ON A CLASS OF NON-SELF-ADJOINT PERIODIC BOUNDARY VALUE PROBLEMS WITH DISCRETE REAL SPECTRUM

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1. Introduction

We study the operator \( L_{\text{per}} \) defined by
\[
L_{\text{per}}u := i\epsilon(f(x)u'(x))' + iu'(x)
\]
in which \( f \) is a given \( 2\pi \)-periodic function having the following properties:
\[
f(x + \pi) = -f(x), \quad f(-x) = -f(x);
\]
and also
\[
f(x) > 0 \quad \text{for } x \in (0, \pi).
\]
In particular it follows that \( f(\pi Z) = 0 \). We assume that \( f \) is continuous, and differentiable except possibly at a finite number of points, the points of non-differentiability excluding \( \pi Z \). We assume that \( f'(0) = 2/\pi \) and that \( 0 < \epsilon < \pi \).

We consider (1.1) on the domain
\[
D = \{ u \in L^2(-\pi, \pi) \mid L_{\text{per}}u \in L^2(-\pi, \pi); \ u(-\pi) = u(\pi) \}.
\]

Remark 1. Of course it is not obvious that functions \( u \in L^2(-\pi, \pi) \) such that \( L_{\text{per}}u \in L^2(-\pi, \pi) \) have boundary values \( u(\pm \pi) \); this was proved in [BLM], where we showed that if \( u \in L^2(-\pi, \pi) \) and \( L_{\text{per}}u \in L^2(-\pi, \pi) \) then \( u \in H^1(-\pi, \pi) \).

The main results of this paper are

Theorem 2. The spectrum of operator \( L_{\text{per}} \) is

(a) real
(b) purely discrete, i.e. it consists only of isolated eigenvalues of finite multiplicity with no accumulation points apart from, possibly, infinity.

Part (a) has been partially proved in [BLM], where we showed that all the eigenvalues (if they exist) are real. The rest of Theorem 2 follows from

Theorem 3. The resolvent \((L_{\text{per}} - \lambda)^{-1}\) is a compact operator on \( L^2(-\pi, \pi) \) if \( \lambda \) is not an eigenvalue of \( L_{\text{per}} \).

Remark 4. The spectrum is always non-empty, as zero is an eigenvalue corresponding to a constant eigenfunction.
In order to prove Theorem 3 we show that when \( \lambda \) is not an eigenvalue of \( L_{\text{per}} \) (which is guaranteed, for instance, if \( \lambda \) is not real) then the boundary value problem
\[
(1.5) \quad i\epsilon (f(x)u'(x))' + iu'(x) - \lambda u(x) = F(x) \quad -\pi < x < \pi
\]
with periodic boundary conditions \( u(-\pi) = u(\pi) \) has a unique solution \( u \in D \) for every \( F \in L^2(-\pi, \pi) \).

The compactness of the resolvent is demonstrated by an “explicit” construction of a bounded Green function \( G(x,s) \) such that
\[
(1.5) \quad u(x) = \int_{-\pi}^{\pi} G(x,s) F(s) \, ds.
\]
The properties of \( G \) are established by studying the solutions of an associated homogeneous equation in Sections 2 and 3.

Motivation and scope of the present paper. Our interest in the operator (1.1) with domain determined by condition (1.2) is primarily motivated by [BeO’BSa] and [Da2], where \( f(x) = (2/\pi) \sin x \), and therefore (1.5) takes the form
\[
(1.6) \quad i\tilde{\epsilon} (\sin(x)u'(x))' + iu'(x) - \lambda u(x) = F(x),
\]
with \( 0 < \tilde{\epsilon} < 2 \). This equation arises in fluid dynamics, and describes small oscillations of a thin layer of fluid inside a rotating cylinder. From a purely theoretical perspective, the eigenvalue problem associated to \( L_{\text{per}} \) has recently drawn a substantial amount of attention: see [ChPe], [We1], [We2], [DaWe] and [ChKaPy]. Despite the fact that (1.6) is highly non-self-adjoint, the spectrum of (1.6) consists exclusively of real eigenvalues of finite multiplicity, it is symmetric with respect to the origin and it accumulates at \( \pm \infty \). It is also known that the eigenfunctions do not form an unconditional basis of \( L^2(-\pi, \pi) \). Moreover, Davies and Weir [DaWe] have recently found explicit asymptotics of the eigenvalues as \( \tilde{\epsilon} \to 0 \).

In [BLM] we established that the eigenvalues of \( L_{\text{per}} \) are all real and form a symmetric set with respect to the origin. Theorem 2 above rules out completely the possibility of a non-empty essential spectrum and it answers an open question posed in our previous paper.

2. A naïve Frobenius analysis

Let \( p \) satisfy the integrating factor equation
\[
(2.1) \quad \frac{p'}{p} = \frac{f'}{f} + \frac{1}{\epsilon f}.
\]
Then \( u(x) \) is a solution of (1.5) iff
\[
(2.2) \quad (p(x)u'(x))' + \frac{i\lambda p}{\epsilon f} u(x) = -\left( \frac{ip}{\epsilon f} F \right)(x) \quad -\pi < x < \pi.
\]
For future use, we also recall the homogeneous differential equation
\[
(2.2') \quad (p(x)u'(x))' + \frac{i\lambda p}{\epsilon f} u(x) = 0 \quad -\pi < x < \pi.
\]
In order to understand how \( p \) behaves near \( x = 0 \) and \( x = \pi \) it is useful to consider a simple model. Suppose that near \( x = 0 \), the function \( f \) satisfies \( f(x) = 2x/\pi \) – this maintains the normalization \( f'(0) = 2/\pi \). Then (2.2') yields \( \log p = \log(2x/\pi) + \pi/(2\epsilon) \log(x) \), whence \( p(x) = Cx^{1+c/\epsilon} \) where \( C \) is an arbitrary non-zero constant and \( c = \pi/2 \). Similarly, near \( x = \pi \), we can consider a simple model \( f(x) = \)
\( \frac{2}{\pi} (\pi - x) \) and obtain \( p(x) = \tilde{C} (\pi - x)^{1-c/\epsilon} \). In [BLM] we proved that, under the minimal assumptions (1.2) on \( f \), the following results hold:

\[
(2.3) \\
p(x) \sim \begin{cases} 
  x^{1+c/\epsilon} & x \sim 0 \\
  (\pi - x)^{1-c/\epsilon} & x \sim \pi,
\end{cases}
\quad \text{and} \quad \\
p(x) \sim \begin{cases} 
  x^{c/\epsilon} & x \sim 0 \\
  (\pi - x)^{-c/\epsilon} & x \sim \pi.
\end{cases}
\]

By considering the model \( f(x) = 2x/\pi \), \( p(x) = x^{1+c/\epsilon} \) in a neighbourhood of the origin and looking for solutions of the differential equation (2.2') one can establish the asymptotic behaviour of solutions for this model in a neighbourhood of the origin; similarly near \( x = \pi \). In [BLM] we show that, under hypotheses (1.2) on \( f \), there exist solutions \( u \) of (2.2') such that

\[
(2.4) \\
\phi(x, \lambda) \sim 1, \quad x \to 0.
\]

The following result is crucial for reducing the problem to the interval \((0, \pi)\); in a sense it plays the same role in the analysis as the orthogonal splitting of the infinite matrix operator in Davies [Da2] in \( \ell^2(\mathbb{Z}) \) into three operators, hence reducing the problem to a problem in \( \ell^2(\mathbb{N}) \).

Lemma 5. The solution \( \phi \) has the symmetry property

\[
(2.5) \\
\phi(-x, \lambda) = \phi(x, -\lambda).
\]

Proof. Define a function \( v(x) = \phi(-x, \lambda) \). A direct calculation shows, thanks to the symmetry conditions (1.2), that \( v \) satisfies (2.2') but with \( \lambda \) on the right hand side replaced by \(-\lambda\). It also satisfies \( v(0) = 1 \). However eqn. (2.2') has only one solution with this property, namely \( \phi(x, -\lambda) \). This proves the result. \( \square \)

We emphasize the important fact that \( \lambda \) is an eigenvalue of (2.2') if and only if \( \phi \) possesses the additional symmetry property

\[
(2.6) \\
\phi(-\pi, \lambda) = \phi(\pi, \lambda).
\]

In [BLM] we also show that there is also a second solution \( \psi(x) = \psi(x, \lambda) \) of (2.2') satisfying

\[
(2.7) \\
\psi(x, \lambda) \sim \begin{cases} 
  |x|^{-c/\epsilon} & x \sim 0 \\
  |x + \pi|^{c/\epsilon} & x \sim \pm\pi
\end{cases}
\]

Observe that \( \psi(-\pi, \lambda) = \psi(\pi, \lambda) = 0 \), and that \( \psi(x, \lambda) \) blows up when \( x \sim 0 \), at least when \( \lambda \) is not an eigenvalue.

Consequently, when \( \lambda \) is not an eigenvalue, we can also normalize \( \psi(x, \lambda) \) by the condition

\[ p(x)\psi'(x)\phi(x) - p(x)\phi'(x)\psi(x) = 1 \quad -\pi < x < \pi. \]

The Wronskian in the right-hand side here is obviously a constant, and below we will always assume that \( \psi(x, \lambda) \) satisfies this condition.
Now, we are back to constructing explicitly the $L^2$ solution $u(x, \lambda)$ of (2.2) assuming that (2.6) fails. By the standard variation of parameters technique the general solution of (2.2) takes the form

$$u(x, \lambda) = \psi(x, \lambda) \int_0^x \phi(s, \lambda) \left( \frac{-i p}{\epsilon f} F(s) \right) ds$$

$$+ \phi(x, \lambda) \int_x^\pi \psi(s, \lambda) \left( \frac{-i p}{\epsilon f} F(s) \right) ds + A\phi(x) + B\psi(x),$$

where $A$ and $B$ are arbitrary complex constants.

It remains to check that one can choose constants $A$ and $B$ in such a way that

(a) $u \in L^2(-\pi, \pi)$,
(b) $L_{\text{per}}u \in L^2(-\pi, \pi)$, and
(c) $u(-\pi, \lambda) = u(\pi, \lambda)$.

It is in fact sufficient, and easier, to check that one can choose constants $A$ and $B$ such that

(a') $u(x, \lambda)$ is continuous at $x = 0$,
(b') $i\epsilon fu' + u$ is continuous at $x = 0$, and
(c) all hold. Indeed, (3.1) together with (a') implies (a), and together with (b') and (c) implies (b).

Note first, that by (a') and the behaviour of $\psi$ near the origin, one is tempted to take $B = 0$ in (3.1). We shall show that this choice is indeed the right one by the careful analysis of the remaining terms in (3.1).

By the Cauchy-Schwarz inequality and (2.3), (2.4), (2.7) we have,

$$\left| \psi(x, \lambda) \int_0^x \phi(s, \lambda) \left( \frac{-i p}{\epsilon f} F(s) \right) ds \right|$$

$$\leq |\psi(x, \lambda)| \left( \int_0^x |\phi(s, \lambda)|^2 \left| \frac{-i p}{\epsilon f} \right|^2 ds \right)^{1/2} \left( \int_0^x |F(s)|^2 ds \right)^{1/2}$$

$$\leq C|x|^{-c/\epsilon} |x|^{c/\epsilon + 1/2} \left( \int_0^x |F(s)|^2 ds \right)^{1/2}$$

$$\leq C|x|^{1/2} \|F\|. \tag{3.2}$$

Here, and throughout the rest of this paper, $C$ denotes a generic positive constant; $\| \cdot \|$ is the standard norm in $L^2(-\pi, \pi)$.

Similarly,

$$\left| \phi(x, \lambda) \int_x^\pi \psi(s, \lambda) \left( \frac{-i p}{\epsilon f} F(s) \right) ds \right|$$

$$\leq |\phi(x, \lambda)| \left( \int_x^\pi |\psi(s, \lambda)|^2 \left| \frac{-i p}{\epsilon f} \right|^2 ds \right)^{1/2} \left( \int_x^\pi |F(s)|^2 ds \right)^{1/2}$$

$$\leq C \cdot 1 \cdot |\pi - x|^{1/2} \left( \int_x^\pi |F(s)|^2 ds \right)^{1/2}$$

$$\leq C|\pi - x|^{1/2} \|F\|. \tag{3.3}$$
so it is bounded. It is also continuous as the integrand is a product of a bounded 
function \( \psi_p/f \) and an \( L^2 \) function \( F \).

Also, \( \phi \) is continuous at zero, and \( \psi \) is not, and therefore \( (a') \) holds if and only 
if \( B = 0 \).

To check the condition \( (b') \), it is now sufficient to verify that \( f u' \) is continuous 
at zero. Differentiating (3.1) with respect to \( x \) gives

\[
f u'(x, \lambda) = f \psi'(x, \lambda) \int_0^x \phi(s, \lambda) \left( \frac{-i p \epsilon f}{\epsilon f} \right) (s) \, ds \\
+ f \phi'(x, \lambda) \int_x^\pi \psi(s, \lambda) \left( \frac{-i p \epsilon f}{\epsilon f} \right) (s) \, ds \\
+ A f \phi'(x),
\]

as the contributions from differentiating the integrals cancel out.

The last two terms go to zero as \( x \to 0 \), and we only need to check the continuity 
of the first term. Similarly to (3.2), we get

\[
\left| f(x) \psi'(x, \lambda) \int_0^x \phi(s, \lambda) \left( \frac{-i p \epsilon f}{\epsilon f} \right) (s) \, ds \right|
\leq |f(x)\psi'(x, \lambda)| \left( \int_0^x |\phi(s, \lambda)|^2 \left| \frac{-i p \epsilon f}{\epsilon f} \right|^2 ds \right)^{1/2} \left( \int_0^x |F(s)|^2 ds \right)^{1/2}
\leq C |x| |x|^{-c'/\epsilon - 1} |x|^{c'/\epsilon + 1/2} \left( \int_0^x |F(s)|^2 ds \right)^{1/2}
\leq C |x|^{1/2} \| F \|,
\]

which proves \( (b') \).

Finally, we need to guarantee that we can choose a value of constant \( A \) to ensure 
that condition \( (c) \) holds. Direct substitution, again taking account of (2.3), (2.4), 
(2.7) gives

\[
u(\pi, \lambda) = A \phi(\pi, \lambda), \\
u(-\pi, \lambda) = A \phi(-\pi, \lambda) + \phi(-\pi, \lambda) \int_{-\pi}^{\pi} \psi(s, \lambda) \left( \frac{-i p \epsilon f}{\epsilon f} \right) (s) \, ds,
\]

and as \( \lambda \) is assumed not to be an eigenvalue, and therefore (2.6) is not satisfied, we 
can choose

\[
A = \int_{-\pi}^{\pi} \psi(s, \lambda) \left( \frac{-i p \epsilon f}{\epsilon f} \right) (s) \, ds / \left( \frac{\phi(\pi, \lambda)}{\phi(-\pi, \lambda)} - 1 \right).
\]

This proves the existence of the resolvent of \( (L_{\text{per}} - \lambda)^{-1} : L_2(-\pi, \pi) \to L_2(-\pi, \pi) \) 
for each \( \lambda \) which is not an eigenvalue of \( L_{\text{per}} \). Thus, the spectrum of \( L_{\text{per}} \) is pure 
point and real.

Now we proceed to writing down the expression for the Green function \( G(x, s) \) 
of \( L_{\text{per}} \). We note that (3.1) can be written, with account of \( B = 0 \) and (3.6), as

\[
u(x, \lambda) = \int_{-\pi}^{\pi} G(x, s) F(s) \, ds = \int_{-\pi}^{\pi} (G_1(x, s) + G_{\text{II}}(x, s) + G_{\text{III}}(x, s)) F(s) \, ds
\]
Thus we need to look at $G$ operator and consider its $(\infty)$ the fact that $\psi$ By virtue of (2.4), (2.7) and (2.3), both $v,w$ $\alpha$ where the singular values orthonormal and not necessarily equal. Here we use the bra-ket notation: where we set $L$ $|x| \geq |s|$, 

$$G_1(x,s) := \begin{cases} \psi(x,\lambda)\phi(s,\lambda) \left( \frac{-ip(s)}{\epsilon f(s)} \right) & \text{if } |x| \geq |s|, \\ 0 & \text{otherwise;} \end{cases}$$

$$G_{II}(x,s) := \begin{cases} \phi(x,\lambda)\psi(s,\lambda) \left( \frac{-ip(s)}{\epsilon f(s)} \right) & \text{if } x \leq s, \\ 0 & \text{otherwise;} \end{cases}$$

$$G_{III}(x,s) := \phi(x,\lambda)\psi(s,\lambda) \left( \frac{-ip(s)}{\epsilon f(s)} \right) / \left( \frac{\phi(\pi,\lambda)}{\phi(-\pi,\lambda)} - 1 \right) \text{ for all } -\pi \leq x, s \leq \pi.$$

The functions $G_{II}(x,s)$ and $G_{III}(x,s)$ are bounded by (2.3) and (2.4), so we need to look at $G_1(x,s)$. The only scope for trouble in the expression for $G_1$ lies in the fact that $\psi(x,\lambda)$ blows up as $x \to 0$. However if $x$ is small then, in the region $|x| \geq |s|$ where $G_1$ is nonzero, (2.4) and (2.3) yield

$$\left| \psi(x,\lambda) \frac{p(s)}{\epsilon f(s)} \right| \leq C|x|^{c/\epsilon}|s|^{c/\epsilon} \leq C.$$

Thus $G$ is bounded and hence is the kernel of a compact operator on $L^2(-\pi, \pi)$. Thus $L_{per}$ has compact resolvent and purely discrete real spectrum.

4. **Schatten class properties of the Green function**

We first recall the standard notion of Schatten class operator. Let $T$ be a compact operator and consider its $(\infty$-dimensional) singular value decomposition (see e.g. (3.8)): 

$$T = \sum_{j=0}^{\infty} \alpha_j |v_j\rangle\langle w_j|$$

where the singular values $\alpha_j \geq 0$ and the two sets of vector $\{v_j\}$ and $\{w_j\}$ are orthonormal and not necessarily equal. Here we use the bra-ket notation: $|v\rangle\langle w|u = \langle u, w\rangle v$. For $p > 0$, we say that $T$ is in the $p$-Schatten class, $T \in \mathcal{C}_p$, if

$$\|T\|_p := \left( \sum_{j=0}^{\infty} \alpha_j^p \right) \frac{1}{p} < \infty.$$

Note that $\mathcal{C}_1$ are the trace class operators and $\mathcal{C}_2$ are the Hilbert-Schmidt operators.

**Theorem 6.** *The resolvent $(\lambda - L_{per})^{-1}$ is in $\mathcal{C}_p$ for all $p > 1$.*

**Proof.** Let $G_1(x,s), G_{II}(x,s), G_{III}(x,s)$ be the components of the Green function $G(x,s)$. It suffices to show that each of the corresponding integral operators $R_1(\lambda), R_{II}(\lambda), R_{III}(\lambda)$ is in $\mathcal{C}_p$.

We start by showing that $R_{III}(\lambda) \in \mathcal{C}_1$. Indeed, $G_{III}(x,s) = v(x)w(s)$ where $v(x) = \phi(x,\lambda)$ and $w(s) = \psi(s,\lambda) \left( \frac{-ip(s)}{\epsilon f(s)} \right) / \left( \frac{\phi(\pi,\lambda)}{\phi(-\pi,\lambda)} - 1 \right)$.

By virtue of (2.4), (2.7) and (2.8), both $v,w \in L^2(-\pi, \pi)$. Then $R_{III}(\lambda) = |v\rangle\langle w|$ and $\|R_{III}(\lambda)\|_1 = \|v\|_{L^2(-\pi, \pi)}\|w\|_{L^2(-\pi, \pi)} < \infty$, as required.
Let us now show that $R_{\Pi}(\lambda) \in C_p$ for all $p > 1$. Let $\Omega_{\Pi} = \{(x, s) \in [-\pi, \pi]^2 : x \leq s\}$, be the supports of $G_{\Pi}(x, s)$. Decompose the characteristic function of $\Omega_{\Pi}$ as

$$I_{\Omega_{\Pi}}(x, s) = \sum_{j=0}^{\infty} \sum_{i=0}^{2^j-1} I_{I_{2i,j}}(x) I_{I_{2i+1,j}}(s)$$

where there are $2^{j+1}$ intervals

$$I_{i,j} = 2\pi \left[ \frac{i}{2^{j+1}}, \frac{i+1}{2^{j+1}} \right]$$

and $\pi = \left[ \frac{i\pi}{2^{j+1}} - \pi, \frac{(i+1)\pi}{2^{j+1}} - \pi \right]$, $i = 0, \ldots, 2^{j+1} - 1$.

Then

$$G_{\Pi}(x, s) = I_{\Omega_{\Pi}}(x, s)v(x)w(s) = \sum_{j=0}^{\infty} \sum_{i=0}^{2^j-1} v(x) I_{I_{2i,j}}(x) w(s) I_{I_{2i+1,j}}(s).$$

Let $S_{i,j} = |v I_{I_{2i,j}}(x)w I_{I_{2i+1,j}}(s)|$. Then

$$\alpha_{i,j} := \|S_{i,j}\| = \|v\|_{L^2(I_{2i,j})} \|w\|_{L^2(I_{2i+1,j})} \leq \frac{m}{2^j}$$

where $m$ is independent of $i$ and $j$. The constant $m$ depends on $\lambda$ and it is finite as a consequence of (2.4), (2.7) and (2.3). Let

$$S_j = \sum_{i=0}^{2^j-1} S_{i,j} = \sum_{i=0}^{2^j-1} \alpha_{i,j} \left\| v I_{I_{2i,j}}(x) \right\| L^2(I_{2i,j}) \left\| w I_{I_{2i+1,j}}(s) \right\| L^2(I_{2i+1,j}) \left\| \right\|.$$ (4.1)

Since the intervals $I_{i,j}$ are pairwise disjoint for a fixed $j$, the right side of (4.1) is a singular value decomposition for $S_j$. Then

$$\| S_j \|_p = \left( \sum_{i=0}^{2^j-1} \alpha_{i,j}^p \right)^{1/p} \leq \frac{m}{\left(2^{j+1}\right)^{1/p}}.$$ (4.1)

By the triangle inequality, this ensures that

$$\| R_{\Pi}(\lambda) \|_p < \infty$$

for all $p > 1$ as required.

The fact that $R_{\Pi}(\lambda) \in C_p$ for all $p > 1$ follows by an analogous decomposition of the support $\Omega_1 = \{(x, s) \in [-\pi, \pi]^2 : |x| \geq |s|\}$ as the union of disjoint rectangles and a very similar argument.

\[\square\]

Remark 7. Let $\lambda_n$ be the eigenvalues of $L_{\text{per}}$ and $\lambda \neq \lambda_n$. By virtue of [DuSc, cor.XI.9.7], the series $\sum_{n=0}^{\infty} (\lambda - \lambda_n)^{-p}$ converges absolutely and

$$\sum_{j=0}^{\infty} |\lambda - \lambda_n|^{-p} \leq \| (\lambda - L_{\text{per}})^{-1} \|_p$$ (4.2)

for all $p > 1$. According to the results of [We2] on the case $f(x) = (2/\pi)\sin x$, it is known that $\lambda_n \sim n^2$ as $n \to \infty$. Hence we know that $(\lambda - L_{\text{per}})^{-1} \not\in C_{1/2}$. As $L_{\text{per}} \neq L_{\text{per}}^{*}$, the inequality in (4.2) can not generally be reverse for any $p > 0$. The question of whether $(\lambda - L_{\text{per}})^{-1} \in C_p$ for $p > 1/2$ will be addressed in subsequent work.
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