Double derivations and Cyclic homology

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Abstract

We give a new construction of cyclic homology of an associative algebra $A$ that does not involve Connes’ differential. Our approach is based on an extended version of the complex $\Omega^* A$, of noncommutative differential forms on $A$, and is similar in spirit to the de Rham approach to equivariant cohomology. Indeed, our extended complex maps naturally to the equivariant de Rham complex of any representation scheme $\text{Rep} A$.

We define cyclic homology as the cohomology of the total complex $(\Omega^* A)[u], d + u \cdot \iota_\Delta$, arising from two anti-commuting differentials, $d$ and $\iota_\Delta$, on $\Omega^* A$. The differential $d$, that replaces the Connes differential $B$, is the Karoubi-de Rham differential. The differential $\iota_\Delta$, that replaces the Hochschild differential $b$, is a map analogous to contraction with a vector field. This new map has no commutative counterpart.

1 Introduction

Throughout, we fix a field $k$ of characteristic 0 and write $\otimes = \otimes_k$. By an algebra we will always mean an associative unital $k$-algebra.

1.1 It is well-known that a regular vector field on a smooth affine algebraic variety $X$ is the same thing as a derivation $k[X] \to k[X]$, of the coordinate ring of $X$. Thus, derivations of a commutative algebra $A$ play the role of vector fields.

It has been commonly accepted until recently that this point of view applies to noncommutative algebras $A$ as well. A first indication towards a different point of view was a discovery by Crawley-Boevey [CB] that, for a smooth affine curve $X$ with coordinate ring $A = k[X]$, the algebra of differential operators on $X$ can be constructed by means of double derivations $A \to A \otimes A$, rather than ordinary derivations $A \to A$. Since then, the significance of double derivations in noncommutative geometry was explored further in [VdB] and [CBEG].

To explain the role of double derivations in more detail we need to introduce, in §§1.2-1.3, some basic definitions involving various kinds of derivations and also free products.
1.2 Derivations and $t$-derivations. Let $B$ be any algebra and $M$ a $B$-bimodule.

Recall that a $k$-linear map $F : B \to M$ is said to be a derivation of $B$ with coefficients in $M$ if $F(b_1 b_2) = F(b_1) b_2 + b_1 F(b_2)$, $\forall b_1, b_2 \in B$. Given a subalgebra $R \subset B$, we let $\text{Der}_R(B, M)$ denote the space of relative derivations of $B$ with respect to the subalgebra $R$, that is, of derivations $B \to M$ that annihilate the subalgebra $R$. In the ‘absolute’ case $R = \mathbb{k}$ we will use simplified notation $\text{Der}(B, M)$.

Now, fix an element $t \in B$. We will be often interested in $k$-linear maps $F : B \to M$ such that

$$F(b_1 t b_2) = F(b_1) t b_2 + b_1 t F(b_2), \quad \forall b_1, b_2 \in B.$$ \hspace{1cm} (1.2.1)

**Definition 1.2.2.** A $k$-linear map $F : B \to M$ is called a $t$-derivation if equation (1.2.1) holds for $F$ and, moreover, we have $F(1) = 0$.

From equation (1.2.1), by induction on $n \geq 1$ one proves

$$F(t^n) = t^n F(1) + t^{n-1} F(1) t + t^{n-2} F(1) t^2 + \ldots + F(1) t^n.$$ 

It follows that, for any $t$-derivation $F$ and an integer $n \geq 1$, one has $F(t^n) = 0$; also one has $F(t b) = t F(b)$ and $F(b t) = F(b) t$, for all $b \in B$.

Let $\mathbb{k}(t)$ be the subalgebra in $B$ generated by the element $t$. It is clear that: *Any derivation $F \in \text{Der}_{\mathbb{k}(t)}(B, M)$ is automatically a $t$-derivation.*

We warn the reader that although any $t$-derivation annihilates the subalgebra $\mathbb{k}(t)$, a $t$-derivation need not necessarily be a derivation, i.e., an element of $\text{Der}_{\mathbb{k}(t)}(B, M)$, in general.

**Lemma 1.2.3.** For $t \in B$, the space of $t$-derivations $B \to B$ is a Lie algebra with respect to the commutator bracket.

**Proof.** For any $t$-derivations $F, G : B \to B$ and $b_1, b_2 \in B$, we compute

$$GF(b_1 t b_2) = G(F(b_1) t b_2 + b_1 t F(b_2)) = GF(b_1) t b_2 + F(b_1) t G(b_2) + G(b_1) t F(b_2) + b_1 t GF(b_2).$$

Performing a similar computation for $FG$, we find that the commutator $(GF - FG)(b_1 t b_2) = [G, F](b_1 t b_2)$ equals

$$[G, F](b_1) t b_2 + F(b_1) t G(b_2) + G(b_1) t F(b_2) - G(b_1) t F(b_2) - F(b_1) t G(b_2) + b_1 t [G, F](b_2) = [G, F](b_1) t b_2 + b_1 t [G, F](b_2). \quad \square$$
1.3 Free product construction. Recall that a free product of two algebras $A$ and $B$, is an associative algebra $A \ast B$ whose elements are formal $\mathbb{k}$-linear combinations of words $a_1 b_1 a_2 b_2 \ldots a_n b_n$, for any $n \geq 1$ and $a_1, \ldots, a_n \in A$, $b_1, \ldots, b_n \in B$. These words are taken up to equivalence imposed by the relation $1_A = 1_B$; for instance, we have $\ldots b_1 a b' \ldots = \ldots b_1 b' \ldots = \ldots (b \cdot b') \ldots$, for any $b, b' \in B$.

In the special case where $B = \mathbb{k}[t]$, a polynomial algebra in one variable, we use simplified notation $A_t := A \ast \mathbb{k}[t]$. It is convenient to view $A_t$ as an even graded algebra $A_t^+ = \oplus_{k \geq 0} A_t^{2k}$ with the grading that counts twice the number of occurrences of the variable $t$, i.e., such that $\deg t = 2$ and such that the subalgebra $A \subset A_t$ is assigned grade degree zero. For each integer $k$, let $\text{Der}^{2k}_{\mathbb{k}[t]}(A_t, A_t)$ denote the space of graded derivations $A_t^+ \rightarrow A_t^{+2k}$, relative to the subalgebra $\mathbb{k}[t] \subset A_t$. Thus, we have a graded Lie algebra

$$\text{Der}^+_t(A_t) = \oplus_{k \in \mathbb{Z}} \text{Der}^{2k}_{\mathbb{k}[t]}(A_t), \quad \text{where} \quad \text{Der}^{2k}_{\mathbb{k}[t]}(A_t) := \text{Der}^{2k}_{\mathbb{k}[t]}(A_t, A_t).$$

View $A$ and $\mathbb{k}[t]$ as two subalgebras in $A_t$.

Lemma 1.3.1. Let $M$ be an $A_t$-bimodule, also viewed as an $A$-bimodule.

(i) Restriction to the subalgebra $A \subset A_t$ provides a vector space isomorphism

$$\left\{ \begin{array}{c} t\text{-derivations} \\ F : A_t \rightarrow M \end{array} \right\} \sim\sim \left\{ \begin{array}{c} \mathbb{k}\text{-linear maps } f : A \rightarrow M \\ \text{such that } f(1) = 0 \end{array} \right\}.$$

(ii) The isomorphism in (i) restricts to a bijection:

$$\text{Der}_{\mathbb{k}[t]}(A_t, M) \cong \text{Der}(A, M).$$

Proof: It is clear that the assignment $F \mapsto f := F|_A$ gives an injective map from the set of $t$-derivations $F : A_t \rightarrow M$ to the set of $\mathbb{k}$-linear maps $f : A \rightarrow M$ such that $f(1) = 0$. We construct a map in the opposite direction by assigning to any $\mathbb{k}$-linear map $f : A \rightarrow M$, such that $f(1) = 0$, a map $f_t : A_t \rightarrow M$ given, for any $a_1, \ldots, a_n \in A$, by the following Leibniz type formula

$$a_1 t a_2 t \ldots t a_n \mapsto \sum_{k=1}^n a_1 t \ldots a_{k-1} t f(a_k) t a_{k+1} t \ldots a_n. \quad (1.3.2)$$

One verifies that the map $f_t$ thus defined satisfies (1.2.1). It is also immediate to check that the maps $F \mapsto F|_A$ and $f \mapsto f_t$ are inverse to each other. This proves (i). Part (ii) is straightforward and is left to the reader. \hfill \Box

Notation 1.3.3. We write $f_t$ for the $t$-derivation $A_t \rightarrow M$ given by formula (1.3.2). Thus, $f_t$ is the $t$-derivation corresponding to a $\mathbb{k}$-linear map $f : A \rightarrow M$ under the bijection inverse to the isomorphism of Lemma 1.3.1.(i). \hfill \diamond
For any derivation \( \theta : A \to A \) (viewed as a derivation \( A \to A_t \)), the corresponding derivation \( \theta_t : A_t \to A_t \) clearly has degree zero, i.e., we have \( \theta_t \in \text{Der}^0_t(A_t) \).

We will use symbolic Sweedler notation to write a \( k \)-linear map \( \Theta : A \to A \otimes A \), as \( a \mapsto \Theta' a \otimes \Theta'' a \), where we systematically suppress the summation symbol. We always view the space \( A \otimes A \) as an \( A \)-bimodule with respect to the outer bimodule structure, which is defined by the formula \( b(a' \otimes a'')c := (ba') \otimes (a''c) \), for any \( a', a'', b, c \in A \).

Notation 1.3.4. We write \( \text{Der} A := \text{Der}(A, A \otimes A) \) for the corresponding vector space of derivations, to be referred to as double derivations.

Next, to any \( k \)-linear map \( \Theta : A \to A \otimes A \), we associate a \( k \)-linear map \( A \to A \) given by the assignment \( a \mapsto \Theta' a \otimes \Theta'' a \).

Assuming further that \( \Theta(1) = 0 \), we may extend the latter map to get a \( t \)-derivation \( \Theta_t : A_t \to A_t \), see Notation 1.3.3 and Lemma 1.3.1.

It is clear that, for any double derivation \( \Theta \in \text{Der} A \), the corresponding derivation \( \Theta_t : A_t \to A_t \) has degree 2, i.e., we have \( \Theta_t \in \text{Der}^2_t(A_t) \).

It is easy to check that converse statements are also true. Specifically, using a graded version of Lemma 1.3.1, one proves the following result that clarifies the relationship between derivations and \( t \)-derivations.

Lemma 1.3.5. (i) The composite map

\[
\text{Der}(A, A) \xrightarrow{A \mapsto A_t} \text{Der}(A, A_t) \xrightarrow{\Theta \mapsto \Theta_t} \text{Der}^0_t(A_t, A_t)
\]

yields a Lie algebra isomorphism \( \text{Der}(A, A) \cong \text{Der}^0_t(A_t) \).

(ii) Similarly, the assignment \( \Theta \mapsto \Theta_t \) yields a vector space isomorphism \( \text{Der} A \cong \text{Der}^2_t(A_t) \).

1.4 The role of double derivations. To explain the role of double derivations, recall first that ordinary derivations of \( A \) may also be thought of, algebraically, as ‘infinitesimal automorphisms’. Specifically, let \( A[t] = A \otimes k[t] \) be the polynomial ring in one variable with coefficients in \( A \). This is a \( k[t] \)-algebra and, for any \( k \)-linear map \( \xi : A \to A \), the assignment \( A := A[t] \), \( a \mapsto t \cdot \xi(a) \) can be uniquely extended to a \( k[t] \)-linear map \( t \xi : A[t] \to A[t] \).

A well known elementary calculation yields

Lemma 1.4.1. The following properties of a \( k \)-linear map \( \xi : A \to A \) are equivalent:

- The map \( \xi \) is a derivation of the algebra \( A \);
- The map \( t \xi : A[t] \to A[t] \) is a derivation of the algebra \( A[t] \);
The map \( \text{Id} + t\xi : \frac{A[t]}{t^2A[t]} \to \frac{A[t]}{t^2A[t]} \) is an algebra automorphism.

All the above holds true, of course, no matter whether the algebra \( A \) is commutative or not. Yet, the element \( t \), playing the role of formal parameter, is by definition a central element of the algebra \( A[t] \).

In noncommutative geometry, the assumption that the formal parameter be central is not quite natural, however. Thus, we are led to consider the algebra \( A_t = A \ast \mathbb{K}[t] \), freely generated by \( A \) and an indeterminate \( t \).

We are going to argue that, once the polynomial algebra \( A[t] \) is replaced by \( A_t \), the free product algebra, it becomes more natural to replace derivations \( A \to A \) by double derivations \( A \to A \otimes A \).

To see this, let \( A^+_t = A_t \cdot t \cdot A_t \) be the two-sided ideal of the algebra \( A_t \) generated by \( t \). Thus, we have \( A_t/A_t^+ = A \). Further, a free product analog of an obvious isomorphism \( \frac{A[t]}{t^2A[t]} = A \oplus tA \) is a natural \( A \)-bimodule isomorphism

\[
A_t/(A_t^+)^2 \cong A \oplus (A \otimes A), \quad a + a' \cdot a'' \mapsto a \oplus (a' \otimes a'').
\] (1.4.2)

Thus, we see that the direct summand \( tA \subset \frac{A[t]}{t^2A[t]} \) gets replaced, in the free product setting, by \( A \otimes A \). Furthermore, with the notation of Lemma 1.3.5, a free product analog of Lemma 1.4.1 reads.

**Lemma 1.4.3.** The following properties of a \( k \)-linear map \( \Theta : A \to A \otimes A \) are equivalent:

- The map \( \Theta \) is a double derivation, i.e., \( \Theta \in \text{Der} A \);
- We have \( \Theta_t \in \text{Der}_t(A_t) \);
- The map \( \text{Id} + \Theta_t : \frac{A_t/(A_t^+)^2}{A_t/(A_t^+)^2} \to \frac{A_t/(A_t^+)^2}{A_t/(A_t^+)^2} \) is an algebra automorphism.

Thus, we conclude that in noncommutative geometry the algebra \( A_t \) should play the role of the polynomial algebra \( A[t] \). Some aspects of this philosophy will be discussed further in subsequent sections.

### 1.5 Layout of the paper

In §2, we recall the definition of the DG algebra of noncommutative differential forms. We also introduce the Karoubi-de Rham complex and an extended Karoubi-de Rham complex, that will play a crucial role later. In §3 we develop the basics of Noncommutative Calculus involving the action of double derivations on the extended Karoubi-de Rham complex, via Lie derivative and contraction operations.

In section 4, we construct a canonical morphism from the extended de Rham complex of an algebra \( A \) to the equivariant de Rham complex of the corresponding representation scheme \( \text{Rep}(A,V) \). In §5 we state two main theorems of the paper. These theorems provide a description, in terms
of the de Rham complex, of Hochschild homology of an algebra $A$ and of cyclic homology of $A$, respectively. The proofs of the theorems are given in §6. The proofs are based on the properties of the Karoubi operator and on Harmonic decomposition of the algebra of noncommutative differential forms introduced by Cuntz-Quillen, [CQ1], [CQ2].

The final section contains an alternative construction of the extended de Rham complex. We also outline a new version of deformation theory of associative algebras with noncentral deformation parameter.

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2 The extended Karoubi-de Rham complex

2.1 The commutator quotient. Let $B = \oplus_{k \in \mathbb{Z}} B^k$ be a $\mathbb{Z}$-graded algebra and $M = \oplus_{k \in \mathbb{Z}} M^k$ a graded $B$-bimodule. A linear map $f : B^r \to M^{r+n}$ is said to be a degree $n$ graded derivation if, for any $u \in B^k$ and $v \in B$, we have $f(uv) = f(u) \cdot v + (-1)^{nk}u \cdot f(v)$. We write $\text{Der}^n(B, M)$ for the vector space of degree $n$ graded derivations. The direct sum $\text{Der}^n(B, B) := \bigoplus_{n \in \mathbb{Z}} \text{Der}^n(B, B)$, of graded derivations of the algebra $B$, has a natural Lie super-algebra structure given by the super-commutator.

We write $[B, B]$ for the super-commutator space of a graded algebra $B$, the $k$-linear span of the set $\{uv - (-1)^{pq}vu \mid u \in B^p, v \in B^q, p, q \in \mathbb{Z}\}$. This is a graded subspace of $B$, and we may consider the commutator quotient space $RB := B/[[B, B]]$, equipped with induced grading $R^*B := \oplus_{k \in \mathbb{Z}} R^kB$. Any degree $n$ graded derivation $f : B^r \to B^{r+n}$ descends to a well-defined linear map of graded vector spaces $R(f) : R^rB \to R^{r+n}B$.

Notation 2.1.1. Given an algebra $A$ and an $A$-bimodule $M$, let $T_AM = \oplus_{n \geq 0} T^n_A M$ be the tensor algebra of $M$ over $A$. Thus, $T^n_A M$ is a graded associative algebra with $T^0_A M = A$. In the special case $A = k$ we will use unadorned notation $TM := T_k M$, and write $M^\otimes n := T^n M$.

Let $T : M^\otimes n \to M^\otimes n$, $u_1 \otimes u_2 \otimes \ldots \otimes u_n \to u_n \otimes u_1 \otimes u_2 \otimes \ldots \otimes u_{n-1}$ denote the generator of the group $\mathbb{Z}/n\mathbb{Z}$ of cyclic permutations of the tensor factors, and write $M^\otimes n_{\text{cyclic}} := M^\otimes n / (\mathbb{Z}/n\mathbb{Z})$ for the space of $\mathbb{Z}/n\mathbb{Z}$-coinvariants. It is clear that, for any $n = 1, 2, \ldots$, we have natural identifications

$$R^n(TM) = M^\otimes n_{\text{cyclic}} = M^\otimes n / \text{Im}(\text{Id} - T),$$

(2.1.2)
for instance, modulo commutators, one has \( x \otimes y \otimes z = z \otimes x \otimes y = y \otimes z \otimes x \).

Later on, we will also use the following space

\[
M_\text{anticyclic}^\otimes := M_\otimes^\otimes / \text{Im}(\text{Id} - (-1)^n T), \quad \forall n = 1, 2, \ldots ,
\]

of coinvariants with respect to a sign-twisted \( \mathbb{Z}/n\mathbb{Z} \)-action on \( M_\otimes^\otimes \). Note that \( T \) is a permutation of parity \( (-1)^{n-1} \).

### 2.2 Commutator quotient for free products.

We will be using free product constructions for \emph{graded} algebras. For a graded algebra \( B \), the algebra \( B_t = B \ast \mathbb{k}[t] \) acquires a natural grading \( B_t^r = \bigoplus_{k \in \mathbb{Z}} B^k_t \), that agrees with the one on the subalgebra \( B \subset B_t \) and such that \( \deg t = 2 \). Let \( R^r(B_t) = \bigoplus_{k \in \mathbb{Z}} R^k(B_t) \) be the corresponding commutator quotient.

There is a graded analog of Lemma \([3.1]\). Specifically, given a graded \( B_t^r \)-bimodule \( M^r = \bigoplus_{k \in \mathbb{Z}} M^k \) and a \( \mathbb{k} \)-linear map \( f : B^r \to M^{n+n} \) such that \( f(1) = 0 \), one introduces a degree \( n \) derivation \( f_t : B^r_t \to M^{n+n} \) defined, for any homogeneous elements \( u_1, \ldots, u_n \in B \), by the formula

\[
f_t(u_1 t u_2 t \ldots t u_r) = \sum_{k=1}^r (-1)^{n(\deg u_1+\ldots+\deg u_{k-1})} u_1 t \ldots u_{k-1} t f(u_k) t u_{k+1} t \ldots t u_r.
\]

We will also consider an \emph{odd} version of free product. Specifically, let \( B_\tau := B \ast \mathbb{k}[\tau] \), and equip this free product with a \( \mathbb{Z} \)-grading \( B_\tau = \bigoplus_{k \in \mathbb{Z}} B^k_\tau \) that agrees with the one on \( B \) and such that \( \deg \tau = -1 \) (as opposed to \( \deg t = 2 \)). Thus, \( B_\tau \) is isomorphic to \( B_t \) as an associative algebra but not as a graded algebra.

The grading on \( B_t \), resp. on \( B_\tau \), induces a grading \( R^r(B_t) = \bigoplus_{k \in \mathbb{Z}} R^k(B_t) \), resp. \( R^r(B_\tau) = \bigoplus_{k \in \mathbb{Z}} R^k(B_\tau) \).

Now, given an algebra \( B \) \emph{without} grading, we may (and will) view it as a graded algebra concentrated in degree zero. In that case, the resulting grading \( B_t = \bigoplus_{k \geq 0} B^k_t \) is the one considered in \([1.3]\) that is, the even grading that counts twice the number of occurrences of the variable \( t \). Similarly, we have the algebra \( B_\tau = \bigoplus_{k \leq 0} B^k_\tau \), with \emph{non-positive} grading, and also, graded commutator quotient spaces \( R^r(B_t) = \bigoplus_{k \geq 0} R^k(B_t) \), resp. \( R^r(B_\tau) = \bigoplus_{k \leq 0} R^k(B_\tau) \).

In particular, we have \( R^0(B_t) = R^0(B_\tau) = RB = B/[B,B] \), where \( [B,B] \subset B \) is the subspace spanned by ordinary commutators. For any \( n \geq 1 \), the space \( R^{2n}(B_t) \), resp. the space \( R^{-n}(B_\tau) \), is spanned by \emph{cyclic} words \( u_1 t u_2 t \ldots t u_n t \), for instance, modulo (graded) commutators, we have \( u_1 t u_2 t = u_2 t u_1 t \), resp. \( u_1 \tau u_2 \tau = -u_2 \tau u_1 \tau \).

This way, one arrives at the following result that generalizes \([1.3]\).
Lemma 2.2.2. For any (non-graded) algebra $B$ and $n \geq 1$, the assignment $u_1 u_2 \cdot \cdot \cdot u_n \mapsto u_1 \otimes u_2 \otimes \cdot \cdot \cdot \otimes u_n$ yields natural vector space isomorphisms $B_t^{2n} \cong B^{\otimes n} \cong B_{\tau}^{-n}$.

The induced isomorphisms of the corresponding graded commutator quotients read

$$R^{2n}(B_t) \cong B_{\text{cyclic}}^{\otimes (n-1)}, \quad \text{resp.} \quad R^{-n}(B_{\tau}) \cong B_{\text{anticyclic}}^{\otimes (n-1)}, \quad \forall n \geq 1.$$ 

For $n = 0, 1$, we have

$$R^0(B_t) = B/[B,B] = R^0(B_{\tau}), \quad \text{and} \quad R^2(B_t) \cong B \cong R^{-1}(B_{\tau}).$$

Observe next that the assignment $\frac{d}{d\tau} : \tau \mapsto 1, b \mapsto 0, \forall b \in B$, extends uniquely to a degree 1 graded derivation $\frac{d}{d\tau} : B_{\tau} \to B_{\tau}$. In particular, we have

$$\frac{d}{d\tau}(\tau^n) = 1 \cdot \tau \cdot \cdots \cdot \tau - \tau \cdot 1 \cdot \tau \cdot \cdots \cdot \tau + \cdot \cdot \cdot + (-1)^{n-1} \tau \cdot \cdots \cdot \tau \cdot 1 = \begin{cases} 0 & \text{if } n \text{ is even} \\ \tau^{n-1} & \text{if } n \text{ is odd}. \end{cases}$$

It follows easily that one has $\frac{d}{d\tau} \cdot \frac{d}{d\tau} = 0$, i.e., the map $\frac{d}{d\tau} : B_{\tau}^* \to B_{\tau}^{*+1}$ is a differential.

Recall further that, associated with any algebra $B$, one has a standard bar complex

$$\ldots \to B \otimes^4 \to B \otimes^3 \to B \otimes^2 \to B.$$ \hspace{1cm} (2.2.3)

Furthermore, it is well known that the bar differential descends to well-defined maps, cf. \cite{Lo}

$$\ldots \to B_{\text{cyclic}}^{\otimes 4} \to B_{\text{cyclic}}^{\otimes 3} \to B_{\text{cyclic}}^{\otimes 2} \to B.$$ \hspace{1cm} (2.2.4)

The resulting complex is known as the cyclic complex for $B$.

Verification of the following result is straightforward.

Proposition 2.2.5. (i) The bar differential $\partial$ in (2.2.3) is transported, via the isomorphisms $B^{\otimes n} \cong B_{\tau}^{-n}$ of Lemma 2.2.2(ii), to the differential $\frac{d}{d\tau} : B_{\tau}^* \to B_{\tau}^{*+1}$.

(ii) The induced isomorphism of Lemma 2.2.2(ii), provides a natural identification of the cyclic complex (2.2.4) with the complex $(R^{<0}(B_{\tau}), R(\frac{d}{d\tau}))$.

I first learned part (i) of the above result, which is a special case $M = 0$ of Proposition 7.2.1 of §7 below, from V. Drinfeld.
2.3 Noncommutative differential forms. We fix an algebra $B$, a subalgebra $R \subset B$, and write $B \otimes_R B \xrightarrow{m} B$ for the multiplication map. Let $\Omega^1_R B := \text{Ker}(m)$ be the $B$-bimodule of noncommutative 1-forms on $B$ relative to the subalgebra $R$, see [CQ1, Sect. 2], and let $i_\Delta : \Omega^1_R B \hookrightarrow B \otimes_R B$ denote the tautological imbedding (this notation will be justified later). We have a fundamental short exact sequence of $B$-bimodules

$$0 \longrightarrow \Omega^1_R B \xrightarrow{i_\Delta} B \otimes_R B \xrightarrow{m} B \longrightarrow 0. \quad (2.3.1)$$

The assignment $b \mapsto db := 1 \otimes b - b \otimes 1$ gives a canonical derivation $d : B \to \Omega^1_R B$. This derivation is ‘universal’ in the sense that, for any $B$-bimodule $M$, we have a bijection

$$\text{Der}_R(B, M) \rightarrow \text{Hom}_{B\text{-bimod}}(\Omega^1_R B, M), \quad \theta \mapsto i_\theta,$$

where the map $i_\theta : \Omega^1_R B \to M$ is defined by the formula $i_\theta(u dv) := u \cdot \theta(v)$. The tensor algebra $\Omega^*_R B := T_B(\Omega^1_R B)$, of the $B$-bimodule $\Omega^1_R B$, is a DG algebra $(\Omega^*_R B, m)$, called the algebra of noncommutative differential forms on $B$ relative to the subalgebra $R$ (we will interchangeably use the notation $\Omega_R B$ or $\Omega^*_R B$ depending on whether we want to emphasize the grading or not). For each $n \geq 1$, there is a standard isomorphism of left $B$-modules, see [CQ1], $\Omega^n_R B = B \otimes_R T^n_B(B/R)$; usually, one writes $b_0 db_1 db_2 \ldots db_n \in \Omega^n_R B$ for the $n$-form corresponding to an element $b_0 (b_1 \otimes \ldots \otimes b_n) \in B \otimes_R T^n(B/R)$ under this isomorphism. The de Rham differential $d : \Omega^*_R B \to \Omega^{*+1}_R B$ is defined by the formula $d : b_0 db_1 db_2 \ldots db_n \mapsto db_0 db_1 db_2 \ldots db_n$.

Following Karoubi, we define the noncommutative de Rham complex of $B$ (relative to the subalgebra $R$) to be the commutator quotient space

$$\text{DR}_R B := R(\Omega^R_B) = \Omega^R_B/[\Omega_R B, \Omega_R B],$$

of the graded algebra $\Omega^*_R B$. The space $\text{DR}_R B$ comes equipped with a natural grading and with the de Rham differential $d : \text{DR}_R^* B \to \text{DR}^{*+1}_R B$, induced from the one on $\Omega^*_R B$, c.f. also [Lo] for more details. In degree zero, we have $\text{DR}_R^0 B = RB = B/[B, B]$.

In the ‘absolute’ case $R = k$ we will use unadorned notation $\Omega^n B := \Omega^n_k B$, $\text{DR} B := \text{DR}_k B$, etc.

Let $\mathbb{C}[\varepsilon]/(\varepsilon^2)$ be the ring of dual numbers. Equip the free product $B_\varepsilon := B \ast (\mathbb{C}[\varepsilon]/(\varepsilon^2))$ with a grading such that $\text{deg} B = 0$ and $\text{deg} \varepsilon = 1$. The graded commutator map $\text{ad} \varepsilon : u \mapsto [\varepsilon, u] := \varepsilon u - u \varepsilon$ gives a degree 1 derivation $\text{ad} \varepsilon : B_\varepsilon \to B_{\varepsilon+1}^*$, moreover, we have $(\text{ad} \varepsilon)^2 = \text{ad}(\varepsilon^2) = 0$. Thus, $(B_\varepsilon, \text{ad} \varepsilon)$ is a DG algebra.
Further, the derivation $B \rightarrow B_{\varepsilon}$, $b \mapsto [\varepsilon, b]$ gives rise, via the universal property of $\Omega' B$, to a DG algebra imbedding $(\Omega' B, d) \hookrightarrow (B_{\varepsilon}, \text{ad } \varepsilon)$, cf. [CQ1] Proposition 1.4], given by the formula

$$b_0 d_1 d_2 \ldots d_n \mapsto b_0 [\varepsilon, b_1] [\varepsilon, b_2] \ldots [\varepsilon, b_n] = b_0 \varepsilon b_1 \varepsilon b_2 \ldots \varepsilon b_n + (-1)^n b_0 b_1 \varepsilon b_2 \ldots \varepsilon b_n \varepsilon.$$

### 2.4 The extended de Rham complex.

Fix an algebra $A$. We introduce an enlargement of the noncommutative de Rham complex, $\text{DR}_A$, as follows.

First, form the free product algebra $A_t = A \ast \mathbb{k}[t]$. This free product contains $\mathbb{k}[t]$ as a subalgebra and we consider $\Omega_t A := \Omega_{\mathbb{k}[t]}(A_t)$, the DG algebra of noncommutative differential forms on $A_t$ relative to the subalgebra $\mathbb{k}[t]$. We define extended de Rham complex of $A$ as

$$\text{DR}_t A := \text{DR}_{\mathbb{k}[t]}(A_t) = \Omega_{\mathbb{k}[t]}(A_t) / [\Omega_{\mathbb{k}[t]}(A_t), \Omega_{\mathbb{k}[t]}(A_t)].$$

On the other hand, one may form a free product $(\Omega A)_t = (\Omega A) \ast \mathbb{k}[t]$, and equip it with the standard even grading that counts twice the number of occurrences of $t$ (and disregards the degrees of differential forms). We have the corresponding direct sum decomposition into grade $d$ components

$$(\Omega A)_t / [(\Omega A)_t, (\Omega A)_t] = \mathbb{R}((\Omega A)_t) = \oplus_{k \geq 0} \mathbb{R}^{2k}((\Omega A)_t). \quad (2.4.1)$$

The differential $d : \Omega A \rightarrow \Omega A \subset (\Omega A)_t$ extends, via formula (2.2.1), to a graded derivation $d_t : (\Omega A)_t \rightarrow (\Omega A)_t$ of degree zero.

A useful alternate point of view on the extended de Rham complex is provided by the following

**Lemma 2.4.2.** (i) There is a canonical DG algebra isomorphism

$$\Omega_t A \xrightarrow{\psi} (\Omega A)_t,$$

such that the de Rham differential on $\Omega_t A = \Omega_{\mathbb{k}[t]}(A_t)$ is transported, via the isomorphism $\psi$, to the map $d_t$, see Notation 1.3.3, on $(\Omega A)_t$.

(ii) The composite isomorphism below

$$\text{DRA} \oplus \Omega A \oplus (\Omega A)_{\text{cyclic}}^2 \oplus (\Omega A)_{\text{cyclic}}^3 \oplus \ldots \xrightarrow{\text{Lemma 2.2.1}} \text{DR}_t A \xrightarrow{\mathbb{R}(\psi)} \mathbb{R}((\Omega A)_t) \xrightarrow{\text{cf. Lemma 2.1}} \oplus_{k \geq 0} \mathbb{R}^{2k}((\Omega A)_t)$$

is compatible with the leftmost and the rightmost direct sum decompositions.
Proof. In general, for any two algebras $A, R$, we have a natural DG algebra isomorphism $\Omega(A \ast R) \cong (\Omega A) \ast (\Omega R)$. Since $\Omega_R(A \ast R)$ is a quotient of the algebra $\Omega(A \ast R)$ by the two-sided ideal generated by the space $dR \subset \Omega^1 R \subset \Omega^1(A \ast R)$, the isomorphism above induces a DG algebra isomorphism

$$\Omega_R(A \ast R) \cong (\Omega A) \ast R.$$

Part (i) of the Lemma follows from the last isomorphism by putting $R = k[t]$. Part (ii) is now an immediate consequence of Lemma 2.2.2 applied to the algebra $B = \Omega A$. \qed

Yet another interpretation of the extended Karoubi-de Rham complex will be given in §7.2.

Next, we observe that the algebra $\Omega_t A = \Omega_k[t](A_t)$ comes equipped with a natural bi-grading $\Omega_t A = \bigoplus_{p,q \geq 0} \Omega^{2p,q}_t A$, where the even $p$-grading is induced from the one on $A_t$, and the $q$-component corresponds to the grading induced by the natural one on $\Omega^* A$. It is easy to see that the $p$-grading corresponds, under the isomorphism of Lemma 2.4.2, to the grading on $(\Omega A) \ast k[t]$ that counts twice the number of occurrences of the variable $t$, and the $q$-component corresponds to the grading induced by the natural one on $\Omega^q A$, e.g. for any $\alpha \in \Omega^k A, \beta \in \Omega^\ell A$ the element $\alpha t^\beta t \in (\Omega A) \ast k[t]$ has bi-degree $(2p = 4, q = k + \ell)$.

The bi-grading on $\Omega_t A$ clearly descends to a similar bigrading on the extended de Rham complex of $A$, with the de Rham differential of bi-degree $(0, 1)$:

$$\text{DR}_t A = \bigoplus_{p,q} \text{DR}^{2p,q}_t A, \quad d : \text{DR}^{2p,q}_t A \to \text{DR}^{2p,q+1}_t A. \quad (2.4.3)$$

3 Noncommutative calculus

3.1 Lie derivative and contraction for derivations. Fix an algebra $B$ and a subalgebra $R \subset B$.

Any derivation $\theta \in \text{Der}_R B$ gives rise to a Lie derivative map $L_\theta : \Omega^*_R B \to \Omega^*_R B$, and also to contraction (with $\theta$) map $i_\theta : \Omega^*_R B \to \Omega^*_{R^{-1}} B$. The map $L_\theta$ is a degree zero derivation of the graded algebra $\Omega^*_R B$. It is defined on 1-forms by the formula $L_\theta(u dv) = (\theta(u)) dv + u d(\theta(v))$, and then extended uniquely to a map $L_\theta : \Omega^*_R B \to \Omega^*_R B$ as a derivation. The contraction map $i_\theta$ is a degree $-1$ graded derivation. It is defined on 1-forms by formula (2.3.2) and is extended to a map $\Omega^*_R B \to \Omega^*_{R^{-1}} B$ as a graded-derivation.
One has the Cartan formula
\[ L_\theta = i_\theta \circ d + d \circ i_\theta, \]
for any \( \theta \in \text{Der}_R B \). The maps \( L_\theta \) and \( i_\theta \) both descend to well-defined operations on the de Rham complex \( DR^*_R B = R'(\Omega_R B) \), satisfying a similar Cartan formula.

Now, let \( A \) be an algebra, \( \theta \in \text{Der} A \) a derivation, and \( L_\theta : \Omega^* A \rightarrow \Omega^* A \) the corresponding Lie derivative.

On one hand, the derivation \( L_\theta \) gives rise, according to Lemma 1.3.1(ii), to a derivation \( (L_\theta)_t : (\Omega^*_t A_t) \rightarrow (\Omega^*_t A_t) \). On the other hand, one may first extend \( \theta \) to a derivation \( \theta_t \in \text{Der}_{k[t]} (A_t) \), and then consider the corresponding Lie derivative
\[ L_{\theta_t} : \Omega^*_t A_t = \Omega^*_t A \rightarrow \Omega^*_t A. \]

Very similarly, we may form graded derivations \( (i_\theta)_t \) and \( i_{\theta_t} \).

An important point is that the two procedures just explained agree with each other in the sense that: 
Under the identification \( \Omega^*_t(A_t) \cong (\Omega^*_A)_t \), provided by the isomorphism \( \psi \) of Lemma 2.4.2(i), we have
\[ L_{\theta_t} = (L_\theta)_t, \quad i_{\theta_t} = (i_\theta)_t. \tag{3.1.1} \]

All the above operations on \( \Omega^*_t A \) descend to the extended de Rham complex \( DR_t A \). The Cartan identity \( L_\theta = i_\theta \circ d + d \circ i_\theta \) induces similar identities for the corresponding operations on \( \Omega_t A \) and on \( DR_t A \).

### 3.2 Lie derivative and contraction for double derivations.

Associated with any double derivation \( \Theta \in \mathbb{D}er A \), one has an \( A \)-bimodule map
\[ i_\Theta : \Omega^1 A \rightarrow A \otimes A, \quad \alpha \mapsto i_\Theta \alpha = (i'_\Theta \alpha) \otimes (i''_\Theta \alpha), \tag{3.2.1} \]
that corresponds to \( \Theta \) under the canonical bijection \( 2.3.2 \). With this understood, we have

**Proposition 3.2.2.** Any double derivation \( \Theta \in \mathbb{D}er A \) gives rise canonically to a degree 0 double derivation \( L_\Theta \in \mathbb{D}er^0(\Omega^* A) \), and also to a degree \(-1\) graded double derivation \( i_\Theta \in \mathbb{D}er^{-1}(\Omega^* A) \).

Explicitly, the double derivation \( L_\Theta \) is given, for any \( a_0, a_1, \ldots, a_n \in A \), by the formula
\[
a_0 da_1 \ldots da_n \mapsto L_\Theta(a_0 da_1 \ldots da_n) := (\Theta' a_0) \otimes (\Theta'' a_0) da_1 \ldots da_n + \sum_{1 \leq k \leq n} \left(a_0 da_1 \ldots da_{k-1} d(\Theta' a_k) \otimes (\Theta'' a_k) da_{k+1} \ldots da_n + a_0 da_1 \ldots da_{k-1} (\Theta' a_k) \otimes d(\Theta'' a_k) da_{k+1} \ldots da_n \right).
\]
The graded double derivation $i_\Theta$ is given, for any $\alpha_1, \ldots, \alpha_n \in \Omega^1A$, by the formula, cf. (3.2.1):

$$\alpha_1 \alpha_2 \ldots \alpha_n \mapsto \sum_{1 \leq k \leq n} (-1)^{k-1} \cdot \alpha_1 \ldots \alpha_{k-1} (i_\Theta^0 \alpha_k) \otimes (i_\Theta^\prime \alpha_k) \alpha_{k+1} \ldots \alpha_n.$$

Thus, we have

$$L_\Theta : \Omega^nA \to \bigoplus_{0 \leq k \leq n} \Omega^kA \otimes \Omega^{n-k}A, \quad i_\Theta : \Omega^nA \to \bigoplus_{1 \leq k \leq n} \Omega^{k-1}A \otimes \Omega^{n-k}A.$$

It is not too difficult to verify by direct computation that the explicit formula for $L_\Theta$, resp. for $i_\Theta$, given in Proposition 3.2.2 does define a degree zero double derivation $\Omega A \to \Omega A \otimes \Omega A$, resp. a degree $-1$ graded double derivation. It is, however, much more rewarding to use a more conceptual approach to the construction of $L_\Theta$ and $i_\Theta$, explained below.

A key point of our approach is that although double derivations do not give rise to natural operations on the DG algebra $\Omega A$ they do give rise to canonical Lie derivative and contraction operations on the extended DG algebra $\Omega_t^*A$. Proposition 3.2.2 will easily follow from this.

To construct Lie derivative and contraction operations associated with a double derivation, we are going to mimic equations (3.1.1). Observe that the left-hand sides of equations (3.1.1) still make sense for double derivations. In more detail, given $\Theta \in \mathfrak{Der} A$, we may first extend it to a free product derivation $\Theta_t : A_t \to A_t$, as in §1.3. After that, we define Lie derivative $L_\Theta$, resp. contraction $i_\Theta$, operation on $\Omega_t A$, as the ordinary Lie derivative, resp. ordinary contraction, operation induced by the derivation $\Theta_t$ on the complex $\Omega_t A = \Omega_{k[t]}(A_t)$, of relative differential forms on the algebra $A_t$. Specifically, we put

$$L_\Theta := L_{\Theta_t}, \quad i_\Theta := i_{\Theta_t} : \Omega_t A \to \Omega_t A. \quad (3.2.3)$$

Now, we use Lemma 2.2.2 to write $\Omega_t A = (\Omega A)_t$. It is immediate from the above formulas that, viewed as maps $(\Omega A)_t \to (\Omega A)_t$, the Lie derivative $L_\Theta$ and contraction $i_\Theta$ are both graded derivations of degree 2 with respect to the grading that counts (twice) the number of occurrences of $t$ (and disregards the degrees of differential forms). Therefore, applying Lemma 1.3.5 to the algebra $B = \Omega A$, we deduce that there exist unique double derivations $L_\Theta, i_\Theta : \Omega A \to (\Omega A) \otimes (\Omega A)$, such that, for the corresponding maps $(\Omega A)_t \to (\Omega A)_t$, we have, cf. (3.2.1)

$$(L_\Theta :=) L_{\Theta_t} = (L_\Theta)_t, \quad \text{resp.} \quad (i_\Theta :=) i_{\Theta_t} = (i_\Theta)_t. \quad (3.2.4)$$
(in the case of contraction operation \(i_\Theta\), one has to repeat the argument using an analogue of Lemma 1.3.5 for graded-derivations).

This completes our proof of Proposition 3.2.2. □

The Lie derivative and contraction operations on \(\Omega_tA\) defined in (3.2.3) satisfy, for any \(\Theta, \Phi \in \Der A\) and \(\xi \in \Der A\), the following standard commutation relations

\[
L_\Theta = d \circ i_\Theta + i_\Theta \circ d, \quad i_\Theta \circ i_\Phi + i_\Phi \circ i_\Theta = 0, \quad i_\xi \circ i_\Theta + i_\Theta \circ i_\xi = 0.
\] (3.2.5)

To prove (3.2.5), one first verifies these identities on the generators of the algebra \(\Omega_tA = (\Omega A)_t\), that is, on differential forms of degrees 0 and 1, which is a simple computation. The general case then follows by observing that any commutation relation between (graded)-derivations that holds on generators of the algebra holds true on all elements of the algebra.

Formulas (3.2.5) imply, in particular, that the Lie derivative \(L_\Theta\) commutes with the de Rham differential \(d\).

Both the Lie derivative and contraction operations on \(\Omega_tA\) descend to the commutator quotient \(\text{DR}_tA\). This way, we obtain the Lie derivative \(L_\Theta\), resp. contraction \(i_\Theta\), operation on the extended de Rham complex. It is immediate that these operations on \(\text{DR}_tA\) also satisfy (3.2.5).

3.3 Reduced Lie derivative and contraction. Let \(B\) be a (possibly graded) algebra and \(F \in \Der^2_t(B_t)\) a degree 2 derivation with respect to the even grading on \(B_t\) that counts twice the number of occurrences of \(t\) and disregards the grading on \(B\). Using the identifications of Lemma 2.2.2, we can interpret the induced linear map \(R(F) : R^*(B_t) \to R^{*+2}(B_t)\) as a chain of maps

\[
B/[B,B] \overset{R(F)}{\longrightarrow} B \xrightarrow{R(F)} B_{\text{cyclic}}^{\otimes 2} \xrightarrow{R(F)} B_{\text{cyclic}}^{\otimes 3} \xrightarrow{R(F)} \ldots. \quad (3.3.1)
\]

We apply the above in the special case where \(B = \Omega A\) and \(F := L_\Theta\), resp. \(F := i_\Theta\), for some \(\Theta \in \Der A\). In this case, the terms in (3.3.1) may be written more explicitly using the identification provided by the composite isomorphism of Lemma 2.4.2(ii). This way, any double derivation \(\Theta \in \Der A\) gives rise to two chains of maps (one for \(L_\Theta\) and one for \(i_\Theta\)) of the form

\[
\text{DR}A \longrightarrow \Omega A \longrightarrow (\Omega A)^{\otimes 2}_{\text{cyclic}} \longrightarrow (\Omega A)^{\otimes 3}_{\text{cyclic}} \longrightarrow \ldots. \quad (3.3.2)
\]

Recall now that the extended de Rham complex \(\text{DR}_tA\) has a natural bigrading, cf. (2.4.3). The second component (the \(q\)-component) of that bigrading induces a grading on each of the spaces \((\Omega A)^{\otimes k}_{\text{cyclic}}, k = 1, 2, \ldots,\)
appearing in (3.3.2). Explicitly, the grading on \((\Omega A)^\otimes_k\) counts the total degree of differential forms involved, i.e., we have \(\deg(\alpha_1 \otimes \ldots \otimes \alpha_k) = \deg \alpha_1 + \ldots + \deg \alpha_k\), for any \(\alpha_1 \otimes \ldots \otimes \alpha_k \in (\Omega A)^\otimes_k\). Further, it is immediate from definitions that the maps \(L_\varnothing\) and \(i_\varnothing\) in (3.2.3) are graded derivations of bi-degrees \((2,0)\) and \((2,-1)\), respectively. We conclude that, in the Lie derivative case, all maps in the corresponding chain (3.3.2) preserve the above defined grading while, in the contraction case, all maps in the corresponding chain (3.3.2) decrease the above defined grading by one.

The leftmost map in (3.3.2), to be denoted \(i_\varnothing\) in the contraction case, resp. \(\mathcal{L}_\varnothing\) in the Lie derivative case, will be especially important for us. In more detail, these maps have the form

\[
i_\varnothing : \text{DR}^*A \to \Omega^{-1}A, \quad \text{resp.}, \quad \mathcal{L}_\varnothing : \text{DR}^*A \to \Omega A. \tag{3.3.3}
\]

One verifies that the map \(i_\varnothing\) in (3.3.3) is given, for any \(\alpha_1, \alpha_2, \ldots, \alpha_n \in \Omega^1 A\), by the following explicit formula, see [CBEG (2.8.4)]:

\[
i_\varnothing(\alpha_1 \alpha_2 \ldots \alpha_n) = \sum_{k=1}^{n} (-1)^{(k-1)(n-k+1)} (i''_\varnothing \alpha_k) \cdot \alpha_{k+1} \ldots \alpha_n \alpha_1 \ldots \alpha_{k-1} \cdot (i'_\varnothing \alpha_k). \tag{3.3.4}
\]

The maps in (3.3.3) have been already introduced, in effect, in [CBEG] under the names of reduced contraction, resp. reduced Lie derivative. An ad hoc definition of these maps given in [CBEG] was based on explicit formulas for \(L_\varnothing\) and \(i_\varnothing\), cf. formulas of Proposition 3.2.2. As a result, verification of commutation relations like (3.2.5) was very painful; it was carried out in [CBEG] by rather long brute force computations. Our present approach based on the free product construction yields the commutation relations almost for free.

3.4 The derivation \(\Delta\). For any algebra \(A\), there is a distinguished derivation

\[
\Delta : A \to A \otimes A, \quad a \mapsto 1 \otimes a - a \otimes 1.
\]

The corresponding contraction map \(i_\Delta : \Omega^1 A \to A \otimes A\) is nothing but the tautological imbedding \(\Omega^1 A \hookrightarrow A \otimes A\). Further, the derivation \(\Delta_t : A_t \to A_t\) associated with \(\Delta\) by formula (1.3.2) clearly equals \(\text{ad} t : u \mapsto t u - u t\). Given \(a \in A\) and \(\omega \in \Omega A\), write \([\omega, a] := \omega a - a \omega\).

Lemma 3.4.1. (i) The Lie derivative \(L_\Delta : \Omega_t A \to \Omega_t A\) is given by the formula \(\omega \mapsto [t, \omega]\). In particular, the induced map \(\text{R}(L_\Delta) : \text{DR}_t A \to \text{DR}_t A\), on the extended de Rham complex, vanishes.
For any $a_0, a_1, \ldots, a_n \in A$, we have

$$
\iota_{\Delta}(a_0 \, da_1 \ldots \, da_n) = \sum_{1 \leq k \leq n} (-1)^k \cdot [da_{k+1} \ldots da_n \, a_0 \, da_1 \ldots \, da_{k-1}, \, a_k].
$$

Proof. To prove (i), it suffices to check the equality $L_{\Delta} = \text{ad} \, t$ on the generators of the algebra $\Omega_t A$, that is, on 0-forms and on 1-forms. This is easy. Part (ii) is a straightforward consequence of formula (3.3.3).

There is also an odd version of the derivation $\Delta_t$. Specifically, one can extend the double derivation $\Delta$ to a degree $-1$ graded derivation $\Delta^\tau$:

$$
B^\tau \to B^\tau \quad \text{(cf. §2.2 for the definition of $B^\tau$),}
$$

such that $\Delta^\tau(b) = \tau b - b \tau$, $\forall b \in B \subset B^\tau$, and $\Delta^\tau(\tau) = 0$. Further, it is straightforward to check that

$$
\frac{d}{d\tau} \circ \Delta^\tau + \Delta^\tau \circ \frac{d}{d\tau} = 0, \quad \text{and} \quad \Delta^\tau \circ \Delta^\tau = -\text{ad}(\tau^2),
$$

where $\text{ad}(\tau^2)$ stands for the graded commutator with $\tau^2$. Observe that $\Delta^\tau \neq \text{ad} \, \tau$, since $\Delta^\tau(\tau) = 0 \neq 2\tau^2 = \text{ad} \, \tau(\tau)$.

We see that the pair $\frac{d}{d\tau}$ and $\Delta^\tau$ gives rise, by Proposition 2.2.5(ii), to two anti-commuting maps of degrees +1 and $-1$, respectively, on the space $R^{<0}(A^\tau)$, hence, on the cyclic complex (2.2.4). Using the isomorphism of Proposition 2.2.5 we find that the corresponding map $A^\otimes n_{\text{anticyclic}} \to A^\otimes (n+1)_{\text{anticyclic}}$, arising from $\frac{1}{2}R(\Delta^\tau)$, is given by the following explicit formula

$$
\frac{1}{2}R(\Delta^\tau)(a_1 \otimes \ldots \otimes a_n) = \sum_{k=1}^n (-1)^k a_1 \otimes \ldots \otimes a_k \otimes 1 \otimes a_{k+1} \otimes \ldots \otimes a_n.
$$

4 The Representation functor.

4.1 Evaluation map. We fix a finite dimensional $k$-vector space $V$, and write $\text{End} := \text{Hom}_k(V, V)$ for the associative algebra of linear endomorphisms of $V$.

Given an algebra $A$, one may consider the set $\text{Hom}_{\text{alg}}(A, \text{End})$ of all algebra maps $\rho : A \to \text{End}$. More precisely, to any finitely generated $k$-algebra $A$ one associates an affine scheme of finite type over $k$, to be denoted $\text{Rep}(A, V)$, such that the set $\text{Hom}_{\text{alg}}(A, \text{End})$ is the set of $k$-points of $\text{Rep}(A, V)$. Write $k[\text{Rep}(A, V)]$ for the coordinate ring of the affine scheme $\text{Rep}(A, V)$, which will be always assumed to be non-empty.

The tensor product $\text{End} \otimes k[\text{Rep}(A, V)]$, is an associative algebra that may also be identified with the algebra of polynomial maps $\text{Rep}(A, V) \to \text{End}$, equipped with pointwise multiplication.
To each element \( a \in A \), one associates the function \( \hat{a} : \text{Rep}(A, V) \to \text{End} \), \( \rho \mapsto \hat{a}(\rho) := \rho(a) \). The assignment \( a \mapsto \hat{a} \) clearly gives an algebra homomorphism, called \textit{evaluation map}, cf. also [Gi] §12,

\[ \text{ev} : A \to \text{End} \otimes \kappa[\text{Rep}(A, V)], \quad a \mapsto \hat{a}. \]

### 4.2 Extended de Rham complex and equivariant cohomology.

In this section, we assume the field \( \kappa \) to be algebraically closed.

Let \( X \) be an affine scheme with coordinate ring \( \kappa[X] \). We write \( T(X) := \text{Der}(\kappa[X]) \) for the Lie algebra of algebraic vector fields, \( \Omega^1(X) \) for the \( \kappa[X] \)-module of Kähler differentials, and let \( \Omega^\ast(X) := \bigwedge^\ast_{\kappa[X]} \Omega^1(X) \) denote the DG algebra of differential forms on \( X \).

Suppose we have a finite dimensional Lie algebra \( \mathfrak{g} \) and a Lie algebra map \( \mathfrak{g} \to T(X) \). Thus, we get a Lie algebra action of \( \mathfrak{g} \) on \( \Omega^\ast(X) \), by the Lie derivative. Also, let \( \mathfrak{g} \) act on \( \mathbb{C}[\mathfrak{g}] \), the polynomial algebra on the vector space \( \mathfrak{g} \), by the adjoint action. We view \( \mathbb{C}[\mathfrak{g}] \) as an even-graded algebra such that the vector space of linear functions on \( \mathfrak{g} \) is assigned degree 2. This makes the tensor product \( \Omega^\ast(X, \mathfrak{g}) := \Omega^\ast(X) \otimes \mathbb{C}[\mathfrak{g}] \) a graded algebra, equipped with \( \mathfrak{g} \)-diagonal action. We write \( \Omega^\ast(X, \mathfrak{g})^\mathfrak{g} \) for the corresponding graded subalgebra of \( \mathfrak{g} \)-invariants.

The \( \mathfrak{g} \)-equivariant algebraic de Rham complex of \( X \) is the complex

\[ (\Omega^\ast(X, \mathfrak{g})^\mathfrak{g}, d_{\text{DR}} + d_{\mathfrak{g}}). \]

Here, the differential \( d_{\text{DR}} = d \otimes \text{id}_{\mathbb{C}[\mathfrak{g}]} \) is induced by the ordinary de Rham differential \( d \) on \( \Omega^\ast(X) \). The differential \( d_{\mathfrak{g}} \) is defined by the formula

\[ d_{\mathfrak{g}} : \omega \otimes f \mapsto \sum_{r=1}^{\dim \mathfrak{g}} (i_{e_r} \omega) \otimes (e_r^* \cdot f), \quad \forall \omega \otimes f \in \Omega^\ast(X) \otimes \mathbb{C}[\mathfrak{g}], \]

where \( \{e_r\} \) and \( \{e_r^*\} \) stand for dual bases of \( \mathfrak{g} \) and \( \mathfrak{g}^* \), respectively, and where \( i_{e_r} \) denotes contraction by the vector field on \( X \) corresponding to the element \( e_r \).

We now return to the setup of §4.1 Thus we fix a finitely generated algebra \( A \), a finite dimensional vector space \( V \), and consider the scheme \( \text{Rep}(A, V) \).

The general linear group \( \text{GL}(V) \) acts naturally on the algebra \( \text{End} = \text{Hom}_\kappa(V, V) \) by inner automorphisms, via conjugation. Hence, given an algebra homomorphism \( \rho : A \to \text{End} \) and \( g \in \text{GL}(V) \), one may define a conjugate homomorphism \( g(\rho) : a \mapsto g \cdot \rho(a) \cdot g^{-1} \). The action \( \rho \mapsto g(\rho) \) makes \( \text{Rep}(A, V) \) a \( \text{GL}(V) \)-scheme.
Let $\mathfrak{g} \coloneqq \text{Lie GL}(V)$ be the Lie algebra of the algebraic group $\text{GL}(V)$. The action of $\text{GL}(V)$ on $\text{Rep}(A, V)$ induces a Lie algebra map

$$\text{act}_A : \mathfrak{g} \longrightarrow \mathfrak{T}(\text{Rep}(A, V)), \quad x \mapsto \text{act}_A(x). \quad (4.2.2)$$

Thus, one may consider $\Omega^\pi_q(\text{Rep}(A, V), \mathfrak{g})^\mathfrak{g}$, the corresponding $\mathfrak{g}$-equivariant algebraic de Rham complex.

Now, we have defined a contraction operation $i_\Delta : \text{DR}_t A \to \text{DR}_t A$, cf. §3.2. The main result of this section is the following

**Theorem 4.2.3.** There is a canonical morphism of complexes compatible with both differentials

$$(\text{DR}_t^* A, d, i_\Delta) \overset{\text{ev}_\pi}{\longrightarrow} (\Omega^\pi_q(\text{Rep}(A, V), \mathfrak{g})^\mathfrak{g}, d_{\text{DR}}, d_\mathfrak{g}).$$

We begin the proof with some general constructions.

### 4.3 Evaluation map on differential forms

Let $B$ be an algebra and $R \subset B$ a subalgebra. The algebra imbedding $R \hookrightarrow B$ induces, by transport of structure, a canonical morphism of schemes $\text{Rep}(B, V) \to \text{Rep}(R, V)$.

Observe next that giving an algebra homomorphism $\rho : k[t] \to \text{End}$ amounts to specifying an element $x = \rho(t) \in \text{End}$, that may be chosen arbitrarily. Thus, we have $\text{Rep}(k[t], V) = \text{End}$.

Similarly, for any algebra $A$, giving an algebra morphism $\rho : A \ast k[t] \to \text{End}$ amounts to giving a homomorphism $A \to \text{End}$ and an arbitrary additional element $x = \rho(t) \in \text{End}$. We see that $\text{Rep}(A_t, V) = \text{Rep}(A, V) \times \text{End}$. With this identification, the canonical morphism $\text{Rep}(A_t, V) \to \text{Rep}(k[t], V)$, induced by the algebra imbedding $k[t] \hookrightarrow A_t$, is nothing but the second projection $\pi : \text{Rep}(A, V) \times \text{End} \to \text{End}$.

We put $G \coloneqq \text{GL}(V)$. This group acts naturally on each of the schemes $\text{Rep}(A_t, V)$, $\text{Rep}(A, V)$, $\text{Rep}(k[t], V)$, and the projection $\pi$ is clearly $G$-equivariant. We will use shorthand notation

$$\text{Rep}_t := \text{Rep}(A_t, V) = \text{Rep}(A, V) \times \text{End}, \quad \text{and} \quad \text{Rep} := \text{Rep}(A, V).$$

Let $\Omega^\pi_\pi(\text{Rep}_t)$ be the DG algebra of relative (with respect to $\pi$) algebraic differential forms on the scheme $\text{Rep}_t$ (in the ordinary sense of commutative algebraic geometry). By definition, we have

$$\Omega^\pi_\pi(\text{Rep}_t) := \Lambda^\ast_{k[\text{Rep}_t]} \Omega^1_{\pi}(\text{Rep}_t) = \Omega^\ast(\text{Rep}) \otimes k[\text{End}]. \quad (4.3.1)$$

Generalizing the construction of §4.1, we now construct an evaluation map $\Omega^\ast_{k[t]}(A_t) \longrightarrow \text{End} \otimes \Omega^\pi_\pi(\text{Rep}_t)$, on relative differential forms as follows.
Thus, by the definition of the extended de Rham complex, \( \text{DR}_q^{(4.3.2)} \) may be viewed as a map \( A_t \to \text{End} \otimes \Omega^1 \text{(Rep}_t) \), which descends to a well-defined map \( d \circ \text{ev} : A_t/k[t] \to \text{End} \otimes \Omega^1 \text{(Rep}_t) \).

Now, given \( n = 0, 1, \ldots \), write \( m : \text{End}^{\otimes (n+1)} \to \text{End} \) for the \((n+1)\)-fold multiplication map in the algebra \( \text{End} \) and \( \text{Alt} : T^n(-) \to \Lambda^n(-) \) for the projection to anti-symmetric tensors. We form the following composite

\[
\begin{align*}
\Omega^n_{k[t]}(A_t) & \xrightarrow{\text{ev}} \text{End} \otimes \Omega^1 \text{(Rep}_t) = (\text{End} \otimes [\text{Rep}_t]) \otimes T^n_{k[t]}(A_t/k[t]) \xrightarrow{(\text{Id} \otimes k[\text{Rep}_t]) \otimes T^n_{k[t]}(\text{End} \otimes \Omega^1 \text{(Rep}_t))} \\
\text{End}^{\otimes n+1} \otimes T^n_{k[t]}(\Omega^1 \text{(Rep}_t)) & \xrightarrow{m \otimes \text{Alt}} \text{End} \otimes (\Lambda^n_{k[\text{Rep}_t]} \Omega^1 \text{(Rep}_t)) \\
& = \text{End} \otimes \Omega^n_{\pi}(\text{Rep}_t).
\end{align*}
\]

All these maps clearly commute with the de Rham differentials. Furthermore, any element in the image of the composite above is easily seen to be \( G \)-invariant with respect to the \( G \)-diagonal action on \( \text{End} \otimes \Omega^n_{\pi}(\text{Rep}_t) \). Thus, the composite yields a well-defined, canonical DG algebra map

\[ \text{ev}_\alpha : \Omega_t A \to (\text{End} \otimes \Omega^*_{\pi}(\text{Rep}_t))^G, \quad \alpha = a_0 a_1 \ldots a_n \mapsto \widehat{\alpha} = \widehat{a_0} \widehat{a_1} \ldots \widehat{a_n}. \]

Further, we have the linear function \( \text{Tr} : \text{End} \to k \), \( x \mapsto \text{Tr}(x) \). We form the following composite

\[ \begin{align*}
\Omega_t A & \xrightarrow{\text{ev}_\alpha} (\text{End} \otimes \Omega^*_{\pi}(\text{Rep}_t))^G \xrightarrow{\text{Tr} \otimes \text{Id}} (k \otimes \Omega^*_{\pi}(\text{Rep}_t))^G = \Omega^*_{\pi}(\text{Rep}_t)^G.
\end{align*} \]

The map \((\text{Id} \otimes \text{Tr}) \circ \text{ev}_\alpha\) vanishes on \([\Omega_t A, \Omega_t A] \subset \Omega_t A\), the (graded) commutator space, due to symmetry of the trace function. Therefore, this map descends to the commutator quotient to give a well-defined \( k \)-linear map of graded vector spaces

\[ (\text{Id} \otimes \text{Tr}) \circ \text{ev}_\alpha : R^*(\Omega_t A) \longrightarrow \Omega^*_{\pi}(\text{Rep}_t)^G, \quad \alpha \mapsto \widehat{\text{Tr} \alpha}. \quad (4.3.2) \]

By construction, this map commutes with the de Rham differentials.

We remark next that the Lie algebra \( \mathfrak{g} \) = \( \text{Lie GL}(V) \) is nothing but the associative algebra \( \text{End} \) viewed as a Lie algebra. Hence, using the isomorphisms in \([4.3.1]\), we can write

\[ \Omega^*_{\pi}(\text{Rep}_t) = \Omega^*(\text{Rep}) \otimes k[\text{End}] = \Omega^*(\text{Rep}) \otimes \mathfrak{g} = \Omega^*(\text{Rep}, \mathfrak{g}). \]

Thus, by the definition of the extended de Rham complex, \( \text{DR}_t^* A \), the map in \((4.3.2)\) may be viewed as a map

\[ \text{ev}_t := (\text{Id} \otimes \text{Tr}) \circ \text{ev}_\alpha : \text{DR}_t^* A \longrightarrow \Omega^*_{\pi}(\text{Rep}_t)^G = \Omega^*(\text{Rep}, \mathfrak{g})^G. \quad (4.3.3) \]
4.4 Proof of Theorem 4.2.3. We are going to show that the map (4.3.3) satisfies the requirements for the map ev in Theorem 4.2.3. Thus, proving the theorem amounts to showing commutativity of the following diagram

\[
\begin{array}{ccc}
DR_t A & \xrightarrow{(\text{Id} \otimes \text{Tr}) \circ ev_\Omega} & \Omega(\text{Rep}, g)^\theta \\
\downarrow i_\Delta & & \downarrow d_\theta \\
DR_t A & \xrightarrow{(\text{Id} \otimes \text{Tr}) \circ ev_\Omega} & \Omega(\text{Rep}, g)^\theta.
\end{array}
\] (4.4.1)

To this end, we need first to recall some generalities about the effect of the Representation functor on derivations, see e.g. [Gi, §12.4] for details.

Let \( B \) be an arbitrary finitely generated algebra. Each derivation \( \delta \in \text{Der} B \) gives rise, in a canonical way, to a vector field on the scheme \( \text{Rep}(B, V) \), that is, to a derivation \( \hat{\delta} : k[\text{Rep}(B, V)] \to k[\text{Rep}(B, V)] \). The assignment \( \delta \mapsto \hat{\delta} \) is a Lie algebra morphism \( \text{Der} B \to \text{Der} k[\text{Rep}(B, V)] \).

This morphism is compatible with the Representation functor in the sense that one has

\[
\hat{\delta}(\text{Tr} \circ ev(b)) = \text{Tr} \circ ev(\delta(b)), \quad \forall b \in B, \, \delta \in \text{Der} B.
\]

Let \( \rho \in \text{Rep}(B, V) \) be a (closed) point. We write \( \hat{\delta}|_\rho \in T_\rho \text{Rep}(B, V) \) for the value of the vector field \( \hat{\delta} \) at the point \( \rho \); thus, \( \hat{\delta}|_\rho \) is an element of the Zariski tangent space to the scheme \( \text{Rep}(B, V) \) at \( \rho \). Similarly, for any \( x \in g = \text{Lie} \text{GL}(V) \), let \( \text{act}_B(x)|_\rho \) denote the value at \( \rho \) of the vector field \( \text{act}_B(x) \), see (4.2.2).

Now, let \( b \in B \) and set \( \delta = \text{ad} b \), an inner derivation. Then, according to [Gi, Example 12.4.1], we have

\[
\hat{\text{ad}} b|_\rho = \text{act}_B(\rho(b))|_\rho, \quad \forall \rho \in \text{Rep}(B, V),
\] (4.4.2)

where \( \rho(b) \in \text{End} \) is viewed as an element of \( g = \text{Lie} \text{GL}(V) = \text{End} \).

**Proof of commutativity of diagram (4.4.1).** We will use the identification \( \text{Rep}(A_t, V) = \text{Rep}(A, V) \times \text{End} = \text{Rep} \times g \) to write an element of \( \text{Rep}(A_t, V) \) as a pair \( (\rho, x) \in \text{Rep} \times g \). The action of \( g \) on \( \text{Rep}(A_t, V) \) corresponds to the \( g \)-diagonal action on \( \text{Rep} \times g \), where the Lie algebra \( g \) acts on the second factor, \( g \), via the adjoint action. Thus, for \( x \in g \), the corresponding vector field on \( \text{Rep}(A_t, V) \) splits into a sum \( \text{act}_A(x) = \text{act}_A(x) \oplus \text{ad}_g x \), of the vector field \( \text{act}_A(x) \) on \( \text{Rep} \) and of the adjoint vector field \( \text{ad}_g x \) on \( g \) defined by the formula \( \text{ad}_g x|_y = [x, y] \), for any \( y \in g \).

We apply formula (4.4.2) to the algebra \( B = A_t \) and the element \( b = t \). Recall that according to §3.3 for the derivation \( \Delta_t : A_t \to A_t \), associated
with the distinguished double derivation $\Delta : A \to A \otimes A$, we have $\Delta_t = \text{ad} t$. Thus, for any point $(\rho, x) \in \text{Rep} \times g$, formula (4.4.2) yields

$$\tilde{\Delta}_t|_{(\rho, x)} = \text{ad} t|_{(\rho, x)} = \text{act}_A(x)|_{(\rho, x)}$$

$$= (\text{act}_A(x)|_{\rho}) \oplus (\text{ad}_g x|_{x}) = (\text{act}_A(x)|_{\rho}) \oplus 0,$$

where in the last equality we have used that $\text{ad}_g x|_x = [x, x] = 0$.

We observe that the vector field on the right of the second line in (4.4.3) is vertical, i.e., is tangent to the fibers of the second projection $\pi : \text{Rep} \times g \to g$. For any vertical vector field $\xi$ on $\text{Rep}$ and any differential form $\omega \in \Omega^q(g)$, we clearly have $i_\xi(\pi^* \omega) = 0$. It follows in particular that, for $\xi = \tilde{\Delta}_t$, the corresponding contraction operation $i_\xi$ descends to a well defined operation on the space $\Omega^*_\pi(\text{Rep}_t)$ of relative differential forms.

To complete the proof we must show that $i_{\tilde{\Delta}_t} = d_g$, as maps $\Omega(\text{Rep}, g)^0 \to \Omega(\text{Rep}, g)^0$. To see this, choose dual bases $\{e_r\}$ and $\{e^r\}$ of the vector spaces $g$ and $g^*$, respectively. Thus, for any $x \in g$, one has $x = \sum_r e^r(x) \cdot e_r$. Hence, we can write $\text{act}_A(x) = \sum_r e^r(x) \cdot \text{act}_A(e_r)$. Therefore, for any differential form $\omega \in \Omega^*(\text{Rep}_t)$, using formula (4.4.3) and the definition of the differential $d_g$, we find

$$(i_{\tilde{\Delta}_t} \omega)|_{(\rho, x)} = (i_{\text{act}_A(x)} \omega)|_{\rho} = \sum_r e^r(x) \cdot (i_{\text{act}_A(e_r)} \omega)|_{\rho} = (d_g \omega)|_{(\rho, x)}.$$

This completes the proof of commutativity of diagram (4.4.1), hence, the proof of the theorem.

5 Applications to Hochschild and Cyclic homology

5.1 Hochschild homology. Let $A$ be a unital $k$-algebra. Given an $A$-bimodule $M$, we let $H_k(A, M)$ denote the $k$-th Hochschild homology group of $A$ with coefficients in $M$. Also, write $[A, M] \subset M$ for the $k$-linear span of the set $\{am - ma \mid a \in A, m \in M\}$. Thus, $[A, M]$ is a vector subspace, and $H_0(A, M) = M/[A, M]$.

It has been shown by Cuntz-Quillen that, in the special case $M = A$, the Hochschild homology groups $H_*(A, A)$ may be computed using noncommutative differential forms. Adapting some of the arguments from [CQ2], we will establish our first result.

Theorem 5.1.1. For any unital $k$-algebra $A$, there is a natural graded space isomorphism

$$H_*(A, A) \cong \text{Ker}[i_\Delta : \text{DR}^* A \to \Omega^{*-1} A].$$
As will be explained later (see discussion after Proposition 5.3.1), this theorem is an easy consequence of Proposition 5.3.1; the latter proposition will be itself proved in Sect. 6.1 below.

To put Theorem 5.1.1 in context, following Cuntz and Quillen, consider a complex \[ \cdots \to \Omega^2 A \xrightarrow{b} \Omega^1 A \xrightarrow{b} \Omega^0 A \to 0 \], where the differential \( b \) is given by the formula
\[ b : \alpha da \mapsto (-1)^n[a, \alpha], \quad \forall \alpha \in \Omega^n A, \; n > 0, \; a \in A/k \] (5.1.2)

It has been explained in [CQ2] that this complex is isomorphic naturally to the Hochschild complex for the algebra \( A \). Therefore, the cohomology groups \( H^q(\Omega A, b) \), of the above complex, may be identified with Hochschild homology groups \( H_q(A, A) \).

5.2 Cyclic homology. We introduce a complex of reduced differential forms by setting \( \Omega^0 := \Omega^0 A/k = A/k \), and \( \Omega^k := \Omega^k A \), for all \( k > 0 \). Let \( \Omega^* := \bigoplus_{k \geq 0} \Omega^k \).

Further, set \( \Omega[t] := \Omega^* \otimes k[t] \), viewed as a graded vector space with respect to the tensor product grading where the variable \( t \) is assigned grade degree 2, as usual.

The Hochschild differential induces a \( k[t] \)-linear differential \( b : \Omega[t] \to \Omega[t] \) of degree \((-1)\). One also has Connes’ differential \( B : \Omega^* \to \Omega^{*+1} \), and we extend it to a \( k[t] \)-linear differential \( \overline{\Omega[t]} \to \overline{\Omega[t]} \) of degree \(+1\). It is known that \( B^2 = b^2 = 0 \) and \( B \circ b + b \circ B = 0 \). Thus, the map \( B + t \cdot b : \Omega[t] \to \Omega[t] \) gives a degree \(+1\) differential on \( \Omega[t] \).

The reduced cyclic homology of \( A \), to be denoted \( \overline{HC}_q(A) \), is defined as homology of the complex \( (\Omega[t], B + t \cdot b) \).

On the other hand, we have the reduced contraction \( i_\Delta : \text{DR}^* A \to \Omega^{*-1} A \). We compose it with the natural projection \( \Omega^* A \to \text{DR}^* A \) to obtain a map \( \Omega^* A \to \Omega^{*-1} A \). The latter map descends to a map \( \Omega^* \to \Omega^{*-1} \). Further, we may extend this last map, resp., de Rham differential \( d : \Omega^* \to \Omega^{*-1} \), to \( k[t] \)-linear maps \( \overline{\Omega[t]} \to \overline{\Omega[t]} \) of degree \(-1\), resp., \(+1\), to be denoted by \( i_\Delta \), resp. \( d \), again. It follows from Lemma 3.4.1(i) and the Cartan formula, cf. also (3.2.5), that one has
\[ i_\Delta \circ d + d \circ i_\Delta = 0, \quad d^2 = (i_\Delta)^2 = 0 \quad \text{on } \text{DR}_t A \] (5.2.1)

Thus, the map \( d + t \cdot i_\Delta \) gives a degree 1 differential on \( \overline{\Omega[t]} \) that may be thought of as some sort of equivariant differential for the ‘vector field’ \( \Delta \).

The following theorem, which is the main result of this paper, shows the importance of the reduced contraction map \( i_\Delta \) for Cyclic homology.
Theorem 5.2.2. The homology of the complex $(\Omega[t], d + t \cdot \iota_\Delta)$ is isomorphic to $HC_*(A)$, the reduced cyclic homology of $A$.

The proof of the theorem is based on various results about the Karoubi operator to be discussed first. Theorem 5.2.2 will be proved after that, in §6.3 below.

5.3 The Karoubi operator. For any algebra $A$ and an $A$-bimodule $M$, we put $M^\# := M/[A, M] = H_0(A, M)$. Now, let $A \to B$ be an algebra homomorphism. Then, $B$ may be viewed as an $A$-bimodule, and one has a canonical projection $B^\# = B/[A, B] \to RB = [B, B]$. In particular, for $B = \Omega^* A$, we get a natural projection $(\Omega^* A)^\# \to DR^* A$ which is not an isomorphism, in general.

Following Cuntz-Quillen [CQ2], we consider a diagram

$$\begin{array}{cccccc}
\Omega^0 A & \xrightarrow{d} & \Omega^1 A & \xrightarrow{d} & \Omega^2 A & \xrightarrow{d} & \cdots
\end{array}$$

where the Hochschild differential $b$ has been defined in (5.1.2).

The maps $d$ and $b$ are related via an important Karoubi operator $\kappa : \Omega^* A \to \Omega^* A$, defined by the formula $\kappa : \alpha da \mapsto (-1)^{\deg \alpha} da \alpha$ if $\deg \alpha > 0$, and $\kappa(\alpha) = \alpha$ if $\alpha \in \Omega^0 A$. One has, see [CQ1],

$$b \circ d + d \circ b = \text{Id} - \kappa.$$

It follows that $\kappa$ commutes with both $d$ and $b$. Further, it is easy to verify, cf. [CQ1] and the proof of Lemma 5.3.2 below, that the Karoubi operator descends to a well-defined map $\kappa : (\Omega^n A)^\# \to (\Omega^n A)^\#, which is essentially a cyclic permutation; specifically, we have

$$\kappa(\alpha_1 \alpha_2 \ldots \alpha_{n-1} \alpha_n) = (-1)^{n-1} \alpha_n \alpha_1 \alpha_2 \ldots \alpha_{n-1}, \quad \forall \alpha_1, \ldots, \alpha_n \in \Omega^1 A.$$

Proposition 5.3.1. For any $n \geq 1$, we have an equality

$$\iota_\Delta = (1 + \kappa + \kappa^2 + \ldots + \kappa^{n-1}) \circ b \quad \text{as maps } \Omega^n A \to \Omega^{n-1} A.$$

Furthermore, the map $\iota_\Delta$ fits into a canonical short exact sequence

$$0 \to H^n(\Omega A, b) \to DR^n A \xrightarrow{\iota_\Delta} [A, \Omega^{n-1} A]^{\kappa} \to 0.$$

Here and below, $(-)^\kappa$ denotes taking $\kappa$-invariants, in particular, we write $(\Omega^* A)^\# := ((\Omega^* A)^\#)^\kappa \subset (\Omega^* A)^\#$. 

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We recall that the cohomology group $H^n(\Omega A, b)$ that occurs in the above displayed short exact sequence is isomorphic, as has been mentioned in §5.1, to the Hochschild homology $H_n(A, A)$. Thus, Theorem 5.1.1 is an immediate consequence of the short exact sequence of the Proposition.

The following result, which was implicit in [CQ2], [Lo], will play an important role in subsequent sections.

**Lemma 5.3.2.**

(i) The projection $(\Omega A)^\kappa_2 \rightarrow DR^1 A$ restricts to a bijection $(\Omega A)^\kappa_2 \rightarrow DR^1 A$.

(ii) The map $b$ descends to a map $b_\kappa : (\Omega A)^\kappa_2 \rightarrow \Omega^{-1} A$.

(iii) The kernel of the map $b_\kappa : (\Omega A)^\kappa_2 \rightarrow \Omega^{-1} A$, the restriction of $b_\kappa$ to the space of $\kappa$-invariants, is isomorphic to $H^n(\Omega A, b)$.

Both Proposition 5.3.1 and Lemma 5.3.2 will be proved later, in §6.1.

**5.4 Special case:** $H_1(A, A)$. Note that the map $\kappa$ acts trivially on $\Omega^0 A \cong A$. Therefore, on 1-forms, we have $\iota_\Delta = b$. Thus, using the identification $H_1(A, A) = H^1(\Omega A, b)$, the short exact sequence of Proposition 5.3.1 reads

$$0 \rightarrow H_1(A, A) \rightarrow DR^1 A \xrightarrow{b=\iota_\Delta} [A, A] \rightarrow 0. \quad (5.4.1)$$

The short exact sequence (5.4.1) may be obtained in an alternate way as follows. We apply the right exact functor $(-)_2$ to the fundamental short exact sequence $0 \rightarrow \Omega^1 A \rightarrow A \otimes A \rightarrow A \rightarrow 0$, see (2.3.1), of $A$-bimodules. The corresponding long exact sequence of Tor-groups reads

$$\ldots \rightarrow H_1(A, A \otimes A) \rightarrow H_1(A, A) \rightarrow (\Omega^1 A)_2 \rightarrow (A \otimes A)_2 \xrightarrow{\iota_\Delta} A_2 \rightarrow 0.$$ 

Now, by definition of Tor, one has $H_k(A, A \otimes A) = 0$ for all $k > 0$. Also, one has natural identifications $(\Omega^1 A)_2 = DR^1 A$, and $(A \otimes A)_2 \cong A$. This way, the map $c$ on the right of the displayed formula may be identified with the natural projection $A \rightarrow A/[A, A]$. Thus, Ker($c$) = $[A, A]$, and the long exact sequence above reduces to the short exact sequence (5.4.1).

Further, it is immediate from definitions that map $b = \iota_\Delta$ in (5.4.1) is given by the formula

$$udv \mapsto [u, v], \text{ hence } (DR^1 A)_{\text{exact}} \subset \text{Ker}(\iota_\Delta) = H_1(A, A).$$

This map $udv \mapsto [u, v]$ has been considered earlier by Cuntz and Quillen, see [CQ1].

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5.5 An application. In this section we use Theorem 5.1.1 in order to prove the following

**Proposition 5.5.1.** Let $A$ be an algebra with $H_2(A, A) = 0$ and such that the following sequence is exact (in which case $A$ may be called ‘connected’)

$$0 \rightarrow \mathbb{k} \rightarrow DR^0 A \xrightarrow{d} DR^1 A.$$  \hfill (5.5.2)

Then, one has

- $H_1(A, A) = (DR^1 A)_{\text{closed}} = (DR^1 A)_{\text{exact}}$.
- There is a natural vector space isomorphism $(DR^2 A)_{\text{closed}} \xrightarrow{\sim} [A, A]$.

**Proof.** We will freely use the notation of [CBEG, §4.1].

According to [CBEG, Proposition 4.1.4], for any algebra $A$ such that the sequence (5.5.2) is exact, one has the following commutative diagram

$$
\begin{array}{ccc}
DR^1 A & \xrightarrow{d} & (DR^2 A)_{\text{closed}} \\
\downarrow{\iota_\Delta} & & \downarrow{\iota_\Delta} \\
[A, A] & \xrightarrow{d} & [A, \Omega^1 A].
\end{array}
$$

(5.5.3)

In this diagram, the map $\tilde{\mu}_{\text{nc}}$ is a lift of the noncommutative moment map that has been introduced in [CBEG].

Assuming that $H^H_2(A) = 0$, we deduce from the short exact sequence of Proposition 5.3.1 for $n = 2$ that the map $\iota_\Delta : DR^2 A \rightarrow [A, \Omega^1 A]$ is injective.

We now exploit diagram (5.5.3). The map $d$ in the bottom row of the diagram is injective due to acyclicity of the de Rham differential on $\Omega^* A$, cf. eg. formula (2.5.1) in [CBEG]. Further, the left vertical map $\iota_\Delta$ in the diagram is surjective by (5.4.1). Therefore, using commutativity of diagram (5.5.3) we deduce by diagram chase that the upper horizontal map $d$ must be surjective, and also the map $\tilde{\mu}_{\text{nc}}$ must be bijective.

This yields both statements of Proposition 5.5.1. \hfill \square

A version of Proposition 5.5.1 applies in the case where $A$ is the path algebra of a quiver with $r$ vertices. In that case, one has to consider algebras over a ground ring $R := \mathbb{k} \oplus \ldots \oplus \mathbb{k}$ ($r$ copies) rather than over the base field $\mathbb{k}$. The corresponding formalism has been worked out in [CBEG].

Write $DR^*_R A$ for the relative de Rham complex of $A$ and let $[A, \Omega^1_R A]^R$ denote the vector space formed by those elements of $[A, \Omega^1_R A]$ which commute with $R$.

An analogue of Proposition 5.5.1 implies the following result.
Corollary 5.5.4. Let $A$ be the path algebra of a quiver. Then there is a natural vector space isomorphism $(\text{DR}_R^2 A)_{\text{closed}} \sim [A, A]^R$. □

6 Proofs

6.1 Proof of Lemma 5.3.2 and Proposition 5.3.1. Our proof of Lemma 5.3.2 follows the proof of Lemma 2.6.8 in [Lo]. Write $\Omega^n := \Omega^n A$ and $\Omega := \bigoplus_n \Omega^n$.

One observes from definitions that $[A, \Omega] = b\Omega$ and $[dA, \Omega] = (\text{Id} - \kappa)\Omega$. Hence, we obtain, cf. [CQ1]:

$$[\Omega, \Omega] = [A, \Omega] + [dA, \Omega] = b\Omega + (\text{Id} - \kappa)\Omega.$$  

We deduce that $\Omega^\ast := \Omega^n / [\Omega, \Omega]$ and $\text{DR}^* A = \Omega^n / (\Omega, \Omega) = \Omega^\ast$. It follows in particular, since $b^2 = 0$, that $b$ descends to a well defined map $b : \Omega^\ast = \Omega^n / b\Omega \to \Omega$.

Further, one has the following standard identities, see [CQ2, §2]

$$\kappa^n - \text{Id} = b \circ \kappa^n \circ d, \quad \kappa^{n+1} \circ d = d \quad \text{hold on } \Omega^n, \quad \forall n = 1, 2, \ldots \quad (6.1.1)$$

The Karoubi operator $\kappa$ commutes with $b$, hence induces a well-defined endomorphism of the vector space $\Omega^n / b\Omega^n, \quad n = 1, 2, \ldots$. Furthermore, from the first identity in (6.1.1) we see that $\kappa^n = \text{Id}$ on $\Omega^n / b\Omega^n$. Hence, we have a direct sum decomposition $\Omega^\ast = (\Omega^\ast)^\kappa \oplus (\text{Id} - \kappa)\Omega^\ast$. It follows that the natural projection $\Omega^\ast = \Omega^n / b\Omega \to \text{DR}^* A = \Omega^\ast / (\text{Id} - \kappa)\Omega^\ast$ restricts to an isomorphism $(\Omega^\ast)^\kappa \to \text{DR}^* A$.

Parts (ii) and (iii) of Lemma 5.3.2 are clear from the proof of Lemma 2.6.8 in [Lo]. □

Proof of Proposition 5.3.1. The equality $\iota_{\Delta} = b \circ (1 + \kappa + \kappa^2 + \ldots + \kappa^{n-1})$, in the statement of the Proposition, is immediate from the formula of Lemma 5.3.2(ii).

To prove the second statement we exploit the first identity in (6.1.1). Using the formula for $\iota_{\Delta}$ and the fact that $b$ commutes with $\kappa$, we compute

$$(\kappa - 1) \circ \iota_{\Delta} = b \circ (\kappa - 1) \circ (1 + \kappa + \kappa^2 + \ldots + \kappa^{n-1}) = b \circ (\kappa^n - 1) = b^2 \circ \kappa^n \circ d = 0.$$  

Hence, we deduce that the image of $\iota_{\Delta}$ is contained in $(b\Omega)^\kappa$. Conversely, given any element $\alpha = b(\beta) \in (b\Omega)^\kappa$, we find

$$\iota_{\Delta}(\beta) = (1 + \kappa + \kappa^2 + \ldots + \kappa^{n-1}) \circ b(\beta) = n \cdot b(\beta) = n \cdot \alpha.$$
Thus, we have \( \text{Im}(\iota_{\Delta}) = (b\Omega)^\kappa = ([A, \Omega])^\kappa \), since \( b\Omega = [A, \Omega] \). Furthermore, it is clear that the two maps \((1 + \kappa + \kappa^2 + \ldots + \kappa^{n-1}) \circ b \) and \( b \) coincide on \((\Omega^*)_b^\kappa \), hence, have the same kernel. The exact sequence of the Proposition now follows from Lemma 6.3.2.

6.2 Harmonic decomposition. Our proof of Theorem 5.2.2 is an adaptation of the strategy used in [CQ2] §2, based on a Harmonic decomposition

\[
\overline{\Omega} = \overline{P\Omega} \oplus \overline{P^\perp \Omega}, \quad \text{where} \quad \overline{P\Omega} := \text{Ker}(\text{Id} - \kappa^2), \quad P^\perp \overline{\Omega} := \text{Im}(\text{Id} - \kappa^2).
\]

The differentials \( B, b, \) and \( d \) commute with \( \kappa \), hence preserve Harmonic decomposition. Moreover, the differentials \( B \) and \( d \) are known to be proportional on \( \overline{P\Omega} \). Specifically, exploiting the second identity in (6.1.1), it has been shown in [CQ2], formula (11), that on \( \overline{P\Omega}^n \) one has

\[
B = (n + 1) \cdot d \quad \text{on} \quad \overline{P\Omega}^n, \quad \forall n = 1, 2, \ldots \quad (6.2.1)
\]

Next, we claim that

(i) \( \iota_{\Delta}|_{\overline{P^\perp \Omega}} = 0 \), and (ii) \( \iota_{\Delta} = n \cdot b \) on \( \overline{P\Omega}^n \), \( \forall n = 1, 2, \ldots \) (6.2.2)

Claim (i) follows from Proposition 5.3.1 since \( \text{Im}(\iota_{\Delta}) \subset \text{Ker}(\text{Id} - \kappa) \subset \text{Ker}(\text{Id} - \kappa^2) = \overline{P\Omega} \).

To prove (ii), let \( \alpha \in \overline{P\Omega} \). From the first identity in (6.1.1), we get \( \alpha - \kappa^n(\alpha) \in b\overline{\Omega} \). Hence, we find \( b\alpha - \kappa^n( b\alpha) \in b^2\overline{\Omega} = 0 \), since \( b^2 = 0 \). Thus, the operator \( \kappa \) has finite order on \( b\overline{\Omega} \), hence, also on \( b(\overline{P\Omega}) \). But, for any operator \( T \) of finite order, one has \( \text{Ker}(\text{Id} - T) = \text{Ker}((\text{Id} - T)^2) \). It follows that if \( \alpha \in \overline{P\Omega}^n \), then we have \( b\alpha \in \text{Ker}((\text{Id} - \kappa)(\text{Id} - \kappa)^2) = \text{Ker}(\text{Id} - \kappa) \). We conclude that the element \( b\alpha \) is fixed by \( \kappa \). Therefore, by Proposition 5.3.1 we obtain

\[
\iota_{\Delta}(\alpha) = (1 + \kappa + \kappa^2 + \ldots + \kappa^{n-1}) \circ b(\alpha) = n \cdot b(\alpha),
\]

and (6.2.2) is proved.

6.3 Proof of Theorem 5.2.2. Harmonic decomposition being stable under all four differentials \( B, b, d, \) and \( \iota_{\Delta} \), we may analyze the homology of each of the direct summands, \( \overline{P\Omega} \) and \( P^\perp \overline{\Omega} \), separately.

First of all, it has been shown by Cuntz-Quillen, see Proposition 4.1(1) in [CQ2], that the complex \((P^\perp \overline{\Omega}[t], B + t \cdot b)\) is acyclic.
Further, the complex \((\Omega, d)\) being acyclic, cf. \([CQ]^2 \S 1\) or \([CHBC\)] formula (2.5.1), we deduce

\[\text{Each of the complexes } (P^{\Omega}, d) \text{ and } (P^{\perp \Omega}, d) \text{ is acyclic.} \quad (6.3.1)\]

Now, the map \(i_{\Delta}\) vanishes on \(P^{\perp \Omega}\) by (6.2.2)(i). Hence, on \(P^{\perp \Omega}[t]\) we have \(d + t \cdot i_{\Delta} = d\). Therefore, we conclude using (6.3.1) that \((P^{\perp \Omega}[t], d)\), hence, \((P^{\perp \Omega}[t], d + t \cdot i_{\Delta})\) is an acyclic complex.

Thus, to complete the proof of the theorem, it suffices to construct an isomorphism of complexes \(F^{q}: (P^{\Omega}[t], d) \sim \rightarrow (P^{\Omega}[t], B + t \cdot b)\). Such an isomorphism is provided by the assignment \(F^{q}(n) : \alpha \mapsto \alpha \cdot n!, \forall \alpha \in (P^{\Omega})^{n}\), since for any \(n = 0, 1, \ldots\) we clearly have \(B \circ F^{q}(n) = F^{q}(n+1) \circ d, \quad \text{and } b \circ F^{q}(n) = F^{q}(n-1) \circ i_{\Delta} \).

\[\blacksquare\]

7 Miscellaneous results

7.1 First order deformations based of free products. Below, we outline a new version of deformation theory of associative algebras, based on the free product construction \(A \ast \kappa [t]\). We are going to study first order deformations of an associative algebra \(A\), in which the deformation parameter \(t\) is not assumed to be central.

Motivated by the classical approach, we first identify the vector space \(A_t/(A_t^+)^2\) with \(A \oplus (A \otimes A)\), using \((1.4.2)\). Thus, we are interested in associative products on the vector space \(A \oplus (A \otimes A)\) that have the following form

\[\left(u \oplus (u' \otimes u'') \right) \times \left(v \oplus (v' \otimes v'') \right) \xrightarrow{\beta} uv \oplus (u' \otimes u'' v + uv' \otimes v'' + \beta(u, v)), \quad (7.1.1)\]

where \(\beta: A \times A \rightarrow A \otimes A\) is a certain \(\kappa\)-bilinear map.

These products are taken up to an equivalence. Specifically, for any \(\kappa\)-linear map \(f: A \rightarrow A \otimes A\), one defines a linear bijection

\[\tilde{f}: A \oplus (A \otimes A) \rightarrow A \oplus (A \otimes A), \quad u \oplus (u' \otimes u'') \mapsto u \oplus (u' \otimes u'' + f(u)).\]

Given a product \(\ast_{\beta}\) and a map \(f\), one defines a new product by transporting the structure via \(\tilde{f}\), that is, by the formula \(x \ast_{\beta} y := f^{-1}(\tilde{f}(x) \ast_{\beta} \tilde{f}(y))\). We say that the products \(\ast_{\gamma}\) and \(\ast_{\beta}\) are equivalent.

Similarly to conventional theory, a classification of equivalence classes of associative products may be given in terms of Hochschild cohomology.
Proposition 7.2.1. A first product $\ast_\beta$ as in (7.1.1) is associative $\iff \beta \in C^2(A, A \otimes A)$ is a Hochschild 2-cocycle with coefficients in $A \otimes A$.

• Two products $\ast_\beta$ and $\ast_\gamma$ corresponding to 2-cochains $\beta$ and $\gamma$ are equivalent $\iff \beta - \gamma$ is a Hochschild coboundary.

Thus, we deduce

**Proposition 7.1.2.** Equivalence classes of associative products, as in (7.1.1), are in one-to-one correspondence with the elements of $H^2(A, A \otimes A)$, the second Hochschild cohomology group of the $A$-bimodule $A \otimes A$. $\square$

### 7.2 Another construction of the extended de Rham complex.

Let $A$ be an algebra and $M$ an $A$-bimodule. We consider $A^{\otimes 2} \oplus M$ as an $A$-bimodule and introduce an $A$-bimodule map $\pi : A^{\otimes 2} \oplus M \to A$, $(a' \otimes a'') \oplus u \mapsto a'a''$.

Further, we equip the bimodule $A^{\otimes 2} \oplus M$ with a grading in two different ways. The first grading assigns $A^{\otimes 2}$ grade degree 2, while the second grading assigns $A^{\otimes 2}$ grade degree $-1$. In both cases, the direct summand $M$ is placed in degree 0. These two gradings on $A^{\otimes 2} \oplus M$ induce the corresponding gradings on the tensor algebra $T_A(A^{\otimes 2} \oplus M)$, to be denoted $T^\prime_A(A^{\otimes 2} \oplus M)$ and $T^\tau_A(A^{\otimes 2} \oplus M)$, respectively.

The following result is a generalization of Proposition 2.2.5.

**Proposition 7.2.1.** (i) For any $A$-bimodule $M$, the map $(a' \otimes a'') \oplus m \mapsto a'ta'' + m$ can be uniquely extended, by multiplicativity, to a graded algebra isomorphism $T^\prime_A(A^{\otimes 2} \oplus M) \to (T_A M)_t$, resp., $T^\tau_A(A^{\otimes 2} \oplus M) \to (T_A M)_\tau$.

(ii) The differential $\frac{d}{dt} : (T_A M)_\tau \to (T_A M)_\tau$, cf. (2.2.5). is transported under the isomorphism in (i) to a degree 1 map $\partial : T^\tau_A(A^{\otimes 2} \oplus M) \to T^\tau_A(A^{\otimes 2} \oplus M)$ given, for any homogeneous $x_1, \ldots, x_n \in A^{\otimes 2} \oplus M$, by

$$
\partial(x_1 \otimes \ldots \otimes x_n) = \sum_{k=1}^{n-1} (-1)^{|x_1| \ldots x_{k-1}|} x_1 \otimes \ldots \otimes x_{k-1} \otimes \pi(x_k)x_{k+1} \otimes \ldots \otimes x_n,
$$

where $|x_1, \ldots, x_{k-1}| := k - 1 + \deg x_1 + \ldots + \deg x_{k-1}$. (7.2.2)

**Proof.** The assignment $(a' \otimes a'') \oplus m \mapsto a'ta'' + m$, clearly gives an $A$-bimodule map $A^{\otimes 2} \oplus M \to (T_A M)_t$. This map can be extended, by the universal properties, to an algebra homomorphism $T_A(A^{\otimes 2} \oplus M) \to (T_A M)_t$. To show that this homomorphism is an isomorphism, we construct an inverse map as follows.
The algebra \((T_A M)_t = (T_A M) \ast \mathbb{k}[t]\) is freely generated by the element \(t\) and the subalgebra \(T_A M \subset (T_A M)_t\). Therefore, constructing an algebra homomorphism \((T_A M)_t \to T_A (A^\otimes 2 \oplus M)\) amounts to giving an algebra homomorphism \(f : T_A M \to T_A (A^\otimes 2 \oplus M)\) and an additional element \(f(t) \in T_A (A^\otimes 2 \oplus M)\), chosen arbitrarily. Thus, we may define the homomorphism \(f : T_A M \to T_A (A^\otimes 2 \oplus M)\) to be the natural algebra imbedding \(T_A M \hookrightarrow T_A (A^\otimes 2 \oplus M)\), induced by the \(A\)-bimodule imbedding \(M = 0 \oplus M \hookrightarrow A^\otimes 2 \oplus M\). Further, we set \(f(t) := 1_A \otimes 1_A \in A^\otimes 2 \subset T_A (A^\otimes 2 \oplus M)\). It is straightforward to check that the resulting homomorphism \((T_A M)_t \to T_A (A^\otimes 2 \oplus M)\) is indeed an inverse of the homomorphism in the opposite direction constructed in the first paragraph of the proof.

This completes the proof of part (i) (compatibility with the corresponding gradings is clear). Part (ii) is proved by a straightforward computation. \(\square\)

In the special case \(M = 0\), part (i) of the above Proposition is due to Van den Bergh (private communication). The differential \(\partial\) in (7.2.2) is a special case of the generalized bar differential introduced by B. Tsygan [T].

Applying Proposition 7.2.1 in the special case \(M = \Omega^1 A\), we obtain

**Corollary 7.2.3.** There is a canonical algebra isomorphism

\[\Omega_t A = (\Omega A)_t \cong T_A^t (A^\otimes 2 \oplus \Omega^1 A).\]

For any double derivation \(\Theta \in \text{Der} A\), the degree \(-1\) graded derivation \(T_A^t (A^\otimes 2 \oplus \Omega^1 A) \to T_A^t (A^\otimes 2 \oplus \Omega^1 A)\), corresponding to the contraction operation \(i_\Theta : \Omega_t A \to \Omega_t A\), is induced by the following \(A\)-bimodule map

\[A^\otimes 2 \oplus \Omega^1 A \to A^\otimes 2 \oplus \Omega^1 A, \quad (a' \otimes a'') \oplus \alpha \mapsto (i'_\Theta \alpha \otimes i''_\Theta \alpha) \oplus 0.\]

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