What can we learn studying black holes?

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Abstract

In this paper we try to answer the main question: what is a quantum black hole?
1 Introduction

What does urge a researcher to investigate quantum black holes? Honestly, his own intrinsic interest. Besides, there are other, more (or less) scientific reasons. It is commonly believed that only small black holes can be considered as quantum objects. Small, what does it mean? To estimate, we should compare the size of the black hole with the corresponding Compton length. The gravitational radius \( r_g \) of the black hole of mass \( m \) equals \( r_g = \frac{2Gm}{c^2} \), where \( G \) is the Newtonian gravitational constant, and \( c \) is the speed of light. The Compton length of such particle is \( \lambda = \frac{\hbar}{mc} \) (\( \hbar \) is the Planck’s constant).

If \( r_g \simeq \lambda \), than the so called Planckian mass is \( m_{pl} = \sqrt{\frac{\hbar c}{G}} \sim 10^{-5}gr \), the Planckian length is \( l_{pl} = \sqrt{\frac{\hbar G}{c^3}} \sim 10^{-33}cm \). In what follows we use the units in which \( \hbar = c = 1 \), so \( m_{pl} = 1/\sqrt{G} \), \( l_{pl} = \sqrt{G} \). Black holes of so small mass and sizes could be created from large metric fluctuation in the very early Universe (primordial black holes), or during vacuum phase transition.

The mankind “knows” about black holes since 1784, when some priest J. Michell recognized [1] that stars becomes invisible if the correspond escape velocity will exceed the speed of light. He called such stars “dark stars” Fifteen years later, in 1799, P.S. Laplace derived a relation between the radius and the mass of these dark stars [2].

Let us consider a spherically symmetric gravitating body of mass \( M \) and of radius \( R_0 \). Then the escape velocity \( v_e \) can be determined from the balance of the kinetic and potential energies of the particle that starts to move away from this gravitating body. Using the non-relativistic mechanics and non-relativistic Newtonian theory of gravity we obtain, following Laplace (\( G \) is the Newton constant, and \( m \) is the mass of the escaping particle)

\[
\frac{mv_e^2}{2} = \frac{GmM}{R_0},
\]

Equating the escape velocity to that of light \( c \) we get for the maximal radius of the invisible body

\[
R_0 = \frac{2GM}{c^2}.
\]
We are used to think of the black holes as of the extremely dense bodies. Indeed, the black hole with the mass of the Earth has the radius about 3 cm and the density much higher than the nuclear density. But due to the linear radius-mass relation the mean density $\rho$ is proportional to the inverse mass squared,

$$\rho = \frac{3c^6}{32\pi G^3 M^2}.$$ 

And for the huge black hole formed, say, of the cluster of galaxies, the density may be less than that of the atmospheric air. Thus the observer falling into such a black hole would not recognize that he never would go out of it (moreover, the free fall time in this case is about 10,000 years).

The history of the “real” black holes started in 1916, when K.Schwarzschild found the, now famous, solution to the Einstein equation for the spherically symmetric vacuum space-time outside gravitating point source [3]. He derived it in the form

$$ds^2 = F dt^2 - F^{-1} dr^2 - r^2 d\Omega^2$$ \hspace{1cm} (1.2) 

$$F = 1 - \frac{2Gm}{r}$$ \hspace{1cm} (1.3)

Here $d\Omega^2$ is the line element of the unit sphere, $r$ - its radius, and $m$ is the total mass (energy) of the system, also known as Schwarzschild mass. The line element (1.2) has a singularity at the so called Schwarzschild radius $r_g = 2Gm$, from where the light cannot escape to the infinity. It is surprising that value of the Schwarzschild radius calculated using General Relativity coincides exactly with non-relativistic (Newtonian) radius of the dark stars.

In classical General Relativity black holes are very special (and, therefore, very interesting) objects. First of all, they are universal in the sense that they are described by only few parameters: their mass, angular momentum and charges. In the process of black hole formation, (i.e., in the process of gravitational collapse) all higher momenta and non conserved charges are radiating away. This feature is formulated as a following conjecture: ”black holes has no hairs”. Thus, a black hole formation results generically in the loss of information about initial states and previous history of collapsing matter. The boundary of the black hole, the so-called event horizon, is the null hyper-surface that acts as a one-way membrane. The matter can fall inside but can not go outside. Because of this the area of black hole horizon can
not decrease. These two features, the loss of information and nondecrease of the horizon area, allowed J. Bekenstein [3] to suggest the analogy between the black hole physics and thermodynamics and identify the area of the horizon with the entropy (up to some factor). He did this for the simplest, spherically symmetric neutral (Schwarzschild) black hole which characterized by only one parameter, Schwarzschild mass. Later the four laws of thermodynamics were derived for a general black hole.

In thermodynamics the appearance of the entropy is accompanied by the temperature. While the nature of the black hole entropy was more or less clear, the notion of its temperature remained mysterious until the revolutionary work by S. Hawking [5]. He showed that the black hole temperature introduced by J. Bekenstein is the real temperature in the sense that the black hole radiates, and this radiation has a Planck's spectrum. The entropy appeared equal one fourth of the event horizon area divided by Planckian length squared. Thus, even large (compared to the Planckian mass and size) black holes exhibit quantum features. It should be stressed that such quantum effect is global, namely, it emerges as a result of nontrivial boundary conditions for the wave function of a quantum field theory in curved space-times nontrivial causal structure (existence of the event horizon(s)).

Due to the process of Hawking’s evaporation any (even super-large) black hole becomes eventually small enough to be considered as a (local) quantum object. At this stage the combination of the global and local quantum effects may result in unpredictable features. That why it is so exciting to try to understand quantum black hole physics.

2 Classical Black Holes

Everybody knows what the classical black hole is. In short, black hole is a region of a space-time manifold beyond an event horizon. In turn, an event horizon is a null surface that separates the region from which null geodesics can escape to infinity and that one from which they cannot. The most general black hole is characterized by only few parameters, its mass, angular momentum, and some conserved charge, similar to the electric one. In general the space-time outside a black hole is stationary and axially symmetric, but for a non-rotating black hole it is spherically symmetric and static. In what follows we are dealing only with spherically symmetric black holes. It is important to stress that the notion of the of the event horizon is global, it
requires knowledge of both past and future histories. To understand this better, let us consider the general structure of spherically symmetric space-time manifolds. The metric of such space-time can always (at least locally) be written in the form

\[ ds^2 = g_{00}dt^2 + 2g_{01}dtdq + g_{11}dq^2 - R^2d\Omega^2 \]  

(2.1)

Four metric coefficients are functions of some time coordinate \( t \) and some radial coordinate \( q \), \( d\Omega = \theta^2 + \sin^2\theta d\phi^2 \) and \( R(t,q) \) is the radius in that sense that the sphere area is \( 4\pi R^2 \). Making use of two allowed coordinate (gauge) transformations it is always possible to transform a two-dimensional \((t,q)\) - part of the metric to the conformally flat metric. Hence any spherically symmetric space-time is (locally) described, actually, by only two invariant functions. One of them is, evidently, the radius \( R(t,q) \). For the other one it is convenient to choose the squared normal vector to the surfaces of constant radius \([7, 8]\)

\[ \Delta = g^{ij}R_iR_j \]  

(2.2)

The function \( \Delta(t,q) \) brings a nontrivial information about a space-time structure. Indeed, in the flat Minkowskian space-time \( \Delta \equiv 1 \), all the surfaces \( R = \text{const.} \) are time-like and therefore, \( R \) can be chosen as spatial coordinate \( q = R \) on the whole manifold. But in the curved space-time \( \Delta \) is no more constant and can in general be both positive and negative. The region with \( \Delta > 0 \) is called the \( R \)-region, and the radius can be chosen as a radial coordinate \( q \). In the region with \( \Delta < 0 \) the surfaces \( R = \text{const.} \) are space-like (the normal vector is time-like), and the radius \( R \) can be chosen as a time coordinate \( t \). Such regions are called the \( T \)-regions. The \( R \)- and \( T \)-regions were introduced by Igor Novikov. But this is not the whole story. It is easy to show that we can not get \( \dot{R} = 0 \) (“dot” means a time derivative) in a \( T \)-region. Hence it must be either \( \dot{R} > 0 \) (such a region of inevitable expansion is called \( T_+ \)-region) or \( \dot{R} < 0 \) (inevitable contraction, a \( T_- \)-region. The same holds for \( R \)-regions. They are divided in two classes, those with \( R' > 0 \) (“prime” stands for a spatial derivative) which are called \( R_+ \)-regions, and \( R_- \)-regions with \( R' < 0 \). These, \( R \)- and \( T \)-regions are separated by the surfaces \( \Delta = 0 \) which are called the apparent horizons. The apparent horizons can be null, time-like or space-like. If surface \( R = 0 \) lies inside a \( T_- \)-region, then boundary of the latter forms the so-called trapped surface from which any light can escape to the infinity. This part of the apparent horizon lies either inside a black hole or coincides partly with an event horizon. Clearly, the notion of the apparent horizon is local.
After all these preparation we consider the following example. Let us imagine that we have a trapped surface and, hence a black hole, and spherically symmetric matter layer outside it. Let such a layer is held at some fixed value of radius with the help, say, of rocket engine. If the engine is abruptly switched off, the layer will start to collapse and, eventually, will cross the apparent horizon. Again a black hole will be formed, but now it will have different parameters (higher mass), and a new event horizon has to be defined. In this case some part of the apparent horizon (that one lying in the past light cone of the crossing point) will appear to be inside the new black hole, while the other part (the new one lying in the future of the crossing point) will coincide with the new event horizon. In principle, we may have many different matter layers, and nobody can foresee, when the astronauts will decide to switch off engines. Our simple gedanken experiment demonstrates clearly the global nature of the event horizon.

All the notions introduced above are extremely important in studying black holes formed during gravitational collapse. But there exist a rather simple and, nevertheless, very important model, called the eternal black hole which we study in the next Subsection.

\subsection*{2.1 Schwarzschild black hole}

All the Schwarzschild metrics \((1.2)\) forms the one-parameter family labeled by the value of total mass \(m\), and there are no other spherically symmetric solutions to the vacuum Einstein equation which asymptotically flat at infinity (this statement is known as Birkhoff’s theorem). It is clear at once that the metric \((1.2)\) has a singularity at the gravitational radius \(r_g = 2Gm\). The nature of this singularity was not understood for a long time. And only after works by M.Kruskal \([10]\) and I.D.Novikov \([9]\) it become clear that it is actually a coordinate singularity, appeared due to the static form of the line element \((1.2)\) and impossibility to synchronize the clocks of static observers at spatial infinity \((r = \text{const} \rightarrow \infty)\) and of the observers (which are inevitable non-static) in the region \(r < 2Gm\).

The manifold, described by the line element \((1.2)\), is not geodesically complete. By definition, in geodesically complete space-time (which also called a maximally analytically extended manifold) any time-like or null geodesics should start and end either at the infinity or the singularity. The structure of maximally extended spherically symmetric manifold very easy to see and investigate using the so-called Carter - Penrose conformal diagrams. On these
diagrams all the infinities lie at the finite distance from the center, and every point represents a sphere.

The conformal diagram of geodesically complete Schwarzschild space-time is shown in Fig.1.

Figure 1:

We see that this manifold has a geometry of the non-traversable wormhole. There are two asymptotically flat $R_\pm$-regions connected by the Einstein-Rosen bridge. If we start from the left spatial infinity at $i_0^\prime$ and go in the direction to the right infinity at $i_0$, the radii of spheres first decrease in the $R_-$-region, reach minimal value on the Einstein-Rosen bridge and then increase in the $R_+$-region. The narrowest part of the bridge called “throat”, it lies at the intersection of two null surfaces $r = 2Gm$, which are the horizon. For the sake of simplicity we will often refer to the $R_+$-region as the ‘our side’, and to the $R_-$-region as the ‘other side’ of the Einstein-Rosen bridge.
The gravitating sources in the eternal Schwarzschild black hole are assumed to be concentrated in the future and past $T$-regions at the singular space-like surfaces of zero radius, $R = 0$. These singularities are the real ones, because (as can be easily shown) the Riemann curvature tensor and, hence the tidal forces, become infinite here. The Schwarzschild geometry in the equatorial plane at the fixed moment of time can also be visualized a geometry of some surface of rotation embedded into three-dimensional flat space. (see Fig.2).

![Figure 2](image_url)

In what follows we will deal sometimes with the electrically charged sources. The space-time outside such sources is described by the Reissner-Nordstrom line element which differs from the metric (1.2) by different function $F$, now it becomes

$$F(r) = 1 - \frac{2Gm}{r} + \frac{Ge^2}{r^2},$$

where $e$ is the electric charge.

### 2.2 Sources

Eternal black hole is the nice and transparent model. It appeared very useful in investigating of classical black holes. But it really physical because of the absence of any dynamical degrees of freedom, namely, there’s no dynamics of
gravitating matter, neither real mass-shell spherically symmetric gravitons. Therefore, having in mind the construction of quantum black hole models, we have to include more realistic gravitating source into consideration. Because of the nonlinearity of the Einstein equations the inclusion of source make the quantization problem very complicated and, in general, unsolvable. Therefore, in order to get some definite results we have to choose the most simple types of sources. We will consider the spherically symmetric thin dust shell which is a direct generalization of a point gravitating mass.

The theory of thin shells in General Relativity was developed by W.Israel [11] and applied then by many authors studying problems of gravitational collapse and cosmology [12, 14]. We don’t need yo consider this theory in details. In our case, both outside and inside the shell we have either vacuum (Schwarzschild) or electro-vacuum (Reissner-Nordstrøm) metric, and and complete information about the shell dynamics is contained in the single equation, namely, in the (0)-Einstein equation for the shell (this is the so called energy constraint). It has the form

$$\sigma_{in} \sqrt{\dot{\rho}^2 + \Delta_{in}} - \sigma_{out} \sqrt{\dot{\rho}^2 + \Delta_{out}} = 4\pi G \rho S^0_0$$

(2.3)

Here $\rho$ is the shell’s radius as a function of the proper time of the observer sitting on the shell, the dot means the proper time derivative, $S^0_0$ is the surface energy density concentrated on the shell, and $\Delta_{in,out}$ is the value of the (introduced above) invariant function just inside and outside the shell. In our case it equals $\Delta = -F$. The step function $\Delta$ take two values $\pm 1$ depending om whether radii increase in the direction of outward shell’s normal vector ($\sigma = +1$) or they decrease ($\sigma = -1$). It is clear that the sign of $\sigma$ coincides with the sign of the $R$-region where the shell moves, and its sign can be changed only in the $T$-regions. For the sake of simplicity we consider here in details only the case when the shell is not electrically charged and there is no other gravitating sources inside it. Then

$$\Delta_{in} = 1, \quad \sigma_{in} = 1$$

$$\Delta_{out} = 1 - \frac{2Gm}{\rho} + \frac{Ge^2}{\rho^2}, \quad m > 0$$

(2.4)

Our shell is made of dust, that is, of noninteracting (except gravitationally) particles. This means that th surface tension is zero, $S^2_2 = S^3_3 = 0$. Then, from the continuity condition,

$$\frac{dS^0_0}{d\tau} + \frac{2\dot{\rho}}{\rho} (S^0_0 - S^2_2) + [T^n_n] = 0$$

(2.5)
it follows that

\[ S_0 = \frac{M}{4\pi\rho^2} \]  

(2.6)

where \( M \) is readily identified as a bare mass of the shell (the sum of the rest masses of particles without the gravitational mass defect).

Finally, we arrive at only one equation which brings all the information about the shell’s motion, namely

\[
\sqrt{\dot{\rho}^2 + 1} - \sigma_{out} \sqrt{\dot{\rho}^2 + 1 - \frac{2Gm}{\rho}} = \frac{GM}{\rho} 
\]

(2.7)

We can solve this equation for the total mass \( m \) to get

\[
m = M\sqrt{\dot{\rho}^2 + 1 - \frac{GM^2}{2\rho}} 
\]

(2.8)

and now it is seen that it is nothing more but the energy conservation equation, the square root being the famous Lorentz factor written in terms of proper time derivatives.

We shall be interested here in bound motion only. Let us denote by \( \rho_0 \) the radius of the shell at the moment of rest, and let \( \rho_0 > R_+ \) (that is, we are outside the event horizon). Then it follows from Eqn.(2.7) and Eqn.(2.8) that \( m < M \) and

\[
m = M - \frac{2GM^2}{2\rho_0} 
\]

(2.9)

\[
\sigma_{out} = \text{Sign} \left( 1 - \frac{GM}{\rho_0} \right) 
\]

(2.10)

Differentiating Eqn.(2.9) with respect to the bare mass \( M \) we obtain

\[
\frac{\partial m}{\partial M} = 1 - \frac{GM}{\rho_0} 
\]

(2.11)

Let us consider the dependence of the total mass \( m \) on the bare mass \( M \) for some fixed value of the turning radius \( \rho_0 \). As seen from Fig.3, the curve has two branches, the increasing and the decreasing ones.
Figure 3:
The total mass $m$ first increase with $M$, and the parameters of the shell in this case are such that

$$\frac{\partial m}{\partial M} > 0$$

$$\sigma_{out} = +1$$

$$\frac{1}{2} < \frac{m}{M} < 1$$

The world lines of these shells start from the past singularity at $R = 0$, pass the $R_+$-region and end at the future singularity at $R = 0$. We shall call such a situation the black hole case (the shell forms the black hole). The complete Carter-Penrose diagram is shown in Figs.4,5.

The spatial geometry at the moment of time symmetry is shown in Fig.6.

The curve $m(M)$ reaches then the maximum and starts to decrease. For the decreasing branch we have

$$\frac{\partial m}{\partial M} < 0$$

$$\sigma_{out} = -1$$

$$\frac{m}{M} < \frac{1}{2}$$

Now the shells pass the $R_-$-region, and we shall call this the wormhole case (the shell forms the wormhole on the other side of the Einstein-Rosen bridge). The corresponding Carter-Penrose diagrams is shown in Fig.8,9. The spatial geometry is shown in Fig.7. It is explicitly seen in the figure that the radii decrease outside the shell, reach the minimal value $2Gm$ at the bifurcation point and then they start to increase. The wormhole region can not be reached or seen from the $R_+$-region. In the marginal case $m/M = 1/2$ and the turning point lies exactly at the bifurcation point $R = 2Gm$.

It is instructive to consider some limiting cases. The first one is the limit $M \to \infty$. When the bare mass is increasing and $\frac{m}{M} < \frac{1}{2}$, the turning point is further and further from the horizon, moving toward the “left” infinity (on the other side Einstein-Rosen bridge). And as a limit we get geometry of the eternal black hole everywhere except at the left infinity which is now singular.

Another limiting case is a completely closed world with zero total mass
Figure 4:
Figure 5:
Figure 6:
Figure 7:
Figure 8:
Figure 9:
\( m = 0 \). We have for such a situation

\[
\begin{align*}
    m &= 0 \\
    M &= 2\rho_0/G
\end{align*}
\]

(2.14)

For completeness let us consider the thin shell with negative total mass \( m < 0 \) (but with positive bare mass \( M \)). From Eqn.(2.7) it follows immediately that \( \sigma_{\text{out}} = -1 \) everywhere, the radii thus decrease outside the shell up to the singularity at \( R = 0 \). The space-time topology is \( S^3 \times R^3 \) rather than \( R^3 \times R^1 \) as it is for positive total masses \( m \). So, in this case we have no spatial infinity at all. In what follows we will consider only positive total masses.

### 2.3 Black hole universality

It was already mentioned that the most general black hole is the stationary axially symmetric space-time manifold, describing by a very limited set of parameters, namely, by its mass, angular momentum and (locally) conserved charges. But what does happen when the initial state of the collapsing matter is not exactly spherically symmetric (here we consider the case of zero angular momentum)? The huge work of many researchers has led to the following result (the detailed review of this topic can be found in the book [15]). In the process of black hole formation everything that can be radiated is radiating away. It concerns all the multipoles (including the monopole moment) of scalar fields, the higher multipoles of electromagnetic, nonabelian and gravitational fields. In some theories the scalar monopole can not be radiated away, but only in those cases when it is rigidly linked to the monopole moment (Coulomb part) of the corresponding nonabelian field. This feature of process gravitational collapse was first understood by J.A.Wheeler and formulated as the (now famous) statement that “The black hole has no hair”. Thus, the process of black hole formation is accompanied, in general, by the loss of information about the initial state and the preceding history of collapsing matter.

From the very definition of the black hole it is clear that nothing can go off the event horizon. But this does not mean automatically that the black hole mass cannot be decreased. The matter is that in case of nonzero angular momentum there exist around the event horizon a special region, called ergosphere, inside which there can be states with negative energy (from
the point of view of a distant observer). Then, the following process can take place. The particles that fall into the ergosphere, may decay into pairs, and one of the component whose energy is negative, may fall into the black hole, while another one would go out of the ergosphere (but not out of the black hole!) and escape to the infinity, thus decreasing the black hole mass. In the literature it is known as the “Penrose process”. If instead of particles, we are dealing with the fields, then the analogous process also will take place, it is called “superradiance”. Both Penrose process and superradiance result in decreasing the black hole angular momentum. And the real universality of black holes is that fact that it is impossible by any classical process to decrease the area of the black hole boundary, i.e. for the event horizon area $A$ we have always

$$dA \geq 0 \quad (2.15)$$
3 Quantum Era of Black Hole Physics

3.1 Black hole thermodynamics

In this section (and only in this) subsection we will use some formulas for a general axisymmetric black hole with total mass \( m \), electric charge \( Q \) and angular momentum \( J \). For such a black hole being the final state of the gravitational collapse, black holes have very interesting properties. It appeared that black holes, during their formation, emit everything that can be emitted. Thus, the black hole resembles the body in thermodynamical equilibrium. In 1972 Jacob Bekenstein suggested that this is not merely a coincidence, and the black hole really brings some amount of entropy and has some definite temperature. This point of view was supported by the following inequality already known at the time, namely, that the area of the black hole horizon cannot decrease. This resembles the second law of thermodynamics. J. Bekenstein suggested that the black hole entropy is proportional to the black hole area \( A \) (more general, the entropy may be equal to some increasing function of the area).

In this section (and only in this) subsection we will use some formulas for a general axisymmetric black hole with total mass \( m \), electric charge \( Q \) and angular momentum \( J \). For such a black hole

\[
A = 4\pi \left( R_+^2 + \frac{J^2}{m^2} \right),
\]

\[
R_+ = G \left( m + \sqrt{m^2 - \frac{Q^2}{G} - \frac{J^2}{m^2 G}} \right)
\]

Moreover, it appeared that the first law of thermodynamics is also valid and can be expressed by the following mass formula.

\[
\delta m = \frac{\kappa}{8\pi} \delta A + \Omega \delta J + \Phi \delta Q
\]

Here \( \delta m \) is the mass difference between two stationary black holes with slightly different areas \( \delta A \), angular momenta \( \delta J \) and electric charges \( \delta Q \), while \( \Omega \) is the black hole’s angular velocity and \( \Phi \) is the value of the Coulomb potential at its surface. The prefactor \( \kappa \) in front of \( \delta A \) is the so-called surface gravity which equals to

\[
\kappa = \frac{4\pi}{A} \sqrt{M^2 - Q^2/G - J^2/M^2 G}
\]
Following J. Bekenstein, the surface gravity should be proportional to the black hole temperature \( \Theta \), \( \kappa = \alpha \Theta \).

Let us consider a Schwarzschild black hole as a thermodynamical system and put it in a thermal bath at some fixed temperature. Then, if the temperature of the black hole is the same as that of the thermal bath, the amounts of the absorbing and emitting radiation are equal, the black hole and the bath are in thermal equilibrium. But this equilibrium is unstable. The reason for this is the negative heat capacity of the Schwarzschild black hole. Indeed, from the definition of the heat capacity we have

\[
C = \Theta \frac{\partial S}{\partial \Theta} = -8\pi Gm^2 < 0 \tag{3.5}
\]

Let us suppose that the black hole is initially in thermal equilibrium with the environment. Then, if by some fluctuation the mass of the black hole will become a little bit larger than its equilibrium value the temperature will become lower than that of the thermal bath causing the absorption of radiation, further increase of the mass and decrease of the temperature, thus resulting in the unbound growth of the black hole size and mass. If the initial fluctuation results in the small decrease of the mass and, consequently, in the increase of the black hole temperature, the black hole starts to emit radiation more than to absorb it, thus further decreasing the mass up to the very end. In this case the black hole disappears completely.

In the case of the Reissner-Nordstrom black hole the situation is different. The temperature is zero for the extreme black hole \( (m = |Q|/G) \), reaches its maximum and then decreases to zero when the mass infinitely increases. The heat capacity is positive near the extreme state, first grows to infinity, then becomes negative like in the Schwarzschild case. We see that the charged nearly extreme black hole is thermally stable like the ordinary bodies.

### 3.2 Hawking radiation

The quantum era of the black hole physics began in 1974 with Hawking’s discovery of the black hole evaporation. Stephen Hawking considered quantum vacuum fluctuations of matter fields on the Schwarzschild black hole back- ground and showed that the very existence of the event horizon leads to the radiation flow from the black hole (violating the classical theorem \( dA \geq 0 \)). The surprising fact is that this radiation is nothing more but the blackbody
radiation with the temperature
\[ \Theta = \frac{k}{2\pi} \]  
(3.6)

Thus, the black hole entropy is exactly one fourth of the dimensionless horizon area
\[ S_{BH} = \frac{1}{4} \frac{A}{l_{pl}^2} \]  
(3.7)

which is in full agreement with the Beckenstein’s conjecture (we use the units in which the Planckian constant \( \hbar \), the light velocity \( c \) and the Boltzmann constant \( k \) are put equal to one; Planckian length equal to \( l_{pl} = \sqrt{\hbar G/c^3} = G^{1/2} \), and Planckian mass is \( m_{pl} = \sqrt{\hbar c/G} = G^{-1/2} \)).

Remarkably enough the same value of the black hole temperature can be obtained in a quite different way. By making a Wick rotation of the \( M_2 \)-part of the Schwarzschild or Reissner-Nordstrom line element written in the curvature coordinates we obtain a two-dimensional Euclidean half-plane \((-\infty < t_E < \infty, R_+ < R < \infty)\). If we place the origin at \( R = R_+ \), assume periodicity of the Euclidean time \( t_E \) and identify points over the period, we then get a two-dimensional surface with, in general, a conical singularity at the origin. This singularity can be removed by a suitable choice of the time period. And the inverse of such a period is equal to the temperature in finite temperature quantum field theories and in our case it can be called a topological temperature. Practically, it can be done as follows. The line element of the two-dimensional Euclidean surface is
\[ dl^2 = F d\tau^2 + \frac{1}{F} dr^2, \]  
(3.8)

\[ F = 1 - \frac{2Gm}{r} + \frac{G\epsilon^2}{r^2} = \frac{1}{r^2} (r - r_+)(r - r_-), \]  
(3.9)

\[ r_\pm = G \left( m \pm \sqrt{m^2 - \epsilon^2/G} \right). \]  
(3.10)

The Euclidean time \( \tau \) is assumed to be a cyclic coordinate. In the vicinity of the horizon \( F = (r - r_+)(r - r_-)/r_+^2 \). Introducing a new coordinate \( \rho = 2r_+ \sqrt{r - r_+}/\sqrt{r_+ - r_-} \), we get
\[ dl^2 = \frac{(r_+ - r_-)^2}{4r_+^4} \rho^2 d\tau^2 + d\rho^2 = d\rho^2 + \rho^2 d\varphi^2. \]  
(3.11)
And this is nothing more but the metric of a locally flat two-dimensional surface, written in the cylindrical coordinates. The point \( \rho = 0 \) is, in general, a conical singularity. To remove it, we have to require the period of the azimuthal angle \( \varphi \) to be \( 2\pi \). Thus,

\[
T = \frac{4\pi r^2}{r_+ - r_-}. \tag{3.12}
\]

Its inverse,

\[
\Theta = \frac{2G\sqrt{m^2 - \frac{e^2}{G}}}{A}. \tag{3.13}
\]

equals exactly the temperature of the Reissner-Nordstrom black hole.

All this means that quantum matter fields in the black hole background are described, in fact, by the finite temperature quantum field theory. In this sense, the appearance of the black hole temperature looks quite naturally.

The violating of the classical law \( dA \geq 0 \) does not mean that the radiation can be emitted off the black hole, because the black hole itself is still considered as the classical object. The calculations of the stress-energy tensor of the vacuum fluctuations show that their energy density is negative. Therefore, the negative energy flow falls into black hole, thus diminishing its mass, while the, equal to it, positive energy flow escapes to infinity. One can consider such a situation as that the quantum fluctuations of matter fields form something like an ergosphere in the vicinity of the event horizon, and inside this ergosphere the Penrose process (for particles) or the superradiance effect (for radiating fields) may take place.

### 3.3 Rindler’s space-time.

Except the black holes there is one more example of the space-time manifold which observers “see” a black body radiation. This is the Rindler’s space-time. It is locally flat, therefore, sufficiently simple and all the quantum field theory equations can be explicitly solved and investigated in details. On the other hand, this example is very important since there is exists a deep relation between Rindler’s and black hole manifolds. Let us consider the two-dimensional Minkowski space-time with the metric

\[
ds^2 = dt^2 - dx^2. \tag{3.14}
\]
The Rindler’s coordinates are obtained by the following transformation

\[ t = \pm \frac{1}{a} e^{a\xi} \sinh a\eta \]  
(3.15)
\[ x = \pm \frac{1}{a} e^{a\xi} \cosh a\eta, \]  
(3.16)

The upper (lower) sign is for \( x > ( < ) 0 \). The line element now reads as follows

\[ ds^2 = e^{2a\xi} (d\eta^2 - d\xi^2) \]  
(3.17)

Rindler’s observer, sitting at \( \xi = \text{const.} \), experience a constant proper acceleration \( \alpha = ae^{-a\xi} \). Being locally flat, the Rindler’s manifold differs, nevertheless, from the Minkowski manifold, it has different boundaries. The boundary of the Minkowski space-time are trivial infinities (spatial, temporal and null). Rindler’s coordinates cover only one half of the Minkowski space-time, corresponding to \( |x| \geq |t| \), and the lines \( x = \pm t \) are the event horizons that serve as new boundaries (in addition to the infinities). Exact quantum field theory calculations show (the details can be found in the book [16]) that the Rindler’s vacuum is, actually, with the black body radiation which temperature equals the so called Unruh’s temperature \( T_0 = \frac{a}{2\pi} \). Every local observer measures its own local temperature \( T = T_0(g_{00})^{-1/2} = T_0 e^{-\alpha} = \frac{\alpha}{2\pi} \).

It should be stressed that the appearance of the temperature (and heat bath) in Rindler’s space-time is a global effect, depending essentially on the boundary condition at the event horizon. And we can do here the same trick as we did with the black hole space-time. Namely, we can introduce the cyclic imaginary time \( \eta \to i\eta = \tau \) and spatial coordinate \( \zeta = \frac{1}{a} e^{a\xi} \) the line Rindler’s element now becomes

\[ dl^2 = a^2 \zeta^2 d\tau^2 + d\zeta^2. \]  
(3.18)

The obtained two-dimensional euclidean surface has, in general, a conical singularity that disappeared only if the period of our imaginary time \( \tau \) equals \( \frac{2\pi}{a} \). The inverse to this period is exactly the Unruh’s temperature. Note, that procedure, described above, is of global character, because the choice of the period in the point dictates actually the period of the “proper” imaginary time for all Rindler’s observers.

Let us consider now a static spherically symmetric space-time the line element of which can always be written in the form

\[ ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2 d\Omega^2, \]  
(3.19)
where \( \nu \) and \( \lambda \) are function of the radius \( r \). By the equivalence principle, the static observer experiences a constant acceleration. And indeed, it can be easily shown by direct calculation that the acceleration equals \( \alpha = \kappa e^{-\nu/2} \), where

\[
\kappa = \frac{\nu'}{2} e^{\frac{\nu - \lambda}{2}} \tag{3.20}
\]

is the so-called surface gravity. A static observer, sitting at \( r = r_0 \neq 0 \) can forget about the angular part of metric (3.19) and consider a 2-dimensional static curved space-time. Such a 2-dimensional surface can be easily transformed to the local Rindler-like coordinates. Indeed,

\[
d s^2 = e^\nu d t^2 - e^\lambda d r^2 = e^\nu (d t^2 - e^{\lambda - \nu} d r^2) = e^\nu (d \eta^2 - d \xi^2), \tag{3.21}
\]

where \( \eta = t \) and \( d \xi = e^{\frac{\lambda - \nu}{2}} \). We can calculate the acceleration parameter \( a \) in the following way. Let \( r = r_0 + \delta r \), then \( \delta \xi = e^{\frac{\lambda - \nu}{2}} \delta r \). The conformal factor near \( r = r_0 \) equals \( e^\nu(r_0)(1 + \nu'\delta r) = e^{2\xi_0}(1 + 2a\delta \xi) \). From this it follows that \( e^{2\xi_0} = e^\nu(r_0) \) and \( a = \frac{\nu'}{2} e^{\frac{\nu - \lambda}{2}}(r_0) \), which is exactly the surface gravity \( \kappa \).

The equivalence of the Rindler’s observers to the static ones in the spherically symmetric space-times does not mean that the latter is heated. This equivalence is only local. To have the global equivalence (of course, with different Rindler’s observers for different static observers) we need the event horizon, and that is exactly what we have in the case of spherically symmetric black holes. But this is not the end of the story. The reduction of the four-dimensional spherically symmetric space-time to 2-dimensional \((t, r)\)-surface can be safely done only if \( r \neq 0 \). The point \( r = 0 \) is the coordinate singularity. Thus, the equivalence described above is valid only if the corresponding euclidean surface does not contain such a point. Otherwise we would have to consider spherically symmetric Rindler’s manifold, which essentially non-static.

And, finally, the last note. The temperature (if it exist) measured by a local static observer obeys the law \( \Theta \sqrt{g_{00}} = \text{const} \). Therefore, the temperature \( \Theta_0 = \frac{1}{\sqrt{g}} \) is measured by observer, sitting at the point where \( g_{00} = 1 \). For the asymptotically flat space-time this is, of course, the point at spatial infinity.
3.4 Quantum mass spectrum

Historically, the discovery of the quantum nature of radiation has led to the construction of the hydrogen atom model and to the explanation of its discrete energy spectrum. Analogously, the quantum nature of radiation suggests that the mass spectrum of black holes is, actually, quantized. Jacob Bekenstein proposed the equidistant spectrum for the black hole horizon area. His motivation is as follows. The entropy value is just the logarithm of the number of ways in which we can construct the black hole with given parameters and related to the amount of information hidden inside the black hole. The amount of information is naturally quantized. Therefore, the black hole area which is proportional to the entropy should have an equidistant spectrum. A more quantitative explanation is due to Slava Mukhanov. We generalized his result originally derived for the Schwarzschild black hole, to the Reissner-Nordstrom charged black hole. Let the black hole emits the quantum energy $\omega$ due to transition from the $n$-th state to the $(n-1)$-th state. The “typical” energy of this quantum minus the work done against the Coulomb attraction should be proportional to the temperature, i.e.

$$\omega_{n,n-1} - \Phi dQ = \pi \alpha \Theta,$$  \hspace{1cm} (3.22)

where $dQ$ is the charge of the quantum in question, and $\alpha$ is a slowly varying function of the integer quantum number $n$. Inserting, then, Eqn.(3.22) into Eqn.(3.3) we have ($\omega_{n,n-1} = dm$)

$$dA = 4\pi G\alpha,$$  \hspace{1cm} (3.23)

or

$$dA = 4\pi G\alpha dn,$$  \hspace{1cm} (3.24)

Integrating we get

$$A = 4\pi G\tilde{\alpha}n + 4\pi GC,$$  \hspace{1cm} (3.25)

where $C$ is a constant of integration, and $\tilde{\alpha}$ is some another slowly varying function of $n$. It is natural to suppose that the value $n = 0$ corresponds to the minimal possible value of the horizon area, e.i., it should correspond to the extreme black hole. This gives us $C = Q^2$. It is clear that the Hawking’s formulae are valid only if the back reaction of the radiation on the spacetime metric can be neglected, e.i., for low (comparing to the black hole mass) temperature. Thus, we are interested in two different cases, in the large black
hole regime and in the nearly extreme regime. In the first case $n \gg Q^2$ and we assume the following expansion

$$
\tilde{\alpha} = \alpha_1 + \beta_1 \frac{Q^2}{n} + \gamma_1 \frac{Q^4}{n^2} + \ldots,
$$

(3.26)

$$
\frac{Q^2}{n} \ll 1,
$$

while in the second case $n \ll Q^2$ and we have

$$
\tilde{\alpha} = \alpha_2 + \beta_2 \frac{n}{Q^2} + \gamma_2 \frac{n^2}{Q^4} + \ldots,
$$

(3.27)

$$
\frac{Q^2}{n} \gg 1.
$$

The corresponding expansions for the black hole mass are

$$
m = \frac{\sqrt{n}}{2\sqrt{G}} \sqrt{\frac{\alpha_1}{\alpha_1}} \left(1 + \frac{\beta_1 + 3 Q^2}{2\alpha_1} \frac{n}{n} \right.

+ \left( \frac{\gamma_1}{2\alpha_1} - \frac{(\beta_1 + 1)(\beta_1 + 4)}{8\alpha_1^2} \right) \frac{Q^4}{n^2} + \ldots \right),
$$

(3.28)

$$
n \gg Q^2
$$

and

$$
m = \frac{|Q|}{2\sqrt{G}} \left(2 + \frac{\alpha_2^2}{4} \frac{n^2}{Q^4} + \ldots \right),
$$

(3.29)

$$
n \ll Q^2
$$

The values of $\alpha$’s, $\beta$’s and $\gamma$’s can and will be determined in the next Section for the specific model.

## 4 Quantum Black Hole Models

### 4.1 Eternal black hole

We have already described in detail the classical eternal black hole and mentioned that it has no dynamical degrees of freedom. In this respect it is analogous to the Coulomb field in electrodynamics. Nevertheless it is instructive
to quantize such a system. Of course, the full program of quantization can not be fulfilled because quantum gravity is not renormalizable perturbatively. But the quantization of the vacuum spherically symmetric space-times can be done in a nonperturbative way in the so-called frozen formalism. In this formalism we quantize only one, radial, degree of freedom. Though not complete, such a procedure can give us rather reliable results, especially in the light of the recent result [18] which support the suggestion that the four-dimensional Einstein gravity may appear nonperturbatively renormalisable [19]. Quantization of the eternal black hole in frozen formalism was done by H.Kastrup and T.Thiemann [20] in the so-called quantum loop gravity and by K.Kuchar [21] who used the conventional geometrodynamics. Even for such a simple space-time manifold the problem appeared nontrivial. But the final result was the same in both approaches, namely, the quantum functional on spherically symmetric vacuum metrics depends only on the Schwarzschild mass. And in the absence of dynamical degrees of freedom it is impossible to extract any information about a black hole spectrum. Physically such a trivial result can be understood if one compares the eternal black hole with its electro-dynamical analog, the hydrogen atom. Let us imagine that we first allow the electron to “fall” classically on the proton and then start quantization procedure. What we would get? Absolutely nothing. Of course, in electrodynamics we have both positive and negative charges, and the mass of the eternal black hole can only be positive. And, instead of “nothing” the curvature singularity is developing in the process of gravitational collapse. But the qualitatively, the picture is the same. Therefore in order to obtain physically meaningful results we have to “switch on” some dynamical degrees of freedom. In subsequent Subsection we consider the quantization of spherically symmetric thin dust shells, for which we already developed the classical theory.

4.2 Naive quantization

The material of this section is based on the works [22, 23, 24]. Now, we investigate a specific quantum black hole model, which classical counterpart is described in the first part of the lectures. It is a self-gravitating spherically symmetric thin dust shell with bare mass $M$, total mass $m$ and electric charge $e$. The “correct” quantization procedure (which starts from the “first principles”) we postpone till the next Chapter and consider here a very simple quantum model based on the classical equation of motion for our thin
shell. Since $m$ is a total mass of the system and the Eqn. (2.8) represents the energy conservation law it is reasonable to consider it equal numerically to the Hamiltonian of the system in question. Thus, we have

$$m = M \sqrt{\dot{\rho}^2 + 1} - \frac{GM^2 - e^2}{2\rho}$$  \hspace{1cm} (4.1)

To quantize a system we should construct first the classical Hamiltonian and then to replace the classical Poisson brackets by the quantum commutation relations. Given the classical expression $E(\dot{\rho}, \rho)$ for the energy of the system as a function of its radius and velocity (actually, rapidity in our case), the conjugate momentum $p$, Lagrangian $L$ and Hamiltonian $H$ can be found from the following relations

$$p = \frac{\partial L}{\partial \dot{\rho}},$$  \hspace{1cm} (4.2)

$$E = p\dot{\rho} - L = \dot{\rho} \frac{\partial L}{\dot{\rho}} - L,$$  \hspace{1cm} (4.3)

$$L = \dot{\rho} \int E \frac{d\dot{\rho}}{\rho^2} = \dot{\rho} \int \frac{\partial E}{\partial \dot{\rho}} \frac{d\dot{\rho}}{\rho} - H$$  \hspace{1cm} (4.4)

which give for the conjugate momentum

$$p = \int \frac{\partial E}{\partial \dot{\rho}} \frac{d\dot{\rho}}{\rho}.$$  \hspace{1cm} (4.5)

Substituting for $E$ the Eqn. (4.1) one gets

$$p = M \log(\dot{\rho} + \sqrt{\dot{\rho}^2 + 1}) + F(\rho),$$  \hspace{1cm} (4.6)

$$L = M(\dot{\rho} \log(\dot{\rho} + \sqrt{\dot{\rho}^2 + 1}) - \sqrt{\dot{\rho}^2 + 1}) + \dot{\rho}F(\rho),$$  \hspace{1cm} (4.7)

where $F(\rho)$ is an arbitrary function. The choice of this function does not affect the Lagrangian equations of motion, and leads to the canonically equivalent systems in the Hamiltonian approach. This choice may become important when one explores the rather complex structure of the maximal analytical extensions of black hole space-times trying to construct a wave function covering the whole manifold. But here we do not need such a complication and put $F(\rho) = 0$. From Eqn. (4.6) we get for $\dot{\rho}$

$$\dot{\rho} = \sinh \frac{p}{M},$$  \hspace{1cm} (4.8)
and, after inserting this into Eqn.(4.1) we obtain the following Hamiltonian $H$.

$$H = M \cosh \frac{p}{M} - \frac{GM^2 - e^2}{2\rho}. \quad (4.9)$$

It is convenient to make a canonical transformation to the new variables ($M$ is the bare mass of the shell)

$$x = M\rho, \quad \Pi = \frac{1}{M}\rho, \quad (4.10)$$

then

$$H = M \left( \cosh \Pi - \frac{GM^2 - e^2}{2x} \right). \quad (4.11)$$

Let us now impose the quantum commutation relation

$$[\Pi, x] = -i. \quad (4.12)$$

Then, using the coordinate representation with $\Pi = -i\partial/\partial x$ and the relation

$$e^{-i\frac{\Pi x}{2}}\Psi(x) = \Psi(x - i) \quad (4.13)$$

we obtain the Schroedinger stationary equation

$$H\Psi(x) = E\Psi(x), \quad (4.14)$$

$$\frac{M}{2} \left( (\Psi(x + i) + \Psi(x - i) - \frac{GM^2 - e^2}{x}\Psi(x) \right) = m\Psi(x). \quad (4.15)$$

Introducing the notation $\epsilon = m/M$ we can rewrite this as

$$\Psi(x + i) + \Psi(x - i) = \left( 2\epsilon + \frac{\kappa M^2 - e^2}{x} \right)\Psi(x). \quad (4.16)$$

Some remarks are in order. First, the Schroedinger equation obtained is not the differential, but the equation in finite differences. This is a direct consequence of the quantization in proper time which is quite natural in the framework of General Relativity. Second, the total mass $m$ is not arbitrary anymore but it is an eigenvalue of the Hamiltonian operator, subject to the condition that for corresponding eigenfunctions the Hamiltonian is Hermitian.
on the positive semi-axes. Third, it should be noted also that the step in our finite difference equation is along the imaginary axes, so the solutions should have certain analyticity to be continued to the complex plane.

The Hamiltonian picture we got is not equivalent to that which could be obtained directly from the Einstein-Hilbert action. But our procedure self-consistent. We started from equations of motion written already in proper time and constructed the new Lagrangian which results in just the equations in proper time with no restrictions or/and constraints. We do not expect that the resulting mass spectrum will be correct. But our model is extremely simple and, as we shall see soon, is exactly solvable, and it seems very instructive to develop here some new methods and ideas.

We consider the non-relativistic limit of the obtained Schrödinger equation. With all dimensional quantities restored the shifted argument \((x \pm i)\) becomes

\[
x \pm i \rightarrow M \left( \rho \pm \frac{1}{M} i \right) \rightarrow \rho \pm \frac{\hbar}{Mc} i,
\]
so for \(\rho \gg \hbar/Mc\) we obtain the non-relativistic limit. Expanding Eqn.(4.16) up to the second order in \((\hbar/Mc)\) we get

\[
-\frac{1}{2M} \frac{d^{2}\Psi}{d\rho^{2}} - \frac{GM^{2} - e^{2}}{2\rho} \Psi = (E - M)\Psi
\]

which is just the non-relativistic Schrödinger equation for the radial s-wave function, \((E - M)\) being the non-relativistic energy of the system. For the negative values of \((E - M)\) acceptable solutions exist only for a discrete spectrum of energies, namely

\[
(E - M)_{n} = -\frac{M(GM^{2} - e^{2})^{2}}{8n^{2}}, \quad n = 1, 2, ...
\]

For \(G = 0\) it reduces to the well known Rydberg formula for the hydrogen atom.

Now, we show how to solve the obtained Schrödinger equation in finite differences. And we will do this first in the momentum representation.

It is convenient to introduce the following parameters, \(\lambda\) and \(\beta\),

\[
\epsilon = \cos \lambda, \quad \alpha = GM^{2} - e^{2} = 2\beta \sin \lambda,
\]
in terms of which our Schrödinger equation becomes

\[
\Psi(x + i) + \Psi(x - i) = 2 \left( \cos \lambda + \frac{\beta \sin \lambda}{x} \right) \Psi(x).
\]
Since for the black holes $\alpha > 0$, the signs of $\lambda$ and $\beta$ are the same. We choose $\lambda > 0$ and, hence, $\beta > 0$. Let us return to the operator form of the equation,
\[(\cosh \hat{p} - \beta \sin \lambda \hat{x}^{-1}) \Psi = \cos \lambda \Psi. \tag{4.20}\]
In the momentum representation operators $\hat{p}$ and $\hat{x}$ act as follows
\[
\hat{p} \Psi_p = p \Psi_p, \quad \hat{x} \Psi_p = i \frac{\partial}{\partial p} \Psi_p, \tag{4.21}
\]
where $|\Psi_p\rangle$ is a wave function in the momentum representation. The operator $\hat{x}^{-1}$ is not well defined. The ambiguity can be removed by adding suitable terms to the potential which are proportional to $\delta$-function and its derivatives at the origin. But, instead, we can multiply the equation by operator $\hat{x}$ from the left. By doing this we get
\[
i \frac{\partial}{\partial p} (\cosh p - \cos \lambda) \Psi_p = \beta \sin \lambda \Psi_p, \tag{4.22}\]
or
\[
\frac{\partial}{\partial p} \log \Psi_p = - \frac{\sinh p + i \beta \sin \lambda}{\cosh p - \cos \lambda}. \tag{4.23}\]
After introducing a new variable, $z = e^p$, the Eqn.\[(4.23)\] takes the form
\[
\frac{\partial}{\partial z} \log \Psi_p = - \frac{z^2 + 2i\beta \sin \lambda z_1}{z(z^2 - 2 \cos \lambda z + 1)} = \frac{1}{z} \left( \frac{\beta + 1}{z - z_0} + \frac{\beta - 1}{z - \bar{z}_0} \right),
\]
where $z_0 = e^{i\lambda}$, $\bar{z}_0 = e^{-i\lambda}$. \tag{4.24}\]
The above equation can be easily solved, the result is
\[
\Psi_p = C \frac{z}{(z - z_0)(z - \bar{z}_0)} \left( \frac{z - z_0}{z - \bar{z}_0} \right)^{\beta}, \tag{4.25}\]
where $z = e^p$.

This solution has very important property, it is periodical with the pure imaginary period $2\pi i$. This property will be explored below.

Now we will transform the solution found in the momentum representation, to the coordinate representation. It can be done by the inverse Fourier transform,
\[
\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixp} \Psi_p dp. \tag{4.26}\]
We obtained before the unique (up to the multiplicative constant) solution for $\Psi_p$, namely, Eqn.(4.26). But due to the periodicity of $\Psi_p$ we can shift the argument $p \rightarrow p + 2\pi ki(k = 0, \pm 1, \pm 2, \ldots)$ which will result in the shift of the path of integration in the complex momentum plane from the real axis to the parallel one. And after such a shift we again obtain a solution in the coordinate representation. But,

$$\Psi_1(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} \Psi_p dp,$$

$$\Psi_2(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(p+2\pi ki)x} \Psi_p dp = e^{-2\pi kx} \Psi_1(x).$$

(4.27)

Thus, given one solution (say, $\Psi_0$), we can construct in this way a countable number of solutions, and, in general,

$$\Psi_{general}(x) = \left( \sum_{k=\infty}^{\infty} c_k e^{-2\pi kx} \right) \Psi_0(x).$$

(4.28)

The infinite sum in parenthesis is nothing more but a Fourier series for a periodical function with an imaginary unit period $i$. And this reproduces the following property of the solutions in coordinate representation. Given some solution, say, $\Psi_0$ we can construct new solution, $\Psi_1$, multiplying it by any periodical function $C(x)$ with the imaginary period $i$.

How many different solutions do we need to construct the general solution? The very fact that in the momentum representation we obtained essentially one solution proves that for this we need only one solution $\Psi_0$ which we call the fundamental solution (though it deserves the name superfundamental). Now we will construct the particular fundamental solution by the suitable choice of the contour of integration in the inverse Fourier transform integral. Note, first of all, that our solution, Eqn.(4.26), in momentum representation has countable number on branching points in the complex momentum plane which can be combined in pairs, $(i\lambda + 2\pi ki, -i\lambda + 2\pi (k+1)i), k =$
0, ±1, ±2, .... Connecting two branching points in each pair by a cut we obtain the complex plane with countable number of cuts. On the corresponding Riemann surface our solution is a single valued analytical function of complex variable. Our choice of the contour of integration is as follows. We will integrate first along the real axes from left to right (i.e., from \(-\infty\) to \(+\infty\)), then along a short curve at the right infinity \((p \to p + 2\pi i, p \to +\infty)\), then along the straight line \(y = 2\pi i\), parallel to the real axis, from left to right, and finally along a short curve at the left infinity back to the real axis. The integration along the short curves at infinities between the straight lines gives zero contributions for positive values of \(x\) (for negative \(x\) we can choose the straight line \(y = -2\pi i\) instead of \(y = 2\pi i\)). The integration along each straight line gives us a solution the linear combination of which is again a solution. Thus, the inverse Fourier integral along such a closed contour gives us a solution. This contour can be distorted to become a contour around the cut \((\lambda i, (2\pi - \lambda)i)\) for \(x > 0\) (or around the cut \((-\lambda i, -(2\pi - \lambda)i)\) for \(x < 0\)). Thus,

\[
\Psi_0(x) = \frac{1}{\sqrt{2\pi}} \oint_{C_+} e^{ipx} \Psi_p dp, \quad x > 0,
\]

\[
\Psi_0(x) = \frac{1}{\sqrt{2\pi}} \oint_{C_-} e^{ipx} \Psi_p dp, \quad x < 0,
\]

where

\[
\Psi_p = C \frac{z}{(z - z_0)(z - \bar{z}_0)} \left(\frac{z - \bar{z}_0}{z - z_0}\right) \beta
\]

\[
z = e^p, \quad z_0 = e^{i\lambda}, \quad \bar{z}_0 = e^{-i\lambda}
\]

In what follows we restrict ourselves to the case of positive \(x\) only. The above integral representation for \(\Psi_0\) can be simplified if we integrate Eqn.(4.29) by parts. Using the fact that

\[
\frac{z dp}{(z - z_0)(z - \bar{z}_0)} = -\frac{1}{2i \sin \lambda} d \left(\log \frac{z - \bar{z}_0}{z - z_0}\right),
\]

we get

\[
\Psi_0(x) = x \oint_{C_+} e^{ipx} \left(\frac{z - \bar{z}_0}{z - z_0}\right) \beta dp, \quad x > 0,
\]
(the extra term vanishes because our contour has no boundary and for convenience we omitted some constant factors). We will use Eqn. (4.33) as an integral representation for the fundamental solution of our finite differences equation, Eqn. (4.19).

Let us investigate the fundamental solution in more details. Our aim is to reduce the integration along the closed contour $C_+$ around the cut to the finite interval between the corresponding branching points. But for $\beta \geq 1$ (remember that we have chosen $\beta$ positive) we have the non-integrable singularity at the lower integration limit. To avoid this difficulty we need some recurrent relation for lowering the parameter $\beta$. To find such a relation we integrate by parts Eqn. (4.33), the result is the following.

$$\Psi_\beta(x) = \Psi_{\beta-1}(x) + \frac{ix}{\beta - 1} \left\{ \Psi_{\text{beta}-1}(x) - e^{-i\lambda} \Psi_{\beta-1}(x + i) \right\}.$$  \hspace{1cm} (4.34)

From the above relation it is easy to derive the structure of $\Psi_\beta$ for general values of $\beta > 1$. Let us take $\beta = n + \tilde{\beta}$, with $\tilde{\beta} \leq 1$. Then, $\Psi_\beta$ is the sum of two terms, which of them is the product of some polynomial of $n$-th degree and of $\Psi_{\tilde{\beta}}(x)$ or $\Psi_{\tilde{\beta}}(x + i)$. Thus, we can proceed assuming $\beta \leq 1$. First of all, consider separately the case $\beta = 1$. We have

$$\Psi_1(x) = x \oint_{C_+} e^{ipx} \frac{e^p - e^{i\lambda}}{e^p - e^{i\lambda}} dp = 2\pi ix \lim_{p \to i\lambda} \frac{e^{ipx}(e^p - e^{-i\lambda})(p - i\lambda)}{e^p - e^{i\lambda}}$$

$$= - (4\pi e^{-i\lambda} \sin \lambda) xe^{-\lambda x}. \hspace{1cm} (4.35)$$

From this it follows that all $\Psi_\beta$ for positive integer $\beta = n$ has the form

$$\Psi_n = P_n(x) e^{-\lambda x}, \hspace{1cm} (4.36)$$

where $P_n(x)$ are some polynomials of $n$-th degree. In the case $0 < \beta < 1$ the integrand in the Eqn. (4.33) is integrable on both ends of the cut in the complex momentum plane and we are able to convert the integral along the closed contour $C_+$ into the integral along the finite interval. We get the following,

$$\Psi_\beta(x) = \oint_{C_+} e^{ipx} f_p dp = \left( 1 - e^{2\pi i/\beta} \right) x \Phi_\beta(x), \hspace{1cm} (4.37)$$

where

$$\Phi_\beta(x) = \int_{i\lambda}^{i(2\pi - \lambda)} e^{ipx} \left( \frac{e^p - e^{-i\lambda}}{e^p - e^{i\lambda}} \right)^\beta dp, \hspace{1cm} x > 0. \hspace{1cm} (4.38)$$
Changing the variables,
\[ e^q - e^{i\lambda} = -(2i \sin \lambda) y, \]  
(4.39)
we get
\[ \Phi_\beta = (1 - e^{-2i\lambda}) e^{-2\lambda x} \int_0^1 (1 - (1 - e^{-2i\lambda}) y)^{ix-1} \left( \frac{y-1}{y} \right)^\beta dy. \]  
(4.40)
Comparing this with the well known integral representation for the Gauss’s hypergeometric function, \( F(a, b; c; z) \),
\[ F(a, b; c; z) = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a} dt, \]
and
\[ B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt, \]
(4.41)
we see that the integral in Eqn.(4.40) is the Gauss’s hypergeometric function \( F(a, b; c; z) \) with the following values of parameters,
\[ a = 1 - ix, \quad b = 1 - \beta; \quad c = 2; \quad z = 1 - e^{-2i\lambda}, \]  
(4.42)
and
\[ B(b, c-b) = B(1 - \beta, 1 + \beta) = \frac{\pi \beta}{\sin \pi \beta}. \]  
(4.43)
Finally, for the fundamental solution \( \Psi_\beta(x) \) we get
\[ \Psi_\beta(x) = \left(-4\pi \beta e^{-i\lambda} \sin \lambda \right) xe^{-\lambda x} F \left(1 - ix, 1 - \beta; 2; 1 - e^{-2i\lambda}\right). \]  
(4.44)
The above expression was derived for \( 0 < \beta, 1 \) only, but the same is valid for any value \( \beta > 0 \). And, finally, it can be easily shown that the original equation in finite differences, Eqn.(4.19), is a direct consequence of the following Gauss’s recurrent relation
\[ (2a - c - az + bz)F(a, b; c; z) + (c-a)F(a-1, b; c; z) + a(z-1)F(a+1, b; c; z) = 0 \]
(4.45)
So, we found the general solution to the equation in finite differences with Coulomb potential,
\[ \Psi(x + i) + \Psi(x - i) = 2 \left( \cos \lambda + \frac{\beta}{x} \sin \lambda \right) \Psi(x). \]  
(4.45)
Now we should remember that this equation is actually the radial Schroedinger equation for zero angular momentum, and the coordinate $x$ runs from zero to infinity, i.e., $x$ takes values in the positive semi-axis only. Thus the Hamiltonian,

$$
\hat{H} = \cosh \left( i \frac{\partial}{\partial x} \right) - \frac{\beta}{2x} \sin \lambda
$$

(4.46)

should be the selfadjoint operator on the positive semi-axis rather than on the whole real axis as dictated by the quantum mechanics postulates. In addition, the wave function should be square integrable on the semi-axis. The corresponding extension of the above Hamiltonian was found by Petr Hajicek. It appeared the wave function should obey the following boundary conditions at the origin, $x = 0$,

$$
\Psi^{(2n)}(0) = 0, \quad n = 0, 1, \ldots
$$

(4.47)

That is, the function itself and all its even derivatives should be zero at the origin. The appearance of infinite number of conditions is due to the infinite order of the operator. P.Hajicek found also a conserved probability current $J(x)$ for the Schroedinger equation in finite differences, Eqn.(4.45). It is given as usual by the equation

$$
J'(x) = i \left( \Psi^* \hat{H} \Psi - \Psi \hat{H} \Psi^* \right)
$$

so that

$$(\Psi^* \Psi)' + J' = 0$$

Writing the Hamiltonian (4.46) as follows

$$
\hat{H} = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \Psi^{(2k)}(x) - \frac{\beta}{2} \frac{\sin \lambda}{x}
$$

(4.48)

we obtain for $J(x)$,

$$
J(x) = i \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \sum_{l=1}^{k} (-1)^{l-1} \left[ \Psi^{*(l-1)} \Psi(2k-l) - \Psi^{(l-1)} \Psi^*(2k-l) \right].
$$

(4.49)

P.Hajicek showed that the boundary condition (4.47) implies that

$$
J(0) = 0.
$$

(4.50)
The boundary condition (4.47) is a direct generalization of the boundary condition in the case of non-relativistic Schrödinger equation which is of second order. In this particular case the boundary conditions (4.47) are reduced to the single one, \( \Psi(0) = 0 \), which is together with the square integrability condition ensures existence of a discrete spectrum for bound states. And, in general, if we expand Hamiltonian (4.46) in series of derivatives and consider the truncated Hamiltonian of \( 2N \)th order, the corresponding Schrödinger equation becomes the differential equation of \( 2N \)th order. It can be shown that the asymptotics at of such an equation at the origin are

\[
\Psi \sim x, x^2, ..., x^{2N-1}, 1 + x^{2N-1} \log x
\]  

and at the infinity they are of the form

\[
\Psi \sim e^{\pm \lambda_k x} e^{\alpha_k \log x}, \quad k = 1, ..., N.
\]

where \( \lambda_k \) are real for bound states. The general solution is defined up to a normalization factor and depends, actually, on \( (2N-1) \) arbitrary constants which are to be determined using boundary conditions. The square integrability condition reduces the number of unknown constants to \( (N-1) \). But for the truncated operator of \( 2N \)th order we have \( N \) conditions at the origin. They can only be satisfied for a discrete spectrum of eigenvalues. Note that the last of the asymptotics (4.51) is excluded by the boundary condition at the origin. We can reverse the above consideration and start from the asymptotic behavior at the origin. To satisfy the boundary conditions there we have to exclude the last asymptotics (4.51) and all the even terms in the expansion up to the \( (2N-2) \)th order. By doing this can determine \( N \) of \( (2N-1) \) unknown constants. Thus we are left with only \( (N-1) \) constants for \( N \) boundary conditions at infinity. But the situation is not so simple in the case of an infinite order operator. First of all, how to get rid of the logarithmic term in asymptotics? The problem is that if we first go to the infinite order limit, then the logarithmic term unconditionally disappears \(|x| < 1!\) leaving 1 as an asymptotics. But the latter can be easily reproduced by some infinite linear superposition of the remained asymptotics. Subtracting one from another we would obtain the solution which would satisfy formally all the boundary conditions at the origin still having enough arbitrary constants to satisfy boundary conditions at infinity. But if we first differentiate \( 2N \) times the asymptotic solution containing logarithmic term we will get \((1/x)\) term. Thus, requiring that the “good” solution should be infinitely differentiable we can reach our goal. Moreover, we have to require the analyticity
of the wave function at least on the real axis. Thus is because we implicitly have used the analyticity of the solutions in transition from the differential equation of infinite order to the finite differences equation. The situation is not good at the infinity either. We saw that in the case of our particular equation we can obtain a new solution multiplying the known one by \( \exp(-2\pi k x) \). Choosing the large enough value of \( k \) we are able to convert the “bad” at infinity solution to the “good” one. We will see in a moment that it is the analyticity condition which solves this problem.

Let us consider the asymptotics of the general solution at \( x \to 0 \) and \( x \to \infty \). The idea is the following. First, we find the asymptotical behavior of the fundamental solution at \( x \to 0 \) and \( x \to \infty \). We have,

\[
x^r, \quad r = 1, 2, \ldots
\]

at \( x \to 0 \) and

\[
\Psi_\beta = -2\pi i \beta e^{-i\lambda \beta} e^{i\pi \beta} \left\{ \frac{(2 \sin \lambda)^\beta}{\Gamma(1+\beta)} x^\beta e^{-\lambda x} - \frac{(2 \sin \lambda)^{-\beta}}{\Gamma(1-\beta)} x^{-\beta} e^{\lambda x} \right\} \phi\left(\frac{1}{x}\right),
\]

\( x \to \infty \)

(4.54)

It is interesting to note that the same asymptotics can be found directly from the original Schrödinger equation in finite differences without knowing the exact solution.

Comparing the asymptotics at \( x = 0 \) and at infinity we see that the fundamental solution is an analytic function only if

\[
\beta = n
\]

(4.55)

(remember that we chose \( \beta > 0 \)). This leads to the discrete spectrum for the total mass \( m \). Indeed, we have

\[
\beta = \frac{\alpha}{2 \sin \lambda} = n, \quad \alpha = \kappa M^2,
\]

\[
\epsilon = \frac{m}{M} = \cos \lambda,
\]

\[
\epsilon = \sqrt{1 - \frac{\alpha^2}{4n^2}}
\]

(4.56)

For \( \beta = n \) the hypergeometric series in Eqn(4.44) terminates and we are left with the following fundamental solution,

\[
\Psi_n(x) = P_n(x) e^{-\lambda x},
\]

(4.57)

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where $P_n$'s are some polynomials of order $n$. They can be calculated directly from the definition, or from the generating function

$$\Phi(z, x) = \sum_{n=0}^{\infty} \frac{\Phi^{(n)}(0, x)}{n!} z^n = \sum_{n=0}^{\infty} \Pi_n(x) z^n = 2\pi e^{2\lambda x} \left( \frac{z + \cot \lambda + i x}{z + \cot \lambda - i} \right)^{ix}$$

(4.58)

At the end of this section we give an example of solving the boundary conditions and find the wave function of the ground state (i.e., for $n = 1$). The fundamental solution for $n = 1$ was found to be

$$\Psi_1 = x e^{-\lambda x}$$

(4.59)

(we omitted here the irrelevant constant factor). The general solution obeying the boundary condition at infinity (exponential falloff at $x \to \infty$) has the form

$$\Psi_{1gen} = x e^{-\lambda x} \sum_{k=0}^{\infty} c_k e^{-2\pi kx}$$

(4.60)

and the coefficients $c_k$ are to be determined by solving boundary conditions at the origin,

$$\Psi^{(2l)}(0) = 0, \quad l = 0, 1, ...$$

(4.61)

Differentiating Eqn.(4.60) $2l$ times we get the following infinite set of linear equations,

$$2l \sum_{k=0}^{\infty} c_k (\lambda + 2\pi k)^{2l-1} = 0, \quad l = 0, 1, ...$$

(4.62)

The determinant of this system is identically zero because the first line consists of zeros. We can truncate this system at some specific value of $l$, calculate all the determinants and minors and then, after appropriate renormalization (say, putting $c_0 = 1$), take the limit $l \to \infty$. In our simplest case of the ground state it is possible to make all the calculations up to the very end with the following result

$$c_k = \frac{(-1)^k}{k!} \frac{\Gamma(\frac{\lambda}{\pi} + k)}{\Gamma(\frac{\lambda}{\pi})} \cdot \quad k = 0, 1, ...$$

(4.63)
Inserting this into Eqn. (4.60) we can write the ground state wave function (up to the normalization factor) in a very simple and nice form,

$$\Psi_1 = xe^{-\lambda x} \sum_{k=0}^{\infty} c_k e^{-2\pi k x} =$$

$$xe^{-\lambda x} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\lambda}{\pi} \left( \frac{\lambda}{\pi} + 1 \right) \ldots \left( \frac{\lambda}{\pi} + k - 1 \right) e^{-2\pi k x} =$$

$$xe^{-\lambda x} \left( 1 + e^{-2\pi x} \right)^{-\frac{1}{2}} =$$

$$2^{-\frac{1}{2}} \frac{x}{(\cosh \pi x)^{\frac{1}{2}}} \quad (4.64)$$

It is easy to see that after shifting the parameter $\lambda \to \lambda + 2\pi l$, where $l$ is a positive integer, we again obtain a solution satisfying all the boundary conditions (technically such a solution can be obtained if put zero first $l$ coefficients $c_m, m = 0, \ldots (l-1)$ in the infinite set of linear equations considered above and normalize to unity $c_l$). Moreover, the introduced earlier conserved current is identically zero for any superposition (with complex coefficients) of these functions. This means that each eigenvalue of a total mass (energy) is infinitely degenerate and the appearance of the infinite number of branching point reflects this fact.

### 4.3 Quantum black holes and Hawking’s radiation.

In the preceding sections we have got a complete solution to the Schroedinger equation in finite differences describing a quantum mechanical behavior of the self-gravitating electrically charged spherically symmetric dust shell. For the bound states we found the discrete spectrum of mass which coincides with Dirac spectrum if we put zero the so called the so called radial quantum number. Similarly to the case of hydrogen atom, to which our spectrum reduces in the non-relativistic limit, we may conclude that the shell does not collapse without radiation. The starting classical situation is however different. The classical hydrogen atom collapses radiating continuously while the classical spherically symmetric shell collapses without radiation. But without such a collapse the event horizon and thus the black hole can not be formed at all. Moreover, if we calculate the mean value of radius of the shell using the wave function we will find that at least for large values of principal quantum number it is far away of classical event horizon, so the shell spends
most of its “lifetime” outside the classical black hole. So, to get a black hole solution we need some new criterion and more detailed investigation of the spectrum is needed. Let us consider the dependence of a total mass $m$ on a bare mass $M$, the other variables, $e$ and $n$ fixed. We see that there are two branches, an increasing one for

$$(GM^2 - e^2)(3GM^2 - e^2) < 4n^2$$

(4.65)

and a decreasing one in the opposite case. Note that now

$$\frac{m}{M} > \sqrt{\frac{2}{3}} \sqrt{1 + \frac{e^2}{3GM^2 - e^2}},$$

(4.66)

and this value is greater than the corresponding classical value. The increasing branch corresponds to the “black hole case” while the decreasing branch - to the “wormhole case”. Using a “quasiclassical” argumentation we can say that for the states obeying inequality (4.65) the expectation value of the radius lies outside the event horizon on the “our” side of the Einstein-Rosen bridge (thus replacing the notion of “classical turning point” by that of “radius expectation value”). For the decreasing branch the same occurs on the “other” side of the Einstein-Rosen bridge. It is clear now why the value of total mass-bare mass ratio in quantum case is greater than the corresponding classical value. The origin of this is just the replacing of the classical turning point by the mean value which is smaller thus giving rise to the increase in the total mass. Following such a line of reasoning we should conjecture that the maximal possible value of a total mass $m$ for charge $e$ and principal quantum number $n$ fixed corresponds to the situation when the mean value of the radius of the shell lies at the event horizon making its collapse possible. The further increase in the bare mass $m$ will lead to the wormhole with smaller total mass. Generalizing, we can introduce the notion of the quantum black hole states. For given values of electrical charge $e$ and principal quantum number $n$ the quantum black hole state is the state with maximal possible total mass $m$. Thus, for the quantum black hole states we have, instead of inequality (4.65) the following equation

$$3(GM^2)^2 - 4GM^2e^2 + e^4 - 4n^2 = 0,$$

(4.67)

the solution of which subject to the condition $GM^2 - e^2 > 0$ is

$$GM^2 = \frac{2}{3}e^2 + \frac{1}{3}\sqrt{e^4 + 12n^2}.$$  

(4.68)
For the black hole mass spectrum we get eventually

\[ m_{BH} = \frac{1}{\sqrt{G}} \sqrt{\frac{4\sqrt{3}}{9}} n \left( 1 + \frac{e^4}{12n^2} \right)^{3/2} + \frac{2}{3} e^2 - \frac{e^6}{54n^2} \]

\[ n = 1, 2, 3, \ldots \]

For the uncharged black hole with \( e = 0 \) we have immediately

\[ m = \frac{2}{\sqrt{27}} \sqrt{nm_{Pl}}. \]  

(4.70)

Two limiting cases are of interest. In the first case

\[ \frac{e^4}{12n^2} \ll 1, \]

\[ m_{BH} = \frac{2}{\sqrt{27}} \sqrt{n} \left( 1 + \frac{\sqrt{3} e^2}{6} n - \frac{1}{32} \frac{e^4}{n^2} + \ldots \right), \]

(4.71)

and the charge of the black hole gives rise only to the small corrections to the mass spectrum of the Schwarzschild black hole. The corresponding expressions for the bare mass \( M \) and the ratio \( m/M \) are, respectively,

\[ M = \sqrt{\frac{4}{3}} \sqrt{n} \left( 1 + \frac{\sqrt{3} e^2}{6} n - \frac{1}{48} \frac{e^4}{n^2} + \ldots \right), \]

\[ m = \sqrt{\frac{2}{3}} \left( 1 + \frac{\sqrt{3} e^2}{12} n - \frac{5}{96} \frac{e^4}{n^2} + \ldots \right). \]

(4.72)

In the second case

\[ \frac{e^4}{12n^2} \gg 1, \]

\[ m_{BH} = \frac{|e|}{\sqrt{G}} \left( 1 + \frac{n^2}{2e^4} + \ldots \right), \]

(4.73)

The corresponding expressions for \( M \) and \( m/M \) are, respectively,

\[ M = \frac{|e|}{\sqrt{G}} \left( 1 + \frac{n^2}{e^4} + \ldots \right), \]

\[ m = \frac{m}{M} = 1 - \frac{n^2}{2e^4} + \ldots. \]

(4.74)
It is interesting that in this second limiting case the black hole mass takes values nearly the mass of a classical extreme Reissner-Nordstrom black hole. It is easily seen that in both limiting cases \( \left| \Delta m/m \right| \ll 1 \), where \( \left| \Delta m \right| \) is the difference in masses for nearby energy (mass) levels. So, these limits are essentially quasiclassical ones, and the corresponding mass spectra should be compatible with the existence of Hawking’s radiation. This is indeed the case. Let us write once more the spectra that follow from the black hole thermodynamics in the same limiting cases.

\[
m = \frac{\sqrt{n}}{2\sqrt{G}} \sqrt{\alpha_1} \left( 1 + \frac{\beta_1 + 3 e^2}{2\alpha_1 n} \right) + \left( \frac{\gamma_1}{2\alpha_1} - \frac{(\beta_1 + 1)(\beta_1 + 4)}{8\alpha_1^2} \right) e^4 n^2 + \ldots, \\
\text{(4.75)}
\]

\( n \gg e^2 \)

and

\[
m = \frac{|e|}{2\sqrt{G}} \left( 2 + \frac{\alpha_2^2 n^2}{4 e^4} + \ldots \right), \\
\text{(4.76)}
\]

\( n \ll e^2 \)

Comparing this with the expansion obtained from the black hole spectrum, we have

\[
\alpha_1 = \frac{16}{3\sqrt{3}}, \quad \beta_1 = -\frac{1}{3}, \quad \gamma_1 = \frac{5\sqrt{3}}{144}, \ldots \\
\alpha_2 = 2, \quad \beta_2 = 2, \quad \gamma_2 = -15, \ldots \quad \text{(4.77)}
\]

Comparing values for \( \alpha_1 \) and \( \alpha_2 \) we see that \( \tilde{\alpha} \) is indeed a slowly varying function. Thus, we see that in the limiting case of low temperature our black hole spectrum is compatible with the existence of Hawking’s radiation. Finally, we would like to note that our black hole spectrum obeys also the third law of thermodynamics. Indeed, for a nearly extreme Reissner-Nordstrom black hole the lowest energy level (for \( n = 1 \)) always exceeds the critical value \( (= |e|/\sqrt{\kappa}) \), so the zero temperature state is not accessible.
5 Geometrodynamics for Black Holes and Wormholes.

This Chapter is devoted to the “correct” quantization of the spherically symmetric gravity with thin shells. It means that we start with the Einstein-Hilbert action reduced to the spherically symmetric space-time, make use of the Arnowitt-Deser-Misner (ADM) formalism in order to obtain the canonical constraints and Hamilton equations of motion and formulate the quantum theory.

Our model remains the same. This is just a self-gravitating spherically symmetric dust thin shell, endowed with a bare mass $M$. The whole space-time is divided into three different regions: the inner part (I), the outer part (II) containing no matter fields separated by thin layer III, containing the dust matter of the shell. The general metric of a spherically symmetric space-time is now convenient to write in the form:

$$ds^2 = -N^2 dt^2 + L^2 (dr + N^r dt)^2 + R^2 (d\theta^2 + \sin^2 \theta d\phi^2)$$ (5.1)

where $(t, r, \theta, \phi)$ are space-time coordinates, $N, N^r, L, R$ are some functions of $t$ and $r$ only. Trajectory of the thin shell is some 3-dimensional surface $\Sigma$ in space-time given by some function $\hat{r}(t)$: $\Sigma^3 = \{(t, r, \theta, \phi) : r = \hat{r}(t)\}$. In region I $r < \hat{r} - \epsilon$, in a region II $r > \hat{r} + \epsilon$, region III is a thin layer $\hat{r} - \epsilon < r < \hat{r} + \epsilon$. We require that metric coefficients $N, N^r, L$ and $R$ are continuous functions but jump discontinuities could appear in their derivatives at the points of $\Sigma$ when the limit $\epsilon \to 0$ is taken. The action functional for the system of spherically symmetric gravitational field and the thin shell is

$$S = S_{gr} + S_{shell} = \frac{1}{16\pi G} \int_{I+II+III}^{(4)} R \sqrt{-g} d^4 x + \text{(surface terms)} - M \int_{\Sigma} d\tau$$ (5.2)

It consists of the standard Einstein-Hilbert action for the gravitational field and matter part of the action describes a thin shell of dust. The surface terms in the gravitational action and the falloff behavior of the metric and its derivatives were studied in details by Karel Kuchar. So we will not consider this question and will use the results of Kuchar when needed. We will be interested in the behavior of the action and constraints on the surface $\Sigma^3$ representing the shell’s trajectory. The complete set of degrees of freedom of
our system consists of the set of \( N(r, t), N^\tau(r, t), L(r, t), R(r, t) \) which describe gravitational field and \( \dot{r}(t) \) which describes the motion of the shell. The metric (5.1) has the standard ADM form for 3+1 decomposition of a space-time with lapse function \( N \), shift vector \( N^i = (N^\tau, 0, 0) \) and space metric \( h_{ik} = \text{diag}(L^2, R^2, R^2\sin^2 \theta) \) given foliation of the manifold on space and time. The scalar curvature density has the form

\[
(4) R\sqrt{-g} = N\sqrt{h} \left( (3) R + ((\text{Tr} K)^2 - \text{Tr} K^2) \right) - 2\left( \sqrt{h} K_{\varphi} \right)_\varphi + 2\left( \sqrt{h}KN^i - \sqrt{h}h^{ij}N_j \right),
\]

where \((3) R\) and \( K^{ij} \) are the scalar curvature of a space metric \( h_{ij} \) and exterior curvature tensor of a surface \( t = \text{const} \). Substituting expression (5.1) for the metric into (5.3) we obtain the expression for internal and external curvatures of the surface \( t = \text{const} \) in the form

\[
(3) R = \frac{2}{R^2} \left( 1 - \frac{(R')^2}{L^2} - \frac{2RR''}{L^2} + \frac{2RR'L'}{L^3} \right)
\]

and

\[
K^i_j = \text{diag}(K^r_r, K^\theta_{\theta}, K^\phi_{\phi}),
\]

\[
K^r_r = \frac{1}{NL} \left( \dot{L} - L'N^\tau - L(N^\tau)' \right),
\]

\[
K^\theta_\theta = K^\phi_{\phi} = \frac{1}{NR} \left( \dot{R} - R'N^\tau \right).
\]

Here dot and prime denote differentiation in \( t \) and \( r \) respectively. Contributions to the gravitational action from the terms containing total derivatives in (5.3) give rise to the surface terms which cancel each other at the common boundaries of regions I, II and II, III. So we are left with the surface terms at infinity. We will turn to them later. The essential part of the action for gravitational field is just the ADM part of the action (5.2) with Lagrangian

\[
L_{\text{gr}} = \frac{1}{16\pi G} NLR^2 \left( (3) R - (\text{Tr} K)^2 - \text{Tr} K^2 \right)
\]

Contribution to the action from the integral over the region III in the limit \( \epsilon \to 0 \) is only due to the term containing second derivative of \( R \), namely

\[
\int_{\Pi_I} \frac{1}{16\pi G} NLR^2 (3) R = -\int_{\Pi_I} \frac{NRR''}{GL} = -\int \hat{N}\dot{R}\left[\hat{R}'\right]/\hat{G}\hat{L}
\]

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We will denote by hats variables on $\Sigma$ and by $[A] = \lim_{\epsilon \to 0} (A(\hat{r} + \epsilon) - A(\hat{r} - \epsilon))$ a jump of variable $A(r)$ on the shell surface. Substituting the expression (5.1) into the shell part of the action we have:

$$S_{\text{shell}} = -M \int_{\Sigma} \sqrt{\hat{N}^2 - \hat{L}^2} \left( \dot{\hat{N}r} + \dot{\hat{r}} \right)^2 dt$$  

(5.8)

The explicit form of the action (5.2) with metric (5.1) becomes

$$S = \frac{1}{G} \int_{I+II+III} \left( \frac{NL}{2} \left( R' \right)^2 - \frac{RR'}{L} \right) + \frac{R}{N} \left( \dot{R} - R' N^r \right) \left( \left( LN^r \right)' - \dot{L} \right) + \frac{L}{2N} \left( \dot{R} - R' N^r \right)^2 \right) \int_{\Sigma} \left( \frac{\hat{N} \hat{R} [R']}{L} - m \sqrt{\hat{N}^2 - \hat{L}^2} \left( \hat{N}r + \dot{\hat{r}} \right)^2 \right) dt$$

(5.9)

The canonical formalism for this action can be described in the following way. Momenta conjugate to corresponding dynamical variables are

$$P_N = \delta S / \delta \dot{\hat{N}} = 0;$$
$$P_{N^r} = \delta S / \delta \dot{\hat{N}r} = 0$$
$$P_L = \delta S / \delta \dot{\hat{L}} = \frac{R}{GN} \left( R' N^r - \dot{\hat{R}} \right)$$
$$P_R = \delta S / \delta \dot{\hat{R}} = \frac{L}{GN} \left( R' N^r - \dot{\hat{R}} \right) + \frac{R}{GN} \left( \left( LN^r \right)' - \dot{\hat{L}} \right)$$
$$P_{\hat{R}} = \delta S / \delta \dot{\hat{R}} = 0$$
$$P_{\hat{L}} = \delta S / \delta \dot{\hat{L}} = 0$$
$$\hat{\pi} = \delta S / \delta \dot{\hat{r}} = \frac{m \hat{L}^2 (N^r + \dot{\hat{r}})}{\sqrt{\hat{N}^2 - \hat{L}^2 (N^r + \dot{\hat{r}})}}$$

(5.10)

The action (5.9) rewritten in the Hamiltonian form becomes

$$S = \int_{I+II} \left( P_L \dot{\hat{L}} + P_R \dot{\hat{R}} - NH - N^r H_r \right) d\tau + \int_{\Sigma} \hat{\pi} \dot{\hat{r}} -$$

$$\hat{N} \left( \dot{\hat{R}} [R'] / (G \hat{L}) + \sqrt{m^2 + \hat{\pi}^2 / \hat{L}^2} \right) - \hat{N}^r \left( -\hat{L} [P_L] - \hat{\pi} \right) dt$$

(5.11)

with

$$H = \frac{LP_L^2 R^2 - P_L P_R}{2R^2} + \frac{1}{G} \left( \frac{L^2}{2} - \frac{\left( R' \right)^2}{2L} + \frac{RR'}{L} \right)$$
$$H_r = P_R R' - LP_L'$$

(5.12)
where $\hat{N}$, $\hat{N}^r$, $\hat{N}$ and $\hat{N}^r$ are Lagrange multipliers in the Hamiltonian formalism. The system of constraints contain two surface constraints in addition to usual Hamiltonian and momentum constraints of the ADM formalism.

ADM constraints:

$$\begin{align*}
H &= 0 \\
H_r &= 0
\end{align*}$$

(5.13)

Shell constraints:

$$\begin{align*}
\hat{H}_r &= \hat{\pi} + \hat{L}[P_L] = 0 \\
\hat{H} &= \frac{R[R']}{GL} + \sqrt{M^2 + \hat{\pi}^2} \frac{1}{L^2} = 0
\end{align*}$$

(5.14)

Karel Kuchar proposed some specific canonical transformation of the variables $(R, P_R, L, P_L)$ to new canonical set $(\bar{R}, \bar{P}_R, M, P_M)$ in which Hamiltonian and momentum constraints given by (5.12) are equivalent to the very simple set of constraints:

$$\begin{align*}
\bar{P}_R &= 0 \\
M' &= 0
\end{align*}$$

(5.15)

The idea is to use the Schwarzschild anzatz for the space-time metric instead of the metric (5.1):

$$ds^2 = -F(R, m)dT^2 + \frac{1}{F(R, m)}dR^2 + R^2(d\theta^2 + \sin^2 \theta d\phi^2)$$

(5.16)

where $T, R$ and $m$ are some functions of $(r, t)$ and $F(R, m) = 1 - 2Gm/R$, and $m$, in general, is a function of $r$, $m = m(r)$. Equating the two forms of the metric (5.1) and (5.16) we obtain the transformation between the two sets of dynamical variables. The explicit form of the transformation is

$$\begin{align*}
L &= \sqrt{\frac{(R')^2}{F} - FP_m^2} \\
P_L &= \frac{RFP_m}{G} \sqrt{\frac{(R')^2}{F} - FP_m^2} \\
R &= R \\
\bar{P}_R &= P_R + \frac{P_m}{2G} + \frac{FP_m}{2G} + \frac{(RFP_m)'RR' - RFP_m(RR')'}{GFR \left(\frac{(R')^2}{F} - FP_m^2\right)}
\end{align*}$$

(5.17)
where $P_m = -T'$. The Liouville form

$$\Theta = \int P_R \dot{R} + P_L \dot{L}$$

(5.18)

can be expressed in the new variables as follows:

$$\Theta = \int P_m \dot{m} + P_R \dot{R} + \frac{\partial}{\partial t} \left( LP_L + \frac{1}{2G} RR' \ln \left| \frac{RR' - LP_L G}{RR' + LP_L G} \right| \right)$$

$$+ \frac{\partial}{\partial r} \left( \frac{1}{2G} R \dot{R} \ln \left| \frac{RR' + LP_L G}{RR' - LP_L G} \right| \right).$$

(5.19)

When there is no shell the total derivatives in (5.19) give rise to some surface terms at infinities. As was shown by K.Kuchar the appropriate falloff conditions at infinities make the last surface term in the Eqn.(5.19) zero. Then it follows from (5.19) that $(R, P_R, m, P_m)$ form the canonical set of variables, and Eqn.(5.17) describes a canonical transformation between $(R, P_R, L, P_L)$ and $(R, \bar{P}_R, m, P_m)$.

In the presence of the thin shell the situation is different. Now surface terms should not be neglected. Let us do the transformation (5.17) in regions I and II of our space-time. The Liouville form of our Hamiltonian system (5.11) has the form

$$\tilde{\Theta} = \int_{I+II} P_R \dot{R} + P_L \dot{L} + \int_{\Sigma} \hat{\pi} \dot{r}.$$  

(5.20)

After integration the total derivatives in (5.19) give some contribution to the Liouville form on $\Sigma$:

$$\Theta = \int_{I+II} \tilde{P}_R \dot{R} + P_m \dot{m} + \int_{\Sigma} \left[ LP_L + \frac{1}{2G} RR' \ln \left| \frac{RR' - LP_L G}{RR' + LP_L G} \right| \right] \dot{r} dr$$

$$- \int_{\Sigma} \left[ \frac{1}{2G} R \dot{R} \ln \left| \frac{RR' + LP_L G}{RR' - LP_L G} \right| \right] + \int_{\Sigma} \hat{\pi} \dot{r}$$

$$= \int_{I+II} P_m \dot{m} + \tilde{P}_R \dot{R} + \int_{\Sigma} \hat{\pi} \dot{r} + \int_{\Sigma} \tilde{P}_R \dot{R}$$

(5.21)

where we denoted

$$\hat{\pi} = \hat{\pi} + L [P_L]$$

$$\tilde{P}_R = \pm \left[ \frac{1}{2G} R \ln \left| \frac{RR' - GLP_L}{RR' + GLP_L} \right| \right].$$

(5.22)
and made use of the identity
\[
\dot{R} = \frac{d}{dt} R(t, \dot{r}(t)) = (\dot{R}(t, r) + R'(t, r) \dot{r}(t)) \big|_{r=\dot{r}(t)}
\] (5.23)

The sign in front of logarithm in the definition of the momentum \(P_\dot{R}\) depends on whether we intersect the shell from “in” to “out” when going along the time coordinate curve from past to future (“+”-sign) or the other way around (“-”-sign). We see that this canonical transformation involves all the set of co-

ordinates in the phase space \(\Pi = \{(R(r, t), P_\dot{R}(r, t), L(r, t), P_L(r, t), \dot{r}(t), \dot{\pi}(t))\}\) according to the formulae (5.17) and (5.22). Moreover it introduces additional pair of canonically conjugate variables \((\dot{R}, \dot{\dot{R}})\) on the shell. In both inner and outer regions I and II constraints are simplified due to the canonical transformation as it was in the absence of the shell (5.13). The surface momentum constraint \(\hat{H}_r = 0\) (5.12) takes the form

\[
\hat{p} = 0
\] (5.24)

To go further we need to know the Hamiltonian constraint on the shell in terms of these new canonical variables. To do this, let us consider more carefully Eqn.(5.23) which is the full time derivative of the shell radius. Using the definition \(P_L = -\frac{R}{G \dot{N}} \left(\dot{R} - R' N^{\dot{r}}\right)\) we get:

\[
\dot{R} = -\frac{G N P_L}{R} + R' (\dot{r} + N^{\dot{r}})
\] (5.25)

Remembering that
\[
\pi \equiv \frac{M L^2 (N^{\dot{r}} + \dot{r})}{\sqrt{N^2 - L^2 (N^{\dot{r}} + \dot{r})^2}}
\]
we can find
\[
L (N^{\dot{r}} + \dot{r}) = \frac{\pi N}{L \sqrt{M^2 + \frac{\pi^2}{L^2}}}
\]
Eqn.(5.25) now reads
\[
\frac{\dot{R}}{N} = -\frac{G P_L}{R} + \frac{R'}{L} \frac{\pi}{L \sqrt{M^2 + \frac{\pi^2}{L^2}}}
\] (5.26)
It is now easy to see that the jump of $\dot{R}$ across the shell is a linear combination of constraints

$$
\begin{align*}
[\dot{R}] &= \frac{G N}{R} \left( \chi H - \frac{H^r}{L} \right) \\
\end{align*}
$$

where

$$
\chi = \frac{\pi}{L \sqrt{M^2 + \frac{\pi^2}{L^2}}}
$$

Now, the definition of the momentum $\dot{R}$ can be rewritten as follows

$$
\beta \equiv e^{\pm G \dot{P}_R R} \sqrt{\frac{F_{\text{out}}}{F_{\text{in}}}} = \left( \frac{\dot{R} + G P_L R (1 + \chi)}{N} \right)_{\text{in}} \equiv \frac{\alpha + y_{\text{in}} (1 + \chi)}{\alpha + y_{\text{out}} (1 + \chi)}
$$

where $\alpha = \frac{\dot{R}}{N}$ and $y = G P_L R$.

The next step is to find the relation between $\alpha$ and $y$. From the definitions of $P_L$ and $\dot{R}$ we have

$$
\frac{R^2}{L^2} = F + y^2 = \left( \frac{\alpha + y}{\chi} \right)
$$

Solving this for $y$ we get

$$
y = \frac{\alpha \pm \sqrt{\alpha^2 - (\chi^2 - 1) (F \chi^2 - \alpha^2)}}{\chi^2 - 1}
$$

Substituting this expression back into Eqn. (5.28) we obtain after simple algebraic transformations

$$
\beta = \frac{\frac{\alpha}{z - \sigma_{\text{in}} \sqrt{z^2 + F_{\text{in}}}}}{\frac{\alpha}{z - \sigma_{\text{out}} \sqrt{z^2 + F_{\text{out}}}}}
$$

where $z^2 = \frac{\alpha^2}{1 - \chi^2}$ (5.31)

Making use of the momentum constraint we can write the jump of $y$ as follows

$$
\sigma_{\text{out}} \chi \sqrt{\alpha^2 + (1 - \chi^2) F_{\text{out}}} = \sigma_{\text{in}} \chi \sqrt{\alpha^2 + (1 - \chi^2) F_{\text{in}}} - G \frac{M}{R} \chi \sqrt{1 - \chi^2}
$$

(5.32)
Note that this is just our starting equation (2.7) if we choose the proper time (substituting \( \dot{R}^2 \) for \( z^2 \).) Squaring this equation we get

\[
\begin{align*}
\sigma_{in} \sqrt{z^2 + \hat{F}}_{in} &= -\frac{R|F|}{2MG} + \frac{MG}{2R} \\
\sigma_{out} \sqrt{z^2 + \hat{F}}_{out} &= -\frac{R|F|}{2MG} - \frac{MG}{2R}
\end{align*}
\]  

(5.33)

\[
z = \pm \sqrt{\left(\frac{R|F|}{2MG}\right)^2 - \frac{1}{2} (F_{out} + F_{in}) + \frac{M^2G^2}{4R^2}}
\]  

(5.34)

The sign for \( z \) is opposite to the sign for \( \dot{R} \). Finally we get for the shell constraint

\[
\beta - z + \frac{R|F|}{2MG} - \frac{MG}{2R} = 0
\]  

(5.35)

It can be easily shown that the above constraint is equivalent to the Hamiltonian constraint (5.30). In what follows we will consider the following squared version of the Hamiltonian constraint (5.30) as the suitable classical counterpart for the quantum constraint for the wave function \( \Psi \)

\[
C = F_{out} + F_{in} - \sqrt{F_{out}} \sqrt{F_{in}} \left[ \exp \left( \frac{G\hat{P}_R}{R} \right) + \exp \left( -\frac{G\hat{P}_R}{R} \right) \right] - \frac{M^2G^2}{4R^2}
\]  

(5.36)

The Hamiltonian constraint (5.30) was derived under the assumption that both \( F_{in} \) and \( F_{out} \) are positive. It is possible, of course to derive analogous constraints in \( T_{\pm} \)-regions, where \( F < 0 \). But, instead, we make the following substitution

\[
\sqrt{F} \rightarrow F^{1/2}
\]  

(5.37)

and consider this function as a function of complex variable. Then the point of the horizon \( F = 0 \) becomes a branching point, and we need the rules of the bypass. We assume the following

\[
F^{1/2} = |F| e^{i\phi}
\]

\[
\begin{align*}
\phi &= 0 \quad \text{in} \ R_+ \text{-region} \\
\phi &= \pi/2 \quad \text{in} \ T_- \text{-region} \\
\phi &= \pi \quad \text{in} \ R_- \text{-region} \\
\phi &= -\pi/2 \quad \text{in} \ T_+ \text{-region}
\end{align*}
\]  

(5.38)
for the black hole case, and

\[ \phi = \pi \quad \text{in } R_+\text{-region} \]
\[ \phi = -\pi/2 \quad \text{in } T_-\text{-region} \]
\[ \phi = 0 \quad \text{in } R_-\text{-region} \]
\[ \phi = \pi/2 \quad \text{in } T_+\text{-region} \]

(5.39)

for the wormhole case. The reason for such analytical continuation is that we are able to get the single equation on the wave function \( \Psi \) which covers all four patches of the complete Penrose diagram for the Schwarzschild spacetime. The Carter-Penrose diagram for \( \sigma = \pm 1 \) (black hole and worm hole case, correspondingly) are shown in Fig.10.

Figure 10:

It should be stress here that we consider the thin selfgravitating shell together with the space-time which it "creates". Thus for different type of shell’s motion we have to choose different forms of Hamiltonian constraints. For the shell escaping to infinity we choose the following constraint (see Fig.8 for the corresponding Carter-Penrose diagram in the case \( F_{in} = 1 \))

\[ e^{\pm \frac{\partial}{\partial R}} = \frac{1}{2} (F_{out} + F_{in}) - \frac{M^2 G^2}{2R^2} \pm \frac{MGz}{R}, \quad (5.40) \]

while for the falling shell that start from infinity (Carter-Penrose diagram
is shown in Fig.9) we have
\[
e^{\pm \frac{G_p}{R}} = \frac{1}{2} \left( F_{\text{out}} + F_{\text{in}} \right) - \frac{M^2 G^2}{2R^2} \pm \frac{M G}{R} z
\]
(5.41)

The bound motion is symmetric to the change of the momentum sign, so for the Hamiltonian constraint we choose the sum of the constraints (5.40) and (5.41), namely
\[
e^{\frac{G_p}{R}} + e^{-\frac{G_p}{R}} = \sigma \frac{F_{\text{out}} + F_{\text{in}} - \frac{M^2 G^2}{R^2}}{\sqrt{F_{\text{in}}} \sqrt{F_{\text{out}}}}
\]
(5.42)

We now turn to the Dirac constraint quantization procedure.

It is convenient to make a canonical transformation from \((\hat{R}, \hat{P}_R)\) to \((\hat{S}, \hat{P}_S)\):
\[
\begin{align*}
\hat{S} &= \frac{\hat{R}^2}{R_0} \\
\hat{P}_S &= R_0^2 \frac{\hat{P}_R}{2R}
\end{align*}
\]
(5.43)

where \(R_0\) is some value radius of the shell. Dimensionless variable \(\hat{S}\) is the surface area of the shell measured in the units of horizon area of the shell of mass \(M\).

The phase space of our model consist of coordinates \((R(r), \hat{P}_R(r), m(r), P_m(r), \hat{S}, \hat{P}_S, \hat{r}, \hat{p}_r)\) \(r \in (-\infty, \hat{r} - \epsilon) \cup (\hat{r} + \epsilon, \infty)\). Then the wave function in coordinate representation depends on configuration space coordinates:
\[
\Psi = \Psi(R(r), m(r), \hat{S}, \hat{r})
\]
(5.44)

and all the momenta become operators of the form
\[
\hat{P}_R(r) = -i \frac{\delta}{\delta R(r)} \quad P_m(r) = -i \frac{\delta}{\delta m(r)} \\
\hat{P}_S = -i \frac{\partial}{\partial \hat{S}} \quad \hat{p}_r = -i \frac{\partial}{\partial \hat{r}}
\]
(5.45)

ADM and shell constraints (5.13) and (5.14) become operator equations on \(\Psi\). The set of ADM constraints is equivalent to the set of constraints (5.15) in Kuchar variables which could be easily solved on quantum level. Indeed, in the regions I and II the equations
\[
\begin{align*}
\partial \Psi/\partial R(r) &= 0 \\
M'(r) \Psi &= 0
\end{align*}
\]
(5.46)
express the fact that wave function does not depend on $R(r)$ and the dependence on $M(r)$ is reduced in each region I and II to $\Psi \equiv \delta(M - M_\pm)$ where $M_\pm$ defined in the regions I (-) and II (+) do not depend on $r$. They equal to Schwarzschild masses in the inner and outer regions $M_{in}$ and $M_{out}$ in (4.19).

The set of shell constraints (5.14) impose further restrictions on $\Psi$. First of them takes the form

$$\frac{\partial \Psi}{\partial \hat{r}} = 0$$

(5.47)

in new variables according to (5.24). So the only nontrivial equation is the shell constraint (5.36) (or (5.30, 5.33),(4.19), which are classically equivalent to the Eqn.(5.36))

$$\hat{C} \left( m_+, m_-, \hat{S}, -i\hbar \partial/\partial \hat{S} \right) = 0$$

(5.48)

$$\Psi = \Psi(m_+, m_-)$$

The operator $\hat{C}$ contains the exponent of the of the momentum $\hat{P}_S$. This exponent becomes an operator of finite displacement when $\hat{P}_S$ becomes differential operator:

$$e^{\frac{G \hat{P}_S}{R_0}} = e^{\frac{P_S}{R_0}} \Psi = e^{-i\zeta \frac{\hbar}{\sqrt{S}}} \Psi = \Psi(m_+, m_-, \hat{S} - \zeta i)$$

(5.49)

$$\zeta = \frac{2G}{R_0}$$

For the sake of simplicity we consider here only the case of bound motion of the single shell having the bare mass $M$. It means that $m_{in} = 0$ and we should be use the symmetric form Hamiltonian constraint. In this case it is naturally to choose for $R_0$ the Schwarzschild radius $R_0 = R_g = 2GM$ ($m = m_{out}$). For $\zeta$ we have now $\zeta = \frac{1}{2} \left( \frac{m_{pl}}{m} \right)^2$, and the Hamiltonian constraint can be written in the form

$$\Psi(S + i\zeta) + \Psi(S - i\zeta) = \frac{2 - \frac{1}{\sqrt{S}} - \frac{M^2}{4m^2S}}{(1 - \frac{1}{\sqrt{S}})^2} \Psi(S).$$

(5.50)

We have mentioned already that the classically equivalent constraints give inequivalent quantum theories. This is well known fact. We suggest that the criterion to choose the correct quantum theory is the behavior of the wave functions in the quasiclassical regime. In our case this means the large black holes limit. Indeed, the parameter $\zeta = \frac{1}{2} \left( \frac{m_{pl}}{m} \right)^2$ becomes small for large masses, and the expansion with respect to this parameter is equivalent to
the expansion in Planckian constant $\hbar$ ($m_{pl} = \sqrt{\hbar/Gc}$). In the next Section we will consider this quasiclassical limit and show that our choice for the quantum constraint is a good one (at least in the case of one thin shell). At the end of this Section we would like to make an important remark.

Our quantum equation (5.50) (which is just a Schrödinger equation) is the equation in finite differences rather than differential equation, and the shift in argument is along an imaginary axis. In the case of differential equation we require the solution to be differentiable sufficiently many times. Similarly, we have to demand the solutions of our finite difference equation (5.50) to be analytical functions. This condition is very restrictive but unavoidable. Our previous experience (see [22]) shows that it is the analyticity of the wave functions and not the boundary conditions that lead to the existence of the discrete mass (energy) spectrum for bound states.

The finite difference equation (5.50) becomes an ordinary differential equation in quasiclassical limit which is the same as the limit of large ($m \gg m_{pl}$) black holes. Indeed the parameter of finite displacement of the argument of $\Psi$ in (5.50) $\zeta$ becomes small and we could cut the Taylor expansion

$$\Psi(\hat{S} + \zeta i) = \Psi(\hat{S}) + i\zeta \Psi'(\hat{S}) - \frac{\zeta^2}{2} \Psi''(\hat{S}) + ...$$

(5.51)

at the second order. Thus,

$$\Psi|_{S \pm \zeta i} \approx \Psi(S) \pm \Psi'(S)\zeta i - \frac{\zeta^2}{2} \Psi''(S)\ldots$$

$$F_{\pm} \bigg|_{S \pm \zeta i} = \sqrt{1 - \frac{1}{\sqrt{s \pm \zeta i}}} \approx F_{\pm} \left(1 \pm \frac{1}{2FS^{3/2}}\zeta i + \frac{3}{8FS^{5/2} + \frac{1}{8F^2S^3}}\zeta^2\right)\ldots$$

(5.52)

This leads to ordinary differential equations of second order, which are different in $R_+, R_-, T_+$ and $T_-$ regions due to the different values of the phases in Eqn.(5.50). The interesting for us singular points of these differential equations are

$$S = \infty \text{ and } S = 1.$$  

(5.53)

In the quasiclassical limit our requirement of the analyticity of the solutions to the exact equation (5.50) transforms into the requirement that the branching points of the leading terms in the solutions to the approximate equations

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should be of the same kind. Thus, we need to keep only those terms in the corresponding equations that give us these leading terms. Below we consider the black hole case only. The results are easily translated to the wormhole case.

The singular point \( S = \infty \) in the region \( R_+ \) lies in a classically forbidden region as far as we restrict ourselves with bound motions of the shell only. In order to analyze the behavior of \( \Psi \) in this region we should take (5.50) with \( \phi = 0 \) and expand all the quantities in terms of \( y \), where \( s = (1 + y)^2 \). The result is

\[
\Psi_{yy} - \frac{1}{y} \Psi_y + \frac{1}{\zeta^2} \left( 1 - \frac{M^2}{m^2} + \frac{1}{2y} (2 - \frac{M^2}{m^2}) \right) \Psi = 0 \quad (5.54)
\]

The leading term of the solution is

\[
\Psi \sim y^2 \left( 1 - \frac{M^2}{m^2} - \frac{2}{4 \mu \zeta^2} \exp(-\mu y) \right), \quad (5.55)
\]

\[
\mu = \frac{1}{\zeta} \sqrt{\frac{M^2}{m^2} - 1}, \quad y \gg \zeta
\]

For another singular point in \( R_+ \) region, that is for \( S \to 1 + 0 \) we have \( (s = (1 + z^2)^2) \)

\[
\Psi_{zz} - 3z \Psi_z + \frac{16z}{\zeta^2} \left( 1 - \frac{M^2}{4m^2} \right) \Psi = 0 \quad (5.56)
\]

with leading term

\[
\Psi \sim 1 - \frac{8}{3 \zeta^2} \left( 1 - \frac{M^2}{4m^2} \right) y^{3/2} \quad (5.57)
\]

\[
y = \sqrt{z}, \quad s \gg \zeta, \quad y \gg \zeta, \quad \zeta \ll 1
\]

Comparing the types of the branching points at \( s \to \infty \) and \( s \to 1 + 0 \) we can conclude that

\[
2 - \frac{M^2}{m^2} = n, \quad n = \text{integer} \quad (5.58)
\]

\[
4 \zeta \sqrt{\frac{M^2}{m^2} - 1}
\]

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This is the first quantization condition. We will not consider here the wormhole case. Note only that, as can be shown, the positive values of quantum number $n$ correspond to black holes while negative $n$ correspond to wormholes.

We do not consider here separately the asymptotics in $R_-$ region near the horizon ($s \to 1 + 0$) because it differs from the corresponding solution in $R_+$-region only by the sign in front of the second term.

Let us now turn to the asymptotics of the solutions in $R_-$-region for $s \to \infty$. Due to the minus sign in front of $F^{1/2}$ the equation for the wave function in a $R_-$-region is quite different from that in a $R_+$-region

$$\Psi_{yy} - \frac{1}{y} \Psi_y - \frac{1}{\zeta^2} \left( 16y^2 + 1 - \frac{M^2}{m^2} \right) \Psi = 0 \quad (5.59)$$

The leading term of the asymptotic is now the following

$$\Psi \sim y \frac{M^2}{m^2} - 1 \frac{1}{8\zeta} \exp\left(-\frac{2}{\zeta} y^2\right) \quad (5.60)$$

Note that the falloff in the $R_-$-region is much faster than it is in the $R_+$-region. This is a quite reasonable result because it means that the quantum shell in the black hole case can penetrate into the $R_-$-region (which is completely forbidden for the classical motion) but the probability of such an event is negligible small.

And, again, comparing the types of the branching points at $s \to 1 + 0$ and $s \to \infty$ in the $R_-$-region we get

$$\frac{M^2}{m^2} - 1 = \frac{1}{2} + p, \quad p = \text{positive integer} \quad (5.61)$$

The appearance of the second quantum number is rather surprising and its origin is in nontrivial casual structure of the Schwarschild space-time. The classical shell in the (e.g.) black hole case cannot move in the $R_-$-region, and simply absent on the complete Carter-Penrose diagram. But the quantum shell with parameters corresponding to the black hole case, “feel” both $R_+$- and $R_-$-regions, this why the second quantum number plays its role. It is instructive to compare our result with the famous Coulomb problem for the
hydrogen atom. The parameters of the Coulomb problem are the proton’s and electron’s electric charge and all of them being quantized from very beginning. Therefore, to obtain a discrete spectrum for bound states it is enough to have only one quantum condition. But, in our case there are two continuous parameters, the total mass $m$ and bare mass $M$. And now, in order to obtain a discrete mass (energy) spectrum, we should have two quantum conditions. And just what we really got!

Now, we would like to consider the behavior of the solutions in the vicinity of the horizon (sub-Planckian deviation), where $|y| \gg \zeta$ ($s \sim 1$). To be specific we will be interested in the solutions $R_+$ and $T_-$ regions. The expansion (5.52) is no more valid for the function $F^{1/2}(s \pm i\zeta)$ but it is still valid for the wave $\Psi$. Keeping the leading terms only we have now

$$\Psi_{ss}(s) - \frac{2}{\zeta} \Psi_s(s) + \left(\frac{4}{\alpha \zeta^{5/2}}(1 - \frac{M^2}{4m^2}) - \frac{2}{\zeta^2}\right)\Psi(s) = 0 \quad (5.62)$$

with the solution

$$\Psi \sim e^{ks}, \quad k \approx -\frac{1}{\zeta} \pm \sqrt{-\frac{4}{\alpha \zeta^{5/2}}(1 - \frac{M^2}{4m^2})} \quad (5.63)$$

The coefficient $\alpha$ equals to 1 in the $R_+$-region and to imaginary unit $i$ in the $T_-$-region.

In the $R_+$ region

$$k \approx -\frac{1}{\zeta} \pm \frac{4i}{\alpha \zeta^{5/2}}(1 - \frac{M^2}{4m^2}) \quad (5.64)$$

and we have superposition of two waves (ingoing and out outgoing) with relatively equal amplitudes.

In the $T_-$ region

$$k \approx -\frac{1}{\zeta} \pm \frac{\sqrt{2}}{\zeta^{5/4}}\sqrt{(1 - \frac{M^2}{4m^2})(1 + i)} \quad (5.65)$$

$$\pm \frac{\sqrt{2}}{\zeta^{5/4}}\sqrt{(1 - \frac{M^2}{4m^2}) - \frac{1}{\zeta}} \pm i \frac{\sqrt{2}}{\zeta^{5/4}}\sqrt{(1 - \frac{M^2}{4m^2})}$$
The existence of two waves in the $T_-$-region reflects the quantum trembling of the horizon. But the outgoing wave is enormously damped relative to the ingoing wave (of course, in the $T_-$-region the situation is exactly inverse one). It is this damping that cause (in a quasiclassical regime) existence of the single ingoing wave in the $T_-$-region at the distances larger than Planckian.

In the end of this Subsection let us discuss briefly some features of the obtained discrete mass spectrum (5.58) and (5.61)

Let us discuss some properties of the spectrum that arises from these conditions.

1. For larger values of quantum number $n$ ($\frac{M^2}{m^2} - 1 << 0$) one can easily derive nonrelativistic Rydberg formula for Kepler’s problem, $E_{\text{nonrel}} = M - m = -\frac{G^2 M^4}{8n^2}$.

2. The role of turning point $\rho_0$ is now played by the integer $n$. Thus, keeping $n$ constant and calculating $\gamma = \frac{\partial m}{\partial M}|_n$ one can distinguish between a black hole case ($\gamma > 0$) and a wormhole case ($\gamma < 0$). It appears that $\frac{\partial m}{\partial M}|_n > 0$ for $n \geq n_0$, negative or zero, and

$$|n_0| = E[\sqrt{2\sqrt{13\sqrt{5} - 29}}(1 + 2p)]$$  \hspace{1cm} (5.66)

where $E[\cdots]$ means the “integer part of ...”. It is interesting, that for $n \geq 0$ the asymptotical behaviour of the wave function is exactly the same as in the Coulomb problem, but for the negative values of $n_0$ there is no such analogy.

3. There exists a minimal possible value for a black hole mass. This occurs if $p = n_0 = 0$,

$$m_{\text{min}} = \sqrt{2}m_{\text{pl}}$$  \hspace{1cm} (5.67)

4. It is interesting to study the limiting case when $M \to \infty$ and total mass $m$ is kept finite. It is correspond to the limit of eternal black hole. From our spectrum it easy to deduce that

$$m = \lim_{n, p \to \infty} \left(\frac{n}{\sqrt{p}}\right) m_{\text{pl}},$$  \hspace{1cm} (5.68)

and, as result, we obtain that mass of the black hole may be arbitrary (but positive).

5. The our spectrum is not universal in the sense that corresponding wave function form two-parameter family $\Psi_{n, p}$. But for quantum Schwarzschild black hole we expect a one-parameter family of wave functions. Quantum black holes should have no hairs, otherwise there will be no smooth limit to
the classical black holes. All this means that our spectrum is not a quantum black hole spectrum, and our shell does not collapse (like an electron in hydrogen atom). Physically, it is quite understandable, because the radiation is yet included into consideration. But before doing this we would like to make one important note.

Till now we considered only the case of zero mass $m_{in}$ inside the shell. The generalization to the general case is not difficult, and result is the following

$$\frac{2(\Delta m)^2 - M^2}{\sqrt{M^2 - (\Delta m)^2}} = \sqrt{2m^2_{Pl}n},$$

$$M^2 - (\Delta m)^2 = 2m^2_{Pl}(1 + 2p),$$

(5.69)

Where $M$ is the bare mass of the shell, $m = m_{out}$ is the total mass of the system, and $\Delta m = m_{out} - m_{in}$. Moreover this spectrum is, in fact, the exact spectrum of our finite difference equation. The reason why the formulas obtained for large black holes appeared the exact ones is pure mathematical and we will not discuss it here. But one important feature of the exact spectrum (5.69) should be mentioned. We know already that all linear homogeneous equation in finite differences obey the following feature, namely, any its solutions multiplied by the periodic function which period equals to the argument shift of the equation, gives again the solution. This leads, in general, to infinite degeneracy of the spectrum. But our spectrum (5.69) can be made nondegenerate. The matter is that as the argument of our equation in finite differences (5.50) we have square of the radius and leading term of the asymptotical solution at the right infinity (in $R_+$-region), in general, has a form $\Psi \sim e^{\alpha r^2}$, but there is one exception. The exceptional solution has an asymptotic $\Psi \sim e^{\lambda r}$, and just the same asymptotically behavior exhibit the wave function of the nonrelativistic Coulomb problem. Thus if we demand the existence of the “correct” non-relativistic limit, then “correct” wave function should have the leading asymptotical term $\Psi \sim e^{\lambda r}$, and such a requirement removes a degeneracy.

### 5.1 Radiation and quantum collapse dynamics

In order to investigate the quantum collapse, we should switch on the process of quantum radiation. For the modelling of such radiation we again make
use of the developed above theory of thin shells, but now shells are null and move with the speed of light. The corresponding classical equation is our nonsymmetric constraint \( (5.40) \) in which the bare mass \( M \) is put zero. The quantum equation looks as follows \[25\]

\[
\Psi(m_{in}, m_{out}, s - i\zeta) = \sqrt{\frac{1 - \mu}{1 - \sqrt{s}}} \Psi(m_{in}, m_{out}, s)
\] (5.70)

here \( \mu = m_{in}/m_{out}, \zeta = \frac{1}{2}m_{pl}^2/m_{out}^2 \). The existence of the second infinity on the other side of the Einstein-Rosen bridge leads to the following quantization condition \( (m = m_{out}) \)

\[
\delta m = m_{out} - m_{in} = -2m + 2 \sqrt{m^2 + km_{pl}^2}.
\] (5.71)

where \( k \) is an integer. It is interesting to note that if we put \( k = 1 \) (minimal radiating energy) and require \( \delta m < m \) (not more than the total mass can be radiated away), then we obtain

\[
m = m_{out} > \frac{2}{\sqrt{5}}m_{pl}.
\] (5.72)

Thus, the black hole with the mass given by Eqn.\( (5.67) \) is not radiating and, therefore, it can not be transformed into semi-closed world (wormhole-like case).

The discrete spectrum of radiation \( (5.71) \) is universal in the sense that it does not depend on the structure and mass spectrum of the gravitating source. This means that the energies of radiating quanta do not coincide with level spacing of the source. The most natural way in resolving such a paradox is to suppose that quanta are created in pairs. One of them is radiated away, while another one goes inside. Thus, the quantum collapse can not proceed without radiating even in the case of spherical symmetry. This radiation is accompanying with creation of new shells inside the primary shell we started with. We see, that the internal structure of quantum black hole is formed during the very process of quantum collapse. And if at the beginning we had one shell and knew everything about it, then already after the first pulse of radiation we have more than one way of creating the inner quantum. So, initially the entropy of the system was zero, it starts to grow during the quantum collapse. If somehow such a process would stop we would
call the resulting object “a quantum black hole”. The natural limit is the transition from black hole to the wormhole-like shell. The matter is that such a transition requires (at least in quasi-classical regime) insertion of an infinitely large volume, and the quasi-classical probability for this process is zero.

Let us write down the spectrum of the shell with nonzero Schwarzschild mass, the total mass inside, \( m_{in} \neq 0 \)

\[
\frac{2(\Delta m)^2 - M^2}{\sqrt{M^2 - (\Delta m)^2}} = \frac{2m^2_{pl}}{\Delta m + m_{in}} n
\]

\( M^2 - (\Delta m)^2 = 2(1 + 2p)m^2_{pl} \)

Here \( \Delta m \) is the total mass of the shell, \( M \) is the bare mass, the total mass of the system equals \( m = m_{out} = \Delta m + m_{in} \). For the black hole case \( M^2 < 4m\Delta m \), or

\[
\frac{\Delta m}{M} > \frac{1}{2}\left(\sqrt{\left(\frac{m_{in}}{M}\right)^2 + 1} - \frac{m_{in}}{M}\right).
\]

After switching on the process of radiation governed by Eqn. 5.71, the quantum collapse starts. Our computer simulations shows that evolves in the “correct” direction, e.g. it becomes nearer and nearer to the threshold (5.74) between the black hole case and wormhole case. The process stops exactly at \( n = 0 \)!

The point \( n = 0 \) in the spectrum is very special. Only in such a state the shell does not “feel” not only the outer regions (what is natural for the spherically symmetric configuration) but it does not know anything about what is going on inside. It “feel” only itself. Such a situation reminds the classical (non-spherical) collapse. Finally when all the shells (both the primary one and newly produced) are in the corresponding states \( n_i = 0 \), the system does not “remember” its own history. And this is a quantum black hole. The masses of all the shells obey the relation

\[
\Delta m_i = \frac{1}{\sqrt{2}} M_i.
\]

The subsequent quantum Hawking’s evaporation can produced only via some collective excitations and formation, e.g., of a long chain of microscopic semi-closed worlds.
5.2 Classical analog of quantum black hole

Let us consider large \((m \gg m_{pl})\) quantum black holes. The number of shells (both primary ones and created during collapse) is also very large, and one may hope to construct some classical continuous matter distribution that would mimic the properties of quantum black holes. First of all, we should translate the “no memory” state \((n = 0\) for all the shells) into “classical language”. To do this let us rewrite the Eqn.(2.7) (energy constraint equation) for the shell, inside which there is some gravitating mass \(m_{in}\),

\[
\sqrt{\dot{\rho}^2 + 1 - \frac{2Gm_{in}}{\rho}} - \sqrt{\dot{\rho}^2 + 1 - \frac{2Gm_{out}}{\rho}} = \frac{GM}{\rho} \tag{5.76}
\]

and consider a turning point, \((\dot{\rho} = 0, \rho = \rho_0)\):

\[
\Delta m = m_{out} - m_{in} = M\sqrt{1 - \frac{2Gm_{in}}{\rho_0}} - \frac{GM^2}{2\rho_0} \tag{5.77}
\]

It is clear now that in order to make parameters of the shell \((\Delta m\) and \(M\)) not depending on what is going on inside we have to put \(m_{in} = a\rho_0\).

Our quantum black hole is in a stationary state. Therefore, a classical matter distribution should be static. We will consider a static perfect fluid with energy density \(\varepsilon\) and pressure \(p\). A static spherically symmetric metric can be written as

\[
ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2) \tag{5.78}
\]

where \(\nu\) and \(\lambda\) are functions of the radial coordinate \(r\) only. The relevant Einstein’s equations are (prime denotes differentiation in \(r\))

\[
8\pi G\varepsilon = -e^\lambda \left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2},
\]

\[
-8\pi Gp = -e^\lambda \left(\frac{1}{r^2} - \frac{\nu'}{r}\right) + \frac{1}{r^2}, \tag{5.79}
\]

\[
-8\pi Gp = -\frac{1}{2}e^\lambda \left(\nu'' + \frac{\nu'^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu'\lambda'}{2}\right)
\]

The first of these equations can be integrated to yield

\[
e^{-\lambda} = 1 - \frac{2Gm(r)}{r}, \tag{5.80}
\]
where
\[ m(r) = 4\pi \int_0^r \varepsilon r'^2 dr' \] (5.81)
is the mass function, that must be identified with \( m_{in} \). Thus, \( m(r) = ar \), and
\[ \varepsilon = \frac{a}{4\pi r^2}, \quad e^{-\lambda} = 1 - 2Ga. \] (5.82)

We can also introduce a bare mass function
\[ M(r) = 4\pi \int_0^r \varepsilon e^{\lambda} r'^2 dr', \] (5.83)
and from Eqn.(5.82) we get
\[ M(r) = \frac{ar}{\sqrt{1 - 2Ga}} \] (5.84)
The remaining two equations can now be solved for \( p(r) \) and \( e^\nu \). The solution for \( p(r) \) that has the correct non-relativistic limit is
\[ p(r) = \frac{b}{4\pi r^2}, \quad b = \frac{1}{G}(1 - 3Ga - \sqrt{1 - 2Ga} \sqrt{1 - 4Ga}), \] (5.85)
and for \( e^\nu \) we have
\[ e^\nu = Cr^{2G - \frac{a+b}{1 - 2Ga}}. \] (5.86)
The constant of integration \( C \) can be found from matching of the interior and exterior metrics at some boundary \( r = r_0 \). Let us suppose that \( r > r_0 \) the space-time is empty, so the interior should be matched to the Schwarzschild metric. Of course, to compensate the jump in pressure \( (\Delta p = p(r_0) = p_0) \) we must introduce some surface tension \( \Sigma \). From matching conditions (see, e.g. [11]) it follow that
\[ C = (1 - 2Ga)r_0^{-2G - \frac{a+b}{1 - 2Ga}}, \]
\[ e^\nu = (1 - 2Ga)(r/r_0)^{2G - \frac{a+b}{1 - 2Ga}}, \]
\[ \Sigma = \frac{2\Delta p}{r_0} = \frac{b}{2\pi r_0^3} \] (5.87)

We would like to stress that the pressure \( p \) in our classical model is not real but only effective because it was introduce in order to mimic the quantum
stationary states. We see, that the coefficient $b$ in Eqn.(5.85) becomes a complex number if $a > 1/4G$. Hence, we must require $a \leq 1/4G$, and in the limiting point we have the stiffest possible equation of state $\varepsilon = p$. It means also that hypothetical quantum collective excitations (phonons) would propagate with the speed of light and could be considered as massless quasiparticles. It is remarkable that in the limiting point we have $m(r) = M(r)/\sqrt{2}$ - the same relation as for the total and bare masses in the “no memory” state $n = 0$! The total mass $m_0 = m(r_0)$ and the radius $r_0$ in this case are related $m_0 = 4Gr_0$ - twice the horizon size.

Calculations of Riemann curvature tensor $R^i_{klm}$ and Ricci tensor $R_{ik}$ show that if $p < \varepsilon$ ($a \neq b$) there is a real singularity at $r = 0$. But, surprisingly enough, both Riemann and Ricci tensors have finite limits at $r \to 0$, if $\varepsilon = p$ ($a=b=1/4G$). Therefore we are allowed to introduce the so-called topological temperature in the same way as for classical black holes. The recipe is the following. One should transform the space-time metric by the Wick rotation to the Euclidean form and smooth out the canonical singularity by the appropriate choice of the period for the imaginary time coordinate. The imaginary time coordinate is considered proportional to some angle coordinate. In our case the point $r = 0$ is already the coordinate singularity. The azimuthal angle $\phi$ has the period equal to $\pi$. Thus, all other angles should be periodical with the period $\pi$. The topological temperature is just the inverse of this period.

The easy exercise shows, that the temperature

$$T = \frac{1}{2\pi r_0} = \frac{1}{8\pi Gm_0} = T_{BH}$$

(5.88)

exactly the same as the Hawking’s temperature $T_{BH}$ [5]. The very possibility of introducing a temperature provides us with the one-parameter family of models with universal distributions of energy density and pressure

$$\varepsilon = p = \frac{1}{16\pi Gr^2}$$

(5.89)

the parameter being the total mass $m_0$ or the size $r_0 = 4Gm_0$. It should be noted that the two-dimensional part of the metric (5.78) in the limiting case $a = b$ is nothing more but the Rindler’s metric, and the null surface $r = 0$ serves as the event horizon. The Carter-Penrose diagrams for this case is shown in Fig.11.
Figure 11:
We can now develop some thermodynamics for our model. First of all we should distinguish between global and local thermodynamic quantities. The global quantities are those measured by a distant observer. He measures the total mass of the system $m_0$ and the black temperature $T_{BH} = T_\infty$ and does not know anything more. Let us assume that this observer is rather educated in order to recognize he is dealing with a black hole and to write the main thermodynamic relation

$$dm = TdS.$$  \hspace{1cm} (5.90)

In this way he ascribes to a black hole some amount of entropy, namely, the Hawking-Bekenstein value \cite{3, 5}

$$S = \frac{1}{4} \left(\frac{4\pi r_g}{l_{pl}^2}\right)^2 = 4\pi G m_0^2 = 4\pi \left(\frac{m_0}{M_{pl}}\right)^2$$  \hspace{1cm} (5.91)

The local observer who measure distribution of energy, pressure and local temperature is also rather educated and writes quite a different thermodynamic relation

$$\varepsilon(r) = T(r)s(r) - p(r) - \mu(r)n(r).$$  \hspace{1cm} (5.92)

Here $\varepsilon(r)$ and $p(r)$ are energy density and pressure, $T(r)$ is the local temperature distribution, $s(r)$ is the entropy density, $\mu(r)$ is the chemical potential, and $n(r)$ is the number density of some (quasi)"particles". For the energy density and pressure the local observer gets, of course, the relation (5.89), and for the temperature - the following distribution

$$T(r) = \frac{1}{\sqrt{2\pi r}},$$  \hspace{1cm} (5.93)

which is compatible with the law $T(r)e^\frac{\varepsilon}{T} = \text{const}$ and the boundary condition $T_\infty = T_{BH}$. Such a distribution is remarkable in that if some outer layer of our perfect fluid would be removed, the inner layers would remain in thermodynamic equilibrium. And what about the entropy density? Surely, the local observer is unable to measure it directly but he can receive some information concerning the total entropy from the distant observer. This information and the measured temperature distribution (5.93) allows him to deduce that

$$s(r) = \frac{1}{8\sqrt{2Gr}}$$  \hspace{1cm} (5.94)

and

$$s(r)T(r) = \frac{1}{16\pi Gr^2}$$  \hspace{1cm} (5.95)
It is interesting to note that in the main thermodynamic equation the contribution from the pressure is compensated exactly by the contribution from the temperature and entropy. It is noteworthy to remind that the pressure in our classical analog model is of quantum mechanical origin as well as the black hole temperature. And what is left actually is the dust matter we started from in our quantum model, namely,

$$\varepsilon = \mu n = \frac{1}{16\pi Gr^2}$$  \hspace{1cm} (5.96)

We may suggest now that the quantum black hole is the ensemble of some collective excitations, the black hole phonons, and \( n(r) \) is just the number density of such phonons.

Knowing equation of state, \( \varepsilon = p \), we are able to construct all the thermodynamical potentials for our system. As an example we show here how to calculate the energy as a function of the entropy \( S \), and the number particles \( N \). By the first law of thermodynamics

$$dE = T dS - pdV + \mu dN$$  \hspace{1cm} (5.97)

where \( T = \frac{\partial E}{\partial S} \bigg|_{V,N} \) is a temperature \( p = \frac{\partial E}{\partial V} \bigg|_{S,N} \) is a pressure, and \( \mu = \frac{\partial E}{\partial N} \bigg|_{S,V} \) is a chemical potential. The energy is additive with respect to the particle number \( N \), hence, \( E = Nf(x,y) \) where \( x = \frac{S}{N} \) and \( y = \frac{N}{V} \). Since \( \varepsilon = \frac{E}{V} = yf(x,y) \) and \( p = y^2 \frac{\partial f}{\partial y} \) from the equation of state we obtain

$$f = \alpha(x) = n\alpha(x)$$

$$\varepsilon = p = n^2 \alpha(x)$$

Further,

$$T = n\alpha'(x)$$

$$\mu = n(2\alpha - x\alpha')$$

But, in any static gravitational field \( T = T_0/\sqrt{g_{00}} \) and \( \mu = \mu_0/\sqrt{g_{00}} \), so \( \mu = \gamma_0 T \), where \( \gamma_0 \)-some numerical factor. Thus,

$$2\alpha - x\alpha' = \gamma_0 \alpha'$$

$$\alpha(x) = C_0(\gamma_0 + x)^2$$
where $C_0$ is a constant of integration. It is easy to see that $p/T = 1/4C_0$. In our specific model $p/T^2 = \pi/8G$, so $C_0 = 2G/\pi$. Moreover, because of the relation $\varepsilon = p = Ts = \mu n$ we know that free energy $F = E - TS$ is zero. From this we have for the entropy

$$S = \gamma_0 N \quad (5.98)$$

The black hole entropy equals one fourth of the dimensionless horizon area, and form this we recover the famous Bekenstein-Mukhanov mass spectrum

$$m = \sqrt{\frac{\gamma_0}{4\pi}} \sqrt{N} m_{pl} \quad (5.99)$$

We can even calculate the remaining unknown coefficient $\gamma_0$ using the phonon model. Indeed, since the free energy $F = 0$ and from the well known relation

$$F = -T \ln \sum_n e^{-\varepsilon_n/T} \quad (5.100)$$

we deduce that the partition function

$$Z = \sum_n e^{-\varepsilon_n/T} = 1 \quad (5.101)$$

Assuming now, that our gravitational phonon are linear oscillators, we have $\varepsilon_n = \omega (n + \frac{1}{2})$

$$Z = \frac{e^{-\frac{\varepsilon_n}{T}}}{1 - e^{-\frac{\varepsilon_n}{T}}} = 1 \quad (5.102)$$

and

$$e^{-\frac{\varepsilon_n}{T}} = \frac{\sqrt{5} - 1}{2} \quad (5.103)$$

The energy level is nothing more but bare mass spectrum. For the black hole mass spectrum we have ($n \gg 1$)

$$m = \sqrt{2T} n \ln \frac{\sqrt{5} + 1}{2} = \frac{2n}{8\pi Gm} \ln \frac{\sqrt{5} + 1}{2} \quad (5.104)$$

Identifying the level number $n$ with the “particle” number $N$ in our thermodynamical model, we obtain finally

$$\gamma_0 = \ln \frac{\sqrt{5} + 1}{2} \approx 0.5$$

and for the black hole mass spectrum

$$m \approx 0.2\sqrt{N} m_{pl} \quad (5.105)$$
6 Conclusion

And again, the question which already have been asked in Introduction. Why should we study black holes, for what? I hope, at least something became clear while reading this paper. The main feature of the black holes is their universality. It seems it is this that should distinct quantum black holes from other quantum objects, irrespective of what kind of theory the future quantum gravity will be - fundamental, effective, induced or some other. In the absence of elaborated quantum theory we have to construct models. It is desirable that such models were exactly solvable, because the exact solutions serve as a background of our physical intuition, which, in turn, allow us to separate the model depending features of the object under consideration (in our case - quantum black holes) from the true fundamental ones. And, in general, the author believes strongly, that in the future the specific features of quantum black hole will play the role of some kind of selection rule for selecting the "correct" quantum theories. It is like the "great Fermat theorem" in mathematics, where numerous attempt to prove it resulted in creation of new fields in the number theory. So, the investigation of quantum black holes is not simply the exciting intellectual game.

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