Phase transition for loop representations of Quantum spin systems on trees

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Abstract

We consider a model of random loops on Galton-Watson trees with an offspring distribution with high expectation. We give the configurations a weighting of \( \theta^n \# \text{loops} \). For many \( \theta > 1 \) these models are equivalent to certain quantum spin systems for various choices of the system parameters. We find conditions on the offspring distribution that guarantee the occurrence of a phase transition from finite to infinite loops for the Galton-Watson tree.

1 Introduction

Loop models are percolation type probabilistic models with intimate connections to the correlation functions of certain quantum spin systems. To describe them, let \( G = (V, E) \) be a graph, and for each edge \( e \in E \), let \( X_e \) be a random variable that takes values in the set of finite collections of points (called ‘links’) inside an interval \([0, \beta]\). The points may be marked, the most important case being that there are two different types of points, called crosses and bars. Given a configuration \((X_e)_{e \in E}\), a loop configuration is constructed in the following way: each vertex \( v \) is assigned a copy of the interval \([0, \beta]\)perc (with end points identified), and is then wired to other vertices by laying wires’ that cross to a neighbouring vertex at those places where \( X_e \), for an edge \( e \) that is incident to \( v \), has a point. If that point is a bar, the wire is in addition continued in the opposite direction on the new edge, otherwise in the same direction. Figure 1 gives an illustration of such a loop configuration. It is easy to see that this prescription indeed results in a configuration of disjoint loops, but that on the other hand a vertex can be contained in more than one loop, and that a single loop may visit a vertex several times. A vertex \( v \) is said to be connected to a vertex \( v' \) if they share a loop. By thinking of the wiring one may also interpret this as the wiring conducting electricity from \( v \) to \( v' \). The basic question is about the existence of percolation in this sense, i.e. the probability of transferring electricity to infinity.

In the simplest models, the joint probability law of the \((X_e)_{e \in E}\) is a product of independent laws for each \( e \in E \). A relevant choice is a pair of independent
Poisson point processes for crosses and bars, respectively. While it is clear that a necessary condition for loop percolation is the existence of an infinite cluster of edges carrying at least one point, the fact that disjoint loops can share an edge makes this condition far from sufficient. Except in cases where reflection positivity can be applied (see below), very little is known about the existence of infinite loops. In the case of random interchange it was shown by Schramm [10] that infinite loops occur on the complete graph. The reader is encouraged to consult the recent review of Ueltschi [13] and references therein for a more complete overview of current results in this direction.

Loop models that correspond to quantum systems are more complicated: they use the independent distribution of the $(X_e)$ as a reference measure, but change it with an energy which (in finite volume) is proportional to the total number of loops in the configuration. We write the corresponding Boltzmann factor in the form $\theta^{\#\text{loops}}$, $\theta \geq 1$. Then an infinite volume limit has to be taken. Relevant quantum systems include the spin-$\frac{1}{2}$ Heisenberg ferromagnet, the quantum XY model and a spin-1 nematic model. The Heisenberg ferromagnet has been investigated using a loop model that is random interchange with $\theta = 2$, see [11]. This representation was extended by Ueltschi [12] to a family of models that, in spin-$\frac{1}{2}$, interpolate between the Heisenberg ferro- and antiferromagnet and the quantum XY model. The case of the antiferromagnet had previously been investigated by Aizenman and Nachtergaele [1] and the two models agree in this case. In all of these examples, one expects that, in the case where the reference measure of the $X_e$ is a standard Poisson point process, a phase transition occurs: below a critical interval length (‘temperature’) $\beta_c$, loops are finite with probability one, but above $\beta_c$ infinite loops appear. This result has been proved in the case of a cubic lattice of sufficiently high dimensions for $\theta$ an integer [12].

The proof relies on the method of reflection positivity and infra-red bounds which are actually properties of the related quantum spin system, and are only available in highly symmetric cases such as $\mathbb{Z}^d$. For the loop models corresponding to the quantum XY model and the Heisenberg antiferromagnet (as well as interpolations between the two models) this result corresponds to results in the papers of Dyson, Lieb and Simon [6] and Kennedy, Lieb and Shastry [9] for the spin-systems.

In [4], Björnberg and Ueltschi investigate the loop model for $\theta = 1$ (independent $X_e$) in the case where $G$ is a $d$-regular tree, and where $d$ is large. The advantage of this setting is that $G$ itself has no loops, so the only complications stem from doubly occupied edges. By making $d$ large and $\beta = O(1/d)$, they can show that in this case the loop percolation threshold corresponds, to first order in $1/d$, to the percolation threshold for occupied edges. What is more, they are able to analyse situations where some doubly occupied edges are present, and thus get upper and lower bounds for the loop percolation threshold that differ from those of edge percolation by a term of order $1/d$ and are sharp up to terms of order $1/d^2$. In addition, these bounds depend on the intensity of crosses and bars, respectively. The case of $\theta = 1$ with only crosses corresponds to random interchange and has been previously studied by Angel [2] and Hammond [7, 8].

In a very recent preprint [5], they extend these results to the case where $\theta \neq 1$. The justification for studying a tree is that for very high space dimensions, the difference between a $d$-regular tree and $\mathbb{Z}^d$ in terms of percolation questions should be small.
In the present paper, we consider the loop model on random trees, more precisely on Galton-Watson trees which are strongly supercritical. The idea is that if we are in \( Z^D \) for very high space dimension \( D \), we can first delete a number of edges that will be unoccupied anyway, and have a graph that is approximately a tree, and which on average has \( 1 \ll d \ll D \) children per vertex. We then assume that it actually is a tree, but it is still random and certainly not \( d \)-regular, and for mathematical convenience we take it to be a Galton-Watson tree. While none of these assumptions are strictly true, they are a slightly better approximation of the truth than the regular tree assumed in the works of Björnberg and Ueltschi. We investigate this model to first order in \( 1/d \), and find that up to this order, the loop percolation threshold is again equal to the occupied edge percolation threshold on the tree. While this result may seem obvious, the presence of the factor \( \theta^{\# \text{loops}} \) and the fact that the tree may have vertices with degree significantly larger than \( d+1 \) makes the proof non-trivial.

It is based on ideas from [4] and estimates on the effects of links on \( \theta^{\# \text{loops}} \).

### 2 Definition and main result

For a graph \( G = (\mathcal{V}(G), \mathcal{E}(G)) \) fix a parameter \( u \in [0,1] \). We construct, independently on each edge \( e \in \mathcal{E}(G) \), two Poisson processes \( N^e,\bar{\ } \) and \( N^e,|| \) over the time interval \( [0,\beta) \) with intensity \( u \) and \( (1-u) \), respectively. Furthermore, for any finite subgraph, \( G_{\text{fin}} \), of \( G \), we denote the joint distribution of \( N^e,* \) with \( e \in \mathcal{E}(G_{\text{fin}}) \) and \(* \in \{\bar{\ },||\} \) by \( \rho_{G_{\text{fin}}} \). For simplicity we write \( N^e := N^e,\bar{\ } + N^e,|| \) and say that there is a link on an edge \( e \in \mathcal{E} \) at time \( t \in [0,\beta) \) iff \( N^e \) has a jump at time \( t \). Finally, the expectation with respect to \( \rho_{G_{\text{fin}}} \) will be denoted by \( \mathbb{E}_{G_{\text{fin}}} \).

For a realisation, \( \omega \), of \( (N^e,*),e \in \mathcal{E}(G_{\text{fin}}),* \in \{\bar{\ },||\} \) and a finite subgraph, \( G_{\text{fin}} \), of \( G \) we construct loops in the usual way (see e.g. [12]). In fact, the construction only depends on the events of \( \omega \) belonging to \( e \in \mathcal{E}(G_{\text{fin}}) \). More precisely, a loop is the support of a parametrisation \( [0,1] \to \mathcal{V}(G_{\text{fin}}) \times [0,\beta)_{\text{per}} \) that respects the links \( \omega_i = (e_i,t_i,*_i) \) of \( \omega = (\omega_i) \) with \( e_i \in \mathcal{E}(G_{\text{fin}}) \). This means that we start at some \( (x,t_0) \) and move along \( x \) in some time direction until we encounter a link on an edge \( e_i = [x,y] \in \mathcal{E}(G_{\text{fin}}) \) at time \( t_i \). Then we jump to \( y \) and continue in the same time direction, if \( *_i = \bar{\ } \), or in the opposite time direction, if \( *_i = || \),
as before. If we reach \((x, 0)\) or \((x, \beta)\) we use periodicity to move to \((x, \beta)\) or \((x, 0)\), respectively, and continue in the same direction. This procedure is best explained by a picture (see figure 1). From this procedure we obtain a partition, \(\mathcal{L}_{G_{\text{fin}}} (\omega)\), of \(V(G_{\text{fin}}) \times [0, \beta)_{\text{per}}\) into loops. By \(L_{G_{\text{fin}}} (\omega) := |\mathcal{L}_{G_{\text{fin}}} (\omega)|\) we denote the number of loops within \(G_{\text{fin}}\) for the realisation \(\omega\). Finally, for \(\theta \geq 1\), the probability measure of interest is given by

\[
P_{\theta}^{G_{\text{fin}}} (B) := \frac{\mathbb{E}_{G_{\text{fin}}} \left[ 1_{B} \theta^{L_{G_{\text{fin}}}} \right]}{\mathbb{E}_{G_{\text{fin}}} \left[ \theta^{L_{G_{\text{fin}}}} \right]}.
\]

(2.1)

Our first theorem concerns the \(d\)-regular tree. We denote by \(T_{r,n}^{x}\) the \(d\)-regular tree rooted at \(x\) and containing \(n\) generations. For a given finite tree \(T_{\text{fin}}^{x}\) and \(x \in V(T_{\text{fin}}^{x})\) we consider the event \(E_{r}^{x \rightarrow m} T_{\text{fin}}^{x}\) that there is a loop within \(L_{T_{\text{fin}}^{x}} (\omega)\) containing \(x\) (at some time) that reaches the \(m\)th generation of the tree.

**Theorem 1.** Let \(b > a > \theta > \eta\) be arbitrary but fixed. Then:

1. There is a \(d_{0} \in \mathbb{N}\), depending on \(a\) and \(b\), such that for all \(d \geq d_{0}\) and all \(\beta = \beta(d) \in \left[\frac{a}{d}, \frac{b}{d}\right]\) we have

\[
\lim_{m \to \infty} \inf_{n \geq m} \frac{\mathbb{P}_{T_{r,n}^{x}}^{E_{r}^{x \rightarrow m} T_{\text{fin}}^{x}}}{\mathbb{E}_{T_{r,n}^{x}} \left[ \theta^{E_{r}^{x \rightarrow m} T_{\text{fin}}^{x}} \right]} > 0.
\]

2. There is a \(d_{0} \in \mathbb{N}\), depending on \(\eta\), such that for all \(d \geq d_{0}\) and all \(\beta = \beta(d) \leq \frac{\eta}{d}\) we have

\[
\lim_{m \to \infty} \sup_{n \geq m} \frac{\mathbb{P}_{T_{r,n}^{x}}^{E_{r}^{x \rightarrow m} T_{\text{fin}}^{x}}}{\mathbb{E}_{T_{r,n}^{x}} \left[ \theta^{E_{r}^{x \rightarrow m} T_{\text{fin}}^{x}} \right]} = 0.
\]

Hence, under the conditions of part 1, with positive probability there is an infinite loop from the root in the limit \(n \to \infty\). Note that the existence of the limit in the theorem is not proved in general (although they certainly exist) and only limited results exist, such as for the case \(u = \frac{1}{2}\) and \(\theta = 2\) \(\cite{3}\). For this reason we take the \(\lim \inf\) and \(\lim \sup\), respectively.

Our next result concerns the Galton-Watson tree. Let \(X\) be a random variable with values in \(\mathbb{N}_{0}\). Denote by \(\mathbb{P}_{\text{GW}}\) and \(\mathbb{E}_{\text{GW}}\) the probability and expectation with respect to the Galton-Watson tree with offspring distribution \(X\), respectively. We denote by \(T_{r,n}^{X}\) a realisation of the Galton-Watson tree with root \(r\) cut at level \(n\) (i.e., the tree has \(n\) generations). We consider the quenched measure

\[
\mathbb{P}_{n}^{\theta,X} := \mathbb{E}_{\text{GW}} \left[ \mathbb{P}_{T_{r,n}^{X}}^{(\cdot)} \right].
\]

(2.2)

We have the following theorem:

**Theorem 2.**

1. If there is an \(\varepsilon > 0\) and a \(\beta > 0\), both depending on the distribution of \(X\), such that

\[
\mathbb{E}_{\text{GW}} \left[ e^{-\frac{\theta}{2} X} \right] \leq 1 - \varepsilon
\]

and

\[
\mathbb{E}_{\text{GW}} \left[ \left( \frac{\theta^2 + \theta \beta}{\theta^2 + e^{\theta \beta} - 1} \right)^{X} - \left( e^{-\frac{\theta}{2} \left( 1 + \frac{\beta}{\theta} (1 - \varepsilon) \right)} \right) \right] \geq \varepsilon.
\]
Then
\[ \liminf_{m \to \infty} \inf_{n \geq m} P_{n}^{\theta,X}[E^{T_{r,n}}_{r} > 0]. \]

2. If there is a \( \beta > 0 \), depending on the distribution of \( X \), such that
\[ E_{GW} \left[ X e^{-\frac{\theta}{\beta}} X \left( 1 + \frac{e^{\beta \theta} - 1}{\theta^2} \right)^{\frac{X-1}{\theta^2}} \right] < 1 \]
holds, then
\[ \limsup_{m \to \infty} \sup_{n \geq m} P_{n}^{\theta,X}[E^{T_{r,n}}_{r} > 0] = 0. \]

To illustrate how to make use of the conditions within Theorem 2, we have the following example.

**Example 3.** Consider Poisson distributed offspring \( X \sim \text{Poi}(\mu) \) with \( \mu > 0 \).

1. Let us fix \( a > \theta \) and choose \( \varepsilon \leq \frac{1}{2} \) such that \( 1 - \exp\left( -\frac{a}{\theta} \varepsilon \right) > \varepsilon \). Then for \( \beta := \frac{a}{\mu} \) it holds
\[ E_{GW} \left[ X e^{-\frac{\theta}{\beta}} \left( 1 + \frac{e^{\beta \theta} - 1}{\theta^2} \right) X \right] \]
(2.3)
\[ = e^{-\mu \left( 1 - \frac{e^{\beta \theta} - 1}{\theta^2} \right)} - e^{-\mu \left( 1 - \frac{e^{\beta \theta} - 1}{\theta^2} \right)} \]
(2.4)
\[ \mu \to \infty \quad \beta = a/\mu \]
(2.5)

Now by choosing \( \mu \geq \frac{a}{\theta} \) large enough and estimating
\[ E_{GW} \left[ X e^{-\frac{\theta}{\beta}} X \right] = \exp\left( -\mu \left( 1 - \frac{1}{\beta} \varepsilon \right) \right) \]
(2.6)
\[ \leq \exp\left( -\mu \frac{\beta \theta}{1 + \frac{\theta}{\beta}} \varepsilon \right) \leq \exp\left( -\frac{a}{\theta} \varepsilon \right) < 1 - \varepsilon \]
(2.7)
we see that – with positive probability – there are long loops.

2. On the other hand, by calculating
\[ E_{GW} \left[ X e^{-\frac{\theta}{\beta}} X \left( 1 + \frac{e^{\beta \theta} - 1}{\theta^2} \right)^{\frac{X-1}{\theta^2}} \right] \]
(2.8)
\[ = \mu e^{-\frac{\beta \theta}{\theta^2}} \exp\left[ \mu \left( 1 + \frac{e^{\beta \theta} - 1}{\theta^2} \right) - 1 \right] \]
(2.9)
for fixed \( \mu \) we see that there is a \( \beta_{0} = \beta_{0}(\mu, \theta) \) such that for all \( \beta \leq \beta_{0} \) there almost surely are no infinite loops.

The following corollary gives further sufficient conditions for the distribution of \( X \) and ranges of \( \beta \) where Theorem 2 is applicable. In particular, we will be able to deduce Theorem 1 from this corollary by considering a deterministic offspring distribution.
Corollary 4.
1. Fix \( b \geq a > \theta \) and let \( X \) be an integrable offspring distribution with \( \mathbb{P}_{GW}[X > 0] > 0 \). Furthermore, choose \( c_2 \geq c_1 > 0 \) such that \( B_X := \left[ c_1 \leq \frac{X}{\mathbb{P}_{GW}[X]} \leq c_2 \right] \) has positive probability. Then there is a \( \lambda_0 \in \mathbb{N} \) such that for all \( \lambda \in \mathbb{N} \) with \( \lambda \geq \lambda_0 \) on the Galton-Watson tree with rescaled offspring distribution \( \lambda \cdot X \) and for every \( \beta \leq \frac{c_1}{\mathbb{P}_{GW}[X]} \cdot \frac{1}{c_2} \mathbb{P}_{GW}(B_X) \) we have
\[
\lim \inf_{m \to \infty} \inf_{n \geq m} \mathbb{P}_{\theta, \lambda \cdot X}^{\theta, \lambda \cdot X} [E^{r \to m}_{T, \lambda \cdot X}] > 0.
\]

2. Fix \( \theta < \beta \). Then there is a \( d_0 \in \mathbb{N} \) such that for every \( d \geq d_0 \), every offspring distribution \( X \) bounded above by \( d \) and for all \( \beta \leq \frac{d}{2} \) it holds
\[
\lim \sup_{m \to \infty} \sup_{n \geq m} \mathbb{P}_{\theta, X}^{\theta, X} [E^{r \to m}_{T, \beta \cdot X}] = 0.
\]

Note that within the first part of this corollary larger choices of \( c_2 \) will make \( \lambda_0 \) larger. Moreover, for given \( X \) and \( c_2 \) one should seek to maximize the quantity \( c_1 \mathbb{P}_{GW}(B_X) \leq 1 \) to show existence of long loops for \( \beta \) just above \( \frac{\theta}{\mathbb{P}_{GW}[X]} \).

3 Proofs
For the next three results we will consider the following setting and notation:

Let \( T_0 \) be a fixed finite tree rooted in \( r \) and denote the children of \( r \) by \( x_1, \ldots, x_d \). Furthermore, write \( T_j \) for the subtree of \( T_0 \) rooted in \( x_j \), \( j = 1, \ldots, d \).

We begin by giving estimates on quantities of \( T_0 \) in terms of the corresponding quantities on its subtrees \( T_j \). In particular, Proposition 5 deals with the number \( L_{T_0} \) of loops and Corollary 6 with estimating the partition function of our model. This will enable us to prove occurrence of long loops with the main ingredients being the recursive estimation in Lemma 8 and a certain self-similarity argument within the proof of Lemma 9.

Similarly, to show absence of long loops we will prove exponential decay of the probability for a loop to reach generation \( m \) by a recursive argument and using the same kind of self-similarity as above in Lemma 10.

The following proposition is a basic observation and we will make use of it multiple times.

**Proposition 5.** Given (3.1), we have
\[
- \sum_{j=1}^{d} N_{\beta}^{(r,x_j)} \leq L_{T_0} - \left( \sum_{j=1}^{d} L_{T_j} + 1 \right) \leq \sum_{j=1}^{d} (|N_{\beta}^{(r,x_j)}| - 1) - 1. \tag{3.2}
\]

Note that by \( L_{T_j}(\omega) \) we mean the number of loops on \( T_j \) in a realisation \( \omega \), i.e. only links on the edges \( e \in \mathcal{E}(T_j) \) are taken into account for constructing these loops.

Furthermore, (3.2) becomes an equality for \( N_{\beta}^{(r,x_j)} \in \{0,1\} \).
Proof. The result is immediate once we understood how adding a link to a realisation changes the number of loops. If there are no links between \( r \) and its children the number of loops is given by

\[
L_T^0 = \sum_{j=1}^{d} L_T^j + 1. \tag{3.3}
\]

Adding the first link to an edge will merge the loops at its end points, reducing the number of loops by 1. Hence, we have \( L_T^0 = \sum_{j=1}^{d} L_T^j + 1 - \sum_{j=1}^{d} N_{\beta}^{(r,x,j)} \) in the case that \( N_{\beta}^{(r,x,j)} \in \{0, 1\} \) holds for all \( j \).

Any further link may either merge two loops, split a loop into two or alter a single loop. Therefore the number of loops changes by at most one and the result follows.

The previous estimations on the loop numbers enable us to give estimates on the partition function:

**Corollary 6.** In the setting of (3.1) we obtain the following bounds

\[
\mathbb{E}_{T_0}[\theta^{L_T^0}] \geq \theta e^{-\beta d + \beta d/\theta} \prod_{j=1}^{d} \mathbb{E}_{T_j}[\theta^{L_T^j}] \tag{3.4}
\]

and

\[
\mathbb{E}_{T_0}[\theta^{L_T^0}] \leq \theta e^{-\beta d} \left( 1 + \frac{e^{\beta \theta} - 1}{\theta^2} \right) \prod_{j=1}^{d} \mathbb{E}_{T_j}[\theta^{L_T^j}] \tag{3.5}
\]

Note that we assumed \( \theta \geq 1 \) in the definition of our model. One could define \( \mathbb{P}_{G_\theta} \) in the same way for \( 0 < \theta \leq 1 \) and up to now this would only exchange the upper and lower bound for the partition function in (3.4) and (3.5).

Proof. We calculate using the lower bound from Proposition 5

\[
\mathbb{E}_{T_0}[\theta^{L_T^0}] = \sum_{n_1 \in \mathbb{N}_0} \cdots \sum_{n_d \in \mathbb{N}_0} \mathbb{E}_{T_0} \left[ \mathbf{1} \left[ N_{\beta}^{(r,x,j)} = n_j \forall j \right] \cdot \theta^{L_T^0} \right] \tag{3.6}
\]

\[
\geq \sum_{n_1 \in \mathbb{N}_0} \cdots \sum_{n_d \in \mathbb{N}_0} \mathbb{E}_{T_0} \left[ \prod_{j=1}^{d} \left( 1 + \mathbf{1} \left[ N_{\beta}^{(r,x,j)} = n_j \right] \right) \prod_{j=1}^{d} \left( \theta^{L_T^j} \right) \right] \theta^{-\sum_{j=1}^{d} n_j + 1}. \tag{3.7}
\]

Now the factors in the expectation are independent, therefore we obtain

\[
\mathbb{E}_{T_0}[\theta^{L_T^0}] \geq \theta \cdot \prod_{j=1}^{d} \left( \sum_{n_j \in \mathbb{N}_0} \rho_{T_0} \left[ N_{\beta}^{(r,x,j)} = n_j \right] \mathbb{E}_{T_0} \left[ \theta^{L_T^j} \right] \theta^{-n_j} \right) \tag{3.8}
\]

\[
= \theta \cdot \left( \sum_{k \in \mathbb{N}_0} \frac{\beta^k}{k!} e^{-\beta \theta} \right) \prod_{j=1}^{d} \mathbb{E}_{T_j} \left[ \theta^{L_T^j} \right] \tag{3.9}
\]

\[
= \theta \cdot e^{-\beta d + \beta d/\theta} \prod_{j=1}^{d} \mathbb{E}_{T_j} \left[ \theta^{L_T^j} \right]. \tag{3.10}
\]
Here we used that by the construction of the model we have \( E_{T_0}[\theta^{L_{T_j}}] = E_{T_j} [\theta^{L_{T_j}}] \) as \( L_{T_j} \) only depends on the links on \( \mathcal{E}(T_j) \).

The upper bound on the partition function follows similarly from the second inequality in (3.2):

\[
E_{T_0}[\theta^{L_{T_0}}] \leq \sum_{n_1 \in \mathbb{N}_0} \cdots \sum_{n_d \in \mathbb{N}_0} \prod_{j=1}^d \left( \rho_{T_0} [N^{{r,x}_j}_\beta] = n_j \right) E_{T_0}[\theta^{L_{T_j}}] \theta^{n_j - 1} \theta
\]

(3.11)

\[
= \theta e^{-\beta d} \left( 1 + \sum_{k=1}^{\infty} \frac{\beta^k}{k!} \theta^{k-2} \right) \prod_{j=1}^d E_{T_j}[\theta^{L_{T_j}}]
\]

(3.12)

\[
= \theta e^{-\beta d} \left( 1 + \frac{e^{\beta \theta} - 1}{\theta^2} \right) \prod_{j=1}^d E_{T_j}[\theta^{L_{T_j}}].
\]

(3.13)

The next corollary gives estimates on three events that are of particular interest.

**Corollary 7.** Given (3.1) we define

\[
A_0 := \left[ N^{{r,x}_j}_\beta = 0 \quad \forall j \leq d \right]
\]

and

\[
A := \left[ N^{{r,x}_j}_\beta \leq 1 \quad \forall j \leq d \right].
\]

(3.14)

Then

\[
P_{T_0}^\theta (A) \leq e^{-\beta d/\theta} \quad \text{and} \quad P_{T_0}^\theta (A_0) \geq \left( \frac{\theta^2 + \beta \theta}{\theta^2 + e^{\beta \theta} - 1} \right)^d.
\]

(3.15)

**Proof.** Using (3.2) and (3.4) we can estimate

\[
P_{T_0}^\theta (A_0) = \frac{E_{T_0} \left[ 1_{A_0} \theta^{L_{T_0}} \right]}{E_{T_0} [\theta^{L_{T_0}}]} \leq \frac{\theta (e^{-\beta})^d \prod_{j=1}^d E_{T_j} [\theta^{L_{T_j}}]}{\theta e^{-\beta d + \beta d/\theta} \prod_{j=1}^d E_{T_j} [\theta^{L_{T_j}}]} = e^{-\beta d/\theta}.
\]

(3.16)

Similarly, from (3.2) and (3.5) we have

\[
P_{T_0}^\theta (A) \geq \frac{\theta (e^{-\beta} + e^{-\beta} \beta \theta - 1)^d \prod_{j=1}^d E_{T_j} [\theta^{L_{T_j}}]}{\theta e^{-\beta d} \left( 1 + \frac{e^{\beta \theta - 1}}{\theta^2} \right) \prod_{j=1}^d E_{T_j} [\theta^{L_{T_j}}]}
\]

(3.17)

\[
= \left( \frac{\theta^2 + \beta \theta}{\theta^2 + e^{\beta \theta} - 1} \right)^d.
\]

(3.18)

**3.1 Occurrence of long loops**

To prove existence of long loops we construct a recursive estimation for the probability to reach level \( m \). The following lemma provides a key relation to obtain this.
Lemma 8. In the setting of [3.1] let us consider the events

\[ B^x_{T_j} := \{ \omega : \text{There is a loop in } \mathcal{L}_{T_j}(\omega) \text{ containing } (x, s) \text{ for some } s \in [0, \beta)_{\per} \text{ that fails to reach generation } m \} \]  

(3.19)

with \( x \in \mathcal{V}(T_j), m \in \mathbb{N} \) and \( 0 \leq j \leq d \). Then

\[
\mathbb{P}_{T_0}(A \cap B^x_{T_j}) \leq \sum_{J \subseteq \{1, \ldots, d\}} e^{-\beta d/\theta} \left( \frac{\beta}{\theta} \right)^{|J|} \prod_{j \in J} \mathbb{P}_{T_j}(B^x_{T_j})^{\wedge m-1} 
\]

(3.20)

holds, where \( A \) is defined by (3.14).

**Proof.** For \((x, t) \in \mathcal{V}(T_0) \times [0, \beta)_{\per}, m \in \mathbb{N} \) and \( 0 \leq j \leq d \) let us define the event

\[ B^{(x,t)}_{T_j} := \{ \omega : \text{The loop within } \mathcal{L}_{T_j}(\omega) \text{ that contains } (x, t) \text{ fails to reach generation } m \}, \]

(3.21)

therefore we have and \( B^{(x,t)}_{T_j} \subset B^x_{T_j} \) for every time \( t \). For any subset \( J \subseteq \{1, \ldots, d\} \) we may also write

\[ A_J := \left[ N^t_{\beta}(r, x) \right] \begin{cases} 1 & \text{if } j \in J \\ 0 & \text{else} \end{cases} \]  

(3.22)

Now let us fix \( \omega \in A_J \) and for \( j \in J \) denote by \( t_j = t_j(\omega) \) the time of the link on \( \{r, x_j\} \). Since the loop \( \gamma_{(r,0)} \in \mathcal{L}_{T_0}(\omega) \) containing \( (r, 0) \) fails to reach generation \( m \) iff for all \( j \in J \) the loop within \( \mathcal{L}_{T_j}(\omega) \) containing \( (x_j, t_j(\omega)) \) fails to reach down \( m - 1 \) generations, we have

\[
1 \left[ B^x_{T_0} \right](\omega) = 1 \left[ B^{(r,0)}_{T_0} \right](\omega) = \prod_{j=1}^k 1 \left[ B^{(x_j,t_j(\omega))}_{T_j} \right](\omega) \leq \prod_{j=1}^k 1 \left[ B^x_{T_j} \right]^{\wedge m-1}(\omega). 
\]

(3.23)

(3.24)

Here, (3.23) holds as, by \( \omega \in A_J \), edges containing \( r \) have at most one link, hence \( \{r\} \times [0, \beta) \) is contained in one single loop. Therefore, using Proposition 5 we obtain

\[
\mathbb{E}_{T_0} \left[ 1_{A_J} 1 \left[ B^x_{T_0} \right] \theta^{L_{T_0}} \right] \leq \mathbb{E}_{T_0} \left[ 1_{A_J} \prod_{j \in J} \left( 1 \left[ B^x_{T_j} \right]^{\wedge m-1} \theta^{L_{T_j}} \right) \prod_{j \in J^c} \left( \theta^{L_{T_j}} \right) \right]. 
\]

(3.25)

By independence, this yields

\[
\mathbb{E}_{T_0} \left[ 1_{A_J} 1 \left[ B^x_{T_0} \right] \theta^{L_{T_0}} \right] \leq \theta^{-|J|} \mathbb{P}_{T_0}(A_J) \cdot \prod_{j \in J} \mathbb{E}_{T_0} \left[ 1 \left[ B^x_{T_j} \right]^{\wedge m-1} \theta^{L_{T_j}} \right] \cdot \prod_{j \in J^c} \mathbb{E}_{T_0} \left[ \theta^{L_{T_j}} \right] 
\]

(3.26)

\[
= \theta^{-|J|} \cdot \mathbb{P}_{T_0}(A_J) \cdot \prod_{j \in J} \mathbb{P}_{T_j} \left( B^x_{T_j} \right) \cdot \prod_{j \in J^c} \mathbb{E}_{T_j} \left[ \theta^{L_{T_j}} \right]. 
\]

(3.27)
Using Corollary 6 we now see that

\[
\mathbb{P}_T^\theta \left(A_J \cap B_{T_0}^{x \rightarrow m}\right) = \frac{\mathbb{E}_T^\theta \left[1_{A_J}, 1_{B_{T_0}^{x \rightarrow m}} \right]}{\mathbb{E}_T^\theta} \tag{3.28}
\]

\[
\leq \frac{\theta^{1-|J|} \cdot \beta^{1-|J|} \cdot \prod_{j \in J} \mathbb{P}_T^\theta (B_{T_j}^{x_j \rightarrow m-1}) \cdot \prod_{j=1}^d \mathbb{E}_{T_j} \left[\theta^{1-T_j}\right]}{\theta \cdot e^{-\beta d+\beta d/\theta} \prod_{j=1}^d \mathbb{E}_{T_j} \left[\theta^{1-T_j}\right]}
\]

\[
= e^{-\beta d/\theta} \left(\frac{\beta}{\theta}\right)^{|J|} \prod_{j \in J} \mathbb{P}_T^\theta (B_{T_j}^{x_j \rightarrow m-1}). \tag{3.29}
\]

Therefore, we are able to conclude

\[
\mathbb{P}_T^\theta \left(A \cap B_r^{x \rightarrow m}\right) = \sum_{J \subseteq \{1, \ldots, d\}} \mathbb{P}_T^\theta \left(A_J \cap B_{T_0}^{x \rightarrow m}\right) \leq \sum_{J \subseteq \{1, \ldots, d\}} e^{-\beta d/\theta} \left(\frac{\beta}{\theta}\right)^{|J|} \prod_{j \in J} \mathbb{P}_T^\theta (B_{T_j}^{x_j \rightarrow m-1}). \tag{3.31}
\]

We now turn our attention to the Galton-Watson tree and show the crucial recurrence relation.

**Lemma 9.** Consider the Galton-Watson tree with offspring distribution $X$. Define

\[
\zeta_{m,n} := \mathbb{P}_n^{X \mid B_{T_{X,n}}^{x \rightarrow m}}. \tag{3.33}
\]

The following recursion holds:

\[
1 - \zeta_{m,n} \geq \mathbb{E}_{GW} \left[ \left( \frac{\theta^2 + \beta \theta}{\theta^2 + e^{\beta \theta} - 1} \right)^X - \left( e^{-\beta/\theta} \left( 1 + \beta \frac{\zeta_{m,n}}{\theta - 1} \right) \right)^X \right]. \tag{3.34}
\]

**Proof.** To begin denote by $X_r$ the number of offspring of the root and fix $n,m \in \mathbb{N}$ with $n \geq m$. For a realisation of the Galton-Watson tree, define $T_0 := T_{X,n}$ to be the (random) tree rooted in $r$ that is cut at level $n$. Then we observe that

\[
1 - \zeta_{m,n} = \sum_{d \in \mathbb{N}_0} \mathbb{E}_{GW} \left[ \mathbb{P}_{T_0}^\theta \left( (B_{T_{X,n}}^{x \rightarrow m})^c \right) \mid X_r = d \right] \mathbb{P}_{GW} [X_r = d]. \tag{3.35}
\]

By using the notation from (3.1) for $X_r = d$ and applying Corollary 7 we obtain

\[
\mathbb{P}_{T_0}^\theta \left( (B_{T_{X,n}}^{x \rightarrow m})^c \right) \geq \mathbb{P}_{T_0}^\theta \left( A \cap (B_{T_0}^{x \rightarrow m})^c \right) \geq \mathbb{P}_{T_0}^\theta (A) - \mathbb{P}_{T_0}^\theta (A \cap B_{T_0}^{x \rightarrow m}) \geq \left( \frac{\theta^2 + \beta \theta}{\theta^2 + e^{\beta \theta} - 1} \right)^d \mathbb{P}_{T_0}^\theta (A \cap B_{T_0}^{x \rightarrow m}). \tag{3.36}
\]
The second term can be estimated by using Lemma 8:
\[ \mathbb{P}_{\theta, T_0} (A \cap B_{T_0}^{x \to m}) \leq \sum_{J \subseteq \{1, \ldots, d\}} e^{-\beta d/\theta} \left( \frac{\beta}{\theta} \right)^{|J|} \prod_{j \in J} \mathbb{P}_{\theta, T_j} \left( B_{T_j}^{x \to m-1} \right). \] (3.39)

Hence by taking the expectation we have
\[ \mathbb{E}_{GW} \left[ \frac{\mathbb{P}_{\theta, T_0} (A \cap B_{T_0}^{x \to m})}{X_r = d} \right] \leq \sum_{J \subseteq \{1, \ldots, d\}} e^{-\beta d/\theta} \left( \frac{\beta}{\theta} \right)^{|J|} \prod_{j \in J} \mathbb{E}_{GW} \left[ \mathbb{P}_{\theta, T_j} \left( B_{T_j}^{x \to m-1} \right) \right]. \] (3.41)

By self-similarity in expectation, this yields
\[ = \sum_{k=0}^{d} \binom{d}{k} e^{-\beta d/\theta} \left( \frac{\beta}{\theta} \right)^k \left( \zeta_{n-1}^m \right)^k \] (3.42)
\[ = e^{-\beta d/\theta} \left( 1 + \frac{\beta}{\theta} \zeta_{n-1}^m \right)^d. \] (3.43)

The result follows.

We can now use this recursion to prove the occurrence of long loops in the Galton-Watson tree under our assumptions on the distribution of $X_r$.

Proof of Theorem 2, part 1. We proceed via induction on $m$ to prove that, for all $m \in \mathbb{N}$ and all $n \geq m$, we have
\[ 1 - \zeta_m^n \geq \varepsilon. \] (3.44)
This is sufficient as we have $(B_{T_n}^{x \to m})^c \subset E_{T_n}^{x \to m}$. For the base step note that, given $X_r = d$, we have $B_{T_n}^{x \to 1} = A \emptyset$. Therefore we have
\[ \zeta_1^n = \sum_{d \in \mathbb{N}_0} \mathbb{E}_{GW} \left[ \frac{\mathbb{P}_{\theta, T_n} (A \emptyset)}{X_r = d} \right] \mathbb{P}_{GW}[X_r = d] \] (3.45)
\[ \leq \sum_{d \in \mathbb{N}_0} e^{-\beta d/\theta} \mathbb{P}_{GW}[X_r = d] \leq 1 - \varepsilon, \] (3.46)
where the first inequality uses Corollary 7 and the last inequality follows from our assumption on the distribution of $X$. Assume now the induction hypothesis holds for $m-1$ and all $\tilde{n} \geq m - 1$. By Lemma 9, our induction hypothesis and our assumptions on the distribution of $X$ we can estimate
\[ 1 - \zeta_m^n \geq \mathbb{E}_{GW} \left[ \left( \frac{\theta^2 + \beta \theta}{\theta^2 + e^{\beta \theta} - 1} \right)^X - \left( e^{-\beta/\theta} \left( 1 + \frac{\beta}{\theta} \zeta_{n-1}^{m-1} \right) \right)^X \right] \] (3.47)
\[ \geq \mathbb{E}_{GW} \left[ \left( \frac{\theta^2 + \beta \theta}{\theta^2 + e^{\beta \theta} - 1} \right)^X - \left( e^{-\beta/\theta} \left( 1 + \frac{\beta}{\theta} (1 - \varepsilon) \right) \right)^X \right] \] (3.48)
\[ \geq \varepsilon. \] (3.49)
Proof of Corollary 4, part 1. For \( \lambda \in \mathbb{N} \) write \( X_\lambda := \lambda \cdot X \), \( d_\lambda := \mathbb{E}_{GW}[X_\lambda] \). Now choose \( \varepsilon > 0 \) such that the inequalities

\[
\mathbb{E}_{GW} \left[ 1 - \exp \left( -\frac{a}{\bar{\theta} c_1 \mathbb{P}_{GW}(B_X)} X \right) \right] \geq \varepsilon \tag{3.50}
\]

and

\[
\exp \left( -\frac{a}{\bar{\theta} \mathbb{P}_{GW}(B_X)} \varepsilon \right) < 1 - \frac{\varepsilon}{\mathbb{P}_{GW}(B_X)} \tag{3.51}
\]

are fulfilled. Using \( \mathbb{P}_{GW}[X > 0] > 0 \) and \( \bar{\theta} > 1 \), this possible. Since

\[
1 \geq \left( 1 + \frac{1}{\bar{\theta}^2} \sum_{k=2}^{\infty} \frac{\bar{\theta}^k}{k!} \right)^{-d_\lambda} \tag{3.52}
\]

holds as \( \bar{d}_\lambda = \lambda \cdot \mathbb{E}_{GW}[X] \xrightarrow{\lambda \to \infty} \infty \), we may find a \( \lambda_0 \in \mathbb{N} \) such that for all \( \lambda \geq \lambda_0 \) we have

\[
\left( 1 + \frac{1}{\bar{\theta}^2} \sum_{k=2}^{\infty} \frac{b_\theta}{c_1 \mathbb{P}_{GW}(B_X) d_\lambda} \right)^{-d_\lambda} \tag{3.53}
\]

\[
\geq 1 - \left( \frac{\varepsilon}{\mathbb{P}_{GW}(B_X)} - \exp \left( -\frac{a}{\bar{\theta} \mathbb{P}_{GW}(B_X)} \varepsilon \right) \right) \tag{3.54}
\]

\[
= \frac{\varepsilon}{\mathbb{P}_{GW}(B_X)} + \exp \left( -\frac{a}{\bar{\theta} \mathbb{P}_{GW}(B_X)} \varepsilon \right) \tag{3.55}
\]

Then, by the lower bound on \( \beta \) and (3.50), we have

\[
\mathbb{E}_{GW} \left[ e^{-\beta X_\lambda} \right] \leq \mathbb{E}_{GW} \left[ \exp \left( -\frac{a}{\bar{\theta} c_1 \mathbb{P}_{GW}(B_X)} \lambda \mathbb{E}_{GW}[X] \right) X_\lambda \right] \leq 1 - \varepsilon, \tag{3.56}
\]

therefore \( X_\lambda \) satisfies the first condition of Theorem 2, part 1. Furthermore, on \( B_X = [c_1 d_\lambda \leq X_\lambda \leq c_2 d_\lambda] \) and taking \( \beta \in \left( \frac{1}{d_\lambda c_1 \mathbb{P}_{GW}(B_X)} [a, b] \right) \), we can estimate

\[
\left( \frac{\theta^2 + \beta \theta}{\theta^2 + \varepsilon \beta} - 1 \right)^{X_\lambda} = \left( 1 + \frac{1}{\bar{\theta}^2 + \beta \theta} \sum_{k=2}^{\infty} \frac{(\beta \theta)^k}{k!} \right)^{-X_\lambda} \tag{3.57}
\]

\[
\geq \left( 1 + \frac{1}{\bar{\theta}^2} \sum_{k=2}^{\infty} \frac{\bar{\theta}^k}{k!} \right)^{-c_2 d_\lambda} \tag{3.58}
\]

\[
\geq \frac{\varepsilon}{\mathbb{P}_{GW}(B_X)} + \exp \left( -\frac{a}{\bar{\theta} \mathbb{P}_{GW}(B_X)} \varepsilon \right) \tag{3.59}
\]
as well as

\[
\left( e^{-\frac{\theta}{\beta}} \left( 1 + \frac{\beta}{\theta} (1 - \varepsilon) \right) \right)^{X_\lambda} \leq \left( e^{-\frac{\theta}{\beta} \varepsilon} e^{-\frac{\theta}{\beta} (1 - \varepsilon)} \left( 1 + \frac{\beta}{\theta} (1 - \varepsilon) \right) \right)^{X_\lambda} \leq \exp \left( -\frac{1}{\theta c_1 p_{GW}(B_X) d_\lambda} \varepsilon c_1 d_\lambda \right) \tag{3.60}
\]

\[
\leq \exp \left( -\frac{a}{\theta p_{GW}(B_X)} \varepsilon \right). \tag{3.61}
\]

This yields that

\[
\mathbb{E}_{GW} \left[ \left( \frac{e^{\theta^2 + \beta \theta}}{\theta^2 + e^{\beta \theta} - 1} \right)^{X_\lambda} - \left( e^{-\frac{\theta}{\beta} (1 + \frac{\beta}{\theta} (1 - \varepsilon))} \right)^{X_\lambda} \right] \geq \left[ \frac{\varepsilon}{\theta p_{GW}(B_X)} + \exp \left( -\frac{a}{\theta p_{GW}(B_X)} \varepsilon \right) - \exp \left( -\frac{a}{\theta p_{GW}(B_X)} \varepsilon \right) \right] \mathbb{P}_{GW}(B_X) \geq \varepsilon \tag{3.62}
\]

and using the first part of Theorem 2 we obtain the result. \(\square\)

**Proof of Theorem 1, part 1.** We can deduce this from Corollary 4 by considering the deterministic offspring distribution \(X = 1\) and setting \(c_1 := c_2 := 1\), hence \(\mathbb{P}_{GW}(B_X) = 1\). \(\square\)

### 3.2 Absence of long loops

We start by considering the Galton-Watson tree. The following lemma is sufficient to prove Theorem 2, part 2, as it even shows exponential decay.

**Lemma 10.** For \(\sigma^m_n := \mathbb{P}^{\theta, X} \left( E^{r \rightarrow m}_{T^X_{r,n}} \right) \),

\[
\tilde{q} := \mathbb{E}_{GW} \left[ X e^{-X^{\beta}/\theta} \left( 1 + \frac{e^{\beta \theta} - 1}{\theta^2} \right)^{X-1} \right] e^{\beta \theta} = \frac{1}{\theta^2} \tag{3.63}
\]

and all \(n, m \in \mathbb{N}\) with \(n \geq m\) we have \(\sigma^m_n \leq \tilde{q}^{n-1}\).

**Proof.** Let us fix \(m, n \in \mathbb{N}\) and write \(T_0 := T^X_{r,n}\) for a realisation of the Galton-Watson tree rooted in \(r\) and cut after \(n\) generations. Furthermore, given such a realisation, consider the setting of (3.1), in particular \(d := X_r\) is the number of children of the root \(r\). Then for \(J \subseteq \{1, \ldots, d\}\) and on

\[
\hat{A}_J := \left[ N^{(r, x_j)}_\beta \begin{cases} 1 & \text{if } j \in J \\ 0 & \text{otherwise} \end{cases} \right] \geq 1 \quad \text{if } J \subseteq \{1, \ldots, d\} \quad \text{and } \quad \text{otherwise} \tag{3.64}
\]

by Proposition 5 we have

\[
L_{T_0} \leq 1 + \sum_{j=1}^{d} L_{T_j} - 2 |J| + \sum_{j \in J} N^{(r, x_j)}_\beta. \tag{3.65}
\]
Furthermore, being given the event $E_{T_0}^{r,m} \cap \hat{A}_J$ means that we can find at least one $i \in J$ such that there is a loop within $T_i$ containing $x_i$ and reaching $m - 1$ generations. Therefore we have

$$E_{T_0}^{r,m} \cap \hat{A}_J \subseteq \bigcup_{i \in J} E_{T_i}^{x_i,m-1} \cap \hat{A}_J.$$  

(3.69)

Using Corollary 6 we obtain

$$\mathbb{P}_{T_0}^\theta \left( E_{T_0}^{r,m} \cap \hat{A}_J \right) \leq \sum_{i \in J} \mathbb{E}_{T_0} \left[ 1_{E_{T_i}^{x_i,m-1}} 1_{\hat{A}_J} \theta^{1-2|J|+\sum_{j \in J} X_j^{(r,x_j)}} \prod_{j=1}^d \theta^{L_{T_j}} \right]$$

(3.70)

$$= e^{-\beta d/\theta} \theta^{-2|J|} \prod_{j=1}^d \left( e^{\beta \theta} - 1 \right)^{|J|} \sum_{i \in J} \mathbb{P}_{T_i}^\theta (E_{T_i}^{r,x_i,m-1}).$$

(3.71)

This yields

$$\mathbb{E}_{GW} \left[ \mathbb{P}_{T_0}^\theta (E_{T_0}^{r,m}) \left| X_r = d \right. \right]$$

(3.72)

$$= \sum_{J \subseteq \{1,\ldots,d\}} \mathbb{E}_{GW} \left[ \mathbb{P}_{T_0}^\theta (E_{T_0}^{r,m} \cap \hat{A}_J) \left| X_r = d \right. \right]$$

(3.73)

$$\leq \sum_{J \subseteq \{1,\ldots,d\}} e^{-\beta d/\theta} \left( \frac{e^{\beta \theta} - 1}{\theta^2} \right)^{|J|} \sum_{i \in J} \mathbb{E}_{GW} \left[ \mathbb{P}_{T_i}^\theta (E_{T_i}^{r,x_i,m-1}) \right]$$

(3.74)

$$= \sum_{J \subseteq \{1,\ldots,d\}} e^{-\beta d/\theta} \left( \frac{e^{\beta \theta} - 1}{\theta^2} \right)^{|J|} \mathbb{E}_{GW} \left[ \mathbb{P}_{T_0}^\theta (E_{T_0}^{r,m}) \left| X_r = d \right. \right]$$

(3.75)

where we used self-similarity in expectation. Therefore we conclude

$$\sigma_m^n = \sum_{d \in \mathbb{N}_0} \mathbb{P}_{GW} [X_r = d] \cdot \mathbb{E}_{GW} \left[ \mathbb{P}_{T_0}^\theta (E_{T_0}^{r,m}) \left| X_r = d \right. \right]$$

(3.76)

$$\leq \sigma_{n-1}^{m-1} \sum_{d \in \mathbb{N}_0} \mathbb{P}_{GW}[X_r = d] \cdot e^{-\beta d/\theta} \sum_{k=0}^d \binom{d}{k} \left( \frac{e^{\beta \theta} - 1}{\theta^2} \right)^k$$

(3.77)

$$= \sigma_{n-1}^{m-1} \sum_{d \in \mathbb{N}_0} \mathbb{P}_{GW}[X_r = d] \cdot e^{-\beta d/\theta} d \frac{e^{\beta \theta} - 1}{\theta^2} \left( 1 + \frac{e^{\beta \theta} - 1}{\theta^2} \right)^{d-1}$$

(3.78)

$$= \sigma_{n-1}^{m-1} \mathbb{E}_{GW} \left[ X e^{-X \beta/\theta} \left( 1 + \frac{e^{\beta \theta} - 1}{\theta^2} \right)^{-X-1} \right] e^{\beta \theta} - 1.$$
Proof of Corollary 4, part 2. By making use of the estimation
\[
e^{-X\beta/\theta} \left(1 + \frac{e^{\beta\theta} - 1}{\theta^2}\right)^{X-1}
\leq \exp \left(-X\frac{\beta}{\theta} + X\frac{e^{\beta\theta} - 1}{\theta^2} \right) \cdot \left[ \exp \left(-\frac{e^{\beta\theta} - 1}{\theta^2} \right) \left(1 + \frac{e^{\beta\theta} - 1}{\theta^2} \right)^{X-1} \right] \leq 1
\] (3.80)
we see that for an offspring distribution $X$ bounded by $d \in \mathbb{N}$ and for $\beta \leq \frac{q}{d}$ we calculate
\[
\mathbb{E}_{GW} \left[ X e^{-X\beta/\theta} \left(1 + \frac{e^{\beta\theta} - 1}{\theta^2}\right)^{X-1} \right] \frac{e^{\beta\theta} - 1}{\theta^2}
\leq d \cdot \exp \left(d\frac{e^{\beta\theta} - \beta\theta - 1}{\theta^2} \right) \frac{e^{\beta\theta} - 1}{\theta^2}
\leq \frac{d}{\theta^2} \left(e^{q\theta/d} - 1\right) \exp \left(\frac{d}{\theta^2} \left(e^{q\theta/d} - \frac{q\theta}{d} - 1\right) \right)
= c_d.
\] (3.85)

Since $\lim_{d \to \infty} c_d = \frac{q}{d} < 1$ holds, we may pick $d_0 \in \mathbb{N}$ such that for all $d \geq d_0$ the condition in the second part of Theorem 2 is fulfilled.

Proof of Theorem 1, part 2. This follows from the second part of Corollary 4 by picking the deterministic offspring distribution $X = d$.

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References

[1] Aizenman, M. and Nachtergaele, B. Geometric aspects of quantum spin states. Comm. Math. Phys., 164(1):17–63, 1994.

[2] Angel, O. Random infinite permutations and the cyclic time random walk. Discrete Math. Theor. Comput. Sci. Proc., :9–16, 2003.

[3] Benassi, C., Lees, B. and Ueltschi, D. Correlation inequalities for the quantum XY model. J. Stat. Phys., 164(1):1157–1166, 2016.

[4] Björnberg, J. E. and Ueltschi, D. Critical parameter of random loop model on trees. arXiv:1608.08473 2017.

[5] Björnberg, J. E. and Ueltschi, D. Critical temperature of Heisenberg models on regular trees, via random loops. arXiv:1803.11430 2018.
[6] Dyson, F. J., Lieb, E. H. and Simon, B. Phase transitions in quantum
spin systems with isotropic and nonisotropic interactions. J. Stat. Phys.,
18(4):335–383, 1978.

[7] Hammond, A. Infinite cycles in the random stirring model on trees. Bull.
Inst. Math. Acad. Sin., 8(4):85–104, 2013.

[8] Hammond, A. Sharp phase transition in the random stirring model on
trees. Probab. Theory Rel. Fields, 161(3-4):429–448, 2015.

[9] Kennedy, T. and Lieb, E.H. and Shastry, B.S. The XY model has long-
range order for all spins and all dimensions greater than one. J. Stat. Phys.,
53(5-6):1019–1030, 1988.

[10] Schramm, O. Compositions of random transpositions. Isreal J. Math., 147
(5-6):221–243, 2005.

[11] Tóth, B. Improved lower bound on the thermodynamic pressure of the spin
1/2 Heisenberg ferromagnet. Lett. Math. Phys., 28(1):75–84, 1993.

[12] Ueltschi, D. Random loop representations for quantum spin systems. J.
Math. Phys., 54, 083301, 2013; arXiv:1301.0811.

[13] Ueltschi, D. Universal behaviour of 3D loop soup models. arXiv:1703.09503.