QUASI–COXETER QUASITRIANGULAR QUASIBIALGEBRAS
AND THE CASIMIR CONNECTION

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ABSTRACT. Let \( \mathfrak{g} \) be a complex, semisimple Lie algebra. We prove the existence of a quasi–Coxeter quasitriangular quasibialgebra structure on the enveloping algebra of \( \mathfrak{g} \), which binds the quasi–Coxeter algebra structure underlying the Casimir connection of \( \mathfrak{g} \) and the quasitriangular quasibialgebra one underlying its KZ equations. This implies in particular that the monodromy of the rational Casimir connection of \( \mathfrak{g} \) is described by the quantum Weyl group operators of the quantum group \( U_q \mathfrak{g} \).

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1. Introduction

1.1. Let \( \mathfrak{g} \) be a complex, semisimple Lie algebra. De Concini [7], and independently the author [24], conjectured that the monodromy of the Casimir connection of \( \mathfrak{g} \) is described by the quantum Weyl group group operators of the quantum group \( U_q \mathfrak{g} \), in a way analogous to the Drinfeld–Kohno theorem [13].

This conjecture was proved in [24] for the Lie algebra \( \mathfrak{sl}_n \). For an arbitrary \( \mathfrak{g} \), it was reduced in [25] to a structural statement about the enveloping algebra \( U_q \mathfrak{g} \), namely the existence of a quasi–Coxeter quasitriangular quasibialgebra structure on \( U_q \mathfrak{g} \) binding the quasi–Coxeter structure underlying the Casimir connection.
of \( g \), to the quasitriangular quasibialgebra structures underlying the Knizhnik–Zamolodchikov connections of its standard Levi subalgebras.

The goal of the present paper is to establish the existence of such a structure, and therefore prove the monodromy conjecture for any semisimple Lie algebra.

1.2. Let \( h \subset g \) be a Cartan subalgebra, \( \Phi \subset h^* \) the corresponding root system, and \( h_{\text{reg}} = h \setminus \bigcup_{\alpha \in \Phi} \text{Ker}(\alpha) \) the set of regular elements in \( h \). Fix an non–degenerate, invariant bilinear form \( \langle \cdot , \cdot \rangle \) on \( g \). The Casimir connection of \( g \) is a connection on the holomorphically trivial bundle \( V \) on \( h_{\text{reg}} \) with fibre a given finite–dimensional representation \( V \) of \( g \). It is given by

\[
\nabla_C = d - \frac{h}{2} \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} \cdot C_\alpha
\]

where \( h \in \mathbb{C} \) is a deformation parameter, \( \alpha \) ranges over a chosen system of positive roots \( \Phi_+ \), and \( C_\alpha \) is the Casimir operator induced by the restriction of \( \langle \cdot , \cdot \rangle \) to the three–dimensional subalgebra \( \mathfrak{sl}_2 \subset g \) corresponding to the root \( \alpha \). The connection \( \nabla_C \) is flat for any \( h \) \([22, 24, 7, 20]\), and can be made equivariant with respect to the Weyl group \( W \) of \( g \) \([22, 24]\). Its monodromy defines a one–parameter family of actions \( \mu_h \) of the braid group \( B_W = \pi(h_{\text{reg}}/W) \) on \( V \) depending on \( h \), which deforms the action of \( (\text{the Tits extension of}) \) \( W \). We denote by \( \mu : B_W \to G(V[h]) \) the formal Taylor series of \( \mu_h \) with respect to the parameter \( h = \pi \hbar \).

1.3. Let \( U_h g \) be the Drinfeld–Jimbo quantum group corresponding to \( g \), thought of as a topological Hopf algebra over \( \mathbb{C}[h] \). Let \( V \) be a quantum deformation of \( V \), that is a topologically free \( \mathbb{C}[h] \)–module such that \( V/hV \) is isomorphic to \( V \) as a \( g \)–module. Since \( V \) is integrable, the braid group \( B_W \) acts on \( V \) though the quantum Weyl group operators of \( U_h g \) \([21]\). The main result of the present paper is the following.

**Theorem.** The monodromy \( \mu : B_W \to GL(V[h]) \) of the Casimir connection on \( V \) is equivalent to the quantum Weyl group action of the braid group \( B_W \) on \( V \).

1.4. Recall that a quasitriangular quasibialgebra is an algebra \( A \) over a commutative ring \( k \) endowed with two morphisms, the coproduct \( \Delta : A \to A \otimes A \) and counit \( \varepsilon : A \to k \), and two distinguished invertible elements, the \( R \)–matrix \( R \in A \otimes A \) and associator \( \Phi \in A \otimes A \otimes A \) \([12]\). The relations satisfied by \( \Delta, \varepsilon, R \) and \( \Phi \) are designed so as to endow the category of \( A \)–modules with the structure of a braided tensor category. In particular, for any \( V \in \text{Mod}(A) \), there is a family of actions \( \rho_b : B_n \to GL(V^{\otimes n}) \) of the \( n \)–strand braid group on the \( n \)–fold tensor product of \( V \), which are labelled by the choice of a complete bracketing \( b \) of the non–associative monomial \( x_1 \cdots x_n \). The actions corresponding to different choices of \( b \) are canonically isomorphic, via intertwiners built out of the associator \( \Phi \). For example, for \( n = 3 \), the action of \( \Phi \) on \( V^{\otimes 3} \) intertwines \( \rho((x_1x_2)x_3) \) and \( \rho(x_1(x_2x_3)) \).

1.5. In a similar spirit, a quasi–Coxeter algebra \( A \) is designed so that a module over it carries a family of canonically equivalent representations of the braid group \( B_W \) of a given irreducible Coxeter group \( W \) \([25]\). Central to this notion are the maximal nested sets on the Coxeter graph \( D \) of \( W \), which generalise complete bracketings to an arbitrary Coxeter type. These were introduced by De Concini–Procesi \([8, 9]\),

\[1\]the connection is independent of the choice of \( \Phi_+ \).
and consist of maximal collections of connected subgraphs of $D$ which are pairwise compatible, that is such that either one is contained in the other, or they are orthogonal, in the sense that they have disjoint vertex sets and are not linked by an edge of $D$. When $W$ is of type $A_{n-1}$, with the vertices of $D$ labelled $1, \ldots, n-1$ as in [4, Planche 1], mapping a connected subdiagram $B$ with vertices $\{i, \ldots, j\}$ to the pair of parentheses $x_1 \cdot \cdot \cdot x_{i-1}(x_i \cdot \cdot \cdot x_j)x_{j+1} \cdot \cdot \cdot x_n$, and noting that $B, B' \subseteq D$ are compatible precisely when the corresponding pairs of parentheses are consistent, yields a bijection between maximal nested sets on $D$ and complete bracketings of the monomial $x_1 \cdot \cdot \cdot x_n$.

1.6. A quasi-Coxeter algebra is endowed with three sets of data. First, a collection of subalgebras $A_B$ labelled by the connected subgraphs $B \subseteq D$, which are such that $A_{B_1} \subseteq A_{B_2}$ if $B_1 \subseteq B_2$, and $[A_{B_1}, A_{B_2}] = 0$ if $B_1$ and $B_2$ are orthogonal. Next, invertible elements $\Phi_{B,F} \in A$ called associators. These are labelled by pairs of maximal nested sets on $D$, and satisfy in particular the transitivity relations $\Phi_{HF}\Phi_{GF} = \Phi_{HG}$. Finally, invertible elements $\Phi_i$ labelled by the vertices of $D$, and called local monodromies. This data satisfies various compatibility relations, in particular a version of the braid relations defining $B_W$. They give rise to a family of actions $\lambda_F : B_W \rightarrow GL(V)$ on any $V \in \text{Mod}(A)$, which are labelled by the maximal nested sets on $D$, with $\Phi_{g,F} \in A$ intertwining $\lambda_F$ and $\lambda_G$.

1.7. A quasi-Coxeter quasitriangular quasibialgebra of type $W$ is a quasi-Coxeter algebra $A$ additionally endowed with a coproduct $\Delta$ and counit $\varepsilon$, such that each diagrammatic subalgebra $A_B$ has a quasitriangular quasibialgebra structure of the form $(\Delta, \varepsilon, R_B, \Phi_B)$, for a given $R$-matrix $R_B \in A_B^{\otimes 2}$ and associator $\Phi_B \in A_B^{\otimes 3}$. This gives rise in particular to a family of commuting representations of the groups $B_n, B_W$ on the tensor power $V^\otimes n$ of any $V \in \text{Mod}(A)$, specifically

$$\rho_{B,b} : B_n \rightarrow GL(V) \quad \text{and} \quad \lambda_{F,b} : B_W \rightarrow GL(V)$$

The first family is determined by the choice of a subdiagram $B \subseteq D$ and a bracketing $b$ of the monomial $x_1 \cdot \cdot \cdot x_n$, and arises by restricting $V$ to the quasitriangular quasibialgebra $A_B$, with the representations $\pi_{B,b}$ and $\pi_{B',b'}$ equivalent provided $B = B'$. The second arises from the action of $A$ on $V^\otimes n$ determined by the choice of $b$, and depends on the choice of a maximal nested set $F$ on $D$, with $\lambda_{b,F}$ equivalent to $\lambda_{b',G}$ for any $b, b'$ and $F, G$.

1.8. A quasi-Coxeter quasitriangular quasibialgebra $A$ possesses an additional piece of data, which binds the associators $\Phi_B$ coming from the quasitriangular quasibialgebra structure on each diagrammatic $A_B$, to the associators $\Phi_{g,F}$ coming from the quasi-Coxeter structure on $A$. This welding of the quasi-Coxeter and quasitriangular quasibialgebra structures is what gives the examples of interest the rigidity required to determine the monodromy of the Casimir connection.

The additional data consists of relative twists, which are elements $F_{(B,\alpha)} \in A_B^{\otimes 2}$ labelled by a connected subdiagram $B$ and a vertex $\alpha \in B$. These twists are required to satisfy two identities. The first is the twist equation

$$(\Phi_B)_{F_{(B,\alpha)}} = \Phi_{B\backslash \alpha} \quad (1.1)$$
together with the requirement that \( F_{(B, \alpha)} \) commute with \( \Delta(A_{B \setminus \alpha}) \). This amounts to asking that \( F_{(B, \alpha)} \) defines a tensor structure on the restriction functor from the monoidal category of \( A_B \)-modules with associativity constraints given by the associator \( \Phi_B \), to that of \( A_{B \setminus \alpha} \)-modules with associativity constraints given by \( \Phi_{B \setminus \alpha} \).

By restricting in stages from \( A_D \) to \( A_\emptyset = k \), the relative twists allow to define a tensor structure on the forgetful functor from the monoidal category of \( A \)-modules with associativity constraints given by \( \Phi_D \) to \( k \)-modules, which depends on the choice of a maximal nested set \( \mathcal{F} \) on \( D \), as follows. For any element \( B \) of \( \mathcal{F} \), the collection of maximal elements of \( \mathcal{F} \) properly contained in \( B \) covers all but one of the vertices \( \alpha_\mathcal{F}^B \) of \( B \) (and consists in fact of the connected components of \( B \setminus \alpha \)).

Define the twist \( F_F \in A^{\otimes 2} \) by

\[
F_F = \prod_{B \in \mathcal{F}} F_{(B, \alpha_\mathcal{F}^B)}
\]

(1.2)

where the product is taken with \( F_{(B, \alpha)} \) to the right of \( F_{(B', \alpha')} \) if \( B' \subset B \).

Then, it follows from (1.1) that \( (\Phi_B)_{F_F} = 1^{\otimes 3} \).

The second identity satisfied by the relative twists requires that the tensor structures \( F_F \) corresponding to the choices of different maximal nested sets be isomorphic, with the isomorphism given by the associators \( \Phi_{\mathcal{G}_\mathcal{F}} \) of the quasi–Coxeter structure, that is

\[
F_{\mathcal{G}_\mathcal{F}} = \Phi_{\mathcal{G}_\mathcal{F}}^{\otimes 2} \cdot F_F \cdot \Delta(\Phi_{\mathcal{G}_\mathcal{F}})^{-1}
\]

(1.3)

1.9. The quantum group \( U_\hbar \mathfrak{g} \) associated to a complex semisimple Lie algebra \( \mathfrak{g} \) is a quasi–Coxeter quasitriangular quasibialgebra in a very simple way. For any subdiagram \( B \) of the Dynkin diagram \( D \) of \( \mathfrak{g} \), the subalgebra \( U_\hbar \mathfrak{g}_B \) is the quantum group corresponding to the generators of \( U_\hbar \mathfrak{g} \) labelled by the vertices of \( B \), endowed with its universal \( R \)-matrix \( R_B \) and trivial associator \( \Phi_B = 1^{\otimes 3} \). The associators \( \Phi_{\mathcal{G}_\mathcal{F}} \) are all trivial, and the local monodromies \( S_i \) are given by Lusztig’s quantum Weyl group group operators. It was proved in [25, Thm 8.3] that this structure can be transferred to an isomorphic one on \( U_\hbar \mathfrak{g} \) (which, however, does not have trivial associators). Moreover, it was also proved in [25, Thm. 9.1] that quasi–Coxeter quasitriangular quasibialgebra structures on \( U_\hbar \mathfrak{g} \) are rigid, that is determined by their \( R \)-matrices \( R_B \) and local monodromies \( S_i \).

Thus, Theorem 1.3 can be proved by showing the existence of a quasi–Coxeter quasitriangular quasibialgebra structure on \( U_\hbar \mathfrak{g} \) which binds the quasi–Coxeter structure underlying the monodromy of the Casimir connection of \( \mathfrak{g} \), to the quasitriangular quasibialgebra structure underlying that of the Knizhnik–Zamolodchikov connections of its standard Levi subalgebras.

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2The twist \( \Phi_F \) of an associator \( \Phi \) by a twist \( F \) is equal to \( 1 \otimes F \cdot \text{id} \otimes \Delta(F) \cdot \Phi \cdot \Delta \otimes \text{id}(F)^{-1} \cdot F^{-1} \otimes 1 \). If \( B \setminus \alpha \) is not connected, the associator \( \Phi_{B \setminus \alpha} \) is taken to be the (commuting) product \( \prod_{B_i} \Phi_{B_i} \), where \( B_i \) runs over the connected components of \( B \setminus \alpha \).

3In type \( A_{n-1} \), if \( B \) is the diagram with vertices \( i, \ldots, j - 1 \), the elements of \( \mathcal{F} \) properly contained in \( B \) give a bracketing of the monomial \( x_i \cdots x_j \), which necessarily contains two maximal pairs of parentheses of the form \( (x_{i+1} \cdots x_{j-1})x_j \), and \( \alpha_\mathcal{F}^B \) is the vertex \( k \).

4This does not specify the order of the factors uniquely, but any two orders satisfying the above requirements are easily seen to give rise to the same product.

5This is not the case of quasi–Coxeter algebra structures on \( U_\hbar \mathfrak{g} \).
1.10. It is a well-known, and beautiful observation of Drinfeld’s that the monodromy of the KZ equations of \( \mathfrak{g} \) gives rise to a quasitriangular quasibialgebra structure on \( U\mathfrak{g}[\hbar] \) [13]. The corresponding \( R \)-matrix is the monodromy \( R_{KZ} = \mathbb{e}^{\hbar \Omega} \) of the KZ equations on \( n = 2 \) points, and the associator \( \Phi_{KZ} \) the ratio \( \Psi_{x_1x_2x_3}^{-1} \cdot \Psi_{x_1(x_2x_3)} \) of the solutions of the KZ equations on \( n = 3 \) points corresponding to the asymptotic zones \( z_1 \gg z_2 \gg z_1 - z_2 \) and \( z_1 \gg z_3 \gg z_2 - z_3 \) respectively. The associativity constraints relating the copies of the \( \Delta \)-fold tensor power \( V^\otimes n \) of a representation \( V \) corresponding to two bracketings \( b,b' \) can be expressed in terms of the associator \( \Phi_{KZ} \), as in any monoidal category, or more directly obtained as the ratio \( \Psi_{b'}^{-1} \cdot \Psi_b \) of the solutions of the KZ equations on \( n \) points corresponding to the asymptotic zones determined by \( b \) and \( b' \).

1.11. Similarly, the monodromy of the Casimir connection of \( \mathfrak{g} \) gives rise to a quasi–Coxeter structure on \( U\mathfrak{g}[\hbar] \). This relies on the De Concini–Procesi construction of a compactification of \( \mathfrak{h}_{\text{reg}} \), where the root hyperplanes are replaced by a normal crossings divisor \([8,9] \). The irreducible components of the divisor which intersect the closure of the Weyl chamber are labelled by the connected subdiagrams of \( D \). The maximal nested sets on the Dynkin diagram \( D \) label the points at infinity, that is the non–empty intersection of a maximal collection of these components. Near each of those, one can construct a canonical fundamental solution \( \Psi_{\nabla} \) having good asymptotics. The associators \( \Phi_{\nabla,F} \) then arise as the ratios \( \Psi_{b'}^{-1} \cdot \Psi_b \).

1.12. The previous paragraphs suggest that the relative twists of a quasi–Coxeter quasitriangular quasibialgebra structure on \( U\mathfrak{g}[\hbar] \) which binds the structures coming from the KZ and Casimir equations might also arise by comparing appropriate solutions of a flat connection. This is indeed the case, as we explain below.

Since the associator \( \Phi_{\nabla,F} \) of the quasi–Coxeter structure underlying the Casimir connection \( \nabla \) is equal to \( \Psi_{\nabla}^{-1} \cdot \Psi_{\nabla,F} \), the latter are the De Concini–Procesi fundamental solutions of \( \nabla \), the compatibility relation (1.3) may be rewritten as

\[
\Psi_{\nabla}^{-1} \cdot \Delta(\Psi_{\nabla}^{-1}) = \Psi_{\nabla,F}^{-1} \cdot \Delta(\Psi_{\nabla,F}^{-1})
\] (1.4)

Either side defines a holomorphic function \( F : \mathfrak{h}_{\text{reg}} \rightarrow U\mathfrak{g}[\hbar] \) which is independent of the choice of a maximal nested set on \( D \) by (1.4), satisfies the differential equation

\[
dF = \frac{\hbar}{2} \sum_{\alpha > 0} \frac{d\alpha}{\alpha} \left( C_{\alpha}^{(1)} + C_{\alpha}^{(2)} \right) F - F \Delta(\alpha)
\] (1.5)

and the twist relation \( (\Phi_{\nabla,F})_F = 1 \otimes^3 \). We shall call such an \( F \) a differential twist. Given \( F \), the twists \( F_{\nabla,F} \) may be recovered as

\[
F_{\nabla,F} = (\Psi_{\nabla,F}^{-1})^{-1} \cdot F \cdot \Delta(\Psi_{\nabla,F})
\]

1.13. We show in Section 4 that the requirement that the twists \( F_{\nabla,F} \) possess the factorised form (1.2), where the factors \( F_{(B,\alpha)} \) satisfy the twist relation (1.1), is equivalent to the following centraliser property of the differential twist \( F \). This assumes that a differential twist \( F_{\mathfrak{g}_B} \) is given for any diagrammatic subalgebra \( \mathfrak{g}_B \subseteq \mathfrak{g} \), and expresses a compatibility between \( F_{\mathfrak{g}_B} \) and \( F_{\mathfrak{g}_{B \cup \alpha}} \), for any vertex \( \alpha \) of \( B \). Specifically, consider the asymptotics \( r \lim_{\alpha \rightarrow \infty} F_{\mathfrak{g}_B}(\mu) \) of \( F_{\mathfrak{g}_B}(\mu) \), as the coordinate \( \alpha \) of \( \mu \in \mathfrak{h}_B \) goes to \( \infty \).\(^6\) These asymptotics are a function of the image \( \overline{\mu} \) of \( \mu \) in

\(^6\)the existence \( r \lim_{\alpha \rightarrow \infty} F_{\mathfrak{g}_B} \) relies on the fact that the Casimir connection has regular singularities.
\( h_{B \setminus \alpha} \), thought of as a quotient of \( h_B \). They solve the Casimir equation (1.5) for the Lie algebra \( g_{B \setminus \alpha} \), which are the limit of the Casimir equations of \( g_B \) as \( \alpha \to \infty \). The centraliser property is the requirement that the element \( F_{(B; \alpha)} \in U g[[\hbar]]^{\otimes 2} \) defined by the equation

\[
\lim_{\alpha \to \infty} F_{\tilde{g}_B} \left( \mu \right) = F_{(B; \alpha)} \cdot F_{\tilde{g}_B^{-\alpha}} \left( \pi \right)
\]

be invariant under \( g_B \). This implies in particular that \( F_{(B; \alpha)} \) does not depend on \( \pi \).

1.14. The construction of a differential twist satisfying the centraliser property can, in turn, be obtained from that of an appropriate fusion operator. The latter is a solution of a dynamical version of the KZ equations in \( n = 2 \) points, namely

\[
\frac{dJ}{dz} = \left( \frac{\hbar}{z} + \text{ad} \mu^{(1)} \right) J
\]

where \( z = z_1 - z_2 \), and \( \mu \in h_{\text{reg}} \). The dynamical KZ equations arise naturally in Conformal Field Theory (see, e.g., [15]). They were studied in more detail by Felder–Markov–Tarasov–Varchenko in [20], where it was shown that they are bispectral to, that is commute with, a dynamical version of the Casimir connection with respect to the variable \( \mu \).

The presence of the dynamical term \( \text{ad} \mu^{(1)} \) creates an irregular singularity at \( z = \infty \) of Poincaré rank 1. Assuming \( \mu \) to be real, so that the Stokes rays of (1.7) all lie on the real axis, we construct in Section 5 two canonical solutions \( J_\pm(z, \mu) \) with values in \( U \hat{g}^{\otimes 2}[\hbar] \), which have the form

\[
J_\pm(z, \mu) = H_\pm(z, \mu) \cdot z^\Omega
\]

where \( H_\pm(z, \mu) \) is a holomorphic function on the upper (resp. lower) half-plane which possesses an asymptotic expansion of the form \( 1^{\otimes 2} + O(\hbar^{-1}) \) as \( z \to \infty \) with \( 0 < \arg z < \pi \), and \( \Omega = t_\mu \otimes t_\mu \), with \( \{ t_\mu \} \) dual bases of \( h \) with respect to \( \langle \cdot , \cdot \rangle \), is the projection of \( \Omega \) on the kernel of \( \text{ad}(\mu^{(1)}) \). The construction of \( J_\pm \) and the study of its analytic properties require some care, since the equation (1.7) takes values in the infinite-dimensional algebra \( U \hat{g}^{\otimes 2}[\hbar] \), and is the main technical contribution of the paper.

Given the fusion operator \( J_\pm(z, \mu) \), a differential twist can be obtained as either of the ratios

\[
F_\pm(\mu) = J_\pm(z, \mu)^{-1} \cdot J_0(z, \mu)
\]

where \( J_0(z, \mu) \) is the unique solution of (1.7) which is asymptotic to \( (1^{\otimes 2} + O(z)) \cdot z^\Omega \) near \( z = 0 \).

1.15. The proof that \( F_\pm(\mu) \) kills the associator is fairly standard (see e.g., [16, 14]), provided an analogue of the fusion operator can be constructed for the dynamical KZ equation in \( n = 3 \) points. Even though the corresponding KZ equations have irregular singularities at \( z_i = \infty \), a solution can still be constructed by simultaneously scaling all variables \( z_1, z_2, z_3 \), and sending the scale to infinity. Here, we crucially exploit a beautiful fact, which is that the dynamical KZ equations in \( n \) variables \textit{abelianise at infinity}. Roughly speaking, this means that the connection

\footnote{The terminology is borrowed from the work of Etingof (see e.g., [16], and references therein). The relation of our construction to [17] is discussed in 1.18.}

\footnote{The existence of \( J_0 \) is straightforward since \( z = 0 \) is a regular singularity of (1.7).}
satisfied by the asymptotics of solutions on the divisor where all $z_i - z_j$ are infinite
is the abelian KZ equations
\[ d - \hbar \sum_{i<j} d \log(z_i - z_j) \Omega_{ij}^b \]
Contrary to its non-abelian analogue, this equation possesses a canonical solution, namely
\[ \prod_{i<j} (z_i - z_j)^{\hbar \Omega_{ij}^b} \]
which leads to the construction of a multicomponent fusion operator of the form
\[ J_{\pm} = H_{\pm}(z_1, \ldots, z_n) \cdot \prod_{i<j} (z_i - z_j)^{\hbar \Omega_{ij}^b} \]
1.16. The fact that the twists $F_{\pm}(\mu)$ satisfy the Casimir equations 1.5 follows from the fact that $J_{\pm}(z, \mu)$ satisfies a dynamical version of the Casimir equations, namely
\[ d_{\hbar} J = \frac{\hbar}{2} \sum_{\alpha > 0} \frac{d\alpha}{\alpha} \left( \Delta(C_{\alpha})J - J(C_{\alpha}(1) + C_{\alpha}(2)) \right) + z \text{ad}(d\mu(1))J \]
which expresses the fact that, when $z_1 - z_2 = \infty$, the dynamical Casimir connection on the tensor product $V_1 \otimes V_2$ of two representations becomes the tensor product of the (non–dynamical) Casimir connections on each factor. The above equation is a consequence of the bispectrality of the dynamical KZ and Casimir equations, together with the fact that $J$ varies smoothly in $\mu \in \mathfrak{h}_{\text{reg}}$, which follows from our analysis.
1.17. In Section 7, we show that the differential twists $F_{\pm}(\mu)$ satisfy the centraliser property. This follows by relating the (irregular) asymptotics of the fusion operator of $\mathfrak{g}$ when one of the simple roots coordinates $\alpha_i$ tends to $\infty$, to the fusion operator of the corank 1 subalgebra $\mathfrak{g}_{D \setminus \alpha_i}$.
We revisit these calculations in Appendix B, and give a direct construction of the relative twists arising from the fusion operator which is similar in spirit to Drinfeld’s construction of the KZ associator. This gives, to the best of our knowledge, the first canonical transcendental construction of a twist killing the KZ associator $\Phi_{\text{KZ}}$.
Specifically, we show that the twist $F_{(D,\alpha_i)}$ defined by the factorisation relation (1.6) can be realised as the constant relating the canonical fundamental solutions at 0 and $\infty$ of an ODE with regular singularities at 0, $-1$ and an irregular singularity at $\infty$. The ODE is defined with respect to the blowup coordinate $v = z\alpha_i$, and is given by
\[ \frac{dG}{dv} = \left( \frac{\hbar \Omega}{v} + \frac{\hbar \Delta(K_D - K_{D \setminus \alpha_i}) - 2\Omega}{v + 1} + \text{ad} \lambda_{\nu(1)} \right) G \]
where, for any subdiagram $B \subseteq D$, $K_B$ is the truncated (Cartan–less) Casimir operator of $\mathfrak{g}_B$, and $\lambda_{\nu}$ is the coroot corresponding to $\alpha_i$.
1.18. Our construction of the differential twist is very close, in spirit at least, to Etingof and Varchenko’s study of the fusion operator $\hat{J}(w, \lambda)$ of the affine Lie algebra $\hat{\mathfrak{g}}$ [17]. The latter satisfies the trigonometric KZ equations with respect to $w$, together with the dynamical difference equations with respect to $\lambda$, where the latter are a system of difference equations which degenerates to the Casimir connection. The regularised limit $\hat{J}(\lambda)$ of $\hat{J}(z, \lambda)$ as $z \to 1$ kills the KZ associator in the shifted sense, that is satisfies
\[ \hat{J}_{12,3}(\lambda) \hat{J}_{1,2}(\lambda - h(3)) = \hat{J}_{1,23}(\lambda) \hat{J}_{2,3}(\lambda) \Phi_{\text{KZ}} \]
A construction of a differential twist might therefore arise by taking an appropriate scaling limit of $\hat{J}(\lambda)$ as $\lambda$ goes to infinity (a process which would kill the shift in the above equation). Controlling the asymptotics of $\hat{J}$ at $\lambda = \infty$ seems difficult, however, since $\hat{J}(\lambda)$ only satisfies a system of difference equations with respect to $\lambda$.

Rather than pursue this path, we give in this paper a direct construction of a solution of the rational dynamical KZ equations, which can be thought of as (and probably is) a degeneration of the fusion operator of $\hat{g}$. Our $J_\pm(z, \mu)$ should in fact be a fusion operator for the current algebra $g[t]$. One further difference with $[17]$ is that, unlike $\hat{J}(w, \lambda)$, $J_\pm(z, \mu)$ is not constructed via representation theory, specifically the fusion construction for loop modules of $\hat{g}$, but purely using differential equations. It seems an interesting question to construct our fusion operator from representation theory. Such a construction should be obtained by replacing Verma modules by the irregular Wakimoto modules for $\hat{g}$ considered in $[19, 18]$, since these give rise to the Casimir connection $[18]$.

1.19. The monodromy theorem proved in this paper is extended to the case of an arbitrary symmetrisable Kac–Moody algebra in $[1, 2, 3]$. The approach is close in spirit to that $[25]$, but differs very significantly in the details of two out of the three steps of the construction, namely the transfer of braided quasi–Coxeter structure from the category $O^\text{int}_h$ of integrable, highest weight $U_h g$–modules to an isomorphic structure on the corresponding category $O^\text{int}[h]$ for $U g[h]$, and the proof that such structures are rigid. The last step, namely the construction of a braided quasi–Coxeter structure on $O^\text{int}[h]$ which accounts for the monodromy of the Casimir equations of $g$ and that of the KZ equations of its Levi subalgebras is carried out in $[3]$ by using the construction of the fusion operator and differential twist given in this paper.

1.20. Outline of the paper. Section 2 contains some preliminary material required to study differential equations with values in infinite–dimensional filtered vector spaces.

In Section 3, we review the definition of quasitriangular quasibialgebras and of quasi–Coxeter algebras together with the fact that the monodromy of the KZ and Casimir connections respectively define such structures on the enveloping algebra $U g$ of a complex, semisimple Lie algebra $g$.

In Section 4, we introduce the notion of differential twist for $g$, and show that it gives rise to a quasi–Coxeter quasitriangular quasibialgebra structure on $U g$, which interpolates between the quasitriangular quasibialgebra structure underlying the monodromy of the KZ connection and the quasi–Coxeter structure underlying that of the Casimir one.

In Section 5, we construct a fusion operator for $g$ as a joint solution of the coupled KZ–Casimir equations on 2 points, with prescribed asymptotics when $z_1 - z_2 \to \infty$.

In Section 6, we obtain a differential twist for $g$ as the regularised limit of the fusion operator when $z \to 0$, and prove that it kills the KZ associator.

In Section 7, we relate the differential twists for $g$ and for a corank 1 Levi subalgebra, and use this to prove that the differential twist of $g$ satisfies the centraliser property described in 1.13.

This property is used in Section 8, to show that the differential twist arising from the fusion operator it gives rise to a quasi–Coxeter quasitriangular quasibialgebra
structure on \( U_\mathfrak{g} \) interpolating between the quasitriangular quasibialgebra structure underlying the KZ equations and the quasi-Coxeter algebra one underlying the Casimir connection.

Section 9 collects some facts about the quantum group \( U_{\hbar}\mathfrak{g} \), and in particular the fact that it possesses a quasi-Coxeter quasitriangular quasibialgebra structure which accounts for both its \( R \)-matrix and quantum Weyl group representations.

Section 10 contains the main result of this paper, namely the equivalence of \( U_{\hbar}\mathfrak{g} \) and \( U_{\mathfrak{g}}[\hbar] \) as quasi-Coxeter quasitriangular quasibialgebras and the immediate corollary that the monodromy of the Casimir connection is described by quantum Weyl group group operators.

Appendix A contains a detailed discussion of the solutions of a linear, scalar ODE with an irregular singularity at \( \infty \), which plays a similar role in this paper than Drinfeld’s ODE
\[
\frac{df}{dz} = (A + B z^{-1})f
\]
does for the construction of the KZ associators.

Appendix B gives an alternative proof that the differential twist obtained in Section 6 possesses the centraliser property. As a corollary, we obtain a canonical, transcendental construction of a twist killing the KZ associator \( \Phi_{KZ} \).

The final Appendix C gives an alternative proof that the coupled KZ–Casimir equations are integrable.

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2. Filtered algebras

2.1. Let \( A \) be an algebra over \( \mathbb{C} \) endowed with an ascending filtration
\[
\mathbb{C} = A_0 \subset A_1 \subset \cdots
\]
such that \( A_m \cdot A_n \subset A_{m+n} \). Let \( o = \{o_k\}_{k \in \mathbb{N}} \) be a sequence of non-negative integers, \( \hbar \) a formal variable, and consider the subspace \( A[\hbar]^o \subset A[\hbar] \) defined by
\[
A[\hbar]^o = \{ \sum_{k \geq 0} a_k \hbar^k | a_k \in A_{o_k} \}
\]
Note that:

(1) \( A[\hbar]^o \) is a (closed) \( \mathbb{C}[\hbar] \)-submodule of \( A[\hbar] \) if \( o \) is increasing.\(^9\)

(2) \( A[\hbar]^o \) is a subalgebra of \( A[\hbar] \) if \( o \) is subadditive, that is such that \( o_k + o_l \leq o_{k+l} \) for any \( k, l \in \mathbb{N} \). This implies in particular that \( o \) is increasing, and that \( o_0 = 0 \).

If the subspaces \( A_k \subset A \) are finite-dimensional, so are the quotients
\[
A[\hbar]^o/(\hbar^{p+1} A[\hbar] \cap A[\hbar]^o) \cong A_{o_0} + \hbar A_{o_1} + \cdots + \hbar^p A_{o_p}
\]
Assuming this, we shall say that a map \( F : X \to A[\hbar]^o \), where \( X \) is a topological space (resp. a smooth or complex manifold), is continuous (resp. smooth or holomorphic) if each of its truncations \( F_p : X \to A[\hbar]^o/(\hbar^{p+1} A[\hbar] \cap A[\hbar]^o) \) are.

\(^9\)\( A[\hbar]^o \) is then the \( \hbar \)-adic completion of the Rees algebra of \( A \) corresponding to the filtration \( A_{o_0} \subset A_{o_1} \subset \cdots \)
2.2. We shall mainly be interested in the following situation: $A = U\mathfrak{g}^\otimes n$ endowed with the standard order filtration given by deg$(x^{(i)}) = 1$ for $x \in \mathfrak{g}$, where $x^{(i)} = 1^\otimes(i-1) \otimes x \otimes 1^\otimes(n-i)$.

The sequence $o$ will be chosen subadditive, and such that $o_1 \geq 2$ in order for $h\Omega_{ij}, h\Delta^{(n)}(K_n) \in A[h]^o$. Note that $\mathfrak{g} \cap U\mathfrak{g}[h]^o = \{0\}$ since $o_0 = 0$, but that the adjoint action of $\mathfrak{g}$ on $U\mathfrak{g}^\otimes n$ induces one on by derivations on $U\mathfrak{g}^\otimes n[h]^o$. Note also that $U\mathfrak{g}[h]^o$ is not a Hopf algebra, since $\Delta : U\mathfrak{g}[h]^o \rightarrow U\mathfrak{g}^\otimes 2[h]^o \supseteq (U\mathfrak{g}[h]^o)^\otimes 2$.

2.3. Let $A$ be a $\mathbb{C}[h]$–module and consider the natural map $\iota : A \rightarrow \lim\inf A/h^nA$. Recall that $A$ is separated if $\iota$ is injective, and complete if it is surjective. By definition, $A$ is topologically free if it is separated, complete and torsion–free.

Consider now the map

$$\iota : \text{End}_{\mathbb{C}[h]}(A) \rightarrow \lim\inf \text{End}_{\mathbb{C}[h]}(A/h^nA)$$

**Lemma.** Assume that $A$ is separated. Then,

1. $\iota$ is injective.
2. If $A$ is complete, $\iota$ is surjective.

**Proof.** (1) If $T \in \text{End}_{\mathbb{C}[h]}$ is such that $\iota T = 0$, then $T(A) \subseteq \bigcap_n h^nA = 0$.

(2) Let $\{T_n\} \subseteq \lim\inf \text{End}_{\mathbb{C}[h]}(A/h^nA)$. For any $a \in A$, the sequence $\{T_n a\}$ lies in $\lim\inf A/h^nA$ and is therefore the image of a unique element $a' \in A$. The assignment $a \rightarrow Ta = a'$ is easily seen to define an element of $\text{End}_{\mathbb{C}[h]}(A)$ which projects to each of the $T_n$. \[\Box\]

**Corollary.** If $A$ is topologically free, the map $\iota : \text{End}_{\mathbb{C}[h]}(A) \rightarrow \lim\inf \text{End}_{\mathbb{C}[h]}(A/h^nA)$ is an isomorphism.

3. Quasi–Coxeter algebras

We review in this section the definition of quasitriangular quasibialgebras following [12], and of quasi–Coxeter algebras following [25], to which we refer for more details. We also explain how the monodromy of the Knizhnik–Zamolodchikov (KZ) and Casimir connections of a complex, semisimple Lie algebra $\mathfrak{g}$ respectively give rise to a quasitriangular quasibialgebra and quasi–Coxeter algebra structure on the enveloping algebra $U\mathfrak{g}[h]$ of $\mathfrak{g}$.

3.1. Quasitriangular quasibialgebras [12].

3.1.1. Recall that a quasibialgebra $(A, \Delta, \varepsilon, \Phi)$ is an algebra $A$ endowed with algebra homomorphisms $\Delta : A \rightarrow A^\otimes 2$ and $\varepsilon : A \rightarrow k$ called the coproduct and counit, and an invertible element $\Phi \in A^\otimes 3$ called the associator which satisfy, for any $a \in A$

- $\text{id} \otimes \Delta(\Delta(a)) = \Phi \cdot \Delta \otimes \text{id}(\Delta(a)) \cdot \Phi^{-1}$
- $\text{id}^\otimes 2 \otimes \Delta(\Phi) \cdot \Delta \otimes \text{id}^\otimes 2(\Phi) = 1 \otimes \Phi \cdot \text{id} \otimes \Delta \otimes \text{id}(\Phi) \cdot \Phi \otimes 1$
- $\varepsilon \otimes \text{id} \circ \Delta = \text{id}$
- $\text{id} \otimes \varepsilon \circ \Delta = \text{id}$
- $\text{id} \otimes \varepsilon \otimes \text{id}(\Phi) = 1$

A twist of a quasibialgebra $A$ is an invertible element $F \in A^\otimes 2$ satisfying

$$\varepsilon \otimes \text{id}(F) = 1 = \text{id} \otimes \varepsilon(F)$$
Given such an $F$, the twisting of $A$ by $F$ is the quasibialgebra $(A, \Delta_F, \varepsilon, \Phi_F)$, where the coproduct $\Delta_F$ and associator $\Phi_F$ are given by
\[
\Delta_F(a) = F \cdot \Delta(a) \cdot F^{-1}
\]
\[
\Phi_F = 1 \otimes F \cdot \text{id} \otimes \Delta(F) \cdot \Phi \cdot \Delta \otimes \text{id}(F^{-1}) \cdot F^{-1} \otimes 1
\]

A strict morphism $\Psi : A \to A'$ of quasibialgebras is an algebra homomorphism satisfying
\[
\varepsilon = \varepsilon' \circ \Psi, \quad \Psi \otimes 2 \circ \Delta = \Delta' \circ \Psi \quad \text{and} \quad \Psi \otimes 3(\Phi) = \Phi'
\]

A morphism $A \to A'$ of quasibialgebras is a pair $(\Psi, F')$ where $F'$ is a twist of $A'$ and $\Psi$ is a strict morphism of $A$ to the twisting of $A'$ by $F'$.

3.1.2. A quasibialgebra $(A, \Delta, \varepsilon, \Phi)$ is quasitriangular if it is endowed with an invertible element $R \in A^{\otimes 2}$ called the $R$–matrix satisfying, for any $a \in A$,
\[
\Delta^{\text{op}}(a) = R \cdot \Delta(a) \cdot R^{-1}
\]
\[
\Delta \otimes \text{id}(R) = \Phi_{312} \cdot R_{13} \cdot \Phi_{132}^{-1} \cdot R_{23} \cdot \Phi_{123}
\]
\[
\text{id} \otimes \Delta(R) = \Phi_{231}^{-1} \cdot R_{13} \cdot \Phi_{213} \cdot R_{12} \cdot \Phi_{123}^{-1}
\]

A twist $F$ of a quasitriangular quasibialgebra $A$ is a twist of the underlying quasibialgebra. The twisting of $A$ by $F$ is the quasitriangular quasibialgebra $(A, \Delta_F, \varepsilon, \Phi_F, R_F)$ where
\[
R_F = F_{21} \cdot R \cdot F^{-1}
\]

A morphism $(\Psi, F') : A \to A'$ of quasitriangular quasibialgebras is a morphism of the underlying quasibialgebras such that $\Psi \otimes 2(R) = R'_{F'}$.

3.1.3. Let $\mathfrak{g}$ be a complex, semisimple Lie algebra, with symmetric, invariant tensor $\Omega \in (S^2 \mathfrak{g})^\Omega$. Then, $U \mathfrak{g} \llbracket \hbar \rrbracket$ is a quasitriangular quasibialgebra with the standard (cocommutative) coproduct $\Delta_0$ and counit $\varepsilon$, $R$–matrix given by $R^{\text{KZ}} = \exp(\hbar \Omega)$ and associator $\Phi^{\text{KZ}}$ constructed as follows.

Consider the differential equation with values in $U \mathfrak{g} \llbracket h \rrbracket^a$ given by
\[
\frac{dG}{dz} = h \left( \frac{\Omega_{12}}{z} + \frac{\Omega_{23}}{z - 1} \right) G
\]
where $\Omega_{12} = \Omega \otimes 1$, $\Omega_{23} = 1 \otimes \Omega$, and $h = \hbar/\pi t$. This equation has a unique solution $G_0$ of the form $H_0(z) \cdot z^{\hbar \Omega_{12}}$, where $H_0$ is holomorphic on the disk $\{ z \in \mathbb{C} \mid |z| < 1 \}$ and such that $H_0(0) = 1$, and $z^{\hbar \Omega_{12}} = \exp(\hbar \Omega_{12} \log z)$, where log is the standard determination of the logarithm. Similarly, there is a unique solution $G_1$ of the form $H_1(z)(1 - z)^{\hbar \Omega_{23}}$, where $H_1$ is holomorphic on $\{ z \in \mathbb{C} \mid |z - 1| < 1 \}$, and such that $H_1(1) = 1$. The KZ associator $\Phi^{\text{KZ}}$ is defined by
\[
\Phi^{\text{KZ}} = G_1^{-1}(x) \cdot G_0(x)
\]
where $x \in (0, 1)$.

3.2. Quasi–Coxeter algebras [25].
3.2.1. \textit{Diagrams.} A diagram is an undirected graph $D$ with no multiple edges or loops. We denote the set of vertices of $D$ by $V(D)$, and set $|D| = |V(D)|$. A subdiagram $B \subset D$ is a full subgraph of $D$, that is a graph consisting of a subset $V(B)$ of vertices of $D$, together with all edges of $D$ joining two elements of $V(B)$. We shall often identify a subdiagram $B$ and its set of vertices $V(B)$.

Two subdiagrams $B_1, B_2 \subseteq D$ are orthogonal if no two vertices $\alpha_1 \in B_1, \alpha_2 \in B_2$ are joined by an edge in $D$. $B_1$ and $B_2$ are compatible if either one contains the other or they are orthogonal.

3.2.2. \textit{Nested sets.} Assume henceforth that $D$ is connected. A nested set on $D$ is a collection $\mathcal{H}$ of pairwise compatible, connected subdiagrams of $D$ containing $D$.

We denote by $\mathcal{N}_D$ the partially ordered set of nested sets on $D$, ordered by reverse inclusion. $\mathcal{N}_D$ has a unique maximal element $\{D\}$. Its minimal elements are the maximal nested sets.

It is easy to see that a maximal nested set $\mathcal{F}$ has the following properties

1. The cardinality of $\mathcal{F}$ is $|D|$.
2. If $B \in \mathcal{F}$, the maximal elements $\{B_i\}$ in $\mathcal{F}$ properly contained in $B$ contain all the vertices of $B$ with the exception of one, which will be denoted $\alpha_B^D$.

The $B_i$ are in fact the connected components of the diagram $B \setminus \alpha_B^D$, and $\mathcal{F}$ may be obtained by taking the connected components $D_i$ of $D \setminus \alpha_B^D$, then those of each $D_i \setminus \alpha_{D_i}^F$ and so on.

3.2.3. \textit{Type $A$.} If $D$ is the Dynkin diagram of type $A_{n-1}$, with vertices labelled $1, \ldots, n-1$, nested sets on $D$ are in bijection with bracketings of the non associative monomial $x_1 \cdots x_n$. The bijection is obtained by mapping a connected subdiagram with vertices $i, \ldots, j-1$ to the pair of parentheses $x_1 \cdots x_{i-1} \left( x_i \cdots x_j \right) x_{j+1} \cdots x_n$ and noting that $B, B' \subseteq D$ are compatible if, and only if the corresponding pairs of parentheses are. It follows that, in this case, the poset $\mathcal{N}_D$ is the face poset of Stasheff’s associahedron.

3.2.4. \textit{$D$–algebras.} Fix henceforth a commutative, unital ring $k$. A $D$–algebra is a $k$–algebra $A$ endowed with subalgebras $A_B$ labelled by the non–empty connected subdiagrams $B \subseteq D$ such that the following holds

- $A_{B'} \subseteq A_{B''}$ whenever $B' \subseteq B''$.
- $A_B$ and $A_{B''}$ commute whenever $B'$ and $B''$ are orthogonal.

If $B_1, B_2 \subseteq D$ are subdiagrams with $B_1$ connected, we denote by $A_{B_1}^{B_2}$ the centraliser in $A_{B_1}$ of the subalgebras $A_{B_2}$ where $B_2$ runs over the connected components of $B_2$.

A morphism of $D$–algebras $A, A'$ is a collection of algebra homomorphisms $\Psi_{\mathcal{F}} : A \to A'$ labelled by the maximal nested sets on $D$ such that for any $\mathcal{F}$ and $B \in \mathcal{F}$, $\Psi_{\mathcal{F}}(A_B) \subseteq A'_B$.

3.2.5. \textit{Pairs of maximal nested sets.} An elementary pair $(\mathcal{F}, \mathcal{G})$ of maximal nested sets on $D$ is one for which $\mathcal{F}$ and $\mathcal{G}$ differ by an element. In this case, there is a unique minimal element $B$ of $\mathcal{F} \cap \mathcal{G}$, called the support of $(\mathcal{F}, \mathcal{G})$, which contains the single element in $\mathcal{F} \setminus \mathcal{G}$ (or $\mathcal{G} \setminus \mathcal{F}$). The central support $\text{supp}(\mathcal{F}, \mathcal{G})$ of $(\mathcal{F}, \mathcal{G})$ is the union of the elements of $\mathcal{F} \cap \mathcal{G}$ properly contained in $\text{supp}(\mathcal{F}, \mathcal{G})$.

In type $A_{n-1}$, an elementary pair $(\mathcal{F}, \mathcal{G})$ corresponds to two complete parenthesisations of $x_1 \cdots x_n$, which are obtained by replacing a pair of parentheses
\[ \cdots (x_i \cdots x_j) \cdots \] with \[ \cdots (x_k \cdots x_\ell) \cdots \], where \( i < k < j < \ell \), and the support of \((F, G)\) to the smallest pair of parentheses \( \cdots (x_i \cdots x_\ell) \cdots \) which is consistent with both pairs.

### 3.2.6. Weak quasi–Coxeter algebras

A weak quasi–Coxeter algebra \( A \) of type \( D \) is a \( D \)–algebra endowed with invertible elements \( \Phi_{G,F} \) called De Concini–Procesi associators labelled by pairs of maximal nested sets satisfying the following axioms\(^{10}\):

- **Transitivity.** For any maximal nested sets \( F, G, H \) on \( D \)
  \[ \Phi_{H,F} = \Phi_{H,G} \cdot \Phi_{G,F} \]

- **Support.** For any elementary pair \((G, F)\) of maximal nested sets on \( D \),
  \[ \Phi_{G,F} \in A_{\supp(G,F)} \]

- **Forgetfulness.** For any elementary pairs \((G, F), (G', F')\) of maximal nested sets on \( D \) such that \( F \setminus G = F' \setminus G' \) and \( G \setminus F = G' \setminus F' \),
  \[ \Phi_{G,F} = \Phi_{G',F'} \]

A morphism of quasi–Coxeter algebras \( A, A' \) of type \( D \) is a morphism \( \{ \Psi_F \} \) of the underlying \( D \)–algebras such that for any elementary pair \((G, F)\) of maximal nested sets on \( D \),
\[ \Psi_G \circ \Ad(\Phi_{G,F}^A) = \Ad(\Phi_{G,F}^{A'}) \circ \Psi_F \]

### 3.2.7. Completion

Let \( \text{Vec}_k \) be the category of finitely–generated, free \( k \)–modules and \( \text{Mod}_{\text{fd}}(A) \) that of \( A \)–modules whose underlying \( k \)–module lies in \( \text{Vec}_k \). By definition, the completion of \( A \) with respect to its finite–dimensional representations is the algebra \( \hat{A} \) of endomorphisms of the forgetful functor
\[ F : \text{Mod}_{\text{fd}}(A) \to \text{Vec}_k \]

An element of \( \hat{A} \) is a collection \( \Theta = \{ \Theta_V \} \), with \( \Theta_V \in \text{End}_k(V) \) for any \( V \in \text{Mod}_{\text{fd}}(A) \), such that for any \( U, V \in \text{Mod}_{\text{fd}}(A) \) and \( f \in \text{Hom}_A(U, V) \)
\[ \Theta_V \circ f = f \circ \Theta_U \]

### 3.2.8. Labelled diagrams

A labelling of \( D \) is the assignment of an integer \( m_{ij} \in \{2, 3, \ldots, \infty\} \) to any pair \( \alpha_i, \alpha_j \) of distinct vertices of \( D \) such that \( m_{ij} = m_{ji} \) for any \( \alpha_i \neq \alpha_j \) and \( m_{ij} = 2 \) if, and only if, \( \alpha_i \) and \( \alpha_j \) are orthogonal.

If \( D \) is labelled, the Artin or braid group \( B_D \) is the group generated by elements \( S_i \) corresponding to the vertices \( \alpha_i \) of \( D \) with relations
\[ S_i S_j \cdots = S_j S_i \cdots \]
for any \( \alpha_i \neq \alpha_j \) such that \( m_{ij} < \infty \).

\(^{10}\)The term weak quasi–Coxeter algebra is borrowed from \([2]\).
3.2.9. Quasi–Coxeter algebras. Let $D$ be a labelled diagram. A quasi–Coxeter algebra of type $D$ is a weak quasi–Coxeter algebra $A$ endowed with an invertible element

$$S^A_i \in \hat{A}_i$$

for any vertex $\alpha_i \in D$, where $\hat{A}_i$ is the completion of $A_{\alpha_i}$ with respect to its finite–dimensional representations. The associators $\Phi_{\mathcal{G}}$ and local monodromies are required to satisfy the following

- **Braid relations.** For any pair $\alpha_i, \alpha_j$ of distinct vertices of $D$ such that $m_{ij} < \infty$, and pair $(\mathcal{G}, \mathcal{F})$ of maximal nested sets on $D$ such that $\alpha_i \in \mathcal{F}$ and $\alpha_j \in \mathcal{G}$,

$$\text{Ad}(\Phi_{\mathcal{G}})(S^A_i) \cdot S^A_j \cdots = S^A_j \cdot \text{Ad}(\Phi_{\mathcal{G}})(S^A_i) \cdots$$

where the number of factors on each side is equal to $m_{ij}$.

A morphism $\{ \Psi_{\mathcal{F}} \}$ of quasi–Coxeter algebras $A, A'$ is one of the underlying weak quasi–Coxeter algebras such that $\Psi_{\mathcal{F}}(S^A_i) = S'^A_i$ for any maximal nested set $\mathcal{F}$ on $D$, and vertex $\alpha_i \in D$ such that $\alpha_i \in \mathcal{F}$.

3.2.10. Braid group representations. A quasi–Coxeter algebra $A$ of type $D$ defines a family of homomorphisms $\pi_{\mathcal{F}} : B_D \to \hat{A}^\times$ of the braid group $B_D$ to the set of invertible elements of the completion of $A$ with respect to finite–dimensional representations. The homomorphisms are labelled by the maximal nested sets on $D$, and are defined as follows.

Let $\mathcal{F}$ be a maximal nested set on $D$. For any $\alpha_i \in D$, choose a maximal nested set $\mathcal{G}_i$ such that $\alpha_i \in \mathcal{G}_i$ and set

$$\pi_{\mathcal{F}}(S_i) = \Phi_{\mathcal{F}} \cdot S^A_i \cdot \Phi_{\mathcal{G}_i, \mathcal{F}}$$

The assignment $S_i \to \pi_{\mathcal{F}}(S_i)$ is independent of the choice of $\mathcal{G}_i$, and extends to a homomorphism $B_D \to \hat{A}^\times$ with the following properties

1. If $\alpha_i \in \mathcal{F}$, then $\pi_{\mathcal{F}}(S_i) = S^A_i$.
2. If $\mathcal{G}$ is another maximal nested set on $D$ then, for any $b \in B_D$,

$$\pi_{\mathcal{G}}(b) = \Phi_{\mathcal{G}, \mathcal{F}} \cdot \pi_{\mathcal{F}}(b) \cdot \Phi_{\mathcal{F}}$$

so that $\pi_{\mathcal{F}}$ and $\pi_{\mathcal{G}}$ are canonically equivalent.
3. If $\{ \Psi_{\mathcal{F}} \} : A \to A'$ is a morphism of quasi–Coxeter algebras, then for any maximal nested set $\mathcal{F}$ and $b \in B_D$, $\Psi_{\mathcal{F}}(\pi^A_{\mathcal{F}}(b)) = \pi^A_{\mathcal{F}}(b)$. In particular, isomorphic quasi–Coxeter algebras yield equivalent representations of $B_D$.

3.3. The Casimir connection.

3.3.1. Let $\mathfrak{g}$ be a complex, simple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, and $\Phi = \{ \alpha \} \subset \mathfrak{h}^*$ the corresponding root system. Set

$$\mathfrak{h}_{\text{reg}} = \mathfrak{h} \setminus \bigcup_{\alpha \in \Phi \setminus \mathfrak{g}} \text{Ker}(\alpha)$$

Choose root vectors $x_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in \Phi$, such that $(x_\alpha, x_{-\alpha}) = 1$, where $(\cdot, \cdot)$ is a fixed non–degenerate, invariant bilinear form on $\mathfrak{g}$, and let

$$K_\alpha = x_\alpha x_{-\alpha} + x_{-\alpha} x_\alpha \in U \mathfrak{g}$$
be the (truncated) Casimir operator of the three–dimensional subalgebra $\mathfrak{sl}_2 \subseteq \mathfrak{g}$ corresponding to $\alpha$.

Let $V$ be a finite–dimensional representation of $\mathfrak{g}$, and $\mathbb{V}$ the holomorphically trivial vector bundle on $\mathfrak{h}_{\text{reg}}$ with fibre $V$. The Casimir connection of $\mathfrak{g}$ is the holomorphism connection on $\mathbb{V}$ given by

$$\nabla_\kappa = d - \frac{h}{2} \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} \cdot K_\alpha \tag{3.4}$$

where $h \in \mathbb{C}$ is a deformation parameter, and $\alpha$ ranges over a chosen system of positive roots $\Phi_+$. The connection $\nabla_\kappa$ has logarithmic singularities on the root hyperplanes and is flat for any $h$ [22, 7, 20].

The flat vector bundle $(\mathbb{V}, \nabla_\kappa)$ defines a one–parameter family of monodromy representations of the pure braid group $P_W = \pi_1(\mathfrak{h}_{\text{reg}})$ on $V$ deforming the trivial action. We explain in the next two paragraphs how to twist $(\mathbb{V}, \nabla_\kappa)$ so that it defines representations of the full braid group $B_W = \pi_1(\mathfrak{h}_{\text{reg}}/W)$.

3.3.2. Tits extension [23]. Let $W \subset GL(h)$ be the Weyl group of $\mathfrak{g}$. It is well–known that $W$ does not act on $V$ in general, but that the triple exponentials

$$\tilde{s}_i = \exp(e_{\alpha_i}) \exp(-f_{\alpha_i}) \exp(e_{\alpha_i})$$

corresponding to the simple roots $\alpha_1, \ldots, \alpha_r$ in $\Phi_+$ and a choice of root vectors $e_{\alpha_i}, f_{\alpha_i}$ such that $[e_{\alpha_i}, f_{\alpha_i}] = h_{\alpha_i}$ give rise to an action of an extension $\tilde{W}$ of $W$ by the sign group $\mathbb{Z}_2^r$.

As an abstract group, $\tilde{W}$ is presented on generators $\tilde{s}_i$ subject to the relations

$$\tilde{s}_i \tilde{s}_j \cdots = \tilde{s}_j \tilde{s}_i \cdots$$

$$\tilde{s}_i \tilde{s}_j \tilde{s}_i = \tilde{s}_j \tilde{s}_i \tilde{s}_j \quad \text{for } i \neq j,$$

$$\tilde{s}_i^2 = 1$$

$$\tilde{s}_i^{-2} \tilde{s}_j \tilde{s}_i^{-1} = \tilde{s}_j^{-2} \tilde{s}_i^{-1}$$

for any $i \neq j$, where $m_{ij}$ is the order of product of simple reflections $s_is_j$ in $W$. It is therefore independent of the choices made, and a quotient of the braid group $B_W$ of $W$. Its action on $V$ factors though that of $N_H \subset G$, where $G$ is the simply–connected complex Lie group with Lie algebra $\mathfrak{g}$, $H \subset G$ its maximal torus with Lie algebra $\mathfrak{h}$, and $N_H$ the normaliser of $H$ in $G$. The image of $\tilde{W}$ in $N_H$, and therefore in $GL(V)$, depends upon the choices of the root vectors, but different choices lead to subgroups of which are canonically conjugate under an element in $H$.

3.3.3. Twisting of $(\mathbb{V}, \nabla_\kappa)$ [22]. The flat vector bundle $(\mathbb{V}, \nabla_\kappa)$ is equivariant under the Tits extension $\tilde{W}$, and may be twisted into a $W$–equivariant, flat vector bundle $(\mathbb{V}, \nabla_\kappa)$ on $\mathfrak{h}_{\text{reg}}$ as follows. Let $\mathfrak{h}_{\text{reg}} \xrightarrow{P} \mathfrak{h}_{\text{reg}}$ be the universal cover of $\mathfrak{h}_{\text{reg}}$ and $\mathfrak{h}_{\text{reg}}/W$. Since $\tilde{W}$ is a quotient of the braid group $B_W$, the latter acts on the flat vector bundle $p^*(\mathbb{V}, \nabla_\kappa)$ on $\mathfrak{h}_{\text{reg}}$. By definition, $(\mathbb{V}, \nabla_\kappa)$ is the quotient $p^*(\mathbb{V}, \nabla_\kappa)/P_W$, where $P_W = \pi_1(\mathfrak{h}_{\text{reg}})$, and carries a residual action of $W = B_W/P_W$.

The flat vector bundle $(\mathbb{V}, \nabla_\kappa)$ may alternatively be described as follows. Let $Z_2^r \cong Z \subset \tilde{W}$ be the subgroup generated by the elements $\{\tilde{s}_i^2\}_{i=1}^r$. Since $Z$ is

\[\text{[Footnote: the connection is independent of the choice of } \Phi_+\]
a quotient of the pure braid group $P_W$, it is the deck group of a Galois cover $\pi: \hat{\mathfrak{h}}_{\text{reg}} \to \mathfrak{h}_{\text{reg}}$ which, via the projection $\mathfrak{h}_{\text{reg}} \to \mathfrak{h}_{\text{reg}}/W$, is also Galois cover of $\mathfrak{h}_{\text{reg}}/W$ with deck group $\hat{W}$. Let $Z \to \mathfrak{h}_{\text{reg}}$ be the direct image of the trivial line bundle over $\hat{\mathfrak{h}}_{\text{reg}}$. Then, $Z$ is a flat, $\hat{W}$–equivariant vector bundle of rank $2^r$ over $\mathfrak{h}_{\text{reg}}$, and $(\mathcal{V}, \nabla)$ is the bundle of $Z$–coinvariants

$$\mathcal{V} = [V \otimes Z]_Z$$

### 3.4. The monodromy of $(\mathcal{V}, \nabla)$, which we shall abusively refer to as the monodromy of the Casimir connection, yields a one–parameter family of representations $\mu^0_\mu$ of $B_W$ on $V$ which is obtained as follows. Fix a base point $\bar{x}_0 \in \hat{\mathfrak{h}}_{\text{reg}}$, and let $x_0, [x_0]$ be its images in $\mathfrak{h}_{\text{reg}}$ and $\mathfrak{h}_{\text{reg}}/W$ respectively. The braid group $\pi_1(\mathfrak{h}_{\text{reg}}/W; [x_0])$ acts on fundamental solutions $\Psi: \mathfrak{h}_{\text{reg}} \to GL(V)$ of $p^*\nabla$ by $b \cdot \Psi(\bar{x}) = b \cdot \Psi(b^{-1} \cdot \bar{x})$. If $\Psi$ is a given fundamental solution, then $\mu^0_\mu(b) = \Psi^{-1} \cdot b \cdot \Psi$ is locally constant function with values in $GL(V)$ and the required monodromy.

It will be shown below that the formal Taylor series of $\mu^0_\mu$ at $\mathfrak{h} = 0$ arises from an appropriate quasi–Coxeter structure on $U\mathfrak{g}[\hbar]$.

### 3.5. The Casimir connection and quasi–Coxeter structure on $U\mathfrak{g}[\hbar]$.

#### 3.5.1. Fundamental solutions of $\nabla$ [9, 5, 6]. Let $D$ be the Dynkin diagram of $\mathfrak{g}$ relative to simple roots $\alpha_1, \ldots, \alpha_r$. An adapted family of $\mathfrak{h}^*$ is a collection $\beta = \{x_B\} \subset \mathfrak{h}^*$ labelled by the connected subdiagrams $B \subseteq D$ such that, for any maximal nested set $\mathcal{F}$ on $D$, and $B \in \mathcal{F}$, the elements $\{x_C\} \subset \mathcal{F} \subseteq B$ form a basis of the subspace $\mathfrak{h}^*_B$ of $\mathfrak{h}^*$ spanned by the simple roots labelled by the vertices of $B$. An example of such a family may be obtained by taking $x_B$ to be the sum of the positive roots in the root subsystem $\Phi_B \subset \Phi$ corresponding to $B$.

For any maximal nested set $\mathcal{F}$, let $\mathcal{U}$ denote the affine space $\mathbb{C}^\mathcal{F}$ with coordinates $\{u_B\} \subset \mathcal{F}$, and consider the map $\rho_\mathcal{F}: \mathcal{U} \to \mathfrak{h}$ given in the coordinates $\{x_B\} \subset \mathcal{F}$ on $\mathfrak{h}$ by

$$x_B = \prod_{F \supseteq C \supseteq B} u_C$$

$\rho_\mathcal{F}$ is a birational map, with inverse

$$u_B = \begin{cases} x_D & \text{if } B = D \\ x_B/x_{C(B)} & \text{otherwise} \end{cases}$$

where $C(B) \in \mathcal{F}$ is the unique minimal element properly containing $B$. It restricts to an isomorphism between the complement in $\mathcal{U}$ of the coordinate hyperplanes $\{u_B = 0\}$, and that in $\mathfrak{h}$ of the hyperplanes $\{x_B = 0\}$. Moreover, $\rho_\mathcal{F}$ maps $\{u_B = 0\}$ into the subspace $\Phi_B^\perp$.

The pull–back to $\mathcal{U}$ of a root $\alpha \in \Phi$ has the following expression in terms of the coordinates $\{u_B\}$. Let $B \in \mathcal{F}$ be the unique minimal element such that $\alpha \in \Phi_B$. Then, there are complex numbers $\{a_{B'}\} \subset \mathcal{F} \supseteq B$, with $a_B \neq 0$, such that

$$\alpha = \sum a_{B'} x_{B'} = a_B x_B \left(1 + \sum_{F \supseteq B' \supseteq B} \frac{a_{B'}}{a_B} x_{B'} \right) = a_B \cdot \prod_{F \supseteq C \supseteq B} u_C \cdot P_\alpha$$

where $P_\alpha$ is a polynomial in the variables $\{u_{B'}\} \subset \mathcal{F}$ such that $P_\alpha(0) = 1$. 
Set $U_F = U \setminus \bigcup_{\alpha \in \Phi} \{ P_\alpha = 0 \}$. The pull–back of the connection $\nabla_\kappa$ to $U_F$ has logarithmic singularities on the divisor $\prod_{B \in F} \{ u_B = 0 \}$, with residue on the hyperplane $u_B = 0$ given by $\frac{h}{2} K_B$, where
\[ K_B = \sum_{\alpha \in \Phi_B \cap \Phi_+} \kappa_\alpha. \]
Let $p_F \in U_F$ be the point with coordinates $u_B = 0$, $B \in F$. Then, for every simply–connected open set $V \subset U_F$ containing $p_F$, there is a unique holomorphic function $H_F : V \to \mathbb{U}[[h]]^o$ such that $H_F(p_F) = 1$ and, for any determination of the logarithm, the function
\[ \Psi_F = H_F \cdot \prod_{B \in F} u_B^{\frac{h}{2} K_B} \]
is a solution of $\nabla_\kappa \Psi_F = 0$. The fundamental solution $\Psi_F$ has good asymptotics near $0 \in \hbar$, when the latter is approached so that, for any $B \subset C \in F$, the roots in $\Phi_B$ go to zero faster than those in $\Phi_C$.

3.5.2. *De Concini–Procesi associators* [9]. Assume that the elements $\{ x_B \}_{B \leq D}$ are real and positive on the fundamental chamber
\[ \mathcal{C} = \{ t \in \hbar | \alpha(t) > 0, \alpha \in \Phi_+ \} \]
For any maximal nested set $F$, let $V_F \subset U_F$ be the complement of the real, codimension one semialgebraic subvarieties $\{ x_B \leq 0 \}$, $B \in F$. The preimage of the chamber $C$ lies in $V_F$ since $x_B > 0$ on $C$. We shall henceforth only consider the standard determination of the logarithm, so that $\log(x_B), \log(u_B), B \in F$ are well–defined and single–valued on $V_F$. The fundamental solution $\Psi_F$ is single–valued on the intersection of a neighborhood of $p_F$ in $U_F$ with $V_F$. Since $p_F$ lies in the closure of $C$, $\Psi_F$ may be continued to a single–valued solution on $C$.

Let now $F, G$ be two maximal nested sets. The De Concini–Procesi associator $\Phi_{GF}$ is the element of $U_g[[h]]$ defined by
\[ \Phi_{GF} = (\Psi_G(y))^{-1} \cdot \Psi_F(y) \]
for any $y \in \mathcal{C}$.

3.5.3. *Quasi–Coxeter structure.*

**Proposition.** Set $h = \pi i \hbar$. Then,
1. The associators $\Phi_{GF}$ and local monodromies
\[ S^\nabla_{i, \kappa} = \tilde{s}_i \cdot \exp (\hbar/2 \cdot \kappa_{\alpha_i}) \]
endow $U_g[[h]]$ with a quasi–Coxeter algebra structure $Q^\nabla_\kappa$ of type $D$.
2. For any finite–dimensional $g$–module $V$ and maximal nested set $F$, the representation
\[ \pi_F : B_W \to GL(V[[h]]) \]
obtained from the quasi–Coxeter structure $Q^\nabla_\kappa$ coincides with the monodromy of $(\nabla, \nabla_\kappa)$ expressed in the fundamental solution $\Psi_F$. 
3.5.4. Modification of $Q^\nabla_{\kappa}$. It will be convenient to alter the local monodromies of the quasi–Coxeter structure $Q^\nabla_{\kappa}$ as follows.

For any root $\alpha \in \Phi$, let $a_\alpha \in \mathbb{C}[h_\alpha][h]$ be an invertible element such that, for any $w \in W$, $wa_\alpha = a_wa_\alpha$. It is easy to see that the local monodromies $(S^\nabla_{i,C})^a = S^\nabla_{i,C} \cdot a_\alpha$, satisfy the braid relations of 3.2.9 with respect to the associators $\Phi_{G,F}$. Indeed, since the $\Phi_{G,F}$ are of weight zero, these relations amount to checking that the following holds for any $i \neq j \in I$

$$s_i(a_\alpha) \cdot s_j(a_\alpha) \cdot s_isjs_i(a_\alpha) \cdots = s_j(a_\alpha) \cdot s_jsjs_j(a_\alpha) \cdots$$

where each side has $m_{ij}$ terms. This identity holds since both sides are equal to the product $\prod_{\alpha \in \Phi_{ij} \cap \Phi_-} a_\alpha$, where $\Phi_{ij} \subset \Phi$ is the rank two root system generated by $\alpha_i$ and $\alpha_j$.

Set now $a_\alpha = \exp\left(\frac{h}{2} \cdot \frac{(\alpha, \alpha)}{2} h_\alpha^2\right)$, so that $(S^\nabla_{i,C})^a$ is given by

$$S^\nabla_{i,C} = \tilde{s}_i \cdot \exp\left(h/2 \cdot C_\alpha\right) \tag{3.6}$$

where $C_\alpha = K_\alpha + \frac{(\alpha, \alpha)}{2} h_\alpha^2$ is the Casimir operator of $\mathfrak{sl}_d$. The following result is a direct consequence of Proposition 3.5.3 and the previous discussion.

**Corollary.** The associators $\Phi_{G,F}$ and local monodromies $S^\nabla_{i,C}$ endow $U \mathfrak{g}[h]$ with a quasi–Coxeter algebra structure $Q^\nabla_C$ of type $D$.

3.5.5. Modification of $\nabla_{\kappa}$. The quasi–Coxeter structure $Q^\nabla_C$ encodes the monodromy of the following connection

$$\nabla_C = d - \frac{h}{2} \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} \cdot C_\alpha \tag{3.7}$$

More precisely, the connection $\nabla_C$ differs from the Casimir connection $\nabla_{\kappa}$ by the addition of the closed, $S^2h$–valued, $W$–equivariant 1–form

$$a = \frac{h}{2} \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} \cdot \frac{(\alpha, \alpha)}{2} h_\alpha^2$$

It follows that $\nabla_C$ is flat and may be twisted to a $W$–equivariant flat connection $\nabla_C$ on $\nabla$. Since $\Psi$ is a horizontal section of $\nabla_{\kappa}$ if, and only if $\Psi \cdot \Theta$ is one of $\nabla_C$, where

$$\Theta = \prod_{\alpha \in \Phi_+} \exp\left(\frac{h}{2} \cdot \frac{(\alpha, \alpha)}{2} h_\alpha^2\right)$$

the following is a direct consequence of Proposition 3.5.3.

**Corollary.** For any finite–dimensional $\mathfrak{g}$–module $V$ and maximal nested set $F$, the representation $\pi_F : BW \to GL(V[h])$ obtained from the quasi–Coxeter structure $Q^\nabla_C$ coincides with the monodromy of $(\nabla, \nabla_C)$ expressed in the fundamental solution $\Psi_F \cdot \Theta$.

**Remark.** One can construct fundamental solutions $\Psi_{F,C}$ of the connection $\nabla_C$ exactly as in 3.5.1 and describe the monodromy of $\nabla_C$ directly in terms of the associators $\Phi_{G,F,C} = \Psi_{G,C}^{-1} \cdot \Psi_{F,C}$. These, however, do not satisfy the central support axiom of 3.2.6 and therefore do not define a quasi–Coxeter structure on $U \mathfrak{g}[h]$. The quasi–Coxeter structure $Q^\nabla_C$ is therefore best seen as arising from a modification of $Q^\nabla_{\kappa}$ rather than from the monodromy of $\nabla_C$. 
3.6. Quasi–Coxeter quasitriangular quasibialgebras.

3.6.1. $D$–quasibialgebras. If $(A, \Delta, \varepsilon)$ is a bialgebra and $n \in \mathbb{N}$, we denote by $\Delta^{(n)} : A \to A^\otimes n$ be the iterated coproduct defined by $\Delta^{(0)} = \varepsilon$, $\Delta^{(1)} = \text{id}$, and

\[ \Delta^{(n+1)} = \Delta \otimes \text{id}^{\otimes n-1} \circ \Delta^{(n)} \]

if $n \geq 1$.

Let $D$ be a connected diagram. A $D$–bialgebra is a $D$–algebra $A$ endowed with a bialgebra structure such that, for any connected $B \subseteq D$, $A_B$ is a subbialgebra of $A$. If $B' \subseteq B \subseteq D$ are subdiagrams, with $B$ connected, and $n$ is an integer, we set

\[ (A_B^\otimes n)^{B'} = \{ a \in A_B^\otimes n | \Delta^{(n)}(a'), a = 0, a' \in A_{B'} \} \]

where $B'$ ranges over the connected components of $B'$.

A $D$–quasibialgebra $(A_B^\otimes n)^{B'} = \{ \Phi_B \}$ is a $D$–bialgebra endowed with the following additional data

- **Associators.** For each connected subdiagram $B \subseteq D$, an invertible element
  \[ \Phi_B \in (A_B^\otimes n)^B \]

- **Structural twists.** For each connected subdiagram $B \subseteq D$ and vertex $\alpha \in B$, a twist
  \[ F_{(B; \alpha)} \in (A_B^\otimes n)^{B \setminus \alpha} \]

satisfying the following axioms

- For any connected $B \subseteq D$, $(A_B, \Delta, \varepsilon, \Phi_B)$ is a quasibialgebra.
- For any connected $B \subseteq D$ and $\alpha \in B$,
  \[ (\Phi_B)_{F_{(B; \alpha)}} = \Phi_B \setminus \alpha \]

where $\Phi_B \setminus \alpha = \prod_{B'} \Phi_{B'}$, with the product ranging over the connected components of $B \setminus \alpha$ if $B \neq \alpha$, and $\Phi_0 = 1^\otimes 3$ otherwise.

The gist of the above axioms is the following. For any subdiagram $B \subseteq D$, let $A_B$ be the algebra generated by the $A_{B_i}$, where $B_i$ runs over the connected components of $B$ and set $\Phi_B = \prod_{B_i} \Phi_{B_i}$. A $D$–quasibialgebra gives rise to a family of tensor categories

\[ C_B = \text{Rep}(A_B, \Delta, \varepsilon, \Phi_B) \]

labelled by the subdiagrams $B \subseteq D$. Moreover, the structural twists give rise to restriction functors $C_B \to C_{B'}$, $B' \subseteq B$ in the following way. For any maximal nested set $F$ on $D$ containing the connected components of $B, B'$, set

\[ F_{B' | B} = \{ C \in F | B_j \subseteq C \subseteq B_i \text{ for some } i, j \} \]

where $B_i, B_j$ are the connected components of $B, B'$. Define the twist $F_{F_{B' | B}} \in A_B^\otimes 2$ by

\[ F_{F_{B' | B}} = \prod_{C \in F_{B' | B}} F_{(C, \alpha_C \varepsilon)} \]

where the product is taken with $F_{(C_1, \alpha_C \varepsilon_{C_1})}$ written to the left of $F_{(C_2, \alpha_C \varepsilon_{C_2})}$ whenever $C_1 \subseteq C_2$. This does not specify the order of the factors uniquely, but two orders satisfying this requirement are readily seen to yield the same product. Then,

\[ (\Phi_B)_{F_{B' | B}} = \Phi_B' \]
so that the restriction functor \( C_B \to C_{B'} \) is endowed with a collection of tensor structures labelled by any such \( \mathcal{F} \).

3.6.2. A weak quasi-Coxeter quasibialgebra of type \( D \) is a set

\[
(A, \{A_B\}, \{\Phi_{G\mathcal{F}}\}, \Delta, \varepsilon, \{\Phi_B\}, \{F_{(B,\alpha)}\})
\]

where

- \((A, \{A_B\}, \{\Phi_{G\mathcal{F}}\})\) is a weak quasi–Coxeter algebra of type \( D \)
- \((A, \{A_B\}, \Delta, \varepsilon, \{\Phi_B\}, \{F_{(B,\alpha)}\})\) is a \( D \)-quasibialgebra

and, for any pair \((G, \mathcal{F})\) of maximal nested sets on \( D \), the following holds

\[
F_G \cdot \Delta(\Phi_{G\mathcal{F}}) = \Phi_{G\mathcal{F}}^2 \cdot F_{\mathcal{F}}
\]

where, in the notation of 3.6.1, \( F_{\mathcal{F}} = F_{\mathcal{F}_{gB}} \).

Thus, in a weak quasi-Coxeter quasibialgebra the tensor structures on the restriction functors \( C_B \to C_{B'} \) are naturally isomorphic via the associators \( \Phi_{G\mathcal{F}} \).

3.6.3. A quasi-Coxeter quasitriangular quasibialgebra of type \( D \) is a weak quasi-Coxeter quasibialgebra \( A \) of type \( D \) endowed with the following additional data

- **\( R \)-matrices.** For any connected \( B \subseteq D \), an invertible element \( R_B \in A_B^{\otimes 2} \) such that \((A_B, \Delta, \varepsilon, \Phi_B, R_B)\) is a quasitriangular quasibialgebra.
- **Local monodromies.** For any vertex \( \alpha_i \in D \), an invertible element \( S_i \in A_{i} \) such that \((A, \{\Phi_{G\mathcal{F}}\}, \{S_i\})\) is a quasi–Coxeter algebra of type \( D \).

The \( R \)-matrices and local monodromies are subject to the following compatibility relation:

- **Coproduct identity.** For any vertex \( \alpha_i \in D \), the following holds

\[
\Delta_{F_{(\alpha_i, \alpha_i)}}(S_i) = (R_{\alpha_i})_{F_{(\alpha_i, \alpha_i)}}^{21} \cdot S_i \otimes S_i
\]

4. Differential twists and quasi-Coxeter structures

We define in this section the notion of differential twist of a semisimple Lie algebra \( g \), and show that it gives rise to a quasi–Coxeter quasitriangular quasibialgebra structure on \( Ug[[h]] \), which interpolates between the quasitriangular quasibialgebra structure underlying the monodromy of the KZ connection and the quasi–Coxeter structure underlying that of the Casimir one. The corresponding structural twists arise by comparing the asymptotics of the differential twist for \( g \) when a given coordinate \( \alpha_i \) tends to \( \infty \), to the differential twist for the subalgebra of \( g \) generated by the root vectors corresponding to the simple roots \( \alpha_j \neq \alpha_i \).

4.1. Notation. For any subdiagram \( B \subseteq D \), let \( g_B \subseteq g \) be the subalgebra generated by the root subspaces \( g_{\pm \alpha_i}, \alpha_i \in B, \Phi_B \subset \Phi \) its root system, and \( I_B = g_B + \mathfrak{h} \) the corresponding Levi subalgebra of \( g \). Denote by

\[
\Omega_B = x_a \otimes x^a, \quad C_B = x_a \cdot x^a \quad \text{and} \quad r_B = \sum_{\alpha \in \Phi_B \cap \Phi_+} x_{-\alpha} \otimes x_\alpha
\]

where \( \{x_a\}, \{x^a\} \) are dual basis of \( g_D \) with respect to \( \langle \cdot, \cdot \rangle \), and \( x_a \in \mathfrak{g}_a \) are root vectors such that \( (x_a, x_{-\alpha}) = 1 \), the corresponding invariant tensor, Casimir operator and standard solution of the modified classical Yang–Baxter equation for \( g_B \) respectively. Let also \( \Phi_B^{ks} \) be the KZ associator for \( g_B \) defined in 3.1.3.
4.2. **Differential twist.** Let $C_R = \{ t \in \mathfrak{h} | \alpha_i(t) > 0, \forall i \in I \} \subset \mathfrak{h}_R$ be the fundamental chamber of $\mathfrak{h}$ and $C = C_R + i\mathfrak{h}_R$ its complexification.

**Definition.** A differential twist for $\mathfrak{g}$ is a holomorphic map $F : C \rightarrow U\mathfrak{g}^{\otimes 2}[\hbar]^o$ such that

1. $\varepsilon \otimes \text{id}(F) = 1 = \text{id} \otimes \varepsilon(F)$.
2. $F \equiv 1^{\otimes 2} \mod \hbar$.
3. $(\Phi_{KZ})_F = 1^{\otimes 3}$.
4. $\text{Alt}_2 F = \hbar r_D \mod \hbar^2$.
5. $F$ satisfies

\[
dF = \frac{\hbar}{2} \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} (\mathcal{K}_\alpha \otimes 1 + 1 \otimes \mathcal{K}_\alpha) F - F \Delta(\mathcal{K}_\alpha)
\]

(4.1)

**Remark.** Condition (5) is compatible with (1) in that it implies that both $\varepsilon \otimes \text{id}(F)$ and $\text{id} \otimes \varepsilon(F)$ satisfy

\[
d\varepsilon = \frac{\hbar}{2} \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} [\mathcal{K}_\alpha, \varepsilon]
\]

which is consistent with $\varepsilon \equiv 1$. It is also compatible with (3)–(4) in the following sense. Define $f : C \rightarrow U\mathfrak{g}^{\otimes 2}$ by $F = 1^{\otimes 2} + \hbar f \mod \hbar^2$. It follows from (3) and the fact that $\Phi_{KZ} = 1^{\otimes 3} \mod \hbar^2$, that $d_H f = 0$, where $d_H : U\mathfrak{g}^{\otimes m} \rightarrow U\mathfrak{g}^{\otimes (m+1)}$ is the Hochschild differential

\[
d_H a = 1 \otimes a + \sum_{i=1}^{m} (-1)^i \text{id}^{\otimes (i-1)} \otimes \Delta \otimes \text{id}^{\otimes (m-i)}(a) + (-1)^{m+1} a \otimes 1
\]

Moreover, (5) and (2) imply that

\[
df = \frac{1}{2} \sum_{\alpha \in \Phi_+} (\mathcal{K}_\alpha \otimes 1 + 1 \otimes \mathcal{K}_\alpha - \Delta(\mathcal{K}_\alpha)) = \frac{1}{2} \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} d_H(\mathcal{K}_\alpha)
\]

It follows that the Hochschild cohomology class of $f$ is constant on $C$ and, by (4), equal to $r_D$.

4.3. **Compatibility with DCP associators.** Fix henceforth a positive, adapted family $\beta = \{ x_B \}_{B \subseteq D} \subset \mathfrak{h}^*$. For any maximal nested set $\mathcal{F}$, let $\Psi_\mathcal{F} : C \rightarrow U\mathfrak{g}[\hbar]^o$ be the fundamental solution of $\nabla_\kappa$ corresponding to $\mathcal{F}$ and $\beta$, and $\Phi_\mathcal{F} = \Psi_\mathcal{F}^{-1}, \Psi_\mathcal{F}$ the corresponding associators.

Let $F : C \rightarrow U\mathfrak{g}^{\otimes 2}[\hbar]^o$ be a differential twist for $\mathfrak{g}$, and set

\[
F_\mathcal{F} = (\Psi_\mathcal{F}^{\otimes 2})^{-1} \cdot F \cdot \Delta(\Psi_\mathcal{F}^{\otimes 2})
\]

**Lemma.** The following holds

1. $F_\mathcal{F}$ is constant on $C$.
2. $\varepsilon \otimes \text{id}(F_\mathcal{F}) = 1 = \text{id} \otimes \varepsilon(F_\mathcal{F})$.
3. $(\Phi_{KZ})_{F_\mathcal{F}} = 1^{\otimes 3}$.
4. $F_\mathcal{F} = \Phi_{\mathcal{F}^2} \cdot F_\mathcal{F} \cdot \Delta(\Phi_{\mathcal{F}^2})^{-1}$. 

Fix $i \in I$, let $\Phi \subset \Phi$ be the root system generated by the simple roots \( \{\alpha_j\}_{j \neq i} \), $\Phi \subset \Phi$ the subalgebra spanned by the root vectors and coroots \( \{x_\alpha, \alpha^\vee\}_{\alpha \in \Phi} \), $\Phi \supset \Phi \subset \Phi$ its Cartan subalgebra, and $\Phi = \Phi \oplus \Phi$ the corresponding Levi subalgebra of $\Phi$.

The inclusion of root systems $\Phi \subset \Phi$ gives rise to a projection $\pi : \Phi \to \Phi$ determined by the requirement that $\alpha(\pi(t)) = \alpha(t)$ for any $\alpha \in \Phi$. The kernel of $\pi$ is the line $C\lambda_i^\vee$ spanned by the $i$th fundamental coweight of $\Phi$.

We shall coordinatise the fibres of $\pi$ by restricting the simple root $\alpha_i$ to them. This amounts to trivialising the fibration $\pi : \Phi \to \Phi$ as $\Phi \to C \times \Phi$ via $(\alpha_i, \pi)$. The inverse of this isomorphism is given by $(w, \overline{\mu}) \to w\lambda_i^\vee + \pi(\overline{\mu})$, where $i : \Phi \to \Phi$ is the embedding with image $\text{Ker}(\alpha_i)$ given by

$$i(\overline{\mu}) = \overline{\mu} - \alpha_i(\overline{\mu})\lambda_i^\vee$$

Denote by

$$K = \sum_{\alpha \in \Phi_+} K_{\alpha} \quad \text{and} \quad \overline{K} = \sum_{\alpha \in \overline{\Phi}_+} K_{\alpha}$$

the (truncated) Casimir operators of $\Phi$ and $\overline{\Phi}$.

### 4.5. Asymptotics of the Casimir connection for $\alpha_i \to \infty$.

Retain the notation of 4.4. Fix $\overline{\mu} \in \overline{\Phi}$, and consider the fiber of $\pi : \Phi \to \Phi$ at $\overline{\mu}$. Since the restriction of $\alpha \in \Phi$ to $\pi^{-1}(\overline{\mu})$ is equal to $\alpha(\lambda_i^\vee)\alpha_i + \alpha(\mu(\overline{\mu}))$, the restriction of the Casimir connection $\nabla_C$ to $\pi^{-1}(\overline{\mu})$ is equal to

$$\nabla_i \overline{\mu} = d - \frac{h}{2} \sum_{\alpha \in \Phi_+ \setminus \overline{\Phi}} \frac{d\alpha_i}{\alpha_i - w_\alpha} K_{\alpha}$$

where $w_\alpha = -\alpha(\mu(\overline{\mu})) / \alpha(\lambda_i^\vee)$. Set

$$R_{\overline{\mu}} = \max\{|w_\alpha|\}_{\alpha \in \Phi \setminus \overline{\Phi}}$$

### Proposition.

1. For any $\overline{\mu} \in \overline{\Phi}$, there is a unique holomorphic function $H_\infty : \{w \in \mathbb{P}^1 | |w| > R_{\overline{\mu}}\} \to U \Phi[h]^\circ$ such that $H_\infty(\infty) = 1$ and, for any determination of $\log(\alpha_i)$, the function $\Upsilon_\infty = H_\infty(\alpha_i) \cdot \alpha_i^{\frac{h}{2}(K - \overline{K})}$ satisfies

$$\left(d - \frac{h}{2} \sum_{\alpha \in \Phi_+ \setminus \overline{\Phi}} \frac{d\alpha_i}{\alpha_i - w_\alpha} K_{\alpha}\right) \Upsilon_\infty = \Upsilon_\infty d$$
(2) The function $H_{\infty}(\alpha_i, \overline{\pi})$ is holomorphic on the simply-connected domain $\mathcal{D}_{\infty} \subset \mathbb{P} \times \mathbb{H}$ given by

$$\mathcal{D}_{\infty} = \{ (w, \overline{\pi}) | |w| > R_{\overline{\pi}} \}$$

(4.5)

and, as a function on $\mathcal{D}_{\infty}$, $\mathcal{Y}_{\infty}$ satisfies

$$
\left( d - \frac{\hbar}{2} \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} K_{\alpha} \right) \mathcal{Y}_{\infty} = \mathcal{Y}_{\infty} \left( d - \frac{\hbar}{2} \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} K_{\alpha} \right)
$$

(3) $H_{\infty}$ satisfies $H_{\infty}(t\alpha_i, t\overline{\pi}) = H_{\infty}(\alpha_i, \overline{\pi})$ for any $t \in \mathbb{C}^*$. 

(4) If $\overline{\pi} = 0$, $H_{\infty}(\alpha_i, 0) = 1$, $\mathcal{Y}_{\infty}(\alpha_i, 0) = \alpha_i^{\frac{1}{2}(K - \overline{K})}$, and both commute with $\mathfrak{T}$.

PROOF. (1) Denote the restriction of $\alpha_i$ to $\pi^{-1}(\overline{\pi})$ by $w$. $H = H_{\infty}$ is required to satisfy the ODE

$$\frac{dH}{dw} = \frac{\hbar}{2} \left( \sum_{\alpha \in \Phi_+} \frac{K_{\alpha}}{w - w_{\alpha}} (H - H) - \frac{K - \overline{K}}{w} \right)
$$

(4.6)

all of whose singularities are contained in the disk $\{ |w| \leq R_{\overline{\pi}} \}$. Writing $H = \sum_{n \geq 0} h^n H_n$ yields the recursive system of ODEs

$$\frac{dH_n}{dw} = \frac{1}{2} \left( \sum_{\alpha \in \Phi_+ \setminus \Phi} \frac{K_{\alpha}}{w - w_{\alpha}} H_{n-1} - H_{n-1} \frac{K - \overline{K}}{w} \right)
$$

where $H_{-1} = 0$, together with the boundary condition $H_n(\infty) = \delta_{n0}$, which clearly possess at most one solution. For $n = 0$, this possesses the solution $H_0 \equiv 1$. For $n \geq 1$, given that

$$\frac{1}{w - w_{\alpha}} = \frac{1}{w} \left( 1 + \frac{w_{\alpha}}{w - w_{\alpha}} \right)
$$

(4.7)

the equation reads

$$\frac{dH_n}{dw} = \frac{1}{2w} \left[ [K - \overline{K}, H_{n-1}] + \sum_{\alpha \in \Phi_+ \setminus \Phi} \frac{w_{\alpha} K_{\alpha}}{w - w_{\alpha}} H_{n-1} \right]
$$

This possesses the holomorphic solution

$$H_n(w) = \int_{\Gamma^w} \frac{1}{2t} \left( [K - \overline{K}, H_{n-1}(t)] + \sum_{\alpha \in \Phi_+ \setminus \Phi} \frac{w_{\alpha} K_{\alpha}}{t - w_{\alpha}} H_{n-1}(t) \right) dt
$$

where $\Gamma^w$ is the ray from $\infty$ to $w$, and $w$ is assumed to be such that $|w| > R_{\overline{\pi}}$, provided the integral converges at $t = \infty$. For $n = 1$, this is the case since $H_0 = 1$ commutes with $K - \overline{K}$, so the integrand is an $O(t^{-2})$ and, for $n \geq 2$, this follows since $H_{n-1}(t) = O(t^{-1})$.

(2) The recursive construction of $H$ clearly shows that it is a holomorphic function of $\overline{\pi}$. The fact that $\mathcal{Y}_{\infty}$ satisfies the claimed PDE follows from the integrability of $\nabla_C$ (a similar argument is given in the proof of part (2) of Theorem 5.4).

(3) Follows by uniqueness.

(4) If $\overline{\pi} = 0$, $w_{\alpha} = 0$ for any $\alpha$, so that (4.6) simply reads

$$\frac{dH}{dw} = \frac{\hbar}{2} \left[ [K - \overline{K}, H] \right]$$


and, by uniqueness, $H(w, 0) \equiv 1$. $\Upsilon_\infty(w, 0) = w^\frac{4}{\hbar}(\mathcal{K} - \overline{\mathcal{K}})$ commutes with $\mathfrak{t}$ because \( \mathcal{K} - \overline{\mathcal{K}} \) does. Indeed, since $\mathcal{K} = C - t_a t^a$, where $C$ is the Casimir operator of $\mathfrak{g}$ and $\{t_a\}, \{t^a\}$ are dual bases of $\mathfrak{h}$,

$$
\mathcal{K} - \overline{\mathcal{K}} = C - \overline{C} - \frac{(\lambda^i)^2}{\|\lambda^i\|^2}
$$

which commutes with $\mathfrak{t}$.

\textbf{Remark.} We later use, we shall need the (obvious) fact that the uniqueness of Proposition 4.5 holds under the weaker assumption that the function $H_\infty(\alpha_i, \overline{\mathfrak{t}})$ is holomorphic on one of the domains

$$
\mathcal{D}^\perp_\infty = \{(w, \overline{\mathfrak{t}}) \in \mathbb{C} \times \overline{\mathfrak{h}} | \text{Im} w \geq 0, |w| > R_{\overline{\mathfrak{t}}})
$$

and is such that $H_\infty(w, \overline{\mathfrak{t}}) \to 1$ as $w \to \infty$ with $0 < \delta < |\arg w| < \pi - \delta < \pi$.

4.6. \textbf{Asymptotics at $\alpha_i = \infty$ and DCP solutions.} Let $\mathcal{F}$ be a maximal nested set on $D$, set $\overline{\mathcal{F}} = \mathcal{F} \setminus \{D\}$ and $\alpha_i = \alpha^D_{\mathcal{F}}$. Let

$$
\Psi_{\mathcal{F}} : \mathcal{C} \to U \mathfrak{g}[\hbar]^o \quad \text{and} \quad \Psi_{\overline{\mathcal{F}}} : \overline{\mathcal{C}} \to U \mathfrak{g}[\hbar]^o
$$

be the fundamental solutions of the Casimir connection for $\mathfrak{g}$ and $\overline{\mathfrak{g}} = \mathfrak{g}_{D \setminus \alpha_i}$ corresponding to $\mathcal{F}, \overline{\mathcal{F}}$ respectively, and a positive, adapted family $\{x_B\}_{B \subseteq D}$. Regard $\Psi_{\overline{\mathcal{F}}}$ as being defined on $\mathcal{C}$ via the projection $\pi : \mathfrak{h} \to \overline{\mathfrak{h}}$. The result below expresses $\Psi_{\mathcal{F}}$ in terms of $\Psi_{\overline{\mathcal{F}}}$ and the solution $\Upsilon_\infty$ given by Proposition 4.5.

\textbf{Proposition.} The following holds

$$
\Psi_{\mathcal{F}} = \Upsilon_\infty \cdot \Psi_{\overline{\mathcal{F}}} \cdot x_D(\lambda^i)^{\frac{4}{\hbar}(\mathcal{K} - \overline{\mathcal{K}})}
$$

\textbf{Proof.} Note first that $\alpha_i$ can be expressed as a linear combination of the form $\sum_{B \in \mathcal{F}} a_B x_B$. Evaluating on $\lambda^i$ shows that $a_D = x_D(\lambda^i)^{-1}$, so that

$$
\alpha_i = \frac{x_D}{x_D(\lambda^i)} \left(1 + \sum_{B \in \mathcal{F}} \frac{x_B}{x_D} \right) = \frac{x_D}{x_D(\lambda^i)} \cdot p_{\alpha_i}(u)
$$

where $p_{\alpha_i}$ is a polynomial in the variables $\{u_B\}_{B \in \mathcal{F}}$, such that $p_{\alpha_i}(0) = 1$. By construction,

$$
\Upsilon_\infty \cdot \Psi_{\overline{\mathcal{F}}} = H_\infty(\alpha_i, \overline{\mathfrak{t}}) \cdot \alpha^D_{\overline{\mathcal{F}}} \cdot \frac{2}{\hbar}(\mathcal{K} - \overline{\mathcal{K}}) \cdot H_{\overline{\mathcal{F}}}(u) \cdot \prod_{B \in \mathcal{F}} x_B^{\frac{2}{\hbar}(\mathcal{K}_B - \mathcal{K}_B \setminus \alpha_i)}
$$

$$
= H_\infty(\alpha_i, \overline{\mathfrak{t}}) \cdot H_{\overline{\mathcal{F}}}(u) \cdot x_D(\lambda^i)^{\frac{4}{\hbar}(\mathcal{K} - \overline{\mathcal{K}})} \cdot \prod_{B \in \mathcal{F}} x_B^{\frac{2}{\hbar}(\mathcal{K}_B - \mathcal{K}_B \setminus \alpha_i)}
$$

where the second equality follows from (4.10) and the fact that $\mathcal{K}, \overline{\mathcal{K}}$ commute with $H_{\overline{\mathcal{F}}}$ since the latter takes values in $U \mathfrak{g}[\hbar]$ and is of weight 0. The claimed result follows from the uniqueness of $\Psi_{\mathcal{F}}$, provided the function $H_\infty(\alpha_i, \overline{\mathfrak{t}}) \cdot H_{\overline{\mathcal{F}}}(u) \cdot p_i(u)^{\frac{2}{\hbar}(\mathcal{K} - \overline{\mathcal{K}})}$ is holomorphic in the neighborhood of $u = \{u_B\}_{B \in \mathcal{F}} = 0$, and equal to 1 at $u = 0$.

This is clearly true for the factor $p_i(u)^{\frac{2}{\hbar}(\mathcal{K} - \overline{\mathcal{K}})}$. For $H_{\overline{\mathcal{F}}}$ note that, for $B \in \overline{\mathcal{F}}, \pi_B$ is equal to $u_B$, if $B$ is not a maximal element of $\mathcal{F}$, and to $x_B = u_B u_B$ otherwise. Thus, $H_{\overline{\mathcal{F}}}$ is holomorphic in the neighborhood of $u = 0$, and equal to 1 at $u = 0$. 

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Finally, note that the coordinates of $\overline{t}$ are $x_B, B \in \mathcal{F}$, with $x_B = x_D \prod_{\mathcal{F} \ni C \supset B} u_C$. Thus, by the homogeneity of $H_\infty$,

$$H_\infty(\alpha, \overline{t}) = H_\infty \left( \frac{x_D}{x_D(\lambda_i^\vee)} p_{\alpha_i}(u), \left\{ x_D \prod_{\mathcal{F} \ni C \supset B} u_C \right\}_{B \in \mathcal{F}} \right) = H_\infty \left( \frac{p_{\alpha_i}(u)}{x_D(\lambda_i^\vee)}, \left\{ \prod_{\mathcal{F} \ni C \supset B} u_C \right\}_{B \in \mathcal{F}} \right)$$

which tends to 1 as $u \to 0$ since $H_\infty(\alpha, 0) = 1$. 

**Corollary.** For any $\alpha \in B \subset D$, let $\Upsilon_B^{\alpha, \delta}$ be the solution of the Casimir connection for $\mathfrak{g}_B$ corresponding to the simple root $\alpha$ given by Proposition 4.5. Then,

$$\Psi_{\mathcal{F}} = \prod_{B \in \mathcal{F}} \Upsilon_B^{\alpha, \delta} \cdot \prod_{B \in \mathcal{F}} x_B(\lambda_i^\vee) \Upsilon_B^{\alpha, \delta}$$

where the first product is ordered with $\alpha$ to the left of $\Upsilon_B^{\alpha, \delta}$ if $B \supset C$.

4.7. **Relative twists.** Let $F$ be a differential twist for $\mathfrak{g}$, $\alpha_i \in D$ a simple root, and $\Upsilon_\infty$ the solution of the Casimir equations given by Proposition 4.5, where we are using the standard determination of log. Define $F_\infty : \mathcal{C} \to U\mathfrak{g}^\otimes \mathfrak{g} \mathfrak{g}$ by

$$F_\infty = (\Upsilon_\infty^{-1} \cdot F \cdot \Delta(\Upsilon_\infty))$$

Then, $F_\infty$ satisfies

1. $\varepsilon \otimes \text{id}(F_\infty) = 1 = \text{id} \otimes \varepsilon(F_\infty)$.
2. $F_\infty \equiv 1 \otimes 2 \text{ mod } h$.
3. $(\Phi_{Kz}) F_\infty = 1 \otimes 3$.
4. $\text{Alt}_2 F_\infty = \hbar r_D \text{ mod } h^2$.
5. $dF_\infty = \hbar \sum_{\alpha \in \mathcal{F}_+} \left( (K_\alpha \otimes 1 + 1 \otimes K_\alpha) \cdot F_\infty - F_\infty \cdot \Delta(K_\alpha) \right)$

Let $\overline{B}$ be the complexified chamber of $\overline{\mathfrak{g}}$, and $\overline{\mathcal{F}} : \overline{\mathcal{C}} \to U\overline{\mathfrak{g}}^\otimes \mathfrak{g}$ a differential twist for $\overline{\mathfrak{g}}$. Since the projection $\pi : \mathfrak{h} \to \overline{\mathfrak{g}}$ maps $\mathcal{C}$ to $\overline{\mathcal{C}}$, we may regard $\overline{\mathcal{F}}$ as a function on $\mathcal{C}$, and define $F'_{\mathcal{C}} : \mathcal{C} \to U\mathfrak{g}^\otimes \mathfrak{g}$ by

$$F'_{\mathcal{C}} = \overline{\mathcal{F}}^{-1} \cdot F_\infty$$

**Proposition.** $F'_{\mathcal{C}}$ satisfies the following properties

1. $\varepsilon \otimes \text{id}(F'_{\mathcal{C}}) = 1 = \text{id} \otimes \varepsilon(F'_{\mathcal{C}})$.
2. $F'_{\mathcal{C}}(\Phi_{Kz}) F'_{\mathcal{C}} = 1 \otimes 2 \text{ mod } h$.
3. $(\Phi_{Kz}) F'_{\mathcal{C}} F'_{\mathcal{C}} = \Phi_{D\setminus \alpha_i}$.
4. $\text{Alt}_2 F'_{\mathcal{C}}(\Phi_{Kz}) = \hbar (r_D - r_{D\setminus \alpha_i}) \text{ mod } h^2$.
5. $dF'_{\mathcal{C}} = \hbar \sum_{\alpha \in \mathcal{F}_+} \left( \Delta(K_\alpha), F'_{\mathcal{C}} \right)$

(6) If $F'_{\mathcal{C}}$ is invariant under $\overline{\mathfrak{g}}$, then it is constant on $\mathcal{C}$. 
4.7 is invariant under 

\[ (\Phi_F)_{F_{\text{red},\alpha}} = (\Phi_F)_{F} = 1_{\otimes^3} \]

whence

\[ (\Phi_F)_{F_{\text{red},\alpha}} = ((\Phi_F)_{\mathcal{T}}, F_{\text{red},\alpha})^{-1} = (1_{\otimes^3})_{\mathcal{T}}^{-1} = \Phi_{F_{\text{red},\alpha}} \]

(5) follows from the PDEs satisfied by \( F_\infty \) and \( \mathcal{T} \).

(6) follows from (5). \( \square \)

4.8. **Centraliser property.** Let \( \{F_B\} \) be a collection of differential twists for the subalgebras \( \mathfrak{g}_B \subseteq \mathfrak{g} \), where \( B \) is a subdiagram of \( D \), such that if \( B \) has connected components \( \{B_i\} \), then \( F_B = \prod_i F_{B_i} \).

**Definition.** The collection \( \{F_B\} \) has the centraliser property if, for any \( \alpha \in B \subseteq D \), the relative twist \( F_{(B,\alpha)} \) defined in 4.7 is invariant under \( \mathfrak{g}_{B\setminus\alpha} \) (and in particular constant).

4.9. **Factorisation.** Let \( \{F_B\}_{B \subseteq D} \) be a collection of differential twists with the centraliser property. For any \( \alpha \in B \subseteq D \), set

\[ F_{(B,\alpha)} = \left( x_B(\lambda^\gamma) - \frac{\Delta}{2}(K_B - K_{B\setminus\alpha}) \right)^{\otimes^2} \cdot F'_{(B,\alpha)} \cdot \Delta \left( x_B(\lambda^\gamma) - \frac{\Delta}{2}(K_B - K_{B\setminus\alpha}) \right) \]

where \( F'_{(B,\alpha)} \in U_{B}^{\otimes^2} [h]^\alpha \) is the relative twist defined in 4.7, and \( \{x_B\}_{B \subseteq D} \) is a positive, adapted family. The (constant) twist \( F_{(B,\alpha)} \) is invariant under \( \mathfrak{g}_{B\setminus\alpha} \), and has the properties (1)–(4) given in Proposition 4.7.

**Lemma.** Let \( F \) be a maximal nested set on \( D \), and \( F_F \) the twist defined in 4.3. Then, the following holds

\[ F_F = \prod_{B \subseteq F} F_{(B,\alpha_B)} \]

where the product is taken with \( F_{(B,\alpha_B)} \) to the right of \( F_{(C,\alpha_C)} \) if \( B \supseteq C \).

**Proof.** By definition of \( F_F \),

\[ F_F = (\Psi_\mathcal{F})^{-1} \cdot F \cdot \Delta (\Psi_\mathcal{F}) \]

\[ = (x_D(\lambda_y) - \frac{\Delta}{2}(K_D - K_{D\setminus\alpha}))^{\otimes^2} \cdot (\Psi_\mathcal{F})^{-1} \cdot (\Psi_\mathcal{F})^{-1} \cdot F \cdot \Delta (\Psi_\mathcal{F}) \cdot \Delta \left( x_D(\lambda_y) - \frac{\Delta}{2}(K_D - K_{D\setminus\alpha}))^{\otimes^2} \right) \]

\[ = (x_D(\lambda_y) - \frac{\Delta}{2}(K_D - K_{D\setminus\alpha}))^{\otimes^2} \cdot F'_{(D,\alpha_D)} \cdot \Delta (\Psi_F) \cdot \Delta \left( x_D(\lambda_y) - \frac{\Delta}{2}(K_D - K_{D\setminus\alpha}))^{\otimes^2} \right) \]

\[ = (\Psi_\mathcal{F})^{-1} \cdot F'_{(D,\alpha_D)} \cdot \Delta (\Psi_F) \cdot \Delta \left( x_D(\lambda_y) - \frac{\Delta}{2}(K_D - K_{D\setminus\alpha}))^{\otimes^2} \right) \]

where the second equality follows by Proposition 4.6, the third one by definition of \( F_{(D,\alpha_D)} \), and the fourth one from the fact that \( F'_{(D,\alpha_D)} \) is invariant under \( \mathfrak{g}_{D\setminus\alpha} \) by the centraliser property, and therefore commutes with \( \Delta (\Psi_F) \), and the fact that, by (4.8), \( K_D - K_{D\setminus\alpha} \) commutes with \( \mathfrak{g}_{D\setminus\alpha} \). The result now follows by induction. \( \square \)
4.10. Quasi–Coxeter quasitriangular quasibialgebra structure. For any connected subdiagram $B \subseteq D$, let $(U\mathfrak{g}[\hbar], \Delta_0, \Phi_{F}^{KZ}, R_{F}^{KZ})$ be the quasitriangular quasibialgebra structure underlying the monodromy of the KZ connection for $\mathfrak{g}_B$ (see 3.1.3). Let also

$$(U\mathfrak{g}[\hbar], \{U\mathfrak{g}_B[\hbar]\}, \{S_{i,C}^{S}\}, \{\Phi_{F}\})$$

be the quasi–Coxeter structure underlying the monodromy of the (untruncated) Casimir connection $\nabla_C$ (see 3.5.3–3.5.4), relative to a choice of positive, adapted family $\{x_B\}_B \subseteq D$. The following is the main result of this section

**Theorem.** Let $\{F_B\}_B \subseteq D$ be differential twists for $\{\mathfrak{g}_B\}_B \subseteq D$ possessing the centraliser property, and such that

- $F_B$ is of weight zero for any $B \subseteq D$.
- The following holds for any simple root $\alpha_i$,

$$\Theta^{\otimes 2}(F_{\alpha}) = F_{\alpha}^{21}$$

where $\Theta$ is an automorphism of $\mathfrak{g}$ acting by $-1$ on $\mathfrak{h}$.

Define the relative twists $\{F_{(B,\alpha)}\}_\alpha \subseteq B \subseteq D$ as in 4.9. Then,

$$(U\mathfrak{g}[\hbar], \{U\mathfrak{g}_B[\hbar]\}, \{S_{i,C}\}, \{\Phi_{F}\}, \Delta_0, \{\Phi_{F}^{KZ}\}, \{R_{F}^{KZ}\}, \{F_{(B,\alpha)}\})$$

is a quasi–Coxeter quasitriangular quasibialgebra such that

$$S_{i,C}^{S} = s_i \cdot \exp(h/2 \cdot C_{\alpha})$$

$$\Phi_{F}^{KZ} = 1 \otimes ^3 \mod h^2$$

$$R_{F}^{KZ} = \exp(h \cdot \Omega_B)$$

$$\text{Alt}_2 F_{(B,\alpha)} = h \cdot (r_B - r_{B,\alpha}) \mod h^2$$

and $\Phi_{F}, F_{(B,\alpha)}$ are of weight 0.

**Proof.** The identity $(\Phi_{F}^{KZ})_{F_{(B,\alpha)}} = \Phi_{F}^{KZ}_{B,\alpha}$ was checked in Proposition 4.7. The identity $F_{\mathfrak{g}} = \Phi_{F}^{KZ} \cdot F_{F} \cdot \Delta_0(\Phi_{F\mathfrak{g}})$ follows from Lemma 4.3 and the factorisation given by Lemma 4.9. Given that

$$\Delta_0(S_{i,C}) = e^{H_{\alpha}} \cdot S_{i,C} \otimes S_{i,C} = R_{F}^{KZ}_{\alpha} \cdot S_{i,C} \otimes S_{i,C}$$

the coproduct identity

$$(\Delta_0)_{F_{(\alpha,\alpha)}}(S_{i,C}) = (R_{F}^{KZ}_{\alpha})^{21}_{F_{(\alpha,\alpha)}} \cdot S_{i,C} \otimes S_{i,C}$$

is readily seen to be equivalent to $\text{Ad}(S_{i,C}^{\otimes 2})(F_{(\alpha,\alpha)}) = F_{(\alpha,\alpha)}^{21}$ and therefore to $\text{Ad}(\tilde{s}_i)(F_{(\alpha,\alpha)}) = F_{(\alpha,\alpha)}^{21}$ since $C_{\alpha}$ is central in $U\mathfrak{g}_B^{\otimes 2}$. Since the restriction to $\mathfrak{sl}_2^{\alpha}$ of $\Theta$ and $\text{Ad}(\tilde{s}_i)$ differ by $\text{Ad}(\exp(t))$ for some $t \in \mathfrak{h}$, and $F_{(\alpha,\alpha)}$ is of weight 0, the coproduct identity is therefore equivalent to $\Theta^{\otimes 2}(F_{(\alpha,\alpha)}) = F_{(\alpha,\alpha)}^{21}$. This in turn follows from the assumption on $F_{\alpha}$, and the fact that

$$F_{(\alpha,\alpha)} = \alpha_i^{-\frac{1}{2}(K_{\alpha_i} \otimes 1 + 1 \otimes K_{\alpha_i})} \cdot F_{\alpha_i} \cdot \alpha_i^{-\frac{1}{2} \Delta_0(K_{\alpha_i})}$$

$\square$
5. The fusion operator

In this section, we construct a fusion operator, that is a joint solution \( J(z, \mu) \) of the coupled KZ-Casimir equations on \( n = 2 \) points, with prescribed asymptotics when \( z = z_1 - z_2 \to \infty \). The coupling gives rise to an irregular singularity in \( z \), and due care needs to be taken to deal with the corresponding Stokes phenomena. We shall in fact construct a multicomponent version of the fusion operator, which solves the coupled Casimir and KZ equations in an arbitrary number of points. The differential twist for \( \mathfrak{g} \), which gives rise to a quasi-Coxeter quasitriangular quasibialgebra structure on \( U_{\mathfrak{g}}[\hbar] \), as explained in Section 4, will be obtained by taking the asymptotics of \( J(z, \mu) \) when \( z \to 0 \).\(^{12}\)

5.1. The joint system. For any \( n \geq 2 \), let

\[
\mathfrak{X}_n = \mathbb{C}^n \setminus \bigcup_{1 \leq i < j \leq n} \{ z_i = z_j \}
\]

be the configuration space of \( n \) ordered points in \( \mathbb{C} \). Consider the following connection on the trivial vector bundle over \( \mathfrak{X}_n \times \mathfrak{h}_{\text{reg}} \) with fibre \( U_{\mathfrak{g}}^{\otimes n}[\hbar] \)

\[
\nabla = d - h \sum_{1 \leq i < j \leq n} \frac{d(z_i - z_j)}{z_i - z_j} \Omega_{ij} - \frac{h}{2} \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} \Delta^{(n)}(\mathcal{K}_\alpha) - d \sum_{i=1}^n z_i \text{ad}(\mu^{(i)})
\]

(5.1)

where \( h \) is a formal parameter, \( \Omega_{ij} = \sum_n X^{(i)}_a X^{(j)}_a \), where \( X^{(i)} = 1^{\otimes (i-1)} \otimes X^{\otimes 1^{\otimes (n-i)}} \), \( \Delta^{(n)} : U_{\mathfrak{g}} \to U_{\mathfrak{g}}^{\otimes n} \) is the iterated coproduct and \( \mu \) is the embedding \( \mathfrak{h} \to U_{\mathfrak{g}} \). The above connection is flat \(^{20}\). We give an alternative proof of this in Appendix C.

5.2. The change of variables. We wish to construct horizontal sections of \( \nabla \) having prescribed asymptotics as \( z_i - z_j \to \infty \) for any \( i \neq j \). Consider to this end the change of variables given by the map

\[
\rho : \mathbb{C}^x \times \mathfrak{X}_n \to \mathfrak{X}_n, \quad \rho(\zeta_1, \ldots, \zeta_n) = (\zeta_1, \ldots, \zeta_n)
\]

Since \( d \log(z_i - z_j) = d \log \zeta + d \log(\zeta_i - \zeta_j) \), the pull–back of \( \nabla \) under \( \rho \) is equal to \( \nabla_\zeta + \nabla \), where

\[
\nabla_\zeta = d_\zeta - \left( \sum_{i=1}^n \zeta_i \text{ad}(\mu^{(i)}) + \frac{h}{\zeta} \Omega \right) d_\zeta
\]

(5.2)

where \( d_\zeta \) is the de Rham differential with respect to \( \zeta \), \( \Omega = \sum_{i<j} \Omega_{ij}, \) and

\[
\nabla = d - h \sum_{1 \leq i < j \leq n} \frac{d(\zeta_i - \zeta_j)}{\zeta_i - \zeta_j} \Omega_{ij} - \frac{h}{2} \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} \Delta^{(n)}(\mathcal{K}_\alpha) - \zeta d \sum_{i=1}^n \zeta_i \text{ad}(\mu^{(i)})
\]

where \( d \) is the de Rham differential with respect to \( (\zeta_1, \ldots, \zeta_n) \in \mathfrak{X}_n \), and \( \mu \in \mathfrak{h}_{\text{reg}}. \)

\(^{12}\)The name fusion operator originates from the work of Etingof and Varchenko \([17]\), where a representation theoretic construction is given for the the fusion operator of the loop algebra \( \mathfrak{g} \). The latter is a joint solution of the trigonometric KZ equations and dynamical difference equations (a difference analogue of the Casimir equations), and should degenerate to our \( J(z, \mu) \).
5.3. **Lemma.** If \((\zeta, \mu) \in X_\mathfrak{n} \times \mathfrak{h}_{\text{reg}}\), the projection \([\Omega_{ij}]\) of \(\Omega_{ij}\) onto the kernel of \(\sum_i \zeta_i \text{ad}(\mu^{(i)})\) is

\[
\Omega^\mathfrak{h}_{ij} = \sum_{a=1}^r (t_a)^{(i)}(t^a)^{(j)}
\]

where \(\{t_a\}, \{t^a\}\) are dual bases of \(\mathfrak{h}\).

**Proof.** By definition, \(\Omega_{ij} = \Omega^\mathfrak{h}_{ij} + \sum_{\alpha \in \Phi} \alpha \cdot x^{(i)}_{\alpha} x^{(j)}_{-\alpha}\), where \(x^{(i)}_{\pm \alpha}\) are root vectors such that \((x_\alpha, x_{-\alpha}) = 1\). Since \(\text{ad}(\mu^{(k)}) x^{(i)}_{\pm \alpha} = \alpha(\mu) (\delta_{ik} - \delta_{kj}) x^{(i)}_{\pm \alpha}\), we have

\[
\sum_k \zeta_k \text{ad}(\mu^{(k)}) x^{(i)}_{\alpha} x^{(j)}_{-\alpha} = \alpha(\zeta_i - \zeta_j) x^{(i)}_{\alpha} x^{(j)}_{-\alpha}
\]

from which the conclusion follows. \(\square\)

5.4. **Holomorphic fundamental solutions of \(\nabla_\zeta\).** The connection \((5.2)\) has an irregular singularity of Poincaré rank one at \(\zeta = \infty\). Let \(\mathbb{H}_\pm = \{\zeta \in \mathbb{C} | \Im \zeta \geq 0\}\). Let \(\Omega^\mathfrak{h} = \sum_{1 \leq i < j \leq n} \Omega^\mathfrak{h}_{ij}\).

Let \(\mathcal{C} = \{t \in \mathbb{H} | \alpha(t) > 0 \text{ for any } \alpha \in \Phi_+\}\) be the fundamental Weyl chamber of \(\mathfrak{g}\), and set \(\mathcal{C}_n = \{\zeta \in X_\mathfrak{n} \cap \mathbb{R}_+^n | \zeta_1 > \zeta_2 > \cdots > \zeta_n\}\). Set \(\imath = \sqrt{-1}\). Let \(H^0 = c\mathbb{1}^\otimes n\) be a multiple of the identity.

**Theorem.**

1. For any \((\zeta, \mu) \in \mathcal{C}_n \times i\mathcal{C}\), there is a unique holomorphic function

\[
H_\pm : \mathbb{H}_\pm \to \mathcal{A}
\]

such that \(H_\pm(\zeta)\) tends to \(H^0\) uniformly as \(\zeta \to \infty\) in any sector of the form \(|\arg(\zeta)| \in (\delta, \pi - \delta), \delta > 0\), and the \(\text{End}_{\mathcal{C}[\mathbb{H}]}(\mathcal{A})\)-valued function

\[
\Psi_\pm(\zeta) = H_\pm(\zeta) e^{\sum_i \zeta_i \text{ad}(\mu^{(i)})} \zeta^\mathfrak{h}
\]

is a fundamental solution of \(\nabla_\zeta\).

Assume henceforth that \(H^0 = 1\).

2. As a function on \(\mathbb{H}_\pm \times i\mathcal{C}_n \times i\mathcal{C}\), \(H_\pm\) is real analytic, and satisfies

\[
\nabla H_\pm = H_\pm \left(\hbar \sum_{i<j} d\log(\zeta_i - \zeta_j) \Omega^\mathfrak{h}_{ij} + \frac{\hbar}{2} \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} (\sum_{i=1}^n K^{(i)}_{\alpha}) + \zeta^\mathfrak{h} \sum_{i=1}^n \zeta_i \text{ad}(\mu^{(i)})\right)
\]

(5.3)

3. The following holds.

\[
\lim_{\zeta_i \to +\infty} H_{\pm}^{(n)} = 1 \otimes H^{(n-1)} \quad \text{and} \quad \lim_{\zeta_n \to -\infty} H_{\pm}^{(n)} = H^{(n-1)} \otimes 1
\]

4. \(H_\pm(t\zeta; \zeta, \mu) = H_\pm(\zeta; t\zeta, \mu)\) for any \(t \in \mathbb{R}_+^n\).

5. Let \(\sigma \in \text{Aut}(Ug^{\otimes n})\) be the algebra automorphism given by \(x_1 \otimes \cdots \otimes x_n \to x_n \otimes \cdots \otimes x_1\). Then \(\sigma(H_{\pm}) = \Theta_{\otimes n}(H_{\pm})\), where \(\Theta \in \text{Aut}(\mathfrak{g})\) is any involution acting as \(-1\) on \(\mathfrak{h}\).

The proof of Theorem 5.4 is given in §5.6 – §5.11, which occupy the rest of this section.
5.5. The multicomponent fusion operator. Let

\[ J^{(n)}_{\pm} : \Lambda_n \times \Lambda \to \text{End}_{\mathbb{C}[t]}(U \mathfrak{g}^{\otimes n} \mathfrak{h},) \]

be the real analytic function given by

\[ J^{(n)}_{\pm}(\underline{\zeta}, \mu) = H_{\pm}(\zeta; \zeta, \mu) e^{\sum_i \zeta_i \text{ad}(\mu(i))} \zeta^H \sum_{i<j} \Omega^b_i \prod_{i<j}(\zeta_i - \zeta_j)^{\Omega^b_{ij}} \]

(5.4)

where \( H_{\pm} \) is given by Theorem 5.4.

**Definition.** The multicomponent fusion operator is the real analytic map

\[ J^{(n)}_{\pm} : \Lambda_n \times \Lambda \to (U \mathfrak{g}^{\otimes n} \mathfrak{h},) \]

given by

\[ J^{(n)}_{\pm}(\underline{\zeta}, \mu) = J^{(n)}_{\pm}(\underline{\zeta}, \mu) 1^{\otimes n} = H_{\pm}(\zeta; \zeta, \mu) \cdot \prod_{i<j}(\zeta_i - \zeta_j)^{\Omega^b_{ij}} \]

It follows from Theorem 5.4 that the fusion operator satisfies

\[ d_{\underline{\zeta}} J^{(n)}_{\pm} = \left( \frac{\hbar}{2} \sum_{1 \leq i < j \leq n} d\log(z_i - z_j) \Omega_{ij} + \sum_{i=1}^n dz_i \text{ad}(\mu(i)) \right) J^{(n)}_{\pm} \]

\[ d_{\hbar} J^{(n)}_{\pm} = \frac{\hbar}{2} \sum_{\alpha \in \Phi^+} \frac{d\alpha}{\alpha} \left( \Delta^{(n)}(\kappa_{\alpha}) \cdot J^{(n)}_{\pm} - J^{(n)}_{\pm} \cdot \left( \sum_{i=1}^n \kappa_{\alpha}^i(i) \right) \right) + \sum_{i=1}^n z_i d(\text{ad}(\mu(i))) J^{(n)}_{\pm} \]

5.6. Proof of Theorem 5.4. (1) We carry out the proofs for \( H_{\pm} \) only, those for \( H_{\mp} \) being identical, and therefore drop the subscript +. \( H \) is required to satisfy

\[ \frac{dH}{d\zeta} = [\sum_i \zeta_i \mu(i), H] + \hbar \frac{\Omega H - H \Omega^b}{\zeta} \]

(5.5)

Expanding \( H(\zeta) = \sum_{n \geq 0} H_n(\zeta) h^n \), yields the recursive inhomogeneous ODEs

\[ \frac{dH_n}{d\zeta} = [\sum_i \zeta_i \mu(i), H_n] + \frac{\Omega H_{n-1} - H_{n-1} \Omega^b}{\zeta} \]

(5.6)

where \( H_{-1} = 0 \) by convention, together with the requirement that \( H_n(\zeta) \to \delta_{n0} H^0 \) as \( \zeta \to \infty \) in \( \mathbb{H} \). We shall treat the cases \( n = 0, \ n = 1 \) and \( n \geq 2 \) separately.

Let \( Q = Z \Phi \subset (\mathfrak{h}^R)^* \) be the root lattice and, for any \( \gamma \in Q \), let \( U \mathfrak{g}_\gamma \subset U \mathfrak{g} \) be the weight space corresponding to \( \gamma \) for the adjoint action of \( \mathfrak{h} \) on \( U \mathfrak{g} \). Thus,

\[ U \mathfrak{g} = \bigoplus_{\gamma \in Q} U \mathfrak{g}_\gamma \]

and

\[ U \mathfrak{g}^{\otimes n} = \bigoplus_{\underline{\gamma} \in Q^n} U \mathfrak{g}_{\underline{\gamma}}^{\otimes n} \]

(5.7)

where, for \( \underline{\gamma} = (\gamma_1, \ldots, \gamma_n) \in Q^n \), \( U \mathfrak{g}_{\underline{\gamma}}^{\otimes n} = U \mathfrak{g}_{\gamma_1} \otimes \cdots \otimes U \mathfrak{g}_{\gamma_n} \). Notation: \( \gamma(\underline{\zeta}, \mu) = \sum_k \zeta_k \gamma_k(\mu) \).
5.7. \( n = 0 \). The equation (5.6) reduces to
\[
\frac{dH_0}{d\zeta} = \sum_i \zeta_i \text{ad}(\mu^{(i)}) H_0
\]
and is therefore equivalent to \( H_0(\zeta) = \exp(\zeta \sum \zeta_i \text{ad}(\mu^{(i)})) C \), where \( C \in U\mathfrak{g}^{\otimes n} \) is some constant element. Write \( C = \sum_{\omega \in \mathcal{Q}^n} C_\omega \) so that \( H_0 = \sum e^{\zeta_2(\zeta, \mu)} C_\omega \). Since \( \gamma(\zeta, \mu) = \sum_k \zeta_k \gamma_k(\mu) \in \mathbb{R} \), the function \( \exp(\zeta \gamma(\zeta, \mu)) \) has a limit as \( \zeta \to \infty \) in \( \mathbb{H} \) if, and only if \( \gamma(\zeta, \mu) = 0 \), so that \( H_0(\zeta) = \sum_{\gamma(\zeta, \mu) = 0} C_\omega \). Thus, \( H_0 \) is a constant function of \( \zeta \), and is therefore equal to its limit as \( \zeta \to \infty \) in \( \mathbb{H} \). This proves the existence and uniqueness of \( H_0 \), as well as the uniqueness of \( H_n \) for \( n \geq 1 \) since the homogeneous equation underlying (5.6) is (5.8).

5.8. \( n = 1 \). Equation (5.6) now yields
\[
\frac{dH_1}{d\zeta} = \sum_i \zeta_i \text{ad}(\mu^{(i)}) H_1 + \frac{\Omega H_0}{\zeta}
\]
where \( \Omega = \Omega - \Omega^8 \). Decomposing this equation along (5.7), and noting that \( \Omega = \sum_{i<j, \alpha \in \Phi} \Omega^\alpha_{ij}, \) where \( \Omega^\alpha_{ij} = x_\alpha^{(i)} x^-_{\alpha} \), yields
\[
\frac{dH^\alpha_{ij}}{d\zeta} = \gamma(\zeta, \mu) H^\alpha_{ij} + \sum_{\alpha \in \Phi, i<j} \delta_{\alpha, (\gamma(\zeta)) - \alpha(i)} \frac{\Omega^\alpha_{ij} H_0}{\zeta}
\]
where \( H^\alpha_{ij} \) is the component of \( H_1 \) along \( U\mathfrak{g}_2^{\otimes n} \), and \( \alpha^{(i)} = (0, \ldots, 0, \alpha, 0, \ldots, 0) \in Q^n \), with the \( \alpha \) in the \( i \)th slot. This equation, together with the requirement that \( H^\alpha_{ij} \to 0 \) as \( \zeta \to \infty \), is clearly solved by setting \( H^\alpha_{ij} = 0 \) if \( \gamma \) is not of the form \( \alpha(i) - \alpha(j) \), and by resorting to (2) of Proposition \( \mathcal{A} \) otherwise since in that case \( \gamma(\zeta, \mu) = \alpha(\mu)(\zeta_i - \zeta_j) \neq 0 \).

This yields a solution \( H_1 \) with values in \( (U\mathfrak{g}^{\otimes n})_{\alpha_1} \), which is a smooth function of \( (\zeta, \mu) \in i\mathcal{C}_n \times i\mathcal{C} \) since, again, \( \gamma(\zeta, \mu) = \alpha(\mu)(\zeta_i - \zeta_j) \neq 0 \) and moreover possesses an asymptotic expansion as \( \zeta \to \infty \) with trivial constant term. Explicitly
\[
H_1(\zeta) = \sum_{i<j, \alpha \in \Phi^+} \left( \int_{-\infty}^0 e^{-t\alpha(\mu)(\zeta_i - \zeta_j)} \frac{\Omega_{ij}^\alpha H_0}{\zeta + t} dt - \int_0^\infty e^{t\alpha(\mu)(\zeta_i - \zeta_j)} \frac{\Omega_{ij}^\alpha H_0}{\zeta + t} dt \right)
\]
which has the asymptotic expansion
\[
H_1 \sim -\sum_{\alpha \in \Phi, i<j} \frac{\Omega^\alpha_{ij}}{\alpha(\mu)(\zeta_i - \zeta_j)} \zeta^{-1} + O(\zeta^{-2})
\]

5.9. \( n \geq 2 \). We now assume inductively that we have constructed \( H_{n-1} \) with values in \( (U\mathfrak{g}^{\otimes n})_{\alpha_{n-1}} \), which is a continuous function of \( (\zeta, \mu) \in i\mathcal{C}_n \times i\mathcal{C} \), smooth away from a finite collection \( \{Q_i\}_{i \in \mathcal{I}_{n-1}} \) of quadrics\(^\text{(13)}\), and possesses an asymptotic expansion in \( \zeta \to \infty \) with leading term of the form \( C_{n-1} \zeta^{-(n-1)} \) away from \( \bigcup_{i \in \mathcal{I}_{n-1}} Q_i \) and \( \zeta^{-1} \) otherwise, and set about constructing \( H_n \) with the same properties.

\(^\text{(13)}\) such a collection is empty if \( n = 2 \), but will be seen to be a priori non–trivial thereafter.
Set $G_n = \Omega H^{n-1} - H^{n-1}\Omega^h \in (U g^{\otimes n})_{\alpha_n}$. Then decomposing equation (5.6) along (5.7) yields, for any $\gamma \in Q^n$,

$$\frac{H^n}{\zeta} = \gamma(\zeta, \mu)H^n + \frac{G^n}{\zeta}$$

Since $G_n \sim \zeta^{-(n-1)}$, part (3) of Proposition A shows the existence of a unique solution $H^n$ with values in $V = (U g^{\otimes n})_{\alpha_n}$. Explicitly

$$H^n(\zeta) = -i \sum_{\gamma \in Q^n} \int_0^\infty e^{-at} \sum_{\zeta k(\mu)} \frac{G^n(\zeta + it)}{\zeta + it} dt$$

This solution is continuous on $\mathcal{C}_n \times \mathcal{C}$, and smooth on the complement of the quadrics $\mathcal{Q}_i$, $i \in \mathcal{I}_{n-1}$ and

$$\mathcal{Q}_2 = \{(\zeta, \mu) \in \mathcal{C}_n \times \mathcal{C} | \gamma(\zeta, \mu) = 0\}$$

where $\gamma$ ranges over the (finitely many) elements of $Q^n$ such that $G^n_\gamma \neq 0$. Moreover, it has an asymptotic expansion in $\zeta$ starting at $\zeta^{-n}$ on the complement of these quadrics, and at $\zeta^{-1}$ elsewhere.

5.10. Our next goal is to show that $H = \sum_{n \geq 0} h^n H_n$ is smooth on $\mathcal{C}_n \times \mathcal{C}$ and satisfies the PDE (5.3). We shall do so for the truncation $\mathcal{H}$ of $H$ mod $h^p$ and then let $p$ tend to $\infty$. By construction, $\mathcal{H}$ is continuous on $\mathcal{C}_n \times \mathcal{C}$, and smooth on the complement $\mathcal{U}$ of finitely many (real) quadrics. Thus, if $\mathcal{D}$ is the flat connection

$$\mathcal{D} = d - \zeta d + \sum_{i=1}^n \zeta_i \text{ad}(\mu^{(i)}) - h \sum_{1 \leq i < j \leq n} d\log(\zeta_i - \zeta_j) \left(\ell(\Omega_{ij}) + r(\Omega_{ij}^h)\right)$$

$$- \frac{h}{2} \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} \left(\ell(\Delta^{(n)}(K_\alpha)) - r(\sum_{i=1}^n K_{ij}^{(i)})\right)$$

where $\ell, r$ denote left and right multiplication respectively, then $G = \mathcal{D} H$ is well–defined on $\mathbb{H} \times \mathcal{U}$. To show that $D_\zeta G = 0$, notice first that

$$D_\zeta G = D_\zeta D H = D D_\zeta H = 0$$

By uniqueness, it therefore suffices to show that, as a function of $\zeta$, $G$ is holomorphic on $\mathbb{H}$ and tends to $0$ as $\zeta \to \infty$ in $\mathbb{H}$. Where $D_\zeta = d_\zeta - \text{ad}(\sum_{i=1}^n \zeta_i \mu^{(i)}) - \ell(h\Omega) + r(h\Omega^h)$ and, by integrability $[D_\zeta, D] = 0$.

Since $H = 1 + H^1 \zeta^{-1} + O(\zeta^{-2})$, it is clear that $G$ has a well–defined limit as $\zeta \to \infty$, given by

$$-\sum_{i=1}^n \zeta_i \text{ad}(\mu^{(i)}) H + \sum_{1 \leq i < j \leq n} d\log(\zeta_i - \zeta_j) (\Omega_{ij} - \Omega_{ij}^h) - \frac{h}{2} \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} \left(\Delta^{(n)}(K_\alpha) - \sum_{i=1}^n K_{ij}^{(i)}\right)$$

Since

$$\Omega_{ij} - \Omega_{ij}^h = \sum_{\alpha} \Omega_{ij}^\alpha$$

and

$$\Delta^{(n)}(K_\alpha) - \sum_{i<j} K_{ij}^{(i)} = 2 \sum_{i<j} (\Omega_{ij}^\alpha + \Omega_{ij}^{-\alpha})$$

the last two summands add up to

$$-h \sum_{i<j} \left(d\log(\zeta_i - \zeta_j) + \frac{d\alpha}{\alpha}\right) \Omega_{ij}^\alpha$$
On the other hand, by (5.10),
\[
[d \sum_{k=1}^{n} \zeta_k \text{ad}(\mu(k)), H^1] = -\hbar \sum_{\alpha \in \Phi} \sum_{i < j}^{n} \left[ d(\zeta_k \text{ad}(\mu(k))), \frac{\Omega^\alpha_{ij}}{\alpha(\mu)(\zeta_i - \zeta_j)} \right] \\
= -\hbar \sum_{\alpha \in \Phi} \left( \frac{d(\zeta_i - \zeta_j)}{\zeta_i - \zeta_j} + \frac{d\alpha}{\alpha} \right) \Omega^\alpha_{ij}
\]
and we are done.

Thus, \( \mathcal{H} \) is a horizontal section of \( D \) on each of the connected components of \( \mathcal{U} \), and therefore extends to a smooth horizontal section on the closure of each connected component in \( iC_n \times iC \). Since however \( \mathcal{H} \) is continuous on \( iC_n \times iC \), this extension coincides with \( \mathcal{H} \) thus showing that \( \mathcal{H} \) is a smooth, horizontal section of \( D \) on the whole of \( iC_n \times iC \).

5.11. Let us prove that \( \lim_{\zeta_1 \rightarrow +\infty} H^{(n)} = 1 \otimes H^{(n-1)} \). By (4) of Proposition A, \( H^{(n)} \) possesses an asymptotic expansion \( H^{(n)} = G_0 + G_1 \zeta_1^{-1} + \cdots \). Plugging this into (5.5), and taking the coefficients of \( \zeta_1 \) and \( \zeta_0 \) respectively gives \( [\mu^{(1)}, G_0] = 0 \) and
\[
\frac{dG_0}{d\zeta} = \left[ M_{\geq 2}, G_0 \right] + \frac{KG_0 - G_0[K]}{\zeta}
\]
Projecting onto \( Z(\mu^{(1)}) \) then yields
\[
\frac{dG_0}{d\zeta} = \left[ M_{\geq 2}, G_0 \right] + \frac{K_0 G_0 - G_0[K]}{\zeta}
\]
where \( K_0 = P_1 K \). Write
\[
K = \sum_{j \geq 2} h \Omega_{1j} + h \Omega_{\geq 2} = \text{id} \otimes \Delta^{(n-1)}(h\Omega) + h\Omega_{\geq 2}
\]
Then \( K_0 = \text{id} \otimes \Delta^{(n-1)}(h\Omega)^0 \) + \( h\Omega_{\geq 2} \). Note that since the coefficients of (5.11) commute with the action of \( h \otimes h \) on \( U\mathfrak{g} \otimes U\mathfrak{g}^{(n-1)} \), so does \( G_0 \) by uniqueness. It follows that \( G_0 \) commutes with \( \text{id} \otimes \Delta^{(n-1)}(h\Omega)^0 \) so the above may be rewritten as
\[
\frac{dG_0}{d\zeta} = \left[ M_{\geq 2}, G_0 \right] + \frac{h\Omega_{\geq 2} G_0 - G_0[h\Omega_{\geq 2}]}{\zeta}
\]
which is the differential equation for the sought for gauge transformation.

The proof that \( \lim_{\zeta_n \rightarrow -\infty} H^{(n)} = H^{(n-1)} \otimes 1 \) is identical.

6. The differential twist

In this section, we construct a differential twist \( J_{\pm}(\mu) \) for \( \mathfrak{g} \) as the regularised limit of the fusion operator \( J_{\pm}(z, \mu) \) when \( z \rightarrow 0 \), and prove that it kills the KZ associator.

6.1. Fundamental solution of the KZ equations near \( z = 0 \).

Proposition.
For any \( \mu \in h \), there is a unique holomorphic function \( H_0 : \mathbb{C} \to A \) such that

\[
H_0(0, \mu) \equiv 1 \text{ and, for any determination of } \log(z), \text{ the } E \text{-valued function }
\]

\[
\Upsilon_0(z, \mu) = e^{z \text{ad}(\mu^{(1)})} \cdot H_0(z, \mu) \cdot \frac{z^{\hbar \Omega}}{z}
\]

satisfies

\[
\left( dz - \left( \frac{\Omega}{z} + \text{ad}(\mu^{(1)}) \right) dz \right) \Upsilon_0 = \Upsilon_0 dz
\]

(2) \( H_0 \) and \( \Upsilon_0 \) are holomorphic functions of \( \mu \), and \( \Upsilon_0 \) satisfies

\[
\left( d_h - \frac{\hbar}{2} \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} \Delta(K_\alpha) - z \text{ad}(d\mu^{(1)}) \right) \Upsilon_0 = \Upsilon_0 \left( d_h - \frac{\hbar}{2} \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} \Delta(K_\alpha) \right)
\]

(3) \( H_0, \Upsilon_0 \) commute with the diagonal action of \( \mathfrak{g} \) if \( \mu = 0 \).

**Proof.** (1) \( H = H_0 \) is required to satisfy

\[
\frac{dH}{dz} = \frac{\hbar}{z} \left( e^{-z \text{ad}(\mu^{(1)})}(\Omega)H - \hbar \Omega \right)
\]

\[
= \hbar \left( \frac{\Omega, H}{z} + \frac{e^{-z \text{ad}(\mu^{(1)})} - 1}{z} \right)(\Omega)H
\]

Writing \( H = \sum_{n \geq 0} h^n H_n \) yields the recursive system of ODEs

\[
\frac{dH_n}{dz} = \left( \frac{\Omega, H_{n-1}}{z} + \frac{e^{-z \text{ad}(\mu^{(1)})} - 1}{z} \right)(\Omega)H_{n-1}
\]

where \( H_{-1} = 0 \), together with the initial value condition \( H_n(0) = \delta_{n0} \), which clearly possesses at most one solution. For \( n = 0 \), the solution is given by \( H_0 \equiv 1 \) and, for \( n \geq 1 \), by

\[
H_n(z) = \int_0^z \frac{\Omega, H_{n-1}(t)}{t} + \frac{e^{-t \text{ad}(\mu^{(1)})} - 1}{t} \Delta(K_\alpha)H_{n-1}(t) dt
\]

provided the integral converges at \( t = 0 \). For \( n = 1 \), this is the case since \( H_0 = 1 \) commutes with \( \Omega \), so the integrand is an \( O(1) \) and, for \( n \geq 2 \), this follows since \( H_{n-1}(t) = O(t) \).

(2) The recursive construction of \( H \) shows that it, and therefore \( \Upsilon_0 \), are holomorphic functions of \( \mu \in h \). The fact that \( \Upsilon_0 \) satisfies the stated PDE follows by integrability.

(3) For \( \mu = 0 \), \( H_0 = 1 \) and \( \Upsilon_0 = z^{\hbar \Omega} \) clearly commute with \( \mathfrak{g} \).

6.2. Differential twist. Let \( J_{\pm}^{(z, \mu)} \) be the fusion operator defined in §5.5.

**Definition.** The differential twist of \( \mathfrak{g} \) is the smooth map \( J_{\pm} : i\mathbb{C} \to U\mathfrak{g}^{S^2}[h] \) given by

\[
J_{\pm}(\mu) = \Upsilon_0(z, \mu)^{-1} \cdot J_{\pm}^{(2)}(z, \mu)
\]

It follows from 5.5 that \( J_{\pm}(\mu) \) is independent of \( z \), and satisfies

\[
d_\mathfrak{g}J_{\pm} = \frac{\hbar}{2} \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} \left( \Delta(K_\alpha) \cdot J_{\pm} - J_{\pm} \cdot (K_\alpha^{(1)} + K_\alpha^{(2)}) \right)
\]

The following is the main result of this section. Its proof will be given in 6.8 after some preparatory work.
Theorem. The following holds
\[ \Phi_{\text{KZ}} \cdot \Delta \otimes \text{id}(J_\pm) \cdot J_\pm \otimes 1 = \text{id} \otimes \Delta(J_\pm) \cdot 1 \otimes J_\pm \]

6.3. The goal of \$6.3–6.5\$ is to show that the constants relating the fundamental solutions of the dynamical KZ equations corresponding to Drinfeld’s asymptotic zones are the same as those relating their non-dynamical counterparts. We begin by constructing the relevant fundamental solutions.

Fix \( n \geq 2 \), and let \( B_n \) be the set of complete bracketings on the monomial \( x_1 \cdots x_n \). By convention, such bracketings always contain the parentheses \((x_1 \cdots x_n)\), and are easily seen to consist of \( n - 1 \) compatible pairs of parentheses. Let \( \mathbb{C}^n_0 = \{ z \in \mathbb{C}^n \mid \sum z_i = 0 \} \). For any \( b \in B_n \), the coordinates \( z_P = z_i - z_j \), where \( P = (x_1 \cdots x_j) \cdots \in b \), are a basis of \((\mathbb{C}^n_0)^*\). Let \( U_b \cong \mathbb{C}^n_0 \) be the affine space with coordinates \( \{ u_P \}_{P \in b} \). Consider the regular map \( \rho_b : U_b \to \mathbb{C}^n_0 \) given by
\[ z_P = \prod_{b_P \supseteq P} u_P \]
\( \rho_b \) is a birational map, with inverse given by \( u_P = z_P/z_Q \), where \( Q \in b \) is the smallest pair of parentheses strictly containing \( P \), and \( z_Q = 1 \) if \( P = (x_1 \cdots x_n) \). It induces an isomorphism
\[ U_b \setminus \bigcup_{P \in b} \{ u_P = 0 \} \sim \mathbb{C}^n_0 \setminus \bigcup_{P \in b} \{ z_P = 0 \} \]
The pull-back \( \rho_b^* \nabla_{\text{DKZ}} \) is of the form
\[ \rho_b^* \nabla_{\text{DKZ}} = d - h \sum_{P \in b} \frac{du_P}{u_P} \Omega_P - \mathcal{R}_b \]
where, for \( P = (x_1 \cdots x_j) \cdots \), \( \Omega_P = \sum_{i < k < j} \Omega_{ij} \), and \( \mathcal{R}_b \) is an \( \text{End}_{\mathcal{C}}[\mathfrak{g} \hat{\otimes} \mathfrak{g}^{\otimes n}][\hbar] \) valued one-form which is regular in the neighborhood of \( 0 \in U_b \). It follows from this, or by direct inspection, that \( [\Omega_P, \Omega_Q] = 0 \) for any \( P, Q \in b \).

Fix now \( b \in B_n \) and, for any \( P \in b \), set \( \Omega_P = \sum Q \Omega_Q \), where \( Q \) ranges over the maximal elements of \( b \) properly contained in \( P \).

Proposition.

1. For any simply-connected neighborhood \( V_b \) of \( 0 \in U_b \), there exists a unique holomorphic function \( H_b : V_b \to \text{End}_{\mathcal{C} \mathfrak{g}}[\mathfrak{g} \hat{\otimes} \mathfrak{g}^{\otimes n}][\hbar] \) such that \( H_b(0) = 1 \), and the function
\[ \Psi_b = H_b \prod_{P \in b} u_P^{\Omega_P} = H_b \prod_{P \in b} z_P^{\Omega_P - \Omega_P} \]
is a fundamental solution of \( \rho_b^* \nabla_{\text{KZ}} \).

2. \( H_b \) is a holomorphic function of \( \mu \in \mathfrak{h} \), and satisfies
\[ \left( d_b - \frac{h}{2} \sum \frac{d\alpha}{\alpha} \Delta^{(n)}(\mathcal{K}_\alpha) - \sum z_i \text{ad}(d_b \mu^{(i)}) \right) H = H \left( \frac{h}{2} \sum \frac{d\alpha}{\alpha} \Delta^{(n)}(\mathcal{K}_\alpha) \right) \]

Proof. (1) is standard, and proved as in Proposition 6.1.

(2) Write the equation as \( DH = 0 \), where \( D \) is the flat connection
\[ D = d_b - \frac{h}{2} \sum \frac{d\alpha}{\alpha} \Delta^{(n)}(\mathcal{K}_\alpha) - \sum z_i \text{ad}(d\mu^{(i)}) \]
Let \( G = DH \). Then \( \nabla_{k^2} G = D \nabla_{k^2} H = 0 \). Moreover, \( G \) is holomorphic near \( u = 0 \) and \( G(0) = 0 \), whence by uniqueness \( G \equiv 0 \) as claimed. \( \square \)

6.4. Let now \( b, b' \in B_n \) be two bracketing on \( n \) letters.

**Lemma.** The following holds

\[
\Upsilon_b = \Upsilon_{b'} \Phi_{KZ}^\ell
\]

where \( X^\ell \) is the operator of left multiplication by \( X \).

**Proof.** Set \( \Phi = \Upsilon_{b'}^{-1} \Upsilon_b \). \( \Phi \) is a holomorphic function of \( \mu \in \mathfrak{h} \) since both \( \Upsilon_b \) and \( \Upsilon_{b'} \) are, and satisfies \( \Phi(0) = \Phi_{KZ}^\ell b \) and

\[
d\Phi = \frac{h}{2} \sum_\alpha \frac{d\alpha}{\alpha} [\Delta(K_\alpha)^\ell, \Phi]
\]

Fix \( \mu \in \mathfrak{h}_{\text{reg}} \), and identify the line \( \mathbb{C} \mu \subset \mathfrak{h}_{\text{reg}} \) with \( \mathbb{C} \ni t \mapsto t\mu \). The restriction of \( \Phi \) to \( \mathbb{C} \mu \) satisfies

\[
\frac{d\Phi}{dt} = \frac{h}{2} [\Delta(K)^\ell, \Phi]
\]

where \( K = \sum_\alpha K_\alpha \). Thus \( \Phi(\mu) = \text{Ad}(\text{Exp}(\frac{h}{2} \Delta(\mathbb{C})(K)^\ell)) \Phi(0) = \Phi_{KZ}^\ell b \) since \( \Phi_{KZ}^\ell b \) is invariant. as claimed. Since \( \mathfrak{h}_{\text{reg}} \subset \mathfrak{h} \) is dense, and \( \Phi \) continuous on \( \mathfrak{h} \) it follows that \( \Phi \equiv \Phi_{KZ}^\ell b \) as claimed. \( \square \)

6.5. For any solution of \( \text{DKZ}_n \) \( \Psi \), and bracketing \( b \in B_n \), set \( r\lim_{b} \Psi = \Upsilon^{-1}_b \cdot \Psi \).

**Corollary.** Let \( \Psi \) be an \( \mathcal{E} \)-valued solution of \( \text{DKZ}_n \), and \( b, b' \) two bracketings on \( n \) letters. Then

\[
r\lim_{b'} \Psi = \Phi_{b\ell}^{b'} r\lim_{b} \Psi
\]

**Proof.** Let \( C_b = r\lim_{b} \Psi \). Then, \( \Psi = \Upsilon_b C_b = \Upsilon_{b'} \Phi_{b\ell}^{b'} C_b \) whence \( \lim_{b'} \Psi = \Phi_{b\ell}^{b'} C_b \) as claimed.

6.6. The two normalised limits.

**Proposition.**

1. The function \( \Upsilon_{\infty(\cdot)} = J(3) \cdot 1 \otimes (J(2))^{-1} \) is regular at \( z_2 = z_3 \), and

\[
\Upsilon_{\infty(\cdot)}(z_1, z_2, z_2; \mu) = \text{id} \otimes \Delta \left( J(2)(z_1, z_2; \mu) \right)
\]

2. The function \( \Upsilon_{(\cdot)\infty} = J(3) \cdot (J(2))^{-1} \otimes 1 \) is regular at \( z_1 = z_2 \), and

\[
\Upsilon_{(\cdot)\infty}(z_1, z_1, z_3; \mu) = \Delta \otimes \text{id} \left( J(2)(z_1, z_3; \mu) \right)
\]
Proof. (1) As a function of $z_1$, $\Upsilon = \Upsilon_{\infty(-)}$ satisfies
\[
\frac{d\Upsilon}{dz_1} = \left( \text{ad} \mu(1) + \hbar \frac{\Omega_{12}}{z_1 - z_2} + \hbar \frac{\Omega_{23}}{z_1 - z_3} \right) \Upsilon
\]
Moreover, by (5.4)
\[
\Upsilon = \mathcal{H}^3(-) \cdot (z_2 - z_1)^{\mathcal{H}^3_1} (z_3 - z_1)^{\mathcal{H}^3_3} \cdot 1 \otimes \mathcal{H}^2(-)^{-1}
\]
where the second equality follows from the fact that $\mathcal{H}^2$ is of weight zero, and the third defines the function $\Xi$. As a function of $z_1$, $\Xi$ is holomorphic and, by (5.4) of Theorem 5.4, tends to 1 as $z_1 \to \infty$.

Corollary. The following holds
\[
r \lim_{\mathcal{J}(\cdot)} J^{(3)} = \Delta \otimes \text{id}(J) \cdot J \otimes 1 \quad \text{and} \quad r \lim_{\mathcal{J}(\cdot)} J^{(3)} = \text{id} \otimes \Delta(J) \cdot 1 \otimes J
\]

Proof. By definition,
\[
r \lim_{\mathcal{J}(\cdot)} J^{(3)} = \lim_{z_1 - z_2, z_3 \to 0} \frac{(z_1 - z_3)^{-h \Delta \otimes \text{id}(\Omega)} (z_1 - z_2)^{-\mathcal{H}^3_1} J^{(3)}}{z_1 - z_3}
\]
Write $J^{(3)} = \Upsilon_{(-)\infty} \cdot \mathcal{J}^{(2)} \otimes 1$, where $\Upsilon_{(-)\infty}$ is defined in Proposition 6.6. Since
\[
\Upsilon_{(-)\infty} (z_1, z_2, z_3) = \Delta \otimes \text{id} \left( \mathcal{J}^{(2)}(z_1, z_2) \right) + (z_1 - z_2) \mathcal{R}
\]
where $\mathcal{R}$ is regular at $z_1 = z_2$, and $\Omega$ commutes with $\Delta(Ug)$, we have
\[
(z_1 - z_2)^{-\mathcal{H}^3_1} \Upsilon_{(-)\infty}(z_1, z_2, z_3) = \Delta \otimes \text{id} \left( \mathcal{J}^{(2)}(z_1, z_2) \right) (z_1 - z_2)^{-\mathcal{H}^3_1} + (z_1 - z_2)^{-\mathcal{H}^3_1} \mathcal{R}
\]
Since the second summand tends to zero as $z_1 - z_2 \to 0$, it follows that
\[
r \lim_{\mathcal{J}(\cdot)} J^{(3)} = \lim_{z_1 - z_2, z_3 \to 0} \Delta \otimes \text{id} \left( (z_1 - z_3)^{-h \Delta} \mathcal{J}^{(2)}(z_1, z_3) \right) \cdot (z_1 - z_2)^{-\mathcal{H}^3_1} J^{(2)}(z_2, z_3) \otimes 1
\]
The second one follows in a similar way. \hfill \Box

6.8. Proof of Theorem 6.2. By Corollary 6.5, $r \lim_{\mathcal{J}(\cdot)} J^{(3)} = \Phi_{KZ} \cdot r \lim_{\mathcal{J}(\cdot)} J^{(3)}$. The result now follows from Corollary 6.7.
7. The centraliser property

In this section, we prove that the differential twist for $\mathfrak{g}$ obtained from the fusion operator possesses the centraliser property. This follows from a detailed analysis of the asymptotics of solutions of the joint Casimir–KZ equations in $n = 2$ points, the regime where $z = z_1 - z_2 \to 0$, and a fixed root coordinate $\alpha_i$ tends to infinity.

7.1. Consider the joint $\text{KZ}$–Casimir connection when $n = 2$. Since we will only consider solutions with values in $(U\mathfrak{g}^{\otimes 2}[h_0^\circ])^b$, the connection reads

$$\nabla = d - h\Omega \frac{dz}{z} - \frac{h}{2} \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} \Delta(K_\alpha) - d(z \text{ ad } \mu^{(1)})$$

(7.1)

where $z = z_1 - z_2 \in \mathbb{C}^\times$. Fix $i \in I$. We construct below a horizontal section of $\nabla$ with prescribed asymptotics as $\alpha_i \to \infty$, $\text{Im } \alpha_i \gtrless 0$.

Retain the notation of $4.4$ and $4.5$, and trivialise the fibration $\mathbb{C} \lambda^\vee_i \to \mathfrak{h} \to \mathfrak{h}$ by $\mathbb{C} \times \mathfrak{h} \ni (w, \bar{\mu}) \to w\lambda^\vee_i + i\bar{\mu}$, where $i : \mathfrak{h} \to \mathfrak{h}$ is given by (4.2). For fixed $\bar{\mu} \in \mathfrak{h}$ and $z \in \mathbb{C}$, the restriction of the connection $\nabla$ to $\pi^{-1}(\bar{\mu}) \times \{ z \}$ is equal to

$$\nabla^i = d_w - \left( \frac{h}{2} \sum_{\alpha \in \Phi_+ \setminus \Phi} \frac{\Delta(K_\alpha)}{w - w_\alpha} + z \text{ ad } \lambda^\vee_i \right) dw$$

(7.2)

where $w_\alpha = -\alpha(\bar{\mu})/\alpha(\lambda^\vee_i)$. Let $R_{\bar{\mu}}$ be given by (4.4), and set

$$\Lambda = \frac{\lambda^\vee_i \otimes \lambda^\vee_i}{\| \lambda^\vee_i \|^2}$$

(7.3)

7.2. Fuchs–Stokes solution. Let $D^\pm_\infty \subset \mathbb{C} \times \mathfrak{h}$ be the domain given by (4.9).

Proposition.

1. For any $\bar{\mu} \in \mathfrak{h}$ and $z \in \mathbb{R}^\times$, there is a unique holomorphic function

$$H^\pm_\infty : \{ w \in \mathbb{C} \mid \text{Im } w \gtrless 0, \ |w| > R_{\bar{\mu}} \} \to A$$

such that $H^\pm_\infty(w, \bar{\mu}, z) \to 1$ as $\alpha_i \to \infty$ with $0 < |\arg w| < \pi$ and, for any determination of $\log(\alpha_i)$, the $\mathcal{E}$–valued function

$$\Psi^\pm_\infty = H^\pm_\infty(\alpha_i, \bar{\mu}, z) \cdot e^{z\alpha_i \text{ ad}(\lambda^\vee_i)} \cdot \alpha_i^\pm((\kappa - \bar{\kappa})^{(1)} + (\kappa - \bar{\kappa})^{(2)})$$

satisfies $\nabla^i \Psi^\pm_\infty = \Psi^\pm_\infty d_w$.

2. The function $H^\pm_\infty(\alpha_i, \bar{\mu}, z)$ is smooth on $D^\pm_\infty \times \mathbb{R}^\times$, and $\Psi^\pm_\infty$ satisfies

$$\left( d - h\Omega \frac{dz}{z} - \frac{h}{2} \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} \Delta(K_\alpha) - d(z \text{ ad } \mu^{(1)}) \right) \Psi^\pm_\infty$$

$$= \Psi^\pm_\infty \left( d - h(\Omega + \Lambda) \frac{dz}{z} - \frac{h}{2} \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} \Delta(K_\alpha) - d(z \text{ ad } i(\bar{\mu})^{(1)}) \right)$$

3. $\Psi^\pm_\infty(\alpha_i, 0, z)$ is invariant under the diagonal action of $\mathfrak{g}$.

4. The function

$$\Psi^\pm_\infty(\alpha_i, \bar{\mu}, z) \cdot e^{-z\alpha_i \text{ ad}(\lambda^\vee_i)} = H^\pm_\infty(\alpha_i, \bar{\mu}, z) \cdot \alpha_i^\pm((\kappa - \bar{\kappa})^{(1)} + (\kappa - \bar{\kappa})^{(2)})$$
admits a limit for $z \to \infty$, which is equal to $\Upsilon_\infty^2$, where the latter is the solution given by Proposition 4.5.

7.3. Proof. (1) For fixed $z \in \mathbb{R}^\times$, and $\underline{\pi} \in \underline{\mathfrak{h}}$, $H = H_\pm^\infty$ is required to satisfy the ODE

$$\frac{dH}{dw} = z[\lambda^{(1)}_\gamma, H] + \frac{\hbar}{2} \left( \sum_{\alpha \in \Phi_+ \setminus \overline{\Phi}} \Delta(K_\alpha)H \frac{w - w_\alpha}{H((K - \overline{K})^{(1)} + (K - \overline{K})^{(2)})} \right)$$

$$= z[\lambda^{(1)}_\gamma, H] + \frac{\hbar}{2w} \left( \Delta(K - \overline{K})H - H((K - \overline{K})^{(1)} + (K - \overline{K})^{(2)}) + \sum_{\alpha \in \Phi_+ \setminus \overline{\Phi}} \frac{w_\alpha \Delta(K_\alpha)}{w - w_\alpha} H \right)$$

where we used (4.7). Writing $H = \sum_{n \geq 0} h^n H_n$, this is equivalent to the recursive system of ODEs

$$\frac{dH_n}{dw} = z[\lambda^{(1)}_\gamma, H_n]$$

$$+ \frac{\hbar}{2w} \left( \Delta(K - \overline{K})H_{n-1} - H_{n-1}((K - \overline{K})^{(1)} + (K - \overline{K})^{(2)}) + \sum_{\alpha \in \Phi_+ \setminus \overline{\Phi}} \frac{w_\alpha \Delta(K_\alpha)}{w - w_\alpha} H_{n-1} \right)$$

where $H_{-1} = 0$, together with the condition that $H_n \to \delta_{n0}$ as $w \to \infty$ in $\mathbb{H}_+$ with $0 < < |\arg(w)| < \pi$.

Let $Q \subset \mathfrak{b}^\times$ be the root lattice, and $U_{\mathfrak{g}}^\otimes 2 = \bigoplus_{\gamma \in Q} U_{\mathfrak{g}}^\otimes 2$ the weight decomposition with respect to the adjoint action of $\mathfrak{h}$ acting on the first tensor copy. In terms of the components $H_\gamma$ of $H_n$, $\gamma \in Q$, the above equation reads

$$\frac{dH_\gamma}{dw} = z\gamma(\lambda^{(1)}_\gamma)H_\gamma$$

$$+ \frac{\hbar}{2w} \left[ \Delta(K - \overline{K})H_{n-1} - H_{n-1}((K - \overline{K})^{(1)} + (K - \overline{K})^{(2)}) + \sum_{\alpha \in \Phi_+ \setminus \overline{\Phi}} \frac{w_\alpha \Delta(K_\alpha)}{w - w_\alpha} H_{n-1} \right]^{\gamma}$$

(7.5)

We shall treat the cases $n = 0$, $n = 1$ and $n \geq 2$ separately.

$n = 0$. In this case, $H_0 \equiv 1$ is a solution of (7.5), which is unique by Proposition A.

$n = 1$. Given that $H_0 = 1$, the equation reads

$$\frac{dH_\gamma}{dw} = z\gamma(\lambda^{(1)}_\gamma)H_\gamma + \frac{\hbar}{2w} \left[ \Delta(K - \overline{K}) - ((K - \overline{K})^{(1)} + (K - \overline{K})^{(2)}) + \sum_{\alpha \in \Phi_+ \setminus \overline{\Phi}} \frac{w_\alpha \Delta(K_\alpha)}{w - w_\alpha} \right]^{\gamma}$$

By Proposition A, this has a unique solution with the required limiting behaviour unless $\left[ \Delta(K - \overline{K}) - ((K - \overline{K})^{(1)} + (K - \overline{K})^{(2)}) \right]^{\gamma} \neq 0$ and $z\gamma(\lambda^{(1)}_\gamma) = 0$. Since $z \neq 0$, this is ruled out by the fact that

$$\Delta(K - \overline{K}) = (K - \overline{K})^{(1)} + (K - \overline{K})^{(2)} + \sum_{\alpha \in \Phi \setminus \overline{\Phi}} x_\alpha \otimes x_{-\alpha}$$

and that $\alpha(\lambda^{(1)}_\gamma) \neq 0$ for any $\alpha \in \Phi \setminus \overline{\Phi}$. 


\( n \geq 2 \). The existence and uniqueness of \( H_n^\gamma \) follows from Proposition A since, by induction, the inhomogeneous term of (7.5) is an \( O(w^{-2}) \).

(2) It follows by Proposition A that \( H \) is a smooth function of \( z \in \mathbb{R}^k \) and \( \mu \in \mathfrak{g}^* \). The fact that \( \Psi_\infty^w \) satisfies the claimed PDE follows by integrability.

(3) When \( \overline{\mu} = 0 \), \( \omega = -\alpha(i(\mu))/\alpha(\lambda_i^\gamma) = 0 \) for any \( \alpha \in \Phi \setminus \overline{\Phi} \), and the connection (7.2) is equal to

\[
d_w - \left( \frac{h}{2} \frac{\Delta(K - \overline{K})}{w} + z \operatorname{ad} \lambda_i^{\gamma(1)} \right) dw
\]

and it follows from (4.8) that \( K - \overline{K} \) is invariant under \( \mathfrak{f} \).

(4) By Proposition A, \( H \) possesses an asymptotic expansion with respect to \( z = \infty \), locally uniformly in \( w \). Plugging \( H(w, \overline{\mu}, z) = H^0(w, \overline{\mu}) + H^1(w, \overline{\mu})z^{-1} + O(z^{-2}) \) into (7.4), and taking the coefficients of \( z \) and \( z^0 \) yields \( [\lambda_i^{\gamma(1)}, H^0] = 0 \) and

\[
dH^0/dw = [\lambda_i^{\gamma(1)}, H^1] + \frac{h}{2} \left( \sum_{\alpha \in \Phi \setminus \overline{\Phi}} \frac{\Delta(K_\alpha)H^0}{w - \omega} - \frac{H^0((K - \overline{K})^{(1)} + (\overline{K} - K)^{(2)})}{w} \right)
\]

which is precisely the differential equation (4.6) satisfied by the holomorphic part \( H_\infty^0(w, \overline{\mu})^\circ \) of \( Y_\infty(w, \overline{\mu})^\circ \). The fact that \( H^0 = H_\infty^0 \) now follows by Remark 4.5. \( \square \)

**Remark.** Note that, by (4.2) the gauge transform of the connection \( \nabla \) by \( Y_\infty^\circ \) can be written as the sum of commuting terms

\[
d - \left( \frac{h^\gamma dz}{z} - \frac{h}{2} \sum_{\alpha \in \Phi \setminus \overline{\Phi}} \frac{d\alpha}{\alpha} \Delta(K_\alpha) - d(z \operatorname{ad} \overline{\mu}(1)) \right) - \left( \frac{h\lambda_i dz}{z} - \operatorname{ad} \lambda_i^{\gamma(1)}(d(z^* \alpha_i)) \right)
\]

where the first two summands are the connection \( \nabla \) for \( \overline{\Phi} \), and \( j : \overline{h} \rightarrow \mathfrak{g} \) is the canonical embedding corresponding to the inclusion \( \overline{\Phi} \subset \Phi \).

7.4. **Recurrence.** Fix \( i \in I \), and let \( Y_\infty, Y_\infty^+ \) be the solutions of the Casimir equations given by Propositions 4.5 and 7.2 respectively. The following result relates the fusion operators of \( \mathfrak{g} \) and \( \overline{\mathfrak{g}} \).

**Theorem.** The following holds for any \( z \in \mathbb{R}_{\geq 0} \) and \( \mu \in iC \)

\[
\mathcal{J}_\mathfrak{g}^+(z, \mu) = Y_\infty^+(\alpha_i, \overline{\mu}, z) \cdot J_\mathfrak{g}^+(z, \overline{\mu}) \cdot e^{-z\alpha_i(\overline{\mu}) \operatorname{ad} \lambda_i^{\gamma(1)}} \cdot (\pm z)^{h\lambda_i} \cdot (Y_\infty(\alpha_i, \overline{\mu})^\circ)^{-1}
\]

**Proof.** By construction, \( J_\mathfrak{g}^+(z, \mu) \) is the unique solution of

\[
\left( d_z - \left( \frac{h}{z} + \operatorname{ad} \mu^{(1)} \right) dz \right) J_\mathfrak{g}^+(z, \mu) = J_\mathfrak{g}^+(z, \mu) dz
\]

which is of the form \( J_\mathfrak{g}^+(z, \mu) = H_\mathfrak{g}^+(z, \mu) \cdot e^{z \operatorname{ad} \mu^{(1)}} \cdot (\pm z)^{h\Omega h} \), where

\[
H_\mathfrak{g}^+: \{ z | \operatorname{Re} z \geq 0 \} \times iC \rightarrow A
\]
is holomorphic in $z$, and such that $H^\pm_{\overline{g}}(z, \mu) \to 1$ as $\mathbb{R} \ni z \to \pm \infty$.

On the other hand, applying $d_z - \left( \hbar \Omega/z + \text{ad } \mu^{(1)} \right) dz$ to the right–hand side of the stated identity yields, by Proposition 7.2, and Remark 7.3

$$
\begin{multline*}
\Upsilon^+_\infty(\alpha_i, \overline{\mu}, z) \cdot \left( d_z - \left( \frac{\Gamma}{z} + \text{ad } \mu^{(1)} \right) dz \right) \cdot J^\pm_{\overline{g}}(z, \overline{\mu}) \\
\cdot e^{-z \alpha_i(\overline{\mu}) \text{ad } \Lambda^{(1)}} \cdot (\Upsilon_\infty(\alpha_i, \overline{\mu}) \otimes 2)^{-1}
\end{multline*}
$$

$$
\begin{multline*}
= \Upsilon^+_\infty(\alpha_i, \overline{\mu}, z) \cdot J^\pm_{\overline{\mu}}(z, \overline{\mu}) \cdot \left( d_z - \left( \frac{\Lambda}{z} - \alpha_i(\overline{\mu}) \text{ad } \Lambda^{(1)} \right) dz \right) \cdot e^{-z \alpha_i(\overline{\mu}) \text{ad } \Lambda^{(1)}} \cdot (\Upsilon_\infty(\alpha_i, \overline{\mu}) \otimes 2)^{-1}
\end{multline*}
$$

where the first equality follows from the fact that $\Lambda$ and $\Lambda^{(1)}$ commute with $\overline{g}$, and the second from the fact that $\Upsilon_\infty(\alpha_i, \overline{\mu})$ is independent of $z$.

Moreover, if $z \in \mathbb{R}_{\geq 0}$, Proposition 7.2 (4) implies that

$$
\begin{multline*}
\Upsilon^+_\infty(\alpha_i, \overline{\mu}, z) \cdot J^\pm_{\overline{\mu}}(z, \overline{\mu}) \cdot e^{-z \alpha_i(\overline{\mu}) \text{ad } \Lambda^{(1)}} \cdot (\Upsilon_\infty(\alpha_i, \overline{\mu}) \otimes 2)^{-1}
\end{multline*}
$$

$$
\begin{multline*}
= (\Upsilon_\infty(\alpha_i, \overline{\mu}) \otimes 2 + O(z^{-1})) \cdot e^{z \alpha_i(\overline{\mu}) \text{ad } \Lambda^{(1)}} \cdot H^\pm_{\overline{\mu}}(z, \overline{\mu}) \cdot e^{z \text{ad } \mu^{(1)}} \cdot (\pm z)^{\hbar \Omega_{\overline{\mu}}}
\end{multline*}
$$

$$
\begin{multline*}
\cdot e^{-z \alpha_i(\overline{\mu}) \text{ad } \Lambda^{(1)}} \cdot (\Upsilon_\infty(\alpha_i, \overline{\mu}) \otimes 2)^{-1}
\end{multline*}
$$

where we used the fact that $\Lambda^{(1)}$ commutes with $\overline{\mu}$, that $\mu = (\alpha_i - \alpha_i(\overline{\mu})) \Lambda^{(1)} + \overline{\mu}$ and that $\Omega_{\overline{\mu}} + \Lambda = \Omega_{\overline{\mu}}$ commute with $\Upsilon_\infty(\alpha_i, \overline{\mu}) \otimes 2$ since $\Upsilon_\infty(\alpha_i, \overline{\mu})$ is of weight $0$.

The result now follows by uniqueness. \hfill $\Box$

**Remark.** The operator $J^\pm_{\overline{\mu}}(z, \overline{\mu}) \cdot e^{-z \alpha_i(\overline{\mu}) \text{ad } \Lambda^{(1)}} \cdot (\pm z)^{\hbar \Omega_{\overline{\mu}}}$ may be thought of as the fusion operator of the Levi subalgebra $\overline{\mathfrak{g}} = \mathfrak{g} + \mathfrak{h} \subset \overline{g}$.

**7.5. Centraliser property.** Retain the notation of 7.4. The following relates the differential twists of $\overline{g}$ and $\overline{\mu}$.

**Theorem.** The following holds

$$
\Delta(\Upsilon_\infty(\alpha_i, \overline{\mu}))^{-1} \cdot J^\pm_{\overline{g}}(\mu) \cdot \Upsilon_\infty(\alpha_i, \overline{\mu}) \otimes 2 = C^\pm \cdot J^\pm_{\overline{g}}(\overline{\mu})
$$

where $C^\pm \in \mathcal{E}$ commutes with the diagonal action of $\overline{\mathfrak{g}}$.

**Proof.** By definition, the left–hand side is equal to

$$
\Delta(\Upsilon_\infty(\alpha_i, \overline{\mu}))^{-1} \cdot \Upsilon_{0, \overline{g}}(z, \mu)^{-1} \cdot J^\pm_{\overline{g}}(z, \mu) \cdot \Upsilon_\infty(\alpha_i, \overline{\mu}) \otimes 2
$$

where $\Upsilon_{0, \overline{g}}(z, \mu)$ is given by Proposition 6.1. On the other hand, by Theorem 7.4,

$$
J^\pm_{\overline{g}}(\overline{\mu}) = \Upsilon_{0, \overline{g}}(z, \overline{\mu})^{-1} \cdot J^\pm_{\overline{g}}(z, \overline{\mu})
$$

$$
= \Upsilon_{0, \overline{g}}(z, \overline{\mu})^{-1} \cdot e^{z \alpha_i(\overline{\mu}) \text{ad } \Lambda^{(1)}} \cdot (\pm z)^{-\hbar \Lambda} \cdot J^\pm_{\overline{g}}(z, \overline{\mu}) \cdot e^{-z \alpha_i(\overline{\mu}) \text{ad } \Lambda^{(1)}} \cdot (\pm z)^{\hbar \Lambda}
$$

$$
= \Upsilon_{0, \overline{g}}(z, \overline{\mu})^{-1} \cdot e^{z \alpha_i(\overline{\mu}) \text{ad } \Lambda^{(1)}} \cdot (\pm z)^{-\hbar \Lambda} \cdot \Upsilon_\infty(\alpha_i, \overline{\mu}, z)^{-1} \cdot J^\pm_{\overline{g}}(z, \mu) \cdot \Upsilon_\infty(\alpha_i, \overline{\mu}) \otimes 2
$$
where the second equality follows from the fact that $\Lambda, \lambda^i$ commute with $\mathfrak{g}$.

We wish to compare the functions

$$\Upsilon_{0\infty} = \Upsilon_{0, \mathfrak{g}}(z, \mu) \cdot \Delta(\Upsilon_{\infty}(\alpha_i, \mathfrak{g}))$$

$$\Upsilon_{0\infty} = \Upsilon_{0, \mathfrak{g}}(z, \mu) \cdot e^{-\lambda(\mathfrak{g}) \cdot \text{ad}(\lambda(\mathfrak{g}))} \cdot (\pm z)^{1/2} \cdot \Upsilon_{0, \mathfrak{g}}(z, \mathfrak{g})$$

It follows from Propositions 6.1 and 4.5 for $\Upsilon_{0\infty}$, and Propositions 7.2 and 6.1 for $\Upsilon_{0\infty}$, that both are holomorphic functions of $\mathfrak{g}$, which satisfy

$$(d \hbar - \frac{\hbar}{2} \sum_{\alpha \in \Phi_+} \frac{d \alpha}{\alpha} \Delta(K_{\alpha}) - z \cdot d(\mu(1))) \Upsilon = \Upsilon \left( d - \frac{\hbar}{2} \sum_{\alpha \in \Phi_+} \frac{d \alpha}{\alpha} \Delta(K_{\alpha}) \right)$$

and are such that their value at $\mathfrak{g} = 0$ commutes with $\mathfrak{g}$. We claim that that $C^\pm$ is a constant function of $\mathfrak{g}$, and therefore that it commutes with $\mathfrak{g}$ for any $\mathfrak{g}$ in $\mathfrak{g}_{\text{reg}}$. Fix $\mathfrak{g} \in \mathfrak{g}_{\text{reg}}$, then $c(t) = C^\pm(\mathfrak{g})$, $t \in \mathbb{C}$, satisfies

$$\frac{dc}{dt} = -\frac{\hbar}{2} \frac{[\Delta(\mathfrak{g}), c]}{t}$$

so that $\text{Ad}(\hbar^{1/2} \Delta(\mathfrak{g}))/c$ is a constant $c_0$. Since $c(t) = e^0 + O(t)$, where $e^0$ commutes with $\mathfrak{g}$, $c_0 = e^0 + \text{Ad}(\hbar^{1/2} \Delta(\mathfrak{g}))/c_0 = e^0 + O(t)$, whence $c_0 = e^0$ and $c(t) = \text{Ad}(\hbar^{1/2} \Delta(\mathfrak{g}))/c_0 = C^\pm(0)$ as claimed. \(\square\)

8. Quasi–Coxeter Quasitriangular Quasibialgebra Structure on $U\mathfrak{g}$

The following is the main result of this paper. It shows the existence of a quasi–Coxeter quasitriangular quasibialgebra structure on $U\mathfrak{g}[\hbar]$ interpolating between the quasitriangular quasibialgebra structure underlying the KZ equations and the quasi–Coxeter algebra one underlying the Casimir connection.

**Theorem.** There exists a quasi–Coxeter quasitriangular quasibialgebra structure on $U\mathfrak{g}[\hbar]$ of the form

$$\left( U\mathfrak{g}[\hbar], \{U\mathfrak{g}[\hbar]\}, \{S_{\mathfrak{g}C}\}, \{\Phi_{\mathfrak{g}F}\}, \Delta_0, \{R_B\}, \{\Phi_B\}, \{F_{(B, \alpha_i)}\} \right)$$

where $\Delta_0$ is the cocommutative coproduct on $U\mathfrak{g}$,

$$S_{\mathfrak{g}C} = \mathfrak{s}_i \cdot \exp(\hbar/2 \cdot C_i)$$

$$R_B = \exp(\hbar \cdot \Omega_B)$$

$$\text{Alt}_2 F_{(B, \alpha_i)} = \hbar \cdot (r_B - r_{\mathfrak{g}B'}(\alpha_i)) \mod \hbar^2$$

and $\Phi_{\mathfrak{g}F}, F_{(B, \alpha_i)}$ are of weight 0. Moreover, $\Phi_B \in \mathfrak{g}_{\text{co}} + \hbar^2 U\mathfrak{g}[\hbar]_{\text{co}}$ is the associator for the KZ equations corresponding to $\mathfrak{g}$, and the $\Phi_{\mathfrak{g}F}$ are the De Concini–Procesi associators of the (truncated) Casimir connection $\nabla_{\kappa}$. 

Proof. Let \( J_+ (\mu) \) be the differential twist obtained from the fusion operator in 6.2. \( J_+ (\mu) \) kills the KZ associator by Theorem 6.2 and satisfies the centraliser property by Theorem 7.5. The result now follows from Theorem 4.10. \( \square \)

9. The Quantum Group \( U_h \mathfrak{g} \)

9.1. Let \( U_h \mathfrak{g} \) be the Drinfeld–Jimbo quantum group corresponding to \( \mathfrak{g} \) and the bilinear form \((\cdot, \cdot)\). Thus, \( U_h \mathfrak{g} \) is the algebra over \( \mathbb{C} [h] \) topologically generated by elements \( E_i, F_i, H_i, i \in \mathcal{I} = \{1, \ldots, r\} \), subject to the relations

\[
[H_i, H_j] = 0 \\
[H_i, E_j] = a_{ij} E_j \\
[H_i, F_j] = -a_{ij} F_j \\
[E_i, F_j] = \delta_{ij} \frac{q_i^{H_i} - q_i^{-H_i}}{q_i - q_i^{-1}}
\]

where \( a_{ij} = \alpha_j (\alpha_i^\vee) \), \( q = e^h \), \( q_i = q^{(\alpha_i, \alpha_i)/2} \), and the \( q \)-Serre relations

\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} i_k E_i^k E_j E_i^{1-a_{ij} - k} = 0 \\
\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} j_k F_i^{k} F_j F_i^{1-a_{ij} - k} = 0
\]

where for any \( k \leq n \),

\[
[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}} \\
[n]_i ! = [n]_i [n-1]_i \cdots [1]_i \quad \text{and} \quad \left[ \frac{n}{k} \right]_i = \frac{[n]_i !}{[k]_i ! [n-k]_i !}
\]

9.2. \( D \)-algebra structure on \( U_h \mathfrak{g} \). For any connected \( B \subseteq D \) with vertex set \( \mathcal{J} \subseteq \mathcal{I} \), let \( U_h \mathfrak{g}_B \subseteq U_h \mathfrak{g} \) be the subalgebra topologically generated by the elements \( \{E_j, F_j, H_j\}_{j \in \mathcal{J}} \). Then, \( U_h \mathfrak{g}_B \) is the Drinfeld–Jimbo quantum group corresponding to Levi subalgebra \( \mathfrak{g}_B \subset \mathfrak{g} \) generated by the root subspaces \( \mathfrak{g}_{a \alpha_j}, \ j \in \mathcal{J} \), and the restriction of the bilinear form \((\cdot, \cdot)\) to it. If \( \mathcal{J} = \{j\} \), we denote \( U_h \mathfrak{g}_B \) by \( U_h \mathfrak{g} \).

It is clear that the assignment \( B \to U_h \mathfrak{g}_B \) defines a \( D \)-algebra structure on \( U_h \mathfrak{g} \).

9.3. Quantum Weyl group operators. For any \( i \in \mathcal{I} \), let \( S^h_{i, \kappa} \) be the operator acting on a finite-dimensional \( U_h \mathfrak{g} \)-module \( \mathcal{V} \) as

\[
S^h_{i, \kappa} v = \sum_{a, b, c \in \mathbb{Z}} (-1)^b q_i^{b-a} E_i^{(a)} F_i^{(b)} E_i^{(c)} v
\]

where

\[
E_i^{(a)} = \frac{E_i^a}{[a]_i !} \quad F_i^{(a)} = \frac{F_i^a}{[a]_i !}
\]

\( \text{we follow here the conventions of [21].} \)

\( \text{the element } S^h_{i, \kappa} \text{ is, in the notation of [21, §5.2.1], the operator } T''_{i, i+1}. \)
and \( v \in V \) if of weight \( \lambda \in \mathfrak{h}^* \). By [21, §39.4], the operators \( S^h_{i,\kappa} \) satisfy the braid relations

\[
S^h_{i,\kappa} S^h_{j,\kappa} \cdots S^h_{m_{ij},\kappa} \cdots = S^h_{j,\kappa} S^h_{i,\kappa} \cdots S^h_{m_{ij},\kappa} \cdots \tag{9.1}
\]

for any \( i \neq j \in I \) such that the order \( m_{ij} \) of \( s_i s_j \) is finite, and therefore define an action of the braid group \( B_W \) on \( V \).

The following modification of the operators \( S^h_{i,\kappa} \) will also be needed. Set

\[ S^h_{i,C} = S^h_{i,\kappa} \cdot q_i^{H_i^2/4} \]

It follows as in 3.5.4 that the operators \( S^h_{i,C} \) also satisfy the braid relations (9.1). We refer to either \( \{ S^h_{i,\kappa} \} \) or \( \{ S^h_{i,C} \} \) as the quantum Weyl group operators of \( U_{h\mathfrak{g}} \).

The subscripts \( \kappa \) and \( C \) are justified by the following. Let \( \exp(\pi t H_i) \) and \( C_i \) be the sign and Casimir operators of \( U_{h\mathfrak{sl}_2} \), that is the central elements of \( U_{h\mathfrak{sl}_2} \) acting on the indecomposable representation \( V_m \) of dimension \( m + 1 \) as multiplication by \((-1)^m\) and \( \frac{1}{2} \cdot \frac{m(m+2)}{2} \) respectively. Let \( \kappa_i = C_i - \langle \alpha_i,\alpha_i\rangle/4 H_i^2 \) be the truncated Casimir operator of \( U_{h\mathfrak{sl}_2} \).

**Lemma.** The following holds

\[
(S^h_{i,\kappa})^2 = \exp(\pi i H_i) \cdot q_i^{C_i} \quad \text{and} \quad (S^h_{i,C})^2 = \exp(\pi i H_i) \cdot q_i^{C_i}.
\]

**Proof.** The first identity is proved in [21, Prop. 5.2.2.(b)], the second is an immediate consequence. \( \square \)

9.4. **Quasi–Coxeter structure on \( U_{h\mathfrak{g}} \).** It follows from 9.2 and 9.3 that the assignments

\[ (U_{h\mathfrak{g}})_B = U_{h\mathfrak{g}}B \quad S^h_{i,\mathfrak{g}} = S^h_{i,\kappa} \quad \text{or} \quad S^h_{i,C} \quad \text{and} \quad \Phi^{U_{h\mathfrak{g}}} = 1 \]

endow \( U_{h\mathfrak{g}} \) with two quasi–Coxeter algebra structures \( Q^h_{\kappa} \) and \( Q^h_{C} \) of type \( D \) respectively.

9.5. **Quasitriangular quasibialgebra structure.** \( U_{h\mathfrak{g}} \) is a topological Hopf algebra coproduct given by

\[
\Delta(E_i) = E_i \otimes 1 + q_i^{H_i} \otimes E_i
\]

\[
\Delta(F_i) = F_i \otimes q_i^{H_i} + 1 \otimes F_i
\]

\[
\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i
\]

For any \( B \subset D \), let

\[ R_{B,h} \in 1 \otimes 2 + h U_{h\mathfrak{g}} \otimes 2 \]

be the universal \( R \)-matrix of \( U_{h\mathfrak{g}} \) [10, 11]. For \( B = \alpha_i \), we denote \( R_{B,h} \) by \( R_{i}^h \). Let \( \alpha_i \in D \), and \( S^h_i \in U_{h\mathfrak{sl}_2} \) the quantum Weyl group element defined in 9.3. It follows by [21, Prop. 5.3.4] that\(^\text{16}\)

\[
\Delta(S^h_{i,\kappa}) = (\overline{R}_{i}^h)^{21} \cdot S^h_{i,\kappa} \otimes S^h_{i,\kappa}
\]

\(^\text{16}\)Specifically, from the fact that \( \Delta(S^h_{i,\kappa}) = (\overline{R}_{i}^h)^{21} \cdot S^h_{i,\kappa} \otimes S^h_{i,\kappa} \), where \( \overline{R}_{i}^h = q_i^{H_i \otimes H_i} \cdot R_{i}^h \).
9.6. Label the Dynkin diagram $D_\alpha$ by attaching to each pair of distinct vertices $\alpha_i \neq \alpha_j$ the order $m_{ij}$ of the product $s_i s_j \in W$. The following is a direct consequence of 9.4 and 9.5

**Proposition.** For any maximal nested sets $\mathcal{F}, \mathcal{G}$ on $D$ and $\alpha_i \in B \subseteq D$, set
\[ \Phi_{\mathcal{G}\mathcal{F}} = 1, \quad F_{(B,\alpha_i)} = 1^{\otimes 2} \quad \text{and} \quad \Phi_B = 1^{\otimes 3} \]
Then,
\[ (U_\hbar \mathfrak{g}, \{U_\hbar \mathfrak{g}_B\}, \{S_{i,C}^h\}, \{\Phi_{\mathcal{G}\mathcal{F}}\}, \Delta, \{R_{B,h}\}, \{F_{(B,\alpha_i)}\}, \{\Phi_B\}) \]
is a quasi–Coxeter quasitriangular quasibialgebra structure $\mathbb{Q}_C^h$ of type $D$ on $U_\hbar \mathfrak{g}$.

10. The Monodromy Theorem

10.1. Let $\mathbb{Q}_C^h$ be the quasi–Coxeter quasitriangular quasibialgebra structure on the quantum group $U_\hbar \mathfrak{g}$ obtained in Section 9, and underlying its $R$–matrix and quantum Weyl group representations. Let $\mathbb{Q}_C^\alpha$ be the quasi–Coxeter quasitriangular quasibialgebra structure on $U_\hbar \mathfrak{g}[\hbar]$ obtained in Section 8 which underlies the monodromy of the KZ and Casimir connections of $\mathfrak{g}$.

**Theorem.** ($U_\hbar \mathfrak{g}, \mathbb{Q}_C^h$) and $(U_\hbar \mathfrak{g}[\hbar], \mathbb{Q}_C^\alpha)$ are isomorphic as quasi–Coxeter, quasitriangular quasibialgebras.

**Proof.** By [25, Thm. 8.3], $\mathbb{Q}_C^h$ is isomorphic to a quasi–Coxeter quasitriangular quasibialgebra structure of type $D$ on $U_\hbar \mathfrak{g}[\hbar]$ of the form
\[ \left( U_\hbar \mathfrak{g}[\hbar], \{U_\hbar \mathfrak{g}_B[\hbar]\}, \{S_{i,C}\}, \{\Phi_{\mathcal{G}\mathcal{F}}\}, \Delta_0, \{\Phi_B\}, \{R_{B}^{\mathrm{KZ}}\}, \{F_{(B,\alpha_i)}\} \right) \]
where $\Delta_0$ is the cocommutative coproduct on $U_\hbar \mathfrak{g}$,
\[ S_{i,C} = \tilde{s}_i \cdot \exp(\hbar/2 \cdot C_i) \]
\[ \Phi_B = 1^{\otimes 3} \mod \hbar^2 \]
\[ R_{B}^{\mathrm{KZ}} = \exp(\hbar \cdot \Omega_B) \]
\[ \text{Alt}_2 F_{(B,\alpha_i)} = \hbar \cdot (r_B - r_{\mathfrak{g}_B(\alpha_i)}) \mod \hbar^2 \]
and $\Phi_{\mathcal{G}\mathcal{F}}, F_{(B,\alpha_i)}$ are of weight 0. Since $\mathbb{Q}_C^\alpha$ is also of this form by Theorem 8, the result follows from the rigidity of such structures [25, Thm. 9.1].

10.2. Monodromy.

**Theorem.**

1. $(U_\hbar \mathfrak{g}[\hbar], \mathbb{Q}_C^\alpha)$ and $(U_\hbar \mathfrak{g}, \mathbb{Q}_C^h)$ are isomorphic as quasi–Coxeter algebras. In particular, if $V$ is a finite–dimensional $\mathfrak{g}$–module, the monodromy of the Casimir connection $\nabla_C$ on $V[\hbar]$ is equivalent to the action of the braid group $B_W$ on any quantum deformation of $V$ given by Lusztig’s quantum Weyl group operators $S_{i,C}^h$.

2. $(U_\hbar \mathfrak{g}[\hbar], \mathbb{Q}_C^\alpha)$ and $(U_\hbar \mathfrak{g}, \mathbb{Q}_C^h)$ are isomorphic as quasi–Coxeter algebras. In particular, if $V$ is a finite–dimensional $\mathfrak{g}$–module, the monodromy of the (truncated) Casimir connection $\nabla_\kappa$ on $V[\hbar]$ is equivalent to the action of the braid group $B_W$ on any quantum deformation of $V$ given by Lusztig’s quantum Weyl group operators $S_{i,C}^h$. 

Proof. The first statement is an immediate consequence of Theorem 10.1. The isomorphism of quasi–Coxeter structures \( Q^\nabla_C \cong Q^\nabla_\kappa \) gives rise to one between \( Q^\nabla_C \) and \( Q^\nabla_\kappa \) since, by construction, the De Concini–Procesi associators of \( Q^\nabla_C \) are the same as those of \( Q^\nabla_\kappa \), and the isomorphism is equivariant for the \( h \)-action, and therefore compatible with the modifications

\[
S^h_{i,C} = S^h_{i,\kappa} \cdot q_i^{H_i^2/4} \quad \text{and} \quad S^{\nabla}_{i,C} = S^{\nabla}_{i,\kappa} \cdot \exp \left( \frac{h}{2} \sum_{i} \langle \alpha_i, \alpha_i \rangle \right)
\]

\[\square\]

Appendix A. The basic ODE

Fix \( R \geq 0 \), and let

\( H^R_\pm = \{ z \in \mathbb{C} | \Im z \geq 0 \text{ and } |z| > R \} \)

be the complements in the upper and lower half–planes \( \mathbb{H}_\pm \) of the closed disk of radius \( R \). Let \( V \) be a finite–dimensional complex vector space, and \( k : \mathbb{H}_R^R \to V \) a holomorphic function possessing an asymptotic expansion of the form \( k \sim \sum_{n \geq 0} k_n z^{-n} \) on \( \mathbb{H}^R_{\pm} \). Thus, for any \( \delta > 0 \) and \( n \in \mathbb{N} \), there is a constant \( C = C(\delta, n) \) such that, for any \( z \in \mathbb{H}^R_{\pm} \) with \( \delta \leq |\arg(z)| \leq \pi - \delta \), the following holds

\[
\| k(z) - \sum_{m=0}^{n} k_m z^{-m} \| \leq C|z|^{-(n+1)}
\]

Let \( \lambda \in \mathbb{R} \) and consider the inhomogeneous ODE

\[
\frac{dh}{dz} = \lambda h + \frac{k}{z}
\] (A.1)

We seek holomorphic solutions \( h : \mathbb{H}^R_\pm \to V \) satisfying the boundary condition

\[
h(z) \to 0 \text{ as } z \to \infty \text{ on any sector } \delta < |\arg(z)| < \pi - \delta
\] (A.2)

For any \( \theta \in [-\pi, \pi] \), let \( \Gamma^\theta_\pm \) be the contour given by the ray \( \pm iR + e^{i\theta} \cdot R_{\geq 0} \), oriented from \( \infty \) to \( \pm iR \), followed by the interval from \( \pm iR \) to 0.

Proposition.

(1) The equation (A.1) has at most one solution such that (A.2) holds.

(2) If \( k_0 \neq 0 \) and \( \lambda = 0 \), no solution to (A.1)–(A.2) exists.

(3) If \( k_0 = 0 \) or \( \lambda \neq 0 \), (A.1)–(A.2) have a unique solution given by the Laplace integral

\[
h(z) = \int_{\Gamma} e^{-\lambda t} \frac{k(z + t)}{z + t} dt
\] (A.3)

where \( \Gamma \) is a path from \( \infty \) to 0 such that \( \Gamma + \mathbb{H}^R_{\pm} \subset \mathbb{H}^R_{\pm} \), and such that the integral is convergent as \( t \) tends to infinity along \( \Gamma \). Specifically,

(a) If \( k_0 \neq 0 \) and \( \lambda \neq 0 \), \( \Gamma \) can be chosen as \( \Gamma^\pm_\theta \), with

\[
\pi > \theta > \pi/2 \text{ if } \lambda < 0 \quad \text{and} \quad \pi/2 > \theta > 0 \text{ if } \lambda > 0
\]

As a function of \( \lambda \), \( h \) is smooth on \( \mathbb{R}^* \).

(b) If \( k_0 = 0 \), \( \Gamma \) can be chosen as \( \Gamma^\pm_0 \), with

\[
\pi > \theta \geq \pi/2 \text{ if } \lambda \leq 0 \quad \text{and} \quad \pi/2 > \theta > 0 \text{ if } \lambda \geq 0
\]

As a function of \( \lambda \), \( h \) is continuous on \( \mathbb{R} \), and smooth on \( \mathbb{R}^* \).
(4) If $\lambda \neq 0$, $h$ has an asymptotic expansion with respect to $z$ which is given by

$$h(z) \sim \sum_{n \geq 1} \frac{(n-1)!}{z^n} \left( \sum_{p=0}^{n-1} \frac{k_p}{p!} \frac{1}{(-\lambda)^{n-p}} \right)$$

(A.4)
and is valid uniformly on compact subsets of $\mathbb{R}^+ \ni \lambda$.

(5) If $\lambda = 0$ and $k_0 = 0$, $h$ has an asymptotic expansion given by

$$h(z) \sim -\sum_{n \geq 1} k_n \frac{z^{-n}}{n}$$

(6) As a function of $\lambda$, $h$ has an asymptotic expansion as $\lambda \to \infty$ given by

$$h \sim -\sum_{n \geq 1} \left( \frac{k(z)}{z} \right)^{(n-1)} \lambda^{-n}$$

which is valid uniformly on compact subsets of $\mathbb{H}_\pm$.

Proof. (1) The solutions of the underlying homogeneous equation are given by $g = e^{\lambda t}g_0$, where $g_0 \in V$ is a constant. If $\lambda = 0$, then $g = g_0$ is equal to its limit as $z \to \infty$, and $g = 0$. If $\lambda \neq 0$, the function $e^{\lambda t}$ does not have a limit as $z \to \infty$ in $\mathbb{H}_\pm$, whence $g_0 = 0$.

(2) Write $k = k_0 + \mathcal{K}$, where $\mathcal{K} = k - k_0 = O(z^{-1})$. Any solution of (A.1) with $\lambda = 0$ is of the form $C + k_0 \ln z - \int \frac{e^{\lambda t}}{t} dt$ where $C$ is a constant, and $\gamma$ is a path in $\mathbb{H}_\pm$ with a fixed starting point and ending in $z$. Since $\int \frac{e^{\lambda t}}{t} dt = O(z^{-1})$, no such solution admits a limit as $z \to \infty$ if $k_0 \neq 0$.

(3) Integration by parts readily shows that the integral (A.3) is a solution of (A.1). If $k_0 \neq 0$ and $\lambda \neq 0$, $\Gamma_{\pm \theta}^\pm$ satisfies the required conditions since integrability at $\infty$ is guaranteed by the exponential factor $e^{-\lambda \Re t}$, so long as $\lambda \cos \theta > 0$. If, on the other hand, $k_0 = 0$, integrability is guaranteed by $k(z)/z = O(z^{-2})$, so long as $e^{-\lambda \Re t}$ remains bounded, and the given $\Gamma_{\pm \theta}^\pm$ satisfy the required conditions.

It is clear that the function defined by (A.3) is a smooth function of $\lambda \in \mathbb{R}^+$, since the convergence of the derivatives of integral is guaranteed by the factor $\exp(-\lambda \Re t)$. If $k_0 = 0$, we may choose $\Gamma = \Gamma_{\pm \theta/2}$ for any $\lambda \in \mathbb{R}$, and the continuity of $h(z)$ at $\lambda = 0$ follows from the Riemann–Lebesgue Lemma.

(4) Set $g(\zeta) = k(\zeta)/\zeta$. If $\lambda \neq 0$, integration by parts shows that, for any $m \geq 0$,

$$h(z) = \int \frac{e^{-\lambda t} g(z + t) dt}{\zeta} = -\sum_{p=0}^{m-1} \frac{1}{\lambda^{p+1}} g^{(p)}(z) + \frac{1}{\lambda^m} \int \frac{e^{-\lambda t} g^{(m)}(z + t) dt}{\zeta}$$

(A.5)

Let $k(z) \sim \sum_{n \geq 0} k_n z^{-n}$ be the asymptotic expansion of $k$. Then, for any $p \geq 0$

$$g^{(p)}(z) \sim (-1)^p \sum_{n \geq 0} k_n \frac{(n+p)!}{n!} z^{-(n+p+1)} = (-1)^p \sum_{n \geq p+1} k_{n-p-1} \frac{(n-1)!}{(n-p-1)!} z^{-n}$$

It follows that

$$-\sum_{p=0}^{m-1} \frac{1}{\lambda^{p+1}} g^{(p)}(z) = -\sum_{p=0}^{m-1} \frac{(-1)^p}{\lambda^{p+1}} \sum_{n \geq p+1} k_{n-p-1} \frac{(n-1)!}{(n-p-1)!} z^{-n} + O(z^{-m-1})$$

$$= \sum_{n=1}^{m} z^{-n} \sum_{p=1}^{n} \frac{(-1)^p}{\lambda^p} k_{n-p} \frac{(n-1)!}{(n-p)!} z^{-n} + O(z^{-m-1})$$
locally uniformly in $\lambda \in \mathbb{R}^*$. 
To estimate the second summand in (A.5), note that

$$\left|\int_{\Gamma} e^{-\lambda t} g^{(m)}(z + t) dt\right| \leq \int_0^\infty e^{-\lambda \Re \varphi(s)} |g^{(m)}(z + \varphi(s))| |\varphi'(s)| ds$$

$$\leq C d(z, -\Gamma)^{m+1} \int_0^\infty e^{-\lambda \Re \varphi(s)} ds$$

$$= C d(z, -\Gamma)^{m+1} \left(R + \frac{1}{\lambda \cos \theta}\right)$$

where $t = \varphi(s)$ is the parametrisation of $\Gamma = \Gamma_{\pm}^\theta$ with the opposite orientation given by $\varphi(s) = \pm ts$ for $s \in [0, R]$ and $\varphi(s) = \pm iR + (s - R)e^{i\theta}$ for $s \geq R$, and the second inequality follows from the fact that $g^{(m)}(z) = O(z^{-m-1})$.

Let $\Gamma_{\pm}^\theta \subset \kappa_{\pm} \subset \mathbb{H}_{\pm}$ be the convex cone bounded by the rays $e^{i\theta}$ and $e^{\pm i(\pi - \theta)}$, and $S_{\pm}^{|z|} \subset \mathbb{H}_{\pm}$ the half-circle with radius $|z|$. Then, $d(z, -\Gamma_{\pm}^\theta) \geq d(S_{\pm}^{|z|}, -\Gamma_{\pm}^\theta) \geq |z| |\sin \theta|$, where the bound is attained when $w \in S_{\pm}^{|z|}$ lies on the real axis. It follows that

$$\left|\int_{\Gamma} e^{-\lambda t} g^{(m)}(z + t) dt\right| \leq C |z|^{-m-1} |\sin \theta| \left(R + \frac{1}{\lambda \cos \theta}\right)$$

(A.6)

where $C$ is independent of $\lambda$.

(5) If $k_0 = 0$ and $\lambda = 0$, then

$$h = \int_{\Gamma} \frac{k(z + t)}{z + t} \sim \sum_{n \geq 1} \int_{\Gamma} k_n(z + t)^{-n-1} dt = -\sum_{n \geq 1} \frac{k_n}{n} z^{-n}$$

(6) The existence of the claimed asymptotic expansion of $h$ with respect to $\lambda$ is guaranteed by (A.5), provided $\int_{\Gamma} e^{-\lambda t} g^{(m)}(z + t) dt = O(\lambda^{-1})$. Integrating by parts, we have

$$\int_{\Gamma} e^{-\lambda t} g^{(m)}(z + t) dt = \frac{1}{\lambda} \left(-g^{(m)}(z) + \int_{\Gamma} e^{-\lambda t} g^{(m+1)}(z + t) dt\right)$$

and the required estimate now follows from (A.6). \[\square\]

**Appendix B. The constant $C^\pm$ revisited**

The goal of this section is to show that the constant $C^\pm$ relating the differential twists of $\mathfrak{g}$ and $\mathfrak{f}$ given by Theorem 7.5 can be computed as the monodromy from 0 to $\infty$ of an ODE on $\mathbb{P}^1$ with regular singularities at 0, 1 and an irregular singularity at $\infty$. This gives in particular a canonical, transcendental construction of a relative twist for the pair of KZ associators $(\Phi, \pi)$, i.e., an element $J \in 1 + h(U_{\mathfrak{g}}[h]^{\otimes 2})\mathfrak{f}$ such that $(\Phi)_J = \pi$, in the spirit of Drinfeld’s construction of the KZ associator.

B.1. Consider the connection $\nabla$ on $\mathbb{C} \times \mathfrak{g}$ given by (7.1). Fix $\pi \in \mathfrak{f}$, and coordinate $\pi^{-1}(\pi)$ by $w = \alpha_1$ as in 7.2. The restriction of $\nabla$ to $\mathbb{C} \times \pi^{-1}(\pi)$ then reads

$$\nabla = d - h\Omega \frac{dz}{z} - \frac{h}{2} \sum_{\alpha \in \Phi_+ \setminus \Phi_{\pm}} \Delta(K_{\pm}) \frac{dw}{w} - \text{ad} \Delta^{(1)}(zdw) - \text{ad} \lambda^{(1)}(zw)$$

Our first goal is to construct two canonical solutions of $\nabla$ with prescribed asymptotic behaviour as $z \to 0$, $w \to \infty$, and $zw \to 0$, $\infty$ respectively.
B.2. Blow–up coordinates. To this end, let \( \rho : (v_1, v_2) \in \mathbb{C} \times \mathbb{P}^1 \to (z, w) \in \mathbb{C} \times \mathbb{P}^1 \) be the rational map given by \(^{17}\)

\[
    z = \frac{v_1 v_2}{v_2 + 1} \quad \text{and} \quad w = \frac{v_2 + 1}{v_1}
\]  

(B.1)

\( \rho \) is a birational isomorphism, with inverse given by

\[
    v_1 = z + 1/w \quad \text{and} \quad v_2 = zw
\]

(B.2)

and restricts to an isomorphism of \( \{(v_1, v_2) \in \mathbb{C} \times \mathbb{P}^1 | v_1 \neq 0, v_2 \neq 1\} \) onto \( \{(z, w) \in \mathbb{C} \times \mathbb{P}^1 | w \neq 0, (z, w) \neq (0, \infty), zw \neq -1\} \). Since \( v_2 = zw \), the asymptotic zones \( z \sim 0, w \sim \infty \) and \( zw \sim 0, \infty \) correspond respectively to the neighborhoods of the points \( (v_1, v_2) = (0,0) \) and \( (0, \infty) \).

Given that \( w - w_\alpha = (v_2 + 1 - w_\alpha v_1)/v_1 \), we get

\[
    d\log(w - w_\alpha) = -\frac{dv_1}{v_1} - w_\alpha\frac{dv_1}{v_2 + 1 - w_\alpha v_1} + \frac{dv_2}{v_2 + 1 - w_\alpha v_1}
\]

It follows that the pulled–back connection \( \rho^* \nabla \) is given by

\[
    \rho^* \nabla = d - \left( \frac{\hbar 2 \Omega - \Delta(K - \bar{K})}{2v_1} - \frac{\hbar}{2} \sum_{\alpha \in \Phi^+_+ \setminus \bar{\Phi}^+} \frac{w_\alpha \Delta(K_\alpha)}{v_2 + 1 - w_\alpha v_1} + \frac{v_2}{v_2 + 1} \text{ad} i(\bar{\pi})^{(1)} \right) dv_1
\]

\[
    - \left( \frac{\hbar \Omega}{v_2} - \frac{\hbar}{2} \sum_{\alpha \in \Phi^+_+ \setminus \bar{\Phi}^+} \frac{\Delta(K_\alpha)}{v_2 + 1 - w_\alpha v_1} + \text{ad} \lambda_i^{(1)} + \frac{v_1}{(v_2 + 1)^2} \text{ad} i(\bar{\pi})^{(1)} \right) dv_2
\]

where \( K, \bar{K} \) are given by (4.3).

B.3. The proof of the following result is similar to that of Proposition 6.1 and therefore omitted.

Proposition.

1. For any \( \bar{\pi} \in \bar{\Gamma} \) and \( v_2 \in \mathbb{P}^1 \setminus \{-1\} \), there is a unique holomorphic function \( I_0 : \{v_1 \in \mathbb{C} | |v_1| < R_{\bar{\pi}^{-1}} \cdot |v_2 + 1|\} \to \mathcal{A} \) such that \( I_0(0, v_2, \bar{\pi}) = 1 \) and, for any determination of \( \log(v_1) \), the \( \mathcal{E} \)-valued function

\[
    \Xi_0(v_1, v_2, \bar{\pi}) = e^{\frac{\pi_0 v_1}{\hbar} \text{ad} i(\bar{\pi})^{(1)}} \cdot I_0(v_1, v_2, \bar{\pi}) \cdot v_1^{-\frac{\hbar}{2} \Delta(K - \bar{K}) - 2\Omega}
\]

satisfies \( \rho^* \nabla_{\partial v_1} \Xi_0 = \Xi_0 d_{v_1} \), where \( \rho^* \nabla_{\partial v_1} \) is given by

\[
    d_{v_1} = -\left( \frac{\hbar 2 \Omega - \Delta(K - \bar{K})}{2v_1} - \frac{\hbar}{2} \sum_{\alpha \in \Phi^+_+ \setminus \bar{\Phi}^+} \frac{w_\alpha \Delta(K_\alpha)}{v_2 + 1 - w_\alpha v_1} + \frac{v_2}{v_2 + 1} \text{ad} i(\bar{\pi})^{(1)} \right) dv_1
\]

(B.3)

2. \( I_0 \) and \( \Xi_0 \) are holomorphic functions of \( v_2 \) and \( \bar{\pi} \), and \( \Xi_0 \) satisfies \( \rho^* \nabla \Xi_0 = \Xi_0 (d_{v_1} + \nabla_0 + \nabla_C) \)

\(^{17}\)note that, unlike previous changes of coordinates, \( \rho \) mixes the configuration coordinate \( z \) with the Cartan coordinate \( w \).
\[\nabla_0 = d_{v_2} - \left( \frac{h\Omega}{v_2} + \frac{h}{2} \frac{\Delta (K - \overline{K})}{v_2 + 1} + \text{ad} \lambda_i^{(1)} \right) dv_2 \quad (B.4)\]

\[\nabla_C = d_{\theta} - \frac{h}{2} \sum_{\alpha \in \Phi_+} \frac{d\alpha}{\alpha} \Delta (K_\alpha) \quad (B.5)\]

B.4. Let \( \nabla_0 \) be the connection given by \((B.4)\).

**Proposition.**

1. The coefficients of the connection \( \nabla_0 \) commute with the diagonal adjoint action of the Levi subalgebra \( \mathfrak{l} \) and with \( \Delta (K - \overline{K}) - 2\Omega \).

2. There is a unique holomorphic function

\[G_0 : \{ v_2 \in \mathbb{C} | |v_2| < 1 \} \to \mathcal{A}\]

such that \( G_0(0) = 1 \) and, for any determination of \( \log v_2 \), the \( \mathcal{E} \)-valued function

\[\Psi_0 = e^{v_2 \text{ad} \lambda_i^{(1)}} \cdot G_0(v_2) \cdot v_2^\Omega \]

satisfies \( \nabla_0 \Psi_0 = \Psi_0 dv_2 \).

3. There is a unique holomorphic function

\[G_\infty^\pm : \{ v_2 \in \mathbb{C} | \text{Im } v_2 \geq 0, |v_2| > 1 \} \to \mathcal{A}\]

such that \( G_\infty^\pm (v_2) \to 1 \) as \( v_2 \to \infty \) with \( 0 < \text{arg}(v_2) < \pi \), and, for any determination of \( \log(v_2) \), the function

\[\Psi_\infty^\pm = G_\infty^\pm (v_2) \cdot e^{v_2 \text{ad} \lambda_i^{(1)}} \cdot v_2^\pm ((K - \overline{K})^{(1)} + (K - \overline{K})^{(2)})\]

satisfies \( \nabla_0 \Psi_\infty^\pm = \Psi_\infty^\pm dv_2 \).

4. The functions \( \Psi_0, \Psi_\infty^\pm \) commute with the diagonal adjoint action of \( \mathfrak{l} \) and with \( \Delta (K - \overline{K}) - 2\Omega \).

**Proof.**

(1) Since \( \Omega \) commutes with the diagonal action of \( \mathfrak{g} \), and \( \lambda_i^{(1)} \) commutes with \( \mathfrak{g} \), it suffices to show that \( \Delta (K - \overline{K}) - 2\Omega \) commutes with \( \mathfrak{l} \) and with \( \lambda_i^{(1)} \). The commutation with \( \lambda_i^{(1)} \) follows from the fact that

\[\Delta (K - \overline{K}) = (K - \overline{K})^{(1)} + (K - \overline{K})^{(2)} + 2 \sum_{\alpha \in \Phi_\mathfrak{g}} x_\alpha \otimes x_{-\alpha} \quad (B.6)\]

The fact that \( K - \overline{K} \) commutes with \( \mathfrak{l} \) follows from \((4.8)\).

(2) is proved in a similar way to Proposition 6.1.

(3) is proved in a similar way to Proposition 7.2, and relies on the fact that the connection \( \nabla_0 \) is of the form

\[d_{v_2} - \left( \text{ad} \lambda_i^{(1)} + \frac{h}{2} \frac{\Delta (K - \overline{K})}{v_2} + O(v_2^{-2}) \right) dv_2\]

and that, by (1), the projection of \( \Delta (K - \overline{K}) \) onto the kernel of \( \text{ad} \lambda_i^{(1)} \) is \( (K - \overline{K})^{(1)} + (K - \overline{K})^{(2)} \).

(4) follows from (1). \( \square \)
B.5. Let $C^\pm \in \mathcal{E}$ be the constant relating the differential twists of $\mathfrak{g}$ and $\overline{\mathfrak{g}}$ given by Theorem 7.5, $\nabla_0$ the connection (B.4), and $\Psi_0, \Psi_\infty^\pm$ its fundamental solutions given by Proposition B.4.

**Theorem.** The following holds,

$$C^\pm = \Psi_0^{-1} \cdot \Psi_\infty^\pm$$

**Proof.** The proof of Theorem 7.5 shows that $C^\pm = \Upsilon_{0,0}^{-1} \cdot \Upsilon_\infty^\pm$, where

$$\Upsilon_{0,0} = \Upsilon_{0,0}(z, \mu) \cdot \Delta(\Upsilon_\infty(\alpha_i, \overline{\mu}))$$

and $\Upsilon_{0,0}(z, \mu), \Upsilon_\infty(\alpha_i, \overline{\mu}), \Upsilon_\infty^\pm(\alpha_i, \overline{\mu}, z)$ are the functions given by Propositions 6.1, 4.5 and 7.2 respectively. Let $\Xi_0$ and $\Psi_0, \Psi_\infty^\pm$ be the functions given by Propositions B.3 and B.4 respectively. We claim that

$$\Upsilon_{0,0} = \Xi_0 \cdot \Psi_0 \quad \text{and} \quad \Upsilon_\infty^\pm = \Xi_0 \cdot \Psi_\infty^\pm$$

so that $C^\pm = \Psi_0^{-1} \cdot \Psi_\infty^\pm$ holds.

To see that the first claimed identity holds, write

$$\Upsilon_{0,0} = \tilde{H}_{0,0}(z, w) \cdot z^{h\Omega} \cdot \Delta(H_\infty(w)) \cdot w^{\frac{1}{2}} \Delta(K-\overline{\mathfrak{g}})$$

where $\tilde{H}_{0,0} = e^{z \cdot \text{ad}(1)} H_{0,0}$, and we have suppressed the dependence in $\overline{\mu} \in \overline{\mathfrak{g}}$ which will be held fixed throughout the argument. We have

$$\Xi_0 \cdot \Psi_0 = \tilde{I}_0(v_1, v_2) \cdot v_1^{\frac{1}{2}}(\Delta(K-\overline{\mathfrak{g}}) - 2\Omega) \cdot G_0(v_2) \cdot v_2^{\frac{1}{2}}$$

where $\tilde{I}_0 = e^{z_1 \cdot v_1 \cdot \text{ad}(1)}(\overline{\mu})^{0(1)} \cdot I_0$, $v_1, v_2$ are expressed in terms of $z, w$ through (B.2), the function $I_0(z, w) = \tilde{I}_0(v_1, v_2) \cdot v_1^{\frac{1}{2}}(\Delta(K-\overline{\mathfrak{g}}) - 2\Omega) \cdot G_0(v_2) \cdot w^{\frac{1}{2}}(\Delta(K-\overline{\mathfrak{g}}) - 2\Omega)$, $I_0(1/w, 0)^{-1}$ is holomorphic near $z = 0$ for fixed $w$ and such that $I_0(0, w) = 1$, and the third equality follows from the fact $\tilde{I}_0(v_1, 0)$ commutes with $\Omega$ since the coefficients of the connection (B.3) of which $\tilde{I}_0(v_1, v_2)$ is a horizontal section do so for $v_1 = 0$. Since $\Xi_0 \cdot \Psi_0$ is a horizontal section of $\nabla_0$, so is $J_0(z, w) \cdot z^{h\Omega} = \Xi_0 \cdot \Psi_0 \cdot \left(\tilde{I}_0(1/w, 0) \cdot w^{\frac{1}{2}} \Delta(K-\overline{\mathfrak{g}})\right)^{-1}$ and it follows by the uniqueness of Proposition 6.1 that $J_0(z, w) = \tilde{H}_{0,0}(z, w)$. We are therefore reduced to proving that $\Delta(H_\infty(w)) \cdot w^{\frac{1}{2}} \Delta(K-\overline{\mathfrak{g}}) = \tilde{I}_0(1/w, 0) \cdot w^{\frac{1}{2}} \Delta(K-\overline{\mathfrak{g}})$, which follows from the uniqueness of Proposition 4.5.

The second identity is proved in a similar way. We may assume that $z \geq 0$. We have

$$\Upsilon_\infty^\pm = H_\infty^\pm(w, z) \cdot e^{z \cdot \text{ad}(1)} \cdot w^{\frac{1}{2}}(\Delta(K-\overline{\mathfrak{g}})^{(1)} + (K-\overline{\mathfrak{g}})^{(2)})$$

$$\cdot e^{-z \cdot \text{ad}(1)} \cdot \left(\pm z\right)^{h\Lambda} \cdot H_{0,0}(z, \overline{\mu}) \cdot z^{h\overline{\mathfrak{g}}}$$
and

\[ \Xi_0 \cdot \Psi^\pm = \tilde{I}_0(v_1, v_2) \cdot v_1 \cdot \Phi^\pm(\Delta(K - K) - 2\Omega) \cdot G^\pm_\infty(v_2) \cdot e^{z w \ad \lambda^{(1)}_i} \cdot v_2 \]

\[ = \tilde{I}_0(v_1, v_2) \cdot G^\pm_\infty(v_2) \cdot v_1 \cdot \Phi^\pm(\Delta(K - K) - 2\Omega) \cdot e^{z w \ad \lambda^{(1)}_i} \cdot (zw)^{\Phi_0(K - K) + (K - K)^{(2)}} \]

\[ = \tilde{J}_0^\pm(w, z) \cdot \tilde{I}_0(z, \infty) \cdot z^{\Phi(w, z)} \cdot (zw)^{\Phi_0(K - K) + (K - K)^{(2)}} \cdot \tilde{I}_0(z, \infty)^{-1} \]

where the second equality follows from the fact that \( G^\pm_\infty \) commutes with \( \Delta(K - K) - 2\Omega \) by Proposition 7.2, the function \( \tilde{J}_0^\pm(w, z) \) is defined by

\[ \tilde{J}_0^\pm(w, z) = \tilde{I}_0(v_1, v_2) \cdot G^\pm_\infty(v_2) \cdot v_1 \cdot \Phi^\pm(\Delta(K - K) - 2\Omega) \cdot \tilde{I}_0(z, \infty)^{-1} \]

and, for fixed \( z \), tends to 1 as \( w \to \infty \) with \( \Im w \geq 0 \) and \( 0 < |\arg w| < \pi \), and the last equality uses the fact that \( \tilde{I}_0(z, \infty) \) commutes with \( \lambda^{(1)}_i \) and \( (K - K)^{(1)} + (K - K)^{(2)} \) since the coefficients of the connection (B.3) do for \( v_2 = \infty \), and (B.6). By the uniqueness of Proposition 7.2, \( \tilde{J}_0^\pm(w, z) = H^\pm_\infty(w, z) \), which reduces the stated claim to proving that

\[ e^{-2\pi \Phi(z, \overline{z}) \ad \lambda^{(1)}_i} \cdot (z^h)^{\Phi_0(z, \overline{z})} \cdot z^{\Phi_0(z, \overline{z})} \cdot \tilde{I}_0(z, \infty)^{-1} \]

In turn, this follows from the uniqueness of Proposition 6.1. \( \Box \)

APPENDIX C. THE DYNAMICAL KZ AND CASIMIR EQUATIONS

For any \( n \geq 2 \), let

\[ \mathbb{X}_n = \mathbb{C}^n \setminus \bigcup_{1 \leq i < j \leq n} \{ z_i = z_j \} \]

be the configuration space of \( n \) ordered points in \( \mathbb{C} \). Consider the following connection on the trivial vector bundle over \( \mathbb{X}_n \times \mathfrak{h}_{\text{reg}} \) with fibre \( U \mathfrak{g} \otimes \mathbb{C}^n \)

\[ \nabla = d - h \sum_{1 \leq i < j \leq n} \frac{d(z_i - z_j)}{z_i - z_j} \Omega_{ij} - \frac{h}{2} \sum_{\alpha \in \Phi^+} \frac{d\alpha}{\alpha} \Delta^{(n)}(K_\alpha) - td(\sum_{i=1}^n z_i \mu^{(i)}) \]

Above, \( h, t \) are complex parameters, \( \Omega_{ij} = \sum_{\alpha} X^{(i)}_{\alpha} X^{(j)}_{\alpha} \), where \( X^{(i)} = 1^{\otimes (i-1)} \otimes X \otimes 1^{\otimes (n-i)} \), \( \Delta^{(n)} : U \mathfrak{g} \to U \mathfrak{g} \otimes \mathbb{C}^n \) is the iterated coproduct and \( \mu \) is the embedding \( \mathfrak{h} \to U \mathfrak{g} \).

The coupling term \( d(\sum_{i=1}^n z_i \mu^{(i)}) \) was shown to lead to a consistent system in [20] when \( t = h \). In the present paper, it is important to keep \( h \) and \( t \) as independent parameters, since we consider \( h \) formal while \( t = 1 \) throughout.

PROPOSITION. The connection is integrable for any \( h, t \in \mathbb{C} \).

PROOF. Consider more generally a connection on the product \( X \times Y \) of two manifolds with values in an algebra \( U \), which is of the form \( \nabla = \nabla^X + \nabla^Y + t\lambda \) where \( \nabla^X \) is an \( A \)-valued connection on \( X \) of the form \( d + hA_X \), with \( A_X \) an \( A \)-valued one-form on \( X \), \( \nabla^Y \) is an \( A \)-valued connection on \( Y \) of the form \( d + hA_Y \), with
\( A_Y \in \Omega^1(X, A) \), and \( \lambda \in \Omega^1(X \times Y, A) \). It is then easy to check that \( \nabla \) is flat for any \( t, h \in \mathbb{C} \) if, and only if

\[
\begin{align*}
    d_X A_X &= 0 = A_X \wedge A_X \\
    d_Y A_Y &= 0 = A_Y \wedge A_Y \\
    [A_X, A_Y] &= 0 \\
    d\lambda &= 0 = \lambda \wedge \lambda \\
    [A_X + A_Y, \lambda] &= 0
\end{align*}
\]

In the case at hand, \( X = X_n \) and \( A_X = \sum_{i<j} d \log(z_i - z_j) \Omega_{ij} \) are well-known to satisfy \( d_X A_X = 0 = A_X \wedge A_X \). Similarly, \( Y = h_{\text{reg}} \) and \( A_Y = \frac{1}{2} \sum_{\alpha \in \Phi^+} d \log \alpha \mathcal{K}_{\alpha} \) satisfy \( d_Y A_Y = 0 = A_Y \wedge A_Y \) [22, 25, 20]. Finally, \( \lambda = d(\sum_i z_i \mu^{(i)}) \) clearly satisfies \( d\lambda = 0 = \lambda \wedge \lambda \) since it is exact and abelian. It therefore remains to check that \( [A_X + A_Y, \lambda] = 0 \). Write to this end \( \lambda = \lambda_X + \lambda_Y \), where

\[
\begin{align*}
    \lambda_X &= \sum_i d z_i \mu^{(i)} \\
    \lambda_Y &= \sum_i z_i d\mu^{(i)}
\end{align*}
\]

Then,

\[
\begin{align*}
[A_X, \lambda_X] &= \sum_{i<j,k} d(z_i - z_j) \wedge \frac{1}{z_i - z_j} d z_k \left[ \Omega_{ij}, \mu^{(k)} \right] \\
&= \sum_{i<j} \left( d z_i \left[ \Omega_{ij}, \mu^{(i)} \right] + d z_j \left[ \Omega_{ij}, \mu^{(j)} \right] \right) \\
&= \sum_{i<j} \left( d z_i - d z_j \right) \left[ \Omega_{ij}, \mu^{(i)} \right] \\
&= 0
\end{align*}
\]

where the second equality follows from the fact that \( \mu^{(k)} \) commutes with \( \Omega_{ij} \) unless \( k \in \{i, j\} \) and the third from the fact that \( \Omega_{ij} \) commutes with \( \mu^{(i)} + \mu^{(j)} \). Next,

\[
\begin{align*}
[A_X, \lambda_Y] &= \sum_{i<j} \frac{d(z_i - z_j)}{z_i - z_j} \wedge \left( z_i [\Omega_{ij}, d\mu^{(i)}] + z_j [\Omega_{ij}, d\mu^{(j)}] \right) \\
&= \sum_{i<j} \left( d(z_i - z_j) \wedge [\Omega_{ij}, d\mu^{(i)}] \right) \\
&= \sum_{i<j} d(z_i - z_j) \wedge [\Omega_{ij}, d\mu^{(i)}]
\end{align*}
\]

which completes the computation of the commutator \( [A_X, \lambda] \).

To compute \( [A_Y, \lambda] \), write

\[
\Delta^{(n)}(\mathcal{K}_{\alpha}) = \sum_{i=1}^n \mathcal{K}_{\alpha}^{(i)} + 2 \sum_{i<j} (\mathcal{K}_{\alpha, +}^{(ij)} + \mathcal{K}_{\alpha, -}^{(ij)})
\]
where \( \mathcal{K}^{(ij)}_{\alpha, \pm} = x^{(i)}_{\pm \alpha} x^{(j)}_{\mp \alpha} \). Then,

\[
[A_Y, \lambda_X] = \sum_{\alpha, i < j} \frac{d\alpha}{\alpha} [\mathcal{K}^{(ij)}_{\alpha, +} + \mathcal{K}^{(ij)}_{\alpha, -}, dz_i \mu^{(i)} + dz_j \mu^{(j)}] = \sum_{\alpha, i < j} \frac{d\alpha}{\alpha} \wedge d(z_i - z_j) [\mathcal{K}^{(ij)}_{\alpha, +} + \mathcal{K}^{(ij)}_{\alpha, -}, \mu^{(i)}] = \sum_{\alpha, i < j} d\alpha \wedge d(z_i - z_j) (-\mathcal{K}^{(ij)}_{\alpha, +} + \mathcal{K}^{(ij)}_{\alpha, -})
\]

where the first equality follows from the fact that \( \mathcal{K}^{(i)}_{\alpha} \) commutes with any \( \mu^{(k)} \), the second from the fact that \( \mathcal{K}^{(ij)}_{\alpha, +} + \mathcal{K}^{(ij)}_{\alpha, -} \) is of weight zero and the third from the fact that \([\mu^{(i)}, \mathcal{K}^{(ij)}_{\alpha, \pm}] = \pm \alpha \mu^{(i)} \). Finally,

\[
[A_Y, \lambda_Y] = \sum_{\alpha, i < j} \frac{d\alpha}{\alpha} [\mathcal{K}^{(ij)}_{\alpha, +} + \mathcal{K}^{(ij)}_{\alpha, -}, z_i d\mu^{(i)} + z_j d\mu^{(j)}] = \sum_{\alpha, i < j} \frac{d\alpha}{\alpha} \wedge (z_i - z_j) [\mathcal{K}^{(ij)}_{\alpha, +} + \mathcal{K}^{(ij)}_{\alpha, -}, d\mu^{(i)}] = \sum_{\alpha, i < j} d\alpha \wedge (z_i - z_j) d\alpha (-\mathcal{K}^{(ij)}_{\alpha, +} + \mathcal{K}^{(ij)}_{\alpha, -}) = 0
\]

To conclude, we need to show that \([A_X, \lambda_Y] + [A_Y, \lambda_X] = 0 \). This follows by writing \( \Omega_{ij} = \sum_{\alpha \in \Phi_+} (\mathcal{K}^{(ij)}_{\alpha, +} + \mathcal{K}^{(ij)}_{\alpha, -}) + \Omega^h_{ij} \), where \( \Omega^h = t_a \otimes t^a \), with \( \{t_a\}, \{t^a\} \) dual bases of \( h \), so that

\[
[A_X, \lambda_Y] = \sum_{i < j} d(z_i - z_j) \wedge [\Omega_{ij}, d\mu^{(i)}] = \sum_{i < j, a} d(z_i - z_j) \wedge [\mathcal{K}^{(ij)}_{\alpha, +} + \mathcal{K}^{(ij)}_{\alpha, -}, d\mu^{(i)}] = \sum_{i < j, a} d(z_i - z_j) \wedge d\alpha (-\mathcal{K}^{(ij)}_{\alpha, +} + \mathcal{K}^{(ij)}_{\alpha, -})
\]

\( \square \)
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