Topological T-duality For Bundles Of Strongly Self-Absorbing $C^*$-Algebras And Some Physical Applications

ASHWIN S. PANDE

We extend the $C^*$-algebraic formalism of Topological T-duality to section algebras of locally trivial bundles of strongly self-absorbing $C^*$-algebras and to a larger class of String Theoretic dualities. We argue that physically this corresponds extending Topological T-duality to Flux Backgrounds of Type II String Theory which possess topologically nontrivial sourceless Ramond-Ramond flux. We demonstrate a map in $K$-theory for the $C^*$-algebras involved in both sides of this generalized duality. We calculate a few examples. We discuss the physical relevance of the above formalism in some detail, in particular, we argue that the above formalism models String Theoretic tree-level dualities found in such Flux Backgrounds such as Fermionic T-duality and Timelike T-duality.

1. Introduction

1.1. Type II T-duality and Continuous-Trace $C^*$-algebras

Consider Type IIA or Type IIB String Theory propagating on a spacetime background $X$. Suppose there is a free circle action $\alpha : S^1 \times X \rightarrow X$ on $X$ and $W = X/S^1$. Then the quotient map $p : X \rightarrow W$ gives $X$ the structure of a principal circle bundle over a base space $W$.

Type IIA and Type IIB String Theories propagating on these backgrounds with sourceless $H$-flux possess a symmetry termed ‘T-duality’ (see Ref. [1] for a good introduction). Roughly speaking, Type IIA String Theory on the spacetime background with underlying topological space $X$ and $H$-flux $[H] \in H^3(X, \mathbb{Z})$ is exactly equivalent to Type IIB String theory on another ‘dual’ spacetime background with underlying dual topological space $X^\#$ with dual $\tilde{H}$-flux $[\tilde{H}] \in H^3(X^\#, \mathbb{Z})$. It can be shown that the dual spacetime $X^\#$ has a dual circle action $\alpha^\# : S^1 \times X^\# \rightarrow X^\#$. In addition $X^\#$ is also a principal circle bundle $p^\# : X^\# \rightarrow W$ due to the dual quotient map by the dual circle action $\alpha^\#$. 

1
Topological T-duality is an attempt to understand the T-duality symmetry of Type II String Theory on the target spacetime $X$ while retaining only the purely topological information about the string theory. (It is unclear as to whether a similar approach will work for the other known types of String Theories). In Topological T-duality, the only information about type II string theory that we retain are the topological quantities associated to the zero energy modes of Type II (A or B) String Theory propagating on the background $X$. It is surprising that just these topological quantities alone are enough to construct a mathematically rigorous, purely topological theory of Type II T-duality termed Topological T-duality (see the monograph Ref. [1] by J. Rosenberg for a detailed exposition).

In Type II String Theory propagating on a spacetime background $X$, the massless modes in the the $NS-NS$ sector are the $B$–field—which corresponds to the data of a smooth gerbe with connection on $X$–the metric and dilaton.

In topological T-duality ([1]) only the following purely topological information about the background $X$ are retained: The characteristic class of the curvature $H$ of the $B$–field together with the characteristic class of the principal circle bundle $p : X \to W$. The characteristic class of the curvature $H$ is an integral three-form on $X$ and we identify it with a class $[H] \in H^3(X, \mathbb{Z})$ in the following. The characteristic class of the principal circle bundle $p : X \to W$ is naturally an element $[p] \in H^2(W, \mathbb{Z})$.

It is remarkable that with such sparse information about the string background, the formalism of Topological T-duality always correctly calculates the topological type of the String Theoretic T-dual spacetime (see Ref. [1, 2] and references therein).

1.2. Ramond-Ramond Fields

Type II String Theory on a spacetime background $X$ possesses excitations termed Ramond-Ramond fields whose field strengths are differential $p$–form valued fields (denoted $G_p$) on $X$. It is well known that Type IIA String Theory has even-dimensional forms

$$G_0, \ldots, G_{2k}, \ldots$$

as Ramond-Ramond field strengths, while Type IIB String theory has odd-dimensional forms

$$G_1, \ldots, G_{2k+1}, \ldots$$

as Ramond-Ramond field strengths.
We usually write the total field strength $G$ of the Ramond-Ramond fields as an inhomogenous sum of field strength forms

$$G = \sum_{i=0}^{5} G_{2i}$$

for Type IIA String Theory and

$$G = \sum_{i=1}^{4} G_{2i+1}$$

for Type IIB String Theory (see Ref. [3] Sec. (1)).

For any value of $p$ above, the field strength $G_p$ is the curvature of a gauge potential $C_{p-1}$, i.e.,

$$G_p = dC_{p-1}.$$

$G_p$ has a gauge invariance

$$C_{p-1} \rightarrow C_{p-1} + dB_{p-2}$$

with $B_{p-2}$ a $(p-2)$-form gauge field (The $(p-2)$–form field $B_{p-2}$ should not to be confused with the Kalb-Ramond field $B$ which is always a 2–form field). $D$–branes are sources of Ramond-Ramond fields.

The above is only an approximation, since it is known (see Ref. [4] Sec. (2), discussion around Eq. (2.19)) that Ramond-Ramond fields may be described by a hierarchy of successively finer descriptions.

At the most basic level, the Ramond-Ramond field strengths $G_i$ on $X$ are closed forms on $X$ with the gauge symmetry described above and should define natural classes in the de Rham cohomology of $X$. Most of these forms correspond to integral cohomology classes, but it may be argued from anomaly cancellation arguments (see Ref. [4]) that some of these classes are nonintegral. Hence, this level of description is incomplete.

At a finer level of description, in Ref. [4] Moore and Witten argued that Ramond-Ramond fields define natural classes in the $K$–theory group of $X$, and that the above field strength forms $G_i$ may be obtained from the associated $K$–theory class by the Minasian-Moore formula (see [4] Eq. (2.17)).

At a still finer level, Freed and Hopkins (see Ref. [5]) argued that Ramond-Ramond fields on $X$ define natural classes in what is termed the differential $K$–theory group of the spacetime background $X$ (see also Ref. [3] Sec. (1) for an overview).

We now give a brief description of the Minasian-Moore formula and its generalizations for flux backgrounds: If the spacetime has no background
sourceless $B$–field Witten and Moore (see Ref. [4,5]) showed that Ramond-Ramond fields naturally define elements of the $K$–theory groups of spacetime. Witten and Moore argued in Ref. [4] that for a Ramond-Ramond field represented by a class $x \in K(X)$ the characteristic class $\omega$ of the Ramond-Ramond field associated to $x$ must be equal to the image of $x$ in the de Rham cohomology of $X$ via the Chern character, i.e., $\text{ch}(x) = [\omega]$. Note that $\omega$ is a sum of differential forms of varying degrees. (For the connection of the above with the differential $K$–theory group of $X$, see Eqs. (10,11) of Ref. [5].)

The inhomogeneous differential form $G$ representing the Ramond-Ramond field is obtained from $\omega$ above by the Minasian-Moore formula

$$ (1.1) \quad G = \sqrt{\hat{A}(X)} \omega. $$

where, as above, $G = \sum G_i$ where $G_i$ are the even or odd dimensional Ramond-Ramond field strength forms $G_i$ on $X$ above. Here, $\hat{A}(X)$ is the $\hat{A}$–genus of $X$. While $\omega$ is an integral form, from the above formula, $G$ (since its obtained by multiplying $\omega$ by the form $\sqrt{\hat{A}(X)}$ which might give rise to nonintegral forms) might not be integral (see Ref. [4]). Thus the Minasian-Moore formula gives an explanation of the nonintegral forms in $G$ which were obtained from anomaly cancellation arguments.

If the spacetime has a background $B$–field with or without $H$–flux, it has been argued that by various authors (see Refs. [3,6,7]) that a generalization of the above argument should hold— in particular, we expect the $B$–field to ‘twist’ various quantities. In Ref. [8], Moore and Saulina show that for a flux background with a $B$–field without $H$–flux there is an analogue of the Minasian-Moore formula twisted by the $B$–field:

$$ (1.2) \quad G(x) = e^{B_2} \text{ch}(X)\sqrt{(A)(X)} $$

When the background $B$–field possesses a nontrivial $H$–flux then Mathai and Sati in Ref. [6] and Brodzki et al in Ref. [7] argue that the Ramond-Ramond fields satisfy a twisted Bianchi identity and are no longer closed under $d$ but vanish under a ‘twisted differential’ $d_H = d + H_3$. The authors define a $H$–twisted de Rham cohomology $H^*_H(X)$ using this twisted differential. It can also be shown (see Ref. [3] and references therein), that there is a twisted Chern character map $\text{ch}_H : K_H^*(X) \to H^*_H(X)$ where $K_H^*(X)$ is the $K$–theory of $X$ twisted by the closed, integral three form $[H]$ naturally defined by the $H$–flux $H$ in $H^3(X, \mathbb{Z})$. 
The authors argue that Ramond-Ramond fields define a class \( y \) in \( K^{*}_{H}(X) \) and the characteristic class of the Ramond-Ramond field in this flux background is the image of \( y \) via the twisted Chern character \( \text{ch}_{H}: K^{*}_{H}(X) \rightarrow H^{*}_{H}(X) \). In addition the authors argue that we should have \( \text{ch}_{H}(y) = [\omega] \) with a twisted version of the Minasian-Moore formula

\[
G = \sqrt{\hat{A}(X)\omega}
\]

(For the generalization of the Minasian-Moore formula to twisted versions of differential \( K \)-theory see Ref. [3] discussion around Eqs. (1.7,1.8) and references therein).

Brodzki et al in Ref. [7] define a noncommutative version of the above for spacetime backgrounds which are ‘noncommutative manifolds’, and in particular, derive a noncommutative Minasian-Moore formula. In Ref. [9], Sec. (4.4) the authors show that the above twisted Minasian-Moore formula for backgrounds with sourceless \( H \)-flux agrees with the result in the previous paragraph for noncommutative manifolds which are described by continuous-trace \( C^{*} \)-algebras with spectrum \( X \) (These were used to describe spacetime backgrounds with sourceless \( H \)-flux in Ref. [1, 2]).

In the rest of this paper, we argue that for spacetimes with a background sourceless \( H \)-flux and background sourceless Ramond-Ramond flux, the background Ramond-Ramond fields should be represented by a class \( z \in K_{T_{w}}(X) \) the generalized twisted \( K \)-theory of \( X \) in the sense of Ref. [10]. The characteristic classes of the Ramond-Ramond fields above should be related to the image of \( z \) via a generalized twisted Chern character \( \text{ch}_{T_{w}} \) taking values in the generalized twisted cohomology of \( X \).

We use the results of Refs. [11, 12] to identify these characteristic classes with the characteristic classes of generalized twisted gerbes on the space \( X \) and further identify these with the generalized Dixmier-Douady characteristic classes of the \( C^{*} \)-algebras of Ref. [11, 12]. We study the effect of T-duality on these backgrounds. We define an extension of the \( C^{*} \)-algebraic formalism of Topological T-duality to these \( C^{*} \)-algebras in Secs. [2, 3] below. We study some physical examples of this in Sec. [4] below.

We further argue below that the noncommutative Minasian-Moore formula of Brodzki et al in Ref. [7] should describe some of the backgrounds in this paper in Subsec. [2,4].
1.3. String Theory in Flux Backgrounds

In this section we discuss Type II String Theory in backgrounds with sourceless $H-$flux or Ramond-Ramond flux—as mentioned in the previous section these are termed **Flux Backgrounds**.

It was originally suspected that Type II String Theory was inconsistent in backgrounds with Ramond-Ramond flux. However, even though these backgrounds were suspected of being inconsistent, nevertheless they were studied using String Field Theory techniques beginning with the work of Bernstein and Leigh (see the references in Ref. [13]). Over the past few years, the remarkable work of Sen and collaborators (see Ref. [14]) has shown that Type II String Theory is stable in these backgrounds: Type II Superstring Field Theory equations of motion exist in these backgrounds, but can’t be obtained from an action. However, the authors obtain gauge invariant nonlocal 1PI equations of motion from String Field Theory and from this they obtain the full $S-$matrix of the String Theory including all quantum corrections.

$D-$branes in Flux backgrounds have also been studied (see Refs. [15] for example): For example, in Ref. [15], Cornalba et al argue that $D-$branes in flux backgrounds can be studied using String Field Theory. They derive a number of interesting results from this which we will discuss below.

In this paper we attempt to study $D-$brane charge in flux backgrounds by extending (or deforming) the $C^*$-algebraic formalism of Topological T-duality mentioned in the previous section.

Using a result of Cornalba et al. (in Ref. [15]), we argue that just as switching on the $H-$flux on a Type II String Background causes the spacetime to become noncommutative, similarly, switching on the Ramond-Ramond flux should cause the spacetime to become a noncommutative manifold in the sense of Brodzki et al (Ref. [7, 9, 16]). We claim that analogously with the formalism of Mathai and Rosenberg this noncommutative manifold may be described by the section algebra of a locally trivial $C^*$-bundle over the spacetime background with fiber a stabilized strongly self-absorbing $C^*$-algebra $A \otimes K$ (Note that the formalism of Mathai and Rosenberg is the case with fiber $\mathbb{C} \otimes K$). We propose that changes in the fiber $C^*$-algebra reflect changes in the String Theory phenomenology. We argue that a generalization the ‘Axiomatic T-duality’ of Sec. (3.1) of Ref. [16] should hold for such backgrounds. We conjecture that the $D-$brane charge formula of Ref. [9] should be valid for these backgrounds.

We define a generalized noncommutative Topological T-dual of the spaces discussed in this paper. We argue that the formalism we propose displays a generalization of the Axiomatic T-duality of Brodzki et al. We relate this
generalized Topological T-dual to two phenomenological Tree-Level dualities in these backgrounds—Fermionic T-duality and Timelike T-duality.

1.4. The $C^*$—algebraic formalism of Topological T-duality

In Ref. [2], Mathai and Rosenberg developed the theory of Topological T-duality using ideas from noncommutative geometry based on the crossed product of continuous-trace $C^*$—algebras by $\mathbb{R}^n$—actions.

Continuous-trace $C^*$—algebras are a very interesting class of $C^*$—algebras which have been studied for a long time. Briefly, a continuous-trace $C^*$—algebra $B$ over a compact metrizable space $X$ is a section algebra of a locally trivial $C^*$—bundle over $X$ with fiber the $C^*$—algebra of compact operators $K$ on a countably infinite dimensional, separable Hilbert space. The space $X$ may be recovered from $B$ as the spectrum of $B$. The set of isomorphism classes of continuous-trace algebras with spectrum $X$ forms a group under $C^0(X)$—balanced tensor product of continuous-trace algebras called the Brauer Group of $X$ denoted $\text{Br}(X)$. It can be proved (see Ref. [1] and references therein) that given a continuous-trace $C^*$—algebra $B$ with spectrum $X$, there is an isomorphism $\delta : \text{Br}(X) \simeq H^3(X, \mathbb{Z})$ and the image $\delta([B])$ of $[B] \in \text{Br}(X)$ under the above isomorphism is termed the Dixmier-Douady invariant of the continuous-trace $C^*$—algebra $B$. A continuous-trace $C^*$—algebra with spectrum $X$ and Dixmier-Douady invariant $\delta$ is denoted $CT(X, \delta)$.

It was argued in Ref. [2] that, from the point of view of Noncommutative Geometry, the presence of a $H$—flux $H$ with characteristic class $[H] \in H^3(X, \mathbb{Z})$ on the spacetime background $X$ corresponds to replacing $C_0(X)$, the algebra of functions on $X$ when there is no $H$—flux by a continuous-trace algebra $B$ with spectrum $X$ with the Dixmier-Douady invariant of $B$ equal to the $H$—flux $[H]$, $\delta([B]) = [H]$—see Ref. [1] for details. Thus, we may say that turning on a $H$—flux with characteristic class $[H]$ causes us to replace $C_0(X)$ by $B = CT(X, [H])$.

The space $X$ possesses a circle action (as explained in Subsec. [1.1]). The Topological T-dual that Mathai and Rosenberg define (see Ref. [2]) depends on lifting the circle action $\alpha$ on $X$ to an action of $\mathbb{R}$ on $A$ covering the circle action on $X$.

It can be shown (see Ref. [1] [2] [18]) that the $S^1$—action $\alpha$ on $X$ lifts to a unique equivalence class (termed exterior equivalence class, see Sec. [2]) of $\mathbb{R}$—actions $\gamma$ on $A$ each of which induces the given $S^1$—action on $X$. This question of existence and uniqueness of lifts of group actions up to
exterior equivalence is part of the study of the crossed product of continuous-
trace $C^*-$algebras which is a vast and well-developed body of work, see for
example Refs. [1] [17] [18] for an overview of the subject.

Mathai and Rosenberg argue that the T-duality operation on $X$ is mod-
elled well by the crossed product of $A$ by the natural lift of the circle action
on $X$ to an $\mathbb{R}-$action $\gamma$ on $A$. More precisely, the T-dual space to $X$ (termed
$X^#$) is given by the spectrum of the $C^*-$algebra $A^#$ which is the crossed
product $C^*-$algebra of $A$ by the $\mathbb{R}-$action $\gamma$, i.e., $A^# \simeq A \rtimes \mathbb{R}$. Varying $\gamma$
in its exterior equivalence class gives isomorphic crossed products.

Note that the topological type of the spectrum of $A$ may be described
in terms of the topological data $[p], [H]$ above associated to the Type II
String Theory background with underlying topological space $X$. Similarly,
the topological type of the spectrum of $A^#$ may be described in terms of
the topological data $[p^#], [H^#]$ associated to the dual Type II String Theory
background with underlying topological space $X^#$.

The $C^*$-algebraic approach to Topological T-duality (see Refs. [1] [2])
postulates that a spacetime background $X$ with a free circle action and a
sourceless $H-$flux $[H]$ (denoted $(X, [H])$ in what follows) can be associ-
ated to an isomorphism class of $C^*-$dynamical systems $[A, \alpha]$. Here $A$
is a continuous-trace algebra with spectrum $X$ and Dixmier-Douady invariant
$[H]$ and $\alpha$ is a lift of the circle action on $X$ to a $\mathbb{R}-$action on $A$. In this
theory, the Dixmier-Douady invariant of the continuous-trace algebra $A$ is
fixed to be equal to $[H]$, the class in $H^3(X, \mathbb{Z})$ induced by the sourceless
$H-$flux $H$.

This is natural physically because of the following argument: The charges
of the $D-$branes in the string background $X$ in the absence of $H-$flux lie in
the the topological $K-$theory group of spacetime. It is conjectured (see Ref. [1] Sec. (4.2.08) ) that if a sourceless $H-$flux $H$ were switched on in
the spacetime $X$, the $D-$brane charges would lie in the twisted topological
$K-$theory group $K_H(X)$ where the magnitude of the twist is given by the
value of the $H-$flux.

In this theory, the $K-$theory of the continuous-trace $C^*-$algebra $A$ is
also a receptacle for $D$-brane charges: The twisted $K$-theory groups of the
background (which are conjectured to be the charge group of $D-$branes
in backgrounds with sourceless $H-$flux) are isomorphic to the operator $K$-
theory groups of a continuous-trace $C^*$-algebra on that background (see for
example, Ref. [1] Sec. (4.2) and references therein).

Further, it was proved in Ref. [19] that the $C^*$-dynamical system associ-
ated to a principal bundle $p : W \times S^1 \to W$ with $H-$flux by the formalism
of Ref. [2] may also be naturally constructed from the data of the string theory background.

Thus, this association of a continuous-trace $C^*$—algebra and associated $C^*$—dynamical system to the spacetime background $(X, [H])$ is physically very natural since it is constructed from both nonperturbative data—the $D$—brane charge group and perturbative data—the equivariant gerbe on spacetime whose gerbe curvature is the characteristic class $[H]$ of the sourceless $H$—flux.

It was observed in Ref. [2] that with the above setup, the T-duality operation corresponds to the crossed-product construction in $C^*$—algebra theory and the underlying topological space to the T-dual is the spectrum of $A \rtimes \mathbb{R}$. Thus, the formalism of Ref. [2] associates the $C^*$—dynamical system $[A \rtimes \mathbb{R}, \alpha]$ to the T-dual of $[A, \alpha]$.

Under T-duality, $D$—branes map by a change of degree—i.e. branes of even degrees go to branes of odd degrees. This phenomenon is visible in the above formalism as explained in Ref. [2]. The mapping of $D$—brane charges under T-duality (see Ref. [1]) was observed to correspond to the Connes-Thom isomorphism in $C^*$—algebra theory $\phi : K_\bullet(A) \to K_{\bullet+1}(A \rtimes \mathbb{R})$.

The formalism of Mathai and Rosenberg has been directly tested in many examples (see Ref. [2] and references therein; see also Ref. [1]). It remarkable that the formalism has always been found to correctly give the underlying topological space of the String Theoretic T-dual spacetime.

1.5. Topological T-duality for Flux Backgrounds

We would like to extend Topological T-duality to the study of dualities in String Theory on backgrounds with sourceless Ramond-Ramond Flux and $H$—flux. To do this, we need to use a generalization of notion of a continuous-trace $C^*$—algebra over $X$ to the notion of a section algebra of a locally trivial bundle of strongly self-absorbing $C^*$—algebras over $X$.

A $C^*$—algebra $A$ is said to be strongly self-absorbing if it is separable unital and there is a $*$—isomorphism $\psi : A \to A \otimes A$ such that $\psi$ is approximately unitarily equivalent to $l : A \to A \otimes A, l(a) = a \otimes 1$, i.e., there is a sequence of unitaries $v_n \in A \otimes A$ such that $\forall x \in A$,

$$||v_n \psi(x) v_n - (x \otimes 1)|| \to 0$$

as $n \to \infty$. These algebras are important in the Elliott programme and arise naturally there in an attempt to classify isomorphism classes of $C^*$—algebras using invariants from $C^*$—algebraic $K$—theory.
There are six known types of strongly self-absorbing $C^*$—algebras. These are listed in Table (1)—Note that $\mathcal{W}$, the Razak-Jacelon algebra (see Ref. [20]) is nonunital and hence does not fit the above definition of a strongly self-absorbing $C^*$—algebra but will be used to calculate an example later.

Table 1: This is a list of all known strongly self-absorbing $C^*$—algebras adapted from Ref. [20], note that $\mathcal{W}$ is not unital.

| Remarks | Stably finite | Purely Infinite | Remarks |
|---------|---------------|-----------------|---------|
| Nawata et al Ref. [21] | $\mathcal{W}$ | $\mathcal{O}_2$ | $\mathcal{O}_\infty$—absorbing |
| Described Below | $M_p\infty$ | $M_p\infty \otimes \mathcal{O}_\infty$ | |
| Sato et al Ref. [22] | $\mathcal{Z}$ | $\mathcal{O}_\infty$ | |

From Ref. [23], every strongly self-absorbing $C^*$—algebra is either stably finite or purely infinite.

First we note the following: A $C^*$—algebra $\mathcal{D}$ as above is said to be $\mathcal{B}$—absorbing for some strongly self-absorbing $C^*$—algebra $\mathcal{B}$, if $\mathcal{D} \otimes \mathcal{B} \simeq \mathcal{D}$. Note that if $\mathcal{A}$ is $\mathcal{B}$—absorbing, so is $\mathcal{D} \in \text{Bun}_X(\mathcal{A} \otimes \mathcal{K})$. In particular, every strongly self-absorbing $C^*$—algebra $\mathcal{A}$ is always Jiang-Su absorbing, so $\mathcal{D}$ above is always $\mathcal{Z}$—absorbing.

There are other absorption results which depend on the specific choice of $\mathcal{A}$. For example, if $\mathcal{A} = \mathcal{O}_2$, we have $\mathcal{O}_2 \otimes M_2\infty \simeq \mathcal{O}_2$ and so every $\mathcal{D}$ in $\text{Bun}_X(\mathcal{O}_2 \otimes \mathcal{K})$ is $M_2\infty$—absorbing.

Dadarlat and Pennig in Ref. [24] considered section algebras of locally trivial fiber bundles with fiber a (stabilized) strongly self-absorbing $C^*$—algebra $\mathcal{A} \otimes \mathcal{K}$ over a compact metrizable space $X$. The isomorphism classes of these $C^*$—algebras over $X$ are denoted $\text{Bun}_X(\mathcal{A} \otimes \mathcal{K})$. In Ref. [24], Thm. (3.8), the authors show that there is a generalized cohomology theory on the category of $CW$—complexes $E^*_\mathcal{A}(X)$ such that

$$E^1(X) = \text{Bun}_X(\mathcal{A} \otimes \mathcal{K}).$$

In addition, in Cor. (4.3) of Ref. [24], the authors calculate $\text{Bun}_X(\mathcal{A} \otimes \mathcal{K})$ for a finite connected $CW$—complex $X$ using the Atiyah-Hirzebruch spectral sequence for $E^*_\mathcal{A}(X)$ when $\mathcal{A} \neq \mathbb{C}$ and $\mathcal{A}$ satisfies the UCT. For such spaces $X$ and $C^*$—algebras $\mathcal{A}$, the authors obtain that

$$\text{Bun}_X(\mathcal{A} \otimes \mathcal{K}) \simeq E^1(\mathcal{X}) \simeq H^1(X, \mathbb{R}_+^\infty) \times \Pi_{k \geq 1} H^{2k+1}(X, \mathbb{R})$$

(1.5)
Note that if $A \simeq \mathbb{C}$, $\text{Bun}_X(\mathbb{C} \otimes \mathcal{K}) \simeq \text{Br}(X) \simeq H^3(X, \mathbb{Z})$ the Brauer Group of Morita equivalence classes of continuous-trace $C^*$-algebras with spectrum $X$.

In Sec. (4) of this paper we claim that $X$ should naturally be viewed as the underlying topological space of a Type II String Theory background with various sourceless fluxes turned on—these are termed flux backgrounds in String Theory. These sourceless fluxes are zero modes of the Type II theory’s spectrum like the $H$-flux or the Ramond-Ramond Flux.

The case $A = \mathbb{C}$ corresponds to the association of a continuous-trace $C^*$-algebra to a Type II String Theory background with a sourceless $H$-flux as described in Refs. [1, 2].

When $A \neq \mathbb{C}$ however, we claim the resulting $C^*$-algebra $D$ still describes a spacetime background, but we should associate locally trivial bundles with fibers of the form $A \otimes \mathcal{K}$ with $A$ as above to spacetime backgrounds with both $H$-flux and $RR$-flux.

We show below that the formalism of Topological T-duality for continuous-trace $C^*$-algebras can be generalized from $A \simeq \mathbb{C}$ to any self-absorbing $C^*$-algebra $A$. One difficulty with this generalization is the lack of a theory of $\mathbb{R}$-actions on the above class of $C^*$-algebras. Hence, the structure of the crossed product by even relatively simple groups such as $\mathbb{R}$ cannot be determined. For continuous-trace $C^*$-algebras there is a satisfactory theory of $\mathbb{R}$-actions which lift circle actions on the spectrum: The structure of the crossed-product $C^*$-algebra by these $\mathbb{R}$-actions is known precisely, and it is always clear when the crossed-product $C^*$-algebra is a continuous-trace $C^*$-algebra at least for elementary group actions such as $\mathbb{R}^n$ (see Refs. [1, 2, 18]). This is not the case for the class of $C^*$-algebras studied in this paper.

We outline this theory in the rest of this paper. We discuss the lifting of circle actions on $X$ to $\mathbb{R}$-actions on $D$ in Sec. (2) below and show that unique ‘Rokhlin’ lifts (up to closed cocycle conjugacy—a generalization of exterior equivalence) of the circle action on $X$ to $\mathbb{R}$-actions on $D$ exist for purely infinite strongly self-absorbing fibers $A$—namely the Cuntz algebras $\mathcal{O}_2, \mathcal{O}_\infty$ and the tensor products of $\mathcal{O}_\infty$ with UHF algebras of infinite type $M_p \otimes \mathcal{O}_\infty$.

When $A$ is stably finite that is, $A$ is any of the three remaining strongly self-absorbing $C^*$-algebras—the Razak-Jacelon algebra $W$, the UHF-algebras of infinite type $M_p \otimes \mathcal{O}_\infty$ or the Jiang-Su algebra $\mathcal{Z}$ there does not seem to be a unique lift of the circle action on $X$ to a $\mathbb{R}$-action on $D$.

We define a generalization of the theory of Topological T-duality for both these classes of $C^*$-algebras in Sec. (2) below and argue that this theory gives the expected T-dual in the continuous-trace case i.e., when $A \simeq \mathbb{C}$.
We calculate the Topological T-dual of several spaces in Sec. (3) below. We discuss the implications for String Theory and $D$–brane charge in Flux backgrounds and also calculate some examples of T-duals using the above in Sec. (4) below. We end the paper with our conclusions in Sec. (5).

2. Lifts of circle actions

In this section we argue about possible lifts of circle actions on the total space of a principal circle bundle $p : X \to W$ to $\mathbb{R}$–actions on $D \in \text{Bun}_X(A \otimes K)$. Let $X$ be a topological space associated to a finite-dimensional manifold. Suppose $p : X \to W$ was a principal circle bundle. Consider $C_0(X)$-algebras $D$ which are locally trivial $C^*$-bundles over $X$ with fiber a strongly self-absorbing $C^*$-algebra $A \otimes K$ as in Ref. [24]. As in Ref. [24], Thm. (A), we denote the isomorphism classes of these $C^*$-algebra bundles over $X$ by $\text{Bun}_X(A \otimes K)$.

In Subsec. (2.1) we calculate the structure of $\text{Prim}(D)$ for $D$ above. In Subsecs. (2.2, 2.3) we briefly discuss Rokhlin $\mathbb{R}$–actions and Rokhlin $\mathbb{Z}$– and $S^1$–actions on the $C^*$–algebras $D$ above. In Subsec. (2.4) we discuss the lifting of circle actions on $X$ to $\mathbb{R}$–actions on $C^*$–algebras $D$ above with purely infinite fiber and Topological T-duality for these $C^*$–algebras. In Subsec. (2.5) we discuss the lifting of circle actions on $X$ to $\mathbb{R}$–actions on $C^*$–algebras $D$ above with stably finite fibers and Topological T-duality for these $C^*$–algebras. In Subsec. (2.6) we discuss maps on $K$–theory induced by the constructions of Subsecs. (2.4, 2.5). In Subsec. (2.7) we discuss some generalizations of the above.

2.1. Structure of $\text{Prim}(D)$

In this paper, to a $C^*$–algebra like $D$ as above, we associate the topological space $\text{Prim}(D)$ as the underlying spacetime background of Type II String theory with background $H$– and $RR$–flux. Also if $\text{Prim}(D)$ does not exist, we view the spacetime background as the noncommutative space given by $D$.

In this section, we calculate $\text{Prim}(D)$ for $D \in \text{Bun}_X(A \otimes K)$. The proof below has been adapted from Prop. (5.36) of I. Raeburn and D. Williams Ref. [25]. The proof there is for continuous-trace $C^*$–algebras and has been generalized to locally trivial $C^*$–bundles with fiber a strongly self-absorbing $C^*$–algebra.

**Theorem 2.1.** Let $A, X, D$ be as at the beginning of this section. For $D \in \text{Bun}_X(A \otimes K)$, $\text{Prim}(D) \simeq X$. 

12
Proof. First note that $\mathcal{A}, \mathcal{K}$ are separable and simple and $\mathcal{K}$ is nuclear. By Thm. (B.45) (c) of Ref. [25], $\text{Prim}(\mathcal{A} \otimes \mathcal{K}) \simeq \text{Prim}(\mathcal{A}) \times \text{Prim}(\mathcal{K}) \simeq \{\ast\}$. Hence, the kernel of any irreducible representation of $\mathcal{A} \otimes \mathcal{K}$ must be $\{0\}$.

By Ref. [24], $\mathcal{D}$ is a $C_0(X)$-algebra, hence, there is an embedding $C_0(X) \to ZM(D)$. If $\pi$ is an irreducible representation of $\mathcal{D}$ on a Hilbert space $\mathcal{H}$, let $\overline{\pi}$ denote the extension of $\pi$ to $M(D)$. Restrict $\overline{\pi}$ to $C_0(X) \subset M(D)$. By the above, $\pi(f)\pi(a) = \pi(fa) = \pi(a)\pi(f)$ and $\pi(f)$ commutes with $\pi(a)$ for every $a \in D$. Since $\pi$ is irreducible, this implies that $\pi(f) \in \mathcal{C}1_{\mathcal{H}}$. Thus, we have an irreducible representation of $C_0(X)$ on $\mathcal{H}$ which implies that there is a $t \in X$ such that $\pi(fa) = f(t)\pi(a), \forall f \in C_0(X), a \in D$.

Let $I_t$ denote the set of sections of the $C^*$-bundle $\mathcal{D}$ which vanish at $t \in X$. Then, by Ex. (A.23) of Ref. [25], second paragraph, we can argue that $I_t \subset \ker(\pi)$. Also, since $a \mapsto a(t)$ induces an isomorphism of $\mathcal{D}/I_t$ onto $\mathcal{D}|_{\{t\}}$, $\pi$ factors through a representation $\tilde{\pi}$ of $\mathcal{D}|_{\{t\}}$. Since $\mathcal{D}|_{\{t\}} \simeq \mathcal{A} \otimes \mathcal{K}$, and $\text{Prim}(\mathcal{A} \otimes \mathcal{K}) \simeq \{0\}$, by the argument at the beginning of the proof, we have that $\ker(\tilde{\pi}) \simeq \{0\}$. Hence, $\ker(\pi) = I_t$.

If $\eta$ is any irreducible representation of $\mathcal{D}|_{\{t\}} \simeq \mathcal{A} \otimes \mathcal{K}$ the evaluation map which sends a section $s \in \mathcal{D}$ to $\eta(s(t))$ gives a representation of $\mathcal{D}$ with kernel $I_t$.

Let $p : Y \to X$ be a bundle with fiber $\mathcal{A} \otimes \mathcal{K}(\mathcal{H})$ and structure group $\text{Aut}(\mathcal{A} \otimes \mathcal{K}(\mathcal{H}))$ over $X$. By Ref. [24], $\mathcal{D}$ is the section algebra of this bundle. Now, the proof of Prop. (4.89) of Ref. [25], second paragraph, with $\mathcal{K}(\mathcal{H})$ replaced by $\mathcal{A} \otimes \mathcal{K}(\mathcal{H})$ shows that for each $y \in Y$, there is a section $s \in \mathcal{D}$ with $s(p(y)) = t$.

Thus, as in Ref. [25], Prop. (5.36), first paragraph, every element of each fiber is the value of some section. We can also multiply by elements of $C_0(X)$.

Hence, ideals $I_t$ corresponding to different $t \in X$ are distinct. Thus, $t \mapsto I_t$ is a bijection of $X$ onto $\text{Prim}(\mathcal{D})$.

Pick an open cover $\{W_\alpha\}$ of $X$ such that for every $W_\alpha$ there is a closed set $V_\alpha \supseteq W_\alpha$ such that Thm. (B) of Ref. [24] lets us pick a projection $p_\alpha \in \mathcal{D}|_{V_\alpha}$. By Thm. (B) of Ref. [24], $[p_\alpha] \in K_0(\mathcal{D}|_{V_\alpha})^\times$ hence, $p_\alpha$ is not zero.

It is enough to show that the map $t \mapsto I_t$ is a homeomorphism. The proof is similar to the one in the last paragraph of Ex. (A.23) of Ref. [25]. It is enough to show that

$$\{I_t : t \in N\} = \{I_t : t \in \overline{N}\}$$

for every subset $N$ of $X$. By definition, the left hand side has the form $\{I_t : t \in M\}$ where, (by Def. (A.19) of Ref. [25]), $M = \{s : I_s \supset \bigcap_{t \in N} I_t\}$, and we have to prove $M = \overline{N}$. Since $\mathcal{D}$ is the algebra of continuous sections
of a locally trivial $C^*$-bundle over $X$, for any $a \in \mathcal{D}$, $a(x) = 0$ for $x \in N$ implies that $a(x) = 0$ for $x \in \overline{N}$ so $\overline{N} \subset M$ by definition of $M$.

If $s \notin \overline{N}$, there exists $f \in C_0(N)$, such that $f|_{\overline{N}} = 0$ and $f(s) = 1$. Pick an $\alpha$ such that $s \in W_\alpha$. Then, by the above, there exists a nonzero projection $p_\alpha \in D|_{V_\alpha}$. Now, $fp_\alpha$ is in $I_t$ for all $t \in \overline{N}$ but not in $I_s$ which implies that $s \notin M$. Thus, $M = \overline{N}$. □

2.2. Introduction to Rokhlin $\mathbb{R}$—Actions

We begin by defining exterior equivalence of two group actions on a $C^*$—algebra and a related notion called cocycle conjugacy in this Subsection. Then we illustrate the idea of a Rokhlin $\mathbb{R}$—action by examining Rokhlin $\mathbb{R}$—actions on a unital $C^*$—algebra. We indicate the generalization of this definition to nonunital $C^*$—algebras.

In this paper we will need to consider lifts of circle actions on $X \cong \text{Prim}(\mathcal{D})$ to what are termed Rokhlin $\mathbb{R}$— or $S^1$—actions on $\mathcal{D}$. We would also like to consider Rokhlin $\mathbb{Z}$—actions on $\mathcal{D}$.

We first recall the notion of exterior equivalent actions and define cocycle conjugate actions:

\textbf{Def 2.1.} Let $\mathcal{D}, \mathcal{D}'$ be two section algebras of locally trivial bundles of strongly self-absorbing $C^*$—algebras with the same primitive spectrum $X$.

- Two actions $\alpha, \alpha'$ of a second countable, locally compact group $G$ on $\mathcal{D}$ are exterior equivalent if there is a continuous map $u : G \to UM(\mathcal{D})$, the group of unitary multipliers in the multiplier algebra of $\mathcal{D}$, with $u_{gh} = u_g \alpha_g(u_h)$ such that $\alpha'_g = \text{Ad} u_g \circ \alpha_g$.

- Two actions $\alpha, \beta$ of a second countable, locally compact group $G$ on $\mathcal{D}$ and $\mathcal{D}'$ are cocycle conjugate if there is an $\alpha$—cocycle $u$ i.e., a continuous map $u : G \to UM(\mathcal{D})$, the group of unitary multipliers in the multiplier algebra of $\mathcal{D}$, with $u_{gh} = u_g \alpha_g(u_h)$ such that $\alpha'_g = \text{Ad} u_g \circ \alpha_g$ is another action. In addition there must be an isomorphism $\phi : \mathcal{D} \to \mathcal{D}'$ such that $\alpha'_g = \phi^{-1} \circ \beta_g \circ \phi, \forall g \in G$.

We argue below that to obtain a T-dual we require a lift of the circle action on $X$ to a $\mathbb{R}$—action on $\mathcal{D}$ or to a $S^1$—action on $\mathcal{D}$. 
Note that if \( A \simeq \mathbb{C} \), that is, \( D \) is a continuous-trace \( C^* \)-algebra the circle action on the spectrum of \( D \) lifts to a unique \( \mathbb{R} \)-action (up to exterior equivalence) on \( D \) but not to a nontrivial circle action on \( D \) (see Ref. [18]), however, this is not the case for general \( D \) in \( \text{Bun}_X(A \otimes K) \) with \( A \neq \mathbb{C} \).

We cannot guarantee a unique lift of circle action on \( X \) to a \( \mathbb{R} \)-action on \( D \) without further conditions on the lifted action. Therefore, instead of lifting the circle action on \( X \) to an arbitrary \( \mathbb{R} \)-action, we lift it to a specific type of action termed a Rokhlin action which we explain below.

It is suspected that there is no way to classify group actions on \( A \)-absorbing \( C^* \)-algebras with any sensible notion of equivalence (including cocycle conjugacy above) unless the actions possess the Rokhlin property. That is a unique (in the sense of cocycle conjugacy) lift of the circle action on \( X \) to a \( \mathbb{R} \)-action on \( D \) might not exist unless the lifted \( \mathbb{R} \)-action possesses the Rokhlin property (see [26] and [27] for more details).

Also for general \( D \), the fibers of \( D \) are not \( K \) but can be any stabilized self-absorbing \( C^* \)-algebra and the lifting problem is difficult to handle without further conditions on the fiber algebra \( A \otimes K \). We divide the lifting problem into two cases, those with \( A \) purely infinite and those with \( A \) stably finite.

Due to this, in this paper we do the following:

1) We principally restrict ourselves to Rokhlin actions of groups on \( C^* \)-algebras and Rokhlin lifts of group actions on topological spaces to \( C^* \)-algebras \( D \) in \( \text{Bun}_X(A \otimes K) \) as far as possible (see Refs. [27–29]).

2) In the absence of Rokhlin actions, in the following we also use actions possessing a weakening of the Rokhlin property called the tracial Rokhlin property (see Ref. [30] and references therein).

3) If Rokhlin actions or actions with the tracial Rokhlin property cannot be proved to exist on \( D \), we will also use another generalization of Rokhlin actions (see Refs. [28, 31, 32] and references therein) to actions with a finite Rokhlin dimension- a Rokhlin action has Rokhlin dimension zero, Rokhlin actions with finite Rokhlin dimension are a generalization of actions with Rokhlin dimension zero.

4) If none of these are possible for a given situation, we will use a natural group action on \( D \), but we will point out that the action is not Rokhlin.

The subject of Rokhlin actions on \( C^* \)-algebras is vast (see Refs. [20–29, 33] and references therein) and cannot be discussed here in any detail. We give below the simplest example of a Rokhlin \( \mathbb{R} \)-action on a unital \( C^* \)-algebra and refer the reader to the literature for more details and for generalizations.
The Rokhlin property for $\mathbb{R}$–actions on a separable, unital $C^*$–algebra (also termed Rokhlin flows) was first defined by Kishimoto in Ref. [27].

**Def 2.2.** Let $\alpha$ be a flow on a separable, unital $C^*$-algebra $A$. We say that $\alpha$ has the Rokhlin property, if for every $p > 0$, there exists an approximately central sequence of unitaries $u_n$ in $A$ satisfying

$$
\lim_{n \to \infty} \max_{|t| \leq 1} ||\alpha_t(u_n) - e^{ipt}u_n|| = 0.
$$

(2.1)

However, the $C^*$–algebras in this paper are always nonunital since $K$ is nonunital and hence section algebras of locally trivial $C^*$–algebra bundles with fiber $A \otimes K$ are nonunital. This is not a problem, since there is a generalization of the idea of a Rokhlin flow to a flow on an arbitrary separable (not necessarily unital) $C^*$–algebra in Ref. [26]-see paragraph before Def. (1.8) of Ref. [26]. We refer the interested reader to that reference.

We suggest two analogies which could give a physical interpretation of a Rokhlin $\mathbb{R}$–action above.

- Suppose that $X \simeq \text{pt}$ so that $D \simeq A \otimes K$. Assume that $A$ was the $C^*$–algebra of observables associated to a quantum field theoretic system, and $\alpha_t$ was the time evolution induced by the Hamiltonian. In particular assume that $A$ was faithfully represented on the Fock space of a quantum field theoretic system. The above definition would then imply that for some interval of time around $t = 0$, for every value of the one-particle momentum $p$, there exist excitations in the system described by $u_n, n = 1, 2, \ldots$ for every value of $p$ which are ‘approximate eigenstates’ of the Hamiltonian and of all other observables for a finite time. In addition, it is clear that these excitations propagate as free particles, at least for finite time. Such excitations are well known in Quantum Field Theory.

- Alternatively, $X$ could be a background for a Type II String Theory. Also, suppose the $\mathbb{R}$–action on $D$ covered the circle action on $X = \text{Prim}(D)$ as above and the $C^*$–algebra $D$ was associated (by some means) to the algebra of observables associated to Strings propagating on $X$. Then the Rokhlin condition would imply that in the Fock space there are orthonormal states which are approximate eigenstates of every operator in $D$ (since $b$ and $u_n$ approximately commute for every $b \in D$) which are nearly invariant under the translation action on $D$. These should correspond to states of wound strings, at least in
the limit \( n \to \infty \). It would be interesting to calculate this from String Theory.

### 2.3. Rokhlin \( \mathbb{Z} \)– and \( S^1 \)–actions

In this subsection we discuss \( \mathbb{Z} \)– and \( S^1 \)–actions on \( \mathcal{D} \). We will use these in the next subsection, SubSec. 2.

If there is no Rokhlin lift of the circle action on \( X \) to a Rokhlin \( \mathbb{R} \)–action on \( \mathcal{D} \in \text{Bun}_X(\mathcal{A} \otimes \mathcal{K}) \) then, we argue below that, in addition to the Rokhlin \( \mathbb{R} \)–actions discussed above, we should consider Rokhlin circle actions and Rokhlin \( \mathbb{Z} \)–actions or Rokhlin automorphisms and some crossed product \( C^* \)–algebras by these actions.

By the above discussion, these actions would have to be on not necessarily unital \( C^* \)–algebras. The reader is referred to Refs. [35, 36] for details on Rokhlin circle actions and crossed products by Rokhlin circle actions on such \( C^* \)–algebras which are relevant to this paper. Note that Ref. [32] studies Rokhlin circle actions on \( \sigma \)–unital \( C^* \)–algebras which applies here since \( \mathcal{D} \) is separable.

We will also need some information about Rokhlin automorphisms on not necessarily unital \( C^* \)–algebras which are studied in Ref. [31]. The Lemma below discusses the existence of Rokhlin \( \mathbb{Z} \)–actions (also called Rokhlin automorphisms or Rokhlin automorphisms of Rokhlin dimension 0) on the \( C^* \)–algebras \( \mathcal{D} \) above.

In Ref. [28], Hirshberg, Zacharias and Winter introduce the idea of Rokhlin dimension for \( \mathbb{Z} \)–actions on \( C^* \)–algebras. Rokhlin automorphisms discussed above are considered to be \( \mathbb{Z} \)–actions with Rokhlin dimension 0.

There is a generalization of Rokhlin Automorphisms in to automorphisms which don’t have the Rokhlin property termed Rokhlin \( \mathbb{Z} \)–actions of dimension greater than 0 in Ref. [31]. The results in Ref. [31] only guarantee the existence of Rokhlin \( \mathbb{Z} \)–actions of dimension \( \leq 1 \) for arbitrary strongly self absorbing \( \mathcal{A} \). Since crossed products by Rokhlin automorphisms of dimension greater than 0 are understood, if there are no results on Rokhlin automorphisms of dimension 0, we will take a crossed product by a Rokhlin automorphism of dimension \( \geq 0 \).

However, if \( \mathcal{A} \) absorbs the universal UHF-algebra there is a stronger result regarding Rokhlin automorphisms of dimension 0.

In the following Lemma we prove that there are Rokhlin \( \mathbb{Z} \)–actions on \( \mathcal{D} \) using the results of Ref. [31] (Note that the results cited to prove Lemma 2.2 don’t need unitality of the \( C^* \)–algebra \( \mathcal{D} \)).
Lemma 2.2. Let $\mathcal{A}$ be a strongly self absorbing $C^*$–algebra. Let $X$ be a compact topological space and $\mathcal{D}$ a section algebra of a locally trivial $C^*$–bundle over $X$ with fiber $\mathcal{A} \otimes K$.

1) For any $C^*$–algebra $\mathcal{D}$ as above, Rokhlin $\mathbb{Z}$–actions on $\mathcal{D}$ with Rokhlin dimension $\leq 1$ are generic in the set of $\mathbb{Z}$–actions on $\mathcal{D}$ (in the sense of Remark (10.2) of Ref. [31]).

2) If $\mathcal{A}$ absorbs the universal UHF algebra, then Rokhlin automorphisms of $\mathcal{D}$ are generic in the set of $\mathbb{Z}$–actions on $\mathcal{D}$ (in the sense of Remark (10.2) of Ref. [31]).

Proof.

1) Since $\mathcal{A}$ is strongly self absorbing, it is Jiang-Su absorbing. Since $\mathcal{D}$ is a section algebra of a $C^*$–bundle over $X$ with fiber $\mathcal{A} \otimes K$ it is clear that $\mathcal{D}$ is Jiang-Su absorbing as well. By Thm. (10.14) and Remark (10.15) of Ref. (31), since $\mathbb{Z}$ is a finitely generated, countable, discrete, residually finite group and $\mathcal{D}$ is separable, Rokhlin $\mathbb{Z}$–actions are generic in the set of all actions of $\mathbb{Z}$ on $\mathcal{D}$ in the sense of Def. (10.1) and Remark (10.2) of Ref. (31).

2) If $\mathcal{A}$ absorbs the universal UHF algebra, then by an argument similar to the previous part $\mathcal{D}$ absorbs the universal UHF algebra as well. By Thm. (10.10) and Remark (10.15) of Ref. (31) actions with Rokhlin dimension 0 are generic in the set of actions of $\mathbb{Z}$ on $\mathcal{D}$ in the sense of Def. (10.1) and Remark (10.2) of Ref. (31).

□

2.4. Topological T-duality for $\mathcal{D}$ with purely infinite fibers

In this section we prove that there are unique lifts of the circle action on $X$ to $\mathbb{R}$–actions on $\mathcal{D}$ for fiber algebras $\mathcal{A}$ which are purely infinite, that is $\mathcal{A}$ is isomorphic to one of the two Cuntz Algebras $\mathcal{O}_2$ or $\mathcal{O}_\infty$ or $\mathcal{A}$ is isomorphic to $\mathcal{M}_\infty \otimes \mathcal{O}_\infty$–a tensor product of an infinite UHF-algebra with the infinite Cuntz algebra. This extends the formalism of Topological T-duality for continuous-trace $C^*$–algebras to these $C^*$–algebras.

We would like to define a theory of Topological T-duality similar to the theory of Topological T-duality for continuous-trace $C^*$–algebras in Ref. [1] for $C^*$–algebras like $\mathcal{D}$ in the previous subsection.

We had argued in Thm. (2.1) above that $\text{Prim}(\mathcal{D}) \simeq X$. Thus, as in the theory of Topological T-duality for continuous-trace algebras, we may try...
to lift the circle action on $X$ to a $\mathbb{R}$–action on $\mathcal{D}$ covering the circle action on $\text{Prim}(\mathcal{D}) \simeq X$. The T-dual should be given by the crossed product $\mathcal{D}$ by the $\mathbb{R}$–action. Note that if there is such a lift as the above, it would have a stabilizer, since the $\mathbb{R}$–action covers the circle action on $X$. However, crossed products of $\mathcal{D}$ by $\mathbb{R}$–actions with stabilizers have not been studied directly.

Currently, it is not clear if $S^1$–actions on $\text{Prim}(\mathcal{D})$ lift to unique Rokhlin $\mathbb{R}$–actions on $\mathcal{D}$ with stabilizers for every self-absorbing $C^*$–algebra $\mathcal{A}$.

We point out below that for purely infinite $\mathcal{A}$ a theorem of Szabo gives a unique lift and hence a well-defined T-dual $C^*$–algebra. In the next section, we prescribe a way to obtain a T-dual $C^*$–algebra even if we cannot find a lift of the circle action on $\text{Prim}(\mathcal{D})$.

Let $\mathcal{D}, X$ be as above and let $\mathcal{A}$ be purely infinite. Then, $\mathcal{A}$ is isomorphic to $\mathcal{O}_2, \mathcal{O}_\infty$ or $M_p \otimes \mathcal{O}_\infty$ and $\mathcal{A}$ is $\mathcal{O}_\infty$–absorbing. Then, Thm. (B) of Szabo, Ref. [26] shows that if there is a Rokhlin lift of an $\mathbb{R}$–action on $\text{Prim}(\mathcal{D})$, this lift will be unique up to cocycle conjugacy. Thus, in this case, if we can find a Rokhlin lift, the circle action on $\text{Prim}(\mathcal{D})$ should lift to a unique Rokhlin $\mathbb{R}$–action on $\mathcal{D}$ up to cocycle conjugacy.

Thus, we would expect a well-defined theory of Topological T-duality to be possible for the $C^*$–algebras $\mathcal{D}$ above with purely infinite fibers.

In particular for this class of $C^*$–algebra we would always have a unique ‘T-duality diamond’ in Eq. (2.2) with $\text{Prim}(\mathcal{D}) \simeq X$:

\[
\begin{array}{ccc}
\mathcal{C} & \simeq & \mathcal{D} \times Z \\
\alpha & \quad & \quad \beta \\
\mathcal{D} & \leftarrow & \leftarrow \mathcal{B} & \simeq & \mathcal{D} \times S^1 \\
\downarrow & \quad & \quad \downarrow & \quad & \quad \downarrow
\end{array}
\]

\[
D^\# \simeq \mathcal{D} \times \mathbb{R} \\
\gamma
\]

(2.2)

Taking Prim of each $C^*$–algebra in the above diagram gives a diamond diagram of spaces (see Eq. (2.3) ) as in ordinary $C^*$–algebraic T-duality (I).
It is not clear that $\mathcal{D}^#$ is a section algebra of a locally trivial bundle of the form $\mathcal{A} \otimes K$ with $\mathcal{A}$ strongly self-absorbing. Due to this it is not clear that $\text{Prim}(\mathcal{D}^#)$ is a nontrivial topological space. However, we may always view it as a noncommutative space. It is clear that a principal circle bundle $\pi : X \to Y$ T-dualizes to a space with a circle action $\pi^# : X^# \to Y^#$. Locally by the Mackey machine it is clear that the T-dual should be of the form $U \times S^1$ where $U \subseteq Y^#$. However, this might not be true globally and there might not be a T-dual topological space.

By the Connes-Thom isomorphism, we will obtain a degree-reversing isomorphism in $K^{-}\text{theory}$ between the $K^{-}\text{theory}$ of $\mathcal{D}$ and the $K^{-}\text{theory}$ of the crossed product $C^\ast$-algebra $\mathcal{D} \rtimes \mathbb{R}$.

Physically, this can be interpreted as a degree-reversing isomorphism between the twisted $K^{-}\text{theory}$ of $\mathcal{X}$ and the $K^{-}\text{theory}$ of Topological T-dual $C^\ast$-algebra $\mathcal{D}^#$. If $\mathcal{D}^#$ is of the form of section algebra of a locally trivial bundle of $C^\ast$-algebras of the form $\mathcal{A} \otimes K$, with $\mathcal{A}$ strongly self-absorbing above, then we would obtain an isomorphism between the corresponding twisted $K^{-}\text{theories}$ on either side. In this paper, we interpret this isomorphism as corresponding to the mapping of $D$-brane charge on both sides of the duality by the T-duality transformation as in the case of continuous-trace $C^\ast$-algebras (see Refs. [1, 2] for details).

Further, as described in Ref. [7], it is possible to define a sensible notion of $D$-brane charge for $D$-branes on spaces which are ‘noncommutative manifolds’. In particular, the $D$-brane charge was found to be given by an explicit formula Eq. (6.20) of Ref. [7] which is a generalization of the Minasian-Moore formula for $D$-brane charge to noncommutative spacetimes.

The above class of $C^\ast$-bundles together with their crossed products by $\mathbb{R}$-actions satisfy the condition for being noncommutative manifolds and the above Topological T-duality theory for the $C^\ast$-algebras $\mathcal{D}$ with purely infinite fibers should be an example of an axiomatic T-duality in the sense of Ref. [16] provided one could define the $KK^{-}\text{equivalence}$ required by that definition for these $C^\ast$-algebras $\mathcal{D}$. It is interesting to conjecture that the
formula for the charge of $D$–branes on spaces described by $D$ above should be given by a formula like Eq. (6.20) of Ref. [7]. Physically for $A \simeq O_2$ this would correspond to a 'noncommutative Minasian-Moore formula' for $D$–branes in non-anticommutative spacetimes with $H$–flux.

Unfortunately the only other physical example should be with fiber $A \simeq \mathbb{M}_{2\infty}$ and for these fiber algebras $D$ is stably finite and, as we will see below in Subsec. (2.5) the crossed product construction will not work for these $C^*$–algebras.

Note that Mahanta (Ref. [34]) has defined a theory of Topological T-duality for $O_{\infty}$–absorbing $C^*$–algebras using the theory of noncommutative motives and the $KK$–theory definition of Topological T-duality. It is interesting to ask whether this theory is related to the theory given above?

### 2.5. Topological T-dual for stably finite fibers

In this section we assume that $D$ has stably finite fibers, i.e. $A$ is of the form $M_{p\infty}$ a UHF-algebra of infinite type or $\mathcal{Z}$ the Jiang-Su algebra. Our remarks should also apply to the case when $A$ is the Razak-Jacelon algebra $\mathcal{W}$.

In the case of stably finite fibers we distinguish two cases depending on the characteristic class of $D$– torsion and non-torsion characteristic class:

1) $\delta(D)$ torsion: When $A$ is stably finite, $X$ is a finite CW–complex and $D$ has torsion characteristic class in $\text{Bun}_X(A \otimes K)$, we will show (see the end of Subsec. (3.2) below) that we may calculate the $T$–dual easily for a restricted class of $\mathbb{R}$–actions.

2) $\delta(D)$ non-torsion: When $A$ is stably finite, there is no way, at present, of guaranteeing that there is a lift or that a given lift is unique. As a there is no way to pick a unique lift of the circle action on $X$ to a $\mathbb{R}$–action, there is no way to guarantee that the T-dual is unique.

However the procedure described in Subsec. (2.5) below lets one pick a unique $\mathbb{R}$–action and a unique T-dual for stably finite $C^*$–algebras possibly with some extra data.

Extending Topological T-duality to locally trivial bundles of purely infinite strongly self-absorbing $C^*$–algebras without arbitrary choices is possible due to Szabo's theorem as discussed above.

I propose that Topological T-duality be extended to locally trivial bundles of stably finite strongly self-absorbing $C^*$–algebras by the following method–which I argue gives a unique, possibly noncommutative T-dual.

As in the previous section, let $p: X \to B$ be a principal circle bundle. Let $D \in \text{Bun}_X(A \otimes K)$ where $A$ is a strongly self-absorbing $C^*$–algebra.
We use the following procedure to find the T-dual when we cannot find an $R$–action covering the circle action on $X$.

1) We suppose given a spectrum-preserving Rokhlin automorphism $\alpha$ of $D$ which commutes with the translation action of the circle. We pick one automorphism in a given equivalence class (usually cocycle conjugacy), perhaps by fixing some datum describing the equivalence class. For example, in some cases, for tracial Rokhlin actions (see Ref. [37] for example), the values of the trace describe the collection of cocycle conjugacy classes.

2) We calculate $B \simeq D \rtimes_{\alpha} \mathbb{Z}$ and take $B$ as the (possibly noncommutative) $C^*$–algebra associated to the correspondence space.

3) The circle action on $X$ gives rise to a circle action $\beta$ on $B$.

4) We take the ‘T-dual’ as the $C^*$–algebra $B \rtimes_{\beta} S^1$ which is the ‘quotient of $B$ by $\beta$’.

The above argument covers all the known stably finite strongly self-absorbing $C^*$–algebras. Thus, we would expect the above procedure to always give a unique T-dual once we fix the equivalence class of the original actions. For example, in certain cases, for actions possessing the tracial Rokhlin property the trace of the automorphism would parametrize such equivalence classes. In such cases we attach the value of the trace of the automorphism as a parameter in the duality. We do not know the physical significance of this at present.

In the above procedure, it is clear that in the above we are calculating the crossed product by a specific $R$–action, namely the $R$–action on $\text{Ind}_{\mathbb{Z}}^R(D \otimes K)$. It is not clear at present that this action is a Rokhlin $R$–action or that there is a unique $R$–action covering the circle action on $X$ in every case. Indeed, we will prove below in a specific example [1] that such an action cannot be Rokhlin.

We propose that the above procedure be used to define the Topological T-dual when a unique lift of the circle action on $X$ to a $R$–action on $D \in \text{Bun}_X(A \otimes K)$ is not available. This ‘Topological T-dual’ will be unique up to varying the cocycle conjugacy class of the $Z$–automorphism and the cocycle conjugacy class of the circle action commuting with it by the above argument.

We now show that this ‘Topological T-dual’ is actually a crossed product $C^*$–algebra of a known type: Let $p : X \to B$ be a principal circle bundle and let $D \in \text{Bun}_X(A \otimes K)$. Let $\alpha$ be a Rokhlin spectrum-fixing $Z$–action on $D$. Let $\delta$ be a lift of the circle action on $X$ to a (not necessarily Rokhlin) circle action on $D$.
Let $H_\alpha$ be $D$ with the usual Hilbert $C^*$-module structure given by twisting by $\alpha$. Let $O_D(H_\alpha)$ be the Cuntz-Pimsner algebra associated to $D$ by the Hilbert $C^*$-module $H_\alpha$.

Note that the circle action $\delta$ lift to a circle action $\zeta$ on $H_\alpha$ (see Ref. [38] and references therein) By Ref. [38], $\zeta$ gives a natural circle action $\gamma$ on the Cuntz-Pimsner algebra $O_D(H_\alpha)$.

We can now calculate the T-dual we would obtain from $D$ if we were to follow the procedure outlined in Subsec. (2.5). It is strange that we can calculate the T-dual explicitly, but it turns out to be a Cuntz-Pimsner algebra naturally associated to $D$.

**Theorem 2.3.**

1) The T-dual of $D$ using the procedure of Subsec. (2.5) is a crossed product of $O_D(H_\alpha)$ by the circle action $\gamma$ and is isomorphic to

\[
O_D(H_\alpha) \rtimes \gamma S^1 \simeq O_D(H_\alpha) \rtimes \zeta S^1
\]

This T-dual does not depend on the cocycle conjugacy class of the $\mathbb{Z}-$action $\alpha$ on $D$.

2) The crossed product of $D$ by a $\mathbb{R}-$action $\beta$ lifting the circle action on $X$ is given by the result in the previous part if $\alpha \simeq \beta|\mathbb{Z}$ and $\delta$ is trivial.

**Proof.** 1) By a result of Pimsner ([39]), the crossed product $D \rtimes \alpha \mathbb{Z} \simeq O_D(H_\alpha)$. Hence, the construction of Subsec. (2.5) gives the T-duality diamond in Eq. (2.5) below

\[
\begin{align*}
C & \simeq D \rtimes \alpha \mathbb{Z} \simeq O_D(H_\alpha) \\
\mathcal{H}_\alpha & \longrightarrow D \\
\uparrow & \downarrow \\
B & \simeq D \rtimes \delta S^1 \\
\uparrow & \downarrow \\
\mathcal{H}_\alpha \rtimes \zeta S^1
\end{align*}
\]

\[
\begin{align*}
\mathcal{D} & \simeq O_D(H_\alpha) \rtimes \gamma S^1 \simeq O_D(H_\alpha) \rtimes \zeta S^1 \\
\mathcal{D}^\# & \simeq O_D(H_\alpha) \rtimes \gamma S^1 \simeq O_D(H_\alpha) \rtimes \zeta S^1
\end{align*}
\]
Since $S^1$ is amenable, by the Hao-Ng Theorem (see Ref. [38] page 6 and Ref. [40] Thm. (2.10))

$$O_D(H_\alpha) \rtimes S^1 \simeq O_D \rtimes S^1(\beta \rtimes \zeta)$$ \hspace{1cm} (2.6)

Note that the T-dual by the above procedure can only depend on the trace conjugacy class of $\alpha$ and on the cocycle conjugacy class of the circle action $\delta$ on $D$.

2) This follows from the previous part since the $\mathbb{R}$ action would give the T-duality diamond-like Diagram (2.5) above because, as discussed in Sec. (2.5), the maps in the above diagram are just the maps induced by induction in stages [41] for a $\mathbb{R}$ action on $D$. Due to this, $\beta$ must restrict to $\alpha$ on $\mathbb{Z} \rightarrow \mathbb{R}$. We can view the circle action $\delta$ as a $\mathbb{R}$ action $\hat{\delta}$ on $D$ by composing $\mathbb{R} \rightarrow \mathbb{R}/\mathbb{Z}$ with $\delta : \mathbb{R}/\mathbb{Z} \rightarrow \text{Aut}(D)$. Then, $\hat{\delta} \circ \beta^{-1}$ is a spectrum-fixing $\mathbb{R}$ action on $D$. 

\[ \square \]

In the above Theorem, the circle action $\delta$ need not be Rokhlin, however Prop. (5.6) of Ref. [42] shows that the $\mathbb{R}$ action $\hat{\delta}$ is Rokhlin if the circle action $\delta$ is Rokhlin.

Note that the above construction is reversible and taking the crossed product of $D\#$ by the correct dual actions will return the original $C^*$-algebra $D$. Thus the construction is involutory as a duality should be.

**Note:** Let $X$ be a point then, then $D \simeq A \otimes K$ with $A$ stably finite. We would like to T-dualize these $C^*$-algebras using the method outlined in this Subsection. For this, we note the following results about automorphisms of stabilized stably finite strongly self-absorbing $C^*$-algebras

- $A \otimes K \simeq \mathcal{W} \otimes K$: In Ref. [21], Nawata has defined a Rokhlin property for automorphisms of the stabilized Razak-Jacelon algebra. This algebra possesses a unique trace $\tau$. The author terms an automorphism $\alpha$ to be trace scaling if there exists an invariant-the trace scaling factor $\lambda(\alpha)$—such that $\lambda(\alpha)\tau(a) = \tau(\alpha(a))$, $\forall a \in \mathcal{W} \otimes K$. He shows that two trace scaling Rokhlin automorphisms $\alpha$ and $\beta$ (i.e. two Rokhlin $\mathbb{Z}$ actions) on $\mathcal{W} \otimes K$ are outer conjugate to each other if and only if $\lambda(\alpha) = \lambda(\beta)$.
- $A \otimes K \simeq M_{p^\infty} \otimes K$: In Ref. [37], Elliott, Evans and Kishimoto classified trace scaling automorphisms of stabilized UHF algebras and showed that all automorphisms $\alpha$ of a stabilized UHF algebra with the same trace scaling factor $s(\alpha) \neq 1$ are outer conjugate. They also determined
the structure of the crossed product of $M_r \otimes K$ by $\alpha$ for some supernatural number $r^\infty$.

- In Ref. [22], Sato has defined a weak Rokhlin property for automorphisms of the Jiang-Su algebra and shows that $\mathbb{Z}$-actions on $\mathbb{Z}$ with this property are all outer conjugate.

If two automorphisms are outer conjugate, the crossed product by these Rokhlin $\mathbb{Z}$-actions are isomorphic. For each of the above classes of stably finite strongly self-absorbing algebras, it follows that the isomorphism classes of the Topological T-dual of $\mathcal{A} \otimes K$ are classified by the trace. Thus, the isomorphism classes of the Topological T-dual of $\mathcal{A} \otimes K$ are classified by the trace as well.

Simple $C^*$-algebras like the Razak-Jacelon and UHF $C^*$-algebras above as correspond to 'noncommutative points'. Also, it is well known that the Cuntz algebra $\mathcal{O}_2$ should be viewed as a 'noncommutative circle'. We will discuss the T-dual of an element of $\mathcal{D} \in \mathcal{Bun}_{\mathbb{Z}}(M_2 \otimes K)$ in Sec. 3 below using the discussion above. We know that $\text{Prim}(\mathcal{D}) \simeq S^1$. We will see that in some cases, the Topological T-dual is a stabilized Cuntz algebra $\mathcal{O}_2 \otimes K$ which we view as a noncommutative circle.

### 2.6. Maps in $K$-theory

In this section we obtain natural maps between the $K$-theory of $\mathcal{D}$ and the $K$-theory of the proposed Topological T-dual $\mathcal{D}^#$ for $\mathcal{D}$ above with $\mathcal{A}$ purely infinite or stably finite. In this Section, we assume that $X, \mathcal{A}, \mathcal{D}, \mathcal{D}^#, \alpha$ and $\gamma$ are as in Thm. 2.3 above.

For Topological T-duality for $\mathcal{D}$ with $\mathcal{A}$ purely infinite, we can lift the circle action on $X$ to a $\mathbb{R}$-action on $\mathcal{D}$ and T-dualize in the sense of Mathai and Rosenberg. The T-dual of $(\mathcal{D}, \alpha)$ will be $(\mathcal{D}^#, \alpha^#)$ and $\mathcal{D}^# \simeq \mathcal{D} \rtimes \mathbb{R}$. As in Topological T-duality for continuous-trace $C^*$-algebras, there will be a degree-reversing isomorphism $K_*(\mathcal{D}) \to K_{*+1}(\mathcal{D}^#)$ induced by the crossed product.

If $\mathcal{D}^#$ is also a section algebra of a locally trivial bundle with fiber a stabilized strongly self-absorbing $C^*$-algebra, then the two $K$-theory groups on either side of the isomorphism correspond to higher twisted $K$-theory of $\text{Prim}(\mathcal{D})$ and $\text{Prim}(\mathcal{D}^#)$ respectively. In this case, we interpret the above isomorphism as an isomorphism on higher twisted $K$-theory of the two sides of the duality induced by Topological T-duality for $\mathcal{A}$ purely infinite.
Is there an isomorphism in $K$–theory for $D$ with stably finite fibers? We now argue that for stably finite fibers there is natural map between $K$–theory of the $C^*$–algebras $D, D^#$ above.

The $K$–theory of the crossed product $O_D(H_\alpha) \rtimes S^1$ which we suggested was the T-dual of $D$ with stably finite fibers in Thm. (2.3) above may be calculated using Theorem (A) of Ref. [43] and depends on the morphism $[H_\alpha]$ induced by $H_\alpha$ on $K_*(D)$.

**Theorem 2.4.** Assume $A, D, X$ and all $\mathbb{Z}$–, $S^1$– and $\mathbb{R}$–actions on them are as in Thm. (2.3)

1) There is a natural map $\phi$ between the $K$–theory of $D$ and the $K$–theory of the 'T-dual' $D^#$. 

2) In case the T-dual is given by crossed product by a Rokhlin $\mathbb{R}$–action lifting the circle action on $X$, $\phi$ is an isomorphism.

**Proof.**

1) By Theorem (A) of Ref. [43], there is a natural isomorphism

$$\lim_{\to}(K_*(D), [H_\alpha]) \cong K_*(O_D(H_\alpha) \rtimes S^1)$$

which is an isomorphism on the $K$–groups which intertwines the natural automorphism $[\hat{\gamma}]$ on the $K$–theory of the crossed product by the natural dual action $\hat{\gamma}$ on the crossed product with the automorphism $[H_\alpha]$ on $K_*(D)$.

We can compose this with the natural morphism $K_*(D) \to \lim_{\to}(K_*(D), [H_\alpha])$ to get the required map $\phi : K_*(D) \to K_*(O_D(H_\alpha) \rtimes S^1)$

2) It is not clear whether the map $\phi$ in the previous item is an isomorphism. However, since the construction in Thm. (2.3) reduces to the crossed product by induction by stages when the original $S^1$–action on $\text{Prim}(D)$ lifts to a Rokhlin action this map will be a degree reversing isomorphism in this case due to the Connes-Thom Isomorphism.

□

The $K$–theory of the $C^*$–algebra $D$ corresponds -see discussion in item (3) of Subsection 4.2 below-to a higher twisted $K$–theory of the space $X = \text{Prim}(D)$. We argue in that Subsection that the higher twisted $K$–theory associated to $D$ above is the $D$–brane charge associated to $D$–branes propagating on the background $X$ with background fluxes present.
When $D^\#$ is also a section algebra of a $C^*-\text{bundle}$ with fiber a strongly self-absorbing $C^*-\text{algebra}$ then we interpret $K_*(D^\#)$ as the higher twisted $K-\text{theory}$ as the $D-\text{brane}$ charge associated to $D-\text{branes}$ propagating on the background $X^\#$ with dual background fluxes present.

We interpret the above isomorphism in $K-\text{theory}$ (for purely infinite fibers) and the above map in $K-\text{theory}$ (for stably finite fibers) as a mapping of $D-\text{brane}$ charges from the original theory to the dual under a 'T-duality like' transformation.

### 2.7. Generalizations

The above formalism defines a T-dual $C^*-\text{algebra}$ for any $C^*-\text{bundle}$ $\mathcal{D}$ of the type defined above. It is interesting to relate this T-dual to the Axiomatic T-duality formalism of Ref. [16] Sec. (5.3).

If the above extension of T-duality was an example of Axiomatic T-duality then, the assignment of the T-dual $C^*-\text{algebra}$ $T(\mathcal{D}) \cong \mathcal{O}_\mathcal{D}(\mathcal{H}_\alpha) \rtimes \gamma S^1$ using the above procedure should give a covariant functor $\mathcal{D} \mapsto T(\mathcal{D})$.

1) If the circle action on $X$ lifts to a Rokhlin $\mathbb{R}-\text{action}$ on $\mathcal{D}$, it is clear that the crossed product by a Rokhlin action of $\mathbb{R}$ gives a covariant functor on the category of locally trivial $C^*-\text{bundles}$ $\mathcal{D}$ with fiber of the form $\mathcal{A} \otimes \mathcal{K}$ above together with Rokhlin $\mathbb{R}-\text{actions}$ and $\mathbb{R}-\text{equivariant}$ morphisms to the diagram category of diagrams of $C^*-\text{algebras}$ of the form in Fig. 2.5. This will be the case for $\mathcal{A}$ purely infinite.

2) If a Rokhlin lift does not exist as above, the procedure given above in Thm. (2.3) is also a covariant functor with the same target as in the previous item, provided one considers the source category the category of locally trivial $C^*-\text{bundles}$ $\mathcal{D}$ with fibers of the form $\mathcal{A} \otimes \mathcal{K}$ above with Rokhlin $\mathbb{Z}-\text{actions}$ and $\mathbb{Z}-\text{equivariant}$ morphisms—this would cause the Hilbert $C^*-\text{module}$ $\mathcal{H}_\alpha$ and the full diamond diagram in Thm. (2.3) to map functorially. Since Rokhlin $\mathbb{Z}-\text{actions}$ should exist when Rokhlin $\mathbb{R}-\text{actions}$ don’t—by the argument in Subsec. (2.5) above—one of these two should always be possible for this class of $C^*-\text{algebras}$.

However, it is not clear if there would be a natural element $\alpha_\mathcal{D}$ in $KK_*(\mathcal{D}, T(\mathcal{D}))$ which is a $KK-\text{equivalence}$ given by the above procedure possessing the properties required by the definition of Axiomatic T-duality. However, we still have the map $\phi$ between the $K-\text{theory}$ of $\mathcal{D}$ and $T(\mathcal{D})$ given by Thm. (2.4) above.
Also, it is interesting to point out in the above, that the Hilbert $C^*$—module need not be the module $H_\alpha$ used above. It is clear that the above T-duality diamond diagram will remain but, it is not clear what relation it would have to String Theory. It would be interesting to see if Hilbert $C^*$—modules apart from $H_\alpha$ above could describe some known phenomenon in string theory.

### 3. Mathematical Examples

#### 3.1. Introduction

Topological T-duality is an attempt to make a model of the T-duality symmetry of Type II string theory using techniques from noncommutative geometry and algebraic topology.

Table (2) summarizes our conjectures for the physical interpretation of $\text{Bun}_X(A \otimes \mathcal{K})$ above for varying $A$. Note the case $A \simeq \mathbb{C}$ corresponds to Topological T-duality for the Type II string theory with supersymmetry ignored. This table is based on the calculations in the following sections.

| Fiber algebra $(A)$ Actual Fiber is $(A \otimes \mathcal{K})$ | Type of T-duality | String Theory associated to T-duality |
|---------------------------------------------------------------|-------------------|--------------------------------------|
| $\mathbb{C}$                                                   | Circle or Torus action | Type II A and Type IIB |
|                                                               | Non-zero $H$—flux | |
|                                                               | Supersymmetry ignored | |
| **Stably Finite Fiber:**                                      |                   |                                      |
| $M_p \infty, \, p = 2$                                        | Supersymmetric Background, $RR$—flux or $H$—flux present | $R_{fiber} \gg l_s$ |
|                                                               | **Fermionic T-duality** | Type II and Type II$^*$ theories |
| $Z$                                                           | String Theories with Compactified Timelike directions i.e. Closed Timelike Loops Timelike T-duality | Background with nonstandard signature $(m, n)$ with $m > 1$ |
| $\mathcal{W}$                                                 | Unknown at present | |
| **Purely Infinite Fiber:**                                    |                   |                                      |
| $\mathcal{O}_2$                                               | Non-anti-commutative Superspace $N = 1/2$ SUSY | $\theta_a \theta_b + \theta_b \theta_a = \epsilon_{ab}$ |
|                                                               | **Fermionic T-duality** | Type II theory on NAC superspace |
| $\mathcal{O}_\infty$                                         | None at present | $R_{fiber} \ll l_s$ |
| $M_p \infty \otimes \mathcal{O}_\infty$                     | None at present | Type IIA, Type IIB string theories |

28
3.2. T-duality with $\mathcal{D}$ of torsion class

We now consider an interesting class of examples for which the Topological T-dual can be calculated exactly.

We study T-duality for $C^\ast$–algebras $\mathcal{D}$ in $\mathcal{Bun}_X(\mathcal{A} \otimes K)$ with $\mathcal{A}$ strongly self-absorbing and $X$ a finite $CW$–complex. In addition, we require that the characteristic class of $\mathcal{D}$, $\delta(\mathcal{D})$ -defined in Thm. (4.5) of Ref. [11]- be torsion, that is, the characteristic class lie in $\text{Tor}(\overline{E}_1^1(X))$ -see Ref. [12], Thm. (2.8) and Thm. (2.10).

By Ref. ([12]), this is equivalent to requiring that each of the rational characteristic classes $\delta_k(\mathcal{D})$ vanish and also requiring that there exist $n > 0$ such that $\mathcal{D} \otimes A \otimes K = C(X) \otimes A \otimes K$.

In addition, by Thm. (2.8)(iii) in Ref. [12], $\mathcal{D}$ is isomorphic to the stabilization of a unital, locally trivial continuous field of $C^\ast$–algebras $\mathcal{B}$ over $X$ with fiber $M_m(\mathcal{A})$ for some $m$. The $C^\ast$–algebra $\mathcal{B}$ is not unique but only unique up to the equivalence given in Thm. (2.9) of Ref. [12].

We consider a spacetime whose underlying topological space is homeomorphic to $\mathbb{R}^n \times X$ with $X$ a finite $CW$–complex. Let $X$ be a principal circle bundle over a base space $W \cong X/S^1$. Suppose we were trying to calculate the T-dual of $X$ with background fluxes whose characteristic classes were torsion. Then, by the above, we should consider $C^\ast$–algebras $\mathcal{D}$ in $\mathcal{Bun}_X(\mathcal{A} \otimes K)$ with torsion characteristic class. Note that $\mathcal{D} \simeq \mathcal{B} \otimes K$ so $\text{Prim}(\mathcal{D})$ and $\text{Prim}(\mathcal{B})$ are homeomorphic. Also, $K$–theory doesn’t change on stabilization, so $K_\ast(\mathcal{D}) \simeq K_\ast(\mathcal{B})$.

We may thus get some idea about the T-dual of $\mathcal{D}$ by considering crossed products of $\mathcal{B}$ by suitable $\mathbb{R}$–actions and stabilizing them. This gives us a large number of examples since $\mathcal{B}$ is unital unlike $\mathcal{D}$. By the above the T-dual $C^\ast$–algebra should be given by the crossed product by the lift of the circle action on $X$ to a $\mathbb{R}$–action when $\mathcal{A}$ is purely infinite. We now concentrate on the case when $\mathcal{A}$ is stably finite.

For Rokhlin $Z$–actions on $\mathcal{B}$, we use Cor. (5.14) and Cor. (5.16) of Ref. [30].

We now show that Rokhlin circle actions on $\mathcal{B}$ induce Rokhlin circle actions on $\mathcal{D}$.

**Lemma 3.1.** Let $X$ be a finite $CW$–complex. Suppose $\mathcal{B}, \mathcal{B}_1, \mathcal{B}_2$ are section algebras of locally trivial $C^\ast$–bundles over $X$ with fibers $M_m(\mathcal{A}), M_{m_i}(\mathcal{A}), i = 1, 2$ such that $\mathcal{B} \otimes K \simeq \mathcal{D}$ with $\mathcal{D}$ of torsion characteristic class and similarly for $\mathcal{B}_1, \mathcal{B}_2$. 

---

29
1) Every Rokhlin action of a circle on $B_1$ induces a Rokhlin action of the circle on $D_1$.

2) If $B \simeq B_1 \otimes B_2$ and $\alpha_1, \alpha_2$ are Rokhlin circle actions on $B_1$ and $B_2$ respectively, then $\alpha_1 \otimes \alpha_2$ is a Rokhlin circle action on $B$.

3) If $A \otimes \mathbb{O}_2 \simeq \mathbb{O}_2$, then the set of Rokhlin circle actions on $B$ are a dense $G_\delta$-set of the space of circle actions on $B$.

Proof. By assumption $D, D_i \in \mathcal{Bun}_X(A \otimes \mathcal{K})$ and the characteristic classes of $D, D_i, i = 1, 2$ are torsion.

1) By the above, $D \simeq B \otimes \mathcal{K}$. Also, $B$ is $\sigma$–unital since it is unital. Let $\alpha$ be a Rokhlin action of the circle on $B$. Let $\xi \mapsto \gamma_\xi$ be the translation action of $S^1$ on $L^2(S^1)$ It can be proved that the translation action lifts to a continuous map $\lambda : S^1 \to \mathcal{U}(L^2(S^1))$ such that $\gamma_\xi(a) = \text{Ad}(\lambda(\xi)) (a), a \in L^2(S^1)$. (for example, see the book by Raeburn and Williams Ref. [25] ). Also it is easy to see that $\alpha \otimes \text{Ad}(\lambda)$ is an action of the circle on $A \otimes \mathcal{K}$. The action $\alpha \otimes \text{Ad}(\lambda)$ is Rokhlin by Remark (2.8) of Ref. [35]- also see remarks before Def. (2.2) in the same paper.

2) This follows from Prop. (2.5) of Ref. [44], since $B_i, i = 1, 2, 3$ are unital.

3) Since $B$ is separable and unital, we use the definition Def. (2.14) for the space of circle actions on $B$. Since the circle is compact, by Cor. (2.22) (i) of Ref. [36], the set of Rokhlin circle actions on $B$ form a dense $G_\delta$–set of the space of circle actions on $A \otimes \mathcal{K}$.

We can also use the unitality of $B$ to find enough $\mathbb{Z}$–actions on $B$.

Lemma 3.2. Let $X$ be a finite CW–complex. Let $B$ be the section algebra of a $C^*$–bundle over $X$ with fiber $M_n(A)$ with $A$ a strongly self-absorbing $C^*$–algebra.

1) For every $B$ above, automorphisms of $B$ of Rokhlin dimension $\leq 1$ are generic.

2) If $A$ is UHF-stable for a UHF-algebra of infinite type, then Rokhlin automorphisms of $B$ are generic.

Proof.

1) Since $A$ is strongly self-absorbing, $A$ is always $\mathbb{Z}$–absorbing, hence, so is $B$. Also, by the above, $B$ is unital, separable and $\mathbb{Z}$–absorbing. By
Thm. (3.4) of Ref. [31], automorphisms of $\mathcal{B}$ of Rokhlin dimension $\leq 1$ are generic in the set of all automorphisms of $\mathcal{B}$ in the sense of Ref. [31].

2) Suppose $\mathcal{A}$ was UHF-stable for a UHF-algebra of infinite type, then so is $\mathcal{B}$. By the previous part, $\mathcal{B}$ is unital, separable and $\mathcal{Z}$-absorbing as well. Hence, by Thm. 31, Rokhlin automorphisms are generic in the set of all automorphisms on $\mathcal{B}$ in the sense of Ref. [31]. 

□

We now try to find a generalized T-duality diagram in the sense of Sec. (2.3) above. If we are given a $C^*$-algebra $\mathcal{D}$ with torsion characteristic class, we pick a $\mathcal{B}$ above such that $\mathcal{B} \otimes \mathcal{K} \simeq \mathcal{D}$. Note that Prim($\mathcal{B}$) $\simeq$ Prim($\mathcal{D}$) as discussed above. Note that $\mathcal{B}$ is not unique as shown in Ref. [12] however, every choice of $\mathcal{B}$ has the same spectrum as $\mathcal{D}$. The only thing which changes is the choice of fiber algebra -each choice of $\mathcal{B}$ corresponds to a fiber algebra of the form $M_n(A)$ with differing values of $n$. This should not affect anything since we are only interested in the result obtained by stabilizing $\mathcal{B}$.

We restrict our choice of group actions on $\mathcal{B}$ to tensor product actions of the type $\alpha \otimes \text{Ad}$ where $\alpha$ is a Rokhlin group action on $\mathcal{B}$. It is always possible to pick such a $\alpha$ by Lemma (3.1) above. By the same Lemma, this gives a natural Rokhlin circle action $\alpha \otimes \text{Ad}(\lambda)$ on $\mathcal{D}$. With this choice of actions, we can restrict ourselves to examining T-duality diagrams of the form in Eq. (3.1) below.

We pick a Rokhlin $\mathcal{Z}$-action $\gamma$ on $\mathcal{B}$ or a Rokhlin $\mathcal{Z}$-action of dimension at most one by Lemma (3.2). We pick the Rokhlin $\mathcal{Z}$-action on $\mathcal{B}$ such that it commutes with $\alpha$ above. Since Rokhlin $\mathcal{Z}$-actions of dimension at most one are dense in the set of all Rokhlin actions on $\mathcal{B}$, this should be possible. This induces a natural $\mathcal{Z}$-action $\theta$ on $\mathcal{B}^\alpha$ which commutes with $\theta$.

Let $\mathcal{B}^\alpha$ be the set of elements of $\mathcal{B}$ fixed by $\alpha$. By Thm. (2.3) of Ref. [45], since $\mathcal{B}$ is unital, there is an automorphism $\theta \in \mathcal{B}^\alpha$ such that $\mathcal{B}$ is isomorphic to $\mathcal{B}^\alpha \times \mathcal{Z}$. Let $\theta'$ be any other automorphism of $\mathcal{B}^\alpha$ such that $(\mathcal{B}^\alpha \times \mathcal{Z}, \theta')$ is conjugate to $(\mathcal{D}, \alpha)$, then by Thm. (2.3)(2) of Ref. [45], $\theta$ is unitarily equivalent to $\theta'$ by a unitary in $\mathcal{B}^\alpha$. Also, by Cor. (2.5) of Ref. [45], $\mathcal{B} \rtimes_{\alpha} S^1 \simeq \mathcal{B}^\alpha \otimes \mathcal{K}$.

Thus the Topological T-duality diagram in Thm. (2.3) is now modified due to the above. In particular, $\mathcal{D}$ is now isomorphic to $\mathcal{B} \otimes \mathcal{K}$ and, by the above, isomorphic to $(\mathcal{B}^\alpha \rtimes \mathcal{Z}) \otimes \mathcal{K}$. Also, we must have that $\mathcal{B}^\alpha$ is isomorphic to the fixed point $C^*$-algebra of $\mathcal{B}^\#$ by the dual automorphism $\alpha^\#$.
for consistency. The T-duality diagram now reduces to the two commuting 
$\mathbb{Z}$–actions $\theta$ and $\theta^#$ on $\mathcal{B}^\alpha$. The T-duality swaps the two $\mathbb{Z}$–actions.

$$\mathcal{C} \simeq \mathcal{B} \rtimes \mathbb{Z}$$

(3.1) $$\begin{align*}
(B^\alpha \rtimes \mathbb{Z})_{\theta}^\alpha & \simeq B^\alpha \\
B^\# & \simeq (B^\alpha \rtimes \mathbb{Z})_{\theta^#}^\alpha \\
B^\alpha & \simeq (B \rtimes S^1)
\end{align*}$$

The data which characterizes the original noncommutative space should be the characteristic class of $\mathcal{D} \simeq \mathcal{B} \otimes K$ together with the automorphism $\theta$ of $\mathcal{B}^\alpha$. The data which characterizes the T-dual space should be the characteristic class of $\mathcal{D}^\# \simeq (B^\alpha \rtimes \mathbb{Z}) \otimes K$ together with the dual automorphism $\theta^#$ on $\mathcal{B}^\alpha$.

Note that $\mathcal{B}, \mathcal{B}^\#$ are unital and $K_*(\mathcal{B})$ is finitely generated since $X$ is finite. Now, $\mathcal{B}^\# \simeq B^\alpha \rtimes \mathbb{Z}$ so $K_*(\mathcal{B}^\#)$ is finitely generated as well.

By Thm. (3.3)(2) of Ref. [45],

$$K_0(\mathcal{B}) \simeq K_1(\mathcal{B}) \simeq K_0(\mathcal{B}^\alpha) \oplus K_1(\mathcal{B}^\alpha)$$

similarly

$$K_0(\mathcal{B}^\#) \simeq K_1(\mathcal{B}^\#) \simeq K_0(\mathcal{B}^\alpha) \oplus K_1(\mathcal{B}^\alpha)$$

Thus, the $K$–theory of $\mathcal{B}$ is isomorphic to the $K$–theory of $\mathcal{B}^\#$ and the isomorphism may be taken to be degree reversing (since $K_*(\mathcal{B})$ and $K_*(\mathcal{B}^\#)$ are isomorphic).

Thus, the same holds for $\mathcal{D} \simeq \mathcal{B} \otimes K$ since stabilization preserves $K$–theory.

### 3.3. Fermionic T-duality

In this subsection we concentrate on $\mathcal{A} \simeq M_{2\infty}$—this corresponds to a fermionic T-duality on the string theory action or to fermionic T-duality composed with bosonic T-duality transformation of the string theory action. A few examples using the proposed T-duality procedure are calculated below and compared with the crossed product by a Rokhlin action of $\mathbb{R}$ whenever possible. Uniqueness of the Rokhlin action and its lifting properties are discussed.
Since it is difficult to find Rokhlin lifts of circle actions on X, we do not have too many examples, but the examples below are interesting.

1) $M_{2\infty}$—bundles over a Circle:
   • We consider a supersymmetric background $\mathbb{R}^{(8,1)} \times S^1$. We place a graviphoton flux on the background and perform a fermionic T-duality. The T-dual should be non-anti-commutative superspace over a circle.
   • For this example let $X$ be $S^1$ viewed as a trivial circle bundle over a point i.e., $X \simeq S^1 \times \{\ast\} \to \{\ast\}$. Suppose we associate to $\{\ast\}$ the $C^*$-algebra $M_{2\infty}$.
     Let $D \in \text{Bun}_X(A \otimes K)$ where $A$ is the CAR algebra $M_{2\infty}$.
     It can be argued that $D$ is an induced algebra of the form $\text{Ind}_{KZ}^R(M_{2\infty} \otimes K)$ for some automorphism $\alpha$ of $M_{2\infty} \otimes K$.
     In Ref. [11] Dadarlat and Pennig show that the characteristic class of such bundles lies in a sum of cohomology groups of the form $H^*(X, R^+_\times)$ where $R = K_0(M_{2\infty} \otimes K)$.
     However, for $M_{2\infty}$, it can be checked (see Ref. [46]) that $R^+_\times$ above is $\mathbb{Z}$, so the characteristic class of the CAR bundle lies in $H^1(S^1, \mathbb{Z}) \simeq \mathbb{Z}$.
   • Now, pick a spectrum-fixing Rokhlin $\mathbb{Z}$—action $\alpha$ on $D$ commuting with the translation action on the circle. Since $\alpha$ is a spectrum-fixing $\mathbb{Z}$—action on $D$, it will act only on the fiber $M_{2\infty} \otimes K$ of $D$.
     The crossed product $\mathcal{B} \simeq D \rtimes \mathbb{Z}$ is a $C(S^1)$—bundle. $\mathcal{B}$ is also $C^*$—bundle over Prim($\mathcal{B}$) with fiber $(M_{2\infty} \otimes K) \rtimes \mathbb{Z}$.
     Every trace scaling automorphism $\alpha$ of $M_{2\infty} \otimes K$ possesses a trace scaling factor $s(\alpha) = p/q$ which is a positive rational number (see [37], pg. 77). Each of these automorphisms is Rokhlin -see Thm. (2.8) of Ref. [47]. Also, by Ref. [37], Thm. (7), automorphisms with the same trace scaling factor are outer conjugate.
     Let $\alpha$ to be an automorphism with trace scaling factor 2. By tensoring $\alpha$ with itself (and similarly for $\alpha^{-1}$) and using the fact that $M_{2\infty} \otimes K$ is isomorphic to its tensor product with itself, we can obtain automorphisms of trace scaling factors $2^k$ for all $k \in \mathbb{Z}$.
     By Remark on pg. (84) of Ref. [37], if $\alpha$ is an automorphism of $M_{(pq)\infty} \otimes K$ with trace $s(\alpha) = p/q$, the crossed product $\mathcal{C} \rtimes \mathbb{Z}$ is isomorphic to $O_{|p-q|+1} \otimes K$. 

33
Of the above automorphisms, it is clear by examining the positive integer solutions to \(pq = 2^m, |p - q| = 1\) that only the conjugacy classes of Rokhlin \(Z\)-actions \(\alpha\) on \(M_{2^\infty} \otimes \mathcal{K}\)-those with trace \(s^*(\alpha) = 2/1\) and those with trace \(s^*(\alpha) = 1/2\)-give \(O_2 \otimes \mathcal{K}\) as a crossed product.

- We require the crossed product to have strongly self-absorbing fiber and by the above, this fiber can only be \(O_2 \otimes \mathcal{K}\). So, changing the characteristic class of \(Z\)-action can only change the trace of \(\alpha\) from 2 to 1/2 or vice versa.
- Thus, the correspondence space is a \(C^*\)-bundle over \(\text{Prim}(\mathcal{D}) \simeq S^1\) with fiber \(O_2 \otimes \mathcal{K}\). Due to the local triviality of \(\mathcal{D}\) and the above argument, this bundle has to be a locally trivial bundle of Cuntz algebras \(O_2\). In Ref. [11] Dadarlat and Pennig show that the characteristic class of such bundles lies in a sum of cohomology groups of the form \(H^*(X, R_\times)\) where \(R = K_0(O_2 \otimes \mathcal{K})\). However, for \(O_2\), it can be checked that \(R\) above is trivial, so the characteristic class of Dadarlat and Pennig is trivial. By Cor. (4.3) and Cor. (4.6) of Ref. [11] the above \(C^*\)-bundle must be trivial as well. Changing the characteristic class of the orginal \(M_{2^\infty}\)-bundle on \(S^1\) won’t change the characteristic class of the \(O_2 \otimes \mathcal{K}\)-bundle above, since it must always be trivial.
- The \(Z\)-action on \(\mathcal{D}\) is spectrum-fixing, and, as a result it acts only on the fiber \(M_{2^\infty}\) of \(\mathcal{D}\). Since it commutes with the translation action on \(S^1\), the crossed product \(\mathcal{D} \rtimes Z\) should then be \(C(S^1) \otimes_{\alpha} O_2 \otimes \mathcal{K}\).
- Thus the correspondence space is a trivial \(O_2 \otimes \mathcal{K}\)-bundle over a topological space \(E \simeq S^1\). The space \(E\) carries a circle action which comes from the original circle action on the original \(S^1\) which only translates \(S^1\). (this can be seen by working out the definition of the crossed product, viewing the algebra \(\mathcal{D}\) as an induced algebra of the form \(T_{\alpha}(M_{2^\infty} \otimes \mathcal{K})\).
- The quotient of the \(C^*\)-bundle by this circle action is \(O_2 \otimes \mathcal{K}\).
- Geometrically, the \(C^*\)-algebra \(O_2\) may be viewed as a noncommutative circle bundle over a CAR algebra. We could view this as the noncommutative geometry of the T-dual.
- Thus, we claim that the fermionic T-dual of a supersymmetric theory over a circle is non-anti-commutative superspace over a point and the T-dual circle action is the rotation of the generators of non-anti-commutative superspace by a phase. This should be matched with the Buscher’s rules for fermionic T-duality: It can be seen that
the physical T-dual is a circle with the fermionic directions ‘twisted’ by the T-dual graviphoton flux into NAC superspace.

We may also argue by calculating the crossed product by $\mathbb{R}$: We note that by Lemma (3.1) and Lemma (3.2) of Ref. [46], $D$ is isomorphic to the induced $C^*$-algebra $T_\alpha(M_{2\infty} \otimes \mathcal{K})$ defined there. By Lemma (5.1) and Lemma (5.2) of [46], the crossed product by the lift of the translation action on $\text{Prim}(D)$ to $\mathbb{R}$ is $O_2 \otimes \mathcal{K}$ which agrees with the above.

**Theorem 3.3.** If the T-dual by a crossed product by a $\mathbb{R}$–action is $O_2 \otimes \mathcal{K}$, then the dual action on $O_2 \otimes \mathcal{K}$ cannot be Rokhlin.

**Proof.** Suppose the dual action on $O_2 \otimes \mathcal{K}$ was Rokhlin.

We first note that $O_2$ is $O_\infty$–absorbing since $O_2$ is purely infinite and we may apply Kirchberg’s theorem to conclude that $O_2 \otimes O_\infty \simeq O_2$.

Thus by Thm. (B) of Szabo ([26]), there is only one Rokhlin lift of the trivial $\mathbb{R}$–action on $*=\text{Prim}(O_2 \otimes \mathcal{K})$ to $O_2 \otimes \mathcal{K}$ up to cocycle conjugacy.

Due to this, the above Rokhlin action must be cocycle conjugate to the following Rokhlin action $\alpha_\lambda$ on $O_2 \otimes \mathcal{K}$ (see Sec. (1) of Jacelon Ref. [20]):

Consider a quasi-free Rokhlin $\mathbb{R}$–action $\alpha_\lambda$ on $O_2$. Let $s_1, s_2$ be generators of $O_2$. Then, define

$$\alpha_\lambda(s_1) = e^{it}s_1$$
$$\alpha_\lambda(s_2) = e^{i\lambda t}s_2$$

This action is Rokhlin if and only if $\lambda$ is irrational. As described in Ref. [20], the isomorphism class of the crossed product depends on the sign of $\lambda$.

\begin{align*}
(3.2) & \quad O_2 \rtimes \mathbb{R} \simeq W \otimes \mathcal{K}, \lambda \in \mathbb{R}_+ - \mathbb{Q} \\
(3.3) & \quad O_2 \rtimes \mathbb{R} \simeq O_2 \otimes \mathcal{K}, \lambda \in \mathbb{R}_- - \mathbb{Q}
\end{align*}

here $W$ is the Razak-Jacelon algebra [20].

We take the crossed product of Eq. (3.3) above by the $\mathbb{R}$–action $\alpha$ which is dual to the $\mathbb{R}$–action $\alpha_\lambda$. Since $O_2$ is unital, separable, nuclear, purely infinite and simple, and $\alpha_\lambda$ is Rokhlin, the induced
dual-\(\mathbb{R}\)-action \(\hat{\alpha}\) on \(\mathcal{O}_2 \rtimes \mathbb{R}\) is also Rokhlin (by Thm. (2.1), part (3) of Ref. [48]). Thus, the action \(\hat{\alpha}\) must be the unique (up to cocycle conjugacy) Rokhlin \(\mathbb{R}\)-action on \(\mathcal{O}_2 \otimes \mathcal{K}\) given by Szabo’s Theorem above.

By Takai duality, the above crossed product is \((\mathcal{O}_2 \otimes \mathcal{K}) \rtimes \mathbb{R} \simeq \mathcal{O}_2 \otimes \mathcal{K}\). Therefore, the action \(\hat{\alpha}\) cannot give the ‘expected’ T-dual \(D \in \text{Bun}_{S^1}(M_2 \otimes \mathcal{K})\). Hence, there cannot be a Rokhlin \(\mathbb{R}\)-action on \(\mathcal{O}_2 \otimes \mathcal{K}\) which gives an element of \(\text{Bun}_{S^1}(M_2 \otimes \mathcal{K})\) as a crossed product. \(\square\)

The ‘reason’ for the above seems to be that the T-dual \(\mathbb{R}\)-action should have a \(\mathbb{Z}\)-stabilizer and the above action does not have any: Since \(\lambda\) is irrational, no value of \(t\) will return the generators \(s_i\) to themselves.

If \(\lambda\) above was rational, the quasi-free action \(\alpha_\lambda\) above would not be Rokhlin. In particular, we would have by Thm. (3.2) of Dean (Ref. [49]), that in the case of rational \(\lambda = p/q\), the crossed product of \(\mathcal{O}_2\) by the quasi-free \(\mathbb{R}\)-action \(\alpha_{p/q}\) would always be a mapping torus \(C^*-\)algebra of an \(AF\)-algebra \(A(p,q)\).

Since we are trying to T-dualize \(M_2\)-bundles over \(S^1\), the above facts would be consistent with Ex. (1), if the action on \(\mathcal{O}_2 \otimes \mathcal{K}\) were cocycle conjugate to a tensor product of a non-Rokhlin quasi-free \(\mathbb{R}\)-action on \(\mathcal{O}_2\) with the identity \(\mathbb{R}\)-action on \(\mathcal{K}\).

The above prescription should extend to \(N\)-torus bundles. We try to apply the above prescription to the simplest such bundle.

2) \(M_{2\infty}\)-bundles over a space \(X\) with a free circle action

- Let \(X\) be an arbitrary supersymmetric Type II string theory background. By Buscher’s rules for fermionic T-duality (see Ref. [50]), the T-dual spacetime has the same \(B\)-field and metric, but the Ramond-Ramond fluxes on the T-dual change. There is a circle action on the superspace fermionic coordinates but the underlying bosonic coordinates are unchanged by the circle action.
- Hence, as argued in the previous section we would expect a duality between \(C^*\)-algebras with \(M_{2\infty}\)-fiber over \(X\) and \(C^*\)-algebras with \(\mathcal{O}_2\)-fiber over \(X\).
- It is difficult to analyze this problem with no further structure to \(X\), so we require that any \(C^*\)-algebra we associate to a space in the problem be a \(C_0(W)\)-algebra where \(W\) is a space over which \(X\) fibers, that is we have a fibration map \(q: X \to W\).

It is clear that in general this is not needed, since the circle action on the associated string theory is only in the fermionic directions so there should be no circle action on \(X\). This is different from
the usual theory of $C^*$–algebraic T-duality (see Ref. [1]), however requiring a fibration $q : X \to W$ helps us work out this example.

- Let $D \in \mathcal{B}un_X(M_{2\infty} \otimes K)$. In addition to requiring $D$ and its dual to be a $C_0(W)$–algebra, we require $X$ to be a principal circle bundle over $p : X \to W$. By Buscher’s rules for fermionic T-duality (see Ref. [50]), the ‘T-dual’ has the same topology and we will show that it can be chosen to have the same $H$–flux.

- We will see that with the method outlined in the previous section we obtain as a ‘T-dual’ of $D \in \mathcal{B}un_X(M_{2\infty} \otimes K)$ a $C^*$–algebra $D' \in \mathcal{B}un_X(O_2 \otimes K)$ which we will argue below should be associated to the fermionic T-dual space.

- If we were to calculate the T–using the crossed product as normal, this crossed product would be difficult to calculate and also, it might not be possible to argue about its uniqueness. With the construction in the above section we get a unique T-duality diagram and hence a unique $C^*$–algebraic T-dual for $D$.

\[
\begin{array}{c}
\text{X} \\
\downarrow \pi \\
\text{W}
\end{array} \quad \begin{array}{c}
\text{C} \\
\downarrow \text{Prim} \\
\text{D} \\
\downarrow \text{Prim} \\
\text{B} \\
\downarrow \text{Prim} \\
\text{D'}
\end{array} \quad \begin{array}{c}
\text{D'} \\
\downarrow \text{Prim} \\
\text{B'} \\
\downarrow \text{Prim} \\
\text{W}
\end{array}
\]

(3.4)

- We associate the $C^*$–algebra $B \simeq CT(W, \delta) \otimes_{C_0(W)} D'$ where $D' \in \mathcal{B}un_W(M_{2\infty} \otimes K)$ to $W$, note that this is a $C_0(W)$–algebra.

- We assume given a on $\mathbb{Z}$–action $\alpha \simeq \alpha_1 \otimes \alpha_2$ on $B$ with $\alpha_1$ an automorphism of $CT(W, \delta)$ and $\alpha_2$ is an automorphism of $D''$ which acts only on the fiber of $D'''$.

- The crossed product by the $\mathbb{Z}$–action induced by $\alpha$ is

\[
D \simeq B \rtimes_{\alpha} \mathbb{Z} \simeq (CT(W, \delta) \rtimes_{\alpha_1} \mathbb{Z}) \otimes_{C_0(W)} (D' \rtimes_{\alpha_2} \mathbb{Z}).
\]

- $D$ is a crossed product of $B$ by the tensor product $\mathbb{Z}$–action generated by $\alpha_1 \otimes \alpha_2$, so $\text{Prim}(D)$ is a fiber product of $X_1 \simeq \text{Prim}(CT(W, \delta) \rtimes_{\alpha_1} \mathbb{Z})$ with $X_2 \simeq \text{Prim}(D' \rtimes_{\alpha_2} \mathbb{Z})$ over $W$.  

37
The crossed product $CT(W, \delta) \rtimes Z$ is isomorphic to $CT(X_1, \eta)$ for some $X_1$ with $\pi_1: X_1 \to W$ a principal circle bundle by the theory of continuous-trace algebras.

Now, $\alpha_2$ acts only on the fiber of $D'$, hence, by Ref. [37], Remark (4.4), the crossed product $D' \rtimes Z$ is a locally trivial $C^*$-bundle with spectrum $W \times S^1$ and fiber $\mathcal{O}_2 \otimes \mathcal{K}$. By Ref. [11], there can be only one such bundle, the trivial one. Thus $D' \rtimes Z$ is isomorphic to $C_0(W \times S^1) \otimes \mathcal{O}_2 \otimes \mathcal{K}$ and hence its spectrum is $W \times S^1$.

Thus, the spectrum of $D$ is the fiber product $X_1 \times_W X_2 \simeq (X_1 \times_W (W \times S^1))$.

• Suppose $\beta \simeq \beta_1 \otimes \beta_2$ was an automorphism of $D$ commuting with the action $\hat{\alpha}$, where $\beta_1$ is an automorphism of $CT(X_1, \eta)$ above and $\beta_2$ a fiber-preserving automorphism of $C_0(W \times S^1) \otimes \mathcal{O}_2 \otimes \mathcal{K}$ of the form $\beta_2 \simeq \text{id} \otimes \beta_3$ where $\beta_3$ is a Rokhlin automorphism of $\mathcal{O}_2 \otimes \mathcal{K}$.

• Note that $D$ is a stabilization of a unital $C^*$-algebra which is a section algebra of a $C^*$-bundle with fiber $\mathcal{O}_2$ and hence, this underlying bundle is $\mathcal{O}_2$-absorbing. By Cor. (2.22) of Ref. [38], the set of Rokhlin circle actions are a dense, $G_4$-set in the set of all continuous circle actions on $D$. Thus we would expect many Rokhlin circle actions on this bundle. Any action $\alpha$ of this type induces an action of the form $\alpha \otimes \text{Ad}(\lambda)$ on the stabilization of this $C^*$-algebra (see proof of item (1) of Lemma (3.1) above, it is clear that the proof will carry over to this situation. Also, note that the symbol $B$ in that proof is different from the $C^*$-algebra $B$ used here). Thus there will be many Rokhlin circle actions on $D$.

• Now consider, $C \simeq D \rtimes Z$: By an argument similar to the above, we find

$$(3.5)\quad \text{Prim}(C) \simeq \text{Prim}(CT(X_1, \eta) \rtimes Z) \times_W \text{Prim}((C_0(W \times S^1)) \otimes \mathcal{O}_2 \otimes \mathcal{K} \rtimes Z)$$

Now $\text{Prim}(CT(X_1, \eta) \rtimes Z)$ is a principal circle bundle over $X_1$ and

$$\text{Prim}((C_0(W \times S^1)) \otimes \mathcal{O}_2 \otimes \mathcal{K} \rtimes Z)$$

is by Ref. [51]

$$(3.6)\quad \text{Prim}(C_0(W \times S^1) \otimes (\mathcal{O}_2 \text{ fibration over } \otimes C_0(X) \otimes C(S^1)))$$

38
(Here we have used Sec. (7.2) of Ref. [51] for the crossed product of $O_2 \otimes K$ by $\beta_3$.) Hence, $\text{Prim}(C)$ is a nontrivial circle bundle over $X$ as expected.

- Since $\beta$ commutes with the circle action $\hat{\alpha}$ it induces a circle action on $C$. Quotienting by this circle action gives the dual $C^*$-algebra.

3) $M_2 \otimes K$-bundles over the torus:

- Let $p : T^N \to \ast$ as a trivial $N$-torus bundle.
- Consider a locally trivial $M_2 \otimes K$-bundle ($D$) over a torus $T^N$.
- As in the previous example, such bundles are classified by elements in $H^{2n+1}(T^N, \mathbb{Z}), n = 1, 2, \ldots$
- Similar to the previous example we suppose given a $\mathbb{Z}^N$-action $\alpha$ on $D$ which fixes $\text{Prim}(D)$ and commutes with translation action on $D$ which is the natural lift of the torus action on $T^N$.
- Since the $\mathbb{Z}^N$-action fixes $\text{Prim}(D)$, it acts on the fiber $M_2 \otimes K$ of $D$ and also commutes with the natural translation action on $T^N$.
- By a theorem of N.C. Phillips Ref. [30], see Thm. (5.15) and item (4) on pg. 1 and the sections from Notation (5.7) to Lemma (5.10) on pg. 29 of Ref. [30] as well-$D$ is $M_2 \otimes K$-absorbing so Rokhlin actions of $\mathbb{Z}^N$ on $D$ are a dense $G_\delta$-set in set of all actions of $\mathbb{Z}^N$ on $D$.
- Thus it should be possible to choose the $\mathbb{Z}^N$-action above to be a Rokhlin $\mathbb{Z}^N$-action.
- In the absence of a characterization of the crossed product of a UHF algebra by $\mathbb{Z}^N$ in the literature the correspondence space can’t be determined. From the literature, it should be a bundle of $\mathbb{Z}$-stable, $\mathbb{Z}$-stable $C^*$-algebras over a point.
- There is no parametrization of cocycle conjugacy classes of $\mathbb{Z}^N$-actions on $C^*$-algebras like $D$ above. Hence it is not possible to determine the structure of the correspondence space or the Topological T-dual.

4. Applications to T-duality in Type II String Theory

In this section we apply the formalism outlined above to study tree-level dualities in flux backgrounds in String Theory, i.e., spacetime backgrounds with sourceless Ramond-Ramond flux and $H$-flux. We begin by discussing the physics of such flux backgrounds in Subsec. (4.1). Then, in Subsec. (4.2) we examine the noncommutative geometry of points in string backgrounds described by $C^*$-algebras $D$ above and make a prediction about $D$–Brane charge. We apply the above formalism to tree-level dualities in String Theory in Subsec. (4.3).
4.1. Flux Backgrounds and $C^*$—algebras

We had described the $C^*$—algebraic formalism of Topological T-duality in Sec. (1) above. We now use the example at the end of the previous Section to study certain phenomena in String Theory which we claim are described by the crossed-product of section algebras of locally trivial bundles of strongly self-absorbing $C^*$—algebras by $\mathbb{R}$—actions.

In Sec. 1 we had mentioned that Topological T-duality describes spacetimes $X$ which are backgrounds for Type II String Theory with a sourceless $H$—flux $H$ together with a circle action. It is not clear whether the usual description of $D$—brane charges on string backgrounds $(X, [H])$ by twisted $K$—theory will work if other fluxes, for example Ramond-Ramond fields, are turned on.

It is interesting to ask if the $C^*$-algebra bundles studied in this paper (i.e., locally trivial bundles of self-absorbing $C^*$ algebras over a base space $X$) can be used to describe these string backgrounds.

We claim that the $C^*$—algebras $D$ in $\mathcal{Bun}_X(A \otimes K)$ describe backgrounds of Type II string theory with a circle action with various fluxes switched on (these are also termed flux backgrounds and were discussed in Sec. (1) above) as described below:

1) $A \simeq \mathbb{C}$: If $A \simeq \mathbb{C}$, the background has a sourceless $H$—flux present and there is an equivariant gerbe with connection present on the spacetime background and the $H$—flux is the (characteristic class of) the curvature of the gerbe connection in $H^3(X, \mathbb{Z})$. There is a natural lift of the $S^1$—action on $X$ to a $\mathbb{R}$—action on the gerbe covering the circle action on $X$ thus giving an equivariant gerbe on $X$.

The equivariant gerbe then defines a $S^1$—equivariant principal $PU$—bundle on the spacetime background up to isomorphism of equivariant $PU$—bundles. The continuous-trace $C^*$—algebra on the spacetime background is the associated bundle to this equivariant $PU$—bundle and has fiber $K$ with $PU$ acting on $K$ by $\text{Ad}$ (see Ref. [1, 52] and references therein).

Thus, if $A = \mathbb{C}$, $D$ corresponds to backgrounds which have a sourceless $H$—flux switched on.

2) $A \neq \mathbb{C}$: Suppose first that $X$ is compact, then, we have the following (heuristic) argument: Cornalba et al in Ref. [15] (Secs. (5.6.2) and (6.2)) consider a $Dp$—brane with a worldvolume gauge field $F$ and argue using String Field Theory that turning on a constant $RR$— and $B$—field backgrounds causes the $Dp$—brane worldvolume gauge field to change due to the backreaction by the open strings. The backreaction has the form $F + \delta F$ where $\delta F$ is of the form $\delta F = (B - *C^{(8)})$. This
implies that the $B-$field and the $RR-$flux together act only to shift the worldvolume gauge field strength. Note that in the above $B-$field is only \textit{shifted} by the $RR-$flux.

We now claim that if the background has a sourceless $H-$flux together and a sourceless Ramond-Ramond flux is turned on, the above implies that the gerbe on the background has been \textit{deformed} into a (possibly different) gerbe with connection form $(B - sC(8))$. We identify this deformed gerbe as one of the higher gerbes of Cor. (3.42) of Ref. [53]. (Recall that Ramond-Ramond fields can also be viewed as curvatures of a gerbe on spacetime (see Sec. (5.1) of Ref. [54])). Note that these higher gerbes are the gerbes corresponding to locally trivial $C^*-$bundles with fiber $A \otimes K$ discussed above.

The twist of this higher gerbe can be used (not uniquely) to obtain a principal $\text{Aut}(A \otimes K)-$bundle over the spacetime background (see Cor. (3.42) of Ref. [53] and following note). Thus the higher gerbe gives an element $D$ in $\mathcal{Bun}_X(A \otimes K)$ by Ref. [24].

Note that with the above argument if the classification scheme of Dadarlat and Pennig is used when $RR-$flux is turned on, the gerbe on $X$ defined by the $B-$field can only change its characteristic class however, it still remains a gerbe. The associated $C^*-$algebra on the other hand, changes from a continuous-trace $C^*-$algebra to a locally trivial $C^*-$bundle with fiber $A \otimes K$.

It would be interesting to generalize the above to equivariant gerbes, but for this, we would need an analogue of the theory of the equivariant Brauer group for $\mathbb{R}-$actions on section algebras of locally trivial bundles with fiber a strongly self-absorbing $C^*-$algebra. In this paper, we have managed to find lifts for various fibers of physical interest based on the current literature, however, the general case is unknown at present.

Hence, the case $\mathcal{A} \neq \mathbb{C}$, for any spacetime background of Type II theory $X$ above the $C^*-$algebra $\mathcal{D}$ in $\mathcal{Bun}_X(A \otimes K)$ corresponds to backgrounds which have other fluxes switched on apart from the $H-$flux (see item (3) in the list below).

Then, the construction proposed in Sec. (2) above should give a duality relation which generalizes Topological T-duality for ordinary spacetime backgrounds with sourceless $H-$flux to a duality for the above spacetime backgrounds with various sourceless $H-$ and $RR-$ fluxes turned on. Under this duality, the $C^*-$algebra $\mathcal{D}^\#$ (see Sec. (2) above) is a 'dual $C^*-$algebra' to $\mathcal{D}$. Now, the dual space $X^\#$ should be recovered from $\mathcal{D}^\#$ as $X^\# = \text{Prim}(\mathcal{D}^\#)$. Note that if $\mathcal{D}^\#$ is a locally trivial bundle of self-absorbing $C^*-$algebras over
then the characteristic classes of the fluxes on it can be recovered from the characteristic class of $D$. If, on the other hand, $D\#$ is not a locally trivial bundle of self-absorbing $C^*$-algebras, techniques from noncommutative geometry might be helpful in determining the dual space $X\#$ (see the last part of Sec. 2 above).

In Refs. [7, 9, 16], the authors define noncommutative spacetimes as $C^*$-algebras satisfying certain conditions. Noncommutative $D$-branes correspond to elements of the $K$-homology of the spacetime and their charges lie in the operator $K$-theory of these $C^*$-algebras. We note that the $C^*$-algebras $A$ satisfy the UCT (by Ref. [23]) and so do the $C^*$-algebras $D$. Hence, by Cor. (4.5) for Ref. [9], it is possible to define a noncommutative analogue of the $D$-brane charge for these spacetimes.

Hence, it is natural to identify a 'generalized $D$-brane charge' of a $D$-brane in $D$ with the operator $K$-theory of $D$.

In Ref. [55], Sec. (6.1.2) Bouwknegt and Mathai calculate the twisted $K$-theory of a space $P$ which is a principal $SU(2)$-fibration over a 4-manifold $M$ twisted by an integral 7-form $H \in H^7(P, \mathbb{Z})$ using the operator $K$-theory of a bundle of Cuntz algebras on $P$. In this paper the authors wish to define a 'T-dual' principal $SU(2)$-bundle using a generalization of Topological T-duality. Since the authors are studying $SU(2)$-bundles over a four-dimensional base space they do not consider crossed products by $\mathbb{R}$ as this paper does. It is interesting to note that in Section (9) item (2) of the same paper, the authors reject the possibility that a crossed-product of the Cuntz algebra bundle above by a $SU(2)$-action would give them the T-dual of the spacetimes they are examining since it turns out there cannot be an isomorphism in twisted $K$-theory for crossed-products by $SU(2)$-actions.

Also, in Ref. [56] pg. 334, Section (1) Macdonald, Mathai and Saratchandran argue that not all higher twisted $K$-theories on a space are obtained by twisting the $K$-theory of a space by cohomology data. As an example of this, they study higher twisted $K$-theory of a space obtained from $O_\infty$-bundles mentioned in the previous paragraph. For an important class of examples, they demonstrate that the twist of the higher twisted $K$-theory obtained naturally defines a class $H : X \to S^n$ in the cohomotopy set of $X$. However, for finite, connected, torsion free spaces these theories do correspond to a twisting by a cohomology class on the space.

In this paper, we argue that the choice of a bundle of self-absorbing $C^*$-algebras on spacetimes with a circle action is a good model for spacetime backgrounds containing $D$-branes with sourceless $H$-flux and possibly $R - R$-flux. In addition, we argue that the Topological T-dual proposed in Sec. 2 gives a physically sensible 'generalized T-duality' (see below for more details).
4.2. Noncommutative Geometry and $K$-theory of Flux Backgrounds

Now we study $X$ together with a fixed algebra $D \in \mathcal{Bun}_X(A \otimes K)$ describing a Type II string theory background with sourceless $H$-flux, and possibly $RR$-flux, from the point of view Noncommutative Geometry and $K$-theory within the above formalism.

1) **Effects on Points:** Consider the inclusion of a point in $\{\ast\}$ in $X$. The pullback of the algebra $D$ along the inclusion map gives a trivial $C^*$-bundle over $\{\ast\}$ with fiber $A \otimes K$. Thus, we may associate the $C^*$-algebra $A \otimes K$ with a point $\{\ast\}$ in this formalism. We may view different types of self-absorbing $C^*$-algebras $A$ as defining various types of noncommutative points on the worldvolume gauge theory of $D$-branes of the space $X$ with $H$-flux or $RR$-flux turned on. *We would expect this to be visible in the worldvolume theory of $D0$-branes on $X$.*

2) **Effects on Branes and Brane Charges:** In Ref. [52], A. Kapustin has argued that in the case of $m$ coincident $D$-branes, the gauge theory on the $D$-brane worldvolume is described by a gauge theory on an Azumaya algebra with fiber $M_m(C)$ whose characteristic class is the restriction of the background $H$-flux to the $D$-brane worldvolume.

   It is well known, (see Ref. [25]) that the stabilization of an Azumaya algebra is a continuous-trace $C^*$-algebra with torsion characteristic class and every continuous-trace $C^*$-algebra with torsion characteristic class arises in this way.

   In a recent paper, Dadarlat and Pennig (see Ref. [12], Thm. (2.8)) have shown that for $A$ a self-absorbing $C^*$-algebra the elements of $\mathcal{Bun}_X(A \otimes K)$ with torsion characteristic class may be obtained from the stabilization of $C^*$-bundles over $X$ with fiber $M_n(A)$ for some $n$. 

   In Sec. (3.2) above, we have calculated the Topological T-dual of such $C^*$-algebras.

   This similarity with the case of continuous-trace $C^*$-algebras is extremely interesting. The change from $M_n(C)$ to $M_n(A)$ in the above may be attributed to ‘noncommutative points’ discussed above.

3) **$K$-theory of $X$:** In String Theory, it is well-known that when the $H$-flux is zero the $K$-theory of $X$ reflects properties of the $D$-brane charge on the space (see Ref. [57] for example).

   It is well-known that when $A = C$, $K_*(D)$ is the twisted $K$-theory of $X$ with the twisting the class of $D$ in $\mathcal{Bun}_X(C \otimes K) \simeq H^3(X, Z)$. In more detail: When the twist vanishes, $D = C(X) \otimes K$ and $K_*(D)$
is the ordinary $K$-theory of $X$. When the twist does not vanish, but $\mathcal{A} = \mathbb{C}$, $K_*(\mathcal{D})$ is the twisted $K$-theory of $X$, with the twist the class determined by $[\mathcal{D}]$ in $H^3(X, \mathbb{Z})$. It is conjectured that $D$-brane charges in String Theory with background $H$-flux take values in twisted $K$-theory of $X$. (see Ref. [1] and references therein).

If we assume that the formalism outlined at the beginning of the current section is true, the $D$–brane charge on $X$ is given by the $K$-theory of $D$ where $D$ is in $\text{Bun}_X(\mathcal{A} \otimes \mathcal{K})$ for any strongly self-absorbing $C^*$-algebra $\mathcal{A}$.

First consider the untwisted case: We consider $D = C_0(X, \mathcal{A}) \simeq C_0(X) \otimes \mathcal{A}$. By Ref. [10] Sec. (4), before Sec. (4.1), for each strongly self-absorbing $C^*$-algebra $\mathcal{A}$, the assignment $X \mapsto K_*(C(X) \otimes \mathcal{A}) \simeq K_*(C(X, \mathcal{A}))$ is a multiplicative generalized cohomology theory on finite CW-complexes. By Ref. [53] $K_0(C(X, \mathcal{A}))$ is the Grothendieck group of isomorphism classes of (finitely generated and projective) Hilbert-$\mathcal{A}$-module bundles over $M$.

In Ref. [58], the author argues that this cohomology theory can be represented by a commutative symmetric spectrum $KU^A_*$ (see Ref. [58] Sec. (1)). In particular, for $\mathcal{A} = \mathbb{Z}$, the Jiang-Su algebra, the resulting theory is topological $K$–theory, while for $\mathcal{A}$ an infinite UHF-algebra of type $p^\infty$, the resulting theory is a localization of $K$–theory at the prime $p$ with spectrum $KU[1/p]$. It would be interesting to see if one could find $D$–brane configurations whose charge group is this group for some choice of $\mathcal{A}$ for a Type II background with the correct sourceless fluxes turned on.

All self-absorbing $C^*$-algebras $\mathcal{A}$ may be written as direct limits of matrix algebras. It is interesting to speculate whether this fact could be used to build such configurations, perhaps as argued in the next example:

**Example (CAR-Bundles):** Consider a trivial $M_{2^\infty}$-bundle over $X$. We can view

$$C_0(X, M_{2^\infty}) \simeq \lim \longrightarrow C(X, M_{2^k}(\mathbb{C}))$$

where each inclusion corresponds to a doubling map (see Ref. [1] and references therein). Each $C_0(X, M_{2^k}(\mathbb{C}))$ factor can be viewed as the Azumaya bundle of the worldvolume gauge theory of a stack of $2^k$ $D$–branes. We view the injection $C_0(X, M_{2^k}(\mathbb{C})) \hookrightarrow C_0(X, M_{4^k}(\mathbb{C}))$ by doubling a matrix as associated to the worldvolume theory of four $D$–branes bound into a 'dimer' of two $D$–branes each. We view the entire direct limit as the worldvolume theory associated to a stack of
such dimers. While this cannot happen if background \( RR \)-flux is not present (since the worldvolume gauge theory would then be given by the Azumaya algebra \( C_0(X, M_4(\mathbb{C})) \)), it would be interesting to see if this or a similar construction would give a volume gauge theory described by \( C_0(X, M_{2\infty}) \) when sourceless background \( RR \)-flux was turned on.

Now consider the twisted case: For the twisted case, we need to consider section algebras of nontrivial \( C^* \)-bundle with fibers stabilized self-absorbing \( C^* \)-algebras. In Ref. [58], the author demonstrates that the operator algebraic \( K \)-theory of these \( C^* \)-algebras is a twisted version of the above, a higher twisted \( K \)-theory.

Thus, for a space with Ramond-Ramond flux concentrated in odd degrees together with \( H \)-flux, we could take \( \mathcal{A} = \mathbb{Z} \) or \( \mathcal{A} = M_{2\infty} \) in the above. Hence, the charge group of \( D \)-branes on a space with Ramond-Ramond flux concentrated in odd degrees should be a higher twisted topological \( K \)-theory or a twisting of the localization of topological \( K \)-theory at the prime 2. It would be interesting if this could be calculated from String Theory.

It would be interesting to speculate which string theory backgrounds would have excitations which have these cohomology theories as charges. Ref. [59] demonstrates that D-branes in a certain flux background geometry have charges in the group \( \mathbb{Z}_p \): Stacks of D-branes in these backgrounds annihilate in sets of \( p \) D-branes at a time. It might be possible to construct backgrounds which have D-brane charges in the \( K \)-groups of the above \( C^* \)-algebras.

### 4.3. Physical Examples and Tree-Level Dualities

We now consider some important examples of the above formalism and the physical transformations associated with them: First note that when all background fluxes are turned off apart from the \( H \)-flux, the formalism above reduces to the \( C^* \)-algebraic T-duality of Mathai and Rosenberg ([2]). However, if some of the background fluxes are turned on, we would obtain the fermionic T-duality of Berkovits and Maldacena ([50]) and the timelike T-duality of Hull ([60]).

In the above, the \( C^* \)-algebraic T-duality of Mathai and Rosenberg models the T-duality symmetry of Type II String theory which is true symmetry of Type II string theory valid to all orders in perturbation theory. However, the Fermionic T-duality of Berkovits and Maldacena and the Timelike T-duality of Hull are currently only tree-level symmetries of Type II string theory, loop corrections to string theory might destroy this symmetry--thus
while the spacetime might have such symmetries on a coarse scale, if examined on fine scales, there need be no such symmetry. We nonetheless consider these tree-level symmetries here, since we are only interested in the coarse features of the spacetime background (in particular, its algebraic topology).

We argue that these three examples correspond the crossed-product construction above for three different choices for $A$—namely $A = \mathbb{C}, M_\infty$ or $\mathbb{Z}$ respectively together with a specific choice of the geometry of $X$. (As was argued above, we associate the $C^*$-algebras $\mathbb{C}, M_\infty, \mathbb{Z}$ with spacetime backgrounds with $H$-flux only, with Ramond-Ramond flux only and with both $H$-flux and Ramond-Ramond flux respectively.)

1) T-duality with $H$-flux: As was explained in detail at the beginning of this paper, the case $A = \mathbb{C}$ corresponds to continuous-trace bundles over $X$ and these describe a space with background sourceless $H$-flux as described by Mathai and Rosenberg in Ref. [2]. The non-stable case (corresponding to $M_n(\mathbb{C})$—bundles over $X$) was examined earlier by Kapustin in Ref. [52].

By Thm. (2.1) above, for any $D \in \mathcal{Bun}_X(A \otimes K)$, $\text{Prim}(D) \simeq X$. For the case $A = \mathbb{C}$, (see Ref. [2, 18]), we have the stronger result $\hat{A} \simeq X$.

2) Fermionic T-duality:

a) Mathematical Example: Now we consider the case $A = M_\infty$ (also called the CAR—algebra). The sections of locally trivial bundles with fiber $M_\infty \otimes K$ are operator-valued fields on $X$ which satisfy the Canonical Anticommutation Relations. The $K$-theory of $M_\infty$ is $K_0(M_\infty) \simeq \mathbb{Z}[1/2]$ and $K_1(M_\infty) \simeq 0$. The positive cone of $M_\infty$ is

$$K_0(M_\infty)_+ \simeq \{a/2^k | a \in \mathbb{Z}, a \geq 0, k \geq 0 \}.$$ 

As a result $K_0(M_\infty)_+ \simeq \mathbb{Z}$ since it is a cyclic group on one generator namely $2 \in \mathbb{Z}[1/2]$. Hence, $\mathcal{Bun}_{S^1}(M_\infty \otimes K) \simeq \mathbb{Z}$ and there can be nontrivial fiberings of $M_\infty$ over $S^1$.

In Sec. (3.3) Item (1), the Topological T-dual-in the sense of Sec. (2)—of $C^*$–algebras in $\mathcal{Bun}_{S^1}(M_\infty \otimes K)$ was calculated. It was shown that the Topological T-dual must be isomorphic $\mathcal{O}_2 \otimes K$ and by the argument in Sec. (2) this must be the unique Topological T-dual as a noncommutative space.

When $X = \{\text{pt}\}$, a space with only one point, it had been remarked in that example that the crossed product should be viewed as a noncommutative space $(\mathcal{O}_2)$ which is a ‘noncommutative fibration’ with circle fibers over a simple algebra (the CAR algebra again, see
Sec. (3.3) Item (1) above)-whose Prim must be a point. We also showed in that example that no Rokhlin $\mathbb{R}$–action can give this space as a crossed product.

We consider a supersymmetric background $\mathbb{R}^{(8,1)} \times S^1$. We place a graviphoton flux on the background and perform a fermionic T-duality. The T-dual should be non-anti-commutative superspace over a circle. We identify the original space before fermionic T-duality with an element of $\text{Bun}_{S^1}(M_{2\infty} \otimes K)$.

We claim that the fermionic T-dual of a supersymmetric theory over a circle is non-anti-commutative superspace over a point-which we identify with the unique Topological T-dual $O_2 \otimes K$ above. The T-dual circle action would then correspond to the rotation of the generators of non-anti-commutative superspace by a phase. This should be matched with the Buscher’s rules for fermionic T-duality:

It can be seen that the physical T-dual is a circle with the fermionic directions ‘twisted’ by the T-dual graviphoton flux into NAC superspace.

In Sec. (3.3) Item (2), the Topological T-dual-in the sense of Sec. (2)-of specific $C^*$–algebras in $\text{Bun}_X(M_{2\infty} \otimes K)$ was calculated for $X$ a space with a free circle action. We can see that the calculation is a ‘fibering’ of the previous calculation in item (1) there.

Due to all the above, we strongly suggest that the Topological T-dual in the sense of Sec. (2) of a $M_{2\infty}$–bundle over a space $X$ with a free circle or torus action is a non-anticommutative superspace over $W \simeq X/S^1$. In our proof we find that the topological type of the T-dual bundle will vary depending on the details of the lifted circle action, but we suggest that this be fixed by matching the result to Buscher’s rules.

In Sec. (3.3) Item (3), we argued that the Topological T-dual in the sense of Sec. (2) may be extended to trivial $T^N$–bundles over a point. We also proved that in this case there is a unique Topological T-dual in this extended sense, but pointed out the difficulty of calculating it. For example, it is clear that the Topological T-dual $C^*$–algebra is a $\mathcal{Z}$–stable, $UHF$–stable $C^*$–bundle over a point, but its not clear what the Topological T-dual is.

b) Relation to Physics: We speculate that the above transformation is the ‘fermionic T-duality’ of Maldacena-Berkovits. In Ref. [50], the authors define a new type of T-duality transformation, at the tree level, which T-dualizes in a fermionic direction. For the duality to be nontrivial, the analogue of the $B$–field in the fermionic direction (the ‘graviphoton flux’) has to be nontrivial.
From the paper of Maldacena-Berkovits, the effect of a fermionic T-duality is to leave the topology of the space alone. The analogue of the Buscher rules for fermionic T-duality are quoted below in the bispinor formalism of Ref. [50] from Sec. (2.4) of the same paper (see Ref. [50] for notation and details):

\[ -\frac{i}{4} e^\phi F'^{\alpha\beta} = -\frac{i}{4} e^\phi F^{\alpha\beta} - \epsilon^\alpha \tilde{\epsilon}^\beta C^{-1} \]

\[ \phi' = \phi + \frac{1}{2} \log C \]

where as usual unprimed symbols denote the original fields while primed ('\) symbols denote the dual fields. Also $\phi$ is the dilaton, $C$ is the $\theta = \tilde{\theta} = 0$ component of the Kalb-Ramond field component $B_{11}$, identified with the graviphoton flux $(\epsilon^\alpha, \tilde{\epsilon}^\alpha)$ are spinors and $F^{\alpha\beta}, F'^{\alpha\beta}$ are bispinors constructed from the Ramond-Ramond Fluxes.

It can be seen from the above that the T-duality transformation does not change the metric or $B-$fields but changes the Ramond-Ramond flux and dilaton on the background.

Fermionic T-duality was later studied for a two dimensional supertorus with $B-$field and graviphoton flux in Ref. [61]. In this paper, the T-dual is the result of an ordinary 'bosonic' T-duality followed by a 'fermionic T-duality'. This is because the isometry being dualized has components in the fermionic direction.

Consider the example in Sec. (3.3) Item (2) above. We associate the $C^\ast-$algebra $\mathcal{D} \in \mathbb{Hom}_{\mathbb{S}}(\mathcal{M}_{2\ast} \otimes \mathcal{K})$ to a superspace with body a point with one fermionic direction with periodization in the fermionic direction. If there is Ramond-Ramond flux present on the original spacetime, it gives a class in $H^1(S^1, \mathbb{Z})$ which is part of the characteristic class of the $\mathcal{M}_{2\ast}-$bundle above.

c) $K-$theory of $\mathcal{D}$ and $\mathcal{D}-$brane charge: The Topological T-dual $C^\ast-$algebra in the sense of Sec. [2] to $\mathcal{D}$ in Part (a) above is -by the Example in Part (a) above-$\mathcal{O}_2 \otimes \mathcal{K}$.

Now, suppose $X \neq \text{pt}$. Since $K_0(\mathcal{O}_2 \otimes \mathcal{K})$ is zero, $K_0(\mathcal{O}_2 \otimes \mathcal{K})_+ \subset \mathcal{D}$ is zero as well, hence the elements of the spectral sequence tableaux of Ref. [24] after Thm. (4.2) are identically zero. The dual bundle is trivial with zero characteristic class since the spectral sequence computing $E_0^\ast(X)$ is identically zero (see Ref. [24] after Thm. (4.2)).

Also, note that since $R^\ast_{\mathbb{C}} = K_0(\mathcal{A})_+ \subset \mathcal{D}$ is zero and the characteristic class of the Ramond-Ramond flux can only lie in $H^1(X, R^\ast_{\mathbb{C}}) \simeq 0$. 

48
there can be no Ramond-Ramond flux on the T-dual. This is consistent with the Buscher’s rules for fermionic T-duality above as well-by Ref. [50] the T-dual has imaginary Ramond-Ramond flux under fermionic T-duality.

We tentatively identify the geometry of the T-dual as a point times ‘non-anti-commutative superspace’ (see Ref. [61]). It is interesting to note that the generators of the Cuntz algebra (which is the T-dual)

\[ S_1 S_1^* + S_2 S_2^* = 1, S_j^* S_j = 1 \]

look remarkably similar to the defining relations for non-anti-commutative variables \( \theta^a \theta^b + \theta^b \theta^a = 1 \).

We tentatively associate the Cuntz algebra to the ring of functions on ‘non-anti-commutative superspace’ - the T-dual of superspace with graviphoton flux turned on.

Consider \( D \) as the trivial element of \( \mathcal{B}un_{S^1} (M_{2\infty} \otimes \mathcal{K}) \). The \( K \)-theory of the space is, by Kunneth’s theorem \( K_0(D) \simeq \mathbb{Z}[1/2], K_1(D) \simeq 0 \).

\( d) \ K\)-theory of \( X \) and twisting: We had remarked above that the space \( X \simeq \text{Prim}(D) \) should be viewed as having ‘noncommutative points’ given by the \( C^* \)-algebra \( A \simeq M_{2\infty} \). We note that this may be viewed as changing the \( K \)-theory of the space \( X \). To calculate the \( K \)-theory of the above space we should actually calculate \( K_*(D) \) (and not \( K_*(C_0(X)) \) as usual).

In the above example, we have a trivial \( (M_{2\infty} \otimes \mathcal{K}) \)-bundle over \( X \) and hence, the \( K \)-theory of the spacetime background should be given by \( K_*(C_0(X, M_{2\infty})) \) and not \( K_*(C_0(X, \mathbb{C})) \) as is usual.

Now, we have that \( C_0(X, M_{2\infty}) \simeq C_0(X) \otimes M_{2\infty} \) hence by the Kunneth theorem for \( C^* \)-algebras (see Blackadar’s book Ref. [62], V.1.5.10, pg. 417), we have that

\[ K_*(C_0(X, M_{2\infty})) \simeq K_*(C_0(X)) \otimes K_*(M_{2\infty}) \]

(the Tor_1^\mathbb{Z} term is zero since \( K_*(M_{2\infty}) \) is torsion free). This gives us the \( K_0 \) group of \( X \) as \( K_0(C_0(X)) \otimes \mathbb{Z}[1/2] \) and \( K_1 \) group of \( X \) as \( K_1(C_0(X)) \otimes \mathbb{Z}[1/2] \). We suspect that the above groups store information about the fermionic part of the supersymmetric background, but it would be interesting to examine this further.

If we restrict ourselves to \( M_{2\infty} \)-bundles over \( X \), we cannot get the ‘higher twists’ of Ref. [53], to see these we need to study other \( C^* \)-algebras \( A \).
Remark: It would be interesting to test the above idea on a better example, perhaps a graded $C^*$--algebra $G$ which was a Cuntz algebra bundle over a space with noncommutative torus fibers. We would also like to make an analogy with Ref. [61] and naturally associate this $C^*$--algebra to the the T-dual of a supertorus with $B$--field and graviphoton flux.

One way to calculate the above $C^*$--algebra $G$ might be as follows: Following the example of Ref. [2], Sec. (5) we could consider $C^*(\mathcal{H}_Z) \otimes \mathcal{O}_2 \otimes \mathcal{K}$. This is a trivial noncommutative $\mathcal{O}_2 \otimes \mathcal{K}$--bundle over the noncommutative space associated to $C^*(\mathcal{H}_Z) \otimes \mathcal{K}$. In Ref. [2], Sec. (5), it was shown that $C^*(\mathcal{H}_Z) \otimes \mathcal{K}$ was a noncommutative two-torus fibration over a circle. It was also argued in Section (5) of Ref. [2] that $C^*(\mathcal{H}_Z) \otimes \mathcal{K}$ was an example of a 'noncommutative T-dual' of a three-torus with $H$--flux.

We would like to construct a $C^*$--algebra $G$ with $\text{Prim}(G) \simeq \mathbb{T}^3$ with $H$--flux and $RR$--flux which would have $C^*(\mathcal{H}_Z) \otimes \mathcal{O}_2$ as a crossed product by a $\mathbb{R}^2$--action lifting the natural circle action on $\mathbb{T}^3$. It would be interesting to generalize the construction on Section (5) of Ref. [2] to this situation.

However, the above lifting result won’t work for lifting $\mathbb{R}^2$--actions, so at present its not clear how to do the above calculation. By analogy with Ref. [61], this noncommutative space $G$ should possess a rich set fermionic and bosonic T-duals which would appear as Morita Equivalences of the $C^*$--algebra $G$.

Remark: Generalizing the above machinery, we conjecture that if the spacetime background possesses $N = k$ supersymmetry we should associate to it $M_{2^k} \otimes \cdots \otimes M_{2^k}$ bundles over $X$. Note that such bundles are isomorphic to $M_{2^k}$--bundles over the space $X$ since

$$M_{2^k} \otimes \cdots \otimes M_{2^k} \simeq M_{2^k}$$

by definition of the CAR algebra as a direct limit of an infinite tensor product of $M_2(\mathbb{C})$ (see Ref. [1] and references therein for details). Thus, we conjecture that we should associate one CAR--algebra factor in $\mathcal{A} \otimes \mathcal{K}$ with each supersymmetric direction.

3) Timelike T-duality:

a) Mathematical Example: From Cor. (4.6) of Ref. [24], if $\mathcal{A} = \mathbb{Z}$ the Jiang-Su algebra, $\text{Bun}_X(\mathbb{Z} \otimes \mathcal{K}) \simeq \bigoplus_{k \geq 1} H^{2k+1}(X, \mathbb{Z})$. Thus, the characteristic class of locally trivial bundles over $X$ with fiber
$\mathbb{Z} \otimes \mathcal{K}$, the stabilized Jiang-Su algebra might encode the characteristic classes of background $RR$-fluxes.

Further, by Cor. (4.7) of Ref. [24], there is a natural map $\mathcal{C} \rightarrow \mathbb{Z}$ which implies that there is a natural transformation of cohomology theories $T : E^*_C(X) \rightarrow E_Z(X)$. By the same Corollary, $E^1(X) \simeq H^3(X, \mathbb{Z})$ is a natural direct summand of $E^1_Z(X) \simeq \text{Bun}_X(\mathbb{Z} \otimes \mathcal{K})$.

Hence locally trivial bundles over $X$ with fiber the stabilized Jiang-Su algebra $\mathbb{Z} \otimes \mathcal{K}$ might describe a space with background $H$-flux together with background $RR$-fluxes.

b) Relation to Physics: We speculate that the above transformation is the ‘Timelike T-duality’ transformation of Hull (see Ref. [60] for details).

Consider spacetime backgrounds of the form $X \times \mathbb{R}$ where the $\mathbb{R}$-direction is the time variable and the $X$ direction has no circle action. To the spacelike part $X$ of $X \times \mathbb{R}$ with sourceless, constant $RR$-flux and $H$-flux we associate $D \in \text{Bun}_X(\mathbb{Z} \otimes \mathcal{K})$. We mimic periodization in the timelike direction by considering $\text{Ind}_\mathbb{R}^\mathbb{Z}(D, \alpha)$— these are stabilized Jiang-Su bundles over the circle. $X \times S^1 \simeq \text{Prim}(\text{Ind}_\mathbb{R}^\mathbb{Z}(D, \alpha))$ where $\alpha$ is an automorphism of $D$ reflecting the properties of the periodization above.

It is not possible to calculate the Topological T-dual in Sec. (2) for $\mathbb{Z} \otimes \mathcal{K}$, not enough is known about Rokhlin group actions on the Jiang-Su algebra. In this case, even though the result won’t cover Rokhlin actions, we attempt to identify the crossed product $\text{Ind}_\mathbb{R}^\mathbb{Z}(D, \alpha) \times \mathbb{R}$ with the timelike T-dual. This is reasonable as this is only a preliminary inquiry.

If we take $\alpha$ to be independent of $x \in X$, then,

$$\text{Ind}_\mathbb{R}^\mathbb{Z}(D \otimes \mathcal{K}, \alpha) \simeq C_0(X) \otimes \text{Ind}_\mathbb{R}^\mathbb{Z}(\mathbb{Z} \otimes \mathcal{K}, \alpha).$$

We calculate this fully in the next part.

c) K-theory and D-brane charge: By Thm. (2.9) of Ref. [63] we find the $K$–theory of $\mathbb{Z}$ as

$$(K_0(\mathbb{Z}), K_0(\mathbb{Z})_+, K_1(\mathbb{Z}), [1_\mathbb{Z}]) \simeq (\mathbb{Z}, \mathbb{Z}_+ \cup \{0\}, \{0\}, 1)$$

Now, by the discussion around Eqs. (1.4, 1.5) above, and by Cor. (4.6) of Ref. [24], $\text{Bun}_{S^1}(\mathbb{Z} \otimes \mathcal{K}) \simeq K_0(\mathbb{Z})_+ \simeq \{1\}$ as an abelian group. Hence there can’t be any nontrivial Jiang-Su bundles over the circle. Thus, $D \simeq C(S^1) \otimes \mathbb{Z} \otimes \mathcal{K}$.

The $K$–theory of $K_*(C_0(X) \otimes T_\alpha(\mathbb{Z} \otimes \mathcal{K}))$ now follows from the previous part. Using the Kunneth theorem and the fact that the
$K$-theory of the Jiang-Su algebra has no torsion (see Ref. [23] and references therein for details) the $K$-theory groups are given by

$$K_*(C_0(X) \otimes C(S^1) \otimes \mathbb{Z} \otimes \mathcal{K}) \simeq K_*(C_0(X \times S^1)) \otimes K_*(\mathbb{Z}).$$

By the previous part, $K_0(\mathbb{Z}) \simeq \mathbb{Z}$ and $K_1(\mathbb{Z}) \simeq 0$. Hence $K_0(C_0(X) \otimes \mathbb{Z} \otimes \mathcal{K}) \simeq K_0(X \times S^1)$ and $K_1(C_0(X) \otimes \mathbb{Z} \otimes \mathcal{K}) \simeq K_1(X \times S^1)$. Thus, the dual has the same $K$-theory as the topological $K$-theory of $X \times S^1$. This is to be expected since the dual gains the timelike direction along which timelike T-duality acted as a new spatial dimension (see Ref. [60]).

d) $K$-theory and twisting: As in the previous example we should calculate $K_*(C_0(X) \otimes \mathcal{Z})$ since the natural $K$-theory of $X$ with $\mathcal{Z}$ present is not given by $K_*(C_0(X))$ but by $K_*(C_0(X) \otimes \mathcal{Z})$. By Ref. [53], this is isomorphic to $K^*(X)$.

Physically, we could interpret this as the following: The twisting of the $D$-brane charges by the $H$-flux seem to vanish when RR-flux is turned on. It would be interesting to see if this can be determined from String Theory calculations.

5. Summary and Conclusions

To summarize, we associate a spacetime background $X$ with $H$-flux and sourceless Ramond-Ramond flux to a section algebra $(\mathcal{D})$ of a locally trivial fiber bundle with fiber a fixed self-absorbing $C^*$-algebra $(\mathcal{A})$.

This has the following consequences:

- The $C^*$-algebra of functions on a point changes from $C \otimes \mathcal{K}$-in the presence of $H$-flux only- to $\mathcal{A} \otimes \mathcal{K}$-where $\mathcal{A}$ depends on which background fields are present: In the presence of Ramond-Ramond flux only $\mathcal{A} = M_{2\infty}$ and in the presence of $H$-flux and Ramond-Ramond flux $\mathcal{A} = \mathbb{Z}$.

  Thus, the presence of sourceless Ramond-Ramond flux causes the appearance of 'noncommutative points' of various types depending on whether other fluxes are present or not. It should be possible to detect this by examining the worldvolume theory of $D0$-branes or the correct matrix model.

- The stabilized $C^*$-algebra of functions on the brane worldvolume changes from being strongly Morita equivalent to $C_0(U)$ locally to being strongly Morita equivalent to $C_0(U, \mathcal{A} \otimes \mathcal{K})$ locally. The obstruction to the global strong Morita equivalence is as follows:
If only $H-$flux is present the $C^*-$algebra of functions on the brane worldvolume is only locally strongly Morita equivalent to $C_0(X)$. The obstruction to it being \textit{globally} strongly Morita equivalent to $C_0(X)$ is the gerbe on the brane worldvolume (see Ref. [53]) whose gerbe curvature is the $H-$flux.

If Ramond-Ramond flux is present the stabilized $C^*-$algebra of functions on the brane worldvolume is locally strongly Morita equivalent to $C_0(X,A)$. The obstruction to it being \textit{globally} strongly Morita equivalent to $C_0(X,A)$ is a \textit{higher gerbe} on the brane worldvolume (see Ref. [53] Sec. (1)).

The work in this paper implies that the ‘twist’ of this higher gerbe encodes all the sourceless background fields present ($H-$flux and Ramond-Ramond flux). It would be interesting to see if a connection-like structure on this higher gerbe gave fields with the correct transformation laws in analogy with gerbe connections on gerbes on the worldvolume of $D-$branes in a background sourceless $H-$flux.

- As argued in Refs. [52] the worldvolume gauge theory for ordinary stack of $N D-$branes may be written using section algebras of locally trivial $M_N(C)-$bundles (Azumaya algebras) over the $D-$brane worldvolume $X$.

Based on the arguments we would expect the worldvolume gauge theory of a possibly infinite stack of $D-$branes in background sourceless $RR-$flux to be expressible in terms of the $C^*-$algebras like $\mathcal{D}$ above which are elements of $\mathcal{Bun}_X(A \otimes K)$. It would also be interesting to see if specific D-brane configurations correspond to specific $C^*-$algebras $D \in \mathcal{Bun}_X(A \otimes K)$.

- In addition we argue about the lift of $S^1-$actions on $X$ to unique Rokhlin $\mathbb{R}-$actions on $D$ up to cocycle conjugacy. We have seen that the lift is possible for a certain class of fiber algebras and might not be possible for other fiber algebras. We propose a procedure for finding the Topological T-dual which always gives a T-dual $C^*-$algebra for any fiber algebra and which agrees with the crossed product $C^*-$algebra T-dual when there is a Rokhlin lift of the circle action on $X$ present. We also characterize the T-dual $C^*-$algebra in Thm. (2.3) above.

We use this formalism to calculate the Topological T-dual $C^*-$algebra for some examples. A more detailed analysis of the structure of the T-dual calculated in Thm. (2.3) might give some more interesting-and possibly physically relevant-examples by analogy with Topological T-duality for continuous-trace $C^*-$algebras.
We propose two physical examples of this extension of Topological T-duality in Subsection (4.3) above: Fermionic T-duality and Timelike T-duality.

6. Acknowledgements

I thank Professor Jonathan M. Rosenberg of the University of Maryland, College Park, for all his advice and help.

I thank the Department of Mathematics, Harish-Chandra Research Institute, Allahabad, for support in the form a postdoctoral fellowship during which the first draft of this paper was written.

I thank the School of Mathematics, NISER, HBNI, Bhubaneshwar for support in the form of a Visiting Assistant Professorship during the writing of part of this paper.

I thank the Mathematical and Physical Sciences Division, School of Arts and Sciences, Ahmedabad University, Ahmedabad for all their help and support.

References

[1] J. Rosenberg ‘Topology, C*-algebras, and String Duality’, *CBMS Regional Conference Series in Mathematics*, 111, American Mathematical Society, (2009).

[2] V. Mathai and J. Rosenberg, ‘T-Duality for Torus Bundles with H-fluxes via Noncommutative Topology’, *Comm. Math. Phys.*, 253, pp. 705-721, (2005).

[3] D. Grady and H. Sati, ‘Ramond-Ramond Fields and Twisted Differential K-Theory’, *Adv. Theor. Math. Phys.*, 26, 5, 1097-1155, (2022).

[4] G. Moore and E. Witten, ‘Self-duality, Ramond-Ramond fields, and K-theory’, *JHEP*, 2000, pp. 032-032, (1999).

[5] D. S. Freed and M. Hopkins, ‘On Ramond-Ramond fields and K-theory’, *JHEP*, 2000, pp. 044-044, (2000).

[6] V. Mathai and H. Sati, ‘Some relations between K—theory and E8 gauge theory’, *JHEP*, 03, pp. 016, (2000).

[7] J. Brodzki, V. Mathai, J. Rosenberg and R. J. Szabo, ‘D-branes, RR—fields and duality on noncommutative manifolds’, *Comm. Math. Phys.*, 277, pp. 643-706, (2008).
[8] G. Moore and N. Saulina, ‘T-duality and K-theoretic Partition Function of Type IIA Superstring Theory’, Nucl. Phys., 670, pp. 27-89, (2002).

[9] J. Brodzki, V. Mathai, J. Rosenberg and R. J. Szabo, ‘Noncommutative Correspondences, Duality and D−branes in Bivariant Theory’, Adv. Theor. Math. Phys., 13, pp. 497-552, (2009).

[10] M. Dadarlat and U. Pennig, ‘Unit Spectra of K-theory from strongly self-absorbing C*-algebras’, Alg. Geom. Topol., 15, pp. 137-168, (2015).

[11] M. Dadarlat and U. Pennig, ‘A Dixmier-Douady theory for locally trivial bundles of self-absorbing C*-algebras’, J. Reine Angew. Math., 718, pp. 153-181, (2016).

[12] M. Dadarlat and U. Pennig, ‘A Dixmier-Douady Theory for strongly self-absorbing C*-algebras II: the Brauer group’, J. Noncommutative Geom., 9, 4, pp. 1137-1154, (2015).

[13] D. Bernstein and R.G. Leigh, ‘Superstring Perturbation Theory and Ramond-Ramond Backgrounds’, Phys. Rev. D, 60, pp. 106002 (1999); D. Bernstein and R. G. Leigh, ‘Quantization of Superstrings in Ramond-Ramond Backgrounds’, Phys.Rev. D, 63, pp. 026004 (2001).

[14] A. Sen, ‘Gauge invariant 1PI effective superstring field theory: inclusion of the Ramond sector’, JHEP, 2015, pp. 25, (2015).

[15] L. Cornalba, M. S. Costa, R. Schiappa, ‘D-brane dynamics in constant Ramond-Ramond potentials, S-duality and noncommutative geometry’ Adv. Theor. Math. Phys. 9, 3, pp. 355-406 (2005).

[16] J. Brodzki, V.Mathai, J. Rosenberg and R. J. Szabo, ‘D-branes, KK−theory and duality on noncommutative spaces’, J. Phys. Conf. Ser., 103, pp. 012004, (2008).

[17] D. Williams, ‘Crossed Products of C*-Algebras’, Surveys and Monographs,134, American Mathematical Society, Providence, (2007).

[18] I. Raeburn and J. Rosenberg, ‘Crossed Products of Continuous-Trace C*-algebras by Smooth Actions’, Trans. Amer. Math. Soc., 305, pp. 1-45, (1988).

[19] P. Bouwknegt and A. Pande, ‘Topological T-duality and T-folds’, Adv. Theor. Math. Phys., 13, 5, (2009).

[20] Bhisan Jacelon, ‘A Stably Finite Analogue of the Cuntz algebra O₂’, Thesis, University of Glasgow, (June 2011).
[21] N. Nawata, ‘Trace Scaling Automorphisms of the Stabilized Razak-Jacelon Algebra’, Proc. Lond. Math. Soc., 118, 3, pp. 545-576, (2019).

[22] Y. Sato, ‘The Rohlin Property for Automorphisms of the Jiang-Su Algebra’, Jour. Funct. Anal., 259, pp. 453-476, (2010).

[23] Andrew S. Toms and Wilhelm Winter, ‘Strongly self-absorbing \( C^* \)-algebras’, Trans. Amer. Math. Soc., 359, pp. 3999-4029, (2007).

[24] M. Dadarlat and U. Pennig, ‘A Dixmier-Douady theory for strongly self-absorbing \( C^* \)-algebras’, J. Reine Angew. Math., 718, pp. 153-181, (2016).

[25] I. Raeburn and D. Williams, ‘Morita Equivalence and Continuous-Trace \( C^* \)-algebras’, Surveys and Monographs, 60, American Mathematical Society, Providence, (1998).

[26] Gabor Szabo, ‘Classification of Rokhlin Flows on \( C^* \)-algebras’, Comm. Math. Phys., 382, pp. 2015-2070, (2021).

[27] A. Kishimoto, ‘A Rohlin property for one-parameter automorphism groups’, Comm. Math. Phys., 179, pp. 599-622, (1996).

[28] I. Hirshberg, W. Winter, J. Zacharias, ‘Rokhlin Dimension and \( C^* \)-dynamics’, Comm. Math. Phys., 335, pp. 637-670, (2015).

[29] I. Hirshberg, G. Szabo, W. Winter and J. Wu, ‘Rohlin dimension for flows’, Comm. Math. Phys., 353, 1, pp. 253-316, (2017).

[30] N. C. Phillips, ‘The Tracial Rokhlin Property is generic’, arXiv:1209.3859

[31] Gabor Szabo, Jianchao Wu, Joachim Zacharias, ‘Rokhlin Dimension for actions of residually finite groups’, Ergod. Th. Dyn. Syst., 39, 8, pp. 2248-2304, (2019).

[32] E. Gardella, ‘Rokhlin Dimension for Compact Group Actions’, Indiana U. Math. J., 66, pp. 659-703, (2017).

[33] Ilan Hirshberg and Wilhelm Winter, ‘Rokhlin Actions and self-absorbing \( C^* \)-algebras’, Pacific Journal of Mathematics, 233, 1, pp. 125-143, (2007).

[34] Snigdhayan Mahanta, ‘Algebraic \( K \)-theory, \( K \)-regularity and \( T \)-duality for \( O_\infty \)-algebras’, Math. Phys. Anal. Geom., 18, 1, pp. 18-12, (2015).
[35] E. Gardella, ‘Crossed Product by Compact Group actions with the Rokhlin Property’, *J. Noncommutative Geom.*, **11**, 4, pp. 1593-1626, (2017).

[36] E. Gardella, ‘Compact Group Actions with the Rokhlin Property’, *Trans. Amer. Math. Soc.*, **371**, pp. 4, (2019).

[37] G. Elliott, D. Evans and A. Kishimoto, ‘Outer Conjugacy Classes of Trace-Scaling Automorphisms of Stable UHF Algebras’, *Math. Scand.*, **83**, 1, pp. 74-86, (1998).

[38] Valentin Deaconu, ‘Symmetries of Cuntz-Pimsner algebras’, Talk at UNL Functional Analysis Seminar, November 3, (2018). Slides at [https://www.math.unl.edu/~adonsig1/NIFAS/1811-Deaconu.pdf](https://www.math.unl.edu/~adonsig1/NIFAS/1811-Deaconu.pdf).

[39] Michael V. Pimsner, ‘A Class of $C^*$—algebras generalizing Cuntz-Kreiger algebras and the Crossed Product by $Z$’, *Fields Inst. Communications*, **12**, (1997).

[40] G. Hao, C-K Ng, ‘Crossed Products of $C^*$—correspondences by amenable group actions’, *J. Math. Anal. Appl.*, **345**, pp. 702-707, (2008).

[41] Dana Williams, ‘The Mackey Machine for Crossed Products’, Slides of Talk at GPOSTS 2009. Slides at [http://math.colorado.edu/gpots2009/WilliamsGPOTS09.pdf](http://math.colorado.edu/gpots2009/WilliamsGPOTS09.pdf).

[42] Gabor Szabo, ‘Strongly Self-Absorbing $C^*$—dynamical Systems’, *Trans. Amer. Math. Soc.*, **370**, pp. 99-130, (2018).

[43] C. Schafhauser, ‘Cuntz-Pimsner Algebras, Crossed Products and K—theory’, *Jour. Funct. Anal.*, **259**, pp. 2927-2946, (2015).

[44] E. Gardella, ‘Circle Actions on $UHF$—absorbing $C^*$—algebras’, *Houston J. Math.*, **44**, 2, pp. 571-601, (2018).

[45] E. Gardella, ‘KK-theory of circle actions with the Rokhlin Property’, *arXiv: 1405.2469*, (2014).

[46] N.C. Phillips, ‘The Classification of Kirchberg Algebras’, Thematic Programme at Fields Institute, Notes by E. Gardella (2014).

[47] D. Evans and A. Kishimoto, ‘Trace scaling automorphisms of certain stable AF algebras’, *Hokkaido Math. Jour.*, **26**, pp. 211-224, (1997).

[48] O. Brattelli, A. Kishimoto, D. W. Robinson, ‘Rohlin flows on the Cuntz algebra $O_\infty$’, *Jour. Funct. Anal.*, **242**, 2, pp. 472-511, (2007).
[49] A. J. Dean, ‘A Continuous Field of Projectionless C*-algebras’, Doctoral Thesis, University of Toronto, (1999).

[50] Nathan Berkovits and Juan Maldacena, ‘Fermionic T-duality, Dual Superconformal Symmetry, and the Amplitude/Wilson Loop Connection’, JHEP, 0809, pp. 062, (2008).

[51] T. Katsura, ‘The ideal structures of Crossed Products of Cuntz algebras by quasi-free actions of abelian groups’, Canad. J. Math., 55, 6, pp. 1302-1338, (2003).

[52] A. Kapustin, ‘D-branes in a topologically nontrivial B-field’, Adv. Theor. Math. Phys., 4, pp. 127-154, (2000).

[53] U. Pennig, ‘Twisted K-theory with coefficients in C*-algebras’, arXiv: 1103.4096, (2011).

[54] Fabio F. Ruffino, ‘Topics on the geometry of D-brane charges and Ramond-Ramond fields’, JHEP, 0911, pp. 012, (2009).

[55] P. Bouwknegt, J. Evslin and V. Mathai, ‘Spherical T-duality’, Comm. Math. Phys., 337, pp. 909-954, (2015).

[56] L. Macdonald, V. Mathai and H. Saratchandran, ‘On the Chern Character in Higher Twisted K-theory and T-duality’, Comm. Math. Phys., 385, pp. 331-368, (2021).

[57] E. Witten, ‘Overview of K-theory applied to Strings’, Int. J. Mod. Phys. A, 16, pp. 693-706, (2001).

[58] U. Pennig, ‘A Noncommutative Model for Higher Twisted K-theory’, J. Topology, 9, 1, pp. 27-50, (2016).

[59] M. Berasaluce-Gonzalez, P.G. Camara, F. Marchesango, A. M. Uranga, ‘Zp charged branes in flux compactifications’, JHEP, 04, pp. 138, (2013).

[60] C. Hull, ‘Timelike T-duality, de Sitter Space, Large N Gauge Theories and Topological Field Theory’, Jour. High Energy Phys., 9807, pp. 021, (1998); C.N. Pope, A. Sadrzadeh, S.R. Scuro, ‘Timelike Hopf Duality and Type IIA* String Solutions’, Class. Quant. Grav., 17, pp. 623-641, (2000).

[61] Ee Chang-Young, Hiroaki Nakajima, Hyeonjoon Shin, ‘Fermionic T-duality and Morita Equivalence’, JHEP, 2011, pp. 2, (2011).

[62] B. Blackadar, ‘Theory of C*-algebras and von Neumann Algebras’, in Operator Algebras and Non-Commutative Geometry, 122, III, Encyclopedia of Mathematical Sciences, Springer-Verlag, (2006).
[63] X. Jiang and H. Su, ‘On a Simple Unital Projectionless C-algebra’, *Am. J. Math.*, 121, 2, pp. 359-413, (Apr. 1999).

**Division of Physical and Mathematical Sciences, School of Arts and Sciences, Ahmedabad University, Ahmedabad, India.**

*E-mail address: ashwin.s.pande@gmail.com, ashwin.pande@ahduni.edu.in*
