Noncommutativity in two-matrix model extension of one-dimensional topological gravity

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Abstract

One-dimensional topological gravity is defined as a Gaussian integral as its partition function. The Gaussian integral supplies a toy model as a simpler version of one-matrix model that is well known to provide a description of two-dimensional topological gravity. The one-dimensional topological gravity inherits an integrable hierarchy structure as with two-dimensional topological gravity, yet it is the Burgers hierarchy rather than the Korteweg–de Vries hierarchy. Making use of this fact, an extension of the one-dimensional topological gravity to an analogue of two-matrix model is investigated and the associated partition function is shown to consist of a pair of partition functions of one-dimensional topological gravity intertwined via the Moyal–Weyl product, which enables to provide an explicit formula for its free energy.

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1 Introduction

One-dimensional topological gravity is formulated by one-dimensional Gaussian integral as a toy model of one-matrix models of being well-known capable of describing two-dimensional topological gravity on closed Riemann surfaces. The two-dimensional quantum gravity, its matrix model descriptions and their applications to statistical physics, string theory, enumerative geometry and so on have been investigated in a variety of literature in both physics and mathematics, having originated from a vast amount of studies since the late 80’s, for example, [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] and for more references, see [12, 13].

The partition function of the one-dimensional topological gravity shows an integrable hierarchical structure called Burgers hierarchy and is subject to a set of Virasoro constraints. This can be thought of as a miniature version of that the partition function of the two-dimensional topological gravity, in other words, the partition function of one-matrix model, is subject to the KdV hierarchy [14, 15, 16] as well as Virasoro constraints [17, 18]. The hierarchy structure of one-matrix model acquires more richness by extending
to two-matrix models [19, 20, 21, 22, 23, 24]. This article investigates what kind of underlying structure appears if the one-dimensional topological gravity is extended to its two-matrix model analogue.

The article is organized as follows. Section 2 starts with introducing a Gaussian integral with a potential given by an infinite set of monomials as a partition function of one-dimensional topological gravity. It is reviewed that the partition function and the free energy obey the Virasoro constraints, and a notion of genus expansion of free energy makes sense by applying a saddle-point method to perform the integral.

Section 3 discusses identification of a recurrence relation for the free energy of one-dimensional topological gravity with the Burgers hierarchy in some detail. The identification provides an implication of the Cole–Hopf transform in the context of one-dimensional topological gravity. It is demonstrated how to find the free energy order by order in the genus expansion making use of the hierarchy structure.

Section 4 addresses the problem of an extension of one-dimensional topological gravity to its two-matrix model analogue. Based on the fact that the partition function of the one-dimensional topological gravity obeys the Burgers hierarchy, the partition function of the extended model is shown to be the Moyal–Weyl product of a pair of the partition functions of one-dimensional topological gravity. The noncommutative parameter turns out to be proportional to a coupling constant describing interaction between two identical one-matrix models, so that under the limit of decoupling the two systems the noncommutativity disappears and the partition function is reduced to an ordinary product of each. With the help of the Baker–Campbell–Hausdorff formula, which can be applied to the Moyal–Weyl product, the free energy is shown to have genus expansions and its explicit formula in terms of the free energy of one-dimensional gravity is obtained. The associated hierarchy structure turns out to contain two sets of Burgers hierarchies.

The final section summarizes the results and is mainly devoted to discussion, and appendices provide some supplemental information.

2 One-dimensional topological gravity

2.1 Gaussian integral as partition function

As reviewed briefly in this section, the Gaussian integral shares many features of two-dimensional topological gravity, as well as matrix model. In this sense, it is considered as a toy model of two-dimensional gravity of matrix model description. Some recent developments on one-dimensional topological gravity can also be found in [25, 26], and supplemental analysis from the viewpoint of quasi-triviality of nonlinear partial differential equations has been made in [27].

The partition function of the topological one-dimensional gravity is defined by

\[ Z^{1D}(t) = \frac{1}{\sqrt{2\pi\lambda}} \int_{-\infty}^{\infty} e^{\frac{1}{\lambda x} S(x)} dx \]  

with action

\[ S(x) = -\frac{1}{2} x^2 + \sum_{n=0}^{\infty} t_n \frac{x^{n+1}}{(n+1)!} \]

It is merely a Gaussian integral with an infinite set of monomial deformations as a potential. Though the integral is not convergent in the presence of such a potential, it is regarded as a formal object and
its perturbative series in terms of the coupling constants is to be understood as a formal expansion. The
normalization factor of the partition function is understood by turning off all the couplings

$$Z_{1D}(t = 0) = \frac{1}{\sqrt{2\pi \lambda}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\lambda}} dx = 1 \quad (3)$$

A formal expansion of the integrand in the coupling constants results in

$$Z_{1D}(t) = \frac{1}{\sqrt{2\pi \lambda}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\lambda}} \sum_{n_0,n_1,n_2,\ldots} \frac{1}{n_0! n_1! n_2! \ldots} \left(\frac{t_0}{\lambda^2}\right)^{n_0} \left(\frac{t_1}{\lambda^2 2!}\right)^{n_1} \left(\frac{t_2}{\lambda^2 3!}\right)^{n_2} \ldots dx \quad (4)$$

By virtue of a formula of the Gaussian integral

$$\frac{1}{\sqrt{2\pi \lambda}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\lambda}} x^n dx = \begin{cases} 0 & (n = \text{odd}) \\ (n-1)!! \lambda^n & (n = \text{even}) \end{cases} \quad (5)$$

(where \((-1)!! \equiv 1\)) the partition function is evaluated as

$$Z_{1D}(t) = \sum_{n_0,n_1,n_2,\ldots} \frac{\lambda^{n_0+2n_1+3n_2+\ldots}}{n_0! n_1! n_2! \ldots} \left(-1 + n_0 + 2n_1 + 3n_2 + \ldots\right)!! \frac{t_0^{n_0} t_1^{n_1} t_2^{n_2} \ldots}{\lambda^{2n_0+2n_1+3n_2+\ldots}} \quad (6)$$

where the summation is taken over \(n_0 + 2n_1 + 3n_2 + \cdots\) being an even number.

The logarithm of the partition function defines the free energy

$$M_{1D}(t) = \log Z_{1D}(t) \quad (7)$$

in other words, the partition function is given by the exponential of the free energy

$$Z_{1D}(t) = e^{M_{1D}(t)} \quad (8)$$

The main focus of the paper is rather on \(M_{1D}(t)\) than \(Z_{1D}(t)\).

2.2 Virasoro constraints

The partition function \(Z_{1D}(t)\) is equipped with an infinity set of constraints

$$L_m Z_{1D}(t) = 0 \quad (m \geq -1) \quad (9)$$

with

$$L_{-1} = \frac{t_0}{\lambda^2} + \sum_{n=1}^{\infty} (t_n - \delta_{n,1}) \frac{\partial}{\partial t_{n-1}}$$

$$L_0 = 1 + \sum_{n=0}^{\infty} (n + 1) (t_n - \delta_{n,1}) \frac{\partial}{\partial t_n}$$

$$L_m = \lambda^2 (m + 1)! \frac{\partial}{\partial t_{m-1}} + \sum_{n=0}^{\infty} \frac{(m + n + 1)!}{n!} (t_n - \delta_{n,1}) \frac{\partial}{\partial t_{m+n}} \quad (m \geq 1) \quad (10)$$

See appendix A for its derivation. The operators form the following (part of the Virasoro) algebra

$$[L_m, L_n] = (m-n) L_{m+n} \quad (m, n \geq -1) \quad (11)$$
The set of constraints can be phrased in terms of the free energy as follows.

\[
0 = t_0 + \sum_{n=1}^{\infty} (t_n - \delta_{n,1}) \frac{\partial M^{1D}(t)}{\partial t_{n-1}}
\]

\[
0 = 1 + \sum_{n=0}^{\infty} (n + 1) (t_n - \delta_{n,1}) \frac{\partial M^{1D}(t)}{\partial t_n}
\]

\[
0 = \lambda^2 (m + 1) \frac{\partial M^{1D}(t)}{\partial t_{m-1}} + \sum_{n=0}^{\infty} \frac{(m + n + 1)!}{n!} (t_n - \delta_{n,1}) \frac{\partial M^{1D}(t)}{\partial t_{m+n}} \quad (m \geq 1)
\]  

\(12\)

2.3 Saddle point method and genus expansion

Although the partition function of one-dimensional topological gravity is simple enough to formally perform the integral as its result is already provided in (6), let us further explore the function from another viewpoint. Eventually, there turns out to be the notion of “genus expansion” of the free energy.

The dominant contribution to the partition function (1) comes from a saddle point defined by

\[
0 = \frac{dS}{dx}(x) = -x + \sum_{n=0}^{\infty} t_n \frac{x^n}{n!}
\]

Let \(x_\infty\) denote a solution to the saddle point equation (13)

\[
x_\infty = \sum_{n=0}^{\infty} t_n \frac{x_\infty^n}{n!}
\]

The Taylor expansion of the action \(S(x)\) around the saddle point \(x_\infty\) gives an alternative expression of the partition function of form

\[
Z^{1D}(t) = \frac{e^{\frac{1}{\lambda} S(x_\infty)}}{\sqrt{2\pi\lambda}} \int e^{-\frac{1}{\lambda} \frac{1}{2} x^2 + \frac{1}{\lambda} \sum_{n=3}^{\infty} I_{n-1} \frac{x^n}{n!}} dy
\]

where a change of variable \(x \to y = x - x_\infty\) is performed, and for \(n \geq 2\)

\[
\frac{d^n S}{dx^n}(x_\infty) = -\delta_{n,2} + \sum_{m=0}^{\infty} t_{n-1+m} \frac{x_\infty^m}{m!} = -\delta_{n,2} + I_{n-1}
\]

with introducing a set of variables [28, 29]

\[
I_n = \sum_{m=0}^{\infty} t_{n+m} \frac{x_\infty^m}{m!}
\]

then in particular \(I_0 = x_\infty\) and thus \(I_n = \sum_{m=0}^{\infty} t_{n+m} \frac{I_m}{m!}\). See appendix B for some detailed relations between \(\{t_k\}\) and \(\{I_k\}\). For later convenience it is worth mentioning that

\[
\frac{\partial x_\infty}{\partial t_0} = \frac{1}{1 - I_1}
\]

and thus

\[
\frac{\partial I_n}{\partial t_0} = \frac{I_{n+1}}{1 - I_1} \quad (n \geq 1)
\]
Expanding the non-Gaussian part of (15) in power series in $I_k$, performing the Gaussian integral yields

$$Z^{1D}(t) = e^{\frac{1}{\lambda^2} S(x_\infty)} \prod_{n_2,n_3,n_4,\cdots} \frac{I_2^{n_2} I_3^{n_3} I_4^{n_4} \cdots}{3^{n_2} 4^{n_3} 5^{n_4} \cdots} \left( \frac{\lambda}{\sqrt{1 - I_1}} \right)^{n_2+2n_3+3n_4+\cdots} (1 + 3n_2 + 4n_3 + 5n_4 + \cdots)!! \right)$$

(20)

with $3n_2 + 4n_3 + 5n_4 + \cdots$ being an even number. Note that the contribution from the summation to $\lambda$ expansion starts with something like $1 + O(\lambda^2)$. Then, using the relation (18), the free energy reads

$$M^{1D}(t) = \frac{S(x_\infty)}{\lambda^2} + \frac{1}{2} \log \left( \frac{\partial x_\infty}{\partial t_0} \right) + \text{(correction of } O(\lambda^2)\text{)}$$

(21)

From the above mentioned observation, the free energy turns out to have an expansion according to the order of $\lambda$ in a form

$$M^{1D}(t) = \sum_{g=0}^{\infty} \lambda^{2g-2} M^{1D}_{(g)}(t)$$

(22)

which can be thought of as “genus” expansion in the same manner as that of two-dimensional topological gravity. Following the notation associated with the notion of genus expansion, $x_\infty$ can be written as

$$x_\infty^n = \frac{\partial M^{1D}_{(0)}(t)}{\partial t_{n-1}} \quad (n \geq 1)$$

(23)

with

$$M^{1D}_{(0)}(t) = S(x_\infty)$$

(24)

and then the relation (23) can be fashionably phrased as a recurrence relation

$$\frac{\partial M^{1D}_{(0)}}{\partial t_n} = \frac{1}{n+1} \frac{\partial M^{1D}_{(0)}}{\partial t_0} \frac{\partial M^{1D}_{(0)}}{\partial t_{n-1}}$$

(25)

The relation (18) is represented as

$$\frac{1}{1 - I_1} = \frac{\partial^2 M^{1D}_{(0)}}{\partial t_0^2}$$

(26)

and then $\lambda^0$-part of the free energy is given by

$$M^{1D}_{(1)}(t) = \frac{1}{2} \log \left( \frac{\partial^2 M^{1D}_{(0)}(t)}{\partial t_0^2} \right)$$

(27)

The set of constraints (12) is as well decomposed according to powers in $\lambda$, in particular the first relation of (12) is written as

$$\frac{\partial M^{1D}_{(g)}}{\partial t_0} = t_0 \delta_{g,0} + \sum_{n=0}^{\infty} t_{n+1} \frac{\partial M^{1D}_{(g)}}{\partial t_n}$$

(28)

One can show that this is true for $g = 0, 1$ by direct evaluation making use of (14) and (23), together with (27). This ensures that the free energy consistently has a $\lambda$-expansion of form (22).
3 Hierarchy structure

3.1 Burgers hierarchy and Cole–Hopf transformation

The recurrence relation (25) is found for the lowest order in \( \lambda \)-expansion of the free energy \( M^{1D} \). It would be natural to expect that there should be similar recurrence relations for any order in \( \lambda \)-expansion. The answer takes the form

\[
\frac{n+1}{\lambda^2} \frac{\partial M^{1D}}{\partial t_n} = \frac{\partial M^{1D}}{\partial t_0} \frac{\partial M^{1D}}{\partial t_{n-1}} + \frac{\partial^2 M^{1D}}{\partial t_0 \partial t_{n-1}}
\]

which can be deduced from the consistency with the string equation (12) as follows. Multiplying (29) with \( t_n \) and summing over \( n \), one finds

\[
\sum_{n=1}^{\infty} t_n \times (\text{l.h.s. of (29)}) = \sum_{n=1}^{\infty} \frac{n+1}{\lambda^2} \frac{\partial M^{1D}}{\partial t_n} = \frac{1}{\lambda^2} \left( -t_0 \frac{\partial M^{1D}}{\partial t_0} - 1 \right) + 2 \frac{\partial M^{1D}}{\partial t_1}
\]

whereas

\[
\sum_{n=1}^{\infty} t_n \times (\text{r.h.s. of (29)}) = \frac{\partial M^{1D}}{\partial t_0} \sum_{n=1}^{\infty} t_n \frac{\partial M^{1D}}{\partial t_{n-1}} + \sum_{n=1}^{\infty} t_n \frac{\partial^2 M^{1D}}{\partial t_0 \partial t_{n-1}}
= \frac{\partial M^{1D}}{\partial t_0} \left( -t_0 \frac{\partial M^{1D}}{\partial t_0} + \frac{\partial}{\partial t_0} \left( -\frac{t_0}{\lambda^2} + \frac{\partial M^{1D}}{\partial t_0} \right) \right)
= \frac{1}{\lambda^2} \left( -t_0 \frac{\partial M^{1D}}{\partial t_0} - 1 \right) + \frac{\partial M^{1D}}{\partial t_0} \frac{\partial M^{1D}}{\partial t_1} + \frac{\partial^2 M^{1D}}{\partial t_0 \partial t_1}
\]

where the first and second relations of (12) are used. The discrepancy is reduced to \( n = 1 \) case of (29).

A direct evaluation with use of an explicit formula (27) for \( M^{1D}_1 \) also ensures that the relation (29) should be valid for the next to the leading order in \( \lambda \)-expansion:

\[
(n+1) \frac{\partial M^{1D}_{(1)}}{\partial t_n} = \sum_{g=0,1} \frac{\partial M^{1D}_{(g)}}{\partial t_0} \frac{\partial M^{1D}_{(1-g)}}{\partial t_{n-1}} + \frac{\partial^2 M^{1D}_{(0)}}{\partial t_0 \partial t_{n-1}}
\]

Crucial relations for its proof are the following

\[
\frac{\partial M^{1D}_{(1)}}{\partial t_n} = \frac{1}{n} \left( \frac{\partial M^{1D}_{(0)}}{\partial t_0} \frac{\partial M^{1D}_{(1)}}{\partial t_{n-1}} + \frac{1}{2} \frac{\partial^2 M^{1D}_{(0)}}{\partial t_0 \partial t_{n-1}} \right) = \frac{\partial M^{1D}_{(1)}}{\partial t_0} \frac{\partial M^{1D}_{(0)}}{\partial t_{n-1}} + \frac{1}{2} \frac{\partial^2 M^{1D}_{(0)}}{\partial t_0 \partial t_{n-1}}
\]

The recurrence relation (29) has an underlying hierarchical structure, which is transparent by rewriting it in a form

\[
\frac{n+1}{\lambda^2} \frac{\partial M^{1D}}{\partial t_n} = \left( \frac{\partial}{\partial t_0} + \frac{\partial M^{1D}}{\partial t_0} \right) \frac{\partial M^{1D}}{\partial t_{n-1}}
\]

and thus

\[
\frac{(n+1)!}{\lambda^{2n}} \frac{\partial M^{1D}}{\partial t_n} = \left( \frac{\partial}{\partial t_0} + \frac{\partial M^{1D}}{\partial t_0} \right)^n \frac{\partial M^{1D}}{\partial t_0}
\]

This set of equations coincides with the Burgers hierarchy formulated by a set of equations

\[
\frac{\partial u}{\partial \tau_n} - \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} + u \right)^n u = 0 \quad n = 0, 1, 2, \ldots
\]
under identifications

\[ x = t_0 \quad \tau_n = \frac{\lambda^{2n}}{(n + 1)!} t_n \quad u = \frac{\partial M^{1D}}{\partial t_0} \]  \hspace{1cm} (37)

It is well-known that the Burgers hierarchy is linearized by the Cole–Hopf transformation

\[ u = (\log \theta)_x \]  \hspace{1cm} (38)

Noting that

\[ \frac{\partial u}{\partial \tau_n} = \frac{\partial \left( \frac{1}{\theta} \frac{\partial \theta}{\partial \tau_n} \right)}{\partial x} \left( \frac{\partial}{\partial x} + u \right)^n u = \frac{\partial^{n+1} \theta}{\partial x^{n+1}} \]  \hspace{1cm} (39)

the Burgers hierarchy is equivalent to a set of linear differential equations

\[ \frac{\partial \theta}{\partial \tau_n} = \frac{\partial^{n+1} \theta}{\partial x^{n+1}} \]  \hspace{1cm} (40)

Recalling back to the identifications and definitions, the function \( \theta \) turns out to be the partition function

\[ \theta = Z^{1D}(t) \]  \hspace{1cm} (41)

keeping which in mind, what the equation (40) is implying is a relation

\[ \frac{(n + 1)!}{\lambda^{2n}} \frac{\partial Z^{1D}}{\partial t_n} = \frac{\partial^{n+1} Z^{1D}}{\partial t_0^{n+1}} \]  \hspace{1cm} (42)

which immediately follows from the definition of the partition function (1). Using the property (42), the set of Virasoro constraints (9) can be written as

\[ L_m Z^{1D}(t) = \lambda^{2(m+1)} \frac{\partial^{m+1}}{\partial t_0^{m+1}} \cdot L_{-1} Z^{1D}(t) \quad (m \geq 0) \]  \hspace{1cm} (43)

so that it can reduce to a single condition \( L_{-1} Z^{1D}(t) = 0 \).

### 3.2 Exact solutions under special conditions

In this subsection, for completeness, let it provide some checks if the partition function is actually a solution to the partial differential equation under some specifically simpler setting.

#### 3.2.1 Burgers equation: Heat-kernel method

Let all the couplings turned off except \( t_0 \) and \( t_1 \): The partition function reduces to the simplest Gaussian integral resulting in

\[ Z^{1D}(t_0, t_1) = \frac{1}{\sqrt{2\pi \lambda}} \int_{-\infty}^{\infty} e^{\frac{1}{\lambda^2} x^2 + t_0 x} dx = \frac{\frac{\partial^2}{2 - 1 - t_1}}{\sqrt{1 - t_1}} \]  \hspace{1cm} (44)

and then the free energy is exactly calculated as

\[ M^{1D}(t_0, t_1) = \frac{1}{\lambda^2} \frac{t_0^2}{2(1 - t_1)} + \frac{1}{2} \log \left( \frac{1}{1 - t_1} \right) \]  \hspace{1cm} (45)
in particular \( M^{1D}(t_0, t_1 = 0) = \frac{t_0^2}{2\lambda^2} \) so that \( \frac{\partial M^{1D}}{\partial t_0}(t_0, t_1 = 0) = \frac{t_0}{\lambda^2} \).

The Burgers equation [30]

\[
\frac{\partial u}{\partial \tau} - \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} + u \right) u = 0 \tag{46}
\]

shortly

\[
u_\tau = u_{xx} + (u^2)_x \tag{47}
\]

is equivalent to the one-dimensional heat equation

\[
\frac{\partial \theta}{\partial \tau} = \frac{1}{2} \frac{\partial^2 \theta}{\partial x^2} \tag{48}
\]

via the Cole–Hopf transformation [31, 32]: \( u = (\log \theta)_x \). The heat equation is solved by the heat-kernel method putting

\[
\theta(x, \tau) = \int dx' G(x - x', \tau) \theta(x', 0) \tag{49}
\]

with the heat-kernel and the initial condition being

\[
G(x - x', \tau) = \frac{1}{\sqrt{4\pi \tau}} e^{-\frac{(x - x')^2}{4\tau}} \quad \theta(x', 0) = e^{\int dx'' u(x'', 0)} \tag{50}
\]

respectively, where a direct evaluation shows

\[
\left( \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial x^2} \right) G(x - x', \tau) = 0 \tag{51}
\]

Performing the integration under initial condition \( u(x, 0) = \frac{x}{\lambda^2} \) yields

\[
\theta(x, \tau) = \int dx' \frac{1}{\sqrt{4\pi \tau}} e^{-\frac{(x - x')^2}{4\tau}} \frac{x^2}{2\lambda^2} = \frac{\sqrt{\lambda^2}}{\sqrt{\lambda^2 - 2\tau}} e^{-\frac{x^2}{4\tau}} \left( 1 - \frac{\lambda^2}{x^2} \right) \tag{52}
\]

Substituting \( \tau \to \frac{\lambda^2}{2} t_1 \) and \( x \to t_0 \) reproduces (44).

### 3.2.2 Burgers equation: \((G'/G)\)-expansion method

A technique for generating a part of exact solutions describing propagating waves governed by nonlinear partial differential equations is the method called \((G'/G)\)-expansion [33]. Although a solution considered in this article will turn out to be out of the range covered by the method, it may be worth explaining what is its obstruction. A brief sketch of the method is given in appendix C.

The method puts ansatz that the solution shall be the form of a (Laurent) polynomial of \((G'/G)\):

\[
u(x, \tau) = u(\xi) = \sum_{n=N_{\min}}^{N_{\max}} a_n \left( \frac{G'}{G} \right)^n \quad G = G(\xi) \tag{53}
\]

with

\[
\xi = x - c\tau \quad G' = \frac{dG}{d\xi} \tag{54}
\]

assuming \( G \) should solve the following ordinary differential equation

\[
G'' + rG' + sG = 0 \tag{55}
\]
Under the ansatz, the Burgers equation $u_\tau = u_{xx} + (u^2)_x$ reduces to an ordinary differential equation

$$-cu' = u'' + (u^2)'$$

which can be integrated as

$$C = cu + u' + u^2$$

where $C$ is the integration constant. Requiring the highest order in $(G'/G)$ shall be cancelled between $u'$ and $u^2$, implies $N_{\text{max}} + 1 = 2N_{\text{max}}$ and thus $N_{\text{max}} = 1$. Applying a similar argument on $N_{\text{min}}$ to the lowest order in $(G'/G)$ is implying $N_{\text{min}} = 0$:

$$u(x, \tau) = u(\xi) = a_1 \left( \frac{G'}{G} \right) + a_0$$

Plugging this ansatz into (57) (equivalently (56)) results in a set of simultaneous equations for the parameters

$$a_1 = 1 \quad c = s + r - 2a_0 \quad C = (s + r - a_0)a_0$$

and one should chose a specific value of parameters consistently with initial conditions.

Coming back to the interest of the current article, imposing the initial condition $u(x, 0) = \frac{x}{\lambda^2}$ implies

$$\left. \frac{G'}{G} \right|_{\tau=0} = \frac{x}{\lambda^2} - a_0$$

so that

$$\frac{G'}{G} = \frac{\xi}{\lambda^2} - a_0$$

and therefore

$$G'' + a_0G' - \frac{1}{\lambda^2} G = -\frac{\xi}{\lambda^2} G'$$

which is out of ansatz. This observation shows that the solution to Burgers equation with the initial condition $u(x, 0) = \frac{x}{\lambda^2}$ would not be written in the form of a closed form as a propagating wave assumed by the ansatz.

### 3.2.3 Sharma–Tasso–Olver equation

The case where all the couplings except $t_0$ and $t_2$ are turned off is associated with the Sharma–Tasso–Olver equation [34, 35]:

$$\frac{\partial u}{\partial \tau} - \frac{\partial}{\partial x} \left( \frac{\partial}{\partial x} + u \right)^2 u = 0$$

shortly

$$u_\tau = u_{xxx} + \frac{3}{2}(u^2)_{xx} + (u^3)_x$$

Correspondingly the partition function (6) reduces to

$$Z^{1D}(t_0, t_2) = \sum_{n_0, n_2} \frac{1}{3^{n_2}} \frac{1}{n_0!n_2!} \lambda^{n_0-n_2} \frac{(-1 + n_0 + 3n_2)!!}{6} \frac{t_0^{n_0} t_2^{n_2}}{m!n!} \lambda^{m+n} =: \theta$$

with

$$x = t_0 \quad \tau = \lambda^4 \frac{t_2}{3} \quad u = \frac{\partial M^{1D}}{\partial t_0}$$

9
The function $\theta$ satisfies

$$\frac{\partial \theta}{\partial \tau} = \sum_{m,n} \frac{(2 + m + 3n)!}{m! n! \lambda^{m+3n+3}} \frac{x^{m+n}}{\partial x^i}$$

which is equivalent to the Sharma–Tasso–Olver equation via $u = (\log \theta)_x$.

### 3.3 Higher order in genus expansion

The first two lowest orders in $\lambda$ expansion of free energy $M^{1D}$ are somewhat trivially given by (24) and (27) because of (20). Non-trivial information about the free energy can be extracted from the hierarchy structure (29) by looking into the next order:

$$(n + 1) \frac{\partial M^{1D}_{(2)}}{\partial t_n} = \frac{\partial M^{1D}_{(0)}}{\partial t_0} \frac{\partial M^{1D}_{(2)}}{\partial t_{n-1}} + \frac{\partial M^{1D}_{(2)}}{\partial t_0} \frac{\partial M^{1D}_{(0)}}{\partial t_{n-1}} + \frac{\partial M^{1D}_{(1)}}{\partial t_0} \frac{\partial M^{1D}_{(1)}}{\partial t_{n-1}} + \frac{\partial^2 M^{1D}_{(1)}}{\partial t_0 \partial t_{n-1}}$$

from which the $\lambda^2$-part of the free energy can be specified as

$$M^{1D}_{(2)} = \left( \frac{\partial^2 M^{1D}_{(0)}}{\partial t_0^2} \right)^{-1} \left[ \left( \frac{\partial^2 M^{1D}_{(1)}}{\partial t_0^2} \right)^2 - \frac{1}{6} \left( \frac{\partial M^{1D}_{(1)}}{\partial t_0} \right)^2 \right]$$

A straightforward evaluation shows that $M^{1D}_{(2)}$ given by (69) satisfies the relation (68), using the recurrence relations for $M^{1D}_{(0)}$ and $M^{1D}_{(1)}$. This result reproduces the expression given in [26]:

$$M^{1D}_{(2)} = \frac{5}{24 (1 - I_1)^3} + \frac{1}{8 (1 - I_1)^2}$$

by noting (19) and (26), together with (27).

Higher order in $\lambda$-expansion will involve more terms, but still it is manageable to proceed to finding the higher order part of the free energy. Let us demonstrate it here taking $M^{1D}_{(3)}$ and $M^{1D}_{(4)}$ as example. From the above mentioned observation and dimensional analysis, one may put ansatz

$$M^{1D}_{(3)} = \left( \frac{\partial^2 M^{1D}_{(0)}}{\partial t_0^2} \right)^{-1} \left\{ a_1 \frac{\partial^2 M^{1D}_{(2)}}{\partial t_0^2} + a_2 \frac{\partial M^{1D}_{(1)}}{\partial t_0} \frac{\partial M^{1D}_{(2)}}{\partial t_0} \right\}$$

$$+ \left( \frac{\partial^2 M^{1D}_{(0)}}{\partial t_0^2} \right)^{-2} \left\{ b_1 \left( \frac{\partial^2 M^{1D}_{(1)}}{\partial t_0^2} \right)^2 + b_2 \frac{\partial M^{1D}_{(1)}}{\partial t_0} \left( \frac{\partial M^{1D}_{(1)}}{\partial t_0} \right)^2 + b_3 \left( \frac{\partial M^{1D}_{(1)}}{\partial t_0} \right)^4 \right\}$$

The coefficients are determined as

$$a_1 = \frac{1}{6} \quad a_2 = - \frac{4}{9} \quad b_1 = - \frac{1}{9} \quad b_2 = \frac{5}{27} \quad b_3 = - \frac{2}{27}$$

by requiring the recurrence relation (29) at the corresponding order in $\lambda$:

$$(n + 1) \frac{\partial M^{1D}_{(3)}}{\partial t_n} = \frac{\partial M^{1D}_{(0)}}{\partial t_0} \frac{\partial M^{1D}_{(3)}}{\partial t_{n-1}} + \frac{\partial M^{1D}_{(3)}}{\partial t_0} \frac{\partial M^{1D}_{(0)}}{\partial t_{n-1}} + \frac{\partial^2 M^{1D}_{(2)}}{\partial t_0 \partial t_{n-1}} + \frac{\partial M^{1D}_{(1)}}{\partial t_0} \frac{\partial M^{1D}_{(1)}}{\partial t_{n-1}} + \frac{\partial M^{1D}_{(2)}}{\partial t_0} \frac{\partial M^{1D}_{(1)}}{\partial t_{n-1}}$$
The next order appears more cumbersome, but one may still be able to put ansatz

\[ M_{(4)}^{1D} \]

\[
= \left( \frac{\partial^2 M_{(0)}^{1D}}{\partial t_0^2} \right)^{-1} \left\{ \alpha_1 \frac{\partial^2 M_{(3)}^{1D}}{\partial t_0^2} + \alpha_2 \frac{\partial M_{(1)}^{1D}}{\partial t_0} \frac{\partial M_{(3)}^{1D}}{\partial t_0} + \alpha_3 \left( \frac{\partial M_{(2)}^{1D}}{\partial t_0} \right)^2 \right\} \\
+ \left( \frac{\partial^2 M_{(0)}^{1D}}{\partial t_0^2} \right)^{-2} \left\{ \beta_1 \frac{\partial^2 M_{(1)}^{1D}}{\partial t_0^2} \frac{\partial^2 M_{(2)}^{1D}}{\partial t_0^2} + \beta_2 \left( \frac{\partial M_{(1)}^{1D}}{\partial t_0} \right)^2 \frac{\partial^2 M_{(2)}^{1D}}{\partial t_0^2} + \beta_3 \left( \frac{\partial M_{(2)}^{1D}}{\partial t_0} \right)^3 + \beta_4 \left( \frac{\partial M_{(1)}^{1D}}{\partial t_0} \right)^{\frac{1}{2}} \frac{\partial^2 M_{(1)}^{1D}}{\partial t_0^2} \right\} \\
+ \left( \frac{\partial^2 M_{(0)}^{1D}}{\partial t_0^2} \right)^{-3} \left\{ \gamma_1 \left( \frac{\partial^2 M_{(1)}^{1D}}{\partial t_0^2} \right)^3 + \gamma_2 \left( \frac{\partial^2 M_{(1)}^{1D}}{\partial t_0^2} \right)^2 \left( \frac{\partial M_{(1)}^{1D}}{\partial t_0} \right)^2 + \gamma_3 \left( \frac{\partial M_{(1)}^{1D}}{\partial t_0} \right)^4 + \gamma_4 \left( \frac{\partial M_{(1)}^{1D}}{\partial t_0} \right)^6 \right\}
\]

which satisfies

\[
(n+1) \frac{\partial M_{(n+1)}^{1D}}{\partial t_n} = \frac{\partial M_{(n)}^{1D}}{\partial t_n} \frac{\partial M_{(0)}^{1D}}{\partial t_{n-1}} + \frac{\partial M_{(n)}^{1D}}{\partial t_{n-1}} \frac{\partial M_{(1)}^{1D}}{\partial t_n} + \frac{\partial M_{(n)}^{1D}}{\partial t_{n-1}} \frac{\partial M_{(2)}^{1D}}{\partial t_n} + \frac{\partial M_{(n)}^{1D}}{\partial t_{n-1}} \frac{\partial M_{(3)}^{1D}}{\partial t_n} + \frac{\partial M_{(n)}^{1D}}{\partial t_{n-1}} \frac{\partial M_{(4)}^{1D}}{\partial t_n}
\]

by setting

\[
\begin{align*}
\alpha_1 &= \frac{1}{8} & \alpha_2 &= \frac{1}{2} & \alpha_3 &= -\frac{2}{5} \\
\beta_1 &= -\frac{5}{24} & \beta_2 &= \frac{7}{36} & \beta_3 &= -\frac{82}{135} & \beta_4 &= \frac{31}{45} \\
\gamma_1 &= \frac{7}{72} & \gamma_2 &= -\frac{13}{45} & \gamma_3 &= \frac{217}{810} & \gamma_4 &= -\frac{32}{405}
\end{align*}
\]

This demonstrates how to put ansatz and find the free energies order by order. Another systematic approach for finding out \( M_{(n)}^{1D} \) using a set of variable \( \{I_n\} \) is found in [26].

### 4 Two-matrix model extension

#### 4.1 Partition function and Moyal–Weyl product

Let it move on to considering an extension of one-dimensional topological gravity. An extension to two-matrix model analogue is one of straightforward possibilities. The partition function can be defined by

\[
Z^\text{TM}(\tilde{t}, \tilde{\kappa}) = \frac{1}{2\pi \lambda^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{1}{\lambda^2} S(x) + \frac{1}{\lambda^2} \tilde{S}(y) + \lambda \tilde{\kappa} x} dx dy
\]

with

\[
S(x) = -\frac{1}{2} x^2 + \sum_{n=0}^{\infty} \frac{x^{n+1}}{(n+1)!} \quad \tilde{S}(y) = -\frac{1}{2} y^2 + \sum_{n=0}^{\infty} \frac{y^{n+1}}{(n+1)!}
\]
The normalization factor is understood when all the couplings are turned off:

\[
Z_{TM}(t = 0, \tilde{t} = 0, \kappa) = \frac{1}{2\pi \lambda^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\lambda^2}} e^{-\frac{y^2}{2\lambda^2}} \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\kappa xy}{\lambda^2} \right)^n dx dy
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\kappa}{\lambda^2} \right)^n \left( \frac{1}{\sqrt{2\pi \lambda}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\lambda^2}} x^n dx \right) \left( \frac{1}{\sqrt{2\pi \lambda}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{2\lambda^2}} y^n dy \right)
\]

\[
= \left( 1 + \sum_{n=1}^{\infty} \frac{1}{(2n)!} \left( \frac{\kappa}{\lambda^2} \right)^n \left( (2n-1)!! \lambda^{2n} \right)^2 \right)
\]

\[
= \left( 1 + \sum_{n=1}^{\infty} \frac{(2n-1)!! \kappa^{2n}}{(2n)!!} \right) = \frac{1}{\sqrt{1 - \kappa^2}}
\]

where the factor involving a summation coincides with the Taylor series of \(1/\sqrt{1 - \kappa^2}\) around \(\kappa = 0\). When one of the sets of couplings, say \(\tilde{t}_n\), is turned off, the partition function is reduced to the one for one-dimensional topological gravity

\[
Z_{TM}(t, \tilde{t} = 0, \kappa) = Z_{1D}(t_0, t_1 + \kappa^2, t_i \geq 2)
\]

(80)

Retaining all the coupling constants, the integral can be written in a form of a product defined on a noncommutative space owing to the fact that the one-dimensional topological gravity is subject to the Burgers hierarchy:

\[
Z_{TM}(t, \tilde{t}, \kappa) = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{\kappa}{\lambda^2} \right)^n \left( \frac{1}{\sqrt{2\pi \lambda}} \int_{-\infty}^{\infty} e^{\frac{1}{\lambda^2}S(x)} x^n dx \right) \left( \frac{1}{\sqrt{2\pi \lambda}} \int_{-\infty}^{\infty} e^{\frac{1}{\lambda^2}\tilde{S}(y)} y^n dy \right)
\]

(81)

\[
\left( 1 + \sum_{n=1}^{\infty} \frac{(2n-1)!! \kappa^{2n}}{(2n)!!} \right) = \frac{1}{\sqrt{1 - \kappa^2}}
\]

(82)

\[
Z_{1D}(t) * Z_{1D}(\tilde{t})
\]

(83)

(84)

where \(\ast\) stands for the noncommutative product known as the Moyal–Weyl product defined by

\[
f(t, \tilde{t}) * g(t, \tilde{t}) = f(t, \tilde{t}) e^{\frac{\theta}{\pi \hbar} \left( \frac{\partial}{\partial t} \frac{\partial}{\partial \tilde{t}} - \frac{\partial}{\partial \tilde{t}} \frac{\partial}{\partial t} \right)} g(t, \tilde{t})
\]

(85)

with the noncommutative parameter

\[
\theta = -2i\kappa \lambda^2
\]

(86)

As \(\kappa \to 0\) the noncommutativity disappears, that accounts for that the system is reduced to a pair of copies of one-dimensional topological gravity under the limit.

Introducing the free energy of two-matrix analogue by

\[
M_{TM}(t, \tilde{t}, \kappa) = \log Z_{TM}(t, \tilde{t}, \kappa)
\]

(87)
then following the Baker–Campbell–Hausdolff formula (for simplicity $M := M^{1D}(t)$ and $\tilde{M} := M^{1D}(\tilde{t})$), the free energy is explicitly given in terms of $M$ and $\tilde{M}$:

$$M^{TM}(t, \tilde{t}, \kappa) = M + \tilde{M} + \frac{1}{2} \left[ [M, \tilde{M}]_* + \frac{1}{12} \left( [M, [M, \tilde{M}]]_* + [\tilde{M}, [M, M]]_* \right) \right] - \frac{1}{24} \left( [M, [M, [M, \tilde{M}]]_*]_* + [\tilde{M}, [M, [M, M]]_*]_* \right) + \frac{1}{720} \left( [M, [M, [M, M]]_*]_* + [\tilde{M}, [M, [M, M]]_*]_* \right)$$

$$- \frac{1}{120} \left( [M, [M, [M, \tilde{M}]]_*]_* + [\tilde{M}, [M, [M, M]]_*]_* \right) - O \left( \lambda^5 \right) \quad (88)$$

where the commutator is defined by

$$[f, g]_* = f \ast g - g \ast f \quad (89)$$

From the genus expansion of $M^{1D}$ of (22) and the formula (88), together with the definition of the Moyal–Weyl product (85), it is transparent that the free energy $M^{TM}$ has a genus expansion in $\lambda$ of form

$$M^{TM}(t, \tilde{t}, \kappa) = \sum_{g=0}^{\infty} \lambda^{2g-2} M^{TM}_{(g)}(t, \tilde{t}, \kappa) \quad (90)$$

Hence, for example, the lowest order in $\lambda$-expansion is given systematically by

$$M^{TM}_{(0)}(t, \tilde{t}, \kappa) = M(0) + \tilde{M}(0) + \frac{1}{2} \left\{ \{M(0), \tilde{M}(0)\}_* + \frac{1}{12} \left\{ \{M(0), \{M(0), \tilde{M}(0)\}_*\}_* + \{\tilde{M}(0), \{M(0), \tilde{M}(0)\}_*\}_* \right\} - \frac{1}{24} \left\{ \{M(0), \{M(0), \tilde{M}(0)\}_*\}_* - O \left( \{ \}_*^4 \right) \right\}$$

$$\quad - \frac{1}{120} \left\{ \{M(0), \{M(0), \tilde{M}(0)\}_*\}_* + \{\tilde{M}(0), \{M(0), \tilde{M}(0)\}_*\}_* \right\} $$

$$- \frac{1}{720} \left\{ \{M(0), \{M(0), \tilde{M}(0)\}_*\}_* + \{\tilde{M}(0), \{M(0), \tilde{M}(0)\}_*\}_* \right\} - O \left( \{ \}_*^4 \right) \quad (91)$$

where $\{\cdot, \cdot\}_*$ stands for the lowest order of the commutator

$$[f, g]_* = \lambda^2 \{f, g\}_* + O(\kappa^2 \lambda^4) \quad (92)$$

being regarded as the Poisson bracket

$$\{f, g\}_* = \kappa \left( \frac{\partial f}{\partial t_0} \frac{\partial g}{\partial \tilde{t}_0} - \frac{\partial f}{\partial \tilde{t}_0} \frac{\partial g}{\partial t_0} \right) \quad (93)$$

A generalization to any order in $\lambda$-expansion is readily applied.
4.2 Extension to generic potentials

It is straightforwardly generalized to the case with a more general interaction

\[ Z_{TM}(t, \bar{t}, \kappa_{m,n}) = \frac{1}{2\pi \lambda^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{1}{\lambda^2} S(x) + \frac{2}{\lambda^2} \bar{S}(y) + \frac{\lambda_{m,n}}{\lambda^2 (m + 1)(n + 1)} x^{m+1} y^{n+1}} \, dx \, dy \]

\[ = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \frac{\kappa_{m,n}}{\lambda^2} \right)^k \left\{ \frac{1}{\sqrt{2\pi \lambda}} \int_{-\infty}^{\infty} e^{\frac{1}{\lambda^2} S(x)} \left( \frac{x^{m+1}}{(m+1)!} \right)^k \, dx \right\} \left\{ \frac{1}{\sqrt{2\pi \lambda}} \int_{-\infty}^{\infty} e^{\frac{1}{\lambda^2} \bar{S}(y)} \left( \frac{y^{n+1}}{(n+1)!} \right)^k \, dy \right\} \]

\[ = \sum_{k=0}^{\infty} \frac{1}{k!} \left( \kappa_{m,n} \lambda^2 \right)^k \left( \frac{\partial^k}{\partial t_m^k} Z^{1D}(t) \right) \left( \frac{\partial^k}{\partial \bar{t}_n^k} Z^{1D}(\bar{t}) \right) = Z^{1D}(t) \ast_{m,n} Z^{1D}(\bar{t}) \]  

with

\[ f(t, \bar{t}) \ast_{m,n} g(t, \bar{t}) = f(t, \bar{t}) e^{\frac{\theta_{m,n}}{2} \left( \frac{\partial}{\partial t_m} - \frac{\partial}{\partial \bar{t}_n} \right)} g(t, \bar{t}) \]

and

\[ \theta_{m,n} = -2i\kappa_{m,n} \lambda^2 \]

The formula for the free energy shall be transparent: it should coincide with (88) but \( \ast \to \ast_{m,n} \).

Finally, the most general case would be

\[ Z_{TM}(t, \bar{t}, \kappa) = \frac{1}{2\pi \lambda^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{1}{\lambda^2} S(x) + \frac{2}{\lambda^2} \bar{S}(y) + \frac{\sum_{m,n \geq 0} \kappa_{m,n} x^{m+1} y^{n+1}}{(m+1)(n+1)}} \, dx \, dy \]

Obviously it is given by

\[ Z_{TM}(t, \bar{t}, \kappa) = Z^{1D}(t) \ast Z^{1D}(\bar{t}) \]

with

\[ f(t, \bar{t}) \ast g(t, \bar{t}) = f(t, \bar{t}) e^{\frac{1}{2} \sum_{m,n \geq 0} \theta_{m,n} \left( \frac{\partial}{\partial t_m} - \frac{\partial}{\partial \bar{t}_n} \right)} g(t, \bar{t}) \]

It is apparent that the partition function has a property

\[ \frac{1}{\lambda^2} \frac{\partial Z^\text{TM}_{m,n}(t, \bar{t}, \kappa)}{\partial \kappa_{m,n}} = \frac{1}{\lambda^2} \frac{\partial Z^{1D}_{m,n}(t)}{\partial t_m} \ast \frac{\partial Z^{1D}_{m,n}(\bar{t})}{\partial \bar{t}_n} \]

and, owing to the fact that \( Z^{1D} \) obeys the Burgers hierarchy,

\[ \frac{1}{\lambda^2} \frac{\partial Z^\text{TM}_{m,n}(t, \bar{t}, \kappa)}{\partial \kappa_{m,n}} = \frac{\lambda^{2(m+n)}}{(m+1)!(n+1)!} \frac{\partial^{n+1} Z^{1D}(t) \ast \partial^{n+1} Z^{1D}(\bar{t})}{\partial t_m^{n+1} \partial \bar{t}_n^{n+1}} \]

identifying \( \kappa_{0,-1} \equiv t_0 \) and \( \kappa_{-1,0} \equiv \bar{t}_0 \)

\[ \frac{(m+1)!(n+1)!}{\lambda^{2(m+n)}} \frac{1}{\lambda^2} \frac{\partial}{\partial \kappa_{m,n}} Z^\text{TM}_{m,n} = \frac{\partial^{n+1} Z^{1D}}{\partial t_m^{n+1} \partial \kappa_{-1,0}^{n+1}} \]  

which is to be thought of as an extended version of the relation (42) that is equivalent to the Burgers hierarchy. This observation shows that the two-matrix analogue of one-dimensional topological gravity contains Burgers hierarchy partly, but has some extended structure. The formula for the free energy shall be given by (88) replacing \( \ast \to \ast \).
5 Conclusion and discussion

A system extended to two-matrix analogue of one-dimensional gravity has been investigated and it has shown that its free energy consists of a pair of one-dimensional topological gravity intertwined by the Moyal–Weyl product. With help of the Baker–Campbell-Hausdorff formula, the formula for the free energy is obtained and it has shown to allow a form of $\lambda$-expansion. The partition function of the extended model is subject to a hierarchy structure partly containing two sets of Burgers hierarchies.

An interesting direction to explore would be specifying the integrable structure of the extended model, although it has been shown at least containing two sets of Burgers hierarchies. The Moyal–Weyl product prevents it from applying a simple argument for figuring out which structure it is, e.g. Cole–Hopf transform would be no more straightforward, whereas it should be simply reduced to the Burgers hierarchy by turning off either $\{t_i\}$ or $\{\tilde{t}_i\}$. A clue might be found in noncommutative Burgers equation [36], where noncommutativity is introduced in $(t_0, t_1)$ (or $(\tilde{t}_0, \tilde{t}_1)$), though, in this sense, it is not the case directly applicable to what has been considered in this article.

A crucial difference of what has been found in this article from two-matrix models is that the interaction between two one-matrix models can be factored by themselves. This factorization has played a key role in representing the partition function as the Moyal–Weyl product. Thus the results obtained in this article is not straightforwardly applied to those of matrix models. An investigation along this line, together with the above mentioned direction, shall be worth exploring.

Another extension of one-dimensional topological gravity

The open KdV hierarchy is an extension of the KdV hierarchy appearing in two-dimensional gravity on closed surfaces to that on surfaces with boundaries [37, 38, 39, 40, 41], for recent developments, see [42]. An additional equation (s-flow equation, and its Laplace transformed version) compatible with the open KdV hierarchy was advocated in [43] and it played an important role in providing an explicit solution to the open KdV hierarchy [44, 45, 46], where its relation to matrix model description was investigated also. However, when it comes to implications of the open KdV hierarchy, together with the s-flow equation, still has it not completely been demystified yet, in a sense of lacking a procedure how to include multiple boundary parameters that accounts for boundary conditions, leading to some redundant description of its matrix model formulation due to combined with bulk coupling constants [47]. Before closing this article, let us here make some comments on another extension of one-dimensional topological gravity to its open analogue in view of a rather simpler setting considered throughout this article.

Resolvent as generating function

The loop operator

$$W_k(z_1, \ldots, z_k; t) = \left\langle \frac{1}{z_1 - x} \ldots \frac{1}{z_k - x} \right\rangle = \frac{1}{\sqrt{2\pi\lambda}} \int \frac{1}{z_1 - x} \ldots \frac{1}{z_k - x} e^{\frac{1}{2\lambda} S(x)} \, dx$$

(103)
has its own importance as generating function: an expansion in $z_i$ gives

$$W_k(z_1, \ldots, z_k; t) = \sum_{n_1, \ldots, n_k=0}^{\infty} \frac{(z_1^{n_1+\cdots+n_k})}{z_1^{n_1+1} \cdots z_k^{n_k+1}}$$

(104)

in particular $k = 1$ provides a generating function of expectation values of monomial in $x$ via

$$W_1(z; t) = \left\langle \frac{1}{z-x} \right\rangle = \sum_{n=0}^{\infty} \frac{\langle x^n \rangle}{z^{n+1}}$$

(105)

Thus applying Cauchy’s residue theorem, the expectation value can be extracted by

$$\langle x^n \rangle = \oint \frac{dz}{2\pi i} z^n W_1(z; t)$$

(106)

**Open Virasoro constraints**

Rewriting the relation (105), $W_1(z; t)$ can be regarded as the Laplace transform

$$W_1(z; t) = \left\langle \int_0^\infty dl \ e^{-l(z-x)} \right\rangle = \int_0^\infty dl \ e^{-lz} Z^\alpha(l; t)$$

(107)

with

$$Z^\alpha(l; t) = \left\langle e^{lz} \right\rangle = \frac{1}{\sqrt{2\pi\lambda}} \int_{-\infty}^{\infty} e^{lx+\frac{i}{\pi}S(x)} dx$$

(108)

and then

$$Z^\alpha(l; t) = Z^{1D}(t_0 + l\lambda^2; t_i \geq 1) = e^{ilx^2/\sigma_0} Z^{1D}(t)$$

(109)

which might be regarded as one-dimensional version of open/closed dualities of two-dimensional topological gravity [48, 49], and for more recent information [50]. Recalling that $Z^{1D}(t)$ is subject to the Virasoro constraints (9), $Z^\alpha(l; t)$ shall also be constrained:

$$0 = e^{i\lambda^2/\sigma_0} \left( L_m Z^{1D}(t) \right) = \left( e^{i\lambda^2/\sigma_0} L_m e^{-i\lambda^2/\sigma_0} \right) Z^\alpha(l; t) =: \tilde{L}_m Z^\alpha(l; t)$$

(110)

where

$$\tilde{L}_m = e^{i\lambda^2/\sigma_0} L_m e^{-i\lambda^2/\sigma_0} = \begin{cases} L_{-1} + l & (m = -1) \\ L_0 + l\lambda^2/\sigma_0 & (m = 0) \\ L_m + l\lambda^2(m+1)!/\sigma_0 & (m \geq 1) \end{cases}$$

(111)

satisfying the relation

$$[\tilde{L}_m, \tilde{L}_n] = (m - n) \tilde{L}_{m+n}$$

(112)

Then on $W_1(z; t)$ these generators can be represented by

$$0 = \int_0^\infty dl \ e^{-lz} \tilde{L}_m Z^\alpha(l; t) = \begin{cases} (L_{-1} - \frac{\partial}{\partial z}) W_1(z; t) & (m = -1) \\ (L_0 - \lambda^2/\sigma_0 \frac{\partial}{\partial z}) W_1(z; t) & (m = 0) \\ (L_m - (m+1)!\lambda^2/\sigma_0 \frac{\partial}{\partial z}) W_1(z; t) & (m \geq 1) \end{cases}$$

(113)
It is noted, using the relation (42), that

$$\frac{\partial}{\partial t_m} W_1(z; t) = \frac{\lambda^m}{(m + 1)!} \frac{\partial^{m+1}}{\partial l_0^{m+1}} W_1(z; t)$$

(114)

then for \(m \geq 1\)

$$L_m^o W_1(z; t) := \left\{ L_m + \left( \lambda^2 \frac{\partial}{\partial l_0} \right)^{m+1} \left( -\frac{\partial}{\partial z} \right) \right\} W_1(z; t) = 0$$

(115)

The generating function \(W_1(z; t)\) obeys also another infinite set of identities

$$\mathcal{L}_m^o W_1(z; t) = 0$$

(116)

with

$$\mathcal{L}_m = L_m + \left( -\frac{\partial}{\partial z} \right)^{m+1} \left( \lambda^2 \frac{\partial}{\partial l_0} \right) \quad \mathcal{L}_m = \left( -\frac{\partial}{\partial z} \right)^{m+1} \circ (-z)$$

(117)

which satisfy the (part of) Virasoro algebra

$$[\mathcal{L}_m^o, \mathcal{L}_n^o] = (m-n) \mathcal{L}_{m+n}^o \quad [\mathcal{L}_m, \mathcal{L}_n] = (m-n) \mathcal{L}_{m+n}$$

(118)

These identities (116) follow from invariance under reparametrization \(l \to l + \varepsilon_m l^{m+1} (m \geq 0)\):

$$W_1(z; t) = \left\langle \int_0^\infty dl \left( 1 + (m + 1)\varepsilon_m l^m - \varepsilon_m l^{m+1} (z - x) \right) e^{-l(z-x)} \right\rangle$$

$$= W_1(z; t) + \varepsilon_m \left\{ (m + 1) \left( \int_0^\infty dl l^m e^{-l(z-x)} \right) - \left( z - \lambda^2 \frac{\partial}{\partial l_0} \right) \left( \int_0^\infty dl l^{m+1} e^{-l(z-x)} \right) \right\}$$

$$= W_1(z; t) + \varepsilon_m \left\{ (m + 1) \left(-\frac{\partial}{\partial z} \right)^m - \left( z - \lambda^2 \frac{\partial}{\partial l_0} \right) \left(-\frac{\partial}{\partial z} \right)^{m+1} \right\} W_1(z; t)$$

$$= W_1(z; t) + \varepsilon_m \left\{ \left(-\frac{\partial}{\partial z} \right)^{m+1} \circ \left( \lambda^2 \frac{\partial}{\partial l_0} - z \right) \right\} W_1(z; t)$$

(119)

Although we have derived two sets of (part of) Virasoro generators, \(\{L_m\}\) and \(\{\mathcal{L}_m\}\), their implications, in particular their associated hierarchy structure (if it exists), are still missing. Further investigation on this aspect shall also be carried out.

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**A Virasoro constraints**

The constraints can be derived by performing an infinitesimal change of variable \(x \to x + \varepsilon_m x^{m+1} (m \geq 1)\)

$$Z^{1D}(t) = \frac{1}{\sqrt{2\pi \lambda}} \int e^{-\frac{(x + \varepsilon_m x^{m+1})^2}{2\lambda^2} + \sum_{n=0}^\infty \frac{\varepsilon_m^n (x + \varepsilon_m x^{m+1})^{n+1}}{(n+1)!} d(x + \varepsilon_m x^{m+1})}$$

$$= Z^{1D}(t) + \varepsilon_m \frac{1}{\lambda^2 \sqrt{2\pi \lambda}} \int e^{-\frac{x^2}{2\lambda^2} + \sum_{n=0}^\infty \frac{\varepsilon_m^n x^{n+1}}{n!} \left( -x^{m+2} + \sum_{n=0}^\infty \frac{t_n x^{m+n+1}}{n!} + \lambda^2 (m+1)x^m \right)} dx + O(\varepsilon_m^2)$$

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noting that
\[
\frac{\partial}{\partial t_n} Z^{1D}(t) = \frac{1}{\sqrt{2\pi \lambda}} \int e^{-\frac{x^2}{2\lambda^2} + \sum_{n=0}^{\infty} \frac{t_n}{\lambda^2} x^{n+1}} \left( \frac{1}{\lambda^2} \frac{x^{n+1}}{(n+1)!} \right) dx
\]
(120)
then one finds an identity
\[
0 = \varepsilon_n \left( -(m + 2)! \frac{\partial}{\partial t_{m+1}} + \sum_{n=0}^{\infty} t_n \frac{(m + n + 1)!}{n!} \frac{\partial}{\partial t_{m+n}} + \lambda^2 (m + 1)! \frac{\partial}{\partial t_{m-1}} \right) \frac{1}{\sqrt{2\pi \lambda}} \int e^{-\frac{x^2}{2\lambda^2} + \sum_{n=0}^{\infty} \frac{t_n}{\lambda^2} x^{n+1}} dx
\]
thus
\[
L_m Z^{1D}(t) = 0
\]
(121)
And \(x \to x + \varepsilon_0 x\) (dilation):
\[
Z^{1D}(t) = \frac{1}{\sqrt{2\pi \lambda}} \int e^{-\frac{(x+\varepsilon_0 x)^2}{2\lambda^2} + \sum_{n=0}^{\infty} \frac{t_n}{\lambda^2} (x+\varepsilon_0 x)^{n+1}} \left( 1 + \varepsilon_0 \right) dx
= Z^{1D}[t] + \varepsilon_0 \frac{1}{\sqrt{2\pi \lambda}} \int e^{-\frac{x^2}{2\lambda^2} + \sum_{n=0}^{\infty} \frac{t_n}{\lambda^2} x^{n+1}} \left( -\frac{x^2}{\lambda^2} + \sum_{n=0}^{\infty} t_n \frac{x^{n+1}}{n!} + 1 \right) dx + O(\varepsilon_0^2)
\]
thus
\[
L_0 Z^{1D}(t) = 0
\]
(122)
Finally, \(x \to x + \varepsilon_{-1}\) (translation):
\[
Z^{1D}(t) = \frac{1}{\sqrt{2\pi \lambda}} \int e^{-\frac{(x+\varepsilon_{-1})^2}{2\lambda^2} + \sum_{n=0}^{\infty} \frac{t_n}{\lambda^2} (x+\varepsilon_{-1})^{n+1}} dx
= Z^{1D}[t] + \varepsilon_{-1} \frac{1}{\sqrt{2\pi \lambda}} \int e^{-\frac{x^2}{2\lambda^2} + \sum_{n=0}^{\infty} \frac{t_n}{\lambda^2} x^{n+1}} \left( -\frac{x}{\lambda^2} + \sum_{n=1}^{\infty} t_n \frac{x^n}{n!} \right) dx + O(\varepsilon_{-1}^2)
\]
thus
\[
L_{-1} Z^{1D}(t) = 0
\]
(123)

B Change of coupling constants

A set of variables is defined by
\[
I_n = \sum_{m=0}^{\infty} t_{n+m} \frac{x^m}{m!}
\]
(124)
which serve an alternative set of couplings \(t_k\). By definition, one finds \(x_\infty = I_0\) and \(I_n = \sum_{m=0}^{\infty} t_{n+m} \frac{I_0^m}{m!}\).

Using the variables \((k \geq 2)\), one finds
\[
\frac{d^k S}{dx^k}(x) = -\delta_{k,2} + I_{k-1}
\]
(125)
Performing a Taylor expansion of \(S(x)\) around the saddle point \(x_\infty\), one obtains
\[
S(x) = S(x_\infty) + (x - x_\infty) \frac{dS}{dx}(x_\infty) + \sum_{k=2}^{\infty} \frac{(x - x_\infty)^k}{k!} \frac{d^k S}{dx^k}(x_\infty)
= -\frac{1}{2} I_0^2 + \sum_{n=0}^{\infty} \frac{t_n I_0^{n+1}}{(n+1)!} + \sum_{k=2}^{\infty} \frac{(I_{k-1} - \delta_{k,2}) (x - I_0)^k}{k!}
\]
(126)
From the definition of $I_k$, it follows that

$$
\sum_{n=0}^{\infty} \frac{t_n I_0^n}{(n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} I_n I_0^n
$$

(127)

with a help of formula

$$
\sum_{k=0}^{n} \frac{(-1)^k}{(k+1)!(n-k)!} = \frac{1}{(n+1)!}
$$

(128)

which is shown by $0 = (1 - 1)^{n+1} = 1 - \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1}$. Hence the Taylor expansion of $S(x)$ around $x_\infty$ reads

$$
S(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1!} (I_n + \delta_{n,1}) I_0^{n+1} + \sum_{n=2}^{\infty} (I_{n-1} - \delta_{n,2}) \frac{(x - I_0)^n}{n!}
$$

(129)

One can switch the description in $\{t_k\}$ to that in $\{I_k\}$, and vice versa, as follows. From

$$
I_0 = \sum_{m=0}^{\infty} t_m \frac{I_0^m}{m!} = t_0 + t_1 \frac{I_0^1}{1!} + t_2 \frac{I_0^2}{2!} + t_3 \frac{I_0^3}{3!} + t_4 \frac{I_0^4}{4!} + t_5 \frac{I_0^5}{5!} + t_6 \frac{I_0^6}{6!} + \ldots
$$

$$
I_1 I_0 = \sum_{m=0}^{\infty} t_{m+1} \frac{I_0^{m+1}}{m!} = t_1 I_0 + t_2 \frac{I_0^2}{1!} + t_3 \frac{I_0^3}{2!} + t_4 \frac{I_0^4}{3!} + t_5 \frac{I_0^5}{4!} + t_6 \frac{I_0^6}{5!} + \ldots
$$

$$
I_2 I_0^2 = \sum_{m=0}^{\infty} t_{m+2} \frac{I_0^{m+2}}{m!} = t_2 I_0^2 + t_3 \frac{I_0^3}{1!} + t_4 \frac{I_0^4}{2!} + t_5 \frac{I_0^5}{3!} + t_6 \frac{I_0^6}{4!} + \ldots
$$

$$
I_3 I_0^3 = \sum_{m=0}^{\infty} t_{m+3} \frac{I_0^{m+3}}{m!} = t_3 I_0^3 + t_4 \frac{I_0^4}{1!} + t_5 \frac{I_0^5}{2!} + t_6 \frac{I_0^6}{3!} + \ldots
$$

(130)

and so forth, it follows that

$$
t_n = \sum_{k=0}^{\infty} \frac{(-1)^k t_k}{(n+1)!} I_{k+n}
$$

(131)

thus

$$
\frac{\partial t_n}{\partial I_m} = \frac{(-1)^{m-n} I_0^{m-n}}{(m-n)!} H(m-n) - t_{n+1} \delta_{0,m}
$$

(132)

where $H(x)$ is the Heaviside step function

$$
H(x) = \begin{cases} 
1 & (x \geq 0) \\
0 & (x < 0) 
\end{cases}
$$

(133)

Hence, from the Leibniz rule, the derivative with respect to $I_m$ can be represented as

$$
\frac{\partial}{\partial I_m} = \sum_{n=0}^{\infty} \frac{\partial t_n}{\partial I_m} \frac{\partial}{\partial t_n} = \sum_{n=0}^{\infty} \left( \frac{(-1)^{m-n} I_0^{m-n}}{(m-n)!} H(m-n) - t_{n+1} \delta_{0,m} \right) \frac{\partial}{\partial t_n}
$$

(134)

so that

$$
\frac{\partial}{\partial I_0} = \frac{\partial}{\partial t_0} - \sum_{n=0}^{\infty} t_{n+1} \frac{\partial}{\partial t_n} \quad \frac{\partial}{\partial I_m} = \sum_{n=0}^{m} \frac{(-1)^{m-n} I_0^{m-n}}{(m-n)!} \frac{\partial}{\partial t_n}
$$

(135)
Conversely, from
\[ I_n = \sum_{m=0}^{\infty} t_{n+m} I_0^m / m! \]  
(136)
one finds
\[ \frac{\partial I_n}{\partial t_k} = \frac{I_0^{k-n}}{(k-n)!} H(k-n) + I_{n+1} \frac{\partial I_0}{\partial t_k} \]  
(137)
in particular
\[ \left( 1 - \sum_{m=0}^{\infty} t_{m+1} I_0^m / m! \right) \frac{\partial I_0}{\partial t_k} = \frac{I_0^k}{k!} \]  
(138)
so that
\[ \frac{\partial I_0}{\partial t_k} = \frac{1}{1 - I_1} \frac{I_0^k}{k!} \]  
(139)
and then
\[ \frac{\partial I_n}{\partial t_k} = \frac{I_0^{k-n}}{(k-n)!} H(k-n) + \frac{I_{n+1}}{1 - I_1} \frac{I_0^k}{k!} \]  
(140)
Therefore
\[ \frac{\partial}{\partial t_k} = \sum_{n=0}^{\infty} \frac{\partial I_n}{\partial t_k} \frac{\partial}{\partial I_n} = \frac{I_0^k}{k!} \left( \frac{1}{1 - I_1} \frac{\partial}{\partial I_0} + \sum_{n=1}^{\infty} \frac{I_{n+1}}{1 - I_1} \frac{\partial}{\partial I_n} \right) + \sum_{n=1}^{\infty} \frac{I_0^{k-n}}{(k-n)!} \frac{\partial}{\partial I_n} \]  
(141)
Thus, knowing the Jacobian, one can switch descriptions in \( \{t_k\} \) and in \( \{I_k\} \).

C \ ((G'/G))-expansion method

The method puts ansatz that the solution shall be the form of a (Laurent) polynomial of \((G'/G)\):
\[ u(x, \tau) = u(\xi) = \sum_{n=N_{\text{min}}}^{N_{\text{max}}} a_n \left( \frac{G'}{G} \right)^n \quad G = G(\xi) \]  
(142)
with
\[ \xi = x - c \tau \quad G' = \frac{dG}{d\xi} \]  
(143)
assuming \( G \) should solve the following ordinary differential equation
\[ G'' + r G' + s G = 0 \]  
(144)
All the parameters \( a_n, r, s, c \) in the ansatz are to be determined later. The solutions to the linear differential equation on \( G \) are classified in three types according to the value of its discriminant \( D = r^2 - 4s \):
\[ G(\xi) = \begin{cases} 
  e^{-\frac{\xi}{2}} \left\{ C_1 \sin \left( \frac{\sqrt{4s-r^2}}{2} \xi \right) + C_2 \cos \left( \frac{\sqrt{4s-r^2}}{2} \xi \right) \right\} & (D < 0) \\
  e^{-\frac{\xi}{2}} \left\{ (C_1 \xi + C_2) \right\} & (D = 0) \\
  e^{-\frac{\xi}{2}} \left\{ C_1 \sinh \left( \frac{\sqrt{4s-r^2}}{2} \xi \right) + C_2 \cosh \left( \frac{\sqrt{4s-r^2}}{2} \xi \right) \right\} & (D > 0)
\end{cases} \]  
(145)
A remarkable property and a reason for putting such ansatz is that a differential with respect to \( \xi \) defines an automorphism on a set of (Laurent) polynomials in \((G'/G)\), because of
\[ \left[ \left( \frac{G'}{G} \right)^n \right]' = -n \left( \frac{G'}{G} \right)^{n+1} - rn \left( \frac{G'}{G} \right)^n - sn \left( \frac{G'}{G} \right)^{n-1} \]  
(146)
where

\[
\frac{G'}{G} = \begin{cases} 
\left(-\frac{1}{2}\right) \left( (rC_1 + \sqrt{4s-r^2}C_2) \sin\left(\frac{\sqrt{4s-r^2}}{2} \xi\right) + (rC_2 - \sqrt{4s-r^2}C_1) \cos\left(\frac{\sqrt{4s-r^2}}{2} \xi\right) \right) & (D < 0) \\
-\frac{rC_1 \xi + (rC_2 - 2C_1)}{2(C_1 \xi + C_2)} \left( (rC_1 - \sqrt{r^2 - 4s}C_2) \sinh\left(\frac{\sqrt{r^2 - 4s}}{2} \xi\right) + (rC_2 - \sqrt{r^2 - 4s}C_1) \cosh\left(\frac{\sqrt{r^2 - 4s}}{2} \xi\right) \right) & (D = 0) \\
\left(-\frac{1}{2}\right) \left( (rC_1 - \sqrt{r^2 - 4s}C_2) \sinh\left(\frac{\sqrt{r^2 - 4s}}{2} \xi\right) + (rC_2 - \sqrt{r^2 - 4s}C_1) \cosh\left(\frac{\sqrt{r^2 - 4s}}{2} \xi\right) \right) & (D > 0)
\end{cases}
\]

This property enables to translate partial differential equations into algebraic relations for parameters to be determined by initial conditions.

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