On bounded pseudodifferential operators in a high-dimensional setting

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Dedicated to the memory of Bernard Lascar

Abstract

This work is concerned with extending the results of Calderón and Vaillancourt proving the boundedness of Weyl pseudo differential operators $O_{h}^{\text{weyl}}(F)$ in $L^{2}(\mathbb{R}^{n})$. We state conditions under which the norm of such operators has an upper bound independent of $n$. To this aim, we apply a decomposition of the identity to the symbol $F$, thus obtaining a sum of operators of a hybrid type, each of them behaving as a Weyl operator with respect to some of the variables and as an anti-Wick operator with respect to the other ones. Then we establish upper bounds for these auxiliary operators, using suitably adapted classical methods like coherent states.

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1. Introduction.

Since the work of Calderón and Vaillancourt [C-V], it is well-known that, if a function $F$, defined on $\mathbb{R}^{2n}$, is smooth and has bounded derivatives, it is possible to associate with it a pseudodifferential operator which is bounded on $L^{2}(\mathbb{R}^{n})$ (see also [HO], [LER], [R], [U2]). This operator is formally defined by

\begin{equation}
(1.1) \quad (O_{h}^{\text{weyl}}(F)f)(x) = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(x-y)\cdot \xi} F \left( \frac{x + \frac{y}{2}, \xi} \right) f(y)dyd\xi
\end{equation}

for $f$ belonging to $L^{2}(\mathbb{R}^{n})$. Moreover, its norm is bounded above by

\begin{equation}
(1.2) \quad \|O_{h}^{\text{weyl}}(F)\|_{L^{2}(\mathbb{R}^{n})} \leq C \sum_{|\alpha + \beta| \leq N} \|\partial_{x}^{\alpha} \partial_{\xi}^{\beta} F\|_{L^{\infty}(\mathbb{R}^{2n})}
\end{equation}

where $N$ and $C$ depend on the dimension $n$.

The aim of this work is to prove that, under certain conditions, the constants appearing in the upper bound do not depend on the dimension. The set of derivation multi-indices which are used depends on the dimension in a way that will be precisely stated.

We shall thus be able to give examples where the dimension goes to infinity and the norm, nevertheless, remains bounded.

In a later work we shall study pseudodifferential operators where the configuration space $\mathbb{R}^{n}$ will be replaced by an infinite dimensional Hilbert space, by a method differing from Bernard Lascar’s (see [LA1]–[LA10]). These results have been announced in a preprint [A-J-N] in September 2012.

We first recall an example in which the constant appearing in the upper bound on the norm does not depend on the dimension. This is the case when the function $F$ is the Fourier transform of a function $G$ belonging to $L^{1}(\mathbb{R}^{n})$:

\[ F(x, \xi) = (2\pi h)^{-2n} \int_{\mathbb{R}^{2n}} e^{\frac{i}{h}(a \cdot x + b \cdot \xi)} G(a, b) dadb. \]

Since the Weyl operator associated with the function

\[ (x, \xi) \rightarrow E_{a, b, h}(x, \xi) = e^{\frac{i}{h}(a \cdot x + b \cdot \xi)} \]
is the operator $W_{a,b,h}$ defined by

$$ (Op_{h}^{Weyl}(E_{a,b,h})f)(u) = (W_{a,b,h}f)(u) = e^{\frac{h}{\pi}a \cdot u + \frac{h}{\pi}a \cdot b} f(u + b), $$

the equality (1.1) may be rewritten in the form

$$ Op_{h}^{Weyl}(F) = (2\pi h)^{-2n} \int_{\mathbb{R}^{2n}} G(a,b)W_{a,b,h}dadb. $$

Since $W_{a,b,h}$ is unitary,

$$ \|Op_{h}^{Weyl}(F)\| \leq (2\pi h)^{-2n} \int_{\mathbb{R}^{2n}} |G(a,b)| dadb. $$

Situations of this kind have been considered by B. Lascar ([LA1]-[LA10]) in an infinite dimensional setting.

Our approach is different, in that we aim at extending the bound (1.2). Let us specify the set of multi-indices which will be used. Cordes [C], Coifman Meyer [C-M], Hwang [HW] noticed that one does not need all the multi-indices to state (1.2) but only the $(\alpha, \beta)$ satisfying $0 \leq \alpha_j \leq 1$ and $0 \leq \beta_j \leq 1$ for each $j$. In this paper we shall use the multi-indices $(\alpha, \beta)$ such that $0 \leq \alpha_j \leq 4$ and $0 \leq \beta_j \leq 4$ for each $j$. We now can state the hypotheses on the function $F$.

Let $(\rho_j)_{1 \leq j \leq n}$ and $(\delta_j)_{1 \leq j \leq n}$ be two sequences satisfying $\rho_j \geq 0$, $\delta_j \geq 0$ and $\rho_j \delta_j \leq 1$ for every $j \leq n$, let $M$ be a nonnegative real number. Suppose that $(H)$ for every multi-index $(\alpha, \beta)$ such that $0 \leq \alpha_j \leq 4$ and $0 \leq \beta_j \leq 4$ for every $j \leq n$, the partial derivative $\partial^{\alpha}_{x} \partial^{\beta}_{\xi} F$ exists, is continuous, bounded and satisfies

$$ |\partial^{\alpha}_{x} \partial^{\beta}_{\xi} F(x,\xi)| \leq M \prod_{j=1}^{n} \rho^{\alpha_j}_{j} \delta^{\beta_j}_{j}. $$

If $\rho_j = 0$ and $\alpha_j = 0$, we set that $\rho^{\alpha_j}_{j} = 1$.

Our main result is the following Theorem.

**Theorem 1.1.** If a function $F$ defined on $\mathbb{R}^{2n}$ satisfies the hypothesis $(H)$, then the operator $Op_{h}^{Weyl}(F)$, defined formally by (1.1), is bounded in $L^2(\mathbb{R}^{n})$ and satisfies

$$ \|Op_{h}^{Weyl}(F)\|_{L^2(\mathbb{R}^{n})} \leq M \prod_{j=1}^{n} (1 + 57\pi h \rho_j \delta_j) $$

if $0 < h \leq 1$.

**Example 1.2.** Let $V \geq 0$ be a real valued bounded function in $C^\infty(\mathbb{R})$, whose derivatives are all bounded. For all integer $n \geq 1$, set

$$ H_n(x,\xi) = \sum_{j \leq n} \xi^2_j + \sum_{\substack{j \leq n,k \leq n \\mid j-k \leq 1}} g_j g_k V(x_j - x_k) $$

where $(g_j)$ is a sequence of positive numbers such that, for some $C_0 > 0$, we have $g_j \leq C_0 g_k$ if $|j - k| \leq 1$. Set:

$$ P_n(x,\xi) = e^{-H_n(x,\xi)} $$
Hypothesis (H) is satisfied, with $M = 1$, with $\rho_j = C_1$, and $\delta_j = C_1 g_j^2$ where $C_1$ is a real constant depending only on $C_0$. In order to apply Theorem 1.1, we assume that $g_j C_1 \leq 1$ for all $j$. By Theorem 1.1, the norm of $Op_h^{Weyl}(P_n)$ satisfies:

$$\|Op_h^{Weyl}(P_n)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq \prod_{j=1}^n (1 + C_2 h g_j^2)$$

where $C_2 > 0$ is a constant. This norm is bounded independently of $n$ if the sum $\sum_{j \geq 1} g_j^2$ converges. It is also bounded if $g_j$ depends on $n$ and if $g_j^2 \leq 1/n$.

**Example 1.3.** The mean-field approximation uses hamiltonians of the form

$$H_n(x, \xi) = \sum_{j \leq n} \xi_j^2 + \frac{1}{n} \sum_{j \leq n, k \leq n} V(x_j - x_k)$$

where $V$ is as in Example 1.2. Let $P_n$ be the function defined as in (1.5). Then hypothesis (H) is satisfied with $M = 1$ and $\rho_j = \delta_j = C_1$, where $C_1$ does not depend on $n$. In this case, Theorem 1.1 shows that, provided $C_1$ is small enough,

$$\|Op_h^{Weyl}(P_n)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq e^{C_2 h n}$$

where $C_2$ is independent of $n$.

**2. Hybrid Weyl anti-Wick quantization.**

In order to prove Theorem 1.1 we shall split the operator into a sum of operators which will behave as Weyl operators with respect to a first subset of the variables (meaning the operators will be defined by a formula analogous to (1.1) in which only these variables appear) and as anti-Wick operators with respect to the other variables.

We first need to recall the anti-Wick quantization. The definition uses the coherent states, which is the family of functions $\Psi_{X, h}$ indexed by $X = (x, \xi) \in \mathbb{R}^{2n}$, depending on $h > 0$ and defined by

$$(2.1) \quad \Psi_{X, h}(u) = (\pi h)^{-n/4} e^{-\frac{|u - x|^2}{2h}} e^{\frac{i}{\hbar} u \cdot \xi - \frac{i}{2\hbar} x \cdot \xi} \quad X = (x, \xi) \in \mathbb{R}^{2n} \quad u \in \mathbb{R}^n.$$ 

Recall that

$$(2.2) \quad < f, g > = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} < f, \Psi_{X, h} > < \Psi_{X, h}, g > dX.$$ 

If $F$ is a function in $L^\infty(\mathbb{R}^{2n})$, one can associate with it an (anti-Wick) operator $Op_h^{AW}(F)$ such that, for all $f$ and $g$ in $L^2(\mathbb{R}^n)$:

$$(2.3) \quad < Op_h^{AW}(F)f, g > = (2\pi h)^{-n} \int_{\mathbb{R}^{2n}} F(X) < f, \Psi_{X, h} > < \Psi_{X, h}, g > dX.$$ 

We then have

$$(2.4) \quad \|Op_h^{AW}(F)\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq \|F\|_{L^\infty(\mathbb{R}^{2n})}. $$

The relationship between Weyl and anti-Wick quantizations is given, for every $F$ in $L^\infty(\mathbb{R}^{2n})$, by:

$$(2.5) \quad Op_h^{AW}(F) = Op_h^{Weyl}(e^{\frac{i}{\hbar} \Delta} F)$$

where

$$(2.6) \quad \Delta = \sum_{j \leq n} \Delta_j \quad \Delta_j = \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial \xi_j^2}. $$


This fact is classical (see Folland [F]). One has an identity decomposition in $L^\infty(\mathbb{R}^{2n})$:

\begin{equation}
I = \sum_{E \subseteq \{1, \ldots, n\}} T_h(E) e^{\frac{h}{4} \Delta_E} \quad T_h(E) = \prod_{j \in E} \left( I - e^{\frac{h}{4} \Delta_j} \right)
\end{equation}

\begin{equation}
\Delta_E = \sum_{j \in E^c} \Delta_j.
\end{equation}

For every subset $E \subseteq \{1, \ldots, n\}$ and every symbol $F$, we define an operator $Op_h^{hyb,E}(F)$ by:

\begin{equation}
Op_h^{hyb,E}(F) = Op_h^{Weyl} \left( e^{\frac{h}{4} \Delta_E} F \right)
\end{equation}

This operator behaves as a Weyl operator with respect to the variables $x_j (j \in E)$ and as an anti-Wick operator with respect to the variables $x_j (j \in E^c)$. If $E = \emptyset$, it is the anti-Wick operator and conversely if $E = \{1, \ldots, n\}$, it is the Weyl operator.

One derives a decomposition of the Weyl operator $Op_h^{Weyl}(F)$:

\begin{equation}
Op_h^{Weyl}(F) = \sum_{E \subseteq \{1, \ldots, n\}} Op_h^{hyb,E}(T_h(E) F).
\end{equation}

We shall now prove an upper bound on the norm of a hybrid operator $Op_h^{hyb,E}(G)$, where the function $G$ is bounded on $\mathbb{R}^{2n}$. The only derivatives of $G$ which will play a role are the derivatives with respect to $x_j$ or $\xi_j$ with $j \in E$. For every integer $m$ we introduce the set of multi-indices

\begin{equation}
I_m(E) = \{ (\alpha, \beta), \quad \alpha_j \leq m, \quad \beta_j \leq m, \quad (1 \leq j \leq n) \quad \alpha_j = \beta_j = 0 \quad \text{if} \quad j \notin E \}.
\end{equation}

We shall prove the following Lemma in Section 3, by adapting classical methods (Unterberger [U2]).

**Lemma 2.1.** If $F$ satisfies hypothesis (H) and if $E \neq \emptyset$, then

\begin{equation}
\left\| Op_h^{hyb,E}(F) \right\|_{L(L^2(\mathbb{R}^n))} \leq \left( \frac{25 \pi}{2} \right)^{|E|} \sum_{(\alpha, \beta) \in I_2(E)} h^{(|\alpha|+|\beta|)/2} \left\| \partial_x^{\alpha} \partial_{\xi}^{\beta} F \right\|_{L^\infty(\mathbb{R}^{2n})}
\end{equation}

We shall establish the following Lemma in Section 4.

**Lemma 2.2.** If $F$ satisfies hypothesis (H) with $\rho_j = \delta_j \leq 1$ for every $j \leq n$, if $0 < h \leq 1$ and if $E \neq \emptyset$, the function $T_h(E) F$ satisfies

\begin{equation}
\sum_{(\alpha, \beta) \in I_2(E)} h^{(|\alpha|+|\beta|)/2} \left\| \partial_x^{\alpha} \partial_{\xi}^{\beta} T_h(E) F \right\|_{L^\infty(\mathbb{R}^{2n})} \leq M \left( \frac{9h}{2} \right)^{|E|} \prod_{j \in E} \rho_j^2
\end{equation}

where $T_h(E)$ is defined in (2.7).

**End of the proof of Theorem 1.1.** Let us consider first the case when $\rho_j = \delta_j$ for every $j \leq n$. According to (2.10), we have

\[ \| Op_h^{Weyl}(F) \|_{L(L^2(\mathbb{R}^n))} \leq \sum_{E \subseteq \{1, \ldots, n\}} \| Op_h^{hyb,E}(T_h(E) F) \|_{L(L^2(\mathbb{R}^n))}. \]
By Lemma 2.1:

\[\|\Omega_{\rho}^{Weyl}(F)\|_{L^2(\mathbb{R}^n)} \leq \sum_{E \subseteq \{1, \ldots, n\}} \frac{25\pi}{2} |E| \sum_{(\alpha, \beta) \in I_2(E)} h^{(|\alpha|+|\beta|)/2} \|\partial_x^\alpha \partial_\xi^\beta T_h(E) F\|_{L^\infty(\mathbb{R}^n)}.\]

If \(\rho_j = \delta_j\) for \(0 < h \leq 1\), Lemma 2.2 shows that

\[\|\Omega_{\rho}^{Weyl}(F)\|_{L^2(\mathbb{R}^n)} \leq M \sum_{E \subseteq \{1, \ldots, n\}} \left(\frac{9 \times 25\pi h}{4}\right)^{|E|} \prod_{j \in E} \rho_j^2\]

It follows easily that

\[\|\Omega_{\rho}^{Weyl}(F)\|_{L^2(\mathbb{R}^n)} \leq M \prod_{j \leq n} \left(1 + 57h\rho_j^2\right)\]

The theorem is proved in the case when \(\rho_j = \delta_j\) for all \(j \leq n\). In the general case, the announced result follows by applying dilations.

3. Proof of Lemma 2.1.

We begin by recalling the results of Unterberger [U1], [U2] concerning the upper bound of \(\langle A\Psi_{X,h}, \Psi_{Y,h} \rangle\), where \(A\) is a pseudodifferential operator and the \(\Psi_{X,h}\) are the coherent states defined by (2.1).

**Lemma 3.1.** For every function \(G\) satisfying the hypothesis (H), for every \(X = (x, \xi)\) and \(Y = (y, \eta)\) in \(\mathbb{R}^{2n}\):

\[
\prod_{j \leq n} \left(1 + |x_j - y_j|^2 \right) \left(1 + |\xi_j - \eta_j|^2 \right) |\langle \Omega_{\rho}^{Weyl}(G) \Psi_{X,h}, \Psi_{Y,h} \rangle| \leq \ldots
\]

\[
\ldots \leq 25^n \sum_{(\alpha, \beta) \in I_2(\{1, \ldots, n\})} h^{(|\alpha|+|\beta|)/2} \|\partial_x^\alpha \partial_\xi^\beta G\|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)}.\]

Indeed, it suffices to follow the proof of [U2], which uses Wigner functions. In particular, the constant 25 appears in the computations of [U2].

**End of the proof of Lemma 2.1.** For every subset \(E\) of \(\{1, \ldots, n\}\), the variable \(x\) of \(\mathbb{R}^n\) can be denoted, in a natural way, by \(x = (x_E, x_{\overline{E}})\) and the variable \(X\) of \(\mathbb{R}^{2n}\) can be denoted by \(X = (X_E, X_{\overline{E}})\). For every \(X_E = (x_E, \xi_E)\) in \(\mathbb{R}^E \times \mathbb{R}^E\) and every \(h > 0\), one can define a function in \(L^2(\mathbb{R}^E)\) by

\[
\Psi_{X_E,h}(u) = \sqrt{\pi h}^{-1/4} e^{-\frac{|x_j - y|^2}{\pi h}} e^{\frac{i}{h} \alpha_j x_j - \frac{1}{h} \beta_j \xi_j} u \in \mathbb{R}^E
\]

One will write \(\Psi_{X,h} = \Psi_{X_E,h} \otimes \Psi_{X_{\overline{E}},h}\). For every function \(F\) on \(\mathbb{R}^{2n}\) and every \(Z_{\overline{E}}\) in \((\mathbb{R}^2)^{\overline{E}}\), we denote by \(F_{Z_{\overline{E}}}\) the function on \((\mathbb{R}^2)^{E}\) defined by \(X_E \mapsto F(X_E, Z_{\overline{E}})\). For every function \(\varphi\) on \((\mathbb{R}^2)^{E}\) satisfying (H), we denote by \(\Omega_{\rho}^{Weyl,E}(\varphi)\) the operator defined as in (1.1), with \(E\) replacing \(\{1, \ldots, n\}\), that is the operator in \(L^2(\mathbb{R}^E)\) formally defined, for every \(f\) in \(L^2(\mathbb{R}^E)\) by

\[
(\Omega_{\rho}^{Weyl,E}(\varphi)f)(u_E) = (2\pi h)^{-|E|} \int_{\mathbb{R}^E \times \mathbb{R}^E} e^{\frac{ix_E - y_E}{2 \rho} \xi_E} \varphi \left(\frac{x_E + y_E}{2}, \xi_E\right) f(y_E)dy_E d\xi_E
\]

for \(u_E\) in \(\mathbb{R}^E\). With these notations the hybrid operator associated with the function \(F\) and the subset \(E\) satisfies
We first note that

\[
\Phi_h(X_E, Y_E, Z_{E^c}) = < f, \Psi_{X,h} \otimes \Psi_{Z_{E^c},h} > < \Psi_{Y,E,h} \otimes \Psi_{Z_{E^c},h}, g > dX_E dY_E dZ_{E^c}
\]

where:

\[
\Phi_h(X_E, Y_E, Z_{E^c}) = (2\pi h)^{-n-|E|} < Op^{weyl,E}_h(F_{Z_{E^c}}) \Psi_{X,E,h}, \Psi_{Y,E,h} > L^2(\mathbb{R}^E)
\]

One then applies Lemma 3.1 with \(\{1, \ldots, n\}\) replaced by \(E, G\) replaced by \(F_{Z_{E^c}}, \Psi_{X,h}\) and \(\Psi_{Y,h}\) replaced by \(\Psi_{X,E,h}\) and \(\Psi_{Y,E,h}\). One gets, using (3.4),

\[
| < Op^{h,y,E}_h(F), g > | \leq I_h(f, g) \sum_{(\alpha, \beta) \in I_2(E)} h^{(|\alpha|+|\beta|)/2} \| \partial_x^\alpha \partial_\xi^\beta F \|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)}
\]

where

\[
I_h(f, g) = \int_{(\mathbb{R}^E) \times (\mathbb{R}^E)} K_h(X_E - Y_E) | < f, \Psi_{X,E,h} \otimes \Psi_{Z_{E^c},h} > | < \Psi_{Y,E,h} \otimes \Psi_{Z_{E^c},h}, g > | dX_E dY_E dZ_{E^c}
\]

and

\[
k_h(X_E - Y_E) = 25^{|E|}(2\pi h)^{-n-|E|} \prod_{j \in E} \left( 1 + \frac{|x_j - y_j|^2}{h} \right)^{-1} \left( 1 + \frac{|\xi_j - \eta_j|^2}{h} \right)^{-1}.
\]

By Schur’s Lemma

\[
|I_h(f, g)| \leq (25h \pi^2)^{|E|}(2\pi h)^{-n-|E|} J_h(f)^{1/2} J_h(g)^{1/2}
\]

\[
J_h(f) = \int_{\mathbb{R}^{2n}} | < f, \Psi_{X,E,h} \otimes \Psi_{Z_{E^c},h} > |^2 dX_E dX_{E^c}.
\]

According to (2.2), \(J_h(f) = (2\pi h)^n \| f \|_{L^2(\mathbb{R}^n)}^2\). Hence

\[
|I_h(f, g)| \leq \left( \frac{25^{|E|}}{2} \right) \| f \|_2 \| g \|_2.
\]

Lemma 2.1 follows from that.

\section*{4. Proof of Lemma 2.2.}

We first note that

\[
e^{\frac{h}{40} \Delta_j} - I = \frac{h}{4} \Delta_j V_{jh}
\]

where \(V_{jh}\) is the operator defined by

\[
(V_{jh} F)(x, \xi) = (\pi h)^{-1} \int_{\mathbb{R}^2 \times [0,1]} e^{-\frac{h}{4}(u^2 + v^2)} 2\theta F(x + \theta \alpha_1, \xi + \theta \alpha_2) du dv d\theta
\]

where \((\alpha_1, \alpha_2)\) is the canonical basis of \(\mathbb{R}^n\). We then observe that the operator \(V_{jh}\) is bounded in \(L^\infty(\mathbb{R}^n \times \mathbb{R}^n)\) and that its norm is smaller than 1. Consequently, for every multi-index \((\alpha, \beta)\) in \(I_2(E)\):

\[
\| \partial_x^\alpha \partial_\xi^\beta T_h(E) F \|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \leq \left( \prod_{j \in E} M_j \right) \| F \|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)}
\]

\text{where:}

\[
\frac{h}{4} \Delta_j = \frac{h}{4} V_{jh}^{-1} V_{jh}
\]
where, for every $j \in E$:

$$M_j = \frac{h}{4} \Delta_j \frac{\partial^{\alpha_j}}{\partial x_j^{\alpha_j}} \frac{\partial^{\beta_j}}{\partial \xi_j^{\beta_j}}$$

Hence, for every multi-index $(\alpha, \beta)$ in $I_2(E)$:

$$\|\partial^\alpha_x \partial^\beta_\xi T_h(E) F \|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \leq 2^{|E|} \sup_{(p,q) \in \tilde{I}_4(E)} \|\partial^p_x \partial^q_\xi F \|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \left( \frac{h}{4} \right)^{|E|}$$

where $\tilde{I}_4(E)$ is the set of multi-indices $(p, q)$ in $I_4(E)$ such that $p_j + q_j \geq 2$ for every $j \in E$. If $F$ satisfies (H), with $\rho_j = \delta_j \leq 1$ for each $j \leq n$, we have, for each $(p, q)$ in $\tilde{I}_4(E)$, provided $0 < h \leq 1$:

$$\|\partial^p_x \partial^q_\xi F \|_{L^\infty(\mathbb{R}^n \times \mathbb{R}^n)} \leq M \prod_{j \in E} \rho_j^2$$

Since the number of elements of $I_2(E)$ is $9^{|E|}$, it gives (2.14), provided $0 < h \leq 1$. This completes the proof of Lemma 2.2.

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