Powerful harmonic charging in quantum batteries

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We consider a classical harmonic driving field as the energy charger for the quantum batteries, which consist of an ensemble of two-level atoms. The maximum stored energy and the final state are derived analytically with the optimal driving frequency. At the end of charging procedure, each of atoms is in the upper state and the batteries are charging completely, which exhibits a substantial improvement over the square-wave charger. Involving the interatomic correlations, we find that the repulsive couplings show an advantage in achieving fully charging with shorter charging period. However, the attractive interactions induce a negative effects on the charging, since the ground state undergoes a quantum phase transition from a separable state to a doubly degenerate state. Approaching to the phase transition regime, the maximum stored energy drops sharply from the fully-charging value. The phase transition favors to suppress the charging of the battery and prevents the final state to be a separable state due to quantum fluctuations in our quantum batteries.

I. INTRODUCTION

Quantum information science develops very quickly in recent years. Various kinds of tasks using quantum information have been studied in detail such as quantum sensing, computations, and communications. Among these proposals, quantum battery (QB) was proposed to use quantum effects such as coherence and entanglement to enhance the charging power and speed up the charging time in comparison with its classical counterpart [1–4]. This topic belongs to a rapidly developing field, the so-called quantum thermodynamics, describing quantum-scale devices. How to make an efficient energy storage by exploiting nonclassical effects is central and practical research subject.

The concept of a QB was originally proposed as a two-level system used to temporarily store energy transferred from external field [1]. In previous studies, the QB is charged via a periodic square-wave driving field, which induces interaction between the battery and the external field to be a constant during the charging process [3–5]. However, an efficient, easy-to-implement solution to a time-dependent Hamiltonian of a QB interacting with a harmonic field remains elusive, for which the interaction is a function of time instead of a constant. Besides a QB charged independently, recent research efforts have been devoted to explore contributions provided by quantum correlations for charging in collective QBs [1, 2]. The collective quantum advantage has been explored in the charging power of the Dicke QBs [7], for which QBs with two-level atoms interact with one common bosonic mode. Moreover, quantum phase transition with the maximal entanglement in the Dicke model can sensibly improve the energy storage [3]. However, one finding is that there always exists a method to extract all possible work without entanglement generation, but the charging power is connected to quantum entanglement [7]. In quantum information theory it is known that correlations and entanglement may induce limitations on the energy storage [11–14]. There remains the open question of quantifying a relation between the quantum correlations and the optimal stored energy in collective QBs.

With the applied field being in a coherent state, a classical harmonic driving is expected to replace the quantized filed [10]. We consider the QBs that consist of N two-level atoms coupled to a classical harmonic driving field instead of a quantized squared-wave field [7], so-called semi-classical Dicke QBs. The optimal stored energy and the final state at the end of charging are derived analytically. In contrast to previous studies with a square-wave field, our Dicke QBs can realize fully charging, for which each of two-level atoms is finally charged to be in the upper state. We further focus on the contributions of the interatomic interactions on maximizing the stored energy and minimizing the charging time in the QBs. We find that the repulsive interactions of the atoms yield an advantage in the maximum stored energy, and the charging process is faster than that they are charged independently. However, for the attractive coupling case, we find that there occurs a quantum phase transition in the ground state. The phase transition suppresses the stored energy with longer charging period and the maximal stored energy drops from the fully-charging value sharply.

The paper is outlined as follows. In Sec. II, we study N two-level atoms charging independently by the classical harmonic field. The maximum stored energy and the optimal driving frequency are given analytically. In Sec. III, we discuss the effects of the repulsive and attractive interatomic coupling in the process of energy storage and the charging period. Finally, a brief summary is given in Sec. IV.
We employ an classical harmonic field as a charger to transfer energy to the battery as much as possible. For a comparison, the driving Hamiltonian with a square-wave charger is $H_1 = A S_x$. It is noted that the coupling between QBs and the harmonic charger is the conchoidal function of time instead of a constant in the square-wave charger. Fig. 1(b) shows the charging procedure is designed to turn on the interaction between $N$ atoms and the classical field during the charging interval $0 < t < T$. Then the interaction is turned off at time $T$, and atoms are isolated from the external field and keep their energy. During the charging step, the total Hamiltonian for $N$ atoms interacting with the harmonic classical field is $H = H_0 + H_1$, which is viewed as the semi-classical Dicke Hamiltonian instead of a quantized field in the Dicke model [17] [19].

To study the charging process with the harmonic driving, $N$ two-level atoms are prepared in the lowest-energy state as $|\varphi_N(0)\rangle = |N/2, -N/2\rangle$, which corresponds that each of atoms is in the ground state $|g\rangle$. According to the evolved state $|\varphi_N(T)\rangle = e^{-iHT}|\varphi_N(0)\rangle$ of the system at time $T$, the energy that moves from the harmonic driving field to the quantum battery can be expressed in terms of the mean local energy of $N$ batteries, i.e.

$$E_N(T) = \langle \varphi_N(T)|H_0|\varphi_N(T)\rangle - \langle \varphi_N(0)|H_0|\varphi_N(0)\rangle.$$

A battery is a physical system that stored energy in atoms, which is transferred from the external driving field. We investigate the maximum stored energy $E_{N,max}(T)$ during the charging process. The advantage of the harmonic driving field lies in the modulated frequency $\omega$, which can be tuned to produce maximum charging work $E_{N,max}$ at an optimal charging period $T_{max} = 2\pi/\omega_{max}$. Fully charging is expected to achieved by $E_{max}(T) = 1$ in units of $N\Delta$, and the corresponding final state at the end of the charging is

$$|\varphi_N(T_{max})\rangle = |e\rangle \otimes N,$$

which is called the fully-charging state in Fig.1 (a).

Inspired by the approximated analytical solution for the driven semi-classical Rabi model for a driving two-level system [20], we extend the approach to solve the semi-classical Dicke Hamiltonian $H$. Using a unitary transformation $U = \exp[i\sum_{m=1}^{N}\xi \sin(\omega t)S_x]$ with the undetermined parameter $\xi$, the transformed Hamiltonian $H = UHU^\dagger - iU H_0 U^\dagger$ is

$$H = \Delta \left\{ \cos \left[ A \frac{\xi \sin(\omega t)}{\omega} \right] S_z + \sin \left[ A \frac{\xi \sin(\omega t)}{\omega} \right] S_y \right\} + A(1 - \xi) \cos(\omega t)S_z.$$

We expand the operator identities $\cos \left[ A \frac{\xi \sin(\omega t)}{\omega} \right] = J_0 \left( A \frac{\xi}{\omega} \right) + 2 \sum_{n=1}^{\infty} J_{2n} \left( A \frac{\xi}{\omega} \right) \cos(2n\omega t)$ and $\sin \left[ A \frac{\xi \sin(\omega t)}{\omega} \right] = 2 \sum_{n=0}^{\infty} J_{2n+1} \left( A \frac{\xi}{\omega} \right) \sin(2n + 1)\omega t$, where $J_n \left( A \frac{\xi}{\omega} \right)$ denotes the Bessel function of integer

![Image of charging protocol](image.png)

FIG. 1: (a) Charging protocol of $N$ two-level atoms as the QBs. At time $t = 0$ each atom is in the ground state $|g\rangle \otimes N$. At the period $T$ the final state for the fully charging is $|e\rangle \otimes N$. (b) Charging protocol with a harmonic driving field $A \cos(\omega t)$. During the charging time $0 < t < T$, the collective batteries interact with an classical harmonic driving field $A \cos(\omega t)$. Finally, the interaction is switched off at the end of charging period $T$. (c) Two-level atoms are polarized in the same direction with an angle $\theta$ induced by the electric field. Repulsive and attractive interactions between two atoms depend on $\theta$. 

II. $N$ BATTERIES INDEPENDENTLY CHARGING

The collective QBs consist of an ensemble of independent two-level atoms, which are charged by a harmonic field in Fig. 1(a). The Hamiltonian of $N$ independent batteries is given as

$$H_0 = \frac{\Delta}{2} \sum_{i=1}^{N} \sigma_i^z = \Delta S_z,$$

where the collective atom operators $S_{\alpha} = \sum_{i=1}^{N} \sigma_i^{\alpha}/2$ ($\alpha = x, y, z$), $\Delta$ is the energy level splitting of the two-level atom. The basis set for representing the atoms system is the Dicke states $|s, m\rangle$ ($m = -s, -s+1, ..., s$), which are eigenstates of $S^2$ and $S_z$. The collection of $N$ two-level atoms is described as a single $(N + 1)$-level system with the large pseudospin $S = N/2$. We set $\Delta = 1$ in the following.

We employ an classical harmonic field as a charger to transfer energy to the battery as much as possible. These two-level atoms are driven by an external harmonic field as

$$H_1 = \frac{A}{2} \cos(\omega t) \sum_{i=1}^{N} \sigma_i^x = A \cos(\omega t)S_x,$$

where $A$ and $\omega$ are the driving amplitude and the modulated frequency.
order \( n \). The Hamiltonian is approximated as

\[
H = \Delta J_0 \left( \frac{A}{\omega} \xi \right) S_z + A(1 - \xi) \cos(\omega t) S_x
+ 2\Delta J_1 \left( \frac{A}{\omega} \xi \right) \sin(\omega t) S_y.
\]

(6)

We neglect the contributions due to virtual process that occurs faster than the time scales of the system by setting \( A(1 - \xi) - 2\Delta J_1 \left( \frac{A}{\omega} \xi \right) = 0 \). Consequently, the transformed Hamiltonian becomes \( H = \Delta J_0 \left( \frac{A}{\omega} \xi \right) S_z + A(e^{i\omega t} S_+ + e^{-i\omega t} S_+) \), where \( \Delta J_0 \left( \frac{A}{\omega} \xi \right) \) is the renormalized atomic transition frequency and the interaction between the QBS and the harmonic charger is

\[
\tilde{A} = \frac{A}{2} (1 - \xi).
\]

(7)

With a rotating \( S = \exp(-i\omega t S_z) \), the effective Hamiltonian for the semi-classical Dicke QBs coupling to the harmonic charger is obtained as

\[
\hat{H} = \tilde{\Delta} S_z + 2\tilde{A} S_x,
\]

(8)

where the effective detuning is

\[
\tilde{\Delta} = \Delta J_0 \left( \frac{A}{\omega} \xi \right) - \omega.
\]

(9)

In the rotating frame we obtain a time-independent model of the charging process similar to the Hamiltonian of \( N \) batteries coupled with a squared-wave charger \( H_s = \Delta S_z + 2A S_x \). The advantage of the harmonic charger lies in the renormalized \( \tilde{\Delta} \) and coupling \( \tilde{A} \), which can be tuned by the driving frequency \( \omega \).

The charging process of \( N \) batteries is equivalent to parallel charging for independent atoms, and the scaled stored energy \( E_N/(N\Delta) \) equals to \( E_1 \) of the single-atom battery. We now solve the effect Hamiltonian in Eq. (8) with the pseudospin \( S = 1/2 \) for the single-atom battery. The eigenvalues are given as \( \varepsilon_{\pm} = \pm\Omega_R/2 \), where \( \Omega_R \) is the effective Rabi frequency

\[
\Omega_R = \sqrt{\Delta^2 + 4\tilde{A}^2}.
\]

(10)

Due to the unitary rotating transformations, the initial state for the transformed Hamiltonian \( \hat{H} \) in Eq. (8) is \( S(T)U(T)|g\rangle \). At the end of the charging protocol, the final state is derived explicitly as

\[
|\varphi_1(T)\rangle = -i\tilde{A} \Omega_R \sin(\varepsilon_+ T)|e\rangle
+ [\cos(\varepsilon_+ T) + i\tilde{\Delta} \Omega_R \sin(\varepsilon_+ T)]|g\rangle.
\]

(11)

According to Eq. (8), the stored energy in the single battery that transfers from the external field is

\[
E_1(T)/\Delta = \frac{2\tilde{A}^2}{\Omega_R^2} [1 - \cos(\Omega_R T)].
\]

(12)

FIG. 2: Stored energy \( E(T)/\Delta \) in the single-atom battery as a function of charging period \( T \) for different driving amplitude \( A \). The QB couples to a harmonic charger \( A \cos(\omega t) \) (red solid line) and a square-wave charger \( A \) in the charging step, respectively. The analytical results are shown in red circle.

At the optimal period \( T_{\text{max}} = n\pi/\Omega_R \) (for odd integer \( n \)), the maximum stored energy is

\[
E_{1,\text{max}}/\Delta = \frac{4\tilde{A}^2}{\Delta^2 + 4\tilde{A}^2}.
\]

(13)

The corresponding optimal driving frequency \( \omega_{\text{max}} \) is determined by

\[
\omega_{\text{max}} = \frac{2\Omega_R}{n}, (n = 1, 3, ...).
\]

(14)

and the final state is reduced as

\[
|\varphi_1(T_{\text{max}})\rangle = -2i\tilde{A} \Omega_R |e\rangle + i\tilde{\Delta} \Omega_R |g\rangle.
\]

(15)

It is obvious that \( E_{1,\text{max}}/\Delta \) ranges from 0 to 1, and depends on the driving frequency \( \omega_{\text{max}} \) in Eq. (14). In particular, by modulating \( \omega \) to satisfy \( \Delta = 0 \), the fully-charging state is obtained

\[
|\varphi_1(T_{\text{max}})\rangle = -i|e\rangle.
\]

(16)
for which the maximal stored energy is $E_{1,\text{max}}/\Delta = 1$. The corresponding driving frequency is derived by $\omega_{\text{max}} = J_0(\Delta/\xi)$, which yields

$$4(\omega_{\text{max}} + \Delta) (\Delta^2 - \omega_{\text{max}}^2) = A^2 \Delta. \quad (17)$$

One can realize fully charging for $E_{1,\text{max}}/\Delta = 1$ when the driving frequency $\omega_{\text{max}}$ satisfies above equations in Eqs. (13) and (17). Consequently, the final fully-charging state of $N$ QBSs is achieved as $|e\rangle^\otimes N$.

Fig. 2 shows the analytical stored energy $E(T)/\Delta$ in Eq. (12) for the single-atom battery, which is consistent with numerical results. The analytical expression for the maximum stored energy $E_{1,\text{max}}/\Delta$ in Eq. (13) and the optimal driving frequency $\omega_{\text{max}}$ in Eq. (14) is valid for a wide range of the charging period $T$. The fully-charged value $E_{\text{max}}/\Delta = 1$ is presented for different driving amplitude $A$, exhibiting the powerful charging by the harmonic charger. The maximum stored energy $E_{\text{max}}/\Delta$ locates at the first peak for $A = 0.5$ and 1, and shifts to the second peak with longer optimal period $T_{\text{max}}$ for a strong value $A = 1.5$.

To clarify the locations of the maximum stored energy $E_{\text{max}}/\Delta$. Fig. 3 displays the optimal driving frequency $\omega_{\text{max}}$ for different driving field $A$. We find the optimal frequency $\omega_{\text{max}}$ locates at the first peak of the stored energy with $n = 1$ in Eq. (14) for a wide range of the driving amplitude. And $E_{\text{max}}/\Delta$ tends to the fully-charged value 1 as $A$ increases to 1 and then decreases. For $A \geq 1.2$, the maximum value of the stored energy shows a jump, and $\omega_{\text{max}}$ changes to a small value with $n = 3$. The optimal frequency $\omega_{\text{max}}$ dependent on $A$ is observed in the contour projection of Fig. 3. It demonstrates that one can achieve fully charging of the battery by modulating the frequency $\omega_{\text{max}}$.

For a comparison, we study the two-level atom coupled with the square-wave charger and the driving Hamiltonian is $H = \Delta S_z + AS_x$. Similarly, the energy stored in the battery at the end of charging time $T$ is

$$E_s(T)/\Delta = \frac{1}{2} \frac{A^2}{\Delta^2 + A^2} \left[1 - \cos(\sqrt{\Delta^2 + A^2} T)\right]. \quad (18)$$

Due to the fixed parameter $\Delta$ and $A$, the maximum stored energy in the battery is determined by $E_{s,\text{max}}/\Delta = A^2/(\Delta^2 + A^2)$ at the driving period $T_{s,\text{max}} = n\pi/\sqrt{\Delta^2 + A^2}$. In contrast to the harmonic charger, it is impossible to reach the fully-charging value $E_{s,\text{max}} = 1$ due to finite $\Delta \neq 0$. As a result, the final state of the atom is not the fully-charging state $|e\rangle$ in Eq. (15). It means that the charging is not completely fulfilled for the battery with the square-wave charger.

III. COLLECTIVE CHARGING WITH INTERATOMIC CORRELATIONS

With the consideration of additional interatomic interactions, a quite natural question follows as to its effect on the charging battery. Quantum correlations in multiparticle systems is connected to energy storage [1, 2, 3, 4]. It is interesting to study the positive and negative effects in the charging process, which is simulated by the interatomic coupling among two-level atoms.

Attributing to the external electric field, each two-level
atom becomes to be polarized, which can be described as an electric dipole operator \( \hat{d} = d_x \hat{x} + d_y \hat{y} \) with the coefficients \( d_x \) and \( d_y \). All dipoles are polarized in the same direction with an angle \( \theta \) down from the z axis in Fig. 4(c). It is reasonable to consider interatomic interactions induced by the dipoles. The battery of \( N \) two-level atoms system with the interatomic interactions is described by

\[
H_0 = \frac{\Delta}{2} \sum_{i=1}^{N} \sigma_i^z + \frac{g}{2N} \sum_{i \neq j} (\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y) + \Delta S_z + \frac{g}{N}(S^2 - S_z^2 - N/2),
\]

where \( g \) is the coupling strength between arbitrary two dipoles and is proportional to \((3 \cos^2 \theta - 1)\) [21, 22]. It demonstrates that the repulsive interaction \( g > 0 \) ranges from \( \theta = -54.7^\circ \) to \( 54.7^\circ \). Otherwise, it is the attractive interatomic interaction with \( g < 0 \). We defined the scaled coupling strength \( \lambda = g/\Delta \).

Fig. 4 shows the polarized value per atom \( \langle S_z \rangle/(N/2) \) in the ground state for finite-size atoms \( N = 200 \). For the repulsive interactions \( \lambda > 0 \), the ground state is always \( | -N/2, -N/2 \rangle \), and \( \langle S_z \rangle/(N/2) \) equals to \(-1\) for arbitrary repulsive coupling strength. However, the polarized value becomes to increase from \(-1\) as the attractive coupling strength exceeds a critical value \( \lambda_c = -1 \). The inset of Fig. 4 displays energy levels in the ground state and the first-excited state. It is observed that the ground state is the doubly degenerate for \( \lambda < \lambda_c \). It means that there occurs a second-order phase transition for the attractive interatomic coupling case \( \lambda < 0 \).

To explore contributions of the phase transition in the attractive interactions, we now use the Hosltein-Primakoff transformation to study the infinite atoms in terms of auxiliary bosonic operators \( b^\dagger \) and \( b \): \( S_z = b^\dagger b - N/2 \) and \( S_+ = b^\dagger \sqrt{N} \). Above the critical coupling value \( \lambda_c \), the bosonic field is expected to shift with a value \( \beta \) as \( b^\dagger \rightarrow b^\dagger + \beta \). We then obtain the approximated Hamiltonian

\[
\hat{H} = \Delta[(b^\dagger + \beta)(b + \beta) - \frac{N}{2}] + \frac{g}{N}\left\{ \frac{N^2}{4} - [(b^\dagger + \beta)(b + \beta) - \frac{N}{2}]^2 \right\}.
\]

To make the linear terms \( (b^\dagger + b) \) vanish, one obtain

\[
\beta^2 = N(g + \Delta)/(2g).
\]

Obviously, the critical value is given by \( g_c = -\Delta \) to make \( \beta^2 \) valid, yielding \( \lambda_c = -1 \). Above the critical value, the expected value of atom polarization in the ground state is given by \( \langle S_z \rangle = \beta^2 - N/2 \). Fig. 4 shows the behavior of \( \langle S_z \rangle/(N/2) \) for infinite atoms \( N \rightarrow \infty \), exhibiting a quantum phase transition at \( \lambda_c = -1 \).

It is interesting to study whether the attractive and repulsive interactions can enhance the charging of a quantum battery, especially the effects induced by the phase transition. We explore how the state evolves form initial ground state \( |N/2, -N/2 \rangle \) to the fully-charged state \( |N/2, N/2 \rangle \) with the interatomic correlations. Due to the intrinsic many-body interactions in the battery, the energy extracted form the harmonic driving field is in general a complicated function of the period \( T \) during the charging process. Powerful charging requires maximum
energy stored in the battery and minimum charging period.

We numerically calculate the stored energy dependent on the coupling strength $\lambda$ for the driving amplitude $A = 1$ in Fig. 5. The maximum value of the stored energy $E_{\text{max}}/(N\Delta)$ increases and approaches a saturated value as $N$ increases. However, for the attractive coupling strength $\lambda = -1.2$ above the critical value, the maximum stored energy increases even for large $N$ in Fig. 5(d). We calculate $E_{\text{max}}/(N\Delta)$ dependent on atom number $N$ in Fig. 6.

In the repulsive coupling case, we find that the maximum value of the stored energy approach to 1 for $\lambda = 0.5$ and 1.2 in Fig. 5(a) and (b). Meanwhile, the corresponding charging period $T_{\text{max}}$ decreases from 6.7 to 5 as $\lambda$ increases from 0.5 to 1.2 in Fig. 5(c)-(d). By comparing with $N$ atoms charging independently in Fig. 5(b), the maximum stored energy is $E_{\text{max}}/(N\Delta)$ is much lower than the fully-charged value 1 in Fig. 5(b). By comparing with results of the repulsive interactions, the charging period $T_{\text{max}}$ is longer in Fig. 5(c)-(d). It reveals that the attractive interatomic interactions with a strong coupling strength $|\lambda| > |\lambda_c|$ suppress the energy transferred from the external field to the batteries. The final state at the charging period $T$ deviates from the fully-charging state $|e\rangle^\otimes N$. The phase transition plays a negative effect in the charging process due to quantum fluctuations, which results in unsaturated value of the maximal stored energy $E_{\text{max}}/(N\Delta)$.

To exhibit the contributions of the coupling strength to the energy storage, Fig. 6 shows the dependence of the maximum stored energy $E_{\text{max}}/(N\Delta)$ and the charging period $T_{\text{max}}$ on the coupling strength. It is observed that $E_{\text{max}}/(N\Delta)$ approaches 1 in a wide range of the repulsive coupling strengths and in the weak attractive coupling regimes. As the attractive coupling strength gets close to the critical value $\lambda_c = -1$, the energy stored in the battery gets worse and a sharp drop of $E_{\text{max}}/N\Delta$ is observed. Meanwhile, we find that the optimal charging period $T_{\text{max}}$ decreases from the attractive to repulsive coupling regimes except for $0 < \lambda < 0.5$ in Fig. 6(b). The charging period with a large repulsive coupling strength is shorter than that of $N$ independent batteries $\lambda = 0$.

It is clearly see two important effects: (i) the repulsive interactions in large $N$ atoms can yield an advantage in charging over the independent batteries, and the optimal charging period is shorter as the repulsive coupling strength increases; (ii) quantum phase transition in the attractive interatomic interactions play a negative effect on the energy storage in the QBs. And the charging period gets longer as the attractive coupling strength increases.

**IV. CONCLUSION**

In summary, we studied the collective charging of the quantum batteries, which consists of $N$ two-level atoms. We have shown the advantage of the charging driven by the harmonic charger instead of a square-wave charger to achieve fully charging. We presented an analytical expression for the optimal driving frequency and the maximum value of the stored energy. By comparing with the square-wave charger, $N$ independent batteries can be charged completely by manipulating the frequency of the harmonic charger.

Considering effects of the interatomic interactions for optimal energy storage, one finding is that the repulsive interactions can enhance the charging power with shorter charging period. Interestingly, we find that the ground state of atoms becomes to be a doubly degenerate state as the attractive coupling strength increases to a critical value. The energy storage in the battery is suppressed in the phase transition regime, where the final state is prevented to be the fully-charging state. Our results provide an alternative charger by choosing an harmonic driving field for different quantum batteries and demonstrate the negative and positive effects of the quantum correlations in interacting systems in the charging process.

![FIG. 7: Maximal stored energy $E_{\text{max}}$ (in unit of $N\Delta$) and the corresponding optimal period $T_{\text{max}}$ dependent on the interatomic coupling strength $\lambda$ for $N = 140$ atoms.](image)
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