On discrete fractional integral operators and related Diophantine equations

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We study discrete versions of fractional integral operators along curves and surfaces. $l^p \to l^q$ estimates are obtained from upper bounds of the number of solutions of associated Diophantine systems. In particular, this relates the discrete fractional integral along the curve $\gamma(m) = (m, m^2, \ldots, m^k)$ to Vinogradov’s mean value theorem. Sharp $l^p \to l^q$ estimates of the discrete fractional integral along the hyperbolic paraboloid in $\mathbb{Z}^3$ are also obtained except for endpoints.

1. Introduction

The classical Hardy-Littlewood-Sobolev inequality gives the $L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ bounds for the fractional integral operator $f \to |\cdot|^{-d\lambda} * f$ for $0 < \lambda < 1$, $1 < p < q < \infty$, and $\frac{1}{q} = \frac{1}{p} - (1 - \lambda)$. The $l^p(\mathbb{Z}^d) \to l^q(\mathbb{Z}^d)$ bounds for its discrete analogue $f \to \sum_{m \in \mathbb{Z}^d \setminus 0} |m|^{-d\lambda} f(\cdot - m)$ follows from the $L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ bounds by a simple comparison argument [15]. However, if one considers discrete analogues of recent variants of fractional integrals, where the integration is taken over a sub-manifold, the argument fails and the problem becomes more interesting.

Let us give a few examples. Consider the operators $I_{k,\lambda}$ and $J_{k,\lambda}$ defined by

$$I_{k,\lambda}(f)(n) = \sum_{m=1}^{\infty} \frac{f(n - m^k)}{m^\lambda},$$

$$J_{k,\lambda}(f)(n_1, \ldots, n_k) = \sum_{m=1}^{\infty} \frac{f(n_1 - m, n_2 - m^2, \ldots, n_k - m^k)}{m^\lambda}.$$  

The study of $l^p \to l^q$ estimates of the operator $I_{k,\lambda}$ and $J_{k,\lambda}$ was initiated by Stein and Wainger [15], where they obtained almost sharp bounds for $k = 2$

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and \(1/2 < \lambda < 1\) by employing the circle method. Oberlin [11] obtained sharp results for \(k = 2\) and \(0 < \lambda < 1\) except for endpoints without using the circle method. The endpoint bounds were fully established in a series of papers by Stein and Wainger [15, 16], and Ionescu and Wainger [9], but only in the case \(k = 2\). The case \(k \geq 3\) seems to be substantially harder. Pierce [13] studied the operator \(I_{k,\lambda}\) for \(k \geq 3\) using tools from number theory, but it is open for \(k \geq 3\) in the full \(p, q\) range. See [14] for a generalization of \(J_{2,\lambda}\) to quadratic forms in \(\mathbb{Z}^d\).

In this paper, we shall consider the operator \(J_{\lambda}(f)\) acting on (initially) compactly supported functions \(f : \mathbb{Z}^d \to \mathbb{C}\) by

\[
J_{\lambda}(f)(n) = \sum_{m=1}^{\infty} \frac{f(n - \gamma(m))}{m^{\lambda}},
\]

where \(\gamma : \mathbb{N} \to \mathbb{Z}^d\) is an injection. Note that \(I_{k,\lambda}\) and \(J_{k,\lambda}\) are special cases of \(J_{\lambda}\).

It is known [15, Proposition (b)] that the operator \(J_{\lambda}\) extends to a bounded operator from \(l^p(\mathbb{Z}^d)\) to \(l^q(\mathbb{Z}^d)\) if \(\frac{1}{q} \leq \frac{1}{p} - (1 - \lambda)\) and \(1 < p < q < \infty\) for any injection \(\gamma\). This result is sharp if \(\gamma(m) = m\), but not in general. We are interested in obtaining a sharper estimate which is sensitive to \(\gamma\).

The previous work described above reveals a close connection between \(l^p \to l^q\) bounds and the number of solutions of Diophantine equations related to the curves at least in two different ways. One approach [13, 15, 16] is via the Fourier multipliers associated with the operators and Parseval’s identity. Another approach [11] is via the combinatorial argument by Christ [4] which originated from the study of an averaging operator along a curve.

The Diophantine equation considered in [13] for the case \(\gamma(m) = m^k\) was

\[
(1.1) \quad x_1^k + \cdots + x_s^k = x_{s+1}^k + \cdots + x_{2s}^k
\]

for \(x_i \in [1, P] \cap \mathbb{Z}, s, P \in \mathbb{N}\). The number of solutions of (1.1) is known as the mean values of Weyl sums, and has applications to Waring’s problem.

Following the combinatorial approach [11], for general \(\gamma\), we shall relate \(l^p \to l^q\) bounds of \(J_{\lambda}\) to the Diophantine system

\[
\gamma(x_1) + \cdots + \gamma(x_s) = \gamma(x_{s+1}) + \cdots + \gamma(x_{2s}).
\]

Moreover, we shall show that it is desirable to obtain estimates on the number of solutions of Diophantine systems with odd-number of unknowns, which turns out to extend the allowable \(\lambda\) range. See Section 2.1 and 4.
Let us introduce a notation. For fixed \( h \in \mathbb{Z}^d, r, P \in \mathbb{N} \), let \( N_\gamma^r(P; h) \) be the number of \( r \)-tuples \((x_1, \ldots, x_r)\) of positive integers \( x_i \leq P \) satisfying the equation

\[
\sum_{i=1}^{r} (-1)^{i+1} \gamma(x_i) = h.
\]

Note that there is a trivial estimate \( N_\gamma^r(P; h) \ll P^r \), where \( \ll \) denotes the Vinogradov’s notation. The following theorem provides \( l^p \to l^q \) estimates for \( J_\lambda^\gamma \) given a non-trivial estimate on \( N_\gamma^r(P; h) \). In what follows, we allow implied constants to depend on \( \gamma, r, \) and \( \epsilon \). Throughout the paper, we assume \( 1 \leq p, q \leq \infty \).

**Theorem 1.1.** Suppose that for fixed \( r \in \mathbb{N} \) and \( \delta > 0 \), we have \( N_\gamma^r(P; h) \ll P^{r-\delta + \epsilon} \) for each \( \epsilon > 0 \) uniformly in \( h \in \mathbb{Z}^d \). Let \( s \in \mathbb{N} \) be the number such that we have either \( r = 2s \) or \( r = 2s - 1 \). Then \( J_\lambda^\gamma \) extends to a bounded operator from \( l^p(\mathbb{Z}^d) \) to \( l^q(\mathbb{Z}^d) \) if \( 1 - \frac{s}{r} < \lambda < 1 \) and \( p, q \) satisfy

(i) \( \frac{1}{q} < \frac{1}{p} - \frac{1-\lambda}{\delta} \) and

(ii) \( \frac{1}{p} > \frac{s}{\delta}(1-\lambda), \frac{1}{q} < 1 - \frac{s}{\delta}(1-\lambda) \).

Several remarks are in order.

1) When \( r = 2s \), it is enough to assume that \( N_\gamma^{2s}(P; 0) \ll P^{2s-\delta + \epsilon} \) since \( N_\gamma^{2s}(P; h) \leq N_\gamma^{2s}(P; 0) \) for all \( h \in \mathbb{Z}^d \). This can be seen by writing

\[
N_\gamma^{2s}(P; h) = \int_{[0,1]^d} |S^\gamma(\alpha)|^{2s} e(-h \cdot \alpha) d\alpha,
\]

where \( e(t) \equiv e^{2\pi it} \) and \( S^\gamma(\alpha) = \sum_{m=1}^{P} e(\gamma(m) \cdot \alpha) \).

2) Given the stronger estimate \( N_\gamma^r(P; h) \ll P^{r-\delta} \), it is possible to replace \( < \) in (i) by \( \leq \) with a slight modification of the proof of Theorem 1.1. This can be done by applying an abstract analogue ([2, Section 6.2]) of an interpolation argument of Bourgain [1], but we shall not pursue it here.

3) If one considers operators with summation over \( m \in \mathbb{Z} \setminus \{0\} \), then the condition \( x_i \in [1, P] \cap \mathbb{Z} \) in (1.2) changes to \( x_i \in [-P, P] \cap \mathbb{Z} \). See Section 4 for an analogous statement in higher dimensions.

Before we turn to the proof of Theorem 1.1, we shall give applications for the case \( \gamma^a(m) = (m^{a_1}, m^{a_2}, \ldots, m^{a_d}) \) by using Vinogradov’s mean value
theorem in Section 2. We prove Theorem 1.1 and a sharp (up to endpoints) \( l^p \to l^q \) bound for the discrete fractional integral along the hyperbolic paraboloid in \( \mathbb{Z}^3 \) in Section 3 and Section 4, respectively. In Appendix 5.3, we generalize the Fourier multiplier approach [13] for operators \( J^a_\lambda \) considered in Section 2, giving an alternative proof of Theorem 2.1.

2. The operator \( J^a_\lambda \)

We study the operator \( J^a_\lambda \equiv J^a_\lambda \), where \( \gamma^a(m) = (m^{a_1}, \ldots, m^{a_d}) \) for a d-tuple of strictly increasing natural numbers \( a = (a_1, \ldots, a_d) \). We define \( \|a\| = a_1 + \cdots + a_d \).

**Conjecture 1.** Let \( 0 < \lambda < 1 \). \( J^a_\lambda \) extends to a bounded operator from \( l^p(\mathbb{Z}^d) \) to \( l^q(\mathbb{Z}^d) \) if and only if \( p \) and \( q \) satisfy

(i) \( \frac{1}{q} \leq \frac{1}{p} - \frac{1}{\|a\|} (1 - \lambda) \) and

(ii) \( \frac{1}{p} > 1 - \lambda \frac{1}{q} < \lambda \).

See Appendix 5.1 for the necessity of conditions (i) and (ii). One is mainly interested in proving Conjecture 1 for the range of \( \frac{\|a\| - 1}{2\|a\| - 1} < \lambda < 1 \), since then one may get the full result by interpolating the result with the trivial \( l^1(\mathbb{Z}^d) \to l^\infty(\mathbb{Z}^d) \) bound for \( \Re(\lambda) \geq 0 \).

In view of Theorem 1.1, the operator \( J^a_\lambda \) is related to the quantity \( N^a_\gamma(P; h) \). For even \( r = 2s \), let us denote \( N^a_{2s}(P; 0) \) by \( J^a_s(P) \), i.e. the number of solutions of the Diophantine system

\[
x_1^{a_1} + \cdots + x_s^{a_j} = x_{s+1}^{a_j} + \cdots + x_{2s}^{a_j}
\]

for \( 1 \leq j \leq d \) with \( x_i \in [1, P] \cap \mathbb{Z} \). By a standard argument (see [6] or [18]), one may show that there is a lower bound

\[
P^s + P^{2s - \|a\|} \ll J^a_s(P).
\]

It is natural to ask if \( P^s + P^{2s - \|a\|} \) is the true order of \( J^a_s(P) \). This question for the case \( a = a_k \equiv (1, 2, \ldots, k) \) has been of great interest. Non-trivial upper bounds for \( J_{s,k}(P) \equiv J^a_s(P) \) are collectively known as Vinogradov’s mean value theorem. Recent work by Wooley [19, 20] and Ford and Wooley [7] report the following substantial progress on Vinogradov’s mean value theorem.
**Theorem A.** Suppose that $s$ and $k \geq 3$ are natural numbers and that $1 \leq s \leq \frac{(k+1)^2}{4}$ or $s \geq k^2 - 1$. Then for each $\epsilon > 0$, one has

$$J_{s,k}(P) \ll P^\epsilon (P^s + P^{2s-\frac{k(k+1)}{2}}).$$

Thus, Theorem A answers the question up to $\epsilon$ if $s$ is sufficiently larger or smaller than $\|a_k\| = \frac{k(k+1)}{2}$. Let $\tilde{V}(k)$ be the least number $s \geq \|a_k\|$ for which (2.2) holds. A crude standard estimate (see Appendix 5.2) gives

$$J_a^s(P) \ll P^{2s-\|a\|+\epsilon}$$

for $s \geq V(a_d)$. However, we expect that (2.3) holds for a larger range of $s$ in view of the lower bound (2.1). We denote by $V(a)$ the least number $s$ for which (2.3) holds. With $r = 2s = 2V(a)$ and $\delta = \|a\|$, Theorem 1.1 implies the following:

**Theorem 2.1.** Let $a_d \geq 3$ and $\lambda = 1 - \frac{\|a\|}{2V(a)}$. Then $J_a^\lambda$ extends to a bounded operator from $l^p(\mathbb{Z}^d)$ to $l^q(\mathbb{Z}^d)$ if $\lambda < \lambda < 1$ and $p, q$ satisfy

(i) $\frac{1}{q} < \frac{1}{p} - \frac{1}{\|a\|}(1 - \lambda)$ and

(ii) $\frac{1}{p} > \frac{V(a)}{\|a\|}(1 - \lambda), \frac{1}{q} < 1 - \frac{V(a)}{\|a\|}(1 - \lambda)$.

If one has $V(a) = \|a\|$, then Theorem 2.1 would imply the nearly sharp result toward Conjecture 1 for $\frac{1}{2} < \lambda < 1$. In order to extend this to the full range of $0 < \lambda < 1$, we need the stronger estimate

$$N_{2\|a\|-1}^{\gamma_a}(P; h) \ll P^\|a\|^{-1+\epsilon}$$

uniformly in $h \in \mathbb{Z}^d$. Indeed, Oberlin’s result for $J_{2,\lambda}$ is based on the estimate (2.4) which is valid for $a = (1, 2)$. See [11] for details.

We record the results for the special case $J_{k,\lambda} = J_{a_k}^{\lambda_k}$, where $a_k = (1, 2, \ldots, k)$ from Theorem A. Theorem 1.1 with $r = 2s = 2(k^2 - 1)$ and $\delta = \frac{k(k+1)}{2}$ gives

**Corollary 2.2.** Let $k \geq 3$ and $\lambda_k = 1 - \frac{k}{k^2-1}$. Then $J_{k,\lambda}$ extends to a bounded operator from $l^p(\mathbb{Z}^k)$ to $l^q(\mathbb{Z}^k)$ if $\lambda < \lambda < 1$ and $p, q$ satisfy

(i) $\frac{1}{q} < \frac{1}{p} - \frac{2}{k(k+1)}(1 - \lambda)$ and

(ii) $\frac{1}{p} > \frac{2(k-1)}{k}(1 - \lambda), \frac{1}{q} < 1 - \frac{2(k-1)}{k}(1 - \lambda)$.
Theorem 1.1 with \( r = 2s = 2\delta = 2\lfloor \frac{(k+1)^2}{4} \rfloor \) implies

**Corollary 2.3.** Let \( k \geq 3 \) and \( \frac{1}{2} < \lambda < 1 \). Then \( J_{k,\lambda} \) extends to a bounded operator from \( l^p(\mathbb{Z}^k) \) to \( l^q(\mathbb{Z}^k) \) if \( p, q \) satisfy

(i) \( \frac{1}{q} < \frac{1}{p} - \frac{1}{\lfloor \frac{(k+1)^2}{4} \rfloor} (1 - \lambda) \) and

(ii) \( \frac{1}{p} > 1 - \lambda, \frac{1}{q} < \lambda \).

Corollary 2.3 complements Corollary 2.2 in the sense that it allows a wider range of applicable \( \lambda \) and that the condition (ii) is optimal at the expense of relaxing condition (i).

Figure 1 illustrates Corollaries 2.2 and 2.3 for the case \( k = 3 \) and \( \lambda = \frac{2}{3} \). It shows the known (shaded) and the conjectured range of exponents \( \left( \frac{1}{p}, \frac{1}{q} \right) \in [0, 1]^2 \) where \( J_{3,\frac{2}{3}} \) is bounded from \( l^p(\mathbb{Z}^3) \) to \( l^q(\mathbb{Z}^3) \). Corollaries 2.2 and 2.3 give the vertices \( \left( \frac{4}{9}, \frac{7}{18} \right) \) and \( \left( \frac{1}{3}, \frac{1}{4} \right) \), respectively.

### 2.1. Relation to Waring’s problem

The purpose of this section is twofold; to give a slight improvement to one of the results on the operator \( I_{k,\lambda} \) in [13] and to give a correction to an interpolation lemma in [13]. We thank Pierce for encouraging us to clarify the issue here.
We denote by $r_{s,k}(l)$ the number of representations of $l$ as a sum of $s$ positive $k$-th powers. Then there is an estimate
\begin{equation}
(2.5) \quad r_{s,k}(l) \ll l^{s/k-1+\epsilon}
\end{equation}
for every sufficiently large $s$ with respect to $k$, see [18]. Let $\hat{G}(k)$ to be the least natural number $s$ for which the estimate (2.5) holds. Since the work by Hardy and Littlewood that $\hat{G}(k) \leq (k-2)2^{k-1} + 5$, numerous improvements have been achieved. We refer the reader to [7] and references therein.

In [13], (2.5) was applied to obtain estimates on the mean values of the Weyl-sums. Instead, we use it to obtain estimates on $N_{2s-1}(P;h)$, which is the number of the solutions of the Diophantine equation of 2 variables
\begin{equation}
(2.6) \quad x_1^k + \cdots + x_s^k = x_{s+1}^k + \cdots + x_{2s-1}^k + h
\end{equation}
for $x_i \in [1,P] \cap \mathbb{Z}$ and $h \in \mathbb{Z}$. Indeed, one has
\begin{equation}
(2.7) \quad N_{2s-1}^\gamma(P;h) \ll P^{2s-1-k+\epsilon}
\end{equation}
for $s \geq \hat{G}(k)$ uniformly in $h \in \mathbb{Z}$.

For the convenience of the reader, we record the argument here. One first considers the number of solutions $(x_1, \ldots, x_s)$ for each fixed $(x_{s+1}, \ldots, x_{2s-1})$. It is $O(P^{s-k+\epsilon})$ uniformly in $h$ by (2.5) since we may assume that the right hand side of (2.6) is $O(P^k)$. We get (2.7) since there are $P^{s-1}$ many choices for $(x_{s+1}, \ldots, x_{2s-1})$.

The estimate (2.7) and Theorem 1.1 with $r = 2\hat{G}(k) - 1$ give the following which slightly improves the allowable range of $\lambda$ in [13].

**Theorem 2.4.** Let $\lambda_k = 1 - \frac{k}{2\hat{G}(k)-1}$. Then $I_{k,\lambda}$ extends to a bounded operator from $l^p(\mathbb{Z})$ to $l^q(\mathbb{Z})$ if $\lambda_k < \lambda < 1$ and $p, q$ satisfy
\begin{enumerate}
(i) $\frac{1}{q} < \frac{1}{p} - \frac{1}{k} (1 - \lambda)$ and
(ii) $\frac{1}{p} > \frac{\hat{G}(k)}{k} (1 - \lambda), \frac{1}{q} < 1 - \frac{\hat{G}(k)}{k} (1 - \lambda)$.
\end{enumerate}

We note that the optimal condition (ii-c) $1/q < \lambda, 1/p > 1 - \lambda$ in the statement of [13, Theorem 4] should be replaced by the weaker condition (ii) in Theorem 2.4. This is due to an error in the interpolation lemma [13, Lemma 2]. The condition (ii) $1/q < \lambda, 1/p > 1 - \lambda$ in the lemma should be corrected to a weaker condition (ii) $1/p > \frac{1-\lambda}{2(1-\eta)}, 1/q < 1 - \frac{1-\lambda}{2(1-\eta)}$. 

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3. Proof of Theorem 1.1

The theorem can be reduced to certain restricted weak type estimates. We decompose the operator $J^\gamma_\lambda$ dyadically. Define $J^\gamma_\lambda,j$ by

$$J^\gamma_\lambda,j(f)(n) = 2^{-\lambda j} \sum_m f(n - \gamma(m))$$

for $j \geq 0$ where $\sum_m$ denotes $\sum_{2^j \leq m < 2^{j+1}}$.

Oberlin [11] used the following lemma contained in the proof of Lemma 1 of [4].

**Lemma B (Christ [4]).** Suppose that $T$ is an operator taking characteristic functions $\chi_E$ onto measurable functions with $T\chi_E \geq 0$ for any measurable set $E$. Given $\alpha > 0$ and $E$ with $0 < |E| < \infty$, take $F = \{x : \alpha < T\chi_E(x) < 2\alpha\}$ and $\beta = |E|^{-1} \langle \chi_F, T\chi_E \rangle$. For $k = 0, 1, \ldots$ there are positive constants $\delta_k$ and $\epsilon_k$ (depend only on $k$) such that the sets $E_k$ and $F_k$ defined by $E_0 = E$, $F_0 = F$,

$$E_{k+1} = \{x \in E_k : T^* \chi_{F_k}(x) \geq \delta_k \beta\},$$

$$F_{k+1} = \{y \in F_k : T\chi_{E_{k+1}}(y) \geq \epsilon_k \alpha\},$$

are nonempty provided that $|F| > 0$.

Let $T = J^\gamma_\lambda,j$, $\alpha > 0$, $E \subset \mathbb{Z}^d$ and $F \equiv F^j$, $\beta \equiv \beta_j$, $E_k \equiv E^j_k$, $F_k \equiv F^j_k$ be as in Lemma B. We may assume that $|F| > 0$.

Note that $n \in E_k$ implies

$$\sum_m \chi_{E_{k-1}}(n + \gamma(m)) \geq 2^{\lambda j} \delta_{k-1} \beta \tag{3.1}$$

and $n \in F_k$ implies

$$\sum_m \chi_{E_k}(n - \gamma(m)) \geq 2^{\lambda j} \epsilon_{k-1} \alpha \tag{3.2}$$

for $k \geq 1$.

When $r = 2s - 1$, we define the sum $S_r$ by

$$S_r = \sum_{m_1} \sum_{m_2} \cdots \sum_{m_r} \chi_E \left(n + \sum_{i=1}^r (-1)^i \gamma(m_i)\right)$$

for a fixed $n \in F_{s-1}$.
Since $n \in F_{s-1}$, there are at least $2^{\lambda j} \epsilon_{s-2} \alpha$ values of $m_1 \in [2^j, 2^{j+1})$ such that $n - \gamma(m_1) \in E_{s-1}$ by (3.2). For each of these $m_1$, there are at least $2^{\lambda j} \delta_{s-2} \beta$ values of $m_2 \in [2^j, 2^{j+1})$ such that $n - \gamma(m_1) + \gamma(m_2) \in F_{s-2}$ by (3.1). Continuing in this manner, we get the following lower bound of $S_r$:

\[
S_r \gtrsim 2^{(2s-1)\lambda j} \alpha \beta^{s-1} \gtrsim 2^r \lambda j \alpha^r \frac{|F|^{s-1}}{|E|^{s-1}}
\]

since $\beta \geq \alpha \frac{|F|}{|E|}$ and $r = 2s - 1$.

Next, we get an upper bound for $S_r$. Let $E' = n - E$.

\[
S_r = \sum_{m_1} \sum_{m_2} \cdots \sum_{m_r} \chi_{E'} \left( \sum_{i=1}^r (-1)^{i+1} \gamma(m_i) \right)
\]

\[
= \sum_{l \in E'} \sum_{i=1}^r (-1)^{i+1} \gamma(m_i) = 1 \leq \sum_{l \in E'} N_r \gamma(2^{j+1}; l) \ll 2^{(r-\delta+\epsilon)j |E|},
\]

where we fix $\epsilon > 0$ so that $\lambda = \frac{r-\delta+2\epsilon}{r}$. (3.3) and (3.4) give

\[
\alpha^r |F|^{s-1} \ll 2^{-\epsilon j} |E|^{s},
\]

or equivalently,

\[
\alpha^r |\{ J^\gamma_{\lambda,j} (\chi_E)(n) > \alpha \}|^{s-1} \ll 2^{-\epsilon j} |E|^{s}.
\]

Since $J^\gamma_{\lambda,j} (\chi_E)(n) \leq \sum_{j=0}^\infty J^\gamma_{\lambda,j} (\chi_E)(n)$, (3.5) implies

\[
\alpha^r |\{ J^\gamma_{\lambda,j} (\chi_E)(n) > \alpha \}|^{s-1} \ll |E|^{s},
\]

which is equivalent to the restricted weak-type $(\frac{2s-1}{s}, \frac{2s-1}{s-1})$ estimate for $J^\gamma_{\lambda,j}$.

When $r = 2s$, we take the sum $S_r$ by

\[
S_r = \sum_{m_1} \sum_{m_2} \cdots \sum_{m_r} \chi_{E} \left( n + \sum_{i=1}^r (-1)^{i+1} \gamma(m_i) \right)
\]

for a fixed $n \in E_s$. After a few similar estimates, we get the restricted weak-type $(\frac{2s}{s+1}, 2)$ estimate for $J^\gamma_{\lambda,j}$.

Moreover, the same estimates are valid for a complex-valued $\lambda$ as long as $\Im(\lambda) > 1 - \frac{\epsilon}{r}$. For each fixed such $\lambda$, we first obtain strong type estimates with bounds uniform in $\Im(\lambda)$ by real interpolation with the $l^1 \to l^\infty$ bound for $\Re(\lambda) \geq 0$. Next, we apply analytic interpolation with the $l^\infty \to l^\infty$ bound
for \( \Re(\lambda) > 1 \). Finally, inclusion property of \( l^p \) spaces completes the proof. We refer the reader to [17, Chapter V] for the interpolation theorems used here.

### 4. Discrete fractional integral along the hyperbolic paraboloid in \( \mathbb{Z}^3 \)

The discrete fractional integrals, where the summation is taken along positive definite quadratic forms in several variables, have been studied by Pierce [14] via studying the Fourier multiplier of the operator. Motivated by [14], we study the discrete fractional integral along the hyperbolic paraboloid in \( \mathbb{Z}^3 \), defined by

\[
P_\lambda f(n_1, n_2, n_3) = \sum_{m \in \mathbb{Z}^2 \setminus 0} \frac{f(n_1 - m_1, n_2 - m_2, n_3 - (m_1^2 - m_2^2))}{|m|^{2\lambda}},
\]

acting on (initially) compactly supported functions \( f : \mathbb{Z}^3 \to \mathbb{C} \).

**Theorem 4.1.** Let \( 0 < \lambda < 1 \). Then \( P_\lambda \) extends to a bounded operator from \( l^p(\mathbb{Z}^3) \) to \( l^q(\mathbb{Z}^3) \) if \( p, q \) satisfy

(i) \( \frac{1}{q} < \frac{1}{p} - \frac{1}{2}(1 - \lambda) \) and

(ii) \( \frac{1}{p} > 1 - \lambda, \frac{1}{q} < \lambda \).

This result is sharp up to endpoint. One can show the necessity of the condition \( \frac{1}{q} \leq \frac{1}{p} - \frac{1}{2}(1 - \lambda) \) by taking \( f(n) = |(n_1, n_2)|^{-\alpha}|n_3|^{-\beta} \) for \( n_j \geq 1 \) and \( f(n) = 0 \) otherwise, for some appropriate \( \alpha, \beta > 0 \). The necessity of condition (ii) can be shown as in Appendix 5.1.

To treat the operator \( P_\lambda \), we consider a variant of \( J_\lambda^r \) defined by

\[
J_\lambda^r(f)(n) = \sum_{m \in \mathbb{Z}^d \setminus 0} \frac{f(n - \gamma(m))}{|m|^{d_0\lambda}},
\]

acting on (initially) compactly supported functions \( f : \mathbb{Z}^d \to \mathbb{C} \), where \( \gamma : \mathbb{Z}^{d_0} \to \mathbb{Z}^d \) is an injection.

For fixed \( h \in \mathbb{Z}^d, r, P \in \mathbb{N} \), let \( N_r^\gamma(P; h) \) denote be the number of solutions of the Diophantine system

\[
\sum_{i=1}^r (-1)^{i+1} \gamma(m_i) = h
\]

for \( m_i \in B_P \), where \( B_P = \{ x \in \mathbb{Z}^{d_0} : |x| \leq P \} \) and \( |x|^2 = x_1^2 + \cdots + x_{d_0}^2 \).
We have the following variant of Theorem 1.1.

**Theorem 4.2.** Suppose that for each \( \epsilon > 0 \) we have \( N^\gamma_3(P; h) \ll P^{d_0(r-\delta)+\epsilon} \) for a fixed \( r \in \mathbb{N} \) and \( \delta > 0 \) uniformly in \( h \in \mathbb{Z}^d \). Let \( s \in \mathbb{N} \) be the number such that we have either \( r = 2s \) or \( r = 2s - 1 \). Then \( J^\lambda_3 \) extends to a bounded operator from \( \ell^p(\mathbb{Z}^d) \) to \( \ell^q(\mathbb{Z}^d) \) if \( 1 - \frac{s}{r} < \lambda < 1 \) and \( p, q \) satisfy

1. \( \frac{1}{q} < \frac{1}{p} - \frac{1-\lambda}{\delta} \) and
2. \( \frac{1}{p} > \frac{s}{\delta}(1-\lambda), \frac{1}{q} < 1 - \frac{s}{\delta}(1-\lambda) \).

The proof of Theorem 4.2 is a straightforward modification of the proof of Theorem 1.1 and will be omitted.

**Proof of Theorem 4.1.** By Theorem 4.2, it is enough to show that

\[
N^\gamma_3(P; h) \ll P^{2+\epsilon}
\]

uniformly in \( h \in \mathbb{Z}^3 \).

Here, \( d_0 = 2, m = (m_1, m_2) \in \mathbb{Z}^2 \) and \( \gamma(m) = (m, \tau(m)) \), where \( \tau(m) = m_1^2 - m_2^2 \). Recall that \( N^\gamma_3(P; h) \) is the number of solutions of the Diophantine system

\[
x + y = z + v \\
\tau(x) + \tau(y) = \tau(z) + t
\]

for \( x, y, z \in B_P \) and \( h = (v, t) \in \mathbb{Z}^{2+1} \).

Since there are \( O(P^2) \) many \( z \) in \( B_P \), it is enough to show that the number of solutions of

\[
x + y = v' \\
\tau(x) + \tau(y) = t'
\]

for \( x, y \in B_P \) is \( O(P^\epsilon) \) uniformly in \( h' = (v', t') \in \mathbb{Z}^{2+1} \).

Suppose that \( (x, y) \) is a solution of (4.2). Then \( |v'| \leq 2P \) and \( |t'| \leq 2P^2 \). We make a change of variables \( X = 2x - v' \) and \( Y = 2y - v' \). Then \( (X, Y) \) is a solution of the Diophantine system

\[
X + Y = 0 \\
\tau(v' + X) + \tau(v' + Y) = 4t',
\]

which is equivalent to the Diophantine equation

\[
(X_1 + X_2)(X_1 - X_2) = N := 2t' - (v'_1^2 - v'_2^2),
\]

where \( X = (X_1, X_2) \) and \( v' = (v'_1, v'_2) \).
Therefore, the number of solutions \((X_1, X_2)\) of (4.3) is \(O(d(N))\), where \(d(N)\) is the number of divisors of \(N\). The fact that \(d(N) = O(|N|^\epsilon)\) (see Chapter 18 of [8]) and \(|N| = O(P^2)\) implies that the number of solutions \((X_1, X_2)\) of (4.3) is \(O(P^\epsilon)\) for any \(\epsilon > 0\), which in turn implies that the number of solutions of (4.2) is \(O(P^\epsilon)\).

Theorem 4.2 with \(r = 3\) and \(\delta = 2\) implies Theorem 4.1 for \(\frac{1}{3} < \lambda < 1\). Interpolating the result with the trivial \(l^1(\mathbb{Z}^3) \to l^\infty(\mathbb{Z}^3)\) bound for \(\Re(\lambda) \geq 0\) finishes the proof. \(\square\)

5. Appendix

5.1. Necessity of conditions in Conjecture 1.

For the necessity of the second condition, we take the example in [15]: \(f(0) = 1, f(n) = 0\) for \(n \neq 0\). Then \(f \in l^p(\mathbb{Z}^d)\) for all \(p\), and

\[
J_\lambda^p(f)(n) = \begin{cases} 
m^{-\lambda} & \text{if } n = \gamma^\alpha(m) \text{ for some } m \in \mathbb{N} \\
0 & \text{otherwise.}
\end{cases}
\]

Thus

\[
\|J_\lambda^p(f)\|_{l^q(\mathbb{Z}^d)}^q = \sum_{m \geq 1} m^{-\lambda q},
\]

where the sum converges only if \(1/q < \lambda\). Duality gives \(1/p > 1 - \lambda\).

For the necessity of the first condition, we take \(f : \mathbb{Z}^d \to \mathbb{C}\) by

\[
f(n) = \begin{cases} 
\prod_{j=1}^d |n_j|^{-\alpha} & \text{if } n_j \neq 0 \text{ for all } 1 \leq j \leq d \\
0 & \text{otherwise}
\end{cases}
\]

for a fixed constant \(\alpha > 1/p\) so that \(f \in l^p(\mathbb{Z}^d)\).

For \(n_1 > 1\) and \(n_j \geq n_1^{a_j/a_1}\) for \(2 \leq j \leq d\),

\[
J_\lambda^a(f)(n) \geq \sum_{1 \leq m^{a_1} < n_1} m^{-\lambda} \prod_{j=1}^d (n_j - m^{a_j})^{-\alpha} \geq n_1^{(1-\lambda)/a_1} \prod_{j=1}^d n_j^{-\alpha}.
\]
Thus,

\[ \| J^{a}_{\lambda}(f) \|_{l^q}^q \gtrsim \sum_{n_1 > 1} n_1^{q(1 - \lambda)/a_1 - qa} \sum_{2 \leq j \leq d} \prod_{j=2}^{d} n_j^{-q\alpha} \]

\[ \gtrsim \sum_{n_1 > 1} n_1^{q(1 - \lambda)/a_1 - qa} \prod_{j=2}^{d} n_j^{a_j(1 - q\alpha)/a_1} \]

\[ \gtrsim \sum_{n_1 > 1} n_1^{q(1 - \lambda)/a_1 + \|a\|(1 - q\alpha)/a_1 - 1}, \]

where the last sum converges only if \( \frac{1}{q} < \alpha - \frac{1}{\|a\|}(1 - \lambda) \). We get the first necessary condition since we may decrease \( \alpha \) to \( 1/p \) as close as we want.

### 5.2. Proof of (2.3)

Let \( a = (a_1, \ldots, a_d) \) be given. Let \( l = a_d - d \) and \( \{b_i\}_{i=1}^{l} \) be the increasing sequence of natural numbers such that \( \{b_1, \ldots, b_l\} = ([1, a_d] \cap \mathbb{Z}) \setminus \{a_1, \ldots, a_d\} \). Considering the underlying Diophantine systems, we have

\[ J^{a}_{\lambda}(P) = \sum_{|h_{b_1}| \leq s P^{a_1}} \cdots \sum_{|h_{b_l}| \leq s P^{a_d}} \int_{[0,1]^{a_d}} |S(\alpha)|^{2s} \prod_{i=1}^{l} e(-h_{b_i} \alpha_{b_i}) d\alpha \]

\[ \ll P^{\sum_{i=1}^{l} b} J_{s,a_d}(P) = P^{a_d(a_d + 1)/2 - \|a\| J_{s,a_d}(P)}, \]

where \( S(\alpha) = \sum_{m=1}^{P} e(\alpha_1 m + \alpha_2 m^2 + \cdots + \alpha_{a_d} m^{a_d}) \).

Combining (2.2) and (5.1), we have

\[ J^{a}_{s}(P) \ll P^{2s - \|a\| + \epsilon} \]

for \( s \geq \tilde{V}(a_d) \).

### 5.3. Fourier multiplier approach for \( J^{a}_{\lambda} \)

Let \( \hat{f}(\alpha) = \sum_{n \in \mathbb{Z}^d} f(n)e(-n \cdot \alpha) \) be the Fourier transform of \( f \in l^1(\mathbb{Z}^d) \), where \( e(t) \equiv e^{2\pi it} \). We shall study the Fourier multiplier \( m^{a}_{\lambda} \) of the operator \( J^{a}_{\lambda} \)

\[ m^{a}_{\lambda}(\alpha) = \sum_{n=1}^{\infty} e(-\gamma^{a}(n) \cdot \alpha) \]

for \( \alpha \in [0,1]^d \), given by the relation \( \hat{J^{a}_{\lambda}}(f)(\alpha) = m^{a}_{\lambda}(\alpha) \hat{f}(\alpha) \).
By closely following the argument in [13, 15], we generalize the Weyl sum approach on the Fourier multipliers (Proposition 5 of [13]) as follows:

**Proposition 5.1.** Let \( s \in \mathbb{N}, 0 < \delta \leq 2s, \) and \( \lambda \in \mathbb{C}. \) \( J^\alpha_s(P) \ll P^{2s-\delta+\epsilon} \) for each \( \epsilon > 0 \) if and only if \( m^\alpha_\lambda \in L^{2s}([0,1]^d) \) for all \( \Re(\lambda) > 1 - \frac{\delta}{2s}. \)

The “folk” lemma (Lemma 2 of [15]) implies the following result.

**Corollary 5.2.** Suppose that one has \( J^\alpha_s(P) \ll P^{2s-\delta+\epsilon} \) for some \( 0 < \delta \leq 2s. \) Then the operator \( J^\alpha_\lambda \) extends to a bounded operator from \( l^{2s}_{s+1}(\mathbb{Z}^d) \) to \( l^2(\mathbb{Z}^d) \) for all \( \Re(\lambda) > 1 - \frac{\delta}{2s}. \)

Note that Theorem 2.1 follows from Corollary 5.2 and a complex interpolation.

**Proof of Proposition 5.1.** For \( l \in \mathbb{Z}^d, \) let \( r^\alpha_s(l) \) be the number of solutions of the Diophantine system
\[
x^\alpha_1 + \cdots + x^\alpha_s = l_j,
\]
where \( l = (l', l_d) = (l_1, \ldots, l_d) \) and \( x_i \geq 1 \) for \( 1 \leq j \leq d. \) In addition, we define \( r^\alpha_s(l; P) \) to be the number of solutions of (5.3) for \( 1 \leq x_i \leq P. \)

Let \( \delta > 0 \) be given. We observe that \( J^\alpha_s(P) \ll P^{2s-\delta+\epsilon} \) is equivalent to
\[
\sum_{l_d=1}^P R^\alpha_s(l_d) \ll P^{(2s-\delta+\epsilon)/a_d}, \quad \text{where} \quad R^\alpha_s(l_d) = \sum_{l' \in \mathbb{Z}^{d-1}} (r^\alpha_s(l', l_d))^2.
\]

This follows from Parseval’s identity applied to \( J^\alpha_s(P) = \int_{[0,1]^d} |S^\gamma_s(\alpha)|^{2s} d\alpha, \) where \( S^\gamma_s(\alpha) = \sum_{m=1}^P e(\gamma^a(m) \cdot \alpha). \) Indeed, one has
\[
J^\alpha_s(P) = \sum_{l \in \mathbb{Z}^d} (r^\alpha_s(l; P))^2 = \sum_{1 \leq l_d \leq s P^{a_d}} \sum_{l' \in \mathbb{Z}^{d-1}} (r^\alpha_s(l', l_d))^2 = \sum_{1 \leq l_d \leq s P^{a_d}} R^\alpha_s(l_d).
\]

Therefore, it is enough to show that (5.4) is equivalent to
\[
m^\alpha_\lambda \in L^{2s}([0,1]^d), \quad \text{for every} \quad \Re(\lambda) > 1 - \frac{\delta}{2s}.
\]

As in [13, 15], one has
\[
m^\alpha_\lambda(\alpha) = C_{a,\lambda} \int_0^1 S^\alpha_y(\alpha) y^{\lambda/a_d} dy + O(1),
\]
where $C_{a,\lambda} = \Gamma(\lambda/a_d)$ and $S_y^a(\alpha) = \sum_{n \geq 1} e^{-n^{a_d}y} e(-\gamma^a(n) \cdot \alpha)$ which is well-defined for each $y > 0$. This follows from the observation

$$\int_0^\infty e^{-n^{a_d}y} y^{\lambda/a_d} \frac{dy}{y} = n^{-\lambda} \Gamma(\lambda/a_d).$$

Thus,

$$\tag{5.6} \|m_\lambda^a\|_{L^2([0,1]^d)} \leq C_{a,\lambda} \int_0^1 \|S_y^a\|_{L^2([0,1]^d)} y^{2\Re(\lambda)/a_d} \frac{dy}{y} + O(1).$$

By Parseval’s identity and summation by parts, (5.4) implies

$$\tag{5.7} \|S_y^a\|_{L^2([0,1]^d)}^{2s} = \sum_{l_d \geq 1} e^{-2l_d y} R_s(l_d) \ll y^{-(2s-\delta+\epsilon)/a_d}.$$

Thus we get (5.5) by (5.6) and (5.7).

For the converse, it is enough to assume (5.5) only for $\lambda \in \mathbb{R}$. Then $m_\lambda^a(\alpha)^s = \sum_{l \in \mathbb{Z}^d} a_l e(-l \cdot \alpha) \in L^2$, where

$$a_l = \sum_{n_1, \ldots, n_s \geq 1} \gamma^a(n_1) + \cdots + \gamma^a(n_s) = l \cdot n_1^{\lambda} \cdots n_s^{\lambda}.$$

Parseval’s identity implies that $\sum_{l \in \mathbb{Z}^d} |a_l|^2$ is finite. Since $n_i^{a_d} \leq l_d$ in the sum, we have $a_l \geq l_d^{-s\lambda/a_d}$ if $l_d \geq 1$. Thus,

$$\infty > \sum_{l \in \mathbb{Z}^d} |a_l|^2 \geq \sum_{l_d \geq 1} \sum_{l \in \mathbb{Z}^d} (l_d^{-s\lambda/a_d})^2 l_d^{-2s\lambda/a_d} = \sum_{l_d \geq 1} R_s(l_d) l_d^{-2s\lambda/a_d}$$

for every $\lambda > 1 - \frac{\delta}{2s}$, which is equivalent to (5.4).

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References

[1] J. Bourgain, Estimations de certaines fonctions maximales. C. R. Acad. Sci. Paris Sér. I Math., 301 (1985), no. 10, 499–502.

[2] A. Carbery, A. Seeger, S. Wainger and J. Wright, Classes of singular integral operators along variable lines. J. Geom. Anal., 9 (1999), no. 4, 583–605.

[3] M. Christ, Endpoint bounds for singular fractional integral operators. Unpublished manuscript, 1988.

[4] M. Christ, Convolution, curvature, and combinatorics: a case study. Internat. Math. Res. Notices, (1998), no. 19, 1033–1048.

[5] H. Davenport, Analytic methods for Diophantine equations and Diophantine inequalities. Cambridge Mathematical Library, Cambridge University Press, Cambridge, second edition (2005).

[6] K. B. Ford, New estimates for mean values of Weyl sums. Internat. Math. Res. Notices, (1995), no. 3, 155–171.

[7] K. B. Ford and T. D. Wooley, On Vinogradov’s mean value theorem: Strongly diagonal behaviour via efficient congruencing. Acta Math., 213 (2014), no. 2, 199–236.

[8] G. H. Hardy and E. M. Wright, An introduction to the theory of numbers. Oxford University Press, Oxford, sixth edition (2008).

[9] A. D. Ionescu and S. Wainger, $L^p$ boundedness of discrete singular Radon transforms. J. Amer. Math. Soc., 19 (2006), no. 2, 357–383.

[10] J. Kim, On discrete fractional integral operators and related Diophantine equations. Master’s thesis, Pohang University of Science and Technology (2011).

[11] D. M. Oberlin, Two discrete fractional integrals. Math. Res. Lett., 8 (2001), no. 1-2, 1–6.

[12] L. B. Pierce, Discrete analogues in harmonic analysis. Ph.D. Thesis, Princeton University (2009).

[13] L. B. Pierce, On discrete fractional integral operators and mean values of Weyl sums. Bull. Lond. Math. Soc., 43 (2011), no. 3, 597–612.

[14] L. B. Pierce, Discrete fractional Radon transforms and quadratic forms. Duke Math. J., 161 (2012), no. 1, 69–106.
[15] E. M. Stein and S. Wainger, *Discrete analogues in harmonic analysis. II. Fractional integration.* J. Anal. Math., 80 (2000), 335–355.

[16] E. M. Stein and S. Wainger, *Two discrete fractional integral operators revisited.* J. Anal. Math., 87 (2002), 451–479.

[17] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces.* Princeton University Press, Princeton, N.J. (1971).

[18] R. C. Vaughan, *The Hardy-Littlewood method.* Cambridge Tracts in Mathematics, Vol. 125, Cambridge University Press, Cambridge, second edition (1997).

[19] T. D. Wooley, *Vinogradov’s mean value theorem via efficient congruencing.* Ann. of Math. (2), 175 (2012), no. 3, 1575–1627.

[20] T. D. Wooley, *Vinogradov’s mean value theorem via efficient congruencing, II.* Duke Math. J., 162 (2013), no. 4, 673–730.

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