EXPOSNENTS OF \([\Omega(S^{r+1}), \Omega(Y)]\)

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Abstract. We investigate the exponents of the total Cohen groups \([\Omega(S^{r+1}), \Omega(Y)]\) for any \(r \geq 1\). In particular, we show that for \(p \geq 3\), the \(p\)-primary exponents of \([\Omega(S^{r+1}), \Omega(S^{2n+1})]\) and \([\Omega(S^{r+1}), \Omega(S^{2n})]\) coincide with the \(p\)-primary homotopy exponents of spheres \(S^{2n+1}\) and \(S^{2n}\), respectively.

We further study the exponent problem when \(Y\) is a space with the homotopy type of \(\Sigma(n)/G\) for a homotopy \(n\)-sphere \(\Sigma(n)\), the complex projective space \(\mathbb{C}P^n\) for \(n \geq 1\) or the quaternionic projective space \(\mathbb{H}P^n\) for \(1 \leq n \leq \infty\).

1. Background and Preliminaries

Let \(X\) be a pointed connected topological space. For any prime \(p\), write \(\pi_k(X; p)\) for the \(p\)-primary component of the \(k\)-th homotopy group \(\pi_k(X)\) for \(k \geq 1\). The \(p\)-primary homotopy exponent of \(X\) is the least integer \(\exp_p(X) = p^t\), if it exists, so that \(\alpha^{p^t} = 1\) for all elements \(\alpha\) in \(\pi_*(X; p)\). If such an integer does not exist or there are no \(p\)-torsions in \(\pi_*(X)\), we set \(\exp_p(X) = 1\). The homotopy exponent problem for spheres has been studied extensively and has a long history. First, James [12] has shown that \(\exp_2(S^{2n+1}) \leq 2^{2n}\) then Toda has proved that \(\exp_p(S^{2n+1}) \leq p^{2n}\) for \(p > 2\) and Selick [18] has shown that \(\exp_p(S^3) = p\).

For \(p = 2\), an upper bound \(\exp_2(S^{2n+1}) \leq 2^{\exp_2(S^{2n-1})}\) has been obtained by Selick [19] which combined with James’ result [12] yields \(\exp_2(S^{2n+1}) \leq 2^{(\frac{3}{2})^{n}+\varepsilon}\), where \(\varepsilon = \begin{cases} 0 & \text{if } n \equiv 0 \text{ (mod 2)}, \\ \frac{1}{2} & \text{if } n \equiv 1 \text{ (mod 2)} \end{cases}\).

We notice that the result of Gray [8] gives a lower bound on the best exponent which combines with the above shows that \(\exp_p(S^{2n+1}) = p^n\) is the best exponent for \(p > 2\).

After that Cohen-Moore-Neisendorfer showed in their seminal work [3] that \(\exp_p(S^{2n+1}) = p^n\) for \(p > 3\) and then subsequently, Neisendorfer [16] established the same exponent for \(p = 3\).
Given a group $G$, the $p$-primary exponent of $G$ is the least integer $\exp_p(G) = p^t$, if it exists, such that $\alpha^{p^t} = 1$ for any $p$-torsion element $\alpha$ in $G$. Again, if such an integer does not exist or there are no $p$-torsions in $G$, we set $\exp_p(G) = 1$.

The so-called total Cohen groups $[J(X), \Omega(Y)]$ play an important role in various aspects of the classical non-stable homotopy theory. First introduced by F. Cohen as a tool to tackle the Barratt conjecture, these Cohen groups have found many connections to other aspects in topology and algebra. For instances, the Cohen groups were used in [20] to show that the functorial homotopy decompositions of loop space of co-H-spaces is equivalent to the functorial coalgebra decompositions of tensor algebra functor. This establishes a fundamental connection between homotopy theory (of loops on co-H-spaces) and the modular representation theory of Lie powers. In [28], modified Cohen groups were used to give bounds on homotopy exponents of $\Omega^2(S^n_{(2)})$. Moreover, the Cohen groups were used in [29] to produce a ring so that double loop spaces are modules over this ring in the homotopy category.

The main objective of this paper is to study $p$-primary exponents of the total Cohen groups $[J(S^r), \Omega(Y)] = [\Omega(S^{r+1}), \Omega(Y)]$. Here, $J(X) = \colim_{n \geq 1} J_n(X)$ denotes the James construction of $X$. It is known that $[J(S^1), \Omega(Y)]$ is in a one-to-one correspondence with the direct product $\prod_{i \geq 2} \pi_i(Y)$ as sets but the group structure of $[J(S^1), \Omega(Y)]$ is far from being abelian. In fact, $\pi_i(Y)$ is not even a subgroup of $[J(S^1), \Omega(Y)]$ in general. Nevertheless, it is natural to ask how the $p$-torsion elements of $\pi^*(Y)$ are related to the $p$-torsion elements of $[J(S^1), \Omega(Y)]$. In particular, we study $\exp_p([J(S^r), \Omega(Y)])$ when $Y$ is the $n$-th sphere $S^n$, a space with the homotopy type of $\Sigma(n)/G$ for a homotopy $n$-sphere $\Sigma(n)$ with a free action of a discrete group $G$, the complex projective space $\mathbb{C}P^n$ for $n \geq 1$ or the quaternionic projective space $\mathbb{H}P^n$ for $1 \leq n \leq \infty$.

Throughout the rest of this paper we do not distinguish between a map and its homotopy class and we follow freely notations from the book of Toda [26].

This paper is organized as follows. Section 1 recalls some known results on the homotopy exponents of spheres, iterated Whitehead products of spheres, and the group structure of $[\Omega(S^{r+1}), \Omega(Y)]$ for $r \geq 1$.

Section 2 is devoted to proving the main result on the exponents of $[\Omega(S^{r+1}), \Omega(S^N)]$. More precisely, we prove:

**Theorem 2.1.** Let $p$ be a prime.

1. If $p \neq 2$ then

\[ \exp_p([\Omega(S^{r+1}), \Omega(S^N)]) = \exp_p(S^N). \]
If \( p = 2 \) then

\[
\exp_2([\Omega(S^{r+1}), \Omega(S^N)]) = \begin{cases} 
\exp_2(S^N) & \text{if } N \text{ is odd;} \\
\leq 2\exp_2(S^N) & \text{if } N \text{ is even.}
\end{cases}
\]

In Section 3, we further investigate the exponents of \([\Omega(S^{r+1}), \Omega(Y)]\) when \( Y \) is a space with the homotopy type of \( \Sigma(n)/G \), the complex projective space \( \mathbb{C}P^n \) or the quaternionic projective space \( \mathbb{H}P^n \) (including \( \mathbb{H}P^\infty \)). These results are stated in Theorems 3.2, 3.6, 3.8 and 3.9.

1.1. Homotopy exponents. After the work of James [13], the first major breakthrough in the homotopy exponent problem is the result of Cohen-Moore-Neisendorfer [3]. Shortly thereafter, Neisendorfer [16] obtained the same result for prime \( p = 3 \). For odd spheres, we have the following:

**Theorem 1.1** ([3], [16]). If \( p \) is an odd prime then

\[\exp_p(S^{2n+1}) = p^n\]

for any \( n \geq 1 \).

For \( p = 2 \), James [13] showed that the odd spheres \( S^{2n+1} \) have the 2-primary exponent less than equal to \( 4^n \). Moreover, he showed that the 2-primary homotopy exponent increases at most by a factor of 4 as one passes from \( S^{2n-1} \) to \( S^{2n+1} \). Selick [19] showed that the 2-primary homotopy exponent increases at most by a factor of 2 as one passes from \( S^{4n-1} \) to \( S^{4n+1} \), thereby improving on the upper bound for \( \exp_2(S^{2n+1}) \) previously obtained by James [13] and by Cohen [2].

James has shown the existence of a weak homotopy equivalence

\[\Sigma J(X) \simeq \bigvee_{i=1}^{\infty} \Sigma X^{(i)},\]

where \( X^{(i)} \) denotes the \( i \)-fold smash power of \( X \). For the \( n \)-th sphere \( X = S^n \) one has \( \Omega(S^{n+1}) \simeq J(S^n) \) and the splitting map above leads to the projection maps \( \Sigma \Omega S^{n+1} \to S^{k+1} \) for \( k \geq 0 \) which are adjoint to the maps

\[H_k : \Omega(S^{n+1}) \to \Omega(S^{kn+1})\]

known as the James-Hopf maps.

Write \( X_{(p)} \) for the localization of a topological space \( X \) at the prime \( p \). Then, recall the fibration

\[\hat{S}_{(2)}^{2n} \to \Omega(S_{(2)}^{2n+1}) \to \Omega(S_{(2)}^{2n+1})\]

found by James [13] and the fibrations

\[\hat{S}_{(p)}^{2n} \to \Omega(S_{(p)}^{2n+1}) \to \Omega(S_{(p)}^{2n+1}),\]
and

\[ S^{2n-1} \to \Omega(S^{2n}) \to \Omega(S_{(p)}^{2np-1}) \]

found by Toda [26] for \( p > 2 \), where \( \hat{S}^{2n} \) is the \((2np - 1)\)-skeleton of the loop space \( \Omega(S^{2n+1}) \). Thus, the fibrations above and the Serre result [21] lead to:

**Theorem 1.2.** (1) The fibre of the James-Hopf map \( H_2 : \Omega(S^{2n}) \to \Omega(S^{4n-1}) \) is \( S^{2n-1} \) and there is an odd primary equivalence (due to Serre)

\[ \Omega(S^{2n}) \simeq S^{2n-1} \times \Omega(S^{4n-1}). \]

(2) The \( p \)-local fibre of \( H_p : \Omega(S^{2n+1}) \to \Omega(S^{2pn+1}) \) is \( J_{p-1}(S^{2n}) \) for any prime \( p \) (due to James for \( p = 2 \) and Toda for \( p > 2 \)).

For even-dimensional spheres, Theorem 1.2(1) gives the torsions at odd primes \( p \) in terms of those of odd-dimensional spheres, using

\[ (1.1) \quad \pi_m(S^{2n}; p) \cong \pi_{m-1}(S^{2n-1}; p) \oplus \pi_m(S^{4n-1}; p) \]

for \( m \geq 2n \), where \( \pi_m(X; p) \) stands for the \( p \)-primary homotopy component of the \( m \)-th homotopy group \( \pi_m(X) \) of a space \( X \). This implies that \( S^{2n} \) has the \( p \)-primary homotopy exponent \( p^{2n-1} \).

The \( EHP \) sequences associated to the fibration

\[ S^{2n-1}_{(2)} \xrightarrow{E} \Omega(S^{2n})_{(2)} \xrightarrow{H} \Omega(S^{4n})_{(2)} \]

and James’ result [13] for \( p = 2 \) show that the \( 2 \)-primary homotopy exponent \( \exp_2(S^{2n}) \) is bounded by \( 4^{2n} \).

For spaces other than the spheres, Neisendorfer [17, Corollary 0.2] showed that \( \exp_p(P^n(p^r)) \leq p^{r+1} \) for Moore spaces \( P^n(p^r) \) of type \((Z_{p^r}, n - 1)\) with an odd prime \( p \) and \( n \geq 3 \). For prime \( p = 2 \), the problem has been investigated by Theriault [24]. Relying on results by James [12] and Toda [25], Stanley reproved in [22] Long’s result [14] that finite \( H \)-spaces have an exponent at any prime.

More recently, the homotopy exponent problem has also been studied for certain homogeneous spaces (see e.g., [9, 30]).

### 1.2. Iterated Whitehead products of spheres.

Given \( \alpha \in \pi_k(X) \) and \( \beta \in \pi_l(X) \) with \( k, l \geq 1 \), write \([\alpha, \beta] \in \pi_{k+l-1}(X)\) for their Whitehead product.

**Proposition 1.3.** Let \( \iota_n \) be the identity map of the \( n \)-sphere \( S^n \). Then:

1. \([\iota_n, \iota_n]\) has infinite order if \( n \) is even, is trivial if \( n = 1, 3, 7 \), and has order 2 otherwise ([7, (1.2)]);
2. \([[[\iota_n, \iota_n], \iota_n]]\) has order 3 if \( n \) is even and is trivial otherwise ([7, Lemma 1.2 and (1.4)]);
3. all Whitehead products in \( \iota_n \) of weight \( \geq 4 \) vanish ([27, Chapter XI, (8.8) Theorem]).
(4) if \( \alpha \in \pi_k(X) \), \( \beta \in \pi_l(X) \) and \( [\alpha, \beta] = 0 \) then \( [\alpha \circ \delta \circ \delta', \beta] = 0 \) for \( \delta \in \pi_S(S^k) \) and \( \delta' \in \pi_S(S^l) \) (\[27\] Chapter X, (8.14) Theorem);

(5) (Jacobi identity) If \( \alpha \in \pi_{p+1}(X) \), \( \beta \in \pi_{q+1}(X) \), \( \gamma \in \pi_{r+1}(X) \), and \( p, q, r \) are all positive, then

\[
(-1)^{(p+1)(r+1)}[\alpha, [\gamma, \beta]] + (-1)^{(q+1)(p+1)}[[\beta, \gamma], \alpha] + (-1)^{(q+1)(r+1)}[[\gamma, \alpha], \beta] = 0
\]

(\[27\] Chapter X, (7.14) Corollary);

(6) if \( \alpha \in \pi_k(X) \), \( \beta \in \pi_l(X) \) and \( \delta \in \pi_S(S^{k-1}) \), \( \delta' \in \pi_S(S^{l-1}) \) then \( [\alpha \circ \Sigma \delta, \beta \circ \Sigma \delta'] = [\alpha, \beta] \circ \Sigma(\delta \wedge \delta') \) (\[27\] Chapter X, (8.18) Theorem).

To state the next results on Whitehead products, first recall from \[1\] Chapter II that the exterior cup products are pairings

\[
\sharp, \sharp : [\Sigma X, \Sigma A] \times [\Sigma Y, \Sigma B] \to [\Sigma X \wedge Y, \Sigma A \wedge B]
\]

defined by the compositions

\[
\alpha \sharp \beta : \Sigma X \wedge Y \overset{\alpha \wedge Y}{\to} \Sigma A \wedge Y = A \wedge \Sigma Y \overset{A \wedge \beta}{\to} A \wedge \Sigma B
\]

and

\[
\alpha \sharp \beta : \Sigma X \wedge Y = X \wedge \Sigma Y \overset{X \wedge \beta}{\to} X \wedge \Sigma B = \Sigma X \wedge B \overset{\alpha \wedge B}{\to} \Sigma A \wedge B,
\]

respectively for \( (\alpha, \beta) \in [\Sigma X, \Sigma A] \times [\Sigma Y, \Sigma B] \), where \( \alpha \wedge Y \) is the map \( \alpha \wedge \text{id}_Y \) and \( A \wedge \beta \) is the map \( \text{id}_{\Sigma A} \wedge \beta \), up to the shuffle of the suspension coordinate. These products are associative.

Let \( h_k : \pi_m(S^{n+1}) \to \pi_m(S^{kn+1}) \) be the map induced by the James-Hopf \( k \)-invariant \( H_k : \Omega(S^{n+1}) \to \Omega(S^{kn+1}) \) for \( k > 1 \). Then, by \[1\] Chapter III, (1.4) Proposition and (1.5) Corollary, we have:

**Proposition 1.4.** Let \( \alpha_i \in \pi_m(S^n) \) for \( i = 1, 2, 3 \). Then we have the following formulas:

1. \([\alpha_1, \alpha_2] = [\tau_n, \tau_n] \circ (\alpha_1 \sharp \alpha_2) + \sum_{n=1}^{-1}[(\alpha_2 \alpha_1) \circ (h_2 \alpha_1) \sharp \alpha_2] + [\tau_n, [\tau_n, \tau_n]] \circ (\alpha_1 \sharp h_2 \alpha_2),\]
2. \([\alpha_2 \alpha_3, \alpha_3] = [[\tau_n, \tau_n], \tau_n] \circ (\alpha_1 \sharp \alpha_2 \sharp \alpha_3).\]

Next, \[1\] Chapter III, (1.8) and (1.9) Corollaries] yield the following result proved by Hilton \([11]\).

**Proposition 1.5.** Let \( \alpha_1 \in \pi_k(S^m) \) for \( m > 1 \) and \( \alpha_2, \alpha_3 \in \pi_m(S^n) \). Then

\[
(\alpha_2 + \alpha_3) \circ \alpha_1 = \alpha_2 \alpha_1 + \alpha_3 \alpha_1 + [\alpha_2, \alpha_3] \circ h_2(\alpha_1).
\]

In particular, if \( t \in \mathbb{Z} \) then

\[
(tm) \circ \alpha = ta + \left( \frac{t(t - 1)}{2} \right) [\tau_m, \tau_m] h_2(\alpha)
\]

for \( \alpha \in \pi_k(S^m). \)
Proposition 1.6. Let $n \geq 2$.

(1) If $\alpha_i \in \pi_{m_i}(S^n)$ with $i = 1, 2, 3$ then $3[[\alpha_1, \alpha_2], \alpha_3] = 0$;

(2) all iterated Whitehead products $[\alpha_1, \ldots, \alpha_k]$ of weight $k \geq 4$ vanish.

Proof. (1): If $n$ is even then by Proposition 1.3(2) we have $3[[\iota_n, \iota_n], \iota_n] = 0$. Hence, for $\alpha_i \in \pi_{m_i}(S^n)$ with $i = 1, 2, 3$, we have

$$0 = (3[[\iota_n, \iota_n], \iota_n]) \circ (\alpha_1 \sharp \alpha_2 \sharp \alpha_3) = (2[[\iota_n, \iota_n], \iota_n] + [[\iota_n, \iota_n], \iota_n]) \circ (\alpha_1 \sharp \alpha_2 \sharp \alpha_3).$$

By Proposition 1.3(3) and Proposition 1.5 we obtain

$$(2[[\iota_n, \iota_n], \iota_n] + [[\iota_n, \iota_n], \iota_n]) \circ (\alpha_1 \sharp \alpha_2 \sharp \alpha_3) = (2[[\iota_n, \iota_n], \iota_n]) \circ (\alpha_1 \sharp \alpha_2 \sharp \alpha_3) + [[\iota_n, \iota_n], \iota_n] \circ (\alpha_1 \sharp \alpha_2 \sharp \alpha_3) = 3([[\iota_n, \iota_n], \iota_n] \circ (\alpha_1 \sharp \alpha_2 \sharp \alpha_3)).$$

Since $[[\alpha_1, \alpha_2], \alpha_3] = [[\iota_n, \iota_n], \iota_n] \circ (\alpha_1 \sharp \alpha_2 \sharp \alpha_3)$ by Proposition 1.4(2), assertion (1) follows.

(2): Let $k = 4$. Then, by Proposition 1.3(3), we have $[[[\iota_n, \iota_n], \iota_n], \iota_n]] = 0$. Using Proposition 1.3(4), we deduce that $[[[[\iota_n, \iota_n], \iota_n]] \circ (\alpha_1 \sharp \alpha_2 \sharp \alpha_3), \alpha_4] = 0$. Hence, Proposition 1.4(2) implies $[[[[\alpha_1, \alpha_2], \alpha_3], \alpha_4] = 0$.

Furthermore, by the Jacobi identity of Proposition 1.3(5), we have

$$[[\alpha_1, \alpha_2], [\alpha_3, \alpha_4]] = \pm [[\alpha_1, \alpha_2], \alpha_3, \alpha_4] \pm [[[\alpha_1, \alpha_2], \alpha_3], \alpha_4].$$

Thus, we conclude that $[[\alpha_1, \alpha_2], [\alpha_3, \alpha_4]] = 0$. Now the rest of the proof is an inductive argument.

Suppose that the result is true for Whitehead products of weight $m$ and let us show for $m+1$, where we will assume that $m+1 \geq 5$. Since $[\alpha_1, \ldots, \alpha_{m+1}]$ is of the form $[[\theta_1, \theta_2]]$, where $\theta_1, \theta_2$ are iterated Whitehead products of weight $k_1$ and $k_2$, respectively with $k_1 + k_2 = m + 1$. If one of the $k_i$’s is $\geq 4$, then the result follows by inductive hypothesis. Otherwise $k_1, k_2 < 4$, and we have two cases, namely both are equal to 3 or one is 3 and the other is 2. In both cases the result follows by inductive hypothesis and the Jacobi identity from Proposition 1.3(5). \qed

The next result is probably also well-known to the experts, but nevertheless, we have decided to state:

Proposition 1.7. Let $n$ be odd and $\alpha_1 \in \pi_k(S^n)$, $\alpha_2 \in \pi_l(S^n)$ and $\alpha_3 \in \pi_t(S^n)$. Then:

(1) $2[\alpha_1, \alpha_2] = 0$;

(2) $[[\alpha_1, \alpha_2], \alpha_3] = 0$. 

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Proof. (1): Let \( \alpha_i \in \pi_m(S^n) \) for \( i = 1, 2 \). By Proposition 1.3(2), we have \([t_n, t_n] = 0\). Hence, Proposition 1.4(1) implies \([\alpha_1, \alpha_2] = [t_n, t_n] \circ (\alpha_1 \circ \alpha_2)\). On the other hand, Proposition 1.3(1) gives \( 2[t_n, t_n] = 0 \). Then, we have \((2[t_n, t_n]) \circ (\alpha_1 \circ \alpha_2) = 0\).

It follows from Proposition 1.4(1) that
\[
0 = (2[t_n, t_n]) \circ (\alpha_1 \circ \alpha_2) = 2((([t_n, t_n]) \circ (\alpha_1 \circ \alpha_2)) + ([t_n, t_n], [t_n, t_n]) \circ h_2(\alpha_1 \circ \alpha_2).
\]
But, the Jacobi identity of Proposition 1.3(5) together with the fact that \([t_n, t_n] = 0\) yield \([t_n, t_n], [t_n, t_n] = 0\). Consequently, \(2[\alpha_1, \alpha_2] = 0\).

We also present another proof of assertion (1) as follows. In view of Proposition 1.3(1) we have \(2[t_n, t_n] = 2[t_n, t_n] = 0\). Then, Proposition 1.3(4) leads to \([2[t_n, \alpha_2] = [t_n, 2\alpha_2] = 0\). Again, by Proposition 1.3(4), we get \([\alpha_1, 2\alpha_2] = 2[\alpha_1, \alpha_2] = 0\) and the assertion follows.

(2): If \( \alpha_i \in \pi_m(S^n) \) for \( i = 1, 2, 3 \) then, it follows from Proposition 1.6(1) that \(3[\alpha_1, \alpha_2, \alpha_3] = 0\).
Together with (1), we conclude that \([\alpha_1, \alpha_2, \alpha_3] = 0\) and the proof is complete. \(\square\)

1.3. Group structure on \([\Omega(S^{r+1}), \Omega(Y)]\). By the group isomorphism \([\Sigma J(S^r), Y] \cong [\Omega(S^{r+1}), \Omega(Y)]\), it follows from \(\Sigma J(S^r) \cong \bigvee_{i \geq 1} S^{r+1}\) that \([\Omega(S^{r+1}), \Omega(Y)]\) is in a one-to-one correspondence with the product \(\prod_{i \geq 1} \pi_{ir+1}(Y)\) as sets. Following [6], we recall how the group multiplication \(\oplus\) on \([\Sigma J(S^r), Y]\) is defined.

Identifying \([\Omega(S^{r+1}), \Omega(Y)]\) with \(\prod_{i \geq 1} \pi_{ir+1}(Y)\) as sets, a typical element of \([\Omega(S^{r+1}), \Omega(Y)]\) is an infinite tuple \(\bar{\alpha} = (\alpha_1, \alpha_2, \ldots)\), where \(\alpha_i \in \pi_{ir+1}(Y)\) for \(i \geq 1\). Denote by \((\bar{\alpha})_j\) the \(j\)-th coordinate of \(\bar{\alpha} \in [\Omega(S^{r+1}), \Omega(Y)]\), i.e., \((\bar{\alpha})_j = \alpha_j \in \pi_{ir+1}(Y)\). Let \(\bar{\alpha} = (\alpha_1, \alpha_2, \ldots), \beta = (\beta_1, \beta_2, \ldots)\) be two elements in \([\Omega(S^{r+1}), \Omega(Y)]\). Then, the product \(\bar{\alpha} \oplus \beta\) is defined to be the element whose \(j\)-th coordinate is given by
\[
(\bar{\alpha} \oplus \beta)_j = \alpha_j + \beta_j + \sum_{i+s=j} \Phi_{i, i+s-1}[\alpha_i, \beta_s],
\]
where the coefficient \(\Phi\) is defined as follows. For any positive integers \(l, k\) with \(1 \leq l \leq k\),
\[
\Phi_{l,k} = \begin{cases} 
-\frac{k}{2} & \text{if } l \text{ is even and } k \text{ is even}; \\
0 & \text{if } l \text{ is odd and } k \text{ is even}; \\
\frac{k-1}{l} & \text{if } l \text{ is odd and } k \text{ is odd}; \\
-\frac{k-1}{l} & \text{if } l \text{ is even and } k \text{ is odd}.
\end{cases}
\]

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Using the multiplication \([1.2]\), the coordinates of any \(p\)-torsion element of \([\Omega(S^{r+1}), \Omega(Y)]\) are \(p\)-torsion elements of \(\pi_*(Y)\) as we show in:

**Proposition 1.8.** Let \(\bar{\alpha} = (\alpha_1, \alpha_2, \ldots) \in [\Omega(S^{r+1}), \Omega(Y)]\). If \(\bar{\alpha}\) is a \(p\)-torsion element then each \(\alpha_i\) is a \(p\)-torsion element of \(\pi_*(Y)\).

**Proof.** Suppose \(\bar{\alpha}^k = 1\) for some \(k = p^t\). It follows from the multiplication \(\oplus\) that \((\bar{\alpha}^k)_1 = k\alpha_1\). This implies that \(\alpha_1\) is a \(p\)-torsion element of \(\pi_{r+1}(Y)\). We proceed by induction as follows. Suppose \(\alpha_i\) is an element in \(\pi_*(Y;p)\) for \(i < n\). Again, using \(\oplus\), \((\bar{\alpha}^k)_n\) is the sum of \(k\alpha_n\) and integer multiples of elements of the form \([\alpha_i, \gamma]\), where \(\gamma \in \pi_{jr+1}(Y)\) with \(i + j = n\). By inductive hypothesis, \(\alpha_i\) is an element in \(\pi_*(Y;p)\) and so is \([\alpha_i, \gamma]\). Since \(\bar{\alpha}^k = 1\), we conclude that \(\alpha_n\) must also be an element in \(\pi_{nr+1}(Y;p)\) and the proof follows. \(\square\)

Let \(\bar{\alpha}, \bar{\beta} \in [\Omega(S^{r+1}), \Omega(S^N);p]\) and write \(\bar{\gamma} = \bar{\alpha} \oplus \bar{\beta} \in [\Omega(S^{r+1}), \Omega(S^N)]\). By \([1.2]\), we have \(\gamma_j = \alpha_j + \beta_j + \sum_{i+s=j} \Phi_{i+s-1}[\alpha_i, \beta_s]\). Suppose \(\bar{\alpha}^{p^t} = 1 = \bar{\beta}^{p^t}\). Then, one can easily show that \(\gamma^{p^{\max(t_1, t_2)}} = 1\). Furthermore, \(\bar{\alpha}^{p^{t_1}} = 1\) implies \((\bar{\alpha}^{-1})^{p^{t_1}} = 1\).

Given a prime \(p\), spaces \(X, Y\) and a co-\(H\)-structure on \(X\), write \([X, Y;p]\) for the set of \(p\)-primary components of \([X, Y]\). Then, using Proposition \([1.8]\) it is straightforward to show the following

**Proposition 1.9.** (1) The bijection \([\Omega(S^{r+1}), \Omega(S^N)] \leftrightarrow \prod_{i \geq 1} \pi_{ni}(S^N)\) restricts to a bijection \([\Omega(S^{r+1}), \Omega(S^N);p] \leftrightarrow \prod_{i \geq 1} \pi_{ni}(S^N; p)\) for \(p \neq 2\);

(2) \([\Omega(S^{r+1}), \Omega(S^N);p]\) is a subgroup of \([\Omega(S^{r+1}), \Omega(S^N)]\) for any prime \(p \geq 2\).

2. Exponents of \([\Omega(S^{r+1}), \Omega(S^N)]\)

Before we state the main theorem of this section, first examine the coordinates of the powers \(\gamma^M\) for \(\gamma \in [\Omega(S^{r+1}), \Omega(S^N)]\). Suppose \(\alpha \in \pi_{n+1}(S^N), \beta \in \pi_{m+1}(S^N)\). Consider the element \((\alpha, \beta) \in [\Omega(S^{r+1}), \Omega(S^N)]\) as an infinite sequence with only two non-zero coordinates in positions \(n\) and \(m\). Then,

\[(\alpha, \beta)^2 = (2\alpha, 2\beta, \Phi_{n, 2n-1}[\alpha, \alpha], \Phi_{m, 2m-1}[\beta, \beta], \Phi_\Delta[\alpha, \beta]).\]

Here \(\Phi_\Delta = \Phi_{n, n+m-1} + (-1)^{(n+1)(m+1)}\Phi_{m, n+m-1}\).

Now, we compute \((\alpha, \beta)^3\). The possible non-zero coordinates are divided into three types: (homogenous) elements, Whitehead products and triple Whitehead products. This is the case because for spheres, all quadruple Whitehead products vanish by Proposition \([1.6, 2]\).

It is easy to see that the first type consists of \(3\alpha\) and \(3\beta\). Second type coordinates are: \(3\Phi_{n, 2n-1}[\alpha, \alpha]\), \(3\Phi_{m, 2m-1}[\beta, \beta]\), and \(3\Phi_\Delta[\alpha, \beta]\). Finally, for the triple Whitehead products, we have:
Φ_{n,2n-1} Φ_{2n,3n-1}[[α, α], Φ_{n,2n-1} Φ_{2n,2n+m-1}[[α, α], β],
Φ_{m,2m-1} Φ_{2m,2m+n-1}[[β, β], Φ_{m,2m-1} Φ_{2m,2m+n-1}[[β, β], β],
Φ_Δ Φ_{n+m,2n+m-1}[[α, β], Φ_Δ Φ_{n+m,2n+m-1}[[α, β], β].

To continue in this fashion, the power \((α, β)^M\) has the following non-zero coordinates:

Type I: \(Mα, Mβ;\)

Type II: \((1+2+3+\ldots+(M-1))Φ_{n,2n-1}[α, α] = \binom{M}{2} Φ_{n,2n-1}[α, α], \binom{M}{2} Φ_{m,2m-1}[β, β], \binom{M}{2} Φ_Δ[α, β];\)

Type III: each of the six triple Whitehead products will have the following as a factor in its coefficient:

\[1 + 3 + (1 + 2 + 3) + (1 + 2 + 3 + 4) + \cdots + (1 + 2 + \cdots + M - 1).\]

This sum in turn is equal to

\[\sum_{l=2}^{M-1} \binom{l}{2} = \binom{M}{3}.\]

By [5, p. 120], we have

\[\sum_{l=2}^{M-1} \binom{l}{2} = \binom{M}{3}.\]

Therefore, for an arbitrary element \(\bar{α} = (α_1, α_2, \ldots)\in [Ω(S^{r+1}), Ω(S^N)]\), the non-zero coordinates of \(\bar{α}^M\) also fall into these three types as described above.

Now, the main result of this section is the following:

**Theorem 2.1.** Let \(p\) be a prime.

1. If \(p \neq 2\) then
   \[\exp_p([Ω(S^{r+1}), Ω(S^N)]) = \exp_p(S^N).\]

2. If \(p = 2\) then
   \[\exp_2([Ω(S^{r+1}), Ω(S^N)]) = \begin{cases} \exp_2(S^N) & \text{if } N \text{ is odd;} \\ \leq 2 \exp_2(S^N) & \text{if } N \text{ is even.} \end{cases}\]

**Proof.** Suppose \(p^t = \exp_p(S^N)\) and \(\bar{α} = (α_1, α_2, \ldots)\) is a \(p\)-torsion element in \([Ω(S^{r+1}), Ω(S^N)]\).

**Case (1):** \(p\) is odd.

Let \(M = p^t\). Since \(M = \exp_p(S^N)\), all coordinates of Type I must be zero. But \(\binom{M}{2} = \frac{p^t(p^t-1)}{2} \text{ and } p\) is odd, so it follows that \(p^t \mid \binom{M}{2}\) and consequently \(M = p^t\) divides all coefficients of coordinates of
Type II in $\bar{\alpha}^M$. Thus, we conclude that these coordinates must be zero. For Type III coordinates, we consider two subcases.

(i) Case $N$ is odd. By Proposition 1.7 all triple Whitehead products vanish so all Type III coordinates must be zero.

(ii) Case $N$ is even. By Proposition 1.6 we have $3[\alpha, [\beta, \gamma]] = 0$ for any $\alpha, \beta, \gamma \in \pi_*(S^N)$. We conclude that if $p \neq 3$ then $[\alpha, [\beta, \gamma]] = 0$ so there are no non-zero coordinates of Type III in $\bar{\alpha}^M$.

If $p = 3$, then $\left(\frac{M}{3}\right) = \frac{3^t(3^t-1)(3^t-2)}{6}$. Since $N$ is even, it follows from [26] that $M = 3^t$, where $t \geq 2$ provided $N > 2$. It follows that $3 | \left(\frac{M}{3}\right)$. Again, by Proposition 1.6 there are no non-zero coordinates of Type III in $\bar{\alpha}^M$.

When $N = 2$, all triple Whitehead products in $\pi_*(S^2)$ vanish so again there are no non-zero coordinates of Type III in $\bar{\alpha}^M$. Hence, we conclude that for $p \neq 2$, $\exp_p([\Omega(S^{r+1}), \Omega(S^N)]) = \exp_p(S^N)$.

Case (2): $p = 2$. We consider two subcases:

(i) Case $N$ is odd. For $N > 3$, it follows from [19] that $\exp_2(S^N) = 2^t$ for some $t > 1$. Since $N$ is odd, by Proposition 1.7 it follows that all triple Whitehead products must vanish. In other words, there are no non-zero coordinates of Type III in $\bar{\alpha}^M$, where $M = 2^t$. Since $\left(\frac{M}{2}\right) = 2^{t-1}(2^t - 1)$ and $t > 1$, $2 | \left(\frac{M}{2}\right)$ so that there are no non-zero coordinates of Type II as $2[\alpha, \beta] = 0$ by Proposition 1.7. For $N = 3$, there are no non-trivial Whitehead products since $S^3$ is a group. Clearly there are no non-zero coordinates of Type I in $\alpha^M$. Thus for $N$ odd, we have $\exp_2([\Omega(S^{r+1}), \Omega(S^N)]) = \exp_2(S^N)$.

(ii) Case $N$ is even. Again for $N > 2$, it follows from [19] and from the EHP sequence that $\exp_2(S^N) = 2^t$ for some $t > 1$. Now let $M = 2^{t+1}$. Then $\left(\frac{M}{2}\right) = 2^t(2^t+1 - 1)$ so $\left(\frac{M}{2}\right)[\alpha, \beta] = 0$ if $\alpha, \beta$ are elements in $\pi_*(S^N; 2)$. Thus, we conclude that there are no non-zero coordinates of Type II. By Proposition 1.6 we have $3[\alpha, [\beta, \gamma]] = 0$. Thus, if $\alpha, \beta$ and $\gamma$ are elements of $\pi_*(S^N; 2)$, it follows that the triple Whitehead product $[\alpha, [\beta, \gamma]] = 0$. We conclude that there are no non-zero coordinates of Type III. Since $M = 2\exp_2(S^N)$, there are no non-zero coordinates of Type I in $\alpha^M$.

Finally for $N = 2$, the same arguments as above show that there are no non-zero coordinates of Types I and II. For Type III coordinates, we note that all triple Whitehead products in $\pi_*(S^2)$ vanish so that there are no non-zero coordinates of Type III. Hence, for $N$ even, we have $\exp_2([\Omega(S^{r+1}), \Omega(S^N)]) \leq 2\exp_2(S^N)$ and the proof is complete.

\[ \square \]

Remark 2.2. When $p = 2$ and $N$ is even, we can further analyze the 2-primary exponent $\exp_2([\Omega(S^{r+1}), \Omega(S^N)])$ as follows. Suppose $N > 2$ is even and let $M = 2^t = \exp_2(S^N)$ with
\( t > 1 \). For Type II coordinates, we have \( \binom{M}{2} = 2^{t-1}(2^t - 1) \);

\[
\Phi_{n,2n-1} = \begin{cases} 
\left( \frac{n - 1}{2} \right) & \text{if } n \text{ is odd;} \\
- \left( \frac{n - 1}{2} \right) & \text{if } n \text{ is even.}
\end{cases}
\]

Now, recall the Lucas’ formula:

\[
\binom{m}{n} \equiv \prod \left( \frac{m_i}{n_i} \right) \pmod{p},
\]

where \( m = m_k p^k + m_{k-1} p^{k-1} + \cdots + m_1 p + m_0 \) and \( n = n_k p^k + n_{k-1} p^{k-1} + \cdots + n_1 p + n_0 \) are the base \( p \) expansions of \( m \) and \( n \) respectively. This uses the convention that \( \binom{m}{n} = 0 \) if \( m < n \) for non-negative integers \( m \) and \( n \) and a prime \( p \).

Then, we get \( \binom{n-1}{2} \equiv 0 \pmod{2} \) and \( \binom{n-1}{2} \equiv \begin{cases} 
1 \pmod{2} & \text{if } n = 2^k, \\
0 & \text{otherwise.}
\end{cases} \)

It follows that \( \binom{M}{2} \Phi_{n,2n-1}[\alpha, \alpha] = 0 \) except possibly when \( n = 2^k \). In the case when \( n = 2^k \), we have \([\alpha, \alpha] = (-1)^{(nr+1)(nr+1)}[\alpha, \alpha] = -[\alpha, \alpha] \). Thus if \( \alpha \) is an element in \( \pi_r(S^N; 2) \), we have \( \binom{M}{2} \Phi_{n,2n-1}[\alpha, \alpha] = 0 \) because \( 2 | \binom{M}{2} \) since \( t > 1 \). Similarly, \( \binom{M}{2} \Phi_{m,2m-1}[\beta, \beta] = 0 \). Thus, the only possible non-zero coordinates of Type II are \( \binom{M}{2} \Phi_\Delta[\alpha, \beta] \).

Now,

\[
\Phi_\Delta = \begin{cases} 
- \left( \frac{n+m}{2} \right) & \text{if } m \text{ is even and } n \text{ is even;} \\
2 \left( \frac{n+m-2}{2} \right) & \text{if } m \text{ is odd and } n \text{ is odd;} \\
- \left( \frac{n-m-1}{2} \right) & \text{if } m \text{ is odd and } n \text{ is even;} \\
- \left( \frac{n-m-1}{2} \right) & \text{if } m \text{ is even and } n \text{ is odd.}
\end{cases}
\]

Therefore, if both \( m \) and \( n \) are odd, \( \binom{M}{2} \Phi_\Delta[\alpha, \beta] = 0 \). This means that if \( \bar{\alpha} = (\alpha_1, \alpha_2, \ldots) \in [\Omega(S^{r+1}), \Omega(S^N)] \) (\( N \) even) such that \( \alpha_{2i} = 0 \) for all \( i \geq 1 \) then \( \alpha^M = 1 \), where \( M = \exp_2(S^N) \).

Based upon the proof of Theorem 2.1 and Remark 2.2, we pose the following:

**Question 2.3.** Let \( \alpha \in \pi_r(S^{2N}; 2) \), \( \beta \in \pi_s(S^{2N}; 2) \) with \( \alpha \neq \beta \), \( r \) or \( s \) is odd. Suppose that the orders \( |\alpha| = |\beta| = 2^t \), the 2-exponent of the sphere \( S^{2N} \).

Is it true that the order \( |[\alpha, \beta]| < 2^t \) for any \( \alpha \in \pi_r(S^{2N}; 2) \), \( \beta \in \pi_s(S^{2N}; 2) \) as above or there are \( \alpha \in \pi_r(S^{2N}; 2) \), \( \beta \in \pi_s(S^{2N}; 2) \) such that \( |[\alpha, \beta]| = 2^t \)?
In this section, we examine $\exp_p([\Omega(S^{r+1}),\Omega(Y)])$ when $Y$ with the homotopy type is of $\Sigma(n)/G$ for a homotopy $n$-sphere $\Sigma(n)$ with a free action of a discrete group $G$, a complex projective space $\mathbb{C}P^n$ for $n \geq 1$ or a quaternionic projective space $\mathbb{H}P^n$ for $1 \leq n \leq \infty$. First, we show certain basic properties about exponents.

Note that $\Omega(Y_1 \times Y_2) \simeq \Omega(Y_1) \times \Omega(Y_2)$ and the space $\Sigma\Omega(S^{r+1})$ is 1-connected for $r \geq 1$. Then, the following result is straightforward.

**Proposition 3.1.** (1) For any prime $p$, we have

$$\exp_p([\Omega(S^{r+1}),\Omega(Y_1 \times Y_2)]) = \max\{\exp_p([\Omega(S^{r+1}),\Omega(Y_1)]), \exp_p([\Omega(S^{r+1}),\Omega(Y_2)])\};$$

(2) A covering map $\tilde{X} \rightarrow X$ induces an isomorphism

$$[\Omega(S^{r+1}),\Omega(\tilde{X})] \cong [\Omega(S^{r+1}),\Omega(X)]$$

and

$$\exp_p([\Omega(S^{r+1}),\Omega(\tilde{X})]) = \exp_p([\Omega(S^{r+1}),\Omega(X)])$$

for any prime $p$.

Recall that a finite dimensional CW-complex $\Sigma(n)$ with the homotopy type of the $n$-th sphere $S^n$ is called a homotopy $n$-sphere. If a discrete group $G$ acts freely and properly discontinuously on $\Sigma(n)$ then the quotient map $\Sigma(n) \rightarrow \Sigma(n)/G$ is a covering map. Let $\mathbb{R}P^n$ be the $n$-th real projective space for $n \geq 1$ and $\gamma_{n,\mathbb{R}} : S^n \rightarrow \mathbb{R}P^n$ be the quotient map. Then, Proposition 3.1(2) yields:

**Theorem 3.2.** The quotient map $\Sigma(n) \rightarrow \Sigma(n)/G$ induces an isomorphism

$$[\Omega(S^{r+1}),\Omega(\Sigma(n))] \xrightarrow{\cong} [\Omega(S^{r+1}),\Omega(\Sigma(n)/G)]$$

for $r \geq 1$. In particular, the quotient map $\gamma_{n,\mathbb{R}} : S^n \rightarrow \mathbb{R}P^n$ induces an isomorphism

$$[\Omega(S^{r+1}),\Omega(S^n)] \xrightarrow{\cong} [\Omega(S^{r+1}),\Omega(\mathbb{R}P^n)]$$

for $r \geq 1$. Consequently,

$$\exp_p[\Omega(S^{r+1}),\Omega(\Sigma(n)/G)] = \exp_p[\Omega(S^{r+1}),\Omega(\Sigma(n))]$$

and

$$\exp_p[\Omega(S^{r+1}),\Omega(\mathbb{R}P^n)] = \exp_p[\Omega(S^{r+1}),\Omega(S^n)]$$

for $n, r \geq 1$ and for any prime $p$. 

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To state the next result, we recall that given a topological group $G$, by the Milnor’s construction, there is a sequence $G \to E_1 \to \cdots \to E_n \to \cdots$, where $E_n = G \ast \cdots \ast G$, the join of $(n+1)$-copies of $G$ and $EG = \text{colim}_n E_n$. Then, we have the universal $G$-fibre bundle

$$G \hookrightarrow EG \xrightarrow{\pi} BG = EG/G.$$ 

Next, consider the pointed suspension $\Sigma G = G \times [-1,1]/\sim$ and write $C_+ = G \times [0,1]/\sim \subseteq \Sigma G$ for the upper and lower cones, respectively. Let $E = C_+ \cup C_- \times G/\sim$, where $((g,0),g') \sim ((g,0),gg')$ for $g, g' \in G$. Then, we get the principal $G$-bundle $G \to E \to \Sigma G$ and let $f : \Sigma G \to BG$ be the corresponding classifying map, and $h : G \to \Omega(BG)$ its adjoint map. Notice that the map $h : G \to \Omega(BG)$ coincides with the composition $G \xrightarrow{\eta} \Omega \Sigma G \xrightarrow{\Omega f} \Omega(BG)$, where $\eta : G \to \Omega \Sigma G$ is determined by the unit map, i.e., the adjoint of the identity map $\text{id}_{\Sigma G} : \Sigma G \to \Sigma G$.

A map $\varphi : (X, \mu) \to (Y, \nu)$ of $H$-spaces is called an $H$-map provided the diagram

$$\begin{array}{ccc}
X \times X & \xrightarrow{\varphi \times \varphi} & Y \times Y \\
\downarrow \mu & & \downarrow \nu \\
X & \xrightarrow{\varphi} & Y
\end{array}$$

is homotopy commutative.

If the group $G$ is a $CW$-complex (e.g., if $G$ is a Lie or discrete group) then the homotopy equivalence $X \ast Y \simeq \Sigma(X \wedge Y)$ imposes a $CW$-structure on $E_n$ and hence, on $EG$ as well. The fact that $G$ is a $CW$-complex implies that $G$ acts cellularly on $EG$. Consequently, $BG = EG/G$ is also a $CW$-complex. By the well-known Milnor’s result [15], the space $\Omega(BG)$ is a $CW$-complex as well.

Let $\partial : \Omega(BG) \to G$ be the connecting map in the Barratt-Puppe sequence

$$\cdots \to \Omega(G) \to \Omega(EG) \to \Omega(BG) \xrightarrow{\partial} G \to EG \to BG$$

associated to the universal $G$-bundle $G \hookrightarrow EG \xrightarrow{\pi} BG$.

**Proposition 3.3.** Suppose the topological group $G$ is a $CW$-complex. Then the connecting map $\partial : \Omega(BG) \to G$ is a homotopy equivalence and is an $H$-map.

**Proof.** By [23, Theorem 8.6], the map $h : G \to \Omega(BG)$ is an $H$-map and is a weak homotopy equivalence. Since $G$ is a $CW$-complex, it follows that $h$ is a homotopy equivalence. Following [10, p. 409], the connecting map $\partial : \Omega(BG) \to G$ is given by $\partial = \rho^{-1} \circ j \circ \varrho$ as in the following
where $F_\pi$ is the homotopy fiber of the map $\pi : EG \to BG$ and $F_i$ is the homotopy fiber of the inclusion map $i : F_\pi \hookrightarrow EG$, both maps $\Phi$ and $\rho$ are the obvious homotopy equivalences. It is easy to verify that the composition $\partial \circ h = \operatorname{id}_G$, the identity map on $G$. Since the map $h$ is an $H$-map and a homotopy equivalence, so is $\partial$ and the proof is complete. □

**Corollary 3.4.** Let $G$ be a compact Lie group and $G \to X \to X/G$ be a principal $G$-fibration where $X/G$ is paracompact. Then the connecting map $\partial_X : \Omega(X/G) \to G$ in the associated Barratt-Puppe sequence

$$\cdots \to \Omega(G) \to \Omega(X) \to \Omega(X/G) \xrightarrow{\partial_X} G \to X \to X/G$$

is an $H$-map.

**Proof.** Since the principal $G$-fibration $X \to X/G$ is classified by the classifying map $\varphi : X/G \to BG$ which can be lifted to a $G$-map $\tilde{\varphi} : X \to EG$, we have the following commutative diagram

$$\begin{array}{cccc}
\cdots & \to & \Omega(X) & \xrightarrow{\partial_X} & \Omega(X/G) & \xrightarrow{G} & X & \xrightarrow{X/G} & X/G \\
\Omega\tilde{\varphi} \downarrow & & \Omega\varphi \downarrow & & \varphi \downarrow & & \varphi \downarrow \\
\cdots & \to & \Omega(EG) & \xrightarrow{\partial} & \Omega(BG) & \xrightarrow{G} & EG & \xrightarrow{BG} & BG.
\end{array}$$

It follows that $\partial_X = \partial \circ \Omega\varphi$, where $\partial$ is as in Proposition 3.3. Since $\Omega\varphi$ and $\partial$ are $H$-maps, by Proposition 3.3 it follows that $\partial_X$ is an $H$-map as well. □

Given a group $G$ and a prime $p$, denote by $\exp_p(G)$ the least positive integer $p^t$ such that $\alpha^{p^t} = 1$ for any $p$-torsion elements $\alpha$ in $G$. If such an integer does not exist or if $G$ has no $p$-torsion elements then we set $\exp_p(G) = 1$. The following result is straightforward.

**Proposition 3.5.** Let $1 \to G' \to G \to G'' \to 1$ be a short exact sequence of groups. Then for any prime $p$,

$$\exp_p(G') \leq \exp_p(G) \leq \exp_p(G') \cdot \exp_p(G'').$$

We now give the $p$-primary exponent of $[\Omega(S^{r+1}), \Omega(\mathbb{C}P^n)]$. 

![Categorical diagram](https://via.placeholder.com/150)
Theorem 3.6. The principal $S^1$-bundle $S^1 \to S^{2n+1} \xrightarrow{\gamma_n \cdot c} \mathbb{C}P^n$ gives rise to a split short exact sequence

$$1 \to [\Omega(S^2), \Omega(S^{2n+1})] \xrightarrow{(\gamma_n \cdot c)^*} [\Omega(S^2), \Omega(\mathbb{C}P^n)] \to \mathbb{Z} \to 0$$

of groups and an isomorphism

$$[\Omega(S^{r+1}), \Omega(\mathbb{C}P^n)] \xrightarrow{\cong} [\Omega(S^{r+1}), \Omega(S^{2n+1})]$$

for $r \geq 2$.

Consequently,

$$\exp_p[\Omega(S^{r+1}), \Omega(\mathbb{C}P^n)] = \exp_p[\Omega(S^{r+1}), \Omega(S^{2n+1})]$$

for $n, r \geq 1$ and any prime $p$.

Proof. Consider the principal $S^1$-bundle $S^1 \hookrightarrow S^{2n+1} \xrightarrow{\gamma_n \cdot c} \mathbb{C}P^n$ and the associated Barratt-Puppe sequence

$$\cdots \to S^1 \to \Omega(S^{2n+1}) \to \Omega(\mathbb{C}P^n) \xrightarrow{\partial} S^1 \to S^{2n+1} \to \mathbb{C}P^n.$$ 

Notice that $[\Omega(S^2), S^1] = H^1(\Omega(S^2), \mathbb{Z}) \cong \mathbb{Z}$ and $[\Omega(S^{r+1}), S^1] = 0$ for $r \geq 2$. Since the inclusion map $S^1 \hookrightarrow S^{2n+1}$ is null-homotopic and, by Corollary 3.4, the connecting map $\partial : \Omega(\mathbb{C}P^n) \to S^1$ is an $H$-map we get a split short exact sequence of groups

$$1 \to [\Omega(S^2), \Omega(S^{2n+1})] \xrightarrow{(\gamma_n \cdot c)^*} [\Omega(S^2), \mathbb{C}P^n] \to [\Omega(S^2), S^1] \cong \mathbb{Z} \to 0.$$

Furthermore, since $[\Omega(S^{r+1}), S^1] = 0$ for $r \geq 2$, the Barratt-Puppe sequence splits and we obtain isomorphisms

$$[\Omega(S^{r+1}), \Omega(\mathbb{C}P^n)] \xrightarrow{\cong} [\Omega(S^{r+1}), \Omega(S^{2n+1})] \quad \text{for } r \geq 2.$$ 

Now, Proposition 3.5 yields

$$\exp_p[\Omega(S^{r+1}), \Omega(\mathbb{C}P^n)] = \exp_p[\Omega(S^{r+1}), \Omega(S^{2n+1})]$$

for $n, r \geq 1$ and any prime $p$, and the proof is complete. \qed

Finally, we consider the quaternionic projective spaces $\mathbb{H}P^n$ with $n \geq 1$ and $\mathbb{H}P^\infty = \colim_n \mathbb{H}P^n$. Because the canonical inclusion map $i : \mathbb{H}P^1 = S^4 \hookrightarrow \mathbb{H}P^\infty$ is the classifying map of the Hopf fibration $\nu_4 : S^7 \to S^4$, there is a map of fibrations

$$
\begin{array}{ccc}
S^3 & \cong & S^3 \\
\downarrow & & \downarrow \\
S^7 & \cong & E\mathbb{H}S^3 \\
\downarrow & & \downarrow \\
S^4 & \xrightarrow{i} & \mathbb{H}P^\infty.
\end{array}
$$
This implies that the induced map $i_* : \pi_k(S^4) \to \pi_k(\mathbb{H}P^\infty)$ is surjective for $k \geq 1$. Since $\pi_k(S^4) \cong \pi_k(S^7) \oplus \Sigma \pi_{k-1}(S^3)$, we derive that the restriction $i_* : \Sigma \pi_{k-1}(S^3) \to \pi_k(\mathbb{H}P^\infty)$ is an isomorphism and there is a splitting short exact sequence

$$0 \to \pi_k(S^7) \to \pi_k(S^4) \to \pi_k(\mathbb{H}P^\infty) \to 0$$

for $k \geq 1$. Now, given $\alpha \in \pi_k(\mathbb{H}P^\infty)$, there is $\alpha' \in \pi_{k-1}(S^3)$ such that $\alpha = i_* \Sigma \alpha'$.

**Proposition 3.7.** (1) If $\alpha = i_* \Sigma \alpha' \in \pi_k(\mathbb{H}P^\infty)$ and $\beta = i_* \Sigma \beta' \in \pi_l(\mathbb{H}P^\infty)$ then $[\alpha, \beta] = i_* (\Sigma \nu^+ \circ \Sigma (\alpha' \wedge \beta'))$ and $12[\alpha, \beta] = 0$.

(2) If $\alpha = i_* \Sigma \alpha' \in \pi_k(\mathbb{H}P^\infty)$, $\beta = i_* \Sigma \beta' \in \pi_l(\mathbb{H}P^\infty)$ and $\gamma = i_* \Sigma \gamma' \in \pi_m(\mathbb{H}P^\infty)$ then $[[\alpha, \beta], \gamma] = \Sigma (\nu^+) \Sigma^4 (\nu^+) \circ \Sigma (\alpha' \wedge \beta' \wedge \gamma')$ and $3[[\alpha, \beta], \gamma] = 0$.

(3) If $\alpha_i = i_* \Sigma \alpha'_i \in \pi_{k_i}(\mathbb{H}P^\infty)$ with $i = 1, \ldots, m$ and $m \geq 4$ then all Whitehead products $[\alpha_1, \ldots, \alpha_m] = 0$.

**Proof.** (1): Given $\alpha \in \pi_k(\mathbb{H}P^\infty)$ and $\beta \in \pi_l(\mathbb{H}P^\infty)$ there are $\alpha' \in \pi_{k-1}(S^3)$ and $\beta' \in \pi_{l-1}(S^3)$ such that $\alpha = i_* \Sigma \alpha'$ and $\beta = i_* \Sigma \beta'$. Hence, by means of Proposition 3.6, we get $[\alpha, \beta] = [i_* \Sigma \alpha', i_* \Sigma \beta'] = i_* ([i_4, i_4] \circ \Sigma (\alpha' \wedge \beta'))$. Since, in view of [7] (1.20), we have $[i_4, i_4] = 2\nu_4 - \Sigma \nu^+$ for $\nu^+ = \nu' - \alpha_1(3)$, we conclude that

$$[\alpha, \beta] = i_* (\Sigma \nu^+ \circ \Sigma (\alpha' \wedge \beta')).$$

Since the order $|\Sigma \nu^+| = 12$, this implies that $12[\alpha, \beta] = 0$.

(2): An element $\gamma \in \pi_m(\mathbb{H}P^\infty)$ leads to $\gamma' \in \pi_{m-1}(S^3)$ with $\gamma = i_* \Sigma \gamma'$. Then, $[[\alpha, \beta], \gamma] = i_* (E \nu^+ \circ \Sigma (\alpha' \wedge \beta'), \Sigma \gamma') = i_* (\Sigma \nu^+, \nu_4) \circ \Sigma (\alpha' \wedge \beta' \wedge \gamma') = \Sigma (\nu^+) \Sigma^4 (\nu^+) \circ \Sigma (\alpha' \wedge \beta' \wedge \gamma')$.

But, $\nu^+ = \nu' - \alpha_1(3)$, and in view of [7] (1.25) and (1.28), we have $\Sigma^2 \nu' = 2\nu_5, \nu' \circ \nu_6 = 0$. Then, we conclude that

$$[[\alpha, \beta], \gamma] = i_* (\alpha(4) \alpha_1(7) \circ \Sigma (\alpha' \wedge \beta' \wedge \gamma')).$$

By [7] (1.8), the order $|\alpha(4) \alpha_1(7)| = 3$, this implies that $3[[\alpha, \beta], \gamma] = 0$.

(3): Finally, $\delta \in \pi_l(\mathbb{H}P^\infty)$ leads to $\delta' \in \pi_{l-1}(S^3)$ with $\delta = i_* \Sigma \delta'$. Then, [[[\alpha, \beta], \gamma], \delta] = i_* ([i_4, i_4] \circ \Sigma (\alpha' \wedge \beta' \wedge \gamma'), \Sigma \delta')) = i_* ([i_4, i_4] \circ \Sigma (\alpha' \wedge \beta' \wedge \gamma' \wedge \delta')) = i_* ([i_4, i_4] \circ \alpha_1(7) \alpha_1(10) \circ \Sigma (\alpha' \wedge \beta' \wedge \gamma' \wedge \delta'))$. Since, in view of [7] (1.8), we have $\alpha_1(7) \alpha_1(10) = 0$, we deduce that

$$[[[\alpha, \beta], \gamma], \delta] = 0.$$

From this we conclude that all Whitehead products $[\alpha_1, \ldots, \alpha_m] = 0$ of weight $m \geq 4$ for $\alpha_i \in \pi_{k_i}(\mathbb{H}P^\infty)$ with $i = 1, \ldots, m$ and the proof follows. \(\square\)

Now, for $Y = \mathbb{H}P^\infty$ we have
Theorem 3.8. If $r \geq 1$ then

\[ \exp_2 [\Omega(S^{r+1}), \Omega(\mathbb{H}P^\infty)] = \exp_2 [\Omega(S^{r+1}), S^3] \leq 2 \exp_2 (S^4) \]

and

\[ \exp_p [\Omega(S^{r+1}), \Omega(\mathbb{H}P^\infty)] = \exp_p [\Omega(S^{r+1}), S^3] = \exp_p (S^4) = p^3 \]

for any odd prime $p$.

Proof. Since $\mathbb{H}P^\infty = B^S S^3$, in view of Proposition 3.3, the connection map $\partial : \Omega(\mathbb{H}P^\infty) \to S^3$ associated with the fibration $S^3 \to ES^3 \to \mathbb{H}P^\infty$ leads to an isomorphism

\[ [\Omega(S^{r+1}), \Omega(\mathbb{H}P^\infty)] \cong [\Omega(S^{r+1}), S^3] \]

for $r \geq 1$ and consequently,

\[ \exp_p [\Omega(S^{r+1}), \Omega(\mathbb{H}P^\infty)] = \exp_p [\Omega(S^{r+1}), S^3] = \exp_p (S^4) = p^3 \]

In the rest of proof we mimic \textit{mutatis mutandis} the ideas of the proof of Theorem 2.1 and sketch below the main facts only.

Case (1): $p = 2$. In view of Proposition 3.7, we get that $4[\alpha, \beta] = 0$ for $\alpha \in \pi_k(\mathbb{H}P^\infty; 2)$ and $\beta \in \pi_l(\mathbb{H}P^\infty; 2)$. Further, all Whitehead products $[\alpha_1, \ldots, \alpha_m] = 0$ of weight $m \geq 3$ for $\alpha_i \in \pi_{k_i}(\mathbb{H}P^\infty; 2)$ with $i = 1, \ldots, m$.

Because, by means of [19] and the EHP sequence, we have $\exp_2 (S^4) = 2^t$ with $t > 1$, we conclude that

\[ \exp_2 [\Omega(S^{r+1}), \Omega(\mathbb{H}P^\infty)] \leq 2 \exp_2 (S^4) = 2^{t+1}. \]

Now let $p$ be an odd prime. Then, by (H.1) we have, $\exp_p (S^4) = p^3$.

Case (2): $p = 3$. In view of Proposition 3.7, we get that $3[\alpha, \beta] = 0$ for $\alpha \in \pi_k(\mathbb{H}P^\infty; 3)$ and $\beta \in \pi_l(\mathbb{H}P^\infty; 3)$. Further, $3[\alpha, \beta, \gamma] = 0$ for $\alpha \in \pi_k(\mathbb{H}P^\infty; 3)$, $\beta \in \pi_l(\mathbb{H}P^\infty; 3)$ and $\gamma \in \pi_l(\mathbb{H}P^\infty; 3)$. Certainly, all Whitehead products $[\alpha_1, \ldots, \alpha_m] = 0$ of weight $m \geq 4$ for $\alpha_i \in \pi_{k_i}(\mathbb{H}P^\infty; 3)$ with $i = 1, \ldots, m$. Because $\exp_3 (S^4) = 27$, we conclude that

\[ \exp_3 [\Omega(S^{r+1}), \Omega(\mathbb{H}P^\infty)] = \exp_3 (S^4) = 27. \]

Case (2): $p > 3$. In view of Proposition 3.7, we get that all Whitehead products $[\alpha_1, \ldots, \alpha_m] = 0$ of weight $m \geq 2$ for $\alpha_i \in \pi_{k_i}(\mathbb{H}P^\infty; p)$ with $i = 1, \ldots, m$. Hence, we conclude that

\[ \exp_p [\Omega(S^{r+1}), \Omega(\mathbb{H}P^\infty)] = \exp_p (S^4) = p^3 \]

and the proof is complete. \qed
Let $\gamma_{n,H}: S^{4n+3} \rightarrow \mathbb{HP}^n$ be the quotient map. Since $S^3 \hookrightarrow S^{4n+3} \xrightarrow{\gamma_{n,H}} \mathbb{HP}^n$ is a principal $S^3$-bundle, Corollary 3.4 and Proposition 3.5 lead to:

**Theorem 3.9.** The principal $S^3$-bundle $S^3 \hookrightarrow S^{4n+3} \xrightarrow{\gamma_{n,H}} \mathbb{HP}^n$ gives rise to a short exact sequence

$$1 \rightarrow [\Omega(S^{r+1}), \Omega(S^{4n+3})] \xrightarrow{\gamma_{n,H}^*} [\Omega(S^{r+1}), \Omega(\mathbb{HP}^n)] \rightarrow [\Omega(S^{r+1}), S^3] \rightarrow 1$$

of groups for $r \geq 1$. Consequently,

$$\exp_p[\Omega(S^{r+1}), \Omega(S^{2n+1})] \leq \exp_p[\Omega(S^{r+1}), \Omega(\mathbb{HP}^n)] \leq \exp_p[\Omega(S^{r+1}), \Omega(S^{2n+1})] \cdot \exp_p[\Omega(S^{r+1}), S^3]$$

for $n,r \geq 1$ and any prime $p$.

**Remark 3.10.** In Theorem 3.9, the bounds can be computed or estimated using Theorem 2.1 together with fact that $\exp_p[\Omega(S^{r+1}), S^3] = \exp_p(S^4) = p^3$ if $p > 2$. While we give bounds for $\exp_p[\Omega(S^{r+1}), \Omega(\mathbb{HP}^n)]$, the calculation of these exponents will be carried out for $n,r \geq 1$ and any prime $p$, in a forthcoming paper.

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