Hyperbolic trigonometry in two-dimensional space-time geometry

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Summary.- By analogy with complex numbers, a system of hyperbolic numbers can be introduced in the same way: \( \{ z = x + h y; h^2 = 1 \} \). As complex numbers are linked to the Euclidean geometry, so this system of numbers is linked to the pseudo-Euclidean plane geometry (space-time geometry).

In this paper we will show how this system of numbers allows, by means of a Cartesian representation, an operative definition of hyperbolic functions using the invariance respect to special relativity Lorentz group. From this definition, by using elementary mathematics and an Euclidean approach, it is straightforward to formalise the pseudo-Euclidean trigonometry in the Cartesian plane with the same coherence as the Euclidean trigonometry.

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1 Introduction

Complex numbers are strictly related to the Euclidean geometry: indeed their invariant (the module) is the same as the Pythagoric distance (Euclidean invariant) and their unimodular multiplicative group is the Euclidean rotation group. It is well known that these properties allow to use complex numbers for representing plane vectors.

In the same way hyperbolic numbers, an extension of complex numbers [1, 2] defined as

\[ \{ z = x + h y; h^2 = 1 \} \]

are strictly related to space-time geometry [2, 3, 4]. Indeed their square module given by\(^1\) \( |z|^2 = z \bar{z} \equiv x^2 - y^2 \) is the Lorentz invariant of two dimensional special relativity, and their unimodular multiplicative group is the special relativity Lorentz group [2]. These relations have been used to extend special relativity [5]. Moreover by using the functions of hyperbolic variable the two-dimensional special relativity has been generalised [3]. These applications make the hyperbolic numbers relevant for physics and stimulate the application of hyperbolic numbers, just as complex numbers are applied to the Euclidean plane geometry [2]. In this paper we present the basic concepts of space-time trigonometry which we derive from the remark that hyperbolic (complex) numbers allow

\(^1\)We call \( \bar{z} = x - h y \) the hyperbolic conjugate of \( z \) as for complex numbers.
to introduce two invariant quantities in respect to Lorentz (Euclid) group. The first invariant is the scalar product, recently considered by Fjelstad and Gal [6] as the basis of their paper. The second one is equivalent to the module of the vector product (i.e., an area). These two invariant quantities allow to define, in a Cartesian representation, the trigonometric hyperbolic functions. These functions are defined in the whole hyperbolic plane and allow to solve triangles having sides of whatever direction\(^2\). Then the space-time trigonometry and geometry can be formalised in a self-consistent axiomatic-deductive way, that can be considered equivalent to Euclidean geometry construction. More precisely we start from the experimental axiom that the Lorentz transformations do hold and we look for the geometrical “deductions”.

It is well known that all the theorems of the Euclidean trigonometry are obtained through elementary geometry observations. In fact once we define in a Cartesian plane the trigonometric functions as a direct consequence of Euclid rotation group (as shown in the appendix A), all the trigonometry theorems follow just as mathematical identities. In our work, since we do not have the same intuitive vision for the pseudo-Euclidean geometry, we make up for this lack of evidence by using an algebraic approach and just checking the validity of the hyperbolic trigonometry theorems by elementary mathematics. This description can be considered similar to the Euclidean representations of the non-Euclidean geometries obtained in XIX century, i.e., to the E. Beltrami’s interpretation, on constant curvature surfaces, of the non-Euclidean geometries [7].

The paper is organised in the following way: in section 2 some properties of hyperbolic numbers are briefly resumed. In section 3 the trigonometric hyperbolic functions are derived as a consequence of invariance with respect to Lorentz group. In section 4 the trigonometry in the pseudo-Euclidean plane is formalised. Three appendixes are added to clarify some specific aspects.

2 Hyperbolic numbers

2.1 Basic concepts

Here we briefly resume some fundamental properties of hyperbolic numbers. This number system has been introduced by S. Lie [8] as a two dimensional example of the more general class of the commutative hypercomplex number systems\(^3\). In more recent years the hyperbolic numbers have been considered by B. Chabat [1] for studying ultrasonic phenomena, and by P. Fjelstad [5] who called them “Perplex numbers”, for extending the special relativity to represent the superluminal phenomena.

Let us now introduce a hyperbolic plane by analogy with the Gauss-Argand plane of the complex variable. In this plane we associate the points \( P \equiv (x, y) \) to hyperbolic numbers \( z = x + h y \). If we represent these numbers on a Cartesian plane, in this plane the square distance of the point \( P \) from \( \ldots \)

\(^2\)We shall see that only side directions parallel to axes bisectors have to be excluded.

\(^3\)Hypercomplex numbers [8, 9] are defined by the expression: \( x = \sum_{\alpha=0}^{N-1} e_{\alpha} x^\alpha \) where \( x^\alpha \in \mathbb{R} \) are called components and \( e_{\alpha} \in \mathbb{R} \) units or versors, as in vector algebra. This expression defines a hypercomplex number if the versors multiplication rule is given by a linear combination of versors: \( e_{\alpha} e_{\beta} = \sum_{\gamma=0}^{N-1} C_{\alpha\beta}^\gamma e_{\gamma} \) where \( C_{\alpha\beta}^\gamma = C_{\beta\alpha}^\gamma \) are real constants, called structure constants, that define the characteristics of the system [9].

The versor product defines also the product of hypercomplex numbers. This product definition makes the difference between vector algebra and hypercomplex systems and allows to relate the hypercomplex numbers to groups. In fact the vector product is not, in general, a vector while the product of hypercomplex numbers is still a hypercomplex number; the same for the division, that for vectors does not exist while for hypercomplex numbers, in general, does exist.
the origin of the coordinate axes is defined as
\[ D = z \bar{z} \equiv x^2 - y^2. \] (1)

The definition of distance (metric element) is equivalent to introduce the bilinear form of the scalar product. The scalar product and the properties of hypercomplex numbers allow to state suitable axioms [2, p. 245] and to give to the pseudo-Euclidean plane the structure of a vector space.

Let us consider the multiplicative inverse of \( z \) that, if it exists, is given by: \( 1/z \equiv \bar{z}/z \bar{z} \). This implies that \( z \) does not have an inverse when \( z \bar{z} \equiv x^2 - y^2 = 0 \), i.e., when \( y = \pm x \), or alternatively when \( z = x \pm h x \). These two straight-lines in the hyperbolic plane, whose elements have no inverses, divide the hyperbolic plane in four sectors that we shall call Right sector (Rs), Up sector (Us), Left sector (Ls), and Down sector (Ds). This property is the same as that of the special relativity representative plane and this correspondence gives a physical meaning (space-time interval) to the definition of distance. Let us now consider the quantity \( x^2 - y^2 \), which is positive in the Rs, Ls (\(|x| > |y|\)) sectors, and negative in the Us, Ds (\(|x| < |y|\)) sectors. This quantity, as known from special relativity, must have its sign and appear in this quadratic form. In the case we should use the linear form (the module of hyperbolic numbers), we follow the definition of I. M. Yaglom [2, pag. 180] and B. Chabat [1, p. 51], [10, p. 72]:
\[ \rho = \sqrt{|z \bar{z}|} \equiv \sqrt{|D|} \] (2)
where \(|D|\) is the absolute value of the square distance.

2.2 Hyperbolic exponential function and hyperbolic polar transformation

The hyperbolic exponential function in the pseudo-Euclidean geometry plays the same important role as the complex exponential function in the Euclidean geometry. By comparing absolutely convergent series it can be written [1, 5]:

if \(|x| > |y|\):
\[ x + h y = \text{sign}(x) \exp[\rho' + h \theta] \equiv \text{sign}(x) \exp[\rho'(\cosh \theta + h \sinh \theta)] \] (3)

if \(|x| < |y|\):
\[ x + h y = \text{sign}(y) \exp[\rho' + h \theta] \equiv \text{sign}(y) \exp[\rho'(\sinh \theta + h \cosh \theta)]. \] (4)

The exponential function allows to introduce the hyperbolic polar transformation. Following [1, 5] we define the radial coordinate as:
\[ \exp[\rho'] \Rightarrow \rho = \sqrt{|x^2 - y^2|} \]
and the angular coordinate as:
for \(|x| > |y|\) : \( \theta = \tanh^{-1}(y/x) \); for \(|x| < |y|\) : \( \theta = \tanh^{-1}(x/y) \).

Then for \(|x| > |y|\), \( x > 0 \) (i.e., \( x, y \in Rs \)), the hyperbolic polar transformation is defined as:
\[ x + h y \Leftrightarrow \rho \exp[h \theta] \equiv \rho(\cosh \theta + h \sinh \theta). \] (5)

Eq. (5) represents the map for \( x, y \in Rs \): the map of the complete \( x, y \) plane is reported in tab. 1.

3 Basis of hyperbolic trigonometry

3.1 Hyperbolic rotation invariants in the pseudo-Euclidean plane geometry

By analogy with the Euclidean trigonometry approach summarised in appendix (A), we can say that the pseudo-Euclidean plane geometry studies the properties that are invariant by Lorentz transfor-
Table 1: Map of the complete $x$, $y$ plane by hyperbolic polar transformation

| Right sector (Rs) | Left sector (Ls) | Up sector (Us) | Down sector (Ds) |
|-------------------|------------------|----------------|------------------|
| $|x| > |y|$          | $|x| < |y|$          |                  |                  |
| $z = \rho \exp[h \theta]$ | $z = -\rho \exp[h \theta]$ | $z = h \rho \exp[h \theta]$ | $z = -h \rho \exp[h \theta]$ |
| $x = \rho \cosh \theta$ | $x = -\rho \cosh \theta$ | $x = \rho \sinh \theta$ | $x = -\rho \sinh \theta$ |
| $y = \rho \sinh \theta$ | $y = -\rho \sinh \theta$ | $y = \rho \cosh \theta$ | $y = -\rho \cosh \theta$ |

Figure 1: For $\rho = 1$ the $x$, $y$ in tab. (1) represent the four arms of equilateral hyperbolas $|x^2 - y^2| = 1$. Here we indicate how each arm is traversed as parameter $\theta$ goes from $-\infty$ to $+\infty$.

motions (Lorentz-Poincarè group of special relativity) corresponding to hyperbolic rotation [2, 3]. We show afterwards, how these properties can be represented by hyperbolic numbers.

Let us define in the hyperbolic plane a hyperbolic vector from the origin to the point $P \equiv (x, y)$, as $v = x + h y$: a hyperbolic rotation of an angle $\theta$ transforms this vector in a new vector $v' \equiv v \exp[h \theta]$. Therefore we can readily verify that the quantity:

$$|v'|^2 \equiv v' \bar{v}' = v \exp[h \theta] \bar{v} \exp[-h \theta] \equiv |v|^2$$

(6)

is invariant by hyperbolic rotation. In a similar way we can find two invariants related to any couple of vectors. Let us consider two vectors $v_1 = x_1 + h y_1$ and $v_2 = x_2 + h y_2$: we have that the real and the hyperbolic parts of the product $v_2 \bar{v}_1$ are invariant by hyperbolic rotation.

In fact $v_2' \bar{v}_1' = v_2 \exp[h \alpha] \bar{v}_1 \exp[-h \alpha] \equiv v_2 \bar{v}_1$. These two invariants allow an operative definition of the hyperbolic trigonometric functions. To show this let us suppose that $|x_1| > |y_1|$, $|x_2| > |y_2|$ and $x_1, x_2 > 0$, and let us represent the two vectors in hyperbolic polar form: $v_1 = \rho_1 \exp[h \theta_1]$, $v_2 = \rho_2 \exp[h \theta_2]$. Consequently we have

$$v_2 \bar{v}_1 \equiv \rho_1 \rho_2 \exp[h (\theta_2 - \theta_1)] \equiv \rho_1 \rho_2 [\cosh(\theta_2 - \theta_1) + h \sinh(\theta_2 - \theta_1)].$$

(7)

As shown in appendix (A) for the Euclidean plane, the real part of the vector product represents the scalar product, while the imaginary part represents the area of the parallelogram defined by the two
vectors. In the pseudo-Euclidean plane, as we know from differential geometry [11], the real part is still the scalar product; as far as the hyperbolic part is concerned, we will see in section (4.2) that it can be considered as a pseudo-Euclidean area.

In Cartesian coordinates we have:

$$v_2 \tilde{v}_1 = (x_2 + h y_2) (x_1 - h y_1) \equiv x_1 x_2 - y_1 y_2 + h(x_1 y_2 - x_2 y_1).$$

By using eqs. (7) and (8) we obtain:

$$\cosh(\theta_2 - \theta_1) = \frac{x_1 x_2 - y_1 y_2}{\rho_1 \rho_2} \equiv \frac{x_1 x_2 - y_1 y_2}{\sqrt{|(x_2^2 - y_2^2)|(x_1^2 - y_1^2)|}},$$

$$\sinh(\theta_2 - \theta_1) = \frac{x_1 y_2 - x_2 y_1}{\rho_1 \rho_2} \equiv \frac{x_1 y_2 - x_2 y_1}{\sqrt{|(x_2^2 - y_2^2)|(x_1^2 - y_1^2)|}}.$$  \hfill (9)  

If we put $v_1 \equiv (1; 0)$ and $\theta_2, x_2, y_2 \to \theta, x, y$ then eqs. (9), (10) become:

$$\cosh \theta = \frac{x}{\sqrt{x^2 - y^2}}; \quad \sinh \theta = \frac{y}{\sqrt{x^2 - y^2}}.$$  \hfill (11)  

The classic hyperbolic functions are defined for $x, y \in R$. We observe that expressions in eq. (11) are valid for $\{x, y \in R | x \neq \pm y\}$ so they allow to extend the hyperbolic functions in the complete $x, y$ plane. This extension is the same as that already proposed in [5, 6], that we recall in appendix (B). In the following of this paper we will denote with $\cosh_e, \sinh_e$ these extended hyperbolic functions.

In tab. 2 the relations between $\cosh_e, \sinh_e$ and traditional hyperbolic functions are reported.

By this extension the hyperbolic polar transformation, eq. (5), is given by

$$x + h y \Rightarrow \rho(\cosh_\theta + h \sinh_\theta),$$

from which, for $\rho = 1$, we obtain the extended hyperbolic Euler formula [5]:

$$\exp_e[h \theta] = \cosh_\theta + h \sinh_\theta.$$  \hfill (13)  

Table 2: Relations between functions $\cosh_e, \sinh_e$ obtained from eq. (11) and classic hyperbolic functions. The hyperbolic angle $\theta$ in the last four columns is calculated referring to semi-axes $x, -x, y, -y$, respectively.

| $|x| > |y|$ | $|x| < |y|$ |
|----------------|----------------|
| $(Rs), x > 0$ | $(Us), y > 0$ |
| $(Ls), x < 0$ | $(Ds), y < 0$ |
| $\cosh_e \theta =$ | $\cosh \theta$ | $- \cosh \theta$ | $\sinh \theta$ | $- \sinh \theta$ |
| $\sinh_e \theta =$ | $\sinh \theta$ | $- \sinh \theta$ | $\cosh \theta$ | $- \cosh \theta$ |

From Tab. 2 or equations (11) it follows that:

$$\text{for } |x| > |y| \Rightarrow \cosh_e^2 - \sinh_e^2 = 1;$$
$$\text{for } |x| < |y| \Rightarrow \cosh_e^2 - \sinh_e^2 = -1$$  \hfill (14)  

5
The complete representation of the extended hyperbolic functions can be obtained by giving to \( x, y \) all the values on the circle \( x = \cos \phi, y = \sin \phi \) for \( 0 \leq \phi < 2\pi \): in this way eqs. (11) become:

\[
\cosh_e \theta = \frac{\cos \phi}{\sqrt{|\cos 2\phi|}} \equiv \frac{1}{\sqrt{|1 - \tan^2 \phi|}}; \quad \sinh_e \theta = \frac{\sin \phi}{\sqrt{|\cos 2\phi|}} \equiv \frac{\tan \phi}{\sqrt{|1 - \tan^2 \phi|}}.
\]

These equations represent a bijective mapping between the points on unit circle (specified by \( \phi \)) and the points on unit hyperbolas (specified by \( \theta \)). From a geometrical point of view eq. (15) represent the projection, from the coordinate axes origin, of the unit circle on the unit hyperbolas. A graph of the function \( \cosh_e \) is reported in fig. 2.

The fact that the extended hyperbolic functions can be represented in terms of just one expression (eqs. 11) allows a direct application of these functions for the solution of triangles with sides in any direction, except the directions parallel to axes bisectors.

### 3.2 The pseudo-Euclidean Cartesian plane

Now we show how some classical definitions and properties of the Euclidean plane must be restated for the pseudo-Euclidean plane.

- **Definitions.**

  Given two points \( P_j \equiv z_j \equiv (x_j, y_j), \ P_k \equiv z_k \equiv (x_k, y_k) \) we define the "square distance" between them by extending eq. (1):

  \[
  D_{j,k} = (z_j - z_k)(\bar{z}_j - \bar{z}_k).
  \]
As a general rule we indicate the square segment lengths by capital letters, and by the same small letters the square root of their absolute value.

\[ d_{j,k} = \sqrt{D_{j,k}}. \]  

(17)

Following [2, pag. 179], a segment or line is said to be of the first (second) kind if it is parallel to a line through the origin located in the sectors containing the axis Ox (Oy). Then the segment \( \overline{P_jP_k} \) is of the first (second) kind if \( D_{j,k} > 0 \) (\( D_{j,k} < 0 \))\(^4\).

- **Straight-lines equations.**

In the pseudo-Euclidean plane, the equation of geodesics (straight-lines) from a point \( P_0 \equiv (x_0, y_0) \) can be obtained from differential geometry [4, 11]. So for \( |x - x_0| > |y - y_0| \), a straight-line (of the first kind) can be written as:

\[(x - x_0) \sinh \theta - (y - y_0) \cosh \theta = 0; \]  

(18)

while for \( |x - x_0| < |y - y_0| \) a straight-line (of the second kind) can be written as\(^5\):

\[(x - x_0) \cosh \theta' - (y - y_0) \sinh \theta' = 0. \]  

(19)

Then from the topological characteristics of the pseudo-Euclidean plane, it follows that the straight-lines will have the two possible expressions given by eqs. (18, 19).

The use of extended hyperbolic functions would give just one equation for all the straight-lines, but in this paragraph and in the next one, we use the classical hyperbolic trigonometric functions which make more evident the peculiar characteristics of the pseudo-Euclidean plane.

- **Pseudo-orthogonality.**

As in the Euclidean plane, two straight-lines in the pseudo-Euclidean plane are said pseudo-orthogonal when the scalar product of their versors is zero [11]. It is easy to show that two straight-lines of the same kind, as given by eqs. (18) or (19), can never be pseudo-orthogonal. Indeed a straight-line of the first kind (eq. 18) has a pseudo-orthogonal line of the second kind (eq. 19) and with the same angle \( \theta \) and \( \theta' \), and conversely\(^6\). Then, as it is well known in special relativity [12, p. 479], [13], two straight-lines are pseudo-orthogonal if they are symmetric with respect to a couple of lines parallel to the axes bisectors (fig. 3).

- **Axis of a segment.**

Let us consider two points \( P_1(x_1, y_1), P_2(x_2, y_2) \). The points that have the same pseudo-Euclidean distance from these two points are determined by the equation:

\[ \overline{PP}_1^2 = \overline{PP}_2^2 \Rightarrow (x - x_1)^2 - (y - y_1)^2 = (x - x_2)^2 - (y - y_2)^2. \]

This implies that:

\[(x_1 - x_2)(2x - x_1 - x_2) = (y_1 - y_2)(2y - y_1 - y_2), \]  

(20)

and in canonical form:

\[ y = \frac{(x_1 - x_2)}{(y_1 - y_2)} x + \frac{(y_1^2 - y_2^2) - (x_1^2 - x_2^2)}{2(y_1 - y_2)}. \]  

(21)

\(^4\)If we give to \( x \) the physical meaning of a time variable, these lines correspond to the spacelike (timelike) lines, as defined in special relativity [13], [2, pag. 251]

\(^5\)the angle \( \theta' \) is referred to \( y \) axis as stated in tab. 2

\(^6\)It is known that in the complex formalism, the equation of a straight-line is given by \( \Re\{x + iy)(\exp[i\phi]) + \phi \} \), and its orthogonal line by \( \Im\{x + iy)(\exp[i\phi]) + \phi \} \). In the pseudo-Euclidean plane in the hyperbolic formalism the equation of a straight-line is \( \Re\{x + hy)(\exp[i\theta]) + \phi \} \), and the hyperbolic part \( \Im\{x + hy)(\exp[i\theta]) + \phi \} \) is its pseudo-orthogonal line. Note that the product of the angular coefficients for two pseudo-orthogonal lines is +1.
eqs. (20) and (21) show that the axis of a segment in the pseudo-Euclidean plane has the same properties as in the Euclidean plane: i.e., it is pseudo-orthogonal to the segment \( P_1P_2 \) in its middle point.

- **Distance of a point from a straight-line.**

Let us take a point \( P(x, y) \) on a straight-line of the second kind \( \gamma : \{ y - mx - q = 0; \ |m| > 1 \} \), and a point \( P_1(x_1, y_1) \) outside the straight-line. The “square distance” \( PP_1^2 = (x - x_1)^2 - (y - y_1)^2 \) has its extreme for \( x \equiv x_2 = (x_1 - my_1 - mq)/(1 - m^2) \), with a square distance:

\[
D_1,2 \equiv P_2P_1^2 = \frac{(y_1 - mx_1 - q)^2}{m^2 - 1} \quad \text{and} \quad d_{1,2} = \frac{|y_1 - mx_1 - q|}{\sqrt{|m^2 - 1|}}.
\]

It is easy to control that this distance correspond to a maximum as it is well known from special relativity [13], [14, p. 315].

From the expression (22) it follows that: the linear distance from the point \( P_1 \) to the straight line \( \gamma \), is proportional to the result of substituting the coordinates of \( P_1 \) in the equation of \( \gamma \), as well as in Euclidean geometry.

The equation of the straight-line \( P_1, P_2 \) is:

\[
(y - y_1) = \frac{1}{m}(x - x_1)
\]

that represents a straight-line pseudo-orthogonal to \( \gamma \).

### 4 Trigonometry in the pseudo-Euclidean plane

#### 4.1 Goniometry

By using the hyperbolic Euler formula (eq. 13) we can derive the hyperbolic angles addition formulas [5, 6]:

\[
\cosh_e[\alpha \pm \beta] + h \sinh_e[\alpha \pm \beta] = (\cosh_e[\alpha] + h \sinh_e[\alpha])(\cosh_e[\beta] \pm h \sinh_e[\beta]).
\]

These formulas allow to obtain for hyperbolic trigonometric functions all the expressions that are equivalent to the Euclidean goniometry ones.

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**Figure 3:** Two pseudo-orthogonal straight-lines
4.2 Trigonometry

Let us consider a triangle in the pseudo-Euclidean plane with no sides parallel to axes bisectors: let us call \( P_n \equiv (x_n, y_n) \) \( n = i, j, k \mid i \neq j \neq k \) the vertices, \( \theta_n \) the hyperbolic angles. The square hyperbolic length of the side opposite to vertex \( P_i \) is defined by eq. (16):

\[
D_i \equiv D_{j,k} = (z_j - z_k)(\bar{z}_j - \bar{z}_k) \text{ and } d_i = \sqrt{|D_i|}.
\]  

(24)

as pointed out before \( D_i \) must be taken with its sign.

Following the conventions of the Euclidean trigonometry we associate to the sides three vectors oriented from \( P_1 \to P_2; \ P_1 \to P_3; \ P_2 \to P_3 \).

From (9), (10) and taking into account the sides orientation as done in the Euclidean trigonometry, we obtain:

\[
\begin{align*}
\cosh_e \theta_1 &= \frac{(x_2 - x_1)(x_3 - x_1) - (y_2 - y_1)(y_3 - y_1)}{d_2 d_3} ; \\
\sinh_e \theta_1 &= \frac{(x_2 - x_1)(y_3 - y_1) - (y_2 - y_1)(x_3 - x_1)}{d_2 d_3} ; \\
\cosh_e \theta_2 &= \frac{(x_3 - x_2)(x_2 - x_1) - (y_3 - y_2)(y_2 - y_1)}{d_1 d_2} ; \\
\sinh_e \theta_2 &= \frac{(x_3 - x_2)(y_3 - y_2) - (y_2 - y_1)(x_3 - x_2)}{d_1 d_2} ; \\
\cosh_e \theta_3 &= \frac{(x_3 - x_1)(x_3 - x_1) - (y_3 - y_2)(y_2 - y_1)}{d_1 d_3} ; \\
\sinh_e \theta_3 &= \frac{(x_3 - x_1)(y_3 - y_2) - (y_2 - y_1)(x_3 - x_2)}{d_1 d_3} .
\end{align*}
\]  

(25)

It is straightforward to verify that all the functions \( \sinh_e \theta_n \) have the same numerator. If we call this numerator:

\[
x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) = 2S
\]  

(26)

we can write:

\[
2S = d_2 d_3 \sinh_e \theta_1 = d_1 d_3 \sinh_e \theta_2 = d_1 d_2 \sinh_e \theta_3.
\]  

(27)

In the Euclidean geometry a quantity equivalent to \( S \) represents the area of the triangle. In the pseudo-Euclidean geometry \( S \) is still an invariant quantity linked to the triangle. For this reason it is appropriate to call \( S \) the pseudo-Euclidean area [2]. We note that the expression of area (eq. 26), in terms of vertices coordinates, is exactly the same as in the Euclidean geometry (Gauss formula for a polygon area applied to a triangle).

Now we are able to restate the trigonometric laws in the pseudo-Euclidean plane.

• Law of sines.

In a triangle the ratio of the hyperbolic sine to the hyperbolic length of the opposite side is constant:

\[
\frac{\sinh_e \theta_1}{d_1} = \frac{\sinh_e \theta_2}{d_2} = \frac{\sinh_e \theta_3}{d_3}.
\]  

(28)

Proof. This theorem follows from eq. (27) if we divide it by \( d_1 d_2 d_3 \).

Then if two hyperbolic triangles have the same hyperbolic angles, they will have their sides proportional.

• Laws of cosines.
From the definitions of the side lengths (eq. 2) and hyperbolic angular functions given by eqs. (25) it is easy to verify that:

\[ D_i = D_j + D_k - 2d_j d_k \cosh_e \theta_i, \]  

(29)

and:

\[ d_i = |d_j \cosh_e \theta_k + d_k \cosh_e \theta_j|. \]  

(30)

- **Pythagoras theorem.**

Let us take in eq. (29) \( \cosh_e \theta_i = 0 \). From eq. (25) it follows:

\[ (x_j - x_i)(x_k - x_i) = (y_j - y_i)(y_k - y_i) \Rightarrow \frac{y_j - y_i}{x_j - x_i} \equiv m_{ij} = \frac{1}{m_{ik}} \equiv \frac{x_k - x_i}{y_k - y_i}. \]  

(31)

The side \( P_i P_j \) is pseudo-orthogonal to the side \( P_i P_k \), i.e. the triangle is a right triangle and we obtain the hyperbolic Pythagoras theorem:

\[ D_i = D_j + D_k. \]  

(32)

We have seen that the topology of the pseudo-Euclidean plane is more complex with respect to the Euclidean one, as well as the relations between \( \sinh_e \) and \( \cosh_e \) and between the side lengths and the square lengths of the sides. For these reasons we could think that the triangle solution would require more information, nevertheless the results of appendix (C) show that as in the Euclidean trigonometry: all the sides and angles of a triangle can be determined if we know three elements (with at least one side).

It also follows the **Theorem** all the elements of a triangle are invariant for hyperbolic rotation.

**Proof.** Let us consider a triangle with vertices in the points

\[ P_1 \equiv (0, 0), P_2 \equiv (x_2, 0) \text{ and } P_3 \equiv (x_3, y_3) : \]  

(33)

since \( P_1 P_3 \equiv d_2 \) and \( P_1 P_2 \equiv d_3 \) are invariant quantities, from eq. (7) follows that \( \theta_1 \) is invariant too. Since these three elements determine all the others, all the elements will be invariant.

From the exposed invariance it follows that, by the coordinate axes translation and hyperbolic rotation, any triangle can be put with a vertex in \( P \equiv (0, 0) \), and a side on one coordinate axis.

Then we do not loss in generality if, from now on, we consider a triangle in a position that will facilitate the control of the theorems which follow. Consequently we will consider the triangle with vertices in the points given by eq. (33) and hyperbolic square lengths:

\[ D_1 = (x_3 - x_2)^2 - y_3^2; \quad D_2 = x_3^2 - y_3^2; \quad D_3 = x_2^2. \]  

(34)

By using eqs. (2), (25) we obtain the other elements:

\[ \cosh_e \theta_1 = \frac{x_2x_3}{d_2d_3}; \quad \cosh_e \theta_2 = \frac{x_2(x_2 - x_3)}{d_1d_3}; \quad \cosh_e \theta_3 = \frac{x_3(x_3 - x_2) - y_3^2}{d_1d_2} \]  

\[ \sinh_e \theta_1 = \frac{x_2y_3}{d_2d_3}; \quad \sinh_e \theta_2 = \frac{x_2y_3}{d_1d_3}; \quad \sinh_e \theta_3 = \frac{x_2y_3}{d_1d_2}. \]  

(35)

\(^7\)Obtained by another method in [6].
4.3 The triangle angles sum

In an Euclidean triangle given two angles \((\phi_1, \phi_2)\), the third one \((\phi_3)\) can be found using the relation
\[\phi_1 + \phi_2 + \phi_3 = \pi.\]
This relation can be expressed in the following form:
\[
\sin(\phi_1 + \phi_2 + \phi_3) = 0, \quad \cos(\phi_1 + \phi_2 + \phi_3) = -1
\]
that allows a direct control for a pseudo-Euclidean triangle.

In fact by eq. (23) and using the relations (35) we obtain:
\[
\sinh_e(\theta_1 + \theta_2 + \theta_3) \equiv \sinh_e \theta_1 \sinh_e \theta_2 \sinh_e \theta_3 + \\
+ \sinh_e \theta_1 \cosh_e \theta_2 \cosh_e \theta_3 + \cosh_e \theta_1 \sinh_e \theta_2 \cosh_e \theta_3 + \cosh_e \theta_1 \cosh_e \theta_2 \sinh_e \theta_3 \equiv \\
x_2^2y_3^2 + x_2x_3(x_2 - x_3) - x_3^2(x_2 - x_3)^2 - y_2^3(x_2 - x_3) = 0,
\]
where:
\[
\cosh_e(\theta_1 + \theta_2 + \theta_3) \equiv \cosh_e \theta_1 \cosh_e \theta_2 \cosh_e \theta_3 + \sinh_e \theta_1 \sinh_e \theta_2 \cosh_e \theta_3 + \\
+ \sinh_e \theta_1 \cosh_e \theta_2 \sinh_e \theta_3 + \cosh_e \theta_1 \cosh_e \theta_2 \sinh_e \theta_3 \equiv \\
x_2^2\left(-x_3^2(x_2 - x_3)^2 + y_3^2[(x_2 - x_3)^2 - y_2^3]\right) \equiv \\
\frac{D_1D_2D_3}{d_1^2d_2^2d_3^2} \equiv +1.
\]

Then: \(\sinh_e(\theta_1 + \theta_2 + \theta_3) = 0\), \(\cosh_e(\theta_1 + \theta_2 + \theta_3) = \pm 1\). By utilising eq. (43) and the formalism exposed in app. (B) we can say that the sum of the triangle angles is given by:
\[
(\theta_1)_k + (\theta_2)_{k'} + (\theta_3)_{k''} \equiv (\theta_1 + \theta_2 + \theta_3)_{k,k',k''} = (0)_{\pm 1}
\]
This result allows to state that if we know two angles we can determine if the Klein group index of the third angle is \(\pm h : \{\theta \in U_s, D_s\}\) or \(\pm 1 : \{\theta \in R_s, L_s\}\). From this we obtain the relation between \(\cosh_e\) and \(\sinh_e\) as stated by eqs. (14). This relation and the condition (37) allow to obtain the hyperbolic functions of the third angle. Then: also for the pseudo-Euclidean triangles if we know two angles we can obtain the third one. In Appendix (C) we show two examples of triangle solutions. The triangle solutions for the other cases not considered there can be easily verified following the standard Euclidean approach.

4.4 Equilateral hyperbolas in the pseudo-Euclidean plane

The unit circle for the definition of trigonometric functions has its counterpart, in the pseudo-Euclidean plane, in the four arms of the unit equilateral hyperbolas \(|x^2 - y^2| = 1\), as shown in [5]. Indeed the equilateral hyperbolas have many of the properties of circles in the Euclidean plane; here we point out some of them.

Definitions. If \(A\) and \(B\) are two points on the equilateral hyperbola, the segment \(AB\) is called a chord of the hyperbola. We define two kinds of chords: if the points \(A, B\) are on the same arm of hyperbola we have “external chords”, if the points are in opposite arms we have “internal chords”. Any internal chord that passes through the centre “\(O\)” of the hyperbola is called a diameter of the hyperbola. We will call \(p\) the semi-diameter.

In particular we follow the definitions stated for segments and straight lines and call hyperbolas of the first (second) kind if the tangent straight lines are of the first (second) kind. If we introduce the
square semi-diameter $P$ with its sign, and $p = \sqrt{|P|}$, we have $P < 0$ ($P > 0$) for hyperbolas of the first (second) kind.

The following theorems hold:

**Theorem 1.** The diameters of the hyperbola are the internal chords of minimum length.

**Proof.** We do not lose in generality if we consider hyperbolas of the second kind, with their centre in the coordinate axes origin; then we have $A \equiv (p \cosh \theta_1, p \sinh \theta_1)$, $B \equiv (-p \cosh \theta_2, -p \sinh \theta_2)$. The square length of the chord is $AB^2 = p^2[(\cosh \theta_1 + \cosh \theta_2)^2 - (\sinh \theta_1 + \sinh \theta_2)^2] \equiv 4p^2 \cosh^2[(\theta_1 - \theta_2)/2]$, i.e., $AB^2 = 4p^2$ for $\theta_1 = \theta_2$ and $AB^2 > 4p^2$ for $\theta_1 \neq \theta_2$.

Now we enunciate for equilateral hyperbolas the pseudo-Euclidean counterpart of well-known Euclidean theorems for circles. These theorems can be demonstrated by elementary analytic geometry, as the previous theorem 1.

**Theorem 2.** The line joining $O$ to the midpoint $M$ of a chord is pseudo-orthogonal to it.

This theorem is also valid in the limiting position when the chord becomes tangent to the hyperbola:

**Theorem 3.** If $M$ is on the hyperbola, the tangent at $M$ is pseudo-orthogonal to the diameter $OM$.

**Theorem 4.** If we have the points $A$ and $B$ on the same arm of the hyperbola, for any point $P$ between $A$ and $B$, the hyperbolic angle $\hat{APB}$ is half the hyperbolic angle $\hat{AOB}$.

**Theorem 5.** If $Q$ is a second point between $A$ and $B$, then $\hat{APB} = \hat{AQB}$.

**Theorem 6.** If a side of a triangle inscribed in an equilateral hyperbola passes through the centre of the hyperbola, the other two sides are pseudo-orthogonal.

**Theorem 7.** If we have three non-aligned points that can be considered the vertices of a triangle, there is always an equilateral hyperbola (circumscribed hyperbola) which passes by these points.

For a circumscribed hyperbola it is quite straightforward to obtain:

$$ P = -\frac{D_1D_2D_3}{4[(x_1 - x_2)(y_1 - y_3) - (y_1 - y_2)(x_1 - x_3)]} \equiv -\frac{D_1D_2D_3}{16S^2} \quad (38) $$

for $P > 0$ we have an equilateral hyperbola of the second kind, while for $P < 0$ we have an equilateral hyperbola of the first kind. Then in relation to the hyperbola type we could say that there are two kinds of triangles depending on the sign of $D_1D_2D_3$. From eqs. (38) and (27) we obtain:

$$ p = \frac{d_1d_2d_3}{4S} \equiv \frac{d_n}{2 \sinh \theta_n} \quad (39) $$

that is the same relation as that for the radius of a circumcircle in an Euclidean triangle.

The theorems shown above indicate that in some cases solutions of equilateral hyperbolas problems may be more easily found by applying the exposed theory.

## 5 Conclusions

In this paper we have shown that the invariant quantities with respect to special relativity Lorentz group allow an operative definition of hyperbolic trigonometric functions. These definitions together with the formal substitution of the pseudo-Euclidean distance and hyperbolic trigonometric functions respectively to Pythagorean distance and circular trigonometric functions, allow to extend the classic trigonometry theorems to the Pseudo-Euclidean plane. From these theorems it follows that the pseudo-Euclidean trigonometry can be formalised with the same coherence as the Euclidean trigonometry does and the hyperbolic triangles can be solved exactly in the same way as the Euclidean triangles. This can be done in spite of the more complex topology of the pseudo-Euclidean

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8Some of these theorems are reported, without demonstrations, in [2]
plane with respect to the Euclidean plane.

This formalisation of the pseudo-Euclidean trigonometry allows also to observe in a more general way [2] the Euclidean geometry and might contribute to give a concrete mathematical meaning to the pseudo-Euclidean plane geometry. As a final conclusion note that our results have been obtained by using the group properties of hyperbolic numbers and by comparing hyperbolic and complex numbers. Moreover as briefly resumed in note (3) and diffusely showed in [9] all systems of commutative hypercomplex numbers can be associated to finite and infinite Lie groups. These properties make these number systems suitable to be used in applied sciences. Unfortunately, none of the geometries associated with number systems of more than two units corresponds to the multidimensional Euclidean geometry or to four dimensional space-time geometry used to describe the physical world. Then, the possibility of applying multidimensional commutative hypercomplex numbers to multidimensional geometries would require some radically new ideas. On the contrary the properties of hyperbolic numbers may allow a more complete insight into the pseudo-Euclidean geometry just by using the formal correspondence between the complex and hyperbolic numbers.

6 Acknowledgments

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A Rotation invariants in the Euclidean plane

Euclidean geometry studies the figure properties that do not depend on their position in a plane. If these figures are represented in a Cartesian plane we can say, in group language, that Euclidean geometry studies the invariant properties by coordinate axes roto-translations. It is well known that these properties can be expressed by complex numbers. Let us consider the Gauss-Argand complex plane where a vector is represented by \( v = x + iy \). The axes rotation of an angle \( \alpha \) transforms this vector in the new vector \( v' = v \exp[i\alpha] \). Therefore we can promptly verify that the quantity (as it is usually done we call \( \bar{v} = x - iy \)) \( |v'|^2 = v'\bar{v}' = v \exp[i\alpha]\bar{v} \exp[-i\alpha] \equiv |v|^2 \) is invariant by axes rotation. In a similar way we find two invariants related to any couple of vectors. If we consider two vectors: \( v_1 = x_1 + iy_1, v_2 = x_2 + iy_2 \); we have that the real and the imaginary part of the product \( v_2\bar{v}_1 \) are invariant by axis rotation. In fact \( v_2\bar{v}'_1 = v_2 \exp[i\alpha]\bar{v}_1 \exp[-i\alpha] \equiv v_2\bar{v}_1 \). Now we will see that these two invariants allow an operative definition of trigonometric functions. Let us represent the two vectors in polar coordinates: \( v_1 \equiv \rho_1 \exp[i\phi_1], v_2 \equiv \rho_2 \exp[i\phi_2] \). Consequently we have:

\[
    v_2\bar{v}_1 = \rho_1\rho_2 \exp[i(\phi_2 - \phi_1)] \equiv \rho_1\rho_2[\cos(\phi_2 - \phi_1) + i \sin(\phi_2 - \phi_1)].
\]

As it is well known the resulting real part of the vector product represents the scalar product, while the imaginary part represents the module of the vector product, i.e., the area of the parallelogram defined by the two vectors.

The two invariants of eq. (40) allow an operative definition of trigonometric functions. In Cartesian coordinates we have:

\[
    v_2\bar{v}_1 = (x_2 + iy_2)(x_1 - iy_1) \equiv x_1x_2 + y_1y_2 + i(x_1y_2 - x_2y_1),
\]

and by using eqs. (40) and (41) we obtain:

\[
    \cos(\phi_2 - \phi_1) = \frac{x_1x_2 + y_1y_2}{\rho_1\rho_2}; \quad \sin(\phi_2 - \phi_1) = \frac{x_1y_2 - x_2y_1}{\rho_1\rho_2}
\]
We know that the theorems of Euclidean trigonometry are usually obtained by following a geometric approach. Now by using the Cartesian representation of trigonometric functions, given by eqs. (42), it is straightforward to control that the trigonometry theorems are simple identities. In fact let us define a triangle in a Cartesian plane by its vertices $P_n \equiv (x_n, y_n)$: from the coordinates of these point we obtain the side lengths and, from the eqs. (42) the angles trigonometric functions. By these definitions it is easy to control the identities defined by the trigonometry theorems.

B Hyperbolic functions on unit equilateral hyperbolas

In the complex Gauss-Argand plane the goniometric circle used for the definition of trigonometric functions is expressed by $x + iy = \exp[i\phi]$. In the hyperbolic plane the hyperbolic trigonometric functions can be defined on the unit equilateral hyperbola, that can be expressed in a way similar to goniometric circle: $x + h y = \exp[h\theta]$. However this expression represents only the arm of unit equilateral hyperbolas in the Right sector ($Rs$). If we want to extend the hyperbolic functions on the whole plane, we must take into account all the unit equilateral hyperbola arms, given by $x + h y = \pm \exp[h\theta]$ and $x + h y = \pm h \exp[h\theta]$ for which $|x^2 - y^2| = 1$.

The aim of this appendix is to summarise the approach followed in [5] to demonstrate how these unit curves allow to extend the hyperbolic functions and to obtain the addition formula for angles in any sector.

These unit curves are the set of points $U$, where $U = \{z \mid \rho(z) = 1\}$. Clearly $U(\cdot)$ is a group, subgroups of this group are for $z \in Rs$, and for $z \in Rs + Us, z \in Rs + Ls, z \in Rs + Ds$. For $z \in Rs$ the group $U(\cdot)$ is isomorphic to $\theta(+) \quad \forall \theta \in R$ is the angular function that for $-\infty < \theta < \infty$ traverses the $Rs$ arm of the unit hyperbolas. Now we can have a complete isomorphism between $U(\cdot)$ and the angular function, extending the last one to other sectors. This can be done by the Klein four group: $k \in K = \{1, h, -1, -h\}$.

Indeed let us consider the expressions of the four arms of the hyperbolas (tab. 1 for $\rho = 1$). We can extend the angular functions as a product between $\exp[h\theta]$ and the Klein group: $k \cdot \exp[h\theta]$. We can write: $U = \{k \exp[h\theta] \mid \theta \in R, k \in K\}$, and we call $U_k = \{k \exp[h\theta] \mid \theta \in R\}$, the hyperbola arm with the value $k$. In the same way we call $\theta_k$ the ordered pair $(\theta, k)$, and we define $\Theta \equiv R \times K = \{\theta_k \mid \theta \in R, k \in K\}$ and $\Theta_k \equiv R \times \{k\} = \{\theta_k \mid \theta \in R\}$. $\Theta_1$ is isomorphic to $R(+)$; then we accordingly think of $\Theta(+)$ as an extension of $R(+)$. To define the complete isomorphism between $\Theta(+) \quad \forall U(\cdot)$, we have to define the addition rule for angles $\theta_k + \theta_{k'}$. This rule is obtained from the isomorphism itself:

$$\theta_k + \theta_{k'} \Rightarrow U_k \cdot U_{k'} \equiv k \exp[h\theta] \cdot k' \exp[h\theta'] \equiv k k' \exp[h(\theta + \theta')] \Rightarrow (\theta + \theta')_{k k'}.$$ \hspace{1cm} (43)

On these basis it is straightforward to obtain the hyperbolic angle $\theta$ and the Klein index $(k)$ from the extended hyperbolic functions $\sinh_e \theta$ and $\cosh_e \theta$:

- If $|\sinh_e \theta| < |\cosh_e \theta|$ \quad $\Rightarrow$ \quad $\theta = \tanh^{-1} \left( \frac{\sinh_e \theta}{\cosh_e \theta} \right)$; \quad $k = \frac{\cosh_e \theta}{|\cosh_e \theta|} \cdot 1$

- If $|\sinh_e \theta| > |\cosh_e \theta|$ \quad $\Rightarrow$ \quad $\theta = \tanh^{-1} \left( \frac{\cosh_e \theta}{\sinh_e \theta} \right)$; \quad $k = \frac{\sinh_e \theta}{|\sinh_e \theta|} \cdot h.$

C Two examples of hyperbolic triangle solutions

In this appendix we report two elementary examples of hyperbolic triangle solution in order to point out analogies and differences with the Euclidean trigonometry. We will note that the Cartesian
representation can give some simplifications in the triangle solution. The Cartesian axes will be chosen so that \( P_1 \equiv (0, 0) \), \( P_2 \equiv (\pm d_3, 0) \), or \( P_2 \equiv (0, \pm d_3) \). The two possibilities for \( P_2 \) depend on the sign of \( D_3 \), the sign plus or minus is chosen so that one goes from \( P_1 \) to \( P_2 \) to \( P_3 \) counter-clockwise. Thanks to relations (35), the triangle is completely determined by the \( P_3 \) coordinates.

- **Given two sides and one opposite angle**
  \( \theta_1; \ D_1; \ D_3 \) with \( D_3 > 0 \).

  Applying the first law of cosine to the side \( d_1 \) we have:
  
  \[
  D_2 - 2d_2d_3 \cosh \theta_1 + D_3 - D_1 = 0,
  \]
  from which we can obtain \( d_2 \). In fact:
  
  for \( \cosh \theta_1 > \sinh \theta_1 \)
  
  \[
  D_2 = d_2^2 \Rightarrow d_2 = d_3 \cosh \theta_1 \pm \sqrt{d_3^2 \sinh^2 \theta_1 + D_1},
  \]
  for \( \cosh \theta_1 < \sinh \theta_1 \)
  
  \[
  D_2 = -d_2^2 \Rightarrow d_2 = -d_3 \cosh \theta_1 \pm \sqrt{d_3^2 \sinh^2 \theta_1 - D_1}.
  \]

  So, as for the Euclidean counterpart, we can have, depending on the value of the square root argument, two, one or no solutions.

  Now the vertex \( P_3 \) coordinates are given by:

  \[
  x_3 = d_2 \cosh \theta_1; \quad y_3 = d_2 \sinh \theta_1.
  \]

  We now use an analytic method distinctive of Cartesian plane. The coordinates of the point \( P_3 \) can be obtained intersecting the straight-line \( y = \tanh \theta_1 x \) with the hyperbola centred in \( P_2 \) and having square semi-diameter \( P = D_1 \), i.e., by solving the system:

  \[
  y = \tanh \theta_1 x; \quad (x - d_3)^2 - y^2 = D_1.
  \]

  The results are the same as those of eq. (44), but now it is easy to understand the geometrical meaning of the solutions. In fact if \( D_1 > 0 \) and \( d_1 > d_3 \) the point \( P_1 \) is included in an hyperbola arm and we will have always two solutions. Otherwise if \( \sinh \theta_1 < d_1/d_3 \) there are no solutions, if \( \sinh \theta_1 = d_1/d_3 \) there is just one solution and if \( \sinh \theta_1 > d_1/d_3 \) two solutions.

  If \( D_3 < 0 \) the \( P_2 \) vertex must be put on the \( y \) axis and we will have the system:

  \[
  x = \tanh \theta_1 y; \quad (y - d_3)^2 - x^2 = -D_1.
  \]

  By comparing this result with the solutions of the system (46) we have to change \( x \leftrightarrow y \).

- **Given two angles and the side between them**
  \( \theta_1; \ \theta_2; \ D_3 \) and \( D_3 > 0 \).

  In the Euclidean trigonometry the solution of this problem is obtained using the condition that the sum of the triangle angles is \( \pi \). We will use this method, that will allow to familiarise with the Klein group defined in appendix (B), as well as an analytical geometry method. Let us start with the classical method. We have:

  \[
  \theta_3 = - (\theta_1 + \theta_2), \quad \text{and Klein group index} \quad k_3 \quad \text{so that} \quad k_1k_2k_3 = \pm 1.
  \]

  Then hyperbolic trigonometric functions will be given by: \( \sinh \theta_3 = \sinh |\theta_1 + \theta_2| \) if \( k_3 = \pm 1 \) and by \( \sinh \theta_3 = \cosh (\theta_1 + \theta_2) \) if \( k_3 = \pm h \). From the law of sines we obtain: 

  \[
  d_2 = d_3 \frac{\sinh \theta_1}{\sinh \theta_3},
  \]

  Eqs. (45) allow to obtain the \( P_3 \) coordinates.

  In the Cartesian representation we can use the following method: let us consider the straight-lines 

  \[
  P_1P_3 \Rightarrow y = \tanh \theta_1 x \quad \text{and} \quad P_2P_3 \Rightarrow y = -\tanh \theta_2 (x - x_2).
  \]

  By solving the system between these straight-lines we obtain the \( P_3 \) coordinates:

  \[
  P_3 \equiv \left( x_2 \frac{\tanh \theta_2}{\tanh \theta_2 + \tanh \theta_1}, \ x_2 \frac{\tanh \theta_1 \tanh \theta_2}{\tanh \theta_2 + \tanh \theta_1} \right).
  \]
If $D_3 < 0$ the straight-lines equations are:

\[ y = \coth_e \theta_1 x \quad \text{and} \quad y - y_2 = -\coth_e \theta_2 x. \]

And the solution will be:

\[ P_3 \equiv \left( y_2 \frac{\tanh_e \theta_1 \tanh_e \theta_2}{\tanh_e \theta_2 + \tanh_e \theta_1}, y_2 \frac{\tanh_e \theta_2}{\tanh_e \theta_2 + \tanh_e \theta_1} \right). \tag{48} \]

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