Lattice fermions obeying the Ginsparg-Wilson relation do correctly represent the physical properties related to chirality. This can be achieved by local fermions, which involve an infinite number of couplings, however. For practical purposes, it is useful to first construct approximate Ginsparg-Wilson fermions within a short range. We report on a successful construction in QCD at $\beta = 6$. The good quality of the approximation is observed from the spectrum, which is situated close to a Ginsparg-Wilson circle. These fermions also provide an excellent approximation to rotational symmetry and they are promising for a good scaling, since they arise from the perfect action framework. Their insertion into the overlap formula renders the Ginsparg-Wilson relation exact. It leads to an improved overlap fermion with a high level of locality. This insertion is statistically on safe grounds at $\beta \gtrsim 5.6$.

1 Ginsparg-Wilson fermions

In a slightly simplified form, the famous Nielsen-Ninomiya theorem states that a local lattice fermion without species doublers cannot be chiral in the sense that the lattice Dirac operator $D$ anti-commutes with $\gamma_5$. Locality means here that the couplings in $D$ decay at least exponentially in the separation between $\psi$ and $\bar{\psi}$. Hence it is an obvious idea to break the chiral symmetry by an irrelevant term, so that it should be restored in the continuum limit. The simplest way to do so is to set $\frac{1}{2} \gamma_5 \{ D, \gamma_5 \}$ equal to some local term of $O(a^2)$, such as the Wilson term $\frac{1}{2} \Delta$, where $\Delta$ is a discretized Laplacian. However, this type of chiral symmetry breaking on the lattice is rather violent; it causes quite some trouble such as additive mass renormalization, $O(a)$ scaling artifacts, renormalization of currents, mixing of matrix elements, etc.

On the other hand, it turned out to be harmless to introduce a non-vanishing anti-commutator as

$$\frac{1}{2} \gamma_5 \{ D^{-1}, \gamma_5 \} = R$$

where $R$ is a local term with $\{ R, \gamma_5 \} \neq 0$. The superiority of this relation can be understood intuitively from the fact that $R$ doesn’t shift the poles in $D^{-1}$.

\textsuperscript{a}Actually the proof in Ref. \cite{1} still holds for an even weaker form of locality.

\textsuperscript{b}Here $a$ is the lattice spacing, but in general we will refer to a hypercubic lattice of unit spacing in Euclidean space.
In the form
\[ \{D_{x,y}, \gamma_5\} = 2(D\gamma_5RD)_{x,y} \] (2)
it is known as the \textit{Ginsparg-Wilson relation} (GWR).\footnote{GWR}

\section*{1.1 Virtues}

Amazingly, it seems that all physical properties related to chirality are represented correctly by a lattice fermion obeying the GWR (a GW fermion). The mass and the vector current (as well as the flavor non-singlet axial vector current) are not renormalized and weak matrix elements do not mix\footnote{Weak matrix elements mixing}. Moreover, the chiral anomalies\footnote{Chiral anomalies}, as well as global anomalies\footnote{Global anomalies}, and the soft pion theorems are reproduced correctly\footnote{Soft pion theorems}. Even the construction of chiral gauge theories on the lattice is feasible based on GW fermions\footnote{Chiral gauge theories on lattice}.

For the understanding of these properties, it is a key observation that GW fermions have an exact — though lattice modified — chiral symmetry at finite lattice spacing\footnote{Lattice modified chiral symmetry}.

It is instructive to consider the spectrum of a GW fermion. For simplicity, we assume \( D^\dagger = \gamma_5D\gamma_5 \) and \( R_{x,y} = \frac{1}{2\mu}\delta_{x,y} (\mu > 0) \), hence the GWR reads \( \mu(D + D^\dagger) = D^\dagger D \). If we introduce the operator \( A = D - \mu \), the GWR simplifies further to
\[ A^\dagger A = \mu^2. \] (3)
Therefore we know that the spectrum of a GW Dirac operator is — with the above assumptions — always situated on a circle in the complex plane, with center and radius \( \mu \). This confirms the absence of additive mass renormalization, and it also rules out “exceptional configurations”\footnote{Exceptional configurations}. Moreover, it provides a well-defined index (since the zero eigenvalues are exact), and together with the index theorem we obtain a sensible definition of the topological charge of a lattice configuration.

\section*{1.2 Limitations}

After celebrating the impressive properties of GW fermions, we now have to address their limitations. One point is that the GWR guarantees a correct chiral behavior, but it does hardly imply anything about other properties, which are also essential for a formulation of lattice fermions, in particular the scaling behavior.

A second point concerns locality: the relaxation of the condition \( \{D, \gamma_5\} = 0 \) to the GWR allows the lattice fermion to be local in the sense that the couplings decay exponentially with the lattice distances. This is sufficient from a conceptual point of view, since there is a decay length of a finite number
of lattice spacings, which ensures the right continuum limit. However, for applications one would like to have even “ultralocality”, which means that the couplings drop to zero beyond a finite number of lattice spacings. Unfortunately, GW fermions cannot have this property, not even in the case of free fermions\textsuperscript{10}. For example, if we want to insert the free Wilson Dirac operator and solve for $R$, then we obtain a pseudo-GW kernel $R$ which decays as $R_{x,y} \propto |x-y|^{-4}$ in $d = 2$, and like $R_{x,y} \propto |x-y|^{-6}$ in $d = 4$. This is non-local, and therefore the Wilson fermion does not obey any GWR, not even in the free case (in the interacting case this is also clear from the mass renormalization).

Of course, in practice one cannot work with couplings over infinite distances. In a finite volume with certain boundary conditions, the GWR — with these boundary conditions implemented — can be solved, but this still requires the coupling of sites (and links) over all distances in this volume.

1.3 Exact and approximate solutions

Regarding the first limitation, there exists a class of lattice actions called “perfect actions” which deserves its name by yielding a scaling identical to the continuum at any lattice spacing. At the same time, perfect actions solve the GWR\textsuperscript{2}, but unfortunately their construction is about as difficult as solving directly the model under consideration, since it requires a functional integration extrapolated to the continuum.

The construction of “classically perfect actions”\textsuperscript{11} is much easier, though still difficult. They also solve the GWR\textsuperscript{3}, and their scaling is still excellent. However, a successful construction and application for interacting fermions could only be achieved in $d = 2$ so far\textsuperscript{12}. From the second limitation we know that such actions need some truncation. Here we truncate to a “hypercube fermion”, HF (with couplings not only to nearest neighbor sites, but to all sites inside a unit hypercube), which is actually applicable in QCD simulations\textsuperscript{13}. We first consider the truncation for the free fermion. There the perfect lattice Dirac operator can be constructed, and the term $R$ occurs in the block variable transformation. Locality is optimal — that is, the exponential decay of the couplings is optimally fast — for $R_{x,y} = \frac{1}{2} \delta_{x,y,3}$ which we denote as the “standard GW kernel”. Hence the spectrum of that perfect fermion is situated on a unit circle with center 1. We now truncate by evaluating the perfect couplings in a small volume of $3^4$ sites, and then we use the same couplings in any volume\textsuperscript{14}. We obtain a lattice Dirac operator of the form

$$D(x, y) = \rho_\mu(x - y) \gamma_\mu + \lambda(x - y) ,$$

where the support of $\rho_\mu(x - y)$, $\lambda(x - y)$ is restricted to $|x_\nu - y_\nu| \leq 1$, ($\nu = \ldots$)
1...4). (These couplings are given in Ref. [13], Table 1.) Of course, in a larger volume it is not exactly perfect any more, and the GWR is violated a little. To probe this truncation effect, we look at the spectrum of the hypercube fermion on a 20^4 lattice. Fig. 1 shows that it is indeed very close to a GW circle. Also

![Figure 1: The spectrum of a truncated perfect, free HF on a 20^4 lattice (plotted in C)]](image)

the scaling artifacts in this truncated perfect free fermion are very small, as we see from the fermion dispersion relation and thermodynamic scaling quantities 15,16.

The same construction can be done in d = 2, but there we prefer to use a “scaling-optimal hypercube fermion” (SO-HF) 17, which is similar to the truncated perfect one, but still a bit improved (with respect to scaling). Again the spectrum for the free fermion is close to a GW unit circle, and its scaling is excellent. To simulate this lattice fermion in the 2-flavor Schwinger model, we performed a “minimal gauging” by hand: we attached the free couplings to the shortest lattice paths only, in equal parts where several shortest paths exist. We also added a clover term with coefficient 1, and used the standard plaquette gauge action. Of course, this simple gauging brings in additional artifacts and a further deviation from the GWR. Hence the eigenvalues spread more and more around the unit circle as β decreases, but for instance at β = 2 the circle is still approximated more or less, see Fig. 2.

As a scaling test we measured the dispersion relations of the two meson-type states, a massless triplet and a massive singlet 18, which we denote as π and η. Indeed, the scaling is dramatically improved over the Wilson fermion (at critical hopping parameter), see Fig. 3. Amazingly, this SO-HF reaches a similar scaling quality as the classically perfect action, which was truncated only very mildly to 123 independent couplings per site 19. In contrast, the
SO-HF involves only 6 independent couplings per site, hence this approach has the potential to be extended and applied in $d = 4$.

Also the rotational symmetry is approximated very well for such HFs. We tested this for free and for interacting fermions in $d = 2$ and $d = 4$. As an example, we show the “speed of light” for the minimally gauged truncated perfect HF in QCD at $\beta = 5$, and compare it to the Wilson fermion in Fig. 4. What is not visible from that figure, however, is the dramatic mass renormalization: what is supposed to be the pion mass amounts to $M = 3.0$. Also in the Schwinger model this problem is serious, as Figs. 2 and 3 show: at $\beta = 6$ we obtain a $\pi$ mass of 0.13. This is the one unpleasant feature of the otherwise successful HFs with minimal gauging. In the next section we discuss a possibility to eliminate this effect completely. In Sec. 4 we are going
to consider further methods to approach the chiral limit.

Figure 4: The “speed of light” $c = p/\sqrt{E^2 - M^2}$ ($M =$ “pion mass”) in QCD at $\beta = 5$.

2 Improved overlap fermions

Let us start from some lattice Dirac operator $D_0$, which obeys the conditions of the Nielsen-Ninomiya theorem (local, no doublers etc.). We assume again $D_0^\dagger = \gamma_5 D_0 \gamma_5$, and we recall that the GWR for $R_{x,y} = \frac{1}{\mu} \delta_{x,y}$ is equivalent to $A_0^\dagger A_0 = \mu^2$, for $A_0 = D_0 - \mu, \ (\mu > 0)$. In general this will not hold, of course, but we can simply enforce it by the overlap formula

$$A_{ov} = \frac{\mu A_0}{\sqrt{A_0^\dagger A_0}} ; \quad D_{ov} = A_{ov} + \mu . \quad (5)$$

$D_{ov}$ represents a GW fermion. A prototype was proposed by H. Neuberger\textsuperscript{19}, starting from the Wilson fermion $D_0 = D_W$ and $\mu = 1$,

$$D_{Ne} = 1 + \frac{A_W}{\sqrt{A_W^\dagger A_W}} ; \quad A_W = D_W - 1 . \quad (6)$$

In the free case and in a smooth gauge background, $D_{Ne}$ is free of doublers and local. The latter has been established analytically assuming a constraint on each plaquette variable\textsuperscript{20,21} and numerically for QCD at $\beta = 6$\textsuperscript{20}. For first simulations in quenched QCD, see Refs.\textsuperscript{22,23}.

However, this is just one example in a large class of overlap fermions, which is obtained by varying $D_0$ and $\mu$. In particular, if it happens that $D_0$ represents a GW fermions already, then it is just reproduced under the overlap formula, $D_{ov} = D_0$ (for a fixed GW kernel $R$). Therefore, also perfect and
classically perfect fermions are special types of overlap fermions. In practice we do not have an exact GW operator $D_0$ at hand, but we can construct an approximation, as discussed in Secs. 1.3 and 4. Then the square root will keep close to the constant $\mu$, and $D_{ov} \approx D_0$. We may say that the overlap formula provides a “GW correction” of $D_0$; in particular it removes the additive mass renormalization.

Based on this property, we suggest the following concept: start from a truncated perfect fermion (or something similar), which scales well and approximates the GWR, and insert it as $D_0$ into the overlap formula. The resulting $D_{ov}$ has exact GW chirality (in particular we are in the chiral limit) and a good scaling behavior can also be expected, due to $D_{ov} \approx D_0$. The latter also suggests that an approximate rotational symmetry of $D_0$ is essentially inherited by $D_{ov}$, and that $D_{ov}$ is very local in the sense of a fast exponential decay: remember that $D_0$ is ultralocal, and — due to the modest modification — the long-range couplings will be turned on just a little bit in $D_{ov}$.

To summarize, this concept aims at combining all desirable properties of a lattice fermion formulation. It has been tested comprehensively in the two-flavor Schwinger model, and the above expectations are observed to hold in a very satisfactory way. In rest of this Section we summarize our 2d results from Ref. \[17\].

We first checked the scaling quality of the free overlap fermions (fermionic dispersion, thermodynamic scaling ratios) and it turns out that the improvement of the SO-HF over the Wilson fermion persists under the overlap formula, i.e. the overlap SO-HF is strongly improved over the Neuberger fermion. By comparing the spectrum of a fixed configuration for the SO-HF and for the overlap SO-HF, we could then literally see that the alteration due to the overlap formula is small; the eigenvalues are moved almost radially onto the unit circle. Next we tested the scaling in the interacting case, and the drastic improvement of the overlap SO-HF over the Neuberger fermion is confirmed again, see Fig. 5. We have used standard operators in both cases, hence the comparison is fair. Of course, both types of overlap fermions may improve if one consistently improves the operators, which can be tedious, however. Then one expects also the Neuberger fermion to scale better (in particular $O(a)$ artifacts are excluded), but our results imply that improved overlap fermions can be used very successfully even with the simple standard operators (as it was also observed for the classically perfect action).

Finally we also tested our prediction regarding the degree of locality, and we found that indeed the exponential decay is much faster for the overlap

\footnote{Also $D_W$ is ultralocal, but not an approximate GW fermion, so its change to $D_{Ne}$ is rather drastic and the argument for a high level of locality does not apply to that case.}
Figure 5: The “meson” dispersion relations for different types of overlap fermions in the 2-flavor Schwinger model at $\beta = 6$.

SO-HF than for the Neuberger fermion, see Fig. 6 and Table 1.

Figure 6: The level of locality for the Neuberger fermion compared to the overlap SO-HF: the decay of the free couplings in all directions (left) and the decay of the “maximal correlation” $\langle \psi |$ over a distance $r$ at $\beta = 6$ (right). The width of the “cones” in the left figure also shows how well rotation invariance is approximated.

3 Overlap fermions and the doubling problem

As we mentioned in Sec. 2, overlap fermions are free of doublers at least in a smooth gauge background. It should be clarified, however, that the overlap formula itself does not remove any doublers. They have to be removed before, in $D_0$, by something like a Wilson term; in our ansatz $D_0 = \rho \gamma_\mu + \lambda$, the scalar term $\lambda$ is crucial for that purpose. Then the overlap formula cures the

$^d$Ref. [24] suggests to use non-standard operators (with $\langle R \rangle$ being subtracted from $\langle D^{-1} \rangle$) even for the free fermion, which does improve e.g. the scaling of the free Neuberger fermion. However, one then deals with a non-local fermion, even though it is used only in an indirect way, which raises conceptual questions concerning the continuum limit. An exception is the perfect fermion at $R = 0$, where for instance the axial anomaly is reproduced correctly [25].

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Table 1: The characteristic radius \( r_\lambda = \left( \sum |\lambda(x)| x^2 \right) / \left( \sum |\lambda(x)| \right) \) and (analogously) \( r_\rho \) for various types of free GW fermions.

|   | \( d = 2 \) | \( d = 2 \) | \( d = 4 \) | \( d = 4 \) |
|---|---|---|---|---|
|   | \( r_\rho \) | \( r_\lambda \) | \( r_\rho \) | \( r_\lambda \) |
| \( \rho \) | 1.268 | 0.871 | 1.635 | 1.187 |
| \( \lambda \) | 1.930 | 1.255 | 1.519 | 1.109 |
| \( \rho \) | 1.255 | 0.888 | 2.530 | 1.708 |
| \( \lambda \) | 1.930 | 1.248 | |

chiral symmetry again in the sense of the GWR, while doublers are supposed not to be re-introduced.

We keep on referring to a point-like GW kernel \( R_{x,y} = \frac{1}{2 \mu} \delta_{x,y} \), where the mass parameter \( \mu > 0 \) can be chosen. It is the center of the circle through zero that the spectrum of the Dirac operator is mapped on by the overlap formula (which also involves the parameter \( \mu \)). Let us focus on the eigenvalues of \( D_0 \) close to the real axis: the small (almost) real eigenvalues have to be mapped on (the vicinity of) zero, whereas the eigenvalues with large real parts have to be mapped on the opposite arc, i.e. on (the vicinity of) \( 2\mu \). For weak up to moderate coupling, there is a (statistically) safe interval where \( \mu \) can be chosen such that this separation is achieved. Inside this interval, \( \mu \) may be optimized with respect to various criteria; the optimal \( \mu \) tends to rise with increasing coupling strength — since it has to adapt to the mass renormalization — as has been observed by optimizing locality [20] or minimizing the mapping effect [17]. For instance, in the Schwinger model such a safe interval still exists at \( \beta = 2 \), as Fig. 2 illustrates.

But how about really strong coupling? At some point, the eigenvalues of typical configurations spread all over inside a certain area for simple (short-ranged) operators \( D_0 \). This is the case for the Wilson fermion and for the HF in the Schwinger model at \( \beta = 1 \), and for QCD at \( \beta = 5 \), as Fig. 6 shows. Wherever we choose \( \mu \), mappings to the “wrong” arc will occur frequently; the possibility of “wrong” projections implies that both, the doubling problem (too many projections to the left arc) as well as mass renormalization (too many projections to the right arc) are back. Hence at really strong coupling, both of these problems return for the overlap fermions based on some simple \( D_0 \), as was also revealed by a strong coupling expansion [21]. Probably the only way out of this problem would be an excellent approximation to a (classically)

\[^{4}\text{The question if also locality breaks down at some point of relatively strong coupling, where the overlap formula is still applicable, is currently under investigation.}^{4}\]

\[^{5}\text{For instance, one can check this for the Neuberger fermion in the Schwinger model at } \beta \lesssim 2, \text{ or in QCD at } \beta \approx 5.6.\]

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perfect fermion, which is, however, very difficult to construct and implement.

It is now of interest to get an idea where this transition occurs. Fig. 8 shows the Wilson spectra for typical (quenched) QCD configurations at $\beta = 5.4$ and $\beta = 5.6$. We see that the existence of a safe interval for $\mu$ seems to set in in between. For the HF the situation is similar, just a little better. Hence $\beta = 6$ can be regarded a really safe regime with respect to this issue — Fig. 8 shows that an isolated left arc is present — and also with respect to locality.

We therefore concentrate on QCD at $\beta = 6$ in the next Section, and we show that we can construct HFs, which approximate the GWR well at that coupling strength. This is the crucial ingredient needed to carry on the program of improved overlap fermions — described in Sec. 3 — to QCD.

Note that in $d = 4$ the notorious square root has to be evaluated by some iterative method, and starting from a good GW fermion approximation instead of the Wilson fermion is also highly profitable with that respect: starting in the right vicinity speeds up the convergence a lot, as we see for instance from the perturbative expansion of the square root around $\mu$.

\footnote{From our experience, even lattices as small as $4^4$ provide a reliable insight into such questions as the applicability of the overlap formula and the level of approximation to the GW circle. A disadvantage is that the left arc close to zero is missing — not due to the lattice action but just due to small volume. Therefore we also evaluated this subset of the spectra (the eigenvalues with smallest real parts) on $8^4$ lattices, see below.}
Figure 8: Typical spectra of the Wilson fermion in QCD on a $4^4$ lattice at $\beta = 5.4$ (left) and $\beta = 5.6$ (right). We recognize the transition to the regime where the overlap projection is statistically safe (for $1.5 < \mu < 2$).

4 Approximate Ginsparg-Wilson Fermions for QCD

Our approach to construct a short-ranged approximate GW fermion for QCD is to stay with the couplings of the truncated perfect free fermion and gauge it by hand, using just very few new parameters to go beyond the “minimal gauging”. This concept was successful in $d = 2$, and in $d = 4$ we already know that the free HF is doing well in scaling, approximating the GWR and approximating rotational invariance (the latter is also checked at strong coupling), see Sec. 1.3. Alternatively, one may try to minimize the GWR violation directly within a limited set of parameters [32,30] or undertake a new effort to parameterize an (approximate) classically perfect action [31].

We are confident that our free HF couplings already provide a good scaling, so the issue is to find a suitable gauging in the sense that the GWR violation is small. As our criterion, we compute the spectra on small lattices and try to arrange for them to be close to a GW unit circle (for typical quenched configurations at $\beta = 6$).

As we see from Fig. [3], the minimally gauged HF suffers from mass renormalization almost as much as the Wilson fermion. On the other hand, the right arc is excellent (see Fig. [7]), but less important. Our first step beyond minimal gauging is the use of fat links: each link in a given configurations is substituted as

$$\text{link} \rightarrow (1 - \alpha) \text{link} + \frac{\alpha}{6} \left[ \sum \text{staples} \right] \quad (\alpha \in \mathbb{R}).$$

(7)
Figure 9: The left arc of typical spectra in QCD at $\beta = 6$ on an $8^4$ lattice for the Wilson fermion and the minimally gauged HF. We show the 600 resp. 300 eigenvalues with smallest real parts, and we see that here the left arc is manifestly isolated.

This is computationally cheap, and since we perform just one such fattening step we do not need to project back onto the gauge group (in contrast to Ref. [32]). The results for different $\alpha$ are shown in Fig. 10 (left). As we observed already in $d = 2$, a strongly negative $\alpha$ is required if one wants to remove the mass renormalization in this way [17]. However, then the spectrum moves far away from the GW circle, so we do not recommend this way to approach the chiral limit. Positive $\alpha$ increases the pion mass further, but it makes the shape of the spectrum more circle-like. We are going to take advantage of that.

Next we attach an amplification factor $1/u$ ($u \lesssim 1$) to each link to compensate the mean suppression by the gauge field. This is related in spirit to tadpole improvement [33], but it can also be viewed as directly generalizing the tuning of the Wilson hopping parameter. For our value of $\beta = 6$, we reach criticality for the minimally gauged HF at $u \simeq 0.8$, see Fig. 10 (right). Once $u$ is fixed, its inclusion is computationally for free, and it does lead already to a decent approximation of the GW circle.

In a next step, we include fat links with positive staples — $\alpha = 0.3$ is a good value — and use again the critical link amplification parameter, which now amounts to $u \simeq 0.76$. Indeed, this helps to move the eigenvalues closer to the GW circle, as Fig. 11 (left) shows.

Still one would like to further reduce the imaginary part of the eigenvalues; in particular the upper (and lower) arc still calls for improvement. Considering the structure of $D_{HF}(x, y, U) = \rho_{\mu}(x, y, U)\gamma_{\mu} + \lambda(x, y, U)$, we recognize that $\rho_{\mu}$ is responsible for the imaginary part, so we multiply a damping factor $v \lesssim 1$ on each link only in the vector term $\rho_{\mu}$. The scalar term $\lambda$, which controls the left and right arc already successfully, remains untouched. The optimal
Figure 10: The effect of fat links (left, \( u = 1 \)) and link amplification (right, \( \alpha = 0, u = 0.8 \)) for a HF spectrum in QCD at \( \beta = 6 \) on a 4\(^4\) lattice.

value for the new parameter is \( v \simeq 0.9 \) without fat links, and \( v \simeq 0.92 \) at \( \alpha = 0.3 \), always at critical \( u \) (which remains practically unchanged). Now also the upper arc follows the GW circle closely, and the fat link helps the eigenvalues to spread less around the circle, see Fig. 11 (right). This is the best approximation achieved so far, and we are confident that this is about the optimum that can be achieved with \( O(10) \) independent parameters. In fact, even if we include fat links we are using just 10 independent parameters.

Of course one might still try further parameters, but they should only be included if they really lead to a significant progress. One could include terms with a new Dirac structure, and we are currently testing the clover term: its inclusion (with a positive coefficient) helps to improve the physically important arc near zero a little, but it distorts the opposite arc (in the sense that the eigenvalues fluctuate stronger around the right half-circle). Generally the clover term tends to attract the eigenvalues closer to the real axis, which is also known from the Wilson fermion.\(^{14}\) Hence the optimal value of \( v \) increases a little: for instance at clover coefficient 0.15 (and \( \alpha = 0.3 \)) it amounts to \( v \simeq 0.94 \), while \( u \simeq 0.767 \) is critical.

The consideration of the “magnetic mass” \( m_B \) suggests that also a term \( \propto \gamma_\mu \gamma_\nu \gamma_\rho \) could be useful\(^{15}\) (we refer to the Pauli term \( \vec{\sigma} \vec{B}/2m_B \) in the low energy expansion).

What is computationally simple and perhaps promising is an extension of the fat link to include also (selected) paths of length 5. All this is currently under investigation and the results will be reported in Ref. [29].
\[\alpha = 0.3, \ u = 0.76, \ v = 0.92 \]
\[\alpha = 0, \ u = 0.8, \ v = 0.9 \]

**5 Summary and Outlook**

Our concept is to first construct a short-ranged approximate GW fermion with a good scaling behavior and quasi rotational invariance. Inserting this fermion into the overlap formula makes the chirality exact. Since the alteration is small, the good scaling and approximate rotational symmetry is essentially preserved, so that we obtain an improved overlap fermion. For the same reason, such an overlap fermion has a high level of locality and its iterative evaluation is fast. This program was tested extensively in the Schwinger model. Furthermore, for very good approximate GW operators the interval (for the mass parameter \(\mu\)) where the overlap formula can be used safely is extended, and such an interval exists up to stronger coupling compared to the Neuberger fermion (which inserts the Wilson fermion).

We add that the same mechanism applies to domain wall fermions: inserting an improved 4d fermion may induce the same advantages.

For QCD, we use a truncated perfect free HF as our building block, and construct a good gauging in the sense of a small GWR violation. At \(\beta = 6\) this is achieved, as we observed from the spectrum: it is close to a GW circle, hence the fermion approximately obeys the GWR with a point-like kernel \(R\).

In our construction, we start form the minimally gauged HF, remove the additive mass renormalization by a critical amplification factor for each link, and suppress the vector term only in order to damp the imaginary part of the eigenvalues, so that the spectrum follows the shape of a GW unit circle.
Fat links help to reduce the fluctuations of the eigenvalues around that circle, and hence provide an approximation on a satisfactory level. It seems quite optimal for a set of about 10 parameters. Perhaps extended staple terms lead to some further progress, and the clover term helps a little with respect to the fine resolution close to zero, which we now focus on. In any case, a good approximation to a GW fermion in QCD has been accomplished, and it is ready to be inserted into the overlap formula 29.

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