The Penrose-Fife phase-field model
with coupled dynamic boundary conditions

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December 12, 2013

Abstract

In this paper we derive, starting from the basic principles of thermodynamics, an extended version
of the nonconserved Penrose-Fife phase transition model, in which dynamic boundary conditions
are considered in order to take into account interactions with walls. Moreover, we study the well-
posedness and the asymptotic behavior of the Cauchy problem for the PDE system associated to the
model, allowing the phase configuration of the material to be described by a singular function.

∗The work of A.M. was partially supported by the FP7-IDEAS-ERC-StG #256872 (EntroPhase). Part of the work has
been developed while A.M. was visiting the University of Milano from February 27 to March 4, 2012.
†The work of E.R. was supported by the FP7-IDEAS-ERC-StG #256872 (EntroPhase).
‡The work of G.S. was supported by the MIUR-PRIN Grant 2008ZKHAHN “Phase transitions, hysteresis and multi-
scaling” and by the FP7-IDEAS-ERC-StG #256872 (EntroPhase).
§The work of A.S. was supported by the MIUR-PRIN Grant 2008ZKHAHN “Phase transitions, hysteresis and multi-
scaling” and by the FP7-IDEAS-ERC-StG #256872 (EntroPhase).
1 Introduction

In this paper we derive a model for phase transitions of Penrose-Fife type settled in a bounded domain $\Omega \subset \mathbb{R}^3$. The peculiarity of our approach consists in the fact that we take into account the relations between $\Omega$ and its exterior, including the effects of the interactions with the boundary into the free energy and entropy functionals $\Psi$ and $S$ (cf. (24)-(25) below) which drive the evolution of the system.

A detailed derivation of the model is carried out in Section 2. The resulting PDEs system couples four nonlinear and singular evolution equations: two for the absolute temperature $\vartheta$ (one in the bulk $\Omega$ and the other on the boundary $\Gamma$) and two for the phase parameter $\chi$, which represents the local proportion of one of the two phases:

\[
\frac{\partial \vartheta}{\partial t} - \Delta \left( -\frac{1}{\vartheta} \right) + \lambda_0' \frac{\partial \chi}{\partial t} = h_b \text{ in } \Omega, \tag{1}
\]

\[
\frac{\partial \vartheta}{\partial t} - \Delta_\Gamma \left( -\frac{1}{\vartheta} \right) + \lambda_0' \frac{\partial \chi}{\partial t} + \partial_\nu \left( -\frac{1}{\vartheta} \right) = h_\Gamma \text{ on } \Gamma, \tag{2}
\]

\[
\frac{\partial \chi}{\partial t} - \Delta \chi - s_0'(\chi) = -\frac{\lambda_0' \chi}{\vartheta} \text{ in } \Omega, \tag{3}
\]

\[
\frac{\partial \chi}{\partial t} - \Delta_\Gamma \chi - s_0'(\chi) + \partial_\nu \chi = -\frac{\lambda_0' \chi}{\vartheta} \text{ on } \Gamma. \tag{4}
\]

Here, $\Delta$ stands for the Laplacian with respect to the space variable $s$ in $\Omega$, $\Delta_\Gamma$ denotes the Laplace-Beltrami operator on $\Gamma$, and $\nu$ is the unit outer normal vector to $\Gamma$. Moreover, $\lambda_0$, $\lambda_\Gamma$, $h_b$, and $h_\Gamma$ are quadratic functions of $\chi$ related to the latent heat of the process, and $\nu$ is the unit outer normal vector to $\Gamma$. The literature devoted to the mathematical features of phase transition models endowed with singular potentials is rather wide, also in specific relation with the Penrose-Fife model. Among the various contributions we quote [31] (for the Cahn-Hilliard equation), [21] (for the Caginalp phase-field model), [40] (for the Penrose-Fife system), and the references in these papers.

A notable feature of system (1)-(4) is the occurrence of dynamic boundary conditions. This type of conditions has been proposed in the literature in different contexts (for instance, in the framework of the Allen-Cahn and Cahn-Hilliard models) with the aim of describing the interactions between the interior of a domain and the walls (cf., e.g. [13], [14], [15], [32], and references therein). In particular, the case of...
singular potentials in the context of the Cahn-Hilliard evolution has been recently analyzed in [17], [18] and [32], whereas the Caginalp phase-field system (cf. [6]) with dynamic boundary conditions on the phase parameter has been considered in several papers (cf., e.g. [7], [8], [9], [16]), while only recently it has been coupled with dynamic boundary conditions for both the phase parameter and the temperature. Such a problem has been considered in [20] and [11], where the well-posedness and the asymptotic behavior (in terms of attractors) of solutions have been studied, also in the case of singular potentials.

As far as the Penrose-Fife model (cf. [34]) is concerned, a vast literature is devoted to the well-posedness (cf., e.g. [10], [23], [28]) and to the long-time behavior of solutions both in term of attractors (cf., e.g. [25], [35], [39]) and of convergence of single trajectories to stationary states (cf. [12]). Most of these contributions deal with Robin boundary conditions for the temperature, i.e.,

$$\frac{\partial (1/\vartheta)}{\partial \nu} = \xi (\vartheta - \vartheta_s) \text{ on } \Gamma,$$

and no-flux conditions for the phase parameter:

$$\frac{\partial \chi}{\partial \nu} = 0 \text{ on } \Gamma.$$  

Relation (6) establishes that the heat flux through the boundary \(\Gamma\) is proportional to the difference between the internal and external temperature \(\vartheta_s > 0\) by a positive coefficient \(\xi\), while (7) prescribes that the boundary has no influence on the phase change process. Finally, we would like to quote the recent paper [26] where a phase separation model of Penrose-Fife type with Signorini boundary condition has been studied.

The main novelties of our contribution stand in the fact that we can consider dynamic boundary conditions for a coupled phase-field system which may display a singular character both in \(\vartheta\) and in \(\chi\) in view of the terms \(1/\vartheta\) and \(s'_0,\varphi(\chi), s'_0,\varphi(\chi)\) in (1)-(4). Physically speaking, the main difference between the dynamic boundary conditions (2), (4) and the “standard” ones (6)-(7) stands in the fact that, in the case of dynamic b.c.’s, the walls of the the container have a significant effect on the phase transition process. This may include the case of two phase changing substances in contact, where one occupies a small layer surrounding the other one and this layer is approximated by a surface in the mathematical model (concentrated capacity phenomenon, see, e.g., [37], [38] for more details).

A rigorous derivation of system (1)-(4) starting from the basic laws of Thermodynamics is carried out in the first part of the paper. Namely, the equations are obtained by combining the free energy balance with the entropy dissipation inequality and imposing physically realistic constitutive expressions for the energy and entropy functionals.

The second part of the paper is devoted to the mathematical analysis of the system in the framework of weak solutions. In comparison with similar models, the main mathematical difficulty consists here in the coupling between

1. the singularity of the model, in particular due to the presence of
   
   (a) the term \(1/\vartheta\) both in the bulk equations (1) and (3) and in the boundary equations (2) and (4),
   
   (b) the (possibly) singular functions \(s'_{0,\varphi}\) in the phase equation (3) in the bulk, and \(s'_{0,\varphi}\) in the phase equation (4) on the boundary,

2. and the occurrence of dynamic boundary conditions.
Actually, as already noted in the case of the Caginalp model with dynamic boundary conditions (see [20]), if we are in presence of singular potentials, a particular care is needed. For instance, in order for equation (3) to make sense a.e. in the space-time domain, the monotone part $f$ of $-s'_{0,b}$ needs to be controlled from above by the monotone part $f_{\Gamma}$ of $-s'_{0,\Gamma}$ (cf. (83) below). In particular, this happens when the potentials $-s'_{0,b}$ and $-s'_{0,\Gamma}$ have the same effective domain (e.g., the set $[-1,1]$, as in the case (5)) and $|s'_{0,b}|$ explodes at most as fast as $|s'_{0,\Gamma}|$ as $\chi$ approaches the boundary of this domain (the values $\pm 1$ in the specific case).

It is worth noticing that, in the case of completely general singular potentials (i.e., without any compatibility condition assumed), a weak solution is still expected to exist; however, the equations (3)-(4) ruling the behavior of $\vartheta$ need to be interpreted in a weaker sense either by means of duality arguments or of variational inequality techniques (cf. [20], [32] for more details). This issue will be addressed in a forthcoming paper, where we also plan to weaken the regularity assumptions on the initial temperature $\vartheta_0$. Actually, in addition to the natural conditions represented by the finiteness of the initial energy and entropy, we will assume here that $\vartheta_0$ has the $L^2$-regularity. This assumption, albeit meaningful, is not required in the physical derivation of the model and can be then considered to be somehow artificial. In other similar contexts (cf., e.g. [40]), the $L^2$-regularity of the initial temperature has been shown to be not necessary and, for this reason, we will try to remove it also for this model.

After proving well-posedness of system (1)-(4), we shall analyze further properties of solutions. Proceeding along the lines of [40] where the case of standard b.c.’s is treated, we shall prove, by using a Moser iteration argument, that the temperature is uniformly bounded from below for strictly positive times. Moreover, if $\vartheta_0$ is slightly more summable (cf. (111) below), we also have a uniform upper bound. This kind of behavior occurs commonly in the framework of parabolic equations with very-fast diffusion terms (see [42] and [3] for the Cauchy problem in the whole space and the recent contributions [40] and [41] for the bounded domain case with Neumann and dynamic boundary conditions, respectively). In particular, in dimension three the exponent $p = 3$ appears to be critical in the sense that solutions starting with initial conditions in $L^p$ with $p > 3$ become $L^\infty$ for strictly positive times. On the other hand, for $p < 3$ the situation is drastically different as the self-similar solution in $\mathbb{R}^3$ (\((\cdot)_+\) denoting the positive part)

$$\Theta(x,t) := \frac{2(T-t)^{1/2}}{|x|}$$

(8)

shows. The smoothing effect for $p = 3$ is currently an open problem (see [42] for further details). Moreover, it is worth noting that, at least when no external source is present, the regularization estimates are also uniform with respect to time and give rise to additional regularity properties for both components of the solution.

Taking advantage of the regularization estimates, in the last part of the paper we finally investigate the long-time behavior of trajectories. Namely, we are able to prove that, at least in case (111) holds, any weak solution admits a non-empty $\omega$-limit set which only contains stationary solutions of the system.

A more precise characterization of the long-time behavior relies on the structure of the set of steady state solutions, which requires some further explanation. Actually, integrating (1) in space, using the boundary condition (2), and assuming zero external source, one readily sees that the value

$$\mu = \int_{\Omega} (\vartheta + \lambda_0(\chi)) \, dx + \int_{\Gamma} (\vartheta + \lambda_\Gamma(\chi)) \, dS,$$

(9)

representing the “total mass” of the internal energy, is conserved in the evolution of the system. Consequently, any limit point of a given weak solution has also to respect the constraint (4), with $\mu$ depending
only on the initial datum. However, when \( \mu \) is small, we are not able to exclude that the set of stationary states satisfying (9) might contain temperatures \( \vartheta \) being arbitrarily close to 0 (note that the stationary formulation of (1)-(2) simply prescribes that \( \vartheta \) is a constant function). Consequently, the right-hand sides of the steady state system associated to (3)-(4) might be arbitrarily large, which considerably weakens the regularity properties of the set of its solutions.

On the contrary, it is easy to show that this situation cannot occur when either \( \mu \) is large enough or \( \lambda' \) satisfies a suitable sign condition (cf. (214) below). If either property holds, the set of stationary states is bounded in a very strong norm, *independently* of the magnitude of the initial data (but depending on the value of \( \mu \)). This fact, together with the precompactness of solution trajectories in the natural energy space and with the existence of a coercive Liapounov functional (namely, the energy \( \Psi \)), implies that the system (11)-(14) admits a smooth global attractor, which is the last result we prove.

We conclude by giving the plan of the paper: in the next Section 2 we detail the physical derivation of the model from the basic thermodynamical principles. In Section 3 we introduce our precise concept of solutions and state our main mathematical results related to well-posedness and regularization properties of weak solutions. The proofs are given in the subsequent Section 4. Finally, the long-time behavior of solutions is separately analyzed in the last Section 5.

## 2 Derivation of the model

The Penrose-Fife model is derived by considering the free energy density \( w \) and the entropy density \( s \), assuming that these quantities depend both on the order parameter \( \chi \) and the absolute temperature \( \vartheta \) (as in the original paper by Penrose and Fife [34]), and imposing that the basic laws of Thermodynamics are satisfied. Assuming that we always have sufficient regularity of the involved variables, we can then write

\[
w = e - \vartheta s \text{ (Gibbs’ relation),} \tag{10}
\]

where

\[
s = -\frac{\partial w}{\partial \vartheta} \tag{11}
\]

and the internal energy density \( e \) is defined by

\[
e = \frac{\partial (\frac{s}{\vartheta})}{\partial (\frac{w}{\vartheta})}. \tag{12}
\]

Moreover, we assume that the total free energy has the expression

\[
\Psi(\chi, \nabla \chi, \vartheta) = \int_{\Omega} \left( \frac{\kappa \vartheta}{2} |\nabla \chi|^2 + w(\vartheta, \chi) \right) \, dx, \tag{13}
\]

whereas the entropy functional is given by

\[
S(\chi, \nabla \chi, e) = \int_{\Omega} \left( -\kappa^2 |\nabla \chi|^2 + s(e, \chi) \right) \, dx, \tag{14}
\]

where \( \Omega \) is the domain occupied by the system and \( \kappa > 0 \) denotes an interfacial energy coefficient.

The evolution equations for \( \chi \) and \( \vartheta \) are then obtained by stating the relations

\[
\frac{\partial \chi}{\partial t} = K^* \frac{\delta S}{\delta \chi}, \quad K^* > 0, \tag{15}
\]
\[
\frac{\partial e}{\partial t} = K \Delta \delta S \frac{\delta e}{\delta e} + h, \quad K > 0,
\]  
(16)

where \( \delta \) denotes a variational derivative and \( h \) is a source term.

Taking finally
\[
e = c_0 \vartheta + \lambda(\chi),
\]  
(17)

where \( c_0 > 0 \) stands for the specific heat of the system (assumed constant) and \( \lambda \) is the latent heat density, typically (cf. [34, p. 53]) given by
\[
\lambda(r) = -ar^2 + br + c, \quad a > 0,
\]  
(18)

we find (see, e.g., [30]; see also below)
\[
s(e, \chi) = c_0 \ln \vartheta + s_0(\chi) + c_1.
\]  
(19)

Here, \(-s_0\) denotes a configuration potential, typically having a double-well character (one can also consider a logarithmic double-well potential of the form \([33]\)), and such that \( s_0'' \leq \delta, \delta \geq 0, \) and \( c_1 \) is a constant. These choices give rise to the Penrose-Fife system:
\[
\frac{\partial \chi}{\partial t} = K^* \left( \kappa \Delta \chi + s'_0(\chi) - \frac{c_0 \lambda'(\chi)}{\vartheta} \right),
\]  
(20)

\[
c_0 \frac{\partial \vartheta}{\partial t} = -K \Delta c_0 \vartheta - \lambda'(\chi) \frac{\partial \chi}{\partial t} + h.
\]  
(21)

These equations are usually endowed with the boundary conditions (cf. the Introduction)
\[
\frac{\partial \chi}{\partial \nu} = 0 \text{ on } \Gamma,
\]  
(22)

\[
\frac{\partial \vartheta}{\partial \nu} = -\xi(\vartheta - \vartheta_s) \text{ on } \Gamma, \quad \xi, \vartheta_s > 0,
\]  
(23)

where \( \Gamma = \partial \Omega \) and \( \nu \) is the unit outer normal vector to \( \Gamma \).

Now, in order to take into account the interactions with the exterior of \( \Omega \) (e.g., the walls), it is natural, following [27] (see also [11] and [19]), to add a boundary contribution to the total free energy and to take, in place of (13),
\[
\Psi = \Psi(\chi, \nabla \chi, \nabla_\Gamma \chi, \vartheta, \vartheta) = \int_\Omega \left( \frac{\kappa_b \vartheta}{2} |\nabla \chi|^2 + w_b(\vartheta, \chi) \right) \, dx + \int_\Gamma \left( \frac{\kappa_\Gamma \vartheta}{2} |\nabla_\Gamma \chi|^2 + w_\Gamma(\vartheta, \chi) \right) \, dS, \quad \kappa_b, \kappa_\Gamma > 0,
\]  
(24)

where \( \nabla_\Gamma \) is the surface gradient and \( w_b \) and \( w_\Gamma \) are the bulk and surface free energy densities, respectively. Similarly, it is reasonable, in view of [24], to introduce the generalized entropy functional
\[
S = S(\chi, \nabla \chi, \nabla_\Gamma \chi, e) = \int_\Omega \left( -\frac{\kappa_b}{2} |\nabla \chi|^2 + s_b(e, \chi) \right) \, dx + \int_\Gamma \left( -\frac{\kappa_\Gamma}{2} |\nabla_\Gamma \chi|^2 + s_\Gamma(e, \chi) \right) \, dS,
\]  
(25)

where \( s_b \) and \( s_\Gamma \) are the bulk and surface entropy densities, respectively.
As before, we assume that Gibbs’ relation holds, namely,

\[ w_b = e - \partial s_b \text{ in } \Omega, \]  
\[ w_\Gamma = e - \partial s_\Gamma \text{ on } \Gamma, \]  

and that

\[ s_b = -\frac{\partial w_b}{\partial \theta} \text{ in } \Omega, \]  
\[ s_\Gamma = -\frac{\partial w_\Gamma}{\partial \theta} \text{ on } \Gamma, \]  
\[ e = \frac{\partial (\frac{w_b}{\theta})}{\partial (\frac{1}{\theta})} \text{ in } \Omega, \]  
\[ e = \frac{\partial (\frac{w_\Gamma}{\theta})}{\partial (\frac{1}{\theta})} \text{ on } \Gamma. \]  

We now note that, in view of (25),

\[ \frac{\delta S}{\delta \chi} = \kappa_b \Delta \chi + \frac{\partial s_b}{\partial \chi} \text{ in } \Omega, \]  
\[ \frac{\delta S}{\delta \chi} = \kappa_\Gamma \Delta_\Gamma \chi + \frac{\partial s_\Gamma}{\partial \chi} - \kappa_b \frac{\partial \chi}{\partial \nu} \text{ on } \Gamma, \]  

where \( \Delta_\Gamma \) is the Laplace-Beltrami operator.

Then, assuming that, as in the classical model,

\[ \frac{\partial \chi}{\partial t} = K^* \frac{\delta S}{\delta \chi}, \quad K^* > 0, \]  

we obtain the equations

\[ \frac{\partial \chi}{\partial t} = K^* \left( \kappa_b \Delta \chi + \frac{\partial s_b}{\partial \chi} \right) \text{ in } \Omega, \]  
\[ \frac{\partial \chi}{\partial t} = K^* \left( \kappa_\Gamma \Delta_\Gamma \chi + \frac{\partial s_\Gamma}{\partial \chi} - \kappa_b \frac{\partial \chi}{\partial \nu} \right) \text{ on } \Gamma. \]  

Next, in order to describe the evolution of the temperature, we generalize (16) as follows. We introduce, for \( U = \left( \frac{u}{v} \right) \) (regular enough at this stage), the linear operator \( A \) defined by

\[ AU = \left( \begin{array}{c} -\Delta u |_{\Omega} \\ -\Delta v |_{\Gamma} + \frac{\partial u |_{\Gamma}}{\partial \nu} \end{array} \right) \]  

and write that

\[ \frac{\partial}{\partial t} \left( \frac{e}{e} \right) = -KA \left( \frac{\delta S}{\delta e} \right) + H, \]  

where \( H = \left( \begin{array}{c} h_b \\ h_\Gamma \end{array} \right) \) is a forcing term, with \( h_b \) and \( h_\Gamma \) standing for the bulk and boundary heat sources, respectively. Introducing the linear operator \( A \) is natural when considering dynamic boundary conditions (see [11] and [19]). Noting that

\[ \frac{\delta S}{\delta e} = \frac{\partial s_b}{\partial e} \text{ in } \Omega, \]
\[ \frac{\delta S}{\delta e} = \frac{\partial s}{\partial e} \] on \( \Gamma \),

we deduce the equations

\[ \frac{\partial e}{\partial t} = K \Delta s + h_b \text{ in } \Omega, \]

\[ \frac{\partial e}{\partial t} = K \Delta e + h_\Gamma - K \frac{\partial}{\partial \nu} \frac{\partial s}{\partial e} \] on \( \Gamma \).

We then assume that (see (17))

\[ e = c_{0,b} \vartheta + \lambda_b(\chi) \text{ in } \Omega, \quad c_{0,b} > 0, \]

\[ e = c_{0,\Gamma} \vartheta + \lambda_\Gamma(\chi) \text{ on } \Gamma, \quad c_{0,\Gamma} > 0. \]

Noting that it follows from (30) and (31) that

\[ e = -\vartheta^2 \frac{\partial (w_b)}{\partial \vartheta} \text{ in } \Omega, \]

\[ e = -\vartheta^2 \frac{\partial (w_\Gamma)}{\partial \vartheta} \text{ on } \Gamma, \]

we have, freezing \( \chi \) and integrating between \( \vartheta_0 \) and \( \vartheta, \vartheta_0, \vartheta > 0, \)

\[ w_b(\vartheta, \chi) - w_b(\vartheta_0, \chi) = -\int_{\vartheta_0}^{\vartheta} \frac{c_{0,b}}{\tau} + \frac{\lambda_b(\chi)}{\tau^2} \, d\tau \text{ in } \Omega, \]

\[ w_\Gamma(\vartheta, \chi) - w_\Gamma(\vartheta_0, \chi) = -\int_{\vartheta_0}^{\vartheta} \frac{c_{0,\Gamma}}{\tau} + \frac{\lambda_\Gamma(\chi)}{\tau^2} \, d\tau \text{ on } \Gamma, \]

which we rewrite in the form

\[ w_b(\vartheta, \chi) = -c_{0,b} \vartheta \ln \vartheta + c_{0,b} \vartheta \ln \vartheta_0 + \lambda_b(\chi) - \vartheta s_{0,b}(\chi) \text{ in } \Omega, \]

\[ w_\Gamma(\vartheta, \chi) = -c_{0,\Gamma} \vartheta \ln \vartheta + c_{0,\Gamma} \vartheta \ln \vartheta_0 + \lambda_\Gamma(\chi) - \vartheta s_{0,\Gamma}(\chi) \text{ on } \Gamma, \]

where we have defined \( s_{0,b} := (w_b(\vartheta_0, \chi) - \lambda_b(\chi))/\vartheta_0 \) and \( s_{0,\Gamma} := (w_\Gamma(\vartheta_0, \chi) - \lambda_\Gamma(\chi))/\vartheta_0. \)

We finally deduce from (28)–(29) and (49)–(50) that

\[ s_b(\epsilon, \chi) = c_{0,b} \ln \vartheta + s_{0,b}(\chi) + c_{1,b}, \quad c_{1,b} = -c_{0,b} \ln \vartheta_0 + c_{0,b} \text{ in } \Omega, \]

\[ s_\Gamma(\epsilon, \chi) = c_{0,\Gamma} \ln \vartheta + s_{0,\Gamma}(\chi) + c_{1,\Gamma}, \quad c_{1,\Gamma} = -c_{0,\Gamma} \ln \vartheta_0 + c_{0,\Gamma} \text{ on } \Gamma. \]

We now note that it follows from (33)–(34) and (51)–(52) that

\[ s_b(\epsilon, \chi) = c_{0,b} \ln \left( \frac{\epsilon - \lambda_b(\chi)}{c_{0,b}} \right) + s_{0,b}(\chi) + c_{1,b} \text{ in } \Omega, \]

\[ s_\Gamma(\epsilon, \chi) = c_{0,\Gamma} \ln \left( \frac{\epsilon - \lambda_\Gamma(\chi)}{c_{0,\Gamma}} \right) + s_{0,\Gamma}(\chi) + c_{1,\Gamma} \text{ on } \Gamma, \]
which yields

\[
\frac{\partial s_b}{\partial t} = \frac{1}{\vartheta}, \quad \frac{\partial s_b}{\partial \chi} = -\frac{\lambda'_b(\chi)}{\vartheta} + s'_{0,b}(\chi) \text{ in } \Omega,
\] (55)

\[
\frac{\partial s_{\Gamma}}{\partial t} = \frac{1}{\vartheta}, \quad \frac{\partial s_{\Gamma}}{\partial \chi} = -\frac{\lambda'_{\Gamma}(\chi)}{\vartheta} + s'_{0,\Gamma}(\chi) \text{ on } \Gamma.
\] (56)

We finally deduce from (39)-(42) and (55)-(56) the following Penrose-Fife system with dynamic boundary conditions:

\[
\frac{\partial \chi}{\partial t} = K^* \left( \kappa_b \Delta \chi + s'_{0,b}(\chi) - \frac{\lambda'_b(\chi)}{\vartheta} \right) \text{ in } \Omega,
\] (57)

\[
\frac{\partial \chi}{\partial t} = K^* \left( \kappa_{\Gamma} \Delta_{\Gamma} \chi + s'_{0,\Gamma}(\chi) - \frac{\lambda'_{\Gamma}(\chi)}{\vartheta} - \kappa_b \frac{\partial \chi}{\partial \nu} \right) \text{ on } \Gamma,
\] (58)

\[
c_{0,b} \frac{\partial \vartheta}{\partial t} = -K \Delta \frac{1}{\vartheta} - \lambda'_b(\chi) \frac{\partial \chi}{\partial t} + h_b \text{ in } \Omega.
\] (59)

\[
c_{0,\Gamma} \frac{\partial \vartheta}{\partial t} = -K \Delta_{\Gamma} \frac{1}{\vartheta} - \lambda'_{\Gamma}(\chi) \frac{\partial \chi}{\partial t} + h_{\Gamma} + K \frac{\partial}{\partial \nu} \frac{1}{\vartheta} \text{ on } \Gamma.
\] (60)

Taking, for simplicity, all constants equal to one, (57)-(60) reduces to

\[
\frac{\partial \chi}{\partial t} = \Delta \chi + s'_{0,b}(\chi) - \frac{\lambda'_b(\chi)}{\vartheta} \text{ in } \Omega,
\] (61)

\[
\frac{\partial \chi}{\partial t} = \Delta_{\Gamma} \chi + s'_{0,\Gamma}(\chi) - \frac{\lambda'_{\Gamma}(\chi)}{\vartheta} - \frac{\partial \chi}{\partial \nu} \text{ on } \Gamma,
\] (62)

\[
c_{0,b} \frac{\partial \vartheta}{\partial t} = -K \Delta \frac{1}{\vartheta} - \lambda'_b(\chi) \frac{\partial \chi}{\partial t} + h_b \text{ in } \Omega,
\] (63)

\[
c_{0,\Gamma} \frac{\partial \vartheta}{\partial t} = -K \Delta_{\Gamma} \frac{1}{\vartheta} - \lambda'_{\Gamma}(\chi) \frac{\partial \chi}{\partial t} + h_{\Gamma} + \frac{\partial}{\partial \nu} \frac{1}{\vartheta} \text{ on } \Gamma,
\] (64)

which can also be rewritten in the following compact form:

\[
\frac{\partial}{\partial t} \left( \begin{array}{c} \chi|_{\Omega} \\ \chi|_{\Gamma} \\ \vartheta|_{\Omega} \\ \vartheta|_{\Gamma} \end{array} \right) = -A \left( \begin{array}{c} \chi|_{\Omega} \\ \chi|_{\Gamma} \\ \vartheta|_{\Omega} \\ \vartheta|_{\Gamma} \end{array} \right) + \left( \begin{array}{c} s'_{0,b}(\chi) \\ s'_{0,\Gamma}(\chi) \\ \lambda'_b(\chi) \frac{\partial \chi}{\partial \vartheta} \\ \lambda'_{\Gamma}(\chi) \frac{\partial \chi}{\partial \vartheta} \end{array} \right),
\] (65)

\[
\frac{\partial}{\partial t} \left( \begin{array}{c} \chi|_{\Omega} \\ \chi|_{\Gamma} \\ \vartheta|_{\Omega} \\ \vartheta|_{\Gamma} \end{array} \right) = A \left( \begin{array}{c} \chi|_{\Omega} \\ \chi|_{\Gamma} \\ \vartheta|_{\Omega} \\ \vartheta|_{\Gamma} \end{array} \right) - \left( \begin{array}{c} \lambda'_b(\chi) \frac{\partial \chi}{\partial \vartheta} \\ \lambda'_{\Gamma}(\chi) \frac{\partial \chi}{\partial \vartheta} \end{array} \right) + H.
\] (66)

The remainder of the paper is devoted to the mathematical analysis of system (65)-(66) in the framework of weak solutions.
3 Main assumptions and preliminary results

We introduce here our main assumptions, together with several mathematical tools which are needed in order to provide a precise analytical statement of our results.

We let \( \Omega \) be a sufficiently smooth, bounded, and connected domain in \( \mathbb{R}^3 \) with boundary \( \Gamma \). We set \( \overline{\Omega} := \Omega \cup \Gamma \), set \( H := L^2(\Omega) \), and denote by \( (\cdot, \cdot) \) the scalar product both in \( H \) and in \( H^3 \) and by \( \| \cdot \| \) the related norm. Next, we set \( V := H^1(\Omega) \) and denote by \( V' \) the (topological) dual of \( V \). The duality between \( V' \) and \( V \) will be indicated by \( (\cdot, \cdot)_{V'} \). Identifying \( H \) with \( H' \) through the scalar product of \( H \), it is then well known that \( V \subset H \subset V' \) with continuous and dense inclusions. In other words, \((V, H, V')\) constitutes a Hilbert triplet (see, e.g., [29]). Such a triplet is usually used for stating weak formulations of elliptic or parabolic problems defined on \( \Omega \).

However, since system (63)-(66) also includes equations defined on \( \Gamma \), we need to introduce some further spaces taking also boundary contributions into account. Thus, we set \( H^\Gamma := L^2(\Gamma) \), \( V^\Gamma := H^1(\Gamma) \), and denote by \( (\cdot, \cdot)_{\Gamma} \) the scalar product in \( H^\Gamma \), by \( \| \cdot \|_{\Gamma} \) the corresponding norm, and by \( (\cdot, \cdot)_{\Gamma} \) the duality between \( V^\Gamma \) and \( V^\Gamma \). In general, the symbol \( \| \cdot \|_X \) indicates the norm in the generic (real) Banach space \( X \) and \( (\cdot, \cdot)_X \) stands for the duality between \( X' \) and \( X \). We also denote by \( \nabla^\Gamma \) the tangential gradient on \( \Gamma \) and by \( \Delta^\Gamma \) the Laplace-Beltrami operator. Thus, we can define the spaces

\[
\mathcal{H} := H \times H^\Gamma \quad \text{and} \quad \mathcal{V} := \{z \in V : z|_\Gamma \in V^\Gamma\}. \tag{67}
\]

Here and in the following, \( z|_\Gamma \), or also \( z_\Gamma \), will denote the trace of \( z \) in the sense of a suitable trace theorem. Next, we introduce the \( \mathcal{H} \)-scalar product in the following natural way:

\[
((k, \kappa), (s, \sigma))_{\mathcal{H}} := (k, s) + (\kappa, \sigma)_{\Gamma}. \tag{68}
\]

Then, we set, respectively on \( \mathcal{V} \) and on \( V \),

\[
((z, w))_{\mathcal{V}} := \int_{\Omega} (\nabla z \cdot \nabla w) \, dx + \int_{\Gamma} (z|_\Gamma w|_\Gamma + \nabla^\Gamma z|_\Gamma \cdot \nabla^\Gamma w|_\Gamma) \, dS, \tag{69}
\]

\[
((z, w))_{V} := \int_{\Omega} (\nabla z \cdot \nabla w) \, dx + \int_{\Gamma} z|_\Gamma w|_\Gamma \, dS, \tag{70}
\]

together with the related norms \( \| \cdot \|_{\mathcal{V}}, \| \cdot \|_V \). It is not difficult to prove (see, e.g., [33, Lemma 2.1]) that the space \( \mathcal{V} \) is dense in \( \mathcal{H} \). Concerning the scalar product in (70), we notice that the related norm \( \| \cdot \|_V \) is obviously equivalent to the usual one. We also set

\[
\mathcal{W} := \{z \in \mathcal{V} : z \in H^2(\Omega), \ z|_\Gamma \in H^2(\Gamma)\} \tag{71}
\]

and endow this space with the graph norm, so that \( \mathcal{W} \subset \mathcal{V} \), continuously and compactly.

The above defined functional spaces allow to introduce some elliptic operators. We set:

\[
A : V \to V', \quad (Az_1, z_2) := \int_{\Omega} \nabla z_1 \cdot \nabla z_2 \, dx, \tag{72}
\]

\[
A^\Gamma : V^\Gamma \to V^\Gamma', \quad (A\Gamma \zeta_1, \zeta_2)_{\Gamma} := \int_{\Gamma} \nabla^\Gamma \zeta_1 \cdot \nabla^\Gamma \zeta_2 \, dS, \tag{73}
\]

\[
\mathcal{A} : \mathcal{V} \to \mathcal{V}', \quad (\mathcal{A}z_1, z_2)_{\mathcal{V}} := (Az_1, z_2) + (A_{\Gamma} z_1, z_2)_{\Gamma}. \tag{74}
\]

We shall also use the operator \( \mathcal{A} \) defined in (67). Note that \( \mathcal{A} \) can be interpreted as an operator defined on \( \mathcal{W} \) and taking values in \( \mathcal{H} \), thanks to the trace theorem for normal derivatives.
In what follows, we will use the following convention: as far as equations on $\Omega$ are concerned, the elements of $V$ will be interpreted as functions defined on $\Omega$ with the proper regularity. When, instead, as in most cases in the paper, a system defined on $\Omega \times \Gamma$ is considered, then the elements of $V$ will be considered as pairs of functions $(z, z_{|\Gamma})$. In other words, $V$ will be identified with a (closed) subspace of the product space $H^1(\Omega) \times H^{1/2}(\Gamma)$. Analogously, $V$ will be identified with a subspace of $H^1(\Omega) \times H^1(\Gamma)$, in view of the trace theorem. If we have, instead, $h \in H$, $h$ will be often thought as a pair of functions belonging, respectively, to $H$ and to $H_{\Gamma}$, and both denoted by the same letter $h$. Of course, if we do not have additional regularity, the second component of $h$ needs not be the trace of the first one. Identifying $H$ with $H'$ through the scalar product (68), we obtain the chain of continuous and dense (thanks to the density of $V$ into $H$, to the density of $H^2(\Omega)$ into $V$, and to the continuous inclusion $H^2(\Omega) \subset V$) embeddings

$$V \subset V \subset H \subset V' \subset V'.$$

(75)

In particular, the relation

$$((k, \kappa), (z, z_{|\Gamma}))_H = \int_{\Omega} k z \, dx + \int_{\Gamma} \kappa z_{|\Gamma} \, dS = ((k, \kappa), z)_V$$

(76)

holds for any $z \in V$ and $(k, \kappa) \in H$. Of course, an analogous relation could be stated for $z \in V$. Next, for any function, or functional $z$, defined on $\Omega$, we set

$$m_\Omega(z) := \frac{1}{|\Omega|} \int_\Omega z \, dx,$$

(77)

where the integral is substituted with the duality $(z, 1)$ in case, e.g., $z \in V'$. We also define the measure $dm$ given by

$$\int v \, dm := \int_{\Omega} v \, dx + \int_{\Gamma} v_{|\Gamma} \, dS,$$

(78)

where $v$ represents a generic function in $L^1(\Omega) \times L^1(\Gamma)$. With some abuse of notation, we will also write

$$m(v) := \frac{1}{|\Omega| + |\Gamma|} \int v \, dm,$$

(79)

i.e., the “mean value” of $v$ w.r.t. the measure $dm$. Here $|\Gamma|$ represents the surface measure of $\Gamma$.

With these functional spaces at our disposal, we can now state our hypotheses on the nonlinear terms. For convenience, we split $-s_0, b$ (respectively, $-s_{0, \Gamma}$) into a sum of a (dominating) monotone part $f$ (respectively, $f_{\Gamma}$) and a linear perturbation. More precisely, we assume that

$$-s_0, b = f(r) - \delta r \quad \forall r \in \text{dom } f, \quad -s_{0, \Gamma}(r) = f_{\Gamma}(r) - \delta r \quad \forall r \in \text{dom } f_{\Gamma}$$

(80)

and for some $\delta \geq 0$, with

$$f \in C^0(\text{dom } f, \mathbb{R}), \quad f_{\Gamma} \in C^0(\text{dom } f_{\Gamma}, \mathbb{R}) \text{ monotone}, \quad f(0) = f_{\Gamma}(0) = 0,$$

(81)

where dom $f$ and dom $f_{\Gamma}$ (i.e., the domains of $f$ and $f_{\Gamma}$) are open intervals of $\mathbb{R}$ containing 0. We will say that, for instance, $-s_{0, b}$ is a “singular” potential if its domain does not coincide with the whole real line (and we will say that it is a “regular” potential otherwise). In both cases, we will assume that

$$\lim_{r \to 0^+ \text{ dom } f} (f(r) - \delta r) \text{ sign } r = \lim_{r \to 0^+ \text{ dom } f_{\Gamma}} (f_{\Gamma}(r) - \delta r) \text{ sign } r = +\infty.$$

(82)
The key assumption that will allow us to obtain a pointwise estimate of the nonlinear terms $f(x)$ and $f_T(x)$ is the following compatibility condition: we ask that there exist two constants $c_s > 0$ and $C_s \geq 0$ such that

$$\text{dom}(f_T) \subseteq \text{dom}(f), \quad f(r)f_T(r) \geq c_s|f(r)|^2 - C_s \quad \forall r \in \text{dom}(f_T). \quad (83)$$

In other words, the boundary nonlinear term $f_T(r)$, up to the sign, has to be larger than the bulk nonlinear term $f(r)$, at least for $r$ far from 0. We also introduce, whenever they make sense, the antiderivatives

$$F(r) := \int_0^r f(s) \, ds, \quad F_T(r) := \int_0^r f_T(s) \, ds. \quad (84)$$

Notice that, in case (for instance) $\text{dom} f$ is bounded, but $f$ is globally summable on $\text{dom} f$, $F$ can (and will) be extended by continuity to $\overline{\text{dom} f}$. This is the case, e.g., of the logarithmic potential $[5]$. Moreover, $F$ (and, analogously, $F_T$) will be thought to be further extended to the whole real line by assuming the value $+\infty$ outside its effective domain. Then, identifying $f$ and $f_T$ with maximal monotone graphs in $\mathbb{R} \times \mathbb{R}$, we have $f = \partial F$ and $f_T = \partial F_T$, $\partial$ representing the subdifferential of convex analysis (here in $\mathbb{R}$). We refer to the monographs [1, 2, 4] for an extensive presentation of the theory of maximal monotone operators and of subdifferentials.

To obtain an estimate of the full $V$-norm of $u$, we will also need a proper form of Poincaré’s inequality (see, e.g., [40, Lemma 3.2]):

**Lemma 3.1.** Assume that $\Omega$ is a bounded open subset of $\mathbb{R}^d$. Suppose $v \in W^{1,1}(\Omega)$ and $v \geq 0$ a.e. in $\Omega$. Then, setting $K := \int_0^1 (\log v)^+ \, dx$, the following estimate holds:

$$\|v\|_{L^1(\Omega)} \leq \|\Omega\| e^{C_1 K} + \frac{C_2}{|\Omega|} \|\nabla v\|_{L^1(\Omega)}, \quad (85)$$

the constants $C_1$ and $C_2$ depending only on $\Omega$.

Besides assumptions (50)-(53), we shall analyze system (54)-(57) under the following hypothoses:

$$\lambda_6 \in C^2(\text{dom } f), \quad \lambda''_\Gamma \in L^\infty(\text{dom } f_T), \quad \lambda' \in C^2(\text{dom } f_T) \text{ with } \lambda'' \in L^\infty(\text{dom } f_T), \quad (86)$$

$$\vartheta_0, \eta_0 \in \mathcal{H}, \quad \vartheta_0, \eta_0 > 0 \text{ a.e., } \quad (\log \vartheta_0, \log \eta_0) \in L^1(\Omega) \times L^1(\Gamma), \quad (87)$$

$$\chi_0 \in \mathcal{V} \text{ with } s_{0,\Gamma}(\chi_0) \in L^1(\Gamma) \text{ and } s_{0,\Gamma}(\chi_0) \in L^1(\Gamma), \quad (88)$$

$$\mathbf{H} \in L^2(0, T; \mathcal{H}), \quad m(\mathbf{H}) = 0 \text{ a.e. in } (0, T). \quad (89)$$

Let us define the energy functional as

$$\mathcal{E}[(\vartheta, \eta), (\chi, \chi_\Gamma)] := \int_{\Omega} \left( \vartheta - \log \vartheta + \lambda_6(\chi) + \frac{\|\nabla \chi\|^2}{2} - s_{0,\Gamma}(\chi) \right) \, dx$$

$$+ \int_{\Gamma} \left( \eta - \log \eta + \lambda_\Gamma(\chi_\Gamma) + \frac{\|\nabla \chi_\Gamma\|^2}{2} - s_{0,\Gamma}(\chi_\Gamma) \right) \, ds, \quad (90)$$

whenever it makes sense. We will often just write $\mathcal{E}[\vartheta, \chi]$ for brevity. Note that we admit $\mathcal{E}$ to take the value $+\infty$. However, we can readily check that the above assumptions (57)-(58) make the initial energy finite. Namely, we have

$$\mathcal{E}_0 := \mathcal{E}[(\vartheta_0, \eta_0), (\chi_0, \chi_{0\Gamma})] < \infty. \quad (91)$$
Together with [86]-[90], we will also need the energy functional $\mathcal{E}$ to be coercive with respect to $(\chi, \chi_\Gamma)$. This is obtained asking that there exist $c_\vartheta > 0$ and $c \in \mathbb{R}$ such that

$$\mathcal{E}[\vartheta, \chi] \geq c_\vartheta \|\chi\|_{V'}^2 - c \quad \text{for all pairs } (\vartheta, \chi).$$

(92)

Such an assumption is satisfied for instance whenever

$$\lambda_b(r) - s_{0,b}(r) \geq c_1 r^2 - c_2 \quad \text{for all } r \in \text{dom } f \quad \text{and} \quad \lambda_\Gamma(r) - s_{0,\Gamma}(r) \geq c_1 r^2 - c_2 \quad \text{for all } r \in \text{dom } f_\Gamma,$$

(93)

and for some $c_1 > 0$ and $c_2 \in \mathbb{R}$. Of course, (92) holds trivially if $-s_{0,b}$, or $-s_{0,\Gamma}$, or both, are singular potentials, or also in the case of standard double-well potentials.

It is worth noting that the first of assumptions [80] on $(\vartheta_0, \eta_0)$ is somehow artificial, in the sense that it is stronger than what would be required in order for the initial energy to be finite. Actually, one could relax (87) by taking just $(\vartheta, \eta)$ in $L^1(\Omega) \times L^1(\Gamma)$ or even $(\vartheta_0, \eta_0) \in \mathcal{V}'$, paying the price of having a weaker (and more delicate to deal with) notion of solution. The Neumann-Neumann Penrose-Fife model with $L^1$ or $\mathcal{V}'$ initial temperature has been recently studied in [40]. In a forthcoming paper, we intend to analyze also the present model in a similar regularity setting.

3.1 Weak solutions

We use the functional framework introduced above to specify a rigorous concept of weak solution to the initial value problem for [85]-[84], named Problem (P) in what follows.

**Definition 3.2.** We say that a quadruplet $(\vartheta, \eta, u, \chi)$ is a weak solution to Problem (P) if there hold

$$\begin{align*}
(\vartheta, \eta) &\in H^1(0, T; \mathcal{V}') \cap L^\infty(0, T; \mathcal{H}), \\
(\log \vartheta, \log \eta) &\in L^\infty(0, T; L^1(\Omega) \times L^1(\Gamma)), \quad \vartheta, \eta > 0 \quad \text{almost everywhere}, \\
u &\in L^2(0, T; \mathcal{V}), \\
\chi &\in H^1(0, T; \mathcal{H}) \cap L^\infty(0, T; \mathcal{V}) \cap L^2(0, T; \mathcal{W}), \\
f(\chi) &\in L^2(0, T; H), \quad f_\Gamma(\chi_\Gamma) \in L^2(0, T; H_\Gamma),
\end{align*}$$

(94) (95) (96) (97) (98)

together with, a.e. in $(0, T)$, the equations

$$\begin{align*}
\frac{\partial}{\partial t} \left( \begin{array}{c} \vartheta \\ \eta \end{array} \right) &\quad = \mathcal{A} \left( \begin{array}{c} u \\ u_\Gamma \end{array} \right) - \left( \begin{array}{c} \lambda_b(\chi)_t \\ \lambda_\Gamma(\chi_\Gamma)_t \end{array} \right) + \left( \begin{array}{c} h_b \\ h_\Gamma \end{array} \right), \quad \text{in } \mathcal{V}', \\
\frac{\partial}{\partial t} \left( \begin{array}{c} \chi \\ \chi_\Gamma \end{array} \right) &\quad + \mathcal{A} \left( \begin{array}{c} \chi \\ \chi_\Gamma \end{array} \right) - \left( \begin{array}{c} s_{0,b}(\chi) \\ s_{0,\Gamma}(\chi_\Gamma) \end{array} \right) = \left( \begin{array}{c} \lambda_b(\chi)_t \\ \lambda_\Gamma(\chi_\Gamma)_t \end{array} \right), \quad \text{in } \mathcal{H},
\end{align*}$$

(99) (101)

and the initial conditions

$$\begin{align*}
(\vartheta, \eta)|_{t=0} &\quad = (\vartheta_0, \eta_0), \quad (\chi, \chi_\Gamma)|_{t=0} = (\chi_0, \chi_\Gamma_0) \quad \text{a.e. in } \Omega \text{ and on } \Gamma.
\end{align*}$$

(102)

Sometimes, for brevity, we shall indicate a solution just as a pair $(\vartheta, \chi)$ rather than as a quadruplet $(\vartheta, \eta, u, \chi)$. 

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Remark 3.3. It is worth giving some explanation on the boundary behavior of \( \vartheta \). Since \( u = -1/\vartheta \in L^2(0; T; \mathcal{V}) \) by (90), it turns out that \( u \) has a trace \( u_\Gamma \) on \( \Gamma \), which belongs to \( V_\Gamma \) for almost every value of the time variable thanks to the definition of \( \mathcal{V} \). On the other hand, we cannot simply write \( \eta = \vartheta_\Gamma \) since the trace of \( \vartheta \) needs not exist. We have, instead, to intend \( \eta \) as (minus) the reciprocal of the trace of \( u \), as specified by the second (100). When we consider smoother solutions (for instance, in the a priori estimates, or in the case when \( \vartheta_0 \) is more summable, cf. (111) below), this problem does not occur since the higher regularity of \( \vartheta \) permits to give sense to its trace. Regarding the phase variable, the situation is simpler; indeed, by (97), \( \chi \) is always smooth enough to have a trace \( \chi_\Gamma \).

3.2 Existence and uniqueness results

In this part we state our main results regarding well posedness of Problem (P) and regularization properties of weak solutions. In what follows we will denote by \( c \) a positive constant, which may vary from line to line (or even in the same formula), depending only on the data of the problem. Specific dependences will be indicated when needed. Moreover, we will denote by \( Q \) a nonnegative-valued, continuous and monotone increasing function of its arguments. Our main result can be stated as follows:

Theorem 3.4. Under assumptions (90)-(93), (89), and (92), there exists a unique weak solution to Problem (P). Moreover, for all \( t > 0 \) there holds the energy identity
\[
\mathcal{E}((\vartheta(t), \eta(t)), (\chi(t), \chi_\Gamma(t))) + \int_0^t \left( \|\nabla u(s)\|_{\mathcal{H}}^2 + \|\chi_t(s)\|_{\mathcal{H}}^2 \right) ds = E_0 + \int_0^t (H(s), u(s))_{\mathcal{H}} ds. \tag{103}
\]
If, in addition to (89), the forcing function satisfies
\[
H = (h_0, h_\Gamma) \in H^1(0, T; \mathcal{V}') \cap L^2(0, T, L^{3+\varepsilon}(\Omega) \times L^{3+\varepsilon}(\Gamma)), \tag{104}
\]
then we also have, for any \( \tau \in (0, T) \), the following regularization properties:
\[
\begin{align*}
\|u\|_{L^\infty(\tau; T; \mathcal{V})} + \|u\|_{L^\infty((\tau, T) \times \Omega)} & \leq Q(E_0, \tau^{-1}), \tag{105} \\
\|\chi\|_{L^\infty(\tau; T; H^2(\Omega))} + \|f(\chi)\|_{L^\infty(\tau; T; H)} & \leq Q(E_0, \tau^{-1}), \tag{106} \\
\|u_\Gamma\|_{L^\infty(\tau; T; \mathcal{V}_\Gamma)} + \|u_\Gamma\|_{L^\infty((\tau, T) \times \Gamma)} & \leq Q(E_0, \tau^{-1}), \tag{107} \\
\|\chi_\Gamma\|_{L^\infty(\tau; T; H^2(\Gamma))} + \|f_\Gamma(\chi_\Gamma)\|_{L^\infty(\tau; T; H)} & \leq Q(E_0, \tau^{-1}). \tag{108}
\end{align*}
\]

The proof of the existence part is based on an approximation-a priori bounds-passage to the limit procedure. Additional estimates lead to (105)-(108). In particular the \( L^\infty \)-bounds in (105) and in (107) are obtained by adapting a Moser iteration scheme with regularization devised in [40]. Note that the existence part generalizes to the dynamic boundary conditions case the result of Ito and Kenmochi [16].

In the case of singular potentials, we can also prove that, at least for strictly positive times, \( \chi \) is uniformly separated from the singularities of the potentials. In order to avoid unnecessary technical complications, we shall state this property under the additional assumption that
\[
\text{dom } f = \text{dom } f_\Gamma = (-1, 1), \quad |f_\Gamma(r)| \geq \kappa_s |f(r)| - C_s \quad \forall r \in (-1, 1) \tag{109}
\]
and for some \( \kappa_s \in (0, 1], C_s \geq 0 \). Namely, we assume the potentials to be normalized so that the pure states are represented by the values \( \pm 1 \), both on the bulk and on the boundary. Moreover, in view of (89), we require \( |f_\Gamma| \) to be larger than (a positive constant times) \( |f| \), at least in proximity of \( \pm 1 \).
Corollary 3.5. Let the assumptions of Theorem 3.4 hold, together with (109). Then, for any \( \tau > 0 \), there exists \( \omega \in (0,1) \), depending on \( \tau \) but independent of \( T \), such that
\[
|\chi(t, x)| \leq 1 - \omega \text{ almost everywhere in } (\tau, T) \times \Omega \text{ and in } (\tau, T) \times \Gamma.
\]
(110)

Next, we discuss the asymptotic regularization of the temperature field. Actually, we have already noted (cf. (105), (107)) that condition (104) on the forcing function \( H \) is sufficient in order for \( \vartheta \) to become uniformly separated from zero for strictly positive times. The following result (which generalizes [40, Thm. 2.7], where Neumann conditions are considered) states that \( \vartheta \) is bounded from above, at least for strictly positive times, provided that (104) holds and the initial temperature enjoys some additional summability property.

Proposition 3.6. Let the assumptions of Theorem 3.4 hold. Let also assume that
\[
(\vartheta_0, \eta_0) \in L^{3+\epsilon}(\Omega) \times L^{3+\epsilon}(\Gamma), \quad \text{for some } \epsilon > 0,
\]
(111)
either (109) holds, or \( \chi_0 \in L^{\infty}(\Omega, dm) \).
(112)

Then, any weak solution to Problem (P) satisfies, for any \( \tau \in (0,T) \),
\[
\|\vartheta\|_{L^\infty(\tau,T;\Omega)} + \|\eta\|_{L^\infty(\tau,T;\Gamma)} \leq Q\left(E_0, \tau^{-1}, \|\vartheta_0\|_{L^{3+\epsilon}(\Gamma)}, \|\eta_0\|_{L^{3+\epsilon}(\Gamma)}, \|\chi_0\|_{L^{\infty}(\Omega, dm)}\right).
\]
(113)

We point out that, in the case of singular potentials, the uniform boundedness of \( \chi_0 \) required by (113) is a direct consequence of (88).

As a byproduct of Prop. 3.6, we can also prove additional regularity of the time derivative of \( \vartheta \):

Corollary 3.7. Under the assumptions of Proposition 3.6, we have
\[
\vartheta \in H^1(\tau, T; \mathcal{H}) \text{ for any } \tau > 0.
\]
(114)

As a consequence, (99) can be decoupled and interpreted in the strong form (66) as a relation in \( \mathcal{H} \).

4 Proofs of the main results

4.1 Proof of Theorem 3.4: a priori estimates

As a first step, we detail the main estimates constituting the core of the existence proof. In order to simplify the exposition, we limit ourselves to perform formal a priori bounds on the solutions of Problem (P). In the next section we will see that these bounds imply weak sequential stability. It is clear that, in a formal proof, the estimates should be performed in the framework of a proper approximation scheme (e.g., a Faedo-Galerkin approximation or a time discretization), possibly combined with some regularization of the data. However, this kind of procedure has been already described in full detail in several papers related to similar models (see, e.g., [17] and [20]) and, actually, the arguments given in these papers could be easily adapted to our case.

That said, we start by presenting the estimates. In all what follows, we shall assume to have sufficient regularity to justify all the computations. In particular, we ask \( \vartheta \) to be smooth enough to have a trace, so that \( \eta = \vartheta_\Gamma \) (and correspondingly \( 1/\eta = 1/\vartheta_\Gamma \)).
On the other hand, we treat the integral over $\Gamma$ in this way:

$$\frac{d}{dt} \mathcal{E}[\vartheta, \chi] + \int \nabla (-1/\vartheta)^2 \, dx + \int |\nabla \chi| \, dS + \int |\chi_t|^2 \, dx + \int |\chi_{\Gamma,t}|^2 \, dS = -(\mathbf{H}, 1/\vartheta)_K. \quad (115)$$

Hence, collecting the above computations, $(115)$ gives the a priori bound

$$\text{Estimate of the nonlinear terms.} \quad \text{Estimate (120) gives that}$$

$$\int \mathcal{E}[\vartheta, \chi](t) + \int_0^T (\|1/\vartheta(s)\|_V^2 + \|\chi_t(s)\|_{H^1_V}^2) \, ds \leq Q(\mathcal{E}_0, T, \|\mathbf{H}\|_{L^2(0,T;\mathcal{H})}) \quad \forall \, t \leq T. \quad (120)$$

Using that $m(\mathbf{H}) = 0$ (cf. (29)), we can write

$$\mathbf{H}, 1/\vartheta)_K = (\mathbf{H}, 1/\vartheta - m_\Omega(1/\vartheta))_K = \int \mathcal{H}_b \left( - m_\Omega \left( \frac{1}{\vartheta} \right) \right) \, dx + \int \mathcal{H}_G \left( - m_\Omega \left( \frac{1}{\vartheta} \right) \right) \, dS. \quad (119)$$

Now, the integral over $\Omega$ is easily estimated, using the Poincaré-Wirtinger inequality, as

$$\int \mathcal{H}_b \left( \frac{1}{\vartheta} - m_\Omega \left( \frac{1}{\vartheta} \right) \right) \, dx \leq c \varepsilon \|h_b\|^2 + \varepsilon \|\nabla (1/\vartheta)\|^2 \quad \forall \varepsilon > 0.$$

On the other hand, we treat the integral over $\Gamma$ in this way:

$$\int \mathcal{H}_G \left( \frac{1}{\vartheta} - m_\Omega \left( \frac{1}{\vartheta} \right) \right) \, dS \leq c \varepsilon \|h_G\|^2 + \varepsilon \|\nabla (1/\vartheta)\|^2 \quad \forall \varepsilon > 0,$$

where in the second and in the fourth inequalities we have used, respectively, the trace theorem and the Poincaré-Wirtinger inequality. Hence, taking $\varepsilon$ small enough and integrating (119), we obtain

$$\mathcal{E}(t) + \frac{1}{2} \int_0^t \left( \|1/\vartheta\|^2_V + \|\nabla (1/\vartheta)\|^2_{H^1} \right) \, ds \leq E_0 + c \|\mathbf{H}\|_{L^2(0,T;\mathcal{H})}^2. \quad (117)$$

To control the full $V$-norm of $1/\vartheta$, we have to provide a bound of its mean value (actually, only the gradient is estimated in (117)). To this aim, we use Lemma (3.1) with $v = -1/\vartheta$, obtaining

$$\|1/\vartheta\|_{L^1(\Omega)} \leq |\Omega| e^{c_1} f_0(\log \vartheta)^{-} + \frac{c_2}{|\Omega|} \|
abla (1/\vartheta)\|_{L^1(\Omega)}, \quad (118)$$

and the first term on the right-hand side is uniformly bounded thanks to (117) and to the expression (90) of the energy functional. Moreover, by the trace theorem, we get

$$\|1/\vartheta\|_{V} \leq c \|1/\vartheta\|_{V} + \|
abla (1/\vartheta)\|_{H^1} \quad (119)$$

Hence, collecting the above computations, (115) gives the a priori bound

$$\mathcal{E}[\vartheta, \chi](t) + \int_0^T (\|1/\vartheta(s)\|_V^2 + \|\chi_t(s)\|_{H^1_V}^2) \, ds \leq Q(\mathcal{E}_0, T, \|\mathbf{H}\|_{L^2(0,T;\mathcal{H})}) \quad \forall \, t \leq T. \quad (120)$$

**Estimate of the nonlinear terms.** Estimate (120) gives that $u = -1/\vartheta \in L^2(0,T;V)$ and $\chi \in L^\infty(0,T;V)$, implying, thanks to (85), that the right-hand side of the phase equation (101) belongs...
to $L^2(0,T;\mathcal{H})$. Now, let us test \ref{101} (both on the bulk and on the boundary) by $f(\chi)$. Using the monotonicity of $f$ and assumption \ref{S0} we then get

$$
\frac{d}{dt} \int_{\Omega} F(\chi) \, dm + \| f(\chi) \|^2 + \int_{\Gamma} f(\chi_{\Gamma}) \left( f_{\Gamma}(\chi_{\Gamma}) - \delta \chi_{\Gamma} + \frac{\lambda_{\Gamma}(\chi_{\Gamma})}{\partial_{\Gamma}} \right) \, dS
\leq \int_{\Omega} f(\chi) \left( \delta \chi - \frac{\lambda_{\Gamma}(\chi_{\Gamma})}{\partial_{\Gamma}} \right) \, dx \leq \frac{1}{2} \| f(\chi) \|^2 + c \| \chi \|^2 + c(1 + \| \chi \|_{L^4(\Omega)}^2) \| u \|_{L^2(\Omega)}^2.
$$

(121)

The key point is represented by the control of the last term on the left-hand side, and here the compatibility condition (83) comes into play. Indeed, using (83), we readily arrive at

$$
\int_{\Gamma} f(\chi_{\Gamma}) \left( f_{\Gamma}(\chi_{\Gamma}) - \delta \chi_{\Gamma} + \frac{\lambda_{\Gamma}(\chi_{\Gamma})}{\partial_{\Gamma}} \right) \geq c_0 \frac{1}{2} \| f(\chi_{\Gamma}) \|^2 - c \| \chi_{\Gamma} \|^2 - c(1 + \| \chi_{\Gamma} \|_{L^4(\Gamma)}^2) \| u_{\Gamma} \|_{L^4(\Gamma)}^2.
$$

(122)

Hence, integrating (121) in time and using (122) and (120), we infer

$$
\| f(\chi) \|_{L^2(0,T;H)} + \| f(\chi_{\Gamma}) \|_{L^2(0,T;H_{\Gamma})} \leq Q(E_0, T).
$$

(123)

Here and below, we allow $Q$ to depend additionally on $\mathcal{H}$. Moreover, we recall that $\| F(\chi_0) \|_{L^1(\Omega, dm)} \leq Q(E_0)$ thanks to (88) and (83).

Next, (120), (123), and a comparison of terms in (61) (i.e., the bulk component of (59) in the strong formulation – recall that we assume the solutions to be smooth at this level), give

$$
\| \Delta \chi \|_{L^2(0,T;H)} \leq Q(E_0, T).
$$

(124)

Consequently, using standard trace and elliptic regularity theorems (cf., e.g. \cite{5} Theorem 2.7.7 and Theorem 3.1.5), it is not difficult to arrive at

$$
\| \partial_{\nu} \chi \|_{L^2(0,T;H_{\Gamma})} \leq Q(E_0, T).
$$

(125)

This allows to test the boundary equation (62) by $f_{\Gamma}(\chi_{\Gamma})$. Proceeding as above (but without using (61)) and controlling the term $\partial_{\nu} \chi$ directly by means of (125), we finally arrive at

$$
\| \Delta_{\Gamma} \chi_{\Gamma} \|_{L^2(0,T;H_{\Gamma})} + \| f_{\Gamma}(\chi_{\Gamma}) \|_{L^2(0,T;H_{\Gamma})} \leq Q(E_0, T).
$$

(126)

Now, let us come to the temperature equation. By (120) and assumption (80), we have

$$
\| \partial_t \lambda_b(\chi) \|_{L^2(0,T;L^3(\Omega))} \leq \| \lambda_{\Gamma}(\chi) \|_{L^\infty(0,T;L^6(\Omega))} \| \chi_t \|_{L^2(0,T;H)} \leq Q(E_0, T),
$$

(127)

and a similar relation on $\Gamma$. Actually, when $-s_{0,b}$ and $-s_{0,\Gamma}$ are singular potentials, we automatically have a uniform $L^\infty$-bound on $\chi$. Hence, we obtain more precisely

$$
\| \partial_t \lambda_b(\chi) \|_{L^2(0,T;H)} \leq \| \lambda_{\Gamma}(\chi) \|_{L^\infty(\Omega)} \| \chi_t \|_{L^2(0,T;H)} \leq Q(E_0, T).
$$

(128)

By a comparison of terms in the (coupled) weak formulation (69), we then get in both cases

$$
\| \partial_t \|_{L^2(0,T;V')} \leq Q(E_0, T).
$$

(129)
Finally, to get the $H$-regularity of $\vartheta$, we test \((99)\) by $\vartheta$. Then, using the first of \((87), (89)\) and \((128)\), noting that
\[
\left| \int_{\Omega} \lambda'_{b}(x) \chi_{t} \vartheta \right| \leq c \left( 1 + \| \chi \|_{L^{\infty}(\Omega)} \| \chi \| \| \vartheta \| \right) \leq c \| \chi \|^{2} + c \left( 1 + \| \chi \|^{2}_{H^{2}(\Omega)} \right) \| \vartheta \|^{2},
\]
thanks also to \((80)\), observing that a similar relation holds on $\Gamma$, and applying Gronwall’s lemma, it is not difficult to arrive at
\[
\| \vartheta \|_{L^{\infty}(0,T;H)} \leq Q(\mathcal{E}_{0}, T).
\]
Relations \((122)\) and \((131)\) give the desired \((133)\).

4.2 Proof of Theorem 3.4: weak sequential stability

In this part, we prove weak sequential stability for Problem (P). Namely, we assume to have a sequence \((\vartheta_{n}, \chi_{n})\) of sufficiently smooth solutions having uniformly bounded initial energies $\mathcal{E}_{0,n}$ and show that, at least up to the extraction of subsequences, this family converges in a suitable sense to a weak solution $u, \chi$.

\textbf{Relations (129) and (131) give the desired (94).}

Then, using the Aubin-Lions lemma, there exist $u, \chi$ weakly in $L^{2}(0,T;\Omega)$ and strongly in $L^{2}(0,T;V)$, $\| \vartheta \|_{L^{\infty}(0,T;\Omega)} \leq Q(\mathcal{E}_{0}, T)$, and to the assumed uniform boundedness of the initial energies, we then obtain
\[
\sup_{n \in \mathbb{N}} \left\{ \| u_{n} \|_{L^{2}(0,T;V)} + \| \partial_{t} \chi_{n} \|_{L^{2}(0,T;H)} + \| \chi_{n} \|_{L^{2}(0,T;W)} + \| s'_{0,b}(\chi_{n}) \|_{L^{2}(0,T;H)} + \| s'_{0,\Gamma}(\chi_{n,\Gamma}) \|_{L^{2}(0,T;H_{\Gamma})} \leq c,
\]
where $c$ is a positive constant independent of $n$. Then, using standard weak compactness arguments and the Aubin-Lions lemma, there exist $u, \chi$ with traces, respectively, $u_{\Gamma}, \chi_{\Gamma}$, and a non-relabelled subsequence of $n$ such that
\[
\begin{align*}
u_{n} & \overset{n \to \infty}{\longrightarrow} u \text{ weakly in } L^{2}(0,T;V), \\
\chi_{n} & \overset{n \to \infty}{\longrightarrow} \chi \text{ weakly in } H^{1}(0,T;\Omega) \cap L^{2}(0,T;W) \text{ and strongly in } L^{2}(0,T;V), \\
s'_{0,b}(\chi_{n}) & \overset{n \to \infty}{\longrightarrow} s'_{0,b} \text{ weakly in } L^{2}(0,T;H), \\
s'_{0,\Gamma}(\chi_{n}) & \overset{n \to \infty}{\longrightarrow} s'_{0,\Gamma} \text{ weakly in } L^{2}(0,T;H_{\Gamma}), \\
\lambda'_{b}(\chi_{n}) & \overset{n \to \infty}{\longrightarrow} \lambda'_{b} \text{ weakly in } L^{2}(0,T;H), \\
\lambda'_{\Gamma}(\chi_{n,\Gamma}) & \overset{n \to \infty}{\longrightarrow} \lambda'_{\Gamma} \text{ weakly in } L^{2}(0,T;H_{\Gamma}).
\end{align*}
\]

Then, thanks to the properties of $\lambda$ and $\lambda_{\Gamma}$, it is not difficult to obtain that $\lambda'(\chi_{n}) \longrightarrow \lambda'(\chi)$ strongly in $L^{2}(0,T;H)$ (similarly for $\lambda'_{\Gamma}$). Moreover, by the usual monotonicity argument [2, Prop. 1.1, p. 42] (recall that $s'_{0,b}$ and $s'_{0,\Gamma}$ verify \((80)\)), we can identify, almost everywhere in $(0,T) \times \Omega$ (or in $(0,T) \times \Gamma$),
$$\overline{s}_{0, b} = s'_{0, b}(\chi), \  \overline{s}_{0, \Gamma} = s'_{0, \Gamma}(\chi_{\Gamma}), \ \overline{\lambda}_b = \lambda_b^*(\chi) u \  \text{and} \  \overline{\lambda}_\Gamma = \lambda_{\Gamma}^*(\chi_{\Gamma}) u_{\Gamma}. \ \text{Now, we can take the limit in } (99).$$

Indeed, by (131) we have

$$(\vartheta_n, \nu_n) \overset{n \to \infty}{\to} (\vartheta, \eta) \ \text{weak star in } L^\infty(0, T; \mathcal{H}). \quad (139)$$

To conclude we have to show that $-1/\vartheta = u$ a.e. in $(0, T) \times \Omega$ and that $-1/\eta = u_{\Gamma}$ a.e. on $(0, T) \times \Gamma$. Actually, these identifications are consequences of the monotonicity of the map $v \mapsto -1/v$ (and of its realization in $L^2(0, T; \mathcal{H})$). Indeed, by (133),

$$u_n \overset{n \to \infty}{\rightharpoondown^w} u \ \text{weakly in } L^2(0, T; \mathcal{V}) \ \text{and consequently in } L^2(0, T; L^0(\Omega) \times L^0(\Gamma)), \quad (140)$$

Moreover, noting that, thanks to (129) and (131),

$$\vartheta_n \overset{n \to \infty}{\rightharpoondown^w} \vartheta \ \text{strongly in } L^2(0, T; \mathcal{V}'), \quad (141)$$

we get the following limsup-inequality:

$$\limsup_{n \to \infty} \int_0^T \int_\Omega \vartheta_n u_n \ dm \ dt = \limsup_{n \to \infty} \int_0^T \langle \vartheta_n, u_n \rangle \nu \ dt = \int_0^T \langle \vartheta, u \rangle \nu \ dt = \int_0^T \int_\Omega \vartheta \ dm \ dt. \quad (142)$$

Combining (139), (140) with (142) and using as above [2, Prop. 1.1, p. 42], we obtain $-1/\vartheta = u$ a.e. in $(0, T) \times \Omega$ and $-1/\eta = u_{\Gamma}$ a.e. on $(0, T) \times \Gamma$. This concludes the proof of weak sequential stability. In particular, this argument could be easily adapted to provide a rigorous existence proof.

Finally, the energy identity (103) can be proved simply by multiplying (99) by $1 - \vartheta/\vartheta$ and (101) by $\chi_i$ and performing standard integrations by parts. Indeed, it is easy to check that, also in the limit, $1 - \vartheta/\vartheta$ and $\chi_i$ are sufficiently smooth to be used as test functions.

### 4.3 Proof of Theorem 3.4: uniqueness and regularity

**Uniqueness.** Uniqueness can be easily obtained as follows: take two weak solutions $(\vartheta_1, \chi_1)$ and $(\vartheta_2, \chi_2)$ and set $(\vartheta, \chi) := (\vartheta_1 - \vartheta_2, \chi_1 - \chi_2)$. Then, write (99) firstly for $(\vartheta_1, \chi_1)$, then for $(\vartheta_2, \chi_2)$, take the difference, integrate it over $(0, t)$, $t \in (0, T]$, and test the result by $u_1 - u_2$, where $u_i = -1/\vartheta_i$. Correspondingly, take the difference of (101) and test it by $\chi$. Using the monotonicity of $f$ and $f_{\Gamma}$ and proceeding as in [39, Sec. 3] in order to estimate the nonlinearities coming from the quadratic terms involving $\lambda_b$ and $\lambda_{\Gamma}$, it is not difficult to achieve the desired result. The details, very similar to the proof given in [36, Sec. 3], are left to the reader.

**Parabolic regularization of solutions.** Now we come to the proof of (103)-(108). As before, we proceed by formal estimates. As noted above, also these estimates should be performed within some approximation scheme, whose details are omitted for simplicity.

**Second estimate.** We test (99) by $t u_i = t \vartheta_i/\vartheta^2$ and add the result to the time derivative of (101) multiplied by $t \chi_i$. Recalling (80) and using the monotonicity of $f$ and of $f_{\Gamma}$, we obtain

$$\frac{d}{dt} \left( t \frac{1}{2} \|\vartheta u\|^2 + \frac{1}{2} \|\nabla u_{\Gamma}\|^2_{L^2}\right) + t \int_{\Omega} \nabla u_{\Gamma} \ n_{\Gamma} \ d\eta_{\Gamma} + t \int_{\Gamma} \vartheta^2 u^2 + t \int_{\Gamma} \vartheta^2 u_{\Gamma}^2 \ \text{and} \ t \cdot \nabla \chi_{\Gamma} \parallel t \|\nabla \chi_{\Gamma}\|^2_{L^2}$$

$$- t \int_{\Omega} h_b u_t - t \int_{\Gamma} h_{\Gamma} u_{\Gamma, t} \leq \frac{1}{2} \left( \|\nabla u\|^2 + \|\nabla u_{\Gamma}\|^2_{L^2}\right) + \left( \frac{1}{2} + \delta \right) \|\chi_i\|^2_{L^2} \ \text{and} \ t \int_{\Gamma} \lambda_{\Gamma}^{n_\Gamma}(\chi_{\Gamma}) \lambda_{\Gamma}^{n_\Gamma, u_{\Gamma}} + t \int_{\Omega} \lambda_b^n(\chi) \chi^2 u. \quad (143)$$

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Now, let us note that, by \( (86) \), we have for any \( \varepsilon > 0 \)
\[
 t \int_\Omega \lambda''(\chi) \chi^2 u \leq t \left( \varepsilon \| \chi_t \|_V^2 + c_\varepsilon \| \chi_t \|_V^2 \right) , \quad t \int_\Gamma \lambda''(\chi) \chi^2 \, t u_G \leq t \left( \varepsilon \| \chi_{t,G} \|_V^2 + c_\varepsilon \| \chi_{t,G} \|_V^2 \right) . \tag{144}
\]
Moreover,
\[
 - t \int h_b u_t - t \int_h u_G,t = - \frac{d}{dt} \left( \int_\Omega t u_b + \int_\Gamma t u_G \right) + \left( \int_\Omega u_{t} + \int_\Gamma u_G \right) . \tag{145}
\]
Now, thanks to the fact that \( H \) has zero mean value (cf. \( \text{(89)} \) ), we can modify all the integrands on the right-hand side by subtracting to \( u \) and \( u_G \) the quantity \( m_\Omega(u) \), as done in \( \text{(116)} \). For instance, the first pair of integrals gives
\[
 - \frac{d}{dt} \left( \int_\Omega u_b(t) + \int_\Gamma u_G(t) \right) = - \frac{d}{dt} \left( \int_\Omega t(u - m_\Omega(u)) h_b(t) + \int_\Gamma t(u_G - m_\Omega(u)) h_G(t) \right) . \tag{146}
\]
The other two pairs of integrals on the right-hand side of \( \text{(145)} \) are managed similarly and then estimated directly in this way:
\[
 \left( \int_\Omega u_b(t) + \int_\Gamma u_G(t) \right) + \left( \int_\Omega t u_{\partial t} h_b(t) + \int_\Gamma t u_{\partial G} h_G(t) \right) \leq c \| u \|_V \| H \|_V + c t \| u \|_V \| \partial H \|_V . \tag{147}
\]
As a consequence, taking \( \varepsilon \) small enough in \( \text{(144)} \), integrating \( \text{(138)} \) between 0 and a generic \( t \in (0, T) \), recalling \( \text{(103)} \) and \( \text{(120)} \), and using Gronwall’s lemma, it is not difficult to arrive at
\[
 \begin{align*}
 & \frac{t}{2} \| \nabla u(t) \|^2 + \frac{t}{2} \| \nabla u_G(t) \|^2 + \frac{t}{2} \| \chi_t(t) \|^2 - \int_\Omega t(u - m_\Omega(u)) h_b(t) \\
 & - \int_\Gamma t(u_G(t) - m_\Omega(u(t))) h_G(t) + \int_0^t s \int_\Omega \partial_t^2 u^2 + \int_0^t s \int_\Gamma \bar{u}_G^2 \\
 & + \int_0^t \int_\Omega s \| \nabla \chi_t \|^2 + \frac{1}{2} \int_0^t s \| \nabla \chi_{t,G} \|^2 \leq Q(\varepsilon_0), \tag{148}
\end{align*}
\]
where, as before, the expression of \( Q \) may depend on the source \( H \) (and in particular on the additional regularity assumptions \( \text{(103)} \)). Note now that
\[
 - \int_\Gamma t(u(t) - m_\Omega(u(t))) h_G(t) \int_\Gamma t(u(t) - m_\Omega(u(t))) h_G(t) \geq - \frac{t}{4} \| \nabla u(t) \|^2 - \frac{t}{4} \| \nabla u_G(t) \|^2 - c, \tag{149}
\]
where \( c = c(H) \) is independent of \( t \). Hence, using again the generalized Poincaré inequality of Lemma \( \text{3.1} \), we finally obtain
\[
 \| u \|^2_{L^\infty(\tau, T; V)} + \| \chi_t \|^2_{L^\infty(\tau, T; H)} + \| \chi_{t,G} \|^2_{L^2(\tau, T; V)} \leq Q(\varepsilon_0, \tau^{-1}, T), \tag{150}
\]
which gives the first of \( \text{(105)} \) and \( \text{(107)} \). Moreover, testing again \( \text{(101)} \) by \( f(\chi) \) we can write (compare with \( \text{(124)} \))
\[
 \| f(\chi) \|^2 + \int_\Gamma f(\chi) \left( f_G(\chi_G) - \delta \chi_G + \frac{\lambda_G(\chi_G)}{\partial G} \right) \leq \int_\Omega f(\chi) \left( - \chi_t + \delta \chi - \frac{\lambda_0(\chi)}{\partial} \right) - \int_\Gamma f(\chi) \chi_{t,G} \leq \frac{1}{2} \| f(\chi) \|^2 + \frac{c_\varepsilon}{4} \| f(\chi) \|^2 + c(\| \chi \|^2 + \| \chi_t \|^2 + \| \chi_{t,G} \|^2) + c(1 + \| \chi \|^2_{L^4(\Omega)} \| u \|^2_{L^4(\Omega)}. \tag{151}
\]
Consequently, recalling (122), taking the essential supremum with respect to time, and using (86), (150), (120), and the compatibility condition (89), it is not difficult to infer
\[ || - ∆τ ||_{L^∞(τ,T;H)} + || s_0,0(τ) ||_{L^∞(τ,T;H)} ≤ Q(ξ_0, τ^{-1}, T), \] for any τ ∈ (0,1).

(152)

Then, using again [5 Theorem 2.7.7 and Theorem 3.1.5], we get \( \partial_τ \chi \in L^∞(τ,T;H_τ) \) for any τ ∈ (0,1). Hence, following the lines of the argument in Section 4.1, we arrive at (compare with (125)-(126))
\[ || - ∆τ \chi ||_{L^∞(τ,T;H_τ)} + || s^0,0(τ) \chi ||_{L^∞(τ,T;H_τ)} ≤ Q(ξ_0, τ^{-1}, T), \] for any τ ∈ (0,1).

(153)

Relations (152), (153) give (106), (108), respectively. Hence, to conclude the proof of Theorem 3.4, it remains to show the second of (105) and of (107). This essentially relies on the following result, which is a variant of [10 Lemma 3.3]:

**Lemma 4.1.** Let \((\theta, \eta)\) be a smooth solution to the problem
\[
\begin{align*}
\partial_τ \theta - ∆u &= g_b, \quad u = -\frac{1}{\theta}, \quad \text{in } Ω, \\
\partial_τ \eta - ∆τ u_τ &= g_τ - ∂_θ u, \quad \eta = -\frac{1}{u_τ} \quad \text{on } Γ,
\end{align*}
\]

over the generic time interval \((S,T)\), where we additionally assume that
\[ ||u||_{L^∞(S,T;L^{3/2}(Ω))} ≤ M, \quad ||g_b||_{L^2(S,T;L^{3/2}(Ω))} + ||g_τ||_{L^2(S,T;L^{3/2}(Γ))} ≤ G, \]

(154)

(155)

for some (given) constants \(M > 0, G > 0\) and some \(ε > 0\). Moreover, let us assume that
\[ u(S) ∈ L^1(Ω), \quad u_τ(S) ∈ L^1(Γ), \quad ||u(S)||_{L^1(Ω)} + ||u_τ(S)||_{L^1(Γ)} ≤ U, \]

(156)

(157)

for some \(U > 0\). Then, we have
\[ ||u||_{L^∞((S+ε,T)×Γ)} + ||u_τ||_{L^∞((S+ε,T)×Γ)} ≤ Q(G, M, U, τ^{-1}) \quad ∀ \tau ∈ (0, T - S). \]

(158)

The proof is mainly based on a Moser iteration scheme with regularization and closely follows the lines of [10 Lemma 3.3], where the case of Neumann boundary conditions is treated. The easy adaptation is left to the reader.

At this point we are in a position to conclude the proof of Theorem 3.4. Actually, assumption (104) and estimate (150) allow us to apply Lemma 4.1 with \(g_b = h_b - (λ_b(χ))_t\) and \(g_τ = h_τ - (λ_τ(χ_τ))_t\) on the generic interval \((S,T)\) with \(S > 0\). This provides the \(L^∞\)-regularization for \(u\) and for its trace \(u_τ\), as desired.

**Proof of Corollary 3.5.** The proof is based on the comparison principle for ODEs, similarly to [21 Sec. 3.2]. Given \(τ > 0\), thanks to the bounds (106), (108) and to assumption (86), there exists \(M > 0\) depending on \(ξ_0\) and \(τ\), but otherwise independent of time, such that
\[ ||δχ + λ_b(χ)u||_{L^∞((τ,T)×Ω)} + ||δχ_τ + λ_τ(χ_τ)u_τ||_{L^∞((τ,T)×Γ)} ≤ M. \]

(159)

Let us now consider the following (forward in time) initial value problem
\[
\begin{align*}
\left\{ \begin{array}{ll}
y' + κ_s f(y) &= M + C_s, \\
y(\tau) &= 1,
\end{array} \right.
\end{align*}
\]

(160)
the constants $\kappa_s$ and $C_s$ being as in (109). Thanks to (82) and to the standard theory of ODE’s, there exists $\omega = \omega(\tau) \in (0, 1)$ such that $y(t) \leq 1 - \omega$ for all $t \geq 2\tau$. Let us then view $y$ as a function also of the space variable $x$ and subtract the first row of (110) from (111), both in the bulk and on the boundary. Then, test the result by $(\chi - y)^+$, $(\cdot)^+$ denoting the positive part. We have

$$
\frac{1}{2} \frac{d}{dt} \int_\Omega |(\chi - y)^+|^2 \, dm + \int_\Omega (f(\chi) - \kappa_s f(y))(\chi - y)^+ \, dx + \int_\Gamma (f_\Gamma(\chi_\Gamma) - \kappa_s f(y) + C_s)(\chi_\Gamma - y)^+ \, dS
$$

$$
= \int_\Omega (\delta \chi + \lambda'_s(\chi)u - M - C_s)(\chi - y)^+ \, dx + \int_\Gamma (\delta \chi_\Gamma + \lambda'_s(\chi_\Gamma)u_\Gamma - M)(\chi_\Gamma - y)^+ \, dS \leq 0. \quad (161)
$$

Recalling (109) and using in particular that $\kappa_s \in (0,1]$, exploiting the monotonicity of $f$, and noting that $(\chi - y)^+|_{t=\tau} = 0$ $\text{dm}$-almost everywhere in $\Omega$ thanks to the fact that $\chi$ takes almost everywhere its values in $\text{dom} f = (-1,1)$, the comparison principle yields that $\chi(t,x) \leq y(t) \leq 1 - \omega$ for all $t \geq 2\tau$ and for $\text{dm}$-almost all $x \in \Omega$. Namely, we have the upper bound in (110). The lower bound is proved similarly.

**Proof of Prop. 3.6.** As before, we get the uniform boundedness of $\vartheta$ by means of a Moser iteration scheme. Namely, we will rely on the following lemma, whose proof can be obtained by suitably modifying the argument given in [10] Lemma 3.5] and is thus left to the reader.

**Lemma 4.2.** Let $(\vartheta, \eta)$ be a smooth solution of problem (154)-(155) over the generic time interval $(S, T)$, where we additionally assume that

$$
\|\vartheta\|_{L^\infty(S,T;L^1(\Omega, \text{dm}))} \leq M, \quad \|\vartheta\|_{L^2(S,T;L^{3+}(\Omega))} + \|\vartheta_\Gamma\|_{L^2(S,T;L^{3+}(\Gamma))} \leq G, \quad (162)
$$

for some (given) constants $M > 0$, $G > 0$ and some $\epsilon > 0$. Moreover, let us assume that

$$
\vartheta(S) \in L^{3+}(\Omega), \quad \eta(S) \in L^{3+}(\Gamma), \quad \|\vartheta(S)\|_{L^{3+}(\Omega)} + \|\eta(S)\|_{L^{3+}(\Gamma)} \leq \Theta, \quad (163)
$$

for some (given) constant $\Theta > 0$. Then, for any $\tau \in (0, T - S)$, we have

$$
\|\vartheta\|_{L^\infty((S+\tau,T) \times \Omega)} \leq Q(M, G, \Theta, \tau^{-1}). \quad (164)
$$

However, to apply the above lemma over the generic interval $(S, T)$, $S > 0$, some care is needed. Actually, taking as above $g_b = h_b - (\lambda_b(\chi))_t$ and $g_\Gamma = h_\Gamma - (\lambda_\Gamma(\chi))_t$, (162) is satisfied thanks to hypothesis (104) and estimate (150). However, it is not a priori obvious that assumption (111) implies (163) for $S > 0$. To prove this fact, we have to provide a uniform control of the $L^{3+}$-norm of $\vartheta$ over the interval $(0,S)$.

To this aim, we just consider the situation (more difficult, here) when $-s_{0,b}$ and $-s_{0,\Gamma}$ are “regular” potentials and (112) holds. In this case, we first have to prove that a uniform control of the $L^p$-norm of $\chi$ holds on small time intervals (this property is, of course, obvious when we have singular potentials). To do this, it suffices to test (101) by $|\chi|^{p-2}\chi$, $p$ to be chosen below, and integrate both in $\Omega$ and on $\Gamma$. Using the monotonicity of $f$ and $f_\Gamma$ together with assumption (86), it is then easy to get, for some $\kappa > 0$,

$$
\frac{d}{dt}\|\chi\|^p_{L^p(\Omega, \text{dm})} + \kappa \int_\Omega |\nabla((|\chi|^{\frac{p-2}{2}})\chi)|^2 + \kappa \int_\Gamma |\nabla_\Gamma((|\chi_\Gamma|^{\frac{p}{2}})\chi_\Gamma)|^2
$$

$$
\leq c \int_\Omega |\chi|^p \, dm + c \int_\Omega (1 + |\chi|^p)|u| \, dm \quad (165)
$$

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and the latter term on the right-hand side can be controlled as follows (we just deal with the bulk component, the boundary one behaving similarly):

\[
\int_{\Omega} |\chi|^p |u| \leq \|u\|_{L^p(\Omega)} \|\chi\|_{L^q(\Omega)}^{\frac{2p}{q}} \leq c \|u\|_{L^p(\Omega)} \|\chi\|_{L^q(\Omega)}^{\frac{2p}{q}} \leq c \|u\|_{L^p(\Omega)} \|\chi\|_{L^q(\Omega)}^{\frac{2p}{q}}.
\]

Then, we first take \( \sigma \) small enough so that the latter term on the right-hand side is controlled by the second term on the left-hand side of (165). Subsequently, we use Gronwall’s Lemma in (165) to obtain, for any \( p \in (1, \infty) \),

\[
\|\chi\|_{L^\infty(0, S; L^p(\overline{\Omega}, dm))} \leq Q(\mathbb{E}_0, \|\chi_0\|_{L^\infty(\mathbb{T}, dm)}, S, p).
\]  

Next, we proceed along the lines of [23] Proof of Theorem 2.7. Namely, we test (99) by \( d\theta/dt \), both on the bulk and on the boundary. We then get

\[
\frac{d}{dt} \|\theta\|_{L^{\infty}(\overline{\Omega}, dm)}^{\infty} \leq c \int_{\Omega} \left| - \lambda\theta + h \theta \right| + c \int_{\Gamma} \left| - \lambda\theta \right| + h \theta \leq c \left( \|\theta\|_{L^\infty(\overline{\Omega})} + \|\theta\|_{L^\infty(\Gamma)} \right) \|\theta\|_{L^{\infty}(\overline{\Omega}, dm)}^{\infty},
\]

whence, clearly,

\[
\frac{d}{dt} \|\theta\|_{L^{\infty}(\overline{\Omega}, dm)}^{\infty} \leq c \left( \|\lambda\theta\|_{L^\infty(\overline{\Omega})} + \|\theta\|_{L^\infty(\overline{\Omega})} + \|\theta\|_{L^\infty(\Gamma)} \right) \|\theta\|_{L^{\infty}(\overline{\Omega}, dm)}^{\infty},
\]

and we have to estimate the right-hand side over the “small” time interval \((0, S)\). Actually, the terms depending on \( H \) are controlled thanks to assumption [23]. We just give an estimate for the bulk term depending on \( \chi_0 \), which is the most difficult one due to worse embeddings holding in 3D. To do this, we use estimate (170) with \( p = 9 + 3\varepsilon \) (this value is selected just for simplicity of computation). Then, using assumption [80] and interpolation,

\[
\|\chi_0\|_{L^{\infty}(\overline{\Omega})} \leq c \left( 1 + \|\chi\|_{L^{9+3\varepsilon}(\overline{\Omega})} \right) \|\chi\|_{L^{6+2\varepsilon}(\overline{\Omega})} \leq c \|\chi\|_{L^{6+2\varepsilon}(\overline{\Omega})},
\]

Thus, using that

\[
H^{6+2\varepsilon}(\overline{\Omega}) \subset L^{6+2\varepsilon}(\overline{\Omega}),
\]

and standard interpolation properties of Sobolev spaces, we obtain

\[
\|\chi\|_{L^{6+2\varepsilon}(\overline{\Omega})} \leq c \|\chi\|_{L^{6+2\varepsilon}(\overline{\Omega})} \leq c \left( 1 + \|\chi\|_{L^{9+3\varepsilon}(\overline{\Omega})} \right) \|\chi\|_{L^{6+2\varepsilon}(\overline{\Omega})} \leq c \|\chi\|_{L^{6+2\varepsilon}(\overline{\Omega})},
\]

whence, using estimate (170),

\[
\|\chi_0\|_{L^{6+2\varepsilon}(\overline{\Omega})} \leq c \|\chi_0\|_{L^{6+2\varepsilon}(\overline{\Omega})} \leq c \|\chi_0\|_{L^{6+2\varepsilon}(\overline{\Omega})} + c \|\chi_0\|_{L^{6+2\varepsilon}(\overline{\Omega})}.
\]

Hence, choosing \( \varepsilon > 0 \) small enough so that the latter exponent of \( t \) is strictly larger than \(-1\), we can integrate (169) over \((0, S)\) to obtain

\[
\|\theta\|_{L^\infty(0, S; L^{6+2\varepsilon}(\overline{\Omega}, dm))} \leq Q(\mathbb{E}_0, \|\theta_0\|_{L^{6+2\varepsilon}(\overline{\Omega}, dm)}, \|\chi_0\|_{L^\infty(\mathbb{T}, dm), S}).
\]
In turn, this provides additional regularity of \( c \) with \( \tau > 0 \) we have
\[
|u(x,t)| + |u_T(x,t)| \geq c(\tau) \quad \text{for a.a. } (x,t) \in \Omega \times (\tau,T),
\]
with \( c(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+ \) possibly going to 0 when \( \tau \searrow 0 \). Hence, using (113), we deduce that
\[
\int_\tau^T \|\partial_t u\|^2 \, ds \leq c(\tau)^{-2} \frac{1}{\tau-1} \int_\tau^T s \int_\Omega \partial_t^2 u^2 \, dx \, ds \leq Q(\mathcal{E}_0, \tau^{-1}, T)
\]
(175) as well as
\[
\int_\tau^T \|\partial_t u\|^2 \, ds \leq c(\tau)^{-2} \frac{1}{\tau-1} \int_\tau^T s \int_\Omega \partial_t^2 u^2 \, dx \, ds \leq Q(\mathcal{E}_0, \tau^{-1}, T).
\]
These relations correspond to (114). To conclude, we need to prove that (99) can be interpreted in the strong form (66). Actually, this follows by reasoning as done in the “Estimate of the nonlinear terms”. In particular, a comparison of terms in the bulk component of the heat equation gives that \( \Delta u \in L^2(\tau,T; H) \). Moreover, it is apparent that the stationary version of (99) simply prescribes that \( \partial_t u \) and, as a consequence, that \( \Delta u_T \in L^2(\tau,T; H_T) \). The details are left to the reader.

5 Long-time behavior

In this section we prove existence and regularity properties of \( \omega \)-limit sets of weak solutions to Problem (P). Moreover, we also prove existence of the global attractor. For simplicity, we will assume that no external heat source is present, i.e., we will take \( H = 0 \). An asymptotically vanishing source could be treated as well, paying the price of technical complications. Moreover, we will restrict ourselves to the case of singular potentials (the case of regular potentials being in fact simpler) and, in particular, we will assume the reinforced compatibility condition (109). Finally, we will assume here the higher summability property (111) on \( \vartheta_0 \) (cf. Remark 5.3 below for a motivation for this choice).

To start with, we write the stationary problem associated to Problem (P). In order to properly state it, we have to notice that, testing (99) by 1, one gets that the quantity
\[
\mu = \mu(\vartheta, \chi) := \int_\Omega \left( \vartheta + \lambda_0(\chi) \right) + \int_\Gamma \left( \eta + \lambda_T(\chi_T) \right),
\]
(177)
representing the “total mass” of the (bulk+boundary) internal energy, is conserved in the time-evolution of the system. Hence, once a solution trajectory evolves from some initial datum \((\vartheta_0, \chi_0)\) having finite energy \( \mathcal{E}_0 \), any point in the \( \omega \)-limit set must respect the constraint (177), with \( \mu \) depending on the given initial datum.

Moreover, it is apparent that the stationary version of (99) simply prescribes that \( u \) is a constant. Hence, we may write the steady state problem associated to Problem (P) as the following system:
\[
\vartheta_\infty \in (0, +\infty), \quad u_\infty = -1/\vartheta_\infty,
\]
(178)
\[
- \Delta \chi_\infty + f(\chi_\infty) - \delta \chi_\infty = \lambda'_0(\chi_\infty) u_\infty,
\]
(179)
\[
- \Delta_T \chi_{\infty, T} + f_T(\chi_{\infty, T}) - \delta \chi_{\infty, T} = \lambda'_T(\chi_{\infty, T}) u_\infty - \partial_n \chi_\infty,
\]
(180)
\[
\int_\Omega \left( \vartheta_\infty + \lambda_0(\chi_\infty) \right) + \int_\Gamma \left( \vartheta_\infty + \lambda_T(\chi_{\infty, T}) \right) = \mu.
\]
(181)
System (178)–(181) will be named as Problem (P_{∞,μ}) in what follows. We can now state and prove our first result regarding existence of nonempty \( ω \)-limit sets:

**Theorem 5.1.** Let the assumptions of Theorem 3.4 hold and let \( H = 0 \). Moreover, let (109) and (111) hold. Let \((θ, χ)\) be the corresponding weak solution to Problem (P). Then, as \( t \nearrow \infty \), \((u(t), χ(t))\) is precompact in \( H × V \). Moreover, any limit point \((θ_∞, χ_∞)\) of (any subsequence of) \((θ(t), χ(t))\) is a solution of Problem \((P_{∞,μ})\), where the quantity \( μ \) in (181) is equal to the “mass” \( μ(θ_0, χ_0) \) of the initial internal energy.

**Proof of Theorem 5.1.** As a first step, we need to go back to the estimates performed in Section 4.

In particular, repeating the energy estimate (115) and noting that now \( H = 0 \), we immediately obtain

\[
\|\vartheta - \log \vartheta\|_{L^∞(0, +∞; L^1(Ω, dm_0))} + \|\nabla u\|_{L^2(0, +∞; H)} + \|\nabla u_t\|_{L^2(0, +∞; H_Γ)} \leq C, \tag{182}
\]

\[
\|χ_t\|_{L^2(0, +∞; H)} + \|χ\|_{L^∞(0, +∞; V)} + \|χ\|_{L^∞((0, +∞) × Ω)} \leq C. \tag{183}
\]

Here and below, \( C > 0 \) denotes suitable constants possibly depending on \( E_0 \) but assumed in any case to be independent of the time variable. Instead, we will denote by \( ε_μ > 0 \) the constants, also independent on time, that are allowed to depend on the initial data only through the conserved value \( μ \).

Thanks to (118), (119), we also get

\[
\|u\|_{L^2(t, t+1; V)} \leq C. \tag{184}
\]

In particular, (182)–(184) imply that

\[
∀ t \geq 0, \exists S ∈ [t, t+1] : \|u(S)\|_V + \|χ_t(S)\|_H \leq C. \tag{185}
\]

Then, we can go back to the “Second estimate” of Subsec. 13. Actually, we test (101) by \( u_t \) and the time derivative of (101) by \( χ_t \) (hence, we do not need here the weight \( t \)). This leads to the analogue of (143), namely,

\[
\frac{d}{dt} \left( \frac{1}{2} \|\nabla u\|^2 + \frac{1}{2} \|\nabla u_t\|^2 + \frac{1}{2} \|χ_t\|^2_H \right) + \int_{Ω} \vartheta^2 u^2 + \int_{Γ} η^2 u^2 + \|\nabla χ_t\|^2 + \|\nabla χ_{Γ,t}\|^2_{Γ} \leq \int_{Ω} η^2 (χ) χ^2 u + \int_{Γ} η^2 (χ_Γ)(χ_{Γ,t})^2 u_Γ. \tag{186}
\]

Then, estimating the above right-hand side as in (144), integrating over \((S, S+2)\), \( S \) as in (185), and using the generalized Poincaré inequality (83), we readily get

\[
\|χ_t\|_{L^∞(1, +∞; H)} + \|u\|_{L^∞(1, +∞; V)} \leq C, \tag{187}
\]

\[
\|\vartheta\|_{L^2(t, t+1; H)} + \|χ_t\|_{L^2(t, t+1; V)} \leq C, \quad ∀ t ∈ [1, +∞). \tag{188}
\]

Going back to (180), and having now (187) at our disposal, we note that the \( V \)-norm of \( u \) on the right-hand side of (144) is controlled uniformly in time. Hence, we can integrate (186) over the whole \((1, +∞)\), so that (180) is improved to

\[
\|χ_t\|_{L^2(1, +∞; V)} + \|\vartheta\|_{L^2(1, +∞; H)} \leq C. \tag{189}
\]

Next, we can repeat estimate (151) over \((S, S+2)\), obtaining

\[
\|f(χ)\|_{L^∞(1, +∞; H)} + \|f_Γ(χ_Γ)\|_{L^∞(1, +∞; H_Γ)} \leq C. \tag{190}
\]
This allows us, as before, to get also
\[ \| \chi \|_{L^\infty(1, +\infty; H^2(\Omega))} + \| \chi \r \|_{L^\infty(1, +\infty; H^2(\Gamma))} \leq C. \] (191)

Next, (182)-(183) and a comparison of terms in (99) allow us to see that
\[ \| \vartheta \|_{L^2(0, +\infty; V')} \leq C. \] (192)

Thanks also to the Aubin-Lions lemma, the first of (187), (191), and the compact embedding \( W \subset V \), immediately give the precompactness of \( \chi(t) \) in \( V \).

Applying Lemma 4.1 over the generic interval \((S, S + 2)\), we also infer that
\[ \| u \|_{L^\infty((1, +\infty) \times \Omega)} \leq C, \] (193)
which, combined with (189), gives
\[ \| u_t \|_{L^2(1, +\infty; H)} \leq C. \] (194)

Thanks to (193), we can repeat the comparison argument of Corollary 3.5. In particular, we get the separation property uniformly in time:
\[ |\chi(t, x)| \leq 1 - \omega \ dm\text{-almost everywhere in } \Omega \text{ for a.e. } t \geq 1, \] (195)
with \( \omega \in (0, 1) \) independent of time.

By (194), (187) and the Aubin-Lions lemma we deduce that \( u(t) \) is precompact in \( H \). With the precompactness of \((u(t), \chi(t))\) and the above estimates at our disposal, we can now prove that the trajectory \((\vartheta(t), u(t))\) admits a nonempty \( \omega \)-limit set.

Namely, we take \( \{t_n\} \) to be a diverging sequences of time and we consider Problem (P) over the time interval \((0, 1)\) and with initial datum \((\vartheta(t_n), \chi(t_n))\). Let us call Problem \( (P_n) \) this problem. Hence, once \((\vartheta, \chi)\) is a weak solution to (P), \((\vartheta_n(t), \chi_n(t)) := (\vartheta(t_n + t), \chi(t_n + t)), t \in (0, 1),\) solves \( (P_n) \).

Thanks to the previous estimates and to the Aubin-Lions lemma, there exist limit functions \( \pi, \chi \) defined over \((0, 1) \times \Omega \) such that, say,
\[ u_n \rightarrow \pi \text{ strongly in } C^0([0, 1]; H^{1-\varepsilon}(\Omega) \times H^{1-\varepsilon}(\Gamma)), \] (196)
\[ \chi_n \rightarrow \chi \text{ strongly in } C^0([0, 1]; V), \] (197)
for all \( \varepsilon > 0 \). As usual, all convergences are to be intended up to the extraction of (nonrelabeled) subsequences of \( n \rightarrow \infty \). Moreover, thanks to (159) and (194), both \( \pi \) and \( \chi \) are constant in time; hence, they coincide with the limit of (the extracted subsequence of) \( u(t_n) \) and \( \chi(t_n) \), respectively. Moreover, thanks to (152), \( \pi \) is also constant in space and, thanks to (158), we have \( |\pi| \leq C \) for some \( C > 0 \) possibly depending on \( E_0 \).

Hence, passing to the limit in (101), we get that \((\pi, \chi)\) solve (179)-(180), which can be written in the strong form thanks to the same considerations on regularity as those made for the evolutionary system. In particular, the limits of the terms \( f(\chi_n) \) and \( f(\chi_n, \Gamma) \) can be identified as before by monotonicity methods.

To conclude the proof, we have to take the limit in the heat equation (99) in order to recover (178) and (181). To do this, we recall that, for all \( t \in [0, 1] \), we have
\[ \vartheta_n(t) = -\frac{1}{u_n(t)}, \quad \eta_n(t) = -\frac{1}{(u_n(t))^\Gamma}, \] (198)
almost everywhere in $\Omega$ and, respectively, on $\Gamma$. Then, by (196), we have
\[ \vartheta_n(t) \to -\frac{1}{|\Omega|}, \quad \eta_n(t) \to -\frac{1}{|\Gamma|}, \] (199)
afterwards everywhere in $\Omega$ and, respectively, on $\Gamma$. Let us now prove that, actually, $\pi$ cannot be 0. Indeed, by (177), we have, for almost all $t \in [0, 1]$,
\[ \int_{\Omega} \vartheta_n(t) \, dm \leq |\mu| + |\Omega| \max_{r \in [-1, 1]} |\lambda_b(r)| + |\Omega| \max_{r \in [-1, 1]} |\lambda_r(r)| \leq c_\mu. \] (200)
Consequently, applying Jensen’s inequality, we deduce that there exists $c_\mu > 0$ depending only on $\mu$, $\lambda_b$ and $\lambda_r$ such that $|\pi| \geq c_\mu$. To characterize the limit of $\vartheta_n$, we need the following lemma:

**Lemma 5.2.** Under the assumptions of Theorem 5.1 if we have in addition
\[ (\vartheta_0, \eta_0) \in L^p(\Omega) \times L^p(\Gamma), \text{ for some } p \geq 3, \] (201)
then it follows that
\[ \| (\vartheta(t), \eta(t)) \|_{L^p(\Omega) \times L^p(\Gamma)} \leq Q_p(\mathbb{E}_0, \| \vartheta_0 \|_{L^p(\Omega)}, \| \eta_0 \|_{L^p(\Gamma)}), \forall t \geq 0, \] (202)
with $Q_p$ independent of $t \in [0, +\infty)$.

**Proof.** For simplicity, we just give the proof in the case when $p = 3$, the general case following by repeating the procedure and applying a simple bootstrap argument. Firstly, let us observe that, by (173) with $\epsilon = 0$, we have
\[ \| (\vartheta(t), \eta(t)) \|_{L^3(\Omega) \times L^3(\Gamma)} \leq Q(\mathbb{E}_0, \| \vartheta_0 \|_{L^3(\Omega)}, \| \eta_0 \|_{L^3(\Gamma)}), \forall t \in [0, 1]. \] (203)
Hence, we can test (199) by $\vartheta^2$. Using that now $\chi$ is uniformly bounded, we obtain (compare with (169)), for some $\kappa > 0$ (which may change from line to line in the computations below),
\[ \frac{d}{dt} \| \vartheta \|^3_{L^3(\Omega, dm)} + \kappa \| \nabla \vartheta^{1/2} \|^2 + \kappa \| \nabla \chi \eta^{1/2} \|^2_{\Gamma} \leq c \int_{\Omega} |\chi \vartheta^2| \, dm. \] (204)
Actually we can notice that, if (201) holds, then the solution is regular enough so that $\eta$ is in fact the trace of $\vartheta$ in this case.

Noting that, due to (177),
\[ \| \vartheta^{1/2}(t) \|^2_{L^3(\Omega, dm)} = \int_{\Omega} \vartheta(t) \, dm \leq c_\mu \quad \forall t \in [0, +\infty), \] (205)
adding (205) to (204), and using standard Sobolev’s embeddings, we infer
\[ \frac{d}{dt} \| \vartheta \|^3_{L^3(\Omega, dm)} + \kappa \| \vartheta \|_{L^3(\Omega, dm)} \leq c \| \chi \|^2_{L^3(\Omega, dm)} \| \vartheta^{1/2} \|^2_{L^2(\Omega, dm)} + c_\mu \leq c \| \chi \|^2_{L^3(\Omega, dm)} \| \vartheta \|^3_{L^3(\Omega, dm)} + \kappa \| \vartheta \|^2_{L^3(\Omega, dm)} + c_\mu. \] (206)
Thus, setting
\[ y(t) := \max \left\{ 1, \| \vartheta(t) \|^2_{L^3(\Omega, dm)} \right\}, \quad m(t) := \| \chi \|^2_{L^3(\Omega, dm)}, \] (207)
it is clear that (206) can be interpreted as the differential inequality
\[ y'(t) + \kappa \leq cm(t)y(t) + c_\mu y(t)^{-1/2}, \] (208)
so that the comparison principle, together with (203), (189) and Gronwall’s lemma, readily imply that
there is nothing to prove, while if \( y > 4c_\mu^2/\kappa^2 \) (208) takes the form \( y' \leq cmy \) and we can use Gronwall’s
lemma since \( m \) is globally summable thanks to (189).

**Remark 5.3.** It is worth noticing that the property stated in the Lemma is likely to be false if \( p \in (1,3) \)
Indeed, in that case the gradient terms in (204) seem to give no help and, consequently, the differential
inequality corresponding to (208) loses its dissipative character (in other words, it provides a control on
the \( L^p \)-norm of \( \vartheta \) which is not uniform in time). This is the reason which led us to assume (111)
in this section. On the other hand, thanks to the conservation of \( \mu \) (cf. (177)), we have in any case a uniform in
time \( L^1 \)-control on \( \vartheta \). However, this seems not enough in order to characterize properly the limit of \( \vartheta_n \).
Indeed, it may happen that
\[ \vartheta_n = \vartheta_{n,1} + \vartheta_{n,2}, \]
where \( \vartheta_{n,1} \) is a “good” function which converges, say, uniformly, to the constant \(-1/\kappa\). Instead, the
sequence \( \{\vartheta_{n,2}\} \) is also bounded in \( L^1 \) but it may converge to some singular measure (e.g., a Dirac mass,
meaning that the support of \( \vartheta_{n,2} \) becomes small and \( \vartheta_{n,2} \) becomes large in that set). However, when one
computes \( u_n \), the contribution of \( \vartheta_{n,2} \) is negligible for large \( n \) because \( \vartheta_{n,2} \) influences the value of \( u_n \) only
in a set that has asymptotically measure 0. Hence, the limit of \( \vartheta_n \) is in this case a measure whose regular
(absolute continuous) part coincides with \(-1/\kappa\), but which may also have a singular component.

Thanks to Lemma 5.2 using pointwise convergence (cf. (199)) and Lebesgue’s theorem, we obtain
\[ \vartheta_n \to -\frac{1}{\kappa} \text{ strongly in } L^p(0,1; L^p(\Omega; dm)) \quad \forall p \in [1,3). \] (209)
In particular, we get the second (178), (181), which concludes the proof of Theorem 5.1.

Finally, we come to the problem of the existence of the global attractor for Problem (P). To address this
issue, we first introduce the natural **phase space** for the dynamical system associated to Problem (P), by setting
\[ X_\mu := \begin{cases}
(\vartheta, \chi) \in (L^{3+\epsilon}(\Omega) \times L^{3+\epsilon}(\Gamma)) \times \mathcal{V} : & \begin{cases}
\vartheta > 0 \text{ a.e. in } \Omega, \log \vartheta \in L^1(\Omega; dm) \\
F(\chi) \in L^1(\Omega), \quad F_T(\chi_\Gamma) \in L^1(\Gamma), \\
\int_{\Omega} (\vartheta + \lambda_0(\chi)) + \int_{\Gamma} (\eta + \lambda_T(\chi_\Gamma)) = \mu
\end{cases}
\end{cases}, \] (210)
where \( \epsilon > 0 \) is given and the conserved value \( \mu \) is also prescribed. It is not difficult to prove (see, e.g.,
[35] for details) that the above space, endowed with the distance
\[ \text{dist}_X((\vartheta_1, \chi_1), (\vartheta_2, \chi_2)) := ||\vartheta_1 - \vartheta_2||_{L^{3+\epsilon}(\Omega; dm)} + ||\log \vartheta_1 - \log \vartheta_2||_{L^1(\Omega; dm)} \\
+ ||\chi_1 - \chi_2||_v + ||F(\chi_1) - F(\chi_2)||_{L^1(\Omega)} + ||F_T(\chi_{1,\Gamma}) - F_T(\chi_{2,\Gamma})||_{L^1(\Gamma)} \] (211)
acquires a complete metric structure.

We will name Problem (P_\mu) the version of Problem (P) where the “mass” \( \mu \) of the initial internal
energy is assigned.
Remark 5.4. In order for the phase space $X_\mu$ not to be empty, we implicitly assume that
\[ \mu > |\Omega| \min_{r \in [-1,1]} \lambda_0(r) + |\Gamma| \min_{r \in [-1,1]} \lambda_\Gamma(r). \] (212)

Actually, if (212) is not satisfied, then from (177) we get that $\vartheta$ needs to be nonpositive, and the problem becomes inconsistent. However, we will see just below that a stronger condition may be needed.

Theorem 5.5. Let the assumptions of Theorem 5.1 hold. Let also assume that, either
\[ \mu > |\Omega| \max_{r \in [-1,1]} \lambda_0(r) + |\Gamma| \max_{r \in [-1,1]} \lambda_\Gamma(r), \] (213)
or
\[ \liminf_{|r| \to 1} \lambda'(r) \text{ sign } r > 0. \] (214)

Then, the dynamical process associated to Problem $(P_\mu)$ admits the global attractor $A_\mu$. Moreover, there exist constants $K > 0$, $\omega \in (0,1)$ and $\alpha \in (0,1)$ depending only on $\mu$ such that, for any $(\vartheta, \chi) \in A_\mu$,
\[ \|\vartheta\|_V + \|\chi\|_W \leq K, \] (215)
\[ \alpha \leq \vartheta \leq \alpha^{-1} \text{ dm-almost everywhere in } \Omega, \] (216)
\[ -1 + \omega \leq \chi \leq 1 - \omega \text{ dm-almost everywhere in } \Omega. \] (217)

Proof. Due to the singular and degenerate character of equation (99), a direct proof of a dissipative estimate (e.g., of the existence of an absorbing set) appears to be out of reach. On the other hand, as in [39], we can take advantage of the fact that our system admits a coercive Liapounov functional (the energy $E$). Thanks to this property, the existence of the global attractor follows by proving the following conditions:

(a) Solution trajectories are precompact with respect to the metrics $\text{dist}_X$ of the phase space $X_\mu$;

(b) The set of stationary states is bounded in the energy space independently of the magnitude of the initial data.

Hence, let us start with the proof of (a). By Theorem 5.1 it is clear that for any weak solution there exists $C$ depending on $E_0$ and on the $L^{3+\epsilon}$-norm of $\vartheta_0$ such that
\[ \|u(t)\|_V + \|\chi(t)\|_W \leq C \quad \forall t \in [1, +\infty). \] (218)

Moreover, thanks also to Lemma 5.2 we have
\[ 0 < \alpha \leq \vartheta(t, x) \leq \alpha^{-1} \text{ dm-a.e. in } \Omega \text{ and for all } t \in [1, +\infty), \] (219)
with $\alpha \in (0,1)$ depending on the same quantities as the above constant $C$. Coupling (218) with (219), we readily get
\[ \|\vartheta(t)\|_V \leq C \quad \forall t \in [1, +\infty). \] (220)

In addition to that, the uniform separation property (195) holds. Hence, both $\vartheta$ and $\chi$ are eventually separated from the singularities. Using (218)-(220) and (195) it is a standard matter to prove that the trajectory is precompact with respect to the metric structure of $X_\mu$. 29
To conclude the proof, let us demonstrate (b). This is more delicate, since we need to prove that the set of stationary states (i.e., of the solutions \((\vartheta_\infty, \chi_\infty)\) to Problem \((P_\infty, \mu)\)) is bounded in a way that depends only on \(\mu\). In other words, all elements of the \(\omega\)-limit of a given solution trajectory are bounded in a way that depends on the initial datum only through the conserved quantity \(\mu\). Actually, we have already seen (see (200)) that \(|\vartheta_\infty| \leq c_\mu\). To prove that the same property holds for \(u_\infty\), we first consider the situation when (213) holds. In this case, combining (213) with (181), \(|u_\infty| \leq c_\mu\) follows immediately. Hence, the terms \(\lambda_b'(\chi_\infty)u_\infty\) and \(\lambda_G'(\chi_\infty, \Gamma)u_\infty, \Gamma\) on the right-hand sides of (179) and, respectively, (180) are uniformly bounded in a way depending only on \(\mu\). Then, by the maximum principle it immediately follows that there exists \(\omega_\mu \in (0, 1)\) such that \(-1 + \omega_\mu \leq \chi_\infty(t, x) \leq 1 - \omega_\mu\) dm-almost everywhere in \(\Omega\). It is immediate to check that the same property holds if (214) is satisfied in place of (213).

Standard elliptic regularity estimates applied to (179)-(180) then give, say,

\[
\|\chi_\infty\|_{W^{1,\infty}} \leq c_\mu \tag{221}
\]

for every stationary solution \((\vartheta_\infty, \chi_\infty)\) (actually, much more is true since \(\chi_\infty\) is separated from the singularities and is, consequently, a classical solution to the elliptic system (179)-(180)). Hence, (b) holds true and the existence of the attractor with the properties (215)-(217) follows from classical results (see, e.g., [22]).

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