PGT ON PSL(2, Z). A SHORT PROOF

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Abstract. Taking the Iwaniec explicit formula as a starting point, we give a short proof of a more precise $\frac{5}{3}$ bound for the exponent in the error term of the Gallagher-type prime geodesic theorem for the modular surface.

1. Introduction

Let $\Gamma = PSL(2, \mathbb{Z})$ be the modular group and $Z_\Gamma$ its Selberg function defined by

$$Z_\Gamma(s) = \prod_{\{P_0\}} \prod_{k=0}^\infty (1 - N(P_0)^{-s-k}), \text{ Re}(s) > 1,$$

and meromorphically continued to the whole complex plane. The product is over hyperbolic conjugacy classes in $\Gamma$. The primitive conjugacy classes $P_0$ correspond to primitive closed geodesics on the modular surface $\Gamma \setminus \mathcal{H}$, where $\mathcal{H}$ is the upper half-plane equipped with the hyperbolic metric. The length of the primitive closed geodesic joining two fixed points, necessarily the same for all representatives of a class $P_0$, equals $\log(N(P_0))$. We are interested in distribution of these geodesics, i.e., in the number $\pi_\Gamma(x)$ of classes $P_0$ such that $N(P_0) \leq x$, for $x > 0$.

It is believed that the error term in the prime geodesic theorem

$$\pi_\Gamma(x) \approx \int_0^x \frac{dt}{\log t} \quad (x \to \infty)$$

is $O(x^{\frac{1}{2} + \varepsilon})$ since $Z_\Gamma$ satisfies the Riemann hypothesis.

Namely, there exists a broad analogy between the prime geodesic theorem and the prime number theorem based on the role played by the distribution of zeros of the Selberg zeta function resp. Riemann zeta function in two respective contexts.

However, due to the fact that $Z_\Gamma$ is a meromorphic function of order 2, as opposed to the Riemann zeta which is of order 1, the best presently available estimate for the exponent in the error term is $\frac{25}{36} + \varepsilon$, obtained by Soundararajan and Young [15]. The generalized Lindelöf hypothesis for Dirichlet $L$-functions would imply $\frac{2}{3} + \varepsilon$ (see [10], [15]).

Inspired by Gallagher’s approach [6] in the Riemann zeta setting, we give a short proof of the following theorem.

Theorem 1. Let $\Gamma = PSL(2, \mathbb{Z})$ be the modular group and $\varepsilon > 0$ arbitrarily small. There exists a set $A$ of finite logarithmic measure such that

$$\pi_\Gamma(x) = \int_0^x \frac{dt}{\log t} + O\left(x^{\frac{1}{2}} (\log \log x)^{\frac{1}{3} + \varepsilon}\right) \quad (x \to \infty, \ x \notin A).$$

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2. Preliminaries. The Iwaniec explicit formula and Gallagher’s lemma.

We shall make use of two lemmas that played important role in [11] and [6]. The first one is the Iwaniec explicit formula [11] with an error term for the Chebyshev function
\[
\psi_{\Gamma}(x) = \sum_{N(P) \leq x} \log N(P_0) = \sum_{N(P_0)^k \leq x} \log N(P_0).
\]

**Lemma A.** For \(1 \leq T \leq \frac{x}{(\log x)^2},\) one has
\[
\psi_{\Gamma}(x) = x + \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} + O \left( \frac{x}{T} (\log x)^2 \right),
\]
where \(\rho = \frac{1}{2} + i\gamma\) denote zeros of \(Z_{\Gamma}\).

The second one is Gallagher’s lemma [5] that enabled him to reduce the error term in the prime number theorem under the Riemann hypothesis.

**Lemma B.** Let \(A\) be a discrete subset of \(\mathbb{R}\) and \(\theta \in (0, 1)\). For any sequence \(c(\nu) \in \mathbb{C}, \nu \in A\), let the series
\[
S(u) = \sum_{\nu \in A} c(\nu) e^{2\pi i \nu u}
\]
be absolutely convergent. Then
\[
\int_{-U}^{U} |S(u)|^2 \, du \leq \left( \frac{\pi \theta}{\sin \pi \theta} \right)^2 \int_{-\infty}^{+\infty} \left| \frac{U}{\theta} \sum_{t \leq \nu \leq t + \frac{\theta}{4}} c(\nu) \right|^2 \, dt.
\]

3. Proof of Theorem

The goal is to find a proper bound for \(\sum_{|\gamma| \leq T} \frac{x^\rho}{\rho}\) in Lemma A. For \(x \in \left[ e^n, e^{n+1} \right)\), we have
\[
(1) \quad \int_{e^n}^{e^{n+1}} \left| \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} \right|^2 \, dx = \int_{e^n}^{e^{n+1}} \left| \sum_{|\gamma| \leq T} \frac{x^{\gamma}}{\rho} \right|^2 \, dx \ll e^{2n} \int_{e^n}^{e^{n+1}} \left| \sum_{|\gamma| \leq T} \frac{x^{\gamma}}{\rho} \right|^2 \, dx.
\]

Through the substitution \(x = e^n \cdot e^{2\pi u(T + \frac{1}{4})}\), the last integral is transformed into
\[
2\pi \int_{-\frac{1}{4\pi}}^{\frac{1}{4\pi}} \left| \sum_{|\gamma| \leq T} e^{(n+\frac{1}{2})i\gamma} \frac{1}{\rho} e^{2\pi i \gamma u} \right|^2 \, du.
\]

Lemma [B], with \(\theta = U = \frac{1}{4\pi}\) and \(c_\gamma = \frac{e^{(n+\frac{1}{2})i\gamma}}{\rho}\) for \(|\gamma| \leq T\), \(c_\gamma = 0\) otherwise, implies
\[
(2) \quad \int_{-\frac{1}{4\pi}}^{\frac{1}{4\pi}} \left| \sum_{|\gamma| \leq T} e^{(n+\frac{1}{2})i\gamma} \frac{1}{\rho} e^{2\pi i \gamma u} \right|^2 \, du \leq \left( \frac{1}{\sin \frac{\pi}{4}} \right)^2 \int_{-\infty}^{+\infty} \left( \sum_{t < |\rho| \leq t + 1} \frac{1}{|\rho|} \right)^2 \, dt.
\]
According to the Weyl law, \( \sum_{t<|\rho| \leq t+1} \frac{1}{|\rho|} = O(1) \). Thus,

(3) \[ \int_{-\infty}^{+\infty} \left( \sum_{t<|\rho| \leq t+1} \frac{1}{|\rho|} \right)^2 dt = O(T) . \]

Combining (1), (2) and (3), we get

(4) \[ e^{n+1} \int_{e^n}^{e^{n+1}} \left| \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} \right|^2 dx = O\left( e^{2nT} \right) . \]

Let \( A_n = \left\{ x \in [e^n, e^{n+1}) : \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} > x^\frac{T}{2} (\log x)^\frac{T}{2} (\log \log x)^\frac{T}{2} \right\} \). Its logarithmic measure is controlled by

\( \mu^* A_n = \int \frac{dx}{e^{2nT} n (\log n)^{1+3\varepsilon}} \left| \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} \right|^2 dx \ll \frac{e^{2nT}}{e^{2nT} n (\log n)^{1+3\varepsilon}} = \frac{1}{n (\log n)^{1+3\varepsilon}} . \)

Thus,

(5) \[ \left| \sum_{|\gamma| \leq T} \frac{x^\rho}{\rho} \right| \leq x^\frac{T}{2} (\log x)^\frac{T}{2} (\log \log x)^\frac{T}{2} \]

outside a set \( A = \cup A_n \) of finite logarithmic measure.

The optimal choice is \( T = \frac{x^{\frac{1}{2}} \log x}{(\log \log x)^{3+\varepsilon}} \). Then, Lemma \( A \) and the relation (5) yield

(6) \[ \psi_\Gamma (x) = x + O\left( x^\frac{T}{2} (\log x)^\frac{T}{2} (\log \log x)^\frac{T}{2} \right) \quad (x \to \infty, x \notin A) . \]

From (6), we obtain the assertion of Theorem \( \text{I} \) in a standard way, making use of the expressions

\[ \pi_\Gamma (x) = \int_{x^{\frac{1}{2}}}^{x} \frac{1}{\log t} d\theta_\Gamma (t) \quad \text{and} \quad \psi_\Gamma (x) = \sum_{n=1}^{\infty} \theta_\Gamma \left( x^{\frac{T}{2}} \right) , \]

where \( \theta_\Gamma (x) = \sum_{N(P_0) \leq x} \log N(P_0) . \)

**Remark 1.** In the case of cofinite Fuchsian groups \( \Gamma \subset PSL(2, \mathbb{R}) \), the best unconditional estimate of the remainder in the prime geodesic theorem is still Randol’s \( O\left( x^{\frac{T}{2}} (\log x)^{\frac{T}{2}} \right) \) (See \( \text{I} \) for a passage from Hejhal’s proof \( \text{II} \) Th. 6.19) of Huber’s bound \( O\left( x^{\frac{T}{2}} (\log x)^{\frac{T}{2}} \right) \) to Randol.) Its analogue is also valid in higher dimensions \( \text{III} \). Randol’s estimate can be reduced to \( \frac{T}{10} + \varepsilon \) outside a set of finite logarithmic
measure \cite{2}, what coincides with the Luo-Sarnak \cite{13} bound that unconditionally holds for $\Gamma = \text{PSL}(2, \mathbb{Z})$.

**Remark 2.** Using a more involved method of Soundararajan and Young, we were able to prove \cite{3} that under the assumption of the generalized Lindelöf hypothesis one has

$$\pi_\Gamma(x) = \int_0^x \frac{dt}{\log t} + O \left( x^{\frac{1}{2} + \varepsilon} \right)$$

as $x \to \infty$ outside a set of finite logarithmic measure.

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