HOMOTOPY LIFTING PROPERTY FOR ACTIONS OF FINITE ABELIAN GROUPS ON HAUSDORFF TOPOLOGICAL SPACES

EDUARDO BLANCO-GÓMEZ

ABSTRACT. In this paper we prove the homotopy lifting property for actions of finite abelian groups on Hausdorff topological spaces.

1. INTRODUCTION

Lifting a continuous function between topological spaces is a question that depends on the topological properties of the two spaces. There are known cases where the lifting can be done relatively easy. One of these situations is when the continuous function is defined over a covering space \( \tilde{X} \) of a topological space \( X \) (see Proposition 1.30 page 69 of [6] and Theorem 17.6 page 157 of [7]). In that case one can prove the important homotopy lifting property, but even there, lifting a continuous function between any topological space and a covering space depends strongly on the homotopy type of the spaces (see Proposition 1.33 page 70 of [6] and theorem 21.2 page 174 of [7]). Another case one has the homotopy lifting property is when working with fibrations. Fibrations in the sense of Hurewicz (see definition 1.2 page 393 of [4] and page 66 of [11]) have the homotopy lifting property by definition, beyond this, fibrations in the sense of Hurewicz are equivalent to fibrations for which one has the homotopy lifting property (see Curtis and Hurewicz’s theorem page 396 of [4]). Again, as it happens with covering maps, lifting a continuous function between any topological space and a fiber space depends strongly on the homotopy type of the spaces (see theorem 5 page 76 of [11]). In fact, fibrations can be treated in most situations as covering maps (something natural as covering spaces are fiber spaces with discrete fiber -see theorem 3 page 67 of [11]-). Out of that cases, there are few general situations but specific ones where a kind of lifting can be done. One of this specific cases is that of the paper of Papakyriakopoulos [10]; he uses the lifting of some maps whose image lies in what he calls prismatic neighborhood and defined in the universal covering of such neighborhood; with these liftings Papakyriakopoulos is able to prove Dehn’s lemma and the sphere theorem. The lifting problem from the general point of view can be studied using obstruction theory, Postnikov towers and Moore-Postnikov towers (see pages 410 and 415 of [6]).
In paper [1] we proved the homotopy lifting property for the symmetric products $SP_m(X)$ and $F_m(X)$ of a topological space, developing the theory of topological puzzles for the $m$-cartesian product $X^m$. In fact, we are going to use some of the ideas showed in that paper to prove the main result of this article.

2. ISOTROPY GROUPS AND DECOMPOSITION OF THE QUOTIENT SPACE

Let $X$ be a topological space and $G$ be a group. We define the action of $g \in G$ on $x \in X$ as $gx \in X$ (remember that the action of $G$ on $X$ is asked to be associative and the identity element in $G$ have to fix every $x \in X$). The maps,

$$\varphi_g : \ X \to X$$

$$x \mapsto \varphi_g(x) = gx,$$

are bijective. This action generates a continuous and onto map, from $X$ over the quotient space $X/G$, defined by,

$$\varphi: \ X \to X/G$$

$$x \mapsto \varphi(x) = [x]_\varphi,$$

with $X/G$ endowed with the quotient topology generates by $\varphi$. The equivalence class $[x]_\varphi$ is defined as,

$$[x]_\varphi = \{x' \in X : \exists g \in G \text{ such that } x' = gx\}.$$

Moreover, if $X$ is a $G$-space (see definition 5.10 page 40 [7]) one has that the maps $\varphi_g$ are homeomorphisms and $\varphi$ is also open (see theorem 5.12 page 40 [7]).

**Definition 2.1.** Let $X$ be a topological space and $G$ acting on $X$. For $x \in X$ we define the isotropy group of $x$ like,

$$G_x = \{g \in G : gx = x\}.$$

First of all, let’s proof that the map $\varphi$ defined in (2) is continuous, open and onto.

**Lemma 2.2.** Let $\varphi$ be the map defined by,

$$\varphi: \ X \to X/G$$

$$x \mapsto \varphi(x) = [x]_\varphi,$$

with $[x]_\varphi$ defined in (3). Then $\varphi$ is continuous, open and onto.

**Proof.** We want to prove that $\varphi_g$, defined in (1) is a homeomorphism. $\varphi_g$ is bijective. Suppose $\varphi_g(x) = \varphi_g(x')$, so $gx = gx'$, then $g^{-1}gx = g^{-1}gx'$ and $x = x'$. Take $x' \in X$. As $G$ is a group then it exists an identity element $g_0$ and for every $g \in G$ it exists $g^{-1} \in G$ such that $gg^{-1} = g_0$. Defining $x = g^{-1}x'$ we have,

$$\varphi_g(g^{-1}x') = gg^{-1}x' = x'.$$
thus \( \varphi_g(x) = x' \) for \( x = g^{-1}x' \) and \( \varphi_g \) is onto.

2. \( \varphi_g \) is continuous. Take \( V \subset X \) an open subset. It is enough to prove that \( \forall g \in G, \ gV \subset X \) is open if \( V \subset X \) is open, being \( gV = \{ x \in X : \exists x' \in V \text{ with } x = gx' \} \). That is easy to see just using the next property: \( x \in V \iff gx \in gV \).

3. \( \varphi_g \) is an open map. Take \( V \subset X \) an open subset. This statement proves similarly to 2.

Now let’s prove the identity,

\[
\varphi^{-1}(\varphi(V)) = \bigcup_{g \in G} \varphi_g(V),
\]

for every \( V \subset X \) an open subset. As the sets of the two members of (3) are into the same topological space endowed with the same topology, we just have to prove the equality between the two sets.

\[\text{Take } x \in \varphi^{-1}(\varphi(V)). \text{ Then it exists } g \in G \text{ such that } gx \in V. \text{ But then } \varphi_g(x) \in V. \text{ As } \varphi_g \text{ is a homeomorphism by the first part of the proof we have } x \in \varphi_g^{-1}(V) = \varphi_{g^{-1}}(V) \text{ so,}
\]

\[
x \in \bigcup_{g \in G} \varphi_g(V) \implies \varphi^{-1}(\varphi(V)) \subset \bigcup_{g \in G} \varphi_g(V).
\]

\[\text{Take } x \in \bigcup_{g \in G} \varphi_g(V). \text{ Then it exists } g \in G \text{ such that } x \in \varphi_g(V). \text{ As } \varphi_g \text{ is a homeomorphism by the first part of the proof we have } \varphi_g^{-1}(x) \in V. \text{ Like } \varphi \circ \varphi_g = \varphi \text{ for every } g \in G, \text{ then } \varphi(x) \in \varphi(V). \text{ As } \varphi(V) \text{ is a saturated set then } x \in \varphi^{-1}(\varphi(V)), \text{ and we conclude that,}
\]

\[
\bigcup_{g \in G} \varphi_g(V) \subset \varphi^{-1}(\varphi(V)).
\]

Thus, we have (3). As we have that \( \varphi \) is continuous and onto by definition 2, putting together (3) and the fact that \( \varphi_g \) is a homeomorphism for every \( g \in G \) we conclude that \( \varphi \) is also an open map.

In this paper we will use extensively the theory of \textit{passings-through} for topological spaces, introduced in section 4 of [1].

Our aim now is to decompose our space \( X \) into subspaces such that we can take a different action (of a quotient group of \( G \) by every isotropy group) over every subspace and with the objective of building a new space for which the existence of the homotopy lifting property is equivalent to the existence of that property for \( X \). Begin with \( X \) a topological space and \( G \) a finite abelian group acting on \( X \). Let \( \{G_j\}_{j \in J} \) be the set of all subgroups of \( G \). Define the subspace,

\[
X_j = \{ x \in X : \ G_x = G_j \},
\]
endowed with the relative topology. The subspaces \( X_j \) are pairwise disjoint by definition. Let \( \tilde{G}_j = G/G_j \) be the quotient set defined as,

\[
\rho_j : G \to G/G_j, \quad g \mapsto \rho_j(g) = [g]_{\rho_j} = \{ \bar{g} \in G : \bar{g}^{-1}g \in G_j \}.
\]

Like \( G \) is abelian then \( G_j \) is normal for every \( j \in J \) and then \( \tilde{G}_j \) is a group, see [2] or [3] or [8]. Furthermore, one can define an action of \( \tilde{G}_j \) on \( X_j \) that is well defined. Let’s see this.

**Lemma 2.3.** Let \( X \) be a topological space and \( G \) a finite abelian group acting on \( X \). Let \( G_j \) be a subgroup of \( G \) and \( \tilde{G}_j = G/G_j \) be the quotient group generated by \( G_j \). Then the action,

\[
\theta_j : X_j \to X_j/\tilde{G}_j, \quad x \mapsto \theta_j(x) = [x]_{\theta_j} = \{ \bar{x} \in X_j : \exists \bar{g} \in \tilde{G}_j \text{ for which } \bar{x} = \bar{g}x \},
\]

of the group \( \tilde{G}_j \) on \( X_j \) is well defined. One understands that if \( \bar{g} = [g]_{\rho_j} \) then \( \bar{gx} = [gx]_{\theta_j} \). Moreover, if \( X \) is Hausdorff then \( \theta_j \) is a covering map for every \( j \in J \).

**Proof.** Take \( g_1, g_2 \in \bar{g} = [g]_{\rho_j} \). Then \( g_2^{-1}g_1 \in G_j \). As \( x \in X_j \) then \( g_2^{-1}g_1x = x \). Operating on the left by \( g_2 \) we get \( g_1x = g_2x \) so if \( g_1, g_2 \in [g]_{\rho_j} \) then \( g_1x, g_2x \in [gx]_{\theta_j} \).

Let’s prove that \( \theta_j \) is a covering map. We begin to show that the action \( \theta_j \) is free. Take \( g_1, g_2 \in G_j \). Then, for every \( x \in X_j \), \( g_1x = g_2x \), so \( g_2^{-1}g_1x = x \) thus \( g_2^{-1}g_1 \in G_j \). This last assertion implies that \( [g_1]_{\rho_j} = [g_2]_{\rho_j} \) so the action \( \theta_j \) is free. All in all, the map \( \theta_j \) is a covering map because it is defined as a free action of a finite group over a Hausdorff space (see theorems 17.1 and 17.2 page 154 of [7]).

Define now the new topological spaces,

\[
\hat{X} = \left( \bigcup_{j \in J} X_j, \Xi_J \right), \quad \tilde{X} = \left( \bigcup_{j \in J} (X_j/\tilde{G}_j), \Xi_J \right)
\]

with \( \Xi_J \) and \( \Xi_J \) the disjoint union topologies. With the definition of these new spaces and with the covering maps \( \text{(6)} \) we can define another function gluing them with the disjoint union topology (see [12] or [5] or [4]),

\[
\theta : \hat{X} \to \tilde{X}, \quad x \mapsto \theta(x) = [x]_{\theta} = [x]_{\theta_j}.
\]

The map \( \theta \) is continuous and onto. Let’s see that it is also open.

**Lemma 2.4.** Let \( \{X_j\}_{j \in J} \) and \( \{Y_j\}_{j \in J} \) be two families of pairwise disjoint topological spaces. Let \( X = \bigcup_{j \in J} X_j \) and \( Y = \bigcup_{j \in J} Y_j \) be topological
spaces both endowed with the disjoint union topology. Suppose we have open \( f_j : X_j \to Y_j \) for every \( j \in J \). From them, define the map,

\[
  f : X \to Y \\
  x \mapsto f(x) = f_j(x),
\]

Then, \( f \) is open \( \iff \) \( f_j \) is open \( \forall j \in J. \)

**Proof.** The right implication is obvious. Let’s see the left one. By an easy argument (see [12] or [5] or [4]) one has that the inclusions \( i_j : Y_j \to Y \) are open (and, in fact, closed too). Take \( U \subset X \) an open subset. Then,

\[
  f(U) = f(\bigcup_{j \in J} (U \cap X_j)) = \bigcup_{j \in J} f(U \cap X_j) = \bigcup_{j \in J} i_j(f_j(U)),
\]

that is a union of open sets in \( Y \). \( \square \)

**Definition 2.5.** Let \( X \) be a topological space and \( n \in \mathbb{N} \). A continuous map \( \gamma : [0, 1]^n \to X \) will be denoted as an \( n \)-region.

From this point we will lead our efforts to prove that the lifting of an \( n \)-region in \( X/G \) to another one in \( X \) is equivalent to the lifting an \( n \)-region in \( \tilde{X} \) to another one in \( \hat{X} \).

**Lemma 2.6.** Let \( X = (X, \mathcal{T}_X) \) be a topological space, \( A \) a set of indexes and \( X_\alpha \subset X \) pairwise disjoint subspaces endowed with the relative topology, \( \alpha \in A \). Let \( \hat{X} = (\bigcup_{\alpha \in A} X_\alpha, \mathcal{T}_A) \) be a topological space endowed with the disjoint union topology. Then,

(a) \( \mathcal{T}_X \subset \mathcal{T}_A. \)

(b) Define,

\[
  \pi : \hat{X} \to X \\
  x \mapsto \pi(x) = x.
\]

Then \( \pi \) is bijective, continuous and for every \( V \in \mathcal{T}_X, \pi(V) = V \in \mathcal{T}_X. \)

(c) Define,

\[
  i : X \to \tilde{X} \\
  x \mapsto i(x) = x.
\]

Then \( i \) is bijective, open and for every \( V \in \mathcal{T}_X, i^{-1}(V) = V \in \mathcal{T}_X. \)

(d) Let \( Y \) be another topological space. Take \( \hat{f} : Y \to \hat{X} \) a function denoting \( \hat{f}(y) = x \). Then \( \hat{f} \) induces a function \( f : Y \to X \), with \( f(y) = x \), such that:

- If \( \hat{f} \) is bijective then \( f \) is bijective.
- If \( \hat{f} \) is continuous then \( f \) is continuous.
- If \( \hat{f} \) is open then \( f \) is open.

(e) Every homeomorphism \( \tilde{f} : \tilde{X} \to \tilde{X} \) induces a homeomorphism \( f : X \to X \) such that \( \tilde{f}|_X = i \circ f \).
Proof. To see (a) we just need the following equality for every \( V \in \mathcal{T}_X \),
\[
V = \bigcup_{\alpha \in A} (V \cap X_\alpha) \in \mathcal{T}_A.
\]
Now (b) and (c) are direct consequences of (a). To prove (d) it is enough to define \( f = \pi \circ \hat{f} \) and use (b). Finally, to prove (e), from \( \hat{f} \) we define \( f = \pi \circ \hat{f} \circ i \) and use (b) and (c). \( \square \)

Remark 2.7. We can not prove that the continuous and bijective map \( \pi \) introduce in lemma 2.6 is always a homeomorphism. However, there are cases when it is. For example, when \( X \) is Hausdorff and the disjoint union is compact (see theorem 8.8 page 58 of [7]).

Lemma 2.8. Let \( X \) be a Hausdorff topological space, \( m \in \mathbb{N} \). Then it exists a continuous and bijective map \( f \),
\[
f : \hat{X} \to X/G,
\]
such that for every open subset \( V \subset X \), \( f(\theta(\pi^{-1}(V))) \) is an open subset of \( X/G \), being \( \theta \) the map defined in (8) and \( \pi \) defined in lemma 2.6 (b).

Proof. Consider the next diagram,
\[
\begin{array}{ccc}
X & \xrightarrow{\pi} & \hat{X} \\
\downarrow{\varphi} & & \downarrow{\theta} \\
X/G & \xleftarrow{f} & \hat{X}
\end{array}
\]

Let’s define,
\[
f : \hat{X} \to X/G \\
[x]_\theta \mapsto f([x]_\theta) = [x]_\varphi,
\]
and \( \pi \) the continuous and bijective map defined in lemma 2.6 (b). The map \( \theta \) is well defined because the subspaces \( X_j \) of (4) are disjoint, and \( f \) is well defined because \([x]_\theta \subset [x]_\varphi \subset X \) for all \( x \in X \). From diagram (10), lemma 2.6 (b) and definitions (8) and (11) we get \( \varphi \circ \pi = f \circ \theta \). We know that \( \varphi \) is continuous and \( \pi \) is continuous. With the last equality, we just need to know that \( \theta \) is continuous to conclude the same for \( f \). But \( \theta \) is continuous by the universal property of the disjoint union topology (see [12] or [5] or [4]) so it is \( f \).

Let’s see now \( f \) is bijective. By definition of \( \hat{X} \), like \( \varphi \) is onto and by lemma 2.6 for every \([x]_\varphi \in X/G \) it exists \( x' \in [x]_\varphi \cap \hat{X} \) such that \((\varphi \circ \pi)(x') = [x]_\varphi \). Thus \( f([x']_\theta) = (f \circ \theta)(x') = (\varphi \circ \pi)(x') = [x]_\varphi \) and we obtain that \( f \) is onto. Suppose now \([x]_\varphi = [x']_\varphi \) for some \( x, x' \in \hat{X} \); then it exists \( g \in G \) such that \( x' = gx \). Like \( G \) is abelian then \( G_{x'} = G_x \) and both \( x \) and \( x' \) are in the same \( X_j \). Thus,
\[
[x]_\theta = [x]_{\theta_j} = [gx]_{\theta_j} = [x']_{\theta_j} = [x']_\theta,
\]
by definition of the action $\theta_j$ from lemma 2.3 and by definition (8) of $\theta$. We conclude that $f$ is injective.

Finally, let’s prove that for every open subset $V \subset X$, $f(\theta(\pi^{-1}(V)))$ is an open subset of $X/G$. But by the previous paragraphs $\varphi \circ \pi = f \circ \theta$. Thus $f(\theta(\pi^{-1}(V))) = \varphi(V)$ and $\varphi$ is open (see theorem 5.12 page 40 [7]).

□

With lemma 2.8 and with the next diagram,

\[\begin{array}{ccc}
Y & \xrightarrow{g_1} & X \\
\downarrow{\bar{g}} & & \downarrow{\pi} \\
X/G & \leftarrow & \hat{X}
\end{array}\]

one can reduce the proof of the homotopy lifting property for $X/G$ to the proof of the homotopy lifting property for $\hat{X}$. More precisely, we just need to lift a path in $\hat{X}$ to a path in $\hat{X}$ to get the lift of a path in $X/G$ to a path in $X$. In the last diagram, $Y$ is a topological space, $g_1$ is a continuous map in $X$, $g_2$ is a continuous map in $\hat{X}$, $\bar{g}$ is a continuous map in $X/G$, $\varphi$ is the map defined in (2), $\pi$ the continuous and bijective map defined in lemma 2.6 (b), $\theta$ is the map defined in (8), $f$ is the continuous and bijective map defined in (11).

**Lemma 2.9.** Let $X$ and $Y$ be Hausdorff topological spaces and $m \in \mathbb{N}$. Let $\bar{g} : Y \to X/G$ a continuous map in $X/G$. Then,

\[\begin{align*}
\text{It exists a continuous map} & \quad \text{It exists a continuous map} \\
g_1 : Y \to X & \iff g_2 : Y \to \hat{X} \\
\text{such that } \bar{g} = \varphi \circ g_1 & \quad \text{such that } \bar{g} = f \circ \theta \circ g_2
\end{align*}\]

Proof. Remembering that the next diagram is commutative,

\[\begin{array}{ccc}
X & \xrightarrow{\pi} & \hat{X} \\
\downarrow{\varphi} & & \downarrow{\theta} \\
X/G & \leftarrow & \tilde{X}
\end{array}\]

we obtain,

\[\begin{align*}
\bar{g} = f \circ \theta \circ g_2 & \iff \bar{g} = \varphi \circ \pi \circ g_2 \iff \bar{g} = \varphi \circ g_1.
\end{align*}\]

□

So now, our efforts will be focused to the proof of the homotopy lifting property for $\tilde{X}$. We are going to work with the next commutative
diagram, for \( n \in \mathbb{N} \),

\[
\begin{array}{ccc}
[0, 1]^n & \xrightarrow{\gamma} & \hat{X} \\
\gamma \downarrow & & \downarrow \theta \\
\tilde{X}. & & \\
\end{array}
\]

Lemma \[ \text{(2.9)} \] gives us an advantage we didn’t have before: instead of working with the map \( \varphi \), we are going to work with the map \( \theta \). Both are continuous, open and surjective maps (\( \varphi \) is open by its definition and by lemma \[ \text{(2.4)} \] but \( \theta \) is also what we call a covering-by-parts map. This is a direct consequence of the definition \[ \text{(8)} \] of theta because \( \theta|_{X_j} = \theta_j \) and \( \theta_j \) is a covering map as stated in lemma \[ \text{(2.3)} \].

Our aim is to “lift by parts” the \( n \)-region \( \tilde{\gamma} : [0, 1]^n \to \hat{X} \) and then glue carefully the lifted pieces.

**Remark 2.10.** Let \( X \) and \( Y \) be topological spaces and \( m \in \mathbb{N} \). Let \( \tilde{\gamma} : Y \to \hat{X} \) be a continuous function. Let \( \theta \) be the map defined in \[ \text{(8)} \]. Take \( y_0 \in Y \). Then for every \( p \in \tilde{\gamma}(y_0) \subset \hat{X} \) the set \( \{(\theta^{-1} \circ \tilde{\gamma})(y) : y \in U_0\} \) is dense in \( p \) for all \( U_0 \subset Y \) an open neighborhood of \( y_0 \). To prove this, suppose not; then it exists \( y_0 \) and \( U_0 \subset Y \) an open neighborhood of \( y_0 \) such that for some \( p \in \tilde{\gamma}(y_0) \subset \hat{X} \) and for some \( V_0 \subset \hat{X} \) an open neighborhood of \( p \), the set \( \{(\theta^{-1} \circ \tilde{\gamma})(y) : y \in U_0\} \) is not dense in \( V_0 \), i.e.,

\[
\{(\theta^{-1} \circ \tilde{\gamma})(y) : y \in U_0\} \cap V_0 = \{p\}.
\]

Having account that \( \theta^{-1}(\tilde{\gamma}(U_0)) \) is a saturated set (see \[ \text{[9]} \] page 155) and from the last equality \[ \text{(15)} \] we have,

\[
U_0 \cap (\tilde{\gamma}^{-1} \circ \theta)(V_0) = (\tilde{\gamma}^{-1} \circ \theta)(\{p\}),
\]

but this is impossible because the left member is an open set and the right one is a closed set as: \( U_0 \) is open, \( (\tilde{\gamma}^{-1} \circ \theta)(V_0) \) is open like \( V_0 \) is open, \( \tilde{\gamma} \) continuous and \( \theta \) open (as stated after diagram \[ \text{(14)} \]), and \( (\tilde{\gamma}^{-1} \circ \theta)(\{p\}) \) is closed as \( \theta(p) = [p]_\theta \) is a point in \( \hat{X} \) and \( \tilde{\gamma} \) is continuous.

Now let \( Y_1 \subset Y \) a subspace of \( Y \). Suppose \( \gamma : Y_1 \to \hat{X} \) is a continuous function such that \( \gamma|_{Y_1} = \theta \circ \gamma \). Take \( y_0 \in \partial Y_1 \). Then the set \( \{\gamma(y) : y \in U_0 \cap Y_1\} \), with \( U_0 \subset Y \) an open neighborhood of \( y_0 \), is dense in some \( p \in \tilde{\gamma}(y_0) \subset \hat{X} \). Take any \( U_0 \subset Y \) an open neighborhood of \( y_0 \). Denote \( Y_2 = Y_1 \cup \{y_0\} \) endowed with the relative topology. By the previous paragraph we have that the set

\[
\{(\theta^{-1} \circ \tilde{\gamma}|_{Y_2})(y) : y \in U_0 \cap Y_2\}
\]

is dense in every \( p \in \tilde{\gamma}(y_0) \). so, for every \( V_0 \subset \hat{X} \) an open neighborhood of \( p \) it exists \( y_p \in Y_1 \) such that \( (\theta^{-1} \circ \tilde{\gamma}|_{Y_2})(y_p) \in V_0 \), i.e.,

\[
(\theta^{-1} \circ \tilde{\gamma}|_{Y_1})(y_p) \in V_0.
\]
If $\gamma(y_p) \notin V_p$ for every $p \in \tilde{\gamma}(y_0)$, then $(\theta \circ \gamma)(y_p) \notin \theta(V_p)$, i.e., $\tilde{\gamma}_1(y_p) \notin \theta(V_p)$, so $(\theta^{-1} \circ \tilde{\gamma}_1)(y_p) \notin (\theta^{-1} \circ \theta)(V_p)$. But as $V_p \subset (\theta^{-1} \circ \theta)(V_p) \setminus \{\tilde{\gamma}_1(y_p)\}$ and this is a contradiction with (16). Furthermore, in the case that the set $\{\tilde{\gamma}_1(y_p)\} \in \theta^{-1}(\tilde{\gamma}(y_0))$ is finite, then the set $\{\gamma(y) : y \in \tilde{\gamma}(y_0)\}$, with $U_0 \subset \gamma$ an open neighborhood of $y_0$, is dense in exactly one and only one $p \in \theta^{-1}(\tilde{\gamma}(y_0)) \subset \hat{X}$ when $X$ is Hausdorff.

**Remark 2.11.** Let $X$ be a Hausdorff topological space, $U \subset X$ an open subset and $x \in U$. Then $U \setminus \{x\}$ is open. To prove that, it is enough to see that every $x' \in U \setminus \{x\}$ is an interior point, i.e., it exists $U' \subset U \setminus \{x\}$ an open neighborhood of $x'$. But this is a direct consequence of the existence of $U'_1 \subset U$ an open neighborhood of $x'$ (that exists because $U$ is an open set) and the existence of $U'_2 \subset X$ an open neighborhood of $x'$ that does not contain $x$ (because $X$ is Hausdorff). Taking $U' = U'_1 \cap U'_2$ we finish.

### 3. Homotopy lifting property for actions of finite abelian groups on Hausdorff topological spaces

We want to prove now the analogue of theorem 5.22 of [1].

**Theorem 3.1.** Let $X$ be a Hausdorff topological space and $n \in \mathbb{N}$. Let $\tilde{\gamma} : [0, 1]^n \to \hat{X}$ be a continuous function. Then it exists $\gamma : [0, 1]^n \to \hat{X}$ continuous such that diagram (14) commutes.

**Proof.** First of all, we will split every piece $X_j$. Denote,

$$\tilde{\gamma}_j = \tilde{\gamma}|_{\tilde{\gamma}^{-1}(X_j/\tilde{G}_j)}.$$

Consider the next diagram,

$$\tilde{\gamma}^{-1}(X_j/\tilde{G}_j) \xrightarrow{\tilde{\gamma}_j} X_j \xleftarrow{\theta_j} X_j/\tilde{G}_j.$$

Take,

$$\tilde{\gamma}^{-1}(X_j/\tilde{G}_j) = \bigcup_{\lambda_j \in \Lambda_j} \Gamma_j^{\lambda_j},$$

the decomposition of $\tilde{\gamma}^{-1}(X_j/\tilde{G}_j)$ in connected and locally path-connected (at the same time) components. To make simpler the notation, when possible, we will denote $\Gamma_j$ a connected and locally path-connected component. Denoting as,

$$\tilde{\Gamma}_j^{\lambda_j} = \tilde{\gamma}(\Gamma_j^{\lambda_j}),$$

and

$$\Theta_j^{\lambda_j} = \theta_j^{-1}(\tilde{\gamma}(\Gamma_j^{\lambda_j})).$$
we can rewrite diagram (17),

\[
\begin{array}{ccc}
\Gamma_j & \longrightarrow & \Theta_j \\
\tilde{\gamma}_j & \downarrow & \theta_j \\
\tilde{\Gamma}_j & \longrightarrow & \tilde{\Theta}_j
\end{array}
\]

As \(\Gamma_j^\lambda\) is connected and \(\tilde{\gamma}_j\) is continuous therefore using 6.1.3. theorem page 352 of [5], \(\tilde{\Gamma}_j^\lambda\) is connected. In fact, \(\tilde{\Gamma}_j^\lambda\) is a connected component of \(X_j/\tilde{G}_j\) because so it is \(\Gamma_j^\lambda\) of \(\tilde{\gamma}_j^{-1}(X_j/\tilde{G}_j)\). As a consequence, every \(y \in \tilde{\Gamma}_j^\lambda\) has got a neighborhood that is disjoint with every \(\tilde{\Gamma}_j^{\lambda'}\) for every \(\lambda' \neq \lambda_j\).

From now til the end of the proof, we will work with the sets \(\Theta_j^\lambda\). Therefore, we will denote as \(\Lambda_j\) the set of indexes associated to \(\lambda_j\), and as \(\mu_j\) an index of \(\Lambda_j\); finally, we will denote \(\hat{\Theta}_j^{\mu_j}\) a path-connected component of \(\Theta_j^\lambda\) or just \(\hat{\Theta}_j\) when possible.

First of all, let’s prove a property of the sets \(\Theta_j^\lambda\),

\[
\text{(20)} \quad \text{There exists no } x \in \Theta_j^\lambda \cup \Theta_j^\mu \text{ such that,}
\]

\[
x \in \Theta_j^\lambda \cap \Theta_j^\mu.
\]

Suppose not. Then \(\tilde{\gamma}_j^{-1}(\theta(x)) \cap (\Gamma_j^\lambda \cup \Gamma_j^\mu)(\Gamma_j^\lambda \cup \Gamma_j^\mu) \neq \emptyset\). But that is impossible because \(\Gamma_j^\lambda\) and \(\Gamma_j^\mu\) are locally path-connected components.

Take now \(x_0 \in \Theta_j\) and \(\hat{\Theta}_j\) the path-connected component including \(x_0\). Let \(\tilde{x}_0 = \theta_j(x_0) \in \tilde{\Gamma}_j\) and \(\tilde{x}_1 \in \tilde{\Gamma}_j\). As \(\tilde{\Gamma}_j\) is connected and locally path connected, then by 21.1 lemma page 175 of [7], \(\tilde{\Gamma}_j\) is path connected. Therefore take a path \(\tilde{\delta} : [0, 1] \rightarrow \tilde{\Gamma}_j\) which holds \(\tilde{\delta}(0) = \tilde{x}_0\) and \(\tilde{\delta}(1) = \tilde{x}_1\); like \(\theta_j\) is a covering map as stated in lemma 2.3, we can apply 17.6 theorem page 157 of [7] to obtain a path \(\delta : [0, 1] \rightarrow \Theta_j\) which holds \(\tilde{\delta} = \theta_j \circ \delta\) and \(\delta(0) = x_0\); moreover, it exists \(x_1 \in \theta_j^{-1}(\tilde{x}_1)\) so that \(\delta(1) = x_1\). As the last deduction can be done with every \(\tilde{x}_1 \in \tilde{\Gamma}_j\), we conclude that \(\hat{\Theta}_j\) holds \(\theta_j(\hat{\Theta}_j) = \tilde{\Gamma}_j\). We obtain the next diagram,

\[
\begin{array}{ccc}
\Gamma_j & \longrightarrow & \hat{\Theta}_j \\
\tilde{\gamma}_j & \downarrow & \theta_j \\
\tilde{\Gamma}_j & \longrightarrow & \tilde{\Theta}_j
\end{array}
\]

Our aim with diagram (21) is to apply 21.2 theorem page 176 of [7] to obtain a lift of \(\tilde{\gamma}_j\). To do that, like \(\Gamma_j\) is connected and locally path
connected by its definition, we also need the next condition,

\[(22) \quad \tilde{\gamma}_j^*(\pi(\Gamma_j)) \subseteq \theta_j^*(\pi(\tilde{\Theta}_j)),\]

where \(\tilde{\gamma}_j^*\) and \(\theta_j^*\) are the induced maps between the fundamental groups. Let \(y_0 \in \tilde{\Theta}_j/G_j\) and,

\[\omega : \pi(\tilde{\Theta}_j/G_j, y_0) \to \tilde{G}_j,\]

\[\delta^*_{H/\Sigma} \mapsto \sigma_{\delta^*},\]

defined in page 165 of [7] (there is defined as \(\varphi\)). For one hand, like \(\tilde{\Theta}_j\) is path connected, we can apply 19.2 theorem page 166 of [7] to obtain that \(\theta_j^*(\pi(\tilde{\Theta}_j)) = \ker \omega\). On the other hand, looking at the definition of \(\tilde{\gamma}_j^*\), we deduce that every loop in \(\Gamma_j\) goes to a loop in \(\tilde{\Gamma}_j\) by the action of \(\tilde{\gamma}_j\), so, by definition of \(\omega\), we conclude that,

\[\tilde{\gamma}_j^*(\pi(\Gamma_j)) \subseteq \ker \omega,\]

that is \((22)\).

All in all, we use 21.2 theorem page 176 of [7] to obtain a continuous lift \(\gamma_j\) of \(\tilde{\gamma}_j\) which holds,

\[(23) \quad \tilde{\gamma}_j = \theta_j \circ \gamma_j.\]

From \(\gamma_j\), we want to build a continuous extension \(\pi_j : \Gamma_j \to \tilde{\Theta}_j\) in this way: denote \(\mathcal{P}_{\Gamma_j}(\tilde{\gamma})\) the set of passings-through of \(\tilde{\gamma}\) which change from a quotient piece to another one from or towards \(\tilde{\Gamma}_j\) (from now til the end of the proof we will use the notation \(\mathcal{P}(\pi_j) = \mathcal{P}_{\Gamma_j}(\tilde{\gamma})\)). Taking \(t \in \mathcal{P}_{\Gamma_j}(\tilde{\gamma})\) and applying remark 2.10, theorem 3.11 page 7 of [1] and 5.1 theorem page 215 of [4] we get a continuous extension of \(\gamma_j\) on \(t\). Let’s make it with every \(t \in \mathcal{P}_{\Gamma_j}(\tilde{\gamma})\) so that we obtain a function,

\[(24) \quad \pi_j : \Gamma_j \cup \mathcal{P}_{\Gamma_j}(\tilde{\gamma}) \to \tilde{\Theta}_j,\]

such that its restrictions to \(\Gamma_j \cup \{t\}\) are continuous functions, for every \(t \in \mathcal{P}_{\Gamma_j}(\tilde{\gamma})\). That function can be defined as \(\pi_j : \Gamma_j \to \tilde{\Theta}_j\) because every \(t \in \partial \Gamma_j\) on which \(\gamma_j\) is not defined, has to be a passing-through by definition 4.1 page 10 of [1] (if \(\Gamma_j\) includes no passing-through in its boundary, then by lemma 4.8 page 12 of [1] \(\gamma_j\) would be the lifting of the whole continuous function \(\tilde{\gamma}\)). Let’s prove that \(\pi_j\) is a continuous function. For one hand, \(\pi_j\) is well defined; this is true due to the next two facts: the first one is that there is no \(p \in \text{Im}_{\gamma_j}(\mathcal{P}_{\Gamma_j}(\tilde{\gamma}))\) such that \(p\) is in another \(\tilde{\Theta}_{j_1,\ldots,j_k}\) by definition. The second fact is that there is no \(t \in \mathcal{P}_{\Gamma_j}(\tilde{\gamma})\) such that \(\pi_j(t)\) can be associated to two or more different points of the same equivalent class. Suppose not; then \(\pi_j(t)\) would not be well defined in the set \(\Gamma_j \cup \{t\}\), but this is a contradiction with
the previous lines. On the other hand, we want to prove that \( \overline{\eta}_j \) is continuous; take the next notation, 
\[
\overline{\eta}_j = \bar{\eta}|_{\overline{\Theta}_j} \quad \text{and} \quad \overline{\theta}_j = \theta|_{\overline{\Theta}_j}.
\]
Let’s prove the following equality,
\[
(25) \quad \overline{\eta}_j^{-1}(V) = \overline{\theta}_j^{-1}(\overline{\Theta}_j(V)) \quad \forall V \subset \overline{\Theta}_j.
\]
But identity \(25\) is true by construction of \( \overline{\eta}_j \); thus, like \( \bar{\eta} \) is continuous and \( \theta \) is open by lemma \(2.4\) then for every open \( V \subset \overline{\Theta}_j \) we have that \( \overline{\eta}_j^{-1}(V) \) is open concluding that \( \overline{\eta}_j \) is continuous.

Our aim now is to glue carefully the liftings obtained in the previous paragraphs. Let’s define a new concept: a shire. We will say that a set \( S = \{ (\Gamma_j^{\lambda_j}, \hat{\Theta}_j^{\mu_j}) \}_{j \in \Lambda_j} \) is a shire if it holds \( \hat{\Gamma}_j^{\lambda_j} = \theta_j(\hat{\Theta}_j^{\mu_j}) \) and the next two conditions,
\[
C1 \forall (\Gamma_j^{\lambda_j}, \hat{\Theta}_j^{\mu_j}) \in S \text{ there exists } (\Gamma_j^{'\lambda_j}, \hat{\Theta}_j^{'\mu_j}) \in S \text{ such that it exists a lifting } \gamma_j \text{ which holds, } \]
\[
\text{Im}_{\gamma_j}(P(\gamma_j)) \cap \partial\hat{\Theta}_j^{\mu_j} \cap \partial\hat{\Theta}_j^{'\mu_j} \neq \emptyset.
\]
\[
C2 \text{ The set of couples } S \text{ cannot be split in a disjoint way with respect to condition } C1.
\]
We will say that \( S \) is a complete shire if,
\[
(26) \quad \bigcup_{\forall (\Gamma_j^{\lambda_j}, \hat{\Theta}_j^{\mu_j}) \in S} \Gamma_j^{\lambda_j} = [0,1]^n.
\]
In other case, we will say that \( S \) is an incomplete shire. We will say that \( S \) is an univalent shire if,
\[
(27) \quad \forall \Gamma_j^{\lambda_j} \exists! \mu_j \text{ such that } (\Gamma_j^{\lambda_j}, \hat{\Theta}_j^{\mu_j}) \in S.
\]
We want now to prove the next result,
\[
(28) \quad \text{If } S = \{ (\Gamma_j^{\lambda_j}, \hat{\Theta}_j^{\mu_j}) \}_{j \in \Lambda_j} \text{ is an univalent shire } \Rightarrow \text{ it exists a continuous function } \gamma_S : \bigcup\Gamma_j^{\lambda_j} \rightarrow \bigcup\hat{\Theta}_j^{\mu_j}.
\]
First of all, take account that the family \( \{ \Gamma_j^{\lambda_j} \}_j \), of 'first coordinates' of the shire, is locally finite; like \( \theta \) is open and \( \bar{\eta} \) continuous, it is enough to prove that the family \( \{ \hat{\Theta}_j^{\mu_j} \}_j \), of 'second coordinates' of the shire, is locally finite; let’s prove it: take \( x \in \hat{\Theta}_j^{\mu_j} \subset \Theta_j^{\lambda_j} \) and \( U_x \subset \hat{X} \) an open neighborhood of \( x \); take another \( \hat{\Theta}_j^{\nu_j} \subset \Theta_j^{\lambda_j} \); it is impossible that \( x \in \partial\hat{\Theta}_j^{\nu_j} \) because \( \hat{\Theta}_j^{\mu_j} \) and \( \hat{\Theta}_j^{\nu_j} \) are path-connected components of \( \Theta_j^{\lambda_j} \).

Suppose now that \( x \in \partial\Theta_j^{\lambda_j} \) with \( \lambda_j \neq \lambda_j \); in that case \( \exists t \in \partial\Gamma_j^{\lambda_j} \cap \partial\Gamma_j^{\lambda_j} \)
with \( t \in \Gamma_j^{\lambda_j} \), being \( \Gamma_j^{\lambda_j} \) associated to a path-connected component of \( \Theta_j^{\lambda_j} \) and \( \Gamma_j^{\lambda_j} \) associated to a path-connected component of \( \Theta_j^{\lambda_j} \); but
that is impossible because $\Gamma_j^{\lambda_j}$ and $\Gamma_j^{\lambda_j'}$ are connected and locally-path connected components. All in all, every $\Theta_j^{e_j}$ can share its boundary with just one $\Theta_j^{e_j'}$ of every piece $X_j'$ different from $X_j$. As the number of pieces of the puzzle $\tilde{X}$ is finite, we conclude that the family $\{\Theta_j^{e_j}\}_j$ is locally finite, thus, so it is the family $\{\Gamma_j^{\lambda_j}\}_j$. Therefore, the family $\{\Gamma_j^{\lambda_j}\}_j$ is locally finite. Applying now exercise 9(c) page 127 of [9], that is a generalization of Pasting lemma (theorem 18.3 page 123 of [9]), and using continuous functions $\gamma_j$ defined in (24), we obtain the continuous function $\gamma$ predicted in (28), that is a function because the shire is univalent, and continuous by construction.

Finally, we need to prove that a maximal univalent shire associated to an $x \in \tilde{X}$ is, in fact, a complete shire. Let’s begin to prove the next statement,

Take $S$ an incomplete shire. Then it exists a shire $S'$ such that $S \subset S'$. Furthermore, if $S$ is univalent, $S'$ can be built as univalent.

As $S$ is an incomplete shire, then,

$$[0, 1]^n \setminus \bigcup_{(\Gamma_j^{\lambda_j}, \tilde{\Theta}_j^{\mu_j}) \in S} \Gamma_j^{\lambda_j} \neq \emptyset.$$  

Like,

$$[0, 1]^n = \tilde{\gamma}^{-1}(\tilde{X}) = \bigcup_{j \in J} \tilde{\gamma}^{-1}(X_j/G_j)$$

remembering (18) we can take a connected and locally path-connected component,

$$\Gamma' \subset [0, 1]^n \setminus \bigcup_{(\Gamma_j^{\lambda_j}, \tilde{\Theta}_j^{\mu_j}) \in S} \Gamma_j^{\lambda_j}$$

that holds $\partial \Gamma' \cap \partial (\bigcup_{(\Gamma_j^{\lambda_j}, \tilde{\Theta}_j^{\mu_j}) \in S} \Gamma_j^{\lambda_j}) \neq \emptyset$. Take now $\Theta'$ a path-connected component of $\tilde{X}$ associated to $\Gamma'$ (in the sense of diagram (21)), that holds, $\partial \Theta' \cap \partial (\bigcup_{(\Gamma_j^{\lambda_j}, \tilde{\Theta}_j^{\mu_j}) \in S} \Theta_j^{e_j}) \neq \emptyset$. That path-connected component exists by construction. Define now the new shire like this,

$$S' = S \cup \{ (\Gamma', \Theta') \}.$$  

Therefore $S'$ is a (univalent if so it is $S$) shire by construction and because so it is $S$. At this point, we have proved statement (29).

To finish the proof of the theorem, take $x \in \tilde{X}$ and $S$ a maximal univalent shire containing $x$ (in its second coordinates). Using statement (29) we conclude that $S$ is a complete univalent shire and applying statement (28) we finish the proof.
Theorem 3.2. Let $X$ be a Hausdorff topological space and $n \in \mathbb{N}$. Let $G$ be a finite abelian group acting on $X$. Let $\tilde{\gamma}$ be an $n$-region over $X/G$. Then it exists $\gamma$ an $n$-region over $X$ such that $\tilde{\gamma} = \varphi \circ \gamma$.

Proof. This theorem is a direct consequence of lemma 2.9 and theorem 3.1. □

References

[1] E. BLANCO-GÓMEZ, Homotopy lifting property in symmetric products, preprint. https://arxiv.org/submit/3057379/view
[2] J. DORRONSORO and E. HERNÁNDEZ, Números, grupos y anillos, Addison-Wesley, 1996
[3] P. DUBREIL, Teoría de grupos, Reverté, 1975
[4] J. DUGUNDJI, Topology, Allyn and Bacon, Inc., 1966.
[5] R. ENGELKING, General topology, Heldermann Verlag, Berlin, 1989.
[6] A. HATCHER, Algebraic topology, http://www.math.cornell.edu/~hatcher/AT/At.pdf
[7] C. KOSNIOWSKI, A first course in algebraic topology, Cambridge University Press, 1980.
[8] S. LANG, Algebra, Springer, 2002
[9] J.R. MUNKRES, Topology, Prentice Hall, 2000.
[10] C.D. PAPAKYRIAKOPOULOS, On Dehn's Lemma and the Asphericity of Knots, Ann. of Math., 66, 1(1957) 1-26.
[11] E. SPANIER, Algebraic Topology, McGraw-Hill, NY, 1966.
[12] S. WILLARD, General Topology, Reading, Massachusetts, Addison-Wesley, 1970.