Stability of closed characteristics on symmetric compact convex hypersurfaces in $\mathbb{R}^{2n}$

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Abstract

In this article, let $\Sigma \subset \mathbb{R}^{2n}$ be a compact convex hypersurface which is symmetric with respect to the origin. We prove that if $\Sigma$ carries finitely many geometrically distinct closed characteristics, then at least $n - 1$ of them must be non-hyperbolic; if $\Sigma$ carries exactly $n$ geometrically distinct closed characteristics, then at least two of them must be elliptic.

Key words: Compact convex hypersurfaces, closed characteristics, Hamiltonian systems, index iteration, stability.

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Running title: Stability of closed characteristics

1 Introduction and main results

In this article, let $\Sigma$ be a fixed $C^3$ compact convex hypersurface in $\mathbb{R}^{2n}$, i.e., $\Sigma$ is the boundary of a compact and strictly convex region $U$ in $\mathbb{R}^{2n}$. We denote the set of all such hypersurfaces by $\mathcal{H}(2n)$. Without loss of generality, we suppose $U$ contains the origin. We denote the set of all compact convex hypersurfaces which are symmetric with respect to the origin by $S\mathcal{H}(2n)$, i.e., $\Sigma = -\Sigma$ for $\Sigma \in S\mathcal{H}(2n)$. We consider closed characteristics $(\tau, y)$ on $\Sigma$, which are solutions of the following problem

$$\begin{cases}
\dot{y} = JN_{\Sigma}(y), \\
y(\tau) = y(0),
\end{cases}$$

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where \( J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \), \( I_n \) is the identity matrix in \( \mathbb{R}^n \), \( \tau > 0 \) and \( N_\Sigma(y) \) is the outward normal vector of \( \Sigma \) at \( y \) normalized by the condition \( N_\Sigma(y) \cdot y = 1 \). Here \( a \cdot b \) denotes the standard inner product of \( a, b \in \mathbb{R}^{2n} \). A closed characteristic \((\tau, y)\) is prime if \( \tau \) is the minimal period of \( y \). Two closed characteristics \((\tau, y)\) and \((\sigma, z)\) are geometrically distinct if \( y(R) \neq z(R) \). We denote by \( \mathcal{J}(\Sigma) \) and \( \tilde{\mathcal{J}}(\Sigma) \) the set of all closed characteristics \((\tau, y)\) on \( \Sigma \) with \( \tau \) being the minimal period of \( y \) and the set of all geometrically distinct ones respectively. Note that \( \mathcal{J}(\Sigma) = \{ \theta \cdot y \mid \theta \in S^1, \ y \text{ is prime} \} \), while \( \tilde{\mathcal{J}}(\Sigma) = \mathcal{J}(\Sigma)/S^1 \), where the natural \( S^1 \)-action is defined by \( \theta \cdot y(t) = y(t + \tau \theta), \ \forall \theta \in S^1, \ t \in \mathbb{R} \).

Let \( j : \mathbb{R}^{2n} \to \mathbb{R} \) be the gauge function of \( \Sigma \), i.e., \( j(\lambda x) = \lambda \) for \( x \in \Sigma \) and \( \lambda \geq 0 \), then \( j \in C^3(\mathbb{R}^{2n} \setminus \{0\}, \mathbb{R}) \cap C^0(\mathbb{R}^{2n}, \mathbb{R}) \) and \( \Sigma = j^{-1}(1) \). Fix a constant \( \alpha \in (1, 2) \) and define the Hamiltonian function \( H_\alpha : \mathbb{R}^{2n} \to [0, +\infty) \) by
\[
H_\alpha(x) = j(x)^\alpha, \quad \forall x \in \mathbb{R}^{2n}.
\]

Then \( H_\alpha \in C^3(\mathbb{R}^{2n} \setminus \{0\}, \mathbb{R}) \cap C^1(\mathbb{R}^{2n}, \mathbb{R}) \) is convex and \( \Sigma = H_\alpha^{-1}(1) \). It is well known that problem (1.1) is equivalent to the following given energy problem of Hamiltonian system
\[
\begin{cases}
\dot{y}(t) = JH'_\alpha(y(t)), \\
y(\tau) = y(0).
\end{cases}
\]

Denote by \( \mathcal{J}(\Sigma, \alpha) \) the set of all solutions \((\tau, y)\) of (1.3) where \( \tau \) is the minimal period of \( y \) and by \( \tilde{\mathcal{J}}(\Sigma, \alpha) \) the set of all geometrically distinct solutions of (1.3). As above, \( \tilde{\mathcal{J}}(\Sigma, \alpha) \) is obtained from \( \mathcal{J}(\Sigma, \alpha) \) by dividing the natural \( S^1 \)-action. Note that elements in \( \mathcal{J}(\Sigma) \) and \( \mathcal{J}(\Sigma, \alpha) \) are one to one correspondent to each other, similarly for \( \tilde{\mathcal{J}}(\Sigma) \) and \( \tilde{\mathcal{J}}(\Sigma, \alpha) \).

Let \((\tau, y) \in \mathcal{J}(\Sigma, \alpha)\). The fundamental solution \( \gamma_y : [0, \tau] \to \operatorname{Sp}(2n) \) with \( \gamma_y(0) = I_{2n} \) of the linearized Hamiltonian system
\[
\dot{w}(t) = JH''_\alpha(y(t))w(t), \quad \forall t \in \mathbb{R},
\]

is called the associate symplectic path of \((\tau, y)\). The eigenvalues of \( \gamma_y(\tau) \) are called Floquet multipliers of \((\tau, y)\). By Proposition 1.6.13 of [Luk], the Floquet multipliers with their multiplicities of \((\tau, y) \in \mathcal{J}(\Sigma)\) do not depend on the particular choice of the Hamiltonian function in (1.3). For any \( M \in \operatorname{Sp}(2n) \), we define the elliptic height \( e(M) \) of \( M \) to be the total algebraic multiplicity of all eigenvalues of \( M \) on the unit circle \( U = \{ z \in \mathbb{C} \mid |z| = 1 \} \) in the complex plane \( \mathbb{C} \). Since \( M \) is symplectic, \( e(M) \) is even and \( 0 \leq e(M) \leq 2n \). As usual \((\tau, y) \in \mathcal{J}(\Sigma) \) is elliptic if \( e(\gamma_y(\tau)) = 2n \). It is non-degenerate if 1 is a double Floquet multiplier of it. It is hyperbolic if 1 is a double Floquet multiplier of it and \( e(\gamma_y(\tau)) = 2 \). It is irrationally elliptic if \( \gamma_y(\tau) \) is suitably homotopic to
the $\diamond$-product of one $N_1(1, 1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $n - 1$ rotation $2 \times 2$ matrices with rotation angles being irrational multiples of $\pi$, more precisely, $N_1(1, 1) \diamond R(\theta_1) \diamond \cdots \diamond R(\theta_{n-1}) \in \Omega^0(\gamma_x(\tau))$ for some $\theta_i \in (0, 2\pi) \setminus (\pi Q)$ with $1 \leq i \leq n - 1$, cf., §3 below for notations. It is well known that these concepts are independent of the choice of $\alpha \in (1, 2)$.

For the existence and multiplicity of geometrically distinct closed characteristics on convex compact hypersurfaces in $\mathbb{R}^{2n}$ we refer to [Rab1], [Wei1], [Ekl1], [EklH1], [Szu1], [Vit1], [HWZ], [LoZ1], [LLZ], [WHL], and references therein.

On the stability problem, in [Eke2] of Ekeland in 1986 and [Lon2] of Long in 1998, for any $\Sigma \in \mathcal{H}(2n)$ the existence of at least one non-hyperbolic closed characteristic on $\Sigma$ was proved provided $\# \mathcal{J}(\Sigma) < +\infty$. Ekeland proved also in [Eke2] the existence of at least one elliptic closed characteristic on $\Sigma$ provided $\Sigma \in \mathcal{H}(2n)$ is $\sqrt{2}$-pinched. In [DDE1] of 1992, Dell’Antonio, D’Onofrio and Ekeland proved the existence of at least one elliptic closed characteristic on $\Sigma$ provided $\Sigma \in S\mathcal{H}(2n)$. In [Lon4] of 2000, Long proved that $\Sigma \in \mathcal{H}(4)$ and $\# \mathcal{J}(\Sigma) = 2$ imply that both of the closed characteristics must be elliptic. In [LoZ1] of 2002, Long and Zhu further proved when $\# \mathcal{J}(\Sigma) < +\infty$, there exists at least one elliptic closed characteristic and there are at least $\left\lfloor \frac{\pi}{2} \right\rfloor$ geometrically distinct closed characteristics on $\Sigma$ possessing irrational mean indices, which are then non-hyperbolic. In [LoW1], Long and the author proved that there exist at least two non-hyperbolic closed characteristics on $\Sigma \in \mathcal{H}(6)$ when $\# \mathcal{J}(\Sigma) < +\infty$. In [Wang], the author proved that on every $\Sigma \in \mathcal{H}(6)$ satisfying $\# \mathcal{J}(\Sigma) < +\infty$, there exist at least two closed characteristics possessing irrational mean indices and if $\# \mathcal{J}(\Sigma) = 3$, then there exist at least two elliptic closed characteristics. It was conjectured by Hofer et al. that $\{ \# \mathcal{J}(\Sigma) | \Sigma \in \mathcal{H}(2n) \} = \{ n \} \cup \{ +\infty \}$ for $n \geq 2$ and it was conjectured by Long et al. in [WHL] that all the closed characteristics on $\Sigma$ are irrationally elliptic for $\Sigma \in \mathcal{H}(2n)$ with $n \geq 2$ whenever $\# \mathcal{J}(\Sigma) < \infty$. Note that both conjectures have been proved in the $n = 2$ case, cf. [HWZ] and [WHL] respectively.

Motivated by these results, we prove the following results in this article:

**Theorem 1.1.** On every $\Sigma \in S\mathcal{H}(2n)$ satisfying $\# \mathcal{J}(\Sigma) < +\infty$ there exist at least $n - 1$ non-hyperbolic closed characteristics in $\mathcal{J}(\Sigma)$.

**Theorem 1.2.** Suppose $\# \mathcal{J}(\Sigma) = n$ for some $\Sigma \in S\mathcal{H}(2n)$ and $n \geq 2$. Then there exist at least two elliptic closed characteristics in $\mathcal{J}(\Sigma)$.

The proofs of these theorems are given in §4. The proofs are motivated by the methods in [LoZ1] and [LLZ] by using the index iteration theory developed by Long and his coworkers, specially the common index jump theorem of Long and Zhu (Theorem 4.3 of [LoZ1], cf. Theorem 11.2.1 of
In §2 and §3, we review briefly the variational structure for closed characteristics and the index iteration theory for symplectic paths respectively.

In this article, let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of natural integers, non-negative integers, integers, rational numbers, real numbers, and complex numbers respectively. Denote by $a \cdot b$ and $|a|$ the standard inner product and norm in $\mathbb{R}^{2n}$. Denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the standard $L^2$-inner product and $L^2$-norm. For an $S^1$-space $X$, we denote by $X_{S^1}$ the homotopy quotient of $X$ module the $S^1$-action, i.e., $X_{S^1} = S^\infty \times_{S^1} X$. We define the functions

$$\begin{align*}
[a] &= \max\{k \in \mathbb{Z} \mid k \leq a\}, \\
E(a) &= \min\{k \in \mathbb{Z} \mid k \geq a\}, \\
\phi(a) &= E(a) - [a],
\end{align*}$$

(1.5)

Specially, $\phi(a) = 0$ if $a \in \mathbb{Z}$, and $\phi(a) = 1$ if $a \notin \mathbb{Z}$. In this article we use only $\mathbb{Q}$-coefficients for all homological modules.

## 2 Variational structure for closed characteristics

In this section, we describe the variational structure for closed characteristics.

As in P.199 of [Eke3], choose some $\alpha \in (1, 2)$ and associate with $U$ a convex function $H_\alpha$ such that $H_\alpha(\lambda x) = \lambda^\alpha H_\alpha(x)$ for $\lambda \geq 0$. Consider the fixed period problem

$$\begin{align*}
\dot{x}(t) &= JH'_\alpha(x(t)), \\
x(1) &= x(0).
\end{align*}$$

(2.1)

Define

$$L_0^{\alpha-1}(S^1, \mathbb{R}^{2n}) = \{u \in L_0^{\alpha-1}(S^1, \mathbb{R}^{2n}) \mid \int_0^1 u dt = 0\}.$$  

(2.2)

The corresponding Clarke-Ekeland dual action functional is defined by

$$\Phi(u) = \int_0^1 \left(\frac{1}{2} Ju \cdot Mu + H^*_\alpha(-Ju)\right) dt, \quad \forall u \in L_0^{\alpha-1}(S^1, \mathbb{R}^{2n}),$$

(2.3)

where $Mu$ is defined by $\frac{d}{dt}Mu(t) = u(t)$ and $\int_0^1 Mu(t) dt = 0$, $H^*_\alpha$ is the Fenchel transform of $H_\alpha$ defined by $H^*_\alpha(y) = \sup\{x \cdot y - H_\alpha(x) \mid x \in \mathbb{R}^{2n}\}$. By Theorem 5.2.8 of [Eke3], $\Phi$ is $C^1$ on $L_0^{\alpha-1}$ and satisfies the Palais-Smale condition. Suppose $x$ is a solution of (2.1). Then $u = \dot{x}$ is a critical point of $\Phi$. Conversely, suppose $u$ is a critical point of $\Phi$. Then there exists a unique $\xi \in \mathbb{R}^{2n}$ such that $Mu - \xi$ is a solution of (2.1). In particular, solutions of (2.1) are in one to one correspondence with critical points of $\Phi$. Moreover, $\Phi(u) < 0$ for every critical point $u \neq 0$ of $\Phi$.

Suppose $u$ is a nonzero critical point of $\Phi$. Then the formal Hessian of $\Phi$ at $u$ on $L_0^2(S^1, \mathbb{R}^{2n})$ is defined by

$$Q(v, v) = \int_0^1 (Jv \cdot Mv + (H^*_\alpha)^n(-Ju)Jv \cdot Jv) dt,$$
which defines an orthogonal splitting \( L_0^\alpha(S^1, \mathbb{R}^{2n}) = E_- \oplus E_0 \oplus E_+ \) of \( L_0^\alpha(S^1, \mathbb{R}^{2n}) \) into negative, zero and positive subspaces. The index of \( u \) is defined by \( i(u) = \dim E_- \) and the nullity of \( u \) is defined by \( \nu(u) = \dim E_0 \). Specially \( 1 \leq \nu(u) \leq 2n \) always holds, cf. P.219 of \([Eke3]\).

We have a natural \( S^1 \)-action on \( L_0^\alpha(S^1, \mathbb{R}^{2n}) \) defined by \( \theta \cdot u(t) = u(\theta + t) \) for all \( \theta \in S^1 \) and \( t \in \mathbb{R} \). Clearly \( \Phi \) is \( S^1 \)-invariant. Hence if \( u \) is a critical point of \( \Phi \), then the whole orbit \( S^1 \cdot u \) is formed by critical points of \( \Phi \). Denote by \( \text{crit}(\Phi) \) the set of critical points of \( \Phi \). Then we make the following definition

**Definition 2.1.** Suppose \( u \) is a nonzero critical point of \( \Phi \), and \( \mathcal{N} \) is an \( S^1 \)-invariant open neighborhood of \( S^1 \cdot u \) such that \( \text{crit}(\Phi) \cap (\Lambda(u) \cap \mathcal{N}) = S^1 \cdot u \). Then the \( S^1 \)-critical modules of \( S^1 \cdot u \) is defined by

\[
C_{S^1, q}(\Phi, S^1 \cdot u) = H_q((\Lambda(u) \cap \mathcal{N})_{S^1}, ((\Lambda(u) \setminus S^1 \cdot u) \cap \mathcal{N})_{S^1}),
\]

where \( \Lambda(u) = \{ w \in L_0^\alpha(S^1, \mathbb{R}^{2n}) \mid \Phi(w) \leq \Phi(u) \} \).

By the proof of Proposition 3.6 of \([Wang]\) we have

\[
C_{S^1, q}(\Phi, S^1 \cdot u) \cong C_{S^1, q}(\Psi_a, S^1 \cdot u_a),
\]

where \( \Psi_a \) is the functional constructed in \([WHL]\) and \( u_a \) is its critical point corresponding to \( u \). By Proposition 3.5 of \([WHL]\), the index and nullity of \( \Psi_a \) at \( u_a \) coincide with those of \( \Phi \) at \( u \). Hence by Propositions 2.3 and 2.6 of \([Wang]\), we have

**Proposition 2.2.** Let \( k_j(u) \equiv \dim C_{S^1, j}(\Phi, S^1 \cdot u) \). Then \( k_j(u) \) equal to 0 when \( j < i(u) \) or \( j > i(u) + \nu(u) - 1 \) and can only take values 0 or 1 when \( j = i(u) \) or \( j = i(u) + \nu(u) - 1 \).

For a closed characteristic \((\tau, y)\) on \( \Sigma \), we denote by \( y^m \equiv (m\tau, y) \) the \( m \)-th iteration of \( y \) for \( m \in \mathbb{N} \). Let \( u^m \) be the unique critical point of \( \Phi \) corresponding to \((m\tau, y)\). Then we define the index \( i(y^m) \) and nullity \( \nu(y^m) \) of \((m\tau, y)\) for \( m \in \mathbb{N} \) by

\[
i(y^m) = i(u^m), \quad \nu(y^m) = \nu(u^m).
\]

The mean index of \((\tau, y)\) is defined by

\[
\hat{i}(y) = \lim_{m \to \infty} \frac{i(y^m)}{m}.
\]

Note that \( \hat{i}(y) > 2 \) always holds which was proved by Ekeland and Hofer in \([Ekh1]\) of 1987 (cf. Corollary 8.3.2 and Lemma 15.3.2 of \([Lon5]\) for a different proof).

Recall that for a principal \( U(1) \)-bundle \( E \to B \), the Fadell-Rabinowitz index (cf. \([Far1]\)) of \( E \) is defined to be \( \sup\{ k \mid c_1(E)^k \neq 0 \} \), where \( c_1(E) \in H^2(B, \mathbb{Q}) \) is the first rational Chern class.
For a $U(1)$-space, i.e., a topological space $X$ with a $U(1)$-action, the Fadell-Rabinowitz index is defined to be the index of the bundle $X \times S^\infty \to X \times_{U(1)} S^\infty$, where $S^\infty \to CP^\infty$ is the universal $U(1)$-bundle.

For any $\kappa \in \mathbb{R}$, we denote by

$$\Phi^{-\kappa} = \{ u \in L^0_0 (S^1, \mathbb{R}^{2n}) \mid \Phi(u) < \kappa \}. \quad (2.8)$$

Then as in P.218 of [Eke3], we define

$$c_i = \inf \{ \delta \in \mathbb{R} \mid \hat{I}(\Phi^{\delta-}) \geq i \}, \quad (2.9)$$

where $\hat{I}$ is the Fadell-Rabinowitz index defined above. Then by Proposition 3 in P.218 of [Eke3], we have

**Proposition 2.3.** Every $c_i$ is a critical value of $\Phi$. If $c_i = c_j$ for some $i < j$, then there are infinitely many geometrically distinct closed characteristics on $\Sigma$.

Comparing with Theorem 4 in P.219 of [Eke3], we have the following property by Proposition 3.5 of [Wang].

**Proposition 2.4.** For every $i \in \mathbb{N}$, there exists a point $u \in L^0_0 (S^1, \mathbb{R}^{2n})$ such that

$$\Phi'(u) = 0, \quad \Phi(u) = c_i, \quad (2.10)$$

$$C_{S^1, 2(i-1)} (\Phi, S^1 \cdot u) \neq 0. \quad (2.11)$$

**Definition 2.5.** A prime closed characteristic $(\tau, y)$ is $(m, i)$-variationally visible: if there exist some $m, i \in \mathbb{N}$ such that $(2.10)$ and $(2.11)$ hold for $y^m$ and $c_i$. We call $(\tau, y)$ infinitely variationally visible: if there exist infinitely many $m, i \in \mathbb{N}$ such that $(\tau, y)$ is $(m, i)$-variationally visible. We denote by $\mathcal{V}(\Sigma, \alpha)$ and $\mathcal{V}_\infty(\Sigma, \alpha)$ the set of variationally visible and infinitely variationally visible closed characteristics respectively.

Recall that the action of a closed characteristic $(\tau, y)$ is defined by (cf. P190 of [Eke3])

$$A(\tau, y) = \frac{1}{2} \int_0^\tau (Jy \cdot \dot{y}) dt. \quad (2.12)$$

Then we have the following

**Theorem 2.6.** Suppose there are only finitely many prime closed characteristics on $\Sigma$. Then for any $(\tau, y) \in \mathcal{V}_\infty(\Sigma, \alpha)$, we have

$$\frac{\dot{i}(y)}{A(\tau, y)} = \frac{1}{\gamma(\Sigma)}, \quad (2.13)$$

where

$$\gamma(\Sigma) = C_\alpha^{-1} \lim_{i \to \infty} (i(-c_i)^{2-\alpha})^{-1} = C_\alpha^{-1} \lim_{i \to \infty} (i(-c_i)^{2-\alpha})^{-1}$$
and \( C_\alpha = \frac{4}{\alpha} (1 - \frac{\alpha}{2} \frac{\alpha^2}{2}) \).

**Proof.** Note that we have \( \hat{i}(y^m) = m\hat{i}(y) \) by (2.7) and \( A(y^m) = mA(y) \) by (2.12). Thus \( \frac{\hat{i}(y^m)}{A(y^m)} = \frac{\hat{i}(y)}{A(y)} \) for any \( m \in \mathbb{N} \). Now the theorem follows from Lemma 5.3.12 and Theorem 5.3.15 of [Eke3].

### 3 A brief review on an index theory for symplectic paths

In this section, we recall briefly an index theory for symplectic paths developed by Y. Long and his coworkers. All the details can be found in [Lon5].

As usual, the symplectic group \( \text{Sp}(2n) \) is defined by

\[
\text{Sp}(2n) = \{ M \in \text{GL}(2n, \mathbb{R}) \mid M^TJM = J \},
\]

whose topology is induced from that of \( \mathbb{R}^{4n^2} \). For \( \tau > 0 \) we are interested in paths in \( \text{Sp}(2n) \):

\[
\mathcal{P}_\tau(2n) = \{ \gamma \in C([0, \tau], \text{Sp}(2n)) \mid \gamma(0) = I_{2n} \},
\]

which is equipped with the topology induced from that of \( \text{Sp}(2n) \). The following real function was introduced in [Lon3]:

\[
D_\omega(M) = (-1)^{n-1} \omega^n \det(M - \omega I_{2n}), \quad \forall \omega \in \mathbb{U}, M \in \text{Sp}(2n).
\]

Thus for any \( \omega \in \mathbb{U} \) the following codimension 1 hypersurface in \( \text{Sp}(2n) \) is defined in [Lon3]:

\[
\text{Sp}(2n)_\omega^0 = \{ M \in \text{Sp}(2n) \mid D_\omega(M) = 0 \}.
\]

For any \( M \in \text{Sp}(2n)_\omega^0 \), we define a co-orientation of \( \text{Sp}(2n)_\omega^0 \) at \( M \) by the positive direction \( \frac{d}{dt} Me^{teJ} \big|_{t=0} \) of the path \( Me^{teJ} \) with \( 0 \leq t \leq 1 \) and \( \epsilon > 0 \) being sufficiently small. Let

\[
\begin{align*}
\text{Sp}(2n)^*_{\omega} &= \text{Sp}(2n) \setminus \text{Sp}(2n)^0_{\omega}, \\
\mathcal{P}^*_{\tau,\omega}(2n) &= \{ \gamma \in \mathcal{P}_\tau(2n) \setminus \gamma(\tau) \in \text{Sp}(2n)^*_{\omega} \}, \\
\mathcal{P}^0_{\tau,\omega}(2n) &= \mathcal{P}_\tau(2n) \setminus \mathcal{P}^*_{\tau,\omega}(2n).
\end{align*}
\]

For any two continuous arcs \( \xi, \eta : [0, \tau] \to \text{Sp}(2n) \) with \( \xi(\tau) = \eta(0) \), it is defined as usual:

\[
\eta \ast \xi(t) = \begin{cases} 
\xi(2t), & \text{if } 0 \leq t \leq \tau/2, \\
\eta(2t - \tau), & \text{if } \tau/2 \leq t \leq \tau.
\end{cases}
\]
Given any two $2m_k \times 2m_k$ matrices of square block form $M_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}$ with $k = 1, 2$, as in [Lon5], the $\diamond$-product of $M_1$ and $M_2$ is defined by the following $2(m_1 + m_2) \times 2(m_1 + m_2)$ matrix $M_1 \diamond M_2:\n$

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$

Denote by $M^\diamond k$ the $k$-fold $\diamond$-product $M \diamond \cdots \diamond M$. Note that the $\diamond$-product of any two symplectic matrices is symplectic. For any two paths $\gamma_j \in \mathcal{P}_\tau(2n_j)$ with $j = 0$ and $1$, let $\gamma_0 \diamond \gamma_1(t) = \gamma_0(t) \diamond \gamma_1(t)$ for all $t \in [0, \tau]$.

A special path $\xi_n \in \mathcal{P}_\tau(2n)$ is defined by

$$\xi_n(t) = \begin{pmatrix} 2 - \frac{t}{\tau} & 0 \\ 0 & (2 - \frac{t}{\tau})^{-1} \end{pmatrix}^{\circ n} \quad \text{for } 0 \leq t \leq \tau. \quad (3.1)$$

**Definition 3.1.** (cf. [Lon3], [Lon5]) For any $\omega \in U$ and $M \in \text{Sp}(2n)$, define

$$\nu_\omega(M) = \dim \ker C(M - \omega I_{2n}). \quad (3.2)$$

For any $\tau > 0$ and $\gamma \in \mathcal{P}_\tau(2n)$, define

$$\nu_\omega(\gamma) = \nu_\omega(\gamma(\tau)). \quad (3.3)$$

If $\gamma \in \mathcal{P}_{\tau, \omega}^*(2n)$, define

$$i_\omega(\gamma) = [\text{Sp}(2n)_\omega^0 : \gamma * \xi_n], \quad (3.4)$$

where the right hand side of (3.4) is the usual homotopy intersection number, and the orientation of $\gamma * \xi_n$ is its positive time direction under homotopy with fixed end points.

If $\gamma \in \mathcal{P}_{\tau, \omega}^0(2n)$, we let $\mathcal{F}(\gamma)$ be the set of all open neighborhoods of $\gamma$ in $\mathcal{P}_\tau(2n)$, and define

$$i_\omega(\gamma) = \sup_{U \in \mathcal{F}(\gamma)} \inf \{i_\omega(\beta) \mid \beta \in U \cap \mathcal{P}_{\tau, \omega}^*(2n)\}. \quad (3.5)$$

Then

$$(i_\omega(\gamma), \nu_\omega(\gamma)) \in \mathbb{Z} \times \{0, 1, \ldots, 2n\},$$

is called the index function of $\gamma$ at $\omega$.

Note that when $\omega = 1$, this index theory was introduced by C. Conley-E. Zehnder in [CoZ1] for the non-degenerate case with $n \geq 2$, Y. Long-E. Zehnder in [LZe1] for the non-degenerate case with $n = 1$, and Y. Long in [Lon1] and C. Viterbo in [Vit2] independently for the degenerate case.
The case for general $\omega \in U$ was defined by Y. Long in [Lon3] in order to study the index iteration theory (cf. [Lon5] for more details and references).

For any symplectic path $\gamma \in P_\tau(2n)$ and $m \in \mathbb{N}$, we define its $m$-th iteration $\gamma^m : [0, m\tau] \to \text{Sp}(2n)$ by

$$
\gamma^m(t) = \gamma(t - j\tau)\gamma(\tau)^j, \quad \text{for} \quad j\tau \leq t \leq (j + 1)\tau, \quad j = 0, 1, \ldots, m - 1.
$$

(3.6)

We still denote the extended path on $[0, +\infty)$ by $\gamma$.

**Definition 3.2.** (cf. [Lon3], [Lon5]) For any $\gamma \in P_\tau(2n)$, we define

$$(i(\gamma, m), \nu(\gamma, m)) = (i_1(\gamma^m), \nu_1(\gamma^m)), \quad \forall m \in \mathbb{N}.
$$

(3.7)

The mean index $\hat{i}(\gamma, m)$ per $m\tau$ for $m \in \mathbb{N}$ is defined by

$$
\hat{i}(\gamma, m) = \lim_{k \to +\infty} \frac{i(\gamma, mk)}{k}.
$$

(3.8)

For any $M \in \text{Sp}(2n)$ and $\omega \in U$, the splitting numbers $S^\pm_M(\omega)$ of $M$ at $\omega$ are defined by

$$
S^\pm_M(\omega) = \lim_{\epsilon \to 0^+} i_{\omega\exp(\pm\sqrt{-1}\epsilon)}(\gamma) - i_{\omega}(\gamma),
$$

(3.9)

for any path $\gamma \in P_\tau(2n)$ satisfying $\gamma(\tau) = M$.

For a given path $\gamma \in P_\tau(2n)$ we consider to deform it to a new path $\eta$ in $P_\tau(2n)$ so that

$$
i_1(\gamma^m) = i_1(\eta^m), \quad \nu_1(\gamma^m) = \nu_1(\eta^m), \quad \forall m \in \mathbb{N},
$$

(3.10)

and that $(i_1(\eta^m), \nu_1(\eta^m))$ is easy enough to compute. This leads to finding homotopies $\delta : [0, 1] \times [0, \tau] \to \text{Sp}(2n)$ starting from $\gamma$ in $P_\tau(2n)$ and keeping the end points of the homotopy always stay in a certain suitably chosen maximal subset of $\text{Sp}(2n)$ so that (3.10) always holds. In fact, this set was first discovered in [Lon3] as the path connected component $\Omega^0(M)$ containing $M = \gamma(\tau)$ of the set

$$
\Omega(M) = \{N \in \text{Sp}(2n) \mid \sigma(N) \cap U = \sigma(M) \cap U \text{ and } \nu_\lambda(N) = \nu_\lambda(M) \forall \lambda \in \sigma(M) \cap U\}.
$$

(3.11)

Here $\Omega^0(M)$ is called the *homotopy component* of $M$ in $\text{Sp}(2n)$.

In [Lon3], [Lon5], the following symplectic matrices were introduced as *basic normal forms*:

$$
D(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \quad \lambda = \pm 2,
$$

(3.12)
\[ N_1(\lambda, b) = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}, \quad \lambda = \pm 1, b = \pm 1, 0, \quad (3.13) \]

\[ R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad \theta \in (0, \pi) \cup (\pi, 2\pi), \quad (3.14) \]

\[ N_2(\omega, b) = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta) \end{pmatrix}, \quad \theta \in (0, \pi) \cup (\pi, 2\pi), \quad (3.15) \]

where \( b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \) with \( b_i \in \mathbb{R} \) and \( b_2 \neq b_3 \).

Splitting numbers possess the following properties:

**Lemma 3.3.** (cf. [Lon3] and Lemma 9.1.5 of [Lon5]) Splitting numbers \( S_M^\pm(\omega) \) are well defined, i.e., they are independent of the choice of the path \( \gamma \in \mathcal{P}_\tau(2n) \) satisfying \( \gamma(\tau) = M \) appeared in (3.12). For \( \omega \in U \) and \( M \in \text{Sp}(2n) \), splitting numbers \( S_N^\pm(\omega) \) are constant for all \( N \in \Omega^0(M) \).

**Lemma 3.4.** (cf. [Lon3], Lemma 9.1.5 and List 9.1.12 of [Lon5]) For \( M \in \text{Sp}(2n) \) and \( \omega \in U \), there hold

\[ S_M^\pm(\omega) = 0, \quad \text{if} \quad \omega \notin \sigma(M). \]

\[ S_{N_1(1,a)}^+(1) = \begin{cases} 1, & \text{if} \quad a \geq 0, \\ 0, & \text{if} \quad a < 0. \end{cases} \quad (3.16) \]

For any \( M_i \in \text{Sp}(2n_i) \) with \( i = 0 \) and \( 1 \), there holds

\[ S_{M_0 \diamond M_1}^\pm(\omega) = S_{M_0}^\pm(\omega) + S_{M_1}^\pm(\omega), \quad \forall \omega \in U. \quad (3.18) \]

We have the following

**Theorem 3.5.** (cf. [Lon4] and Theorem 1.8.10 of [Lon5]) For any \( M \in \text{Sp}(2n) \), there is a path \( f : [0, 1] \to \Omega^0(M) \) such that \( f(0) = M \) and

\[ f(1) = M_1 \diamond \cdots \diamond M_k, \quad (3.19) \]

where each \( M_i \) is a basic normal form listed in (3.12)-(3.15) for \( 1 \leq i \leq k \).

Let \( \Sigma \in \mathcal{H}(2n) \). Using notations in §1, for any \( (\tau, y) \in \mathcal{J}(\Sigma, \alpha) \) and \( m \in \mathbb{N} \), we define its \( m \)-th iteration \( y^m : \mathbb{R}/(m\tau \mathbb{Z}) \to \mathbb{R}^{2n} \) by

\[ y^m(t) = y(t - j\tau), \quad \text{for} \quad j\tau \leq t \leq (j + 1)\tau, \quad j = 0, 1, 2, \ldots, m - 1. \quad (3.20) \]

Note that this coincide with that in §2. We still denote by \( y \) its extension to \([0, +\infty)\).

We define via Definition 3.2 the following

\[ S^+(y) = S_{\gamma_y(\tau)}^+(1), \quad (3.21) \]

\[ (i(y, m), \nu(y, m)) = (i(\gamma_y, m), \nu(\gamma_y, m)), \quad (3.22) \]

\[ \hat{i}(y, m) = \hat{i}(\gamma_y, m), \quad (3.23) \]
for all $m \in \mathbb{N}$, where $\gamma_y$ is the associated symplectic path of $(\tau, y)$. Then we have the following.

**Theorem 3.6.** (cf. Lemma 1.1 of [LoZ1], Theorem 15.1.1 of [Lon5]) Suppose $(\tau, y) \in J(\Sigma, \alpha)$. Then we have

$$i(y^m) \equiv i(m\tau, y) = i(y, m) - n, \quad \nu(y^m) \equiv \nu(m\tau, y) = \nu(y, m), \quad \forall m \in \mathbb{N},$$

(3.24)

where $i(y^m)$ and $\nu(y^m)$ are the index and nullity defined in §2. In particular, (2.7) and (3.8) coincide, thus we simply denote them by $\hat{i}(y)$.

### 4 Proofs of the main theorems

In the rest of this article, we fix a $\Sigma \in \mathcal{SH}(2n)$ and assume the following condition on $\Sigma$:

(F) There exist only finitely many geometrically distinct closed characteristics

$\{(\tau_j, y_j)\}_{1 \leq j \leq k}$ on $\Sigma$.

We denote by $\gamma_j \equiv \gamma_{y_j}$ the associated symplectic path of $(\tau_j, y_j)$ on $\Sigma$ for $1 \leq j \leq k$. Then by Lemma 1.3 of [LoZ1] or Lemma 15.2.4 of [Lon5], there exist $P_j \in \text{Sp}(2n)$ and $M_j \in \text{Sp}(2n - 2)$ such that

$$\gamma_j(\tau_j) = P_j^{-1}(N_1(1, 1) \circ M_j)P_j, \quad \forall 1 \leq j \leq k,$$

(4.1)

here we use notations in §3.

Firstly we have the following property, cf., Lemma 4.2 of [LL Z].

**Lemma 4.1.** Suppose $(\tau, y) \in J(\Sigma, \alpha)$, then $(\tau, -y) \in J(\Sigma, \alpha)$ and either $O(y) = O(-y)$ or $O(y) \cap O(-y) = \emptyset$, where $O(\pm y) = \{\pm y(t) | t \in \mathbb{R}\}$. Moreover, if $O(y) \cap O(-y) \neq \emptyset$, then we have

$$y(t) = -y\left(t + \frac{\tau}{2}\right), \quad \forall t \in \mathbb{R}.$$

In the following we call a closed characteristic $(\tau, y)$ on $\Sigma \in \mathcal{SH}(2n)$ symmetric if $O(y) \cap O(-y) \neq \emptyset$, non-symmetric if $O(y) \cap O(-y) = \emptyset$. Thus if $(\tau, y)$ is non-symmetric, then $(\tau, y)$ and $(\tau, -y)$ are geometrically distinct; if $(\tau, y)$ is symmetric, then $(\tau, y)$ and $(\tau, -y)$ are geometrically the same.

We have the following property, cf., Lemma 15.6.4 of [Lon5].

**Lemma 4.2.** Suppose $(\tau, y) \in J(\Sigma, \alpha)$ is a symmetric closed characteristic on $\Sigma \in \mathcal{SH}(2n)$. Then we have

$$i(y, 1) + 25^+(y) - \nu(y, 1) \geq n.$$

(4.2)

Now we can give:
Proof of Theorem 1.1. Since $\hat{i}(y_j) > 2$ for $1 \leq j \leq k$, we can use the common index jump theorem (Theorems 4.3 and 4.4 of [LoZ1], Theorems 11.2.1 and 11.2.2 of [Lon5]) to obtain infinitely many $(T, m_1, \ldots, m_k) \in \mathbb{N}^{k+1}$ such that the following hold:

\begin{align}
\nu(y_j, 2m_j - 1) &= \nu(y_j, 1), \\
i(y_j, 2m_j) &\geq 2T - \frac{e(\gamma_j(\tau_j))}{2} \geq 2T - n, \\
i(y_j, 2m_j) + \nu(y_j, 2m_j) &\leq 2T + \frac{e(\gamma_j(\tau_j))}{2} - 1 \leq 2T + n - 1, \\
i(y_j, 2m_j + 1) &= 2T + i(y_j, 1). \\
i(y_j, 2m_j - 1) + \nu(y_j, 2m_j - 1) &= 2T - (i(y_j, 1) + 2S^+(y_j) - \nu(y_j, 1)).
\end{align}

(4.3) \quad (4.4) \quad (4.5) \quad (4.6)

Note that (4.5) holds by Theorem 4.4 of [LoZ1], other parts follow by Theorem 4.3 of [LoZ1]. More precisely, by Theorem 4.1 of [LoZ1] (in (11.1.10) in Theorem 11.1.1 of [Lon5], with $D_j = \hat{i}(y_j)$), we have

\begin{align}
m_j &= \left(\left\lfloor \frac{T}{M\hat{i}(y_j)} \right\rfloor + \chi_j \right) M, \quad 1 \leq j \leq k, 
\end{align}

(4.8)

where $\chi_j = 0$ or 1 for $1 \leq j \leq k$ and $M \in \mathbb{N}$ is fixed such that $\frac{M\theta}{\pi} \in \mathbb{Z}$, whenever $e^{\sqrt{-1}\theta} \in \sigma(\gamma_j(\tau_j))$ and $\frac{\theta}{\pi} \in \mathbb{Q}$ for some $1 \leq j \leq k$. Moreover, we have the following

\begin{align}
\frac{T}{M\hat{i}(y_j)} \in \mathbb{N} \quad \text{and} \quad \chi_j = 0, \quad \text{if} \quad \hat{i}(y_j) \in \mathbb{Q},
\end{align}

(4.9)

which can be seen from the proof of Theorem 4.1 of [LoZ1], cf. the proof of Theorem 5.3 of [LoZ1].

By Corollary 1.2 of [LoZ1], we have

\begin{align}
i(y_j, 1) \geq n, \quad 1 \leq j \leq k.
\end{align}

(4.10)

Note that $e(\gamma_j(\tau_j)) \leq 2n$ for $1 \leq j \leq k$. Hence Theorem 2.3 of [LoZ1] yields

\begin{align}
i(y_j, m) + \nu(y_j, m) &\leq i(y_j, m + 1) - i(y_j, 1) + \frac{e(\gamma_j(\tau_j))}{2} - 1 \\
&\leq i(y_j, m + 1) - 1. \quad \forall m \in \mathbb{N}, \ 1 \leq j \leq k.
\end{align}

(4.11)

Specially, we have

\begin{align}
i(y_j, m) < i(y_j, m + 1), \quad \forall m \in \mathbb{N}, \ 1 \leq j \leq k.
\end{align}

(4.12)

By Theorem 3.5 we have

\begin{align}
\gamma_j(\tau_j) \approx N_1(1,1)^{op_{j,-}} \circ I_2^{op_{j,0}} \circ N_1(1,-1)^{op_{j,+}} \circ G_j, \quad 1 \leq j \leq k
\end{align}
for some nonnegative integers $p_{j,-}, p_{j,0}, p_{j,+}$, and some symplectic matrix $G_j$ satisfying $1 \notin \sigma(G_j)$. By (4.11), (4.12) and Lemma 3.4 we obtain

$$2S^+(y_j) = 2(p_{j,-} + p_{j,0}) \geq 2, \quad 1 \leq j \leq k. \quad (4.13)$$

By (4.3), (4.7) and (4.10) we have

$$i(y_j, 2m_j - 1) = 2T - (i(y_j, 1) + 2S^+(y_j)) \leq 2T - n - 2. \quad (4.14)$$

Now by (4.4)-(4.7), (4.10), (4.11), (4.14) and Theorem 3.6 we have

$$i(y_{2m_j}) \geq 2T - 2n, \quad (4.15)$$

$$i(y_{2m_j} + 1) \leq 2T - 2, \quad (4.16)$$

$$i(y_{2m_j - m}) + \nu(y_{2m_j - m}) - 1 \leq 2T - 2n - 4, \quad \forall m \geq 2. \quad (4.17)$$

$$i(y_{2m_j - 1} + \nu(y_{2m_j - 1}) - 1 = 2T - (i(y_j, 1) + 2S^+(y_j) - \nu(y_j, 1)) - n - 1. \quad (4.19)$$

By Proposition 2.4, For every $1 \leq i \leq n$, there exists a point $u \in L_{\rho(i)}^0(S^1, R^{2n})$ such that

$$\Phi'(u) = 0, \quad \Phi(u) = c_{T-i+1}, \quad C_{S^1, 2(T-i)}(\Phi, S^1 \cdot u) \neq 0. \quad (4.20)$$

Let $y_{\rho(i)}^\lambda$ for $1 \leq i \leq n$ be a set of closed characteristics satisfying (4.20), where $\rho: \{1, \ldots, n\} \to \{1, \ldots, k\}$ and $\lambda: \{1, \ldots, n\} \to \mathbb{N}$ are integer valued functions. Note that by Condition (F) and the infiniteness of the tuples $(T, m_1, \ldots, m_k) \in \mathbb{N}^{k+1}$, we can assume

$$y_{\rho(i)} \in V_\infty(S, \alpha), \quad 1 \leq i \leq n. \quad (4.21)$$

By (4.17), (4.18), (4.20) and Proposition 2.2, we have

$$\lambda(i) \in \{2m_{\rho(i)} - 1, 2m_{\rho(i)}\}, \quad (4.22)$$

for each $1 \leq i \leq n$.

**Claim 1.** If $y_{\rho(i)}$ is symmetric, then $\lambda(i) = 2m_{\rho(i)}$.

In fact, by Lemma 4.2 and (4.19), we have

$$i(y_{\rho(i)}^{2m_{\rho(i)} - 1}) + \nu(y_{\rho(i)}^{2m_{\rho(i)} - 1}) - 1 \leq 2T - 2n - 1. \quad (4.23)$$

Thus Claim 1 holds by (4.20) and Proposition 2.2.

**Claim 2.** If $\lambda(i) = 2m_{\rho(i)} - 1$, then $y_{\rho(i)}$ is non-symmetric and non-hyperbolic.
The first statement follows directly from Claim 1. We prove the latter. In fact, suppose \( y_{p(i)} \) is hyperbolic. Thus by (4.1), (4.10), Lemma 3.4 and Theorem 3.5 we have
\[
i(y_{p(i)}, 1) + 2S^+(y_{p(i)}) - \nu(y_{p(i)}, 1) \geq n + 1. \tag{4.24}
\]
Hence we have
\[
i(2m_{p(i)} - 1) + \nu(y_{p(i)}^{2m_{p(i)} - 1}) - 1 \leq 2T - 2n - 2. \tag{4.25}
\]
This contradict to (4.20) and Proposition 2.2. Thus Claim 2 holds.

Claim 3. If \( \lambda(i_1) = 2m_{p(i_1)} \) and \( \lambda(i_2) = 2m_{p(i_2)} \), then either \( \hat{i}(y_{p(i_1)}) \in \mathbb{R} \setminus \mathbb{Q} \) or \( \hat{i}(y_{p(i_2)}) \in \mathbb{R} \setminus \mathbb{Q} \).

Suppose the contrary, i.e., both \( \hat{i}(y_{p(i_1)}) \in \mathbb{Q} \) and \( \hat{i}(y_{p(i_2)}) \in \mathbb{Q} \). Then by (4.8) and (4.9) we have
\[
2m_{p(i_1)}\hat{i}(y_{p(i_1)}) = 2 \left( \left\lfloor \frac{T}{M\hat{i}(y_{p(i_1)})} \right\rfloor + \chi_{p(i_1)} \right) M\hat{i}(y_{p(i_1)})
\]
\[
= 2 \left( \frac{T}{M\hat{i}(y_{p(i_1)})} \right) M\hat{i}(y_{p(i_1)}) = 2T = 2 \left( \frac{T}{M\hat{i}(y_{p(i_2)})} \right) M\hat{i}(y_{p(i_2)})
\]
\[
= 2 \left( \left\lfloor \frac{T}{M\hat{i}(y_{p(i_2)})} \right\rfloor + \chi_{p(i_2)} \right) M\hat{i}(y_{p(i_2)}) = 2m_{p(i_2)}\hat{i}(y_{p(i_2)}). \tag{4.26}
\]
On the other hand, by (4.20) we have
\[
\Phi(y_{p(i_1)}^{2m_{p(i_1)}}) = cT-i_1+1 \neq cT-i_2+1 = \Phi(y_{p(i_2)}^{2m_{p(i_2)}}). \tag{4.27}
\]
By (4.21) and Theorem 2.6 we have
\[
\frac{\hat{i}(y_{p(i_1)})}{A(y_{p(i_1)})} = \frac{1}{\gamma(\Sigma)} = \frac{\hat{i}(y_{p(i_2)})}{A(y_{p(i_2)})}. \tag{4.28}
\]
Note that we have the relations
\[
\hat{i}(m^y) = m\hat{i}(y), \quad A(m^y) = mA(y), \quad \Phi(y) = - \left( 1 - \frac{\alpha}{2} \right) \left( \frac{2}{\alpha} A(y) \right)^{\frac{\alpha}{\alpha-2}}, \quad \forall m \in \mathbb{N}, \tag{4.29}
\]
for any closed characteristic \( y \) on \( \Sigma \) by (2.7), (2.12) and (45) in P.221 of [Eke3].

Hence we have
\[
2m_{p(i_1)}\hat{i}(y_{p(i_1)}) = \gamma(\Sigma)^{-1} \cdot 2m_{p(i_1)}A(y_{p(i_1)}) = \gamma(\Sigma)^{-1} \cdot A(y_{p(i_1)}^{2m_{p(i_1)}})
\]
\[
= 2 (\gamma(\Sigma)C_1)^{-1} (-\Phi(y_{p(i_1)}^{2m_{p(i_1)}}))^{-\frac{\alpha-2}{\alpha}} = 2 (\gamma(\Sigma)C_1)^{-1} (-cT-i_1+1)^{-\frac{\alpha-2}{\alpha}}
\]
\[
\neq 2 (\gamma(\Sigma)C_1)^{-1} (-cT-i_2+1)^{-\frac{\alpha-2}{\alpha}} = 2 (\gamma(\Sigma)C_1)^{-1} (-\Phi(y_{p(i_2)}^{2m_{p(i_2)}}))^{-\frac{\alpha-2}{\alpha}}
\]
\[
= \gamma(\Sigma)^{-1} \cdot A(y_{p(i_2)}^{2m_{p(i_2)}}) = \gamma(\Sigma)^{-1} \cdot 2m_{p(i_2)}A(y_{p(i_2)})
\]
\[
= 2m_{p(i_2)}\hat{i}(y_{p(i_2)}), \tag{4.30}
\]
where $C_\alpha$ is the constant given in Theorem 2.6. This contradict to (4.26). Hence Claim 3 holds.

Now we prove Theorem 1.1 as follows.

For each $j \in \mathcal{I}_\rho$, we have $\#\{\rho^{-1}(j)\} \in \{1, 2\}$ by (4.22). Then we have the following three cases.

**Case 1.** We have $j \in \Theta_1 \equiv \{\#\{\rho^{-1}(l)\} = 2\}$.

In this case $y_j$ is non-symmetric and non-hyperbolic by (4.22) and Claim 2. Hence $y_j$ and $-y_j$ are geometrically distinct by Lemma 4.1. Thus we obtain two non-hyperbolic closed characteristics in $\tilde{J}(\Sigma)$ for each $j$ such that $\#\{\rho^{-1}(j)\} = 2$. Hence we have $2\#\Theta_1$ non-hyperbolic closed characteristics in $\tilde{J}(\Sigma)$ in this case.

**Case 2.** We have $j \in \Theta_2 \equiv \{\#\{\rho^{-1}(l)\} = 1\}$.

In this case $\rho|_{\rho^{-1}(\Theta_2)} : \rho^{-1}(\Theta_2) \to \Theta_2$ is a bijection. By Claim 2, $y_j$ is non-symmetric and non-hyperbolic for $j \in \Theta_2$. Thus we obtain two non-hyperbolic closed characteristics for each $j \in \Theta_2$ as in Case 1. Hence we have $2\#\Theta_2$ non-hyperbolic closed characteristics in $\tilde{J}(\Sigma)$ in this case.

**Case 3.** We have $j \in \Theta_3 \equiv \{\#\{\rho^{-1}(l)\} = 1\}$.

Note that in this case $\rho|_{\rho^{-1}(\Theta_3)} : \rho^{-1}(\Theta_3) \to \Theta_3$ is a bijection. By Claim 3, there exists at most one $j \in \Theta_3$ such that $\hat{i}(y_j) \in Q$. Thus there are at least $\#\Theta_3 - 1$ non-hyperbolic closed characteristics in $\tilde{J}(\Sigma)$ in this case.

Clearly $\Theta_1$, $\Theta_2$, $\Theta_3$ are pairwise disjoint and $\{1, \ldots, n\} = \rho^{-1}(\Theta_1) \cup \rho^{-1}(\Theta_2) \cup \rho^{-1}(\Theta_3)$. Thus we have $2\#\Theta_1 + \#\Theta_2 + \#\Theta_3 = n$ since $\rho|_{\rho^{-1}(\Theta_1)} : \rho^{-1}(\Theta_1) \to \Theta_1$ is a two to one map and $\rho|_{\rho^{-1}(\Theta_i)} : \rho^{-1}(\Theta_i) \to \Theta_i$ are bijections for $i = 2, 3$. By Claims 1-3, the number of non-hyperbolic closed characteristics in $\tilde{J}(\Sigma)$ is at least

$$2\#\Theta_1 + 2\#\Theta_2 + \#\Theta_3 - 1 \geq 2\#\Theta_1 + \#\Theta_2 + \#\Theta_3 - 1 = n - 1.$$ 

The proof of Theorem 1.1 is complete.

**Proof of Theorem 1.2.** As Definition 1.1 of [LoZ1], for $\alpha \in (1, 2)$, we define a map $\varrho_\alpha : \mathcal{H}(2n) \to \mathbb{N} \cup \{+\infty\}$

$$\varrho_\alpha(\Sigma) = \begin{cases} +\infty, & \text{if } \#\mathcal{V}(\Sigma, \alpha) = +\infty, \\ \min \left\{ \left[ \frac{i(x, 1) + 2S^+(x) - v(x, 1) + n}{2} \right] \right\}, & \text{if } \#\mathcal{V}(\Sigma, \alpha) < +\infty, \end{cases} \quad (3.31)$$

where $\mathcal{V}(\Sigma, \alpha)$ and $\mathcal{V}_\infty(\Sigma, \alpha)$ are variationally visible and infinite variationally visible sets respectively given in Definition 2.5.

By Theorem 1.4 of [LoZ1], if $\#\tilde{J}(\Sigma) \leq 2\varrho_\alpha(\Sigma) - 2$, then there exist at least two elliptic closed characteristics in $\tilde{J}(\Sigma)$. By Theorem 1.1 of [LoZ1] we have $\varrho_\alpha(\Sigma) \geq \left\lceil \frac{n}{2} \right\rceil + 1$. Thus Theorem 1.2 holds when $n$ is even.
In the following, we prove Theorem 1.2 for $n$ being odd. We have the following two cases.

**Case 1.** All the closed characteristics on $\Sigma$ are symmetric.

In this case, by Lemma 4.2 and (4.31), we have $\varrho_n(\Sigma) \geq n$. Thus Theorem 1.2 holds by Theorem 1.4 of [LoZ1].

**Case 2.** At least one closed characteristic on $\Sigma$ is non-symmetric.

We may assume without loss of generality that $(\tau_2, y_2) = (\tau_1, -y_1)$. Since $\Sigma = -\Sigma$, we have $H_\alpha(x) = H_\alpha(-x)$. Then it is easy to see that $(\tau_1, y_1)$ and $(\tau_1, -y_1)$ have the same properties

\[(i(y_1^m), \nu(y_1^m)) = (i((-y_1)_1^m), \nu((-y_1)_1^m)), \quad \Phi(y_1^m) = \Phi((-y_1)_1^m), \quad \forall m \in \mathbb{N} \quad (4.32)\]

\[C_{S_1, q}(\Phi, S_1 \cdot u_1^m) \cong C_{S_1, q}(\Phi, S_1 \cdot (-u_1)_1^m), \quad \forall m \in \mathbb{N}, \forall q \in \mathbb{Z}, \quad (4.33)\]

where we denote by $(\pm u)_1^m$ the critical point of $\Phi$ corresponding to $(\pm y)_1^m$. In fact, we have a nature $\mathbb{Z}_2$-action on $L_{q-1}(S^1, \mathbb{R}^n)$ defined by $u \mapsto -u$ and the functional $\Phi$ defined in (2.3) is $\mathbb{Z}_2$-invariant. Thus (4.32) and (4.33) hold.

Now we consider the set of closed characteristics: $\Delta \equiv \{(\tau_1, y_1), (\tau_3, y_3), \ldots, (\tau_n, y_n)\}$, i.e., we remove $(\tau_2, y_2)$ from the set $\{(\tau_j, y_j)\}_{1 \leq j \leq n}$. Then one can use the proof of Theorem 1.4 of [LoZ1] to obtain Theorem 1.2. In fact, we have $\# \Delta = n - 1 = 2\lfloor \frac{n}{2} \rfloor$. While the proof of Theorem 1.4 of [LoZ1] depend only on the index iteration theory, hence it remains valid if we replace the set $\{(\tau_j, y_j)\}_{1 \leq j \leq n}$ there by $\Delta$. The proof of Theorem 1.2 is complete.

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**References**

[CoZ1] C. Conley and E. Zehnder, Morse-type index theory for flows and periodic solutions for Hamiltonian equations. *Comm. Pure. Appl. Math.* 37 (1984) 207-253.

[DDE1] Dell’Antonio, G., B. D’Onofrio and I. Ekeland, Les systèm hamiltoniens convexes et pairs ne sont pas ergodiques en general. *C. R. Acad. Sci. Paris*. Series I. 315 (1992), 1413-1415.

[Eke1] I. Ekeland, Une théorie de Morse pour les systèmes hamiltoniens convexes. *Ann. IHP. Anal. non Linéaire*. 1 (1984) 19-78.

[Eke2] I. Ekeland, An index theory for periodic solutions of convex Hamiltonian systems. *Proc. Symp. in Pure Math.* 45 (1986) 395-423.
[Eke3] I. Ekeland, Convexity Methods in Hamiltonian Mechanics. Springer-Verlag. Berlin. 1990.

[EkH1] I. Ekeland and H. Hofer, Convex Hamiltonian energy surfaces and their closed trajectories. *Comm. Math. Phys.* 113 (1987) 419-467.

[EkL1] I. Ekeland and L. Lassoued, Multiplicité des trajectoires fermées d’un système hamiltonien sur une hypersurface d’énergie convexe. *Ann. IHP. Anal. non Linéaire.* 4 (1987) 1-29.

[FaR1] E. Fadell and P. Rabinowitz, Generalized comological index theories for Lie group actions with an application to bifurcation equations for Hamiltonian systems.

[GrM1] D. Gromoll and W. Meyer, On differentiable functions with isolated critical points. *Topology.* 8 (1969) 361-369.

[HWZ] H. Hofer, K. Wysocki and E. Zehnder, The dynamics on three-dimensional strictly convex energy surfaces. *Ann. of Math.* 148 (1998) 197-289.

[LLZ] C. Liu, Y. Long and C. Zhu, Multiplicity of closed characteristics on symmetric convex hypersurfaces in $\mathbb{R}^{2n}$. *Math. Ann.* 323 (2002), 201-215.

[Lon1] Y. Long, Maslov-type index, degenerate critical points and asymptotically linear Hamiltonian systems. *Science in China.* Series A. 33(1990), 1409-1419.

[Lon2] Y. Long, Hyperbolic closed characteristics on compact convex smooth hypersurfaces in $\mathbb{R}^{2n}$. *J. Diff. Equa.* 150 (1998), 227-249.

[Lon3] Y. Long, Bott formula of the Maslov-type index theory. *Pacific J. Math.* 187 (1999), 113-149.

[Lon4] Y. Long, Precise iteration formulae of the Maslov-type index theory and ellipticity of closed characteristics. *Advances in Math.* 154 (2000), 76-131.

[Lon5] Y. Long, Index Theory for Symplectic Paths with Applications. Progress in Math. 207, Birkhäuser. Basel. 2002.

[Lon6] Y. Long, Index iteration theory for symplectic paths with applications to nonlinear Hamiltonian systems. *Proc. of Inter. Congress of Math. 2002.* Vol.II, 303-313. Higher Edu. Press. Beijing. 2002.

[LoW1] Y. Long and W. Wang, Stability of closed characteristics on compact convex hypersurfaces, *Memory Volume for Professor S. S. Chern.* Ed. by P. Griffiths. Nankai Tracts in Mathematics Vol. 11, World Scientific. 313-333.
Y. Long and E. Zehnder, Morse theory for forced oscillations of asymptotically linear Hamiltonian systems. In *Stoc. Proc. Phys. and Geom.*, S. Albeverio et al. ed. World Sci. (1990) 528-563.

Y. Long and C. Zhu, Closed characteristics on compact convex hypersurfaces in $\mathbb{R}^{2n}$. *Ann. of Math.* 155 (2002) 317-368.

J. Mawhin and M. Willem, Critical Point Theory and Hamiltonian Systems. Springer. New York. 1989.

P. H. Rabinowitz, Periodic solutions of Hamiltonian systems. *Comm. Pure Appl. Math.* 31 (1978) 157-184.

A. Szulkin, Morse theory and existence of periodic solutions of convex Hamiltonian systems. *Bull. Soc. Math. France.* 116 (1988) 171-197.

C. Viterbo, Equivariant Morse theory for starshaped Hamiltonian systems. *Trans. Amer. Math. Soc.* 311 (1989) 621-655.

C. Viterbo, A new obstruction to embedding Lagrangian tori. *Invent. Math.* 100 (1990) 301-320.

W. Wang, X. Hu and Y. Long, Resonance identity, stability and multiplicity of closed characteristics on compact convex hypersurfaces. *Duke Math. J.* Volume 139, Number 3 (2007), 411-462.

W. Wang, Stability of closed characteristics on compact convex hypersurfaces in $\mathbb{R}^6$. *math.SG/0701673* to appear in *J. Eur. Math. Soc.*

A. Weinstein, Periodic orbits for convex Hamiltonian systems. *Ann. of Math.* 108 (1978) 507-518.