CUSP KÄHLER-RICCI FLOW ON COMPACT KÄHLER MANIFOLD

JIawei Liu and xi Zhang

Abstract. In this paper, by limiting twisted conical Kähler-Ricci flows, we prove the long-time existence and uniqueness of cusp Kähler-Ricci flow on compact Kähler manifold which carries a smooth hypersurface $D$ such that the twisted canonical bundle $K_M + D$ is ample. Furthermore, we prove that this flow converge to a unique cusp Kähler-Einstein metric.

1. Introduction

In this paper, we study a type of singular Kähler-Ricci flows which are obtained by limiting (twisted) conical Kähler-Ricci flows. Our motivation for considering the limit flows of conical flows is to study the existence of singular Kähler-Einstein metrics as the cone angles tend to 0. In [32], Tian anticipated that the complete Tian-Yau Kähler-Einstein metric on the complement of a divisor should be the limit of Kähler-Einstein metrics as the cone angles tend to 0.

Let $M$ be a compact Kähler manifold with complex dimension $n$ and $D \subset M$ be a smooth hypersurface. Here, by supposing that the twisted canonical bundle $K_M + D$ is ample, we prove the long-time existence, uniqueness and convergence of cusp Kähler-Ricci flow by limiting twisted conical Kähler-Ricci flows. As an application, we show the existence of cusp Kähler-Einstein metric [16, 33] by using cusp Kähler-Ricci flow.

The conical Kähler-Ricci flow was introduced to attack the existence problem of conical Kähler-Einstein metric. This equation was first proposed in Jeffres-Mazzeo-Rubinstein’s paper (Section 2.5 in [13]). Song-Wang made some conjectures on the relation between the convergence of conical Kähler-Ricci flow and the greatest Ricci lower bound of $M$ (conjecture 5.2 in [30]). The long-time existence, regularity and limit behaviour of conical Kähler-Ricci flow have been widely studied, see the works of Liu-Zhang [19, 20], Chen-Wang [4, 5], Wang [38], Shen [27, 28], Edwards [6], Nomura [26], Liu-Zhang [18] and Zhang [39].

By saying a closed positive $(1,1)$-current $\omega$ is conical Kähler metric with cone angle $2\pi\beta \,(0 < \beta \leq 1)$ along $D$, we mean that $D$ is locally given by $\{z^n = 0\}$ and $\omega$ is asymptotically equivalent to model conical metric

$$\sqrt{-1} \sum_{j=1}^{n-1} dz^j \wedge d\overline{z}^j + \frac{\sqrt{-1} dz^n \wedge d\overline{z}^n}{|z^n|^{2(1-\beta)}}.$$  

And by saying a closed positive $(1,1)$-current $\omega$ is cusp Kähler metric along $D$, we mean that $D$ is locally given by $\{z^n = 0\}$ and $\omega$ is asymptotically equivalent to

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model cusp metric
\begin{equation}
\sqrt{-1} \sum_{j=1}^{n-1} dz^j \wedge \overline{dz}^j + \frac{\sqrt{-1} dz^n \wedge \overline{dz}^n}{|z^n|^2 \log^2 |z^n|^2}.
\end{equation}

Let $\omega_0$ be a smooth Kähler metric on $M$ and satisfy $c_1(K_M) + c_1(D) = [\omega_0]$. We pick a section $s$ of $L_D$ cutting out this hypersurface, then we fix a smooth hermitian metric $h$ on $L_D$ and let $\theta$ be its curvature form. In [20], we proved the long-time existence, uniqueness, regularity and convergence of conical Kähler-Ricci flow with weak initial data $\omega_{\varphi_0} \in \mathcal{E}_p(M, \omega_0)$ when $p > 1$, where
\begin{align*}
\mathcal{E}_p(M, \omega_0) &= \{ \varphi \in \mathcal{E}(M, \omega_0) \mid \frac{(\omega_0 + \sqrt{-1} \partial \overline{\partial} \varphi)^p}{\omega_0^p} \in L^p(M, \omega_0^n) \}, \\
\mathcal{E}(M, \omega_0) &= \{ \varphi \in PSH(M, \omega_0) \mid \int_M (\omega_0 + \sqrt{-1} \partial \overline{\partial} \varphi)^n = \int_M \omega_0^n \}.
\end{align*}

Let $\alpha$ be a smooth closed $(1,1)$-form and $\hat{\omega}_\beta = \omega_0 + \sqrt{-1} k \partial \overline{\partial} |s|_h^{2\beta}$. When $c_1(M) = \mu [\omega_0] + (1 - \beta) c_1(D) + [\alpha]$ ($\mu \in \mathbb{R}$), by our arguments in [20], there exists a unique long-time solution of twisted conical Kähler-Ricci flow
\begin{equation}
\begin{cases}
\frac{\partial \bar{\omega}_{\beta}(t)}{\partial t} = - \text{Ric}(\bar{\omega}_{\beta}(t)) + \mu \bar{\omega}_{\beta}(t) + (1 - \beta) [D] + \alpha, \\
\bar{\omega}_{\beta}(t)|_{t=0} = \omega_{\varphi_0},
\end{cases}
\end{equation}
in the following sense:

- For any $[\delta, T]$ ($\delta, T > 0$), there exists constant $C$ such that
  \[ C^{-1} \omega_{\beta} \leq \omega_{\beta}(t) \leq C \omega_{\beta} \quad \text{on} \quad [\delta, T] \times (M \setminus D); \]
- on $(0, \infty) \times (M \setminus D)$, $\omega_{\beta}(t)$ satisfies smooth twisted Kähler-Ricci flow;
- on $(0, \infty) \times M$, $\omega_{\beta}(t)$ satisfies equation (1.3) in the sense of currents;
- there exists metric potential $\varphi_{\beta}(t) \in C^0([0, \infty) \times M) \cap C^\infty((0, \infty) \times (M \setminus D))$ such that $\omega_{\beta}(t) = \omega_0 + \sqrt{-1} \partial \overline{\partial} \varphi_{\beta}(t)$ and $\lim_{t \to 0^+} \| \varphi_{\beta}(t) - \varphi_0 \|_{L^\infty(M)} = 0$;
- on $[\delta, T]$, there exist constant $\alpha \in (0, 1)$ and $C^*$ such that the above metric potential $\varphi_{\beta}(t)$ is $C^\alpha$ on $M$ with respect to $\omega_0$ and $\| \frac{\partial \varphi_{\beta}(t)}{\partial t} \|_{L^\infty(M \setminus D)} \leq C^*$. From Guenancia’s result (Lemma 3.1 in [10]),
\begin{equation}
\omega_{\beta} = \omega_0 - \sqrt{-1} \partial \overline{\partial} \log \frac{1 - |s|_h^{2\beta}}{\beta} := \omega_0 + \sqrt{-1} \partial \overline{\partial} \psi_{\beta}
\end{equation}
is a conical Kähler metric with cone angle $2\pi \beta$ along $D$. Hence, $\omega_{\beta} \in \mathcal{E}_p(M, \omega_0)$ for $p \in (1, \frac{1}{1 - \beta})$. By direct calculations, it is obvious that $\omega_{\beta} \geq \frac{1}{2} \omega_0$ for choosing suitable hermitian metric $h$ and $\omega_{\beta}$ converge to cusp Kähler metric
\begin{equation}
\omega_{cusp} = \omega_0 - \sqrt{-1} \partial \overline{\partial} \log |s|_h^2 := \omega_0 + \sqrt{-1} \partial \overline{\partial} \psi_0
\end{equation}
as $\beta \to 0$. After choosing $\mu = -1$, $\alpha = \beta \theta$ and $\omega_{\varphi_0} = \omega_{\beta}$ in (1.3), we obtain twisted conical Kähler-Ricci flow
\begin{equation}
\begin{cases}
\frac{\partial \omega_{\beta}(t)}{\partial t} = - \text{Ric}(\omega_{\beta}(t)) - \omega_{\beta}(t) + (1 - \beta) [D] + \beta \theta, \\
\omega_{\beta}(t)|_{t=0} = \omega_{\beta}
\end{cases}
\end{equation}
Then by proving uniform estimates (independent of $\beta$) for twisted conical Kähler-Ricci flows (1.6), we obtain a long-time solution to cusp Kähler-Ricci flow

\[
\begin{cases}
\frac{\partial \omega(t)}{\partial t} = -\text{Ric}(\omega(t)) - \omega(t) + [D]. \\
\omega(t)|_{t=0} = \omega_{\text{cusp}}
\end{cases}
\]

(1.7)

**Definition 1.1.** We call $\omega(t)$ a long-time solution to cusp Kähler-Ricci flow (1.7) if it satisfies the following conditions.

1. For any $|\delta, T| (|\delta, T| > 0)$, there exists constant $C$ such that
   \[C^{-1}\omega_{\text{cusp}} \leq \omega(t) \leq C\omega_{\text{cusp}} \text{ on } [\delta, T] \times (M \setminus D);\]
2. On $(0, \infty) \times (M \setminus D)$, $\omega(t)$ satisfies smooth Kähler-Ricci flow;
3. On $(0, \infty) \times M$, $\omega(t)$ satisfies equation (1.7) in the sense of currents;
4. There exists $\varphi(t) \in C^0([0, \infty) \times (M \setminus D)) \cap C^\infty((0, \infty) \times (M \setminus D))$ such that
   \[\omega(t) = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi(t) \text{ and } \lim_{t \to 0^+} \|\varphi(t) - \psi_0\|_{L^1(M)} = 0;\]
5. On $(0, T)$, $\|\varphi(t) - \psi_0\|_{L^\infty(M \setminus D)} \leq C$;
6. On $[\delta, T]$, there exist constant $C$ such that $\|\frac{\partial \varphi(t)}{\partial t}\|_{L^\infty(M \setminus D)} \leq C$.

There are some important results on Kähler-Ricci flows (as well as its twisted versions with smooth twisting forms) from weak initial data, such as Chen-Ding [2], Chen-Tian-Zhang [3], Guedj-Zeriahi [9], Lott-Zhang [21, 22], Nezza-Lu [25], Song-Tian [29], Székelyhidi-Tosatti [31] and Zhang [40] etc. In particular, on $M \setminus D$, Lott-Zhang [21] thoroughly studied the existence and convergence of Kähler-Ricci flow whose initial metric is finite volume Kähler metric with “superstandard spatial asymptotics” (Definition 8.10 in [21]). Their flow keeps “superstandard spatial asymptotics” and this type of metrics contain cusp Kähler metrics. Here we consider Kähler-Ricci flow with non-smooth twisting form and weak initial data on $M$ globally, which can be seen as Lott-Zhang’s case in some sense when we restrict it on $M \setminus D$. There are also some significant results on singular Ricci flows, see Ji-Mazzeo-Sesum [14], Kleiner-Lott [15], Mazzeo-Rubinstein-Sesum [24], Topping [35] and Topping-Yin [37] etc.

In [21], we studied conical Kähler-Ricci flow which is twisted by non-smooth twisting form and starts from weak initial data with $L^p$-density for $p > 1$. Here, by approximating twisted conical Kähler-Ricci flows (1.4), we study cusp Kähler-Ricci flow with initial data $\omega_{\text{cusp}}$ which only admits $L^1$-density. For obtaining flow (1.7), in addition to getting uniform estimates (independent of $\beta$) of flows (1.6), it is important to prove that $\varphi(t)$ converge to $\psi_0$ globally in $L^1$-sense and locally in $L^\infty$-sense outside $D$ as $t \to 0^+$. In this process, we need to construct auxiliary function, and we also need a key observation (Proposition 2.7 and 2.8) that both $\psi_\beta$ and $\varphi_\beta(t)$ are monotone decreasing as $\beta \searrow 0$. Then we obtain a uniqueness result of cusp Kähler-Ricci flow. In fact, we obtain the following theorem.

**Theorem 1.2.** Let $M$ be a compact Kähler manifold and $\omega_0$ be a smooth Kähler metric. Assume that $D \subset M$ is a smooth hypersurface which satisfies $c_1(K_M) + c_1(D) = [\omega_0]$. Then there exists a unique long-time solution $\omega(t) = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi(t)$ to cusp Kähler-Ricci flow (1.7).
Remark 1.3. The uniqueness need to be understood in the following sense: if $\phi(t) \in C^0([0, \infty) \times (M \setminus D)) \cap C^\infty((0, \infty) \times (M \setminus D))$ is a solution to equation
\begin{equation}
\begin{aligned}
\frac{\partial \phi(t)}{\partial t} &= \log \frac{\omega_0 + \sqrt{-1} \partial \bar{\partial} \phi(t)}{\omega_0^n} - \phi(t) + h_0 + \log |s|^2_h \\
\phi(0) &= \psi_0
\end{aligned}
\end{equation}
on $(0, \infty) \times (M \setminus D)$ and satisfies (1), (4), (5) and (6) in Definition 1.1, then $\phi(t)$ lies below $\varphi(t)$ which is obtained by limiting twisted conical Kähler-Ricci flows $\omega(t)$ in Theorem 1.2. When $n = 1$, this uniqueness property is called “maximally stretched” in Topping’s \cite{Topping} and Giesen-Topping’s \cite{Giesen-Topping} works.

Remark 1.4. Since $K_M + D$ is ample, $K_M + (1 - \beta)D$ is also ample for sufficiently small $\beta$. Guenancia \cite{Guenancia} proved that cusp Kähler-Einstein metric is the limit of conical Kähler-Einstein metrics with background metrics $\omega_0 - \beta \theta$ as $\beta \to 0$. The cohomology classes are changing in this process. But in the flow case, we can not obtain a uniqueness result of cusp Kähler-Ricci flow (1.7) if we choose the approximating flows that are conical Kähler-Ricci flows with background metrics $\omega_0 - \beta \theta$. In fact, if we choose the approximating flows that are conical Kähler-Ricci flows
\begin{equation}
\begin{aligned}
\frac{\partial \tilde{\omega}_\beta(t)}{\partial t} &= -\text{Ric}(\tilde{\omega}_\beta(t)) - \tilde{\omega}_\beta(t) + (1 - \beta)[D] \\
\tilde{\omega}_\beta(t)|_{t=0} &= \omega_0 - \beta \theta + \sqrt{-1} \partial \bar{\partial} \psi_\beta
\end{aligned}
\end{equation}
with background metrics $\omega_0 - \beta \theta$, that is, $\tilde{\omega}_\beta(t) = \omega_0 - \beta \theta + \sqrt{-1} \partial \bar{\partial} \tilde{\varphi}_\beta(t)$, we can also get a long-time solution $\tilde{\omega}(t) = \omega_0 + \sqrt{-1} \partial \bar{\partial} \tilde{\varphi}(t)$ to equation (1.7). But we do not know whether $\tilde{\varphi}(t)$ is unique or maximal. We can only prove $\tilde{\varphi}_\beta(t) + \beta \log |s|^2_h \not\nearrow \tilde{\varphi}(t)$ outside $D$ as $\beta \nearrow 0$. However, by the uniqueness result in Theorem 1.2, $\tilde{\varphi}(t)$ must lie below $\varphi(t)$. Therefore, we set the background metric to $\omega_0$ in this paper.

At last, we prove the convergence of cusp Kähler-Ricci flow (1.7).

Theorem 1.5. Cusp Kähler-Ricci flow (1.7) converge to a Kähler-Einstein metric with cusp singularity along $D$ in $C^\infty_{\text{loc}}$-topology outside hypersurface $D$ and globally in the sense of currents.

Kobayashi \cite{Kobayashi} and Tian-Yau \cite{Tian-Yau} asserted that if the twisted canonical bundle $K_M + D$ is ample, then there is a unique (up to constant multiple) complete cusp Kähler-Einstein metric with negative Ricci curvature on $M \setminus D$. The above convergence result recovers the existence of this cusp Kähler-Einstein metric.

The paper is organized as follows. In section 2, we prove the long-time existence and uniqueness of cusp Kähler-Ricci flow (1.7) by limiting twisted conical Kähler-Ricci flows (1.6) and constructing auxiliary function. In section 3, we prove the convergence theorem.

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2. The long-time existence of cusp Kähler-Ricci flow

In this section, we prove the long-time existence of cusp Kähler-Ricci flow by limiting twisted conical Kähler-Ricci flows (1.6), and we also prove the uniqueness theorem. For further consideration in the following arguments, we shall pay attention to the estimates which are independent of $β$.

From our arguments in [20], we know that there exists a unique long-time solution $ϕ_β(0) = ψ_β$ on $(0, ∞) \times (M \setminus D)$, where $h_0$ satisfies $-\text{Ric}(ω_0) + θ - ω_0 = \sqrt{-1}∂\bar{∂}h_0$. Let $ϕ_β(t) = ϕ_β(t) - ψ_β$, we write the equation (2.1) as

\[
\begin{cases}
\frac{∂ϕ_β(t)}{∂t} = \log \frac{(ω_0 + \sqrt{-1}∂\bar{∂}ϕ_β(t))^n}{ω_0^n} - ϕ_β(t) + h_0 + (1 - β) \log |s|_h^2 + (1 - β) ω_0^n

φ_β(0) = ψ_β
\end{cases}
\]

on $(0, ∞) \times (M \setminus D)$, where $h_β = -ψ_β + h_0 + \log \frac{|s|_h^{(1-β)ω_0^n}}{ω_0^n}$ is uniformly bounded by constant $C$ independent of $β$.

Lemma 2.1. There exists constant $C$ independent of $β$ and $t$ such that

\[
\|ϕ_β(t)\|_{L^∞(M)} ≤ C.
\]

Proof: For any $ε > 0$, we let $χ_{β,ε}(t) = ϕ_β(t) + ε \log |s|_h^2$. Since $χ_{β,ε}(t)$ is smooth on $M \setminus D$, bounded from above and goes to $-∞$ near $D$, it achieves its maximum on $M \setminus D$. Let $(t_0, x_0)$ be the maximum point of $χ_{β,ε}(t)$ on $[0, T] \times M$ with $x_0 \in M \setminus D$. If $t_0 = 0$, then we have

\[
ϕ_β(t) ≤ -ε \log |s|_h^2.
\]

If $t_0 ≠ 0$. At $(t_0, x_0)$, we have

\[
0 ≤ \frac{∂χ_{β,ε}(t)}{∂t} = \log \frac{(ω_β + \sqrt{-1}∂\bar{∂}ϕ_β(t))^n}{ω_β^n} - ϕ_β(t) + h_β
\]

\[
= \log \frac{(ω_β + \sqrt{-1}∂\bar{∂}χ_{β,ε}(t) + ϵθ)^n}{ω_β^n} - ϕ_β(t) + h_β
\]

\[
≤ n \log 2 - ϕ_β(t) + C.
\]

Hence, $ϕ_β(t_0, x_0) ≤ C$ and

\[
ϕ_β(t) ≤ C - ε \log |s|_h^2,
\]

where constant $C$ independent of $β$, $t$ and $ε$. Let $ε → 0$, we have $ϕ_β(t) ≤ C$ on $M \setminus D$. Since $ϕ_β(t)$ is continuous, $ϕ_β(t) ≤ C$ on $M$.

For the minimum, we can reproduce the same arguments with $χ_{β,ε}(t) = ϕ_β(t) - ε \log |s|_h^2$, and get $ϕ_β(t) ≥ C$ on $M$. \qed

We now prove the uniform equivalence of volume forms along complex Monge-Ampère equations (2.2).
Lemma 2.2. For any $T > 0$, there exists constant $C$ independent of $\beta$ such that for any $t \in (0, T]$,
\begin{equation}
\left(\frac{t^n}{C}\right) \leq \left(\frac{\omega_\beta + \sqrt{-1} \partial \overline{\partial} \phi_\beta(t)}{\omega_\beta^n}\right)^n \leq e^{\frac{t}{\delta T}} \quad \text{on } M \setminus D.
\end{equation}

**Proof:** For any $t > 0$, we assume that $t \in [\delta, T]$ with $\delta > 0$. Let $\Delta_{\beta,t}$ be the Laplacian operator associated to $\omega_\beta(t) = \omega_\beta + \sqrt{-1} \partial \overline{\partial} \phi_\beta(t)$. Straightforward calculations show that
\begin{equation}
\left(\frac{\partial}{\partial t} - \Delta_{\beta,t}\right) \phi_\beta(t) = -\dot{\phi}_\beta(t).
\end{equation}

Let $H_{\beta,\varepsilon}^+(t) = (t - \delta) \phi_\beta(t) - \phi_\beta(t) + \varepsilon \log |s|_h^2$. Since $H_{\beta,\varepsilon}^+(t)$ is smooth on $M \setminus D$, bounded from above and goes to $-\infty$ near D, it achieves its maximum point on $M \setminus D$. Let $(t_0, x_0)$ be the maximum point of $H_{\beta,\varepsilon}^+(t)$ on $[\delta, T] \times M$ with $x_0 \in M \setminus D$. If $t_0 = \delta$, then
\begin{equation}
(t - \delta) \phi_\beta(t) \leq C - \varepsilon \log |s|_h^2,
\end{equation}
where constant $C$ independent of $\beta$, $\delta$, $t$ and $\varepsilon$. If $t_0 \neq \delta$, then we have
\begin{equation}
\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta_{\beta,t}\right) H_{\beta,\varepsilon}^+(t) &= - (t - \delta) \dot{\phi}_\beta(t) + n + tr_{\omega_\beta(t)} (-\omega_\beta + \varepsilon \theta) \\
&\leq - (t - \delta) \dot{\phi}_\beta(t) + n
\end{aligned}
\end{equation}
for sufficiently small $\varepsilon$. By the maximum principle, we have
\begin{equation}
(t - \delta) \dot{\phi}_\beta(t) \leq C - \varepsilon \log |s|_h^2,
\end{equation}
where constant $C$ independent of $\beta$, $\delta$, $t$ and $\varepsilon$. Let $\varepsilon \to 0$ and then $\delta \to 0$, we have
\begin{equation}
\dot{\phi}_\beta(t) \leq \frac{C}{\delta} \quad \text{on } (0, T) \times (M \setminus D),
\end{equation}
where constant $C$ independent of $\beta$ and $t$.

Let $H_{\beta,\varepsilon}^-(t) = \phi_\beta(t) + 2 \phi_\beta(t) - n \log(t - \delta) - \varepsilon \log |s|_h^2$. Then $H_{\beta,\varepsilon}^-(t)$ tend to $+\infty$ either $t \to 0^+$ or $x \to D$.
\begin{equation}
\left(\frac{\partial}{\partial t} - \Delta_{\beta,t}\right) H_{\beta,\varepsilon}^-(t) \geq \frac{\dot{\phi}_\beta(t)}{t - \delta} - 2n - \frac{n}{t - \delta} + tr_{\omega_\beta(t)} \omega_\beta.
\end{equation}
Assume that $(t_0, x_0)$ is the minimum point of $H_{\beta,\varepsilon}^-(t)$ on $[\delta, T] \times M$ with $t_0 > \delta$ and $x_0 \in M \setminus D$. There exists constant $C_1$ and $C_2$ such that
\begin{equation}
\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta_{\beta,t}\right) H_{\beta,\varepsilon}^-(t)|_{(t_0, x_0)} &\geq \left(C_1 \frac{\omega_\beta^n}{\omega_\beta^n(t)} \right)^{\frac{1}{2}} + \log \left(\frac{\omega_\beta^n}{\omega_\beta^n(t)} \right)^{\frac{1}{2}} - \frac{C_2}{t - \delta})|_{(t_0, x_0)} \\
&\geq \left(C_1 \frac{\omega_\beta^n}{\omega_\beta^n(t)} \right)^{\frac{1}{2}} - \frac{C_2}{t - \delta})|_{(t_0, x_0)}.
\end{aligned}
\end{equation}
where constant $C_1$ depends only on $n$, $C_2$ depends only on $n$, $\omega_0$ and $T$. In inequality (2.13), without loss of generality, we assume that $\frac{\omega_\beta^n(t)}{\omega_\beta^n(t)} > 1$ and $\frac{C_1}{2} \frac{\omega_\beta^n(t)}{\omega_\beta^n(t)} > 0$ at $(t_0, x_0)$. By the maximum principle, we have
\begin{equation}
\omega_\beta^n(t_0, x_0) \geq C_4 (t_0 - \delta)^n \omega_\beta^n(x_0),
\end{equation}
where $C_4$ independent of $\beta$, $\varepsilon$ and $\delta$. Then it easily follows that
\begin{equation}
\dot{\phi}_\beta(t) \geq -C + n \log(t - \delta) + \varepsilon \log |s|_h^2,
\end{equation}
\begin{equation}
\frac{\partial}{\partial t} - \Delta_{\beta,t} - |\partial t| \geq -C + n \log(t - \delta) + \varepsilon \log |s|_h^2.
\end{equation}
where constant $C$ independent of $\beta$, $\varepsilon$ and $\delta$. Let $\varepsilon \to 0$ and then $\delta \to 0$, we have

$$
\dot{\phi}_\beta(t) \geq -C + n \log t \quad \text{on} \quad (0, T] \times (M \setminus D),
$$

where constant $C$ independent of $\beta$. By (2.11) and (2.10), we obtain (2.6). \qed

We first recall Guenancia’s results about the curvature of $\omega_\beta$ (Theorem 3.2 [10]).

**Lemma 2.3.** There exists a constant $C$ depending only on $M$ such that for all $\beta \in (0, \frac{1}{2}]$, the holomorphic bisectional curvature of $\omega_\beta$ is bounded by $C$.

Next, we prove the uniform equivalence of metrics along twisted conical Kähler-Ricci flows (1.6) by Chern-Lu inequality.

**Lemma 2.4.** For any $T > 0$, there exists constant $C$ independent of $\beta$ such that for any $t \in (0, T]$,

$$
e^{-C_1} \omega_\beta \leq \omega_\beta(t) \leq e^{C_2} \omega_\beta \quad \text{on} \quad M \setminus D.
$$

**Proof:** By Chern-Lu inequality [11][23], on $M \setminus D$, we have

$$
\Delta_{\beta, t} \log \tr_{\omega_\beta(t)} \omega_\beta = \frac{\bar{g}_{\beta j} \bar{g}_{\beta k} R_{\beta i j k \ell} g_{\ell k i \ell}}{\tr_{\omega_\beta(t)} \omega_\beta} - \frac{1}{\tr_{\omega_\beta(t)} \omega_\beta} \frac{\bar{g}_{\beta j} \bar{g}_{\beta k} R_{\beta i j k \ell}}{\tr_{\omega_\beta(t)} \omega_\beta} \frac{\tr_{\omega_\beta(t)} \omega_\beta}{(\tr_{\omega_\beta(t)} \omega_\beta)^2} + \frac{\bar{g}_{\beta j} \bar{g}_{\beta k} \bar{g}_{\beta q l \ell} \bar{g}_{\beta k p q} \bar{g}_{\ell p q}}{\tr_{\omega_\beta(t)} \omega_\beta}.
$$

At the same time, on $M \setminus D$,

$$
\frac{\partial}{\partial t} \log \tr_{\omega_\beta(t)} \omega_\beta = \frac{g_{\beta j} \bar{g}_{\beta k} (R_{\beta i j k \ell} + g_{\beta k i \ell} - \beta \theta_{k i \ell})}{\tr_{\omega_\beta(t)} \omega_\beta}.
$$

By using Cauchy-Schwarz inequality and Lemma 2.3, we have

$$
\left(\frac{\partial}{\partial t} - \Delta_{\beta, t}\right) \log \tr_{\omega_\beta(t)} \omega_\beta \leq C \tr_{\omega_\beta(t)} \omega_\beta + 1,
$$

where constant $C$ independent of $\beta$.

Let $H_{\beta, x}(t) = (t - \delta) \log \tr_{\omega_\beta(t)} \omega_\beta - A \dot{\phi}_\beta(t) + \varepsilon |s|^2_\beta$, $A$ be a sufficiently large constant and $(t_0, x_0)$ be the maximum point of $H_{\beta, x}(t)$ on $[\delta, T] \times (M \setminus D)$. We know that $x_0 \in M \setminus D$ and we need only consider $t_0 > \delta$. By direct calculations,

$$
\left(\frac{\partial}{\partial t} - \Delta_{\beta, t}\right) H_{\beta, x}(t) \leq \log \tr_{\omega_\beta(t)} \omega_\beta + C \tr_{\omega_\beta(t)} \omega_\beta - A \dot{\phi}_\beta(t) - A \tr_{\omega_\beta(t)} \omega_\beta + \varepsilon t \tr_{\omega_\beta(t)} \theta + C
$$

$$
\leq -\frac{A}{2} \tr_{\omega_\beta(t)} \omega_\beta + \log \tr_{\omega_\beta(t)} \omega_\beta - A \log \frac{\omega_\beta(t)}{\omega_\beta} + C,
$$

where constant $C$ independent of $\beta$ and $\delta$.

Without loss of generality, we assume that $-\frac{A}{4} \tr_{\omega_\beta(t)} \omega_\beta + \log \tr_{\omega_\beta(t)} \omega_\beta \leq 0$ at $(t_0, x_0)$. Then at $(t_0, x_0)$, by Lemma 2.2 we have

$$
\left(\frac{\partial}{\partial t} - \Delta_{\beta, t}\right) H_{\beta, x}(t) \leq -\frac{A}{4} \tr_{\omega_\beta(t)} \omega_\beta - A \log(t - \delta) + C.
$$

By the maximum principle, at $(t_0, x_0)$,

$$
\tr_{\omega_\beta(t)} \omega_\beta \leq C \log \frac{1}{t - \delta} + C,
$$
which implies that
\[(2.22) (t - \delta) \log tr_{\omega_\beta(t)} \omega_\beta \leq (t_0 - \delta) \log(C \log \frac{1}{t_0 - \delta} + C) + C - \varepsilon \log |s|^2.
\]
Let \(\varepsilon \to 0\) and then \(\delta \to 0\), on \((0, T) \times (M \setminus D)\),
\[(2.23) tr_{\omega_\beta(t)} \omega_\beta \leq e^\frac{C}{t}.
\]
By using inequality
\[(2.24) tr_{\omega_\beta(t)} \omega_\beta \leq \frac{1}{(n-1)!} (tr_{\omega_\beta(t)} \omega_\beta)^{n-1} \frac{\omega_\beta^n(t)}{\omega_\beta^t},
\]
we have
\[(2.25) tr_{\omega_\beta(t)} \omega_\beta \leq e^\frac{C}{t},
\]
where \(C\) independent of \(\beta\). From (2.23) and (2.25), we prove the lemma. □

By the argument as that in [20], we get the following local Calabi's \(C^3\)-estimates and curvature estimates.

**Lemma 2.5.** For any \(T > 0\) and \(B_r(p) \subset M \setminus D\), there exist constants \(C, C'\) and \(C''\) depend only on \(n, T, \omega_0\) and \(\operatorname{dist}_{\omega_0}(B_r(p), D)\) such that
\[\begin{align*}
S_{\omega_\beta(t)} &\leq \frac{C'}{r^2} e^{\frac{C}{t}}, \\
|\operatorname{Rm}_{\omega_\beta(t)}|_{\omega_\beta(t)}^2 &\leq \frac{C''}{r^4} e^{\frac{C}{t}}
\end{align*}
\]
on \((0, T) \times B_r(p)\).

By using the standard parabolic Schauder regularity theory [17], we obtain the following proposition.

**Proposition 2.6.** For any \(0 < \delta < T < \infty\), \(k \in \mathbb{N}^+\) and \(B_r(p) \subset M \setminus D\), there exists constant \(C_{\delta, T, k, p, r}\) depends only on \(n, \delta, k, T, \omega_0\) and \(\operatorname{dist}_{\omega_0}(B_r(p), D)\) such that for \(\beta \in (0, \frac{3}{4}]\),
\[(2.26) \|\phi_\beta(t)\|_{C^k([\delta, T] \times B_r(p))} \leq C_{\delta, T, k, p, r}.
\]

Through a further observation to \(\psi_\beta\) and equation (2.3), we prove the monotonicity of \(\psi_\beta\) and \(\varphi_\beta(t)\) with respect to \(\beta\).

**Proposition 2.7.** For any \(x \in M\), \(\psi_\beta(x)\) is monotone decreasing as \(\beta \searrow 0\).

**Proof:** By direct computations, for any \(x \in M \setminus D\), we have
\[(2.27) \frac{d\psi_\beta}{d\beta} = 2\beta |s|^2 \beta \log |s|^2 |x| + 1 - |s|^2 |x|^2.
\]
Denote \(f_\beta(x) = \beta x^\beta \log x + 1 - x^\beta\) for \(\beta > 0\) and \(x \in [0, 1]\),
\[(2.28) f'_\beta(x) = \beta^2 x^{\beta-1} \log x \leq 0.
\]
Hence \(f_\beta(x) \geq f_\beta(1) = 0\) and we have \(\frac{d\psi_\beta}{d\beta} \geq 0\). □

**Proposition 2.8.** For any \((t, x) \in (0, \infty) \times M\), \(\varphi_\beta(t, x)\) is monotone decreasing as \(\beta \searrow 0\).
Proof: By the arguments in [20], we obtain (2.1) by approximating equations
\[
\frac{\partial \varphi_{\beta}(t)}{\partial t} = \log \left( \frac{\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_{\beta}(t) + \omega_0^\alpha}{\omega_0^\alpha} \right) - \varphi_{\beta}(t) + h_0 + \log(\varepsilon^2 + |s|^2_h)^{1-\beta}
\]
(2.29)
\[
\varphi_{\beta}(0) = \psi_{\beta}
\]
For \( \beta_1 < \beta_2 \), let \( \psi_{1,2}(t) = \varphi_{\beta_1}(t) - \varphi_{\beta_2}(t) \). On \([\eta, T] \times M\) with \( \eta > 0 \) and \( T < \infty \),
\[
\frac{\partial}{\partial t} (e^{\varepsilon \eta} \psi_{1,2}(t)) \leq e^{\varepsilon \eta} \log \left( \frac{e^{\varepsilon \eta} \omega_0 + \sqrt{-1} \partial \bar{\partial} e^{\varepsilon \eta} \varphi_{\beta_1}(t) + \sqrt{-1} \partial \bar{\partial} e^{\varepsilon \eta} \varphi_{\beta_2}(t))}{e^{\varepsilon \eta} \omega_0 + \sqrt{-1} \partial \bar{\partial} e^{\varepsilon \eta} \varphi_{\beta_1}(t)} \right)\]
(2.30)

Let \( \tilde{\psi}_{1,2}(t) = e^{\varepsilon \eta} \psi_{1,2}(t) - \delta(t - \eta) \) with \( \delta > 0 \) and \((t_0, x_0)\) be the maximum point of \( \psi_{1,2}(t) \) on \([\eta, T] \times M\). If \( t_0 > \eta \), by the maximum principle, at this point,
\[
0 \leq \frac{\partial}{\partial t} \tilde{\psi}_{1,2}(t) = \frac{\partial}{\partial t} (e^{\varepsilon \eta} \psi_{1,2}(t)) - \delta \leq -\delta
\]
which is impossible, hence \( t_0 = \eta \). So for any \((t, x) \in [\eta, T] \times M\),
\[
\psi_{1,2}(t, x) \leq e^{-\varepsilon \eta} \sup_M \psi_{1,2}(\eta, x) + T \delta.
\]

Since \( \lim_{t \to 0^+} \| \varphi_{\beta}(t) - \chi_{\beta} \|_{L^\infty(M)} = 0 \), let \( \eta \to 0 \), we get
\[
\psi_{1,2}(t, x) \leq e^{-\varepsilon \eta} \sup_M (\psi_{\beta_1} - \psi_{\beta_2}) + T \delta \leq T \delta.
\]
(2.33)

Let \( \delta \to 0 \) and then \( \varepsilon \to 0 \), we conclude that \( \varphi_{\beta_1}(t, x) \leq \varphi_{\beta_2}(t, x) \). \( \square \)

For any \([\delta, T] \times K \subset (0, \infty) \times M \setminus D\) and \( k \geq 0 \), \( \| \varphi_{\beta}(t) \|_{C^k([\delta, T] \times K)} \) is uniformly bounded by Proposition [20]. Let \( \delta \) approximate to 0, \( T \) approximate to \( \infty \) and \( K \) approximate to \( M \setminus D \), by diagonal rule, we get a sequence \( \{ \beta_n \} \), such that \( \varphi_{\beta_n}(t) \) converge in \( C^\infty_{\text{loc}} \)-topology on \((0, \infty) \times (M \setminus D)\) to a function \( \varphi(t) \) that is smooth on \( C^\infty((0, \infty) \times (M \setminus D))\) and satisfies equation
\[
\frac{\partial \varphi(t)}{\partial t} = \log \left( \frac{\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi(t) + \omega_0^\alpha}{\omega_0^\alpha} \right) - \varphi(t) + h_0 + \log |s|^2_h \]
(2.34)
on \((0, \infty) \times (M \setminus D)\). Since \( \varphi_{\beta}(t) \) is monotone decreasing as \( \beta \to 0 \), \( \varphi_{\beta}(t) \) converge in \( C^\infty_{\text{loc}} \)-topology on \((0, \infty) \times (M \setminus D)\) to \( \varphi(t) \). For any \( T > 0 \),
\[
e^{-\varphi} \omega_{\text{cusp}} \leq \omega(t) \leq e^{\varphi} \omega_{\text{cusp}} \text{ on } (0, T) \times (M \setminus D),
\]
where \( \omega(t) = \omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi(t) \), constants \( C \) depend only on \( n, \omega_0 \) and \( T \).

Next, by using the monotonicity of \( \varphi_{\beta}(t) \) with respect to \( \beta \) and constructing auxiliary function, we prove the \( L^1 \)-convergence of \( \varphi(t) \) as \( t \to 0^+ \) as well as \( \varphi(t) \) converge to \( \varphi_0 \) in \( L^\infty \)-norm as \( t \to 0^+ \) on any compact subset \( K \subset (0, \infty) \times (M \setminus D) \).

Lemma 2.9. There exists a unique \( \varphi_{\beta} \in PSH(M, \omega_0) \cap L^\infty(M) \) to equation
\[
(\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi_{\beta})^n = e^{\varphi_{\beta} - h_0} \frac{\omega_0^n}{|s|^2_h^{(1-\beta)}}.
\]
(2.36)
Furthermore, \( \varphi_{\beta} \in C^{2,\alpha,\beta}(M) \) and \( \| \varphi_{\beta} - \psi_{\beta} \|_{L^\infty(M)} \) can be uniformly bounded by constant \( C \) independent of \( \beta \).
exists a unique continuous solution $\varphi_\beta$ to equation (2.36). Then by Guenancia-Păun’s regularity estimates [11] (see also Liu-Zhang [18]), $\varphi_\beta \in C^{2,\alpha,\beta}(M)$. Next, we prove $\|\varphi_\beta - \psi_\beta\|_{L^\infty(M)}$ can be uniformly bounded. Let $u_\beta = \varphi_\beta - \psi_\beta$, we write equation (2.36) as
\begin{equation}
(\omega_\beta + \sqrt{-1} \partial \bar{\partial} u_\beta)^n = e^{u_\beta + h_\beta} \omega_\beta^n,
\end{equation}
where $h_\beta = \psi_\beta - h_0 + \log \frac{\omega_0}{|s|_h^{n-1-\beta}}$ is uniformly bounded independent of $\beta$. Define $\chi_{\beta,\varepsilon} = u_\beta + \varepsilon \log |s|_h^2$. Then $\sqrt{-1} \partial \bar{\partial} \chi_{\beta,\varepsilon} = \sqrt{-1} \partial \bar{\partial} u_\beta - \varepsilon \theta$. Since $\chi_{\beta,\varepsilon}$ is smooth on $M \setminus D$, bounded from above and goes to $-\infty$ near $D$, it achieves its maximum on $M \setminus D$. Let $x_0$ be the maximum point of $\chi_{\beta,\varepsilon}$ on $M$ with $x_0 \in M \setminus D$.
\begin{equation}
(\omega_\beta + \sqrt{-1} \partial \bar{\partial} u_\beta)^n(x_0) = (\omega_\beta + \sqrt{-1} \partial \bar{\partial} \chi_{\beta,\varepsilon} + \varepsilon \theta)^n(x_0) \leq 2^n \omega_\beta^n(x_0).
\end{equation}
By the maximum principle, $u_\beta \leq C - \varepsilon \log |s|_h^2$, where constant $C$ independent of $\beta$ and $\varepsilon$. Let $\varepsilon \to 0$, we get the uniform upper bound of $u_\beta$. By the similar arguments, we can obtain the uniform lower bound of $u_\beta$.

**Proposition 2.10.** $\varphi(t) \in C^0([0, \infty) \times (M \setminus D))$ and
\begin{equation}
\lim_{t \to 0^+} \|\varphi(t) - \psi_0\|_{L^1(M)} = 0.
\end{equation}

**Proof:** By the monotonicity of $\varphi_\beta(t)$ with respect to $\beta$, for any $(t, z) \in (0, T] \times (M \setminus D)$, we have
\begin{equation}
\varphi(t, z) - \psi_0(z) \leq \varphi_\beta(t, z) - \psi_0(z)
\end{equation}
\begin{equation}
\leq |\varphi_\beta(t, z) - \psi_\beta(z)| + |\psi_\beta(z) - \psi_0(z)|.
\end{equation}
Since $\psi_\beta$ converge to $\psi_0$ in $C^\infty_{\text{loc}}$-sense outside $D$ as $\beta \to 0$, and
\begin{equation}
\lim_{t \to 0^+} \|\varphi_\beta(t, z) - \psi_\beta\|_{L^\infty(M)} = 0.
\end{equation}
For any $\varepsilon > 0$ and $K \subset M \setminus D$, there exists $N$ such that for $\beta_1 < \frac{1}{N}$,
\begin{equation}
\|\psi_{\beta_1}(z) - \psi_0(z)\|_{L^\infty(K)} < \frac{\varepsilon}{2}.
\end{equation}
Fix such $\beta_1$, there exists $0 < \delta_1 < T$ such that
\begin{equation}
\sup_{[0, \delta_1] \times M} |\varphi_{\beta_1}(t, z) - \psi_{\beta_1}| < \frac{\varepsilon}{2}.
\end{equation}
Combining the above estimates together, for any $t \in (0, \delta_1]$ and $z \in K$
\begin{equation}
\sup_{[0, \delta_1] \times K} (\varphi(t, z) - \psi_0(z)) < \varepsilon.
\end{equation}
We define function
\begin{equation}
H_\beta(t) = (1 - te^{-t})\psi_\beta + te^{-t}\varphi_\beta + h(t)e^{-t},
\end{equation}
where $\varphi_\beta$ and $u_\beta = \varphi_\beta - \psi_\beta$ are obtained in Lemma 2.9 and
\begin{equation}
h(t) = (1 - e^t - t)\|u_\beta\|_{L^\infty(M)} + n(t \log t - t)e^t - n \int_0^t e^s \log s ds.
\end{equation}
Combining the above inequalities, which is equivalent to Straightforward calculations show that
\[
\partial_t H_\beta(t) + H_\beta(t) = \psi_\beta + e^{-t}u_\beta - e^{-t}\|u_\beta\|_{L^\infty(M)} - \|u_\beta\|_{L^\infty(M)} + n \log t - nt
\]
\[
\leq \psi_\beta + u_\beta + n \log t - nt
\]
\[
\leq \varphi_\beta + n \log t - nt
\]
Therefore, we have
\[
e^{nH_\beta(t)+H_\beta(t)} \omega_0^n \leq t^n e^{-nt} e^{\varphi_\beta} \omega_0^n.
\]

When \( t \) is sufficiently small,
\[
\omega_0 + \sqrt{-1} \bar{\partial} \bar{\partial} H_\beta(t) = (1 - te^{-t})(\omega_0 + \sqrt{-1} \bar{\partial} \bar{\partial} \psi_\beta) + te^{-t}(\omega_0 + \sqrt{-1} \bar{\partial} \bar{\partial} \varphi_\beta)
\]
\[
\geq te^{-t}(\omega_0 + \sqrt{-1} \bar{\partial} \bar{\partial} \varphi_\beta).
\]

Combining the above inequalities,
\[
(\omega_0 + \sqrt{-1} \bar{\partial} \bar{\partial} H_\beta(t))^n \geq r^n e^{-nt}(\omega_0 + \sqrt{-1} \bar{\partial} \bar{\partial} \varphi_\beta)^n
\]
\[
\geq e^{-h_0 + \hat{a}H_\beta(t) + H_\beta(t)} \left( \frac{\omega_0^n}{|s_h^{(1-\beta)}} \right),
\]

which is equivalent to
\[
\left\{ \begin{array}{l}
\frac{\partial}{\partial t} H_\beta(t) \leq \log \frac{\omega_0 + \sqrt{-1} \bar{\partial} \bar{\partial} H_\beta(t)^n}{\omega_0^n} - H_\beta(t) + h_0 + \log |s_h^{2(1-\beta)}|.
\end{array} \right.
\]

Next, we prove \( H_\beta(t) \leq \varphi_\beta(t) \) for sufficiently small \( t \) by using Jeffres’ trick [12]. For any \( 0 < t_1 < T < \infty \) and \( a > 0 \).

Denote \( \Psi(t) = H_\beta(t) + a|s_h^{2q} - \varphi_\beta(t) \) and \( \hat{\Delta} = \int_0^1 \sqrt{g_{\bar{\partial} \bar{\partial}} H_\beta(t) + (1-s) \varphi_\beta(t)} \frac{\partial^2}{\partial z \partial \bar{z}} ds \),

where \( 0 < q < 1 \) is determined later. \( \Psi(t) \) evolves along the following equation
\[
\frac{\partial \Psi(t)}{\partial t} = \hat{\Delta} \Psi(t) - a \Delta |s_h^{2q} - \Psi(t) + a|s_h^{2q}.
\]

Since
\[
\omega_0 + \sqrt{-1} \bar{\partial} \bar{\partial} H_\beta(t) \geq (1 - te^{-t})(\omega_0 + \sqrt{-1} \bar{\partial} \bar{\partial} \psi_\beta) \geq \frac{1}{4} \omega_0,
\]
\[
\omega_0 + \sqrt{-1} \bar{\partial} \bar{\partial} \varphi_\beta(t) \geq C(t_1) \omega_0 \geq \frac{C(t_1)}{2} \omega_0.
\]

\[
\sqrt{-1} \bar{\partial} \bar{\partial}|s_h^{2q} = q^2|s_h^{2q} \sqrt{-1} \bar{\partial} \log |s_h|^q \wedge \bar{\partial} \log |s_h|^2 + q |s_h|^{2q} \sqrt{-1} \bar{\partial} \bar{\partial} \log |s_h|^2,
\]

we obtain the estimate
\[
\hat{\Delta} |s_h^{2q} \geq q |s_h|^{2q} \int_0^1 g_{H_\beta(t) + (1-s) \varphi_\beta(t)} \frac{\partial^2}{\partial z \partial \bar{z}} \log |s_h|^q ds
\]
\[
= -q |s_h|^{2q} \int_0^1 g_{H_\beta(t) + (1-s) \varphi_\beta(t)} \partial_{ij} ds
\]
\[
\geq -C(t_1) q |s_h|^{2q} g_{ij} \bar{g}_{0,ij} \geq -C(t_1)
\]
on \( M \setminus D \), where constant \( C(t_1) \) independent of \( a \). Then we obtain
\[
\frac{\partial \Psi(t)}{\partial t} \leq \hat{\Delta} \Psi(t) - \Psi(t) + aC(t_1).
\]
Let \( \tilde{\Psi} = e^{(t-t_1)} \Psi - aC(t_1)e^{(t-t_1)} - \varepsilon(t-t_1) \). By choosing suitable \( 0 < q < 1 \), we can assume that the space maximum of \( \tilde{\psi} \) on \([t_1, T] \times M\) is attained away from \( D \). Let \((t_0, x_0)\) be the maximum point. If \( t_0 > t_1 \), by the maximum principle, at \((t_0, x_0)\), we have

\[
0 \leq (\frac{\partial}{\partial t} - \hat{\Delta}) \tilde{\Psi}(t) \leq -\varepsilon,
\]

which is impossible, hence \( t_0 = t_1 \). Then for \((t, x) \in [t_1, T] \times M\), we obtain

\[
H_\beta(t) - \varphi_\beta(t) \leq \|H_\beta(t_1, x) - \varphi_\beta(t_1, x)\|_{L^\infty(M)} + aC(t_1) + \varepsilon T
\]

Since \( \lim_{t \to 0^+} \|H_\beta(\delta, z) - \psi_\beta\|_{L^\infty(M)} = 0 \) and (2.41), let \( a \to 0 \) and then \( t_1 \to 0^+ \),

\[
H_\beta(t) - \varphi_\beta(t) \leq \varepsilon T.
\]

It shows that \( H_\beta(t) \leq \varphi_\beta(t) \) after we let \( \varepsilon \to 0 \). For any \((t, z) \in (0, T] \times (M \setminus D)\)

\[
\varphi_\beta(t, z) - \psi_0(z) \geq t e^{-\varepsilon t} u_\beta + h(t)e^{-t} + \psi_\beta - \psi_0
\]

(2.47)

where \( h_1(t) = n(t \log t - t)e^t - n \int_0^t e^s \log s ds \), constant \( C \) independent of \( \beta \). Let \( \beta \to 0 \), we have

\[
\varphi(t, z) - \psi_0(z) \geq -Ct - C(1 - e^{-t}) + h_1(t)e^{-t},
\]

There exists \( \delta_2 \) such that for any \( t \in [0, \delta_2] \) and \( z \in M \setminus D \),

\[
\varphi(t, z) - \psi_0(z) > -\frac{\varepsilon}{2}.
\]

Let \( \delta = \min(\delta_1, \delta_2) \), then for any \( t \in (0, \delta) \) and \( z \in K \),

\[
-\varepsilon < \varphi(t, z) - \psi_0(z) < \varepsilon.
\]

Hence, \( \varphi(t) \in C^0([0, \infty) \times (M \setminus D)) \). Since \( \psi_\beta \) converge to \( \psi_0 \) in \( L^1 \)-sense on \( M \), for sufficiently small \( \beta_2 \), we have

\[
\int_M |\psi_{\beta_2}(z) - \psi_0(z)| \omega_0^n < \frac{\varepsilon}{2}
\]

By (2.40), (2.41) and (2.49), there exists \( \delta \) such that for any \( t \in (0, \delta) \),

\[
\int_M |\varphi(t) - \psi_0(z)| \omega_0^n < \varepsilon,
\]

which implies (2.39). \( \square \)

**Theorem 2.11.** \( \omega(t) = \omega_0 + \sqrt{-1} \partial \bar{\partial} \psi(t) \) is a long-time solution to cusp Kähler-Ricci flow (1.1).

**Proof:** We should only prove that \( \omega(t) \) satisfies equation (1.1) in the sense of currents on \((0, \infty) \times M \).

Let \( \eta = \eta(t, x) \) be a smooth \((n-1, n-1)\)-form with compact support in \((0, \infty) \times M \). Without loss of generality, we assume that its compact support included in \((\delta, T) \) \((0 < \delta < T < \infty) \). On \([\delta, T] \times (M \setminus D)\),

\[
\frac{\omega_0^n(t) |\Psi|^{2(1-\beta)}}{\omega_0^n} - \psi_\beta, \log \frac{\omega^n(t) |\Psi|^{2(1-\beta)}}{\omega_0^n} - \psi_0,
\]
\(\varphi_\beta(t) - \psi_\beta\) and \(\varphi(t) - \psi_0\) are uniformly bounded. On \([\delta, T]\), we have
\[
\int_M \frac{\partial \omega_\beta(t)}{\partial t} \wedge \eta = \int_M \sqrt{-1 \partial \partial \varphi_\beta(t)} \wedge \eta
\]
\[
= \int_M \left( \log \frac{\omega_\beta(t)}{\omega_0} \right)^{(1-\beta)} - \psi_\beta - (\varphi_\beta(t) - \psi_\beta) + h_0 \right) \sqrt{-1 \partial \partial \eta}
\]
\[
\xrightarrow{\varepsilon \to 0} \int_M \left( \log \frac{\omega(t)}{\omega_0} \right)^2 - \psi_0 - (\varphi(t) - \psi_0) + h_0) \sqrt{-1 \partial \partial \eta}
\]
\[\tag{2.53} = \int_M (-\text{Ric}(\omega(t)) - \omega(t) + [D]) \wedge \eta.\]

At the same time, there also holds
\[
\int_M \omega_\beta(t) \wedge \frac{\partial \eta}{\partial t} = \int_M \omega_0 \wedge \frac{\partial \eta}{\partial t} + \int_M \varphi_\beta(t) \sqrt{-1 \partial \partial \varphi_\beta(t)} \wedge \eta
\]
\[
\xrightarrow{\beta \to 0} \int_M \omega_0 \wedge \frac{\partial \eta}{\partial t} + \int_M \varphi(t) \sqrt{-1 \partial \partial \varphi(t)} \wedge \eta
\]
\[\tag{2.54} = \int_M \omega(t) \wedge \frac{\partial \eta}{\partial t}.\]

On the other hand,
\[
\frac{\partial}{\partial t} \int_M \omega_\beta(t) \wedge \eta = \int_M \varphi_\beta(t) \sqrt{-1 \partial \partial \varphi_\beta(t)} \wedge \eta + \int_M \omega_0 \wedge \frac{\partial \eta}{\partial t} + \int_M \frac{\partial \varphi_\beta(t)}{\partial t} \sqrt{-1 \partial \partial \eta}
\]
\[
\xrightarrow{\varepsilon \to 0} \int_M \varphi(t) \sqrt{-1 \partial \partial \varphi(t)} \wedge \eta + \int_M \omega_0 \wedge \frac{\partial \eta}{\partial t} + \int_M \frac{\partial \varphi(t)}{\partial t} \sqrt{-1 \partial \partial \eta}
\]
\[\tag{2.55} = \frac{\partial}{\partial t} \int_M \omega(t) \wedge \eta.\]

Combining equality
\[
\frac{\partial}{\partial t} \int_M \omega_\beta(t) \wedge \eta = \int_M \frac{\partial \omega_\beta(t)}{\partial t} \wedge \eta + \int_M \omega_\beta(t) \wedge \frac{\partial \eta}{\partial t}
\]
with equalities (2.53) - (2.55), on \([\delta, T]\), we have
\[
\frac{\partial}{\partial t} \int_M \omega(t) \wedge \eta = \int_M (-\text{Ric}(\omega(t)) - \omega(t) + [D]) \wedge \eta
\]
\[\tag{2.56} + \int_M \omega(t) \wedge \frac{\partial \eta}{\partial t}.\]

Integrating form 0 to \(\infty\) on both sides,
\[
\int_{(0,\infty) \times M} \frac{\partial \omega(t)}{\partial t} \wedge \eta dt = - \int_{(0,\infty) \times M} \omega(t) \wedge \frac{\partial \eta}{\partial t} dt = - \int_0^\infty \int_M \omega(t) \wedge \frac{\partial \eta}{\partial t} dt
\]
\[
= \int_0^\infty \int_M (-\text{Ric}(\omega(t)) - \omega(t) + [D]) \wedge \eta dt
\]
\[= \int_{(0,\infty) \times M} (-\text{Ric}(\omega(t)) - \omega(t) + [D]) \wedge \eta dt.\]

By the arbitrariness of \(\eta\), we prove that \(\omega(t)\) satisfies cusp Kähler-Ricci flow (1.7) in the sense of currents on \((0, \infty) \times M\). \(\square\)

Now we prove the uniqueness theorem.
Theorem 2.12. Let $\tilde{\varphi}(t) \in C^0([0, \infty) \times (M \setminus D)) \cap C^\infty((0, \infty) \times (M \setminus D))$ be a long-time solutions to parabolic Monge-Ampère equation

$$\frac{\partial \varphi(t)}{\partial t} = \log \left( \frac{\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi(t)}{\omega_0^n} \right) - \varphi(t) + h_0 + \log |s|^2_h \tag{2.57}$$
onumber

on $(0, \infty) \times (M \setminus D)$. If $\tilde{\varphi}$ satisfies

- For any $0 < \delta < T < \infty$, there exists uniform constant $C$ such that
  $C^{-1}\omega_{cusp} \leq \omega_0 + \sqrt{-1}\partial\bar{\partial}\tilde{\varphi}(t) \leq C\omega_{cusp}$ on $[\delta, T] \times (M \setminus D)$;
- on $(0, T)$, $\|\tilde{\varphi}(t) - \psi_0\|_{L^\infty(M \setminus D)} \leq C$;
- on $[\delta, T]$, there exists constant $C^*$ such that $\|\frac{\partial \tilde{\varphi}(t)}{\partial t}\|_{L^\infty(M \setminus D)} \leq C^*$;
- $\lim_{t \to 0^+} \|\tilde{\varphi}(t) - \psi_0\|_{L^1(M)} = 0$.

Then $\tilde{\varphi}(t) \leq \varphi(t)$.

Proof: For any $0 < t_1 < T < \infty$ and $a > 0$. Denote $\Psi(t) = \tilde{\varphi}(t) + a \log |s|^2_h - \varphi_\beta(t)$ and $\hat{\Delta} = \int_0^1 g^{-1}_{\tilde{\varphi}(t)+(1-s)\varphi_\beta} \frac{\partial^2}{\partial^2 \varphi} ds$. We note that $\tilde{\varphi}(t)$ is bounded from above. $\Psi(t)$ evolves along the following equation

$$\frac{\partial \Psi(t)}{\partial t} = \hat{\Delta} \Psi(t) - a \hat{\Delta} \log |s|^2_h - \Psi(t) + (a + \beta) \log |s|^2_h.$$ 

Since $\sqrt{-1}\partial\bar{\partial}\log |s|^2_h = 0$, we obtain

$$-\hat{\Delta} \log |s|^2_h = \int_0^1 g^{-1}_{\tilde{\varphi}(t)+(1-s)\varphi_\beta} \theta_{ij} ds \leq C(t_1)$$
onumber

on $M \setminus D$. Then we obtain

$$\frac{\partial \Psi(t)}{\partial t} \leq \hat{\Delta} \Psi(t) - \Psi(t) + aC(t_1).$$

Then by the arguments as that in Proposition 2.10 on $[t_1, T] \times (M \setminus D)$,

$$\tilde{\varphi}(t) - \varphi_\beta(t) \leq e^{-(t-t_1)} \sup_M (\tilde{\varphi}(t_1) - \varphi_\beta(t_1)).$$

Since $\tilde{\varphi}(t_1)$ converge to $\psi_0$ in $L^1$-sense and $\varphi_\beta(t)$ converge to $\psi_\beta$ in $L^\infty$-sense as $t_1 \to 0^+$, by Hartogs Lemma, we have

$$\tilde{\varphi}(t) - \varphi_\beta(t) \leq e^{-t} \sup_M (\psi_0 - \psi_\beta) \leq 0,$$

after we let $t_1 \to 0$. Hence $\tilde{\varphi}(t) \leq \varphi(t)$ on $(0, \infty) \times (M \setminus D)$.

Remark 2.13. If $M$ is a compact Kähler manifold with smooth hypersurface $D$. We can also consider unnormalized cusp Kähler-Ricci flow

$$\begin{cases}
\frac{2\varphi(t)}{\partial t} = -\text{Ric}(\hat{\omega}(t)) + [D], \\
\hat{\omega}(t)|_{t=0} = \omega_{cusp}
\end{cases} \tag{2.58}$$

If we define $\omega(t) = e^{-t}\hat{\omega}(e^t - 1)$, then flow (2.58) is actually the same as normalized cusp Kähler-Ricci flow (1.7) only moduli a scaling. Let

$$T_0 = \sup \{ t \mid |\omega_0| - t(c_1(M) - c_1(D)) > 0 \}. \tag{2.59}$$
Combining the arguments of Tian-Zhang [34] and Liu-Zhang [20] with the arguments in this paper, there exists a unique solution to flow (2.55) on \([0, T_0]\) in some weak sense which is similar as Definition 1.1.

3. **The Convergence of cusp Kähler-Ricci flow**

In this section, we prove the convergence theorem of cusp Kähler-Ricci flow (1.7).

**Proof of Theorem 1.5** Differentiating equation (2.57) in time \(t\), we have

\[
\frac{d}{dt} - \Delta_t \frac{\partial \varphi}{\partial t} = -\frac{\partial \varphi}{\partial t}
\]

on \([\delta, T] \times (M \setminus D)\) with \(\delta > 0\). For any \(\varepsilon > 0\),

\[
\frac{d}{dt} - \Delta_t \left( \frac{\partial \varphi}{\partial t} + \varepsilon \log |s|_h^2 \right) = \frac{\partial \varphi}{\partial t} + \varepsilon \text{tr} \omega(t) \theta
\]

\[
\leq - \left( \frac{\partial \varphi}{\partial t} + \varepsilon \log |s|_h^2 \right) + \varepsilon C(\delta, T),
\]

where constant \(C(\delta, T)\) independent of \(\varepsilon\). For any \(\eta > 0\), let \(H = e^{t - \delta} \left( \frac{\partial \varphi}{\partial t} + \varepsilon \log |s|_h^2 \right) - e^{t - \delta} C(\delta, T) - \eta(t - \delta)\). Since \(\frac{\partial \varphi}{\partial t}\) is bounded on \([\delta, T] \times (M \setminus D)\), the maximum point \((t_0, x_0)\) of \(H\) satisfies \(x_0 \in M \setminus D\). By the maximum principle, \(t_0 = \delta\). Hence,

\[
\frac{\partial \varphi}{\partial t} \leq C(\delta)e^{-t} - \varepsilon \log |s|_h^2 + \varepsilon C(\delta, T) + \eta T.
\]

Let \(\varepsilon \to 0, \eta \to 0\) and then \(T \to \infty\), we obtain

\[
\frac{\partial \varphi}{\partial t} \leq C(\delta)e^{-t} \quad \text{on} \quad [\delta, \infty) \times (M \setminus D).
\]

By the same arguments, we can get the lower bound of \(\frac{\partial \varphi}{\partial t}\). In fact, we obtain

\[
\frac{\partial \varphi}{\partial t} \leq C(\delta)e^{-t} \quad \text{on} \quad [\delta, \infty) \times (M \setminus D).
\]

For \(\delta < t < s\),

\[
|\varphi(t) - \varphi(s)| \leq C(\delta)(e^{-t} - e^{-s}) \quad \text{on} \quad [\delta, \infty) \times (M \setminus D).
\]

Therefore, \(\varphi(t)\) converge exponentially fast in \(L^\infty\)-sense to \(\varphi_\infty\) on \(M \setminus D\). And \(\varphi(t)\) converge to \(\varphi_\infty\) in \(C^\infty_{\text{loc}}\)-sense on \(M \setminus D\). At the same time, For any smooth \((n - 1, n - 1)\)-form \(\eta\),

\[
\int_M \frac{\partial \omega(t)}{\partial t} \wedge \eta = \int_M \frac{\partial \varphi}{\partial t} \sqrt{-1} \partial \bar{\partial} \eta \xrightarrow{t \to \infty} 0
\]

while

\[
\int_M \frac{\partial \omega(t)}{\partial t} \wedge \eta = \int_M \sqrt{-1} \partial \bar{\partial} \log \left( \frac{|s|^2_h^2}{\omega_0^2} (\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi(t))^n \right) - \varphi(t) + h_0 \wedge \eta
\]

\[
= \int_M \left( \log \frac{|s|^2_h^2}{\omega_0^2} (\omega_0 + \sqrt{-1} \partial \bar{\partial} \varphi(t))^n \right) - \psi_0 - (\varphi(t) - \psi_0 + h_0) \sqrt{-1} \partial \bar{\partial} \eta
\]

\[
= \int_M (-\text{Ric}(\omega_\infty) - \omega_\infty + [D]) \wedge \eta.
\]

which implies the convergence in the sense of currents.
REFERENCES

[1] S. S. Chern, On holomorphic mappings of Hermitian manifolds of the same dimension, Proc. Symp. Pure Math., 11 (1968), 157–170.
[2] X. X. Chen and W. Y. Ding, Ricci flow on surfaces with degenerate initial metrics, J. Partial Differential Equations, 20 (2007), 193–202.
[3] X. X. Chen, G. Tian and Z. Zhang, On the weak Kähler-Ricci flow, Trans. Amer. Math. Soc. 363 (2011), 2849–2863.
[4] X. X. Chen and Y. Q. Wang, Bessel functions, Heat kernel and the Conical Kähler-Ricci flow, Journal of Functional Analysis, 269 (2015), 551–632.
[5] X. X. Chen and Y. Q. Wang, On the long-time behaviour of the Conical Kähler- Ricci flows, Journal für die reine und angewandte Mathematik, 2014.
[6] G. Edwards, A scalar curvature bound along the conical Kähler-Ricci flow, arXiv: 1505.02083.
[7] P. Eyssidieux, V. Guedj and A. Zeriahi, Singular Kähler-Einstein metrics, J. Amer. Math. Soc. 22 (2009), 607–63.
[8] G. Giesen and P. Topping, Existence of Ricci flows of incomplete surfaces. Communications in Partial Differential Equations, 36 (2011) 1860–1880.
[9] V. Guedj and A. Zeriahi, Regularizing properties of the twisted Kähler-Ricci flow, Journal für die reine und angewandte Mathematik, (2013).
[10] H. Guenancia, Kähler-Einstein metrics: from cones to cusps, arXiv: 150 4.01947.
[11] H. Guenancia and M. Paun, Conic singularities metrics with perscribed Ricci curvature: the case of general cone angles along normal crossing divisors, Journal of Differential Geometry, 103 (2016), 15–57.
[12] T. Jeffres, Uniqueness of Kähler-Einstein cone metrics, Publ. Mat. 44 (2000), 437–448.
[13] T. Jeffres, R. Mazzeo and Y. Rubinstein, Kähler-Einstein metrics with edge singularities, Annals of Mathematics, 183 (2016), 95–176.
[14] J. Ji, R. Mazzeo and N. Sesum, Ricci flow on surfaces with cusps. Mathematische Annalen, 345 (2009), 819–834.
[15] B. Kleiner and J. Lott, Singular Ricci flows I, arXiv: 1408.2271.
[16] R. Kobayashi, Kähler-Einstein metric on an open algebraic manifolds, Osaka Journal of Mathematics, 21 (1984), 399–418.
[17] G. Lieberman, Second Order Parabolic Differential Equations, World Scientific, Singapore New Jersey London Hong Kong, 1996.
[18] J. W. Liu and C. J. Zhang, The conical complex Monge-Ampère equations on Kähler manifolds, arXiv: 1609.03821.
[19] J. W. Liu and X. Zhang, The conical Kähler-Ricci flow on Fano manifolds, Advances in Mathematics, 307 (2017), 1324–1371.
[20] J. W. Liu and X. Zhang, The conical Kähler-Ricci flow with weak initial data on Fano manifold, International Mathematics Research Notices, doi:10.1093/imrn/rnw171.
[21] J. Lott and Z. Zhang, Ricci flow on quasi-projective manifolds, Duke Math. J. 156 (2011), 87–123.
[22] J. Lott and Z. Zhang, Ricci flow on quasiprojective manifolds II, Journal of the European Mathematical Society, 18 (2016), 1813–1854.
[23] Y. C. Lu, Holomorphic mappings of complex manifolds, Journal of Differential Geometry, 2 (1968), 299–312.
[24] R. Mazzeo, Y. Rubinstei and N. Sesum, Ricci flow on surfaces with conic singularities, Analysis & PDE, 8 (2015), 839–882.
[25] E. D. Nezza and C. H. Lu, Uniqueness and short time regularity of the weak Kähler-Ricci flow, Advances in Mathematics, 305 (2017), 953–993.
[26] R. Nomura, Blow-up behavior of the scalar curvature along the conical Kähler-Ricci flow with finite time singularities, arXiv: 1607.03004.
[27] L. M. Shen, Unnormalize conical Kähler-Ricci flow, arXiv: 1411.7284.
[28] L. M. Shen, C2,α-estimate for conical Kähler-Ricci flow, arXiv: 1412.2420.
[29] J. Song and G. Tian, The Kähler-Ricci flow through singularities, Invent. math., doi: 10.1007/s00222-016-0674-4.
[30] J. Song and X. W. Wang, The greatest Ricci lower bound, conical Einstein metrics and the Chern number inequality, Geom. Topol., 20 (2016), 49–102.

[31] G. Székelyhidi and V. Tosatti, Regularity of weak solutions of a complex Monge-Ampère equation, Anal. PDE, 4 (2011), 369–378.

[32] G. Tian, Kähler-Einstein metrics on algebraic manifolds, Transcendental methods in algebraic geometry (Cetraro 1994), Lecture Notes in Math. 1646, 143–185.

[33] G. Tian and S. T. Yau, Existence of Kähler-Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry, Advanced Series in Mathematical Physics 11 (1987), 574–628.

[34] G. Tian and Z. Zhang, On the Kähler-Ricci flow on projective manifolds of general type, Chinese Annals of Mathematics, Series B, 27 (2006), 179–192.

[35] P. Topping, Ricci flow compactness via pseudolocality, and flows with incomplete initial metrics, Journal of the European Mathematical Society, 12 (2010), 1429–1451.

[36] P. Topping, Uniqueness and nonuniqueness for Ricci flow on surfaces: Reverse cusp singularities, International Mathematics Research Notices, 12 (2012), 2356–2376.

[37] P. Topping and H. Yin, Rate of curvature decay for the contracting cusp Ricci flow, arXiv: 1606.07877.

[38] Y. Q. Wang, Smooth approximations of the conical Kähler-Ricci flows, Mathematische Annalen, 365 (2016), 835–856.

[39] Y. S. Zhang, A note on conical Kähler-Ricci flow on minimal elliptic Kähler surfaces, arXiv: 1610.09880.

[40] Z. Zhang, Kähler-Ricci flow with degenerate initial class, Trans. Amer. Math. Soc. 366 (2014), 3389–3403.

Jiawei Liu, Beijing International Center for Mathematical Research, Peking University, Beijing 100871, China,
E-mail address: jwliu@math.pku.edu.cn

Xi Zhang, Key Laboratory of Wu Wen-Tsun Mathematics, Chinese Academy of Sciences, School of Mathematical Sciences, University of Science and Technology of China, Hefei, 230026, China,
E-mail address: mathzx@ustc.edu.cn