\mu\nu}(0)|H_2(p)\rangle &= \lambda_{H_2} \varepsilon_{\mu\nu},
\end{align*}
\]
where the $\lambda_{H_{0/1/2}}$ are the pole residues, the $\varepsilon_{\mu}$ and $\varepsilon_{\mu\nu}$ are the polarization vectors of the axialvector and tensor hexaquark states, respectively. The $H$’s have three diquarks,

$$
H_0 = A_{bc} A_{de} S_{ac}, \\
H_1 = S_{be} S_{dc} A_{ac}, \\
H_2 = A_{bc} A_{de} S_{ac},
$$

(6)

where the subscripts $uc$ and $dc$ stand for the quark constituents. The axialvector charmed diquark states have slightly larger masses than the scalar charmed diquark states, or they have almost degenerated masses $[17]$, the masses of the triply-charmed hexaquark states may have the hierarchy $M_1 \leq M_0 \leq M_2$. It is horrible to carry out the operator product expansion for the triply heavy hexaquark states, in this article, we choose the axialvector current $J_\mu(x)$ to study the lowest state $H_1$ to estimate the magnitude of the masses of the triply-charmed hexaquark states.

In the following, we write down the two-point correlation function $\Pi_{\mu\nu}(p)$ in the QCD sum rules,

$$
\Pi_{\mu\nu}(p) = i \int d^4 x e^{ipx} \langle 0 | T \left\{ J_\mu(x) J^\dagger_\nu(0) \right\} | 0 \rangle.
$$

(7)

At the hadron side of the correlation function $\Pi_{\mu\nu}(p)$, we isolate the contribution of the lowest axialvector triply-charmed hexaquark state,

$$
\Pi_{\mu\nu}(p) = \frac{\lambda_H^2}{M_H^2 - p^2} \left( -g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2} \right) + \cdots,
$$

$$
= \mathcal{P}(p^2) \left( -g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2} \right) + \cdots,
$$

(8)

thereafter we will smear the subscript $1$. In this article, we choose the tensor structure $-g_{\mu\nu} + \frac{p_\mu p_\nu}{p^2}$ and study the component $\Pi(p^2)$ to explore the axialvector hexaquark state.

At the QCD side of the correlation function $\Pi_{\mu\nu}(p)$, we contract the $u$, $d$ and $c$ quark fields with Wick theorem and obtain the result,

$$
\Pi_{\mu\nu}(p) = i \varepsilon^{abc} \varepsilon_{dij} \varepsilon^{bkl} \varepsilon^{cmn} \varepsilon^{e' b' c'} \varepsilon^{i' j' k' l'} \varepsilon^{m' n'} \int d^4 x e^{ipx}
$$

$$
\left\{ \text{Tr} \left[ \gamma_5 C_{jj'}(x) \gamma_5 C_{ij'}(x) C \right] \text{Tr} \left[ \gamma_5 C_{jj'}(x) \gamma_5 C_{ij'}(x) C \right] \text{Tr} \left[ \gamma_5 C_{nn'}(x) \gamma_5 C_{mn'}(x) C \right] + \text{Tr} \left[ \gamma_5 C_{jj'}(x) \gamma_5 C_{ij'}(x) C \right] \text{Tr} \left[ \gamma_5 C_{jj'}(x) \gamma_5 C_{ij'}(x) C \right] \text{Tr} \left[ \gamma_5 C_{nn'}(x) \gamma_5 C_{mn'}(x) C \right]
$$

$$
+ \text{Tr} \left[ \gamma_5 C_{jj'}(x) \gamma_5 C_{ij'}(x) C \right] \text{Tr} \left[ \gamma_5 C_{jj'}(x) \gamma_5 C_{ij'}(x) C \right] \text{Tr} \left[ \gamma_5 C_{nn'}(x) \gamma_5 C_{mn'}(x) C \right] \right\} \left\{ \text{Tr} \left[ \gamma_5 C_{jj'}(x) \gamma_5 C_{ij'}(x) C \right] \text{Tr} \left[ \gamma_5 C_{jj'}(x) \gamma_5 C_{ij'}(x) C \right] \text{Tr} \left[ \gamma_5 C_{nn'}(x) \gamma_5 C_{mn'}(x) C \right]
$$

$$
+ \text{Tr} \left[ \gamma_5 C_{jj'}(x) \gamma_5 C_{ij'}(x) C \right] \text{Tr} \left[ \gamma_5 C_{jj'}(x) \gamma_5 C_{ij'}(x) C \right] \text{Tr} \left[ \gamma_5 C_{nn'}(x) \gamma_5 C_{mn'}(x) C \right] \right\} \left\{ \text{Tr} \left[ \gamma_5 C_{jj'}(x) \gamma_5 C_{ij'}(x) C \right] \text{Tr} \left[ \gamma_5 C_{jj'}(x) \gamma_5 C_{ij'}(x) C \right] \text{Tr} \left[ \gamma_5 C_{nn'}(x) \gamma_5 C_{mn'}(x) C \right]
$$

$$
+ \text{Tr} \left[ \gamma_5 C_{jj'}(x) \gamma_5 C_{ij'}(x) C \right] \text{Tr} \left[ \gamma_5 C_{jj'}(x) \gamma_5 C_{ij'}(x) C \right] \text{Tr} \left[ \gamma_5 C_{nn'}(x) \gamma_5 C_{mn'}(x) C \right] \right\}, \quad (9)
$$
where \( S_{ij}(x) = U_{ij}(x) \) and \( D_{ij}(x) \),

\[
S_{ij}(x) = \frac{i \delta_{ij} x}{2\pi^2 x^4} - \frac{\delta_{ij} \langle \bar{q}q \rangle}{12} - \frac{\delta_{ij} x^2 \langle \bar{q}g_s \sigma G q \rangle}{192} - \frac{i g_s G_{\alpha \beta} t^a_{ij} (\sigma^{\alpha \beta} + \sigma^{\beta \alpha} \not{x})}{32\pi^2 x^2} - \frac{\delta_{ij} x^4 \langle \bar{q}q \rangle (g_s^2 GG)}{27648} - \frac{1}{8} \langle \bar{q}j \sigma^{\mu \nu} q_i \rangle \sigma_{\mu \nu} + \cdots ,
\]

(10)

\[
C_{ij}(x) = \frac{i}{(2\pi)^4} \int d^4k e^{-ik \cdot x} \left\{ \delta_{ij} - \frac{g_s G_{\alpha \beta} t^a_{ij} \sigma^{\alpha \beta} (\not{k} + \not{m}_c)}{4 (k^2 - m_c^2)^2} \left( \frac{f^{\alpha \beta \mu \nu} + f^{\alpha \mu \beta \nu} + f^{\alpha \mu \beta \nu} + f^{\alpha \beta \mu \nu}}{4(k^2 - m_c^2)^2} + \cdots \right) \right\} ,
\]

\[f^{\alpha \beta \mu \nu} = (\not{k} + \not{m}_c) \gamma^\alpha (\not{k} + \not{m}_c) \gamma^\beta (\not{k} + \not{m}_c) \gamma^\mu (\not{k} + \not{m}_c) \gamma^\nu ,
\]

(11)

and \( \gamma^\mu = \frac{\lambda^\mu}{2} \), the \( \lambda^\mu \) is the Gell-Mann matrix \[12, 20, 21\]. In the full light quark propagator, see Eq. \[11\], we add the term \( \langle \bar{q}j \sigma^{\mu \nu} q_i \rangle \), which comes from Fierz rearrangement of the quark-antiquark pair \( \langle \bar{q}j \rangle \) to absorb the gluons emitted from other quark lines, to extract the mixed condensates \( \langle \bar{q}g_s \sigma G q \rangle, \langle \bar{q}g_s \sigma G q \rangle^2 \) and \( \langle \bar{q}g_s \sigma G q \rangle^3 \), respectively \[12\]. There are three light quark lines (or propagators) and three heavy quark lines (or propagators) in the correlation function \( \Pi_{\mu \nu}(p) \), see Eq. \[10\], if each heavy quark line emits a gluon and each light quark line contributes quark-antiquark pair, we obtain a quark-gluon operator \( g_s G_{\mu \nu} g_s G_{\alpha \beta} G_{\lambda \gamma} \langle \bar{q}q \rangle \langle \bar{q}q \rangle \langle \bar{q}q \rangle \), which is of dimension 15, and leads to the vacuum condensates \( \langle \alpha_{GG} G \rangle \langle \bar{q}q \rangle^2 \langle \bar{q}g_s \sigma G q \rangle, \langle \bar{q}g_s \sigma G q \rangle \langle \bar{q}g_s \sigma G q \rangle^2 \) and \( \langle \bar{q}g_s \sigma G q \rangle^3 \).

In the QCD sum rules for the tetraquark (molecular) states, pentaquark (molecular) states and hexaquark states (or dibaryon states), we take into the vacuum condensates, which are vacuum expectations of the quark-gluon operators of the order \( O(\alpha_s^k) \) with \( k \leq 1 \) in a consistent way \[12, 13, 23, 24, 25, 28, 29\]. In the present case, if take the truncation \( k \leq 1 \), the highest dimension vacuum condensates are \( \langle \bar{q}q \rangle^3 \langle \alpha_{GG} G \rangle \) and \( \langle \bar{q} \bar{q} \rangle \langle \bar{q}g_s \sigma G q \rangle^2 \), the vacuum condensates \( \langle \alpha_{GG} G \rangle \langle \bar{q}q \rangle^2 \langle \bar{q}g_s \sigma G q \rangle, \langle \bar{q}g_s \sigma G q \rangle \langle \bar{q}g_s \sigma G q \rangle^2 \) and \( \langle \bar{q}g_s \sigma G q \rangle^3 \) come from the quark-gluon operators of the order \( O(\alpha_s^3) \) and should be discarded. In this article, we take into account the vacuum condensate \( \langle \bar{q}q \rangle \langle \bar{q}g_s \sigma G q \rangle^3 \) and neglect the vacuum condensates \( g_s^2 GGG \langle \bar{q}q \rangle^3 \) and \( \langle \alpha_{GG} G \rangle \langle \bar{q}q \rangle^2 \langle \bar{q}g_s \sigma G q \rangle \) due to their small values. All in all, we carry out the operator product expansion to the vacuum condensates up to dimension-15, and take into account the vacuum condensates \( \langle \bar{q}q \rangle, \langle \alpha_{GG} G \rangle \langle \bar{q}q \rangle^3 \), \( \langle \bar{q}q \rangle \langle \bar{q}g_s \sigma G q \rangle^2 \), \( \langle \bar{q} \bar{q} \rangle^2 \langle \alpha_{GG} G \rangle \), \( \langle \bar{q}g_s \sigma G q \rangle^2 , \langle \bar{q}q \rangle \langle \bar{q}g_s \sigma G q \rangle^3 \), \( \bar{q}q \rangle \langle \bar{q}g_s \sigma G q \rangle^2 \), \( \langle \bar{q}g_s \sigma G q \rangle^3 \).

Then we obtain the analytical expression of the QCD spectral density through dispersion relation, and match the hadron side with the QCD side of the correlation function \( \Pi(p^2) \) below the continuum threshold \( s_0 \) and perform the Borel transformation in regard to \( P^2 = -p^2 \) to obtain the QCD sum rules:

\[
\lambda_H^2 \exp \left( -\frac{M_H^2}{T^2} \right) = \int_{s_0}^{\infty} ds \rho_{QCD}(s) \exp \left( -\frac{s}{T^2} \right) .
\]

(12)

We neglect the lengthy expression of the QCD spectral density \( \rho_{QCD}(s) \) for simplicity.

We derive Eq. \[12\] in regard to \( \tau = \frac{1}{T} \), then eliminate the pole residue \( \lambda_H \) and obtain the QCD sum rules for the mass of the triply-charmed hexaquark state,

\[
M_H^2 = -\frac{4}{\pi} \int_{s_0}^{\infty} ds \rho_{QCD}(s) \exp \left( -s \tau \right) \int_{s_0}^{\infty} ds \rho_{QCD}(s) \exp \left( -s \tau \right)
\]

(13)
We choose the standard values of the vacuum condensates $\langle \bar{q}q \rangle = -(0.24 \pm 0.01 \text{ GeV})^3$, $\langle \bar{q}g_5 G q \rangle = m_0^2 \langle \bar{q}q \rangle$, $m_0^2 = (0.8 \pm 1) \text{ GeV}^2$, $\langle \frac{\sigma_g G}{\pi} \rangle = (0.33 \text{ GeV})^4$ at the energy scale $\mu = 1 \text{ GeV}$ [26] [30] [31], and choose the $\overline{\text{MS}}$ mass $m_c(m_c) = (1.275 \pm 0.025) \text{ GeV}$ from the Particle Data Group [32], and set $m_u = m_d = 0$. We take into account the energy-scale dependence of the input parameters,

$$
\langle \bar{q}q \rangle (\mu) = \langle \bar{q}q \rangle (1 \text{ GeV}) \left[ \frac{\alpha_s(1 \text{ GeV})}{\alpha_s(\mu)} \right]^\frac{1}{32},
$$

$$
\langle \bar{q}g_5 G q \rangle (\mu) = \langle \bar{q}g_5 G q \rangle (1 \text{ GeV}) \left[ \frac{\alpha_s(1 \text{ GeV})}{\alpha_s(\mu)} \right]^\frac{1}{32},
$$

$$
m_c(\mu) = m_c(m_c) \left[ \frac{\alpha_s(\mu)}{\alpha_s(m_c)} \right]^\frac{1}{32},
$$

$$
\alpha_s(\mu) = \frac{1}{b_0 t} \left[ 1 - \frac{b_1}{b_0^2} \log t + \frac{b_2}{b_0} (\log^2 t - \log t - 1) + b_3 b_2 \right],
$$

where $t = \log \frac{t^2}{\Lambda^2}$, $b_0 = \frac{23 - 2n_f}{12}$, $b_1 = \frac{153 - 19n_f}{24n_f^2}$, $b_2 = \frac{2857 - 4033n_f + 152n_f^2}{128n_f^4}$, $\Lambda = 210 \text{ MeV}$, $292 \text{ MeV}$ and $332 \text{ MeV}$ for the flavors $n_f = 5$, $4$ and $3$, respectively [32] [33], and evolve all the input parameters to the pertinent energy scale $\mu$ to extract the mass of the triply-charmed hexaquark state with the flavor $n_f = 4$.

The continuum threshold parameters are not entirely free parameters, we often consult the experimental data to choose them. Now let us borrow some ideas from the exotic $X$, $Y$ and $Z$ states. We usually assign the $Z^+_c(4430)$ to be the first radial excited state of the $Z^{++}_c(3900)$ according to the analogous decays,

$$
Z^+_c(3900) \rightarrow J/\psi \pi^+, 
$$

$$
Z^{++}_c(4430) \rightarrow \psi' \pi^+, 
$$

and the analogous mass gaps $M_{Z_c(4430)} - M_{Z_c(3900)} = 591 \text{ MeV}$ and $M_{\psi'} - M_{J/\psi} = 589 \text{ MeV}$ from the Particle Data Group [32] [35] [36] [37]. On the other hand, we can assign the $Z_c(4600)$ to be the vector tetraquark state with $J^{PC} = 1^{--}$ [37], or the first radial excited state of the axialvector tetraquark state candidate $Z_c(4020)$ with $J^{PC} = 1^{+-}$ [38] [39], the energy gap between the ground state $Z_c(4020)$ and the first radial excited state $Z_c(4600)$ is about $M_{Z_c(4600)} - M_{Z_c(4020)} = 576 \text{ MeV}$ from the Particle Data Group [32]. In calculations, we choose the continuum threshold parameter as $\sqrt{s_0} = M_H + 0.59 \text{ GeV} \pm 0.10 \text{ GeV}$, and get a constraint to obey.

There are two basic criteria which have to be satisfied in the QCD sum rules, the one is pole dominance at the hadron side, the other is convergence of the operator product expansion at the QCD side. Firstly, let us define the pole contribution PC,

$$
\text{PC} = \int_{m_0}^{s_0} ds \rho_{QCD}(s) \exp \left(-\frac{s}{\Lambda^2} \right),
$$

and define the contributions of the vacuum condensates of dimension $n$,

$$
D(n) = \int_{m_0}^{s_0} ds \rho_{QCD,n}(s) \exp \left(-\frac{s}{\Lambda^2} \right),
$$

where the $\rho_{QCD,n}(s)$ is the QCD spectral density containing the vacuum condensates of dimension $n$.

For the six-quark states, the largest power of the QCD spectral densities $\rho_{QCD}(s) \propto s^7$, while for the four-quark states, the largest power of the QCD spectral densities $\rho_{QCD}(s) \propto s^4$, the
continuum contributions cannot be suppressed efficiently if the Borel parameters are not small enough. However, small Borel parameters lead to bad convergent behavior of the operator product expansion. Furthermore, for the six-quark states, the pole dominance criterion is more difficult to satisfy compared to the cases for the four-quark states. We have to take some methods to enhance the pole contributions.

In this article, we study the diquark-diquark-diquark type hexaquark states, which have three charmed diquarks. Such triply-charmed six-quark systems are characterized by the effective charmed quark mass or constituent quark mass $M_c$ and the virtuality $\sqrt{\mu^2 - (3M_c)^2}$, while the hidden-charm (or doubly-charmed) four-quark systems are characterized by the effective mass $M_c$ and the virtuality $\sqrt{\mu^2 - (2M_c)^2}$ [13, 14]. We set the energy scales of the QCD spectral densities to be $\mu = \sqrt{\mu^2 - (2M_c)^2}$ suggested in the QCD sum rules for the hidden-charm tetraquark states to the triply-charmed hexaquark states [13, 14]. In this article, we choose the updated value $M_c = 1.82$ GeV [20], and take the energy scale formula,

$$\mu = \sqrt{\mu^2 - (3M_c)^2},$$

as a powerful constraint to satisfy. In previous works, we observed that the energy scale formula can enhance the pole contribution remarkably for the tetraquark (molecular) states and pentaquark (molecular) states [13, 14, 23, 28, 29].

Now let us optimize the continuum threshold parameter $s_0$ and choose the best Borel parameter $T^2$ via trial and error, and finally obtain the Borel window, the continuum threshold parameter, the best energy scale of the QCD spectral density, and the pole contribution, which are shown in Table 1. From the Table, we can see that the pole contribution is as large as (40 - 60)%, it is large enough to extract the hexaquark mass reliably.

We use the energy scale formula shown in Eq. (18) to enhance the pole contribution significantly so as to satisfy the pole dominance criterion. In Fig 1, we plot the pole contribution with variation of the energy scale $\mu$ of the QCD spectral density for the Borel parameter $T^2 = 4.0$ GeV$^2$ and continuum threshold parameter $\sqrt{s_0} = 6.4$ GeV. From the figure, we can see that the pole contribution increases monotonously and quickly with the increase of the energy scale $\mu$ at the region $\mu < 2$ GeV, then the pole contribution increases monotonously and slowly with the increase of the energy scale $\mu$. It is very important and necessary to choose the pertinent energy scale $\mu$.

We can rewrite the energy scale formulas as

$$M_{X/Y/Z/H} = \sqrt{\mu^2 + 4M_c^2},$$

$$M_H = \sqrt{\mu^2 + 9M_c^2},$$

where the $X, Y, Z$ and $H$ denote the hidden-charm or doubly-charmed or triply-charmed tetraquark states and hexaquark states. In Fig 2, we plot the predicted masses $M_{Z_c(3900)}$, $M_{Z_c(4020)}$, $M_{H_{cc}}$, and $M_{H_{ccc}}$ with variations of the energy scales $\mu$ of the QCD spectral densities, where we have taken the central values of the input parameters, see Refs. 39, 41, and Table 1, and use the subscripts $cc$ and $ccc$ to stand for the doubly-charmed and triply-charmed hexaquark states, respectively. In Ref. 24, we construct the interpolating current $J(x)$,

$$J(x) = \varepsilon^{abc}\varepsilon^{aij}\varepsilon^{blk}\varepsilon^{cmn} u_i^T(x)C\gamma_5d_j(x)u_k^T(x)C\gamma_5c_l(x)d_m^T(x)C\gamma_5c_n(x),$$

(20)

to study the scalar-diquark-scalar-diquark-scalar-diquark type doubly-charmed hexaquark state $uuddcc$ or $H_{cc}$ with the QCD sum rules. After the article was published, we checked the calculations again and observed that the QCD spectral densities $\rho_i(s)$ with $i = 3, 5, 9, 11$ and 13 should change a minus sign, $\rho_i(s) \rightarrow -\rho_i(s)$. Now we recalculate the mass of the doubly-charmed hexaquark state $uuddcc$ or $H_{cc}$ with all the updated parameters in a consistent way, the relevant parameters and the numerical results are also presented in Table 1.
Figure 1: The pole contribution with variation of the energy scale $\mu$ of the QCD spectral density.

From Fig.2, we can see that the predicted masses of the tetraquark states and hexaquark states decrease monotonically with the increase of the energy scales $\mu$, the line $M = \sqrt{\mu^2 + 4 \times (1.82 \text{ GeV})^2}$ intersects with the lines of the masses of the $Z_c(3900)$, $Z_c(4020)$ and $H_{cc}$ tetraquark or hexaquark states at the energy scales about $\mu = 1.4 \text{ GeV}$, $1.7 \text{ GeV}$ and $2.7 \text{ GeV}$, respectively, which happen to reproduce the experimental values of the masses of the $Z_c(3900)$ and $Z_c(4020)$, respectively. The energy scale formula serves a milestone to choose the pertinent energy scales of the QCD spectral densities. Accordingly, the line $M = \sqrt{\mu^2 + 9 \times (1.82 \text{ GeV})^2}$ intersects with the line of the mass of the triply-charmed hexaquark state $H_{ccc}$ at the energy scale $\mu = 2.0 \text{ GeV}$, which is expected to be the pertinent energy scale of the QCD spectral density and leads to the ideal mass. The mass gap $M_{H_{ccc}} - M_{H_{cc}} = 1.25 \text{ GeV}$ happens to be the $\overline{MS}$ mass of the $c$-quark, $m_c(m_c)$. We can draw the conclusion tentatively that the energy scale formula can be applied to study the hexaquark states in a consistent way.

In Fig.3, we plot the absolute values of the $D(n)$ for the central values of the input parameters shown in Table 1. From the figure, we can see that the contributions of the vacuum condensates with the dimensions $n \leq 8$ vibrate, the contribution of the perturbative term or $D(0)$ is small, the contributions $D(3)$ and $D(6)$ are very large, the contributions $D(4)$ and $D(7)$ are tiny, however, such vibrations cannot destroy the convergence of the operator product expansion. The vacuum condensate $\langle \bar{q}q \rangle \langle \bar{q}g_s \sigma Gq \rangle$ with the dimension 8 serves as a milestone, the absolute values of the contributions $|D(n)|$ with $n \geq 8$ decrease monotonically and quickly with the increase of the dimensions $n$, the value $|D(15)| \approx 0$, the operator product expansion is convergent.

In Fig.4, we plot the contributions of the vacuum condensates $D(n)$ with $n \geq 8$ for the central values of the input parameters shown in Table 1. From the figure, we can see that the contributions of the higher dimensional vacuum condensates decrease monotonously and quickly with the increase of the Borel parameter $T^2$ at the region $T^2 \leq 3.3 \text{ GeV}^2$, then they decrease monotonously and slowly with the increase of the Borel parameter $T^2$. It is reasonable to choose the value $T^2 > 3.3 \text{ GeV}^2$. The higher dimensional vacuum condensates play a minor important role in the Borel windows, but they play an important role in determining the Borel windows.

In Fig.5, we plot the predicted triply-charmed hexaquark masses with variation of the Borel parameter $T^2$ for the truncations of the operator product expansion up to the vacuum condensates of dimensions $n = 8, 9, 10, 11, 13$ and 15 respectively with the central values of the other parameters shown in Table 1. From the figure, we can see that the predicted masses change greatly with
Figure 2: The predicted masses with variations of the energy scales $\mu$ of the QCD spectral densities, where the ESF1 and ESF2 represent the formulas $M = \sqrt{\mu^2 + 4 \times (1.82 \text{ GeV})^2}$ and $\sqrt{\mu^2 + 9 \times (1.82 \text{ GeV})^2}$, respectively.

Figure 3: The absolute values of the contributions of the vacuum condensates of dimension $n$ for central values of the input parameters.
the contributions of the higher dimensional vacuum condensates with variation of the Borel parameter $T^2$.

Table 1: The Borel parameters, continuum threshold parameters, energy scales, pole contributions, masses and pole residues for the triply-charmed and doubly-charmed hexaquark states.

| $J^P$ | $T^2$(GeV$^2$) | $\sqrt{s_0}$(GeV) | $\mu$(GeV) | pole | $M$(GeV) | $\lambda$(10$^{-5}$GeV$^8$) |
|-------|----------------|-------------------|-------------|-------|---------|-----------------------------|
| $1^+$ (ccuudd) | 3.8 - 4.2 | 6.40 ± 0.10 | 2.0 | (40 - 60)% | 5.81 ± 0.10 | 1.60 ± 0.30 |
| $0^+$ (ccuudd) | 3.0 - 3.4 | 5.15 ± 0.10 | 2.7 | (40 - 63)% | 4.56 ± 0.11 | 0.78 ± 0.15 |

From Figs.6-7, we can see that there appear flat platforms in the Borel windows both for the masses and pole residues, it is reliable to extract the hexaquark masses. From Table 1 we can see that the central values of the hexaquark masses satisfy the energy scale formula $\mu = \sqrt{M_H^2 - (3M_c)^2}$ and $\sqrt{M_H^2 - (2M_c)^2}$, respectively. Also from Table 1 we can obtain reasonably relation between the doubly-charmed and triply-charmed hexaquark states, $M_{H_{ccc}} - M_{H_{cc}} = m_c(m_c)$, which validates the present calculations.

In Ref. [25], we construct the color-singlet-color-singlet type currents to study the scalar and axialvector $\Xi_{cc}\Sigma_c$ dibaryon states with QCD sum rules, and obtain the masses $M_{\Xi_{cc}\Sigma_c(0^+)} = 6.05 \pm 0.13$ GeV and $M_{\Xi_{cc}\Sigma_c(1^+)} = 6.03 \pm 0.13$ GeV, which lie above mass of the diquark-diquark-diquark type triply-charmed hexaquark state, $M_{H_{ccc}}(1^+) = 5.81 \pm 0.10$ GeV. The decays of the triply-charmed hexaquark state $H_{ccc}$ to the final states $\Xi_{cc}\Sigma_c$, $\Xi_{cc}\Lambda_c$ and $\Omega_{ccc}p$ are kinematically
Figure 5: The predicted masses with variation of the Borel parameter $T^2$ for the truncations of the operator product expansion up to the vacuum condensates of dimensions $n = 8, 9, 10, 11, 13$ and 15, the region between the two vertical lines is the Borel window.

Figure 6: The masses of the triply-charmed and doubly-charmed hexaquark states with variation of the Borel parameter $T^2$, the regions between the two vertical lines are the Borel windows.
4 Conclusion

In this article, we construct the diquark-diquark-diquark type current to interpolate the triply-charmed axialvector hexaquark state, and study its mass and pole residue with the QCD sum rules in details by carrying out the operator product expansion up to the vacuum condensates of dimension 15. In calculations, we choose the best energy scale of the QCD spectral density with the energy scale formula $\mu = \sqrt{M_H^2 - (3M_c)^2}$, which can enhance the pole contribution remarkably to satisfy the pole dominance criterion at the hadron side and improve the convergent behavior of the operator product expansion by suppressing the contributions of the higher dimensional vacuum condensates at the QCD side. Finally, we obtain the mass and pole residue of the triply-charmed axialvector hexaquark state, the predicted mass lies below the two-baryon thresholds, which indicates that the triply-charmed hexaquark state $H_{ccc}$ decays weakly, the width is small. If the spin-breaking effects are small, the ground state hexaquark states with the spin-parity $J^P = 0^+, 1^+$ and $2^+$ have almost degenerated masses, and analogous narrow widths. Furthermore, we re-analyze the mass of the diquark-diquark-diquark type scalar doubly-charmed hexaquark state, and obtain reasonable relation between the doubly-charmed and triply-charmed hexaquark states. We can search for the triply-charmed and doubly-charmed hexaquark states at the LHCb, Belle II, CEPC, FCC, ILC in the future.

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