Any star network of bipartite pure entangled states is genuine multipartite nonlocal

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Quantum entanglement and nonlocality are inextricably linked. However, while entanglement is necessary for nonlocality, it is not always sufficient in the standard Bell scenario. We derive sufficient conditions for entanglement to give rise to genuine multipartite nonlocality in networks. We find that a star network where Alice shares a pure bipartite entangled state with every other party is genuine multipartite nonlocal, independently of the amount of entanglement in the shared states. We also show that all genuine multipartite entangled states are genuine multipartite nonlocal in the sense that measurements can be found on finitely many copies of any genuine multipartite entangled state to yield a genuine multipartite nonlocal behaviour. Our results pave the way towards feasible manners of generating genuine multipartite nonlocality based only on topological considerations.

Correlations between quantum particles may be much stronger than those between classical particles. Their applications are manifold: quantum cryptography [1, 2], randomness extraction, amplification and certification [3], communication complexity reduction [4], etc., and the study of these nonlocal correlations has led to the growing field of device-independent quantum information processing [5, 6] (see also [8]).

A necessary condition to achieve nonlocality is quantum entanglement. Indeed, this is one of the reasons why entangled states are useful for communication-related tasks. However, not all entangled states are nonlocal: from some bipartite entangled states, only local distributions can be extracted [9, 10]. Still, for pure bipartite states, entanglement is also sufficient for nonlocality, which is the content of Gisin’s theorem [11, 12]. In this Letter we explore the relationship between entanglement and nonlocality in the multipartite regime.

While bipartite entanglement and nonlocality have been well researched in the past three decades, much less is known about the multipartite case. The study of correlations in quantum multicomponent systems is gaining attention lately together with applications in multiparty cryptographic protocols [13], the development of quantum networks [14, 22] and the understanding of condensed matter physics [23, 24]. Interestingly, the distribution of certain bipartite entangled states in multipartite networks has been shown to give rise to nonlocality even if the involved states are individually local [15, 18, 23, 28].

Moreover, the multipartite setting has a richer structure than the bipartite one, as different forms of entanglement and nonlocality can be identified. Full separability (full locality) refers to systems that do not display any form of entanglement (locality) at all. However, falsifying these models does not necessarily imply truly multipartite quantum correlations since spreading them among two parties is sufficient. Hence, a stronger, genuine multipartite, notion which inextricably relates all parties together is more often considered. Here, a state is genuine multipartite entangled (GME) if it is not a tensor product of states of two subsets of parties, M and its complement \( \overline{M} \), i.e., of the form \( |\psi\rangle = |\psi_M\rangle \otimes |\psi_{\overline{M}}\rangle \), or a convex combination of such states \( |\psi\rangle = \sum_k \alpha_k |\psi_k\rangle \) across all bipartitions. By analogy, a probability distribution \( \{ P(\alpha_1, \ldots, \alpha_n | \chi_1, \ldots, \chi_n ) \}_{\alpha_1, \ldots, \alpha_n = \text{arbitrary}} \) (with input \( \chi_i \) and output \( \alpha_i \), for party \( i \)) which is not of the form

\[
P(\alpha_1, \ldots, \alpha_n | \chi_1, \ldots, \chi_n ) = \sum_{M \subseteq \{1, \ldots, n\}} \sum_{\lambda} q_M(\lambda) P_M(\{\alpha_i \}_{i \in M} | \{\chi_i \}_{i \in M})
\]

\[
\times P_{\overline{M}}(\{\alpha_i \}_{i \in \overline{M}} | \{\chi_i \}_{i \in \overline{M}}),
\]

where \( q_M(\lambda) \geq 0 \forall \lambda, M, \sum_{\lambda, M} q_M(\lambda) = 1 \) and \( |n| := \{1, \ldots, n\} \), is genuine multipartite nonlocal (GMNL) [29–31], and a state is GMNL if measurements exist which give rise to a GMNL distribution. The distributions \( P_M, P_{\overline{M}} \) are usually assumed nonsignalling as this captures most physical situations better than unrestricted \( P_M, P_{\overline{M}} \) [32, 33].

In this Letter we show that the nonlocality arising from certain networks of bipartite entangled states is a generic property and manifests in its strongest form, GMNL. Specifically, we obtain that any star network of bipartite pure entangled states is GMNL. It was already known that a star network of maximally entangled states is GMNL [15], but we show the result still holds when each party shares any pure entangled state with the central node. Thus, this is a property of the network topology, independent of how much entanglement each pair of nodes share.

Further, there are known mixed GME states that are not GMNL [40, 41]—some are even fully local [42]. Moreover, while multipartite entangled pure states are never fully local [42, 43], it is not known whether Gisin’s theorem extends to the genuine multipartite regime. Recent results in this direction show that, for pure \( n \)-qubit symmetric states [45] and all pure 3-qubit states [46], GME
implies GMNL (at the single-copy level). Our second result uses the above property of star networks to establish that all pure GME states are GMNL in the sense that measurements can be found on finitely many copies of any GME state to yield a GMNL behaviour. We thus tighten the relationship between multipartite entanglement and nonlocality.

Our construction exploits the fact that the set of non-GME states is not closed under tensor products, i.e. GME can be superactivated by taking tensor products of states that are unentangled across different bipartitions. Thus, GME can be achieved by distributing bipartite entangled states among different pairs of parties. To obtain our results, we extend the superactivation property \[47–49\] from the level of states to that of probability distributions, i.e. GMNL can be superactivated by taking Cartesian products of probability distributions that are local across different bipartitions. In fact, when considering copies of quantum states, we only consider local measurements performed on each copy separately, thus pointing at a stronger notion of superactivation to achieve GMNL.

It is worth clarifying that, when searching for nonlocality in entangled states, we only consider measurements performed directly on the state. If local operators can be applied prior to measurements, states that are entangled but local may display ‘hidden nonlocality’. This has been studied in Refs. \[42, 57, 54\] and is exploited in \[46\] to extend the equivalence of GME and GMNL for three parties beyond qubits.

**Definitions and preliminaries** We consider distributions arising from GME states, and ask whether they are of the form \([1]\). The set of such distributions forms a polytope: indeed, the set of local distributions across each bipartition \(M|M\) is a polytope, and taking convex combinations preserves that structure. We call this \(n\)-partite polytope \(B_n\). We call an inequality

\[
\sum_{\alpha_1,\ldots,\alpha_n,\chi_1,\ldots,\chi_n} c_{\alpha_1,\ldots,\alpha_n,\chi_1,\ldots,\chi_n} P(\alpha_1\ldots\alpha_n|\chi_1\ldots\chi_n) \leq c_0
\]

which holds for all distributions \(P\) of the form \([1]\) a GMNL inequality.

We use results from \[55\] to lift inequalities to account for more parties, inputs and outputs. It is noteworthy that they consider the fully local polytope, denoted \(\mathcal{L}\), which only includes distributions

\[
P(\alpha|\beta|\chi) = \sum_{\lambda} q(\lambda)P_A(\alpha|\chi,\lambda)P_B(\beta|\chi,\lambda)
\]

where each party may have different numbers of inputs and outputs (more parties may be considered by adding more distributions correlated only by \(\lambda\)). Polytope \(B_n\) is more general as it includes convex combinations of these distributions, but the lifting results in \[55\] still hold. Indeed, to check that an inequality is valid for a polytope, it is sufficient by convexity to check it holds for the extremal points. As all extremal points in \(B_n\) are extremal points in some polytope \(\mathcal{L}\) (splitting the parties in two as per the bipartition \(M|M\)), lifting results for \(\mathcal{L}\) can be straightforwardly extended to \(B_n\).

We also use the EPR2 decomposition \[56\] and its multipartite extension \[57\]: any distribution \(P\) can be expressed (nonuniquely) as

\[
P(\alpha_1\ldots\alpha_n|\chi_1\ldots\chi_n) = \sum_{M \subseteq [n]} p^M_L P^M_M(\alpha_1\ldots\alpha_n|\chi_1\ldots\chi_n)
\]

\[
+ p_{NS} P_{NS}(\alpha_1\ldots\alpha_n|\chi_1\ldots\chi_n)
\]

Figure 1. Star network of bipartite entanglement. For each \(i \in [n-1]\), Alice’s \(i\)th particle is entangled to party \(B_i\)’s particle, Alice has input \(x_i\) and output \(a_i\) for particle \(i\), and \(B_i\) has input \(y_i\) and output \(b_i\).

where \(\sum_{M \subseteq [n]} p^M_L + p_{NS} = 1\), \(P^M_L\) is local across \(M|M\) (i.e. satisfies equation \([3]\) with parties grouped as per \(M|M\)), and \(P_{NS}\) is nonsignalling. \(P\) is GMNL if all such decompositions have \(p_{NS} > 0\), and fully-GMNL if all such decompositions have \(p_{NS} = 1\). A state \(\rho\) is fully-GMNL if, for all \(\varepsilon > 0\), there exist local measurements which give rise to a distribution \(P\) such that any decomposition \([3]\) has \(p_{NS} > 1 - \varepsilon\). Bipartite distributions and states may be nonlocal or fully-nonlocal \[58\] analogously.

**GMNL from bipartite entanglement** Our first result shows that any star network of pure bipartite entanglement (see Figure \([1]\)) is GMNL.

**Theorem 1.** For any star network where Alice shares a bipartite entangled pure state with each of \((n-1)\) parties there exist local measurements giving rise to an \(n\)-partite GMNL distribution.

We outline the tripartite proof and leave the full \(n\)-partite proof to \[59\]. Since it turns out to be sufficient
to consider individual measurements on Alice’s different particles (see Figure 4 for the n-partite structure), the shared distribution takes the form

\[ P(a_1a_2, b_1, b_2|x_1x_2, y_1, y_2) = P_1(a_1b_1|x_1y_1)P_2(a_2b_2|x_2y_2) \]  

(5)

where Alice’s input and output are \(x_1x_2, a_1a_2\) respectively, in terms of their digits; Bob1’s input and output are \(y_1, b_1\), and Bob2’s are \(y_2, b_2\). Depending on whether or not the shared states are maximally entangled, the distributions \(P_{1,2}\) will be different, therefore the proof considers three possible cases.

If both states are less-than-maximally entangled, we derive an inequality that detects GMNL and find measurements on the shared states to violate it. To construct the inequality, we take bipartite inequalities between Alice and each of the other parties, lift them to three parties and then combine them, using Refs. [59, 60], to obtain the following GMNL inequality:

\[ I_3 = I_{00}^{AB_1} + I_{00}^{AB_2} + P(00, 0, 0|00, 0, 0) \]

\[ - \sum_{a_2=0,1} P(0a_2, 0, 0|00, 0, 0) \]

\[ - \sum_{a_1=0,1} P(a_10, 0, 0|00, 0, 0) \leq 0 . \]

(6)

Here,

\[ I_{00}^{AB_1} = \sum_{a_2=0,1} (P(0a_2, 0, 0|00, 0, 0) - P(0a_2, 1, 0|00, 1, 0) \]

\[ - P(1a_2, 0, 0|10, 0, 0) - P(0a_2, 0, 0|10, 10, 0)) \leq 0; \]

(7)

\[ I_{00}^{AB_2} = \sum_{a_1=0,1} (P(a_10, 0, 0|00, 0, 0) - P(a_10, 0, 1|00, 1, 0) \]

\[ - P(a_11, 0, 0|01, 0, 0) - P(a_10, 0, 0|01, 0, 0)) \leq 0 \]

(8)

are liftings of

\[ I^{AB} = P(00|00) - P(01|01) - P(10|10) - P(00|11) \leq 0 \]

(9)

to three parties with Alice having 4 inputs and 4 outputs. Inequality (9) is equivalent to the CHSH inequality [61] for nonsignalling distributions [60]. Thus, inequalities (7), (8) are satisfied by distributions that are local across \(AB_1\) and \(AB_2\) respectively. To see that equation (9) is a GMNL inequality it is sufficient to check it holds for distributions that are local across some bipartition. This is straightforwardly done by observing the cancellations that occur when either \(I_{00}^{AB_1}\) and/or \(I_{00}^{AB_2}\) are \(\leq 0\).

Since both states are less-than-maximally entangled, Alice can satisfy Hardy’s paradox [62, 63] with each other party, achieving

\[ P_1(00|00) > 0 = P_1(01|01) = P_1(10|10) = P_1(00|11) \]

(10)

for both \(i\) (the proof for qubits in [62, 63] can be extended to qudits by measuring on a 2-dimensional subspace, see [60]). Then, each negative term in \(I_{00}^{AB_1}\) and \(I_{00}^{AB_2}\) is zero, as

\[ \sum_{a_2=0,1} P(0a_2, 1, 0|00, 1, 0) = P_1(01|01) \sum_{a_2=0,1} P_2(a_2|00) \]

(11)

and similarly for the others. Hence, only

\[ P(00, 0, 0|00, 0, 0) = P(00|00)P_2(00|00) > 0 \]

(12)

survives, violating the inequality.

The case where both states are maximally entangled was proven in Refs. [58, 59].

Finally, if Alice shares a maximally entangled state with Bob1, and a less-than-maximally entangled state with Bob2, then \(AB_2\) can choose measurements so that \(P_2\) satisfies Hardy’s paradox, so that there exists \(\varepsilon > 0\) such that its local component in any EPR2 decomposition satisfies

\[ p_{L,2} \leq 1 - \varepsilon. \]

(13)

Since the maximally entangled state is fully-nonlocal [64], for this \(\varepsilon, AB_1\) can choose measurements such that any EPR2 decomposition of \(P_1\) satisfies

\[ p_{L,1} < \varepsilon. \]

(14)

Then, we assume for a contradiction that \(P(\alpha, b_1b_2|\chi, y_1y_2)\) is not GMNL and decompose it in its bipartite splittings,

\[ P(\alpha, b_1b_2|\chi, y_1y_2) = \sum_{\lambda} (p_L(\lambda)P_{AB}(\alpha, b_1|\chi, y_1, \lambda)P_{C}(b_2|y_2, \lambda) \]

\[ + q_L(\lambda)P_{AC}(\alpha, b_2|\chi, y_2, \lambda)P_{B}(b_1|y_1, \lambda) \]

\[ + r_L(\lambda)P_A(\alpha|\chi, \lambda)P_{BC}(b_1, b_2|y_1, y_2, \lambda)) \]

(15)

where \(\sum_{\lambda} (p_L(\lambda) + q_L(\lambda) + r_L(\lambda)) = 1\).

Summing equation (25) over \(a_2, b_2\) and using equation (15), we get an EPR2 decomposition of \(P_1\) with local components \(q_L, r_L\). By equation (27), this entails \(\sum_{\lambda} (q_L(\lambda) + r_L(\lambda)) < \varepsilon\), so

\[ \sum_{\lambda} p_L(\lambda) > 1 - \varepsilon. \]

(16)

Summing, instead, equation (28) over \(a_1, b_1\), we obtain an EPR2 decomposition of \(P_2\) whose only nonnegligible component, \(\sum_{\lambda} p_L(\lambda)\), is local in \(AB_2\), contradicting equation (25). Therefore, \(P\) must be GMNL.

**GMNL from GME** We have seen that a star network where the central node shares pure entanglement with all others is GMNL. We now address the question of whether all GME states are GMNL (i.e. the genuine multiparticle extension of Gisin’s theorem). We show \((n - 1)\) copies
Theorem 2. Any GME state |Ψ⟩ ∈ ℋ1 ⊗ ... ⊗ ℋn ≈ (C^d)^⊗n is such that |Ψ⟩⊗(n-1) is GMNL.

The full proof is given in [39], and we presently outline the tripartite case. Hence, we must consider two copies of the state. For each copy, we derive measurements for Bob1 and Bob2 that leave Alice bipartitely entangled with Bob2 and Bob1 respectively. We yield a network as in equation [4] but post-selected on the inputs and outputs of these measurements. We then generalise Theorem 1 to show that any such network is also GMNL.

For i, j = 1, 2, on copy i, Bob1’s measurements have input y_j^i and output b_j^i and Alice’s measurement has input x_i and output a_i. We denote B_j^i’s inputs and outputs in terms of their digits as v_j^i = y_j^i y_j^i and β_j = b_j^i b_j^i. Then, after measurement, the parties share a distribution

\[ P(\alpha_1 \beta_2 | x_1 v_2) = P_1(a_1, b_1^i b_2^i | x_1, y_1^i y_2^i) P_2(a_2, b_2^i b_2^i | x_2, y_1^i y_2^i) \]  

(17)

We assume, for each i, j = 1, 2, i ≠ j, that Bob_j uses input 0_j^i and output 0_j^i to project the ith copy of |Ψ⟩ onto |φ_i⟩_AB_i, as shown in Figure 2 for n parties. Then, Refs. [43, 44] and a continuity argument serve to show that we only have two possibilities for each i: either there exists an input and output per party such that |φ_i⟩_AB_i is less-than-maximally entangled, or there exists an input per party such that, for all outputs, |φ_i⟩_AB_i is maximally entangled. In each case we generalise the proof in Theorem 1 to show |Ψ⟩⊗(n-2) is GMNL.

If both |φ_i⟩_AB_i, i = 1, 2 are less-than-maximally entangled, we use the following expression, which is a GMNL inequality by the same reasoning as in Theorem 1

\[ I_3 = \sum_{i=1}^{2} I_{AB_i}^{B_1} + P(00, 00, 00|00, 00, 00) \]

(18)

\[ \sum_{i=1}^{2} \sum_{a_j, b_j} P(0, a_j, 0, b_j^i, 0, b_j^i | 0, 0, 0, 0) \leq 0, \]

where

\[ I_{AB_i}^{B_1} = \sum_{a_j, b_j} \left( P(0, a_j, 0, b_j^i, 0, b_j^i | 0, 0, 0, 0) - P(0, a_j, 1, b_j^i, 0, b_j^i | 0, 0, 0, 0) - P(1, a_j, 0, b_j^i, 0, b_j^i | 0, 0, 0, 0) + P(0, a_j, 0, b_j^i, 0, b_j^i | 0, 0, 0, 0) \right) . \]

(19)

Evaluating the inequality on the distribution (19), we find again that all negative terms in each I_{AB_i}^{B_1} can be sent to zero. For each i we get, for example,

\[ \sum_{a_j, b_j} P(0, a_j, 1, b_j^i, 0, b_j^i | 0, 0, 0, 0) = P_i(0, 0, 1, 0, 0, 1) \]

(20)

as the sum over P_j is 1. But, conditioned on B_j’s input and output being 0_j^i, parties AB_i can measure so that P_i satisfies Hardy’s paradox, so this term is zero, and similarly for the other two negative terms. This means all terms in I_3 are zero except P(00, 00, 00|00, 00, 00) > 0, so the inequality is violated. Therefore, |Ψ⟩⊗2 is GMNL.

If, for both i = 1, 2, there exists a local measurement for party B_j, j ≠ i such that, for all outputs, |Ψ⟩ is projected onto a maximally entangled state |φ_i⟩_AB_i, then |Ψ⟩ satisfies Theorem 2 in [37], so |Ψ⟩ itself is GMNL. Therefore so is |Ψ⟩⊗2.

Finally, if |φ_1⟩_AB_1 is maximally entangled for all of B_2’s outputs, and |φ_2⟩_AB_2 is less-than-maximally entangled, we can use Refs. [43, 57] to deduce that the bipartite EPR2 components of the distributions P_1,2 across A|B_1,2 respectively have the same bounds as in Theorem 1. That is, there exists ε > 0 such that the local component of any EPR2 decomposition across A|B_2 satisfies

\[ p_{L, A|B_2} \leq 1 - \varepsilon \]

(21)

and, given this ε, parties AB_1 can measure locally such that all bipartite EPR2 decompositions across A|B_1 have
a local component

\[ p_{L,1}^{A|B_1} < \varepsilon. \]  

(22)

Then, we assume \( P(\alpha_1\beta_2|\chi_1\nu_2) \) is not GMNL and decompose it in local terms across different bipartitions, like in equation (28) in Theorem 1. Summing over \( a_2, b_2^j, j = 1, 2 \) gives an EPR2 decomposition of \( P_1 \) whose local components can be bounded using equation (22). Summing over \( a_1, b_1^j, j = 1, 2 \) instead gives an EPR2 decomposition of \( P_2 \). But the bound on the local component of \( P_1 \) entails a bound on that of \( P_2 \) which contradicts equation (21), proving that \( P \) is GMNL.

**Conclusions** We have shown that GMNL can be obtained from the distribution of arbitrary pure bipartite entanglement, which paves the way towards a feasible way of generating GMNL in networks based only on topological considerations. In practical applications, the entanglement shared by the nodes would unavoidably degrade to mixed-state form. By continuity, the GMNL in the star network of pure bipartite entanglement considered here must be robust to some noise. To quantify this tolerance is interesting for future work.

Finding out which other geometries of entanglement in networks, aside from the star configuration, give rise to GMNL is worthy of further study.

Further, we have shown that a tensor product of finitely many GME states is always GMNL. The question of whether all single-copy pure GME states are GMNL remains open.

Very recently, Ref. [65] proposed the concept of “genuine network entanglement”, a stricter notion than GME which rules out states which are a tensor product of non-GME states. One might hope that states that are GME but not genuine network entangled might be detected device-independently by not passing GMNL tests. However, our results show this will not work. At least in the star configuration, any distribution of pure bipartite states, even with arbitrarily weak entanglement, always displays GMNL. This further motivates searching for an analogous concept of genuine network nonlocality that may detect genuine network entanglement.

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Supplemental material

We prove Theorems 1 and 2 in the main text. For the reader’s convenience, we also restate some definitions and the results.

We say a probability distribution \( \{ P(\alpha_1\alpha_2...\alpha_n|\chi_1\chi_2...\chi_n) \} \) (with input \( \chi_i \) and output \( \alpha_i \) for party \( i \)) is genuine multipartite nonlocal (GMNL) if there exist measurements which give rise to a distribution that cannot be written in the form

\[
P(\alpha_1\alpha_2...\alpha_n|\chi_1\chi_2...\chi_n) = \sum_{M \subseteq [n]} \sum_{\lambda} q_M(\lambda) P_M(\{\alpha_i\} \in M | \{\chi_i\} \in M, \lambda) P_M(\{\alpha_i\} \in \overline{M} | \{\chi_i\} \in \overline{M}, \lambda),
\]

(1)

where \( q_M(\lambda) \geq 0 \) for each \( \lambda, M, \sum_M q_M(\lambda) = 1 \) and the distributions \( P_M, P_{\overline{M}} \) on each bipartition are nonsignalling. A state is GMNL if there exist measurements which give rise to a distribution that cannot be written as (1).

Given an inequality

\[
\sum_{\alpha_1,\alpha_n,\chi_1,...,\chi_n} c_{\alpha_1...\alpha_n,\chi_1,...,\chi_n} P(\alpha_1...\alpha_n|\chi_1...\chi_n) \leq c_0
\]

(2)

which holds for all distributions \( P \) of the form (1), we refer to it as a GMNL inequality. The set of points \( P(\alpha_1...\alpha_n|\chi_1...\chi_n) \) which satisfy it with equality is a face of the polytope \( B_n \) of \( n \)-partite distributions \( \mathbf{1} \). Inequalities are said to support faces of the polytope. Faces \( F \neq B_n \) of maximal dimension are facets, and inequalities which support facets are called facet inequalities.

The multipartite EPR2 decomposition \([56, 57]\) of a probability distribution \( P \) is

\[
P(\alpha_1...\alpha_n|\chi_1...\chi_n) = \sum_{M \subseteq [n]} p_L^M p_M^M (\alpha_1...\alpha_n|\chi_1...\chi_n) + p_{NS} P_{NS}(\alpha_1...\alpha_n|\chi_1...\chi_n),
\]

(3)

where \( p_L^M \geq 0 \) for every \( M, p_{NS} \geq 0 \) and

\[
\sum_{M \subseteq [n]} p_L^M + p_{NS} = 1,
\]

(4)

\( p_L^M \) is local across the bipartition \( M|\overline{M} \), and \( P_{NS} \) is nonsignalling. We are interested in decompositions which maximise the local EPR2 components, in order to deduce properties about the distributions. For a distribution \( P \), we define

\[
EPR2(P) = \max \left\{ \sum_{M \subseteq [n]} p_L^M : P = \sum_{M \subseteq [n]} p_L^M p_M^M + p_{NS} P_{NS}, \sum_{M \subseteq [n]} p_L^M + p_{NS} = 1 \right\}
\]

(5)

and, for a state \( \rho \), we define (with a slight abuse of notation)

\[
EPR2(\rho) = \inf \left\{ EPR2(P) : P = \text{tr} \left( \bigotimes_{i=1}^n E_{\alpha_i|\chi_i}^i \rho \right) \right\},
\]

(6)

where the infimum is taken over local measurements \( E_{\alpha_i|\chi_i}^i \) on each particle such that

\[
E_{\alpha_i|\chi_i}^i \geq 0 \forall \alpha_i, \chi_i, \sum_{\alpha_i} E_{\alpha_i|\chi_i}^i = 1 \forall \chi_i, \forall i \in [n],
\]

(7)

with any number of inputs and outputs. Then, a distribution \( P \) or a state \( \rho \) are GMNL if \( EPR2(\cdot) < 1 \), while they are fully-GMNL if \( EPR2(\cdot) = 0 \). When considering bipartite distributions and states, the analogous property is termed full-nonlocality. Notice that the optimisation for probability distributions yields a maximum since the number of inputs and outputs is fixed. Instead, the optimisation for a state may involve measurements with an arbitrarily large number of inputs or outputs, as is the case for the maximally entangled state \([64]\). In this work, the number of inputs and outputs is always finite, and this will become relevant when bounding the EPR2 components of distributions arising from maximally entangled states in Theorems 1 and 2.
GMNL from bipartite entanglement

**Theorem 1.** For any star network where Alice shares a bipartite entangled pure state with each of \((n - 1)\) parties there exist local measurements which give rise to an \(n\)-partite GMNL distribution.

**Proof.** Let Alice be labelled \(A\) and the rest of the parties \(B_1, ..., B_{n - 1}\). It will be enough to consider separate measurements on each of Alice’s particles. Let Alice have input \(x_i\) and output \(a_i\) for each particle \(i \in [n - 1]\), and denote \(\chi \equiv x_1 \ldots x_{n-1}, \alpha \equiv a_1 \ldots a_{n-1}\) in terms of these digits. Let the input and output of each other party \(B_i, i \in [n - 1]\) be \(y_i\) and \(b_i\) respectively. Thus, we denote the global \(n\)-partite distribution shared between the parties by

\[
P(\alpha, b_1 \ldots b_{n-1} | \chi, y_1 \ldots y_{n-1}) = \prod_{i=1}^{n-1} P_i(a_i, b_i | x_i, y_i),
\]

where

\[
P_i(a_i, b_i | x_i, y_i)
\]

is the distribution arising from the state shared by Alice and party \(B_i\).

Depending on the nature of the shared states, we consider three cases:

(i) every shared state is less-than-maximally entangled;

(ii) every shared state is maximally entangled;

(iii) some shared states are maximally entangled, some are not.

**Case (i)**: if all states are less-than-maximally entangled, we prove the result by deriving an inequality that detects GMNL and finding measurements on the shared states to violate it. To derive the inequality, we will find bipartite inequalities that Alice can violate with each other party, lift them to more inputs, outputs and parties using the techniques in Ref. \[55\] and combine them to obtain a GMNL inequality using tools in Ref. \[60\]. We will consider 2-input 2-output measurements on each particle. Thus, the global distribution will have \(2^n - 1\) inputs and outputs for Alice, and 2 inputs and outputs for the other parties.

Consider the following bipartite inequality for a distribution between Alice and party \(B_i\):

\[
I_{AB_i} = P(00|00) - P(01|01) - P(10|10) - P(00|11) \leq 0.
\]

This is a facet inequality which, for nonsignalling distributions, can be shown to be equivalent to the well-known \[61\]

\[
CHSH = \langle A_0 B_0 \rangle + \langle A_0 B_1 \rangle + \langle A_1 B_0 \rangle - \langle A_1 B_1 \rangle \leq 2,
\]

where \(\langle A_0 B_0 \rangle = \sum_{a,b} P(a = b|xy) - P(a \neq b|xy).

Inequality (11) can be lifted to more parties, inputs and outputs, and the bound remains the same. To get inequality (11) to account for an extra party (while keeping the number of inputs and outputs fixed), it is sufficient to fix the input and output of the extra party to a certain value (wlog, 0 for both) and the bound does not change. Moreover, if the original inequality supports a facet of the polytope, so does the lifted one.

To lift inequality (11) to all \(n\) parties, we fix the inputs and outputs of every party \(B_j, j \neq i\) to zero and denote this by \(\vec{0}_i\), to obtain

\[
\tilde{I}_{AB_i}^\alpha_{\vec{0}_i} = P(0,0, \vec{0}_i|0,0, \vec{0}_i) - P(0,1, \vec{0}_i|0,1, \vec{0}_i) - P(1,0, \vec{0}_i|1,0, \vec{0}_i) - P(0,0, \vec{0}_i|1,1, \vec{0}_i) \leq 0,
\]

where \(i = [n - 1] \setminus \{i\}\). To account for \(2^n - 1\) inputs for Alice, we can ignore all but 2 of them. We would like to select the 2 inputs that will give a violation for the distribution \(P_i\) that Alice and party \(B_i\) share. But these inputs are precisely \(x_i = 0, 1\) for each \(i\), which correspond, wlog, to \(\chi = 0, \vec{0}_i, 1, \vec{0}_i\). Then, inequality (12) becomes

\[
\tilde{I}_{AB_i}^\alpha_{\vec{0}_i} = P(0,0, \vec{0}_i|0, \vec{0}_i, \vec{0}_i) - P(0,1, \vec{0}_i|0 \vec{0}_i, 1, \vec{0}_i) - P(1,0, \vec{0}_i|1 \vec{0}_i, 0, \vec{0}_i) - P(0,0, \vec{0}_i|1 \vec{0}_i, 1, \vec{0}_i) \leq 0.
\]

Next, we can combine the \(2^n - 1\) outputs Alice has in the target distribution into two groups, to get an effective distribution. It is convenient to do it differently for each \(i\) so that each distribution \(P_i\) violates the corresponding inequality. For each \(i\), we add over the outputs corresponding to the uncorrelated parties. We obtain the following
inequality which detects bipartite nonlocality between Alice and party $B_i$, in an $n$-partite distribution where Alice has $2^{n-1}$ inputs and outputs and every other party has 2 inputs and outputs:

\[
I_{\hat{0}}^{AB_i} = \sum_{\vec{a}_i} \left( P(0, \vec{a}_i, 0, \vec{0}_i|0 \vec{0}_i) - P(0, \vec{a}_i, 1, \vec{0}_i|0 \vec{0}_i) \right) + \left( P(1, \vec{a}_i, 0, \vec{0}_i|1 \vec{0}_i) - P(0, \vec{a}_i, 0, \vec{0}_i|1 \vec{0}_i) \right) \leq 0, \tag{14}
\]

where the sum is over each binary digit $a_j$ of Alice’s output, except digit $a_i$ which is fixed to $0_i$ or $1_i$ in each term.

We can now combine the expressions $I_{\hat{0}}^{AB_i}$ for each $i$ into a GMNL inequality:

\[
I_n = \sum_{i=1}^{n-1} I_{\hat{0}}^{AB_i} + P(0, 0|0, 0) - \sum_{i=1}^{n-1} \sum_{\vec{a}_i} P(0, \vec{a}_i, 0, \vec{0}_i|0 \vec{0}_i) \leq 0. \tag{15}
\]

To prove that this is indeed a GMNL inequality, notice that all distributions that are not GMNL must be local across $A|B_j$ for some $j = 1, ..., n-1$, so these are the only bipartitions we need to consider. Notice that any distribution which is local across one such bipartition satisfies the inequality

\[
I_{\hat{0}}^{AB_i} \leq 0 \tag{16}
\]

and so

\[
I_n \leq \sum_{i=1}^{n-1} I_{\hat{0}}^{AB_i} + P(0, 0|0, 0) - \sum_{i=1}^{n-1} \sum_{\vec{a}_i} P(0, \vec{a}_i, 0, \vec{0}_i|0 \vec{0}_i). \tag{17}
\]

Now, for each $i \neq j$, the only nonnegative term in $I_{\hat{0}}^{AB_i}$ gets subtracted in the final summation. The term $P(0_j, \vec{0}_j, 0, \vec{0}_j|0 \vec{0}_j, 0, \vec{0}_j)$ then cancels out with the first term of the final summation for $i = j$, leaving only nonpositive terms in the expression as required.

To complete the proof, we find local measurements for each party to violate inequality (15). Since all shared states are inseparable and less-than-maximally entangled, the parties can choose local measurements on each particle such that all resulting distributions satisfy Hardy’s paradox:

\[
P_i(00|00) > 0 = P_i(01|01) = P_i(10|10) = P_i(00|11) \tag{18}
\]

for each $i \in [n-1]$. This was proven for qubits in Refs. 62, 63, and we show the extension to any local dimension in Proposition 11 below. However, no local distribution can satisfy equation (18). Notice that, because $P$ is of the form (8), each term in the inequality (11) simplifies significantly. For example, we have for a particular $i$

\[
\sum_{\vec{a}_i} P(0, \vec{a}_i, 1, \vec{0}_i|0 \vec{0}_i, 1, \vec{0}_i) = P_i(0_i, 1_i|0_i, 1_i) \prod_{j=1}^{n-1} P_j(a_j, 0_j|0_j, 0_j) \prod_{j \neq i}^{n-1} P_j(b_j = 0|y_j = 0) = P_i(0_i, 1_i|0_i, 1_i) p_i, \tag{19}
\]

for some $p_i \in [0, 1]$, and where the marginal $P_j(b_j = 0|y_j = 0)$ is well-defined as all distributions are assumed to be nonsignalling. In a similar way, the other two nonnegative terms in $I_{\hat{0}}^{AB_i}$ yield

\[
\sum_{\vec{a}_i} P(1, \vec{a}_i, 0, \vec{0}_i|1 \vec{0}_i, 0, \vec{0}_i) = P_i(1_i, 0_i|1_i, 0_i) p_i, \tag{20}
\]
The first term in each \( \frac{I_{AB_1}}{\partial | \partial } \) cancels with the last summation, and the only remaining term in \( I_n \) gives

\[
P(\emptyset , \emptyset | \emptyset , \emptyset ) = \prod_{i=1}^{n-1} P_i(0_i, 0_i | 0_i, 0_i).
\] (21)

However, because all distributions \( P_i \) satisfy Hardy’s paradox, the summation terms \([19], [20]\) are zero for all \( i \), while the only surviving term, \( P(\emptyset , \emptyset | \emptyset , \emptyset ) \), is strictly greater than zero. This gives the desired violation.

- **Case (i)** this case follows from Ref. [15], since the present network meets the requirements of Theorem 2 in [57]. In particular, this means that the distribution is fully-GMNL.

- **Case (ii)** assume wlog that Alice shares less-than-maximally entangled states with parties \( B_1, ..., B_m \) and maximally entangled states with parties \( B_{m+1}, ..., B_{n-1} \). Then, there exist measurements giving rise to a distribution which can be written as

\[
P(\alpha, b_1...b_{n-1}|\chi, y_1...y_{n-1}) = P_H(a_1...a_m, b_1...b_m | x_1...x_m, y_1...y_m)P_+ (a_{m+1}...a_{n-1}, b_{m+1}...b_{n-1} | x_{m+1}...x_{n-1}, y_{m+1}...y_{n-1}),
\] (22)

where

\[
P_H(a_1...a_m, b_1...b_m | x_1...x_m, y_1...y_m) = \prod_{i=1}^{m} P_i(a_i, b_i | x_i, y_i)
\]
\[
P_+ (a_{m+1}...a_{n-1}, b_{m+1}...b_{n-1} | x_{m+1}...x_{n-1}, y_{m+1}...y_{n-1}) = \prod_{i=m+1}^{n-1} P_i(a_i, b_i | x_i, y_i).
\] (23)

The terms \( P_i \) are of two kinds: for \( i = 1, ..., m \), i.e. the components of \( P_H \), satisfy Hardy’s paradox (equation [15]) as they arise from the measurements performed in Case (i), while for \( i = m + 1, ..., n - 1 \), i.e. the components of \( P_+ \), arise from the maximally entangled state, whose measurements will be specified later (subscript + is chosen because the maximally entangled state is usually denoted \( \phi^+ \)).

Because \( P_H \) satisfies Hardy’s paradox in \( A|B_i \) for all \( i = 1, ..., m \), it is an \((m + 1)\)-way nonlocal distribution (i.e., it is GMNL when restricted to parties \( A, B_1, ..., B_m \)), as we have seen in Case (i). Therefore, there exists an \( \varepsilon > 0 \) such that any EPR2 decomposition of \( P_H \) (see equation [3]) as

\[
P_H = \sum_{M} P_{L,H}^M P_{L,H}^M + p_{NS,H} P_{NS,H}
\] (24)

satisfies

\[
\sum_{M} P_{L,H}^M \leq 1 - \varepsilon.
\] (25)

Given this \( \varepsilon \), we know from Ref. [57] that Alice and parties \( B_i, i = m + 1, ..., n - 1 \) can choose local measurements to define \( P_+ \) such that, for all EPR2 decompositions of the form [3], their distribution

\[
P_+ = \sum_{M} P_{L,+}^M P_{L,+}^M + p_{NS,+} P_{NS,+}
\] (26)

has

\[
\sum_{M} P_{L,+}^M < \varepsilon.
\] (27)

In particular, \( P_+ \) is \((n - m)\)-way nonlocal (i.e., it is GMNL when restricted to parties \( A, B_{m+1}, ..., B_{n-1} \)).

We want to show that \( P \) is GMNL. We now assume the converse, i.e. that \( P \) can be written in the form [13], and reach a contradiction by using the bounds derived above for \( P_+ \) and \( P_H \). Because \( P \) is assumed to be of the form [13], it can be written as a sum of local components across all bipartitions \( M | M, M \in [n-1], P(\alpha, b_1...b_{n-1}|\chi, y_1...y_{n-1}) = \sum_{\lambda, M} p_{L}^M(\lambda) P_M(\alpha, \{b_i\}_{i \in M} | \chi, \{y_i\}_{i \in M}, \lambda) P_{M}^\lambda(\{b_i\}_{i \in M}, \{y_i\}_{i \in M}, \lambda),
\] (28)
where \( p_L^M(\lambda) \) are nonnegative numbers for every \( M, \lambda \) such that
\[
\sum_{\lambda, M} p_L^M(\lambda) = 1. \tag{29}
\]
Notice that we always take Alice as belonging to the first element of each bipartition, in order to avoid duplications. Then, summing equation (28) over \( a_i, b_i \) for \( i = 1, ..., m \) gives \( P_+ \) on the left-hand side, from equation (22). The right-hand side will contain two types of terms: one type corresponds to bipartitions \( M|\overline{M} \) where all indices \( i = m+1, ..., n-1 \) belong to \( M \) (though \( M \) may contain more indices too). Let \( I_1 \) be the set of such \( M \). Then, \( \overline{M} \) contains only indices \( i \in \{1, ..., m\} \), so that \( P_{\overline{M}} \) adds up to 1. This leaves only
\[
\sum_{\lambda} p_L^M(\lambda) P_M(a_{m+1}...a_{n-1}, \{b_i\}_{i>m}\chi, \{y_i\}_{i>m}, \lambda), \tag{30}
\]
for \( M \in I_1 \). On the other hand, if some of \( i \in \{m+1, ..., n-1\} \) belong to \( \overline{M} \), then we get terms that are local across some bipartition \( A|B_i, i = m+1, ..., n-1 \). Let \( I_2 \) be the set of the corresponding \( M \); then the terms in \( M \in I_2 \) are of the form
\[
\sum_{\lambda} p_L^M(\lambda) P_M(a_{m+1}...a_{n-1}, \{b_i\}_{i\in\overline{M}}\chi, \{y_i\}_{i\in\overline{M}}, \lambda)P_{\overline{M}}(\{b_i\}_{i\in\overline{M}}\{y_i\}_{i\in\overline{M}}, \lambda). \tag{31}
\]
Hence we have
\[
P_+(a_{m+1}...a_{n-1}, b_{m+1}...b_{n-1}|x_{m+1}...x_{n-1}, y_{m+1}...y_{n-1}) =
\sum_{\lambda, M \in I_2} p_L^M(\lambda) P_M(a_{m+1}...a_{n-1}, \{b_i\}_{i\in\overline{M}}\chi, \{y_i\}_{i\in\overline{M}}, \lambda)P_{\overline{M}}(\{b_i\}_{i\in\overline{M}}\{y_i\}_{i\in\overline{M}}, \lambda)
+ \sum_{\lambda, M \in I_1} p_L^M(\lambda) P_M(a_{m+1}...a_{n-1}, \{b_i\}_{i>m}\chi, \{y_i\}_{i>m}, \lambda). \tag{32}
\]
Since \( P_M(a_{m+1}...a_{n-1}, \{b_i\}_{i>m}\chi, \{y_i\}_{i>m}, \lambda) \) is nonsignalling for every \( M \in I_1 \) and \( \lambda \), and \( P_M(a_{m+1}...a_{n-1}, \{b_i\}_{i\in\overline{M}}\chi, \{y_i\}_{i\in\overline{M}}, \lambda)P_{\overline{M}}(\{b_i\}_{i\in\overline{M}}\{y_i\}_{i\in\overline{M}}, \lambda) \) is local in \( M|\overline{M} \) for every \( M \in I_2 \) and \( \lambda \), then equation (22) is an EPR2 decomposition of \( P_+ \) as in equation (26) 68. Hence, by the choice of measurements in parties \( A, B_{m+1}, ..., B_{n-1} \) which generated \( P_+ \), from equation (27) we have
\[
\sum_{\lambda, M \in I_2} p_L^M(\lambda) < \varepsilon \tag{33}
\]
and so
\[
\sum_{\lambda, M \in I_1} p_L^M(\lambda) > 1 - \varepsilon. \tag{34}
\]
This entails that the only nonnegligible terms in the global distribution \( P \) in equation (28) are those where \( M \in I_1 \).

If, instead, we sum equation (28) over \( a_i, b_i, i = m+1, ..., n-1 \) and use equation (22) again, we get \( P_H \) on the left-hand side. On the right-hand side, recall that the terms where \( M \in I_1 \) are such that \( M \) contains all indices \( i = m+1, ..., n-1 \). Then, \( \overline{M} \) must contain at least some \( i \in \{1, ..., m\} \), as \( \overline{M} \) cannot be empty. Hence, these terms are local in \( A|B_i \) for some \( i = 1, ..., m \). However, the terms where \( M \in I_1 \) are the only nonnegligible ones in \( P_H \), as we have just seen. Therefore, the terms
\[
\sum_{\lambda} p_L^M(\lambda) P_M(a_1...a_m, \{b_i\}_{i\in\overline{M}, i\leq m}\chi, \{y_i\}_{i\in\overline{M}, i\leq m}, \lambda)P_{\overline{M}}(\{b_i\}_{i\in\overline{M}, i\leq m}\{y_i\}_{i\in\overline{M}, i\leq m}, \lambda) \tag{35}
\]
for \( M \in I_1 \) satisfy
\[
\sum_{\lambda, M \in I_1} p_L^M(\lambda) > 1 - \varepsilon. \tag{36}
\]
But this contradicts the fact that \( P_H \) is \((m+1)\)-way nonlocal, as the sum of the terms (35) over \( M \in I_1 \) can be seen as the local component of an EPR2 decomposition of \( P_H \). So from equation (25) and our choice of \( \varepsilon \) we have
\[
\sum_{\lambda, M \in I_1} p_L^M(\lambda) \leq 1 - \varepsilon. \tag{37}
\]
Thus, we reach a contradiction, and so the distribution \( P \) must be GMNL.
In Theorem 11 we assumed that all less-than-maximally entangled states satisfy Hardy’s paradox. This is shown for qubits in 63, and we now extend the proof to any dimension.

**Proposition 1.** Let $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \cong (\mathbb{C}^d)^{\otimes 2}$ be a nonseparable and less-than-maximally entangled pure state. Then, $|\psi\rangle$ satisfies Hardy’s paradox.

**Proof.** Let $|\psi\rangle$ be as in the statement of the Proposition. We present 2-input, 2-output measurements for $|\psi\rangle$ to generate a distribution which satisfies Hardy’s paradox 62, 63 using tools from Ref. 60.

Consider the Schmidt decomposition

$$|\psi\rangle = \sum_{i=0}^{d-1} \lambda_i^{1/2} |ii\rangle$$  \hspace{1cm} (38)

and assume the coefficients are ordered such that $0 \neq \lambda_0 \neq \lambda_1 \neq 0$, which is always possible if the state is nonseparable and less-than-maximally entangled. Wlog assume the Schmidt basis of the state is the canonical basis. Let $\alpha \in [0,\pi/2]$ and $\delta \in \mathbb{R}$ and consider the dual vectors

$$\langle e_{0|0} | = \cos \alpha \langle 0 | + e^{i\delta} \sin \alpha \langle 1 |$$
$$\langle e_{1|1} | = \lambda_0 \cos \alpha \langle 0 | + \lambda_1 e^{i\delta} \sin \alpha \langle 1 |$$
$$\langle f_{0|0} | = \lambda_0^{3/2} e^{i\delta} \sin \alpha \langle 0 | - \lambda_0^{3/2} \cos \alpha \langle 1 |$$
$$\langle f_{1|1} | = \lambda_1^{1/2} e^{i\delta} \sin \alpha \langle 0 | - \lambda_1^{1/2} \cos \alpha \langle 1 |$$  \hspace{1cm} (39)

(one can write the projectors in the Schmidt basis of the state instead of assuming the state decomposes into the canonical basis). Define the measurements $E_{a|x}$ for Alice, with input $x$ and output $a$, and $F_{b|y}$ for Bob, with input $y$ and output $b$, given by

$$E_{0|0} \propto \langle e_{0|0} | e_{0|0} \rangle$$
$$E_{1|0} \propto \langle e_{0|0} | e_{1|1} \rangle^\perp$$
$$E_{0|1} \propto \langle e_{1|1} | e_{0|0} \rangle^\perp$$
$$E_{1|1} \propto \langle e_{1|1} | e_{1|1} \rangle$$
$$F_{0|0} \propto \langle f_{0|0} | f_{0|0} \rangle$$
$$F_{1|0} \propto \langle f_{0|0} | f_{1|1} \rangle^\perp$$
$$F_{0|1} \propto \langle f_{1|1} | f_{1|1} \rangle$$
$$F_{1|1} \propto \langle f_{1|1} | f_{1|1} \rangle$$  \hspace{1cm} (40)

where $\langle e_{0|0} | e_{0|0} \rangle^\perp$ denotes the density matrix corresponding to the vector orthogonal to $|e_{0|0}\rangle$ when restricted to the subspace spanned by $\{|0\rangle, |1\rangle\}$, and $\mathbb{I}_{2\ldots d-1}$ is the identity operator on the subspace spanned by $\{|i\rangle\}_{i=2}^{d-1}$, for either Alice or Bob. Note that, since we are only interested in whether some probabilities are equal or different from zero, normalisation will not play a role.

We now show that the distribution given by

$$P(\alpha \beta | xy) = \text{tr} (E_{a|x} \otimes F_{b|y} |\psi\rangle \langle \psi|)$$  \hspace{1cm} (41)

satisfies Hardy’s paradox. Indeed, because of the probabilities considered and the form of the measurements, only the terms in $i = 0, 1$ contribute to the probabilities that appear in Hardy’s paradox, therefore

$$P(10|10) \propto \left| \sum_{i=0}^{d-1} \lambda_i^{1/2} \langle e_{1|1} | \otimes \langle f_{0|0} | \rangle_{ii} \right|^2 = 0$$
$$P(01|01) \propto \left| \sum_{i=0}^{d-1} \lambda_i^{1/2} \langle e_{0|0} | \otimes \langle f_{1|1} | \rangle_{ii} \right|^2 = 0$$
$$P(00|11) \propto \left| \sum_{i=0}^{d-1} \lambda_i^{1/2} \langle e_{0|0} | \otimes \langle f_{0|0} | \rangle_{ii} \right|^2 = 0.$$  \hspace{1cm} (42)
For $P(00|00)$, we find

$$P(00|00) \propto \left| \sum_{i=0}^{1} \lambda_i^{1/2} (\langle e_0| \otimes \langle f_0| ) |ii\rangle \right|^2$$

$$= |e^{i\delta} \sin \alpha \cos \lambda \lambda_0^{1/2} (\lambda_1 - \lambda_0)|^2,$$

which is strictly greater than zero when $\alpha \in [0, \pi/2]$ and $0 \neq \lambda_0 \neq \lambda_1 \neq 0$, like we assumed. This proves the claim. \(\Box\)

**GMNL from GME**

We fix some notation that we will use in Theorem 2. Both results consider a GME state $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_{n-1}} \cong (\mathbb{C}^d)^{\otimes n}$, $n-1$ copies of which are shared between $n$ parties $A, B_1, \ldots, B_{n-1}$. Each party measures locally on each particle, like in Theorem 1. As before, we denote Alice’s input and output, respectively, as $\chi = x_1 \cdots x_{n-1}$, $\alpha = a_1 \cdots a_{n-1}$ in terms of the digits $x_i, a_i$ corresponding to each particle $i \in [n-1]$. Now, however, all other parties also have $n-1$ particles to measure on. We let the measurement made by party $B_j$ on copy $i$ have input $y_j^i$ and output $b_j^i$, where $i,j = 1, \ldots, n-1$, and for each $j$ we denote $v_j = y_j^1 \cdots y_j^{n-1}$ and $\beta_j = b_j^1 \cdots b_j^{n-1}$ digit-wise. Then, after measurement, the parties share a distribution

$$\{ P(\alpha \beta_1 \cdots \beta_{n-1} | \chi v_1 \cdots v_{n-1}) \}_{\alpha \beta_1 \cdots \beta_{n-1}}.$$  

(44)

Because we are considering local measurements made on each particle, this distribution is of the form

$$P(\alpha \beta_1 \cdots \beta_{n-1} | \chi v_1 \cdots v_{n-1}) = \prod_{i=1}^{n-1} P_i(a_i b_i^1 \cdots b_i^{n-1} | x_i y_i^1 \cdots y_i^{n-1}) ,$$

(45)

where each $P_i$ is the distribution arising from copy $i$ of the state $|\Psi\rangle$.

**Theorem 2.** Any GME state $|\Psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_{n-1}} \cong (\mathbb{C}^d)^{\otimes n}$ is such that $|\Psi\rangle^{\otimes(n-1)}$ is GMNL.

**Proof.** For each copy $i = 1, \ldots, n-1$ of the state $|\Psi\rangle$, we will find measurements for parties $\{ B_j \}_{j \neq i}$ that leave Alice and party $B_i$ with a bipartite entangled state. This will yield a network in a similar configuration to Theorem 1 but conditionalised on the inputs and outputs of these measurements. We will generalise the result of Theorem 1 to show that this network is also GMNL.

Let $i \in [n-1]$ and consider the $i$th copy of $|\Psi\rangle$. Suppose each party $B_j$, $j \neq i$, performs a local, projective measurement onto a basis $\{ |b_j\rangle \}_{b_j=0}^{d-1}$. We pick the computational basis on each party’s Hilbert space to be such that the measurement performed by the parties $B_j$, $j \neq i$, leave Alice and $B_i$ in state $|\phi_{\overrightarrow{b}}\rangle_{AB_i}$, where $\overrightarrow{b} = b_1 \cdots b_{i-1} b_{i+1} \cdots b_{n-1}$ denotes the output obtained by the parties $B_j$, $j \neq i$ (we briefly omit the script $i$ referring to the copy of the state, for readability). This means that we can write the state $|\Psi\rangle$ as

$$|\Psi\rangle = \sum_{\overrightarrow{b}} \lambda_{\overrightarrow{b}} \left| \phi_{\overrightarrow{b}} \right\rangle_{AB_i} \left\langle \overrightarrow{b} \right|_{B_1 \cdots B_{i-1} B_{i+1} \cdots B_{n-1}}.$$  

(46)

Ref. [43], whose proof was completed in Ref. [44], showed that there always exist measurements (i.e. bases) $\{ |b_j\rangle \}_{b_j=0}^{d-1}$ such that $|\phi_{\overrightarrow{b}}\rangle_{AB_i}$ is entangled for a certain output $\overrightarrow{b}$. We now show that this opens up only two possibilities for each $i$: either there exists an output such that $|\phi_{\overrightarrow{b}}\rangle_{AB_i}$ is less-than-maximally entangled, or for all outputs $\overrightarrow{b}$, $|\phi_{\overrightarrow{b}}\rangle_{AB_i}$ is maximally entangled. Indeed, the only option left to discard is one where, for some $\overrightarrow{b} = \overrightarrow{b}^* \left| \phi_{\overrightarrow{b}^*} \right\rangle_{AB_i}$ is maximally entangled, and for some other $\overrightarrow{b} = \overrightarrow{b}^\perp \left| \phi_{\overrightarrow{b}^\perp} \right\rangle_{AB_i}$ is separable. But it is easy to see, by using a continuity argument, that in this case the bases $\{ |b_j\rangle \}_{b_j=0}^{d-1}$ can be modified so that there exists one output for which $AB_i$ are projected onto a less-than-maximally entangled state: it suffices to consider one (normalised) element of the measurement basis to be $c_0 \left| b_j^* \right\rangle + c_1 \left| b_j^\perp \right\rangle$ for some values $c_0, c_1 \in \mathbb{C}$, for each $j$.

Therefore, we consider the following cases:

(i) for all $i \in [n-1]$, there exists an input and output for each $B_j, j \neq i$ such that $|\phi_{i}\rangle_{AB_i}$ is less-than-maximally entangled;
(ii) for all \( i \in [n - 1] \), there exists an input for each \( B_j, j \neq i \) such that \( |\phi_i\rangle_{AB_i} \) is maximally entangled for all outputs;

(iii) there exist \( i, k \in [n - 1] \) such that \( |\phi_i\rangle_{AB_i} \) is as in Case (ii) and \( |\phi_k\rangle_{AB_k} \) is as in Case (i)

**Case (i):** let \( i \in [n - 1] \). Suppose parties \( \{B_j\}_{j \neq i} \) perform the measurements explained above that leave Alice and \( B_i \) less-than-maximally entangled. Then, Alice and \( B_i \) can perform local measurements on the resulting state to satisfy Hardy’s paradox. We will modify the inequality in Theorem (1) and show that these measurements on \( |\Psi\rangle^{\otimes (n-1)} \) give a distribution which violates the inequality.

To modify the inequality in Theorem (1) we observe that parties \( B_i, i = 1, ..., n - 1 \) now have more inputs and outputs, which need to be accounted for. We import the strategy of Theorem (1) to lift inequality (11) to \( n \) parties, each with \( 2^{n-1} \) inputs and outputs. We want \( I^{AB_i} \) to detect bipartite nonlocality between Alice’s ith particle and \( B_i \)’s ith particle, that is, nonlocality in \( a_i b_i^i |x_i y_i^i) \). Therefore, for each \( i \) we now need to fix all other inputs \( x_j, y_i^j, y_j^i \) and add over all other outputs \( a_j, b_j^i, b_j^j, j \neq i \), so that

\[
I_n = \sum_{i=1}^{n-1} I^{AB_i}_{\theta | \theta} + P(\bar{0}, \bar{0} | \bar{0}, \bar{0}) - \sum_{i=1}^{n-1} \sum_{a_i, b_i^i, b_i^j=0,1} P(0, a_i, 1, b_i^i, 0, b_i^j | 0, 0, 0, 0) \leq 0
\]

where the outputs in the first term are denoted as follows: 0, \( a_i \) denotes output \( a = a_1...a_{i-1}a_{i+1}...a_{n-1} \), 0, \( b_i \) denotes output \( b = b_1...b_{i-1}b_{i+1}...b_{n-1} \) for all \( j \neq i \). Inputs are denoted similarly, and the notation is similar for the other three terms. Then, the inequality

\[
I_n = \sum_{i=1}^{n-1} I^{AB_i}_{\theta | \theta} + P(\bar{0}, \bar{0} | \bar{0}, \bar{0}) - \sum_{i=1}^{n-1} \sum_{a_i, b_i^i, b_i^j=0,1} P(0, a_i, 1, b_i^i, 0, b_i^j | 0, 0, 0, 0) \leq 0
\]

is a GMNL inequality, by the same reasoning as in Theorem (1).

Evaluating the inequality on the distribution (45), we find again that each term simplifies. For each \( i \) we get, for example,

\[
P(0, a_i, 1, b_i^i, 0, b_i^j | 0, 0, 0, 0) = P_i(0, 1, 0, 0) = P_i(0, 1, 0, 0) \prod_{j=1}^{n-1} P_j(a_j b_{j+1}^{j+1} b_{j+2}^{j+1} ... b_{n-1}^{j+1} | 0_j 0_j ... 0_j 0_j) \]

and, similarly,

\[
P(0, a_i, 1, b_i^i, 0, b_i^j | 1, 0, 0, 0) = P_i(0, 0, 0) = P_i(0, 0, 0) \prod_{j=1}^{n-1} P_j(0, 0, 0, 1) \]

Also,

\[
P(0, 0, 0, 0, 0, 0) = \prod_{i=1}^{n-1} P_i(0, 0, 0, 0, 0, 0) .
\]

Now each \( P_i \) in equation (45) arises from measurements by \( \{B_j\}_{j \neq i} \) to create a less-than-maximally entangled state between Alice and \( B_i \), who can then choose measurements to satisfy Hardy’s paradox. Hence all terms are zero except \( P(\bar{0}, \bar{0} | \bar{0}, \bar{0}) > 0 \), and so the inequality is violated. Therefore, \(|\Psi\rangle^{\otimes (n-1)} \) is GMNL.
Case (ii) we assumed that, for all $i \in [n - 1]$, there exist local measurements on $|\Psi\rangle$ for parties $\{B_j\}_{j \neq i}$ that, for all outcomes, create a maximally entangled state $|\phi_i\rangle_{AB_i}$ shared between Alice and $B_i$. Since all bipartitions can be expressed as $A|B_i$ for some $i$, we find that $|\Psi\rangle$ meets the requirements of Theorem 2 in [57], and so $|\Psi\rangle$ is GMNL. That is, one copy of the shared state $|\Psi\rangle$ is already GMNL, and therefore so is $|\Psi\rangle^{\otimes(n-1)}$.

Case (iii) assume wlog that the state $|\phi_i\rangle_{AB_i}$ is less-than-maximally entangled for $i = 1, \ldots, m$ and maximally entangled for $i = m + 1, \ldots, n - 1$. We will show that $|\Psi\rangle^{\otimes(m+1)}$ is GMNL, which implies that $|\Psi\rangle^{\otimes(n-1)}$ is so too.

For each $i = 1, \ldots, m$, parties $A_i$ can perform measurements on their shared state $|\phi_i\rangle_{AB_i}$, which, together with the measurements of parties $\{B_j\}_{j \neq i}$ that projected $|\Psi\rangle$ onto $|\phi_i\rangle_{AB_i}$, give rise to a distribution

$$P_i(a_i b_1^i \ldots b_{n-1}^i | x_i y_1^i \ldots y_{n-1}^i)$$

which satisfies Hardy’s paradox when post-selected on the inputs and outputs of parties $\{B_j\}_{j \neq i}$. Then, the distribution arising from the first $m$ copies of $|\Psi\rangle$ is

$$P_H(\{a_i\}_{i \leq m} | \{b_j^i\}_{i \leq m, j \in [n-1]} | \{x_i\}_{i \leq m} | \{y_j^i\}_{i \leq m, j \in [n-1]}) = \prod_{i=1}^{m} P_i(a_i b_1^i \ldots b_{n-1}^i | x_i y_1^i \ldots y_{n-1}^i),$$

with $P_i$ as in equation (52). This distribution is similar to that in Case (i) when post-selected on the inputs and outputs of parties $\{B_j\}_{j > m}$. More precisely, by the nonsignalling condition, we have

$$P_H(\{a_i\}_{i \leq m} | \{b_j^i\}_{i \leq m, j \leq m} | \{b_j^i = 0\}_{i \leq m, j \geq m} | \{x_i\}_{i \leq m} | \{y_j^i\}_{i \leq m, j \leq m} | \{y_j^i = 0\}_{i \leq m, j > m}) = P_{A_1 B_2 \ldots B_m}(\{a_i\}_{i \leq m} | \{b_j^i\}_{i \leq m, j \leq m} | \{b_j^i = 0\}_{i \leq m, j \geq m} | \{y_j^i\}_{i \leq m, j \leq m} | \{y_j^i = 0\}_{i \leq m, j > m})$$

(54)

where by Case (ii) we know that $P_{A_1 B_2 \ldots B_m}$ is GMNL in its parties. Then, $P_H$ must be $(m + 1)$-way nonlocal (i.e., GMNL when restricted to parties $A, B_1, \ldots, B_m$). Indeed, if this were not the case, by equation (54) we could obtain a decomposition of the form (ii) for $P_{A_1 B_2 \ldots B_m}$, which would contradict the fact that this distribution is GMNL.

Therefore, there exists an $\varepsilon > 0$ such that any EPR2 decomposition of $P_H$ as

$$P_H = \sum_{M} p_{H}^{M} P_{L,H}^{M} + p_{NS,H} P_{NS,H}$$

(55)

satisfies

$$\sum_{M} p_{L,H}^{M} \leq 1 - \varepsilon,$$

(56)

where the summation runs over every bipartition $M \otimes M$ that splits $A|B_i$ for $i = 1, \ldots, m$.

On the other hand, $|\Psi\rangle$ satisfies Theorem 1 in Ref. [57] for all bipartitions $A|B_i$ for $i = m + 1, \ldots, n - 1$, hence it is fully-nonlocal across all such bipartitions. This means that, for any $\varepsilon > 0$, there exist local measurements on $|\Psi\rangle$ (which depend on $i$) that lead to a distribution

$$P_+(ab_1 \ldots b_{n-1} | xy_1 \ldots y_{n-1})$$

(57)

such that any bipartite EPR2 decomposition across a bipartition $A|B_i$, for $i = m + 1, \ldots, n - 1$,

$$P_+ = p_{L,+}^{A|B_i} P_{L,+}^{A|B_i} + (1 - p_{L,+}^{A|B_i}) P_{NS,+}^{A|B_i}$$

(58)

satisfies

$$p_{L,+}^{A|B_i} < \delta_i,$$

(59)

Thus, considering the possibility of implementing all the above measurements for each $i$ leads to a distribution of the form (57) in which equation (58) holds for every $i = 1, \ldots, m$.

Therefore, given the $\varepsilon$ above, the parties can choose suitable $\delta_i$ to bound the bipartitely local components and hence ensure that any multipartite EPR2 decomposition of $P_+$,

$$P_+ = \sum_{M} p_{L,+}^{M} P_{L,+}^{M} + p_{NS,+} P_{NS,+}$$

(60)
such that every bipartition $M|\bar{M}$ splits $A|B_i$ for some $i = m + 1, \ldots, n - 1$, satisfies

$$\sum_M p^M_{L,+} < \varepsilon. \quad (61)$$

Since we only need to consider $(m + 1)$ copies of the state, we denote the inputs and outputs of Alice and each party $B_j, j \in [n-1]$ by $\chi = x_1 \ldots x_{m+1}, \upsilon_j = y^j_1 \ldots y^j_{m+1}; \alpha = a_1 \ldots a_{m+1}, \beta_j = b^1_j \ldots b^{m+1}_j$ respectively. Then, the global distribution obtained from $|\Psi\rangle^{\otimes (m+1)}$ is

$$P(\alpha_1 \ldots \beta_{n-1}|\chi \psi_1 \ldots \psi_{n-1}) = P_H(\{a_i \}_{i \leq m}\{b^i_j \}_{i \leq m, j \in [n-1]}|\{x_i \}_{i \leq m}\{y^i_j \}_{i \leq m, j \in [n-1]}) \times P_+(a_{m+1}b^{m+1}_1 \ldots b^{m+1}_{n-1}|x_{m+1}y^{m+1}_1 \ldots y^{m+1}_{n-1}), \quad (62)$$

where $P_H$ comes from equation 55 and the EPR2 components of $P_H, P_+$ are as per equations 60, 61.

We now follow a similar strategy to that in Theorem 1. To prove that the global distribution $P$ is GMNL, we assume the converse, and we derive a contradiction from the nonlocality properties of $P_H$ and $P_+$. Assuming $P$ is not GMNL, we can express the distribution as

$$P(\alpha_1 \ldots \beta_{n-1}|\chi \psi_1 \ldots \psi_{n-1}) = \sum_{\lambda, M} p^M_L(\lambda) P_M(\alpha_1 \beta_j \{j \}_{j \in M}|\chi \psi_j \{j \}_{j \in M}, \lambda) P_{\bar{M}}(\{\psi_j \}_{j \in M}|\{v_j \}_{j \in M}, \lambda), \quad (63)$$

where

$$\sum_{\lambda, M} p^M_L(\lambda) = 1, \quad (64)$$

for each $\alpha, \beta_j, \chi, \upsilon_j, j = 1, \ldots, n-1$, where we recall that each $\beta_j = b^1_j \ldots b^{m+1}_j$ and similarly for $\upsilon_j$. Notice that Alice always belongs to the first element of each bipartition, in order to avoid duplications.

Now, if we sum equation (63) over $a_i, b^i_j$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n-1$ (that is, we sum over the $i$th digit, $i \leq m$, of Alice and all parties $B_j$), we obtain $P_+$ on the left-hand side, from equation 62. On the right-hand side, we obtain, for each $M$, 70

$$\sum_{\lambda} p^M_L(\lambda) P_M(a_{m+1}\{b^j_{m+1} \}_{j \in M}|\chi \psi \{j \}_{j \in M}, \lambda) P_{\bar{M}}(\{\psi_j \}_{j \in M}|\{v_j \}_{j \in M}, \lambda), \quad (65)$$

where we can distinguish two kinds of terms, denoted by sets $I_1$ and $I_2$: for $M \in I_1$, $\bar{M}$ contains only indices $j \in \{1, \ldots, m\}$. For $M \in I_2$, there is some $j \in \{m+1, \ldots, n-1\}$ which belongs to $\bar{M}$, so there is some digit $b^{m+1}_j$ with $j > m$ appearing in $P_{\bar{M}}$. Then, the latter terms are local across $A|B_j$ for some $j > m$, and so summing equation (63) over $M$ gives an EPR2 decomposition of $P_+$ with local component

$$\sum_{\lambda, M \in I_2} p^M_L(\lambda). \quad (66)$$

Therefore, the choice of measurements which generated $P_+$ ensures (by equation (61)) that

$$\sum_{\lambda, M \in I_2} p^M_L(\lambda) < \varepsilon \quad (67)$$

and hence

$$\sum_{\lambda, M \in I_1} p^M_L(\lambda) > 1 - \varepsilon. \quad (68)$$

Going back now to equation (63), we sum over $a_{m+1}, b^{m+1}_j$ for $j = 1, \ldots, n-1$ (that is, we sum over the $(m+1)$th digit of Alice and all parties $B_j$). Then, we obtain $P_H$ on the left-hand side, from equation 62. On the right-hand side, we obtain for each $M$, 84

$$\sum_{\lambda} p^M_L(\lambda) P_M(\{a_i \}_{i \leq m}\{b^i_j \}_{i \leq m, j \in M}|\chi \psi \{j \}_{j \in M}, \lambda) P_{\bar{M}}(\{\psi_j \}_{i \leq m, j \in M}|\{v_j \}_{j \in M}, \lambda). \quad (69)$$
Now, from equation (68), the only nonnegligible terms are those where $M \in I_1$ and, in these terms, all $j \in \{m + 1, \ldots, n - 1\}$ are in $M$. Hence, there must be some $j \leq m$ in $M$, otherwise $M$ would be empty. Therefore, $P_{\not\emptyset\emptyset}$ always contains at least one digit $b_j$ for some $j \leq m$, and so the terms where $M \in I_1$ are local across the bipartition $A|B_j$ for some $j \leq m$. Therefore, there is a possible EPR2 decomposition of $P_H$ where

$$\sum_{\lambda, M \in I_1} p^M_L (\lambda)$$

(70)

is the local component. Then, by equation (68), our choice of $\epsilon$ implies that

$$\sum_{\lambda, M \in I_1} p^M_L (\lambda) \leq 1 - \epsilon,$$

(71)

but this is in contradiction with equation (68). □

[1] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, Quantum Cryptography, Rev. Mod. Phys. 74, 145 (2002), arXiv:quant-ph/0101098.
[2] S. Pirandola, U. L. Andersen, L. Banchi, M. Berta, D. Bunandar, R. Colbeck, D. Englund, T. Gehring, C. Lupo, C. Ottaviani, J. Pereira, M. Razavi, J. S. Shaari, M. Tomamichel, V. C. Usenko, G. Vallone, P. Villoresi, and P. Wallden, Advances in Quantum Cryptography, arXiv Preprint (2019), arXiv:1906.01645.
[3] A. Acín and L. Masanes, Certified randomness in quantum physics, Nature 540, 213 (2016), arXiv:1708.00265.
[4] H. Buhrman, R. Cleve, S. Massar, and R. de Wolf, Nonlocality and communication complexity, Rev. Mod. Phys. 82, 665 (2010), arXiv:0907.3584 [quant-ph].
[5] D. Mayers and A. Yao, Quantum Cryptography with Imperfect Apparatus, in Proceedings of the 39th Annual Symposium on Foundations of Computer Science, FOCS ’98 (IEEE Computer Society, USA, 1998) p. 503, arXiv:quant-ph/9809039.
[6] A. Acín, N. Brunner, N. Gisin, S. Massar, S. Pironio, and V. Scarani, Device-independent security of quantum cryptography against collective attacks, Phys. Rev. Lett. 98, 230501 (2007), arXiv:quant-ph/0702152.
[7] R. Colbeck, Quantum And Relativistic Protocols For Secure Multi-Party Computation, Ph.D. thesis (2011), arXiv:0911.3814.
[8] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Bell nonlocality, Rev. Mod. Phys. 86, 419 (2014), arXiv:1303.2849.
[9] R. F. Werner, Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model, Phys. Rev. A 40, 4277 (1989).
[10] J. Barrett, Nonsequential positive-operator-valued measurements on entangled mixed states do not always violate a Bell inequality, Physical Review A 65, 10.1103/PhysRevA.65.042302 (2002), arXiv:quant-ph/0107045.
[11] N. Gisin, Bell’s inequality holds for all non-product states, Physics Letters A 154, 201 (1991).
[12] N. Gisin and A. Peres, Maximal violation of Bell’s inequality for arbitrarily large spin, Physics Letters A 162, 15 (1992).
[13] L. Aolita, R. Gallego, A. Cabello, and A. Acín, Fully nonlocal, monogamous, and random genuinely multipartite quantum correlations, Phys. Rev. Lett. 108, 100401 (2012), arXiv:1109.3163.
[14] C. Branciard, N. Gisin, and S. Pironio, Characterizing the Nonlocal Correlations Created via Entanglement Swapping, Physical Review Letters 104, 10.1103/PhysRevLett.104.170401 (2010), arXiv:0911.1314 [quant-ph].
[15] D. Cavalcanti, M. L. Almeida, V. Scarani, and A. Acín, Quantum networks reveal quantum nonlocality, Nat Commun 2, 184 (2011), arXiv:1010.0000.
[16] C. Branciard, D. Rosset, N. Gisin, and S. Pironio, Bilocal versus non-bilocal correlations in entanglement swapping experiments, Phys. Rev. A 85, 032119 (2012), arXiv:1112.4502.
[17] N. Gisin, Q. Mei, A. Tavakoli, M. O. Renou, and N. Brunner, All entangled pure quantum states violate the bilocality inequality, Phys. Rev. A 96, 020304 (2017), arXiv:1702.00333.
[18] I. Šupić, P. Skrzypczyk, and D. Cavalcanti, Measurement-device-independent entanglement and randomness estimation in quantum networks, Phys. Rev. A 95, 042340 (2017).
[19] A. Tavakoli, M. O. Renou, N. Gisin, and N. Brunner, Correlations in star networks: From Bell inequalities to network inequalities, New J. Phys. 19, 073003 (2017), arXiv:1702.03866.
[20] M.-O. Renou, E. Bäumer, S. Boreiri, N. Brunner, N. Gisin, and S. Beigi, Genuine Quantum Nonlocality in the Triangle Network, Phys. Rev. Lett. 123, 140401 (2019), arXiv:1905.04902 [quant-ph].
[21] N. Gisin, J.-D. Bancal, Y. Cai, A. Tavakoli, E. Z. Cruzeiro, S. Popescu, and N. Brunner, Constraints on nonlocality in networks from no-signaling and independence, arXiv Preprint (2019), arXiv:1906.06495.
[22] T. Kriváčy, Y. Cai, D. Cavalcanti, A. Tavakoli, N. Gisin, and N. Brunner, A neural network oracle for quantum nonlocality problems in networks, arXiv Preprint (2019), arXiv:1907.10552.
[23] J. Tura, R. Augusiak, A. B. Sainz, T. Vértesi, M. Lewenstein, and A. Acín, Detecting nonlocality in many-body quantum states, Science 344, 1256 (2014), arXiv:1306.6860 [quant-ph].
R. Augusiak, M. Demianowicz, J. Tura, and A. Acín, Entanglement and nonlocality are inequivalent for any number of 

D. Collins, N. Gisin, S. Popescu, D. Roberts, and V. Scarani, Bell-Type Inequalities to Detect True n-Body Nonseparability, 

D. Schmid, D. Rosset, and F. Buscemi, Quantifying Bell nonclassicality across arbitrary resource types, arXiv Preprint 

J. Bowles, J. Francfort, M. Fillettaz, F. Hirsch, and N. Brunner, Genuinely multipartite entangled quantum states with fully 

J. I. de Vicente, On nonlocality as a resource theory and nonlocality measures, J. Phys. A: Math. Theor. 

R. Augusiak, M. Demianowicz, and J. Tura, Construction of genuinely entangled multipartite states with applications to 

M. Seevinck and G. Svetlichny, Bell-Type Inequalities for Partial Separability in N-Particle Systems and Quantum Mechanical Violations, Physical Review Letters 89, 10.1103/PhysRevLett.89.060401 (2002), [arXiv:quant-ph/0201046] 

D. Collins, N. Gisin, S. Popescu, D. Roberts, and V. Scarani, Bell-Type Inequalities to Detect True n-Body Nonseparability, Physical Review Letters 88, 10.1103/PhysRevLett.88.170405 (2002), [arXiv:quant-ph/0201058] 

F. Buscemi, All Entangled Quantum States Are Nonlocal, Phys. Rev. Lett. 108, 200401 (2012), [arXiv:1106.6095 [quant-ph]] 

R. Gallego, L. E. Würflinger, A. Acín, and M. Navascués, Operational Framework for Nonlocality, Phys. Rev. Lett. 109, 070401 (2012) [arXiv:1112.2602 [quant-ph]] 

J.-D. Bancal, J. Barrett, N. Gisin, and S. Pironio, Definitions of multipartite nonlocality, Phys. Rev. A 88, 014102 (2013), [arXiv:1112.2626 [quant-ph]] 

J. Geller and M. Piani, Quantifying non-classical and beyond-quantum correlations in the unified operator formalism, J. Phys. A: Math. Theor. 47, 424030 (2014) [arXiv:1401.8197] 

R. Gallego and L. Aolita, Nonlocality free wirings and the distinguishability between Bell boxes, Phys. Rev. A 95, 032118 (2017), [arXiv:1611.06932 [quant-ph]] 

D. Schmid, D. Rosset, and F. Buscemi, Quantifying Bell nonclassicality across arbitrary resource types, arXiv Preprint (2019), [arXiv:1909.04065] 

E. Wolfe, D. Schmid, A. B. Sainz, R. Kunjwal, and R. W. Spekkens, Quantifying Bell: The Resource Theory of Nonclassicality of Common-Cause Boxes, arXiv Preprint (2019), [arXiv:1903.06311] 

R. Augusiak, M. Demianowicz, J. Tura, and A. Acín, Entanglement and nonlocality are inequivalent for any number of particles, Phys. Rev. Lett. 115, 030404 (2015) [arXiv:1407.3114] 

R. Augusiak, M. Demianowicz, and J. Tura. Constructions of genuinely entangled multipartite states with applications to local hidden variables (LHV) and states (LHS) models, Phys. Rev. A 98, 012321 (2018), [arXiv:1803.00279] 

J. Bowles, J. Francfort, M. Fillettaz, F. Hirsch, and N. Brunner, Genuinely multipartite entangled quantum states with fully local hidden variable models and hidden multipartite nonlocality, Phys. Rev. Lett. 116, 130401 (2016) [arXiv:1511.08401] 

S. Popescu and D. Rohrlich, Generic quantum nonlocality, Physics Letters A 166, 293 (1992) 

M. Gachechiladze and O. Gühne, Completing the proof of “Generic quantum nonlocality”, Physics Letters A 381, 1281 (2017), [arXiv:1607.02948 [quant-ph]] 

Q. Chen, S. Yu, C. Zhang, C. H. Lai, and C. H. Oh, Test of Genuine Multiparticle Nonlocality without Inequalities, Phys. Rev. Lett. 112, 140404 (2014) [arXiv:1305.4472 [quant-ph]] 

S. Yu and C. H. Oh, Tripartite entangled pure states are tripartite nonlocal, arXiv:1306.5330 [quant-ph] (2013), [arXiv:1306.5330 [quant-ph]] 

M. Navascués and T. Vértesi, Activation of Nonlocal Quantum Resourses, Physical Review Letters 106, 10.1103/PhysRevLett.106.060403 (2011), [arXiv:1010.5191 [quant-ph]] 

C. Palazuelos, Super-activation of quantum non-locality, Phys. Rev. Lett. 109, 190401 (2012) [arXiv:1205.3118] 

P. Caban, A. Molenda, and K. Trzcinka, Activation of the violation of the Svetlichny inequality, Physical Review A 92, 10.1103/PhysRevA.92.032119 (2015). 

N. Gisin, Hidden quantum nonlocality revealed by local filters, Physics Letters A 210, 151 (1996) 

L. Masanes, Asymptotic Violation of Bell Inequalities and Distillability, Phys. Rev. Lett. 97, 050503 (2006), [arXiv:quant-ph/0512153] 

L. Masanes, Y.-C. Liang, and A. C. Doherty, All bipartite entangled states display some hidden nonlocality, Phys. Rev. Lett. 100, 090403 (2008), [arXiv:quant-ph/0703268] 

F. Hirsch, M. T. Quintino, J. Bowles, and N. Brunner, Genuine Hidden Quantum Nonlocality, Physical Review Letters 111, 10.1103/PhysRevLett.111.160402 (2013), [arXiv:1307.4404 [quant-ph]] 

F. Hirsch, M. T. Quintino, J. Bowles, T. Vértesi, and N. Brunner, Entanglement without hidden nonlocality, New Journal of Physics 18, 113019 (2016), [arXiv:1606.02215 [quant-ph]] 

S. Pironio, Lifting Bell inequalities, Journal of Mathematical Physics 46, 062112 (2005), [arXiv:quant-ph/0503179] 

A. C. Elitzur, S. Popescu, and D. Rohrlich, Quantum nonlocality for each pair in an ensemble, Physics Letters A 162, 25 (1992) 

M. L. Almeida, D. Cavalcanti, V. Scarani, and A. Acín, Multiparticle fully-nonlocal quantum states, Phys. Rev. A 81, 052111 (2010), [arXiv:0911.3559]
Not to be confused with “nonfully local”, which is the opposite of “fully local”. “Fully-nonlocal” is a particular case of “nonfully local”.

See Supplemental Material for full proofs of the results in the main text.

F. J. Curchod, M. L. Almeida, and A. Acín, A versatile construction of Bell inequalities for the multipartite scenario, New J. Phys. 21, 023016 (2019), arXiv:1808.10688 [quant-ph]

J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Proposed Experiment to Test Local Hidden-Variable Theories, Phys. Rev. Lett. 23, 880 (1969)

L. Hardy, Quantum mechanics, local realistic theories, and Lorentz-invariant realistic theories, Phys. Rev. Lett. 68, 2981 (1992)

L. Hardy, Nonlocality for two particles without inequalities for almost all entangled states, Phys. Rev. Lett. 71, 1665 (1993)

J. Barrett, A. Kent, and S. Pironio, Maximally Nonlocal and Monogamous Quantum Correlations, Physical Review Letters 97, 10.1103/PhysRevLett.97.170409 (2006), arXiv:quant-ph/0605182

M. Navascues, E. Wolfe, D. Rosset, and A. Pozas-Kerstjens, Genuine Network Multipartite Entanglement, arXiv Preprint (2020), arXiv:2002.02773.

Note that when summing over only some of the digits of Alice’s output, $P_M$ still depends on the whole input $\chi$ as the distribution may be signalling in the different digits of Alice’s input.

However, while each of the terms on the right-hand side may depend on the whole of Alice’s input $\chi$, the left-hand side does not, because the distribution is of the form (22). That is, equation (32) holds for any fixed value of the inputs $\{x_i\}_{i \leq m}, \{y_i\}_{i \leq m}$ on the left- and right-hand sides.

Note that in equation (32) the left-hand side does not depend on $x_1, \ldots, x_m$ while the right-hand side does. We can fix $x_1 = \cdots = x_m = 0$ on the right-hand side and the equality remains true.

Again, these may depend on $x_{m+1}, \ldots, x_{n-1}$.

Note that, once more, the distribution obtained by summing over only some of the digits of a party’s output still depends on the whole input as it may be signalling in the different digits of the party’s input. However, as in Theorem 1 these extra inputs can be fixed to an arbitrary value as the left-hand side is independent of them.