Von Neumann Modules, Intertwiners and Self-Duality*

Michael Skeide

Università degli Studi del Molise
Dipartimento S.E.G.e S.
Via de Sanctis
86100 Campobasso, Italy
E-mail: skeide@math.tu-cottbus.de
Homepage: http://www.math.tu-cottbus.de/INSTITUT/lswas/_skeide.html

Bangalore, July 2003

Abstract

We apply the ideas of Muhly, Skeide and Solel [MSS03] of considering von Neumann $\mathcal{B}$–modules as intertwiner spaces for representations of $\mathcal{B}'$ to obtain a new, simple and self-contained proof for self-duality of von Neumann modules. This simplifies also the approach of [MSS03].

*This work is supported by DAAD and ISI Bangalore
1 Introduction

Let $E$ be a Hilbert module over a von Neumann algebra $\mathcal{B} \subset \mathcal{B}(G)$ acting (non-degenerately) on the Hilbert space $G$. We define the Hilbert space $H = E \otimes G$ as the interior tensor product over $\mathcal{B}$ of the right $\mathcal{B}$–module $E$ and the $\mathcal{B}$–$\mathcal{C}$–module $G$ with inner product $\langle x_1 \otimes g_1, x_2 \otimes g_2 \rangle = \langle g_1, (x_1, x_2)g_2 \rangle$. Every $x \in E$ gives rise to a mapping $L_x : g \mapsto x \otimes g$ in $\mathcal{B}(G, H)$ and it is easy to verify that $L_x b = L_x b$ and $L_x^* L_y = \langle x, y \rangle$.

We, therefore, may and will identify every Hilbert $\mathcal{B}$–module over a von Neumann algebra $\mathcal{B} \subset \mathcal{B}(G)$ as a concrete submodule $E \subset \mathcal{B}(G, H)$ of operators, where $H = E \otimes G$. Following Skeide [Ske00, Ske01] we say $E$ is a von Neumann $\mathcal{B}$–module, if $E$ is strongly closed in $\mathcal{B}(G, H)$.

On $H$ we define a (normal unital) representation $\rho' : \mathcal{B}' \to \mathcal{B}(H)$ of the commutant $\mathcal{B}'$ of $\mathcal{B}$ by $\rho'(b')(id_E \otimes b')$. (This is well-defined, because $b'$ is a bilinear mapping on the $\mathcal{B}$–$\mathcal{C}$–module $G$, and also checking normality is routine.) In the special case when $E$ is the GNS-module of a completely positive mapping with values in $\mathcal{B}$ (see Paschke [Pas73]), $\rho'$ is known as commutant lifting (Arveson [Arv69]).

Following Skeide [Ske98, Ske00], the $\mathcal{B}'$–center of the $\mathcal{B}'$–$\mathcal{B}'$–module $\mathcal{B}(G, H)$ is defined as

$$C_{\mathcal{B}'}(\mathcal{B}(G, H)) = \{ x \in \mathcal{B}(G, H) : \rho'(b')x = xb' (b' \in \mathcal{B}') \}.$$ 

As observed, for instance, by Goswami and Sinha [GS99], it is easy to check that $C_{\mathcal{B}'}(\mathcal{B}(G, H))$ is a itself a von Neumann $\mathcal{B}$–module.

Clearly, $E$ is contained in $C_{\mathcal{B}'}(\mathcal{B}(G, H))$.

It is the starting point in [MSS03] to show that $E$ is all of $C_{\mathcal{B}'}(\mathcal{B}(G, H))$. Once known that von Neumann modules are self dual, i.e. every bounded right linear mapping $\Phi : E \to \mathcal{B}$ (so-called $\mathcal{B}$–functionals) has the form $\langle x, \bullet \rangle$ for a (unique) $x \in E$, (see [Ske00, Ske01] for proof using complete quasi orthonormal systems, a suitable geralization of orthonormal bases in Hilbert spaces) this task is easy: Like for Hilbert spaces a strongly closed (and, therefore, self-dual) submodule with zero-complement is all. And since $EG$ is total in $H$ the complement of $E$ in $C_{\mathcal{B}'}(\mathcal{B}(G, H))$ is, indeed, $\{0\}$.

1.1 Remark. The other important observation in [MSS03] is that for an arbitrary (normal unital) representation $\rho'$ of $\mathcal{B}'$ on a Hilbert space, $C_{\mathcal{B}'}(\mathcal{B}(G, H))$ is a von Neumann $\mathcal{B}$–module acting totally on $G$, what gives a one-to-one correspondence between von Neumann $\mathcal{B}$–modules contained in $\mathcal{B}(G, H)$ and representations $\rho'$ of $\mathcal{B}'$ on $H$.

This approach becomes particularly fruitful, when the von Neumann modules are two-sided so that there is arround another (normal unital) representation $\rho$ on $H$ of a second von Neumann algebra $\mathcal{A}$. Switching the roles of $\mathcal{B}$ and $\mathcal{A}'$, the result is a duality between $\mathcal{A}$–$\mathcal{B}$–modules and...
\( B' - \mathcal{A}' \)-modules generalizing the duality between a von Neumann algebra and its commutant. One application is a complete theory of normal representations of the adjointable operators on a von Neumann \( B \)-module on a von Neumann \( A \)-module (this can be, e.g., a Hilbert space).

In this short note we show that \( E = C_{B'}(B(G, H)) \) and we show that \( C_{B'}(B(G, H)) \) is self-dual, thus, showing that \( E \) is self-dual. The only prerequisite for the first statement is von Neumann’s double commutant theorem, the only prerequisite for the second statement is a technical lemma which asserts that every \( B \)-functional \( \Phi \) is represented by an operator in \( B(H, G) \) (see below). The simplicity of the proofs improves also accessibility of [MSS03] and, therefore, of the whole theory of von Neumann modules.

2 \( E = C_{B'}(B(G, H)) \)

Every \( a \in B^a(E) \) (the algebra of adjointable operators on \( E \)) gives rise to a bounded operator \( x \odot g \mapsto ax \odot g \) on \( H \). In that way, we identify \( B^a(E) \) as a \( * \)-subalgebra of \( B(H) \). It is easy to see that \( B^a(E) \) is a von Neumann subalgebra of \( B(H) \).

It follows that the matrix \( * \)-algebra

\[
\mathcal{M} = \begin{pmatrix}
B & E^* \\
E & B^a(E)
\end{pmatrix}
\]

with the obvious operations is a von Neumann algebra on \( G \oplus H \). Let us compute its commutant.

2.1 Proposition. The commutant of \( \mathcal{M} \) is \( \mathcal{M}' = \{ (b' 0) : b' \in B' \} \).

Proof. Suppose \( \left( \begin{smallmatrix} b' & y' \\ x' & a' \end{smallmatrix} \right) \in B(G \oplus H) \) is an element in \( \mathcal{M}' \). As it must commute with \( \left( \begin{smallmatrix} b & 0 \\ 0 & 0 \end{smallmatrix} \right) \) (\( b \in B \)) we find

\[
\left( \begin{smallmatrix} b'b & 0 \\ x'b & 0 \end{smallmatrix} \right) = \left( \begin{smallmatrix} bb' & by'^* \\ 0 & 0 \end{smallmatrix} \right) .
\]

As this must hold for all \( b \in B \) (in particular also for \( b = 1 \)), we find \( x' = y' = 0 \) and \( b' \in B' \). The remaining part \( \left( \begin{smallmatrix} b' & 0 \\ 0 & a' \end{smallmatrix} \right) \) must commute with \( \left( \begin{smallmatrix} 0 & 0 \\ x & 0 \end{smallmatrix} \right) (x \in E) \). Therefore,

\[
\left( \begin{smallmatrix} 0 & a'x \\ 0 & 0 \end{smallmatrix} \right) = \left( \begin{smallmatrix} 0 & xb' \\ 0 & 0 \end{smallmatrix} \right) .
\]

We find \( a'(x \odot g) = a'xg = xb'g = \rho'(b')(x \odot g) \) for all \( x \in E, g \in G \) and, therefore, \( a' = \rho'(b') \). □

The commutant of \( \mathcal{M}' \) is, clearly,

\[
\mathcal{M}'' = \begin{pmatrix}
B & C_{B'}(B(H, G)) \\
C_{B'}(B(G, H)) & \rho'(B')'
\end{pmatrix} .
\]
By the *double commutant theorem* $\mathcal{M}'' = \mathcal{M}$. Therefore, we do not only show the statement of this section’s headline, but, as an additional benefit, we identify also $\mathcal{B}^a(E)$ as the commutant of the image of $\mathcal{B}'$ under $\rho'$. (This can also be done by using *Morita equivalence* for von Neumann algebras; see Rieffel [Rie74].)

2.2 Proposition. $E = C_{\mathcal{B}'}(\mathcal{B}(G,H))$ and $\mathcal{B}^a(E) = \rho'((\mathcal{B}')')$.

3. **$C_{\mathcal{B}'}(\mathcal{B}(G,H))$ is self-dual**

A $\mathcal{B}$–functional $\Phi \in \mathcal{B}'(C_{\mathcal{B}'}(\mathcal{B}(G,H)), \mathcal{B})$ gives rise to a linear mapping

$$L_\Phi : \text{span} C_{\mathcal{B}'}(\mathcal{B}(G,H)) G \to G \quad L_\Phi(x \otimes g) = (\Phi x)g.$$  

The proof of the following lemma consists, essentially, in showing that for computing the operator norm of $L_\Phi$ it is sufficient to take the supremum only over elementary tensors $x \otimes g$ ($\|x\| \leq 1, \|g\| \leq 1$).

3.1 Lemma. $\|L_\Phi\| = \|\Phi\|$. Therefore, $L_\Phi$ extends to a bounded operator in $\mathcal{B}(H,G)$ identified with $\Phi$.

PROOF. (Sketch only. See [Ske00] [Ske01] for details.) Suppose that there is a cyclic vector $g_0 \in G$, i.e. $\mathcal{B}g_0$ is dense in $G$. (Otherwise, use a decomposition of $G$ into subspaces $G_\alpha$ cyclic for $\mathcal{B}$ and take into account the facts, firstly, that also $H$ decomposes accordingly into cyclic subspaces $H_\alpha$ and, secondly, that the norm of an element in a direct sum of operator spaces $\mathcal{B}(G_\alpha, H_\alpha)$ is just the supremum over the single norms.) Then every element in $H$ can be approximated by those of the form $h = x \otimes g_0$. Use polar decomposition $x = x_0 |x|$ of $x$ and put $g = |x|g_0$. Then $\|h\| = \|g\|$ because $g \in |x|G$. In particular, every unit vector in $H$ can be approximated by $x \otimes g$ where $x$ is a partial isometry in $E$ and $g$ is a unit vector in $G$.

3.2 Proposition. $C_{\mathcal{B}'}(\mathcal{B}(G,H))$ is self-dual.

PROOF. From $\Phi \rho'(b') (x \otimes g) = \Phi (x \otimes b'g) = \Phi x b'g = b'\Phi x g = b'\Phi(x \otimes g)$ we see that $\Phi$ intertwines $b'$ and $\rho'(b')$. Therefore the adjoint $y = \Phi^*$ of $\Phi$ is an element in $C_{\mathcal{B}'}(\mathcal{B}(G,H))$ such that $\Phi x = \langle y, x \rangle$ for all $x \in C_{\mathcal{B}'}(\mathcal{B}(G,H))$.

4. **Synthesis**

4.1 Theorem. *Every von Neumann $\mathcal{B}$–module is self-dual.*

PROOF. $E = C_{\mathcal{B}'}(\mathcal{B}(G,H))$ and $C_{\mathcal{B}'}(\mathcal{B}(G,H))$ is self-dual.
4.2 Remark. It seems that Lemma 3.1 forms always an essential part of the proofs of self-duality, which cannot be replaced by simpler arguments.

4.3 Remark. To be honest, we should mention that (unlike the approach by complete quasi orthonormal systems) the preceding arguments cannot be used to show the Riesz representation theorem (Hilbert spaces are self-dual), but, actually, reduce the statement about von Neumann modules to that about Hilbert spaces. An equivalent form of the Riesz representation theorem is that all bounded operators between Hilbert spaces have an adjoint. Without this, in the proof of Proposition 3.2 it was not possible to pass from $\Phi$ to $\Phi^*$. One may see the failure of the argument clearly, by taking $B = \mathbb{C}$ and $G = \mathbb{C}$ and for $H$ only a pre-Hilbert space. This is, actually, the only place in these notes, where we are not able to write down an adjoint explicitly on the algebraic domain $\text{span} \, EG$. (The adjoint of $x: \, g \mapsto x \odot g$ is, of course, $x^*: \, y \odot g \mapsto \langle x, y \rangle g$.)

Acknowledgement. These notes were written during a two months stay of the author at ISI Bangalore. The author wishes to express his gratitude to Prof. B.V.R. Bhat and the ISI for warm hospitality and to the DAAD for travel support.

References

[Arv69] W. Arveson, Subalgebras of $C^*$–algebras, Acta Math. 123 (1969), 141–224.

[GS99] D. Goswami and K.B. Sinha, Hilbert modules and stochastic dilation of a quantum dynamical semigroup on a von Neumann algebra, Commun. Math. Phys. 205 (1999), 377–403.

[MSS03] P.S. Muhly, M. Skeide, and B. Solel, (Tentative title) On product systems of $W^*$–modules and their commutants, Preprint, Campobasso, in preparation, 2003.

[Pas73] W.L. Paschke, Inner product modules over $B^*$–algebras, Trans. Amer. Math. Soc. 182 (1973), 443–468.

[Rie74] M.A. Rieffel, Morita equivalence for $C^*$–algebras and $W^*$–algebras, J. Pure Appl. Algebra 5 (1974), 51–96.

[Ske98] M. Skeide, Hilbert modules in quantum electro dynamics and quantum probability, Commun. Math. Phys. 192 (1998), 569–604.

[Ske00] ______, Generalized matrix $C^*$–algebras and representations of Hilbert modules, Mathematical Proceedings of the Royal Irish Academy 100A (2000), 11–38.

[Ske01] ______, Hilbert modules and applications in quantum probability, Habilitationsschrift, Cottbus, 2001, Available at [http://www.math.tu-cottbus.de/INSTITUT/lwas/_skeide.html](http://www.math.tu-cottbus.de/INSTITUT/lwas/_skeide.html)