FINITE-DIMENSIONALITY OF THE
SPACE OF CONFORMAL BLOCKS

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Abstract. Without using Gabber’s theorem, the finite-dimensionality of the space
of conformal blocks in the Wess-Zumino-Novikov-Witten models is proved.

§0 Introduction.

Conformal field theory with non-abelian gauge symmetry, called the Wess-Zumino-
Novikov-Witten (WZNW) model, has been studied by many physicists and math-
ematicians. A mathematical formulation of this model over the projective line is
given in [TK], and it is generalized in [TUY] over algebraic curves of arbitrary
genus. In (chiral) conformal field theory, the main objects are $N$-point functions,
and in [TUY] they are regarded as sections of a certain vector bundle over the
moduli space of $N$-pointed stable curves. The fiber of this bundle at a stable curve
$X$ is called the space of conformal blocks (or the space of vacua) attached to $X$.
The finite-dimensionality of this space is essential to the mathematical treatment
of conformal field theory, and is proved in [TUY] as a consequence of Gabber’s
theorem [Ga] stating the involutivity of characteristic varieties.

The aim of the present paper is to give a proof of the finite-dimensionality
without using Gabber’s theorem.

In §1 we summarize some basic facts on affine Lie algebras following [Ka]. In §2
and §3, we recall the definition of pointed stable curves and the space of conformal
blocks. The essence in our proof is explained in §4 and the complete proof is given
in §5.

§1. Integrable highest weight modules of Affine Lie algebras.

By $\mathbb{C}[t]$ and $\mathbb{C}(t)$, we mean the ring of formal power series in $t$ and the field
of formal Laurent series in $t$, respectively. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, $\mathfrak{h}$
its Cartan subalgebra and $\mathfrak{h}^*$ the dual space of $\mathfrak{h}$ over $\mathbb{C}$. By $\Delta$, we denote the root
system of $(\mathfrak{g}, \mathfrak{h})$, and for $\alpha \in \Delta$ we denote the root vector corresponding to $\alpha$ by
$X_\alpha$. Let

$$(\ , ) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$$

be the Cartan-Killing form normalised by the condition

$$(H_\theta, H_\theta) = 2,$$
where $\theta$ is the maximal root of $\mathfrak{g}$ and $H_\theta$ is the element of $\mathfrak{h}$ defined by

$$\theta(H) = (H_\theta, H)$$

for all $H \in \mathfrak{h}$.

The affine Lie algebra $\hat{\mathfrak{g}}$ associated with $\mathfrak{g}$ is defined by

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[[t]] \oplus \mathbb{C}c,$$

where $c$ is an central element of $\hat{\mathfrak{g}}$ and the Lie algebra structure is given by

$$[X \otimes f(t), Y \otimes g(t)] = [X, Y] \otimes f(t)g(t) + c \cdot (X, Y) \operatorname{Res}_{t=0} (g(t) \cdot df(t)),$$

for $X, Y \in \mathfrak{g}$, $f(t), g(t) \in \mathbb{C}((\xi))$. We define the subalgebras $\hat{\mathfrak{g}}_+, \hat{\mathfrak{g}}_-$ of $\hat{\mathfrak{g}}$ by

$$\hat{\mathfrak{g}}_+ = \mathfrak{g} \otimes \mathbb{C}[t[t], \hat{\mathfrak{g}}_- = \mathfrak{g} \otimes \mathbb{C}[t^{-1}]t^{-1}.$$

Let $P_+$ be the set of all dominant integral weights of $\mathfrak{g}$ and for a fixed positive integer $\ell$ (called the level) put

$$P_\ell = \{ \lambda \in P_+ \mid 0 \leq \lambda(H_\theta) \leq \ell \}.$$

**Proposition 1.1.** For each $\lambda \in P_\ell$ there exists a unique irreducible left $\hat{\mathfrak{g}}$-module $\mathcal{H}_\lambda$ (called the integrable highest weight $\hat{\mathfrak{g}}$-module) satisfying the following properties:

1. $V_\lambda := \{ |v\rangle \in \mathcal{H}_\lambda \mid \hat{\mathfrak{g}}_+ |v\rangle = 0 \}$ is the irreducible left $\mathfrak{g}$-module with highest weight $\lambda$.
2. The central element $c$ acts on $\mathcal{H}_\lambda$ as $\ell \cdot \text{id}$.
3. $\mathcal{H}_\lambda$ is generated by $V_\lambda$ over $\hat{\mathfrak{g}}_-$ with only one relation

$$|X_\theta \otimes t^{-1}|^{\ell-(\theta, \lambda)+1} |\lambda\rangle = 0,$$

where $|\lambda\rangle$ is the highest weight vector.

Similarly we define the integrable highest weight right $\hat{\mathfrak{g}}$-module $\mathcal{H}^\dagger_\lambda$. There is a perfect bilinear pairing

$$\langle , \rangle : \mathcal{H}^\dagger_\lambda \times \mathcal{H}_\lambda \to \mathbb{C},$$

which is $\hat{\mathfrak{g}}$-invariant:

$$\langle u|av \rangle = \langle ua|v \rangle$$

for all $|u\rangle \in \mathcal{H}^\dagger_\lambda$, $|v\rangle \in \mathcal{H}_\lambda$ and $a \in \hat{\mathfrak{g}}$.

§2. Pointed Stable Curves.

**Definition 2.1.** A set of data $\mathfrak{X} = (C; Q_1, Q_2, \ldots, Q_N)$ is called an $N$-pointed stable curve of genus $g$, if the following conditions are satisfied:

1. $C$ is a semi-stable curve of genus $g$.
2. $Q_1, Q_2, \ldots, Q_N$ are non-singular points of the curve $C$.
3. The $N$-pointed curve $\mathfrak{X}$ has no infinitesimal automorphisms.

In the following we also assume the following condition (*):

(*) Each irreducible component of $C$ contains at least one $Q_j$. 

For a curve $C$ and a non-singular point $Q$ on $C$, an isomorphism

$$s : \hat{O}_{C,Q} \xrightarrow{\sim} \mathbb{C}[[t]]$$

is called a formal neighborhood of $C$ at $Q$. Here $\hat{O}_{C,Q}$ is the ring of formal power series at $Q$.

**Definition 2.2.** A set of data $\mathcal{X} = (C; Q_1, \ldots, Q_N; s_1, \ldots, s_N)$ is called an $N$-pointed stable curve of genus $g$ with formal neighborhoods if the following conditions are satisfied:

1. $(C; Q_1, \ldots, Q_N)$ is an $N$-pointed stable curve of genus $g$.
2. $s_j$ is a formal neighborhood at $Q_j$.

Introducing a parameter $\xi_j$ for each $j$, we regard $s_j$ as an isomorphism

$$s_j : \hat{O}_{C,Q_j} \xrightarrow{\sim} \mathbb{C}[[\xi_j]].$$

**Lemma 2.3.** Let $\mathcal{X} = (C; Q_1, \ldots, Q_N; s_1, \ldots, s_N)$ be an $N$-pointed stable curve of genus $g$ with formal neighborhoods which satisfies the condition $(\ast)$. Then $s_1 \ldots s_N$ induce the following injective homomorphism:

$$s : H^0(C, \mathcal{O}_C(\ast \sum_{j=1}^{N} Q_j)) \rightarrow \bigoplus_{j=1}^{N} \mathbb{C}((\xi_j)).$$

§3. The space of conformal blocks attached to $\mathcal{X}$.

We define a Lie algebra

$$\hat{g}_N = \bigoplus_{j=1}^{N} (g \otimes \mathbb{C}((\xi_j))) \oplus Cc$$

with the following commutation relations:

$$[\bigoplus_{j=1}^{N} X_j \otimes f_j, \bigoplus_{j=1}^{N} Y_j \otimes g_j] =$$

$$\bigoplus_{j=1}^{N} [X_j, Y_j] \otimes f_j g_j + \sum_{j=1}^{N} (X_j, Y_j)_{\xi_j=0} \text{Res}(g_j \cdot df_j) \cdot c,$$

$$c \in \text{center}.$$

We also put

$$\hat{g}(\mathcal{X}) = g \otimes \mathcal{C} H^0(C, \mathcal{O}_C(\ast \sum_{j=1}^{N} Q_j))$$

and regard it as a subspace of $\hat{g}_N$ by the mapping $s$ given in Lemma 2.3.

Then by the residue theorem we have the following lemma.
Lemma 3.1. The space $\hat{g}(X)$ is a Lie subalgebra of $\hat{g}_N$. 

For each $\vec{\lambda} = (\lambda_1, \ldots, \lambda_N) \in (P_{\ell \circ})^N$ a left $\hat{g}_N$-module $\mathcal{H}_{\vec{\lambda}}$ are defined by
\[ \mathcal{H}_{\vec{\lambda}} = \mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_N} . \]

Similarly a right $\hat{g}_N$-module $\mathcal{H}_{\vec{\lambda}}^\dagger$ is defined by
\[ \mathcal{H}_{\vec{\lambda}}^\dagger = \mathcal{H}_{\lambda_1}^\dagger \otimes \cdots \otimes \mathcal{H}_{\lambda_N}^\dagger . \]

We use the notation
\[ | u_1 \otimes \cdots \otimes u_N \rangle = | u_1 \rangle \otimes \cdots \otimes | u_N \rangle \]
for $| u_j \rangle \in \mathcal{H}_{\lambda_j}$ ($j = 1, \ldots, N$). The $\hat{g}_N$-action on $\mathcal{H}_{\vec{\lambda}}$ is given by
\[ (\oplus_{j=1}^N X_j \otimes f_j) | u_1 \otimes \cdots \otimes u_N \rangle = \sum_{j=1}^N | u_1 \otimes \cdots \otimes u_{j-1} \otimes (X_j \otimes f_j) u_j \otimes u_{j+1} \otimes \cdots \otimes u_N \rangle . \]

The right action on $\mathcal{H}_{\vec{\lambda}}^\dagger$ is defined similarly. There is a $\hat{g}_N$-invariant perfect bilinear pairing
\[ \langle \langle \cdot \rangle \rangle : \mathcal{H}_{\vec{\lambda}}^\dagger \times \mathcal{H}_{\vec{\lambda}} \to \mathbb{C} . \]

Definition 3.2. Put
\[ V_{\vec{\lambda}}(X) = \mathcal{H}_{\vec{\lambda}} / \hat{g}(X) \mathcal{H}_{\vec{\lambda}} , \]
\[ V_{\vec{\lambda}}^\dagger(\mathcal{X}) = \{ \langle \Psi | \in \mathcal{H}_{\vec{\lambda}}^\dagger ; \langle \Psi | \hat{g}(X) = 0 \} \]
\[ \cong \text{Hom}_C(V_{\vec{\lambda}}(X), \mathbb{C}) . \]

We call $V_{\vec{\lambda}}^\dagger(X)$ the space of conformal blocks (or the space of vacua) attached to $\mathcal{X}$.

Theorem 3.3. The spaces $V_{\vec{\lambda}}(X)$ and $V_{\vec{\lambda}}^\dagger(X)$ are finite dimensional vector spaces. 

The above theorem is fundamental to the formulation of the WZNW-models over algebraic curves and proved in [TUY]. We will give an alternative proof of Theorem 3.3 in §5.

§4. Main idea.

The main idea in our proof of Theorem 3.3 is to substitute the equality (4.1) for Gabber’s theorem. We explain this point in the following without details. For simplicity, we consider the 1-point case, and let $\mathcal{X}$ be a 1-pointed stable curve and $\lambda$ be a weight. We want to show the finite-dimensionality of the space
\[ V_\lambda(\mathcal{X}) = \mathcal{H}_\lambda / \hat{g}(\mathcal{X}) \mathcal{H}_\lambda . \]

First, we introduce the filtration $\{ \mathcal{E}_\bullet \}$ on $\mathcal{H}_\lambda$ by
\[ \mathcal{E}_m \mathcal{H}_\lambda = \{ 0 \} \ (m < 0), \mathcal{E}_0 \mathcal{H}_\lambda = V_\lambda , \]
\[ \mathcal{E}_m \mathcal{H}_\lambda = \mathcal{E}_{m-1} \mathcal{H}_\lambda + U(\hat{g}) \mathcal{E}_{m-1} \mathcal{H}_\lambda \ (m > 0) , \]
and introduce the induced filtration on $\mathcal{V}_\lambda(\mathfrak{X})$.

Then what we must show is the following:

(i) $\dim_{\mathbb{C}} E_m \mathcal{V}(\mathfrak{X}) < \infty$ for any $m \in \mathbb{Z}$.
(ii) $\dim_{\mathbb{C}} E_m \mathcal{V}(\mathfrak{X}) = \dim_{\mathbb{C}} E_{m+1} \mathcal{V}(\mathfrak{X})$ for sufficiently large $m \in \mathbb{Z}$.

As we shall see in the next section, (i) follows from the Riemann-Roch theorem and the proof of (ii), which is more crucial, is reduced to the following:

(ii)' For each $X \in \mathfrak{g}$ and $n \in \mathbb{Z}$, there exists an integer $k$ such that

$$(X \otimes \xi^n)^k|u) \equiv 0 \mod \hat{\mathfrak{g}}(\mathfrak{X}) \mathcal{H}_\lambda + E_{m-1} \mathcal{H}_\lambda \text{ for any } m \in \mathbb{Z} \text{ and } |u) \in E_{m-k} \mathcal{H}_\lambda.$$ 

For a root vector $X_\alpha (\alpha \in \Delta)$, we can easily show (ii)' by the nilpotency of $X_\alpha \otimes \xi^n$ on $V_\lambda$, but for a Cartan element $H_\alpha = [X_\alpha, X_{-\alpha}]$ we need some trick. We put $H = H_\alpha, E = X_\alpha, F = X_{-\alpha}$. Then a simple calculation implies the following equality

$$((H \otimes \xi^n)^s(F \otimes \xi^n)^t|u) \equiv$$

$$\frac{1}{t+1} \{ 2(s-1)(H \otimes \xi^n)^{s-2}(F \otimes \xi^n)^{t+1}E \otimes \xi^n|u) + (H \otimes \xi^n)^{s-1}(F \otimes \xi^n)^{t+1}E|u) \} \mod \hat{\mathfrak{g}}(\mathfrak{X}) \mathcal{H}_\lambda + E_{m-1} \mathcal{H}_\lambda$$

for any $m \in \mathbb{Z}$ and $|u) \in E_{m-s-1} \mathcal{H}_\lambda$, where $s$ and $t$ are integers such that $s \geq 1, t \geq 0$. By this formula we can increase the number $t$ by decreasing the number $s$, and hence (ii)' for $X = H$ follows from the nilpotency of $F \otimes \xi^n$.

§5. Proof of Theorem 3.3.

We consider the following Lie subalgebras

$$\hat{\mathfrak{g}}_N^M = \bigoplus_{j=1}^N \mathfrak{g} \otimes \mathbb{C}[\xi_j^{-1}]|\xi_j^{-M} \quad M = 0, 1, \ldots ,$$

$$\mathfrak{g}[1] = \{ \bigoplus_{j=1}^N X \otimes \xi_j^0 \mid X \in \mathfrak{g} \}$$

of $\hat{\mathfrak{g}}_N$. Note that for a positive integer $M$, the direct sum of $\hat{\mathfrak{g}}_N^M$ and $\mathfrak{g}[1]$ is again a Lie subalgebra of $\hat{\mathfrak{g}}_N$. Put

$$\mathcal{W}_M = \mathcal{H}_\lambda / (\mathfrak{g}_N^M \oplus \mathfrak{g}[1]) \mathcal{H}_\lambda.$$ 

We first prove the following lemma.

Lemma 5.1. There exists a positive integer $M$ such that the finite-dimensionality of $\mathcal{W}_M$ implies that of $\mathcal{V}_\lambda(\mathfrak{X})$.

Proof. We introduce a filtration $\{ \mathcal{F}_p \}$ on $\hat{\mathfrak{g}}_N$ as follows.

$$\mathcal{F}_p \hat{\mathfrak{g}}_N = \bigg\{ \bigoplus_{j=1}^N \mathfrak{g} \otimes \mathbb{C}[\xi_j][\xi_j^{-p}] \oplus \mathbb{C} \cdot c \quad \text{for } p \geq 0 ,$$

$$\bigoplus_{j=1}^N \mathfrak{g} \otimes \mathbb{C}[\xi_j][\xi_j^{-p}] \quad \text{for } p < 0 .$$

Then $\mathcal{H}_\lambda$ have the natural filtration induced from that of $\hat{\mathfrak{g}}_N$:

$$\mathcal{F}_p \mathcal{H}_\lambda = \mathcal{F}_p U(\hat{\mathfrak{g}}_N) \cdot V_\lambda,$$
where
\[
\mathcal{F}_p U(\hat{\mathfrak{g}}_N) = \sum_{p_1 + p_2 + \ldots + p_i \leq p} \mathcal{F}_{p_1} \hat{\mathfrak{g}}_N \mathcal{F}_{p_2} \hat{\mathfrak{g}}_N \ldots \mathcal{F}_{p_i} \hat{\mathfrak{g}}_N,
\]
\[
V_\lambda = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_N}.
\]
We have
\[
\mathcal{F}_p U(\hat{\mathfrak{g}}_N) \cdot \mathcal{F}_q \mathcal{H}_\lambda \subset \mathcal{F}_{p+q} \mathcal{H}_\lambda,
\]
\[
\mathcal{F}_p \mathcal{H}_\lambda = \{0\} \text{ for } p < 0.
\]
We introduce the induced filtrations on subalgebras of \(\hat{\mathfrak{g}}_N\) and the quotient filtrations on \(V_\lambda(X)\) and \(W_M\), and consider the associated graded objects \(\text{gr}^F(\ )\). Then we have the following exact sequences of graded vector spaces over \(\mathbb{C}\):
\[
0 \to \text{gr}^F(\hat{\mathfrak{g}}_M) \to \text{gr}^F \mathcal{H}_\lambda \to \text{gr}^F V_\lambda(X) \to 0,
\]
\[
0 \to \text{gr}^F((\hat{\mathfrak{g}}_M \oplus \mathfrak{g}[1]) \mathcal{H}_\lambda) \to \text{gr}^F \mathcal{H}_\lambda \to \text{gr}^F W_M \to 0.
\]
The Riemann-Roch theorem implies that, for a sufficiently large integer \(M\), we have
\[
\text{gr}^F (\hat{\mathfrak{g}}_M \oplus \mathfrak{g}[1]) \mathcal{H}_\lambda \subset \text{gr}^F (\hat{\mathfrak{g}}_0 \mathcal{H}_\lambda)
\]
as subspaces of \(\text{gr}^F \mathcal{H}_\lambda\). Therefore for such an integer \(M\), we have the following surjective homomorphism:
\[
\text{gr}^F W_M \to \text{gr}^F V_\lambda(X) \to 0.
\]
This proves Lemma 5.1. \(\square\)

In the following we fix an integer \(M\) which satisfies (5.1) and put \(W = W_M\).

The rest of this section is devoted to prove the finite-dimensionality of \(W\).

We define the finite dimensional vector space \(\mathfrak{b}\) by
\[
\mathfrak{b} = \oplus_{j=1}^N (\mathfrak{g} \otimes (\oplus_{n=0}^{M-1} \mathbb{C} \xi_j^{-n})).
\]
Since \(\hat{\mathfrak{g}}_M^0\) is an ideal of \(\hat{\mathfrak{g}}_N^0\), a Lie algebra structure on \(\mathfrak{b}\) is defined through the isomorphism
\[
\mathfrak{b} \cong \hat{\mathfrak{g}}_N^0/\hat{\mathfrak{g}}_M^0.
\]
Put
\[
\mathcal{N} = \mathcal{H}_\lambda/\hat{\mathfrak{g}}_M^0 \mathcal{H}_\lambda.
\]
Then \(\mathcal{N}\) has a structure of \(U(\mathfrak{b})\)-module and it is generated by the image \(\tilde{V}_\lambda\) of \(V_\lambda\) on \(\mathcal{N}\):
\[
(5.2) \quad \mathcal{N} \cong U(\mathfrak{b}) \cdot \tilde{V}_\lambda.
\]

**Remark.** In [TUY] the finite-dimensionality of \(\mathcal{N}\), which implies that of \(W\), is proved by Gabber’s theorem.
Now we define other filtrations \( \{E_m\} \) on \( U(\hat{g}^0_N) \) and \( \mathcal{H}_{\bar{X}} \) as follows.

\[
E_m U(\hat{g}^0_N) = \begin{cases} 
\{0\} & m < 0, \\
\mathbb{C} \cdot 1 & m = 0, \\
E_{m-1} U(\hat{g}^0_N) + \hat{g}^0_N E_{m-1} U(\hat{g}^0_N) & m > 0,
\end{cases}
\]

\[
E_m \mathcal{H}_{\bar{X}} = E_m U(\hat{g}^0_N) \cdot \mathcal{V}_{\bar{X}}.
\]

Then we have

\[
[E_m U(\hat{g}^0_N), E_n U(\hat{g}^0_N)] \subset E_{m+n-1} U(\hat{g}^0_N),
\]

\[
E_m U(\hat{g}^0_N) E_n \mathcal{H}_{\bar{X}} \subset E_{m+n} \mathcal{H}_{\bar{X}}.
\]

On subspaces of \( \mathcal{H}_{\bar{X}} \) we introduce the induced filtrations from that of \( \mathcal{H}_{\bar{X}} \). On \( \mathcal{V}_{\bar{X}}(\mathcal{X}), \mathcal{W} \) and \( \mathcal{N} \) we introduce the quotient filtrations from that of \( \mathcal{H}_{\bar{X}} \). On \( U(\mathfrak{b}) \cong U(\hat{g}^0_N) / \hat{g}^0_N U(\hat{g}^0_N) \) we introduce the quotient filtrations from that of \( U(\hat{g}^0_N) \).

We denote the associated graded objects by \( \text{gr}_E(\cdot) \). Then we have the following isomorphism as graded algebras:

\[
\text{gr}_E U(\mathfrak{b}) \cong S(\mathfrak{b}),
\]

where \( S(\mathfrak{b}) \) is the symmetric algebra of \( \mathfrak{b} \). By (5.2), the space \( \text{gr}_E \mathcal{N} \) is generated by \( \text{gr}_0 \mathcal{N} \) as a \( S(\mathfrak{b}) \)-module:

\[
\text{gr}_E \mathcal{N} \cong S(\mathfrak{b}) \cdot \text{gr}_0 \mathcal{N}.
\]

We denote the degree \( m \)-part of \( S(\mathfrak{b}) \) by \( S^m(\mathfrak{b}) \). Then we have the following lemma.

**Lemma 5.2.** For each integer \( m \), there exists a commutative diagram of surjective homomorphisms:

\[
\begin{array}{ccc}
E_m \mathcal{N} & \xrightarrow{\rho_m} & S_m(\mathfrak{b}) \cdot \text{gr}_0 \mathcal{N} \\
\downarrow & & \Downarrow \varphi_m \\
E_m \mathcal{W} & \xrightarrow{\text{gr}_m} & \text{gr}_m \mathcal{W}.
\end{array}
\]

\[
\square
\]

This lemma implies that the space \( \text{gr}_E \mathcal{W} = \varphi_m (S^m(\mathfrak{b}) \cdot \text{gr}_0 \mathcal{N}) \) is finite dimensional over \( \mathbb{C} \), since \( \mathfrak{b} \) and \( \text{gr}_0 \mathcal{N} \) are finite dimensional. Hence, to prove the finite-dimensionality of \( \mathcal{W} \), it is sufficient to prove the following.

**Claim 1.** For a sufficiently large integer \( m \), we have

\[
\text{gr}_m \mathcal{W} = \{0\}.
\]

Put

\[
\pi_m = \varphi_m \circ \rho_m : E_m \mathcal{N} \to \text{gr}_m \mathcal{W}.
\]

To prove Claim 1, it is sufficient to prove the following.
Claim 2. For each $X \in \mathfrak{g}$, there exists an integer $K$ such that we have
\begin{equation}
(5.3) \quad k \geq K \Rightarrow \pi_m((X \otimes \xi_j^{-n})^k|u)) = 0
\end{equation}
for any $m \in \mathbb{Z}$, $j = 1, \ldots, N$, $n = 0, 1, \ldots, M - 1$ and $|u) \in \mathcal{E}_{m-k}\mathcal{N}$.

Before proving Claim 2, let us show that Claim 1 follows from Claim 2. Assuming Claim 2, we can take an integer $K$ for which (5.3) holds for any $X \in \mathfrak{g}, m \in \mathbb{Z}, j = 0, \ldots, N$, $n = 0, \ldots, M - 1$ and $|u) \in \mathcal{E}_{m-k}\mathcal{N}$. Fix an integer $m$ larger than $K \times \dim \mathfrak{b} = K \times \dim \mathfrak{g} \times N \times M$, and take an element
\[ a = \prod_{i=1}^{\dim \mathfrak{g}} \prod_{j=1}^{N} \prod_{n=0}^{M-1} (J^i \otimes \xi_j^{-n})^{k_i,j,n} \]
of $S^m(\mathfrak{b})$ with $\sum_{i,j,n} k_{i,j,n} = m$, where $\{J^i; i = 1, \ldots, \dim \mathfrak{g}\}$ is a basis of $\mathfrak{g}$. Then we can find at least one index $k_{i',j',n'}$ which is larger than $K$, and by the assumption we have $\varphi_m(a|v)) = 0$ for any $|v) \in \mathfrak{g}_0^\mathfrak{g}\mathcal{N}$. Therefore we have $\mathfrak{g}_m^\mathfrak{g} \mathcal{W} = \varphi_m(S^m(\mathfrak{b}) \cdot \mathfrak{g}_0^\mathfrak{g}\mathcal{N}) = \{0\}$.

Let us prove Claim 2. Fix integers $j = 1, \ldots, N, n = 0, \ldots, M - 1$ and a positive root $\alpha \in \Delta_+$, and put
\[ E = X_\alpha, \quad F = X_{-\alpha}, \quad H = H_\alpha = [X_\alpha, X_{-\alpha}]. \]
For $X = E$ or $F$, it is easy to show Claim 2, since $X \otimes \xi_j^{-n}$ acts nilpotently on $V_j$. In order to prove Claim 2 for $X = H$, we consider the element $|u)$ in $\mathcal{E}_{m}\mathcal{N}$ of the form
\[ |u) = (H \otimes \xi_j^{-n})^{s-1}(F \otimes \xi_j^{-n})^{t+1}|v), \]
where $s$ and $t$ are integers such that $s \geq 1$ and $t \geq 0$, and $|v) \in \mathcal{E}_{m-s-t}\mathcal{N}$.

Put
\[ E[1] = \bigoplus_{j=1}^{N} E \otimes \xi_j^0. \]
Then we have
\begin{equation}
(5.4) \quad E[1]|u) = \bigoplus_{j=1}^{N} E \otimes \xi_j^0|u) = \]
\[ [E \otimes \xi_j^0,(H \otimes \xi_j^{-n})^{s-1}(F \otimes \xi_j^{-n})^{t+1}|v) + (H \otimes \xi_j^{-n})^{s-1}(F \otimes \xi_j^{-n})^{t+1}E[1]|v). \]

Note that
\[ E[1]|v) \in \mathcal{E}_{m-s-t}\mathcal{N}. \]
Sending (5.4) by $\pi_m$ after calculating the commutator, we have
\[ \pi_m(E[1]|u)) = -2(s-1)\pi_m((H \otimes \xi_j^{-n})^{s-1}(F \otimes \xi_j^{-n})^{t+1}E \otimes \xi_j^{-n}|v)) \]
\[ + (t+1)\pi_m((H \otimes \xi_j^{-n})^{s-1}(F \otimes \xi_j^{-n})^{t+1}|v)) \]
\[ + \pi_m((H \otimes \xi_j^{-n})^{s-1}(F \otimes \xi_j^{-n})^{t+1}E[1]|v)). \]

On the other hand, by the definition of $\mathcal{W}$, we have $\pi_m(E[1]|u)) = 0$. Therefore we get the following formula:
\[ \pi_m((H \otimes \xi_j^{-n})^{s-1}(F \otimes \xi_j^{-n})^{t+1}|v) = \]
\[ \frac{1}{t+1}\{2(s-1)\pi_m((H \otimes \xi_j^{-n})^{s-2}(F \otimes \xi_j^{-n})^{t+1}E \otimes \xi_j^{-n}|v)) \]
\[ + \pi_m((H \otimes \xi_j^{-n})^{s-1}(F \otimes \xi_j^{-n})^{t+1}E[1]|v)). \]
By this formula we can increase the number $t$ by decreasing the number $s$. Hence if we take an integer $K'$ such that

$$k \geq K' \Rightarrow \pi_m \left( (F \otimes \xi_j^{-n})^k |v\rangle \right) = 0 \text{ for any } m \in \mathbb{Z} \text{ and } |v\rangle \in \mathcal{E}_{m-kN},$$

then we have

$$k \geq 2K' \Rightarrow \pi_m \left( (H \otimes \xi_j^{-n})^k |v\rangle \right) = 0 \text{ for any } m \in \mathbb{Z} \text{ and } |v\rangle \in \mathcal{E}_{m-kN}.$$ 

This proves Claim 2 for $X = H$. Now, it is easy to prove Claim 2 for any $X$.  

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