Investigation on the Analysis of Bending and Buckling for FGM Euler-Bernoulli Beam Resting on Winkler- Pasternak Elastic Foundation

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Abstract. In this paper, Functionally Graded Material (FGM) has been analyzed to examine bending and buckling of simply supported beams. Using Euler-Bernoulli beam theory (EBT), these beams that rested on Winkler- Pasternak elastic foundation are exposed to two types of loads that are axial compressive force and distributed transverse load. Here, based on power-law distributions, the properties of the material of FGM beam is assumed to be varied at the direction of the thickness. The derivation of the FGM beams’ governing equations was done using the total potential energy principle. The transverse deflection and the critical buckling of the FGM beam were determined using the Navier-type solution method with simple boundary conditions. A validation study for numerical results was carried out here with previous results from the literature and they are said to be in excellent agreement. It is shown by the numerical results that critical buckling load is decreasing with increasing both, slenderness ratio and values of power-law exponent and vice versa for transverse deflection.

Keywords. Bending analysis, Buckling analysis, functionally graded materials, Euler-Bernoulli beam theory, Winkler- Pasternak elastic foundation

1. Introduction
Advanced multilayered composite materials are considered dominant materials in the majority of mechanical and space engineering structures because of their high strength and high stiffness [1]. The manufacture of Functionally Graded Materials (FGMs) is simply the mixing of metals and ceramics. The main property of these materials is the change that is occurring smoothly and continuously from one surface to another. The reason for that could be the volume fraction of ingredient materials that is gradually changing [2]. Different materials are combined with specific physical properties that in turn permit a tailoring material design. This leads to broadening the structural design space in a way that detects an increase in the minimum weight as a result of the implementation of a multi-functional response [3]. The reduction of the stress concentration, residual and thermal stresses can occur in the FGM. This is located in interference of the layer in conventional laminated composites. In addition, the change of gradient properties across the interface can result in different the stress intensity factor at the crack tip. The addition of ceramic to the metal in FGMs enhances many parameters like higher fracture resistance. Therefore, it increases toughness owing to crack bridging in a graded volume fraction. The even variation of composition offers a much-improved boundary bonding as the interface is crossed [4].

Many of engineering structures are considered as beams when they are modelled. These are such as propellers of the ship, the blades of the turbine, base frames for rotating machinery, helicopter rotor, railroad tracks and in the design of aircraft structures [5]. The reason for the destructive effects as an
initial defect in the material which the structural elements are subjected to could be due to fatigue or stress concentration. The structure initial cracking exhibits local flexibility over which stiffness reduction and changes of the buckling, static and dynamic behavior of the structure are observed [6]. The failure of accurate prediction of the mechanical behavior of nanostructures using the classical continuum theories can be ascribed to the effects of small-scale that might be existed. Hence, to predict an accurate size dependency of nanostructures, new continuum models that contain parameters of material length scale were extensively suggested [7]. Much research has been done on beam theories to analyse the structural behaviour of the composite beams during the last decade. For example, the use of Euler-Bernoulli beam theory (EBT) to solve the bending behaviour of the thin beams by using the total potential energy principle. The assumption of composite materials that were used for (FGMs) was eliminated by one of the developed theories which have considered them as advanced materials [8]. Recently, the widespread of FG materials in the application in nano-structures. The bending response was investigated by Zenkour et al for an (FG) viscoelastic sandwich beam that is simply supported and functionally graded. This beam had an elastic core that rests on elastic foundations of Pasternak. FG viscoelastic material was used to make the sandwich beam faces the elasticity, however, was preserved in the core. The theory of the refined sinusoidal shear deformation beam was compared against its counterpart theories of the simple, higher- and first-order in terms of the numerical results of deflections and. Parameters that affecting deflection and stresses such as time, the stiffness range of foundation, distribution of material and the fraction of span with respect to thickness has been reported [9]. Akbaş et al analysed the post-buckling behaviour of an FGM cantilever beam cracked at the edge that was exposed to axial compressive loads. The analysis was based on the use of the total Lagrangian Timoshenko beam element approximation. In the post-buckling analysis, it was noted that the cracked FGM beam was significantly affected by the material distributions. In addition, during buckling loads, the negative effects that are accompanied by cracks can be easily deducted when using FGMs [6]. Using the theory of hyperbolic shear deformation (HYSĐT), another post-buckling analysis was done by Bouazza et al for a thick FGM rectangular beam. Hamilton principle and Navier type were used to find the governing equations for an FGM beam and to solve the post-buckling problem. Then, a validation test for FG beam in terms of the results of post-buckling was made against that of other theories in order to check their agreement [10]. While Hadji et al investigated the bending analysis of FG beams using the theory of refined exponential shear deformation beam. It has been found that the proposed theory almost has shown solutions that are identical with solutions of other theories [11]. The theory of exponential shear deformation with different boundary conditions that is simply modified for the analysis of FG beams in terms of bending, buckling, and free vibration was studied by Sayyad et al. The derivation of associated boundary conditions and equations of motion was done according to the principle of Hamilton [12]. Using the theory of third-order shear deformation, Karaman et al devoted an effort to provide Ritz-type analytical solutions of two-directional functionally graded beams for buckling behavior [13]. The Hyperbolic shear deformation theory was used by Ghugal et al to investigate the response of buckling, bending and free vibration for higher-order FG beams. Analytical solutions and equations of motion were derived using of Hamilton’s principle and Navier’s solution technique. Their results were compared against previous studies and excellent agreement was found between them [14]. While Xia et al studied the correlation between the solutions of the homogenous Euler–Bernoulli beams and those of the static bending of FGM "Reddy–Bickford Beams". The relationships that can exactly relate the FGM "Reddy–Bickford Beams" with the beams of the static bending solutions with material properties changing continuously across the depth were sought. The derivation of the equilibrium equations of static bending for the FGM beam and shear deformation beam theory has been achieved based on the rotation and the deflection of the cross-section and the axial displacement depending on the energy-variation principle [15]. This paper, hence, aims to introduce a unified approach for the analysis of bending and buckling behaviour of FGM by using Euler-Bernoulli beam theory (EBT) that is resting on Winkler-Pasternak elastic foundation. The solution methods of the analytical equations will be presented in detail in the next section. Moreover, the results obtained will be discussed and validated with previous work.
2. Theory and Formulation

Let it be a simply supported beam made from FGM in the top metallic (Aluminum) and the bottom ceramic (Alumina) with length L, width b and thickness h, resting on Winkler- Pasternak elastic foundation spring constant Kw and shear constant KG. The FGM beam subjected to distributed transverse load \( q(x) \) and axial compressive force \( N_x \) as shown in figure 1.

Figure 1. A simple supported FGM beam subjected to a uniform distributed load and axial compressive force resting on elastic foundation with cross-section.

Young's modulus of FGM beam changes in the thickness direction according to the power-law distribution that is given by:

\[
E(z) = E_m + (E_c - E_m)(z/h + 0.5)^n
\]

where \( E_m \) is Young's modulus of elasticity in the top surface and \( E_c \) Young's modulus of elasticity in the bottom surface, \( E_z \)The variation of Young’s modulus of FGM, \( n \) is the power-law Index of FGM. Poisson's ratio \( v \) is assumed to be constant in the thickness direction. The displacement fields are expressed based on Euler-Bernoulli beam theory (EBT) as follows:

\[
\begin{align*}
&u(x, y, z) = u_0(x) - z \frac{\partial w_0}{\partial x} \quad (2a) \\
v(x, y, z) = 0 \quad (2b) \\
w(x, y, z) = w_0(x) \quad (2c)
\end{align*}
\]

Where, \( u \), is the displacement of the beam in the x-direction, and \( v, w \) are transverse displacements of the beam in y-z directions respectively, \( w_0(x) \) is the transverse deflection of the beam. The axial strain can be calculated by:

\[
\varepsilon_{xx} = \frac{\partial u}{\partial x} - z \frac{\partial^2 w}{\partial x^2}
\]

The strain energy (potential energy) is given by:

\[
U = \frac{1}{2} \int \sigma_y \varepsilon_y dV
\]

\[
\delta U = \int \sigma_y \delta \varepsilon_y dV
\]

\[
\begin{align*}
\delta U &= \left[ \int \sigma_{yy} \delta \varepsilon_{yy} + \sigma_{yy} \delta \varepsilon_{yz} + \sigma_{yy} \delta \varepsilon_{zy} + \sigma_{yz} \delta \varepsilon_{yz} + \sigma_{yz} \delta \varepsilon_{zy} + \sigma_{zz} \delta \varepsilon_{zz} \right] dV \\
&\quad + \sigma_{yy} \delta \varepsilon_{yx} + \sigma_{yy} \delta \varepsilon_{zy} dV
\end{align*}
\]

The virtual strain energy of the beam can be defined by using the axial strain and the axial stress as follows:
\[ \delta U = \int (\sigma_{xx} \delta e_{xx}) \, dA \, dx \]  

(6)

Where \( \delta \) is the variable parameter and \( \sigma_{xx} \) is the normal stress component in the x-x plane, and \( e_{xx} \) is the normal strain in "x" direction. Substituting Eq. (3) into Eq. (6) and using Eqs. (8) & (9) yields the final strain energy for Euler-Bernoulli beam theory (EBT).

\[ \delta U = \int ( - \frac{\partial N_x}{\partial x} \delta u_0 - \frac{\partial^2 M_x}{\partial x^2} \delta w_0 ) \, dx \]  

(7)

Where:

\[ N_x = \int \sigma_{xx} \, dA \]  

(8)

\[ M_x = \int \sigma_{xx} \, z \, dA \]  

(9)

\( N_x \) is the axial normal force, and \( M_x \) is the bending moment. The external work can be done as follows:

\[ W_{ext} = -\frac{1}{2} \int \left( F_{ext} w_0 \right) \, dA \]  

(10)

\[ \delta W_{ext} = -b \int \left( F_{ext} \delta w_0 \right) \, dx \]  

(11)

The final external work of Euler-Bernoulli beam theory (EBT) is given by:

\[ \delta W_{ext} = -b \int \left[ (-k_w w_0 + k_g \frac{\partial^2 w_0}{\partial x^2}) + N_{x0} \frac{\partial^2 w}{\partial x^2} + q(x) \delta w_0 \right] \, dx \]  

(12)

The governing equations are derived by using total potential energy:

\[ \Pi = (U + W_{ext}) \]  

(13a)

\[ \delta \Pi = 0 \]  

(13b)

\[ \delta U + \delta W_{ext} = 0 \]  

(14)

By substituting Eqs. (7) and (12) into Eq. (14), we have:

\[ \int \left( - \frac{\partial N_x}{\partial x} \right) \delta u_0 - \left( \frac{\partial^2 M_x}{\partial x^2} + b k_w w_0 - b k_g \frac{\partial^2 w_0}{\partial x^2} - b N_{x0} \frac{\partial^2 w}{\partial x^2} - b q(x) \right) \delta w_0 \, dx = 0 \]  

(15)

\[ \delta u_0 : \quad - \frac{\partial N_x}{\partial x} = 0 \]  

(16a)

\[ \delta w_0 : \quad - \frac{\partial^2 M_x}{\partial x^2} + b k_w w_0 - b k_g \frac{\partial^2 w_0}{\partial x^2} - b N_{x0} \frac{\partial^2 w}{\partial x^2} - b q(x) = 0 \]  

(16b)

The axial normal force and bending moment:

\[ N_x = A_x b \frac{\partial u_0}{\partial x} - B_x b \frac{\partial^2 w_0}{\partial x^2} \]  

(17a)

\[ M_x = B_x b \frac{\partial u_0}{\partial x} - D_x b \frac{\partial^2 w_0}{\partial x^2} \]  

(17b)

The governing equations of FGM beam can be obtained by substituting for \( N_x, M_x \) from Eqns. (17a) & (17b) into Eqns. (16a) & (16b) as follows:
\[ A_{xx} \frac{\partial^2 H_0}{\partial x^2} - B_{xx} \frac{\partial^3 W_0}{\partial x^3} = 0 \]  
(18a)

\[ B_{xx} \frac{\partial^2 H_0}{\partial x^2} - D_{xx} \frac{\partial^4 W_0}{\partial x^4} - k_w w_0 + k_c \frac{\partial^2 W_0}{\partial x^2} + N_{x0} \frac{\partial^2 w}{\partial x^2} + q(x) = 0 \]  
(18b)

Where:
\[ A_{xx} = \int_{-h/2}^{h/2} E(z) dz \]  
(19a)

\[ B_{xx} = \int_{-h/2}^{h/2} E(z) z dz \]  
(19b)

\[ D_{xx} = \int_{-h/2}^{h/2} E(z) z^2 dz \]  
(19c)

Axx, Bxx, Dxx are the stiffness coefficient.

3. Analysis of Static Bending and Buckling for a Simply-Supported FGM Beam
The static buckling and bending of a simply-supported FG beam are analytically solved by the governing equations. The simply-supported FG beam boundary conditions of at x=0, x=L is given as:
\[ x = 0 \Rightarrow (w = 0), (\frac{\partial w}{\partial x} = 0), (\frac{\partial u}{\partial x} = 0) \]  
(20a)

\[ x = L \Rightarrow (w = 0), (\frac{\partial w}{\partial x} = 0), (\frac{\partial u}{\partial x} = 0) \]  
(20b)

A semi-analytical method for bending and buckling analysis of a simply supported FGM beam exposed to the axial compressive force and distributed transverse load resting on Winkler- Pasternak elastic foundation by using Navier’s solution method is defined as follows:
\[ u_{(x)} = \sum_{m=1,2,3}^\infty U_m \cos \left( \frac{m\pi x}{L} \right) \]  
(21a)

\[ w_{(x)} = \sum_{m=1,2,3}^\infty W_m \sin \left( \frac{m\pi x}{L} \right) \]  
(21b)

Where Um and Wm are the unknown Fourier coefficients. For the static bending problem, the applied transverse load q(x) can be expanded in Fourier series as
\[ q_{(x)} = \sum_{m=1,2,3}^\infty q_m \sin \left( \frac{m\pi x}{L} \right) \]  
(22)

Where:
\[ q_m = \frac{2}{L} \int_0^L q(x) \sin m\pi x \, dx \]  
(23)

Where qm are the Fourier coefficients and are given for uniform load as follows:
\[ q_m = \frac{4q_0}{m\pi}, \quad m = 1, 3, 5 \]  
(24)

Where q0 is the intensity of the uniformly distributed load. Substituting Eqns. (21a), (21d) and Eqns. (22) into Eqns. (18a) & (18b) the final matrix form of FGM Euler-Bernoulli beam theory (EBT) can be obtained as follows:
4. Numerical results and discussions

The critical buckling load and transverse deflection by using Euler-Bernoulli beam theory (EBT) based on Navier’s type solution method for functionally graded materials (FGM) simply supported boundary conditions. The beam consists of two materials at the top surface material of beam as pure metallic and the bottom surface material of the beam as pure ceramic. The beam is subjected to distributed transverse load \( q(x) \), and axial compressive force \( N_x \). The Young modulus, and Poisson's ratio of elasticity for ceramic \( E_c, \nu_c \) and similar of elasticity for metal \( E_m, \nu_m \) the longitudinal wave number \( m \). The width of the beam is \( b \), the length of the beam is \( L \) and the height of the beam is \( h \), with other properties that are shown in Table 1.

| Property                      | FGM  |
|-------------------------------|------|
|                               | Metallic (Aluminum) | Ceramic (Alumina) |
| Young's modulus \( E \) (GPa) | 70   | 380   |
| Poisson's ratio \( \nu \)     | 0.23 | 0.23  |
| Length (m)                    | 1m   | 1m    |
| Thickness \( h \) (m)         | 0.1m | 0.1m  |
| Width \( b \) (m)             | 0.1m | 0.1m  |
| Longitudinal wave number (m)  | 1    | 1     |

Table 2 shows the values of the critical buckling load \( P_{cr} \) of FGM beam with the different the values of the power-law index \( n \) for constant \( \delta=h/L=1/5 \) based on Euler-Bernoulli beam theory EBT. It has been found that the values of critical buckling load \( P_{cr} \) are an excellent agreement with the solution of Li et al [16], at simply supported boundary conditions. It concluded that the critical buckling load \( P_{cr} \) of the FGM beam decreases with the increase of the values of the power-law index \( n \).

| \( n \) | Present work \( P_{cr} \) for EBT  | Li et al. \( P_{cr} \) for EBT |
|--------|----------------------------------|-------------------------------|
| 0.0    | 53.578                           | 53.578                        |
Table 3 shows the critical buckling load $P_{cr}$ of the Euler-Bernoulli theory EBT FGM beam for simply supported boundary conditions and various values of power-law index $n$ for constant slenderness ratio $L/h=5$. It has been shown that the present critical buckling load $P_{cr}$ is an excellent agreement with the solution of Li and Batra [17]. It can be noted from this Table that the power-law index $n$ value is inversely related to the value of the critical buckling load $P_{cr}$ of the FGM beam.

Table 3. Dimensionless critical buckling loads $P_{cr} = p^*L^2/E_b^*I$ of FGM beams for constant slenderness ratio $L/h=5$ and the various values of the power-law index $n$.

| $n$   | Present work $(P_{cr})$ for EBT | Li and Batra $(P_{cr})$ for EBT |
|-------|---------------------------------|---------------------------------|
| 0.0   | 53.578                          | 53.578                          |
| 0.5   | 34.731                          | 34.731                          |
| 1     | 26.705                          | 26.705                          |
| 2     | 20.838                          | 20.838                          |
| 5     | 17.623                          | 17.623                          |
| 7     | 16.899                          | 16.899                          |
| 10    | 16.052                          | 16.052                          |
| 100   | 11.066                          | 11.066                          |
| $10^{10}$ (∞) | 9.8696                         | 9.8696                          |

Table 4 shows the critical buckling load $P_{cr}$ of the Euler-Bernoulli theory EBT FGM beam for simply supported boundary conditions and various values of power-law index $n$ and constant $h/L=0.1$. It has been found the critical buckling load $P_{cr}$ is an excellent agreement for simply supported boundary conditions with the solution of Torki and Reddy [18]. It is seen from this Table that the critical buckling load $P_{cr}$ decreases with the increase in the power-law index $n$.

Table 4. Dimensionless critical buckling loads $P_{cr} = p^*L^2/E_m^*h/12$ of FGM beam for constant $h/L=0.1$ and the various values of the power-law index $n$.

| $n$   | Present work                   | Torki and Reddy                  |
|-------|--------------------------------|---------------------------------|
|       |                                |                                 |
| 0.0   |                                |                                 |
| 0.5   |                                |                                 |
| 1     |                                |                                 |
| 2     |                                |                                 |
| 5     |                                |                                 |
| 7     |                                |                                 |
| 10    |                                |                                 |
| 100   |                                |                                 |
| $10^{10}$ (∞) |                                |                                 |

Table 3 shows the critical buckling load $P_{cr}$ of the Euler-Bernoulli theory EBT FGM beam for simply supported boundary conditions and various values of power-law index $n$ for constant slenderness ratio $L/h=5$. It has been shown that the present critical buckling load $P_{cr}$ is an excellent agreement with the solution of Li and Batra [17]. It can be noted from this Table that the power-law index $n$ value is inversely related to the value of the critical buckling load $P_{cr}$ of the FGM beam.

Table 3. Dimensionless critical buckling loads $P_{cr} = p^*L^2/E_b^*I$ of FGM beams for constant slenderness ratio $L/h=5$ and the various values of the power-law index $n$.

| $n$   | Present work $(P_{cr})$ for EBT | Li and Batra $(P_{cr})$ for EBT |
|-------|---------------------------------|---------------------------------|
| 0.0   | 53.578                          | 53.578                          |
| 0.5   | 34.731                          | 34.731                          |
| 1     | 26.705                          | 26.705                          |
| 2     | 20.838                          | 20.838                          |
| 5     | 17.623                          | 17.623                          |
| 7     | 16.899                          | 16.899                          |
| 10    | 16.052                          | 16.052                          |
| 100   | 11.066                          | 11.066                          |
| $10^{10}$ (∞) | 9.8696                         | 9.8696                          |

Table 4 shows the critical buckling load $P_{cr}$ of the Euler-Bernoulli theory EBT FGM beam for simply supported boundary conditions and various values of power-law index $n$ and constant $h/L=0.1$. It has been found the critical buckling load $P_{cr}$ is an excellent agreement for simply supported boundary conditions with the solution of Torki and Reddy [18]. It is seen from this Table that the critical buckling load $P_{cr}$ decreases with the increase in the power-law index $n$.

Table 4. Dimensionless critical buckling loads $P_{cr} = p^*L^2/E_m^*h/12$ of FGM beam for constant $h/L=0.1$ and the various values of the power-law index $n$.

| $n$   | Present work | Torki and Reddy |
|-------|--------------|-----------------|
|       |              |                 |
| 0.0   |              |                 |
| 0.5   |              |                 |
| 1     |              |                 |
| 2     |              |                 |
| 5     |              |                 |
| 7     |              |                 |
| 10    |              |                 |
| 100   |              |                 |
| $10^{10}$ (∞) |              |                 |
Figure 2 shows that the effect of shear constant KG on the critical buckling load $P_{cr}$ with the slenderness ratio $L/h$ for the functionally graded materials (FGM) beam. It has been shown that the critical buckling load $P_{cr}$ decreases with the increase of the slenderness ratio $L/h$. It is also seen from Figure 2 that the value of the shear constant KG plays an important role in the critical buckling load $P_{cr}$ of the FGM beam, where the increase in the shear constant KG leads to increases the critical buckling load $P_{cr}$ since, with increasing the shear constant KG the beam gets stiffer.

Figure 3 shows the effect of spring constant Kw on the critical buckling load $P_{cr}$ with the slenderness ratio $L/h$ for the FGM Euler-Bernoulli simply supported beam. The results show that an increase in the slenderness ratio $L/h$ leads to the decreasing of the critical buckling load $P_{cr}$. As can be seen in this Figure, the spring constant Kw increased with increases the critical buckling load $P_{cr}$. This is because with increasing the spring constant Kw the beam gets high stiffer.

| $L/h$ | $k_G=0$ | $k_G=3\times10^9$ | $k_G=6\times10^9$ |
|-------|---------|-------------------|-------------------|
| 0     | 53.578  | 53.578            | 53.578            |
| 0.5   | 34.731  | 34.731            | 34.731            |
| 1     | 26.705  | 26.705            | 26.705            |
| 2     | 20.838  | 20.838            | 20.838            |
| 5     | 17.623  | 17.623            | 17.623            |
| 7     | 16.899  | 16.899            | 16.899            |
| 10    | 16.052  | 16.052            | 16.052            |
| 100   | 11.066  | 11.066            | 11.066            |
| $10^{11}(\infty)$ | 9.8696 | 9.8696 | 9.8696 |

**Figure 2.** The effect of shear constant $K_G$ on the critical buckling load $P_{cr}$ FGM Euler-Bernoulli beam with slenderness ratio $L/h$.  

**Figure 3.** Critical Buckling Load $P_{cr}$ versus Slenderness Ratio $L/h$. The figure shows the effect of spring constant $K_w$ on the critical buckling load $P_{cr}$ with the slenderness ratio $L/h$. The results show that an increase in the slenderness ratio $L/h$ leads to the decreasing of the critical buckling load $P_{cr}$. As can be seen in this Figure, the spring constant $K_w$ increased with increases the critical buckling load $P_{cr}$. This is because with increasing the spring constant $K_w$ the beam gets high stiffer.
Figure 3. The effect of Winkler spring constant $K_w$ on the critical buckling load $P_{cr}$ of FGM beam with slenderness ratio $L/h$.

In figure 4 the effect of the longitudinal wave number $m$ is shown on the critical buckling load $P_{cr}$ of FGM Euler-Bernoulli beam with a ratio of slenderness $L/h$. It can be seen in this Figure the slenderness ratio $L/h$ increases with the decrease in the critical buckling load $P_{cr}$. It is also seen in this Figure that the critical buckling load $P_{cr}$ is proportionally related to the longitudinal wave number $m$.

Figure 4. The effect of longitudinal wave number $m$ on the critical buckling load $P_{cr}$ of FGM beam with slenderness ratio $L/h$.

Figure 5 explains that the effect of power-law index $n$ on the critical buckling load $P_{cr}$ and a slenderness ratio $L/h$. It is seen in this figure that the power-law index $n$ and slenderness ratio $L/h$
increases with the decrease of the critical buckling load \( P_{cr} \). Since with an increase in the power-law index, \( n \) leads to a decrease in the stiffness of the FGM beam.

**Figure 5.** The effect of power-law index \( n \) on the critical buckling load \( P_{cr} \) of FGM beam with slenderness ratio \( L/h \).

Figures 6-7 shows the effects of both shear constant \( K_G \) and spring constant \( K_w \) on the maximum deflection \( w \) and slenderness ratio \( L/h \). It is found that the maximum deflection \( w \) decreases with the increase both of shear constant \( K_G \) and spring constant \( K_w \). Since with increases in the shear and spring constants lead to an increase in the stiffness of the FGM beam. Also, can be seen that the maximum deflection \( w \) increases with an increase in the slenderness ratio \( L/h \) according to the three different values of the shear and spring constants.

**Figure 6.** The effect of shear constant \( K_G \) on the maximum deflection \( w \) of FGM Euler-Bernoulli beam with slenderness ratio \( L/h \).
Figure 7. The effect of spring constant $K_w$ on the maximum deflection $w$ of FGM beam with slenderness ratio $L/h$.

Figures 8 and 9 show the effect both of longitudinal wave number $m$ and power-law index $n$ on the maximum deflection $w$ and slenderness ratio $L/h$. It is found that the maximum deflection $w$ increases with the increase both of longitudinal wave number $m$ and power-law index $n$. Also, can be seen that the maximum deflection $w$ increasing with an increase in the slenderness ratio $L/h$ according to the five different values of the longitudinal wave number $m$ and power-law index $n$.

Figure 8. The effect of longitudinal wave number $m$ on the maximum deflection $w$ of FGM beam with slenderness ratio $L/h$
5. Conclusions
The analysis of Functionally Graded Material (FGM) has been carried out to study bending and buckling of simply supported beams. Euler-Bernoulli beam theory (EBT) was applied for the beams that rested on Winkler- Pasternak elastic foundation with the exposure to axial compressive force and distributed transverse loads. In this paper, based on power-law distributions, the variation of the properties of the material of FGM beam is assumed to be in the direction of the thickness. Also, the theory of total potential energy was used in order to derive the FGM beams’ governing equations. Additionally, the transverse deflection and the critical buckling of the FGM beam was determined using the Navier-type solution method with simple boundary conditions. Moreover, the factors that are affecting the transverse deflection and critical buckling load such as the power-law exponent of FGM, and the spring constant with the shear constant of the elastic foundation were studied. The numerical results were validated here with previous results from the literature and their agreement was superb. It is concluded that the critical buckling load is inversely related to both, slenderness ratio and values of power-law exponent and proportionally with transverse deflection.

1. The critical buckling load of FGM beam increases with the increasing both, of shear and spring constants, and the maximum deflection of FGM beam decreases with the increasing both, of shear and spring constants.
2. The critical buckling load of FGM beam decreases with the increase both, slenderness ratio and values of the power-law exponent of FGM, and maximum deflection of FGM beam increases with increasing in the slenderness ratio and increases with increase in the values of the power-law exponent.
3. The critical buckling load and maximum deflection of FGM beam increase with increasing of longitudinal wave number.

Reference

[1]. Li, X.-F., 2008, Journal of Sound and vibration, A unified approach for analyzing static and dynamic behaviors of functionally graded Timoshenko and Euler–Bernoulli beams, 318(4-5): p. 1210-1229.

[2]. Zhang, D.-G., 2013, Composite Structures, Nonlinear bending analysis of FGM beams based on physical neutral surface and high order shear deformation theory. 100: p. 121-126.
[3]. Giunta, G., S. Belouettar, and E. Carrera, 2010, Mechanics of Advanced Materials and Structures, Analysis of FGM beams by means of classical and advanced theories. 17(8): p. 622-635.

[4]. Alexraj, S., N. Vasrajab, and P. Nagarajc, 2013, International Conference on Energy Efficient Technologies for Sustainability. IEEE, Static behaviour of Functionally Graded Material beam using Finite Element Method.

[5]. Singh, A. and P. Kumari, 2018, Journal of Solid Mechanics. Two-Dimensional Elasticity Solution for Arbitrarily Supported Axially Functionally Graded Beams, 10(4): p. 719-733.

[6]. Akbaş, Ş.D., 2015, International Journal of Structural Stability and Dynamics, On post-buckling behavior of edge cracked functionally graded beams under axial loads. 15(04): p. 1450065.

[7]. Eltaher, M., et al., 2014, Applied Mathematics and Computation, Static and buckling analysis of functionally graded Timoshenko nanobeams. 229: p. 283-295.

[8]. Karamanlı, A., 2018, Politeknik Dergisi, Bending Analysis of Two Directional Functionally Graded Beams Using A Four- Unknown Shear and Normal Deformation Theory. 21(4): p. 861-874.

[9]. Zenkour, A.M., M. Allam, and M. Sobhy, 2010, Acta Mechanica, Bending analysis of FG viscoelastic sandwich beams with elastic cores resting on Pasternak’s elastic foundations, 212(3-4): p. 233-252.

[10]. Bouazza, M., et al., 2015, Review of Information Engineering and Applications, Postbuckling analysis of functionally graded beams using hyperbolic shear deformation theory, 2(1): p. 1-14.

[11]. Hadji, L., et al., 2015, Coupled systems mechanics, Static bending and free vibration of FGM beam using an exponential shear deformation theory, 4(1): p. 99-114.

[12]. Sayyad, A.S. and Y.M. Ghugal, 2018, Asian Journal of Civil Engineering, Asian Journal of Civil Engineering, Analytical solutions for bending, buckling, and vibration analyses of exponential functionally graded higher order beams. 19(5): p. 607-623.

[13]. Karamanlı, A., 2018, Akademik Platform Mühendislik ve Fen Bilimleri Dergisi. Analytical Solutions for Buckling Behavior of Two Directional Functionally Graded Beams Using a Third Order Shear Deformable Beam Theory, 6(2): p. 164-178.

[14]. Sayyad, A.S. and Y.M. Ghugal, 2018, Mechanics of Advanced Composite Structures, Bending, buckling and free vibration responses of hyperbolic shear deformable FGM beams, 5(1): p. 13-24.

[15]. Xia, Y.-M., S.-R. Li, and Z.-Q. Wan, 2019, Acta Mechanica Solida Sinica. Bending solutions of FGM Reddy–Bickford beams in terms of those of the homogenous Euler–Bernoulli beams. 32(4): p. 499-516.

[16]. Li, Xuan Wang Zeqing Wan , 2015, Acta Mechanica Solida Sinica, Classical and Homogenized Expressions Forbuckling Solutionsof Functionally Gradedmaterial Levinsonbeams, 28 (5 ): p. 593-604.

[17]. Shi-Rong Li, Romesh. Batra, 2013, Composite Structures, Relations between buckling loads of functionally graded Timoshenko and homogeneous Euler–Bernoulli beams, 95: p. 5-9.

[18]. Mohammad Ebrahim Torki, 2016, International Journal of Structural Stability and Dynamics, Buckling of Functionally-Graded Beams with Partially Delaminated Piezoelectric Layers, 16(1): p. 1-25.