A CRITERION FOR THE EXISTENCE OF PERIODIC POINTS
BASED ON THE EIGENVALUES OF MAPS INDUCED IN COHOMOLOGY

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Abstract. We present a criterion for the existence of periodic points based on the eigenvalues of maps induced in cohomology for spaces with rational cohomology isomorphic to a tensor product of a graded exterior algebra with generators in odd dimensions and a graded algebra with all elements of even degree. We give a number of natural examples of such spaces and provide some non-trivial ones. We also give a counterexample to a claim in [4] given there without proof.

1. Introduction

In this short paper we generalize the results from [3] and [4] provided for rational exterior spaces to spaces with rational cohomology isomorphic to a tensor product of a graded exterior algebra with generators in odd dimensions and a graded algebra with all elements of even degree. In doing so we expand the methods presented in these paper to for example manifolds arising as blow ups of $CP^n$, all products of spheres and some bundles over said spheres. By the Künneth Theorem our results are automatically extended to products of such spaces. It has also came to our attention that in [4] the author claims without proof that a tower of odd dimensional sphere bundles is a rational exterior space. This claim has been quoted in several papers without verification. We provide two examples showing that it is in fact false.

We start the paper by recalling some of the results and methods used in the aforementioned articles. In the subsequent section we provide our main results. We follow this with some examples of non-trivial towers of sphere bundles to which our theorems apply (this can be found via some work with the Serre spectral sequence). We finish the paper with a section containing counterexamples to the claim in [4].

2. Preliminaries

Let $X$ be a connected euclidean neighborhood retract, in short ENR, $f : X \to X$ a continuous map and $f_k := f \circ \ldots \circ f$ a composition of $f$ with itself $k$ times (we use this convention also for morphism of algebras). Function $f$ induces a graded ring homomorphism $f^* : H^*(X, \mathbb{Q}) \to H^*(X, \mathbb{Q})$, $f^* = f^0 \oplus f^1 \oplus \ldots$, where $f^n$ is a map induced in the n-th cohomology group.

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The number
\[ \mathcal{L}(f) := \sum_{n=0}^{\infty} (-1)^n \text{tr}(f^n) \]
is called a Lefschetz number of the map \( f \).

The Lefschetz fixed point theorem implies that if \( \mathcal{L}(f) \neq 0 \), then \( f \) has a fixed point. Therefore, if \( \mathcal{L}(f_k) \neq 0 \) then \( f \) has a periodic point of period \( k \). However, we do not know what happens when \( \mathcal{L}(f) = 0 \).

Observe that we can define a Lefschetz number for any ring endomorphism \( g : H^\ast(X, \mathbb{Q}) \to H^\ast(X, \mathbb{Q}) \). Lefschetz fixed point theorem loses its meaning in this context, but later we will see that such extended definition is useful.

**Definition 1.** We say that a map \( f \) is Lefschetz periodic point free, we will call it \( \text{LPPF} \), if \( \mathcal{L}(f_k) = 0 \) for all \( k \leq 1 \). We will call such a function \( \text{LPPF} \) map.

In this section we recall the theory given in [4] and reformulate it in language of ring endomorphisms of cohomology ring.

We say that the element \( a \in H^r(X, \mathbb{Q}) \) for some \( r > 0 \) is decomposable if one can find pairs of cohomology classes
\[ (b_1, c_1) \in H^p(X, \mathbb{Q}) \times H^q(X, \mathbb{Q}), \]
where \( p, q > 0 \) and \( p + q = r \), such that \( a = \sum b_i \sim c_i \).

The set of all such elements, lets call it \( D^r(X, \mathbb{Q}) \), is a subspace of \( H^r(X, \mathbb{Q}) \). So the quotient \( A^r(X, \mathbb{Q}) := H^r(X, \mathbb{Q})/D^r(X, \mathbb{Q}) \) is a vector space over \( \mathbb{Q} \). Now, for a graded ring homomorphism \( f : H^\ast(X, \mathbb{Q}) \to H^\ast(Y, \mathbb{Q}) \) we have \( f(D^r(X, \mathbb{Q})) \subset D^r(Y, \mathbb{Q}) \), so passing to the quotient we get a homomorphism \( A^r(f) : A^r(X, \mathbb{Q}) \to A^r(Y, \mathbb{Q}) \). Denote
\[ A(X, \mathbb{Q}) := A^1(X, \mathbb{Q}) \oplus A^2(X, \mathbb{Q}) \oplus \ldots \]
\[ A(f) := A^1(f) \oplus A^2(f) \oplus \ldots \]

**Definition 2.** Rational exterior space \( X \) is an ENR for which we can find such homogeneous elements \( x_1, \ldots, x_s \in H^{\text{odd}}(X, \mathbb{Q}) \) that the inclusions \( x_i \hookrightarrow H^\ast(X, \mathbb{Q}) \) for \( i \in \{1, \ldots, s\} \) give rise to a ring homomorphism \( \Lambda_\mathbb{Q}(x_1, \ldots, x_s) \simeq H^\ast(X, \mathbb{Q}) \).

**Example 3.** Simplest examples of such spaces are products of spheres and Lie groups.

The main result of [4] implies (Corollary 1. in [4])

**Theorem 4.** If \( f : X \to X \) is a continuous map of rational exterior space and \( \lambda_1, \ldots, \lambda_n \) are eigenvalues of \( A(f) \), then \( L(f_k) = \prod_{i=1}^{n} (1 - \lambda_i^k) \).

The following corollary is a simple consequence of the previous theorem

**Corollary 5.** If \( f : X \to X \) is a continuous map of rational exterior space and \( \lambda_1, \ldots, \lambda_n \) are eigenvalues of \( A(f) \), then \( f \) is \( \text{LPPF} \) if and only if there is \( i_0 \in \{1, \ldots, n\} \) such that \( \lambda_{i_0} = 1 \).

We note that the methods used in [4] are algebraic in nature and hence the formula can be proven in the exact same fashion for a ring endomorphism of an exterior algebra.
**Theorem 6.** If \( f : H^*(X, \mathbb{Q}) \to H^*(X, \mathbb{Q}) \) is a ring endomorphism of cohomology of a rational exterior space \( X \) and \( \lambda_1, \ldots, \lambda_n \) are eigenvalues of \( A(f) \), then \( L(f^k) = \prod_{i=1}^{n} (1 - \lambda_i^k) \).

**Corollary 7.** If \( f : H^*(X, \mathbb{Q}) \to H^*(X, \mathbb{Q}) \) is a ring endomorphism of cohomology of a rational exterior space \( X \) and \( \lambda_1, \ldots, \lambda_n \) are eigenvalues of \( A(f) \), then \( f \) is LPPF if and only if there is \( t_0 \in \{1, \ldots, n\} \) such that \( \lambda_{t_0} = 1 \). In particular if such a \( \lambda_{t_0} = 1 \) does not exist then \( f \) has a periodic point.

Lefschetz zeta function is defined as a formal series

\[
\zeta_f(t) := \exp\left( \sum_{k=1}^{\infty} \frac{L(f^k)}{k} t^k \right).
\]

There is an equivalent form of Lefschetz zeta function

**Lemma 8.**

\[
\zeta_f(t) = \prod_{k=0}^{n} \det(I - tf)^{(-1)^{k+1}}.
\]

Its proof can be found in [1]. Note that all the Lefschetz numbers in the series vanish if and only if \( \zeta_f(t) \) is equal to 1 for all \( t \). Hence, the map is LPPF if and only if \( \zeta_f(t) \) is equal to 1 for all \( t \) and consequently if this condition does not hold then \( f \) has a periodic point.

The following theorem appeared in [3] for products of even dimensional spheres. Using the same method one can prove it for a wider class of spaces.

**Theorem 9.** Let \( X \) be a space with only even cohomology groups nonzero, \( f : H^*(X, \mathbb{Q}) \to H^*(X, \mathbb{Q}) \) a ring homomorphism. Then there is some \( n \in \mathbb{N} \) such that \( L(f^n) \neq 0 \). In particular a map between such spaces always has a periodic point.

**Proof.** In our case

\[
\zeta_f(t) = \prod_{k \in \{0, \ldots, n\}} \det(I - tf(L(f^k))^{(-1)}.
\]

When \( H^k(X, \mathbb{Q}) \) is nonzero, the expression \( \det(I - tf) \) is a polynomial of degree greater than one. So \( \zeta_f(t) \) is different than one for some \( t \).

3. MAIN RESULT

Let \( X = R \times E \) be a product of a rational exterior space \( R \) and a space \( E \) with only even cohomology nonzero and let \( f : X \to X \) be a continuous map.

\[
\mathcal{L}(f_k) = \sum_{n=0}^{\infty} (-1)^n \text{tr} f_k^n = \sum_{n=0}^{\infty} \text{tr}(-1)^n f_k^n = \text{tr} \sum_{n=0}^{\infty} (-1)^n f_k^n = \text{tr} T_k.
\]

Where \( T_k : H^*(X, \mathbb{Q}) \to H^*(X, \mathbb{Q}) \) is defined as \( T_k := \sum_{n=0}^{\infty} (-1)^n f_k^n \).

**Lemma 10.** Let \( F : H^*(X, \mathbb{Q}) \to H^*(X, \mathbb{Q}) \) be a ring homomorphism, then

\[
\text{tr}(F) = \text{tr}(F|_{\mathcal{H}_{\text{odd}}} \otimes \pi_{\mathcal{H}_{\text{ev}}} F|_{\mathcal{H}_{\text{ev}}}).
\]

where \( \mathcal{H}_{\text{odd}} := H^*(R, \mathbb{Q}) \otimes H^0(E, \mathbb{Q}) \) and \( \mathcal{H}_{\text{ev}} := H^0(R, \mathbb{Q}) \otimes H^*(E, \mathbb{Q}) \).
Proof. Using the Künneth Theorem we get that $H^*(X, \mathbb{Q}) \simeq H^*(R, \mathbb{Q}) \otimes H^*(E, \mathbb{Q})$. Denote by $\alpha_d(\alpha)\ldots, \alpha_d(\alpha)$ a basis of $H^*(R, \mathbb{Q})$ as graded vector space, where $d(\alpha)$ denotes degree of $\alpha$ as cohomology class. Similarly denote by $\beta_d(\beta)\ldots, \beta_d(\beta)$ a basis of $H^*(E, \mathbb{Q})$. We arrange the basis so that the sequences $d(\alpha_i)$ and $d(\beta_i)$ are nondecreasing. We want to compute the coefficients of $F(\alpha_d(\alpha) \otimes \beta_d(\beta))$. Let $\eta := d(\alpha_i), \eta := d(\beta_j)$.

$$F(\alpha_i^\eta \otimes \beta_j^\mu) = F(\alpha_i^\eta \otimes \beta_j^\mu)$$

$$= \sum_k a_k \alpha_k^\eta \sum_{d(\alpha_m) + d(\beta_i) = \mu} b_{m,l} \alpha_m^{d(\alpha_m)} \otimes \beta_l^{d(\beta_i)}$$

$$= \sum_k a_k \alpha_k^\eta \left( \sum_{d(\alpha_m) + d(\beta_i) = \mu} b_{m,l} \alpha_m^{d(\alpha_m)} \otimes \beta_l^{d(\beta_i)} + \sum_p b_{0,p} \beta_p^\mu \right)$$

$$= A + \sum_k a_k b_{0,p} \alpha_k^\eta \otimes \beta_p^\mu. $$

Where

$$A = \sum_k a_k \alpha_k^\eta \sum_{d(\alpha_m) + d(\beta_i) = \mu} b_{m,l} \alpha_m^{d(\alpha_m)} \otimes \beta_l^{d(\beta_i)}. $$

When we calculate a trace of $F$ we are interested only in the coefficient $a_i b_{0,j}$ which appears in the final line of the computation. It is the same number as corresponding coefficient for $F|_{H_{odd} \otimes \pi_{H_{ev}} F|_{H_{ev}}}$.

**Theorem 11.** Let $X = R \times E$ be a product of a rational exterior space $R$ and a space $E$ with only even cohomology nonzero and let $f : X \to X$ be a continuous map. If $f$ is LPPF then there is an eigenvalue of $A^{odd}(f)$ equal to a root of one. In particular $f$ has a periodic points if it does not have such an eigenvalue.

Proof. From the above lemma we can conclude

$$\mathcal{L}(f_k) = \text{tr}(T_k) = \text{tr}(T_k |_{H_{odd}}) \text{tr}(\pi_{H_{ev}} T_k |_{H_{ev}})$$

$$= \text{tr}(\sum_{n=0}^{\infty} (-1)^n f_k^n |_{H_{odd}}) \text{tr}(\pi_{H_{ev}} \sum_{n=0}^{\infty} f_k^n |_{H_{ev}})$$

$$= \text{tr}(\sum_{n=0}^{\infty} (-1)^n f_k^n |_{H_{odd}}) \text{tr}(\sum_{n=0}^{\infty} (\pi_{H_{ev}} f_k^n) |_{H_{ev}}).$$

We get a product of Lefschetz numbers of two maps, the first is a ring endomorphism of cohomology ring of $R$, and the second a ring endomorphism of cohomology ring of $E$. The second map cannot always vanish as it is shown in Theorem [9] hence for all these numbers to be zero the first part must vanish at least for some $k$. This by Theorem [3] can only happen when there is a root of unity among the eigenvalues of $A^{odd}(f)$.

$\square$
Observation 12. In the Theorem we do not need an extended rational exterior space.

We note that for a slightly better behaved class of spaces the criterion can be strengthened.

Definition 13. Let $X$ be as in the above observation. Let us assume that the algebra with even degree elements has at most one algebra generator in each degree. Moreover, each such generator squares to zero and the product of all these generators is the generator of the highest nontrivial degree. We call such a space an extended rational exterior space.

Example 14. A product of a rational exterior space with $S^{k_1} \times \ldots \times S^{k_l}$ for pairwise different even natural numbers $k_1, \ldots, k_l$ forms such a space.

Following the exposition in [4] (with appropriate modifications) we choose a base $A = \{x_1, \ldots, x_k, y_1, \ldots, y_l\}$ of $A(X)$ consisting of homogeneous elements where $x_i$ denote elements of odd degree and $y_i$ denote elements of even degree. Under our assumptions this gives a base $H$ of the rational cohomology ring $H^*(X, \mathbb{Q})$ consisting of 1 and the cup products of the above generators. Consider the odd length function $l$ on elements of $H$ defined by $l(1) = 1$ and $l(x_{i_1} \cup \ldots \cup x_{i_r} \cup y_{j_1} \cup \ldots \cup y_{j_s}) = r$ as well as the standard length operator $\tilde{l}$ defined by $\tilde{l}(1) = 1$ and $\tilde{l}(x_{i_1} \cup \ldots \cup x_{i_r} \cup y_{j_1} \cup \ldots \cup y_{j_s}) = r + q$.

Observation 15. For $\alpha \in H$ we have $(-1)^{\deg \alpha} = (-1)^{\tilde{l}(\alpha)}$.

Consider next the duality operator

$$d(x_{i_1} \cup \ldots \cup x_{i_r} \cup y_{j_1} \cup \ldots \cup y_{j_s}) := x_{i'_1} \cup \ldots \cup x_{i'_m} \cup y_{j'_1} \cup \ldots \cup y_{j'_n},$$

where $i'$ and $j'$ denote the complement of $i$ and $j$ (preserving the order). Let $s : H \to \{0, 1\}$ be the sign operator defined by $s(x_{i_1} \cup \ldots \cup x_{i_r} \cup y_{j_1} \cup \ldots \cup y_{j_s}) = 0$ when the sign of the permutation $i_1, \ldots, i_r, j_1, \ldots, j_s$ is even and 1 otherwise ($j$ and $j'$ are omitted here since they correspond to the even degree generators which commute with any other element in the cohomology ring).

Observation 16. Given $(\alpha, \beta) \in H \times H$ with $l(\alpha) \geq l(d(\beta))$ we have

$$\alpha \prec \beta = \begin{cases} (-1)^{s(\alpha)} x_1 \cup \ldots \cup x_k \cup y_1 \cup \ldots \cup y_l, & \text{if } \beta = d(\alpha), \\
0, & \text{otherwise}. \end{cases}$$

Let $X$ be an extended rational exterior space. For a map $f : X \to X$ we have

$$f^*(\alpha) = \lambda_\alpha \alpha + \sum_{\substack{\beta \in H \setminus \{\alpha\} \\ \ l(\alpha) \leq l(\beta)}} \sigma_{\alpha, \beta}\beta.$$  

Due to the previous observation

$$f^*(\alpha) \prec d(\alpha) = (-1)^{s(\alpha)} \lambda_\alpha x_1 \cup \ldots \cup x_k \cup y_1 \cup \ldots \cup y_l.$$  

Note that the definition of Lefschetz number $L(f)$ implies

$$L(f) = \sum_{\alpha \in H} (-1)^{\deg \alpha} \lambda_\alpha = \sum_{\alpha \in H} (-1)^{\tilde{l}(\alpha)} \lambda_\alpha.$$
Which together with the preceding discussion gives us
\[ L(f)x_1 \sim \ldots \sim x_k \sim y_1 \sim \ldots \sim y_l = \sum_{\alpha \in H} (-1)^{l(\alpha) + s(\alpha)} f^*(\alpha) \sim d(\alpha). \]

The right hand side of this equality can also be given by
\[ (x_1 - f^*(x_1)) \sim \ldots \sim (x_k - f^*(x_k)) \sim (y_1 + f^*(y_1)) \sim \ldots \sim (y_l + f^*(y_l)). \]

Hence, we arrive at the following lemma:

**Lemma 17.** Let \( X \) be an extended rational exterior space. If we treat \( A(X) \) as a subspace of \( H^*(X, \mathbb{Q}) \) (this can be done due to the chosen basis \( A \) and \( H \)) we have the formula
\[ L(f)x_1 \sim \ldots \sim x_k \sim y_1 \sim \ldots \sim y_l = (x_1 - A(f)(x_1)) \sim \ldots \sim (x_k - A(f)(x_k)) \sim (y_1 + A(f)(y_1)) \sim \ldots \sim (y_l + A(f)(y_l)). \]

**Proof.** Let \( H(r) \) denote the subspace of \( H^*(X, \mathbb{Q}) \) generated by those \( \alpha \in H \) with \( l(\alpha) \geq r \). Note that \( H(r) = 0 \) when \( r > k+l \). Note also that the cup product gives a map \( -\circ H(r) \times H(p) \rightarrow H(r+p) \). By the definition of \( A \) we have \( f_*(x_i) - A(f)(x_i) \in H(2) \). This implies that the difference
\[ (x_1 - f_*(x_1)) \sim \ldots \sim (x_k - f_*(x_k)) \sim (y_1 + f_*(y_1)) \sim \ldots \sim (y_l + f_*(y_l)) - (x_1 - A(f)(x_1)) \sim \ldots \sim (x_k - A(f)(x_k)) \sim (y_1 + A(f)(y_1)) \sim \ldots \sim (y_l + A(f)(y_l)) \]
lies in \( H(k + l + 1) = 0 \). Now the desired equality follows from the preceding discussion. \( \square \)

Now we are ready to prove the strengthening of our criterion for extended rational exterior spaces

**Theorem 18.** Let \( X \) be an extended rational exterior space and let \( f : X \rightarrow X \) be continuous. Then \( L(f^n) = \Pi_{i=1}^r (1 - \lambda_i^n) \Pi_{j=1}^s (1 + \sigma_j^n) \) where \( \lambda_i \) are the eigenvalues of \( A^{odd}(f) \) and \( \sigma_j \) are the eigenvalues of \( A^{ev}(f) \). Hence, \( f \) is Lefschetz periodic point free if and only if one of the following conditions hold
1. there exists an integer \( i \) such that \( \lambda_i = 1 \),
2. there exist integers \( i \) and \( j \) such that \( \lambda_i = \sigma_j = -1 \).

In, particular if neither of the conditions above is satisfied the \( f \) has a periodic point.

**Proof.** By standard linear algebra and the preceding discussion we have
\[ L(f)x_1 \sim \ldots \sim x_k \sim y_1 \sim \ldots \sim y_l = \det(Id - A^{odd}(f))x_1 \sim \ldots \sim x_k \sim (y_1 + A(f)(y_1)) \sim \ldots \sim (y_l + A(f)(y_l)) \]
However, since \( y_i \) are the only generators in each degree of \( A^{ev}(X) \) we have that \( A(f)(y_l) = \sigma_i y_l. \) With this we can change the right hand side to
\[ \det(Id - A^{odd}(f))(1 + \sigma_1) \ldots (1 + \sigma_l)x_1 \sim \ldots \sim x_k \sim y_1 \sim \ldots \sim y_l \]
hence after substituting \( f_\ast \) for \( f \) we arrive at the formula:
\[ L(f_n) = \Pi_{i=1}^r (1 - \lambda_i^n) \Pi_{j=1}^s (1 + \sigma_j^n) \]
To see the further claim note that \( L(f) \) vanishes if and only if there exists \( \lambda_i \) which is equal to \( 1 \) or \( \sigma_j \) which is equal to \( -1 \). If \( 1 \) is among the eigenvalues of \( A^{odd}(f) \) then all the numbers \( L(f_n) \) vanish. On the other hand if \( 1 \) is not among these eigenvalues but \( -1 \) is among the eigenvalues of \( A^{ev}(f) \) then it is necessary for \( L(f_2) \) to vanish
that there is a $-1$ among the eigenvalues of $A^{\text{od}}(f)$ since $i$ cannot be among the values of $A^{\text{ev}}(f)$ since this is a diagonal matrix on a rational vector space. One readily checks that in this case all of the $L(f^n)$ vanish. The final claim is automatic from the Lefschetz fixed point theorem.

We finish this section by pinpointing a large class of sphere bundles to which our results apply.

**Proposition 19.** Let $B$ be a simply connected manifold of dimension $n$ with rational cohomology ring isomorphic to a tensor product of an exterior algebra on odd degree generators and an algebra with only even degree elements and let $k \in \mathbb{N}$ be odd and such that $k \geq n$. Then any bundle $E$ with fiber $S^k$ over $B$ has rational cohomology isomorphic to a tensor product of an exterior algebra on odd degree generators and an algebra with only even degree elements.

**Proof.** The second page of the Serre spectral sequence in cohomology is well known (see e.g. chapter 5 in [2]) to be of the form

$$E_2^{p,q} = H^p(B, H^q(S^k, \mathbb{Q})).$$

From this we conclude that the only non-vanishing rows of the second page are the 0th and $k$-th row. So only the boundary operator on page $k + 1$ can be non-zero. But on that page the derivative jumps $(k + 1)$ steps to the right and since $k + 1 > n$ this means that it hits a column of zeroes. This shows that the cohomology of $E$ (which are isomorphic to the final page of $E_2^{p,q}$) are isomorphic as $\mathbb{Q}$ vector spaces to the cohomology of $B \times S^k$. We choose a class $\alpha \in H^k(E, \mathbb{Q})$ corresponding to a non-zero class in $H^0(B, H^k(S^k, \mathbb{Q}))$. Then the ring structure is defined by the following properties

1. $\alpha^2 = 0$ since $k$ is odd.
2. The cup product on classes corresponding to classes in $H^p(B, H^0(S^k, \mathbb{Q}))$ is prescribed by the mapping induced by the projection map $\pi : E \to B$.
3. Multiplying classes corresponding to classes in $H^p(B, H^0(S^k, \mathbb{Q}))$ by $\alpha$ is prescribed by the multiplicative structure of the spectral sequence.

This implies that the ring structure is also as in the tensor product (where we add the generator $\alpha$ in the $k$-th degree to the exterior algebra).

**Observation 20.** Note that this proof does not hold in general when $k$ is even since then $\alpha^2$ might not be zero as is the case for example in the twisted bundle of $S^2$ over $S^2$.

Combining this proposition with our main result we get the following corollary.

**Corollary 21.** Take $E$ as above and let $f : E \to E$ be continuous. If there are no roots of unity among the eigenvalues of $A^{\text{od}}(f)$ then $f$ has periodic points.

4. COUNTEREXAMPLES

In this section we give two counterexamples to the claim in [4], that a tower of odd dimensional sphere bundles is a rational exterior space.

The first example is the Kodaira-Thurston manifold, which is a bundle of $S^1$ over $T^3$. 

7
This manifold is obtained by identifying the points of $\mathbb{T}^2 \times \mathbb{R} \times \mathbb{S}^1$ in the following way

$$(x, y, t, z) \sim (x, y, t + j, z + jx), \ j \in \mathbb{Z}.$$  

One can easily see that this is equivalent to taking $\mathbb{T}^2 \times I \times \mathbb{S}^1$ with the boundaries identified by the function:

$$f(x, y, 0, z) = (x, y, 1, z + x).$$

From this we can conclude that the projection onto the first three coordinates gives this manifold the structure of a circle bundle over $\mathbb{T}^3$. The Kodaira-Thurston manifold (denoted $X$) has a trivialization of the cotangent bundle given by the forms: $dx, dy, dt, dz - tdx$. Computing the de Rham cohomology of $X$ we get

$$H^0(X, \mathbb{Q}) = \mathbb{Q},\ H^1(X, \mathbb{Q}) = \mathbb{Q}^3,\ H^2(X, \mathbb{Q}) = \mathbb{Q}^4,\ H^3(X, \mathbb{Q}) = \mathbb{Q}^3,\ H^4(X, \mathbb{Q}) = \mathbb{Q}.$$  

We can see that $X$ cannot be a rational exterior space, because it has inappropriate dimensions of cohomology groups.

The next example will be a bit more complicated, but simply connected. We are going to construct a $\mathbb{S}^5$ bundle over $\mathbb{S}^5 \times \mathbb{S}^3$ which will not be a rational exterior space.

Assume $X$ is a bundle we are looking for. We can write it as a union of two sets $A := \mathbb{S}^3 \times D_5^2 \times \mathbb{S}^5$ and $B := \mathbb{S}^3 \times D_2^2 \times \mathbb{S}^5$ which are glued using some function $f : \mathbb{S}^3 \times S^2 \times S^5 \to \mathbb{S}^5 \times S^2 \times S^5$. We will look for such $f$, that produces a bundle fitting our condition. Let us use Mayer-Vietoris sequence

$$0 \to H^5(X) \xrightarrow{(i^*, j^*)} H^5(A) \oplus H^5(B) \xrightarrow{i^* - j^*} H^5(A \cap B) \to H^6(X) \to 0.$$  

The first place on the left is $H^4(\mathbb{S}^3 \times S^2 \times S^5)$ and the last right place is $H^6(A) \oplus H^6(B)$, both are equal to zero according to Künneth theorem. Group $H^5(A \cap B)$ is equal to $\mathbb{Q}^2$ and is generated by two classes corresponding to terms in the product, $[\mathbb{S}^5]$ and $[\mathbb{S}^3 \times S^2]$ (via the Universal Coefficients Theorem). Whereas the term $H^5(A) \oplus H^5(B)$ is equal to $\mathbb{Q}^2$ generated by two classes $[\mathbb{S}^5]$ and $[\mathbb{S}^5]'$ deriving from the highest cohomology of two copies of $\mathbb{S}^5$ lying in $A$ and $B$. Two previous sentences are also implied by Künneth theorem. Taking $A$ as a reference point we can take $i$ to be the identity while $j$ is a map given by the transition functions of our bundle. Hence, $j$ is of the form:

$$(x, y, z) \mapsto (x, y, \phi(x, y, z))$$

de$\phi$ as a composition of two maps:

$$\mathbb{S}^3 \times \mathbb{S}^2 \times \mathbb{S}^5 \xrightarrow{g \times \text{id}_{\mathbb{S}^5}} \mathbb{S}^5 \times \mathbb{S}^5 \xrightarrow{h} \mathbb{S}^5.$$  

Let $g : \mathbb{S}^3 \times \mathbb{S}^2 \to \mathbb{S}^5$ be a map collapsing 3-, and 2-cell to a point.

Let us use the one-point compactification on every fiber of the open disc bundle of $T\mathbb{S}^5$ with radius $\pi$. It is easily proven that this is isomorphic to the unit sphere bundle of $T\mathbb{S}^5 \oplus NS\mathbb{S}^5$ and so it is trivial. Hence, we got a space homeomorphic to $\mathbb{S}^5 \times \mathbb{S}^5$. We identify this with our sphere product by identifying the second copy of $\mathbb{S}^5$ with the fibers of this bundle. Now, using exponential map induced by the standard Riemannian metric on the sphere, we can project the fiber spheres onto the base in such a way that the mapping restricted to each fiber is a homeomorphism. The map defined in such a way is the desired map $h$.

We show, that the map $i^* - j^*$ is an isomorphism. It is enough to show that the generators of $H^5(A) \oplus H^5(B)$ map to a set of generators of $H^5(A \cap B)$. Class $[\mathbb{S}^5]'$
is mapped to class $[S^5]$, because on $A$ we set identity. The class $[S^5]'$ is mapped to $[S^3 \times S^2] + [S]^5$. Now, since the sequence above is exact we get that $H^5(X)$ and $H^6(X)$ are equal to zero.

The other cohomology classes can be computed using similar argument about Serre spectral sequence as in Proposition 19. The non-zero cohomology groups are as follows $H^0(X) = \mathbb{Q}$, $H^3(X) = \mathbb{Q}^2$, $H^8(X) = \mathbb{Q}^2$, $H^{11}(X) = \mathbb{Q}$.

Again we can see that $X$ cannot be a rational exterior space because of dimensions of the cohomology classes.

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