Tight Decomposition Functions for Continuous-Time Mixed-Monotone Systems with Disturbances

Matthew Abate, Maxence Dutreix, and Samuel Coogan

Abstract—The vector field of a mixed-monotone system is decomposable via a decomposition function into increasing (cooperative) and decreasing (competitive) components, and this decomposition allows for, e.g., efficient computation of reachable sets and forward invariant sets. A main challenge in this approach, however, is identifying an appropriate decomposition function. In this work, we show that any continuous-time dynamical system with Lipschitz continuous vector field is mixed-monotone, and we provide an explicit construction for the decomposition function that provides the tightest approximation of reachable sets possible with the theory of mixed-monotonicity. Our construction is similar to that recently proposed by Yang and Ozay for computing decomposition functions of discrete-time systems [1] where we make appropriate modifications for the continuous-time setting and also extend under case with unknown disturbance inputs. As in [1], our decomposition function construction requires solving an optimization problem for each point in the state-space; however, we demonstrate through two examples how tight decomposition functions can be calculated in closed form. As a second contribution, we show how under-approximations of reachable sets can be efficiently computed via the mixed-monotonicity property considering the backward time dynamics. We demonstrate our results with an example and a case study.

I. INTRODUCTION

Mixed-monotone systems are characterized by vector fields that are decomposable into increasing (cooperative) and decreasing (competitive) interactions. This allows for embedding the system dynamics into a higher dimensional system with twice as many states but for which the dynamics are monotone [2]–[4]. Thus, decomposing the system dynamics enables one to apply the powerful theory of monotone dynamical systems to the higher dimensional embedding system to conclude properties of the original system. For example, mixed-monotonicity allows for: efficiently approximating reachable sets by evaluating only one trajectory of the embedding system [5], [6]; identifying forward invariant and attractive sets by identifying equilibria in the embedding space [6]; concluding global asymptotic stability by proving the nonexistence of equilibria of the embedding system except in a certain lower dimensional subspace [7]. See also [8], [9] for fundamental results on monotone dynamical systems.

A primary challenge in applying the theory of mixed-monotone systems is in identifying an appropriate decomposition function. There exists certain special cases for which a decomposition function can be readily identified, e.g., when each off-diagonal entry of the systems Jacobian matrix is uniformly upper or lower bounded [10]–[13], however, identifying decomposition functions generally relies on domain knowledge of the underlying physical system.

The question of existence of decomposition functions was recently explored in [1] in the discrete-time setting. In discrete-time, a decomposition function for an update map $F$ leads to an embedding system that over-approximates the image of $F$ when evaluated on a hyperrectangular set. It is observed in [1] that it is possible to define a decomposition function as the function that returns a tight hyperrectangular over-approximation of the image of $F$ evaluated on any given hyperrectangular set, and thus all discrete-time systems are mixed-monotone with a decomposition function that tightly approximates one-step reachable sets. While this result is constructive in that it provides an explicit decomposition function construction applicable to all discrete-time systems, evaluating the decomposition function at any point in the embedding space requires computing a reachable set itself. Nonetheless, knowing that a decomposition function does exist means that a search directed by, e.g., domain expertise, is not generally unreasonable.

In this paper, we study an analogous question regarding existence of decomposition functions in the continuous-time setting, and we additionally consider systems with disturbances. Our main result is to show that any continuous-time system possessing a vector field that is Lipschitz continuous in state and disturbance admits a Lipschitz continuous decomposition function. Moreover, we provide a construction for the decomposition function that provides the tightest possible reachable set approximations via the mixed-monotonicity property. Thus, our results complement those from [1] by answering similar questions in the continuous-time setting, however, we emphasize that the results and tools here are different as compared to the discrete-time setting of [1]. In particular, unlike decomposition functions for continuous-time systems, decomposition functions for discrete-time systems do not need to be Lipschitz continuous, or even continuous. Moreover, we allow for disturbance inputs and define a different notion of tightness to accommodate the fact that it is generally not possible to obtain tight hyperrectangular reachable set approximations.
in continuous-time over any horizon. As in [1], our construction is defined as an optimization problem, and thus not practically useful for applications other than system verification via simulation [14], [15]. However, we demonstrate through examples how tight decomposition functions can be calculated in closed form in certain instances.

As a second contribution, we show how under-approximations of reachable sets can be efficiently computed from a decomposition function for the backward-time dynamics. Mixed-monotonicity in the backward-time setting was first considered in [6] where it is shown how finite-time backward reachable sets can be approximated using an analogous technique to that of the forward-time case. Here, we extend these results and specifically show that (a) a backward-time decomposition function can be used to compute under-approximations of forward reachable sets, (b) in certain instances, a tight backward-time decomposition function can be efficiently derived from a tight forward-time decomposition function, and (c) a tight backward-time decomposition function provides the tightest, in a certain sense, under-approximations of forward reachable sets.

II. NOTATION

Let \( \mathbb{R}_{\geq 0} \) and \( \mathbb{R}_{\leq 0} \) denote the nonnegative and nonpositive real numbers respectively, and let \( \mathbb{R} := \mathbb{R} \cup \{-\infty, \infty\} \) denote the extended real numbers. Let \( x_i \) for \( i \in \{1, \ldots, n\} \) denote the \( i \)th entry of \( x \in \mathbb{R}^n \).

Let \( (x, y) \) denote the vector concatenation of \( x, y \in \mathbb{R}^n \), i.e., \( (x, y) := [x^T \ y^T]^T \in \mathbb{R}^{2n} \), and let \( \preceq \) denote the componentwise vector order, i.e. \( x \preceq y \) if and only if \( x_i \leq y_i \) for all \( i \). Given \( x, y \in \mathbb{R}^n \) with \( x \preceq y \),

\[
[x, y] := \{ z \in \mathbb{R}^n \mid x \preceq z \text{ and } z \preceq y \}
\]
denotes the hyperrectangle defined by the endpoints \( x \) and \( y \).

We also allow \( x_i, y_i \in \mathbb{R} \) so that \( [x, y] \) defines an extended hyperrectangle, that is, a hyperrectangle with possibly infinite extent in some coordinates. Given \( a := (x, y) \in \mathbb{R}^{2n} \) with \( x \preceq y \), we denote by \([a]\) the hyperrectangle formed by the first and last \( n \) components of \( a \), i.e., \([a] := [x, y] \).

Let \( \preceq_{SE} \) denote the southeast order on \( \mathbb{R}^{2n} \) defined by

\[
(x, x') \preceq_{SE} (y, y') \iff x \preceq y \text{ and } y' \preceq x'
\]

where \( x, y, x', y' \in \mathbb{R}^n \). In the case that \( x \preceq x' \) and \( y \preceq y' \), note that

\[
(x, x') \preceq_{SE} (y, y') \iff [y, y'] \subseteq [x, x'] \tag{1}
\]

III. PRELIMINARIES

We consider the system

\[
\dot{x} = F(x, w) \tag{2}
\]

with state \( x \in \mathcal{X} \subset \mathbb{R}^n \) and disturbance input \( w \in \mathcal{W} \subset \mathbb{R}^m \). We assume that the vector field \( F : \mathcal{X} \times \mathcal{W} \rightarrow \mathbb{R}^n \) is locally Lipschitz continuous. Additionally, we assume that \( \mathcal{X} \) is an extended hyperrectangle with nonempty interior and \( \mathcal{W} \) is a hyperrectangle defined by \( \mathcal{W} := [\underline{w}, \overline{w}] \) for some \( \underline{w}, \overline{w} \in \mathbb{R}^m \) with \( \underline{w} \preceq \overline{w} \). Let \( \Phi(T; x, w) \in \mathcal{X} \) denote the state of (2), reached at time \( T \) when starting at the initial state \( x \in \mathcal{X} \) at time 0 and when subjected to the disturbance input \( w : [0, T] \rightarrow \mathcal{W} \).

In this work, we are specifically interested in mixed-monotone systems. Define by

\[
\mathcal{T}_X := \{(x, \tilde{x}) \in \mathcal{X} \times \mathcal{X} \mid x \preceq \tilde{x}\},
\]

\[
\mathcal{T}_W := \{(w, \tilde{w}) \in \mathcal{W} \times \mathcal{W} \mid w \preceq \tilde{w}\},
\]

the sets of ordered points in \( \mathcal{X} \) and \( \mathcal{W} \), respectively. Additionally, define

\[
\mathcal{T} := \{(x, w, \tilde{x}, \tilde{w}) \in \mathcal{X} \times \mathcal{W} \times \mathcal{X} \times \mathcal{W} \mid
(x, \tilde{x}) \in \mathcal{T}_X \text{ and } (w, \tilde{w}) \in \mathcal{T}_W\} \tag{3}
\]

or

\[
(\tilde{x}, x) \in \mathcal{T}_X \text{ and } (\tilde{w}, w) \in \mathcal{T}_W\} \tag{4}
\]

Definition 1. Given a locally Lipschitz continuous function \( d : \mathcal{T} \rightarrow \mathbb{R}^n \), the system (2) is mixed-monotone with respect to \( d \) if

1) For all \( x \in \mathcal{X} \) and all \( w \in \mathcal{W} \) we have
\[
d(x, w, x, w) = F(x, w).
\]

2) For all \( i, j \in \{1, \ldots, n\} \), with \( i \neq j \), we have
\[
\frac{\partial d}{\partial x_i} (x, w, x, w) \geq 0 \text{ for all } (x, w, x, w) \in \mathcal{T} \text{ such that } \frac{\partial d}{\partial x_i} \text{ exists.}
\]

3) For all \( i, j \in \{1, \ldots, n\} \), with \( i \neq j \), we have
\[
\frac{\partial d}{\partial w_j} (x, w, x, w) \leq 0 \text{ for all } (x, w, x, w) \in \mathcal{T} \text{ such that } \frac{\partial d}{\partial w_j} \text{ exists.}
\]

4) For all \( i \in \{1, \ldots, n\} \) and all \( k \in \{1, \ldots, m\} \), we have
\[
\frac{\partial d}{\partial w_k} (x, w, x, w) \geq 0 \text{ and } \frac{\partial d}{\partial w_k} (x, w, x, w) \text{ for all } (x, w, x, w) \in \mathcal{T} \text{ such that } \frac{\partial d}{\partial w_k} \text{ and } \frac{\partial d}{\partial w_k} \text{ exist.}
\]

Remark 1. Conditions 2–4 of Definition 1 are equivalent to the following two conditions:

C1) For all \( i \in \{1, \ldots, n\} \),
\[
d_i(x, w, \tilde{x}, \tilde{w}) \leq d_i(y, v, \tilde{x}, \tilde{w}) \tag{5}
\]

for all \( x, y \in \mathcal{X} \) such that \( x \preceq y \) and \( x_i = y_i \), and for all \( w, v \in \mathcal{W} \) such that \( w \preceq v \).

C2) For all \( i \in \{1, \ldots, n\} \),
\[
d_i(x, w, \tilde{y}, \tilde{v}) \leq d_i(x, w, \tilde{x}, \tilde{w}) \tag{6}
\]

for all \( x, \tilde{x} \in \mathcal{X} \) such that \( \tilde{x} \preceq \tilde{y} \) and for all \( w, \tilde{w}, \tilde{v} \in \mathcal{W} \) such that \( \tilde{w} \preceq \tilde{v} \).

These are the so-called Kamke or type K conditions for monotonicity, modified for the mixed-monotone setting [8, Section 3]. It is sometimes more convenient to use these alternative conditions, while the conditions provided in the above definition are generally more intuitive.

If (2) is mixed-monotone with respect to \( d \), \( d \) is said to be a decomposition function for (2). Typically, decomposition functions are defined over the entire cross-space \( \mathcal{X} \times \mathcal{W} \times \mathcal{X} \times \mathcal{W} \); this is the case, for instance, in [5], [12]. Here, we note that the typical tools for reachability analysis, including those provided later in this work, only require that \( d \) be defined on \( \mathcal{T} \), and thus we restrict the domain of \( d \) to \( \mathcal{T} \) without compromising the usefulness of the mixed-monotonicity property.
Construct
\[
\dot{x} = e(x, \hat{x}) = \begin{bmatrix}
d(x, w, \hat{x}, \hat{w}) \\
d(\hat{x}, w, x, w)
\end{bmatrix}
\] (7)
with state \((x, \hat{x}) \in T_X\). We refer to \(\mathcal{E}\) as the embedding system relative to \(d\) and \(e\) the embedding function relative to \(d\). Let \(\Phi^e(T; (x, \hat{x}))\) be the state transition function for \(\mathcal{E}\), that is, \(\Phi^e(T; (x, \hat{x}))\) denotes the state of \(T\) at time \(T\) when initialised at state \((x, \hat{x})\) \(\in T_X\). The set \(T_X\) is forward invariant for \(\mathcal{E}\), meaning that trajectories of \(\mathcal{E}\) will not evolve from \(T_X\) to \((X' \times X')\backslash T_X\) [6]. However, trajectories of the embedding system may leave \(X' \times X'\), and this is true even when \(X'\) is robustly forward invariant for \(\mathcal{E}\). Therefore, \(\Phi^e(T; (x, \hat{x}))\) is understood to exist only when \(\Phi^e(t; (x, \hat{x})) \in T_X\) for all \(0 \leq t \leq T\), and statements involving \(\Phi^e(T; (x, \hat{x}))\) are understood to apply only when \(\Phi^e(T; (x, \hat{x}))\) exists. Importantly, \(\mathcal{E}\) is monotone with respect to the southeast order; that is, for all \((x, \hat{x}), (y, \hat{y}) \in T_X\) and all \(T \geq 0\) we have
\[(x, \hat{x}) \preceq_SE (y, \hat{y}) \Rightarrow \Phi^e(T; (x, \hat{x})) \preceq_SE \Phi^e(T; (y, \hat{y})).\]

We next recall how reachable sets for \(\mathcal{E}\) are over-approximated by trajectories of \(\mathcal{E}\). To that end, denote by
\[
R^E(T; X_0) = \{\Phi(T; x, w) \in X' \mid x \in X_0, \text{ for some } w : [0, T] \to W\}\] (8)
the forward reachable set of \(\mathcal{E}\) over the time horizon \(T\) from the set of initial conditions \(X_0 \subseteq X\). The following fundamental result connects reachable sets to the dynamics of the embedding system [5], [12].

**Proposition 1.** Let \(\mathcal{E}\) be mixed-monotone with respect to \(d\), and let \(X_0 = [\underline{x}, \overline{x}]\) for some \(\underline{x} \leq \overline{x}\). Then \(R^E(T; X_0) \subseteq [\Phi^e(T; [\underline{x}, \overline{x}])\].

Next, note that the system
\[
\dot{x} = -F(x, w)
\] (9)
with \(x \in X'\) and \(w \in W\) encodes the backward time dynamics of \(\mathcal{E}\); that is, if \(x_1 = \Phi(T; x_0, w)\) for \(w : [0, T] \to W\), then \(x_0 = \Phi'(T; x_1, w')\) for \(w'(t) = w(T - t)\), where \(\Phi'\) denotes the state transition function of \(\mathcal{E}\). Therefore, in the instance that \(\Phi'\) is mixed-monotone, finite-time backward reachable sets of \(\mathcal{E}\) can be approximated using a procedure analogous to that of Proposition 1 Let
\[
S(T; X_1) := \{x \in X' \mid \Phi(T; x, w) \in X_1, \text{ for some } w : [0, T] \to W\}\] (10)
denote the set of initial conditions for which there exists a \(w : [0, T] \to W\) capable of driving \(\mathcal{E}\) to the set \(X_1 = [\underline{x}, \overline{x}]\) in time \(T \geq 0\). The following is proved in [6].

**Proposition 2.** Let \(D\) be a decomposition function for \(\Phi'\) and let \(X_1 = [\underline{x}, \overline{x}]\) for some \(\underline{x} \leq \overline{x}\). Let \(E\) denote the embedding function relative to \(D\), and let \(\Phi^E\) denote the state transition function of the embedding system \(\mathcal{E}\) relative to \(E\). Then \(S(T; X_1) \subseteq [\Phi^E(T; [\underline{x}, \overline{x}])]\).
In the following theorem, we show that all continuous-time systems with disturbances as in (2) are mixed-monotone and we present a construction for tight decomposition functions.

**Theorem 1.** Any system of the form (2) is mixed-monotone with respect to \( \delta : T \to \mathbb{R}^n \) constructed elementwise according to

\[
\delta_i(x, w, \hat{x}, \hat{w}) = \begin{cases} 
\min_{y \in [x, z], y \in \mathcal{W}, z \in \mathcal{W}} F_i(y, z) & \text{if } x \leq \hat{x} \text{ and } w \leq \hat{w}, \\
\max_{y \in [x, z]} F_i(y, z) & \text{if } \hat{x} \leq x \text{ and } \hat{w} \leq w.
\end{cases}
\]  

(17)

Moreover, \( \delta \) is a tight decomposition function for (2).

**Proof.** We begin by establishing that \( \delta \) from (17) is Lipschitz continuous; this is done by showing that \( \delta_i \) is Lipschitz in \( x \), and Lipschitz continuity holds with respect to other arguments by analogous reasoning. Let \( x^1, x^2, \hat{x} \in \mathcal{X}, w, \hat{w} \in \mathcal{W} \), where we assume without loss of generality that \( x^1 \leq \hat{x}, x^2 \leq \hat{x} \), and \( w \leq \hat{w} \). Observe that for any \( y^1 \in [x^1, \hat{x}] \) with \( y^1 = x^1 \), there exists \( y^2 \in [x^2, \hat{x}] \) with \( y^2 = x^2 \) such that \( \|y^1 - y^2\| \leq \|x^1 - x^2\| \) and vice-versa, where \( \|\cdot\| \) denotes the usual one-norm on \( \mathbb{R}^n \). In particular, for any minimizer \((y^1, z)\) that achieves the value of \( \delta_i(x^1, w, \hat{x}, \hat{w}) \) in the definition (17), there exists a point \( y^2 \) so that \( F_i(y^2, z) \) upper bounds \( \delta_i(x^2, w, \hat{x}, \hat{w}) \) with \( \|y^1 - y^2\| \leq \|x^1 - x^2\| \), and vice-versa. It follows then that \( \|\delta_i(x^1, w, \hat{x}, \hat{w}) - \delta_i(x^2, w, \hat{x}, \hat{w})\| \leq L\|x^1 - x^2\| \) where \( L \) is a Lipschitz constant for \( F \) applicable on a neighborhood of \([x^1, \hat{x}] \cup [x^2, \hat{x}]\). Thus \( \delta_i \) is Lipschitz in \( x \), and therefore \( \delta \) is Lipschitz in \( x, \hat{x}, w, \hat{w} \).

We next show that \( \delta \) is a decomposition function for (2). Trivially, \( \delta_i(x, w, x, w) = F_i(x, w) \) for all \( i \) and all \( x \in \mathcal{X}, w \in \mathcal{W} \), and thus \( \delta \) satisfies Condition 1 of Definition 1. We show that \( \delta \) satisfies the equivalent Kanke conditions in Remark 1. Specifically, choose \( x, y, \hat{x} \in \mathcal{X} \) and \( w, v, \hat{w} \in \mathcal{W} \) such that \( x \leq y, x_i = y_i, \) and \( w \leq v \). Then \( \delta_i(x, w, \hat{x}, \hat{w}) \leq \delta_i(y, v, \hat{x}, \hat{w}) \) follows from the min/max construction of (17). This proves C1, and C2 is proven analogously. Therefore, (2) is mixed-monotone with respect to \( \delta \).

Lastly, we show that \( \delta \) is a tight decomposition function for (2). Let \( \delta : T \to \mathbb{R}^n \) be another decomposition function for (2) and choose \((x, \bar{w}) \in T_x \) and \((w, \bar{w}) \in T_w \). Additionally, choose \( x \in [x, \bar{x}] \) and \( w \in [w, \bar{w}] \). Then \((x, \bar{w}) \preceq \delta (x, w, \bar{x}, \bar{w}) \) and \((w, \bar{w}) \preceq \delta (x, w, \bar{x}, \bar{w}) \), therefore

\[
\begin{align*}
\min_{y \in [x, \bar{x}], z \in [w, \bar{w}]} & F_i(y, z) \\
& \leq \max_{y \in [x, \bar{x}], z \in [w, \bar{w}]} F_i(y, z)
\end{align*}
\]

(18)

Since (18) holds for all \( x \in [x, \bar{x}] \) and all \( w \in [w, \bar{w}] \) we now have

\[
\begin{align*}
\min_{y \in [x, \bar{x}], z \in [w, \bar{w}]} F_i(y, z) & \leq \max_{y \in [x, \bar{x}], z \in [w, \bar{w}]} F_i(y, z)
\end{align*}
\]

(19)

and thus

\[
\begin{align*}
\left[ d(x, w, \bar{x}, \bar{w}) \right] & \preceq \delta (x, w, \bar{x}, \bar{w}) \\
\left[ d(x, w, \bar{x}, \bar{w}) \right] & \preceq \delta (x, w, \bar{x}, \bar{w})
\end{align*}
\]

Therefore \( \delta \) is a tight decomposition function for (2) as \((x, \bar{x}) \in T_x \) and \((w, \bar{w}) \in T_w \) were selected arbitrarily. This completes the proof.

We next demonstrate the applicability of Theorem 1 through an example.

**Example 1.** Consider the system

\[
\begin{align*}
\dot{x}_1 &= F(x) = \frac{x_2^2}{x_1} \\
\dot{x}_2 &= x_1
\end{align*}
\]

(20)

with \( \mathcal{X} = \mathbb{R}^2 \). We compute a tight decomposition function for (20) via Theorem 1. Specifically, let \( \delta : T \to \mathbb{R}^2 \) be given by

\[
\begin{align*}
\delta_1(x, \bar{x}) & = \begin{cases} 
\bar{x}_2 & \text{if } x \geq 0 \text{ and } \bar{x} \leq \bar{x}_1, \\
-\bar{x}_1 & \text{if } x \leq 0 \text{ and } \bar{x} \leq \bar{x}_1,
\end{cases} \\
\delta_2(x, \bar{x}) & = x_1.
\end{align*}
\]

Then \( \delta \) satisfies (17) and is therefore a tight decomposition function for (20).

V. UNDER-APPROXIMATING REACHABLE SETS VIA MIXED-MONOTONICITY

In this section, we show how under-approximations of \( R^F(T; x_0) \) are computed via the mixed-monotonicity property.

**Theorem 2.** Let (9) be mixed-monotone with respect to \( D \), and let \( x_0 = \langle x, \bar{x} \rangle \) for some \( x \preceq \hat{x} \). Construct the system

\[
\begin{align*}
\dot{\hat{x}} & = \Gamma(x, \hat{x}) = \left[ \begin{array}{c}
-D(x, w, \hat{x}, \bar{w}) \\
-D(x, \bar{w}, \hat{x}, \bar{w})
\end{array} \right] \\
\dot{(x,w)} & = (x,w)
\end{align*}
\]

(21)

with state transition function \( \Phi^F \). If \( \Phi^F(t; \langle x, \bar{x} \rangle) \in T_x \) for all \( 0 \leq t \leq T \) then \( \Phi^F(T; \langle x, \bar{x} \rangle) \) is Lipschitz linearly related to \( \Phi(T; x_0) \).

**Proof.** Let \( E \) denote the embedding function relative to \( D \) and let \( \Phi^E \) denote the state transition function of its embedding system. Then for all \( x, \bar{x} \in \mathcal{X} \) we have \( E(x, \bar{x}) = -\Gamma(x, \bar{x}) \). Moreover, \( \Phi^E \) and \( \Phi^F \) are related in the following way: if \( \Phi^E(t; (x, \bar{x})) = (y, \bar{y}) \) then \( \Phi^F(T; (y, \bar{y})) = (x, \bar{x}) \), where \((x, \bar{x}), (y, \bar{y}) \in T_x \) and \( T \geq 0 \).

We prove this result by showing that for all \( y \in \Phi^E(\langle T, (x, \bar{x}) \rangle) \) there exists an \( x \in x_0 \) and a disturbance input \( w : [0, T] \to \mathcal{W} \) such that \( y = \Phi^T(x, w) \). Define \( \varphi(t) := \Phi^E(t; (x, \bar{x})) \) where we let \( \varphi(t) = (\varphi(t), \bar{\varphi}(t)) \). Choose \( T \geq 0 \) and choose \( y \in \Phi^T(\langle T; (x, \bar{x}) \rangle) \) \( (\varphi(t), \bar{\varphi}(t)) \) where we have \( \varphi(T), \bar{\varphi}(T) \) by (1). Then

\[
\Phi^E(T; (\varphi(T), \bar{\varphi}(T))) \preceq \Phi^E(T; (y, \bar{y}))
\]

(22)

follows from the monotonicity of the embedding system relative to \( E \). As a result of Proposition 1 we have that for any \( w' : [0, T] \to \mathcal{W}, \Phi^E(T; w') \in F^E(T; (\varphi(T), \bar{\varphi}(T))) = \]
where \( \Phi^t \) is taken to be the state transition function of (9). Take \( x = \Phi^t(T; y, w) \) and define \( w(t) := w^t(T - t) \). Then \( y = \Phi(T; x, w) \). This completes the proof. \( \Box \)

In Theorem 2 we show how the system (24), which is constructed from a decomposition function for (9), is used to under-approximate forward reachable sets for the system (2). As a consequence of Theorem 1 we have that (9) is mixed-monotone, and we next provide a special case for which a tight decomposition function for the backward-time system (9) can be computed from a tight decomposition function for the forward-time system (2).

**Special Case 1.** If
1) \( \delta \) is a tight decomposition function for (2), and
2) \( F_i \) does not depend on \( x_i \) for all \( i \in \{1, \cdots, n\} \),
then \( \Delta(x, w, \hat{x}, \hat{w}) := -\delta(\hat{x}, \hat{w}, x, w) \) is a tight decomposition function for (9).

**Proof.** From Theorem 1 we have that a tight decomposition function for (9) is defined element wise by

\[
\Delta_i(x, w, \hat{x}, \hat{w}) = \begin{cases} 
\max_{y \in [x, \hat{x}], \hat{y} \in [w, \hat{w}]} F_i(y, z) & \text{if } x \leq \hat{x} \text{ and } w \leq \hat{w}, \\
\min_{y \in [x, \hat{x}], \hat{y} \in [w, \hat{w}]} F_i(y, z) & \text{if } \hat{x} \leq x \text{ and } \hat{w} \leq w.
\end{cases}
\]

(23)

Therefore, if \( F_i \) does not depend on \( x_i \) for all \( i \) then \( \Delta(x, w, \hat{x}, \hat{w}) := -\delta(\hat{x}, \hat{w}, x, w) \) is a tight decomposition function for (9).

We compute a tight decomposition function for (24) by solving the optimization problem (17). Specifically, using critical point analysis we find that a tight decomposition function for (24) is given by

\[
\delta_1(x, w, \hat{x}, \hat{w}) = \begin{cases} 
\frac{-1}{4w_1} + w_2 & \text{if } w_1x_2 \leq \frac{1}{2} \leq w_1\hat{x}_2, \\
w_1x_2^2 - x + w_2 & \text{if } \frac{1}{2} \leq w_1x_2 \leq w_1\hat{x}_2, \\
w_1\hat{x}_2^2 - x + w_2 & \text{if } w_1x_2 \leq w_1\hat{x}_2 \leq \frac{1}{2}, \\
1 \leq w_1(x_2 + \hat{x}_2) & \text{if } w_1(x_2 + \hat{x}_2) \leq 1,
\end{cases}
\]

(25)

\[
\delta_2(x, w, \hat{x}, \hat{w}) = x_3 + 2, \\
\delta_3(x, w, \hat{x}, \hat{w}) = x_1 - \hat{x}_2 - \hat{w}_3^3,
\]

(26)

where we note that \( \delta \) satisfies (17).

We next demonstrate how forward reachable sets are over-approximated via Proposition 1 and under-approximated via Theorem 2. Specifically, we take \( \mathcal{W} = [-1/4, 0] \times [0, 1/4] \) and approximate \( \mathcal{R}^F(1/2; \mathcal{X}_0) \) for \( \mathcal{X}_0 = [-1/2, 1/2]^3 \). An over-approximation of \( \mathcal{R}^F(1/2; \mathcal{X}_0) \) is computed by simulating the system (7), here taken relative to \( \delta \), forward in time for \( T = 1/2 \). Simulation results are provided in Figure 1a. Additionally, note that (24) satisfies the hypothesis of Special Case 1 and thus \( \Delta(x, w, \hat{x}, \hat{w}) = -\delta(\hat{x}, \hat{w}, x, w) \) is a tight decomposition function for the backward-time system \( \hat{x} = -F(x, w) \). An under-approximation of \( \mathcal{R}^F(1/2; \mathcal{X}_0) \) is computed by simulating the system (21), here taken relative to \( \Delta \), forward in time for \( T = 1/2 \). This experiment is depicted in Figure 1b.

**VII. DISCUSSION AND CONCLUSION**

A mixed-monotone system is generally mixed-monotone with respect to many decomposition functions and, as such, we can expect the system (2) to induce decomposition functions other than that constructed in (17). However, some decomposition functions may be more/less conservative than others when used with Proposition 1 and we have shown that (17) is the least conservative in the sense that it provides the tightest rectangular approximations of reachable sets using the theory of mixed-monotonicity.

As discussed above and demonstrated in Example 1 and the Case Study, a closed form expression for the optimal decomposition function via (17) is sometimes possible. However, even in these cases, the optimal decomposition function is generally characterized piecewise and the number of pieces scales exponentially in the dimension of the state space and disturbance spaces in general. This explains an apparent contradiction: as argued in, e.g., [16], a significant feature of mixed-monotone systems theory is that computing reachable sets for an \( n \)-dimension system requires computing trajectories of a \( 2n \)-dimensional embedding system, and thus reachable set computations apparently scale linearly in the state dimension. However, evaluating the vector field of the embedding system for, e.g., simulation, requires evaluating the decomposition function, and when this is the tightest...
decomposition function given by (17), this evaluation itself might scale exponentially in state and disturbance dimension.

Fig. 1: Approximating forward reachable sets of (24) from the set of initial conditions $\mathcal{X}_0 = [-1/2, 1/2]^3$ where the disturbance bound is given by $\mathcal{W} = [-1/4, 0] \times [0, 1/4]$.

For the aforementioned reasons, in certain instances it may be preferable to use alternate decomposition functions, even when the tightest decomposition function is available in closed form. A decomposition function construction resulting from bounds on the state and disturbance Jacobian matrices is presented in [11]–[13], and in [6] a basic procedure is presented for generating decomposition functions for polynomial systems.

While (17) is not always attainable in closed form, Theorem 1 does suggest a general theory as to how decomposition functions should be formed in the general setting of (2); in particular, we observe that a decomposition function $d$ should be large when its first two inputs are larger than its second two inputs and small when its first two inputs are smaller than its second two inputs. This is because $d(x, w, \tilde{x}, \tilde{w})$ governs the movement of the first $n$ entries of $\Phi^e$ when $x \leq \tilde{x}$ and $w \leq \tilde{w}$ and therefore should be large in order to attain tight approximations. Likewise, $d(\tilde{x}, \tilde{w}, x, w)$ governs the movement of the second $n$ entries of $\Phi^e$ and therefore should be small. Note however that there is an intrinsic maximum/minimum evaluation of $d(x, \tilde{x}, w, \tilde{w})$, and this bound is determined by the location of the state and disturbance relative to the manifold where $x = x$ and $w = w$. As shown in Theorem 1 this bound is attained only if $d$ is tight.

REFERENCES

[1] L. Yang and N. Ozay, “Tight decomposition functions for mixed monotonicity,” in 2019 IEEE 58th Conference on Decision and Control (CDC), pp. 5318–5322, Dec 2019.
[2] G. Enciso, H. Smith, and E. Sontag, “Nonmonotone systems decomposable into monotone systems with negative feedback,” Journal of Differential Equations J. Differential Equations, vol. 22405007, pp. 205–227, 05 2006.
[3] M. Kulesov and O. Merino, “A global attractivity result for maps with invariant boxes,” Discrete and Continuous Dynamical Systems Series B, vol. 6, no. 1, p. 97, 2006.
[4] J.-L. Gouze and K. P. Hadeler, “Monotone flows and order intervals,” Nonlinear World, vol. 1, no. 1, pp. 23–34, 1994.
[5] S. Coogan and M. Arcak, “Stability of traffic flow networks with a polytree topology,” Automatica, vol. 66, pp. 246–253, Apr. 2016.
[6] M. Abate and S. Coogan, “Computing robustly forward invariant sets for mixed-monotone systems,” 2020. Submitted to 2020 IEEE 59th Conference on Decision and Control (CDC). Preprint available: https://arxiv.org/abs/2003.05912.
[7] H. Smith, “Global stability for mixed monotone systems,” Journal of Difference Equations and Applications, vol. 14, no. 10-11, pp. 1159–1164, 2008.
[8] H. Smith, Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems. Mathematical surveys and monographs, American Mathematical Society, 2008.
[9] D. Angeli and E. D. Sontag, “Monotone control systems,” IEEE Transactions on Automatic Control, vol. 48, pp. 1684–1698, Oct 2003.
[10] S. Coogan, M. Arcak, and A. A. Kurzhanskiy, “Mixed monotonicity of partial first-in-first-out traffic flow models,” in 2016 IEEE 55th Conference on Decision and Control (CDC), pp. 7611–7616, Dec 2016.
[11] L. Yang, O. Mickelin, and N. Ozay, “On sufficient conditions for mixed monotonicity,” IEEE Transactions on Automatic Control, vol. 64, pp. 5080–5085, Dec 2019.
[12] P.-J. Meyer, A. Devonport, and M. Arcak, “Tira: Toolbox for interval reachability analysis,” in Proceedings of the 22nd ACM International Conference on Hybrid Systems: Computation and Control, HSCC 19, p. 224229, Association for Computing Machinery, 2019. An extended version of this work appears on ArXive https://arxiv.org/abs/1902.05204.
[13] P. Meyer and D. V. Dimarogonas, “Hierarchical decomposition of LTL synthesis problem for nonlinear control systems,” IEEE Transactions on Automatic Control, vol. 64, pp. 4676–4683, Nov 2019.
[14] M. Dutreix and S. Coogan, “Efficient verification for stochastic mixed monotone systems,” in 2018 ACM/IEEE 9th International Conference on Cyber-Physical Systems (ICCPPS), pp. 150–161, April 2018.
[15] M. Dutreix and S. Coogan, “Specification-guided verification and abstraction refinement of mixed monotone stochastic systems,” 2019. In submission. Preprint available: https://arxiv.org/abs/1903.02191.
[16] S. Coogan and M. Arcak, “Efficient finite abstraction of mixed monotone systems,” in Proceedings of the 18th International Conference on Hybrid Systems: Computation and Control, pp. 58–67, 2015.