OSCILLATORY STATES AND PATTERNS FORMATION IN A TWO-CELL CUBIC AUTOCATALYTIC REACTION-DIFFUSION MODEL SUBJECTED TO THE DIRICHLET CONDITIONS

H. M. SRIVASTAVA*

Department of Mathematics and Statistics, University of Victoria
Victoria, British Columbia V8W 3R4, Canada
Department of Medical Research, China Medical University Hospital
Taichung 40204, Taiwan
Department of Mathematics and Informatics, Azerbaijan University
71 Jeyhun Hajibeyli Street, AZ1007 Baku, Azerbaijan

H. I. ABDEL-GAWAD

Department of Mathematics, Faculty of Science, Cairo University
Al Orman, Giza 12613, Egypt

KHALED MOHAMMED SAAD

Department of Mathematics, College of Arts and Sciences, Najran University
Najran, Kingdom of Saudi Arabia
Department of Mathematics, Faculty of Applied Science, Taiz University
Taiz, Yemen

Abstract. The approximate solutions of a two-cell reaction-diffusion model equation subjected to the Dirichlet conditions are obtained. The reaction is assumed to occur in the presence of cubic autocatalyst which decays to an inert compound in the first cell. Coupling with the reactant is assumed to be cubic in the concentrations. A linear exchange in the concentration of the reactant is taken between the two cells. The formal exact solution is found analytically. Here, in this investigation, use is made of the Picard iterative scheme which is constructed and applied after the exact one. The results obtained are compared with those found by means of a numerical method. It is observed that the solution obtained here is symmetric with respect to the mid-point of the container. The travelling wave is expected due to the parity of the space operator and the symmetric boundary conditions. Symmetric patterns, including among them a parabolic one, are observed for a large time.

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* Corresponding author: H. M. Srivastava.
When the initial conditions are periodic, the most dominant modes travel at a constant speed for a large time. This phenomenon is highly affected by the rate of decay of the autocatalyst to an inert compound. The present work is of remarkably significant interest in chemical engineering as well as in other physical sciences. For example, in chemical industry, the objective is to achieve a great yield of a given product, which is carried by controlling the initial concentration of the reactant. Furthermore, in the last section on conclusions, we have cited many potentially useful recent works related to the subject-matter of this investigation in order to provide incentive and motivation for making further advances by using space-time fractional derivatives along the lines of the problem of finding approximate analytical solutions of the reaction-diffusion model equations which we have discussed in this article.

1. Introduction, motivation and preliminaries. Experimental studies of the pattern dynamics in Faraday waves (see, for example, [5] and [6]) reveal an ordered time-average, which is metric that measures the average over a defined period of time. The form of the average pattern is induced by the symmetry imposed by the boundary conditions. The numerical simulations [4] on the Kuramoto-Shivashinsky equation under free boundary conditions show that the system has a parabolic-averaged pattern. The stability of spatial patterns, near the stationary-state solutions, in diffusion reactions subjected to appropriate boundary conditions has been studied currently in the literature (see, for example, [17] to [12]). In the presence of the decay of the autocatalyst, spatial patterns are expected to be stable. The results found in the present case consolidate the last claim. In these reactions, waves oscillations, which are induced by the boundary conditions, are produced. Similar phenomena have been observed in biological systems such as in enzyme reactions (see [23]).

Our aim here is to study the oscillations which are induced by the boundary of a container and are produced in a cubic two-cell reaction in an open system. The mutual interaction between the two cells is studied by considering the linear diffusion interchange of the reactant and the autocatalyst. This may occur, practically, via a semi-permeable membrane interface between the two regions. Here, in this investigation, we consider the following model studied in (for example) [18] to [19] (see also [20]) for a system in an infinite domain. In these earlier works, attention was focused upon the initial-value problem. We mention that the model which will be studied here holds true for open systems. The reaction is described as follows. In the first cell, it is given by a cubic autocatalyst which decays to an inner product $C$:

$$A + 2B \rightarrow 3B \quad \text{(rate = $k_1ab^2$)}$$

and

$$B \rightarrow C \quad \text{(rate = $k_2b$)}.$$

In the second cell, the reaction is given purely by interchange of the reactant through the two cells while the autocatalyst concentration is taken to be zero. Also, it is assumed that $A$ interchanges from one cell to the other through mass exchange rate $k_3$. We mention that, in closed systems, the model will be completely different from the equations (1.1), (1.2) and (1.3) below. The mass exchange will be expressed by a condition at the frontiers of the two regions.

The aim of this work is to investigate the self-oscillation produced by the reactions in a system of finite size. It will be shown that the effects of the size of the system plays an eminent role in increasing the number of oscillations. observed in the figures that show the evolution of the concentrations reactant and the autocatalyst. Here, in this paper, we confine ourselves to the one-dimensional model equations which describe such autocatalytic reactions. They are given, in the first cell, by

$$a_{t} = D_Aa_{XX} + k_3(a_2 - a_1) - k_1a_1b^2 \quad (1.1)$$

and

$$b_{t} = D_Bb_{XX} - k_2b + k_1a_1b^2, \quad (1.2)$$
and, in the second cell, by
\[ a_{2r} = D_A a_2XX + k_3(a_1 - a_2) \quad ((X, t) \in (0, \ell) \times (0, \infty)). \] (1.3)

In the above equations (1.1), (1.2) and (1.3), \( a_1, b \) and \( a_2 \) are the concentrations of the reactant \( A \) and the autocatalyst \( B \) in the two cells, respectively. The coefficients \( D_A \) and \( D_B \) are the diffusion coefficients of the species \( a \) and \( b \) in the two regions. The decay rates \( k_1, k_2 \) and \( k_3 \) are assumed to be positive.

The Dirichlet boundary conditions are taken as follows:
\[ a_i(0, t) = a_i(\ell, t) = a_0 \quad (i = 1, 2) \quad \text{and} \quad b(0, t) = b(\ell, t) = 0. \] (1.4)

We mention that the boundary values in (1.4) are taken to be the equilibrium values of the concentrations. We refer the reader to the equations (1.1), (1.2) and (1.3), which will be found in the next section (Section 2). We note also that the mechanism of the cubic autocatalysis is similar to those in many reaction-diffusion models in mathematical biology (see, for example, [7] and [11]).

We remark in passing that, in the last section (Section 5) on conclusions, we have cited many potentially useful recent works related to the subject-matter of this investigation in order to provide incentive and motivation for making further advances by using space-time fractional derivatives along the lines of the problem of finding approximate analytical solutions of the reaction-diffusion model equations which we have discussed in our present investigation.

2. Stability analysis. The linear stability analysis of the equilibrium states (ES) of the system, given by the equations (1.1), (1.2) and (1.3) against homogeneous and inhomogeneous disturbances, is studied here. In the absence of the diffusion term in the equations (1.1), (1.2) and (1.3), the ES of these equations is obtained as the solution of the following algebraic equations:
\[ f_1 \equiv k_3(a_2 - a_1) - k_1 a_1 b^2 = 0, \] (2.1)
\[ f_2 \equiv -k_2 b + k_1 a_1 b^2 = 0 \] (2.2)
and
\[ f_3 \equiv k_3(a_1 - a_2) = 0. \] (2.3)

These equations lead to the solution given by
\[ b = 0 \quad \text{and} \quad a_1 = a_2 = a_0, \] (2.4)

where
\[ a_0 = \frac{k_2}{k_1} \]
or otherwise.
The linear stability analysis requires the finding of the eigenvalues of the Jacobian matrix, which are evaluated from the following equation:

\[
\begin{vmatrix}
\frac{\partial f_1}{\partial a_1} & \frac{\partial f_1}{\partial b} & \frac{\partial f_1}{\partial a_2} \\
\frac{\partial f_2}{\partial a_1} & \frac{\partial f_2}{\partial b} & \frac{\partial f_2}{\partial a_2} \\
\frac{\partial f_3}{\partial a_1} & \frac{\partial f_3}{\partial b} & \frac{\partial f_3}{\partial a_2}
\end{vmatrix} - \lambda I = det
\begin{pmatrix}
-k_3 - \lambda & 0 & k_3 \\
0 & -k_2 - \lambda & 0 \\
-k_3 & 0 & -k_3 - \lambda
\end{pmatrix} = 0, \tag{2.5}
\]

where \( f_i \) \((i = 1, 2, 3)\) are given by (2.1), (2.2) and (2.3). The eigenvalues of the matrix in (2.5) are 0, \(-k_2\), and \(-2k_3\). The reason is that \( a_1 = a_2 \) can take on any value: That their values have to be equal to \( a_0 \) comes in only through the boundary conditions, but not through the functions \( f_i \) \((i = 1, 2, 3)\). Thus, for the homogeneous equations, there is a family of equilibria, and they are all stable.

We now study the stability of the HSSS of the equations (2.1), (2.2) and (2.3) against spatial disturbance of small amplitude by taking into account the diffusion term. We mention that, if a stable HSSS is linearly unstable against spatial disturbance, then the Turing instability occurs (see [13]–[30]). We assume that HSSS undergoes an initial disturbance as follows:

\[
a_i(X,0) = a_{is} + \epsilon \overline{a_i}(X), \tag{2.6}
\]

\[
b(X,0) = \epsilon \overline{b}(X) \tag{0 < \epsilon \ll 1} \tag{2.7}
\]

and

\[
|\overline{a_i}(X)|, |\overline{b}(X)| < 1 \quad (i = 1, 2; \ X \in [0, \ell]).
\]

For \( t > 0 \), the disturbance in the concentrations are taken as follows:

\[
a_i(X,t) = a_{is} + \epsilon \overline{a_i}(X,t) \quad (i = 1, 2) \tag{2.8}
\]

and

\[
b(X,t) = \epsilon \overline{b}(X,t). \tag{2.9}
\]

We assume that the disturbances \( \overline{a_i}(X,t) \) \((i = 1, 2)\) and \( \overline{b}(X,t) \) satisfy the Dirichlet boundary conditions:

\[
\overline{a_i}(0,t) = \overline{a_i}(\ell,t) = \overline{b}(0,t) = \overline{b}(\ell,t) = 0. \tag{2.10}
\]

Upon substituting from the (2.6) and (2.7) into the equations (1.1), (1.2) and (1.3), if we neglect the terms of order \( \epsilon^2 \) and higher, we get

\[
\overline{a}_{1t} = D_A \overline{a}_{1XX} + k_3(\overline{a}_2 - \overline{a}_1), \tag{2.11}
\]

\[
\overline{b}_t = D_B \overline{b}_{XX} - k_2 \overline{b} \tag{2.12}
\]

and

\[
\overline{a}_{2t} = D_A \overline{a}_{2XX} + k_3(\overline{a}_1 - \overline{a}_2), \tag{2.13}
\]

The condition (2.10) suggests the assumption that the disturbance \( \overline{a}_i \) \((i = 1, 2)\) and \( \overline{b} \) take the following form:

\[
\overline{a}_i = \sum_{n=1}^{\infty} a_{in} e^{\omega_n t} \sin\left(\frac{n\pi}{\ell}X\right) \quad (i = 1, 2) \tag{2.14}
\]
and
\[ \tilde{f} = \sum_{n=1}^{\infty} b_n e^{\omega_n t} \sin \left( \frac{n\pi}{L} X \right). \]  
(2.15)

We now substitute from the equations (2.14) and (2.15) into the equations (2.11), (2.12) and (2.13). For a non-zero \( a_i \) in \((1, 2)\) and \( b_n \), we thus find that
\[
\begin{vmatrix}
D_A \lambda_n^2 + \omega_n + k_3 & 0 & -k_3 \\
0 & D_B \lambda_n^2 + \omega_n + k_2 & 0 \\
-k_3 & 0 & D_A \lambda_n^2 + \omega_n + k_3 \\
\end{vmatrix} = 0,
\]
(2.16)

where \( \lambda_n = \frac{n\pi}{L} \).

The equation (2.16) can be solved to get
\[
\omega_n = -(D_A \lambda_n^2, -(D_B \lambda_n^2 + k_2), -(D_A \lambda_n^2 + 2k_3)).
\]
(2.17)

Since \( \omega_n \) is negative for all \( n \), the HSSS is stable under a small spatial disturbance. We, therefore, conclude that the ES is stable against perturbations of the homogeneous and inhomogeneous cases. That is, no Turing instability occurs in either of these cases. This may be argued by using the fact that we have normalized the nonlinear coupling rate \( k_1 \) to be 1. We observe that, if the rate \( k_1 \) is left non-normalized and the linear stability analysis is carried out by using the approach in [15], different patterns formation may be revealed.

In what follows, we shall carry out a nonlinear analysis of the system given by the equations (1.1), (1.2) and (1.3) and thereby investigate the effects of taking into account for the nonlinear coupling terms on spatial configurations.

3. Nonlinear analysis. In this section, in order to carry out the nonlinear analysis of the equations (1.1), (1.2) and (1.3) subjected to the boundary conditions (3.4), we use the method proposed in [2] to find an approximate analytical solution for this system.

First of all, we give a brief account for the method proposed in [2].

3.1. Formulation of the method. We propose the above-mentioned method to analytically find the formal exact solution of the equations (1.1), (1.2) and (1.3) that satisfies the boundary conditions (3.4). The Picard iterative sequence of the solutions is then constructed and applied. We truncate the sequence of iterations at the first approximation. The results of Section 2 suggest that, without loss of generality, we can take \( D_A = D_B \).

We proceed by making the scaling transformations given by
\[
a_i = a_0 \alpha_i \quad (i = 1, 2), \quad b = a_0 \beta, \quad t = k_1 a_0^2 \tau \quad \text{and} \quad x = \left( \frac{k_1^2 a_0^2}{D} \right)^{\frac{1}{2}} X.
\]

Then the equations (1.1), (1.2) and (1.3) become
\[
\alpha_1 t = \alpha_{1xx} + \gamma (\alpha_2 - \alpha_1) - \alpha_1 \beta^2,
\]
(3.1)
\[
\beta_t = \beta_{xx} - k \beta + \alpha_1 \beta^2
\]
(3.2)
and
\[
\alpha_2 t = \alpha_{2xx} + \gamma (\alpha_1 - \alpha_2) \quad ((x, t) \in (0, L) \times (0, \infty)),
\]
(3.3)

where \( L = \left( \frac{k_1^2 a_0^2}{D} \right)^{\frac{1}{2}} \).

The boundary conditions are given by
\[
\alpha_i (0, t) = \alpha_i (L, t) = 1 \quad (i = 1, 2) \quad \text{and} \quad \beta (0, t) = \beta (L, t) = 0.
\]
(3.4)
In the equations (3.1), (3.2) and (3.3), the dimensionless constants $k$ and $\gamma$ are given by

$$k = \frac{k_2}{k_1a_0} \quad \text{and} \quad \gamma = \frac{k_3}{k_1a_0}.$$ 

These dimensionless constants $k$ and $\gamma$ represent the strength of the autocatalyst decay and the coupling between the two regions, respectively. For convenience, we make the transformation $\alpha_i = 1 - u_i$ ($i = 1, 2$) in the equations (3.1), (3.2) and (3.4). We thus find that

$$u_{1t} = u_{1xx} + \gamma(u_2 - u_1) + (1 - u_1)\beta^2, \quad \beta_t = \beta_{xx} - k\beta + (1 - u_1)\beta^2$$ 

and

$$u_{2t} = u_{2xx} + \gamma(u_1 - u_2) \quad ((x, t) \in (0, L) \times (0, \infty)).$$ 

The boundary conditions become

$$u_i(0, t) = u_i(L, t) = 0 \quad (i = 1, 2) \quad \text{and} \quad \beta(0, t) = \beta(L, t) = 0.$$ 

We rewrite the set of the equations (3.5), (3.6) and (3.7) by using the notion of vectors and matrices as follows:

$$\vec{U}_t = \hat{M}\vec{U} + \vec{S}(\vec{U}), \quad (3.9)$$

$$\hat{M} = \begin{pmatrix} \partial_{xx} - \gamma & 0 & \gamma \\ 0 & \partial_{xx} - k & 0 \\ \gamma & 0 & \partial_{xx} - \gamma \end{pmatrix} \quad (3.10)$$

and

$$\vec{S}(\vec{U}) = \begin{pmatrix} (1 - u_1)\beta^2 \\ (1 - u_1)\beta^2 \end{pmatrix},$$

$$\vec{U} = \vec{U}_L + \int_0^t e^{(t-t_1)\hat{M}}\vec{S}(\vec{U}(x, t_1))dt_1.$$ 

(3.11)

As it was observed in [1, 19], the formal exact solution of the equations (3.9), (3.10) and (3.11) is given by

$$\vec{U}_t = \vec{U}_L + \int_0^t e^{(t-t_1)\hat{M}}\vec{S}(\vec{U}(x, t_1))dt_1.$$ 

(3.12)

Furthermore, the Picard iteration scheme is given by

$$\vec{U} = \vec{U}_L + \int_0^t \int_0^t e^{(t-t_1)\hat{M}}\vec{S}(\vec{U}(x, t_1))dt_1, \quad \vec{U} = \begin{pmatrix} u_1 & \beta & u_2 \end{pmatrix}^T.$$ 

(3.13)

The solution of the linear problem is given by

$$\vec{U} = e^{\hat{M}t}\vec{U}(0).$$ 

(3.14)

We require that $\vec{U}_L$ satisfies the boundary conditions

$$u_i(0, t) = u_i(L, t) = 0 \quad (i = 1, 2) \quad \text{and} \quad \beta(x = 0, t) = \beta(x = L, t) = 0.$$ 

We take $\vec{U}(x, 0)$ to satisfy the boundary conditions, namely,

$$\vec{U}(0) = \sum_{n=0}^\infty \begin{pmatrix} a_n & b_n & c_n \end{pmatrix}^T [\sin(nx) + \sin(n(L - x))].$$ 

(3.15)

Thus, clearly, the linear solution (3.14) becomes

$$\vec{U}(0) = \sum_{n=0}^\infty \psi_n(t) [\sin(nx) + \sin(n(L - x))] e^{-\bar{n}^2t} \quad \text{and} \quad \bar{n} = \frac{n\pi}{L}.$$ 

(3.16)
where

\[
\vec{\psi}_n(t) = \begin{pmatrix} \lambda_+(t)a_n + \lambda_-(t)c_n \\ \rho(t)b_n \\ \lambda_-(t)a_n + \lambda_+(t)c_n \end{pmatrix},
\]

(3.17)

and

\[
\lambda_\pm(t) = \frac{1}{2} (1 \pm e^{-2\gamma t})
\]

(3.18)

We note that all modes generated in the container of size $L$ are damping. By using the Picard iterative scheme given by (3.13), we get the first approximation as follows:

\[
\vec{U}^{(1)}(x,t) = \vec{U}^{(0)}(x,t) + \int_0^t e^{(t-t_1)\vec{M}} \vec{S}^{(0)}(\vec{U}^{(0)}(x,t_1)) dt_1,
\]

(3.23)

where

\[
\vec{S}^{(0)}(\vec{U}^{(0)}(x,t_1)) = \begin{pmatrix} \gamma \left( 1 - u_{11}^{(0)}(x,t_1) (\beta^{(0)})^2(x,t_1) \right) \\ \gamma \left( 1 - u_{11}^{(0)}(x,t_1) (\beta^{(0)})^2(x,t_1) \right) \\ 0 \end{pmatrix}.
\]

(3.24)

3.2. Approximate analytical solutions for the Dirichlet problem. We distinguish two cases when $\gamma < 1$ and $\gamma > 1$. We first consider the case when $\gamma < 1$ and make the scaling transformation $t \to \gamma t$ in the equation (3.16).

In the original set of variables, we have for the concentration $\alpha_i^{(0)}$ and $\beta^{(0)}$ given by

\[
\alpha_i^{(0)} = 1 - \frac{1}{2} \sum_{n=1}^{\infty} (\lambda_+(t) a_n + \lambda_-(t) c_n) \sin \left( \frac{n \pi}{L} \right) \cos \pi \left( \frac{L}{2} - x \right) e^{-\pi^2 \gamma t},
\]

(3.20)

\[
\beta^{(0)} = \rho(\gamma t) \sum_{n=0}^{\infty} b_n \sin \left( \frac{n \pi}{L} \right) \cos \pi \left( \frac{L}{2} - x \right) e^{-\pi^2 \gamma t}
\]

(3.21)

and

\[
\alpha_2^{(0)} = 1 - \frac{1}{2} \sum_{n=0}^{\infty} (\lambda_-(t) a_n + \lambda_+(t) c_n) \sin \left( \frac{n \pi}{L} \right) \cos \pi \left( \frac{L}{2} - x \right) e^{-\pi^2 \gamma t}.
\]

(3.22)

We note that all modes generated in the container of size $L$ are damping. By introducing the Fourier sine series, the first approximation is given by

\[
\vec{U}^{(1)}(x,t) = \vec{U}^{(0)}(x,t) + \int_0^t e^{(t-t_1)\vec{M}} \vec{S}^{(0)}(\vec{U}^{(0)}(x,t_1)) dt_1,
\]

(3.25)

and

\[
\vec{\phi}_r = \frac{2}{L} \int_0^L \vec{\psi}_r (\gamma (t-t_1) \gamma (1 - u_{11}^{(0)}(x,t_1)) (\beta^{(0)})^2(x,t_1) \sin(\tilde{r}x) \, dx,
\]

(3.26)

where

\[
\tilde{r} = \frac{r \pi}{L}
\]

and $\vec{\psi}_r$ is given by the equations (3.17), (3.18) and (3.19). In the equations (3.25) and (3.26), we take the initial conditions for $\beta$ near its ES (namely, $\beta = 0$). That is, we take $b_n$ to be sufficiently small for the system in order to attain its HSSS. On the other hand,
through numerical calculations, we shall consider the perturbation in the concentrations $a_n$ and $c_r$ as follows:

$$O(a_n) = O(c_r) \leq 4 \times 10^{-2},$$

where the terms of cubic order, namely,

$$\sum_{n,m,s} O(a_nb_mc_s) < 10^{-6}$$

will be neglected.

In order to find approximate analytical solutions of the problem under consideration, we turn to the equations (3.1), (3.2) and (3.3), providing that $\gamma > 1$. We make use of the transformation of the old $t \rightarrow \gamma t$ into the equations (3.1), (3.2) and (3.3). The results of the first approximation of the concentrations found here when $\alpha_i = 1 - u_i$ ($i = 1, 2$) and $\beta$ are given by the equations (3.16), (3.25) and (3.26). These results hold true also in the new situation when $t \rightarrow \frac{1}{\gamma^2}$.

These above-derived results are displayed in a set of figures by varying the set of relevant parameters in order to visualize the scope of the different aspects of the problem.

3.3. Numerical results. The results given by the equations (3.16), (3.25) and (3.26) for $\alpha_i$ ($i = 1, 2$) and $\beta$ are displayed against $x$ in Figures 1, 2 and 3 for different values of $t$.

In Figure 1, the concentration $\alpha_1$ is displayed for when $a_n = 0.1$, $b_n = 0.002$, $c_n = 0.1$, $a_r = 0.001$ and $c_r = 0.001$. The other parameters are taken as follows: $L = 100$, $k = 0.01$ and $\gamma = 0.1$.

Figures 1(a), 1(b) and 1(c) correspond to the values of $t = 30$ and $t = 100$, respectively.

In Figures 2 and 3, the concentrations $\beta$ and $\alpha_2$ are displayed against $x$ for the same caption as in Figures 1(a), 1(b) and 1(c). From these figures, we find that the variation of the concentration is more significant near the boundaries, while it is very small far from the boundaries. In the initial stage, the amplitude of the oscillations decays rapidly with time and with increasing distance from the boundaries. Experimental studies [5] and [6] and numerical simulation [4] show the same behavior. The most dominant modes are generated near the sidewalls of the container with the wave number $\lambda$, satisfying the condition $\pi \lambda = 4$.

These figures show that the oscillations of small amplitudes disappear in the central region, the symmetric pattern with respect to the mid-point of the container, namely, $x = \frac{L}{2}$, propagates towards a parabolic pattern [9]. The two dominant modes generated near the sidewalls, which travel, one from the left to the right, which is near $x = 0$, and the other one, near $x = L$, that travels from the right to the left (see Figures 1(d) and 3(d)). These two waves are of initial wave number $\lambda = \frac{\pi}{4}$ and their minimum values are located at $x = 4$ and 96, respectively. At $t = 100$, their minimum values are located at $x = 6$ and 94, respectively. Further more, their wave number becomes $\frac{\pi}{15}$. They are enlarged and joined by a plateau in the central region. Thus, clearly, the permanent traveling waves solution exists in the systems of finite size with periodic initial conditions and for moderate values of the size of the system.

The effects of varying the normalized rate of interchange of the reactant through the membrane of the second cell $\gamma$ lead to no resulting change of the autocatalyst. The effects of the decay rate $k$ and the size of the container $L$ on the space-time variation of the concentration of the reactant and the autocatalyst are visualized numerically and these effects are not produced here. We have found that the oscillation of the concentration is significantly diminished in the central region when the values of $\gamma$ increase. When varying the parameter $k$, we find that it has a weak effect on the concentration $\alpha_1$. We remark also that the variation in the concentration in the second cell is more significant than that in the first cell. This may be argued by noting the presence of the decay of the autocatalyst
The concentration of the reactant $\alpha_1$ displayed against $x$ for $\gamma = 0.1$, $k = 0.01$ and $L = 100$ in (a), (b) and (c) for $t = 30$, and $t = 100$, respectively.

In the first cell. Indeed, if $k \to 0$, then $\alpha_2$ behaves mainly as $\alpha_1$. By studying the effects of the size of the system $L$, we find that, for smaller values of the size of the container, the amplitude of the oscillations in the central region is very small, while it is higher for larger values of $L$. Again, the variation of the concentration $\alpha_2$ is more significant in a container of larger size than for smaller values.

In Figure 4, the concentrations $\alpha_i$ ($i = 1, 2$) and $\beta$ are displayed against $x$ for

$$k = 0.09, \quad \gamma = 0.6, \quad L = 100, \quad a_n = \frac{0.8}{n^2}, \quad b_n = \frac{0.5}{n^2}, \quad b_m = \frac{0.001}{n^2}.$$  

The dotted, dashed and solid curves correspond to the values $t = 0$, $t = 30$ and $t = 100$, respectively. Here, we find that the initial condition is not periodic and the concentration is still not periodic at later times. The minimum value of the initial concentration varies with the time variable $t$, but not with the space variable $x$. In this case, the solutions exhibit standing waves. We mention that these results agree with work presented in [3].

In Figure 5, the contour plots of the concentrations are presented for

$$k = 0.01, \quad \gamma = 0.09, \quad a_n = 0.2, \quad b_n = 0.01, \quad b_m = 0.002,$$

$$c_n = 0.1, \quad a_r = 0.001 \quad \text{and} \quad c_r = 0.001.$$  

Light regions are of high concentrations, while the dark ones are of low concentrations. This figure shows that small islands of low concentrations are formed in the initial stage. Waves are observed in the concentrations $\alpha_1$, $\beta$ and $\alpha_2$ up to time $t < 800$. This is a transient behavior of the solution which reaches asymptotically to its homogeneous steady state value.
Figure 2. The concentration of the autocatalyst $\beta$ displayed against $x$ for $\gamma = 0.1$, $k = 0.01$ and $L = 100$ in (a), (b) and (c) for $t = 30$, and $t = 100$, respectively.

Figure 3. The concentration of the reactant $\alpha_2$ displayed against $x$ for $\gamma = 0.1$, $k = 0.01$ and $L = 100$ in (a), (b) and (c) for $t = 30$, and $t = 100$, respectively.
4. Comparison between the approximate analytical and the numerical solutions. Here, in this section, we make comparison between the results found previously by the approximate analytic solutions and those found by the numerical scheme built in Mathematica (using NDSolve). Indeed this scheme is based upon the Crank-Nicolson method.

Approximate analytical solutions and numerical solutions for $\alpha_1(x,t)$, $\beta(x,t)$ and $\alpha_2(x,t)$ are shown in Figure 6 and Figure 7, respectively.

In Figure 6 and Figure 7, the concentrations $\alpha_1(x,t)$, $\beta(x,t)$ and $\alpha_2(x,t)$, which are evaluated by using the analytical results (solid line) and by using the numerical scheme of Mathematica (dashed line) are displayed against $x$. For instance, in Figure 6(a), 6(b) and 6(c), we set

$$ t = 50, \quad k = 0.09, \quad \gamma = 0.03, \quad L = 10, \quad a_n = \frac{0.5}{n^2}, \quad c_n = \frac{0.5}{n^2}, \quad b_n = \frac{0.001}{n^2}, $$

$$ b_m = \frac{0.001}{m^2}, \quad a_r = \frac{0.001}{r^2} \quad \text{and} \quad c_r = \frac{0.001}{r^2}. $$

In Figure 7(a), 7(b) and 7(c), we take

$$ t = 50, \quad k = 0.01, \quad \gamma = 0.001, \quad L = 100, \quad a_n = 0.01, \quad c_n = 0.01, \quad b_n = 0.05, \quad b_m = 0.002, \quad a_r = 0.001 \quad \text{and} \quad c_r = 0.001. $$

Figure 4. The concentration of $\alpha_1$, $\beta$ and $\alpha_2$ displayed against $x$ for $\gamma = 0.6$, $k = 0.09$ and $L = 100$. (•••) for $t = 0$, (−−) for $t = 30$ and (−) for $t = 100$, respectively.
Figure 5. Contour plots of $\alpha_1$, $\beta$ and $\alpha_2$, respectively, for $\gamma = 0.09$, $k = 0.01$ and $L = 100$. 
FIGURE 6. Approximate analytical and numerical solutions of $\alpha_i(x,t)$ and $\beta(x,t)$ displayed against $x$ for $t = 50$, $k = 0.09$, $\gamma = 0.03$ and $L = 10$. (−) for approximate analytical solutions; (−−) for numerical solutions. The error estimate is of order $(10^{-8})$.

Figures 4(a), 4(b) and 4(c) show the absolute error between the approximate analytical and numerical solutions of $\alpha_i(x,t)$ and $\beta(x,t)$ for $k = 0.09$, $\gamma = 0.1$ and $L = 100$ with $t = 30$, $t = 50$ and $t = 100$, respectively.

After the above-mentioned figures, we have the following remarks:

(i) The solution evaluated by means of the two methods exhibits the same qualitative behavior.

(ii) The error between the two solutions is only relevant in a small domain.

(iii) The maximum relative error involved is of order $5 \times 10^{-4}$. 
Figure 7. Approximate analytical and numerical solutions of $\alpha_i(x,t)$ and $\beta(x,t)$ displayed against $x$ for $t = 50$, $k = 0.01$, $\gamma = 0.001$ and $L = 100$. (−) for approximate analytical solutions; (−−) for numerical solutions. The error estimate is of order $10^{-4}$. 
Figure 8. The absolute error between the approximate analytical and numerical solutions of $\alpha_i(x,t)$ and $\beta(x,t)$ displayed against $x$, $\alpha_1(x,t)$ (solid line), $\beta(x,t)$ (dotted line) and $\alpha_2(x,t)$ (dashed line) for $k = 0.09$, $\gamma = 0.1$ and $L = 100$. (a) $t = 30$, (b) $t = 50$ and (c) $t = 100$. The error estimate is of order $10^{-3}$. 
5. **Conclusions.** An approach for finding approximate analytical solutions of the reaction-diffusion model equations is presented. It is based upon finding the formal exact solution of the Dirichlet problem for the equation under consideration. It is found that the solution of the one-dimensional diffusion model for periodic boundary conditions is symmetric with respect to the mid-point of the size of the container. This symmetric behavior is induced by the parity order of the spatial (diffusion) operator and the symmetric Dirichlet boundary values. It is a global behavior for systems obeying the same conditions with periodic initial conditions. Concentric waves are generated in systems of finite size. However, if the initial condition is not periodic, the wave may be of the standing kind.

In our future works, we propose to focus our attention on the use of the space-time fractional derivatives, especially in the reaction-diffusion equations and the travelling-wave equations. In fact, space-time fractional PDEs occupy a wide area of research in the literature on the reaction-diffusion equations, travelling-wave equations, and so on (see the recent works [24] to [21]). To this end, the usage of the space-time fractional derivatives in (for example) the two-cell equation deserves to be investigated. In this connection, some of the most commonly used fractional-order derivatives include (for example) the Riemann-Liouville fractional derivatives and the Liouville-Caputo fractional derivatives (see, for details, [10] and [28]).

**References**

[1] H. I. Abdel-Gawad and K. M. Saad, On the behaviour of solutions of the two-cell cubic autocatalator, *ANZIAM J.*, 44 (2002), E1–E32.

[2] H. I. Abdel-Gawad and H. A. Abdusalam, Approximate solutions of the Kuramoto-Sivashinsky equation for periodic boundary value problems and chaos, *Chaos Solitons Fract.*, 12 (2001), 2039–2050.

[3] P. Arcuri and J. D. Murray, Pattern sensitivity to boundary conditions in reaction-diffusion models, *J. Math. Biol.*, 24 (1986), 141–165.

[4] N. F. Britton, *Reaction-Diffusion Equations and Their Applications to Biology*, Academic Press, New York, 1986.

[5] L. Debnath, *Nonlinear Partial Differential Equations for Scientists and Engineers*, Birkhäuser, Basel and Boston, 1997.

[6] I. R. Epstein and J. A. Pojman, *An Introduction to Nonlinear Chemical Dynamics: Oscillations, Waves, Patterns and Chaos*, Clarendon (Oxford University) Press, Oxford, London and New York, 1998.

[7] R. FitzHugh, Impulses and physiological states in theoretical models of nerve membrane, *Biophys. J.*, 1 (1961), 445–466.

[8] L. K. Forbes, On stability and uniqueness of stationary one-dimensional patterns in the Belousov-Zhabotinskii reaction, *Physica D: Nonlinear Phenom.*, 50 (1991), 42–58.

[9] W. Jager, J. Moser and R. Renmert, Modelling of Patterns in Space and Time, Springer Lectures in Biomathematics, Springer-Verlag, Berlin, Heidelberg and New York, 1984.

[10] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematical Studies, 204, Elsevier (North-Holland) Science Publishers, Amsterdam, London and New York, 2006.

[11] A. Kolmogoroff, I. Petrovsky and N. Piscounoff, Etude de l’équation de la diffusion avec croissance de la quantité de matière et son application a un problème biologique, *Moscow Univ. Bull. Math.*, 1 (1937), 1–25.

[12] J. A. Leach, J. H. Merkin and S. K. Scott, Oscillations and waves in the Belousov-Zhabotinsky reaction in a finite medium, *J. Math. Chem.*, 16 (1994), 115–124.

[13] D. Luss, M. Golubitsky and S. Strogatz, *Pattern Formation in Continuous and Coupled Systems*, IMA Volumes in Mathematics and Its Applications, Springer-Verlag, Berlin, Heidelberg and New York, 1999.

[14] P. K. Maini, Spatial and spatiotemporal pattern formation in generalised Turing systems, *Comput. Math. Appl.*, 32 (1996), 71–77.

[15] T. R. Marchant, Cubic autocatalytic reaction-diffusion equations: Semi-analytical solutions, *Proc. Roy. Soc. London Ser. A Math. Phys. Engry. Sci.*, 458 (2002), 1–16.
[16] J. H. Merkin, D. J. Needham and S. K. Scott, Coupled reaction-diffusion waves in an isothermal autocatalytic chemical system, *IMA J. Appl. Math.*, 50 (1993), 43–76.

[17] J. D. Murray, *Mathematical Biology*, Springer-Verlag, Berlin, Heidelberg and New York, 1989.

[18] I. Prigogine and R. Lefever, Symmetry breaking instabilities in dissipative system. II, *AIP J. Chem. Phys.*, 48 (1968), 1–7.

[19] K. M. Saad and A. M. El-Shrae, Travelling waves in a cubic autocatalytic reaction, *Adv. Appl. Math. Sci.*, 8 (2011), 87–104.

[20] K. M. Saad and E. H. F. Al-Sharif, Comparative study of a cubic autocatalytic reaction via different analysis methods, *Discrete Continuous Dyn. Syst. Ser. S*, 12 (2019), 665–684.

[21] K. M. Saad, H. M. Srivastava and J. F. Gómez-Aguilar, A fractional quadratic autocatalysis associated with chemical clock reactions involving linear inhibition, *Chaos Solitons Fractals*, 132 (2020), 1–9.

[22] R. A. Satnoianu, M. Menzinger and P. K. Maini, Turing instabilities in general systems, *J. Math. Biol.*, 41 (2000), 493–512.

[23] E. E. Sel’kov, Self-oscillations in glycolysis. 1: A simple kinetic model, *European J. Biochem.*, 4 (1968), 79–86.

[24] H. M. Srivastava and K. M. Saad, Some new models of the time-fractional gas dynamics equation, *Adv. Math. Models Appl.*, 3 (2018), 5–17.

[25] H. M. Srivastava and K. M. Saad, New approximate solution of the time-fractional Nagumo equation involving fractional integrals without singular kernel, *Appl. Math. Inform. Sci.*, 14 (2020), 1–8.

[26] H. M. Srivastava, K. M. Saad and E. H. F. Al-Sharif, New analysis of the time-fractional and space-time fractional-order Nagumo equation, *J. Inform. Math. Sci.*, 10 (2018), 545–561.

[27] H. M. Srivastava, H. I. Abdel-Gawad and K. M. Saad, Stability of traveling waves based upon the Evans function and Legendre polynomials, *Appl. Sci.*, 10 (2020), 1–16.

[28] H. M. Srivastava, Fractional-order derivatives and integrals: Introductory overview and recent developments, *Kyunpook Math. J.*, 60 (2020), 73–116.

[29] A. M. Turing, The chemical basis of morphogenesis, *Philos. Trans. Roy. Soc. London Ser. B Biol. Sci.*, 237 (1952), 37–72.

[30] J. J. Tyson, Classification of instabilities in chemical reaction systems, *AIP J. Chem. Phys.*, 62 (1975), 1–7.

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E-mail address: harimsri@math.uvic.ca
E-mail address: hamdyig@yahoo.com
E-mail address: khaledma_sd@hotmail.com