Montaldo, S.; Oniciuc, C.; Ratto, A.
On the second variation of the biharmonic Clifford torus in $S^4$. (English) [Zbl 07601373] 
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Summary: The flat torus $T = S^1 \left( \frac{1}{2} \right) \times S^1 \left( \frac{1}{2} \right)$ admits a proper biharmonic isometric immersion into the unit 4-dimensional sphere $S^4$ given by $\Phi = i \circ \varphi$, where $\varphi : T \to S^4 \left( \frac{1}{\sqrt{2}} \right)$ is the minimal Clifford torus and $i : S^1 \left( \frac{1}{\sqrt{2}} \right) \to S^4$ is the biharmonic small hypersphere. The first goal of this paper is to compute the biharmonic index and nullity of the proper biharmonic immersion $\Phi$. After, we shall study in the detail the kernel of the generalised Jacobi operator $I_2^\phi$. We shall prove that it contains a direction which admits a natural variation with vanishing first, second and third derivatives, and such that the fourth derivative is negative. In the second part of the paper, we shall analyse the specific contribution of the equivariant second variation $\tilde{\Phi}$. We shall prove that it contains a direction which admits a natural variation with vanishing first, second and third derivatives, and such that the fourth derivative is negative. In the second part of the paper, we shall analyse the specific contribution of $\Phi$ to the biharmonic index and nullity of $\Phi$. In this context, we shall study a more general composition $\Phi = i \circ \varphi$, where $\varphi : M^m \to S^{n-1} \left( \frac{1}{\sqrt{2}} \right)$, $m \geq 1$, $n \geq 3$, is a minimal immersion and $i : S^{n-1} \left( \frac{1}{\sqrt{2}} \right) \to S^n$ is the biharmonic small hypersphere. First, we shall determine a general sufficient condition which ensures that the second variation of $\Phi$ is nonnegatively defined on $\mathcal{C}(\tilde{\varphi}^{-1} T^{n-1} \left( \frac{1}{\sqrt{2}} \right))$. Then, we complete this type of analysis on our Clifford torus and, as a complementary result, we obtain the $p$-harmonic index and nullity of $\varphi$. In the final section, we compare our general results with those which can be deduced from the study of the equivariant second variation.

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53C43 Differential geometric aspects of harmonic maps

Keywords:
biharmonic immersions; second variation; index; nullity

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