Observing Quantum Tunneling in Perturbation Series

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It is well-known that the quantum tunneling makes conventional perturbation series non-Borel summable. We use this fact reversely and attempt to extract the decay width of the false-vacuum from the actual perturbation series of the vacuum energy density (vacuum bubble diagrams). It is confirmed that, at least in quantum mechanical examples, our proposal provides a complimentary approach to the conventional instanton calculus in the strong coupling region.

1. Introduction

Usually the quantum tunneling is regarded as a purely non-perturbative phenomenon. This widely accepted picture emerges from the observation like this: Consider a simple Hamiltonian of a metastable system

\[ H = \frac{1}{2} p^2 + \frac{1}{2} x^2 - \frac{1}{4!} gx^4. \]  

(1)

Since the potential energy is not bounded from below, the quantum system should be defined by the analytic continuation from \( g < 0 \). This continuation produces an imaginary part of the quasi-ground state energy eigenvalue, that is physically interpreted as the total decay width of the quasi-ground state due to quantum tunneling. In the WKB (or instanton) approximation, we find

\[ \text{Im} E(g) \sim -2\sqrt{3} \left( \frac{S_0}{2\pi g} \right)^{1/2} e^{-S_0/g} (1 + O(g)), \]  

(2)

where \( S_0 \) is the instanton action and \( S_0 = 8 \) in this model. Since eq. (2) vanishes to all orders of the expansion on \( g \), the tunneling effect is invisible in a simple expansion with respect to the coupling constant. In other words, a simple (truncated) sum of the conventional Rayleigh-Schrödinger perturbation series of the ground state energy,

\[ E(g) \sim \sum_{n=0}^{\infty} c_n g^n, \]  

(3)

where \( c_0 = 1/2, \ c_1 = -1/32, \ c_2 = -7/1536, \cdots \), is always real for \( g \) real. Therefore, one has to utilize a certain non-perturbative technique to evaluate the tunneling amplitude (2): This is the conventional wisdom.

However, it is interesting to note that extensive studies started in the seventies [2] have revealed the close relationship between the tunneling amplitude in the weak coupling region (2) and the large order behavior of the perturbation series (3). In fact, it is possible to show that [2],

\[ c_n \sim - \frac{2\sqrt{3}}{\pi (2\pi)^{1/2}} \left( \frac{1}{S_0} \right)^n \Gamma(n+1/2)(1+O(1/n)). \]  

(4)

In general, at least in super-renormalizable theories, one finds the following correspondence:

Tunneling rate in the weak coupling \( \Leftrightarrow \) Higher order perturbation coefficients.

(5)

This relation is interesting because the right hand side is hard to be evaluated, while the left hand side is tractable. The large order behavior of perturbation series in various systems has been investigated on the basis of the connection (4). Furthermore it can also be shown that the \( O(g) \) correction in (2) gives rise to the \( O(1/n) \) correction for the large order behavior (4). Therefore, by extrapolating the relation (4), we expect
the following:

Tunneling rate in the \textit{strong} coupling \hspace{1em} (6)
\leftrightarrow \textit{Lower} order perturbation coefficients.

This possibility seems astonishing, because, although it is very difficult to systematically evaluate the left hand side, we can certainly compute the right hand side: This is the basic idea behind our approach \cite{34}. (A similar approach was first proposed in \cite{5} on the basis of another kind of resummation method.) We stress that the higher order corrections to the leading instanton approximation \cite{3} is difficult: The difficulty is not only technical but even a principal one—one has to resolve the “mixing” between non-trivial configurations and perturbative fluctuations to avoid the double-counting. Also such an instanton expansion, i.e., an expansion with respect to \(\exp(-S_0/g)\), is expected to be asymptotic at best. Therefore if a certain method can bypass these complications of the instanton calculus, it will provide a complementary approach to the tunneling phenomenon in the strong coupling region. This is what we want to propose in the next section.

2. Borel resummation of vacuum bubbles

As the natural extension of \cite{3}, we consider the \(O(N)\) symmetric \(\phi^4\) model whose action is given by

\[
S[\phi] = \int d^D x \left[ \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} \phi^2 + \frac{1}{4!} g (\phi^2)^2 \right], \tag{7}
\]

where \(\phi^2 = \phi \cdot \phi\). Note that the system is metastable for \(g > 0\). We consider the super-renormalizable cases \((D = 1, 2, \text{and } 3)\) and assume the appropriate renormalization. Then the standard instanton calculus \cite{3} gives the following expression for the imaginary part of the false-vacuum energy density,

\[
\text{Im} \mathcal{E}(g) \sim \frac{-A_N C_{D,N}}{2\pi g} \left( \frac{S_0}{2\pi g} \right)^{(D+N-1)/2} e^{-S_0/g}, \tag{8}
\]

where \(S_0\) denotes the instanton action,

\[
S_0 = \begin{cases} 
8 & \text{for } D = 1, \\
35.10269 & \text{for } D = 2, \\
113.38351 & \text{for } D = 3,
\end{cases} \tag{9}
\]

and \(A_N\) and \(C_{D,N}\) are some numbers arising from the Gaussian integration around the instanton and the collective coordinate integrations \cite{3}.

As noted in Introduction, the leading instanton approximation \cite{3} is reliable only for \(g \ll 1\). Therefore, following the basic idea \cite{3}, we start with the conventional perturbative expansion of the vacuum energy density, i.e., a sum of the vacuum bubble diagrams

\[
\mathcal{E}(g) \sim \sum_{n=0}^{\infty} c_n g^n. \tag{10}
\]

Once the perturbation series \(\{c_n\}\) is obtained, we construct the Borel-Leroy transform:

\[
B(z) \equiv \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(n + (D + N)/2)} z^n. \tag{11}
\]

Then we define the vacuum energy density by the Borel integral

\[
\mathcal{E}(g) = \frac{1}{g(D+N)/2} \int_0^{\infty} dz \, e^{-z/g} z^{(D+N)/2-1} \times B(z \pm i\varepsilon). \tag{12}
\]

This representation is justified by the Borel summability of \(g < 0\) case \cite{3} combined with several plausible assumptions \cite{3}.

Now, the fact that all the perturbative coefficients in \(\{c_n\}\) (except \(c_0\)) have the same signs is important (see \(\{3\}\) and \(\{4\}\)). The perturbation series of non-alternating sign is usually called non-Borel summable because the Borel transform \(\{1\}\) develops singularities on the positive real axis, and thus the integration \(\{12\}\) is ill-defined without the \(\varepsilon\)-separation. However, these singularities are welcome for us because these are precisely the points where the imaginary part emerges—Borel singularities detect the quantum tunneling. In fact, the leading instanton approximation \cite{3} for \(g \ll 1\) implies the nearest singularity from the origin is a square-root branch point

\[
B(z) = -\frac{A_N C_{D,N} S_0^{1/2}}{\sqrt{\pi}(2\pi)^{(D+N-1)/2}} (S_0 - z)^{-1/2 + \ldots}. \tag{13}
\]
When substituted in (12), the branch cut reproduces the imaginary part (8) by choosing the upper contour.

Thus, in principle, we construct the Borel transform (11) from the actual perturbation coefficients $c_n$ and substitute it in (12) to extract the “non-perturbative” information. However, this way of working is impossible in practice because the radius of convergence of the series (11) is finite ($= S_0$) due to the singularity (13). In order to perform the Borel integration (12), we have to analytically continue the series (11) outside the convergence circle: This is impossible without knowing all the perturbative coefficients. Fortunately, as is well-known [2], this difficulty can be avoided by the conformal mapping trick. To apply this trick, the knowledge on the position of the nearest singularity (13), i.e., the instanton action $S_0$, is important.

In this way, we obtain the $P$th order approximation of the imaginary part (3,4),

$$[\text{Im } E(g)]_P = \left(\frac{S_0}{g}\right)^{(D+N)/2} \int_0^\pi d\theta \exp \left( -\frac{S_0}{g} \frac{1}{\cos^2 \theta/2} \right) \times \frac{\sin \theta/2}{\cos^{D+N+1} \theta/2} \sum_{k=0}^{P} d_k \sin k\theta, $$

where

$$d_k \equiv \sum_{n=0}^{k} (-1)^{k-n} \Gamma(k+n)(4S_0)^n \times \frac{\Gamma(k+n)(2n)!\Gamma(n+(D+N)/2)}{(k-n)!\Gamma(2n)\Gamma(n+(D+N)/2)}c_n. $$

Note that eq. (14) is solely expressed by the first $P$ perturbative coefficients $c_n$ and the value of the instanton action $S_0$ (8). In this integration, the contribution around the origin $\theta = 0$ is proportional to $\exp(-S_0/g)$, the leading instanton behavior. Other parts might be regarded as the higher order corrections to it: This is true at least in quantum mechanical cases (3,4).

3. Numerical tests

The validity of our formula (14) has been tested numerically in quantum mechanics $D = 1$ and $D = 2$, tunneling on line (3,4). For $D = 1$, perturbative coefficients of the vacuum energy, i.e., the ground state energy, to very high orders are available. The first several coefficients are

$$c_0 = \frac{N}{2}, \quad c_1 = -\frac{N(N+2)}{96},$$

$$c_2 = -\frac{N(N+2)(2N+5)}{4068},$$

$$c_3 = -\frac{N(N+2)(8N^2 + 43N + 60)}{221184}, \ldots. $$

With the aid of computer, it is not difficult to compute $c_n$ to, say, $n = 50$. The exact complex quasi-ground state energy is also available by a numerical diagonalization of the Hamiltonian. Therefore, we can compare our formula (14) and the instanton result (3) with the exact value of the imaginary part. For the detail, we refer the reader to Refs. (3,4). For $D = 1$, we see an excellent convergence of (14) to the exact value in a whole range of the coupling constant; the proposal in fact gives rise to the improvement of instanton calculus, as we announced. For example, for $g = 4$, which belongs to the strong coupling region (4), eq. (14) with $P = 5$ (thus only the first five perturbative coefficients!) gives only a few percent error, while the leading instanton approximation (3) is about two times larger than the correct value.

For $D = 2$, the situation is yet not obvious. We have calculated vacuum bubble diagrams to five loop orders under the same renormalization condition assumed in (8). We found (with an appropriate shift of the origin of the vacuum energy) (9):

$$c_0 = 0, \quad c_1 = 0,$$

$$c_2 = -\frac{N(N+2)}{3} \times 8.833 \times 10^{-5},$$

$$c_3 = -\frac{N(N+2)(N+8)}{27} \times 3.012 \times 10^{-6},$$

and

$$c_4 = -\frac{N(N+2)(N^2 + 6N + 20)}{81} \times 5.657 \times 10^{-8} - \frac{N(N+2)^2}{9} \times 1.006 \times 10^{-7}.$$
$$-rac{N(N+2)(5N+22)}{81} \times 2 \times 10^{-7}.$$ 

Unfortunately, any converging behavior could not be observed and it was impossible to draw a definite conclusion on the convergence of (14) [4]. It is not clear whether this is due to the lack of orders of the perturbation series, or there exists a fundamental obstruction for our approach which we did not encounter in quantum mechanics. To clarify this point, much higher order perturbative calculation is certainly desirable.

4. Discussion

We have proposed a new approach to the tunneling phenomenon in super-renormalizable field theories. Our approach utilizes only the information of the conventional perturbation series around the naive vacuum $\phi = 0$ (and the value of the instanton action). We have verified numerically that, at least in quantum mechanical cases ($D = 1$), we can extract a very accurate tunneling rate from the perturbation series. The procedure thus provides a complimentary approach to the instanton calculus in the strong coupling region. In $D = 2$, unfortunately, the number of orders of the perturbation series we computed is yet insufficient to draw a definite conclusion on the convergence.

As another test of our proposal, one of us recently applied it to the Gaussian propagator model, in which the perturbative calculations in $D = 1, 2, 3$ and $4$, up to the ninth order of the loop expansion are known [9]. It was found that eq. (14) rapidly converges in the whole range of the coupling constant for lower dimensional cases, $D = 1$ and 2, and, at least in the strong coupling region for higher dimensional cases, $D = 3$ and 4.

As a general question, one might ask “To which extent we can expect the applicability of such a perturbative approach to tunneling phenomenon?” For example, we see that our approach cannot work for the double-well potential model (i.e., no decaying process involved). This fake imaginary part is believed to be canceled by the multi-instanton contributions [2]. In other words, in this system, the information of the perturbation series around the naive vacuum is not sufficient to specify the original physical quantities. Therefore the natural answer to the above question is: Such a perturbative approach to tunneling phenomena is expected to be applicable if the system can be defined by an analytic continuation of another system in which the Borel summability (of the energy density) is guaranteed. Eq. (7) with $D = 1, 2$ and 3 is expected to be precisely such a system.

Stated differently, for our approach to be workable, it is important that there is no non-trivial topological sector in the configuration space. The instanton in the system (7) is “non-topological”—the so-called bounce solution—and it can mix with the trivial perturbative sector. Our approach systematically counts the contribution of the perturbative fluctuations while avoiding the double counting of the bounce configuration. If there exist a non-trivial topological sector which does not mix with the trivial perturbative sector, such as the instanton in the double well potential, we will have to supplement the “true” non-perturbative information. This is the principal limitation of our approach.

When one considers a renormalizable field theory, such as (7) with $D = 4$, new difficulty may arise [2]. The renormalon—another known source of the Borel singularity—emerges in general. The renormalon has two effects: i) The contribution of renormalons to the Borel singularity produces new imaginary part besides the quantum tunneling. ii) The UV renormalon suggests the triviality of the model. In (7) with $D = 4$, when $g > 0$, there is no renormalon singularity on the positive real $z$ axis, because the model is asymptotically free. However the model with $g < 0$ has the UV renormalon and is expected to be trivial. Therefore the meaning of the true tunneling amplitude, which should be defined by the analytic continuation from $g < 0$, is not obvious. Presumably, in renormalizable field theories, our proposal is workable only with an UV cutoff.

Finally, we would like to comment that our
approach is applicable not only to the quantum tunneling problem, but also to metastable problems in statistical mechanics. For example, the free energy of the Ising model below the critical temperature acquires the imaginary part when it is analytically continued from the positive $H$ to the negative $H$ ($H$ is the external magnetic field and we assumed the Ising spins are originally aligned to the positive direction). The imaginary part might be physically interpreted as the inverse of the relaxation time of the metastable state. Our approach is applicable to the perturbation series on $H$, which can be computed from the low temperature expansion of the free energy. Then we expect the validity of our approach in the strong $H$ which will provide a complimentary approach to the conventional droplet calculation. A study along this line is in progress.

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