Regularity of initial data in dynamical massless scalar field models

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We discuss here the issue of regularity of initial data for dynamical spherically symmetric massless scalar field models in a spacetime. Generalizing the known solutions of Einstein equations given in this case by Wyman and Roberts, we examine the issue of regularity on a given spacelike surface, especially when the gradient of the field is spacelike. In particular, we isolate the class of models which would have necessarily a singularity at the center, and therefore these would be unsuitable for studying either gravitational collapse or dynamical evolutions in cosmology.

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I. INTRODUCTION

Massless scalar fields coupled to gravity are of particular interest in both gravitational collapse situations as well as for the cosmological scenarios. In cosmology, special importance is attached to the evolution of a scalar field, which has attracted a great deal of attention in past decades. This is because one would like to know the behaviour for fundamental matter fields towards understanding the transition from matter dominated regime to dark energy domination (see e.g. and references therein). Scalar fields are of much interest in view of the inflationary scenarios that govern the early universe dynamics, because such a field can act as an ‘effective’ cosmological constant in driving the inflation.

In gravitational collapse studies, the nature of singularity for the evolving massless scalar fields has been examined and a number of numerical and analytical works have been done in recent years, mainly on spherical collapse models. The perspective here is mainly that of the cosmic censorship hypothesis, in the sense that one would like to understand the nature of the spacetime singularity developing in collapse in order to know whether it is within a black hole or it could be a naked singularity not covered within an event horizon, in violation to the censorship conjecture.

Both the collapse and cosmological models, however, share one important physical feature, namely that except for the possible initial (in the case of cosmology) and the final (can be applicable for both the cases of collapse and cosmology) singularities, the other epochs should be regular in these models. Therefore, for any physical model, we expect to have a regular initial data without any singularities from which there is a dynamical evolution. If any solution to Einstein equations fails to satisfy this criterion, then that would be unsuitable as a model for gravitational collapse or cosmological scenarios.

In this note, we examine this question in some detail, and in particular we isolate a rather wide class of massless scalar field models which are unsuitable as either collapse models or in cosmology, because they do not have a regular initial data and admit a singularity necessarily. We do not assume here or discuss any particular solution, but a general class of models is considered. Clearly, information such as this would be useful as one needs to avoid such classes where initial data is singular, for any physically realistic discussion. In what follows, we shall examine whether it is possible to have regular initial data for the general class of spherically symmetric massless scalar field models, where the gradient of the scalar field is spacelike on the initial spacelike hypersurface. The consideration would also hold as long as the gradient remains spacelike in the neighborhood of the center, but could be timelike in other regions of the initial surface.

The static limit of this class of models is the class of solutions given by Wyman. The solutions given by Wyman are for spherically symmetric static models for a massless scalar field coupled to gravity, where the gradient of the scalar field is spacelike. We shall show that, in general, even for any dynamical class of such models, if the matter is distributed in a ball around the center of spherical symmetry, then there is necessarily a singularity at the center. In other words, one cannot have a regular initial spacelike surface in this case, which is necessary for a physical collapse model. Therefore this class of models is not suitable for studying collapse. As an example of the violation of regularity, we can consider the static solutions given by Wyman, all of which have a singularity at the center. Of course, these models, being static, cannot provide any insight on gravitational collapse or dynamical evolutions. Unfortunately, not many dynamical solutions of the massless scalar field coupled to gravity are known. Among the existing few, the Roberts solution is the one which has been studied in some detail in literature. It has again a central singularity at all epochs. In this case, however, the gradient of the scalar field is timelike in some part and spacelike in other parts of the spacetime.

The existence of a central singularity in these models, when the gradient of the scalar field is spacelike, is in agreement with our work in this paper. In this sense, the result presented here may be considered as providing a generalization of this important feature observed in these known classes of explicitly known solutions for the
case of a massless scalar field. For the sake of clarity, we emphasize that we shall neither give here any explicit solutions nor outline any method for doing so in the present paper. Our aim is in fact to show that even when it is not possible to completely solve the Einstein equations, it is possible to know about certain important physical features of the solutions, such as the regularity of the models, as in the present case.

The plan of this paper is as follows. First, we set up the basic formalism by specifying the coordinate system which we work with. Then we briefly review the two classes of solutions given by Wyman and Roberts, and we note the presence of central curvature singularities in both these cases. In the next section, we then generalize this feature to demonstrate the breakdown of the regularity at the center, whenever the gradient of the scalar field is spacelike. The existence of central singularities in both the Wyman and Roberts solution are also discussed in view of this result. Finally, we consider the implications of this result in the context of the dynamical evolution of a massless scalar field coupled to gravity.

II. MASSLESS SCALAR FIELDS

In our analysis here, we consider a four-dimensional spacetime manifold which has spherical symmetry. The massless scalar field \( \phi(x^a) \) on such a spacetime manifold \((M, g_{ab})\) is described by the Lagrangian,

\[
\mathcal{L} = -\frac{1}{2} \phi_{,a} \phi_{,b} g^{ab}. \tag{1}
\]

The corresponding Euler-Lagrange equation is then given by,

\[
\phi_{,ab} g^{ab} = 0, \tag{2}
\]

and the energy-momentum tensor for the massless scalar field, as calculated from the above Lagrangian, is given as

\[
T_{ab} = \phi_{,a} \phi_{,b} - \frac{1}{2} g_{ab} \left( \phi_{,c} \phi_{,d} g^{cd} \right). \tag{3}
\]

The massless scalar field is a Type I matter field [14], i.e., it admits one timelike and three spacelike eigenvectors. At each point \( q \in M \), we can express the tensor \( T^{ab} \) in terms of an orthonormal basis \( (E_0, E_1, E_2, E_3) \), where \( E_0 \) is a timelike eigenvector with the eigenvalue \( \rho \). The vectors \( E_\alpha \) \( (\alpha = 1, 2, 3) \) are three spacelike eigenvectors with eigenvalues \( p_\alpha \). The eigenvalue \( \rho \) represents the energy density of the scalar field as measured by an observer whose world line at \( q \) has an unit tangent vector \( E_0 \), and the eigenvalues \( p_\alpha \) represent the principal pressures in three spacelike directions \( E_\alpha \).

We now choose the spherically symmetric coordinates \((t, r, \theta, \phi)\) along the eigenvectors \((E_0, E_\alpha)\). Such a reference frame describes comoving coordinates which we use here. As discussed in [13], the general spherically symmetric metric in comoving coordinates can be written as,

\[
ds^2 = e^{2\nu(t,r)} dt^2 - e^{2\psi(t,r)} dr^2 - R^2(t,r) d\Omega^2, \tag{4}
\]

where \( d\Omega^2 \) is the metric on a unit two-sphere, and we have used the two gauge freedoms of two variables, namely, \( t' = f(t, r) \) and \( r' = g(t, r) \), to make the \( g_{tr} \) term in the metric and the radial velocity of the matter field to vanish. That means that the energy-momentum tensor is diagonal in such a coordinate system, and therefore the eigenvalues \( \rho \) and \( p_\alpha \) have definite physical meaning of mass-energy density and pressures. We note that we still have two scaling freedoms of one variable available here, namely \( t \rightarrow f(t) \) and \( r \rightarrow g(r) \). We note that the metric function \( R \) here denotes the physical radius of the matter cloud.

As we are considering here spherically symmetric spacetimes, we have \( \psi = \phi(t, r) \) necessarily. Furthermore, from equation (3), we can easily see that in the comoving reference frame with the metric given by (4), \( T_{00} = \phi' \phi \). It follows therefore that we must have necessarily \( \phi(t, r) = \phi(t) \) or \( \phi(t, r) = \phi(r) \), because the energy-momentum tensor here is necessarily diagonal.

As noted above, we have adopted here the comoving coordinates. This coordinate system had been used earlier to discuss the massless scalar fields, by Wyman [12], Xanthopoulos and Xianas [13], and others. Some of these authors used this coordinate system to study the static case, and also some dynamical cases were examined. However, it is also the case that this coordinate system is not used at times to study the dynamical evolution of a massless scalar field coupled to gravity. The main reason for that is, the comoving coordinate system breaks down if and when the gradient of the scalar field becomes null as the field evolves. However, this does not pose any problem as far as our purpose is concerned, because what we want to show here is that, for a massless scalar field with a spacelike gradient, there cannot exist any regular spacelike hypersurface from which to start the dynamical evolution of the gravitational collapse in a regular or non-singular manner. What we examine below is to consider the scalar field configuration only at an epoch of constant time, when the gradient of the field is spacelike.

III. THE WYMAN AND ROBERTS SOLUTIONS

We first discuss here the massless scalar field solutions given by Wyman [12], and Roberts [6, 7, 8], respectively. As mentioned above, Wyman used comoving coordinates in his analysis of the static massless scalar field spacetimes. Then there are two possible cases that arise, namely, \( \phi(t, r) = \phi(t) \) or \( \phi(t, r) = \phi(r) \). In the first case, the gradient of the scalar field is timelike everywhere, whereas it is spacelike in the latter case. For the first case, the Einstein equations are difficult to solve and can be solved only for a special case. The Einstein equations
can be solved, however, when the gradient of the scalar field is spacelike, and a general solution can be obtained in this case. In this second case, the line element given by Wyman is

\[
\begin{align*}
\text{ds}^2 = & -e^{\alpha r} dt^2 - e^{-\alpha r} \left[ \frac{\gamma^r - 1}{\sinh \gamma r} \right]^4 dr^2 \\
& -e^{-\alpha r} \left[ \frac{\gamma^r - 1}{\sinh \gamma r} \right]^2 r^2 d\Omega^2
\end{align*}
\] (5)

This generalises most of the previous static solutions obtained for a massless scalar field, which were given by various authors such as Buchdal [16] and Yilmaz [17]. Newman, Janis, and Winicour [18] also discussed this solution in the comoving coordinate system. But they used a scaling of the coordinate radius \( r \) which was different from the one that was used by Wyman.

In this class of models given by Wyman, there is always a strong curvature singularity present at the center \( R = 0 \) at all epochs. The Ricci scalar diverges at the singularity at \( R = 0 \). From the line element (5), we see that the metric component \( g_{00} \) diverges at the center if \( \alpha > 0 \) but is finite for \( \alpha = 0 \).

The dynamical solution given by Roberts [6], Brady [7] and Oshiro et al. [8] has the following form when written in double null coordinates,

\[
\text{ds}^2 = -dudv + R(u, v)^2 d\Omega^2
\]

\[
R(u, v) = \frac{1}{2} \sqrt{(1-p^2)\nu^2 - 2\nu u + u^2}.
\] (6)

The Ricci scalar here takes the form

\[
R_c = \frac{p^2uv}{2R^4}
\] (7)

There are three qualitatively different subclasses of this solution. For all the three subclasses, however, a singularity is always present at the center of the coordinates, where the Ricci scalar diverges. For the subclass \( 0 < p < 1 \), the gradient of the scalar field is first timelike, and then it changes over to a spacelike vector. These two different regions are separated by a null hypersurface. In both the regions, one can set up different comoving coordinate systems, which however, would break down at the null hypersurface joining them. In the region where the gradient of the scalar field is spacelike, the spacetime metric in the comoving coordinates has the form,

\[
\text{ds}^2 = dt^2 - \frac{1}{4} \left( \frac{(1-p)^2}{p^2} \right)^2 \frac{(1-p^2)r + 2p}{l^2} dr^2
\]

\[
- \frac{(1-p)}{2} \left[ \frac{(1-p^2)r + 2p}{l^2} \right] d\Omega^2
\] (8)

For the other two subclasses of the solution, the gradient of the field is always timelike. Because of the ever-present spacetime singularity at the center \( r = 0 \), it is not possible to find a regular initial spacelike hypersurface in this model where one can define the regular initial data from which to develop and evolve the gravitational collapse of the massless scalar field.

**IV. THE DYNAMICAL MODELS WITH A SPACELIKE GRADIENT FOR THE SCALAR FIELD**

In this section, we consider the issue of the regularity for the dynamical models for which the gradient of the massless scalar field is spacelike during the evolution. We note that, even if the evolution of the field began in such a manner that the gradient of the scalar field is spacelike everywhere on the given initial spacelike surface, it is possible that later in the evolution it may change to being timelike, either everywhere on a given later spacelike surface, or in some regions of the same. The conclusions given below remain valid as long as the gradient is spacelike in some neighbourhood of the center. What we show below is, as long as a neighbourhood of the center has a strong curvature gradient for the scalar field, the center must admit a curvature singularity, and therefore an evolution from a regular initial data is not possible.

To do this, we first write down the Einstein equations in the comoving coordinates. For the metric (4), and using the following definitions,

\[
G(t, r) = e^{-2\psi(R^2)}^2, \quad H(t, r) = e^{-2\varphi(R^2)}^2,
\] (9)

\[
F = R(1 - G + H),
\] (10)

we can write the independent Einstein equations for the spherical massless scalar field (in the units \( 8\pi G = c = 1 \)) as below (see [19]),

\[
\rho = \frac{F'}{R^2 R'},
\] (11)

\[
P_r = -\frac{\dot{F}}{R^2 R'},
\] (12)

\[
\nu' (\rho + P_r) = 2 (P_\theta - P_r) \frac{R'}{R} - P_r',
\] (13)

\[
- 2 \dot{R} + \frac{R' G}{G} + \dot{R} \frac{H'}{H} = 0,
\] (14)

In the above, the function \( F(t, r) \) has the interpretation of the mass function for the matter field, in that it represents the total mass contained within a coordinate radius \( r \).

For the class of models, \( \phi = \phi(r) \), the components of the energy-momentum tensor are given by,

\[
T^t_t = -T^r_r = T^\theta_\theta = T^\phi_\phi = \frac{1}{2} e^{-2\psi} \dot{\phi}^2
\] (15)

It follows that the equation of state in this case, which relates the scalar field energy density and pressures is then given by,

\[
\rho = P_r = -P_\theta
\] (16)
Putting the expressions of density and pressure in the Einstein equations, we have,

$$\frac{1}{2} e^{-2\phi} \dot{\phi}^2 = \frac{F'}{R^2 R'} = -\frac{\dot{R}}{R^2 R}$$  \hspace{1cm} (17)$$

We can also write from (13),

$$\phi'' = (\psi' - 2 \frac{R'}{R} - \nu') \phi'$$  \hspace{1cm} (18)$$

and the equations (10), (14) and (18) can be integrated once to give,

$$\phi' = \frac{e^{\psi - \nu + b(t)}}{R^2},$$  \hspace{1cm} (19)$$

where $b(t)$ is an arbitrary function of the time coordinate. The Ricci scalar $R_c = g^{\mu \nu} R_{\mu \nu}$ can be written as $R_c = -2e^{-2\phi} \dot{\phi}^2$. Using (19) we now get,

$$R_c = -\frac{e^{-2\nu + b(t)}}{R^2}.$$  \hspace{1cm} (20)$$

In a physically reasonable collapse model, the initial spacelike surface should be regular. This implies that $g_{\mu \nu}$ and $g^{\mu \nu}$ are finite and regular on that surface. From (20), it is clear that on that surface, at $R = 0$, there will be a singularity. Unless we are considering the collapse of a thick shell of scalar field, the physical radius $R$ of the spherical distribution of scalar field must be zero at the center. Therefore, there will always be a singularity at $R = 0$ during the time evolution of a scalar field of the form $\phi = \phi(r)$, unless $e^{-2\nu}$ goes to zero at the center $R = 0$ at least as fast as $R^4$. This later situation implies a breakdown of the coordinate system at the center $R = 0$. For the static solutions given by Wyman, as $R$ goes to zero, $e^{-2\nu}$ never goes to zero as fast as $R^4$. Therefore there is a singularity always present at the center for these models. In the dynamical solution given by Roberts, $e^{-2\nu} = 1$ always, and therefore from (20), it follows that the Ricci scalar diverges at the center, $R = 0$.

It follows that our consideration here generalizes the result to the cases where the gradient of the scalar field is spacelike in a neighbourhood of the center. It thus follows that this class cannot be used for any considerations related to collapse or cosmology, where dynamical evolution from a regular initial data is a necessary condition.

We noted above that a breakdown of the coordinates at $R = 0$ could be an alternative to singularity. Let us now consider what such a breakdown of the comoving coordinate system at $R = 0$ would signify in the case, if there is no singularity. For $R_c$ to be finite at $R = 0$, $e^{-2\nu} \sim R^4$ as $R$ goes to zero. If $\phi_\mu$ is spacelike, then a comoving shell with non-zero physical radius $R$ cannot evolve in such a way so that the physical radius becomes zero without hitting the singularity. In that case, we can consider the center as a comoving world line. Therefore, as one goes towards the center, along the comoving world lines $r = \text{const.}$, the proper time must necessarily diverge. In other words, the length of the comoving world lines diverges as one approaches the center. While this may not be as serious as having a singularity at the center, it does imply some kind of violation of regularity at the center. In such a case, this may not mean a violation of the regularity of the initial data, but rather that the dynamical evolution is not regular in some sense. The collapse or the evolution at the center then never reaches a singularity in future for an infinite proper length and the collapse is ‘freezing’ in the sense above.

V. DISCUSSION

It is seen that for the class of models considered here, the center is not regular at any epoch. In fact, this feature is seen in the static spacetimes described by Wyman’s solution also. All such static solutions with $\phi = \phi(r)$ are singular at $R = 0$. It follows that the class of massless scalar field models with $\phi = \phi(r)$ is thus not adequate for the investigation of dynamical gravitational collapse evolution of the field from a regular initial data. Therefore, whenever we are interested in such time-dependent models, or key issues related to the gravitational collapse phenomena, we must focus on the $\phi = \phi(t)$ class of massless scalar fields only.

Another important point to note here is that the central singularity could also exist even if $\phi_\mu$ is spacelike at a neighbourhood of the center. This is because in that case, we could set up a comoving coordinate system which covers this neighbourhood during the dynamical evolution. This implies that in that patch, the Einstein equations can be written in the same form as given in this paper. Then it follows that at the center, there would be a violation of regularity. This means that one can extend the result to the cases where the gradient of the scalar field is spacelike at the center. In those cases also, it is not possible to have regular initial data, from which the collapse would evolve.

Another interesting outcome of this result is that during the evolution, if $\phi_\mu$ changes to a spacelike vector from a timelike one, the center of the scalar field cloud would cease to be a regular point. It would also be interesting to ask whether the singularity in question is naked or covered within an event horizon. While the answer can be found in some special cases, we do not address this issue here presently, which would in general require more information on the nature of the solutions for the massless scalar field case.
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