Energy-momentum Tensors in Matrix Theory
and in Noncommutative Gauge Theories

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Abstract. The energy-momentum tensor of Matrix Theory is derived by computing disk amplitudes with one closed string and an arbitrary number of open strings and by taking the DKPS limit. We clarify its relation to the energy-momentum tensor of the noncommutative gauge theory derived in our previous paper.

1. Introduction

Noncommutative gauge theories can be realized by considering branes in string theory with a strong NS-NS two-form field $\Omega^{1,1}$. In our previous paper $[12]$, we derived the energy-momentum tensors of these theories by computing disk amplitudes with one closed string and an arbitrary number of open strings and by taking the Seiberg-Witten limit $[10]$ of these amplitudes. We found that the energy-momentum tensors involve the open Wilson lines $[13]-[18]$. However, they do not reduce to the ones in the commutative theories in the limit $\theta \to 0$, where $\theta$ is the noncommutative parameter. This is because the Seiberg-Witten limit does not commute with the commutative limit $[2]$. We also found that the energy-momentum tensors are conserved in interesting ways. In particular, in theories derived from bosonic string, the energy-momentum tensors are kinematically conserved, namely, the conservation law holds identically for any field configuration irrespective of the equations of motion.

It turns out that we can use the same method to derive the energy-momentum tensor of Matrix Theory $[33]$. If we perform the dimensional reduction of the noncommutative theories along their noncommutative directions, the Seiberg-Witten limit reduces to the DKPS limit $[35]$ (or the Sen-Seiberg limit $[36, 37]$ in the context of Matrix Theory $[33, 38]$). Therefore, the energy-momentum tensor of Matrix Theory can be derived as a special case of the computation in $[12]$ where there are no noncommutative directions. In $[39, 40]$, the energy-momentum tensor of Matrix Theory was deduced from computations of one-loop amplitudes in Matrix Theory and from their comparison with graviton exchange amplitudes. The energy-momentum tensor we derive from the disk amplitudes perfectly agrees with that obtained in $[39, 40]$ including the structure of the higher moments.

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1 Various aspects of the open Wilson lines and related issues were discussed in $[19]-[31]$.
2 In this paper, we study the energy-momentum tensor coupled to gravity in the bulk. The energy-momentum tensor derived in the Noether procedure, such as the one discussed in $[32]$, would couple to the open string metric on the branes.
3 See also a recent review $[44]$. 

It has been shown in \cite{41, 33, 42} and \cite{43}–\cite{59} that noncommutative
gauge theories are realized as certain backgrounds of Matrix Theory and of the IIB matrix
model. This implies a natural correspondence of gauge invariant observables
between Matrix Theory and noncommutative gauge theories. In fact, it was in this
context \cite{13} that the open Wilson lines were discovered as a basis for observables in
noncommutative theories. In \cite{18}, it was proposed to use this correspondence to de-
rive the energy-momentum tensors of the noncommutative theories. In this paper,
we carry out this proposal explicitly. We find that the resulting energy-momentum
tensors agree exactly with those derived in \cite{12}, both in the superstring case and
in the bosonic string case. The correspondence between Matrix Theory and non-
commutative theories also leads to a simpler proof of the conservation law of the
energy-momentum tensors, which came out rather miraculously in our earlier work
\cite{12}.

This paper is organized as follows. In Section 2, we will derive the ener-
gy-momentum tensor of Matrix Theory by computing disk amplitudes with one closed
string and an arbitrary number of open strings and by taking the DKPS limit. As
we mentioned in the above, this can be regarded as a special case of the computa-
tion in \cite{12}. In Section 3, we use the correspondence between Matrix Theory and
noncommutative gauge theories to derive the energy-momentum tensors in non-
commutative theories and find an agreement with our earlier result. We will study
the case of bosonic string in Section 4.

Let us summarize the notations used in this paper. The coordinates in the
bulk are denoted by $x^M$. For the discussion regarding the DKPS limit of D-branes,
we use the late Greeks ($\mu, \nu, ...$) for the directions along the branes and $I, J, ...$ for
those transverse to the branes:

$$x^M = \begin{cases} 
  x^\mu & \text{for the directions along the branes}, \\
  x^I & \text{for the transverse directions}.
\end{cases}$$

In the context of noncommutative gauge theories, we use the romans ($i, j, ...$) for
the noncommutative directions on the branes, the late Greeks ($\mu, \nu, ...$) for the
commutative directions, and the early Greeks ($\alpha, \beta, ...$) for the directions transverse
to the branes:

$$x^M = \begin{cases} 
  x^\mu & \text{for the commutative directions along the branes}, \\
  x^i & \text{for the noncommutative directions along the branes}, \\
  x^\alpha & \text{for the transverse directions}.
\end{cases}$$

As we will see, the transverse directions $x^I$ transmute into $x^i$ and $x^\alpha$.

In the noncommutative directions, the coordinates obey the Heisenberg rela-
tion,

$$[x^i, x^j] = -i\theta^{ij}.$$

The closed string metric is denoted by $g_{MN}$ and the open string metric $G_{MN}$ in
the zero-slope limit is given by

$$G^{ij} = \frac{1}{(2\pi \alpha')^2} \theta^{ij} g_{mn}, \quad G^{\mu\nu} = g^{\mu\nu}.\footnote{See also \cite{30}.}$$
The noncommutative gauge theory is defined by taking the zero-slope limit, $\alpha' \to 0$, while keeping $G^{ij}$ and $\theta^{ij}$ finite (the Seiberg-Witten limit \[10\]). This means that the closed string metric along the noncommutative direction $g_{ij}$ is scaled as $\alpha'^2$.

## 2. Energy-momentum tensor of Matrix Theory

In \[12\], the energy-momentum tensors of noncommutative gauge theories are derived by computing the disk amplitudes with one closed string and an arbitrary number of open strings. It turns out that this method is applicable to the computation of the actions of multiple D-branes in weakly curved background, or equivalently, the energy-momentum tensor of Matrix Theory.

The action of a single D$p$-brane in curved space is given by

$$S = -\frac{1}{g^2_{YM}} \int dx \sqrt{-g} \left[ \frac{1}{4} g^{\mu\nu} g^{\sigma\tau} F_{\mu\nu} F_{\sigma\tau} + \frac{1}{2(2\pi\alpha')^2} g^{\mu\nu} g_{IJ} \partial_{\mu} Y^{I} \partial_{\nu} Y^{J} \right],$$

where the scalar fields $Y^{I}(x)$ describe the location of the brane in the transverse directions and the Yang-Mills coupling constant $g_{YM}$ is given in terms of the string coupling constant $g_s$ and $\alpha'$ as follows:

$$g^2_{YM} = g_s (2\pi)^{p-2} \alpha^{p-2}. \quad (6)$$

The integral $\int dx$ is over the $p+1$ dimensional worldvolume of the D-brane. The action (5) is obtained as a zero-slope limit $\alpha' \to 0$ known as the DKPS limit \[35\] (or the Sen-Seiberg limit \[36,37\] in the context of Matrix Theory \[33,38\]) given by

$$Y^{I} \sim O(\alpha'), \quad k_{I} \sim O(\alpha'^{-1}), \quad (7)$$

while the other quantities $A_{\mu}, k_{\mu}, g_{\mu\nu}, g_{IJ}$ and $g_{YM}$ are kept finite as $\alpha' \to 0$. Here $k_{M} = (k_{\mu}, k_{I})$ is the Fourier mode conjugate to the coordinate $x^{M} = (x^{\mu}, x^{I})$. In this limit, $\Phi^{I}$ defined by

$$\Phi^{I}(x) = \frac{Y^{I}(x)}{2\pi\alpha'}, \quad (8)$$

is kept finite. In the case of multiple D-branes, where $A_{\mu}$ and $Y^{I}$ become matrix valued, a complete action in curved space is not known although its various aspects have been studied in \[39,40\] and \[60-70\]. Here we will study the first term in the expansion,

$$g_{IJ}(x, Y(x)) = \eta_{IJ} + h_{IJ} \exp \left[ i k_{\mu} x^{\mu} + i k_{I} Y^{I}(x) \right] + O(h^2). \quad (9)$$

The DKPS limit \[6\] defined in this way is the same as the Seiberg-Witten limit of noncommutative gauge theories as far as the commutative directions are concerned. Therefore, we can derive the energy-momentum tensors of multiple D-branes by considering the energy-momentum tensors of noncommutative gauge theories found in \[12\] and by taking their dimensional reductions in the noncommutative directions.

It would be instructive to recapitulate the derivation of the energy-momentum tensor in \[12\], in the language of Matrix Theory. The worldsheet action is

$$S = \frac{1}{2\pi} \int d^2 \sigma \left[ \frac{2}{\alpha'} g_{MN} \partial X^{M} \partial X^{N} + \frac{1}{\alpha' g_{MN}} \bar{\psi}^{M} \partial \psi^{N} + \frac{1}{\alpha' g_{MN}} \bar{\psi}^{M} \partial \bar{\psi}^{N} \right]. \quad (10)$$
and those for the fermions are
\[
\langle \psi^M(z) \psi^N(w) \rangle = -\alpha' \left[ b^{MN} \log |z-w| - b^{MN} \log |z-\bar{w}| + 2b^{MN} \log |z-\bar{w}| \right],
\]
and those for the fermions are
\[
\langle \psi^M(z) \bar{\psi}^N(w) \rangle = \frac{\alpha'}{z-w} b^{MN},
\]
\[
\langle \psi^M(z) \bar{\psi}^N(\bar{w}) \rangle = \frac{\alpha'}{z-w} (-b^{MN} + 2b^{MN}),
\]
\[
\langle \bar{\psi}^M(z) \psi^N(w) \rangle = \frac{\alpha'}{z-w} (-b^{MN} + 2b^{MN}),
\]
\[
\langle \bar{\psi}^M(z) \bar{\psi}^N(\bar{w}) \rangle = \frac{\alpha'}{z-w} b^{MN},
\]
where
\[
G_{\mu\nu} = g_{\mu\nu}, \quad G_{IJ} = 0.
\]
The fermions on the boundary $\Psi^M(t)$ are defined by
\[
\Psi^\mu(t) = \frac{1}{2} [\psi^\mu(t) + \bar{\psi}^\mu(t)], \quad \Psi^I(t) = \frac{1}{2} [\psi^I(t) - \bar{\psi}^I(t)],
\]
and the boundary-boundary and bulk-boundary propagators for the fermions are given by
\[
\langle \Psi^\mu(t) \Psi^{\mu'}(t') \rangle = \frac{\alpha'}{t-t'} g^{\mu\nu}, \quad \langle \Psi^I(t) \Psi^{I'}(t') \rangle = \frac{\alpha'}{t-t'} g^{IJ},
\]
\[
\langle \psi^\mu(z) \bar{\psi}^{\nu}(t) \rangle = \frac{\alpha'}{z-t} g^{\mu\nu}, \quad \langle \bar{\psi}^I(z) \psi^J(t) \rangle = \frac{\alpha'}{z-t} g^{IJ}, \quad \langle \bar{\psi}^I(z) \bar{\psi}^J(t) \rangle = \frac{\alpha'}{z-t} g^{IJ},
\]
where $t$ and $t'$ are on the boundary.

It is convenient to use the vertex operator in the $(−1, −1)$-picture for the closed string,
\[
V^{(-1,-1)}(z) = \frac{1}{2} \delta(\gamma) \delta(\tilde{\gamma}) h_{MN}(k) \psi^M(z) \bar{\psi}^N(\tilde{z}) e^{ikX(z)},
\]
where $\gamma$ and $\tilde{\gamma}$ are bosonic ghosts, and the operator in the 0-picture for open string,
\[
U^{(0)}(t) = A_\mu \frac{dX^\mu}{dt} + A^I(t) g_{IJ} \partial_1 X^J - F_{\mu\nu}(X) \psi^\mu \psi^\nu - 2D_\mu \Phi^I(X) g_{IJ} \psi^\mu \psi^J - i g_{IK} g_{JL} [\Phi^I(X), \Phi^J(X)] \psi^K \psi^L.
\]
Here we used $\Phi^I$ defined in (8) instead of $Y^I$ to make it clear that the open string vertex operator is $O(1)$ in the zero-slope limit. The correlation function which we need to evaluate is then
\[
(z-\bar{z})^2 \langle V^{(-1,-1)}(z) tr P \exp \left( i \int_{-\infty}^{t} dt' U^{(0)}(t') \right) \rangle,
\]
with a proper gauge-fixing of the SL(2, R) invariance. Here the symbol $P$ denotes the path-ordering with respect to $t$. 

By definition, all the propagators are $O(\alpha')$ whereas the open vertex operator is $O(1)$ in the zero-slope limit. Therefore, it costs $\alpha'$ each time we use a propagator. However, when we make a contraction between an open string vertex operator and the $e^{ik_j X^j}$ part of the graviton vertex operator, the factor $\alpha'$ is cancelled by the fact that the momentum $k_I$ scales as $O(\alpha'^{-1})$, so the contraction remains finite for any number of open string vertex operators.

\begin{equation}
\langle e^{ikX(z)} \tr \prod_a \Phi^I(X(t_a)) g_{I,J} \partial_\perp X^J(t_a) \rangle = \int dx \ e^{ik_\mu x^\mu} \ tr \prod_a 2\pi \alpha' k_I \Phi^I(x) \frac{\partial \tau(t_a, z)}{\partial t_a}.
\end{equation}

Here the function $\tau(t, z)$ is given by

\begin{equation}
\tau(t, z) = \frac{1}{2\pi i} \log \left( \frac{t - z}{t - \bar{z}} \right),
\end{equation}

and we used that

\begin{equation}
i\partial_\perp \log |z - t|^2 = \frac{z - \bar{z}}{(z - t)(\bar{z} - t)} = 2\pi i \frac{\partial \tau(t, z)}{\partial t}.
\end{equation}

For a fixed value of $z$, $\tau(t, z)$ is a monotonically increasing function of $t$ and

\begin{equation}
\tau(\infty, z) - \tau(-\infty, z) = 1.
\end{equation}

Therefore, we have

\begin{equation}
\langle e^{ikX(z)} \ tr P \exp \left( i \int_{-\infty}^{\infty} dt \ \Phi^I(X(t)) g_{I,J} \partial_\perp X^J(t) \right) \rangle = \int dx \ e^{ik_\mu x^\mu} \ tr P \exp \left( i \int_{\infty}^{1} d\tau \ 2\pi \alpha' k_I \Phi^I(x) \right).
\end{equation}

Note that the path-ordering with respect to $\tau$ on the right-hand side is inherited from that with respect to $t$ on the left-hand side because $\tau$ is a monotonically increasing function of $t$. Note also that the $A_\mu(X) dX^\mu / dt$ part of the open string vertex operator does not contribute to the energy-momentum tensor in the zero-slope limit because there is no divergent factor like $k_I$ which can compensate the factor of $\alpha'$ coming from the propagator.

We also need to contract the fermions in the closed string vertex operator to obtain a non-vanishing result. The contribution to the graviton vertex operator starts at $O(\alpha'^3)$ with three contractions of the fermions.

\begin{equation}
(z - \bar{z}) \left( \frac{1}{2} \left[ \psi^I(z) \bar{\psi}^J(\bar{z}) + \psi^I(\bar{z}) \bar{\psi}^J(z) \right] \ tr P \exp \left( i \int_{-\infty}^{\infty} dt \ U^{(0)}(t) \right) \right)
= 8\pi^2 \alpha'^3 \ tr \left[ g^{\mu\nu} \int_{0}^{1} d\tau_1 D_\mu \Phi^I(x) \int_{0}^{1} d\tau_2 D_\nu \Phi^J(x) + g_{KL} \int_{0}^{1} d\tau_1 [\Phi^I(x), \Phi^K(x)] \int_{0}^{1} d\tau_2 [\Phi^L(x), \Phi^I(x)] \right] + O(\alpha'^4)
+ \text{the dilaton part.}
\end{equation}
After taking into account the contributions from the ghosts, we obtain the components $T^{IJ}$ of the energy-momentum tensor in the case of superstring as follows:

\begin{equation}
2\alpha' \int dx \ e^{ikx} \ tr \left[ \exp \left( i \int_0^1 d\tau \ k_1 Y^I(x) \right) \right. \\
\left. \times \left[ g^{\mu \nu} \int_0^1 d\tau_1 D_\mu Y^I(x) \int_0^1 d\tau_2 D_\nu Y^J(x) \right. \\
\left. + \frac{g_{KL}}{(2\pi \alpha')^2} \int_0^1 d\tau_1 [Y^I(x), Y^K(x)] \int_0^1 d\tau_2 [Y^L(x), Y^J(x)] \right] \right],
\end{equation}

where we used $Y^I$ instead of $\Phi^I$ in (8). Note that the ordering of the fields is determined by the values of $\tau$, $\tau_1$, and $\tau_2$ which are inherited from the positions of open string vertex operators. We can transform the expression for $T^{IJ}$ into a more convenient form. For any operators $A$ and $B$,

\begin{equation}
tr \left[ \exp \left( i \int_0^1 d\tau \ kY \right) \right. \\
\left. \int_0^1 d\tau_1 A \int_0^1 d\tau_2 B \right]
\end{equation}

\begin{align*}
= & \int_0^1 d\tau_1 \int_0^1 d\tau_2 \ tr \ A e^{i(\tau_2 - \tau_1)kY} B e^{i(\tau_1 - \tau_2)kY} \\
& + \int_0^1 d\tau_1 \int_0^{\tau_1} d\tau_2 \ tr \ A e^{i(1 - \tau_1 + \tau_2)kY} B e^{i(\tau_1 - \tau_2)kY} \\
= & \int_0^1 d\tau \left[ \int_0^{1-\tau} d\tau' + \int_1^{1-\tau} d\tau' \right] \ tr \ A e^{i\tau kY} B e^{i(1-\tau)kY} \\
= & \int_0^1 d\tau \ tr \ A e^{i\tau kY} B e^{i(1-\tau)kY}.
\end{align*}

Therefore, we can write

\begin{equation}
T^{IJ} = 2\alpha' \int dx \ e^{ikx} \ tr \int_0^1 d\tau \left[ g^{\mu \nu} D_\mu Y^I e^{i\tau k_1 Y^I} D_\nu Y^J e^{i(1-\tau)k_1 Y^I} \right. \\
\left. + \frac{g_{KL}}{(2\pi \alpha')^2} [Y^I, Y^K] e^{i\tau k_1 Y^I} [Y^L, Y^J] e^{i(1-\tau)k_1 Y^I} \right].
\end{equation}

Furthermore, we can carry out the remaining $\tau$-integral and the result is expressed in terms of the symmetrized trace \cite{71}. To see this we note that, for any operators $A$ and $B$,

\begin{equation}
\int_0^1 d\tau \ tr \left[ A e^{i\tau Y} B e^{i(1-\tau)Y} \right]
\end{equation}

\begin{align*}
= & \sum_{n,m=0} \int_0^1 d\tau \left[ A \frac{1}{n!} (i\tau Y)^n B \frac{1}{m!} (i(1-\tau)Y)^m \right] \\
= & \sum_{n,m=0} \frac{n!m!}{(n+m+1)!} \ tr \left[ A \frac{1}{n!} (iY)^n B \frac{1}{m!} (iY)^m \right] \\
= & \sum_{p=0}^\infty \frac{1}{p!} \sum_{q=0}^p \ tr \left[ A (iY)^q B (iY)^{p-q} \right].
\end{align*}

\footnote{We thank S. Das, W. Taylor and S. Trivedi for discussion on this point.}
Since
\[ \text{Str} (C^p AB) = \frac{1}{(p+1)!} \sum_{q=0}^{p} p! \text{tr} [AC^q BC^{p-q}] \]
\[ = \frac{1}{p+1} \sum_{q=0}^{p} \text{tr} [AC^q BC^{p-q}], \]
where \( \text{Str} \) is the symmetrized trace, we can write
\[ \int_{0}^{1} d\tau \text{tr} \left[ A e^{i\tau kY} B e^{i(1-\tau)kY} \right] = \text{Str} (e^{ikY} AB). \]
Thus \( T^{IJ} \) is expressed as
\[ T^{IJ} = 2\alpha' \int dx e^{ikx} \text{Str} \left[ e^{ik_i Y^I} g^\mu\nu D_\mu Y^I D_\nu Y^J \right. \]
\[ + \left. e^{ik_i Y^I} \frac{g_{KL}}{(2\pi \alpha')^2} [Y^I, Y^K] [Y^L, Y^J] \right], \]
where we symmetrized over all possible orderings of \( Y^I, D_\mu Y^I \) and \([Y^I, Y^J]\) inside the trace. Similarly, we can derive the expressions for \( T^{\mu J} \) and \( T^{\mu \nu} \).
\[ T^{\mu J} = 2\alpha' g^{\mu}\nu_1 \int dx e^{ikx} \text{tr} \left[ e^{ik_i Y^I} D_\mu Y^J \right], \]
\[ T^{\mu \nu} = 2\alpha' g^{\mu}\nu \int dx e^{ikx} \text{tr} \left[ e^{ik_i Y^I} \right]. \]
These expressions agree with those derived in [39, 40, 67, 68] using one-loop amplitudes of Matrix Theory. Our result for the graviton can be extended to all other closed string states including massive ones as discussed in [12].

The appearance of the symmetrized trace is not an obvious consequence of string theory computation. To the contrary, in the case of bosonic string, the ordering of operators in the energy-momentum tensor does not obey the symmetrized trace prescription as we will see in Section 4.

The conservation of the energy-momentum tensor,
\[ k_\mu T^{\mu J} + k_1 T^{IJ} = 0, \]
has been verified by Van Raamsdonk [72]. For the D(-1)-branes, it goes as
\[ k_1 T^{IJ} \propto \text{tr} \int_{0}^{1} d\tau \left[ k \cdot Y, Y^K \right] e^{i\tau k \cdot Y} [Y^I, Y^J] e^{i(1-\tau)k \cdot Y} \]
\[ = i \text{tr} \int_{0}^{1} d\tau \frac{d}{d\tau} \left( Y^K e^{i\tau k \cdot Y} [Y^I, Y^J] e^{i(1-\tau)k \cdot Y} \right) \]
\[ = -i \text{tr} [Y^K, [Y^I, Y^J]] e^{ik \cdot Y} \]
\[ = 0, \]
where \( k \cdot Y \equiv k_i Y^I \) and \( Y^K \equiv g_{KL} Y^L \). In the last line, we used the equation of motion of Matrix Theory,
\[ [Y^K, [Y^I, Y^J]] = 0. \]
It is easily generalized to the case of Dp-branes using the identity

\begin{equation}
D_\mu(e^A) = \int_0^1 d\tau \ e^{(1-\tau)A} D_\mu A \ e^{-\tau A}.
\end{equation}

3. Relation to noncommutative gauge theory

Given the energy-momentum tensor of Matrix Theory, we can derive those of noncommutative gauge theories following the suggestion in [18]. Let us carry this out explicitly here. We divide the transverse directions \(x^i\) into two, \(x^i = (x^\alpha, x^i)\). To fit with the standard notation in noncommutative gauge theories, we rescale the coordinates \(x^i\) such that \(g_{ij} \sim O(\alpha^2)\). On the other hand, \(g_{\alpha\beta}\) as well as \(g_{\mu\nu}\) remains finite. Consider the background

\begin{equation}
Y^\alpha = 0, \quad Y^i = x^i \quad \text{such that} \quad [x^i, x^j] = -i\theta^{ij},
\end{equation}

and expand the scalar fields around this background as follows:

\begin{equation}
Y^\alpha = 2\pi\alpha' \Phi^\alpha \sim O(\alpha'), \quad Y^i = x^i + \theta^{ij} A_j \sim O(1).
\end{equation}

We take the limit \(k_\alpha \sim O(\alpha'^{-1})\) and \(g_{ij} \sim O(\alpha'^2)\) keeping \((\Phi^\alpha, A^i, A_\mu), (k_i, k_\mu)\) and \((g_{\alpha\beta}, G_{\mu\nu} = g_{\mu\nu})\) finite. This is identical to the limit taken in [12] following [10]. An open Wilson line, which constitutes a basic ingredient of the energy-momentum tensors of the noncommutative gauge theories, emerges in this background as follows [13]:

\begin{equation}
tr \ exp \left[i k_\alpha Y^\alpha\right] = tr \ exp \left[i k_\alpha x^\alpha + il^i A_i + iy_\alpha \Phi^\alpha\right]
\end{equation}

\begin{equation}
\propto \int dx \ \star \left[ \exp \left(i \int_0^1 d\tau \left(l^i A_i(x + l\tau) + y_\alpha \Phi^\alpha(x + l\tau)\right)\right) e^{ik_\alpha x^\alpha}\right],
\end{equation}

where \(l^i = k_j \theta^{ij}, y_\alpha = 2\pi\alpha' k_\alpha\). Here we mapped the matrices into the functions on noncommutative space with the star product on the right-hand side. The symbol \(\star[\ldots]\) means that we take the star product in the expression in \([\ldots]\) taking into account the path-ordering along the open Wilson line between \(x^i\) and \(x^i + l^i\). The integral \(\int dx\) here is over the noncommutative directions \(x^i\) along the branes which constitutes the integral over the whole worldvolume of the branes together with the previous integral over the commutative directions \(x^\mu\). In the zero-slope limit, the on-shell condition of the graviton is

\begin{equation}
G_{ij} l^i l^j + g_{\alpha\beta} y_\alpha y_\beta = 0.
\end{equation}

For the cases with operator insertions along the Wilson line, the following formulas are useful:

\begin{equation}
D_\mu Y^I = \begin{cases}
2\pi\alpha' D_\mu \Phi^\alpha & \text{when } I = \alpha, \\
\theta^{ij} F_{\mu j} & \text{when } I = i,
\end{cases}
\end{equation}

\begin{equation}
[Y^I, Y^J] = \begin{cases}
(2\pi\alpha')^2 [\Phi^\alpha, \Phi^\beta] & \text{when } (I, J) = (\alpha, \beta), \\
2\pi\alpha' \theta^{ij} D_j \Phi^\alpha & \text{when } (I, J) = (\alpha, j), \\
-i\theta^{ii} \theta^{ij} (F_{ij} - \theta^{-1}_{ij}) & \text{when } (I, J) = (i, j),
\end{cases}
\end{equation}
and

\[(45) \quad tr \int_0^1 d\tau \ Ae^{i r k_i Y^i} B e^{i(1-\tau)k_i Y^i} \]
\[
\propto \int dx * \left[ \exp \left( i \int_0^1 d\tau \left( l^i A_i(x + l\tau) + y_\alpha \Phi^\alpha(x + l\tau) \right) \right) \right.
\times \left. \int_0^1 d\tau_1 A(x + l\tau_1) \int_0^1 d\tau_2 B(x + l\tau_2) e^{ik_i x^i} \right].
\]

It is straightforward to verify that all the components of energy-momentum tensor of the noncommutative gauge theory in the case of superstring derived in [12] are correctly reproduced including the relative coefficients. For example, the expression for \( T^{ij} \) is given by

\[(46) \quad T^{ij} = 2\alpha' \theta^{-1}_{ij} \int dx * \left[ e^{ikx} \exp \left( i \int_0^1 d\tau \left( l^i A_i(x + l\tau) + y_\alpha \Phi^\alpha(x + l\tau) \right) \right) \right.
\times \left. \int_0^1 d\tau_1 \int_0^1 d\tau_2 \left[ G^{mn} (F'_{i'm}(x + l\tau_1) - \theta^{-1}_{i'm}) (F'_{j'n}(x + l\tau_2) - \theta^{-1}_{j'n}) \right.
\left. + G^{\mu\nu} F'_{i'\mu}(x + l\tau_1) F'_{j'\nu}(x + l\tau_2) + g_{\alpha\beta} D_i \Phi^\alpha(x + l\tau_1) D_j \Phi^\beta(x + l\tau_2) \right] \right]
\]

and the other components are presented in the Appendix.

Two important checks have been made in [12] to demonstrate that this is indeed the correct expression for the energy-momentum tensor. One is the consistency with the way the action of the noncommutative gauge theory [10] depends on the bulk metric \( g_{MN} \).

\[(47) \quad S = \frac{1}{g_{YM}} \int dx \sqrt{-\det G} \left[ \frac{1}{4} G^{MN} G^{PQ} (F'_{MP} - \theta^{-1}_{MP}) (F'_{NQ} - \theta^{-1}_{NQ}) + \frac{1}{2} G^{MN} g_{\alpha\beta} D_M \Phi^\alpha D_N \Phi^\beta - \frac{1}{4} g_{\alpha\beta} g_{\gamma\delta} [\Phi^\alpha, \Phi^\gamma] [\Phi^\beta, \Phi^\delta] \right].
\]

Here \( \theta^{-1}_{MN} = 0 \) unless \( (M, N) = (i, j) \). Since the action is derived assuming that the bulk metric is flat, we can compare the variation \( \partial S / \partial g_{MN} \) with the zero-momentum limit of the energy-momentum tensor.

The other one is the conservation of the energy-momentum tensor,

\[(48) \quad k_M T^{MN} = 0.
\]

We now have a simpler proof of this in the language of the matrix model discussed in the preceding section.

One may have expected that the energy-momentum tensor of the noncommutative theory should reduce to that of the commutative theory (theory defined on the commutative space) in the limit \( \theta_{MN} \to 0 \).

\[(49) \quad T_{comm}^{ij} = G^{ii'} G^{jj'} (F'_{i'm} F'_{j'n} G^{mn} - \frac{1}{4} G_{i'j'} F^2 + \cdots), \quad \text{etc.}
\]

Here \( \cdots \) denotes terms containing the scalar field \( \Phi^\alpha \) etc. It turned out that they are different in the following three accounts:
1. In the commutative case (49), the metric $G^{ii'}G^{jj'}$ is used to raise the indices of the energy-momentum tensor, whereas in the noncommutative case (46) we use $\theta^{ii'}\theta^{jj'}$ in the corresponding term.

2. In the noncommutative case, the field strength $F_{ij}$ is shifted by $\theta^{-1}_{ij}$.

3. In the noncommutative case, there are no terms that correspond to the term $-\frac{1}{4}G^{ij}F^2$ in (49).

All these differences are important in order to maintain the consistency with the metric dependence of the Seiberg-Witten action (47) and the conservation of the energy-momentum tensor.

4. Bosonic string

We can use the same technique to derive the energy-momentum tensor of multiple D-branes in the bosonic string theory in the zero-slope limit. The closed string vertex operator is

\[ V(z) = h_{MN}(k)\partial X^M \bar{\partial} X^N e^{ikX(z)}, \]

and the open string vertex operator is nothing but the bosonic part of $U^{(0)}(t)$ in (18). In the superstring case, the contribution to the graviton coupling starts at $O(\alpha'^3)$. In the bosonic case, on the other hand, it starts at $O(\alpha'^2)$ coming from the product of the following contractions:

\[ \langle \partial X^I(z) \partial_\perp X^K(t) \rangle = \frac{\alpha'g^{IK}}{(z-t)^2} e^{-2\pi i\tau(t,z)} \frac{\partial \tau(t,z)}{\partial t}, \]

\[ \langle \bar{\partial} X^J(\bar{z}) \partial_\perp X^L(t) \rangle = \frac{\alpha'g^{JL}}{(\bar{z}-\bar{t})^2} e^{2\pi i\tau(t,z)} \frac{\partial \tau(t,z)}{\partial t}. \]

The expression for $T^{IJ}$ is then

\[ T^{IJ} = -\int dx e^{ikx} \text{tr} \left[ \exp \left( i \int_0^1 dt \kappa_I Y^I(x) \right) \right. \]

\[ \times \left[ \int_0^1 d\tau_1 e^{-2\pi i\tau_1 Y^I(x)} \int_0^1 d\tau_2 e^{2\pi i\tau_2 Y^J(x)} + (I \leftrightarrow J) \right] \]

\[ = -\int dx e^{ikx} \text{tr} \int_0^1 d\tau \left[ e^{2\pi i\tau Y^I} e^{i\kappa_I Y^I Y^J} e^{i(1-\tau)\kappa_I Y^J} + (I \leftrightarrow J) \right]. \]

It is interesting to note that this leading term in the zero-slope limit vanishes in the case of a single brane since all the scalar fields commute and the only $\tau$ dependence of the integrand is $e^{2\pi i\tau}$. Thus $\int_0^1 d\tau e^{2\pi i\tau} = 0$. Another consequence of the existence of the factor $e^{2\pi i\tau}$ is that the ordering of the scalar fields no longer obeys the symmetrized trace prescription,

\[ T^{IJ} \neq -2 \int dx e^{ikx} \text{Str} \left[ e^{ik_I Y^I Y^J} \right]. \]

One can see this easily by considering the case of a single D-brane where $T^{IJ} = 0$ in this limit as we just mentioned whereas the symmetrized trace on the right-hand side is non-vanishing.

To see the relation of this energy-momentum tensor to those of the noncommutative gauge theories derived in \[12\] (reproduced in the Appendix of this paper),
the following identity is useful, which can be derived by integration by parts,

\begin{equation}
\text{tr} \int_0^1 d\tau e^{\pm 2\pi i \tau} A e^{i\tau k \cdot Y} B e^{i(1-\tau) k \cdot Y} = \text{tr} \int_0^1 d\tau \frac{1}{\pm 2\pi i} \left( e^{\pm 2\pi i \tau} A e^{i\tau k \cdot Y} B e^{i(1-\tau) k \cdot Y} \right) \\
= -\frac{1}{\pm 2\pi i} \text{tr} [A, B] e^{ik \cdot Y} \\
- \frac{1}{\pm 2\pi i} \text{tr} \int_0^1 d\tau e^{\pm 2\pi i \tau} A e^{i\tau k \cdot Y} [ik \cdot Y, B] e^{i(1-\tau) k \cdot Y} \\
= -\frac{1}{\pm 2\pi i} \text{tr} [A, B] e^{ik \cdot Y} \\
- \frac{1}{(2\pi)^2} \text{tr} [A, [ik \cdot Y, B]] e^{ik \cdot Y} \\
+ \frac{1}{(2\pi)^2} \text{tr} \int_0^1 d\tau e^{\pm 2\pi i \tau} [ik \cdot Y, A] e^{i\tau k \cdot Y} [ik \cdot Y, B] e^{i(1-\tau) k \cdot Y}.
\end{equation}

After we symmetrize with respect to $A$ and $B$, the first term on the right-hand side vanishes. Combining this with

\begin{equation}
[i k_1 Y^I, x^j + \theta^{jm} A_m] = -\theta^{jm} \left( l^m (F_{mn} - \theta^{-1}_{mn}) + y_{\beta} D_m \Phi^{\beta} \right), \\
[x^j + \theta^{jm} A_m, O] = -i \theta^{jm} D_m O,
\end{equation}

it is then straightforward to see that (52) reproduces the energy-momentum tensors of the noncommutative theories (52)–(54).

The proof of the conservation of the energy-momentum tensor $k_1 T^{IJ} = 0$ in the matrix form (52) of $T^{IJ}$ is given by

\begin{equation}
\text{tr} \int_0^1 d\tau e^{\pm 2\pi i \tau} k \cdot Y e^{i\tau k \cdot Y} Y^J e^{i(1-\tau) k \cdot Y} \\
= \text{tr} \int_0^1 d\tau e^{\pm 2\pi i \tau} e^{i\tau k \cdot Y} k \cdot Y Y^J e^{i(1-\tau) k \cdot Y} \\
= \int_0^1 d\tau e^{\pm 2\pi i \tau} \text{tr} k \cdot Y Y^J e^{ik \cdot Y} \\
= 0 \quad \text{because of the } \tau \text{ integration.}
\end{equation}

Note that the conservation holds without using the equation of motion. This proof is significantly simpler than the one given in [12].

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Appendix A. Energy-momentum tensors of noncommutative gauge theories

A.1. Superstring.

(56) \[ T^{ij} = 2\alpha' \theta^{ij} \]
\[
\times \int dx * \left[ e^{ikx} \exp \left( i \int_0^1 d\tau \left( l^i A_i(x + l\tau) + y_{\alpha} \Phi^\alpha(x + l\tau) \right) \right) \right.
\]
\[
\times \int_0^1 d\tau_1 \int_0^1 d\tau_2 \left[ G^{mn} (F_{\nu m}(x + l\tau_1) - \theta_{-1}^{\nu m}(x + l\tau_2) - \theta_{-1}^{\nu m}) \right.
\]
\[
+ G^{\mu\nu} F_{\nu\mu}(x + l\tau_1) F_{\nu\nu}(x + l\tau_2) + g_{\alpha\beta} D_{\alpha}\Phi^\alpha(x + l\tau_1) D_{\beta}\Phi^\beta(x + l\tau_2) \bigg],
\]

(57) \[ T^{ij} = 2\alpha' G^{ij} \theta^{ij} \]
\[
\times \int dx * \left[ e^{ikx} \exp \left( i \int_0^1 d\tau \left( l^i A_i(x + l\tau) + y_{\alpha} \Phi^\alpha(x + l\tau) \right) \right) \right.
\]
\[
\times \int_0^1 d\tau' F_{\nu\mu}(x + l\tau'), \bigg],
\]

(58) \[ T^{ij} = -4\pi \alpha'^2 \theta^{ij} \]
\[
\times \int dx * \left[ e^{ikx} \exp \left( i \int_0^1 d\tau \left( l^i A_i(x + l\tau) + y_{\alpha} \Phi^\alpha(x + l\tau) \right) \right) \right.
\]
\[
\times \int_0^1 d\tau_1 \int_0^1 d\tau_2 \left[ G^{mn} D_{\alpha}\Phi^\alpha(x + l\tau_1) (F_{\nu m}(x + l\tau_2) - \theta_{-1}^{\nu m}) \right.
\]
\[
+ G^{\mu\nu} D_{\mu}\Phi^\alpha(x + l\tau_1) F_{\nu\nu}(x + l\tau_2) + g_{\alpha\beta} i[\Phi^\beta, \Phi^\alpha](x + l\tau_1) D_{\nu}\Phi^\nu(x + l\tau_2) \bigg],
\]

(59) \[ T^{\mu\nu} = 2\alpha' G^{\mu\nu} \]
\[
\times \int dx * \left[ e^{ikx} \exp \left( i \int_0^1 d\tau \left( l^i A_i(x + l\tau) + y_{\alpha} \Phi^\alpha(x + l\tau) \right) \right) \right],
\]

(60) \[ T^{\mu\nu} = 4\pi \alpha'^2 G^{\mu\nu} \]
\[
\times \int dx * \left[ e^{ikx} \exp \left( i \int_0^1 d\tau \left( l^i A_i(x + l\tau) + y_{\alpha} \Phi^\alpha(x + l\tau) \right) \right) \right.
\]
\[
\times \int_0^1 d\tau' D_{\mu}\Phi^\nu(x + l\tau'), \bigg].
(61) \[ T^{\alpha\beta} = 8\pi^2 \alpha^3 \]
\[ \times \int dx \left[ e^{ikx} \exp \left( i \int_0^1 d\tau \left( l^i A_i(x + l\tau) + y_\alpha \Phi^\alpha(x + l\tau) \right) \right) \right. \]
\[ \times \int_0^1 d\tau_1 \int_0^1 d\tau_2 \left[ G^{ij} D_i \Phi^\alpha(x + l\tau_1) D_j \Phi^\beta(x + l\tau_2) \right. \]
\[ + G^{\mu\nu} D_\mu \Phi^\alpha(x + l\tau_1) D_\nu \Phi^\beta(x + l\tau_2) \]
\[ \left. - g_{\gamma\delta} [\Phi^\gamma, \Phi^\delta](x + l\tau_1) [\Phi^\delta, \Phi^\beta](x + l\tau_2) \right]. \]

A.2. Bosonic string.

(62) \[ T^{ij} = \frac{\theta^{ij} g_{ij'} + \theta^{ji} g_{ij'}}{(2\pi)^2} \]
\[ \times \int dx \left[ e^{ikx} \exp \left( i \int_0^1 d\tau \left( l^i A_i(x + l\tau) + y_\alpha \Phi^\alpha(x + l\tau) \right) \right) \right. \]
\[ \times \left\{ i \int_0^1 d\tau_1 e^{-2\pi i\tau_1} \left( l^m F_{\nu m}(x + l\tau_1) + y_\alpha D_\nu \Phi^\alpha(x + l\tau_1) \right) \right. \]
\[ \times i \int_0^1 d\tau_2 e^{2\pi i\tau_2} \left( l^n F_{\nu n}(x + l\tau_2) + y_\beta D_\nu \Phi^\beta(x + l\tau_2) \right) \]
\[ + i \int_0^1 d\tau \left( l^m D_i F_{ij'n}(x + l\tau') + y_\alpha D_i D_j \Phi^\alpha(x + l\tau') \right) \left. \right\}. \]

(63) \[ T^{ij} = \alpha' \theta^{jm} \int dx \left[ e^{ikx} \exp \left( i \int_0^1 d\tau \left( l^i A_i(x + l\tau) + y_\alpha \Phi^\alpha(x + l\tau) \right) \right) \right. \]
\[ \times \left\{ i \int_0^1 d\tau_1 e^{-2\pi i\tau_1} \Phi^\alpha(x + l\tau_1) \right. \]
\[ \times i \int_0^1 d\tau_2 e^{2\pi i\tau_2} \left( l^n F_{mn}(x + l\tau_2) + y_\beta D_m \Phi^\beta(x + l\tau_2) \right) \]
\[ - i \int_0^1 d\tau_1 e^{2\pi i\tau_1} \Phi^\alpha(x + l\tau_1) \]
\[ \times i \int_0^1 d\tau_2 e^{-2\pi i\tau_2} \left( l^n F_{mn}(x + l\tau_2) + y_\beta D_m \Phi^\beta(x + l\tau_2) \right) \left. \right\}. \]

(64) \[ T^{\alpha\beta} = -(2\pi\alpha')^2 \]
\[ \times \int dx \left[ e^{ikx} \exp \left( i \int_0^1 d\tau \left( l^i A_i(x + l\tau) + y_\alpha \Phi^\alpha(x + l\tau) \right) \right) \right. \]
\[ \times \left\{ i \int_0^1 d\tau_1 e^{-2\pi i\tau_1} \Phi^\alpha(x + l\tau_1) \int_0^1 d\tau_2 e^{2\pi i\tau_2} \Phi^\beta(x + l\tau_2) + (\alpha \leftrightarrow \beta) \right\}. \]
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