Abstract

We introduce an efficient class of high order schemes for the Klein–Gordon equation from low to high frequency regimes. The new schemes resolve the oscillations triggered by the input term and allow for second order convergence in time uniformly in the high frequencies $\omega_n$ and fourth order convergence under the natural scaling $\Delta t \sim 1/\sqrt[3]{|\omega_n|}$. The construction is based on Magnus expansions tailored to the structure of the input term. Numerically experiments underline our theoretical findings and show the efficiency of the new schemes.
Effective high order integrators for linear Klein-Gordon equations in low to highly oscillatory regimes

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1 Introduction

We consider the linear Klein-Gordon equation

\[ \partial^2_t \psi(x, t) = \Delta \psi(x, t) + f(x, t) \psi(x, t) \]  

with initial condition \( \psi(x, 0) = \psi_0(x) \) equipped with periodic boundary conditions (that is \( x \in \mathbb{T}^d \)). Here, \( f(x, t) \) is a given periodic function under the form

\[ f(x, t) = \alpha(x, t) + \sum_n a_n(x, t) e^{i \omega_n t}. \]

The latter includes the case of non-oscillatory input terms \( \alpha(x, t) \), or purely oscillatory input terms of type

\[ \sum_n a_n(x, t) e^{i \omega_n t}, \]

as well as the combination of the two. It is enough to assume smoothness of the functions \( \alpha \) and \( a_n \), for the precise assumptions, see Section 3.

The aim of this paper lies in developing high order efficient numerical methods which are computationally cheap and allow for small error constants. The new class of schemes will in particular allow us to handle highly oscillatory input terms with \( |\omega_n| \gg 1 \). Note that classical methods for (1) introduce a time error of type \( (\Delta t \cdot \omega_n)^p \) leading to severe step size restrictions \( \Delta t < \frac{1}{\sup_n |\omega_n|} \), loss of convergence and huge computational costs. In contrast, the time (and spatial) step sizes in our new schemes can be chosen significantly larger than the high frequencies \( \omega_n \) of the input term. Our new schemes allow the following error scaling (see Theorem 2); second order convergence \( O((\Delta t)^2) \) holds uniformly in \( \omega_n \) and fourth order convergence \( O((\Delta t)^4) \) under the scaling

\[ \Delta t \gtrsim 1/\sqrt[n]{\min_n |\omega_n|}. \]

In practice we assume, that there is only a finite number of non-zero coefficients \( a_n(x, t) \), so that we are working with a certain number of high frequencies \( \omega_n \), where \( -N \leq n \leq N \). In the construction and analysis of the new methods we will use, as provided in Section 3, the definition of \( \omega = \min_{|n| \leq N} |\omega_n| \).

Linear and nonlinear Klein-Gordon equations have recently gained a lot attention in computational mathematics, see for instance [1], [2], [3], [4], [5], and the references therein. The highly oscillatory case \( \omega \gg 1 \) is thereby particularly challenging as classical methods in general fail to resolve the underlying oscillatory structure. Up to our knowledge there are only two publications devoted to computational approaches in case of space- and time-dependent input terms. Namely, fourth and sixth order splittings proposed in [6] which are based on commutator-free, Magnus type expansions. The latter perform very well in non-oscillatory regimes, however, introduce large error constants in case of high frequencies \( \omega \gg 1 \). The recent work [7] on the other hand proposes a method specialized to highly oscillatory input terms, however, fails in non-oscillatory regimes. The aim of this paper is to close this gap.

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Note that equation (1) is a special case of Klein-Gordon equation with atomic scaling \( \hbar = c = 1 \)

\[
(\partial_t^2 - \mathcal{B}) \psi(x, t) = 0, \quad \mathcal{B} := \Delta - m^2,
\]

where mass function \( m^2 = m^2(x, t) \) is time and space dependent. This type of Klein-Gordon equations with time dependent mass was first proposed by M. Znojil in [8], [9], as a significant contribution to the investigations presented in [10], [11] by Mostafazadeh. Due to Znojil’s improvement, not only non-negativity of probability density is assured but also Lorentz covariance of the interaction is not violated. Moreover, this type of equation can be extended to quantum cosmology, see [10], [12].

To illustrate the idea behind our method we will rewrite the Klein–Gordon equation (1) as follows. Let us set

\[
A(x, t) = \begin{bmatrix} 0 & N(x, t) \\ N(x, t) & 1 \end{bmatrix} \quad \text{and} \quad N(x, t) = \Delta + f(x, t).
\]

Then, we can easily observe that (1) is equivalent to

\[
\partial_t z(x, t) = A(x, t) z(x, t), \quad \text{where} \quad z(t, x) = \begin{bmatrix} \psi(x, t) \\ \frac{\partial}{\partial t} \psi(x, t) \end{bmatrix}.
\]

Analytical solutions of (3) can be presented for example as infinite Fer, Magnus or Dyson expansion. In this manuscript we base our analysis on truncation of Magnus series. Magnus expansions have been broadly studied, see, e.g., [13], [14] or [15]. We need to be careful, however, because the special structure of our operator \( A(x, t) \) defined in (4) causes that odd components of Magnus expansion form anti-diagonal matrices while even components from diagonal ones and, needless to say, need to be treated separately. Similar structure of the operator \( A(x, t) \) was analysed in [16]. The difference, however is in the function \( f(x, t) \), which instead of being highly oscillatory like in our case, is growing to infinity along with time (in [16]). So the problem both in [16] and in our case is highly oscillatory, but the oscillations have a different source so require different approaches.

Unlike in [13], [14], [16] or [15], we deal with Magnus expansion which components do not scale only in terms of time step \( \Delta t = \Delta h \), but also in terms of the oscillation frequency \( \omega \). In this paper we construct schemes which allow us to control these terms efficiently up to high order.

**Outline of the paper.** In Section 2 we will present the full derivation of the methods. Section 3 deals with rigorous estimates of the error terms, which were abandoned in the derivation of the schemes. In Section 4 we provide the structure of error committed by the proposed schemes. In Section 5 we underline our theoretical findings with numerical experiments. In addition, we compare various existing methods with the new approach and highlight the behaviour of the new methods with respect to high frequencies \( \omega \). Appendices A and B abound with calculations which may be of interest while reading Section 3. In Appendix C we provide detailed derivations of 4-th order compact schemes (15) and (16).

## 2 Derivation of methods

### 2.1 Truncation of Magnus expansion

It is well known that the solution of (3) can be presented via an infinite Magnus series

\[
z(x, t + h) = e^{\Theta(x, t + h, t)} z(x, t),
\]

where \( \Theta(x, t + h, t) = \sum_{k=1}^{\infty} \Theta_k(x, t + h, t) \), and \( \Theta_k(x, t + h, t) \) are \( k \)-th times nested integrals of \( (k - 1) \) times nested commutators. Below we write the first few terms of such an expansion,

\[
\Theta_1(x, t + h, t) = \int_0^h A(x, t + t_1) \, dt_1
\]

\[
\Theta_2(x, t + h, t) = -\frac{1}{2} \int_0^h \int_0^{t_1} [A(x, t + t_2), A(x, t + t_1)] \, dt_2 \, dt_1
\]

\[
\Theta_3(x, t + h, t) = \frac{1}{6} \int_0^h \int_0^{t_1} \int_0^{t_2} [A(x, t + t_3), [A(x, t + t_2), A(x, t + t_1)]] \, dt_3 \, dt_2 \, dt_1
\]

\[
+ \frac{1}{6} \int_0^h \int_0^{t_1} \int_0^{t_2} [A(x, t + t_3), [A(x, t + t_2), A(x, t + t_1)]] \, dt_3 \, dt_2 \, dt_1
\]
\[ \Theta_4(x, t+h, t) = \frac{1}{12} \int_0^h \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} \left[ [[A(x, t+t_1), A(x, t+t_2), A(x, t+t_3), A(x, t+t_4)] dt_4 dt_3 dt_2 dt_1 \right] \] (10)

In our approach we will take the approximation obtained by the two first components only, that is

\[ z(x, t+h) \approx e^{\int_0^h A(x, t+t_1) dt_1 - \frac{1}{2} \int_0^h \int_0^{t_1} [A(x, t+t_2), A(x, t+t_1)] dt_2 dt_1} e^{\int_0^h \int_0^{t_1} [A(x, t+t_3), A(x, t+t_1)] dt_3 dt_1} \quad (11) \]

Rigorous estimates of the cut-off terms \( \Theta_3(x, t+h, t), \Theta_4(x, t+h, t) \) and \( \Theta_5(x, t+h, t) \) will be presented in Subsection 3.1. The terms \( \Theta_r(x, t+h, t), r \geq 6 \) are neglected, because most pessimistic estimates indicate scaling at least like \( h^r \). Indeed, it is enough to observe that the summands \( \Theta_r(x, t+h, t) \) are \( r \)-times nested integrals.

### 2.2 Strang splitting

We start with illustrating the Strang splitting. The exponent appearing on the right hand side of (11) is computationally costly. As we will see in Sections 2.3 and 2.4 the integrals

\[ \int_0^h A(x, t+t_1) dt_1 \]

and

\[ -\frac{1}{2} \int_0^h \int_0^{t_1} [A(x, t+t_2), A(x, t+t_1)] dt_2 dt_1 \]

differ not only because of their structure (the first one is anti-diagonal, while the second is diagonal), but also due to the magnitude of order with which they scale (in terms of \( h \) and \( \omega \)).

For this reason we will exploit the Strang splitting, which for \( X = O(h^a) \) and \( Y = O(h^b) \) reads

\[ \exp(X + Y) = \exp(\frac{1}{2}X) \exp(\frac{1}{2}Y) + \mathcal{O}(h^{a+b+1}). \]

### 2.3 Numerical treatment of the outer term

Although the outer term \( e^{\int_0^h \int_0^{t_1} [A(x, t+t_2), A(x, t+t_1)] dt_2 dt_1} \) in the approximation (13) seems to be computationally complicated (because of the double integral of a commutator), we will show that its numerical treatment is unexpectedly straightforward. We start out by computing the commutator in

\[ \int_0^h \int_0^{t_1} [A(x, t+t_2), A(x, t+t_1)] dt_2 dt_1 \]

\[ = \int_0^h \int_0^{t_1} \left[ \begin{array}{cc} 0 & 1 \\ N(x, t+t_2) & 0 \end{array} \right] \left[ \begin{array}{cc} 0 & 1 \\ N(x, t+t_1) & 0 \end{array} \right] dt_2 dt_1 \]

\[ = \frac{1}{2} \int_0^h \int_0^{t_1} \left[ N(x, t+t_1) - N(x, t+t_2) \\ 0 \right] \left[ N(x, t+t_2) - N(x, t+t_1) \\ 0 \right] dt_2 dt_1 \]

\[ = \left[ \int_0^h \int_0^{t_1} (f(x, t+t_1) - f(x, t+t_2)) dt_2 dt_1 \right] \int_0^h \int_0^{t_1} (f(x, t+t_2) - f(x, t+t_1)) dt_2 dt_1 \]
Remark 1 By simple integration by parts we can observe, that the problem of two-dimensional quadratures boils down to the far less computationally costly one-dimensional quadrature:

$$\int_0^h \int_0^{t_1} (f(x, t + t_1) - f(x, t + t_2)) \, dt_2 \, dt_1 = 2 \int_0^h (t + t_1 - \frac{h}{2}) f(x, t + t_1) \, dt_1.$$  

Let us notice that after semi-discretization, $x = (x_1, \ldots, x_M) \otimes \cdots \otimes (x_1^d, \ldots, x_M^d)$, the considered integral $\int_0^h \int_0^{t_1} [A(x, t + t_2), A(x, t + t_1)] \, dt_2 \, dt_1$ can be approximated by a diagonal matrix, as a tensor product of diagonal matrices

$$\begin{bmatrix}
O(x^1_1, t) & \ldots & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \ldots & O(x^M_1, t) & 0 & \ldots \\
\vdots & \ldots & 0 & -O(x^1_1, t) & \ldots \\
0 & \ldots & 0 & \ldots & -O(x^M_1, t)
\end{bmatrix} \otimes \cdots \otimes
\begin{bmatrix}
O(x^1_1, t) & \ldots & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \ldots & O(x^M_1, t) & 0 & \ldots \\
\vdots & \ldots & 0 & -O(x^1_1, t) & \ldots \\
0 & \ldots & 0 & \ldots & -O(x^M_1, t)
\end{bmatrix},$$

where

$$O(x^k_i, t) = 2 \int_0^h (t + t_1 - \frac{h}{2}) f(x^k_i, t + t_1) \, dt_1, \quad k = 1, \ldots, d, i = 1, \ldots, M.$$  

Obviously taking the exponential of a diagonal matrix is computationally straightforward.

### 2.4 Numerical treatment of the inner term

In this subsection we will tackle the inner term

$$e^{h \int_0^h A(x, t + t_1) \, dt_1} = \exp \begin{bmatrix} 0 & h \\ D + F & 0 \end{bmatrix},$$  

where for sake of clarity we used the notation

$$D := h \Delta, \quad \text{and} \quad F := \int_0^h f(x, t + t_1) \, dt_1.$$  

Exponent of the anti-diagonal matrix \([14]\) can be performed by applying hyperbolic functions, more precisely,

$$\exp \begin{bmatrix} 0 & h \\ D + F & 0 \end{bmatrix} = \begin{bmatrix} \cosh \left( \sqrt{h(D + F)} \right) & \sqrt{D + F} \sinh \left( \sqrt{h(D + F)} \right) \\ \sqrt{D + F} \sinh \left( \sqrt{h(D + F)} \right) & \cosh \left( \sqrt{h(D + F)} \right) \end{bmatrix}.$$  

Needless to say the computation of hyperbolic sine or cosine of $\sqrt{h(D + F)}$ is very complicated, because it is neither diagonal nor circulant symmetric.

Therefore, we will propose two methods on how to overcome the computational difficulties arising at this stage. In all our approaches we will be willing to split the troublesome matrix $\begin{bmatrix} 0 & h \\ D + F & 0 \end{bmatrix}$ in such a way, that each components obtained after the splitting could be easily exponentiated. Let us start with the observations, that each entry of the matrix we have to exponentiate scales like $h$. For this reason Strang decomposition would result in second order splitting only, so it is out of considerations while deriving 4-th order methods. Instead we will resort to the higher order splittings of the following kind:

$$e^{hX + hY} = e^{hX} e^{hY} = e^{hX} e^{hY} e^{\frac{h^2}{2} hY X + \frac{h^2}{2} hX Y} e^{\frac{h^2}{2} hY} e^{\frac{h^2}{6} hY^2 X^2 + \frac{h^2}{6} hY^2 Y^2} e^{\frac{h^2}{24} hY^3 X^3} e^{O(h^5)}$$

$$e^{hX + hY} = e^{h^{\frac{3}{2}} hX} e^{h^{\frac{3}{2}} hY} e^{\frac{h^2}{2} h^{\frac{3}{2}} Y X + \frac{h^2}{2} h^{\frac{3}{2}} X Y} e^{\frac{h^2}{2} h^{\frac{3}{2}} Y} e^{\frac{h^2}{6} h^{\frac{3}{2}} Y^2 X^2 + \frac{h^2}{6} h^{\frac{3}{2}} Y^2 Y^2} e^{\frac{h^2}{24} h^{\frac{3}{2}} Y^3 X^3} e^{O(h^5)},$$

which can be derived using two times Baker–Campbell–Hausdorff formula and compute order conditions. The derivation of the above splittings is detailed in Appendix C. Despite of the choice of components $X$ and $Y$ (where $X + Y = \begin{bmatrix} 0 & h \\ D + F & 0 \end{bmatrix}$) the inner exponent in (16), that is $\exp \left( \sqrt{\frac{h^2}{3}} X - \left( \frac{2 - \sqrt{2}}{24} \right) h^2 \langle [X, Y], Y \rangle \right)$ turns out to be computationally costly. Hence, we are left with option (15) only.
2.4.1 Inner term - towards scheme $\Gamma_1^{[4]}$

In the first presented time splitting we separate the Laplacian part $\begin{bmatrix} 0 & 0 \\ D & 0 \end{bmatrix}$ from the potential one $\begin{bmatrix} 0 & h \\ F & 0 \end{bmatrix}$ and apply (15), concluding with the following splitting of order 4 with respect to time step $h$

$$\exp\left[\begin{bmatrix} 0 & 0 \\ D + F & 0 \end{bmatrix}\right] = \exp\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) + \begin{bmatrix} 0 & h \\ F & 0 \end{bmatrix} = O(h^5) +$$

$$\exp\left[\begin{bmatrix} 0 & 0 \\ \frac{1}{2}D & 0 \end{bmatrix}\right] \exp\left[\begin{bmatrix} 0 & \frac{1}{2}h \\ F & 0 \end{bmatrix}\right] \exp\left[\begin{bmatrix} \frac{2}{3}D + \frac{1}{2}hD^2 & 0 \\ 0 & 0 \end{bmatrix}\right] \exp\left[\begin{bmatrix} 0 & \frac{1}{2}h \\ F & 0 \end{bmatrix}\right] \exp\left[\begin{bmatrix} 0 & 0 \\ \frac{1}{2}D & 0 \end{bmatrix}\right].$$

Note that each component is computationally friendly. Indeed, let us observe, that hyperbolic functions of $\sqrt{\frac{hF}{2}}$ appearing in

$$\exp\left[\begin{bmatrix} 0 & \frac{1}{2}h \\ F & 0 \end{bmatrix}\right] = \begin{bmatrix} \cosh\left(\sqrt{\frac{hF}{2}}\right) & \frac{h}{2} \sinh\left(\sqrt{\frac{hF}{2}}\right) \\ \sqrt{F} \sinh\left(\sqrt{\frac{hF}{2}}\right) & \cosh\left(\sqrt{\frac{hF}{2}}\right) \end{bmatrix}$$

can be computed easily because $\sqrt{\frac{hF}{2}}$ becomes a diagonal matrix after semi-discretization. Also the other two matrices can be exponentiated cheaply, as

$$\exp\left[\begin{bmatrix} 0 & 0 \\ \frac{1}{6}D & 1 \end{bmatrix}\right] \quad \text{and} \quad \exp\left[\begin{bmatrix} \frac{2}{3}D + \frac{1}{2}hD^2 & 0 \\ 0 & 0 \end{bmatrix}\right] = \begin{bmatrix} \frac{2}{3}D + \frac{1}{2}hD^2 & 0 \\ 0 & 0 \end{bmatrix}.$$

2.4.2 Inner term - towards scheme $\Gamma_2^{[4]}$

An alternative splitting may be obtained by keeping the kinetic and potential parts together, for which the sum $D + F$ is the only nonzero entry of the matrix. After applying the splitting (15) we have to exponentiate matrices with only one nonzero input

$$\exp\left[\begin{bmatrix} 0 & 0 \\ D + F & 0 \end{bmatrix}\right] = \exp\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = O(h^5) +$$

$$\exp\left[\begin{bmatrix} 0 & 0 \\ \frac{1}{6}(D + F) & 0 \end{bmatrix}\right] \exp\left[\begin{bmatrix} \frac{2}{3}(D + F) + \frac{1}{2}h(D + F)^2 & 0 \\ 0 & 0 \end{bmatrix}\right] \exp\left[\begin{bmatrix} 0 & \frac{1}{2}h \\ F & 0 \end{bmatrix}\right] \exp\left[\begin{bmatrix} 0 & 0 \\ \frac{1}{6}(D + F) & 0 \end{bmatrix}\right]$$

which makes the scheme extremely fast computationally.

2.5 Complete numerical schemes $\Gamma_1^{[4]}$ and $\Gamma_2^{[4]}$

Now, after taking care of numerical treatments of the outer term in Section 2.3 and inner terms in Subsections 2.4.1 and 2.4.2 we are ready to build up upon local approximation presented in (13). Adopting simplifications in the notation introduced earlier we define

$$D := h\Delta, \quad F_k := \int_0^h f(x, t_k + t_1) dt_1 \quad \text{and} \quad \mathcal{F}_k := \frac{1}{2} \int_0^h (t_k + t_1 - \frac{h}{2}) f(x, t_k + t_1) dt_1.$$

Assuming that the solution $z(x, t_k)$ is known at $t_k = kh$, we present the scheme $\Gamma_1^{[4]}$ as follows:

$$z(x, t_k + h) \approx \begin{bmatrix} \exp(-\mathcal{F}_k) & 0 \\ 0 & \exp(\mathcal{F}_k) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{1}{6}D & 1 \end{bmatrix} \begin{bmatrix} \cosh\left(\frac{\sqrt{hF}}{2}\right) & \frac{h}{2} \sinh\left(\frac{\sqrt{hF}}{2}\right) \\ \sqrt{F} \sinh\left(\frac{\sqrt{hF}}{2}\right) & \cosh\left(\frac{\sqrt{hF}}{2}\right) \end{bmatrix} \begin{bmatrix} \frac{2}{3}D + \frac{1}{2}hD^2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cosh\left(\frac{\sqrt{hF}}{2}\right) & \frac{h}{2} \sinh\left(\frac{\sqrt{hF}}{2}\right) \\ \sqrt{F} \sinh\left(\frac{\sqrt{hF}}{2}\right) & \cosh\left(\frac{\sqrt{hF}}{2}\right) \end{bmatrix} \begin{bmatrix} \exp(-\mathcal{F}_k) & 0 \\ 0 & \exp(\mathcal{F}_k) \end{bmatrix} z(x, t_k).$$

The Algorithm for the scheme $\Gamma_1^{[4]}$ is presented in Table [1].
expansion and applying the Strang splitting to gain in computational efficiency. The following theorem will be
specified.

Table 1: Algorithm \( \Gamma^{[4]}_1 \) for finding the approximate solution on the time interval \([t_0, t_T]\) with \( T \) time steps \( h = (t_T - t_0)/T \). Note that \( \mathbf{z}(\mathbf{x}, t) = [z_1(\mathbf{x}, t), z_2(\mathbf{x}, t)]^T \), where the discretisation in space is not yet applied or specified.

In a similar way we present final scheme \( \Gamma^{[4]}_2 \), and its algorithm is presented in Table 2.

z(\mathbf{x}, t_k + h) \approx \left[ \begin{array}{c}
\exp\left(-\mathcal{F}_k\right) \\
0
\end{array} \right] \left[ \begin{array}{c}
\frac{1}{2} (D + F_k) \\
1
\end{array} \right] \left[ \begin{array}{c}
\exp\left(\mathcal{F}_k\right) \\
0
\end{array} \right] z(\mathbf{x}, t_k).

Table 2: Algorithm \( \Gamma^{[4]}_2 \) for finding the approximate solution on the time interval \([t_0, t_T]\) with \( T \) time steps \( h = (t_T - t_0)/T \). Note that \( \mathbf{z}(\mathbf{x}, t) = [z_1(\mathbf{x}, t), z_2(\mathbf{x}, t)]^T \), where the discretisation in space is not yet applied or specified.

3 Estimates of the cut off terms

In this Section we will provide details on estimates of the cut off terms obtained while truncating the Magnus expansion and applying the Strang splitting to gain in computational efficiency. The following theorem will be exploited in these estimates frequently.

T steps of 4th order algorithm \( \Gamma^{[4]}_1 \)

\[
\text{do } k = 0, T - 1 \\
q_0 = \exp(-\mathcal{F}_k)z_1; \quad p_0 = \exp(\mathcal{F}_k)z_2 \\
p_1 = \frac{1}{6} (D + F_k)q_0 + p_0 \\
q_1 = q_0 + \frac{1}{2} hp_1 \\
p_2 = \left(\frac{1}{12} (D + F_k) + \frac{1}{6} h(D + F_k)^2\right) q_1 + p_1 \\
q_2 = q_1 + \frac{1}{2} hp_2 \\
p_3 = \frac{1}{6} (D + F_k)q_2 + p_2 \\
q_3 = \exp(-\mathcal{F}_k)q_2; \quad p_4 = \exp(\mathcal{F}_k)p_3 \\
z_1 := q_3; \quad z_2 := p_4 \\
\text{end do}
\]

T steps of 4th order algorithm \( \Gamma^{[4]}_2 \)

\[
\text{do } k = 0, T - 1 \\
q_0 = \exp(-\mathcal{F}_k)z_1; \quad p_0 = \exp(\mathcal{F}_k)z_2 \\
p_1 = \frac{1}{6} (D + F_k)q_0 + p_0 \\
q_1 = q_0 + \frac{1}{2} hp_1 \\
p_2 = \left(\frac{1}{12} (D + F_k) + \frac{1}{6} h(D + F_k)^2\right) q_1 + p_1 \\
q_2 = q_1 + \frac{1}{2} hp_2 \\
p_3 = \frac{1}{6} (D + F_k)q_2 + p_2 \\
q_3 = \exp(-\mathcal{F}_k)q_2; \quad p_4 = \exp(\mathcal{F}_k)p_3 \\
z_1 := q_3; \quad z_2 := p_4 \\
\text{end do}
\]
Theorem 1 Let \( a \in C^1([0,h]) \) be a real function and \( h \leq 1 \) and \( \omega \geq 1 \). Then the following estimate holds

\[
\left| \int_0^h \int_0^{t_1} \int_0^{t_2} \ldots \int_0^{t_{m-1}} a(t_k)e^{i\omega t_k} dt_m dt_{m-1} \ldots dt_1 \right| \leq C A \min \left\{ h^m, \frac{h^{m-1}}{\omega} \right\}, \quad 1 \leq k \leq m, \tag{17}
\]

where \( C \) is a constant and \( A = \max_{\xi \in [0,h]} \{ |a(\xi)|, |a'(\xi)| \} \).

Proof: Let us start with an observation that for \( r = 0, 1, 2, \ldots \)

\[
\left| \int_0^{t_m-1} t_r^r a(t_m)e^{i\omega t_m} dt_m \right| \leq (r+2)CA \min \left\{ h^{r+1}, \frac{h^r}{\omega} \right\}, \quad 0 \leq t_m < t_{m-1} \leq h. \tag{18}
\]

Indeed, the immediate estimate is

\[
\left| \int_0^{t_m-1} t_r^r a(t_m)e^{i\omega t_m} dt_m \right| \leq CAh^{r+1}, \tag{19}
\]

but by simple integration by parts we observe that for \( r = 0 \)

\[
\int_0^{t_m-1} a(t_m)e^{i\omega t_m} dt_m = \frac{1}{i\omega} a(t_{m-1})e^{i\omega t_{m-1}} - \frac{1}{i\omega} a(0) - \frac{1}{i\omega} \int_0^{t_m-1} a'(t_m)e^{i\omega t_m} dt_m,
\]

and that for \( r = 1, 2, \ldots \) we have

\[
\int_0^{t_m-1} t_r^r a(t_m)e^{i\omega t_m} dt_m = \frac{1}{i\omega} t_r^{r-1} a(t_{m-1})e^{i\omega t_{m-1}} - \frac{1}{i\omega} \int_0^{t_m-1} r t_r^{r-1} a(t_m)e^{i\omega t_m} dt_m - \frac{1}{i\omega} \int_0^{t_m-1} t_r^r a'(t_m)e^{i\omega t_m} dt_m.
\]

we can obtain the more subtle result

\[
\left| \int_0^{t_m-1} t_r^r a(t_m)e^{i\omega t_m} dt_m \right| \leq \frac{h^r}{\omega} CA + \frac{h^r}{\omega} CA + \frac{h^{r+1}}{\omega} CA, \quad r = 0, 1, 2, \ldots \tag{20}
\]

Combining (19) and (20) we can conclude inequality (18). Inequality (17) is the immediate consequence of (18). \( \square \)

Remark 2 Let us assume, that there are \( 2N+1 \) known, non-zero, real functions \( a_n \in C^1([0,h]) \), and let \( h \leq 1 \) and \( \min |\omega_n| \geq 1 \). Then we can show a result similar to the one obtained in Theorem 4, namely, that

\[
\left| \int_0^h \int_0^{t_1} \int_0^{t_2} \ldots \int_0^{t_{m-1}} a_n(t_k)e^{i\omega t_k} dt_m dt_{m-1} \ldots dt_1 \right| \leq C A \min \left\{ h^m, \frac{h^{m-1}}{\min |\omega_n|} \right\}, \quad 1 \leq k \leq m, \tag{21}
\]

where \( C \) is a generic constant and \( A = \max_{\xi \in [0,h]} \{ |a_n(\xi)|, |a_n'(\xi)| \} \).

Remark 3 For clarity of exposition we will abuse the notation and write

\[
f(t_k) = a(t_k) + \sum_n a_n(t_k)e^{i\omega t_k}
\]

instead of

\[
f(x, t + t_k) = a(x, t + t_k) + \sum_n a_n(x, t + t_k)e^{i\omega(t + t_k)}.
\]

In further part of the paper we denote by \( \| \cdot \|_s \) the standard \( H^s \) Sobolev norm on the torus \( \mathbb{T}^d \) and assume that the following Assumption holds

Assumption 1 Let us assume that

1. \( \exists N \geq 0 \) such that \( a_n(x, t) \equiv 0 \),
2. \( a_n \in C^1([0,T], H^{2+s}(\mathbb{T}^d)) \),
3. \( a \in C^1([0,T], H^{2+s}(\mathbb{T}^d)) \),
4. \( s \geq 0, s + 2 > \frac{d}{2} \),
5. \( \omega = \min_{|n| \leq N} |\omega_n| \).
Definition 1 In all calculations of this article we understand that $C$ is generic constant. Let us define

$$\tilde{A} = \max_{\xi \in [0,T]} \left\{ \| a_n(\cdot, \xi) \|_{s+2}, \| a_n^2(\cdot, \xi) \|_s, \| \partial_t a_n(\cdot, \xi) \|_{s+2} ; \, |n| \leq N \right\},$$

$$\tilde{E} = \max_{\xi \in [0,T]} \left\{ \| a(\cdot, \xi) \|_{s+2}, \| a^2(\cdot, \xi) \|_s, \| \partial_t a(\cdot, \xi) \|_{s+2}, \| \partial_t^2 a(\cdot, \xi) \|_{s+2} \right\},$$

$$\tilde{A} \tilde{E} = \max_{\xi \in [0,T]} \left\{ \| a^p(\cdot, \xi) \|_s \cdot \| a_n^q(\cdot, \xi) \|_s ; \, p + q = 5, |n| \leq N \right\}.

3.1 Error committed by the Magnus truncation

The commutators appearing in [4]-[10] are calculated in Appendix A. Let us start with the first truncated term of Magnus expansion:

$$\Theta_3(x, t + h, t) = \frac{1}{6} \int_0^h \int_0^{t_1} \int_0^{t_2} [A(x, t + t_1), [A(x, t + t_2), A(x, t + t_1)]] dt_3 dt_2 dt_1

+ \frac{1}{6} \int_0^h \int_0^{t_1} \int_0^{t_2} [A(x, t + t_3), [A(x, t + t_2), A(x, t + t_1)]] dt_3 dt_2 dt_1

= \frac{1}{6} \int_0^h \int_0^{t_1} \int_0^{t_2} \left[ \begin{array}{cc} 0 & H_1 + H_3 \\ H_2 + H_4 & 0 \end{array} \right] dt_3 dt_2 dt_1,$$

where

$$H_1 + H_3 = 2(-f(t_1) + 2f(t_2) - f(t_3));$$

$$H_2 + H_4 = \Delta (f(t_3) - 2f(t_2) + f(t_1)) + (f(t_3) - 2f(t_2) + f(t_1)) \Delta 

+ 4f(t_1)f(t_3) - 2f(t_1)f(t_2) - 2f(t_3)f(t_2).$$

In the estimates below we separate the non-oscillatory part from the oscillatory and obtain

$$\left\| \int_0^h \int_0^{t_1} \int_0^{t_2} (H_1 + H_3) dt_3 dt_2 dt_1 \right\| \leq \int_0^h \int_0^{t_1} \int_0^{t_2} 2(-\alpha(t_1) + 2\alpha(t_2) - \alpha(t_3)) dt_3 dt_2 dt_1

+ \sum_n \int_0^h \int_0^{t_1} \int_0^{t_2} 2(-a_n(t_1)e^{i\omega n t_1} + 2a_n(t_2)e^{i\omega n t_2} - a_n(t_3)e^{i\omega n t_3}) dt_3 dt_2 dt_1

\leq Ch^5 \max_{\mu \in [0,h]} |a''(\mu)| + C \tilde{A} \min \left\{ h^3, \frac{h^2}{\omega} \right\}

\leq Ch^5 \tilde{E} + C \tilde{A} \min \left\{ h^3, \frac{h^2}{\omega} \right\}.$$
The computations of the commutators appearing in $\Theta_4$ are more complicated (details can be found in Appendix [A]), but the estimates of obtained formulas are similar to these appearing in $\Theta_3$:

$$
\Theta_4(x, t + h, t) = \frac{1}{12} \int_0^h \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} \left[ [A(x, t + t_1), A(x, t + t_2), A(x, t + t_3), A(x, t + t_4)] dt_4 dt_3 dt_2 dt_1
\right.
+ \frac{1}{12} \int_0^h \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} [A(x, t + t_1), [A(x, t + t_2), A(x, t + t_3), A(x, t + t_4)] dt_4 dt_3 dt_2 dt_1
\right.
+ \frac{1}{12} \int_0^h \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} [A(x, t + t_1), [A(x, t + t_2), [A(x, t + t_3), A(x, t + t_4)] dt_4 dt_3 dt_2 dt_1
\right.
+ \frac{1}{12} \int_0^h \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} [A(x, t + t_2), [A(x, t + t_3), [A(x, t + t_4), A(x, t + t_1)] dt_4 dt_3 dt_2 dt_1
\left.
= \frac{1}{12} \int_0^h \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} \left[ \mathcal{H}_1 \mathcal{H}_2 \right] dt_4 dt_3 dt_2 dt_1,
\right.
$$

where

$$
\mathcal{H}_1 = \Delta (2f(t_2) - 2f(t_3)) + 3(2f(t_2) - 2f(t_3)) \Delta + 4f(t_4)(f(t_2) - f(t_3)) + 4f(t_1)(f(t_2) - f(t_3));
\mathcal{H}_2 = -3\Delta (2f(t_2) - 2f(t_3)) - (2f(t_2) - 2f(t_3)) \Delta - 4f(t_4)(f(t_2) - f(t_3)) - 4f(t_1)(f(t_2) - f(t_3)).
$$

Thanks to Theorem [1] we thus obtain for any $\varphi$ sufficiently smooth

$$
\left\| \int_0^h \int_0^{t_1} \int_0^{t_2} \int_0^{t_3} \mathcal{H}_1 dt_4 dt_3 dt_2 dt_1 \varphi \right\| \leq C \ h^5 \tilde{L} \| \varphi \|_{s+2} + C \tilde{A} \min \left\{ h^4, \frac{h^4}{\omega} \right\} \| \varphi \|_{s+2}
+ C \tilde{L} \tilde{A} \min \left\{ h^4, \frac{h^4}{\omega} \right\} \| \varphi \|_s.
$$

The estimate of the triple integral in $\mathcal{H}_2$ follows similar calculations as above.

It can be shown that

$$
\| \Theta_5(x, t + h, t) \varphi \|_s \leq \left( C \ h^6 \tilde{L} + \tilde{A} \min \left\{ h^5, \frac{h^4}{\omega} \right\} \right) \| \varphi \|_{s+2} + C \tilde{L} \tilde{A} \min \left\{ h^5, \frac{h^4}{\omega} \right\} \| \varphi \|_s,
$$

but the easiest estimate

$$
\| \Theta_5(x, t + h, t) \varphi \|_s \leq C \ h^5 \tilde{A} \tilde{L}_5 \| \varphi \|_{s+2},
$$

can be obtained immediately by the observation, that $\Theta_5(x, t + h, t)$ is a five times nested integral of four times nested commutators.

All together, the error committed by truncating the Magnus expansion can be bounded for any $\varphi$ sufficiently smooth by

$$
\| (\Theta_3(x, t + h, t) + \Theta_4(x, t + h, t) + \Theta_5(x, t + h, t)) \varphi \|_s \leq C \ h^5 \tilde{L} \| \varphi \|_{s+2} + C \tilde{A} \min \left\{ h^5, \frac{h^4}{\omega} \right\} \| \varphi \|_{s+2} + C \tilde{L} \tilde{A} \min \left\{ h^5, \frac{h^4}{\omega} \right\} \| \varphi \|_s + C \ h^5 \tilde{A} \tilde{L}_5 \| \varphi \|_{s+2}.
$$

### 3.2 Error committed by the Strang splitting

To estimate the error of the approximation (13) we exploit Baker-Campbell-Hausdorff formula, which reads

$$
\exp(X + Y) = \exp\left( \frac{1}{2} X \right) \exp(Y) \exp\left( \frac{1}{2} X \right) + \frac{1}{12} [Y, [Y, X]] - \frac{1}{24} [X, [Y, X]]
- \frac{1}{720} ([Y, [Y, [Y, X]]] + [X, [X, [X, Y]]]) + \frac{1}{360} ([X, [Y, [Y, [Y, X]]]] + [Y, [X, [X, [X, Y]]]]) + \cdots
$$

where

$$
Y = \int_0^h \left[ \Delta + f(x, t_1) \right] dt_1,
$$
and

\[ X = \int_{0}^{h} \int_{0}^{t_1} [A(x, t_1), A(x, t_2)] dt_2 dt_1 = \int_{0}^{h} \int_{0}^{t_1} \left[ \frac{f(x, t_1) - f(x, t_2)}{0} \right] dt_2 dt_1. \]

Obviously, five-times nested commutators scale at least like \( O(h^5) \), so it is enough to consider \([Y, X], X\) + \([Y, X], Y\).

Denoting

\[ F = \int_{0}^{h} f(t + t_1) dt_1, \]

\[ F = \int_{0}^{h} \int_{0}^{t_1} (f(t + t_2) - f(t + t_1)) dt_2 dt_1 \]

underlying commutators (as is calculated in Appendix [B]) have the following form

\[
[Y, [Y, X]] = \begin{bmatrix}
-h^2 \Delta F - 3h^2 F \Delta - 4h F F^2 & 0 \\
0 & 3h^2 \Delta F + h^2 F \Delta + 4h F F^2
\end{bmatrix};
\]

\[
[X, [X, Y]] = \begin{bmatrix}
2h F \Delta F + h F^2 \Delta + h \Delta F^2 + 4F^2 F & 0 \\
0 & 4h F^2
\end{bmatrix}.
\]

Using Theorem [D] we obtain the estimates

\[ |F| \leq C h \max_{\mu \in [0, T]} \left( \|\alpha(\cdot, \mu)\|_s \right) + C \min \left\{ h, \frac{1}{\omega} \right\} \max_{\mu \in [0, T]} \left( \|a_n(\cdot, \mu)\|_s \right) \]

\[ \leq C h \tilde{C} + C \tilde{A} \min \left\{ h, \frac{1}{\omega} \right\} \]

\[ |F| \leq C h^2 \max_{\mu \in [0, T]} \left( \|\partial_t \alpha(\cdot, \mu)\|_s \right) + C \min \left\{ h, \frac{1}{\omega} \right\} \max_{\mu \in [0, T]} \left( \|a_n(\cdot, \mu)\|_s \right) \]

\[ \leq C h^2 \tilde{C} + C \tilde{A} \min \left\{ h^2, \frac{h}{\omega} \right\}. \]

\[ \|\Delta F \|_s \leq C h^2 \max_{\mu \in [0, T]} \left( \|\partial_t \alpha(\cdot, \mu)\|_s \right) + C \min \left\{ h, \frac{1}{\omega} \right\} \max_{\mu \in [0, T]} \left( \|a_n(\cdot, \mu)\|_s \right) \]

\[ \|\Delta F \|_s \leq C h^2 \max_{\mu \in [0, T]} \left( \|\partial_t \alpha(\cdot, \mu)\|_s \right) + C \min \left\{ h^2, \frac{h}{\omega} \right\} \max_{\mu \in [0, T]} \left( \|a_n(\cdot, \mu)\|_s \right) \]

\[ \|\Delta F \|_s \leq C h^3 \tilde{C} + C \tilde{A} \min \left\{ h^2, \frac{h}{\omega} \right\} \|\|_{\|s+2} \]

\[ \|\Delta F \|_s \leq C h^3 \tilde{C} + C \tilde{A} \min \left\{ h^2, \frac{h}{\omega} \right\} \|\|_{\|s+2} \]

\[ \|F \|_s \leq C h^2 \tilde{C} + C \tilde{A} \min \left\{ h^2, \frac{h}{\omega} \right\} \|\|_{\|s+2} \]

All together,

\[ \left| \frac{1}{12} [Y, [Y, X]] - \frac{1}{24} [X, [X, Y]] \right| \leq \begin{bmatrix}
G_1 & G_2 \\
G_3 & G_4
\end{bmatrix} ,\]

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Another interesting observation is that for purely oscillatory input term to obtain an order 4 method. This phenomena is also observed in the numerical examples, see Section 5.

Let us discuss the error bound of Theorem 2 in more detail. Note that if the input function satisfies

\[
\|G_1\|_s \leq C \left( h^5 \tilde{C} + \tilde{A} \min \left\{ h^4, h^3, \frac{h^2}{\omega} \right\} \right) \|\varphi\|_{s+2} + C \left( h^4 \tilde{C} + \tilde{A} \min \left\{ h^3, h^2, \frac{h}{\omega} \right\} \right) \left( h\tilde{C} + \tilde{A} \min \left\{ h, \frac{1}{\omega} \right\} \right) \|\varphi\|_s
\]

\[
\|G_2\|_s \leq C \left( h^5 \tilde{C} + \tilde{C} \tilde{A} \min \left\{ h^4, h^3, \frac{h^2}{\omega} \right\} \right) \|\varphi\|_s
\]

\[
\|G_3\|_s \leq C \left( h^4 \tilde{C} + \tilde{A} \min \left\{ h^3, h^2, \frac{h}{\omega} \right\} \right) \left( h\tilde{C} + \tilde{A} \min \left\{ h^2, h \right\} \right) \|\varphi\|_{s+2} + h \left( h^2 \tilde{C} + \tilde{A} \min \left\{ h, \omega \right\} \right) \|\varphi\|_s
\]

\[
\|G_4\|_s \leq C \left( h^4 \tilde{C} + \tilde{A} \min \left\{ h^3, h^2, \frac{h}{\omega} \right\} \right) \|\varphi\|_{s+2} + C \left( h^3 \tilde{C} + \tilde{A} \min \left\{ h^2, h \right\} \right) \left( h\tilde{C} + \tilde{A} \min \left\{ h, \omega \right\} \right) \|\varphi\|_s.
\]

Taking into account the error committed by Magnus expansion, it is enough to observe that Strang splitting also satisfies

\[
\| \frac{1}{12} [Y, [Y, X]] - \frac{1}{24} [X, [X, Y]] \|_s \leq \left[ \frac{1}{1} \frac{1}{1} \frac{1}{1} \right] C \left( h^5 \tilde{C} + \tilde{A} \min \left\{ h^3, \frac{h^2}{\omega} \right\} \right) \|\varphi\|_{s+2} + \tilde{C} \tilde{A} \min \left\{ h^3, \frac{h^2}{\omega} \right\} \|\varphi\|_s + h^5 \tilde{A} \tilde{C}_5 \|\varphi\|_s.
\]

4 Structure of the error

Thanks to the previous two section we can conclude the following local error scaling of the proposed schemes scales.

Theorem 2 (Error scaling) Let us assume that Assumption [4] holds and let us denote by \( \phi^f \) the exact flow, i.e., \( z(\mathbf{x}, t) = \phi^f(t_0) \) and by \( \Phi^h \) the numerical flow, i.e.,

\[
z^{n+1} = \Phi^h(z^n)
\]

where \( \Phi^h \) corresponds to Algorithm \( \Gamma_1^{[4]} \) or \( \Gamma_2^{[4]} \). Then, for \( \varphi \in H^{s+2}(\mathbb{T}^d) \) we have that

\[
\|\phi^h(\varphi) - \Phi^h(\varphi)\|_s \leq C \left( h^5 \tilde{C} + \tilde{A} \min \left\{ h^3, \frac{h^2}{\omega} \right\} \right) \|\varphi\|_{s+2} + C \left( h^5 \tilde{A} \tilde{C}_5 + \tilde{C} \tilde{A} \min \left\{ h^3, \frac{h^2}{\omega} \right\} \right) \|\varphi\|_s
\]

where

\[
\tilde{A} = \max_{\xi \in [0,7]} \left\{ \|a_n(\xi, \xi)\|_{s+2}, \|a_n^2(\xi, \xi)\|_s, \|a_n(\xi, \xi)\|_{s+2} : |n| \leq N \right\},
\]

\[
\tilde{C} = \max_{\xi \in [0,7]} \left\{ \|a(\xi, \xi)\|_{s+2}, \|a^2(\xi, \xi)\|_s, \|a(\xi, \xi)\|_{s+2} : |\xi| \leq N \right\},
\]

\[
\tilde{A} \tilde{C}_5 = \max_{\xi \in [0,7]} \left\{ \|a^p(\xi, \xi)\|_s \cdot \|a^q(\xi, \xi)\|_s : p + q = 5, |n| \leq N \right\}.
\]

Remark 4 Let us discuss the error bound of Theorem 2 in more detail. Note that if the input function \( f(\mathbf{x}, t) \) does not involve the highly oscillatory term, that is \( a_n(\mathbf{x}, t) \equiv 0 \), then all derived methods are of order 4. If, however, some of the coefficients \( a_n(\mathbf{x}, t) \) are non-zero, then it is enough to take time steps \( \Delta t \geq 1/\sqrt{\min_n |\omega_n|} \) to obtain an order 4 method. This phenomena is also observed in the numerical examples, see Section 5. Another interesting observation is that for purely oscillatory input term \( f(\mathbf{x}, t) \) (that is when \( a(\mathbf{x}, t) \equiv 0 \)), we can obtain a method of order \( r \), that is of local error \( O((\Delta t)^r+1) \) by choosing time step \( \Delta t \geq 1/\sqrt[r]{\min_n |\omega_n|} \).

5 Numerical experiments

In this section we compare newly constructed numerical approaches with several schemes from the literature. In this respect, the following methods are considered:

- BBCK\(^{[4]}\): 4-th order method \( \Sigma_3^{[4]} \) from \( \Sigma_3^{[4]} \);
- BBCK\(^{[6]}\): 6-th order method \( \Sigma_3^{[6]} \) from \( \Sigma_3^{[6]} \);
- Asympt\(^{[3]}\): 3-rd order asymptotic method from \( \Sigma_3^{[3]} \).

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For our experiments, we assume that the solution is confined in a region \([x_0, x_M]\), and periodic boundary conditions are imposed. We divide the spatial region into \(M = 200\) intervals of length \(\Delta x = (x_M - x_0)/M\) and, after spatial discretization, we obtain an equation similar to the first order system \(\Delta x = (z_1, z_2)^T\) and \(z_1, z_2(t) \approx z(x_i, t)\), which can be understood as semiclassical_approximation.

As a reference solution we take the 6-th order method based on self-adjoint basis of Munthe-Kaas & Owren \cite{15} with a step size \(h = 10^{-5}\), and then we carry out the numerical integration with each method using different time steps and measure the \(l_2\) error at the final time. This error is plotted in double-logarithmic scale versus the order of the method and time of calculation expressed in seconds. For Laplacian discretization we use Fourier methods, described in detail in \cite{17}.

Example 1.
In first example we take a wave equation with time-dependent potential and with frequency \(\omega\), namely

\[
\partial_t^2 u = \partial_x^2 u - \sigma \left(1 + \frac{1}{5} \cos(\omega t)\right)x^2 u, \quad x \in [-\pi, \pi], \quad t \in [0, 1];
\]

\[
u(x, 0) = e^{-\frac{1}{2}(x - \Delta x)^2} + e^{-\frac{1}{2}(x + \Delta x)^2}, \quad u'(x, 0) = 0;
\]

\[
u(-\pi, t) = u(\pi, t), \quad t \in [0, 1].
\]

Example 2.
Let us consider example featuring large disproportion between the laplacian part and the influx-term part, which can be understood as semiclassical_like regime, has a negative effect on all considered methods. For \(\omega = 1\) method \(\Gamma_1^{[4]}\) is less computationally costly and obtains better accuracy than \(\Gamma_2^{[4]}\). Moreover, in this case method \(\Gamma_1^{[4]}\) achieves an error only slightly greater than 6-th order method BBCK\([6]\). However, as the oscillation increases, the \(\Gamma_1^{[4]}\) and \(\Gamma_2^{[4]}\) methods require much smaller time steps than the BBCK\([6]\) method, but larger than BBCK\([4]\) method. Indeed, for \(\omega = 500\) and time,

\[
\sigma = 1/5, \quad \omega = 50
\]

\[
\sigma = 1, \quad \omega = 50
\]
In this example we consider a wave equation in two dimensions, namely

\begin{equation}
\Gamma^2\partial_t u = \partial_x^2 u - (1 + \varepsilon \cos(\omega t)) x^2 u, \quad x \in [-10, 10], \quad t \in [0, 1],
\end{equation}

\begin{align*}
u(x, 0) &= e^{-x^2}, \quad \partial_t u(x, 0) = 0, \\
u(-10, t) &= u(10, t), \quad t \in [0, 1].
\end{align*}

On the first graph in Figure 2 we present the initial condition (blue line), solution at final time step for \( \varepsilon = 10, \omega = 10 \) (yellow line) and solution at final time step for \( \varepsilon = 0.1, \omega = 100 \) (red line). Next two graphs show the evolutions in time of the solutions for \( \varepsilon = 0.1, \omega = 100 \) and for \( \varepsilon = 10, \omega = 10 \).

Comparisons of costs and accuracy for equation (29) are presented in Figures 5 and 6. First of all let us observe, that asymptotic method Asympt propose [7] is unbeatable for equation with extremely oscillatory influx terms. Indeed - asymptotic method was designed especially for this type of equations. For small \( \omega \) asymptotic method Asympt is ineffective. In the less oscillatory case, when \( \omega = 10 \), methods \( \Gamma^1 \) and \( \Gamma^2 \) preforms predictably - they achieve a worse error than the 6-th order BBCK method, but delivers significantly smaller error than 4-th order method BBCK and asymptotic method Asympt. As the oscillations get larger, \( \omega = 1000 \), asymptotic method Asympt starts behaving extraordinary - as it was designed - especially and only for extremely large oscillations. Methods BBCK and BBCK require truly small time steps to handle the oscillations, while our new methods deliver expected errors for all time steps \( h \). Moreover, for \( h > 10^{-2} \) method BBCK coincides with the method BBCK. In case of extremely high oscillations, \( \omega = 10^6 \), methods BBCK and BBCK present the same order of convergence, close to the second order, while the methods \( \Gamma^1 \) and \( \Gamma^2 \) achieve the 4-th order of convergence from the first, largest, time step \( h = 1 \), what is consistent with the theory. Computational cost of methods \( \Gamma^1 \) and \( \Gamma^2 \) are impressing in case of all frequencies \( \omega \).

Example 3.

Now we consider the equation evolving on wider space interval

\begin{align*}
n(1 + \varepsilon \cos(\omega t)) x^2 u, \quad x \in [-10, 10], \quad t \in [0, 1],
\end{align*}

\begin{align*}
n(x, 0) &= e^{-x^2} , \quad \partial_t u(x, 0) = 0, \\
u(-10, t) &= u(10, t), \quad t \in [0, 1].
\end{align*}

step \( h = 10^{-4} \) 6-th order method BBCK obtains error scaling like \( 10^{-11} \), while 4-th order method BBCK achieves error scaling like \( 10^{-4} \) and new 4-th order methods \( \Gamma^1 \) and \( \Gamma^2 \) obtain error scaling like \( 10^{-8} \). Still, methods \( \Gamma^1 \) and \( \Gamma^2 \) are less computationally costly then methods BBCK and BBCK.

Example 4.

In this example we consider a wave equation in two dimensions, namely

\begin{equation}
\Gamma^2\partial_t u = \Delta u - \frac{1}{5} \cos(\omega t) x^2 y^2 u, \quad x, y \in [-\pi, \pi], \quad t \in [0, 1],
\end{equation}

\begin{align*}
u(x, y, 0) &= e^{-\frac{1}{2}(x-3)^2} + e^{-\frac{1}{2}(x+3)^2} + e^{-\frac{1}{2}(y-3)^2} + e^{-\frac{1}{2}(y+3)^2} , \quad u'(x, y, 0) = 0; \\
u(-\pi, -\pi, t) &= u(\pi, \pi, t) = u(\pi, -\pi, t) = u(\pi, -\pi, t), \quad t \in [0, 1].
\end{align*}
The semiclassical-like regime shows superiority of $\Gamma_1^{[4]}$ to $\Gamma_2^{[4]}$, especially in the non-oscillatory case where $\omega = 1$. Both new methods are competitive to 4-th order method BBCK$^{[4]}$, but 6-th order method BBCK$^{[6]}$ obtains, as expected, better accuracy.

The above example is used to show that methods $\Gamma_1^{[4]}$ and $\Gamma_2^{[4]}$ can also be used for problems in higher spatial dimensions. Although we present an example in two spatial dimensions, these methods can be extended to higher dimensions, but as the number of spatial dimensions $d$ increases, the size of the matrix resulting from semidiscretization grows as $M^d$. Thus, such calculations require computers with very high computing power.

**Example 5.** In the last example we consider a wave equation with frequency $\omega$, namely

$$\begin{align*}
\partial_t^2 u &= \partial_x^2 u - \frac{1}{1+t^2} \left(1 + \frac{1}{5} \cos(\omega t)\right) x^2 u, \quad x \in [-\pi, \pi], \quad t \in [0, 1]; \\
u(x, 0) &= e^{-\frac{1}{2}(x-3)^2} + e^{-\frac{1}{2}(x+3)^2}, \quad u'(x, 0) = 0; \\
u(-\pi, t) &= u(\pi, t), \quad t \in [0, 1].
\end{align*}$$

In example (31) we consider the most complicated function, where non-oscillatory part $-\frac{1}{1+t^2} x^2$ and oscillatory part $\frac{1}{5} \frac{1}{1+t^2} \cos(\omega t)x^2$ are time and space dependent. Methods $\Gamma_1^{[4]}$ and $\Gamma_2^{[4]}$ present the same accuracy and error constant. Like in Example (27), $\Gamma_1^{[4]}$ is slightly more computationally costly than $\Gamma_2^{[4]}$, but both methods are much more accurate than method BBCK$^{[4]}$ and much less computationally costly than 6-th order method BBCK$^{[6]}$. 
Asymptotic method Asympt\textsuperscript{3} fails for small $\omega$, but as the oscillations get higher, the Asympt\textsuperscript{3} method becomes unbeatable - this method was designed especially for case of high oscillations. In case of small $\omega$, 6-th order method BBCK\textsuperscript{6} achieves better error and 4-th order method BBCK\textsuperscript{4} worse error than methods $\Gamma_1\textsuperscript{4}$ and $\Gamma_2\textsuperscript{4}$, as expected. For medium oscillations, $\omega = 10^3$, methods BBCK\textsuperscript{4} and BBCK\textsuperscript{6} require a very small time step to perform theoretical orders of their convergence. In case of extremely high oscillations, methods BBCK\textsuperscript{4} and BBCK\textsuperscript{6} fail, while methods $\Gamma_1\textsuperscript{4}$ and $\Gamma_2\textsuperscript{4}$ achieve the 4-th order of convergence for all presented time steps. It is also worth noting that for each oscillation $\omega$ methods $\Gamma_1\textsuperscript{4}$ and $\Gamma_2\textsuperscript{4}$ are very cheap computationally.

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Asymptotic method Asympt[3] fails for small $\omega$, but as the oscillations get higher, the Asympt[3] method becomes unbeatable - this method was designed especially for case of high oscillations. In case of small $\omega$, 6-th order method BBCK[6] achieves better error and 4-th order method BBCK[4] worse error than methods $\Gamma_1[4]$ and $\Gamma_2[4]$, as expected. For medium oscillations, $\omega = 10^3$, methods BBCK[4] and BBCK[6] require a very small time step to perform theoretical orders of their convergence. In case of extremely high oscillations, methods BBCK[4] and BBCK[6] fail, while methods $\Gamma_1[4]$ and $\Gamma_2[4]$ achieve the 4-th order of convergence for all presented time steps. It is also worth noting that for each oscillation $\omega$ methods $\Gamma_1[4]$ and $\Gamma_2[4]$ are very cheap computationally.

Appendices

A Commutators erasing in Magnus expansion, in equations (7)-(10)

Let us recall that

$$ f(t) = \alpha(t) + \sum_n a_n(t)e^{i\omega nt}. $$

Then

$$ A(t_1) = \begin{bmatrix} 0 & \Delta + f(t_1) \\ \Delta + f(t_1) & 0 \end{bmatrix} $$

and

$$ [A(t_2), A(t_1)] = \begin{bmatrix} f(t_1) - f(t_2) & 0 \\ 0 & f(t_2) - f(t_1) \end{bmatrix}. $$

In the following part we are calculating two- and three-times nested commutators.

$$ [A(t_1), [A(t_2), A(t_3)]] = \begin{bmatrix} 0 & 1 \\ \Delta + f(t_1) & 0 \end{bmatrix} \begin{bmatrix} f(t_3) - f(t_2) & 0 \\ 0 & f(t_2) - f(t_3) \end{bmatrix} - \begin{bmatrix} f(t_3) - f(t_2) & 0 \\ 0 & f(t_2) - f(t_3) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \Delta + f(t_1) & 0 \end{bmatrix} $$

$$ = \begin{bmatrix} 0 & 1 \\ \Delta + f(t_1) & 0 \end{bmatrix} \begin{bmatrix} f(t_3) - f(t_2) & 0 \\ 0 & f(t_2) - f(t_3) \end{bmatrix} $$

$$ = \begin{bmatrix} 0 & H_1 \\ H_2 & 0 \end{bmatrix}, $$

where

$$ H_1 = 2(f(t_2) - f(t_3)) $$

$$ H_2 = (\Delta + f(t_1))(f(t_3) - f(t_2)) - (f(t_2) - f(t_3))(\Delta + f(t_1)) $$

$$ = \Delta f(t_3) + f(t_3)\Delta - \Delta f(t_2) - f(t_2)\Delta + 2f(t_1)f(t_3) - 2f(t_1)f(t_2). $$

Analogously

$$ [A(t_3), [A(t_2), A(t_1)]] = \begin{bmatrix} 0 & H_3 \\ H_4 & 0 \end{bmatrix}, $$

$$ Figure 6: Comparison of orders (first row) and of costs in seconds (second row) for equation (29) for $\varepsilon = 1$. Asymptotic method Asympt[3] fails for small $\omega$, but as the oscillations get higher, the Asympt[3] method becomes unbeatable - this method was designed especially for case of high oscillations. In case of small $\omega$, 6-th order method BBCK[6] achieves better error and 4-th order method BBCK[4] worse error than methods $\Gamma_1[4]$ and $\Gamma_2[4]$, as expected. For medium oscillations, $\omega = 10^3$, methods BBCK[4] and BBCK[6] require a very small time step to perform theoretical orders of their convergence. In case of extremely high oscillations, methods BBCK[4] and BBCK[6] fail, while methods $\Gamma_1[4]$ and $\Gamma_2[4]$ achieve the 4-th order of convergence for all presented time steps. It is also worth noting that for each oscillation $\omega$ methods $\Gamma_1[4]$ and $\Gamma_2[4]$ are very cheap computationally.
where

\[ H_3 = 2(f(t_2) - f(t_1)) \]
\[ H_4 = \Delta f(t_1) + f(t_1)\Delta - \Delta f(t_2) - f(t_2)\Delta + 2f(t_3)f(t_1) - 2f(t_3)f(t_2). \]

For three-time nested commutators we have

\[
[A(t_4), [[A(t_1), A(t_2)], A(t_3)]] = \begin{bmatrix}
0 & H_5 & 0 & 0 \\
H_6 & 0 & \Delta + f(t_4) & 0 \\
0 & \Delta + f(t_4) & 0 & H_6 \\
0 & 0 & H_6 & 0
\end{bmatrix}
\]

where

\[ H_5 = 2(f(t_2) - f(t_1)) \]
\[ H_6 = \Delta(f(t_1) - f(t_2)) + (f(t_1) - f(t_2))\Delta + 2f(t_3)(f(t_1) - f(t_2)) \]

\[
[A(t_1), [[A(t_2), A(t_3)], A(t_4)]] = \begin{bmatrix}
0 & 0 & H_7 & 0 \\
H_8 & 0 & \Delta + f(t_1) & 0 \\
0 & \Delta + f(t_1) & 0 & H_8 \\
0 & 0 & H_8 & 0
\end{bmatrix}
\]

where

\[ H_7 = 2(f(t_3) - f(t_2)) \]
\[ H_8 = \Delta(f(t_2) - f(t_3)) + (f(t_2) - f(t_3))\Delta + 2f(t_4)(f(t_2) - f(t_3)) \]

\[
[A(t_1), [A(t_2), [A(t_3), A(t_4)]]] = \begin{bmatrix}
0 & 0 & H_9 & 0 \\
H_{10} & 0 & \Delta + f(t_1) & 0 \\
0 & \Delta + f(t_1) & 0 & H_{10} \\
0 & 0 & H_{10} & 0
\end{bmatrix}
\]
Figure 8: Comparison of orders (first row) and of costs in seconds (second row) for equation (30). As in previous examples, increasing the oscillation does not affect the order of the methods $\Gamma_1^{[3]}$ and $\Gamma_2^{[4]}$, it only slightly raises the error constant.

where

$$H_9 = 2(f(t_3) - f(t_4))$$

$$H_{10} = \Delta (f(t_4) - f(t_3)) + (f(t_4) - f(t_3)) \Delta + 2f(t_2) (f(t_4) - f(t_3))$$

$$[A(t_2), [A(t_3), [A(t_4), A(t_1)]]] = \begin{bmatrix}
\Delta + f(t_2) & 0 & H_{11} \\
0 & H_{12} & 0 \\
H_{11} & 0 & \Delta + f(t_2)
\end{bmatrix} - \begin{bmatrix}
0 & H_{11} & 0 \\
H_{12} & 0 & 0 \\
0 & \Delta + f(t_2) & H_{11} - H_{12}
\end{bmatrix}$$

(32)

where

$$H_{11} = 2(f(t_4) - f(t_1))$$

$$H_{12} = \Delta (f(t_1) - f(t_4)) + (f(t_1) - f(t_4)) \Delta + 2f(t_3) (f(t_1) - f(t_4))$$

To estimate term $\Theta_4$ we need to sum the matrices resulting from individual commutators. The result is a matrix

$$\begin{bmatrix}
H_1 & 0 \\
0 & H_2
\end{bmatrix},$$

where

$$H_1 = -H_6 + H_5 (\Delta + f(t_4)) + H_8 - H_7 (\Delta + f(t_1)) + H_{10} - H_9 (\Delta + f(t_4)) + H_{12} - H_{11} (\Delta + f(t_2))$$

$$= \Delta (f(t_2) - f(t_3)) + (f(t_2) - f(t_1)) \Delta + 2f(t_3) (f(t_2) - f(t_1)) - 2 (f(t_1) - f(t_2)) (\Delta + f(t_4))$$

$$+ \Delta (f(t_2) - f(t_3)) + (f(t_2) - f(t_4)) \Delta + 2f(t_4) (f(t_2) - f(t_3)) - 2 (f(t_3) - f(t_2)) (\Delta + f(t_1))$$

$$+ \Delta (f(t_4) - f(t_3)) + (f(t_4) - f(t_1)) \Delta + 2f(t_3) (f(t_4) - f(t_3)) - 2 (f(t_3) - f(t_4)) (\Delta + f(t_1))$$

$$+ \Delta (f(t_4) - f(t_1)) + (f(t_4) - f(t_3)) \Delta + 2f(t_3) (f(t_4) - f(t_3)) - 2 (f(t_3) - f(t_4)) (\Delta + f(t_2))$$

$$= \Delta (2f(t_2) - 2f(t_3)) + 3 (2f(t_2) - 2f(t_3)) \Delta + 4f(t_4) (f(t_2) - f(t_3)) + 4f(t_1) (f(t_2) - f(t_4))$$

and

$$H_2 = -3\Delta (2f(t_2) - 2f(t_3)) - (2f(t_2) - 2f(t_3)) \Delta - 4f(t_4) (f(t_2) - f(t_3)) - 4f(t_1) (f(t_2) - f(t_3)),$$

where

$$f(t_k) f(t_l) = \left(\alpha(t_k) + \sum_n a_n(t_k)e^{i\omega_n t_k}\right)\left(\alpha(t_l) + \sum_n a_n(t_l)e^{i\omega_n t_l}\right)$$

$$= \alpha(t_k)\alpha(t_l) + \alpha(t_k) \sum_n a_n(t_k)e^{i\omega_n t_k} + \alpha(t_l) \sum_n a_n(t_k)e^{i\omega_n t_k} + \sum_n a_n(t_k)e^{i\omega_n t_k} \sum_m a_m(t_l)e^{i\omega_m t_l}$$
B Calculations erasing in Strang splitting error in Section 3.2

Taking

\[
X = -\int_0^h \int_0^{t_1} [A(t + t_2), A(t + t_1)] dt_1 dt_2 = \begin{bmatrix} -F & 0 \\ 0 & F \end{bmatrix},
Y = \int_0^h \begin{bmatrix} \Delta + f(t + t_1) & 1 \\ 0 & h \end{bmatrix} dt_1 = \begin{bmatrix} 0 & h \\ h\Delta + F & 0 \end{bmatrix},
\]

we have

\[
[Y, X] = \begin{bmatrix} 0 & h \\ h\Delta + F & 0 \end{bmatrix} \begin{bmatrix} -F & 0 \\ 0 & F \end{bmatrix} - \begin{bmatrix} F & 0 \\ 0 & -F \end{bmatrix} \begin{bmatrix} 0 & h \\ h\Delta + F & 0 \end{bmatrix}
= \begin{bmatrix} 0 & h F \\ -(h\Delta + F) F & 0 \end{bmatrix} - \begin{bmatrix} F(h\Delta + F) & 0 \\ 0 & -h F \end{bmatrix} = \begin{bmatrix} 0 & 2h F \\ -h\Delta F - h\Delta\Delta - 2FF & 0 \end{bmatrix}
\]

\[
[Y, [Y, X]] = \begin{bmatrix} 0 & h \\ h\Delta + F & 0 \end{bmatrix} \begin{bmatrix} -h\Delta F - h\Delta\Delta - 2FF & 0 \\ 0 & h \end{bmatrix} - \begin{bmatrix} 0 & 2h F \\ -h\Delta F - h\Delta\Delta - 2FF & 0 \end{bmatrix} \begin{bmatrix} 0 & h \\ h\Delta + F & 0 \end{bmatrix}
= \begin{bmatrix} 0 & 2h F \\ -h^2\Delta F - h^2\Delta\Delta - 2FF & 0 \end{bmatrix} - \begin{bmatrix} 0 & h\Delta + F \\ (h\Delta + F)2h F & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2h F \\ -h^2\Delta F - h^2\Delta\Delta - 2FF & 0 \end{bmatrix}
\]

\[
[X, Y] = -[Y, X] = \begin{bmatrix} 0 & 2h F \\ h\Delta F + h\Delta\Delta + 2FF & 0 \end{bmatrix}
\]

\[
[X, [X, Y]] = \begin{bmatrix} -F & 0 \\ 0 & F \end{bmatrix} \begin{bmatrix} 0 & 2h F \\ h\Delta F + h\Delta\Delta + 2FF & 0 \end{bmatrix} - \begin{bmatrix} -F & 0 \\ 0 & F \end{bmatrix} \begin{bmatrix} 0 & 2h F \\ h\Delta F + h\Delta\Delta + 2FF & 0 \end{bmatrix}
= \begin{bmatrix} 0 & 2h F \\ h\Delta F + h\Delta\Delta + 2FF & 0 \end{bmatrix} - \begin{bmatrix} 0 & 2h F \\ (h\Delta F + h\Delta\Delta + 2FF)(-F) & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2h F \\ 4h F^2 & 0 \end{bmatrix}
\]

Hence

\[
[Y, [Y, X]] = \begin{bmatrix} -h^2\Delta F - 3h^2F\Delta - 4hFF & 0 \\ 0 & 3h^2\Delta F + h^2\Delta\Delta + 4hFF \end{bmatrix}
\]

\[
[X, [X, Y]] = \begin{bmatrix} 2h F\Delta F + h F^2\Delta + h\Delta F^2 + 4FF^2 & 0 \\ 0 & 4h F^2 \end{bmatrix}
\]

C Fourth order splittings

In section 2.4 we introduced the 4-th order splitting (15) and (16) of the following form

\[
e^{X+Y} = e^{aX} X e^{bY} e^{aX+\alpha[[X,Y],X]+\beta[[X,Y],Y]} X e^{bY} e^{aX} + O(h^5)
\]

where \(a\) or \(b\) vanishes.

In the below calculations we will derive coefficients \(c_0, c_1, c_2\) for both cases : where \(a = 0\) and where \(b = 0\).

Let us recall sBCH formula

\[
\exp[Y] \exp[X] \exp[Y] = \exp[sBCH(Y, X)], \text{ where}
\]

\[
sBCH(Y, X) = 2Y + X - \frac{1}{6} [[X, Y], Y] - \frac{1}{6} [[X, Y], X] + O(h^5)
\]

1.

\[
e^{c_1Y} e^{aX} e^{c_1Y} \overline{w}_1 = \exp[sBCH(c_1 Y, c_0 X)]
\]

\[
sBCH(c_1 Y, c_0 X) = 2c_1 Y + c_0 X - \frac{1}{6} c_0 c_1^2 [[[X, Y], Y] - \frac{1}{6} c_0^2 c_1 [[X, Y], X]}
\]
2. \[ e^{c_2 X} e^{c_1 Y} e^{c_0 X} e^{c_1 Y} e^{c_2 X} = \exp[sBCH(c_2 X, W_1)] \]

\[ sBCH(c_2 X, W_1) = 2c_2 X + W_1 - \frac{1}{6} c_2^2[[W_1, X], X] - \frac{1}{6} c_2 [[W_1, X], W_1] \]

\[ = 2c_2 X + 2c_1 Y + c_0 X - \frac{1}{6} c_0 c_1^2 [[X, Y], Y] - \frac{1}{6} c_0^3 c_1 [[X, Y], X] \]

\[ - \frac{1}{3} c_1 c_2^2 [[Y, X], X] - \frac{2}{3} c_1^2 c_2 [[Y, X], Y] - \frac{1}{3} c_0 c_1 c_2 [[Y, X], X] \]

\[ = (c_0 + 2c_2) X + 2c_1 Y + (\frac{2}{3} c_1^2 c_2 - \frac{1}{6} c_0^3 c_1) [[X, Y], Y] \]

\[ + \left( \frac{1}{3} c_1^2 c_2^2 - \frac{1}{6} c_0^2 c_1 + \frac{1}{3} c_0 c_1 c_2 \right) [[X, Y], X] \]

because

\[ - \frac{1}{6} c_2^2[[W_1, X], X] = - \frac{1}{6} c_2^2[[2c_1 Y, X], X] - \frac{1}{6} c_2^2[[c_0 X, X], X] = - \frac{1}{3} c_1 c_2^2[[Y, X], X] \]

\[ - \frac{1}{6} c_2[[W_1, X], W_1] = - \frac{1}{6} c_2[[2c_1 Y + c_0 X, X], 2c_1 Y + c_0 X] = - \frac{2}{3} c_1 c_2^2[[Y, X], Y] - \frac{1}{3} c_0 c_1 c_2 [[Y, X], X], \]

where we neglected terms with commutators nested more than two times (that is scaling at least like \( O(h^3) \)).

Depending on which of the expressions is more computationally complicated \( ([[X, Y], Y] \) or \( [[X, Y], X] \) we will obtain two systems of equations for the coefficients \( c_0, c_1, c_2 \):

| Elimination of \([X, Y], Y\] towards (15) | Elimination of \([X, Y], X\] towards (16) |
|------------------------------------------|------------------------------------------|
| \[
\begin{align*}
  c_0 + 2c_2 &= 1 \\
  2c_1 &= 1 \\
  a &= \frac{1}{2} c_1^2 c_2 - \frac{1}{6} c_0 c_1 + \frac{1}{3} c_0 c_1 c_2 \\
  b &= \frac{2}{3} c_1 c_2 - \frac{1}{6} c_0 c_2 \\
  b &= 0 \\
\end{align*}
\] | \[
\begin{align*}
  c_0 + 2c_2 &= 1 \\
  2c_1 &= 1 \\
  a &= \frac{1}{2} c_1^2 c_2 - \frac{1}{6} c_0 c_1 + \frac{1}{3} c_0 c_1 c_2 \\
  b &= \frac{2}{3} c_1 c_2 - \frac{1}{6} c_0 c_2 \\
  a &= 0 \\
\end{align*}
\] |

\[
\begin{align*}
  c_0 &= \frac{2}{3}, c_1 &= \frac{1}{2}, c_2 &= \frac{1}{6} \quad \text{and} \quad a = -\frac{1}{72} \\
\end{align*}
\]

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