Electromagnetic Nucleon Properties and Quark Sea Polarization

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Abstract: In this paper we present the derivation as well as the numerical results for all electromagnetic form factors of the nucleon within the chiral quark soliton model in the semiclassical quantization scheme. The model is based on a semibosonized Nambu – Jona-Lasinio lagrangean where the boson fields are treated as classical ones. Other observables, namely the nucleon mean squared radii, the magnetic moments and the nucleon–Δ splitting are calculated as well. The calculations have been done taking into account the quark sea polarization effects. The final results, including rotational $1/N_c$ corrections, are compared with the existing experimental data and they are found to be in a good agreement for the constituent quark mass of 400 – 420 MeV.

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1 Introduction

In the last years one of the most challenging problems in elementary particle physics seems to be the solution of QCD in the low energy region. The main difficulties are due to the non-perturbative effects caused by the growing effective coupling constant of the fundamental theory in the low energy limit. This prevents one from using the well-known main tool of theoretical physics — the perturbation theory. Because of this, the most intriguing features of QCD — confinement and chiral symmetry breaking — still remain conceptual and practical problems.

The above mentioned obstacles have initiated an increasing interest among the physicists in non-perturbative methods and effective low-energy models of hadrons. The effective models are expected to mimic the behaviour of QCD at energies below $\sim 1 \text{ GeV}$ (confinement and/or chiral symmetry breaking) and to reproduce experimental data in this region. In principle, these models could be related to QCD by integrating out gluonic fields and reparameterizing fermionic degrees of freedom.

The simplest purely fermionic Lorentz invariant model with spontaneous chiral symmetry breaking is the Nambu–Jona-Lasinio (NJL) model \[^1\]. It contains chirally invariant local four-fermion interaction terms. The NJL Lagrangian in its simplest SU(2) form has the following structure:

$$L_{NJL} = \bar{\Psi} \left[ i \slashed{D} - m_0 - G \pi \left( \sigma + i \vec{\tau} \cdot \gamma_5 \right) \right] \Psi + \frac{G^2}{2} \left[ (\bar{\Psi} \Psi)^2 + (\bar{\Psi} i \vec{\tau} \gamma_5 \Psi)^2 \right] ,$$  

where $G$ is the coupling constant, $m_0$ is the current quark mass and $\vec{\tau}$ are the Pauli matrices in the isospin space. We consider the up and down quarks to be degenerated in mass.

The NJL model is generally solved after applying the well known bosonization procedure following Eguchi \[^2\] to arrive at the model expressed in terms of auxiliary meson fields $\sigma, \vec{\pi}$:

$$L_{NJL} = \bar{\Psi} \left[ i \slashed{D} - m_0 - g_\pi (\sigma + i \vec{\pi} \cdot \gamma_5) \right] \Psi - \frac{\mu^2}{2} (\sigma^2 + \vec{\pi}^2) ,$$  

where $G = g_\pi^2 / \mu^2$. Here, $g_\pi$ is the physical pion–quark coupling constant implying that $\vec{\pi}$ is the physical pion field. The meson fields are constrained to the chiral circle:

$$\sigma^2(\vec{x}) + \vec{\pi}^2(\vec{x}) = f_\pi^2 ,$$  

where $f_\pi = 93 \text{ MeV}$ is the pion decay constant.

In the chiral quark soliton model \[^3\] based on the lagrangean \[^4\] (frequently
referred simply as NJL model) the baryons appear as a bound state of \(N_c\) valence quarks coupled to the polarized Dirac sea. Operationally the baryon sector of the model is solved in two steps. In the first step, motivated by the large \(N_c\) limit, a static localized solution (soliton) is found by solving the corresponding equations of motion in an iterative self-consistent \[5\] procedure assuming that the \(\sigma\) and \(\vec{\pi}\) fields have hedgehog structure. However, this hedgehog soliton does not preserve the spin and isospin. In order to describe the nucleon properties one needs nucleon states with the proper spin and isospin numbers. To this end making use of the rotational zero modes the soliton is quantized. In fact, such a cranking procedure was elaborated in refs. \[6,3\] and problems arising in this scheme due to the regularization are considered at first in ref. \[4\]. Within the semiclassical quantization scheme quite successful calculations including Dirac sea polarization effects have been reported for the nucleon-delta mass splitting \[8\], magnetic moments \[9\] and axial-vector form factor \[10\] as well as some results for the nucleon electric form factors \[11\]. Very recently, in the semiclassical quantization procedure important \(1/N_c\) rotational corrections have been derived \[12\] which improve considerably the theoretical values for the isovector magnetic moment.

Because of several reasons the model, described by lagrangean \(2\), is considered as one of the most promising effective theories describing low energy QCD phenomena.

First, the model is the simplest quark model which provides mechanism for spontaneous breaking of the chiral symmetry — the basic feature of QCD. The mesons appear as composite states of quark-antiquark pairs. The philosophy behind this approach is based on the hypothesis that the chiral symmetry breaking is the dominant mechanism which determines the structure of the low-mass hadrons whereas the confinement has no direct impact on them \[3\].

Second, the self-consistent solitonic solution has been found to exist in the large \(N_c\) limit of the model within the physically acceptable range of parameters \[3\] fixed in the meson sector. The chiral soliton provides a good description of the nucleon and gives us possibility to take into account vacuum polarization effects from the Dirac quark sea. Calculations done in the semiclassical quantization procedure yield quite reasonable results for quantities like the nucleon radii, \(\Sigma\)–terms, axial vector coupling constant, \(\Delta\)–nucleon splitting as well as the mass splittings within the octet and decuplet.

Third, there are various hints \[13,15\] that the NJL-type Lagrangian can be obtained from QCD in various low-energy approximations. It should be stressed that the large \(N_c\) limit plays a prominent role in those considerations.

The aim of this paper is to calculate the electromagnetic nucleon form factors within the \(SU(2)\) chiral quark soliton model based on the semibosonized NJL-
type lagrangean with the vacuum polarization effects taken into account. The calculations done in the semiclassical quantization scheme will include the rotational $1/N_c$ corrections, which have been shown \cite{12} to be important for the isovector magnetic moment. Calculations of this sort give us possibility to calculate the nucleon electromagnetic form factors, as well as such quantities like the electric mean squared radii, the magnetic moments, the nucleon–Δ energy splitting, and the electric and magnetic charge distributions of the nucleon.

The paper is organized as follows: we begin in Section 2 with the electromagnetic current in the model defined in terms of path integrals and introduce the rotational zero modes treating the angular velocity as a perturbation. After that we consider the problem of regularization. In Section 3 we derive the expressions for the form factors including terms up to the linear order in angular velocity ($1/N_c$). In Section 4 we present and discuss our numerical results.

\section{The current and regularized action}

Our main goal in this section is to introduce the matrix element of the electromagnetic currents in terms of path integrals and to evaluate it within the semiclassical quantization scheme. We follow the valence picture for the nucleon in which it appears as a bound state of $N_c$ valence quarks coupled to the Dirac sea. Since the NJL model is not renormalizable a special attention will be paid to the problem of regularization.

We start with the definitions of the electromagnetic current of a fermion field $\Psi(x)$:

$$J_\mu(x) = \Psi^\dagger(x)\gamma_0\gamma_\mu\hat{Q}\Psi(x).$$

Here $\hat{Q}$ is the quark charge matrix

$$\hat{Q} \equiv \frac{1}{6}\mathbb{1} + \frac{1}{2}\tau^3.$$  \hspace{1cm} (5)

Using the partition function of the model in Minkowski space

$$Z_{NJL} = \int DU \int D\Psi D\Psi^\dagger \ e^{i\int d^4x L_{NJL}(x)},$$

we express the nucleon matrix element of current $J_\mu$ as a path integral:

$$\langle N(p) | J_\mu(0) | N(p) \rangle = \lim_{T \to -i\infty} \frac{1}{Z} \int d^3x \ d^3y \ e^{ip\cdot x} \ e^{-ip\cdot y} \int DU \int D\Psi D\Psi^\dagger \times J_N(T/2, x) J_N^\dagger(-T/2, y) \Psi^\dagger(0)\gamma_0\gamma_\mu\hat{Q}\Psi(0) \ e^{i\int d^4x \mathcal{L}(U)\Psi}.$$ \hspace{1cm} (7)
written in terms of quark $\Psi, \Psi^\dagger$ and meson $U$ fields. The equality (7) should be understood as a limit at large Euclidean time separation. Here $Z$ is the normalization factor which is related to the same path integral but without the electromagnetic current $J_\mu$. In fact, the latter represents the correlation function of two nucleon currents $J_N(T/2, \vec{x})$. $D$ is the operator

$$D(U) = i\partial_t - h(U),$$

which includes the one-particle hamiltonian

$$h(U) = \vec{\alpha} \cdot \vec{\nabla} \tau + \beta MU^{\gamma_5} + m_0\beta. \tag{9}$$

Here $\vec{\alpha}$ and $\beta$ are the Dirac matrices and the constituent quark mass $M = g f_\pi$. The meson fields can be equivalently written as

$$U = e^{i\vec{\pi} \cdot \hat{\pi}}$$

where

$$\hat{\pi} = \frac{\vec{\pi}}{||\vec{\pi}||} \quad \text{and} \quad ||\vec{\pi}|| = f_\pi \sin(P(\vec{r})) \tag{11}$$

It can be easily checked that the hamiltonian is hermitian: $h^\dagger = h$. The composite $N_c$ quark operator with the quantum numbers $J_3, T T_3$ (spin, isospin) of the nucleon can be chosen to be one of the Ioffe currents $[16]$

$$J_N(x) = \frac{1}{N_c!} \epsilon^{\beta_1 \cdots \beta_{N_c}} \Gamma^{f_{J_3, T T_3} f_{J_3, T T_3}} \bar{\Psi}_{\beta_1 f_1} (x) \cdots \bar{\Psi}_{\beta_{N_c} f_{N_c}} (x), \tag{12}$$

where $\beta_i$ is the color index, and $\Gamma^{f_{J_3, T T_3} f_{J_3, T T_3}}$ is a matrix in $f_i$ standing for both the flavor and the spin indices.

In eq.(7) the quarks can be integrated out:

$$\langle N(\vec{p}) | J_\mu(0) | N(\vec{p}') \rangle = \frac{1}{Z} N_c \Gamma^{f_{J_3, T T_3} f_{J_3, T T_3}} \int d^3x \ d^3y \ e^{i\vec{p} \cdot \vec{x}} e^{-i\vec{p}' \cdot \vec{y}} \int DU \times \left\{ \langle T/2, \vec{x} | i \int \frac{D}{D} | 0, 0 \rangle f_{1f'} (\gamma_0 \gamma_\mu \hat{Q}) f_{g'f} \langle 0, 0 | i \int \frac{D}{D} | 0, 0 \rangle \gamma_0 \gamma_\mu \hat{Q} \right\}$$

$$\times \langle T/2, \vec{x} | i \int \frac{D}{D} | -T/2, \vec{y} \rangle f_{1g_1} \prod_{i=2}^{N_c} \langle T/2, \vec{x} | i \int \frac{D}{D} | -T/2, \vec{y} \rangle f_{ig_1} e^{iS_{\text{eff}}}, \tag{13}$$

where the effective action

$$S_{\text{eff}} = -iN_c \text{Tr} \log D(U). \tag{14}$$

Since we treat the meson fields classically (i.e. at the 0-loop level) and also they are constrained on the chiral circle (3), the only non-trivial part of the
action is the fermionic part Tr log \(D(U)\). In eq. (13) in a natural way the current matrix element is split in a valence – the first term, and a Dirac sea contribution – the second one (see the diagrams on the l.h.s. of Fig.2 a) and b). Since the model is not renormalizable, in principle, the contribution of the Dirac sea includes divergences and should be regularized. The second part of this section will be devoted to the problem of regularization.

In eq. (13) we are left with the integral over the meson fields \(U\). In leading order in \(N_c\) it can be done in a saddle point approximation. To that end we look for a stationary localized meson configuration (soliton) of hedgehog structure

\[
\bar{U}(x) = e^{i\vec{r} \cdot \vec{x} \frac{P(x)}{}} ,
\]

which minimizes the effective action. As it was mentioned before, the hedgehog soliton \(\bar{U}(x)\) does not preserve the spin and isospin. As a next step, making use of the rotational zero modes we quantize it. Since the fluctuations which correspond to the zero modes are not small, they have to be treated “exactly” in the meaning of path integral. It can be done assuming a rotating meson hedgehog fields of the form

\[
U(\vec{x}, t) = R(t) \bar{U}(\vec{x}) R^+(t) ,
\]

with \(R(t)\) being a time-dependent rotation SU(2) matrix in the isospin space and

\[
R^\dagger R = \mathbf{1} .
\]

It is easy to see that for such an Ansatz one can transform the effective action

\[
\text{Tr} \log D(U) = \text{Tr} \log(D(\bar{U}) - \Omega)
\]

in order to separate the angular velocity matrix:

\[
\Omega = -ir^+(t)\dot{R}(t) = \frac{1}{2}\Omega_a \tau_a ,
\]

from \(D(\bar{U})\). Similar to the effective action the quark propagator in the background meson fields \(U\) can be rewritten as

\[
\langle T/2, \vec{x} | \frac{i}{D(U)} | -T/2, \vec{y} \rangle = R(T/2) \langle T/2, \vec{x} | \frac{i}{D(\bar{U}) - \Omega} | -T/2, \vec{y} \rangle R^+(-T/2) .
\]

Here, the operator \(D(\bar{U}) - \Omega\) corresponds to the body-fixed frame of the soliton in which the quark fields are transformed as

\[
\Psi \rightarrow R(t)\Psi \quad \text{and} \quad \Psi^\dagger \rightarrow \Psi^\dagger R^+(t) .
\]

Since (as can be seen below from the canonical quantization rules) \(\Omega \sim \frac{1}{N_c}\) one
can consider Ω as perturbation and evaluate any observable as a perturbation series in Ω which is actually an expansion in $\frac{1}{N_c}$.

In this scheme the matrix element (7) of the current can be written as

$$\langle N(\vec{p}^\prime) | J_\mu(0) | N(\vec{p}) \rangle = \frac{1}{Z} N_c \int d^3x d^3y d^3z \ e^{i\vec{p}^\prime \cdot \vec{x}} \ e^{-i\vec{p} \cdot \vec{y}} \ e^{-i(\vec{p}^\prime - \vec{p}) \cdot \vec{z}}$$

$$\times \int DR \ D_{-T_3J_3}[R(T/2)] D_{-T_3J_3}[R(-T/2)] \langle T/2, \vec{x} | \frac{i}{D - \Omega} \ | - T/2, \vec{y} \rangle^{N_c-1}$$

$$\times \left\{ \langle T/2, \vec{x} \ | \frac{i}{D - \Omega} \ | 0, \vec{z} \rangle R^+(0) \gamma_0 \gamma_\mu \hat{Q} R(0) \langle 0, \vec{z} | \frac{i}{D - \Omega} \ | - T/2, \vec{y} \rangle \right\}$$

$$- \text{Tr} \left( \langle 0, \vec{z} | \frac{i}{D - \Omega} \ | 0, \vec{z} \rangle R^+(0) \gamma_0 \gamma_\mu \hat{Q} R(0) \right) \langle T/2, \vec{x} | \frac{i}{D - \Omega} \ | - T/2, \vec{y} \rangle \right\}$$

$$\times e^{N_c \text{Tr} \log(D - \Omega)}.$$ \hfill (21)

Here, the finite rotation matrix (Wigner D-function) $D^{(J)}_{-T_3J_3}$, which carries the spin and isospin quantum numbers of the nucleon, appears due to the rotations $R(t)$ of the valence quark propagators in eq.(19) correlated by the $\Gamma^{(g)}_{JJ^3,T^3}$ matrices. The D-functions represent the collective part of the nucleon wave function in the semiclassical quantization scheme. The integral over $\vec{z}$ is due to the translational zero modes treated in the leading order.

Now we are ready to make an expansion in Ω. For the effective action it yields \([3,7]\) up to the second order in Ω:

$$S_{eff} \approx N_c \text{Tr} \log D + \frac{\Theta}{2} \int dt \Omega^2.$$ \hfill (22)

Here Θ is the moment of inertia. However, this quantity includes a divergent contribution from the Dirac sea which should be regularized. Later we will come back to this point. A derivation of the moment of inertia including regularization can be found in ref. \([7]\) and it has been calculated numerically in ref. \([8,9]\). The first term in eq.(22) will be absorbed in $Z$ whereas the second one drives the evolution in the space of matrix $R$. Expanding the quark propagator

$$\frac{1}{D - \Omega} \rightarrow \frac{1}{D} + \frac{1}{D} \Omega \frac{1}{D} + ...$$ \hfill (23)

we can separate the zero order ($\sim N^0_c$) and the linear order ($\sim \frac{1}{N_c}$) corrections in Ω. The expansion in Ω is illustrated in Fig.[4] a) and b) for the valence contribution and for the Dirac sea one, respectively.

After the expansion in Ω we should deal with the path integral over $R$. In the
case of the zero order term we have

\[
\int dR_1 \, dR_2 \, D_{-T_3J_3}^J(R_1) \, D_{-T_3J_3}^{J^*}(R_2) \int_{R_1 = R(-T/2)}^{R_2 = R(T/2)} DR \, \left( R^+(0) \hat{Q} R(0) \right)_{fg} \times e^{i \frac{\Theta}{2} \int dt \Omega^2(t')} ,
\]

the linear in \( \Omega \) terms are left with a more complicated integral:

\[
\int dR_1 \, dR_2 \, D_{-T_3J_3}^J(R_1) \, D_{-T_3J_3}^{J^*}(R_2) \int_{R_1 = R(-T/2)}^{R_2 = R(T/2)} DR \, \left( R^+(0) \hat{Q} R(0) \right)_{fg} \Omega_c(t) \times e^{i \frac{\Theta}{2} \int dt' \Omega^2(t')} .
\]

For the isoscalar part \( \frac{1}{6} \hat{1} \) of the charge matrix \( \hat{Q} \) the integrals (24) and (25) reduce simply to the normalization of the finite rotation matrix \( D_{-T_3J_3}^{(J)} \). In the case of the isovector part \( \frac{T_3}{2} \) we use the identity

\[
\left( R^+(0) \tau^a R(0) \right)_{fg} = \frac{1}{2} \text{Tr} \left( R^+(0) \tau^a R(0) \tau^b \right) (\tau^b)_{fg}
\]

in order to separate the \( R(t) \) dependent part of the current which does not carry flavor indices \( f, g \). The path integrals (24) and (25) can be taken rigorously [3]. We essentially made use of the basic feature of the path integral:

\[
\int_{q_1 = q(T_1)}^{q_2 = q(T_2)} Dq \, F_1(q(t_1)) \cdots F_n(q(t_n)) \, e^{iS} = \langle q_2, T_2 \, | \, T \{ \hat{F}_1(q(t_1)) \cdots \hat{F}_n(q(t_n)) \} | q_1, T_1 \rangle,
\]

namely that the path integral, which contains \( F_i(q(t_1)) \), can be equivalently written as a matrix element of the time ordered product \( T \) of the corresponding operators \( \hat{F}_i(q(t_1)) \). In our case we have the well-known canonical quantization rule

\[
\Omega_c \rightarrow \frac{\hat{J}_c}{\Theta},
\]

where \( \hat{J}_a \) is the spin operator. For the zero order the collective path integral (24) is reduced to an ordinary integral of three Wigner D-functions whereas the final result for (25) is a time ordered product:

\[
\vartheta(-t) D_{ab}(R(0)) J_c + \vartheta(t) J_c D_{ab}(R(0)),
\]

sandwiched between the nucleon rotational wave functions. Now using the standard spectral representation of the quark propagator

\[
\langle t', \vec{x}' | \frac{1}{D} | t, \vec{x} \rangle = -i \delta(t' - t) \sum_{\epsilon_n > 0} e^{-i \epsilon_n (t' - t)} \Phi_n(\vec{x}') \Phi_n^\dagger(\vec{x})
\]
\[ i \Gamma(t - t') \sum_{\epsilon_n \leq 0} e^{-i\epsilon_n(t' - t)} \Phi_n(\vec{x}') \Phi_n^\dagger(\vec{x}), \quad (29) \]

It is straightforward to evaluate the matrix element of the current \( J_\mu \) – eq. (21). Here \( \Phi_n \) and \( \epsilon_n \) are the eigenfunctions and eigenvalues of the single-particle hamiltonian \( h \):

\[ \left( \frac{\vec{\alpha} \cdot \vec{\nabla}}{i} + \beta M U^\gamma + m_0 \beta \right) \Phi_n(\vec{x}) = \epsilon_n \Phi_n(\vec{x}). \quad (30) \]

Now we come to the problem of regularization. Since the NJL model is a non-renormalizable theory, a regularization scheme of an appropriate cut-off \( \Lambda \) is needed to make the theory finite. In our case it concerns the divergencies in the Dirac sea contribution to the matrix element of the electromagnetic current \( J_\mu \) - diagrams in Fig. 1 b). In Minkowski space-time it can be equivalently written as:

\[
\langle N(\vec{p}') | J_\mu(0) | N(\vec{p}) \rangle_{sea} \equiv i \frac{1}{Z} \int d^3 \vec{z} \ e^{-i(\vec{p}' - \vec{p}) \cdot \vec{z}} \int DR D^{(J)_\mu}_{-T^2J_3} [R(T/2)] \times \delta A^\mu(0, \vec{z}) e^{iS_{eff}[\Omega, A^\mu, R]} \bigg|_{A^\mu = 0}. \quad (31)
\]

Here in the effective action \( S_{eff} \) (14) we include an explicit electromagnetic coupling in a minimal way:

\[ D(\vec{U}) = i \partial_t - h(\vec{U}) - \Omega - \hat{Q} \gamma_0 \gamma_\mu A^\mu, \quad (32) \]

where \( A^\mu \) is an external electromagnetic field.

We will work in Euclidean space-time where we use the metric tensor’s signature transformed from \((+ - - -)\) to \((-- --)\), i.e. the Euclidean metric tensor \( g^E_{\mu\nu} \equiv -\delta_{\mu\nu} \) and the general formulae to perform the transformation from Minkowski to Euclidean space are:

\[
\begin{align*}
A^4_E &= iA^0_M, & A^k_E &= A^k_M, \\
\gamma^4_E &= i\gamma^0_M, & \gamma^k_E &= \gamma^k_M, \\
\tau \equiv x^4_E &= ix^0_M \equiv it, & x^k_E &= x^k_M, \\
\int d^4x_E &= i \int d^4x_M, & \partial_t &= -i\partial_t, \\
\{\gamma^E_\mu, \gamma^E_\nu\} &= -2\delta_{\mu\nu}, & \{\gamma^E_\mu, \gamma^E_5\} &= 0, \\
(\gamma^E_5)^\dagger &= -\gamma^E_5, & (\gamma^E_5)^\dagger &= \gamma^E_5 = \gamma^E_M. \quad (33)
\end{align*}
\]

The transformation of the electromagnetic field \( A_\mu \) under the Wick rotation (transformation to the Euclidean space-time) is defined in (33) and in order to preserve the gauge invariance the field \( A_\mu \) should be hermitian in Euclidean space-time. Then in body-fixed frame the Dirac operator of the rotating soliton
\( D(U) \) can be expressed in the form:

\[
D(\bar{U}) = R^\dagger D(U) R = \partial_\tau + h(\bar{U}) + i\Omega + iA_4 R^\dagger \bar{QR} - i\gamma_4 \gamma_k A_k R^\dagger \bar{QR} .
\] (34)

Here, the rotation velocity matrix \( \Omega \) in the Euclidean space-time is given by:

\[
\Omega \equiv \frac{1}{2} \tau^a \Omega^a = -i R^\dagger \dot{R},
\] (35)

where the derivative \( \dot{R} \) is with respect to the Euclidean time \( \tau \) and as usual, summation over repeated indices is assumed. Similar to ref. [7] the angular velocity \( \Omega \) is hermitian in Euclidean space-time. The collective variable \( \Omega \) will be quantized according to the canonical quantization rule (27) which in Euclidean space-time is:

\[
\Omega_a \rightarrow -i \hat{J}_a \Theta.
\] (36)

where \( \hat{J}_a \) is the spin operator.

Now we can write down the hermitian conjugate to (34):

\[
D^\dagger(\bar{U}) = -\partial_\tau + h(\bar{U}) - i\Omega - iA_4 R^\dagger \bar{QR} - i\gamma_4 \gamma_k A_k R^\dagger \bar{QR},
\] (37)

As we can see from (34), (37), the operator (34) is non-hermitian. Hence, in general, the Euclidean action (14) will have the real and imaginary part. To make clear distinction we write down both parts explicitly:

\[
S_{\text{eff}}^F = \Re S_{\text{eff}}^F + \Im S_{\text{eff}}^F,
\] (38)

where

\[
\Re S_{\text{eff}}^F \equiv -\frac{1}{2} N_c \text{Tr} \log(D^\dagger D),
\] (39)

\[
\Im S_{\text{eff}}^F \equiv \frac{i}{2} N_c \text{Tr} \log(D/D^\dagger),
\] (40)

The imaginary part is finite and does not need regularization. In addition, any regularization of this part of the action would break several important features of the model. On the other hand, the real part is infinite and a regularization is necessary. We introduce the Schwinger proper-time regularization of the action. In general, for an operator \( A \hat{B}^{-1} \) its proper-time regularized form reads [17]:

\[
\text{Tr} \log(A \hat{B}^{-1}) \Rightarrow - \text{Tr} \int \frac{du}{u} \left( e^{-u\hat{A}} - e^{-u\hat{B}} \right).
\] (41)
Applying (41) to the real part of the action (39) we obtain:
\[ \Re S_{\text{eff}}^F = \frac{N_c}{2} \text{Tr} \int_1^\infty \frac{du}{u} \left( e^{-u D^\dagger D} - e^{-u D^\dagger D_v} \right), \] (42)

The cut-off \( \Lambda \) is fixed in the meson sector to reproduce the physical pion decay constant \( f_\pi \). We also subtract the vacuum contribution where the corresponding operator:
\[ D_v = \partial_\tau + h_v \] (43)
includes the unperturbed (vacuum) hamiltonian \( h_0 \):
\[ h_v = \frac{\vec{\alpha} \cdot \vec{V}}{i} + \beta M + m_0 \beta. \] (44)

Thus the effective action includes the regularized real part (39) and finite imaginary part (40). In fact, for the contribution from the imaginary part (40) one can use the non-regularized expression (21).

Now, our task is to evaluate the contribution (31) coming from the regularized real part (39). In this case the difficulty lies in that the expression (42) contains rather complicated operator exponent \( \exp(-u D^\dagger D) \) in the integrand. However, because afterwards we take derivative over the fields \( A_\mu \) and set \( A_\mu = 0 \), the only non-zero contribution will come from the part of the integrand linear in \( A_\mu \). We shall treat the angular velocity \( \Omega \sim \frac{1}{N_c} \) as a small perturbation that is consistent with the large \( N_c \) limit philosophy behind the NJL model. Hence, for the electromagnetic current we can neglect terms \( \Omega^2 \) and higher. However, as can be seen later the term \( \sim \Omega^2 \) must be taken into account to evaluate the moment of inertia (see also ref. [7]).

In the next step we expand the integrand in (31) in terms of \( A_\mu \) and \( \Omega \). It is useful to separate
\[ D^\dagger D = D_0^\dagger D_0 + \Omega^2 + V, \] (45)
where
\[ D_0^\dagger D_0 = -\partial_\tau^2 + h^2, \] (46)
and the sum of all perturbative terms we denote by \( V \):
\[ V = W_0 + W_1 + W_2 \] (47)
with
\[ W_0 = i [h, \Omega] - i \{ \Omega, \partial_\tau \}, \] (48)
\[ W_1 = [h, i A_4 R^\dagger \hat{Q} R] + A_4 \{ R^\dagger \hat{Q} R, \Omega \} - i \{ A_4 R^\dagger \hat{Q} R, \partial_\tau \}, \] (49)
and

\[ W_2 = -i\gamma_4\gamma_k A_k \{ h, R^i \hat{Q} R \} + \gamma_4 \gamma_k A_k [ R^i \hat{Q} R, \Omega ] - i\gamma_4\gamma_k [ A_k R^i \hat{Q} R, \partial_r ]. \]  

(50)

Here \[ , \] and \{ , \} denote commutators and anticommutators, respectively.

After that for the operator exponent \( \text{Tr} \exp (-u D^1 D) \) in eq.(42) we apply an expansion

\[ e^{\hat{A} + \hat{B}} = e^{\hat{A}} + \int_0^1 d\alpha \ e^{\alpha \hat{A}} \hat{B} \ e^{(1-\alpha)\hat{A}} + \int_0^1 d\beta \ \int_0^{1-\beta} d\alpha \ e^{\alpha \hat{A}} \hat{B} \ e^{\beta \hat{A}} \hat{B} \ e^{(1-\alpha-\beta)\hat{A}} + \ldots \]  

(51)

There for the operator \( \hat{A} \) we take the part \( -u D_0^i D_0 \), which does not contain \( A_\mu \) and \( \Omega \), and using the cyclic properties of \( \text{Tr} \) we get:

\[ \text{Tr} \ e^{-u D^1 D} = \text{Tr} \ e^{-u D_0^i D_0} - u \ \text{Tr} \left( e^{-u D_0^i D_0} V \right) + \frac{u^2}{2} \int_0^1 d\beta \ \text{Tr} \left( e^{-u (1-\beta) D_0^i D_0} V \ e^{-u \beta D_0^i D_0} V \right) + \ldots . \]  

(52)

We compute the trace over the Euclidean time in the \( \omega \)-space, the Fourier conjugate to \( \tau \):

\[ \langle \tau | f(\partial_\tau, \ldots) | \tau \rangle = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} f(-i\omega, \ldots) . \]  

(53)

Applying the operator expansion (52), there are, in general, two types of terms in (52) – linear and bilinear in \( V \). Obviously, in order to give a non-zero contribution to eq.(31) all those terms should be linear \( A_\mu \). This implies that only \( W_{1(2)} \)-terms will contribute to the first order in (52):

\[ -u \ \text{Tr} \left( e^{-u D_0^i D_0} \frac{\delta W_{1(2)}}{\delta A_\mu} \bigg|_{A_\mu=0} \right) \]  

(54)

One should also keep in mind that \( W_{1(2)} \) includes also terms linear in \( \Omega \).

The terms contributing to the second order in (51) have a more complicated structure:

\[ \text{Tr} \left( e^{-u (1-\beta) D_0^i D_0} W_0 \ e^{-u \beta D_0^i D_0} \frac{\delta W_{1(2)}}{\delta A_\mu} \bigg|_{A_\mu=0} \right) + W_0 \leftrightarrow \frac{\delta W_{1(2)}}{\delta A_\mu} . \]  

(55)

In the case that the integral over \( R \) includes both \( \Omega_c \) and \( D_{ab} \) we are left with an additional integral over Euclidean time which includes their time-ordered
\[
\int d\tau e^{i(\omega - \omega')\tau} \left[ \vartheta(-\tau) D_{ab} \dot{J} + \vartheta(\tau) \dot{J} D_{ab} \right] = \mathcal{P} \frac{i}{\omega - \omega'} [D_{ab}, \dot{J}],
\]

where in r.h.s. \( \mathcal{P} \) means a principal value.

To summarize, in the scheme presented above the matrix element of the current \( J_\mu \) includes non-regularized as well as regularized contributions. In general, there are two types of non-regularized contributions: the valence parts (for all form factors) and the sea parts coming from the imaginary part of the action. To compute those terms we use the non-regularized expression (21). For the contribution from the real part of the action we follow the expressions (54) - (56) which include an explicit proper-time regulator (41).

To begin with explicit evaluation of the particular current matrix elements and the corresponding form factors we will sketch briefly the derivation of the Dirac sea contribution to the moment of inertia within the present scheme. The valence part which does not need regularization and can be found in ref. [3]:

\[
\Theta_{ab}^{\text{val}} = \frac{N_c}{2} \sum_{n \neq \text{val}} \frac{\langle \text{val} | \tau_a | n \rangle \langle n | \tau_b | \text{val} \rangle}{\epsilon_n - \epsilon_{\text{val}}}. \tag{57}
\]

Here \( |n\rangle \) and \( \epsilon_n \) are the eigenfunctions and eigenenergies of the hamiltonian \( h \) and \( \text{val} \) stands for the valence level.

The Dirac sea part of the moment of inertia \( \Theta \) originates from the regularized real part \( \Re S_{\text{eff}} \) of the effective action (39) with electromagnetic field \( A_\mu \) set to zero. The imaginary part does not contribute. In first order operator expansion (51) only the \( \Omega^2 \) term contributes:

\[
- \frac{N_c}{2} \int_1^{\infty} du \text{Tr} \left( e^{-u D_0^\dagger D_0} \tau_a \tau_b \right) = - \frac{N_c}{4\sqrt{\pi}} \sum_{n,m} \int_1^{\infty} \frac{du}{\sqrt{u}} \frac{e^{-u \epsilon_n^2} + e^{-u \epsilon_m^2}}{2} \langle m | \tau_a | n \rangle \langle n | \tau_b | m \rangle, \tag{58}
\]

where for convenience we introduce the complete set \( |m\rangle \langle m| \) and symmetrize with respect to \( n \leftrightarrow m \). The second order term in expansion (51) includes only \( W_0 \):

\[
\frac{N_c}{4} \int_1^{\infty} du \left[ \frac{1}{\sqrt{\pi}} \int_0^1 d\beta \text{Tr} \left( e^{-u(1-\beta) D_0^\dagger D_0} \frac{\delta W_0}{\delta \Omega_a} e^{-u\beta D_0^\dagger D_0} \frac{\delta W_0}{\delta \Omega_b} + a \leftrightarrow b \right) \right]. \tag{59}
\]

It is easy to be seen that the above expression is symmetric in \( a \leftrightarrow b \). After
some straightforward calculations one gets:

\[
\frac{N_c}{2} \sum_{n,m}^{\infty} \int \frac{du}{\sqrt{u}} \frac{e^{-u\epsilon_n^2} - e^{-u\epsilon_m^2}}{\epsilon_m^2 - \epsilon_n^2} \left[ \frac{1}{u} + \frac{1}{2} (\epsilon_m - \epsilon_n)^2 \right] \langle m | \tau_a | n \rangle \langle n | \tau_b | m \rangle.
\]  

(60)

Combining (58) and (60) the final result is:

\[
\Theta_{sea} = \frac{N_c}{2} \sum_{n \neq m} \langle n | \tau_a | m \rangle \langle m | \tau_b | n \rangle \mathcal{R}_\Theta^A(\epsilon_m, \epsilon_n).
\]  

(61)

Here \(\mathcal{R}_\Theta^A\) is the regularization function:

\[
\mathcal{R}_\Theta^A(\epsilon_m, \epsilon_n) = \frac{1}{4\sqrt{\pi}} \int_{\frac{1}{\sqrt{\pi}}}^{\infty} \frac{du}{\sqrt{u}} \left( \frac{1}{u} \frac{e^{-u\epsilon_n^2} - e^{-u\epsilon_m^2}}{\epsilon_m^2 - \epsilon_n^2} - \frac{\epsilon_n e^{-u\epsilon_n^2} + \epsilon_m e^{-u\epsilon_m^2}}{\epsilon_m + \epsilon_n} \right),
\]  

(62)

which coincides with ref. [7]. It is easy to see that the moment of inertia is diagonal:

\[
\Theta_{ab} \equiv \delta_{ab} \Theta.
\]  

(63)

### 3 Computing the form factors

The nucleon electromagnetic Sachs form factors are related to the matrix element of the electromagnetic current by:

\[
\langle N(p') | J_0(0) | N(p) \rangle = G_E(q^2),
\]  

(64)

\[
\langle N(p') | \vec{J}_i(0) | N(p) \rangle = \frac{1}{2\mathcal{M}_N} G_M(q^2) i \langle \bar{N} | (\bar{\sigma} \times \bar{q}) | N \rangle,
\]  

(65)

where \(|N(p)\rangle\) is the nucleon state of a four momentum \(p\) and proper spin and isospin quantum numbers. The four momentum transfer \(q = p' - p\). The electromagnetic current \(J_\mu(x)\) is in the Minkowski space-time and \(\mathcal{M}_N\) is the nucleon mass. The expressions for the matrix element of the current have been derived in the previous Section.

The isoscalar and isovector parts of the form factors are defined by:

\[
G_{E(M)} \equiv \frac{1}{2} G_{E(M)}^{T=0} + \hat{T}_3 G_{E(M)}^{T=1}.
\]  

(66)

The isoscalar form factors (electric and magnetic) come from the scalar part of the quark current matrix \(\hat{Q} (\sim \bar{1})\) while the isovector form factors is from
the triplet part ($\sim \tau^3$). Since the imaginary part of the effective action in Euclidean space-time is due to the isoscalar part of the charge matrix $\hat{Q}$, one should expect that for both isoscalar (electric and magnetic) form factors the Dirac sea contribution originates from the imaginary part of the effective action. Indeed, as can be checked directly the contribution from the regularized real part of the action to both isoscalar form factors is exactly zero. This means that for the calculation of these form factors the non-regularized expression (21) can be used.

We start with the electric scalar form factor. Since only the isoscalar part of the operator $\hat{Q}$ contributes ($\hat{J}_0 = \frac{1}{6} \hat{1}$) the rotational matrices $R(t)$ vanish from the action. Hence, the $D R$ integration in (21) becomes trivial. It can be seen easily from (22) and (23) using the explicit form of $W_0$ and $W_1$ that because of symmetry considerations there is no contribution from the real part of the action and the Dirac sea contribution comes only from the imaginary part (40).

We start with the non-regularized eq.(21), expand the effective action and the quark propagator in $\Omega$, (22) -(23), and insert the spectral representation (29). In this case the contribution from the terms linear in $\Omega$ is exactly zero. Finally we arrive at:

$$G_T^{E=0}(q^2) = \int d^3 z \ e^{i\vec{q} \cdot \vec{z}} \ N_c \ \frac{3}{\Theta} \ \{ \Phi_{val}^{\dagger}(\vec{z}) \ \Phi_{val}(\vec{z}) - \frac{1}{2} \ \sum_n \ \text{sign}(\epsilon_n) \ \Phi_n^{\dagger}(\vec{z}) \ \Phi_n(\vec{z}) \} \ , \quad (67)$$

The first term in eq.(67) comes from the valence quarks and the second term is due to the quark vacuum polarization.

At $q^2 = 0$ the isoscalar electric form factor reproduces the baryon number $B = 1$: it easy to see that the contribution from the sea vanishes exactly whereas the valence term gives one (normalization of $\Phi_{val}$).

As the next step, we compute the electric isovector form factor. Using the formulae (21)-(29) the evaluation of the valence part is straightforward (see ref. [3]) and one gets

$$\langle N(p') | \Phi_{val}^{\dagger} \tau^3 | \Psi | N(p) \rangle_{val} = -\frac{N_c}{2\Theta} \ \int d z \ e^{-i\vec{q} \cdot \vec{z}} \ \sum_{m \neq val} \ \frac{\langle \text{val}|\tau_m|m\rangle (\Phi_m^{\dagger}(\vec{x}) \tau_3 \Phi_{val}(\vec{x}))}{\epsilon_m - \epsilon_{val}} \ \times \langle 1/2, J_3 T_3 | \{ \hat{J}_c, D_{3b} \} | 1/2, J_3 T_3 \rangle \ . \quad (68)$$

The anticommutator $\{ \hat{J}_c, D_{3b} \}$ appears due to the symmetry of the $\tau$ matrix element product under $n \leftrightarrow m$.

The contribution of the Dirac sea comes from the real part of the effective action and a regularization is necessary. Contributions to the first order in expansion come only from the term of $W_1$ containing both $A_4$ and $\Omega$:
\{A_4 R^\dagger \tau^3 R, \Omega\}:

\[
\text{Tr}\left( e^{-u D^\dagger_0 D_0} \frac{\delta W_1}{\delta A_4(0, \bar{z})} \right|_{A_4=0} = \frac{N_c}{4\sqrt{\pi}} \sum_{n,m} \int \frac{du}{\sqrt{u}} \frac{e^{-u \epsilon_n^2} + e^{-u \epsilon_m^2}}{2} \times \langle n|\tau_c|m\rangle \left( \Phi^\dagger_{m}(\bar{z}) \tau^3 \Phi_{n}(\bar{z}) \right) \langle 1/2, J_3 T_3 |{\hat{J}}_c, D_{3b}| 1/2, J_3 T_3 \rangle.
\]

(69)

The other two terms in \( W_1, i[h, A_4 R^\dagger \tau^3 R] \) and \(-i\{A_4 R^\dagger \tau^3 R, \partial\tau\}\), give no contribution.

The second order in (52) includes terms from \( W_0 \) and \( W_1 \):

\[
-\frac{N_c}{4} \int_0^1 d\beta (1 - \beta) \int \frac{du}{\sqrt{u}} \text{Tr}\left( e^{-u D^\dagger_0 D_0} W_0 e^{-u\beta D^\dagger_0 D_0} \frac{\delta W_1}{\delta A_4(0, \bar{z})} \right|_{A_4=0} + W_0 \leftrightarrow \delta W_1 \frac{\delta A_4}{\delta A_4}\)
\]

(70)

Because of the symmetry of the \( \tau \) matrix element product under \( n \leftrightarrow m \) only the symmetric part of the r.h.s. of the integral (56) survives. After integrating over \( \omega \) and \( \beta \) (the integrals are the same like in the case of the moment of inertia) one obtains:

\[
- \frac{N_c}{4\sqrt{\pi \Theta}} \int \frac{du}{\sqrt{u}} \frac{e^{-u \epsilon_n^2} - e^{-u \epsilon_m^2}}{\epsilon_m^2 - \epsilon_n^2} \left[ \frac{1}{u} + \frac{1}{2} (\epsilon_m - \epsilon_n)^2 \right] \langle n|\tau_c|m\rangle \times \left( \Phi^\dagger_{m}(\bar{z}) \tau_3 \Phi_{n}(\bar{z}) \right) \langle 1/2, J_3 T_3 |{\hat{J}}_c, D_{3b}| 1/2, J_3 T_3 \rangle.
\]

(71)

Finally, using the quantization rule (36) the collective matrix element can be easily calculated:

\[
\langle 1/2, J'_3 T'_3 |{\hat{J}}_c, D_{3b}| 1/2, J_3 T_3 \rangle = -\frac{1}{3} \delta_{cb} (\tau^a)_{T'_3 T_3} \delta_{J'_3 J_3}.
\]

(72)

Combining (68) - (71) we arrive at the following expression for the electric isovector form factor in Minkowski space-time:

\[
G^{T=1}_E(q^2) = \frac{N_c}{6\Theta} \int d^3 x e^{-i \vec{q} \vec{x}} \left\{ \sum_n \frac{\left( \Phi^\dagger_{\text{val}}(\bar{z}) \tau^a \Phi_n(\bar{z}) \right) \langle n|\tau^a|\text{val}\rangle}{\epsilon_n - \epsilon_{\text{val}}} \right. \\
+ \left. \sum_{m,n} \mathcal{R}_{\Theta}(\epsilon_m, \epsilon_n) \left( \Phi^\dagger_{m}(\bar{z}) \tau^a \Phi_n(\bar{z}) \right) \langle m|\tau^a|n \rangle \right\},
\]

(73)

where the regularization function \( \mathcal{R}_{\Theta} \) is exactly the same as in the case of moment of inertia. Hence, formula (73) for \( q^2 = 0 \) is equivalent to the quotient
of the moment of inertia by itself:

$$G_{E}^{T=1} \left( q^2 = 0 \right) = 1, \quad (74)$$

as it should be.

Now we proceed to compute the magnetic form factors. We start with the isoscalar one. In this case, the quark charge matrix $\hat{Q}$ contributes via the isoscalar term, $\sim \frac{1}{6} \gamma_0 \gamma_k$. Similar to the isoscalar electric form factor, the Dirac sea contribution from the real part of the action is zero: the term from the first order in $(52)$ is proportional to $\langle n \mid \gamma_0 \gamma_k \mid n \rangle = 0$ and in the second order the terms are odd in $\omega$ and after integration vanish. Hence, the Dirac sea contribution comes solely from the imaginary part of the action and the current matrix element can be calculated directly from $(21)$. However, since the current does not include rotation matrices $R(t)$ the first (zero order in $\Omega$) term from $(23)$ gives no contributions. In the leading order (linear in $\Omega$) the collective part includes only $\Omega$:

$$\langle N(p') \mid \Psi^\dagger \gamma_0 \gamma_k \Psi \mid N(p) \rangle = -\frac{N_c}{2\Theta} \int d^3z \, e^{-i \vec{q} \cdot \vec{z}} \sum_{\substack{n > \text{val} \\ m \leq \text{val}}} \frac{\Phi^\dagger_m(\vec{z}) \gamma_0 \gamma_k \Phi_n(\vec{z})}{\epsilon_n - \epsilon_m} \langle n \mid \tau^a \mid m \rangle \epsilon_{kja} q_j q_k q^2 \sum_{\substack{n > \text{val} \\ m \leq \text{val}}} \langle n \mid \tau^a \mid m \rangle, \quad (75)$$

From $(65)$ and $(75)$ resolving the spin structure we get for the magnetic isoscalar form factor the following expression:

$$G_{M}^{T=0}(q^2) = i \frac{N_c}{6\Theta} \mathcal{M}_N \int d^3z \, e^{-i \vec{q} \cdot \vec{z}} \epsilon_{kja} \frac{q_j q_k q^2}{q^2} \sum_{\substack{n > \text{val} \\ m \leq \text{val}}} \frac{\Phi^\dagger_m(\vec{z}) \gamma_0 \gamma_k \Phi_n(\vec{z})}{\epsilon_n - \epsilon_m} \langle n \mid \tau^a \mid m \rangle, \quad (76)$$

where, as usually, summation over repeated indices $k, j, a$ is assumed.

For the magnetic isovector form factor the current $J_\mu$ includes the isotriplet part $\tau^3/2$ of the quark charge matrix $\hat{Q}$. Because of this we cannot get rid of the rotation matrix $R(t)$ in the action. The valence part is obtained from eq. $(21)$. Expanding the quark propagator in $\Omega$ and using the spectral representation $(29)$ it is straightforward to calculate the contributions from the first as well as from the second order terms in expansion $(23)$. For the first order (zero in $\Omega$) there is no problem with time-ordering and the collective integral $(25)$ is reduced to an ordinary integral of three Wigner D-functions:

$$\langle 1/2, J_3 T_3 \mid D_{ab} \mid 1/2, J_3 T_3 \rangle = -\frac{1}{3} \left( \tau^a \right)_{T_3 T_3} \left( \tau^b \right)_{J_3 J_3} \quad (77)$$

In the second order (linear in $\Omega$) we have a time-ordered product $(28)$. In contrast to the isoscalar magnetic form factor the quark matrix element
\langle n | \gamma_0 \gamma_k \tau^3 | m \rangle \langle m | \tau^a | n \rangle \) is asymmetric under \( n \leftrightarrow m \) and we end up with a commutator in the collective part:

\[
\langle 1/2, J^3 T_3 \mid [\hat{J}_c, D_{ab}] \rangle 1/2, J_3 T_3 \rangle = i \varepsilon^{cib} \langle 1/2, J^3 T'_3 | D_{ad} \rangle 1/2, J_3 T_3 \rangle
\]  
(78)

Summing up the first and the second order terms we get for the valence part:

\[
\langle N(p') \mid \Psi \gamma_0 \gamma_k \frac{\tau^3}{2} \Psi \mid N(p) \rangle_{val} = N_c \int d^3 z \; e^{-i \vec{q} \cdot \vec{z}}
\]

\[
\times \left\{ \left( \Phi_{val}(\vec{z}) \gamma_0 \gamma_k \tau^a \Phi_{val}(\vec{z}) \right) \langle 1/2, J_3 T_3 \mid D_{ab} \rangle 1/2, J_3 T_3 \rangle + \frac{1}{2 \Theta} \sum_{n \neq val} \text{sign}(\epsilon_n)
\]

\[
\times \left( \Phi_{val}^+(\vec{z}) \gamma_0 \gamma_k \tau^b \Phi_n(\vec{z}) \right) \langle n \mid \tau^a \rangle_{val} \langle 1/2, J_3 T_3 \mid [\hat{J}_c, D_{ab}] \rangle 1/2, J_3 T_3 \rangle \right\}.
\]  
(79)

The Dirac sea part originates from the regularized real part of the effective action (83) (in Euclidean space-time). In the first order in (52) the only term from \( W_2 \) (53), which gives non-zero contribution, is \( i \gamma_4 \gamma_k \{ \hat{h}, R^\dagger \hat{Q} R \} \). The other two, \( i \gamma_4 \gamma_k [R^\dagger \hat{Q} R, \partial \tau] \) and \( -i \gamma_4 \gamma_k [R^\dagger \hat{Q} R, \Omega] \) simply vanish. Using (77) one gets

\[
N_c \int_0^\infty du \; \text{Tr} \left( e^{-u D_0^+ D_0} \frac{\delta W_2}{\delta A_k(0, \vec{z})} \right) = -N_c \sum_n \mathcal{R}^A_{M1}(\epsilon_n)
\]

\[
\times \left( \Phi_{n}^+(\vec{z}) \gamma_4 \gamma_k \tau^3 \Phi_{n}(\vec{z}) \right) \langle 1/2, J_3 T_3 \mid D_{ab} \rangle 1/2, J_3 T_3 \rangle,
\]  
(80)

with \( \mathcal{R}^A_{M1} \) being the corresponding regulator:

\[
\mathcal{R}^A_{M1}(\epsilon_n) = \frac{1}{2\sqrt{\pi}} \int \frac{du}{\sqrt{u}} \epsilon_n \; e^{-ue_n^2}.
\]  
(81)

To the second order

\[
- \frac{N_c}{4} \int_0^1 d\beta \int \frac{du}{\pi} \int du \; \text{Tr} \left( e^{-u(1-\beta) D_0^+ D_0} W_0 \; e^{-u\beta D_0^+ D_0} \frac{\delta W_1}{\delta A_k(0, \vec{z})} \right) \bigg|_{A_k=0}
\]

\[
+ W_0 \leftrightarrow \frac{\delta W_1}{\delta A_k},
\]  
(82)

contribute \( i \gamma_4 \gamma_k \{ \hat{h}, R^\dagger \hat{Q} R \} \) and \( i \gamma_4 \gamma_k [R^\dagger \hat{Q} R, \partial \tau] \). Using the asymmetry of the quark matrix element \( \langle n \mid \gamma_4 \gamma_k \tau^3 | m \rangle \langle m \mid \tau^a | n \rangle \) under \( n \leftrightarrow m \) one can show that the symmetric part of the r.h.s. of the integral (56) vanishes. Only the asymmetric part survives:

\[
\frac{N_c}{2 \Theta} \int d\beta \int du \int \frac{d\omega}{2\pi} \int \frac{d\omega'}{2\pi} \; e^{-u(1-\beta)(\omega_0^2+\varepsilon_n^2)} \; e^{-u\beta(\omega_0^2+\varepsilon_n^2)} \frac{1}{\omega - \omega'} (\omega_0 + \omega' \varepsilon_n)
\]
Using (78) and performing the integrations over $\omega$ and $u$ we get for the second order contribution in the form:

$$
i N_c \sum_{n,m} \mathcal{R}_{M2}^{A}(\epsilon_m, \epsilon_n) \left( \Phi_m^\dagger(z) \gamma_0 \gamma_k \tau^b \Phi_n(z) \right) \langle n | \tau^c | m \rangle \langle 1/2, J_3 T_3 | [\hat{J}_c, D_{ab}] | 1/2, J_3 T_3 \rangle,
$$

with a second regulator $\mathcal{R}_{M2}^{A}$:

$$
\mathcal{R}_{M2}^{A}(\epsilon_m, \epsilon_n) = \frac{1}{4\pi} \int_{0}^{1} \frac{d\beta}{\sqrt{(1-\beta)\beta}} \frac{\epsilon_m - \beta (\epsilon_n + \epsilon_m)}{(1-\beta)\epsilon_m^2 + \beta \epsilon_n^2} e^{-[(1-\beta)\epsilon_m^2 + \beta \epsilon_n^2]/\Lambda^2}.
$$

Summing all obtained contributions (79)-(84) and using eqs. (65) and (66), we obtain the final expression for the magnetic isovector form factor in the following form:

$$
G_T^{M} = i M N_c \sum_{n,m} \mathcal{R}_{M2}^{A}(\epsilon_m, \epsilon_n) \left( \Phi_m^\dagger(z) \gamma_0 \gamma_k \tau^b \Phi_n(z) \right) \langle n | \tau^c | m \rangle \langle 1/2, J_3 T_3 | [\hat{J}_c, D_{ab}] | 1/2, J_3 T_3 \rangle,
$$

The proton and neutron form factors are defined, respectively, as sum and difference of the isoscalar and isovector form factors:

$$
G_E,M^p = \frac{1}{2} \left[ G_T^{E,M} + G_T^{T=0} \right],
$$

$$
G_E,M^n = \frac{1}{2} \left[ G_T^{E,M} - G_T^{T=0} \right],
$$

and the numerical results for them are presented in Section 4.

4 Numerical results

To compute observables we use and numerical method of Ripka and Kahana [18] for solving the eigenvalue problem in finite quasi–discrete basis. We consider a spherical box of a large radius $D$ and the basis is made discrete by imposing a boundary condition at $r = D$. Also, it is made finite by restricting momenta of the basis states to be smaller than the numerical cut-off $K_{\text{max}}$.  

\[19\]
Both quantities have no physical meaning and the results should not depend on them. The typical values used are \( D \sim 20/M \) and \( K_{\text{max}} \sim 7M \).

In addition, all checks concerning the numerical stability of the solution with respect to varying box size and choice of the numerical cut-off have been done and the actual calculation is completely under control.

The actual calculations are done by fixing the parameters in meson sector in the well known way \([5]\) to have \( f_\pi = 93 \text{ MeV} \) and \( m_\pi = 139.6 \text{ MeV} \). This leaves the constituent quark mass \( M \) as the only free parameter.

The proton and neutron electric and magnetic form factors are displayed in Figs.2-5. The theoretical curves resulting from the model are given for the following four values the constituent quark mass: 370, 400, 420 and 450 MeV. The magnetic form factors are normalized to the experimental values of the corresponding magnetic moments at \( q^2 = 0 \). With one exception of the neutron electric form factor (Fig.3), all other form factors fit to the experimental data quite well. The best fit is for the constituent quark mass around 400–420 MeV.

As can be seen the only form factor which deviates from the experimental data is the neutron electric form factor and this requires some explanation. Obviously, this form factor is the most sensitive for numerical errors. According to the formula \([8]\) the form factor has been calculated as a difference of the electric isoscalar and electric isovector form factors that were the direct output of our code. Both form factors were of order 1.00 and calculated by the code with a high enough accuracy. However, the resulting neutron form factor has the proper (experimental) value of order 0.04, \( i.e. \) about 4\% of the value of its components. This means that a small numerical error for one of the components can be enhanced by a factor 50. Hence, small numerical errors together with the applied approximations (like the \( 1/N_c \) approximation behind the model and neglecting the boson–loop effects as well as higher order fermion loops) are strongly magnified resulting in a deviation from the experimental data for momentum transfers above 100 MeV/c.

As the next step, one can compute other electromagnetic observables: the mean squared radii, the magnetic moments and the nucleon–∆ splitting. The
static nucleon properties, in particular the charge radii and the magnetic moments can be obtained from the form factors:

\[
\langle r^2 \rangle_{T=0,1} = - \frac{6}{G^T_{E}} \frac{dG^T_{E}}{dq^2} \bigg|_{q^2=0},
\]

(89)

\[
\mu^T_{0,1} = G^T_{M} \bigg|_{q^2=0}.
\]

(90)

For the quark masses 370, 420 and 450 MeV and pion mass \( m_\pi = 140 \text{MeV} \) the calculated values are presented in Table 1. The values for the axial vector coupling constant \( g_A \), presented for completeness, are from ref. [12]. For comparison, in Table 2 we give the theoretical values of the same quantities but with the physical pion mass set to zero.

The chiral limit \( (m_\pi \to 0) \) mostly influences the isovector charge radius. Indeed, as it should be expected [13][14], in our calculations with zero pion mass the isovector charge radius diverges in chiral limit. It is illustrated in Fig. 6, where the isovector charge radius is plotted vs. the box size \( D \). The Dirac sea contribution to the radius grows linearly with \( D \) and diverges as \( D \to \infty \).

Because of this that quantity (and the relative quantities) is not included in Table 2. The other observable strongly influenced by the chiral limit is the neutron electric form factor. For the \( m_\pi \to 0 \) the discrepancy from the experiment is by almost a factor two larger than in the case \( m_\pi \neq 0 \). The other observables differ in the chiral limit by about 30\%. The comparison of the values in the two tables indicate that taking the physical pion mass gives us a best fit with a much better agreement with the experimental data. To be particular, in chiral limit \( (m_\pi = 0) \) the calculations for the electric observables suggest a high value \( (M \sim 450 \text{MeV}) \) while for the magnetic ones indicate \( M \sim 370 \text{MeV} \) which is not the case with physical pion mass. In addition, in chiral limit we observe much larger contribution from the sea effects, about 50\% of the total value.

| Table 2 |
| --- |

The results of Table 1 \( (m_\pi = 140 \text{MeV}) \) again indicate the value \( \sim 400–420 \text{MeV} \) for the constituent quark mass, in agreement with the conclusion drawn from the form factor curves. The same has been suggested earlier [8][11], where smaller number of observables has been taken into account. With the exception of the neutron electric squared radii, to which remarks similar to the case of the neutron electric form factor are applicable, the contribution of the valence quarks is dominant. However, the contribution of the Dirac sea is non-negligible, it lies within the range 15 – 40\%.  

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One can notice, that the numerical results for the nucleon–Δ mass splitting ($M_\Delta - M_N$), the mean squared proton, isoscalar and isovector electric radii and the axial coupling constant ($g_A$), as well as the proton electric and magnetic and neutron magnetic form factor, differ from the experimental data by no more then about ±5%. Finally, for the magnetic moments we have got results 20–25% below their experimental values. Despite of this underestimation of both magnetic moments we have found surprisingly good result for the ratio $\mu_p/\mu_n$ which is far better than in other models. The agreement with the experimental value is better than 3% for the constituent quark mass 420 $MeV$. So good result has been obtained neither in the Skyrme model nor in the linear sigma model.

In section 3 we have found that the magnetic isovector form factor is the only one which includes non-zero contributions in both leading ($N_c^0$) and next to leading order ($1/N_c$) in $\Omega$. However, the numerical calculations show that the ($1/N_c$) rotational corrections do not affect the $q^2$-dependence (slope) of the form factor but rather the value at the origin $(G_{M}^{T=1}(q^2 = 0) = \mu^{T=1})$ which is the isovector magnetic moment. It is not surprising since (as can be seen from eq.(86)) in both leading and next to leading order terms the shape of the wave functions $\Phi_n$ determines the $q^2$-dependence of the form factor whereas the value at $q^2 = 0$ depends on the particular matrix elements included. In leading order the isovector magnetic moment is strongly underestimated [9]. As has been shown in ref. [12] the enhancement for this quantity due to the $1/N_c$ corrections is of order $(N_c + 2)/N_c$ and it improves considerably the agreement with experiment. However, as can be seen from Table 1 it is still below the experimental value by 25%.

The isoscalar and isovector electric mean squared radii are shown in Figs.7,8 as functions of the constituent quark mass. The same plot but for the experimentally measured quantities: the proton and neutron electric charge radii, is given in Fig.9. In these plots the valence and sea contributions are explicitly given (dashed and dash-dotted lines, respectively). As could be expected, for the isoscalar electric charge radius the valence part is dominant (about 85%), due to the fact that there is no $1/N_c$ rotational contributions. This is not true for the isovector electric charge radius, where the sea part contributes to about 45% of the total value (Fig.7). Also, this effect can be seen from the proton and neutron charge radii (Fig.9) which are linear combinations of the isoscalar and isovector ones. For proton the sea contribution is about 30%. However, for the neutron charge radius the negative sea part is dominating and the valence contribution is negligible.
In addition, in Fig. 10 we plot the proton and neutron charge distribution for the constituent quark mass \( M = 420 \) MeV. For the proton we have a positive definite charge distribution completely dominated by the valence contribution whereas in the case of the neutron the Dirac sea is dominant. In accordance with the well accepted phenomenological picture one realizes a positive core coming from the valence quarks and a long negative tail due to the polarization of the Dirac sea. Using the gradient expansion the latter can be expressed in terms of the dynamical pion field – pion cloud.

For completeness, in Fig. 11 we present also the magnetic moment density distribution for proton and neutron for the constituent quark mass \( M = 420 \) MeV. The sea contribution becomes non-negligible for distances greater than 0.5 fm. Due to the relatively large tail contribution to the magnetic moments the sea contribution to these quantities is about 30%.

5. Summary and conclusions

Our numerical results support the view that the chiral quark soliton model of the Nambu–Jona-Lasinio type offers relatively simple but quite successful description of some low-energy QCD phenomena and, in particular, of the electromagnetic properties of nucleons. In parameter free calculations we have obtained overall good results for the electromagnetic form factors, the mean squared radii, the magnetic moments and the nucleon–\( \Delta \) splitting. Using \( f_\pi = 93 \) MeV and \( m_\pi = 139.6 \) MeV and a constituent quark mass of \( M = 420 \) MeV the isoscalar and isovector electric radii are reproduced within 15%. The magnetic moments are underestimated by about 25%, however, their ratio \( \mu_p/\mu_n \) is almost perfectly reproduced. The \( q \)-dependence of \( G_E^p(q^2) \), \( G_M^p(q^2) \) and \( G_M^n(q^2) \) is very well reproduced up to momentum transfer of 1 GeV. The neutron form factor \( G_E^n(q^2) \) is by a factor of two too large for \( q^2 > 100 \) MeV^2. One should note, however, that \( G_E^n(q^2) \) is more than an order of magnitude smaller than \( G_E^p(q^2) \) and as such it is extremely sensitive to both the model approximations and the numerical techniques used. Here, it turns out that the agreement is noticeably worse if the chiral limit \( (m_\pi \to 0) \) is used. This can be easily understood since the pion mass determines the asymptotic behaviour of the pion field. Altogether we conclude that the chiral quark soliton model based on a bosonized NJL type lagrangean is quite appropriate for the evaluation of nucleonic electromagnetic properties.
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Fig. 1. Diagrams corresponding to the expansion in $\Omega$ of the electromagnetic current matrix element: a) the valence contribution and b) the Dirac sea contribution.

Fig. 2. The proton electric form factor for the momentum transfers below 1 GeV.

Fig. 3. The neutron electric form factor for the momentum transfers below 1 GeV.

Fig. 4. The proton magnetic form factor normalized to the experimental value of the proton magnetic moment at $q^2 = 0$ for the momentum transfers below 1 GeV. The normalization factor can be extracted from Table 1.

Fig. 5. The neutron magnetic form factor normalized to the experimental value of the neutron magnetic moment at $q^2 = 0$ for the momentum transfers below 1 GeV. The normalization factor can be extracted from Table 1.

Fig. 6. The isovector charge radius as a function of the box size $D$ for $m_\pi = 0$ and $m_\pi = 139.6$ MeV.

Fig. 7. The isoscalar electric charge radius as a function of the constituent quark mass $M$. The valence and sea parts are marked by the dashed and dashed-dotted lines.

Fig. 8. The isovector electric charge radius as a function of the constituent quark mass $M$. The valence and sea parts are marked by the dashed and dashed-dotted lines.

Fig. 9. The electric charge radii of proton and neutron as functions of the constituent quark mass $M$. The valence and sea parts are marked by the dashed lines.

Fig. 10. The charge density distribution of the proton (lower) and neutron (upper) for the constituent quark mass $M = 420$ MeV.

Fig. 11. The magnetic moment density of proton and neutron for the constituent quark mass $M = 420$ MeV.
### Table 1
Nucleon observables calculated with the physical pion mass.

| Constituent Quark Mass | 370 MeV | 420 MeV | 450 MeV | Exper. |
|------------------------|---------|---------|---------|--------|
|                        | total   | sea     | total   | sea    | total   | sea    |
| $< r^2 >_{T=0}$ [fm$^2$] | 0.63    | 0.05    | 0.52    | 0.07   | 0.48    | 0.09   | 0.62   |
| $< r^2 >_{T=1}$ [fm$^2$] | 1.07    | 0.33    | 0.89    | 0.41   | 0.84    | 0.45   | 0.86   |
| $< r^2 >_p$ [fm$^2$]    | 0.85    | 0.19    | 0.70    | 0.24   | 0.66    | 0.27   | 0.74   |
| $< r^2 >_n$ [fm$^2$]    | -0.22   | -0.14   | -0.18   | -0.17  | -0.18   | -0.18  | -0.12  |
| $\mu_{T=0}$ [n.m.]     | 0.68    | 0.09    | 0.62    | 0.03   | 0.59    | 0.05   | 0.88   |
| $\mu_{T=1}$ [n.m.]     | 3.56    | 0.77    | 3.44    | 0.97   | 3.16    | 0.80   | 4.71   |
| $\mu_p$ [n.m.]         | 2.12    | 0.43    | 2.03    | 0.50   | 1.86    | 0.43   | 2.79   |
| $\mu_n$ [n.m.]         | -1.44   | -0.34   | -1.41   | -0.47  | -1.29   | -0.38  | -1.91  |
| $|\mu_p/\mu_n|$         | 1.47    | —       | 1.44    | —      | 1.44    | —      | 1.46   |
| $M_\Delta - M_N$ [MeV] | 213     | —       | 280     | —      | 314     | —      | 294    |
| $g_A$                  | 1.26    | 0.08    | 1.21    | 0.11   | 1.13    | 0.06   | 1.26   |

### Table 2
Nucleon observables calculated with the zero pion mass.

| Constituent Quark Mass | 370 MeV | 420 MeV | 450 MeV | Exper. |
|------------------------|---------|---------|---------|--------|
|                        | total   | sea     | total   | sea    | total   | sea    |
| $< r^2 >_{T=0}$ [fm$^2$] | 0.88    | 0.20    | 0.66    | 0.26   | 0.61    | 0.23   | 0.62   |
| $\mu_{T=0}$ [n.m.]     | 0.66    | 0.07    | 0.59    | 0.09   | 0.57    | 0.09   | 0.88   |
| $\mu_{T=1}$ [n.m.]     | 4.61    | 1.59    | 4.38    | 1.89   | 3.91    | 1.48   | 4.71   |
| $\mu_p$ [n.m.]         | 2.63    | 0.83    | 2.49    | 0.99   | 2.24    | 0.79   | 2.79   |
| $\mu_n$ [n.m.]         | -1.97   | -0.76   | -1.89   | -0.90  | -1.67   | -0.70  | -1.91  |
| $|\mu_p/\mu_n|$         | 1.34    | —       | 1.34    | —      | 1.34    | —      | 1.46   |
| $M_\Delta - M_N$ [MeV] | 221     | —       | 261     | —      | 301     | —      | 294    |