Research Article

Right and Left Weyl Operator Matrices in a Banach Space Setting

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Abstract

Let \( X_i, Y_i \) (\( i = 1, 2 \)) be Banach spaces. The operator matrix of the form \( M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \) acting between \( X_1 \oplus X_2 \) and \( Y_1 \oplus Y_2 \) is investigated. By using row and column operators, equivalent conditions are obtained for \( M_C \) to be left Weyl, right Weyl, and Weyl for some \( C \in \mathcal{B}(X_2, Y_1) \), respectively. Based on these results, some sufficient conditions are also presented. As applications, some discussions on Hamiltonian operators are given in the context of Hilbert spaces.

1. Introduction

Throughout this paper, \( X, Y, Z \) and \( X_i, Y_i, (i = 1, 2) \) are always reserved to denote some Banach spaces. If \( T \) is a bounded linear operator from \( X \) to \( Y \), we write \( T \in \mathcal{B}(X, Y) \), and if \( X = Y \), we write \( \mathcal{B}(X) \) instead of \( \mathcal{B}(X, X) \). For \( T \in \mathcal{B}(X, Y) \), the range and the kernel of \( T \) are denoted by \( R(T) \) and \( N(T) \), respectively. Write \( \alpha(T) = \dim R(T) \) and \( \beta(T) = \dim (X/R(T)) \). The sets of all left and right Fredholm operators are, respectively, defined as \( \mathcal{F}_L(X, Y) \) and \( \mathcal{F}_R(X, Y) \). They are natural extensions of Hilbert space cases since every closed subspace is complemented in the whole space.

The set of Fredholm operators is defined as

\[
\Phi(X, Y) = \Phi_L(X, Y) \cup \Phi_R(X, Y) = \{ T \in \mathcal{B}(X, Y) : \alpha(T) < \infty, \beta(T) < \infty \}.
\]

An operator \( T \) is said to be semi-Fredholm if \( T \in \Phi_L(X, Y) \cup \Phi_R(X, Y) \), for which the index is defined as \( \text{ind}(T) = \alpha(T) - \beta(T) \). The left Weyl, right Weyl, and Weyl operators are collected in

\[
\Phi_l(X, Y) = \{ T \in \mathcal{B}(X, Y) : \alpha(T) < \infty, \beta(T) < \infty, \text{ind}(T) \leq 0 \}.
\]

\[
\Phi_r(X, Y) = \{ T \in \mathcal{B}(X, Y) : \alpha(T) < \infty, \beta(T) < \infty, \text{ind}(T) \geq 0 \}.
\]

\[
\Phi(X, Y) = \{ T \in \mathcal{B}(X, Y) : \text{ind}(T) = 0 \}.
\]

An operator \( T \) is said to be left (resp. right) invertible if there exists an operator \( S \in \mathcal{B}(Y, X) \) such that \( ST = I_X \) (resp. \( TS = I_Y \)). If \( T \) is both left and right invertible, then \( T \) is invertible. As it is well known, the sets of all left invertible, right invertible, and invertible operators are as follows:
\[
\mathcal{G}_1(\mathcal{X}, \mathcal{Y}) = \{T \in \mathcal{B}(\mathcal{X}, \mathcal{Y}) : T \text{ is injective}, \mathcal{R}(T) \text{ is complemented in } \mathcal{Y}\},
\]
\[
\mathcal{G}_r(\mathcal{X}, \mathcal{Y}) = \{T \in \mathcal{B}(\mathcal{X}, \mathcal{Y}) : T \text{ is surjective}, \mathcal{N}(T) \text{ is complemented in } \mathcal{X}\},
\]
\[
\mathcal{G}(\mathcal{X}, \mathcal{Y}) = \{T \in \mathcal{B}(\mathcal{X}, \mathcal{Y}) : T \text{ is a bijection}\}.
\]

Again, we have the abbreviations \(\mathcal{G}_1(\mathcal{X}), \mathcal{G}_r(\mathcal{X}), \mathcal{G}(\mathcal{X})\) of the above classes of operators like \(\mathcal{B}(\mathcal{X})\). Now let \(T \in \mathcal{B}(\mathcal{X})\). As usual, the spectrum \(\sigma(T)\), left spectrum \(\sigma_l(T)\), right spectrum \(\sigma_r(T)\), Fredholm spectrum \(\sigma_\text{F}(T)\), left Fredholm spectrum \(\sigma_l(\text{F})\), right Fredholm spectrum \(\sigma_r(\text{F})\), Weyl spectrum \(\sigma_w(\text{F})\), left Weyl spectrum \(\sigma_l(w)\), and right Weyl spectrum \(\sigma_r(w)\) will be the collections of the numbers \(\lambda \in \mathbb{C}\) such that \(T - \lambda I\) does not belong to the corresponding classes of operators, i.e., \(\mathcal{G}(\mathcal{X}), \mathcal{G}_1(\mathcal{X}), \mathcal{G}_r(\mathcal{X}), \Phi(\mathcal{X}), \Phi_l(\mathcal{X}), \Phi_r(\mathcal{X}), \Phi(w)(\mathcal{X}), \Phi_l(w)(\mathcal{X}), \Phi_r(w)(\mathcal{X})\), and \(\Phi_w(\mathcal{X})\), respectively.

We say that \(T \in \mathcal{B}(\mathcal{X})\) is relatively regular or simply regular if there exists some \(S \in \mathcal{B}(\mathcal{X})\) such that \(TST = T\). In this case, \(S\) is called an inner generalized inverse of \(T\). Obviously, the classes of left or right invertible, invertible, semi-Fredholm, and Fredholm operators are all regular. If \(M\) is a closed subspace in Banach space \(\mathcal{X}\), then \(M\) is said to be topologically complemented or simply complemented if there exists another closed subspace \(N\) of \(\mathcal{X}\) such that \(\mathcal{X} = M + N\); for such complementary subspaces \(M\) and \(N\), we write \(\mathcal{X} = M \oplus N\). It is well known that \(T\) is regular if and only if \(\mathcal{R}(T)\) and \(\mathcal{N}(T)\) are complemented subspaces of \(\mathcal{X}\). Denote by \(R_T\) and \(G_T\) the complementary subspaces with \(\mathcal{N}(T)\) and \(\mathcal{R}(T)\), respectively.

For given \(A \in \mathcal{B}(\mathcal{X})\) and \(B \in \mathcal{B}(\mathcal{Y})\), define
\[
M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \in \mathcal{B}(\mathcal{X} \oplus \mathcal{Y}),
\]
where \(C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})\) is an unknown element. The spectrum and its various subdivisions of \(M_C\) are considered in many papers such as \([1–7]\) and the references therein. Although most of these papers worked in the context of Hilbert spaces, some results on the invertibility and Fredholm theory were established in Banach spaces \([5–7]\). In this note, we investigate the left and right Weyl spectra of upper triangular operator matrices in a Banach space setting. Our main tools are the regularity of an operator and its equivalent form, which are closely related to some appropriate space decompositions.

2. Preliminaries

This section is devoted to collecting some basic results. Although most of them are well-known standard results on Fredholm operators, we list them here for convenience of later proofs.

Lemma 1. Let \(T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})\) be a semi-Fredholm operator and \(S \in \mathcal{B}(\mathcal{X}, \mathcal{Y})\) be a compact operator. Then, \(T + S\) is semi-Fredholm and \(\text{ind}(T + S) = \text{ind}(T)\).

The following lemma is obvious, so its proof will be omitted.

Lemma 2. Assume that \(L \in \mathcal{B}(\mathcal{Y})\) and \(R \in \mathcal{B}(\mathcal{X})\) have bounded inverses defined on \(\mathcal{Y}\) and \(\mathcal{X}\), respectively. Then, \(\mathcal{R}(LT)\) and \(\mathcal{R}(TR)\) (and hence \(\mathcal{R}(LTR)\)) have the same closedness as \(\mathcal{R}(T)\), where \(T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})\).

Lemma 3. For \(T \in \mathcal{B}(\mathcal{X})\), we have
\[
\mathcal{N}(T^\prime) = \mathcal{R}(T)^\perp, \\
\mathcal{N}(T) = \perp \mathcal{R}(T)^\perp, \\
\mathcal{R}(T) \in \mathcal{N}(T)^\perp, \\
\mathcal{R}(T^\prime) \in \mathcal{N}(T)^\perp.
\]

Note that the last two inclusions are strict in general, and due to the closed range theorem, equality holds here precisely when \(\mathcal{R}(T)\) is closed.

The sets of all upper and lower semi-Fredholm operators are defined by
\[
\Phi(u)(\mathcal{X}, \mathcal{Y}) = \{T \in \mathcal{B}(\mathcal{X}, \mathcal{Y}) : \mathcal{R}(T) \text{ is closed and } \alpha(T) < \infty\}, \\
\Phi_l(u)(\mathcal{X}, \mathcal{Y}) = \{T \in \mathcal{B}(\mathcal{X}, \mathcal{Y}) : \beta(T) < \infty\},
\]
respectively. It is obvious that \(\Phi(u)(\mathcal{X}, \mathcal{Y}) \subseteq \Phi_l(u)(\mathcal{X}, \mathcal{Y}) \subseteq \Phi_r(u)(\mathcal{X}, \mathcal{Y})\).

Lemma 4. Let \(T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})\) with closed range. Then, \(\alpha(T^\prime) = \beta(T)\), \(\beta(T^\prime) = \alpha(T)\), and
\[
T \in \Phi(u)(\mathcal{X}, \mathcal{Y}) \iff T^\prime \in \Phi'(u')(\mathcal{X}', \mathcal{X}'), \\
T \in \Phi_r(u)(\mathcal{X}, \mathcal{Y}) \iff T^\prime \in \Phi_r'(u')(\mathcal{X}', \mathcal{X}'), \\
T \in \Phi_l(u)(\mathcal{X}, \mathcal{Y}) \iff T^\prime \in \Phi_l'(u')(\mathcal{X}', \mathcal{X}').
\]

Lemma 5. Let \(T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})\) and \(S \in \mathcal{B}(\mathcal{Y}, \mathcal{X})\).

(i) If \(ST\) is right Fredholm, then \(S\) is right Fredholm.
(ii) If \(ST\) is left Fredholm, then \(T\) is left Fredholm.
(iii) If \(ST\) is Fredholm, then \(S\) is right Fredholm and \(T\) is left Fredholm.

Lemma 6. Let \(T \in \mathcal{B}(\mathcal{X}, \mathcal{Y})\) and \(S \in \mathcal{B}(\mathcal{Y}, \mathcal{X})\). If both \(S\) and \(T\) are upper semi-Fredholm or both are lower semi-Fredholm, then \(\text{ind}(ST) = \text{ind}(S) + \text{ind}(T)\).

3. Main Results and Proofs

In this section, we present the main results of this paper and their proofs. First, we establish the right Weylness of \(M_C\).
**Theorem 1.** Let $A \in \mathcal{B}(\mathcal{X}_1, \mathcal{Y}_1)$ and $B \in \mathcal{B}(\mathcal{X}_2, \mathcal{Y}_2)$ be given operators. Then, there exists $C \in \mathcal{B}(\mathcal{X}_2, \mathcal{Y}_1)$ such that $M_C$ is right Weyl if and only if $B$ is right Fredholm, and one of the following statements is fulfilled:

(i) $\alpha(B) = \infty$, and there exists $S \in \mathcal{B}(\mathcal{N}(B), \mathcal{Y}_1)$ such that the row operator $[A \ S]$ is a right Fredholm operator with $\text{ind}([A \ S]) \geq \beta(B)$.

(ii) $M_0 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is right Weyl.

**Proof.** Let $B$ be right Fredholm. If assertion (ii) holds, then taking $C = 0$ gives the conclusion. We now assume that assertion (i) is true. There are two possible cases depending on the dimension of $\mathcal{Y}_1$.

(i) Case 1: $\dim \mathcal{Y}_1 < \infty$. In this case,

$$
\begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} : \mathcal{X}_1 \oplus \mathcal{X}_2 \rightarrow \mathcal{Y}_1 \oplus \mathcal{Y}_2,
$$

is obviously a right Weyl operator. By Lemma 1, we see that $M_C$ is a right Weyl operator for any $C \in \mathcal{B}(\mathcal{X}_2, \mathcal{Y}_1)$.

(ii) Case 2: $\dim \mathcal{Y}_1 = \infty$. In this case, $\mathcal{R}(B)$ is complemented in $\mathcal{Y}_1$ and $\mathcal{N}(B)$ is complemented in $\mathcal{X}_2$, since $B$ is right Fredholm. Then, $\mathcal{X}_2$ and $\mathcal{Y}_2$ have the decompositions

$$
\mathcal{X}_2 = \mathcal{P}_B \oplus \mathcal{N}(B),
$$

$$
\mathcal{Y}_2 = \mathcal{R}(B) \oplus \mathcal{Q}_B.
$$

Consequently, $B$ can be written as

$$
B = \begin{bmatrix} B_1 & 0 \\ 0 & 0 \end{bmatrix} : \mathcal{P}_B \oplus \mathcal{N}(B) \rightarrow \mathcal{R}(B) \oplus \mathcal{Q}_B.
$$

Define operator $C$ by

$$
C = \begin{bmatrix} 0 & S \\ \end{bmatrix} : \mathcal{P}_B \oplus \mathcal{N}(B) \rightarrow \mathcal{Y}_1.
$$

As an operator from $\mathcal{X}_1 \oplus \mathcal{P}_B \oplus \mathcal{N}(B)$ to $\mathcal{Y}_1 \oplus \mathcal{R}(B) \oplus \mathcal{Q}_B$, $M_C$ has the following matrix form:

$$
M_C = \begin{bmatrix} A & 0 & S \\ 0 & B_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
$$

It is not difficult to see that $\mathcal{R}(M_C)$ is complemented in $\mathcal{Y}_1 \oplus \mathcal{Y}_2$ and $\mathcal{N}(M_C)$ is complemented in $\mathcal{X}_1 \oplus \mathcal{X}_2$, and that

$$
\alpha(M_C) = \alpha([A \ S]),
$$

$$
\beta(M_C) = \beta([A \ S]) + \beta(B).
$$

Thus, $\text{ind}(M_C) = \text{ind}([A \ S]) - \beta(B) \geq 0$. This shows that $M_C$ is right Weyl.

Conversely, assume that there exists $C \in \mathcal{B}(\mathcal{X}_2, \mathcal{Y}_1)$ such that $M_C$ is right Weyl. Obviously, $B$ is right Fredholm.

In view of the space decompositions in (10), as an operator from $\mathcal{X}_1 \oplus \mathcal{P}_B \oplus \mathcal{N}(B)$ to $\mathcal{Y}_1 \oplus \mathcal{R}(B) \oplus \mathcal{Q}_B$, $M_C$ can be represented as

$$
M_C = \begin{bmatrix} A & C_1 & C_2 \\ 0 & B_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
$$

Evidently, $B_1 \in \mathcal{C}(\mathcal{P}_B, \mathcal{R}(B))$ and

$$
E = \begin{bmatrix} I & -C_1 B_1^{-1} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \in \mathcal{C}(\mathcal{Y}_1 \oplus \mathcal{R}(B) \oplus \mathcal{Q}_B).
$$

Then,

$$
EM_C = \begin{bmatrix} I & -C_1 B_1^{-1} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A & C_1 & C_2 \\ 0 & B_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} A & 0 & C_2 \\ 0 & B_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
$$

If $\alpha(B) < \infty$, then $C_2$ is of finite rank, and hence, from (17) and Lemmas 1 and 2, it follows that $M_0 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is right Weyl, i.e., assertion (ii) is valid. We now assume $\alpha(B) = \infty$. Since $M_C$ is right Weyl, from (17), we equivalently have that $\mathcal{R}(EM_C)$ is right Weyl. Thus, $C_2 \in \mathcal{B}(\mathcal{N}(B), \mathcal{Y}_1)$ is exactly an operator such that the row operator $[A \ C_2]$ is right Fredholm, and $\text{ind}([A \ C_2]) - \beta(B) = \text{ind}(EM_C) \geq 0$, i.e., $\text{ind}([A \ C_2]) \geq \beta(B)$. Taking $S = C_2$ yields assertion (i).

From Theorem 1, the following perturbation result of right Weyl spectrum is obvious. \(\Box\)

**Corollary 1.** Let $A \in \mathcal{B}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{Y})$ be given operators. Then,

$$
\bigcap_{C \in \mathcal{B}(\mathcal{X}, \mathcal{Y})} \sigma_{rw}(M_C) = \sigma_{rc}(B) \cup \left( \sigma_{rw}(\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}) \cap (\Omega_1 \cup \Omega_2) \right),
$$

where

$$
\Omega_1 = \{ \lambda \in \mathbb{C} : \text{a}(B - \lambda I) < \infty \},
$$

$$
\Omega_2 = \{ \lambda \in \mathbb{C} : \text{there is no} S \in \mathcal{B}(\mathcal{N}(B - \lambda I), \mathcal{Y}) \text{such that} \begin{bmatrix} A - \lambda I & S \end{bmatrix} \text{is a right Fredholm operator with} \text{ind}([A - \lambda I \ S]) \geq \beta(B - \lambda I) \}.
$$

If having certain special properties for the given diagonal entries, one can further analyze the right Weylness of $M_C$ on the basis of Theorem 1.

**Corollary 2.** Let $A \in \mathcal{B}(\mathcal{X}_1, \mathcal{Y}_1)$ and $B \in \mathcal{B}(\mathcal{X}_2, \mathcal{Y}_2)$ be given operators. If $A$ is right Fredholm, then there exists $C \in \mathcal{B}(\mathcal{X}_2, \mathcal{Y}_1)$ such that $M_C$ is right Weyl if and only if

$$
M_0 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \text{is right Weyl}.
$$
Proof. Assume that there exists \( C \in \mathcal{B}(\mathcal{X}_2, \mathcal{Y}_1) \) such that \( M_C \) is right Weyl. From the assumption and the right Fredholmness of \( A \), it follows that \( \beta(A) < \infty \) and \( \beta(B) < \infty \), and hence if \( \alpha(B) = \infty \), as required in assertion (i) of Theorem 1, we still conclude that \( M_0 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \) is right Weyl, i.e., assertion (ii) holds. By virtue of Theorem 1, the desired result is obvious.

\( \square \)

**Corollary 3.** Let \( A \in \mathcal{B}(\mathcal{X}_1, \mathcal{Y}_1) \) and \( B \in \mathcal{B}(\mathcal{X}_2, \mathcal{Y}_2) \) be given operators. If \( A \) is Weyl, then there exists \( C \in \mathcal{B}(\mathcal{X}_2, \mathcal{Y}_1) \) such that \( M_C \) is right Weyl if and only if \( B \in \mathcal{B}(\mathcal{X}_2, \mathcal{Y}_2) \) is right Weyl.

Proof. From the proof of Corollary 2, it is sufficient to notice that \( M_0 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \) is right Weyl if and only if \( B \) is right Weyl when \( A \) is Weyl.

Dually, we have the following left Weyl description.

**Theorem 2.** Let \( A \in \mathcal{B}(\mathcal{X}_1, \mathcal{Y}_1) \) and \( B \in \mathcal{B}(\mathcal{X}_2, \mathcal{Y}_2) \) be given operators. Then, there exists \( C \in \mathcal{B}(\mathcal{X}_2, \mathcal{Y}_1) \) such that \( M_C \) is left Weyl if and only if \( A \) is left Fredholm, and one of the following statements is fulfilled:

(i) \( \beta(A) = \infty \), and there exists \( J \in \mathcal{B}(\mathcal{X}_2, \mathcal{A}_A) \) such that the column operator \( \begin{bmatrix} J \\ B \end{bmatrix} \) is a left Fredholm operator with \( \text{ind} \left( \begin{bmatrix} J \\ B \end{bmatrix} \right) \leq -\alpha(A) \).

(ii) \( M_0 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \) is left Weyl.

Proof. Let \( A \) be left Fredholm. Similar to the proof of Theorem 2, it suffices to consider the case that assertion (i) is true under the condition \( \dim \mathcal{X}_2 = \infty \). Obviously, \( \mathcal{R}(A) \) is complemented in \( \mathcal{Y}_1 \) and \( \mathcal{N}(A) \) is complemented in \( \mathcal{X}_1 \), and hence

\[
\mathcal{X}_1 = \mathcal{P}_A \oplus \mathcal{N}(A), \\
\mathcal{Y}_1 = \mathcal{R}(A) \oplus \mathcal{C}_A.
\]

(20)

Then,

\[
A = \begin{bmatrix} A_1 & 0 \\ 0 & B \end{bmatrix} : \mathcal{P}_A \oplus \mathcal{N}(A) \longrightarrow \mathcal{R}(A) \oplus \mathcal{C}_A.
\]

(21)

Define operator \( C \) by

\[
C = \begin{bmatrix} 0 \\ J \end{bmatrix} : \mathcal{Y}_2 \longrightarrow \mathcal{R}(A) \oplus \mathcal{C}_A.
\]

(22)

At this point, \( M_C \) has the following matrix form:

\[
M_C = \begin{bmatrix} A_1 & 0 \\ 0 & B \end{bmatrix} : \mathcal{P}_A \oplus \mathcal{N}(A) \oplus \mathcal{X}_2 \longrightarrow \mathcal{R}(A) \oplus \mathcal{C}_A \oplus \mathcal{Y}_2.
\]

(23)

Evidently, \( \mathcal{R}(M_C) \) is complemented in \( \mathcal{Y}_1 \oplus \mathcal{Y}_2 \) and \( \mathcal{N}(M_C) \) is complemented in \( \mathcal{X}_1 \oplus \mathcal{X}_2 \). In addition, \( \alpha(M_C) = \alpha \left( \begin{bmatrix} J \\ B \end{bmatrix} \right) + \alpha(A) \) and \( \beta(M_C) = \beta \left( \begin{bmatrix} J \\ B \end{bmatrix} \right) \), which implies

\[
\text{ind}(M_C) = \text{ind} \left( \begin{bmatrix} J \\ B \end{bmatrix} \right) + \alpha(A) \leq 0.
\]

(24)

Thus, \( M_C \) is left Weyl.

Conversely, assume that there exists \( C \in \mathcal{B}(\mathcal{X}_2, \mathcal{Y}_1) \) such that \( M_C \) is left Weyl. It is clear that \( A \) is left Fredholm, and as an operator from \( \mathcal{P}_A \oplus \mathcal{N}(A) \oplus \mathcal{X}_2 \) to \( \mathcal{R}(A) \oplus \mathcal{C}_A \oplus \mathcal{Y}_2 \), \( M_C \) can be written as

\[
M_C = \begin{bmatrix} A_1 & 0 & C_1 \\ 0 & 0 & C_2 \\ 0 & 0 & B \end{bmatrix}.
\]

(25)

Then, with the aid of the operators \( A_1 \in \mathcal{G}(\mathcal{P}_A, \mathcal{R}(A)) \) and

\[
F = \begin{bmatrix} I & 0 & -A_1^{-1}C_1 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \in \mathcal{G}(\mathcal{P}_A \oplus \mathcal{N}(A) \oplus \mathcal{X}_2),
\]

we get

\[
M_CF = \begin{bmatrix} A_1 & 0 & C_1 \\ 0 & 0 & C_2 \\ 0 & 0 & B \end{bmatrix} \begin{bmatrix} I & 0 & -A_1^{-1}C_1 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} A_1 & 0 & 0 \\ 0 & 0 & C_2 \\ 0 & 0 & B \end{bmatrix}.
\]

(27)

If \( \beta(A) < \infty \), then \( C_2 \) is of finite rank, which implies that \( M_0 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \) is left Weyl, i.e., assertion (ii) is valid. We now assume that \( \beta(A) = \infty \). Since \( M_C \) is left Weyl, it follows from (27) that \( \mathcal{R}(M_CF) \) is left Weyl. Thus, \( C_2 \in \mathcal{B}(\mathcal{X}_2, \mathcal{C}_A) \) is a desired operator such that the column operator \( \begin{bmatrix} C_2 \\ B \end{bmatrix} \) is left Fredholm, and

\[
\text{ind} \left( \begin{bmatrix} C_2 \\ B \end{bmatrix} \right) + \alpha(A) = \text{ind}(M_CF) \leq 0, \text{ i.e.,}
\]

\[
\text{ind} \left( \begin{bmatrix} C_2 \\ B \end{bmatrix} \right) \leq -\alpha(A). \text{ Taking } J = C_2 \text{ yields assertion (i).} \quad \square
\]
Remark 1. Theorem 2 is dual to Theorem 1 to some extent. However, we cannot directly resort to conjugate relationship, since Lemma 4 is not valid for left and right Fredholm operators. Actually, $T \in \Phi_1(X, Y)$ implies $T^* \in \Phi_1(Y, X^*)$, but the converse is not true in general (see, e.g., [10] for more details).

### Corollary 5
Let $A \in \mathcal{B}(X, Y)$ and $B \in \mathcal{B}(X_2, Y_2)$ be given operators. If $B$ is left Fredholm, then there exists $C \in \mathcal{B}(X_2, Y_2)$ such that $M_C$ is left Weyl if and only if $M_0 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is left Weyl.

### Corollary 6
Let $A \in \mathcal{B}(X, Y)$ and $B \in \mathcal{B}(X_2, Y_2)$ be given operators. If $B$ is Weyl, then there exists $C \in \mathcal{B}(X_2, Y_2)$ such that $M_C$ is left Weyl if and only if $A \in \mathcal{B}(X_2, Y_2)$ is left Weyl.

In [16, Theorem 3.6], the author described the Weylness of $M_{C}$ on a Banach space. Using the row operator and the column operator in Theorems 1 and 2, we will give a different statement.

### Theorem 3
Let $A \in \mathcal{B}(X, Y)$ and $B \in \mathcal{B}(X_2, Y_2)$ be given operators. Then, there exists $C \in \mathcal{B}(X_2, Y_2)$ such that $M_C$ is Weyl if and only if $A$ is left Fredholm, $B$ is right Fredholm, and one of the following statements is fulfilled:

1. $\alpha(B) = \infty = \beta(A)$, and there exists $S \in \mathcal{B}(\mathcal{N}(B), Y_1)$ and $J \in \mathcal{B}(X_2, \mathcal{O}_A)$ such that $[A, S]$ is right Fredholm with $\text{ind}([A, S]) \geq \beta(B)$, $\begin{bmatrix} J \\ B \end{bmatrix}$ is left Fredholm with $\text{ind}([J, B]) \leq -\alpha(A)$, and $P_{Q_1} S_{\|J\|}(B) = P_{Q_1} I_{\|J\|}(B)$.
2. $M_0 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is Weyl.

**Proof.** Assume that $A$ is left Fredholm and $B$ is right Fredholm. If assertion (ii) is valid, then taking $C = 0$ yields the desired sufficiency. We now assume that assertion (i) holds.

Obviously, $\alpha(A) < \infty$, $\beta(B) < \infty$, and hence decompositions (10) and (20) are still satisfied. Then, the operators $S$ and $J$ in assertion (i) can be written as

$$\Omega'_1 = \{ \lambda \in \mathbb{C} : \beta(A - \lambda I) < \infty \},$$

$$\Omega'_2 = \{ \lambda \in \mathbb{C} : \text{there is no } J \in \mathcal{B}(Y, \mathcal{O}_{A - \lambda I}) \text{ such that } \begin{bmatrix} J \\ B - \lambda I \end{bmatrix} \text{ is a left Fredholm operator with ind} \left( \begin{bmatrix} J \\ B - \lambda I \end{bmatrix} \right) \leq -\alpha(A - \lambda I) \}.$$
are invertible operators such that

\[
EM_C F = \begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & 0 & \Delta & 0 \\ 0 & 0 & B_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.
\]  

(36)

From Lemmas 5 and 6, we infer that \( M_C \) has the same semi-Fredholmness and index as the operator matrix on the right hand side of (36). Thus, \( \mathcal{R}(M_C) \) is clearly closed, and \( \text{ind}(M_C) = \alpha(A) + \alpha(\Delta) - \beta(B) - \beta(\Delta), \) and hence \( \text{ind}(M_C) = 0 \) by (32). This shows that \( M_C \) is Weyl, as claimed.

Conversely, assume that there exists \( C \in \mathcal{R}(\mathcal{X}_2, \mathcal{Y}_1) \) such that \( M_C \) is Weyl. Obviously, \( A \) is left Fredholm, and \( B \) is a right Fredholm. Then, decompositions (10) and (20) still hold true, and as an operator from \( \mathcal{P}_A \oplus \mathcal{N}(A) \oplus \mathcal{P}_B \oplus \mathcal{N}(B) \) to \( \mathcal{R}(A) \oplus \mathcal{L}_A \oplus \mathcal{R}(B) \oplus \mathcal{L}_B, \) \( M_C \) has a new block representation:

\[
M_C = \begin{bmatrix} A_1 & 0 & C_1 & C_2 \\ 0 & 0 & C_3 & C_4 \\ 0 & 0 & B_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\]  

(37)

where \( A_1 \in \mathcal{R}(\mathcal{X}_2, \mathcal{A}(A)) \) and \( B_1 \in \mathcal{R}(\mathcal{P}_B, \mathcal{R}(B)). \) Then,

\[
E = \begin{bmatrix} I & 0 & -C_1 B_1^{-1} & 0 \\ 0 & I & -C_2 B_1^{-1} & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \in \mathcal{R}(\mathcal{R}(A) \oplus \mathcal{L}_A \oplus \mathcal{R}(B) \oplus \mathcal{L}_B),
\]

\[
F = \begin{bmatrix} I & 0 & 0 & -A_1^{-1} C_2 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \in \mathcal{R}(\mathcal{P}_A \oplus \mathcal{N}(A) \oplus \mathcal{P}_B \oplus \mathcal{N}(B)).
\]  

(38)

If either \( \alpha(B) < \infty \) or \( \beta(A) < \infty \) holds, then \( C_4 \) is of finite rank, which together with (39) implies that \( M_0 = \begin{bmatrix} A & 0 & 0 \\ 0 & 0 & B \end{bmatrix} \) is Weyl, i.e., assertion (ii) is valid. We now assume that \( \alpha(B) = \infty = \beta(A) \). From (39), we equivalently have that \( EM_C F \) is Weyl, which implies that \( \alpha(A) + \alpha(C_4) - \beta(C_4) - \beta(B) = 0 \). Taking \( S = \begin{bmatrix} 0 & C_4 \end{bmatrix} : \mathcal{N}(B) \rightarrow \mathcal{A}(A) \oplus \mathcal{L}_A \) gives a right Fredholm row operator \( \begin{bmatrix} A & S \end{bmatrix} \) with \( \text{ind}(\begin{bmatrix} A & S \end{bmatrix}) = \beta(B) \), while taking \( J = \begin{bmatrix} 0 & C_4 \end{bmatrix} : \mathcal{A}(A) \oplus \mathcal{N}(A) \rightarrow \mathcal{L}_A \) yields a left Fredholm column operator \( \begin{bmatrix} J & B \end{bmatrix} \) with \( \text{ind}(\begin{bmatrix} J & B \end{bmatrix}) \leq -\alpha(A) \). Also, \( P_{Q_\mathcal{A}} S |_{\mathcal{Y}(B)} = P_{Q_\mathcal{A}} J |_{\mathcal{Y}(B)} \). This proves assertion (i). More generally, \( S = \begin{bmatrix} C_2 \\ C_4 \end{bmatrix} \in \mathcal{R}(\mathcal{N}(B), \mathcal{Y}_1) \) and \( J = \begin{bmatrix} C_3 & C_4 \end{bmatrix} \in \mathcal{R}(\mathcal{X}_2, \mathcal{L}_A) \) also serve as the desired operators, since \( A_1 \) and \( B_1 \) are invertible.

As a direct consequence of Theorem 3, we have the following.

**Corollary 7.** Let \( A \in \mathcal{B}(\mathcal{X}) \) and \( B \in \mathcal{B}(\mathcal{Y}) \) be given operators. Then,

\[
\bigcap_{C \in \mathcal{B}(\mathcal{Y}, \mathcal{X})} \sigma_w(M_C) = \sigma_w(A) \cup \sigma_w(B) \cup \left( \sigma_w\left( \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \right) \right) \cap (\Omega_1 \cup \Omega_2 \cup \Omega_3),
\]

(40)

where \( \Omega_1 \) is the collection of all \( \lambda \in \mathcal{C} \) for which there is not any pair \( (S, J) \) of operators such that \( \begin{bmatrix} A & S \end{bmatrix} \) is right Fredholm with \( \text{ind}(\begin{bmatrix} A & S \end{bmatrix}) \geq \beta(\mathcal{B} - \lambda I) \), \( J \) is left Fredholm with \( \text{ind}(\begin{bmatrix} J & B \end{bmatrix}) \leq -\alpha(A - \lambda I) \), and \( P_{Q_\mathcal{A}} S |_{\mathcal{Y}(B - \lambda I)} = P_{Q_\mathcal{A}} J |_{\mathcal{X}(B - \lambda I)} \). Note that \( S \in \mathcal{R}(\mathcal{N}(B - \lambda I), \mathcal{Y}) \), \( J \in \mathcal{B}(\mathcal{Y}, \mathcal{A}_\mathcal{L}(\mathcal{A})) \), and \( \Omega_1, \Omega_2, \Omega_3 \) are defined as in Corollaries 1 and 4.

**Corollary 8.** Let \( A \in \mathcal{B}(\mathcal{X}_1, \mathcal{Y}_1) \) and \( B \in \mathcal{B}(\mathcal{X}_2, \mathcal{Y}_2) \) be given operators. If \( A \) is right Fredholm or \( B \) is left Fredholm, then there exists \( C \in \mathcal{B}(\mathcal{X}_2, \mathcal{Y}_1) \) such that \( M_C \) is Weyl if and only if \( M_0 = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \) is Weyl.
proof. Assume that there exists \( C \in \mathcal{B}(X_2, Y_1) \) such that \( M_C \) is Weyl. Then, we always see that the relations \( \alpha(B) = \infty = \beta(A) \) do not hold, whenever \( A \) is right Fredholm or \( B \) is left Fredholm. From Theorem 3, it follows that \( M_0 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \) is Weyl. The rest of the proof is trivial.

\[ \square \]

Corollary 9. Let \( A \in \mathcal{B}(X_1, Y_1) \) and \( B \in \mathcal{B}(X_2, Y_2) \) be given operators. If \( A \) (resp. \( B \)) is Weyl, then there exists \( C \in \mathcal{B}(X_2, Y_1) \) such that \( M_C \) is Weyl if and only if \( B \) (resp. \( A \)) is Weyl.

proof. By symmetry of the result, we only prove the case when \( A \) is Weyl. The sufficiency is obvious. For necessity, let \( M_C \) be Weyl for some \( C \in \mathcal{B}(X_2, Y_1) \). Since \( A \) is Weyl, \( \beta(A) = \alpha(A) < \infty \). Then, assertion (i) of Theorem 3 cannot be satisfied. Hence, \( \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \) is Weyl, which is equivalent to the fact that \( B \) is Weyl.

Based on the embedded relationship of certain spaces, the sufficient conditions under which \( M_C \) is left and right Weyl in a Banach space setting are given, respectively.

\[ \square \]

Definition 1 (see 6, Definition 4.2]). For two Banach spaces \( X \) and \( Y \), we say that \( X \) can be embedded in \( Y \) and write \( X \subset Y \) if there exists a left invertible operator \( J : X \rightarrow Y \). Note that \( X \subset Y \) if and only if there exists a right invertible operator \( S : Y \rightarrow X \).

Theorem 4. Let \( A \in \mathcal{B}(X_1, Y_1) \) and \( B \in \mathcal{B}(X_2, Y_2) \) be given operators such that \((\mathcal{Y}/\mathcal{R}(A)) \oplus \mathcal{Y}/\mathcal{R}(B) \subset \mathcal{N}(A) \oplus \mathcal{N}(B)\). Then, there exists \( C \in \mathcal{B}(X_2, Y_1) \) such that \( M_C \) is right Weyl if the following statements are fulfilled:

(i) \( B \) is right Fredholm.

(ii) \( \mathcal{R}(A) \) is complemented in \( Y_1 \).

(iii) There exists \( S \in \mathcal{B}(\mathcal{N}(B), Y_1) \) such that \( [A \ S] \) is right Fredholm.

proof. Assume that assertions (i), (ii), and (iii) hold. If \( \alpha(B) < \infty \), then \( [A \ S] \) is right Fredholm if and only if \([A \ 0] \) is right Fredholm, and hence \( \beta(A) < \infty \) which together with the right Fredholmness of \( B \) and the relation \((\mathcal{Y}/\mathcal{R}(A)) \oplus \mathcal{Y}/\mathcal{R}(B) \subset \mathcal{N}(A) \oplus \mathcal{N}(B)\) implies that \( \alpha(A) + \alpha(B) \geq \beta(A) + \beta(B) \), i.e., \( \text{ind}(M_0) = \text{ind}(A) + \text{ind}(B) \geq 0 \). This means that \( M_0 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \) is right Weyl, and taking \( C = 0 \) gives the theorem.

Now let \( \alpha(B) = \infty \). If \( \beta(A) < \infty \), again, \( \text{ind}(M_0) = \text{ind}(A) + \text{ind}(B) \geq 0 \), and \( C = 0 \) is the desired operator. It remains to consider the case of \( \beta(A) = \infty \). In this case, the decompositions in (10) and the second equality of (20) are still valid, and we further have

\[
[A \ S] = \begin{bmatrix} A_1 & S_1 \\ 0 & S_2 \end{bmatrix} : X_1 \oplus N(B) \rightarrow R(A) @ \mathcal{Q}_A. \quad (41)
\]

Obviously, \( A_1 \in \mathcal{B}(X_1, \mathcal{R}(A)) \) is right invertible and write \( A_1^{-1} \) for some left inverse of \( A_1 \). Since

\[
\begin{bmatrix} A_1 & S_1 \\ 0 & S_2 \end{bmatrix} \begin{bmatrix} I & -A_1^{-1}S_1 \\ 0 & I \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & S_2 \end{bmatrix}, \quad (42)
\]

the operator \( S_2 \in \mathcal{B}(\mathcal{N}(B), \mathcal{Q}_A) \) is right Fredholm. Hence,

\[
\mathcal{N}(B) = \mathcal{R}(S_2) \oplus \mathcal{N}(S_2),
\]

\[
\mathcal{Q}_A = \mathcal{R}(S_2) \oplus \mathcal{Q}_A,
\]

with \( \beta(S_2) < \infty \) and \( \mathcal{R}(S_2) \equiv \mathcal{R}(S_2) \). Since \((\mathcal{Y}/\mathcal{R}(A)) \oplus (\mathcal{Y}/\mathcal{R}(B)) \subset \mathcal{N}(A) \oplus \mathcal{N}(B)\), from \((\mathcal{N}(B)/\mathcal{R}(S_2)) \equiv \mathcal{R}(S_2) \), it follows that

\[
\left( \mathcal{Q}_A \oplus \mathcal{Y}/\mathcal{R}(B) \right) \subset \mathcal{N}(A) \oplus \mathcal{N}(S_2). \quad (44)
\]

It is clear that \( \alpha(A) + \alpha(S_2) \geq \beta(B) + \beta(S_2) \). This together with (42) implies that

\[
\text{ind}([A \ S]) = \alpha(A) + \alpha(S_2) - \beta(B) - \beta(S_2). \quad (45)
\]

Therefore, operator \( S \) is the desired one satisfying assertion (i) of Theorem 1, and hence we get the conclusion that there exists \( C \in \mathcal{B}(X_2, Y_1) \) such that \( M_C \) is right Weyl.

\[ \square \]

Theorem 5. Let \( A \in \mathcal{B}(X_1, Y_1) \) and \( B \in \mathcal{B}(X_2, Y_2) \) be given operators such that \( \mathcal{N}(A) \oplus \mathcal{N}(B) \subset (\mathcal{Y}/\mathcal{R}(A)) \oplus (\mathcal{Y}/\mathcal{R}(B)) \). Then, there exists \( C \in \mathcal{B}(X_2, Y_1) \) such that \( M_C \) is left Weyl if the following statements are fulfilled:

(i) \( A \) is left Fredholm.

(ii) \( \mathcal{N}(B) \) is complemented in \( X_2 \).

(iii) There exists \( J \in \mathcal{B}(X_2, \mathcal{Q}_A) \) such that \( [J \ B] \) is left Fredholm.

proof. Assume that assertions (i), (ii), and (iii) hold. If \( \dim \mathcal{Q}_A < \infty \), then \( [J \ B] \) is left Fredholm if and only if \( [0 \ B] \) is left Fredholm, and hence \( \alpha(B) < \infty \) which together with the left Fredholmness of \( A \) and the relation \( \mathcal{N}(A) \oplus \mathcal{N}(B) \subset \mathcal{N}(A) \oplus \mathcal{N}(B) \) implies that \( \alpha(A) + \alpha(B) \leq \beta(A) + \beta(B) \), i.e., \( \text{ind}(M_0) = \text{ind}(A) + \text{ind}(B) \leq 0 \). This means that \( M_0 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \) is left Weyl, and taking \( C = 0 \) proves the theorem.

Now let \( \dim \mathcal{Q}_A = \infty \). If \( \alpha(B) < \infty \), again, \( \text{ind}(M_0) = \text{ind}(A) + \text{ind}(B) \geq 0 \), and \( C = 0 \) is the desired operator. It remains to consider the case of \( \alpha(B) = \infty \). In this case, the decompositions in the first equality of (17) and (20) are still valid, and we further have

\[
\begin{bmatrix} J \\ B \end{bmatrix} = \begin{bmatrix} J_1 & J_2 \\ B_1 & 0 \end{bmatrix} : \mathcal{P}_B @ N(B) \rightarrow \mathcal{Q}_A @ \mathcal{Y}_2. \quad (46)
\]

Obviously, \( B_1 \in \mathcal{B}(\mathcal{P}_B, \mathcal{Y}_2) \) is left invertible and write \( B_1^{-1} \) for some left inverse of \( B_1 \). Since
the operator \( J_2 \in \mathcal{B}(\mathcal{N}(B), \mathcal{O}_A) \) is left Fredholm. Hence, \( \mathcal{N}(B) = \mathcal{P}_1 \oplus \mathcal{N}(J_2) \), \( \mathcal{O}_A = \mathcal{P}(J_2) \oplus \mathcal{O}_1 \), \( \mathcal{O}_A = \mathcal{O}(J_2) \oplus \mathcal{O}_1 \), with \( \alpha(J_2) < \infty \) and \( \mathcal{P}_1 \oplus \mathcal{B} \). Since \( \mathcal{N}(A) \oplus \mathcal{N}(B) < (\mathcal{W} \mathcal{A}(\mathcal{N}(A)) \oplus (\mathcal{W} \mathcal{A}(\mathcal{N}(B))) \), from \( \mathcal{N}(B)/\mathcal{N}(J_2) \equiv \mathcal{P}_1 \equiv \mathcal{P}(J_2) \), it follows that \( \mathcal{N}(A) \oplus \mathcal{N}(J_2) < \frac{(\mathcal{O} \oplus \mathcal{W}) \mathcal{A}(\mathcal{B})}{\mathcal{R}(J_2)} \). It is clear that \( \alpha(A) + \alpha(J_2) \leq \beta(B) + \beta(J_2) \). This together with (47) implies that \( \text{ind} \begin{bmatrix} I \\ A \end{bmatrix} = \alpha(J_2) - (\beta(J_2) + \beta(J_2)) \leq - \alpha(A) \). The following is a dual result of Corollary 10, and it gets proved directly by Corollary 10.

Corollary 11. Let \( A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1) \) and \( B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_2) \) be given operators. Then, there exists \( C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1) \) such that \( M_C \) is left Weyl if and only if \( A \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1) \) is left Fredholm, and one of the following holds:

1. \( d(A) = \infty \).
2. \( M_0 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \) is left Weyl.

Corollary 12. Let \( A \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1) \) and \( B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_2) \) be given operators. Then, there exists \( C \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1) \) such that \( M_C \) is Weyl if and only if \( A \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1) \) is right Fredholm, and one of the following holds:

1. \( n(B) = \infty = \text{dim} A \).
2. \( M_0 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \) is Weyl.

proof. Assume that \( A \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1) \) is left Fredholm, \( B \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_2) \) is right Fredholm, and \( n(B) = \infty = \text{dim} A \). To complete the proof, it suffices to show that there exists \( \mathcal{H} \), \( \mathcal{N}(A), \mathcal{N}(B) \), and \( J \in \mathcal{B}(\mathcal{N}(A), \mathcal{N}(B)) \) such that \( A S \) is right Fredholm with \( \text{ind}(A S) \geq \text{dim} B \), \( \begin{bmatrix} J \\ A \end{bmatrix} \), \( \text{ind} \begin{bmatrix} J \\ A \end{bmatrix} \leq - \text{dim} A \), and \( \mathcal{P}(\mathcal{A}) \), \( \mathcal{P}(\mathcal{B}) \), \( \mathcal{P}(\mathcal{A}) \), \( \mathcal{P}(\mathcal{B}) \) are complemented, \( \mathcal{N}(A), \mathcal{N}(B) \) are complemented, and one has the following decompositions:

\[
\begin{align*}
\mathcal{H}_1 &= \mathcal{N}(A) \oplus \mathcal{A}, \\
\mathcal{H}_2 &= \mathcal{N}(B) \oplus \mathcal{B}, \\
\mathcal{H}_1 &= \mathcal{R}(A) \oplus \mathcal{A}, \\
\mathcal{H}_2 &= \mathcal{R}(B) \oplus \mathcal{B}.
\end{align*}
\]

Since \( n(B) = \infty = \text{dim} A \), we have

\[
\begin{align*}
\mathcal{N}(B) &= \mathcal{N}(B) \oplus \mathcal{N}(B), \\
\mathcal{R}(A) &= \mathcal{R}(B) \oplus \mathcal{B},
\end{align*}
\]

where \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are infinite dimensional closed subspaces of \( \mathcal{N}(B) \) and \( \mathcal{R}(A) \), respectively. Pick an arbitrary bounded operator \( \Delta \) from \( \mathcal{N}_2 \) into \( \mathcal{M}_2 \) such that \( \text{dim} \Delta = \text{dim} (B) - \text{dim} (A) \). Take

\[
\begin{align*}
\mathcal{S} &= \begin{bmatrix} S_1 & S_2 \\ U & 0 \end{bmatrix} : \mathcal{N}_1 \oplus \mathcal{M}_2 \longrightarrow \mathcal{R}(A) \oplus \mathcal{M}_1 \oplus \mathcal{M}_2, \\
\mathcal{J} &= \begin{bmatrix} J_1 & 0 \\ J_2 & 0 \end{bmatrix} : \mathcal{N}(B) \oplus \mathcal{N}_1 \oplus \mathcal{N}_2 \longrightarrow \mathcal{M}_1 \oplus \mathcal{M}_2,
\end{align*}
\]
where $U$ is some unitary operator from $\mathcal{N}_1$ to $\mathcal{M}_1$, and $S_i$ and $J_i$ ($i = 1, 2$) can be chosen as certain bounded operators between the corresponding spaces. Evidently, $[A, S]$ is right Fredholm, \[
abla \text{ is left Fredholm,}\]
ind([A, S]) = n(\alpha) + \text{ind}(\Delta) = d(B),
\]
\[
\text{ind}\left(\begin{bmatrix} J \\ B \end{bmatrix}\right) = \text{ind}(\Delta) - d(B) = -n(\alpha), \tag{54}
\]
and $P_{\mathcal{M}(A)^+} S_{\mathcal{N}(B)}$ = \[
\begin{bmatrix} U \\ 0 \\ \Delta \end{bmatrix} = P_{\mathcal{M}(A)^+} H_{\mathcal{N}(B)}. \tag{55}
\]
The proof is completed.

Recall that for an upper or lower semi-Fredholm operator $T$, i.e., $T \in \Phi_+(\mathcal{X}) \cup \Phi_-(\mathcal{X})$, we can also define $\text{ind}(T) = \alpha(T) - \beta(T)$. Let $\Phi_+^c(\mathcal{X})$ be the class of all $T \in \Phi_+(\mathcal{X})$ with ind($T$) $\leq 0$ and $\Phi^-_+(\mathcal{X})$ be the class of all $T \in \Phi_-(\mathcal{X})$ with ind($T$) $\geq 0$; write $\Phi_0(\mathcal{X}) = \Phi_+^c(\mathcal{X}) \cap \Phi^-_+(\mathcal{X})$, and hence $\Phi_0(\mathcal{X}) = \mathcal{W}(\mathcal{X})$. Particularly, if $\mathcal{X}$ is a Hilbert space, then $\Phi_0(\mathcal{X}) = \Phi_+(\mathcal{X})$ and $\Phi_+(\mathcal{X}) = \Phi_-(\mathcal{X})$.

The next results’ proofs will be omitted. Actually, the corollaries listed above are, respectively, the reformulations of Theorems 2.1, 2.3, and 2.4 in [11], and the following are their initial forms.

\[\Box\]

Corollary 13 (see 11, Theorem 2.1). A $2 \times 2$ operator matrix $M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \in \Phi_0^r(H \oplus K)$ for some $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ if and only if $A \in \Phi_+(H)$ and

\[
\begin{cases}
(n(B) < \infty \text{ and } n(A) + n(B) \leq d(A) + d(B), & \text{if } \mathcal{R}(B) \text{ is closed}, \\
(n(B) = \infty = d(A), & \text{if } \mathcal{R}(B) \text{ is not closed}.
\end{cases}
\]

Corollary 14 (see 11, Theorem 2.3). A $2 \times 2$ operator matrix $M_C = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \in \Phi_0^r(H \oplus K)$ for some $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ if and only if $A \in \Phi_+(H)$, $B \in \Phi_-(H)$ and one of the cases exists:

(i) $d(A) < \infty, n(B) < \infty,$ and $n(A) + n(B) = d(A) + d(B)$, or $n(B) = \infty = d(A)$,

(ii) $d(A) = n(B) = \infty$.

Let $A \in \mathcal{B}(\mathcal{H})$. We denote by $H_C$ the operator on $\mathcal{H} \oplus \mathcal{K}$ of the form $H_C = \begin{bmatrix} A & C \\ 0 & -A^* \end{bmatrix}$, \tag{57}

where $C \in \mathcal{B}(\mathcal{H})$ is an unknown self-adjoint operator, which is clearly a Hamiltonian operator. One can naturally find its appearance in linear Hamiltonian systems and operator Riccati equations (see, e.g., [12, 13]). As applications, we now present the analogues of Theorems 1, 2, and 3 for Hamiltonian operators.
Corollary 17. Let \( A \in \mathcal{B}(\mathcal{H}) \) be a given operator. Then, there exists a self-adjoint operator \( C \in \mathcal{B}(\mathcal{H}) \) such that \( H_C \) is right Weyl.

\[
C = [0 \ S]: \mathcal{N}(-A^*)^\perp \oplus \mathcal{N}(-A^*) \longrightarrow \mathcal{H} \text{ is a self-adjoint operator such that } H_C \text{ is right Weyl.} \quad \square
\]

\[\text{Corollary } 18. \text{ Let } A \in \mathcal{B}(\mathcal{H}) \text{ be a given operator. Then, there exists a self-adjoint operator } C \in \mathcal{B}(\mathcal{H}) \text{ such that } H_C \text{ is Weyl if and only if } A \text{ is left Fredholm.}\]

\[
\text{proof. The necessity is obvious, and we now prove the sufficiency. If } d(A) < \infty, \text{ then } H_0 = \begin{bmatrix} A & 0 \\ 0 & -A^* \end{bmatrix} \text{ is still a Weyl operator, and from condition (ii) of Theorem 2, the conclusion is clear. However, if } d(A) = \infty, \text{ then }\begin{bmatrix} 0 & I \\ -A^* & 0 \end{bmatrix}: \mathcal{N}(-A^*)^\perp \oplus \mathcal{N}(-A^*) \longrightarrow \mathcal{R}(A)^\perp \oplus \mathcal{R}(A)^\perp \text{ is left Fredholm, and hence, the column operator } \begin{bmatrix} f \\ -A^* \end{bmatrix} \text{ is left Fredholm, and } \text{ind}\left( \begin{bmatrix} f \\ -A^* \end{bmatrix} \right) = -d(-A^*) = -n(A). \text{ Thus, from the proof of Theorem 2, we see that } C = \begin{bmatrix} 0 \\ 0 \end{bmatrix}: \mathcal{R}(A) \oplus \mathcal{R}(A)^\perp \longrightarrow \mathcal{H} \text{ is a self-adjoint operator such that } H_C \text{ is left Weyl.} \quad \square
\]

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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