LOWER BOUNDS FOR THE EIGENVALUE ESTIMATES OF THE
SUBMANIFOLD DIRAC OPERATOR

YONGFA CHEN

ABSTRACT. We get optimal lower bounds for the eigenvalues of the submanifold Dirac operator on locally reducible Riemannian manifolds in terms of intrinsic and extrinsic expressions. The limiting-cases are also studied. As a corollary, one gets several known results in this direction.

1. Introduction

It is well known that the spectrum of the Dirac operator on closed spin manifolds detects subtle information on the geometry and the topology of such manifolds (see [15]). A fundamental tool to get estimates for eigenvalues of the basic Dirac operator $D$ acting on spinors is the Schrödinger-Lichnerowicz formula

$$D^2 = \nabla^* \nabla + \frac{1}{4} R^M,$$  \hfill (1.1)

where $\nabla^*$ is the formal adjoint of $\nabla$ with respect to the natural Hermitian inner product on spinor bundle $\Sigma M^m$ and $R^M$ stands for the scalar curvature of the closed spin manifold $(M^m, g)$.

The problem of finding optimal lower bounds for the eigenvalues of the Dirac operator on closed manifolds was for the first time considered in 1980 by Friedrich. Using the Schrödinger-Lichnerowicz formula and a modified spin connection, he proved the following sharp inequality:

$$\lambda^2 \geq c_m \min_M R^M,$$  \hfill (1.2)

where $c_m = \frac{m}{4(m-1)}$. The case of equality in (1.2) occurs iff $(M^m, g)$ admits a nontrivial spinor field $\psi$ called a real Killing spinor, satisfying the following overdetermined elliptic equation

$$\nabla_X \psi = -\frac{\lambda}{m} X \cdot \psi,$$  \hfill (1.3)

where $\lambda \in \mathbb{R}$, $\forall X \in \Gamma(TM)$ and the dot “.” indicates the Clifford multiplication.

The manifold must be a locally irreducible Einstein manifold. Note complete simply-connected Riemannian spin manifolds $(M^m, g)$ carrying a non-zero space of real Killing spinors have been completely classified by Bär [3]. The limiting manifold

Key words and phrases. Dirac operator, eigenvalue, mean curvature, scalar curvature.
must be either a stand $n$-sphere, an Einstein-Sasaki manifold, a 6-dimensional nearly Kähler manifold or a 7-dimensional manifold with 3-form $\nabla \eta = *\eta$.

The dimension dependent coefficient $c_m = \frac{m}{4(m-1)}$ in the estimate can be improved if one imposes geometric assumptions on the metric. Kirchberg [13, 14] showed that for Kähler metrics $c_m$ can be replaced by $\frac{m+2}{4m}$ if the complex dimension $\frac{n}{2}$ is odd, and by $\frac{m}{4(m-2)}$ if $\frac{n}{2}$ is even. Alexandrov, Grantcharov, and Ivanov [2] showed that if there exists a non-trivial parallel 1-form on $M^m$, then $c_m$ can be replaced by $c_{m-1}$. Later, Moroianu and Ornea [17] weakened the assumption on the 1-form from parallel to harmonic with constant length. Note the condition that the norm of the 1-form being constant is essential, in the sense that the topological constraint alone (the existence of a non-trivial harmonic 1-form) does not allow any improvement of Friedrich’s inequality (see [4]). The generalization of [2] to locally reducible Riemannian manifolds was achieved by Alexandrov [1], extending earlier work by Kim [12].

On the other hand, motivated by the using of the hypersurface Dirac operator in the proof of the positive energy conjecture by Witten, Zhang first investigated the corresponding eigenvalue estimates problem. Optimal lower bounds for the hypersurface Dirac operator in terms of the scalar curvature, the mean curvature and energy-momentum tensor were obtained. Follow-up related results can be seen in [21, 10, 11, 16, 9].

In this paper, based on the Schrödinger-Lichnerowicz formula, by defining appropriate modified connections, we shall get optimal lower bounds for the eigenvalues of the submanifold Dirac operator on any locally reducible Riemannian manifolds in terms of the scalar curvature as well as a normal curvature term which only appears in codimension greater than one. These results can be also translated in an intrinsic way for a twisted Dirac-Schrödinger operator. As a corollary, one recovers several classical results in [1, 2, 9, 11, 17]. The discussion of the limiting cases of these estimates give rise to two new field equations generalizing the Killing equation.

The remainder of the article is organized as follows. Section 2 and 3 describe some geometric conventions and preliminaries about “partial” Dirac operator, the $\beta$-twist $D_\beta$ and the submanifold Dirac operator on locally reducible Riemannian manifolds. The main result and its proof are given in section 4. In section 5, we consider the intrinsic estimates for the twisted Dirac-Schrödinger operator. In the final section, we shall extend the our techniques to other generalized hypersurface Dirac operators which appear in General Relativity.

2. On locally reducible Riemannian manifolds

Let $M^m$ be a closed Riemannian spin manifold with positive scalar $R^M$. Suppose $TM = T_1 \oplus \cdots \oplus T_k$ is orthogonal sum, where $T_i$ are parallel distributions of dimension $m_a, a = 1, \cdots, k,$ and $m_1 \geq m_2 \geq \cdots \geq m_k$. One important consequence of the parallelism of $T_a$ is that $R(X,Y) = 0$ whenever $X \in T_a, Y \in T_b$ with $a \neq b$. Then
one can define a locally decomposable Riemannian structure $\beta$ as follows

$$\beta|_{T_i} = \text{Id}, \quad \beta|_{T_i^\perp} = -\text{Id}.$$  

(2.1)

Suppose

$$\{e_1, \cdots, e_{m_1}, e_{m_1+1}, \cdots, e_{m_1+m_2}, \cdots, e_{m_1+m_2+\cdots+m_k-1+1}, \cdots, e_m\}$$

is an adapted local orthonormal frame, i.e., such that $\{e_{m_1+m_2+\cdots+m_a-1+1}, \cdots, e_{m_1+m_2+\cdots+m_a}\}$ spans the subbundle $T_a$. Let $I_a = \{m_{a-1}+1, \cdots, m_a\}$, then

$$D_a = \sum_{i \in I_a} e_i \cdot \nabla_i$$  

(2.2)

is the “partial” Dirac operator of subbundle $T_a$, which is formally self-adjoint operator. Hence the Dirac operator $D$ and the $\beta$-twist $D_\beta$ can be expressed as the following

$$D = \sum_{k=1}^m e_k \cdot \nabla_k = D_1 + D_2 + \cdots + D_k$$  

(2.3)

$$D_\beta := \sum_{k=1}^m \beta(e_k) \cdot \nabla_k = D_1 - (D_2 + \cdots + D_k)$$  

(2.4)

Note for $a \neq b$, one has

$$D_a D_b + D_b D_a = 0,$$  

(2.5)

that is,

$$D_\beta^2 = D^2 = \sum_{a=1}^k D_a^2.$$  

(2.6)

In addition, we also have

$$DD_\beta + D_\beta D = 2(D_1^2 - D_2^2 - \cdots - D_k^2)$$

Let $R_a^M$ be the “scalar curvature” of $T_a$, i.e.,

$$R_a^M = \sum_{s,t \in I_a} \langle R(e_s, e_t)e_t, e_s \rangle.$$  

Hence the scalar curvature $R^M$ of $M^m$ is $R^M = \sum_{a=1}^k R_a^M$.

3. The submanifold Dirac operator

Now let $(\bar{M}^{m+n}, \bar{g})$ be an $(m+n)$-dimensional Riemannian spin manifold and $M^m$ be an $m$-dimensional submanifold in $\bar{M}^{m+n}$ with its induced Riemannian structure. Assume that two manifolds are equipped with a spin structure so that there exists unique spin structure on the normal bundle of $M^m$ such that the sum of the spin structures of the tangent bundle $TM^n$ and of the normal bundle $NM^n$ of $M^n$.
is exactly the spin structure of $\mathbf{M}^{m+n}$ restricted to $M^m$. Note that in particular $M^m, \mathbf{M}^{m+n}$ are oriented. Denote $\Sigma\mathbf{M}$ the spinor bundle of $\mathbf{M}^{m+n}$, then $\Sigma := \Sigma\mathbf{M}|_M$ is globally defined along $M^m$.

Denote the Levi-Civita connections of $M^m$ and $\mathbf{M}^{m+n}$ by $\nabla$ and $\nabla$ respectively and denote by the same symbol their corresponding lift to the spinor bundle $\Sigma$. Consider the Dirac operator $D$ of $M^m$ defined by $\nabla$ on $\Sigma$ and $D$ the submanifold Dirac Operator defined by $\nabla$ on $\Sigma$. It is known that there exists a positive definite Hermitian metric on $\Sigma$ which satisfies, for any $X \in \Gamma(TM)$, any spinor fields $\phi, \varphi \in \Gamma(\Sigma)$, the relation

$$\langle X \cdot \phi, X \cdot \varphi \rangle = |X|^2 \langle \phi, \varphi \rangle,$$

where "·" denotes Clifford multiplication on $\mathbf{M}^{m+n}$. This metric is globally defined along $M^m$. The connection is compatible with the metric $\langle \cdot, \cdot \rangle$. Fix a point $p \in M^m$ and a local orthonormal basis $\{e_\alpha\}$ of $T_p \mathbf{M}^{m+n}$ with $\{e_A\}$ normal to $M^m$ and $\{e_i\}$ tangent to $M^m$ such that for $1 \leq i, j \leq m$

$$(\nabla_i e_j)_p = 0.$$ 

All the computations will be made in such charts. Then, at point $p$, for $1 \leq i, j \leq m$ and $m + 1 \leq A, B \leq m + n$,

$$\nabla_i e_A = -\sum_j h_{Aij} e_j + \nabla_i^\perp e_A, \quad (3.1)$$

$$\nabla_i e_j = \sum_A h_{Aij} e_A, \quad (3.2)$$

$$\nabla_i^\perp e_A := \sum_B a_{iAB} e_B, \quad (3.3)$$

where

$$h_{Aij} = h_{Aji} = -\langle \nabla_i e_A, e_j \rangle, \quad a_{iAB} = -a_{iBA} = \langle \nabla_i e_A, e_B \rangle$$

are the components of the second fundamental form $h$ and normal connection $\nabla^\perp$ at $p$ respectively. The spinorial Gauss formula says that, for $1 \leq i, j \leq m$ and $m + 1 \leq A \leq m + n$,

$$\nabla_i = \nabla_i + \frac{1}{2} \sum_{j,A} h_{Aij} e_j \cdot e_A^*, \quad (3.4)$$
which implies that the connection $\nabla$ is compatible with the metric $\langle \cdot, \cdot \rangle$ too, and also for any spinor fields $\phi \in \Gamma(S)$,
\[
\nabla_i(e_A \cdot \phi) = \nabla_i(e_A \cdot \phi) - \frac{1}{2} \sum_{j,B} h_{Bij} e_j \cdot e_B \cdot e_A \cdot \phi
\]
\[
= \nabla_i(e_A \cdot \phi) + e_A \cdot \nabla_i \phi + \frac{1}{2} \sum_{j,B} h_{Bij} e_j \cdot (e_A \cdot e_B + 2 \delta_{AB}) \cdot \phi
\]
\[
= \left( -\sum_j h_{Aij} e_j + \nabla_i e_A \right) \cdot \phi + e_A \cdot \nabla_i \phi - e_A \cdot \frac{1}{2} \sum_{j,B} h_{Bij} e_j \cdot e_B \cdot \phi
\]
\[
+ \sum_j h_{Aij} e_j \cdot \phi
\]
\[
= \nabla_i e_A \cdot \phi + e_A \cdot \nabla_i \phi.
\]

Note that in the above orthonormal frame $\{e_i\}$ of $M^m$, the Dirac operator and the submanifold Dirac operator is defined as follow,
\[
D = \sum_{i=1}^{m} e_i \cdot \nabla_i, \quad \overline{D} = \sum_{i=1}^{m} e_i \cdot \nabla_i,
\]
respectively. Contrast to the basic Dirac operator, $\overline{D}$ is in general not self-adjoint w.r.t. positive definite Hermitian metric on $S$. In fact, it is easy to see that
\[
\overline{D} = D - \frac{1}{2} \sum_A H_A e_A^*, \quad (3.7)
\]
\[
\overline{D}^* = D + \frac{1}{2} \sum_A H_A e_A^*, \quad (3.8)
\]
where $H_A := \sum_i h_{Aii}$ is the component of the mean curvature vector field of $M^m$.

From now on, we always denote $\sum_A H_A e_A$ by $\tilde{H}$.

Recall $\omega_n = i^{\frac{m+1}{2}} e_{m+1} \cdots e_{m+n}$ is the complex volume element of the normal bundle $NM^n$. There is an operator $\omega_\perp$ on $S$ defined by
\[
\omega_\perp := \begin{cases} 
\omega_n & \text{if } n \text{ is even} \\
-i\omega_n & \text{if } n \text{ is odd}
\end{cases}
\]
(see [9, 11]). Then one can check the following
\[
\omega_\perp = (-1)^n, \\
\langle \omega_\perp \cdot \phi, \varphi \rangle = (-1)^n \langle \phi, \omega_\perp \cdot \varphi \rangle, \\
\omega_\perp \cdot e_A^* = (-1)^{n-1} \omega_\perp \cdot e_A^*.
\]
\[ \nabla_i (\omega_\perp \cdot \phi) = \omega_\perp \cdot \nabla_i \phi, \]
\[ D (\omega_\perp \cdot \phi) = (-1)^n \omega_\perp \cdot D \phi. \]

Hence
\[ \mathcal{D}^* = \omega_\perp \cdot \mathcal{D} \omega_\perp . \quad (3.10) \]

One can also prove that
\[ D_H := (-1)^n \omega_\perp \cdot \mathcal{D} \quad (3.11) \]
\[ = (-1)^n \omega_\perp \cdot D + \frac{1}{2} \bar{H} \cdot \omega_\perp . \quad (3.12) \]
is formally self-adjoint with respect to the metric \( \langle \cdot , \cdot \rangle \) and, hence, \( D_H \) has real eigenvalues. This first-order Dirac operator arises in apparent horizons in the spinor proof of the positive mass theorem for black holes. Moreover, we have the following Schrödinger-Lichnerowicz type formula
\[ \mathcal{D}^* \mathcal{D} \phi = D^2 \phi \]
\[ = \left( (-1)^n \omega_\perp \cdot D + \frac{1}{2} \bar{H} \cdot \omega_\perp \right) \left( (-1)^n \omega_\perp \cdot D \phi + \frac{1}{2} \bar{H} \cdot \omega_\perp \cdot \phi \right) \]
\[ = D^2 \phi + \frac{(-1)^n}{2} \omega_\perp \cdot D (\bar{H} \cdot \omega_\perp \cdot \phi) + \frac{1}{2} \bar{H} \cdot D \phi + \frac{1}{4} \bar{H}^2 \phi \]
\[ = D^2 \phi + \frac{1}{4} |\bar{H}|^2 \phi - \frac{1}{2} D^\perp \bar{H} \cdot \phi + \bar{H} \cdot D \phi \]
\[ = \nabla^* \nabla \phi + \frac{1}{4} (R^M + \mathfrak{R}_\phi^N + |\bar{H}|^2) \phi - \frac{1}{2} D^\perp \bar{H} \cdot \phi + \bar{H} \cdot D \phi, \quad (3.13) \]

Here \( D^\perp := \sum_{i=1}^{m} e_i \cdot \nabla^\perp_i \) and for any spinor field \( \phi \in \Gamma(\mathbb{S}) \), the real function
\[ \mathfrak{R}_\phi^N := -\frac{1}{2} \sum_{i,j,A,B} R_{ijAB} (e_i \cdot e_j \cdot e_A \cdot e_B \cdot \phi, \phi/|\phi|^2) \quad (3.14) \]
is defined on subset \( M_\phi := \{ x \in M^m | \phi(x) \neq 0 \} \), where \( R_{ijAB} \) is the curvature tensor of the normal bundle \( NM^m \).

4. Estimates for the eigenvalues of the submanifold Dirac operator

In this section, we introduce some modified connections to get the lower bounds for the eigenvalues of the operator \( D_H \).

**Theorem 1.** Let \( M^m \subset \overline{M}^{m+n} \) be a closed spin submanifold of dimension \( m \geq 2 \) whose normal bundle is also spin. Suppose \( TM^m = T_1 \oplus \cdots \oplus T_k \), where \( T_a \) are parallel and pairwise orthogonal distributions of dimension \( m_a, a = 1, \cdots, k \), and
where $q$ is some real function, $q \neq \frac{1}{m_1}$ if $\vec{H} \neq 0$. If $\lambda^2_H$ achieves its minimum and the normal bundle is flat, the scalar curvature $R^M$ and $|\vec{H}|$ are both constants.

Proof. First, one can define a locally decomposable Riemannian structure $\beta$ as follows

$$\beta |_{T_1} = \text{Id}, \quad \beta |_{T_1^\perp} = -\text{Id}.$$ 

and we also define the following modified connection

$$T_i \phi = \nabla_i \phi + \frac{1}{2} (\beta(e_i) + e_i) \cdot \left( \frac{1}{2} p\vec{H} + q\lambda_H \omega_\perp \right) \cdot \phi + \frac{1}{2} \nabla_{(\beta - \text{Id})(e_i)} \phi$$

(4.2)

where $p, q$ are smooth real-valued functions that are specified later. Then, a direct computation gives

$$|T \psi|^2 = |\nabla \psi|^2 + (m_1 pq - p - q) \Re \langle \vec{H} \cdot \psi, D \psi \rangle + (m_1 q^2 - 2q)|D_H \psi|^2$$

$$+ \frac{|\vec{H}|^2}{4} (m_1 p^2 + 2q - 2m_1 pq)|\psi|^2$$

$$+ \Re \left\langle \left( \frac{1}{2} p\vec{H} + q\lambda_H \omega_\perp \right) \cdot \psi, (D - D_\beta) \psi \right\rangle - \sum_{s=m_1+1}^m |\nabla_s \psi|^2,$$ (4.3)

where $D_H \psi = \lambda_H \psi$, for a non-trivial spinor field $\psi$.

Since $\Re \langle D^+ \vec{H} \cdot \phi, D \phi \rangle = 0$, integrating (3.13) over $M^m$ yields, for any spinor $\phi$

$$\int_M |D_H \phi|^2 = \int_M |\nabla \phi|^2 + \frac{1}{4} (R^M + \Re^N + |\vec{H}|^2)|\phi|^2 - \Re \langle \vec{H} \cdot \phi, D \phi \rangle.$$ (4.4)

Hence we obtain, for the eigenspinor $\psi$ of $D_H$

$$\int_M |T \psi|^2 + \sum_{s=m_1+1}^m |\nabla_s \psi|^2 - \Re \left\langle \left( p\vec{H}/2 + q\lambda_H \omega_\perp \right) \cdot \psi, (D - D_\beta) \psi \right\rangle$$

$$= \int_M [m_1 pq - p - q + 1] \Re \langle \vec{H} \cdot \psi, D \psi \rangle + [m_1 q^2 - 2q + 1] \lambda^2_H |\psi|^2$$

$$- \frac{1}{4} (R^M + \Re^N)|\psi|^2 + \frac{|\vec{H}|^2}{4} [m_1 p^2 + 2q - 2m_1 pq - 1] |\psi|^2.$$ (4.5)

If $\vec{H} \neq 0$, let $m_1 pq - p - q + 1 = 0$, that is,

$$p = \frac{1 - q}{1 - m_1 q}.$$
Then

\[ \text{R.H.S. of (4.5)} \]

\[ = \int_M (m_1 q^2 - 2q + 1) \left[ \lambda_H^2 - \frac{1}{4} \left( \frac{R^M + \mathcal{M}^N}{m_1 q^2 - 2q + 1} - \frac{(m_1 - 1) |\bar{H}|^2}{(1 - m_1 q)^2} \right) \right] |\psi|^2. \quad (4.6) \]

Now we turn to deal with the L.H.S of (4.5). First, observe that for any spinor field \( \phi \in \Gamma(S) \)

\[ \sum_{i=1}^{m} e_i \cdot T_i \phi = \frac{1}{2} (D_\beta + D) \phi - m_1 \left( \frac{1}{2} p\bar{H} + q\lambda_H \omega_\perp \right) \cdot \phi \quad (4.7) \]

Hence by \( D_\beta^2 = D^2 \), we can deduce the relation, for any spinor \( \phi \):

\[ \Re \int_M \langle \sum_{i=1}^{m} e_i \cdot T_i \phi, -\frac{1}{2} (D_\beta \phi - D\phi) \rangle \]

\[ = -\frac{m_1}{2} \int_M \Re \left( \left( \frac{p\bar{H}}{2} + q\lambda_H \omega_\perp \right) \cdot \phi, D\phi - D_\beta \phi \right). \quad (4.8) \]

And also note for any spinor \( \phi \) and \( s > m_1, T_s \phi = 0 \). Hence

\[ \Re \left( \sum_{i=1}^{m} e_i \cdot T_i \phi, \frac{1}{2} (D_\beta \phi - D\phi) \right) \]

\[ = \left| \sum_{i=1}^{m_1} e_i \cdot T_i \phi \right| \left| \frac{1}{2} (D_\beta \phi - D\phi) \right| \]

\[ \leq \sqrt{m_1} |T\phi| \left( \frac{1}{2} |D_\beta \phi - D\phi| \right) \]

\[ \leq \frac{\sqrt{m_1}}{2} \left( \varepsilon^{-1} |T\phi|^2 + \frac{\varepsilon}{4} |D_\beta \phi - D\phi|^2 \right), \]

where \( \varepsilon \) is to be a fixed positive constant.

Hence, plugging Equation (4.8) into (4.5) and using the Cauchy-Schwarz inequality and the equality (2.6), we have that

\[ \text{L.H.S. of (4.5)} \]

\[ = \int_M \left( 1 - \frac{1}{\varepsilon \sqrt{m_1}} \right) |T\psi|^2 + \sum_{s=m_1+1}^{m} |\nabla_s \psi|^2 - \frac{\varepsilon}{\sqrt{m_1}} (|D_2 \psi|^2 + \cdots + |D_k \psi|^2) \]

\[ \geq \int_M \left( 1 - \frac{1}{\varepsilon \sqrt{m_1}} \right) |T\psi|^2 + \left( \frac{1}{m_2} - \frac{\varepsilon}{\sqrt{m_1}} \right) |D_2 \psi|^2 \]

\[ + \cdots + \left( \frac{1}{m_k} - \frac{\varepsilon}{\sqrt{m_1}} \right) |D_k \psi|^2. \quad (4.9) \]
Now we take $\varepsilon = \frac{1}{\sqrt{m}}$, hence, (4.9), combined with (4.6), yields that
\[
0 \leq \int_M \left( 1 - \sqrt{\frac{m}{m_1}} \right) |T\psi|^2 + \left( \frac{1}{m_2} - \frac{1}{\sqrt{m_1m_2}} \right) |D_2\psi|^2 \\
+ \cdots + \left( \frac{1}{m_k} - \frac{1}{\sqrt{m_1m_2}} \right) |D_k\psi|^2 \\
\leq \int_M \left( m_1q^2 - 2q + 1 \right) \left[ \lambda_H^2 - \frac{1}{4} \left( R^M + \mathfrak{R}_\psi^N - \frac{m_1 - 1}{m_1q^2 - 2q + 1} \right) \right] |\psi|^2. \tag{4.10}
\]
Therefore, the first part of the theorem follows.

If $\lambda_H^2$ achieves its minimum, then
\[
\sum_{i \in \mathcal{I}_1} e_i \cdot T_i\psi = \frac{1}{2} (D_\beta\psi - D\psi), \tag{4.11}
\]
which, together with (4.7), implies that,
\[
D\psi = m_1 \left( \frac{1}{2} p\tilde{H} + q\lambda_H\omega_\perp \right) \cdot \psi. \tag{4.12}
\]
But $p = \frac{1-q}{1-m_1q}$ and
\[
D\psi = \left( \omega_\perp \cdot D_H + \frac{1}{2} \tilde{H} \right) \psi = \lambda_H\omega_\perp \cdot \psi + \frac{1}{2} \tilde{H} \cdot \psi.
\]
This implies that
\[
(m_1 - 1) \tilde{H} \cdot \psi = 2(1 - m_1q)^2 \lambda_H\omega_\perp \cdot \psi \tag{4.13}
\]
and moreover from (4.12),
\[
D\psi = m_1 \left( \frac{1}{2} p\tilde{H} + q\lambda_H\omega_\perp \right) \cdot \psi = \frac{m_1}{2} \mathcal{H} \cdot \psi, \tag{4.14}
\]
where $\mathcal{H} := \frac{1 + m_1q^2 - 2q}{(1 - m_1q)^2} \tilde{H}$.

**Case 1.** If $m_1 > m_2$, the eigenspinor corresponding to the smallest eigenvalue of $D_H^2$ satisfies the following generalized Killing type equations
\[
\nabla_{m_1+1}\psi = \nabla_{m_1+2}\psi = \cdots = \nabla_m\psi = 0,
\]
as well as $T\psi = 0$, which is equivalent to, for $i = 1, \cdots, m_1$,
\[
\nabla_i\psi = -e_i \cdot \left( \frac{1}{2} p\tilde{H} + q\lambda_H\omega_\perp \right) \cdot \psi = -\frac{1}{2} e_i \cdot \mathcal{H} \cdot \psi. \tag{4.15}
\]
Obviously, \( d|\psi|^2 = 0 \). And for any \( i, j = 1, \cdots, m_1, s, t = m_1 + 1, \cdots, m \), one has \( \nabla_i \nabla_s \psi = \nabla_s \nabla_i \psi = 0 \),

\[
\nabla_s \nabla_i \psi = -\frac{1}{2} e_i \cdot \nabla_s^\perp \mathcal{H} \cdot \psi \quad (4.16)
\]
as well as

\[
\nabla_j \nabla_i \psi = -\frac{1}{2} e_i \cdot \nabla_j^\perp \mathcal{H} \cdot \psi + \frac{1}{4} |\mathcal{H}|^2 e_i \cdot e_j \cdot \psi. \quad (4.17)
\]

If the normal bundle is flat, then

\[
0 = -\frac{1}{2} \text{Ric}(e_s) \cdot \psi = \sum_{j=1}^{m_1} e_j \cdot \nabla_s \nabla_j \psi = \sum_{j=1}^{m_1} e_j \cdot \left(-\frac{1}{2} e_j \cdot \nabla_s^\perp \mathcal{H} \cdot \psi \right) = \frac{m_1}{2} \nabla_s^\perp \mathcal{H} \cdot \psi,
\]

which implies that

\[
\nabla_s^\perp \mathcal{H} = 0, \text{ for } s > m_1. \quad (4.18)
\]

Moreover

\[
-\frac{1}{2} \text{Ric}(e_i) \cdot \psi = \sum_{j=1}^{m_1} e_j \cdot \mathcal{R}_{e_i, e_j} \psi = \frac{1}{2} \left(-D^\perp \mathcal{H} \cdot e_i \cdot \psi + m_1 \nabla_i^\perp \mathcal{H} \cdot \psi\right) - \frac{m_1-1}{2} |\mathcal{H}|^2 e_i \cdot \psi. \quad (4.19)
\]

Using the fact \( \sum_{i \in I_1} e_i \cdot D^\perp \mathcal{H} \cdot e_i = -(m_1-2)D^\perp \mathcal{H} \), one gets

\[
\frac{1}{2} R^M = (m_1 - 1) D^\perp \mathcal{H} \cdot \psi + \frac{m_1(m_1 - 1)}{2} |\mathcal{H}|^2 \psi. \quad (4.20)
\]

Take the inner product of the above equality with \( \psi \) and compare its real and imaginary parts to obtain

\[
D^\perp \mathcal{H} \cdot \psi = 0, \quad R^M = m_1(m_1 - 1) |\mathcal{H}|^2. \quad (4.21)
\]

As a consequence, \( R^M \) is a constant, due to (4.10).

Furthermore, note

\[
D^\perp \mathcal{H} \cdot e_i \cdot \psi \quad = \quad -\sum_{j=1}^{m} e_j \cdot e_i \cdot \nabla_j^\perp \mathcal{H} \cdot \psi \quad = \quad \sum_{j=1}^{m} (e_i \cdot e_j + 2 \delta_{ij}) \cdot \nabla_j^\perp \mathcal{H} \cdot \psi \quad = \quad e_i \cdot D^\perp \mathcal{H} \cdot \psi + 2 \nabla_i^\perp \mathcal{H} \cdot \psi = 2 \nabla_i \mathcal{H} \cdot \psi,
\]
which yields
\[ \frac{1}{2} \text{Ric}(e_i) \cdot \psi = -\frac{m_1 - 2}{2} \nabla_i^\perp \mathcal{H} \cdot \psi + \frac{m_1 - 1}{2} |\mathcal{H}|^2 e_i \cdot \psi. \] (4.22)

Hence from (4.18) and (4.22), one gets
\[ \nabla_i^\perp \mathcal{H} = 0, \]
if \( m_1 \neq 2 \) and
\[ \text{Ric}(e_i) = (m_1 - 1)|\mathcal{H}|^2 e_i, \quad \text{for } i = 1, \ldots, m_1 \]
as well as
\[ \lambda_2^2_\mathcal{H} = (m_1 - 1)^2 4(1 + m_1 q^2 - 2q^2)|\mathcal{H}|^2. \]

**Case 2.** If \( m_1 = m_2 = \cdots = m_l < m_{l+1} \leq \cdots \leq m_k, \) then
\[ D_{l+1} \psi = D_{l+2} \psi = \cdots = D_k \psi = 0 \] (4.23)
and we also have, for \( i, j \in I_1, \) the two spinor fields are proportional, i.e.,
\[ e_i \cdot T_i \psi = e_j \cdot T_j \psi \] (4.24)
which, in turn implies that
\[ e_i \cdot \nabla_i \psi = e_j \cdot \nabla_j \psi. \] (4.25)
And also for any \( a \in \{2, \cdots, l\}, \) and any \( s, t \in I_a, \)
\[ e_s \cdot \nabla_s \psi = e_t \cdot \nabla_t \psi. \] (4.26)
and \( \alpha \in (I_1 \cup I_2 \cup \cdots \cup I_l)\c, \)
\[ \nabla_\alpha \psi = 0. \] (4.27)

Therefore, for \( i \in I_1, \)
\[ \nabla_i \psi = T_i \psi - e_i \cdot \left( \frac{1}{2} p_{\mathcal{H}} + q \lambda_\mathcal{H} \mathcal{H} \right) \cdot \psi \]
\[ = -\frac{1}{2m_1} e_i \cdot (D_\beta \psi - D_\psi) - \frac{1}{m_1} e_i \cdot D_\psi \]
\[ = -\frac{1}{4} e_i \cdot \mathcal{H} \cdot \psi - \frac{1}{2m_1} e_i \cdot D_\beta \psi, \] (4.28)
and while, for any \( b \in \{2, \cdots, l\}, s \in I_b, \)
\[ \nabla_s \psi = -\frac{1}{m_1} e_s \cdot D_b \psi \]
\[ = \frac{1}{2m_1} e_s \cdot (D_\beta \psi - D_\psi) + \frac{1}{m_1} \sum_{a=2, \neq b}^l e_s \cdot D_a \psi \]
\[ = -\frac{1}{4} e_s \cdot \mathcal{H} \cdot \psi + \frac{1}{2m_1} e_s \cdot D_\beta \psi + \frac{1}{m_1} \sum_{a=2, \neq b}^l e_s \cdot D_a \psi. \] (4.29)
Using the facts, for $s, t \in I_b$,
\[
\sum_{s \in I_b} e_s \cdot (e_s \cdot e_t - e_t \cdot e_s) = (-2m_1 + 2)e_t,
\]
and
\[
\sum_{s \in I_b} e_s \cdot (e_s \cdot \nabla_t - e_t \cdot \nabla_s) = -m_1 \nabla_t + \sum_{s \in I_b} (e_t \cdot e_s \cdot \nabla_s + 2\delta_{st}\nabla_s)
\]
\[
= (-m_1 + 2)\nabla_t + e_t \cdot D_b,
\]
it follows
\[
-\frac{1}{2} Ric(e_t) \cdot \psi = \sum_{s \in I_b} e_s \cdot R_{e_t,e_s} \psi
\]
\[
= \sum_{s \in I_b} e_s \cdot [\nabla_t, \nabla_s] \psi
\]
\[
= \frac{1}{4} (m_1 \nabla_t^2 H - D_b^\perp H \cdot e_t) \cdot \psi - \frac{m_1 - 1}{8} |H|^2 e_t \cdot \psi
\]
\[
- \frac{m_1 - 1}{4c} e_t \cdot H \cdot D_\beta \psi + \frac{1}{2m_1} [(2 - m_1) \nabla_t (D_\beta \psi) + e_t \cdot D_b (D_\beta \psi)]
\]
\[
- \frac{m_1 - 1}{2m_1} e_t \cdot H \cdot \sum_{a=2, \neq b}^l D_a \psi + \frac{1}{m_1} \sum_{a=2, \neq b}^l [(2 - m_1) \nabla_t (D_a \psi) + e_t \cdot D_b (D_a \psi)].
\]
Performing its Clifford multiplication by $e_t$ and summing over $t \in I_b, b \in \{2, \cdots, l\}$ yields,
\[
\frac{1}{2} R^M_{b} \psi = \frac{1}{2} \sum_{t \in I_b} e_t \cdot Ric(e_t) \cdot \psi
\]
\[
= \frac{m_1 - 1}{2} D_b^\perp H \cdot \psi + \frac{m_1 (m_1 - 1)}{8} |H|^2 \psi + \frac{m_1 - 1}{4} H \cdot D_\beta \psi - \frac{m_1 - 1}{m_1} D_b (D_\beta \psi)
\]
\[
+ \frac{m_1 - 1}{2} H \cdot \sum_{a=2, \neq b}^l D_a \psi - \frac{2(m_1 - 1)}{m_1} \sum_{a=2, \neq b}^l D_b (D_a \psi).
\] (4.30)
On the other hand,
\[
\frac{1}{2} R^M_{1} \psi = -\frac{1}{2} \sum_{i \in I_1} e_i \cdot Ric(e_i) \cdot \psi
\]
\[
= \frac{m_1 - 1}{2} D^1_\perp (H) \cdot \psi + \frac{m_1 (m_1 - 1)}{8} |H|^2 \psi
\]
\[
- \frac{m_1 - 1}{4} H \cdot D_\beta \psi + \frac{m_1 - 1}{m_1} D_1 (D_\beta \psi).
\] (4.31)
Combining Eqs. (4.31) and (4.30), by (2.5) one gets

\[
\frac{1}{2} R^M \psi = \frac{1}{2} \sum_{a=1}^{l} R^M_a \psi
\]

\[
= \frac{m_1 - 1}{2} \sum_{a=1}^{l} D^\perp_a \mathcal{H} \cdot \psi + l \cdot \frac{m_1(m_1 - 1)}{8} |\mathcal{H}|^2 \psi
\]

\[
+ (l - 2) \cdot \frac{m_1 - 1}{4} \mathcal{H} \cdot D_\beta \psi + \frac{m_1 - 1}{m_1} \left( D_1 - \sum_{b=2}^{l} D_b \right) (D_\beta \psi)
\]

\[
+ (l - 2) \cdot \frac{m_1 - 1}{2} \mathcal{H} \cdot \sum_{b=2}^{l} D_b \psi
\]

\[
= \frac{m_1 - 1}{2} \sum_{a=1}^{l} D^\perp_a \mathcal{H} \cdot \psi + \frac{m_1(m_1 - 1)}{4} |\mathcal{H}|^2 \psi
\]

\[
+ \frac{m_1 - 1}{m_1} \left( D_1 - \sum_{b=2}^{l} D_b \right) (D_\beta \psi),
\] (4.32)

where in the last step we used the relation

\[
\sum_{b=2}^{l} D_b \psi = \frac{1}{2} (D\psi - D_\beta \psi) = \frac{m_1}{4} \mathcal{H} \cdot \psi - \frac{1}{2} D_\beta \psi.
\]

Note, for \( \alpha \in (I_1 \cup I_2 \cup \cdots \cup I_l)^c \), we have \( \nabla_\alpha \psi = 0 \). Thus

\[
\left( D_1 - \sum_{b=2}^{l} D_b \right) (D_\beta \psi) = D^2_\beta \psi = D^2 \psi
\]

\[
= D \left( \frac{m_1}{2} \mathcal{H} \cdot \psi \right)
\]

\[
= \frac{m_1}{2} (D^\perp \mathcal{H} \cdot \psi - \mathcal{H} \cdot D \psi)
\]

\[
= \frac{m_1}{2} D^\perp \mathcal{H} \cdot \psi + \frac{m_1^2}{4} |\mathcal{H}|^2 \psi.
\] (4.33)

Therefore,

\[
\frac{1}{2} R^M \psi = \frac{m_1 - 1}{2} \sum_{a=1}^{l} D^\perp_a \mathcal{H} \cdot \psi + \frac{m_1 - 1}{2} \mathcal{H} \cdot D^\perp \psi + \frac{m_1(m_1 - 1)}{2} |\mathcal{H}|^2 \psi.
\] (4.34)

This implies that \( R^M = m_1(m_1 - 1)|\mathcal{H}|^2 \). The whole proof of Theorem 1 is complete.

Q.E.D.
Furthermore, now assume that $m_1(R^M + \mathfrak{R}_\psi^N) > (m_1 - 1)|\vec{H}|^2$ on $M_\psi$, the complement of which in $M^m$ is of zero measure, then one can choose $q$ such that

$$
(1 - m_1 q)^2 = \frac{(m_1 - 1)|\vec{H}|}{\sqrt{\frac{m_1}{m_1 - 1}(R^M + \mathfrak{R}_\psi^N) - |\vec{H}|}} \quad \text{on } M_\psi, \quad (4.35)
$$

in (4.10), to obtain

$$
0 \leq \int_M \left(1 - \sqrt{\frac{m_2}{m_1}}\right) |T\psi|^2 + \left(\frac{1}{m_2} - \frac{1}{\sqrt{m_1 m_2}}\right) |D_2\psi|^2 + \cdots + \left(\frac{1}{m_k} - \frac{1}{\sqrt{m_1 m_2}}\right) |D_k\psi|^2
$$

$$
\leq \int_M (m_1 q^2 - 2q + 1) \left[\lambda_H^2 - \frac{1}{4} \left(\sqrt{\frac{m_1}{m_1 - 1}(R^M + \mathfrak{R}_\psi^N) - |\vec{H}|}\right)^2\right] |\psi|^2. \quad (4.36)
$$

At the same time by (4.35), (4.13) can be simplified as

$$
\vec{H} \cdot \psi = \text{sign}(\lambda_H)|\vec{H}|\omega_\perp \cdot \psi \quad (4.37)
$$

in the limiting case.

This implies the following theorem

**Theorem 2.** Let $M^m \subset \overline{M}^{m+n}$ be a closed spin submanifold of a Riemannian spin manifold $\overline{M}^{m+n}$. Suppose $TM^m = T_1 \oplus \cdots \oplus T_k$, where $T_a$ are parallel and pairwise orthogonal distributions of dimension $m_a$, $a = 1, \cdots, k$, and $m_1 \geq m_2 \geq \cdots \geq m_k \geq 0$. Consider a non-trivial spinor field $\psi \in \Gamma(S)$ such that $D_H \psi = \lambda_H \psi$, and also assume that $m_1(R^M + \mathfrak{R}_\psi^N) > (m_1 - 1)|\vec{H}|^2$ on $M_\psi$, then

$$
\lambda_H^2 \geq \frac{1}{4} \min_{M_\psi} \left(\sqrt{\frac{m_1}{m_1 - 1}(R^M + \mathfrak{R}_\psi^N) - |\vec{H}|}\right)^2. \quad (4.38)
$$

If $\lambda_H^2$ achieves its minimum and the normal bundle is flat, the scalar curvature $R^M$ and $|\vec{H}|$ are both constants.

**Remark 1.**

(1) If $M^m \hookrightarrow \overline{M}^{m+1}$ is not minimal, then the normal bundle is an oriented real line bundle, hence trivial. (4.13) becomes into

$$
(m_1 - 1)H = 2(1 - m_1 q)^2 \lambda_H. \quad (4.39)
$$

Therefore, we also get $\text{sign}(\lambda_H) = \text{sign}(H)$ in the limiting cases of Theorem 1 and Theorem 2.

(2) If $m(R^M + \mathfrak{R}_\psi^N) > (m - 1)|\vec{H}|^2$ on $M_\psi$, we just take $\beta = \text{Id}$ in the proof of Theorem 2, then the result was obtained in [21, 10, 11, 16, 9].

(3) If $M^m \hookrightarrow \overline{M}^{m+1}$ is a minimal closed spin hypersurface, the estimate (4.38) reduces to the Alexandrov’s result [11]. Moreover by (4.25), (4.26) and (4.27), following
the arguments in [1], one knows that, if (4.38) is an equality, the universal cover \( \tilde{M} \) of \( M \) is isometric to a product \( M_1 \times \cdots \times M_k \), where \( \dim M_s = m_s, s = 1, \ldots, k \). \( M_1 \) has a real Killing spinor and \( M_s \) has a parallel spinor if \( m_s < m_1 \), and \( M_s \) has a parallel spinor or a real Killing spinor if \( m_s = m_1 \).

Note our approach can be also used to recover the estimate in [17]. First, it is not difficult to check that \( D_\beta \) is a formally self-adjoint elliptic operator with respect to \( L^2 \)-product on closed manifold, only assumption that \( \text{div} \beta := (\nabla_{e_i} \beta)(e_i) = 0 \) is needed. So if \( \theta \) is a harmonic vector field of unit length, we can define the \( \beta \)-twist Dirac operator
\[
D_\beta \phi := e_i \cdot \nabla_{\beta(e_i)} \phi = \beta(e_i) \cdot \nabla_{e_i} \phi,
\]
where \( \beta(X) := X - 2\theta(X)\theta^\flat \), for \( X \in \Gamma(TM) \). The spectrum of \( D_\beta \) is still discrete and real. Moreover, we also have the following Lichnerowicz-type formula (see, [5])
\[
D_\beta^2 \phi = -\theta \cdot \nabla^\star \nabla(\theta \cdot \phi) + \frac{R}{4} \phi.
\]

**Theorem 3. ([17], Moroianu and Ornea.)** Let \( M^m \subset \tilde{M}^{m+1} \) be a minimal closed spin hypersurface of dimension \( m \geq 3 \), and if on \( M^m \) there exists a harmonic 1-form \( \theta \) of unit length. Consider a non-trivial spinor field \( \psi \in \Gamma(S) \) such that \( D\psi = \lambda \psi \), then
\[
\lambda^2 \geq \frac{m-1}{4(m-2)} \min_M R^M.
\]

If \( \lambda^2 \) achieves its minimum, \( \theta \) is in fact a parallel 1-form and the eigenspinor \( \psi \) corresponding to \( \lambda \) satisfies that \( D(\theta \cdot \psi) = \lambda \theta \cdot \psi \).

**Proof.** In this case, we choose \( q = \frac{1}{m-1} \) and for the first eigenspinor \( \psi \) for of \( D \), the min-max principle gives
\[
\int_M |D_\beta \psi|^2 \overset{\text{4.41}}{=} \int_M |D(\theta \cdot \psi)|^2 \geq \lambda^2 \int_M |\psi|^2.
\]

Therefore, (4.43) turns to be
\[
\Re \int_M \left\langle \sum_{i=1}^m e_i \cdot T_i \phi, \frac{1}{2}(D \phi - D_\beta \phi) \right\rangle \leq \frac{\lambda}{2} \int_M \Re \langle e_0 \cdot \phi, D_\beta \phi - D \phi \rangle.
\]

Hence, the same argument still works and the estimate (4.42) can be also obtained and in the limiting case, \( \nabla_{\theta} \psi = 0 \), and \( T\psi = 0 \), that is, for \( i = 1, \cdots, m-1 \),
\[
\nabla_i \psi = -\frac{\lambda}{m-1} e_i \cdot e_0 \cdot \psi.
\]
Therefore, using the \( \frac{1}{2}Ricci \)-formula yields
\[
\frac{1}{2}Ric(\theta) \cdot \psi = D(\nabla_\theta \psi) - \nabla_\theta (D\psi) - \sum_{i=1}^{m} e_i \cdot \nabla_{\nabla_i \theta} \psi
\]
\[
= -\sum_{i=1}^{m} e_i \cdot \nabla_{\nabla_i \theta} \psi
\]
\[
= \frac{\lambda}{m-1}D\theta \cdot e_0 \cdot \psi = 0.
\]
Hence, \( Ric(\theta) = 0 \). Furthermore, one obtains that \( \theta \) is in fact parallel by Bochner-Weitzenböck formula on 1-forms
\[
\Delta = \nabla^* \nabla + Ric,
\]
and then we can apply Theorem 3.1 in [2] to know that the universal covering space of \( M^m \) is a Riemannian product of the form \( M_1 \times \mathbb{R} \), where \( M_1 \) admits a real Killing spinor. At last, for any 1-form \( \theta \) and any spinor \( \phi \), we always have
\[
D(\theta \cdot \phi) = -\theta \cdot D\phi - 2\nabla_\theta \phi + (D\theta) \cdot \phi.
\]
Q.E.D.

5. Intrinsic estimates for the twisted Dirac-Schrödinger operator

In fact, the normal bundle of the submanifold can be replaced by an auxiliary arbitrary vector bundle on the submanifold and all the preceding computations can be done in an intrinsic way to obtain results for a twisted Dirac-Schrödinger operator on the manifold.

Let \( (M_m, g) \) be a closed Riemannian spin manifold, and let \( \Sigma M \) be the spinor bundle for \( M^m \) with the canonical Riemannian connection \( \nabla_{\Sigma M} \). Let \( E \) be any vector bundle over \( M^m \) equipped with a metric connection \( \nabla_E \) and a spin structure. Set
\[
\Sigma := \Sigma M \otimes \Sigma E.
\]
Recall that Clifford multiplication on \( \Gamma(\Sigma) \) by a tangent vector field \( X \in \Gamma(TM^m) \) is given by:
\[
X \cdot \phi := (X \cdot \sigma) \otimes \varepsilon \in \Gamma(\Sigma),
\]
for \( \forall \phi = \sigma \otimes \varepsilon \in \Gamma(\Sigma) \) and the canonical tensor product connection on \( \Gamma(\Sigma) \) is defined by the formula
\[
\nabla(\sigma \otimes \varepsilon) := (\nabla_{\Sigma M} \sigma) \otimes \varepsilon + \sigma \otimes (\nabla_E \varepsilon),
\]
where \( \nabla_{\Sigma M} \) and \( \nabla_{\Sigma E} \) denote the covariant derivatives on \( \Sigma M \) and \( \Sigma E \) respectively. A direct verification shows that the curvature transformation \( \mathcal{R}_{X,Y} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \) of \( \Sigma M \otimes \Sigma E \) is also a derivation, i.e,
\[
\mathcal{R}(\sigma \otimes \varepsilon) = (\mathcal{R}^M \sigma) \otimes \varepsilon + \sigma \otimes (\mathcal{R}^E \varepsilon),
\]
where $R^M$ and $R^E$ denote the curvature transformations of $\Sigma M$ and $\Sigma E$ respectively.

The Dirac operator on $M^m$ twisted with the bundle $\Sigma E$ is given by

$$D^{\Sigma E}: \Gamma(\Sigma) \rightarrow \Gamma(\Sigma)$$

$$\phi \mapsto D^{\Sigma E} \phi = \sum_{i=1}^{m} e_i \cdot \nabla_i \phi.$$

It is straightforward to check that we have the following formula Schrödinger-Lichnerowicz-type formula for the twisted Dirac operator $D^{\Sigma E}$ on the twisted spinor bundle $\Sigma = \Sigma M \otimes \Sigma E$ over $M^m$

$$(D^{\Sigma E})^2 = \nabla^* \nabla + \frac{1}{2} \sum_{i,j=1}^{m} e_i \cdot e_j \cdot R_{e_i e_j}$$

$$= \nabla^* \nabla + \frac{1}{2} \sum_{i,j=1}^{m} e_i \cdot e_j \cdot (R^M_{e_i e_j} \otimes \text{Id} + \text{Id} \otimes R^E_{e_i e_j})$$

$$= \nabla^* \nabla + \frac{1}{2} \sum_{i,j=1}^{m} \left[ (e_i \cdot e_j \cdot R^M_{e_i e_j}) \otimes \text{Id} + (e_i \cdot e_j \cdot \text{Id}) \otimes R^E_{e_i e_j} \right]$$

$$= \nabla^* \nabla + \frac{1}{4} (R^M + \mathcal{R}^E), \quad (5.1)$$

where $\mathcal{R}^E: \Sigma \rightarrow \Sigma$ is a smooth symmetric bundle endomorphism defined by the formula

$$\mathcal{R}^E(\sigma \otimes \varepsilon) := 2 \sum_{i,j=1}^{m} (e_i \cdot e_j \cdot \sigma) \otimes R^E_{i,j} \varepsilon \quad (5.2)$$

on vectors $\sigma \otimes \varepsilon \in \Gamma(\Sigma)$ of simple type.

For any smooth real function $F$ on $M^m$, define the twisted Dirac-Schrödinger operator by

$$D_F = D^{\Sigma E} - \frac{1}{2} F. \quad (5.3)$$

Now suppose $TM^m = T_1 \oplus \cdots \oplus T_k$, where $T_a$ are parallel and pairwise orthogonal distributions of dimension $m_a, a = 1, \cdots, k$, and $m_1 \geq m_2 \geq \cdots \geq m_k$. And also assume

$$m_1 (R^M + \kappa_1) > (m_1 - 1)F^2 > 0 \quad (5.4)$$

on $M^m$, where $\kappa_1$ be the lowest eigenvalue of the endomorphism $\mathcal{R}^E$. Then define the modified connection on $\Sigma = \Sigma M \otimes \Sigma E$

$$T_i \phi = \nabla_i \phi + \left( \frac{1 - q}{2(1 - m_1 q)} F + q\lambda_F \right) \frac{1}{2} \left[ (\beta(e_i) + e_i) \cdot \phi + \frac{1}{2} \nabla_{(\beta - \text{Id})(e_i)} \phi \right], \quad (5.5)$$
where the smooth real function \( q \) satisfies
\[
(1 - m_1q)^2 = \frac{(m_1 - 1)|F|}{\sqrt{\frac{m_1}{m_1 - 1}(R^M + \kappa_1)} - |F|}.
\]  
(5.6)

The same computations as in the proof of Theorem 2 lead to the following estimate,
\[
\int_M \left(1 - \sqrt{\frac{m_2}{m_1}}|T\psi|^2 + \left(\frac{1}{m_2} - \frac{1}{\sqrt{m_1m_2}}\right)|D_2\psi|^2 + \cdots + \left(\frac{1}{m_k} - \frac{1}{\sqrt{m_1m_2}}\right)|D_k\psi|^2\right)
\leq \int_M \left(m_1q^2 - 2q + 1\right) \left[\lambda_F^2 - \frac{1}{4} \left(\sqrt{\frac{m_1}{m_1 - 1}(R^M + \kappa_1)} - |F|\right)^2\right] |\psi|^2,
\]  
(5.7)

where \( D_F\psi = \lambda_F\psi \) for a non-trivial spinor field \( \psi \). Therefore we can conclude the following theorem.

**Theorem 4.** Let \((M^m, g)\) be a closed Riemannian spin manifold and \( E \to M^m \) be a Riemannian and spin vector bundle of rank \( n \) over \( M^m \). Assume \( TM^m = T_1 \oplus \cdots \oplus T_k \), where \( T_a \) are parallel and pairwise orthogonal distributions of dimension \( m_a, a = 1, \cdots, k \), and \( m_1 \geq m_2 \geq \cdots \geq m_k \geq 0 \). Let \( \lambda_F \) be any eigenvalue of the Dirac-Schrödinger operator \( D_F = D^\Sigma E - \frac{1}{2} F \), associated with the eigenspinor \( \psi \).
Assume \( m_1(R^M + \kappa_1) > (m_1 - 1)F^2 \) on \( M^m \), then
\[
\lambda_H^2 \geq \frac{1}{4} \min_M \left(\sqrt{\frac{m_1}{m_1 - 1}(R^M + \kappa_1)} - |F|\right)^2.
\]  
(5.8)

If \( \lambda_F \) achieves its minimum, \( F \) are constant and also \( \mathcal{R}^E\psi = \kappa_1\psi \); In fact, the universal cover \( \tilde{M} \) of \( M^m \) is isometric to a product \( M_1 \times \cdots \times M_s \), where \( \dim M_s = m_s, M_1 \) has a real Killing spinor and \( M_s \) has a parallel spinor if \( m_s < m_1 \), and \( M_s \) has a parallel spinor or a real Killing spinor if \( m_s = m_1 \).

Note there is a way to give an intrinsic meaning to the modified connection \( \tilde{\nabla} \) in the previous section, assuming an additional condition
\[
\tilde{H} \cdot \psi = F\omega_\perp \cdot \psi
\]  
(5.9)

In fact, there exists an identification of the restricted spinor bundle \( S := \Sigma M|_M \) with the twisted spinor bundle \( \Sigma := \Sigma M \otimes \Sigma NM \) or its direct sum \( \Sigma \oplus \Sigma \):
\[
S \to \begin{cases}
\Sigma, & \text{if } m \text{ or } n \text{ is even} \\
\Sigma \oplus \Sigma, & \text{otherwise},
\end{cases}
\]  
(5.10)

which sends \( \phi \in \Gamma(S) \) to \( \phi^* \in \Gamma(\Sigma) \), for example, if \( m \) or \( n \) is even. Moreover, with respect to this identification, Clifford multiplication by a vector field \( X \in \Gamma(TM^m) \), is given (see [9]) by
\[
\forall \phi \in \Gamma(S), \quad X \cdot \phi^* = (X \cdot \omega_\perp \cdot \phi)^*.
\]  
(5.11)
6. Other generalized hypersurface Dirac operators

In this section, we shall extend the above techniques to other generalized hypersurface Dirac operators which appear in General Relativity. Let \( p_{ij} \) be a-tensor on \( \mathbb{R}^{m+1} \) and \( P := \bar{g}^{ij} p_{ij} |_M \), then we consider the following operator

\[
D_P = e_0 \cdot D - \frac{\sqrt{-1}}{2} P e_0. 
\]  

(6.1)

Note that \( D_P \) is also formally self-adjoint with respect to \( \int_M \langle \cdot, \cdot \rangle \) and, hence, \( D_P \) has real eigenvalues. Suppose \( TM^m = T_1 \oplus \cdots \oplus T_k \), where \( T_a \) are parallel and pairwise orthogonal distributions of dimension \( m_a \), \( a = 1, \ldots, k \), and \( m_1 \geq m_2 \geq \cdots \geq m_k \geq 0 \). We can define

\[
T_i \phi = \nabla_i + \frac{\sqrt{-1}}{4} p \left( \beta + \text{Id} \right) (e_i) \cdot \phi + \frac{1}{2} q \lambda_P e_0 \cdot \left( \beta + \text{Id} \right) (e_i) \cdot \phi + \frac{1}{2} \nabla_{(\beta - \text{Id})(e_i)} \phi. 
\]  

(6.2)

Then for \( D_P \psi = \lambda_P \psi \),

\[
|T \phi|^2 = \vert \nabla \phi \vert^2 + \left( m_1 p q - p - q \right) \Re e \langle D \psi, \sqrt{-1} \psi \rangle + \left( m_1 q^2 - 2q \right) |D_P \psi|^2 + \frac{p^2}{4} \left( m_1 p^2 + 2q - 2m_1 pq \right) |\psi|^2 - \sum_{s=m_1+1}^m |\nabla_s \psi|^2
\]  

(6.3)

In this case, we still have for any spinor field \( \phi \in \Gamma(S) \)

\[
\sum_{i=1}^m e_i \cdot T_i \phi = D \phi + m_1 \left( -\frac{\sqrt{-1}}{2} p \phi + q \lambda_P e_0 \cdot \phi \right) + \frac{1}{2} (D_\beta \phi - D \phi). 
\]  

(6.4)

Hence

\[
\Re e \int_M \left( \sum_{i=1}^m e_i \cdot T_i \phi - \frac{1}{2} (D_\beta \phi - D \phi) \right) = \frac{m_1}{2} \Re e \int_M \left( -\frac{\sqrt{-1}}{2} p \phi + q \lambda_P e_0 \cdot \phi \right) \phi, D \phi - D_\beta \phi \]  

(6.5)

At the same time

\[
\int_M |D_P \psi|^2 = \int_M |\nabla \phi|^2 + \frac{1}{4} (R^M + P^2) |\phi|^2 - P \Re e \langle D \phi, \sqrt{-1} \phi \rangle
\]  

Now assume \( m_1 R^M > (m_1 - 1) P^2 \) on \( M^m \), then

\[
\lambda_P^2 \geq \frac{1}{4} \min_M \left( \sqrt{\frac{m_1}{m_1 - 1}} R^M - |P| \right)^2. 
\]  

(6.6)
References

[1] B. Alexandrov, The first eigenvalue of the Dirac operator on locally reducible Riemannian manifolds, J. Geom. Phys. 57, 467–472(2007).
[2] B. Alexandrov, G. Grantcharov, S. Ivanov, An estimate for the first eigenvalue of the Dirac operator on compact Riemannian spin manifold admitting a parallel one-form, J. Geom. Phys. 28, 263–270(1998).
[3] C. Bär, Real Killing spinors and holonomy, Comm. Math. Phys. 154, 509–521(1993).
[4] C. Bär, M. Dahl, The first Dirac eigenvalues on manifolds with positive scalar curvature, Proc. Amer. Math. Soc. 132, 3337–3344(2004).
[5] Y. Chen, The Dirac operator on manifold admitting parallel one-form, J. Geom. Phys. 117, 214–221 (2017).
[6] T. Friedrich, Dirac operators in Riemannian geometry, Graduate Studies in Mathematics 25, American Mathematical Society(2000).
[7] G. Gibbons, S. Hawking, G. Horowitz, M. Perry, Positive mass theorems for black holes. Commun. Math. Phys. 88, 295–308 (1983).
[8] M. Herzlich, The positive mass theorems for black holes revisited. J. Geom. Phys. 26, 97–111 (1998).
[9] N. Ginoux, B. Morel, On eigenvalue estimate for the submanifold Dirac operator, Int. J. Math. 13, 533–548 (2002).
[10] O. Hijazi, X. Zhang, Lower bounds for the eigenvalues of Dirac operator, Part I. The hypersurface Dirac operators. Ann Glob Anal Geom. 19, 355–376(2001).
[11] O. Hijazi, X. Zhang, Lower bounds for the eigenvalues of Dirac operator, Part II. The submanifold Dirac operators. Ann Glob Anal Geom. 19, 163–181(2001).
[12] E. C. Kim, Lower bounds of the Dirac eigenvalues on compact Riemannian spin manifolds with locally product structure, arXiv:math.DG/0402427 (2004).
[13] K.-D. Kirchberg, An estimation for the first eigenvalue of the Dirac operator in closed Kähler manifolds of positive scalar curvature, Ann. Glob. Ann. Geom. 3, 291–325 (1986).
[14] K.-D. Kirchberg, The first eigenvalue of the Dirac operator on Kähler manifolds, J. Geom. Phys. 7, 449–468(1990).
[15] H. B. Lawson, M. L. Michelsohn, Spin Geometry, Princeton Math Series. 38, Princeton University Press(1989).
[16] B. Morel, Eigenvalue estimates for the Dirac-Schrödinger operators, J. Geom. Phys. 38, 1–18(2001).
[17] A. Moroianu, L. Ornea, Eigenvalue estimates for the Dirac operator and harmonic 1-forms of constant length, C. R. Math. Acad. Sci. Paris. 338, 561–564 (2004).
[18] Mc.K.Y. Wang, Parallel spinors and parallel forms, Ann. Glob. Anal. Geom. 7, 59–68(1989).
[19] K. Yano, M. Kon, Structures on manifolds. Singapore: World Sci.(1984).
[20] X. Zhang, Lower bounds for eigenvalues of hypersurface Dirac operators. Math Res Lett. 5, 199–210 (1998); A remark: Lower bounds for eigenvalues of hypersurface Dirac operators. Math Res Lett. 6, 465–466(1999).
[21] X. Zhang, Angular momentum and positive mass theorem, Commun. Math. Phys. 206, 137–155 (1999).

Yongfa Chen
School of Mathematics and Statistics, Central China Normal University, Wuhan 430079, P.R.China.

Email address: yfchen@mail.ccnu.edu.cn