QUASILINEAR SCHRODINGER EQUATIONS

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Abstract. In this paper we prove local well-posedness for Quasi-linear Schrödinger equations with initial data in unweighted Sobolev Spaces. For small initial data with minimal smoothness this has addressed by J. Marzuola, J. Metcalfe and D. Tataru [15], [16]. This work does not attempt to address the minimal regularity for initial data, but instead builds on the previous results of C. Kenig, G. Ponce, and L. Vega [13], [12] to remove the smallness condition in unweighted spaces. This is accomplished by developing a non-centered version of Doi’s Lemma, which allows one to prove Kato type smoothing estimates. These estimates make it possible to achieve the necessary a priori linear results.

1. Introduction

We are interested in the local solvability of the IVP

\[
\begin{align*}
\partial_t u &= i a_{jk}(x, t, u, \bar{u}, \nabla u, \nabla \bar{u}) \partial_{x_j} \partial_{x_k} u + \bar{b}_1(x, t, u, \bar{u}, \nabla u, \nabla \bar{u}) \cdot \nabla u \\
&\quad + \bar{b}_2(x, t, u, \bar{u}, \nabla u, \nabla \bar{u}) \cdot \nabla \bar{u} + c_1(x, t, u, \bar{u}) u + c_2(x, t, u, \bar{u}) \bar{u} + f(x, t) \\
\end{align*}
\]

u(x, 0) = u_0(x).

Quasi-linear Schrödinger equations have been studied extensively in recent years. The aim of the current work is to extend some results of C. Kenig, G. Ponce, and L. Vega in [12]. In particular we aimed to remove the assumption that the initial data \(\langle x \rangle^2 \partial^\alpha u_0 \in L^2\) for suitable \(\alpha\).

As pointed out in [12], other forms of these equations have been extensively studied. In [9], the same authors show that the equation

\[
\partial_t u + i \mathcal{L} u = P(u, \bar{u}, \nabla u, \nabla \bar{u})
\]

with

\[
\mathcal{L} = \sum_{i=1}^{k} \partial_{x_i} - \sum_{i=k-1}^{n} \partial_{x_k}
\]

and \(P(\cdot)\) a non-linearity, is locally well posed for small initial data in \(H^s\). The smallness condition was first removed in \(n = 1\) by N. Hayashi and T. Ozawa in [6]. After a change of variables they were able to write the equation as an equivalent system that did not involve first order terms in \(u\). For this system can be handled by the energy method.

For the case elliptic case when \(\mathcal{L} = \Delta\), H. Chihara [1] was able to remove the smallness condition in all dimensions. Again, the main idea here was to use a transformation which eliminates the first order terms in \(u\) so that the energy method applies. For the change of variables to cancel the first order terms it was necessary to first diagonalize the system for \((u, \bar{u})\). In order to diagonalize the system, as we will see below, the ellipticity of \(\mathcal{L}\) is essential.
In [13], Kenig et al. removed the smallness condition in all dimensions. They construct a pseudo-differential operator $C$ so that $C^T = C\tau$, and because of this they are able to avoid the diagonalization argument needed in [1]. The construction of $C$ produces a symbol in the Calderón-Vaillancourt class.

As one moves to variable coefficient second order terms it becomes necessary to introduce non-trapping conditions on the coefficients. Consider the equation

$$\left\{ \begin{array}{l} \partial_t u = i\partial_x a_{kj}(x)\partial_x u + \vec{b}_1(x) \cdot \nabla u + f \\ u|_{t=0} = u_0 \end{array} \right.$$  

where $a_{kj}$ elliptic and asymptotically flat. Ichinoise [7] show that

$$\sup_{x_0 \in \mathbb{R}^n} \left| \int_0^{t_0} \Im\vec{b}_1(X(s,x_0,\xi_0)) : \Xi(s,x_0,\xi_0) \, ds \right|$$

is a necessary condition for the estimate

$$\sup_{0 < t < T} \|u\|_{L^2} \leq C_T \left( \|u_0\|_{L^2} + \|f\|_{L^1_T L^2_x} \right).$$

The non-trapping assumption is closely related to local smoothing estimates, which are key to the linear theory. This can be seen from the work of S. Doi ([3], [4]), Craig et al. [2], and others. From their work it can be seen that, under appropriate smoothness, ellipticity and asymptotic flatness assumptions, the non-trapping condition for

$$\left\{ \begin{array}{l} \partial_t u = i\partial_x a_{kj}(x)\partial_x u \\ u|_{t=0} = u_0 \end{array} \right.$$  

verify local smoothing estimates. That is, estimates of the form

$$\|J^{1/2}u\|_{L^2(\mathbb{R}^n \times [0,T], \langle x \rangle^m \, dx \, dt)} \leq C_T \|u_0\|_{L^2}. \quad \text{In addition, Doi [5] also showed that, under the same conditions, if the above estimate holds then the non-trapping assumption must hold.}$$

C. Kenig et al in ([10], [11]) have extended the results of their previous work in the variable coefficient case by removing ellipticity assumptions. Their work assumes that the initial data is in a weighted Sobolev space. It will be the subject of future work to extend the methods here to remove the weights in this cases.

Recently, in both the elliptic and hyperbolic settings J. Marzuola, J. Metcalfe and D. Tataru [16] have established low regularity local well-posedness for for small initial data in $H^s$ for $s > (n+5)/2$. Having a smallness condition on the initial data allows the authors to avoid explicit non-trapping assumptions. In [15] the above authors also considered the situation in which quadratic interactions are present and establish low regularity well-posedness results.

Our own contribution to this body of research is remove the smallness condition for the work of Kenig, Ponce, Vega without imposing any smallness on conditions on the initial data.

Specifically, we assume the following conditions on the coefficients. Let $B^k_M(0) = \{ z \in \mathbb{C}^k : |z| < M \}$.

(NL1) There exist $\hat{N} = \hat{N}(n) \in \mathbb{N}$ such that, for any $M > 0$, $a_{jk}, b_{1,k}, b_{2,j} \in C^\infty_0(\mathbb{R}^n \times \mathbb{R} \times B^k_M(0))$ for $j,k = 1,2,\ldots,n$, and $c_1, c_2 \in C^\infty_0(\mathbb{R}^n \times \mathbb{R} \times B^2_M(0))$. 

In [13], Kenig et al. removed the smallness condition in all dimensions. They construct a pseudo-differential operator $C$ so that $C^T = C\tau$, and because of this they are able to avoid the diagonalization argument needed in [1]. The construction of $C$ produces a symbol in the Calderón-Vaillancourt class.
Let $(x, t, \vec{z}) \in \mathbb{R}^n \times \mathbb{R} \times C^{2n+2}$. The matrix $A(x, t, \vec{z}) = (a_{jk}(x, t, \vec{z}))_{j,k=1,\ldots,n}$ is real valued.

The matrix $A(x, t, \vec{z})$ is symmetric for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $\vec{z} \in B_2^{2n+2}(0)$.

For $\vec{z} \in B_2^{2n+2}(0)$ the matrix $A(x, t, \vec{z})$ is uniformly elliptic. That is, there exists $\gamma_M > 0$ such that

$$\gamma_M |\xi|^2 \leq \sum_{j,k=1}^n a_{jk}(x, t, \vec{z})\xi_j \xi_k \leq \gamma_M^{-1} |\xi|^2,$$

for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ and $\vec{z} \in B_2^{2n+2}(0)$.

Here and throughout we let $\mathbb{R}^n = \bigcup_{\mu \in \mathbb{Z}^n} Q_{\mu}$ where $Q_{\mu}$ are unit cubes with vertices in $\mathbb{Z}^n$ and centers $x_{\mu}$. Suppose that, for $j = 1, 2, b_j(x, t, 0, 0, 0, 0) = 0$, and $\partial_x b_j(x, t, 0, 0, 0, 0) = 0$.

Also, for some $C_M > 0$, we have that $\partial_x a_{jk}(x, t, \vec{z}) = \sum_{\mu \in \mathbb{Z}^n} a_\mu \phi^{(j)k} (x, t, \vec{z})$ with $a_\mu > 0$, $\sum a_\mu < C_M$, $\phi^{(j)k}(x, t, \vec{z}) \in C^\infty(\mathbb{R}^n)$ with $\|\phi^{(j)k}\|_{C^\infty} \leq 1$ and uniformly for $x \in \mathbb{R}$ and $\vec{z} \in B_2^{2n+2}(0)$ we have $\text{supp } \phi^{(j)k} \subseteq Q_{n+1}^* (the \text{ double of } Q_{\mu})$ for $k, j = 1 \ldots n$. Similarly for $\partial_x a_{jk}$, $\partial_{\vec{z}m} a_{jk}$, and $\partial_x \partial_{\vec{z}m} a_{jk}$.

We associate to our coefficients and our initial data the symbol $h(x, \xi) = -a_{jk}(x, 0, u_0, \nabla u_0, \nabla \bar{u}_0)\xi_j \xi_k$.

We assume the bicharacteristic flow obtained from $h$ is non-trapping. That is the solution to the system of ODE’s

$$\begin{aligned}
\frac{d}{dt} X_j(s, x, \xi) &= \frac{\partial h}{\partial \xi_j}(X, \Xi) \\
\frac{d}{dt} \Xi_j(s, x, \xi) &= -\frac{\partial h}{\partial x_j}(X, \Xi) \\
X(0, x, \xi) &= x \quad \text{and} \quad \Xi(0, x, \xi) = \xi
\end{aligned}$$

satisfies $\{X(s, x, \xi) \mid s > 0\}$ and $\{X(s, x, \xi) \mid s < 0\}$ are unbounded for all $(x, \xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$.

**Theorem 1.1.** Under these assumptions there exist $N$, $s$ depending only on the dimension so that if we are given $u_0 \in H^s$ and $f \in L^\infty([0,1]; H^s)$, then there is a $T_0 < 1$ depending on the norms of $u_0$ and $f$ and \[\text{NL1}\] \[\text{NL7}\] so that there is a unique solution to $u(x,t)$, on the interval $[0, T_0]$ satisfying

$u \in C([0, T_0]; H^{s+1}) \cap L^\infty([0, T_0]; H^s)$.

The remainder of the paper is organized as follows. In section 2 we establish an uncentered version of Doi’s lemma necessary to later results. In section 3 we establish a priori linear results. Finally, in section 4 we give the proof of Theorem 1.1.

2. Doi’s Lemma

Doi’s lemma is a key estimate that allows us to obtain local smoothing. It is the local smoothing estimates that allow us to handle the first order terms in the linear...
theory. In this section we present two variants of Doi’s lemma, one due to S. Doi that holds in the elliptic setting and one due to C. Kenig, G. Ponce, C. Rolvung, and L. Vega in [10], that also holds when the coefficients are not necessarily elliptic. We then show how to extend these results to corresponding “non-centered” versions that we need for the precise form of our local smoothing estimates.

We consider the symbol \( h(x, \xi) \in S^2_{1,0} \) defined by \( h(x, \xi) = \sum_{j,k=1}^{n} a_{jk}(x) \xi_j \xi_k \).

Let \( A(x) \) denote the matrix \( (a_{jk}(x))_{j,k=1}^{n} \). We impose the following assumptions on \( A(x) \):

(D1) There exist \( N = N(n) \in \mathbb{N} \) and \( C > 0 \) so that \( a_{jk}(x) \in C^N_R(\mathbb{R}^n) \) for \( j, k = 1, 2, \ldots, n \), with norm controlled by \( C \).

(D2) The functions \( a_{jk}(x) \) are real valued and the matrix \( A(x) = (a_{jk}(x))_{j,k=1,\ldots,n} \) is symmetric.

(D3) The matrix \( A(x) \) is uniformly elliptic. That is, for all \( x \in \mathbb{R}^n \) there is a positive number \( \gamma \) so that

\[
C^{-1} |\xi|^2 \leq \sum_{j,k=1}^{n} a_{jk}(x) \xi_j \xi_k \leq C |\xi|^2.
\]

(D4) The matrix \( A(x) \) is asymptotically flat. That is

\[
|I - A(x)| \leq \frac{C}{(x)^2} \quad \text{and} \quad |\nabla_x a_{jk}(x)| \leq \frac{C}{(x)^2}.
\]

(D5) Let \( X(s, x, \xi) \) and \( \Xi(s, x, \xi) \) be the Hamiltonian flow associated to \( h \). That is \( X \) and \( \Xi \) are solutions to the following ODEs:

\[
\begin{align*}
\frac{d}{dt} X_j(s, x, \xi) &= 2 a_{jk}(X(s, x, \xi)) \Xi_k(s, x, \xi) \\
\frac{d}{dt} \Xi_j(s, x, \xi) &= - \frac{\partial a_{jk}}{\partial x_j}(X(s, x, \xi)) \Xi_i(s, x, \xi) \Xi_k(s, x, \xi) \\
X(0, x, \xi) &= x \quad \text{and} \quad \Xi(0, x, \xi) = \xi.
\end{align*}
\]

Then for each pair \( x, \xi \) with \( \xi \neq 0 \) we assume that the sets \( \{X(s, x, \xi) \mid s > 0\} \) and \( \{X(s, x, \xi) \mid s < 0\} \) are unbounded.

**Lemma 2.1 (S. Doi [4])**. With \( a_{jk}(x) \) satisfy \((D1)-(D3)\) there exist a symbol \( p \in S^0_{1,0} \), with semi-norms bounded in terms of \( C \), and a constant \( B \in (0,1) \) depending on \( C \) and \((D5)\) such that

\[
H_h p := \{h, p\} = \sum_{i=1}^{n} \frac{\partial h}{\partial \xi_i} \frac{\partial p}{\partial x_i} - \frac{\partial h}{\partial x_i} \frac{\partial p}{\partial \xi_i} \geq \frac{B |\xi|}{(x)^2} - \frac{1}{B} \quad \text{for all } x, \xi \in \mathbb{R}^n.
\]

It is worth noting that in the case that the coefficients \( a_{jk}(x) \) are elliptic, one can use the fact that the symbol \( h \) is preserved under the Hamiltonian flow together with the ellipticity to deduce that \( C^{-2} |\xi|^2 \leq |\Xi(s, x, \xi)|^2 \leq C^2 |\xi|^2 \). This implies that the solutions \( X \) and \( \Xi \) exist for all times.

For our purposes we need a version of Doi’s lemma that is not centered at the origin. As before we let \( \mathbb{R}^n = \bigcup_{\mu \in \mathbb{Z}^n} Q_{\mu} \) with \( Q_{\mu} \) unit cubes with vertices in the lattice \( \mathbb{Z}^n \) (indexed by some corner), and let \( x_{\mu} \) be the center of \( Q_{\mu} \).

**Lemma 2.2.** Suppose \( a_{jk} \) satisfies \((D1)-(D3)\). Then there exists a symbol \( p_{\mu} \in S^0_{1,0} \) such that \( H_h p_{\mu}(x, \xi) \geq C_1 \frac{\xi^2}{(x-x_{\mu})^2} - C_2 \), where \( C_1, C_2 \) and the semi-norms of \( p_{\mu} \) can be bounded independently of \( \mu \).
Proof. For $|x_\mu| < 10$ we take $p_\mu = p$. The content of the lemma is for $|x_\mu| >> 0$. Let \( h(x, \xi) = |\xi|^2 \), applying Lemma 2.1, to the Laplacian can find a symbol \( r(x, \xi) \) so that \( H_r r(x, \xi) \geq C_1 |\xi|^2 - \tilde{C}_2 \).

We take \( p_\mu(x, \xi) = Np(x, \xi) + r(x - x_\mu, \xi) \), with \( N \) to be determined. Let \( r_\mu(x, \xi) = r(x - x_\mu, \xi) \). We calculate

\[
H_h r_\mu(x, \xi) = \sum_{i=1}^n (2a_{ik}(x) \xi_k) \frac{\partial r_\mu}{\partial x_i} + \frac{\partial a_{ik}(x) \xi_k}{\partial x_i} \frac{\partial r_\mu}{\partial \xi_i} \\
= \sum_{i=1}^n 2 \xi_i \frac{\partial r_\mu}{\partial x_i} + 2(a_{ik}(x) - \delta_{ik}) \xi_k \frac{\partial r_\mu}{\partial x_i} + \frac{\partial a_{ik}(x) \xi_k}{\partial x_i} \frac{\partial r_\mu}{\partial \xi_i}.
\]

For the second term we have that \( 2(a_{ik} - \delta_{ik}) \xi_k \frac{\partial r_\mu}{\partial x_i} \leq C_1 |\xi|^2 \) from (D4) and the bounds for the semi-norms of \( r_\mu \). Similarly for the third term we have that \( \frac{\partial a_{ik}(x) \xi_k}{\partial x_i} \frac{\partial r_\mu}{\partial \xi_i} \leq C_1 |\xi|^2 \). The first term gives that \( H_h r_\mu \geq \tilde{C}_1 \frac{|\xi|}{(x - \xi)^2} - \tilde{C}_2 \).

Finally,

\[
H_h(p_\mu) = NH_h(p) + H_h(r_\mu) \geq NC_1 \frac{|\xi|}{(x - \xi)^2} - C_1 \frac{|\xi|}{(x - x_\mu)^2} + \tilde{C}_1 \frac{|\xi|}{(x - x_\mu)^2} - NC_2 - \tilde{C}_2.
\]

So we choose \( N \) large enough so that \( NC_1 \frac{|\xi|}{(x - \xi)^2} - C_1 \frac{|\xi|}{(x - x_\mu)^2} \geq 0 \) and we get

\[
H_h(p_\mu) \geq C_1 \frac{|\xi|}{(x - x_\mu)^2} - \tilde{C}_2.
\]

\[\square\]

Remark 2.3. We remark that, perhaps by increasing our choice of \( N \) to \( 2N \), we can ensure that \( H_h p_\mu \geq C_1 \frac{|\xi|}{(x - x_\mu)^2} + C_2 \frac{|\xi|}{(x - x_\mu)^2} - C_3 \). This will be important to us when we want to consider linear estimates where the coefficients of the second order term depend on time.

3. Linear Results

In this section we consider the system

\[
\begin{aligned}
\partial_t u &= -\epsilon \Delta u + i \partial_x a_{jk}(x, t) \partial_{x_k} u + b_{1}(x, t, D) u + \tilde{b}_2(x, t) \cdot \nabla \hat{u} \\
&\quad + c_{1}(x, t, D) u + c_{2}(x, t, D) \hat{u} + f(x, t), \\
\hat{u}(x, 0) &= u_0(x).
\end{aligned}
\]

(5)

First we set some notation. We denote by \( A(x, t) \) the matrix \( (a_{jk}(x, t))_{j,k=1}^n \) and the symbol \( h(x, t, \xi) = \sum_{j,k=1}^n a_{jk}(x, t) \xi_j \xi_k \). For a function \( u(x, t) \) the Fourier transform of \( u \) in the \( x \) variable will be denoted by \( \hat{u}(\xi, t) \). For a time varying symbol \( q(x, t, \xi) \) we use the following notation

\[
\Psi_q u(x, t) = \frac{1}{(2\pi)^n} \int e^{ix \cdot \xi} q(x, t, \xi) \hat{u}(\xi, t) d\xi.
\]

We let \( \mathbb{R}^n = \bigcup_{\mu \in \mathbb{Z}^n} Q_\mu \) with \( Q_\mu \) unit cubes with vertices in the lattice \( \mathbb{Z}^n \). We let \( x_\mu \) denote the the center of \( Q_\mu \) and \( Q_\mu^* \) denote its concentric double.

When we use the linear estimates in the non-linear problem we will evaluate our coefficients at some local solution. For this reason it will be important for the constant appearing in our final inequality to depend only on the coefficients at \( t = 0 \). We therefore take the convention that constants related to our coefficients at
$t = 0$ will be denoted by $C_0$ and constants depending on our coefficients at times other then 0 will be generically denoted by $C$.

We place the following assumptions on the coefficients.

(L1) There exist $M_L = M_L(n) \in \mathbb{N}$, and $C > 0$ so that $a_{jk}(\cdot, t), b_{2,j}(\cdot, t), \partial_t a_{jk}(\cdot, t), \partial_t b_{2,j}(\cdot, t) \in \mathcal{C}^{M_L}_b(\mathbb{R}^n)$ for $j, k = 1, 2, \ldots, n$, with norm controlled by $C$. We assume that, uniformly in $t$, the symbols $c_1(x, t, \xi), c_2(x, t, \xi) \in S^0_{1,0}$ and $b_1(x, t, \xi) \in S^1_{1,0}$ with seminorms controlled by $C$. In addition, we have that the norms of $a_{jk}(x, 0), b_{2,j}(x, 0)$ for $j, k = 1, \ldots, n$ in $\mathcal{C}^{M_L}_b$ together with the seminorms of $b_1(x, 0, \xi), c_1(x, 0, \xi)$ and $c_2(x, 0, \xi)$ are controlled by $C_0$.

(L2) The matrix $A(x, t) = (a_{jk}(x, t))_{j,k=1 \ldots n}$ has real valued entries, is symmetric, and is uniformly elliptic. That is, there is a positive number $C$ so that

$$C^{-1} |\xi|^2 \leq \sum_{j,k=1}^n a_{jk}(x, t) \xi_j \xi_k \leq C |\xi|^2.$$

Further at $t = 0$ we have

$$C_0^{-1} |\xi|^2 \leq \sum_{j,k=1}^n a_{jk}(x, 0) \xi_j \xi_k \leq C_0 |\xi|^2.$$

(L3) The matrix $A(x, t)$ is asymptotically flat. That is,

$$|I - A(x, t)| + |\nabla_x A(x, t)| + |\partial_t a_{jk}(x, t)| + |\partial_t b_{2,j}(x, t)| \leq \frac{C}{(x)^2}$$

and

$$|I - A(x, 0)| + |\nabla_x A(x, 0)| + |\partial_t a_{jk}(x, 0)| + |\partial_t b_{2,j}(x, 0)| \leq \frac{C_0}{(x)^2}.$$

(L4) The symbol $b_1(x, t, \xi)$ satisfies an estimate of the form

$$|\text{Re} b_1(x, 0, \xi)| \leq \sum_{\mu \in \mathbb{Z}^n} \beta^0_{\mu} |\varphi_{\mu}(x)| \xi|,$$

where $\beta^0_{\mu} \geq 0$, $\sum_{\mu \in \mathbb{Z}^n} \beta^0_{\mu} \leq C_0$, $\|\varphi_{\mu}\|_{C^{M_L}} \leq 1$ and supp $\varphi_{\mu} \subseteq Q^*_\mu$. Also assume that

$$\partial_t (\text{Re} b_1(x, t, \xi)) = \sum_{\mu \in \mathbb{Z}^n} \tilde{\beta}_\mu(t) \varphi_{\mu}(x, t, \xi).$$

where $\tilde{\beta}_\mu(t) \geq 0$, and $\sum_{\mu \in \mathbb{Z}^n} \tilde{\beta}_\mu(t) \leq C$. For all $t \in \mathbb{R}_+$ the time varying symbols $\varphi_{\mu}(x, t, \xi) \in S^1_{1,0}$ with seminorms bounded by 1, independently of $t$ and $\mu$, and supp $\varphi_{\mu}(\cdot, t, \xi) \subseteq Q^*_\mu$.

(L5) We assume that Hamiltonian flow associated to $h_0(x, \xi) := h(x, 0, \xi)$ is non-trapping. Let $p_{\mu}(x, \xi)$ be the Doi symbol for cube $Q^*_\mu$ associated to as constructed in the previous section. We assume that these symbols satisfy

$$H_{h_0} p_{\mu} \geq \frac{1}{C_0} \left( \frac{|\xi|^2}{(x)^2} + \frac{|\xi|^2}{(x - x_\mu)^2} \right) - C_0.$$

The bounds in our arguments also depend on a finite number of seminorms of $p_{\mu}$ in $S^0_{1,0}$ and we assume these seminorms are controlled by $C_0$. See Remark 2.3 in Section 2 for this version of Doi’s Lemma.
**Theorem 3.1.** Suppose that $u_0 \in L^2$ and there exists a solution $u(x, t)$ to (3) in $C([0, T]; L^2)$, where the coefficients satisfy $[L1][L5]$. Then there exist real numbers $T = T(C, C_0, \beta_0^\mathbb{Z})$ and $A = A(C_0, \beta_0^\mathbb{Z})$ such that if $f \in L^1([0, T]; L^2)$, then $u$ satisfies

$$
\sup_{0 \leq t \leq T} \|u(\cdot, t)\|_2^2 + \sup_{\mu \in \mathbb{Z}} \|J^{1/2}u\|_{L^2(\mathbb{T}^n \times [0, T])}^2 \leq A \left( \|u_0\|_2^2 + \left( \int_0^T \|f(\cdot, t)\|_2^2 \, dt \right)^2 \right).
$$

*Proof.* We break the proof of this theorem into several steps.

**Step 1. Reduction to a system.**

Let $\tilde{w} = \begin{pmatrix} u \\ \bar{u} \end{pmatrix}$, $\bar{f} = \begin{pmatrix} f \\ 0 \end{pmatrix}$, and $w_0 = \begin{pmatrix} u_0 \\ \bar{u}_0 \end{pmatrix}$. Let $\mathcal{L}(x, t)$ denote the operator $\partial_{x_j}(a_{jk}(x, t)\partial_{x_k})$. Then using the equations for $u$ and $\bar{u}$, we see that $w$ satisfies

$$
\begin{cases}
\partial_t \bar{w} = -i\Delta^2 \bar{w} + (iH + B + C) \bar{w} + \bar{f}(x, t), \\
\bar{w}(x, 0) = \bar{w}_0(x),
\end{cases}
$$

where

$$
H = \begin{pmatrix} \mathcal{L}(x, t) & 0 \\ 0 & -\mathcal{L}(x, t) \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \begin{pmatrix} b_{11}(x, t, D) & b_{12}(x, t, \nabla) \\ b_{21}(x, t, \nabla) & b_{22}(x, t, D) \end{pmatrix},
$$

and $C = \begin{pmatrix} c_{11}(x, t, D) & c_{12}(x, t, D) \\ c_{21}(x, t, D) & c_{22}(x, t, D) \end{pmatrix}$.

Note that for the rest of this chapter $\langle \bar{u}, \bar{v} \rangle = \int u_1 \bar{v}_1 + u_2 \bar{v}_2 \, dx$ and $\|\bar{w}\|_2^2 = \langle \bar{u}, \bar{u} \rangle$.

**Step 2. Diagonalize the first order terms.**

We now define an operator $S = \begin{pmatrix} 0 & s_{12} \\ s_{21} & 0 \end{pmatrix}$ where $s_{12}$ and $s_{21}$ will be defined to be time varying PDO’s and have symbols in $S^{-1}_{1,0}$ uniformly in $t$. We being by choosing $\phi \in \mathcal{C}^\infty_0(\mathbb{R}^n)$ so that $\phi(y) = 1$ for $|y| < 1$ and $\phi(y) = 0$ for $|y| \geq 2$. Let $\theta_R(\xi) = 1 - \phi(\xi/R)$ and $\theta(\xi) = \theta_1(\xi)$.

Let $\hat{h}(x, t, \xi) = \theta_R(\xi)\hat{h}^{-1}(x, t, \xi)$. Notice that, by ellipticity $h(x, t, \xi) \geq C^{-1} |\xi|^2$, hence $\hat{h}$ is a smooth function. Let $\hat{\mathcal{L}} = \Psi_{\hat{h}}$, then we see that $\hat{\mathcal{L}}\mathcal{L} = I + \Psi_{r_1}$, with $r_1 \in S^{-1}_{1,0}$ uniformly in $t$.

We define $S_{12} = \frac{1}{2}iB_{12}\hat{\mathcal{L}}$ and $S_{21} = -\frac{1}{2}iB_{21}\hat{\mathcal{L}}$. We denote the symbols of $S_{12}$ and $S_{21}$ by $s_{12}(x, t, \xi)$ and $s_{21}(x, t, \xi)$ respectively. Clearly $s_{ij}(x, t, \xi) \in S^{-1}_{1,0}$ uniformly in $t$. Let $\Lambda = I - S$, if we choose $R$ large enough, then we can control the norms of $\Lambda$ and $\Lambda^{-1}$ by constants (see Kenig’s Park City Lecture 2 [8]).

We will use $\Lambda$ to change variables, and the resulting system will have diagonal first order terms. We first perform some calculations that are necessary to rewrite the system in terms of $\Lambda u$.

$$
-i\Lambda H + i\Lambda H = -iH + iH - i\mathcal{L} = -i(\mathcal{L} - \mathcal{L}) + \begin{pmatrix} 0 & s_{12} \\ s_{21} & 0 \end{pmatrix} \begin{pmatrix} \mathcal{L} & 0 \\ 0 & -\mathcal{L} \end{pmatrix} \begin{pmatrix} 0 & s_{12} \\ s_{21} & 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 & iS_{12} \mathcal{L} + i\mathcal{L}S_{12} \\ -iS_{21} \mathcal{L} - i\mathcal{L}S_{21} \end{pmatrix}.
$$


Notice that \( \mathcal{L}S_{12} = S_{12} \mathcal{L} + E_{12}^0 \), where \( E_{12}^0 \) is an error of order 0. Similarly \( \mathcal{L}S_{21} = S_{21} \mathcal{L} + E_{21}^0 \) with \( E_{21}^0 \) of order 0.

Hence \( iS_{12} \mathcal{L} + iE_{12}^0 = -B_{12} \mathcal{L} + iE_{12}^0 = -B_{12} + E_{12}^0 \) and \( -iS_{21} \mathcal{L} - iE_{21}^0 = -B_{21} \mathcal{L} + iE_{21}^0 = -B_{21} + E_{21}^0 \) with \( E_{12}^0 \) and \( E_{21}^0 \) errors of order 0.

We write \( B = B_d + B_{ad} \) where \( B_d = (B_{12}^0, 0, 0, 0) \) and \( B_{ad} = (0, B_{12}^0, 0, 0) \). Now,
\[
\Lambda B_{ad} = IB_{ad} - SB_{ad} - \left( \begin{array}{cccc} S_{12}B_{21} & 0 & 0 & 0 \\ 0 & S_{21}B_{12} & 0 & 0 \end{array} \right) = B_{ad} + E_{ad}^0
\]
with \( E_{ad}^0 \) of order 0.

For the other terms we will want to commute \( \Lambda \) and our operators, in order to derive the equation for \( \bar{w} \).

Starting with \( \Lambda \partial_t = \partial_t \Lambda + \partial_t \Lambda \) where this last expression is the matrix of \( \Psi \text{DO's} \) whose symbols are given by \( \partial \mathring{\psi} \). Using the bounds for \( \partial \psi \) and \( \partial a_{jk} \) we see that these symbols are uniformly in \( S_{1,0} \).

Notice that \( \Lambda B_d = B_d - SB_d \) and \( B_d = B_d \Lambda + B_d S \), so that \( \Lambda B_d = B_d \Lambda + B_d S - SB_d = B_d \Lambda + E_{d}^0 \) with \( E_{d}^0 \) is of order 0.

\[\Lambda \epsilon I \Delta^2 - \epsilon \left( \begin{array}{cccc} 0 & \Delta^2 S_{12} - S_{12} \Delta^2 & 0 \\ \Delta^2 S_{21} - S_{21} \Delta^2 & 0 & 0 \end{array} \right) = \epsilon \Delta^2 I \Lambda + \epsilon \bar{R}\]
where \( \bar{R} \) is a matrix whose entries are operators of order 2.

We set \( R = R \Lambda^{-1} \), which is still of order 2. We write
\[\Lambda C + \partial_t S + E_{ad}^0 + E_d^0 = (\Lambda C \Lambda^{-1} + \partial_t \Lambda \Lambda^{-1} + E_{ad}^0 \Lambda^{-1} + E_d^0 \Lambda^{-1}) \Lambda =: \tilde{\Lambda} \Lambda\]
with \( \tilde{\Lambda} \) of order 0.

Lastly set \( \tilde{F} = \Lambda \tilde{F} \). Define \( \tilde{z} = \Lambda \bar{w} \) and apply \( \Lambda \) to our equation. We have
\[\Lambda \partial_t \bar{w} = i\epsilon \Delta^2 I \bar{w} + (i\Lambda H + \Lambda B + \Lambda C) \bar{w} + \Lambda \bar{F}\]
Using our calculations above we have
\[\partial_t \tilde{z} = -\epsilon \Delta^2 I \tilde{z} + \epsilon R \tilde{z} + iH \tilde{z} - \left( \begin{array}{cccc} 0 & B_{12} & 0 & 0 \\ B_{21} & 0 & 0 & 0 \end{array} \right) \bar{w} + B_d \tilde{z} + B_{ad} \bar{w} + \tilde{C} \tilde{z} + \tilde{F}.
Hence if we set \( \tilde{z}_0 = \Lambda \bar{w}_0 \) to arrive at a system with diagonal first order terms, namely
\[
\begin{cases}
\partial_t \tilde{z} = -\epsilon \Delta^2 I \tilde{z} + \epsilon R \tilde{z} + iH \tilde{z} + B_d \tilde{z} + \tilde{C} \tilde{z} + \tilde{F}, \\
\tilde{z}(x,0) = \bar{z}_0(x).
\end{cases}
\]
As we pointed out earlier we have control of the norms of \( \Lambda \) and \( \Lambda^{-1} \), so deriving our desired estimates for \( \tilde{z} \) will imply the estimates for \( \bar{w} \).

Since we work in slightly unusual norms it seems a good time to recall them and justify this last statement.

**Definition 3.2.** Let \( \mathbb{R}^n = \bigcup_{\mu \in \mathbb{Z}^n} Q_\mu \) as usual. Let \( f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C} \) be measurable function. We define
\[
\|||f|||_T = \sup_{\mu \in \mathbb{Z}^n} \|f\|_{L^2(Q_\mu \times [0,T])}
\]
and
\[ |||f|||_T' = \sum_{\mu \in \mathbb{Z}^n} \|f\|_{L^2(Q_\mu \times [0,T])}. \]

**Theorem 3.3.** For \( a \in S^0_{1,0} \) there is an \( N(n) \) so that \( |||\Psi_a f|||_T \leq C|||f|||_T \) and \( |||\Psi_a f|||_T \leq C|||f|||_T \).

**Proof.** See Kenig’s Park City Lecture notes, Lecture 2 [8]. \( \square \)

Now we return to our linear estimates. It is important for the non-linear theory that we only make our non-trapping assumptions at \( t = 0 \). The following lemma allows us to handle time varying leading coefficients.

**Lemma 3.4.** Let \( h(x, \xi) = a_{jk}(x, 0)\xi_j\xi_k \) and let \( p_\mu \) be the Doi symbol corresponding to \( h \) centered at cube \( Q_\mu \). Then there exists a \( T_1 = T_1(C, C_0) \) so that uniformly for all \( t < T_1 \) the time varying symbol \( h_t(x, \xi) = a_{jk}(x, t)\xi_j\xi_k \) satisfies
\[ H_{h_t}p_\mu(x, \xi) \geq \frac{1}{C_0} \frac{|\xi|}{\langle x \rangle^2} - C_0. \]

**Proof.** By direct calculation we have
\[
H_{h_t}p_\mu = \sum_{i=1}^n \frac{\partial h}{\partial \xi_i} \frac{\partial p_\mu}{\partial x_i} - \frac{\partial h}{\partial x_i} \frac{\partial p_\mu}{\partial \xi_i} + \left( \frac{\partial h_i}{\partial \xi_i} - \frac{\partial h}{\partial \xi_i} \right) \frac{\partial p_\mu}{\partial x_i} - \left( \frac{\partial h_i}{\partial x_i} - \frac{\partial h}{\partial x_i} \right) \frac{\partial p_\mu}{\partial \xi_i}
\]
\[
= H_{h_t}p_\mu + \sum_{i=1}^n 2 \left( a_{ik}(x, t) - a_{ik}(x, 0) \right) \xi_k \frac{\partial p_\mu}{\partial x_i} - \left( \frac{\partial a_{jk}(x, t)}{\partial x_i} - \frac{\partial a_{jk}(x, 0)}{\partial x_i} \right) \xi_j \xi_k \frac{\partial p_\mu}{\partial \xi_i}.
\]

From the asymptotic flatness condition (L3) there is a \( T_1 = T_1(C, C_0) \) so that if \( t < T_1 \) we have that we have that
\[
\left| \sum_{i=1}^n 2 \left( a_{ik}(x, t) - a_{ik}(x, 0) \right) \xi_k \frac{\partial p_\mu}{\partial x_i} - \left( \frac{\partial a_{jk}(x, t)}{\partial x_i} - \frac{\partial a_{jk}(x, 0)}{\partial x_i} \right) \xi_j \xi_k \frac{\partial p_\mu}{\partial \xi_i} \right| \leq \frac{1}{C_0} \frac{|\xi|}{\langle x \rangle^2}.
\]

Now using (L5) we get that
\[
H_{h_t}p_\mu \geq H_{h_t}p_\mu - \frac{1}{C_0} \frac{|\xi|}{\langle x \rangle^2} \geq \frac{1}{C_0} \frac{|\xi|}{\langle x - x_\mu \rangle^2} - C_0.
\]

**Step 3. Energy Estimates.**

The goal of this section is to conclude the proof. The program is to again introduce an invertible change of variables, this time based on Doi’s Lemma. It is Doi’s lemma that allows us to absorb the first order terms.

Set
\[
\Psi_M = \begin{pmatrix} \Psi_{q_1} & 0 \\ 0 & \Psi_{q_2} \end{pmatrix}
\]
where \( \Psi_{q_1}, \Psi_{q_2} \) are invertible \( \Psi \)DO’s of order 0 that will be defined below.

First we compute the necessary commutators that arise in the change of variables. For the leading order terms
\[
\Psi_M \epsilon \Delta^2 I = \epsilon \Delta^2 I \Psi_M + \epsilon \hat{R}^3 \Psi_M
\]
with $\tilde{R}^3$ of order 3. The second order remainder term yields. $\Psi_M\varepsilon R = \varepsilon R\Psi_M + \varepsilon\tilde{R}^1\Psi_M$ We collect these remainder terms by setting $R^3 = \tilde{R}^3 + R + \tilde{R}^1$, which is of order 3. The remaining terms pose no difficulty. The first and zeroth order terms simply give $\Psi$ order 3. The remaining terms pose no difficulty. The first and zeroth order terms yield $\Psi_M = \tilde{C}\Psi_M + E_0^0$ and lastly we set $\tilde{C} = \Psi_M\tilde{F}$.

Again we absorb the error terms of order 0 into $\tilde{C}$. By setting $\tilde{a} = \Psi_M\tilde{z}$ and $\tilde{a}_0 = \Psi_M\tilde{z}_0$ we arrive at the system

$$
\begin{align*}
\partial_t \tilde{a} &= -\epsilon \Delta^2 I \tilde{a} + \epsilon R^3 \tilde{a} + i[H, \Psi_M] \tilde{z} + iH\tilde{a} + B_d \tilde{a} + \tilde{C}\tilde{a} + \tilde{G} \\
\tilde{a}(x, 0) &= \tilde{a}_0(x).
\end{align*}
$$

(7)

To construct $\Psi_M$ we let $\mathbb{R}^n = \bigcup_{\mu \in \mathbb{Z}^n} Q_\mu$ as usual. Fix a cube $Q_{\mu_0}$ and let

$$
\gamma_{\mu_0}(x, \xi) = p_{\mu_0}(x, \xi) + \sum_{\mu \in \mathbb{Z}^n} \beta_{\mu} q_{\mu}(x, \xi)
$$

with $\beta_{\mu}$ as in (L4). Notice that $\gamma_{\mu_0} \in S_{1,0}^0$ with seminorms controlled in terms of $C_0$.

Let $q_1(x, \xi) = \exp(\theta_R(\xi)\tilde{C}_0\gamma_{\mu_0}(x, \xi))$ and $q_2(x, \xi) = \exp(-\theta_R(\xi)\tilde{C}_0\gamma_{\mu_0}(x, \xi))$. Where $\tilde{C}_0$ depends on $C_0$ will be chosen below. Notice that again, if we take $R$ large we may ensure that $\Psi_M$ is invertible uniformly in $\mu_0$. We now compute

$$
-i[H, \Psi_M] = -i \begin{pmatrix} \mathcal{L} q_1 - \Psi_{q_1} \mathcal{L} & 0 \\ 0 & -\mathcal{L} q_2 + \Psi_{q_2} \mathcal{L} \end{pmatrix}.
$$

Let $\ell(x, t, \xi)$ be the symbol for $\mathcal{L}$, then $\ell(x, t, \xi) = a_{jk}(x, t)\xi_j \xi_k + \partial_x a_{jk}(x, t)\xi_k = h_t(x, \xi) + \ell_1(x, t, \xi)$. Note that $\{\ell_1, q_1\} \in S_{1,0}^0$ uniformly in $t$ for $i = 1, 2$. Hence,

$$
-i (\mathcal{L} q_1 - \Psi_{q_1} \mathcal{L}) = \Psi_{\{q_1, q_1\}} + E_7^0,
$$

with $E_7^0$ an operator of order 0.

It follows that

$$
\{h_t, q_1\} = \left( \frac{\partial h_t}{\partial \xi_k} \theta_R(\xi) \frac{\partial \gamma_{\mu_0}}{\partial x_i} - \frac{\partial h_t}{\partial x_i} \theta_R(\xi) \frac{\partial \gamma_{\mu_0}}{\partial \xi_k} - \frac{\partial h_t}{\partial \xi_k} \theta_R(\xi) \frac{\partial \gamma_{\mu_0}}{\partial x_i} \right) e^{\theta_R(\xi)\gamma_{\mu_0}},
$$

where the last term in the parentheses is in $S_{1,0}^{-\infty}$. Therefore

$$
-i (\mathcal{L} q_1 - \Psi_{q_1} \mathcal{L}) = -\Psi h_t \gamma_{\mu_0} \Psi q_1 + E_8^0,
$$

with $E_8^0$ of order 0. In the same way

$$
-i (-\mathcal{L} q_2 + \Psi_{q_2} \mathcal{L}) = -\Psi h_t \gamma_{\mu_0} \Psi q_2 + E_9^0.
$$

Thus our system (after absorbing errors into $\tilde{C}$) looks like

$$
\begin{align*}
\partial_t \tilde{a} &= -\epsilon \Delta^2 I \tilde{a} + \epsilon R^3 \tilde{a} + \left( -\Psi h_t \gamma_{\mu_0} \Psi_{q_1} + \Psi h_t \gamma_{\mu_0} \Psi_{q_2} \right) \tilde{a} \\
&\quad + iH\tilde{a} + B_d \tilde{a} + \tilde{C}\tilde{a} + \tilde{G}, \\
\tilde{a}(x, 0) &= \tilde{a}_0(x).
\end{align*}
$$
We now proceed to derive energy estimates for \(\alpha\). Consider

\[
\partial_t \langle \vec{\alpha}, \vec{\alpha} \rangle = \langle \partial_t \vec{\alpha}, \vec{\alpha} \rangle + \langle \vec{\alpha}, \partial_t \vec{\alpha} \rangle
\]

\[
= \big( -\epsilon \Delta^2 \vec{\alpha} - \vec{\alpha} \big) + \langle \vec{\alpha}, -\epsilon \Delta^2 \vec{\alpha} \rangle + \langle \vec{\alpha}, \epsilon R^3 \vec{\alpha} \rangle + \langle \vec{\alpha}, \epsilon R^3 \vec{\alpha} \rangle + \langle iH \vec{\alpha}, \vec{\alpha} \rangle + \langle \vec{\alpha}, iH \vec{\alpha} \rangle + \langle B_d \vec{\alpha}, \vec{\alpha} \rangle + \langle \vec{\alpha}, B_d \vec{\alpha} \rangle + \langle \vec{C} \vec{\alpha}, \vec{\alpha} \rangle + \langle \vec{\alpha}, \vec{C} \vec{\alpha} \rangle + \langle \vec{\alpha}, \vec{C} \vec{\alpha} \rangle + \langle \vec{\alpha}, \vec{C} \vec{\alpha} \rangle + \langle \vec{\alpha}, \vec{C} \vec{\alpha} \rangle
\]

\[
\left\langle \left( -\Psi_{\theta R}(\xi) H_{\eta} \gamma_{\mu_0}, \gamma_{\mu_0} \right) \vec{\alpha}, \vec{\alpha} \right\rangle + \left\langle \left( -\Psi_{\theta R}(\xi) H_{\eta} \gamma_{\mu_0}, \gamma_{\mu_0} \right) \vec{\alpha}, \vec{\alpha} \right\rangle.
\]

In the first two terms we have we have that 

\[ -\epsilon \langle \Delta^2 \vec{\alpha}, \vec{\alpha} \rangle - \epsilon \langle \vec{\alpha}, \Delta^2 \vec{\alpha} \rangle = -2\epsilon \langle \Delta I \vec{\alpha}, \Delta I \vec{\alpha} \rangle = -2\epsilon \| \Delta \vec{\alpha} \|^2. \]

The second two terms contribute 

\[ \langle \vec{\alpha}, \epsilon R^3 \vec{\alpha} \rangle + \langle \vec{\alpha}, \epsilon R^3 \vec{\alpha} \rangle = 2\epsilon \text{Re} \langle R^3 \vec{\alpha}, \vec{\alpha} \rangle = 2\epsilon \text{Re} \langle J^{-3/2} R^3 \vec{\alpha}, J^{3/2} \vec{\alpha} \rangle. \]

As both \( J^{3/2} \) and \( J^{-3/2} R^3 \) are operators of order 3/2 we can bound this by

\[ C \| \vec{\alpha} \|^2_{L^{3/2}}. \]

Now by interpolation we have that 

\[ \| \vec{\alpha} \|^2_{L^{3/2}} \leq \eta_0 \| \Delta I \vec{\alpha} \|^2_2 + \frac{2}{\eta_0} \| \vec{\alpha} \|^2_2. \]

Hence

\[ |2\epsilon \text{Re} \langle R^3 \vec{\alpha}, \vec{\alpha} \rangle| \leq 2\epsilon C \eta_0 \| \Delta I \vec{\alpha} \|^2_2 + \frac{4\epsilon C \| \vec{\alpha} \|^2_2}{\eta_0}. \]

By setting \( \eta_0 = 1/(2C) \) we can absorb the first term into 

\[ -\epsilon \| \Delta \vec{\alpha} \|^2 \] to get the first four terms are bounded by 

\[ -\epsilon \| \Delta \vec{\alpha} \|^2 + 8\epsilon C^2 \| \vec{\alpha} \|^2_2. \]

We now turn our attention to first order terms. That is, the last two terms and the terms involving \( B_d \). Consider the matrix of symbols

\[
F := \begin{pmatrix} -\theta R(\xi) H_{\eta} \gamma_{\mu_0}(x, \xi) + b_1(x, t, \xi) & 0 \\ 0 & -\theta R(\xi) H_{\eta} \gamma_{\mu_0}(x, \xi) + b_2(x, t, -\xi) \end{pmatrix}
\]

We will need to control \( F + F^* \) to apply the vector valued Gårding’s inequality.

Let \( \phi Q^*_\mu \) be a smooth cut off to the double of \( Q_\mu \) and let \( \chi Q^*_\mu = \phi^2 Q^*_\mu \). By our construction of \( \gamma_{\mu_0} \) we have that

\[
-\hat{C}_0 \theta R(\xi) H_{\eta} \gamma_{\mu_0} \leq \hat{C}_0 \theta R(\xi) \left( -\frac{1}{C_0} \frac{|\xi|}{(x - x_{\mu_0})^2} + C_0 \\ -\sum_{\mu \in \mathbb{Z}^n} \beta^0_{\mu} \left( \frac{1}{C_0} \frac{|\xi|}{(x - x_{\mu})^2} - C_0 \right) \right)
\]

\[
\leq \hat{C}_0 \theta R(\xi) \left( -C'_0 |\xi| \chi Q^*_\mu - \sum_{\mu \in \mathbb{Z}^n} \beta^0_{\mu} C_0' |\xi| \chi Q^*_\mu \right) + \hat{C}_0 C_0''
\]

\[
\leq \hat{C}_0 \theta R(\xi) \left( -C'_0 |\xi| \chi Q^*_\mu - C'_0 |\text{Re} b_1(x, 0, \xi)| \right) + \hat{C}_0 C_0''.
\]
Here we choose \( \tilde{C}_0 \) so that \( \tilde{C}_0 C'_0 \geq 2 \). Now we have that

\[
- \theta_R(\xi) H_{\gamma \mu \nu} + 2 \text{Re} \, b_1(x, t, \xi) = \\
(\theta_R(\xi) H_{\gamma \mu \nu} + 2 \text{Re} \, b_1(x, 0, \xi)) + 2 \text{Re} \, \int_0^t \partial_t b_1(x, s, \xi) \, ds \\
\leq - \tilde{C}_0' \theta_R(\xi) |\xi| \chi_{Q_{\nu_0}} + \tilde{C}_0'' + 2 \int_0^t \left( \sum_{\mu \in \mathbb{Z}^n} \tilde{c}_\mu^*(s) \varphi_\mu(x, s, \xi) \right) \, ds.
\]

Let \( p(x, t, \xi) = 2 \int_0^t \left( \sum_{\mu \in \mathbb{Z}^n} \tilde{c}_\mu^*(s) \varphi_\mu(x, s, \xi) \right) \, ds \). Apply the vector valued Gårding inequality (see [14, 17]) to get

\[
\text{Re} \, \left\langle \left( -\Psi_{\theta_R}(\xi) H_{\gamma \mu \nu} + b_1(x, t, D) - \psi_{\theta_R}(\xi) H_{\gamma \mu \nu} + b_1(x, t, D) \right) \tilde{\alpha}, \tilde{\alpha} \right\rangle \\
\leq C \|\tilde{\alpha}\|^2_2 - \text{Re} \, \left\langle \left( \begin{array}{c} \Psi_{\theta_R}(\xi) \chi_{Q_{\nu_0}} \\ 0 \end{array} \right), \tilde{\alpha} \right\rangle \left\langle \left( \begin{array}{c} 0 \\ \Psi_{\theta_R}(\xi) \chi_{Q_{\nu_0}} \end{array} \right), \tilde{\alpha} \right\rangle + \\
\text{Re} \, \left\langle \left( \begin{array}{c} \Psi_{p(x, t, \xi)} \\ 0 \end{array} \right), \tilde{\alpha} \right\rangle \left\langle \left( \begin{array}{c} 0 \\ \Psi_{p(x, t, -\xi)} \end{array} \right), \tilde{\alpha} \right\rangle.
\]

We denote this last term by \( \text{Re} \langle E_M \tilde{\alpha}, \tilde{\alpha} \rangle \). The symbol of the operator \( \Psi_{\theta_R}(\xi) \chi_{Q_{\nu_0}} - J^{1/2} \chi_{Q_{\nu_0}} J^{1/2} \) is in \( S_{0, 0}^0 \). Hence we have

\[
\text{Re} \, \left\langle \left( \begin{array}{c} \Psi_{\tilde{c}_0 \theta_R}(\xi) \chi_{Q_{\nu_0}} \\ 0 \end{array} \right), \tilde{\alpha} \right\rangle \left\langle \left( \begin{array}{c} 0 \\ \Psi_{\tilde{c}_0 \theta_R}(\xi) \chi_{Q_{\nu_0}} \end{array} \right), \tilde{\alpha} \right\rangle \\
\geq \langle \phi_{Q_{\nu_0}} J^{1/2} I \tilde{\alpha}, \phi_{Q_{\nu_0}} J^{1/2} I \tilde{\alpha} \rangle - C \|\tilde{\alpha}\|^2_{L^2} \\
\geq \|\phi_{Q_{\nu_0}} J^{1/2} I \tilde{\alpha}\|_{L^2(Q_{\nu_0})} - C \|\tilde{\alpha}\|^2_{L^2}.
\]

To handle the terms involving \( H \) notice that \( \int L \alpha_1 \alpha_1 = \int \alpha_1 L \alpha_1 \), and hence \( i \langle H \tilde{\alpha}, \tilde{\alpha} \rangle - i \langle \tilde{\alpha}, H \tilde{\alpha} \rangle = 0 \). For the terms involving \( \tilde{C} \) we use Cauchy-Schwartz inequality, \( \langle \tilde{C} \tilde{\alpha}, \tilde{\alpha} \rangle \leq \|\tilde{C}\| \|\tilde{\alpha}\| \leq C \|\tilde{\alpha}\|^2_2 \).

Putting all this together we see that

\[
\frac{d}{dt} \|\tilde{\alpha}\|^2_2 \leq -\varepsilon \|\Delta I \tilde{\alpha}\|^2_2 + C \|\tilde{\alpha}\|^2_2 - \|J^{1/2} I \tilde{\alpha}\|^2_{L^2(Q_{\nu_0})} + 2 \text{Re} \left\langle \tilde{\alpha}, \tilde{G} \right\rangle + \text{Re} \left\langle E_M \tilde{\alpha}, \tilde{\alpha} \right\rangle.
\]

Integrating in time we find that

\[
\|\tilde{\alpha}(t)\|_2^2 + \|\phi_{Q_{\nu_0}} J^{1/2} I \tilde{\alpha}\|^2_{L^2(Q_{\nu_0} \times [0, t])} \leq \|\tilde{\alpha}(0)\|_2^2 + C \int_0^t \|\tilde{\alpha}\|^2_2 \, ds + \\
2 \int_0^t \text{Re} \left\langle \tilde{\alpha}, \tilde{G} \right\rangle \, ds + \int_0^t \text{Re} \left\langle E_M \tilde{\alpha}, \tilde{\alpha} \right\rangle \, ds.
\]

In order to handle the terms \( \int_0^t \text{Re} \left\langle E_M \tilde{\alpha}, \tilde{\alpha} \right\rangle \, ds \), we have that
\[
\int_0^t \int_0^s \sum_{\mu \in \mathbb{Z}^n} \beta_\mu(r) \psi_\mu(x,r,\xi) \alpha_1(s) \alpha_1(s) \, dx \, dr \, ds = \\
\int_0^t \sum_{\mu \in \mathbb{Z}^n} \beta_\mu(r) \int_r^t \int \psi_\mu(x,r,\xi) \alpha_1(s) \alpha_1(s) \, ds \, ds \, dr.
\]

Our estimates on \( \varphi(x,s,\xi) \) give us that \( \int \psi_\mu(x,s,\xi) \alpha_1 \, dx \leq \| J^{1/2} \alpha \|_{L^2(Q^n_\mu)}^2 + C \| \alpha_1 \|^2 \). Thus we have

\[
\left| \int_0^t \langle E_M \bar{\alpha}, \bar{\alpha} \rangle \right| \leq C t \sup_{\mu \in \mathbb{Z}^n} \| J^{1/2} I \bar{\alpha} \|_{L^2([0,t] \times Q^n_\mu)} + C t \sup_{0 \leq s \leq t} \| \bar{\alpha} \|^2,
\]

Hence, after taking a supremum over \( 0 \leq t \leq T \), we arrive at

\[
\sup_{0 \leq t \leq T} \| \bar{\alpha}(t) \|^2 + \sup_{0 \leq t \leq T} \| \phi_{Q_{\mu_0}} J^{1/2} I \bar{\alpha} \|^2_{L^2(Q^n_{\mu_0} \times [0,T])} \leq \| \bar{\alpha}(0) \|^2 + C T \sup_{0 \leq t \leq T} \| \bar{\alpha} \|^2 + 2 \int_0^T \left| \Re \left\langle \bar{\alpha}, \bar{G} \right\rangle \right| \, ds + C T \sup_{\mu \in \mathbb{Z}^n} \| J^{1/2} I \bar{\alpha} \|_{L^2([0,T] \times Q^n_\mu)}.
\]

By choosing \( T \) small we may make \( C T \sup_{0 \leq t \leq T} \| \bar{\alpha} \|^2 \leq \frac{1}{2} \sup_{0 \leq t \leq T} \| \bar{\alpha} \|^2 \) and absorb this term into the left hand side. In this way we get

\[
(8) \quad \sup_{0 \leq t \leq T} \| \bar{\alpha}(t) \|^2 + \| J^{1/2} I \bar{\alpha} \|^2_{L^2(Q^n_{\mu_0} \times [0,T])} \leq 2 \| \bar{\alpha}(0) \|^2 + 4 \int_0^T \left| \Re \left\langle \bar{\alpha}, \bar{G} \right\rangle \right| \, ds + C T \sup_{\mu \in \mathbb{Z}^n} \| J^{1/2} I \bar{\alpha} \|_{L^2([0,T] \times Q^n_\mu)}.
\]

In terms of \( \bar{z} \) our estimates now tell us

\[
\sup_{0 \leq t \leq T} \| \Psi_M \bar{z}(t) \|^2 + \| \phi_{Q_{\mu_0}} J^{1/2} I \Psi_M \bar{z} \|^2_{L^2(Q^n_{\mu_0} \times [0,T])} \leq 2 \| \Psi_M \bar{z}(0) \|^2 + 4 \int_0^T \left| \Re \left\langle \Psi_M \bar{z}, \bar{G} \right\rangle \right| \, ds + C T \sup_{\mu \in \mathbb{Z}^n} \| J^{1/2} I \Psi_M \bar{z} \|_{L^2([0,T] \times Q^n_\mu)}.
\]

But notice that, \( J^{1/2} I \Psi_M = \Phi_M J^{1/2} I + E \), where \( E \) is of order 0. Hence

\[
\int_0^T \| \phi_{Q_{\mu_0}} J^{1/2} I \Psi_M \bar{z} \|^2_{L^2(Q^n_{\mu_0})} \, dt \geq \int_0^T C_0 \| J^{1/2} I \bar{z} \|^2_{L^2(Q^n_{\mu_0})} \, dt - C T \sup_{0 \leq t \leq T} \| \bar{z} \|^2.
\]

Thus, possibly after another restriction in \( T \), we arrive at

\[
\sup_{0 \leq t \leq T} \| \bar{z}(t) \|^2 + \| J^{1/2} I \bar{z} \|^2_{L^2(Q^n_{\mu_0} \times [0,T])} \leq C_0 \left( \| \bar{z}(0) \|^2 + \int_0^T \left| \Re \left\langle \Psi_M \bar{z}, \bar{G} \right\rangle \right| \, ds \right. \]
\[
\left. + C T \sup_{\mu \in \mathbb{Z}^n} \| J^{1/2} I \bar{z} \|_{L^2([0,T] \times Q^n_\mu)} \right).
\]
Now estimate the term involving $\bar{G}$.

$$\int_0^T \left| \operatorname{Re} \left< \Psi_M \bar{z}, \bar{G} \right> \right| ds \leq \int_0^T C_0 \left\| \bar{z} \right\|_2 \left\| \bar{G} \right\|_2 dt \leq C_0 \sup_{0 \leq s \leq t} \left\| \bar{z}(s) \right\|_2 \left\| G \right\|_{L_1^1 L_2^2}$$

$$\leq C_0 \eta \sup_{0 \leq s \leq t} \left\| \bar{z}(t) \right\|_2 + \frac{C_0}{\eta} \left\| G \right\|_{L_1^1 L_2^2}^2$$

Choosing $\eta$ small enough to absorb the term involving $\bar{z}$ to the left hand side. Our estimate now is of the form

$$\sup_{0 \leq t \leq T} \left\| \bar{z}(t) \right\|_2^2 + \left\| J^{1/2} I \bar{z} \right\|_{L^2(Q_{\mu} \times [0,T])}^2 \leq C_0 \left( \left\| \bar{z}(0) \right\|_2^2 + \left\| G \right\|_{L_1^1 L_2^2}^2 + CT \sup_{\mu \in \mathbb{Z}^n} \left\| J^{1/2} I \bar{z} \right\|_{L^2([0,T] \times Q_{\mu})} \right).$$

Finally to get Theorem 3.1 we take a supremum in $\mu_0$, then after a suitable restriction in $T$ we may absorb $CT \sup_{\mu \in \mathbb{Z}^n} \left\| J^{1/2} I \bar{z} \right\|_{L^2([0,T] \times Q_{\mu})}$ into the left hand side. Keeping in mind that estimates for $\bar{z}$ will imply the corresponding estimates in $u$. □

We now turn to a perturbation result. It is possible to weaken the non-trapping condition [L5]. It is enough to assume that the second order coefficients are “close” to coefficients that are non-trapping.

To this end, we again consider equation [5]. We still assume that the coefficients satisfy conditions [L1]–[L4] instead of [L5] suppose that $A(x,t) = A_0(x,t) + \eta A_1(x,t)$. Assume that $h_0(x,\xi) = \langle A_0(x,0)\xi, \xi \rangle$ satisfies the non-trapping condition [L5]. In addition, assume that $|A_1(x,t)| + |D_x A_1(x,t)| \leq \frac{C_0}{\eta}$ uniformly in $t$. Then for $\eta$ sufficiently small, depending on $C$ and $C_0$, the conclusion of Theorem 3.1 holds.

To see this, notice that we only use the non-trapping condition [L5] in the proof of Lemma 3.3. We will now prove this lemma in the under these slightly more general assumptions.

**Lemma 3.5.** Let $h_0(x,\xi)$ be as above and let $p_\mu$ be the Doi symbol corresponding to $h_0$ centered at cube $Q_\mu$. Then there exists a $T_1 = T_1(C,C_0)$ so that, uniformly for all $t < T_1$, the time varying symbol $h_t(x,\xi) = a_{jk}(x,t)\xi_j \xi_k$ satisfies

$$H_{h_t p_\mu}(x,\xi) \geq \frac{1}{C_0} \frac{|\xi|}{(x-x_\mu)^2} - C_0.$$

**Proof.** To facilitate calculations we use the following notations for the matrix entries $(a_{jk}^0(x,t))_{j,k=1,...,n} := A_0(x,t)$ and $(a_{jk}^1(x,t))_{j,k=1,...,n} := A_1(x,t)$. It is also convenient to denote $k_0(x,t,\xi) = \langle A_0(x,t)\xi, \xi \rangle$, and $k_1(x,t,\xi) = \langle A_1(x,t)\xi, \xi \rangle$. 


Proceeding with our calculation as before we have

\[
H_{h_i \mu} = \sum_{i=1}^{n} \eta \frac{\partial p_{\mu}}{\partial x_i} - \left( \frac{\partial h_0}{\partial x_i} - \frac{\partial h_1}{\partial x_i} + \eta \frac{\partial k_1}{\partial x_i} \right) \frac{\partial p_{\mu}}{\partial x_i}
\]

\[
= H_{h_\mu} + \sum_{i=1}^{n} \left( a_{ik}^0(x, t) - a_{ik}^0(x, 0) + b_{ik}^1(x, t) \right) \eta \frac{\partial p_{\mu}}{\partial x_i}
\]

As before the asymptotic flatness condition \([L3]\) there is a \(T_1 = T_1(C, C_0)\) so that if \(t < T_1\) we have that \(h^0\) satisfies

\[
\sum_{i=1}^{n} 2 \left( a_{ik}^0(x, t) - a_{ik}^0(x, 0) \right) \eta \frac{\partial p_{\mu}}{\partial x_i}
\]

\[
- \left( \frac{\partial a_{ik}^0(x, t)}{\partial x_i} - \frac{\partial a_{ik}^0(x, 0)}{\partial x_i} + \eta \frac{\partial a_{ik}^0(x, t)}{\partial x_i} \right) \eta C \xi \frac{\partial p_{\mu}}{\partial x_i} \leq \frac{1}{2C_0} |\xi|^2.
\]

Our conditions on \(A_1\), together with the control of the seminorms of \(p_{\mu}\) give that

\[
\eta \left| \sum_{i=1}^{n} a_{ik}^0(x, t) \xi \frac{\partial p_{\mu}}{\partial x_i} + \frac{\partial a_{ik}^0(x, 0)}{\partial x_i} \xi \frac{\partial p_{\mu}}{\partial x_i} \right| \leq \eta C \xi \frac{|\xi|}{(x^2)^2}.
\]

We choose \(\eta\) so that \(\frac{\eta C |\xi|}{(x^2)^2} \leq \frac{1}{2C_0} |\xi|^2\). Now using the assumption that \(h^0\) satisfies \([L5]\) we get that

\[
H_{h_\mu} \geq H_{h_\mu} - \frac{1}{C_1} \frac{|\xi|}{(x^2)^2} \geq \frac{1}{C_1} \frac{|\xi|^2}{(x - x_\mu)^2} - C_1.
\]

Using the same version of Doi’s lemma as before.

4. Nonlinear Results

In this section we approach \([\text{11}]\) by the artificial viscosity method. Hence, we are interested in the system

\[
\begin{cases}
\partial_t u = -\epsilon \Delta^2 u + ia_{jk}(x, t, u, \bar{u}, \nabla u, \nabla \bar{u}) \partial_{x_j} \partial_{x_k} u \\
+ b_1(x, t, u, \bar{u}, \nabla u, \nabla \bar{u}) \cdot \nabla u + b_2(x, t, u, \bar{u}, \nabla u, \nabla \bar{u}) \cdot \nabla \bar{u} \\
+ c_1(x, t, u, \bar{u})u + c_2(x, t, u, \bar{u})\bar{u} + f(x, t), \\
u(x, 0) = u_0(x).
\end{cases}
\]
where

\[ \mathcal{L}(u)v = ia_{jk}(x,t,u,\bar{u},\nabla u, \nabla \bar{u})\partial_{x_j} \partial_{x_k} v + \bar{b}_1(x,t,u,\bar{u},\nabla u, \nabla \bar{u}) \cdot \nabla v + \nabla v + c_1(x,t,u,\bar{u})v + c_2(x,t,u,\bar{u})v. \]

**Theorem 4.1.** Take \( s > n + 3 \). For \( v_0 \in H^s \) and \( f \in L^\infty([0,1];H^s) \), define \( \lambda = \|v_0\|_s + \int_0^1 \|f(t)\|_{H^s} \, dt \). Define

\[ X_{M_0,T} = \{ v : \mathbb{R}^n \times [0, T] \to \mathbb{C} \mid v(x,0) = v_0, v \in C([0,T];H^s), \|v\|_{L^\infty_t H^s_x} \leq M_0 \}. \]

If \( \lambda < M_0/2 \), then there exists \( T_\epsilon, 1 > T_\epsilon > 0 \), so that equation (9) with initial data \( v_0 \) has a unique solution \( v^\epsilon \in X_{M_0,T^\epsilon} \).

**Proof.** For \( t < 1 \), consider the integral form the equation

\[ \Gamma v(t) = e^{-t\Delta^2} v_0 + \int_0^t e^{-\epsilon(t-t')\Delta^2} (\mathcal{L}(v)v(t') + f(\cdot,t')) \, dt'. \]

We show that \( \Gamma \) is a contraction mapping on the space \( X_{M_0,T} \) after a suitable restriction of \( T \). So let \( \alpha \) be a multi-index such that \( |\alpha| = s \) and consider

\[ \partial^\alpha_x \Gamma v(t) = e^{-t\Delta^2} \partial^\alpha_x v_0 + \int_0^t \partial^\alpha_x e^{-\epsilon(t-t')\Delta^2} \partial^\beta_x (\mathcal{L}(v)v(t')) \, dt' \]

Choose multi-indices \( \beta \) and \( \beta' \) so that \( |\beta'| = 2, |\beta| = s - 2, \) and \( \alpha = \beta + \beta' \).

\[ \partial^\alpha_x \Gamma v(t) = e^{-t\Delta^2} \partial^\alpha_x v_0 + \int_0^t \partial^\beta_x e^{-\epsilon(t-t')\Delta^2} \partial^\beta_x (\mathcal{L}(v)v(t')) \, dt' \]

\[ + \int_0^t e^{-\epsilon(t-t')\Delta^2} \partial^\beta_x f(\cdot,t') \, dt'. \]

Hence,

\[ \| \partial^\alpha_x \Gamma v \|_2 \leq C \left( \| \partial^\alpha_x v_0 \|_2 + \int_0^t \left( \| \partial^\beta_x e^{-\epsilon(t-t')\Delta^2} \partial^\beta_x (\mathcal{L}(v)v) \|_2 + \int_0^t \| \partial^\beta_x f(t') \|_2 dt' \right) \right) \]

\[ \leq C \left( \| v_0 \|_{H^s} + \int_0^t \frac{1}{(t-t')^{1/2}} \| \partial^\beta_x (\mathcal{L}(v)v(t')) \|_2 dt' + \int_0^t \| f(t') \|_{H^s} dt' \right). \]

In order to proceed further we need to turn our attention to \( \partial^\beta_x \mathcal{L}(v)v \).

**Lemma 4.2.** Let \( u, \bar{v} \in X_{M,T} \) and suppose that \( |\beta| = 2s - 2 \) for \( s > n + 3 \), then there exists a \( P \in \mathbb{N} \) so that \( \| \partial^\beta_x \mathcal{L}(u)v \|_{2} \leq C \| v \|_{H^s} \left( 1 + \| u \|_{H^s} + \| u \|_{H^s}^P \right) \) with \( 0 \leq t \leq T \) and \( C = C(M,n,s) \).

**Proof.** We estimate term by term,

\[ \partial^\beta_x \mathcal{L}(u)v = \partial_x (a_{jk}(x,t,u,\bar{u},\nabla u, \nabla \bar{u})\partial_{x_j} \partial_{x_k} v) + \partial^\beta_x (\bar{b}_1(x,t,u,\bar{u},\nabla u, \nabla \bar{u}) \cdot \nabla v) + \nabla v + c_1(x,t,u,\bar{u})v + c_2(x,t,u,\bar{u})v. \]

We start with \( c_1(x,t,u,\bar{u}) \). Let \( \tilde{c}_1(x,t,u,\bar{u}) = c_1(x,t,u,\bar{u}) - c_1(x,t,0,\bar{u}) \) so that \( \tilde{c}_1(x,t,0,0) = 0 \). Then \( \partial^\beta_x (c_1(x,t,u,\bar{u})v) = \partial^\beta_x (\tilde{c}_1(x,t,u,\bar{u})v) + \partial^\beta_x (c_2(x,t,0,0)v), \)
and the $H^s$ norm of the second term is clearly bounded by $C \|v\|_{H^s}$ where $C$ depends on $c_1$ and $\beta$. We have,
\[
\|\partial_x^\beta (\tilde{c}_1(x, t, u, \bar{u})v)\|_2 \leq \sum_{\gamma+\delta=\beta} \|\partial_x^\gamma (\tilde{c}_1(x, t, u, \bar{u})) \partial_x^\delta v\|_2.
\]

If $|\delta| < s - n/2$, then $\|\partial_x^\delta v\|_\infty \leq C\|\partial_x^\delta v\|_{H^s-|\delta|} \leq C \|v\|_{H^s}$. It follows that $\|\partial_x^\beta \tilde{c}_1(x, t, u, \bar{u}) \partial_x^\delta v\|_2 \leq C \|v\|_{H^s} \|\partial_x^\beta \tilde{c}_1(x, t, u, \bar{u})\|_2$.

As $\tilde{c}_1(x, t, 0, 0) = 0$ and $\tilde{c}_1 \in C_b^\infty$ it follows that $\tilde{c}_1(x, t, u, \bar{u}) \in H^s$. Hence $\|\partial_x^\beta \tilde{c}_1(x, t, u, \bar{u})\|_2 \leq |c_1(x, t, u, \bar{u})|_{H^s} \leq C (\|u\|_{H^s} + \|u_p\|_{H^s})$ for some $p \in \mathbb{N}$.

On the other hand, if $|\delta| \geq s - n/2$, then we may not estimate $\partial_x^\delta v$ in $L^\infty$. Instead we estimate the other factor in $L^\infty$. Because $|\gamma| + |\delta| = |\beta| = s - 2$, we have that $|\gamma| \leq n/2 - 2$. Since $s > n - 2$, we have that $s - |\gamma| > n/2$. Therefore,
\[
\|\partial_x^\beta \tilde{c}_1(x, t, u, \bar{u})\|_{H^s-|\gamma|} \leq C \|\partial_x^\beta \tilde{c}_1(x, t, u, \bar{u})\|_{H^s} \leq C (\|u\|_{H^s} + \|u_p\|_{H^s})
\]
with $P$ as before. The estimates for $c_2$ work in exactly the same way.

To estimate $\partial_x^{\beta} \left( \tilde{b}_1(x, t, u, \bar{u}, \nabla u, \nabla \bar{u}) \cdot \nabla v \right)$ note that our assumptions imply $b_1(x, t, 0, 0, \bar{0}, \bar{0}) = 0$. Again we have
\[
\|\partial_x^{\beta} \left( \tilde{b}_1(x, t, u, \bar{u}, \nabla u, \nabla \bar{u}) \cdot \nabla v \right)\|_2 \leq \sum_{i=1}^n \sum_{\gamma+\delta=\beta} \|\partial_x^\gamma b_1(x, t, u, \bar{u}, \nabla u, \nabla \bar{u}) \partial_x^\delta v\|_2.
\]

In this case, if $|\delta| < s - n/2 - 1$, then we proceed by estimating $\partial_x^\delta \partial_x v$ in $L^\infty$. If instead $|\delta| \geq s - n/2 - 1$, then we get that $|\gamma| \leq n/2 - 1$. We have that $s > n + 2$, so that $s - |\gamma| > n/2 + 1$. Hence we may estimate $\partial_x^\beta b_1(x, t, u, \bar{u}, \nabla u, \nabla \bar{u})$ in $L^\infty$. Again the estimates for the terms involving $b_2$ work in the same way as the terms involving $b_1$.

The estimates for terms involving $a_{jk}$ are essentially identically to those for $c_i$ and $b_i$ except that we need to require $s > n + 3$.

So we know that
\[
\|\partial_x^\beta \Gamma v\|_2 \leq C \|v_0\|_{H^s} + C_{M_0} \frac{2^{1/2}}{\epsilon^{1/2}} M_0(1 + M_0^p) + \int_0^1 \|f(t)\|_{H^s} \, dt
\]
and therefore
\[
\|\Gamma v\|_{H^s} \leq C \|v_0\|_{H^s} + C_{M_0} \frac{2^{1/2}}{\epsilon^{1/2}} M_0(1 + M_0^p) + \int_0^1 \|f(t)\|_{H^s} \, dt.
\]

By choosing $T$ so that $C_{M_0} \left( T^{1/2}/\epsilon^{1/2} + T \right) M_0(1 + M_0^p) < \lambda$ then we get that $\Gamma$ maps $X_{M_0,T}$ to itself.

Now take $u, v \in X_{M_0,T}$. We wish to show that $\Gamma$ is a contraction mapping. We have that
\[
\Gamma u(t) - \Gamma v(t) = \int_0^t e^{-\epsilon(t-t')} \Delta^2 (\mathcal{L}(u)u - \mathcal{L}(v)v)(t') \, dt' =
\int_0^t e^{-\epsilon(t-t')} \Delta^2 (\mathcal{L}(u) - \mathcal{L}(v))(u + \mathcal{L}(v)(u - v)) \, dt'.
\]
To estimate the terms that arise from $\mathcal{L}(v)(u - v)$ we may use Lemma 4.2 to conclude that
\[
\left\| \int_0^t e^{-\epsilon(T-t)} \Delta^2 \mathcal{L}(v)(u - v) \right\|_{H^s} dt \leq C \|u - v\|_{H^s} \left( \frac{t^{1/2}}{\epsilon^{1/2}} + t \right) (1 + M_0 + M_0^p).
\]
So by choosing $T_\epsilon < T$ this last expression is less then $1/4 \|u - v\|_{H^s}$.

To estimate terms involving $(\mathcal{L}(u) - \mathcal{L}(v))u$ we proceed in essentially the same way. For example, to estimate $\|(c_1(x, t, u, \bar{u}) - c_1(x, t, v, \bar{v}))u\|_{H^s}$, rewrite the difference as follows
\[
c_1(x, t, u, \bar{u}) - c_1(x, t, v, \bar{v}) = c_1(x, t, v, \bar{u}) + c_1(x, t, v, \bar{v}) - c_1(x, t, v, \bar{v}) = \partial_{z_1} c_1(x, t, su - (1-s)v, \bar{u})u(u - v) + \partial_{z_2} c_1(x, t, su + (1-r)v, \bar{u})u(u - v).
\]
We can see the above two terms are bounded by $C(M_0 + M_0^p) \|u - v\|_{H^s}$.

The other terms work similarly. We conclude that
\[
\|(\Gamma u - \Gamma v)(t)\|_{H^s} \leq C \left( \frac{T^{1/2}}{\epsilon^{1/2}} + T \right) (1 + M_0 + M_0^p) \|u - v\|_{H^s}.
\]
Choosing $T_\epsilon < T$ appropriately, we see $\Gamma$ is a contraction mapping. Hence there is a unique $v^\epsilon \in X_{T_\epsilon, M_0}$ such that $v^\epsilon$ solves (9) with initial data $v_0$. \hfill \Box

The following lemma is useful in verifying the conditions for our linear estimates which help us get a uniform time of existence.

**Lemma 4.3.** Let $v \in X_{T, M_0}$ with $v(0) = u_0$, and suppose that $v$ satisfies (9) then for $s > N + n/2 + 4$ the coefficients $a_{j,k}(x, t, v, \nabla v, \nabla \bar{v})$ satisfies (L1), (L2), (L3) and (L5). Where the constant $C$ that appears in these conditions depends on $M_0$ and $C_1$ depends on $u_0$.

**Proof.** Take $s > N + n/2 + 4$, then $v$ together with all of it’s derivatives to order $N+1$ are in $L^\infty$. This, together with (NL1) allows us to verify (L1). The assumptions (L2) and (L5) follow immediately from (NL2), (NL3), (NL4) and (NL7).

It remains to verify (L3) and (L4). Clearly, $|I - a_{j,k}(x, t, v, \nabla v, \nabla \bar{v})| \leq C/\langle x \rangle^2$ follows from (NL5) and our $L^\infty$ bounds just as in the cases above.

Let * denote $(x, t, v, \nabla v, \nabla \bar{v})$, and consider
\[
\partial_{z_1} a_{j,k}(*) = \frac{\partial a}{\partial x_i}(*) + \frac{\partial a_{j,k}}{\partial v}(*) \frac{\partial v}{\partial x_i} + \cdots + \frac{\partial a_{j,k}}{\partial \partial_{x_n} v}(*) \frac{\partial^2 \bar{v}}{\partial x_i \partial x_n}.
\]
The first and second order derivatives of $v$ are in $L^\infty$ because $s > n/2 + 2$. Hence by using (NL5) we can bound each term by $C/\langle x \rangle^2$.

The estimate for $\partial_{t} a_{j,k}(x, t, v, \nabla v, \nabla \bar{v})$ is similar. The primary difference is that we have to estimate terms of the form $\partial_{t} v$ and $\partial_{t} (\partial_{x_i} v)$ in $L^\infty$. To handle $\partial_{t} v$ it is enough to notice that $\partial_{t} v$ is equal to the right hand side of (9). Each term of $\mathcal{L}(v)$ is in $L^\infty$ by (NL1) and our $L^\infty$ bound on $v$ and it’s derivatives. Since $f(x, t) \in L^\infty_{t, x} H^s_x$ it is in $L^\infty_{t, x}$. To handle the final term $\partial_{t} \partial_{x_i} v$ we apply $\partial_{x_i}$ to our
equation and get

\[ \partial_t \partial_x v = -\epsilon \Delta^2 \partial_x v + i a_{jk}(\ast) \partial_{x,x_k} (\partial_x v) + i \frac{\partial a_{jk}}{\partial x_i} (\ast) \partial_{x,x_k} v + i \frac{\partial a_{jk}}{\partial v} (\ast) \partial_x v \\
+ i \frac{\partial a_{jk}}{\partial v} (\ast) \partial_{x,v} \bar{v} + i \sum_{l=1}^m \left( \frac{\partial a_{jk}}{\partial \partial v} (\ast) \partial_{x,x_k} v \right) \partial_{x,x_l} \bar{v} \\
+ \bar{b}_l (\ast) \cdot \nabla \partial_x v + \cdots + \partial_x f(x,t). \]

We find that each of these terms may again be handled by our \( L^\infty \) bounds for \( v \) and its derivatives and \([\text{NL1}]\). Again \( \partial_x f \in L^\infty_t H^s_x \) so we may bound \( ||\partial_x f||_\infty \).

Lastly, to bound \( \partial_t \partial_{x,a_{jk}}(x,t,v,\bar{v},\nabla v \nabla \bar{v}) \) we proceed in the same way. We additionally have to estimate terms of the form \( \partial_t \partial_x, \partial_x \partial v \) in \( L^\infty \), but we simply apply \( \partial_x, \partial_t \) to our equation, and again we only encounter terms involving the derivatives of our coefficients evaluated at \( v \) multiplied by derivatives of \( v \) of order less than 4, so for \( s > n/2 + 4 \) we may estimate these terms as before. \( \square \)

To see that our first order terms will satisfy the conditions of our linear theory we first need two lemma’s.

**Lemma 4.4.** Suppose \( b(x,t,z_1,\ldots,z_{2n+2}) \in \mathcal{C}_b^N(\mathbb{R} \times \mathbb{R} \times B^{2n+2}_M(0)) \) satisfies \( b(x,t,0,0,0,0) = 0 \) and \( \partial_t b(x,t,0,0,0,0) = 0 \). For any \( M \in \mathbb{N} \), if \( s > n/2 + M + 1 \) and \( \tilde{N} > s + 2 \) then \( b(x,t,u,\bar{u},\nabla u,\nabla \bar{u}) \in W^{1,M} \) for \( u \in L^\infty_t H^s_x \).

**Proof.** First to see it is in \( L^1 \), we set \( f'(r) = b(x,t,ru,ru,ru) \). Then we calculate

\[ f'(r) = \frac{\partial b}{\partial z_1}(x,t,ru,ru,ru) + \frac{\partial b}{\partial z_2}(x,t,ru,ru,ru) + \sum_{i=1}^n \frac{\partial b}{\partial z_{i+2}^n}(x,t,ru,ru,ru) \frac{\partial u}{\partial x_i}. \]

Clearly \( f(0) = f'(0) = 0 \), and \( f(1) = b_1(x,t,u,\bar{u},\nabla u,\nabla \bar{u}) \). Now,

\[ ||b||_1 = ||f(1)||_1 = \left| \int_0^1 (1-r)f''(r) \, dr \right|_1 \leq \int_0^1 (1-r) ||f''(r)||_1 \, dr. \]

With in \( f''(r) \) are terms of the form \( (\partial^2 b)u^2, (\partial^2 b)u \bar{u}, (\partial^2 b)u \partial u, (\partial^2 b)\bar{u} \partial \bar{u}, \) etc. The key observation is that they all involve exactly of degree two when looked at as polynomials in the derivatives of \( u \). So we may apply Cauchy-Schwartz and integrate in \( s \). For example

\[ \int \left| \frac{\partial^2 b}{\partial z_1 \partial z_3}(x,t,u,\bar{u},\nabla u,\nabla \bar{u}) u \frac{\partial u}{\partial x_1} \right| dx \leq \left| \int \left| \frac{\partial^2 b}{\partial z_1 \partial z_3} \right|_\infty \left| \frac{\partial u}{\partial x_1} \right| dx \right| \leq \left| \int \left| \frac{\partial^2 b}{\partial z_1 \partial z_3} \right|_\infty ||u||_2 \left| \frac{\partial u}{\partial x_1} \right|_2 \right| \leq C_b ||u||_H^s. \]

Estimates of \( \partial^2 b \) work similarly. In fact,

\[ \partial_{x,b} = \frac{\partial b}{\partial x_1} + \frac{\partial b}{\partial z_1}(\cdot) \frac{\partial u}{\partial x_1} + \frac{\partial b}{\partial z_2}(\cdot) \frac{\partial \bar{u}}{\partial x_1} + \sum_{j=1}^n \frac{\partial b}{\partial z_{j+2}^n}(\cdot) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^n \frac{\partial b}{\partial z_{j+n+2}^n}(\cdot) \frac{\partial^2 u}{\partial x_i \partial x_j}. \]
Each term has the property that if it is evaluated at \( u = 0 \) it is 0, as well a
derivative in the \( z_1 \sim z_{2n+2} \). Hence we may apply the argument above to bound the
\( L^1 \) norm of each of these. \( \Box \)

The above lemma together with the following observation of Kenig et. al. \cite{13}
allow us to see that \( b_1(x,t,u,\bar{u},\nabla u,\nabla \bar{u}) \) satisfies our linear assumptions.

**Lemma 4.5.** For \( M > N + n \) if \( b(x,t) \in W^{1,M} \) uniformly in \( t \), then one can find
\( \varphi_\mu(x,t) \) so that \( \text{supp} \varphi_\mu(\cdot,t) \subset Q_\mu^* \),
\( \| \varphi_\mu(\cdot,t) \|_{C^N} \leq 1 \), and
\[
b(x,t) = \sum_{\mu \in \mathbb{Z}^n} \alpha_\mu(t) \varphi_\mu(x,t) \text{ with } \sum_{\mu \in \mathbb{Z}^n} |\alpha_\mu| \leq c \| b \|_{W^{1,M}}.
\]

**Proof.** By the Sobolev Imbedding theorem if \( M > N + n \) then \( \| b(\cdot,t) \|_{C^N(Q_\mu^*)} \leq C \| b \|_{W^{1,M}(Q_\mu^*)} \) with \( C \) independent of \( \mu \). Let \( \eta_\mu \) be a \( C^\infty \) partition of unity subor-
dinate to \( Q_\mu^* \) with \( \| \eta_\mu \|_{C^N} \) independent of \( \mu \). Then \( b(x,t) = \sum_{\mu \in \mathbb{Z}^n} \eta_\mu b(x,t) \) and
\( \| \eta_\mu b(\cdot,t) \|_{C^N} \leq C \| b \|_{W^{1,M}(Q_\mu^*)} \). Since \( Q_\mu^* \) have bounded overlap \( \sum_{\mu \in \mathbb{Z}^n} \| b \|_{W^{1,M}(Q_\mu^*)} \leq C \| b \|_{W^{1,M}} \). Hence we just have to set
\[
\varphi_\mu(x,t) = \frac{b(x,t)\eta_\mu(x)}{\| \eta_\mu b(\cdot,t) \|_{C^N}} \text{ and } \alpha_\mu(t) = \| \eta_\mu b(\cdot,t) \|_{C^N}.
\]

\( \Box \)

Let \( J = (1 + \Delta)^{\frac{1}{2}} \). In order to get the necessary estimates on \( \| u(t) \|_{H^s} \) we
inductively estimate \( J^{2m} u \). Again let \( * \) denote \( (x,t,u^*,\bar{u}^*,\nabla u^*,\nabla \bar{u}^*) \). As in \cite{12}
we consider the following systems, for \( m = 1, 2, \ldots/s/2 \),
\[
\partial_t J^{2m} u^* = -\epsilon \Delta^2 J^{2m} u^* + L(u^*) J^{2m} u^* + 2m \nu \partial_{\lambda} (a_{jk}(\star) \partial_{\lambda} J^{2m-1} u^* \\
+ i \partial_{jk} \partial_{\lambda} a_{jk}(\star) \partial_{\lambda} J^{2m-1} u^* + i \partial_{jk} \partial_{\lambda} a_{jk}(\star) \partial_{\lambda} J^{2m-1} u^* \\
+ \partial_{\lambda} J^{2m} u^* (\partial_{\lambda} b_{1,j}(\star) \partial_{\lambda} J^{2m} u^* + \partial_{\lambda} b_{1,j}(\star) \partial_{\lambda} J^{2m} u^* \\
+ \partial_{\lambda} J^{2m} \bar{u}^* (\partial_{\lambda} b_{2,j}(\star) \partial_{\lambda} J^{2m} \bar{u}^* + \partial_{\lambda} b_{2,j}(\star) \partial_{\lambda} J^{2m} \bar{u}^*) \\
+ c_{1,2m}(x,t, (\partial^\beta \bar{u}^*)_{|\beta| \leq 4}, (\partial^\beta \bar{u}^*)_{|\beta| \leq 4}) R_{2m,1} J^{2m} u^* \\
+ c_{2,2m}(x,t, (\partial^\beta \bar{u}^*)_{|\beta| \leq 2m-2}, (\partial^\beta \bar{u}^*)_{|\beta| \leq 2m-2}) + J^{2m} f(x,t)
\]
Or more briefly,
\[
\partial_t J^{2m} u^* = -\epsilon \Delta^2 J^{2m} u^* + L_{2m}(u^*) J^{2m} u^* \\
+ f_{2m}(x,t, (\partial^\beta \bar{u}^*)_{|\beta| \leq 2m-2}, (\partial^\beta \bar{u}^*)_{|\beta| \leq 2m-2})
\]

Where
\[
L_{2m}(u)v = i a_{jk}(x,t,u,\bar{u},\nabla u,\nabla \bar{u}) \partial_{jk} v + b_{1,1,j}(x,t, (\partial^\alpha u)_{|\alpha| \leq 2}, (\partial^\alpha \bar{u})_{|\alpha| \leq 1}) \partial_{\lambda} x_j v + \\+ b_{1,k,1}(x,t,u,\bar{u},\nabla u,\nabla \bar{u}) R_{k} \partial_{\lambda} x_j v + b_{2,2m,2}(x,t, (\partial^\alpha u)_{|\alpha| \leq 2}, (\partial^\alpha \bar{u})_{|\alpha| \leq 1}) \cdot \nabla \bar{u}^* \\
+ c_{1,2m}(x,t, (\partial^\beta u^*)_{|\beta| \leq 4}, (\partial^\beta \bar{u}^*)_{|\beta| \leq 4}) R_{2m,1} v \\
+ c_{2,2m}(x,t, (\partial^\beta u^*)_{|\beta| \leq 4}, (\partial^\beta \bar{u}^*)_{|\beta| \leq 4}) R_{2m,2} \bar{v},
\]
with
\[ b_{1,1,j}(x, t, u, \bar{u}, \nabla u, \nabla \bar{u}) = b_{1,j}(x, t, u, \bar{u}, \nabla u, \nabla \bar{u}) + \partial_t u \partial_{x_j} b_{1,j}(x, t, u, \bar{u}, \nabla u, \nabla \bar{u}), \]
\[ R_{ik} = \partial_{i}^2 J^{-2}, \quad \text{and} \]
\[ b_{2m,2,j}(x, t, u, \bar{u}, \nabla u, \nabla \bar{u}) = b_{2,j}(x, t, u, \bar{u}, \nabla u, \nabla \bar{u}) + \partial_t u \partial_{x_j} b_{2,j}(x, t, u, \bar{u}, \nabla u, \nabla \bar{u}). \]

The same observations from [12] apply. The principal part of \( L_{2m}(u^\prime) \) is independent of \( m \). The coefficients \( b_{1,1,j}, b_{2m,2,j} \), and \( b_{ijk} \) depend on the coefficients \( a_{jk}, \tilde{b}_j \) and their first derivatives, \( u \) and the derivatives of \( u \), but only on \( m \) as a multiplicative constant. Notice that here both \( a_{jk} \) and \( b_2 \) generate first order terms but the \( \Psi DO \)'s \( R_{ik} \) are independent of \( m \).

We need to verify that these coefficients satisfy the conditions for our linear theory when we evaluate them at any solution \( v \in X_{T,M_0} \) with \( v(0) = u_0 \). Since the leading order coefficients have not changed, Lemma 4.3 still assures us that our linear assumptions are verified. Because \( s > N + n/2 + 4 \) our \( H^s \) bounds on \( v \) together with (NL1) give us that the other coefficients satisfy (L1). Now we just need to verify (L4). Notice that in our linear theory we had the equation in divergence form and hence we have to add an additional first order term to be able to apply the theory.

**Lemma 4.6.** The first order coefficients \( \tilde{b}_{1,1,j}(x, t, v, \bar{v}, \nabla v, \nabla \bar{v}), \)
\( \tilde{b}_{ik,j}(x, t, v, \bar{v}, \nabla v, \nabla \bar{v}) \) and \( \partial_{x_j}(a_{jk}(x, t, v, \bar{v}, \nabla v, \nabla \bar{v})) \) satisfy (L4).

**Proof.** Let \( M \) be as in Lemma 4.3. Then by (NL6) we may apply directly apply Lemma 4.3 to get that \( b_{1,j} \in W^{1-1-M} \). Similarly we may apply Lemma 4.3 to \( \partial b_{1,j} \). And hence \( b_{1,j} \) has the necessary decomposition for (L4). For the terms involving \( \partial_{x_i} b_{k,j} \) (\( k = 1, 2 \)) notice that if \( b_{k}(x, t, 0, 0, \bar{0}, \bar{0}) = \partial_{z_k} b_{k}(x, t, 0, 0, \bar{0}, \bar{0}) = 0 \) then \( (z_{i} \partial_{x_{i}} b_{k}(x, t, \bar{z}))_{|z=0} = 0 \). So we may again apply Lemma 4.4 and in the same way as for \( b_{1,j} \) get the desired decomposition for these terms.

The bounds for \( \partial_{x_j} a_{jk}(x, t, v, \bar{v}, \nabla v, \nabla \bar{v}) \) and \( \partial_{x_j} a_{jk}(x, t, v, \bar{v}, \nabla v, \nabla \bar{v}) \) follow from (NL6) together with the \( L^\infty \) bounds for \( \partial_{x_j} u, \partial_{x_j} \bar{u}, \partial_{x_j} v, \partial_{x_j} \bar{v} \) and \( \partial_{x_j} u, \partial_{x_j} \bar{v} \). Similarly with \( b_{l,j}(x, t, v, \bar{v}, \nabla v, \nabla \bar{v}) R_{ik} \) and \( i \partial_{x_i} v \partial_{x_j} a_{ik}(x, t, v, \bar{v}, \nabla v, \nabla \bar{v}) \). \( \square \)

For \( J_{2m} u^\prime \), observe that if we evaluate our coefficients at any \( v \in X_{M_0,T} \) with \( v(0) = u_0 \) we arrive at a linear equation whose solution satisfies Theorem 3.1 with \( A_m \) depending on \( u_0 \) and the behavior of the coefficients for the system of \( J_{2m}^\prime u^\prime \) at \( t = 0 \). Let \( A = \max A_m \) and take \( M_0 = 20 A \). Notice at each stage the terms that come from \( f_{2m} \) depend only on terms of order strictly less then \( 2(m-1) \), which have been estimated in a previous step in \( L^\infty_T L^2_x \) and so appear with a factor of \( T \) in front when we apply our a priori estimate.

Thus there is a \( T' \) independent of \( \epsilon \) so that for a fixed increasing function \( R \), so that
\[ \sup_{[0,T']} \| u^\prime(\cdot, t) \|_s \leq A (\lambda + T'R(M_0)) \]
We may choose $T'$ small enough so that $A(\lambda + T'R(M_0)) \leq M_0/4 = 5A\lambda$. Then, by our remark after Theorem 4.1, we can reapply our contraction mapping theorem with initial data $u(T_i)$. We obtain a solution until time $2T_i$, if we apply our linear theory again (on the whole interval $[0, 2T_i]$) we see that $\|u(2T_i)\|_s \leq M_0/4$. Then we may continue $k$ times as long as $kT_i < T'$.

We thereby extend $u^\epsilon$ to a solution on $[0, T_0]$ with $u^\epsilon \in X_{T_0, M_0}$ for any $\epsilon$. Finally we come to the last result.

**Theorem 4.7.** There exists $u \in C([0, T^*]; H^{s-1}(\mathbb{R}^n)) \cap L^\infty([0, T^*]; H^s(\mathbb{R}^n))$ such that $u^\epsilon \to u$ as $\epsilon \to 0$ in $C([0, T^*]; H^{s-1})$.

**Proof.** Let $\epsilon, \epsilon' \in (0, 1)$ with $\epsilon' < \epsilon$. Let $v = u^\epsilon - u^{\epsilon'}$. Then $v$ satisfies

\[
\begin{aligned}
\partial_t v &= -(\epsilon - \epsilon')\Delta^2 u^\epsilon - \epsilon'\Delta^2 v + \mathcal{L}(u^{\epsilon'}) v + \left(\mathcal{L}(u^\epsilon) - \mathcal{L}(u^{\epsilon'})\right) u^\epsilon \\
v(0, x) &= 0
\end{aligned}
\]

Now we rewrite $\left(\mathcal{L}(u^\epsilon) - \mathcal{L}(u^{\epsilon'})\right) u^\epsilon$ we proceed term by term

\[
\begin{aligned}
&\left\{\begin{array}{l}
\partial_t v = -(\epsilon - \epsilon')\Delta^2 u^\epsilon - \epsilon'\Delta^2 v + \mathcal{L}(u^{\epsilon'}) v + \left(\mathcal{L}(u^\epsilon) - \mathcal{L}(u^{\epsilon'})\right) u^\epsilon \\
v(0, x) = 0
\end{array}\right.
\end{aligned}
\]

So we get zeroth and first order terms in $v$. The first order terms have coefficients $\partial_2 a_{jk}(x, t, u^\epsilon, \bar{u}', \nabla u^\epsilon)$ and $\partial_2 a_{jk}(x, t, u^{\epsilon'}, \bar{u}', \nabla u^{\epsilon'})$. By [NL6] we assumed the necessary decomposition of $\partial_2 a_{jk}$ and $\partial_2 \partial_2 a_{jk}$ so that these terms satisfy [L4].

We apply the same idea to the $b_{ij}$, $l = 1, 2$ and also get zeroth and first order terms in $v$. To see that our first order terms still satisfy the required estimates we remark that the conclusion of Lemma 4.4 still holds for $\partial_2 b_{1,j}(x, t, u^\epsilon, \bar{u}', \nabla u^\epsilon) \partial_j u^\epsilon$. Indeed when we estimate the $L^1$ norm will still have the product of two elements of $H^s$ whose norm is controlled by $M_0$. Similarly with the first order terms.

Thus we arrive at a system whose coefficients satisfy the conditions for our linear estimates.

Applying our linear estimates we conclude that

\[
\sup_{[0, T^*]} \|v\| \leq C(\epsilon - \epsilon') \int_0^{T^*} \|\Delta^2 u^\epsilon\|_2 dt \leq C(\epsilon - \epsilon') T_0 M_0.
\]

Hence as $\epsilon - \epsilon' \to 0$ we have $u^\epsilon - u^{\epsilon'} \to 0$ in $C([0, T^*]; L^2)$. So there is a $u \in C([0, T^*]; L^2)$ such that $u^\epsilon \to u$. Since $u^\epsilon \in L^\infty([0, T^*]; H^s)$ and $L^\infty([0, T^*]; H^s)$ is the dual of $L^1([0, T^*]; H^{-s})$ we know there is a subsequence that has a limit in $L^\infty([0, T^*]; H^s)$. But by our first estimate this could only be $u$.

To see that $u \in C([0, T^*]; H^{s-1})$ we simply notice that

\[
\|u(t) - u(t')\|_{H^{s-1}} \leq \|u(t) - u(t')\|_2^{1/s} \|u(t) - u(t')\|_{H^{s-1}/s}. \]
The first term in the right hand side tends to 0 and the second is bounded. Hence $u \in C([0, T^*]; H^{s-1})$.

To see that $u$ is unique we reapply the last argument with $\epsilon = \epsilon' = 0$. We will end up with

$$\sup_{[0,T^*]} \|v\|_2 = 0.$$ 

and therefore $u$ is unique. □

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