ISOMETRY CLASSES OF GENERALIZED ASSOCIAHEDRA

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Abstract. Let \((W,S)\) be a finite Coxeter system acting by reflections on an \(R\)-Euclidean space with simple roots \(\Delta = \{\alpha_s | s \in S\}\) of the same length, and fundamental weights \(\Delta^* = \{\nu_s | s \in S\}\) of the same length. We set \(M(e) = \sum_{s \in S} \kappa_s v_s, \kappa_s > 0\) and for \(w \in W\) we set \(M(w) = w(M(e))\). The permutahedron \(\text{Perm}(W)\) is the convex hull of the set \(\{M(w) | w \in W\}\). Given a Coxeter element \(c \in W\), we have defined in a previous work a generalized associahedron \(\text{Asso}_c(W)\) whose normal fan is the corresponding \(c\)-Cambrian fan \(F_c\) defined by N. Reading. By construction, \(\text{Asso}_c(W)\) is obtained from \(\text{Perm}(W)\) by removing some half-spaces according to a rule prescribed by \(c\). In this work, we classify the isometry classes of these realizations. More precisely, for \((W,S)\) an irreducible finite Coxeter system and \(c, c'\) two Coxeter elements in \(W\), we have that \(\text{Asso}_c(W)\) and \(\text{Asso}_{c'}(W)\) are isometric if and only if \(\mu(c') = c\) or \(\mu(c') = w_0 c^{-1} w_0\) for \(\mu\) an automorphism of the Coxeter graph of \(W\) such that \(\kappa_s = \kappa_{\mu(s)}\) for all \(s \in S\). As a byproduct, we classify the isometric Cambrian fans of \(W\).

1. Introduction.

Studying binary operations, Stasheff [10, 11] remarked that there was a cell complex whose vertices correspond to the possible compositions of \(n\) binary operations. Furthermore this cell complex can be realized as a simple polytope, the associahedron. There is a natural relation between the permutahedron (weak lattice on permutations) and the associahedron: the permutahedron can naturally be written as an intersection of halfspaces indexed by weights for the \(A_n\) root system. If one intersects a certain, carefully chosen subset of these halfspaces, one can obtain the associahedron (see Example 2.2 below). Shnider-Sternberg [9] and Loday [5] give us a beautiful explicit construction of the associahedron from the permutahedron. The facets of both objects encode the algebraic structure of the corresponding free operads.

Generalized associahedra were introduced by S. Fomin and A. Zelevinsky in their work on cluster algebras [2]. The geometry of these objects encode nice algebraic structures. Therefore one important question is to find good polytopal realizations of the generalized associahedra. This was first answered in [1]. Then, N. Reading [8] constructed a family of fans, the Cambrian fans \(\{F_c\}\) indexed by Coxeter elements \(c\) of a given finite Coxeter group \(W\). More recently, we have constructed [3] a family of generalized associahedra \(\text{Asso}_c(W)\), one for each Cambrian fan \(F_c\). These generalized associahedra are realized from the corresponding permutahedron by...

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removing some halfspaces according to a rule specified by $c$, linking questions about
generalized associahedra to questions about the better known permutahedron.
It is now natural to ask how many distinct (up to isometry) generalized associa-
hedra we have. This is what we answer here. Our main theorem (Theorem 2.3) is
to describe completely the isometry classes of generalized associahedra as realized
in $[3]$ . The isometry classes depend on the choice of the starting permutahedron.
As a byproduct we obtain a classification of the isometry classes of Cambrian fans
(Corollary 2.6): the Cambrian fans indexed by Coxeter elements $c$ and $c'$ are iso-
metric if and only if $\mu(c') = c$ or $\mu(c') = c^{-1}$ for some $\mu$ an automorphism of the
Coxeter graph of $W$. In Section 2 we introduce the necessary definitions and state
our main theorem (Theorem 2.3). The proof is found in Section 4. Section 3 is
dedicated to some auxiliary results needed for this proof. For most of the paper, we
make the simplifying assumption that the Coxeter system in question is irreducible;
in Section 5, we explain how to deal with the reducible case.

2. BACKGROUND AND MAIN THEOREM.

We assume some basic familiarity with Coxeter groups and root systems and
follow the notation of $[4]$. Let $(W, S)$ be a finite Coxeter system acting by reflections
on an $\mathbb{R}$-euclidean space $(V, \langle \cdot , \cdot \rangle )$ with length function $\ell : W \to \mathbb{N}$. Without loss of
generality, we assume that the action of $W$ is essential relative to $V$, that is, has
no non trivial space fixed pointwise.

Let $\Phi$ be a root system corresponding to $(W, S)$, with all roots having equal
length. (In particular, we do not assume $\Phi$ is crystallographic.) The simple roots
$\Delta$ are a basis of $V$, and the reflection $s$ maps $\alpha$ to $-\alpha$ and fixes the hyperplane
$H_s = \{ v \in V \mid \langle v, \alpha_s \rangle = 0 \}$. Let $\Delta^* = \{ v_s \mid s \in S \}$ be the set of fundamental weights
of $\Delta$, that is, $\langle v_s, \alpha_s \rangle = 1$ if $s = t$ and 0 otherwise. As $V$ is finite dimensional we
identify $V$ and $V^*$.

We now aim for a definition of the $W$-permutahedron and pick a point $u \in V$
contained in the complement of the reflection hyperplanes of $W$. Without loss of
generality, we choose $u := \sum_{s \in S} \kappa_s v_s$, $\kappa_s > 0$.

For $w \in W$ we write

$M(e) := u$ \quad and \quad $M(w) := w(M(e))$

and obtain the permutahedron $\text{Perm}_u(W)$ as convex hull of $\{ M(w) \mid w \in W \}$. The
index $u$ will often be omitted for brevity. Equivalently, we have

$\text{Perm}(W) = \bigcap_{s \in S} \bigcap_{x \in W} \mathcal{H}_{(x, s)}$

where

$\mathcal{H}_{(x, s)} := \{ v \in V \mid \langle v, x(v_s) \rangle \leq \langle M(e), v_s \rangle \}.$

We also make use of the hyperplane $H_{(x, s)} = \{ v \in V \mid \langle v, x(v_s) \rangle = \langle M(e), v_s \rangle \}$.

Denote by $W_I$ the standard parabolic subgroup of $W$ generated by $I \subseteq S$. Note
that $H_{(w, s)} = H_{(x, s)}$ if and only if $w \in xW_{S \setminus \{ s \}}$. Also, $M(w) \in H_{(x, s)}$ if and only
if $H_{(x, s)} = H_{(w, s)}$. Hence we have a simple way to describe the vertices:

$\{ M(w) \} = \bigcap_{s \in S} H_{(w, s)}.$
Example 2.1 (Realization of $\text{Perm}(A_2)$). We consider the Coxeter group $W = S_3$ of type $A_2$ acting on $\mathbb{R}^2$. The reflections $s_1$ and $s_2$ generate $W$. The simple roots that correspond to $s_1$ and $s_2$ are $\alpha_1$ and $\alpha_2$. They are normal to the reflection hyperplanes $H_{s_1}$ and $H_{s_2}$. The dual vectors to the simple roots correspond to the vectors $v_1$ and $v_2$. Fix a ray $L = \{\mu(\kappa_1 v_1 + \kappa_2 v_2) \mid \mu > 0\}$ where $\kappa_1, \kappa_2 > 0$. We choose $M(e) \in L$ and obtain the permutahedron as convex hull of the $W$-orbit of $M(e)$. Alternatively, the permutahedron can be described as intersection of the half spaces $\mathcal{H}(x,s)$ with bounding hyperplanes $H(x,s)$ for $x \in W$ and $s \in S$. All the objects are indicated in Figure 1.

For $c$ a Coxeter element in $W$, that is to say, the product of the simple reflections of $W$ taken in some order, and $I \subseteq S$, we denote by $c(I)$ the subword of $c$ obtained by taking only the simple reflections in $I$. So $c(I)$ is a Coxeter element of $W_I$. Reading defined the c-sorting word of $w \in W$ in [6] Section 2 as the unique subword of the infinite word $c^\omega = cccccc\ldots$ that is a reduced expression for $w$ and is the lexicographically smallest sequence of positions occupied by this subword. In particular, the c-sorting word of $w$ is such that $w = c(K_1)c(K_2)\ldots c(K_p)$ with non-empty $K_i \subseteq S$ and $\ell(w) = \sum_{i=1}^p |K_i|$. As example we consider the Coxeter group $W = S_4$ of type $A_3$ generated by the simple reflections $S = \{s_1, s_2, s_3\}$, where $s_1, s_3$ commute, and the Coxeter element $c = s_2s_1s_3$. The c-sorting word of the longest element $w_0 \in W$ is $s_2s_1s_3s_2s_1s_3 = c_1c_2c_3$. If we choose the Coxeter element $c = s_1s_2s_3$ instead of $s_2s_1s_3$, then the c-sorting word of $w_0$ is $s_1s_2s_3s_1s_2s_1 = c(s)c_1c_2c_3$.

The sequence $c(K_1), \ldots, c(K_p)$ associated to the c-sorting word for $w$ is called the c-factorization of $w$. The c-factorization of $w$ is independent of the chosen reduced word for $c$ but depends on the Coxeter element $c$. In general the c-factorization does not yield a nested sequence $K_1, \ldots, K_p$ of subsets of $S$. An element $w \in W$ is called c-sortable if $K_1 \supseteq K_2 \supseteq \ldots \supseteq K_p$. Reading proves in [6] that the longest element $w_0 \in W$ is c-sortable for any chosen Coxeter element $c$.

Given a specific reduced word $v$, we say that $u$ is a prefix up to commutation of $v$ if some reduced word for $u$ appears as a prefix of a word which can be obtained from
For example $s$ is a prefix up to commutation because it appears as a prefix after commuting the simple reflections $s_1$, $s_3$.

The halfspace $\mathcal{H}_{(x,s)}$ is said to be \textit{c-admissible} if the hyperplane $H_{(x,s)}$ contains $M(w)$ for some $c$-singleton $w$. We have shown in [3] that the intersection of all $c$-admissible halfspaces $\mathcal{H}_{(x,s)}$ is a \textit{generalized associahedron} $\text{Asso}_c(W)$ whose normal fan is the $c$-Cambrian fan $\mathcal{F}_c$ (see [2] for a definition of $\mathcal{F}_c$).

\textbf{Example 2.2.} The Coxeter group $W = S_3$ generated by the reflections $s_1$ and $s_2$ has two Coxeter elements: $c_1 = s_1s_2$ and $c_2 = s_2s_1$. The $c_1$-singletons are $e$, $s_1$, $s_1s_2$, and $s_1s_2s_1$ while the $c_2$-singletons are $e$, $s_2$, $s_2s_1$, and $s_2s_1s_2$. Starting with the permutahedron $\text{Perm}(S_3)$, we obtain the two associahedra $\text{Asso}_{c_1}(S_3)$ and $\text{Asso}_{c_2}(S_3)$ shown in Figure 2 as intersection of the $c_1$- and $c_2$-admissible halfspaces.

For most of the paper, we will assume that $(W,S)$ is irreducible. The case where $(W,S)$ is reducible requires a straightforward (but not immediate) extension of the results in the irreducible case; we describe this in the final section.

An automorphism of the Coxeter graph associated to $(W,S)$ is a bijection $\mu$ on $S$ such that the order of $\mu(s)\mu(t)$ equals the order of $st$ for all $s, t \in S$. In particular, $\mu$ induces an automorphism on $W$.

Let $u = \sum_{s \in S} \kappa_s v_s$ be a point in $V$. We will say that $u$ is \textit{balanced} if $\kappa_s = \kappa_t$ for all $s, t \in S$. An automorphism $\mu$ of the Coxeter graph is a $u$-\textit{automorphism} if $\kappa_s = \kappa_{\mu(s)}$ for all $s \in S$. In particular, if $u$ is balanced, then any automorphism of a Coxeter graph is a $u$-automorphism.

\textbf{Theorem 2.3.} Let $(W,S)$ be an irreducible finite Coxeter system and $c_1$, $c_2$ be two Coxeter elements in $W$. Suppose that $u = \sum_{s \in S} \kappa_s v_s$ for some linear isometry $\varphi$ on $V$.
There is an $u$-automorphism $\mu$ of the Coxeter graph of $(W,S)$ such that $\mu(c_2) = c_1$ or $\mu(c_2) = w_0c_1^{-1}w_0$.

Observe that $w_0c_1^{-1}w_0$ may or may not equal $c_1$ (for instance in $A_3$ take $c = s_1s_3s_2$). So the second condition in Theorem 2.3 may be redundant and the associahedra may actually be identical (not just isometric). Moreover, if the $\kappa_s$ are chosen generically, that is distinct, then the isometry classes are of cardinality 1 or 2. As stated in the next corollary, the isometry classes reach their maximal cardinality if $u$ is balanced.

**Corollary 2.4.** Let $(W,S)$ be an irreducible finite Coxeter system and $c_1$, $c_2$ be two Coxeter elements in $W$. If $u$ is balanced, then the following statements are equivalent.

1. $\text{Assoc}_{c_1}(W) = \varphi(\text{Assoc}_{c_2}(W))$ for some linear isometry $\varphi$ on $V$.
2. There is an automorphism $\mu$ of the Coxeter graph of $(W,S)$ such that $\mu(c_2) = c_1$ or $\mu(c_2) = c_1^{-1}$.

**Proof.** It follows from Theorem 2.3 and a rewriting of the second assumption in accordance with the fact that the map $s \mapsto w_0sw_0$ is an automorphism of the Coxeter graph.

If $u$ is balanced, then $\text{Assoc}_{c_1}(W) = \varphi(\text{Assoc}_{c_2}(W))$ for some linear isometry $\varphi$ on $V$ if and only if there is an $u$-automorphism $\theta$ of the Bruhat ordering of $(W,S)$ such that $\theta(c_2) = c_1$. This follows by inspection for $|S| = 1, 2$ and, for $|S| \geq 3$, from a characterization of automorphisms of Bruhat orderings due to van den Hombergh, see Section 8.8 of [4] and the fact that a Coxeter element $c$ defines an orientation of the Coxeter graph $\Gamma$: Orient the edge $\{s_i, s_j\}$ from $s_i$ to $s_j$ if and only if $s_i$ is to the left of $s_j$ for any reduced word for $c$.

![Figure 3. Six Coxeter elements and their associated oriented Coxeter graphs of the Coxeter group of type $D_4$ that yield isometric associahedra.](image-url)
Theorem 2.3 combined with the classification of irreducible finite Coxeter groups yields that the cardinality of an isometry class in the case where $u$ is balanced is either two, four, or six. We briefly discuss the situation.

Example 2.5. Use the notation and hypothesis of Corollary 2.4.

1. Let $(W,S)$ be a Coxeter system of type $A_n$ ($n \geq 2$), $E_6$, $F_4$, or $I_2(m)$. Then there is precisely one non-trivial automorphism $\mu$ of the Coxeter graph. Hence there are either two or four elements in the isometry class of $\text{Assoc}(W)$. The cardinality equals two if $\mu(c) \in \{c, c^{-1}\}$ and equals four if $\mu(c) \notin \{c, c^{-1}\}$.

2. Let $(W,S)$ be a Coxeter system of type $B$, $E_7$, $E_8$, $H_3$, or $H_4$. Then the conjugation by $w_0$ is the identity, and $\text{Id}$ is the only automorphism of the associated Coxeter graph. So each isometry class has cardinality two, only the Coxeter elements $c$ and $c^{-1}$ yield isometric associahedra.

3. Let $(W,S)$ be a Coxeter system of type $D$. If $|S| > 4$ then there is only one non-trivial automorphism $\mu$ of the Coxeter graph and the isometry class of $\text{Assoc}(W)$ has cardinality two if $\mu(c) \in \{c, c^{-1}\}$ and four otherwise. If $|S| = 4$, the group of automorphisms of the Coxeter graph is generated by the non-trivial automorphisms $\mu$ and $\nu$ with $\mu^2 = \text{Id}$ and $\nu^3 = \text{Id}$. The isometry class of $\text{Assoc}(W)$ consists either of two or six elements, see Figure 3 for six Coxeter elements that yield isometric associahedra.

Theorem 2.3 allows a classification of the isometric Cambrian fans as well. The proof will be given at the end of Section 4.

Corollary 2.6. The following propositions are equivalent:

1. The Cambrian fans $\mathcal{F}_c$ and $\mathcal{F}_{c'}$ are isometric;
2. $\text{Assoc}(W)$ and $\text{Assoc}(W')$ are isometric if $u$ is balanced;
3. there is an automorphism $\mu$ of the Coxeter graph of $(W,S)$ such that $\mu(c') = c$ or $\mu(c') = c^{-1}$.

Remark 2.7. In fact, the condition that $u$ be balanced in (2) above can be weakened to require only that $u$ satisfies $\kappa_s = \kappa_{w_0sw_0}$, for all $s \in S$ (see Eq. [1] in the proof).

3. Preliminary results

From now on, we fix $u = \sum_{s \in S} \kappa_s v_s$ for some constants $\kappa_s > 0$. Remember that $M(e) = u$.

Let $(W,S)$ be a Coxeter system and consider the polyhedron $P$ defined by:

$$P := \bigcap_{s \in S} \mathcal{H}_{(e,s)} \cap \bigcap_{s \in S} \mathcal{H}_{(w_0,s)}.$$

Remark 3.1. In fact, $P$ is a full-dimensional convex polytope because $W$ acts essentially on $V$ and the cones $\bigcap_{s \in S} \mathcal{H}_{(e,s)}$ and $\bigcap_{s \in S} \mathcal{H}_{(w_0,s)}$ are strictly convex, pointed with apex $M(e)$ and $M(w_0)$, and both contain $\text{Perm}(W)$. In other words, $P$ is obtained from $\text{Perm}(W)$ by removing, from the definition of $\text{Perm}(W)$ as an intersection of halfspaces, all halfspaces $\mathcal{H}_{(x,s)}$ that satisfy $M(e) \notin H_{(x,s)}$ and $M(w_0) \notin H_{(x,s)}$. 
Proposition 3.2. Let \( \varphi : V \to V \) be a linear isometry that maps \( P \) to itself and has the fixed point \( M(e) \). Then \( \varphi \) induces an \( u \)-automorphism \( \mu \) of the Coxeter graph of \( (W,S) \) such that \( \varphi(v_s) := v_{\mu(s)} \) for every \( s \in S \).

Proof. For \( \varphi \) satisfying our hypothesis, we have that \( \varphi(M(e)) = M(e) \in H_{(e,r)} \). Hence \( \varphi \) induces a bijection on the set \( \{ \mathcal{H}(e) \mid s \in S \} \). Since \( v_s \) is a normal vector to \( \mathcal{H}(e) \) for any \( s \in S \), we have that \( \varphi(v_s) = k_s v_{t_s} \) for some \( k_s > 0 \) and \( t_s \in S \). Hence \( \sum_{s \in S} k_s \kappa_s v_s = \varphi(M(e)) = M(e) = \sum_{s \in S} \kappa_s v_s \) and then \( \kappa_s = k_s \kappa_s \).

On the other hand, since \( \varphi \) is an isometry fixing \( P \) and \( M(e) \), it induces a bijection on the set of edges of \( P \) which have \( M(e) \) as one of their vertices. Each of these edges is contained in a line \( l_s := \bigcap_{r \in S \setminus \{s\}} H_{(e,r)} \), for \( s \in S \). So \( \varphi \) induces a bijection on the set \( \{ l_s \mid s \in S \} \). Since \( v_r \) is a fundamental weight of \( \Delta \), and is a normal vector of \( H_{(e,r)} \), the hyperplane \( H_{(e,r)} \) is spanned by the simple roots \( \{ \alpha_s \mid u \in S \setminus \{r\} \} \), and thus \( l_s \) is spanned by the simple root \( \alpha_s \). As the simple roots are all of the same length and \( \varphi \) preserves the norm of vectors, \( \varphi(\alpha_s) = \pm \alpha_r \) for some \( r, s \in S \). Now

\[
1 = \langle v_s, \alpha_s \rangle = \langle \varphi(v_s), \varphi(\alpha_s) \rangle = k_s \langle v_{t_s}, \alpha_r \rangle.
\]

We conclude that \( r_s = r, k_s = 1 \) and \( \kappa_r = \kappa_t \) for all \( r \in S \). Therefore \( \varphi \) induces a bijection on the set \( \{ v_s \mid s \in S \} \). In other words \( \varphi(\Delta^* ) = \Delta^* \) and, since \( \phi \) is a vector isometry, \( \varphi(\Delta) = \Delta \) and the angle between \( \alpha_s, \alpha_r \) is preserved. That is, \( \varphi \) induces an \( u \)-automorphism of the Coxeter graph of \( W \), since the order of \( st \) in \( W \) is entirely determined by the angle between \( \alpha_s \) and \( \alpha_t \) and since \( \kappa_s = \kappa_t \) for all \( s \in S \).

Remark 3.3. In the proof of the previous proposition, we have made use of the assumption (stated when we introduced the root system \( \Phi \)) that all its roots are of equal length. Note that this is not an important restriction, since any root system can be rescaled to have all its roots of equal length. For example, the vectors \((1,0)\) and \((-1,1)\) are simple roots for the crystallographic root system \( B_2 \). Instead of these, we would take \((1,0)\) and \((-\sqrt{2},1,\sqrt{2})\) as the simple roots. Attention, the assumption that the simple roots are of the same length does not imply that the fundamental weights are of the same length, see for instance in \( A_3 \).

Proposition 3.4. For every \( u \)-automorphism \( \mu \) of the Coxeter graph, there is a unique linear isometry \( \varphi_\mu \) that fixes \( P \) and \( M(e) \) defined by \( \varphi_\mu(\alpha_s) := \alpha_{\mu(s)} \) for every \( s \in S \).

Proof. The map \( \varphi_\mu \) is well-defined since \( \Delta \) is a basis of \( V \). As \( \mu \) is an automorphism of the Coxeter graph and \( (\alpha_{\mu(s)}, \alpha_{\mu(t)}) \) depends only on the order of \( st \), we have \( \langle \alpha_{\mu(s)}, \alpha_{\mu(t)} \rangle = \langle \alpha_s, \alpha_t \rangle \) for \( s, t \in S \). In other words \( \varphi_\mu \) is an isometry since \( \Delta \) is a basis of \( V \).

From duality it is clear that \( v_s = \sum_{r \in S} \langle v_r, v_s \rangle \alpha_r \), for all \( s \in S \). Also, the matrices \( [(v_r, v_s)]_{s,t} \) and \( [(\alpha_r, \alpha_s)]_{s,t} \) are inverse to each other and the permutation \( \mu : S \to S \) is such that \( [(\alpha_{\mu(r)}, \alpha_{\mu(s)})]_{s,t} = [(\alpha_r, \alpha_s)]_{s,t} \). Hence \( [(v_{\mu(r)}, v_{\mu(s)})]_{s,t} = [(v_r, v_s)]_{s,t} \). Thus, for \( s \in S \) we have

\[
\varphi_\mu(v_s) = \sum_{r \in S} \langle v_s, v_r \rangle \alpha_{\mu(r)} = \sum_{r \in S} \langle v_{\mu(s)}, v_{\mu(r)} \rangle \alpha_{\mu(r)} = \sum_{r' \in S} \langle v_{\mu(s)}, v_{r'} \rangle \alpha_{\mu(r')} = v_{\mu(s)}. \]

Now as \( \kappa_s = \kappa_{\mu(s)} \) for all \( s \in S \), \( \varphi_\mu \) fixes \( M(e) \), and therefore \( P \).

Similarly, for every isometry \( \varphi \) that fixes \( P \) and \( M(e) \) there is an \( u \)-automorphism \( \mu \) such that \( \varphi(v_s) := v_{\mu(s)} \) for every \( s \in S \), by Proposition 3.2.
Corollary 3.5. Let $\mu$ be an automorphism of the Coxeter graph of $(W, S)$ and $\varphi$ be a linear isometry that maps $P$ to itself and has $M(e)$ as fixed point. Suppose that $\mu$ and $\varphi$ are related via $\varphi(v_s) = v_{\mu(s)}$ for all $s \in S$. Then $\varphi = \varphi_\mu$ and $\mu$ is an $u$-automorphism. Moreover,

$$\varphi(w(v_s)) = (\mu(w))(v_{\mu(s)}) \quad \text{and} \quad \varphi(\mathcal{H}(w, s)) = \mathcal{H}(\mu(w), \mu(s))$$

for $w \in W$, $s \in S$. In particular, $\varphi(\text{Perm}(W)) = \text{Perm}(W)$.

Proof. As $\Delta^*$ is a basis of $V$, $\varphi = \varphi_\mu$. Moreover, since $\varphi$ fixes $M(e)$ we have

$$\sum_{\mu(s) \in S} \kappa_{\mu(s)}(v_{\mu(s)}) = \sum_{s \in S} \kappa_s v_s = M(e) = \varphi(M(e)) = \sum_{s \in S} \kappa_s v_{\mu(s)}.$$  

By identification, we have $\kappa_s = \kappa_{\mu(s)}$ for all $s \in S$, which proves that $\mu$ is an $u$-automorphism.

We prove the first remaining claim by induction on the length of $w$. If $w = e$ the claim is $\varphi(v_e) = v_{\mu(e)}$ and was shown in the proof of Proposition 3.2. Now assume $\ell(w) > 0$. There is $t \in S$ such that $w = w't$ with $\ell(w') < \ell(w)$. The action of $t$ on $V$ is a reflection with reflection hyperplane $H_t$. Hence we have

$$t(v_s) = v_s - \frac{2\langle v_s, \alpha_t \rangle}{\langle \alpha_t, \alpha_t \rangle} \alpha_t.$$  

Also, since $\varphi$ is an isometry that maps $\alpha_e$ to $\alpha_{\mu(e)}$ and $v_r$ to $v_{\mu(r)}$, we have

$$\varphi_\mu(w(v_s)) = \varphi_\mu(w'(t(v_s))) = \mu(w') \left( v_{\mu(s)} - \frac{2\langle v_{\mu(s)}, \alpha_{\mu(t)} \rangle}{\langle \alpha_{\mu(t)}, \alpha_{\mu(t)} \rangle} \alpha_{\mu(t)} \right) = \mu(w') \mu(t) \left( v_{\mu(s)} \right) = \mu(w) \left( v_{\mu(s)} \right).$$

We now prove the second claim.

$$\varphi_\mu(\mathcal{H}(w, s)) = \{ \varphi_\mu(v) \in V \mid \langle v, w(v_s) \rangle \leq \langle M(e), v_s \rangle \} = \{ v \in V \mid \langle v, \varphi_\mu(w(v_s)) \rangle \leq \langle \varphi_\mu(M(e)), \varphi_\mu(v_s) \rangle \} = \{ v \in V \mid \langle v, \mu(w)(v_{\mu(s)}) \rangle \leq \langle M(e), v_{\mu(s)} \rangle \} = \mathcal{H}(\mu(w), \mu(s))$$

The $c$-singletons form a distributive sublattice of the right weak order, see [3]. We denote the Hasse diagram of this poset by $G_c$, see Figure 4 for examples. They also form a sublattice of the $c$-Cambrian lattice.

There is an important linear isometry on $V$ that fixes $P$ and interchanges $M(e)$ and $M(w_0)$: The map $g$ defined by $v \mapsto w_0(v)$.

Reading proves in [7, Proposition 1.3] that the map $w \mapsto w w_0$ is an anti-isomorphism from the $c$-Cambrian lattice to the $c^{-1}$-Cambrian lattice. In particular, the map $w \mapsto w w_0$ is an anti-isomorphism between the lattices of $c$-singletons and $c^{-1}$-singletons by restriction, that is, $w$ is a $c$-singleton if and only if $w w_0$ is a $c^{-1}$-singleton. Since the map $w \mapsto w_0 w w_0$ is an isomorphism from the $c$-Cambrian lattice to the $w_0 c w_0$-Cambrian lattice, the map $w \mapsto w_0 w = (w_0 w w_0) w_0$
Let \( w \) and \( uw \) be both \( c \)-sortable and \( c \)-antisortable by Proposition 2.2 of [3], that is, the element \( w \) is \( c \)-sortable and \( uw_0 \) is \( c^{-1} \)-sortable. Hence \( w \) is \( c \)-sortable and \( c^{-1} \)-sortable. From [6, Theorem 4.1] we know that \( w \in W \) is \( c \)-sortable and \( c^{-1} \)-sortable if and only if \( I(g) \cap (W' \setminus \{ t_i \}) \neq \emptyset \) implies \( t_2 \in I(g) \) for any irreducible dihedral parabolic subgroup \( W' \) of \( W \) (that is \( |W'| > 4 \)) with simple generators \( t_1, t_2 \in T \).
Assume that \( w \neq e \). There exists \( s \in I(w) \cap S \). Pick \( t \in S \) such that the standard parabolic subgroup \( W' \) generated by \( \{s, t\} \) is dihedral and of cardinality > 4. We first show that \( s, t \in I(w) \). We have to distinguish two cases:

1. If \( I(w) \cap (W' \setminus \{s\}) \neq \emptyset \) then \( t \in I(w) \) because \( w \) is \( c \)- sortable and \( c^{-1} \)- sortable. Hence \( s, t \in I(w) \).
2. Assume \( I(w) \cap (W' \setminus \{s\}) = \emptyset \). We first remark \( I(ww_0) = I(w_0) \setminus I(w) \). Hence \( I(ww_0) \cap (W' \setminus \{t\}) \neq \emptyset \) which implies \( s \in I(ww_0) \) since \( ww_0 \) is also \( c \)- sortable and \( c^{-1} \)- sortable. In particular, \( s \in I(w) \cap I(ww_0) \) which is impossible.

Since \( (W, S) \) is irreducible, the Coxeter graph associated to \( (W, S) \) is connected. Now repeat this process along paths starting at \( s \) to conclude that \( S \subseteq I(w) \). Hence \( w = w_0 \).

(ii) \( \Rightarrow \) (iii): For \( w = e \) the result is clear. Recall that the conjugation by \( w_0 \) is an automorphism \( \varphi \) of the Coxeter system \( (W, S) \). So the \( w_0 \)- factorization of \( w_0 \) is induced by \( \varphi \) from the \( c \)- factorization of \( w_0 \). The claim for \( w = w_0 \) follows.

(iii) \( \Rightarrow \) (iv): Set \( c' := wcw^{-1} \). (iv) \( \Rightarrow \) (i): Since \( e \) and \( w_0 \) are \( c' \)- singletons, we conclude that \( w^{-1} \) and \( w^{-1}w_0 \) are both \( c \)- singletons. Apply (i) \( \Rightarrow \) (ii) to deduce that \( w^{-1} = e \) or \( w^{-1} = w_0 \). In particular, \( w \) and \( w_0 \) are both \( c \)- singletons.

\[ \square \]

**Remark 3.8.** Two maximal cones \( C \) and \( C' \) in the \( c \)- Cambrian fan \( F_c \) are antipodal if \( C = -C' \). Theorem 3.7 implies that a pair of antipodal maximal cones that correspond to \( c \)- singletons is unique and the corresponding elements are \( e \) and \( w_0 \).

4. **Proof of Theorem 3.3**

Assume there is an isometry \( \varphi \) on \( V \) such that \( \text{Asso}_{c_1}(W) = \varphi(\text{Asso}_{c_2}(W)) \).

Let \( w \) be a \( c_2 \)- singleton. Then

1. \( M(w) = \bigcap_{s \in S} H_{(w,s)} \) is a vertex of \( \text{Asso}_{c_2}(W) \),
2. \( \varphi(M(w)) = \bigcap_{s \in S} \varphi(H_{(w,s)}) \) is a vertex of \( \text{Asso}_{c_1}(W) \),
3. \( \varphi(M(w)) = M(w') \) for some \( c_1 \)-singleton \( w' \) since \( \varphi \) is an isometry.

(For 3, note that \( c \)-singleton cones are the only cones in the Cambrian fan which consist of a single chamber from the Coxeter fan, and thus an isometry must take singleton cones to singleton cones.)

Apply these results to \( w = e \) to obtain a \( c_1 \)-singleton \( w'_e \) with \( M(w'_e) = \varphi(M(e)) \). Moreover, \( w'_e w_0 \) is also a \( c_1 \)-singleton with \( M(w'_e w_0) = \varphi(M(w_0)) \). Hence \( w'_e \in \{e, w_0\} \) by Theorem 3.7 and \( \varphi \) is a linear isometry of \( V \) that fixes \( P \) and either fixes \( M(e) \) and \( M(w_0) \) or interchanges \( M(e) \) and \( M(w_0) \). If \( \varphi \) fixes \( M(e) \) and \( P \) then there is an induced \( u \)- automorphism \( \mu \) of the Coxeter graph of \( (W, S) \) by Proposition 3.2 and \( \mu(c_2) = c_1 \). If \( \varphi \) interchanges \( M(e) \) and \( M(w_0) \) we consider \( \varphi := g \circ \varphi \). We have \( \varphi(\text{Asso}_{c_1}(W)) = \text{Asso}_{w_0 c_1^{-1} w_0}(W) \) by Proposition 3.6 and \( \varphi \) is an isometry that fixes \( P, M(e), \) and \( M(w_0) \). Hence \( \varphi \) induces an \( u \)- automorphism \( \mu \) of the Coxeter system \( (W, S) \) by Proposition 3.2 and \( \mu(c_2) = w_0 c_1^{-1} w_0 \).

Assume there is an \( u \)- automorphism \( \mu \) of the Coxeter graph \( (W, S) \). Without loss of generality we may assume that \( \mu(c_2) = c_1 \) because \( \text{Asso}(W) \) and \( \text{Asso}_{w_0 c_1^{-1} w_0}(W) \) are isometric via \( g \) by Proposition 3.6.

We have to specify an isometry \( \varphi_{\mu} \) on \( V \) such that \( \varphi_{\mu}(\text{Asso}_{c_2}(W)) = \text{Asso}_{c_1}(W) \).

This is done according to Proposition 3.4. Define \( \varphi_{\mu}: V \to V \) by \( \varphi_{\mu}(\alpha_s) := \alpha_{\mu(s)} \) for all \( s \in S \) or equivalently by \( \varphi_{\mu}(v_s) := v_{\mu(s)} \).
It remains to show that \( \varphi_\mu \) maps \( c_2\)-admissible halfspaces to \( c_1\)-admissible halfspaces. From Corollary 3.5 we know how the facet defining halfspaces \( \mathcal{H}(w,s) \) are permuted by the isometry \( \varphi_\mu \), that is, \( \varphi_\mu(\mathcal{H}(w,s)) = \mathcal{H}(\mu(w),\mu(s)) \) for \( w \in W \) and \( s \in S \). The automorphism \( \mu \) on \( W \) preserves the length function \( \ell \), so we have \( \mu(w_0) = w_0 \) and any prefix of the \( c \)-factorization of \( w_0 \) up to commutation is a prefix of the \( \mu(c) \)-factorization of \( w_0 \) up to commutation. In other words, \( \mu \) induces a lattice isomorphism between the \( c_2\)-singletons and the \( \mu(c_2) \)-singletons. Hence \( \mathcal{H}(x,s) \) is \( c_2\)-admissible if and only if \( \mathcal{H}(\mu(x),\mu(s)) \) is \( \mu(c_2) \)-admissible. This shows that \( \varphi_\mu(\text{Assoc}_c(W)) = \text{Assoc}_\mu(c)(W) \) and ends the proof of Theorem 2.3.

Proof of Corollary 2.6 (2) equivalent to (3) is Theorem 2.3 (2) implies (1) follows from the definition of normal fans.

Now, we show (1) implies (3). As the \( c \)-Cambrian fan \( \mathcal{F}_c \) and the \( c' \)-Cambrian fan \( \mathcal{F}_{c'} \) are isometric, there is an isometry \( \varphi \) such that the image under \( \varphi \) of each cone in \( \mathcal{F}_c \) is a cone of \( \mathcal{F}_{c'} \). By Remark 3.8 the pair of antipodal singleton cones \( C(e), C(w_0) \) corresponding to \( e, w_0 \) are the unique singleton antipodal cones in both Cambrian fans, and either \( \varphi \) fixes them or exchanges them. If \( w \) is balanced, then apply Corollary 3.5 with \( \mu \) to be the conjugation by \( w_0 \) to obtain

\[
M(w_0) = \sum_{s \in S} \kappa w_0(v_s) = \sum_{s \in S} \kappa(-v_{w_0,s w_0}) = -M(e).
\]

As \( C(e) \) is \( \mathbb{R}_{>0} \)-spanned by \( \Delta^* \) and \( C(w_0) \) is \( \mathbb{R}_{>0} \)-spanned by \(-\Delta^* \), either \( \varphi \) fixes \( M(e) \) and \( M(w_0) \) or interchanges them. In both cases, \( \varphi(P) = P \). So either \( \varphi \) or \( g \circ \varphi \) fixes \( M(e) \) and \( P \). Conclude by Proposition 3.2 as in the first part of the proof of Theorem 2.3.

5. The reducible case

The reducible case does not follow immediately from an application of the irreducible case. Rather, one goes through the same steps as in the proof of the irreducible case, but with some slight added complication. We sketch the process below.

Let \( D \) denote the set of irreducible components of the Coxeter graph of \((W,S)\). For any \( A \subset D \), let \( w_A \) be the longest word for the subgroup generated by the components in \( A \). Let \( L = \{ w_A \mid A \subset D \} \). All the elements of \( L \) are \( c \)-singletons (for any \( c \)).

Up to just before Proposition 3.6 the argument goes in exactly the same way. Then, instead of constructing a single isometry \( g \), we construct an isometry \( g_A \) for each \( A \subset D \), defining \( g_A(v) = w_A(v) \). Write \( c^A \) for the Coxeter element obtained from \( c \) by reversing the order of the reflections in \( c \) coming from components in \( A \). The generalization of Proposition 3.6 then asserts that \( \text{Assoc}_c(W) = g_A(\text{Assoc}_{c^{A \subset A}}(W)) \) and, in particular, these associahedra are isometric. Theorem 3.7 goes through with condition (ii) replaced by the condition that \( w \in L \).

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