VALUE DISTRIBUTION FOR THE GAUSS MAPS OF VARIOUS CLASSES OF SURFACES

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Dedicated to Professor Ryoichi Kobayashi on the occasion of his sixtieth birthday

ABSTRACT. We present in this article a survey of recent results in value distribution theory for the Gauss maps of several classes of immersed surfaces in space forms, for example, minimal surfaces in Euclidean $n$-space ($n=3$ or 4), improper affine spheres in the affine 3-space and flat surfaces in hyperbolic 3-space. In particular, we elucidate the geometric background of their results.

1. INTRODUCTION

The geometric nature of value distribution theory of complex analytic maps is well-known. One of the most notable results is the geometric interpretation of the precise maximum ‘2’ for the number of exceptional values of a nonconstant meromorphic function on the complex plane $\mathbb{C}$. Here we call a value that a function or map never attains an exceptional value of the function or map. In fact, Ahlfors [2] and Chern [7] proved that the least upper bound for the number of exceptional values of a nonconstant holomorphic map from $\mathbb{C}$ to a closed Riemann surface $\Sigma_\gamma$ of genus $\gamma$ coincides with the Euler characteristic of $\Sigma_\gamma$ by using Nevanlinna theory (see also [28, 42, 43, 44, 48]). In particular, for a nonconstant meromorphic function on $\mathbb{C}$, the geometric meaning of the maximal number ‘2’ of exceptional values is the Euler characteristic of the Riemann sphere $\mathbb{C} := \mathbb{C} \cup \{\infty\}$. We remark that if the closed Riemann surface is of $\gamma \geq 2$, then such a map does not exist because the Euler characteristic is negative.

There exist several classes of immersed surfaces in 3-dimensional space forms whose Gauss maps have value-distribution-theoretic property. For instance, Fujimoto (Theorem I), [14]) proved that the Gauss map of a nonflat complete minimal surface in Euclidean 3-space $\mathbb{R}^3$ can omit at most 4 values. Moreover, Fujimoto [12] obtained a unicity theorem for the Gauss maps of nonflat complete minimal surfaces in $\mathbb{R}^3$, which is analogous to the Nevanlinna unicity theorem (19) for meromorphic functions on $\mathbb{C}$. On the other hand, the author and Nakajo [27] showed that the maximal number of exceptional values of the Lagrangian Gauss map of a weakly complete improper affine front in the affine 3-space is 3, unless it is an elliptic paraboloid. Moreover, the author [24] gave a similar result for flat fronts in hyperbolic 3-space $\mathbb{H}^3$.

The purpose of this review paper is to give geometric interpretation of value-distribution-theoretic property for their Gauss maps. The paper is organized as follows: In Section 2, we first give a curvature bound for the conformal metric $ds^2 = (1 + |g|^2)^m|\omega|^2$ on an open Riemann
2. Main results

We first give the following curvature bound for the conformal metric

\[ ds^2 = (1 + |g|^2)^m |\omega|^2 \]

on an open Riemann surface \( \Sigma \).

**Theorem 2.1 ([23])**. Let \( \Sigma \) be an open Riemann surface with the conformal metric

\[ ds^2 = (1 + |g|^2)^m |\omega|^2, \]

where \( \omega \) is a holomorphic 1-form, \( g \) is a meromorphic function on \( \Sigma \), and \( m \) is a positive integer. Assume that \( g \) omits \( q \geq m+3 \) distinct values. Then there exists a positive constant \( C \), depending on \( m \) and the set of exceptional values, but not the surface, such that for all \( p \in \Sigma \), we have

\[ |K_{ds^2}(p)|^{1/2} \leq \frac{C}{d(p)}, \]

where \( K_{ds^2}(p) \) is the Gaussian curvature of \( ds^2 \) at \( p \) and \( d(p) \) is the geodesic distance from \( p \) to the boundary of \( \Sigma \), that is, the infimum of the lengths of the divergent curves in \( \Sigma \) emanating from \( p \).

More generally, when all of the multiple values of the meromorphic function \( g \) in the metric \( \Pi \) are totally ramified, the following theorem holds.

**Theorem 2.2 ([25])**. Let \( \Sigma \) be an open Riemann surface with the conformal metric given by \( \Pi \). Let \( q \) be a positive integer, \( \alpha_1, \ldots, \alpha_q \in \overline{\mathbb{C}} \) be distinct and \( \nu_1, \ldots, \nu_q \in \mathbb{Z}_+ \cup \{\infty\} \). Assume that

\[ \gamma := \sum_{j=1}^{q} \left( 1 - \frac{1}{\nu_j} \right) > m + 2. \]

If \( g \) satisfies the property that all \( \alpha_j \)-points of \( g \) have multiplicity at least \( \nu_j \), then there exists a positive constant \( C \), depending on \( m \), \( \gamma \) and \( \alpha_1, \ldots, \alpha_q \) but not the surface, such that for all \( p \in \Sigma \) inequality \( \Xi \) holds.
This is a generalization of Theorem 2.1. Indeed, we can show it by setting $\nu_1 = \cdots = \nu_{q=m+3} = \infty$. As an application of this theorem, we obtain an analogue of a special case of the Ahlfors islands theorem (See [4] for details of this theorem) for the meromorphic function $g$ on $\Sigma$ with the complete conformal metric $ds^2$. For more details, see [25].

As a corollary of these theorems, we give the least upper bound for the number of exceptional values of the meromorphic function $g$ on $\Sigma$ with the complete conformal metric given by (1).

**Corollary 2.3 ([23]).** Let $\Sigma$ be an open Riemann surface with the conformal metric given by (1). If the metric $ds^2$ is complete and the meromorphic function $g$ is nonconstant, then $g$ can omit at most $m + 2$ distinct values.

**Proof.** By way of contradiction, suppose that $g$ omits $m + 3$ distinct values. By Theorem 2.1 (1) holds. If $ds^2$ is complete, then we may set $d(p) = +\infty$ for all $p \in \Sigma$. Thus $K_{ds^2} \equiv 0$ on $\Sigma$. On the other hand, the Gaussian curvature with respect to the metric $ds^2$ is given by

$$K_{ds^2} = -\frac{2m |g'|^2}{(1 + |g|^2)^{m+2}|\omega|^2},$$

where $\omega = \omega z dz$, $g_z' = dg/dz$. Thus $K_{ds^2} \equiv 0$ if and only if $g$ is constant. This contradicts the assumption that $g$ is nonconstant. \qed

We give examples which ensure that Corollary 2.3 is optimal.

**Proposition 2.4.** Let $\Sigma$ be either the complex plane punctured at $q - 1$ distinct points $\alpha_1, \ldots, \alpha_{q-1}$ or the universal cover of that punctured plane. We set

$$\omega = \frac{dz}{\prod_{i=1}^{q-1}(z - \alpha_i)}, \quad g = z.$$

Then $g$ omits $q$ distinct values and the metric $ds^2 = (1 + |g|^2)^m|\omega|^2$ is complete if and only if $q \leq m + 2$. In particular, there exist examples whose metric $ds^2$ is complete and $g$ omits $m + 2$ distinct values.

**Proof.** We can easily show that $g$ omits the $q$ distinct values $\alpha_1, \ldots, \alpha_{q-1}$ and $\infty$ on $\Sigma$. A divergent curve $\Gamma$ in $\Sigma$ must tend to one of the points $\alpha_1, \ldots, \alpha_{q-1}$ or $\infty$. Thus we have

$$\int_{\Gamma} ds = \int_{\Gamma} (1 + |g|^2)^m|\omega| = \int_{\Gamma} \frac{(1 + |z|^2)^{m/2}}{\prod_{i=1}^{q-1}|z - \alpha_i|} |dz| = +\infty,$$

when $q \leq m + 2$. \qed

**Remark 2.5.** The geometric interpretation of the ‘2’ in ‘$m + 2$’ is the Euler characteristic of the Riemann sphere. Indeed, if $m = 0$ then the metric $ds^2 = (1 + |g|^2)^0|\omega|^2 = |\omega|^2$ is flat and complete on $\Sigma$. We thus may assume that $g$ is a meromorphic function on $\mathbb{C}$ because $g$ is replaced by $g \circ \pi$, where $\pi: \mathbb{C} \to \Sigma$ is a holomorphic universal covering map. On the other hand, Ahlfors [2] and Chern [7] showed that the best possible upper bound ‘2’ of the number of exceptional values of nonconstant meromorphic functions on $\mathbb{C}$ coincides with the Euler characteristic of the Riemann sphere. Hence we get the conclusion. Remark that Ros [47] gave a different approach of this fact by using ‘Bloch-Zalcman principle’.
We next provide another type of value-distribution-theoretic property of the meromorphic function \( g \) on an open Riemann surface \( \Sigma \) with the conformal metric given by (1). Nevanlinna \( [41] \) showed that two nonconstant meromorphic functions on \( \mathbb{C} \) having the same images for 5 distinct values must identically equal to each other. We obtain the following analogue to this unicity theorem.

**Theorem 2.6** \( [25] \). Let \( \Sigma \) be an open Riemann surface with the conformal metric

\[
ds^2 = (1 + |g|^2)^m |\omega|^2,
\]

and \( \hat{\Sigma} \) another open Riemann surface with the conformal metric

\[
\hat{d}s^2 = (1 + |\hat{g}|^2)^m |\hat{\omega}|^2,
\]

where \( \omega \) and \( \hat{\omega} \) are holomorphic 1-forms, \( g \) and \( \hat{g} \) are nonconstant meromorphic functions on \( \Sigma \) and \( \hat{\Sigma} \) respectively, and \( m \) is a positive integer. We assume that there exists a conformal diffeomorphism \( \Psi: \Sigma \to \hat{\Sigma} \). Suppose that there exist \( q \) distinct points \( \alpha_1, \ldots, \alpha_q \in \mathbb{C} \) such that

\[
g^{-1}(\alpha_j) = (\hat{g} \circ \Psi)^{-1}(\alpha_j) \quad (1 \leq j \leq q).
\]

If \( q \geq m + 5 = \left( m + 4 \right) + 1 \) and either \( ds^2 \) or \( \hat{d}s^2 \) is complete, then \( g \equiv \hat{g} \circ \Psi \).

We remark that Theorem 2.6 coincides with the Nevanlinna unicity theorem when \( m = 0 \). The maps \( g \) and \( \hat{g} \circ \Psi \) are said to share the value \( \alpha \) (ignoring multiplicity) when \( g^{-1}(\alpha) = (\hat{g} \circ \Psi)^{-1}(\alpha) \).

**Theorem 2.6** is optimal for an arbitrary even number \( m \) \( (\geq 2) \) because there exist the following examples.

**Proposition 2.7** \( [12, 25] \). For an arbitrary even number \( m \) \( (\geq 2) \), we take \( m/2 \) distinct points \( \alpha_1, \ldots, \alpha_{m/2} \) in \( \mathbb{C} \setminus \{0, \pm 1\} \). Let \( \Sigma \) be either the complex plane punctured at \( m + 1 \) distinct points \( 0, \alpha_1, \ldots, \alpha_{m/2}, 1/\alpha_1, \ldots, 1/\alpha_{m/2} \) or the universal covering of that punctured plane. We set

\[
\omega = \frac{dz}{z \prod_{i=1}^{m/2} (z - \alpha_i)(\alpha_i z - 1)}, \quad g(z) = z,
\]

and

\[
\hat{\omega} (= \omega) = \frac{dz}{z \prod_{i=1}^{m/2} (z - \alpha_i)(\alpha_i z - 1)}, \quad \hat{g}(z) = \frac{1}{z}.
\]

We can easily show that the identity map \( \Psi: \Sigma \to \Sigma \) is a conformal diffeomorphism and the metrics \( ds^2 = (1 + |g|^2)^m |\omega|^2 \) is complete. Then the maps \( g \) and \( \hat{g} \) share the \( m + 4 \) distinct values

\[
0, \infty, 1, \alpha_1, \ldots, \alpha_{m/2}, 1/\alpha_1, \ldots, 1/\alpha_{m/2}
\]

and \( g \neq \hat{g} \circ \Psi \). These show that the number ‘\( m+5 \)’ in Theorem 2.6 cannot be replaced by ‘\( m+4 \)’.

### 3. Applications

In this section, as applications of the main results, we give some value-distribution-theoretic properties for the Gauss maps of several classes of surfaces.
3.1. Gauss map of a complete minimal surface in \( \mathbb{R}^3 \). We first recall some basic facts of minimal surfaces in Euclidean 3-space \( \mathbb{R}^3 \). Details can be found, for example, in [13], [34] and [46]. Let \( X = (x^1, x^2, x^3): \Sigma \to \mathbb{R}^3 \) be an oriented minimal surface in \( \mathbb{R}^3 \). By associating a local complex coordinate \( z = u + \sqrt{-1}v \) with each positive isothermal coordinate system \((u, v)\), \( \Sigma \) is considered as a Riemann surface whose conformal metric is the induced metric \( ds^2 \) from \( \mathbb{R}^3 \). Then

\[
\triangle_{ds^2} X = 0
\]

holds, that is, each coordinate function \( x^i \) is harmonic. With respect to the local complex coordinate \( z = u + \sqrt{-1}v \) of the surface, (6) is given by

\[
\bar{\partial}\partial X = 0,
\]

where \( \partial = (\partial/\partial u - \sqrt{-1}\partial/\partial v)/2, \quad \bar{\partial} = (\partial/\partial u + \sqrt{-1}\partial/\partial v)/2 \). Hence each \( \phi_i := \partial x^i dz \) \((i = 1, 2, 3)\) is a holomorphic 1-form on \( \Sigma \). If we set that

\[
\omega = \phi_1 - \sqrt{-1}\phi_2, \quad g = \frac{\phi_3}{\phi_1 - \sqrt{-1}\phi_2},
\]

then \( \omega \) is a holomorphic 1-form and \( g \) is a meromorphic function on \( \Sigma \). Moreover the function \( g \) coincides with the composition of the Gauss map and the stereographic projection from \( S^2 \) onto \( \mathbb{C} \), and the induced metric is given by

\[
ds^2 = (1 + |g|^2)^2|\omega|^2.
\]

Applying Theorem 2.1 to the metric \( ds^2 \), we can obtain the Fujimoto curvature bound for a minimal surface in \( \mathbb{R}^3 \).

**Theorem 3.1.** [11, Theorem I and Corollary 3.4] Let \( X: \Sigma \to \mathbb{R}^3 \) be an oriented minimal surface whose Gauss map \( g: \Sigma \to \mathbb{C} \) omits greater than or equal to 5 (= 2 + 3) distinct values. Then there exists a positive constant \( C \) depending on the set of exceptional values, but not the surface, such that for all \( p \in \Sigma \) inequality (2) holds. In particular, the Gauss map of a nonflat complete minimal surface in \( \mathbb{R}^3 \) can omit at most 4 (= 2 + 2) values.

We note that this theorem is a generalization of the Bernstein theorem, stating that the only solution to the minimal surface equation over the whole plane is the trivial solution: a linear function \((3, 6)\).

**Remark 3.2.** For the Gauss maps of complete embedded minimal surfaces in \( \mathbb{R}^3 \), there exists an interesting conjecture called ‘Four Point Conjecture’. For more details, see [37].

Moreover, by applying Theorem 2.6, we can get the Fujimoto unicity theorem for the Gauss maps of complete minimal surfaces in \( \mathbb{R}^3 \).

**Theorem 3.3 (\[13, Theorem I\]).** Let \( X: \Sigma \to \mathbb{R}^3 \) and \( \hat{X}: \hat{\Sigma} \to \mathbb{R}^3 \) be two nonflat minimal surfaces and assume that there exists a conformal diffeomorphism \( \Psi: \Sigma \to \hat{\Sigma} \). Let \( g: \Sigma \to \mathbb{C} \) and \( \hat{g}: \hat{\Sigma} \to \mathbb{C} \) be the Gauss maps of \( X(\Sigma) \) and \( \hat{X}(\hat{\Sigma}) \), respectively. If \( g \neq \hat{g} \circ \Psi \) and either \( X(\Sigma) \) or \( \hat{X}(\hat{\Sigma}) \) is complete, then \( g \) and \( \hat{g} \circ \Psi \) share at most 6 (= 2 + 4) distinct values.
3.2. Lagrangian Gauss map of a weakly complete improper affine front in $\mathbb{R}^3$. Improper affine spheres in the affine 3-space $\mathbb{R}^3$ also have similar properties to minimal surfaces in Euclidean 3-space (for example, see [6]). Recently, Martínez [36] discovered the correspondence between improper affine spheres and smooth special Lagrangian immersions in the complex 2-space $\mathbb{C}^2$ and introduced the notion of \textit{improper affine fronts}, that is, a class of (locally strongly convex) improper affine spheres with some admissible singularities in $\mathbb{R}^3$. We note that this class is called \textquote{improper affine maps} in [36], but we call this class \textquote{improper affine fronts} because all of improper affine maps are wave fronts in $\mathbb{R}^3$ ([40], [51]). The differential geometry of wave fronts is discussed in [50]. Moreover, Martínez gave the following holomorphic representation for this class.

**Theorem 3.4 ([36, Theorem 3]).** Let $\Sigma$ be a Riemann surface and $(F, G)$ a pair of holomorphic functions on $\Sigma$ such that $\text{Re}(F dG)$ is exact and $|dF|^2 + |dG|^2$ is positive definite. Then the induced map $\psi: \Sigma \rightarrow \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ given by

$$\psi := \left( G + \overline{F}, \frac{|G|^2 - |F|^2}{2} + \text{Re}\left(G F - 2 \int F dG\right) \right)$$

is an improper affine front. Conversely, any improper affine front is given in this way. Moreover we set $x := G + \overline{F}$ and $n := \overline{F} - G$. Then $L_\psi := x + \sqrt{-1}n: \Sigma \rightarrow \mathbb{C}^2$ is a special Lagrangian immersion whose induced metric $d\tau^2$ from $\mathbb{C}^2$ is given by

$$d\tau^2 = 2(|dF|^2 + |dG|^2).$$

In addition, the affine metric $h$ of $\psi$ is expressed as $h := |dG|^2 - |dF|^2$ and the singular points of $\psi$ correspond to the points where $|dF| = |dG|$.

We remark that Nakajo [40] constructed a representation formula for indefinite improper affine spheres with some admissible singularities. The nontrivial part of the Gauss map of $L_\psi: \Sigma \rightarrow \mathbb{C}^2 \cong \mathbb{R}^4$ (see [9]) is the meromorphic function $\nu: \Sigma \rightarrow \overline{\mathbb{C}}$ given by

$$\nu := \frac{dF}{dG},$$

which is called the \textit{Lagrangian Gauss map} of $\psi$. An improper affine front is said to be \textit{weakly complete} if the induced metric $d\tau^2$ is complete. We note that

$$d\tau^2 = 2(|dF|^2 + |dG|^2) = 2(1 + |\nu|^2)|dG|^2.$$

Applying Theorem 2.1 to the metric $d\tau^2$, we can get the following theorem.

**Theorem 3.5 ([23, Theorem 4.6]).** Let $\psi: \Sigma \rightarrow \mathbb{R}^3$ be an improper affine front whose Lagrangian Gauss map $\nu: \Sigma \rightarrow \overline{\mathbb{C}}$ omits greater than or equal to 4 (= 2 + 2) distinct values. Then there exists a positive constant $C$ depending on the set of exceptional values, but not $\Sigma$, such that for all $p \in \Sigma$ we have

$$|K_{d\tau^2}(p)|^{1/2} \leq \frac{C}{d(p)},$$

where $K_{d\tau^2}(p)$ is the Gaussian curvature of the metric $d\tau^2$ at $p$ and $d(p)$ is the geodesic distance from $p$ to the boundary of $\Sigma$. In particular, if the Lagrangian Gauss map of a weakly complete improper affine front in $\mathbb{R}^3$ is nonconstant, then it can omit at most 3 (= 1 + 2) values.
Since the singular points of $\psi$ correspond to the points where $|\nu| = 1$, we can obtain a simple proof of the parametric affine Bernstein theorem ([5], [21]) for improper affine spheres from the viewpoint of value-distribution-theoretic properties of the Lagrangian Gauss map.

**Corollary 3.6 ([5, 21]).** Any affine complete improper affine sphere in $\mathbb{R}^3$ must be an elliptic paraboloid.

**Proof.** Since an improper affine sphere has no singularities, the complement of the image of its Lagrangian Gauss map $\nu$ contains at least the circle $\{ |\nu| = 1 \} \subset \mathbb{C}$. Thus, by exchanging roles of $dF$ and $dG$ if necessarily, $|\nu| < 1$ holds, that is, $|dF| < |dG|$. On the other hand, we have

$$h = |dG|^2 - |dF|^2 < 2(|dF|^2 + |dG|^2) = d\tau^2.$$ 

Thus if an improper affine sphere is affine complete, then it is also weakly complete. From Theorem 3.5 and [27, Proposition 3.1], it is an elliptic paraboloid. \square

By applying Theorem 2.6, we give the following unicity theorem for the Lagrangian Gauss maps of weakly complete improper affine fronts in $\mathbb{R}^3$.

**Theorem 3.7 ([25, Theorem 4.24]).** Let $\psi: \Sigma \to \mathbb{R}^3$ and $\hat{\psi}: \hat{\Sigma} \to \mathbb{R}^3$ be two improper affine fronts and assume that there exists a conformal diffeomorphism $\Psi: \Sigma \to \hat{\Sigma}$. Let $\nu: \Sigma \to \mathbb{C}$ and $\hat{\nu}: \hat{\Sigma} \to \mathbb{C}$ be the Lagrangian Gauss maps of $\psi(\Sigma)$ and $\hat{\psi}(\hat{\Sigma})$ respectively. Suppose that there exist $q$ distinct points $\alpha_1, \ldots, \alpha_q \in \mathbb{C}$ such that $\nu^{-1}(\alpha_j) = (\hat{\nu} \circ \Psi)^{-1}(\alpha_j)$ ($1 \leq j \leq q$). If $q \geq 6 (= (1 + 4) + 1)$ and either $\psi(\Sigma)$ or $\hat{\psi}(\hat{\Sigma})$ is weakly complete, then either $\nu \equiv \hat{\nu} \circ \Psi$ or $\nu$ and $\hat{\nu}$ are both constant, that is, $\psi(\Sigma)$ and $\hat{\psi}(\hat{\Sigma})$ are both elliptic paraboloids.

3.3. **Ratio of canonical forms of a weakly complete flat front in $H^3$.** For a holomorphic Legendrian immersion $L: \Sigma \to SL(2, \mathbb{C})$ on a simply connected Riemann surface $\Sigma$, the projection

$$f := LL^*: \Sigma \to H^3$$

gives a flat front in $H^3$. Here, flat fronts in $H^3$ are flat surfaces in $H^3$ with some admissible singularities (see [30], [33] for the definition of flat fronts in $H^3$). We call $L$ the holomorphic lift of $f$. Since $L$ is a holomorphic Legendrian map, $L^{-1}dL$ is off-diagonal (see [16], [32], [33]). If we set

$$L^{-1}dL = \begin{pmatrix} 0 & \theta \\ \omega & 0 \end{pmatrix},$$

then the pull-back of the canonical Hermitian metric of $SL(2, \mathbb{C})$ by $L$ is represented as

$$ds_L^2 := |\omega|^2 + |\theta|^2$$

for holomorphic 1-forms $\omega$ and $\theta$ on $\Sigma$. A flat front $f$ is said to be weakly complete if the metric $ds_L^2$ is complete ([31], [51]). We define a meromorphic function on $\Sigma$ by the ratio of canonical forms

$$\rho := \frac{\theta}{\omega}.$$ 

Then a point $p \in \Sigma$ is a singular point of $f$ if and only if $|\rho(p)| = 1$ ([29]). We note that

$$ds_L^2 = |\omega|^2 + |\theta|^2 = (1 + |\rho|^2)|\omega|^2.$$
Applying Theorem 2.1 to the metric $ds^2_L$, we can get the following theorem.

**Theorem 3.8** ([23, Theorem 4.8]). Let $f: \Sigma \to \mathbf{H}^3$ be a flat front on a simply connected Riemann surface $\Sigma$. Suppose that the ratio of canonical forms $\rho: \Sigma \to \mathbb{C}$ omits greater than or equal to 4 ($= 2 + 2$) distinct values. Then there exists a positive constant $C$ depending on the set of exceptional values, but not $\Sigma$, such that for all $p \in \Sigma$ we have

$$|K_{ds^2_L}(p)|^{1/2} \leq \frac{C}{d(p)},$$

where $K_{ds^2_L}(p)$ is the Gaussian curvature of the metric $ds^2_L$ at $p$ and $d(p)$ is the geodesic distance from $p$ to the boundary of $\Sigma$. In particular, if the ratio of canonical forms of a weakly complete flat front in $\mathbf{H}^3$ is nonconstant, then it can omit at most 3 ($= 1 + 2$) values.

If $\Sigma$ is not simply connected, then we consider that $\rho$ is a meromorphic function on its universal covering surface $\tilde{\Sigma}$. As an application of Theorem 3.8, we can obtain a simple proof of the classification of complete nonsingular flat surfaces in $\mathbf{H}^3$. For the proof, see [24, Corollary 3.5].

**Corollary 3.9** ([49, 52]). Any complete nonsingular flat surface in $\mathbf{H}^3$ must be a horosphere or a hyperbolic cylinder.

Finally, by applying Theorem 2.6 we provide the following unicity theorem for the ratios of canonical forms of weakly complete flat fronts in $\mathbf{H}^3$.

**Theorem 3.10** ([25, Theorem 4.29]). Let $f: \Sigma \to \mathbf{H}^3$ and $\hat{f}: \hat{\Sigma} \to \mathbf{R}^3$ be two flat fronts on simply connected Riemann surfaces and assume that there exists a conformal diffeomorphism $\Psi: \Sigma \to \hat{\Sigma}$. Let $\rho: \Sigma \to \mathbb{C} \cup \{\infty\}$ and $\hat{\rho}: \hat{\Sigma} \to \mathbb{C} \cup \{\infty\}$ be the ratio of canonical forms $f(\Sigma)$ and $\hat{f}(\hat{\Sigma})$ respectively. If $\rho \neq \hat{\rho} \circ \Psi$ and either $f(\Sigma)$ or $\hat{f}(\hat{\Sigma})$ is weakly complete, then $\rho$ and $\hat{\rho} \circ \Psi$ share at most 5 ($= 1 + 4$) distinct values.

**Remark 3.11.** The hyperbolic Gauss map of a weakly complete or complete flat front in $\mathbf{H}^3$ has also interesting geometric property. For more details, see [29], [32], [33] and [39].

4. **Further topics**

In this section, we give geometric interpretations of the maximal number of exceptional values and unicity theorem for the Gauss maps of complete minimal surfaces in $\mathbf{R}^3$. We also provide an effective estimate for the maximal number of exceptional values of the Gauss map of a nonflat complete minimal surface of finite total curvature in $\mathbf{R}^3$.

4.1. **Gauss map of a complete minimal surface in $\mathbf{R}^4$.** We first give an optimal estimate for the size of the image of the holomorphic map $G = (g_1, \ldots, g_n): \Sigma \to (\mathbb{C})^n := \mathbb{C} \times \cdots \times \mathbb{C}$ on an open Riemann surface $\Sigma$ with the complete conformal metric

$$ds^2 = \prod_{i=1}^n (1 + |g_i|^2)^{m_i} |\omega|^2.$$
Theorem 4.1 ([17 Theorem 2.1]). Let $\Sigma$ be an open Riemann surface with the conformal metric
\[ ds^2 = \prod_{i=1}^{n} (1 + |g_i|^2)^{m_i} |\omega|^2, \]
where $G = (g_1, \ldots, g_n): \Sigma \to (\mathbb{C})^n$ is a holomorphic map, $\omega$ is a holomorphic 1-form on $\Sigma$ and each $m_i$ ($i = 1, \ldots, n$) is a positive integer. Assume that $g_{i_1}, \ldots, g_{i_k}$ ($1 \leq i_1 < \cdots < i_k \leq n$) are nonconstant and the others are constant. If the metric $ds^2$ is complete and each $g_{i_l}$ ($l = 1, \cdots, k$) omits $q_{i_l} > 2$ distinct values, then we have
\[ \sum_{l=1}^{k} \frac{m_{i_l}}{q_{i_l} - 2} \geq 1. \]

We note that Theorem 4.1 also holds for the case where at least one of $m_1, \ldots, m_n$ is positive and the others are zeros. For instance, we assume that $g := g_{i_1}$ is nonconstant and the others are constant. If $m := m_{i_1}$ is a positive integer and the others are zeros, then inequality (9) coincides with
\[ \frac{m}{q - 2} \geq 1 \iff q \leq m + 2, \]
where $q := q_{i_1}$. The result corresponds with Corollary 2.3.

We next give a unicity theorem for the holomorphic map $G = (g_1, \ldots, g_n): \Sigma \to (\mathbb{C})^n$ on an open Riemann surface $\Sigma$ with the complete conformal metric defined by (9).

Theorem 4.2 ([17 Theorem 2.1]). Let $\Sigma$ be an open Riemann surface with the conformal metric
\[ ds^2 = \prod_{i=1}^{n} (1 + |g_i|^2)^{m_i} |\omega|^2, \]
and $\hat{\Sigma}$ another open Riemann surface with the conformal metric
\[ d\hat{s}^2 = \prod_{i=1}^{n} (1 + |\hat{g}_i|^2)^{m_i} |\hat{\omega}|^2, \]
where $\omega$ and $\hat{\omega}$ are holomorphic 1-forms, $G$ and $\hat{G}$ are holomorphic maps into $(\mathbb{C})^n$ on $\Sigma$ and $\hat{\Sigma}$ respectively, and each $m_i$ ($i = 1, \ldots, n$) is a positive integer. We assume that there exists a conformal diffeomorphism $\Psi: \Sigma \to \hat{\Sigma}$, and $g_{i_1}, \ldots, g_{i_k}$ and $\hat{g}_{i_1}, \ldots, \hat{g}_{i_k}$ ($1 \leq i_1 < \cdots < i_k \leq n$) are nonconstant and the others are constant. For each $i_l$ ($l = 1, \cdots, k$), we suppose that $g_{i_l}$ and $\hat{g}_{i_l} \circ \Psi$ share $q_{i_l} > 4$ distinct values and $g_{i_l} \neq \hat{g}_{i_l} \circ \Psi$. If either $ds^2$ or $d\hat{s}^2$ is complete, then we have
\[ \sum_{l=1}^{k} \frac{m_{i_l}}{q_{i_l} - 4} \geq 1. \]

We remark that Theorem 4.2 also holds for the case where at least one of $m_1, \ldots, m_n$ is positive and the others are zeros. For instance, we assume that $g := g_{i_1}$ and $\hat{g} := \hat{g}_{i_1}$ are nonconstant and the others are constant. If $m := m_{i_1}$ is a positive integer and the others are zeros, then inequality (11) coincides with
\[ \frac{m}{q - 4} \geq 1 \iff q \leq m + 4, \]
where \( q := q_1 \). The result corresponds with Theorem 2.6.

We will apply these results to the Gauss maps of complete minimal surfaces in \( \mathbb{R}^4 \). We briefly summarize here basic facts on minimal surfaces in \( \mathbb{R}^4 \). For more details, we refer the reader to [8, 18, 19, 45]. Let \( X = (x^1, x^2, x^3, x^4) : \Sigma \to \mathbb{R}^4 \) be an oriented minimal surface in \( \mathbb{R}^4 \). By associating a local complex coordinate \( z = u + \sqrt{-1}v \) with each positive isothermal coordinate system \((u, v)\), \( \Sigma \) is considered as a Riemann surface whose conformal metric is the induced metric \( ds^2 \) from \( \mathbb{R}^4 \). With respect to the local complex coordinate \( z = u + \sqrt{-1}v \) of the surface, it holds that

\[
\partial \partial X = 0,
\]

where \( \partial = (\partial/\partial u - \sqrt{-1}\partial/\partial v)/2 \), \( \bar{\partial} = (\partial/\partial u + \sqrt{-1}\partial/\partial v)/2 \). Hence each \( \phi_i := \partial x^i dz \) (\( i = 1, 2, 3, 4 \)) is a holomorphic 1-form on \( \Sigma \). If we set

\[
\omega = \phi_1 - \sqrt{-1}\phi_2, \quad g_1 = \frac{\phi_3 + \sqrt{-1}\phi_4}{\phi_1 - \sqrt{-1}\phi_2}, \quad g_2 = \frac{-\phi_3 + \sqrt{-1}\phi_4}{\phi_1 - \sqrt{-1}\phi_2},
\]

then \( \omega \) is a holomorphic 1-form, and \( g_1 \) and \( g_2 \) are meromorphic functions on \( \Sigma \). Moreover the holomorphic map \( G := (g_1, g_2) : \Sigma \to \overline{\mathbb{C}} \times \overline{\mathbb{C}} \) coincides with the Gauss map of \( X(\Sigma) \). We remark that the Gauss map of \( X(\Sigma) \) in \( \mathbb{R}^4 \) is the map from each point of \( \Sigma \) to its oriented tangent plane, the set of all oriented (tangent) planes in \( \mathbb{R}^4 \) is naturally identified with the quadric

\[
Q^2(\mathbb{C}) = \{[w^1 : w^2 : w^3 : w^4] \in \mathbb{P}^3(\mathbb{C}) ; (w^1)^2 + \cdots + (w^4)^2 = 0\}
\]

in \( \mathbb{P}^3(\mathbb{C}) \), and \( Q^2(\mathbb{C}) \) is biholomorphic to the product of the Riemann spheres \( \overline{\mathbb{C}} \times \overline{\mathbb{C}} \). Furthermore the induced metric from \( \mathbb{R}^4 \) is given by

\[
(12) \quad ds^2 = (1 + |g_1|^2)(1 + |g_2|^2)|\omega|^2.
\]

Applying Theorem 4.1 to the induced metric, we can obtain the Fujimoto theorem for the Gauss map of a complete minimal surface in \( \mathbb{R}^4 \).

**Theorem 4.3.** [11, Theorem II] Let \( X : \Sigma \to \mathbb{R}^4 \) be a complete nonflat minimal surface and \( G = (g_1, g_2) : \Sigma \to \overline{\mathbb{C}} \times \overline{\mathbb{C}} \) the Gauss map of \( X(\Sigma) \).

(i) Assume that \( g_1 \) and \( g_2 \) are both nonconstant and omit \( q_1 \) and \( q_2 \) distinct values respectively. If \( q_1 > 2 \) and \( q_2 > 2 \), then we have

\[
(13) \quad \frac{1}{q_1 - 2} + \frac{1}{q_2 - 2} \geq 1.
\]

(ii) If either \( g_1 \) or \( g_2 \), say \( g_2 \), is constant, then \( g_1 \) can omit at most 3 distinct values.

**Proof.** We first show (i). Since \( g_1 \) and \( g_2 \) are both nonconstant and \( m_1 = m_2 = 1 \) from \( 12 \), we can prove inequality \( 13 \) by Theorem 4.1. Next we show (ii). If we set that \( g_1 \) omits \( q_1 \) values, then we obtain

\[
\frac{1}{q_1 - 2} \geq 1
\]

from Theorem 4.1 because \( m_1 = 1 \). Thus we have \( q_1 \leq 3 \). □

Hence we reveal that the Fujimoto theorem depends on the orders of the factors \((1 + |g_1|^2)\) and \((1 + |g_2|^2)\) in the induced metric from \( \mathbb{R}^4 \) and the Euler characteristic of the Riemann sphere \( \overline{\mathbb{C}} \). In [1], we give some applications of this theorem, for example, to provide optimal results for
the maximal number of exceptional values of the nontrivial part of the Gauss map of a complete minimal Lagrangian surface in the complex 2-space \( \mathbb{C}^2 \) and the generalized Gauss map of a complete nonorientable minimal surface in \( \mathbb{R}^4 \).

In the same way, by Theorem 4.2, we obtain a unicity theorem for the Gauss maps of complete minimal surfaces in \( \mathbb{R}^4 \).

**Theorem 4.4.** [17, Theorem 1.2] Let \( X: \Sigma \rightarrow \mathbb{R}^4 \) and \( \hat{X}: \hat{\Sigma} \rightarrow \mathbb{R}^4 \) be two nonflat minimal surfaces, and \( G = (g_1, g_2): \Sigma \rightarrow \overline{\mathbb{C}} \times \overline{\mathbb{C}}, \hat{G} = (\hat{g}_1, \hat{g}_2): \hat{\Sigma} \rightarrow \overline{\mathbb{C}} \times \overline{\mathbb{C}} \) the Gauss maps of \( X(\Sigma) \), \( \hat{X}(\hat{\Sigma}) \) respectively. We assume that there exists a conformal diffeomorphism \( \Psi: \Sigma \rightarrow \hat{\Sigma} \) and either \( X(\Sigma) \) or \( \hat{X}(\hat{\Sigma}) \) is complete.

(i) Assume that \( g_1, g_2, \hat{g}_1, \hat{g}_2 \) are nonconstant and, for each \( i (i = 1, 2) \), \( g_i \) and \( \hat{g}_i \circ \Psi \) share \( p_i > 4 \) distinct values. If \( g_1 \neq \hat{g}_1 \circ \Psi \) and \( g_2 \neq \hat{g}_2 \circ \Psi \), then we have

\[
\frac{1}{p_1 - 4} + \frac{1}{p_2 - 4} \geq 1.
\]

In particular, if \( p_1 \geq 7 \) and \( p_2 \geq 7 \), then either \( g_1 \equiv \hat{g}_1 \circ \Psi \) or \( g_2 \equiv \hat{g}_2 \circ \Psi \), or both hold.

(ii) Assume that \( g_1, \hat{g}_1 \) are nonconstant, and \( g_1 \) and \( \hat{g}_1 \circ \Psi \) share \( p \) distinct values. If \( g_1 \neq \hat{g}_1 \circ \Psi \) and \( g_2 \equiv \hat{g}_2 \circ \Psi \) is constant, then we have \( p \leq 5 \). In particular, if \( p \geq 6 \), then \( G \equiv \hat{G} \circ \Psi \).

4.2. **Gauss map of a complete minimal surface of finite total curvature in \( \mathbb{R}^3 \).** We review some of the standard facts on complete minimal surfaces of finite total curvature in \( \mathbb{R}^3 \). Let \( X = (x^1, x^2, x^3): \Sigma \rightarrow \mathbb{R}^3 \) be an oriented minimal surface in \( \mathbb{R}^3 \). Set \( \phi_i := \partial x^i \right.dz \) \( (i = 1, 2, 3) \). These satisfy

(C) \( \sum \phi_i^2 = 0 \): conformal condition,

(R) \( \sum \left| \phi_i \right|^2 > 0 \): regularity condition,

(P) For every loop \( \gamma \in H_1(\Sigma, \mathbb{Z}) \), \( \Re \int_\gamma \phi_i = 0 \): period condition.

For the meromorphic function \( g \) and holomorphic 1-form \( \omega \) given by (7),

\[
\phi_1 = \frac{1}{2}(1 - g^2)\omega, \quad \phi_2 = \frac{\sqrt{-1}}{2}(1 + g^2)\omega, \quad \phi_3 = g\omega
\]

hold. We call \((\omega, g)\) the Weierstrass data (W-data, for short). If we are given the W-data on \( \Sigma \), we get \( \phi_j \)'s by this formula. They satisfy condition (C) automatically, and condition (R) is interpreted as the poles of \( g \) of order \( l \) coincides exactly with the zeros of \( \omega \) of order \( 2k \), because the induced metric \( ds^2 \) is given by (8). In general, for a given meromorphic function \( g \) on \( \Sigma \), it is not so hard to find a holomorphic 1-form \( \omega \) satisfying condition (R). However, the period condition (P) always causes trouble. The total curvature of \( X(\Sigma) \) is given by

\[
\tau(\Sigma) := \int_\Sigma K ds^2 dA = - \int_\Sigma 2\sqrt{-1}dg \wedge d\bar{g} \over (1 + |g|^2)^2,
\]

where \( dA \) is the area element with respect to the metric \( ds^2 \). Note that \( |\tau(\Sigma)| \) is the area of \( \Sigma \) with respect to the metric induced from the Fubini-Study metric of the Riemann sphere \( \overline{\mathbb{C}} \) by \( g \).

**Theorem 4.5.** A complete minimal surface of finite total curvature \( X: \Sigma \rightarrow \mathbb{R}^3 \) satisfies
(i) $\Sigma$ is conformally to $\Sigma_\gamma \setminus \{p_1, \ldots, p_k\}$, where $\Sigma_\gamma$ is a closed Riemann surface of genus $\gamma$ and $p_1, \ldots, p_k \in \Sigma$ \cite{20},

(ii) The W-date $(\omega, g)$ can be extended meromorphically to $\Sigma_\gamma$ \cite{45}.

From this fact, we call such surfaces algebraic minimal surfaces. Osserman proved the following result for the number of exceptional values of the Gauss map of a complete minimal surface of finite total curvature in $\mathbb{R}^3$.

**Theorem 4.6.** \cite{45, Theorem 3} The Gauss map of a nonflat complete minimal surface of finite total curvature omits at most 3 values.

The author, Kobayashi and Miyaoka refined Theorem 4.6 and give the following estimate for the number of exceptional values of the Gauss map of a complete minimal surface of finite total curvature in $\mathbb{R}^3$.

**Theorem 4.7.** \cite{26, Theorem 3.3} Let $X: \Sigma = \Sigma_\gamma \setminus \{p_1, \ldots, p_k\} \to \mathbb{R}^3$ be a nonflat complete minimal surface of finite total curvature, $g: \Sigma \to \mathbb{C}$ its Gauss map, $d$ the degree of $g$ considered as a map on $\Sigma_\gamma$. Then the number $D_g$ of exceptional values of $g$ satisfies

\begin{equation}
D_g \leq 2 + \frac{2}{R}, \quad R = \frac{d}{\gamma - 1 + (k/2)} > 1.
\end{equation}

**Proof.** By a suitable rotation of the surface in $\mathbb{R}^3$, we may assume that the Gauss map $g$ has neither zero nor pole at $p_j$ and that the zeros and poles of $g$ are simple. The simple poles of $g$ coincide with the double zeros of $\omega$ because the surface satisfies the regularity condition (R). By the completeness of the surface, $\omega$ has a pole at each end $p_j$ \cite{35, 46, Lemma 9.6}). Moreover, since the surface satisfies the period condition (P), $\omega$ has a pole of order $\mu_j \geq 2$ at $p_j$ \cite{45}. Applying the Riemann-Roch theorem to $\omega$ on $\Sigma_\gamma$, we obtain that

\[ 2d - \sum_{j=1}^{k} \mu_j = 2\gamma - 2. \]

Thus we have

\begin{equation}
D_g \leq 2 + \frac{2}{R}, \quad R = \frac{d}{\gamma - 1 + (k/2)} > 1.
\end{equation}

On the other hand, we assume that $g$ omits $D_g$ values. Let $n_0$ be the sum of the branching orders at the image of exceptional values. Then we have

\[ k \geq dD_g - n_0. \]

Let $n_g$ be the total branching order of $g$ on $\Sigma_\gamma$. Then applying the Riemann-Hurwitz formula to the meromorphic function $g$ on $\Sigma_\gamma$, we have

\begin{equation}
n_g = 2(d + \gamma - 1).
\end{equation}

Hence we have

\[ D_g \leq \frac{n_0 + k}{d} \leq \frac{n_g + k}{d} = 2 + \frac{2}{R}. \]

$\square$
Remark 4.8. More precisely, \((15)\) holds for the totally ramified value number \(\nu_g\) for the Gauss map of a complete minimal surface of finite total curvature in \(\mathbb{R}^3\). One of the most important results for the number is to discover nonflat complete minimal surfaces of finite total curvature in \(\mathbb{R}^3\) with \(\nu_g = 2.5\). For the details, see [22].

By the proof of Theorem 4.7 we reveal that the reason why the upper bound for \(D_g\) changes from ‘4’ to ‘3’ is that the order \(\mu_j\) of a pole of \(\omega\) at each end \(p_j\) changes from ‘\(\mu_j \geq 1\)’ to ‘\(\mu_j \geq 2\)’. Remark that this principle is equivalent to the distinction between the Cohn-Vossen inequality and the Osserman inequality on the total curvature of a complete minimal surface of finite total curvature in \(\mathbb{R}^3\).

There still remains the following question.

**Problem 4.9 ([45]).** Does there exist a complete minimal surface of finite total curvature in \(\mathbb{R}^3\) whose Gauss map omits 3 values?

If so, Theorems 4.6 and 4.7 are optimal. If not, the maximum is ‘2’ and is attained by the catenoid and examples constructed by Miyaoka and Sato [38]. In regards to this problem, the following facts are well-known.

**Proposition 4.10.** For a nonflat complete minimal surface of finite total curvature in \(\mathbb{R}^3\),

- (i) When \(\gamma = 0\), the Gauss map omits at most 2 values ([15], [26]),
- (ii) When \(\gamma = 1\) and the surface has a non-embedded end, the Gauss map omits at most 2 values ([15], [10], [26]),
- (iii) If the Gauss map omits 3 values, then \(\gamma \geq 1\) and the total curvature \(\tau(\Sigma) \leq -20\pi\) ([15], [53], [10]).

By virtue of Theorem 4.7 and this proposition, if there exists a complete minimal surface of finite total curvature whose Gauss map omits 3 values, then it has the complexity of topological data. Thus it is very hard to solve the period condition (P) of the surface. This is the difficulty of this problem.

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