Hidden Variables and Nonlocality in Quantum Mechanics

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October 1996
Abstract

At the present time, most physicists continue to hold a skeptical attitude toward the proposition of a ‘hidden variables’ interpretation of quantum theory, in spite of David Bohm’s successful construction of such a theory and John S. Bell’s strong arguments in favor of the idea. Many are convinced either that it is impossible to interpret quantum theory in this way, or that such an interpretation would actually be irrelevant. There are essentially two reasons behind such doubts. The first concerns certain mathematical theorems (von Neumann’s, Gleason’s, Kochen and Specker’s, and Bell’s) which can be applied to the hidden variables issue. These theorems are often credited with proving that hidden variables are indeed ‘impossible’, in the sense that they cannot replicate the predictions of quantum mechanics. Many who do not draw such a strong conclusion nevertheless accept that hidden variables have been shown to exhibit prohibitively complicated features. The second reason hidden variables are disregarded is that the most sophisticated example of a hidden variables theory—that of David Bohm—exhibits *nonlocality*, i.e., it can happen in this theory that the consequences of events at one place propagate to other places instantaneously. However, as we shall show in the present work, neither the mathematical theorems in question nor the attribute of nonlocality serve to detract from the importance of a hidden variables interpretation of quantum theory. The theorems imply neither that hidden variables are impossible nor that they must be overly complex. As regards nonlocality, this feature is present in quantum mechanics itself, and is a required characteristic of *any* theory that agrees with the quantum mechanical predictions.

In the present work, the hidden variables issue is addressed in the following ways. We first discuss the earliest analysis of hidden variables—that of von Neumann’s theorem—and review John S. Bell’s refutation of von Neumann’s ‘impossibility proof’. We recall and elaborate on Bell’s arguments regarding the theorems of Gleason, and Kochen and Specker. According to Bell, these latter theorems do not imply that hidden variables interpretations are untenable, but instead that such theories must exhibit *contextuality*, i.e., they must allow for the dependence of measurement results on the characteristics of both measured system and measuring apparatus. We demonstrate a new way to understand the implications of both Gleason’s theorem and Kochen and Specker’s theorem by noting that they prove a result we call “spectral incompatibility”. We develop further insight into the concepts involved in these two theorems by investigating a special quantum mechanical experiment which was first described by David Albert. We review the Einstein–Podolsky–Rosen paradox, Bell’s theorem, and Bell’s later argument that these imply that quantum mechanics is irreducibly nonlocal. We present this discussion in a somewhat more gradual fashion than does Bell so that the logic of the argument may be more transparent.
The paradox of Einstein, Podolsky, and Rosen was generalized by Erwin Schrödinger in the same paper where his famous ‘cat paradox’ appeared. We develop several new results regarding this generalization. We show that Schrödinger’s conclusions can be derived using a simpler argument—one which makes clear the relationship between the quantum state and the ‘perfect correlations’ exhibited by the system. We use Schrödinger’s EPR analysis to derive a wide variety of new quantum nonlocality proofs. These proofs share two important features with that of Greenberger, Horne, and Zeilinger. First, they are of a deterministic character, i.e., they are ‘nonlocality without inequalities’ proofs. Second, as we shall show, the quantum nonlocality results we develop may be experimentally verified in such a way that one need only observe the ‘perfect correlations’ between the appropriate observables; no further tests are required. This latter feature serves to contrast these proofs with EPR/Bell nonlocality, the laboratory confirmation of which demands not only the observation of perfect correlations, but also an additional set of observations, namely those required to test whether ‘Bell’s inequality’ is violated. The ‘Schrödinger nonlocality’ proofs we give differ from the GHZ proof in that they apply to two-component composite systems, while the latter involves a composite system of at least three-components. In addition, some of the Schrödinger proofs involve classes of observables larger than that addressed in the GHZ proof.
Acknowledgments

I would like to express my profound gratitude to my advisor, Sheldon Goldstein, for his very kind and generous guidance. It is impossible to imagine a research advisor who is more inspiring to work with than Shelly. I would also like to express thanks to Joel Lebowitz for his support and encouragement. It has been a most rewarding experience to work within the friendly and stimulating environment of the Rutgers mathematical physics group. For many helpful discussions, I thank some of my peers with whom I worked at Rutgers: Karin Berndl, Martin Daumer, Mahesh Yadav, Nadia Topor, and Subir Ghoshal. My thanks are extended also to Richard McKenzie, Andrew Pica, Asif Shakur, David Kanarr, and Gail Welsh of Salisbury State University for their interest in this work, and for providing an opportunity to present talks on quantum mechanics to physics student audiences. I would like to thank especially Asif Shakur, who has helped proofread the manuscript and has given valuable advice toward the presentation of the material. For their continuous support and zealous help with the proofreading, I thank my parents. Finally, I would like to thank Melissa Thomas, Tom Hemmick, and Raju Farouk for aiding me with the computer communications techniques which made it possible to complete this work while living away from campus. This work was supported by a Rutgers University Excellence Fellowship.
Dedication

This dissertation is dedicated to the memory of Geordie Williams, whose energy and enthusiasm made him an inspiration to all of his friends from the old Joppatowne group.
“Attention has recently been called to the obvious but very disconcerting fact that even though we restrict the disentangling measurements to one system, the representative obtained for the other system is by no means independent of the particular choice of observations which we select for that purpose and which by the way are entirely arbitrary.”

Erwin Schrödinger [92]
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Chapter 1

Introduction

1.1 The issue of hidden variables

1.1.1 Contextuality, nonlocality, and hidden variables

The aim of this thesis is to contribute to the issue of hidden variables as a viable interpretation of quantum mechanics. Our efforts will be directed toward the examination of certain mathematical theorems that are relevant to this issue. We shall examine the relationship these theorems bear toward hidden variables and the lessons they provide regarding quantum mechanics itself. The theorems in question include those of John von Neumann [101], A. M. Gleason [57], J.S. Bell [7], and S. Kochen and E. P. Specker [74]. Arguments given by John Stewart Bell\(^1\) demonstrate that the prevailing view that these results disprove hidden variables\(^2\) is actually a false one. According to Bell, what is shown\(^3\) by these theorems is that hidden variables must allow for two important and fundamental

\(^1\)See references by Bell: [8, 16] on the theorems of von Neumann, Gleason, and Kochen and Specker. Bell’s works emphasize the conclusion that these theorems would place no serious limitations on a hidden variables theory, and he declares that [16] “What is proved by impossibility proofs [these theorems] . . . . is lack of imagination.” A recent work by Mermin [80] also addresses the impact of these theorems on hidden variables. Mermin does not accept Bell’s conclusions, however, and states “Bell . . . is unreasonably dismissive of the importance of . . . impossibility proofs [theorems]. . . . Bell’s criticism [of these theorems] undervalues the importance of defining limits to what speculative theories can or cannot be expected to accomplish.” In the present work, we argue for Bell’s position on these matters.

\(^2\)The following references found in the bibliography present discussions of all four theorems along with their relationship to hidden variables: [6, 69, 70, 80]. Von Neumann’s theorem is claimed to disprove hidden variables in [2, 35, 71, 101]. This conclusion is reached for both Gleason’s theorem and the theorem of Kochen and Specker in [74, 83]. The following references state that Bell’s theorem may be regarded in this way: [24], [55, p. 172], [104].

\(^3\)The theorem due to von Neumann is not as closely connected with these concepts as the others. We include it for the sake of completeness and as introduction to the discussion of the other theorems. In addition, we will point out the presence in a work by Schrödinger [91] of an analysis leading to essentially the same conclusion as von Neumann’s.
quantum features: contextuality and nonlocality.

To allow for contextuality, one must consider the results of a measurement to depend on the attributes of both the system, and the measuring apparatus. The concept of contextuality is evidently in contrast to the tendency to consider the quantum system in isolation, without considering the configuration of the experimental apparatus\(^4\). Such a tendency conflicts with the lesson of Niels Bohr concerning [90, page 210] “the impossibility of any sharp separation between the behavior of atomic objects and the interaction with the measuring instruments which serve to define the conditions under which the phenomena appear.” In any attempt to develop a theory of hidden variables, one must take contextuality into account.

The concept of contextuality lies at the heart of both Gleason’s theorem and the theorem of Kochen and Specker. However, typical expositions\(^5\) of these arguments give either no discussion of the concept, or else a very brief treatment that fails to convey its full meaning and importance. The conclusion drawn by authors giving no discussion of contextuality is that the mathematical impossibility of hidden variables has been proved. References which give only a brief mention of the concept tend to leave their readers with the impression that it is not of much significance and is somewhat artificial. Readers of these works would be led to the conclusion that Gleason’s and Kochen and Specker’s theorems demonstrate that the prospect of hidden variables is a dubious one, at best.

The concept of nonlocality is well known through Bell’s famous mathematical theorem [7] in which he addresses the problem of the Einstein–Podolsky–Rosen paradox\(^6\). In a system exhibiting nonlocality, the consequences of events at one place propagate to other places instantaneously. Although Einstein, Podolsky and Rosen were attempting to demonstrate a different conclusion (the incompleteness of the quantum theory), their analysis served to point out the conditions under which (as would become evident after Bell’s work) nonlocality arises. The existence of such conditions in quantum mechanics was regarded by Erwin Schrödinger as being of great significance, and in a work [92] in which he generalized the EPR argument, he called these conditions “the characteristic trait of quantum mechanics, the one that forces its entire departure from classical lines.” Bell’s work essentially completed the proof that the quantum phenomenon discovered by EPR and elaborated by Schrödinger truly does entail nonlocality.

Although the EPR paradox and Bell’s theorem are quite well known, there

\(^4\)See papers in the recent book by Bell [8, 16] for a good discussion of contextuality.

\(^5\)The exception to this is two works by Bell: bibliography references [8, 16]. In these works, contextuality is made quite clear, and it is concluded that neither theorem proves the impossibility of hidden variables. References that discuss the Kochen and Specker theorem without treating contextuality are [74, 78, 83]. The following references briefly discuss the relationship of Gleason’s and Kochen and Specker’s theorems to contextuality, but fail to convey its importance: [6, 69, 70].

\(^6\)The EPR paper appears in [51], and it is reprinted in [103].
exists a misperception regarding the relationship these arguments bear to non-locality. Some authors (mistakenly) conclude\(^7\) that the EPR paradox and Bell’s theorem imply that a conflict exists \textit{only} between local theories of hidden variables and quantum mechanics, when a more general conclusion than this follows. When taken together, the EPR paradox\(^8\) and Bell’s theorem imply that \textit{any local theoretical explanation whatsoever must disagree with quantum mechanics.} In Bell’s words: \cite{14} “It now seems that the non-locality is deeply rooted in quantum mechanics itself and will persist in any completion.” We can conclude from the EPR paradox and Bell’s theorem that the quantum theory is irreducibly nonlocal\(^9\).

As we have mentioned, Erwin Schrödinger regarded the situation brought out in the Einstein, Podolsky, Rosen paper as a very important problem. His own work published just after the EPR paper \cite{91} provided a generalization of this analysis. This work contains Schrödinger’s presentation of the well known paradox of ‘Schrödinger’s cat’. However, a close look at the paper reveals much more. Schrödinger’s work contains a very extensive and thought provoking analysis of quantum theory. He begins with a statement of the nature of theoretical modeling and a comparison of this to the framework of quantum theory. He continues with the cat paradox, the measurement problem, and finally his generalization of the EPR paradox and its implications. In addition, he derives a result quite similar to that of von Neumann’s theorem\(^10\).

Subsequent to our discussion of the issues of the EPR paradox, Bell’s theorem, and nonlocality, we will discuss a new form of quantum nonlocality proof\(^11\) based on Schrödinger’s generalization of the EPR paradox. The type of argument we shall present falls into the category of a “nonlocality without inequalities” proof\(^12\). Such analyses—of which the first\(^13\) to be developed was that of Greenberger, Horne, and Zeilinger \cite{60}—differ from the nonlocality proof derived from the EPR paradox and Bell’s theorem in that they do not possess the statistical character of the latter.

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\(^7\)See for example, \cite[p 48]{40}, \cite{62, 80, 83}. Clauser and Shimony \cite{38} regard the EPR paradox and Bell’s theorem as proof that quantum mechanics must conflict with any ‘realistic’ local theory. Their view seems to fall short of Bell’s (see Bell \cite{12, 15}) contention that it is \textit{locality itself} which leads to this clash with quantum theory. 

\(^8\)Bohm’s spin singlet version \cite{25} of EPR.

\(^9\)See the following: \cite{1, 12, 15, 30, 64, 77, 85}.

\(^10\)Schrödinger does not explicitly mention von Neumann’s work, however.

\(^11\)A nonlocality proof similar to that presented in this thesis was developed by Heywood and Redhead \cite{66} and by Brown and Svetlichny \cite{36}. These authors present arguments addressed to one particular example of the general class of maximally entangled states we consider. Their argumentation does not bring out all the conclusions we do in the present work, as they do not draw the \textit{general} conclusion of nonlocality, but instead only the nonlocality of the particular class of hidden variables they address.

\(^12\)See Greenberger, Horne, Shimony and Zeilinger in \cite{61}.

\(^13\)See Mermin in \cite{79}.
David Bohm’s theory of hidden variables

It must be mentioned that the problem of whether hidden variables are possible is more than just conjectural. Louis de Broglie’s 1926 ‘Pilot Wave’ theory is, in fact, a viable interpretation of quantum phenomena. The theory was more systematically presented by David Bohm in 1952. This encouraged de Broglie himself to take up the idea again. We shall refer to the theory as “Bohmian mechanics”, after David Bohm.

The validity of Bohmian mechanics is a result of the first importance, since it restores objectivity to the description of quantum phenomena. The quantum formalism addresses only the properties of quantum systems as they appear when the system is measured. An objective description, by contrast, must give a measurement-independent description of the system’s properties. The lack of objectivity in quantum theory is something which Albert Einstein disliked, as is clear from his writings: [90, page 667]

What does not satisfy me, from the standpoint of principle, is (quantum theory’s) attitude toward that which appears to be the programmatic aim of all physics: the complete description of any (individual) real situation (as it supposedly exists irrespective of any act of observation or substantiation).

Moreover, Einstein saw quantum theory as incomplete, in the sense that it omits essential elements from its description of state: [90, page 666] “I am, in fact, firmly convinced that the essentially statistical character of contemporary quantum theory is solely to be ascribed to the fact that this (theory) operates with an incomplete description of physical systems.” Bohmian mechanics essentially fulfills Einstein’s goals for quantum physics: to complete the quantum description and to restore objectivity to the theoretical picture.

Besides the feature of objectivity, another reason the success of Bohmian mechanics is important is that it restores determinism. Within this theory one may predict, from the present state of the system, its form at any subsequent time. In particular, the results of quantum measurements are so determined from the present state. Determinism is the feature with which the hidden variables program has been traditionally associated.

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14See [41, 42, 43]. A summary of the early development of the theory is given by de Broglie in [44].
15See [28]. See also Bell in [16, 19]. A recent book on Bohmian mechanics is [68].
16It is for this reason that Bohm refers to his theory as an ‘Ontological Interpretation’ of quantum mechanics: [32, page 2] “Ontology is concerned primarily with that which is and only secondarily with how we obtain our knowledge about this. We have chosen as the subtitle of our book “An Ontological Interpretation of Quantum Theory” because it gives the clearest and most accurate description of what the book is about....the primary question is whether we can have an adequate conception of the reality of a quantum system, be this causal or be it stochastic or be it of any other nature.”
17This is the motivation mentioned by von Neumann in his no-hidden variables proof. See reference [101]
The importance of the existence of a successful theory of hidden variables in the form of Bohmian mechanics is perhaps expressed most succinctly by J.S. Bell: [16]

Why is the pilot wave picture [Bohmian mechanics] ignored in textbooks? Should it not be taught, not as the only way, but as an antidote to the prevailing complacency? To show that vagueness, subjectivity, and indeterminism are not forced upon us by experimental facts, but by deliberate theoretical choice?

1.2 Original results to be presented

The analysis we present will consist of two main parts: one devoted to contextuality and the theorems of Gleason, and Kochen and Specker, and the other devoted to nonlocality, Bell’s theorem, and Schrödinger’s generalization of the Einstein–Podolsky–Rosen paradox. We will review and elaborate on the reasons why these theorems do not disprove hidden variables, but should instead be regarded as illustrations of contextuality and nonlocality. Our presentation will link topics that have most often been addressed in isolation, as we relate the various mathematical theorems to one another. Our analysis of contextuality and the theorems of Gleason and Kochen and Specker goes beyond that of other authors\textsuperscript{18} in the following ways. First, we will show that the theorems of Gleason, and Kochen and Specker imply a result we call “spectral-incompatibility”. Regarded as proofs of this result, the implications of these theorems toward hidden variables become much more transparent. Second, through discussion of an experimental procedure first given by David Albert [1], we show that contextuality provides an indication that the role traditionally ascribed to the Hermitian operators in the description of a quantum system is not tenable\textsuperscript{19}.

After reviewing the EPR paradox, Bell’s theorem, and the argument for quantum nonlocality that follows, we then present what is perhaps the most distinctive feature of the thesis, as we focus on Erwin Schrödinger’s generalization of the EPR paradox. First, we show that the conclusions of ‘Schrödinger’s paradox’ may be reached using a more transparent method than Schrödinger’s which allows one to relate the form of the quantum state to the perfect correlations it exhibits. Second, we use Schrödinger’s paradox to derive a broad spectrum of new instances of quantum nonlocality. Like the proof given by Greenberger, Horne, and Zeilinger, these ‘Schrödinger nonlocality’ proofs are of a deterministic character, i.e., they are ‘nonlocality without inequalities’ proofs. The quantum nonlocality result based on the EPR paradox and Bell’s theorem is, on the other hand, essentially statistical, since it concerns the average

\textsuperscript{18}Bell [8, 16] and Mermin [80].

\textsuperscript{19}A work by Daumer, Dürr, Goldstein, and Zanghí ([39]) argues for the same conclusion, but does not do so by focusing on the Albert experiment.
of the results from a series of quantum measurements. We demonstrate that just as is true for the GHZ nonlocality, Schrödinger nonlocality differs from the EPR/Bell demonstration in that it may be experimentally confirmed simply by observing the ‘perfect correlations’ between the relevant observables. To verify the EPR/Bell nonlocality result, on the other hand, one must observe not only that the perfect correlations exist, but also that Bell’s inequality is violated\(^{20}\). The test of Schrödinger nonlocality is simpler than that of GHZ in that the perfect correlations in question are between observables of two subsystems, rather than between those of three or more. Moreover, some of the Schrödinger nonlocality proofs are stronger than the GHZ proof in that they involve larger classes of observables than that addressed by the latter.

### 1.3 Review of the formalism of quantum mechanics

#### 1.3.1 The state and its evolution

The formalism of a physical theory typically consists of two parts. The first describes the representation of the state of the system; the second, the time evolution of the state. The quantum formalism contains besides these a prescription\(^{21}\) for the results of measurement.

We first discuss the quantum formalism’s representation of state. The quantum formalism associates with every system a Hilbert space \(H\) and represents the state of the system by a vector \(\psi\) in that Hilbert space. The vector \(\psi\) is referred to as the *wave function*. This is in contrast to the classical description of state which for a system of particles is represented by the coordinates \(\{q_1, q_2, q_3, \ldots\}\) and momenta \(\{p_1, p_2, p_3, \ldots\}\). We shall often use the Dirac notation: a Hilbert space vector may be written as \(|\psi\rangle\), and the inner product of two vectors \(|\psi_1\rangle, |\psi_2\rangle\) as \(\langle \psi_1 | \psi_2 \rangle\). The quantum mechanical state \(\psi\) in the case of a non-relativistic \(N\)-particle system is given\(^{22}\) by a complex-valued function on configuration space, \(\psi(q)\), which is an element of \(L^2\)—the Hilbert space of square-integrable functions. Here \(q\) represents a point \(\{q_1, q_2, \ldots\}\) in the configuration space \(\mathbb{R}^{3N}\). The fact that \(\psi\) is an element of \(L^2\) implies

\[
\langle \psi | \psi \rangle = \| \psi \|^2 = \int dq \, \psi^* (q) \psi (q) < \infty \tag{1.1}
\]

where \(dq = (dq_1 dq_2 \ldots)\). The physical state is defined only up to a multiplicative constant, i.e., \(c|\psi\rangle\) and \(|\psi\rangle\) (where \(c \neq 0\)) represent the same state. We may

\(^{20}\)See [3, 4, 5].

\(^{21}\)It is the existence of such rules in the quantum mechanical formalism that marks its departure from an objective theory, as we discussed above.

\(^{22}\)In the absence of spin.
therefore choose to normalize $\psi$ so that 1.1 becomes

$$\langle \psi | \psi \rangle = 1.$$  \hfill (1.2)

The time evolution of the quantum mechanical wave function is governed by Schrödinger’s equation:

$$i\hbar \frac{\partial \psi}{\partial t} = H\psi,$$  \hfill (1.3)

where $H$ is an operator whose form depends on the nature of the system and in particular on whether or not it is relativistic. To indicate its dependence on time, we write the wave function as $\psi_t$. For the case of a non-relativistic system, and in the absence of spin, the Hamiltonian takes the form

$$H = -\frac{1}{2}\hbar^2 \sum_j \frac{\nabla_j^2}{m_j} + V(q)$$  \hfill (1.4)

where $V(q)$ is the potential energy\textsuperscript{23} of the system. With the time evolution so specified, $\psi_t$ may be determined from its form $\psi_{t_0}$ at some previous time.

1.3.2 Rules of measurement

As stated above, the description of the state and its evolution does not constitute the entire quantum formalism. The wave function provides only a formal description and does not by itself make contact with the properties of the system. Using only the wave function and its evolution, we cannot make predictions about the typical systems in which we are interested, such as the electrons in an atom, the conduction electrons of a metal, and photons of light. The connection of the wave function to any physical properties is made through the rules of measurement. Because the physical properties in quantum theory are defined through measurement, or observation, they are referred to as ‘observables’. The quantum formalism represents the observables by Hermitian operators\textsuperscript{24} on the system Hilbert space.

Associated with any observable $O$ is a set of eigenvectors and eigenvalues. These quantities are defined by the relationship

$$O | \phi \rangle = \mu | \phi \rangle,$$  \hfill (1.5)

where $\mu$ is a real constant. We refer to $| \phi \rangle$ as the eigenvector corresponding to the eigenvalue $\mu$. We label the set of eigenvalues so defined as $\{\mu_a\}$. To each member of this set, there corresponds a set of eigenvectors, all of which are elements of a subspace of $\mathcal{H}$, sometimes referred to as the eigenspace belonging to $\mu_a$. We label this subspace as $\mathcal{H}_a$. Any two such distinct subspaces are orthogonal, i.e., every vector of $\mathcal{H}_a$ is orthogonal to every vector of $\mathcal{H}_b$ if $\mu_a \neq \mu_b$.

\textsuperscript{23}We ignore here the possibility of an external magnetic field.

\textsuperscript{24}In the following description, we use the terms observable and operator interchangeably.
It is important to develop explicitly the example of a *commuting set* of observables \( (O_1, O_2, \ldots) \) i.e., where each pair \( (O_i, O_j) \) of observables has commutator zero:

\[
[O_i, O_j] = O_i O_j - O_j O_i = 0
\]  

(1.6)

For this case, we have a series of relationships of the form 1.5

\[
O^i |\phi \rangle = \mu^i |\phi \rangle 
\]

(1.7)

\[i = 1, 2, \ldots\]

defining the set of *simultaneous* eigenvalues and eigenvectors, or *joint-eigenvalues* and *joint-eigenvectors*. Note that 1.5 may be regarded as a vector representation of the set of equations 1.7 where \( O = (O_1, O_2, \ldots) \) and \( \mu = (\mu^1, \mu^2, \ldots) \) are seen respectively as ordered sets of operators and numbers. We use the same symbol, \( \mu \), to denote a joint-eigenvalue, as was used to designate an eigenvalue. We emphasize that for a commuting set \( \mu \) refers to an *ordered set* of numbers. The correspondence of the joint-eigenvalues to the joint-eigenvectors is similar to that between the eigenvalues and eigenvectors discussed above; to each \( \mu_a \) there corresponds a set of joint-eigenvectors forming a subspace of \( \mathcal{H} \) which we refer to as the *joint-eigenspace* belonging to \( \mu_a \).

We denote by \( P_a \) the projection operator onto the eigenspace belonging to \( \mu_a \).

It will be useful for our later discussion to express \( P_a \) in terms of an orthonormal basis of \( \mathcal{H}_a \). If \( \phi_k \) is such a basis, we have

\[
P_a = \sum_k |\phi_k \rangle \langle \phi_k |,
\]

(1.8)

which means that the operator which projects onto a subspace is equal to the sum of the projections onto the one-dimensional spaces defined by any basis of the subspace. If we label the eigenvalues of the observable \( O \) as \( \mu_a \) then \( O \) is represented in terms of the \( P_a \) by

\[
O = \sum_a \mu_a P_a.
\]

(1.9)

Having developed these quantities, we now discuss the rules of measurement. The first rule concerns the possible outcomes of a measurement and it states that they are restricted to the eigenvalues \( \{\mu_a\} \) for the measurement of a single observable\(^{25}\) \( O \) or *joint-eigenvalues* \( \{\mu_a\} \) for measurement a commuting set of observables \( (O_1, O_2, \ldots) \).

The second rule provides the probability of the measurement result equaling one particular eigenvalue or joint-eigenvalue:

\[
P(O = \mu_a) = |\langle \psi | P_a |\psi \rangle|^2
\]

(1.10)

\[
P((O^1, O^2, \ldots) = \mu_a) = |\langle \psi | P_a |\psi \rangle|^2
\]

\(^{25}\)A measurement may be classified either as *ideal* or *non-ideal*. Unless specifically stated, it will be assumed whenever we refer to a measurement process, what is said will apply just as well to *either* of these situations.
where the former refers to the measurement of a single observable and the latter
to a commuting set. As a consequence of the former, the expectation value for
the result of a measurement of $O$ is given by

$$E(O) = \sum_a \mu_a \langle \psi | P_a | \psi \rangle = \langle \psi | O | \psi \rangle$$  \hspace{1cm} (1.11)$$

where the last equality follows from 1.9.

The third rule governs the effect of measurement on the system’s wave
function. It is here that the measurement is governed by a different rule depending
on whether one performs an ideal or non-ideal measurement. An ideal measure-
ment is defined as one for which the wave function’s form after measurement
is given by the (normalized) projection of $\psi$ onto the eigenspace $\mathcal{H}_a$ of the
measurement result $\mu_a$

$$P_a \psi / \| P_a \psi \|$$  \hspace{1cm} (1.12)$$

The case of a non-ideal measurement arises when an ideal measurement of a com-
muting set of observables ($O_1, O_2, ...$) is regarded as a measurement of an indi-
vidual member of the set. An ideal measurement of the set of observables leaves
the wave function as the projection of $\psi$ onto their joint-eigenspace. For an
arbitrary vector $\psi$, the projection onto the eigenspace of an individual member
of the set does not generally equal the projection onto the set’s joint-eigenspace.
Therefore an ideal measurement of a commuting set cannot be an ideal mea-
surement of an individual member of the set. We refer to the procedure as a non-ideal
measurement of the observable. This completes the presentation of the general rules of measurement. We discuss below two special cases for which
these rules reduce to a somewhat simpler form.

The first special case is that of a non-degenerate observable. The eigenspaces
of such an observable are one-dimensional and are therefore spanned by a single
normalized vector called an eigenvector. We label the eigenvector corresponding
to eigenvalue $\mu_a$ as $|a\rangle$. The operator which projects onto such a one-dimensional
space is written as:

$$P_a = |a\rangle \langle a|.$$  \hspace{1cm} (1.13)$$

It is easy to see that the projection given in 1.8 reduces to this form. The form
of the observable given in 1.9 then reduces to:

$$O = \sum_a \mu_a |a\rangle \langle a|.$$  \hspace{1cm} (1.14)$$

The probability of a measurement result being equal to $\mu_a$ is then

$$P(O = \mu_a) = |\langle a | \psi \rangle|^2,$$  \hspace{1cm} (1.15)$$
in which case the wave function subsequent to the measurement is given by $|a\rangle$. These rules are perhaps the most familiar ones since the non-degenerate case is
usually introduced first in presentations of quantum theory.
Finally, there is a case for which the measurement rules are essentially the same as these, and this is that of a set of commuting observables which form a complete set. The joint-eigenspaces of a complete set are one-dimensional. The rules governing the probability of the measurement result $\mu_i$ and the effect of measurement on the wave function are perfectly analogous to those of the non-degenerate observable. This completes our discussion of the rules of measurement.

1.4 Von Neumann’s theorem and hidden variables

1.4.1 Introduction

The quantum formalism contains features which may be considered objectionable by some. These are its subjectivity and indeterminism. The aim of the development of a hidden variables theory is to give a formalism which, while being empirically equivalent to the quantum formalism, does not possess these features. In the present section we shall present and discuss one of the earliest works to address the hidden variables question, which is the 1932 analysis of John von Neumann. We shall also review and elaborate on J.S. Bell’s analysis of this work, in which he made clear its limitations.

Von Neumann’s hidden variables analysis appeared in his now classic book *Mathematical Foundations of Quantum Mechanics*. This book is notable both for its exposition of the mathematical structure of quantum theory, and as one of the earliest works to systematically address both the hidden variables issue and the measurement problem. The quantum formalism presents us with two different types of state function evolution: that given by the Schrödinger equation and that which occurs during a measurement. The latter evolution appears in the formal rule given above in equation 1.12. The measurement problem is the problem of the reconciliation of these two types of evolution.

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26 J.S. Bell goes further than this: he refers to quantum mechanics as unprofessional ([18] and [40, p. 45]) in its lack of clarity.

27 See the discussion of Bohmian mechanics above. The great success of this theory is that it explains quantum phenomena without such features. The general issue of hidden variables is, of course, discussed in several references. Bell’s work is the most definitive: see [8, 9, 16] in [21]. A recent review was published by N.D. Mermin, who has done much to popularize Bell’s theorem through articles in *Physics Today* and through popular lectures. Discussions may be also be found in Bohm [28, 32], Belinfante [6], Hughes [69] and Jammer [70].

28 The original work is [101]. Discussions of von Neumann’s hidden variables analysis may be found within [2, 8, 16, 71].

29 Within chapter 4 of the present work, we shall discuss a 1935 analysis by Erwin Schrödinger [91]. This is the paper in which the ‘Schrödinger’s cat’ paradox first appeared, but it is not generally appreciated that it contains other results of equal or perhaps greater significance, such as Schrödinger’s generalization of the Einstein–Podolsky–Rosen paradox. We believe that this remarkable paper could have done much to advance the study of the foundations of quantum mechanics, had these latter features been more widely appreciated.
In his analysis of the hidden variables problem, von Neumann proved a mathematical result now known as von Neumann’s theorem and then argued that this theorem implied the very strong conclusion that no hidden variables theory can provide empirical agreement with quantum mechanics: (preface p. ix.x) “...such an explanation (by ‘hidden parameters’) is incompatible with certain qualitative fundamental postulates of quantum mechanics.” The author further states: [101, p. 324] “It should be noted that we need not go further into the mechanism of the ‘hidden parameters,’ since we now know that the established results of quantum mechanics can never be re-derived with their help.” The first concrete demonstration\(^{30}\) that this claim did not follow was given in 1952 when David Bohm constructed a viable theory of hidden variables [28]. Then in 1966, J.S. Bell [8] analyzed von Neumann’s argument against hidden variables and showed it to be in error. In this section, we begin by discussing an essential concept of von Neumann’s analysis: the state representation of a hidden variables theory. We then present von Neumann’s theorem and no-hidden-variables argument. Finally, we show where the flaw in this argument lies.

The analysis of von Neumann is concerned with the description of the state of a system, and the question of the incompleteness of the quantum formalism’s description. The notion of the incompleteness of the quantum mechanical description was particularly emphasized by Einstein, as we noted in section 1.1.2. The famous Einstein–Podolsky–Rosen paper was designed as a proof of such incompleteness and the authors concluded this work with the following statement: [51] “While we have thus shown that the wave function does not provide a complete description of the physical reality, we left open the question of whether or not such a description exists. We believe, however, that such a theory is possible.” The hidden variables program, which is an endeavor to supplement the state description, is apparently exactly the type of program Einstein was advocating. A complete state description might be constructed to remove some of the objectionable features of the quantum theoretical description.

The particular issue which von Neumann’s analysis addressed was the following: is it possible to restore determinism to the description of physical systems by the introduction of hidden variables into the description of the state of a system. The quantum formalism’s state representation given by \(\psi\) does not generally permit deterministic predictions regarding the values of the physical quantities, i.e. the observables. Thus results obtained from performance of measurements on systems with identical state representations \(\psi\) may be expected to vary. (The statistical quantity called the dispersion is used to describe this variation quantitatively.). While it generally does not provide predictions for each individual measurement of an observable \(O\), the quantum formalism does

\(^{30}\)See Jammer’s book [70, page 272] for a discussion of an early work by Grete Hermann that addresses the impact of von Neumann’s theorem.
give a prediction for its average or expectation value\(^{31}\):

\[
E(O) = \langle \psi \mid O \mid \psi \rangle.
\] (1.17)

Von Neumann’s analysis addresses the question of whether the lack of determinism in the quantum formalism may be ascribed to the fact that the state description as given by \(\psi\) is incomplete. If this were true, then the complete description of state—consisting of both \(\psi\) and an additional parameter we call \(\lambda\)—should allow one to make predictions regarding individual measurement results for each observable. Note that such predictability can be expressed mathematically by stating that for every \(\psi\) and \(\lambda\), there must exist a “value map” function, i.e. a mathematical function assigning to each observable its value. We represent such a value map function by the expression \(V^\psi_\lambda(O)\). Von Neumann referred to a hypothetical state described by the parameters \(\psi\) and \(\lambda\) as a “dispersion-free state”, since results obtained from measurements on systems with identical state representations in \(\psi\) and \(\lambda\) are expected to be identical and therefore exhibit no dispersion.

Von Neumann’s theorem is concerned with the general form taken by a function \(E(O)\), which assigns to each observable its expectation value. The function addressed by the theorem is considered as being of sufficient generality that the expectation function of quantum theory, or of any empirically equivalent theory must assume the form von Neumann derived. In the case of quantum theory, \(E(O)\) should take the form of the quantum expectation formula 1.16. In the case of a dispersion-free state, the average over a series of \(E(O)\) should return the value of each observable. When one analyzes the form of \(E(O)\) developed in the theorem, it is easy to see that it cannot be a function of the latter type. Von Neumann went on to conclude from this that no theory involving dispersion-free states can agree with quantum mechanics. However, since the theorem places an unreasonable restriction on the function \(E(O)\), this conclusion does not follow.

### 1.4.2 Von Neumann’s theorem

The assumptions regarding the function \(E(O)\) are as follows. First, the value \(E\) assigns to the ‘identity observable’ \(1\) is equal to unity:

\[
E(1) = 1
\] (1.18)

The identity observable is the projection operator associated with the entire system Hilbert space. All vectors are eigenvectors of \(1\) of eigenvalue 1. The

\(^{31}\)When generalized to the case of a mixed state, this becomes

\[
E(O) = \text{Tr}(UO)
\] (1.16)

where \(U\) is a positive operator with the property \(\text{Tr}(U) = 1\). Here \(U\) is known as the “density matrix”. See for example, [89, p. 378]
second assumption is that $E$ of any real-linear combination of observables is the same linear combination of the values $E$ assigns to each individual observable:

$$E(aA + bB + ...) = aE(A) + bE(B) + ...$$  \hspace{1cm} (1.19)

where $(a, b, ...)$ are real numbers and $(A, B, ...)$ are observables. Finally, it is assumed that $E$ of any projection operator $P$ must be non-negative:

$$E(P) \geq 0.$$  \hspace{1cm} (1.20)

For example, in the case of the value map function $V_{\psi}^\lambda$, $P$ must be assigned either 1 or 0, since these are its possible values. According to the theorem, these premises imply that $E(O)$ must be given by the form

$$E(O) = \text{Tr}(UO)$$  \hspace{1cm} (1.21)

where $U$ is a positive operator with the property $\text{Tr}(U) = 1$.

The demonstration\footnote{A proof of the theorem may be found in von Neumann’s original work [101]. Albertson presented a simplification of this proof in 1961 [2]. What we present here is a further simplification.} of this conclusion is straightforward. We begin by noting that any operator $O$ may be written as a sum of Hermitian operators. Define $A$ and $B$ by the relationships $A = \frac{1}{2}(O + O^\dagger)$ and $B = \frac{i}{2}(O - O^\dagger)$, where $O^\dagger$ is the Hermitian conjugate of $O$. Then it is easily seen that $A$ and $B$ are Hermitian and that

$$O = A + iB.$$  \hspace{1cm} (1.22)

We define the function $E^*(O)$ by

$$E^*(O) = E(A) + iE(B),$$  \hspace{1cm} (1.23)

where $E(O)$ is von Neumann’s $E(O)$, and $A$ and $B$ are defined as above. From the equations 1.23 and 1.19 we have that $E^*(O)$ has the property of complex linearity. Note that $E^*(O)$ is a generalization of von Neumann’s $E(O)$: the latter is a real-linear function on the Hermitian operators, while the former is a complex-linear function on all operators. The general form of $E^*$ will now be investigated for the case of a finite-dimensional operator expressed as a matrix in terms of some orthonormal basis. We write the operator $O$ in the form

$$O = \sum_{m,n} |m\rangle\langle m|O|n\rangle\langle n|,$$  \hspace{1cm} (1.24)

where the sums over $m, n$ are finite sums. This form of $O$ is a linear combination of the operators $|m\rangle\langle n|$, so that the complex linearity of $E^*$ implies that

$$E^*(O) = \sum_{m,n} \langle m|O|n\rangle E^*\left(|m\rangle\langle n|\right).$$  \hspace{1cm} (1.25)
Now we define the operator $U$ by the relationship $U_{nm} = E^*(|m\rangle\langle n|)$ and the 1.25 becomes

$$E^*(O) = \sum_{m,n} O_{mn} U_{nm} = \sum_n (UO)_{nn} = \text{Tr}(UO).$$

(1.26)

Since von Neumann’s $E(O)$ is a special case of $E^*(O)$, 1.26 implies that

$$E(O) = \text{Tr}(UO).$$

(1.27)

We now show that $U$ is a positive operator. It is a premise of the theorem that $E(P) \geq 0$ for any projection operator $P$. Thus we write $E(P_{\chi}) \geq 0$ where $P_{\chi}$ is a one-dimensional projection operator onto the vector $\chi$. Using the form of $E$ found in 1.27, we have

$$\text{Tr}(UP_{\chi}) = \langle \chi | U | \chi \rangle \geq 0.$$  

(1.28)

Since $\chi$ is an arbitrary vector, it follows that $U$ is a positive operator. The relation $\text{Tr}(U) = 1$ is shown as follows: from the first assumption of the theorem 1.18 together with the form of $E$ given by 1.27 we have $\text{Tr}(U) = \text{Tr}(U1) = 1$. This completes the demonstration of von Neumann’s theorem.

1.4.3 Von Neumann’s impossibility proof

We now present von Neumann’s argument against the possibility of hidden variables. Consider the function $E(O)$ evaluated on the one-dimensional projection operators $P_{\phi}$. For such projections, we have the relationship

$$P_{\phi} = P_{\phi}^2.$$  

(1.29)

As mentioned above, in the case when $E(O)$ is to correspond to a dispersion free state represented by some $\psi$ and $\lambda$, it must map the observables to their values. We write $V^{\psi}_{\lambda}(O)$ for the value map function corresponding to the state specified by $\psi$ and $\lambda$. Von Neumann noted $V^{\psi}_{\lambda}(O)$ must obey the relation:

$$f(V^{\psi}_{\lambda}(O)) = V^{\psi}_{\lambda}(f(O)),$$  

(1.30)

where $f$ is any mathematical function. This is easily seen by noting that the quantity $f(O)$ can be measured by measuring $O$ and evaluating $f$ of the result. This means that the value of the observable $f(O)$ will be $f$ of the value of $O$.

---

33The equality $\text{Tr}(UP_{\chi}) = \langle \chi | U | \chi \rangle$ in 1.28 is seen as follows. The expression $\text{Tr}(UP_{\chi})$ is independent of the orthonormal basis $\phi_n$ in terms of which the matrix representations of $U$ and $P_{\chi}$ are expressed, so that one may choose an orthonormal basis of which $|\chi\rangle$ itself is a member. Since $P_{\chi} = |\chi\rangle\langle\chi|$, and $P_{\chi} |\phi_n\rangle = 0$ for all $|\phi_n\rangle$ except $|\chi\rangle$, we have $\text{Tr}(UP_{\chi}) = \langle \chi | U | \chi \rangle$. 

---
Thus, if $\Psi_\lambda(O)$ maps each observable to a value then we must have 1.30. Hence

$$V_\lambda(P_\phi) = (V_\lambda(P_\phi))^2$$

which together with 1.29 implies

$$V_\lambda(P_\phi) = (V_\lambda(P_\phi))^2 \quad (1.31)$$

This last relationship implies that $V_\lambda(P_\phi)$ must be equal to either 0 or 1.

Recall from the previous section the relation $E(P_\phi) = \langle \phi | U | \phi \rangle$. If $E(O)$ takes the form of a value map function such as $V_\lambda(P_\phi)$, then it follows that the quantity $\langle \phi | U | \phi \rangle$ is equal to either 0 or 1. Consider the way this quantity depends on vector $| \phi \rangle$. If we vary $| \phi \rangle$ continuously then $\langle \phi | U | \phi \rangle$ will also vary continuously. If the only possible values of $\langle \phi | U | \phi \rangle$ are 0 and 1, it follows that this quantity must be a constant, i.e. we must have either $\langle \phi | U | \phi \rangle = 0 \forall \phi \in \mathcal{H}$, or $\langle \phi | U | \phi \rangle = 1 \forall \phi \in \mathcal{H}$. If the former holds true, then it must be that $U$ itself is zero. However, with this and the form 1.27, we find that $E(1) = 0$; a result that conflicts with the theorem assumption that $E(1) = 1$ (1.18). Similarly, if $\langle \phi | U | \phi \rangle = 1$ for all $\phi \in \mathcal{H}$ then it follows that $U = 1$. This result also conflicts with the requirement 1.18, since it leads to $E(1) = \text{Tr}(1) = n$ where $n$ is the dimensionality of $\mathcal{H}$.

From the result just obtained, one can conclude that any function $E(O)$ which satisfies the constraints of von Neumann’s theorem (see 1.18, 1.19, and 1.20) must fail to satisfy the relationship 1.30, and so cannot be a value map function on the observables. From this result, von Neumann concluded that it is impossible for a deterministic hidden variables theory to provide empirical agreement with quantum theory: [101, p. 325]

It is therefore not, as is often assumed, a question of a re-interpretation of quantum mechanics — the present system of quantum mechanics would have to be objectively false in order that another description of the elementary processes than the statistical one be possible.

### 1.4.4 Refutation of von Neumann’s impossibility proof

While it is true that the mathematical theorem of von Neumann is a valid one, it is not the case that the impossibility of hidden variables follows. The invalidity of von Neumann’s argument against hidden variables was shown by Bohm’s development [28] of a successful hidden variables theory (a counter-example to von Neumann’s proof) and by J.S. Bell’s explicit analysis [8] of von Neumann’s proof. We will now present the latter.

The no hidden variables demonstration of von Neumann may be regarded as consisting of two components: a mathematical theorem and an analysis of

\[\begin{align*}
\text{1.30 It should be noted that the same result may be proven without use of 1.30 since the fact that } V_\lambda(P_\phi) \text{ must be either 0 or 1 follows simply from the observation that these are the eigenvalues of } P_\phi.\]
its implications toward hidden variables. As we have said, the theorem itself is correct when regarded as pure mathematics. The flaw lies in the analysis connecting this theorem to hidden variables. The conditions prescribed for the function $E$ are found in equations 1.18, 1.19, and 1.20. The theorem of von Neumann states that from these assumptions follows the conclusion that the form of $E(O)$ must be given by 1.27. When one considers an actual physical situation, it becomes apparent that the second of the theorem’s conditions is not at all a reasonable one. As we shall see, the departure of this condition from being a reasonable constraint on $E(O)$ is marked by the case of its application to non-commuting observables.

We wish to demonstrate why the assumption 1.19 is an unjustified constraint on $E$. To do so, we first examine a particular case in which such a relationship is reasonable, and then contrast this with the case for which it is not. The assumption itself calls for the real-linearity of $E(O)$, i.e. that $E$ must satisfy $E(aA + bB + \ldots) = aE(A) + bE(B) + \ldots$ for any observables $\{A, B, \ldots\}$ and real numbers $\{a, b, \ldots\}$. This is in fact, a sensible requirement for the case where $\{A, B, \ldots\}$ are commuting observables. Suppose for example, the observables $O_1, O_2, O_3$ form a commuting set and that they obey the relationship $O_1 = O_2 + O_3$. We know from the quantum formalism that one may measure these observables simultaneously and that the measurement result $\{o_1, o_2, o_3\}$ must be a member of the joint-eigenspectrum of the set. By examining the relation 1.7 which defines the joint-eigenspectrum, it is easily seen that any member of the joint eigenspectrum of $O_1, O_2, O_3$ must satisfy $o_1 = o_2 + o_3$. This being the case, one might well expect that the function $E(O)$—which in the case of a dispersion free state must be a value map $V_{\lambda}^{X}(O)$ on the observables—should be required to satisfy $E(O_1) = E(O_2) + E(O_3)$. On the other hand, suppose we consider a set $\{O, P, Q\}$ satisfying $O = P + Q$, where the observables $P$ and $Q$ fail to commute, i.e. $[P, Q] \neq 0$. It is easy to see that $O$ commutes with neither $P$ nor $Q$. It is therefore impossible to perform a simultaneous measurement of any two of these observables. Hence, measurements of these observables require three distinct experimental procedures. This being so, there is no justification for the requirement that $E(O) = E(P) + E(Q)$ for such cases.

As an example one may consider the case of a spin $\frac{1}{2}$ particle. Suppose that the components of the spin given by $\sigma_x$, $\sigma_y$ and $\sigma'$ where

$$\sigma' = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_y),$$

(1.32)

are to be examined. The measurement procedure for any given component of the spin of a particle is performed by a suitably oriented Stern-Gerlach magnet. For example, to measure the $x$-component, the magnet must be oriented along the $x$-axis; for the $y$-component it must be oriented along the $y$-axis. A measurement of $\sigma'$ is done using a Stern-Gerlach magnet oriented along an axis in yet another
direction. The relationship 1.32 cannot be a reasonable demand to place on the expectation function $E(O)$ of the observables $\sigma_x, \sigma_y, \sigma'$, since these quantities are measured using completely distinct procedures.

Thus von Neumann’s hidden variables argument is seen to be an unsound one. That it is based on an unjustified assumption is sufficient to show this. It should also be noted that the presence of the real-linearity postulate discussed above makes von Neumann’s entire case against hidden variables into an argument of a rather trivial character. Examining the above example involving the three spin components of a spin $\frac{1}{2}$ particle, we find that the eigenvalues of these observables $\pm \frac{1}{2}$ do not obey 1.32, i.e.

$$\pm \frac{1}{2} \neq \frac{1}{\sqrt{2}} (\pm \frac{1}{2} \pm \frac{1}{2})$$

(1.33)

Since $E(O)$ by hypothesis must satisfy 1.32, it cannot map the observables to their eigenvalues. Hence, with the real-linearity assumption one can almost immediately ‘disprove’ hidden variables. It is therefore apparent that Von Neumann’s case against hidden variables rests essentially upon the arbitrary requirement that $E(O)$ obey real linearity—an assumption that gives immediate disagreement with the simple and natural demand that $E$ agree with quantum mechanics in giving the eigenvalues as the results of measurement.

### 1.4.5 Summary and further remarks

In our discussion of von Neumann’s no hidden variables argument, we found that the argument may be regarded as consisting of two components: a theorem which concerns the general form for an expectation function $E(O)$ on the observables, and a proof that the function $E(O)$ so developed cannot be a value map function. Because the assumption of the real-linearity of $E(O)$ is an unjustified one, the work of von Neumann does not imply the general failure of hidden variables. Finally, we noted that assuming only the real-linearity of $E$, one may easily arrive at the conclusion that such a function cannot be a map to the observables’ eigenvalues. Ultimately, the lesson to be learned from von Neumann’s theorem is simply that there exists no mathematical function from observables to their values obeying the requirement of real-linearity.

Abner Shimony has reported [96] that Albert Einstein was aware of both the von Neumann analysis itself and the reason it fails as a hidden variables impossibility proof. The source of Shimony’s report was a personal communication with Peter G. Bergmann. Bergmann reported that during a conversation with Einstein regarding the von Neumann proof, Einstein opened von Neumann’s book to the page where the proof is given, and pointed to the linearity assumption. He then said that there is no reason why this premise should hold.

---

35It is not difficult to show that $\sigma'$ defined in this way is the spin component along an axis which is in the $x, y$ plane and lies at 45° from both the $x$ and $y$ axis.
in a state not acknowledged by quantum mechanics, if the observables are not simultaneously measurable. Here the “state not acknowledged by quantum mechanics” seems to refer to von Neumann’s dispersion-free state, i.e. the state specified by \( \psi \) and \( \lambda \). It is almost certain that Erwin Schrödinger would also have realized the error in von Neumann’s impossibility proof, since in his 1935 paper [91] he gives a derivation which is equivalent to von Neumann’s theorem in so far as hidden variables are concerned, yet he does not arrive at von Neumann’s conclusion of the impossibility of hidden variables. We shall discuss Schrödinger’s derivation in the next section. In view of the scarcity\(^{37}\) of early responses to von Neumann’s proof, it is valuable to have such evidence of Einstein’s and Schrödinger’s awareness of the argument and its shortcomings. In addition, this affirms the notion that Einstein regarded the problem of finding a complete description of quantum phenomena to be of central importance (see again the Einstein quotes given in sections 1.1.2 and 1.4.1).

In our introduction to von Neumann’s theorem, we stated that the existence of a deterministic hidden variables theory led to the result that for each \( \psi \) and \( \lambda \), there exists a value map on the observables. We represented such value maps by the expression \( V^\psi_\lambda(O) \). If one considers the question of hidden variables more deeply, it is clear that the agreement of their predictions with those of quantum mechanics requires an additional criterion that beyond the existence of a value map for each \( \psi \) and \( \lambda \): it requires agreement with the statistical predictions of the quantum formalism. To make possible the empirical agreement of quantum theory, in which only statistical predictions are generally possible with the deterministic description of a hidden variables theory, we regard their descriptions of a quantum system in the following way. The quantum mechanical state given by \( \psi \) corresponds to a statistical ensemble of the states given by \( \psi \) and \( \lambda \); the members of the ensemble being described by the same \( \psi \), but differing in \( \lambda \).

The variation in measurement results found for a series of quantum mechanical systems with identical \( \psi \) is to be accounted for by the variation in parameter \( \lambda \) among the ensemble of \( \psi, \lambda \) states. For precise agreement in this regard, we require that for all \( \psi \) and \( O \), the following relationship must hold:

\[
\int_{-\infty}^{\infty} d\lambda \rho(\lambda) V^\psi_\lambda(O) = \langle \psi | O | \psi \rangle,
\]  

(1.34)

where \( \rho(\lambda) \) is the probability distribution over \( \lambda \).

We have seen from von Neumann’s result and from our simple examination of the spin \( \frac{1}{2} \) observables \( \sigma_x, \sigma_y, \frac{1}{\sqrt{2}}(\sigma_x + \sigma_y) \), that it is impossible to develop

\(^{36}\)In fact, if Schrödinger had interpreted his result this way, this—in light of his own generalization of the EPR paradox presented in the same paper—would have allowed him to reach a further and very striking conclusion. We shall discuss this in chapter 4.

\(^{37}\)See Max Jammer’s book [70, p. 265]. Jammer mentions that not only was there very little response to von Neumann’s impossibility proof, but the book itself was never given a review before 1957, with the exception of two brief works by Bloch and Margenau (see the Jammer book for these references).
a linear function mapping the observables to their eigenvalues. We shall also find that an impossibility proof may be developed showing that the criterion of agreement with the quantum statistics, i.e., the agreement with 1.34, cannot be met by functions of the form $V^\psi(O)$. Bell’s theorem is, in fact, such a proof. We will present the proof of Bell’s result along with some further discussion in chapter 3.

1.4.6 Schrödinger’s derivation of von Neumann’s ‘impossibility proof’

As mentioned above, in his famous “cat paradox” paper [91], Schrödinger presented an analysis which, as far as hidden variables are concerned, was essentially equivalent to the von Neumann proof. Schrödinger’s study of the problem was motivated by the results of his generalization of the Einstein–Podolsky–Rosen paradox. While EPR had concluded definite values on the position and momentum observables only, Schrödinger was able to show that such values must exist for all observables of the state considered by EPR. We will discuss both the original Einstein–Podolsky–Rosen analysis, and Schrödinger’s generalization thereof in much more detail in chapter 4. To probe the possible relationships which might govern the values assigned to the various observables, Schrödinger then gave a brief analysis of a system whose Hamiltonian takes the form

$$H = p^2 + a^2 q^2.$$  \hfill (1.35)

We are aware from the well-known solution of the harmonic oscillator problem, that this Hamiltonian’s eigenvalues are given by the set \{a\hbar, 3a\hbar, 5a\hbar, 7a\hbar, \ldots\}. Consider a mapping $V(O)$ from observables to values. If we require that the assignments $V$ makes to the observables $H, p, q$ satisfy 1.35 then we must have

$$V(H) = (V(p))^2 + a^2 (V(q))^2,$$ \hfill (1.36)

which implies

$$((V(p))^2 + a^2 (V(q))^2)/a\hbar = \text{an odd integer}. \hfill (1.37)$$

This latter relationship cannot generally be satisfied by the eigenvalues of $q$, and $p$—each of which may be any real number—and an arbitrary positive number $a$.

The connection of this result to the von Neumann argument is immediate. In the discussion of section 1.4.3, we noted that the value of the observable $f(O)$ will be $f$ of the value of $O$, so that any value map must satisfy $f(V(O)) = V(f(O))$, as given in equation 1.30. Here $f$ may be any mathematical function. It follows from this that 1.36 is equivalent to a relation between the observables $H, p^2$, and $q^2$ given by:

$$V(H) = V(p^2) + a^2 V(q^2).$$ \hfill (1.38)

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With the known eigenvalues of $H$, this leads to

\[ (V(p^2) + a^2V(q^2))/\hbar = \text{an odd integer}, \quad (1.39) \]

which cannot generally be satisfied by the eigenvalues of $q^2$, and $p^2$—each of which may be any positive real number—and an arbitrary positive number $a$. We have here another example leading to a demonstration of von Neumann’s result that there is no linear value map on the observables (Recall the example of the spin component observables—$\sigma_x, \sigma_y$ and $\sigma' = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_y)$ given above.). If we consider the von Neumann function $E(O)$, the real-linearity assumption requires it to satisfy 1.38. Therefore, $E(O)$ cannot map the observables to their eigenvalues. Schrödinger did not regard this as proof of the impossibility of hidden variables, as von Neumann did, but concluded only that the relationships such as 1.35 will not necessarily be satisfied by the value assignments made to the observables constrained by such a relation. Indeed, if Schrödinger had made von Neumann’s error of interpretation, this would contradict results he had developed previously according to which such hidden variables must exist. Such a contradiction would have allowed Schrödinger to reach a further conclusion which is quite striking, and which we shall discuss in chapter 4.
We have seen in the previous section that the analysis of von Neumann has little impact on the question of whether a viable hidden variables theory may be constructed. However, further mathematical results were developed by Andrew M. Gleason [57] in 1957 and by Simon Kochen and E.P. Specker in 1967 [74], which were claimed by some to imply the impossibility of hidden variables. In the words of Kochen and Specker [74, p. 73]: “If a physicist X believes in hidden variables... ...the prediction of X contradicts the prediction of quantum mechanics”. The Gleason, and Kochen and Specker arguments are in fact, stronger than von Neumann’s in that they assume linearity only for commuting observables. Despite this, a close analysis reveals that the impossibility proofs of Gleason and of Kochen and Specker share with von Neumann’s proof the neglect of the possibility of a hidden variables feature called contextuality [8, 16]. We will find that this shortcoming makes these theorems inadequate as proofs of the impossibility of hidden variables.

We begin this chapter with the presentation of Gleason’s theorem and the theorem of Kochen and Specker. We then discuss a more recent theorem discovered by Mermin2, which is similar to these, and yet admits a much simpler proof. This will be followed by a discussion of contextuality and its relevance to these analyses. We will make clear in this discussion why the theorems in question fail as impossibility proofs. As far as the question of what conclusions do follow, we show that these theorems’ implications can be expressed in a simple and concise fashion, which we refer to as “spectral-incompatibility”. We conclude the chapter with the discussion of an experimental procedure first discussed by Albert3 which provides further insight into contextuality.

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1See [74, 78, 83].
2See Mermin in [78, 80].
3See D. Albert [1].
2.1 Gleason’s theorem

Von Neumann’s theorem addressed the question of the form taken by a function $E(O)$ of the observables. Gleason’s theorem essentially addresses the same question, the most significant difference being that the linearity assumption is relaxed to the extent that it is demanded that $E$ be linear on only commuting sets of observables. Besides this, Gleason’s theorem involves a function $E$ on only the projection operators of the system, rather than on all observables. Finally, Gleason’s theorem contains the assumption that the system’s Hilbert space is at least three dimensional. As for the conclusion of the theorem, this is identical to von Neumann’s: $E(P)$ takes the form $E(P) = \text{Tr}(UP)$ where $U$ is a positive operator and $\text{Tr}(U) = 1$.

Let us make the requirement of linearity on the commuting observables somewhat more explicit. First we note that any set of projection $\{P_1, P_2, \ldots\}$ onto mutually orthogonal subspaces $\{\mathcal{H}_1, \mathcal{H}_2, \ldots\}$ will form a commuting set. Furthermore, if $P$ projects onto the direct sum $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \ldots$ of these subspaces, then $\{P, P_1, P_2, \ldots\}$ will also form a commuting set. It is in the case of this latter type of set that the linearity requirement comes into play, since these observables obey the relationship

$$P = P_1 + P_2 + \ldots$$

(2.1)

The condition on the function $E$ is then

$$E(P) = E(P_1) + E(P_2) + \ldots$$

(2.2)

The formal statement of Gleason’s theorem is expressed as follows. For any quantum system whose Hilbert space is at least three dimensional, any expectation function $E(P)$ obeying the conditions 2.2, $0 \leq E(P) \leq 1$, and $E(1) = 1$ must take the form

$$E(P) = \text{Tr}(UP),$$

(2.3)

where $\text{Tr}(U) = 1$ and $U$ is a positive operator. We do not present the proof of this result here. In the next section, we give the outline the proof of Kochen and Specker’s theorem. The same impossibility result derived from Gleason’s theorem follows from this theorem.

It is straightforward to demonstrate that the function $E(P)$ considered within Gleason’s theorem cannot be a value map function on these observables.\footnote{The original form presented by A.M. Gleason referred to a probability measure on the subspaces of a Hilbert space, but the equivalence of such a construction with a value map on the projection operators is simple and immediate. This may be seen by considering that there is a one-to-one correspondence of the subspaces and projections of a Hilbert space and that the values taken by the projections are 1 and 0, so that a function mapping projections to their eigenvalues is a special case of a probability measure on these operators.}

\footnote{See Bell [8]. Bell proves that any function $E(P)$ satisfying the conditions of Gleason’s theorem cannot map the projection operators to their eigenvalues.}
To demonstrate this, one may argue in the same fashion as was done by von Neumann, since the form developed here for $E(P)$ is the same as that concluded by the latter. (See section 1.4.3). We recall that if $E(O)$ is to represent a dispersion free state specified by $\psi$ and $\lambda$, it must take the form of such a value map, and $E(O)$ evidently cannot be the expectation function for such a state. It is on this basis that the impossibility of hidden variables has been claimed to follow from Gleason’s theorem.

### 2.2 The theorem of Kochen and Specker

As with Gleason’s theorem, the essential assumption of Kochen and Specker’s theorem is that the expectation function $E(O)$ must be linear on commuting sets of observables. It differs from the former only in the set of observables considered. Gleason’s theorem was addressed to the projection operators on a Hilbert space of arbitrary dimension $N$. Kochen and Specker consider the squares $\{s_{\theta,\phi}^2\}$ of the spin components of a spin 1 particle. One may note that these observables are formally identical to projection operators on a three-dimensional Hilbert space. Thus, the Kochen and Specker observables are a subset of the “$N = 3$” case of the Gleason observables. Among the observables $\{s_{\theta,\phi}^2\}$ any subset $\{s_x^2, s_y^2, s_z^2\}$ corresponding to mutually orthogonal component directions $x, y, z$ will be a commuting set. Each such set obeys the relationship

$$s_x^2 + s_y^2 + s_z^2 = 2.$$  \tag{2.4}

For every such subset, Kochen and Specker require that $E(s_{\theta,\phi}^2)$ must obey

$$E(s_x^2) + E(s_y^2) + E(s_z^2) = 2.$$  \tag{2.5}

Kochen and Specker theorem states that there exists no function $E(s_{\theta,\phi}^2)$ on the squares of the spin components of a spin 1 particle which maps each observable to either 0 or 1 and which satisfies 2.5.

We now make some comments regarding the nature of this theorem’s proof. The problem becomes somewhat simpler to discuss when formulated in terms of a geometric model. Imagine a sphere of unit radius surrounding the origin in $\mathbb{R}^3$. It is easy to see that each point on this sphere’s surface corresponds to a direction in space, which implies that each point is associated with one observable of the set $\{s_{\theta,\phi}^2\}$. With this, $E$ may be regarded as a function on the surface of the unit sphere. Since the eigenvalues of each of these spin observables are 0 and 1, it must be that $E(O)$ must take on these values, if it is to assume the form of a value map function. Satisfaction of 2.5 requires that for each set of mutually orthogonal directions, $E$ must assign to one of them 0 and to the other directions 1. To gain some understanding of why such an assignment

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6 The set of projections on a three-dimensional space is actually a larger class of observables.

7 We follow here the argument given in Belinfante [6, p. 38]
of values must fail, we proceed as follows. To label the points on the sphere, we imagine that each point on the sphere to which the number 0 is assigned is painted red, and each point assigned 1 is painted blue. We label each direction as by the unit vector $\hat{n}$. Since one direction of every mutually perpendicular set is assigned red, then in total we require that one-third of the sphere is painted red. If we consider the components of the spin in opposite directions $\theta, \phi$, and $180^\circ - \theta, 180 + \phi$, these values are always opposite, i.e., if $s_x$ takes the value +1, then $s_{-x}$ takes the value $-1$. This implies that the values of $s_{\theta, \phi}^2$ and $s_{180^\circ - \theta, 180 + \phi}^2$ will be equal. There, we must have that points on the sphere lying directly opposite each other, i.e. the “antipodes”, must receive the same assignment from $E$. Suppose that one direction and its antipode are painted red. These points form the two poles of a great circle, and all points along this circle must then be painted blue, since all such points represent directions orthogonal to the directions of our two ‘red’ points. For every such pair of red points on the sphere, there must be many more blue points introduced, and we will find this makes it impossible to make one-third of the sphere red, as would be necessary to satisfy 2.5.

Suppose we paint the entire first octant of the sphere red. In terms of the coordinates used by geographers, this is similar to the region in the Northern hemisphere between 0 degrees and 90 degrees longitude. If the point at the north pole is painted red then the great circle at the equator must be blue. Suppose that the 0 degree and 90 degree meridians are also painted red. Then the octant which is the antipode of the first octant must also be painted red. This octant would be within the ‘southern hemisphere’ between 180 degrees and 270 degrees longitude. If we now apply the condition that for every point painted red, all points lying on the great circle defined with the point at its pole, we find that all remaining points of the sphere must then be colored blue, thereby preventing the addition of more red points. This assignment fails to meet the criterion that less than one-third of the sphere is blue, so there are some sets of mutually perpendicular directions which are all colored blue. An example of such a set of directions is provided by the points on the sphere’s surface lying in the mutually orthogonal directions represented by $(.5,.5,-.7071)$, $(-.1464,.8535,.5)$, $(.8535,-.1464,.5)$. Each of these points lies in a quadrant to which we have assigned the color blue by the above scheme. This assignment of values to the spatial directions must therefore fail to meet the criteria demanded of the function $E(s_{\theta, \phi}^2)$ in Kochen and Specker’s premises: that $E(s_{\theta, \phi}^2)$ satisfies 2.5 and maps the observables to their values.

In their proof, Kochen and Specker show that for a discrete set of 117 different directions in space, it is impossible to give appropriate value assignments to the corresponding spin observables\textsuperscript{8}. Kochen and Specker then assert that hid-

\textsuperscript{8}Since their presentation, the proof has been simplified by Peres in 1991 [86] whose proof is based on examination of 33 such $\hat{n}$ vectors. We also note that a proof presented by Bell [8] may be shown to lead easily to a proof of Kochen and Specker. See Mermin in this connection.
den variables cannot agree with the predictions of quantum mechanics. Their conclusion is that if some physicist ‘X’, mistakenly decides to accept the validity of hidden variables then “the prediction of X (for some measurements) contradicts the prediction of quantum mechanics.” [74, p 73] The authors cite a particular system on which one can perform an experiment they claim reveals the failure of the hidden variables prediction. We will demonstrate in the next section of this work that what follows from Kochen and Specker’s theorem is only that a non-contextual hidden variables theory will conflict with quantum mechanics, so that the general possibility of hidden variables has not been disproved. Furthermore, we show that if the type of experiment envisioned by these authors is considered in more detail, it does not indicate where hidden variables must fail, but instead serves as an illustration indicating that the requirement of contextuality is a quite natural one.

2.2.1 Mermin’s theorem

We would like to discuss here a much more recent theorem discovered by N. David Mermin\(^9\) in 1990, which is of the same character as those considered above. Mermin makes essentially the same assumption regarding the function \(E(O)\) as did Gleason, and Kochen and Specker: that \(E(O)\) must obey all relationships among the commuting sets of observables. This theorem is more straightforward in its proof and simpler in form than those of Gleason and of Kochen and Specker.

The system addressed by Mermin’s theorem is that of a pair of spin \(\frac{1}{2}\) particles. The observables involved are the \(x\) and \(y\) components of these spins, and six other observables which are defined in terms of these four. We begin with the derivation of the expression

\[
\sigma_x^{(1)} \sigma_y^{(1)} \sigma_x^{(2)} \sigma_y^{(2)} \sigma_x^{(1)} \sigma_y^{(1)} \sigma_y^{(2)} \sigma_y^{(2)} = -1, \tag{2.6}
\]

since this is actually quite crucial to the theorem. For simplicity, we normalize the spin eigenvalues from \(\pm \frac{1}{2}\) to \(\pm 1\). To demonstrate 2.6, we make use of the commutation rules for the \(x\) and \(y\) components of two spin \(\frac{1}{2}\) particles. Any pair of such observables associated with different particles will commute, so that we have \([\sigma_x^{(1)}, \sigma_x^{(2)}] = 0\) and \([\sigma_x^{(1)}, \sigma_y^{(2)}] = 0\), for example. Any pair which involves the same component will also commute, thus we have \([\sigma_x^{(1)}, \sigma_x^{(1)}] = 0\) and \([\sigma_x^{(2)}, \sigma_x^{(2)}] = 0\). Note that commutation of two observables \(O_1, O_2\) implies that \(O_1 O_2 = O_2 O_1\). Those pairs associated with the same particle but different components do not commute, but *anticommutate*. For two anticommuting observables \(O_1, O_2\), it follows that their anticommutator \([O_1, O_2]^+ = O_1 O_2 + O_2 O_1\) is equal to 0. This implies that \(O_1 O_2 = -O_2 O_1\). Using these rules, we may manipulate the expression on the left hand side of 2.6 by sequentially interchanging the first

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\(^9\)See [78] and [80]
$\sigma_x^{(1)}$ with the operator appearing to its right. If we exchange $\sigma_x^{(1)}$ with $\sigma_y^{(2)}$, $\sigma_y^{(1)}$ and $\sigma_x^{(2)}$ the expression becomes $-\sigma_y^{(2)} \sigma_y^{(1)} \sigma_x^{(2)} \sigma_x^{(1)} \sigma_y^{(2)} \sigma_x^{(1)} \sigma_y^{(2)}$. The overall minus sign results from its interchange with $\sigma_y^{(1)}$. At this point, it is straightforward to simplify the expression using that the square of any component of the spin has the value 1, i.e. $(\sigma^{(i)})^2 = 1$. If we apply this to the expression in question, we can easily see that the entire expression reduces to $-1$, thereby verifying 2.6.

Motivated by the expression 2.6, we introduce six additional observables

$$\{A, B, C, X, Y, Z\}. \quad (2.7)$$

If we group the observables in the left hand side of 2.6 by pairs, we can rewrite this relationship as $ABXY = -1$ where $A, B, X, Y$ are defined as

$$A = \sigma_x^{(1)} \sigma_y^{(2)},$$
$$B = \sigma_y^{(1)} \sigma_x^{(2)},$$
$$X = \sigma_x^{(1)} \sigma_x^{(2)},$$
$$Y = \sigma_y^{(1)} \sigma_y^{(2)}.$$  \hfill (2.8)

Defining observables $C$ and $Z$ as

$$C = AB \quad (2.9)$$
$$Z = XY$$

then allows one to write 2.6 as

$$CZ = -1. \quad (2.10)$$

It is important to note that the equation 2.10 is equivalent to 2.6, given that $A, B, C, X, Y, Z$ are defined as given above in 2.9 and 2.8.

If we examine the first equation in 2.8, we see that the three observables involved are a commuting set: $[\sigma_x^{(1)}, \sigma_x^{(2)}] = 0$, $[\sigma_x^{(1)}, A] = [\sigma_x^{(1)}, \sigma_x^{(1)} \sigma_y^{(2)}] = 0$ and $[\sigma_y^{(2)}, A] = [\sigma_y^{(2)}, \sigma_x^{(1)} \sigma_y^{(2)}] = 0$. Examination of the other equations in 2.8 reveals that the same holds true for these, i.e. the observables in each form a commuting set. Repeated application of the commutation rules reveals that the sets $\{C, A, B\}$, $\{Z, X, Y\}$, and $\{C, Z\}$ are also commuting sets. As was done in Gleason’s theorem and Kochen and Specker’s theorem, we consider the question of a function $E(O)$ on the observables $\sigma_x^{(1)}, \sigma_y^{(1)}, \sigma_x^{(2)}, \sigma_y^{(2)}, A, B, C, X, Y, Z$ which returns for each observable its value. We require that $E(O)$ satisfy all constraining relationships on each commuting set of observables. The relationships in question differ from those of Gleason (2.2), and of Kochen and Specker (2.5) only in that they involve products of observables, rather than just linear combinations. However, this distinction is not a significant one—the essential
feature of all of these theorems is simply that they require $E(O)$ to satisfy the relationships constraining each commuting set; no relations on non-commuting observables are regarded as necessary constraints. The relationships in question in the present analysis are the defining equations in 2.8, 2.9 and 2.10. Given that $E(O)$ satisfies these, the Mermin theorem implies that this function cannot map the observables to their eigenvalues.

We will now present the proof of the theorem. Since $\sigma_j^{(i)} = \pm 1 \forall i, j$, the eigenvalues of each of the ten observables will be $\pm 1$. This is readily seen for the observables $A, B, X, Y$ defined by 2.8: each of these is the product of two commuting observables whose eigenvalues are 1 and $-1$. With this, it follows similarly that $C$ and $Z$ each have eigenvalues of $\pm 1$. We therefore require that $E(O)$ must assign either $-1$ or $1$ to each observable. Now let us recall an important result which we saw above: the relationship $CZ = -1$ is equivalent to the relationship 2.6, provided that the observables $A, B, C, X, Y, Z$ are defined by 2.9, and 2.8.

Consider the function $E(O)$. Since the assignments made by this function have been required to satisfy all the commuting relationships 2.10, 2.9, and 2.8, it follows that they must also satisfy 2.6. However, if we examine 2.6, it follows that no assignment of the values $-1$ and $+1$ can possibly be made which will satisfy this equation. This follows since each of the spin observables appears twice in the left hand side of 2.6, so that any such assignment must give a value of 1 to the entire expression, but the right hand side of 2.6 is $-1$. Therefore, there is no function $E(O)$ which maps each observable of the set $\sigma_x^{(1)}, \sigma_y^{(1)}, \sigma_z^{(1)}, \sigma_y^{(2)}, A, B, C, X, Y, Z$ to an eigenvalue, if we insist that $E(O)$ on every commuting set must obey all the constraint equations. This completes the proof of Mermin’s theorem.

2.3 Contextuality and the refutation of the impossibility proofs of Gleason, Kochen and Specker, and Mermin

We have seen that the theorems of Gleason, Kochen and Specker, and Mermin all demonstrate the impossibility of value maps on some sets of observables such that the constraining relationships on each commuting set of observables are obeyed. One might be at first inclined to conclude with Kochen and Specker that such results imply the impossibility of a hidden variables theory. However, if we consider that there exists a successful theory of hidden variables, namely Bohmian mechanics [28] (see section 1.1.2), we see that such a conclusion is in error. Moreover, an explicit analysis of Gleason’s theorem has been carried out by Bell [8, 16] and its inadequacy as an impossibility proof was shown. Bell’s
argument may easily be adapted\(^\text{10}\) to provide a similar demonstration regarding Kochen and Specker’s theorem. The key concept underlying Bell’s argument is that of *contextuality*, and we now present a discussion of this notion.

Essentially, contextuality refers to the dependence of measurement results on the detailed experimental arrangement being employed. In discussing this notion, we will find that an inspection of the quantum formalism suggests that contextuality is a natural feature to expect in a theory explaining the quantum phenomena. Furthermore, we shall find that the concept is in accord with Niels Bohr’s remarks regarding the fundamental principles of quantum mechanics. According to Bohr, [33] “a closer examination reveals that the procedure of measurement has an essential influence on the conditions on which the very definition of the physical quantities in question rests.” In addition, he stresses [90, p. 210] “the impossibility of any sharp distinction between the behavior of atomic objects and the interaction with the measuring instruments which serve to define the conditions under which the phenomena appear.” The concept of contextuality represents a concrete manifestation of the quantum theoretical aspect to which Bohr refers. We will first explain the concept itself in detail, and then focus on its relevance to the theorems of Gleason, Kochen and Specker, and Mermin.

We begin by recalling a particular feature of the quantum formalism. In the presentation of this formalism given in chapter one, we discussed the representation of the system’s state, the rules for the state’s time evolution, and the rules governing the measurement of an observable. The measurement rules are quite crucial, since it is only through measurement that the physical significance of the abstract quantum state (given by \(\psi\)) is made manifest. Among these rules, one finds that any commuting set of observables may be measured simultaneously. With a little consideration, one is led to observe that the possibility exists for a *variety of different experimental procedures* to measure a single observable. Consider for example, an observable \(O\) which is a member of the commuting set \(\{O, A_1, A_2, \ldots\}\). We label this set as \(C\). A simultaneous measurement of the set \(C\) certainly gives among its results a value for \(O\) and thus may be regarded as providing a measurement of \(O\). It is possible that \(O\) may be a member of another commuting set of observables \(C’ = \{O, B_1, B_2, \ldots\}\), so that a simultaneous measurement of \(C’\) also provides a measurement of \(O\). Let us suppose further that the members of set \(\{A_i\}\) fail to commute with those of \(\{B_i\}\). It is then clear that experiments measuring \(C\) and \(C’\) are quite different, and hence must be distinct. A concrete difference appears for example, in the effects of such experiments on the system wave function. The measurement rules tell us that the wave function subsequent to an ideal measurement of a commuting set is prescribed by the equation 1.12, according to which the post-measurement

\(^{10}\)As we have mentioned, since the Kochen and Specker observables are formally equivalent to projections on a three-dimensional Hilbert space, this theorem is actually a special case of Gleason’s. Therefore, Bell’s argument essentially addresses the Kochen and Specker theorem as well as Gleason’s.
wave function is calculated from the pre-measurement wave function by taking the projection of the latter into the joint-eigenspace of that set. Since the members of $C$ and $C'$ fail to commute, the joint-eigenspaces of the two are necessarily different, and the system wave function will not generally be affected in the same way by the two experimental procedures. Apparently the concept of ‘the measurement of an observable’ is ambiguous, since there can be distinct experimental procedures for the measurement of a single observable.

There are, in fact, more subtle distinctions between different procedures for measuring the same observable, and these also may be important. We will see such an example in studying Albert’s experiment at the conclusion of this chapter. To introduce the experimental procedure of measurement into our formal notation, we shall write $E(O)$, $E'(O)$, etc., to represent experimental procedures used to measure the observable $O$. From what we have seen here, it is quite natural to expect that a hidden variables theory should allow for the possibility that different experimental procedures, e.g., $E(O)$ and $E'(O)$, for the measurement of an observable might yield different results on an individual system. This is contextuality.

Examples of observables for which there exist incompatible measurement procedures are found among the observables addressed in each of the theorems of Gleason, Kochen and Specker, and Mermin. Among observables addressed by Gleason are the one-dimensional projection operators $\{P_\phi\}$ on an $N$-dimensional Hilbert space $\mathcal{H}_N$. Consider a one-dimensional projection $P_\phi$ where $\phi$ belongs to two sets of orthonormal vectors given by $\{\phi, \psi_1, \psi_2, \ldots\}$ and $\{\phi, \chi_1, \chi_2, \ldots\}$. Note that the sets $\{\psi_1, \psi_2, \ldots\}$ and $\{\chi_1, \chi_2, \ldots\}$ are constrained only in that they span $\mathcal{H}_\phi^\perp$ (the orthogonal complement of the one-dimensional space spanned by $\psi$). Given this, there exist examples of such sets for which some members of $\{\psi_1, \psi_2, \ldots\}$ are distinct from and not orthogonal to the vectors in $\{\chi_1, \chi_2, \ldots\}$. Since any distinct vectors that are not orthogonal correspond to projections which fail to commute, the experimental procedures $E(P_\phi)$ measuring $\{P_\phi, P_{\psi_1}, P_{\psi_2}, \ldots\}$ and $E'(P_\phi)$ measuring $\{P_\phi, P_{\chi_1}, P_{\chi_2}, \ldots\}$ are incompatible. The argument just given applies also to the Kochen and Specker observables (the squares of the spin components of a spin 1 particle), since these are formally identical to projections on a three-dimensional Hilbert space. To be explicit, the observable $s^2_z$ is a member of the commuting sets $\{s^2_x, s^2_y, s^2_z\}$ and $\{s^2_x, s^2_y, s^2_{z'}\}$ where the $y'$ and $z'$ are oblique relative to the $y$ and $z$ axes. In this case, $s^2_y, s^2_{z'}$ do not commute with $s^2_y, s^2_z$. Thus, the experimental procedures to measure these sets are incompatible. If we examine the Mermin observables, we find that here also, each is a member of two incompatible commuting sets. For example $\sigma^{(1)}_x$ belongs to $\{\sigma^{(1)}_x, \sigma^{(2)}_y, A\}$, and to $\{\sigma^{(1)}_x, \sigma^{(2)}_x, X\}$. Here, the observables $\sigma^{(2)}_y, A$ do not commute with $\sigma^{(2)}_x, X$, so that the experimental procedures $E(\sigma^{(1)}_x)$ and $E'(\sigma^{(1)}_x)$ that entail respectively the measurement of $\{\sigma^{(1)}_x, \sigma^{(2)}_y, A\}$ and $\{\sigma^{(1)}_x, \sigma^{(2)}_x, X\}$ are incompatible.
While it is true that the arguments against hidden variables derived from these theorems are superior to von Neumann’s, since they require agreement only with operator relationships among commuting sets, these arguments nevertheless possess the following shortcoming. Clearly, the mathematical functions considered in each case, \( E(P), E(s^2_{\theta, \phi}) \), and \( E(\sigma^{(1)}_x, \sigma^{(1)}_y, \sigma^{(2)}_x, \sigma^{(2)}_y, A, B, C, X, Y, Z) \) do not allow for the possibility that the results of measuring each observable using different and possibly incompatible procedures may lead to different results. What the theorems demonstrate is that no hidden variables formulation based on assignment of a unique value to each observable can possibly agree with quantum mechanics. But this is a result we might well have expected from the fact that the quantum formalism allows the possibility of incompatible experimental procedures for the measurement of an observable. For this reason, none of the theorems here considered—Gleason’s theorem, Kochen and Specker’s theorem and Mermin’s theorem—imply the impossibility of hidden variables, since they fail to account for such a fundamental feature of the quantum formalism’s rules of measurement.

2.3.0.1 Discussion of a procedure to measure the Kochen and Specker observables

In a discussion of the implications of their theorem, Kochen and Specker mention a system for which well-known techniques of atomic spectroscopy may be used to measure the relevant spin observables. Although these authors mention this experiment to support their case against hidden variables, the examination of such an experiment actually reinforces the assertion that one should allow for contextuality—the very concept that refutes their argument against hidden variables.

Kochen and Specker note\(^{11}\) that for an atom of orthohelium\(^{12}\) which is subjected to an electric field of a certain configuration, the first-order effects of this field on the electrons may be accounted for by adding a term of the form \( aS^2_x + bS^2_y + cS^2_z \) to the electronic Hamiltonian. Here \( a, b, c \) are distinct constants, and \( S^2_x, S^2_y, S^2_z \) are the squares of the components of the total spin of the two electrons with respect to the Cartesian axes \( x, y, z \). The Cartesian axes are defined by the orientation of the applied external field. For such a system, an experiment measuring the energy of the electrons also measures the squares of the three spin components. To see this, note that the value of the perturbation

\(^{11}\)One can derive the analogous first-order perturbation term arising for a charged particle of orbital angular momentum \( L = 1 \) in such an electric field using the fact that the joint-eigenstates of \( L^2_x, L^2_y, L^2_y \) are the eigenstates of the potential energy due to the field. This latter result is shown in Kittel [73, p. 427].

\(^{12}\)Orthohelium and parahelium are two species of helium which are distinguished by the total spin \( S \) of the two electrons: for the former we have \( S = 1 \), and for the latter \( S = 0 \). There is a rule of atomic spectroscopy which prohibits atomic transitions for which \( \Delta S = 1 \), so that no transitions from one form to the other can occur spontaneously.
energy will be \((a + b), (a + c), \) or \((b + c)\) if the joint values of the set \(S_x^2, S_y^2, S_z^2\) equal respectively \(\{1, 1, 0\}, \{1, 0, 1\}, \) or \(\{0, 1, 1\}\).

To understand why the external electric field affects the orthohelium electrons in this way, consider the ground state of orthohelium\(^{13}\). The wave function of this state is given by a spatial part \(\phi(r_1, r_2)\), depending only on \(r_1, r_2\) (the radial coordinates of the electrons), multiplied by the spin part, which is a linear combination of the eigenvectors \(\psi_{\pm 1}, \psi_0, \psi_{-1}\) of \(S_z\), corresponding to \(S_z = +1, 0, -1\), respectively. Thus, the ground state may be represented by any vector in the three-dimensional Hilbert space spanned by the vectors \(\phi(r_1, r_2)\psi_{\pm 1}, \phi(r_1, r_2)\psi_0, \phi(r_1, r_2)\psi_{-1}\). The external electric field will have the effect of “lifting the degeneracy” of the state, i.e. the new Hamiltonian will not be degenerate in this space, but its eigenvalues will correspond to three unique orthogonal vectors. Suppose that we consider a particular set of Cartesian axes \(x, y, z\). We apply an electric field which is of orthorhombic symmetry\(^{14}\) with respect to these axes. It can be shown\(^{15}\) that the eigenvectors of the Hamiltonian due to this field are

\[
v_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \quad (2.11)
\]

\[
v_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad (2.12)
\]

\[
v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad (2.13)
\]

\(^{13}\)Using spectroscopic notation, this state would be written as the \(^2^3S\) state of orthohelium. The \(^2\) refers to the fact that the principle quantum number \(n\) of the state equals 2, \(^3\) denotes that the total orbital angular momentum is zero, and the \(^S\) superscript means that it is a spin triplet state. Orthohelium has no state of principle quantum number \(n = 1\), since the Pauli exclusion principle forbids the \(^1^3S\) state.

\(^{14}\)Orthorhombic symmetry is defined by the criterion that rotation about either the \(x\) or \(y\) axis by 180° would bring such a field back to itself.

\(^{15}\)A straightforward way to see this is by analogy with a charged particle of orbital angular momentum \(L = 1\). The effects of an electric or magnetic field on a charged particle of spin 1 are analogous to the effects of the same field on a charged particle of orbital angular momentum 1. To calculate the first-order effects of an electric field of orthorhombic symmetry for such a particle, one can examine the spatial dependence of the \(L_z = 1, 0, -1\) states \(\psi_{-1}, \psi_0, \psi_{+1}\), together with the spatial dependence of the perturbation potential \(V(r)\), to show that the states \(1/\sqrt{2}(\psi_1 - \psi_{-1})\), \(1/\sqrt{2}(\psi_1 + \psi_{-1})\), and \(\psi_0\) are the eigenstates of such a perturbation. A convenient choice of \(V\) for this purpose is \(V = Ax^2 + By^2 + Cz^2\). See Kittel in [73, p. 427].
If we then express $S_x^2, S_y^2, S_z^2$ as matrices in terms of the same basis, then by elementary matrix multiplication, one can show that $v_1$ corresponds to the joint-eigenvalue $\mu = \{0, 1, 1\}$, $v_2$ corresponds to $\mu = \{1, 0, 1\}$, and $v_3$ corresponds to $\mu = \{1, 1, 0\}$. Thus, the eigenvectors $v_1, v_2, v_3$ of the Hamiltonian term $H'$ which arises from a perturbing electric field (defined with respect to $x, y, z$) are also the joint-eigenvectors of the set $\{S_x^2, S_y^2, S_z^2\}$. This implies that we can represent $H'$ by the expression $aS_x^2 + bS_y^2 + cS_z^2$, where $H'$'s eigenvalues are $\{(b + c), (a + c), (a + b)\}$.

All of this leads to the following conclusion regarding the measurement of the spin of the orthohelium ground state electrons. Let the system be subjected to an electric field with orthorhombic symmetry with respect to a given set of Cartesian axes $x, y, z$. Under these circumstances, the measurement of the total Hamiltonian will yield a result equal (to first-order approximation) to the (unperturbed) ground state energy plus one of the perturbation corrections $\{(b + c), (a + c), (a + b)\}$. If the measured value of the perturbation energy is $(a + b), (a + c), (b + c)$ then the joint values of the set $\{S_x^2, S_y^2, S_z^2\}$ are given respectively by $\{1, 1, 0\}$, $\{1, 0, 1\}$, or $\{0, 1, 1\}$.

Suppose we consider two such experiments$^{16}$, one which measures the set $\{S_x^2, S_y^2, S_z^2\}$ and the other of which measures $\{S_x^2, S_y^2, S_z^2\}$, i.e. the squares of the components in the $x, y, z'$ system. The former experiment involves an electric field with orthorhombic symmetry with respect to $x, y, z$, while the latter involves an electric field with a different orientation in space, since it must have symmetry with respect to the axes $x, y, z'$. Although both procedures can be regarded as measurements of $S_x^2$, they involve quite different experiments.

It is quite apparent from this example that it would be unreasonable to require that a hidden variables theory must assign a single value to $S_x^2$, independent of the experimental procedure.

### 2.4 Contextuality theorems and spectral-incompatibility

We saw in our discussion of von Neumann’s theorem that its implications toward hidden variables amounted to the assertion that there can be no mathematical function $E(O)$ that is linear on the observables and which maps them to their

---

$^{16}$The type of experiments mentioned by Kochen and Specker are actually non-ideal measurements of sets such as $\{S_x^2, S_y^2, S_z^2\}$, since the post-measurement wave function of the electrons is not equal to the projection of the pre-measurement wave function into an eigenspace of the set $\{S_x^2, S_y^2, S_z^2\}$ (see section 1.3). In particular, they envision a spectroscopic analysis of the atom, i.e., the observation of photons emitted during transitions between the stationary states of the electrons. The fact that they consider non-ideal measurements of each set $\{S_x^2, S_y^2, S_z^2\}$ rather than ideal only serves to widen the range of possibilities for different experimental procedures, and so strengthens our point that distinct procedures exist for what Kochen and Specker consider to be a ‘measurement of $S_x^2$’.

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eigenvalues. This was neither a surprising, nor particularly enlightening result, since it follows also from a casual observation of some example of linearly related non-commuting observables, as we saw in examining the observables \( \frac{1}{\sqrt{2}} (\sigma_x + \sigma_y), \sigma_x, \sigma_y \) of a spin \( \frac{1}{2} \) particle. The theorems of Gleason, Kochen and Specker, and Mermin imply a somewhat less obvious type of impossibility: there exists no function \( E(O) \) mapping the observables to their eigenvalues, which obeys all relationships constraining commuting observables. What we develop here is a somewhat simpler expression of the implication of these theorems. We will find that they imply the spectral-incompatibility of the value map function: there exists no mathematical function that assigns to each commuting set of observables a joint-eigenvalue of that set.

We begin by recalling the notions of joint-eigenvectors and joint-eigenvalues of a commuting set of observables \( \{O_1, O_2, \ldots\} \). For a commuting set, the eigenvalue equation \( O|\psi\rangle = \mu|\psi\rangle \) is replaced by a set of relationships 1.7: \( O_i|\psi\rangle = \mu_i|\psi\rangle \) \( i = 1, 2, \ldots \), one for each member of the commuting set. If a given \( |\psi\rangle \) satisfies this relationship for all members of the set, it is referred to as a joint-eigenvector. The set of numbers \( (\mu_1, \mu_2, \ldots) \) that allow the equations to be satisfied for this vector are collectively referred to as the joint-eigenvalue corresponding to this eigenvector, and the symbol \( \mu = (\mu_1, \mu_2, \ldots) \) is used to refer to this set. The set of all joint-eigenvalues \( \{\mu_a\} \) is given the name 'joint-eigenspectrum'.

In general, the members of any given commuting set of observables might not be independent, i.e., they may be constrained by mathematical relationships. We label the relationships for any given commuting set \( \{O_1, O_2, \ldots\} \) as

\[
\begin{align*}
f_1(O_1, O_2, \ldots) &= 0 \\
f_2(O_1, O_2, \ldots) &= 0 \\
\vdots
\end{align*}
\]

The equations 2.1 in Gleason’s theorem, 2.4 in Kochen and Specker’s theorem, and 2.8, 2.9 and 2.10 in Mermin’s theorem, are just such relations. We now demonstrate the following two results. First, that every member of the joint-eigenspectrum must satisfy all relationships 2.14. Second, that any set of numbers \( \xi_1, \xi_2, \ldots \) satisfying all of these relationships is a joint-eigenvalue.

To demonstrate the first of these, we suppose that \( \mu = (\mu_1, \mu_2, \ldots) \) is a joint-eigenvalue of the commuting set \( \{O_1, O_2, \ldots\} \), with joint-eigenspace \( \mathcal{H} \). We then consider the operation of \( f_i(O_1, O_2, \ldots) \) on a vector \( \psi \in \mathcal{H} \) where \( f_i(O_1, O_2, \ldots) = 0 \) is one of the relationships constraining the commuting set. We find

\[ f_i(O_1, O_2, \ldots) \psi = f_i(\mu_1, \mu_2, \ldots) \psi = 0. \]  

The second equality implies that \( f_i(\mu_1, \mu_2, \ldots) = 0 \). Since \( f_i(O_1, O_2, \ldots) = 0 \) is an arbitrary member of the relationships 2.14 for the commuting set \( \{O_1, O_2, \ldots\}, \)
it follows that every joint-eigenvalue \( \mu \) of the set must satisfy all such relationships.

We now discuss the demonstration of the second point. Suppose that the numbers \( \{\xi_1, \xi_2, \ldots\} \) satisfy all of 2.14 for some commuting set \( \{O^1, O^2, \ldots\} \).

We consider the following relation:

\[
((O^1 - \mu_1^1)^2 + (O^2 - \mu_1^2)^2 + \ldots)\psi = 0. \tag{2.16}
\]

Here, we operate on the vector \( \psi \) with a product whose factors each consist of a sum of various operators. The product is taken over all joint-eigenvalues \( \{\mu_i\} \). We represent each joint-eigenvalue \( \mu_i \) by a set \( (\mu_1^i, \mu_2^i, \ldots) \). The validity of 2.16 is easily seen. Since the joint-eigenspaces of any commuting set are complete, the vector \( \psi \) must lie within such a space. Suppose that \( \psi \in \mathcal{H}_i \), where \( \mathcal{H}_i \) is the joint-eigenspace corresponding to \( \mu_i \). Then the \( i \)th factor in the product of operators in 2.16 must give zero when operating on \( \psi \). Therefore, the entire product operating on \( \psi \) must also give zero. Since \( \psi \) is an arbitrary vector, it follows that

\[
((O^1 - \mu_1^1)^2 + (O^2 - \mu_1^2)^2 + \ldots)((O^1 - \mu_2^1)^2 + (O^2 - \mu_2^2)^2 + \ldots)\psi = 0. \tag{2.17}
\]

Note that 2.17 is itself a constraining relationship on the commuting set, so that it must be satisfied by the numbers \( (\xi_1, \xi_2, \ldots) \). This can only be true if these numbers form a joint-eigenvalue of \( \{O^1, O^2, \ldots\} \), and this is the result we were to prove.

From this, we can discern a simple way to regard the implications of the theorems of Gleason, Kochen and Specker, and Mermin toward the question of a value map. The requirement 2.2 Gleason’s theorem places on the function \( E(P) \) can be re-stated as the requirement that for each commuting set, \( E(P) \) must satisfy all the relationships constraining its members. From the above argument, it follows that this assumption is equivalent to the constraint that \( E(P) \) must assign to each commuting set a joint-eigenvalue. The same is also true of Kochen and Specker’s assumption that \( E(s^2_{\theta, \phi}) \) satisfy equation 2.5, and the Mermin requirement that \( E \) satisfy 2.8, 2.9 and 2.10.

Thus, all three of these theorems can be regarded as proofs of the impossibility of a function mapping the observables to their values such that each commuting set is assigned a joint-eigenvalue. An appropriate name for such a proof would seem to be ‘spectral-incompatibility theorem.’

### 2.5 Albert’s example and contextuality

In thinking about any given physical phenomenon, it is natural to try to picture to oneself the properties of the system being studied. In using the quantum formalism to develop such a picture, one may tend to regard the ‘observables’
of this formalism, i.e., the Hermitian operators (see section 1.3), as representative of these properties. However, the central role played by the experimental procedure $E(O)$ in the measurement of any given observable $O$ seems to suggest that such a view of the operators may be untenable. We describe an experiment originally discussed by David Albert [1] that indicates that this is indeed the case: the Hermitian operators cannot be regarded as representative of the properties of the system\textsuperscript{17}. Albert considers two laboratory procedures that may be used to measure the $z-$component of the spin of a spin $\frac{1}{2}$ particle. Although the two procedures are quite similar to one another, they cannot not be regarded as identical when considered in light of the hidden variables theory known as Bohmian mechanics. This is a particularly striking instance of contextuality, and it indicates the inadequacy of the conception that the spin operator $\sigma_z$ represents an intrinsic property of the particle. From Albert’s example, one can clearly see that the outcome of the $\sigma_z$ measurement depends not only on the parameters of the particle itself, but also on the complete experimental setup.

The Albert example is concerned with the measurement of spin\textsuperscript{18} as performed using a Stern-Gerlach magnet. The schematic diagram given in figure 2.1 exhibits the configuration used in both of the measurement procedures to be described here. Note that we use a Cartesian coordinate system for which the $x$-axis lies along the horizontal direction with positive $x$ directed toward the right, and the $z$-axis lies along the vertical direction with positive $z$ directed upward. The $y$-axis (not shown) is perpendicular to the plane of the figure, and—since we use a right-handed coordinate system—positive $y$ points into this plane. The long axis of the Stern-Gerlach magnet system is oriented along the $x$-axis, as shown. The upper and lower magnets of the apparatus are located in the directions of positive $z$ and negative $z$. We define the Cartesian system further by requiring that $x$-axis (the line defined by $y = 0, z = 0$) passes through the center of the Stern-Gerlach magnet system. In each experiment, the spin $\frac{1}{2}$ particle to be measured is incident on the apparatus along the positive $x-$axis. In the region of space the particle occupies before entering the Stern-Gerlach apparatus, its wave function is of the form

$$\psi_t(r) = \phi_t(r)(|\uparrow\rangle + |\downarrow\rangle),$$  \hspace{1cm} (2.18)$$

where the vectors $|\uparrow\rangle$ and $|\downarrow\rangle$ are the eigenvectors of $\sigma_z$ corresponding to eigenvalues $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively. Here $\phi_t(r)$ is a localized wave packet moving in the positive $x$ direction toward the magnet.

We wish to consider two experiments that differ only in the orientation of the magnetic field inside the Stern-Gerlach apparatus. In the experiment 1, the upper magnet has a strong magnetic north pole toward the region of particle

\textsuperscript{17}This idea has been propounded by Daumer, Dür, Goldstein, and Zanghí in [39]. See also Bell in [22].

\textsuperscript{18}As is usual in discussions of Stern-Gerlach experiments, we consider only those effects relating to the interaction of the magnetic field with the magnetic moment of the particle. We consider the electric charge of the particle to be zero.
passage, while the lower has a somewhat weaker magnetic south pole toward this region. In the experiment 2, the magnets are such that the gradient of the field points in the opposite direction, i.e., the upper magnet has a strong magnetic south pole toward the region of passage, while the lower has a weak magnetic north pole towards it. After passing the Stern-Gerlach apparatus, the particle will be described by a wave function of one of the following forms:

\[
\psi_1(t)(\mathbf{r}) = \frac{1}{\sqrt{2}}(\phi_1^+(\mathbf{r})|↑⟩ + \phi_1^-(\mathbf{r})|↓⟩) \tag{2.19}
\]

\[
\psi_2(t)(\mathbf{r}) = \frac{1}{\sqrt{2}}(\phi_2^-(\mathbf{r})|↑⟩ + \phi_2^+(\mathbf{r})|↓⟩).
\]

Here \(\psi_1(t)(\mathbf{r})\) corresponds to experiment 1, and \(\psi_2(t)(\mathbf{r})\) corresponds to experiment 2. In both cases, the function \(\phi_1^+(\mathbf{r})\) represents a localized wave packet moving obliquely upward and \(\phi_1^-(\mathbf{r})\) represents a localized wave packet moving obliquely downward. To measure \(\sigma_z\), one places detectors in the paths of these wave packets. Examination of the first equation of 2.19 shows that for experiment 1, if the particle is detected in the upper path, the result of our \(\sigma_z\) measurement is \(+\frac{1}{2}\). If the particle is detected in the lower path, the result is \(-\frac{1}{2}\). For experiment 2, the second equation of 2.19 leads to the conclusion that similar detections are associated with results opposite in sign to those of experiment 1. Thus, for experiment 2, detection in the upper path implies \(\sigma_z = -\frac{1}{2}\), while detection in the lower implies \(\sigma_z = +\frac{1}{2}\).

We make here a few remarks regarding the symmetry of the system. We constrain the form of the wave packet \(\varphi_t(\mathbf{r})\) of 2.18, by demanding that it has no dependence on \(y\), and that it exhibits reflection symmetry through the plane defined by \(z = 0\), i.e., \(\varphi_t(x, z) = \varphi_t(x, -z)\). Moreover, the vertical extent of this wave packet is to be the same size as the vertical spacing between the upper and lower magnets of the apparatus. As regards the wave packets \(\phi_1^+(\mathbf{r})\) and \(\phi_1^-(\mathbf{r})\) of 2.19, if the magnetic field within the apparatus is such that \(\partial B_z/\partial z\)
is constant\textsuperscript{19}, then for both experiments, these packets move at equal angles above and below the horizontal (See figure 2.1). Thus, the particle is described both before and after it passes the Stern-Gerlach magnet, by a wave function which has reflection symmetry through the plane defined by \( z = 0 \).

2.5.0.2 Bohmian Mechanics and Albert’s example

We have mentioned that the hidden variables theory developed by David Bohm [28] gives an explanation of quantum phenomena which is empirically equivalent to that given by the quantum formalism. Bohmian mechanics allows us to regard any given system as a set of particles having well-defined (but distinctly non-Newtonian [49]) trajectories. Within Bohmian mechanics, it is the configuration of the system \( q = (q_1, q_2, q_3, \ldots) \) which plays the role of the hidden variables parameter \( \lambda \). Thus, the state description in this theory consists of both \( \psi \) and \( q \). Bohmian mechanics does not involve a change in the mathematical form of \( \psi \); just as in the quantum formalism, \( \psi \) is a vector in the Hilbert space associated with the system, and it evolves with time according to the Schrödinger equation:

\[
\frac{i\hbar}{\partial t} \psi = H \psi. \tag{2.20}
\]

The system configuration \( q \) is governed by the equation:

\[
\frac{dq}{dt} = \left( \frac{\hbar}{m} \right) \text{Im} \left( \frac{\psi^* \nabla \psi}{\psi \psi^*} \right). \tag{2.21}
\]

In the case of a particle with spin, we make use of the spinor inner-product in this equation. For example, in the case of a spin \( \frac{1}{2} \) particle whose wave function is \( \psi = \chi_+(\mathbf{r})|\uparrow\rangle + \chi_-\mathbf{(r)}|\downarrow\rangle \), the equation 2.21 assumes the form

\[
\frac{dq}{dt} = \left( \frac{\hbar}{m} \right) \text{Im} \left( \frac{\chi_+^*\mathbf{r} \nabla \chi_+ (\mathbf{r}) + \chi_-^*\mathbf{r} \nabla \chi_- (\mathbf{r})}{\chi_+ (\mathbf{r}) \chi_+ (\mathbf{r}) + \chi_- (\mathbf{r}) \chi_- (\mathbf{r})} \right). \tag{2.22}
\]

\textsuperscript{19}The term added to the particle’s Hamiltonian to account for a magnetic field is \( g_s \cdot \mathbf{B} \), where \( s \) is the spin, \( \mathbf{B} \) is the magnetic field and \( g \) is the gyromagnetic ratio. To determine the form of this term in the case of a Stern-Gerlach apparatus, we require the configuration of the magnetic field. A Stern-Gerlach magnet apparatus has a “long axis” which for the example of figure 2.1 lies along the \( x \)-axis. Since the component of the magnetic field along this axis will vanish except within a small region before and after the apparatus, the effects of \( B_x \) may be neglected. Furthermore, \( B_y \) and \( B_z \) within the apparatus may be regarded as being independent of \( x \). The component of the magnetic field along this axis does not involve any dependence on \( y \). The results we discuss in the present section are those which arise from taking account of the magnetic field by adding to the Hamiltonian term of the form \( g \sigma_z B_z (z = 0) + g \sigma_z (\frac{d}{dz} (z = 0)) z \).
As we expect from the fact that this theory is in empirical agreement with quantum theory, Bohmian mechanics does not generally provide, given $\psi$ and $q$, a mapping from the observables to their values. In other words, it does not provide a non-contextual value map for each state. As we shall see, the choice of experimental procedure plays such a pronounced role in the Bohmian mechanics description of Albert’s spin measurements, that one cannot possibly regard the spin operator as representative of an objective property of the particle.

We first discuss the Bohmian mechanics description of the Albert experiments. Since the wave function and its evolution are the same as in quantum mechanics, the particle’s wave function $\psi$ is taken to be exactly as described above. As far as the configuration $q$ is concerned, there are two important features of the Bohmian evolution equations to be considered: the uniqueness of the trajectories and the equivariance of the time evolution. The first feature refers to the fact that each initial $\psi$ and $q$ leads to a unique trajectory. Since the particle being measured has a fixed initial wave function, its initial conditions are defined solely by its initial position. The equivariance of the system’s time evolution is a more complex property. Suppose that at some time $t$, the probability that the system’s configuration is within the region $dq$ about $q$ obeys the relationship:

$$P(q' \in dq) = |\psi(q)|^2 dq.$$  \hspace{1cm} (2.23)

According to equivariance, this relationship will continue to hold for all later times $t' > t$. In considering the Bohmian mechanics description of a system, we assume that the particle initially obeys 2.23. By equivariance, we then have that for all later times the particle will be guided to “follow” the motion of the wave function. Thus, after it passes through the Stern-Gerlach apparatus, the particle will enter either the upward or downward moving packet. From consideration of the uniqueness of the trajectory and the equivariance of the time evolution, it follows that the question of which branch of the wave function the particle enters depends solely on its initial position.

If we consider the situation in a little more detail, we find a simple criterion on the initial position of the particle that determines which branch of the wave function it will enter. Recall that the initial 2.18 and final 2.19 wave function have no dependence on the $y$ coordinate, and that they exhibit reflection symmetry through the $z = 0$ plane. From this symmetry together with the uniqueness of Bohmian trajectories, it follows that the particle cannot cross the $z = 0$ plane. In conjunction with equivariance, this result implies that if the particle’s initial $z$ coordinate is greater than zero, it must enter the upper branch, and if its initial $z$ is less than zero the particle must enter the lower branch.

If we now consider the above described spin measurements, we find a somewhat curious situation. For any given initial configuration $q$, the question of whether the measurement result is $\sigma_z = +\frac{1}{2}$ or $\sigma_z = -\frac{1}{2}$ depends on the configuration of the experimental apparatus. Suppose that the particle has an initial
$z > 0$, so that according to the results just shown, it will enter the upper branch of the wave function. If the magnetic field inside the Stern-Gerlach apparatus is such that $\partial B_z / \partial z < 0$, then the particle’s final wave function is given by the first equation in 2.19, and its detection in the upper branch then implies that $\sigma_z = +\frac{1}{2}$. If, on the other hand, the magnetic field of the Stern-Gerlach magnet has the opposite orientation, i.e., $\partial B_z / \partial z > 0$, then the second equation in 2.19 obtains and the detection in the upper branch implies that $\sigma_z = -\frac{1}{2}$. Thus, we arrive at the conclusion that the “measurement of $\sigma_z$” gives a different result for two situations that differ only in the experimental configuration.

The quantum formalism’s rules for the measurement of observables strongly suggest that the Hermitian operators represent objective properties of the system. Moreover, such a conception is a common element of the expositions given in quantum mechanics textbooks. On the other hand, the fact that the result of the “measurement” of $\sigma_z$ can depend on properties of both system and apparatus contradicts this conception. In general, one must consider the results of the “measurement of an observable” to be a joint-product of system and measuring apparatus. Recall Niels Bohr’s comment that [33] “a closer examination reveals that the procedure of measurement has an essential influence on the conditions on which the very definition of the physical quantities in question rests.” For further discussion of the role of Hermitian operators in quantum theory, the reader is directed to Daumer, Dürr, Goldstein, and Zanghí in [39]. According to these authors: “the basic problem with quantum theory . . . more fundamental than the measurement problem and all the rest, is a naive realism about operators . . . by (this) we refer to various, not entirely sharply defined, ways of taking too seriously the notion of operator-as-observable, and in particular to the all too casual talk about ‘measuring operators’ which tends to occur as soon as a physicist enters quantum mode.”
Chapter 3

The Einstein–Podolsky–Rosen paradox and nonlocality

In contrast to contextuality, nonlocality is quite an unexpected and surprising feature to meet with in the quantum phenomena. As seen in the previous chapter, contextuality is essentially the dependence of the hidden variables predictions on the different possible experimental procedures for the measurements of the observable. Although the nonlocality of a quantum system is not something one would regard as natural, it may be proved by a mathematical analysis that this feature is inevitable. The demonstration in question arises from the well-known Einstein–Podolsky–Rosen paradox [51], in combination with Bell’s theorem [7]. There exists, however, a misperception among some authors that what follows from these analyses is only that local hidden variables theories must conflict with quantum mechanics. In fact, a more careful examination shows that the conjunction of the EPR paradox with Bell’s theorem implies that any local theoretical explanation must disagree with quantum mechanics. To demonstrate this conclusion, we now review the EPR paradox and Bell’s theorem argument. We begin with a discussion of the spin singlet version of the EPR argument. This will lead us to the presentation of Bell’s theorem and its proof. We then discuss the conclusion that follows from the conjunction of the EPR paradox and Bell’s theorem.
3.1 Review of the Einstein-Podolsky-Rosen analysis

3.1.1 Rotational invariance of the spin singlet state and perfect correlations between spins

The well-known work of Einstein, Podolsky, and Rosen, first published in 1935, was not designed to address the possibility of nonlocality, as such. The title of the paper was “Can Quantum Mechanical Description of Physical Reality be Considered Complete?”, and the goal of these authors was essentially the opposite of authors such as von Neumann: Einstein, Podolsky, and Rosen wished to demonstrate that the addition of hidden variables to the description of state is necessary for a complete description of a quantum system. According to these authors, the quantum mechanical state description given by $\psi$ is incomplete, i.e. it cannot account for all the objective properties of the system. This conclusion is stated in the paper’s closing remark: “While we have thus shown that the wave function does not provide a complete description of the physical reality, we left open the question of whether such a description exists. We believe, however, that such a theory is possible.”

Einstein, Podolsky, and Rosen arrived at this conclusion having shown that for the system they considered, each of the particles must have position and momentum as simultaneous “elements of reality”. Regarding the completeness of a physical theory, the authors state: [51] (emphasis due to EPR) “Whatever the meaning assigned to the term complete, the following requirement for a complete theory seems to be a necessary one: every element of the physical reality must have a counterpart in the physical theory”. This requirement leads them to conclude that the quantum theory is incomplete, since it does not account for the possibility of position and momentum as being simultaneous elements of reality. To develop this conclusion for the position and momentum of a particle, the authors make use of the following sufficient condition for a physical quantity to be considered as an element of reality: “If without in any way disturbing a system, we can predict with certainty (i.e. with probability equaling unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this quantity”.

What we shall present here1 is a form of the EPR paradox which was developed in 1951, by David Bohm2. Bohm’s version involves the properties of the spin singlet state of a pair of spin $\frac{1}{2}$ particles. Within his argument, Bohm shows that various components of the spin of a pair of particles must be elements of reality in the same sense as the position and momentum were for Einstein, Podolsky, and Rosen. We begin with a discussion of the formal properties of the

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1The Bohm spin singlet version and the original version of the EPR paradox differ essentially in the states and observables with which they are concerned. We shall consider the original EPR state more explicitly in chapter 4, section 4.1.1.

2[25, p. 611-623]. A recent reprint appears within [103, p. 356-368]
spin singlet state, and then proceed with our presentation of Bohm’s version of the EPR argument.

Because the spin and its components all commute with the observables associated with the system’s spatial properties, one may analyze a particle with spin by separately analyzing the spin observables and the spatial observables. The spin observables may be analyzed in terms of a Hilbert space $H_s$ (which is a two-dimensional space in the case of a spin $\frac{1}{2}$ particle) and the spatial observables in terms of $L_2(\mathbb{R})$. The full Hilbert space of the system is then given by tensor product $H_s \otimes L_2(\mathbb{R})$. Hence we proceed to discuss the spin observables only, without explicit reference to the spatial observables of the system. We denote each direction in space by its $\theta$ and $\phi$ coordinates in spherical polar coordinates, and (since we consider spin $\frac{1}{2}$ particles) the symbol $\sigma$ denotes the spin. To represent the eigenvectors of $\sigma_{\theta,\phi}$ corresponding to the eigenvalues $+\frac{1}{2}$ and $-\frac{1}{2}$, we write $|\uparrow \theta, \phi\rangle$, and $|\downarrow \theta, \phi\rangle$, respectively. For the eigenvectors of $\sigma_z$, we write simply $|\uparrow\rangle$ and $|\downarrow\rangle$. Often vectors and observables are expressed in terms of the basis formed by the eigenvectors of $\sigma_z$. The vectors $|\uparrow \theta, \phi\rangle$, and $|\downarrow \theta, \phi\rangle$ when expressed in terms of these are

$$
|\uparrow \theta, \phi\rangle = \cos(\theta/2)|\uparrow\rangle + \sin(\theta/2)e^{i\phi}|\downarrow\rangle
$$

$$
|\downarrow \theta, \phi\rangle = \sin(\theta/2)e^{-i\phi}|\uparrow\rangle - \cos(\theta/2)|\downarrow\rangle.
$$

We now discuss the possible states for a system consisting of two spin $\frac{1}{2}$ particles. For such a system, the states are often classified in terms of the total spin $S = \sigma^{(1)} + \sigma^{(2)}$ of the particles, where $\sigma^{(1)}$ is the spin of particle 1 and $\sigma^{(2)}$ is the spin of particle 2. The state characterized by $S = 1$ is known as the spin triplet state, $\psi_{ST}$; it is so named because it consists of a combination of the three eigenvectors of $S_z$—the $z$-component of the total spin—which correspond to the eigenvalues $-1, 0, 1$. The spin singlet state, in which we shall be interested, is characterized by $S = 0$. The name given to the state reflects that it contains just one eigenvector of the $z$-component $S_z$ of the total spin: that corresponding to the eigenvalue 0. In fact, as we shall demonstrate, the spin singlet state is an eigenvector of all components of the total spin with an eigenvalue 0. We express the state in terms of the eigenvectors $|\uparrow\rangle^{(1)}, |\downarrow\rangle^{(1)}$ of $\sigma_z^{(1)}$ and $|\uparrow\rangle^{(2)}, |\downarrow\rangle^{(2)}$ of $\sigma_z^{(2)}$, as follows:

$$
\psi_{ss} = |\uparrow\rangle^{(1)}|\downarrow\rangle^{(2)} - |\downarrow\rangle^{(1)}|\uparrow\rangle^{(2)}.
$$

For simplicity, we have suppressed the normalization factor $1/\sqrt{2}$. Note that each of the two terms consists of a product of an eigenvector of $\sigma_z^{(1)}$ with an eigenvector of $\sigma_z^{(2)}$ such that the corresponding eigenvalues are the negatives of each other.

\(^3\)See for example, Messiah [81], and Shankar [94].

\(^4\)Note that a term such as $|a\rangle^{(1)}|b\rangle^{(2)}$ represents a tensor product of the vector $|a\rangle^{(1)}$ of the Hilbert space associated with the first particle with the vector $|b\rangle^{(2)}$ of the Hilbert space associated with the second. The formal way of writing such a quantity is as: $|a\rangle^{(1)} \otimes |b\rangle^{(2)}$. For simplicity of expression, we shall omit the symbol “$\otimes$” here.
If we invert the relationships 3.1 we may then re-write the spin singlet state 3.2 in terms of the eigenvectors \(|↑\theta,\phi\rangle\) and \(|↓\theta,\phi\rangle\) of the component of spin in the \(\theta,\phi\) direction. Inverting 3.1 gives:

\[
|↑\rangle = \cos(\theta/2)|↑\rangle + \sin(\theta/2)e^{i\phi}|↓\rangle \quad (3.3)
\]

\[
|↓\rangle = \sin(\theta/2)e^{-i\phi}|↑\rangle - \cos(\theta/2)|↓\rangle.
\]

Now let us consider re-writing the spin singlet state 3.2 in the following manner. We substitute for \(|↑\rangle\) and \(|↓\rangle\) expressions involving \(|↑\theta_2,\phi_2\rangle\) and \(|↓\theta_2,\phi_2\rangle\) by making use of 3.3. Doing so gives us a new expression for the spin singlet state:

\[
ψ_{ss} = |↑⟩(1)\left[\sin(\theta_2/2)e^{-i\phi_2}|↑\theta_2,\phi_2\rangle(2) - \cos(\theta_2/2)|↓\theta_2,\phi_2\rangle(2)\right] − |↓⟩(1)\left[\cos(\theta_2/2)|↑\theta_2,\phi_2\rangle(2) + \sin(\theta_2/2)e^{i\phi_2}|↓\theta_2,\phi_2\rangle(2)\right]
\]

\[
= -\left[\cos(\theta_2/2)|↑⟩(1) + \sin(\theta_2/2)e^{i\phi_2}|↓⟩(1)\right]|↑\theta_2,\phi_2\rangle(2) + \sin(\theta_2/2)e^{-i\phi_2}|↓⟩(1) - \cos(\theta_2/2)|↓⟩(1)|↑\theta_2,\phi_2\rangle(2).
\]

Examining the second of these relationships, we see from 3.1 that \(ψ_{ss}\) reduces:

\[
ψ_{ss} = |↑\theta_2,\phi_2\rangle(1)|↓\theta_2,\phi_2\rangle(2) − |↓\theta_2,\phi_2\rangle(1)|↑\theta_2,\phi_2\rangle(2). \quad (3.5)
\]

Note that this form is similar to that given in 3.2. Each of its terms is a product of an eigenvector of \(σ(1)_{\theta,\phi}\) and an eigenvector of \(σ(2)_{\theta,\phi}\) such that the factors making up the product correspond to eigenvalues that are just the opposites of each other. We drop the ‘2’ from \(\theta\) and \(\phi\) giving

\[
ψ_{ss} = |↑\theta,\phi\rangle(1)|↓\theta,\phi\rangle(2) − |↓\theta,\phi\rangle(1)|↑\theta,\phi\rangle(2). \quad (3.6)
\]

Suppose that we consider the implications of 3.6 for individual measurements of \(σ(1)_{\theta,\phi}\) of particle 1 and the same spin component of particle 2. In each term of this spin singlet form, we have a product of eigenvectors of the two spin components such that the eigenvalues are simply the negatives of one another. Therefore, it follows that measurements of \(σ(1)_{\theta,\phi}\) and \(σ(2)_{\theta,\phi}\) always give results that sum to zero, i.e. if measurement of \(σ(1)_{\theta,\phi}\) gives the result \(±\frac{1}{2}\), then the measurement of \(σ(2)_{\theta,\phi}\) always gives \(±\frac{1}{2}\). We say that these observables exhibit perfect correlation. Note that this holds true for pairs of spin observables \(σ(1)_{\theta,\phi}\) and \(σ(2)_{\theta,\phi}\) where \(\theta,\phi\) refer to an arbitrary direction.

\[5\]If we multiply a wave function by any constant factor \(c\), where \(c \neq 0\) the resulting wave function represents the same physical state. We multiply \(ψ_{ss}\) by \(-1\) to facilitate comparison with 3.2.
3.1.2 Incompleteness argument

Consider a situation in which the two particles described by the spin singlet state are spatially separated from one another, and spin-component measurements are to be carried out on each. Since there exist perfect correlations, it is possible to predict with certainty the result of a measurement of $\sigma_x^{(1)}$ from the result of a previous measurement of $\sigma_x^{(2)}$. Suppose for example, if we measure $\sigma_x^{(2)}$ and find the result $\sigma_x^{(2)} = \frac{1}{2}$. A subsequent measurement of $\sigma_x^{(1)}$ must give $\sigma_x^{(1)} = -\frac{1}{2}$. If we assume locality then the measurement of $\sigma_x^{(2)}$ cannot in any way disturb particle 1, which is spatially separated from particle 2. Using the Einstein-Podolsky-Rosen criterion, it follows that $\sigma_x^{(1)}$ is an element of reality.

Similarly, one can predict with certainty the result of a measurement of $\sigma_y^{(1)}$ from a previous measurement of $\sigma_y^{(2)}$. Again, locality implies that measurement of $\sigma_y^{(2)}$ does not disturb particle 1, and by the EPR criterion, $\sigma_y^{(1)}$ must be an element of reality. In total we have shown that both $\sigma_x^{(1)}$ and $\sigma_y^{(1)}$ are elements of reality. On the other hand, from the quantum formalism’s description of state given by $\psi$, one can deduce, at most, one of two such non-commuting quantities. Therefore, we may conclude that this description of state is incomplete. Moreover, since we have perfect correlations between all components of the spins of the two particles, a similar argument may be given for any component of $\sigma^{(1)}$. Therefore, particle 1’s spin component in an arbitrary direction $\theta, \phi$ must be an element of reality.

Note that we can interchange the roles of particle 1 and 2 in this argument to show the same conclusion for all components of particle 2’s spin, as well.

The EPR paradox thus has the following structure. The analysis begins with the observation that the spin singlet state exhibits perfect correlations such that for any direction $\theta, \phi$, measurements of $\sigma_{\theta,\phi}^{(1)}$ and $\sigma_{\theta,\phi}^{(2)}$ always give results which sum to zero. Together with the assumption of locality, this leads to the conclusion that all components of the spins of both particles must be elements of reality. Since the quantum state description given by $\psi$ does not allow for the simultaneous reality of non-commuting quantities such as $\sigma_x^{(1)}$ and $\sigma_y^{(1)}$, it follows that this description is incomplete.

3.2 Bell’s theorem

The incompleteness of the quantum mechanical state description concluded by EPR implies that one must consider a theoretical description of state consisting of $\psi$ and some additional parameter, in order to fully account for a system’s properties. In terms of such a state description, one ought to be able to mathematically represent the definite values concluded by EPR using a function\(^6\)

\(^6\)Since we are considering a system of fixed $\psi$, namely that of the spin singlet state, no $\psi$ dependence need be included in $V$
\( V_\lambda(O) \) mapping each component of the spin of each particle to a value. Here we have denoted the supplemental state parameter as \( \lambda \) in accordance with the discussion of von Neumann’s theorem in chapter 1. In 1965, John S. Bell presented a famous theorem which addressed the possibility of just such a function on the spin observables. Bell was able to show that this formulation must conflict with the statistical predictions of quantum mechanics for various spin measurements. We now present Bell’s theorem.

Let us first fix our notation. To denote directions in space, we write unit vectors such as \( \hat{a}, \hat{b}, \hat{c} \) instead of \( \theta \) and \( \phi \). Rather than using the form \( V_\lambda(O) \), we shall write \( A(\lambda, \hat{a}) \) and \( B(\lambda, \hat{b}) \) to represent functions on the spin components of particles 1 and 2, respectively. Since the two particles are each of spin \( \frac{1}{2} \), we should have \( A = \pm \frac{1}{2} \) and \( B = \pm \frac{1}{2} \), however for simplicity we rescale these to \( A = \pm 1 \) and \( B = \pm 1 \).

### 3.2.1 Proof of Bell’s theorem

The key feature of the spin singlet version of the EPR paradox was its analysis of the perfect correlations arising when the two particles of a spin singlet pair are subject to measurements of the same spin component. Thus, it may not be surprising that Bell’s theorem is concerned with a correlation function, which is essentially a measure of the statistical correlation between the results of spin component measurements of the two particles. The correlation function is to be determined as follows: we set the apparatus measuring particle 1 to probe the component in the \( \hat{a} \) direction, and the apparatus measuring 2 is set for the \( \hat{b} \) direction. We make a series of measurements of spin singlet pairs using this configuration, recording the product \( \sigma^{(1)}_a \sigma^{(2)}_b \) of the results on each trial. The average of these products over the series of measurements is the value of the correlation function.

In general, we expect the value of the average determined in this way to depend on the directions \( \hat{a}, \hat{b} \) with respect to which the spin components are measured. According to the quantum formalism, we may predict the average, or expectation value of any observable using the formula \( E(O) = \langle \psi | O \psi \rangle \). For the series of experiments just described, we take the expectation value of product of the appropriate spin component observables, giving:

\[
P_{QM}(\hat{a}, \hat{b}) = \langle \sigma^{(1)}_a \sigma^{(2)}_b \rangle = -\hat{a} \cdot \hat{b}.
\]

(3.7)

In the case of the predetermined values, the average of the product of the two spin components \( \sigma^{(1)}_a \sigma^{(2)}_b \) is obtained by taking an average over \( \lambda \):

\[
P(\hat{a}, \hat{b}) = \int d\lambda \rho(\lambda) A(\lambda, \hat{a}) B(\lambda, \hat{b}),
\]

(3.8)
where $\rho(\lambda)$ is the probability distribution over $\lambda$. $\rho(\lambda)$ is normalized by:

$$\int d\lambda \rho(\lambda) = 1.$$  (3.9)

We will now examine the question of whether the correlations function given by 3.8 is compatible with the quantum mechanical prediction 3.7 for this function.

Crucial to the EPR analysis is the fact that there is a perfect correlation between the results of the measurement of any component of particle 1’s spin in a given direction with the measurement of the same component of particle 2’s spin, such that the results are of opposite sign. To account for this, the correlation function must give

$$P(\hat{a}, \hat{a}) = -1 \forall \hat{a}. \quad (3.10)$$

It is easy to see that the quantum correlation function satisfies this condition. If the prediction derivable using the predetermined values is to reflect this, we must have

$$A(\lambda, \hat{a}) = -B(\lambda, \hat{a}) \forall \hat{a}, \lambda. \quad (3.11)$$

At this point, we have enough information to derive the conclusion of the theorem. Using 3.8 together with 3.11 and the fact that $[A(\lambda, \hat{a})]^2 = 1$, we write

$$P(\hat{a}, \hat{b}) - P(\hat{a}, \hat{c}) = -\int d\lambda \rho(\lambda) [A(\lambda, \hat{a})A(\lambda, \hat{b}) - A(\lambda, \hat{a})A(\lambda, \hat{c})] \quad (3.12)$$

$$= -\int d\lambda \rho(\lambda) A(\lambda, \hat{a})A(\lambda, \hat{b})[1 - A(\lambda, \hat{b})A(\lambda, \hat{c})]$$

Using $A, B = \pm 1$, we have that

$$|P(\hat{a}, \hat{b}) - P(\hat{a}, \hat{c})| \leq \int d\lambda \rho(\lambda)|1 - A(\lambda, \hat{b})A(\lambda, \hat{c})|; \quad (3.13)$$

then using the normalization 3.9, and 3.11 we have

$$|P(\hat{a}, \hat{b}) - P(\hat{a}, \hat{c})| \leq 1 + P(\hat{b}, \hat{c}), \quad (3.14)$$

and this relation, which is commonly referred to as “Bell’s inequality”, is the theorem’s conclusion.

Thus, the general framework of Bell’s theorem is as follows. The definite values of the various components of the two particles’ spins are represented by the mathematical functions $A(\lambda, \hat{a})$, and $B(\lambda, \hat{b})$. The condition

$$A(\lambda, \hat{a}) = -B(\lambda, \hat{a}) \forall \hat{a}, \lambda \quad (3.15)$$

(equation 3.11), placed on the functions $A(\lambda, \hat{a})$, $B(\lambda, \hat{b})$ ensures the agreement of these functions with the perfect correlations. Bell’s theorem tells us that
from these conditions it follows that the theoretical prediction for the correlation function, \( P(\hat{a}, \hat{b}) \), must satisfy the Bell inequality, 3.14.

Based on the fact that the Bell inequality is not satisfied by the quantum mechanical correlation function 3.7 (as we shall see below), some authors\(^7\) have concluded that Bell’s theorem proves the impossibility of hidden variables. As we shall see, however, this conclusion does not follow.

### 3.3 The EPR paradox, Bell’s theorem, and non-locality

Recall our discussion of the EPR paradox. We found that for a system described by the spin singlet state, each component of the spin of one particle is perfectly correlated with the same component of the spin of the other. Such perfect correlations seem to give the appearance of nonlocality, since the measurement of one spin seems capable of immediately influencing the result of a measurement of its distant partner. The conclusion of nonlocality can be avoided only if all components of the spins of each particle possess definite values. This much is developed from the EPR analysis. According to Bell’s theorem, these definite values lead to the prediction that the correlation function \( P \) must satisfy the inequality 3.14. Since Bell’s theorem assumes nothing beyond what can be concluded from the spin singlet EPR paradox, one can deduce from the conjunction of EPR with Bell that any local theoretical description which accounts for the perfect correlations of the spin singlet state leads to a correlation function satisfying Bell’s inequality.

Consider now the quantum mechanical prediction for the correlation function 3.7. If we examine this function, we find that it does not in general satisfy Bell’s inequality. Suppose, for example, that we have defined some angular orientation such that \( \hat{a}, \hat{b}, \hat{c} \) all lie in the \( x,y \) plane (so that \( \theta = 90^\circ \)), with \( \hat{a} \) along \( \phi = 60^\circ \), \( \hat{b} \) along \( \phi = 0^\circ \), and \( \hat{c} \) along \( \phi = 120^\circ \). With this, we have \( P_{QM}(\hat{a}, \hat{b}) = \frac{1}{2} \), \( P_{QM}(\hat{a}, \hat{c}) = \frac{1}{2} \) and \( P_{QM}(\hat{b}, \hat{c}) = -\frac{1}{2} \), so that \( |P_{QM}(\hat{a}, \hat{b}) - P_{QM}(\hat{a}, \hat{c})| = 1 \) and \( 1 + P_{QM}(\hat{b}, \hat{c}) = \frac{1}{2} \), which is in violation of 3.14. From what we have found above, the disagreement of the quantum mechanical prediction for this correlation function with Bell’s inequality implies that quantum mechanics must disagree with any local theoretical description. In the words of Bell: [20] “It is known that with Bohm’s example of EPR correlations, involving particles with spin, there is an irreducible nonlocality.”

Concerning the claim that Bell’s theorem is an ‘impossibility proof’, the falsity of this conclusion is already indicated by the success of Bohmian mechanics (section 1.1.2). On the other hand, Bell’s theorem shows that from the existence of definite values for the spins follows a conclusion that is in conflict with

\(^7\)This is the conclusion reached by the following authors: [24], [55, p. 172], [103, Wigner p. 291]
the quantum mechanical predictions. This prompts the question of just what feature Bohmian mechanics possesses which allows it to succeed where the quite general looking formulation of hidden variables analyzed by Bell does not. If we examine the functions $A(\lambda, \hat{a})$ and $B(\lambda, \hat{b})$ analyzed by Bell, we see that they do not possess the feature we have just seen is present in the quantum theory itself: nonlocality. This follows since the value of $A$ does not depend on the setting $\hat{b}$ of the apparatus measuring particle 2, nor does $B$ depend on the setting $\hat{a}$ of the apparatus measuring particle 1. Thus, Bell analyzes a local theory of hidden variables. Bohmian mechanics, on the other hand, is nonlocal, and it is precisely this that allows it to “escape” disproof by Bell’s theorem. Hence, Bell’s theorem does not constitute a disproof of hidden variables in general, but only of local hidden variables.

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8This feature is not “accidental”: the hidden variables analyzed by Bell are precisely those given by the spin singlet EPR analysis—which itself is based on the locality assumption. Moreover, Bell did not regard his analysis as a hidden variables impossibility proof, since he was aware that: [7] “…a hidden variable interpretation (Bohmian mechanics) has been explicitly constructed”. Instead, Bell viewed his theorem as a proof that the “grossly nonlocal structure” inherent in Bohmian mechanics “is characteristic . . . of any such (hidden variables) theory which reproduces exactly the quantum mechanical predictions”.

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Chapter 4

Schrödinger’s paradox and nonlocality

4.1 Schrödinger’s paradox

In addition to his seminal role in the development of quantum theory itself, Erwin Schrödinger made important contributions [91, 92, 93] to its interpretation, both in his development of the paradox of “Schrödinger’s cat” and in his generalization of the Einstein–Podolsky–Rosen paradox. In this latter analysis, Schrödinger demonstrated that the state considered by EPR leads to a much more general result than these authors had concluded. According to the incompleteness argument given by Schrödinger, for any system described by a certain class of quantum state\(^1\), all observables of both particles must be elements of reality. As was the case with EPR, Schrödinger’s argument was based on the existence of perfect correlations exhibited by the state. In this chapter, we show that Schrödinger’s result can be developed in a simpler way which allows one to determine which observables are perfectly correlated with one another using the form of the maximally entangled state in question. Moreover, we derive a wide variety of new quantum nonlocality proofs based on Schrödinger’s generalization of EPR in conjunction with the theorems discussed in chapter 2, Gleason’s theorem, Kochen and Specker’s theorem, and Mermin’s theorem. We show that any such “spectral-incompatibility” theorem (see section 2.4) when taken together with the Schrödinger paradox provides a proof of quantum nonlocality. Since the spectral-incompatibility theorems involve the quantum predictions for individual measurements, the nonlocality proofs to which they lead are of a deterministic, rather than statistical, character. A further noteworthy feature of this type of quantum nonlocality is that its experimental confirmation need involve only the verification of the perfect correlations, with no further obser-

\(^1\)A ‘maximally entangled state’.
vations required. Before addressing these matters, we discuss the original form of the EPR paradox.

4.1.1 The Einstein–Podolsky–Rosen quantum state

The Bohm version of the EPR paradox presented in chapter 3 was addressed to the spin components of two particles represented by the spin singlet state. While this argument was concerned with perfect correlations in the spin components, the original form of the EPR paradox involved perfect correlations of a slightly different form for the positions and momenta. For a system described by the original EPR state, we find that measurements of the positions \(x_1\) and \(x_2\) of the two particles give equal results\(^2\)\(^3\), while measurements of the momenta give results which sum to zero.

To develop the perfect correlations in position and momentum, we first recall how the spin singlet state leads to the spin correlations. The spin singlet state takes the form\(^3\):

\[
\psi_{ss} = |\uparrow, \theta, \phi\rangle \otimes |\downarrow, \theta, \phi\rangle - |\downarrow, \theta, \phi\rangle \otimes |\uparrow, \theta, \phi\rangle,
\]

where we have suppressed the normalization for simplicity. We have here a sum of two terms, each being a product of an eigenfunction of \(\sigma_{\theta, \phi}^{(1)}\) with an eigenfunction of \(\sigma_{\theta, \phi}^{(2)}\) such that the corresponding eigenvalues sum to zero. In the first term of 4.1 for example, the factors are eigenvectors corresponding to \(\sigma_{\theta, \phi}^{(1)} = \frac{1}{2}\) and \(\sigma_{\theta, \phi}^{(2)} = -\frac{1}{2}\). With this, it is clear that we will have perfect correlations between the results of measurement of \(\sigma_{\theta, \phi}^{(1)}\) and \(\sigma_{\theta, \phi}^{(2)}\), i.e. measurements of these observables will give results which sum to zero. By analogy with this result, the perfect correlations in position and momentum emphasized by Einstein, Podolsky, and Rosen will follow if the wave function assumes the forms:

\[
\psi = \int_{-\infty}^{\infty} dp \ |\phi_{-p}\rangle \otimes |\phi_p\rangle \tag{4.2}
\]

\[
\psi = \int_{-\infty}^{\infty} dx \ |\varphi_x\rangle \otimes |\varphi_x\rangle.
\]

Here \(|\phi_p\rangle\) is the eigenvector of momentum operator corresponding to a momentum of \(p\). \(|\varphi_x\rangle\) is the eigenvector of position. We now show that the the wave

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\(^2\)EPR discuss a form in which the difference between the positions is equal to a constant they call \(d\). The case we consider differs from this only in that the points of origin from which the positions of the two particles are measured are different, so that the quantity \(x_2 - x_1 + d\) for EPR becomes \(x_2 - x_1\) for our case.

\(^3\)It is clear that the Einstein–Podolsky–Rosen quantum state 4.3 is not normalizable. Nevertheless, as we saw in the example of the spin singlet state, it is possible to carry out a similar argument for states which can be normalized. The ‘maximally entangled states’, which are the subject of the Schrödinger paradox, contain among them a large class of normalizable states, as we shall see.
function given by the first equation in 4.2:

$$\psi_{EPR} = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp \left| \phi_{-p} \right\rangle \otimes \left| \phi_p \right\rangle$$  \hspace{1cm} (4.3)

assumes also the form of the second equation in 4.2.

To see this we expand the first factor in the summand of \(\psi_{EPR}\) in terms of \(\left| \phi_x \right\rangle\):

$$\psi_{EPR} = \int_{-\infty}^{\infty} dp \left( \int_{-\infty}^{\infty} dx \left| \phi_x \right\rangle \langle \phi_x | \phi_{-p} \right\rangle \otimes \left| \phi_p \right\rangle. \hspace{1cm} (4.4)$$

Using \(\langle \phi_x | \phi_{-p} \rangle = \langle \phi_p | \phi_x \rangle\), this becomes

$$\psi_{EPR} = \int_{-\infty}^{\infty} dx \left| \phi_x \right\rangle \otimes \int_{-\infty}^{\infty} dp \left| \phi_p \right\rangle \langle \phi_p | \phi_x \rangle. \hspace{1cm} (4.5)$$

Since \(\int_{-\infty}^{\infty} dp \left| \phi_p \right\rangle \langle \phi_p \) is a unit operator, it follows that

$$\psi_{EPR} = \int_{-\infty}^{\infty} dx \left| \phi_x \right\rangle \otimes \left| \phi_x \right\rangle. \hspace{1cm} (4.6)$$

This is the result we had set out to obtain.

### 4.1.2 Schrödinger’s generalization

#### 4.1.2.1 Maximal perfect correlations

Schrödinger’s work essentially revealed the full potential of the quantum state which Einstein, Podolsky, and Rosen had considered. In his analysis, Schrödinger demonstrated that the perfect correlations the EPR state exhibits are not limited to those in the positions and momenta. For two particles described by the EPR state, every observable of each particle will exhibit perfect correlations with an observable of the other. What we present here and in subsequent sections is a simpler way to develop such perfect correlations than that given by Schrödinger. This result may be seen as follows. The EPR state 4.3 is rewritten as

$$\langle x_1, x_2 | \psi_{EPR} \rangle = \int_{-\infty}^{\infty} dp_1 dp_2 \langle x_1, x_2 | p_1, p_2 \rangle \langle p_1, p_2 | \psi_{EPR} \rangle = \delta(x_2 - x_1). \hspace{1cm} (4.7)$$

We may then use a relationship known as the completeness relationship. According to the completeness relation, if \(\{\phi_n(x)\}\) is any basis for the Hilbert

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4See [92]

5The reader may object that for this state, the two particles lie ‘on top of one another’, i.e., the probability that \(x_1 \neq x_2\) is identically 0. However, in the following sections, we show that the same conclusions drawn for the EPR state also follow for a more general class of ‘maximally entangled states’, of which many do not restrict the positions in just such a way.

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space $L_2$, then we have $\sum_{n=1}^{\infty} \phi_n^*(x_1)\phi_n(x_2) = \delta(x_2 - x_1)$. The EPR state may be rewritten using this relation as:

$$\psi_{EPR}(x_1, x_2) = \sum_{n=1}^{\infty} \phi_n^*(x_1)\phi_n(x_2), \quad (4.8)$$

where $\{\phi_n(x)\}$ is an arbitrary basis of $L_2$. Since the form 4.8 resembles the spin singlet form 3.6, one might anticipate that it will lead to the existence of perfect correlations between the observables of particles 1 and 2. As we shall see, this is true not only for quantum systems described by 4.8, but also for a more general class of states as well.

Let us consider a Hermitian operator $A$ (assumed to possess a discrete spectrum) on the Hilbert space $L_2$. At this point, we postpone defining an observable of the EPR system in terms of $A$—we simply regard $A$ as an abstract operator. Suppose that $A$ can be written as

$$A = \sum_{n=1}^{\infty} \mu_n \langle \phi_n | \phi_n \rangle, \quad (4.9)$$

where $|\phi_n\rangle\langle \phi_n|$ is the one-dimensional projection operator associated with the vector $|\phi_n\rangle$. The eigenvectors and eigenvalues of $A$ are respectively the sets $\{|\phi_n\rangle\}$, and $\{\mu_n\}$. Suppose that another Hermitian operator, called $\tilde{A}$, is defined by the relationship

$$\tilde{A}_{x,x'} = A_{x,x'}^*, \quad (4.10)$$

where $\tilde{A}_{x,x'}$, and $A_{x,x'}$ are respectively the matrix elements of $\tilde{A}$ and $A$ in the position basis $\{\varphi_x\}$, and the superscript ‘*’ denotes complex conjugation. Note that for any given $A$, the ‘complex conjugate’ operator $\tilde{A}$ defined by 4.10 is unique. Writing out $A_{x,x'}$ gives

$$A_{x,x'} = \langle \varphi_x | \left( \sum_{n=1}^{\infty} \mu_n \langle \phi_n | \phi_n \rangle \right) \varphi_{x'} \rangle \quad (4.11)$$

$$= \sum_{n=1}^{\infty} \mu_n \langle \varphi_x | \phi_n \rangle \langle \phi_n | \varphi_{x'} \rangle.$$

Using 4.10, we find

$$\tilde{A}_{x,x'} = \sum_{n=1}^{\infty} \mu_n \langle \varphi_x | \phi_n \rangle^* \langle \phi_n | \varphi_{x'} \rangle^* \quad (4.12)$$

$$= \sum_{n=1}^{\infty} \mu_n \langle \varphi_x | \phi_n^* \rangle \langle \phi_n^* | \varphi_{x'} \rangle,$$
where \(|\phi^*_n\rangle\) is just the Hilbert space vector corresponding\(^6\) to the function \(\phi^*_n(x)\).

From the second equation in 4.12, it follows that \(\tilde{A}\) takes the form

\[
\tilde{A} = \sum_{n=1}^{\infty} \mu_n |\phi^*_n\rangle\langle\phi^*_n|.
\]

(4.15)

The eigenvectors and eigenvalues of \(\tilde{A}\) are respectively the sets \(|\phi^*_n\rangle\), and \(\{\mu_n\}\).

We now consider the observables \(1 \otimes A\) and \(\tilde{A} \otimes 1\) on the Hilbert space of the EPR state, \(L_2 \otimes L_2\), where \(1\) is the identity operator on \(L_2\). Put less formally, we consider \(A\) to be an observable of particle 2, and \(\tilde{A}\) as an observable of particle 1, just as \(\sigma^{(1)}_z\) represented ‘the spin component of particle 1’ in our above discussion of the spin singlet state. Examining 4.8, we see that each term is a product of an eigenvector of \(A\) with an eigenvector of \(\tilde{A}\) such that the eigenvalues are equal. For these observables, we have that \(\psi_{EPR}\) is an eigenstate of \(A - \tilde{A}\) of eigenvalue zero, i.e., \((A - \tilde{A})\psi_{EPR} = 0\). Thus, \(\psi_{EPR}\) exhibits perfect correlation between \(A\) and \(\tilde{A}\) in that the measurements of \(A\) and \(\tilde{A}\) give results that are equal.

Recall that the \(L_2\) basis \(|\phi_n(x)\rangle\) appearing in 4.8 is an arbitrary \(L_2\) basis. With this, and the fact that the eigenvalues \(\mu_n\) chosen for \(A\) are arbitrary, it follows that the operator \(A\) can represent any observable of particle 2. We can thus conclude that for any observable \(A\) of particle 2, there exists a unique observable \(\tilde{A}\) (defined by 4.10) of particle 1 which exhibits perfect correlations with the former.

We may interchange the roles of particles 1 and 2 in the above argument, if we note that the EPR state is symmetric in \(x_1\) and \(x_2\). This latter is proved as follows. Since the Dirac delta function is an even function, we have

\[
\delta(x_2 - x_1) = \delta(x_1 - x_2).
\]

(4.16)

Thus the EPR state assumes the form

\[
\psi_{EPR} = \delta(x_2 - x_1) = \delta(x_1 - x_2) = \sum_{n=1}^{\infty} \phi_n(x_1)\phi^*_n(x_2),
\]

(4.17)

---

\(^6\)More formally, we develop the correspondence of Hilbert space vectors \(|\phi\rangle\) to functions \(\phi(x)\) by expanding \(|\psi\rangle\) in terms of the position eigenvectors \(|\varphi_x\rangle\):

\[
|\phi\rangle = \int_{-\infty}^{\infty} dx |\varphi_x\rangle \langle\varphi_x|\phi\rangle.
\]

(4.13)

The function \(\phi(x)\) can then be identified with the inner product \(\langle\varphi_x|\phi\rangle\). Then the Hilbert space vector corresponding to \(\phi^*(x)\) is just

\[
|\phi^*\rangle = \int_{-\infty}^{\infty} dx |\varphi_x\rangle \langle\varphi_x|\phi\rangle^*
\]

\[
= \int_{-\infty}^{\infty} dx |\varphi_x\rangle \langle\phi|\varphi_x\rangle.
\]
where \( \{ \phi_n(x) \} \) is an arbitrary basis of \( L_2 \), and the second equality is the completeness relation. This establishes the desired symmetry.

Suppose now that we consider the observables \( A \otimes 1 \) and \( 1 \otimes \tilde{A} \), i.e., we consider \( A \) as an observable of particle 1, and \( \tilde{A} \) as an observable of particle 2. Then from the symmetry of the EPR state, it follows that we can reverse the roles of the particles in the above discussion to show that for any observable \( A \) of particle 1, there exists a unique observable \( \tilde{A} \) of particle 2 defined by 4.10, which exhibits perfect correlations with the former.

The perfect correlations in position and momentum originally noted by Einstein, Podolsky and Rosen are a special case of the perfectly correlated observables \( A \) and \( \tilde{A} \) given here. EPR found that the measurement of the positions \( x_1 \) and \( x_2 \) of the two particles must give results that are equal. The momenta \( p_1 \) and \( p_2 \) showed slightly different perfect correlations in which their measurements give results which sum to zero. To assess whether the EPR perfect correlations in position and momentum are consistent with the scheme given above, we derive using 4.10 the forms of \( \tilde{x} \) and \( \tilde{p} \). Since the position observable \( x \) is diagonal in the position eigenvectors and has real matrix elements, 4.10 implies that \( \tilde{x} = x \). As for the momentum, this operator is represented in the position basis by the differential operator \(-i\hbar \frac{d}{dx}\). Using 4.10 we see that \( \tilde{p} \) is just equal to the negative of \( p \) itself. Hence the perfect correlations developed in the present section, according to which \( A \) and \( \tilde{A} \) are equal, imply that measurements of the positions of the two particles must give equal results, whereas measurements of the momenta \( p_1 \) and \( p_2 \) must give results which sum to zero. This is just what EPR had found.

### 4.1.2.2 Incompleteness argument

The existence of such ubiquitous perfect correlations allows one to develop an argument demonstrating that all observables of particle 1 and all observables of particle 2 are “elements of reality”, i.e., they possess definite values. This argument is similar to the incompleteness argument given above for the spin singlet EPR case. We consider the possibility of separate measurements of observables \( \tilde{A}, A \) being performed respectively on particles 1 and 2 of the EPR state. As in the spin singlet EPR analysis, we assume locality, so that the properties of each particle must be regarded as independent of those of its spatially separated partner. Suppose we perform an experimental procedure \( \mathcal{E}(\tilde{A}) \) which constitutes a measurement of the observable \( \tilde{A} \) of particle 1. Since \( \tilde{A} \) is perfectly correlated with \( A \), the result we find allows us to predict with certainty the result of any subsequent experiment \( \mathcal{E}(A) \) measuring \( A \) of particle 2. For example, if \( \mathcal{E}(\tilde{A}) \) gives the result \( \tilde{A} = \mu_a \) then we can predict with certainty that \( \mathcal{E}(A) \) will give \( A = \mu_a \). Since we have assumed locality, \( A \) must be an element of reality, i.e., there exists some definite value \( E(A) \). Furthermore, since the same number is predicted no matter what experimental procedure \( \mathcal{E}(A) \) is used in \( A \)'s measurement, \( E(A) \) cannot depend on the choice of procedure,
i.e., $E(A)$ must be non-contextual.

We saw in the discussion above that for any observable $A$ of particle 2, there exists an observable $\tilde{A}$ of particle 1 with which the former shows perfect correlation. Hence, the above argument may be applied to any observable of particle 2, and we can therefore conclude that all observables of particle 2 are elements of reality, and that the definite value of each is non-contextual. Of course, if we consider the set of all observables of particle 2, there are some pairs among the set which are non-commuting. Since, from the quantum formalism’s description of state, one can deduce, at most, one observable of such a pair, we may conclude that this description of state is incomplete. Thus, any theoretical structure which hopes to capture these definite values must contain a state description which extends that of quantum mechanics.

As we noted above, it is possible to interchange the roles of the two particles when deriving the perfect correlations. Thus, to every observable $A$ of particle 1, there is a unique observable $\tilde{A}$ of particle 2 with which the former is perfectly correlated. This being the case, one can construct a similar incompleteness argument to show that all observables of particle 1 possess non-contextual definite values.

Finally, note that if we consider the possibility of an experiment measuring a commuting set of observables of either particle, the quantum theory predicts that the result is one of the joint-eigenvalues of that set. For agreement with this prediction, the values $E(O)$ assigned to the observables of this particle must map each commuting set to a joint-eigenvalue.

The Schrödinger paradox is seen to have the following structure. The analysis begins with the observation that the EPR state 4.8 exhibits maximal perfect correlations such that for any observable $A$ of either particle, there is a unique observable $\tilde{A}$ of the other with which $A$ is perfectly correlated. If we assume locality, then these perfect correlations imply the existence of a non-contextual value map on all observables of both particles. These value maps must assign to every commuting set a joint-eigenvalue.

The key to the above incompleteness argument is that one can, by measurement of an observable of particle 2, predict with certainty the result of any measurement of a particular observable of particle 1. In his presentation of the EPR incompleteness argument (which concerns position and momentum) Schrödinger uses a colorful analogy to make the situation clear. He imagines particles 1 and 2 as being a student and his instructor. The measurement of particle 2 corresponds to the instructor consulting a textbook to check the answer to an examination question, and the measurement of particle 1 to the response the student gives to this question. Since he always gives the correct answer, i.e., the same as that in the instructor’s textbook, the student must have known the answer beforehand. Schrödinger presents the situation as follows: [91] (emphasis by original author)

Let us focus attention on the system labeled with small letters
and call it for brevity the “small” system. Then things stand as follows. I can direct one of two questions to the small system, either that about \( q \) or that about \( p \). Before doing so I can, if I choose, procure the answer to one of these questions by a measurement on the fully separated other system (which we shall regard as auxiliary apparatus), or I may take care of this afterwards. My small system, like a schoolboy under examination, cannot possibly know whether I have done this or for which questions, or whether or for which I intend to do it later. From arbitrarily many pretrials I know that the pupil will correctly answer the first question I put to him. From that it follows that in every case, he knows the answer to both questions. . . . No school principal would judge otherwise . . . He would not come to think that his, the teacher’s, consulting a textbook first suggests to the pupil the correct answer, or even, in the cases when the teacher chooses to consult it only after ensuing answers from the pupil, that the pupil’s answer has changed the text of the notebook in the pupil’s favor.

4.1.2.3 Generalized form of the EPR state

Consideration of the structure of the EPR state 4.8 suggests the possibility that a more general class of states might also exhibit maximal perfect correlations. Before developing this, we first note that the formal way to write 4.8 is such that each term consists of a tensor product of a vector of particle 1’s Hilbert space with a vector of particle 2’s Hilbert space:

\[
\psi_{EPR} = \sum_{n=1}^{\infty} |\phi_n^* \rangle \otimes |\phi_n \rangle.
\]  

(4.18)

We shall often use this convenient notation in expressing the maximally entangled states. The operation of complex conjugation in the first factor of the tensor product may be regarded as a special case of a class of operators known as “anti-unitary involutions”—operators which we shall denote by \( C \), to suggest complex conjugation. We will find that any state of the form

\[
\psi_C = \sum_{n=1}^{\infty} C|\phi_n \rangle \otimes |\phi_n \rangle,
\]  

(4.19)

will exhibit ubiquitous perfect correlations leading to the conclusion of definite values on all observables of both subsystems.

An anti-unitary involution operation represents the generalization of complex conjugation from scalars to vectors. The term “involution” refers to any operator \( C \) whose square is equal to the identity operator, i.e. \( C^2 = 1 \). Anti-
unitarity entails two conditions, the first of which is anti-linearity:

\[ C(c_1|\psi_1\rangle + c_2|\psi_2\rangle + \ldots) = c_1^*C|\psi_1\rangle + c_2^*C|\psi_2\rangle + \ldots, \]

where \( \{c_i\} \) are constants and \( \{|\psi_i\rangle\} \) are vectors. The second condition is the anti-linear counterpart of unitarity:

\[ \langle C\psi|C\phi \rangle = \langle \psi|\phi \rangle^* = \langle \phi|\psi \rangle \quad \forall \psi, \phi, \]

which tells us that under the operation of \( C \), inner products are replaced by their complex conjugates. Note that this property is sufficient to guarantee that if the set \( \{|\phi_n\rangle\} \) is a basis, then the vectors \( \{C|\phi_n\rangle\} \) in 4.19 must also form a basis. For each anti-unitary involution \( C \), there is a special Hilbert space basis whose elements are invariant under \( C \). The operation of \( C \) on any given vector \( |\psi\rangle \) can be easily obtained by expanding the vector in terms of this basis. If \( \{|\varphi_n\rangle\} \) is this special basis then we have

\[ C|\psi\rangle = C \sum_{i=1}^{\infty} |\varphi_i\rangle \langle \varphi_i|\psi \rangle = \sum_{i=1}^{\infty} |\varphi_i\rangle \langle \varphi_i|\psi \rangle^*. \]

When one is analyzing any given state of the form 4.19, it is convenient to express the state and observables using this special basis. The EPR state is a special case of the state 4.19 in which the anti-unitary involution \( C \) is such that the position basis \( \{|\varphi_n\rangle\} \) plays this role.

The state 4.19 shows an invariance similar to that we developed for the EPR state: the basis \( \{|\phi_n\rangle\} \) in terms of which the state is expressed, is arbitrary. We now develop this result. Note that the expression 4.19 takes the form

\[ \psi_C = \sum_{n=1}^{\infty} C \left( \sum_{i=1}^{\infty} |\chi_i\rangle \langle \phi_n|\chi_i \rangle \right) \otimes \left( \sum_{j=1}^{\infty} |\chi_j\rangle \langle \chi_j|\phi_n \rangle \right), \]

if we expand the vectors in terms of an arbitrary basis \( \{|\chi_i\rangle\} \). Applying the \( C \) operation in the first factor and rearranging the expression, we obtain

\[ \psi_C = \sum_{n=1}^{\infty} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \langle \chi_j|\phi_n\rangle \langle \phi_n|\chi_i \rangle C|\chi_i \rangle \otimes |\chi_j \rangle \]

\[ = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \langle \chi_j|\phi_n\rangle \langle \phi_n|\chi_i \rangle C|\chi_i \rangle \otimes |\chi_j \rangle, \]

\[ \text{7This term may appear confusing for the following reason. It does not not refer to an operator which is “not unitary”, but instead the prefix “anti” refers to anti-linearity.} \]
where the first equality follows from the anti-linearity of $C$. Since the expression $\sum_{n=1}^{\infty} |\phi_n\rangle\langle\phi_n|$ is the identity operator, the orthonormality of the set $\{\chi_i\}$ implies that

$$
\psi_C = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \delta_{ij} C|\chi_i\rangle \otimes |\chi_j\rangle = \sum_{i=1}^{\infty} C|\chi_i\rangle \otimes |\chi_i\rangle.
$$

Thus, the form of the state 4.19 is invariant under any change of basis, and we have the desired result.

Note then the role played by the properties of anti-linear unitarity and anti-linearity: from the former followed the result that the vectors $\{C|\phi_n\rangle\}$ form a basis if the vectors $\{|\phi_n\rangle\}$ do so, and from the latter followed the invariance just shown.

Consider a Hermitian operator $A$ on $L^2$ which can be written as

$$
A = \sum_{n=1}^{\infty} \mu_n |\phi_n\rangle \langle\phi_n|.
$$

Note that $A$'s eigenvectors and eigenvalues are given by the sets $\{|\phi_n\rangle\}$ and $\{\mu_n\}$, respectively. If we define the observable $\hat{A}$ by the relationship

$$
\hat{A} = CAC^{-1},
$$

then $\hat{A}$'s eigenvalues are the same as $A$'s, i.e., $\{\mu_n\}$, and its eigenvectors are given by $\{C|\phi_n\rangle\}$. To see this, note that

$$
CAC^{-1}C|\phi_n\rangle = C\mu_n |\phi_n\rangle = \mu_n C|\phi_n\rangle,
$$

where the first equality follows from $CC^{-1} = 1$ and $A|\phi_n\rangle = \mu_n |\phi_n\rangle$. Since $C^2 = 1$, it follows that $C = C^{-1}$, and we may rewrite 4.27 as:

$$
\hat{A} = CAC.
$$

Note that for any given $A$, the observable $\hat{A}$ defined by 4.31 is unique.

If we identify $A$ as an observable of subsystem 2, and $\hat{A}$ as an observable of subsystem 1, then examination of the state 4.19 shows that these exhibit perfect correlations such that their measurements always yield results that are equal.

Note that to evaluate $\hat{A}$, one can express it using its matrix elements with respect to the invariant basis $\{\varphi_n\}$ of $C$. If we evaluate the matrix element $CAC_{ij}$, we find

$$
\langle\varphi_i|CAC|\varphi_j\rangle = \langle\varphi_i|CA|\varphi_j\rangle = \langle\varphi_i|A|\varphi_j\rangle^*,
$$

where the second equality follows from 4.22. From 4.29 follows the relationship

$$
\hat{A}_{ij} = A^*_{ij},
$$

which is a convenient form one may use to evaluate $\hat{A}$, as we will see in section 4.1.3. Note that 4.30 reduces to 4.10 when the invariant basis $\{\varphi_n\}$ of $C$ is the position basis $|\varphi_x\rangle$. 

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\hat{A}_{ij} = A^*_{ij},
$$

which is a convenient form one may use to evaluate $\hat{A}$, as we will see in section 4.1.3. Note that 4.30 reduces to 4.10 when the invariant basis $\{\varphi_n\}$ of $C$ is the position basis $|\varphi_x\rangle$. 

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From the invariance of the state, we can then conclude that for any observable \( A \) of subsystem 2, there is a unique observable \( \tilde{A} \) of subsystem 1 defined by 4.31 which exhibits perfect correlations with \( A \). Since, as can easily be proved\(^9\), the state 4.19 assumes the form

\[
\psi_C = \sum_{n=1}^{\infty} |\phi_n⟩ \otimes C |\phi_n⟩,
\]

one may develop the same results with the roles of the subsystems reversed, i.e., for any observable \( A \) of subsystem 1, there exists a unique observable \( \tilde{A} \) of subsystem 2 which exhibits perfect correlations with \( A \).

An incompleteness argument similar to that given above can be given, and one can show the existence of a value map \( E(O) \) on all observables of both subsystems.

### 4.1.2.4 The general form of a maximally entangled state

States exhibiting ubiquitous perfect correlations are not limited to those of the form 4.19. If we examine any composite system whose subsystems are of the same dimensionality\(^{10}\) and which is represented by a state\(^{11}\)

\[
\psi_{ME} = \sum_{n=1}^{N} |ψ_n⟩ \otimes |φ_n⟩,
\]

where \( \{ |ψ_n⟩ \} \) is any basis of subsystem 1 and \( \{ |φ_n⟩ \} \) is any basis of subsystem 2, we find that each observable of either subsystem exhibits perfect correlations with some observable of the other subsystem. To examine the properties of the

\(^9\)The proof of this result follows similar lines as the invariance proof given above. One expands the vectors of the state 4.19 in terms of the invariant basis \( \{ ϕ_n \} \). Doing so leads to an expression similar to the first equality in 4.25, with \( ϕ_i \) and \( ϕ_j \) replacing \( χ_i \) and \( χ_j \), respectively. The delta function is then replaced using the relationship

\[
δ_{ij} = ⟨ϕ_i| \left( \sum_{n=1}^{∞} |ϕ_n⟩⟨ϕ_n| \right) |ϕ_j⟩,
\]

and one can then easily develop 4.33

\(^{10}\)The derivations of this section can be carried out for an entangled system whose subsystems are either finite or infinite dimensional. Infinite sums may be substituted for the finite sums written here to develop the same results for the infinite dimensional case.

\(^{11}\)This is equivalent to the form

\[
ψ_{ME} = \sum_{n=1}^{N} c_n |ψ_n⟩ \otimes |φ_n⟩,
\]

where \( \{ |ψ_n⟩ \} \) is any basis of subsystem 1 and \( \{ |φ_n⟩ \} \) is any basis of subsystem 2 and the \( |c_n|^2 = 1 \ ∀n \).
state 4.35, we note that it may be rewritten\(^\text{12}\) as

\[
\psi_{ME} = \psi = \sum_{n=1}^{N} U|\phi_n\rangle \otimes |\phi_n\rangle, \tag{4.37}
\]

where \(U\) is the anti-unitary operator defined by \(U|\phi_n\rangle = |\psi_n\rangle\). Recall from the above discussion that (anti-linear) unitarity and anti-linearity are sufficient to guarantee both that \(\{U|\phi_n\rangle\}\) is a basis if \(\{|\phi_n\rangle\}\) is, and that the state shows the invariance we require. We conclude our presentation of the Schrödinger paradox by discussing this general form.

We consider a Hermitian operator \(A\) on \(\mathcal{H}_N\) which can be written as

\[
A = \sum_{n=1}^{N} \mu_n |\phi_n\rangle \langle \phi_n|. \tag{4.38}
\]

We then define the observable \(\tilde{A}\) by\(^\text{13}\)

\[
\tilde{A} = UAU^{-1}. \tag{4.40}
\]

For any given \(A\), this defines a unique operator \(\tilde{A}\). Using \(A|\phi_n\rangle = \mu_n|\phi_n\rangle\) and \(UU^{-1} = 1\), we have

\[
UAU^{-1}|\phi_n\rangle = \mu_n U|\phi_n\rangle, \tag{4.41}
\]

so that the eigenvectors and eigenvalues of \(\tilde{A}\) are given by \(\{|U\phi_n\rangle\}\) and \(\{\mu_n\}\).

Identifying \(A\) as an observable of subsystem 2 and \(\tilde{A}\) as an observable of subsystem 1, we can see from the form of 4.37 that these exhibit perfect correlations for such a state. As in the EPR case and its generalization, the basis-invariance and symmetry of the state imply that for any observable \(A\) of either subsystem, there is a unique observable \(\tilde{A}\) of the other with which \(A\) is perfectly correlated. An argument similar to that given for the EPR state leads to the existence of a non-contextual value map \(E(O)\) on all observables of subsystem 1 and all observables of subsystem 2.

\(^{12}\)At this point, one may address the objection that the EPR state 4.8 makes the positions of the two particles coincide. For example, one can consider an anti-unitary operator \(U_d\) defined by

\[
U_d|\psi\rangle = \int_{-\infty}^{\infty} dx |\varphi_d\rangle \langle \psi|\varphi_{x+d}|, \tag{4.36}
\]

where \(d\) is an arbitrary constant. Then, in the state 4.37, the two particles are separated by a distance \(d\).

\(^{13}\)For comparison with the form 4.10, and 4.30, we note that if we express the operator \(U\) as \(U = CU\) with \(C\) an anti-unitary involution, and \(U\) a unitary matrix, then 4.40 leads to

\[
\tilde{A}_{ij} = (UAU^{-1})_{ij}, \tag{4.39}
\]

where the \(ij\) subscript indicates the \(ij\)th matrix element in terms of \(\{\varphi_n\}\), the basis that is invariant under \(C\).
4.1.3 The spin singlet state as a maximally entangled state

It is instructive to make an explicit comparison of the properties given above for a general maximally entangled state with those of the spin singlet state. The spin singlet state takes the form

\[ \psi_{ss} = |\uparrow, \theta, \phi\rangle \otimes |\downarrow, \theta, \phi\rangle - |\downarrow, \theta, \phi\rangle \otimes |\uparrow, \theta, \phi\rangle, \quad (4.42) \]

when expressed in terms of the eigenvectors of \( \sigma_{\theta, \phi} \). We have suppressed the normalization for simplicity. Inspection of this form leads one to conclude that the state must exhibit correlations such that the results of measurement of \( \sigma_{\theta, \phi}^{(1)} \) and \( \sigma_{\theta, \phi}^{(2)} \) will give results which sum to zero. These might be better named “perfect anti-correlations”, since the measurement results are the negatives of one another. On examining the form given in 4.34, one can see immediately that the spin singlet state 4.42 is a maximally entangled state.

This being the case, it follows that the spin singlet state must assume the form 4.37. To see this, we write the anti-unitary operator \( U = C\bar{U} \), where \( \bar{U} \) is a unitary operator given by

\[ \bar{U} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]

and \( C \) is an anti-unitary involution under which the \( \sigma_z \) eigenvectors are invariant, i.e., \( C|\uparrow\rangle = |\uparrow\rangle \) and \( C|\downarrow\rangle = |\downarrow\rangle \). Using \( |\uparrow\rangle, |\downarrow\rangle \) as the basis in the maximally entangled state expression 4.37, we obtain

\[ \psi = U|\uparrow\rangle \otimes |\uparrow\rangle + U|\downarrow\rangle \otimes |\downarrow\rangle. \quad (4.43) \]

This reduces\(^{14}\) to the familiar spin singlet form

\[ \psi = |\uparrow\rangle \otimes |\downarrow\rangle - |\downarrow\rangle \otimes |\uparrow\rangle, \quad (4.44) \]

when we write out the operation of \( \bar{U} \) as a matrix multiplication.

The perfect correlations between spin components of the two particles can be regarded as a special case of the maximally entangled state perfect correlations, which hold between any observables \( A \) of one subsystem and \( \tilde{A} \) of the other, where \( \tilde{A} = UAU^{-1} \). To develop this, we recall that the observable \( \tilde{A} \) exhibits perfect correlations with the observable \( A \) such that measurements of \( A \) and \( \tilde{A} \) give results that are equal. In the case of the spin singlet state, we have what one might call “perfect anti-correlations”, i.e., measurements of \( \sigma_{\theta, \phi}^{(1)} \) and \( \sigma_{\theta, \phi}^{(2)} \) give results which sum to zero. Thus, for the case of the spin singlet state, we expect that \( \overline{\sigma_{\theta, \phi}} = -\sigma_{\theta, \phi} \), or

\[ U\sigma_{\theta, \phi}U^{-1} = -\sigma_{\theta, \phi}, \quad (4.45) \]

where \( U \) is the operator described above. Using \( C^{-1} = C \), one can see that the left hand side of 4.45 reduces to \( C\bar{U}\sigma_{\theta, \phi}\bar{U}^{-1}C \). In deriving 4.45, we begin by evaluating the expression \( \bar{U}\sigma_{\theta, \phi}\bar{U}^{-1} \).

\(^{14}\) To within an overall minus sign, which can be ignored.
Recall that the form of the observable $\sigma_{\theta,\phi}$ is given by
\[
\sigma_{\theta,\phi} = \begin{pmatrix}
\cos(\theta) & e^{-i\phi} \sin(\theta) \\
e^{i\phi} \sin(\theta) & -\cos(\theta)
\end{pmatrix}.
\] (4.46)
Note that one can write this as $\sigma_{\theta,\phi} = \begin{pmatrix} a & b \\ b^* & -a \end{pmatrix}$, where $a = \cos(\theta)$ and $b = e^{-i\phi} \sin(\theta)$. A simple calculation involving matrix multiplication shows that
\[
\bar{U} \begin{pmatrix} a & b \\ b^* & -a \end{pmatrix} \bar{U}^{-1} = \begin{pmatrix} -a & -b^* \\ -b & a \end{pmatrix}.
\] (4.47)
To complete the calculation, we must evaluate the expression $CAC$, where $A$ is given by the matrix $\begin{pmatrix} -a & -b^* \\ -b & a \end{pmatrix}$. We saw in section 4.1.2.3 an expression of the form $CAC$ can be evaluated in terms of its matrix elements in $\{\varphi_n\}$, the basis that is invariant under $C$. To do so, we use the relation 4.29:
\[
(CAC)_{ij} = A^*_{ij}.
\] (4.48)
Using 4.48, it follows that $C \begin{pmatrix} -a & -b^* \\ -b & a \end{pmatrix} C = \begin{pmatrix} -a & -b \\ -b^* & a \end{pmatrix}$. In comparing this relation to the form of $\sigma_{\theta,\phi}$, one can see that $C \bar{U} \sigma_{\theta,\phi} \bar{U}^{-1} C = -\sigma_{\theta,\phi}$, and we have arrived at 4.45.

### 4.1.4 Bell’s theorem and the maximally entangled states

We have seen, in chapter 3, how quantum nonlocality may be proved for one particular maximally entangled state, which is the spin singlet state. Here, we show how Bell’s theorem may be applied in proofs of the nonlocality of a larger class of maximally entangled states. A more general form of the proof we give here is given by Popescu and Rohrlich [87], who demonstrate that the nonlocality of any maximally entangled state follows from Bell’s theorem.

Consider a general maximally entangled state given by
\[
\psi_{ME} = \sum_{n=1}^{N} |\varphi_n\rangle \otimes |\psi_n\rangle = |\phi_1\rangle \otimes |\psi_1\rangle + |\phi_2\rangle \otimes |\psi_2\rangle + \sum_{n=3}^{N} |\phi_n\rangle \otimes |\psi_n\rangle.
\] (4.49)
Let us define $\mathcal{H}_1$ and $\mathcal{H}_2$ as the subspaces of particle 1 and 2’s Hilbert spaces spanned respectively by $\phi_1, \phi_2$ and $\psi_1, \psi_2$. We define the sets of observables $\{\xi_{\theta,\phi}^{(1)}\}$ and $\{\xi_{\theta,\phi}^{(2)}\}$ as follows. The set $\{\xi_{\theta,\phi}^{(1)}\}$ is formally identical\textsuperscript{15} to the set $\{\sigma_{\theta,\phi}\}$ on $\mathcal{H}_1$, and all its members give zero when operating on any vector in the orthogonal complement of $\mathcal{H}_1$. Similarly, the set $\{\xi_{\theta,\phi}^{(2)}\}$ is formally identical\textsuperscript{15} since any two Hilbert spaces of the same dimension are isomorphic, it is possible to define such formally identical observables.

\textsuperscript{15}
to \{\sigma_{\theta,\phi}\} on \mathcal{H}_2, and its members give zero when operating on any vector in its orthogonal complement. We now select a class of states smaller than that given by 4.49 by making the following substitutions: the vectors \ket{\uparrow} and \ket{\downarrow} replace \phi_1 and \phi_2 and the vectors \ket{\downarrow} and \ket{-\uparrow} replace \psi_1 and \psi_2. Here, \ket{\uparrow} and \ket{\downarrow} are the eigenvectors of \sigma_z. The state 4.49 then becomes

\[ \psi = \ket{\uparrow} \otimes \ket{\downarrow} - \ket{\downarrow} \otimes \ket{\uparrow} + \sum_{n=3}^{N} \phi_n \otimes \psi_n. \] (4.50)

The sets of observables \{\xi^{(1)}_{\theta,\phi}\} and \{\xi^{(2)}_{\theta,\phi}\} are zero on every term of 4.50 with the exception of the first two. Since the first two terms are identical to the spin singlet state, it follows that \psi is an eigenstate of \xi^{(1)}_{\theta,\phi} + \xi^{(2)}_{\theta,\phi} of eigenvalue zero, \forall \theta, \phi, and therefore perfect correlations between all such observables. We may exploit this situation to derive a nonlocality proof with some further effort, as we now show.

Let \mathcal{P}^{(1)} and \mathcal{P}^{(2)} be the projections operators which project respectively onto \mathcal{H}_1 and \mathcal{H}_2. Since the state 4.50 exhibits maximal perfect correlations, one can give an incompleteness argument to show that \mathcal{P}^{(1)}, \mathcal{P}^{(2)}, and the sets \{\xi^{(1)}_{\hat{a}}\} and \{\xi^{(2)}_{\hat{b}}\} must all possess definite values. However, the existence of such values cannot agree with the statistical predictions of quantum mechanics, as one may see. We now examine what might be called the “conditional correlation function”, which we define as follows. As was done in the presentation of Bell’s theorem, we represent the predetermined values of the observables \{\xi^{(1)}_{\theta,\phi}\} and \{\xi^{(2)}_{\theta,\phi}\} using the mathematical functions \(A(\lambda, \hat{a})\) and \(B(\lambda, \hat{b})\). Suppose that we consider measurements of the set \{\xi^{(1)}_{\hat{a}}, \mathcal{P}^{(1)}\} of subsystem 1 and \{\xi^{(2)}_{\hat{b}}, \mathcal{P}^{(2)}\} of subsystem 2. In every such case, we discard those results for which either \mathcal{P}^{(1)} or \mathcal{P}^{(2)} (or both) equals zero. We consider the correlation in measurements of \xi^{(1)}_{\hat{a}} and \xi^{(2)}_{\hat{b}} in those cases for which \mathcal{P}^{(1)} = \mathcal{P}^{(2)} = 1. Under these conditions, the quantum mechanical prediction for the correlation function \(P_{QM}(\hat{a}, \hat{b})\) is given by

\[ P_{QM}(\sigma^{(1)}_{\hat{a}}, \sigma^{(2)}_{\hat{b}}) = \langle \psi_{ss} | \xi^{(1)}_{\hat{a}} \xi^{(2)}_{\hat{b}} | \psi_{ss} \rangle = -\hat{a} \cdot \hat{b}. \] (4.51)

On the other hand, the prediction derived from the predetermined values \(A(\lambda, \hat{a})\) and \(B(\lambda, \hat{b})\) can be shown to satisfy Bell’s inequality 3.14. Therefore, the statistics of these values—which themselves follow from the assumption of locality—are in conflict with the predictions of quantum mechanics and so we must conclude quantum nonlocality.
4.2 Schrödinger nonlocality, and a discussion of experimental verification

4.2.1 Schrödinger nonlocality

We have seen that the Schrödinger paradox gives a more general result than follows from either version of the EPR paradox. Because of this greater generality, it is possible to construct a nonlocality proof using Schrödinger’s paradox in conjunction with any one of a wide variety of theorems besides that of Bell. This is due mainly to the fact that the Schrödinger paradox predicts definite values for such a large class of observables that the theorems required need not address more than one particle or subsystem—even value map impossibility proofs which are concerned with single particle systems may be sufficient. Consider the case of Kochen and Specker’s theorem. Suppose that some system is described by a maximally entangled state whose subsystems are of dimensionality three:

$$\sum_{n=1}^{3} \phi_n \otimes \psi_n.$$  (4.52)

Among the set of all observables on both $\mathcal{H}_1$ and $\mathcal{H}_2$ are the squares of the various spin components$^{16}$ of a spin 1 particle, $\{s_{\theta,\phi}^2\}$. The state 4.52 exhibits perfect correlations between all the observables $\{s_{\theta,\phi}^2\}$ of one subsystem, and their counterparts $\{\overline{s}_{\theta,\phi}^2\}$ of the other. As we saw in the incompleteness argument given in section 4.1.2.2, if we assume locality, then the perfect correlations imply the existence of a non-contextual value map $E(O)$ on the sets $\{s_{\theta,\phi}^2\}$ and $\{\overline{s}_{\theta,\phi}^2\}$. However, we know from the Kochen and Specker theorem that there exists no value map $E(s_{\theta,\phi}^2)$ such that a joint-eigenvalue is assigned to every commuting set. This conclusion contradicts the quantum mechanical prediction that the measurements of any commuting set always give one of its joint-eigenvalues. Thus, if we demand that the perfect correlations of the state 4.52 are to be explained through a local theory, we are led to a conclusion that is in conflict with quantum mechanics. Therefore the quantum description of any system described by a state such as 4.52 must entail nonlocality. Note that the development of the contradiction between the quantum predictions and those of the definite values is expressed using the observables of one subsystem, namely the set $\{s_{\theta,\phi}^2\}$.

Moreover, if we consider any maximally entangled state of dimensionality $N$

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16Of course, such a system need not consist of two spin 1 particles. If it does not, then the same conclusion as is given here holds for those observables which are formally equivalent to the sets $\{s_{\theta,\phi}^2\}$ and $\{\overline{s}_{\theta,\phi}^2\}$. Since any two Hilbert spaces of the same dimension are isomorphic, it follows that such formally equivalent observables must exist.
of at least three:

\[ \psi = \sum_{n=1}^{N} \phi_n \otimes \psi_n, \quad (4.53) \]

then we can evidently develop a nonlocality proof for this state by using the Kochen and Specker theorem. To see this, we re-write 4.53 as

\[ \psi = \phi_1 \otimes \psi_1 + \phi_2 \otimes \psi_2 + \phi_3 \otimes \psi_3 + \sum_{n=4}^{N} \psi_n \otimes \psi_n, \quad (4.54) \]

and we define \( \mathcal{H}_2 \) to be the subspace of particle 2’s Hilbert space spanned by the vectors \( \psi_1, \psi_2, \psi_3 \), and the operator \( P \) as the projection operator of \( \mathcal{H}_2 \). Let us define the set of observables \( \{ \zeta_{\theta, \phi} \} \) such that they are formally identical to the squares of the spin components \( \{ \zeta_{\theta, \phi} \} \) on \( \mathcal{H}_2 \) and give zero when operating on any vector in its orthogonal complement. Then the Kochen and Specker theorem tells us that the existence of definite values for the set \( \{ P, \{ \zeta_{\theta, \phi} \} \} \) must conflict with quantum mechanics. To see this, consider those joint-eigenvalues of \( \{ P, \{ \zeta_{\theta, \phi} \} \} \) for which \( P = 1 \). The corresponding values of the set \( \{ \zeta_{\theta, \phi} \} \) in this case are equal to joint-eigenvalues on the spin observables themselves. Since the Kochen and Specker theorem implies the impossibility of an assignment of values to the spin components, then the same follows for the observables \( \{ \zeta_{\theta, \phi} \} \) given that \( P = 1 \).

If we consider the conjunction of Schrödinger’s paradox with either Gleason’s theorem or Mermin’s theorem, we again find that quantum nonlocality follows. From Gleason’s theorem we have that a value map on the set of all projections \( \{ P \} \) must conflict with quantum mechanics. One can show that for any maximally entangled state

\[ \sum_{n=1}^{N} \phi_n^{(1)} \otimes \psi_n^{(2)}, \quad (4.55) \]

where \( N \) is at least three, we have quantum nonlocality. This result also holds true for for maximally entangled states of infinite-dimensionality, in which case the sum in 4.55 is replaced by an infinite sum. In the case of Mermin’s theorem, we can show that for any maximally entangled state whose subsystems are at least four dimensional, the quantum predictions must entail nonlocality.

Thus, the nonlocality of the maximally entangled states may be proved not only by the analyses involving Bell’s theorem (see section 4.1.4), but also by consideration of any of the theorems of Gleason, Kochen and Specker, or Mermin, taken together with the Schrödinger paradox. We have seen that the latter type of argument, which we have called “Schrödinger nonlocality”, differs from that involving Bell’s theorem in several ways. First, the Schrödinger nonlocality it is of a deterministic, rather than statistical character. This is due to the fact that the conflict of the Schrödinger incompleteness with quantum
mechanics is in terms of the quantum prediction for individual measurements, i.e., the prediction that measurements of a commuting set always give one of that set’s joint-eigenvalues. Second, such quantum nonlocality proofs can be developed from these theorem’s implications regarding the observables of just one subsystem. Finally, the Schrödinger nonlocality can be proved for a larger class of observables than can the EPR/Bell nonlocality. Note that beyond the arguments presented here, there remains the possibility of further instances of Schrödinger nonlocality, since any ‘spectral incompatibility theorem’ (see section 2.4) leads to such a demonstration. If the theorem in question concerns a class of observables on an N-dimensional Hilbert space, then the proof can be applied to any maximally entangled states whose subsystem’s are at least N-dimensional.

4.2.2 Discussion of the experimental tests of EPR/Bell and Schrödinger nonlocality

In chapter three, we reviewed the demonstration that the spin singlet version of the EPR paradox in conjunction with Bell’s theorem leads to the conclusion of quantum nonlocality. That quantum theory entails such an unusual feature as nonlocality invited physicists to perform laboratory experiments to test whether the quantum predictions for the relevant phenomena are actually borne out. The experimental tests\(^\text{17}\) inspired by the discovery of quantum nonlocality have focused mainly\(^\text{18}\) on a maximally entangled system of two photons rather than two spin \(\frac{1}{2}\) particles. The results of these experiments are in agreement with the quantum mechanical predictions\(^\text{19}\). Since we have described the spin singlet version of the EPR/Bell nonlocality in some detail (chapter 3), we address this case rather than the two-photon maximally entangled state.

Let us briefly recall how the EPR argument and Bell’s theorem give rise to the conclusion of quantum nonlocality. According to the EPR argument, the perfect correlations exhibited by the spin singlet state can be explained under locality only if there exist definite values for all components of the spins of both particles. Bell’s theorem shows that any such definite values lead to the prediction that the ‘correlation function’ \(P(\hat{a}, \hat{b})\) must satisfy Bell’s inequality. Thus, if we combine the EPR paradox with Bell’s theorem we obtain the argument that any local theoretical explanation of the perfect correlations must give a prediction for \(P(\hat{a}, \hat{b})\) that satisfies the Bell inequality. On the other hand,\(^\text{17}\) See for example, Freedman and Clauser [53], Fry and Thompson [54], and Aspect, et. al. [3, 4, 5]. For the two-photon systems studied by these authors, an analysis first given by Clauser, Horne, Holt and Shimony [37] plays the role of Bell’s theorem: the assumption of locality together with the existence of the perfect correlations in the photon polarizations leads to the CHHS inequality, which relationship is not generally satisfied by the quantum mechanical predictions.

\(^{18}\)The experiment of Lamehi-Rachti, and Mittig [75] involves a pair of protons described by the spin singlet state. These results are in support of the quantum predictions.

\(^{19}\)For a discussion of the experimental results, see following: [10, 12, 64].

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the quantum mechanical predictions for \( P(\hat{a}, \hat{b}) \) violate this inequality. Thus, any local theoretical description of the spin singlet state must conflict with the quantum mechanical one.

We now consider what such a laboratory test of quantum nonlocality would entail. The EPR/Bell analysis depends on the correctness of the quantum mechanical predictions of perfect correlations. If these quantum predictions are not confirmed, i.e., the experimental test does not reveal perfect correlations, one could not interpret the experiment in terms of EPR/Bell. It is only with the observation of the perfect correlations that further tests can be performed to make an experimental judgment between quantum mechanics and the family of local theories. Such a judgment can be made based on the appropriate measurements of the correlation function \( P \), and the comparison of the results with the Bell inequality. One must measure the correlation function for just those angles \( \hat{a}, \hat{b}, \hat{c} \) for which the quantum mechanical prediction violates the Bell inequality (see section 3.3). If the results of these tests agree with the Bell inequality, then the quantum predictions are refuted, and the experiment may be interpreted in terms of a theory consistent with the concept of locality. If the results disagree with the Bell inequality, this would confirm the quantum mechanical predictions and would support the notion of nonlocality.

### 4.2.2.1 Test of Schrödinger nonlocality by perfect correlations only

If one examines the various proofs of quantum nonlocality, one finds that all have a somewhat similar structure. The first part of such a proof consists of an EPR-like incompleteness argument, according to which the perfect correlations exhibited by some system together with the assumption of locality are shown to imply the existence definite values for certain observables. The second part of a nonlocality proof is a demonstration that such definite values must conflict with the predictions of quantum mechanics. The two parts can be combined to produce a proof of quantum nonlocality. When we come to consider the laboratory confirmation of such quantum nonlocality, it is natural to expect that the two part structure of its proof will be reflected in the various stages of the experiment itself. As we have seen above, this is indeed true for the test of the EPR/Bell nonlocality: one must observe the perfect correlations, and the violation of the Bell inequality to verify that the spin singlet state reflects quantum nonlocality. When we examine the experimental test of what we have called ‘Schrödinger nonlocality’, however, we find that it is possible to perform the experiment in such a way that the perfect correlations by themselves are sufficient to verify that the system being studied shows nonlocal effects of this sort. We now turn to the question of such an experimental test.

For the sake of definiteness, let us consider a particular example of Schrödinger nonlocality, namely that which arises from the conjunction of Schrödinger’s paradox with Mermin’s theorem (see section 2.2.1). This proof is constructed as follows. The Schrödinger paradox incompleteness argument implies that the
perfect correlations exhibited by the maximally entangled state lead to the existence of a non-contextual value map on all observables of both subsystems. If the subsystems of the state in question are associated with Hilbert spaces that are at least 4 dimensional (see section 4.2.1), then there must exist such a map on the Mermin observables (or a set of formally equivalent observables). Mermin’s theorem, however, contradicts the possibility of such a map. If we combine this theorem with the Schrödinger incompleteness argument the result is a quantum nonlocality proof.

For the reader’s convenience, we recall Mermin’s theorem here. The observables addressed by this theorem are the $x$ and $y$ components of two spin $\frac{1}{2}$ particles, and the observables $A, B, C, X, Y, Z$ which are defined in terms of these through the relations

\begin{align*}
A &= \sigma^{(\alpha)}_{x}\sigma^{(\beta)}_{y} \\
B &= \sigma^{(\alpha)}_{y}\sigma^{(\beta)}_{x} \\
X &= \sigma^{(\alpha)}_{x}\sigma^{(\beta)}_{x} \\
Y &= \sigma^{(\alpha)}_{y}\sigma^{(\beta)}_{y},
\end{align*}

and

\begin{align*}
C &= AB \\
Z &= XY.
\end{align*}

Note that the symbols $\sigma^{(\alpha)}$ and $\sigma^{(\beta)}$ are used here\(^{20}\), rather than $\sigma^{(1)}$ and $\sigma^{(2)}$, which we employed in the presentation of Mermin’s theorem given in section 2.2.1. We will refer to these observables as the ‘Mermin observables’, and the notation $M_i, i = 1 \ldots 10$ shall refer to an arbitrary member of the set. In the discussion of Mermin’s theorem given in section 2.2.1, we showed that the commutation relationships among the spin components led to the relation

\begin{equation}
\sigma^{(\alpha)}_{x}\sigma^{(\beta)}_{y}\sigma^{(\alpha)}_{y}\sigma^{(\beta)}_{x}\sigma^{(\alpha)}_{x}\sigma^{(\beta)}_{y}\sigma^{(\alpha)}_{y}\sigma^{(\beta)}_{x} = -1. \tag{4.58}
\end{equation}

From this and the definitions 4.56 and 4.57, it is easy to see $C$ and $Z$ satisfy

\begin{equation}
CZ = -1. \tag{4.59}
\end{equation}

Mermin’s theorem implies that there exists no function $E(O)$ on the observables $M_i$ that satisfies all the relationships constraining the commuting observables.

On inspection of the first equation in 4.56, we see that the three observables involved are a commuting set: $[\sigma^{(\alpha)}_{x}, \sigma^{(\beta)}_{y}] = 0$, $[\sigma^{(\alpha)}_{x}, A] = [\sigma^{(\alpha)}_{x}, \sigma^{(\alpha)}_{x}\sigma^{(\beta)}_{y}] = 0$ and $[\sigma^{(\beta)}_{y}, A] = [\sigma^{(\beta)}_{y}, \sigma^{(\alpha)}_{x}\sigma^{(\beta)}_{y}] = 0$. Examination of the other equations in 4.56

\(^{20}\)In the present section, we consider the entire set of Mermin observables as being associated with one subsystem of a maximally entangled state. Use of the notation $\sigma^{(1)}$, $\sigma^{(2)}$ suggests observables belonging to separate subsystems, and might have led to confusion.
shows that the same holds true for these, i.e. the observables in each form a commuting set. Repeated application of the commutation rules may be used to show that the sets \{C, A, B\}, \{Z, X, Y\}, and \{C, Z\} are also commuting sets. For convenience, we list these sets here:

\[
\begin{align*}
\{A, \sigma_x^{(\alpha)}, \sigma_y^{(\beta)}\} & \quad \{B, \sigma_y^{(\alpha)}, \sigma_x^{(\beta)}\} & \quad \{X, \sigma_x^{(\alpha)}, \sigma_x^{(\beta)}\} \\
\{Y, \sigma_y^{(\alpha)}, \sigma_y^{(\beta)}\} & \quad \{C, A, B\} & \quad \{Z, X, Y\} \\
\{C, Z\} & \quad & 
\end{align*}
\]

The relationships which Mermin’s theorem requires the function \(E(O)\) to satisfy are the defining equations in 4.56, 4.57 and 4.59.

We now recall the Schrödinger incompleteness argument in detail. In this argument, one considers the possibility of separate experimental procedures being performed to measure observables of the two subsystems of any given maximally entangled state. Suppose that one measures \(A\) of subsystem 2 and \(\tilde{A}\) of subsystem 1, through experimental procedures \(\mathcal{E}(A)\), and \(\mathcal{E}(\tilde{A})\). The perfect correlation between these observables implies that from the value of \(A\) found in any procedure \(\mathcal{E}(A)\), we can predict with certainty the value of \(\tilde{A}\) found in any procedure \(\mathcal{E}(\tilde{A})\). With this and the assumption of locality, we must conclude that \(\tilde{A}\) possesses a definite value \(V(\tilde{A})\), which cannot depend on its measurement context. The symmetry of the system allows one to argue in a similar fashion to show that \(A\) also possesses a non-contextual value \(V(A)\). The invariance and symmetry of the maximally entangled state then imply that such an argument can be performed to show that all observables of subsystems 1 and 2 must possess definite values.

To confirm the ‘Schrödinger-Mermin’ nonlocality, one must perform perfect correlation tests in such a way that—in light of the above argument—it follows that all of the Mermin observables possess non-contextual values. Let us examine the following series of experiments. For each of the Mermin observables, perform the following series of experiments. First, we simultaneously perform \(\mathcal{E}(M_i)\) on subsystem 2 and \(\mathcal{E}(\tilde{M}_i)\) on subsystem 1 (The precise experiments required for each \(M_i\) are to be defined below.). Second, we simultaneously perform \(\mathcal{E}'(M_i)\) on subsystem 2 and \(\mathcal{E}'(\tilde{M}_i)\). If the perfect correlations are verified in both cases and for all Mermin observables then it follows that all of the Mermin observables of subsystem 2 must have definite values, and that these must be noncontextual. The latter follows since the perfect correlations are observed under conditions where the context of \(M_i\) is varied (i.e., from \(\mathcal{E}(M_i)\) to \(\mathcal{E}'(M_i)\)) while that of \(\tilde{M}_i\) is not.

In this way, one performs that part of the experiment which corresponds to the incompleteness argument of the nonlocality proof. One might suppose that what is required next is the performance of a separate series of tests to judge between the existence of the definite values implied by the perfect correlations, and the quantum mechanical predictions. As we saw above, Bell’s inequality
provides for the empirical difference between these two approaches in the case
of EPR/Bell nonlocality. In the present case, we have that Mermin’s theorem
provides such a difference: the noncontextual values must fail to satisfy the
relationships 4.56, 4.57, and 4.59, while quantum mechanics predicts that such
relationships are satisfied. To confirm the quantum mechanical predictions and
the presence of nonlocality in the maximally entangled state in question, it would
appear that one must check whether or not these relationships are satisfied.
However, as we shall see, if the measurements involved in the perfect correlations
test are done in a particular way, then the existence of perfect correlations is
sufficient in itself to prove such a conclusion. We now demonstrate this.

Consider the measurement of the commuting set \{A, \sigma_x^{(\alpha)}, \sigma_y^{(\beta)}\}. These ob-
servables are related by the first equation in 4.56, which we repeat here for
convenience:

\[ A = \sigma_x^{(\alpha)} \sigma_y^{(\beta)}. \]  

(4.61)

Suppose that to measure the set in question, we first measure the set \{\sigma_x^{(\alpha)}, \sigma_y^{(\beta)}\}. Then the values for \sigma_x^{(\alpha)} and \sigma_y^{(\beta)} so obtained are simply multiplied together to
determine the value of \( A \). One can, of course measure any commuting set
that obeys a constraining relationship in this way, i.e., any commuting s-
set \{O, O_1, O_2, \ldots\} where

\[ O = f(O_1, O_2, \ldots), \]  

(4.62)

can be measured by first performing an experiment to measure \{O_1, O_2, \ldots\},
and then evaluating \( f \) of the resulting values obtained, to obtain the value of
\( O \). If the set \{A, \sigma_x^{(\alpha)}, \sigma_y^{(\beta)}\} is measured in such a way then it is a priori that
the relationship 4.61 will be satisfied by the measurement results, since this
relationship is “built into” the very procedure itself, i.e., in such a procedure,
we use 4.61 in determining these results.

Now let us suppose that in all of procedures \( \mathcal{E}(M_i) \) involved in the perfect
correlation tests discussed above, the above described method is followed. That
it is possible to use this method in each case is clear from the fact that every
commuting set in 4.60 is constrained by one of the relationships of 4.56, 4.57,
and 4.59. If we follow such a method to measure the perfect correlations, then
it is a priori that the measurement results will obey the commuting relationships
4.56, 4.57, and 4.59. Since, as we have mentioned, it is the judgment of
whether or not these relationships are obeyed which is required to complete our
laboratory test of nonlocality, it follows that the perfect correlation tests by
themselves are sufficient for this test.

If, in fact, the perfect correlation test described above gives a ‘positive’
result, then the conclusion of quantum nonlocality necessarily follows. The
only local theoretical interpretation of such a result—the existence of definite
values—is immediately ruled out since (through Mermin’s theorem) it follows
that such values cannot satisfy the commuting relationships, which are a priori
obeyed when one performs the commuting set measurements in the fashion just discussed.

Moreover, the Schrödinger nonlocality that follows from either of the other theorems studied in chapter 2 (Gleason’s, and Kochen and Specker’s) can also be tested through the confirmation of perfect correlations. This follows since every observable among the set addressed by each theorem is a member of two ‘incompatible commuting sets’, where each set obeys a relationship of the form $O = f(O_1, O_2, \ldots)$. Thus, each observable can be measured by two distinct procedures $\mathcal{E}(O_i)$ and $\mathcal{E}'(O_i)$ for which the commuting relationships are a priori.

4.3 Schrödinger’s paradox and von Neumann’s no hidden variables argument

Lastly, we would like to make a few observations which are of historical interest. We now focus on the line of thought Schrödinger followed subsequent to his generalization of the Einstein–Podolsky–Rosen paradox. In addition, we consider the question of the possible consequences had he repeated the mistake made by a well-known contemporary, or had he anticipated any of several mathematical theorems that were developed somewhat later.

Having concluded the existence of definite values on the observables, Schrödinger considered the question of the type of relationships which might govern these values. As we discussed in section 1.4.6, he was able to show that no such values can obey the same relationships that constrain the observables themselves. In particular, he observed that the relationship

$$H = p^2 + a^2 q^2, \quad (4.63)$$

is not generally obeyed by the eigenvalues of the observables $H, p^2, q^2$, so that no value map $V(O)$ can satisfy this equation. From this it follows that there exists no value map which is linear on the observables. Thus we see that Schrödinger’s argument essentially leads to the same conclusion regarding hidden variables as von Neumann’s theorem. Schrödinger did not, however, consider this as proof of the impossibility of hidden variables.

Instead, the fact that the definite values of his EPR generalization must fail to obey such relationships prompted Schrödinger to consider the possibility that no relationship whatsoever serves to constrain them: [91] (emphasis due to original author) “Should one now think that because we are so ignorant about the relations among the variable-values held ready in one system, that none exists, that far-ranging arbitrary combination can occur?” Note, however, that if the ‘variable-values’ obey no constraining relationship, each must be an independent parameter of the system. Thus, Schrödinger continues with

\[\text{To show this, one need only note that the value of any observable } f(O) \text{ will be } f \text{ of the value of } O, \text{ where } f \text{ is any mathematical function (see section 1.4.6).}\]
the statement: “That would mean that a system of ‘one degree of freedom’ would need not merely two numbers for adequately describing it, as in classical mechanics, but rather many more, perhaps infinitely many.”

Recall the discussion of the hidden variables theory known as Bohmian mechanics presented in section 2.5. There we saw that the state description of a system is given in this theory by the wave function $\psi$ and the system configuration $\mathbf{q}$. The mathematical form of $\psi$ is the same as in the quantum formalism, i.e., $\psi$ is a vector in the Hilbert space associated with the system. Consider a spinless particle constrained to move in 1 dimension. The Bohmian mechanics state description would consist of the wave function $\psi(x) \in L^2$ and position $x \in \mathbb{R}$. Since any $L^2$ function $\psi(x)$ is infinite dimensional, i.e. it is an assignment of numbers to the points $x \in (-\infty, \infty)$, the Bohmian mechanics state description is infinite-dimensional. Thus, a theory of hidden variables such as Bohmian mechanics provides just the type of description that Schrödinger’s speculations had led him to conclude.

Let us now suppose that rather than reasoning as he did, Schrödinger instead committed the same error as von Neumann. In other words, we consider the possible consequences had Schrödinger regarded the failure of the definite values to satisfy the same relations as the observables as proof that no such values can possibly agree with quantum mechanics. This false result would appear to refute the conclusion of Schrödinger’s generalization of EPR: it would seem to imply that the Schrödinger paradox leads to a conflict with the quantum theory. The Schrödinger paradox, like the EPR paradox, assumes only that the perfect correlations can be explained in terms of a local theoretical model. Hence, if Schrödinger had concluded with von Neumann that hidden variables must conflict with quantum mechanics, then he would have been led to deduce quantum nonlocality.

As we saw above, such a conclusion actually follows from the combination of Schrödinger’s paradox with any of the spectral incompatibility theorems, for example, those of Gleason, Kochen and Specker, and Mermin. Since Gleason’s theorem involves the proof of the trace relation 2.3, it seems reasonable to regard it as being the most similar of these theorems to that of von Neumann, which features a derivation of this same formula\(^{22}\). Note that to whatever degree one might regard Gleason’s theorem as being similar to von Neumann’s, one must regard Schrödinger as having come to within that same degree of a proof of quantum nonlocality.

4.4 Summary and conclusions

Our investigation of the hidden variables issue was motivated by several mathematical results that have been interpreted as proofs either of the incompatibility

\(^{22}\)Von Neumann’s derivation is based on quite different assumptions, as we saw.
of hidden variables with the quantum theory, or of the existence of serious limitations on such theories. We have reviewed the arguments first presented by J.S. Bell, according to which not only do these theorems fail to demonstrate the impossibility of hidden variables, but the restrictions they place on such theories—contextuality and nonlocality—are quite similar to particular features of the quantum theory itself. When considering the theorems of Gleason and of Kochen and Specker, we found that they imply essentially that within any hidden variables theory, the way in which values are assigned each observable \( O \) must allow for contextuality—an attribute reflecting the quantum formalism’s rules of measurement. Nonlocality is certainly surprising and unexpected, yet it has been proved as a feature intrinsic to quantum mechanics from the combination of Bell’s theorem with the spin singlet version of EPR. We found also that from Erwin Schrödinger’s extensive generalization of the EPR paradox, it follows that every spectral-incompatibility theorem (such as the theorems of Gleason, Kochen and Specker, and Mermin) will give a new proof of “nonlocality without inequalities.”

We began our exposition by discussing the earliest work on hidden variables, John von Neumann’s. Von Neumann considered the possibility of a theory whose state description would supplement that of the quantum formalism with a parameter we called \( \lambda \). He regarded this scheme as a way to introduce determinism into the quantum phenomena. Mathematically, such determinism is represented by requiring that for each \( \psi \) and \( \lambda \), i.e., for each state, there exists a function assigning to each observable its value. Von Neumann showed that no function \( E(O) \) on the observables satisfying his assumptions can be such a value map. From this, he concluded that empirical agreement between any hidden variables theory and the quantum theory is impossible. That this conclusion is unjustified follows since one of von Neumann’s assumptions—that \( E(O) \) be linear on the observables—is quite unreasonable. There is no basis for the demand that \( E(O) \) obey \( O = X + Y \) where \([X, Y] \neq 0\), since each of these observables is measured by a distinct procedure.

The theorems of Gleason, Kochen and Specker, and Mermin seem at first to succeed where von Neumann failed, i.e., to prove the impossibility of hidden variables. Nevertheless, as Bell has shown [8, 16], these theorems also fail as arguments against hidden variables, since they do not account for contextuality. This concept is easily illustrated by examination of the quantum formalism’s rules of measurement. We find that the ‘measurement of an observable \( O \)’ can be performed using distinct experimental procedures \( \mathcal{E}(O) \) and \( \mathcal{E}'(O) \). That \( \mathcal{E}(O) \) and \( \mathcal{E}' \) are distinct is especially obvious if these measure the commuting sets \( \mathcal{C} \) and \( \mathcal{C}' \) where \( \mathcal{C} \) and \( \mathcal{C}' \) both contain \( O \), but the members of \( \mathcal{C} \) fail to commute with those of \( \mathcal{C}' \). It is therefore quite reasonable to expect that a hidden variables theory should allow for the possibility that different procedures for some observable’s measurement might yield different results for an individual system. That it is necessary to account for the detailed experimental arrangement recalls the views of Niels Bohr, who warns us of [90, page 210]
“the impossibility of any sharp separation between the behavior of atomic objects and the interaction with the measuring instruments which serve to define the conditions under which the phenomena appear.”

Examples of just such incompatible commuting sets are found among the observables in each of the theorems we addressed in chapter 2: Gleason’s, Kochen and Specker’s, and Mermin’s. For example, in the theorem of Kochen and Specker, the commuting sets are simply of the form \( \{ s_x^2, s_y^2, s_z^2 \} \), i.e., the squares of the spin components of a spin 1 particle taken with respect to some Cartesian axis system \( x, y, z \). Here one can see that a given observable \( s_x^2 \) belongs to both \( \{ s_x^2, s_y^2, s_z^2 \} \), and \( \{ s_x^2, s_y^2', s_z^2' \} \), where the \( y \) and \( z \) axes are oblique relative to the \( y \) and \( z \) axes. Since the theorems of Gleason, Kochen and Specker, and Mermin consider a function \( E(O) \) which assigns a single value to each observable, they cannot account for the possibility of incompatible measurement procedures which the quantum formalism’s rules of measurement allows us. Clearly, the approach taken by these theorems falls far short of addressing the hidden variables issue properly. Thus, we come to concur with J.S. Bell’s assessment that [16]

“What is proved by impossibility proofs . . . . . . is lack of imagination.”

To address the general question of hidden variables, one must allow for this important feature of contextuality. The theorems of Gleason, Kochen and Specker, and Mermin may be seen as explicit proofs that this simple and natural feature is necessary in any hidden variables theory. In particular, any attempt to construct a value map that neglects this feature will fail in that it cannot satisfy the requirement that the relationships constraining the commuting sets must be obeyed. We have seen that there is a simpler way to express this result: there exists no value map on the observables mapping every commuting set to one of its joint-eigenvalues.

We were able to gain some measure of additional insight into contextuality in examining Albert’s example. As one can see from the experiments considered by Albert, the Hermitian operators cannot be considered as representing properties intrinsic to the quantum system itself. Instead, the results of the “measurement of a quantum observable” must be considered as the joint-product of system and measuring apparatus.

When we come to consider Bell’s theorem, we must do so in the context of the spin singlet version of the Einstein–Podolsky–Rosen paradox. This argument essentially shows that locality necessarily leads to the existence of definite values for all components of the spin of both particles of the spin singlet state. Since the quantum mechanical description of the state does not account for such values, EPR conclude that this description is incomplete. Bell’s theorem essentially continues where the spin singlet EPR analysis concluded. The fixed values for the spin components of the two particles are represented by functions \( A(\lambda, \hat{a}) \), \( B(\lambda, \hat{b}) \) where \( \hat{a}, \hat{b} \) are unit vectors in the directions of the axis of the spin component for particle 1 and 2 respectively. In his analysis, Bell considers the statistical correlation between spin component measuring experiments
carried out on the two particles. According to Bell’s theorem, the theoretical prediction for this correlation derivable from the variables \( A(\lambda, \hat{a}) \) and \( B(\lambda, \hat{b}) \) must satisfy ‘Bell’s inequality’. The prediction given by quantum mechanics does not generally agree with this inequality. Bell’s theorem in itself provides a proof that local hidden variables must conflict with quantum mechanics.

It is important to note that what might appear to be Bell’s assumption—the existence of definite values for all spin components—is identical to the conclusion of the spin singlet version of the Einstein–Podolsky–Rosen argument. In fact, Bell assumes nothing beyond what follows from EPR. Therefore the proper way to assess the implications of these arguments is to combine them into a single analysis that will begin with the assumptions of the spin singlet EPR paradox, and end with the conclusion of Bell’s theorem; i.e., the assumption of locality leads to the conclusion of Bell’s inequality. Since this inequality, as we saw, disagrees with the quantum theory, we finally have an argument that locality implies disagreement with the predictions of quantum mechanics. This agrees with Bell’s own assertion in the matter: [14] “It now seems that the non-locality is deeply rooted in quantum mechanics itself and will persist in any completion.”

We have now come to understand the issues of contextuality and nonlocality as features of a hidden variables interpretation of quantum theory. Contextuality in hidden variables is a natural feature to expect since it reflects the possibility of distinct experimental procedures for measurement of a single observable. As J.S. Bell expresses it: [8] “The result of an observation may reasonably depend not only on the state of the system (including hidden variables) but also on the complete disposition of the apparatus.” According to the above results, the fact that nonlocality is required of hidden variables does not in any way diminish the prospect of these types of theories. As we have mentioned, there has existed a successful theory of hidden variables since 1952, and there is no reason not to consider this theory (Bohmian mechanics) as a serious interpretation of quantum mechanics. We noted in section 1.1.2 that Bohmian mechanics possesses the advantages of objectivity and determinism.

In the fourth and final chapter, we addressed Erwin Schrödinger’s generalization of the EPR paradox. Besides greatly extending the incompleteness argument of EPR, Schrödinger’s analysis provides for a new set of “nonlocality without inequalities” proofs, which have several important features. The Schrödinger paradox concerns the perfect correlations exhibited not just by a single quantum state, but for a general class of states called the maximally entangled states. A maximally entangled state is any state of the form

\[
\sum_{n=1}^{N} |\phi_n\rangle \otimes |\psi_n\rangle,
\]

where \( \{ |\phi_n\rangle \} \) and \( \{ |\psi_n\rangle \} \) are bases of the (N-dimensional) Hilbert spaces of subsystems 1 and 2, respectively. As in the EPR analysis (both the spin singlet and the original version) Schrödinger gives an incompleteness argument,
according to which there must exist precise values for all observables of both subsystems. It may be shown using Gleason’s theorem, Kochen and Specker’s theorem, Mermin’s theorem, or any other ‘spectral incompatibility’ proof, that the definite values concluded in the Schrödinger’s paradox must conflict with the empirical predictions of quantum mechanics. The implication of such a disagreement is quantum nonlocality. This conflict in empirical predictions differs from that developed within Bell’s theorem, in that it involves predictions for individual measurements, rather than the statistics of a series of measurements. Thus, combining Schrödinger’s paradox with any spectral incompatibility theorem provides a ‘nonlocality without inequalities’ proof. Moreover, as we observed in section 4.2.1, the conflict between the Schrödinger incompleteness and such a theorem exists even when one considers only the observables of one of the two subsystems. We have referred to this type of proof by the name “Schrödinger nonlocality.”

When we consider the experimental verification of Schrödinger nonlocality, we find a curious result. The measurement of a commuting set satisfying an equation of constraint may be carried out in such a way that the set of values obtained will satisfy this constraint a priori. Since the definite values concluded in the Schrödinger paradox cannot satisfy these constraining relationships, performing an experimental test using such measurement procedures provides that the perfect correlations themselves are sufficient to imply nonlocality.

All this leads us to inquire how Schrödinger himself regarded his results, and what further conclusions he drew within his remarkable paper. Clearly, it would have been possible for him to argue for quantum nonlocality had he anticipated the results of any of a wide variety of theorems including at least Bell’s (see section 4.1.4), Gleason’s, Kochen and Specker’s, Mermin’s, or any other spectral incompatibility theorem. Instead, as we saw in chapter 1, Schrödinger essentially reproduced the von Neumann argument against hidden variables, in his observation of a set of observables that obey a linear relationship not satisfied by the set’s eigenvalues. What may be seen from von Neumann’s result is just what Schrödinger noted—relationships constraining the observables do not necessarily constrain their values. Schrödinger continued this line of thought by speculating on the case for which no relation whatsoever constrained the values of the various observables. In light of his generalization of the EPR paradox, this line of thought led Schrödinger to the idea that the quantum system in question might possess an infinite number of degrees of freedom, which concept is actually quite similar to that of Bohmian mechanics.

If, on the other hand, Schrödinger had made von Neumann’s error, i.e., had concluded the impossibility of a map from observables to values, this mistaken line of reasoning would have permitted him to arrive at the concept of quantum nonlocality. Thus, insofar as one might regard the von Neumann proof as “almost” leading to the type of conclusion that follows from Gleason’s theorem, one must consider Schrödinger as having come precisely that close to a proof of quantum nonlocality.
It is quite interesting to see that many of the issues related to quantum mechanical incompleteness and hidden variables were addressed in the 1935 work [91, 92, 93] of Erwin Schrödinger. Schrödinger’s work seems to be the most far reaching of the early analyses addressed to the subject. Not only did he see deeper into the problem than did von Neumann, but Schrödinger developed results beyond those of the Einstein, Podolsky, Rosen paper—an extension of their incompleteness argument, and an analysis of incompleteness in terms of the implication of von Neumann’s theorem. It seems clear that the field of foundations of quantum mechanics might have been greatly advanced had these features of Schrödinger’s paper been more widely appreciated at the time it was first published.
Chapter 5

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