ON THE $K$-THEORY OF PULLBACKS

MARKUS LAND AND GEORG TAMME

Abstract. To any pullback square of $E_1$-ring spectra, for example a Milnor square of discrete rings, we associate a pullback square of $K$-theory spectra in which one corner is the $K$-theory of a new $E_1$-ring naturally associated to the original pullback square. The proof is categorical and applies for any localizing invariant in place of $K$-theory.

As immediate consequences we obtain an improved version of Suslin’s excision result in $K$-theory, generalizations of results of Geisser and Hesselholt on torsion in (bi)relative $K$-groups, and a generalized version of pro-excision for $K$-theory. Furthermore, we show that any truncating invariant satisfies excision, nilinvariance, and cdh-descent. Examples of truncating invariants include the fibre of the cyclotomic trace, the fibre of the rational Goodwillie–Jones Chern character, and periodic cyclic homology in characteristic zero.

Various of the results we obtain have been known previously, though most of them in weaker forms and with less direct proofs.

Introduction

The classical excision theorem in algebraic $K$-theory due to Milnor, Bass, and Murthy says that any Milnor square of rings

$$
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
A' & \longrightarrow & B'
\end{array}
$$

i.e. a pullback square of rings with $B \to B'$ surjective, gives rise to a long exact sequence of $K$-groups (the excision sequence)

$$(1) \quad K_1(A) \longrightarrow K_1(A') \oplus K_1(B) \longrightarrow K_1(B') \longrightarrow K_0(A) \longrightarrow K_0(A') \oplus K_0(B) \longrightarrow \ldots$$

starting at $K_1(A)$, see [Bas68, Theorem XII.8.3]. This sequence is extremely useful for computations of negative $K$-groups. Unfortunately, for a long time it seemed to be impossible to extend this sequence to the left in a natural way, which made computations of higher $K$-groups much harder. For instance, Swan proved that there is no functor $'K_2'$ from rings to abelian groups such that the exact sequence $'(1)$ could be extended to an exact sequence

$$(2) \quad \ldots \longrightarrow K_i(A) \longrightarrow K_i(A') \oplus K_i(B) \longrightarrow K_i(B') \longrightarrow K_{i-1}(A) \longrightarrow \ldots \quad (i \in \mathbb{Z})$$

see [Swan] Corollary 1.2]. Our main insight is that to any Milnor square (1) one can in fact functorially associate a connective ring spectrum $A' \otimes_B^L B$ together with natural maps from $A'$ and $B$ and to $B'$ for which there is a long exact sequence

This sequence coincides with the sequence (1) in degrees $\leq 1$.

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In fact, our main result is more general. Before we formulate it, we remark two things. Firstly, a Milnor square is a particular example of a pullback square of $\mathbb{E}_1$-ring spectra. Secondly, non-connective algebraic $K$-theory is a particular example of a localizing invariant. A localizing invariant $E$ gives rise to a functor on $\mathbb{E}_1$-ring spectra by evaluating $E$ on the $\mathbb{E}_1$-category of perfect modules: $E(A) = E(\text{Perf}(A))$. Our main result is the following.

**Main Theorem.** Assume that (□) is a pullback square of $\mathbb{E}_1$-ring spectra. Associated to this square there exists a natural $\mathbb{E}_1$-ring spectrum $A' \otimes_B B'$ sitting in a natural commutative diagram of $\mathbb{E}_1$-ring spectra

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
A' & \longrightarrow & A' \otimes_B B'
\end{array}
\]  

such that any localizing invariant sends the square (3) to a pullback square.

Moreover, there is a natural $\mathbb{E}_1$-map $A' \otimes_B B' \to B'$, and the composites $A' \to A' \otimes_B B' \to B'$ and $B \to A' \otimes_B B' \to B'$ are canonically equivalent to the given maps $A' \to B'$ and $B \to B'$, respectively. The underlying spectrum of $A' \otimes_B B'$ is equivalent to $A' \otimes_A B$, and the underlying maps of the $\mathbb{E}_1$-maps above are the canonical ones.

**Remark.** If (□) is a diagram of $\mathbb{E}_\infty$-ring spectra, then the derived tensor product $A' \otimes_A B$ also carries a natural $\mathbb{E}_\infty$-structure. Its underlying $\mathbb{E}_1$-structure in general differs from the one on $A' \otimes_B B'$. We work out an illuminating example in Section 4.

As mentioned earlier, the absence of an excision long exact sequence of $K$-groups in positive degrees makes computations for positive $K$-groups difficult. To remedy this situation, there have been essentially three approaches:

The first approach is due to Suslin and Wodzicki [SW92, Sus95]. Suslin proves [Sus95, Theorem A] that for a Milnor square the sequence (1) in fact does extend up to degree $n$ provided the Tor groups $\text{Tor}^\mathbb{Z}_I(\mathbb{Z}, \mathbb{Z})$ vanish for $i = 1, \ldots, n - 1$. Here $I$ denotes the kernel of the map $A \to A'$, and $\mathbb{Z} \times I$ its unitalization. As a simple consequence of our Main Theorem we get the following generalization and strengthening of Suslin’s result.

**Theorem A.** Assume that (□) is a pullback square of $\mathbb{E}_1$-ring spectra all of which are connective. If the multiplication map $A' \otimes_A B \to B'$ is $n$-connective for some $n \geq 1$, then the induced diagram of $K$-theory spectra

\[
\begin{array}{ccc}
K(A) & \longrightarrow & K(B) \\
\downarrow & & \downarrow \\
K(A') & \longrightarrow & K(B')
\end{array}
\]

is $n$-cartesian (see Definition 2.1). There is also a more general statement for $K$-theory with coefficients and for $k$-connective localizing invariants, see Theorems 2.7, 2.8.

If (□) is a Milnor square, then $A' \otimes_A B \to B'$ is $n$-connective if and only if $\text{Tor}^A_i(A', B) = 0$ for $i = 1, \ldots, n - 1$. Thus Theorem A strengthens Suslin’s result as the condition only depends on the given Milnor square and not on the ideal $I = \ker(A \to A')$ as an abstract non-unital
ring. For instance, for $i = 1$, Suslin’s condition is that $I = I^2$, whereas $\text{Tor}_1^A(A/I, B)$ vanishes for example if $B = A/J$ and $I \cap J = 0$, a strictly weaker condition.

Our result also applies more generally than Suslin’s, for instance to ‘analytic isomorphism squares’ and to affine Nisnevich squares, respectively, see Example 2.9 for details. In both cases, the multiplication map $A' \otimes_A B \to B'$ is an equivalence, hence $A' \otimes_A^B B \simeq B'$, and the long exact sequence (2) is the classical one obtained from the work of Karoubi [Kar74], Quillen [Qui73], Vorst [Vor79], and Thomason [TT90], respectively.

The second approach is to use trace methods. Here, the decisive step was done by Cortiñas.

Combining a pro-version of the result of Suslin–Wodzicki [SW92] with ideas of Cuntz and Quillen from their proof of excision for periodic cyclic homology over fields of characteristic zero [CQ97] Theorem 5.3, he proves that the failure of excision in $K$-theory is rationally the same as that in (negative) cyclic homology [Cor06, Main Theorem]. Geisser and Hesselholt [GH06, Theorem 1] proved the analog of Cortinas’ result with finite coefficients, replacing negative cyclic homology by topological cyclic homology. Using these results, Dundas and Kittang finally proved that the failure of excision in $K$-theory is also integrally the same as that in topological cyclic homology under the additional assumption that both maps $A' \to B'$ and $B \to B'$ in the Milnor square are surjective [DK13, Theorem 1.1].

All these results are special cases of the following theorem, which itself is a simple direct consequence of our Main Theorem. We call a localizing invariant $E$ truncating, if the canonical map $E(A) \to E(\pi_0(A))$ is an equivalence for every connective $E_1$-ring spectrum $A$. Examples of truncating invariants are periodic cyclic homology $HP(-/k)$ over commutative $\mathbb{Q}$-algebras, the fibre $K_{Q}^{\text{inf}}$ of the Goodwillie–Jones Chern character $K(-)_Q \to \text{HN}(- \otimes \mathbb{Q}/\mathbb{Q})$ from rational $K$-theory to negative cyclic homology, and the fibre $K^{\text{inv}}$ of the cyclotomic trace $K \to \text{TC}$ from $K$-theory to topological cyclic homology.

**Theorem B.** Any truncating invariant satisfies excision and nilinvariance. More precisely: assume that $\square$ is a pullback square of $E_1$-ring spectra all of which are connective and that the induced map $\pi_0(A' \otimes_A B) \to \pi_0(B')$ is an isomorphism. If $E$ is a truncating invariant, then the induced square

$$
\begin{array}{ccc}
E(A) & \to & E(B) \\
\downarrow & & \downarrow \\
E(A') & \to & E(B')
\end{array}
$$

is a pullback. Moreover, if $I$ is a nilpotent ideal in the discrete ring $A$, then $E(A) \to E(A/I)$ is an equivalence.

We refer to Corollaries 3.1, 3.8 and 3.9 for the results of Cuntz–Quillen, Cortiñas, and Dundas–Kittang. In particular, this removes the additional surjectivity assumption in the result of Dundas and Kittang.

The third approach is based on the pro-version of Suslin’s excision theorem due to Cortiñas [Cor06, Theorem 3.16] and Geisser–Hesselholt [GH11, Theorem 3.1] and the observation of Morrow that the sufficient pro-Tor vanishing condition for excision is automatically satisfied for ideals in noetherian commutative rings [Mor18, Theorem 0.3]. A similar pro-excision result for noetherian simplicial commutative rings was established by Kerz–Strunk–Tamme in order to prove pro-descent for abstract blow-up squares in $K$-theory [KST18, Theorem 4.11]. Again, our Main Theorem easily implies a common generalization of all of these pro-excision results: Theorem A holds mutatis mutandis for pro-systems of $E_1$-ring spectra, see
Theorem 2.28 for details. In particular, if \( A \to B \) is a map of discrete rings sending the ideal \( I \subseteq A \) isomorphically onto the ideal \( J \subseteq B \), then the diagram of pro-rings

\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
\{A/I^\lambda\} & \longrightarrow & \{B/J^\lambda\}
\end{array}
\]

where \( \lambda \in \mathbb{N} \) induces a weakly cartesian square of \( K \)-theory pro-spectra provided the pro-Tor-groups \( \{\text{Tor}_i^A(A/I^\lambda, B)\} \) vanish for \( i > 0 \). Again, this condition is strictly weaker than the one occurring in \([\text{GH11}, \text{Theorem 3.1}]\) and is automatic if \( A \) is commutative and noetherian. See Corollaries 2.29 and 2.30 for details.

A key feature of our Main Theorem applied to \( K \)-theory is that it describes the failure of excision as the relative \( K \)-theory of the \( E_1 \)-map \( A' \otimes_A B \to B' \). Note that for a Milnor square, this map is just a 0-truncation. Besides the applications already mentioned, we can use this observation to obtain qualitative statements about the torsion in birelative algebraic \( K \)-groups improving a previous result of Geisser and Hesselholt \([\text{GH11}, \text{Theorem C}]\). We are grateful to Akhil Mathew for pointing out that our method should give a direct proof of this result avoiding topological cyclic homology.

**Theorem C.** Assume that \( \square \) is a pullback square of \( E_1 \)-ring spectra all of which are connective. If the map \( A' \otimes_A B \to B' \) is 1-connective, and if the homotopy groups of its fibre in degrees \( \leq n \) are annihilated by \( N \) for some integer \( N > 0 \), then the birelative \( K \)-groups \( K_i(A, B, A', B') \) are annihilated by some power of \( N \) for every \( i \leq n \).

If \( \square \) is a Milnor square, the condition in the theorem is that the groups \( \text{Tor}_i^A(A', B) \) are killed by multiplication by \( N \). Using our Main Theorem again, we can reduce questions about relative \( K \)-groups of nilpotent ideals to questions about relative \( K \)-groups of 0-truncations. We obtain the following generalization of a result of Geisser–Hesselholt \([\text{GH11}, \text{Theorem A}]\).

**Theorem D.** Let \( A \) be a discrete ring, and let \( I \) be a two-sided nilpotent ideal in \( A \). Assume that \( N \cdot I = 0 \) for some integer \( N > 0 \). Then for every integer \( i \), the relative \( K \)-groups \( K_i(A, A/I) \) are annihilated by some power of \( N \).

In both theorems the assumptions are automatically satisfied if \( \square \) is a Milnor square of \( \mathbb{Z}/N \)-algebras. In this situation the previous two theorems are due to Geisser and Hesselholt. Their method does not apply directly to our more general setting.

Finally, our methods combined with an argument of \([\text{KST18}]\) give the following cdh-descent result, which we prove in Appendix A.

**Theorem E.** Any truncating invariant satisfies cdh-descent on the category of schemes of finite type over some noetherian base scheme.

Together with \([\text{KST18}, \text{Theorem A}]\) this directly implies that topological cyclic homology satisfies pro-descent for abstract blow-up squares of noetherian schemes, see Corollary A.3. For noetherian, \( F \)-finite \( \mathbb{Z}_{(p)} \)-schemes of finite dimension, and with finite coefficients, the latter result was previously obtained by Morrow \([\text{Mor16}, \text{Theorem 3.5}]\).

To end this introduction, we sketch the proof of the Main Theorem. The condition that \( \square \) is a pullback square implies that \( \text{Perf}(A) \) embeds as a full subcategory in the lax pullback

\[ \text{Perf}(A') \rightleftarrows_{\text{Perf}(B')} \text{Perf}(B). \]
It turns out that the Verdier quotient of this lax pullback by \( \text{Perf}(A) \) is generated by a single object, and that the underlying spectrum of the endomorphisms of this generator is equivalent to \( A' \otimes_A B \). This easily implies the result.

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1. The main theorem

**Notation 1.1.** We write \( \text{Cat}^\text{ex}_\infty \) for the \( \infty \)-category of small stable \( \infty \)-categories and exact functors. Given an \( \mathbb{E}_1 \)-ring spectrum \( A \), we denote by \( \text{RMod}(A) \) its presentable stable \( \infty \)-category of \( A \)-right modules. The \( \infty \)-category of **perfect** \( A \)-modules, \( \text{Perf}(A) \in \text{Cat}^\text{ex}_\infty \), is by definition the smallest stable full subcategory of \( \text{RMod}(A) \) which contains \( A \) and is closed under retracts. It coincides with the compact objects in \( \text{RMod}(A) \) so that \( \text{Ind}(\text{Perf}(A)) \) is equivalent to \( \text{RMod}(A) \).

**Definition 1.2.** A sequence \( A \rightarrow B \rightarrow C \) in \( \text{Cat}^\text{ex}_\infty \) is called **exact** if the composite is zero, \( A \rightarrow B \) is fully faithful, and the induced functor on the Verdier quotient \( B/A \rightarrow C \) becomes an equivalence after idempotent completion. For \( T \) a stable \( \infty \)-category, a \( T \)-valued **localizing invariant** is a functor

\[
E : \text{Cat}^\text{ex}_\infty \rightarrow T,
\]

which sends exact sequences in \( \text{Cat}^\text{ex}_\infty \) to fibre sequences in \( T \).

Let us recall the setup for our main theorem. We consider a commutative diagram of \( \mathbb{E}_1 \)-ring spectra as follows.

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
A' & \rightarrow & B'
\end{array}
\]

For convenience we recall the statement of the Main Theorem.

**Theorem 1.3** (Main Theorem). Assume that \( \square \) is a pullback square of \( \mathbb{E}_1 \)-ring spectra. Associated to this square there exists a natural \( \mathbb{E}_1 \)-ring spectrum \( A' \otimes_A B' \) sitting in a natural commutative diagram of \( \mathbb{E}_1 \)-ring spectra

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
A' & \rightarrow & A' \otimes_A B'
\end{array}
\]

such that any localizing invariant sends the square \( \square \) to a pullback square.

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1 Sometimes this property is called **exact up to factors**.

2 Such an invariant was called **weakly localizing** in [Tam18], as in [BGT13] a localizing invariant is further required to commute with filtered colimits. However, we think that the decisive property is that of sending exact sequences to fibre sequences and propose to use another adjective, e.g. finitary, for the localizing invariants that preserve filtered colimits.
Moreover, there is a natural $E_1$-map $A' \circ_B B \to B'$, and the composites $A' \to A' \circ_B B' \to B'$ and $B \to A' \circ_B B' \to B'$ are canonically equivalent to the given maps $A' \to B'$ and $B \to B'$, respectively. The underlying spectrum of $A' \circ_B B$ is equivalent to $A' \otimes_A B$, and the underlying maps of the $E_1$-maps above are the canonical ones.

The following is an immediate consequence of the Main Theorem.

**Corollary 1.4.** Assume that $\square$ is a pullback square of $E_1$-ring spectra. If the multiplication map $A' \otimes_A B \to B'$ is an equivalence, then the square

\[
\begin{array}{ccc}
E(A) & \to & E(B) \\
\downarrow & & \downarrow \\
E(A') & \to & E(B')
\end{array}
\]

is cartesian for any localizing invariant $E$.

If the map $A \to A'$ is Tor-unital, i.e. if the multiplication $A' \otimes_A A' \to A'$ is an equivalence, then also the map $A' \otimes_A B \to B'$ is an equivalence, see Lemma 2.14. Thus Corollary 1.4 recovers a previous result of the second author, see [Tam18, Theorem 28].

For the proof of Theorem 1.3 we need some preparations. We begin by recalling the notion of lax pullbacks of $\infty$-categories [Tam18, Definition 5] and refer to [Tam18, Section 1] for more details and references. Let us consider a diagram

\[
\begin{array}{ccc}
B & \to & C \\
q \downarrow & & \downarrow \\
A & \xrightarrow{p} & C
\end{array}
\]

of $\infty$-categories. The *lax pullback* of (5), denoted by $A \times^L C$, is defined via the pullback diagram

\[
\begin{array}{ccc}
A \times^L C & \to & \text{Fun}(\Delta^1, C) \\
\downarrow & & \downarrow \\
A \times B & \xrightarrow{p \times q} & C \times C
\end{array}
\]

in simplicial sets. Since the source-target map $\text{Fun}(\Delta^1, C) \to C \times C$ is a categorical fibration, $A \times^L C$ is in fact an $\infty$-category, and (6) is a pullback diagram of $\infty$-categories. Thus, the objects of $A \times^L C$ are triples $(X, Y, f)$ with $X \in A$, $Y \in B$, and $f : p(X) \to q(Y)$ a morphism in $C$.

If (5) is a diagram of stable $\infty$-categories and exact functors, then $A \times^L C$ is stable, and the projection functors to $A$, $B$, and $\text{Fun}(\Delta^1, C)$, respectively, are exact [Tam18, Lemma 8]. Moreover, colimits in $A \times^L C$ are formed component-wise. There are fully faithful inclusions $j_1 : A \to A \times^L C$ and $j_2 : B \to A \times^L C$ given by $X \mapsto (X, 0, 0)$ and $Y \mapsto (0, Y, 0)$, respectively, and any object $(X, Y, f)$ of $A \times^L C$ sits in a fibre sequence

\[
(0, Y, 0) \to (X, Y, f) \to (X, 0, 0).
\]

The following lemma will be used in the proof of Proposition 1.14 below.

**Lemma 1.5.** Assume that (5) is a diagram of stable $\infty$-categories and exact functors and suppose that $p$ admits a right adjoint $u$. Then the functor $j_1 : A \to A \times^L C$ has a right adjoint
given by \((X, Y, f) \mapsto \text{fib}(X \to uq(Y))\) where the map in the fibre is the composition of the unit \(X \to up(X)\) with \(u(f)\): \(up(X) \to uq(Y)\).

**Proof.** A unit transformation is given by the canonical equivalence \(X \simeq \text{fib}(X \to 0)\). Indeed, this follows easily from the formula for mapping spaces in the lax pullback \([\text{Tam18, Remark 6}]\). □

**Lemma 1.6.** The functor

\[ i: \text{RMod}(A) \longrightarrow \text{RMod}(A') \times_{\text{RMod}(B')} \text{RMod}(B) \]

induced by extension of scalars admits a right adjoint \(s\). Explicitly, for an object \((M, N, f) \in \text{RMod}(A') \times_{\text{RMod}(B')} \text{RMod}(B)\) we have

\[ s(M, N, f) \simeq M \times N \otimes_{B} B' \]

where the map \(M \to N \otimes_{B} B'\) is the composite \(M \to M \otimes_{A'} B' \xrightarrow{f} N \otimes_{B} B'\).

**Proof.** Both categories are presentable, and the functor \(i\) preserves colimits \([\text{Tam18, Lemma 8}]\), thus \(i\) admits a right adjoint \(s\). To obtain the explicit formula for \(s\), we observe that we have canonical equivalences of mapping spectra

\[ s(M, N, f) \simeq \text{map}_{A}(A, s(M, N, f)) \simeq \text{map}_{A'}(i(A), (M, N, f)) \]

where the second equivalence comes from the adjunction. We may identify \(i(A) = (A', B, \text{id})\).

By the formula for mapping spaces in a lax pullback \([\text{Tam18, Remark 6}]\) we have a pullback square

\[ \begin{array}{ccc}
\text{map}_{A'}(i(A), (M, N, f)) & \longrightarrow & \text{map}_{\text{RMod}(B') A'}(A, f) \\
\downarrow & & \downarrow \\
\text{map}_{A'}(A', M) \times \text{map}_{B}(B, N) & \longrightarrow & \text{map}_{B'}(B', M \otimes_{A'} B') \times \text{map}_{B'}(B', N \otimes_{B} B')
\end{array} \]

In this square, the lower left corner identifies with \(M \times N\), the lower right corner with \(M \otimes_{A'} B' \times N \otimes_{B} B'\). The mapping spectrum in the upper right corner canonically identifies with \(M \otimes_{A'} B'\). Under these identifications, the right vertical map is given by \((\text{id}, f)\). We thus get the left pullback square in the following diagram.

\[ \begin{array}{ccc}
s(M, N, f) & \longrightarrow & M \otimes_{A'} B' \\
\downarrow & & \downarrow \text{id, f} \\
M \times N & \longrightarrow & M \otimes_{A'} B' \times N \otimes_{B} B' \xrightarrow{f-\text{id}} N \otimes_{B} B'
\end{array} \]

Obviously, the right-hand square is also a pullback square, whence is the combined square. This proves the claim. □

**Lemma 1.7.** Assume that \(\square\) is a pullback square of \(E_{1}\)-ring spectra. Then the functor \(i\) in Lemma \([\text{Tam18}]\) and hence its restriction to perfect modules

\[ i: \text{Perf}(A) \longrightarrow \text{Perf}(A') \times_{\text{Perf}(B')} \text{Perf}(B) \]

are fully faithful.

**Proof.** We have to show that the unit \(\text{id} \to si\) is an equivalence. Using the concrete formulas, this follows directly from the fact that \(\square\) is a pullback diagram. □
Definition 1.8. A small stable ∞-category \( \mathcal{A} \) is said to be \textit{generated by a set of objects} \( S \), if \( \mathcal{A} \) coincides with the smallest stable full subcategory of \( \mathcal{A} \) that contains \( S \) and is closed under retracts in \( \mathcal{A} \).

Lemma 1.9. Consider a diagram as \( \bullet \) in \( \text{Cat}^\Sigma_\infty \). If \( \mathcal{A} \) is generated by the set of objects \( S \) and \( \mathcal{B} \) is generated by the set of objects \( T \), then the lax pullback \( \mathcal{A} \times^\Sigma_\mathcal{C} \mathcal{B} \) is generated by the set of objects

\[
\{(X,0,0) \mid X \in S\} \cup \{(0,Y,0) \mid Y \in T\}.
\]

Proof. It suffices to recall that every object \((M,N,f)\) in \( \mathcal{A} \times^\Sigma_\mathcal{C} \mathcal{B} \) sits in a cofibre sequence

\[
(0,N,0) \to (M,N,f) \to (M,0,0).
\]

Suppose again that the diagram of \( \mathbb{E}_1 \)-rings \( \bullet \) is a pullback. We let \( Q \) be the Verdier quotient of the fully faithful embedding of perfect modules of Lemma 1.7. We thus obtain an exact sequence

\[
(7)\quad \text{Perf}(A) \xrightarrow{i} \text{Perf}(A') \overset{\times}{\to} \text{Perf}(B) \xrightarrow{\pi} Q
\]
in \( \text{Cat}^\Sigma_\infty \).

Lemma 1.10. The small stable ∞-category \( Q \) is generated by the single object \( \overline{\mathcal{B}}=\pi(0,B,0) \). Moreover, for any \( M \in \text{Perf}(A) \) there is a natural equivalence

\[
\Omega\pi(M \otimes_A A',0,0) \simeq \pi(0,M \otimes_A B,0).
\]

Proof. By definition, \( \text{Perf}(A') \) and \( \text{Perf}(B) \) are generated by \( A' \) and \( B \), respectively. Thus, by Lemma 1.9 the lax pullback \( \text{Perf}(A') \times_{\text{Perf}(B')} \text{Perf}(B) \) is generated by \((0,B,0)\) and \((A',0,0)\). It follows that their images \( \overline{B} \) and \( \pi(A',0,0) \) generate \( Q \). Now observe that the functor \( i: \text{Perf}(A) \to \text{Perf}(A') \times_{\text{Perf}(B')} \text{Perf}(B) \) takes the object \( M \) to the triple \((M \otimes_A A',M \otimes_A B,\text{id})\), and that this object sits inside a fibre sequence

\[
(0,M \otimes_A B,0) \to (M \otimes_A A',M \otimes_A B,\text{id}) \to (M \otimes_A A',0,0)
\]
in the lax pullback. Thus, after applying \( \pi \) we see that \( \Omega\pi(M \otimes_A A',0,0) \simeq \pi(0,M \otimes_A B,0) \). In particular, \( \Omega\pi(A',0,0) \simeq \overline{B} \), which proves the lemma.

Notation 1.11. We denote the \( \mathbb{E}_1 \)-ring \( \text{End}_Q(\overline{\mathcal{B}}) \) by \( A' \otimes^\mathcal{A}_A B \).

We prove Theorem 1.3 in two parts: in part (1) we establish the commutative diagram of \( \mathbb{E}_1 \)-rings and prove that any localizing invariant sends it to a pullback square; in part (2) we establish the identifications of the \( \mathbb{E}_1 \)-ring \( A' \otimes^\mathcal{A}_A B \) and the maps involved.

We write \( i_1: \text{Perf}(A) \to \text{Perf}(A') \) and \( i_2: \text{Perf}(A) \to \text{Perf}(B') \) for the respective extension of scalars functors, and

\[
\begin{align*}
\eta_1: \text{Perf}(A') &\to \text{Perf}(A') \times_{\text{Perf}(B')} \text{Perf}(B), \\
\eta_2: \text{Perf}(B) &\to \text{Perf}(A') \times_{\text{Perf}(B')} \text{Perf}(B)
\end{align*}
\]
for the inclusion functors. The latter are split by the projection functors \( \text{pr}_1 \) and \( \text{pr}_2 \), respectively.

Proof of Theorem 1.3 part (1). Recall the exact sequence (7). By Lemma 1.10 the Verdier quotient \( Q \) is generated by the single object \( \overline{\mathcal{B}} \). It follows that its ind-completion \( \text{Ind}(Q) \) is a presentable stable ∞-category which is compactly generated by the image of \( \overline{\mathcal{B}} \) in \( \text{Ind}(Q) \).
By the Schwede–Shipley theorem \[\text{Lur17, Theorem 7.1.2.1}\] we have a canonical equivalence
\[
\text{Ind}(Q) \simeq \text{RMod}(\text{End}_Q(B)) = \text{RMod}(A' \circledast_A B).
\]
Passing to compact objects we find that
\[
\text{Idem}(Q) \simeq \text{Perf}(A' \circledast_A B)\]
where \text{Idem} denotes the idempotent completion. We thus get an exact sequence
\[
\text{Perf}(A) \rightarrow \text{Perf}(A' \times_{\text{Perf}(B')} \text{Perf}(B)) \rightarrow \text{Perf}(A' \circledast_A B)
\]
of small stable \(\infty\)-categories. From Lemma 1.10 we get a commutative diagram of small stable \(\infty\)-categories
\[
\begin{array}{ccc}
\text{Perf}(A) & \rightarrow & \text{Perf}(A' \circledast_A B) \\
\downarrow^{i_1} & & \downarrow^{\pi j_2} \\
\text{Perf}(A') & \xrightarrow{\Omega \pi j_1} & \text{Perf}(A' \circledast_A B)
\end{array}
\]
in which all functors respect our preferred generators \(A, A', B,\) and \(\overline{B},\) respectively. It follows that we obtain a commutative diagram of \(E_1\)-rings
\[
\begin{array}{ccc}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
A' & \rightarrow & A' \circledast_A B
\end{array}
\]
which induces diagram (9) on categories of perfect modules. Given any localizing invariant \(E,\) we want to show that \(E\) applied to the square (9) or equivalently (10) gives a pullback square. Since \(E(\Omega)\) is multiplication by \(-1,\) an equivalent statement is that the sequence
\[
E(\text{Perf}(A)) \rightarrow E(\text{Perf}(A')) \oplus E(\text{Perf}(B)) \rightarrow E(\text{Perf}(A' \circledast_A B))
\]
is a fibre sequence. But applying \(E\) to the exact sequence (8) yields a fibre sequence
\[
E(\text{Perf}(A)) \rightarrow E(\text{Perf}(A' \times_{\text{Perf}(B')} \text{Perf}(B))) \rightarrow E(\text{Perf}(A' \circledast_A B))
\]
and using \[\text{Tam18, Proposition 10}\] we get an equivalence
\[
E(A') \oplus E(B) \xrightarrow{\sim} E(\text{Perf}(A') \times_{\text{Perf}(B')} \text{Perf}(B))
\]
induced by \(j_1\) and \(j_2\) with inverse induced by \(\text{pr}_1\) and \(\text{pr}_2.\) Thus the sequence (11) is in fact equivalent to the sequence (12), and thus a fibre sequence. This finishes the proof of part (1) of Theorem 1.3. \(\square\)

In the following we establish the identification of \(A' \circledast_A B\) and all involved maps. To compute \(A' \circledast_A B = \text{End}_Q(B)\) we may pass to ind-completions. Note that by \[\text{Tam18, Proposition 13}\], \(\text{Ind}(\text{Perf}(A') \times_{\text{Perf}(B')} \text{Perf}(B)) \simeq \text{RMod}(A') \times_{\text{RMod}(B')} \text{RMod}(B).\)

**Lemma 1.12.** The functor
\[
\pi: \text{RMod}(A') \times_{\text{RMod}(B')} \text{RMod}(B) \rightarrow \text{Ind}(Q)
\]
is a Bousfield localization, i.e. it admits a fully faithful right adjoint \(p.\) The localization functor \(L = p\pi\) is given by the cofibre
\[
\text{cofib}(i s(-) \rightarrow (-))
\]
of the counit transformation of the adjunction \((i, s)\) from Lemma 1.10.
Proof. Since $Q$ is the Verdier quotient of the lax pullback of perfect modules by $\text{Perf}(A)$, the functor $\pi$ on ind-completions is a Bousfield localization and has kernel $\text{Ind}(\text{Perf}(A)) \simeq \text{RMod}(A)$, see [NS17 Proposition 1.3.5]. The local objects are given by the image of $\rho$. It follows that the pair $(i(\text{RMod}(A)), \rho(\text{Ind}(Q)))$ is a semi-orthogonal decomposition of the $\infty$-category $\text{RMod}(A') \times_{\text{RMod}(B')} \text{RMod}(B)$ in the sense of [Lur18 Definition 7.2.0.1], see Corollary 7.2.1.7 there. The desired formula for $L$ now follows immediately from [Lur18 Remark 7.2.0.2].

The following observations will be used in the proofs below. We consider a functor $F: C \rightarrow D$ between stable presentable $\infty$-categories and we fix an object $c \in C$. Then we can form the endomorphism spectrum $\text{End}_C(c)$, and $F(c)$ carries canonically the structure of an $\text{End}_C(c)$-left module in $D$. If $G$ is a second functor $C \rightarrow D$, we have the two $\text{End}_C(c)$-left modules $F(c), G(c)$ in $D$ and the mapping spectrum map$_D(F(c), G(c))$ acquires a canonical $\text{End}_C(c)$-bimodule structure, where the left module structure is induced by post-composition via $G$ and the right module structure is induced by pre-composition via $F$.

Given a pullback square of $\mathbb{E}_1$-ring spectra $\square$, we denote the fibre of $B \rightarrow B'$ by $I$. Thus $I$ is naturally a $B$-bimodule, and the fibre of $A \rightarrow A'$ is obtained from $I$ by the forgetful functor to $A$-bimodules.

**Proposition 1.13.** The underlying spectrum of $A' \odot_A^{B'} B$ is canonically equivalent to $A' \odot_A B$. Under this identification the ring map $B \rightarrow A' \odot_A^{B'} B$ induced from the functor $\pi j_2$ is the obvious map $B \rightarrow A' \odot_A B$. Moreover, the underlying $B$-bimodule of $A' \odot_A^{B'} B$ sits in a cofibre sequence

$$I \otimes_A B \longrightarrow B \longrightarrow A' \odot_A^{B'} B.$$ 

**Proof.** It follows from Lemma 1.12 that we have an equivalence of $\mathbb{E}_1$-rings

$$A' \odot_A^{B'} B \simeq \text{End}_Q(B) \simeq \text{End}_{\mathbb{E}_1}(L(0, B, 0)).$$

We also have an equivalence of $\mathbb{E}_1$-rings $B \simeq \text{End}_B(B)$ given by left multiplication. We also write $j_2$ for the functor

$$\text{RMod}(B) \longrightarrow \text{RMod}(A') \times_{\text{RMod}(B')} \text{RMod}(B)$$

that sends $M$ to $(0, M, 0)$. Then the $\mathbb{E}_1$-map $B \rightarrow A' \odot_A^{B'} B$ is the map $B \simeq \text{End}_B(B) \rightarrow \text{End}_{\mathbb{E}_1}(L(0, B, 0)) \simeq \text{End}_{\mathbb{E}_1}(L_j(B)) \simeq A' \odot_A^{B'} B$ induced by the functor $L j_2$. Since $L j_2(B)$ is a local object, the canonical map $j_2(B) \rightarrow L j_2(B)$ induces an equivalence

$$\text{End}_{\mathbb{E}_1}(L j_2(B)) \simeq \text{map}_{\mathbb{E}_1}(j_2(B), L j_2(B)).$$

Using Lemma 1.12 we see that there is a commutative diagram

$$\begin{align*}
\text{map}_{\mathbb{E}_1}(j_2(B), L j_2(B)) & \longrightarrow \text{map}_{\mathbb{E}_1}(j_2(B), L j_2(B)) \\
\text{map}_{\mathbb{E}_1}(j_2(B), L j_2(B)) & \longrightarrow \text{map}_{\mathbb{E}_1}(j_2(B), L j_2(B))
\end{align*}
$$

where the top row is a cofibre sequence of $B \simeq \text{End}_B(B)$-bimodules, and the lower horizontal map is the $\mathbb{E}_1$-map we are interested in. We can identify the top row further. The functor $j_2$ is left adjoint to the projection $\text{pr}_2$ from the lax pullback to $\text{RMod}(B)$ [Tam18, Proposition 10]. From the formula for $s$ in Lemma 1.6 we see that $s j_2(B) \simeq I$ considered as an $A$-right module. Moreover, for any $B$-right module $M$, we have a canonical equivalence $\text{map}_B(B, M) \simeq M$.
and if $M$ is a $B$-bimodule this equivalence refines to a $B$-bimodule equivalence. Hence the first map in the top row of (13) canonically identifies with the $B$-bimodule map $I \otimes_A B \to B$ which is given by $I \to B$ and the multiplication in $B$. This establishes the last claim of the proposition.

On the other hand, tensoring the cofibre sequence of $A$-right modules $I \to A \to A'$ with $B$ we deduce that the above map sits in a cofibre sequence

$$I \otimes_A B \to B \to A' \otimes_A B$$

where the second map is the obvious one. From this we immediately deduce the first two claims of the proposition. \hfill \Box

We now prove the analog of the previous proposition for the ring map $A' \to A' \otimes_A^R B$ induced by the functor $\Omega\pi_j$ appearing in diagram (13). We denote the fibre of $A' \to B'$ in $A'$-bimodules by $J$. Again, after forgetting to $A$-bimodules, $J$ becomes equivalent to the fibre of $A \to B$.

**Proposition 1.14.** Under the identification of Proposition 1.13, the ring map $A' \to A' \otimes_A^R B$ induced by the functor $\Omega\phi$ is the obvious map $A' \to A' \otimes_A B$. Moreover, the underlying $A'$-bimodule of $A' \otimes_A^R B$ sits in a cofibre sequence

$$A' \otimes_A J \to A' \to A' \otimes_A^R B.$$ 

**Proof.** From the canonical fibre sequence $(0, B, 0) \to (A', B, \mathrm{id}) \to (A', 0, 0)$ in the lax pullback, we get a map

$$\sigma : \Omega(A', 0, 0) \to (0, B, 0),$$

which by Lemma 1.10 becomes an equivalence upon applying the localization functor $L$. Now we consider the following commutative diagram of mapping spectra in the lax pullback.

\[
\begin{array}{ccc}
\map(\Omega(A', 0, 0), L\Omega(A', 0, 0)) & \xrightarrow{\sigma^*} & \map((0, B, 0), L\Omega(A', 0, 0)) \\
\uparrow & & \uparrow \\
\map(\Omega(A', 0, 0), \Omega(A', 0, 0)) & \xrightarrow{\sigma^*} & \map((0, B, 0), \Omega(A', 0, 0)) \\
\uparrow & & \uparrow \\
\map(\Omega(A', 0, 0), is\Omega(A', 0, 0)) & \xleftarrow{\sigma^*} & \map((0, B, 0), is\Omega(A', 0, 0))
\end{array}
\]

Note that the columns in this diagram are cofibre sequences. The inclusion of $\text{RMod}(A')$ in the lax pullback and the functor $\Omega$ induce a canonical equivalence

$$A' \simeq \text{End}_A(A') \simeq \map(\Omega(A', 0, 0), \Omega(A', 0, 0)).$$

In the proof of Proposition 1.13 we have identified

$$A' \otimes_A^R B \simeq \map((0, B, 0), L(0, B, 0)) \simeq A' \otimes_A B.$$

Under these identifications, the map $A' \to A' \otimes_A^R B$ induced by the functor $\Omega\phi$ is the map from the middle term in the left column to the top right corner in the above diagram. We now identify all terms in the above diagram. Firstly, $s\Omega(A', 0, 0)$ is $\Omega A'$ viewed as $A$-right module. Using Lemma 1.13 and the canonical identification $\map_A(\Omega A', \Omega X) \simeq X$ for any $A'$-right module (or bimodule) $X$, we identify the lower left vertical map as the canonical map $A' \otimes_A J \to A'$ in $A'$-bimodules. The underlying spectrum of its cofibre is canonically equivalent to $A' \otimes_A B$ via the obvious map $A' \to A' \otimes_A B$. Hence the proof of the proposition
is finished once we show that under the respective identifications the top row of the above diagram becomes the identity of $A' \otimes_A B$. To see this, we observe that the two lower pullback squares of the diagram canonically identify with the pullback squares

$$
\begin{array}{ccc}
A' & \xleftarrow{0} & B \\
\uparrow & & \uparrow \\
A' \otimes_A J & \xleftarrow{\Omega A' \otimes_A B} & I \otimes_A B
\end{array}
$$

where the left-hand square is the cofibre sequence $\Omega B \to J \to A$ tensored with $A'$, the right-hand square is the cofibre sequence $\Omega A' \to I \to A$ tensored with $B$. This implies that the induced map on vertical cofibres is the identity of $A' \otimes_A B$, as desired. \hfill $\square$

We now address the map $A' \otimes_{A'} B' \to B'$ which arises as follows. There is a functor

$$
cofib: \text{Perf}(A') \xleftarrow{\times} \text{Perf}(B') \to \text{Perf}(B')
$$

sending an object $(M, N, f)$ to $\text{cofib}(f)$. Since the composition of $\text{cofib}$ with the inclusion of $\text{Perf}(A)$ in the lax pullback vanishes, there is an essentially unique functor $Q \to \text{Perf}(B')$ such that the diagram

$$
\begin{array}{ccc}
\text{Perf}(A') & \xleftarrow{\times} & \text{Perf}(B') \\
\downarrow^{\pi} & & \downarrow \\
\text{Perf}(B) & \xrightarrow{\text{cofib}} & \text{Perf}(B')
\end{array}
$$

commutes. This functor sends the generator $\overline{B}$ to $\text{cofib}(0, B, 0) \simeq B'$ and hence induces an $E_1$-map $A' \otimes_{A'} B \to B'$ such that the functor $\text{Idem}(Q) \to \text{Perf}(B')$ is given by extension of scalars along this $E_1$-map.

**Proposition 1.15.** The compositions $A' \to A' \otimes_{A'} B \to B'$ and $B \to A' \otimes_{A'} B \to B'$ are naturally equivalent to the given maps $A' \to B'$ and $B \to B'$, respectively. Moreover, under the identification of Proposition 1.13, the $E_1$-map $A' \otimes_{A'} B \to B'$ is given by the map $A' \otimes_A B \to B'$ induced by the maps $A' \to B'$, $B \to B'$, and the multiplication in $B'$.

**Proof.** The composite $A' \to A' \otimes_{A'} B \to B'$ is induced by the functor $\text{cofib}(\Omega(-) \otimes_A B' \to 0) \simeq (-) \otimes_{A'} B'$ and is thus the given map $A' \to B'$. Similarly, the composite $B \to A' \otimes_{A'} B \to B'$ is the given map $B \to B'$.

Since as an $E_1$-map $A' \otimes_{A'} B \to B'$ is in particular $A'$-left linear, and since under the identification of $A' \otimes_{A'} B$ with $A' \otimes_A B$, the $A'$-left module structure of $A' \otimes_{A'} B$ is the canonical one (Proposition 1.13), the map $A' \otimes_{A'} B \to B'$ is the unique $A'$-linear extension of the composite $B \to A' \otimes_{A'} B \to B'$. Together with the above, this finishes the proof. \hfill $\square$

**Remark 1.16.** An argument similar to the proof of Proposition 1.15 applies to show that the $E_1$-ring $A' \otimes_{A'} B$ is natural in the diagram (□) in the obvious way.

We are now ready to finish the proof of Theorem 1.3.

**Proof of Theorem 1.3, part (2).** The remaining identifications are done in Propositions 1.13, 1.14, 1.15 and in Remark 1.16. \hfill $\square$

**Remark 1.17.** Let $k$ be an arbitrary $E_\infty$-ring spectrum. Then the category $\text{Perf}(k)$ is, as a stably symmetric monoidal $\infty$-category, an object of $\text{CAlg}(\text{Cat}_{\infty}^e)$. As such, one can consider the $\infty$-category $\text{Mod}_{\text{Perf}(k)}(\text{Cat}_{\infty}^e)$ of $k$-linear stable $\infty$-categories. We denote the $\infty$-category
of $k$-linear stable $\infty$-categories by $\text{Cat}_{\infty}^k$. Note that $\text{Cat}_{\infty}^k = \text{Cat}_{\infty}^{ex}$. A typical example of a $k$-linear stable $\infty$-category is the category $\text{Perf}(A)$ of perfect $A$-right modules for a $k$-algebra $A$. In a $k$-linear stable $\infty$-category, the mapping spectra canonically refine to $k$-module spectra, i.e. any $k$-linear stable $\infty$-category is enriched in the presentably symmetric monoidal stable $\infty$-category $\text{Mod}(k)$. There is a canonical forgetful functor

$$\text{Cat}_{\infty}^k \to \text{Cat}_{\infty}^{ex}$$

which preserves finite limits and finite colimits. In particular, a sequence $A \to B \to C$ of $k$-linear stable $\infty$-categories is exact if and only if it is so after forgetting the $k$-linear structure. For an auxiliary stable $\infty$-category $\mathcal{T}$, a $\mathcal{T}$-valued localizing invariant of $k$-linear stable $\infty$-categories is then just a functor

$$\text{Cat}_{\infty}^k \to \mathcal{T}$$

that sends exact sequences of $k$-linear stable $\infty$-categories to fibre sequences in $\mathcal{T}$. Examples are provided by restricting localizing invariants in the sense of Definition 1.2 along the above functor $\text{Cat}_{\infty}^k \to \text{Cat}_{\infty}^{ex}$.

If $\square$ is a pullback diagram of $k$-algebras, then all arguments in the proof of Theorem 1.3 can be made after replacing $\text{Cat}_{\infty}^{ex}$ by $\text{Cat}_{\infty}^k$. In particular, $A' \otimes_{B'} B$ then carries a canonical $k$-algebra structure and Theorem 1.3 holds for any localizing invariant of $k$-linear stable $\infty$-categories.

The usefulness of this extra generality comes from the fact that topological Hochschild homology relative to $k$ is an example of a localizing invariant of $k$-linear stable $\infty$-categories which does not arise in the previously mentioned manner. We will exploit this in Section 3.

2. Applications to $K$-theory

2.1. Preliminaries.

**Definition 2.1.** A map $f : X \to Y$ of spectra is said to be $n$-connective if its fibre $F$ is $n$-connective. If $\Lambda$ is an abelian group, then $f$ is said to be $\Lambda$-$n$-connective if $F \otimes M\Lambda$ is $n$-connective, where $M\Lambda$ is the Moore spectrum for $\Lambda$. A commutative diagram

$$
\begin{array}{ccc}
X & \to & Y \\
\downarrow & & \downarrow \\
Z & \to & W 
\end{array}
$$

is said to be $n$-cartesian if the canonical map $X \to Z \times_W Y$ is $n$-connective, or, equivalently, if the induced map on horizontal fibres is $n$-connective. Similarly, the above diagram is said to be $\Lambda$-$n$-cartesian, if the diagram obtained by tensoring with $M\Lambda$ is $n$-cartesian, or, equivalently, if the canonical map $X \to Z \times_W Y$ is $\Lambda$-$n$-connective.

**Remark 2.2.** From the equivalence

$$\Omega \text{fib}(Z \sqcup_X Y \to W) \simeq \text{fib}(X \to Z \times_W Y)$$

we deduce that a diagram is $\Lambda$-$n$-cartesian if and only if the canonical map $Z \sqcup_X Y \to W$ is $\Lambda$-$(n + 1)$-connective.

**Remark 2.3.** An $n$-cartesian diagram as above in particular gives rise to a long exact Mayer–Vietoris sequence

$$
\pi_n(X) \to \pi_n(Y) \oplus \pi_n(Z) \to \pi_n(W) \to \pi_{n-1}(X) \to \ldots
$$
For future reference, we record the following well-known statement, see [Wal78, Proposition 1.1] for a similar result.

**Lemma 2.4.** Let $\Lambda$ be a localisation of $\mathbb{Z}$ or $\mathbb{Z}/m$ for some integer $m$. Assume that $A \to B$ is a map of connective $E_1$-rings which induces an isomorphism $\pi_0(A) \xrightarrow{\sim} \pi_0(B)$ and is $\Lambda$-$n$-connective for some $n \geq 1$. Then the induced map

$$K(A) \to K(B)$$

is $\Lambda$-($n+1$)-connective. In other words, the map $K(A; \Lambda) \to K(B; \Lambda)$ of $K$-theories with coefficients in $\Lambda$ is $(n+1)$-connective.

**Proof.** As non-positive $K$-groups of a connective $E_1$-ring $A$ only depend on $\pi_0(A)$, see [BGT13, Theorem 9.53], it suffices to prove the result for connected $K$-theory. For this we recall the plus-construction of algebraic $K$-theory. For any connective $E_1$-ring $A$, there is a group-like $E_1$-monoid $GL(A)$ which satisfies

$$\Omega_0^\infty(K(A)) \simeq BGL(A)^+$$

where $(-)^+$ denotes Quillen’s plus-construction and the subscript 0 denotes the component of the trivial element of $K_0(A)$, see e.g. [BGT13, Lemma 9.39]. By construction of $GL$ and the assumption that $A \to B$ induces an isomorphism on $\pi_0$, there is an equivalence

$$\text{fib}(GL(A) \to GL(B)) \xrightarrow{\sim} \text{fib}(M(A) \to M(B))$$

where $M$ denotes the $E_\infty$-space of matrices. This implies that the map of spaces

$$BGL(A) \to BGL(B)$$

is an isomorphism on fundamental groups and $\Lambda$-($n+1$)-connective. It follows that the fibre $F$ of this map is a nilpotent space (we may pass to universal covers without changing the fibre of the map) with abelian fundamental group which is in addition $\Lambda$-($n+1$)-connective, i.e. that $\pi_i(F; \Lambda) = 0$ for $i \leq n$. We claim that also $H_i(F; \Lambda) = 0$ for $i \leq n$. If $\Lambda$ is a localisation of $\mathbb{Z}$, this follows by Serre class theory and for $\Lambda = \mathbb{Z}/m$ from [Nei10, Theorem 9.7] and deduce the claim. It follows that the map

$$H_i(BGL(A); \Lambda) \to H_i(BGL(B); \Lambda)$$

is an isomorphism for $i \leq n$ and a surjection for $i = n+1$. Since the plus-construction is a homology equivalence it follows that also the map

$$H_i(BGL(A)^+; \Lambda) \to H_i(BGL(B)^+; \Lambda)$$

is an isomorphism for $i \leq n$ and a surjection for $i = n+1$. Hence the map $BGL(A)^+ \to BGL(B)^+$ is $\Lambda$-($n+1$)-connective: For $\Lambda$ a localization of $\mathbb{Z}$ this again follows by Serre class theory and for $\Lambda = \mathbb{Z}/m$ from [Nei10, Corollary 9.15]. □
2.2. Excision results in algebraic $K$-theory. In this section, we will apply our main theorem to localizing invariants which satisfy a certain connectivity assumption. We thank Thomas Nikolaus for suggesting to introduce the following definition. As earlier, let $\Lambda$ be a localisation of $\mathbb{Z}$ or $\mathbb{Z}/m$ for some integer $m$.

**Definition 2.5.** A localizing invariant $E$ is said to be $\Lambda$-$k$-connective if for every map $A \to B$ of connective $E_1$-rings which is a $\pi_0$-isomorphism and $\Lambda$-$n$-connective for some $n \geq 1$ the induced map $E(A) \to E(B)$ is $\Lambda$-$(n+k)$-connective. A $\mathbb{Z}$-$k$-connective localizing invariant is simply called $k$-connective.

Notice that $\Lambda$-$k$-connective localizing invariants are closed under extensions, i.e. if $E' \to E \to E''$ is a fibre sequence of localizing invariants where $E'$ and $E''$ are $\Lambda$-$k$-connective, then so is $E$.

**Example 2.6.** Many localizing invariants are $k$-connective for some $k$. In the following examples, $\Lambda$ is as before a localization of $\mathbb{Z}$ or $\mathbb{Z}/m$.

(i) $K$-theory is $\Lambda$-1-connective by Lemma 2.4,
(ii) topological Hochschild homology THH is $\Lambda$-0-connective,
(iii) truncating invariants (see Definition 3.1) are $k$-connective for every $k$,
(iv) topological cyclic homology $TC$ and rational negative cyclic homology $HN(- \otimes \mathbb{Q}/\mathbb{Q})$ are 1-connective, as each sits in a fibre sequence with $K$-theory and a truncating invariant.

With this notion we have the following consequence of Theorem 1.3.

**Theorem 2.7.** Assume that $\square$ is a pullback square of $E_1$-ring spectra all of which are connective. If the map $A' \otimes_A B \to B'$ is a $\pi_0$-isomorphism and $\Lambda$-$n$-connective for some $n \geq 1$, and if $E$ is a $\Lambda$-$k$-connective localizing invariant, then the diagram

$$
\begin{array}{ccc}
E(A) & \longrightarrow & E(B) \\
\downarrow & & \downarrow \\
E(A') & \longrightarrow & E(B')
\end{array}
$$

is $\Lambda$-$(n+k-1)$-cartesian.

**Proof.** By Theorem 1.3 and Remark 2.2 we need to show that the map $E(A' \otimes_A B') \to E(B')$ is $\Lambda$-$(n+k)$-connective. This follows directly from the assumptions. \hfill $\Box$

As $K$-theory is $\Lambda$-1-connective we obtain the following theorem.

**Theorem 2.8.** Assume that $\square$ is a pullback square of $E_1$-ring spectra all of which are connective. If the map $A' \otimes_A B \to B'$ is a $\pi_0$-isomorphism and $\Lambda$-$n$-connective for some $n \geq 1$, then the diagram

$$
\begin{array}{ccc}
K(A) & \longrightarrow & K(B) \\
\downarrow & & \downarrow \\
K(A') & \longrightarrow & K(B')
\end{array}
$$

is $\Lambda$-$n$-cartesian.

**Example 2.9.** To illustrate this theorem, consider the case of a diagram $\square$ consisting of discrete rings. In all of the following cases the diagram $\square$ is a pullback square of $E_1$-ring spectra and the map $A' \otimes_A B \to B'$ is a $\pi_0$-isomorphism.
(i) The diagram \( \square \) is a Milnor square, i.e. the maps \( A \to A' \) and \( B \to B' \) are surjective and the map \( A \to B \) sends the ideal \( I = \ker(A \to A') \) isomorphically to \( \ker(B \to B') \). We have:

\[
\pi_0(A' \otimes_A B) \cong \text{Tor}_0^A(A/I, B) \cong B/IB \cong B'.
\]

Since \( B' \) is discrete, the map \( A' \otimes_A B \to B' \) is always 1-connective, and we deduce the classical Mayer-Vietoris sequence of a Milnor square (see [Bas68, Theorem XII.8.3])

\[
K_1(A) \to K_1(A') \oplus K_1(B) \to K_1(B') \xrightarrow{\partial} K_0(A) \to K_0(A') \oplus K_0(B) \to \ldots
\]

(ii) The map \( A \to B \) is an analytic isomorphism along a multiplicatively closed set \( S \) of central elements of \( A \), i.e. it maps \( S \) to central elements of \( B \) and induces isomorphisms on the kernel and cokernels by \( s \) for every \( s \in S \), and we have \( A' = S^{-1}A, B' = S^{-1}B \). These conditions immediately imply that the square \( \square \) is a pullback of \( \mathbb{E}_1 \)-ring spectra. Moreover, \( A \to A' \) is flat and hence \( A' \otimes_A B \to B' \) is an equivalence, and in particular \( A \)-\( n \)-connective for any \( n \). We deduce the classical long exact sequence of an analytic isomorphism due to Karoubi, Quillen, and Vorst [Vor79, Proposition 1.5]

\[
\ldots \to K_i(A) \to K_i(A') \oplus K_i(B) \to K_i(B') \xrightarrow{\partial} K_{i-1}(A) \to \ldots
\]

(iii) The diagram \( \square \) is an affine Nisnevich square, i.e. all rings in it are commutative, the map \( \text{Spec}(A') \to \text{Spec}(A) \) is an open immersion, \( B' \cong A' \otimes_A B \), and \( A \to B \) is étale and induces an isomorphism on the closed complements (\( \text{Spec}(B) \setminus \text{Spec}(B') \)) \( \xrightarrow{\sim} (\text{Spec}(A) \setminus \text{Spec}(A')) \) with the reduced subscheme structure. In this case all maps in the diagram are flat, hence \( B' \cong A' \otimes_{A'} B' \), and we deduce the long exact Nisnevich Mayer-Vietoris sequence [TT90, Theorem 10.8].

If \( \square \) is a Milnor square of discrete rings (see Example 2.9(i)), the connectivity condition in Theorem 2.8 may be expressed as a condition on Tor-groups as in the following corollary. For simplicity, we formulate the result only for \( K \)-theory; for an arbitrary \( k \)-connective localizing invariants the resulting square is \( \Lambda-(n+k-1) \)-cartesian.

**Corollary 2.10.** Assume that \( \square \) is a Milnor square of discrete rings.

(i) If \( \Lambda \) is a localisation of \( \mathbb{Z} \) and \( \text{Tor}_i^A(A', B \otimes_\mathbb{Z} \Lambda) = 0 \) for \( i = 1, \ldots, n-1 \), or

(ii) if \( \Lambda = \mathbb{Z}/m \) and the \( m \)-multiplication on \( \text{Tor}_i^A(A', B) \) is an isomorphism for \( i = 1, \ldots, n-2 \) and surjective for \( i = n-1 \),

then the diagram

\[
\begin{array}{ccc}
K(A) & \rightarrow & K(B) \\
\downarrow & & \downarrow \\
K(A') & \rightarrow & K(B')
\end{array}
\]

is \( \Lambda \)-\( n \)-cartesian.

**Remark 2.11.** Assume that \( \square \) is a Milnor square. If \( \text{Tor}_i^A(A', B) = 0 \) for \( i = 1, \ldots, n-1 \), then Corollary 2.10 says that the induced map on relative \( K \)-groups (see Definition 2.17)

\[
K_i(A, A') \to K_i(B, B')
\]

\footnote{Often, this condition is replaced by the stronger condition that \( S \) consists of nonzerodivisors which map to nonzerodivisors in \( B \).}
is bijective for \( i < n \) and surjective for \( i = n \). In fact, one can say a little bit more: It follows from Theorem 1.3 and \cite[Proposition 1.2]{Wal78} that there is an exact sequence

\[
K_{n+1}(A, A') \rightarrow K_{n+1}(B, B') \rightarrow \text{Tor}^A_n(A', B)/(bx - xb) \rightarrow K_n(A, A') \rightarrow K_n(B, B') \rightarrow 0
\]

where the middle term denotes the symmetrization of the \( B' \)-bimodule \( \text{Tor}^A_n(A', B) \). For example, if \( A \) and \( B \) are commutative and \( I = \ker(A \rightarrow A') \), then the symmetrization of \( \text{Tor}^A_1(A', B) \) is isomorphic to \( \Omega B/A \otimes B/I/I^2 \) where \( \Omega B/A \) denotes the \( B \)-module of Kähler differentials. For \( n = 1 \), the part of the above exact sequence starting with the Tor-term was obtained by Swan \cite[Corollary 4.7]{Swa71} (and Vorst, see \cite[Exercise III.2.6]{Wei13}).

Let us now explain how this implies Suslin’s excision result for non-unital rings. Let \( I \) be a not necessarily unital ring and denote by \( Z \ltimes I \) its unitalisation. It admits a natural augmentation \( Z \times I \rightarrow Z \) and we set

\[
K(I) = \text{fib}(K(Z \times I) \rightarrow K(Z)).
\]

**Definition 2.12 (Suslin).** A non-unital ring \( I \) is said to satisfy excision in \( K \)-theory with coefficients in \( \Lambda \) up to degree \( n \) if for any unital ring \( B \) containing \( I \) as a two-sided ideal, the diagram

\[
\begin{array}{ccc}
K(Z \times I) & \rightarrow & K(B) \\
\downarrow & & \downarrow \\
K(Z) & \rightarrow & K(B/I)
\end{array}
\]

is \( \Lambda \)-\((n + 1)\)-cartesian.

We find the following consequence of Theorem 2.8, which is \cite[Theorem A]{Su95}. Here, as before, \( \Lambda \) is a localisation of \( Z \) or \( Z/m \) for some integer \( m \).

**Theorem 2.13.** The non-unital ring \( I \) satisfies excision in \( K \)-theory with coefficients in \( \Lambda \) up to degree \( n \) if \( \text{Tor}^Z_{\times I}(Z, \Lambda) = 0 \) for \( i = 1, \ldots, n \).

**Proof.** Let \( B \) be any ring containing \( I \) as a two-sided ideal. We observe that the diagram

\[
\begin{array}{ccc}
Z \times I & \rightarrow & B \\
\downarrow & & \downarrow \\
Z & \rightarrow & B/I
\end{array}
\]

is a Milnor square. The assumption implies that the multiplication map \( Z \otimes Z \times I Z \rightarrow Z \) is \( \Lambda \)-\((n + 1)\)-connective. Here we use that \( Z \otimes M \Lambda \simeq \Lambda \) as \( Z \) is torsion free. Lemma 2.14 below together with the fact that \( B/I \) is discrete imply that \( Z \otimes Z \times I B \rightarrow B/I \) is \( \Lambda \)-\((n + 1)\)-connective. Obviously, it is also a \( \pi_0 \)-isomorphism. Now Theorem 2.8 implies that the diagram of \( K \)-theory spectra induced from the Milnor square \( \square \) is \( \Lambda \)-\((n + 1)\)-cartesian, as desired. \( \square \)

**Lemma 2.14.** Suppose that \( \square \) is a pullback square of \( E_1 \)-rings all of which are connective.

If the map \( A' \otimes_A A' \rightarrow A' \) is \( \Lambda \)-\( n \)-connective, then the map \( A' \otimes_A B \rightarrow B' \) is \( \Lambda \)-\((n - 1)\)-connective and in addition induces an isomorphism on \( \pi_{n-1} \) with coefficients in \( \Lambda \). If \( n = \infty \), then the same conclusion holds for possibly non-connective \( E_1 \)-rings.

**Proof.** Since \( A' \otimes_A A' \rightarrow A' \) is \( \Lambda \)-\( n \)-connective, the same holds for the map \( A' \otimes_A B' \rightarrow B' \) obtained by base change along \( A' \rightarrow B' \). It thus suffices to show that after tensoring with
the map $A' \otimes_A B \to A' \otimes_A B'$ is $(n-1)$-connective and induces an isomorphism on $\pi_{n-1}$. For this we consider the pullback square

\[
\begin{array}{ccc}
A' \otimes M \Lambda & \to & A' \otimes_A B \otimes M \Lambda \\
\downarrow & & \downarrow \\
A' \otimes_A A' \otimes M \Lambda & \to & A' \otimes_A B' \otimes M \Lambda \\
\end{array}
\]

obtained from \(\square\) by tensoring with $A'$ over $A$ and with $MA$. The left vertical map is split by the multiplication map. By assumption, this split is $n$-connective. A little diagram chase using the long exact sequences of homotopy groups associated to the vertical maps now gives the claim. The assertion about non-connective $E_1$-rings follows analogously. \(\square\)

**Remark 2.15.** It is worthwhile to clarify the relation between the various Tor-vanishing conditions. Suppose \(\square\) is a Milnor square. By Corollary 2.10, excision in $K$-theory holds for this particular square provided $\text{Tor}_i^A(A', B) = 0$ for $i \geq 1$. Suslin proved that excision holds for all Milnor squares with fixed ideal $I$ if $\text{Tor}_i^{Z \otimes I}(Z, Z) = 0$ for $i \geq 1$. Morrow showed in \[Mor18, Theorem 0.2\] that Suslin’s condition is in fact equivalent to the vanishing of $\text{Tor}_i^A(A', A')$ for $i \geq 1$ and some Milnor square \(\square\) with ideal $I$. Lemma 2.14 and the following example show that the condition of Corollary 2.10 is the most general of the above.

**Example 2.16.** Let $A'$ be a commutative ring, and $f \in A'$. We let $A'_f = A'[1/f]$ be the localization and consider the Milnor square

\[
\begin{array}{ccc}
A & \to & A'_f \\
\downarrow & & \downarrow \\
A' & \to & A'_f \\
\end{array}
\]

where the right vertical map sends the variable $x$ to zero. On the one hand, the map $A \to A'_f[x]$ is a localization and thus flat. On the other hand $\text{Tor}_1^A(A'_f, A'_f) \cong I/I^2$, where $I = \ker(A \to A')$, and the variable $x$ obviously gives rise to a non-zero element in $I/I^2$.

**2.3. Torsion in birelative and relative $K$-groups.** In this section we explain how to use our main theorem to prove stronger versions of the results obtained by Geisser–Hesselholt in \[GH11\]. We thank Akhil Mathew for pointing out that our main theorem should give a direct proof these results.

**Definition 2.17.** The _relative $K$-theory_ of a map of $E_1$-rings $A \to B$ is defined as the fibre of the induced map of $K$-theory spectra,

$$K(A, B) = \text{fib}(K(A) \to K(B)),$$

its homotopy groups are denoted by $K_i(A, B)$. Similarly, if \(\square\) is a commutative square of $E_1$-rings, the _birelative $K$-theory_ is

$$K(A, B, A', B') = \text{fib}(K(A, B) \to K(A', B')).$$

The following is an immediate consequence of Theorem 1.3

**Lemma 2.18.** If the commutative square \(\square\) is a pullback square of $E_1$-rings, then there is a canonical equivalence

$$K(A, B, A', B') \simeq \Omega \text{fib}(K(A' \otimes_A B') \to K(B')).$$
Fix an integer \( N \geq 1 \). We say that an abelian group \( A \) is a bounded \( N \)-torsion group if \( A \) is killed by a power of \( N \).

**Proposition 2.19.** Let \( A \to B \) be a 1-connective map of connective \( \mathbb{E}_1 \)-ring spectra. Assume that \( \pi_i(\text{fib}(A \to B)) \) is a bounded \( N \)-torsion group for \( i \leq n \). Then the relative \( K \)-group \( K_i(A, B) \) is a bounded \( N \)-torsion group for each \( i \leq n + 1 \).

**Proof.** The proof is similar to that of Lemma 2.18. Since the map \( K(A) \to K(B) \) is 2-connective, we may again use the plus-construction of \( \text{BGL} \) to compute the relative \( K \)-groups. Consider the fibre sequences

\[
F \to \text{BGL}(A) \to \text{BGL}(B),
\]

\[
\tilde{F} \to \text{BGL}(A)^+ \to \text{BGL}(B)^+,
\]

and note that \( K_i(A, B) \cong \pi_i(\tilde{F}) \). The assumptions imply that \( F \) is simply connected, and that \( \pi_i(F) \) is a bounded \( N \)-torsion group for \( i \leq n + 1 \). As the bounded \( N \)-torsion groups form a Serre class of abelian groups, it follows that the reduced homology \( \tilde{H}_i(F; \mathbb{Z}) \) is a bounded \( N \)-torsion group for every \( i \leq n + 1 \). The relative Serre spectral sequence

\[
E^2_{p,q} = H_p(\text{BGL}(B); \tilde{H}_q(F; \mathbb{Z})) \implies H_{p+q}(\text{BGL}(A), \text{BGL}(B); \mathbb{Z})
\]

then implies that \( H_i(\text{BGL}(A), \text{BGL}(B); \mathbb{Z}) \) is a bounded \( N \)-torsion group for \( i \leq n + 1 \). Now consider the relative Serre spectral sequence for the fibration of plus-constructions. Using that the map to the plus construction is acyclic we find a spectral sequence

\[
E^2_{p,q} = H_p(\text{BGL}(B); \tilde{H}_q(\tilde{F}; \mathbb{Z})) \implies H_{p+q}(\text{BGL}(A), \text{BGL}(B); \mathbb{Z}).
\]

whose abutment is a bounded \( N \)-torsion group in degrees \( \leq n + 1 \) by the above. Since \( \tilde{F} \) is also simply connected, again by Serre class theory it follows that also \( \pi_i(\tilde{F}) \) for \( i \leq n + 1 \) are bounded \( N \)-torsion groups. \( \square \)

**Theorem 2.20.** Assume that \( [\square] \) is a pullback square of \( \mathbb{E}_1 \)-ring spectra all of which are connective. If the map \( A' \otimes_A B \to B' \) is 1-connective, and if the homotopy groups of its fibre are bounded \( N \)-torsion groups in degrees \( \leq n \), then the birelative \( K \)-groups \( K_i(A, B, A', B') \) are bounded \( N \)-torsion groups for every \( i \leq n \).

**Proof.** The claim follows directly from the assumptions, Lemma 2.18 and Proposition 2.19. \( \square \)

In the case of a Milnor square, we may again express the assumptions in the previous theorem in terms of Tor-groups as follows.

**Corollary 2.21.** Assume that \( [\square] \) is a Milnor square of discrete rings such that \( \text{Tor}^A_i(A', B) \) is a bounded \( N \)-torsion group for all \( i = 1, \ldots, n \). Then the birelative \( K \)-group \( K_i(A, B, A', B') \) is a bounded \( N \)-torsion group for every \( i \leq n \).

**Remark 2.22.** The assumptions of the corollary are automatically satisfied for any integer \( n \) if \( A' \) or \( B \) is a \( \mathbb{Z}/N \)-algebra. If \( A \) itself is a \( \mathbb{Z}/N \)-algebra, then the corollary is due to Geisser–Hesselholt [GH11] Theorem C, see also Remark 2.26.

Here is an example where Theorem C of [GH11] cannot be applied, but Corollary 2.21 does apply. Let \( G \) be a finite group of order \( N \). Let \( \mathfrak{M} \) be a maximal \( \mathbb{Z} \)-order of the rational
group ring $\mathbb{Q}[G]$ containing $\mathbb{Z}[G]$. Then $I = N \cdot \mathfrak{M}$ is a common ideal in $\mathbb{Z}[G]$ and $\mathfrak{M}$ [Bas68, Corollary XI.1.2] so that we get a Milnor square

\[
\begin{array}{ccc}
\mathbb{Z}[G] & \longrightarrow & \mathfrak{M} \\
\downarrow & & \downarrow \\
\mathbb{Z}[G]/I & \longrightarrow & \mathfrak{M}/I.
\end{array}
\]

**Corollary 2.23.** For a finite group $G$ the birelative $K$-groups $K_*(\mathbb{Z}[G], \mathfrak{M}, \mathbb{Z}[G]/I, \mathfrak{M}/I)$ are bounded $N$-torsion groups, and for $i \geq 1$, the relative $K$-groups $K_i(\mathbb{Z}[G], \mathfrak{M})$ are torsion groups of bounded exponent.

**Proof.** Since $\mathbb{Z}[G]/I$ is a $\mathbb{Z}/N$-algebra, Corollary 2.21 implies the statement about birelative $K$-groups. Since $\mathbb{Z}[G]/I$ and $\mathfrak{M}/I$ are both finite, their $K$-groups in positive degrees are also finite by a result of Kuku [Wei13, Proposition IV.1.16]. Hence the relative $K$-groups $K_i(\mathbb{Z}[G]/I, \mathfrak{M}/I)$ are finite for $i \geq 1$. This implies the claim. \qed

**Remark 2.24.** From the localization sequence for $\mathfrak{M}$ [Oli88, Theorem 1.17] we see that also the relative $K$-groups $K_i(\mathfrak{M}, \mathbb{Q}[G])$ are torsion for $i \geq 1$. It follows that $K_i(\mathbb{Z}[G]) \otimes \mathbb{Q} \rightarrow K_i(\mathbb{Q}[G]) \otimes \mathbb{Q}$ is an isomorphism for $i \geq 2$, a result due to Borel [Wei13, Theorem IV.1.17], and injective for $i = 1$.

**Theorem 2.25.** Let $A$ be a discrete ring, and let $I$ be a two-sided nilpotent ideal in $A$. Assume that, as an abelian group, $I$ is a bounded $N$-torsion group. Then the relative $K$-groups $K_*(A, A/I)$ are bounded $N$-torsion groups.

**Proof.** Let $n$ be the smallest natural number such that $I^n = 0$. Consider the two projections $A \rightarrow A/I^2 \rightarrow A/I$ and the associated fibre sequence of relative $K$-theory spectra

\[
K(A, A/I^2) \rightarrow K(A, A/I) \rightarrow K(A/I^2, A/I).
\]

By induction on $n$ we may assume that the claim holds for $K(A, A/I^2)$. Since the kernel of $A/I^2 \rightarrow A/I$ is a square zero ideal, it hence suffices to treat the case $n = 2$.

In this case $I$ is canonically an $A/I$-bimodule, so we may form the two differential graded algebras

\[C(I, A) = [I \xrightarrow{i} A] \quad \text{and} \quad C(I, A/I) = [I \xrightarrow{0} A/I] \]

concentrated in degrees 1 and 0. Note that $C(I, A) \simeq A/I$. There is a canonical map $C(I, A) \rightarrow C(I, A/I)$ given by the identity on $I$ and the projection on $A$. We now consider the commutative diagram

\[\begin{array}{ccc}
A & \longrightarrow & A/I \\
\downarrow & & \downarrow \\
C(I, A) & \longrightarrow & C(I, A/I)
\end{array}\]

in which the vertical maps are the canonical inclusions in degree zero. As the induced map on horizontal fibres is an equivalence, this is a pullback diagram. One checks that the map $C(I, A) \otimes_{A/I} A/I \rightarrow C(I, A/I)$ is a 1-truncation. For $i \geq 2$, the homotopy groups of the fibre of this map are thus given by

\[
\pi_i(C(I, A) \otimes_{A/I} A/I) \cong \text{Tor}_i^A(A/I, A/I) \cong \text{Tor}_{i-1}^A(I, A/I).
\]
These are bounded $N$-torsion groups by the assumption on $I$. Hence the fibre of the map $K(C(I, A) \oplus A/I) \to K(C(I, A/I))$ has bounded $N$-torsion homotopy groups by Proposition 2.19. Since by Lemma 2.18 this fibre is the birelative $K$-theory of diagram (16), it is enough to show that the fibre of the map $K(C(I, A)) \to K(C(I, A/I))$ has bounded $N$-torsion homotopy groups. Since the composition $C(I, A) \to C(I, A/I) \to A/I$ is an equivalence, it suffices to argue that the fibre of $K(C(I, A)) \to K(A/I)$ has bounded $N$-torsion homotopy groups. This follows again from Proposition 2.19 since $C(I, A) \to A/I$ is a 0-truncation with fibre $I[1]$ which has bounded $N$-torsion homotopy groups by assumption. □

Remark 2.26. The conditions of Corollary 2.21 and Theorem 2.25 are automatically satisfied if $A$ is a $\mathbb{Z}/N$-algebra. In this form, these results are due to Geisser–Hesselholt [GH11, Theorem C & A]. Geisser and Hesselholt first prove the relative case through topological cyclic homology. Using the relative case, and a pro-excision result in $K$-theory, which we address in Corollary 2.29, Geisser and Hesselholt deduce the birelative case. To the contrary, we first prove the birelative case and deduce the relative case from the birelative one and no use of topological cyclic homology is made.

2.4. Pro-excision results. To state the next result, we recall the notion of a weak equivalence of pro-spectra: A pro-spectrum is a diagram in spectra indexed by a small cofiltered $\infty$-category. Pro-spectra naturally form an $\infty$-category $\text{Pro}(\text{Sp})$, and the homotopy group functor $\pi_* : \text{Sp} \to \text{Ab}$ induces a ‘homotopy pro-group’ functor $\text{Pro}(\text{Sp}) \to \text{Pro}(\text{Ab})$. A map of pro-spectra, or more generally pro-$E_1$-rings etc., is called a weak equivalence if it induces isomorphisms on all homotopy pro-groups. Thus any equivalence in $\text{Pro}(\text{Sp})$ is a weak equivalence, but the converse is not true. Similarly, we use the terms weakly $n$-cartesian and weakly $n$-connective. The notation $\{X_\lambda\}_{\lambda \in \Lambda}$, or simply $\{X_\lambda\}$, indicates a pro-object indexed by the small cofiltered $\infty$-category $\Lambda$. The following is a variant of [KST18, Lemma 4.1].

**Lemma 2.27.** Let $\{A_\lambda\}$ be a pro-system of connective $E_1$-rings. Suppose $f : \{M_\lambda\} \to \{N_\lambda\}$ is a weak equivalence of pro-systems of connective $\{A_\lambda\}$-right modules and let $\{B_\lambda\}$ be a pro-system of connective $\{A_\lambda\}$-left modules. Then also the induced map

$$\{M_\lambda \otimes_{A_\lambda} B_\lambda\} \to \{N_\lambda \otimes_{A_\lambda} B_\lambda\}$$

is a weak equivalence.

**Proof.** There is a strongly convergent spectral sequence

$$\{\text{Tor}^*_{\pi_*}(A_\lambda)(\pi_*(M_\lambda), \pi_*(B_\lambda))\} \Rightarrow \{\pi_*(M_\lambda \otimes_{A_\lambda} B_\lambda)\}$$

and likewise for $\{N_\lambda\}$ in place of $\{M_\lambda\}$. The map $f$ induces a morphism of spectral sequences which is a pro-isomorphism on $E_2$-pages in any finite range by assumption. □

We now consider a pro-system of commutative diagrams of connective $E_1$-rings indexed by $\Lambda$ as follows.

$$\begin{array}{ccc}
A_\lambda & \to & B_\lambda \\
\downarrow & & \downarrow \\
A'_\lambda & \to & B'_\lambda
\end{array}$$

(17)

**Theorem 2.28.** Fix an integer $n \geq 1$. Assume that the diagram of connective pro-$E_1$-rings given by (17) is weakly cartesian and that the canonical map $\{A'_\lambda \otimes_{A_\lambda} B_\lambda\} \to \{B'_\lambda\}$ is weakly
\( n \)-connective. Then the diagram of pro-spectra

\[
\begin{array}{ccc}
K(A_\lambda) & \to & K(B_\lambda) \\
\downarrow & & \downarrow \\
K(A'_\lambda) & \to & K(B'_\lambda)
\end{array}
\]

is weakly \( n \)-cartesian.

**Proof.** We define the \( E_1 \)-ring \( C_\lambda \) via the pullback

\[
C_\lambda \to B_\lambda
\]

\[
A'_\lambda \to B'_\lambda
\]

The first assumption immediately implies that the canonical map \( \{A_\lambda\} \to \{C_\lambda\} \) is a weak equivalence of pro-\( E_1 \)-rings. We claim that also \( \{A'_\lambda \otimes A_\lambda \} \to \{A'_\lambda \otimes C_\lambda \} \) is a weak equivalence. To see this, we write the source of this map as \( \{A'_\lambda \otimes A_\lambda C_\lambda \otimes C_\lambda B_\lambda\} \). Under this identification, the map is given by extension of scalars along \( \{C_\lambda\} \to \{B_\lambda\} \) of the multiplication map \( \{A'_\lambda \otimes A_\lambda C_\lambda \} \to \{A'_\lambda \} \). By Lemma \(2.27\) it then suffices to show that the latter is a weak equivalence. This map has a section given by \( \{A'_\lambda\} \approx \{A'_\lambda \otimes A_\lambda A_\lambda\} \to \{A'_\lambda \otimes A_\lambda C_\lambda\} \), and as \( \{A_\lambda\} \to \{C_\lambda\} \) is a weak equivalence, so is the former by Lemma \(2.27\) again.

We set \( R_\lambda = A'_\lambda \otimes_{C_\lambda} B_\lambda \). By Theorem \(1.3\) we have to show that the map of pro-\( E_1 \)-rings \( \{R_\lambda\} \to \{B'_\lambda\} \) induces a weakly \((n+1)\)-connective map on \( K \)-theory pro-spectra. We may again use the plus-construction. Since \( \{R_\lambda\} \to \{B'_\lambda\} \) is weakly \( n \)-connective, the induced map \( \{BGL(R_\lambda)\} \to \{BGL(B'_\lambda)\} \) is weakly \((n+1)\)-connective, hence an isomorphism in integral homology pro-groups in degrees \( \leq n \), an epimorphism in degree \( n+1 \). Then the same is true for the map on plus-constructions, and by the Whitehead theorem for pro-spaces \([Sin81, \text{Theorem 4.1.1}]\), the map on plus-constructions is weakly \((n+1)\)-connective, as desired. \(\square\)

In the special case of a Milnor square of discrete rings, the theorem in particular gives the following.

**Corollary 2.29.** Let \( A \to B \) be a ring homomorphism sending the two-sided ideal \( I \subseteq A \) isomorphically onto the two-sided ideal \( J \subseteq B \). If the pro-Tor-groups \( \{\Tor^A_i(A/I^\lambda, B)\}_{\lambda \in \mathbb{N}} \) vanish for \( i = 1, \ldots, n-1 \), then the diagram of pro-spectra

\[
\begin{array}{ccc}
K(A) & \to & K(B) \\
\downarrow & & \downarrow \\
K(A/I^\lambda) & \to & K(B/J^\lambda)
\end{array}
\]

is weakly \( n \)-cartesian.

**Proof.** Apply Theorem \(2.28\) with \( A_\lambda = A, A'_\lambda = A/I^\lambda, B_\lambda = B, B'_\lambda = B/J^\lambda \). \(\square\)

For \( n = \infty \) and under the stronger assumption that the ideal \( I \) is (rationally) pro-Tor-unital, i.e. that the pro-Tor-groups \( \{\Tor^Z_i(I^\lambda Z, Z)\} \) vanish (rationally) for \( i > 0 \), the same result was first proven rationally by Cortiñas \([Cor06, \text{Theorem 3.16}]\) and later integrally by Geisser–Hesselholt \([GH11, \text{Theorem 3.1}]\), \([GH06, \text{Theorem 1.1}]\). They use pro-versions of the method of Suslin and Wodzicki \([SW92, Sus95]\), which is based on the homology of affine groups. Their results can in fact be deduced from ours, similarly as in the proof of
Theorem 2.13 Morrow [Mor18, Theorem 0.2] proves that the ideal $I$ is pro-Tor-unital if and only if the pro-Tor groups $\{\text{Tor}_i^A(A/I^λ, A/I^λ)\}$ vanish for $i > 0$. Our condition is still weaker.

If we moreover assume that $A$ is commutative and noetherian, then the condition of the previous corollary is automatically satisfied. The following was proven before by Morrow [Mor18, Corollary 2.4] by noting that ideals in commutative noetherian rings satisfy the above vanishing condition for pro-Tor-groups and using Geisser–Hesselholt’s pro-excision theorem.

Corollary 2.30. Let $A$ be a commutative noetherian ring, and let $A \to B$ be a ring homomorphism sending the ideal $I \subseteq A$ isomorphically onto the two-sided ideal $J \subseteq B$. Then the diagram of pro-spectra

$$
\begin{array}{ccc}
K(A) & \longrightarrow & K(B) \\
\downarrow & & \downarrow \\
\{K(A/I^λ)\} & \longrightarrow & \{K(B/J^λ)\}
\end{array}
$$

is weakly cartesian.

Proof. Since $A$ is noetherian, the pro-group $\{\text{Tor}_i^A(A/I^λ, M)\}$ vanishes for $i > 0$ and any finitely generated $A$-module $M$ [And74, Lemme X.11]. This implies that $\{A/I^λ \otimes_A A/I\} \to A/I$ is a weak equivalence. Extending scalars along $A/I \to B/J$ we deduce, using Lemma 2.27 that also $\{A/I^λ \otimes_A B/J\} \to B/J$ is a weak equivalence. Using the above vanishing of pro-Tor-groups for $M = J \cong I$ and the long exact sequence of Tor-groups associated with $J \to B \to B/J$ we conclude that $\{\text{Tor}_i^A(A/I^λ, B)\}$ vanishes for every $i > 0$ as desired. □

Similarly, we can derive a pro-excision result for noetherian simplicial commutative rings from Theorem 2.28. Recall that a simplicial commutative ring $A$ is called noetherian if $π_0(A)$ is noetherian and each $π_0(A)$ is a finitely generated $π_0(A)$-module. We now fix a commutative ring $R$ and a finite sequence $\bar{c} = (c_1, \ldots, c_r)$ of elements in $R$. If $A$ is a simplicial commutative $R$-algebra, we set

$$
A/\bar{c} = A \otimes_R [x_1, \ldots, x_r] R
$$

where the ring map $R[x_1, \ldots, x_r] \to R$ sends the variable $x_i$ to $c_i$, and the tensor product is, as always, derived. If $A$ is discrete, the homotopy groups $π_i(A/\bar{c})$ are the Koszul homology groups $H_i(A; \bar{c})$. For a positive integer $μ$ we denote by $\bar{c}(μ)$ the sequence $(c_1^μ, \ldots, c_r^μ)$, and for an $R$-module $M$ we write $\bar{c}(μ)M$ for the submodule $c_1^μ M + \cdots + c_r^μ M$ of $M$. The following corollary is precisely [KST18, Theorem 4.11] whose proof is based on an adaption of Suslin’s and Wodzicki’s method to the noetherian and pro-simplicial setting. In [KST18] this is used to prove that $K$-theory satisfies pro-descent for abstract blow-up squares.

Corollary 2.31. Consider a morphism of pro-systems of noetherian simplicial commutative $R$-algebras $φ: \{A_λ\} \to \{B_λ\}$. Assume that $φ$ induces an isomorphism

$$
\{\bar{c}(μ)π_i(A_λ)\}_{λ,μ} \xrightarrow{\cong} \{\bar{c}(μ)π_i(B_λ)\}_{λ,μ}
$$

for all $i ≥ 0$. Then the diagram of pro-spectra

$$
\begin{array}{ccc}
\{K(A_λ)\}_λ & \longrightarrow & \{K(B_λ)\}_λ \\
\downarrow & & \downarrow \\
\{K(A_λ/\bar{c}(μ))\}_{λ,μ} & \longrightarrow & \{K(B_λ/\bar{c}(μ))\}_{λ,μ}
\end{array}
$$

is weakly cartesian.
Proof. In order to apply Theorem 2.28, we view \( \{ A_\lambda \} \) as a pro-system indexed by \( (\lambda, \mu) \in \Lambda \times \mathbb{N} \) which is constant in the \( \mu \)-direction and consider the square

\[
\begin{array}{ccc}
\{ A_\lambda \} & \longrightarrow & \{ B_\lambda \} \\
\downarrow & & \downarrow \\
\{ A_\lambda / \overline{c}(\mu) \} & \longrightarrow & \{ B_\lambda / \overline{c}(\mu) \}
\end{array}
\]

(18)

Since \( A_\lambda \) is noetherian, [KST18, Lemma 4.10] implies that

\[
\{ \overline{c}(\mu) \pi_i(\lambda(A_\lambda)) \}_{\mu} \cong \{ \pi_i(\text{fib}(\lambda(A_\lambda) \to \lambda(A_\lambda) / \overline{c}(\mu))) \}_{\mu}
\]

and similarly for \( \{ B_\lambda \} \). Hence the square (18) is weakly cartesian. Furthermore, since \( (\lambda(A_\lambda) / \overline{c}(\mu)) \otimes_{A_\lambda} B_\lambda \simeq (\lambda(A_\lambda \otimes_{\mathbb{R}[x_1, \ldots, x_r]} \mathbb{R}) \otimes_{A_\lambda} B_\lambda \simeq B_\lambda \otimes_{\mathbb{R}[x_1, \ldots, x_r]} \mathbb{R} \simeq B_\lambda / \overline{c}(\mu) \),

the second assumption in Theorem 2.28 is trivially satisfied. \( \square \)

Remark 2.32. We remark that the proof of Theorem 2.28 in the special case \( n = \infty \) remains valid for any localizing invariant \( E \) with values in spectra that preserves weak equivalences of pro-systems of connective \( E_1 \)-rings. This is for example the case for topological Hochschild homology, as one can see for instance from the Bar construction, the fact that weak equivalences are closed under tensor products, and that the geometric realization of a simplicial diagram of weak equivalences between pro-systems of connective spectra is again a weak equivalence of pro-spectra. It follows that Corollary 2.30 and Corollary 2.31 remain valid when replacing \( K \)-theory with topological Hochschild homology. We will make use of this in Theorem A.5.

3. Applications to truncating invariants

The goal of this section is to show that truncating invariants satisfy excision and nilinvariance, and to reprove the excision theorems of Cuntz–Quillen, Cortiñas, Geisser–Hesselholt and Dundas–Kittang. Recall that for a localizing invariant \( E \) and an \( E_1 \)-ring \( A \), we write \( E(A) \) for \( E(\text{Perf}(A)) \).

Definition 3.1. Let \( E : \text{Cat}^{\text{ex}}_{\infty} \to \mathcal{T} \) be a localizing invariant. Then \( E \) is said to be \textit{truncating} if for every connective \( E_1 \)-ring spectrum \( A \), the canonical map \( E(A) \to E(\pi_0(A)) \) is an equivalence. It is said to be \textit{nilinvariant} if for every nilpotent two-sided ideal \( I \subseteq A \) in a discrete unital ring \( A \), the canonical map \( E(A) \to E(A/I) \) is an equivalence. Finally, \( E \) is said to be \textit{excisive}, or to satisfy \textit{excision}, if it sends the diagram of \( E_1 \)-ring spectra (\( \square \)) to a pullback square

\[
\begin{array}{ccc}
E(A) & \longrightarrow & E(B) \\
\downarrow & & \downarrow \\
E(A') & \longrightarrow & E(B')
\end{array}
\]

in \( \mathcal{T} \), provided the square (\( \square \)) satisfies the following two conditions:

(E1) The square (\( \square \)) is cartesian and all \( E_1 \)-rings in it are connective.

(E2) The induced map \( \pi_0(A' \otimes_A B) \to \pi_0(B') \) is an isomorphism.
Remark 3.2. Conditions (E1) and (E2) in Definition 3.1 are satisfied for all classes of squares discussed in Example 2.9, in particular for Milnor squares. Hence an excisive invariant sends a Milnor square to a pullback square. This is what is often called excision classically.

More generally, assume that \( \square \) is a pullback diagram of connective \( E_1 \)-ring spectra such that \( \pi_0(B) \to \pi_0(B') \) is surjective. Then conditions (E1) and (E2) are satisfied. Indeed, the connectivity assumption implies that the canonical map \( \pi_0(A') \otimes \pi_0(A) \to \pi_0(A') \otimes_\pi_0(B) \) is an isomorphism. Moreover, as \( \square \) is a pullback square, and as the map \( \pi_0(B) \to \pi_0(B') \) is surjective, also the map \( \pi_0(A) \to \pi_0(A') \) is surjective and the kernel of the latter surjects onto the kernel of the former. This implies

\[
\pi_0(A') \otimes \pi_0(A) \to \pi_0(B) \cong \ker(\pi_0(B) \to \pi_0(B')) \cong \pi_0(B')
\]

as needed.

Theorem 3.3. Any truncating invariant satisfies excision.

Proof. Clearly, for any truncating invariant \( E \) and any square of \( E_1 \)-ring spectra \( \square \) satisfying (E1) and (E2) the map \( E(A' \otimes_\pi_0(A) B) \to E(B') \) is an equivalence. We thus conclude by Theorem 1.3.

Remark 3.4. If \( k \) is some \( E_\infty \)-ring, and if \( E \) is a truncating invariant defined on \( k \)-linear stable \( \infty \)-categories as in Remark 1.17 then \( E \) sends any diagram of connective \( k \)-algebras satisfying (E1) and (E2) to a pullback square. We say that \( E \) satisfies excision on \( k \)-algebras.

Corollary 3.5. Any truncating invariant is nilinvariant.

Proof. The proof is similar to that of Theorem 2.25. By induction we may assume that \( I^2 = 0 \). Then we form the connective differential graded algebras \( C(I, A) \) and \( C(I, A/I) \) as in (15) and the pullback diagram (16). Since the map \( C(I, A) \to C(I, A/I) \) is a \( \pi_0 \)-isomorphism, we may apply Theorem 3.3 to get the following pullback diagram.

\[
\begin{array}{ccc}
E(A) & \longrightarrow & E(A/I) \\
\downarrow & & \downarrow \\
E(C(I, A)) & \longrightarrow & E(C(I, A/I))
\end{array}
\]

Using that \( E \) is truncating and the fact that \( C(I, A) \to C(I, A/I) \) is a \( \pi_0 \)-isomorphism again, we find that the lower horizontal map in this pullback is an equivalence. Thus so is the upper horizontal map as claimed.

From Theorem 3.3 we obtain simple direct proofs of several previously known excision results. Our main new application is the following. We denote by \( K^{inv} \) the fibre of the cyclotomic trace \( K \to TC \) from \( K \)-theory to integral topological cyclic homology.

Corollary 3.6. The fibre of the cyclotomic trace \( K^{inv} \) satisfies excision.

Proof. Both \( K \)-theory and TC are localizing invariants and the cyclotomic trace is a natural transformation of such, see Corollary 19 in Nikolaus’s lecture and Corollary 16 in Gepner’s lecture in [HS18]. Thus \( K^{inv} \) is localizing. The main result of Dundas–Goodwillie–McCarthy [DGM13, Theorem 7.0.0.2] implies that \( K^{inv} \) is truncating.
Remark 3.7. After profinite completion the same result was proven by Dundas–Kittang [DK08] building on work of Geisser–Hesselholt [GH06] in the discrete case. Using also Cortiñas’ rational analogue of Corollary 3.6 (see Corollary 3.8 below), Dundas and Kittang prove a slightly weaker integral result in [DK13], namely that $K^{\text{inf}}$ sends the pullback diagram of connective $E_1$-ring spectra to a pullback if both maps $A' \to B'$ and $B \to B'$ are surjective on $\pi_0$. For technical reasons, our result is not obtainable with their method, see [DK13] Remark 1.2(3)].

Next we want to deduce Cortiñas’ rational analogue of Corollary 3.6 of which it was an important precursor, from Theorem 3.3. To do so, we first have to explain how cyclic homology and its variants fit into our framework. For this, let $k$ be an $\mathbb{F}_\infty$-ring and consider the category $\text{Cat}_k^\infty$ of $k$-linear stable $\infty$-categories as in Remark 1.17. As discussed in [Hoy18], there is a functor

$$\text{HH}(-/k) : \text{Cat}_k^\infty \to \text{Fun}(B\mathbb{T}, \text{Mod}_k)$$

sending a $k$-linear stable $\infty$-category $C$ to its Hochschild homology relative to $k$, equipped with its canonical $\mathbb{T}$-action. Following the argument of Keller [Kel99] or Blumberg–Mandell [BM12] Theorem 7.1 one proves that $\text{HH}(-/k)$ takes exact sequences of small $k$-linear stable $\infty$-categories to fibre sequences in $\text{Mod}(k)$, so $\text{HH}(-/k)$ is a localizing invariant for $k$-linear stable $\infty$-categories. It follows that $HC(-/k)$, $HN(-/k)$, respectively $HP(-/k)$ are also localizing invariants for $k$-linear stable $\infty$-categories: They are obtained by applying the exact functors of orbits $(-)_k\mathbb{T}$, fixed points $(-)^{\mathbb{T}}$, respectively the Tate construction $(-)^{\mathbb{T}T}$ under the circle group $\mathbb{T}$ to $\text{HH}(-/k)$. If $k$ is a discrete commutative ring and $C = \text{Perf}(A)$ for some $k$-algebra $A$, then these constructions reproduce Hochschild, cyclic, respectively periodic cyclic homology of $A$ relative to $k$, see [Hoy18] Theorem 2.1, where we always mean the derived version of these theories.

The composition

$$\text{Cat}_k^\infty \to \text{Cat}_k^\infty \xrightarrow{\text{HN}(-/\mathbb{Q})} \text{Mod}(\mathbb{Q})$$

of extension of scalars from $\mathbb{S}$ to $\mathbb{Q}$ and $\text{HN}(-/\mathbb{Q})$ is thus a localizing invariant and will be denoted by $\text{HN}_\mathbb{Q}$. We will write $K_\mathbb{Q}$ for rational $K$-theory, i.e. for the localizing invariant that sends a small stable $\infty$-category $C$ to $K(C) \otimes \mathbb{Q}$.

The Goodwillie–Jones Chern character gives a natural transformation $K_\mathbb{Q} \to \text{HN}_\mathbb{Q}$. We let $K^{\text{inf}}_\mathbb{Q}$ be its fibre and call it rational infinitesimal $K$-theory. For discrete rings, the following result is due to Cortiñas, see [Cor06 Main Theorem 0.1].

**Corollary 3.8.** Rational infinitesimal $K$-theory $K^{\text{inf}}_\mathbb{Q}$ satisfies excision.

**Proof.** Since both $K_\mathbb{Q}$ and $\text{HN}(-/\mathbb{Q})$ are localizing, so is $K^{\text{inf}}_\mathbb{Q}$. By Goodwillie’s theorem [Goo86 Main Theorem], the map $K^{\text{inf}}_\mathbb{Q}(R) \to K^{\text{inf}}_\mathbb{Q}(\pi_0(R))$ is an equivalence for every connective $E_1$-ring $R$ (see Remark 3.9 below). That is, $K^{\text{inf}}_\mathbb{Q}$ is truncating. We thus conclude by Theorem 3.3. □

**Remark 3.9.** Note that [Goo86 Main Theorem] is a statement about simplicial rings rather than connective $E_1$-algebras. The more general case may be reduced to the statement about simplicial rings by applying [Goo86 Main Theorem] to $A \otimes H\mathbb{Z}$ instead of the $E_1$-ring $A$, see for instance [AR12 Corollary 2.2].

For discrete rings and $k$ a field of characteristic zero the following corollary is a result of Cuntz and Quillen [CQ97 Theorem 5.3]. Besides the work of Suslin and Wodzicki [SW92]
the ideas of Cuntz and Quillen played a crucial role in the previous approaches to excision results by Cortiñas, Geisser–Hesselholt, and Dundas–Kittang. The extension of the Cuntz–Quillen theorem to arbitrary commutative base rings $k$ containing $\mathbb{Q}$, as we present it here, was also obtained by different techniques by Morrow [Mor18, Theorem 3.15].

**Corollary 3.10.** Let $k$ be a discrete commutative ring containing $\mathbb{Q}$. Then periodic cyclic homology $\text{HP}(-/k)$ satisfies excision for $k$-algebras.

*Proof.* As explained before, $\text{HP}(-/k)$ is a localizing invariant for $k$-linear stable $\infty$-categories. Since $k$ contains the rational numbers, $\text{HP}(-/k)$ is truncating: If $k$ is a field, this is [Goo85, Theorem IV.2.1], the more general case follows from [Goo85, Theorem IV.2.6]. We conclude from the $k$-linear version of Theorem [3.3] see Remark [3.4].

4. An Example

In this section we discuss the example of the following family of pullback diagrams of discrete rings. Let $k$ be a commutative unital ring, and let $\alpha$ be an element of $k$. Consider the following commutative diagram in which all maps are the canonical ones.

\[
\begin{array}{ccc}
k & \rightarrow & k[y] \\
\downarrow & & \downarrow \\
k[x] & \rightarrow & k[x, y]/(yx - \alpha)
\end{array}
\]

This is a pullback of $\mathbb{E}_1$-rings, and thus Theorem [1.3] provides an $\mathbb{E}_1$-ring $k[x] \circ_k^{k[x,y]/(yx-\alpha)} k[y]$ with underlying spectrum $k[x] \otimes_k k[y]$.

**Proposition 4.1.** The $\mathbb{E}_1$-ring $k[x] \circ_k^{k[x,y]/(yx-\alpha)} k[y]$ in the above example is discrete and isomorphic to $k\langle x, y \rangle/(yx - \alpha)$, the free $k$-algebra on two non-commuting variables $x$ and $y$ modulo the relation $yx = \alpha$.

*Proof.* Since the map $k \rightarrow k[x]$ is flat, the $\mathbb{E}_1$-ring $k[x] \circ_k^{k[x,y]/(yx-\alpha)} k[y]$ is discrete with underlying $k$-module $k[x] \otimes_k k[y]$. From Propositions [1.13] and [1.14] we know that under this identification the ring maps $k[x] \rightarrow k[x] \otimes_k k[y]$ and $k[y] \rightarrow k[x] \otimes_k k[y]$ as well as the $k[x]$-left module structure and the $k[y]$-right module structure are the canonical ones. From the formula

\[
(x^k \otimes y^l) \cdot (x^m \otimes y^n) = (x^k \otimes 1) \cdot (1 \otimes y^l) \cdot (x^m \otimes 1) \cdot (1 \otimes y^n)
\]

we see that it remains to describe the $k[y]$-left module structure on $k[x] \otimes_k k[y]$. Let $I$ be the fibre of the right vertical map in \([19]\). By Proposition [1.19] the $k[y]$-left module structure on $k[x] \otimes_k k[y]$ is determined by the cofibre sequence of $k[y]$-left modules

\[
I \otimes_k k[y] \rightarrow k[y] \rightarrow k[x] \otimes_k k[y].
\]

We represent $\Sigma I$ by the complex $[k[y] \rightarrow k[x, y]/(yx - \alpha)]$ concentrated in degrees 1 and 0. Then the rotation of the previous cofibre sequence is represented by the following diagram of complexes of left $k[y]$-modules:

\[
\begin{bmatrix}
0 \\
k[y]
\end{bmatrix} \rightarrow \begin{bmatrix}
0 \\
k[x] \otimes_k k[y]
\end{bmatrix} \rightarrow \begin{bmatrix}
k[y] \otimes_k k[y] \\
k[x, y]/(yx - \alpha) \otimes_k k[y]
\end{bmatrix}
\]
Since the map $j$ is injective and $k[y]$-left linear, we may calculate the left-multiplication by $y$ after applying the map $j$. For any $m \geq 1$ we get
\[
j((1 \otimes y) \cdot (x^m \otimes 1)) = (yx^m) \otimes 1 = j(\alpha \cdot x^{m-1} \otimes 1).
\]
This finishes the proof. \hfill \Box

We can use the previous proposition and Theorem 1.3 to compute $E(k(x, y)/(yx - \alpha))$ for any localizing invariant $E$. We denote by $NE(k)$ the cofibre of the canonical map $E(k) \to E(k[x])$, so that $E(k[x]) \simeq E(k) \oplus NE(k)$. We get:

**Corollary 4.2.** There is a canonical equivalence
\[
E(k(x, y)/(yx - \alpha)) \simeq E(k) \oplus NE(k) \oplus NE(k).
\]

**Remark 4.3.** Let us consider the case where $\alpha = 1$, so that $k[x, y]/(yx - \alpha) \cong k[x, x^{-1}]$ is the ring of Laurent polynomials. There is a diagram of exact sequences in $\text{Cat}_\text{ex}^\otimes$
\[
\begin{array}{ccc}
\text{Perf}(k) & \rightarrow & \text{Perf}(k[x]) \\
\downarrow & & \downarrow \\
\text{Perf}(\mathbb{P}^1_k) & \rightarrow & \text{Perf}(k[x]) \\
\end{array}
\]
\[
\begin{array}{ccc}
\text{Perf}(k[x]) & \rightarrow & \text{Perf}(k[x, x^{-1}]) \\
\downarrow & & \downarrow \\
\text{Perf}(k[x]) & \rightarrow & \text{Perf}(k[x, x^{-1}]) \\
\end{array}
\]
and it follows that the canonical functor
\[
\text{Perf}(k(x, y)/(yx - 1)) \longrightarrow \text{Perf}(k[x, x^{-1}])
\]
is a Verdier quotient map up to idempotent completion. By [Tam18, Lemma 22] this is equivalent to the statement that the map $k(x, y)/(yx - 1) \to k[x, x^{-1}]$ is Tor-unital, which is also a nice exercise in (non)-commutative algebra to work out explicitly. It follows from the above diagram and the computation $K(\mathbb{P}^1_k) \cong K(k) \oplus K(k)$ that there is a fibre sequence of $K$-theory spectra
\[
K(k(x, y)/(yx - 1)) \longrightarrow K(k[x, x^{-1}]) \longrightarrow \Sigma K(k).
\]
Via this fibre sequence, the decomposition of $K(k(x, y)/(yx - 1))$ in Corollary 1.2 is compatible with the decomposition
\[
K(k[x, x^{-1}]) \cong K(k) \oplus NK(k) \oplus NK(k) \oplus \Sigma K(k)
\]
given by the fundamental theorem of algebraic $K$-theory.

**Remark 4.4.** Let us consider also the case where $\alpha = 0$, so that $k[x, y]/(yx - \alpha)$ is the coordinate ring of the coordinate axes in the affine $k$-plane. The kernel of the canonical map
\[
k(x, y)/(yx) \longrightarrow k[x, y]/(yx)
\]
is the two-sided ideal generated by the element $xy$. Since $(xy)^2 = 0$, this is a square zero ideal. So [DGM13, Theorem 7.0.0.2] implies that the following is a pullback square
\[
\begin{array}{ccc}
K(k(x, y)/(yx)) & \longrightarrow & \text{TC}(k(x, y)/(yx)) \\
\downarrow & & \downarrow \\
K(k[x, y]/(yx)) & \longrightarrow & \text{TC}(k[x, y]/(yx))
\end{array}
\]
and one can calculate the top two terms as in Corollary 1.2. Thus, in order to calculate the $K$-theory of the coordinate axes in the plane it suffices to calculate the fibre of the right vertical map. Let us mention that Hesselholt calculated the $K$-theory of the coordinate axes
relative to $k$ for $k$ a regular $\mathbb{F}_p$-algebra using the normalization Milnor square, excision for $K^{\text{inv}}$, and finally a calculation in topological cyclic homology [Hes07].

**APPENDIX A. CDH-DESCENT FOR TRUNCATING INVARIANTS**

In this appendix we prove that any truncating invariant satisfies cdh-descent. We fix a noetherian base scheme $S$, and we denote by $\text{Sch}_S$ the category of $S$-schemes of finite type. Recall that an abstract blow-up square is a pullback square

$$
\begin{array}{ccc}
D & \rightarrow & \tilde{X} \\
\downarrow & & \downarrow^p \\
Y & \hookrightarrow & X \\
\end{array}
$$

in $\text{Sch}_S$ where $i$ is a closed immersion, and $p$ is proper and an isomorphism over $X \setminus Y$. The cdh-topology on $\text{Sch}_S$ is generated by the Nisnevich coverings and the families $\{Y \rightarrow X, \tilde{X} \rightarrow X\}$ for any abstract blow-up square as above.

As it will become relevant shortly, let us recall the notion of a derived blow-up square from [KST18, §3.1]. Given a finite sequence $\bar{a} = (a_1, \ldots, a_r)$ of elements of the commutative noetherian ring $A$, we choose a commutative noetherian ring $A'$ together with a regular sequence $\bar{a}' = (a'_1, \ldots, a'_r)$ and a ring map $A' \rightarrow A$ sending $\bar{a}'$ to $\bar{a}$. We set $X = \text{Spec}(A)$, $Y = V((\bar{a})) \subseteq X$, and $X', Y'$ analogously and consider the following blow-up square.

$$
\begin{array}{ccc}
D' & \rightarrow & \text{Bl}_{Y'}(X') \\
\downarrow & & \downarrow \\
Y' & \hookrightarrow & X' \\
\end{array}
$$

The derived pullback of the square (21) along the map $X \rightarrow X'$ is a square of derived schemes for which we use the notation

$$
\begin{array}{ccc}
D & \rightarrow & \tilde{X} \\
\downarrow & & \downarrow \\
Y & \hookrightarrow & X \\
\end{array}
$$

Such a square is called a derived blow-up square associated to the sequence of elements $\bar{a}$. For any derived scheme $X$, we denote by $tX$ its underlying scheme. Then $tY \cong Y$, the square $t(22)$ is cartesian, and $t\tilde{X}$ contains the ordinary blow-up $\text{Bl}_Y(X)$ as a closed subscheme.

If $E$ is a localizing invariant, and if $X$ is a quasi-compact, quasi-separated (derived) scheme, we define $E(X)$ to be $E(\text{Perf}(X))$ where $\text{Perf}(X)$ denotes the $\infty$-category of perfect $O_X$-modules. We call a commutative square of schemes $E$-cartesian if $E$ sends it to a pullback square. Similarly, we call a morphism of schemes an $E$-equivalence if $E$ sends it to an equivalence. For convenience we record the following statements as a lemma.

**Lemma A.1.**

(i) Every localizing invariant satisfies Nisnevich descent.

(ii) A localizing invariant $E$ satisfies cdh-descent if and only if every abstract blow-up square is $E$-cartesian.

(iii) For every localizing invariant $E$, a derived blow-up square is $E$-cartesian.

**Proof.** By a theorem of Voevodsky, a presheaf $E$ on $\text{Sch}_S$ satisfies Nisnevich, respectively cdh-descent if and only if elementary Nisnevich squares, respectively elementary Nisnevich squares and abstract blow-up squares are $E$-cartesian, see e.g. [AHW17, Theorem 3.2.5]. Using this,
(i) follows from Thomason-Trobaugh, [TT90] and (ii) from (i). For (iii), one notes that the proof of [KST18, Theorem 3.7] applies for any localizing invariant in place of $K$-theory. □

**Theorem A.2.** Let $E$ be a truncating invariant. Then $E$ satisfies cdh-descent on $\text{Sch}_S$. Equivalently, it sends any abstract blow-up square (20) of noetherian schemes to a pullback square.

The proof follows the arguments in the proof of [KST18, Theorem A].

**Proof of Theorem A.2.** We show that any abstract blow-up square (20) is $E$-cartesian.

**Step 1.** If the map $p$ in diagram (20) is a closed immersion, then (20) is $E$-cartesian.

By Zariski descent we may assume that $X$ and hence all schemes in (20) are affine, say $X = \text{Spec}(A)$, $Y = \text{Spec}(A/I)$, $\tilde{X} = \text{Spec}(B)$ with $B = A/J$, and thus $D = \text{Spec}(B/I^N)$. Since the map $p$ is an isomorphism outside $Y$, there is an integer $N > 0$ such that $J \cdot I^N = 0$. By the Artin-Rees lemma we may assume that $J \cap I^N = 0$, so that the square

$$
\begin{array}{c}
A \\
\downarrow
\end{array}
\begin{array}{c}
A/I^N \\
\downarrow
\end{array}
\begin{array}{c}
B \\
\downarrow
\end{array}
\begin{array}{c}
B/I^N B
\end{array}
$$

is a Milnor square. The claim then follows from the fact that $E$ is satisfies excision by Theorem 3.3 and is nilinvariant by Corollary 3.5.

**Step 2.** If (20) is a blow-up square, i.e. if $\tilde{X} = \text{Bl}_Y(X)$, then (20) is $E$-cartesian.

We may again assume that $X$ is affine. Pick a finite sequence of generators of the ideal defining $Y \subseteq X$ and form an associated derived blow-up square as in (22). We obtain the following commutative diagram:

$$
\begin{array}{c}
D' \\
\downarrow
\end{array}
\begin{array}{c}
\text{Bl}_Y(Y) \\
\downarrow
\end{array}
\begin{array}{c}
\tilde{X} \\
\downarrow
\end{array}
\begin{array}{c}
\tilde{Y} \\
\downarrow
\end{array}
\begin{array}{c}
Y \\
\downarrow
\end{array}
\begin{array}{c}
X
\end{array}
$$

The left square is $E$-cartesian by Step 1, the middle square is $E$-cartesian because $E$ is truncating, and the right square is a derived blow-up square and thus $E$-cartesian by Lemma A.1. Finally, the map $Y \to \tilde{Y}$ is an $E$-equivalence, again since $E$ is truncating.

**Step 3.** If $\tilde{X} = \text{Bl}_Y(Y)$ for some closed subscheme $Y' \to Y$ of $Y$, then (20) is $E$-cartesian.

Consider the diagram of pullback squares:

$$
\begin{array}{c}
D'' \\
\downarrow (1)
\end{array}
\begin{array}{c}
\text{Bl}_Y(Y) \\
\downarrow (2)
\end{array}
\begin{array}{c}
\tilde{X} \\
\downarrow (3)
\end{array}
\begin{array}{c}
Y' \\
\downarrow
\end{array}
\begin{array}{c}
Y \\
\downarrow
\end{array}
\begin{array}{c}
X
\end{array}
$$

Square (1) is $E$-cartesian by Step 1. The square obtained by combining (1) and (2) is $E$-cartesian by Step 2, as is the square obtained by combining (2) and (3). Thus so is square (3).

**Step 4.** Any abstract blow-up square (20) is $E$-cartesian.
First, by Step 1 we may replace $X$ by the scheme theoretic closure of $X \setminus Y$ in $X$ and $Y$ by its pullback without changing the relative term $E(X,Y)$ up to equivalence. Similarly we may replace $\tilde{X}$ and $D$ and thus assume that $X \setminus Y$ is scheme-theoretically dense in $X$ and $\tilde{X}$. In this situation one can use Raynaud–Gruson’s platification par éclatements to find a closed subscheme $Y' \to X$ which is set-theoretically contained in $Y$ and such that there is a factorization $\text{Bly'}(X) \to \tilde{X} \to X$, see [Tem08, Lemma 2.1.5]. Since $E$ is nilinvariant by Corollary 3.5 we can replace $Y$ by an infinitesimal thickening of $Y$ so that $Y'$ becomes a closed subscheme of $Y$ without changing the value $E(Y)$. We then consider the following diagram of pullback squares.

$$
\begin{array}{ccc}
D' & \longrightarrow & D \\
\downarrow & & \downarrow \\
\text{Bly'}(X) & \longrightarrow & \tilde{X} \\
\end{array}
$$

The composite of relative $E$-terms

$$
E(X,Y) \overset{\alpha}{\longrightarrow} E(\tilde{X},D) \overset{\beta}{\longrightarrow} E(\text{Bly'}(X),D')
$$

is an equivalence by Step 3. Thus, we have now proven that for any abstract blow-up square, the induced map $\alpha$ of relative $E$-terms has a left-inverse. Since diagram (1) is itself an abstract blow-up square, it follows that $\beta$ itself has a left-inverse, and thus is an equivalence. Thus also $\alpha$ is an equivalence as needed.

**Corollary A.3.** The fibre $K^{\text{inv}}$ of the cyclotomic trace $K \to \text{TC}$ satisfies cdh-descent.

For schemes essentially of finite type over an infinite perfect field admitting strong resolution of singularities, a similar result was proven by Geisser and Hesselholt [GH10, Theorem B] using an argument of Haesemeyer [Hae04].

The following is the analog of [KST18, Theorem A] for topological cyclic homology.

**Corollary A.4.** Topological cyclic homology satisfies ‘pro-descent’ for abstract blow up squares of noetherian schemes, i.e. for any abstract blow-up square of noetherian schemes \[(20),\] the square of pro-spectra

$$
\begin{array}{c}
\text{TC}(X) \\
\downarrow \\
\{\text{TC}(Y_n)\} \\
\end{array}
\quad \begin{array}{c}
\longrightarrow \\
\downarrow \\
\longrightarrow \\
\end{array}
\begin{array}{c}
\text{TC}(\tilde{X}) \\
\{\text{TC}(D_n)\} \\
\end{array}
$$

where $Y_n$ and $D_n$ denote the $n$-th infinitesimal thickening of $Y$ in $X$ and $D$ in $\tilde{X}$, respectively, is weakly cartesian.

**Proof.** This follows directly from Corollary A.3 and the corresponding statement for $K$-theory which is [KST18, Theorem A].

One checks that the only arguments in the proof of [KST18, Theorem A] that are seemingly specific to $K$-theory are the following two:

(i) The fact that $K$-theory satisfies Nisnevich descent and descent for derived blow-up squares — as does any localizing invariant by Lemma A.1 and

(ii) the validity of [KST18, Theorem 4.11], which is precisely Corollary 2.31, which for instance also holds for topological Hochschild homology by Remark 2.32.

We thus obtain the following result.
Theorem A.5. If $E$ is a localizing invariant which satisfies Corollary [2.31] with $E$ in place of $K$-theory, then $E$ satisfies pro-descent for abstract blow-up squares of noetherian schemes. In particular, topological Hochschild homology $THH$ satisfies pro-descent for abstract blow-up squares of noetherian schemes.

Previously, Morrow proved that $THH$ and $TC$ with finite coefficients satisfy pro-descent for abstract blow-up squares of noetherian, $F$-finite $\mathbb{Z}(p)$-schemes of finite dimension, see [Mor16, Theorem 3.5].

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Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany

E-mail address: markus.land@ur.de

Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany

E-mail address: georg.tamme@ur.de