Relative topological surgery exact sequence and additivity of relative higher ρ invariants

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Abstract

In this paper, we define the relative higher ρ invariant for orientation preserving homotopy equivalence between PL manifolds with boundary in $K$-theory of the relative obstruction algebra, i.e. the relative analytic structure group. We also show that the map induced by the relative higher ρ invariant is a group homomorphism from the relative topological structure group to the relative analytic structure group. For this purpose, we generalize Weinberger, Xie and Yu’s definition of the topological structure group in their article "Shmuel Weinberger, Zhizhang Xie, and Guoliang Yu. Additivity of higher rho invariants and nonrigidity of topological manifolds. Communications on Pure and Applied Mathematics, to appear." to make the additive structure of the relative topological structure group transparent.

1 Introduction

The surgery exact sequence is a powerful tool in the study of the classification of manifolds. It is usually defined geometrically (Wall [26], Quinn [18]). Ranicki developed the algebraic surgery exact sequence ([19]). In light of the higher index theory, Higson and Roe indicated that the exact sequence of the $K$-theory of the geometric $C^*$-algebras is a candidate for the analytic surgery exact sequence. In fact, they constructed a transformation from the smooth surgery exact sequence to the exact sequence of the $K$-theory of the geometric $C^*$-algebras in a series of articles named “Mapping surgery to analysis I, II, III" (cf. [9], [10], [11]). After that, the exact sequence of the $K$-theory of the geometric $C^*$-algebras is usually referred to as analytic surgery exact sequence. The transformation constructed by Higson and Roe consists of maps induced by the higher signature class, the $K$-homology class of the signature operator, the relative higher ρ invariant for smooth or PL homotopy equivalence. In particular, the higher ρ invariant, a variant of the higher η invariant (cf. [13]), induces a map from the smooth structure set (generally not an abelian group) to the $K$-theory of a certain $C^*$-algebra. In 2016, Piazza and Schick developed an index theoretic approach to map the smooth surgery exact sequence to the analytic surgery exact sequence, by giving a different construction of the higher ρ invariant for smooth homotopy equivalence ([17]). Zenobi defined the higher
\(\rho\) invariant for topological homotopy equivalence, and thus defined a map on the topological structure set, merely as a set map (30).

The topological structure set is actually an abelian group, whose abelian group structure can be described by the Siebenmann periodicity map (cf. Cappell and Weinberger, [2]). It is natural to ask whether the higher \(\rho\) invariant actually defines an additive map on the topological structure group. In 2019, Weinberger, Xie and Yu answered this problem positively in their breakthrough work [27]. The major novelty of their work is that they made the group structure of the topological structure group transparent by giving the topological structure group a new description. More precisely, for a topological manifold \(X\) with dimension \(n \geq 5\), they defined a new group \(S_n(X)\) geometrically, whose group structure is given by disjoint union, and proved that

\[ST_{\text{Top}}(X) \cong S_n(X),\]

where \(ST_{\text{Top}}(X)\) is the usual topological structure group. Based on this new description, they showed that the higher \(\rho\) invariant, living in \(K\)-theory of Yu’s obstruction algebra, induces an additive map on \(S_n(X)\). They then managed to transform the topological surgery exact sequence to the analytic one by group homomorphisms, and to estimate to what extent a manifold is topological nonrigid.

In the present article, we generalize Weinberger, Xie and Yu’s work to transform the relative topological surgery exact sequence to the relative analytic surgery exact sequence, i.e. the exact sequence of the \(K\)-theory of the relative geometric \(C^*\)-algebras.

Let \((X, \partial X)\) be a topological manifold with boundary, with \(\pi_1(X) = G\) and \(\pi_1(\partial X) = \Gamma\). Then there is the following relative topological surgery exact sequence (see Section 9 in Wall’s book [26]):

\[
\cdots \rightarrow N_{\partial x}^{T_{\text{Top}}}(X \times I, (X \times \partial I \cup X \times I)) \xrightarrow{\partial_{T_{\text{Top}}}} L_{n+1}(\pi_1 X, \pi_1(\partial X); \omega) \xrightarrow{\iota_{T_{\text{Top}}}} \overline{N}_{\partial_{T_{\text{Top}}}}(X, \partial X) \xrightarrow{\partial_{T_{\text{Top}}}} N_{T_{\text{Top}}}(X, \partial X) \xrightarrow{\iota_{T_{\text{Top}}}} L_n(\pi_1 X, \pi_1(\partial X); \omega).\]

The first main result of this article is to define abelian group \(S_n(X, \partial X; \omega)\), the group structure of which is given by disjoint union, and to obtain the following commutative diagram

\[
\begin{array}{ccc}
N_{\partial x}^{T_{\text{Top}}}(X \times I, (X \times \partial I \cup X \times I)) & \xrightarrow{\cong} & N_{n+1}(X, \partial X; \omega) \\
\downarrow & & \downarrow \\
L_{n+1}(\pi_1 X, \pi_1(\partial X); \omega) & \xrightarrow{=} & L_{n+1}(\pi_1 X, \pi_1(\partial X); \omega) \\
\downarrow & & \downarrow \\
S_{T_{\text{Top}}}(X, \partial X) & \xrightarrow{\cong} & S_n(X, \partial X; \omega) \\
\downarrow & & \downarrow \\
N_{T_{\text{Top}}}(X, \partial X) & \xrightarrow{\cong} & N_n(X, \partial X; \omega)
\end{array}
\]

Here \(N_n(X, \partial X; \omega)\) is controlled version of the relative \(L\) group \(L_n(\pi_1 X, \pi_1(\partial X); \omega)\).
Our second main result is involved with the $K$-theory of the relative Roe algebra, the relative localization algebra, and the relative obstruction algebra (denoted as $C^*(\tilde{X}, \partial \tilde{X})^G, \Gamma$, $C^*_L(\tilde{X}, \partial \tilde{X})^G, \Gamma$, and $C^*_L(\tilde{X}, \partial \tilde{X})^G, \Gamma$ respectively). More precisely, we obtain the following commutative diagram of groups (Theorem 6.8):

$$
\begin{align*}
N_{n+1}(X, \partial X) \xrightarrow{\text{relInd}_L} & K_{n+1}(C^*_L(\tilde{X}, \partial \tilde{X})^G, \Gamma) \\
L_{n+1}(\pi_1 X, \pi_1 \partial X) \xrightarrow{\text{relInd}} & K_{n+1}(C^*(\tilde{X}, \partial \tilde{X})^G, \Gamma) \\
S_n(X, \partial X) \xrightarrow{k_n \text{rel}} & K_n(C^*_L(\tilde{X}, \partial \tilde{X})^G, \Gamma) \\
N_n(X, \partial X) \xrightarrow{\text{relInd}_L} & K_n(C^*_L(\tilde{X}, \partial \tilde{X})^G, \Gamma)
\end{align*}
$$

Let us briefly introduce the group homomorphisms relInd, relInd$_L$ and rel$\rho$. All of the groups in the above diagram is reviewed or defined in Section 2.

The group homomorphism

$$
\text{relInd} : L_n(\pi_1 X, \pi_1 \partial X) \to K_n(C^*(\tilde{X}, \partial \tilde{X})^G, \Gamma)
$$

is induced by the relative signature class living in the $K$-theory of the relative Roe algebra, $K_n(C^*(\tilde{X}, \partial \tilde{X})^G, \Gamma)$, which can also be viewed as the relative index of the signature operator on a manifold with boundary. Block and Weinberger proposed to investigate the relative index in [1]. In 2015, Chang, Weinberger and Yu defined the relative Roe algebra and the relative index of the Dirac operator on a spin manifold with boundary in [3].

The relative localization algebra $C^*_L(\tilde{X}, \partial \tilde{X})^G, \Gamma$ was first defined and considered in [3]. The $K$-theory group $K_n(C^*_L(\tilde{X}, \partial \tilde{X})^G, \Gamma)$ is actually isomorphic to the relative $K$-homology $K_n(X, \partial X)$. The definitions of the relative Roe and localization algebras will be reviewed in Section 3. The group homomorphism

$$
\text{relInd}_L : N_{n+1}(X, \partial X) \to K_{n+1}(C^*_L(\tilde{X}, \partial \tilde{X})^G, \Gamma)
$$

is induced by the relative $K$-homology classes of signature operators.

The algebra $C^*_L(\tilde{X}, \partial \tilde{X})^G, \Gamma$ is the so called relative obstruction algebra (see Section 3 for definition), whose $K$-theory is the resident of the relative higher $\rho$ invariant. In this article, we show that the relative higher $\rho$ invariant induces a group homomorphism rel$\rho$ from $S_n(X, \partial X)$ to $K_n(C^*_L(\tilde{X}, \partial \tilde{X})^G, \Gamma)$, which can be fitted into the commutative diagram (1.1).

We point out that the relative $K$-homology class and the relative index are related to the relative assembly map. The injectivity of the relative assembly
map has various implications in geometry and topology (cf. [1], [3], [25]), on which there have been fruitful results (cf. [25], [22], [23], [24], [7], [6]). Our research in this article has deep connection to this topic.

This paper is organized as follows. In Section 2 we generalize Weinberger, Xie, and Yu’s results in [27], to give a new definition of the relative topological structure group of a topological manifold with boundary. In light of this new definition, the group structure of the relative topological structure group is simply given by disjoint union. In Section 3 we recall the definitions of the relative Roe, localization and obstruction algebras. In Section 4 we introduce the signature class of the Hilbert-Poincaré complex defined by Higson and Roe. Our construction has its roots in the definition of signature class of the Hilbert-Poincaré complex. In Section 5 we define the relative signature class and the relative K-homology class of the signature operator on a manifold with boundary, and show that they induce well defined group homomorphisms on $L^-(\pi_1(X, \pi_1\partial X)$ and $N^-(X, \partial X)$ respectively. In Section 6 we define the relative higher $\rho$ invariant and prove that it induces a group homomorphism from $S^-(X, \partial X)$ to $K^-(C^*_L(X, \partial X)^G, F)$. We will also show the commutativity of the diagram (1.1) in this very last section.

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2 Surgery

In this section, we give a new description of the relative surgery group and the relative surgery exact sequence.

We first recall some definitions related to the infinitesimally controlled homotopy equivalence.

Let $X$ be a closed topological manifold. Fix a metric on $X$ that agrees with the topology of $X$.

**Definition 2.1.** Let $Y$ be a topological space. We call a continuous map $\phi : Y \to X$ a control map of $Y$.

**Definition 2.2.** Let $Y$ and $Z$ be two compact Hausdorff spaces equipped with control maps $\psi : Y \to X$ and $\phi : Z \to X$. A continuous map $f : Y \to Z$ is said to be a controlled homotopy equivalence over $X$, if

1. $\phi = \psi f$;
2. there exists a continuous map $g : Z \to Y$ such that $\psi = \phi g$;
3. $fg \sim_h I_Y$ and $gf \sim_h I_Z$.

**Definition 2.3** (Infinitesimally controlled homotopy equivalence, Definition 3.3 of [27]). Let $Y$ and $Z$ be two compact Hausdorff spaces equipped with control maps $\psi : Y \to X$ and $\phi : Z \to X$. A continuous map $f : Y \to Z$ is said to be...
an infinitesimally controlled homotopy equivalence over $X$, if there exist proper continuous maps

$$\Phi : Z \times [1, \infty) \to X \times [1, \infty) \quad \text{and} \quad \Psi : Y \times [1, \infty) \to X \times [1, \infty),$$

$$F : Y \times [1, \infty) \to Z \times [1, \infty) \quad \text{and} \quad Z \times [1, \infty) \to Y \times [1, \infty)$$

satisfying the following conditions:

1. $\Phi F = \Psi$;
2. $F|_{Y \times \{1\}} = f, \Phi|_{Z \times \{1\}} = \phi, \Psi|_{Y \times \{1\}} = \psi$;
3. there is a proper continuous homotopy $\{H_s\}_{0 \leq s \leq 1}$ between

$$H_0 = FG \quad \text{and} \quad H_1 = \text{id} : Z \times [1, \infty) \to Z \times [1, \infty)$$

such that the diameter of the set $\Phi(H(z, t)) = \{\Phi(H_s(z, t)) | 0 \leq s \leq 1\}$ goes uniformly (i.e. independent of $z \in Z$) to zero, as $t \to \infty$;
4. there is a proper continuous homotopy $\{R_s\}_{0 \leq s \leq 1}$ between

$$R_0 = GF \quad \text{and} \quad H_1 = \text{id} : Y \times [1, \infty) \to Y \times [1, \infty)$$

such that the diameter of the set $\Psi(R(y, t)) = \{\Psi(R_s(y, t)) | 0 \leq s \leq 1\}$ goes uniformly (i.e. independent of $y \in Y$) to zero, as $t \to \infty$;

Let $X$ be a compact manifold with boundary $\partial X$. The definition of relative $L$-group follows from Wall’s work in [26].

**Definition 2.4** (Objects for the definition of $L_n(\pi_1 X, \pi_1(\partial X); \omega)$). An object

$$\theta = \{M, \partial_{\pm} M, \phi, N, \partial_{\pm} N, \psi, f\}$$

in $L_n(\pi_1 X, \pi_1(\partial X); \omega)$ consists of the following data

1. two manifold 2-ads $(M, \partial_{\pm} M)$ and $(N, \partial_{\pm} N)$ with $\dim M = \dim N = n$, with $\partial M = \partial_{+} M \cup \partial_{-} M$ (resp. $\partial N = \partial_{+} N \cup \partial_{-} N$) the boundary of $M$ (resp. $\partial N$). In particular, $\partial_{+} M \cap \partial_{-} M = \partial\partial_{\pm} M$ and $\partial_{+} N \cap \partial_{-} N = \partial\partial_{\pm} N$;
2. continuous maps $\phi : (M, \partial_{-} M) \to (X, \partial X)$ and $\psi : (N, \partial_{-} N) \to (X, \partial X)$ so that $\phi^*(\omega)$ and $\psi^*(\omega)$ describe the orientation characters of $M$ and $N$;
3. a degree one normal map of manifold 2-ads $f : (N, \partial_{\pm} N) \to (M, \partial_{\pm} M)$ such that $\phi \circ f = \psi$;
4. the restriction $f|_{\partial_{+} N} : (\partial_{+} N, \partial_{+} N) \to (\partial_{+} M, \partial_{+} M)$ is a homotopy equivalence of pairs over $(X, \partial X)$;
5. the restriction $f|_{\partial_{-} N} : \partial_{-} N \to \partial_{-} M$ is a degree one normal map over $\partial X$.  

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Figure 1: An object $\theta = \{M, \partial \pm M, \phi, N, \partial \pm N, \psi, f\}$ in $L_n(\pi_1 X, \pi_1(\partial X); \omega)$.

**Definition 2.5** (Equivalence relation for the definition of $L_n(\pi_1 X, \pi_1(\partial X); \omega)$). Let

$$\theta = \{M, \partial \pm M, \phi, N, \partial \pm N, \psi, f\}$$

be an object in $L_n(\pi_1 X, \pi_1(\partial X); \omega)$. We write $\theta \sim 0$ if the following conditions are satisfied.

1. There exists a manifold 3-ads $(W, \partial W)$ of dimension $(n+1)$ with a continuous map $\Phi : (W, \partial_3 W) \to (X, \partial X)$ so that $\Phi^*(\omega)$ describes the orientation character of $W$, where $\partial W = M(= \partial_1 W) \cup \partial_2 W \cup \partial_3 W$. Moreover, we have decompositions $\partial M = \partial_+ M \cup \partial_- M$, $\partial(\partial_3 W) = \partial\partial_{2,+} W \cup \partial\partial_{2,-} W$, and $\partial(\partial_3 W) = \partial\partial_{3,+} W \cup \partial\partial_{3,-} W$ such that

$$\partial_+ M = \partial\partial_{2,+} W, \quad \partial_- M = \partial\partial_{3,-} W \quad \text{and} \quad \partial\partial_{2,-} M = \partial\partial_{3,+} W.$$

Furthermore, we have

$$\partial_+ M \cap \partial_- M = \partial\partial_{2,+} W \cap \partial\partial_{2,-} W = \partial\partial_{3,+} W \cap \partial\partial_{3,-} W.$$ 

2. Similarly, we have a manifold 3-ads $(V, \partial V)$ of dimension $(n+1)$ with a continuous map $\Psi : (V, \partial_3 V) \to (X, \partial X)$ so that $\Psi^*(\omega)$ describes the orientation character of $V$, where $\partial V = N(= \partial_1 V) \cup \partial_2 V \cup \partial_3 V$ satisfying similar conditions as $W$. 

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3. There is a degree one normal map of manifold 3-ads $F : (V, \partial V) \to (W, \partial W)$ such that $\Phi \circ F = \Psi$. Moreover, $F$ restricts to $f$ on $N \subseteq \partial V$.

4. The restriction $F|_{\partial^2 V} : \partial^2 V \to \partial^2 W$ is a homotopy equivalence over $X$.

We denote by $L_n(\pi_1 X, \pi_1 (\partial X); \omega)$ the set of equivalence classes from Definition 2.5. Note that $L_n(\pi_1 X, \pi_1 (\partial X); \omega)$ is an abelian group with the sum operation being disjoint union. We call $L_n(\pi_1 X, \pi_1 (\partial X); \omega)$ the relative $L$-group.

In the following, we give a controlled version of $L_n(\pi_1 X, \pi_1 (\partial X); \omega)$.

**Definition 2.6 (Objects for the definition of $N_n(X, \partial X; \omega)$).** An object

$$\theta = \{M, \partial_{\pm} M, \phi, N, \partial_{\pm} N, \psi, f\}$$

in $N_n(X, \partial X; \omega)$ consists of the following data

1. two manifold 2-ads $(M, \partial_{\pm} M)$ and $(N, \partial_{\pm} N)$ with $\dim M = \dim N = n$, with $\partial M = \partial_{\pm} M \cup \partial_\mp M$ (resp. $\partial N = \partial_+ N \cup \partial_- N$) the boundary of $M$ (resp. $\partial N$). In particular, $\partial_{\pm} M \cap \partial_\mp M = \partial \partial_{\pm} M$ and $\partial_{\pm} N \cap \partial_{\pm} N = \partial \partial_{\pm} N$;

2. continuous maps $\phi : (M, \partial_{\pm} M) \to (X, \partial X)$ and $\psi : (N, \partial_{\pm} N) \to (X, \partial X)$ so that $\phi^* (\omega)$ and $\psi^* (\omega)$ describe the orientation characters of $M$ and $N$;

3. a degree one normal map of manifold 2-ads $f : (N, \partial_{\pm} N) \to (M, \partial_{\pm} M)$ such that $\phi \circ f = \psi$;

4. the restriction $f|_{\partial_{\pm} N} : \partial_{\pm} N \to \partial_{\pm} M$ is an infinitesimally controlled homotopy equivalence over $X$;

5. the restriction $f|_{\partial_{\mp} N} : \partial_{\mp} N \to \partial_{\mp} M$ is a degree one normal map over $X$. 
Definition 2.7 (Equivalence relation for the definition of $N_n(X, \partial X; \omega)$). Let

$$\theta = \{M, \partial_{\pm} M, \phi, N, \partial_{\pm} N, \psi, f\}$$

be an object in $N_n(X, \partial X; \omega)$. We write $\theta \sim 0$ if the following conditions are satisfied.

1. There exists a manifold 3-ads $(W, \partial W)$ of dimension $(n+1)$ with a continuous map $\Phi : (W, \partial_3 W) \to (X, \partial X)$ so that $\Phi^*(\omega)$ describes the orientation character of $W$, where $\partial W = M(= \partial_1 W) \cup \partial_2 W \cup \partial_3 W$. Moreover, we have decompositions $\partial M = \partial_+ M \cup \partial_- M$, $\partial(\partial_3 W) = \partial \partial_2 W \cup \partial \partial_2 W$, and $\partial(\partial_3 W) = \partial \partial_3 + W \cup \partial \partial_3 - W$ such that

$$\partial_+ M = \partial \partial_2 + W; \partial_- M = \partial \partial_3 - W \text{ and } \partial \partial_2 - M = \partial \partial_3 + W.$$ 

Furthermore, we have

$$\partial_+ M \cap \partial_- M = \partial \partial_2 + W \cap \partial \partial_2 - W = \partial \partial_3 + W \cap \partial \partial_3 - W.$$ 

2. Similarly, we have a manifold 3-ads $(V, \partial V)$ of dimension $(n+1)$ with a continuous map $\Psi : (V, \partial_3 V) \to (X, \partial X)$ so that $\Psi^*(\omega)$ describes the orientation character of $V$, where $\partial V = N(= \partial_1 V) \cup \partial_2 V \cup \partial_3 V$ satisfying similar conditions as $W$.

3. There is a degree one normal map of manifold 3-ads $F : (V, \partial V) \to (W, \partial W)$ such that $\Phi \circ F = \Psi$. Moreover, $F$ restricts to $f$ on $N \subseteq \partial V$.

4. The restriction $F|_{\partial_3 V} : \partial_3 V \to \partial_3 W$ is an infinitesimally controlled homotopy equivalence over $X$.

We denote by $N_n(X, \partial X; \omega)$ the set of equivalence classes from Definition 2.7 which is actually an abelian group with the sum operation being disjoint union.

Now we introduce the new description of relative topological surgery group.

Definition 2.8 (Objects for the definition of $S_n(X, \partial X; \omega)$). An object

$$\theta = \{M, \partial_{\pm} M, \phi, N, \partial_{\pm} N, \psi, f\}$$

in $S_n(X, \partial X; \omega)$ consists of the following data

1. two manifold 2-ads $(M, \partial_{\pm} M)$ and $(N, \partial_{\pm} N)$ with $\dim M = \dim N = n$, with $\partial M = \partial_{\pm} M \cup \partial_- M$ (resp. $\partial N = \partial_{\pm} N \cup \partial_- N$) the boundary of $M$ (resp. $\partial N$). In particular, $\partial_+ M \cap \partial_- M = \partial \partial_{\pm} M$ and $\partial_+ N \cap \partial_- N = \partial \partial_{\pm} N$;

2. continuous maps $\phi : (M, \partial_- M) \to (X, \partial X)$ and $\psi : (N, \partial_- N) \to (X, \partial X)$ so that $\phi^*(\omega)$ and $\psi^*(\omega)$ describe the orientation characters of $M$ and $N$;

3. a homotopy equivalence of manifold 2-ads $f : (N, \partial_{\pm} N) \to (M, \partial_{\pm} M)$ such that $\phi \circ f = \psi$;
4. the restriction \( f|_{\partial_{+} N} : \partial_{+} N \to \partial_{+} M \) is an infinitesimally controlled homotopy equivalence over \( X \);

5. the restriction \( f|_{\partial_{-} N} : \partial_{-} N \to \partial_{-} M \) is a homotopy equivalence over \( X \).

**Definition 2.9** (Equivalence relation for the definition of \( S_n(X,\partial X;\omega) \)). Let

\[
\theta = \{ M, \partial_{\pm} M, \phi, N, \partial_{\pm} N, \psi, f \}
\]

be an object in \( S_n(X,\partial X;\omega) \). We write \( \theta \sim 0 \) if the following conditions are satisfied.

1. There exists a manifold 3-ads \((W,\partial W)\) of dimension \((n+1)\) with a continuous map \( \Phi : (W,\partial_3 W) \to (X,\partial X) \) so that \( \Phi^*(\omega) \) describes the orientation character of \( W \), where \( \partial W = M(=\partial_1 W) \cup \partial_2 W \cup \partial_3 W \). Moreover, we have decompositions \( \partial M = \partial_{+} M \cup \partial_{-} M \), \( \partial(\partial_2 W) = \partial_{2,+} W \cup \partial_{2,-} W \), and \( \partial(\partial_3 W) = \partial_{3,+} W \cup \partial_{3,-} W \) such that

\[
\partial_{+} M = \partial_{2,+} W; \quad \partial_{-} M = \partial_{3,-} W \quad \text{and} \quad \partial_{2,-} M = \partial_{3,+} W.
\]

Furthermore, we have

\[
\partial_{+} M \cap \partial_{-} M = \partial_{2,+} W \cap \partial_{2,-} W = \partial_{3,+} W \cap \partial_{3,-} W.
\]

2. Similarly, we have a manifold 3-ads \((V,\partial V)\) of dimension \((n+1)\) with a continuous map \( \Psi : (V,\partial_3 V) \to (X,\partial X) \) so that \( \Psi^*(\omega) \) describes the orientation character of \( V \), where \( \partial V = N(=\partial_1 V) \cup \partial_2 V \cup \partial_3 V \) satisfying similar conditions as \( W \).

3. There is a homotopy equivalence of manifold 3-ads \( F : (V,\partial V) \to (W,\partial W) \) such that \( \Phi \circ F = \Psi \). Moreover, \( F \) restricts to \( f \) on \( N \subseteq \partial V \).

4. The restriction \( F|_{\partial_2 V} : \partial_2 V \to \partial_2 W \) is an infinitesimally controlled homotopy equivalence over \( X \).

We denote by \( S_n(X,\partial X;\omega) \) the set of equivalence classes from Definition 2.9. It is not difficult to see that \( S_n(X,\partial X;\omega) \) is an abelian group with the sum operation being disjoint union.

We need the following auxiliary group to form the new description of the relative surgery exact sequence.

**Definition 2.10** (Objects for the definition of \( L_n(\pi_1 X,\pi_1(\partial X),X;\omega) \)). An object

\[
\theta = \{ M, \partial_{k} M, \phi, N, \partial_{k} N, \psi, f ; k = 1, 2, 3 \}
\]

in \( L_n(\pi_1 X,\pi_1(\partial X),X;\omega) \) consists of the following data

1. two manifold 3-ads \((M,\partial_{k} M;k = 1, 2, 3)\) and \((N,\partial_{k} N;k = 1, 2, 3)\) with \( \dim M = \dim N = n \), with \( \partial M = \partial_1 M \cup \partial_2 M \cup \partial_3 M \) (resp. \( \partial N = \partial_1 N \cup \partial_2 N \cup \partial_3 N \)) the boundary of \( M \) (resp. \( \partial N \)). Moreover, \( \partial(\partial_2 M) = \cup_{j \neq i} \partial \partial_{i,j} M \) for each \( i = 1, 2, 3 \) and \( \partial \partial_{i,j} M = \partial_i M \cap \partial_j M \) for any \( i \neq j \);
2. continuous maps $\phi : (M, \partial_3M) \to (X, \partial X)$ and $\psi : (N, \partial_3N) \to (X, \partial X)$ so that $\phi^*(\omega)$ and $\psi^*(\omega)$ describe the orientation characters of $M$ and $N$;

3. a degree one normal map of manifold 3-ads $f : (N, \partial N) \to (M, \partial M)$ such that $\phi \circ f = \psi$;

4. the restriction $f|_{\partial_1N} : \partial_1N \to \partial_1M$ is a degree one normal map over $X$;

5. the restriction $f|_{\partial_2N} : \partial_2N \to \partial_2M$ is a homotopy equivalence over $X$ and it restricts to an infinitesimally controlled homotopy equivalence $f|_{\partial\partial_1,2N} : \partial\partial_1,2N \to \partial\partial_1,2M$ over $X$;

6. the restriction $f|_{\partial_3N} : \partial_3N \to \partial_3M$ is a degree one normal map over $X$.

**Definition 2.11** (Equivalence relation for the definition of $L_n(\pi_1X, \pi_1(\partial X), X; \omega)$).

Let
\[ \theta = \{ M, \partial_kM, \phi, N, \partial_kN, \psi, f; k = 1, 2, 3. \} \]

be an object in $L_n(\pi_1X, \pi_1(\partial X), X; \omega)$. We write $\theta \sim 0$ if the following conditions are satisfied.

1. There exists a manifold 4-ads $(W, \partial W)$ of dimension $(n + 1)$ with a continuous map $\Phi : (W, \partial_kW) \to (X, \partial X)$ so that $\Phi^*(\omega)$ describes the orientation character of $W$, where $\partial W = M(= \partial_1W) \cup \partial_2W \cup \partial_3W \cup \partial_4W$. Moreover, we have decompositions $\partial M = \partial_1M \cup \partial_2M \cup \partial_3M,$ $\partial(\partial_2W) = \partial\partial_2,1W \cup \partial\partial_2,3W \cup \partial\partial_2,4W,$ $\partial(\partial_3W) = \partial\partial_3,1W \cup \partial\partial_3,2W \cup \partial\partial_3,4W,$ and $\partial(\partial_4W) = \partial\partial_4,1W \cup \partial\partial_4,2W \cup \partial\partial_4,3W$ such that
\[ \partial_1M = \partial\partial_1,3W, \quad \partial_2M = \partial\partial_1,3W, \quad \text{and} \quad \partial_3M = \partial\partial_1,4W \]

and
\[ \partial\partial_1,jW = \partial\partial_1,jW = \partial_1W \cap \partial_jW \quad \text{for any} \quad i, j = 1, 2, 3, 4. \]

Furthermore, we have
\begin{align*}
\partial_1M \cap \partial_2M &= \partial_2,1W \cap \partial_1,3W = \partial_2,1W \cap \partial_2,3W = \partial\partial_2,1W \cap \partial\partial_1,2W \\
&= \partial_2W \cap \partial_3W = \partial\partial_1,2,3W, \\
\partial_1M \cap \partial_3M &= \partial_2,1W \cap \partial_1,4W = \partial_2,1W \cap \partial_2,4W = \partial\partial_2,1W \cap \partial\partial_4,2W \\
&= \partial_3W \cap \partial_4W = \partial\partial_1,2,4W \\
\partial_2M \cap \partial_3M &= \partial_1,3W \cap \partial_1,4W = \partial\partial_1,3W \cap \partial\partial_1,4W = \partial\partial_1,4W \cap \partial\partial_4,3W \\
&= \partial_1W \cap \partial_3W \cap \partial_4W = \partial\partial_1,3,4W,
\end{align*}

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and
\[
\partial_1 M \cap \partial_2 M \cap \partial_3 M = \partial \partial_{1,2} W \cap \partial \partial_{1,3} W \cap \partial \partial_{1,4} W \\
= \partial \partial_{2,1} W \cap \partial \partial_{2,3} W \cap \partial \partial_{2,4} W \\
= \partial \partial_{3,1} W \cap \partial \partial_{3,2} W \cap \partial \partial_{3,4} W \\
= \partial \partial_{4,1} W \cap \partial \partial_{4,2} W \cap \partial \partial_{4,3} W \\
= \partial_1 W \cap \partial_2 W \cap \partial_3 W \cap \partial_4 W \\
= \partial \partial \partial \partial_{1,2,3,4} W.
\]

2. Similarly, we have a manifold 4-ads \((V, \partial V)\) of dimension \((n + 1)\) with a continuous map \(\Psi : (V, \partial_4 V) \to (X, \partial X)\) so that \(\Psi^*(\omega)\) describes the orientation character of \(V\), where \(\partial V = N(= \partial_1 V) \cup \partial_2 V \cup \partial_3 V \cup \partial_4 V\) satisfying similar conditions as \(W\).

3. There is a degree one normal map of manifold 4-ads \(F : (V, \partial V) \to (W, \partial W)\) such that \(\Phi \circ F = \Psi\). Moreover, \(F\) restricts to \(f\) on \(N \subseteq \partial V\).

4. The restriction \(F|_{\partial_k V} : \partial_k V \to \partial_k W\) is a degree one normal map over \(X\) for \(k = 1, 2, 4\).

5. The restriction \(F|_{\partial_3 V} : \partial_3 V \to \partial_3 W\) is a homotopy equivalence over \(X\) and it restricts to an infinitesimally controlled homotopy equivalence \(F|_{\partial_2,3 V} : \partial \partial_{2,3} V \to \partial \partial_{2,3} W\) over \(X\).

The set of equivalence classes from Definition 2.11 is denoted by \(L_n(\pi_1 X, \pi_1(\partial X); X; \omega)\), which is actually a group with the sum operation being disjoint union.

Now let us form our description of the relative topological surgery exact sequence.

Note that there is a natural group homomorphism
\[
i_* : N_n(X, \partial X; \omega) \to L_n(\pi_1 X, \pi_1(\partial X); \omega)
\]
by forgetting control.

Define
\[
j_* : L_n(\pi_1 X, \pi_1(\partial X); \omega) \to L_n(\pi_1 X, \pi_1(\partial X), X; \omega)
\]
by
\[
j_*(\theta) = \{M, (\emptyset, \partial_+ M, \partial_- M), \phi, N, (\emptyset, \partial_+ N, \partial_- N), \psi, f\}
\]
for \(\theta = \{M, \partial_\pm M, \phi, N, \partial_\pm N, \psi, f\}\), and define
\[
\partial_* : L_{n+1}(\pi_1 X, \pi_1(\partial X), X; \omega) \to N_n(X, \partial X; \omega)
\]
by
\[
\partial_* (\theta) = \partial_1 (\theta) = \theta_1 = \{\partial_1 M, (\partial \partial_1, M, \partial \partial_{1,3} M), \phi, \partial_1 N, (\partial \partial_1, M, \partial \partial_{1,3} M), \psi, f\}
\]
for any \(\theta = \{M, \partial_k M, \phi, N, \partial_k N, \psi, f; k = 1, 2, 3\}\). Furthermore, we call \(\theta_1\) the \(\partial_1\)-boundary of \(\theta\) and we may define \(\partial_k\)-boundary similarly.
Theorem 2.12. We have the following long exact sequence

\[ \cdots \to L_{n+1}(\pi_1X, \pi_1(\partial X), X; \omega) \xrightarrow{\partial} N_n(X, \partial X; \omega) \xrightarrow{i} L_n(\pi_1X, \pi_1(\partial X); \omega) \]

Then \( L_n(\pi_1X, \pi_1(\partial X); \omega) \)

Proof. (I) Exactness at \( N_n(\pi_1X, \pi_1(\partial X); \omega) \). Let \( \theta \in N_n(\pi_1X, \pi_1(\partial X); \omega) \). Then \( i_* (\theta) = 0 \) if and only if there exists an element

\[ \eta = \{W, \partial_kW, \Phi, V, \partial_kV, \Psi, F; k = 1, 2, 3\} \]

satisfying the conditions in 2.5. Note that \( \eta \) is an element in \( L_{n+1}(\pi_1X, \pi_1(\partial X), X; \omega) \) and is mapped to \( \theta \) under \( \partial_* \). This proves the exactness at \( N_n(\pi_1X, \pi_1(\partial X); \omega) \).

(II) Exactness at \( L_n(\pi_1X, \pi_1(\partial X); \omega) \). Let

\[ \xi = \{M, \partial_\pm M, \phi, N, \partial_\pm N, \psi, f\} \in N_n(\pi_1X, \pi_1(\partial X); \omega) \]

Then \( j_* i_* (\xi) = 0 \) since \( \xi \times I \) is a cobordism of \( \xi \) to the empty set where \( I \) is the unit interval. More precisely, \( \xi \times I \) consists of the following data.

1. \( W = M \times I \) with continuous map

\[ \Phi = \phi \circ p_1 : (W, \partial_4W) \xrightarrow{\partial_3} (M, \partial_-M) \xrightarrow{\phi} (X, \partial X), \]

where \( p_1 : W \to M \) is the natural projection, \( \partial W = \partial_1W(= M \times \{0\}) \cup \partial_2W \cup \partial_3W \cup \partial_4W \) with \( \partial_3W = M \times \{1\} \), \( \partial_2W = \partial_+M \times I \) and \( \partial_1W = \partial_-M \times I \).

2. There is a similar picture for \( (V, \partial V) \) with \( \partial V = \partial_1V(= N \times \{0\}) \cup \partial_2V \cup \partial_3V \cup \partial_4V \), where \( \partial_2V = N \times \{1\} \), \( \partial_3V = \partial_+N \times I \) and \( \partial_4V = \partial_-N \times I \).

3. A degree one normal map of manifold 4-ads, \( F = f \times Id : (V, \partial V) \to (W, \partial W) \). Obviously, \( \Phi \circ F = \Psi \) and \( F \) restricts to \( f \) on \( N \subseteq \partial V \).

4. \( F|_{\partial_2V} : \partial_2V = \partial_+N \times I \to \partial_3W = \partial_+M \times I \) is a homotopy equivalence. This is because \( f : \partial_+N \to \partial_+M \) is an infinitesimally controlled homotopy equivalence.

5. Moreover, \( F|_{\partial_23V} : \partial_23V = \partial_+N \to \partial_23W = \partial_+M \) is an infinitesimally controlled homotopy equivalence over \( X \).

Conversely, suppose an element

\[ \theta = \{M, \partial_\pm M, \phi, N, \partial_\pm N, \psi, f\} \in L_n(\pi_1X, \pi_1(\partial X); \omega) \]

is mapped to zero in \( L_n(\pi_1X, \pi_1(\partial X), X; \omega) \). Then

\[ j_* (\theta) = \{M, (\emptyset, \partial_\pm M, \partial_-M), \phi, N, (\emptyset, \partial_\pm N, \partial_-N), \psi, f\} \]

is cobordant to empty set in \( L_n(\pi_1X, \pi_1(\partial X), X; \omega) \). More precisely, we have the following data:

1. There exists a manifold 4-ads \( (W, \partial W) \) of dimension \( (n+1) \) with a continuous map \( \Phi : (W, \partial_4W) \to (X, \partial X) \) so that \( \Phi^*(\omega) \) describes the orientation character of \( W \), where \( \partial W = M(= \partial_1W) \cup \partial_2W \cup \partial_3W \cup \partial_4W \).
2. We have decompositions \( \partial M = \partial_1 M(= \emptyset) \cup \partial_2 M(= \partial_\pm M) \cup \partial_3 M(= \partial_- M) \), \( \partial(\partial_2 W) = \partial\partial_{2,1} W \cup \partial\partial_{2,3} W \cup \partial\partial_{2,4} W \), \( \partial(\partial_3 W) = \partial\partial_{3,1} W \cup \partial\partial_{3,2} W \cup \partial\partial_{3,4} W \), and \( \partial(\partial_4 W) = \partial\partial_{4,1} W \cup \partial\partial_{4,2} W \cup \partial\partial_{4,3} W \) such that
\[
\partial_1 M = \emptyset = \partial\partial_{1,2} W, \quad \partial_2 M = \partial_+ M = \partial\partial_{1,3} W, \quad \text{and} \quad \partial_3 M = \partial_- M = \partial\partial_{1,4} W.
\]
Moreover, we have \( \partial\partial_{1,3} W \cap \partial\partial_{2,3} W = \emptyset \).

3. Similarly, we have a manifold 4-ads \( (V, \partial V) \) of dimension \((n + 1)\) with a continuous map \( \Psi : (V, \partial V) \to (X, \partial X) \) so that \( \Psi^* (\omega) \) describes the orientation character of \( V \), where \( \partial V = N(= \partial V) \cup \partial_2 V \cup \partial_3 V \cup \partial_4 V \) satisfying similar conditions as \( W \).

4. There is a degree one normal map of manifold 4-ads \( F : (V, \partial V) \to (W, \partial W) \) such that \( \Phi \circ F = \Psi \). Moreover, \( F \) restricts to \( f \) on \( N \subseteq \partial V \).

5. The restriction \( F|_{\partial_k V} : \partial_k V \to \partial_k W \) is a degree one normal map over \( X \) for \( k = 1, 2, 4 \).

6. The restriction \( F|_{\partial_3 V} : \partial_3 V \to \partial_3 W \) is a homotopy equivalence over \( X \) and it restricts to an infinitesimally controlled homotopy equivalence \( F|_{\partial\partial_{2,3} V} : \partial\partial_{2,3} V \to \partial\partial_{2,3} W \) over \( X \).

Consequently, \( F : (V, \partial V) \to (W, \partial W) \) provides a cobordism between \( \theta \) and
\[
\eta = \{ \partial_0 W, (\partial\partial_{2,3} W, \partial\partial_{3,4} W), \Phi|_{\partial_0 W}, \partial_3 V, (\partial\partial_{2,3} V, \partial\partial_{3,4} V), \Psi|_{\partial_0 V}, F \}.
\]
Note that \( \eta \) is an element in \( N_n(\pi_1 X, \pi_1 (\partial X); \omega) \). This finishes the proof.

\textbf{(III) Exactness at} \( L_n(\pi_1 X, \pi_1 (\partial X); X; \omega) \). It is obvious that \( \partial_0 j_* = 0 \) by definition. On the other hand, if an element
\[
\theta = \{ M, \partial_k M, \phi, N, \partial_k N, \psi, f; k = 1, 2, 3 \} \in L_n(\pi_1 X, \pi_1 (\partial X), X; \omega)
\]
such that \( \partial_+ (\theta) = 0 \), then there is a cobordism of \( \partial_+ (\theta) \) to the empty set, i.e.
\[
\eta = \{ W, \partial_k W, \Phi, V, \partial_k V, \Psi, F; k = 1, 2, 3 \}
\]
following from Definition 2.7. Consequently, Let \( \theta' = \eta \cup_{\partial_+ (\theta)} \theta \). Then a cobordism of \( \theta' \) to \( \theta \) is provided by \( \theta' \times I \) with \( \partial_1 (\theta' \times I) = \theta' \times \{ 0 \} \cup \theta \times \{ 1 \} \), \( \partial_2 (\theta' \times I) = \eta \times \{ 1 \} \), \( \partial_3 (\theta' \times I) = \partial_0 \theta' \times I \) and \( \partial_4 (\theta' \times I) = \partial_0 \theta' \times I \). Note that the \( \partial_0 \)-boundary of \( \theta' \) is empty, so \( \theta' \) is the image of \( j_* \) of some element in \( L_n(\pi_1 X, \pi_1 (\partial X); \omega) \). This proves the exactness at \( L_n(\pi_1 X, \pi_1 (\partial X), X; \omega) \).

There is a natural group homomorphism
\[
c_* : S_n(\pi_1 X, \pi_1 (\partial X); \omega) \to L_{n+1}(\pi_1 X, \pi_1 (\partial X), X; \omega)
\]
by mapping
\[
\theta = \{ M, \partial_\pm M, \phi, N, \partial_\pm N, \psi, f \} \mapsto \theta \times I
\]

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where \( \theta \times I \) consists of the following data:

1. a manifold 3-ad \((M \times I, \partial_k(M \times I); k = 1, 2, 3)\) with \(\partial_1(M \times I) = (M \times \{0\}) \cup_{\partial_k(M \times I)} (\partial_+ M \times I)\), \(\partial_2(M \times I) = M \times \{1\}\) and \(\partial_3(M \times I) = \partial_- M \times I\); in particular, \(\partial \partial_2(M \times I) = \partial_k M\);

2. similarly, another manifold 3-ad \((N \times I, \partial_k(N \times I); k = 1, 2, 3)\) with \(\partial_1(N \times I) = (N \times \{0\}) \cup_{\partial_k(N \times I)} (\partial_+ N \times I)\), \(\partial_2(N \times I) = N \times \{1\}\) and \(\partial_3(N \times I) = \partial_- N \times I\);

3. a continuous map

\[
\tilde{\phi} := \phi \circ p_1 : (M \times I, \partial_3(M \times I)) \xrightarrow{\partial} (M, \partial_- M) \xrightarrow{\phi} (X, \partial X)
\]

such that \((\phi \circ p_1)^\ast(\omega)\) describes the orientation character of \(M \times I\), where \(p_1\) is the canonical projection map from \(M \times I\) to \(M\); similarly, a continuous map

\[
\tilde{\psi} := \phi \circ p_2 : (N \times I, \partial_3(N \times I)) \xrightarrow{\partial} (N, \partial_- N) \xrightarrow{\psi} (X, \partial X)
\]

describes the orientation character of \(N \times I\), where \(p_2\) is the canonical projection map from \(N \times I\) to \(N\);

4. a degree one normal map of manifold 3-ads

\[
\bar{f} := f \times Id : (N \times I, \partial_k(N \times I); k = 1, 2, 3) \to (M \times I, \partial_k(M \times I); k = 1, 2, 3)
\]

such that \(\tilde{\phi} \circ \bar{f} = \tilde{\psi}\);

5. the restriction \(\bar{f}|_{\partial_1(N \times I)} : \partial_1(N \times I) \to \partial_1(M \times I)\) is a degree one normal map (homotopy equivalence) over \(X\);

6. the restriction \(\bar{f}|_{\partial_2(N \times I)} : \partial_2(N \times I) \to \partial_2(M \times I)\) is a homotopy equivalence over \(X\) and it restricts to an infinitesimally controlled homotopy equivalence \(\bar{f}|_{\partial_1,2(N \times I)} : \partial_1,2(N \times I) \to \partial_1,2(M \times I)\) over \(X\);

7. the restriction \(\bar{f}|_{\partial_3(N \times I)} : \partial_3(N \times I) \to \partial_3(M \times I)\) is a degree one normal map over \(X\).

Define

\[
\rho_* : L_{n+1}(\pi_1 X, \pi_1(\partial X); \omega) \to S_n(\pi_1 X, \pi_1(\partial X); \omega)
\]

by

\[
\rho_*(\theta) = \partial_2(\theta) = \theta_2 = \{\partial_2 M, (\partial \partial_1,2 M, \partial \partial_2,3 M), \phi, \partial_2 N, (\partial \partial_1,2 N, \partial \partial_2,3 N), \psi, f\},
\]

for \(\theta = \{M, \partial_k M, \phi, N, \partial_k N, \psi, f; k = 1, 2, 3\}\), where \(\partial \partial_1,2 M\) means \(\partial_+(\partial_2 M)\) and \(\partial \partial_2,3 M\) means \(\partial_-(\partial_2 M)\) (resp. for \(N\)).

**Theorem 2.13.** The homomorphisms \(c_*\) and \(r_*\) are inverse of each other. In particular, we have \(S_n(\pi_1 X, \pi_1(\partial X); \omega) \cong L_{n+1}(\pi_1 X, \pi_1(\partial X), X; \omega)\).

**Proof.** First, it is obvious that

\[
r_* \circ c_* = Id : S_n(\pi_1 X, \pi_1(\partial X); \omega) \to S_n(\pi_1 X, \pi_1(\partial X); \omega).
\]
Conversely, for any
\[ \theta = \{ M, \partial_k M, \phi, N, \partial_k N, \psi; k = 1, 2, 3 \} \in L_{n+1}(\pi_1 X, \pi_1(\partial X), X; \omega), \]
c_\ast r_\ast(\theta) is cobordant to \( \theta \) in \( L_{n+1}(\pi_1 X, \pi_1(\partial X), X; \omega) \). Indeed, Consider the element
\[
(\theta \times I) \bigcup_{(\theta_2 \times I) \times \{0\} \subseteq \theta \times \{1\}} (\theta_2 \times I \times I)
\]
where \((\theta_2 \times I) \times \{0\}\) is glued to the subset \((\theta_2 \times I) \subseteq \theta \times \{1\}\). This produces a cobordism between \( c_\ast r_\ast(\theta) \) and \( \theta \), which completes the proof.

In summary, we have the following long exact sequence:
\[
\cdots \to S_n(X, \partial X; \omega) \xrightarrow{\partial} N_n(X, \partial X; \omega) \xrightarrow{i} L_n(\pi_1 X, \pi_1(\partial X); \omega) \xrightarrow{\partial} S_{n-1}(X, \partial X; \omega) \xrightarrow{i} N_{n-1}(X, \partial X; \omega) \to \cdots.
\]

The rest of this section is devoted to prove the above exact sequence is isomorphic to the classical topological surgery exact sequence. We will first briefly recall the definition of topological surgery exact sequence. For more details, see [26] and [16].

Let \( X \) be an \( n \)-dimensional compact manifold with boundary \( \partial X \).

**Definition 2.14 (Relative normal group).** An element in the relative normal group \( N^{TOP}(X, \partial X) \) consists of \((M, \partial M, f)\) where \( f : (M, \partial M) \to (X, \partial X) \) is a degree one normal map. Two elements \((M_1, \partial M_1, f_1)\) and \((M_2, \partial M_2, f_2)\) are equivalent if they are normal cobordant in the sense of

1. There exists a degree one normal map of manifold with boundary \( f : V \to \partial X \times [0, 1] \) such that \( \partial V = \partial M_1 \cup \partial M_2 \), \( f \) equals to \( f_1 \cup f_2 \) restricting to \( \partial V \), and maps \( \partial M_1 \) to \( X \times \{0\} \), \( \partial M_2 \) to \( X \times \{1\} \).
2. There is a manifold with boundary \((W, \partial W)\) such that \( \partial W = M_1 \cup V \cup M_2 \).
3. \( F \) is a degree one normal map from \( W \) to \( X \times [0, 1] \) such that \( F = f_1 \cup f \cup f_2 \) restricting to \( \partial W \), and maps \( M_1 \) to \( X \times \{0\} \), \( M_2 \) to \( X \times \{1\} \).

Obviously, there is a natural map
\[
\alpha_\ast : N^{TOP}(X, \partial X) \to N_n(X, \partial X; \omega)
\]
defined by
\[
[f : (M, \partial M) \to (X, \partial X)] \\
\mapsto \theta = \{ X, (\partial_+ X = \emptyset, \partial_- X = \partial X), Id, M, (\partial_+ M = \emptyset, \partial_- M = \partial M), f, f \}.
\]
Definition 2.15 (\(\partial_+\)-Relative normal group). An element in \(\partial_+\)-relative normal group \(N^{TOP}_{\partial_+}(X \times D^i, \partial (X \times D^i))\) consists of \((M, \partial \pm M, f)\) where

\[
f : (M, (\partial_+ M, \partial_- M)) \to (X \times D^i, (X \times \partial D^i, \partial X \times D^i))
\]

is a degree one normal map of manifold 2-ads and it restricts to a homeomorphism \(f|_{\partial \pm M} : \partial \pm M \to X \times \partial D^i\). Two elements

\[(M_1, \partial \pm M_1, f_1) \text{ and } (M_2, \partial \pm M_2, f_2)\]

are equivalent if they are normal cobordant in the sense of

1. There exists a degree one normal map of manifold with boundary \(f : V \to \partial (X \times D^i) \times [0, 1]\) with \(V = \partial_+ V \cup \partial_- V\) and \(\partial \pm V = \partial \pm M_1 \cup \partial \pm M_2\), such that \(f\) restricts to a homeomorphism \(f|_{\partial \pm V} : \partial \pm V \to (X \times \partial D^i) \times [0, 1]\), \(f\) equals to \(f_1 \cup f_2\) restricting to \(\partial V\), and maps \(\partial M_1\) to \(\partial (X \times D^i) \times \{0\}\), \(\partial M_2\) to \(\partial (X \times D^i) \times \{1\}\).

2. There is a manifold with boundary \((W, \partial W)\) such that \(\partial W = M_1 \cup V \cup M_2\).

3. \(F\) is a degree one normal map from \(W\) to \((X \times D^i) \times [0, 1]\) such that \(F = f_1 \cup f \cup f_2\) restricting to \(\partial W\), and maps \(M_1\) to \((X \times D^i) \times \{0\}\), \(M_2\) to \((X \times D^i) \times \{1\}\).

It follows from [5] that we can define the addition on \(\partial_+\)-Relative normal group in an explicit way. Denote \(D^i = D^i_u \cup_{D^i-1} D^i_d\), where

\[
D^i_u = \{ (t_1, \ldots, t_i) \in \mathbb{R}^i_+; \sum_{k=1}^i t_k^2 \leq 1 \text{ and } t_i \geq 0 \}
\]

and

\[
D^i_d = \{ (t_1, \ldots, t_i) \in \mathbb{R}^i_+; \sum_{k=1}^i t_k^2 \leq 1 \text{ and } t_i \leq 0 \}.
\]

Let \(S^{i-1}_u = \partial(D^i) \cap D^i_u\) and \(S^{i-1}_d = \partial(D^i) \cap D^i_d\). Then one can choose suitable homeomorphisms

\[
(D^i, S^{i-1}_u, S^{i-1}_d) \cong (D^i_u, S^{i-1}_u, D^{i-1}) \text{ and } (D^i, S^{i-1}_d, S^{i-1}_d) \cong (D^i_d, S^{i-1}_d, D^{i-1}).
\]

Note that for any \((M, \partial \pm M, f) \in N^{TOP}_{\partial_+}(X \times D^i, \partial (X \times D^i))\), we have a decomposition \(\partial_+ M = \partial_+^u M \cup \partial_+^d M\) such that \(\partial_+^u f = f|_{\partial_+^u M} : \partial_+^u M \to X \times S^{i}_u\) and \(\partial_+^d f = f|_{\partial_+^d M} : \partial_+^d M \to X \times S^{i}_d\) are homeomorphisms.

For any two elements \((M_k, \partial \pm M_k, f_k), k = 1, 2\), in \(N^{TOP}_{\partial_+}(X \times D^i, \partial (X \times D^i))\), define \((M_1 + M_2, \partial \pm (M_1 + M_2), f_1 + f_2)\) by

\[
f_1 + f_2 = f_1 \cup f_2 : M_1 + M_2 = M_1 \cup_{\partial} M_2 \to X \times D^i = X \times D^i_u \cup X \times D^i_d,
\]

where \(g : \partial_+^u M_1 \to \partial_+^d M_2\) is given by \(g = (\partial_+^d f_2)^{-1} \circ \partial_+^u f_1\).

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There is a natural map
\[ \alpha_\ast : \mathcal{N}^{TOP}_{\partial_+}(X \times D^i, \partial(X \times D^i)) \to \mathcal{N}_{n+i}(X, \partial X; \omega) \]
defined by
\[ [f : (M, \partial M) \to (X \times D^i, \partial(X \times D^i))] \mapsto \theta = \{ X \times D^i, (X \times \partial D^i, \partial X \times D^i), p, M, (\partial_+ M, \partial_- M), p \circ f, f \}, \]
where \( p : X \times D^i \to X \) is the natural projection.

By the idea of control topology \([8, 28]\), we have the following theorem similar to a result in \([27]\).

**Theorem 2.16.** If \( \dim X = n \geq 6 \), the map \( \alpha_\ast : \mathcal{N}^{TOP}_{\partial_+}(X \times D^i, \partial(X \times D^i)) \to \mathcal{N}_{n+i}(X, \partial X; \omega) \) is an isomorphism.

**Proof.** For any \( i \geq 0 \), there is a commutative diagram
\[
\begin{align*}
\mathcal{N}^{TOP}_{\partial_+}(X \times D^i, \partial(X \times D^i)) & \xrightarrow{\alpha_\ast} \mathcal{N}_{n+i}(X, \partial X; \omega) \\
\cong & \downarrow \cong \downarrow \\
H_{n+i}(X, \partial X; \mathbb{L}_\ast) & = H_{n+i}(X, \partial X; \mathbb{L}_\ast)
\end{align*}
\]
where the vertical isomorphisms are the corresponding algebraic normal invariant maps. This completes the proof. \( \square \)

**Definition 2.17 (Relative structure set).** An element in relative structure set \( S^{TOP}(X, \partial X) \) consists of \((M, \partial M, f)\) where \( f : (M, \partial M) \to (X, \partial X) \) is a homotopy equivalence. Two elements \((M_1, \partial M_1, f_1)\) and \((M_2, \partial M_2, f_2)\) are equivalent if they are \( h \)-cobordant.

1. There exists a homotopy equivalence of manifold with boundary \( f : V \to \partial X \) such that \( \partial V = \partial M_1 \cup \partial M_2 \), \( f \) equals to \( f_1 \cup f_2 \) restricting to \( \partial V \), and maps \( \partial M_1 \) to \( X \times \{0\} \), \( \partial M_2 \) to \( X \times \{1\} \).

2. There is a manifold with boundary \((W, \partial W)\) such that \( \partial W = M_1 \cup V \cup M_2 \).

3. \( F \) is a homotopy equivalence from \( W \) to \( X \times [0, 1] \) such that \( F = f_1 \cup f \cup f_2 \) restricting to \( \partial W \), and maps \( M_1 \) to \( X \times \{0\} \), \( M_2 \) to \( X \times \{1\} \).

The map
\[ \iota_\ast : S^{TOP}(X, \partial X) \to \mathcal{S}_n(X, \partial X; \omega) \]
defined by
\[ [f : (M, \partial M) \to (X, \partial X)] \mapsto \theta = \{ X, (\partial_+ X = \emptyset, \partial_- X = \partial X), Id, M, (\partial_+ M = \emptyset, \partial_- M = \partial M), f, f \} \]
is a natural map.
Definition 2.18 ($\partial_+$-relative structure set). An element in the $\partial_+$-relative structure set $STOP_{\partial_+}(X \times D^i, \partial(X \times D^i))$ consists of $(M, \partial_+ M, f)$ where

$$f : (M, (\partial_+ M, \partial_- M)) \to (X \times D^i, (X \times \partial D^i, \partial X \times D^i))$$

is a homotopy equivalence of manifold 2-ads and it restricts to a homeomorphism $f|_{\partial_+ M} : \partial_+ M \to X \times \partial D^i$.

Two elements $(M_1, \partial M_1, f_1)$ and $(M_2, \partial M_2, f_2)$ are equivalent if they are $h$-cobordant.

1. There exists a homotopy equivalence of manifold with boundary $f : V \to \partial(X \times D^i) \times [0, 1]$ with $V = V_+ \cup V_-$ and $\partial V_\pm = \partial_+ M_1 \cup \partial_\pm M_2$, such that $f$ restricts to a homeomorphism $f|_{V_+} : V_+ \to (X \times \partial D^i) \times [0, 1]$, and $f$ maps $V_-$ into $(\partial X \times D^i) \times [0, 1]$ $f$ equals to $f_1 \cup f_2$ restricting to $\partial V$, and maps $\partial M_1$ to $\partial(X \times D^i) \times \{0\}$, $\partial M_2$ to $\partial(X \times D^i) \times \{1\}$.

2. There is a manifold with boundary $(W, \partial W)$ such that $\partial W = M_1 \cup V \cup M_2$.

3. $F$ is a homotopy equivalence from $W$ to $X \times [0, 1]$ such that $F = f_1 \cup f \cup f_2$ restricting to $\partial W$, and maps $M_1$ to $X \times \{0\}$, $M_2$ to $X \times \{1\}$.

Moreover, for $i \geq 1$, the addition on $STOP_{\partial_+}(X \times D^i, \partial(X \times D^i))$ is given as what we do for $N_{\partial_+}(X \times D^i, \partial(X \times D^i))$.

There is obvious a natural map

$$\beta_* : S_{\partial_+} TOP(X \times D^i, (X \times \partial D^i, \partial X \times D^i)) \to S_{n+i}(X, \partial X; \omega)$$

by

$$[f : (M, \partial M) \to (X, \partial X)]$$

$$\mapsto \theta = \{X \times D^i, (X \times \partial D^i, \partial X \times D^i), p, M, (\partial_+ M, \partial_- M), p \circ f, f\},$$

where $p : X \times D^i \to X$ is the natural projection.

**Lemma 2.19.** For $i \geq 1$, the map $\beta_* : S_{\partial_+} TOP(X \times D^i, (X \times \partial D^i, \partial X \times D^i)) \to S_{n+i}(X, \partial X; \omega)$ is a group homomorphism.

**Proof.** Given any two elements $(M_1, \partial M_1, f_1)$ and $(M_2, \partial M_2, f_2)$ in $S_{\partial_+} TOP(X \times D^i, (X \times \partial D^i, \partial X \times D^i))$. Let

$$\theta = \beta_*(M_1 + M_2, \partial(M_1 + M_2), f_1 + f_2)$$

$$= \{X \times D^i, (X \times \partial D^i, \partial X \times D^i), p, M_1 + M_2, (\partial_+(M_1 + M_2), \partial_-(M_1 + M_2)), p \circ (f_1 + f_2), f_1 + f_2\}$$

and

$$\eta = \beta_*(M_1, \partial M_1, f_1) + \beta_*(M_2, \partial M_2, f_2)$$

$$= \{X \times D^i \cup X \times D^i, (X \times \partial D^i \cup X \times \partial D^i, \partial X \times D^i \cup \partial X \times D^i), p \cup p, M_1 \cup M_2, (\partial_+(M_1 \cup M_2), \partial_-(M_1 \cup M_2)), (p \circ f_1 \cup p \circ f_2), f_1 \cup f_2\}.$$
Following from the construction of the addition in $S_{\partial_x}^{\text{TOP}}(X \times D^i, (X \times \partial D^i, \partial X \times D^i))$, we can obtain a map $g : \eta \to \theta$. Then, consider the element

$$ (\eta \times I) \bigcup_{\eta \times \{1\} \to \theta \times \{0\}} (\theta \times I). $$

This produces a cobordism between $\eta$ and $\theta$. This completes the proof.

The following is the classical topological surgery exact sequence:

$$ \cdots \to N_{\partial_x}^{\text{TOP}}(X \times I, (X \times \partial I, \partial X \times I)) \overset{\varepsilon_{\partial_x}}{\longrightarrow} L_{n+1}(\pi_1 X, \pi_1(\partial X); \omega) \to S_{\partial_x}^{\text{TOP}}(X, \partial X) \overset{\varepsilon_{\partial_x}}{\longrightarrow} N^{\text{TOP}}(X, \partial X) \overset{\varepsilon_{\partial_x}}{\longrightarrow} L_n(\pi_1 X, \pi_1(\partial X); \omega). $$

For details, see [26].

By the above argument, we have the following theorem.

**Theorem 2.20.** The following diagram

$$
\begin{array}{cccc}
\vdots & \vdots & \vdots \\
N_{\partial_x}^{\text{TOP}}(X \times I, (X \times \partial I, \partial X \times I)) & \overset{\varepsilon_{\partial_x}}{\longrightarrow} & N_{n+1}(X, \partial X; \omega) \\
\downarrow & & \downarrow \\
L_{n+1}(\pi_1 X, \pi_1(\partial X); \omega) & \overset{\varepsilon_{\partial_x}}{\longrightarrow} & L_{n+1}(\pi_1 X, \pi_1(\partial X); \omega) \\
& & \downarrow \\
S_{\partial_x}^{\text{TOP}}(X, \partial X) & \overset{\varepsilon_{\partial_x}}{\longrightarrow} & S_n(X, \partial X; \omega) \\
\downarrow & & \downarrow \\
N^{\text{TOP}}(X, \partial X) & \overset{\varepsilon_{\partial_x}}{\longrightarrow} & N_n(X, \partial X; \omega) \\
\downarrow & & \downarrow \\
L_n(\pi_1 X, \pi_1(\partial X); \omega) & \overset{\varepsilon_{\partial_x}}{\longrightarrow} & L_n(\pi_1 X, \pi_1(\partial X); \omega).
\end{array}
$$

is commutative.

Consequently, by the standard five lemma argument, we obtain the following results.

**Lemma 2.21.** If $\dim X = n \geq 6$, the map $i_* : S_{\partial_x}^{\text{TOP}}(X, \partial X) \to S_n(X, \partial X; \omega)$ is a bijection. Moreover, for $i \geq 1$, the map $\beta_* : S_{\partial_x}^{\text{TOP}}(X \times D^i, (X \times \partial D^i, \partial X \times D^i)) \to S_{n+1}(X, \partial X; \omega)$ is a group isomorphism.

**Lemma 2.22.** If $\dim X = n \geq 6$, the map $i_* : S_{\partial_x}^{\text{TOP}}(X, \partial X) \to S_n(X, \partial X; \omega)$ is a group homomorphism.

**Proof.** The Siebenmann’s periodicity theorem plays an important role in the present proof (from [21] with a correction [14]). S. Cappell and S. Weinberger [2] gave a geometric interpretation of the Siebenmann periodicity phenomena. Now, Siebenmann’s periodicity theorem can be stated in terms of a exact sequence,
and it also works for oriented connected topological manifolds with boundary (see also [12]):

\[ 0 \rightarrow S_{\partial \tau}^{TOP}(X, \partial X) \xrightarrow{\text{CW}} S_{\partial \tau}^{TOP}(X \times D^4, (X \times \partial D^4, \partial X \times D^4)) \xrightarrow{\sigma} \mathbb{Z}. \]

This is because for any homotopy equivalence \( f : (M, \partial M) \rightarrow (X, \partial X) \), the construction of CW yields \( f \times \text{id} : \partial M \times D^4 \rightarrow \partial X \times D^4 \) on the boundary so that the map to \( \mathbb{Z} \) is trivial. Then following from [27], we have the commutative diagram

\[
\begin{array}{ccc}
S_{\partial \tau}^{TOP}(X, \partial X) & \xrightarrow{\text{CW}} & S_{\partial \tau}^{TOP}(X \times D^4, (X \times \partial D^4, \partial X \times D^4)) \\
\downarrow \iota_* & & \downarrow \beta_* \\
S_n(X, \partial X; \omega) & \xrightarrow{\text{CP}^2} & S_{n+4}(X, \partial X; \omega).
\end{array}
\]

This finishes the proof.

\[ \Box \]

Combing Lemma 2.21 and Lemma 2.22, one can obtain the following theorem.

**Theorem 2.23.** If \( \dim X = n \geq 6 \), the map \( \iota_* : S_{\partial}^{TOP}(X, \partial X) \rightarrow S_n(X, \partial X; \omega) \) is an isomorphism.

### 3 K-theory preparation

In this section, we introduce the definitions of several relative geometric C*-algebras involved in this paper. We start with the definitions of the maximal Roe, localization and obstruction algebras. We then introduce the relative version of maximal Roe, localization and obstruction algebras in light of [3]. We also introduce some results in the K-theory of the relative obstruction algebras briefly for later purpose.

#### 3.1 Basic notions

We first recall the definitions of the maximal Roe, localization and obstruction algebras. Let \( X \) be a proper metric space with bounded geometry. A discrete group \( G \) acts freely and properly on it. A \( G \)-equivariant \( X \)-module \( H_X \) is separable Hilbert space equipped with a *-representation \( \phi \) of \( C_0(X) \) and a covariant \( G \) action \( \pi \) such that

\[
\pi(g)(\phi(f)v) = \phi(f^g)(\pi(g)(v)), \quad \forall g \in G, f \in C_0(X) \text{ and } v \in H_X,
\]

where \( f^g(x) = f(g^{-1}x) \). \( H_X \) is said to be standard if no nonzero function in \( C_0(X) \) acts as a compact operator. We call \( H_X \) non-degenerate if any *-representation of \( C_0(X) \) is non-degenerate.

**Definition 3.1** (cf. [20]). Let \( H_X \) be a \( G \)-equivariant standard \( X \)-module.
1. The support \( \text{supp}(T) \) of a bounded linear operator \( T \in B(H_X) \) is defined to be the complement of the set of all points \((x, y) \in X \times X\) for which there exist \( f, g \in C_0(X) \) such that \( gTf = 0, f(x) \neq 0, g(y) \neq 0 \).

2. A bounded linear operator \( T \in B(H_X) \) is said to have finite propagation if
\[
\sup \{ d(x, y) : (x, y) \in \text{Supp}(T) \} < \infty.
\]
This number will be called the propagation of \( T \), and denoted as propagation\((T)\).

3. A bounded linear operator \( T \in B(H_X) \) is said to be locally compact if the operators \( fT \) and \( Tf \) are compact for all \( f \in C_0(X) \).

Denote by \( C[X]^G \) the set of all locally compact, finite propagation \( G \)-invariant operators on a standard non-degenerate \( X \)-module \( H_X \).

**Definition 3.2.** Let \( X \) be a proper metric space with bounded geometry. \( G \) acts on \( X \) freely and properly.

1. The maximal Roe algebra \( C_{\text{max}}^*(X)^G \) is the completion of \( C[X]^G \) with respect to the \( C^* \)-norm
\[
\| T \|_{\text{max}} := \sup \{ \| \phi(T) \|_{B(H_\phi)} : \phi : C[X]^G \to H_\phi, \text{ a } *-\text{representation} \},
\]
where \( H_\phi \) is a \( G \)-equivariant standard non-degenerate \( X \)-module with the \( * \)-representation \( \phi \).

2. The maximal localization algebra \( C_{L,\text{max}}^*(X)^G \) is the \( C^* \)-algebra generated by all bounded and uniformly norm-continuous functions \( f : [0, \infty) \to C_{\text{max}}^*(X)^G \) such that the propagation of \( f(t) \to 0 \), as \( t \to \infty \).

3. The maximal obstruction algebra \( C_{L,0,\text{max}}^*(X)^G \) is the kernel of the evaluation at 0 map
\[
ev : C_{L,\text{max}}^*(X)^G \to C_{\text{max}}^*(X)^G, \ ev(f) = f(0).
\]

4. If \( Y \) is a subspace of \( X \) and \( G \) acts on \( Y \) freely and properly, then \( C_{L,\text{max}}^*(Y, X)^G \) (resp. \( C_{L,0,\text{max}}^*(Y; X)^G \)) is defined to be the closed subalgebra of \( C_{L,\text{max}}^*(X)^G \) (resp. \( C_{L,0,\text{max}}^*(X)^G \)) generated by all elements \( f \) such that there exists \( c_t > 0 \) satisfying \( \lim_{t \to \infty} c_t = 0 \) and
\[
\text{Supp}(f(t)) \subset \{ (x, y) \in X \times X | d((x, y), Y \times Y) \}
\]
for all \( t \).

### 3.2 Relative \( C^* \)-algebras

In this subsection, we recall the definition of the relative Roe algebra, the relative localization algebra and the relative obstruction algebra in light of [3]. We start with the following construction.
**Definition 3.3.** Let $i : A \to B$ be a $C^*$-algebra homomorphism. We define $C_{i:A\to B}$ to be the $C^*$-algebra generated by

$$\{(a,f) : f \in C_0([0,1), B), a \in A, f(0) = i(a)\}.$$ 

For a manifold with boundary $(M, \partial M)$, let $p : \tilde{M} \to M$ and $p' : \partial \tilde{M} \to \partial M$ be the universal covering maps of $M$ and $\partial M$ respectively, and let $\tilde{\partial M}'$ be $p^{-1}\partial M$. Let

$$i : \partial M \to M$$

be the embedding map and

$$j : \pi_1(\partial M) \to \pi_1(M)$$

be the inclusion of the fundamental groups induced by $i$. Let $\tilde{\partial M}''$ be the Galois covering space of $\partial M$ whose Deck transformation group is $j\pi_1(\partial M)$. We have $\partial \tilde{M}'' = \pi_1(M) \times j\pi_1(\partial M) \tilde{\partial M}''$. This decomposition naturally gives rise to a $*$-homomorphism

$$\phi' : C^*_\text{max}(\tilde{\partial M}'')^{\pi_1(\partial M)} \to C^*_\text{max}(\partial \tilde{M}'').$$

Lemma 2. 12 of \cite{3} shows that there is a natural $*$-homomorphism

$$\phi'' : C^*_\text{max}(\tilde{\partial M})^{\pi_1(\partial M)} \to C^*_\text{max}(\tilde{\partial M}'')^{\pi_1(\partial M)}.$$ 

Thus $i : \partial M \to M$ induces a $C^*$-algebra homomorphism

$$\phi'\phi'' : C^*_\text{max}(\tilde{\partial M})^{\pi_1(\partial M)} \to C^*_\text{max}(\tilde{\partial M})^{\pi_1(\partial M)}.$$ 

With a little abuse of notation, we denote $\phi'\phi''$ still as $i$. Similarly, one can see that $i : \partial M \to M$ also induces the following two $*$-homomorphisms

$$i_L : C^*_\text{max}(\tilde{\partial M})^{\pi_1(\partial M)} \to C^*_\text{max}(\tilde{M})^{\pi_1(M)},$$

$$i_{L,0} : C^*_{L,0,\text{max}}(\tilde{\partial M})^{\pi_1(\partial M)} \to C^*_\text{max}(\tilde{M})^{\pi_1(M)}.$$ 

For any $C^*$-algebra $A$, let $SA$ be its suspension algebra.

**Definition 3.4** (Relative maximal algebras).

1. The relative maximal Roe algebra is then defined as

$$C^*_\text{max}(\tilde{M}, \tilde{\partial M})^{\pi_1(M), \pi_1(\partial M)} := SC_i.$$ 

2. The relative maximal localization algebra is then defined as

$$C^*_{L,\text{max}}(\tilde{M}, \tilde{\partial M})^{\pi_1(M), \pi_1(\partial M)} := SC_{i_L}.$$
In this subsection, we introduce the quantitative obstruction algebra. In the following, we convene that $\epsilon$ is a positive number.

The relative maximal obstruction algebra is then defined as

$$C_{L,0,\text{max}}(\tilde{M}, \partial \tilde{M})_{\pi_1(M), \pi_1(\partial M)} := SC_{L,0}.$$ 

All the Roe algebras, localization algebras and obstruction algebras considered in this paper are maximal ones. For the sake of conciseness, we oppose the subscription $\text{max}$ in the following. The relative algebras defined above are then denoted as $C^*(\tilde{M}, \partial \tilde{M})_{\pi_1(M), \pi_1(\partial M)}$, $C^*_L(\tilde{M}, \partial \tilde{M})_{\pi_1(M), \pi_1(\partial M)}$ and $C^*_{L,0}(\tilde{M}, \partial \tilde{M})_{\pi_1(M), \pi_1(\partial M)}$ respectively. No confusion should be arose.

Let $G$ be $\pi_1(M)$ and $\Gamma$ be $\pi_1(\partial M)$. The following $K$-theory six exact sequence is routine:

$$
\begin{array}{cccc}
K_0(C^*_{L,0}((\tilde{M}, \partial \tilde{M})^{G, \Gamma})) & \to & K_0(C^*_L((\tilde{M}, \partial \tilde{M})^{G, \Gamma})) & \to & K_0(C^*(\tilde{M}, \partial \tilde{M})^{G, \Gamma}) \\
\downarrow & & \downarrow & & \downarrow \\
K_1(C^*(\tilde{M}, \partial \tilde{M})^{G, \Gamma}) & \to & K_1(C^*_L((\tilde{M}, \partial \tilde{M})^{G, \Gamma})) & \to & K_1(C^*_{L,0}((\tilde{M}, \partial \tilde{M})^{G, \Gamma}))
\end{array}
$$

Moreover, let $I$ and $I'$ be intervals with $I \subseteq I' \subseteq (-\infty, \infty)$. Note that there is a natural embedding:

$$C^*_L(\tilde{M} \times I; \tilde{\partial M} \times I') \to C^*_L(\tilde{M} \times I; \tilde{\partial M} \times I')^{G, \Gamma}.$$ 

Similarly as above, one can define $C^*_L(\tilde{M} \times I, \tilde{\partial M} \times I', \tilde{M} \times I, \tilde{\partial M} \times I')^{G, \Gamma}$. By discussion in [29] and an argument of the five lemma, one can see that

$$C^*_L(\tilde{M} \times [1, m], \tilde{\partial M} \times [1, m]; \tilde{M} \times [1, \infty), \tilde{\partial M} \times [1, \infty))^{G, \Gamma}$$

$$\cong C^*_L(\tilde{M} \times [1, m], \tilde{\partial M} \times [1, m])^{G, \Gamma}$$

$$\cong C^*_L(\tilde{M}, \tilde{\partial M})^{G, \Gamma}.$$ 

and

$$C^*_L(\tilde{M} \times [-m, m], \tilde{\partial M} \times [-m, m]; \tilde{M} \times (-\infty, \infty), \tilde{\partial M} \times (-\infty, \infty))^{G, \Gamma}$$

$$\cong C^*_L(\tilde{M} \times [-m, m], \tilde{\partial M} \times [-m, m])^{G, \Gamma}$$

$$\cong C^*_L(\tilde{M}, \tilde{\partial M})^{G, \Gamma}.$$ 

### 3.3 Quantitative $K$ theory of the relative obstruction algebra

In this subsection, we introduce the quantitative $K$-theory of the obstruction algebra $C^*_L(\tilde{M}, \tilde{\partial M})^{G, \Gamma}$, which plays a central role in proving that the relative higher $\rho$ invariant induces a well defined map on the relative topological structure group in Section [3]. We start with the quantitative $K$-theory of the regular obstruction algebra. In the following, we convene that $\epsilon$ is a positive number.
less than $\frac{1}{100}$. Let $(M, \partial M)$ be a PL manifold with boundary, with $\pi_1(M) = G$ and $\pi_1(\partial M) = \Gamma$.

For any positive number $r$, set $C^*_{L,0}(\widetilde{M})^G_r$ the linear subspace of $C^*_{L,0}(\widetilde{M})^G$ defined by

$$C^*_{L,0}(\widetilde{M})^G_r = \{ f \in C^*_{L,0}(\widetilde{M})^G | \sup_{t \in I} \{ \text{propagation}(f(t)) \} \leq r \}.$$  

The subspaces $\{C^*_{L,0}(\widetilde{M})^G_r\}_{r \geq 0}$ form a filtration of $C^*_{L,0}(\widetilde{M})^G$ and $C^*_{L,0}(\partial \widetilde{M})^G$ in the sense of [15]. The $(\epsilon, r)$-K$_0$-theory of $C^*_{L,0}(\widetilde{M})^G$, $K^*_{e,r}(C^*_{L,0}(\widetilde{M})^G)$, is then the abelian group generated by elements in

$$P^*_{\infty}(C^*_{L,0}(\widetilde{M})^G) \triangleq \{ p \mid p \in M_\infty(C^*_{L,0}(\widetilde{M})^G), \| p^2 - p \| \leq \epsilon \}$$

under equivalent relationship

$$p_1 \sim p_2 \iff \exists h : [0, 1] \to P^*_{\infty}(C^*_{L,0}(\widetilde{M})^G), \text{ s.t. } h(0) = p_1 \text{ and } h(1) = p_2.$$  

In the meanwhile, the $(\epsilon, r)$-K$_1$-theory of $C^*_{L,0}(\widetilde{M})^G$, $K^*_{e,r}(C^*_{L,0}(\widetilde{M})^G)$, is then the abelian group generated by elements in

$$U^*_{\infty} \triangleq \{ u \in M_\infty(C^*_{L,0}(\widetilde{M})^G) \mid u \text{ is invertible and } \| u - u^* \| \leq \epsilon \}$$

under the equivalence relationship

$$u_1 \sim u_2 \iff \exists h : [0, 1] \to U^*_{\infty}(C^*_{L,0}(\widetilde{M})^G), \text{ s.t. } h(0) = u_1 \text{ and } h(1) = u_2.$$  

For more details of the quantitative K-theory, see [15]. Similarly, one can define $K^*_{e,r}(C^*_{L,0}(\partial \widetilde{M})^\Gamma)$.

In [4], Chen, Yu and the second author proved the following result.

**Proposition 3.5** (Corollary 4.2 of [4]). Let $X$ be an $m$ dimensional complete manifold. There is a peper, free and cocompact G-action by isometries on $X$. For any $0 < \epsilon < \frac{1}{100}$, and $r > 0$, there exist $0 < \epsilon_1 \leq \epsilon$, $0 < r_1 \leq r$, such that every element in $K^*_{e_1,r_1}(C^*_{L,0}(X)^G)$ (resp. $K^*_{e_1,r_1}(C^*_{L,0}(X)^G)$) equals to the trivial element in $K^*_{e,r}(C^*_{L,0}(X)^G)$ (resp. $K^*_{e,r}(C^*_{L,0}(X)^G)$), where $\epsilon_1$ depends only on $\epsilon$, $r_1$ depends only on $r$.

The argument for Corollary 4.2 of [4] is based on the CW complex structure of a complete manifold. However, it can be applied verbatim to a PL manifold.

Recall the embedding

$$i : \partial M \to M$$

induces a $\ast$-homomorphism

$$i_{L,0} : C^*_{L,0}(\partial \widetilde{M})^\Gamma \to C^*_{L,0}(\widetilde{M})^G.$$  

To introduce the quantitative K-theory of the relative obstruction algebra $C^*_{L,0}(\widetilde{M}, \partial \widetilde{M})^{G, \Gamma}$, set a filtration of $C^*_{L,0}(\widetilde{M}, \partial \widetilde{M})^{G, \Gamma}$ as follows

$$C^*_{L,0}(\widetilde{M}, \partial \widetilde{M})^{G, \Gamma}_r = SC_{i_{L,0}, r},$$
where $C_{iL,o}$ is a linear subspace of $C_{iL,o}$ defined by
\[{(a, f) \in C_{iL,o} \mid \sup_{t \geq 0} \{\text{propagation}(a(t))\} \leq r} \}\}
\[{(a, f) \in C_{iL,o} \mid \sup_{t \geq 0} \{\text{propagation}(f(t))\} \leq r} \}\].

Note that the subalgebras $\{C_{iL,o,r}\}_{r \geq 0}$ form a filtration of the algebra $C_{iL,o}$ in the sense of [15].

We thus can define $K_{*}^{r}r(C_{L,0}(\tilde{M}, \tilde{\partial M})^{G, \Gamma})$, and have the following result.

**Theorem 3.6.** Let $(M, \partial M)$ be an $m$ dimensional PL manifold with boundary, and $G$ be fundamental group of $M$, and $\Gamma$ be the fundamental group of $\partial M$. For any $0 < \epsilon < \frac{1}{100}$, $r > 0$, there exist $0 < \epsilon_{1} \leq \epsilon$, $0 < r_{1} \leq r$, such that every element in $K_{1}^{r_{1}}(C_{L,0}(\tilde{M}, \tilde{\partial M})^{G, \Gamma})$ (resp. $K_{0}^{r_{1}}(C_{L,0}(\tilde{M}, \tilde{\partial M})^{G, \Gamma})$) is equal to the trivial element in $K_{1}^{r_{1}}(C_{L,0}(\tilde{M}, \tilde{\partial M})^{G, \Gamma})$ (resp. $K_{0}^{r_{1}}(C_{L,0}(\tilde{M}, \tilde{\partial M})^{G, \Gamma})$), where $\epsilon_{1}$ depends only on $\epsilon$, $r_{1}$ depends only on $r$.

**Proof.** This theorem follows immediately from Proposition 3.5 and the following short exact sequence.

$K_{*}^{r}r(C_{L,0}(\tilde{M})^{G}) \otimes K_{1}(C(S)) \rightarrow K_{*}^{r}r(C_{iL,o}) \rightarrow K_{*}^{r}r(C_{L,0}(\tilde{\partial M})^{G}).$

\[\square\]

### 3.4 A hybrid $C^{*}$-algebra

In this subsection, we define a series of $C^{*}$-algebras which is useful to the proof of the additivity of the relative higher $\rho$ invariant.

In Section 4 of [27], Weinberger, Xie and Yu introduced a certain hybrid $C^{*}$-algebras. Let $Y$ be a proper metric space equipped with a proper and free $G$-action.

**Definition 3.7 (Definition 4. 1 of [27]).** The $C^{*}$-algebra $C^{*}(Y)^{G}$ is defined to be the $C^{*}$-subalgebra of $C^{*}(Y)^{G}$ generated by elements $\alpha \in C^{*}(Y)$ of the following form: for any $\epsilon > 0$, there exists a $G$-invariant cocompact subset $K \subset Y$ such that the propagation of $a\chi_{(Y-K)}$ and $\chi_{(Y-K)}\alpha$ are both less than $\epsilon$. Here $\chi_{(Y-K)}$ is the characteristic function of $Y-K$.

**Definition 3.8 (Definition 4. 5 of [27]).** The $C^{*}$-algebra $C_{L,c}^{*}(Y)^{G}$ is defined to be the $C^{*}$-subalgebra of $C_{L,c}^{*}(Y)^{G}$ generated by elements $\alpha \in C_{L,c}^{*}(Y)$ of the following form: for any $\epsilon > 0$, there exists a $G$-invariant cocompact subset $K \subset Y$ such that
\[\sup_{t \geq 0} \{\text{propagation}(\alpha(t)\chi_{(Y-K)})\} \text{ and } \sup_{t \geq 0} \{\text{propagation}(\chi_{(Y-K)}\alpha(t))\}\]
are both less than $\epsilon$. Here $\chi_{(Y-K)}$ is the characteristic function of $Y-K$. 

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Definition 3.9 (Definition 4.2 of [27]). The C*-algebra $C^*_{L,0,c}(Y)^G$ is defined to be the C*-subalgebra of $C^*_{L,0,c}(Y)^G$ generated by elements $\alpha \in C^*_{L,0,c}(Y)$ of the following form: for any $\epsilon > 0$, there exists a $G$ invariant $G$-cocompact subset $K \subset Y$ such that

$$\sup_{t \geq 0}\{\text{propagation}(\alpha(t)\chi_{(Y-K)}(t))\} \text{ and } \sup_{t \geq 0}\{\text{propagation}(\chi_{(Y-K)}\alpha(t))\}$$

are both less than $\epsilon$. Here $\chi_{(Y-K)}$ is the characteristic function of $Y - K$.

Consider a manifold with boundary $(M, \partial M)$. The embedding map $i : \partial M \to M$ certainly induces the following embedding maps,

$$i_c : C_*^c(\tilde{\partial M} \times [1, \infty))^G \to C_*^c(\tilde{\partial M} \times [1, \infty))^G,$$

$$i_{L,c} : C_*^c(\tilde{\partial M} \times [1, \infty))^G \to C_*^c(\tilde{\partial M} \times [1, \infty))^G,$$

$$i_{L,0,c} : C_*^c(\tilde{\partial M} \times [1, \infty))^G \to C_*^c(\tilde{\partial M} \times [1, \infty))^G.$$

Thus we can define

$$C_*^c(\tilde{\partial M} \times [1, \infty), \tilde{\partial M} \times [1, \infty))^G \equiv SC_{i_c},$$

$$C_*^{L,c}(\tilde{\partial M} \times [1, \infty), \tilde{\partial M} \times [1, \infty))^G \equiv SC_{i_{L,c}},$$

$$C_*^{L,0,c}(\tilde{\partial M} \times [1, \infty), \tilde{\partial M} \times [1, \infty))^G \equiv SC_{i_{L,0,c}}.$$

Temporally, we denote $C_*^c(\tilde{\partial M} \times [1, \infty), \tilde{\partial M} \times [1, \infty))^G$ (resp. $C_*^{L,c}(\tilde{\partial M} \times [1, \infty), \tilde{\partial M} \times [1, \infty))^G$, $C_*^{L,0,c}(\tilde{\partial M} \times [1, \infty), \tilde{\partial M} \times [1, \infty))^G$) as $C_*^c$ (resp. $C_*^{L,c}$, $C_*^{L,0,c}$) for short. By definition and a standard Eilenberg swindle argument, one can see that

$$K_*(C_*^{c,c}) \cong K_*(C_*^{c}(\tilde{\partial M} \times [1, \infty), \tilde{\partial M} \times [1, \infty))^G) \cong \{0\}.$$  

Thus we have the following isomorphism

$$\partial_* : K_{n+1}(C_*^c) \to K_n(C_*^{L,0,c}),$$

where $\partial_*$ is the connecting map in the following $K$-theory six exact sequence

$$K_0(C_*^{L,0,c}) \to K_*(C_*^{L,c}) \to K_0(C_*^c) \to K_1(C_*^c) \to K_1(C_*^{L,c}) \to K_1(C_*^{L,0,c}).$$

Proposition 4.4 of [27] showed that

$$K_n(C_*^{L,0,c}(\tilde{\partial M} \times [1, \infty))^G) \cong K_n(C_*^{L,0}(\tilde{M})^G),$$

$$K_n(C_*^{L,0,c}(\tilde{\partial M} \times [1, \infty))^G) \cong K_n(C_*^{L,0}(\tilde{M})^G).$$
Applying a five lemma argument, we have
\[ K_i(C^*_0(L,0,t(M,\partial M \times [1,\infty)))) \cong K_i(C^*_0(\tilde{M},\partial M \times [1,\infty))^{G,F}) \]

In summary, we obtain the following result

**Theorem 3.10.** With the notions as above, we have
\[ K_{i+1}(C^*_0(\tilde{M},\partial M \times [1,\infty]))^{G,F}) \cong K_i(C^*_0(\tilde{M},\partial M \times [1,\infty))^{G,F}) \]

4 Geometrically controlled Hilbert-Poincaré complex

In this section we introduce the geometrically controlled Hilbert-Poincaré complex. Generally, one can define the signature class for a geometrically controlled Hilbert-Poincaré complex (c.f. [9], [10]). A certain triangulation of a PL manifold \( X \) gives rise to a geometrically controlled Hilbert-Poincaré complex over \( X \). In this case, one can further define the \( K \)-homology class of the signature operator (c.f. [27]). At last, we recite the definition of the higher \( \rho \) invariant for a homotopy equivalence between two PL manifolds. The readers are referred to [9], [10] and [27] for more details.

**Definition 4.1 (Geometrically controlled module).** Let \( X \) be a proper metric space. A complex vector space \( V \) is geometrically controlled over \( X \) if it is provided with a basis \( B \subset V \) and a function \( c : B \to X \) with the following property: for every \( R > 0 \), there is an integer \( N < \infty \) such that if \( S \subset X \) has diameter less than \( R \) then \( c^{-1}S \) has cardinality less than \( N \). We call such \( V \) a geometrically controlled \( X \)-module.

Note that each geometrically controlled \( X \)-module \( V \) can be completed into a Hilbert space \( \overline{V} \). Let \( V^*_f = \text{Hom}(V;\mathbb{C}) \) be the vector space of finitely support linear functions on \( V \). The vector space \( V^*_f \) is identified with \( V \) by the inner product on \( V \).

We now introduce an example of the geometrically controlled module arose naturally from topology which will play a central role in this article.

**Definition 4.2 (Bounded geometry complex).** A simplicial complex \( M \) is of bounded geometry if there is a positive integer \( k \) such that each of the vertices lies in at most \( k \) different simplex.

Let \( M \) be a manifold. Take a triangulation of \( M \), one obtain a simplicial complex. If the complex is of bounded geometry, the \( L^2 \) completion of it provides a geometrically controlled module over \( M \).

**Definition 4.3 (Geometrically controlled map).** A linear map \( T : V \to W \) is geometrically controlled over \( X \) if

1. \( V \) and \( W \) are geometrically controlled,
2. the matrix coefficients of $T$ with respect to the given basis of $V$ and $W$ are uniformly bounded,

3. and there is a constant $K > 0$ such that the $(v, w)$-matrix coefficients are zero whenever $d(c(v), c(w)) > K$. The smallest such $K$ is called the propagation of $T$.

**Definition 4.4 (Geometrically controlled complex).** A chain complex

$$E_0 \xleftarrow{b_1} E_1 \xleftarrow{b_2} \cdots \xleftarrow{b_n} E_n$$

is called a geometrically controlled complex over $X$ if each $E_p$ is geometrically controlled over $X$ and each $b_p$ is a geometrically controlled linear map.

We now introduce the geometrically controlled chain homotopy.

**Definition 4.5 (Controlled chain homotopy).** Let $f_1, f_2 : (E, b) \to (E', b')$ be two geometrically controlled chain maps between two geometrically controlled complexes $(E, b)$ and $(E', b')$. We say $f_1$ and $f_2$ are geometrically controlled homotopic to each other if there exists a family of geometrically controlled linear maps $h : (E_*, b) \to (E'_*, b')$ such that

$$f_1 - f_2 = b'h + hb.$$  

The family of linear maps $h$ is called the geometrically controlled homotopy between $f_1$ and $f_2$.

**Definition 4.6 (Geometrically controlled Hilbert-Poincaré complex).** An $n$-dimensional Hilbert-Poincaré complex over $X$ is a complex of geometrically controlled $X$-modules

$$E_0 \xleftarrow{b_1} E_1 \xleftarrow{b_2} \cdots \xleftarrow{b_n} E_n$$

together with a family of geometrically controlled linear maps $T : E_p \to E_{n-p}$ such that

1. if $v \in E_p$, then $T^*v = (-1)^{p(n-p)}Tv$,
2. if $v \in E_p$, then $Tb^*(v) + (-1)^pbT(v) = 0$,
3. $T$ is geometrically controlled chain homotopy equivalence from the dual complex

$$E_n \xleftarrow{b_n} \cdots E_2 \xleftarrow{b_2} E_1 \xleftarrow{b_1} E_0$$

to the complex $(E, b)$.

**Definition 4.7 (Geometrically controlled homotopy equivalence).** Given two geometrically controlled $n$-dimensional Hilbert-Poincaré complexes $(E, b, T)$ and $(E', b', T')$. A geometrically controlled homotopy equivalence between them consists of two geometrically controlled chain maps $f : (E, b) \to (E', b')$ and $g : (E', b') \to (E, b)$, such that:
1. \(gf\) and \(fg\) are geometrically controlled homotopic to the identity,

2. \(fTf^*\) is geometrically controlled homotopic to \(T'\), where \(f^*\) is the adjoint of \(f\).

For all the notions above, one can define the \(G\)-equivariant version, when there is a proper and free \(G\)-action on \(X\) by isometries.

### 4.1 Signature class

In this subsection, we briefly recall the definition of the signature class of \(G\)-equivariant geometrically controlled Hilbert-Poincaré complex over \(X\), given by Higson and Roe in [9] and [10]. Let \((E, \partial, T)\) be an \(n\)-dimensional \(G\)-equivariant geometrically controlled Hilbert-Poincaré complex. We denote \(l\) to be the integer such that
\[
n = \begin{cases} 
2l & \text{if } n \text{ is even}, \\
2l + 1 & \text{if } n \text{ is odd}.
\end{cases}
\]
Define the bounded operator \(S : E \to E\) by
\[
S(v) = \sqrt{-1}^{(p-1)+l} T(v)
\]
for \(v \in E_p\). Direct computation shows that \(S = S^*\) and \(\partial S + S \partial = 0\). Moreover, we have \(\partial + \partial^* \pm S\) are self-adjointable invertible operators ([9]). In the following, we set \(B := \partial + \partial^*\).

**Definition 4.8** (Signature class). 1. Let \((E, \partial, T)\) be an odd dimensional \(G\)-equivariant geometrically controlled Hilbert-Poincaré complex over \(X\). It was shown in [9] that the following operator
\[
\frac{B + S}{B - S} : E_{ev} \to E_{ev}
\]
belongs to \((C^*(X)^G)^+\). The signature class of \((E, \partial, T)\) is defined to be \(K_1(C^*(X)^G)\) class represented by
\[
\frac{B + S}{B - S} : E_{ev} \to E_{ev}.
\]

2. Let \((E, \partial, T)\) be an even dimensional \(G\)-equivariant geometrically controlled Hilbert-Poincaré complex over \(X\). It was shown in [9] that \(P_+(B \pm S)\), the positive spectral projection of \(B \pm S\) can be approximated by finite propagation operators, and
\[
P_+(B + S) - P_+(B - S) \in C^*(X)^G,
\]
thus the formal difference \([P_+(B + S)] - [P_+(B - S)]\) determines a class in \(K_0(C^*(X)^G)\). The signature class of \((E, \partial, T)\) is defined to be the class in \(K_0(C^*(X)^G)\) determined by
\[
[P_+(B + S)] - [P_+(B - S)].
\]
Without loss of generality, we assume that both 
\[
\frac{B + S}{B - S} \quad \text{and} \quad P_+(B \pm S)
\]
are of finite propagation.

For a compact PL manifold \(X\) with fundamental group \(G\), the triangulation of \(X\) gives rise to a simplicial complex with bounded geometry, and thus to a geometrically controlled module. Considering the lifted triangulation for \(\tilde{X}\), the universal covering of space \(X\), one obtain a \(G\)-equivariant geometrically controlled Hilbert-Poincaré complex over \(\tilde{X}\). Thus we can define the signature class for this PL manifold as in Definition 4.8. We denote this class as \(\text{Ind}(X)\).

### 4.2 Signature class as homotopy equivalence invariant

In this subsection, we recall the proof of the fact that the signature class of compact PL manifold is invariant under homotopy equivalence.

Let \(f : N \to M\) be a homotopy equivalence between two PL manifolds, \(G\) be the fundamental group of \(M\) and \(N\). Then

\[
(E_{\tilde{M}} \oplus E_{\tilde{N}}, \begin{pmatrix} \partial_{\tilde{M}} & 0 \\ 0 & \partial_{\tilde{N}} \end{pmatrix}, \begin{pmatrix} T_{\tilde{M}} & 0 \\ 0 & -T_{\tilde{N}} \end{pmatrix})
\]

is a \(G\)-equivariant geometrically controlled Hilbert-Poincaré complex over \(\tilde{N}\). In the following we will denote the signature class of the complex \((4.1)\) as

\[
\text{Ind}(M \cup -N) \in K_\ast(C^\ast(M)G).
\]

Higson and Roe built an explicit homotopy path connecting \(\text{Ind}(M \cup -N)\) to the trivial element in \([9]\). We describe this homotopy path in details only for odd dimensional case. The even dimensional case is completely similar. Set

\[
B = \begin{pmatrix} \partial_{\tilde{M}} & 0 \\ 0 & \partial_{\tilde{N}} \end{pmatrix} + \begin{pmatrix} \partial^*_{\tilde{M}} & 0 \\ 0 & \partial^*_N \end{pmatrix}, S = \begin{pmatrix} S_{\tilde{M}} & 0 \\ 0 & -S_{\tilde{N}} \end{pmatrix}.
\]

The signature class of complex \((4.1)\) then equals

\[
V = \frac{B + S}{B - S}.
\]

Let \(g : M \to N\) be the corresponding homotopy equivalence inverse.

**Definition 4.9** ([9], Definition 4.4). Let \((E, b)\) be Hilbert modules. An operator homotopy of Hilbert-Poincaré complex structure is norm continuous family of adjointable operators \(T_s, (s \in [0, 1])\) such that each \((E, b, T_s)\) is Hilbert-Poincaré complex.

**Lemma 4.10** ([9], Lemma 4.6). If a duality operator \(T\) on a Hilbert-Poincaré complex is operator homotopic to \(-T\), then the signature of \((E, b, T)\) is trivial.
Let \( g \) be a homotopy inverse of \( f \).
From [9] and [27], we know that the following are all \( G \)-equivariant geometrically controlled Hilbert-Poincaré complexes over \( N \):

\[
\left( E_{\tilde{M}} \oplus E_{\tilde{N}}, \frac{\partial_{\tilde{M}}}{\partial_{\tilde{N}}} \right), T_1(s) = \begin{pmatrix} T_{\tilde{M}} & 0 \\ 0 & (s-1)T_{\tilde{N}} - sgT_{\tilde{M}}g^* \end{pmatrix}, \text{ } s \in [0, 1], \tag{4.2}
\]

\[
\left( E_{\tilde{M}} \oplus E_{\tilde{N}}, \frac{0}{\partial_{\tilde{N}}} \right), T_2(s) = \begin{pmatrix} \cos(s)T_{\tilde{M}} & \sin(s)T_{\tilde{M}}g^* \\ \sin(s)gT_{\tilde{M}} & -\cos(s)gT_{\tilde{M}}g^* \end{pmatrix}, \text{ } s \in [0, \frac{\pi}{2}]. \tag{4.3}
\]

Now we have constructed a path in \( C^*(\tilde{N})^G \)

\[
\begin{cases}
\frac{B+S(s)}{B-S(s)} & s \in [0, 1] \\
\frac{B+S((s-1)}{B-S(s-1)} & s \in [1, 1 + \frac{\pi}{2}]
\end{cases}
\]

connecting \( V \) to

\[
V' = \frac{B + \begin{pmatrix} 0 & S_{\tilde{M}}g^* \\ gS_{\tilde{M}} & 0 \end{pmatrix}}{B - \begin{pmatrix} 0 & S_{\tilde{M}}^*g^* \\ gS_{\tilde{M}}^* & 0 \end{pmatrix}}.
\]

Note that the following are still \( G \)-equivariant geometrically controlled Hilbert-Poincaré complexes:

\[
\left( E_{\tilde{M}} \oplus E_{\tilde{N}}, \frac{0}{\partial_{\tilde{N}}} \right), \text{ } \begin{pmatrix} 0 & e^{-is}S_{\tilde{M}}^* \\ e^{-is}gS_{\tilde{M}} & 0 \end{pmatrix}, \text{ } s \in [0, \pi]. \tag{4.4}
\]

Thus we can connect \( V' \) to the identity by the path

\[
\frac{B + \begin{pmatrix} 0 & S_{\tilde{M}}^* \\ gS_{\tilde{M}} & 0 \end{pmatrix}}{B - \begin{pmatrix} 0 & e^{-is}S_{\tilde{M}}^* \\ e^{-is}gS_{\tilde{M}}^* & 0 \end{pmatrix}}, \text{ } s \in [0, \pi].
\]

In a word, parameterizing the above path properly, we have a path connecting \( V \) to the identity in \( C^*(\tilde{N})^G \). In the following, we will denote this path by

\[
\frac{B_f + S_f}{B_f - S_f}(t), \text{ } t \in [0, 1]
\]

where

\[
\frac{B_f + S_f}{B_f - S_f}(0) = \frac{B + S}{B - S}, \text{ } \frac{B_f + S_f}{B_f - S_f}(1) = I.
\]

In even case, the path will be denoted as

\[
P_+(B_f + S_f) - P_+(B_f - S_f).
\]
4.3 Signature class as bordism invariant

In [9], Higson and Roe built an explicit homotopy path to show that the signature class is a cobordism invariant. In this subsection, we briefly recall the construction of this homotopy path. Our definition of the relative signature class, the relative $K$-homology class of the signature operator and the relative higher $\rho$ invariant have their roots in this construction. We start with the following definition.

**Definition 4.11** (Geometrically controlled Hilbert-Poincaré complex pair). An $n+1$ dimensional geometrically controlled Hilbert-Poincaré complex pair is a geometrically controlled complex

$$E_0 \xleftarrow{b_1} E_1 \xleftarrow{b_2} \cdots \xleftarrow{b_n} E_n$$

together with a family of geometrically controlled operators $T : E_p \to E_{n+1-p}$ and a family of geometrically controlled projections $P : E_p \to E_p$ such that

1. The orthogonal projection $P$ determines a subcomplex of $(E, b)$, that is $PbP = bP$.
2. The range of the operator $Tb^* + (-1)^pbT : E_p \to E_{n-p}$ is contained within the range of $P : E_{n-p} \to E_{n-p}$.
3. $T^* = (-1)^p(n + 1 - p)pT : E_p \to E_{n+1-p}$.
4. $P^\perp T$ is a geometrically controlled chain homotopy equivalence from the dual complex $(E, b^*)$ to $(P^\perp E, P^\perp b)$.

**Lemma 4.12** ([27], Lemma 5.4). Let $(E, b, T, P)$ be an $n+1$ dimensional geometrically controlled Hilbert-Poincaré complex pair. Then $T_0 = Tb^* + (-1)^pbT : E_p \to E_{n-p}$ satisfies the following conditions:

1. $T_0^* = (-1)^{(n-p)p}T_0 : E_p \to E_{n-p}$.
2. $T_0 = PT_0 = T_0P$.
3. $T_0b^* + (-1)^pbT_0 = 0 : PE_p \to PE_p$.
4. $T_0$ induces a geometrically controlled homotopy equivalence from $(PE, Pb^*)$ to $(PE, Pb)$.

The above lemma asserts that $(PE, Pb, T_0)$ is a geometrically controlled Hilbert-Poincaré complex. We call it the boundary of the geometrically controlled Hilbert-Poincaré complex pair $(E, b, T, P)$.

**Theorem 4.13** (Theorem 5.7. [27]). Let $(E, b, T, P)$ be an $n+1$ dimensional $\Gamma$-equivariant geometrically controlled Hilbert-Poincaré pair over $X$, then the signature class of $(PE, Pb, T_0)$ is trivial in $K_n(C^*(X)^\Gamma)$.
Denote
\[ \tilde{E}_p = E_p \oplus P^\perp E_{p+1}, \quad \tilde{b}_\lambda = \begin{pmatrix} b & 0 \\ \lambda P^\perp & P^\perp b \end{pmatrix}, \]
which consist the mapping cone complex of \( \lambda P^\perp : (E, b) \to (P^\perp E, P^\perp b) \). The following operator
\[ \tilde{T} = \begin{pmatrix} 0 & T P^\perp \\ (-1)^p P^\perp T & 0 \end{pmatrix} : \tilde{E}_p \to \tilde{E}_{n-p}. \]
is a Hilbert-Poincaré duality operator. The triple \((\tilde{E}, \tilde{b}, \tilde{T})\) is then a geometrically controlled Hilbert-Poincaré complex. Note that
\[ A : E_p \to E_p \oplus P^\perp E_{p+1} \]
\[ A(v) = v \oplus 0 \]
defines a geometrically controlled chain homotopy equivalence
\[ A : (PE, Pb, T_0) \to (\tilde{E}, \tilde{b}_{-1}, \tilde{T}). \]
Moreover, for \((\tilde{E}, \tilde{b}_0)\), \(\tilde{T}\) is operator homotopic to \(-\tilde{T}\) along the path
\[ \left( \begin{array}{cc} 0 & e^{is\pi TP^\perp} \\ (-1)^p e^{-is\pi P^\perp T} & 0 \end{array} \right), \quad s \in [0, 1]. \]
Thus, we have constructed a path connected the element representing the signature class of \((PE, Pb, T_0)\) to the trivial element. When \(n\) is odd, we denote this path by
\[ \frac{B_P + S_P}{B_P - S_P}, \]
where
\[ \frac{B_P + S_P}{B_P - S_P}(t) \]
equals
\[ \left( \begin{array}{cc} b + b^* & -(1 - 2t)P^\perp \\ -(1 - 2t)P^\perp & P^\perp b + P^\perp b^* \end{array} \right) + \left( \begin{array}{cc} 0 & SP^\perp \\ SP^\perp & 0 \end{array} \right) \]
\[ \left( \begin{array}{cc} b + b^* & -(1 - 2t)P^\perp \\ -(1 - 2t)P^\perp & P^\perp b + P^\perp b^* \end{array} \right) - \left( \begin{array}{cc} 0 & SP^\perp \\ SP^\perp & 0 \end{array} \right) \]
when \(t \in [0, \frac{1}{2}]\), and equals
\[ \left( \begin{array}{cc} b + b^* & 0 \\ 0 & P^\perp b + P^\perp b^* \end{array} \right) + \left( \begin{array}{cc} 0 & SP^\perp \\ SP^\perp & 0 \end{array} \right) \]
\[ \left( \begin{array}{cc} b + b^* & 0 \\ 0 & P^\perp b + P^\perp b^* \end{array} \right) - \left( \begin{array}{cc} 0 & SP^\perp \\ SP^\perp & 0 \end{array} \right) \]
equal to
\[ e^{i(2t-1)\pi \frac{SP^\perp}{SP^\perp}}, \]
when \(t \in [\frac{1}{2}, 1]\).
Similarly, in even case, the path will be denoted as
\[ P_+(B_P + S_P) - P_+(B_P - S_P). \]
### 4.4 Geometrically controlled Hilbert-Poincaré complex 2-ads

In this subsection we introduce the notion of geometrically controlled Hilbert-Poincaré complex 2-ads, which is necessary for us to prove the bordism invariant of the relative invariants we defined in Section 5 and 6.

Let $X$ be a topological space and $G$ be the fundamental group of $X$.

**Definition 4.14.** [Hilbert-Poincaré 2-ads] An $(n+2)$ dimensional $G$-equivariant $\tilde{X}$ controlled Hilbert-Poincaré complex 2-ads consists of a $G$-equivariant $\tilde{X}$ controlled complex $(E, b)$, a family of geometrically controlled maps $T : E_p \to E_{n+2-p}$ and a family of geometrically controlled projections $P_{\pm} E_p \to E_p$ such that

1. $P_{\pm} b P_{\pm} = b P_{\pm}$.
2. $P_\gamma = P_+ \vee P_-$, and $(E, b, T, P_\gamma) \tilde{X}$ is an $n$ dimensional controlled Hilbert-Poincaré pair, where $(P E, P b, T_0) \tilde{X}$ is its boundary.
3. $P_\wedge = P_+ \wedge P_-$, and $(P_\pm P_\gamma E, P_\pm P_\gamma b, P_\pm T_0 P_\pm, P_\wedge \tilde{X})$ are $n$ dimensional controlled Hilbert-Poincaré pairs. The boundary complexes of them are $\tilde{X}$ controlled homotopy equivalence to each other.
4. $P_{\perp} TP_{\perp} : (P_{\perp} E, P_{\perp} b) \to (P_{\perp} E, P_{\perp} b)$ is a controlled homotopy equivalence of complex.

**Remark 4.15.** Note that in general, there is no $P_\pm b P_{\pm} = b P_{\pm}$, however, there is $P_{\perp} b P_{\perp} = b P_{\perp}$.

**Lemma 4.16.** The triple $(\tilde{E}, b_{\lambda, \mu, s}, T_{s}^')_{+,\tilde{X}}$, where

\[
\tilde{E}_{+,\tilde{X},p} = E_{\tilde{X},p} \oplus P_+ E_{\tilde{X},p+1} \oplus P_+ E_{\tilde{X},p+1} \oplus P_+ E_{\tilde{X},p+2}
\]

\[
b_{\lambda, \mu, s, +, \tilde{X}} = \begin{pmatrix}
b_{\tilde{X}} & 0 & 0 & 0 \\
\mu P_{\perp} & -P_{\perp} b_{\tilde{X}} & 0 & 0 \\
0 & -\lambda P_{\perp} & \mu P_{\perp} & P_{\perp} b_{\tilde{X}} \\
0 & 0 & -\lambda P_{\perp} & \mu P_{\perp}
\end{pmatrix}
\]

\[
T_{s, +, \tilde{X}}' = \begin{pmatrix}
0 & 0 & 0 & e^{-i\pi s} P_{\perp} T \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
e^{i\pi s} T P_{\perp}
\end{pmatrix}
\]

are $G$-equivariant Hilbert-Poincaré complexes controlled over $\tilde{X}$ as long as
1. $\lambda, \mu \in [-1, 0], s \in [0, 1]$.
2. $\lambda s = 0$.
3. $\lambda = 0$ if $\mu = 0$.

**Proof.** By direct computation, one can see that $(\tilde{E}, b_{\lambda, \mu})_{+, \tilde{X}}$ is a controlled complex. Thus it remains only to show that $T'_{s+}, \tilde{X}$ are controlled Hilbert-Poincaré dualities. The $\lambda = 0$ case is trivial. We will focus on $\lambda \neq 0$ case only.

(1) First one can see by direct computation that $T''_{0+, \tilde{X}} = (-1)^{(n-p)p} T_{0+, \tilde{X}}$.

(2) One also need to show that $(-1)^p b_{\lambda, \mu, +, \tilde{X}} T''_{0+, \tilde{X}} + T'_{0+, \tilde{X}} b^*_{\lambda, \mu, +, \tilde{X}} = 0$.

However, this can be proved by the following direct computations. First note

$$
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\lambda P_\perp & 0 & 0 & 0 \\
0 & -\lambda P_\perp & 0 & 0
\end{pmatrix}
T'_{0+, \tilde{X}} =
\begin{pmatrix}
0 & 0 & 0 & \lambda P_\perp T \\
0 & 0 & -\lambda (-1)^p P_\perp T P_\perp & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\lambda P_\perp & 0 & 0 & 0 \\
0 & -\lambda P_\perp & 0 & 0
\end{pmatrix}
T'_{0+, \tilde{X}} =
\begin{pmatrix}
0 & 0 & 0 & -(-1)^p \lambda P_\perp T \\
0 & 0 & \lambda P_\perp T P_\perp & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

In the meanwhile, there are

$$
\begin{pmatrix}
\mu P_\perp & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \mu P_\perp & 0 \\
0 & 0 & 0 & \mu P_\perp
\end{pmatrix}
T'_{0+, \tilde{X}} =
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mu P_\perp T P_\perp \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

and

$$
\begin{pmatrix}
\mu P_\perp & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \mu P_\perp & 0 \\
0 & 0 & 0 & \mu P_\perp
\end{pmatrix}
T'_{0+, \tilde{X}} =
\begin{pmatrix}
0 & 0 & 0 & -(-1)^p \mu P_\perp T P_\perp \\
0 & 0 & \mu P_\perp T P_\perp & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

$$
\begin{pmatrix}
\mu P_\perp & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \mu P_\perp & 0 \\
0 & 0 & 0 & \mu P_\perp
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
T'_{0+, \tilde{X}}.
$$
At last, we have

\[
\begin{pmatrix}
0 & -P_\mp b_{\tilde{X}} & 0 & 0 \\
0 & 0 & -P_\mp b_{\tilde{X}} & 0 \\
0 & 0 & 0 & P_\mp b_{\tilde{X}} \\
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & P_\mp T \\
-1)^p P_\mp T P_\pm & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0 & 0 & 0 & P_\mp T b_{\tilde{X}}^p \\
0 & 0 & 0 & 0 \\
b_{\tilde{X}} T P_\pm & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & P_\mp T b_{\tilde{X}}^p \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0 & 0 & 0 & P_\mp T b_{\tilde{X}}^p \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & P_\mp T b_{\tilde{X}}^p \\
0 & 0 & 0 & 0 \\
-1)^p P_\mp T b_{\tilde{X}}^p & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
= -1)^p \begin{pmatrix}
0 & 0 & 0 & P_\mp b_{\tilde{X}} \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 0 & P_\mp T b_{\tilde{X}}^p \\
0 & 0 & 0 & 0 \\
-1)^p P_\mp b_{\tilde{X}} T P_\pm & 0 & 0 & 0 \\
\end{pmatrix}
\]

In summary, we have that

\[
(-1)^p b_{\lambda,\mu,+,\tilde{X}}^T T_{0,+}^{T,0,+,\tilde{X}} + T_{0,+}^{T,0,+,\tilde{X}} b_{\lambda,\mu,+,\tilde{X}}^* = 0.
\]

We claim that \( T_{0,+}^{T,0,+,\tilde{X}} \) is a homotopy equivalence. In fact, we decompose \( \tilde{E}_{+,\tilde{X}} \) as \( E_1 \oplus E_2 \), where

\[
E_{1,p} = E_{X,p} \oplus P_+^p E_{X,p+1}, \quad E_{2,p} = \oplus_p P_+^p E_{X,p+1} \oplus P_\mp E_{\tilde{X},p+2}.
\]

Set

\[
b_1 = \begin{pmatrix} b_{\tilde{X}} & 0 \\
\mu P_\mp b_{\tilde{X}} & -P_\mp b_{\tilde{X}} \\
\end{pmatrix}, \quad b_2 = \begin{pmatrix} -P_\mp b_{\tilde{X}} & 0 \\
\mu P_\mp b_{\tilde{X}} & P_\mp b_{\tilde{X}} \\
\end{pmatrix}.
\]

Set

\[
T_1 = \begin{pmatrix} 0 & P_\mp T P_\pm \\
-1)^p P_\mp T & 0 \\
\end{pmatrix}, \\
T_2 = \begin{pmatrix} 0 & (-1)^p P_\mp T P_\pm \\
P_\mp T P_\pm & 0 \\
\end{pmatrix}.
\]

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It is direct to see that we have

\[ 0 \longrightarrow (E_1, b_1) \longrightarrow (\tilde{E}, b_{\lambda, \mu})_{+ \tilde{X}} \longrightarrow (E_2, b_2) \longrightarrow 0. \]

By basic topology theory, we know that \( T_1 : (E_1, b_1) \rightarrow (E_2^*, b_2^*) \) and \( T_2 : (E_2, b_2) \rightarrow (E_1^*, b_1^*) \) are both chain homotopy equivalences, so be \( T_{0, \pm} \tilde{X} \) by Lemma 4.2 of [9]. Combining with some other routine computations, we obtain the lemma.

\[ \square \]

In the same reason, we have

**Lemma 4.17.** The triple \((\tilde{E}, b_{\lambda, \mu}, T'_s)_{- \tilde{X}}\), where

\[
\begin{align*}
\tilde{E}_{- \tilde{X}, p} &= E_{\tilde{X}, p} \oplus P_{\tilde{X}, p+1} \oplus P_{\tilde{X}, p+2} \\
b_{\lambda, \mu, - \tilde{X}} &= \begin{pmatrix}
b_{\tilde{X}} & 0 & 0 & 0 \\
\mu P_{\tilde{X}} & -P_{\tilde{X}} b_{\tilde{X}} & 0 & 0 \\
\lambda P_{\tilde{X}} & 0 & -P_{\tilde{X}} b_{\tilde{X}} & 0 \\
0 & \lambda P_{\tilde{X}} & -\mu P_{\tilde{X}} & P_{\tilde{X}} b_{\tilde{X}}
\end{pmatrix}
\end{align*}
\]

\[
T'_{s, - \tilde{X}} = \begin{pmatrix}
0 & 0 & 0 & e^{-i\pi s} P_{\tilde{X}} T \\
0 & 0 & 0 & 0 \\
(1)^{p+1} e^{-i\pi s} P_{\tilde{X}} T & 0 & 0 & 0 \\
e^{i\pi s T} P_{\tilde{X}} & 0 & 0 & 0
\end{pmatrix}
\]

are \( G \)-equivariant Hilbert-Poincaré complexes controlled over \( \tilde{X} \) as long as

1. \( \lambda, \mu \in [-1, 0], s \in [0, 1] \).
2. \( \lambda s = 0 \).
3. \( \lambda = 0 \) if \( \mu = 0 \).

**Proof.** This lemma follows the computation of Lemma 4.16 and a unitary equivalence induced by

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

\[ \square \]

Let \((E, b, T)_{\tilde{X}}\) be an odd dimensional Hilbert-Poincaré complex 2-ads. We can define an element in \([1, 0] \times C^* (\tilde{X})^G \) by considering the signature classes of
families of odd dimensional Hilbert-Poincaré complexes

\[
\begin{cases}
(\tilde{E}, b_{-(1-2t)}, T_0^t)_{+, \tilde{X}} & t \in [0, \frac{1}{2}]
\end{cases}
\]

\[
\begin{cases}
(\tilde{E}, b_{0, T_0^t})_{+, \tilde{X}} & t \in \left[\frac{1}{2}, 1\right]
\end{cases}
\]

and

\[
\begin{cases}
(\tilde{E}, b_{-(1-2t)}, T_0^t)_{-, \tilde{X}} & t \in [0, \frac{1}{2}]
\end{cases}
\]

\[
\begin{cases}
(\tilde{E}, b_{0, T_0^t})_{-, \tilde{X}} & t \in \left[\frac{1}{2}, 1\right]
\end{cases}
\]

We will denote these two elements by

\[
B_{\mu, P_+} + S_{\mu, P_+} \in C_0([1, 0), C^*(\tilde{X})^G)
\]

and

\[
\frac{B_{\mu, P_+} + S_{\mu, P_+}}{B_{\mu, P_+} - S_{\mu, P_+}} \in C_0([1, 0), C^*(\tilde{X})^G)
\]

respectively. When \((E, b, T)_{\tilde{X}}\) is even dimensional complex 2-ads, the elements will be denoted by

\[
P_+(B_{\mu, P_+} + S_{\mu, P_+}) - P_+(B_{\mu, P_+} - S_{\mu, P_+}) \in C_0([1, 0), C^*(\tilde{X})^G)
\]

and

\[
P_+(B_{\mu, P_+} + S_{\mu, P_+}) - P_+(B_{\mu, P_+} - S_{\mu, P_+}) \in C_0([1, 0), C^*(\tilde{X})^G)
\]

similarly.

**Lemma 4.18.** The controlled complex \((P_+ \tilde{E}, (P_+ b)_\lambda, (P_+ T_0)_{s})_{\tilde{X}}\), where

\[
P_+ \tilde{E}_p = (P_+ E)_p \oplus P_\lambda^\perp (P_+ E)_{p+1}
\]

\[
(P_+ b)_\lambda = \left(\begin{array}{c}
(P_+ b) \\
\lambda P_\lambda^\perp - P_\lambda^\perp (P_+ b)
\end{array}\right), \lambda \in [-1, 0],
\]

\[
(P_+ T_0 P_+)_s = \left(\begin{array}{c}
(P_+ T_0 P_+)_s \\
(-1)^p e^{is\pi} (P_+ T_0 P_+)_s P_\lambda^\perp
\end{array}\right), s \in [0, 1],
\]

is \(G\)-equivariantly homotopy equivalent to \((\tilde{E}, b_{\lambda, -1}, T_0^s)_{+, \tilde{X}}\) under the the controlled chain map

\[
A : P_+ \tilde{E}_p \to \tilde{E}_{+, \tilde{X}, p}
\]

\[
(v, w) \to (v, 0, w, 0)
\]

**Proof.** Again, it is sufficient to prove the lemma for \((P_+ \tilde{E}, (P_+ b)_\lambda, (P_+ T_0)_s)_{\tilde{X}}\). Obviously, \(A\) is a controlled chain map. By observation, one have the following
commutative diagram

\[
\begin{array}{ccc}
0 & \xrightarrow{I} & (P_+ \tilde{E}, (P_+ b)_{\lambda}) \overset{I}{\rightarrow} (\tilde{E}, b_{\lambda,-1})_{+, \tilde{X}} \\
& & \downarrow \Lambda \\
0 & \xrightarrow{A} & (P_+ \tilde{E}, (P_+ b)_{\lambda}) \overset{A}{\rightarrow} (\tilde{E}, b_{\lambda,-1})_{+, \tilde{X}} \\
& & \downarrow T_2 \\
& & 0
\end{array}
\]

where

\[
\tilde{E}_{+, p} = E_{X, p} \oplus P_+^1 E_{X, p+1} \oplus P_-^1 E_{X, p+1} \oplus P_\vee^1 E_{X, p+2}
\]

and

\[
b_{\lambda,-1,+, \tilde{X}} = \begin{pmatrix}
\lambda P_\vee^1 & 0 & 0 & 0 \\
-1P_+^1 & -P_+^1 b_{\tilde{X}} & 0 & 0 \\
0 & \lambda P_\vee^1 & -P_+^1 & 0 \\
0 & 0 & -1P_+^1 & P_\vee^1 b_{\tilde{X}}
\end{pmatrix}.
\]

Then by Lemma 4.2 of [9], one can see that \(A : (P_+ \tilde{E}, (P_+ b)_{\lambda}) \rightarrow (\tilde{E}, b_{\lambda,-1})_{+, \tilde{X}}\) is a chain homotopy. It remains to show that \(A(P_+ T_0 P_+)'s A^*\) and \(T'_s, +, \tilde{X}\) are geometrically controlled homotopy equivalent to each other. However, this can be seen by simply verifying

\[
A(P_+ T_0 P_+)'s A^* - T'_s, +, \tilde{X} = h_{p+1} b_{\lambda,-1} + (-1)^p b_{\lambda,-1} h_p,
\]

where the operator \(h_p\) on

\[
\tilde{E} = E_{X, p} \oplus P_+^1 E_{X, p+1} \oplus P_-^1 E_{X, p+1} \oplus P_\vee^1 E_{X, p+2}
\]

is

\[
\begin{pmatrix}
0 & 0 & P_+ T P_-^1 & 0 \\
0 & 0 & 0 & 0 \\
(-1)^p P_-^1 T P_+ & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

However, by Remark 4.15 we have that

\[
b P_+ T P_-^1 + (-1)^p P_+ T P_-^1 b^* = P_+ b T P_-^1 + (-1)^p P_+ b^* P_-^1 = P_+ T_0 P_+ P_-^1.
\]

\[\square\]

**Corollary 4.19.** The controlled complex \((P_+ \tilde{E}, (P_+ b)_{\lambda})_X, (P_+ T_0 P_+)_{0}X\) is \(G\)-equivariantly homotopy equivalent to \((\tilde{E}, b_{\lambda,-1}, T'_s)_{+, \tilde{X}}\), with the homotopy factor through the \(G\)-equivariantly homotopy equivalence between \((P_+ \tilde{E}, (P_+ b)_{\lambda})_X, (P_+ T_0 P_+)_{0}X\) and \((\tilde{E}, b_{\lambda,-1}, T'_s)_{+, \tilde{X}}\).

In the same reason, we have
Lemma 4.20. The controlled complex \((P_-, \tilde{E}, (P_-, b)_\lambda, (P_-, T_0 P_-)_{\gamma_0})_X\), where

\[
\begin{align*}
P_-, \tilde{E}_p &= (P_-, E)_p \oplus P_+^\perp (P_-, E)_{p+1} \\
(P_- b)_\lambda &= \begin{pmatrix}
(P_- b) \\
\lambda P_+^\perp \\
-P_+^\perp (P_- b)
\end{pmatrix}, \lambda \in [-1, 0], \\
(P_-, T_0 P_-)'_s &= \begin{pmatrix}
0 \\
\lambda e^{-is\pi} P_+^\perp (P_- T_0 P_-) \\
e^{is\pi} (P_- T_0 P_-) P_+^\perp
\end{pmatrix}, s \in [0, 1], \\
\lambda s &= 0.
\end{align*}
\]

is \(G\)-equivariantly homotopy equivalent to \((\tilde{E}, b_\lambda, -1, T'_s)_{-\tilde{X}}\), under the controlled chain map

\[
A : P_-, \tilde{E}_p \rightarrow \tilde{E}_-, \tilde{X}_p \\
(v, w) \rightarrow (v, 0, w, 0)
\]

Corollary 4.21. The controlled complex \((P_-, \tilde{E}, (P_- b)_\lambda, (P_-, T_0 P_-)_{\gamma_0})_X\) is \(G\)-equivariantly homotopy equivalent to \((\tilde{E}, b_\lambda, -1, T'_s)_{-\tilde{X}}\), with the homotopy factor through the \(G\)-equivariant homotopy equivalence of \((P_-, \tilde{E}, (P_- b)_\lambda, (P_-, T_0 P_-)_{\gamma_0})_X\) and \((\tilde{E}, b_\lambda, -1, T'_s)_{-\tilde{X}}\).

5 Relative invariant of PL manifold with boundary

In this section, we define the relative signature class and the relative \(K\)-homology class of the signature operator for PL manifolds with boundary. These invariants provide group homomorphisms from the relative \(L\) group and the relative Normal group to the \(K\)-theory of the relative Roe algebra and the relative Localization algebra respectively. In this section, \((M, \partial M)\) is a manifold with boundary, with \(\pi_1(M) = G\) and \(\pi_1(\partial M) = \Gamma\).

5.1 Relative signature class

We equip \((M, \partial M)\) with a triangulation. The triangulation in turn produces a simplicial complex of bounded geometry, thus gives rise to a \(G\)-equivariant \(\tilde{M}\) controlled Hilbert-Poincaré complex pair \((E, b, T, P)_{\tilde{X}}\) and a \(\Gamma\)-equivariant \(\tilde{M}\) controlled Hilbert-Poincaré complex \((E, b, T)_{\tilde{\partial M}}\). Let \(i : \partial M \rightarrow M\) be the embedding. We are now ready to define the relative signature class for \((M, \partial M)\) in \(K_n(C^*(\tilde{M}, \partial M)^{G, \Gamma})\):

\textbf{Definition 5.1.} Let \([v]\) be the generator class of \(K_1(C(S^1))\)

1. When \(n\) is odd, then

\[
[(P_+ (B_{\partial M} + S_{\partial M}), P_+ (B_P + S_P))] - [(P_+ (B_{\partial M} - S_{\partial M}), P_+ (B_P - S_P))]
\]
defines an element in $K_{n-1}(C_i)$, thus

$$([P_+(B_{\partial M} + S_{\partial M}), P_+(B_P + S_P)] - [P_+(B_{M,\partial M} - S_{\partial M}), P_+(B_{M,\partial M} - S_{M,\partial M})]) \otimes [v]$$

defines an element in $K_n(C^*(\tilde{M}, \tilde{\partial M})^{G,\Gamma})$.

2. When $n$ is even, then

$$\left[\frac{B_{\partial M} + S_{\partial M}}{B_{\partial M} - S_{\partial M}}, \frac{B_P + S_P}{B_P - S_P}\right]$$

defines an element in $K_{n-1}(C_i)$, thus

$$\left[\frac{B_{M,\partial M} + S_{M,\partial M}}{B_{M,\partial M} - S_{M,\partial M}}, \frac{B_P + S_P}{B_P - S_P}\right] \otimes [v]$$

defines an element in $K_n(C^*(\tilde{M}, \tilde{\partial M})^{G,\Gamma})$.

We will call the element defined above the relative signature class of $(M, \partial M)$, and denote it by $\text{relInd}(M, \partial M)$. For convenience, we will denote the representative elements as

$$([P_+(B_{M,\partial M} + S_{M,\partial M})] - [P_+(B_{M,\partial M} - S_{M,\partial M})]) \otimes [v]$$

when $M$ is odd dimensional and

$$\left[\frac{B_{M,\partial M} + S_{M,\partial M}}{B_{M,\partial M} - S_{M,\partial M}}\right] \otimes [v]$$

when $M$ is even dimensional. It is obviously to see that the relative index of $(M, \partial M)$ does not depend on the choice of the triangulation.

### 5.2 Relative signature class as homotopy equivalence invariant

In this subsection, we prove that the relative signature class is a homotopy equivalence invariant. Let $f : (M, \partial M) \to (N, \partial N)$ be a homotopy equivalence of PL manifolds with boundary. We will denote the homotopy equivalence from $\partial M$ to $\partial N$ obtained by restricting $f$ to the boundary as $\partial f$.

**Theorem 5.2.** Let $f : (N, \partial N) \to (M, \partial M)$ be a homotopy equivalence of PL manifolds with boundary. Set $G = \pi_1(M)$ and $\Gamma = \pi_1(\partial M)$. We then have

$$\text{relInd}(M, \partial M) = \text{relInd}(N, \partial N) \in K_n(C^*(\tilde{M}, \tilde{\partial M})^{G,\Gamma})$$

**Proof.** We only prove the even dimensional case in detail. The argument for odd case is totally the same. By the definition and the argument behind Lemma 4.10, $\partial f$ defines a homotopy path

$$\frac{B_{\partial f} + S_{\partial f}}{B_{\partial f} - S_{\partial f}} : [0, 1] \to C^*(\tilde{\partial M})^\Gamma$$
where
\[
\frac{B_{\partial f} + S_{\partial f}(0)}{B_{\partial f} - S_{\partial f}} = \frac{B_{\partial M} + S_{\partial M} B_{\partial N} - S_{\partial N}}{B_{\partial M} - S_{\partial M} B_{\partial N} - S_{\partial N}}
\]
\[
\frac{B_{\partial f} + S_{\partial f}(1)}{B_{\partial f} - S_{\partial f}} = I.
\]

In the meantime, one can construct a continuous path
\[
\frac{B_{f,P} + S_{f,P}}{B_{f,P} - S_{f,P}} : \{0, 1\} \to C_0([0, 1), C^*(\tilde M)^G)
\]
with
\[
\frac{B_{f,P} + S_{f,P}(0)}{B_{f,P} - S_{f,P}} = \frac{B_P + S_P B_P - S_P}{B_P - S_P B_P + S_P}
\]
\[
\frac{B_{f,P} + S_{f,P}(1)}{B_{f,P} - S_{f,P}} = I.
\]

These together give the homotopy path between
\[
\left( \frac{B_{M,\partial M} + S_{M,\partial M}}{B_{M,\partial M} - S_{M,\partial M}} \right) \left( \frac{B_{N,\partial N} - S_{N,\partial N}}{B_{N,\partial N} + S_{N,\partial N}} \right)
\]
and the identity, which proves the theorem.

In the following, for \( f : (N, \partial N) \to (M, \partial M) \), when \((M, \partial M)\) is an even dimensional manifold with boundary, we denote the path of invertible elements we constructed in the proof of Theorem 5.2 connecting
\[
\frac{B_{M,\partial M} + S_{M,\partial M}}{B_{M,\partial M} - S_{M,\partial M}} \left( \frac{B_{N,\partial N} - S_{N,\partial N}}{B_{N,\partial N} + S_{N,\partial N}} \right)
\]
to identity as
\[
\frac{B_{f,\partial f} + S_{f,\partial f}}{B_{f,\partial f} - S_{f,\partial f}} : \{0, 1\} \to C_1.
\]

While \((M, \partial M)\) is an odd dimensional manifold with boundary, we denote the corresponding path of difference of projections as
\[
P_+(B_{f,\partial f} + S_{f,\partial f}) - P_+(B_{f,\partial f} - S_{f,\partial f}) : \{0, 1\} \to C_1.
\]

### 5.3 Relative signature class as bordism invariant

In this subsection, we prove the following theorem which guarantees that the relative signature class is a bordism invariant.

**Theorem 5.3.** Let \((M, \partial_\pm M)\) be an \(n\) dimensional PL manifold 2-ads, with \(\pi_1(M) = G\), \(\pi_1(\partial_\pm M) = \Gamma_\pm\) and \(\pi_1(\partial_\pm M) = \Gamma\). Let \(i_+: (\partial_+ M, \partial_+ M) \to (M, \partial_- M)\) be the embedding of the positive part of the boundary. Then we have
\[
i_+^*(\text{relInd}(\partial_+ M, \partial_+ M)) = 0 \in K_{n-1}(\tilde{\tilde{M}}, \tilde{\tilde{M}})^{G, \Gamma_-})
\]
Proof. Let $i_- : (\partial_- M, \partial \partial_- M) \to (M, \partial_- M)$ be the embedding of the negative part of the boundary. By definition we have that

$$i_-^*\left(\text{relInd}(\partial_- M, \partial \partial_- M)\right) = 0 \in K_{n-1}(\tilde{M}, \tilde{\partial_- M})^G, $$

where $K_n$ is the $n$th relative $K$-group. Our strategy of the proof is to show that

$$i_+^*\left(\text{relInd}(\partial_+ M, \partial \partial_+ M)\right) = i_-^*\left(\text{relInd}(\partial_- M, \partial \partial_- M)^{-1}\right).$$

We will go through the detail for $n = 2k + 1$ case only. By Lemma 4.18 and Corollary 4.19, we can see that

$$i_+^*(\text{relInd}(\partial_+ M, \partial \partial_+ M)) = [(B_{\partial \partial_- M} + S_{\partial \partial_- M}, B_{\mu, P_-} + S_{\mu, P_-})], \mu \in [-1, 0].$$

In the same time, by Lemma 4.20 and Corollary 4.21, we have

$$i_-^*(\text{relInd}(\partial_- M, \partial \partial_- M)^{-1}) = [(B_{\partial \partial_- M} + S_{\partial \partial_- M}, B_{\mu, P_-} + S_{\mu, P_-})], \mu \in [-1, 0].$$

Obviously,

$$\left(\frac{B_{\partial \partial_- M} + S_{\partial \partial_- M}}{B_{\partial \partial_- M} - S_{\partial \partial_- M}}, \frac{B_{\mu, P_-} + S_{\mu, P_-}}{B_{\mu, P_-} - S_{\mu, P_-}}\right) \sim_h \left(\frac{B_{\partial \partial_- M} + S_{\partial \partial_- M}}{B_{\partial \partial_- M} - S_{\partial \partial_- M}}, f_+\right),$$

where by $\sim_h$ we mean homotopy equivalence of elements, and

$$f_{\pm}(t) = \begin{cases} \frac{B_{\partial \partial_- M} + S_{\partial \partial_- M}}{B_{\partial \partial_- M} - S_{\partial \partial_- M}}, & t \in [0, \frac{1}{2}] \\ \frac{B_{\partial \partial_- M} + S_{\partial \partial_- M}}{B_{\partial \partial_- M} - S_{\partial \partial_- M}}, & t \in [\frac{1}{2}, 1]. \end{cases}$$

However, set

$$U = \left(\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right),$$

$$U^*(\frac{B_{\partial \partial_- M} + S_{\partial \partial_- M}}{B_{\partial \partial_- M} - S_{\partial \partial_- M}}, f_+) = \left(\frac{B_{\partial \partial_- M} + S_{\partial \partial_- M}}{B_{\partial \partial_- M} - S_{\partial \partial_- M}}, f_-\right).$$

This finishes our proof.

\[\square\]

5.4 Relative signature class and $L$ group

The relative signature class induces a group homomorphism from the relative $L$-theory to $K$-theory the relative Roe algebra. The homomorphism is defined as follows.
Definition 5.4. Let $\theta = (M, \partial M, N, \partial N, f, \phi)$ be an element in $L_\pi^\pi(X, \partial X)$. We first fix triangulation $\text{Tri}_{(N, \partial N)}$ for $(N, \partial N)$ and triangulation $\text{Tri}_{(M, \partial M)}$ for $(M, \partial M)$. One can paste $M$ and $-N$ along $\partial + N$ and $\partial + M$ by the homotopy equivalence $f$. Although the resulted topological spaces $M \sqcup -N$ and $\partial (M \sqcup -N) := \partial -M \sqcup -\partial N$ and are not PL manifold with boundary and PL manifold any more, they still form Hilbert-Poincaré complex pair and Hilbert-Poincaré complex respectively. Thus the triangulations $\text{Tri}_{(N, \partial N)}$ and $\text{Tri}_{(M, \partial M)}$ still generate a controlled Hilbert-Poincaré complex $(E, b, T, \widetilde{\partial X})$ and a pair $(E, b, T, P, \widetilde{X})$. When $n$ is even, we define the relative index of $\theta$ to be

$$\left[ \frac{B_{M \sqcup -N, \partial M \sqcup -N} + S_{M \sqcup -N, \partial M \sqcup -N}}{B_{M \sqcup -N, \partial M \sqcup -N} - S_{M \sqcup -N, \partial M \sqcup -N}} \cdot [v] \right]$$

where

$$\frac{B_{M \sqcup -N, \partial M \sqcup -N} + S_{M \sqcup -N, \partial M \sqcup -N}}{B_{M \sqcup -N, \partial M \sqcup -N} - S_{M \sqcup -N, \partial M \sqcup -N}}$$

represents

$$B_{\partial M \sqcup -N} - S_{\partial M \sqcup -N} : B_P + S_P, B_P - S_P.$$ 

Similarly, when $n$ is odd, the relative index is

$$\left[ P_+(B_{M \sqcup -N, \partial M \sqcup -N} + S_{M \sqcup -N, \partial M \sqcup -N}) - \frac{P_+(B_{M \sqcup -N, \partial M \sqcup -N} - S_{M \sqcup -N, \partial M \sqcup -N})}{[v]} \right]$$

where

$$P_+(B_{M \sqcup -N, \partial M \sqcup -N} \pm S_{M \sqcup -N, \partial M \sqcup -N})$$

represents

$$(P_+(B_{\partial M \sqcup -N} \pm S_{\partial M \sqcup -N}), P_+(B_P \pm S_P)).$$

Theorem 5.2 and Theorem 5.3 guarantee that this is a well defined map

$$\text{relInd} : L_\pi^\pi(X, \partial X) \to K_n(C^*(\widetilde{X}, \partial X)^{G, T}).$$

The fact that the relative signature class is well defined, i.e. independent from the choice of the triangulation, follows from the next subsection.

5.5 Relative $K$-homology class of the signature operator

In this subsection let us consider the definition of the relative $K$-homology class of the signature operator for PL manifold with boundary $(M, \partial M)$. We give the definition in detail only for the case where $M$ is of even dimensional. The odd case is totally parallel.

Equip $(M, \partial M)$ with a triangulation $\text{Sub}(M, \partial M)$. Recall that Section 4.2 of [27] described a procedure of refinement of $\text{Sub}(M, \partial M)$, $\text{Sub}^n(M, \partial M)$, $n \in \mathbb{N}_+$, such that $\text{Sub}^n(M, \partial M)$ has uniformly bounded geometry.

Recall that every locally finite simplicial complex carries a natural metric, whose restriction to each $n$-simplex is the Riemannian metric obtained by identifying the $n$-simplex with the standard $n$-simplex in the Euclidean space $\mathbb{R}^n$. Such metric is called a simplicial metric.
Equip $\cup_n(M, \partial M)$ with the simplicial metric, we denote the new metric space as $\cup_n(M^n, \partial M^n)$. Note that the metric of $(M^n, \partial M^n)$ increases by $n$. Thus the triangulation $\cup_n \text{Sub}^n(M, \partial M)$ of $\cup_n(M^n, \partial M^n)$ defines a geometrically controlled Hilbert-Poincaré complex

$$\oplus (E^n, b^n, T^n)_{\partial M^n}$$

and a controlled Hilbert-Poincaré pair

$$\oplus (E^n, b^n, T^n, P^n)_{\widetilde{M}^n}.$$ 

The relative signature class

$$[\frac{B_{\cup_n M^n, \cup_n \partial M^n} + S_{\cup_n M^n, \cup_n \partial M^n}}{B_{\cup_n M^n, \cup_n \partial M^n} - S_{\cup_n M^n, \cup_n \partial M^n}}] \otimes [v]$$

defines an element in

$$K_n(\star((\cup_n \widetilde{M}^n, \cup_n \widetilde{\partial M}^n)^{G, \Gamma}).$$

Note that $\oplus (E^n, b^n, T^n)_{\partial M^n}$ and $\oplus (E^n, b^n, T^n, P^n)_{\widetilde{M}^n}$ are geometrically controlled homotopy equivalent to

$$\oplus (E^{n+1}, b^{n+1}, T^{n+1})_{\partial M^{n+1}}$$

and $\oplus (E^{n+1}, b^{n+1}, T^{n+1}, P^{n+1})_{\widetilde{M}^{n+1}}$ respectively. Thus, one can construct paths

$$\frac{(B_{M^n, \partial M^n} + S_{M^n, \partial M^n})}{B_{M^n, \partial M^n} - S_{M^n, \partial M^n}}(s), s \in [n, n+1]$$

with

$$\frac{B_{M^n, \partial M^n} + S_{M^n, \partial M^n}}{B_{M^n, \partial M^n} - S_{M^n, \partial M^n}}(n) = \frac{B_{M^n, \partial M^n} + S_{M^n, \partial M^n}}{B_{M^n, \partial M^n} - S_{M^n, \partial M^n}}(n+1)$$

and the propagation is uniformly controlled for $n$ and $s$.

Now scaling $(M^n, \partial M^n)$ back to $(M, \partial M)$, one can get a path $[0, \infty)$ to $C_i$, where $i$ is the embedding of the boundary. This path will be denoted as

$$\frac{(B_{M, \partial M} + S_{M, \partial M})_L}{B_{M, \partial M} - S_{M, \partial M}} : [0, \infty) \rightarrow C_i.$$ 

By construction, one can see that

$$\lim_{s \to \infty} \text{propagation of } \frac{(B_{M, \partial M} + S_{M, \partial M})_L(s)}{B_{M, \partial M} - S_{M, \partial M}} = 0.$$ 

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Thus
\[ \left[ \frac{B_{M,\partial M} + S_{M,\partial M}}{B_{M,\partial M} - S_{M,\partial M}} \right] L \otimes [v] \]
defines an element in \( K_n(C^*_L(\widetilde{M}, \widetilde{\partial M})^{G,\Gamma}) \). We call it the relative \( K \)-homology class of the signature operator for \((M, \partial M)\), which will be denoted as \( \text{relInd}_L(M, \partial M) \).

When \( n \) is odd, the relative \( K \)-homology class of signature operator will be denoted as
\[ ([P_+(B_{M,\partial M} + S_{M,\partial M})L] - [P_+(B_{M,\partial M} - S_{M,\partial M})L]) \otimes [v]. \]

One can easily see that if \( f : (N, \partial N) \to (M, \partial M) \) is an infinitesimally controlled homotopy equivalence, we have
\[ \text{relInd}_L(N, \partial N) = \text{relInd}_L(M, \partial M) \in K_n(C^*_L(\widetilde{M}, \widetilde{\partial M})^{G,\Gamma}). \]

This can be seen directly by the construction of the relative \( K \)-homology class of signature operator and a similar argument to the proof of Theorem 5.2. Thus the definition of the relative \( K \)-homology class of the signature operator is independent from the choice of the triangulation.

Furthermore, the following theorem guarantees that the relative \( K \)-homology class is a bordism invariant.

**Theorem 5.5.** Let \((M, \partial \pm M)\) be a manifold with corner with \( \pi_1(M) = G, \pi_1(\partial \pm M) = \Gamma_{\pm} \) and \( \pi_1(\partial \partial \pm M) = \Gamma \). Let \( i : (\partial_+ M, \partial \partial_+ M) \to (M, \partial_- M) \) be the embedding. Then we have
\[ i_*(\text{relInd}_L(M, \partial M)) = 0 \in K_n(C^*_L(\widetilde{M}, \widetilde{\partial_- M})^{G,\Gamma_-}). \]

**Proof.** It could be proved by applying Theorem 5.3 to the complex
\[ \oplus(E^n, b^n, T^n, P^n)_{(\widetilde{M}_n, \widetilde{\partial M}_n)}. \]

Note that all maps involved in the proof of Theorem 5.3 most of which appear in Lemma 4.16, 4.17, 4.18, 4.20 and Corollary 4.19, 4.21 are all geometrically controlled.

5.6 Relative \( K \)-homology class of the signature operator and the relative Normal group

In this subsection, we show that the relative \( K \)-homology class of the signature operator actually gives rise to a well defined map from the relative Normal group \( N_n(X, \partial X) \) to \( K_n(C^*_L(\widetilde{X}, \widetilde{\partial X})^{G,\Gamma}) \). Generally, let
\[ \theta = (N, \partial_\pm N, M, \partial_\pm M, f, \phi), \]
be an element in \( N_n(X, \partial X) \). Recall \( f : \partial_+ N \rightarrow \partial_+ M \) is an infinitesimally controlled homotopy equivalence, and \( f \) induces an analytically controlled homotopy equivalence

\[
\cup_n f^{(n)} : \cup_n (\partial_+ N^n, \partial \partial_+ N^n) \rightarrow \cup_n (\partial_+ M^n, \partial \partial_+ M^n).
\]

In the following, we define \( \partial_{\pm} f \) to be the map obtained from restricting \( f \) to \( \partial_{\pm} N \). We then consider \((M^n \sqcup -N^n, \partial_- M^n \sqcup \partial_- N^n)\) obtained by gluing \(-N^n\) and \(M^n\) by \(\partial_{\pm} f^{(n)}\) along \(\partial_{\pm} N^n\) and \(\partial_{\pm} M^n\) by the controlled homotopy equivalence \(f^{(n)} : \partial_+ N^n \rightarrow \partial_+ M^n\). Although \((M^n \sqcup -N^n, \partial_- M^n \sqcup \partial_- N^n)\)

is not a manifold with boundary, we can still make sense of its signature class. Then by the same procedure we have depicted above, one can define the relative \( K \)-homology class of the signature operator for \((M \sqcup -N, \partial_- M \sqcup \partial_- N)\).

The relative \( K \)-homology class of the signature operator \( \text{relInd}_L(\theta) \) of \( \theta \) is then defined to be the relative \( K \)-homology class of the signature operator of \((M \sqcup -N, \partial_- M \sqcup -\partial_- N)\). Denote the representative class of \( \text{relInd}_L(\theta) \) we obtained as above as

\[
\left[ [\left( \frac{B_{M \sqcup -N, \partial_- M \sqcup -\partial_- N} + S_{M \sqcup -N, \partial_- M \sqcup -\partial_- N}}{B_{M \sqcup -N, \partial_- M \sqcup -\partial_- N} - S_{M \sqcup -N, \partial_- M \sqcup -\partial_- N}} \right)_L] \otimes [v]. \right.
\]

By definition, it is direct to see that \( \text{relInd}_L(\theta) \) is trivial if \( f \) is an infinitesimally controlled homotopy equivalence. By Theorem [5.2] and the discussion behind it, one can show that the relative \( K \)-homology class of the signature operator induces a well defined group homomorphism from \( N_n(X, \partial X) \) to \( K_n(C^*_L(X, \partial X)^{\mathbb{G}, \mathbb{T}}) \).

## 6 Relative mapping surgery to analysis

In this section, we define the relative higher \( \rho \) invariant for elements of the relative structure group \( S_n(X, \partial X) \). We also show that the relative higher \( \rho \) invariant induces a well defined group homomorphism from \( S_n(X, \partial X) \) to \( K_n(C^*_L, 0(\tilde{X}, \partial \tilde{X})^{\mathbb{G}, \mathbb{T}}) \). For this purpose, we introduce a group homomorphism \( \text{rel} \rho \) from \( L_n(\pi(X), \pi_1(\partial X), X) \) to \( K_n(C^*_L, 0(\tilde{X}, \partial \tilde{X})^{\mathbb{G}, \mathbb{T}}) \). We finally address the problem of the relative mapping surgery to analysis in subsection [6.4].

### 6.1 Relative higher \( \rho \) invariant

We first define the relative higher \( \rho \) invariant of a homotopy equivalence \( f : (N, \partial N) \rightarrow (M, \partial M) \). When \((M, \partial M)\) is an even dimensional manifold with boundary, we have defined the relative \( K \)-homology class of the signature operator of \((M \sqcup -N, \partial M \sqcup -\partial N)\), denoted as

\[
\left( \frac{B_{M \sqcup -N, \partial M \sqcup -\partial N} + S_{M \sqcup -N, \partial M \sqcup -\partial N}}{B_{M \sqcup -N, \partial M \sqcup -\partial N} - S_{M \sqcup -N, \partial M \sqcup -\partial N}} \right)_L \otimes [v].
\]
Lemma 6.1. Let \( \rho \) shows that the relative higher invariant of \( \rho \) in \( L(0) \).

Concatenate the path
\[
\frac{B_{f, \partial f} + S_{f, \partial f}}{B_{f, \partial f} - S_{f, \partial f}}
\]
with
\[
\frac{B_{M \cup \partial M \cup \partial N} - S_{M \cup \partial M \cup \partial N} \otimes \rho}{B_{M \cup \partial M \cup \partial N} + S_{M \cup \partial M \cup \partial N} \otimes \rho} L(0)
\]
one obtain an invertible element in \( C_{L, 0} \). We denote this element simply as \( \tilde{\rho}(f, \partial f) \). Then \( [\tilde{\rho}(f, \partial f)] \otimes [v] \) defines an element in \( K_0(C^*_L(M, \partial M)_{G, \Gamma}) \), which we call the relative higher \( \rho \) invariant of \( f \), and denote it as \( \text{rel}(f, \partial f) \). Certainly \( \text{rel}(f, \partial f) \) does not depend on the choice of the triangulation. Similarly, one can define the relative higher \( \rho \) invariant of \( f : (N, \partial N) \to (M, \partial M) \) when \( (M, \partial M) \) is an odd dimensional manifold with boundary. The following lemma shows that the relative higher \( \rho \) invariant is a bordism invariant.

**Lemma 6.1.** Let \( f : (N, \partial N) \to (N', \partial N') \) be a homotopy equivalence of PL manifold 2-ads, with \( \pi_1(N') = G, \pi_1(\partial N') = \Gamma \) and \( \pi_1(\partial \partial N') = \Gamma \). Let \( i_\pm : (\partial \pm N', \partial \partial \pm N') \to (N', \partial \partial N') \) be the embedding. Let \( \partial \pm f \) be \( f \) restricting to \( \partial \pm N' \), and \( \partial \pm f \) be \( f \) restricting to \( \partial \partial \pm N' \). Then \( i_+^*(\text{rel}(\partial_+ f, \partial \partial_+ f)) = I(0) \in K_n(C^*_L(N', \partial \partial N')_{G, \Gamma}) \).

**Proof.** We prove only for the case that \( N \) is of even dimension. Similarly to Theorem 5.3 we need only to prove
\[
i_+^*(\text{rel}(\partial_+ f, \partial \partial_+ f)) = i_-^*(\text{rel}(\partial_- f, \partial \partial_- f)).
\]
Recall that we have constructed a path connecting
\[
\frac{B_{\partial_+ N' \cup \partial N, \partial \partial_+ N' \cup \partial \partial N} - S_{\partial_+ N' \cup \partial N, \partial \partial_+ N' \cup \partial \partial N}}{B_{\partial_+ N' \cup \partial N, \partial \partial_+ N' \cup \partial \partial N} \otimes \rho}
\]
to
\[
\frac{B_{\partial_- N' \cup \partial N, \partial \partial_- N' \cup \partial \partial N} - S_{\partial_- N' \cup \partial N, \partial \partial_- N' \cup \partial \partial N}}{B_{\partial_- N' \cup \partial N, \partial \partial_- N' \cup \partial \partial N} \otimes \rho} L(0)
\]
We denote the path simply as \( \frac{B + S}{B - S} L : [0, 1] \to C_{L, 0} \). Note that for any \( s \in [0, 1] \), \( \frac{B + S}{B - S} L(s) \in C_{L, 0} \). Applying the construction in Theorem 5.2, the homotopy
equivalence $f$ simultaneously produces paths connecting $(\frac{B_n+S}{B_n+S})_L(s)(0)$ to identity for any $s \in [0,1]$. Thus we obtained a path realizing the homotopy equivalence

$$i^*_+(\text{rel}\rho(\partial_+ f, \partial_+ f)) \sim_k i^*_-(\text{rel}\rho(\partial_- f, \partial_- f)).$$

This finish the proof. \qed

Let us prove that the relative higher $\rho$ invariant for homotopy equivalence between manifolds with boundary is an obstruction to the homotopy equivalence being infinitesimally controlled.

**Lemma 6.2.** If $f : (N, \partial N) \to (M, \partial M)$ is an infinitesimally controlled homotopy equivalence, then $\text{rel}\rho(f)$ is trivial in $K_*(C_{L,0}(\tilde{M}, \partial \tilde{M})^{G,\Gamma})$.

**Proof.** We prove only the even case. As in Section 5.5, we consider $\bigcup_n (N^n, \partial N^n)$ and $\bigcup_n (M^n, \partial M^n)$. Since $f : (N, \partial N) \to (M, \partial M)$ is an infinitesimally controlled homotopy equivalence, we have

$$\bigcup_n f^{(n)} : \bigcup_n (N^n, \partial N^n) \to \bigcup_n (M^n, \partial M^n)$$

is still an analytically controlled homotopy equivalence over $\bigcup_n (M^n, \partial M^n)$.

Thus one can define

$$\tilde{\rho}(\bigcup_n f^{(n)}, \partial \bigcup_n f^{(n)}) \in C_{L,n}^*,$$

where $i^n : \partial \bigcup_n M^n \to \bigcup_n M^n$ is the corresponding embedding. Recall that we have assumed $\tilde{\rho}(\bigcup_n f^{(n)}, \partial \bigcup_n f^{(n)})$ to be of finite propagation. Scale the metric on $(M^n, \partial M^n)$ back to the original metric on $(M, \partial M)$. Then $\tilde{\rho}(f^{(n)}, \partial f^{(n)})$ produces a series of elements in $C_{L,0}^*$ with propagation as small as possible as $n$ tends to infinity. With a little abuse of notation, we still denote them as $\tilde{\rho}(f^{(n)}, \partial f^{(n)})$. It is obvious to see that for all $n$,

$$[\tilde{\rho}(f^{(n)}, \partial f^{(n)})] \otimes [v] = \text{rel}\rho(f, \partial f).$$

The above argument shows that $\text{rel}\rho(f, \partial f)$ actually lies in

$$K_1^*(C_{L,0, i; \partial M \to M}) \otimes K_1(C(S^1)).$$

Thus, $[\tilde{\rho}(f^{(n)}, \partial f^{(n)})] \otimes [v]$ is trivial. \qed

### 6.2 Relative higher $\rho$ invariant and relative structure group

Generalizing the definition of the higher $\rho$ invariant for homotopy equivalence of manifold with boundary, one can define the relative higher $\rho$ map from $S_n(X, \partial X)$ to $K_n(C_{L,0}^*(\tilde{X}, \partial \tilde{X})^{G,\Gamma})$. Let

$$\theta = (N, \partial \pm N, M, \partial \pm M, f, \phi),$$

be an element of $S_n(X, \partial X)$. We consider even case in details only. Recall that we have defined the relative $K$-homology class of the signature operator

$$\text{relInd}_L(\theta),$$
which is represented by
\[ \left[ \frac{B_{M \cup N, \partial} \cup - N + S_{M \cup N, \partial} \cup - N}{B_{M \cup N, \partial} \cup - N - S_{M \cup N, \partial} \cup - N} \right]_L \otimes [v]. \]

Since \( f \) is a homotopy equivalence, similarly to the discussion in 4.1, \( (B_{M \cup N, \partial} \cup - N + S_{M \cup N, \partial} \cup - N) \) can be connected to the identity through a path of invertible elements. Concatenating this path to \( (B_{M \cup N, \partial} \cup - N - S_{M \cup N, \partial} \cup - N) \), one can obtain an element in \( C_{i, L, 0} \), where \( i \) is the embedding \( \partial X \to X \).

We denote this element as \( \rho(f, \partial X) \). Then
\[ [\text{rel} \rho(\theta)] \triangleq [\rho(f, \partial X)] \otimes [v] \]
defines an element in \( K_0(C_{L, 0}^*(\widetilde{X}, \widetilde{(\partial X)}^{G, \Gamma})) \). The following theorem is an immediate consequence of Lemma 6.1 and Lemma 6.2.

**Theorem 6.3.** If \( \theta_1 \) and \( \theta_2 \) are two equivalent objects for the definition of \( S_n(X, \partial X) \), then
\[ \text{rel} \rho(\theta_1) = \text{rel} \rho(\theta_2) \in K_n(C_{L, 0}^*(\widetilde{X}, \widetilde{(\partial X)}^{G, \Gamma})). \]

That is, \( \text{rel} \rho \) is a well defined map from \( S_n(X, \partial X) \) to \( K_n(C_{L, 0}^*(\widetilde{X}, \widetilde{(\partial X)}^{G, \Gamma})) \). We call this map the relative higher \( \rho \) map.

### 6.3 Group homomorphism \( \text{rel} \hat{\rho} \)

We need to show that the relative higher \( \rho \) map is a group homomorphism. For this purpose, we introduce a group homomorphism
\[ \text{rel}(\hat{\rho}) : L_n(\pi_1(X), \pi_1(\partial X), X) \to K_{n-1}(C_{L, 0}^*(\widetilde{X}, \widetilde{(\partial X)}^{G, \Gamma})). \]

Equip \((X \times [1, \infty), \partial X \times [1, \infty))\) with the product metric. By using the standard subdivision of Section 4.2 of [27], there exists a triangulation \( \text{Tri}(X \times [1, \infty), \partial X \times [1, \infty)) \) of \((X \times [1, \infty), \partial X \times [1, \infty))\) such that

1. \( \text{Tri}(X \times [1, \infty), \partial X \times [1, \infty)) \) has uniformly bounded geometry;
2. the sizes of simplexes in \( \text{Tri}(X \times [1, \infty), \partial X \times [1, \infty)) \) uniformly go to zero, as we approach to infinity along the cylindrical direction.
**Definition 6.4.** Equip $X \times [1, \infty)$ with the triangulation $\text{Tri}(X \times [1, \infty), \partial X \times [1, \infty))$ from above. Define the simplicial metric cone of $(X, \partial X)$, denoted by $(CX, \partial CX)$, to be the manifold with boundary $(X \times [1, \infty), \partial X \times [1, \infty))$ equipped with the simplicial metric determined by $\text{Tri}(X \times [1, \infty), \partial X \times [1, \infty))$.

From now on, $(X \times [1, \infty), \partial X \times [1, \infty))$ stands for the space $(X \times [1, \infty), \partial X \times [1, \infty))$ with the product metric. In the following, we set $\tau$ to be the natural map

$$\tau : (CX, \partial CX) \to (X \times [1, \infty), \partial X \times [1, \infty)).$$

Let $\theta = \{M, \partial_k M, \phi, N, \partial_k N, \psi, f ; k = 1, 2, 3,\}$ be an element in the group $L_n(\pi_1(X), \pi_1(\partial X), X)$. Consider the manifold 2-ads $CM = M \cup \partial_2 M \times [1, \infty)$, where $\partial_1 CM = \partial_1 M \cup \partial \partial_1 M \times [1, \infty)$ and $\partial CM = \partial_2 M \cup \partial \partial_2 M \times [1, \infty)$. Similarly, consider $CN = N \cup \partial_2 N \times [1, \infty)$. Furthermore, consider the pull back triangulations on $CM$ and $CN$ of $\text{Tri}(X \times [1, \infty), \partial X \times [1, \infty))$. Equip $CM$ and $CN$ with the corresponding simplicial metric. Since $\partial f : \partial \partial_1 N \to \partial \partial_1 C$ is an infinitesimally controlled homotopy equivalence, $\partial CM = \partial_1 M \cup \partial \partial_1 M \times [1, \infty)$ induces an analytically controlled homotopy equivalence from $\partial CM$ to $\partial CM$. Let $CZ$ be $CM \cup_{\partial CM} CN$. Then the relative signature class of $CZ$, $\text{relInd}(CZ, \partial CZ)$ lies in $K_n(C^*(\widetilde{CX}, \partial \widetilde{CX})^{G, \Gamma})$. Thus we have

$$\tau(\text{relInd}(CZ, \partial CZ)) \in K_n(C^*_c(\widetilde{X} \times [1, \infty), \partial \widetilde{X} \times [1, \infty))^{G, \Gamma})$$

$$\cong K_{n-1}(C^*_L,\partial_c(\widetilde{X} \times [1, \infty), \partial \widetilde{X} \times [1, \infty))^{G, \Gamma})$$

$$\cong K_{n-1}(C^*_L,\partial(\widetilde{X}, \partial \widetilde{X})^{G, \Gamma}).$$

We call $\tau(\text{relInd}(CZ, \partial CZ))$ the relative $\hat{\rho}$ class of $\theta$, denoted as $\text{rel}\hat{\rho}(\theta)$. It is direct to see from the above discussion that $\text{rel}\hat{\rho}$ is well defined and induces a group homomorphism form $L_n(\pi_1(X), \pi_1(\partial X), X)$ to $K_{n-1}(C^*_L,\partial(\widetilde{X}, \partial \widetilde{X})^{G, \Gamma})$.

Recall that we have the following natural isomorphism

$$c_* : S_n(X, \partial X) \to L_{n+1}(\pi_1(X), \pi_1(\partial X), X)$$

by taking the product with the unit interval $\theta \to \theta \times I$. It follows that $\text{rel}\hat{\rho}$ also induces a group homomorphism from $S_n(X, \partial X)$ to $K_{n-1}(C^*_L,\partial(\widetilde{X}, \partial \widetilde{X})^{G, \Gamma})$.

We intend to show that $\text{rel}\hat{\rho}$ equals to $k_n \text{rel}\rho$, where $k_n = 1$ if $n$ is odd and $k_n = 2$ if $n$ is even. For this purpose, we need to establish a product formula for the relative higher $\rho$ invariant.

Given $\theta = (M, \partial_k M, \phi, N, \partial_k N, \psi, f) \in S_n(X, \partial X)$. Let $\theta \times \mathbb{R} \in S_{n+1}(X \times \mathbb{R}, \partial X \times \mathbb{R})$ be the product of $\theta$ and $\mathbb{R}$. The relative higher $\rho$ invariant of $\theta \times \mathbb{R}$, $\text{rel}\rho(\theta \times \mathbb{R})$, belongs to $K_{n+1}(C^*_L,\partial(\widetilde{X} \times \mathbb{R}, \partial \widetilde{X} \times \mathbb{R})^{G, \Gamma})$. Recall that there is a natural homomorphism

$$\alpha : C^*_L,\partial(\widetilde{X}, \partial \widetilde{X})^{G, \Gamma} \otimes C^*_L,\partial(\mathbb{R}, \partial \mathbb{R})^{G, \Gamma} \to C^*_L,\partial(\widetilde{X} \times \mathbb{R}, \partial \widetilde{X} \times \mathbb{R})^{G, \Gamma},$$

which induces a $K$-theory isomorphism.
Theorem 6.5. With the notations above, we have
\[ k_n\alpha_*(\relrho(\theta) \otimes \Ind_L(\mathbb{R})) = \relrho(\theta \times \mathbb{R}). \]
where $\Ind_L(\mathbb{R})$ is the $K$-homology class of the signature operator on $\mathbb{R}$.

Proof. The proof is elementary and exactly the same with the proof of Theorem 6.8 of [27] (Appendix D of [27]). We thus omit the details for the sake of conciseness.

We further introduce some notations. Let $\mathcal{A}$ be $C^*_L(\partial X, \partial X)^{G,F}$. We define
\[
\mathcal{A}_+ = \bigcup_{n \in \mathbb{N}} C^*_L(\partial X \times [-n, \infty), \partial X \times [-n, \infty); \partial X \times \mathbb{R})^{G,F},
\]
\[
\mathcal{A}_- = \bigcup_{n \in \mathbb{N}} C^*_L(\partial X \times (-\infty, n], \partial X \times (-\infty, n]; \partial X \times \mathbb{R})^{G,F},
\]
\[
\mathcal{A}_\cap = \bigcup_{n \in \mathbb{N}} C^*_L(\partial X \times [-n, n]; \partial X \times \mathbb{R})^{G,F}.
\]
It is clear that $\mathcal{A}_\pm$ and $\mathcal{A}_\cap$ are closed two-sided ideals of $\mathcal{A}$. Moreover, we have $\mathcal{A}_+ + \mathcal{A}_- = \mathcal{A}$ and $\mathcal{A}_+ \cap \mathcal{A}_- = \mathcal{A}_\cap$. Thus, we have the following Mayor-Vietoris sequence
\[
\begin{array}{c}
K_0(\mathcal{A}_\cap) \xrightarrow{\partial_{MV}} K_0(\mathcal{A}_+) \oplus K_0(\mathcal{A}_-) \xrightarrow{\partial_{MV}} K_0(\mathcal{A}) \ .
\end{array}
\]
Similarly, consider the $C^*$-algebra $\mathcal{B} = C^*_L(\mathbb{R})$ and its closed two-sided ideals
\[
\mathcal{B}_+ = \bigcup_{n \in \mathbb{N}} C^*_L([n, \infty); \mathbb{R})
\]
\[
\mathcal{B}_- = \bigcup_{n \in \mathbb{N}} C^*_L((-\infty, n]; \mathbb{R})
\]
\[
\mathcal{B}_\cap = \bigcup_{n \in \mathbb{N}} C^*_L([n, n]; \mathbb{R}) = \mathcal{B}_+ \cap \mathcal{B}_-.
\]
The above $C^*$-algebras give rise to the following Mayor-Vietoris sequence
\[
\begin{array}{c}
K_0(\mathcal{B}_\cap) \xrightarrow{\partial_{MV}} K_0(\mathcal{B}_+) \oplus K_0(\mathcal{B}_-) \xrightarrow{\partial_{MV}} K_0(\mathcal{B}) \ .
\end{array}
\]
Note that the homomorphism $\alpha : C^*_L(\partial X, \partial X)^{G,F} \otimes \mathcal{B} \to \mathcal{A}$ restricts to the homomorphisms
\[
\alpha : C^*_L(\partial X, \partial X)^{G,F} \otimes \mathcal{B}_\pm \to \mathcal{A}_\pm \quad \text{and} \quad \alpha : C^*_L(\partial X, \partial X)^{G,F} \otimes \mathcal{B}_\cap \to \mathcal{A}_\cap
\]
such that the following diagram commutes
\[
\begin{array}{c}
K_n(C^*_L(\partial X, \partial X)^{G,F}) \otimes K_1(\mathcal{B}) \xrightarrow{\cong} K_{n+1}(C^*_L(\partial X, \partial X)^{G,F}) \quad
\\
\downarrow \otimes \partial_{MV} \quad \quad \downarrow \partial_{MV}
\end{array}
\]
\[
K_n(C^*_L(\partial X, \partial X)^{G,F}) \otimes K_0(\mathcal{B}_\cap) \xrightarrow{\cong} K_n(\mathcal{A}_\cap) = K_n(C^*_L(\partial X, \partial X)^{G,F})
\]
Theorem 6.6. The following diagram commutes

\[
\begin{array}{ccc}
L_{n+1}(\pi_1X,\pi_1\partial X, X) & \overset{\text{rel}^p}{\longrightarrow} & K_{n+1}(C^*_c(\overline{X} \times [1, \infty), \overline{\partial X} \times [1, \infty))^{G,\Gamma}) \\
\downarrow_{\varepsilon_*} & & \downarrow_{\partial_*}
\end{array}
\]

\[
S_n(X, \partial X) \overset{k_n \cdot \text{rel}^p}{\longrightarrow} K_n(C^*_L(\overline{X} \times [1, \infty), \overline{\partial X} \times [1, \infty))^{G,\Gamma}) \cong K_n(C^*_L(\overline{X}, \overline{\partial X})^{G,\Gamma})
\]

where the connecting map

\[
\partial_* : K_{n+1}(C^*_c(\overline{X} \times [1, \infty), \overline{\partial X} \times [1, \infty))^{G,\Gamma}) \to K_n(C^*_L(\overline{X} \times [1, \infty), \overline{\partial X} \times [1, \infty))^{G,\Gamma})
\]

is the isomorphism we mentioned in Subsection 3.4.

Proof. Recall that a standard way to construct the \(\partial_*\) is to lift a projection (resp. invertible) in \(C^*_c(\overline{X} \times [1, \infty), \overline{\partial X} \times [1, \infty))^{G,\Gamma}\) to an element in \(C^*_L(\overline{X} \times [1, \infty), \overline{\partial X} \times [1, \infty))^{G,\Gamma}\). For \(\theta \in S_n(X, \partial X)\), recall \(c_\theta(\theta) = \theta \times [0,1]\). Consider \(\text{rel}^p(\theta \times [0,1]) \in C^*_c(\overline{X} \times [1, \infty), \overline{\partial X} \times [1, \infty))^{G,\Gamma}\). There is an explicit lifting of \(\text{rel}^p(\theta \times [0,1])\), \(a_\theta \times [0,1] \in C^*_L(\overline{X} \times [1, \infty), \overline{\partial X} \times [1, \infty))^{G,\Gamma}\), defined as follows. Set

\[
a_\theta \times [0,1](n) = \chi_n \text{rel}^p(\theta) \chi_n,
\]

where \(\chi_n\) is the characteristic function on \((\overline{X} \times [n, \infty), \overline{\partial X} \times [n, \infty))\). We define

\[
a_\theta \times [0,1](t) = (n+1-t)a_\theta \times [0,1](n) + (t-n)a_\theta \times [0,1](n+1)
\]

for all \(n \leq t \leq n+1\). It is clear that \(a_\theta \times [0,1]\) lies in \(C^*_L(\overline{X} \times [1, \infty), \overline{\partial X} \times [1, \infty))^{G,\Gamma}\) and is a lifting of \(\text{rel}^p(\theta \times [0,1])\).

On the other hand, set

\[
a_\theta \times \mathbb{R}(n) = \chi_n \text{rel}^p(\theta \times \mathbb{R}) \chi_n
\]

and

\[
a_\theta \times \mathbb{R}(t) = (n+1-t)a_\theta \times \mathbb{R}(n) + (t-n)a_\theta \times \mathbb{R}(n+1)
\]

for all \(t \in [n, n+1]\). Then \(a_\theta \times \mathbb{R}\) is a lifting of \(\text{rel}^p(\theta \times \mathbb{R})\) in \(A_+ \oplus A_-\) for the connecting map

\[
\partial_{MV} : K_{n+1}(A_+ + A_-) \to K_n(A_+) \cong K_n(C^*_L(\overline{X}, \overline{\partial X})^{G,\Gamma}).
\]

Now one can compute \(\partial_* (\text{rel}^p(\theta \times [0,1]))\) and \(\partial_{MV} (\text{rel}^p(\theta \times \mathbb{R}))\) by a standard formula. By direct comparison, it turns out that

\[
\partial_* (\text{rel}^p(\theta \times [0,1])) = \partial_{MV} (\text{rel}^p(\theta \times \mathbb{R})).
\]

Moreover, we have

\[
\partial_{MV} (\text{rel}^p(\theta \times \mathbb{R})) = \partial_{MV} (k_n \text{rel}^p(\theta) \otimes \text{Ind}_L(\mathbb{R})) = k_n \text{rel}^p(\theta) \otimes \partial_{MV} (\text{Ind}_L(\mathbb{R})) = k_n \text{rel}^p(\theta).
\]
**Corollary 6.7.** $\text{rel} \rho$ is a well defined group homomorphism from $S_n(X, \partial X)$ to $K_n(C^*_L,0)(\widetilde{X}, \widetilde{\partial X})^{G, \Gamma}$.

### 6.4 Commutativity

Combining the above discussions, we have the following main result of this article.

**Theorem 6.8.** The following diagram commutes

\[
\begin{array}{ccc}
N_{n+1}(X, \partial X) & \xrightarrow{\text{rel} \text{Ind}_L} & K_{n+1}(C^*_L(\widetilde{X}, \widetilde{\partial X})^{G, \Gamma}) \\
\downarrow & & \downarrow \\
L_{n+1}(\pi_1 X, \pi_1 \partial X) & \xrightarrow{\text{rel} \text{Ind}_L} & K_{n+1}(C^*(\widetilde{X}, \widetilde{\partial X})^{G, \Gamma}) \\
\downarrow & & \downarrow \\
S_n(X, \partial X) & \xrightarrow{k_n \text{rel} \rho} & K_n(C^*_{L,0}(\widetilde{X}, \widetilde{\partial X})^{G, \Gamma}) \\
\downarrow & & \downarrow \\
N_n(X, \partial X) & \xrightarrow{\text{rel} \text{Ind}_L} & K_n(C^*_L(\widetilde{X}, \widetilde{\partial X})^{G, \Gamma}).
\end{array}
\]

**Proof.** The commutativity of the upper square and the lower square follows immediately from the definition. In the meanwhile, the commutativity of the middle square is an immediate consequence of Theorem 6.6. \hfill \Box

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