One-loop Feynman integrals with Carlson hypergeometric functions

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Abstract. In this paper, we present analytic results for scalar one-loop two-, three-, four-point Feynman integrals with complex internal masses. The calculations are considered in general space-time dimension $D$ for two- and three-point functions and $D = 4$ for four-point functions. The analytic results are expressed in terms of the Carlson hypergeometric functions ($\mathcal{R}$-functions) and valid for both real and complex internal masses.

1 Introduction

In order to confront particle physics theory with high-precision of experimental data at future colliders, theoretical predictions including high-order corrections are required. In general framework for computing high-order corrections, detailed calculations for one-loop multi-leg and higher-loop are necessary for building blocks. When we compute scattering processes which Feynman diagrams involve internal unstable particles that can be on-shell, we have to resume Feynman propagators with a complex mass term in the denominator. In other words, one has to perform the perturbative renormalization in the Complex-Mass Scheme [1]. Therefore, the calculations for Feynman loop integrals with complex internal masses are of great interest. Furthermore, within the general framework for computing two-loop or higher-loop corrections scalar one-loop integrals in general space-time dimension play a crucial role for several reasons. Higher-terms in the $\varepsilon$-expansion from one-loop integrals are necessary for building blocks. In additional, one-loop integrals at higher space-time dimension $D > 4$ may be taken into account in the framework.

There have been available many calculations for scalar one-loop integrals in $D = 4 - 2\varepsilon$ dimensions at $\varepsilon^0$-expansion [2–11]. Scalar one-loop integrals in general dimension $D$ have performed in [12–16]. However, not all of these calculations cover general dimension $D$ with a general $\varepsilon$-expansion at general scale and complex internal masses. In this paper, based on the method in [5–8], we present analytic results for scalar one-loop two-, three-, four-point Feynman integrals with complex internal masses. The calculations are considered in general space-time dimension $D$ for two- and three-point functions and $D = 4$ for four-point functions. The analytic results are expressed in terms of the Carlson hypergeometric functions.

The layout of the paper is as follows: In section 2, we present in detail the method for evaluating scalar one-loop functions. In this section, analytic results for one-loop two-, three- and four-point functions are presented. Conclusions and outlooks are devoted in section 3. Several useful formulas used in this calculation can be found in the appendix.

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2 The calculations

Based on the method introduced in Refs. [5–7], we present the calculations for scalar one-loop functions with complex internal masses. Scalar one-loop $N$-point functions are defined

$$ J_N = \int d^D l \frac{1}{\mathcal{P}_1\mathcal{P}_2\cdots\mathcal{P}_N}. $$ (1)

Where inverse Feynman propagators are given

$$ \mathcal{P}_k = (l + q_k)^2 - m_k^2 + i\rho, \quad \text{with} \quad k = 1, 2, \cdots, N. $$ (2)

The Feynman prescription is $i\rho$. We use momenta $q_k = \sum_{j=1}^k p_j, p_j$ are external momenta and they are inward as shown in Fig. 1. The internal masses in the Complex-Mass scheme are taken the form of

$$ m_k^2 = m_{0k}^2 - im_{0k} \Gamma_k, \quad \text{for} \quad \Gamma_k \geq 0. $$ (3)

The $\Gamma_k$ are decay widths of unstable particles. The momenta $q_k$ may take the following configuration

$$ q_1 = q_1(q_{10}, q_{11}, 0, \cdots, 0, \overrightarrow{0}_{D-J}), $$ (4)

$$ q_2 = q_2(q_{20}, q_{21}, 0, \cdots, 0, \overrightarrow{0}_{D-J}), $$ (5)

$$ q_3 = q_3(q_{10}, q_{31}, q_{32}, 0, \cdots, 0, \overrightarrow{0}_{D-J}), $$ (6)

$$ \cdots = \cdots, $$

$$ q_{N-1} = q_{N-1}(q_{(N-1)0}, q_{(N-1)1}, \cdots, q_{(N-1)(J-1)}, \overrightarrow{0}_{D-J}) $$ (7)

which have $J$ non-zero components. Here, $q_{10} = 0$ for $q_1^2 < 0$ and $q_{11} = 0$ for $q_1^2 > 0$. As a result, scalar product of external and internal momenta are obtained

$$ q_k^2 = q_{k0}^2 - q_{k1}^2 - \cdots - q_{k(J-1)}^2, $$ (8)

$$ l^2 = l_0^2 - l_1^2 - \cdots - l_{J-1}^2 - l_\perp^2, $$ (9)

$$ l \cdot q_k = l_0 \cdot q_{k0} - l_1 \cdot q_{k1} - \cdots - l_{J-1} \cdot q_{k(J-1)}. $$ (10)

In parallel space which is the linear span of the external momenta and its orthogonal space (POS) [5, 6], scalar one-loop $N$-point functions are taken the form of:

$$ J_N = \frac{2\pi^{n_D/2}}{\Gamma^{(n_D/2)}} \int_{-\infty}^{\infty} dl_0 dl_1 \cdots dl_{J-1} \int_0^{\infty} dl_\perp \frac{p_{D-J-1}}{\mathcal{P}_1\mathcal{P}_2\cdots\mathcal{P}_N}. $$ (11)

The propagators now become

$$ \mathcal{P}_k = (l_0 + q_{k0})^2 - (l_1 + q_{k1})^2 - \cdots - (l_{J-1} + q_{k(J-1)})^2 - l_\perp^2 - m_k^2 + i\rho, $$ (12)

for $k = 1, 2, \cdots, N$. The calculations can be summarized as follows. We first make the partition for the integrand of $J_N$ as

$$ \frac{1}{\mathcal{P}_1\mathcal{P}_2\cdots\mathcal{P}_N} = \sum_{k=1}^N \frac{1}{\mathcal{P}_k \prod_{l=1 \atop l \neq k}^N (\mathcal{P}_l - \mathcal{P}_k)}. $$ (13)
with
\[
\mathcal{P}_l - \mathcal{P}_k = (l_0 + q_{l0})^2 - (l_0 + q_{k0})^2 + (l_1 + q_{l1})^2 - (l_1 + q_{k1})^2 + \cdots + (l_{J-1} + q_{l(J-1)})^2 - (l_{J-1} + q_{k(J-1)})^2 + m_k^2 - m_l^2
\]
(14)
\[
= a_{lk} l_0 + b_{lk} l_1 + \cdots + c_{lk} l_{J-1} + \delta_{lk}.
\]
(15)

Where we have introduced the following kinematic variables
\[
a_{lk} = 2(q_{l0} - q_{k0}), \quad b_{lk} = -2(q_{l1} - q_{k1}), \quad \cdots,
\]
(16)
\[
c_{lk} = -2(q_{l(J-1)} - q_{k(J-1)}), \quad \delta_{lk} = q_l^2 - q_k^2 + m_k^2 - m_l^2.
\]
(17)

Making a shift
\[
l_0 \to l_0 + q_{l0}, \quad l_1 \to l_1 + q_{k1}, \cdots, \quad l_{J-1} \to l_{J-1} + q_{k(J-1)},
\]
(18)
we convert all \(\mathcal{P}_k\) in (13) to \(\mathcal{P}_N\). As a matter of this fact, the \(l_\perp\)-integral then yields a simple form which can be taken easily as follows:
\[
\int_0^\infty dl_\perp \frac{p_{D-1}}{[l_0^2 - l_1^2 - \cdots - l_{J-1}^2 - l_\perp^2 - m_k^2 + ip]} = \Gamma \left( \frac{D-1}{2} \right) \left( -l_0^2 + l_1^2 + \cdots + l_{J-1}^2 + m_k^2 - ip \right)^{\frac{D-3}{2}}.
\]
(19)

We then arrive at the \((J-1)\)-fold integrals
\[
\frac{J_N}{\Gamma \left( \frac{J+2-D}{2} \right)} = \pi^{\frac{D-2}{2}} \sum_{k=1}^N \int_{-\infty}^\infty dl_0 dl_1 \cdots dl_{J-1} \frac{(-l_0^2 + l_1^2 + \cdots + l_{J-1}^2 + m_k^2 - ip)^{\frac{D-3}{2}}}{\prod_{l=1}^N \prod_{k \neq l} [a_{lk} l_0 + b_{lk} l_1 + \cdots + c_{lk} l_{J-1} + \delta_{lk}]}.
\]
(20)

In this formula \(a_{lk}, b_{lk}, \cdots, c_{lk} \in \mathbb{R}\) and \(d_{lk} = (q_{l} - q_{k})^2 - (m_l^2 - m_k^2) \in \mathbb{C}\) which is obtained from \(\delta_{lk}\) after applying the shift (18). The integrals in (20) can be carried out with the help of residue theorem. For that purpose, one first linearizes the \(l_0\) for example, 
\(l_0' = l_1 + l_0\). The result reads
\[
\frac{J_N}{\Gamma \left( \frac{J+2-D}{2} \right)} = \pi^{\frac{D-2}{2}} \sum_{k=1}^N \int_{-\infty}^\infty dl_0 dl_1 \cdots dl_{J-1} \frac{(-2l_0 l_1 + l_1^2 + \cdots + l_{J-1}^2 + m_k^2 - ip)^{\frac{D-3}{2}}}{\prod_{l=1}^N \prod_{k \neq l} [AB_{lk} l_0 + b_{lk} l_1 + \cdots + c_{lk} l_{J-1} + d_{lk}]}.
\]
(21)
with $AB_{lk} = a_{lk} - b_{lk}$. The singularity poles of the integrand in (21) are obtained:

$$l_0 = l_1^2 + \cdots + l_{J-1}^2 + m_k^2 - i\rho \over 2l_1, \quad \text{Im}(l_0) = -m_0\Gamma_k + \rho \over 2l_1,$$

and

$$f_0^{(l)} = -b_{lk}l_1 + \cdots + c_{lk}l_{J-1} + d_{lk} \over AB_{lk}, \quad \text{Im}[f_0^{(l)}] = \text{Im} \left( -d_{lk} \over AB_{lk} \right).$$

The pole $l_0$ in (22) locates upper (lower) in $l_0$-complex plane if $l_1 < 0$ ($l_1 > 0$) respectively. We plan to close the contour integration for $l_0$ that $l_0$-poles in (22) locate outside the contour, seen Fig. 2 for more detail. As a result, the poles in (23) are only taken into account to the residue contributions for $l_0$-integration. The resulting reads

$$J_N \over \Gamma \left( {J + 2 - D \over 2} \right) = \pi \over 2 \sum_{k=1}^{N} \sum_{l_1=1}^{N} \left\{ f_{lk}^+ \int_0^\infty d l_1 + f_{lk}^- \int_{-\infty}^0 d l_1 \right\} \cdots \int_{-\infty}^\infty d l_{J-1} \left[ 1 - \delta(AB_{lk}) \right]$$

$$\times \left[ \left( 1 - 2b_{lk} \over AB_{lk} \right)f_1^2 + \cdots + f_{J-1}^2 - 2c_{lk}l_1l_{J-1} - 2d_{lk}l_1 + m_k^2 - i\rho \over 2 \right] \sum_{m_{lk} = 1}^{N} \prod_{m_{lk} \neq l} \left[ \tilde{A}_{mlk} + \cdots + \tilde{C}_{mlk}l_{J-1} + \tilde{F}_{mlk} \right].$$

Where the $\delta$-function is defined as

$$\delta(x) = \begin{cases} 0, & \text{if } x \neq 0; \\ 1, & \text{if } x = 0. \end{cases}$$

New kinematic variables $\tilde{A}_{mlk}, \cdots, \tilde{C}_{mlk} \in \mathbb{R}$ and $\tilde{F}_{mlk} \in \mathbb{C}$ are obtained from residue contributions of the poles in (23). The functions $f_{lk}^\pm$ indicate the location of the poles in (23) in the $l_0$ complex plane:

$$f_{lk}^+ = \begin{cases} 0, & \text{if } \text{Im} \left( -d_{lk} \over AB_{lk} \right) < 0; \\ 1, & \text{if } \text{Im} \left( -d_{lk} \over AB_{lk} \right) = 0; \quad \text{and} \quad f_{lk}^- = \begin{cases} 0, & \text{if } \text{Im} \left( -d_{lk} \over AB_{lk} \right) > 0; \\ 1, & \text{if } \text{Im} \left( -d_{lk} \over AB_{lk} \right) = 0; \\ 2, & \text{if } \text{Im} \left( -d_{lk} \over AB_{lk} \right) < 0. \end{cases} \end{cases}$$

Figure 2. We close the contour integration for $l_0$ that the poles in (22) locate outside the contour.
We continue to linearize $l_1$ in numerator of the integrand of (24) by applying a Euler shift $l_1 \to l_1 + \beta_k l_2$. $\beta_k$ can be chosen in such a way of the disappearance of $l_1^2$-term. The residue theorem is applied against for $l_1$-integration. At the final stage, the resulting integrals can be expressed in terms of $\mathcal{R}$-functions [18] which is defined as

$$
\int_{r}^{\infty} (x - r)^{q-1} \prod_{i=1}^{k} (z_i + w_i x)^{-b_i} dx
= \mathcal{B}(\beta - \alpha, \alpha) \mathcal{R}_{\alpha-\beta} \left( b_1, \ldots, b_k, r + \frac{z_1}{w_1}, \ldots, r + \frac{z_k}{w_k} \right) \prod_{i=1}^{k} w_i^{-b_i},
$$

with $\beta = \sum_{i=1}^{k} b_i$. In next subsections, we present analytic results for scalar one-loop two-, three- and four-point functions. Detailed calculations for these functions have published in Ref. [17].

### 2.1 One-loop two-point functions

In POS, $J_2$ takes the form of [5, 6]

$$
J_2 = \frac{2\pi^{D-1}}{\Gamma \left( \frac{D-1}{2} \right)} \int_{-\infty}^{\infty} dl_0 \int_{0}^{\infty} dl_+ \mathcal{B}\left( \frac{4 - D}{2}, \frac{1}{2} \right) \frac{p_+^{D-2}}{[(l_0 + q_{10})^2 - l_+^2 - m_1^2 + ip][l_0^2 - l_+^2 - m_2^2 + ip]}. 
$$

Here $q = q(\vec{q}_{10}, \vec{0}_{D-1})$ for $q^2 > 0$. If $q^2 < 0$, we refer [17] for detailed evaluations. The results in [17] have shown that the below formulas for $J_2$ are valid for both above cases. The $\mathcal{R}$-function representation for two-point integrals is as follows [17]:

$$
\frac{J_2}{\Gamma \left( 3 - \frac{D}{2} \right)} = \pi^{(D-1)/2} e^{i\pi(3-D)/2} \frac{2}{2} \frac{\mathcal{B}\left( \frac{4 - D}{2}, \frac{1}{2} \right)}{\mathcal{B}\left( \frac{3 - D}{2}, 1 \right)} \times \left\{ \left( \frac{q^2 + m_1^2 - m_2^2}{2q^2} \right) \mathcal{R}_{\frac{D-4}{2}} \left( \frac{3 - D}{2}, 1; -m_1^2 + ip, -\frac{(q^2 + m_1^2 - m_2^2)^2}{4q^2} \right) + \left( \frac{q^2 - m_1^2 + m_2^2}{2q^2} \right) \mathcal{R}_{\frac{D-4}{2}} \left( \frac{3 - D}{2}, 1; -m_2^2 + ip, -\frac{(q^2 - m_1^2 + m_2^2)^2}{4q^2} \right) \right\}. 
$$

We can derive other representations for $J_2$ by employing the transformations in appendix for $\mathcal{R}$-functions from (46) to (51). For example, using Euler’s transformation (50) for $\mathcal{R}$-functions.
(50), Eq. (29) becomes
\[
\frac{J_2}{\Gamma\left(3 - \frac{D}{2}\right)} = -\pi^{(D-1)/2} B\left(\frac{4 - D}{2}, \frac{1}{2}\right) \times \left\{ \frac{(m_2^2 - ip)^{\frac{D-2}{2}}}{q^2 + m_1^2 - m_2^2} R_{\frac{1}{2}} \left(\frac{5 - D}{2}, 2; \frac{-1}{m_1^2 - ip}, \frac{-4q^2}{(q^2 + m_1^2 - m_2^2)^2}\right) \right. \\
+ \left. \frac{(m_2^2 - ip)^{\frac{D-2}{2}}}{q^2 - m_1^2 + m_2^2} R_{\frac{1}{2}} \left(\frac{5 - D}{2}, 2; \frac{-1}{m_2^2 - ip}, \frac{-4q^2}{(q^2 - m_1^2 + m_2^2)^2}\right) \right\}. 
\tag{30}
\]

It can be seen that the right hand sides of Eqs. (29,30) are symmetric under the interchange of \(m_1^2 \leftrightarrow m_2^2\). From Eqs. (29,30) we can take the limits of \(m_1^2 = m_2^2 \to 0\) and \(q^2 \to 0\) respectively, see Ref. [17] for more detail.

### 2.2 One-loop three-point functions

The momenta \(q_1, q_2\) take the following configuration \(q_1 = q_1(0_{10}, 0_{21}, \theta\_D-2)\), \(q_2 = q_2(0_{20}, 0_{21}, \theta\_D-2)\). Here \(q_{10} = 0\) for \(q_{11}^2 < 0\) and \(q_{11} = 0\) for \(q_{11}^2 > 0\). The results for \(J_3\) in this paper cover both the above cases. The integral \(J_3\) in POS takes the form of [5, 6]

\[
J_3 = \frac{\pi^{D-2}}{\Gamma\left(\frac{D-2}{2}\right)} \int_{-\infty}^{\infty} dl_0 \int_{-\infty}^{\infty} dl_1 \int_{-\infty}^{\infty} l_\perp^{D-3} dl_\perp \frac{1}{[(l_0 + q_{10})^2 - (l_1 + q_{11})^2 - l_\perp^2 - m_1^2 + ip][(l_0 + q_{20})^2 - (l_1 + q_{21})^2 - l_\perp^2 - m_2^2 + ip][l_0^2 - l_1^2 - l_\perp^2 - m_3^2 + ip]}.
\tag{31}
\]

Scalar one-loop three-point functions are also expressed in terms of \(R\)-functions [18] as

![Figure 4. Triangle diagrams.](image-url)
follows \cite{17}

\[
\frac{J_3}{\Gamma\left(2 - \frac{D}{2}\right)} = -\pi^2 i \mathcal{B}(4 - D, 1) \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{[1 - \delta(AB_{lk})]}{A_{mlk}} \left(\alpha_{lk} - ip\right)^{\frac{D}{2}} \nonumber
\]
\[
\times \left\{ S_{lk}^+ f_{lk}^+ \mathcal{R}_{D-4} \left(\frac{4 - D}{2}, \frac{4 - D}{2}, 1; +Z_{lk}^{(1)}, +Z_{lk}^{(2)}, +F_{mlk}\right) \right. \nonumber
\]
\[
+ S_{lk}^- f_{lk}^- \mathcal{R}_{D-4} \left(\frac{4 - D}{2}, \frac{4 - D}{2}, 1; -Z_{lk}^{(1)}, -Z_{lk}^{(2)}, -F_{mlk}\right) \} ,
\]

(32)

for \( m \neq l \). When all internal masses are real, \( f_{lk}^+ = f_{lk}^- = 1 \) and \( S_{lk}^\pm = 1 \), Eq. (32) confirms
the results of, for instance, \( J_3 \) in the Eq. (11) of \cite{6}. We can derive other represents for \( J_3 \) by applying several transformations for \( \mathcal{R} \)-functions, as shown in appendix. For example, with the help of (50), one obtains

\[
\frac{J_3}{\Gamma\left(2 - \frac{D}{2}\right)} = -\pi^2 i \mathcal{B}(4 - D, 1) \sum_{k=1}^{3} \sum_{l=1}^{3} \frac{[1 - \delta(AB_{lk})]}{C_{mlk}} (m_{lk}^2)^{(D-4)/2} \nonumber
\]
\[
\times \left\{ f_{lk}^+ \mathcal{R}_{-1} \left(\frac{6 - D}{2}, \frac{6 - D}{2}, 2; +\frac{1}{Z_{lk}^{(1)}}, +\frac{1}{Z_{lk}^{(2)}}, +\frac{1}{F_{mlk}}\right) \right. \nonumber
\]
\[
- f_{lk}^- \mathcal{R}_{-1} \left(\frac{6 - D}{2}, \frac{6 - D}{2}, 2; -\frac{1}{Z_{lk}^{(1)}}, -\frac{1}{Z_{lk}^{(2)}}, -\frac{1}{F_{mlk}}\right) \} ,
\]

(33)

for \( m \neq l \). The kinematic variables appear in subsection are listed:

\[
a_{lk} = 2(q_{l0} - q_{k0}), \quad b_{lk} = -2(q_{l1} - q_{k1}), \nonumber
\]
\[
AB_{lk} = a_{lk} - b_{lk}, \quad c_{lk} = (q_{k} - q_{l})^2 + m_{lk}^2 - m_{lk}^2, \nonumber
\]
\[
A_{mlk} = -AB_{km} b_{lk} + AB_{lk} b_{km}, \quad C_{mlk} = -AB_{km} c_{lk} + AB_{lk} c_{km}, \nonumber
\]
\[
F_{mlk} = C_{mlk}/A_{mlk}, \quad Z_{lk}^{(1,2)} = \frac{c_{lk}}{a_{lk} + b_{lk}} \pm \sqrt{\left(\frac{c_{lk}}{a_{lk} + b_{lk}}\right)^2 - \frac{m_{lk}^2 - ip}{a_{lk}}} .
\]

The factor \( S_{lk}^\pm \) is given by

\[
S_{lk}^\pm = \text{Exp} \left[ \pi i \theta(-\alpha_{lk}) \theta(\pm \text{Im}(Z_{lk}^{(1)})) \theta(\pm \text{Im}(Z_{lk}^{(2)}))(D - 4) \right] \nonumber
\]
\[
\times \text{Exp} \left[ -\pi i \theta(\alpha_{lk}) \theta(\pm \text{Im}(Z_{lk}^{(1)})) \theta(\pm \text{Im}(Z_{lk}^{(2)}))(D - 4) \right] .
\]

(34)

We turn our attention into the analytic results for scalar one-loop four-point functions in next subsection.

### 2.3 One-loop four-point functions

At present, the calculations for four-point functions are performed in \( D = 4 \). We set configuration of external momenta as follows \( q_1 = (q_{10}, q_{11}, 0, 0), q_2 = (q_{20}, q_{21}, 0, 0), q_3 = (q_{30}, q_{31}, q_{32}, 0) \). Where \( q_{10} = 0 \) for \( q_{11}^2 < 0 \) and \( q_{11} = 0 \) for \( q_{11}^2 > 0 \). Our result presented in this paper cover all the above cases. In POS, \( J_4 \) takes the form of

\[
J_4 = 2 \int_{-\infty}^{\infty} dl_0 dl_1 dl_2 \int_{0}^{\infty} dl_3 \frac{1}{\mathcal{P}_1 \mathcal{P}_2 \mathcal{P}_3 \mathcal{P}_4} , \quad \mathcal{P}_k = (l + q_k)^2 - m_k^2 + ip \text{ for } k = 1, 2, \cdots, 4. \nonumber
\]

(35)

Scalar one-loop four-point functions are
written as one-fold integrals [17] as follows

\[
\frac{J_4}{\pi^2} = \sum_{k=1}^{4} \sum_{l=1}^{4} \sum_{m=1}^{4} \frac{1 - \delta(AC_{lk})(1 - \delta(B_{mlk}))}{AC_{lk}B_{mlk}A_{mlk} - B_{mlk}A_{mlk}} \times \\
\left\{ \int_0^\infty dz \, G(z) \left[ (f^+_{lk} g^+_{mlk} + f^-_{lk} g^-_{mlk}) \ln \left( \frac{F_{nmlk}}{\beta_{mlk}} \right) - f^-_{lk} g^-_{mlk} \ln \left( z + F_{nmlk} \right) \right] \right. \\
\left. - f^+_{lk} g^+_{mlk} \ln \left( -z + F_{nmlk} \right) - (f^+_{lk} g^+_{mlk} + f^-_{lk} g^-_{mlk}) \ln \left( \frac{S(\sigma_{mlk}, z)}{P_{mlk} z + Q_{mlk}} \right) \right. \\
\left. + f^+_{lk} g^+_{mlk} \ln \left( \frac{S(\sigma_{mlk} = 0, z)}{P_{mlk} z + Q_{mlk}} \right) + f^-_{lk} g^-_{mlk} \ln \left( \frac{S(\sigma_{mlk} = 0, z)}{P_{mlk} z + Q_{mlk}} \right) \right] \\
+ \int_{-\infty}^0 dz \, G(z) \left[ -f^-_{lk} g^-_{mlk} \ln \left( \frac{F_{nmlk}}{\beta_{mlk}} \right) + (f^+_{lk} g^+_{mlk} + f^-_{lk} g^-_{mlk}) \ln \left( z + F_{nmlk} \right) \right] \\
\left. - f^+_{lk} g^+_{mlk} \ln \left( -z + F_{nmlk} \right) - (f^+_{lk} g^+_{mlk} + f^-_{lk} g^-_{mlk}) \ln \left( \frac{S(\sigma_{mlk}, z)}{P_{mlk} z + Q_{mlk}} \right) \right. \\
\left. + f^+_{lk} g^+_{mlk} \ln \left( \frac{S(\sigma_{mlk}, z)}{P_{mlk} z + Q_{mlk}} \right) + f^-_{lk} g^-_{mlk} \ln \left( \frac{S(\sigma_{mlk} = 0, z)}{P_{mlk} z + Q_{mlk}} \right) \right\} \\
\]

Where the related kinematic variables are given:

\[
a_{lk} = 2(q_1 - q_{k1}), \quad c_{lk} = -2(q_{l2} - q_{k2}), \quad b_{lk} = -2(q_1 - q_k), \quad d_{lk} = (q_1 - q_k)^2 - (m_l^2 - m_k^2), \\
AC_{lk} = a_{lk} + c_{lk}, \quad \alpha_{lk} = b_{lk} / AC_{lk}, \\
A_{mlk} = a_{mlk} - \frac{d_{mlk}}{AC_{mlk}} A_{mlk}, \quad B_{mlk} = b_{mlk} - \frac{b_{lk}}{AC_{lk}} A_{mlk}, \\
C_{mlk} = d_{mlk} - \frac{d_{mlk}}{AC_{mlk}} A_{mlk}, \quad D_{mlk} = -4(q_1 - q_k)^2 / AC_{lk}^2, \\
F_{nmlk} = \frac{C_{mlk} B_{mlk} - B_{mlk} C_{mlk}}{A_{mlk} B_{mlk} - B_{mlk} A_{mlk}} + i p', \quad p^{(1,2)}_{\beta_{hlk}} = \frac{\alpha_{hlk} - \alpha_{hlk}}{D_{mlk}}, \\
Q_{mlk} = -2 \left( \frac{d_{mlk}}{AC_{mlk}} \right) B_{mlk}, \quad P_{mlk} = -2 \left( \frac{\alpha_{hlk}}{B_{mlk}} - \alpha_{hlk} - \beta_{hlk} D_{mlk} \right), \\
E_{mlk} = -2d_{lk} / AC_{lk}, \quad S_{\eta_{mlk}} = D_{mlk} + P_{mlk} \sigma_{mlk},
\]
with $\sigma_{mlk} = 0, -11/\beta_{mlk}$. The $S(\sigma_{mlk}, z)$ and $G(z)$ are obtained:

$$S(\sigma_{mlk}, z) = S_{mlk}(\sigma_{mlk})^2 + (E_{mlk} + Q_{mlk}\sigma_{mlk})z - m_k^2 + i\rho,$$

$$G^{-1}(z) = Z_{mlk}z^2 + K_{mlk}z - \beta_{mlk}(m_k^2 - i\rho) - F_{nnlk}Q_{mlk},$$

with $Z_{mlk} = D_{mlk}\beta_{mlk} - P_{mlk}$ and $K_{mlk} = E_{mlk}\beta_{mlk} - Q_{mlk} - P_{mlk}F_{nnlk}$. The functions $f_{\pm lk}^\pm$ (and $g_{\pm mlk}^\pm$) are defined as in (26) with replacing $c_{lk}/AB_{lk}$ by $d_{lk}/AC_{lk}$ (and $C_{mlk}/B_{mlk}$) respectively.

The $J_4$ in (36) is decomposed into two basic integrals as follows:

$$I_1 = \int_0^\infty \frac{1}{(z + T_1)(z + T_2)} dz = \mathcal{R}_{-1}(1, 1; T_1, T_2),$$

$$I_2 = \int_0^\infty \ln(1 + z/T_3) \frac{dz}{(z + T_1)(z + T_2)} = \lim_{\omega \to 0} \frac{1}{\omega} \left\{ \int_0^\infty \frac{1}{(z + T_1)(z + T_2)} dz - \int_0^\infty \frac{(1 + z/T_3)^{-\omega}}{(z + T_1)(z + T_2)} dz \right\}$$

$$= \lim_{\omega \to 0} \frac{1}{\omega} \left\{ \mathcal{R}_{-1}(1, 1; T_1, T_2) - \frac{B(1 + \omega, 1)}{T_3^{\omega}} \mathcal{R}_{-1-\omega}(1, 1, \omega; T_1, T_2, T_3) \right\}.$$  

The $\varepsilon$-expansions for all $\mathcal{R}$-functions appear in this paper have devoted in Ref. [17]. The numerical checks for all analytic formulas in this paper and applications of this work to compute Feynman diagrams in real scattering processes have shown in [17].

3 Conclusions

We have presented the analytic results for scalar one-loop two-, three-, four-point Feynman integrals with complex internal masses. The analytic results in this paper are valid for both real and complex internal masses. The calculations have carried out in general space-time dimension for two- and three-point functions. At present work, the four-point functions have performed in $D = 4$. The analytic formulas have expressed in terms of the $\mathcal{R}$-functions. In future work, we will extend this work to tensor one-loop integrals (to be published).

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Appendix: Useful relations for $\mathcal{R}$-functions

Useful relations for $\mathcal{R}$-functions are also listed in this appendix. The formulas shown here are collected from Ref. [18]. We denote that $b, z$ and $e_i$ are $k$-tuple

$$b = (b_1, b_2, \cdots, b_k),$$
$$z = (z_1, z_2, \cdots, z_k),$$
$$e_i = (0, 0, \cdots, 1, 0, \cdots, 0) \text{ where the 1 is located at the } i \text{th entry.}$$
The relations are presented as follows

\[ \mathcal{R}_t(b, z) = \sum_{i=1}^{k} \frac{b_i}{\beta} \mathcal{R}_t(b + e_i, z), \quad (46) \]

\[ \mathcal{R}_{t+1}(b, z) = \sum_{i=1}^{k} \frac{b_i}{\beta} z_i \mathcal{R}_t(b + e_i, z), \quad (47) \]

\[ \beta \mathcal{R}_t(b, z) = (\beta + t) \mathcal{R}_t(b + e_i, z) - tz_i \mathcal{R}_{t-1}(b + e_i, z), \quad (48) \]

\[ \partial_z \mathcal{R}_t(b, z) = \frac{b_i}{\beta} t \mathcal{R}_t(b + e_i, z), \quad (49) \]

\[ \mathcal{R}_t(b, z) = \prod_{i=1}^{k} z_i^{-b} \mathcal{R}_{\beta, t}(b + e_i, z^{-1}), \quad \text{Euler’s transformation} \quad (50) \]

\[ \mathcal{R}_t(b, \lambda z) = \lambda^t \mathcal{R}_t(b, z) \quad \text{scaling law.} \quad (51) \]

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