An FPRAS and Polynomial-Time Uniform Sampler for Tree Automata

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Abstract

In this work, we introduce the first fully polynomial time randomized approximation scheme (FPRAS) for counting the number of trees of size $n$ accepted by a tree automaton, where $n$ is given in unary, and the first polynomial time algorithm for sampling uniformly from this set of trees. Our results improve over the prior quasi-polynomial time randomized approximation scheme (QPRAS) and sampling algorithm of Gore, Jerrum, Kannan, Sweedyk, and Mahaney '97. At the heart of our algorithm is a reduction to the problem of estimating the number of strings of length $n$ accepted by a succinct non-deterministic finite automaton (NFA), which is an NFA where the transitions are labeled by succinctly encoded sets of symbols, whose sizes can be exponential in the encoding length. Assuming these sets of symbols can be efficiently sampled from, and their sizes approximated, we show that there is an FPRAS and polynomial time almost uniform sampler for succinct NFAs, which may be of independent interest.

We demonstrate that, by applying our FPRAS for tree automata, we can obtain an FPRAS for many hitherto open problems in the fields of constraint satisfaction problems (CSPs), database systems, software verification, and knowledge compilation. Specifically, we obtain an FPRAS for counting solutions for CSPs that are acyclic or, more generally, that have bounded hypertree-width, which results in an FPRAS for counting the number of answers to conjunctive queries that are acyclic or which have bounded hypertree-width. Moreover, these results can be extended to unions of acyclic conjunctive queries, and to the more general class of unions of conjunctive queries with bounded hypertree-width. Finally, we also obtain FPRAS for the problems of counting the number of error threads in programs with nested call subroutines, and counting valid assignments to structured DNNF circuits.

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1 Introduction

Automata theory is a core component of computer science, having a large number of applications in both theory and practice [HMIU07]. The quintessential object of study in this area are word automata, as they form a computational model expressive enough for a large number of applications, while having many desirable computational properties. The extension of these automata to trees has been a widely studied topic, since tree automata also have a remarkable capacity to model problems, while retaining many of the desirable computational properties of word automata [Sei90, Tho97]. Indeed, the popularity of tree automata has been fueled by the large number of applications they have found over the years. Beginning with the strong decidability result established by Rabin [Rab69], many important problems have been shown to be decidable via tree automata, and they have even resulted in extended and simplified proofs of classical results such as the decidability of Presburger arithmetic [FR98, Opp78, MV96]. The fact that tree automata are equivalent to monadic second order-logic [TW68] is a basic component of the proof of Courcelle’s theorem [Cou090], which states that every graph property definable in monadic second-order logic can be decided in linear time on graphs with bounded treewidth. Further applications of tree automata, among others, include model checking [EJ91, Var95], program analysis [AEM04, AM04, AM09], databases [Nev02, Sch07], and knowledge representation [Ter99, CDGL99, BCM+03]. We refer the reader to the surveys [Tho97, CDG+07] for a further discussion of these applications.

Tree automata run over labeled ordered trees, which are trees where every node has a label from a fixed alphabet, and the children of each node are totally ordered (e.g. one can talk about the $i$-th child of a node). Roughly speaking, a tree automaton $\mathcal{T}$ is a tuple $(S, \Sigma, \Delta, s_{init})$, where $S$ is a set of states, $\Sigma$ is a finite alphabet, $s_{init} \in S$ is an initial state, and $\Delta$ is a set of transitions of the form $(s, a, (s_1, \ldots, s_k))$ where $s, s_1, \ldots, s_k \in S$, $k \geq 0$, and $a \in \Sigma$. Intuitively, the transition $(s, a, (s_1, \ldots, s_k))$ is a rule which specifies that from a node $u$ in a tree, labeled with the symbol $a$ and with children $u_1, \ldots, u_k$, the automaton in state $s$ over the node $u$ assigns the state $s_i$ to each child $u_i$, and continues its run over each branch in parallel (i.e. it runs top-down over the tree). If $k = 0$, then this means that $u$ is a leaf, and the run ends satisfactorily over this branch whenever $u$ is labeled with $a$. A tree $t$ is said to be accepted by $\mathcal{T}$ if there is run of the automaton which agrees with the labels and structure of $t$ as just described. The set of trees accepted by $\mathcal{T}$ is denoted by $\mathcal{L}(\mathcal{T})$, and is known as a regular tree language.

Recall the classic definition of a non-deterministic finite automaton $\mathcal{A}$ (NFA) that runs over words: $\mathcal{A}$ is a tuple $(S, \Sigma, \Delta, s_{init}, F)$ where $S$, $\Sigma$, and $s_{init}$ are defined in the same way as for tree automata, $\Delta \subseteq S \times \Sigma \times S$ is the transition relation, and $F \subseteq S$ is the final set of states. The language accepted by an NFA is known as a regular language. Notice that an NFA can be seen as a special case of a tree automaton that runs over trees where every node has at most one child (i.e., it is a path). In fact, every NFA $\mathcal{A}$ is equivalent to a tree automaton $(S, \Sigma, \Delta', s_{init})$ where $\Delta' = \Delta \cup \{(s, a, s') \mid \exists s' \in F: (s, a, s') \in \Delta\}$, namely, the leaves transitions (i.e., $k = 0$) in the tree automaton correspond to the transitions into a final state in $\mathcal{A}$. Thus, tree automata are a non-trivial generalization of NFA from words to trees.

It is important to notice that, although some desirable algorithmic properties extend from NFAs to tree automata [Tho97, CDG+07], there are problems that have been solved for NFAs but which are still open for tree automata. A remarkable example is the development of an algebraic characterization of regular languages [Sch65], which has been used to prove fundamental results such as the decidability of the problem of verifying whether a regular language is definable in first-order logic [MP71]. No such a result has been established for the case of tree automata.

Counting and Uniform Sampling. Given a tree automaton $\mathcal{T}$ and an integer $n$, define the $n$-slice of $\mathcal{L}(\mathcal{T})$ as $\mathcal{L}_n(\mathcal{T}) = \{t \in \mathcal{L}(\mathcal{T}) \mid |t| = n\}$, where $|t|$ is the number of nodes of $t$. In this work,
we consider two fundamental and intimately related tasks. First, the problem of estimating the size of $L_n(T)$, and second, the problem of sampling a tree uniformly from $L_n(T)$. These two problems in the context of formal languages have a long history. For *unambiguous* context free languages, where each string $w$ has exactly one possible derivation, relatively simple polynomial-time algorithms exist [Mai94]. Since context free languages generalize both NFAs and tree automata, these algorithms can also be used for unambiguous word and tree automata. These results can be further extended to the case where the ambiguity of the language (the number of possible derivations) is bounded [BGS00, KSM95]. However, even for the restrictive class of regular languages, if unbounded ambiguity is allowed then counting becomes an intractable problem. More precisely, given an NFA $A$ and a number $n$ specified in unary as $0^n$, the problem of computing $|L_n(A)|$ is #P-hard [ÂJ93], where here $L_n(A)$ is the set of all strings of length $n$ accepted by $A$. Since tree automata generalize NFAs, it follows that computing $|L_n(T)|$ for a tree automata $T$ is also #P-hard.

However, hardness results for exact counting of the $n$-slice do not rule out the possibility of efficient approximation algorithms. This observation was first exploited by Kannan, Sweedyk, and Mahaney, who gave a quasi polynomial-time approximation scheme (QPRAS) for regular languages [KSM95]. This QPRAS was later extended by the aforementioned authors, Gore, and Jerrum [GJK+97], to the class of context-free languages, which implies a QPRAS for regular tree languages. In particular, if $T$ is a tree automaton and $m = |T|$ is the size of the description of $T$, then the algorithm of [GJK+97] runs in time $\epsilon^{-2}(nm)^{O(\log(n))}$, where $\epsilon$ is the error parameter. In addition, [GJK+97] gave a quasi polynomial-time algorithm to sample *almost-uniformly* from the $n$-slice of a context free language. Here, an almost-uniform sampler for a set $S$ with error $\epsilon$ is a sampler which outputs each $t \in S$ with probability within the interval $(1 \pm \epsilon)|S|^{-1}$.\(^1\)

The algorithms of [GJK+97] and [KSM95] are based on a recursive, approximate form of Karp-Luby sampling [KLM89], which is a type of rejection sampling. This approach has the drawback that the probability a sample is chosen is exponentially small in the depth of the recursion. Recently, using a different approach based on recursively partitioning the language, it was shown that a *fully polynomial-time randomized approximation scheme* (FPRAS) and a *uniform sampler* exist for the $n$-slice of a regular word language [ACJR19].\(^2\) However, the techniques in [ACJR19] break down in several fundamental ways (discussed in the following) when applied to tree automata. This motivates the following important open problem:

*Do FPRAS and polynomial-time uniform samplers exist for the languages accepted by more general classes of automata, such as tree automata?*

### 1.1 Our Contribution

We begin to answer this open question by obtaining the first FPRAS and uniform sampler from the $n$-slice of a tree automata. Our contribution involve a sampling scheme based on growing trees along a carefully constructed and evolving path, and a reduction to the problem of estimating the $n$-slice of a *succinct* NFA, which is an NFA whose transition are succinctly encoded (see below). Formally:

**Theorem 1.1** (Theorem 3.7 abbreviated). *Given a tree automaton $T$ and $n \geq 1$ (given in unary as $0^n$), there is an algorithm which runs in time $\text{poly}(|T|, n, \epsilon^{-1}, \log(\delta^{-1}))$ and with probability $1 - \delta$, outputs a value $\tilde{N}$ such that: $(1 - \epsilon)|L_n(T)| \leq \tilde{N} \leq (1 + \epsilon)|L_n(T)|$.  

\(^1\)The runtime dependency on $\epsilon$ in [GJK+97] is the same as in their QPRAS.

\(^2\)The notion of a uniform sampler in [ACJR19] allows some probability of not outputting a sample and some probability $\delta$ of failure, which will be the case for our result as well (see Theorem 1.2).
Theorem 1.2 (Theorem 3.8 abbreviated). Given a tree automaton $\mathcal{T}$, $n \geq 1$ (given in unary as $0^n$) and $\delta \in (0, 1/2)$, there is an algorithm such that, after a pre-processing phase which runs in time $\text{poly}(|\mathcal{T}|, n, \log(\delta^{-1}))$ and succeeds with probability $1 - \delta$, the algorithm proceeds to an output phase, where it can now generate uniformly random trees $t \in \mathcal{L}_n(\mathcal{T})$, each in expected time $\text{poly}(|\mathcal{T}|, n, \log(\delta^{-1}))$.

The value $\delta$ in Theorem 1.2 only depends on the randomness in a pre-processing step that is run exactly once. Conditioned on the success of this step, every call to the resulting algorithm generates a uniform sample. Observe that this is a strictly stronger than being an almost-uniform sampler. In particular, Theorem 1.2 implies the existence of an almost-uniform sampler that always runs in $\text{poly}(|\mathcal{T}|, n, \log(\epsilon^{-1}))$ time, which is still stronger than the standard definition of an almost-uniform sampler (which generally has at least polynomial dependency on $\epsilon$). We remark that Theorem 1.1 is actually implied by Theorem 1.2, while Theorem 1.1 only implies the existence of a standard almost-uniform sampler. However the two algorithms depend on one another, and are more aptly seen as a single algorithm.

Along the way to designing the algorithms of Theorem 1.1 and 1.2, we solve the intermediate problems of counting and sampling words accepted by a succinct NFA $\mathcal{N}$, which is an NFA with transitions that are succinctly encoded. Formally, a succinct NFA $\mathcal{N}$ is a 5-tuple $(S, \Sigma, \Delta, s_{\text{init}}, s_{\text{final}})$, where $S$ is a set of states, $\Sigma$ is an alphabet, $s_{\text{init}}, s_{\text{final}} \in S$ are the initial and final states, and $\Delta \subseteq S \times 2^{\Sigma} \times S$ is the transition relation, where each transition is labeled by a subset $A \subseteq \Sigma$. For each transition $e = (s, A, s') \in \Delta$, we assume that the set $A$ is succinctly encoded via some representation (e.g. a DNF formula) and, therefore, the number of elements in $A$ (and, in particular, the size of the alphabet $\Sigma$) can be exponentially large in the representation of $\mathcal{N}$. A word $w = w_1 w_2 \ldots w_n \in \Sigma^*$ is accepted by $\mathcal{N}$ if there is a sequence $s_{\text{init}} = s_0, s_1, \ldots, s_n = s_{\text{final}}$ of states such that there is a transition $(s_{i-1}, A, s_i) \in \Delta$ with $w_i \in A$ for each $i = 1, 2, \ldots, n$. Notice that a succinct NFA $\mathcal{N}$ can be described by a standard NFA with at most $O(|\Sigma| \cdot |s|^2)$ transitions, but this representation will be exponential in the size of $\mathcal{N}$. Also, observe that the special case where each transition $(s, A, s') \in \Delta$ must satisfy $|A| = 1$ is precisely the standard definition of an NFA. To solve the aforementioned problems for succinct NFA, we must assume that the encodings of the label sets satisfy some basic conditions. Roughly, we require that for each transition $(s, A, s')$, we are given an oracle which can test membership in $A$, produces estimates of the size of $|A|$, and generates almost-uniform samples from $A$. An abbreviated version of our theorem, discussed formally in Section 4, is given below.

Theorem 1.3 (Theorem 4.3 informal). Let $\mathcal{N} = (S, \Sigma, \Delta, s_{\text{init}}, s_{\text{final}})$ be a succinct NFA and $n \geq 1$ (given in unary as $0^n$), where $m = |\Delta| + |S|$ and $|\Sigma| = \exp((nm)^{O(1)})$. Suppose that the sets $A$ in each transition $(s, A, s') \in \Delta$ satisfy the properties described above. Then there is a $\text{poly}(n, m, \epsilon^{-1}, \log(\delta^{-1}))$-time algorithm that with probability $1 - \delta$ outputs an estimate $\tilde{N}$ such that $(1 - \epsilon)\mathcal{L}_n(\mathcal{N}) \leq \tilde{N} \leq (1 + \epsilon)\mathcal{L}_n(\mathcal{N})$, and there is an almost-uniform sampler for $\mathcal{L}_n(\mathcal{N})$ with error $\epsilon$ running in $\text{poly}(n, m, \epsilon^{-1})$ time.

We remark that while standard (non-succinct) NFAs were known to admit an FPRAS by the results of [ACJR19], Theorem 1.3 is a strong extension of the main result of [ACJR19], and requires several non-trivial additional insights beyond the techniques in [ACJR19].

Lastly, we demonstrate that our FPRAS for tree automata results in the first polynomial-time randomized approximation algorithms for a myriad of previously open problems in the fields of constraint satisfaction problems, database query evaluation, verification of correctness of programs with nested calls to subroutines, and knowledge compilation. We give a brief overview of such problems in Section 1.3, and we described them in detail in Section 5.
In order to capture the essence of the problem, in this section we will consider a simplified version of tree automata. Specifically, we restrict the discussion to unlabeled binary ordered trees, which are sufficient to present the main ideas of the algorithm. A binary ordered tree $t$ (or just tree) is a rooted binary tree where the children of each node are ordered; namely, one can distinguish between the left and right child of each non-leaf node. For a non-leaf node $u$ of $t$, we write $u_1$ and $u_2$ to denote the left and right children of $u$, respectively, and we will denote the root of any tree $t$ by $\lambda$. We will write $u \in t$ to denote that $u$ is a node of $t$, and $|t|$ to denote the number of nodes of $t$. For example, Figure 1 depicts a binary ordered tree $t_1$ with $|t_1| = 9$, and another tree $t_2$ with $|t_2| = 13$, where the root is always the top node and the ordering on the children is given from left to right.

A tree automaton $T$ (over binary ordered trees) is a tuple $(S, \Delta, s_{\text{init}})$ where $S$ is a finite set of states, $\Delta \subseteq (S \times S \times S) \cup S$ is the transition relation, and $s_{\text{init}} \in S$ is the initial state. A run $\rho$ of $T$ over a tree $t$ is a function $\rho : t \to S$ mapping nodes to states that respects the transition relation. Namely, for every node $u$ of $t$ we have $\rho(u) \in \Delta$ whenever $u$ is a leaf, and $(\rho(u), \rho(u_1), \rho(u_2)) \in \Delta$, otherwise. We say that $T$ accepts $t$ if there exists a run $\rho$ of $T$ over $t$ such that $\rho(\lambda) = s_{\text{init}}$, and such a run $\rho$ is called an accepting run of $T$ over $t$. The set of all trees accepted by $T$ is denoted by $L(T)$, and the $n$-slice of $L(T)$, denoted by $L_n(T)$, is the set of trees $t \in L(T)$ with size $n$. For the sake of presentation, in the following we write $s \rightarrow q \rho$ to represent the transition $(s, q, \rho) \in \Delta$ and $s \rightarrow \cdot$ to represent $s \in \Delta$. Note that transitions of the form $s \rightarrow \cdot$ correspond to leaves that have no children, and can be thought as “final states” of a run.

In Figure 1 we see an example for a tree automaton $T$ with states $\{s, r, q\}$. At the right-hand side of Figure 1 we can also see an example of an accepting run of $T$ over $t_2$. One can easily check from the transitions of $T$ in this example that a tree $t$ is in $L(T)$ if, and only if, there exists a node $v \in t$ such that both children of $v$ are internal (non-leaf) nodes. For example, $t_2$ satisfies this property and $t_2 \in L(T)$. On the other hand, all nodes $v \in t_1$ have at least one child that is a leaf, and thus there is no accepting run of $T$ over $t_1$, so $t_1 \notin L(T)$. Given such a tree automaton $T$ over binary unlabeled trees and an integer $n \geq 1$, we can now consider the problems of approximating $|L_n(T)|$ and sampling from $L_n(T)$.

Unrolling the Automaton. Fix $n \geq 1$ and a tree automaton $T = (S, \Delta, s_{\text{init}})$ as defined above. Our first step will be to unroll the automaton, so that each state is restricted to only producing trees of a fixed size. Specifically, we construct an automaton $T = (S, \Delta, s_{\text{init}}^n)$, where each state $s \in S$ is duplicated $n$ times into $s_i, s_2, \ldots, s^n \in S$, and where $s_i$ is only allowed to derive trees of size $i$. To enforce this, each transition $s \rightarrow rq$ in $\Delta$ is replaced with $s^i \rightarrow r^j q^k \in \Delta$ for all $j, k > 0$.

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1.2 Technical Overview

Figure 1: A tree automata $T$, binary ordered trees $t_1$ and $t_2$, and a run of $T$ over $t_2$.
such that \( i = j + k + 1 \), and each transition \( s \to \cdot \) in \( \Delta \) is replaced with \( s^1 \to \cdot \in \Delta \). Now for every \( s \in S \), let \( T(s^i) \) be set of trees that can be derived beginning from the state \( s^i \) (all of which have size \( i \)). When \( i > 1 \), we can then define \( T(s^i) \) via the relation

\[
T(s^i) = \bigcup_{(s^i \to r)q^k \in \Delta} \left( T(r^j) \otimes T(q^k) \right)
\]

where \( T(r^j) \otimes T(q^k) \) is a shorthand to denote the set of all trees that can be created by taking every \( t_1 \in T(r^j) \) and \( t_2 \in T(q^k) \) and forming the tree \( t_1 \cdot t_2 \). Namely, \( T(s^i) \) is the union of the set of trees which can be produced by each of the transitions \( s^i \to r_j q^k \) from \( s^i \). This fact allows us to define each set \( T(s^i) \) recursively as a union of “products” of other such sets. Our problem is now to estimate \( |T(s^i_{\text{init}})| \) and to sample from the set \( T(s^i_{\text{init}}) \).

We remark that for so-called “bottom-up deterministic” automata \( T \) [CDG+07], the sets \( T(r^j) \otimes T(q^k) \) in the union in Equation (1) are disjoint, so \( |T(s^i)| = \sum_{(s^i \to r)q^k \in \Delta} |T(r^j)| \cdot |T(q^k)| \) and one can then compute the values \( |T(s^i)| \) exactly via dynamic programming. Thus, the core challenge is the ambiguity of the problem: namely, the fact that trees \( t \in T(s^i_{\text{init}}) \) may admit exponentially many derivations in the automata. For example, the tree automaton \( T \) from Figure 1 can accept \( t_2 \) by two different runs. In what follows, we will focus on the problem of uniform sampling from such a set \( T(s^i) \), since given a uniform sampler the problem of size estimation is routine.

**A QPRAS via Karp-Luby Sampling.** In order to handle the problem of sampling with ambiguous derivations, Gore, Jerrum, Kannan, Sweedyk, and Mahaney [GJK+97] utilized a technique known as Karp-Luby sampling. This technique is a form of rejection sampling, where given sets \( T_1, \ldots, T_k \) and \( T = \cup_i T_i \), one can sample from \( T \) by: (1) sampling a set \( T_i \) with probability proportional to \( |T_i| \), (2) sampling an element \( t \) uniformly from \( T_i \), (3) accepting \( t \) with probability \( 1/m(t) \), where \( m(t) \) is the total number of sets \( T_j \) which contain \( t \). The QPRAS of [GJK+97] applied this procedure recursively, using approximations \( \tilde{N}(T_i) \) in the place of \( |T_i| \), where the union \( T = \cup_i T_i \) in question is just the union in Equation (1), and each \( T_i \) is a product of smaller sets \( T_i = T_{i1} \otimes T_{i2} \) which are themselves unions of sets at a lower depth. So to carry out (2), one must recursively sample from \( T_{i1} \) and \( T_{i2} \). The overall probability of rejection in (3) is now exponential in the sampling depth. Using a classic depth reduction technique of Valiant, Skyum, Berkowitz, and Rackoff [VS81], they are able to reduce the depth to \( \log(n) \), but since \( m(t) \) can be \( \Omega(n \cdot |T|) \) at each step, the resulting acceptance probability is quasi-polynomially small.

**A Partition Based Approach.** The difficult with Karp-Luby sampling is that it relies on a rejection step to compensate for the fact that some elements can be sampled in multiple ways. Instead, our approach will be to partition the sets in question, so that no element can be sampled in more than one way. Simply put, to sample from \( T \), we will first partition \( T \) into disjoint subsets \( T'_1, \ldots, T'_{\ell} \). Next, we sample a set \( T'_i \) with probability (approximately) proportional to \( |T'_i| \), and lastly we set \( T \leftarrow T'_i \) and now recursively sample from the new \( T \). The recursion ends when the current set \( T \) has just one element. Clearly no rejection procedure is needed now for the sample to be approximately uniform. To implement this template, however, there are two main implementation issues which we must address. Firstly, how to partition the set \( T \), and secondly, how to efficiently estimate the size of each part \( T_i \). In the remainder, we will consider these two issues in detail.
Our High-Level Sampling Template

Input: Arbitrary set $T$

1. If $|T| = 1$, return $T$. Otherwise, find some partition $T = \bigcup_{i=1}^{t} T_i'$.
2. Call subroutine to obtain estimates $\tilde{N}(T_i') \approx |T_i'|$.
3. Set $T \leftarrow T_i'$ with probability $\frac{\tilde{N}(T_i')}{\sum_j \tilde{N}(T_j)}$, and recursively sample from $T$.

Algorithmic Overview and Setup. For the rest of the section, fix some state $s^i$. It will suffice to show how to generate a uniform sample from the set $T(s^i)$. To implement the above template, we will rely on having inductively pre-computed estimates of $|T(r^j)|$ for every $r \in S$ and $j < i$. Specifically, our algorithm proceeds in rounds, where on the $j$-th round we compute an approximation $\tilde{N}(r^j) \approx |T(r^j)|$ for each state $r \in S$. In addition to these estimates, a key component of our algorithm is that, on the $i$-th round, we also store sketches $\tilde{T}(r^j)$ of each set $T(r^j)$ for $j < i$, which consist of polynomially many uniform samples from $T(r^j)$. One can uses these sketches $\tilde{T}(r^j)$ to aid in the generation of uniform samples for the larger sets $T(s^i)$ on the $i$-th round. For instance, given a set of trees $T = \bigcup_{j=1}^{k} T_j$ for some sets $T_1, \ldots, T_k$ where we have estimates $\tilde{N}(T_j) \approx |T_j|$ and sketches $\tilde{T}_j \subseteq T_j$, one could estimate $|T|$ by the value

$$\sum_{j=1}^{k} \tilde{N}(T_j) \left( \frac{|\tilde{T}_j \setminus \bigcup_{j < j} T_j'|}{|\tilde{T}_j|} \right)$$

(2)

Here, the term in parenthesis in 2 estimates the fraction of the set $\tilde{T}_j$ which is already contained in the earlier sets $T_j'$.

The Partition Scheme for Word Automata. The above insight of sketching the intermediate subproblems $T(r^j)$ of the dynamic program and applying 2 was made by [ACJR19] in their FPRAS for non-deterministic finite automata. Given an NFA $\mathcal{N}$ with states $S$, $\Sigma = \{0, 1\}$, and any state $s \in S$ of $\mathcal{N}$, one can similarly define the intermediate subproblem $W(s^i)$ as the set of words of length $i$ that can be derived starting at the state $s$. The FPRAS of [ACJR19] similarly pre-computes sketches for these sets in a bottom-up fashion. To sample a string $w = w_1 \cdots w_i \in W(s^i)$, they sampled the symbols in $w$ bit by bit, effectively “growing” a prefix of $w$.

First, $W(s^i)$ is partitioned into $W(s^i, 0) \cup W(s^i, 1)$, where $W(s^i, b) \subseteq W(s^i)$ is the subset of strings $x = x_1 \cdots x_i \in W(s^i)$ with first bit equal to $b$. If for any prefix $w'$, we define $R_{w'} \subseteq S$ to be the set of states $r$ such that there is a path of transitions from $s$ to $r$ labeled by $w'$, then observe that $W(s^i, b) = \{b\} \cup_{r \in R_{w'}} W(r^{i-1})$, where $\cdot$ is the concatenation operation for words, and $X_1 \cdot X_2 = \{x_1 \cdot x_2 \mid x_1 \in X_1 \text{ and } x_2 \in X_2\}$ for two sets $X_1, X_2$ of words. Thus $W(s^i, b)$ can be estimated directly by Equation 2 in polynomial time. After the first bit $w_1 = b$ is sampled, they move on to sample the second bit $w_2$ conditioned on the prefix $w_1 = b$. By partitioning the strings again into those with prefix equal to either $b0$ or $b1$, each of which is described compactly as $\{bb'\} \cup_{r \in R_{wb}} W(r^{i-2})$ for $b' \in \{0, 1\}$, one can use Equation 2 again to sample $w_2$ from the correct distribution, and so on.

The key “victory” in the above approach is that for NFAs, one can compactly condition on a prefix $w'$ of a word $w \in W(s^i)$ as a union $\cup_{r \in R_{wb}} W(r^{i-|w'|})$ taken over some easy to compute subset of states $R_{w'} \subseteq S$. In other words, to condition on a partial derivation of a word, one need only remember a subset of states. This is possible because, for NFAs, the overall configuration of the automata at any given time is specified only by a single current state of the automata. However, this fact breaks down fundamentally for tree automata. Namely, at any intermediate point in the

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4We use $W$ to denote sets of words, and $T$ for sets of trees.

5We remark in [ACJR19], the algorithm instead grows a suffix of $w$, however the two approaches are equivalent.
derivation of a tree, the configuration of a tree automata is described not by a single state, but rather by the combination of states \((r_1^{j_1}, \ldots, r_k^{j_k})\) assigned to the (possibly many) leaves of the partially derived tree. So the number of possible configurations is exponential in the number of leaves of the partial tree. Consequently, the number of sets in the union of Equation 2 is exponentially large.\footnote{By being slightly clever about the order in which one derives the tree, one can reduce the number of “active” leaves to \(O(\log n)\), which would result in a quasi-polynomial \(|S|^O(\log n)\) time algorithm following the approach of [ACJR19], which in fact is a slight improvement on the \((|S|n)^{O(\log n)} \) obtained from [GJK+97].}

Handling this lack of a compact representation is the main challenge for tree automata, and will require a substantially different approach to sampling.

**The Partition Scheme for Tree Languages.** Similarly at a high level to the word case, our approach to sampling will be to “grow” a tree \(t\) from the root down. However, unlike in the word case, there is no longer any obvious method to partition the ways to grow a tree (for words, one just partitions by the next bit in the prefix). Our solution to this first challenge is to partition based on the sizes of the subtrees of all the leaves of \(t\). Namely, at each step we expand one of the leaves \(\ell\) of \(t\), and choose what the final sizes of the left and right subtrees of \(\ell\) will be. By irrevocably conditioning on the final sizes of the left and right subtrees of a leaf \(\ell\), we partition the set of possible trees which \(t\) can grow into based on the sizes that we choose. Importantly, we do **not** condition on the states which will be assigned to any of the vertices in \(t\), since doing so would no longer result in a partition of \(T(s^i)\).

More formally, we grow a partial tree \(\tau\), which is an ordered tree with the additional property that some of its leaves are labeled with positive integers, and these leaves are referred to as holes. For an example, see the leftmost tree in Figure 2. A partial tree \(\tau\) is called complete if it has no holes. For a hole \(H\) of \(\tau\), we denote its integral label by \(\tau(H) \geq 1\), and call \(\tau(H)\) the final size of \(H\), since \(\tau(H)\) will indeed be the final size of the subtree rooted at \(H\) once \(\tau\) is complete. Intuitively, in order to complete \(\tau\), we must replace each hole \(H\) of \(\tau\) with a subtree of size exactly \(\tau(H)\). Notice that because no states are involved in this definition, a partial tree \(\tau\) is by itself totally independent of the automata.

We can now define the set \(T(s^i, \tau) \subseteq T(s^i)\) of completions of \(\tau\) as the set of trees \(t \in T(s^i)\) such that \(\tau\) is a subtree of \(t\) sharing the same root, and such that for every hole \(H \in t\) the subtree rooted at the corresponding node \(H \in t\) has size \(\tau(H)\). Equivalently, \(t\) can be obtained from \(\tau\) by replacing each hole \(H \in \tau\) with a subtree \(t_H\) of size \(\tau(H)\). If \(\tau(i)\) is the partial tree consisting of a single hole with final size \(i\), then we have \(T(s^i, \tau(i)) = T(s^i)\). So at each step in the construction of \(\tau\), beginning with \(\tau = \tau(i)\), we will attempt to sample a tree \(t\) uniformly from \(T(s^i, \tau)\). To do so, we can pick any hole \(H \in \tau\), and expand it by adding left and right children and fixing the final sizes of the subtrees rooted at those children. If \(H\) has final size \(\tau(H) = d\), then there are \(d\) ways of doing this; namely, we can fix the final size of the left and right subtrees of \(H\) to be \(j\) and \(d - j - 1\) respectively, for each \(j \in \{0, 1, 2, \ldots, d - 1\}\). So let \(\tau_j\) be the partial tree resulting from fixing these final sizes to be \(j\) and \(d - j - 1\), and notice that \(T(s^i, \tau_j)\) forms a partition of the set \(T(s^i, \tau)\). Thus it will now suffice to efficiently estimate the sizes \(|T(s^i, \tau_j)|\) of each piece in the partition.

**Estimating the number of completions of a partial tree \(\tau\) via the Main Path.** The remaining challenge can now be rephrased in following way: given any partial tree \(\tau\), design a subroutine to estimate the number of completions \(|T(s^i, \tau)|\). The key tool in our approach to doing this is a reduction which allows us to represent the set \(T(s^i, \tau)\) as the regular language generated by a succinct NFA, whose transitions are labeled by large sets which are succinctly encoded (see earlier definition before Theorem 1.3). In our reduction, the alphabet \(\Sigma\) of the succinct NFA will be the set of all ordered trees of size at most \(i\). Note that this results in \(\Sigma\) and the label sets \(A\) being
Figure 2: Two examples of partial trees. The left-hand side tree shows the label of each hole written inside the node. The right-hand side tree illustrates the main path, where non-white (red) nodes and thick arcs are used to highlight the vertices and edges on the main path. Here, holes are indicated by the letter \( H \), and the white unlabeled nodes are neither holes nor on the main path.

exponentially large in \( n \), preventing one from applying the algorithm of [ACJR19].

Our first observation in the reduction is that by always choosing the hole \( H \) at the lowest depth to expand in the partitioning scheme, the resulting holes \( H_1, \ldots, H_k \in \tau \) will be nested within each other. Namely, for each \( i > 1 \), \( H_i \) will be contained in the subtree rooted at the sibling of \( H_{i-1} \). Using this fact, we can define a distinguished path \( P \) between the parent of \( H_1 \) and the parent of \( H_k \). Observe that each hole \( H_j \) must be a child of some node in \( P \). We call \( P \) the main path of \( \tau \) (see Figure 2 for an example). For simplicity, assume that each vertex \( v \in P \) has exactly one child that is a hole of \( \tau \),\(^7\) and label the vertices of the path \( P = \{v_1, v_2, \ldots, v_k\} \), so that \( H_j \) is the child of \( v_j \). Notice by the nestedness property described above, the holes \( H_j, H_{j+1}, \ldots, H_k \) are all contained in the subtree rooted at \( v_j \).

Now observe that any completed tree \( t \in T(s^i, \tau) \) can be uniquely represented by the trees \( (t_1, \ldots, t_k) \), such that \( t \) is obtained from \( \tau \) by replacing each hole \( H_i \in \tau \) by the tree \( t_i \). Thinking of each tree \( t_i \) as a symbol in the alphabet \( \Sigma \) of all ordered trees, we can thus specify the tree \( t \) by a word \( t_1t_2\cdots t_k \in \Sigma^* \). So our goal is to show that the set of words \( T(s^i, \tau) = \{t_1\cdots t_k \in \Sigma^* \mid t_1\cdots t_k \in T(s^i, \tau)\} \) is the language accepted by an succinct NFA \( N \) over the alphabet of ordered trees \( \Sigma \) with polynomially many states and set-labeled transitions.

Intuitively, our reduction to a succinct NFA is possible for the following reasons. Generally speaking, NFAs can only express sets of paths (corresponding to words or sequences) and not trees. However, if we restrict ourselves to the main path \( P \), then the sequence of states of \( T \) which can occur on \( P \) can indeed be expressed by an NFA. Every transition between two states \( s \to s' \) in this NFA corresponds to a transition \( s \to s'r \) or \( s \to rs' \) in the tree automata \( T \). Since all the holes of \( \tau \) lie immediately off of \( P \), such a transition implies that the state \( r \) is placed inside of a hole \( H_j \). Since tree automata admit a “Markov Property” which states that the set of possible subtrees rooted at a given vertex \( v \) depends only on the state \( r \) which is assigned to \( v \), by labeling such a transition \( s \to s' \) in the NFA by the set of trees which could be derived by \( r \) when assigned to the hole \( H_j \), the resulting succinct (set-labeled) NFA now fully captures the set of completions of \( \tau \).

More formally, our construction is roughly as follows; the full details can be found in Section 4. We create a state in \( N \) for every tuple \((v, r)\), where \( v \) is a vertex on the main path \( P \) and \( r \in S \) is a state in the tree automaton \( T \) (recall that \( T = (S, \Delta, s_{\text{init}}) \)). Then if \( v \in P \) is any vertex with the hole \( H_j \) as a left child and the vertex \( v' \in P \) as a right child, and \( r \to qr' \) is a transition in

\(^7\)Extra care should be taken when this is not the case.
\( T \), then we create a transitions from \((v, r)\) to \((v', r')\) in \( N \) labeled (implicitly) by the set of trees \( T(q^\tau(H_j)) \). The start and ending states of \( N \) are then chosen as distinguished states corresponding to the first and last nodes on the path \( P \). Then, if \( v_j \in P \) and \( p > j \), any sequence of transitions \((v_j, r_j) \rightarrow (v_{j+1}, r_{j+1}) \rightarrow \cdots \rightarrow (v_p, r_p)\) will correspond to a set of trees \( T(q_j^\tau(H_j)) \times \cdots \times T(q_p^\tau(H_p)) \), for some states \( q_j, \ldots, q_p \in S \) which could be placed in the holes \( H_j, H_{j+1}, \ldots, H_p \). Notice that by construction, every transition is labeled implicitly by a set of the form \( T(r^j) \) for some \( r \in S \) and \( j < i \). Thus, our succinct encoding of the set \( T(r^j) \) can include not just the implicit identity of the set, but also sketches and estimates \( \tilde{T}(r^j), \tilde{N}(T^j) \) of \( T(r^j) \), which have already been computed by the algorithm at this point.

**An FPRAS for succinct NFAs.** Now that we have constructed the succinct NFA \( N \) which recognizes the language \( T(s^t, \tau) \) as its \( k \)-slice, we must devise a subroutine to approximate the number of words of length \( k \) accepted by \( N \). Let \( S', \Delta' \) be the states and transitions of \( N \). In order to estimate \( |L_k(N)| \), we once again mimic the overall inductive, dynamic programming approach of our “outside” algorithm.\(^8\) Namely, we define the partial states of a dynamic program on \( N \), by setting \( W(x^\ell) \) to be the set of words of length \( \ell \) accepted in the NFA \( N \) starting from the state \( x \in S' \). We then similarly divide the computation of our algorithm into rounds, where on round \( \ell \leq k \) of the subroutine, we inductively pre-compute new NFA sketches \( \tilde{W}(x^\ell) \) of \( W(x^\ell) \) and estimates \( \tilde{N}(x^\ell) \) of \( |W(x^\ell)| \) for each state \( x \in S' \) and size \( \ell \leq k \). Given these estimates and sketches, our procedure for obtaining the size estimates \( \tilde{N}(x^\ell) \) is standard. Thus, similar to the outside algorithm, the central challenge in designing an FPRAS is to design a polynomial time algorithm to sample from the set \( W(x^\ell) \), allowing us to construct the next sketch \( W(x^{\ell+1}) \).

For a string \( u \in \Sigma^* \), define \( W(x^\ell, u) \) to be the set of strings \( w \in W(x^\ell) \) with prefix equal to \( u \). Recall the high-level approach of [ACJR19] to solving this problem for standard NFAs began by partitioning \( W(x^\ell) \) into \( \bigcup_{\alpha \in \Sigma} W(x^\ell, \alpha) \) and estimating the size \( |W(x^\ell, \alpha)| \) for each \( \alpha \in \Sigma \). Then one chooses \( \alpha \) with probability (approximately) \( \Pr[\alpha] = |W(x^\ell, \alpha)| / \sum_{\beta \in \Sigma} |W(x^\ell, \beta)| \) and recurses into the set \( W(x^\ell, \alpha) \). Clearly we can no longer follow this strategy, as \( |\Sigma| \) is of exponential size with respect to \( N \). Specifically, we cannot estimate \( |W(x^\ell, \alpha)| \) for each \( \alpha \in \Sigma \). Instead, our approach is to approximate the behavior of the “idealistic” algorithm which does estimate all these sizes, by sampling from \( \Sigma \) without explicitly estimating the sampling probabilities \( \Pr[\alpha] \). Namely, in general, for a prefix \( u \) we must sample a string \( v \sim W(x^\ell, u) \), by first sampling the next symbol \( \alpha \sim \Sigma \) from a distribution \( \tilde{D}(u) \) which is still close to the exact distribution \( D(u) \) over \( \Sigma \) given by \( \Pr[\alpha] = |W(x^\ell, u \cdot \alpha)| / |W(x^\ell, u)| \) for each \( \alpha \in \Sigma \).

To do this, first note that we can write \( W(x^\ell, u) = \{ u \} \cup \bigcup_{y \in \overline{R(x, u)}} W(y^{(\ell-|u|)}) \), where \( R(x, u) \subseteq S' \) is the set of states \( y \) such that there is a path of transitions from \( x \) to \( y \) labeled by sets \( A_1 \ldots A_{|u|} \) with \( u_j \in A_j \) for each \( j \in \{1, \ldots, |u|\} \). Thus the set of possible symbols \( \alpha \) that we can append to \( u \) is captured by the sets of labels of the transitions out of some state \( y \in R(x, u) \). Now consider the set of transitions \( \{(y, A, z) \in \Delta' \mid y \in R(x, u)\} \), namely, all transitions out of some state in \( R(x, u) \). Furthermore, suppose for the moment that we were given an oracle which generates uniform samples from each label set \( A \) of a transition \( (y, A, z) \), and also provided estimates \( \tilde{N}(A) \) of the size of that set \( |A| \). Given such an oracle, we design a multi-step rejection procedure to sample a symbol \( \alpha \) approximately from \( \tilde{D}(u) \), based on drawing samples from the external oracle and then rejecting them based on intersection ratios of our pre-computed NFA sketches \( \tilde{W}(y^{(\ell-|u|)}) \). Since \( \alpha \) is generated by a transition out of \( R(x, u) \), we first sample such a transitions with probability proportional to the number of remaining suffixes which could be derived by taking that transition. More specifically, the number of suffixes that can be produced by following a transition \( (y, A, z) \) is given by \( |A| \cdot |W(z^{(\ell-|u|)-1})| \), which can be approximated by \( \tilde{N}(A) \cdot \tilde{N}(z^{(\ell-|u|)-1}) \) using the oracle.

\(^8\)We think of this subroutine to estimate \( |T(s^t, \tau)| \) as being the “inner loop” of the FPRAS.
and our internal estimates. Then if \( Z \) is the sum of the estimates \( \tilde{N}(A) \cdot \tilde{N}(z^{\ell-|u|-1}) \) taken over all transitions \( \{ (y, A, z) \in \Delta' \mid y \in R(x, u) \} \), we choose a transition \( (y, A, z) \) with probability \( \tilde{N}(A) \cdot \tilde{N}(z^{\ell-|u|-1})/Z \) and then call the oracle to obtain a sample \( \alpha \sim A \). The sample \( \alpha \) now defines a piece \( W(x^\ell, u \cdot \alpha) \) of the partition of \( W(x^\ell, u) \) which the idealistic algorithm would have estimated and potentially chosen. However, at this point \( \alpha \) is not drawn approximately from the correct distribution \( D(u) \), since the sample from the oracle does not taken into account any information about the other transitions which could also produce \( \alpha \). To remedy this, we show that it suffices to accept the symbol \( \alpha \) with probability:

\[
\frac{\tilde{W}(z^{\ell-|u|-1}) \setminus \bigcup_{\zeta \in \mathcal{B}(\alpha) : \zeta \prec z} W(\zeta^{\ell-|u|-1})}{\tilde{W}(z^{\ell-|u|-1})} \tag{\dagger}
\]

where \( \prec \) is an ordering over \( S' \) and \( \mathcal{B}(\alpha) \) is the set of all states that can be reached from \( R(x, u) \) by reading \( \alpha \), namely, all states \( \zeta \) such that there exists a transition \( (\eta, B, \zeta) \in \Delta' \) with \( \eta \in R(x, u) \) and \( \alpha \in B \). Otherwise, we reject \( \alpha \). Intuitively, probability \( \dagger \) is small when the sets of suffixes which could be derived following transitions \( \mathcal{B}(\alpha) \) that could also produce \( \alpha \) intersect heavily. If this is the case, we have in some sense “overcounted” the contribution of the set \( W(x^\ell, u \cdot \alpha) \) in the partition, and so the purpose of the probability \( \dagger \) is to compensate for this fact. One can then show that this procedure results in samples \( \alpha \) drawn from a distribution \( D(u) \) which is close in statistical distance to the exact distribution \( D(u) \) (and moreover, which is close to the distribution obtained by the idealistic algorithm which explicitly estimated all of the sampling probabilities \( \Pr[\alpha] \)). Furthermore, one can bound the rejection probability by \( \dagger \geq 1/\operatorname{poly}(n) \) in expectation over the choice of \( \alpha \), so that one need only repeat the call to the oracle polynomially many times before obtaining an accepted sample \( \alpha \). Once \( \alpha \) is accepted, we can condition on it and move to the next symbol, avoiding any recursive rejection sampling.

We now return to the assumption that each transition \( (y, A, z) \) admitted an external oracle to generate samples from \( A \). Recall that the set \( A \) is indeed given by the set of trees \( T(s^j) \) for some \( s \in S \) and \( j < i \) on step \( i \) of the external algorithm, for which we have a succinct encoding via a sketch and estimate \( \tilde{T}(s^j), \tilde{N}(s^j) \). To simulate this oracle, we reuse the samples within the sketches \( \tilde{T}(s^j) \) for each call to the succinct NFA sub-routine, pretending that they are being generated fresh and on the fly. Without conditioning on the identities of the samples \( \tilde{T}(s^j) \), within a single call to the succinct NFA subroutine, the samples procedure by this oracle are indeed uniform and independent. However, since the same sketches must be reused on each call to the subroutine, this results in the loss of independence between the samples and sketches used within subsequent calls to the subroutine. To handle this, we show that one can condition on a deterministic property of the sketches \( \{ \tilde{T}(s^j) \}_{s \in S, j < i} \), so that every possible run of the succinct NFA subroutine will yield a good approximation, allowing us to assume independence within the analysis for each call.

Lastly, one must handle the propagation of error resulting from the statistical distance between \( \tilde{D}(u) \) and \( D(u) \) for each prefix \( u \) during the production of a sample \( w \sim W(x^\ell) \). This error in statistical distance feeds into the error for the size estimates \( \tilde{N}(x^{\ell+1}) \) on the next step, both of which feed back into the distributional error when sampling from \( W(x^{\ell+1}) \), doubling the error at each step. We handle this by introducing an approximate rejection sampling step for each sample \( w \), inspired by an exact rejection sampling technique due to Jerrum, Valiant, and Vazirani [JVV86b] (the exact version was also used in [ACJR19]). This approximately corrects the distribution of each sample \( w \), causing the error to increases linearly in the rounds instead of geometrically, which will be acceptable for our purposes.
1.3 Applications of the FPRAS

**Constraint satisfaction problems.** Constraint satisfaction problems (CSPs) offer a general and natural setting to represent a large number of problems where solutions must satisfy some constraints, and which can be found in different areas [Var00, CKS01, RVBW06, HN04, BHvMW09, RN16]. The most basic task associated to a CSP is the problem of verifying whether it has a solution, which corresponds to an assignment of values to the variables of the CSP that satisfies all the constraints of the problem. Tightly related with this task is the problem of counting the number of solution to a CSP. In this work, we consider this counting problem in the usual setting where a projection operator for CSPs is allowed, so that it is possible to indicate the output variables of the problem. We denote this setting as ECSP.

As counting the number of solutions of an ECSP is \#P-complete and cannot admit an FPRAS (unless \(NP = RP\)), we focus on two well known notions of acyclicity that ensure that solutions can be found in polynomial time [GLS00, GLS02]. More precisely, we define \#AECSP as the problem of counting, given an acyclic ECSP \(\mathcal{E}\), the number of solutions to \(\mathcal{E}\). Moreover, given a fixed \(k \geq 0\), we define \#\(k\)-HW-ECSP as the problem of counting, given an ECSP \(\mathcal{E}\) whose hypertree-width is at most \(k\), the number of solution for \(\mathcal{E}\). Although both problems are known to be \#P-complete [PS13], we obtain as a consequence of Theorem 1.1 that both \#AECSP and \#\(k\)-HW-ECSP admit FPRAS.

**Database systems.** Conjunctive queries (CQs) are the most common class of queries used in database systems. Concerning this work, we are interested in the fundamental problem of counting the number of answers to a CQ over a database. In general, this problem is \#P-complete and cannot admit an FPRAS (unless \(NP = RP\)), so as for the case of constraint satisfaction problems we focus on two notions of acyclicity. More precisely, \#ACQ is the problem of counting, given an acyclic CQ \(Q\) and a database \(I\), the number of answers of \(Q\) over \(I\), while for a fixed \(k \geq 0\), \#\(k\)-HW is the problem of counting, given a CQ \(Q\) whose hypertree-width is at most \(k\) and a database \(I\), the number of answers of \(Q\) over \(I\). Although both problems are also known to be \#P-complete [PS13], we obtain as a consequence of Theorem 1.1 that both \#ACQ and \#\(k\)-HW admit FPRAS.

An important extension of the class of CQs is obtained by adding the union operator. More precisely, \#UACQ is the problem of counting, given a database \(I\) and a union conjunctive queries \(Q_1 \cup Q_2 \cup \cdots \cup Q_n\) such that each CQ \(Q_i\) is acyclic, the number of answers of \(Q_1 \cup Q_2 \cup \cdots \cup Q_n\) over \(I\). As expected, \#UACQ is \#P-complete [PS13] and cannot admit an FPRAS (unless \(NP = RP\)). However, the complexity of \#UACQ is considerably harder as even when the projection operator (that provides the list of output variables) is removed, \#UACQ is still \#P-complete, while in this case the counting problem for acyclic CQs can be solved in polynomial time [PS13]. By combining the fact that \#ACQ admits an FPRAS with a relaxation used in [GJK+97] of the techniques proposed in [KL83], we show that \#UACQ admits an FPRAS. Moreover, define \#\(k\)-UHW as the problem of counting, given a database \(I\) and union of conjunctive queries \(Q_1 \cup Q_2 \cup \cdots \cup Q_n\) such that the hypertree-width of each CQ \(Q_i\) is at most \(k\), the number of answers of \(Q_1 \cup Q_2 \cup \cdots \cup Q_n\) over \(I\). Then we can also show that \#\(k\)-UHW admits an FPRAS.

**Software verification.** Nested words have been proposed as a model for the formal verification of correctness of structured programs that can contain nested calls to subroutines [AEM04, AM04, AM09]. In particular, the execution of a program is viewed as a linear sequence of states, but where a matching relation is used to specify the correspondence between each point during the execution at which a procedure is called with the point when we return from that procedure call. This idea gives rise to the notion of nested word, which is defined as a regular word accompanied by a matching relation. Moreover, properties of programs to be formally verified are specified by using nested word automata (NWA). The emptiness problem for nested word automata ask whether, given a NWA \(\mathcal{N}\), there exists a nested word accepted by \(\mathcal{N}\). This is a fundamental problem when
looking for faulty executions of a program with nested calls to subroutines; if \( \mathcal{N} \) is used to encode the complement of a property we expect to be satisfied by a program, then a nested word accepted by \( \mathcal{N} \) encodes a bug of this program. In this sense, the following is also a very relevant problem for understanding how faulty a program is. Define \#NWA as the problem of counting, given a nested word automaton \( \mathcal{N} \) and a string \( 0^n \), the number of nested words of length \( n \) accepted by \( \mathcal{N} \). As expected, \#NWA is a \#P-complete problem. Interestingly, from Theorem 1.1 and the results in [AM09] showing how nested word automata can be represented by using tree automata over binary trees, it is possible to prove that \#NWA admits an FPRAS.

**Knowledge compilation.** Model counting is the problem of counting the number of satisfying assignments given a propositional formula. Although this problem is \#P-complete [Val79], there have been several approaches to tackle it [GSS09]. One of them comes from the field of knowledge compilation, a subarea in artificial intelligence [DM02]. Roughly speaking, this approach consists in dividing the reasoning process in two phases. The first phase is to compile the formula into a target language (e.g. Horn formulae, BDDs, circuits) that has good algorithmic properties. The second phase is to use the new representation to solve the problem efficiently. The main goal then is to find a target language that is expressive enough to encode a rich set of propositional formulae and, at the same time, that allows for efficient algorithms to solve the counting problem.

A target language for knowledge compilation that has attracted a lot of attention is the class of DNNF circuits [Dar01a]. DNNF has good algorithmic properties in terms of satisfiability and logical operations. Furthermore, DNNF can be seen as a generalization of DNF formulae and, in particular, of binary decision diagrams (BDD), in the sense that every BDD can be transformed into a DNNF circuit in polynomial time. Moreover, DNNF is exponentially more succinct than DNF or BDD, and then it is a more appealing language for knowledge compilation. Regarding model counting, DNNF circuits can easily encode \#P-complete problems (e.g. \#DNF) and, therefore, researchers have look into subclasses of DNNF where counting can be done more efficiently. One such a class that has recently received a lot of attention is the class of structured DNNF which has been used for efficient enumeration [ABJM17, ABMN19], and has proved to be appropriate to compile propositional CNF formulae with bounded width (e.g. CV-width) [OD14]. Unfortunately, the problem of computing the number of propositional variable assignments that satisfy a structured DNNF circuit is a \#P-complete problem, as these circuits include the class of DNF formulae. However, and in line with the idea that structured DNNF circuits allow for more efficient counting algorithms, we prove that the counting problem of structured DNNF circuits admits a fully-polynomial time randomized approximation schema as a consequence of Theorem 1.1.

2 Preliminaries

In this section, we introduce the main terminology used in this paper.

2.1 Intervals, strings, trees and tree automata

Given \( m \leq n \) with \( n, m \in \mathbb{N} \), we use notation \([m,n]\) for the set \( \{m, m+1, \ldots, n\} \), and notation \([n]\) for the set \( \{1, n\} \). Moreover, given \( u, \epsilon \in \mathbb{R} \) with \( \epsilon \geq 0 \), let \((u, u+\epsilon)\) denote the real interval \([u-\epsilon, u+\epsilon]\). In general, we consider real intervals of the form \((1, \epsilon)\), and we use \( x(1, \epsilon) \) to denote the range \([x-x\epsilon, x+x\epsilon]\), and \( x = (1, \epsilon)y \) to denote the containment \( x \in [y-\epsilon y, y+\epsilon y] \).

Given a finite alphabet \( \Sigma \), a finite string over \( \Sigma \) is a sequence \( w = w_1 \ldots w_n \) such that \( n \geq 0 \) and \( w_i \in \Sigma \) for every \( i \in [n] \). Notice that if \( n = 0 \), then \( w \) is the empty word, which is denoted by \( \lambda \). We write \(|w| = n \) for the length of \( w \). As usual, we denote by \( \Sigma^n \) all strings over \( \Sigma \). For two
sets \( A, B \subseteq \Sigma^* \) we denote by \( A \cdot B = \{ u \cdot v \mid u \in A, v \in B \} \), where \( u \cdot v \) is the concatenation of two strings \( u \) and \( v \), and by \( A^i \) the concatenation of \( A \) with itself \( i \) times, that is, \( A^0 = \{ \lambda \} \) and \( A^{i+1} = A \cdot A^i \) for every \( i \in \mathbb{N} \).

Fix \( k \in \mathbb{N} \) with \( k \geq 1 \). A finite ordered \( k \)-tree (or just a \( k \)-tree) is a prefix-closed non-empty finite subset \( t \subseteq [k]^* \); namely, if \( w \cdot i \in t \) with \( w \in [k]^* \) and \( i \in [k] \), then \( w \in t \) and \( w \cdot j \in t \) for every \( j \in [i] \). For a \( k \)-tree \( t \), \( \lambda \in t \) is the called the root of \( t \) and every maximal element in \( t \) (under prefix order) is called a leaf. We denote by \( \text{leaves}(t) \) the set of all leaves of \( t \). For every \( u, v \in t \), we say that \( u \) is a child of \( v \), or that \( v \) is the parent of \( u \), if \( u = v \cdot i \) for some \( i \in [k] \). We say that \( v \) has \( n \) children if \( v \cdot 1, \ldots, v \cdot n \in t \) with \( n = \max_{i \in t} \{ i \} \). We denote by \( v = \text{parent}(u) \) when \( v \) is the parent of \( u \) (if \( u \) is the root, then \( \text{parent}(u) \) is undefined). Furthermore, we say that \( v \) is an ancestor of \( u \), or \( u \) is a descendant of \( v \), if \( v \) is a prefix of \( u \). The size of \( t \), i.e. the number of nodes, is denoted by \( |t| \).

Let \( \Sigma \) be a finite alphabet and \( t \) be a \( k \)-tree. Slightly abusing notation, we also use \( t \) to denote a \( k \)-tree labeled over \( \Sigma \). That is, we also consider \( t \) as a function such that for every \( u \in t \), it holds that \( t(u) \in \Sigma \) is the label assigned to node \( u \). For \( a \in \Sigma \), we denote just by \( a \) the tree consisting of a one node labeled with \( a \). For labeled \( k \)-trees \( t \) and \( t' \), and a leaf \( \ell \in t \), we define \( t[\ell \to t'] \) the labeled \( k \)-tree resulting from “hanging” \( t' \) on the node \( \ell \) in \( t \). Formally, we have that \( t[\ell \to t'](u) = t(u) \) whenever \( u \in (t \setminus \{ \ell \}) \) and \( t[\ell \to t']((\ell \cdot u)) = t'(u) \) whenever \( u \in t' \). Note that the leaf \( \ell \) takes in \( t[\ell \to t'] \) the label on \( t' \) instead of its initial label on \( t \). When \( t \) consists of just one node with label \( a \) and with two children, we write \( a(t_1, t_2) \) for the tree defined as \( t[1 \to t_1][2 \to t_2] \), namely, the tree consisting of a root \( a \) with \( t_1 \) and \( t_2 \) hanging to the left and right, respectively.

In particular, \( t = a(b, c) \) is the tree with three nodes such that \( t(\lambda) = a \), \( t(1) = b \), and \( t(2) = c \). Finally, we denote by \( \text{Trees}_k[\Sigma] \) the set of all \( k \)-trees labeled over \( \Sigma \) (or just \( k \)-trees over \( \Sigma \)).

**Tree automata.** A (top-down) tree automaton \( T \) over \( \text{Trees}_k[\Sigma] \) is a tuple \( (S, \Sigma, \Delta, s_{\text{init}}) \) where \( S \) is a finite set of states, \( \Sigma \) is the finite alphabet, \( \Delta \subseteq S \times \Sigma \times (\cup_{i=0}^k S^i) \) is the transition relation, and \( s_{\text{init}} \in S \) is the initial state. We will usually use \( s, q, \) and \( r \) to denote states in \( S \). A run \( \rho \) of \( T \) over a \( k \)-tree \( t \) is a function \( \rho : t \to S \) that assigns states to nodes of \( t \) such that for every \( u \in t \), if \( u \cdot 1, \ldots, u \cdot n \) are the children of \( u \) in \( t \), then \( (\rho(u), t(u), \rho(u \cdot 1)\rho(u \cdot 2) \ldots \rho(u \cdot n)) \in \Delta \). In particular, if \( u \) is a leaf, then it holds that \( (\rho(u), t(u), \lambda) \in \Delta \). We say that \( T \) accepts \( t \) if there exists a run of \( T \) over \( t \) with \( \rho(\lambda) = s_{\text{init}} \), and we define \( \mathcal{L}(T) = \text{Trees}_k[\Sigma] \) as the set of all \( k \)-trees over \( \Sigma \) accepted by \( T \). We write \( \mathcal{L}_n(T) \) to denote the \( n \)-slice of \( \mathcal{L}(T) \), namely \( \mathcal{L}_n(T) = \{ t \in \mathcal{L}(T) \mid |t| = n \} \) of all \( k \)-trees of size \( n \) in \( \mathcal{L}(T) \).

Give a state \( s \in S \), we will usually parameterize \( T \) by the initial state \( s \), specifically, we write \( T[s] = (S, \Sigma, \Delta, s) \) for the modification of \( T \) where \( s \) is the new initial state. Furthermore, let \( \tau = (s, a, w) \in \Delta \) be any transition. We denote by \( T[\tau] = (S, \Sigma, \Delta \cup \{(s^*, a, w)\}, s^* \) where \( s^* \) is a fresh state not in \( Q \). In other words, \( T[\tau] \) is the extension \( T \) that recognizes trees where runs are forced to start with transition \( \tau \).

**Binary trees.** A binary labeled tree \( t \) is a labeled 2-tree such that every node has two children or is a leaf. Notice that 2-trees are different from binary trees, as in the former a node can have a single child, while this is not allowed. For every non-leaf \( u \in t \), we denote by \( u \cdot 1 \) and \( u \cdot 2 \) the left and right child of \( u \), respectively. Similar than for \( k \)-trees, we denote by \( \text{Trees}_B[\Sigma] \) the set of all binary trees. We say that a tree automaton \( T = (S, \Sigma, \Delta, s_{\text{init}}) \) is over \( \text{Trees}_B[\Sigma] \) if \( \Delta \subseteq S \times \Sigma \times ((\lambda) \cup S^2) \).
2.2 Approximate counting and a notion of reduction

A randomized approximation scheme (RAS) for a function \( f : \Sigma^* \to \mathbb{N} \) is a randomized algorithm \( A: \Sigma^* \times (0,1) \to \mathbb{N} \) such that for every \( w \in \Sigma^* \) and \( \epsilon \in (0,1) \):

\[
\Pr[|A(w,\epsilon) - f(w)| \leq \epsilon \cdot f(w)] \geq \frac{3}{4}.
\]

Moreover, if there exists a polynomial \( q(x,y) \) such that for every \( w \in \Sigma^* \) and \( \epsilon \in (0,1) \), the number of steps needed to compute \( A(w,\epsilon) \) is at most \( q(|w|, \epsilon^{-1}) \), then \( A \) is said to be a fully polynomial-time randomized approximation scheme (FPRAS) for \( f \) \cite{JVV86a}. Thus, if \( A \) is an FPRAS for \( f \), then \( A(w,\epsilon) \) approximates the value \( f(w) \) with a relative error of \( \epsilon \), and it can be computed in polynomial time in the size of \( w \) and the value \( \epsilon^{-1} \).

Given functions \( f, g : \Sigma^* \to \mathbb{N} \), a polynomial-time parsimonious reduction from \( f \) to \( g \) is a polynomial-time computable function \( h : \Sigma^* \to \Sigma^* \) such that, for every \( w \in \Sigma^* \), it holds that \( f(w) = g(h(w)) \). If such a function \( h \) exists, then we use notation \( f \leq_{\text{PAR}} g \). Notice that if \( f \leq_{\text{PAR}} g \) and \( g \) admits an FPRAS, then \( f \) admits an FPRAS.

2.3 The counting problems

The following is the main counting problem studied in this paper:

| Problem: \( \#\text{TA} \) | Input: A tree automaton \( T \) over Trees_{k}[\Sigma] and a string \( 0^n \) | Output: \( |L_n(T)| \) |

By the results in \cite{CDG+07} about encoding \( k \)-trees as binary trees using an extension operator \( @ \), it is possible to conclude the following:

**Lemma 2.1.** Let \( \Sigma \) be a finite alphabet and \( @ \notin \Sigma \). Then there exists a polynomial-time algorithm that, given a tree automata \( T \) over Trees_{k}[\Sigma], produces a tree automaton \( T' \) over Trees_{B}[\Sigma \cup \{@\}] such that, for every \( n \geq 1 \):

\[
|\{t \mid t \in L(T) \text{ and } |t| = n\}| = |\{t' \mid t' \in L(T') \text{ and } |t'| = 2n - 1\}|.
\]

Therefore, we also consider in this paper the following problem:

| Problem: \( \#\text{BTA} \) | Input: A tree automaton \( T \) over Trees_{B}[\Sigma] and a string \( 0^n \) | Output: \( |L_n(T)| \) |

As we know from Lemma 2.1 that there exists a polynomial-time parsimonious reduction from \( \#\text{TA} \) to \( \#\text{BTA} \), we can show that \( \#\text{TA} \) admits an FPRAS by proving that \( \#\text{BTA} \) admits an FPRAS.

3 Fully Polynomial-Time Randomized Approximation Scheme for \( \#\text{BTA} \) and \( \#\text{TA} \)

In this section, we provide an FPRAS for \( \#\text{BTA} \). Thus, we obtain as well that \( \#\text{TA} \) admits an FPRAS, given that there exists a polynomial-time parsimonious reduction from \( \#\text{TA} \) to \( \#\text{BTA} \).

Fix a tree automaton \( T = (S, \Sigma, \Delta, s_{\text{init}}) \) over binary trees and let \( n \geq 1 \) be a natural number given in unary. We assume that every state in \( S \) is mentioned in \( \Delta \), and that every symbol in \( \Sigma \) is
mentioned in $\Delta$ (if it not the case, then the elements that are not mentioned in $\Delta$ can just be removed form the tree automaton). Let $m$ be the size of the tree automaton $T$, defined as $m = \|\Delta\|$, where $\|\Delta\|$ is the size of the transition relation $\Delta$ (represented as a string over an appropriate alphabet). In the following, fix an error parameter $\epsilon > 0$. Since our algorithm will run in time $\text{poly}(n, m, 1/\epsilon)$, we can assume $\epsilon < \frac{1}{(4nm)^2}$ without loss of generality. Note that if we are only interested in uniform sampling, we can just fix $\epsilon = 1/\text{poly}(nm)$. Finally, recall that $L_n(T) = \{|t| \in L(T) \text{ and } |t| = n\}$.

**Remark 3.1.** We can assume that $m, n = \omega(1)$, since if $n = O(1)$, then the number of unlabeled trees is constant, so the number of labeled trees is a polynomial in $m$, and we can check whether each such a tree is in $L(T)$ to compute $|L_n(T)|$ in polynomial time. If $m = O(1)$, then we can transform $T$ into a constant sized deterministic bottom-up tree automaton,\(^9\) and then $|L_n(T)|$ can be computed in polynomial time by dynamic programming. Thus, for the remainder we can now assume that $n \geq 2$ and $m \geq 3$.

Unfolding of the tree automaton $T$. We begin by making a number of copies of the states in $T$ in order to “unfold” $T$ into $n$ levels. For this, let the new set of states be $\overline{S} = \{s^i \mid i \in [n], s \in S\}$. Intuitively, from $s^i$ we only want to accept trees of size $i$. This will allow us to define a natural partition scheme for the sampling procedure. To enforce this constraint, we build a new tree automaton $\overline{T} = (\overline{S}, \Sigma, \overline{\Delta}, s^n_{\text{init}})$ such that for every transition $(s, a, q \cdot r) \in \Delta$ and $i \in [2, n]$, we add the transition $(s^i, a, q^i \cdot r^{i-j-1})$ to $\overline{\Delta}$ for every $j \in [1, i-2]$. Also, for every transition $(s, a, \lambda) \in \Delta$ we add $(s^i, a, \lambda)$ to $\overline{\Delta}$. We say that $i$ is the level of $s^i$. Note that one can construct the set $\overline{S}$ and the automaton $\overline{T}$ in polynomial time in the size of $T$ [CDG+07].

Given the definition of $\overline{T}$, one can easily check that $L(\overline{T}[s]) = \{t \in L(T[s]) \mid |t| = i\}$ for every $s^i \in \overline{S}$. In particular, we have that $|L(T)| = |\{t \in L(T) \mid |t| = n\}|$ and, thus, the goal becomes to estimate $|L(T)| = |L(\overline{T}[s^n_{\text{init}}])|$. For clarity of notation, we write $T(s^i)$ for $L(\overline{T}[s^i])$ and $\tilde{N}(s^i)$ for $|T(s^i)|$. Note that the goal becomes to estimate $N(s^n_{\text{init}})$.

**Remark 3.2** (Proviso on the sizes of trees). Every binary tree has an odd number of nodes. Thus, we will have that $T(s^{2i}) = \emptyset$ and $N(s^{2i}) = 0$ for each $i \geq 1$. However, to make the notation simpler, we do not limit ourselves to the trees of odd sizes. On the contrary, the algorithms provided in this article are able to compute $N(s^{2i}) = 0$, and also to realize that no sample has to be produced from $T(s^{2i})$.

**Two basic properties, and the estimation of $N(s^i)$.** Our algorithm simultaneously computes estimates $\tilde{N}(s^i)$ for the set sizes $N(s^i)$, as well as sketches $\tilde{T}(s^i)$ of $T(s^i)$ which consist of polynomially many uniform samples from $T(s^i)$. Specifically, at each level $i$ and for every $s \in S$, our algorithm will store an estimate which satisfies $\tilde{N}(s^i) = (1 \pm \epsilon)N(s^i)$. At step $i$, for each $j < i$, our algorithm will also store $i$ distinct independent uniformly sampled subsets sets $\tilde{T}_1(s^j), \tilde{T}_2(s^j), \ldots, \tilde{T}_i(s^j)$ of $T(s^j)$ which satisfy certain deterministic criteria that will result in the correctness of our sampling algorithm on nodes $\tilde{s}^i$ (see Lemma 3.5). Using these estimates $\tilde{N}(s^i)$ and sketches $\tilde{T}_i(s^j)$ for $j < i$ as input, we will construct a procedure that allows us to obtain fresh, independent samples from the sets $T(s^i)$ for all $s \in S$. Formally, the properties we need to inductively condition on are as follows:

**Property 1:** For a fixed $i \in [n]$, we have $\tilde{N}(s^i) = (1 \pm \epsilon)N(s^i)$ for all $s \in S$.\(^9\)

\(^9\)A tree automaton is bottom-up if it assigns states to a labeled tree $t$ starting from the leaves, and moving toward the root [CDG+07]. In particular, if $t$ is a binary tree, then the transition function is of the form $\Delta : S \times S \times \Sigma \rightarrow S$, that is, a state is assigned to a node depending on the states of its two children and its label.
Property 2: For a fixed $i \in [n]$, we have an oracle which returns uniform, independent samples $t \sim T(s^j)$ for every $j \leq i$ and $s \in S$, and runs in $\text{poly}(n, m, 1/\epsilon, \gamma)$ time, for some fixed parameter $\gamma$ which we will later choose. The oracle is allowed to fail with probability at most $3/4$, in which case it outputs no sample.

Fix an arbitrary $i \in [n]$, and suppose we have computed $\tilde{N}(q^j)$ and $\tilde{T}_k(q^j)$ for all $q \in S$, $j < i$ and $k \in [i]$. Fix now a state $s$. We first show how to compute the estimate $\tilde{N}(s^i)$.

Proposition 3.3. Fix $\delta \in (0, 1)$. If Property 1 and 2 hold for all levels $j < i$, then with probability $1 - \delta$ and time $\text{poly}(n, m, 1/\epsilon, \log(1/\delta))$ we can compute a value $\tilde{N}(s^i)$ such that $\tilde{N}(s^i) = (1 + \epsilon)N(s^i)$. In other words, Property 1 holds for level $i$.

Proof. If $i = 1$, we can compute $N(s^1)$ exactly in time $O(m)$, and we make $\tilde{N}(s^i) = N(s^i)$. Thus, assume that $i \geq 2$. For each transition $\tau = (s^i, a, q^j \cdot r^{i-j-1}) \in \Sigma$, recall the definition of the extension $T[\tau]$ (see Section 2), which recognizes trees where runs are forced to start with transition $\tau$. We now define $N(\tau) = |\mathcal{L}(T[\tau])|$, and observe that $N(\tau) = |T(q^j) \times T(r^{i-j-1})| = N(q^j) \cdot N(r^{i-j-1})$. Thus, we obtain an estimate $\tilde{N}(\tau)$ of $N(\tau)$ via:

$$\tilde{N}(\tau) = \tilde{N}(q^j) \cdot \tilde{N}(r^{i-j-1}) = (1 + \epsilon)(1 + (i - j - 1)\epsilon) \cdot N(q^j) \cdot N(r^{i-j-1}) = (1 + \epsilon(i - j) + 1\epsilon) \cdot N(\tau) = \left(1 + \epsilon(i - j)\right) \cdot N(\tau)$$

Where in the last equation, we used our assumption that $\epsilon < 1/(4nm)^{18} < 1/n^3$ and then applied the fact that $(i - j)\epsilon^2 \leq n\epsilon^2 \leq \epsilon/n$. Also, notice that we are using the fact that Property 1 holds for all sizes $j < i$. Now let $\tau_1, \tau_2, \ldots, \tau_\ell \in \Sigma$ be all the transitions of the form $\tau_j = (s^i, a_j, q_j \cdot r_j)$ with $q_j, r_j \in \overline{S}$ and $a_j \in \Sigma$. Observe that $N(s^i) = |\bigcup_{j=1}^\ell \mathcal{L}(T[\tau_j])|$. Now for each $j \in [\ell]$, let $p_j$ be the probability that a uniform sample $t \sim \mathcal{L}(T[\tau_j])$ is not contained in $\mathcal{L}(T[\tau_j'])$ for all $j' < j$. Then $N(s^i) = \sum_{j=1}^\ell N(\tau_j)p_j$, so in order to estimate $N(s^i)$ it suffices to estimate the values $p_j$. Since Property 2 holds for all levels less than $i$, by making calls to oracles $t_q \sim \mathcal{L}(T[q_j])$ and $t_r \sim \mathcal{L}(T[r_j])$ we can obtain an i.i.d. sample $a_j(t_q, t_r)$ from $\mathcal{L}(T[\tau_j])$ (recall the notation for trees introduced in Section 2). By repeating this process, we can obtain i.i.d. samples $t_1, t_2, \ldots, t_h \sim \mathcal{L}(T[\tau_j])$ uniformly at random, where $h = O(\log(4m/\delta)m^2/\epsilon^2)$. Now let $\tilde{p}_j$ be the fraction of the samples $t_k$ such that $t_k \notin \mathcal{L}(T[\tau_j'])$ for each $j' < j$. Note that checking if $t_k \notin \mathcal{L}(T[\tau_j'])$ can be done in $\text{poly}(n, m)$ time via a membership query for tree automata. Thus if we let

$$X_k = \begin{cases} 1 & \text{if } t_k \notin \mathcal{L}(T[\tau_j']) \text{ for each } j' < j \\ 0 & \text{otherwise.} \end{cases}$$

then we have $\tilde{p}_j = h^{-1} \sum_{k=1}^h X_k$. Then setting $p_j = \mathbb{E}[X_k]$, by Hoeffding’s inequality we have $|\tilde{p}_j - p_j| \leq \frac{\epsilon}{4m}$ with probability at least $1 - \delta/(2m)$, so we can union bound over all $j \in [\ell]$ and obtain $|\tilde{p}_j - p_j| \leq \frac{\epsilon}{4m}$ for all $j \in [\ell]$ with probability at least $1 - \delta$. Putting all together, we can derive an estimate $\tilde{N}(s^i)$ for $N(s^i)$ by using the estimates $\tilde{N}(\tau_j)$ and $\tilde{p}_j$ of $N(\tau_j)$ and $p_j$, respectively, as follows:

$$\tilde{N}(s^i) = \sum_{j=1}^\ell \tilde{N}(\tau_j)\tilde{p}_j$$
\[ = \left(1 \pm \left(i - 1 + \frac{1}{n}\right) \epsilon\right) \sum_{j=1}^\ell N(\tau_j) \tilde{p}_j \]

\[ = \left(1 \pm \left(i - 1 + \frac{1}{n}\right) \epsilon\right) \left(\sum_{j=1}^\ell N(\tau_j) p_j \pm \frac{\epsilon}{4m} \sum_{j=1}^\ell N(\tau_j)\right) \]

\[ = \left(1 \pm \left(i - 1 + \frac{1}{n}\right) \epsilon\right) \left(N(s^i) \left(1 \pm \frac{\epsilon}{4}\right)\right) \]

\[ = \left(1 \pm i \epsilon\right) N(s^i). \]

Where we use that \( \sum_{j=1}^\ell N(\tau_j) \leq \sum_{j=1}^\ell N(s^i) = \ell N(s^i) \leq mN(s^i) \) in the second to last step, and the fact that \( n \geq i \geq 2 \) in the last step. For runtime, notice that the key result \( \Pr[|\tilde{p}_j - p_j| \leq \epsilon/(4m)] \geq 1 - \delta/(2m) \) is conditioned on the event that we were able to obtain \( h \) samples \( t_k \) using the oracle. Recall that the sampling oracle can fail with probability at most \( 3/4 \). Then, the required number of calls \( h' \) to the poly-time sampling oracle is at most \( 4h/3 \) in expectation. For our purposes, \( h' = O(h) \) will also be enough, as we now show. For \( j \in [\ell] \) call \( G_j \) the event that we obtain \( h \) samples from \( \mathcal{L}(T[\tau_j]) \) and \( H_j \) the event that \( |\tilde{p}_j - p_j| \leq \epsilon/(4m) \). Then, as we showed above,

\[ \Pr[\tilde{N}(s^i) = (1 \pm i \epsilon)N(s^i)] \geq \Pr\left[\bigcap_{j=1}^\ell (H_j \cap G_j)\right] = 1 - \Pr\left[\bigcup_{j=1}^\ell (H_j \cup G_j)\right] \geq 1 - m \Pr[\overline{H_{j_0}} \cup \overline{G_{j_0}}], \]

where the last inequality is due to a union bound obtained considering \( j_0 = \arg\max_{j \in [\ell]} \Pr[\overline{H_{j_0}} \cup \overline{G_{j_0}}]. \) Recall that we want \( \Pr[\tilde{N}(s^i) = (1 \pm i \epsilon)N(s^i)] \geq 1 - \delta, \) hence it suffices to show

\[ 1 - m \Pr[\overline{H_{j_0}} \cup \overline{G_{j_0}}] \geq 1 - \delta \iff \frac{\delta}{m} \geq \Pr[\overline{H_{j_0}} \cup \overline{G_{j_0}}] \iff \Pr[H_{j_0} \cap G_{j_0}] \geq 1 - \frac{\delta}{m} \quad (3) \]

By Hoeffding’s inequality, as we showed before, \( \Pr[H_{j_0} \mid G_{j_0}] \geq 1 - \delta/(2m) \). Suppose that we also have that \( \Pr[G_{j_0}] \geq 1 - \delta/(2m) \). Then,

\[ \Pr[H_{j_0} \cap G_{j_0}] = \Pr[H_{j_0} \mid G_{j_0}] \cdot \Pr[G_{j_0}] \geq \left(1 - \frac{\delta}{2m}\right)^2 \geq 1 - 2 \cdot \frac{\delta}{2m} = 1 - \frac{\delta}{m}, \]

as required by equation (3). Thus, it suffices to show \( \Pr[G_{j_0}] \geq 1 - \delta/(2m) \). Letting \( X_i \) be the random variable that indicates whether the \( i \)-th call to the sampling procedure was successful, then the total number of samples obtained is \( X = \sum_{i=1}^{h'} X_i \), where \( \mathbb{E}[X] \geq h'/4 \), so by a Chernoff bound we have

\[ \Pr[G_{j_0}] = 1 - \Pr[\overline{G_{j_0}}] = 1 - \Pr[X < h] \geq 1 - \exp\left(-\frac{h'}{8} \left(1 - \frac{4h}{h'}\right)^2\right) \geq 1 - \exp\left(-\frac{h'}{8} \left(1 - \frac{4h}{h'}\right)\right) \]

assuming that \( 4h < h' \). Hence,

\[ 1 - \exp\left(-\frac{h'}{8} \left(1 - \frac{4h}{h'}\right)\right) \geq 1 - \frac{\delta}{2m} \iff \frac{\delta}{2m} \geq \exp\left(-\frac{h'}{8} \left(1 - \frac{4h}{h'}\right)\right) \iff h' \geq 4h + 8 \ln\left(\frac{2m}{\delta}\right). \]

so by definition of \( h \), it is sufficient to set \( h' = 5h \), which completes the proof. \( \blacksquare \)
The notion of a partial tree. We need to demonstrate how to obtain uniform samples from $T(s^i)$ to build the sets $T_j(s^i)$. To do this, we will provide an algorithm that recursively samples a tree $t \in T(s^i)$ from the top down. But before showing this procedure, we need to introduce the notion of a partial tree. In the following, recall that $\Sigma$ is a finite alphabet and assume, without loss of generality, that $\Sigma \cap [n] = \emptyset$.

**Definition 3.4.** A partial tree is a binary labeled tree $t$ over $\Sigma \cup [n]$. A node $u$ labeled by $t(u) \in [n]$ is called a hole of $t$, and we assume that holes can appear only at the leaves of $t$. The full size of $t$, denoted by $fsize(t)$, is defined as $|\{u \mid t(u) \in \Sigma\}| + \sum_{u : t(u) \in [n]} t(u)$. Moreover, a partial tree $t$ is said to be complete if $t$ contains no holes.

Intuitively, in a partial tree $t$, a hole $u$ represents a placeholder where a subtree of size $t(u)$ is going to be hanged. That is, partial tree $t$ is representing all trees over $\Sigma$ that have the same trunk as $t$ and, for each hole $u$, the subtree rooted at $u$ is of size $t(u)$. Notice that all trees represented by $t$ will have the same size $|\{u \mid t(u) \in \Sigma\}| + \sum_{u : t(u) \in [n]} t(u)$ and, therefore, we define the full size of $t$ as this quantity. Finally, observe that if a partial tree $t$ is complete, then $t$ contains no holes and, hence, no extension is needed. For an example of a partial tree, see Figure 3.

For every partial tree $t$ and node $x \in t$, write $t_x$ to denote the partial subtree of $t$ rooted at $x$. For each hole $u \in t$ with size $t(u) = i$, we say that $t'$ is an immediate extension of $t$ over $u$ if $t' = t[u \rightarrow a(j,i-j-1)]$ for some $a \in \Sigma$ and $j \in [i-2]$. That is, $t$ is extended by replacing the label of $u$ with $a$ and hanging from $u$ two new holes whose sizes sum to $i - 1$ (note that the resulting partial subtree $t'_u$ has full size $i$). In case that $i = 1$, then it must hold that $t' = t[u \rightarrow a]$ for some $a \in \Sigma$. We define the set of all immediate extensions of $t$ over $u$ as $\text{ext}(t,u)$. Note that $|\text{ext}(t,u)| = (i-2)|\Sigma|$. Finally, given two partial trees $t$ and $t'$, we write $t \rightsquigarrow u t'$ if $t'$ is an immediate extension of $t$ over $u$, and $t \rightsquigarrow t'$ if $t'$ is an immediate extension of $t$ over some hole $u \in t$. We then define the reflexive and transitive closure $\rightsquigarrow^*$ of $\rightsquigarrow$, and say that $t'$ is an extension of $t$ if $t \rightsquigarrow^* t'$. In other words, $t \rightsquigarrow^* t'$ if either $t' = t$ or $t'$ can be obtained from $t$ via a non-empty sequence of immediate extensions $t \rightsquigarrow t_1 \rightsquigarrow t_2 \rightsquigarrow \cdots \rightsquigarrow t'$. We say that $t'$ is a completion of $t$ when $t \rightsquigarrow^* t'$ and $t'$ is complete.
Obtaining uniform samples from $T(s^i)$. Given a partial tree $t$ with $\text{fs}ize(t) = i$, consider now the set $T(s^i, t)$ of all completions $t'$ of $t$ derivable with $s^i$ as the state in the root node, namely, $T(s^i, t) = \{t' \in T(s^i) \mid t'$ is a completion of $t$\}. Further, define $N(s^i, t) = |T(s^i, t)|$. To obtain a uniform sample from $T(s^i)$, we start with a partial tree $t = i$ (i.e. $t$ is a partial tree with one node, which is a hole of size $i$). At each step, we choose the hole $u \in t$ with the smallest size $t(u)$, and consider an immediate extension of $t$ over $u$. Note that the set $T(s^i, t)$ can be partitioned by the sets $\{T(s^i, t')\}_{t \rightarrow u, t'}$ of such immediate extensions. The fact that $T(s^i, t') \cap T(s^i, t'') = \emptyset$, whenever $t \rightarrow u, t' \rightarrow u$ and $t' \neq t''$, follows immediately from the fact that $t'$ and $t''$ have different labels from $\Sigma$ in the place of $u$ or unequal sizes of the left and right subtrees of $u$. We then will sample each partition $T(s^i, t')$ with probability approximately proportional to its size $N(s^i, t')$, set $t' \leftarrow t$, and continue like that recursively. Formally, the procedure to sample a tree in $T(s^i)$ is shown in Algorithm 1.

Algorithm 1: $\text{SAMPLE}(s^i, \{\tilde{T}_i(\ell^j)\}_{\ell \in S, j < i}, \{\tilde{N}(\ell^j)\}_{\ell \in S, j \leq i}, \epsilon, \delta)$

1. Initialize a partial tree $t = i$, and set $\varphi = 1$
2. while $t$ is not complete do
3. Let $u$ be the hole of $t$ with the minimum size $t(u)$. If more than one node reaches this minimum value, choose the first such a node according to a prespecified order on the holes of $t$.
4. Let $\text{ext}(t, u) = \{t_1, \ldots, t_l\}$ be the set of immediate extensions of $t$ over $u$.
5. For each $k \in [l]$, call $\text{ESTIMATEPARTITION}(t_k, s^i, \{\tilde{T}_i(\ell^j)\}_{\ell \in S, j < i}, \{\tilde{N}(\ell^j)\}_{\ell \in S, j \leq i}, \epsilon, \delta)$ to obtain an estimate $\tilde{N}(s^i, t_k)$ of $N(s^i, t_k)$. // Recall that $T = (S, \Sigma, \Delta, s_{\text{init}})$
6. Sample partition $k \in [l]$ with probability $\frac{\tilde{N}(s^i, t_k)}{\sum_{k' = 1}^{l} \tilde{N}(s^i, t_{k'})}$.
7. Set $\varphi \leftarrow \varphi \cdot \frac{\tilde{N}(s^i, t_k)}{\sum_{k' = 1}^{l} \tilde{N}(s^i, t_{k'})}$.
8. Set $t \leftarrow t_k$.
9. end
10. return $t$ with probability $\frac{1}{2\varphi N(s^i)}$, otherwise output $\text{FAIL}$.

Notice that $\text{SAMPLE}(s^i, \{\tilde{T}_i(\ell^j)\}_{\ell \in S, j < i}, \{\tilde{N}(\ell^j)\}_{\ell \in S, j \leq i}, \epsilon, \delta)$ uses the precomputed values $\tilde{T}_i(\ell^j)$ for every $r \in S$ and $j \in [i - 1]$, and the precomputed values $\tilde{N}(\ell^j)$ for every $r \in S$ and $j \in [i]$. This procedure first selects a hole $u$ with the minimum size $t(u)$, and then calls a procedure $\text{ESTIMATEPARTITION}$ to obtain an estimate $\tilde{N}(s^i, t_k)$ of $N(s^i, t_k)$ for every immediate extensions $t_k$ of $t$ over $u$. Thus, to prove our main theorem about the procedure $\text{SAMPLE}$, we first need the following lemma about the correctness of the partition size estimates. The proof of Lemma 3.5 is the main focus of Section 4.

Lemma 3.5. Let $\delta \in (0, 1/2)$, and fix independent and uniform samples sets $\tilde{T}_i(s^j)$ of $T(s^i)$ each of size $O(\log^2(1/\delta)(nm)^{13}/\epsilon^5)$, for every $s \in S$ and $j < i$. Suppose further that we have values $\tilde{N}(s^j) = (1 \pm \epsilon)N(s^j)$ for every $s \in S$ and $j \leq i$ Then with probability $1 - \delta_{nm}$, the following holds: for every state $s \in S$ and for every partial tree $t$ with $\text{fs}ize(t) = i$, the procedure $\text{ESTIMATEPARTITION}(t, s^i, \{\tilde{T}_i(\ell^j)\}_{\ell \in S, j < i}, \{\tilde{N}(\ell^j)\}_{\ell \in S, j \leq i}, \epsilon, \delta)$ runs in $\text{poly}(n, m, 1/\epsilon, \log(1/\delta))$-
time and returns a value $\tilde{N}(s^i, t)$ such that
\[
\tilde{N}(s^i, t) = (1 \pm (4nm)^{17} \epsilon)N(s^i, t).
\]

Notice that $\delta$ is the parameter that controls the success probability of the EstimatePartition procedure. In the statement of Lemma 3.5, we ask that $\delta < 1/2$. This is due to technical reasons in some proofs, but notice that it can be safely assumed, because a smaller $\delta$ will yield a better success probability. So as long as it does not cause an exponential runtime blowup, it is never a problem to assume an upper bound for $\delta$, as we do now and in subsequent results.

**Lemma 3.6.** Given $\delta \in (0, 1/2)$, $\{\tilde{T}_i(r^j)\}_{r \in S, j \leq i}$ and $\{\tilde{N}(r^j)\}_{r \in S, j \leq i}$, suppose that the procedure EstimatePartition$(t, s^i, \{\tilde{T}_i(r^j)\}_{r \in S, j \leq i}, \{\tilde{N}(r^j)\}_{r \in S, j \leq i}, \epsilon, \delta)$ produces an estimate $\tilde{N}(s^i, t)$ with $\tilde{N}(s^i, t) = (1 \pm (4nm)^{17} \epsilon)N(s^i, t)$ for every partial tree $t$ of size $i$ and state $s^i$.

Suppose that Property 1 holds for all $j \leq i$ (see page 15). Moreover, assume that $n \geq 2$. Then conditioned on not outputting FAIL, each call to the procedure Sample$(s^i, \{\tilde{T}_i(r^j)\}_{r \in S, j < i}, \{\tilde{N}(r^j)\}_{r \in S, j < i}, \epsilon, \delta)$ produces an independent, uniform sample $t \sim T(s^i)$. Moreover, the probability that a given call to Sample$(s^i, \{\tilde{T}_i(r^j)\}_{r \in S, j < i}, \{\tilde{N}(r^j)\}_{r \in S, j < i}, \epsilon, \delta)$ outputs FAIL is at most $3/4$, and the number of times EstimatePartition is called in each iteration of the loop is at most $nm$.

**Proof.** Fix a tree $t \in T(s^i)$. Then there is a unique sequence of partial trees $i = t_0, t_1, t_2, \ldots, t_i = t$ such that $T(s^i) = T(s^i, t_0) \supseteq T(s^i, t_1) \supseteq T(s^i, t_2) \supseteq \cdots \supseteq T(s^i, t_i) = \{t\}$, which gives a sequence of nested partitions which could have been considered in the call Sample$(s^i, \{\tilde{T}_i(r^j)\}_{r \in S, j < i}, \{\tilde{N}(r^j)\}_{r \in S, j < i}, \epsilon, \delta)$. For $j \in [i]$, let $p_j$ be the true ratio of $\frac{N(s^i, t_j)}{N(s^i, t_{j-1})}$, which is the probability that we should have chosen partition $T(s^i, t_j)$ conditioned on being in partition $T(s^i, t_{j-1})$. Note that $\prod_{j=1}^{i} p_j = \frac{1}{N(s^i)} = \frac{1}{N(s^i)}$. Now assuming EstimatePartition always returns an estimate with at most $(1 \pm (4nm)^{17} \epsilon)$-relative error, it follows that conditioned on being in partition $T(s^i, t_{j-1})$, we chose the partition $T(s^i, t_j)$ with probability $\tilde{p}_j = (1 \pm (4nm)^{17} \epsilon)p_j$. Thus the probability that we choose $t$ at the end of the loop in step 2 of the Sample procedure is:

\[
\varphi = \prod_{j=1}^{i} \tilde{p}_j = (1 \pm (4nm)^{17} \epsilon)^i \prod_{j=1}^{i} p_j = (1 \pm 2/n) \prod_{j=1}^{i} p_j = (1 \pm 2/n) \frac{1}{N(s^i)} = (1 \pm 2/n) \frac{1}{N(s^i)}.
\]

Notice that we use the fact that $\epsilon < (4nm)^{-18}$ and that Property 1 holds for all $j \leq i$. The probability that we do not output FAIL can be bounded by

\[
\frac{1}{2\varphi N(s^i)} = \frac{1}{2N(s^i)} \prod_{j=1}^{i} \frac{1}{\tilde{p}_j} \geq \frac{1}{2N(s^i)(1 + 2/n)} \geq \frac{1}{2(1 + 2/n)} \geq 1/4
\]

since $n \geq 2$, which completes the proof that the probability that the call Sample$(s^i, \{\tilde{T}_i(r^j)\}_{r \in S, j < i}, \{\tilde{N}(r^j)\}_{r \in S, j < i}, \epsilon, \delta)$ outputs FAIL is at most $3/4$. For the uniformity claim, note that we accept $t$ at the end with probability $\varphi \cdot \frac{1}{2\varphi N(s^i)} = \frac{1}{2N(s^i)}$, which is indeed uniform conditioned on not outputting FAIL, as it does not depend on $t$. Finally, notice that EstimatePartition is called at most $(t(u) - 2) \cdot |\Sigma| \leq nm$ times in each iteration of the loop. 

\[
\boxed{
}
\]
Fully polynomial-time approximation schemata for \#BTA and \#TA. We show in Algorithm 2 a fully polynomial-time approximation schema for \#BTA, which puts together the different components mentioned in this section. The correctness of this algorithm is shown in the following theorem. Notice that the algorithm receives a parameter $\delta$, which controls the success probability. Making $\delta = 1/4$ would constitute an FPRAS as defined in Section 2.

Algorithm 2: FprasBTA($T, 0^n, \epsilon, \delta$)

1. Set $m \leftarrow |T|$
2. if $n < 2$ or $m < 3$ then
   3. Edge case, $|L_n(T)|$ can be exactly computed (Remark 3.1)
   4. end
5. Construct the tree automaton $\overline{T}$
6. Set $\epsilon \leftarrow \min\{\epsilon, 1/(4mn)^{18} - 1\}$
7. Set $\gamma = \log(1/\delta) + 2n$
8. Set $\alpha \leftarrow O(\log^2(1/\delta)(nm)^{13}/\epsilon^5)$, $\text{total} \leftarrow O(\alpha)$
9. For each $s \in S$, compute $N(s^1)$ exactly and set $\tilde{N}(s^1) \leftarrow N(s^1)$
10. For each $s \in S$, create set $\tilde{T}_2(s^1)$ with $\alpha$ uniform, independent samples from $T(s^1)$
11. for $i = 2, \ldots, n$ do
12.   For each $s \in S$, compute $\tilde{N}(s^i)$ such that $\Pr[\tilde{N}(s^i) = (1 \pm i\epsilon)N(s^i)] \geq 1 - \exp(-\gamma n^{20})$
13.   if $i < n$ then
14.      for each $s \in S$ and $j = 1, \ldots, i$ do
15.         Set $\tilde{T}_{i+1}(s^j) \leftarrow \emptyset$, $\text{counter} \leftarrow 1$
16.         while $|\tilde{T}_{i+1}(s^j)| < \alpha$ and $\text{counter} \leq \text{total}$ do
17.            Call the procedure SAMPLE($s^j, \{\tilde{T}_i(r^k)\}_{r \in S, k < j}, \{\tilde{N}(s^k)\}_{r \in S, k \leq j}, \epsilon, 2^{-2n}\delta)$
18.            If this procedure returns a tree $t$, then set $\tilde{T}_{i+1}(s^j) \leftarrow \tilde{T}_{i+1}(s^j) \cup \{t\}$
19.         end
20.      end
21.      if $|\tilde{T}_{i+1}(s^j)| < \alpha$ then
22.         return FAIL
23.      end
24.   end
25. end
26. end
27. return $\tilde{N}(s^{|S|\text{init}})$.

Theorem 3.7. Let $\epsilon, \delta \in (0, 1/2)$, $n \geq 1$, $T = (S, \Sigma, \Delta, s_{\text{init}})$ be a tree automaton, and $m = \|\Delta\|$ be the size of $T$. Then the call FprasBTA($T, 0^n, \epsilon, \delta$)\(^{10}\) returns, with probability at least $1 - \delta$, where $\delta$ holds.

\(^{10}\)Here we write $0^n$ as the unary representation of $n$. Since our algorithms are polynomial in $n$, the algorithm is
a value $\widetilde{N}$ such that $\widetilde{N} = (1 \pm \epsilon)|L_n(T)|$. Moreover, the runtime of the algorithm FPRASBTA is \(\text{poly}(n, m, 1/\epsilon, \log(1/\delta))\).

**Proof.** Set \(\alpha = O(\log^2(1/\delta)(nm)^{13}/\delta^2)\). For every \(j \in [n]\), let \(E_j^1\) denote the event that Property 1 holds for level \(j\), and similarly define \(E_j^2\) for Property 2. Set \(\gamma = \log(1/\delta)\). We prove inductively that

\[
\Pr\left[\bigwedge_{j \leq i} (E_j^1 \land E_j^2)\right] \geq 1 - 2^{-\gamma + 2i}
\]

for each \(i \in [n]\). Since \(N(s^1)\) is computed exactly in step 9 of FPRASBTA \((T, 0^n, \epsilon, \delta)\) and the size of each tree in \(T(s^1)\) is 1, the base case \(i = 1\) trivially holds. Now at an arbitrary step \(i \geq 2\), suppose \(E_j^1 \land E_j^2\) holds for all \(j < i\). By considering \(\exp(-\gamma n^{20})\) as the value for the parameter \(\delta\) in Proposition 3.3, it follows that \(E_i^1\) holds with probability at least \(1 - \exp(-\gamma n^{20})\), and the runtime to obtain Property 1 is \(\text{poly}(n, m, 1/\epsilon, \gamma)\). Thus,

\[
\Pr\left[E_i^1 \mid \bigwedge_{j < i} (E_j^1 \land E_j^2)\right] \geq 1 - \exp(-\gamma n^{20}).
\]

We now must show that \(E_i^2\) holds – namely, that we can obtain uniform samples from all sets \(T(s^i)\). By Lemma 3.5, if we can obtain fresh uniform sample sets \(\widehat{T}_i(s^j)\) of \(T(s^i)\) for each \(s \in S\) and \(j < i\), each of size \(\alpha\), then with probability at least \(1 - 2^{-\gamma nm}\), we have that for every partial tree \(t'\) of size \(i\) (that is, \(\text{fsize}(t') = i\)) and state \(s \in S\), the procedure \textsc{EstimatePartition}(\(t', s^i\), \(\{\widehat{T}_i(s^j)\}_{s \in S, j < i}\), \(\{\widetilde{N}(s^j)\}_{s \in S, j < i}\), \(\epsilon, \delta\)) produces an estimate \(\widetilde{N}(s^i, t')\) such that \(\widetilde{N}(s^i, t') = (1 \pm (4nm)^{17} \epsilon)N(s^i, t')\). Since we have to call \textsc{EstimatePartition} at most \(\gamma nm\) times (see Lemma 3.6), after a union bound we get that the conditions of Theorem 3.6 are satisfied with probability at least \(1 - 2^{-\gamma}\), and it follows that we can sample uniformly from the set \(T(s^i)\) for each \(s \in S\) in polynomial time.

It remains to show that we can obtain these fresh sample sets \(\widehat{T}_i(s^j)\) of \(T(s^i)\) for each \(s \in S\) and \(j < i\) in order to condition on the above. But the event \(E_{i-1}^2\) states precisely that can indeed obtain such samples in \(\text{poly}(n, m, 1/\epsilon, \gamma)\) time per sample. Thus the conditions of the above paragraph are satisfied, so we have

\[
\Pr\left[E_i^2 \mid E_i^1 \land \bigwedge_{j < i} (E_j^1 \land E_j^2)\right] \geq 1 - 2^{-\gamma}.
\]

Therefore, we conclude that

\[
\Pr\left[E_i^1 \land E_i^2 \mid \bigwedge_{j < i} (E_j^1 \land E_j^2)\right] \geq 1 - 2^{-\gamma} - \exp(-\gamma n^{20}) + 2^{-\gamma} \exp(-\gamma n^{20}) \geq 1 - 2^{-\gamma + 1}
\]

Hence, by induction hypothesis:

\[
\Pr\left[\bigwedge_{j \leq i} (E_j^1 \land E_j^2)\right] = \Pr\left[E_i^1 \land E_i^2 \mid \bigwedge_{j < i} (E_j^1 \land E_j^2)\right] \cdot \Pr\left[\bigwedge_{j < i} (E_j^1 \land E_j^2)\right]
\]

\[
\geq (1 - 2^{-\gamma + 1})(1 - 2^{-\gamma + 2(i-1)})
\]

\[
= 1 - 2^{-\gamma + 1} - 2^{-\gamma + 2i - 2} + 2^{-2\gamma + 2i - 1}
\]

\[
\geq 1 - 2^{-\gamma + 1} - 2^{-\gamma + 2i - 1}
\]

polyomial in the size of the input.
\[ \geq 1 - 2^{-\gamma + 2i - 1} - 2^{-\gamma + 2i - 1} \\
= 1 - 2^{-\gamma + 2i} \]

which completes the inductive proof. Redefining \( \gamma = \log(1/\delta) + 2n \) (see Line 7 of Algorithm 17) and considering \( 2^{-2n} \delta \) when using Lemma 3.5 (see Line 17 of Algorithm 17), we obtain that the success probability of the overall algorithm is \( 1 - \delta \) as needed.

For runtime, note that by Lemma 3.6, the expected number of trials to obtain \( \alpha \) samples \( T_i(s^j) \) for each \( s^j \in \mathcal{F} \) and \( i \in [n] \) is \( O(\alpha) \), and thus is \( O(\alpha) \) with probability \( 1 - 2^{-\alpha} > 1 - 2^{-mn\gamma} \) by Chernoff bounds. That is, with \( O(\alpha) \) trials, we have probability at least \( 1 - 2^{-mn\gamma} \) of not failing in step 22 of Algorithm 2. Since we go through that step at most \( O(n^2 m) \) times during the whole run of the algorithm, that means that the overall probability of returning FAIL can be bounded by \( 1 - 2^\gamma = 1 - \delta \), which is a loose bound but enough for our purposes. Moreover, by Lemma 3.6, the runtime of each sampling trial in step 17 of Algorithm 2 is polynomial in \( n, m, 1/\epsilon \) and \( \log(1/(2^{-2n} \delta)) = \gamma \). It follows that the entire algorithm runs in \( \text{poly}(n, m, 1/\epsilon, \log(1/\delta)) \) time, which completes the proof. \( \blacksquare \)

**Theorem 3.8.** Let \( \delta \in (0, 1/2) \), \( n \geq 1 \), \( \mathcal{T} = (S, \Sigma, \Delta, s_{\text{init}}) \) be a tree automaton, and \( m = \|\Delta\| \) be the size of \( \mathcal{T} \). Then there is a sampling algorithm \( \mathcal{A} \) and a pre-processing step with the following property. The pre-processing step runs in \( \text{poly}(n, m, \log(1/\delta)) \) time, and with probability \( 1 - \delta \) over the randomness used in this pre-processing step,\(^{11} \) each call to the algorithm \( \mathcal{A} \) runs in expected \( \text{poly}(n, m, \log(1/\delta)) \) time,\(^{12} \) and returns either a uniform samples \( t \sim \mathcal{L}_n(\mathcal{T}) \) or FAIL. Moreover, the probability that the sampler returns FAIL is at most 3/4.

The above implies that there is a sampler that always runs in time \( \text{poly}(n, m, \log(1/\delta)) \) time, and outputs samples \( t \sim_D \mathcal{L}_n(\mathcal{T}) \) from a distribution \( \mathcal{D} \) over \( \mathcal{L}_n(\mathcal{T}) \) such that

\[ \mathcal{D}(t) = \frac{(1 \pm \delta)}{|\mathcal{L}_n(\mathcal{T})|} \]

for all \( t \in \mathcal{L}_n(\mathcal{T}) \).

**Proof.** The pre-processing step here is just the computation of the estimates \( \tilde{N}(s^i) \) and sketches \( \tilde{T}(s^i) \) for all \( i \leq n \), which can be thought of as a call to Theorem 3.7 using a fixed \( \epsilon = 1/\text{poly}(n, m) \) small enough. Then the first result follows from Lemma 3.6 and Theorem 3.7. In particular, if we condition on the success of Theorem 3.7, which hold with probability \( 1 - \delta \), then by the definition of Property 2, and the fact that Property 2 holds for the size \( n \) conditioned on Theorem 3.7, this is sufficient to guarantee that the samples produced by our sampling procedure are uniform.

For the second claim, note that the probability \( \delta \) of failure induces an additive \( \delta \) difference in total variational distance from the uniform sampler. We can then run the algorithm with \( \delta_0 = \delta|\mathcal{L}_n(\mathcal{T})|^{-1} = \delta \exp(-\text{poly}(n, m)) \), which does not affect the stated polynomial runtime. This results in

\[ \mathcal{D}(t) = \frac{1}{|\mathcal{L}_n(\mathcal{T})|} \pm \delta_0 \]
\[ = \frac{1}{|\mathcal{L}_n(\mathcal{T})|} \pm \delta|\mathcal{L}_n(\mathcal{T})|^{-1} \]
\[ = \frac{(1 \pm \delta)}{|\mathcal{L}_n(\mathcal{T})|} \]

\(^{11}\)Note that we cannot detect if the event within the preprocessing step that we condition on here fails, which occurs with probability \( \delta \).

\(^{12}\)Note that the runtime has exponential tails: the probability that the runtime is a factor of \( K \) larger than the expectation is \( e^{-\Theta(K)} \).
as desired. To deal with the 3/4 probability that the output of our sampler (as defined by property 2) is FAIL, we can run the sampler for a total of \(\Theta(\log(1/\delta)|L_n(T)|) = \text{poly}(n, m, \log(1/\delta))\) trials, and return the first sample obtained from an instance that did not return FAIL. If all trails output FAIL, then we can also output FAIL. By doing so, this causes another additive \(\delta|L_n(T)|^{-1}\) error in the sampler, which is dealt with in the same way as shown in Equation (4) above.

We conclude this section by pointing out that from Theorem 3.7 and the existence of a polynomial-time parsimonious reduction from \#TA to \#BTA, we obtain the following corollary:

**Corollary 3.9.** Both \#BTA and \#TA admit fully polynomial-time randomized approximation schema.

### 4 Estimating the Partition Sizes

The goal of this section is to prove Lemma 3.5, namely, to show how to implement the procedure \textsc{EstimatePartition}. To this end, we show next how \textsc{EstimatePartition} can be implemented by reducing it to a problem about counting words accepted by succinct NFAs, which we introduced in Subsection 1.2 and now formally define.

**Succinct NFAs.** Let \(\Gamma\) be a finite set of labels. A succinct NFA over \(\Gamma\) is a 5-tuple \(N = (S, \Gamma, \Delta, s_{\text{init}}, s_{\text{final}})\) where \(S\) is the set of states and each transition is labeled by a subset of \(\Gamma\), namely \(\Delta \subseteq S \times 2^\Gamma \times S\). Thus each transition is of the form \((s, A, s')\), where \(A \subseteq \Gamma\). For each transition \((s, A, s') \in \Delta\), the set \(A \subseteq \Gamma\) is given in some representation (e.g. a tree automaton, a DNF formula, or an explicit list of elements), and we write \(\|A\|\) to denote the size of the representation. Note that while the whole set \(A\) is a valid representation of itself, generally the number of elements of \(A\), denoted by \(|A|\), will be exponential in the size of the representation \(\|A\|\). We define the size of the succinct NFA \(N\) as \(|N| = |S| + |\Delta| + \sum_{(s, A, s') \in \Delta} |A|\). For notational simplicity, we will sometimes write \(r = |N|\).

Given a succinct NFA \(N\) as defined above and elements \(w_1, \ldots, w_n \in \Gamma\), we say that \(N\) accepts the word \(w_1 w_2 \ldots w_n\) if there exist states \(s_0, s_1, \ldots, s_n \in S\) and sets \(A_1, \ldots, A_n \subseteq \Gamma\) such that:

- \(s_0 = s_{\text{init}}\) and \(s_n = s_{\text{final}}\)
- \(w_i \in A_i\) for all \(i = 1 \ldots n\)
- \((s_{i-1}, A_i, s_i) \in \Delta\) for all \(i = 1 \ldots n\)

We denote by \(L_k(N)\) the set of all words of length \(k\) accepted by \(N\). We consider the following general counting problem:

| Problem: | \#SuccinctNFA |
|----------|----------------|
| Input:   | \(k \geq 1\) given in unary and a succinct NFA \(N\) |
| Output:  | \(|L_k(N)|\) |

**Reduction to Unrolled Succinct NFAs** Our algorithm for approximating \(|L_k(N)|\) first involves unrolling \(k\) times the NFA \(N = (S, \Gamma, \Delta, s_{\text{init}}, s_{\text{final}})\), to generate an unrolled NFA \(N^k_{\text{unroll}}\). Specifically, for every state \(p \in S\) create \(k - 1\) copies \(p^1, p^2, \ldots, p^{k-1}\) of \(p\), and include them as states of the unrolled NFA \(N^k_{\text{unroll}}\). Moreover, for every transition \((p, A, q)\) in \(\Delta\), create the edge \((p^\alpha, A, q^{\alpha+1})\) in \(N^k_{\text{unroll}}\) for every \(\alpha \in \{1, \ldots, k-2\}\). Finally, if \((s_{\text{init}}, A, q)\) is a transition in \(\Delta\), then
$(s_{\text{init}}, A, q^1)$ is a transition in $N_{\text{unroll}}^k$, while if $(p, A, s_{\text{final}})$ is a transition in $\Delta$, then $(p^{k-1}, A, s_{\text{final}})$ is a transition in $N_{\text{unroll}}^k$. In this way, we keep $s_{\text{init}}$ and $s_{\text{final}}$ as the initial and final states of $N_{\text{unroll}}^k$, respectively. Since $k$ is given in unary, it is easy to see that $N_{\text{unroll}}^k$ can be constructed in polynomial time from $N$. Thus, for the remainder of the section, we will assume that the input succinct NFA $N$ has been unrolled according to the value $k$. Thus, we consider the following problem.

**Problem:** $\#\text{UnrolledSuccinctNFA}$

**Input:** $k \geq 1$ given in unary and an unrolled succinct NFA $N_{\text{unroll}}^k$

**Output:** $|L_k(N_{\text{unroll}}^k)|$

Clearly in the general case, without any assumptions on our representation $\|A\|$ of $|A|$, it will be impossible to obtain polynomial in $|N|$ time algorithms for the problem above. In order to obtain polynomial time algorithms, we require the following four properties of the label sets $A$ to be satisfied. The properties state that the sizes $|A|$ are at most singly exponential in $|N|$, we can efficiently test whether an element $a \in \Gamma$ is a member of $A$, we can obtain approximations of $|A|$, and that we can generate almost uniform samples from $A$.

**Definition 4.1** (Required properties for a succinct NFA). Fix $\epsilon_0 > 0$. Then for every label set $A$ present in $\Delta$, we have:

1. **Size bound:** There is a polynomial $g(x)$ such that $|A| \leq 2^{g(|N|)}$.

2. **Membership:** There is an algorithm that given any $a \in \Gamma$, verifies in time $T = \text{poly}(|N|)$ whether $a \in A$.

3. **Size approximations:** We have an estimate $\tilde{N}(A) = (1 \pm \epsilon_0)|A|$.

4. **Almost uniform samples:** We have an oracle which returns independent samples $a \sim A$ from a distribution $D$ over $A$, such that for every $a \in A$:

   $$D(a) = (1 \pm \epsilon_0) \frac{1}{|A|}$$

The reason for the first condition is that our algorithms will be polynomial in $\log(N)$, where $N$ is an upper bound on the size of $|L_k(N)|$. We remark that for the purpose of our main algorithm, we actually have truly uniform samples from each set $A$ that is a label in a transition. However, our results may be applicable in other settings where this is not the case. In fact, along with the first two conditions from Definition 4.1, a sufficient condition for our algorithm to work is that the representations of each set $A$ allows for an FPRAS and a polynomial time almost uniform sampler.

**The Main Path of a Partial Tree** Next we show that if we can approximate the number of words of a given length accepted by a succinct NFA, then we can implement the procedure $\text{ESTIMATEPARTITION}$. But first we need to introduce the notion of main path of a partial tree. Let $t$ be a partial tree constructed via the partitioning procedure of Algorithm 1 (see page 19). Given that we always choose the hole with the minimal size in Line 3, one can order the holes of $t$ as $u_1, u_2, \ldots, u_k$, such that for each $i$, parent($u_i$) is an ancestor of $u_{i+1}$, namely, $u_{i+1}, \ldots, u_k$ are contained in the subtree rooted at the parent of $u_i$. Note that by definition of the loop of Algorithm 1, it could be the case that two holes $u$ and $v$ share the same parent (e.g. the last step produced a subtree of the form $a(i, j)$). If this is the case, we order $u$ and $v$ arbitrarily. Then we
define the main path $\pi$ of $t$ considering two cases. If no two holes share the same parent, then $\pi$ is the path $\text{parent}(u_1), \text{parent}(u_2), \ldots, \text{parent}(u_k)$ (from the most shallow node $u_1$ to the deepest node $u_k$). On the other hand, if two holes share the same parent, then by definitions of Algorithm 1 and sequence $u_1, \ldots, u_k$, these two nodes must be $u_{k-1}$ and $u_k$. In this case, we define $\pi$ as the path $\text{parent}(u_1), \text{parent}(u_2), \ldots, \text{parent}(u_{k-1}), u_k$, (again, from the most shallow node $u_1$ to the deepest node $u_k$).

13 Strictly speaking, $\text{parent}(u_1), \ldots, \text{parent}(u_k)$ is a sequence and is not necessarily a path in the tree, because there could be missing nodes between the elements of the sequence. However, for the purpose of the proof the missing nodes do not play any role and will be omitted.

Figure 4: Two examples of a partial tree. The holes are indicated by the letters $H$ and $H'$, while the nodes that are not holes have labels $a, b, c \in \Sigma$. Non-white nodes and thick arcs are used to highlight the main paths.

We illustrate the notion of main path in Figure 4. For the partial tree in the left-hand side, we have that the main path is $\text{parent}(H_1), \text{parent}(H_2), \text{parent}(H_3), \text{parent}(H_4)$ as no two holes share the same parent. On the other hand, the main path for the partial tree in the right-hand side is $\text{parent}(H'_1), \text{parent}(H'_2), \text{parent}(H'_3), H'_4$, as in this case holes $H'_3$ and $H'_4$ share the same parent.

Lemma 4.2. There exists a polynomial-time algorithm that, given a tree automaton $T$, a partial tree $t$ with $k$ holes constructed via the partitioning procedure of Algorithm 1 with $i = \text{fsize}(t)$ and state $s$ of $T$, returns a succinct NFA $N$ such that

$$|T(s^i, t)| = |\mathcal{L}_k(N)|.$$

Moreover, $|N| \leq 3(im)^4$, where $m$ is the size of $T$.

Proof. Let $u_1, \ldots, u_k$ be the holes of $t$. Assume first that no two holes of $t$ share the same parent, so that $\pi = p_1, p_2, \ldots, p_k$ is the main path of $t$ with $p_i = \text{parent}(u_i)$. Counting the number of elements of $T(s^i, t)$ is the same as counting all sequences of trees $t_1, \ldots, t_k$ over $\Sigma$ such that there exists a run $\rho$ of $T$ over the tree $t[u_1 \rightarrow t_1] \cdots [u_k \rightarrow t_k]$ with $\rho(\lambda) = s^i$. In other words, we hang $t_1$ on $u_1$, $\ldots$, $t_k$ on $u_k$ to form a tree that is accepted by $T$ when $s^i$ is the initial state. Then the plan of the reduction is to produce a succinct NFA $N$ such that all words accepted by $N$ are of the form $t_1 \cdots t_k$ with $t[u_1 \rightarrow t_1] \cdots [u_k \rightarrow t_k] \in T(s^i, t)$.

13 Strictly speaking, $\text{parent}(u_1), \ldots, \text{parent}(u_k)$ is a sequence and is not necessarily a path in the tree, because there could be missing nodes between the elements of the sequence. However, for the purpose of the proof the missing nodes do not play any role and will be omitted.
For the construction of \( \mathcal{N} \) it will be useful to consider the following extension of \( \mathcal{T} \) over partial trees. Recall the definition of \( \mathcal{T} = (\mathcal{S}, \Sigma, \Delta, s_{\text{init}}) \) from Section 3, but assuming here that the unfolding is done for \( i \) levels. Then define \( \mathcal{T}' = (\mathcal{S}, \Sigma \cup \{i\}, \Delta^*, s_{\text{init}}^i) \) such that \( \Delta^* = \Delta \cup \{(s^j, j, \lambda) \mid s^j \in \mathcal{S}^j\} \), namely we add to \( \mathcal{T} \) special transitions over holes when the level \( j \) of \( s^j \) coincides with the value of the hole. Intuitively, if we have a run \( \rho \) of \( \mathcal{T}' \) over \( t \) with \( \rho(\lambda) = s^j \) and \( T(\rho(u)) \neq \emptyset \) for every \( \ell \in [k] \), then \( t \) can be completed with trees \( t_1 \in T(\rho(u_1)), \ldots, t_k \in T(\rho(u_k)) \) such that \( t[u_1 \rightarrow t_1] \cdots [u_k \rightarrow t_k] \in T(s^j, t) \).

Let \( i_1, \ldots, i_k \) be the sizes \( t(u_1), \ldots, t(u_k) \) on the holes \( u_1, \ldots, u_k \), respectively. Furthermore, let \( j_1, \ldots, j_k \) be the final sizes of the subtrees of \( t \) hanging from nodes \( p_1, \ldots, p_k \), respectively. That is, if \( t_1 \) is the subtree hanging from \( p_1 \) in \( t \), then \( j_1 = \text{size}(t_1) \), and so on. Note that by the definition of the main path \( \pi \), we have that \( j_1 > j_2 > \ldots > j_k \) (since each \( p_i \) is the parent of \( p_{i+1} \)). We now have the ingredients to define the succinct NFA \( \mathcal{N} = (\mathcal{S}_N, \Gamma, \Delta_N, s_0, s_e) \). The set \( S_N \) of states will be a subset of the states of \( \mathcal{S} \), plus two additional states \( s_0 \) and \( s_e \), formally, \( S_N = \bigcup_{i=1}^{k} \{q^i \in \mathcal{S} \mid q \in S\} \cup \{s_0, s_e\} \). The set \( \Delta_N \) of transitions is defined as follows: for every states \( q_1, q_2, r \in S \) and \( \ell \in [k-1] \), we add a transition \( (q_1^\ell, T(r^\ell), q_2^{\ell+1}) \in \Delta_N \) if there exists a run \( \rho \) of \( \mathcal{T}' \) over \( t \) such that \( \rho(p_\ell) = q_1^{\ell}, \rho(p_{\ell+1}) = q_2^{\ell+1} \), and \( \rho(u_\ell) = r^\ell \). Moreover, assuming that \& is a fresh symbol, we add transition \( (u_0, \{\&\}, q^1) \) to \( \Delta_N \) if there exists a run \( \rho \) of \( \mathcal{T}' \) over \( t \) such that \( \rho(p_1) = q^1 \), and \( \rho(\lambda) = s^1 \). Finally, we add transition \( (q^k, T(r^k), u_e) \) to \( \Delta_N \).

Note that all transitions in the succinct NFA are directed from level \( j_{\ell-1} \) to level \( j_\ell \) with \( j_{\ell-1} > j_\ell \), for some \( \ell \in \{1, 2, \ldots, k-1\} \), which implies that \( \mathcal{N} \) is unrolled. Here, the level \( j_\ell \) is defined as the set of states \( \{q^i \in \mathcal{S} \mid q \in S\} \). Furthermore, note that transitions are labeled by sets \( T(r^\ell) \) where \( i_\ell < i \), which are represented by tree automaton \( \mathcal{T}[r^\ell] \) for \( i_\ell < i \). Thus, the conditions required by Definition 4.1 are satisfied since for each transition label \( T(r^\ell) \), it holds that \( |T(r^\ell)| \) is at most exponential in the size of \( \mathcal{T}[r^\ell] \), and by Algorithm 1, we have already precomputed values such that we can check membership, approximate its size, and obtain an almost uniform sample from \( T(r^\ell) \). Finally, the existence of the run \( \rho \) for the definition of each transition in \( \Delta_N \) can be checked in polynomial time in the size of \( t \) \cite{CDG+07} and, thus, \( \mathcal{N} \) can be constructed from \( t \) and \( \mathcal{T} \) in polynomial time.

It’s only left to show that \( |L_k(\mathcal{N})| = |T(s^j, t)| \). For this, note that every word accepted by \( \mathcal{N} \) is of length \( k+1 \) and of the form \&t_1t_2 \cdots t_k. Then consider the function that maps words \&t_1t_2 \cdots t_k to the tree \( t[u_1 \rightarrow t_1] \cdots [u_k \rightarrow t_k] \). One can show that each such a tree is in \( T(s^j, t) \), and then the function goes from \( L_k(\mathcal{N}) \) to \( T(s^j, t) \). Furthermore, the function is a bijection. Clearly, if we take two different words, we will produce different trees in \( T(s^j, t) \), and then the function is injective. To show that the function is surjective, from a tree \( t' \in T(s^j, t) \) and a run \( \rho \) of \( \mathcal{T} \) over \( t' \), we can build the word \&t_1t_2 \cdots t_k where each \( t_i \) is the subtree hanging from the node \( u_i \) in \( t' \). Also, this word is realized by the following sequence of transitions in \( \mathcal{N} \):

\[
(s_0, \{\&\}, \rho(p_1)), (\rho(p_1), T(\rho(u_1)), \rho(p_2)), \ldots, (\rho(p_{k-1}), T(\rho(u_{k-1})), \rho(p_k)), (\rho(p_k), T(\rho(u_k)), s_e).
\]

Thus, the function is surjective. Hence, from the existence of a bijection from \( L_k(\mathcal{N}) \) to \( T(s^j, t) \), we conclude that \( |L_k(\mathcal{N})| = |T(s^j, t)| \).

Recall that the size of succinct NFA \( \mathcal{N} \) is defined as \( |\mathcal{N}| = |S_N| + |\Delta_N| + \sum_{(s, A, s') \in \Delta_N} \|A\| \). Thus, given that \( |S_N| = im \), each set label \( A = T(\rho(u_i)) \) is represented by the tree automaton \( \mathcal{T}[\rho(u_i)] \) and the size of \( \mathcal{T}[\rho(u_i)] \) is bounded by \( (im)^2 \), we conclude that \( |\Delta_N| \leq (im)^3 \) and \( \sum_{(s, A, s') \in \Delta_N} \|A\| \leq (im)^4 \). Putting everything together, we conclude that \( |\mathcal{N}| \leq 3(im)^4 \), which was to be shown.

To finish with the proof, we need to consider the sequence \( u_1, \ldots, u_k \) of holes of \( t \), and assume that two holes of \( t \) share the same parent, so that \( \pi = p_1, p_2, \ldots, p_k \) is the main path of \( t \), with
$p_i = \text{parent}(u_i)$ for each $i \in [k-1]$ and $p_k = u_k$. The proof for this case can be done in a completely analogous way.

In the following theorem, we show how to estimate $|L_k(N)|$ for a given unrolled succinct NFA $N$ and integer $k \geq 1$ given in unary (recall the definition of unrolled succinct NFA from the beginning of this section).

**Theorem 4.3.** Let $N$ be an unrolled succinct NFA, $k \geq 1 \in (100|N|^4\epsilon_0, 1)$, where $\epsilon_0$ is as in Definition 4.1. Moreover, fix $\delta \in (0, 1/2)$ and assume that $N$ satisfies that $|L_k(N)| \leq N$. Then there exists an algorithm that with probability at least $1 - \delta$ outputs a value $\tilde{N}$ such that $\tilde{N} = (1 \pm \epsilon)|L_k(N)|$. The algorithm runs in time

$$O\left(T \cdot \frac{\log(N/\epsilon) \log^2(1/\delta)|N|^{18}}{\epsilon^4}\right),$$

where $T$ is as in Definition 4.1, and makes at most

$$O\left( \frac{\log^2(1/\delta) \log(N/\epsilon)|N|^{18}}{\epsilon^4} \right)$$

queries to the sampling oracle. Furthermore, there is an almost uniform sampler which returns elements of $L_k(N)$ such that

$$\Pr[\text{outputs } \pi] = (1 \pm \epsilon) \frac{1}{|L_k(N)|}$$

for every $\pi \in L_k(N)$, and has the same runtime and oracle complexity as above.

As a corollary based on the reduction described earlier, we obtain the following.

**Corollary 4.4.** Let $N$ be a succinct NFA, $k \geq 1$, $u, v \in V$, $\epsilon \in (100((k+1)|G|)^4\epsilon_0, 1)$, where $\epsilon_0$ is as in Definition 4.1. Moreover, fix $\delta \in (0, 1/2)$ and assume that $N$ satisfies that $|L_k(N)| \leq N$. Then there exists an algorithm that, with probability at least $1 - \delta$ outputs a value $\tilde{N}$ such that $\tilde{N} = (1 \pm \epsilon)|L_k(N)| \leq N$. The algorithm runs in time

$$O\left(T \cdot \frac{\log(N/\epsilon) \log^2(1/\delta)(k|N|)^{18}}{\epsilon^4}\right),$$

where $T$ is as in Definition 4.1, and makes at most

$$O\left( \frac{\log^2(1/\delta) \log(N/\epsilon)(k|N|)^{18}}{\epsilon^4} \right)$$

queries to the sampling oracle. Furthermore, there is an almost uniform sampler which returns elements of $L_k(N)$ such that

$$\Pr[\text{outputs } \pi] = (1 \pm \epsilon) \frac{1}{|L_k(N)|}$$

for every $\pi \in L_k(N)$, and has the same runtime and oracle complexity as above.

**Proof.** The result follows from Theorem 4.3, as well as the reduction described earlier from arbitrary succinct NFAs to unrolled succinct NFAs. Notice that in this reduction the size of $N$ increases by a factor of $O(k)$, which completes the proof. ■
Using Theorem 4.3, we can prove Lemma 3.5.

Proof of Lemma 3.5. Recall that we assume given a tree automaton \( \mathcal{T} = (S, \Sigma, \Delta, s_{\text{init}}) \) over binary trees, a natural number \( n \geq 1 \) given in unary, the relative error \( \epsilon \in (0, 1) \) and a value \( \delta \in (0, 1/2) \). Moreover, we assume that \( m \geq 3 \) is the size of \( \mathcal{T} \), which we define as \( m = \| \Delta \| \).

Let \( s \in S \) and \( t \) be a partial tree such that \( \text{fsize}(t) = i \). By using Lemma 4.2, we can construct in polynomial-time a succinct NFA \( \mathcal{N} \) such that \( N(s', t) = |T(s', t)| = |\mathcal{L}(\Lambda)\). Moreover, by the reduction of Lemma 4.2, each label set \( A \) of a transition in \( \mathcal{N} \) is of the form \( A = T(s^j) \) for some state \( s \) and some \( j < i \) in the graph. Thus, if we assume \( \epsilon_0 = \epsilon 4nm \), then this gives us \( \hat{N}(s^j) = (1 \pm \epsilon_0)N(s^j) \) as \( \mathcal{N} \) is required to satisfy the properties of Definition 4.1. Moreover, define \( \epsilon_1 = \epsilon_0 (4nm)^{16} \), where \( \epsilon_1 \) is the precision parameter from Theorem 4.3. Then we have that \( \epsilon_1 > 300(nm)^{16} \epsilon_0 \geq 100|\mathcal{N}|^{4} \epsilon_0 \) as required by Theorem 4.3, where here we used the fact that \(|\mathcal{N}| \leq 3(nm)^4 \) in the reduction of Lemma 4.2. Moreover, \( \epsilon_1 = (4nm)^{17} \epsilon < 1 \), as also required by Theorem 4.3, since we assume that \( \epsilon < 1/(4nm)^{18} \). Finally, we also set \( \delta_0 = \delta(nm)^3 \) to be the failure probability as in Theorem 4.3. Thus, by Theorem 4.3, using at most \( O(\log^2(1/\delta_0) \log(N/\epsilon_1)(nm)^{4-18}/\epsilon_1^4) \) samples, we obtain a \((1 \pm \epsilon_1)\)-estimate of the size of the number of labeled paths with probability \( 1 - \delta_0 = 1 - \delta(nm)^3 \). By Lemma 4.2, we therefore obtain the same estimate of the partition \( t \), for a given partial tree \( t \).

We now bound the number of ordered, rooted, labeled trees of size \( n \). By Cayley’s formula, we can bound the number of unlabeled, unordered, undirected trees by \( n^{n-2} \). The number of rooted, unordered, undirected, unlabeled trees can then be bounded by \( n^{n-1} \). For each tree, every vertex has \( |\Sigma| \leq m \) choices of a labeling, thus there are \( m^n n^{n-1} < (nm)^{nm} \) labeled unordered, undirected rooted trees (recall that \( m \geq 3 \)). Finally, for each such a tree, we can bound the number of ways to transform it into an ordered and directed tree by \( (2n)! \), which gives a bound of \( (2n)! \cdot (nm)^{nm} \leq (nm)^{(nm)^2} \) (recall again that \( m \geq 3 \)). Note that this also implies that \( N \leq (nm)^{(nm)^2} \), which gives a total sample complexity bound of \( O(\log^2(1/\delta)(nm)^{13}/\epsilon^5) \). Observe that the number of partial trees \( t \) such that \( \text{fsize}(t) = i \) is bounded by \( n(nm)^{(nm)^2} \). In particular, this bound is obtained by considering that \( m \geq 3 \) and the fact that the number of labels for partial trees is at most \( |\Sigma| + n \leq m + n \). Now by a union bound and considering that \( \delta < 1/2 \), with probability

\[
1 - (nm)^{(nm)^2} \delta(nm)^3 \geq 1 - \delta(nm)^3 - ((nm)^2 + 1) \log(nm) \geq 1 - \delta(nm)^3,
\]

EstimatePartition\( (t, s^j, \{ \hat{T}_i(s^j) \}_{s \in S, j \leq i}, \{ \hat{N}(s^j) \}_{s \in S, j \leq i}, \epsilon, \delta) \) returns a \((1 \pm \epsilon_1) = (1 \pm (4nm)^{17}\epsilon)\) estimate for all trees \( t \) such that \( \text{fsize}(t) = i \) and for all states \( s^j \) such that \( s \in S \). Finally, note that the runtime is \( \text{poly}(n, m, 1/\epsilon, \log(1/\delta)) \) since it is bounded by a polynomial in the sample complexity, which is polynomial in \( n, m, 1/\epsilon \) and \( \log(1/\delta) \) by Lemma 4.7 and Theorem 4.3. \( \blacksquare \)

4.1 Approximate Counting of Accepted Words in succinct NFAs

The goal of this section is to prove Theorem 4.3. In what follows, fix a succinct NFA \( \mathcal{N} = (S, \Gamma, \Delta, s_{\text{init}}, s_{\text{final}}) \) over a finite set of labels \( \Gamma \) and recall that the label sets of \( \Delta \) have to satisfy the conditions of Definition 4.1. Besides, assume that \( \mathcal{N} \) is unrolled, and recall the definition of unrolled succinct NFA from the beginning of this section. Without loss of generality, assume that \( S \) only contains states which lie on a path from \( s_{\text{init}} \) to \( s_{\text{final}} \). Furthermore, let \( s_0, \ldots, s_n \) be a topological order of the states in \( S \) such that \( s_0 = s_{\text{init}} \) and \( s_n = s_{\text{final}} \), where \(|S| = n + 1 \). In other words, every path from \( s_0 \) to \( s_n \) can be written in the form \( s_0, s_{i_1}, s_{i_2}, \ldots, s_n \), where \( 1 \leq i_1 \leq i_2 \leq \cdots \leq n \). Finally, for brevity, we write \( r = |\mathcal{N}| \).

For the sake of presentation, for every state \( s_i \), let \( W(s_i) = \mathcal{L}(\mathcal{N}_{s_i}) \) and \( N(s_i) = \mathcal{L}(\mathcal{N}_{s_i}) \), where \( \mathcal{N}_{s_i} \) is an exact copy of \( \mathcal{N} \) only with the final state changed to \( s_i \). Then our goal is to estimate
is unrolled. It is now straightforward to check that $N(s_i) \leq N$ for each node $s_i$, for some $N \leq 2^{\text{poly}(r)}$. Set $\gamma = \log(1/\delta)$. Finally, since the FPRAS must run in time $\text{poly}(r, 1/\epsilon)$, we can assume $\epsilon < 1/(300r)$ without loss of generality.

We first observe that membership in $W(s_i)$ is polynomial-time testable given polynomial-time membership tests for each label $A$.

**Proposition 4.5.** Suppose that given a sequence $a_1 \ldots a_t \in \Gamma^t$, we can test in time $T$ whether $a_i \in A$ for each transition label $A$ ($T$ is the membership time in Definition 4.1). Then given any state $s_j$, we can test whether $a_1 \ldots a_t \in W(s_j)$ in time $O(|\Delta|T)$

**Proof.** We can first remove all transitions not contained in a run of length exactly $t$ from $s$ to $s_j$ in time $O(|\Delta|)$ by a BFS. Then for each transition $e = (s', A, s'')$ remaining which is on the $i$-th step from $s$ to $s_j$, with $i \leq t$, we keep $e$ if and only if $a_i \in A$. Note that $i$ is unique for $e$ as $\mathcal{N}$ is unrolled. It is now straightforward to check that $s_j$ is reachable from $s$ with the remaining transitions if and only if $a_1 \ldots a_t \in W(s_j)$. It is easy to check that the time needed by the entire procedure is $O(|\Delta|T)$.

Now, analogous to the prior section, we define the following properties for each state $s_i$. Recall that we use $r = |\mathcal{N}|$ to denote the size of $\mathcal{N}$ for brevity.

**Property 3:** We say that $s_i$ satisfies Property 3 if $\tilde{N}(s_i) = (1 \pm i\epsilon/r)N(s_i)$.

**Property 4:** We say that $s_i$ satisfies Property 4 if for every subset $L \subseteq \{0, 1, 2, \ldots, i - 1\}$, we have that

$$\frac{|\tilde{W}(s_i) \setminus \left( \bigcup_{j \in L} W(u_j) \right)|}{|W(s_i)|} \leq \frac{\epsilon}{r}$$

Moreover, we have that the subsets $\tilde{W}(s_i)$ are of size $|\tilde{W}(s_i)| = O\left(\frac{r^3}{\epsilon^2} \right)$.

**Property 5:** We say that $s_i$ satisfies Property 5 if we have a polynomial-time algorithm which returns independent samples from $W(s_i)$, such that for all $w \in W(s_i)$:

$$\Pr\{\text{outputs } w \mid \text{FAIL} \} = \left( 1 \pm \frac{\epsilon}{3r^2} \right) \frac{1}{N(s_i)}$$

The algorithm is allowed to fail with probability at most $1/4$, in which case it returns \text{FAIL} (and returns no element). Finally, each run of the algorithm is allowed to use at most $O\left(\frac{\log(N(s_i)\gamma)^{17}}{\epsilon^2} \right)$ oracle calls to the sampling oracle of Definition 4.1.

**Lemma 4.6.** Suppose that Properties 3, 4 and 5 hold for all $s_j$ with $j < i$. Then with probability at least $1 - 2^{-\gamma r}$ we can return an estimate $\tilde{N}(s_i) = (1 \pm i\epsilon/r)N(s_i)$. In other words, under these assumptions it follows that Property 3 holds for $s_i$. Moreover, the total number of calls to the sampling oracle of Definition 4.1 is $O\left(\frac{\log(N(s_i)\gamma)^{17}}{\epsilon^2} \right)$, and the total runtime can be bounded by $O\left(T \frac{\log(N(s_i)\gamma)^{17}}{\epsilon^2} \right)$, where $T$ is the membership test time in Definition 4.1.
Proof. First note that for \( i = 0 \), Property 3 trivially holds since \( W(s_0) = \emptyset \). Otherwise, let \( i \geq 1 \) and \((v_1, A_1, s_1), \ldots, (v_k, A_k, s_k) \in \Delta\) be the set of all transitions going into \( s_i \) (recall that we sorted the \( \{s_i\}_{i \in [0,n]} \) by a topological ordering, so \( v_1, \ldots, v_k \in \{s_0, \ldots, s_{i-1}\} \)). Observe \( W(s_i) = \bigcup_{j=1}^{k}(W(v_j) \cdot A_j) \). Fix a transition \((v_j, A_j, s_i)\) and assume that \( \epsilon_0 = \epsilon/(100r^4) \) in Definition 4.1, so that we are given estimates \( \tilde{N}(A_j) = (1 \pm \epsilon_0)|A_j| \) since \( N \) satisfies the conditions in this definition. Then the number of words reaching \( s_i \) through \((v_j, A_j, s_i)\) is given by \( N(v_j)|A_j| \), and it can be estimated as follows assuming that \( v_j = s_k \) with \( k < i \):

\[
\tilde{N}(v_j) \cdot \tilde{N}(A_j) = (1 \pm k\epsilon/r)(1 \pm \epsilon_0)N(v_j)|A_j| = (1 \pm (i - 1)\epsilon/r)(1 \pm \epsilon/(100r^4))N(v_j)|A_j| = (1 \pm (i - 1 + 1/r^2)(\epsilon/r))N(v_j)|A_j|.
\]

Notice that in this deduction we use the fact that \( i - 1 \leq r \). Let \( p_j \) denote be the probability that a uniformly drawn \( s \sim (W(v_j) \cdot A_j) \) is not contained in \( \bigcup_{j' < j}(W(v_{j'}) \cdot A_{j'}) \). Then we can write \( W(s_i) = \sum_{j=1}^{k} N(v_j)|A_j|p_j \). We now estimate \( p_j \) via \( \tilde{p}_j \). By Property 5 and the assumptions from Definition 4.1, we can obtain nearly uniform samples \( w \sim W(v_j) \) and \( a \in A_j \) in polynomial time, such that the probability of sampling a given \( w \) and \( a \) are \((1 \pm \epsilon/(3r^2))N(v_j)^{-1}\) and \((1 \pm \epsilon_0)|A_j|^{-1}\), respectively. Moreover, \( w \cdot a \) is a sample from \( W(v_j) \cdot A_j \), such that for any \( w' \cdot a' \in W(v_j) \cdot A_j \):

\[
\Pr[w \cdot a = w' \cdot a'] = \left(1 \pm \frac{\epsilon}{3r^2}\right)
\]

Note that the relative error \((1 \pm \epsilon/(3r^2))(1 \pm \epsilon_0)\) can be bounded in the range \((1 \pm 2\epsilon/(5r^2))\) using the fact that \( \epsilon_0 = \epsilon/(100r^4) \). We repeat this sampling process \( d = O(\gamma r^5/\epsilon^2) \) times, obtaining samples \( w_1a_1, \ldots, w_da_d \sim W(v_j) \cdot A_j \) and set \( \tilde{p}_j \) to be the fraction of these samples not contained in \( \bigcup_{j' < j}(W(v_{j'}) \cdot A_{j'}) \). Then by Hoeffding’s inequality, with probability \( 1 - 2^{-\gamma r} \) we have that \( \tilde{p}_j = p_j \pm 2\epsilon/(3r^2) \) (here we use (5), which tell us that the expectation of \( \tilde{p}_j \) is at most \( 2\epsilon/(5r^2) \) far from the correct expectation \( p_j \)). Then we set:

\[
\tilde{N}(s_i) = \sum_{j=1}^{k} \tilde{N}(v_j) \cdot \tilde{N}(A_j) \cdot \tilde{p}_j
\]

\[
= (1 \pm (i - 1 + 1/r^2)(\epsilon/r)) \sum_{j=1}^{k} N(v_j) \cdot |A_j| \cdot \left( p_j \pm \frac{2\epsilon}{3r^2} \right)
\]

\[
= (1 \pm (i - 1 + 1/r^2)(\epsilon/r)) \left[ \sum_{j=1}^{k} N(v_j)|A_j|p_j \pm \frac{2\epsilon}{3r^2} \sum_{j=1}^{k} N(v_j)|A_j| \right]
\]

\[
= (1 \pm (i - 1 + 1/r^2)(\epsilon/r)) \left[ N(s_i) \pm \frac{2\epsilon}{3r^2} \sum_{j=1}^{k} N(s_i) \right]
\]

\[
= (1 \pm (i - 1 + 1/r^2)(\epsilon/r)) \left[ N(s_i) \pm 2\epsilon/(3r)N(s_i) \right]
\]

\[
= (1 \pm i\epsilon/r)N(s_i)
\]

as desired. Note that we need only compute \( \tilde{p}_j \) for at most \( r \) values of \( v_j \), thus the total number of samples required is \( O(\gamma r^6/\epsilon^2) \). By Property 5, each sample required \( O(\log(N/\epsilon)\gamma^2r^{11}) \) oracle calls, thus the total oracle complexity is \( O(\log(N/\epsilon)\gamma^2r^{17}) \) as needed. By Proposition 4.5, each membership
test required while computing the probabilities \( \tilde{p}_j \) required \( O(Tr) \) time, thus the total runtime can be bounded by \( O(T \frac{\log(N/\epsilon)^2 r^{11}}{\epsilon^2}) \), which was to be shown.

We now describe our sampling procedure. To do so, we will first develop some notation. We extend our previous notation and use \( \mathcal{N}_{s, s'} \) to denote an exact copy of \( \mathcal{N} \) but with \( s \) as the initial state and \( s' \) as the final state. Let \( s_i \) be a vertex and \( w \in \Gamma^* \) a sequence of symbols. If \( w \) contains at least one symbol, then let \( \mathcal{F}(s_i, w) = \{(s_a, A, s_b) \in \Delta \mid w \in L_k(\mathcal{N}_{s_i, s_b})\} \), namely, \( \mathcal{F}(s_i, w) \) is the set of all transitions \((s_a, A, s_b)\) incident to a state \( s_b \) from which we can reach \( s_i \) by a path labeled by \( w \). Otherwise, we have that \( w = \lambda \), where \( \lambda \) is the empty string, and \( \mathcal{F}(s_i, \lambda) \) is defined as \( \{(s_a, A, s_b) \in \Delta \mid s_b = s_i\} \). Moreover, let \(|w|\) be the length of \( w \) and \( d : S \times S \rightarrow \mathbb{N} \) be the distance metric between states of \( \mathcal{N} \) when considered as a graph, i.e. \( d(s, s') \) is the number of transitions that we need to make to get from \( s \) to \( s' \). Without loss of generality (by unrolling the succinct NFA if needed), we can make sure that \( d \) is well defined.

We now present our main sampling algorithm of this section: Algorithm 3. For ease of presentation, Algorithm 3 is written as a Las Vegas randomized algorithm, which could potentially have unbounded runtime. However, by simply terminating the execution of the algorithm after a fixed polynomial runtime and outputting an arbitrary string of bits, the desired correctness properties of the sampler will hold. The analysis of Algorithm 3, along with the finite-time termination procedure, is carried out in the proof of Lemma 4.7 below.

**Lemma 4.7.** Suppose Property 3 holds for all levels \( j \leq i \), and Property 4 holds for all levels \( j < i \). If \( W(s_i) = \emptyset \), then \( \text{SAMPLEFROMSTATE}(s_i, \tilde{N}(s_i)) \) return \( \bot \) with probability 1. Otherwise, conditioned on not outputting \( \text{FAIL} \), \( \text{SAMPLEFROMSTATE}(s_i, \tilde{N}(s_i)) \) returns \( w \sim W(s_i) \) from a distribution \( D \) over \( W(s_i) \) such that

\[
D(w) = \left(1 \pm \frac{\epsilon}{3r^2}\right) \frac{1}{|W(s_i)|}
\]

for all \( w \in W(s_i) \). Moreover, the algorithm uses at most \( O\left(\frac{\log(N/\epsilon)^2 r^{11}}{\epsilon^2}\right) \) calls to the uniform sampling oracle of Definition 4.1, runs in time \( O(T \frac{\log(N/\epsilon)^2 r^{11}}{\epsilon^2}) \), and outputs \( \text{FAIL} \) with probability at most 3/4.

**Proof.** Assume first \( W(s_i) = \emptyset \), so that \( N(s_i) = 0 \). Then given that \( \tilde{N}(s_i) = (1 \pm \epsilon)N(s_i) \) by Property 3, we conclude that \( \tilde{N}(s_i) = 0 \) and the algorithm returns \( \bot \) in line 2. Notice that if \( W(s_i) \neq \emptyset \), then \( N(s_i) > 0 \) and, therefore, \( \tilde{N}(s_i) > 0 \) by Property 3. Thus, if \( W(s_i) \neq \emptyset \), then the algorithm does not return \( \bot \).

Assume that \( W(s_i) \neq \emptyset \), and notice that this implies \( r \geq 4 \). Consider an element \( w \) sampled so far at any intermediate state of the execution of \( \text{SAMPLEFROMSTATE}(s_i, \tilde{N}(s_i)) \). Let \( W(s_i, w) = \{t \in W(s_i) \mid t = w' \cdot w\} \). In other words, \( W(s_i, w) \subseteq W(s_i) \) is the subset of words with suffix equal to \( w \). We now want to sample the next symbol \( a \in \Gamma \) conditioned on having sampled the suffix \( w \) of a path so far. In other words, we want to sample \( a \) with probability proportional to the number of words in \( W(s_i, w) \) which have the suffix \( aw \), meaning we want to choose \( a \) with probability:

\[
\frac{|W(s_i, aw)|}{|W(s_i, w)|}.
\]

However, we do not know these sizes exactly, so we must approximately sample from this distribution. Let us consider the probability that our algorithm samples \( a \) on this step (given \( w \)). For
Algorithm 3: SampleFromState($s_i, \bar{N}(s_i)$)

1. if $\bar{N}(s_i) = 0$ then
   2. return $\bot$  // $\bot$ indicates that $W(s_i)$ is empty
3. end

4. Initialize $w \leftarrow \lambda$, $q \leftarrow 1$.  // $\lambda$ is the empty string
5. for $\beta = 1, 2, \ldots, d(s_0, s_i)$ do
   6. while $|w| < \beta$ do
      7. Let $F(s_i, w) = \{(x_1, A_1, y_1), \ldots, (x_k, A_k, y_k)\}$.
      8. Let $Z_j = \bar{N}(x_j)\bar{N}(A_j)$ for each $j \in [k]$, and $Z = \sum_{j=1}^{k} Z_j$.
      9. Order the $Z_i$'s so that $Z_1 \geq Z_2 \geq \cdots \geq Z_k$.
     10. Sample $j \sim [k]$ with probability $\frac{Z_j}{Z}$.
     11. Obtain an almost uniform sample $a \sim A_j$.  // via Definition 4.1
     12. Let $B(a) = \{j' \in [k] \mid a \in A_{j'}\}$ and accept $a$ with probability:
        $$q_{a,j} = \frac{|\bar{W}(x_j) \setminus (\bigcup_{j' \in B(a): j' \leq j} W(x_{j'}))|}{|W(x_j)|}$$
   13. if $a$ is accepted then
      14. $\rho \leftarrow 0$, $M \leftarrow \Theta\left(\frac{\log(N/\epsilon)}{\epsilon^2}\right)$.
        // $\rho$ approximates the probability that a trial fails to accept some $a$
     15. for $h = 1, 2, \ldots, M$ do
         16. Sample $j \sim [k]$ with probability $\frac{Z_j}{Z}$, then sample $a_h \sim A_j$.
         17. With probability $1 - q_{a_h,j}$ increment $\rho \leftarrow \rho + 1$.
     18. end
     19. $\rho \leftarrow \frac{\rho}{M}$, and update:
        $$q \leftarrow q \cdot \left(\frac{\sum_{j' \in B(a)} \frac{\bar{N}(x_{j'})}{Z} q_{a,j'}}{1 - \frac{\rho}{M}}\right)$$
   20. $w \leftarrow aw$
   21. end
   22. end
23. end  // $q$ approximates the probability that $w$ was sampled up to this point
24. With probability $\frac{1}{2q\bar{N}(s_i)}$ return $w$, otherwise return $\text{FAIL}$.  // $q$ approximates the probability that $w$ was sampled up to this point

the algorithm to sample $a$, it must first choose to sample a transition $(x_j, A_j, y_j)$ from the set $F(s_i, w) = \{(x_1, A_1, y_1), (x_2, A_2, y_2), \ldots, (x_k, A_k, y_k)\}$ such that $a \in A_i$, which occurs with probability $Z_j/Z$ with $Z = \sum_{j'=1}^{k} Z_{j'}$ and $Z_j = \bar{N}(x_j)\bar{N}(A_j)$. Then, on the call to the oracle on line 11,
it must obtain \( a \sim A_j \) as the almost uniform sample, which occurs with probability \((1 \pm \epsilon_0)|A_j|\) by Definition 4.1. Finally, it must choose to keep \( a \) on line 12, which occurs with probability 
\[
\frac{|\tilde{W}(x_j)\backslash(\bigcup_{j'\in\mathcal{B}(a)\cup_{j'\prec j}W(x_{j'})}|}{|W(x_j)|},
\]
where \( \mathcal{B}(a) = \{j' \in [k] \mid a \in A_{j'}\} \). Thus, altogether, the probability that we choose \( a \in \Gamma \) on this step is

\[
\sum_{j \in \mathcal{B}(a)} \frac{\tilde{N}(x_j)\tilde{N}(A_j)}{Z} \cdot \frac{1 \pm \epsilon_0}{|A_j|} \cdot \frac{|\tilde{W}(x_j)\backslash(\bigcup_{j'\in\mathcal{B}(a)\cup_{j'\prec j}W(x_{j'})}|}{|W(x_j)|}
= (1 \pm 3\epsilon_0) \sum_{j \in \mathcal{B}(a)} \frac{\tilde{N}(x_j)}{Z} \cdot \frac{|\tilde{W}(x_j)\backslash(\bigcup_{j'\in\mathcal{B}(a)\cup_{j'\prec j}W(x_{j'})}|}{|W(x_j)|}
= (1 \pm 3\epsilon_0) \sum_{j \in \mathcal{B}(a)} \frac{(1 \pm \epsilon)|W(x_j)|}{Z} \cdot \frac{|\tilde{W}(x_j)\backslash(\bigcup_{j'\in\mathcal{B}(a)\cup_{j'\prec j}W(x_{j'})}|}{|W(x_j)|}
= (1 \pm 2\epsilon) \sum_{j \in \mathcal{B}(a)} \frac{|W(x_j)|}{Z} \cdot \frac{|\tilde{W}(x_j)\backslash(\bigcup_{j'\in\mathcal{B}(a)\cup_{j'\prec j}W(x_{j'})}|}{|W(x_j)|} \pm \frac{\epsilon}{r}
= (1 \pm 2\epsilon) \frac{1}{Z} \sum_{j \in \mathcal{B}(a)} \left| W(x_j) \backslash (\bigcup_{j'\in\mathcal{B}(a)\cup_{j'\prec j}W(x_{j'})} \right| \pm \frac{|W(x_j)| \epsilon}{r}
= (1 \pm 2\epsilon) \frac{1}{Z} \left( \sum_{j \in \mathcal{B}(a)} |W(x_j)\backslash(\bigcup_{j'\in\mathcal{B}(a)\cup_{j'\prec j}W(x_{j'})}| \pm \sum_{j \in \mathcal{B}(a)} |W(x_j)| \frac{\epsilon}{r} \right)
= (1 \pm 2\epsilon) \frac{1}{Z} \left( |W(s_i, aw)| \pm |W(s_i, aw)| \epsilon \right)
= (1 \pm 4\epsilon) \frac{1}{Z} |W(s_i, aw)|
\]

Where equation (6) uses the fact that \( \tilde{N}(A_j) = (1 \pm \epsilon_0)|A_j| \), and equation (7) uses the fact that 
\( |W(s_i, aw)| = (\bigcup_{j\in\mathcal{B}(a)} W(x_j)) \cdot \{aw\} \). The above demonstrates that on a single trial of the inner while loop in lines 6 to 22, conditioned on having chosen the sample \( w \) so far, for each \( a \in W(s_i, w) \) we choose \( a \) with probability \((1 \pm 4\epsilon)\frac{|W(s_i, aw)|}{Z}\). However, we do not break out of the while loop on line 22 and move to the next step in the outer for loop in line 5 until we have chosen an \( a \in W(s_i, w) \) to append to \( w \). If on a given trial of the loop in line 6, the algorithm does not choose some element to append to \( w \), we say that it outputs no sample. Call the event that we output some sample \( \mathcal{E}_i \), and let \( \mathcal{E}_i(a) \) denote the event that we specifically output \( a \in \Gamma \). Then

\[
\Pr[\mathcal{E}_i] = \sum_{a \in W(s_i, w)} \Pr[\mathcal{E}_i(a)]
= (1 \pm 4\epsilon) \sum_{a \in W(s_i, w)} \frac{|W(s_i, aw)|}{Z}
= (1 \pm 4\epsilon) \frac{|W(s_i, w)|}{Z}.
\]
Therefore,

\[
\Pr[\mathcal{E}_i(a) \mid \mathcal{E}_i] = \frac{\Pr[\mathcal{E}_i(a)]}{\Pr[\mathcal{E}_i]} = \frac{(1 + 4\epsilon)}{(1 + 4\epsilon)} \left( \frac{Z}{|W(s_i, w)|} \right) \frac{|W(s_i, aw)|}{Z} = (1 + 10\epsilon) \frac{|W(s_i, aw)|}{|W(s_i, w)|}.
\]

Thus, conditioned on outputting a sample at this step, we choose \( a \in \Gamma \) with probability

\[
(1 + 10\epsilon) \frac{|W(s_i, aw)|}{|W(s_i, w)|} \tag{8}
\]

Observe the above is within \((1 \pm 10\epsilon)\) of the correct sampling probability.

**Estimating the probability that we sample a given** \( w \in W(s_i) \). We now analyze the quantity \( q \) in the algorithm, and argue that at the point where line 24 is executed, \( q \) is a good approximation of the probability that our algorithm sample \( w \) at this point. Now let \( \rho^*_\beta \) be the probability that, within step \( \beta \in \{1, 2, \ldots, d(s_0, s_i)\} \) of the outer for loop on line 5, a given run of the inner while loop between lines 6 to 22 fails to append a new sample \( a \) to \( w \). Let \( \rho_\beta \) be the value that we assign to the variable \( \rho \) at the end of the for loop in line 15 (note that this loop is executed at most once within step \( \beta \) of the outer loop 5). The variable \( \rho_\beta \) will be our estimate of \( \rho^*_\beta \).

Note that each trial of the inner while loop is independent, so \( \rho^*_\beta \) only depends on the \( \beta \) from the outer loop, and the value of \( w \) sampled so far. Let \( D'_\beta(aw) \) be the exactly probability that entry \( a \) is chosen on step \( \beta \) of the outer loop of our algorithm, conditioned on having chosen \( w \) so far. Being in step \( \beta \) of the outer loop then implies that \( |aw| = \beta \). Now fix any \( w = w_1w_2 \ldots w_{d(s_0, s_i)} \in W(s_i) \). Let \( D'(w) \) be the exact probability that \( w \) is sampled at this point right before the execution of line 24. By definition we have

\[
D'(w) = D'_\emptyset(w_{d(s_0, s_i)} \cdot \left( d(s_0, s_i) - 1 \right) \prod_{j=1}^{d(s_0, s_i) - 1} D'_j(w_{d(s_0, s_i) - j} \cdots w_{d(s_0, s_i)})
\]

so via (8) we obtain:

\[
D'(w) = \frac{1}{|W(s_i)|} \prod_{j=1}^{d(s_0, s_i)} (1 + 10\epsilon) = \frac{(1 + 10\epsilon)^r}{|W(s_i)|} = \frac{(1 + 20r\epsilon)}{|W(s_i)|} \tag{9}
\]

**Claim 4.8.** If \( q \) is the value the variable \( q \) takes at the point where line 24 is executed, given that \( w = w_1 \ldots w_{d(s_0, s_i)} \) is the value of \( e \) at this point, then

\[
D'(w) = \left( 1 + \frac{\epsilon}{50r^2} \right) q
\]

with probability at least \( 1 - r(\epsilon/N)^{2r} \).

**Proof.** To see this, consider step \( \beta \in \{1, 2, \ldots, d(s_0, s_i)\} \) of the for outer loop in line 5. We first claim that \( \rho^*_\beta \leq 1 - \frac{\epsilon}{\beta} \). To see this, note that the probability that \( Z_1 \) is chosen is at least \( \frac{1}{\beta} \geq \frac{1}{\beta} \), since we ordered \( Z_1 \geq Z_2 \geq \cdots \geq Z_k \), and if \( Z_1 \) is chosen the sample \( a \sim A_1 \) is never rejected,
which completes the claim. Now each iteration of the for loop in line 15 defines a random variable \( Z \) which indicates if a random trial of the inner loop in line 6 would result in a failure. Here, if \( Z = 1 \) (a trials fails), then we increment \( \rho = \rho + 1 \), otherwise we do not. Thus \( \mathbb{E}[Z] = \rho^*_\beta \), and by Hoeffding’s inequality, after repeating \( M = \Theta(N \log^2(N)) \) times, it follows that with probability \( 1 - (\epsilon/N)^{2t} \) that we have \( \rho_\beta = \rho^*_\beta \pm \epsilon/(400r^5) \) and, therefore, \( 1 - \rho_\beta = (1 \pm \epsilon/(400r^4))(1 - \rho^*_\beta) \) since \( 1/r \leq 1 - \rho^*_\beta \). Thus, it holds that

\[
\frac{1}{1 - \rho^*_\beta} = \left(1 \pm \frac{\epsilon}{200r^4}\right) \frac{1}{1 - \rho_\beta}
\]

(10)

Let \( \tau = d(s_0, s_i) - \beta + 1 \), so that on step \( \beta \) of the for outer loop in line 5 we are considering the probability that we sample a \( w_\tau \in \Gamma \) given that we have already sampled \( w_{\tau+1} \ldots w_{d(s_0, s_i)} \). Now as shown above, the probability that \( w_\tau \) is accepted on one trial of the while loop is precisely:

\[
q^*(w_\tau) = \sum_{j \in B(w_\tau)} \frac{\widetilde{N}(x_j) N(A_j)}{Z} \cdot \frac{\epsilon'}{|A_j|} \cdot \left| \frac{\widetilde{W}(x_j) \setminus \left( \bigcup_{j' \in B(w_\tau)} W(x_j') \right) }{ W(x_j) } \right|.
\]

Notice that we are not trying to bound \( q^*(w_\tau) \) in this expression, we are computing the exact value of \( q^*(w_\tau) \), but based on an unknown value \( \epsilon' \). However, we know by Definition 4.1 that \( 1 - \epsilon_0 \leq \epsilon' \leq 1 + \epsilon_0 \). Thus, although we do not know the exact value of \( q^*(w_\tau) \), we do know that \( 1 - 3\epsilon_0 \leq \widetilde{N}(A_j) \cdot \epsilon'/|A_j| \leq 1 + 3\epsilon_0 \) by the assumptions of Definition 4.1. Thus, we can estimate \( q^*(w_\tau) \) by

\[
\hat{q}(w_\tau) = \sum_{j \in B(w_\tau)} \frac{\widetilde{N}(x_j) |\widetilde{W}(x_j) \setminus (\bigcup_{j' \in B(w_\tau)} W(x_j'))|}{|\widetilde{W}(x_j)|}.
\]

so that \( q^*(w_\tau) = (1 \pm 3\epsilon_0) \hat{q}(w_\tau) \). The probability that \( w_\tau \) is accepted overall before moving to the next step of the loop is \( \sum_{j=1}^\infty q^*(w_\tau)(\rho^*_\beta)^{j-1} = q^*(w_\tau)(1/(1 - \rho^*_\beta)) \), for which by equation (10) we have a \( (1 \pm \epsilon/(200r^4))(1 \pm 3\epsilon_0) = (1 \pm \epsilon/(100r^4)) \) estimate of via the value \( \hat{q}(w_\tau)/(1 - \rho_\beta) \) (recall that \( r \geq 4 \)). Note that this is precisely the value which we scale the variable \( q \) by after an iteration of the inner loop that appends a new sample \( a \) to \( w \) in line 20 of the algorithm. It follows that at the end of the main loop, we have:

\[
\mathcal{D}'(w) = \left(1 \pm \frac{\epsilon}{100r^4}\right) d(s_0, s_i) \cdot q = \left(1 \pm \frac{\epsilon}{50r^2}\right) \cdot q
\]

as needed. Notice that this equality holds under the condition that for every \( \beta = 1, \ldots, d(s_0, s_i) \), it holds that \( \rho_\beta = \rho^*_\beta \pm \epsilon/(400r^5) \), which occurs with probability \( 1 - (\epsilon/N)^{2t} \) for each \( \beta \). By a union bound, we obtain the desired success probability of at least \( 1 - r(\epsilon/N)^{2t} \).

Thus, by rejecting with probability \( \frac{1}{2q\widetilde{N}(s_i)} \), it follows from Claim 4.8 that the true probability \( \mathcal{D}^*(w) \) that we output a given \( w \in W(s_i) \) is

\[
\mathcal{D}^*(w) = \frac{\mathcal{D}'(w)}{2q\widetilde{N}(s_i)} = \left(1 \pm \frac{\epsilon}{50r^2}\right) \frac{1}{2\widetilde{N}(s_i)}
\]

(11)

Note that for the above fact to be true, we need that \( \frac{1}{2q} \leq \widetilde{N}(s_i) \), else the above rejection probability could be larger than 1. But again by Claim 4.8 we have that

\[
\frac{1}{2q\widetilde{N}(s_i)} \leq \left(1 \pm \frac{\epsilon}{50r^2}\right) \frac{1}{2\widetilde{N}(s_i)\mathcal{D}'(w)}
\]
\[
(1 + 2\epsilon) \frac{1}{2|W(s_i)|D'(w)} \leq (1 + 122r\epsilon) \frac{1}{2} \leq \frac{3}{4}
\]

where the second to last inequality holds applying (9), and the last inequality holds give that \(\epsilon < 1/(300r)\). Therefore, the rejection probability is always a valid probability. Similarly:

\[
\frac{1}{2qN(s_i)} \geq \left(1 - \frac{\epsilon}{50r^2}\right) \frac{1}{2N(s_i)D'(w)} \geq (1 - 2\epsilon) \frac{1}{2|W(s_i)|D'(w)} \geq (1 - 42r\epsilon) \frac{1}{2} \geq 1/4
\]

Thus, by the above, we can bound the probability that we output **FAIL** on this last step by \(3/4\) as required. Now, we are ready to analyze the true output distribution \(D\) over \(W(s_i)\), which is given by the distribution \(D^*\) conditioned on not outputting **FAIL**. Now for any \(w \in W(s_i)\), we can apply equation (11) to compute \(D(w)\) via:

\[
D(w) = \frac{\Pr[\text{output } w \in W(s_i) \mid \neg \text{FAIL}]}{\Pr[\text{output } w \in W(s_i) \cap \neg \text{FAIL}]} = \frac{\Pr[\text{not output FAIL}]}{\Pr[\neg \text{FAIL}]} = \sum_{w \in W(s_i)} D^*(w) = \frac{1}{2|W(s_i)|} \sum_{w \in W(s_i)} \left(1 \pm \epsilon/(50r^2)\right) = \frac{1}{|W(s_i)|} \left(1 \pm \frac{3\epsilon}{50r^2}\right) = \frac{1}{|W(s_i)|} \left(1 \pm \frac{\epsilon}{10r^2}\right)
\]

which is the desired result.

**Oracle complexity and runtime** For the complexity of the sample procedure, note that each iteration to sample a \(a \in \Gamma\) has failure probability at most \(\frac{1}{r}\) independently, thus with probability \(1 - (\epsilon/(rN))^22^{-10r\gamma}\) it requires at most \(10^{-3}\log(Nr/\epsilon)\gamma\) iterations. Thus with probability \(1 - (\epsilon/(N))^22^{-10r\gamma}\), the total number of iterations required to produce a single sample (or output **FAIL**
at the end) is \(10r^4 \log(Nr/\epsilon)\gamma\). Note that each iteration that fails to accept an \(a \in \Gamma\) produces one call to the unit oracle. Once an \(a\) is accepted, we run an experiment \(M\) times, which produces \(M = O(\frac{\log(N/\epsilon)\gamma r^10}{\epsilon^2})\) oracle calls. Since this occurs at most \(r\) times, the total number of oracle calls is \(O(\frac{\log(N/\epsilon)\gamma r^{11}}{\epsilon^2})\). Note that the runtime is dominant by the cost of the \(\rho\) estimation procedure, wherein the probability \(q_{ah,j}\) is computed at each step of line 15. Note that to compute \(q_{ah,j}\), we must test for each sample in \(s \in W(x_j)\) if \(s\) is contained in the union of at most \(r\) sets, which requires at most \(Tr\) runtime by the assumptions of Definition 4.1. Note that each set has size at most \(O(2^3 r^4)\). Thus the total runtime can be bounded by \(O(T\frac{\log(N/\epsilon)\gamma r^{17}}{\epsilon^4})\).

In summary, with probability \(1 - (\epsilon/(N))^{2-10r\gamma}\), the total number of samples (unit oracle calls) required is \(O(\frac{\log(N/\epsilon)\gamma r^{11}}{\epsilon^2})\) (and the runtime is as stated above). Now if the sample complexity becomes too large we can safely output anything we would like (specifically, we can output \text{FAIL}, or even an arbitrary sequence of bits). The probability that this occurs, or that any of our \(O(r)\) estimate of the inner failure probabilities \(\rho\) fails to be within our desired bounds, is at most \((\epsilon/(N))^{2-10r\gamma} + r(\epsilon/N)^{2-2\gamma} \leq (\epsilon/N)^{2-\gamma}\). Call the event that the sample complexity becomes too large \(Q\), and let \(P\) be the event that any of our \(O(r)\) estimate of \(\rho\) fail to be within our desired bounds. We have just proven that

\[
\Pr[\text{we output } w \in W(s_i) \mid \neg P] = (1 \pm \epsilon/(10r^2)) \frac{1}{|W(s_i)|}.
\]

Now since \(\Pr[P \cup Q] \leq (\epsilon/N)^{2-\gamma}\), we have

\[
\Pr[\text{we output } w \in W(s_i) \mid \neg Q] = \Pr[\text{we output } w \in W(s_i) \mid \neg Q, \neg P] \pm (\epsilon/N)^{2-\gamma}
= \Pr[\text{we output } w \in W(s_i) \mid \neg P] \pm 3(\epsilon/N)^{2-\gamma},
\]

so it follows that for each \(w \in W(s_i)\), we have

\[
\Pr[\text{we output } w \in W(s_i) \mid \neg Q] = (1 \pm \epsilon/(10r^2)) \frac{1}{|W(s_i)|} \pm 3(\epsilon/N)^{2-\gamma}
= (1 \pm \epsilon/(10r^2)) \frac{1}{u_i \epsilon} \pm \frac{3}{|W(s_i)|^{2-\gamma}}
= (1 \pm \epsilon/(3r^2)) \frac{1}{|W(s_i)|},
\]

which shows that our sampler is still correct even if we output random bits whenever \(Q\) fails to hold, which is the desired result taking \(\gamma = \Omega(\log(r/\epsilon))\).

\[\square\]

We can use the above sampling regime to now show that having properties 3, 4, 5 for \(s_j\) with \(j < i\) will imply them for \(s_i\).

**Lemma 4.9.** Fix any \(\gamma > 0\). Suppose Properties 3, 4, 5 hold for all \(s_j\) with \(j < i\). Then with probability \(1 - 2^{-10\gamma}\), properties 3, 4 and 5 hold for \(s_i\). Moreover, the total number of oracle calls is at most \(O(\gamma^2 \log(N/\epsilon) r^{17}/\epsilon^4)\), and the total runtime is \(O(T\frac{\log(N/\epsilon)\gamma^2 r^{17}}{\epsilon^4})\).

**Proof.** We obtain property 3 with probability \(1 - 2^{-10\gamma}\) by Lemma 4.6, which uses \(O(\frac{\log(N/\epsilon)\gamma r^{17}}{\epsilon^4})\) sampling oracle calls. By Lemma 4.7, conditioned on property 4 holding for all levels \(j < i\) and property 3 holding for all \(j \leq i\), we now have a procedure which can sample each \(w \sim W(s_i)\) with probability in the range \((1 \pm \epsilon/(3r^2))\frac{1}{|W(s_i)|}\), and such that the sampler satisfies the other conditions of Property 5. Thus property 5 for level \(i\) now holds deterministically conditioned on property 3 holding for \(i\) and all \(j < i\).
Now for property 4, we can take \( s' = \Theta(\gamma^3/\epsilon^2) \) samples to build \( \widetilde{W}(s_i) \). By Lemma 4.7, each run of the algorithm requires \( O(\gamma^{11}\log(N/\epsilon)/\epsilon^2) \) oracle calls, and fails to return a sample with probability at most 3/4. Applying Hoeffding’s inequality on the required number of trials of the sampling algorithm to obtain \( s' \) independent samples, this requires \( O(\gamma^2r^{15}\log(N/\epsilon)/\epsilon^4) \) oracle calls with probability \( 1 - 2^{-100}\gamma r \). Given this, we have that each sample in \( \widetilde{W}(s_i) \) is a \((1\pm\epsilon/(3r^2))\)-relative error almost uniform sample. Applying Hoeffding’s inequality again, it follows that for a fixed set \( L \subset \{s_0, \ldots, s_{i-1}\} \), we have

\[
\left| \frac{\left| \mathcal{W}(s_i) \setminus \left( \bigcup_{s_j \in L} \mathcal{W}(s_j) \right) \right|}{|\mathcal{W}(s_i)|} - \frac{|\mathcal{W}(s_i) \setminus \left( \bigcup_{s_j \in L} \mathcal{W}(s_j) \right)|}{|\mathcal{W}(s_i)|} \right| \leq \frac{\epsilon}{3r^2} + \frac{\epsilon}{2r} \leq \frac{\epsilon}{r}
\]

with probability \( 1 - 2^{-100\gamma r} \), and since there are only at most \( 2^r \) such subsets \( L \), by a union bound this holds for all such subsets with probability \( 1 - 2^{-100\gamma r - r} \). Thus the overall probability of success is \( 1 - 2^{-100\gamma r - r} - 2^{-\gamma r} > 1 - 2^{-\gamma r} \). Note that the runtime is dominated by the time required to obtain Property 3 via Lemma 4.6, which is \( O(T\log(N/\epsilon)^2\gamma^2r^{17}) \).

We are now ready the prove the main theorem.

**Proof of Theorem 4.3.** By Lemma 4.9, conditioned on having Properties 3, 4, and 5 for a level \( i \), we get it for \( i + 1 \) with probability \( 1 - 2^{-10\gamma} \) with at most \( O(\gamma^2\log(N/\epsilon)r^{17}/\epsilon^4) \) oracle calls. It follows inductively that with probability \( 1 - r2^{-10\gamma} \), we have Property 3 and 5 for all levels with at most \( O(\gamma^2\log(N/\epsilon)r^{18}/\epsilon^4) \) oracle calls, which completes the proof after recalling that \( \gamma := \log(1/\delta) \). The runtime for each level is \( O(T\log(N/\epsilon)^2\gamma^2r^{17}) \) by Lemma 4.9, thus the total runtime is \( O(T\log(N/\epsilon)^2\gamma^2r^{18}) \).

5 Some Applications of our Main Results

5.1 Constraint satisfaction problems

Constraint satisfaction problems offer a general and natural setting to represent a large number of problems where solutions must satisfy some constraints, and which can be found in different areas such as artificial intelligence, satisfiability, programming languages, temporal reasoning, scheduling, graph theory, and databases [Var00, CKS01, RVBW06, HN04, BHvMW09, RN16]. Formally, a constraint satisfaction problem (CSP) is a triple \( \mathcal{P} = (V, D, C) \) such that \( V = \{x_1, \ldots, x_m\} \) is a set of variables, \( D \) is a set of values and \( C = \{C_1, \ldots, C_n\} \) is a set of constraints, where each constraint \( C_i \) is a pair \((\bar{t}_i, R_i)\) such that \( \bar{t}_i \) is a tuple of variables from \( V \) of arity \( k \), for some \( k \geq 1 \), and \( R_i \subseteq D^k \). Moreover, an assignment \( \nu : V \rightarrow D \) is said to be a solution for \( \mathcal{P} \) if for every \( i \in [n] \), it holds that \( \nu(\bar{t}_i) \in R_i \) [RN16], where \( \nu(\bar{t}_i) \) is obtained by replacing each variable \( x_j \) occurring in \( \bar{t}_i \) by \( \nu(x_j) \). The set of solution for CSP \( \mathcal{P} \) is denoted by sol(\( \mathcal{P} \)).

The two most basic tasks associated to a CSP are the evaluation and the satisfiability problems. In the evaluation problem, we are given a CSP \( \mathcal{P} \) and an assignment \( \nu \), and the question to answer is whether \( \nu \in \text{sol}(\mathcal{P}) \). In the satisfiability problem, we are given a CSP \( \mathcal{P} \), and the question to answer is whether \( \text{sol}(\mathcal{P}) \neq \emptyset \). Clearly, these two problems have very different complexities, as in the former we only need to verify the simple condition that \( \nu(t) \in R \) for every constraint \((t, R)\) in \( \mathcal{P} \), while in the latter we need to search in the space of all possible assignments for one that satisfies all the constraints. In fact, these two problems also look different in terms of our interest in the specific values for the variables of the CSP; in the former we are interested in the value of each one of them that is given in the assignment \( \nu \), while in the latter the variables of \( \mathcal{P} \) are considered as existential.
quantifiers, as we are interested in knowing whether there exists a solution for $\mathcal{P}$ even if we do not know how to construct it. As a way to unify these two problems, and to indicate for which variables we are interested in their values, a projection operator has been used in the definition of CSPs [CJ06, Wil10]. Notice that the definition of this operator has also played an important role when classifying the complexity of CSPs in terms of algebraic properties of relations [CJ06]. Formally, an existential CSP (ECSP) is defined as a pair $\mathcal{E} = (U, \mathcal{P})$, where $\mathcal{P} = (V, D, C)$ is a CSP and $U \subseteq V$. Moreover, the set of solution for $\mathcal{E}$ is defined as

$$\text{sol}(\mathcal{E}) = \{ \nu|_{U} \mid \nu \in \text{sol}(\mathcal{P}) \},$$

where $\nu|_{U}$ is the restriction of function $\nu$ to the domain $U$. Notice that both the evaluation and the satisfiability problems for a CSP $\mathcal{P}$ can be reduced to the evaluation problem for an ECSP. In fact, the satisfiability problem for $\mathcal{P}$ corresponds to the problem of verifying whether the assignment with empty domain belongs to $\text{sol}(\mathcal{E})$, where $\mathcal{E}$ is the ECSP $(\emptyset, \mathcal{P})$. Moreover, the evaluation and satisfiability problems are polynomially interreducible for ECSPs, so ECSPs provide a uniform framework for these two problems allowing us to focus only on the evaluation problem.

Clearly the satisfiability problem for CSPs, as well as the evaluation problem for ECSPs, is NP-complete; in particular, NP-hardness is a consequence that the satisfiability of 3-CNF propositional formulae can be easily encoded as a constraint satisfaction problem. Thus, a large body of research has been devoted to understanding the complexity of the evaluation problem for ECSPs, and finding tractable cases. In particular, two prominent approaches in this investigation have been based on the idea of viewing an ECSP as a homomorphism problem where the target structure is fixed [FV98, Bul17, Zhu17] or on the use of decomposition methods that require of some acyclicity conditions on an ECSP to be satisfied [GLS00, GLS02]. In this section, we focus on the latter class of methods, and show how the main results of this article can be used to deal with the fundamental problem of counting the number of solutions to an ECSP.

### 5.1.1 A first notion of acyclicity

To define the first notion of acyclicity for ECSPs considered in this article, we need to introduce some terminology. Assume that $\mathcal{P} = (V, D, C)$ is a CSP such that $V = \{x_1, \ldots, x_m\}$, $C = \{C_1, \ldots, C_n\}$ and $C_i = (\bar{t}_i, R_i)$ for every $i \in [n]$. Moreover, assume that $\text{var} (\bar{t}_i)$ is the set of variables mentioned in $\bar{t}_i$, for every $i \in [n]$. A join tree $T = (N, E)$ for $\mathcal{P}$ is a rooted tree such that $N = \{\bar{t}_1, \ldots, \bar{t}_n\}$, and which satisfies the following connectedness condition:

- for every $x \in V$, the set $\{\bar{t}_i \mid i \in [n] \text{ and } x \in \text{var}(\bar{t}_i)\}$ induces a (connected) subtree of $T$.

Notice that definition of the notion of join tree does not take into consideration the set $D$ of possible values for the variables, and the relations $R_i$’s used in the definition of the constraints $C_1, \ldots, C_n$. As shown in the following example, some CSPs admit join trees, while others do not.

**Example 5.1.** Consider a CSP $\mathcal{P}_1$ with a set of variables $\{u, v, w, x, y\}$ and a set of constraints $\{C_1, \ldots, C_6\}$ such that each $C_i = (\bar{t}_i, R_i)$, $\bar{t}_1 = (x, u, v)$, $\bar{t}_2 = (x, v, w)$, $\bar{t}_3 = (u, y, u)$, $\bar{t}_4 = (u, y)$, $\bar{t}_5 = (v, w)$ and $\bar{t}_6 = (x, y, u)$. Then the following is a join tree $T$ for $\mathcal{P}_1$:  

40
Notice that the connectedness condition is satisfied for every variable in the set \( \{u, v, w, x, y\}\). For example, the set of atoms mentioning variable \( y \) is \( \{\bar{t}_3, \bar{t}_4, \bar{t}_6\} \), and this set induces a (connected) subtree of \( T \):

On the other hand, assume that \( \mathcal{P}_2 \) is a CSP with a set of variables \( \{x, y, z, w\} \) and a set of constraints \( \{C'_1, \ldots, C'_4\} \) such that each \( C'_i = (\bar{t}'_i, R'_i) \), \( \bar{t}'_1 = (x, y) \), \( \bar{t}'_2 = (y, z) \), \( \bar{t}'_3 = (z, w) \) and \( \bar{t}'_4 = (w, x) \). Then we have that \( \mathcal{P}_2 \) does not admit a join tree.

A CSP \( \mathcal{P} \) is acyclic if and only if \( \mathcal{P} \) admits a join tree [Yan81, GLS00]. For instance, in Example 5.1, we have that CSP \( \mathcal{P}_1 \) is acyclic, while CSP \( \mathcal{P}_2 \) is not acyclic. Intuitively, this latter problem is not acyclic as it encodes a cycle of length four. Moreover, an ECSP \( (U, \mathcal{P}) \) is acyclic if and only if \( \mathcal{P} \) is an acyclic CSP.

The notion of acyclic CSP coincides with the notion of \( \alpha \)-acyclicity for hypergraphs [Fag83, BFMY83], and it has played an important role in finding tractable cases for ECSPs [GLS00]. In fact, if \( \text{AECSP} = \{(\mathcal{E}, \nu) \mid \mathcal{E} \text{ is an acyclic ECSP and } \nu \in \text{sol}(\mathcal{E})\} \), then it holds that AECSP is \( \text{LogCFL} \)-complete under many-to-one logspace reductions [GLS98]. Recall that \( \text{LogCFL} \) consists of all decision problems that are logspace reducible to a context-free language, and it holds that \( \text{NL} \subseteq \text{LogCFL} \subseteq \text{AC}^1 \). Thus, we have that all problems in \( \text{LogCFL} \) can be solved in polynomial time and are highly parallelizable.

Concerning to our investigation, we are interested in the fundamental problem of counting the number of solutions to an ECSP. In the most general version of this problem, and even if we focus on CSPs, such a problem is \#P-complete and cannot admit an FPRAS (unless \( \text{NP} = \text{RP} \), given that the satisfiability problems for CSP and ECSP are both \( \text{NP} \)-complete). Thus, in this section we focus on this counting problem for acyclic ECSPs:

| Problem: | \#AECSP |
|----------|----------|
| Input:   | An acyclic ECSP \( \mathcal{E} \) |
| Output:  | \( |\text{sol}(\mathcal{E})| \) |

It has been shown that \#AECSP can be solve in polynomial time if we focus on CSPs, that is, if we focus on ECSPs of the form \( (V, \mathcal{P}) \) where \( V \) is the set of variables of \( \mathcal{P} \) [PS13]. However, in the general case we have that \#AECSP is \#P-complete [PS13]. Thus, a natural and important question is whether \#AECSP admits an FPRAS. By using the existence of an FPRAS for \#TA (see Corollary 3.9), we are able to give a positive answer to this question.

**Proposition 5.2.** \#AECSP admits an FPRAS.
This proposition can be proved by providing a polynomial-time parsimonious reduction from \#AECSP to \#TA, as we conclude that \#AECSP admits an FPRAS from the fact that \#TA admits an FPRAS. However, as shown in Section 5.1.2, Proposition 5.2 is a corollary of Proposition 5.4, so we only provide here an intuitive explanation of how the parsimonious reduction from \#AECSP to \#TA works.

Assume that \(E_1 = \{(x, y) \mid P_1\}\) is an acyclic ECSP, where \(P_1\) is the acyclic CSP defined in Example 5.1. We first notice that it is proved in [GLS98] that a canonical join tree for an acyclic CSP can be constructed by a logarithmic space Turing Machine with oracle in SL (Symmetric Logspace [LP82]). Thus, given that SL = L [Rei08], it is possible to conclude that a canonical join tree for an acyclic CSP can be constructed in logarithmic space. Hence, the join tree for \(P_1\) in Example 5.1 can be considered as the starting point to show how the reduction from \#AECSP to \#TA works.

Given that the join tree for \(P_1\) in Example 5.1 has six nodes, our task is to construct a tree automaton \(\mathcal{T}_{E_1} = (S, \Sigma, \Delta, S_0)\) such that \(|\text{sol}(E_1)|\) is equal to \(|\{t \in \mathcal{L}(\mathcal{T}_{E_1}) \mid |t| = 6\}|\). Notice that in the general form of the reduction for an acyclic ECSP \(E = (U, P)\), the input of \#TA will be a tree automaton \(\mathcal{T}_E\) and a number \(n\) given in unary, which will be equal to the number of constraints in \(P\). Besides, notice that for the sake of presentation, we are assuming that \(\mathcal{T}_E\) has a set of initial states, instead of a single initial state. Then the set of states \(S\) is used to encode the possible assignments to variables according to the nodes in the join tree for \(E\):

\[
\begin{align*}
\{[\bar{t}_1 \mapsto (a, b, c)] &\mid (a, b, c) \in R_1\} \cup \{[\bar{t}_6 \mapsto (a, b, c)] \mid (a, b, c) \in R_6\} \cup \\
\{[\bar{t}_2 \mapsto (a, b, c)] &\mid (a, b, c) \in R_2\} \cup \{[\bar{t}_3 \mapsto (a, b, a)] \mid (a, b, a) \in R_3\} \cup \\
\{[\bar{t}_4 \mapsto (a, b)] &\mid (a, b) \in R_4\} \cup \{[\bar{t}_5 \mapsto (a, b)] \mid (a, b) \in R_5\}
\end{align*}
\]

Moreover, the set \(S_0\) of initial states is defined as the set of possible assignments for the root of the join tree: \(S_0 = \{[\bar{t}_1 \mapsto (a, b, c)] \mid (a, b, c) \in R_1\}\), and the alphabet \(\Sigma\) is defined as the set of possible assignments for the set \(\{x, y\}\) of output variables of \(E\):

\[
\begin{align*}
\{[\bar{t}_1 \mapsto (a, *, *)] &\mid \exists b \exists c : (a, b, c) \in R_1\} \cup \{[\bar{t}_6 \mapsto (a, b, *)] \mid \exists c : (a, b, c) \in R_6\} \cup \\
\{[\bar{t}_2 \mapsto (a, *, *)] &\mid \exists b \exists c : (a, b, c) \in R_2\} \cup \{[\bar{t}_3 \mapsto (*, b, *)] \mid \exists a : (a, b, a) \in R_3\} \cup \\
\{[\bar{t}_4 \mapsto (*, b)] &\mid \exists a : (a, b) \in R_4\} \cup \{[\bar{t}_5 \mapsto (*, *)] \mid \exists a \exists b : (a, b) \in R_5\}
\end{align*}
\]

The symbol \(*\) is used in a position of a tuple if we are not assigning a value for an output variable in such a position. Notice that \(S, S_0\) and \(\Sigma\) can be constructed in polynomial time since they are defined from the relations in the constraints of \(E_1\). In other words, something that we cannot do in the reduction is to use generic assignments of the form \((x_1, . . . , x_\ell) \mapsto (a_1, . . . , a_\ell)\), where \(x_1, . . . , x_\ell\) is the sequence of variables from \(P_1\) and \(a_1, . . . , a_\ell\) is a sequence of possible values for these variables, as we can have an exponential number of such assignments. Finally, the transition relation \(\Delta\) is defined to represent a top-down evaluation of the join tree for the ECSP \(E\). For instance, if we have that \((a, b, c) \in R_1, (a, c, e) \in R_6\) and \((a, d, b) \in R_2\), then the following is a tuple in \(\Delta\):

\[
([\bar{t}_1 \mapsto (a, b, c)]; [\bar{t}_1 \mapsto (a, *, *)]; [\bar{t}_6 \mapsto (a, d, b)]; [\bar{t}_2 \mapsto (a, c, e)]).
\]

This tuple indicate that if we are in the state \([\bar{t}_1 \mapsto (a, b, c)]\) in the root of the tree and the label of this node is \([\bar{t}_1 \mapsto (a, *, *)]\), then we assign the states \([\bar{t}_6 \mapsto (a, d, b)]\) and \([\bar{t}_2 \mapsto (a, c, e)]\) to the left and right children of the root, respectively. That is, this tuple indicates that if \((a, b, c) \in R_1\), then we can assign to \(x\) value \(a\) according to the root of the join tree, and then given that \((a, d, b) \in R_6\) and \((a, c, e) \in R_2\), we can continue with the computation of a solution for \(E\) by using the assignments.
Thus, the transition relation encodes a top-down evaluation of the join tree for ECSP $E$. In particular, for each node $p$ and each children $p'$ of $p$ in the join tree, if a variable $z$ is mentioned in both, then its value is inherited from $p$ to $p'$, and if $z$ appears in $p'$ but not in $p$, then its value is guessed. It is important to notice that we have a consistent way of assigning values to the variables of $E$ given the inheritance rule just described and the connectedness condition satisfied by the join tree of $E$ (that is, it is not possible to assign two different values to the same variable).

As a final remark, it is important to notice that $T_{E_1}$ can be constructed in polynomial time, and that each possible solution to $E_1$ is associated with exactly with one tree $t$ such that $t$ is accepted by $T_{E_1}$ and $|t|=6$. In particular, observe that all trees accepted by $T_{E_1}$ have six nodes by definition of this automaton. For example, assuming that $\nu(x)=a$ and $\nu(b)=b$ is an assignment such that $\nu \in \text{sol}(E_1)$, we have that the following tree accepted by $T_{E_1}$ is associated to $\nu$:

$$
\begin{array}{c}
\bar{t}_1 \mapsto (a,*,*) \\
\bar{t}_6 \mapsto (a,b,*) \\
\bar{t}_3 \mapsto (*,b,*) \\
\bar{t}_4 \mapsto (*,b) \\
\bar{t}_5 \mapsto (*,*)
\end{array}
$$

Hence, we conclude that the reduction from $\#\text{AECSP}$ to $\#\text{TA}$ just described is a parsimonious reduction that can be computed in polynomial time.

### 5.1.2 A more general notion of acyclicity

In this section, we consider a more general notion of acyclicity for ECSPs. More precisely, let $P = (V, D, C)$ be a CSP such that $C = \{C_1, \ldots, C_n\}$ and $C_i = (\bar{t}_i, R_i)$ for each $i \in [n]$. A hypertree for $P$ is a triple $(T, \chi, \xi)$ such that $T = (N, E)$ is a rooted tree, and $\chi$ and $\xi$ are node-labelling functions such that for every $p \in N$, it holds that $\chi(p) \subseteq V$ and $\xi(p) \subseteq \{\bar{t}_1, \ldots, \bar{t}_n\}$. Moreover, $T$ is said to be a hypertree decomposition for $P$ [GLS02] if the following conditions hold:

- for each atom $i \in [n]$, there exists $p \in N$ such that $\text{var}(\bar{t}_i) \subseteq \chi(p)$;
• for each variable $x \in V$, the set $\{p \in N \mid x \in \chi(p)\}$ induces a (connected) subtree of $T$;
• for each $p \in N$, it holds that

$$\chi(p) \subseteq \bigcup_{\bar{t} \in \xi(p)} \var(i)$$

• for each $p \in N$, it holds that

$$\left( \bigcup_{\bar{t} \in \xi(p)} \var(i) \right) \cap \left( \bigcup_{p' : p' \text{ is a descendant of } p \text{ in } T} \chi(p') \right) \subseteq \chi(p)$$

The width of the hypertree decomposition $\langle T, \chi, \xi \rangle$ is defined as the maximum value of $|\xi(p)|$ over all vertices $p \in N$. Finally, the hypertree-width $hw(P)$ of CSP $\mathcal{P}$ is defined as the minimum width over all its hypertree decompositions [GLS02].

**Example 5.3.** Consider again the CSPs $\mathcal{P}_1$ and $\mathcal{P}_2$ defined in Example 5.1. The following is a hypertree decomposition for $\mathcal{P}_1$, where the values of $\chi(p)$ and $\xi(p)$ are shown on the left- and right-hand sides of the rectangle for node $p$:

$$\{x, u, v\}, \{\bar{t}_1\}$$

$$\{x, y, u\}, \{\bar{t}_6\}$$

$$\{x, v, w\}, \{\bar{t}_2\}$$

$$\{y, u\}, \{\bar{t}_3\}$$

$$\{u, y\}, \{\bar{t}_4\}$$

$$\{v, w\}, \{\bar{t}_5\}$$

Notice that the width of this hypertree decomposition is 1, as $|\xi(p)| = 1$ for every node $p$ in it. Thus, we conclude that $hw(\mathcal{P}_1) = 1$. On the other hand, no hypertree decomposition of width 1 can be constructed for $\mathcal{P}_2$. In fact, $hw(\mathcal{P}_2) = 2$, which is witnessed by the following hypertree decomposition for $\mathcal{P}_2$ of width 2:

$$\{x, y, z\}, \{\bar{t}'_1, \bar{t}'_2\}$$

$$\{x, z, w\}, \{\bar{t}'_3, \bar{t}'_4\}$$

It was shown in [GLS02] that a CSP $\mathcal{P}$ is acyclic if and only if $hw(\mathcal{P}) = 1$. Thus, the notion of hypertree-width generalizes the notion of acyclicity given in the previous section. More importantly, it was shown in [GLS02] that the evaluation problem for ECSPs of bounded hypertree-width can be solved efficiently. More precisely, given an ECSP $\mathcal{E} = (U, \mathcal{P})$, define $hw(\mathcal{E})$ as $hw(\mathcal{P})$, and for $k \geq 1$ define the language $k$-HW-ECSP = $\{(\mathcal{E}, \nu) \mid \mathcal{E} \text{ is an ECSP such that } hw(\mathcal{E}) \leq k \text{ and } \nu \in \text{sol(\mathcal{E})}\}$. Then it holds that $k$-HW-ECSP is LogCFL-complete under many-to-one logspace reductions [GLS02], which, as mentioned before, implies that $k$-HW-ECSP can be solved in polynomial time and is highly parallelizable. Concerning to our investigation, we are interested in the following fundamental problem associated to the evaluation problem for $k$-HW-ECSP:
Problem: \(\text{#}_k\text{-HW-ECSP}\)
Input: An ECSP \(\mathcal{E}\) such that \(\text{hw}(\mathcal{E}) \leq k\)
Output: \(|\text{sol}(\mathcal{E})|\)

As for the case of acyclic ECSPs, it was shown in [PS13] that for every \(k \geq 1\), the counting problem \(\text{#}_k\text{-HW-ECSP}\) is \(#P\)-complete, and it can be solved in polynomial time if we restrict to the case of CSPs (that is, ECSPs of the form \((V, \mathcal{P})\) where \(V\) is the set of variables of \(\mathcal{P}\)). Thus, a fundamental question is whether \(\text{#}_k\text{-HW-ECSP}\) admits an FPRAS. In the following proposition, we give a positive answer to this question. Notice that from this result we obtain Proposition 5.2 as a corollary given that \(#\text{AECSP} = \text{#}_1\text{-HW-ECSP} \).

**Proposition 5.4.** For every \(k \geq 1\), it holds that \(\text{#}_k\text{-HW-ECSP}\) admits an FPRAS.

The proof of this proposition is given in Appendix A.1.

### 5.2 Conjunctive queries

 Conjunctive queries are the most common class of queries used in database systems, as well as the most studied in the database literature. They correspond to the Select-Project-Join queries of SQL. Formally, a conjunctive query (CQ) is an expression of the form:

\[
Q(\bar{x}) \leftarrow S_1(\bar{u}_1) \land \cdots \land S_n(\bar{u}_n),
\]

where for every \(i \in [n]\), \(S_i\) is a \(k\)-ary relation symbol and \(\bar{u}_i\) is a \(k\)-ary tuple of variables and constants, and \(\bar{x}\) is a tuple of variables such that each variable in \(\bar{x}\) occurs in some \(\bar{u}_i\). The symbol \(Q\) is used as the name of the query. Intuitively, the left-hand side \(S_1(\bar{u}_1) \land \cdots \land S_n(\bar{u}_n)\) of \(Q\) is used to specify a pattern over a database, while the tuple \(\bar{x}\) is used to store the answer to the query when such a pattern is found. More precisely, a database \(I\) is a set of facts of the form \(T(\bar{a})\), which indicates that \(\bar{a}\) is a tuple in the table \(T\) in \(I\). Then a homomorphism from \(Q\) to \(I\) is a function from the set of variables occurring in \(Q\) to the constants in \(I\) such that for every \(i \in [1, n]\), it holds that \(S_i(h(\bar{u}_i))\) is a fact in \(I\), where \(h(\bar{u}_i)\) is obtained by applying \(h\) to each component of \(\bar{u}_i\) leaving the constants unchanged. Moreover, given such a homomorphism \(h\), the tuple of constants \(h(\bar{x})\) is said to be an answer to \(Q\) over the database \(I\), and \(\text{answer}(Q, I)\) is defined as the set of answers of \(Q\) over \(I\).

**Example 5.5.** Consider a university database consisting of the following relations: \(\text{Enrolled}(\text{sno}, \text{cno})\) indicates that a student with identifier \(\text{sno}\) is taking a course with identifier \(\text{cno}\), \(\text{Course}(\text{cno}, \text{dno})\) indicates that a course with identifier \(\text{cno}\) is given by a department with identifier \(\text{dno}\), and \(\text{Dept}(\text{dno}, \text{name})\) indicates that \(\text{name}\) is the name of the department with identifier \(\text{dno}\). The following is a simple university database \(I\):

| Enrolled | sno  | cno |
|----------|------|-----|
| st1      | mat101 |     |
| st1      | stat101 |   |
| st2      | mat101 |     |
| st2      | cs101 |     |

| Course | cno | dno |
|--------|-----|-----|
| cs101  | dept1 |     |
| stat101 | dept3 |   |
| mat101 | dept2 |     |

| Dept | dno | name          |
|------|-----|---------------|
| dept1 | Computer Science |
| dept2 | Mathematics |
| dept3 | Statistics |

The following CQ can be used to retrieve the list of students that are taking at least one course:

\[Q_1(x) = \exists y \text{Enrolled}(x, y).\]
In fact, we have that st1 is an answer to \( Q_1 \) over \( I \) since the function \( h \) defined as \( h(x) = \text{st1} \) and \( h(y) = \text{mat101} \) is a homomorphism from \( Q_1 \) to \( I \) such that \( h(x) = \text{st1} \); in particular, we have that \( h \) is a homomorphism since \( \text{Enrolled}(h(x), h(y)) = \text{Enrolled}(\text{st1}, \text{mat101}) \) is a fact in \( I \). In the same way, it can be shown that st2 is an answer to \( Q_1 \) over \( I \). In fact, we have that \( \text{answer}(Q, I) = \{ \text{st1}, \text{st2} \} \).

Moreover, the following CQ can be used to retrieve the list of students that are taking at least one course in the Department of Computer Science:

\[
Q_2(x) = \exists y \exists z \text{Enrolled}(x, y) \land \text{Course}(y, z) \land \text{Dept}(z, \text{“Computer Science”}).
\]

Notice that \( Q_2 \) includes the constant “Computer Science”.

The problem of evaluating a conjunctive query over a database is a fundamental problem that has been widely studied. In its decision version, it can be defined as the problem of verifying, given a CQ \( Q \), a database \( I \) and a tuple of constants \( \bar{a} \), whether \( \bar{a} \) is an answer to \( Q \) over \( I \) (that is, whether \( \bar{a} \in \text{answer}(Q, I) \)). It is well-known that such a problem is NP-complete [CM77], so many restrictions ensuring tractability have been explored in the literature [Yan81, CR97, GSS01, GLS02, GGLS16]. In particular, starting with the work in [Yan81], a line of research that has been particularly fruitful in finding tractable cases is based on analyzing the degree of cyclicity of the underlying structure of a conjunctive query.

The evaluation problem for CQs is equivalent to the evaluation problem for existential CSPs [KV00], which are studied in Section 5.1. To see why this is the case, we first notice that we can focus on CQs not mentioning constants, as a constant \( a \) in a query \( Q \) can always be encoded by adding a fresh table \( T_a \) that consists only of the constant \( a \), replacing \( a \) by a fresh variable \( x_a \) in \( Q \) and adding to \( Q \) an atom \( T_a(x_a) \). For instance, to evaluate \( Q(x) \leftarrow R(x, a) \land O(x, y, a) \) over a database \( I \) is equivalent to evaluate \( \text{CQ} \) \( Q'(x) \leftarrow R(x, x_a) \land O(x, y, x_a) \land T_a(x_a) \) over a database \( I' \) constructed by adding to \( I \) a table \( T_a \) consisting only of the constant \( a \). Thus, when showing how to transform a CQ \( Q \) and a database \( I \) into an ECSP \( \mathcal{E}_{Q, I} \),\(^{14}\) we can assume that \( Q \) is of the (14) but without mentioning any constants. More precisely, we define \( \mathcal{E}_{Q, I} = (U, \mathcal{P}) \), where \( U \) is the set of variables occurring in \( \bar{x} \), \( \mathcal{P} = (V, D, C) \), \( V \) is the set of variable occurring in \( S_1(\bar{u}_1) \land \cdots \land S_n(\bar{u}_n) \), \( D \) is the set of constants occurring in \( I \) and \( C \) is the following set of constraints. For each \( i \in [n] \), define a constraint \( C_i = (\bar{a}, R_i) \), where \( R_i = \{ \bar{a} \mid S_i(\bar{a}) \text{ is a fact in } I \} \). Moreover, assuming that \( \bar{x} = (x_1, \ldots, x_m) \), and that \( \bar{a} = (a_1, \ldots, a_m) \) is a tuple of constants from \( D \) that is consistent with \( \bar{x} \) in the sense that \( a_i = a_j \) whenever \( x_i = x_j \), define an assignment \( \nu_a \) as \( \nu_a(x_i) = a_i \) for every \( i \in [m] \). Then it is easy to see that for every tuple \( \bar{a} \) of constants, it holds that:

\[
\bar{a} \in \text{answer}(Q, I) \quad \text{if and only if} \quad \nu_a \in \text{sol}(\mathcal{E}_{Q, I}).
\]

This tight connection can be used to extend the notions of acyclicity given in Sections 5.1.1 and 5.1.2 to the case of CQs. More precisely, given a CQ \( Q \), notice that by definition of the notions of acyclicity in Section 5.1.1 and hypertree-width in Section 5.1.2, we have that for every pair of databases \( I_1, I_2 \), it holds that \( \mathcal{E}_{Q, I_1} \) is acyclic if and only if \( \mathcal{E}_{Q, I_2} \) is acyclic, and \( \text{hw}(\mathcal{E}_{Q, I_1}) = \text{hw}(\mathcal{E}_{Q, I_2}) \). Thus, given an arbitrary database \( I \), conjunctive query \( Q \) is said to be acyclic if and only if \( \mathcal{E}_Q \) is acyclic [Yan81, GLLS08], and \( \text{hw}(Q) \) is defined as \( \text{hw}(\mathcal{E}_Q) \) [GLS02].

Concerning to our investigation, we are interested in the fundamental problem of counting the number of answers to a conjunctive query over a database. In general, this problem is \#P-complete and cannot admit an FPRAS (unless \( \text{NP} = \text{RP} \), given that the evaluation problem for CQs is NP-complete). Thus, we focus on the following fundamental problems where the degree of cyclicity of conjunctive queries is bounded.

\(^{14}\) The transformation in the opposite direction is done analogously.
Problem: \#ACQ
Input: An acyclic conjunctive query \( Q \) and a database \( I \)
Output: \(|\text{answer}(Q, I)|\)

Problem: \#k-HW
Input: A conjunctive query \( Q \) such that \( \text{hw}(Q) \leq k \) and a database \( I \)
Output: \(|\text{answer}(Q, I)|\)

It is important to notice that \#ACQ = \#1-HW, as it is proved in [GLS02] that a CQ is acyclic if and only if \( \text{hw}(Q) = 1 \). However, we will keep both languages for historical reasons, as acyclic conjunctive queries were defined two decades earlier and are widely used and more popular in databases. Both \#ACQ and \#k-HW, for a fixed \( k \geq 1 \), are known to be \#P-complete [PS13].

On the positive side, from the characterization of the evaluation problem of CQs in terms of the evaluation problem for ECSPs and Propositions 5.2 and 5.4, we conclude that these problems admit FPRAS.

**Proposition 5.6.** \#ACQ admits an FPRAS, and for every \( k \geq 1 \), \#k-HW admits an FPRAS.

### 5.3 Union of conjunctive queries

An important and well-studied extension of the class of conjunctive queries is obtained by adding the union operator. A union of conjunctive queries (UCQ) is an expression of the form:

\[
Q(\bar{x}) \leftarrow Q_1(\bar{x}) \lor \cdots \lor Q_m(\bar{x}),
\]

where \( Q_i(\bar{x}) \) is a conjunctive query for each \( i \in \{1, \ldots, m\} \), and the same tuple \( \bar{x} \) of output variables is used in the CQs \( Q_1(\bar{x}), \ldots, Q_m(\bar{x}) \). As for the case of CQs, the symbol \( Q \) is used as the name of the query. For instance, if we consider again the CQ \( Q_2(x) \) and the database given in Example 5.5, and if we define \( Q_6(x) \) as the following CQ:

\[
Q_6(x) = \exists y \exists z \ Enrolled(x, y) \land Course(y, z) \land Dept(z, \text{“Statistics”}),
\]

then the union of conjunctive queries \( Q(x) \leftarrow Q_2(x) \lor Q_6(x) \) can be used to retrieve the list of students that are taking at least one course in the Department of Computer Science or in the Department of Statistics.

A tuple \( \bar{a} \) is said to be an answer of UCQ \( Q \) in (15) over a database \( I \) if and only if \( \bar{a} \) is an answer to \( Q_i \) over \( I \) for some \( i \in \{1, \ldots, m\} \). Thus, we have that:

\[
\text{answer}(Q, I) = \bigcup_{i=1}^{m} \text{answer}(Q_i, I).
\]

As for the case of CQs, without loss of generality we can focus on UCQs without constants. As expected, the problem of verifying, given a UCQ \( Q \), a database \( I \) and a tuple of constants \( \bar{a} \), whether \( \bar{a} \) is an answer to \( Q \) over \( I \) is an NP-complete problem [CM77]. Also as expected, the evaluation problem for union of acyclic conjunctive queries can be solved in polynomial time, given that the evaluation problem for acyclic CQs can be solved in polynomial time. Concerning to our investigation, we are interested in the following fundamental problem associated to the evaluation problem for union of acyclic conjunctive queries:
As expected from the result for conjunctive queries, \#UACQ is \#P-complete [PS13]. However, \#UACQ remains \#P-hard even if we focus on the case of UCQs without existentially quantified variables, that is, UCQs of the form (15) where \( \bar{x} \) consists of all the variables occurring in CQ \( Q_i(\bar{x}) \) for each \( i \in [m] \). Notice that this is in sharp contrast with the case of CQs, where \#ACQ can be solved in polynomial time if we focus on case of CQs without existentially quantified variables [PS13]. However, by using Proposition 5.6, we are able to provide a positive result about the possibility of efficiently approximating \#UACQ.

**Proposition 5.7.** \#UACQ admits an FPRAS.

**Proof.** We need to prove that there exists a randomized algorithm \( A \) and a polynomial \( p(x,y) \) such that \( A \) receives as input a union of acyclic conjunctive queries \( Q \), a database \( I \) and \( \epsilon \in (0,1) \), \( A \) works in time \( p(||Q||+||I||,\epsilon^{-1}) \), where \( ||Q||+||I|| \) is the size of \( Q \) and \( I \), and \( A \) satisfies the following condition:

\[
\Pr[|A(Q,I,\epsilon)-|\text{answer}(Q,I)|| \leq \epsilon \cdot |\text{answer}(Q,I)|] \geq \frac{3}{4}.
\]

Assume that \( Q \) is of the form (15), from which we have that \( \text{answer}(Q,I) = \bigcup_{i=1}^{m} \text{answer}(Q_i,I) \) and, therefore, \( |\text{answer}(Q,I)| = |\bigcup_{i=1}^{m} \text{answer}(Q_i,I)| \). Thus, we know from [KL83] that the algorithm \( A \) can be constructed if three conditions are satisfied: (a) there exists a polynomial-time algorithm that verifies whether \( \bar{a} \in \text{answer}(Q_i,I) \); (b) there exists a randomized polynomial-time algorithm that generates an element in \( \text{answer}(Q_i,I) \) with uniform distribution; and (c) there exists a polynomial-time algorithm that computes \( |\text{answer}(Q_i,I)| \). In our case, property (a) holds as each \( Q_i(\bar{x}) \) is an acyclic conjunctive query, while condition (c) cannot hold unless \( \text{FP} = \#\text{P} \), given that \#ACQ is \#P-complete. However, as shown in [GJK+97], the existence of algorithm \( A \) can still be guaranteed under condition (a) and the existence of an FPRAS for the function \( (Q_i,I) \mapsto |\text{answer}(Q_i,I)| \), as this latter condition also implies the existence of a fully polynomial-time almost uniform generator for \( \text{answer}(Q_i,I) \) [JVV86a]. Therefore, we conclude that algorithm \( A \) exists from Proposition 5.6.

As a final fundamental problem, we consider the problem of counting the number of solutions of a union of conjunctive queries of bounded hypertree-with.

### Problem:
\#k-UHW

**Input:** A union of conjunctive query \( Q(\bar{x}) \leftarrow Q_1(\bar{x}) \lor \cdots \lor Q_m(\bar{x}) \) such that \( \text{hw}(Q_i) \leq k \) for every \( i \in [m] \), and a database \( I \).

**Output:** \( |\text{answer}(Q,I)| \)

By using the same ideas as in the proof of Proposition 5.7, we obtain from Proposition 5.6 that:

**Proposition 5.8.** For every \( k \geq 1 \), it holds that \#k-UHW admits an FPRAS.

### 5.4 Nested words

Nested words have been proposed as a model for the formal verification of correctness of structured programs that can contain nested calls to subroutines [AEM04, AM04, AM09]. In particular, the execution of a program is viewed as a linear sequence of states, but where a matching relation is
used to specify the correspondence between each point during the execution at which a procedure is called with the point when we return from that procedure call.

Formally, a binary relation \( \mu \) on an interval \([n]\) is a matching if the following conditions hold:

(a) if \( \mu(i, j) \) holds then \( i < j \); (b) if \( \mu(i, j) \) and \( \mu(i, j') \) hold then \( j = j' \), and if \( \mu(i, j) \) and \( \mu(i', j) \) hold then \( i = i' \); (c) if \( \mu(i, j) \) and \( \mu(i', j') \) hold, where \( i \neq i' \) and \( j \neq j' \), then either \([i, j] \cap [i', j'] = \emptyset \) or \([i, j] \subseteq [i', j'] \) or \([i', j'] \subseteq [i, j] \). Moreover, given a finite alphabet \( \Sigma \), a nested word of length \( n \) over \( \Sigma \) is a tuple \( \bar{w} = (w, \mu) \), where \( w \in \Sigma^* \) is a string of length \( n \), and \( \mu \) is a matching on \([n]\).

A position \( i \) in a nested word \( \bar{w} \) is a call (resp., return) position if there exists \( j \) such that \( \mu(i, j) \) (resp., \( \mu(j, i) \)) holds. If \( i \) is neither a call nor a return position in \( \bar{w} \), then \( i \) is said to be an internal position in \( \bar{w} \). Figure 5 shows a nested word (without the labeling with alphabet symbols). Solid lines are used to draw the linear edges that define a standard word, while nesting edges are drawn using dashed lines. Thus, the relation \( \mu \) is \( \{(2,4),(5,6),(1,7)\} \), the set of call positions is \( \{1,2,5\} \), the set of return positions is \( \{4,6,7\} \) and the set of internal positions is \( \{3,8\} \).

Properties to be formally verified are specified by using nested word automata. Such automata have the same expressiveness as monadic second order logic over nested words [AM09], so they are expressive enough to allow the specification and automatic verification of a large variety of properties over programs with nested calls to subroutines. Formally, a (nondeterministic) nested word automaton (NWA) \( \mathcal{N} \) is a tuple \((\Sigma, S_0, F, P, \Delta_C, \Delta_I, \Delta_R)\) consisting of a finite set of states \( S \), an alphabet \( \Sigma \), a set of initial states \( S_0 \subseteq S \), a set of final states \( F \subseteq S \), a finite set of hierarchical symbols \( P \), a call-transition relation \( \Delta_C \subseteq S \times \Sigma \times S \times P \), an internal-transition relation \( \Delta_I \subseteq S \times \Sigma \times S \), and a return-transition relation \( \Delta_R \subseteq S \times P \times \Sigma \times S \).

An NWA \( \mathcal{N} = (\Sigma, S_0, F, P, \Delta_C, \Delta_I, \Delta_R) \) works as follows with input a nested word \( \bar{w} \). \( \mathcal{N} \) starts in an initial state in \( S_0 \) and reads \( \bar{w} \) from left to right. The state is propagated along the linear edges of \( \bar{w} \) as in case of a standard word automaton. However, at a call position in \( \bar{w} \), the nested word automaton propagates a state along the linear edge together with a hierarchical symbol along the nesting edge of \( \bar{w} \). At a return position in \( \bar{w} \), the new state is determined based on the state propagated along the linear edge as well as the symbol along the incoming nesting edge. Formally, a run \( \rho \) of the automaton \( \mathcal{N} \) over a nested word \( \bar{w} = (a_1 \cdots a_n, \mu) \) is a sequence \( s_0, s_1, \ldots, s_n \) of states along the linear edges, and a sequence \( p_i \), for every call position \( i \), of hierarchical symbols along the nesting edges, such that: (a) \( s_0 \in S_0 \); (b) for each call position \( i \), it holds that \( (s_{i-1}, a_i, s_i, p_i) \in \Delta_C \); (c) for each internal position \( i \), it holds that \( (s_{i-1}, a_i, s_i) \in \Delta_I \); and (d) for each return position \( i \) such that \( \mu(j, i) \) holds, we have that \( (s_{i-1}, p_j, a_i, s_i) \in \Delta_R \). Moreover, the run \( \rho \) is accepting if \( s_n \in F \), and

\[
\mathcal{L}(\mathcal{N}) = \{ \bar{w} \mid \bar{w} \text{ is a nested word over } \Sigma^* \text{ and there exists an accepting run of } \mathcal{N} \text{ with input } \bar{w} \}.
\]

The emptiness problem for nested word automata ask whether, given a NWA \( \mathcal{N} \), there exists a nested word \( \bar{w} \) accepted by \( \mathcal{N} \). This is a fundamental problem when looking for faulty executions of a program with nested calls to subroutines; if \( \mathcal{N} \) is used to encode the complement of a property we expect to be satisfied by a program, then a nested word \( \bar{w} \in \mathcal{L}(\mathcal{N}) \) encodes a bug of this program. In this sense, the following is also a very relevant problem for understanding how faulty a program is:

| Problem: | \#NWA |
| Input: | A nested word automaton \( \mathcal{N} \) and a string \( 0^n \) |
| Output: | \( \{|\bar{w} \in \mathcal{L}(\mathcal{N}) \mid |\bar{w}| = n\| \} \) |

As there exists a trivial polynomial-time parsimonious reduction from \#NFA to \#NWA, we have that \#NWA is \#P-complete. Interestingly, from the existence of an FPRAS for \#BTA (see Corollary 3.9) and the results in [AM09] showing how nested word automata can be represented by using tree automata over binary trees, it is possible to prove that:
Theorem 5.9. \#NW A admits an FPRAS

5.5 Knowledge compilation

Model counting is the problem of counting the number of satisfying assignments given a propositional formula. Although this problem is \#P-complete [Val79], there have been several approaches to tackle it [GSS09]. One of them comes from the field of knowledge compilation, a subarea in artificial intelligence [DM02]. Roughly speaking, this approach consists in dividing the reasoning process in two phases. The first phase is to compile the formula into a target language (e.g. Horn formulae, BDDs, circuits) that has good algorithmic properties. The second phase is to use the new representation to solve the problem efficiently. The main goal then is to find a target language that is expressive enough to encode a rich set of propositional formulae and, at the same time, that allows for efficient algorithms to solve the counting problem.

One of the most used formalism in knowledge compilation are circuits in Negation Normal Form (NNF for short). An NNF circuit \( C = (V, E, g_0, \mu) \) is a directed acyclic graph \( (V, E) \) where \( V \) are called gates, edges \( E \) are called wires, and \( g_0 \in V \) is a distinguished gate called the output gate. The function \( \mu \) assigns a type to each gate that can be \( \land \) (AND), \( \lor \) (OR), or a literal (i.e. a variable or the negation of a variable). We assume that all literals have in-degree 0 and we call them input gates. Without loss of generality, we assume that all \( \land \)-gate and \( \lor \)-gate have in-degree two (if not, we can convert any NNF circuit in poly-time to binary gates). For a gate \( g \) we define the set \( \text{Vars}(g) \) of all variables whose value can alter the value of \( g \), formally, \( v \in \text{Vars}(g) \) if and only if there exists an input gate \( g' \) with variable \( v \) (i.e. \( \mu(g) = v \) or \( \mu(g) = \bar{v} \)) and there is a path from \( g' \) to \( g \) in \( (V, E) \). A valuation for \( C \) is a mapping \( \nu \) from the variables of \( C \) to \{0, 1\}. The valuation of \( C \) with \( \nu \), denoted by \( \nu(C) \), is the value (i.e. 0 or 1) taken by \( g_0 \) when \( C \) is evaluated in a bottom up fashion.

A target language for knowledge compilation that has attracted a lot of attention is the class of DNNF circuits. An NNF circuit \( C \) is called decomposable [Dar01a] if and only if for every \( \land \)-gate \( g \) with incident gates \( g_1, g_2 \) it holds that \( \text{Vars}(g_1) \cap \text{Vars}(g_2) = \emptyset \). In other words, if the incident gates of every \( \land \)-gate share no variables. For example, one can easily check that the NNF circuit of Figure 6 is decomposable. DNNF is the set of all NNF circuits that are decomposable. DNNF has good algorithmic properties in terms of satisfiability and logical operations. Furthermore, DNNF can be seen as a generalization of DNF formulae and, in particular, of binary decision diagrams (BDD), in the sense that every BDD can be transformed into a DNNF circuit in polynomial time. Nevertheless, DNNF is exponentially more succinct than DNF or BDD, and then it is a more appealing language for knowledge compilation.

Regarding model counting, DNNF circuits can easily encode \#P-complete problems (e.g. \#DNF)
and, therefore, researchers have look into subclasses of DNNF with efficient counting properties. Deterministic DNNF (d-DNNF for short) is a subclass of DNNF where the counting problem can be solved in polynomial time (see [Dar01b] for a definition of d-DNNF). Indeed, several problems can be compiled into d-DNNF circuits, finding applications in probabilistic reasoning [CDJ06], query evaluation [BLRS17], planning [BG06], among others.

However, as pointed out in [PD08] the compilation into d-DNNF circuits usually satisfies a structural property between variables, which naturally brings the class of structured DNNF circuits.

A v-tree is a binary tree \( t \) whose leaves are in one-to-one correspondence with a set of variables. Similar than for circuits, for a node \( u \) in a v-tree, we denote by \( \text{Vars}(u) \) the set of all variables in the leaves of the subtree rooted at \( u \). Then we say that a DNNF circuit \( C \) respects a v-tree \( t \) if for every \( \land \)-gate \( g \) and the two incident gates \( g_1 \) and \( g_2 \) of \( g \), there exists a node \( u \) in \( t \) such that \( \text{Vars}(g_1) \subseteq \text{Vars}(u_1) \) and \( \text{Vars}(g_2) \subseteq \text{Vars}(u_2) \), where \( u_1 \) and \( u_2 \) are the left and right child of \( u \) in \( t \), respectively. We say that a DNNF circuit \( C \) is structured if and only if there exists a v-tree \( t \) such that \( C \) respects \( t \). For example, in the right-hand side of Figure 6, we show a v-tree for variables \( \{x, y, z, u, v, w\} \). The red dashed lines show how \( \land \)-gates have to be assigned to the nodes in the v-tree in order for the circuit to respect this v-tree. Structured DNNF is the class of all DNNF circuits that are structured. As it was already mentioned, the compilation into d-DNNF circuits usually produces circuits that are also structured [PD08]. Structured DNNF have been recently used for efficient enumeration [ABJM17, ABMN19] and in [OD14] it was shown that CNF formulae with bounded width (e.g. CV-width) can be efficiently compiled into structured DNNF circuits. Unfortunately, structured DNNF circuits includes the class of DNF formulae and, therefore, its underlying counting problem is also \#P-complete. Specifically, consider the following problem:

| Problem: | \#StructuredDNNF |
|----------|------------------|
| Input:   | A DNNF circuit \( C \) and a v-tree \( t \) such that \( C \) respects \( t \). |
| Output:  | \( |\{\nu | \nu(C) = 1\}| \) |

By using the existence of an FPRAS for \#TA, we can show that \#StructuredDNNF also admits an FPRAS.

**Proposition 5.10.** \#StructuredDNNF admits an FPRAS.

**Proof.** The connection between structured DNNF and tree automata was already used in [ABJM17, ABMN19], so this connection is not new. Here, we show that there exists a parsimonious reduction from \#StructuredDNNF into \#TA, which proves the FPRAS for structured DNNF.
Let $t$ be a $v$-tree and $C = (V,E,g_0,\mu)$ be a DNNF circuit such that $C$ respects $t$. Let $V_{\land}$ be all gates in $V$ that are $\land$-gates or input gates. Furthermore, let $f : V_{\land} \rightarrow t$ be the function that realizes that $C$ respects $t$, namely, for every $\land$-gate $g$ and gates $g_1$ and $g_2$ incident to $g$ it holds that $\text{Vars}(g_1) \subseteq \text{Vars}(f(g) \cdot 1)$ and $\text{Vars}(g_2) \subseteq \text{Vars}(f(g) \cdot 2)$. We also assume that if $g \in V_{\land}$ is a literal, then $f(g)$ is the leaf in $t$ that has the same variable as $g$ in $C$. One can easily check that for two gates $g_1, g_2 \in V_{\land}$ in $C$, if there is a path from $g_2$ to $g_1$ in $C$, then $f(g_2)$ is a descendant of $f(g_1)$ in $t$. Without loss of generality, we assume that $g_0$ is a $\land$-gate in $C$ and $f(g_0) = \epsilon$. Finally, for any $\land$-gate $g$ we define the set $D(g)$ of all gates $g' \in V_{\land}$ such that there exists a path from $g'$ to $g$ in $(V,E)$ passing only through $\lor$-gates. Intuitively, if $g' \in D(g)$ then $g'$ is directly affecting the value of $g$ in the sense that if $g'$ is true, then at least one of the incident wires to $g$ is true.

The idea for the parsimonious reduction is to construct a tree automaton $\mathcal{T}_C$ that will accept trees that encodes valuations that makes $C$ true. To encode a valuation, $\mathcal{T}_C$ will only accept binary trees having exactly the same tree-shape as $t$, but its leaves are labeled with 0 or 1 (internal node will have any symbol, e.g. @$\@$). Given that leaves of $t$ are in one-to-one correspondence with the variables in $C$, then these encodings are in one-to-one correspondence with valuations in $C$. So, for a valuation $\nu$ let $t_\nu$ be the tree that has the same tree-shape as $t$ and whose leaves encode $\nu$. Furthermore, suppose that $t_\nu$ is an input tree for the tree automaton $\mathcal{T}_C$. For checking that $\nu(C) = 1$, the states of $\mathcal{T}_C$ will be either nodes $u \in t$ or pairs of the form $(u,g)$ where $u \in t$, $g \in V_{\land}$, and $f(g)$ is a descendant of $u$ in $t$. A node $u$ in a state (e.g. in the pair $(u,g)$) will take care of checking that the tree-shape of $t_\nu$ is the same as $t$. On the other hand, the gate $g$ in the pair $(u,g)$ will be use to navigate $C$ and find whether $g$ is evaluated to 1 given the valuation encoded by $t_\nu$. When $f(g)$ is a strict descendant of $u$ (i.e. $f(g) \neq u$) we will continue down $t_\nu$ trying to find a node $u'$ such that $f(g) = u'$. When a node $u$ with $f(g) = u$ is found, then to evaluate $g$ to 1 we need to find two gates $g_1, g_2 \in D(g)$ that are also evaluated to 1 with $\nu$. Given that $C$ respects $t$ we know that $f(g_1)$ and $f(g_2)$ must be descendants of $f(g)$ in $t_\nu$, and then $\mathcal{T}_C$ will recurse into the states $(u_1, g_1)$ and $(u_2, g_2)$ continuing into $g_1$ and $g_2$. If the non-deterministic decisions of $\mathcal{T}_C$ are taken correctly, $\mathcal{T}_C$ will reach the leaves $u$ of $t_\nu$ that has the same variable as the gate $g$ and it will check if the value $t_\nu(u)$ is correct with respect to $g$.

Let $\mathcal{T}_C = (Q, \Sigma, \Delta, s_{\text{init}})$ be the tree automaton constructed from $C$ and $t$ such that $\Sigma = \{@, 0, 1\}$, $Q = t \cup \{(u,g) \in t \times V_{\land} \mid \text{f(g) is a descendant of u in t}\}$, and $s_{\text{init}} = (\lambda, g_0)$ (recall that we assume that $g_0$ is a $\land$-gate and $f(g_0) = \lambda$). We define the transition relation $\Delta$ by case analysis:

- For $u \in t$ and $u$ is not a leaf, then $(u, @, u1 \cdot u2) \in \Delta$.
- For $u \in t$ and $u$ is a leaf, then $(u, a, \lambda) \in \Delta$ for every $a \in \{0, 1\}$.
- For $(u,g) \in Q$ such that $f(g) \neq u$, if $f(g)$ is a descendant of $u1$, then $((u,g), @, (u1,g) \cdot u2) \in \Delta$. Otherwise, if $f(g)$ is a descendant of $u2$, then $((u,g), @, u1 \cdot (u2,g)) \in \Delta$.
- For $(u,g) \in Q$ such that $f(g) = u$ and $u$ is not a leaf in $t$, then $((u,g), @, (u1, g1) \cdot (u2,g2)) \in \Delta$ for every $g1,g2 \in D(g)$ with $f(g1)$ is a descendant of $u1$ and $f(g2)$ is a descendant of $u2$.
- For $(u,g) \in Q$ such that $f(g) = u$ and $u$ is a leaf in $t$, then $((u,g), 1, \lambda) \in \Delta$ iff $\mu(g) = t(u)$, that is, if $g$ is a positive literal and its variable coincide with the variable assign to $u$ in $t$. Similarly, $((u,g), 0, \lambda) \in \Delta$ iff $\mu(g) = -t(u)$.

Finally, given a circuit $C$ the reduction produces the tree automaton $\mathcal{T}_C$ and the value $0^{|t|}$. From the construction, it is straightforward to check that $\mathcal{T}_C$ will accept trees that have the same tree-shape as $t$ and whose leaves encode a valuation of $C$. Furthermore, for a valuation $\nu$ and its tree $t_\nu$ one can check that $\nu(C) = 1$ if, and only if, $t_\nu \in L(\mathcal{T}_C)$. Therefore, the reduction is parsimonious.
Finally, the number of states and transitions of $T_C$ is polynomial in the size of $C$ and $t$, and thus the reduction can be computed in polynomial time.

6 Conclusion Remarks

In this work, we provide the first polynomial time approximation algorithms for the $n$-slice of regular tree languages, and also the first polynomial time uniform samplers from this class. In addition to the recent result of [ACJR19], which provides polynomial time algorithms for the restricted class of regular languages, to the best of our knowledge this work demonstrates only the second instance of an ambiguous formal language for which an FPRAS and polynomial time sampler exist. Thus, a natural open question is which other formal languages admit efficient approximation and sampling algorithms. It would be particularly interesting to obtain negative results: namely, examples of formal languages which do not admit an FPRAS under known complexity assumptions. Such a candidate assumption would be a reduction from the problem #BIS of counting the number of independent sets in a bipartite graph, which is complete for an intermediate class of counting problems for which it is unknown whether polynomial approximation algorithms exists [DGGJ04].

One particularly interesting question is whether our results can be obtained by Monte-Carlo Markov Chain approaches. Despite being an extremely successful technique in the design of randomized approximation algorithms, so far it appears that formal languages are resistant to this approach, and do not admit any naturally convergent chains (for instance, [GJK+97] reports a similar observation). Thus, the discovery of a quickly mixing Markov chain for tree languages, or even the subclass of regular languages, would be a highly interesting and potentially enlightening result.

A Appendix: Proofs and Intermediate Results

A.1 Proof of Proposition 5.4

Fix $k \geq 1$. We provide a polynomial-time parsimonious reduction from $\#k$-HW-ECSP to $\#TA$, as it is known that the latter problem admits an FPRAS (see Corollary 3.9). Let $\mathcal{E} = (U, P)$ be an ECSP such that $P = (V, D, C)$ and $\text{hw}(\mathcal{E}) \leq k$. We have from [GLS02] that there exists a polynomial-time algorithm that, given $P$, produces a hypertree decomposition $(T, \chi, \xi)$ for $P$ of width $k$, where $T = (N, E)$. Moreover, tuple($C$) is used to denote the set of tuples occurring in the constraints in $C$, and for every $t \in \text{tuple}(C)$, notation $R_t$ is used for the relation such that $(t, R_t) \in C$. Finally, we can assume that $(T, \chi, \xi)$ is a complete hypertree decomposition in the sense that for every $t \in \text{tuple}(C)$, there exists $p \in N$ such that $\text{var}(t) \subseteq \chi(p)$ and $t \in \xi(p)$ [GLS02]. Finally, assume that $n = |N|$.

In what follows, we define a tree automaton $T = (S, \Sigma, \Delta, S_0)$ such that

$$|\text{sol}(\mathcal{E})| = \{|t \in \mathcal{L}(T) \mid |t| = n\}|.$$

Notice that for the sake of presentation, we are assuming that $T$ has a set $S_0$ of initial states, instead of a single initial state. Such an automaton can be translated in polynomial time into a tree automaton with a single initial state. Given a tuple of variables $\bar{x} = (x_1, \ldots, x_r)$ and a tuple of constants $\bar{a} = (a_1, \ldots, a_r)$, we use notation $\bar{x} \mapsto \bar{a}$ to indicate that variable $x_i$ is assigned value $a_i$ for every $i \in [r]$. Notice that $\bar{x}$ can contain repeated variables, and if this is the case then each occurrence of a repeated variable is assigned the same value. For example, $(x, y, x, y) \mapsto (a, b, a, b)$ is an assignment, while $(x, y, x, y) \mapsto (a, b, a, c)$ is not an assignment if $b \neq c$. Besides, notice that $\emptyset \mapsto \emptyset$ is an assignment. Moreover, two such assignments $\bar{x} \mapsto \bar{a}$ and $\bar{y} \mapsto \bar{b}$ are said to be consistent.
if for every variable \( z \) that occurs both in \( \bar{x} \) and \( \bar{y} \), it holds that the same value is assigned to \( z \) in \( \bar{x} \mapsto \bar{a} \) and in \( \bar{y} \mapsto \bar{b} \). Then for every \( p \in N \) such that:

\[
\begin{align*}
\chi(p) &= \{ x_1, \ldots, x_r \} \\
\xi(p) &= \{ \bar{t}_1, \ldots, \bar{t}_s \},
\end{align*}
\]

and assuming that \( \chi(p) \cap U = \{ z_1, \ldots, z_o \} \), \( \bar{x} = (x_1, \ldots, x_r) \) and \( \bar{z} = (z_1, \ldots, z_o) \), we define

\[
S(p) = \{ [p, \bar{x} \mapsto \bar{a}, \bar{z} \mapsto \bar{b}, \bar{t}_1 \mapsto \bar{c}_1, \ldots, \bar{t}_s \mapsto \bar{c}_s] \mid \\
\quad \bar{c}_i \in R_{t_i} \text{ for every } i \in [s], \\
\quad \bar{x} \mapsto \bar{a} \text{ is consistent with } \bar{z} \mapsto \bar{b}, \\
\quad \bar{x} \mapsto \bar{a} \text{ is consistent with } \bar{t}_i \mapsto \bar{c}_i \text{ for every } i \in [s], \\
\quad \text{and } \bar{t}_i \mapsto \bar{c}_i \text{ is consistent with } \bar{t}_j \mapsto \bar{c}_j \text{ for every } i, j \in [s] \},
\]

and

\[
\Sigma(p) = \{ [p, \bar{z} \mapsto \bar{b}] \mid \exists \bar{a} \exists \bar{c}_1 \cdots \exists \bar{c}_s : [p, \bar{x} \mapsto \bar{a}, \bar{z} \mapsto \bar{b}, \bar{t}_1 \mapsto \bar{c}_1, \ldots, \bar{t}_s \mapsto \bar{c}_s] \in S(p) \}
\]

With this terminology, we define \( S_0 = S(p_0) \), where \( p_0 \) is the root of the hypertree decomposition \( \langle T, \chi, \xi \rangle \), and we define:

\[
\begin{align*}
S &= \bigcup_{p \in N} S(p) \\
\Sigma &= \bigcup_{p \in N} \Sigma(p)
\end{align*}
\]

Finally, the transition relation \( \Delta \) is defined as follows. Assume again that \( p \in N \) satisfies (16) and (17). If \( p \) has children \( p_1, \ldots, p_\ell \) in \( T \), where \( \ell \geq 1 \) and for every \( i \in [\ell] \):

\[
\begin{align*}
\chi(p_i) &= \{ u_{i,1}, \ldots, u_{i,r_i} \} \\
\xi(p_i) &= \{ \bar{t}_{i,1}, \ldots, \bar{t}_{i,s_i} \},
\end{align*}
\]

with \( s_i \leq k \). Then assuming that \( \chi(p_i) \cap U = \{ w_{i,1}, \ldots, w_{i,o_i} \} \), \( \bar{u}_i = (u_{i,1}, \ldots, u_{i,r_i}) \) and \( \bar{w}_i = (w_{i,1}, \ldots, w_{i,o_i}) \) for each \( i \in [\ell] \), the following tuple is included in \( \Delta \)

\[
( [p, \bar{x} \mapsto \bar{a}, \bar{z} \mapsto \bar{b}, \bar{t}_1 \mapsto \bar{c}_1, \ldots, \bar{t}_s \mapsto \bar{c}_s], [p, \bar{z} \mapsto \bar{b}], \\
[p_1, \bar{u}_1 \mapsto \bar{d}_1, \bar{w}_1 \mapsto \bar{e}_1, \bar{t}_{1,1} \mapsto \bar{f}_{1,1}, \ldots, \bar{t}_{1,s_1} \mapsto \bar{f}_{1,s_1}], \ldots, \\
[p_\ell, \bar{u}_\ell \mapsto \bar{d}_\ell, \bar{w}_\ell \mapsto \bar{e}_\ell, \bar{t}_{\ell,1} \mapsto \bar{f}_{\ell,1}, \ldots, \bar{t}_{\ell,s_\ell} \mapsto \bar{f}_{\ell,s_\ell}] )
\]

whenever the following conditions are satisfied: (a) \( [p, \bar{x} \mapsto \bar{a}, \bar{z} \mapsto \bar{b}, \bar{t}_1 \mapsto \bar{c}_1, \ldots, \bar{t}_s \mapsto \bar{c}_s] \in S(p) \); (b) \( [p_i, \bar{u}_i \mapsto \bar{d}_i, \bar{w}_i \mapsto \bar{e}_i, \bar{t}_{i,1} \mapsto \bar{f}_{i,1}, \ldots, \bar{t}_{i,s_i} \mapsto \bar{f}_{i,s_i}] \in S(p_i) \) for each \( i \in [\ell] \); (c) \( \bar{t}_i \mapsto \bar{c}_i \) is consistent with \( \bar{t}_{j_1,j_2} \mapsto \bar{f}_{j_1,j_2} \) for every \( i \in [s] \), \( j_1 \in [\ell] \) and \( j_2 \in [s_{j_1}] \); and (d) \( \bar{t}_{j_3,j_4} \mapsto \bar{f}_{j_3,j_4} \) is consistent with \( \bar{t}_{j_3,j_3} \mapsto \bar{f}_{j_3,j_3} \) for every \( j_1 \in [\ell] \), \( j_2 \in [s_{j_1}] \), \( j_3 \in [\ell] \), \( j_4 \in [s_{j_3}] \). On the other hand, if \( p \) has no children in \( T \), then the following tuple is included in \( \Delta \)

\[
( [p, \bar{x} \mapsto \bar{a}, \bar{z} \mapsto \bar{b}, \bar{t}_1 \mapsto \bar{c}_1, \ldots, \bar{t}_s \mapsto \bar{c}_s], [p, \bar{z} \mapsto \bar{b}], \lambda )
\]

whenever \( [p, \bar{x} \mapsto \bar{a}, \bar{z} \mapsto \bar{b}, \bar{t}_1 \mapsto \bar{c}_1, \ldots, \bar{t}_s \mapsto \bar{c}_s] \in S(p) \).
It is straightforward to see that there exists a polynomial-time algorithm that generates \( S, S_0, \Sigma \) and \( \Delta \) from ECSP \( \mathcal{E} \) and the hypertree decomposition \( \langle T, \chi, \xi \rangle \) for \( P \). In particular, we have that \( |S(p)| = O(|\mathcal{E}|^k) \), where \( |S(p)| \) is the number of elements in \( S(p) \) and \( |\mathcal{E}| \) is the size of ECSP \( \mathcal{E} \), by definition of \( S(p) \) and the fact that \( \chi(p) \subseteq \bigcup_{i \in \xi(p)} \text{var}(i) \). Notice that this implies that each \( S(p) \) is of polynomial size given that \( k \) is fixed and each tuple in \( S \) is of polynomial size in \( |\mathcal{E}| \). Moreover, observe that as \( n = |N| \), we can construct the input \( 0^n \) for the problem \( \#TA \) in polynomial time in the size of \( \mathcal{E} \), given that the hypertree decomposition \( \langle T, \chi, \xi \rangle \) is of polynomial size in the size of \( P \).

Finally, we need to prove that \( |\text{sol}(\mathcal{E})| = \{|t \in L(T) | |t| = n|\} \). To see that this is the case, for every assignment \( \nu \in \text{sol}(\mathcal{E}) \), define a labelled tree \( t_{\nu} \) as follows. Tree \( t_{\nu} \) has the same structure as \( T \), but every node \( p \in N \) is assigned the following label in \( \Sigma \). Assume that \( \chi(p) \cap U = \{z_1, \ldots, z_r\} \) and \( \overline{z} = (z_1, \ldots, z_r) \). Moreover, assume that \( \nu(z_i) = a_i \) for every \( i \in [r] \). Then the label of \( p \) in \( t_{\nu} \) is \( [p, \overline{z} \mapsto \overline{a}] \), where \( \overline{a} = (a_1, \ldots, a_r) \). By definition of \( T \), we have that \( L(T) = \{ t_\nu \mid \nu \in \text{sol}(\mathcal{E}) \} \). Therefore, given that \( t_\nu \neq t_{\nu'} \) for every \( \nu, \nu' \in \text{sol}(\mathcal{E}) \) such that \( \nu \neq \nu' \), we conclude that \( |\text{sol}(\mathcal{E})| = \{|t \in L(T) | |t| = n|\} \), as every tree accepted by \( T \) has \( n \) nodes.

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