Effect of surface tension on the Rayleigh-Taylor and Richtmyer-Meshkov instability induced nonlinear structure at two fluid interface and their stabilization

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Abstract. The effect of surface tension on the Rayleigh-Taylor (RT) instability and Richtmyer – Meshkov (RM) instability induced development of nonlinear structures like spike and bubble at two fluid interface have been investigated. In case of RT instability, the surface tension reduces the velocity of the tip of both the bubble and the spike by the factor \((1-k_c^2/3k_c^2)\) where, \(k_c^2 = (\rho_h-\rho_l)g/T\). For \(k^2 > 3k_c^2\), RT instability is stabilized. Any perturbation, whatever be its magnitude, results in stable large amplitude nonlinear oscillations. The RM instability is always stabilized by surface tension leading to nonlinear oscillation. Both the amplitude and the period of oscillation decreases monotonically as \(S = 12k^2T/ (\rho_h+\rho_l)\) increases where \(k\) is the wave number. The maximum height attained by the peak of the nonlinear interfacial structures (spike or bubble) increases approximately logarithmically with the initial interface velocity resulting from the shock.

1. Introduction
The interface between two fluids becomes unstable when driven by an impulsively acting force such as a shock wave or a continuously acting (with respect to time) force, e.g., gravity. In the former case it is called Richtmyer-Meshkov (RM) instability induced development of nonlinear structures like spike and bubble at two fluid interface have been investigated. In case of RT instability, the surface tension reduces the velocity of the tip of both the bubble and the spike by the factor \((1-k_c^2/3k_c^2)\) where, \(k_c^2 = (\rho_h-\rho_l)g/T\). For \(k^2 > 3k_c^2\), RT instability is stabilized. Any perturbation, whatever be its magnitude, results in stable large amplitude nonlinear oscillations. The RM instability is always stabilized by surface tension leading to nonlinear oscillation. Both the amplitude and the period of oscillation decreases monotonically as \(S = 12k^2T/ (\rho_h+\rho_l)\) increases where \(k\) is the wave number. The maximum height attained by the peak of the nonlinear interfacial structures (spike or bubble) increases approximately logarithmically with the initial interface velocity resulting from the shock.
the effect of surface tension both in the RT and RM case demonstrating their influence in regard to the dynamics of bubble and spikes. In case of RT instability the growth rate of the tip of the nonlinear structures is reduced by the factor \( \frac{1-k^2}{3k_c^2} \). If \( \frac{k^2}{3k_c^2} > 1 \) which occurs for sufficiently strong surface tension \( T \), there is no RT instability even when the heavier fluid overlies the lighter one; any perturbation only results in nonlinear surface oscillation.

On the other hand, for all Atwood numbers and for all wave numbers \( k \), the RM instability is seen to disappear. This may be understood from the following observation: the action of the shock wave being impulsive the initial disturbance grows under the influence of no force in the form of a bubble or a spike depending on whether the lighter or heavier fluid penetrates into the heavier or lighter one. In consequence of presence of surface tension which has a stabilizing influence and which is the only continuously (with respect to time) acting force on the system the growth of the bubble or spike peak height is limited to a finite value. The restoring force tends to bring down the surface displacement to its initial zero value but due to inertia it overshoots in the opposite direction and as a result the nonlinear oscillatory motion ensues. Numerical integration of the approximate dynamical equations shows that the maximum height attained by the peak of the bubble (spike) depends approximately logarithmically on the strength of the shock.

Section 2 briefly presents the basic equations describing the instability dynamics for uniform density fluid as investigated earlier. This helps us to analyze the effect of surface tension on RT and RM as done in Section 3. Since the governing equations are not analytically integrable, the results derived are based on numerical solution of the governing equations. Finally the results are briefly summarized in Section 4.

### 2. Basic Equations

Let us assume that the undisturbed surface is \( y = 0 \), the transverse coordinate being represented by \( x \). The heavier fluid (density \( \rho_h = \text{constant} \)) occupies the region \( y > 0 \) while the lighter fluid (density \( \rho_l \)) is in the region \( y < 0 \); gravity is taken along negative \( y \)-axis.

The finger shaped interface perturbation is taken to have a parabolic form:

\[
y(x,t) = \eta_0(t) + \eta_2(t) x^2
\]  

(1)

Corresponding to the above stated assumption regarding the regions occupied by the heavier and lighter fluids we have

for bubble: \( \eta_0 > 0 \) and \( \eta_2 < 0 \)

(2)

for bubble: \( \eta_0 < 0 \) and \( \eta_2 > 0 \)

(3)

Following Goncharov [5], the velocity potentials describing the irrotational motion for the heavier and lighter fluid (The suffixes \( h \) and \( l \) will signify them) are assumed to be given by

\[
\phi_h(x,y,t) = a_1(t) \cos(kx)e^{-k(y-\eta_0(t))} ; \quad y > 0
\]

(4)

\[
\phi_l(x,y,t) = b_1(t) y + b_0(t) \cos(kx)e^{k(y-\eta_0(t))} ; \quad y < 0
\]

(5)

The introduction of three unknown functions — (two first harmonics \( a_1 \) and \( b_1 \) and one zeroeth harmonic \( b_0(t) \)) is required to equalize the total number of unknown functions to the total number of equations obtained from the kinematical and boundary conditions describing the dynamics. This was pointed out earlier by Hecht et. al. [1].

Corresponding to interfacial surface perturbation \( y(x,t) = \eta(x,t) = \eta_0(t) + \eta_2(t) x^2 \) the kinematic boundary conditions are:

\[
v_{hx} \frac{\partial \eta}{\partial x} - v_{hy} = v_{lx} \frac{\partial \eta}{\partial x} - v_{ly}
\]

(6)

\[
\frac{\partial \eta}{\partial t} + v_{lx} \frac{\partial \eta}{\partial x} = v_{ly}
\]

(7)
The irrotational fluid motion is described by Bernoulli’s equation
\[-\frac{\partial \phi}{\partial t} + \frac{1}{2} (\nabla \phi)^2 + gy = -\frac{p}{\rho} + f(t)\]

The condition of continuity of pressure at the interface is
\[p_b = p_i\] (8)

This results in the following dynamical boundary condition, provided the fluid densities are constant
\[\rho_b \left[ -\frac{\partial \phi_b}{\partial t} + \frac{1}{2} (\nabla \phi_b)^2 + gy \right] - \rho_i \left[ -\frac{\partial \phi_i}{\partial t} + \frac{1}{2} (\nabla \phi_i)^2 + gy \right] = \rho_b f_i(t) - \rho_i f_b(t)\] (9)

Substituting for \(\phi_b\) and \(\phi_i\) from (4) and (5), expanding in powers of the transverse coordinate \(x\), neglecting term \(O(x^i)\) \((i \geq 3)\) and finally eliminating \(b_0(t)\) and \(b_1(t)\) leads to the following set of equation describing the time development of the parameters for the bubbles [5]
\[\frac{dx}{dt} = x_3\] (10)
\[\frac{dx}{dt} = -3x_3\left(x_2 + \frac{1}{6}\right)\] (11)
\[\frac{dx}{dt} = -\frac{N(x_3)^3}{D(x_2)} \left( x_2 + \frac{1}{6} \right) + 2(r-1)kg \frac{x_1(6x_2-1)}{D(x_2)}\] (12)

where
\[N(x_3) = 36(1-r)x_3^3 + 12(r+4)x_3 + (7-r)\] (13)
\[D(x_2) = 12(1-r)x_2^3 + 4(1-r)x_2 + (1+r)\] (14)
\[r = \frac{\rho_b}{\rho_i} \text{ when } r > 1; \quad r = \frac{\rho_i}{\rho_b} \text{ when } r < 1\] (15)
\[x_1 = k\eta_0; \quad x_2 = \frac{\eta_2}{k}; \quad x_3 = k^2a_i\] (16)

Equation (12) shows that if \(x_2 > 0\), then \(x_2\) continues to decrease monotonically from any value of \(x_2 > -\frac{1}{6}\) to \(x_2 = -\frac{1}{6}\).

This happens in either of the following two cases:

(i) \(g=0\); in this case both \(x_3\) and consequently \(x_3 \rightarrow 0\) asymptotically with \(x_2 \rightarrow -\frac{1}{6}\) as \(t \rightarrow \infty\). In figure 1 this is demonstrated through numerical integration of equations (10)-(12).

(ii) \(g \neq 0\) and \(r>1\); in this case \(x_3 \rightarrow 0\) asymptotically as \(t \rightarrow \infty\) for a finite value of \(x_3\) as \(x_2 \rightarrow -\frac{1}{6}\). This is also demonstrated by numerical integration of equations (10)-(12) in figure 2.

For \(g = 0\), \(x_3(t)\) increases logarithmically with \(t\) as \(x_3 \rightarrow 0\) describing the motion of the tip of RM instability associated bubble:
\[(x_3(t))_{\text{asymp}} \approx \frac{1}{3} \left( 1 + \frac{2}{1 + A} \right) \log (tf_0)\]

where \(A = (\rho_b - \rho_i)/(\rho_b + \rho_i)\) is the Atwood number and \(t_0\) is a constant introduced for nondimensionalization.
Figure 1. Variation of $x_1$ with $\tau$ as obtained by solution of equation (10)-(12) neglecting the second term in equation (12) (RM instability) with initial values $x_1 = 0$, $x_2 = 0$, $x_j/\sqrt{k_g} = 0.1$ and $r = 1.5$

Figure 2. Variation of $x_1$ with $\tau$ as obtained by solution of equation (10)-(12) (RT instability) with initial values $x_1 = 0$, $x_2 = 0$, $x_j/\sqrt{k_g} = 0.1$ and $r = 1.5$

In the second case which corresponds to RT instability, the height of the tip of the bubble $x_1(t)$ increases linearly with $t$. Approximate analytic expression for $(x_j)_{asymp}$ were given by Goncharov [5]:

For RT bubble

$$(x_j)_{asymp} = \sqrt{\frac{-2A}{3(1+A)}} k_g$$

For RT spike one obtains $(x_j)_{asymp}$ by following Goncharov’s prescription [5]:

$${\eta_0} \rightarrow -{\eta_0}, \quad {\eta_2} \rightarrow -{\eta_2}, \quad g \rightarrow -g, \quad \text{and} \quad r \rightarrow \frac{l}{r}$$

Then $\dot{x}_j = 0$ with $\eta_2 = -\frac{l}{6}$ yields for RT spike:

$$(x_j)_{asymp} = \sqrt{\frac{2A}{3(1-A)}} k_g$$

For RM spikes the same prescription yields

$$(x_j(t))_{asymp} \approx \frac{1}{3} \left(1 + \frac{2}{l-A}\right) \log \left(\frac{t}{t_0}\right) \text{ for } t/t_0 \gg 1$$

If $r < l$ (i.e., the lighter fluid is above the heavier liquid) it is well known that the RT system is stable with respect to small perturbation which results in small oscillation. The outcome when the perturbation is no longer small can be investigated here.
If \( r < 1 \) the second term in equation (12) changes sign. As a result, from whatever initial positive value \( x_{i0} \) of \( x_2 \) one may start integrating, \( \dot{x}_2 \) will continue to be less than zero for \(-\frac{1}{6} < x_2 < 0\) and thus a stage of temporal evolution will reach when \( x_2 < 0 \) and hence \( \dot{x}_2 \) will become greater than zero. \( x_2 \) will now increase and ultimately become greater than zero. Thus \( x_2 \) will oscillate in the range \(-\frac{1}{6} < x_2 < \frac{1}{6}\); this will also lead to concomitant oscillation of \( x_1 \) and \( x_3 \). These are stable nonlinear oscillation and are of arbitrarily large amplitude permissible within the range of validity of the applicability of Layzer’s method. The result of numerical integration of equations (10)-(12) exhibiting such oscillations are shown in figure 3. This may be called finite amplitude oscillating bubble (spike) as opposed to the usual bubble (spike) for which \( x_2(t) \to \infty \) either logarithmically or linearly with time \( t \).

**Figure 3.** Variation of \( x_1 \) with \( \tau \) as obtained by solution of equation (10)-(12) (RT instability) with initial values \( x_1 = -0.1 \), \( x_2 = 0.1 \), \( x_3 / \sqrt{kg} = 0.1 \) and \( r=0.5 \)

**Figure 4.** Variation of \( x_1 \) with \( \tau \) as obtained by solution of equation (10), (11) and (29) with initial values \( x_1 = 0.1 \), \( x_2 = -0.1 \), \( x_3 / \sqrt{kg} = 0.1 \) and \( r=1.5262, S=0.1 \)

### 3. Effect of surface tension

When surface tension effect is included the pressure continuity condition (8) is to be replaced by

\[
p_s - p_i = \frac{T}{R}
\]

where \( R \) is the radius of curvature. Since

\[
y(x,t) \equiv \eta(x,t) = \eta_\eta(t) + \eta_\eta(t)x^2
\]

\[
\frac{T}{R} = 2\eta_\eta(l + 4\eta_\eta^2 x^2)^{-\frac{3}{2}} \approx 2\eta_\eta(l - 6\eta_\eta^2 x^2)
\]

on retaining terms \( O(x^3) \) consistent with the order of approximation used in Layzer’s method. So
the dynamical boundary condition (9) will be replaced by
\[
\rho_h \left[ -\frac{\partial \phi_h}{\partial t} + \frac{1}{2} \nabla (\phi_h)^2 + g y \right] - \rho_l \left[ -\frac{\partial \phi_l}{\partial t} + \frac{1}{2} \nabla (\phi_l)^2 + g y \right] = -2T \eta_x (1 - 6\eta_x x^2)
\]
\[
+ \rho_n f(t) - \rho_l f(t)
\]
(21)
Incorporation of surface tension term will replace equation (12) by the following equation
\[
\frac{dx_j}{dt} = -\frac{N(x_j)}{D(x_j)} x_j^2 + 2(r-1)kg \frac{x_j (6x_j -1)}{D(x_j)} \left( 1 - 12x_j^2 \frac{k^2}{k_c^2} \right)
\]
where \( k_c^2 = (\rho_n - \rho_l) g / T \)
(22)
This equation will apply to the Rayleigh-Taylor instability case.

Let \( r > 1 \); if \( k^2 < 3k_c^2 \), then \( 1 - 12x_j^2 \frac{k^2}{k_c^2} > 0 \) for \( -\frac{1}{6} < x_j < 0 \)
we proceed as in section 2 and find that \( x_j \to 0 \) and \( x_j \to -\frac{1}{6} \) asymptotically as \( t \to \infty \). This yields the asymptotic bubble velocity given by
\[
\frac{(x_j)_{\text{asymp}}}{k} = \sqrt{\frac{2(r-1)}{3rkg}} \left( 1 - \frac{k^2}{3k_c^2} \right) = \sqrt{\frac{2A}{3(1+A)(1 - \frac{k^2}{3k_c^2})}} \frac{g}{k}
\]
(25)
This is less than \( \frac{(x_j)_{\text{asymp}}}{k} \) in absence of surface tension and thus stabilizing effect of surface tension is demonstrated.

If \( k^2 = 3k_c^2 \), equilibrium is attained, i.e.,
\[
x_j = 0 \quad \text{and} \quad x_j \to -\frac{1}{6}
\]
(26)
If \( k^2 > 3k_c^2 \), the sign of the second term in equation (22) is reversed. Hence oscillatory state will emerge even for \( r > 1 \).

Thus RT instability is stabilized when
\[
k^2 \geq 3k_c^2
\]
(27)
The instability however persists but with reduced growth rate for
\[
k^2 < 3k_c^2
\]
(28)
When gravity is neglected, i.e., we are considering the RM instability equation (22) is replaced by
\[
\frac{d(x_j/\sqrt{kg})}{d\tau} = -\frac{N(x_j)}{D(x_j)} \left( x_j / \sqrt{kg} \right)^2 - 2S(r+1) \frac{x_j (6x_j -1)}{D(x_j)}
\]
\[
\text{where} \quad S = \frac{12k^2T}{(\rho_n + \rho_l) g}; \quad \tau = t \sqrt{kg}
\]
(29)
The same argument as stated in the discussion of the large amplitude interfacial oscillation in the closing paragraph of the last section will apply here. For all values of \( S \) integration of (10), (11) and (29) shows the existence of large amplitude oscillation as shown in figure 4 for \( r = 1.5262 \) (SF$_6$-Air). These are bubbles as \( (x_j)_{\text{max}} > 0 \) which implies that the lighter fluid (\( \rho_l \)) is pushing into heavier fluid (\( \rho_n \)).

The temporal evolution of spike states is exhibited in figure 5; the results follow from the numerical integration of equations (10), (11) and (29) using the transformation \( \eta_0 \to -\eta_0, \eta_2 \to -\eta_2 \) and \( r \to \frac{1}{r} \). The pattern of nonlinear oscillation is identical to that for the bubble
states except that the peak displacement (which occurs in the negative direction for spiking) is greater than that of the bubble for the same values of \( r, S \) and initial value parameters.

\[ x_1 \]

Figure 5. Variation of \( x_1 \) with \( \tau \) as obtained by solution of equation (10), (11) and (29) using the transformation
\[ \eta_0 \rightarrow -\eta_0, \eta_2 \rightarrow -\eta_2 \text{ and } r \rightarrow \frac{r}{r} \] with initial values \( x_1 = -0.1, x_2 = 0.1, \frac{x_1}{\sqrt{k_0}g} = 0.1 \) and \( r=1.5, \eta=0.1 \)

Thus surface tension always stabilizes RM instability. Note that the initial value \( \left( x_1 / \sqrt{k_0g} \right)_{\text{initial}} \) of the interfacial surface is a measure of the strength of the impinged shock. Figure 6 shows that the maximum interfacial surface displacement \( |(x_{1})_{\text{max}}| \) both for bubbles and spikes increases approximately logarithmically with shock strength.

An interesting point is to be noted in this connection is that the amplitude and the period of oscillation diminish as \( S \) increases which is caused either as the magnitude of surface tension increases or as \( k \) increases. This is shown in figure 7 and figure 8. It is seen that both \( |(x_{1})_{\text{max}}| \) and the period of oscillation diminishes with increasing \( S \). This is expected as surface tension and small wavelength perturbation are more likely to suppress the amplitude of oscillation.

4. Concluding remarks
Finally we briefly summarize the results.
(i) In the RT case small amplitude oscillations are known to be stable when the lighter fluid lies above the heavier fluid (i.e., when \( r < 1 \)). Here it is seen that large amplitude oscillations are also stable (Figure 3) in the same situation.
(ii) The asymptotic velocity of the tip of the bubble (spike) is shown to be reduced for \( k^2 < 3k_0^2 \).
\[
\frac{(x_i)_{asym}}{k} = \frac{2}{3} \frac{A(1-k^2)}{1+A(1-3k^2)} \quad \text{for bubble}
\]
\[
\frac{(x_i)_{asym}}{k} = \frac{2}{3} \frac{A(1-k^2)}{1-A(1-3k^2)} \quad \text{for spike.}
\]

For \( k^2 > 3k_c^2 \), RT instability is nonlinearly stabilized.

\[ S \]

\[ S \]

Figure 7. Variation of \( |(x_i)_{max}| \) with \( S \) for \( x_i = 0.1 \) and \( r = 1.5262 \).

Figure 8. Variation of period of oscillation with \( S \) for \( x_i = 0.1 \) and \( r = 1.5262 \).

(iii) Richtmyer-Meshkov instability is always suppressed by surface tension; any interfacial perturbation results in nonlinear oscillation. (Figure 4 for bubble (lighter fluid pushing into heavier one) & Figure 5 for spike (heavier fluid pushing into lighter one))

(iv) The maximum interfacial surface displacement \( |(x_i)_{max}| \) increases approximately logarithmically with \( \left( \frac{x_i}{\sqrt{kg}} \right)_{\text{init}} \), i.e., the shock strength. (Figure 6).

(v) The amplitude and period of oscillation decreases monotonically when \( S = \frac{12k^2T}{(\rho_h + \rho_l)g} \) increases, i.e., surface tension \( T \) or the wave number \( k \) increases. (Figure 7 & 8).

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