11-Dimensional Supergravity Compactified on Calabi–Yau Threefolds

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Abstract

We consider generic features of eleven dimensional supergravity compactified down to five dimensions on an arbitrary Calabi–Yau threefold.

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Recently, it has been suggested that eleven dimensional supergravity may arise as an effective theory of some string theories in their strong coupling regime \[1\]. In particular, compactification of \( D = 11 \) supergravity to diverse dimensions \( D < 11 \) contains a Kaluza–Klein spectrum which is a natural candidate, as shown by Witten, for some non perturbative BPS states of string theories. In this note we fill a gap in this analysis by considering \( D = 11 \) supergravity compactified to five dimensions on an arbitrary Calabi–Yau manifold with Hodge numbers \((h_{(1,1)}, h_{(2,1)})\) and intersection matrix \( d_{\Lambda \Sigma \Delta} \) \((\Lambda, \Sigma, \Delta = 1, \ldots, h_{(1,1)})\). Here we will only report the generic structure which emerges in doing this analysis, while the complete action of the theory will be given elsewhere.

The five dimensional theory obtained in this way happens to contain the gravity multiplet\[2\]

\[
(e_{a\mu}, \psi_{\mu I}, A_\mu) \quad (I = 1, 2),
\]

\( h_{(1,1)} - 1 \) vector multiplets

\[
(A^A_\mu, \lambda^A_I, \phi^A) \quad (A = 1, \ldots, h_{(1,1)} - 1),
\]

and \( h_{(2,1)} + 1 \) hypermultiplets

\[
(\zeta^m, A^m_I) \quad (m = 1, \ldots, 2(h_{(2,1)} + 1)).
\]

It is convenient to introduce a vector index \( \Lambda = 1, \ldots, h_{(1,1)} \) which covers also the graviphoton. Then, the entire coupling of vector multiplets to five dimensional supergravity is specified, as shown in \[3\], by the intersection numbers \( d_{\Lambda \Sigma \Delta} \) which, in particular, express the coupling of the 5D topological term

\[
\int d^5 x \ d_{\Lambda \Sigma \Delta} F^\Lambda \wedge F^\Sigma \wedge A^\Delta.
\]

Before deriving the results, let us first show how the counting of degrees of freedom for the bosonic fields is obtained. In \( D = 11 \) we have a pure geometrical theory (with no coupling constant) containing the metric \( G_{\hat{\mu} \hat{\nu}} \) and a three-form gauge field \( A_{\hat{\mu} \hat{\nu} \hat{\rho}} \). On a Calabi–Yau threefold with Hodge numbers \((h_{(1,1)}, h_{(2,1)})\) we obtain the following degrees of freedom\[4\] (the fermions, that we neglect here, just complete the multiplets), splitting \( \hat{\mu} = (\mu, i, \bar{\tau}) \), \((\mu = 1, \ldots, 5, i, \bar{\tau} = 1, 2, 3)\): the graviton \( G_{\mu \nu} \), \( h_{(2,1)} \) complex scalars \( G_{ij} \), \( h_{(1,1)} \) real scalars \( G_{i\bar{\tau}} \), one real scalar \( A_{\mu \nu \rho} \), \( h_{(1,1)} \) vectors \( A_{\mu i} \), \( h_{(2,1)} \) complex scalars \( A_{ijk} \) and one complex scalar \( A_{ijk} = \epsilon_{ijk} C \).
So we get, as promised, $h_{(2,1)} + 1$ hypermultiplet scalars $(G_{ij}, A_{ijk}, A_{\mu \nu \rho}, V, C)$, $h_{(1,1)} - 1$ vector multiplet scalars $(G_{i\overline{j}}$ except the volume) and $h_{(1,1)}$ vector fields $A_{\mu i\overline{j}}$.

In the decomposition of the Kähler form\[5\]

$$J = \sum_{\Lambda=1}^{h_{(1,1)}} M_{\Lambda} V^{\Lambda} \quad (V^{\Lambda} \in H^{(1,1)})$$

one can extract the volume modulus

$$\mathcal{V} = \frac{1}{3!} \int J \wedge J \wedge J$$

and then consider moduli coordinates

$$(t_{\Lambda} = \frac{M_{\Lambda}}{\mathcal{V}^{3}}, \mathcal{V})$$

such that $\mathcal{V}(t_{\Lambda}) = 1$. This is the natural splitting in five dimensions. In fact, it is easy to see that $(\mathcal{V}, A_{\mu \nu \rho}, C)$ then becomes an universal hypermultiplet, present in any Calabi–Yau compactification \[6\], which of course has its counterpart in the dimensionally reduced theory in $D = 4$. If $h_{(2,1)} = 0$, this multiplet belongs to the $SU(2)\times U(1)$ quaternionic manifold, as shown in \[3\].

In five dimensions, the vector multiplet moduli space will just be the hypersurface $\mathcal{V} = 1$ of the classical Kähler cone\[3\], which is related to the moduli space of the $M_{\Lambda}$. This is precisely a space of the general form allowed by 5D supergravity studied in \[4\]. Note that the quantum moduli space for the $H^2$-cohomology (obtained by mirror symmetry\[4\]) is not allowed by 5D supergravity\* because of the absence of the antisymmetric tensor $B_{\mu \nu}$. This is similar to the analysis made in \[4\], in the compactification of $D = 11$ supergravity on $K3$ down to $D = 7$. There, the $K3$ moduli space was also the classical one, i.e. $SO(3,19)/SO(3)\times SO(19)$ for precisely the same reason. On the other hand, the quaternionic manifold is compatible with the moduli space of the complex structure of the Calabi–Yau and then will in fact agree with what is usually obtained by the c-map \[3,17\]. Therefore, even in 5D, the quaternionic metric will be

\* The absence of world-sheet instanton corrections to the Kähler metric in $D = 5$ seems to be related, by duality, to the absence of logarithmic infrared singularities in the heterotic counterpart. Infact, an holomorphicity argument, by further reduction to $D = 4$, confirms the above statement. This would give even more evidence that world-sheet instantons on Calabi–Yau \[8,13\] are dual, in $D = 4$, to non perturbative singularities of $N = 2$ microscopic Yang–Mills theories\[16\] (described by heterotic strings on $N = 2$ vacua).
parametrized in terms of the prepotential $F$ of the special geometry of the deformation of the three-form complex cohomology. It is then obvious that the asymmetry between $H^2$ and $H^3$ is a pure five dimensional phenomenon.

Let us briefly mention how the above results are actually derived. One starts with the eleven dimensional geometrical theory of Cremmer, Julia and Scherk[18]

$$L_{11} = -\frac{1}{2} \hat{e}_{11} \hat{R} - \frac{1}{48} \hat{e}(\hat{F}_{\mu_1 \mu_2 \mu_3 \mu_4})^2 + \frac{\sqrt{2}}{124} \hat{e}_{\mu_1 \ldots \mu_11} \hat{F}_{\mu_1 \ldots \mu_4} \hat{F}_{\mu_5 \ldots \mu_8} \hat{A}_{\mu_9 \ldots \mu_{11}}.$$ (8)

Compactification on a Calabi–Yau threefold is obtained, using the results of ref. [19]. For instance, from the Einstein term we obtain

$$L_5 = e_5 \left\{ -\frac{1}{2} \mathcal{V}_5(M) R_5 + \mathcal{V}_5(M) \partial_\mu z^\alpha \partial_\mu \bar{z}^\beta G_{\alpha \beta} - \frac{1}{2} \partial_\mu M^\Lambda \partial_\mu M^\Sigma [\mathcal{V}_5(M) G_{\Lambda \Sigma} + K_{\Lambda \Sigma}] \right\},$$ (9)

where we have used the decomposition[20][19]

$$i \delta G_{ij} = \sum_{\Lambda=1}^{h_{(1,1)}} M^\Lambda \mathcal{V}_{ij}^\Lambda$$

$$\delta G_{ij} = \sum_{\alpha=1}^{h_{(2,1)}} \bar{z}^\alpha \delta_{\alpha ij},$$ (10)

and we have defined

$$3! \mathcal{V}(M) \equiv K = d_{\Lambda \Sigma \Delta} M^\Lambda M^\Sigma M^\Delta$$

$$G_{\Lambda \Sigma} = -\frac{1}{2} \partial M^\Lambda \partial M^\Sigma \log K = -3 \frac{K_{\Lambda \Sigma}}{K} + \frac{9 K_{\Lambda} K_{\Sigma}}{2 K^2}$$

$$(G_{\Lambda \Sigma} M^\Sigma = \frac{3 K_{\Lambda}}{2 K}, G_{\Lambda \Sigma} M^\Lambda M^\Sigma = \frac{3}{2}).$$ (11)

By making a Weyl rescaling to bring the Einstein term to the canonical form, the previous term becomes

$$e_5^{-1} L_5 = -\frac{R_5}{2} + G_{\alpha \beta} \partial_\mu z^\alpha \partial_\mu \bar{z}^\beta - \frac{1}{2} G_{\Lambda \Sigma} \partial_\mu t^\Lambda \partial_\mu t^\Sigma$$

$$- \frac{1}{4} (\partial_\mu \log \mathcal{V}_5(M))^2$$

(12)
with $V_5(t) = 1$ and $M^\Lambda = t^A V_5(M)^{1/3}$. The $(h_{(1,1)} - 1) t^A$ fields are precisely the coordinates $\phi^A$ of the moduli space of 5D supergravity with vector multiplet coupling fixed in terms of the $d_{\Lambda \Sigma \Delta}$ symmetric symbols of ref. [3]. The other terms are part of the hypermultiplets kinetic term, which now also include the volume modulus $V(M)$.

It is easy to see, repeating a calculation similar to that of ref. [21], that in absence of the $z^\alpha$ scalars ($h_{(2,1)} = 0$), the coupling from the $\hat{F}$ four-forms, after dualizing $A_{ijk}$ to a scalar field, reproduces, together with $A_{ijk} = \epsilon_{ijk} C$, the one-dimensional quaternionic space $SU(2,1)/SU(2) \times SU(1)$, which was inferred in ref. [6] and explicitly constructed in refs. [21,17].

When the complex structure scalars are turned on, then one obtains a quaternionic manifold identical to that discussed in refs. [6,19] ⋆.

To make contact with string theories, we must proceed to further compactify the 5D theory on $S_1$, so that we obtain a 4D theory which is the compactification of $D = 11$ supergravity on $CY \times S_1$ †.

By introducing the 5-dimensional radius $\phi_5$, and confining our discussion to the $H^2$ moduli, we obtain after Weyl rescaling

$$ e^{-1} \mathcal{L}_4 = -\frac{1}{2} R_4 - \frac{1}{2} G_{\Lambda \Sigma}(t) \partial_\mu t^\Lambda \partial_\mu t^\Sigma - \frac{1}{12} \left[ \partial_\mu (\log \phi_5^3) \right]^2 - \frac{1}{4} (\partial_\mu \log V_5(M))^2. \tag{13} $$

We can now compare this lagrangian with the one obtained by compactifying 10D Type IIA theory on a Calabi–Yau threefold to $D = 4$ [19]. By splitting in this case the $H^2$ moduli in $(t^A, \sqrt{V}(v))$, $(\sqrt{V}(t) = 1)$ we get

$$ -\frac{1}{2} R_4 - \frac{1}{2} G_{\Lambda \Sigma}(t) \partial_\mu t^\Lambda \partial_\mu t^\Sigma - \frac{1}{12} (\partial_\mu \log V(v))^2 - \frac{1}{4} \left[ \partial_\mu (\log V(v) \phi^{-3}) \right]^2, \tag{14} $$

where $\phi$ is the 10D dilaton field, related to the 10 dimensional Yang–Mills heterotic gauge coupling through the relation $g^{-2}_{YM} \sim \phi^{-3/4}$. We see that $\phi_3^3 = V(v)$, while the four dimensional dilaton is $V_5(M)$. By observing that [19]

$$ v^A = \phi^{3/4} M^\Lambda, \quad V(v) = \phi^{3/4} V_4(M) \tag{15} $$

⋆ We observe that the full lagrangian, including hypermultiplets, can be obtained in a straightforward way either by reduction from $D = 6$ [22], or directly in $D = 5$ using the techniques developed in ref. [23]. Quaternionic spaces encompassing the dynamics of hypermultiplets are the same for $D = 6,5$ and 4. However, one difference in the coupling to fermions is that in $D = 5$, being the scalar manifold real and the fermions non-chiral, there is no Kähler connection in the covariant derivative of fermions [24].

† Notice that in $D = 4$ the $h_{(1,1)}$ moduli can be complexified by the additional $A_{5,\gamma}$ scalars, and a new vector multiplet comes from the metric massless degrees of freedom $G_5, G_{55}$. 

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we then obtain

\[ \phi_5 = \phi^{3/4} \mathcal{V}_4(M)^{1/3}, \quad \mathcal{V}_5(M) = \phi^{-3/4} \mathcal{V}_4(M). \]  

(16)

These formulae show that \( \phi_5, \mathcal{V}_5(M) \) are the generalization of \( \text{Re}T, \text{Re}S \) introduced in ref. \( [25] \).

If one assumes S-T duality\[26\], the Calabi–Yau moduli are related to the string coupling constants of heterotic strings and the previous theory may be used to investigate some non-perturbative properties of heterotic string theories. In particular, it was shown in ref.\[1\] that some BPS states of 5D heterotic strings have quantum numbers related to Yang–Mills instanton charge. On the 11D supergravity side, these states should come from two-branes\[27\]\[28\][12] wrapping around closed two-surfaces \( A \) on Calabi–Yau manifolds

\[ \int_{A \times S^2} F \]  

(17)

where \( A \) is a two-cycle and \( S_2 \) is a two-sphere on \( M_4 \). This gives another hint that space-time instantons and world-sheet instantons are actually different descriptions of the same physical entities.

Let us further consider the moduli space of vector multiplets in 5D heterotic string theory. For a string compactified on \( K3 \times S_1 \) this space is \[3\] *

\[ \frac{SO(1,n-1)}{SO(n-1)} \times SO(1,1), \]  

(18)

where the \( SO(1,1) \) is the dilaton vector multiplet (including the \( B_{\mu\nu} \) field, dual to a vector) and the total number of vector multiplets is \( n \). By further reduction to \( D = 4 \)\[3\] this manifold becomes the special Kähler manifold \[32, 9\]

\[ \frac{SO(2,1)}{SO(2)} \times \frac{SO(2,n)}{SO(2) \times SO(n)}, \]  

(19)

including \( n + 1 \) vector multiplets. In heterotic string theory, the \( d_{\Lambda\Sigma\Delta} \) symbol is just \( d_{1AB} = \eta_{AB} \), with signature \( (1, n - 1) \), and vanishes otherwise. It would then seem

* Note that this manifold should not be confused with the \( SO(1,n)/SO(n) \) manifold of \( n \) tensor multiplets coupled to \( N = 1, D = 6 \) supergravity\[25\]. In heterotic strings on \( K3 \)[30], there is only one tensor multiplet (containing the dilaton) which yields the \( SO(1,1) \) part. The \( SO(1,n-1)/SO(n-1) \) space in \( D = 5 \) is the Narain moduli space \[31\] (modulo global modifications) of a circle compactification.

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that the dual Calabi–Yau manifold should have the same intersection form, possibly restricting the allowed Calabi–Yau manifolds which are dual candidates to heterotic theories. This fact has recently been suggested in [10] and verified for some dual candidates in [33,13].

As a final remark, we would like to speculate that if $D = 11$ supergravity is taken as a serious non perturbative description of strings, then a mechanism of supersymmetry breaking may be possible, that is the Scherk-Schwarz mechanism [34], which has been already used in heterotic string models [35]. Since the gravitino gets in this case a Kaluza–Klein BPS mass [36], by string duality one would expect that this reflects in a non perturbative gravitino mass term in the heterotic string coupling. This may also suggest that space-time instantons, in heterotic strings, may induce supersymmetry breaking.

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