Study of Gauge Symmetry of Some Field Theoretical Models Through the Lagrangian Formulation

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Study of gauge symmetry is carried over the different interacting and noninteracting field theoretical models through a prescription based on lagrangian formulation. It is found that the prescription is capable of testing whether a given model posses a gauge symmetry or not. It can successfully formulate the gauge transformation generator in all the cases whatever subtleties are involved in it. It is found that the prescription has the ability to show a direction how to extend the phase space using auxiliary fields to restore the gauge invariance of a theory. Like the usual phase space the prescription is found to be equally powerful in the extended phase space of a theory.

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I. INTRODUCTION

Every basic interaction is supposed to have their origin from the gauge principle and understanding of the gauge symmetry of a physical theory is a very important problem which has received much attention to the physicist from the long past. In a gauge theory, their exists some transformation that leaves physical content of the theory invariant. It even stands as a fundamental principle that determines the form of lagrangian of a theory. Two main approaches have been followed in the literature to study the local symmetry of the lagrangian of the gauge theories. The oldest one is the hamiltonian formulation based on Dirac conjecture [1–3]. Several authors have tried to find out the answer of several interesting questions related to the gauge symmetry using hamiltonian formulation [4–13]. The most general form of gauge transformation generator too can be determined with that hamiltonian formulation. To study BRST symmetry, hamiltonian approach also has been found to be instrumental [14–21].

It is true that unitarity of a theory can not be well understood without hamiltonian approach. However, hamiltonian embedding of constrained system has some drawbacks. It does not always lead to Lorentz covariant generating functional. This drawback indeed has the remedy in the lagrangian formulation. So the importance of the study of gauge symmetric property through the formalism based on lagrangian formulation can not be ignored. Therefore, gauge symmetry related studies on dynamical theory should be extended with equal intensity in both the approaches. Few studies using lagrangian approach are available in the literature [22–24]. However, before Shirzad very little was achieved to understand the fundamental question related to the gauge symmetry in the lagrangian formulation. In [24], Shirzad gave a systematic development of gauge symmetry related study for an arbitrary lagrangian and applied it to the so called generalized Schwinger model [23]. It is fair to admit that not much attention have been paid in this direction after that. So application of this formalism on the different field theoretical models in order to test whether a given model does have gauge symmetry or it is lacking in it and hence to find out the appropriate gauge transformation for that theory would be of interest. It would be much more interesting to apply this formalism in the extended phase space needed to restore the gauge invariance and to verify whether this formalism works there in an appropriate manner as it was found to work in the usual phase space in [24] would certainly be an interesting subject of study.

The aim of this paper is therefore, to study the different gauge symmetric and gauge non symmetric (anomalous) models with the prescription based on lagrangian formulation developed by Shirzad in [24]. We are intended to investigate whether Shirzad’s formalism enables one to verify the presence or absence of gauge symmetry in a given theory. One reasons behind the consideration of anomalous model is to ensure whether this scheme is capable of testing the absence of gauge symmetry when it is lacking in a given model. The another reason is to study the power of this approach towards its applicability in the extended phase space. Thus the application of this approach on

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anomalous models when shows the absence of gauge symmetry, it is immediately modified in such a way that the
gauge symmetry gets restored and hence apply the formalism to find out the correct gauge transformations in that
situation.
The paper is organized as follows. In Sec. II, we have presented briefly the general formalism to study the gauge
symmetry of a singular lagrangian developed by Shirzad in [24]. In Sec. III, we apply the above formulation on
few non-interacting models. The models which are not gauge invariant, are made gauge invariant by adding some
appropriate terms involving auxiliary fields and hence calculate the appropriate gauge transformation generators. Sec.
IV, is dealt with two interacting field theoretical model e. g., Schwinger model with mass like term for gauge field
[12, 13] and Chiral Schwinger model with Fadeevian [10, 11] anomaly in the same perspective. The last section is
containing a conclusion.

II. A BRIEF DISCUSSION OF SHIRZAD’S FORMALISM

In order to make this paper self contained we would like give a brief account of the formalism developed by Shirzad
[24] in this section. If a dynamical system with N degrees of freedom is considered which is described by the lagrangian,

\[ L = L(q_i, \dot{q}_i), \]  

the Euler equations of motion for that lagrangian will be

\[ L_i = w_{ij} \dot{q}_j + \alpha_i. \]  

Here i=1,2.....N. The matrix \( w \) stands for the Hessian matrix of the system. The Hessian matrix \( w_{ij} \) and \( \alpha_i \) of equation \( 2 \) respectively are

\[ w_{ij} = \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}, \]

\[ \alpha_i = \frac{\partial^2 L}{\partial \dot{q}_i \partial q_i} \dot{q}_i - \frac{\partial L}{\partial q_i}. \]

For a singular lagrangian \( \text{det}[w_{ij}] = 0 \). The equations of motion in this situation can not be solved for all accelerations.

If the rank of \( w \) is \( N - A_1 \) then \( A_1 \) number of null eigen vector will be found for the matrix \( w_{ij} \).

\[ \lambda_a^{i1} w_{ij} = 0, \]

where \( a_1 = 1.........A_1 \). Here \( \lambda_a^{i1} \) indicates the null eigen vector. When equation \( 2 \) is multiplied by the null eigen

\[ \gamma^{a1} = \lambda_a^{i1} L_i = \lambda_a^{i1} \alpha_i = 0. \]

This indicates the presence of \( A_1 \) number of lagrangian constraints of velocity and coordinates, but all of these

\[ \gamma^{a1}(q, \dot{q}) = \sum_{a_1=1}^{A_1} C^{a1}_{a_1}(q, \dot{q}) \gamma^{a1}(q, \dot{q}), \]

where \( a_1 = 1,......A_1 \), and \( C^{a1}_{a_1} \) represents the coefficients which may depend on \( q_i \) and \( \dot{q}_i \). These set of lagrangian

\[ \sum_{a_1=1}^{A_1} C^{a1}_{a_1}(q, \dot{q}) \gamma^{a1}(q, \dot{q}) = 0, \]

where \( a_1 = 1,......A_1 \). These are of course linear combinations of \( \gamma^{a1} \)'s and their number will be \( \tilde{A}_1 = A_1 - A_1 \). This

\[ \lambda^{a1}(q, \dot{q}) = \sum_{a_1=1}^{A_1} C^{a1}_{a_1}(q, \dot{q}) \gamma^{a1}(q, \dot{q}). \]
Using equation (9), primary constraints can be calculated and these are given by

\[ \gamma_{\hat{a}i}(q, \dot{q}) = \lambda_{i1}^{\hat{a}i} L_i. \]  

(10)

Equating equation (10) and (8) one obtain

\[ \lambda_{\hat{a}i} = \sum_{a_1=1}^{A_1} A_{a_1}^{i} C_{a_1}^{\hat{a}i} (q, \dot{q}) \lambda_{a_1}^{\hat{a}i} (q, \dot{q}). \]  

(11)

Equation (11), represents \( \hat{A}_1 \) number of null eigen vector. Now identities of the Euler derivatives appears as

\[ \lambda_{i1}^{\hat{a}i} L_i = 0. \]  

(12)

In order to get a consistent theory, the time derivatives of the primary constraint (10) is to be added to the equation of motion (2). Therefore, one gets \( N + \hat{A}_1 \) number of equations that contain accelerations which can be written in a combined manner as follows

\[ L_{i1} = w_{i1,1} \dot{q}_j + \alpha_{i1} = 0, \]  

(13)

where \( i_1 = 1 \ldots \ldots N + \hat{A}_1 \). Here \( \hat{A}_1 \) represents the rank of time derivatives of \( \gamma_{\hat{a}i} \). The matrix \( w_{i1,1} \) may also contain some other null eigen vector like the previous one. Following the previous process one gets new null eigen vector \( \lambda_{a}^{\hat{a}2} \), and the expressions of it is

\[ \gamma_{\hat{a}2}(q, \dot{q}) = \lambda_{i2}^{\hat{a}2} L_i^{1} = 0, \]  

(14)

where \( a_2 = 1 \ldots \ldots \hat{A}_2 \). So, one finds \( \hat{A}_2 \) number of independent functions \( \gamma_{\hat{a}2} \) and \( \hat{A}_2 \) number of identities \( \gamma_{\hat{a}2} \) for \( \gamma_{\hat{a}2} \), standing in equation (14). In the next step the time derivative of secondary constraint is to be added to the equation (13), as it is done for the former set of constraint in order to maintain consistency. This gives,

\[ L_{i2} = w_{i2,1} \dot{q}_j + \alpha_{i2} = 0, \]  

(15)

where \( i_2 = 1 \ldots \ldots N + \hat{A}_1 + \hat{A}_2 \). Here \( \hat{A}_2 \) stands for the rank of time derivative of secondary constraint. In this way one needs to proceed step by step. In each step some identities along with some new constraints may results. Finally, in the \( n^{th} \) stage the equations of motion for the system will be of the form

\[ L_n = w_{in,1} \dot{q}_j + \alpha_{in} = 0, \]  

(16)

where \( i_n = 1 \ldots \ldots N + \hat{A}_1 + \hat{A}_n \). If in this case one finds new null eigen vector \( \lambda_{i+1}^{a_n+1} \) for \( w^n \) and multiplication of \( \lambda_{i+1}^{a_n+1} \) with equation (16) provides \( \hat{A}_n+1 \) number of lagrangian constraints and \( \hat{A}_n+1 \) number of identities and these will hold the following relation.

\[ \lambda_{i+1}^{a_n+1} L_i + \lambda_{i+1}^{a_n+1} \frac{d^2 \gamma_{l1}}{dt^2} + \ldots + \lambda_{i+1}^{a_n+1} \frac{d^2 \gamma_{l2}}{dt^2} = 0. \]  

(17)

Equation (17) can be written down in the form of a total derivative as follows

\[ \sum_{s=0}^{n} \frac{d^s}{dt^s} (\phi_{si} L_i) = 0, \]  

(18)

where \( \phi_{si} \) are some functions of coordinate and their derivatives. That can be determined with a judicious choice. If \( w^n \) does not give any new eigen vector, it indicates that the process gets terminated. Another way of testing the termination of the procedure is to check whether the \( n^{th} \) step gives any new constraint or not. The appearance of no new constraint too indicate the termination of the process. For the lagrangian \( L(q_i, \dot{q}_i) \) the action is found to be invariant under the following transformation,

\[ \delta q_i = \sum_{s=0}^{n} (-1)^s \frac{d^s}{dt^s} f \phi_{si}, \]  

(19)

if \( \phi_{si} \) exists for that particular dynamical system represented by the \( L(q_i, \dot{q}_i) \), where \( f(t) \) is an arbitrary function of time. The variation of lagrangian under the transformation (19) is given by

\[ \delta L = \left[ \sum_{s=0}^{n} \frac{d^s}{dt^s} \phi_{si} L_i \right] f = 0. \]  

(20)
If the Lagrangian of a dynamical system is described by the set of fields \( q_i(x, t) \) the general form of the lagrangian reads

\[
L = \int dx L(q_i(x, t), \partial_x q_i(x, t), \partial_t q_i(x, t)).
\]  

(21)

The equations of motion in this situation becomes

\[
L_i(x, t) = \int dy w_{ij}(x, y) \dddot{q}_j(y, t) + \alpha_i(x, t),
\]

(22)

where \( i = 1, \ldots, N \). \( N \) here represents the number of fields describing the dynamical system. \( w(x, y, t) \) and \( \alpha(x, t) \) in this situation takes the form

\[
w_{ij}(x, y, t) = \delta^2 L \frac{\delta}{\delta q_j(x, t)} \frac{\delta}{\delta q_i(x, t)},
\]

(23)

\[
\alpha_i(x, t) = \int dy \left( \frac{\delta^2 L}{\delta q_j(y, t)} \frac{\delta}{\delta q_i(x, t)} - \frac{\delta L}{\delta q_i(x, t)} \right).
\]

(24)

The null eigen vector of the Hessian matrix \( w(x, y, t) \) here looks

\[
\lambda^a_x(x) = \lambda^a \delta(z - x).
\]

(25)

Multiplication of \( \lambda^a_x(x) \) with equation of motion gives primary lagrangian constraint as follows

\[
\gamma^a = \int dx \lambda^a_x \delta(z - x)L_i(x, t) = \lambda^a L_i(z, t).
\]

(26)

If the process continues in the similar manner as it is described earlier keeping in mind that the system is described by field then one will arrive at the following gauge transformation formula

\[
\delta q_i(x, t) = \sum_{\alpha=1}^m n_s \sum_{s=0}^n (-1)^s \int dz \frac{\delta f_{\alpha}(z, t)}{\delta t^s} \phi^\alpha_{si}(z, x).
\]

(27)

III. NON-INTERACTING FIELDS E. G., FREE MAXWELL LAGRANGIAN, MAXWELL LAGRANGIAN WITH MASS LIKE TERM FOR THE GAUGE FIELD AND FREE CHIRAL BOSON

Let us consider the lagrangian density of Free Maxwell field

\[
\mathcal{L}_{FM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}.
\]

(28)

In \((1 + 1)\) dimension the lagrangian density reads

\[
\mathcal{L}_{FM} = \frac{1}{2}(\dot{A}_1^2 + \dot{A}_0^2 - 2A_0' \dot{A}_1).
\]

(29)

The equations of motion for the field \( A_0 \) and \( A_1 \) that come out from equation (2) for the lagrangian \( \mathcal{L}_{FM} \) are

\[
L_{A_0} = A_0'' - \dot{A}_1',
\]

(30)

\[
L_{A_1} = \dot{A}_1 - \dot{A}_0'..
\]

(31)

For the lagrangian (29) \( w \) and \( \alpha \) respectively are

\[
w = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \delta(z - x),
\]

(32)
\[ \alpha = \begin{pmatrix} A_0'' - A_1' \\ -A_0' \end{pmatrix}. \]  

(33)

It is found that Hessian matrix \( w \) has a null eigen vector

\[ \lambda^1(z) = (1, 0) \delta(z - x). \]  

(34)

Multiplying the equation (32) from the left by \( \lambda^1 \), we get the primary lagrangian constraint

\[ \gamma^1(z, t) = \int dx \delta(z - x)L_{A_0}(x, t) = L_{A_0}(z, t) = (A_0'' - A_1')(z, t). \]  

(35)

Time derivatives of \( \gamma^1 \) yields

\[ \frac{\partial}{\partial t} \gamma^1(x, t) = (A_0'' - A_1')(x, t). \]  

(36)

According to Shirzad’s prescription equation (36) is to be added with (29) in order to maintain consistency condition of the primary constraint and that results

\[ L_1(x, t) = (L_{FM} + \frac{\partial}{\partial t} \gamma^1)(x, t). \]  

(37)

\( w^1 \) and \( \alpha^1 \) as standing in equation (33) and (34) are found out for this system as

\[ w^1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & -\partial/\partial x \end{pmatrix} \delta(z - x). \]  

(38)

\[ \alpha^1 = \begin{pmatrix} A_0'' - A_1' \\ -A_0' \\ A_0'' \end{pmatrix}. \]  

(39)

A straight forward calculation shows that \( w^1 \) also has a null eigen vector

\[ \lambda^2(z) = (0, \partial/\partial x, 1) \delta(z - x). \]  

(40)

Multiplying the equation (37) from left by \( \lambda^2 \) we find that \( \gamma^2 \) comes out to be zero. Explicitly,

\[ \gamma^2(z, t) = \int dx \lambda^2 L_{i_1}(x, t) = \lambda^2 \alpha_1(z, t) = 0. \]  

(41)

So, \( \lambda^2 \) does not give rise to any new constraint. As a result we can not increase the the rank of equation for accelerations and \( \gamma^2 \) can be expressed in the following form

\[ \gamma^2(x, t) = (\frac{\partial}{\partial t} L_{A_0} + \frac{\partial}{\partial x} L_{A_1})(x, t). \]  

(42)

Comparing (12) with equation (18) we get the non vanishing \( \phi_{s_1} \)’s:

\[ \phi_{1, 1}(z, x) = \delta(z - x), \]  

(43)

\[ \phi_{0, 2}(z, x) = \frac{\partial}{\partial z} \delta(z - x). \]  

(44)

The gauge transformation formula (27) gives the following gauge transformation for the field \( A_0 \) and \( A_1 \):

\[ \delta A_0 = - \int dz \frac{\partial}{\partial t} f(z, t) \delta(z - x) = - \frac{\partial}{\partial t} f(x, t), \]  

(45)
\[ \delta A_1 = - \int dz \left( \frac{\partial}{\partial z} \delta(z-x) \right) f(z,t) = \frac{\partial}{\partial x} f(x,t). \]  

(46)

A little algebra shows that the variation of \( L_{FM} \) is

\[ \delta L_{FM}(x,t) = - \sum_{s=0}^{n} \frac{\partial}{\partial t^s} (\Phi_{x_i} L_i)(x,t) \]

\[ = - \left[ \frac{\partial}{\partial t} L_{A_0} + \frac{\partial}{\partial x} L_{A_1} \right](x,t) = 0. \]

It is shows that the lagrangian (29) is invariant under the gauge transformation (45) and (46). It is the expected result since it is known that the lagrangian (29) is invariant under the transformation \( A_\mu \rightarrow A_\mu - \partial_\mu f \).

### A. Maxwell lagrangian with mass like term

Let us now add the mass like term \( \frac{a^2}{2} A_\mu A^\mu \) with the Maxwell lagrangian and apply the formalism to test whether it has the gauge symmetry or not. So lagrangian with which we are going to start start our analysis is

\[ L_{MM} = \int \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{a^2}{2} A_\mu A^\mu \right) dx. \]

(47)

In (1 + 1) dimension the lagrangian density takes the following form

\[ L_{MM} = \frac{1}{2} (\dot{A}_1 - A_0')^2 + \frac{a^2}{2} (A_0^2 - A_1^2). \]

(48)

The equations of motion for the field \( A_0 \) and \( A_1 \) that come out from equation (2) for the lagrangian under consideration are

\[ L_{A_0} = A_0'' - A_1' - a^2 A_0, \]

(49)

\[ L_{A_1} = \dot{A}_1 - A_0' + a^2 A_1. \]

(50)

The matrices \( w \) and \( \alpha \) for this modified lagrangian come out to be

\[ w = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \delta(z-x), \]

\[ \alpha = \begin{pmatrix} A_0'' - \dot{A}_1' - a^2 A_0 \\ \dot{A}_0' + a^2 A_1 \end{pmatrix}. \]

(51)

(52)

The Hessian matrix \( w \) here gives the following null eigen vector

\[ \lambda^1(z) = (1, 0) \delta(z-x). \]

(53)

Equation (48) when multiplied from left by \( \lambda^1 \), it results the primary lagrangian constraint

\[ \gamma^1(x,t) = (A_0'' - \dot{A}_1' - a^2 A_0)(x,t) = L_{A_0}(x,t). \]

(54)

We need the time derivatives of \( \gamma^1 \) to calculate \( L_1(x,t) \):

\[ \frac{\partial}{\partial t} \gamma^1(x,t) = (\dot{A}_0'' - \ddot{A}_1' - a^2 \dot{A}_0)(x,t). \]

(55)

Now adding equation (55) with \( L_{MM} \) we get

\[ L_1(x,t) = (L_{MM} + \frac{\partial}{\partial t} \gamma^1)(x,t). \]

(56)
This $L_1(x, t)$ is to be used for further analysis in order to maintain consistency of the primary constraint (54). Using equation (3) and (4) the matrices $w^1$ and $\alpha^1$ for this system are calculated as follows.

$$w^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\partial/\partial x & 0 \end{pmatrix} \delta(z - x),$$

$$\alpha^1 = \begin{pmatrix} A_0'' - \dot{A}_1' - a^2 A_0 \\ -\dot{A}_1' + a^2 A_1 \\ A_0'' - a^2 \dot{A}_0 \end{pmatrix}.$$  \hspace{1cm} (57)\hspace{1cm} (58)

A straightforward calculation shows that $w^1$ has another null eigenvector

$$\lambda^2(z) = (0, \partial/\partial x, 1) \delta(z - x).$$

Multiplying the equation (56) from left by $\lambda^2$, we get

$$\gamma^2(x, t) = \int dx \lambda^2_2(x, t) = a^2 (A_1' - \dot{A}_0)(x, t).$$

which is non vanishing one. This $\lambda^2$ is nothing but the secondary constraint for this system. For maintaining consistency we have added the time derivative of secondary constraint to the equation (56) and obtain

$$L_2(x, t) = (L_1 + \partial/\partial t \gamma^2)(x, t).$$

The matrix $w^2$ and $\alpha^2$ standing in equation (15) will be the following for this particular situation.

$$w^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -a^2 & -\partial/\partial x & 0 \end{pmatrix} \delta(z - x),$$

$$\alpha^2 = \begin{pmatrix} A_0'' - \dot{A}_1' - a^2 A_0 \\ -\dot{A}_1' + a^2 A_1 \\ A_0'' - a^2 \dot{A}_0 \end{pmatrix}.$$  \hspace{1cm} (62)\hspace{1cm} (63)

We find that $w^2$ again gives a null eigenvector

$$\lambda^3(z) = (0, \partial/\partial x, 1, 0) \delta(z - x).$$

Multiplying the equation (61) from left by $\lambda^3$, we obtain

$$\gamma^3(x, t) = a^2 (A_1' - \dot{A}_0)(x, t) = \gamma^2(x, t).$$

Note that $\gamma^3(x, t)$ and $\gamma^2(x, t)$ are identical. So multiplication of $\lambda^3$ with $L_2$ does not provide any new constraint. Therefore, we can not increase the the rank of equation for accelerations. Let us now try to write $\gamma^3$ in the form of equation (17).

$$\gamma^3(x, t) = (\partial/\partial t L_{A_0} + \partial/\partial x L_{A_1})(x, t).$$

Equating equation (66) with (18) we get $\phi_{si}$’s and the non vanishing $\phi_{si}$’s are found out to be

$$\phi_{1,1}(z, x) = \delta(z - x),$$

$$\phi_{0,2}(z, x) = \partial/\partial z \delta(z - x).$$
The gauge transformation formula (27) gives the following gauge transformation for the field $A_0$ and $A_1$

\[ \delta A_0 = \int dz \frac{\partial}{\partial t} f(z, t) \delta(z - x) = - \frac{\partial}{\partial t} f(x, t), \]

(69)

\[ \delta A_1 = - \int dz \left( \frac{\partial}{\partial z} \delta(z - x) \right) f(z, t) = \frac{\partial}{\partial x} f(x, t). \]

(70)

The variation of the lagrangian density (48) under the variation of the fields (69) and (70) is

\[ \delta L(x, t) = - \sum_{s=0}^{n} \frac{d^s}{dt^s} (\Phi_{si} L_i) f(x, t) \]

\[ = - \left( \frac{d}{dt} L_{A_0} + \frac{d}{dx} L_{A_1} \right) f(x, t) \]

\[ = - a^2 (A_1' - A_0) f(x, t). \]

(71)

Since $\delta L_{MM}$ does not vanish, there is no gauge symmetry of the lagrangian density (48). The result here to does not go beyond our expectation since it is known that the presence of mass like term breaks the gauge invariance of the free Maxwell theory. In the following section we will proceed to study the application of the Shirzad’s formalism in the extended phase space of this system.

B. Maxwell Lagrangian with mass like term made Gauge invariant

We have seen in the previous Section that the lagrangian density (48) is not invariant under the Gauge transformation (69) and (70). So we add some terms involving auxiliary fields $\theta$ with the lagrangian (48) in order to make right hand side of the equation (71) zero. Lagrangian density under consideration along with with the appropriate terms needed to make equation (71) zero is

\[ L_{EM} = \frac{1}{2} (A_1 - A_0')^2 + \frac{ae^2}{2} (A_0^2 - A_1^2) + \frac{a}{2} (\theta^2 - \theta'^2) + ae(\dot{\theta} A_0 - \theta' A_1). \]

(72)

Note that the term is nothing but the Wess-Zunino term which we have needed to add to make equation (71) zero. The equations of motion for the field $A_0$, $A_1$ and $\theta$ are

\[ L_{A_0} = A_0'' - \ddot{A}_1' - ae^2 A_0 - ae\dot{\theta}, \]

(73)

\[ L_{A_1} = \ddot{A}_1 - \ddot{A}_0' + ae^2 A_1 + ae\theta', \]

(74)

\[ L_{\theta} = a\ddot{\theta} - a\theta'' + ae\dot{A}_0 - ae\dot{A}_1'. \]

(75)

Here we repeat the same calculation as before and $w$ and $\alpha$ for this lagrangian are found out to be

\[ w = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \end{pmatrix} \delta(z - x), \]

(76)

\[ \alpha = \begin{pmatrix} A_0'' - A_1' - ae^2 A_0 - ae\dot{\theta} \\ -\dot{A}_0' + ae^2 A_1 + ae\theta' \\ -a\theta'' + ae\dot{A}_0 - ae\dot{A}_1' \end{pmatrix}. \]

(77)

We also find that the Hessian matrix $w$ has the null eigen vector

\[ \lambda^1(z) = \left( 1, 0, 0 \right) \delta(z - x). \]

(78)
Like the massless situation we calculate the primary lagrangian constraint in this situation too:

$$\gamma^1(x, t) = (A_0'' - \dot{A}_1 - ae^2 A_0 - ae\dot{\theta})(x, t).$$  (79)

$L_1$ in this situation is obtained as

$$L_1(x, t) = (L_{EM} + \frac{\partial}{\partial t} \gamma^1)(x, t).$$  (80)

Here $w^1$ and $\alpha^1$ too are found out using equation (3) and (4).

$$w^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a \\ 0 & -\partial/\partial x & -ae \end{pmatrix} \delta(z - x),$$  (81)

$$\alpha^1 = \begin{pmatrix} A_0'' - \dot{A}_1 - ae^2 A_0 - ae\dot{\theta} \\ -\dot{A}_1 + ae^2 A_0 + ae\dot{\theta}' \\ -ae^2 A_0 + ae\dot{A}_1 \\ A_0'' - ae^2 A_0 \end{pmatrix}.$$  (82)

A little algebra shows that $w^1$ has the following null eigen vector

$$\lambda^2(z) = \{ 0, \partial/\partial x, +e, 1 \} \delta(z - x).$$  (83)

Multiplying the equation (80) from left by $\lambda^2$, we find that

$$\gamma^2(x, t) = 0.$$  (84)

So, $\lambda^2$ does not give rise to any new constraint. Therefore, the process gets terminated. Thus the increase of the rank of equation for acceleration is not possible here. $\gamma^2$ here also can be written in the form of equation (17) as follows

$$\gamma^2(x, t) = (\frac{\partial}{\partial t} L_{A_0} + \frac{\partial}{\partial x} L_{A_1} + eL_{\theta})(x, t).$$  (85)

Comparing equation (85) with (18) we get the non vanishing components of $\phi_{si}$'s:

$$\phi_{1,1}(z, x) = \delta(z - x),$$  (86)

$$\phi_{0,2}(z, x) = \frac{\partial}{\partial z} \delta(z - x),$$  (87)

$$\phi_{0,3}(z, x) = +e\delta(z - x).$$  (88)

Finally we find the gauge transformation of the fields for the system with the help of equation (27).

$$\delta A_0 = -\frac{\partial}{\partial t} f(x, t),$$  (89)

$$\delta A_1 = -\frac{\partial}{\partial x} f(x, t),$$  (90)

$$\delta \theta = +ef(x, t).$$  (91)

The variation of the lagrangian density (72) under the transformation (89), (90) and (91) comes out to be

$$\delta L_{EM} = -\sum_{s=0}^{n} \frac{d}{dt} (\phi_{si} L_i)f(x, t)$$

$$= -[\frac{\partial}{\partial t} L_{A_0} + \frac{\partial}{\partial x} L_{A_1} + eL_{\theta}]f(x, t) = 0.$$  (92)

Equation (92) confirms that the action is invariant under the gauge transformation (89), (90) and (91). Note that the formalism shows its successful application in the extended phase space of this simple non interacting system. In the following section the formalism is again applied to another noninteracting field theory e.g., Chiral boson which is known as a basic ingredient of heterotic string theory.
C. Free Chiral Boson

Free Chiral Boson[7, 26–28] though a very simple field theory the study of Gauge symmetry for this system is very subtle and interesting because the lagrangian of Chiral Boson contains a second class constraint \( (\partial_0 + \partial_1)\phi = 0 \). So it is studied here using Shirzad’s formalism. Lagrangian density of free Chiral Boson as described in [28, 29] is given by

\[
L_{CB} = \frac{1}{2} (\dot{\phi}^2 - \phi''^2) + \eta(\dot{\phi} - \phi') \tag{93}
\]

Here \( \eta \) stands for the lagrange multiplier field. The equations of motion for the field \( \phi \) and \( \eta \) are

\[
L_\phi = (\dddot{\phi} - \phi''') + \dot{\eta} - \eta' \tag{94}
\]

\[
L_\eta = - (\dot{\phi} - \phi') \tag{95}
\]

In this case \( w \) and \( \alpha \) are,

\[
w = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \delta(z - x) \tag{96}
\]

\[
\alpha = \begin{pmatrix} -\phi'' + (\dot{\eta} - \eta') \\ -\dot{\phi} + \phi' \end{pmatrix} \tag{97}
\]

A little algebra shows that Hessian matrix \( w \) has the null eigen vector, \( \lambda^1(z) = (0, 1) \delta(z - x) \). Multiplying \( \lambda^1 \) with equation (93) from left we obtain the primary constraint of the theory as usual.

\[
\gamma^1(x, t) = (-\dot{\phi} + \phi')(x, t). \tag{98}
\]

The consistency of this primary constraint with time needs to be maintained which necessities to calculate \( L_1 \) as follows for further analysis

\[
L_1(x, t) = (L_{CB} + \frac{\partial \gamma^1}{\partial t})(x, t). \tag{99}
\]

After a little algebra, we have \( w^1 \) and \( \alpha^1 \):

\[
w^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ -1 & 0 \end{pmatrix} \delta(z - x), \tag{100}
\]

\[
\alpha^1 = \begin{pmatrix} -\phi'' + (\dot{\eta} - \eta') \\ -\dot{\phi} + \phi' \end{pmatrix} \tag{101}
\]

We find that \( w^1 \) has another null eigen vector \( \lambda^2 = (1, 0, 1) \delta(z - x) \). So secondary lagrangian constraint is now obtained, multiplying \( \lambda^2 \) with \( \gamma^1 \).

\[
\gamma^2(x, t) = (-\phi'' + \dot{\eta} + \dot{\phi}')(x, t). \tag{102}
\]

Adding the time derivatives of \( \gamma^2 \) with \( L_1, L_2 \) is obtained to maintain the consistency of the secondary constraint with time.

\[
L_2(x, t) = (L_1 + \frac{\partial \gamma^2}{\partial t})(x, t). \tag{103}
\]

For \( L_2 \), the matrices \( w^2 \) and \( \alpha^2 \) are found out as

\[
w^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \partial/\partial x & 1 \end{pmatrix} \delta(z - x), \tag{104}
\]
\[ \alpha^2 = \begin{pmatrix} -\phi'' + (\dot{\eta} - \eta') \\ -\dot{\phi} + \phi' \\ +\dot{\phi}' \\ -\dot{\phi}'' - \dot{\eta}' \end{pmatrix}. \quad (105) \]

It is found that \( w^2 \) is also having a null eigen vector \( \lambda^3(z) = (1, 0, 1, 0) \delta(z - x) \). Multiplying the equation \( 103 \) from left by \( \lambda^3 \) we find
\[ \gamma^3(x, t) = \gamma^2(x, t). \quad (106) \]

So from the previous step it can be concluded that there is no further constraint and we can not increase the rank of equations for accelerations. The process is thus terminated. We now write \( \gamma^3 \) in the following form.
\[ \gamma^3(x, t) = (L \phi + \frac{\partial}{\partial t} L \eta)(x, t) \quad (107) \]

At this stage we need to compare \( 107 \) with equation \( 18 \) to compute the following non vanishing \( \phi_{si} \)'s.
\[ \phi_{0,1}(z, x) = \delta(z - x), \quad (108) \]
\[ \phi_{1,2}(z, x) = \delta(z - x). \quad (109) \]

Finally we obtain the gauge transformation of the field \( \phi \) and \( \eta \) for the system using equation \( 27 \).
\[ \delta \phi = f(x, t), \quad (110) \]
\[ \delta \eta = -\frac{\partial}{\partial t} f(x, t). \quad (111) \]

Let us now calculate the variation of \( \mathcal{L} \) under the above transformations of the fields \( 110 \) and \( 111 \).
\[ \delta \mathcal{L}_{CB} = -(L \phi + \frac{\partial L \eta}{\partial t})(x, t) = -(\phi'' + \dot{\eta}' + \dot{\phi}'). \quad (112) \]

This shows that the lagrangian \( 123 \) is not invariant under the above gauge transformations. The result of course have not gone beyond our expectation because chiral boson is known not to possess any gauge symmetry. This shows that the formalism is capable of testing the gauge symmetric property of this simple system having subtlety in many respects.

**D. Free Chiral Boson in the Extended phase space**

Let us add some appropriate terms involving auxiliary fields \( \theta \) to the lagrangian density of free Chiral Boson that makes the right hand side of the equation \( 112 \) zero. It is found that Lagrangian density that satisfy the above requirement is
\[ \mathcal{L}_{ECB} = \frac{1}{2} (\ddot{\phi}^2 - \phi'^2) + \eta(\dot{\phi} - \phi') - \frac{1}{2} (\dot{\theta}^2 + \theta'^2) + \phi' \theta' + \dot{\theta} \theta - \dot{\theta} \phi' - \eta(\dot{\theta} - \theta'). \quad (113) \]

What follows next is to study the gauge symmetric property of the lagrangian \( 113 \) using the formalism given in section (2). To this end we calculate the equations of motion corresponding to the field \( \phi, \eta \) and \( \theta \)
\[ L_{\phi} = (\ddot{\phi} - \phi'' + \dot{\eta}' + \theta'' - \dot{\theta}'), \quad (114) \]
\[ L_{\eta} = -\dot{\phi} + \phi', \quad (115) \]
\[ L_{\theta} = -\dot{\theta} - \theta'' + \phi'' + 2 \dot{\theta}' - \dot{\phi}' - \dot{\eta} + \eta'. \quad (116) \]
In this situation \( w \) and \( \alpha \) are

\[
\begin{align*}
  w &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \delta(z - x), \\
  \alpha &= \begin{pmatrix} -\phi'' + (\dot{\eta} - \eta') + \theta'' - \dot{\theta}' \\ -\dot{\phi} + \phi' + \dot{\theta} - \theta' \\ -\theta'' + 2\dot{\theta}' + \phi'' - \dot{\phi}' + \eta' - \dot{\eta} \end{pmatrix}.
\end{align*}
\]

(117)

(118)

We find that \( w \) has the following null eigen vector

\[
\lambda^1(z) = (0, 1, 0) \delta(z - x).
\]

(119)

Multiplying \( \lambda^1 \) with equation (113) from left, we obtain the primary constraint in this situation

\[
\gamma^1(x, t) = (-\dot{\phi} + \phi' + \dot{\theta} - \theta')(x, t).
\]

(120)

It is needed to add the time derivatives of \( \gamma^1 \) with \( L \) for further analysis otherwise we will fail to maintain consistency of the primary constraint.

\[
L_1(x, t) = (L_{ECB} + \frac{\partial}{\partial t} \gamma^1)(x, t).
\]

(121)

\( w^1 \) and \( \alpha^1 \) here are

\[
\begin{align*}
  w^1 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \delta(z - x), \\
  \alpha^1 &= \begin{pmatrix} -\phi'' + (\dot{\eta} - \eta') + \theta'' - \dot{\theta}' \\ -\dot{\phi} + \phi' + \dot{\theta} - \theta' \\ -\theta'' + 2\dot{\theta}' + \phi'' - \dot{\phi}' + \eta' - \dot{\eta} \end{pmatrix}.
\end{align*}
\]

(122)

(123)

A little algebra shows that \( w^1 \) has a new null eigen vector

\[
\lambda^2(z) = (1, 0, 1) \delta(z - x).
\]

(124)

Multiplying \( \lambda^2 \) with equation (121) we get the following vanishing condition:

\[
\gamma^2(x, t) = 0.
\]

(125)

If we proceed to write \( \gamma^2 \) in the form of equation (17) we reach at

\[
\gamma^2(x, t) = (L_{\phi} + \frac{\partial}{\partial t} L_{\phi} + L_{\theta})(x, t).
\]

(126)

Comparing the above equation (126) with equation (18) we get the non vanishing \( \phi_{si} \)'s.

\[
\begin{align*}
  \phi_{0,1}(z, x) &= \delta(z - x), \\
  \phi_{1,2}(z, x) &= \frac{\partial}{\partial t} \delta(z - x), \\
  \phi_{0,3}(z, x) &= \delta(z - x).
\end{align*}
\]

(127)

(128)

(129)
We are now in a position to compute the gauge transformation for the fields describing the system:

\[ \delta \phi = f(x, t), \]

\[ \delta \eta = -\frac{\partial}{\partial t} f(x, t), \]

\[ \delta \theta = -f(x, t). \]

A little algebra shows that \( \mathcal{L}_{ECB} \) will show the following variation under the above set of transformations (130), (131) and (132).

\[ \delta \mathcal{L}_{ECB} = -(L_\phi + \frac{\partial}{\partial t} L_\eta + L_\theta) f(x, t) = 0. \]

It shows that the lagrangian (113) is invariant under the transformation (130), (131) and (132). It is the expected result because the terms which we are forced to add to make equation (112) zero is nothing but the Wess-Zumino term that has brought back the gauge symmetry in the system. Thus the formalism is found to work successfully in the extended phase space of this system too. So for free field theories the formalism is found to works equally well both in the usual and extended phase space. In the following sections we will consider some interacting system to test how well it woks there.

**IV. VECTOR SCHWINGER MODEL WITH MASS LIKE TERM FOR THE GAUGE FIELD**

Vector Schwinger model [30–32] is an interesting field theoretical model which though posses gauge symmetry however, the same model with mass like term for the gauge field as studied in [12, 13, 20] does not posses that symmetry. It is an interesting model where Gauge boson acquires mass in the same way as it acquires in the Chiral Schwinger model [4–6] and the fermion gets liberated. So it would be of worth to test the gauge symmetric property of this model through this prescription. We would like to mention here that the usual vector Schwinger model was tested by Shirzad through his prescription as a limiting case [24] of the generalized Schwinger model [25]. The lagrangian density in our present consideration is

\[ \mathcal{L}_{SM} = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \alpha e^2 A_\mu A^\mu + \epsilon^{\mu \nu} A_\mu \partial_\nu \phi. \]

When it is written explicitly the lagrangian density reads

\[ \mathcal{L}_{SM} = \frac{1}{2} (\dot{A}_1 - A'_0)^2 + \frac{\alpha e^2}{2} (A^2_0 - A^2_1) + \frac{1}{2} (\dot{\phi}^2 - \phi'^2) + (\phi' A_0 - \dot{\phi} A_1). \]

The equations of motion for the field \( \phi, A_0 \) and \( A_1 \) for the lagrangian density under consideration are

\[ L_\phi = (\ddot{\phi} - \phi'') + A'_0 - \dot{A}_1, \]

\[ L_{A_0} = A''_0 - \dot{A}_1 - \phi' - \alpha e^2 A_0, \]

\[ L_{A_1} = \ddot{A}_1 - A'_0 + \alpha e^2 A_1 + \dot{\phi}. \]

For this system the matrices \( w \) and \( \alpha \) are

\[ w = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \delta(z - x), \]

\[ \alpha = \begin{pmatrix} -\phi'' - \dot{A}_1 + A'_0 \\ A''_0 - \dot{A}_1 - \phi' - \alpha e^2 A_0 \\ -\dot{A}_0' + \alpha e^2 A_1 + \dot{\phi} \end{pmatrix}. \]
It is found that Hessian matrix \( w \) has a null eigen vector which is given by
\[
\lambda^1(z) = (0, 1, 0) \delta(z - x),
\]
and we get the primary lagrangian constraint multiplying equation (134) by \( \lambda^1 \).
\[
\gamma^1(x, t) = (A_0'' - \dot{A}_1 - ae^2A_0 - \phi')(x, t).
\]
To maintain consistency of the primary constraint with time the time derivative of \( \gamma^1 \) is added with \( L_{SM} \) that gives \( L_1 \), which contains \( w_1 \) and \( \alpha_1 \) and those are
\[
L_1(x, t) = (L_{SM} + \frac{\partial \gamma^1}{\partial t})(x, t),
\]
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -\partial/\partial x
\end{pmatrix} \delta(z - x),
\]
\[
\begin{pmatrix}
-\phi'' - \dot{A}_1 + A_0' \\
A_0'' - \dot{A}_1 - \phi' - ae^2A_0 \\
-A_0' + ae^2A_1 + \phi \\
A_0'' - ae^2\dot{A}_0 - \phi'
\end{pmatrix}.
\]
The matrix \( w^1 \) is found to have a different null eigen vector
\[
\lambda^2(z) = (0, 0, \partial/\partial x, 1) \delta(z - x).
\]
Multiplying equation (133) from left by \( \lambda^2 \) we get the secondary constraint
\[
\gamma^2(x, t) = ae^2(A_1' - \dot{A}_0)(x, t).
\]
The process is again repeated since in this situation it does not lead to any terminating condition. Addition of the time derivative of \( \gamma^2 \) with \( L_1 \), we get \( L_2 \) as usual.
\[
L_2(x, t) = (L_1 + \frac{\partial}{\partial t} \gamma^2)(x, t).
\]
It contains \( w^2 \) and \( \alpha^2 \) those are found out as
\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -ae^2 & 0
\end{pmatrix} \delta(z - x),
\]
\[
\begin{pmatrix}
-\phi'' - \dot{A}_1 + A_0' \\
A_0'' - \dot{A}_1 - \phi' - ae^2A_0 \\
-A_0' + ae^2A_1 + \phi \\
A_0'' - ae^2\dot{A}_0 - \phi'
\end{pmatrix}.
\]
The matrix \( w^2 \) also is found to give another null eigen vector
\[
\lambda^3(z) = (0, 0, \partial/\partial x, 1, 0) \delta(z - x).
\]
On multiplying equation (138) from left by \( \lambda^3 \) we find
\[
\gamma^3(x, t) = ae^2(A_1' - \dot{A}_0)(x, t) = \gamma^2(x, t).
\]
Equation (152) indicates a terminating condition. So multiplication of $\lambda^3$ with $L_2$ does not gives rise to any new constraint. The rank of equation for accelerations therefore, does not increase here. One can express $\gamma^3$ in the form of equation (17) in a straightforward manner.

$$\gamma^3(x,t) = \left(\frac{\partial}{\partial t} L_{A_0} + \frac{\partial}{\partial x} L_{A_1}\right)(x,t).$$

(153)

In order to get the non vanishing $\phi_{si}$’s we equate equation (153) with (155) and we find that

$$\phi_{0,1}(x,z) = 0,$$

(154)

$$\phi_{1,2}(z,x) = \delta(z-x),$$

(155)

$$\phi_{0,3}(z,x) = \frac{\partial}{\partial x} \delta(z-x).$$

(156)

The gauge transformation formula (27) here renders the following gauge transformation for the field $\phi$, $A_0$ and $A_1$:

$$\delta \phi = 0,$$

(157)

$$\delta A_0 = -\frac{\partial}{\partial t} f(x,t),$$

(158)

$$\delta A_1 = -\frac{\partial}{\partial x} f(x,t).$$

(159)

The variation of $L_{SM}$ under the transformations (157), (158) and (159) is

$$\delta L_{SM} = -\sum_{s=0}^{n} \frac{\partial}{\partial t s}(\Phi_{si} L_i) f(x,t)$$

$$= -\left(\frac{\partial}{\partial t} L_{A_0} + \frac{\partial}{\partial x} L_{A_1}\right) f(x,t)$$

$$= -ae^2 (A_1' - A_0) f(x,t).$$

(160)

The lagrangian density (135) is therefore, not invariant under the transformation (157), (158) and (159). Here also it appears that the formalism has tested correctly that the lagrangian (135) has no gauge invariance in the usual phase space.

A. Vector Schwinger model with mass like term made Gauge invariant in the extended phase space

In the preceding section we have found that the Shirzad’s prescription have successfully tested that the lagrangian of the Schwinger model with mass like term has no gauge invariance under the transformation generated there. Let us add the appropriate term to the lagrangian (134) which certainly extend the phase space of the theory and investigate whether the process can infer that gauge invariance has restored in the extended phase space. The lagrangian under our present consideration is

$$L_{ESM} = \int dx \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} a e^2 A_\mu A^\mu \
+ \epsilon^{\mu\nu} \partial_\mu \phi A_\nu - \frac{a}{2} \partial_\mu \theta \partial^\mu \theta + a e A_\mu \partial^\mu \theta \right].$$

(161)

Lagrangian density takes the following form in $(1+1)$ dimension

$$L_{ESM} = + \frac{1}{2} (A_1 - A'_0)^2 + \frac{ae^2}{2} (A_0^2 - A_1^2) + \frac{1}{2} (\dot{\phi}^2 - \phi^2)$$

$$+ (\phi' A_0 - \dot{\phi} A_1) + \frac{a}{2} (\dot{\theta}^2 - \theta^2) + a e (A_0 \dot{\theta} - A_1 \theta').$$

(162)
The field \( \theta \) represents an auxiliary field. The equations of motion for the field \( \phi \), \( A_0 \), \( A_1 \) and \( \theta \) that come out using equation (2) are

\[
L_\phi = \ddot{\phi} - \phi'' + A_0' - \dot{A}_1, \tag{163}
\]

\[
L_{A_0} = A_0'' - \dot{A}_1 - \phi' - ae^2 A_0 - ae \dot{\theta}, \tag{164}
\]

\[
L_{A_1} = \ddot{A}_1 - \dot{A}_0' + ae^2 A_1 + \dot{\phi} + ae \theta', \tag{165}
\]

\[
L_\theta = a(\ddot{\theta} - \theta'') + ae \dot{A}_0 - ae \dot{A}_1'. \tag{166}
\]

For the lagrangian (162), \( w \) and \( \alpha \) are found out as

\[
w = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & +a
\end{pmatrix} \delta(z - x), \tag{167}
\]

\[
\alpha = \begin{pmatrix}
-\phi'' - \dot{A}_1 + A_0' \\
A_0'' - \dot{A}_1 - \phi' - ae^2 A_0 - ae \dot{\theta} \\
-\dot{A}_0' + ae^2 A_1 + \dot{\phi} + ae \theta' \\
-a \theta'' + ae \dot{A}_0 - ae \dot{A}_1'
\end{pmatrix}. \tag{168}
\]

We see that Hessian matrix \( w \) possesses the following null eigen vector

\[
\lambda^1(z) = (0, 1, 0, 0) \delta(z - x), \tag{169}
\]

and the primary lagrangian constraint is obtained here in the same way by Multiplying \( \lambda^1 \) with (161) from the left.

\[
\gamma^1(x, t) = (A_0'' - \dot{A}_1' - ae^2 A_0 - \phi' - ae \dot{\theta})(x, t). \tag{170}
\]

In order to maintain consistency of the above primary constraint we add the time derivatives of \( \gamma^1 \) with \( L_{ECM} \) and obtain \( L_1(x, t) \) as follows.

\[
L_1(x, t) = (L_{ESM} + \frac{\partial}{\partial \theta} \gamma^1)(x, t). \tag{171}
\]

The above \( L_1(x, t) \) contains \( w^1 \) and \( \alpha^1 \) as usual which are given by

\[
w^1 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & +a
\end{pmatrix} \delta(z - x), \tag{172}
\]

\[
\alpha^1 = \begin{pmatrix}
-\phi'' - \dot{A}_1 + A_0' \\
A_0'' - \dot{A}_1' - \phi' - ae^2 A_0 - ae \dot{\theta} \\
-\dot{A}_0' + ae^2 A_1 + \dot{\phi} + ae \theta' \\
-a \theta'' + ae \dot{A}_0 - ae \dot{A}_1'
\end{pmatrix}. \tag{173}
\]

The matrix \( w^1 \) in this situation is found to give the following null eigen vector.

\[
\lambda^2(z) = (0, 0, \partial/\partial x, e, 1, ) \delta(z - x). \tag{174}
\]
Multiplying the equation \((171)\) from left by \(\lambda^2\), we get

\[
\gamma^2(x, t) = 0. \tag{175}
\]

The above vanishing condition establishes that multiplication of \(\lambda^2\) with \(L_1\) will not give new constraint. Naturally, rank of equation for acceleration will not be increased. If we write equation \((175)\) in the form of equation \((17)\) as done in earlier different cases it looks

\[
\gamma^2(x, t) = \left( \frac{\partial}{\partial t} L_{A_0} + \frac{\partial}{\partial x} L_{A_1} + eL_{\theta} \right)(x, t). \tag{176}
\]

One needs to compare equation \((176)\) with \((18)\) to obtain \(\phi_s\)'s and that results the following.

\[
\phi_{0,1}(z, x) = 0, \tag{177}
\]

\[
\phi_{1,2}(z, x) = \delta(z - x), \tag{178}
\]

\[
\phi_{0,3}(z, x) = \frac{\partial}{\partial z} \delta(z - x), \tag{179}
\]

\[
\phi_{0,4}(z, x) = e\delta(z - x). \tag{180}
\]

The gauge transformation for the field \(\phi, A_0, A_1, \theta\) formula that follows from the formula \((27)\) are

\[
\delta\phi = 0, \tag{181}
\]

\[
\delta A_0 = -\frac{\partial}{\partial t} f(x, t), \tag{182}
\]

\[
\delta A_1 = -\frac{\partial}{\partial x} f(x, t), \tag{183}
\]

\[
\delta \theta = e f(x, t). \tag{184}
\]

The variation of \(L_{ECM}\) under the above transformations \((181), (182), (183)\) and \((184)\) is found out to be

\[
\delta L(x, t)_{ESM} = \sum_{s=0}^{n} \frac{\partial}{\partial t} (\Phi_s L_i) f(x, t) + \left( \frac{\partial}{\partial t} L_{A_0} + \frac{\partial}{\partial x} L_{A_1} + eL_{\theta} \right) f(x, t) = 0. \tag{185}
\]

It shows that in the extended phase space the lagrangian \((162)\) is invariant under the gauge transformation \((181), (182), (183)\) and \((184)\). Therefore, it is found that the formalism is capable of inferring about the gauge symmetric property of this interacting system in the extended phase space too. We will now proceed to consider the another interacting system the so called Chiral Schwinger model with the Faddevian anomaly in the following section.

### B. Chiral Schwinger Model with Faddeevian anomaly

Let us consider the lagrangian of the so called Chiral Schwinger model with Faddeevian anomaly \([10, 11]\) and apply the same formalism to study its gauge symmetric property. The gauss law of this theory shows a special type of non vanishing commutation relation because of the presence of anomaly in the system. This is commonly known as Faddeevian type of anomaly \([33, 34]\). This model is interesting in different respect \([10, 11]\). So study of this model with this formalism would certainly be of interest.

\[
L_{CSM} = \int \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi + e(g_{\mu\nu} - \epsilon_{\mu\nu}) \partial^{\mu} \phi A^{\nu} + \frac{1}{2} e^2 (A_0^2 - 2 A_0 A_1 - 3 A_1^2) \right) dx. \tag{186}
\]
Implicitly the lagrangian can be written down as follows

\[
L_{CSM} = \int \left[ \frac{1}{2} (\dot{A}_1 - A_0')^2 + \frac{1}{2} (\ddot{\phi} - \ddot{\phi})^2 + \frac{1}{2} e^2 (A_0^2 - 2A_0A_1 - 3A_1^2) + e(\dot{\phi} + \phi') (A_0 - A_1) \right] dx.
\]

(187)

The equations of motion for the field \( \phi \), \( A_0 \) and \( A_1 \) are

\[
L_\phi = (\ddot{\phi} - \ddot{\phi}') + e(A_0' - A_1') + e(\dot{A}_0 - \dot{A}_1),
\]

(188)

\[
L_{A_0} = A_0'' - \dot{A}_1' - e(\phi' + \dot{\phi}) - e^2 A_0 + e^2 A_1,
\]

(189)

\[
L_{A_1} = \ddot{A}_1 - \dot{A}_0' + 3e^2 A_1 + e(\phi' + \dot{\phi}) + e^2 A_0.
\]

(190)

For this lagrangian the Hessian matrix \( w \) and \( \alpha \) are

\[
w = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 1
\end{pmatrix} \delta(z - x),
\]

(191)

\[
\alpha = \begin{pmatrix}
  -\phi'' + e(\dot{A}_0 - \dot{A}_1) + e(A_0' - A_1') \\
  A_0'' - \dot{A}_1' - e(\dot{\phi} + \phi') - e^2 A_0 + e^2 A_1 \\
  -\dot{A}_0' + 3e^2 A_1 + e(\dot{\phi} + \phi') + e^2 A_0
\end{pmatrix}.
\]

(192)

A little algebra shows that the Hessian matrix \( w \) bears the following null eigen vector

\[
\lambda^1(z) = \left(0, 1, 0 \right) \delta(z - x).
\]

(193)

Here too multiplying equation (187) from left by \( \lambda^1 \) we get the primary constraint,

\[
\gamma^1(x, t) = (A_0'' - \dot{A}_1' - e^2 A_0 + e^2 A_1 - e(\dot{\phi} + \phi'))(x, t).
\]

(194)

The constraint has to be consistent with time. So we add time derivatives of \( \gamma^1 \) with \( L_{CSM} \), and it results

\[
L_1(x, t) = (L_{CSM} + \frac{\partial}{\partial t} \gamma^1)(x, t).
\]

(195)

In matrix \( w^1 \) and \( \alpha^1 \) that occurs in equation (187) are found out as

\[
w^1 = \begin{pmatrix}
  1 & 0 & 0 \\
  0 & 0 & 0 \\
  0 & 0 & 1
\end{pmatrix} \delta(z - x),
\]

(196)

\[
\alpha^1 = \begin{pmatrix}
  -\phi'' + e(\dot{A}_0 - \dot{A}_1) + e(A_0' - A_1') \\
  A_0'' - \dot{A}_1' - e(\dot{\phi} + \phi') - e^2 A_0 + e^2 A_1 \\
  -\dot{A}_0' + 3e^2 A_1 + e(\dot{\phi} + \phi') + e^2 A_0
\end{pmatrix}.
\]

(197)

Let us now calculate null vector \( \lambda^2(z) \) which the matrix \( w^1 \) is having.

\[
\lambda^2(z) = \left(e, 0, \partial/\partial x, 1 \right) \delta(z - x).
\]

(198)

We get secondary lagrangian constraint multiplying equation (195) from left by \( \lambda^2 \):

\[
\gamma^2(x, t) = 2e^2 (A_1' + A_0')(x, t).
\]

(199)
In order to maintain consistency again the time derivatives of \( \gamma^2 \) is added with \( L_1 \) which results

\[
L_2(x, t) = (L_1 + \frac{\partial}{\partial t} \gamma^2)(x, t).
\]

(200)

In this situation the matrices \( w^2 \) and \( \alpha^2 \) for \( L_2(x, t) \) are found out as

\[
w^2 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
-\varepsilon & 0 & -\partial / \partial x \\
0 & 0 & 0
\end{pmatrix}
\delta(z - x),
\]

(201)

\[
\alpha^2 = \begin{pmatrix}
\phi'' + e(\dot{A}_0 - \dot{A}_1) + e(A'_0 - A'_1) \\
A_0'' - \dot{A}_1' - e(\dot{\phi} + \phi') - e^2A_0 + e^2A_1 \\
-\dot{A}_0' + 3e^2A_1 + e^2A_0 + e(\dot{\phi} + \phi') \\
A_0'' - e^2A_0 - e^2\dot{\phi}' + e^2\dot{A}_1 \\
2e^2(A_1' + A_0')
\end{pmatrix}
\]

(202)

A little algebra shows that \( w^2 \) also has a new null eigen vector

\[
\lambda^3(z) = \{ e, 0, \partial / \partial x, 1, 0 \} \delta(z - x).
\]

(203)

If we multiplying the equation \( (200) \) from left by \( \lambda^3 \) and get

\[
\gamma^3(x, t) = 2e^2(A_1' + A_0')(x, t) = \gamma^2(x, t).
\]

(204)

The mapping of \( \gamma^3(x, t) \) onto \( \gamma^2(x, t) \) indicates that there is no other constraint. Thus the process is terminated. As it is done in this previous cases \( \gamma^3 \) here too is expressed in the form of equation \( (17) \).

\[
\gamma^3(x, t) = \left( \frac{\partial}{\partial t} L_{A_0} + \frac{\partial}{\partial x} L_{A_1} + eL_{\phi} \right)(x, t).
\]

(205)

Comparing \( \gamma^3(x, t) \) with equation \( (18) \) we get non vanishing \( \phi_3 \)'s which will be useful to calculate gauge transformations of the fields describing the system.

\[
\phi_0,1(z, x) = e\delta(z - x),
\]

(206)

\[
\phi_1,2(z, x) = \delta(z - x),
\]

(207)

\[
\phi_0,3(z, x) = \frac{\partial}{\partial x} \delta(z - x).
\]

(208)

We are now in a state to find the gauge transformation of the field describing the system by using equation \( (27) \).

\[
\delta \phi = e f(x, t),
\]

(209)

\[
\delta A_0 = -\frac{\partial}{\partial t} f(x, t),
\]

(210)

\[
\delta A_1 = -\frac{\partial}{\partial x} f(x, t).
\]

(211)

The variation of \( L_{CSM} \) under the transformations \( (209), (210) \) and \( (211) \) are found out to be

\[
\delta L_{CSM} = -\sum_{i=0}^{n} \frac{\partial}{\partial s}(\Phi_s L_i) f(x, t)
\]

\[
= -(eL_{\phi} + \frac{\partial}{\partial t} L_{A_0} + \frac{\partial}{\partial x} L_{A_1}) f(x, t)
\]

\[
= -2e^2(A_1' + A_0') f(x, t).
\]

(212)

So it is found that the lagrangian \( (187) \) is not invariant under the above transformations. The formalism here too gives the expected result because it is an anomalous model with Faddeevian type of anomaly and the appearance of gauge non invariance for this model is obvious.
C. Chiral Schwinger Model with Fadeevian anomaly made Gauge invariant in the extended phase space

The lagrangian of Chiral Schwinger model with Fadeevian anomaly is found gauge non invariant under the transformation generated in the previous section by Shirzad’s formalism. So we add some terms with the previous lagrangian (187) to bring back its symmetry and apply the prescription to verify whether we get the expected result in the extended phase space like the previous case. Lagrangian density with appropriate terms involving the auxiliary field $\theta$ that helps to make equation (212) zero reads

$$L_{ESM} = + \frac{1}{2}(\dot{A}_1 - A'_0)^2 + \frac{1}{2}(\dot{\phi}^2 - \phi'^2) + \frac{1}{2}e^2(A_0^2 - 2A_0A_1 - 3A_1^2)$$

$$+ e(\dot{\phi} + \phi')(A_0 - A_1) + \frac{1}{2}(\dot{\theta}^2 - 2\dot{\theta}' - 3\theta'^2)$$

$$- \frac{1}{2}(\dot{\theta}^2 - \theta'^2) - e(A_0\dot{\theta} - A_0\theta' - A_1\dot{\theta} - 3A_1\theta')$$

$$+ e(A_0\theta' - A_1\dot{\theta}) + e(-A_1\theta' + A_0\dot{\theta}).$$

(213)

The equations of motion corresponding to the field $\phi$, $A_0$, $A_1$ and $\theta$ are

$$L_{\phi} = (\ddot{\phi} - \phi'') + e(A'_0 - A'_1) + e(\dot{A}_0 - \dot{A}_1),$$

(214)

$$L_{A_0} = A''_0 - \dot{A}_1' - e(\phi' + \dot{\phi}) - e^2A_0 + e^2A_1 - 2e\theta',$$

(215)

$$L_{A_1} = \ddot{A}_1 - \dot{A}_0' + 3e^2A_1 + e(\phi' + \dot{\phi}) + 2eA_0 - 2e\theta',$$

(216)

$$L_{\theta} = -2\theta'' - 2\theta' + 2eA'_0 + 2eA'_1.$$  

(217)

The matrices $w$ and $\alpha$ in this situation are

$$w = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \delta(z - x),$$

(218)

$$\alpha = \begin{pmatrix} -\phi'' + e(\dot{A}_0 - \dot{A}_1) + e(A'_0 - A'_1) \\ A''_0 - \dot{A}_1' - e(\phi' + \dot{\phi}) - 2eA_0 + e^2A_1 - 2e\theta' \\ -A'_0 + e(\dot{\phi} + \phi') + 2eA_0 - 2e\theta' + 3e^2A_1 \\ -2\theta'' - 2\theta' + 2e(A'_0 + A'_1) \end{pmatrix}. $$

(219)

It is found that there exists a null vector within the the Hessian matrix $w$ which is given by

$$\lambda^1(z) = \{0, 1, 0, 0\} \delta(z - x).$$

(220)

Multiplying the equation (213) with $\lambda^1$ from the left we obtain the following primary constraint

$$\gamma^1(x, t) = (A''_0 - \dot{A}_1' - e^2A_0 + e^2A_1 - 2e\theta')(x, t).$$

(221)

When the time derivative of $\gamma^1$ is added with $L_{ESM}$ it gives $L_1$ which is the requirement for the primary constraint to be consistent with time.

$$L_1(x, t) = (L_{ESM} + \frac{\partial}{\partial t} \gamma^1)(x, t).$$

(222)

It is now needed to find out $w^1$ and $\alpha^1$ contained in $L_1(x, t)$ for this situation.

$$w^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -e & 0 & -\partial/\partial x & 0 \end{pmatrix} \delta(z - x),$$

(223)
\[
\alpha^1 = \begin{pmatrix}
-\phi'' + e(\dot{A}_0 - \dot{A}_1) + e(A_0' - A_1') \\
-\dot{A}_0'' - \dot{A}_1'' - e(\dot{\phi} + \dot{\phi}') - e^2 A_0 + e^2 A_1 - 2e\theta' \\
\dot{A}_0' + e(\dot{\phi} + \dot{\phi}') + e^2 A_0 - 2e\theta' + 3e^2 A_1 \\
-2\theta'' - 2\dot{\theta}' + 2e(A_0' + A_1') \\
\dot{A}_0' - e\phi' - e^2 A_0 + e^2 A_1 - 2e\theta'
\end{pmatrix}.
\] (224)

Our next step is to find out whether the matrix do have any null eigen vector and it is seen that matrix \( w^1 \) has the following null eigen vector.

\[
\lambda^2(z) = \begin{pmatrix} e, 0, \partial/\partial x, -e, 1 \end{pmatrix} \delta(z - x).
\] (225)

Now we multiply the equation (222) from left by \( \lambda^2 \) to find \( \gamma^2 \) which turns out to be zero here.

\[
\gamma^2(x, t) = \lambda^2 \alpha_2(x, t) = 0.
\] (226)

So it is not possible to increase the rank of equation for accelerations. Since \( \lambda^2 \) does not give rise to any new constraint one needs to express \( \gamma^2 \) in the form of equation (17).

\[
\gamma^2(x, t) = \left( \frac{\partial}{\partial t} L_{A_0} + \frac{\partial}{\partial x} L_{A_1} + eL_\phi - eL_\theta \right)(x, t).
\] (227)

Equating (227) with equation (18) we find that the non vanishing \( \phi_i \)'s in this situation are

\[
\begin{align*}
\phi_{0,1}(z, x) &= e\delta(z - x), \\
\phi_{1,2}(z, x) &= \delta(z - x), \\
\phi_{0,3}(z, x) &= \frac{\partial}{\partial z} \delta(z - x), \\
\phi_{0,4}(z, x) &= -e\delta(z - x).
\end{align*}
\] (228) (229) (230) (231)

It is now straightforward to compute the gauge transformations (27) for the field \( \phi, A_0, A_1 \) and \( \theta \).

\[
\begin{align*}
\delta\phi &= e f(x, t), \\
\delta A_0 &= -\frac{\partial}{\partial t} f(x, t), \\
\delta A_1 &= -\frac{\partial}{\partial x} f(x, t), \\
\delta\theta &= -e f(x, t).
\end{align*}
\] (232) (233) (234) (235)

A little algebra shows that the variation of \( L_{ECSM} \) under the above gauge transformations is

\[
\delta L_{ECSM} = \left( \frac{\partial}{\partial t} L_{A_0} + \frac{\partial}{\partial x} L_{A_1} + eL_\phi - eL_\theta \right)f(x, t) = 0.
\] (236)

It shows that the lagrangian (213) is invariant under the gauge transformation equations (232), (233), (234) and (235). Therefore, we again observe the real ability of this formalism for testing the gauge symmetric property in the extended phase space of the so called chiral Schwinger model with Faddeevian anomaly.
V. CONCLUSION

An instrument for testing gauge symmetry as well as generating gauge transformation of a theory through lagrangian formulation developed by Shirzad in [24] has been applied on different interacting and non-interacting field theoretical model. Some of the model had gauge symmetry to start with and in some model it was lacking. The formalism is found instrumental to study the gauge symmetric property for all the cases whatever subtleties are involved in these. Using Shirzad's formalism, we have successfully tested whether a given model does posses gauge symmetry or not. When a model is found gauge non-invariant it is made gauge invariant by adding some auxiliary fields with the lagrangian of that model and investigation is carried over using Shirzad's prescription to test whether gauge symmetric gets restored in it. The process of adding auxiliary fields though extends the phase space the physical content of the theory remains unaltered because the fields required for the extension keep themselves allocated in the unphysical sector of the theory. More importantly, it has been possible to generate gauge transformation generator in the extended phase space too. So it is found that the formalism is not only useful in the usual phase space of the theory but also it is equally powerful in the extended phase space. In this context, we should mention that in [24], Shirzad kept himself confined within the usual phase space of the theory. One important aspect of this formalism which we have noticed here is that one can have a guess about the Weiss-Zumino term needed to bring back the symmetry of a gauge non invariant theory. In every cases of our studies we have noticed that the terms involving auxiliary field needed for making the variation of a particular gauge non invariant lagrangian zero under the respective transformations generated through Shirzad's formalism leads to the Weiss-Zumino term for the respective theory. But it is fair to admit that the formalism is still lacking the mechanism to make a theory gauge invariant in a straightforward manner, i.e., the automatic generation of Wess-Zumino term as it has been found to be generated during the BRST invariant reformulation through Batalin-Fradkin-Vilkovisky formalism [14–16]. However the formalism has enough room for improvement towards this end. More serious and intense investigation is needed in that direction.

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