QUASILINEAR TROPICAL COMPACTIFICATIONS

NOLAN SCHOCK

Abstract. The prototypical examples of tropical compactifications are compactifications of complements of hyperplane arrangements, which possess a number of remarkable properties not satisfied by more general tropical compactifications of closed subvarieties of tori. We introduce a broader class of tropical compactifications, which we call quasilinear (tropical) compactifications, and which continue to satisfy the desirable properties of compactifications of complements of hyperplane arrangements. In particular, we show any quasilinear compactification is schön, and its intersection theory is described entirely by the intersection theory of the corresponding tropical fan. As applications, we prove the quasilinearity of the moduli spaces of 6 lines in \( \mathbb{P}^2 \) and marked cubic surfaces, obtaining results on the geometry of the stable pair compactifications of these spaces.

1. Introduction

Let \( Y \) be a \( d \)-dimensional closed subvariety of an \( n \)-dimensional torus \( T \cong (\mathbb{C}^*)^n \). It is natural to attempt to compactify \( Y \subset T \) by taking its closure \( \overline{Y} \) in a toric variety \( X(\Sigma) \) with torus \( T \). If \( \overline{Y} \) is proper and the induced multiplication map \( T \times \overline{Y} \to X(\Sigma) \) is flat and surjective, then \( \overline{Y} \) is called a tropical compactification of \( Y \) \cite{tcs07}. The definition implies that if \( \overline{Y} \subset X(\Sigma) \) is a tropical compactification, then it intersects each torus orbit in \( X(\Sigma) \) nontrivially in the expected dimension; therefore, one can pullback the stratification of \( X(\Sigma) \) by torus orbits to a stratification of \( \overline{Y} \), giving one a method to compare the geometry of \( \overline{Y} \) to the geometry of the ambient toric variety \( X(\Sigma) \).

The goal of the present article is to study how close the geometry of a tropical compactification is to the geometry of the ambient toric variety. Tropical compactifications can be quite general (see \cite[Chapter 6]{ms15} for many interesting examples), so at first glance one should not expect to be able to say much. On the other hand, the prototypical example of a tropical compactification is when \( Y \) is a complement of a hyperplane arrangement, in which case tropical compactifications of \( Y \) satisfy a number of remarkable properties:

- \( Y \) is schön, i.e., there is a tropical compactification \( \overline{Y} \subset X(\Sigma) \) of \( Y \) such that the multiplication map \( T \times \overline{Y} \to X(\Sigma) \) is smooth and surjective. This implies that any tropical compactification of \( Y \) has smooth multiplication map \cite[Theorem 1.4]{tcs07}, and if \( \overline{Y} \subset X(\Sigma) \) is any such tropical compactification, then all strata of \( \overline{Y} \) are smooth \cite[Lemma 2.7]{hac08}. In fact, when \( \overline{Y} \subset X(\Sigma) \) is a complement of a hyperplane arrangement, all strata of \( \overline{Y} \) are also complements of hyperplane arrangements \cite{kp11}.
- If \( \overline{Y} \subset X(\Sigma) \) is a tropical compactification of \( Y \), then there is an isomorphism of Chow rings

\[
A^*(\overline{Y}) \cong A^*(X(\Sigma)).
\]

If in addition the toric variety \( X(\Sigma) \) is nonsingular, then \( \overline{Y} \) is also nonsingular, and the Chow ring of \( \overline{Y} \) agrees with its cohomology ring \cite{gro15, adh22, ahk18}.

We call complements of hyperplane arrangements linear varieties, and their tropical compactifications linear (tropical) compactifications.

We define a broader class of closed subvarieties of tori, called quasilinear varieties, and their tropical compactifications quasilinear (tropical) compactifications, which includes the class of linear varieties (and linear tropical compactifications), for which the analogues of the above properties hold. Our main results are the following.

---

2020 Mathematics Subject Classification. Primary 14T90, 14J10; Secondary 14C15.

Key words and phrases. tropical compactification, tropical modification, del Pezzo surface, hyperplane arrangement, Chow ring, moduli space.
Theorem 1.1 (Theorems 4.16 and 5.2 and Corollary 5.3). (1) Quasilinear varieties are smooth, irreducible, rational, Chow-free, and linearly stratified (see Definitions 4.8 and 4.11). (2) Every stratum of a quasilinear tropical compactification is a quasilinear variety.

In particular, quasilinear varieties are schön.

Theorem 1.2 (Theorem 5.12). Let \( \Sigma \) be a quasilinear tropical compactification. Then for all \( k \), the pullback \( i^*: A^k(X(\Sigma)) \to A^k(Y) \) is an isomorphism, inducing an isomorphism of Chow rings \( A^*(Y) \cong A^*(X(\Sigma)) \).

If in addition the toric variety \( X(\Sigma) \) is nonsingular, then \( Y \) is also nonsingular, and the cycle class map induces an isomorphism \( A^*(Y) \cong H^*(Y) \).

Remark 1.3. We emphasize in particular that the above theorem yields an isomorphism of Chow rings even in the case where \( \Sigma \) is not simplicial. This is a generalization of previous results even in the linear case.

The definition of quasilinearity is inspired by recent work of Amini and Piquerez on Hodge theory for tropical fans [AP20, AP21]. Combinatorial Hodge theory, which has recently been developed to resolve a number of long-standing conjectures in combinatorics (cf. [AHK18, ADH22, Huh19]), begins from the observation that the isomorphism of cohomology rings \( H^*(Y) \cong A^*(X(\Sigma)) \) for linear tropical compactifications endows an a priori purely combinatorial object (the Chow ring of a realizable matroid) with the rich structure of the cohomology ring of a smooth projective algebraic variety. In particular, this ring satisfies Poincaré duality, Hard Lefschetz, and the Hodge-Riemann relations (the so-called “Kähler package”), which implies a number of remarkable combinatorial consequences. It was soon observed that even Chow rings of non-realizable matroids are Kähler [AHK18] and inspired by this Amini and Piquerez have defined a broader class of shellable tropical fans, continuing to satisfy the Kähler package [AP21].

We define in a slightly different fashion the notion of a quasilinear tropical fan, which it turns out is the same as a shellable tropical fan in the sense of Amini and Piquerez (Theorem 3.3). Namely, we say a reduced tropical fan is quasilinear if it is isomorphic to a complete tropical fan, or a fan supported on a tropical modification of a quasilinear tropical fan along a quasilinear tropical divisor (Definition 3.1). (A reduced tropical fan is one whose weights are all 1. Tropical modifications, introduced by Mikhalkin [Mik07], are the tropical versions of graphs of piecewise linear polynomials—the classical graph is unbalanced, so one must modify it by adding some additional cones. For precise definitions, see Sections 2 and 3. We say a closed subvariety \( Y \) of an algebraic torus \( T \) is quasilinear if its tropicalization is (the support of) a quasilinear tropical fan, and we define a quasilinear tropical compactification to be any tropical compactification of a quasilinear algebraic variety. This inductive definition of quasilinearity allows one to prove the main results Theorems 1.1 and 1.2 by a (careful) inductive procedure. Indeed, a key step in proving Theorems 1.1 and 1.2 is to prove the following main properties of quasilinear fans.

Theorem 1.4 (Theorems 5.3, 5.5, and 3.9). (1) Let \( \Sigma \) and \( \Sigma' \) be two reduced tropical fans. Then \( \Sigma \times \Sigma' \) is quasilinear if and only if both \( \Sigma \) and \( \Sigma' \) are quasilinear.

(2) Star fans of quasilinear tropical fans are quasilinear.

(3) Quasilinear tropical fans are Poincaré. (In fact, quasilinear tropical fans satisfy the Kähler package.)

1.1. Examples and applications. The latter half of this article is devoted to giving criteria for checking whether a given algebraic variety is quasilinear, and applying this criteria to describe some interesting nontrivial examples. Our main criteria are the following.

Theorem 1.5 (Theorem 6.1). Let \( Y \subset T \) be a quasilinear variety, and \( f \) a regular function on \( Y \) such that either \( f \) is nonvanishing or \( D = V(f) \subset Y \) is also quasilinear in \( T \). Then \( \tilde{Y} = \{ z = f \} \subset T \times \mathbb{C}^*_z \) is quasilinear.

Recall that the moduli space \( M_{0,n} \) of \( n \) points in \( \mathbb{P}^1 \) is a linear variety, the complement in \( \mathbb{P}^{n-3} \) of the hyperplanes \( x_i = 0 \) and \( x_i = x_j \). In particular, the (previously known) analogues of Theorems 1.1 and 1.2 imply that the moduli space \( \overline{M}_{0,n} \) of stable \( n \)-pointed rational curves is the log canonical compactification of \( M_{0,n} \) (cf. [KT06, HKT06, Section 2]), as well as quickly recover Keel’s results on the Chow ring of \( \overline{M}_{0,n} \) [Kee92].

There are two natural higher-dimensional analogues of \( \overline{M}_{0,n} \): the moduli space \( \overline{M}(r,n) \) of stable hyperplane arrangements, compactifying the moduli space \( M(r,n) \) of arrangements of \( n \) hyperplanes in general
position in $\mathbb{P}^{r-1}$ \cite{HKT06, Ale15}, and the moduli space $\overline{Y}(E_n)$ of (weighted) stable marked del Pezzo surfaces of degree $9-n$ \cite{HKT09}, compactifying the moduli space $Y(E_n)$ of smooth marked del Pezzo surfaces of degree $9-n$. Using the notion of quasilinearity, we obtain the analogues of the aforementioned results on $\overline{M}_{0,n}$ for the first cases of these higher-dimensional moduli spaces.

**Theorem 1.6** (Theorem 7.1 and Corollaries 7.2 and 7.3). (1) The moduli space $M(3,6)$ of 6 lines in $\mathbb{P}^2$ is quasilinear, and in particular schön.

(2) The stable pair compactification $\overline{M}(3,6)$ of $M(3,6)$ is the log canonical compactification.

(3) Let $\overline{M}^E(3,6) \subset X(\Sigma)$ be any tropical compactification of $M(3,6)$. Then there is an isomorphism of Chow rings $A^*(\overline{M}^E(3,6)) \cong A^*(X(\Sigma))$. If $X(\Sigma)$ is nonsingular, then $\overline{M}^E(3,6)$ is a resolution of singularities of $\overline{M}(3,6)$, and $A^*(\overline{M}^E(3,6)) \cong H^*(\overline{M}(3,6))$.

**Remark 1.7.** Keel and Tevelev have conjectured that (the normalization of the main irreducible component of) $\overline{M}(r,n)$ is the log canonical compactification of $M(r,n)$ precisely in the cases $(r,n) = (2,n)$ (in which case $\overline{M}(2,n) = \overline{M}_{0,n}$), and $(r,n) = (3,6),(3,7),(3,8)$ (as well as those cases obtained by duality $\overline{M}(r,n) \cong \overline{M}(n-r,n)$). This conjecture has recently been proven by work of Luxton \cite{Lux08}, Corey \cite{Cor21}, and Corey-Luber \cite{CL22}. The above theorem gives an alternative proof of the main step in Luxton’s argument for $\overline{M}(3,6)$. (For another alternative argument in this case, see \cite{Sch22b, Chapter 7}.)

The work of Corey and Luber \cite{CL22} implies that the moduli space $M(3,8)$ is not quasilinear, in contrast to an erroneous statement in the first preprint version of this article. On the other hand, the moduli space $M(3,7)$ can still be shown to be quasilinear, although the argument is quite involved—see \cite{Sch22a, Chapter 7} for details.

**Remark 1.8.** Part (3) of the above theorem also quickly recovers the main results of \cite{Sch22b}.

**Theorem 1.9** (Theorem 7.1 and Corollaries 7.10 and 7.11). (1) The moduli space $Y(E_6)$ of smooth marked cubic surfaces is quasilinear, and in particular schön.

(2) Let $\overline{Y}(E_6) \subset X(\Sigma)$ be any tropical compactification of $Y(E_6)$. Then $A^*(\overline{Y}(E_6)) \cong A^*(X(\Sigma))$. If $X(\Sigma)$ is nonsingular, then so is $\overline{Y}(E_6)$, and $A^*(\overline{Y}(E_6)) \cong H^*(\overline{Y}(E_6))$.

**Remark 1.10.** The log canonical compactification $\overline{Y}(E_6)$ of $Y(E_6)$ is described in \cite{HKT09}, following work of Naruki \cite{Nar82}. It is not the the moduli space of stable marked cubic surfaces, but it is a tropical compactification, and admits an interpretation as a moduli space of weighted stable marked cubic surfaces \cite{GKS21, Sch23}. The moduli space of stable marked cubic surfaces is also an explicitly described tropical compactification of $Y(E_6)$ \cite{HKT09}. In particular, the above theorem allows one to write down explicit presentations of the Chow rings of moduli of weighted stable marked cubic surfaces. The intersection theory of $\overline{Y}(E_6)$ has also previously been studied in \cite{Cv04}.

1.2. **Outline.** This paper is organized as follows.

In Section 2 we study the geometry of tropical fans. This section is largely a self-contained review of known results and definitions on tropical fans, following mainly \cite{GKM09, AR10, AP21}. We extend previous results by studying tropical intersection theory (and in particular Poincaré duality) for possibly non-simplicial fans. Our hope is that this section, while lengthy, could serve as a coherent introduction to tropical fans and their intersection theory for readers with little to no prior background in this area.

In Sections 3 to 5 we define respectively quasilinear fans, varieties, and compactifications and prove the main results concerning them. Section 5 also contains a quite general discussion on the Chow rings of tropical compactifications, including some criteria for isomorphisms of Chow rings which potentially hold outside of the quasilinear case, and may be of independent interest.

In Section 6 we study criteria for determining when a given closed subvariety of a torus is quasilinear, and apply this criteria to give some basic examples and non-examples of quasilinear varieties.

Finally, in Section 7 we apply the notion of quasilinearity to study the moduli spaces $M(3,6)$ of 6 lines in $\mathbb{P}^2$ and $Y(E_6)$ of marked cubic surfaces, proving the last of the main results.

**Acknowledgements.** This work has benefited greatly from conversations with Valery Alexeev, Omid Amini, Daniel Corey, Emanuele Delucchi, Alex Fink, and Diane Maclagan. I am also grateful to the anonymous referees for very helpful feedback, and for suggesting the terminology “linearly stratified” for varieties which are linear in the sense of \cite{Tot14}—improving the previous name of “weakly linear.”
2. Tropical preliminaries

2.0.1. Notation. Throughout we fix a lattice $N$ with dual $M = \text{Hom}(N, \mathbb{Z})$, and write $N_\mathbb{R} = N \otimes \mathbb{R} \cong \mathbb{R}^n$ (likewise for $M_\mathbb{R}$). We will often denote $N_\mathbb{R}$ just by $\mathbb{R}^n$. By a fan in $N_\mathbb{R}$ we mean a strongly convex, rational, polyhedral fan, so that all fans correspond to toric varieties [Ful93 Section 1.4]. We write $\Sigma_k$ for the set of $k$-dimensional cones of $\Sigma$. We write $\tau \prec \sigma$ if a cone $\tau$ is a face of a cone $\sigma$. We write $N_\sigma$ for the sublattice of $N$ generated by the cone $\sigma$. If $\tau \prec \sigma$ and $\dim \sigma = \dim \tau + 1$, then we write $n_{\sigma, \tau}$ for any lattice point in the relative interior of $\sigma$ whose image generates the 1-dimensional quotient lattice $N_\sigma / N_\tau$. The star fan of $\Sigma$ at a cone $\sigma$ is the fan in $N_\mathbb{R}/N_\sigma \mathbb{R}$ whose cones are the images of the cones of $\Sigma$ containing $\sigma$.

We denote by $X(\Sigma)$ the toric variety corresponding to a fan $\Sigma$, with torus $T = N \otimes \mathbb{C}^*$. We write $O(\sigma)$ for the torus orbit of $X(\Sigma)$ corresponding to the cone $\sigma$, and $V(\sigma)$ for its closure; recall that $V(\sigma)$ is isomorphic to the toric variety $X(\Sigma')$.

2.1. Basic definitions.

**Definition 2.1** ([GKM09, AR10]). A tropical fan in $N_\mathbb{R}$ is a pair $(\Sigma, \omega)$ consisting of a pure $d$-dimensional fan $\Sigma$ in $N_\mathbb{R}$, together with a function $\omega : \Sigma_d \to \mathbb{Z}_{\geq 0}$ satisfying the balancing condition: for every $(d - 1)$-dimensional cone $\tau$, one has

$$\sum_{\sigma \in \Sigma_d \atop \sigma \supset \tau} \omega(\sigma)n_{\sigma, \tau} = 0 \mod N_\tau.$$

The function $\omega$ is called the weight of the tropical fan.

**Example 2.2.** Any complete fan is tropical, with weight one on all top-dimensional cones.

**Example 2.3.** If $(\Sigma, \omega)$ and $(\Sigma', \omega')$ are two tropical fans, then $(\Sigma \times \Sigma', \omega \times \omega')$ is also tropical, where $(\omega \times \omega')(\sigma \times \sigma') = \omega(\sigma)\omega(\sigma')$ [GKM09 Example 2.9].

Any refinement $\tilde{\Sigma}$ of a tropical fan $(\Sigma, \omega)$ is also tropical, with the natural weight function $\tilde{\omega}(\tilde{\sigma}) = \omega(\sigma)$ for all $\tilde{\sigma} \in \tilde{\Sigma}_d$, where $\sigma \in \Sigma_d$ is the unique inclusion-minimal cone of $\Sigma$ containing $\tilde{\sigma}$. The fan $(\tilde{\Sigma}, \tilde{\omega})$ is then called a tropical refinement of $(\Sigma, \omega)$. Two tropical fans are said to be tropically equivalent if they have a common tropical refinement; this is indeed an equivalence relation [GKM09 Example 2.11].

**Definition 2.4** ([AR10]). A tropical fan cycle $(\mathcal{F}, \omega)$ in $N_\mathbb{R}$ is an equivalence class of tropical fans in $N_\mathbb{R}$ up to common tropical refinement. A (tropical) fan supported on $(\mathcal{F}, \omega)$, or a (tropical) fan structure on $(\mathcal{F}, \omega)$ is any representative of $(\mathcal{F}, \omega)$.

We typically think of a tropical fan cycle $(\mathcal{F}, \omega)$ as the support $\mathcal{F}$ of a pure-dimensional fan, such that any fan supported on $\mathcal{F}$ is a tropical fan, with the weight $\omega$. Thus, for instance, when we refer to the support of a tropical fan, we view it as a tropical fan cycle.

**Example 2.5.** If $(\Sigma, \omega)$ is a tropical fan of dimension $d$, and $\sigma \in \Sigma_k$, then the star fan $\Sigma^\sigma$ is a tropical fan of dimension $d - k$, with the natural weight $\omega^\sigma$ inherited from $\omega$.

If $(\mathcal{F}, \omega)$ is a tropical fan cycle, and $v \in \mathcal{F}$ (viewing $\mathcal{F}$ as the support of a fan in $N_\mathbb{R}$, as above), then set [Gub13 A.6]

$$\mathcal{F}^v = \{ u \in N_\mathbb{R} \mid u + \epsilon v \in \mathcal{F} \text{ for all sufficiently small } \epsilon \}.$$ 

Then $\mathcal{F}^v$ is also a tropical fan cycle with weight inherited from $\omega$. If $\Sigma$ is a fan structure on $\mathcal{F}$ such that $v$ is in the relative interior of a cone $\sigma \in \Sigma_k$, then

$$\mathcal{F}^v = |\Sigma^\sigma| \times \mathbb{R}^k.$$

**Definition 2.6.** A tropical fan $(\Sigma, \omega)$ is reduced if $\omega(\sigma) = 1$ for all top-dimensional cones $\sigma$ of $\Sigma$.

**Definition 2.7.** A tropical fan $(\Sigma, \omega)$ is irreducible if there is no tropical fan $(\Sigma', \omega')$ of the same dimension, such that $|\Sigma'| \subsetneq |\Sigma|$. A tropical fan $(\Sigma, \omega)$ is locally irreducible if all of its star fans are irreducible.

**Definition 2.8.**

- A property $\mathcal{P}$ of tropical fans is intrinsic to the support if whenever $(\Sigma, \omega)$ and $(\Sigma', \omega')$ are two representatives of the same tropical fan cycle, $(\Sigma, \omega)$ satisfies $\mathcal{P}$ if and only if $(\Sigma', \omega')$ satisfies $\mathcal{P}$.

- A property $\mathcal{P}$ of tropical fans is local if whenever $(\Sigma, \omega)$ satisfies $\mathcal{P}$, all of its star fans (viewed as tropical fans by Example 2.5) also satisfy $\mathcal{P}$. 
A property $\mathcal{P}$ of tropical fans which is intrinsic to the support is stably invariant if a tropical fan $(\Sigma, \omega)$ satisfies $\mathcal{P}$ if and only if $\Sigma \times \Delta$ satisfies $\mathcal{P}$ for all complete fans $\Delta$ of dimension $\geq 1$ [AP21 3.2.3].

The point of stable invariance is to ensure properties of tropical fans which are intrinsic to the support and local also give well-behaved local properties of tropical fan cycles. Indeed, continuing with the notation of Example 2.5, if $v \in \mathcal{F}$ and $\Sigma, \Sigma'$ are two fan structures on $\mathcal{F}$ such that $v$ is in the relative interior of $\sigma \in \Sigma'$ and $\sigma' \in \Sigma'$, then

$$\mathcal{F}^v = |\Sigma'| \times \mathbb{R}^{\dim \sigma} = |(\Sigma')^{\sigma'}| \times \mathbb{R}^{\dim \sigma'},$$

so in order to, for instance, obtain a property $\mathcal{P}$ for $(\Sigma')^{\sigma'}$ from the corresponding property for $\Sigma^\sigma$, we ask that the property is also satisfied by $|\Sigma^\sigma| \times \mathbb{R}^{\dim \sigma - \dim \sigma'}$.

Following the above discussion, suppose $\mathcal{P}$ is a property of tropical fans which is intrinsic to the support, local, and stably invariant. Then a tropical fan $\Sigma$ supported on a tropical fan cycle $\mathcal{F}$ satisfies $\mathcal{P}$ if and only if any tropical fan supported on $\mathcal{F}$ satisfies $\mathcal{P}$. Thus in this case we can also say that the tropical fan cycle $\mathcal{F}$ satisfies the property $\mathcal{P}$.

**Proposition 2.9.**
1. The property of being reduced is intrinsic to the support, local, and stably invariant.
2. The property of being irreducible is intrinsic to the support and stably invariant. In fact, if $\Sigma$ and $\Sigma'$ are two tropical fans, then $\Sigma \times \Sigma'$ is irreducible if and only if $\Sigma$ and $\Sigma'$ are both irreducible.
3. The property of being locally irreducible is intrinsic to the support, local, and stably invariant.

**Proof.** Immediate. \qed

In particular, by the preceding discussion, it makes sense to speak of tropical fan cycles being reduced or (locally) irreducible.

**Remark 2.10.** We warn the reader that while our definition of irreducibility given above agrees with the one given in [GKM09, GS12], our definition of local irreducibility differs from that of [GS12 Definition 2.21], which we might instead call irreducible in codimension one.

### 2.2. Intersection theory.

**Definition 2.11.** Let $\Sigma$ be any fan. The *Chow ring* (or ring of tropical cocycles) $A^*(\Sigma)$ of $\Sigma$ is the (operational [Ful98 Chapter 17]) Chow ring $A^*(X(\Sigma))$ of the corresponding toric variety $X(\Sigma)$.

We recall the following well-known presentation of the Chow ring of a unimodular fan.

**Theorem 2.12** ([Bri96, KP08]). Let $\Sigma$ be a unimodular fan in $\mathbb{N}_R$. Then

$$A^*(\Sigma) \cong \frac{\mathbb{Z}[x_\rho \mid \rho \in \Sigma_1]}{L + I},$$

where

- $L$ is the ideal of linear relations, generated by $\sum_{\rho \in \Sigma_1} \langle u, v_\rho \rangle x_\rho$ for $u \in M$.
- $I$ is the Stanley-Reisner ideal, generated by $x_{\rho_1} \cdots x_{\rho_k}$ whenever $\rho_1, \ldots, \rho_k$ do not span a cone of $\Sigma$.

If $\Sigma$ is simplicial, the same presentation holds with $\mathbb{Q}$-coefficients.

**Definition 2.13** ([FS97]). Let $\Sigma$ be any fan. The *group of k-dimensional Minkowski weights* (or *k-dimensional tropical fan cycles*) on $\Sigma$ is the group $M_k(\Sigma)$ of functions $\omega : \Sigma_k \to \mathbb{Z}$ satisfying the balancing condition: for any cone $\tau \in \Sigma_{k-1}$,

$$\sum_{\sigma \in \Sigma_k \atop \sigma \cap \tau} \omega(\sigma) n_{\sigma, \tau} = 0 \mod N_\tau.$$

In particular, in the definition of a tropical fan $(\Sigma, \omega)$, the weight $\omega$ is nothing more than a $d$-dimensional Minkowski weight on $\Sigma$. The group of $k$-dimensional Minkowski weights on $\Sigma$ is thus naturally interpreted as the group of $k$-dimensional tropical fan subcycles of $\Sigma$, *with fan structures induced by $\Sigma$*.

It is a standard fact that for any toric variety $X(\Sigma)$, the Chow group $A_{n-k}(X(\Sigma))$ is generated by the classes of the torus orbit closures $V(\sigma)$ for $\sigma \in \Sigma_k$, see Corollary [10]. The relations on $A_{n-k}(X(\Sigma))$ are described by the balancing condition, leading to the following fundamental description of $M_k(\Sigma)$.
Lemma 2.14 (FS97 Proposition 2.1]). For any fan $\Sigma$ in $N_\mathbb{R} \cong \mathbb{R}^n$, one has

$$M_k(\Sigma) \cong \text{Hom}(A_{n-k}(X(\Sigma)), \mathbb{Z}).$$

Definition 2.15. Let $(\Sigma, \omega)$ be a tropical fan. Define the (tropical) cap product

$$A^k(\Sigma) \times M_j(\Sigma) \to M_{j-k}(\Sigma),$$

as

$$\cap: A^k(X(\Sigma)) \times \text{Hom}(A_{n-j}(X(\Sigma)), \mathbb{Z}) \to \text{Hom}(A_{n-j+k}(X(\Sigma)), \mathbb{Z}),$$

$$(\alpha, \eta) \mapsto (\beta \mapsto \eta(\alpha \cap \beta)),$$

where $\alpha \cap \beta \in A_{n-j}(X(\Sigma))$ is the usual cap product between $\alpha \in A^k(X(\Sigma))$ and $\beta \in A_{n-j+k}(X(\Sigma))$.

Remark 2.16. When $\Sigma$ is unimodular or simplicial, $A^*(-\Sigma)$ can be described in terms of piecewise polynomials, and there is a purely combinatorial interpretation of the above cap product, see for instance [Fra12]. However, if $\Sigma$ is not simplicial, then piecewise polynomials do not give the correct notion for $A^*(\Sigma)$, cf. [KP08].

Remark 2.17. For a tropical fan cycle $\mathcal{F}$, one may define $A^*(-\mathcal{F})$ (resp. $M_*(\mathcal{F})$) as the direct limit over all fan structures $\Sigma$ on $\mathcal{F}$ of $A^*(-\Sigma)$ (resp. $M_*(\Sigma)$), cf. e.g. [GS21]. This will not play a significant role for us.

Definition 2.18. A tropical fan $(\Sigma, \omega)$ of dimension $d$ is Poincaré if $-\cap \omega: A^k(\Sigma) \to A_{d-k}(\Sigma)$ is an isomorphism for all $k$. The tropical fan $(\Sigma, \omega)$ is star-Poincaré if all of its star fans are Poincaré.

Remark 2.19. If $\Sigma$ is unimodular, then the toric variety $X(\Sigma)$ is nonsingular, so $A_{n-d+k}(X(\Sigma)) \cong A^{d-k}$. Poincaré duality in this case is therefore equivalent to the Chow ring $A^*(\Sigma)$ being a Poincaré duality ring of dimension $d$.

Example 2.20. Any complete fan is star-Poincaré by [FS97].

Minkowski weights lead to a very useful characterization of irreducible tropical fans.

Proposition 2.21 (GS12 Lemma 2.20]). A tropical fan $(\Sigma, \omega)$ is irreducible if and only if $M_d(\Sigma) \cong \mathbb{Z}$.

Definition 2.22. The fundamental weight of an irreducible tropical fan $\Sigma$ is the (unique) positive generator of $M_d(\Sigma) \cong \mathbb{Z}$.

Lemma 2.23. If $(\Sigma, \omega)$ is a Poincaré tropical fan, then $\Sigma$ is irreducible and $\omega$ is the fundamental weight. If $(\Sigma, \omega)$ is star-Poincaré, then it is reduced and locally irreducible.

Proof. Since for any fan $\Sigma$, the corresponding toric variety $X(\Sigma)$ is irreducible, one has $A^0(\Sigma) \cong \mathbb{Z}$. Poincaré duality implies that $A^0(\Sigma) \to M_d(\Sigma)$, $1 \mapsto \omega$ is an isomorphism, hence $\Sigma$ is irreducible by Proposition 2.21 and $\omega$ is the fundamental weight. It follows immediately that a star-Poincaré tropical fan is locally irreducible.

If $\sigma$ is a top-dimensional cone of $\Sigma$, then $\Sigma^\sigma$ is a 0-dimensional fan, so $M_0(\Sigma^\sigma) \cong \mathbb{Z}$, generated by the weight $w(0) = 1$. In particular, if $\Sigma$ is star-Poincaré at $\sigma$, then $\omega(\sigma) = w(0) = 1$. \qed

Proposition 2.24. Let $(\Sigma, \omega), (\Sigma', \omega')$ be two tropical fans. Then $A^*(\Sigma \times \Sigma') \cong A^*(\Sigma) \otimes A^*(\Sigma')$ and $M_*(\Sigma \times \Sigma') \cong M_*(\Sigma) \otimes M_*(\Sigma')$. In particular, $\Sigma \times \Sigma'$ is Poincaré, if and only if $\Sigma$ and $\Sigma'$ are both Poincaré.

Proof. The first part follows from the well-known fact that toric varieties satisfy the Chow-Künneth property: if $X(\Sigma)$ is a toric variety and $Z$ is any finite-type scheme, then $A_*(X(\Sigma)) \otimes A_*(Z) \to A_*(X(\Sigma) \times Z)$ is an isomorphism ([MSS95 Theorem 2]. (See also Lemma 5.6 below.))

For the second part, say $\text{dim } \Sigma = d$, $\text{dim } \Sigma' = d'$. Then

$$A^k(\Sigma \times \Sigma') \cong \bigoplus_{i+j=k} A^i(\Sigma) \otimes A^j(\Sigma')$$

and

$$M_{d+d'-k}(\Sigma \times \Sigma') \cong \bigoplus_{i+j=k} M_{d-i}(\Sigma) \otimes M_{d'-j}(\Sigma'),$$

and the cap product with the weight $\omega \times \omega'$ is given by

$$\bigoplus_{i+j=k} A^i(\Sigma) \otimes A^j(\Sigma') \Theta_{(-\cap \omega, -\cap \omega')} \bigoplus_{i+j=k} M_{d-i}(\Sigma) \otimes M_{d'-j}(\Sigma').$$
It follows that $\Sigma \times \Sigma'$ is Poincaré $\iff$ $\Sigma$ and $\Sigma'$ are both Poincaré.

**Theorem 2.25.** The property of being a star-Poincaré tropical fan is intrinsic to the support.

**Proof.** This is by now a well-known (and nontrivial) result in the unimodular case, see [AHK18] [ADH22] [AP21] [GS21]. The proof uses the weak factorization theorem for toric varieties [Wei97]. With our definition of Poincaré duality for non-unimodular fans, a similar proof also works for the non-unimodular case, cf. [GS21 Proposition 3.5].

**Corollary 2.26.** The property of being a star-Poincaré tropical fan is intrinsic to the support, local, and stably invariant.

**Proof.** “Local” is by definition, while “stably invariant” and “intrinsic to the support” follow by Proposition 2.24 and Theorem 2.25.

In particular, it makes sense to speak of star-Poincaré tropical fan cycles.

### 2.3. Morphisms

For a fan $\Sigma \subset N_\mathbb{R}$ (resp. fan cycle $F$), let $N_{\Sigma, \mathbb{R}}$ (resp. $N_{F, \mathbb{R}}$) be the vector subspace of $N_\mathbb{R}$ spanned by $\Sigma$ (resp $F$), and let $N_\Sigma = N \cap N_{\Sigma, \mathbb{R}}$ (resp. $N_F = N \cap N_{F, \mathbb{R}}$).

By definition, $N_\Sigma$ is spanned by a subset of a basis of $N$, so we do not need to worry about finite index sublattices (cf. [CLS11 Proof of Proposition 3.3.9]). We can view $\Sigma$ either as a fan in $N_\Sigma$ or as a fan in $N$; write $X(\Sigma; N_\Sigma)$ and $X(\Sigma; N)$ for the corresponding toric varieties. Extending a basis of $N_\Sigma$ to a basis of $N$ implies that [CLS11 Proposition 3.3.9]

$$X(\Sigma; N) \cong X(\Sigma; N_\Sigma) \times (\mathbb{C}^*)^{rk N - rk N_\Sigma}.$$ 

This implies in particular that $A^*(X(\Sigma; N)) \cong A^*(X(\Sigma; N_\Sigma))$ and $A_*(X(\Sigma; N)) \cong A_*(X(\Sigma; N_\Sigma))$, so both $A^*(\Sigma)$ and $M_*(\Sigma)$ are independent of the choice of ambient space between $N_\Sigma$ and $N$.

**Definition 2.27.** Let $\Sigma \subset N_\mathbb{R}$ and $\Sigma' \subset N'_\mathbb{R}$ be two fans. A morphism $f : \Sigma \to \Sigma'$ is a map $f : |\Sigma| \to |\Sigma'|$ induced by a linear map $f : N_\Sigma \to N_{\Sigma'}$, such that $f(\sigma)$ is contained in a cone of $\Sigma'$ for all $\sigma \in \Sigma$ ([CLS11 Section 3.3]).

Let $F \subset N_\mathbb{R}$ and $F' \subset N'_\mathbb{R}$ be two tropical fan cycles. A morphism $f : F \to F'$ is a map $f : F \to F'$ induced by a morphism of fans $\Sigma \to \Sigma'$ for some fan structures $\Sigma$ on $F$ and $\Sigma'$ on $F'$.

**Remark 2.28.** There is no condition on the fundamental weights for a morphism of tropical fans or tropical fan cycles, cf. [ARI10 Definition 4.1].

**Definition 2.29.** Let $f : (\Sigma, \omega) \to (\Sigma', \omega')$ be a morphism of tropical fans. The pullback of tropical fan cocycles is the usual pullback morphism $f^* : A^*(\Sigma') = A^*(X(\Sigma')) \to A^*(X(\Sigma)) = A^*(\Sigma)$.

**Definition 2.30.** Let $f : (\Sigma, \omega) \to (\Sigma', \omega')$ be a morphism of tropical fans, and assume that the image of each cone of $\Sigma$ is a cone of $\Sigma'$. The pushforward of tropical fan cycles is the morphism $f_* : M_k(\Sigma) \to M_k(\Sigma')$ sending $\eta \in M_k(\Sigma)$ to $f_*\eta \in M_k(\Sigma')$, defined by

$$f_*\eta(\sigma') = \sum_{\sigma \in \Sigma_k, f(\sigma) = \sigma'} \eta(\sigma)[N_{\sigma'} : f(N_\sigma)].$$

**Remark 2.31.** Recall the Chow groups of an arbitrary toric variety are generated by the classes of the torus orbit closures. If $f : (\Sigma, \omega) \to (\Sigma', \omega')$ is a morphism of tropical fans such that the image of each cone of $\Sigma$ is a cone of $\Sigma'$, then all fibers of the corresponding morphism of toric varieties $f : X(\Sigma) \to X(\Sigma')$ have the same dimension [AK00 Lemma 4.1]. Therefore one can define a pullback morphism $f^* : A^*_{n-k}(X(\Sigma')) \to A^*_n(X(\Sigma))$ by

$$f^*[V(\sigma')] = [f^{-1}(V(\sigma'))] = \sum_{\sigma \in \Sigma_k, f(\sigma) = \sigma'} [V(\sigma)] \cdot [N_{\sigma'} : f(N_\sigma)],$$

and $f_* : M_k(\Sigma) \to M_k(\Sigma')$ is the dual to this pullback under the isomorphism of Lemma 2.14 cf. [FS97 Proposition 3.7, Corollaries 4.6, 4.7].

If $\Sigma'$ is unimodular then $f : X(\Sigma) \to X(\Sigma')$ is flat [Har77 Exercise III.10.9], and the above pullback is the usual flat pullback as in [Ful98 Section 1.7].
Lemma 2.32 (Projection formula). Let \( f : (\Sigma, \omega) \to (\Sigma', \omega') \) be a morphism of tropical fans such that the image of every cone of \( \Sigma \) is a cone of \( \Sigma' \). Let \( \alpha \in A^d(\Sigma') \) and \( \eta \in M_j(\Sigma) \). Then
\[
f_*(f^* \alpha \cap \eta) = \alpha \cap f_* \eta.
\]

Proof. Let \( \beta \in A_{n-j+k}(X(\Sigma')) \). Then
\[
(\alpha \cap f_* \eta)(\beta) = f_* \eta(\alpha \cap \beta) = \eta(f^*(\alpha \cap \beta)) = \eta(f^* \alpha \cap f^* \beta) = (f^* \alpha \cap \eta)(f^* \beta) = f_*(f^* \alpha \cap \eta).
\]
(Here the second and last equality follow from the discussion of the above remark.) \( \square \)

Definition 2.33. Let \( (\Sigma, \omega) \subset N_\mathbb{R} \) and \( (\Sigma', \omega') \subset N'_\mathbb{R} \) be two tropical fans. An isomorphism \( f : (\Sigma, \omega) \simto (\Sigma', \omega') \) is a morphism induced by an isomorphism \( f : N_\Sigma \simto N_{\Sigma'} \), such that \( f(\sigma) \) is a cone of \( \Sigma' \) for all \( \sigma \in \Sigma \), \( f^{-1}(\sigma') \) is a cone of \( \Sigma \) for all \( \sigma' \in \Sigma' \), \( f^* \omega = \omega' \), and \( (f^{-1})^* \omega = \omega \).

Let \( (F, \omega) \subset N_\mathbb{R} \) and \( (F', \omega') \subset N'_\mathbb{R} \) be two tropical fan cycles. An isomorphism \( f : (F, \omega) \simto (F', \omega') \) is a morphism \( f : F \to F' \) induced by an isomorphism of tropical fans for some fan structures \( \Sigma \) on \( F \) and \( \Sigma' \) on \( F' \).

Remark 2.34. In our definition, an isomorphism of fans \( \Sigma \subset N_\mathbb{R} \) and \( \Sigma' \subset N'_\mathbb{R} \) does not necessarily induce an isomorphism of toric varieties \( X(\Sigma; N) \cong X(\Sigma'; N') \), but instead an isomorphism of toric varieties \( X(\Sigma; N_\Sigma) \cong X(\Sigma'; N_{\Sigma'}) \). Note this implies in particular that isomorphic tropical fans have isomorphic rings of tropical cocycles and groups of tropical cycles.

2.4. Divisors and tropical modifications. Tropical modifications, introduced by Mikhalkin [Mik07], play a fundamental role in this article. They are used in, e.g., [Sha13] to describe intersection theory of linear tropical fans, and in [AP21] to understand homology of tropical fans. Our exposition is based on [AR10, AP21].

2.4.1. Divisors and piecewise integral linear functions.

Definition 2.35. Let \( \Sigma \) be a fan. A linear function on \( \Sigma \) is a continuous function \( \ell : |\Sigma| \to \mathbb{R} \) which is the restriction of an integral linear function \( \ell \in M = \text{Hom}(N, \mathbb{Z}) \). A piecewise integral linear function on \( \Sigma \) is a continuous function \( \varphi : |\Sigma| \to \mathbb{R} \) that is linear on each cone \( \sigma \in \Sigma \), \( \varphi|_\sigma \) is identified with the restriction of an integral linear function \( \varphi_\sigma \in M_\sigma = \text{Hom}(N_\sigma, \mathbb{Z}) \). The group of piecewise integral linear functions on \( \Sigma \) is denoted by \( PP^1(\Sigma) \).

Proposition 2.36 ([KP08 Theorem 4.5, Corollary 4.6]). Let \( \Sigma \) be any fan. Then
\[
\text{Pic}(X(\Sigma)) \cong A^1(\Sigma) \cong PP^1(\Sigma) / M.
\]

Definition 2.37 ([AR10]). Let \( \Sigma \) be a tropical fan. A tropical Cartier divisor on \( \Sigma \) is an element of \( A^1(\Sigma) \).

Remark 2.38. While on the algebraic side \( A^1(X(\Sigma)) \) is really the group of Cartier divisor classes, there is no such distinction on the tropical side. This is because, linear functions on \( \Sigma \) actually define trivial tropical divisors, as opposed to potentially nontrivial principal Cartier divisors on \( X(\Sigma) \). See [AR10, AHR16] for more details.

Definition 2.39. Let \( \Sigma \) be a tropical fan of dimension \( d \). A tropical Weil divisor on \( \Sigma \) is an element of \( M_{d-1}(\Sigma) \).

Definition 2.40 ([AR10]). Let \( \varphi \) be a piecewise integral linear function on a \( d \)-dimensional tropical fan \( (\Sigma, \omega) \). The order of vanishing of \( \varphi \) along a cone \( \tau \in \Sigma_{d-1} \) is
\[
\text{ord}_\tau(\varphi) = \varphi_\tau \left( \sum_{\sigma \in \Sigma_d, \sigma \succ \tau} \omega(\sigma)n_{\sigma, \tau} \right) - \sum_{\sigma \in \Sigma_d, \sigma \succ \tau} \varphi_\sigma(\omega(\sigma)n_{\sigma, \tau})
\]
Lemma 2.41 ([AR10 Proposition 3.7]). The function $\Sigma_{d-1} \to \mathbb{Z}$, $\tau \mapsto \text{ord}_\tau(\varphi)$ is a well-defined (independent of the choice of $n_{\sigma,\tau}$) $(d-1)$-dimensional Minkowski weight on $\Sigma$.

Definition 2.42 ([AR10 Definition 3.4]). Let $\varphi$ be a piecewise integral linear function on a $d$-dimensional tropical fan $(\Sigma, \omega)$. The principal tropical Weil divisor associated to $\varphi$ is the tropical fan cycle $\text{div}(\varphi) \in M_{d-1}(\Sigma)$ defined by $\text{div}(\varphi)(\tau) = \text{ord}_\tau(\varphi)$.

When we wish to view $\text{div}(\varphi)$ as a tropical fan rather than a tropical fan cycle, we write it as $\text{div}(\varphi) = (\Delta, \delta)$, where $\Delta$ is the support of $\text{div}(\varphi)$, i.e. the $(d-1)$-dimensional fan consisting of the cones $\tau \in \Sigma_{d-1}$ for which $\text{ord}_\tau(\varphi) \neq 0$, and $\delta(\tau) = \text{ord}_\tau(\varphi)$. Thus by abuse of notation $\text{div}(\varphi)$ could refer to either the tropical fan cycle $\text{div}(\varphi) \in M_{d-1}(\Sigma)$ or the tropical fan $(\Delta, \delta)$.

Proposition 2.43 ([AP21 Proposition 4.8]). Let $(\Sigma, \omega)$ be a $d$-dimensional tropical fan and let $\varphi \in A^1(\Sigma)$. Then

$$\varphi \cap \omega = - \text{div}(\varphi)$$

Proof. This is proved in the reduced unimodular case in [AP21 Proposition 4.8]. A similar proof works in the general case, using in particular that the description of elements of $A^1(\Sigma)$ as piecewise integral linear functions modulo globally linear functions holds for arbitrary fans $\Sigma$ (Proposition 2.36).

Remark 2.44. Our definition of $\text{ord}_\tau(\varphi)$ agrees with the one in [AP21] (in the reduced case), but is the negative of the one in [AR10]. This is because we work with the “min” convention in tropical geometry, while [AR10] works with the “max” convention.

2.4.2. Tropical modifications. The definition of the tropical divisor associated to a piecewise integral linear function is motivated by the following observation.

Let $\varphi : \Sigma \to \mathbb{R}$ be a piecewise integral linear function on a tropical fan $(\Sigma, \omega)$ in $N_\mathbb{R}$. The graph of $\varphi$ gives a fan $\Gamma_{\varphi}(\Sigma) \subset \overline{N_\mathbb{R}} = \overline{N_\mathbb{R}} \times \mathbb{R}$ with cones

$$\overline{\sigma} = \{(x, \varphi(x)) \mid x \in \sigma\}.$$

A direct computation [AR10 Construction 3.3] shows that, for a codimension one cone $\tau$ of $\overline{\sigma}$,

$$\sum_{\dim \sigma = \dim \tau + 1} (n_{\sigma,\tau}, \varphi_\sigma(n_{\sigma,\tau})) = (0, -\text{ord}_\tau(\varphi)) \mod N_\tau.$$

In particular, $\Gamma_{\varphi}(\Sigma)$ fails the balancing condition around the cones $\overline{\tau}$ for which $\text{ord}_\tau(\varphi)$ is nonzero. Balancing is restored by adding the cones

$$\tau_\geq = \overline{\tau} + \mathbb{R}_{\geq 0}(0, 1)$$

for $\tau \in \text{div}(\varphi)$, with weight $\text{ord}_\tau(\varphi)$.

Definition 2.45. The tropical modification of a tropical fan $\Sigma \subset N_\mathbb{R}$ with respect to a piecewise integral linear function $\varphi$ is the tropical fan $\mathcal{T}M_{\varphi}(\Sigma) \subset \overline{N_\mathbb{R}} = \overline{N_\mathbb{R}} \times \mathbb{R}$ with cones

- $\overline{\sigma} = \{(x, \varphi(x)) \mid x \in \sigma\}$, for $\sigma \in \Sigma$,
- $\tau_\geq = \overline{\tau} + \mathbb{R}_{\geq 0}(0, 1)$ for $\tau \in \text{div}(\varphi)$, and weights $\tilde{\omega}(\overline{\sigma}) = \omega(\sigma)$ and $\tilde{\omega}(\tau_\geq) = \text{ord}_\tau(\varphi)$.

Tropical modifications are indeed tropical fans by [AR10 Proposition 3.7]—see [MR18 Section 4.5] and [AP21 Sections 4-5] (in the reduced case) for more details. See Section 4.2 for examples.

Remark 2.46. An important special case of tropical modifications occurs when the divisor $\Delta = \text{div}(\varphi)$ is trivial, in which case the tropical modification is said to be degenerate [AP21 5.1.4]. In general a degenerate tropical modification may differ from the original fan [AP21 Example 11.4]; however, this does not occur if $\varphi$ is a globally integral linear function, in which case the modification is simply a linear re-embedding of $\Sigma$, and therefore isomorphic to $\Sigma$. In particular, degenerate tropical modifications of Poincaré tropical fans are isomorphic to the original fan, see [AP21 5.1.4].

Remark 2.47. Tropical modifications depend both on the divisor $\Delta$ and the piecewise integral linear function $\varphi$. In [BL12 Example 4.3], the authors give an example of two different tropical modifications along the same divisor. On the other hand, if $\Sigma$ is Poincaré, then $\text{div} : A^1(\Sigma) \to M_{d-1}(\Sigma)$ is an isomorphism, so any two piecewise integral linear functions which define the same tropical Weil divisor differ by a linear function.
This linear function induces an isomorphism between the two tropical modifications, cf. [AP21 5.1.5]. Thus when \( \Sigma \) is Poincaré, we write \( \mathcal{T}\mathcal{M}_\Delta(\Sigma) \) to denote any tropical modification of \( \Sigma \) with respect to a piecewise integral linear function \( \varphi \) such that \( \text{div}(\varphi) = \Delta \).

**Lemma 2.48.** If \( \Sigma \) is a unimodular tropical fan, then any tropical modification of \( \Sigma \) is also unimodular.

**Proof.** Clearly if \( \sigma \in \Sigma \) is unimodular, then the cone \( \tilde{\sigma} \) is also unimodular. The possible new cones of \( \Sigma \) have the form \( \tilde{\tau} + \mathbb{R}_{\geq 0}(0, 1) \) for some \( \tau \in \Sigma \), hence these are clearly unimodular as well. \( \square \)

**Lemma 2.49.** A tropical modification of a reduced tropical fan along a trivial or reduced tropical divisor is reduced.

**Proof.** Immediate. \( \square \)

**Proposition 2.50** ( [AP21 Proposition 5.2]). Let \( \varphi \) be a rational function on a tropical fan \( \Sigma \), and let \( \widetilde{\Sigma} = \mathcal{T}\mathcal{M}_\varphi(\Sigma) \). Let \( \Delta = \text{div}(\varphi) \) (possibly trivial). The star fans of \( \widetilde{\Sigma} \) are described as follows.

1. If \( \sigma \in \Sigma \), then \( \widetilde{\Sigma}^\sigma \cong \mathcal{T}\mathcal{M}_{\varphi^\sigma}(\Sigma^\sigma) \), where \( \varphi^\sigma \) is any piecewise integral linear function on \( \Sigma^\sigma \) induced by \( \varphi \). In particular, if \( \sigma \nsubseteq \Delta \), then \( \widetilde{\Sigma}^\sigma \) is a degenerate tropical modification of \( \Sigma^\sigma \).

2. If \( \tau \in \Delta \), then \( \widetilde{\Sigma}^\tau \cong \Delta^\tau \).

**Proposition 2.51.** Let \( p : \tilde{\Sigma} = \mathcal{T}\mathcal{M}_\varphi \Sigma \to \Sigma \) be a tropical modification along a tropical divisor \( \Delta = \text{div}(\varphi) \). Then \( p_* : M_k(\tilde{\Sigma}) \to M_k(\Sigma) \) is injective for all \( k \).

**Proof.** Given a cone \( \sigma \in \Sigma_k \), the only \( k \)-dimensional cone \( \tau \in \tilde{\Sigma} \) such that \( p(\tau) = \sigma = \tau = \tilde{\sigma} \). It follows that for any \( w \in M_k(\Sigma) \) and any \( \sigma \in \Sigma_k \), \( p_*(w(\sigma)) = w(\tilde{\sigma}) \).

Suppose \( p_*w = 0 \), i.e., \( p_*w(\sigma) = w(\tilde{\sigma}) = 0 \) for all \( \sigma \in \Sigma_k \). The only other possible \( k \)-dimensional cones of \( \tilde{\Sigma} \) are \( \tau_{\geq} \) for \( \tau \in \Delta_{k-1} \), so we just need to show that \( w(\tau_{\geq}) = 0 \) for all such \( \tau \). But given \( \tau \in \Delta_{k-1} \), the \( k \)-dimensional cones of \( \tilde{\Sigma} \) containing \( \tilde{\tau} \) are precisely the cones \( \tilde{\sigma} \) for \( \sigma \in \Sigma_k \), \( \sigma \triangleright \tau \), and \( \tau_{\geq} \). So the balancing condition of \( w \in M_k(\Sigma) \) at \( \tilde{\tau} \in \Sigma_{k-1} \) says

\[
\sum_{\sigma \triangleright \tau} w(\tilde{\sigma})n_{\tilde{\sigma}, \tilde{\tau}} + w(\tau_{\geq})n_{\tau_{\geq}, \tilde{\tau}} = 0 \mod N_\tilde{\tau}.
\]

Since by assumption \( w(\tilde{\sigma}) = 0 \) for all \( \sigma \in \Sigma_k \), this reduces to the equation

\[
w(\tau_{\geq})n_{\tau_{\geq}, \tilde{\tau}} = 0 \mod N_\tilde{\tau}.
\]

By definition \( n_{\tau_{\geq}, \tilde{\tau}} \neq 0 \mod N_\tilde{\tau} \), so we conclude that \( w(\tau_{\geq}) = 0 \), hence \( p_* \) is injective. \( \square \)

**Corollary 2.52.** Any tropical modification of an irreducible tropical fan is irreducible. Tropical modifications of locally irreducible tropical fans along trivial or locally irreducible divisors are locally irreducible.

**Proof.** Suppose \( p : (\tilde{\Sigma}, \tilde{\omega}) \to (\Sigma, \omega) \) is a tropical modification and \( \Sigma \) is irreducible of dimension \( d \). Then \( p_* : M_d(\tilde{\Sigma}) \to M_d(\Sigma) \cong \mathbb{Z} \cdot \omega \) is injective by Proposition 2.51 and \( p_* \tilde{\omega} = \omega \) (cf. the proof of Proposition 2.41). Thus \( p_* : M_d(\tilde{\Sigma}) \to M_d(\Sigma) \) is an isomorphism, so \( \tilde{\Sigma} \) is irreducible by Proposition 2.31. The local case follows from Proposition 2.10. \( \square \)

**Proposition 2.53.** Let \( p : \tilde{\Sigma} = \mathcal{T}\mathcal{M}_\varphi \Sigma \to \Sigma \) be a tropical modification along a tropical divisor \( \Delta = \text{div}(\varphi) \). Then \( p^* : A^*(\Sigma) \to A^*(\tilde{\Sigma}) \) is a surjective morphism of graded rings, inducing a surjection \( p^* : A^k(\Sigma) \to A^k(\tilde{\Sigma}) \) for all \( k \).

**Proof.** In the case that \( \Sigma \), hence \( \tilde{\Sigma} \), is unimodular, this is a direct verification using the presentation of the Chow ring of a unimodular fan in Theorem 2.12. If the modification is degenerate, the result is immediate. Otherwise, \( A^*(\tilde{\Sigma}) \) is generated over \( A^*(\Sigma) \) by the class \( x_0 \) corresponding to the unique additional ray \( 0_{\geq} \), and the lattice point \( \tilde{m} = (0, 1) \in \tilde{M} \cong M \times \mathbb{Z} \) gives the relation

\[
x_0 = -\sum_{\rho \in \Sigma_1} \langle \tilde{m}, v_\rho \rangle x_\rho.
\]

See [AP21 Proposition 6.5] for more details.
If $\Sigma$ is not unimodular, let $\Sigma'$ be a unimodular refinement of $\Sigma$. This induces a unimodular refinement $\widetilde{\Sigma}'$ of $\widetilde{\Sigma}$ as well, allowing one to easily reduce to the unimodular case by using Kimura’s description of the Chow ring of a singular variety in terms of a resolution \cite[cf. \cite{Pay 06} Section 2]{Shu92}.

**Theorem 2.54.** Let $p : (\widetilde{\Sigma}, \widetilde{\omega}) \to (\Sigma, \omega)$ be a tropical modification.

1. If $- \cap \omega : A^k(\Sigma) \to M_{d-k}(\Sigma)$ is injective (resp. surjective) then $- \cap \widetilde{\omega} : A^k(\widetilde{\Sigma}) \to M_{d-k}(\widetilde{\Sigma})$ is injective (resp. surjective).
2. If $- \cap \omega : A^k(\Sigma) \to M_{d-k}(\Sigma)$ is injective, then $p^* : A^k(\Sigma) \to A^k(\widetilde{\Sigma})$ is injective, hence an isomorphism by Proposition 2.55.
3. If $- \cap \omega : A^k(\Sigma) \to M_{d-k}(\Sigma)$ is surjective, then $p_* : M_{d-k}(\widetilde{\Sigma}) \to M_{d-k}(\Sigma)$ is surjective, hence an isomorphism by Proposition 2.57.

In particular if $(\Sigma, \omega)$ is a Poincaré tropical fan, then $(\widetilde{\Sigma}, \widetilde{\omega})$ is a Poincaré tropical fan, and $p^* : A^*(\Sigma) \to A^*(\widetilde{\Sigma})$ and $p_* : M_*(\Sigma) \to M_*(\widetilde{\Sigma})$ are isomorphisms. Furthermore, a tropical modification of a star-Poincaré tropical fan along a trivial or star-Poincaré tropical divisor is star-Poincaré.

**Proof.** Everything follows easily from Propositions 2.50, 2.51 and 2.53 and the diagram

\[
\begin{array}{ccc}
A^k(\Sigma) & \xrightarrow{- \cap \omega} & M_{d-k}(\Sigma) \\
| & & | \\
p^* & \uparrow & p_* \\
A^k(\Sigma) & \xrightarrow{- \cap \widetilde{\omega}} & M_{d-k}(\Sigma),
\end{array}
\]

which commutes by the projection formula.

Recall that the properties of being reduced, locally irreducible, or star-Poincaré are intrinsic to the support, local, and stably invariant Proposition 2.49 and Corollary 2.26. The above results show that these properties are also preserved by tropical modifications, in the following sense.

**Definition 2.55.** A property $\mathcal{P}$ of tropical fans is preserved by tropical modifications if whenever $\Sigma$ is a tropical fan satisfying $\mathcal{P}$ and $\varphi$ is a piecewise integral linear function on $\Sigma$ such that $\text{div}(\varphi)$ is either trivial or satisfies $\mathcal{P}$, then $T_\mathcal{P}(\Sigma)$ satisfies $\mathcal{P}$.

**Corollary 2.56.** The properties of being reduced, locally irreducible, or star-Poincaré, are preserved by tropical modifications.

**Proof.** This is Lemma 2.49 Corollary 2.52 and Theorem 2.54.

**Proposition 2.57.** Let $(\Sigma, \omega), (\Sigma', \omega')$ be two tropical fans. Suppose $(\Sigma, \omega)$ is reduced. Let $\varphi$ be a piecewise integral linear function on $\Sigma'$ and define a piecewise integral linear function $\psi$ on $\Sigma \times \Sigma'$ by $\psi(x, y) = \varphi(y)$. Then

\[
\text{div}(\psi) = \Sigma \times \text{div}(\varphi) \quad \text{and} \quad T\mathcal{M}_\psi(\Sigma \times \Sigma') = \Sigma \times T\mathcal{M}_\varphi(\Sigma').
\]

**Proof.** Say $\dim \Sigma = d$, $\dim \Sigma' = d'$. Then a codimension one cone of $\Sigma \times \Sigma'$ either looks like $\sigma \times \tau'$ for $\sigma \in \Sigma_d$, $\tau' \in \Sigma_{d-1}'$, or $\tau \times \sigma'$ for $\tau \in \Sigma_{d-1}$, $\sigma' \in \Sigma_{d-1}'$.

In the former case, the top-dimensional cones containing $\sigma \times \tau'$ are all of the form $\sigma \times \sigma'$, where $\sigma'$ is a top-dimensional cone of $\Sigma'$ containing $\tau'$. Let $n_{\sigma \times \sigma', \sigma \times \tau'} = (v, n_{\sigma', \tau'})$, where $v$ is any lattice point in the relative interior of $\sigma$, and $n_{\sigma', \tau'}$ is any lattice point in the relative interior of $\sigma'$ whose image generates the one-dimensional quotient lattice $N_{\sigma'}/N_{\tau'}$. Then the image of $n_{\sigma \times \sigma', \sigma \times \tau'}$ generates the one-dimensional...
quotient lattice $N_{\sigma \times \sigma'}/N_{\tau \times \tau'} \cong 0 \times N_{\sigma'}/N_{\tau'}$. Therefore,

$$\text{ord}_{\sigma \times \tau'}(\psi) = \sum_{\sigma' \in \Sigma_d} \omega(\sigma') \omega(\sigma')(v, n_{\sigma', \tau'}) - \sum_{\sigma' \in \Sigma_d} \psi_{\sigma \times \sigma'}(\omega(\sigma') \omega(\sigma') (v, n_{\sigma', \tau'}))$$

$$= \varphi_{\sigma'} \left( \sum_{\sigma' \in \Sigma_d} \omega(\sigma') n_{\sigma', \tau'} \right) - \sum_{\sigma' \in \Sigma_d} \varphi_{\sigma'}(\omega(\sigma') n_{\sigma', \tau'})$$

$$= \text{ord}_{\tau'}(\varphi).$$

(The second-to-last equality is because $(\Sigma, \omega)$ is reduced, so $\omega(\sigma) = 1$ for all $\sigma \in \Sigma_d$.)

In the latter case, the top-dimensional cones of $\Sigma \times \Sigma'$ containing $\tau \times \sigma'$ are those of the form $\sigma \times \sigma'$ for $\sigma$ a top-dimensional cone of $\Sigma$ containing $\tau$. Let $n_{\sigma, \tau \times \sigma'} = (n_{\sigma, \tau}, v)$, where $n_{\sigma, \tau}$ is any lattice point in the relative interior of $\sigma$ whose image generates the one-dimensional quotient lattice $N_{\sigma}/N_{\tau}$, and $v$ is any lattice point in the relative interior of $\sigma'$. Then the image of $n_{\sigma, \tau \times \sigma'}$ generates the one-dimensional quotient lattice $N_{\sigma \times \sigma'}/N_{\tau \times \sigma'} \cong N_{\sigma}/N_{\tau} \times 0$. Therefore,

$$\text{ord}_{\tau \times \sigma'}(\psi) = \psi_{\tau \times \sigma'} \left( \sum_{\sigma \in \Sigma_d} \omega(\sigma) \omega(\sigma') (n_{\sigma, \tau}, v) \right) - \psi_{\sigma \times \sigma'}(\omega(\sigma) \omega(\sigma') (n_{\sigma, \tau}, v))$$

$$= \varphi_{\sigma'} \left( \sum_{\sigma \in \Sigma_d} \omega(\sigma) \omega(\sigma') v \right) - \varphi_{\sigma'}(\omega(\sigma) \omega(\sigma') v)$$

$$= 0$$

by linearity of $\varphi_{\sigma'}$. It follows that $\text{div}(\psi) = \Sigma \times \text{div}(\varphi)$.

Now if $\sigma \times \sigma'$ is any cone of $\Sigma \times \Sigma'$, then

$$\sigma \times \sigma' = \{ (x, y, \psi(x, y)) \mid (x, y) \in \sigma \times \sigma' \}$$

$$= \{ (x, y, \varphi(y)) \mid (x, y) \in \sigma \times \sigma' \}$$

$$= \sigma \times \sigma'. $$

If $\sigma \times \tau \in \text{div}(\psi) = \Sigma \times \text{div}(\varphi)$, then

$$\sigma \times \tau = \sigma \times \tau + \mathbb{R}_{\geq} (0, 0, 1)$$

$$= \sigma \times \tau + \mathbb{R}_{\geq} (0, 0, 1)$$

$$= \sigma \times \tau. $$

It follows that $\mathcal{T}M_{\psi}(\Sigma \times \Sigma') = \Sigma \times \mathcal{T}M_{\varphi}(\Sigma')$.

It will be useful to us to speak of tropical modifications without referring to a specific tropical fan structure.

**Definition 2.58.** Let $F$ be a tropical fan cycle. A **piecewise integral linear function** on $F$ is a continuous function $\varphi : F \to \mathbb{R}$ which is piecewise integral linear for some fan structure $\Sigma$ on $F$. The **tropical modification** $\mathcal{T}M_{\varphi}(\Sigma)$ of $F$ with respect to $\varphi$ is the tropical fan cycle in $\bar{N}_\mathbb{R} = N_\mathbb{R} \times \mathbb{R}$ defined by $\mathcal{T}M_{\varphi}(\Sigma)$. 
3. Quasilinear fans

**Definition 3.1.** A reduced tropical fan cycle $F$ is *quasilinear* if it is isomorphic to a complete tropical fan cycle, or a tropical modification of a quasilinear tropical fan cycle along a quasilinear tropical divisor.

A tropical fan $\Sigma$ is *quasilinear* if it is supported on a quasilinear tropical fan cycle.

**Remark 3.2.** The definition of quasilinearity is inductive on $\text{rk } N_F$.

**Theorem 3.3.** Quasilinear tropical fans are reduced, locally irreducible, and star-Poincaré.

**Proof.** Recall from Corollaries 2.26 and 2.56 and Proposition 2.9 that these properties are intrinsic to the support, local, stably invariant, and preserved by tropical modifications. The result follows by the definition of quasilinearity. □

**Corollary 3.4.** Let $\tilde{\Sigma}$ be a degenerate tropical modification of a quasilinear tropical fan $\Sigma$. Then $\tilde{\Sigma} \cong \Sigma$, and in particular $\tilde{\Sigma}$ is also quasilinear.

**Proof.** Since quasilinear tropical fans are star-Poincaré, this follows from the discussion of Remark 2.46. □

**Theorem 3.5.** Let $F$ and $G$ be two reduced tropical fan cycles. Then $F \times G$ is quasilinear if and only if $F$ and $G$ are both quasilinear.

**Proof.** Note that $F \times G$ is complete $\iff$ $F$ and $G$ are both complete. Therefore, we can assume without loss of generality that $G$ and $F \times G$ are not complete. We will prove the result by induction on the rank $n$ of the ambient lattice of $F \times G$. Note the base case $n \leq 2$ is trivial.

( $\iff$ ) Suppose $F$ and $G$ are both quasilinear. By assumption $G$ is a non-complete quasilinear tropical fan cycle, so $G \cong TM_{DG'}$ for some quasilinear tropical fan cycle $G'$ and quasilinear tropical divisor $D$ on $G'$. Then by Proposition 2.57

$$F \times G \cong F \times TM_{DG'} \cong TM_{F \times D}(F \times G'),$$

so by induction $F \times G$ is quasilinear.

( $\implies$ ) Suppose $F \times G$ is quasilinear. By assumption both $F \times G$ and $G$ are non-complete, so (after composing with an isomorphism if necessary) we can assume that the projection to the last coordinate realizes $F \times G$ as a tropical modification of a quasilinear tropical fan of the form $F \times G'$ along a quasilinear tropical divisor $D$. By induction $G'$ is quasilinear. Since by Proposition 2.21 we have

$$M_{d+d'-1}(F \times G') = (M_d(F) \otimes M_{d'-1}(G')) \oplus (M_{d-1}(F) \otimes M_d(G')),$$

it follows that $D = F \times D' + D'' \times G'$ for some (possibly trivial) divisors $D'$ on $G'$ and $D''$ on $F$. Since $D$ is quasilinear, it is irreducible. Both $F \times D'$ and $D'' \times G'$ are contained in $D$ and if nontrivial then they have the same dimension as $D$. It follows by irreducibility that one of them must be trivial and the other must be $D$ itself. Since $F \times G$ looks like the tropical modification of $F \times G'$ along $D$, the only possibility is that $D = F \times D'$ and $D'' \times G'$ is trivial. So we conclude that

$$F \times G \cong TM_{F \times D'}(F \times G') \cong F \times TM_{D'}G',$$

hence $G = TM_{D'}G'$. Since both $D'$ and $G'$ are quasilinear by induction, it follows that $G$ is quasilinear. □

**Theorem 3.6.** Star fans of quasilinear tropical fans are quasilinear.

**Proof.** Since star fans of complete fans are complete, and the star fans of a tropical modification $TM_{\Delta}(\Sigma)$ look either like (possibly degenerate) modifications of star fans of $\Delta$ along star fans of $\Sigma$, or star fans of $\Sigma$ (Proposition 2.50), it follows by induction on the rank of the ambient lattice that if star fans of $\Sigma$ and $\Delta$ are quasilinear, then so are star fans of $TM_{\Delta}(\Sigma)$. The result now follows by Theorem 3.5 and the discussion following Definition 2.8. □

**Corollary 3.7.** The property of being a quasilinear tropical fan is intrinsic to the support, local, stably invariant, and preserved by tropical modifications.

**Proof.** “Intrinsic to the support” and “preserved by tropical modifications” are by definition, and “local” and “stably invariant” are by Theorems 3.5 and 3.6. □
3.1. Shellable fans.

**Definition 3.8 (AP21 Definitions 5.3, 5.4).** The class of (tropically) shellable tropical fans is the smallest class of reduced tropical fans containing the 0-fan and the (unique) complete fan in $\mathbb{R}^1$, and which is closed under products, tropical modifications along trivial or shellable fans, and refinements and coarsenings along rays in the relative interior of cones whose star fans are shellable.

The motivation for this terminology comes from the observation that the simplicial complex associated to a tropically shellable tropical fan is shellable in the usual sense.

**Theorem 3.9.** A reduced tropical fan is quasilinear if and only if it is shellable.

**Proof.** ($\Rightarrow$) The class of quasilinear tropical fans includes the 0 fan and the unique complete fan in $\mathbb{R}^1$, is closed under products by Theorem 3.5, and is closed under refinements/coarsenings and tropical modifications by definition. Therefore, any reduced quasilinear fan is shellable.

($\Leftarrow$) Let $\Sigma$ be a reduced tropical fan. We will prove by induction on $n = \text{rk} N_\Sigma$ that $\Sigma$ is shellable if $\Sigma$ is quasilinear. In the case $n = 1$ the result is immediate, as the shellable tropical fans in $\mathbb{R}^1$ are precisely the (unique) complete fan and the 0 fan, and these are also precisely the quasilinear tropical fans in $\mathbb{R}^1$. Now assume the result for all $k < n$. Let $\Sigma$ be a shellable tropical fan in $\mathbb{R}^n$. Then $\Sigma$ is supported on some fan which is obtained either by a tropical modification of a shellable tropical fan in $\mathbb{R}^{n-1}$ along a trivial or shellable tropical divisor, or as the product of two shellable tropical fans in smaller ambient spaces. By induction, all of these smaller fans are quasilinear. Since products and tropical modifications of quasilinear fans are quasilinear, it follows that $\Sigma$ is quasilinear. □

**Corollary 3.10.** Quasilinear tropical fans are smooth in the sense of [AP21], and simplicial quasilinear tropical fans satisfy hard Lefschetz and the Hodge-Riemann relations.

**Proof.** This follows from the above theorem and [AP21 Theorem 5.7]. □

3.2. Examples. Recall that by a linear fan we mean any (reduced) tropical fan supported on the Bergman fan of a matroid.

**Theorem 3.11.** Linear fans are quasilinear.

**Proof.** Let $\Sigma_M$ be the Bergman fan associated to the matroid $M$. Then either $\Sigma_M$ is complete, or $\Sigma_M$ is a tropical modification of $\Sigma_{M\backslash i}$ along $\Sigma_{M/i}$ [Sha13 Proposition 2.25]. The result follows by induction. □

**Example 3.12.** We classify (up to isomorphism) all quasilinear tropical fan cycles in $\mathbb{R}^n$ for $n \leq 3$. Note the trivial fan is quasilinear, and complete fans are quasilinear. These are all of the possible fans in $\mathbb{R}^n$ for $n \leq 1$. For $n = 2, 3$, we only have to consider nontrivial, non-complete fans.

1. There are (up to isomorphism) two one-dimensional quasilinear fans in $\mathbb{R}^2$. They are pictured in Fig. 1. We call the first fan the “classical line” and the second fan the “(standard) tropical line” (in $\mathbb{R}^2$). The trivial line is a degenerate modification of $\mathbb{R}^1$; the tropical line is a modification of $\mathbb{R}^1$ along 0.

   ![Figure 1](image)

   (a) The classical line.  (b) The standard tropical line.

2. There are three types of one-dimensional quasilinear fans in $\mathbb{R}^3$. The first is a classical line $\mathbb{R}^1 \subset \mathbb{R}^3$.

   The second is the fan shown in Fig. 2a; it can be viewed either as a degenerate modification of the standard tropical line in $\mathbb{R}^2$, or as a nondegenerate modification of the classical line in $\mathbb{R}^2$.

   The third is the fan shown in Fig. 2b, known as the “standard tropical line in $\mathbb{R}^3$.”
(3) There are three types of two-dimensional quasilinear fans in $\mathbb{R}^3$. The first is a classical plane $\mathbb{R}^2 \subset \mathbb{R}^3$, which can be viewed as a degenerate tropical modification of $\mathbb{R}^2$.

The second is a tropical modification of $\mathbb{R}^2$ along a classical line, pictured in Fig. 3a. It is also isomorphic to the product of the standard tropical line in $\mathbb{R}^2$ with $\mathbb{R}^1$.

The third is a tropical modification of $\mathbb{R}^2$ along the standard tropical line, pictured in Fig. 3b. This fan is known as the “standard tropical plane (in $\mathbb{R}^3$).”

**Example 3.13.** The line $2y = 3x$ in $\mathbb{R}^2$ is isomorphic to $\mathbb{R}^1$ via the linear map $\mathbb{R}^2 \to \mathbb{R}^1$, $(x, y) \mapsto y - x$. Thus this line is a quasilinear tropical fan cycle, even though neither coordinate projection $\mathbb{R}^2 \to \mathbb{R}^1$ realizes it as a tropical modification.

**Example 3.14.** More generally, let $a, b \in \mathbb{Z} \setminus \{0\}$ and let $\Sigma$ be the line $ax = by$ in $\mathbb{R}^2$. Let $d = \gcd(a, b)$. Then there exist integers $x, y$ such that $ax + by = d$, and so the linear map

$$N_\Sigma \to \mathbb{Z}, \quad (a, b) \mapsto \frac{a}{d}x + \frac{b}{d}y = 1$$

induces an isomorphism of $\Sigma$ with $\mathbb{R}^1$. Thus $\Sigma$ is quasilinear.

**Example 3.15.** The fan $\Sigma$ in $\mathbb{R}^2$ pictured in Fig. 4 is not irreducible, and therefore not quasilinear. On the other hand, if $\varphi(x, y) = x - y$ for $x + y \geq 0$ and 0 otherwise, then the tropical modification of $\Sigma$ with respect to $\varphi$ is isomorphic to the standard tropical line in $\mathbb{R}^3$ [AP21, Example 11.4]. This is a nontrivial degenerate modification of $\Sigma$.

**Figure 4.** A reducible fan in $\mathbb{R}^2$.

**Example 3.16.** The fan $\Sigma$ in $\mathbb{R}^2$ pictured in Fig. 5 is tropical with weights 1, 2, 1 on the rays through (1, 0), (0, 1), (−1, −2), respectively. In particular this fan is not reduced, and therefore not quasilinear. Note $\Sigma$ is the tropical modification of $\mathbb{R}^1$ with respect to the function $\varphi(x) = \min\{2x, 0\}$; the divisor of this function is the origin with weight 2.
4. QUASILINEAR VARIETIES

4.1. Preliminaries. We will assume some familiarity with the basics of tropicalizations and tropical compactifications of closed subvarieties of tori, see for instance [Tev07, LQ11, Gub13], [MS15, Chapter 6]. We quickly summarize here the relevant definitions and facts.

Let $T$ be an algebraic torus and $X(\Sigma)$ a toric variety with torus $T$. Recall that the closure $\overline{Y} \subset X(\Sigma)$ of a pure-dimensional closed subscheme $Y \subset T$ is a tropical compactification if $\overline{Y}$ is proper and the multiplication map $m: \overline{Y} \times T \to X(\Sigma)$ is faithfully flat (i.e., flat and surjective) [Tev07, Definition 1.1]. This implies in particular that $\overline{Y}$ intersects each torus orbit $O(\sigma)$ of $X(\Sigma)$ nontrivially in the expected dimension, inducing a stratification of $\overline{Y}$ in dimension $k$ by strata of the form $Y_\sigma = \overline{Y} \cap O(\sigma)$ for $\sigma \in \Sigma_k$. The tropical compactification $\overline{Y} \subset X(\Sigma)$ is called schön if the multiplication map $m: \overline{Y} \times T \to X(\Sigma)$ is in addition smooth [Tev07, Definition 1.3], or equivalently all strata $Y_\sigma$ of $\overline{Y}$ are smooth [Hac08, Lemma 2.7].

The tropicalization $1 \text{ trop}(Y)$ of $Y$ is the tropical fan cycle defined by the fan $\Sigma$ of any tropical compactification $\overline{Y} \subset X(\Sigma)$ of $Y$, where the weight $\omega(\sigma)$ of a top-dimensional cone $\sigma$ of $\Sigma$ is equal to the length of the zero-dimensional scheme $Y_\sigma$. We refer to $\omega$ as the induced weight (from $\text{trop}(Y)$) when we wish to emphasize that it is the weight coming from $\text{trop}(Y)$, rather than some other weight on $\text{trop}(Y)$ as an abstract tropical fan cycle.

While it is not the case that every tropical fan supported on $\text{trop}(Y)$ yields a tropical compactification of $Y$ [ST08, Example 3.10], it is true that if $\Sigma$ gives a tropical compactification of $Y$, then so does any refinement of $\Sigma$ [Tev07, Proposition 2.5]; furthermore, if $Y$ admits a schön tropical compactification, then any fan supported on $\text{trop}(Y)$ also gives a schön tropical compactification of $Y$, hence it makes sense to call $Y$ itself schön [LQ11, Theorem 1.5].

Lemma 4.1. If $Y \subset T$ is any pure-dimensional closed subscheme such that $\text{trop}(Y)$ is irreducible and the induced weight is the fundamental weight, then $Y$ is irreducible and generically reduced. In particular, if $\text{trop}(Y)$ is reduced and irreducible, then $Y$ is irreducible and generically reduced.

Proof. Let $Y = Y_1 \cup \cdots \cup Y_k$ be the decomposition of $Y$ into its irreducible components, with multiplicities $a_1, \ldots, a_k$. Then $\text{trop}(Y) = \text{trop}(Y_1) \cup \cdots \cup \text{trop}(Y_k)$ by [OP13 Definition 2.5.2, Proposition 2.7.2]. Each $\text{trop}(Y_i)$ is a tropical fan cycle of the same dimension as $\text{trop}(Y)$, with support contained in $\text{trop}(Y)$, so by irreducibility of $\text{trop}(Y)$ we have $\text{trop}(Y) = \text{trop}(Y_1) = \cdots = \text{trop}(Y_k)$ as sets. Furthermore,

$$\omega = a_1 \omega_1 + \cdots + a_k \omega_k,$$

where $\omega, \omega_i$ are the induced weights on $\text{trop}(Y)$, $\text{trop}(Y_i)$. Since $\text{trop}(Y_i) = \text{trop}(Y)$ for all $i$, we see that $\omega_i(\sigma) \neq 0$ for every cone $\sigma$ in $\text{trop}(Y)$. Since $\text{trop}(Y)$ is irreducible and $\omega$ is the fundamental weight on $\text{trop}(Y)$, it follows by Proposition [221] that for $i = 1, \ldots, k$ there is some integer $b_i > 0$ such that $\omega_i = b_i \omega$. Then

$$\omega = (a_1 b_1 + \cdots + a_k b_k) \omega.$$
hence
\[ 1 = a_1 b_1 + \cdots + a_k b_k. \]
Since the \( a_i \)'s and \( b_i \)'s are all positive integers, it follows that \( k = 1 \) and \( a_1 = b_1 = 1 \), so \( Y \) is irreducible and generically reduced. The last sentence follows from the first, since the statement that \( \text{trop}(Y) \) is reduced and irreducible means that \( \text{trop}(Y) \) is irreducible and reduced with the induced weight, thus the induced weight is the fundamental weight. \qedhere

**Lemma 4.2.** Let \( \mathcal{Y} \subset X(\Sigma) \) be a tropical compactification of \( Y \subset T \). Then for every \( \sigma \in \Sigma, Y_\sigma = \mathcal{Y} \cap O(\sigma) \) is nonempty of pure dimension \( \dim \mathcal{Y} - \dim \sigma \), and \( \text{trop}(Y_\sigma \subset O(\sigma)) \) is a tropical fan cycle supported on \( \Sigma^\sigma \), where the weight on \( \Sigma^\sigma \) induced by \( Y_\sigma \) is the same as the weight induced by \( \Sigma \).

**Proof.** See [Gub13, Proposition 13.13, 14.3, 14.7]. \qedhere

**Remark 4.3.** The tropicalization of \( Y \) can equivalently be defined as the closure in \( N_\mathbb{R} \) of the set of points \( w \in N \) such that the initial degeneration \( \text{in}_w(Y) \) is nonempty, see for instance [Gub13]. If \( \Sigma \) is a fan structure on \( \text{trop}(Y) \) giving a tropical compactification \( \mathcal{Y} \subset X(\Sigma) \), and \( w \) is in the relative interior of a cone \( \sigma \in \Sigma \), then \( \text{in}_w(Y) \cong Y_\sigma \times (\mathbb{C}^*)^{\dim \sigma} \) [HK12, Lemma 3.6], reflecting the fact that \( \text{trop}(Y)^w = |\Sigma^\sigma| \times \mathbb{R}^{\dim \sigma} \) (cf. Example 2.5).

**Corollary 4.4.** Suppose \( Y \subset T \) is a pure-dimensional closed subscheme such that \( \text{trop}(Y) \) is irreducible and the induced weight is the fundamental weight. If the reduced induced subscheme structure \( Y_{\text{red}} \) on \( Y \) is normal, then \( Y \) is reduced and irreducible, i.e., \( Y = Y_{\text{red}} \).

**Proof.** By Lemma 4.1 \( Y \) is irreducible and generically reduced, hence \( Y_{\text{red}} \) is reduced and irreducible. The result now follows by [Carl12, Theorem 11] (using the identification \( \text{in}_0(Y) \cong Y \) as in the above remark). \qedhere

### 4.2. Quasilinear varieties.

**Definition 4.5.** A closed subvariety \( Y \) of an algebraic torus \( T \) is called **quasilinear** if \( \text{trop}(Y) \) is a quasilinear tropical fan cycle.

**Remark 4.6.** We emphasize in particular that this definition implies \( \text{trop}(Y) \) is irreducible, and reduced with the induced weight from \( Y \), cf. Theorem 3.3.

For examples of quasilinear varieties, see Sections 6.2 and 7.

**Lemma 4.7.** Fix an algebraic torus \( \mathcal{T} \) with cocharacter lattice \( \mathcal{N} \). Let \( \mathcal{Y} \subset \mathcal{T} \) be an irreducible closed subvariety such that \( \text{trop}(\mathcal{Y}) \) is irreducible and the induced weight is the fundamental weight. If there is a linear map \( p : \mathcal{N} \rightarrow \mathbb{Z}^\vee \), \( \text{rk } \mathcal{N} = \text{rk } \mathcal{N} - 1 \), realizing \( \text{trop}(\mathcal{Y}) \) as a tropical modification of an irreducible tropical fan cycle \( \mathcal{F} \) in \( \mathcal{N} \), then the corresponding projection of tori \( \pi : \mathcal{T} \rightarrow \mathcal{T} \) restricts to a birational morphism with finite fibers \( \pi : \mathcal{Y} \rightarrow Y \) to an irreducible closed subvariety \( Y \subset T \) such that \( \text{trop}(Y) = \mathcal{F} \) as tropical fan cycles.

**Proof.** Let \( Y = \pi(\mathcal{Y}) \). Then \( Y \) is an irreducible closed subvariety of \( T \), and \( \pi : \mathcal{Y} \rightarrow Y \) is dominant, so by [ST08, Proposition 2.8],
\[ \text{trop}(Y) = p(\text{trop}(\mathcal{Y})) = \mathcal{F}. \]
In particular,
\[ \dim Y = \dim \text{trop}(Y) = \dim \text{trop}(\mathcal{Y}) = \dim \mathcal{Y}, \]
so \( \pi : \mathcal{Y} \rightarrow Y \) is a dominant morphism of irreducible varieties of the same dimension, hence is generically finite, say of degree \( \delta \). Let \( \omega \) and \( \omega \) be the induced weights on \( \text{trop}(\mathcal{Y}) \) and \( \text{trop}(Y) \). By [ST08, Theorem 3.12],
\[ \omega = \frac{1}{\delta} p_\omega \omega. \]
But since \( \omega \) is the fundamental weight on \( \text{trop}(\mathcal{Y}) \), it pushes forward to the fundamental weight on \( \text{trop}(Y) \) by Corollary 2.5.2. Thus \( \omega = p_\omega \omega \) and \( \delta = 1 \). Therefore, \( \pi : \mathcal{Y} \rightarrow Y \) is dominant and generically finite of degree one, i.e., it is birational.

Let \( y \in Y \). Then \( \pi^{-1}(y) \subset \mathcal{Y} \) is a closed subscheme of \( \mathcal{Y} \) of dimension \( \leq 1 \). Suppose \( \pi^{-1}(y) \) is nonempty. Then \( \pi : \pi^{-1}(y) \rightarrow y \) is dominant, so by [ST08, Proposition 2.8], \( p(\text{trop}(\pi^{-1}(y))) = \text{trop}(y) = 0 \in \mathbb{N}_R. \)
Thus if \( \Sigma \) is a fan structure on \( \trop(Y) \) inducing a fan structure on \( \trop(\pi^{-1}(y)) \), then a cone \( \sigma \in \Sigma \) lies in \( \trop(\pi^{-1}(y)) \) if and only if \( p(\sigma) = 0 \). Since \( p \) realizes \( \Sigma \) as a tropical modification of a fan structure on \( F \), we see that the only possible cones of \( \trop(\pi^{-1}(y)) \) are the 0-cone, and \( 0_\Sigma \). But there is no way for the one-dimensional fan consisting of just the ray \( 0_\Sigma \) to be balanced, so we conclude that \( \trop(\pi^{-1}(y)) = 0 \), and in particular \( \dim\pi^{-1}(y) = \dim\trop(\pi^{-1}(y)) = 0 \). Thus every fiber of \( \pi : \tilde{Y} \to Y \) is either empty or zero-dimensional. \( \square \)

We introduce two general properties of varieties which will be useful later (Section 5.2).

**Definition 4.8.** An irreducible variety \( Y \) is Chow-free if \( A_{\dim Y}(Y) = \mathbb{Z} \cdot [Y] \) and \( A_k(Y) = 0 \) for \( k \neq \dim Y \).

**Lemma 4.9.** Any open subvariety of a Chow-free variety is Chow-free.

**Proof.** Let \( Y \) be Chow-free, \( U \subset Y \) any (nonempty) open subvariety, and \( Z = Y \setminus U \). Then \( \dim Z < \dim Y \), so the result is immediate from the standard exact sequence

\[
A_k(Z) \to A_k(Y) \to A_k(U) \to 0.
\]

**Example 4.10.** Affine space is Chow-free, so any open subvariety of affine space is Chow-free. In particular, any open subvariety of an algebraic torus is Chow-free.

**Definition 4.11.** A variety \( Y \) is linearly stratified if it is isomorphic to an affine space or the complement of a linearly stratified variety in a linearly stratified variety, or if it contains a linearly stratified variety \( Z \) such that the complement \( Y \setminus Z \) is also linearly stratified.

**Remark 4.12.** Linearly stratified varieties are called “linear” in [Tot14, Ian06]. We rename them “linearly stratified” to avoid confusion with our notion of linear varieties.

It follows from the definition that a variety stratified by linearly stratified varieties is itself linearly stratified. Thus the class of linearly stratified varieties is strictly larger than both the class of linear varieties (in our sense) and the class of quasilinear varieties.

Recall from Section 2.3 that if \( F \) is a tropical fan cycle in \( N_R \), then \( N_F = N_{F,R} \cap N \), where \( N_{F,R} \) is the vector subspace of \( N_R \) spanned by \( F \).

**Definition 4.13.** Let \( Y \subset T \) be a closed subvariety. The minimal torus of \( Y \) is the torus \( T_Y = N_{\trop Y} \otimes \mathbb{C}^* \subset T \).

**Definition 4.14.** Let \( Y \subset T \) and \( Y' \subset T' \) be closed subvarieties of algebraic tori \( T \) and \( T' \) with respect character lattices \( N \) and \( N' \). A tropical isomorphism \( Y \xrightarrow{\sim} Y' \) is a linear isomorphism \( N_{\trop Y} \xrightarrow{\sim} N_{\trop Y'} \), inducing an isomorphism \( \trop(Y) \xrightarrow{\sim} \trop(Y') \), and such that the corresponding isomorphism of minimal tori \( T_Y \xrightarrow{\sim} T_{Y'} \) restricts to an isomorphism \( Y \xrightarrow{\sim} Y' \).

**Remark 4.15.** By [Jel20] Proposition 5.3, if \( Y \subset T \) and \( T_Y \) is the minimal torus of \( Y \), then some translate \( Y' \cong Y \) of \( Y \) is contained in \( T_Y \), and \( \trop(Y' \subset T_Y) \cong \trop(Y \subset T) \). Thus \( Y \subset T \) is tropically isomorphic to \( Y' \subset T_Y \), so there is no harm in replacing the original embedding \( Y \subset T \) with \( Y' \subset T_Y \). For the rest of this section we will therefore assume that \( T \) is itself the minimal torus of \( Y \).

**Theorem 4.16.** Quasilinear varieties are smooth, irreducible, rational, Chow-free, and linearly stratified.

**Proof.** Let \( \tilde{Y} \subset \tilde{T} \) be a quasilinear variety. Following Remark 4.15 we reduce to the case where \( \tilde{T} \) is the minimal torus of \( \tilde{Y} \). If \( \trop(\tilde{Y}) = \mathbb{N}_R \), then \( \tilde{Y} = \tilde{T} \) and we are done. Otherwise, \( \trop(\tilde{Y}) \) is a tropical modification of a quasilinear tropical fan cycle \( F \) in \( N_R \), \( \rk N = \rk \mathbb{N} - 1 \), along a quasilinear tropical divisor \( D \). By Lemma 4.7 the corresponding projection of tori restricts to a birational morphism with finite fibers \( \pi : \tilde{Y} \to Y \) to some variety \( Y \subset T \) with \( \trop(Y) = F \) (as tropical fan cycles). Then \( Y \) is quasilinear, so by induction on the dimension of the ambient torus, \( Y \) is smooth, irreducible, rational, Chow-free, and linearly stratified. In particular, \( Y \) is normal, so by a version of Zariski’s main theorem, \( \pi : \tilde{Y} \to Y \) is an open embedding [Liu02, Ch. 4, Corollary 4.6]. Since \( \pi \) is a morphism of affine varieties, it is also affine, so \( \pi(\tilde{Y}) \cong \tilde{Y} \) is a dense affine open subvariety of \( Y \). Therefore, by [SPA22, Lemma 0BCV], \( D = Y \setminus \pi(\tilde{Y}) \) is either empty or has codimension one in \( Y \). If \( D \) is empty, then \( \pi : \tilde{Y} \to Y \) is an isomorphism, and \( \trop(\tilde{Y}) \to \trop(Y) \) is a degenerate modification, so we’re done. If \( D \) is nonempty, then \( \trop(D) = \mathbb{D} \)
is quasilinear, hence \( D \) is irreducible and generically reduced by Lemma 4.1. Now if \( D_{\text{red}} \) is the reduced induced subscheme structure on \( D \), then \( \text{trop}(D_{\text{red}}) = \text{trop}(D) = D \) (with the same weights, since \( \text{trop}(D) \) is reduced), so \( D_{\text{red}} \) is a quasilinear variety. By induction \( D_{\text{red}} \) is smooth, irreducible, rational, Chow-free, and linearly stratified. In particular \( D_{\text{red}} \) is normal, so by Corollary 4.14 \( D = D_{\text{red}} \), and the result follows. \( \square \)

We can extract from the above theorem and its proof some more detailed facts about quasilinear varieties.

**Corollary 4.17.** Let \( Y \subset T \) be any pure-dimensional closed subscheme such that \( \text{trop}(Y) \) is quasilinear. Then \( Y \) is actually a quasilinear variety (i.e., \( Y \) is reduced and irreducible).

**Proof.** By assumption, since \( \text{trop}(Y) \) is quasilinear, it is reduced and irreducible Theorem 3.3. Thus \( Y \) is irreducible and generically reduced. If \( Y_{\text{red}} \) is the reduced induced subscheme structure on \( Y \), it follows that \( \text{trop}(Y) = \text{trop}(Y_{\text{red}}) \), so \( Y_{\text{red}} \) is a quasilinear variety, and in particular normal by the theorem. Thus \( Y = Y_{\text{red}} \) by Corollary 4.14. \( \square \)

**Definition 4.18.** Let \( Y \subset T \) be a closed subvariety of an algebraic torus \( T \), and \( f \) a regular function on \( Y \). The very affine graph of \( f \) on \( Y \) is the closed subvariety \( \Gamma_f^Y = \Gamma_f \cap (Y \times \mathbb{C}^*) \subset T \times \mathbb{C}^* \), where \( \Gamma_f \subset Y \times \mathbb{A}^1 \) is the usual graph of \( f \).

In equations, \( \Gamma_f^Y = V(z - f) \subset Y \times \mathbb{C}^* \). Recall that the usual graph \( \Gamma_f \) of \( f \) is isomorphic to \( Y \) via the projection \( Y \times \mathbb{A}^1 \to Y \); likewise \( \Gamma_f^Y \) is isomorphic to \( Y \setminus V(f) \) via the projection \( Y \times \mathbb{C}^* \to Y \).

**Corollary 4.19.** Let \( \tilde{Y} \subset \tilde{T} \) be a quasilinear variety. Then, up to tropical isomorphism, either \( \tilde{Y} \) is an algebraic torus, or there is a projection of tori \( \pi : \tilde{T} \to T \), where \( \dim T = \dim \tilde{T} - 1 \), realizing \( \tilde{Y} \) as the very affine graph of a regular function \( f \) on a quasilinear variety \( Y \subset T \), such that \( D = V(f) \) is also quasilinear.

**Proof.** Continuing with the notation from the proof of Theorem 4.16, we know that if \( \tilde{Y} \) is not an algebraic torus, then the projection \( \pi : \tilde{T} \to T \) induces an isomorphism \( \pi : \tilde{Y} \approx \tilde{Y} \setminus D_\text{red} \), where \( Y \) is a quasilinear variety in \( T \) and \( D \) is a quasilinear divisor on \( Y \). By the theorem, \( Y \) is Chow-free, meaning in particular that every divisor on \( Y \) is principal, so \( D = V(f) \) for some regular function \( f \) on \( Y \). But then it follows that (up to multiplying \( f \) by a nonvanishing regular function), \( \tilde{Y} \) is the very affine graph of \( f \) on \( Y \). \( \square \)

5. **Quasilinear tropical compactifications**

5.1. **Strata.**

**Definition 5.1.** A tropical compactification \( \overline{Y} \subset X(\Sigma) \) of a closed subvariety \( Y \subset T \) is quasilinear if \( Y \subset T \) is a quasilinear variety.

**Theorem 5.2.** Suppose \( \overline{Y} \subset X(\Sigma) \) is a quasilinear tropical compactification. Then for every cone \( \sigma \in \Sigma \), the stratum \( Y_\sigma = \overline{Y} \cap O(\sigma) \) of \( \overline{Y} \) is quasilinear, and in particular smooth, irreducible, rational, Chow-free, and linearly stratified.

**Proof.** By Lemma 4.12 \( Y_\sigma \) is pure-dimensional and \( \text{trop}(Y_\sigma) \) is a tropical fan cycle supported on \( \Sigma^\sigma \), with the induced weight from \( Y_\sigma \) the same as the induced weight from \( \Sigma \). Since star fans of quasilinear tropical fans are quasilinear Theorem 3.6 it follows by Corollary 4.17 that \( Y_\sigma \) is a quasilinear variety. The last part follows by Theorem 4.16. \( \square \)

**Corollary 5.3.** Quasilinear varieties are schön.

**Proof.** By the theorem, if \( \overline{Y} \subset X(\Sigma) \) is any quasilinear tropical compactification, then all strata \( Y_\sigma \) of \( \overline{Y} \) are quasilinear, and in particular smooth. The result follows by [Hac08, Lemma 2.7]. \( \square \)

5.2. **Chow and cohomology rings.** We now explain our interest in Chow-free and linearly stratified varieties, following [Ful93, Section 5.2] and [PMSS95, Tot14].

The following lemma is well-known.

**Lemma 5.4.** Let \( Z \) be a variety with a stratification by locally closed strata which are all Chow-free. Then \( A_k(Z) \) is generated by the classes of the closures of the \( k \)-dimensional strata.
Proof. We induct on \( n = \dim Z \). When \( n = 0 \) the claim is obvious. Suppose the result holds for all \( i < n \).

Let \( Z_i \) be the union of the closed strata of dimension \( \leq i \). This gives a filtration

\[
\emptyset = Z_{-1} \subset Z_0 \subset Z_1 \subset \cdots \subset Z_n = Z.
\]

Each \( Z_i \setminus Z_{i-1} \) is the disjoint union of the strata of dimension \( i \).

Consider the exact sequence

\[
A_k(Z_{i-1}) \to A_k(Z_i) \to A_k(Z_i \setminus Z_{i-1}) \to 0.
\]

If \( k = i \), then for dimension reasons \( A_i(Z_{i-1}) = 0 \), so \( A_i(Z_i) \cong A_i(Z_i \setminus Z_{i-1}) \), hence by assumption is generated by the classes of the closures of the \( i \)-dimensional strata.

If \( k < i \), then by assumption \( A_i(Z_i \setminus Z_{i-1}) = 0 \), so \( A_k(Z_i) \) is generated by \( A_k(Z_{i-1}) \).

By induction, \( A_k(Z) \) is generated by \( A_k(Z_i) \), which is generated by the classes of the closures of the \( k \)-dimensional strata. \( \square \)

Corollary 5.5.  

(1) For a toric variety \( X(\Sigma) \), \( A_{n-d+k}(X(\Sigma)) \) is generated by the classes of the torus orbit closures \( V(\sigma) \), for \( \sigma \in \Sigma_{d-k} \).

(2) For a tropical compactification \( \bar{Y} \subset X(\Sigma) \), if all strata \( Y_\sigma \) are irreducible and Chow-free, then \( A_k(\bar{Y}) \) is generated by the classes of the closures \( \bar{Y}_\sigma = \bar{Y} \cap V(\sigma) \) for \( \sigma \in \Sigma_{d-k} \).

Proof. Immediate from the lemma. \( \square \)

Lemma 5.6 (\cite{Tot14,FMS95}). Let \( \bar{Y} \) be a linearly stratified variety.

(1) (Chow-Kuenneth) If \( Z \) is any finite-type scheme, then \( A_*(\bar{Y}) \otimes A_*(Z) \to A_*(\bar{Y} \times Z) \) is an isomorphism.

(2) Assume \( \bar{Y} \) is proper.

(a) (Kronecker-Poincaré duality) The natural map \( A^k(\bar{Y}) \to \text{Hom}(A_*(\bar{Y}), Z) \) is an isomorphism for all \( k \). In particular, if \( \bar{Y} \) is smooth, then \( A^*(\bar{Y}) \) is a Poincaré duality ring of dimension \( \dim \bar{Y} \).

(b) (Homology isomorphism) If \( \bar{Y} \) is nonsingular, then the cycle class map \( A_*(\bar{Y}) \cong H_{2k}(\bar{Y}) \) is an isomorphism and \( H_{2k+1}(\bar{Y}) = 0 \) for all \( k \), inducing an isomorphism \( A_*(\bar{Y}) \cong H_*(\bar{Y}) \).

Proof. The first two parts are \cite{Tot14} Propositions 1.2. (The second part in fact follows from the first part. Totaro’s proof of the first part is actually for a slightly narrower class of varieties, but his proof also works for linearly stratified varieties as we have defined them, cf. \cite{Gon15} Comments after Definition 2.3.) The last part is also a consequence of the first part \cite{FMS95} Corollary to Theorem 2. \( \square \)

For the remainder of this section fix any \( d \)-dimensional closed subvariety \( Y \subset T \cong (\mathbb{C}^*)^n \) and a tropical compactification \( i : \bar{Y} \to X(\Sigma) \) of \( Y \), and let \( \omega \) be the weight on \( \Sigma \) induced by \( Y \). Recall that \( \omega(\sigma) \) can be defined as \( \int_{\bar{Y}} i^*[V(\sigma)] \) \cite{MS15} Theorem 6.7.5, and via the isomorphism \( M_0(\Sigma) \cong \text{Hom}(A_{n-d}(X(\Sigma)), Z) \), we view \( \omega \) as a homomorphism \( A_{n-d}(X(\Sigma)) \to Z \).

Since \( i : \bar{Y} \to X(\Sigma) \) is a tropical compactification, the multiplication map \( m : \bar{Y} \times T \to X(\Sigma) \) is flat, of relative dimension \( d \). Therefore, there is a flat pullback morphism \cite{Ful98} Section 1.7

\[
m^* : A_{n-d+k}(X(\Sigma)) \to A_{k+n}(\bar{Y} \times T),
\]

\[
[V(\sigma)] \mapsto [m^{-1}(V(\sigma))] = ([\bar{Y} \cap V(\sigma)] \times T).
\]

There is a natural isomorphism \( A_*(\bar{Y} \times T) \cong A_*(\bar{Y}) \otimes A_*(T) \cong A_*(\bar{Y}) \), under which the above map is identified with

\[
i^* : A_{n-d+k}(X(\Sigma)) \to A_k(\bar{Y}),
\]

\[
[V(\sigma)] \mapsto [\bar{Y} \cap V(\sigma)].
\]

Corollary 5.7. If all strata \( Y_\sigma \) of \( \bar{Y} \subset X(\Sigma) \) are irreducible and Chow-free, then \( i^* : A_{n-d+k}(X(\Sigma)) \to A_k(\bar{Y}) \) is surjective for all \( k \).

Proof. Immediate from the above discussion and Corollary 5.6. \( \square \)

Taking duals, we get a morphism

\[
i_* : \text{Hom}(A_k(\bar{Y}), Z) \to \text{Hom}(A_{n-d+k}(X(\Sigma)), Z) \cong M_{d-k}(\Sigma).
\]
Proposition 5.8. The following diagram commutes

\[
\begin{array}{ccc}
A^k(\Sigma) & \xrightarrow{-\cap \omega} & M_{d-k}(\Sigma) \\
\downarrow i^* & & \downarrow \iota_* \\
A^k(\overline{Y}) & \xrightarrow{\partial_Y} & \text{Hom}(A_k(\overline{Y}), \mathbb{Z}),
\end{array}
\]

where the maps are defined as follows.

1. \(- \cap \omega : A^k(\Sigma) \to M_{d-k}(\Sigma)\) is the tropical cap product.
2. \(i^* : A^k(\Sigma) = A^k(X(\Sigma)) \to A^k(\overline{Y})\) is the natural pullback of Chow cohomology groups.
3. \(\partial_Y : A^k(\overline{Y}) \to \text{Hom}(A_k(\overline{Y}), \mathbb{Z})\) is the “Kronecker-Poincaré” duality map, \(\alpha \mapsto (\beta \mapsto \int_Y \alpha \cap \beta)\).
4. \(i_* : \text{Hom}(A_k(\overline{Y}), \mathbb{Z}) \to \text{Hom}(A_{n-d+k}(X(\Sigma)), \mathbb{Z}) \cong M_{d-k}(\Sigma)\) is the dual to the pullback morphism \(A_{n-d+k}(X(\Sigma)) \to A_{k+n}(Y \times T) \xrightarrow{\sim} A_k(\overline{Y})\) defined above.

Proof. Unwinding definitions, we are simply asking that for any \(\alpha \in A^k(X(\Sigma)), \beta \in A_{n-d+k}(X(\Sigma))\), one has

\[\omega(\alpha \cap \beta) = \int_Y i^* \alpha \cap i^* \beta,\]

where \(i^* \alpha\) is the pullback on Chow cohomology groups and \(i^* \beta\) is the pullback on Chow homology groups as defined above. But \(i^* \alpha \cap i^* \beta = i^* (\alpha \cap \beta)\), where now \(i^* (\alpha \cap \beta)\) is the pullback \(i^* : A_{n-d}(X(\Sigma)) \to A_0(\overline{Y})\).

The desired equality now follows from the definition of \(\omega\).

\[\square\]

Corollary 5.9. If \((\Sigma, \omega)\) is Poincaré, then \(i^* : A^k(\Sigma) \to A^k(\overline{Y})\) is injective and \(i_* : \text{Hom}(A_k(\overline{Y}), \mathbb{Z}) \to M_{d-k}(\Sigma)\) is surjective for all \(k\).

Proof. Immediate from the proposition.

\[\square\]

Corollary 5.10. If \((\Sigma, \omega)\) is Poincaré and all strata \(Y_\sigma\) of \(\overline{Y}\) are irreducible and Chow-free, then \(i_* : \text{Hom}(A_k(\overline{Y}), \mathbb{Z}) \to M_{d-k}(\Sigma)\) is an isomorphism for all \(k\). If in addition, either all strata \(Y_\sigma\) are linearly stratified, or both \(\overline{Y}\) and \(X(\Sigma)\) are nonsingular, then \(i^* : A^k(\Sigma) \to A^k(\overline{Y})\) and \(\partial_Y : A^k(\overline{Y}) \to \text{Hom}(A_k(\overline{Y}), \mathbb{Z})\) are both isomorphisms, i.e., all arrows in \((11)\) are isomorphisms.

Proof. The first sentence follows from Corollaries 5.7 and 5.9. For the second, first suppose all strata \(Y_\sigma\) are linearly stratified. Then by Lemma 5.6, the Kronecker-Poincaré duality map \(\partial_Y : A^k(\overline{Y}) \to \text{Hom}(A_k(\overline{Y}), \mathbb{Z})\) is an isomorphism, i.e., all arrows in \((11)\) are isomorphisms, except possibly \(i^* : A^k(\Sigma) \to A^k(\overline{Y})\). But of course, it now follows that this is also an isomorphism. If instead both \(\overline{Y}\) and \(X(\Sigma)\) are nonsingular, then \(A_{n-d+k}(X(\Sigma)) \cong A^{d-k}(\Sigma)\) and \(A_k(\overline{Y}) \cong A^{d-k}(\overline{Y})\), so the map \(i_* : \text{Hom}(A_k(\overline{Y}), \mathbb{Z}) \to M_{d-k}(\Sigma)\) is just the dual to \(i^* : A^{d-k}(\Sigma) \to A^{d-k}(\overline{Y})\), and the result follows.

\[\square\]

Corollary 5.11. Suppose all the conditions of Corollary 5.10 hold, i.e., \(\Sigma\) is Poincaré, all strata \(Y_\sigma\) of \(\overline{Y}\) are irreducible, Chow-free, and linearly stratified, and both \(\overline{Y}\) and \(X(\Sigma)\) are nonsingular. Then the cycle class map \(A_k(\overline{Y}) \to H_{2k}(\overline{Y})\) is an isomorphism and \(H_{2k+1}(\overline{Y}) = 0\) for all \(k\), inducing isomorphisms

\[H^*(\overline{Y}) \cong \mathbb{A}^*(\overline{Y}) \cong \mathbb{A}^*(X(\Sigma)).\]

Proof. Immediate from Lemma 5.6 and Corollary 5.10.

\[\square\]

Theorem 5.12. If \(i : \overline{Y} \hookrightarrow X(\Sigma)\) is a quasilinear tropical compactification, then \(i^* : A^k(\Sigma) \to A^k(\overline{Y})\) is an isomorphism for all \(k\), inducing an isomorphism of Chow rings

\[A^*(\overline{Y}) \cong A^*(X(\Sigma)).\]

If in addition \(X(\Sigma)\) is nonsingular, then so is \(\overline{Y}\), and we have \(H^{2k}(\overline{Y}) = 0\) and \(H^{2k}(\overline{Y}) \cong A^k(\overline{Y}) \cong A^k(X(\Sigma))\) for all \(k\), inducing isomorphisms

\[H^*(\overline{Y}) \cong \mathbb{A}^*(\overline{Y}) \cong \mathbb{A}^*(X(\Sigma)).\]

Proof. By Theorem 5.2, all strata \(Y_\sigma\) of \(\overline{Y}\) are smooth, irreducible, Chow-free, and linearly stratified, and by Theorem 5.3, \(\Sigma\) is (star-)Poincaré. The first statement is now immediate from Corollary 5.10. For the second statement, if \(X(\Sigma)\) is nonsingular then since \(\overline{Y} \subset X(\Sigma)\) is a schön tropical compactification, \(\overline{Y}\) is also nonsingular, and the result follows by Corollary 5.11.

\[\square\]
6. Quasilinear criteria and examples

6.1. General criteria.

**Theorem 6.1.** Let $Y \subset T$ be a quasilinear variety, and $f$ a regular function on $Y$ such that either $f$ is nonvanishing or $D = V(f) \subset Y$ is also quasilinear in $T$. Let $\tilde{Y} \subset T \times \mathbb{C}^*$ be the very affine graph of $f$ on $Y$. Then trop$(\tilde{Y})$ is a tropical modification of trop$(Y)$ along trop$(D)$, and in particular $\tilde{Y}$ is quasilinear.

**Proof.** We prove the result in the case that $D = V(f)$ is nontrivial, leaving the (easier) case that $f$ is nonvanishing to the reader.

Choose unimodular fan structures $\Sigma$ on trop$(Y)$ and $\Delta$ on trop$(D)$ such that every cone of $\Delta$ is a cone of $\Sigma$. Since by Corollary 6.3 quasilinear varieties are schön, it follows that the closures $\overline{\Sigma} \subset X(\Sigma)$ and $\overline{\Delta} \subset X(\Delta)$ are schön tropical compactifications [LQ11, Theorem 1.5], and in particular (since $X(\Sigma)$ and $X(\Delta)$ are smooth), $Y \subset \overline{\Sigma}$ and $D \subset \overline{\Delta}$ are smooth compactifications with simple normal crossings boundary [Tev07, Theorem 1.4]. By Theorem 6.2 all boundary strata of $\overline{\Sigma}$ and $\overline{\Delta}$ are quasilinear, and in particular reduced and irreducible. Therefore the fans $\Sigma$ and $\Delta$ can be recovered from $\overline{\Sigma}$ and $\overline{\Delta}$ via geometric tropicalization [HKT09, Section 2], [Cue12, Theorem 2.8]. Namely, a ray $\rho$ of $\Sigma$ is recovered from the corresponding irreducible boundary divisor $D_\rho = \overline{\rho}_\Sigma$ as the ray through the vector $[\text{val}_{D_\rho}] = ([\text{val}_{D_\rho}(m_1|_Y), \ldots, \text{val}_{D_\rho}(m_n|_Y)])$, where $m_1, \ldots, m_n$ are a basis of $M$, and a collection of such rays forms a cone precisely when the corresponding boundary divisors intersect; likewise for recovering $\Delta$ from $\overline{\Delta}$. Furthermore, $\overline{\Delta}$ is also obtained as the closure of $D$ in $\overline{\Sigma}$, and $\overline{\Delta}$ intersects a stratum $\gamma_\sigma$ of $\overline{\Sigma}$ if and only if $\sigma \in \Delta$. It follows that $\overline{\Sigma}$ is also a smooth simple normal crossings compactification of $Y \cong Y \setminus D$, so we can also recover trop$(\tilde{Y})$ from the boundary of $Y \subset \overline{\Sigma}$ via geometric tropicalization. Now the irreducible components of the boundary are $\overline{\Delta}$, and $D_\rho = \overline{\rho}_\Sigma$ for $\rho$ a ray of $\Sigma$, a collection of boundary divisors $D_{\rho_1}, \ldots, D_{\rho_k}$ intersect $\iff \rho_1, \ldots, \rho_k$ form a cone of $\Sigma$, and a collection of boundary divisors $\overline{D_{\rho_1}}, \ldots, \overline{D_{\rho_k}}$ intersect $\iff \rho_1, \ldots, \rho_k$ form a cone of $\Delta$. So geometric tropicalization implies that trop$(\tilde{Y})$ is the support of the fan $\Sigma \subset \overline{\Sigma} = N_\mathbb{R} \times \mathbb{R}$, whose rays are

$$\tilde{\rho} = \mathbb{R}_{\geq 0}(\text{val}_{D_{\rho}}(m_1|_Y), \ldots, \text{val}_{D_{\rho}}(m_n|_Y), \text{val}_{D_{\rho}}(m_{n+1}|_Y))$$

$$= \{(x, \varphi(x)) \mid x \in \rho\},$$

$$0_\geq = \mathbb{R}_{\geq 0}(\text{val}_{\overline{D}}(m_1|_Y), \ldots, \text{val}_{\overline{D}}(m_n|_Y), \text{val}_{\overline{D}}(m_{n+1}|_Y))$$

where $\varphi : |\Sigma| \to \mathbb{R}$ is the piecewise integral linear function defined by setting $\varphi(\rho_\sigma) = \text{val}_{D_{\rho_\sigma}}(m_{n+1}|_Y)$ for each ray $\rho_\sigma$ and extending by linearity on each cone of $\Sigma$. From the above description of the boundary complex of $\tilde{Y} \subset \overline{\Sigma}$, we see that a collection of rays $\rho_1, \ldots, \rho_k$ form a cone of $\Sigma$ precisely if the corresponding rays $\tilde{\rho}_1, \ldots, \tilde{\rho}_k$ form a cone of $\tilde{Y}$ precisely if the corresponding rays $\varphi(\rho_1), \ldots, \varphi(\rho_k)$ form a cone of $\Delta$. It follows that $\Sigma$ is precisely the tropical modification of $\Sigma$ with respect to the piecewise integral linear function $\varphi$, and $\text{div}(\varphi) = \Delta$.

**Corollary 6.2.** Let $Y \subset T$ be a quasilinear variety and $D_1, \ldots, D_k \subset Y$ quasilinear hypersurfaces on $Y$ such that all nonempty intersections of the $D_i$ are also quasilinear in $T$. Then $Y \setminus (D_1 \cup \cdots \cup D_k) \subset T \times T^k$ is also quasilinear.

**Proof.** The proof is by induction on $k$. The base case $k = 1$ is immediate from Theorem 6.1. Assume the result for $k-1$. Then $Y' = Y \setminus (D_1 \cup \cdots \cup D_{k-1})$ is quasilinear by induction, and $Y' \setminus (D_1 \cup \cdots \cup D_k) = Y' \setminus D'_k$, where $D'_k = D_k \setminus (D_k \cap D_1 \cup \cdots \cup (D_k \cap D_{k-1}))$. Since $D_k$ and all intersections of the $D_i$’s are quasilinear, $D'_k$ is also quasilinear by induction. Thus $Y' \setminus D'_k$ is quasilinear.

**Theorem 6.3.** Let $Y_1 \subset T_1$ and $Y_2 \subset T_2$ be two closed subvarieties of tori. Then $Y_1 \times Y_2 \subset T_1 \times T_2$ is quasilinear, if and only if both $Y_1 \subset T_1$ and $Y_2 \subset T_2$ are quasilinear.

**Proof.** We have trop$(Y_1 \times Y_2) = \text{trop}(Y_1) \times \text{trop}(Y_2)$ [Cue10, Theorem 3.3.4], so the result is immediate from Theorem 3.5.

6.2. Examples. Recall a closed subvariety $Y \subset T$ is *linear* if the equations defining $Y$ are linear, or equivalently $Y$ is isomorphic to the complement of a hyperplane arrangement.

**Theorem 6.4.** Linear varieties are quasilinear.
Proof. The tropicalization of a linear variety is the (support of the) Bergman fan of the matroid defining the corresponding hyperplane arrangement, so the result follows by Theorem 3.13. □

Example 6.5. Let \( a, b \in \mathbb{Z} \setminus \{0\} \), \( m \in \mathbb{C}^* \), and let \( Y = \{x^a = my^b\} \subset (\mathbb{C}^*)^2 \). Then trop\((Y)\) is the line \( \{ax = by\} \subset \mathbb{R}^2 \), hence \( Y \) is quasilinear by Example 3.14.

Similarly, if \( c \in \mathbb{Z} \setminus \{0\} \), then \( Y = \{z^c = mx^ay^b\} \subset (\mathbb{C}^*)^3 \) is quasilinear; its tropicalization is a classical plane in \( \mathbb{R}^3 \).

Example 6.6. Let \( a, b \in \mathbb{Z} \setminus \{0\}, m, n \in \mathbb{C}^* \). Then \( Y = \{z = mx^a - ny^b\} \subset (\mathbb{C}^*)^3 \) is quasilinear; its tropicalization is a modification of \( \mathbb{R}^2 \) along the line \( \{ax = by\} \).

Example 6.7. Let \( H_1 = \{x = y\} \), \( H_2 = \{xy = 1\} \), and let \( Y = (\mathbb{C}^*)^2 \setminus (H_1 \cup H_2) \). Then
\[
Y = \{z = x - y, w = xy - 1\} \subset (\mathbb{C}^*)^4
\]
is not quasilinear—its tropicalization has a top-dimensional cone of with weight two, corresponding to the two intersection points of \( H_1 \) and \( H_2 \).

Example 6.8. Continuing with the previous example, let \( H_1 = \{x = y\} \), \( H_2 = \{xy = 1\} \), \( H_3 = \{x = 1\} \), and let \( Y = (\mathbb{C}^*)^2 \setminus (H_1 \cup H_2 \cup H_3) \). Then \( Y \) is quasilinear, even though \( Y = (\mathbb{C}^*)^2 \setminus (H_1 \cup H_2) \) is not. To see this, let \( Y' = (\mathbb{C}^*)^2 \setminus (H_1 \cup H_3) \). Then
\[
Y' = \{z = x - y, w = x - 1\} \subset (\mathbb{C}^*)^4
\]
is linear, hence quasilinear. Let
\[
H'_2 = H_2 \cap Y' = \{xy = 1, z = x - y, w = x - 1\} \subset (\mathbb{C}^*)^4.
\]
Then \( \tilde{Y} = Y' \setminus H'_2 \), so we just need to show that \( H'_2 \) is quasilinear. But the projection \( (\mathbb{C}^*)^4 \to (\mathbb{C}^*)^3 \) dropping the \( z \) coordinate realizes \( H'_2 \) as the graph of the regular function \( f = x - y \) on the variety
\[
H''_2 = \{xy = 1, w = x - 1\} \subset (\mathbb{C}^*)^3.
\]
The projection \( (\mathbb{C}^*)^3 \to (\mathbb{C}^*)^2 \) dropping the coordinate \( y \) realizes \( H''_2 \) as the graph of the nonvanishing regular function \( 1/x \) on the (quasi)linear variety
\[
\{w = x - 1\} \subset (\mathbb{C}^*)^2,
\]
thus \( H''_2 \) is quasilinear. Now \( V = V(f) \subset H''_2 \) is given by
\[
\{x = y, xy = 1, w = x - 1\} = \{x = y = -1, w = -2\} \subset (\mathbb{C}^*)^3,
\]
hence \( V \) is (quasi)linear. (Indeed, it is a single point.) It follows that \( H'_2 \) is quasilinear, hence \( \tilde{Y} = Y' \setminus H'_2 \) is quasilinear.

The above example is representative of our general procedure for determining when a variety is quasilinear, see Section 7.

7. Applications

7.1. Moduli of line arrangements. For \( n \geq 4 \), let \( M(3, n) \) denote the moduli space of arrangements of \( n \) lines in general position in \( \mathbb{P}^2 \). By fixing the last 4 lines and scaling the equations of the remaining lines, we can write \( M(3, n) \) as the complement in \( A^2(\mathbb{N}^2-4) \) of the \( 3 \times 3 \) minors of the matrix
\[
\begin{bmatrix}
1 & 1 & \cdots & 1 & 1 & 1 & 0 & 0 \\
x_1 & x_2 & \cdots & x_{n-4} & 1 & 0 & 1 & 0 \\
y_1 & y_2 & \cdots & y_{n-4} & 1 & 0 & 0 & 1
\end{bmatrix}.
\]
The \( 3 \times 3 \) minors of this matrix include the hyperplanes \( x_i = 0 \) and \( y_i = 0 \), inducing the embedding of \( M(3, n) \) in its intrinsic torus \( (\mathbb{C}^*)^3 \).

Theorem 7.1. The moduli space \( M(3, 6) \) is quasilinear.
Proof. Observe that $M = M(3, 6)$ is the complement in $(\mathbb{C}^*)^4$ of the 10 hypersurfaces defined by the following 10 equations.

$$
\begin{align*}
&x_1 = 1, \quad x_2 = 1, \quad x_1 = x_2, \quad y_1 = 1, \quad y_2 = 1, \quad y_1 = y_2, \\
&x_1 = y_1, \quad x_2 = y_2, \quad x_1 y_2 = x_2 y_1, \quad (x_1 - 1)(y_2 - 1) = (x_2 - 1)(y_1 - 1).
\end{align*}
$$

Let $M_1 \subset (\mathbb{C}^*)^{12}$ be the complement of the linear hypersurfaces. Then $M_1$ is linear, and in particular quasilinear. Equations for $M_1 \subset (\mathbb{C}^*)^{12}$ can be written as follows.

$$
\begin{align*}
&z_1 = x_1 - 1, \quad z_2 = x_2 - 1, \quad z_{12} = x_1 - x_2, \\
&w_1 = y_1 - 1, \quad w_2 = y_2 - 1, \quad w_{12} = y_1 - y_2, \\
&u_1 = x_1 - y_1, \quad u_2 = x_2 - y_2.
\end{align*}
$$

Let $Q_1 = \{x_1 y_2 = x_2 y_1\} \subset M_1$ and $Q_2 = \{(x_1 - 1)(y_2 - 1) = (x_2 - 1)(y_1 - 1)\} = \{z_1 w_2 = z_2 w_1\} \subset M_1$. Then $M = M_1 \setminus (Q_1 \cup Q_2)$, so to show $M$ is quasilinear it suffices to show that $Q_1$, $Q_2$, and $Q_1 \cap Q_2$ are all quasilinear. A direct computation shows that in fact $Q_1 \cap Q_2 = \emptyset$, so it is enough to show that $Q_1$ and $Q_2$ are quasilinear. But by the symmetry of the equations, we see that it is enough to check that $Q_1$ is quasilinear.

Let $Q = \{x_1 y_2 = x_2 y_1\} \subset (\mathbb{C}^*)^4$. Then $Q_1$ is the complement in $Q$ of the linear hypersurfaces. So to show $Q_1 \subset (\mathbb{C}^*)^{14}$ is quasilinear, it suffices to show $Q \subset (\mathbb{C}^*)^4$ intersects any collection of the linear hypersurfaces quasilinearly. Up to symmetry, there are three types of intersections of $Q$ with a single linear hypersurface:

1. $Q \cap \{x_1 = y_1\} = \{x_1 = y_1, x_2 = y_2\}$ is linear.
2. $Q \cap \{x_1 = x_2\} = \{x_1 = x_2, y_1 = y_2\}$ is linear.
3. $Q \cap \{x_1 = 1\} = \{x_1 = 1, y_2 = x_2 y_1\}$. The projection dropping the coordinate $y_2$ realizes this as the graph of the nonvanishing regular function $x_2 y_1$ on the (quasi)linear variety $\{x_1 = 1\} \subset (\mathbb{C}^*)^3$, hence $Q \cap \{x_1 = 1\}$ is quasilinear.

Finally, any intersection of $Q$ with two or more linear hypersurfaces is linear. Thus, $Q_1$ is linear. □

As an immediate consequence of Theorem 7.1, we recover the main result of [Lux08], describing the log canonical compactification of $M(3, 6)$.

**Corollary 7.2.** The moduli space $M(3, 6)$ is schön, and its stable pair compactification $\overline{M}(3, 6)$ is the log canonical compactification.

**Proof.** Schönnness of $M(3, 6)$ is immediate from Theorem 7.1 and Corollary 5.3. The stable pair compactification $\overline{M}(3, 6)$ is obtained as the closure of $M(3, 6)$ in the toric variety associated to the Dressian $Dr(3, 6)$ (see [HKT06,Ale15] for more details). The Dressian $Dr(3, 6)$ is a convexly disjoint fan supported on $\text{trop}(M(3, 6))$ (meaning that any convex subset of $Dr(3, 6)$ is contained in a cone of $Dr(3, 6)$). This together with the schönnness of $M(3, 6)$ implies that $\overline{M}(3, 6)$ is the log canonical compactification, see [HKT09, Proof of Theorem 9.14], [Lux08]. □

**Remark 7.3.** In general, let $M(r, n)$ denote the moduli space of arrangements of $n$ hyperplanes in $\mathbb{P}^{r-1}$ in general position, and $\overline{M}(r, n)$ the normalization of the main irreducible component of its stable pair compactification $\overline{M}(r, n)$ [HKT06]. In [KT06], Keel and Tevelev show that $\overline{M}(r, n)$ is not log canonical except possibly in the cases $(r, n) = (2, n)$, $(3, 6)$, $(3, 7)$, $(3, 8)$ (and those obtained by duality $\overline{M}^\vee(r, n)$) and they conjecture that in these cases $\overline{M}(r, n)$ is indeed the log canonical compactification. They prove this for the case $r = 2$, in which case $\overline{M}(2, n) \cong \overline{M}_{0,n}$ [KT06]. The case $(3, 6)$ was shown by Luxton [Lux08]; the above corollary gives a much shorter proof of Luxton’s result. (For another proof of this case, without using any tropical geometry, see [Sch22, Section 7].) The cases $(3, 7)$ and $(3, 8)$ have recently been proven by Corey [Cor21], and Corey-Luber [CL22].

**Remark 7.4.** Corey and Luber show that for $(r, n) = (3, 8)$ (and hence for any larger case), $\text{trop}(M(3, 8))$ is not reduced, and in particular $M(3, 8)$ is not quasilinear [CL22]. The nonreducedness of $\text{trop}(M(3, 8))$ comes from zero-dimensional strata of $\overline{M}(3, 8)$ parameterizing (stable replacements of) configurations of 8 lines in $\mathbb{P}^2$ with 8 triple intersection points. Since the dual of such a configuration admits the same description, one sees that these strata consist of 2 distinct points, and thus are not irreducible. See [CL22] for more details.
On the other hand, such examples do not occur for $(r, n) = (3, 7)$, and one can show by similar arguments to the proof of Theorem 7.1 that $M(3, 7)$ is quasilinear. The arguments in this case are quite subtle and do not offer much insight into the structure of $M(3, 7)$ so we omit them, referring to [Sch22a, Chapter 7] for more details.

As another immediate consequence of Theorem 7.1 we recover the main results of [Sch22b], describing the Chow rings of tropical compactifications of $M(3, 6)$.

**Corollary 7.5.** Let $\overline{M}(3, 6) \subset X(\Sigma)$ be any tropical compactification of $M(3, 6)$. Then $A^*(\overline{M}(3, 6)) \cong A^*(X(\Sigma))$. If $\Sigma$ is unimodular, then $\overline{M}(3, 6)$ is a resolution of singularities of the stable pair compactification $\overline{M}(3, 6)$, and $A^*(\overline{M}(3, 6)) \cong H^*(\overline{M}(3, 6))$.

**Proof.** Everything is immediate from Theorem 7.1 and Theorem 5.12 except for the statement that if $\Sigma$ is unimodular then $\overline{M}(3, 6)$ is a resolution of singularities of $\overline{M}(3, 6)$; this in turn follows from the observation that $\overline{M}(3, 6)$ is the tropical compactification associated to the coarsest fan structure on trop($M(3, 6)$) (cf. the proof of Corollary 7.2 and [Lux08]).

**Remark 7.6.** The moduli space $\overline{M}(3, 6)$ has 15 singular points, each locally isomorphic to the cone over $\mathbb{P}^1 \times \mathbb{P}^2$ (see [Lux08, Sch22b]). There are $2^{15}$ small resolutions of $\overline{M}(3, 6)$, whose fibers over each singular point are either $\mathbb{P}^3$ or $\mathbb{P}^2$. In [Sch22b] we give an explicit presentation of the Chow ring of each small resolution, and as a consequence also obtain a presentation of the Chow ring of $\overline{M}(3, 6)$ itself. Since each small resolution of $\overline{M}(3, 6)$ is also a tropical compactification of $M(3, 6)$, the above result gives an alternative proof of these presentations.

### 7.2. Moduli of marked cubic surfaces.

Let $Y(E_6)$ denote the moduli space of smooth marked cubic surfaces. A marking of a cubic surface $S$ is equivalent to a realization of $S$ as the blowup of 6 points in general position in $\mathbb{P}^2$, where here general position means no 2 points coincide, no 3 lie on a line, and no 6 lie on a conic. Projective duality therefore identifies $Y(E_6)$ as the complement in $M(3, 6)$ of the hypersurface $Q$ parameterizing the locus of 6 points on a conic. In the coordinates of Section 7, equations for $Q$ are given by

$$Q = \left\{ \det \begin{pmatrix} 1 & 1 & 1 \\ x_1 & y_1 & x_1 y_1 \\ x_2 & y_2 & x_2 y_2 \end{pmatrix} = 0 \right\}$$

(see, for instance [Yos00]). Note that $Q$ is also naturally isomorphic to the moduli space $M_{0,6}$ of 6 points on $\mathbb{P}^1$.

**Theorem 7.7.** The moduli space $Y(E_6)$ is quasilinear.

**Proof.** Since $Y(E_6)$ is the complement in $M(3, 6)$ of the hypersurface $Q$, and $M(3, 6)$ is quasilinear by Theorem 7.1 all we need to do is show that $Q$ is quasilinear (in the embedding $Q \subset M(3, 6) \subset (\mathbb{C}^*)^{14}$). Since $Q \cong M_{0,6}$ is quasilinear (indeed, linear) in its intrinsic torus, it suffices to show that the tropicalization of $Q$ in $(\mathbb{C}^*)^{14}$ is isomorphic to the tropicalization of $M_{0,6}$. In fact, this follows from the results of Luxton [Lux08], which show that there is a tropical compactification $\tilde{M}(3, 6)$ of $M(3, 6)$ such that the closure of $Q$ in $M(3, 6)$ is isomorphic to $\overline{M}_{0,6}$.

**Remark 7.8.** Recall that the Segre cubic $\mathcal{S}$ is the unique cubic threefold in $\mathbb{P}^4$ with 10 nodes. It is a birational model of $\overline{M}_{0,6}$ obtained by contracting the 10 boundary divisors of the form $D_{ijk}$ to the nodes. The open Segre cubic $\mathcal{S}^o$ is the complement in $\mathcal{S}$ of the images of the remaining 15 boundary divisors $D_{ij}$ of $\overline{M}_{0,6}$, note that $\mathcal{S}^o \cong M_{0,6}$. This induces an embedding of $M_{0,6}$ into a 14-dimensional torus, which, up to an appropriate change of coordinates, agrees with our embedding $Q \subset M(3, 6) \subset (\mathbb{C}^*)^{14}$ above (cf. [RSS14, Section 2]). By [RSS14, Theorem 2.4], the tropicalization of this embedding is isomorphic to the tropicalization of $M_{0,6}$ in its intrinsic torus, giving another proof that $Q \subset (\mathbb{C}^*)^{14}$ is quasilinear.

It is also interesting to note that the closure $\overline{Q}$ of $Q$ in $\overline{M}(3, 6)$ is in fact isomorphic to the Segre cubic. The 10 nodes of the Segre cubic are cut out by the intersections of $\overline{Q}$ with the codimension two boundary strata of $\overline{M}(3, 6)$ described by intersections $D_{ijk} \cap D_{lmn}$, where $ijk \subset [6]$ and $lmn \subset [6]$ form a partition of $[6] = \{1, \ldots, 6\}$, and the divisors $D_{ijk} \cong \overline{M}_{0,6}$ are described in [Sch22b, Section 2] (see also [Lux08]). Note that $\overline{Q}$ is not a tropical compactification of $Q$ because of its intersections with these strata. Blowing up
\(M(3,6)\) at the 10 intersections \(D_{ijk} \cap D_{i'j'k'}\) amounts to blowing up \(Q\) at the 10 nodes, giving the resolution of singularities \(M_{0,6} \to \tilde{Q}\). (This description of \(Q\) is implicit in [Lux08].)

**Remark 7.9.** For another proof that \(Q \subset M(3,6) \subset (\mathbb{C}^*)^{14}\) (hence \(Y(E_6)\)) is quasilinear, see [Sch22a, Chapter 8]. The proof there works explicitly with the equations of \(Q\).

A particularly nice, smooth projective compactification \(\overline{Y}(E_6)\) of \(Y(E_6)\), with simple normal crossings boundary, was constructed by Naruki in [Nar82]. Using this compactification, Hacking-Keel-Tevelev show that \(Y(E_6)\) is schön and \(\overline{Y}(E_6)\) is the log canonical compactification [HKT09]. They also show that a natural blowup \(\tilde{Y}(E_6)\) of \(\overline{Y}(E_6)\) is the moduli space of stable marked cubic surfaces. (In fact, \(\overline{Y}(E_6)\) also has an interpretation as a moduli space of weighted stable marked cubic surfaces, see [GKS21, Sch23a].) Theorem 7.7 gives an alternative proof that \(Y(E_6)\) is schön, and allows one to describe the Chow rings of \(\overline{Y}(E_6)\) and \(\tilde{Y}(E_6)\).

**Corollary 7.10.** The moduli space \(Y(E_6)\) is schön.

**Proof.** Immediate from Theorem 7.7 and Corollary 5.3.

From Theorem 7.7 we also obtain a description of the Chow rings of tropical compactifications of \(Y(E_6)\), including Naruki’s compactification \(\overline{Y}(E_6)\) and the moduli space of stable marked cubic surfaces \(\tilde{Y}(E_6)\).

**Corollary 7.11.** Let \(\overline{Y}(E_6) \subset X(\Sigma)\) be any tropical compactification of \(Y(E_6)\). Then there is an isomorphism \(\ast^* : (\overline{Y}(E_6)) \cong \ast^*(X(\Sigma))\). If \(\Sigma\) is unimodular, then \(\overline{Y}(E_6)\) is nonsingular and \(\ast^*(\overline{Y}(E_6)) = H^*(\overline{Y}(E_6))\). In particular, this applies to Naruki’s compactification \(\overline{Y}(E_6)\), and to the moduli space of stable marked del Pezzo surfaces \(\tilde{Y}(E_6)\).

**Proof.** Immediate from Theorems 5.12 and 7.7 and Corollary 5.3. (For the last sentence, note that both \(\overline{Y}(E_6)\) and \(\tilde{Y}(E_6)\) are tropicalcompactifications by [HKT09].)

**Remark 7.12.** (1) The intersection theory of \(\overline{Y}(E_6)\) has previously been studied by Colombo and van Geemen [CV04]. The above result goes a step further, by allowing one to write down an explicit presentation for the Chow ring of \(\overline{Y}(E_6)\), and indeed of any tropical compactification of \(Y(E_6)\).

(2) We expect similar results to hold for moduli of marked del Pezzo surfaces of degrees 1 and 2. (Note these are the only remaining interesting cases, since the moduli space of marked del Pezzo surfaces of degree 4 is isomorphic to \(M_{0,5}\), and the moduli spaces of marked del Pezzo surfaces of degrees > 4 are trivial.) Indeed, note for instance that Hacking-Keel-Tevelev also show that the moduli space \(Y(E_7)\) of marked del Pezzo surfaces of degree 2 is schön and the log canonical compactification \(\overline{Y}(E_7)\) is a tropical compactification [HKT09]. The results of Sekiguchi [Sek10] explicitly describing the strata of \(\overline{Y}(E_7)\) imply that each open stratum is irreducible, rational, and Chow-free.

A more detailed study of the geometry, in particular the intersection theory, of the moduli space of marked del Pezzo surfaces will appear in future work.

**References**

[ADH22] Federico Ardila, Graham Denham, and June Huh, *Lagrangian geometry of matroids*, Journal of the American Mathematical Society (2022).

[AHK18] Kareem Adiprasito, June Huh, and Eric Katz, *Hodge theory for combinatorial geometries*, Annals of Mathematics 188 (2018), no. 2.

[AHR16] Lars Allermann, Simon Hampe, and Johannes Rau, *On Rational Equivalence in Tropical Geometry*, Canadian Journal of Mathematics 68 (2016), no. 2, 241–257.

[AK00] D. Abramovich and K. Karu, *Weak semistable reduction in characteristic 0*, Inventiones mathematicae 139 (2000), no. 2, 241–273.

[Ale15] Valery Alexeev, *Moduli of Weighted Hyperplane Arrangements*, Advanced Courses in Mathematics - CRM Barcelona, Springer Basel, Basel, 2015.

[AP20] Omid Amini and Matthieu Piquerez, *Hodge theory for tropical varieties*, arXiv:2007.07826 [math] (2020).

[AP21] , *Homology of tropical fans*, arXiv:2105.01504 [math] (2021).

[AR10] Lars Allermann and Johannes Rau, *First Steps in Tropical Intersection Theory*, Mathematische Zeitschrift 264 (2010), no. 3, 633–670.

[BL12] Erwan A. Brugallé and Lucia M. López de Medrano, *Inflection points of real and tropical plane curves*, Journal of Singularities 4 (2012), 74–103.
Grigory Mikhalkin, *Tropical geometry and its applications*, Proceedings of the International Congress of Mathematicians Madrid, August 22–30, 2006 (Marta Sanz-Solé, Javier Soria, Juan Luis Varona, and Joan Verdera, eds.), European Mathematical Society Publishing House, Zuerich, Switzerland, May 2007, pp. 827–852.

Grigory Mikhalkin and Johannes Rau, *Tropical Geometry*, https://www.math.uni-tuebingen.de/user/jora/downloads/main.pdf, 2018.

Diane Maclagan and Bernd Sturmfels, *Introduction to tropical geometry*, Graduate Studies in Mathematics, vol. 161, American Mathematical Society, Providence, RI, 2015.

I. Naruki, *Cross Ratio Variety as a Moduli Space of Cubic Surfaces*, Proceedings of the London Mathematical Society s3-45 (1982), no. 1, 1–30.

Brian Osserman and Sam Payne, *Lifting Tropical Intersections*, Documenta Mathematica (2013), 56.

Sam Payne, *Equivariant Chow cohomology of toric varieties*, Mathematical Research Letters 13 (2006), no. 1, 29–41.

Qingchun Ren, Steven V. Sam, and Bernd Sturmfels, *Tropicalization of Classical Moduli Spaces*, Mathematics in Computer Science 8 (2014), no. 2, 119–145.

Nolan Schock, *Geometry of tropical compactifications of moduli spaces*, Ph.D. thesis, University of Georgia, May 2022.

Nolan Schock, *Intersection theory of the stable pair compactification of the moduli space of six lines in the plane*, European Journal of Mathematics 8 (2022), 139–192.

Nolan Schock, *Moduli of weighted stable marked del Pezzo surfaces*, May 2023.

Jiro Sekiguchi, *Cross ratio varieties for root systems II: The case of the root system of type E_7*, Kyushu Journal of Mathematics 54 (2000), 7–37.

Kristin M. Shaw, *A Tropical Intersection Product in Matroidal Fans*, SIAM Journal on Discrete Mathematics 27 (2013), no. 1, 459–491.

Kimura Shun-ichi, *Fractional intersection and bivariant theory*, Communications in Algebra 20 (1992), no. 1, 285–302.

The Stacks Project Authors, *Stacks Project*, https://stacks.math.columbia.edu/, 2022.

Bernd Sturmfels and Jenia Tevelev, *Elimination Theory for Tropical Varieties*, Mathematical Research Letters 15 (2008), no. 3, 543–562.

Jenia Tevelev, *Compactifications of subvarieties of tori*, American Journal of Mathematics 129 (2007), no. 4, 1087–1104.

Burt Totaro, *Chow groups, chow cohomology, and linear varieties*, Forum of Mathematics, Sigma 2 (2014), e17.

Jaroslaw Wlodarczyk, *Decomposition of Birational Toric Maps in Blow-Ups and Blow-Downs*, Transactions of the American Mathematical Society 349 (1997), no. 1, 373–411.

Masaaki Yoshida, *A W(E_6)-equivariant projective embedding of the moduli space of cubic surfaces*, arXiv:math/0002102 (2000).

Email address: nschoc2@uic.edu

Department of Mathematics, University of Illinois at Chicago, Chicago IL, 60607, USA