Unfactorizing Polychromatic Penguins.

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Firstly, I report on a recent work –carried out with T. Hambye and E. de Rafael– on how to use the large-$N_c$ expansion to calculate unfactorized contributions from the strong penguin operators, and their impact on observables such as $\epsilon'/\epsilon$ and the $\Delta I = 1/2$ rule. Secondly, based on work done with M. Golterman, I explain how this calculation allows one to predict some rather dramatic consequences for quenched QCD. This may help explain the present discrepancy between lattice and experimental results for $\epsilon'/\epsilon$. The emphasis of this article is put on the explanation of the method of calculation used, which is fully analytic. This allows one to build some intuition and understand the role played by the different hadronic scales in determining the size of the different contributions.

Due to the disparity of scales between the kaon and the W masses, the decay process $K \rightarrow \pi\pi$ has very large logarithms, such as $\log M_W/M_K$. The technique of Effective Field Theory allows one to resum all these large logarithms as the renormalization group running of certain new couplings (the so-called Wilson coefficients). These coefficients appear in the Effective Lagrangian where the W and all the other heavy particles of the Standard Model are no longer present, i.e. they have been “integrated out”. Therefore, in this Effective Lagrangian only the u,d, s quarks and the gluon fields appear explicitly, the effect of the heavy particles being encoded in a set of 10 Wilson coefficients and equal number of four-quark operators, i.e.

$$\mathcal{L}_{\text{eff}} = \sum_{AB} C_{AB}(\mu) \, Q_{AB}(\mu) ,$$

where $Q_{AB}(\mu) = \bar{q} \Gamma_A q(x) \, \bar{q} \Gamma_B q(x)$ and $\Gamma_{A,B}$ are matrices in Dirac space (color indices are suppressed). This Effective Lagrangian, $\mathcal{L}_{\text{eff}}$, changes the short-distance properties of Green’s functions with respect to the full Standard Model, where the heavy particles were explicit fields in the Lagrangian. A set of matching conditions ensure that the physics does not change, however. These matching conditions define Wilson coefficients and operators and make the whole construction meaningful so that the scheme dependence appearing in the Wilson coefficients cancels that appearing in the four-quark operators’ matrix elements.

In perturbation theory all this is very well understood. Two groups have been able to compute the Effective Lagrangian even to two loops. However, perturbation theory may be a sensible approximation down to the integration of the charm quark, but certainly not to describe the matrix elements of the kaon. In fact, since the kaon is a (pseudo) Goldstone boson, its dynamics is appropriately described in terms of yet another Effective Field Theory. In the $\Delta S = 1$ sector, this is given by the Chiral Lagrangian:

$$\mathcal{L}_{\Delta S=1} = -\frac{G_F}{\sqrt{2}} V_{ud} V_{us}^*$$

$$\left[ g_8 \mathcal{L}_8 + g_{27} \mathcal{L}_{27} + e^2 g_{ew} \text{Tr} \left( U_L^{(32)} U_R^\dagger Q_R \right) \right] ,$$

where

$$\mathcal{L}_8 = \sum_{i=1,2,3} (\mathcal{L}_\mu)_{2i} (\mathcal{L}_\mu)_{i3} \quad \text{and} \quad \mathcal{L}_{27} = \frac{2}{3} (\mathcal{L}_\mu)_{21} (\mathcal{L}_\mu)_{13} + (\mathcal{L}_\mu)_{23} (\mathcal{L}_\mu)_{11} ,$$

with

$$\mathcal{L}_\mu = -i F_0^2 U(x)^\dagger D_\mu U(x) , \quad \lambda_{L}^{(32)} = \delta_{33} \delta_{22} \quad (5)$$

$^{1}$See, e.g., ref. for the full list of $\Delta S = 1$ operators. $^{2}$In the large-$N_c$ expansion one has to include the $\eta'$ in a nonet.
and

\[ Q_L = Q_R = Q = \text{diag}(2/3, -1/3, -1/3) \]  \hspace{1cm} (6)

The pion decay coupling constant \( F_0 \) is in the chiral limit, i.e., \( F_0 \approx 87 \text{ MeV} \). The matrix field \( U \) contains the Goldstone fields of the corresponding spontaneously broken chiral symmetry of QCD, and \( D_\mu U \) stands for the covariant derivative \( D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu \) in the presence of external chiral sources \( l_\mu \) and \( r_\mu \) of left- and right-handed currents. The Lagrangian \( \text{Eq.} (2) \) can be thought of as the one obtained from \( \text{Eq.} (1) \) after integrating out all the hadrons but the Goldstone octet. How does one go about matching the two Effective Field Theories \( \text{Eq.} (1) \) and \( \text{Eq.} (2) \)? At this point we find it very convenient to use the large-\( N_c \) expansion to organize the calculation because, being a systematic expansion in QCD, it can be carried out either with quarks and gluons or with mesons as degrees of freedom. So, in a way, the large-\( N_c \) expansion plays the role of a dictionary between these two languages.

In the following I will concentrate on how to impose the matching condition for the case of the strong penguin operators

\[ Q_6 = -8 \sum_{q=u,d,s} (\bar{q}_L q_R) (\bar{q}_R d_L) \quad \text{and} \quad Q_4 = 4 \sum_{q=u,d,s} (\bar{q}_L q_R) d_L \quad \text{(7)} \]

where \( q_{L,R} = \frac{1}{2} (1 \mp \gamma_5) q \) and sum over color indices within brackets is understood. Notice that, since four-quark operators mix among themselves, it makes no sense to consider the contribution from one of them in isolation. However, as it turns out, if we are willing to restrict ourselves to the leading logarithmic approximation and only to those subleading contributions in \( 1/N_c \) which are enhanced by the number of flavors \( n_F \), there is an important simplification since the \( Q_{4,6} \) system closes and no other operator needs be considered through mixing. I will adopt this simplification in the following.

Firstly, let us define the constant \( g_8 \) in the Lagrangian \( \text{Eq.} (2) \) as the coupling governing the non-diagonal “mass term” \( r_\mu \frac{\partial}{\partial x} \bar{q}_L q_R g_{\alpha \beta} \) for any flavor \( q = u,d,s \). For definiteness, we shall take \( q = u \). The matching condition stating that the same term be obtained with the Lagrangian \( \text{Eq.} (1) \) reads

\[ g_8 Q_4, Q_6 = C_0(\mu) \left\{ -\frac{16L_5 \langle \bar{q} \gamma_\mu q \rangle^2}{F_0^3} - \frac{8n_f}{16\pi^2 F_0^4} \int_0^\infty dQ^2 Q^2 W_{\text{DGRR}}(Q^2) \right\}_{\text{MS}} \]  \hspace{1cm} (8)

\[ + C_4(\mu) \left\{ 1 - \frac{4n_f}{16\pi^2 F_0^4} \int_0^\infty dQ^2 Q^2 W_{\text{LLRR}}(Q^2) \right\}_{\text{MS}} . \]

The subscript \( \text{MS} \) reminds one that these integrals are UV divergent and have to be regularized and renormalized, using the same scheme as for the Wilson coefficients \( C_{4,6} \). The parameters \( \langle \bar{q} \gamma_\mu q \rangle \) and \( L_5 \) are also renormalized accordingly. In \( \text{Eq.} (5) \) all the unfactorized contributions, of \( \mathcal{O}(n_F/N_c) \), are contained in the terms proportional to the functions \( W_{\text{DGRR}} \) and \( W_{\text{LLRR}} \). The terms proportional to \( L_5 \) and unity correspond to the factorized contribution from \( Q_6 \) and \( Q_4 \) respectively and, formally, are of \( \mathcal{O}(N_c^0) \).

The functions \( W_{\text{DGRR}} \) and \( W_{\text{LLRR}} \) are defined through the connected four-point Green’s functions

\[ W^{\alpha \beta}_{\text{DGRR}}(q) = i^3 \int d^4x \, d^4y \, d^4z \, e^{iq \cdot x} \langle 0 | T \{ D_{\bar{s}q}(x) G_{\bar{q}d}(y) R^3_{du}(y) R^0_{\bar{q}u}(z) \} | 0 \rangle_{\text{conn}} , \]

\[ W^{\alpha \beta}_{\text{LLRR}}(q) = i^3 \int d^4x \, d^4y \, d^4z \, e^{iq \cdot x} \langle 0 | T \{ L_{\bar{s}q}(x) L^0_{\bar{q}d}(y) R^3_{du}(y) R^0_{\bar{q}u}(z) \} | 0 \rangle_{\text{conn}} , \]  \hspace{1cm} (9)

after integration over the solid angle in \( q \)-momentum space in the manner specified in \( \text{Eq.} (5) \). In \( \text{Eq.} (6) \) \( D_{\bar{s}q} = \bar{q}_L q_R \), \( G_{\bar{q}d} = \bar{q}_R d_L \), \( L_{\bar{s}q} = \bar{q}_L q_R \), \( R^{3}_{du} = \bar{d}_R \gamma^\alpha u_R \), and similarly all the others. One then recognizes the pair of fermion bilinears which make up the operators \( Q_{6,4} \) in \( \text{Eq.} (6) \) except that they are located at different spacetime points. It is the integral over \( Q \) in \( \text{Eq.} (5) \) which brings them back to the same point so that, in fact, the matching condition \( \text{Eq.} (5) \) is nothing but the statement that in the four-quark Effective Lagrangian of \( \text{Eq.} (1) \) one can only generate this.
two-point correlator between $r^\alpha_{\mu}$ and $r^\beta_{\nu}$ by inserting the combination $c_6 Q_6 + c_4 Q_4$. Since the same two-point correlator is proportional to $g_8$ in the language of the Chiral Effective Lagrangian, this explains the origin of the condition (8).

It is clear that if one knew the functions $W_{DG(\langle L \rangle \langle R \rangle)}$ one immediately would be able to compute $g_8$ through Eq. (8). The problem is of course that these functions are not known. It is known, however, how they behave both at low and at high values of $Q^2$ thanks to Chiral Perturbation Theory and the Operator Product Expansion, respectively. So, if we can find a reliable way to interpolate between the two, our job is done. At this point, large-$N_c$ comes to help. Since in the large-$N_c$ limit both functions $W_{DG(\langle L \rangle \langle R \rangle)}$ are made out of an infinity of zero-width resonances, they are meromorphic functions. In other words, they only have isolated poles, but no cut. Furthermore, they are order parameters of chiral symmetry and would vanish were it not for its spontaneous breakdown. It follows, therefore, that perturbation theory yields a vanishing result to all orders in $\alpha_s$ and, in particular, there is no parton-model logarithm. With all these considerations, our choice for the interpolator is then the most natural one, namely a meromorphic function with a finite number of poles. The position of these poles will be identified with the resonance masses, and the unknown residues will be determined so that the interpolator reproduces the low- and high-$Q^2$ expansions given by ChPT and the OPE. In mathematics this is known as a rational approximant. Therefore, our interpolator constitutes an approximation to the large-$N_c$ curve.

I would like to emphasize that this approximation is systematic: the more terms in the OPE and ChPT are known, the more resonances can be included in the interpolator\(^4\). Although the solution to large-$N_c$ QCD is not known, it is plausible that such an approximation may do a nice job. For one thing, we are interpolating a QCD Green's function in the euclidean. Therefore one should not expect a lot of “structure”; i.e. resonances do not show up as “peaks”, unlike in the minkowski region. For another thing, we interpolate in the gap between the regime governed by ChPT and the OPE, so the gap does not seem very large! Of course, in the end, one will have to judge by the results obtained.

What are the generic properties which one can expect for functions such as $W_{DG(\langle L \rangle \langle R \rangle)}(Q^2)$? Figure 1 is a schematic view of the expected typical profile for one such generic QCD Green's function, here called $G(Q^2)$. As we can see, at $Q^2 = 0$ the function reaches a value determined by a typical chiral parameter, such as $F_0$ or $\langle \bar{\psi} \psi \rangle$, whereas at large $Q^2$ it falls off like an inverse power of $Q^2$. The turning point is given by a typical resonance mass, of the order of 1 GeV.

\[^4\text{The interpolator which matches just the first term in the OPE is what we have sometimes called the “Minimal Hadronic Approximation” because it is the simplest one which guarantees the correct short distance properties for the matrix element.}\]

\[ F_0 \ll \langle \bar{\psi} \psi \rangle^{1/3} \ll M_R \sim 1 \text{ GeV}, \quad (10) \]

and this determines the size of the final contribution. For instance, let us now estimate the size of the unfactorized contributions relative to the factorized ones in the case of $g_8$. Equation (8) shows that the factorized contribution from $Q_4$ is
of order unity. In order to estimate its unfactorized contribution one can use that the shape of \( \mathcal{W}_{LLRR}(Q^2) \) is like that of \( G(Q^2) \) in Fig. 1 and therefore

\[
\int dQ^2 Q^2 \mathcal{W}_{LLRR}(Q^2) \sim F_0^2 M_R^2, \tag{11}
\]

up to logarithmic factors. Consequently the ratio of contributions

\[
\frac{\text{unfactorized}}{\text{factorized}} \sim \frac{M_R^2}{16\pi^2 F_0^2}, \tag{12}
\]

which, although suppressed at large \( N_c \) like \( O(1/N_c) \), is a number of order unity. The important point is that this effect is very generic, and does not depend on the particular form of the four-quark operator. For instance, despite appearances, the contribution from \( Q_6 \) also obeys \( \mathcal{W}_{LLRR}(Q^2) \): the factorized contribution is proportional to \( L_5(\bar{\psi}\psi)^2/F_0^2 \), but the function \( W_{DGRR}(Q^2) \) has the shape of \( G(Q^2) \) in Fig. 1 multiplying its value at the origin by \( (L_5 - 5L_3/2)(\bar{\psi}\psi)^2/F_0^4 \). This is a consequence of \( Q_6 \) being made up of scalar and pseudoscalar densities rather than left and right currents.

One simple reason behind the estimate given in (12) is that unfactorized contributions know about the scale \( M_R \sim 1 \text{ GeV} \) whereas factorization is dominated by chiral parameters, like \( F_0 \) or \( \langle \bar{\psi}\psi \rangle \), whose typical scales are smaller. Factorized contributions do not have “access” to the scale \( M_R \). In this sense it is something similar to the opening-up of a “new channel” in a scattering process. Therefore, there is no reason to expect the next-to-next-to-leading contribution to be even bigger, with the consequent breakdown of the large-\( N_c \) expansion. Since the large contribution is due to the coming into play of a new scale, once all the scales have already appear in the problem there should be no more surprises.

There is another (independent) reason why, in the particular case of \( Q_6 \), one could have expected large unfactorized corrections: the factorized contribution depends on \( L_5(\mu) \). Although the running of \( L_5 \) with \( \mu \) is a \( 1/N_c \) effect, in fact, a small change in scale such as going from the rho mass to 1 GeV already changes the factorized contribution by roughly a factor of 2! If one takes this as a naive estimate for the \( 1/N_c \) corrections, one also concludes that it is not unnatural for these to be large. Indeed, it was found that the unfactorized contributions were actually larger than the factorized ones\(^4\). Within the context of a model, this was also found in Refs. \([6]\).

Using that the Wilson coefficients can be decomposed as\(^2\)

\[
C_i(\mu) = z_i(\mu) + \tau y_i(\mu), \quad \tau = \frac{V_{ts}^\ast V_{td}}{V_{us}^\ast V_{ud}}, \tag{13}
\]

the coupling constant \( g_8 \) in Eq. \( (8) \) picks up an imaginary part due to \( \tau \). We obtain\(^4\)

\[
\text{Im} g_8 \simeq (3 \pm 1) \times \text{Im} \tau, \tag{14}
\]

where the error has been estimated by varying the quark condensate (which is the source of the biggest uncertainty), \( \langle \bar{\psi}\psi \rangle^{1/3} (2 \text{ GeV}) = (0.240 \pm 0.260) \text{ GeV} \). In Ref. \([7]\) it was also obtained that\(^5\)

\[
\text{Im} \left( \epsilon^2 g_{ew} \right) = (1.6 \pm 0.4) \times 10^{-6} \text{ GeV}^6 \times \text{Im} \tau. \tag{15}
\]

With \([15]\) and \([14]\) one can get to an estimate for \( \epsilon'/\epsilon \)\(^1\):

\[
\frac{\epsilon'}{\epsilon} = \frac{\text{Im}(V_{ts}^\ast V_{td}) G_F \omega}{2|\omega||\text{Re} A_0|} \left[ P^{(0)}(1 - \Omega_{\text{IB}}) - \frac{1}{\omega} P^{(2)} \right], \tag{16}
\]

where

\[
P^{(0,2)} = \sum_{i=4,6,8} y_i(\mu) \langle (\pi\pi)_{0,2} | Q_i(\mu) | K^0 \rangle, \tag{17}
\]

and \( \Omega_{\text{IB}} = 0.16 \pm 0.03 \)\(^9\) is a term induced by isospin breaking. Using the above value for the condensate, \( 1/\omega = 22.2 \) and the physical value for \( \text{Re} A_0 \), we obtained in Ref. \([4]\)

\[
\frac{\epsilon'}{\epsilon} \simeq (2 \pm 2) \times 10^{-3}. \tag{18}
\]

However, using the recent determination \( \Omega_{\text{IB}} = 0.06 \pm 0.08 \)\(^10\) and including final state interactions as in Ref. \([11]\) one obtains instead,

\[
\frac{\epsilon'}{\epsilon} \simeq (5 \pm 3) \times 10^{-3}. \tag{19}
\]

This is to be compared to the experimental number, i.e. \( \epsilon'/\epsilon = (1.66 \pm 0.16) \times 10^{-3} \)\(^12\). The lesson is that, due to the difference between the two

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\(^5\)See Ref. \([8]\) for other analyses.
smaller terms in [10], small corrections to either term get amplified making this observable extremely difficult to control. Matching the level of precision of the experimental result is going to be a very arduous task.

At present, the lattice determination of $\varepsilon'/\varepsilon$ is smaller than the experimental result by roughly a factor of three and differs in sign. Our previous considerations could be of help in understanding this situation. First of all, on the lattice one does not really have QCD, but a quenched version of it. This quenched version is currently done.

As a consequence, an unquenched QCD and turn to the $\Delta I = 1/2$ rule. This rule can be expressed by the fact that $n_F$ is effectively zero, as a consequence of the quark-gluon cancelation. One predicts, therefore, that the result is given by the factorized contribution, which is a much smaller value for $\alpha_q^{(8,1)}$ than in the unquenched theory:

\begin{equation}
\alpha_q^{(8,1)} = -16L_5F_0^2B_0^2 \left(1 + O\left(\frac{1}{N_c^2}\right)\right). \tag{23}
\end{equation}

Moreover, if one compares $\alpha_q^{NS}$ with $\alpha_q^{(8,1)}$ one finds

\begin{equation}
\frac{\alpha_q^{NS}}{\alpha_q^{(8,1)}} = \frac{1}{16L_5} \sim 60, \tag{24}
\end{equation}

where we have used that $L_5 \sim 10^{-3}$. This is not a small number and obviously questions the validity of neglecting $\alpha_q^{NS}$ in lattice simulations of $\varepsilon'/\varepsilon$ as is currently done.

I would like to conclude by coming back to unquenched QCD and turn to the $\Delta I = 1/2$ rule. This rule can be expressed by the fact that Re $g_{\pi N}^{\text{exp}} \simeq 3.3$ is much larger than $g_{\pi N}^{\text{exp}} \simeq 0.23$, after subtraction of the chiral corrections. Any systematic framework must also face the calculation of these values.
Although due to mixing below the charm mass all operators contribute to Re $g_8$, at $\mu = m_c$ only $Q_{2,1}$ given by
\[
Q_2 = 4(\bar{s}_L \gamma^\mu u_L)(\bar{u}_L \gamma^\mu d_L),
\]
\[
Q_1 = 4(\bar{s}_L \gamma^\mu d_L)(\bar{u}_L \gamma^\mu u_L)
\]
have non-vanishing contributions. In fact, these contributions from $Q_{1,2}$ can be related (neglecting penguins\[18\]) to the coupling constant $g_{S=2}$ governing the local $K^0 \leftrightarrow \bar{K}^0$ transition\[19\]. Now, our calculations allow us to add to this the contributions coming from penguin configurations. The point is that the penguin-like contribution from $Q_2$ which is obtained by contracting the two $u$ quarks is the same as the unfactorized contribution from $Q_1$ in Eq. (8), via the replacement $n_F \rightarrow 1$. One then obtains
\[
\text{Re} g_8 = \frac{1.33 \pm 0.40}{(Q_{2,1} \text{ non-penguin})} + \frac{0.8 \pm 0.4}{(Q_2 \text{ penguin})} = 2.1 \pm 0.8. \tag{26}
\]
In spite of the large errors involved I find this result quite encouraging.

Furthermore, $g_{27}$ can also be calculated. This is due to the celebrated relation to $g_{S=2}$ and $B_K$\[20\]. Using the results of Ref. \[19\], one finds
\[
g_{27} = 0.29 \pm 0.12 \leftrightarrow \hat{B}_K = 0.36 \pm 0.15, \tag{27}
\]
in the chiral limit, with a nice agreement with $g_{27}^{\exp}$.

Other interesting applications of our approximation to large-$N_c$ QCD include the decays $\pi^0 \rightarrow e^+e^-$ and $\eta \rightarrow \mu^+\mu^-$\[21\], hadronic vacuum polarization\[22\] and the calculation of the hadronic light-by-light contribution to $g_{\mu} - 2$\[23\].

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