Bloch oscillations in tight-binding lattices with defects

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Defects in tight-binding lattices generally destroy the onset of Bloch oscillations (BOs) and the periodic self-imaging of the wave packet due to the lack of an equally-spaced Wannier-Stark ladder spectrum. Here it is shown that localized and extended defects in the lattice can be engineered to be transparent for BOs. Such lattices are synthesized from the defect-free lattice by the technique of intertwining operators generally employed in supersymmetric quantum mechanics. The energy spectrum of the synthesized lattices differs from the Wannier-Stark ladder of the defect-free lattice because of the missing of pairs of resonances in the ladder, thus ensuring the persistence of BOs.

I. INTRODUCTION

Bloch oscillations (BOs) represent a fundamental coherent transport phenomenon originally predicted for quantum particles in periodic potentials driven by an external dc force. BOs have been observed in a wide variety of quantum and classical physical systems, including semiconductor superlattices [1], ultracold atoms [2], Bose-Einstein condensates [3], arrays of evanescently-coupled optical waveguides [4, 5], photonic superlattices [6], and acoustical superlattices [7]. In the absence of dephasing and scattering processes and for a negligible Zener tunneling, the particle accelerated by the external force undergoes an oscillatory (rather than a translational) motion owing to Bragg scattering off the lattice. The periodicity of the motion is basically ascribed to the transition of the energy spectrum from a continuous band with delocalized Bloch eigenstates in absence of the external force, to a discrete ladder spectrum with localized Wannier-Stark (WS) eigenstates when the external force is applied and Zener tunneling is negligible. A single-band tight-binding lattice in one spatial dimension provides the simplest model to describe the onset of BOs and the formation of WS ladder spectrum with localized Wannier-Stark eigenstates. As is well known, its spectrum is purely discrete and the allowed energy levels are given linearly with \( \kappa_n \), i.e. \( V_n = -F n \), where \( F \) is the potential gradient. The corresponding tight-binding Hamiltonian will be denoted in the following by \( \mathcal{H}_0 \) and referred to as the WS Hamiltonian. As is well known, its spectrum is purely discrete and the allowed energy levels are given by \( E_l = l F \), where \( l = 0, \pm 1, \pm 2, \ldots \). Our goal is to synthesize a lattice Hamiltonian, of the form given by Eq.(1), with an energy spectrum which differs from

In this work it is shown that localized and extended defects in the lattice can be engineered to be transparent for BOs. Such lattices are synthesized starting from the defect-free lattice Hamiltonian by application of the technique of intertwining operators, successfully employed in supersymmetric quantum mechanics [17, 18] to add or delete energy states of a given Hamiltonian. In our case, the spectrum of the synthesized lattices differ from the Wannier-Stark ladder of the defect-free lattice because of the missing of pairs of resonances in the ladder, which ensures the persistence of BOs and self-imaging of a wave packet. The paper is organized as follows. In Sec.II, the technique of intertwining operators for spectral engineering of tight-binding Hamiltonians is presented. In Sec.III, the technique is then applied to synthesize lattice models with defects that support BOs. Finally, Sec.IV outlines the main conclusions.

II. TIGHT-BINDING LATTICE ENGINEERING

Let us consider a one-dimensional tight-binding lattice described by the Hamiltonian

\[
\mathcal{H} = \sum_n \kappa_n (|n-1\rangle\langle n| + |n\rangle\langle n-1|) + \sum_n V_n |n\rangle\langle n| \tag{1}
\]

where \(|n\rangle\) is a Wannier state localized at site \( n \) of the lattice, \( \kappa_n \) is the hopping rate between sites \(|n-1\rangle \) and \(|n\rangle\), and \( V_n \) is the energy of Wannier state \(|n\rangle\) in presence of the external applied force. To ensure the Hermiticity of \( \mathcal{H} \), the hopping amplitudes \( \kappa_n \) and site energies \( V_n \) must assume real values. For a defect-free lattice and a homogenous force, the intersite coupling \( \kappa_n \) is independent of \( n \) (\( \kappa_n = \kappa \)) and the site energy \( V_n \) increases linearly with \( n \), i.e. \( V_n = -F n \), where \( F \) is the potential gradient. The corresponding tight-binding Hamiltonian will be denoted in the following by \( \mathcal{H}_0 \) and referred to as the WS Hamiltonian. As is well known, its spectrum is purely discrete and the allowed energy levels are given by \( E_l = l F \), where \( l = 0, \pm 1, \pm 2, \ldots \). Our goal is to synthesize a lattice Hamiltonian, of the form given by Eq.(1), with an energy spectrum which differs from

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that of the WS Hamiltonian $H_0$ because of the missing of some resonances in the WS ladder spectrum. This spectral engineering problem can be solved by the technique of intertwining operators generally employed in supersymmetric quantum mechanics [17], which is also valid for matrix Hamiltonians (see, for instance, [18] and references therein). The technique is briefly described in a rather general manner in this section.

Let us indicate by $H_1$ the tight-binding Hamiltonian defined by Eq.(1) with hopping amplitudes and site energies given by $\{\kappa_n, V_n^{(1)}\}$, and let us assume that $\kappa_n \to \kappa$ as $n \to \pm \infty$ and that $|V_n|$ is bounded or diverges with an algebraic law $|V_n| \sim n$ as $n \to \pm \infty$; note that such assumptions are satisfied for the WS Hamiltonian $H_0$. Let us indicate by $\mu_1, \mu_2, \mu_3, \ldots$ the point spectrum of $H_0$ and let $|\phi^{(1)}_n| = \sum_n |\phi^{(1)}_n| (n)$ be the (proper) eigenfunction of $H_1$ corresponding to the energy $\mu_1$, i.e. [19]

$$\kappa_n^{(1)} \phi^{(1)}_{n-1} + \kappa_{n+1}^{(1)} \phi^{(1)}_{n+1} + V_n^{(1)} \phi^{(1)}_n = \mu_1 \phi^{(1)}_n,$$

with $|\phi^{(1)}_n| \to 0$ for $n \to \pm \infty$. It can be readily shown that, provided that $|\phi^{(1)}_n|$ does not vanish, the following factorization for $H_1$ holds

$$H_1 = Q_1 R_1 + \mu_1$$

where

$$Q_1 = \sum_n \left( \epsilon_n^{(1)} |n\rangle \langle n| + \epsilon_{n-1}^{(1)} |n-1\rangle \langle n| \right)$$

and

$$R_1 = \sum_n \left( \epsilon_n^{(1)} |n\rangle \langle n| + \epsilon_{n+1}^{(1)} |n+1\rangle \langle n| \right)$$

Using Eqs.(4-9), from Eq.(10) it can be readily shown that $H_2$ describes the Hamiltonian of a tight-binding lattice [i.e., it is of the form (1)] with hopping amplitudes and site energies $\{\kappa_n^{(2)}, V_n^{(2)}\}$ given by

$$\kappa_n^{(2)} = \kappa_n^{(1)} \frac{r_{n-1}^{(1)}}{r_n^{(1)}}$$

$$V_n^{(2)} = V_n^{(1)} + \kappa_{n+1}^{(1)} \frac{\phi_{n+1}}{\phi_n} - \kappa_n^{(1)} \frac{\phi_{n-1}}{\phi_n}.$$  

An interesting property of the new Hamiltonian $H_2$ is that its energy spectrum is the same as that of $H_1$, apart from the lack of the discrete energy level $E = \mu_1$. In fact, let us indicate by $|\psi_E| = \sum_n v_n(E)|n\rangle$ a proper (or improper) eigenfunction of $H_1$ with energy $E$. Note that, if $E$ belongs to the point spectrum of $H_1$, $|\psi_n(E)| \to 0$ as $n \to \pm \infty$, whereas if $E$ belongs to the continuous spectrum of $H_1$, $|\psi_n(E)|$ remains bounded as $n \to \pm \infty$. Let us first assume that $E \neq \mu_1$. Using the factorization (3) for $H_1$, the eigenvalue equation $H_1|\psi_E| = E|\psi_E|$ reads explicitly

$$Q_1 R_1 |\psi_E| = (E - \mu_1)|\psi_E|$$

from which it follows that $R_1|\psi_E| \neq 0$ since $E \neq \mu_1$. Applying the operator $R_1$ to both sides of Eq.(13), one obtains

$$R_1 Q_1 |\tilde{\psi}_E| = (E - \mu_1)|\tilde{\psi}_E|,$$

i.e. $H_2|\tilde{\psi}_E| = E|\tilde{\psi}_E|$, where we have set $|\tilde{\psi}_E| = R_1 |\psi_E|$ or, explicitly [see Eq.(5)]

$$\tilde{\psi}_n(E) = v_n^{(1)} \psi_n(E) + v_{n+1}^{(1)} \psi_{n+1}(E).$$

Therefore, $|\tilde{\psi}_E|$ is an eigenfunction of $H_2$ corresponding to the energy $E$. Also, from Eqs.(6), (7), (15) and from the assumed asymptotic behavior of $\kappa_n$ and $V_n^{(1)}$, it follows that $|\tilde{\psi}_E|$ is a proper (improper) eigenfunction of $H_2$ in the same way as $|\psi_E|$ is a proper (improper) eigenfunction of $H_1$. In a similar way, one can show that any eigenvalue $E$ of $H_2$, belonging to the continuous or to the point spectrum (with $E \neq \mu_1$), is also an eigenvalue of $H_2$. Therefore the continuous and point spectra of $H_1$ and $H_2$ do coincide, apart from the $E = \mu_1$ eigenvalue which needs a separate analysis. For $E = \mu_1$, the (proper) eigenfunction of $H_1$ is by construction $|\phi^{(1)}_1\rangle$ [see Eq.(2)], which satisfies the condition $R_1 |\phi^{(1)}_1\rangle = 0$. On the other hand, from Eq.(10) it follows that the eigenvalue equation $H_2 |\tilde{\psi}_1| = \mu_1 |\tilde{\psi}_1|$ is satisfied for either $|\tilde{\psi}_1| = |\phi^{(1)}_1\rangle$ or $|\tilde{\psi}_1| = |\phi^{(2)}_1\rangle$, where $Q_1 |\phi^{(1)}_1\rangle = 0$ and $Q_1 |\phi^{(2)}_1\rangle = |\phi^{(1)}_1\rangle$.

The equation $Q_1 |\phi^{(1)}_1\rangle = 0$ reads explicitly

$$q_n^{(1)} \psi_n^{(1)} + q_{n+1}^{(1)} \psi_{n+1}^{(1)} = 0.$$  

Using the expressions of $q_n^{(1)}$ and $q_{n+1}^{(1)}$ given by Eqs.(6-9), the difference equation (16) for $\psi^{(1)}_n$ can be solved in a closed form, yielding

$$\psi^{(1)}_n = \frac{1}{\sqrt{\kappa_n^{(1)} \phi_n^{(1)} \phi_{n-1}^{(1)}}}.$$  

In view of the asymptotic behavior of $\phi_n^{(1)}$ and $\kappa_n$ as $n \to \pm \infty$, it turns out that $\psi^{(1)}_n$ is unbounded as $n \to \pm \infty$, i.e. it is not an eigenfunction (neither proper nor improper) of $H_2$. Similarly, as $\phi_n^{(1)} \to 0$ as $n \to \pm \infty$, the equation $Q_1 |\phi^{(2)}_1\rangle = |\phi^{(1)}_1\rangle$ for $|\phi^{(2)}_1\rangle$ reduces to Eq.(16) in
the asymptotic limit $n \to \pm \infty$, and thus also $\psi^{(2)}$ is unbounded at $n \to \pm \infty$. Therefore, none of the two linearly independent solutions $|\psi^{(1)}\rangle$ and $|\psi^{(2)}\rangle$ of the second-order difference equation $H_2|\psi\rangle = \mu_1|\psi\rangle$ are bounded, i.e. $\mu_1$ does not belong neither to the point spectrum nor to the continuous spectrum of $H_2$. The factorization method can be iterated to construct new Hamiltonians $\mathcal{H}_3$, $\mathcal{H}_4$, $\mathcal{H}_5$, ... whose energy spectra differ from that of $H_1$ owing to the missing of the discrete energy levels $\{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5, \}$. For instance, to construct the Hamiltonian $\mathcal{H}_3$, let $|\psi\rangle$ be the (proper) eigenfunction of $H_2$ corresponding to the energy $E = \mu_2$, and let us set $|\phi^{(2)}\rangle = \mathcal{R}_1|\psi\rangle$. From the previous analysis, it follows that $|\phi^{(2)}\rangle$ is the (proper) eigenfunction of $H_2$ corresponding to the energy $E = \mu_2$, i.e.

$$\kappa_n(2) \phi_n^{(2)} + \kappa_{n+1}^{(2)} \phi_{n+1}^{(2)} + V_n^{(2)} \phi_n^{(2)} = \mu_2 \phi_n^{(2)}.$$  

Let us then construct the new operators $\mathcal{R}_2$ and $\mathcal{Q}_2$, defined as in Eqs. (4-9) but with $\phi_n^{(1)}$ and $\kappa_n^{(1)}$ replaced by $\phi_n^{(2)}$ and $\kappa_n^{(2)}$, respectively. The factorization $H_3 = \mathcal{Q}_2 \mathcal{R}_2 + \mu_2$ then holds. Reversing the order of the $\mathcal{R}$ and $\mathcal{Q}$ operators, one obtains the new Hamiltonian $\mathcal{H}_3 = \mathcal{R}_2 \mathcal{Q}_2 + \mu_2$, which possesses the same energy spectrum of $H_0$, except for the missing of the two energy levels $E = \mu_1$ and $E = \mu_2$. The hopping amplitudes $\kappa_n^{(3)}$ and site energies $V_n^{(3)}$ of the lattice corresponding to the Hamiltonian $\mathcal{H}_3$ are given by Eqs. (11) and (12), with $\phi_n^{(1)}$, $\kappa_n^{(1)}$, and $V_n^{(1)}$ replaced by $\phi_n^{(2)}$, $\kappa_n^{(2)}$, and $V_n^{(2)}$, respectively. It should be noted that the technique of intertwining operators so far described could generate lattice Hamiltonians with complex-valued hopping rates $\kappa_n$ or site energies $V_n$; for instance, the hopping amplitudes of $H_3$ might become complex-valued when $\phi_n^{(1)}$ does not have a defined sign [see Eq. (6) and (11)]. This situation, corresponding to a non-Hermitian lattice with real-valued energy spectrum, will not be considered in this work. It should be nevertheless observed that iteration of the intertwining operator technique could finally restore the Hermiticity of the lattice Hamiltonian, in spite of some of the intermediate Hamiltonians are not self-adjoint. This is precisely the case of our interest that will be discussed in the next section.

### III. BLOCH OSCILLATIONS

Let us consider the WS Hamiltonian $\mathcal{H}_0$, defined by Eq. (1) with $\kappa_n = \kappa$ and $V_n = -Fn$. As is well known (see, e.g., [8]), $\mathcal{H}_0$ has a purely point spectrum (the WS ladder spectrum) with energies [see Fig. 1(a)]

$$E_l = lF$$  

and corresponding localized eigenstates

$$|\psi^{(l)}\rangle = \sum_{n} J_{n+l}(\gamma)|n\rangle,$$

where $\gamma = 2\kappa/F$ and $l = 0, \pm 1, \pm 2, \ldots$ is the quantum number $\mathcal{R}$. Note that the WS state with quantum number $l$ is localized at around the lattice site $n = -l$. Owing to the equal spacing of WS modes, the temporal evolution of any wave packet is periodic and self-imaging is attained at times multiples of the Bloch period $T_B = 2\pi/F$. This effect is clearly visible by observing the breathing or oscillatory modes $\mathcal{R}$ corresponding to either an initial single-site or broad-site excitation of the lattice, as shown in Figs. 1(b) and (c). The introduction of some defects in the lattice, in either the hopping amplitudes $\kappa_n$ or site energies $V_n$, generally breaks the self-imaging property of the lattice [see, as an example, the simulations shown in Figs. 1(d) and (e)]. In this section we aim to construct tight-binding lattices with defects in which the self-imaging phenomenon of the WS (defect-free) Hamiltonian $\mathcal{H}_0$ is maintained. This goal can be achieved by the application of the intertwining operator technique described in the previous section assuming as the starting Hamiltonian $\mathcal{H}_1$ the WS Hamiltonian $\mathcal{H}_0$, i.e. $\mathcal{H}_1 = \mathcal{H}_0$. The main idea is that any new Hamiltonian $\mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4, \ldots$ obtained from $\mathcal{H}_0$ by successive application of the
The intertwining operator technique has a spectrum which differs from that of $\mathcal{H}_0$ by the missing of some of the WS resonances, and thus a periodic temporal dynamics of the wave packet is maintained with the same period $T_B$ of the original WS Hamiltonian. As a first step, let us construct the Hamiltonian $\mathcal{H}_2$ by removing from the spectrum of the WS Hamiltonian $\mathcal{H}_0$ one WS resonance, for instance the one with energy $\mu_1 = 0$ corresponding to the quantum number $l = 0$ in Eq.(19). According to Eqs.(11) and (12) with $\kappa_n^{(1)} = \kappa$, $\phi_n^{(1)} = -F_n$ and $\phi_n^{(1)} = J_n(\gamma)$, the hopping amplitudes and site energies of the lattice associated to $\mathcal{H}_2$ read explicitly

$$
\kappa_n^{(2)} = \kappa \sqrt{\frac{J_n(\gamma)J_{n-2}(\gamma)}{J_n^2(\gamma)}} \tag{21}
$$

$$
V_n^{(2)} = -F_n + \kappa \left( \frac{J_{n+1}(\gamma)}{J_n(\gamma)} - \kappa \frac{J_n(\gamma)}{J_{n-1}(\gamma)} \right) \tag{22}
$$

A typical behavior of $|\kappa_n^{(2)}|^2$ and $\Delta V_n^{(2)} \equiv V_n^{(2)} - V_n^{(1)} = V_n^{(2)} + F_n$, as predicted by Eqs.(21) and (22), is shown in Fig.2. Note that $\kappa_n^{(2)} \to \kappa$ and $\Delta V_n^{(2)}$ settles down to constant values as $n \to \pm \infty$, so that far from the defect near $n = 0$ the lattice described by $\mathcal{H}_2$ behaves like a defect-free WS lattice. Note also that, owing to the asymptotic behavior of Bessel functions $J_n$ at large indices, one has $\Delta V_n^{(2)} \to 0$ as $n \to +\infty$ but $\Delta V_n^{(2)} \to 2\kappa/\gamma = F$ as $n \to -\infty$ [see Fig.2(b)]. Unfortunately, due to the oscillating behavior of Bessel functions $J_n(\gamma)$, the hopping rates $\kappa_n^{(2)}$ can become complex-valued and the Hamiltonian $\mathcal{H}_2$, correspondingly, ceases to be non-Hermitian. This is clearly shown in Fig.2(a), where $|\kappa_n^{(2)}|^2$ becomes negative at a few lattice sites near $n = 0$. This circumstance indicates that, for such indices, the hopping rates are purely imaginary. In spite of the non-Hermiticity of $\mathcal{H}_2$, its spectrum remains real-valued and BOs with the same period $T_B$ as that of the WS Hamiltonian are found [20]. Fortunately, a second application of the intertwining operator technique, assuming as a second energy level $\mu_2$ of $\mathcal{H}_0$ one of the two WS resonances adjacent to $\mu_1$, i.e. $\mu_2 = \pm F$ [corresponding to the quantum number $l = \pm 1$ in Eq.(19)], the Hermiticity of $\mathcal{H}_3$ is restored.

As an example, Fig.3 shows the behaviors of $|\kappa_n^{(3)}|^2$ and $\Delta V_n^{(3)} \equiv V_n^{(3)} - V_n^{(1)}$ corresponding to $\mu_2 = -F$ and for two values of $\gamma$. Note that, for $n \to +\infty$, $\kappa_n^{(3)} \to 1$ and $\Delta V_n^{(3)}$ settles down to constant values $|\Delta V_n^{(3)}| = 0$ for $n \to +\infty$, $|\Delta V_n^{(3)}| \to 2F$ for $n \to -\infty$, so that the synthesized lattice described by $\mathcal{H}_3$ behaves like the WS (defect-free) lattice for site indices far from $n = 0$, i.e. far from the lattice sites of WS localized states removed by the intertwining operator technique. As $F$ is decreased, i.e. the localization length of the WS states is increased [see (20)], the localization length of the defect in the lattice is increased, as one can see by comparing Figs.3(a),(b) with Figs.3(c),(d). The persistence of BOs in such synthesized lattices, corresponding to either single-site excitation at $t = 0$ or to a multiple-site excitation with a broad Gaussian wave packet, is demonstrated in Figs.3(e-h). In the figures, the numerically-computed evolutions of $|c_n(t)|^2$ versus $t$, as obtained by solving the Schrödinger equation for the Hamiltonian $\mathcal{H}_3$ with the initial condition $c_n(0) = \delta_{n,0}$ (for the breathing BO eigenvalues, Figs.3(e) and (g)) and $c_n(0) = \exp[-(n-1)^2/64]$ (for the oscillatory BO modes, Figs.3(f) and (h)) are depicted for two values of $F$.

Tight-binding lattices with engineered hopping rates and energy sites corresponding to the ones shown in Fig.3 could be realized using arrays of evanescently-coupled optical waveguides with engineered size and distance, in which the distances between adjacent waveguides control the hopping rates $\kappa_n$ whereas the channel widths (or refractive index changes) of the guides set the values of the site energies $E_n$ (see, for instance, [4, 16, 21] and references therein).

The technique of intertwining operators can be iterated by removing additional resonances from the WS ladder of $\mathcal{H}_0$. Extended numerical simulations show that the resulting Hamiltonians turn out to be Hermitian provided that couples of adjacent WS resonances are removed, whereas in the other cases the hopping rates can become imaginary at some lattice sites. As a general rule, the removal of a new WS resonance couple at energies $E = \pm lF, (l\pm 1)F$ introduces in the lattice new defects localized at around the lattice sites $n = l$. For the Hamiltonian $\mathcal{H}_{2s+1}$ obtained by removing from $\mathcal{H}_0$ $s$ couples of adjacent WS resonances, one has $\kappa_n^{(s+1)} \to 1$ for $n \to \pm \infty$, $\Delta V_n^{(s+1)} \to 0$ for $n \to +\infty$, and $\Delta V_n^{(s+1)} \to 2sF$ for $n \to -\infty$. As an example, Fig.4 shows the behavior of hopping amplitudes and site energy offsets for a synthesized lattice obtained by removal of the four WS resonances with energies $E = 0, -F, -25F, -26F$. As the number of removed WS resonances (and hence of defects in the lattice) increases, the behaviors of hopping amplitudes $\kappa_n$ and site energies $V_n$ become highly irregular, as shown in the example of Fig.5. Here the Hamiltonian

![FIG. 2: (color online) Behavior of (a) the square of hopping amplitudes $|\kappa_n^{(2)}|^2$, and (b) of site energy offset $\Delta V_n^{(2)} = V_n^{(2)} + F_n$ for the Hamiltonian $\mathcal{H}_2$ synthesized from the WS Hamiltonian $\mathcal{H}_0$ by removal of the WS resonance $\mu_1 = 0$ (see the inset). Parameter values are $\kappa = 1$ and $F = 0.6$.](image-url)
$\mathcal{H}_{13}$ is synthesized by successive application of the intertwining operator technique that removes from the WS ladder spectrum the 12 resonances $E = 42F$, 41$F$, 26$F$, 25$F$, 11$F$, 10$F$, 0, $-F$, $-11F$, $-12F$, $-30F$, $-31F$. It is remarkable that, in such a rather irregular lattice with extended defects, BOs still persists and exact self-imaging is attained, as shown in Figs.5(c) and (d).

As a final comment, it should be noted that the hopping rates and site energies of the synthesized lattices described by the Hermitian Hamiltonians $\mathcal{H}_3$, $\mathcal{H}_5$, $\mathcal{H}_7$, ... depend on $\gamma$, i.e. on the amplitude of the forcing $F$ entering in $\mathcal{H}_0$. Therefore, a change of the forcing parameter $F$ in the original WS Hamiltonian $\mathcal{H}_0$ gives different lattice realizations for $\mathcal{H}_3$, $\mathcal{H}_5$, $\mathcal{H}_7$, ... (see, for instance, Fig.3). This means that, as an additional term $H_p = -fn|n\rangle$ to $\mathcal{H}_0$ simply changes the BO period (just because the external forcing is changed from $F$ to $F + f$), the addition of $H_p = -fn|n\rangle$ to $\mathcal{H}_3$, $\mathcal{H}_5$, $\mathcal{H}_7$, ... destroys the onset of BOs. This is shown in Fig.6(a), where the temporal evolution of the site occupation probabilities $|c_n(t)|^2$ is shown for the Hamiltonian $\mathcal{H}_{13}$ of Fig.5, with an added perturbation term $H_p = -fn|n\rangle$ for $F = 0.2$ and $F = -0.1$. For comparison, the evolution of site occupation probabilities for the lattice with the WS Hamiltonian $\mathcal{H}_0$ perturbed with $H_p = -fn|n\rangle$ (for the same values of $F$ and $f$) is depicted in Fig.6(b). Therefore, while in a WS ladder Hamiltonian BOs are observed for any value of forcing $F$ (a change of $F$ corresponds to a change of the BO period), in the synthesized lattices $\mathcal{H}_3$, $\mathcal{H}_5$, $\mathcal{H}_7$, ... a change of the forcing is detrimental for the onset of BOs. Yet, it is remarkable that at a fixed forcing strength BOs can be observed in tight-binding lattices with defects, and even in greatly irregular lattices (like the one shown in Fig.5).

IV. CONCLUSIONS

In conclusion, in this work it has been shown that single and multiple defects in a tight-binding Wannier-Stark lattice can be introduced such that BOs and the self-imaging property of the WS lattice are not destroyed. Such lattices are synthesized from the defect-free lattice by the technique of intertwining operators generally employed in supersymmetric quantum mechanics to engineer the spectrum of Hermitian Hamiltonians. The energy spectrum of the synthesized lattices differs from the Wannier-Stark ladder of the defect-free lattice because of the missing of pairs of resonances in the ladder, thus ensuring the persistence of BOs. It is envisaged that the lattice engineering technique proposed in this work could be extended to other coherent dynamical regimes, such as dynamic localization in presence of an ac (time-periodic) force [22]. It is also envisaged that the possibility to realize non-Hermitian tight-binding lattices with real-valued energies—mentioned in this work—would deserve further investigation and could stimulate novel studies in the framework of the rapidly developing field of non-Hermitian quantum mechanics [21, 22].

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FIG. 3: (color online) (a-d) Behavior of the square of hopping amplitudes $|\kappa_n^{(3)}|^2$ and of site energy offset $\Delta V_n^{(3)} = V_n^{(3)} + Fn$ for the Hamiltonian $H_3$ synthesized from the WS Hamiltonian $H_0$ by removal of the WS resonances $\mu_1 = 0$ and $\mu_2 = -F$ (see the inset at the top of the figure). Parameter values are $\kappa = 1$ and $F = 0.6$ in (a) and (b), and $\kappa = 1$ and $F = 0.2$ in (c) and (d). (e) and (f): persistent BOs (breathing and oscillatory modes) in the synthesized lattice corresponding to $F = 0.6$. (g) and (h): Same as (e) and (f), but for the synthesized lattice with $F = 0.2$. 
FIG. 4: (color online) Same as Fig.2, but for the Hamiltonian $H_5$ synthesized from $H_0$ by removing the four WS resonances $E = 0, -F, -25F, -26F$. Parameter values are $\kappa = 1$ and $F = 0.6$.

FIG. 5: (color online) (a) and (b) Same as Fig.4, but for the Hamiltonian $H_{13}$ synthesized from $H_0$ by removing twelve WS resonances (parameter values are $\kappa = 1$ and $F = 0.2$). In (c) and (d), the persistence of breathing and oscillatory BO modes, corresponding to either single-site excitation $c_n(0) = \delta_{n,0}$ [(c)] and to a broad Gaussian wave packet excitation $c_n(0) = \exp[-(n - 10)^2/64]$ [(d)], are demonstrated.

FIG. 6: (color online) (a) Destruction of BOs (breathing modes) for the lattice Hamiltonian $H_{13}$ of Fig.5 due to the addition of the perturbation $H_p = -fn|n\rangle$. Parameter values are $\kappa = 1$, $F = 0.2$ and $f = -0.1$. In (b) the BOs with varied period, corresponding to the addition of $H_p$ to the WS lattice Hamiltonian $H_0$, are shown for comparison.