Lectures on Twistors

Itzhak Bars

Department of Physics and Astronomy
University of Southern California, Los Angeles, CA 90089-0484, USA

Abstract

In these lectures I will discuss the following topics

- Twistors in 4 flat dimensions.
  - Massless particles, constrained phase space \((x^\mu, p^\mu)\) versus twistors.
  - Physical states in twistor space.

- Introduction to 2T-physics and derivation of 1T-physics holographs and twistors.
  - Emergent spacetimes & dynamics, holography, duality.
  - Sp(2,R) gauge symmetry, constraints, solutions and \((d,2)\).
  - Global symmetry, quantization and the SO(\(d,2\)) singleton.
  - Twistors for particle dynamics in \(d\) dimensions, particles with mass, relativistic, non-relativistic, in curved spaces, with interactions.

- Supersymmetric 2T-physics, gauge symmetries & twistor gauge.
  - Coupling \(X, P, g\), gauge symmetries, global symmetries.
  - Covariant quantization, constrained generators & representations of \(G_{\text{super}}\).
  - Twistor gauge: supertwistors dual to super phase space. Examples in \(d=4,6,10,11\).

- Supertwistors and some field theory spectra in \(d=4,6\).
  - Super Yang-Mills \(d=4, N=4\); Supergravity \(d=4, N=8\).
  - Self-dual tensor supermultiplet and conformal theory in \(d=6\).

- Twistor superstrings
  - \(d + 2\) view of twistor superstring in \(d = 4\).
  - Worldsheet anomalies and quantization of twistor superstring.
  - Open problems.

---

Lectures delivered at the “2005 Summer School on String/M Theory” in Shanghai, China, and the International Symposium QTS4, “Quantum Theory and Symmetries IV”, Varna, Bulgaria.
1 Twistors in d=4 flat dimensions

A massless spinless relativistic particle in 4 space-time dimensions is described by the action
\[ S(x,p) = \int d\tau \left( \partial_\tau x^\mu p_\mu - \frac{1}{2} \epsilon_{\mu\nu} p_\mu p_\nu \right). \]

It has a gauge symmetry under the transformations \( \delta e = \partial_\tau \varepsilon(\tau), \delta x^\mu = \varepsilon(\tau) p^\mu \), \( \delta p_\mu = 0 \). The generator of the gauge symmetry is \( p^2/2 \), and it vanishes as a consequence of the equation of motion for the gauge field \( \delta S/\delta e = p^2/2 = 0 \). This equation of motion is interpreted as demanding that the solution space must be gauge invariant (since the generator must vanish).

In the covariant quantization of this system one defines the physical states as those that satisfy the constraint \( p^2|\phi\rangle = 0 \), so that they are gauge invariant. A complete set of physical states is found in momentum space \( |k\rangle \) on which the gauge generator is simultaneously diagonal with the momentum operator \( p_\mu |k\rangle = |k\rangle k_\mu \), and \( p^2|k\rangle = |k\rangle k^2 = 0 \). The probability amplitude of a physical state in position space \( \langle x|\phi \rangle \) satisfies the condition \( \langle x|p^2|\phi \rangle = 0 \) which gives the Klein-Gordon equation \( \partial_\tau^2 \phi(x) = 0 \). The general solution is a superposition of plane waves, which are the probability amplitudes of physical states with definite momentum
\[ \text{General solution: } \phi(x) = \int \frac{d^4k}{(2\pi)^4} \delta(k^2) \left[a(k) e^{ik\cdot x} + h.c. \right] \]

Plane wave with definite momentum \( k^\mu: \phi_k(x) = \langle x|k\rangle \sim e^{ik\cdot x}, k^2 = 0 \).

A similar treatment for spinning particles leads to the spinning free field equations, such as the Dirac equation, Maxwell equation, linearized Einstein equation, etc.

1.1 Twistors

The following shows several ways of solving the constraint \( p^2 = 0 \) or \( k^2 = 0 \) that enter in these equations
\[ p^2 = 0 : \ p^0 = \pm \sqrt{p^2} \quad p^- = p_\mu^2/2p^+ \quad \text{or} \quad p_{\alpha\beta} = \pm (\lambda\lambda^\dagger)_{\alpha\beta} = \frac{1}{\sqrt{2}} p^\mu (\sigma_\mu)_{\alpha\beta} \]

In the second form, the matrix \( p_{\alpha\beta} \) is constructed from two complex numbers \( \lambda_1, \lambda_2 \) that form a doublet of \( \text{SL}(2,C) = \text{SO}(3,1) \)
\[ p = \pm \begin{pmatrix} \lambda_1 & \lambda_2^\ast \\ \lambda_1^\ast & \lambda_2 \end{pmatrix} = \pm \frac{1}{2\sqrt{2}} \begin{pmatrix} p^0 + p^3 & p_1 - ip_2 \\ p_1 + ip_2 & p^0 - p^3 \end{pmatrix} \]

This has automatically zero determinant \( \det(p) = (\lambda_1\lambda_2^\ast) (\lambda_2\lambda_1^\ast) - (\lambda_1\lambda_2^\ast) (\lambda_2\lambda_1^\ast) = 0 = (p^0 + p^3)(p^0 - p^3) - (p_1 - ip_2)(p_1 + ip_2) = p_0^2 - p^2 \), which imposes the desired solution \( p^\mu p_\mu = 0 \) automatically. Note that the overall phase \( e^{i\phi} \) of \( \lambda_\alpha \) drops out, so the matrix \( p_{\alpha\beta} \) really has only 3 real parameters, as it should.
1.1 Twistor

The reader is reminded of a bit of group theory for $\text{SL}(2, C) = \text{SO}(3, 1)$

spinors: \[
\begin{aligned}
\lambda_\alpha &= \left(\frac{1}{2}, 0\right) \\
\bar{\lambda}_\alpha &= \lambda_\dagger_{\bar{\alpha}} (0, \frac{1}{2})
\end{aligned}
\]

invariant tensors: \[
\varepsilon^{\alpha\beta} \text{ or } \varepsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

metric, raise / lower indices 

\[
\begin{aligned}
\left(\sigma_\mu\right)_{\alpha\beta} &= (1, i\sigma)_{\alpha\beta} \\
\left(\bar{\sigma}_\mu\right)_{\dot{\alpha}\dot{\beta}} &= (-1, \bar{\sigma})_{\dot{\alpha}\dot{\beta}} \\
\left(\frac{1}{\sqrt{2}}\sigma^\mu\right)_{\alpha\beta} &= \left(\frac{1}{\sqrt{2}}\bar{\sigma}^\mu\right)_{\dot{\alpha}\dot{\beta}}
\end{aligned}
\]

Finally a twistor is defined as \[
\mu^\alpha = -i \bar{x}^{\dot{\alpha}\dot{\beta}}\lambda_\beta, \quad \text{a “line” in spinor space.} \tag{6}
\]

Finally, a twistor is defined as $Z_A = \left(\mu^\alpha, \lambda_\alpha\right)$, $A = 1, 2, 3, 4$, that bundles together $\mu$ and $\lambda$ as a quartet. If $\mu$ satisfies the Penrose relation, then the pair $\mu, \lambda$ is equivalent to the phase space of the massless particle

\[
Z_A = \begin{pmatrix} \mu^\alpha \\ \lambda_\alpha \end{pmatrix} = \begin{pmatrix} (-i\bar{x}\lambda)^\dagger \\ \lambda_\alpha \end{pmatrix} \quad \text{on-shell phase space} \quad (x^\mu, p_\mu) \tag{7}
\]

Although not manifest, the massless particle action above has a hidden conformal symmetry $\text{SO}(4, 2)$. This symmetry can be made manifest through the twistor since $\text{SO}(4, 2) = \text{SU}(2, 2)$ and the quartet $Z_A$ can be classified as the fundamental representation 4 of $\text{SU}(2, 2)$. This non-compact group has a metric which can be taken as $C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1 \times 1$. Using the metric we define the other fundamental representation 4 of $\text{SU}(2, 2)$ and relate it to the complex conjugate of $Z_A$ as follows

\[
\bar{Z}^A = Z^C = \left(\lambda^\dagger_{\bar{\alpha}} \mu^{1\alpha}\right) = \left(\lambda^\dagger_{\bar{\alpha}} \left(i\lambda^\dagger \bar{x}\right)^\alpha\right), \quad C = \sigma_1 \times 1 \tag{8}
\]

So $\bar{Z}^A Z_A$ is invariant under $\text{SU}(2, 2)$. We remind the reader that the 4 and $\bar{4}$ of $\text{SU}(2, 2)$ correspond to the two Weyl spinors of $\text{SO}(4, 2)$. Now, with $\mu$ as given above, we have

\[
\bar{Z}^A Z_A = \lambda^\dagger_{\bar{\alpha}} \mu^{1\alpha} + \mu^{1\alpha} \lambda_\alpha = -i\lambda^\dagger \bar{x}\lambda + i\lambda^\dagger \bar{x}\lambda = 0. \tag{9}
\]

So, by construction the $Z_A$ are 4 constrained complex numbers. But we can reverse this reasoning, and realize that the definition of twistors is just the statement that $Z_A$ is a quartet that has an overall irrelevant phase and that is constrained by $\bar{Z}^A Z_A = 0$. Then the form of $\mu$ in terms of $\lambda$ can be understood as one of the possible ways of parameterizing a solution. The solution $\mu^\alpha = -i \bar{x}^{\dot{\alpha}\dot{\beta}}\lambda_\beta$ is interpreted as the massless particle. This is the conventional interpretation of twistors.

However, recently it has been realized that there are many other ways of parameterizing solutions for the same $Z_A$ in terms of phase spaces that have many other different interpretations [8]. For any solution, if we count the number of independent real degrees of freedom, we find

\[
\text{Independent: } 8 \text{ real } Z - (1 \text{ real constraint}) = 6 \text{ real } = \frac{\text{same as}}{3\bar{x} + 3\bar{p}} \tag{10}
\]
This is the right number not only for the massless particle, but also the massive particle, relativistic or non-relativistic, in flat space or curved space, interacting or non-interacting.

Next, we compute the canonical structure for the pair \((Z_A,\bar{Z}^A)\), and we find that it is equivalent to the canonical structure in phase space for the massless particle, iff we use the solution \(\mu^\alpha = -i\bar{x}^{\alpha\beta}\lambda_\beta\)

\[
L = i\bar{Z}^A \partial_\tau Z_A = i\bar{\lambda}_\alpha \partial_\tau \mu^\alpha + i\bar{\mu}^\alpha \partial_\tau \lambda_\alpha \\
= \lambda_\alpha^\dagger \partial_\tau (\bar{x}\lambda)^\alpha - (\bar{\lambda}^\dagger \bar{x})^\alpha \partial_\tau \lambda_\alpha \\
= \lambda_\alpha^\dagger (\partial_\tau \bar{x}^{\alpha\beta}) \lambda_\beta = Tr(p\partial_\tau \bar{x}) = p_\mu \partial_\tau x^\mu
\]

So the canonical pairs \((Z_A, i\bar{Z}^A)\) or \((\lambda^\alpha, i\mu_\alpha^\dagger)\) or \((x^\mu, p_\mu)\) are equivalent as long as they satisfy the respective constraints \(\bar{Z}^A Z_A = 0\) and \(p^2 = 0\). If we use some of the other solutions given in [8] then the correct canonical structure emerges for the massive particle, etc., all from the same twistor (see below).

Just like the constraint \(p^2 = 0\) followed from an action principle in Eq.(1), the constraints \(\bar{Z}^A Z_A = 0\) can also be obtained from the following action principle by minimizing with respect to \(V\)

\[
S(Z) = \int d\tau \left(\bar{Z}^A iD_\tau Z_A - 2hV\right) = \int d\tau \left(\bar{Z}^A i\partial_\tau Z_A + V\bar{Z}^A Z_A - 2hV\right).
\] (11)

Here \(D_\tau Z_A = \partial_\tau Z_A - iV Z_A\) is a covariant derivative for a U(1) gauge symmetry \(Z_A(\tau) \rightarrow Z_\alpha(\tau) = e^{i\omega(\tau)} Z_A(\tau)\). The gauge symmetry is precisely what is needed to remove the unphysical overall phase noted above.

For spinning particles, an extra term \(-2hV\) is included in the action (missing in former literature). This term is gauge invariant by itself under the U(1) gauge transformation of \(V\). We have been discussing the spinless particle \(h = 0\), but twistors can be generalized to spinning particles by taking \(h \neq 0\). The equation of motion with respect to \(V\) gives the constraint \(\bar{Z}^A Z_A = 2h\). If the twistor transform for massless particles, appropriately modified to include spin, is used to solve this constraint [2] [3] [4], then \(h\) is interpreted as the helicity of the spinning massless particle. But if the more general transforms in [8] are used, then \(h\) is not helicity, but is an eigenvalue of Casimir operators of SU(2, 2) in a representation for spinning particles\(^2\).

We have argued that the twistor action \(S(Z)\) is equivalent to the spinless massless particle action \(S(x, p)\) (at least in one of the possible ways of parameterizing its solutions). But note that \(S(Z)\) is manifestly invariant under the global symmetry SU(2, 2). This is the hidden conformal symmetry SO(4, 2) of the massless particle action \(S(x, p)\). Applying Noether’s theorem we derive the conserved current, which

\(^2\)This point will be discussed in detail in a future paper.
in turn is written in terms of $x^\mu, p_\mu$ as follows

$$J^A_B = Z_A \bar{Z}^B - \frac{1}{4} Z_C \bar{Z}^C \delta^B_A = \left( -\frac{i \bar{x} \lambda}{\lambda} \right) (\lambda^\dagger i \lambda^\dagger \bar{x})$$  \hspace{1cm} (12)

$$= \left( -\frac{i \bar{x} \lambda \lambda^\dagger \bar{x} \lambda \lambda^\dagger \bar{x}}{\lambda \lambda^\dagger} \right) \left( \frac{i \bar{x} p \bar{x} p \bar{x}}{p - ip \bar{x}} \right) = \frac{1}{4i} \Gamma_{MN} L_{MN}$$  \hspace{1cm} (13)

$$= \frac{1}{2i} \left( -\Gamma^{+\alpha' -} + \frac{1}{2} L_{\mu \nu} \Gamma^{+\mu - \nu} - \Gamma_{\mu}^{+\mu -} - \Gamma_{-\mu}^{+\mu -} \right)$$  \hspace{1cm} (14)

In the last line the traceless $4 \times 4$ matrix $\left( \begin{smallmatrix} i \bar{x} p \bar{x} p \bar{x} \\ p - ip \bar{x} \end{smallmatrix} \right)$ is expanded in terms of the following complete set of SO(4, 2) gamma matrices $\Gamma_{MN}$ ($M = \pm, \mu, \nu$, see footnote 5)

$$\Gamma^{+\alpha' -} = \left( \begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right), \quad \Gamma^{\mu \nu} = \begin{array}{cc} \bar{\sigma}^{\mu \nu} & 0 \\ 0 & \sigma^{\mu \nu} \end{array}, \quad \bar{\sigma}^{\mu \nu} \equiv \bar{\sigma}^{[\mu \sigma \nu]} \sigma^{\mu \nu} \equiv \sigma^{[\mu \sigma \nu]}$$  \hspace{1cm} (15)

$$\Gamma^{+\mu -} = i \sqrt{2} \left( \begin{array}{cc} 0 & \bar{\sigma}^{\mu} \\ \sigma^{\mu} & 0 \end{array} \right), \quad \Gamma_{-\mu}^{+\mu -} = -i \sqrt{2} \left( \begin{array}{cc} 0 & 0 \\ 0 & \sigma^{\mu} \end{array} \right).$$  \hspace{1cm} (16)

This identifies the generators of the conformal group $L_{MN}$ as the coefficients

$$L^{+\mu -} = x \cdot p, \quad L^{\mu \nu} = x^\mu p^\nu - x^\nu p^\mu, \quad L^{+\mu + \nu} = p^\mu, \quad L^{-\mu -} = \frac{x^2}{2} p^\mu - x^\mu x \cdot p. \quad (17)$$

It can be checked that this form of $L_{MN}$ are the generators of the hidden SO(4, 2) conformal symmetry of the massless particle action. The SO(4, 2) transformations are given by the Poisson brackets $\delta x^\mu = \frac{1}{2} \omega_{MN} \{L_{MN}, x^\mu\}$ and $\delta p_\mu = \frac{1}{2} \omega_{MN} \{L_{MN}, p^\mu\}$, and these $L_{MN}$ are the conserved charges given by Noether’s theorem. Furthermore they obey the SO(4, 2) Lie algebra under the Poisson brackets. This result is not surprising once we have explained that $S(Z) = S(x, p)$ via the twistor transform.

The same SU(2, 2) symmetry of the twistor action $S(Z)$ has other interpretations as the hidden symmetry of an assortment of other particle actions when other forms of twistor transform is used, as explained in [8]. This recent broader result may seem surprising because it is commonly unfamiliar.

1.2 Physical states in twistor space

In covariant quantization a physical state for a particle of any helicity should satisfy the helicity constraint $\frac{1}{2} (Z_A \bar{Z}^A + \bar{Z}^A Z_A) |\psi\rangle = 2h |\psi\rangle$. This is interpreted as meaning that the physical state $|\psi\rangle$ is invariant under the U(1) gauge transformation generated by the constraint that followed from the twistor action $S(Z)$. The probability amplitude in $Z$ space is $|\psi\rangle \equiv \langle Z |\psi\rangle$, so we can write $\bar{Z}^A \psi(Z) = \langle Z |\bar{Z}^A |\psi\rangle = -\frac{\partial}{\partial Z} \psi(Z).$

Then the helicity constraint $\frac{1}{2} (Z_A \bar{Z}^A + \bar{Z}^A Z_A) |\psi\rangle = 2h |Z |\psi\rangle$ produces the physical state condition,

$$Z_A \frac{\partial}{\partial Z_A} \psi(Z) = (2h - 2) \psi(Z) \quad (18)$$
for a particle of helicity $h$. So a physical wavefunction in twistor space $\psi(\lambda, \mu)$ that describes a particle with helicity $h$ must be homogeneous of degree $(-2h-2)$ under the rescaling $Z \rightarrow \i Z$ or $(\mu, \lambda) \rightarrow (t \mu, t \lambda)$ [2] [3]. This is the only requirement for a physical state $\psi(Z)$ in twistor space, and it is easily satisfied by an infinite set of functions.

If we use the twistor transform for massless particles $\mu = -i\vec{x}\lambda$ and $p = \lambda \bar{\lambda}$, then any homogeneous physical state in twistor space should be a superposition of massless particle wavefunctions since $p^2 = 0$ is automatically satisfied. A similar statement would hold for any of the other twistor transforms given in [8], so a physical state in twistor space can also be expanded in terms of the wavefunctions of the particle systems discussed in [8] [9].

Let us now consider the expansion of a physical state $|\psi\rangle$ in terms of momentum eigenstates $p^\mu |k\rangle = k^\mu |k\rangle$, for a massless particle with $k^2 = 0$. We parameterize $k_{\alpha\beta} = \pi_{\alpha} \bar{\pi}_\beta$ as in Eq. (4), where $\pi_{\alpha}$ can be redefined up to a phase $\pi \rightarrow e^{i\gamma} \pi$ without changing the physical state $|k\rangle$. In position space such a physical state gave the plane wave as in Eq. (3), which we can rewrite as $\phi_k(x) = \langle x|k\rangle \sim \exp(i\i k \cdot x) = \exp(iT r \vec{x} \pi \bar{\pi})$. The twistor space analog is $\phi_k(\lambda, \mu) = \langle Z|k\rangle = \langle \lambda, \mu|\pi, \bar{\pi}\rangle$. Since $|k\rangle$ is a complete set of states, it is possible to write a general physical state in twistor space as an infinite superposition of the $\langle Z|k\rangle$ with arbitrary coefficients, in the same way as the general solution of the Klein-Gordon equation in Eq. (2).

$$\psi(Z) = \int d^2\pi d^2\bar{\pi} \left[ a(\pi, \bar{\pi}) \langle Z|\pi, \bar{\pi}\rangle + h.c. \right]$$ (19)

To determine $\langle Z|k\rangle = \langle \lambda, \mu|\pi, \bar{\pi}\rangle$, first note that the eigenstate of $\lambda_\alpha$ is proportional to $\pi_{\alpha}$, so there must be an overall delta function $\langle \lambda, \mu|\pi, \bar{\pi}\rangle \sim \delta(\langle \lambda \pi\rangle)$. The argument of the delta function is the SL(2, $C$) invariant dot product defined by the symbol $\langle \lambda \pi \rangle \equiv \lambda_\alpha \pi_\beta = \delta^{\alpha\beta}$. The vanishing of $\langle \lambda \pi \rangle = 0$ requires $\lambda_\alpha \propto \pi_{\alpha}$, hence in the wavefunction $\langle \lambda, \mu|\pi, \bar{\pi}\rangle$ we can replace $\lambda_\alpha = \frac{\bar{\lambda}}{\lambda} \pi_{\alpha}$ up to an overall constant $c$ symbolized by $c = \frac{\bar{\lambda}}{\lambda}$. This is the ratio of either component $\frac{\bar{\lambda}}{\lambda} = \bar{\lambda}_{\frac{\lambda}{\lambda}} = \frac{\bar{\lambda}}{\lambda}$. So we can write $\langle Z|k\rangle = \langle \lambda, \mu|\pi, \bar{\pi}\rangle = \delta(\langle \lambda \pi\rangle) f(\pi, \bar{\pi}, \frac{\bar{\lambda}}{\lambda}, \mu)$. Next examine the matrix elements of the twistor transform $p_{\alpha\beta} - \lambda_\alpha \bar{\lambda}_\beta = 0$ and apply the operators on either the ket or the bra as follows ($\bar{\lambda}_\beta$ acts as a derivative $-\frac{\partial}{\partial \mu^\beta}$ on the eigenvalue of $\mu^\beta$)

$$0 = \langle Z| p_{\alpha\beta} - \lambda_\alpha \bar{\lambda}_\beta |k\rangle = \left( k_{\alpha\beta} + \lambda_\alpha \frac{\partial}{\partial \mu^\beta} \right) \langle \lambda, \mu|\pi, \bar{\pi}\rangle$$ (20)

$$= \delta(\langle \lambda \pi\rangle) \pi_{\alpha} \left( \bar{\lambda}_\beta + \frac{\lambda}{\pi} \frac{\partial}{\partial \mu^\beta} \right) f(\pi, \bar{\pi}, \frac{\bar{\lambda}}{\lambda}, \mu).$$ (21)

The solution is $f(\pi, \bar{\pi}, \lambda, \mu) = g(\pi, \bar{\pi}, \frac{\lambda}{\bar{\lambda}})$ of the plane wave $\exp(iT r \vec{x} \pi \bar{\pi})$ by using $\pi = \frac{\bar{\lambda}}{\lambda}$ and then setting $\mu = -i\vec{x}\lambda$.

Finally, we determine $g(\pi, \bar{\pi}, \frac{\lambda}{\bar{\lambda}})$ for a particle with any helicity $h$. According to the previous paragraph, since $\langle Z|k\rangle$ is a physical wavefunction, it should be homogeneous of degree $(-2h-2)$ under a rescaling $(\mu, \lambda) \rightarrow (t \mu, t \lambda)$. It should also
be phase invariant under the phase transformations $\pi \to e^{i\gamma} \pi$, $\bar{\pi} \to e^{-i\gamma} \bar{\pi}$ since the momentum state $|k\rangle$ labeled by $k_{\alpha\beta} = \pi_{\alpha} \bar{\pi}_{\beta}$ is phase invariant. The exponential $\exp\left(-\frac{\pi_{\alpha} \bar{\pi}_{\mu} k^\mu}{\lambda}\right)$ is homogeneous as well as phase invariant, while the delta function satisfies $\delta\left(t \lambda e^{i\gamma} \pi\right) = t^{-1} e^{-i\gamma} \delta\left(\lambda \pi\right)$. These considerations determine $g\left(\pi, \bar{\pi}, \frac{1}{\lambda}\right) = \left(\frac{1}{\pi}\right)^{-1-2h} \phi_h\left(\pi, \bar{\pi}\right)$, with $\phi_h\left(e^{i\gamma} \pi, e^{-i\gamma} \bar{\pi}\right) = e^{-2ih\gamma} \phi_h\left(\pi, \bar{\pi}\right)$.

The specific $\phi_h\left(\pi, \bar{\pi}\right)$ for each helicity are determined as follows. $\phi_h\left(\pi, \bar{\pi}\right)$ must have SL$(2, C)$ spinor indices for the representation $(j_1, j_2)$ since for a spinning particle the complete set of labels includes Lorentz indices $|k, j_1, j_2, \cdots\rangle$ in addition to momentum. The chirality of the SL$(2, C)$ labels must be compatible with the spin $j_1 + j_2 = |h|$. So this determines the Lorentz indices on the wavefunction $\phi_h\left(\pi, \bar{\pi}\right)$ as well as the coefficients $a\left(\pi, \bar{\pi}\right)$ in Eq. (19).

Examples of the overall wavefunction $\langle Z|k\rangle$ is given in the table below.

| particle | $(j_1, j_2)$ | $\phi_h\left(\pi, \bar{\pi}\right)$ |
|----------|--------------|-----------------------------------|
| scalar   | $(0, 0)$     | $\phi_0\left(\pi, \bar{\pi}\right) = 1$ |
| quark    | $(0, \frac{1}{2})$ | $\psi_\alpha^{h=+1/2}\left(\pi, \bar{\pi}\right) = \pi_\alpha$ |
|          | $(\frac{1}{2}, 0)$ | $\psi_\alpha^{h=-1/2}\left(\pi, \bar{\pi}\right) = \pi_\alpha$ |
| gauge potential $A_\mu$ | $(\frac{1}{2}, \frac{1}{2})$ | $A_{h=+1, \alpha\beta}\left(\pi, \bar{\pi}\right) = \frac{\psi_\alpha^{h=+1/2} \psi_\beta^{h=-1/2}}{(\pi \bar{\pi})^{1/2}}$ |
|          | $(\frac{1}{2}, -\frac{1}{2})$ | $A_{h=-1, \alpha\beta}\left(\pi, \bar{\pi}\right) = \frac{\pi_{\alpha} \bar{\pi}_{\beta}}{(\pi \bar{\pi})^{1/2}}$ |
| field strength $F_{\mu\nu}$ | $(0, 1)$ | $F_{\alpha\beta}^{h=+1}\left(\pi, \bar{\pi}\right) = \pi_\alpha \bar{\pi}_\beta$ |
|          | $(1, 0)$ | $F_{\alpha\beta}^{h=-1}\left(\pi, \bar{\pi}\right) = \pi_\alpha \bar{\pi}_\beta$ |
| metric $g_{\mu\nu}$ | $(1, 1)$ | $g_{\alpha\beta \gamma\delta}^{h=+2}\left(\pi, \bar{\pi}\right) = \frac{\pi_{\alpha} \pi_{\beta} \bar{\pi}_{\gamma} \bar{\pi}_{\delta}}{(\pi \bar{\pi})^2}$ |
|          | $(1, -1)$ | $g_{\alpha\beta \gamma\delta}^{h=-2}\left(\pi, \bar{\pi}\right) = \frac{\pi_{\alpha} \bar{\pi}_{\beta} \bar{\pi}_{\gamma} \pi_{\delta}}{(\pi \bar{\pi})^2}$ |
| curvature $H_{\mu\nu\lambda\sigma}$ | $(0, 2)$ | $H_{\alpha\beta \gamma\delta}^{h=+2}\left(\pi, \bar{\pi}\right) = \pi_\alpha \bar{\pi}_\beta \bar{\pi}_\gamma \pi_\delta$ |
|          | $(2, 0)$ | $H_{\alpha\beta \gamma\delta}^{h=-2}\left(\pi, \bar{\pi}\right) = \pi_\alpha \bar{\pi}_\beta \pi_\gamma \bar{\pi}_\delta$ |

The field strength $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$ can be written in terms of the gauge potential in momentum and spinor space for an arbitrary combination of both helicities as follows

$$A_{\alpha\beta} = a^+ A^+_{\alpha\beta}(\pi, \bar{\pi}) + a^- A^-_{\alpha\beta}(\pi, \bar{\pi})$$

(23)

$$F_{\alpha\beta \gamma\delta} = k_{\alpha\beta} A_{\gamma\delta} - k_{\gamma\delta} A_{\alpha\beta} = \varepsilon_{\alpha\gamma} a^+ F^+_{\beta\delta}(\pi, \bar{\pi}) + \varepsilon_{\beta\delta} a^- F^-_{\alpha\gamma}(\pi, \bar{\pi})$$

(24)

which is consistent with the wavefunctions $A^\pm(\pi, \bar{\pi}), F^\pm(\pi, \bar{\pi})$ given in the table above. Note that all wavefunctions are automatically transverse to $k_{\alpha\beta} = \pi_{\alpha} \bar{\pi}_{\beta}$ under the Lorentz invariant dot product using the metric in spinor space $\varepsilon_{\alpha\beta} \otimes \varepsilon_{\alpha\beta}$.

In field theory computations that use twistor techniques [17], the twistor space wavefunctions above are used for the corresponding physical external particles with definite momentum, up to overall normalizations.
2 2T-physics

As mentioned above, it has been discovered recently that there are many ways of solving the same constraints on the twistor $Z_A$ and derive other relations between $\mu, \lambda$ and phase space [8]. These other solutions describe not only the massless particle, but also massive particle, relativistic or non-relativistic, in flat space or curved space, interacting or non-interacting, as shown in the examples in Fig.1. These new twistors were discovered by using two time physics (2T-physics) as a technique.

2T-physics was also used to obtain the generalization of twistors to higher dimensions, to supersymmetry, and to D-branes. In the rest of these lectures I will first give a brief outline of the main aspects of 2T-physics and then summarize these new results.

2.1 Emergent spacetimes & dynamics, holography, duality.

2T-physics can be viewed as a unification approach for one-time physics (1T-physics) systems through higher dimensions. It is distinctly different than Kaluza-Klein theory because there are no Kaluza-Klein towers of states, but instead there is a family of 1T systems with duality type relationships among them. The 2T theory is in $(d+2)$ dimensions, but has enough gauge symmetry to compensate for the extra $1 + 1$ dimen-
2.2 Sp(2,R) gauge symmetry, constraints, solutions and (d,2)

The essential ingredient in 2T-physics is the basic gauge symmetry Sp(2,R) acting on phase space $X^M, P_M$ in $d + 2$ dimensions. The two timelike directions is not an input, but is one of the outputs of the Sp(2,R) gauge symmetry. A consequence of this gauge symmetry is that position and momentum become indistinguishable at any instant, so the symmetry is of fundamental significance. The transformation of $X^M, P_M$ is generally a nonlinear map that can be explicitly given in the presence of background fields [18], but in the absence of backgrounds the transformation reduces to a linear doublet action of Sp(2,R) on $(X^M, P^M)$ for each $M$ [5]. The physical phase space is the subspace that is gauge invariant under Sp(2,R). Since Sp(2,R) has 3 generators, to reach the physical space we must choose 3 gauges and solve 3 constraints. So, the gauge invariant subspace of $d + 2$ dimensional phase space $X^M, P_M$ is a phase space with six fewer degrees of freedom in $(d - 1)$ space dimensions $(x^i, p_i)$, $i = 1, 2, \cdots (d - 1)$.

In some cases it is more convenient not to fully use the three Sp(2,R) gauge symmetry parameters and work with an intermediate space in $(d - 1) + 1$ dimensions $(x^\mu, p_\mu)$, that includes time. This space can be further reduced to $d - 1$ space dimensions $(x^i, p_i)$ by a remaining one-parameter gauge symmetry.

There are many possible ways to embed the $(d - 1) + 1$ or $(d - 1)$ phase space in $d + 2$ phase space, and this is done by making Sp(2,R) gauge choices. In the resulting gauge fixed 1T system, time, Hamiltonian, and in general curved spacetime, are emergent concepts. The Hamiltonian, and therefore the dynamics as tracked by the emergent time, may look quite different in one gauge versus another gauge in terms of the remaining gauge fixed degrees of freedom. In this way, a unique 2T-physics action
gives rise to many 1T-physics systems.

A particle interacting with various backgrounds in \((d - 1) + 1\) dimensions (e.g. electromagnetism, gravity, high spin fields, any potential, etc.), usually described in a worldline formalism in 1T-physics, can be equivalently described in 2T-physics.

The general 2T theory for a particle moving in any background field has been constructed \[18\]. For a spinless particle it takes the form

\[
S = \int d\tau \left( \dot{X}^i M P_M - \frac{1}{2} A^{ij} Q_{ij} (X, P) \right),
\]

where the symmetric \(A^{ij}(\tau), i, j = 1, 2\), is the \(\text{Sp}(2, R)\) gauge field, and the three \(\text{Sp}(2, R)\) generators \(Q_{ij}(X(\tau), P(\tau))\), which generally depend on background fields that are functions of \((X(\tau), P(\tau))\), are required to form an \(\text{Sp}(2, R)\) algebra. The background fields must satisfy certain conditions to comply with the \(\text{Sp}(2, R)\) requirement. An infinite number of solutions to the requirement can be constructed \[18\]. So any 1T particle worldline theory, with any backgrounds, can be obtained as a gauge fixed version of some 2T particle worldline theory.

The 1T systems which appear in the diagram above are obtained by considering the simplest version of 2T-physics without any background fields. The 2T action for a “free” 2T particle is \[5\]

\[
S_{2T} = \frac{1}{2} \int d\tau D^i X^M_i X^N_j \eta_{MN} \varepsilon^{ij} = \int d\tau \left( \dot{X}^M_i P^i - \frac{1}{2} A^{ij} X^i M X^N_j \right) \eta_{MN}.
\]

Here \(X^M_i = (X^M, P^M)_i, i = 1, 2\), is a doublet under \(\text{Sp}(2, R)\) for every \(M\), the structure \(D^i X^M_i = \partial_\tau X^M_i - A^i_j X^M_j\) is the \(\text{Sp}(2,R)\) gauge covariant derivative, \(\text{Sp}(2,R)\) indices are raised and lowered with the antisymmetric \(\text{Sp}(2, R)\) metric \(\varepsilon^{ij}\), and in the last expression an irrelevant total derivative \(- (1/2) \partial_\tau (X \cdot P)\) is dropped from the action. This action describes a particle that obeys the \(\text{Sp}(2, R)\) gauge symmetry, so its momentum and position are locally indistinguishable due to the gauge symmetry. The \((X^M, P^M)\) satisfy the \(\text{Sp}(2, R)\) constraints

\[
Q_{ij} = X_i \cdot X_j = 0 : X \cdot X = P \cdot P = X \cdot P = 0,
\]

that follow from the equations of motion for \(A^{ij}\). The vanishing of the gauge symmetry generators \(Q_{ij} = 0\) implies that the physical phase space is the subspace that is \(\text{Sp}(2, R)\) gauge invariant. These constraints have non-trivial solutions only if the metric \(\eta_{MN}\) has two timelike dimensions. So when position and momentum are locally indistinguishable, to have a non-trivial system, two timelike dimensions are necessary as a consequence of the \(\text{Sp}(2, R)\) gauge symmetry.

Thus the \((X^M, P^M)\) in Eq. \[26\] are \(\text{SO}(d, 2)\) vectors, labeled by \(M = 0', 1', \mu\) or \(M = \pm', \mu\), and \(\mu = 0, 1, \cdots, (d - 1)\) or \(\mu = \pm, 1, \cdots, (d - 2)\), with lightcone type definitions of \(X^{\pm'} = \frac{1}{\sqrt{2}} (X^{0'} \pm X^{1'})\) and \(X^{\pm} = \frac{1}{\sqrt{2}} (X^0 \pm X^3)\). The \(\text{SO}(d, 2)\)
metric $\eta^{MN}$ is given by

$$ds^2 = dX^M dX^N \eta_{MN} = -2dX^+ dX^- + dX^\mu dX^\nu \eta_{\mu\nu}$$

(28)

$$= -(dX^0)' + (dX^1)' - (dX^0)'' + (dX^1)'' + (dX_\perp)^2$$

(29)

$$= -2dX^+ dX^- - 2dX^+ dX^- + (dX_\perp)^2.$$  

(30)

where the notation $X_\perp$ indicates SO($d-2$) vectors.

2.3 SO($d,2$) global symmetry, quantization and the singleton

The target phase space $X^M, P_M$ is flat in $d+2$ dimension, and hence the system in Eq.(26) has an SO($d,2$) global symmetry. The conserved generators of SO($d,2$)

$$L_{MN} = X^M P^N - X^N P^M, \quad \partial_\tau L_{MN} = 0,$$  

(31)

commute with the SO($d,2$) invariant Sp(2,$\mathbb{R}$) generators $X \cdot X, P \cdot P, X \cdot P$. It will be useful to consider the matrix

$$(L)_{\;B}^{\;A} = \frac{1}{4i} L_{MN} \left(\Gamma^{MN}\right)_{\;B}^{\;A}$$

(32)

constructed by using the $d$-dimensional analogs of the gamma matrices in Eqs.(15,16) (see footnotes 5,6 for details).

If the square of the matrix $L^2$ is computed at the classical level, i.e. not caring about the orders of generators $L_{MN}$, then one finds that $(L^2)_{\;B}^{\;A} = \frac{1}{8} L_{MN} \cdot L_{MN}$ 1. Furthermore by computing, still at the classical level $\frac{1}{2} L_{MN} L_{MN} = X^2 P^2 - (X \cdot P)^2$, and imposing the classical constraints $X^2 = P^2 = (X \cdot P) = 0$, one finds that $L^2 = 0$ in the space of gauge invariants of the classical theory. By taking higher powers of $L$, we find $L^n = 0$ for all positive integers $n \geq 2$. This is a very special non-trivial representation of the non-compact group SO($d,2$)$_L$, and all classical gauge invariants, which are functions of $L_{MN}$, can be classified as irreducible multiplets of SO($d,2$)$_L$.

We now consider the SO($d,2$) covariant quantization of the theory. In the quantum theory the $L_{MN}$ form the Lie algebra of SO($d,2$), therefore if the square of the matrix $L$ is computed at the quantum level, by taking into account the orders of the operators $L_{MN}$, one finds

$$L^2 = \left(\frac{1}{4i} \Gamma_{MN} L_{MN}\right)^2 = \frac{d}{2} \left(\frac{1}{4i} \Gamma_{MN} L_{MN}\right) + \frac{1}{8} L_{MN} L_{MN} 1.$$  

(33)

In this computation we used the properties of gamma matrices

$$\Gamma_{MN} \Gamma_{RS} = \Gamma_{MNR} R + (\eta_{NR} \eta_{MS} - \eta_{MR} \eta_{NS})$$

$$+ (\eta_{NR} \Gamma_{MS} - \eta_{MR} \Gamma_{NS} - \eta_{NS} \Gamma_{MR} + \eta_{MS} \Gamma_{NR}).$$

The term $\Gamma_{MNR} L_{MN} L_{RS}$ vanishes for $L_{MN} = X^{[MPN]}$ due to a clash between symmetry/antisymmetry. The term “$\eta_{NR} \Gamma_{MS} \cdots$” turns into a commutator, and after
2.4 Twistors for other particle dynamics

We now introduce the general twistor transform that applies not only to massless particles, but to other particle systems shown in Fig.1 in any dimension \( d \). The basic idea [7] follows from the formalism in the following section that includes supersymmetry. Here we give the result for spinless particles and without supersymmetry in \( d \) dimensions [9] and comment on its properties. The general twistor transform is [9]

\[
Z = \begin{pmatrix} \mu \\ \lambda \end{pmatrix}, \quad \mu = -iX_\mu \bar{\gamma}_\mu = \bar{X}_\mu \gamma^\mu, \quad \lambda = \frac{1}{\sqrt{2}} \left( X^{+\mu} P^\mu - P^{+\mu} X^\mu \right) \gamma_\mu, \quad (37)
\]

Here \( Z^a_A \) is a \( s_d \times \frac{s_d}{2} \) matrix with \( A = 1, 2, \ldots, s_d \) and \( a = 1, 2, \ldots, \frac{s_d}{2} \), and \( \mu, \lambda \) are \( \frac{s_d}{2} \times \frac{s_d}{2} \) matrices, where \( s_d = 2^{d/2} \) is the dimension of the Weyl spinor for \( \text{SO}(d, 2) \) for even \( d \). For \( d = 4 \) this reduces to the quartet \( Z_A \) that we discussed in section 1.1. The \( \gamma_\mu, \bar{\gamma}_\mu \) are the gamma matrices for even \( d \) dimensions in the two Weyl bases (analogy of Pauli matrices in section 1.1). The general twistor \( Z^a_A \) automatically satisfies the following \( \left( \frac{s_d}{2} \right)^2 \) constraints by construction

\[
(\bar{Z}Z)^b_a = (\bar{\lambda} \bar{\mu}) (\mu_\lambda)^b_a = (\bar{\mu} \lambda + \bar{\lambda} \mu)^b_a = 0. \quad (38)
\]
2.4 Twistors for other particle dynamics

We regard this constraint as the generator of a gauge symmetry that acts on the $a$ index, and introduce a gauge field $V^b_a$ associated with this constraint.

The $A$ index on $Z_b^a$ is the basis for the SO($d$, 2) spinor. This is the global symmetry whose generators $L^{MN}$ can be constructed from either the twistors $\bar{Z}$, $Z$ or from phase space $X^M$, $P^M$. Indeed the twistor transform above is constructed to satisfy the relation [9]

$$\frac{1}{4i} L^{MN} (\Gamma_{MN})^B_A = L^B_A = (Z \bar{Z})^B_A - \frac{1}{8d} Tr (Z \bar{Z}) \delta^B_A. \quad (39)$$

The trace term automatically vanishes if $Z$ is constructed to satisfy $(Z \bar{Z})^b_a = 0$ as above.

By inserting the twistor transform into the following twistor action (in which $\bar{Z}^b_a Z^a_b$ already vanishes) we derive the phase space action that determines the canonical structure for the phase space $(X^M, P^M)$ in $d + 2$ dimensions

$$S(Z) = \frac{4}{s_d} \int d\tau \left( i \partial_\tau Z^a_b \bar{Z}_b^A + Z^a_b V^b_a \bar{Z}_b^A \right)$$

$$= \frac{4}{s_d} \int d\tau \frac{1}{\sqrt{2}} \left( X^\mu P^\mu - P^\mu X^\mu \right) Tr \left( \partial_\tau \left( \frac{X^\mu \bar{\gamma}_\mu}{\sqrt{2X^\tau}} \right) \bar{\gamma}_\tau \right) \quad (40)$$

$$= \frac{4}{s_d} \int d\tau \left( X^\mu P^\mu - P^\mu X^\mu \right) \partial_\tau \left( \frac{X^\mu}{X^\tau} \right) \quad (41)$$

$$= \int d\tau \left( \partial_\tau X^\mu - \partial_\tau X^\mu + \partial_\tau X^\mu \right) \left( P^\mu - P^\mu \right) \quad (42)$$

$$= \int d\tau \left( \partial_\tau X^\mu P^\mu - \partial_\tau X^\mu P^\mu - \partial_\tau X^\mu P^\mu \right) = \int d\tau \partial_\tau X^M P_M. \quad (43)$$

The last line follows thanks to the constraints $X^2 = P^2 = X \cdot P = 0$ that are satisfied in the Sp($2, R$) invariant physical sector. This shows the consistency of our twistor transform of Eq. (37) for spinless particles in all dimensions. Hence the 2T-physics system in $d + 2$ dimensions is reproduced by the twistor $Z^a_b$ with the given properties.

Now we can choose explicitly the Sp($2, R$) gauges that reduce the 2T-physics system to the various holographic pictures given in Fig.1. By inserting the gauge fixed forms of $(X^M, P^M)$ we will obtain the twistor transforms for all the holographic pictures.

The SO($d - 1, 1$) covariant massless particle emerges if we choose the two gauges, $X^{+\tau}$($\tau$) = 1 and $P^{-\tau}$($\tau$) = 0, and solve the two constraints $X^2 = X \cdot P = 0$ to obtain the $(d - 1) + 1$ dimensional phase space $(x^\mu, p_\mu)$ embedded in $(d + 2)$ dimensions

$$X^M = \left( x^\tau, x^\tau / 2, x^\mu \right), \quad (44)$$

$$P^M = \left( 0, x \cdot p, p^\mu \right). \quad (45)$$
The remaining constraint, \( P^2 = -2P^\mu P_\mu + P^\mu P_\mu = p^2 = 0 \), which is the third \( \text{Sp}(2, R) \) generator, remains to be imposed on the physical sector. In this gauge the 2T action reduces to the relativistic massless particle action in Eq. (41):

\[
S = \int dt \left( \dot{X}^M P^N - \frac{1}{2} A^{ij} X_i^M X_j^N \right) \eta_{MN} = \int dt \left( \dot{x}^\mu p_\mu - \frac{1}{2} A^{22} p^2 \right). \tag{48}
\]

Furthermore, the \( \text{Sp}(2, R) \) gauge invariant \( L^{MN} = X^M P^N - X^N P^M \) take the following nonlinear form

\[
L_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \quad L^+ = x \cdot p, \quad L^± = \frac{x^2}{2} p^\mu - x^\mu x \cdot p. \tag{49}
\]

These are recognized as the generators of \( \text{SO}(d, 2) \) conformal transformations of the \((d - 1) + 1\) dimensional phase space at the classical level. Thus the hidden conformal symmetry of the massless system is understood as the Lorentz symmetry in \( d + 2 \) dimensions.

Inserting the gauge fixed form of \((X^M, P^M)\) of Eqs. (46,47) into the general twistor transform in Eq. (37), and specializing to four dimensions \( d = 4 \), we obtain the Penrose version of twistor transform Eqs. (4,6) for the massless spinless particle. Note that for \( d \geq 4 \) the rectangular twistor \( Z_{\alpha}^A \) has several columns, which is a structure that is absent in the Penrose transform in \( d = 4 \). The columns of the higher dimensional twistor satisfy many relations among themselves since they only depend only on the vectors \( X^M, P^M \).

The parent theory can be gauge fixed in many ways that produce not only the massless particle, but also an assortment of other particle dynamical systems [5] [6] [8]. To emphasize this point we give also the massive relativistic particle gauge by fixing two gauges and solving the constraints \( X^2 = X \cdot P = 0 \) explicitly as follows

\[
X^M = \begin{pmatrix}
1 + a, & x^2a, & \mu \\
2a, & 1 + a, & x^\mu
\end{pmatrix}, \quad a \equiv \sqrt{1 + \frac{m^2 x^2}{(x \cdot p)^2}} \tag{50}
\]

\[
P^M = \begin{pmatrix}
\frac{-m^2}{2(x \cdot p)a}, & (x \cdot p) a, & p^\mu
\end{pmatrix}, \quad P^2 = p^2 + m^2 = 0. \tag{51}
\]

In this gauge the 2T action reduces to the relativistic massive particle action

\[
S = \int dt \left( \dot{X}^M P^N - \frac{1}{2} A^{ij} X_i^M X_j^N \right) \eta_{MN} = \int dt \left( \dot{x}^\mu p_\mu - \frac{1}{2} A^{22} (p^2 + m^2) \right). \tag{52}
\]

The twistor for the massive particle follows from Eqs. (50) [5] [8], as shown in [8]

\[
\mu^\alpha = -i x^\alpha_{\beta} \lambda^\beta, \quad \lambda_{\alpha} \tilde{\lambda}_{\beta} = \frac{1 + a}{2a} p_{\alpha \beta} + \frac{m^2}{2(x \cdot p)a} x_{\alpha \beta}. \tag{53}
\]

A little recognized fact is that this action is invariant under \( \text{SO}(d, 2) \). This \( \text{SO}(d, 2) \) does not have the form of conformal transformations of Eq. (49), but is a deformed
2.4 Twistors for other particle dynamics

version of it, including the mass parameter. Its generators are obtained by inserting the massive particle gauge into the gauge invariant $L^{MN} = X^M P^N - X^N P^M$

$$L^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu, \quad L^{+_{\mu \nu}} = (x \cdot p) a,$$

$$L^{-_{\mu \nu}} = \frac{x^2 a}{1 + a} p^\mu - (x \cdot p) ax^\mu$$

It can be checked explicitly that the massive particle action above is invariant under the SO($d, 2$) transformations generated by the Poisson brackets $\delta x^\mu = \frac{1}{2} \omega_{MN} \{ L^{MN}, x^\mu \}$ and $\delta p^\mu = \frac{1}{2} \omega_{MN} \{ L^{MN}, p^\mu \}$, up to a reparametrization of $A^{22}$ by a scale and an irrelevant total derivative.

Since both the massive and massless particles give bases for the same representation of SO($d, 2$), we must expect a duality transformation between them. Of course this transformation must be an Sp($2, R$) $=\text{SL}(2, R)$ local gauge transformation $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} (\tau)$ with unit determinant $\alpha \delta - \beta \gamma = 1$, that transforms the doublets $\begin{pmatrix} X^M \\ p_M \end{pmatrix}$ into $\begin{pmatrix} x^\mu \\ p^\mu \end{pmatrix}$. The $\alpha, \beta, \gamma, \delta$ are fixed by focussing on the doublets labeled by $M = +^\sigma$

$$\begin{pmatrix} \frac{1+a}{2a} \\ \frac{2a}{2(x \cdot p)a} \end{pmatrix} \begin{pmatrix} 1+a \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1+a}{2a} \\ \frac{2a}{2(x \cdot p)a} \end{pmatrix} \begin{pmatrix} 1+a \\ 2 \end{pmatrix}.$$  

Applying the inverse of this transformation on the doublets labeled by $M = \mu$ gives the massless particle phase space (re-labeled by $(\tilde{x}^\mu, \tilde{p}^\mu)$ below) in terms of the massive particle phase space (labeled by $(x^\mu, p^\mu)$)

$$\begin{pmatrix} \frac{2a}{1+a} \\ \frac{m^2}{2(x \cdot p)a} \end{pmatrix} \begin{pmatrix} x^\mu \\ p^\mu \end{pmatrix} = \begin{pmatrix} \frac{2a}{1+a} \\ \frac{m^2}{2(x \cdot p)a} \end{pmatrix} \begin{pmatrix} 1+a \\ 2 \end{pmatrix} \begin{pmatrix} \tilde{x}^\mu \\ \tilde{p}^\mu \end{pmatrix}.$$  

This duality transformation is a canonical transformation $\{ \tilde{x}^\mu, \tilde{p}^\mu \} = \eta^{\mu\nu} = \{ x^\mu, p^\nu \}$.

Also note that the time coordinate $\tilde{x}^0$ is different than the time coordinate $x^0$, and so are the corresponding Hamiltonians for the massless particle $\tilde{p}^0 = \sqrt{\tilde{p}^i \tilde{p}^i}$ versus the massive particle $p^0 = \sqrt{p^i p^i} + m^2$.

The same reasoning applies among all gauge choices of the 2T theory in Eq. (56). All resulting 1T dynamical systems are holographic images of the same parent theory. Some of the images are illustrated in the diagram above. In the quantum theory we have already shown by covariant quantization that the global symmetry SO($d, 2$) of the 2T-physics action is realized in the singleton representation. All the emergent lower dimensional theories obtained by different forms of gauge fixing, either in the form of twistors, or in the form of phase space, must also be realized in the same singleton representation.

4 At the classical level all Casimir eigenvalues vanish for the various form of the SO($d, 2$) generators.
3 Supersymmetric 2T-physics and twistor gauge

The 2T-physics action [26] and the twistor action [11] are related to one another and can both be obtained as gauge choices from the same theory in the 2T-physics formalism. This formalism was introduced in [7] and developed further in the context of the twistor superstring [14] [15] and to derive the general twistor transform [8] [9].

3.1 Coupling X, P, g, gauge symmetries, global symmetries.

In addition to the phase space SO(d, 2) vectors (X^M, P^M) (τ), we introduce a group element g(τ). The group G is chosen so that it contains SO(d, 2) as a subgroup in the spinor representation. The simplest case is G = SO(d, 2), and in that case

\[ g(τ) = \exp \left( \frac{i}{4} \Gamma^{MN} \omega_{MN}(τ) \right) \]

where \( \Gamma^{MN} \) are the gamma matrices\(^5\) for SO(d, 2). Table 1 shows the smallest possible bosonic groups G that contain SO(d, 2) in the spinor representation for various dimensions \( 3 \leq d \leq 12 \).

The table lists all the generators of G as represented by antisymmetrized products of gamma matrices \( \Gamma^{M_1 \ldots M_n} \equiv \frac{1}{n!} (\Gamma^{M_1} \Gamma^{M_2} \ldots \Gamma^{M_n} \mp \text{permutations}) \). The criterion for choosing G is that G is the smallest group whose smallest fundamental representation has the same dimension \( s_d \) as the spinor of SO(d, 2). Then \( \Gamma^{MN} \) (i.e. SO(d, 2) generators in the spinor basis) must be included among the generators of G. The number of generators represented by \( \Gamma^{M_1 \ldots M_n} \) in \( d + 2 \) dimensions is indicated as the subscript. This number is given by the binomial coefficient \( \frac{(d+2)!}{n!(d+2-n)!} \) in general, but is divided by 2 for the case of \( \Gamma^{M_1 \ldots M_6} \) because this one is self dual for \( d+2 = 12 \).

The total number of gamma matrices listed is equal to the number of generators in G. Taken together these form the Lie algebra of G under matrix commutation. The following column gives information on whether the gamma matrices occur in the symmetric or antisymmetric products of the spinors of SO(d, 2), when both spinor indices A, B are lowered or raised in the form \( (\Gamma^{M_1 \ldots M_n})_{AB} \) by using the metric C in spinor space.

But at the quantum level, due to ordering of factors that are needed for the correct closure of the algebra, the Casimir eigenvalues are non-zero and agree with Eq.(77), the singleton representation. The ordering of the quantum factors has been explicitly performed in the majority of the holographic images given in Fig.1 [5] [6].

\(^5\)The trace in spinor space gives the dimension of the spinor \( \text{Tr} \left( 1 \right) = s_d \) and \( \text{Tr} \left( \Gamma^{M} \Gamma^{N} \right) = s_d \delta^{MN} \).

For even dimensions \( s_d = 2^{d/2} \) for the Weyl spinor of SO(d, 2), and the \( \Gamma^{M} \), \( \Gamma^M \) are the gamma matrices in the bases of the two different spinor representations. The correctly normalized generators of SO(d, 2) in the spinor representation are \( S^{MN} = \frac{1}{2} \Gamma^{MN} \), where the gamma matrices satisfy \( \Gamma^M \Gamma^N + \Gamma^N \Gamma^M = 2 \eta^{MN} \), while \( \Gamma^{MN} = \frac{1}{2} (\Gamma^M \Gamma^N - \Gamma^N \Gamma^M) \). \( \Gamma^{M} \Gamma^{N} \Gamma^{K} = \frac{1}{4} (\Gamma^M \Gamma^N \Gamma^K \mp \text{permutations}) \), etc. There exists a metric C of SO(d, 2) in the spinor representation such that when combined with hermitian conjugation it gives \( C^{-1} (\Gamma^{M})^{\dagger} C = -\Gamma^{M} \) and \( C^{-1} (\Gamma^{MN})^{\dagger} C = -\Gamma^{MN} \). So the inverse \( g^{-1} \) is obtained by combining hermitian and C-conjugation \( g^{-1} = C^{-1} (g)^{\dagger} C \equiv \tilde{g} \). In odd number of dimensions the even-dimension gamma matrices above are combined to a larger matrix \( \tilde{\Gamma}^{M} = \begin{pmatrix} 0 & \Gamma^{M} \\ \Gamma^{-M} & 0 \end{pmatrix} \) for \( \tilde{M} = 0', 1', 0, 1, \ldots, (d-2) \) and add one more matrix for the additional last dimension \( \Gamma^{d-1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \).

The text is written as if d is even; for odd dimensions we replace everywhere \( \tilde{\Gamma}^{M} \) for both \( \Gamma^{M} \) and \( \Gamma^{-M} \).
The subset of gamma matrices $\Gamma^{MN}$ represent the $SO(d,2)$ subgroup in $G$. The gamma matrices $\Gamma^{M_1 \cdots M_n}$ with $n \neq 2$ lead to degrees of freedom in the model that correspond to D-branes as explained in [9]. Only the cases of $d = 3, 4, 5, 6$ can be constructed purely with particle degrees of freedom without any D-branes.

| $d$ | $SO(d,2)$ spinor $s_d$ | $G$ | $G$ in spin(d,2) basis | $\Gamma^{M_1 \cdots M_n}$ in product $s_d \times s_d$ | $G_{\text{super}}$ |
|-----|----------------|-----|---------------------|---------------------|----------------|
| 3   | $4$            | $\text{Sp}(4, R)$ | $\Gamma_{10}^{MN}$ | $(4 \times 4)_a$ | $\text{OSp}(N|4)$ |
| 4   | $4, 4$         | $\text{SU}(2, 2)$ | $\Gamma_{15}^{MN}$ | $4 \times 4$ | $\text{SU}(2, 2|N)$ |
| 5   | $8$            | $\text{SO}^*(8)$ | $\Gamma_{21}^{MN} \oplus \Gamma_{16}^{M}$ | $(8 \times 8)_a$ | $\text{F}(4)$ |
|     | $8_+$          | $\text{SO}^*(8)$ | $\Gamma_{28}^{MN}$ | $(8 \times 8)_a$ | $\text{OSp}(8|2N)$ |
| 7   | $16$           | $\text{SO}^*(16)$ | $\Gamma_{64}^{MN} \oplus \Gamma_{24}^{MNKL}$ | $(16 \times 16)_a$ | $\text{OSp}(16|2N)$ |
| 8   | $16, 16_1$    | $\text{SU}^*(16)$ | $\Gamma_{45}^{MN} \oplus \Gamma_{210}^{MNKL}$ | $16 \times 16_1$ | $\text{SU}(16|N)$ |
| 9   | $32$           | $\text{Sp}^*(32)$ | $\Gamma_{55}^{MN} \oplus \Gamma_{11}^{M} \oplus \Gamma_{462}^{M}$ | $(32 \times 32)_a$ | $\text{OSp}(N|32)$ |
| 10  | $32_+$         | $\text{Sp}^*(32)$ | $\Gamma_{95}^{MN} \oplus \Gamma_{462}^{M}$ | $(32 \times 32)_a$ | $\text{OSp}(N|32)$ |
| 11  | $64$           | $\text{Sp}^*(64)$ | $\Gamma_{78}^{MN} \oplus \Gamma_{286}^{MNKL} \oplus \Gamma_{1716}^{M}$ | $(64 \times 64)$ | $\text{OSp}(N|64)$ |
| 12  | $64, 64_1$    | $\text{SU}^*(64)$ | $\Gamma_{91}^{MN} \oplus \Gamma_{1601}^{MNKL} \oplus \Gamma_{3003}^{M}$ | $(64 \times 64)$ | $\text{SU}(64|N)$ |

Table 1: Smallest group $G$ that contains $\text{Spin}(d,2)$; supergroups $G_{\text{super}}$: D-branes.

Groups that are larger than the listed $G$ may also be considered in our scheme in every dimension (e.g. $\text{SU}(8)$ instead of $\text{SO}(8)$ in $d = 6$). In that case the number of generators $\Gamma^{M_1 \cdots M_n}$ increases compared to the ones listed in the table for each $d$. Furthermore the corresponding D-brane degrees of freedom also get included in the model.

The last column of the table lists the smallest supergroups $G_{\text{super}}$ that contain $G$. The number of supersymmetries depend on the value of $N = 0, 1, 2, \cdots$. For $N = 0$ we just have $G$. In fact for $N = 0$, the smallest group is $SO(d,2)$ itself for every $d$, and this would include only particle degrees of freedom without D-branes for every $d$. The $N = 0$ case for either $SO(d,2)$ or $G$ is discussed in detail in [8] [9].

For $N \neq 0$, the supergroup listed is the smallest supergroup that contains $SO(d,2)$ in the spinor representation. For physical purposes the total number of real fermionic generators in $G_{\text{super}}$ cannot exceed 64 (32 ordinary supercharges and 32 conformal supercharges). For example, for $d = 4$ we can go as far as $N = 8$, since $G_{\text{super}} = \text{SU}(2, 2|8)$ has 64 real fermionic parameters. Similarly for $d = 11$, we cannot have more than $N = 1$, hence $\text{OSp}(1|64)$. For a given $N$, the $N$-dependent bosonic subgroup is then the $R$-symmetry group that acts on the supercharges. Thus, for $d = 4$ and $N = 4$, the $R$-symmetry subgroup of $\text{PSU}(2, 2|4)$ is $\text{SU}(4)$.

The supergroup element $g(\tau)$ can be written by exponentiating the Lie superalgebra in the form

\begin{equation}
\frac{1}{4} \Gamma^{MN} \omega_{MN} + \cdots \Theta_{\text{spinor fermi}}^{\tau = 1 \cdots N} \tilde{g}^{\text{fermi}} \Theta_{\text{R-symmetry}}^{\text{R}^a \omega_a}\end{equation}

for $d \leq 6$ the spacetime part of $g$ has only $SO(d,2)$ degrees of freedom

for $d \geq 7$ supergroups contain more $\Gamma^{M_1 \cdots M_n}$, which give D-branes.
where \( \cdots \) stands for the contributions of the \( \Gamma^{M_1 \cdots M_n} \) in Table 1, while the \( R^a \) are the generators of the R-symmetry group. The generalized 2T superparticle action

\[
S_{2T} (X, P, g) = \int dt \left[ \frac{1}{2} e^{ij} \partial_t X_i \cdot X_j - \frac{1}{2} A^{ij} X_i \cdot X_j + \frac{4}{s_d} \text{Str} \left( ig^{-1} \partial_t g L \right) \right]
\]

includes the degrees of freedom \( (X^M, P^M) \) and those of the supergroup \( g \), namely \( \omega_{MN}, \cdots, \omega_{M_1 \cdots M_n}, \omega_a, \) and \( G_{\text{spinor}}^{1 \cdots N} \). Here the matrix \( L \) has the following form

\[
L = \frac{1}{4i} \begin{pmatrix} \Gamma^{MN} & 0 \\ 0 & 0 \end{pmatrix} \quad L_{MN} = \frac{1}{4i} \begin{pmatrix} (\Gamma \cdot X \vec{\Gamma} \cdot P - \Gamma \cdot P \vec{\Gamma} \cdot X) & 0 \\ 0 & 0 \end{pmatrix}
\]

(61)

It has been established [7] that for \( d = 3, 4, 5, 6 \) and any \( N \), this action descends to the usual superparticle action in \( d = 3, 4, 5, 6 \) dimensions by using the massless particle gauge in Eq. (64):

\[
S_{d=3,4,5,6}^{\text{gauge fixed}} = S^{\text{superparticle}}_{2T} (x, p, \theta)
\]

\[
= \int dt \left[ \dot{x} \cdot p + i \bar{\theta}^i \gamma^\mu \dot{\theta}_i p_\mu - \frac{1}{2} e^2 \theta^2 \right], \quad i = 1, 2, \ldots, N.
\]

(63)

So, in this gauge, the supergroups \( G_{\text{super}} \) given in the table above, namely \( \text{OSp}(N|4) \) in \( d = 3 \), \( \text{SU}(2,2|N) \) in \( d = 4 \), \( \text{F}(4) \) in \( d = 5 \) and \( \text{OSp}(8|2N) \) in \( d = 6 \), are interpreted as the hidden superconformal symmetries of the superparticle action [7]. The fermions \( \theta \) are half of the fermions \( \Theta \) that appear in \( g \). This action has a remaining kappa supersymmetry and can remove half of the \( \theta \), so only 1/4 of the original fermions \( \Theta \) are physical.

In \( d \geq 7 \) the action \( S_{2T} (X, P, g) \) has D-brane degrees of freedom in addition to the particle (D0) degrees of freedom [9]. In particular in \( d = 11 \) the brane degrees of freedom are the D2-brane and D5-brane of M-theory [28].

This action has several local and global symmetries given by [7]

local: \( \text{Sp}(2, R) \times \left( \begin{array}{c} \text{SO} (d, 2) \\frac{3}{4} \text{kappa} \\ \frac{3}{4} \text{kappa} \\text{R-symm} \end{array} \right) \), global: \( G_{\text{super}} \)

(64)

The global symmetry \( G_{\text{super}} \) is evident when \( g(\tau) \) is transformed from the left side as \( g \rightarrow g^\prime(\tau) = g \Gamma_\tau g^{-1} \) with a \( g \Gamma_\tau \in G_{\text{super}} \) that is \( \tau \) independent. Then the Cartan connection \( g^{-1} \partial_\tau g \) remains invariant. Noether’s theorem gives the conserved \( G_{\text{super}} \) charges in the form

\[
global: G_{\text{super}}: J^B_A = (g L g^{-1})^B_A = \left( \begin{array}{cc} G^{\text{super}} & \text{R-symm} \end{array} \right)^B_A
\]

(65)

The local symmetries that act on the right side of \( g \) have the matrix form in Eq. (64) which is different than the matrix form in Eq. (65) for the global symmetries that act on the left side of \( g \). Before giving details in the next paragraph, we mention that, by \( \frac{3}{4} \text{kappa} \) we imply that the local kappa supersymmetry can remove as much as \( 3/4 \) of the fermions in \( g \). The \( \text{SO}(d, 2) \) local symmetry can remove the \( \text{SO}(d, 2) \) parameters...
3.1 Coupling $X, P, g$, gauge symmetries, global symmetries.

$\omega_{MN}$ in $g$ but cannot remove the additional parameters in $G$ associated with the other generators $\Gamma^{M_1 \cdots M_n}$ listed in Table 1. The local R-symmetry can remove from $g$ all of the subgroup parameters $\omega_a$. Thus for $d = 3, 4, 5, 6$ cases listed in Table 1, all the bosons in $g$ can be eliminated and $3/4$ of the fermions can be eliminated, if we wish to choose such a gauge. For $d \geq 7$ some of the bosonic degrees of freedom in $g$ cannot be eliminated, those are related to D-branes.

As in the purely bosonic theory, the local $\text{Sp}(2, R) \times (\text{SO}(d, 2) \frac{1}{4}\kappa_{\text{symm}})$ symmetries can be gauge fixed in a variety of ways to descend to supersymmetric 1T-systems that are dual to each other, and holographically represent the same 2T superparticle. The hidden symmetries of any holographic image is the original global symmetry $G_{\text{super}}$. Some of these gauge choices are outlined in the following figure.

![Figure 2 - More dualities, holographic images of 2T superparticle.](image)

Let’s outline the properties of the local symmetries $\text{Sp}(2, R) \times (\text{SO}(d, 2) \frac{1}{4}\kappa_{\text{symm}})$. The local symmetry $\text{Sp}(2, R)$ is straightforward since the first two terms of the action $S_{2T} (X, P, g)$ are the same as Eq. [26]. These terms are invariant under $\text{Sp}(2, R)$ which acts on $X_i^M = (X^M, P^M)$ as a doublet for every $M$, and on $A^{ij}$ as the gauge field. Furthermore, by taking $g (\tau)$ as a $\text{Sp}(2, R)$ singlet while noting that $L^M = e^{ij} X_i^M X^N_j = X^M P^N - X^N P^M$ is $\text{Sp}(2, R)$ gauge invariant, we see that the full action is gauge invariant under $\text{Sp}(2, R)$. To see the local symmetry under
When both \( X_i^M \) and \( g ( \tau ) \) are transformed under local SO\((d, 2) \times \text{R-symm}\) transformations as \( \delta_R X_i^M = \varepsilon_R^{MN} X_iN \) and \( \delta_R g = -\frac{i}{2} A^{ij} X_i \cdot X_j \), one can see that the structures \( g \left( \begin{array}{cc} X_0 & 0 \\ 0 & 0 \end{array} \right) g^{-1} \) and \( X_i \cdot X_j \) are gauge invariant under \( \delta_R \). Note that \( g \) transforms from the right side under \( \delta_R \). The local kappa supersymmetry also acts on \( g \) from the right as \( \delta_\kappa g = g K \) with \( K = \left( \begin{array}{c} 0 \\ \kappa M \end{array} \Gamma_{MN} \right) X_i^M \), and on \( \delta_\kappa A^{ij} \neq 0 \) as follows. Under this \( \delta_\kappa \) transformation the action in the form of Eq. (66) gives \( \delta_\kappa S_{2T} = \int d\tau \left[ -\frac{i}{2} \kappa M \partial_\tau X_i \cdot X_j + \frac{i}{2} \kappa M \delta_{\kappa} \left( g^{-1} \partial_\tau g \right) \right] , \) where the second term takes the form \( \kappa M \partial_\tau \left( ig^{-1} \partial_\tau g \right) \) with \( \sigma = \left( \Gamma_{NK} \Gamma_{MN} \right) X_i^M X_j^N X_k^R \) and \( \Gamma_{NK} X_k^R \). The important thing is that \( \kappa M \partial_\tau \left( ig^{-1} \partial_\tau g \right) \) is proportional to \( X_i \cdot X_j \), and therefore it can be canceled by choosing \( \delta_\kappa A^{ij} \) in front of the same coefficient \( X_i \cdot X_j \), so that \( \delta_\kappa S_{2T} = 0 \).

The action \( S_{2T} (X, \hat{P}, g) \) can be generalized by increasing the number of dimensions, but keeping the same \( g \in G_{\text{super}} \). We will denote the new action as \( S_{2T} (\hat{X}, \hat{P}, g) \). To describe its content, first we recall that the group element \( g \in G_{\text{super}} \) was chosen by considering the number of dimensions \( d \) and a group \( G \supset SO(d, 2) \) as listed in Table 1. Now we extend the \( d + 2 \) dimensions \( X_i^M = (X^M, P^M) \) by adding \( 2d' \) more spacelike dimensions \( X_i^M = (X^I, P^I), \) \( I = 1, 2, \ldots, d' \). We associate the SO\((d') \) acting on the \( d' \) dimensions with the R-symmetry group, just as the \( d + 2 \) dimensions are associated with the group \( G \). Namely we choose the number of dimensions \( d' \) such that the dimension \( s_{d'} \) of the spinor representation of SO\((d') \) coincides with the fundamental representation of the R-symmetry group. Then, instead of the \( L \) in Eq. (66) we define an extended \( \hat{L} \)

\[
\hat{L} = \frac{1}{4 \epsilon} \left( \begin{array}{cc} \Gamma_{MN} L_{MN} & 0 \\ 0 & -\alpha \Gamma^{IJ} L_{IJ} \end{array} \right)
\] (67)

where \( \alpha = \frac{2d}{d'} \) is the ratio of the dimensions of the spinor representations for SO\((d + 2) \) and SO\((d') \). Now define \( \hat{X}^M_i = (X^M, X^I) \) and \( \hat{X}^M_i = (P^M, P^I) \) as the phase space in \( d + d' + 2 \) dimensions, and write the same form of action as Eq. (66) in the extended dimensions \( \hat{S} (\hat{X}, \hat{P}, g) \), but use the new form of \( \hat{L} \) given above. The coupling \( A^{ij} \hat{X}_i \cdot \hat{X}_j \) leads to the Sp\((2, R) \) constraints that includes all dimensions at an equal footing. In the coupling \( \kappa M \partial_\tau (ig^{-1} \partial_\tau g L) \) the first \( d + 2 \) dimensions couple to the SO\((d, 2) \) in \( G \) and the last \( d' \) dimensions couple to SO\((d') \) in the R-symmetry group. This extended action has the following local and global symmetries

local: \( \text{Sp}(2, R) \times \left( \begin{array}{c} \text{SO}(d, 2) \\ \frac{1}{2} \kappa \text{super} \end{array} \right) \), global: \( G_{\text{super}} \) (68)
3.2 Covariant quantization & representations of $G_{\text{super}}$

Note that the global symmetry is the same as before the extension since the Cartan connection $g^{-1}\partial_{\tau}g$ is still invariant, but the local symmetry is less as seen by comparing to Eq. (64). In particular now we have the $kappa$ so more fermions in $g$ are physical. Also there may be more bosons if $\text{SO}(d')$ is smaller than the R-symmetry in $G_{\text{super}}$, as labeled by $N$ in Table 1. So, the extended model is expected to have a different physical set of states. In this scheme we obtain interesting cases, depending on the gauge choices, such as superparticles with compactified $d'$ dimensions, and without D-branes for the cases $d = 3, 4, 5, 6$. With the right choice of dimensions and groups the emerging space is quite interesting from the point of view of M-theory.

For example [24] [14] [15] the superparticle on $\text{AdS}_5\times S^5$ with a total of 10 dimensions ($d = 4, d' = 6$, and $d + d' + 2 = 12$) is obtained by taking the supergroup $\text{SU}(2, 2|4)$, and then specializing to a particular gauge. It was shown in [24] that this approach gives a particle spectrum that is identical to the Kaluza-Klein towers of $\text{d}=10$ supergravity compactified on $\text{AdS}_5\times S^5$. This was discussed by choosing the spacetime gauge as shown in the first branch of Fig.2. The same theory can be brought to the form of a sigma model for the coset $\text{SU}(2, 2|4)/\text{SO}(4,1)\times \text{SO}(5)$ [30], as shown in the third branch of Fig.2, or can be put to the form of a twistor theory as shown in the second branch of Fig.2.

Similarly the superparticle on $\text{AdS}_4\times S^7$ ($d = 3, d' = 8$) or $\text{AdS}_7\times S^4$ ($d = 6, d' = 5$) with a total of 11 dimensions, and no D-branes, emerges by taking the supergroup $\text{OSp}(8|4)$ [14] [15]. More details will be given in a separate publication.

3.2 Covariant quantization & representations of $G_{\text{super}}$

As seen from the form of $J$ in Eq. (65), it is gauge invariant under $\left(\begin{array}{cc} \text{SO}(d,2)_{\text{super}} & \text{R-symmetry} \end{array}\right)$ as well as $\text{Sp}(2, R)$ transformations. Therefore the $G_{\text{super}}$ charges $J^B_A$ are physical observables that classify the physical states under $G_{\text{super}}$ representations. With this in mind we study the properties of $J$. In particular the square of the matrix $J$, given by $(J^2)_A^B = (gL^{-1}gL^{-1})_A^B = (gL^2g^{-1})_A^B$, contains important information about the physical states. At the classical level $L^2 = 0$ as discussed in section 2.8, and therefore $(J^2)_A^B = 0$ at the classical level. At the quantum level we must be careful not only about the computation of $L^2$ as discussed in section 2.8, but also about the order of operators in $gLg^{-1}$ because, unlike section 2.3, $g$ cannot be fully eliminated by the available gauge symmetries. Then $J^2$ is not necessarily of the form $gL^2g^{-1}$ except for the simplest case of $G = \text{SO}(d, 2)$.

The details of the quantum discussion will be given in a separate paper, but suffice it to mention that we obtain the following general form of algebraic constraints among the generators of $G_{\text{super}}$

$$JJ = \alpha J + \beta 1$$  \hspace{1cm} (69)

where the coefficients $\alpha, \beta$ depend on $G_{\text{super}}$. This equation is to be compared to the simpler case of $\text{SO}(d, 2)$ in Eq. (2.4). By taking a super trace we learn that $\beta$ gives the quadratic Casimir operator. The absence of any quadratic term in $J$ on the right hand side of Eq. (69) not proportional to 1 is very nontrivial. This does not happen
for generic representations of \( G_{\text{super}} \). But for the representations generated by the from \( J = gLg^{-1} \) it is expected since \( J^2 \) vanished at the classical level.

So the algebraic constraints above must determine the representation of \( G_{\text{super}} \). Indeed, by using Eq. (69) repeatedly we obtain

\[
J^n = \alpha_n J + \beta_n 1, \quad C_n \sim \text{Str} ((2J)^n) \sim 2^n \beta_n.
\]

(70)

where all coefficients \( \alpha_n, \beta_n \) and the Casimir eigenvalues \( C_n \) are determined in terms of \( \alpha, \beta \) as follows

\[
\alpha_n = \left( \alpha + \sqrt{\alpha^2 + 4\beta} \right)^{n+1} - \left( \alpha - \sqrt{\alpha^2 + 4\beta} \right)^{n+1} \over 2^{n+1}\sqrt{\alpha^2 + 4\beta}, \quad \beta_n = \beta \alpha_{n-2}.
\]

(71)

These properties of the Casimir eigenvalues completely determine the representation of \( G_{\text{super}} \).

This is the representation that classifies the physical states of the theory \( S_{2T} (X, P, g) \), or the extended one \( S_{2T} (X, \hat{P}, g) \), under the global symmetry \( G_{\text{super}} \). No matter which gauge we choose to describe the physical content of the theory we cannot change the group theoretical content of the physical states. In the particle gauge, in position space these physical states correspond to an on-shell free field in a field theory. Some cases of interest are listed in the table below. These were obtained by covariant quantization of the 2T particles or superparticles in various dimensions without choosing a gauge. Details of the computation will appear elsewhere.

| particle/superparticle | \( G_{\text{super}} \) algebraic constraints | Field theory |
|------------------------|------------------------------------------|-------------|
| SO(\(d,2\), any \(d\) massless, spinless) | \( JJ = -d^2 J + \frac{1}{4} \left( 1 - \frac{d^2}{4} \right) \) | Scalar Klein-Gordon |
| SO(\(4,2\), \(d=4\) massless, any helicity \(h\)) | \( JJ = (h - 2) J + \frac{1}{4} (h^2 - 1) \) | massless field, any spin Dirac, Maxwell, Einstein, etc. |
| OSp(\(N\mid4\)), \(d=3\) | \( \begin{align*} \frac{N=8}{N=16} J & = -\frac{2}{3} J + \beta_8 \\ \frac{N=16}{N=16} J & = -\frac{2}{3} J + \beta_{16} \end{align*} \) | \( N=8 \) Super Yang-Mills \( N=16 \) SUGRA |
| SU(\(2,2\mid4\)), \(d=4\) | \( \begin{align*} \frac{N=4}{N=8} J & = -2 J + \frac{1}{4} \\ \frac{N=8}{N=8} J & = -2 J + \frac{1}{4} \end{align*} \) | \( N=4 \) Super Yang-Mills \( N=8 \) SUGRA |
| OSp(\(8\mid4\)), \(d=6\) | \( \begin{align*} \frac{N=4}{N=8} J & = -3 J + \beta_4 \\ \frac{N=8}{N=8} J & = -3 J + \beta_8 \end{align*} \) | \( N=4 \) self-dual CFT \( N=8 \) |
| SU(\(2,2\mid4\)), \(d'=4\) | \( JJ = -2 J + \frac{l(l+1)}{4}, \ l = 1, 2, 3, \ldots \) | type IIB, AdS_5 \times S^5 |
| \( SU(2,2\mid4\)), \(d'=6\) | \( JJ = -2 J + \frac{l(l+1)}{4}, \ l = 1, 2, 3, \ldots \) | compactified SUGRA |

Table 2 - Algebraic constraints \( JJ = \alpha J + \beta 1 \) satisfied by the generators of \( G_{\text{super}} \)

If we choose other gauges than the particle gauge, we find other holographic images of the same 2T-theory. The other images are dual to the particle or the field theory image included in the table above. The various gauges will yield the same representation of \( G_{\text{super}} \) since the Casimir eigenvalues are gauge invariant and cannot change. The quantum states in various images can differ from one image to another only by the set of operators that are simultaneously diagonal on the physical state (usually including the Hamiltonian for that image) beyond the Casimir operators. Since a gauge transformation is a duality transformation from one image to another, this duality transformation is a unitary transformation within the unitary representation of \( G_{\text{super}} \) fixed above by \( \alpha, \beta \).
3.3 Twistor gauge: supertwistors dual to super phase space

There are different ways of choosing gauges to express the theory given by $S_{2T}(X, P, g)$, or the extended one $S_{2T}(X, P, g)$, in terms of the physical sector. One extreme in gauge space is to eliminate all of the SO(d,2)×R-symm subgroup of $g$ completely, while another extreme is to eliminate $(X, P)$ completely. When most of $g$ is eliminated we obtain the phase space description, and when $(X, P)$ is eliminated we obtain the twistor description.

To obtain the twistor description for the action $S_{2T}(X, P, g)$ we eliminate $(X^M, P^M)$ completely and keep only $g$ as discussed in [7]. This is done by using the Sp(2,R) and the SO(d,2) local symmetries to completely fix $X^M, P^M$ to the convenient form $X^+ = 1$ and $P^+ = 1$, while all other components vanish

$$X^M = (1, 0, \ldots, 0), \quad P^M = (0, 1, \ldots, 0), \quad i = 1, \ldots, (d - 2). \quad (72)$$

These $X^M, P^M$ already satisfy the constraints $X^2 = P^2 = X \cdot P = 0$. In this gauge the only non-vanishing component of $L^{MN}$ is $L^+ = 1$, so that

$$L_{fixed} = \frac{-2}{4i} \left( \begin{array}{cc} \Gamma^- & 0 \\ 0 & 0 \end{array} \right) L^+ = \frac{i}{2} \left( \begin{array}{cc} \Gamma^- & 0 \\ 0 & 0 \end{array} \right) \equiv \Gamma. \quad (73)$$

Hence the physical content of the theory is now described only in terms of $g$ and the fixed matrix $\Gamma$ embedded in the Lie algebra of SO(d,2).

The matrix $\Gamma$ has very few non-zero entries as seen by choosing a convenient form of gamma matrices$^6$ for SO(d,2). Then, up to similarity transformations, $\Gamma$ can be brought to the form

$$\Gamma = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) = \left( \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right). \quad (74)$$

The identity matrix 1, and the small 0’s in the last expression are $\frac{s_d}{4} \times \frac{s_d}{4}$ square block matrices embedded in the $s_d \times s_d$ spinor representation of SO(d,2). Then the gauge

$^6$An explicit form of SO(d,2) gamma matrices that we find convenient in even dimensions, is given by

$$\Gamma^0 = -1 \times 1, \quad \Gamma^i = \sigma_3 \times \gamma^i, \quad \Gamma^\pm = -i \sqrt{2} \sigma_2 \times 1 \quad (note \quad \Gamma^\pm = -i \sigma_1 \times 1 \quad and \quad \Gamma^\tau = \sigma_2 \times 1)$$

where $\gamma^i$ are the SO(d−1) gamma matrices. The $\Gamma^M$ are the same as the $\Gamma^M$ for $M = \pm, i$, but for $M = 0$ we have $\Gamma^0 = -1 \times 1$. From these we construct the traceless $\Gamma^{\mu, \nu}$, $\Gamma^{\mu, \nu} = i \sqrt{2} \left( \begin{array}{ll} 0 & s_\mu \\ 0 & 0 \end{array} \right)$. $\Gamma^{\mu, \nu} = -i \sqrt{2} \left( \begin{array}{ll} 0 & 0 \\ 0 & 0 \end{array} \right)$, $\Gamma^{\mu, \nu} = \Gamma^{\nu, \mu}$, with $\gamma^\mu = (1, \gamma^i)$ and $\gamma^\mu = (-1, \gamma^i$).

Then $\frac{1}{2} \Gamma_{MN} J^{MN} = -\Gamma^{\mu, \nu} J_{\mu, \nu} + \frac{1}{2} J_{\mu, \nu} \Gamma^{\mu, \nu} - \Gamma_{\mu, \nu} J^{\nu, \mu}$ takes the matrix form given in Eq.(??). We can further write $\gamma^\tau = \tau^3 \times 1$, $\gamma^2 = \tau^2 \times 1$ and $\gamma^r = \tau^3 \times \rho^r$ for $\rho^r$ the gamma matrices for SO(d−3). These gamma matrices are consistent with the metric $C = \sigma_1 \times 1 \times 1$ of Eq.(??), and

$^7$The gamma matrices $\Gamma^M$ of footnote 4 can be redefined differently for the left or right sides of $g$ up to similarity transformations. Thus, for the right side of $g$ we apply a similarity transformation so that

$\gamma^1 = \tau^3 \times 1$, etc., to obtain $\gamma^- = (\gamma^0 - \gamma^1) / \sqrt{2}$ in the form given in Eq.(??).
invariant 2T action in Eq. (61), and the gauge invariant \( \text{SO}(d,2)_L \) charges in Eq. (65), take the twistor form similar to Eq. (11)

\[
S_{2T}(X,P,g) = \frac{4}{s_d} \int d\tau \text{Str} \left(i \partial_\tau g \Gamma g^{-1}\right) = \frac{4}{s_d} \int d\tau \ i \bar{Z}_a^A \partial_\tau Z_\alpha^A \equiv S_{\text{twistor}},
\]  

(75)

\[
J_B^A = (g \Gamma g^{-1})_B^A = \left(Z_\alpha^A \bar{Z}_\alpha^B - \frac{1}{s_d - s_{d'}} \text{Str} (Z \bar{Z}) \delta_B^A\right),
\]  

(76)

The \( Z_\alpha^A, \bar{Z}_\alpha^A \), as constructed from the group element \( g \), are supertwistors that already obey constraints as we explain below, so \( \frac{4}{s_d} \int d\tau i \bar{Z}_a^A \partial_\tau Z_\alpha^A \) is the full supertwistor action. Due to the form of \( \Gamma \) it is useful to think of \( g \) as written in the form of \( \frac{s_d}{2} \times \frac{s_d}{2} \) square blocks. Then \( Z_\alpha^A \) with \( A = \text{fundamental of } G_{\text{super}} \) and \( \alpha = 1,2,\ldots, \frac{s_d}{2} \) emerges as the rectangular supermatrix that corresponds to the fourth block of columns of the matrix \( g \), and similarly \( \bar{Z}_\alpha^A \) corresponds to the second block of rows of \( g^{-1} \). Since

\[
g^{-1} = \left( \begin{array}{cc} C^{-1} & 0 \\ 0 & 1 \end{array} \right) g^\dagger \left( \begin{array}{cc} C & 0 \\ 0 & 1 \end{array} \right),
\]

we find that \( \bar{Z} = c^{-1} Z^\dagger \left( \begin{array}{cc} C & 0 \\ 0 & 1 \end{array} \right) \), where

\[
C = \sigma_1 \times 1 \times c \text{ is given in footnote (6). Furthermore, as part of } g, g^{-1}, \text{ the } Z_\alpha^A, \bar{Z}_\alpha^B \text{ must satisfy the constraint } \bar{Z}_\alpha^A Z_\beta^B = 0 \text{ since the product } \bar{Z}_\alpha^A Z_\beta^B \text{ contributes to an off-diagonal block of the matrix } 1 \text{ in } g^{-1} g = 1,
\]

(77)

A constraint such as this one must be viewed as the generator of a gauge symmetry that operates on the \( a \) index (the columns) of the supertwistor \( Z_\alpha^A \). For the purely bosonic version of this process see [9] where twistors in any dimension are obtained.

In \( d = 4,6 \) the supertwistors that emerge from this approach coincide with supertwistors previously known in the literature if we work in the massless particle gauge [7] [14] [15]. However, if we work in one of the other gauge choices that lead to the holographic images depicted in Figs. 1, 2, then we obtain new results for the twistor description for those cases. Some applications of the \( d = 4,6 \) twistors will be given in the next section.

Similarly, to obtain the twistor description for the extended action \( S_{2T}(\hat{X}, \hat{P}, g) \) we eliminate \( (\hat{X}^M, \hat{P}^M) \) completely and keep only \( g \) as discussed in [14] [15]. Thus, we first use the local \( \text{SO}(d,2) \times \text{SO}(d') \subset G_{\text{super}} \) to rotate the \( d + d' + 2 \) components to the form

\[
\hat{M} = (0' 0 1 \cdots d' , I = 1 2 3 \cdots d')
\]

\[
\hat{X}^M(\tau) = (1 0 0 \cdots 0 , 1 0 0 \cdots 0)
\]

(78)

\[
\hat{P}^M(\tau) = (0 1 0 \cdots 0 , 0 1 0 \cdots 0)
\]

(79)

These solve also the \( \text{Sp}(2,R) \) constraints. In this gauge the extended matrix \( \hat{L} \) simplifies
In this gauge the action and the $G_{\text{super}}$ symmetry current are expressed only in terms of the group element

\[
S_{2T} \left( \hat{X}, \hat{P}, g \right) \sim \int \text{Str} \left( g^{-1} \hat{\Gamma} i \partial g \right), \quad J_A^B = \left( g^{-1} \hat{\Gamma} g \right)_A^B \tag{81}
\]

\[
g \in G_{\text{super}} / H_{\tilde{\Gamma}}. \tag{82}
\]

Due to the form of $\hat{\Gamma}$ there are gauge symmetries $H_{\tilde{\Gamma}}$ that correspond to all generators of $G_{\text{super}}$ that commute with $\hat{\Gamma}$. The gauge symmetries remove degrees of freedom so that the physical degrees of freedom that remain in $g$ corresponds to the coset $G_{\text{super}} / H_{\tilde{\Gamma}}$. As shown in [14] [15] in the case of $G_{\text{super}} = \text{PSU}(2|2)$ the coset is $\text{PSU}(2|2) / \text{PSU}(2|2)$, as seen for $\hat{\Gamma} = \text{diag}(1, 1, 1, -1, -1, -1, -1)$ after a rearrangement of rows and columns. These coset degrees of freedom are equivalent to the superparticle moving on AdS$_5 \times$S$^5$ space.

Furthermore, by an appropriate parametrization of $g$, including the gauge degrees of freedom, this action can be written in twistor form. The twistors in this case is a gauged super grassmanian $Z_A^a$ described as follows:

| $a = 1, \ldots, 8$ | $g = 1, 2$ | $n = 4$ |
|------------------|--------|-----|
| $Z_A^a$ = (bosonic fermionic) 8x4 rectangular matrix 4 fundamental reps of $\text{PSU}(2|4)$ |
| $L = Z_A^a (\partial + V) Z_A^a$, $V_a^b = \text{PSU}(2|2) \times U(1)$ gauge field |
| $J_A^B = (ZZ - l)_A^B$, $l \equiv \frac{1}{2} \text{Str} \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} ZZ \right)$ global symmetry $\text{PSU}(2|2)$ classifies physical states |
| $U(1): \Delta \equiv \text{Str} (ZZ) = \text{Str} (ZZ) = 0$ vanish on gauge invariants in twistor space |
| $\text{PSU}(2|2): G_a^b \equiv Z_a^A Z_A^b - 2 l \delta_a^b - \frac{1}{2} \left( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right)_a^b = 0$ vanish on gauge invariants in twistor space |

The $\text{PSU}(2|2) \times U(1)|_{\text{local}}$ gauge invariant physical space described by this twistor is the full Kaluza-Klein spectrum of of type-IIB $d=10$ supergravity compactified on AdS$_5 \times$S$^5$, which is classified by $\text{PSU}(2|2)|_{\text{global}}$ representations labeled by the eigenvalues of the operator $l = 0, 1, 2, \ldots$. This result was already obtained in [24] in a different gauge of the same 2T-physics action $S_{2T} \left( \hat{X}, \hat{P}, g \right)$.

In a similar way, when $G_{\text{super}} = \text{OSp}(8|4)$, the coset is $\text{OSp}(8|4) \times \text{OSp}(4|2) \times U(1) / U(1)$ for $\Gamma = \text{diag}(1, 1, 1, -1, -1, -1, -1, -2, -2, 2, 2)$. The twistor equivalent version
is a gauged super Grassmanian $Z_A^a$ described as follows

$$Z_A^a = \frac{8_B}{4_F} \left( \frac{(8 \times 4)_B}{(4 \times 4)_F} \right) \left( \frac{(8 \times 2)_F}{(4 \times 2)_B} \right), \quad \text{pseudo-real } Z_A^a$$

\(\text{OSp}(8|4)\) global acting on \(A\), \(\text{OSp}(4|2) \times R\) local acting on \(a\)

\[L = i \text{Str} (\bar{Z} D Z), \quad D Z = \partial_\tau Z + AZ, \quad A = \text{OSp}(4|2) \text{ gauge field}\]

The physical space for this twistor corresponds to 11-dimensional supergravity compactified on \(\text{AdS}_7 \times \text{S}_4\) or \(\text{AdS}_4 \times \text{S}_7\).

We have used 2T-physics as a tool to obtain supertwistors that describe various systems in higher dimensions. Some of the properties of these more exotic twistors have been outlined in [14] [15], and more will be discussed elsewhere.

4 Supertwistors and some field theory spectra in d=4,6

4.1 Supertwistors for d=4, N=4 Super Yang-Mills

Consider the twistor obtained from the 2T-physics approach using \(S_{2T} (X, P, g)\) in \(d = 4\) and \(G_{\text{super}} = \text{PSU}(2,2|4)\) as given in Table 1. The same theory in a different gauge gives the \(d = 4\) superparticle with \(N = 4\) supersymmetries, as described in Eq.(63).

The quantum spectra of both descriptions, corresponding to the physical states, must coincide. Let’s see how this is obtained explicitly.

To begin the superparticle in Eq.(63) has \(4x 4p\) and \(16\theta\) real degrees of freedom in super phase space. In the lightcone gauge we remove \(1x\) and \(1p\), due to \(\tau\) reparametrization and the corresponding \(p^2 = 0\) constraint. We also remove 8 fermionic degrees of freedom due to kappa supersymmetry. We are left over with \(3x, 3p, 8\theta\) physical degrees of freedom. With these we construct the physical quantum states as an arbitrary linear combination of the basis states in momentum space \(|\vec{p}, \alpha\rangle\), where \(\alpha\) is the basis for the Clifford algebra satisfied by the \(8\theta\). This basis has 8 bosonic states and 8 fermionic states labeled by \(\alpha\). Viewed as probability amplitudes in position space \((x, \alpha|\psi\rangle\) these are equivalent to fields \(\psi(x)_{8_B+8_F}\) which correspond to the independent solutions of all the constraints. One finds that these are the same as the 8 bos and 8 fermi fields of the Super Yang-Mills (SYM) theory which are the solutions of the linearized equations of motion in the lightcone gauge. They consist of two helicities of the gauge field \(A_{\pm1}(x)\), two helicities for the gauginos \(\psi_{\pm \frac{1}{2}}^{\alpha}(x)\), \(\tilde{\psi}_{-1,\alpha}(x)\) in the 4, \(\bar{4}\) of \(\text{SU}(4)\), and six scalars \(\phi^{[ab]}(x)\) in the 6 of \(\text{SU}(4)\).

Now we count the physical degrees of freedom for the twistors. As seen from Eq.(75), for \(d = 4\) we have one complex twistor \(Z_A\) in the fundamental representation of \(\text{PSU}(2,2|4)\), with a Lagrangian and a conserved current given by

\[
L = i\bar{Z}^A \partial_\tau Z_A, \quad J_A^B = Z_A \bar{Z}^B, \quad \text{and } \bar{Z}^A Z_A = 0
\]

\(Z_A\) is in fundamental representation of \(\text{PSU}(2,2|N) \leftrightarrow \text{CP}^3|N\)

This is recognized as the particle version of the \(d = 4, N = 4\), twistor superstring [10]-[15]. To start \(Z_A\) has 4 complex bosons and 4 complex fermions, i.e. \(8_B + 8_F\) real
4.1 Supertwistors for \(d=4, N=4\) Super Yang-Mills

degrees of freedom. However, there is one constraint \(\bar{Z} A Z_A = 0\) and a corresponding \(U(1)\) gauge symmetry, which remove 2 bosonic degrees of freedom. The result is \(6_B + 8_F\) physical degrees of freedom which is equivalent to \(\mathbb{CP}^{3/4}\), and the same number as \(3x, 3p, 8\theta\) for the superparticle, as expected.

To construct the spectrum in twistor space we could resort to well known twistor techniques by working with fields \(\phi (Z)\) that are holomorphic in \(Z_A\) on which \(\bar{Z} A\) acts as a derivative \(\bar{Z} A \phi (Z) = - \partial \phi (Z) / \partial Z_A\), as dictated by the canonical structure that follows from the Lagrangian \([89]\). Imposing the constraint amounts to requiring \(\phi (Z)\) to be homogeneous of degree 0, namely

\[
Z_A \frac{\partial \phi (Z)}{\partial Z_A} = 0, \quad Z_A = \text{PSU}(2,2|4) \text{ supertwistor.} \tag{87}
\]

Quantum ordering does not change the homogeneity degree because there are an equal number of bosons and fermions in the case of \(N = 4\). We write \(Z_A = (\xi_\alpha)\) with \(z_i = (\mu, \lambda)\) the 4 of \(SU(2,2) \subset \text{PSU}(2,2|4)\) and \(\xi_\alpha\) the 4 of \(SU(4) \subset \text{PSU}(2,2|4)\). Then the superfield \(\phi (Z)\) can be expanded in powers of the fermions \(\xi_\alpha\)

\[
\phi (Z) = \sum_{n=0}^{4} (\xi_{a_1} \cdots \xi_{a_n}) \phi^{a_1 \cdots a_n} (z). \tag{88}
\]

The equation \(0 = \frac{\partial \phi (Z)}{\partial z_i} = z_i \frac{\partial \phi (Z)}{\partial z_i} + \xi_\alpha \frac{\partial \phi (Z)}{\partial \xi_\alpha} = 0\) becomes a homogeneity condition for the coefficients \(\phi^{a_1 \cdots a_n} (z)\)

\[
\frac{\partial \phi^{a_1 \cdots a_n} (z)}{\partial z_i} = -n \phi^{a_1 \cdots a_n} (z) \tag{89}
\]

Comparing to Eq. \([13]\) we see that the helicity of the wavefunction \(\phi^{a_1 \cdots a_n} (z)\) is \(h_\alpha = \frac{n}{2} - 1\). So we will label the wavefunction by its helicity as well as its \(SU(4)\) labels, by including a subscript \(\frac{n}{2} - 1\) that corresponds to the helicity. More explicitly, the wavefunction \(\phi (Z)\) takes the form

\[
\phi (Z) = A_{-1} (z) + \xi_a \psi_{-1/2} (z) + \xi_\alpha \xi_b \phi^{ab} (z) \tag{90}
\]

\[
+ \frac{\varepsilon^{abcd} \xi_c \xi_d \xi_a \xi_b}{3!} \psi_{+1/2, d} (z) + \frac{\varepsilon^{abcd} \xi_c \xi_d \xi_a \xi_b \xi_e A_{+1} (z)}{4!}, \tag{91}
\]

where we gave suggestive names to the coefficients \(\phi^{a_1 \cdots a_n} (z)\)

\[
\phi^{a_1 \cdots a_n} (z) : \left( A_{-1}, \psi_{-1/2}, \phi^{ab}, \psi_{+1/2, d}, A_{+1} \right). \tag{92}
\]

These are precisely the helicity fields, including \(SU(4)\) representation content, that correspond to the vector supermultiplet in \(N = 4, d = 4\) SYM theory. They each are homogeneous of degree \(-2h - 2\) where \(h\) corresponds to the helicity indicated by the subscript. For example, \(A_{-1} (z)\) is homogeneous of degree 0, while \(A_{+1} (z)\) is homogeneous of degree \(-4\), etc. Thus the \(\phi (Z)\) is the degree zero wavefunction \(\phi (Z)\) described in [13]. The entire wavefunction can be expanded in terms of momentum eigenstates as in Eq. \([19]\) using the results for \(\langle z|k \rangle\) listed in Eq. \([22]\).
The superfield $\phi(Z)$ is a representation basis of $\mathrm{PSU}(2,2|4)$ which is an evident global symmetry of the twistor action (56). The symmetry current $J^B_A = Z_A Z^B$ acts as $J^B_A \phi(Z) = -Z_A \frac{\partial \phi(Z)}{\partial Z^B}$, and this induces the symmetry transformations on the individual fields. This is the hidden superconformal symmetry of the $d = 4$, $N = 4$ SYM field theory in the twistor version. Recall that the twistor form of $J^B_A$ followed by gauge fixing the original gauge invariant form given in Eq. (65), so the $\mathrm{PSU}(2,2|4)$ superconformal symmetry of SYM theory is understood as the global symmetry of the underlying $4 + 2$ dimensional superparticle.

Recall that in [13] there are also twistor wavefunctions $f(Z), g(Z)$ that describe the spectrum of conformal gravity; those can arise also in our twistor formalism, but for a different superparticle model that gives a different degree $c \neq 0$ in the $\mathrm{PSU}(2,2|4)$ supertwistor homogeneity equation $Z_A \frac{\partial \phi(Z)}{\partial Z^A} = c \bar{\partial} \phi_k(Z)$. Since only the value of $c = 0$ is permitted in the $N = 4$, $d = 4$ superparticle model, only SYM states $\phi(Z)$ are present. Of course, this is no surprise in the 2T setting. We have simply compared two gauges, and we must agree.

4.1.1 Oscillators, supertwistors, and unitarity of $d=4,N=4$ spectrum

It is also worth analyzing the quantum system in terms of oscillators related to twistors and understand the unitarity of the physical space. We emphasize that $\bar{Z}^A$ is obtained from $Z_A$ by Hermitian conjugation and multiplying by the $\mathrm{PSU}(2,2|4)$ metric as given following Eq. (70). To see the oscillator formalism clearly we work in a basis of $\mathrm{SU}(2,2|4)$ in which the group metric is diagonal of the form $\text{diag}(1_2, -1_2, 1_2)$. The block $\text{diag}(1_2, -1_2) = \sigma_2 \times 1$ part is the $\mathrm{SU}(2,2)$ metric in the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ basis, to be contrasted with the $\mathrm{SL}(2,C)$ basis in which the metric $C = \sigma_1 \times 1$ is off-diagonal as in footnote 6. The two bases are simply related by a linear transformation that diagonalizes the $\mathrm{SU}(2,2)$ metric $C = \sigma_1 \times 1 \rightarrow \sigma_2 \times 1$. In this diagonal basis we work with compact $\mathrm{SU}(2) \times \mathrm{SU}(2)$ oscillators $z = \left(\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}\right)$ and $\bar{z} = \left(\begin{smallmatrix} \bar{z}_1 \\ -\bar{z}_2 \end{smallmatrix}\right) = z^4 C$, which are combinations of the $\mathrm{SL}(2,C)$ doublet twistor components $(\mu^a_\alpha, \bar{\lambda}_\alpha, \lambda^a_\alpha)$ we discussed before. A bar over the symbol means Hermitian conjugation. In terms of these, the twistor Lagrangian and the current $J$ take the form

$$L = i \bar{Z}^A \partial_r Z_A = i \bar{a}^i \partial_r a_i - i b_I \partial_r b_I + i \bar{\xi}^r \partial_r \xi_r, \quad \text{SU}(2), \quad i = 1, 2, \quad \text{SU}(2), \quad I = 1, 2, \quad r = 1, \ldots, 4 \text{ SU}(4)$$

$$Z_A = \begin{pmatrix} a_i \\ b_I \\ \xi_r \end{pmatrix}, \quad \bar{Z}^A = \begin{pmatrix} \bar{a}^i \\ -\bar{b}_I \end{pmatrix}$$

$$J^B_A = Z_A \bar{Z}^B = \begin{pmatrix} a_i \bar{a}^j & -a_i b_j & a_i \bar{\xi}^s \\ -b_I \bar{a}^j & -b_I \bar{b}_j & b_I \bar{\xi}^s \\ -\xi_r \bar{a}^j & -\xi_r \bar{b}_j & \xi_r \bar{\xi}^s \end{pmatrix}$$

It is significant to note that, after taking care of the metric in $\bar{Z}$ as above, the usual canonical rules applied to this Lagrangian identifies the oscillators as being all positive norm oscillators $[a_i, \bar{a}^j] = \delta_i^j$, $[b_I, \bar{b}^J] = \delta_I^J$ and $\{\xi_r, \bar{\xi}^s\} = \delta_r^s$. Therefore all Fock
states have positive norm. In the Fock space of these oscillators we must identify the physical states as only those that satisfy the constraints
\[ 0 = \bar{Z}^A Z_A = \bar{a}^i a_i - b^f \bar{b}_f + \bar{\xi}^r \xi_r = \bar{a}^i a_i - (\bar{b}^f b_f + 2) + \bar{\xi}^r \xi_r \]

This physical state condition is written in terms of the number operators for the oscillators \( N_a, N_\xi, \bar{N}_b \) as
\[ \Delta = N_a + N_\xi - N_b = 2 \quad \text{(96)} \]

This setup is precisely the Bars-Gunaydin oscillator formalism for unitary representations of noncompact groups [31] for a single “color”, supplemented with the constraint \( \Delta = 2 \) as discussed in [24]. All Fock space states that satisfy \( \Delta = 2 \) are easily classified under the compact subgroup \( \text{SU}(2) \times \text{SU}(2) \times \text{SU}(4) \) of \( \text{PSU}(2, 2|4) \). They are organized through the unitary infinite dimensional representations of the subgroup \( \text{SU}(2, 2) \) by identifying the so-called “lowest” states that are annihilated by the double annihilation generators \( a_i b_j \) which is part of \( J^B_{\Delta} \) in the conformal subgroup \( \text{SU}(2, 2) \) given in Eq. (95). The list of the \( \Delta = 2 \) lowest states is easily identified and classified under \( \text{SU}(2) \times \text{SU}(2) \times \text{SU}(4) \subset \text{PSU}(2, 2|4) \) as
\[ \Delta = 2 : \begin{pmatrix} A_{-1} & \psi^{r/2}_{-1} & 0 \\ \bar{a}^i \bar{a}^j & \bar{a}^i \bar{\xi}^j & \bar{\xi}^j \xi^i \\ (1,0,1) & (1,0,1) & (0,0,0) \end{pmatrix} \begin{pmatrix} b^f b_f + 2 \\ \bar{\xi}^r \xi_r \\ (0,1,1) \end{pmatrix} \begin{pmatrix} A_{+1} \\ \bar{\psi}^{r/2}_{+1} \\ (0,0,0) \end{pmatrix} |0\rangle \quad \text{(97)} \]

The notation in the last line is \( (j_1, j_2, \text{dim} (\text{SU}(4))) \), where \( (j_1, j_2) \) is for \( \text{SU}(2) \times \text{SU}(2) \) while \( \text{dim} (\text{SU}(4)) \) is the dimension of the \( \text{SU}(4) \) representation. In arriving at the \( \text{SU}(2) \times \text{SU}(2) \times \text{SU}(4) \) representation labels in the third line, we took into account that \( \bar{a}^i \bar{a}^j \) is symmetric while \( \bar{\xi}^r \xi^i \) is antisymmetric, etc. All other states with \( \Delta = 2 \) are descendants of these, and are obtained by applying arbitrary powers of the double creation generator \( \bar{a}^i \bar{b}^j \) in \( \text{SU}(2, 2) \). All states have positive norm by virtue of the positive norm oscillators we identified above. So, the towers of states generated on each lowest state is an irreducible infinite dimensional unitary representation of \( \text{SU}(2, 2) \). The full collection of states is a single irreducible representation of \( \text{PS}(2, 2|4) \) called the doubleton representation of \( \text{PSU}(2, 2|4) \) (sometimes it is also called the singleton, so the name is not so important).

We have shown that the list above is equivalent to a classification under \( \text{SU}(2, 2) \times \text{SU}(4) \), so the lowest states should be sufficient to identify the SYM fields, and the descendants should be analogous to applying multiple derivatives on a field since \( \bar{a}^i \bar{b}^j \) is a vector \((1/2,1/2)\) under \( \text{SU}(2) \times \text{SU}(2) \). Indeed, we can imagine now an analytic continuation back to the \( \text{SL}(2, C) \) basis instead of the \( \text{SU}(2) \times \text{SU}(2) \) basis and reinterpret the \( (j_1, j_2) \) as the \( \text{SL}(2, C) \) labels for the field. In this analytic continuation the spin subgroup \( \text{SO}(3) \) is a common subgroup in both \( \text{SL}(2, C) \) = \( \text{SO}(3, 1) \) and \( \text{SU}(2) \times \text{SU}(2) = \text{SO}(4) \), therefore the spin of the state is \( \text{spin} = j_1 + j_2 \). The helicity and chirality in \( \text{SL}(2, C) \) are related, so by using the spin and chirality we can identify the helicity. Using this, in Eq. (97) we have identified the SYM fields with their helicities \( h \) above each of the oscillator combination. Although we have indicated the gauge field as \( A_{\pm 1} \), as if it is only two states, the full \( \text{SU}(2) \times \text{SU}(2) \) or \( \text{SL}(2, C) \) set of
oscillator states (1,0) and (0,1) really correspond to all the 6 components of the gauge invariant field strength $F_{\mu\nu}$ in SL(2, C) notation. These comments are consistent with the helicities of the $A_\mu$ versus $F_{\mu\nu}$ listed in Eq. (22). Similar comments apply to all the other spin $(j_1,j_2)$ multiplets.

Although we gave a list of lowest states above as a supermultiplet, there really is only one lowest oscillator state for the entire unitary representation of PSU(2, 2|4). That one is simply $\xi^r\xi^s|0\rangle$, which satisfies $\Delta = 2$ and is annihilated not only by $a_i b_j$ but also by the supersymmetry generators $\xi^r b_j, a_i \xi^s$ that are part of $J$. This is the lowest state from which all other states with $\Delta = 2$ listed can be obtained as descendants by applying all powers of $J^B_A$ on this state (note $[\Delta, J^B_A] = 0$). This entire tower is the doubleton of PSU(2, 2|4). If we watch carefully the orders of the oscillators we can show that the generators $J = ZZ$ of PSU(2, 2|4) in the doubleton representation satisfy [24] [14] in the following nonlinear constraints as listed in Table 2:

$$(JJ)^B_A - 2 (J)^B_A + 0 \delta^B_A = 0.$$ (98)

The linear $J$ follows from the commutation rules among the generators, the coefficient $-2$ is related to the commutation rules among the $J$’s but also to the overall normalization of $J$ (taken differently in [24] [14]), while the coefficient 0 is the PSU(2, 2|4) quadratic Casimir eigenvalue $C_2 = 0$. We see that the renormalized operator $(-J/2)$ acts as a projection operator on physical states. This equation should be viewed as a set of constraints on the generators that are satisfied only in the doubleton representation, and as such this relation identifies uniquely the representation only in terms of the generators $J$. If the theory is expressed in any other form (such as particle description, or field theory) the doubleton representation can be identified in terms of the global symmetry as one that must satisfy the constraints (98), automatically requiring the 6 scalars $\phi^{[ab]}$ as the lowest SU(4) multiplet. This is a completely PSU(2, 2|4) covariant and gauge invariant way of identifying the physical and unitary states of the theory. Of course, the $d = 4, N = 4$ SYM fields satisfy this criterion as seen above.

4.2 Twistor for $d = 4, N = 8$ SUGRA

We can repeat the $N = 4$ analysis of the superparticle or twistors for other values of $N$, and still $d = 4$. Twistor or 1T-supersparticle are different gauge choices of the 2T superparticle action $S_{2T} (X, P, g)$, so we expect the same physical spectrum. In the 1T-supersparticle gauge of Eq. (65) we start out with $4N$ real $\theta$’s in the action, but due to kappa supersymmetry only $2N$ real $\theta$’s can form physical states in the lightcone gauge. The quantum algebra among the physical $\theta$’s is the 2N dimensional Clifford algebra which is equivalent to N creation and N annihilation fermionic oscillators. With the creation operators we construct $2^N$ physical states, half are bosons and the other half are fermions. Therefore for the superparticle the lightcone spectrum is $2^{N-1} + 2^{N-1}$.

Since each fermionic $\theta$ carries spin 1/2, and SU($N$) quantum numbers in the fundamental representation, we can obtain the SU($N$) and helicity quantum numbers of the physical states by assigning a Young tableau $\square_{1/2}$ for the $N$ creation operators with helicity 1/2, and take antisymmetric products (fermions) to construct the $2^{N-1} + 2^{N-1}$ physical states. If we start from a helicity $-h$ for the $\theta$ vacuum, then the physical states
have helicity and SU(\(N\)) quantum numbers given by the following Young tableaux (this is for \(N=\text{even}\))

\[
\text{even: } \left( 0_{-h} + \square_{-h+1} + \cdots + \square_{h-1} \right) ; \text{ odd: } \left( \square_{-h+\frac{1}{2}} + \square_{-h+\frac{3}{2}} + \cdots + \square_{h+\frac{1}{2}} \right)
\]

A CPT invariant spectrum emerges provided for the even number of boxes (similarly for the odd) the top helicity is the opposite of the bottom helicity \(-h + \frac{N}{2} = h\). So the top and bottom helicities are \(h = \pm N/4\). This applies for \(N=\text{even}\), and is consistent with half integer quantized helicity. If \(N\) is odd we do not get a CPT invariant spectrum. So, let us consider the even \(N = 2, 4, 6, 8\) cases. For \(N = 4, 8\) the even number of boxes have integer helicities hence they are bosons, and the odd number of boxes have half-integer helicities hence they are fermions. For \(N = 2, 6\), the even number of boxes are fermions, and the odd number of boxes are bosons.

Besides the \(N = 4\) case that gave the SYM spectrum, a most interesting case is \(N = 8\). This gives the top/bottom helicities for the bosons \(h = \pm (8/4) = \pm 2\), which correspond to the graviton. Therefore the full spectrum gives the \(N = 8\), \(d = 4\) supergravity spectrum.

We now analyze the theory in Eq. (75), for \(d = 4\), and any \(N\) from the point of view of twistors. For general \(N\) the quantum ordered constraint is

\[
\frac{1}{2} \langle Z | \left( Z_A Z^A + (1)^{A} \tilde{Z}^A Z_A \right) | \phi \rangle = 0,
\]

where \((-1)^A = \pm 1\) is inserted for the bose/fermi components since \(Z_A\) is a PSU(\(2, 2|N\)) supertwistor. In \(Z\) space this is rearranged to the form

\[
Z_A \frac{\partial \phi (Z)}{\partial Z_A} + \frac{1}{2} \text{Str} (1) \phi (Z) = 0, \text{Str} (1) = 4 - N.
\]

We write \(Z_A = (z^i, \xi)\) with \(z^i = \left( \xi^a \right)\) the 4 of SU(\(2, 2\)) \(\subset\) PSU(\(2, 2|N\)) and \(\xi_a\) the \(N\) of SU(\(N\)) \(\subset\) PSU(\(2, 2|N\)). Then the superfield \(\phi (Z)\) can be expanded in powers of the fermions \(\xi_a\)

\[
\phi (Z) = \sum_{n=0}^{N} (\xi_{a_1} \cdots \xi_{a_n}) \phi^{a_1 \cdots a_n} (z).
\]

The equation \(\frac{1}{2} (N - 4) \phi (Z) = Z_A \frac{\partial \phi (Z)}{\partial Z_A} = z_i \frac{\partial \phi (z \xi)}{\partial z_i} + \xi_a \frac{\partial \phi (z \xi)}{\partial \xi_a} = 0\) becomes a homogeneity condition for the coefficients \(\phi^{a_1 \cdots a_n} (z)\)

\[
\frac{\partial \phi^{a_1 \cdots a_n} (z)}{\partial z_i} = - \left( n + \frac{4 - N}{2} \right) \phi^{a_1 \cdots a_n} (z)
\]

Comparing to Eq. (15) we see that the helicity of the wavefunction \(\phi^{a_1 \cdots a_n} (z)\) is \(h_n = \frac{n}{2} - \frac{N}{4}\). So, for a given \(N\) the lowest helicity is \(h_{\text{min}} = -\frac{N}{4}\) and the top helicity is \(h_{\text{min}} = \frac{N}{4}\). This is consistent with quantization of spin only for \(N=\text{even}\). For \(N = 8\)
we obviously get the gravity supermultiplet in the form of a field in twistor space

\[
\phi(Z) = g_2(z) + \xi_a \Lambda_a^{(−3/2)}(z) + \xi_b \bar{V}^{ab}_{−1}(z) + \frac{\xi_a \xi_b \xi_c \xi_d}{3!} \psi_{a b c}(z) + \phi_0^{ab}(z) + \cdots + \xi^8 g_{+2}(z),
\]

where in the last line the SU(8) representation and the helicity are indicated. This is in agreement with the superparticle spectrum discussed above.

If we are interested in a physical state |ϕ⟩ of definite momentum |k⟩, we can use the table in Eq. (22) to write the wavefunction for each component of the superfield above, for \( \phi_h(z) = (g_{−2}(z), \cdots, g_{−2}(z)) \)

\[
\phi_h(z) = \delta(⟨λπ⟩) \exp \left(−\frac{π}{\lambda} \bar{π}_α μ^α \right) \left(\frac{λ}{π}\right)^{1−2h} \phi_h(π, \bar{π})
\]

We can also approach the twistor quantum theory from the point of view of oscillators. The formalism of Eqs. (93, 95) applies just by taking \( N = 8 \) instead of \( N = 4 \). Then we obtain the spectrum of lowest states in Fock space with \( \Delta = 4 \) as follows

\[
\begin{pmatrix}
g_{−2}, \bar{A}_{(2,0)}, \bar{A}_{(4,0)} \\
g_{−2}, \bar{A}_{(0,8)}, \bar{A}_{(1,0,28)} \\
g_{−2}, \bar{A}_{(0,56)}, \bar{A}_{(0,56)}, \bar{A}_{(0,1,28)} \\
g_{−2}, \bar{A}_{(0,8)}, \bar{A}_{(0,2,1)}
\end{pmatrix}
\]

As explained in the case of \( N = 4 \), the representations under \( g_{+2} \) labeled as \( (2, 0, 1) \) and \( (0, 2, 1) \) are really the components of the gauge invariant curvature tensor \( R_{a\beta γ δ} \) and \( R_{a\beta γ δ} \) in agreement with the table in Eq. (22). Similar comments apply to the other states labeled as \( (j_1, j_2, \dim(\text{SU}(8))) \).

### 4.3 Super twistors for d=6 and self dual super multiplet

The superparticle in \( d = 6 \) and \( N = 4 \), derived from the 2T-physics theory \( S (X, P, g) \) as in Eq. (63), starts out with \( 6x, 6P, 16θ \) real degrees of freedom. Fixing \( τ \), and kappa local gauges and solving constraints, reduces the physical degrees of freedom down to \( 5x, 5P, 8θ \). The superparticle action has a hidden global superconformal symmetry \( \text{OSp}(8^∗|4) \) [7], therefore the physical states should be classified as a unitary representation under this group [15].

If we quantize in the lightcone gauge we find \( 8_B + 8_F \) states, which should be compared to the physical fields of a six dimensional field theory

\[
\text{SO}(5,1) \times \text{Sp}(4): F_{[μνλ]}^+, \psi_{[α]}^+, \phi^{[ab]}
\]

taken in the lightcone gauge. Indeed we have the following 8 bosonic fields in the lightcone gauge: a self dual antisymmetric tensor \( A_{ij} = ε_{ijkl} A^{kl} \) in SO(4) ⊂ SO(5, 1)
consistent with expectation. An approach may be helpful.

\[ Z_{Aa} = \left(\begin{array}{c} Z_{a}^1 \end{array}\right)_{Aa} \]

\[ Z_{Aa} = (12, 2) \text{ of } OSp(8^*|4) \text{ global } \times SU(2) \text{ local} \]

\[ L = Z^{Aa} ((\partial + V) Z)_{Aa}, \quad V = SU(2) \text{ gauge field} \]

\[ \hat{Z}^{aA} = (Z^T \eta)^{aA} = \left(\begin{array}{c} a_{11} \bar{a}_{11} - a_{21} \bar{a}_{21} \xi_{1\alpha} \xi_{1\alpha} \bar{a}_{11} - a_{21} \bar{a}_{21} \xi_{1\alpha} \xi_{2\alpha} \xi_{2\alpha} \bar{a}_{11} - a_{21} \bar{a}_{21} \xi_{1\alpha} \xi_{2\alpha} \xi_{2\alpha} \bar{a}_{11} - a_{21} \bar{a}_{21} \xi_{1\alpha} \xi_{2\alpha} \xi_{2\alpha} \bar{a}_{11} - a_{21} \bar{a}_{21} \xi_{1\alpha} \xi_{2\alpha} \xi_{2\alpha} \bar{a}_{11} - a_{21} \bar{a}_{21} \xi_{1\alpha} \xi_{2\alpha} \xi_{2\alpha} \bar{a}_{11} - a_{21} \bar{a}_{21} \xi_{1\alpha} \xi_{2\alpha} \xi_{2\alpha} \bar{a}_{11} - a_{21} \bar{a}_{21} \xi_{1\alpha} \xi_{2\alpha} \xi_{2\alpha} \bar{a}_{11} - a_{21} \bar{a}_{21} \xi_{1\alpha} \xi_{2\alpha} \xi_{2\alpha} \bar{a}_{11} - a_{21} \bar{a}_{21} \xi_{1\alpha} \xi_{2\alpha} \xi_{2\alpha} \bar{a}_{11} - a_{21} \bar{a}_{21} \xi_{1\alpha} \xi_{2\alpha} \xi_{2\alpha} \bar{a}_{11} - a_{21} \bar{a}_{21} \xi_{1\alpha} \xi_{2\alpha} \xi_{2\alpha} \bar{a}_{11} - a_{21} \bar{a}_{21} \xi_{1\alpha} \xi_{2\alpha} \xi_{2\alpha} \bar{a}_{11} - a_{21} \bar{a}_{21} \xi_{1\alpha} \xi_{2\alpha} \xi_{2\alpha} \bar{a}_{11} - a_{21} \bar{a}_{21} \xi_{1\alpha} \xi_{2\alpha} \xi_{2\alpha} \bar{a}_{11} - a_{21} \bar{a}_{21} \xi_{1\alpha} \xi_{2\alpha} \xi_{2\alpha} \end{array}\right) \]

\[ \hat{Z}_{Aa} \text{ is pseudo real, } \hat{Z}^{aA} = \varepsilon^{ab} (Z^T)_{bB} C^{BA}, \quad C = \left(\begin{array}{cc} \sigma_1 \times 14 & 0 \\ 0 & -i\sigma_2 \times 12 \end{array}\right) \]

Let us now examine the twistors that emerge in Eq. 75 for this case. Writing the twistor in the oscillator basis we have the results listed above. The pseudo-reality property follows from the fact that \( Z_{Aa} \) is part of the group element \( g \in OSp(8^*|4) \) that satisfies \( g^{-1} = C^{-1} g^{ST} C \) with the \( C^{BA} \) used above. Then \( Z_{Aa} \) takes the form above in a natural basis. Thus the second column is related to the first one, but still consistent with a local \( SU(2) \) applied on \( a = 1, 2 \).

When \( Z, \hat{Z} \) of these forms are inserted in the Lagrangian \( L = \hat{Z}^{Aa} ((\partial + V) Z)_{Aa} \) we get (after integration by parts and dropping total derivatives, and taking care of boson/fermi statistics in interchanging factors)

\[ L = \bar{a_i} \partial a_i + \bar{a_i} \partial a_{2i} + \bar{\xi_i} \partial \xi_{1\alpha} + \bar{\xi_i} \partial \xi_{2\alpha} - \frac{1}{2} Tr (V G), \quad V = SU(2) \text{ gauge field} \]

\[ G = (Z \bar{Z}) a^b \]

\[ J = ZZ - \frac{1}{4} Str (Z \bar{Z}), \quad 12 \times 12 \text{ supermatrix of } OSp(8^*|4) \text{ charges} \]

It is seen that according to the canonical formalism, the oscillators identified above all have positive norm

\[ [a_{1i}, a_{1\bar{i}}] = \delta^i_\bar{i}, \quad [a_{2i}, a_{2\bar{i}}] = \delta^i_\bar{i}, \quad \{ \xi_{1\alpha}, \bar{\xi}_1 \} = \delta^{-i}_{\alpha}, \quad \{ \xi_{2\alpha}, \bar{\xi}_2 \} \]

We count the degrees of freedom before the constraints, and find that \( Z_{Aa} \) has \( 8B + 4F \times 2 = 16B + 8F \) real parameters (namely the complex \( a_{1i}, a_{2i}, \xi_{1\alpha}, \xi_{2\alpha} \)). The constraints are due to a SU(2) gauge symmetry acting on the index \( a = 1, 2 \) (although...
it seems like SU(2) × U(1), the U(1) part is automatically satisfied because of the pseudoreal form of $Z_{Ab}$. The 3 gauge parameters and 3 constraints remove 6 bosonic degrees of freedom, and we remain with $10B + 8F$ physical degrees of freedom. This is the same as the count for the superparticle ($5x, 5p, 8θ$). It is obvious we have the same number of degrees of freedom and the same symmetries $\text{OSp}(8^*|4)$, with the symmetry being much more transparent in the twistor basis.

The quantum theory can proceed in terms of the oscillators. There is only one lowest physical state, namely the Fock vacuum $|0\rangle$ that is gauge invariant $G|0\rangle = 0$, and is also annihilated by the double annihilation operators of $\text{SO}(6,2)$ in $J = ZZ$. All other $G = 0$ physical states are descendants of this one by applying all powers of $J$. The resulting representation is precisely the doubleton of $\text{OSp}(8^*|4)$ which is equivalent to the fields in Eq. (105) and Table 3 below. This oscillator representation was worked out long ago in [32] using again the Bars-Günaydin method [31].

The details of the doubleton are found as follows. First construct the SU(2) gauge singlet (i.e., $G = 0$) ground states in Fock space as in the first column of Table 3. The ground states are the states annihilated by the double annihilation bosonic generator $(aa) = a_{ai}a_{bj}\varepsilon^{ab} = a_{11}a_{22} - a_{12}a_{21}$. This is one of the generators in $J = ZZ - \frac{1}{4}\text{Str}(ZZ)$ that sits in the conformal subgroup $\text{SO}^*(8)$ (i.e. spinor of $\text{SO}(6,2)$). All the states that are annihilated by this generator are included in the first column below (note zero or one power of $\bar{a}$ is obviously annihilated, the two powers of $\bar{a}$ in the last item is possible only because of an appropriate symmetrization as described below, higher powers of $\bar{a}$ cannot be annihilated by this generator if we also require annihilation by $G$).

| Fock space lowest state | SU(4) × SU(2) Young tableau | SU(4) × Sp(4) dimensions | SU(4) × Sp(4) field |
|-------------------------|----------------------------|---------------------------|----------------------|
| $|0\rangle$             | $(0,0) = (1,1)$            | $(1,5)$                   | $\phi^{[ab]}$        |
| $(\xi\bar{\xi})|0\rangle$ | $(0,\square\square) = (1,3)$ | $(4,4)$                   | $\psi^{[\alpha]}$   |
| $(\bar{a}\xi)$ $|0\rangle$ | $(\square\square,\bar{a}) = (4,2)$ | $(10,1)$                 | $\partial_\lambda A_{\mu\nu} + \text{self dual}$ |

Table 3 - The $\text{OSp}(8^*|4)$ doubleton.

To insure that these are also annihilated by the SU(2) gauge generators $G^a_b$, we must combine the SU(2) gauge indices on the oscillators into SU(2) singlets. This guarantees that all of these states are physical. So $(\xi\bar{\xi})$, $(\bar{a}\xi)$ and the creation generator $(\bar{a}\bar{a})$ stand for

$$(\xi\bar{\xi}) = \bar{\xi}^a\bar{\xi}^b\varepsilon^{ab} = (0,\square\square), \quad (\bar{a}\xi) = \bar{a}_i\bar{\xi}^\beta\varepsilon^{ab} = (\square,\square), \quad (\bar{a}\bar{a}) = \bar{a}_i\bar{a}_j\varepsilon^{ab} = \left(\square,\bar{\square}\right)$$

while the annihilation $(aa)$ generator stands for $(aa) = a_{ai}a_{bj}\varepsilon^{ab} = \left(\bar{\square},0\right)$, where a dotted box represents the complex conjugate representation (but for SU(4), $\bar{\square} = \square$). The boxes in the Young tableaux represent the un-summed indices $i, \alpha$ which stand for the fundamental representations of SU(4) × SU(2) where SU(4) ⊂ SO$^*(8)$ and SU(2) ⊂ Sp(4). To keep track of these indices we use the Young tableau notation as in
Eq. (108) for the operators, and use that property to figure out the second column of Table 3. In both Eq. (108) and Table 3 we take into account that the \( a \) oscillators are bosons and the \( \xi \) oscillators are fermions, so under permutations of the \( a \)'s there must be symmetry and under permutation of the \( \xi \)'s there must be antisymmetry. These properties lead uniquely to the Young tableaux listed in the table. Next, for each SU(4) representation we combine the SU(2) representations into complete Sp(4) representations as in the third column of Table 3. From this we can easily read off the corresponding fields as in the last column of Table 3. Having established all possible ground states for the operator \( (aa) \), we apply all possible powers of the generator \( (\bar{a}\bar{a}) \) on those ground states in order to obtain all the states of the irreducible representation. Applying the powers of \( (\bar{a}\bar{a}) \) just gives the descendants of the ground states. The collection of all these states is the same as starting with the single ground state \( |0\rangle = (0,0) \) and then applying all the powers of the OSp(8|4) generators \( J = ZZ - \frac{1}{2} \text{Str} (ZZ) \). The reason for organizing the states as in Table 3, is to read off the standard supersymmetry multiplet that corresponds to the fields in field theory as described in the next paragraph (Poincaré supersymmetry is a subgroup of OSp(8|4)).

In the last step of Table 3 we interpret SU(4) as the spacetime SO(5, 1) after an analytic continuation. The number of states in the field theory notation must match the number of states in the third column. For example the 5 states \( (1,5) \) corresponds to \( \phi^{(ab)} \) which is a 4 \( \times \) 4 antisymmetric, and traceless tensor under the Sp(4) metric, which has just 5 components. Also \( \partial_X A_{\mu\nu} \) is a 10-component, 3-index antisymmetric and self-dual tensor, using SO(5, 1) vector indices \( \mu, \nu, \lambda \), instead of the spinor indices \( \Box \Box \). These fields form the basis for the self-dual tensor supermultiplet under supersymmetry in 6 dimensions.

Of course they are also the basis of the infinite dimensional unitary representation of OSp(8|4). The generators of the latter are of course the \( J = ZZ - \frac{1}{2} \text{Str} (ZZ) \), constructed from the twistors in the form of a12\( \times \)12 supermatrix of global OSp(8\( ^* \)|4) charges.

## 5 2T superstring descends to twistor superstring

So far in these lectures I discussed superparticles and the associated supertwistors, and their physical spectra. These have a direct generalization to superstrings via the 2T superstring formalism given in [14]. Briefly, the action is

\[
S = \int d\tau d\sigma \left( L^T_{2T} + L^-_{2T} \right),
\]

and \( L^T_{2T} \) are defined on the worldsheet as follows

\[
L^T_{2T} = \partial_+ X \cdot P^+ - \frac{1}{2} AX \cdot X - \frac{1}{2} B_{\pm \pm} P^+ \cdot P^- - C_+ P^+ \cdot X
- \frac{1}{2^{d/2-1}} \text{Str} \left( \partial_{+ \pm} g \bar{g} \left( L^\pm_{MN} \Gamma^{MN} 0 0 \right) \right) + L_G
\]

\( X_M(\tau, \sigma) \), \( P^T_M(\tau, \sigma) \), \( L^T_{MN} = X_M P^T_N \), \( g(\tau, \sigma) \) are now string fields, and \( \partial_{\pm} = \frac{1}{2} \left( \partial_\tau \pm \partial_\sigma \right) \). Here \( L^T_{2T} \) represent left/right movers, and there is open string boundary conditions. \( L_G \) describes additional degrees of freedom that may be needed to insure a
critical worldsheet theory that is conformally exact. In the case of the $d = 4, N = 4$ twistor superstring $L_G$ describes an internal current algebra for some SYM group $G$, with conformal central charge $c = 28$ [11].

The local and global symmetries are similar to those of the particle and are described in [14]. The global symmetry is $G_{super}$ chosen for various $d$ as in Table 1. One must insure that there are no anomalies in the operator products of the local symmetry currents. Such details will be discussed elsewhere.

In the twistor gauge the 2T superstring above reduces to twistor superstrings, with the twistors described in the previous sections. In $d = 4$, and $G_{super} = SU(2,2|4)$ the 2T superstring descends to the twistor superstring in the Berkovits version. There are open problems that remain to be resolved in this theory [13], in particular the conformal supergravity sector is undesirable and some constraint that projects out this sector would be interesting to find.

For other values of $d$ and $G_{super}$ these theories remain to be investigated. We know of course the particle limit of these worldsheet theories as discussed in these lectures. In particular the $d = 4, N = 8$ twistors lead to $N = 8$ supergravity in the particle limit, so this provides a starting point for an investigation similar to [16] [17] of a twistor superstring for $d = 4, N = 8$ supergravity.

Similarly to the $d + d' = 10, 11$ particle case we also consider the $d + d' = 10, 11$ 2T superstrings

$$\hat{L}_{2T}^\pm = \partial_{\pm}\hat{X} \cdot \hat{P}^\pm - \frac{1}{2}A\hat{X} \cdot \hat{X} - \frac{1}{2}B_{\pm\pm} \hat{P}^\pm \cdot \hat{P}^\pm - C_{\pm} \hat{P}^\pm \cdot \hat{X}$$

$$- \frac{1}{8} \text{Str} \left( \partial_{\pm\pm} g \left( L_{MN}^\pm \Gamma^{MN} 0 0 -\alpha L_{IJ}^\pm \Gamma^{IJ} \right) \right)$$

where $SO((d + d'),2) \to SO(d,2) \times SO(d')$, $\hat{X}^M = (X^m, X^I)$, $\hat{P}^\pm_M = (P^\pm_m, P^\pm_I)$, $g(\tau, \sigma) \in SU(2,2|4)$ or OSp$(8|4)$, and $L_{MN}^\pm = X_M^P P^\pm_P$, $L_{IJ}^\pm = X_M^P P^\pm_P$. The local and global symmetries are discussed in the second part of section 3.3. In the particle-type gauge, the spectrum in the particle limit is the same as linearized type IIB SUGRA compactified on AdS$_5 \times S^5$ [24] for $d + d' = 10$, and 11D SUGRA compactified to AdS$_4 \times S^7$ or AdS$_7 \times S^4$ for $d + d' = 11$. In the twistor gauge this theory is currently being investigated by using the twistors in Eqs. (83, 84).

In the 2T philosophy each one of these 2T superstrings have many duals that can be found and investigated by choosing various gauges. This is a completely open field of investigation at this time, and it could be quite interesting from the point of view of M-theory.

The analogies to certain aspects of M-theory are striking. Dualities in M-theory appear to be analogs of the Sp(2,R) and its generalizations discussed above, and illustrated in Fig.1 for the simplest model of 2T-physics. Taking into consideration that 2T-physics correctly describes 1T-physics, and provides a framework for a deeper view of spacetime and a new unification of 1T-dynamics, we are tempted to take the point of view that it probably applies also to M-theory. So possibly M-theory would eventually be most clearly formulated as a 13-dimensional theory with signature $(11, 2)$ and global supersymmetry OSp$(1|64)$. This is consistent with certain attractive features of the supergroup OSp$(1|64)$ as a hidden symmetry of M-theory [25]- [28].
The twistor approach discussed in these lectures may be a useful tool for further progress in this deeper direction as well as for developing new computational tools in conventional theories.

6 Acknowledgments

I would like to thank the organizers of the “2005 Summer School on String/M Theory” in Shanghai, China, and the organizers of the International Symposium QTS4, “Quantum Theory and Symmetries IV”, in Varna, Bulgaria, for their hospitality and support. This research was supported by the US Department of Energy under grant No. DE-FG03-84ER40168.

References

[1] I. Bars, B. Orcal, and M. Picon, in preparation, I. Bars and B. Orcal, in preparation.

[2] R. Penrose, “Twistor Algebra,” J. Math. Phys. 8 (1967) 345; “Twistor theory, its aims and achievements, in Quantum Gravity”, C.J. Isham et. al. (Eds.), Clarendon, Oxford 1975, p. 268-407; “The Nonlinear Graviton”, Gen. Rel. Grav. 7 (1976) 171; “The Twistor Program,” Rept. Math. Phys. 12 (1977) 65.

[3] R. Penrose and M.A. MacCallum, “An approach to the quantization of fields and space-time”, Phys. Rept. C6 (1972) 241; R. Penrose and W. Rindler, Spinors and space-time II, Cambridge Univ. Press (1986).

[4] T. Shirafuji, “Lagrangian Mechanics of Massless Particles with Spin,” Prog. Theor. Phys. 70, (1983) 18.

[5] I. Bars, C. Deliduman and O. Andreev, “Gauged Duality, Conformal Symmetry and Spacetime with Two Times”, Phys. Rev. D58 (1998) 066004 [arXiv:hep-th/9803188]. For reviews of subsequent work see: I. Bars, “Two-Time Physics”, in the Proc. of the 22nd Intl. Colloq. on Group Theoretical Methods in Physics, Eds. S. Corney at. al., World Scientific 1999, [arXiv:hep-th/9809034]; “Survey of two-time physics,” Class. Quant. Grav. 18, 3113 (2001) [arXiv:hep-th/0008164]; “2T-physics 2001,” AIP Conf. Proc. 589 (2001), pp.18-30; AIP Conf. Proc. 607 (2001), pp.17-29 [arXiv:hep-th/0106021].

[6] I. Bars, “Conformal symmetry and duality between free particle, H-atom and harmonic oscillator”, Phys. Rev. D58 (1998) 066006 [arXiv:hep-th/9804028]; “Hidden Symmetries, AdSd×S^n, and the lifting of one-time physics to two-time physics”, Phys. Rev. D59 (1999) 045019 [arXiv:hep-th/9810025].

[7] I. Bars, “2T physics formulation of superconformal dynamics relating to twistors and supertwistors,” Phys. Lett. B 483, 248 (2000) [arXiv:hep-th/0004090].
REFERENCES

[8] I. Bars and M. Picon, “Single twistor description of massless, massive, AdS, and other interacting particles,” arXiv:hep-th/0512091.

[9] I. Bars and M. Picon, “Twistor Transform in d Dimensions and a Unifying Role for Twistors,” arXiv:hep-th/0512348.

[10] E. Witten, “Perturbative gauge theory as a string theory in twistor space”, Commun. Math. Phys. 252 (2004) 189 [arXiv:hep-th/0312171]; “Parity invariance for strings in twistor space”, hep-th/0403199.

[11] N. Berkovits, “An Alternative string theory in twistor space for N=4 Super Yang-Mills”, Phys. Rev. Lett. 93 (2004) 011601 [arXiv:hep-th/0402045].

[12] N. Berkovits and L. Motl, “Cubic twistorial string field theory”, JHEP 0404 (2004) 56, [arXiv:hep-th/0403187].

[13] N. Berkovits and E. Witten, “Conformal supergravity in twistor-string theory”, JHEP 0408 (2004) 009, [arXiv:hep-th/0406051].

[14] I. Bars, “Twistor superstring in 2T-physics,” Phys. Rev. D70 (2004) 104022, [arXiv:hep-th/0407239].

[15] I. Bars, “Twistors and 2T-physics,” AIP Conf. Proc. 767 (2005) 3, [arXiv:hep-th/0502065].

[16] F. Cachazo, P. Svrcek and E. Witten, “MHV vertices and tree amplitudes in gauge theory”, JHEP 0409 (2004) 006 [arXiv:hep-th/0403047]; “Twistor space structure of one-loop amplitudes in gauge theory”, JHEP 0410 (2004) 074 [arXiv:hep-th/0406177]; “Gauge theory amplitudes in twistor space and holomorphic anomaly”, JHEP 0410 (2004) 077 [arXiv:hep-th/0409245].

[17] For a review of Super Yang-Mills computations and a complete set of references see: F.Cachazo and P.Svrcek, “Lectures on twistor strings and perturbative Yang-Mills theory,” PoS RTN2005 (2005) 004, [arXiv:hep-th/0504194].

[18] I. Bars, “Two time physics with gravitational and gauge field backgrounds”, Phys. Rev. D62, 085015 (2000) [arXiv:hep-th/0002140]; I. Bars and C. Deliduman, “High spin gauge fields and two time physics”, Phys. Rev. D64, 045004 (2001) [arXiv:hep-th/0103042].

[19] Z. Perjés, Rep. Math. Phys. 12, 193 (1977).

[20] L.P. Hughston, “Twistors and Particles”, Lecture Notes in Physics 97, Springer Verlag, Berlin (1979).

[21] P.A. Tod, Rep. Math. Phys. 11, 339 (1977).

[22] S. Fedoruk and V.G. Zima, “Bitwistor formulation of massive spinning particle,” arXiv:hep-th/0308154.
[23] A. Bette, J.A. de Azcarraga, J.Lukierski and C. Miquel-Espanya, “Massive relativistic particle model with spin and electric charge from two-twistor dynamics,” Phys. lett. B 595, 491 (2004), [arXiv:hep-th/0405166]; J.A.de Azcarraga, A.Frydryszak, J.Lukierski and C.Miquel-Espanya, “Massive relativistic particle model with spin from free two-twistor dynamics and its quantization,” arXiv:hep-th/0510161.

[24] I. Bars, “Hidden 12-dimensional structures in AdS$_5$ x S$^5$ and M$^4$ x R$^6$ supergravities,” Phys. Rev. D 66, 105024 (2002) [arXiv:hep-th/0208012]; “A mysterious zero in AdS$_5$ x S$^5$ supergravity,” Phys. Rev. D 66, 105023 (2002) [arXiv:hep-th/0205194].

[25] I. Bars, “S Theory”, Phys. Rev. D 55 (1997) 2373 [arXiv:hep-th/9607112].

[26] I. Bars, C. Deliduman and D. Minic, “Lifting M-theory to Two-Time Physics”, Phys. Lett. B 457 (1999) 275 [arXiv:hep-th/9904063].

[27] P. West, “Hidden superconformal symmetry in M theory,” JHEP 0008 (2000) 007 [arXiv:hep-th/0005270].

[28] I. Bars, “Toy M-model”, to be published. For a brief discussion see [26] and the second review paper in [5].

[29] I. Bengtsson and M. Cederwall, “Particles, Twistors And The Division Algebras,” Nucl. Phys. B 302 (1988) 81. M. Cederwall, “AdS twistors for higher spin theory,” AIP Conf. Proc. 767 (2005) 96, [arXiv:hep-th/0412222].

[30] R.R. Metsaev and A.A. Tseytlin, “Type IIB superstring action in AdS(5) x S(5) background,” Nucl. Phys. B 533 (1998) 109 [arXiv:hep-th/9805028].

[31] I. Bars and M. G"unaydin, Comm. Math. Phys. 91 (1983) 31.

[32] M. G"unaydin, P. van Nieuwenhuizen, N.P. Warner, Nucl. Phys. B 255 (1985) 63.