On ruin probabilities with investments in a risky asset with a switching regime price

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Abstract We investigate the asymptotic of ruin probabilities when the company invests its reserve in a risky asset with a switching regime price. We assume that the asset price is a conditional geometric Brownian motion with parameters modulated by a Markov process with a finite number of states. Using the technique of the implicit renewal theory we obtain the rate of convergence to zero of the ruin probabilities as the initial capital tends to infinity.

Keywords Ruin probabilities · Risky investments · Stochastic volatility · Hidden Markov model · Regime switching · Implicit renewal theory

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1 Introduction

Models, where an insurance company invests its reserve (or a part of it) in a risky asset, constitutes an important class currently under an extensive study. Considering a single risky asset is justified by the common practice of investing in a market portfolio or in an index (a fund which simulates an index like the DAX or the S&P500) which is an economically reasonable strategy. Since the insurance contacts usually are of a long duration and the return may depend, e.g., on the business cycles of economy, models with regime switching are now more and more popular. The main question is the rate of decay of the ruin probability as the initial reserve tends to infinity.

In this note we extend the recent result of Ellanskaya and Kabanov, established for a model with characteristics of the asset price depending on a telegraph process, i.e. on a

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Markov process with two states, 0 and 1. Here we study the case where the characteristics depend on the ergodic Markov process $\theta$ with a finite number of states. When $\theta_t = k$, the asset price evolves as a geometric Brownian motion with drift $\alpha_k$ and volatility $\sigma_k$. It is well known, see, e.g. [9, 13, 15], that in the case of a single regime, i.e. when the price process is the classic GBM with drift $\alpha$ and volatility $\sigma$, the ruin probability decreases to zero as the initial capital $u$ tends to infinity with the rate $\beta := 2\alpha / \sigma - 1$ provided that $\beta > 0$. In [7] it was shown that in the model with two regimes, 0 and 1, the ruin probabilities decrease with a rate $\beta$ where $\beta$ is a number between the values $\beta_k := 2\alpha_k / \sigma_k - 1$, $k = 0, 1$, assumed to be strictly positive. This $\beta$ is the root of an algebraic equation of third order and does not depend on the initial value of $\theta$.

In the present paper we extend this result to the case where the number of states of the hidden Markov process $\theta$ is $K \geq 2$. It happens that, provided all $\beta_k > 0$, $k = 0, \ldots, K - 1$, the rate of convergence to zero of the ruin probabilities, depending, in general, on the initial state $i$, is a root of the cumulant generating function of the value of log price process at the first return time of $\theta$ to the state $i$. It is worth to note that the switching by telegraph signal is rather specific: the latter returns to the initial state after the second jump while for a general Markov process the return may happen after arbitrary number of jumps.

Though the main idea is again based on the implicit renewal theory, it happens that the analysis of the considered model is much more complicated and the calculation of the rate parameter, depending, in general, on the initial state, is not so straightforward. We hope that the result of this paper elucidate challenging problems of estimation of ruin probabilities for other stochastic volatility models.

2 The model

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$ be a stochastic basis with a Wiener process $W = (W_t)$, a Poisson random measure $\pi(dt, dx)$ on $\mathbb{R}_+ \times \mathbb{R}$ with the mean $\bar{\pi}(dt, dx) = \Pi(dx)dt$, and a piecewise constant right-continuous Markov process $\theta = (\theta_t)$. For the latter we assume that it takes values in the finite set $\{0, 1, \ldots, K - 1\}$, has the $K \times K$ transition intensity matrix $\Lambda = (\lambda_{jk})$ with communicating states, and the initial value $\theta_0 = i$ (so that $\theta = \theta'$). The $\sigma$-algebras generated by $W$, $\pi$, and $\theta$ are independent.

Recall that $\lambda_{jj} = -\sum_{k \neq j} \lambda_{jk}$ and $\lambda_{ii} := -\lambda_{ii} > 0$ for each $i$.

Let $T_n$ be the successive jumps of the Poisson process $N = (N_t)$ with $N_t := \pi([0,t], \mathbb{R})$ and let $\tau_n$ be the successive jumps of $\theta$ with the convention $T_0 = 0$ and $\tau_0 = 0$.

Recall that the lengths of the intervals between the consecutive jumps of $\theta$ are independent exponentially distributed random variables.

The reserve $X = X^W$ of an insurance company evolves not only due to the business activity part, described as in the classical Cramér–Lundberg model, but also due to the stochastic interest rate. We assume that the reserve is fully invested in a risky asset whose price $S$ is a conditional geometric Brownian motion given the Markov process $\theta$. That is, $S$ is given by a so-called hidden Markov model with

$$dS_t = S_t(a_\theta dt + \sigma_\theta dW_t), \quad S_0 = 1,$$

where $a_k \in \mathbb{R}$, $\sigma_k > 0$, $k = 0, \ldots, K - 1$. In this case, $X$ is of the form

$$X_t = u + \int_0^t X_s dR_s + dP_t$$

(1)
where \(dR_t = a_\theta dt + \sigma_\theta dW_t = dS_t/S_t\), that is \(R\) is the relative price process, and

\[
P_t = ct + \int_0^t \int x\pi(dt, dx) = ct + x \ast \pi_t.
\]  

(2)

So, the reserve evolution is described by the process \((X^u, \theta) = (X^u, \theta')\) where \(u > 0\) is the initial capital and \(i\) is the initial regime, i.e. the initial value of \(\theta\).

We assume that \(P\) is not an increasing process: otherwise the probability of ruin is zero.

We also assume that \(\Pi(\mathbb{R}) < \infty\), that is \(\Pi(dx) = \alpha_1 F_1(dx) + \alpha_2 F_2(dx)\) where \(F_1(dx)\) is a probability distribution on \([-\infty, 0]\) and \(F_2(dx)\) is a probability distribution on \([0, \infty]\). In this case the integral with respect to the jump measure is simply a difference of two independent compound Poisson processes with intensities \(\alpha_1, \alpha_2\) of jumps downwards and upwards and whose absolute values have the distributions \(F_1(dx)\) and \(F_2(dx)\), respectively.

The solution of the linear equation (1) can be represented as

\[
X_t^u = \mathcal{E}_t(R)(u - Y_t) = e^{V_t}(u - Y_t)
\]  

where \(Y_t := -\int_{[0,t]} \mathcal{E}_s^{-1}(R)dP_s = -\int_{[0,t]} e^{-V_s}dP_s = -e^{-V_t} \cdot P_t\),

(4)

the stochastic exponential \(\mathcal{E}_t(R)\) is equal to \(S_t\), and the log price process \(V = \ln \mathcal{E}(R)\) admits the stochastic differential

\[
dV_s = \sigma\theta_s dW_s + (a\theta_s - (1/2)\sigma^2\theta_s)s, \quad V_0 = 0.
\]

Of course, \(S, R, Y, \) and \(V\) depend on \(i\) (we omitted the superscript \(i\) in the above formulae).

Let \(\tau_{u,i} := \inf\{t > 0 : X_t^{u,i} \leq 0\}\) be the instant of ruin corresponding to the initial capital \(u\) and the initial regime \(i\). Then \(\Psi_t(u) := \mathbb{P}[\tau_{u,i} < \infty]\) is the ruin probability and \(\Phi_t(u) := 1 - \Psi_t(u)\) is the survival probability. It is clear that \(\tau_{u,i} = \inf\{t \geq 0 : Y_t^{u,i} \geq u\}\).

Recall that the constant parameter values \(a = 0, \sigma = 0\), correspond to the Cramér–Lundberg model for which the process \(X^{u,i} = u + P_t\). In the actuarial literature the compound Poisson process \(P\) is usually written in the form

\[
P_t = ct - \sum_{k=1}^{N_t} \xi_k
\]  

(5)

where either \(\xi_k \geq 0, c > 0\) (i.e. \(F_2 = 0\) — jumps only downwards — the case of non-life insurance) or \(\xi_k \leq 0, c < 0\) (i.e. \(F_1 = 0\) — jumps only upwards — the case of life insurance or annuity payments). Models with both kinds of jumps are also considered in the literature, see, e.g. [1] and references therein. For the classical models with a positive average trend and \(F\) having a "non-heavy" tail, the Lundberg inequality asserts that the ruin probability decreases exponentially as the initial capital \(u\) increases to infinity. For the exponentially distributed claims the ruin probability admits an explicit expression, see [3], Ch. IV.3b, or [11], Section 1.1.

For the models with investment in a risky asset the situation is completely different. For example, for the model with exponentially distributed jumps and price following a geometric Brownian motion with the drift coefficient \(a\) and the volatility \(\sigma > 0\) in the case where \(2a/\sigma^2 - 1 > 0\), the ruin probability as a function of the initial capital \(u\), decreases as \(Cu^{1-2a/\sigma^2}\). If \(2a/\sigma^2 - 1 \leq 0\) the ruin happens with probability one, see [2], [13], [21], [15].
To formulate our result for the model where the volatility and drift are modulated by a finite-state Markov process we assume throughout this note that except Section 7 that

\[ \beta_j := 2a_j / \sigma_j^2 - 1 > 0, \quad j = 0, ..., K - 1, \tag{6} \]

(in other words, \( \beta_* := \min_j \beta_j > 0 \)).

Let \( v_i^1 := \inf\{ t > 0; \theta_{t-}^i \neq i, \theta_t^i = i \} \) be the first return time of the (continuous-time) Markov process \( \theta = \theta^t \) to its initial state \( i \). We consider further the consequent return times defined recursively:

\[ v_k^i := \inf\{ t > v_{k-1}^i; \theta_{t-}^i \neq i, \theta_t^i = i \}, \quad k = 2, ... \]

We introduce the random variable \( M_{i1} := e^{-Vv_i^1} \) and define the moment generating function \( T_i : \mathbb{R}_+ \to \mathbb{R}_+ := \mathbb{R}_+ \cup \{ \infty \} \) with

\[ T_i(q) := \mathbb{E}[M_{i1}^q]. \]

**Proposition 1** The function \( T_i \) is strictly convex, continuous, and there is unique \( \gamma_i > 0 \) such that \( T_i(\gamma_i) = 1 \).

Note that one can characterize \( \gamma_i \) also as the strictly positive roots of the cumulant generating functions \( H_i(q) := \ln \mathbb{E} e^{-Vv_i^1} \) strictly convex and continuous.

Postponing the proof of Proposition 1 to the next section we formulate our main result:

**Theorem 1** Suppose that \( H(|x|^{\gamma_i}) := \int |x|^{\gamma_i} H(dx) < \infty \) Then

\[ 0 < \lim \inf_{u \to \infty} u^{\gamma_i} \psi_i(u) \leq \lim \sup_{u \to \infty} u^{\gamma_i} \psi_i(u) < \infty. \]

**Important remark.** In the case where \( \theta \) is a telegraph signal, i.e. a two-state Markov process, the values \( \gamma_0 \) and \( \gamma_1 \) coincide (see (7)). In the considered general case, \( \gamma_i \) may depend on the initial value \( i \). To alleviate formulae, we fix the initial value \( i = 0 \) and omit the index \( i = 0 \) when this does not lead to ambiguity.

The proof of Theorem 1 is based on the implicit renewal theory. To apply it, we verify that the random variables \( Q = Q_i := -e^{-V} \cdot P_{\theta_i} \) belong to \( L^\gamma(\Omega) \), the process \( Y \) has at infinity a finite limit \( Y_\infty \) and show that \( Y_\infty \) is a random variable unbounded from above with the same law as \( Q + M Y_\infty \) where \( M = M_1 \) and \( \mathbb{E}[M_1^\gamma] = 1 \). Also \( \mathbb{E}[M_1^{\gamma+\delta}] < \infty \) for some \( \delta > 0 \). Clearly, the law of the random variable \( \ln M_1 \) is not arithmetic. We establish the bounds \( G(u) \leq \Psi_i(u) \leq CG(u) \) where \( G_i(u) = \mathbb{P}[Y_\infty > u] \) and a constant \( C > 0 \), Lemma 5.

With these facts Theorem 1 follows from Theorem 2 below which is the Kesten–Goldie theorem, see Th. 4.1 in [10], combined with a statement on strict positivity of \( C_\gamma \) due to Guiivanh Le Page, [5] (for a simpler proof of the latter see Buraczewski and Damek, [2], and an extended discussion in Kabanov and Pergamenshchikov, [14]).

**Theorem 2** Let \( Y_\infty \) has the same law as \( Q + M Y_\infty \) where \( M > 0 \). Suppose that \( (M, Q) \) is such that the law of \( \ln M \) is non-arithmetic and, for some \( \gamma > 0 \),

\[ \mathbb{E}[M^\gamma] = 1, \quad \mathbb{E}[M^\gamma (\ln M)^+] < \infty, \quad \mathbb{E}[Q^\gamma] < \infty. \tag{7} \]
where $C_+ + C_- > 0$.

If the random variable $Y_\infty$ is unbounded from above, then $C_+ > 0$.

3 Properties of the moment generating function: the proof of Proposition\[1\]

Recall that $\tau_n$ are the moments of consecutive jumps of $\theta$, that is, $\tau_0 := 0$,

$$\tau_n := \inf \{ t > \tau_{n-1} : \theta_t - \theta_t \neq \theta_t \}, \quad n \geq 1.$$  

We introduce the imbedded Markov chain $\vartheta_n := \tau_{}\vartheta_n$, $n = 0, 1, \ldots$ with transition probabilities $P_{kl} = \lambda_{ik}/\lambda_k$, $k \neq l$, and $P_{kk} = 0$. Then $\tau_w := \inf \{ j \geq 2 : \vartheta_j = 0 \}$ is the first return time of the (discrete-time) Markov chain $\vartheta$ to the starting point 0 and $v_1 = \tau_{\tau_w}$.

Put

$$Z^j_t := \sigma_j W_t + (\alpha_j - \frac{\alpha_j^2}{2}) t = \sigma_j W_t + (1/2) \sigma_j^2 \beta_j t.$$  

The random variable $M_1$ admits the representation

$$M_1 = \sum_{k \geq 2} \sum_i \sum_{i, \theta_i \neq 0, \theta_i \neq 0, \ldots, i_k = 0} I_{\{\vartheta_1 = i_1, \vartheta_2 = i_2, \ldots, \vartheta_k = i_k\}} e^{\vartheta_i \sigma_j \beta_j} \ldots e^{\vartheta_k \sigma_j \beta_j},$$

where

$$\vartheta^0 := Z^{i_1}_1 - Z^{i_1}_0, \quad \vartheta^{i_1}_1 := Z^{i_1}_2 - Z^{i_1}_1, \ldots, \vartheta^{i_{k-1}}_k := Z^{i_{k-1}}_k - Z^{i_{k-1}}_{k-1}.$$  

The conditional law of random variables $\vartheta^0_1, \ldots, \vartheta^{i_{k-1}}_k$ given $\vartheta_1 = i_1, \vartheta_2 = i_2, \ldots, \vartheta_k = i_k$ is the same as the unconditional law of independent random variables $\vartheta^0_1, \ldots, \vartheta^{i_{k-1}}_k$. For any $m$ the law $\mathcal{L}(\vartheta^0_1) = \mathcal{L}(\sigma_j W_T + (1/2) \sigma_j^2 \beta_j \tau)$ where an exponential random variable $\tau$ with parameter $\lambda_j$ is independent on the Wiener process $W$.

It follows that

$$\mathcal{T}(q) := E[M_j^q] = \sum_{k \geq 2} \sum_{i, i, \theta_i \neq 0, \theta_i \neq 0, \ldots, i_k = 0} P_{0i_1} \cdot P_{i_1 i_2} \ldots P_{i_{k-2} i_{k-1}} f_0(q) f_1(q) \ldots f_{k-1}(q).$$

(8)

where

$$f_j(q) = E[e^{\vartheta^0_q}] = \lambda_j E \left[ \int_0^\infty e^{-q(S_j W_T + (1/2) \sigma_j^2 \beta_j t)} e^{-\lambda_j t} dt \right] = \frac{\lambda_j}{\lambda_j + (1/2) \sigma_j^2 q(\beta_j - q)}.$$  

if the denominator is positive, and $f_j(q) = \infty$ otherwise.

Clearly, $f_j(q) < \infty$, if $q \in [0, r_j]$, $f_j(q) = \infty$, if $q \in [r_j, \infty]$, and $f_j(r_j -) = \infty$, where $r_j$ is the positive root of the equation

$$q^2 - \beta q - 2\lambda_j \sigma_j^2 = 0,$$

that is, $r_j = \sqrt{\beta^2 / 4 + 2\lambda_j \sigma_j^2}.$

(9)
Note that the formula (8) can be written in a shorter form

\[ T(q) = E \left[ \sum_{k=2}^{\infty} f_0(q) f_{\theta_1}(q) \cdots f_{\theta_{k-1}}(q) I_{\{\varpi_k = 0\}} \right]. \] (10)

If \( q \leq \beta_* \) := \min_j \beta_j, then all \( f_j(q) \leq 1 \) and \( T(q) \) is dominated the probability of return of \( \theta \) to the initial state (equal to unit), that is, \([0, \beta_*] \subseteq \text{dom } T \). Also, \( f_j(\beta_* / 2) < 1 \) for all \( j \) and, therefore, \( T(\beta_* / 2) < 1 \). Since we assume that any state of \( \theta \) can be reached from any other state, \( \text{dom } T \subseteq [0, r_*] \) where \( r_* := \min_j r_j \).

More precise information gives the following lemma.

**Lemma 1** We have \( \text{dom } T = [0, r_*] \) and \( \lim_{q \uparrow r_*} T(q) = \infty. \)

**Proof.** To explain the idea let us consider first the case \( K = 3 \). Regrouping terms in the formula (8) according to 4 pairs “exit from 0 to l, return back from \( m \)” we get the representation

\[ T(q) = \left[ P_{0,1} f_0(q) P_{1,0} f_1(q) + P_{0,2} f_0(q) P_{2,0} f_2(q) + P_{0,1} P_{1,2} P_{2,0} f_0(q) f_1(q) f_2(q) \right. \\
\left. + P_{0,2} P_{1,2} P_{2,0} f_0(q) f_2(q) f_1(q) \right] \sum_{k=0}^{\infty} (P_{1,2} f_1(q) P_{2,1} f_2(q))^k. \]

Note that if \( P_{1,2} f_1(q) P_{2,1} f_2(q) < 1 \), then

\[ \sum_{k=0}^{\infty} (P_{1,2} f_1(q) P_{2,1} f_2(q))^k = \frac{1}{1 - P_{1,2} f_1(q) P_{2,1} f_2(q)}, \]

otherwise the above sum is equal to infinity. Thus, \( T \) is a product of two continuous functions with values in \( \mathbb{R}_+ \), hence, has the same property and the result follows.

For a model with an arbitrary \( K \) we get the continuity of \( T \) from the continuity result for more general functions.

Let us consider a subset \( A \subseteq \{0, 1, \ldots, K - 1\} \). For \( i, k \notin A \) we denote by \( \Gamma_{ik}^A \) the set of vectors \((i, i_1, i_2, \ldots, i_m, k) \), \( i_j \in A, j = 1, \ldots, m, m \in \mathbb{N} \). The elements of \( \Gamma_{ik}^A \) are interpreted as cuts of sample paths of the Markov chain entering to \( A \) from the state \( i \), evolving in \( A \), and living to the state \( k \).

Putting \( h_{i,j}(q) = P_{i,j} f_j(q) \) with the natural convention \( 0 \times \infty = 0 \) we associate with elements of \( \Gamma_{ik}^A \) the continuous functions

\[ q \mapsto h_{i_1,i_2}(q) \cdots h_{i_{m-1},i_m}(q) h_{i_m,k}(q) \]

with values in \( \mathbb{R}_+ \) and consider the sum of all these functions

\[ U_{ik}^A : q \mapsto \sum h_{i_1,i_2}(q) \cdots h_{i_{m-1},i_m}(q) h_{i_m,k}(q). \]

Since \( f_j < 1 \) on the interval \([0, \beta_*]\), also \( U_{ik}^A < 1 \) on this interval.

We show by induction that \( U_{ik}^A : \mathbb{R}_+ \to \mathbb{R}_+ \) is a continuous function with \( U_{ik}^A(0) \leq 1 \). Since \( T = U_{00}^A \), this gives the assertion of the lemma.
The idea of the proof consists in representing $U_{ik}^A$ as a sum of finite number of positive continuous functions using an appropriate partition of $\Gamma_{ik}^A$. Namely, for $i_1 \in A$ and $n \geq 0$ we define the set
\[
\Delta_{ik}^{1,n} := \{i\} \times \Gamma_{i_1,i_1}^A \times \cdots \times \Gamma_{i_1,1}^A \times \Gamma_{i_1,k}^A,
\]
composed by the vectors with the first component $i$, followed by $n \geq 0$ blocks formed by vectors from $\Gamma_{i_1,i_1}^A$, and completed by vectors from $\Gamma_{i_1,k}^A$. Clearly, the countable family $\Delta_{ik}^{1,n}$, $i_1 \in A$, $n \geq 0$, is a partition of $\Gamma_{ik}^A$ and
\[
U_{ik}^A(q) = \sum_{i_1 \in A} h_{i,i_1}(q) U_{i_1,k}^{A \setminus \{i_1\}}(q) \sum_{n=0}^{\infty} \left[ U_{i_1,i_1}^{A \setminus \{i_1\}}(q) \right]^n.
\] (11)

The result is obvious when $A$ is a singleton, i.e. $|A| = 1$. Supposing that the assertion is already proven for the case where $|A| = K_1 - 1$ we consider the case where $|A| = K_1$. By the induction hypothesis $U_{i_1,m}^A \colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function for every $i_1 \in A$, $m \notin A \setminus i_1$. The result follows from (11) and the formula for the geometric series. \(\square\)

The strictly convex function $\Upsilon$ is less or equal to unit on $[0, \beta_s]$, finite on $[0, r_s]$, and tends to infinity at $r_s$. Hence, there is unique $\gamma \in [\beta_s, r_s]$ such that $\Upsilon(\gamma) = 1$. Moreover, $\Upsilon(\gamma + \epsilon) < \infty$ for some $\epsilon > 0$.

4 Integrability of $Q_1$

The following identity is obvious:
\[
Y_{\upsilon_\alpha} = -\sum_{k=1}^{n-1} \prod_{j=1}^{k-1} e^{-(V_{\upsilon_j} - V_{\upsilon_{j-1}})} \int_{|\upsilon_{k-1}, \upsilon_k|} e^{-(V_s - V_{\upsilon_{k-1}})} dP_s.
\]

Using the abbreviations
\[
Q_k := -\int_{|\upsilon_{k-1}, \upsilon_k|} e^{-(V_s - V_{\upsilon_{k-1}})} dP_s, \quad M_j := e^{-(V_{\upsilon_j} - V_{\upsilon_{j-1}})}
\]
we rewrite it in a more transparent form as
\[
Y_{\upsilon_\alpha} = Q_1 + M_1Q_2 + M_1M_2Q_3 + \ldots + M_1\ldots M_{n-1}Q_n.
\] (12)

Note that the random variables $\upsilon_k - \upsilon_{k-1}$, that is, the lengths of intervals between successive returns to the initial state, form and i.i.d. sequence. The random variables $(Q_k, M_k)$ have the same law and are independent on the $\sigma$-algebra $\sigma\{Q_1, Q_2, \ldots, Q_{K-1}\}$.

First, we study integrability properties of $Y$. For this we need a general lemma involving parameters $\beta, \sigma > 0$ and an independent on $W$ exponential random variable $\tau$ with parameter $\lambda > 0$.

Lemma 2 Let $0 < q < r(\lambda, \beta, \sigma)$ where $r(\lambda, \beta, \sigma)$ is given by (9). Then
\[
C(q, \lambda, \beta, \sigma) := \mathbb{E} \left[ \left( \int_0^\tau e^{-[(\sigma W_s + (1/2)\sigma^2 \beta) \tau]} ds \right)^q \right] < \infty.
\]
Lemma 3
Let \( \tau \) be a Wiener process independent on \( \rho, \rho' < r \).

Corollary 1
above lemma implies the following useful property we get from (10) with the abbreviation \( q > 0 \), \( \beta > 0 \), \( \sigma > 0 \), \( \lambda > 0 \), \( \rho > 0 \), \( \rho' > 0 \) such that \( 1/\rho + 1/\rho' = 1 \) and \( \rho q < r(\lambda, \beta, \sigma) \).

The lemma is proven. \( \Box \)

Proof. Put \( W_{\tau}^{(\sigma/2)} := W_{\tau} + (1/2)\sigma \beta s \). Take \( \rho, \rho' > 1 \) such that \( 1/\rho + 1/\rho' = 1 \) and \( \rho q < r(\lambda, \beta, \sigma) \). Dominating the integrant by its supremum and using the Hölder inequality we get that

\[ C(q, \lambda, \beta, \sigma) \leq E \left[ \tau \sup_{s \leq \tau} e^{-\sigma q W_{\tau}^{(\sigma/2)}} \right] \leq \left( E \left[ \tau \right] \right)^{1/\rho'} \left( E \left[ \sup_{s \leq \tau} e^{-\sigma q W_{\tau}^{(\sigma/2)}} \right] \right)^{1/\rho}. \]

Since an exponential random variable has moments of any order, the first multiplier in the right-hand side is finite. According to the formula (1.2.1) in Ch. 2 of the reference book [4],

\[ E \left[ \sup_{s \leq \tau} e^{-\sigma q W_{\tau}^{(\sigma/2)}} \right] = E \left[ e^{-\sigma q \inf_{s \leq \tau} W_{\tau}^{(\sigma/2)}} \right] = \frac{r(\lambda, \beta, \sigma)}{r(\lambda, \beta, \sigma) - \rho q} < \infty. \]

The lemma is proven. \( \Box \)

Note that the condition \( T(q) < \infty \) holds only \( q < r_\ast := \min_j r(\lambda_j, \beta_j, \sigma_j) \) and the above lemma implies the following useful

Corollary 1
If \( T(q) < \infty \), then \( C^{*}(q) := \max_j C(q, \lambda_j, \beta_j, \sigma_j) < \infty. \)

Lemma 3
Let \( q > 0 \) be such that \( T(q) < \infty \). Then

\[ E \left( \int_{0}^{T(q)} e^{-q V_s} ds \right) < \infty, \quad E \left( \left( \int_{0}^{T(q)} e^{-V_s} ds \right)^q \right) < \infty. \quad (13) \]

Proof. Let \( \tau \) be a random variable exponentially distributed with parameter \( \lambda > 0 \) and let \( W \) be a Wiener process independent on \( \tau \). Then

\[ E \left[ \int_{0}^{T(q)} e^{-q W_{\tau} + (1/2)\sigma^2 \beta s} ds \right] = \int_{0}^{\infty} P(\tau \leq s) E \left[ e^{-q W_{\tau} + (1/2)\sigma^2 \beta s} ds \right] = \int_{0}^{\infty} \frac{1}{\lambda} e^{-\lambda (1/2)\sigma^2 \beta s} ds = \frac{1}{\lambda + (1/2)\sigma^2 \beta}. \]

if the denominator in the right-hand side is strictly greater than zero, and infinity otherwise.

Using the conditioning with respect to \( \mathcal{F}_{k-1} = \sigma \{ \theta_1, \ldots, \theta_{k-1} \} \) and the Markov property we get from (10) with the abbreviation \( f_k(q) := f_0(q) f_{\theta_1}(q) \cdots f_{\theta_{k-1}}(q), k \geq 1, \) that

\[ T(q) = \sum_{k=2}^{\infty} E \left[ f_k(q) I_{\{ \omega \geq k, \theta_{k-1} = 0 \}} \right] = \sum_{k=2}^{\infty} E \left[ f_k(q) I_{\{ \omega \geq k \}} \right] E\left[ I_{\{ \theta_{k-1} = 0 \}} \right] \]

where \( p_\ast > 0 \) is the minimal of values \( P_{j, 0} \) different from zero.

Note that

\[ \sum_{j=2}^{\infty} \tilde{f}_j(q) = \sum_{j=2}^{\infty} I_{\{ \omega = j \}} \sum_{k=2}^{j} \tilde{f}_k(q) = \sum_{k=2}^{\infty} \tilde{f}_k(q) \sum_{j=k}^{\infty} I_{\{ \omega = j \}} = \sum_{k=2}^{\infty} \tilde{f}_k(q) I_{\{ \omega \geq k \}}. \]

It follows that

\[ E \left[ \sum_{j=2}^{\infty} \tilde{f}_j(q) \right] = E \left[ \sum_{k=2}^{\infty} \tilde{f}_k(q) I_{\{ \omega \geq k \}} \right] \leq \frac{1}{p_\ast} T(q). \]
Since
\[
E \left[ \int_0^{\tau_1} e^{-qV_s} \, ds \right] = E \left[ \sum_{k=2}^{\infty} I_{\{\tau=k\}} \sum_{j=0}^{k-1} e^{-qV_{\tau_j}} \int_{\tau_j}^{\tau_{j+1}} e^{-q(V_s-V_{\tau_j})} \, ds \right]
\]
\[
= E \left[ \sum_{k=2}^{\infty} I_{\{\tau=k\}} \sum_{j=1}^{k-1} f_j(q) \right] \leq \frac{1}{\lambda_*} \left( f_0(q) + E \left[ \sum_{j=2}^{\infty} f_j(q) \right] \right),
\]
where \( \lambda_* := \min_i \lambda_i \), we obtain from the above inequalities that
\[
E \left[ \int_0^{\tau_1} e^{-qV_s} \, ds \right] \leq \frac{f_0(q)}{\lambda_*} + \frac{1}{\lambda_* \mu} \Upsilon(q) < \infty. \tag{14}
\]
The first integrability property of (13) is proven.

To prove the second property of (13) we start with the case \( q \leq 1 \). Using the elementary inequality \( \sum |x_i|^q \leq \sum |x_i|^p \) and Corollary 1 we get that
\[
E \left[ \left( \int_0^{\tau_1} e^{-qV_s} \, ds \right)^q \right] \leq E \left[ \sum_{k=2}^{\infty} I_{\{\tau=k\}} \sum_{j=0}^{k-1} e^{-qV_{\tau_j}} \left( \int_{\tau_j}^{\tau_{j+1}} e^{-q(V_s-V_{\tau_j})} \, ds \right)^q \right]
\]
\[
= E \left[ \sum_{k=2}^{\infty} I_{\{\tau=k\}} \sum_{j=0}^{k-1} f_k(q) E \left[ \int_{\tau_j}^{\tau_{j+1}} e^{-q(V_s-V_{\tau_j})} \, ds \right]^q \right]
\]
\[
\leq C^*(q) E \left[ \sum_{k=2}^{\infty} I_{\{\tau=k\}} \sum_{j=0}^{k-1} f_k(q) \right]
\]
\[
\leq C^*(q) \left( 1 + f_0(q) + E \left[ \sum_{k=2}^{\infty} f_k(q) \right] \right)
\]
\[
\leq C^*(q) \left( 1 + f_0(q) + \frac{1}{\mu} \Upsilon(q) \right) < \infty.
\]

Let \( q > 1 \). Due to the continuity of \( \Upsilon \) there exists \( q' > q \) such that \( \Upsilon(q') < \infty \). Applying, first, the Hölder inequality with the conjugate exponents \( q \) and \( p := q/(q - 1) \) and then the Young inequality for products with the conjugate exponents \( q/p \) and \( q'/\left(q'-q\right) \), we get that
\[
E \left[ \left( \int_0^{\tau_1} e^{-qV_s} \, ds \right)^q \right] \leq E \left[ \int_0^{\tau_1} e^{-qV_s} \, ds \right]
\]
\[
\leq (1 - q/q') E \left[ \int_0^{\tau_1} e^{-q'V_s} \, ds \right] + (q/q') E \left[ \int_0^{\tau_1} e^{-qV_s} \, ds \right],
\]
where \( q_1 := (q - 1)q'/\left(q'-q\right) \). The expectation of the integral in the right-hand side is finite in virtue of the first inequality in (13), applied with \( q' \). It remains to recall that the first return time of the finite state Markov process \( \theta \) has moments of any order.

For the reader convenience we give the proof of the above fact. Take an arbitrary \( m > 1 \) and let denote by \( \tau_j := \tau_j - \tau_{j-1} \) the interjump times of the Markov process \( \theta \). Recall that the conditional distribution of the vector \((\vartheta_1, \ldots, \vartheta_k)\) given \( \vartheta_1 = i_1, \ldots, \vartheta_{k-1} = i_{k-1} \) is the same as the distribution of the vector \((\vartheta_1, \ldots, \vartheta_k)\) with independent components having,
respectively, exponential distributions with the parameters $\lambda_0$, $\lambda_1$, $\ldots$, $\lambda_{k-1}$. Using the H"older inequality (now for the sum) and this fact we get that

$$
E \left[ \nu_1^m \right] = E \left[ \left( \sum_{j=1}^\infty \theta_j \right)^m \right] \leq E \left[ \varpi^{m-1} \sum_{j=1}^\infty \theta_j^m \right]
$$

$$
= E \left[ \sum_{k \geq 2} \sum_{i_1 \neq 0,i_2 \neq 0,\ldots,i_k=0} I \{ \theta_1=i_1,\ldots,\theta_{k-1}=i_{k-1},\theta_k=0 \} k^{m-1} \sum_{j=1}^k \theta_j^m \right]
$$

$$
= \Gamma(m+1) E \left[ \varpi^{m-1} \sum_{j=1}^\infty \frac{1}{\lambda_{\theta_{j-1}}} \right] \leq \Gamma(m+1) \lambda_{\varpi}^{-m} E \left[ \varpi^m \right]
$$

where $\Gamma$ is the Gamma function and $\lambda_{\varpi} := \min_j \lambda_j$. It remains to make a reference to the fact that the first return time $\varpi$ for the Markov chain $\vartheta$ has moments of any order, see, e.g. [8], Ch. XV, exer. 18-20.

The following lemma provides the required integrability property of $Q_1$.

**Lemma 4** Suppose that $H(|x|^\gamma) := \int |x|^\gamma \mu(dx) < \infty$ then $E[|Q_1|^\gamma] < \infty$.

**Proof.** Case where $\gamma \leq 1$. The inequality $(|x| + |y|)^\gamma \leq |x|^\gamma + |y|^\gamma$ allows us to check separately finiteness of moments of the integral of $e^{-V}$ with respect to the Lebesgue measure (this is already done, see the second property in (13)) and the integral with respect to the jump component of the process $P$. The latter integral is just a sum. Since the jump measure $\pi(dt, dx)$ has the compensator $\pi(dt, dx) = H(dx)dt$, we have that

$$
E \left[ e^{-V} \pi_1^m \right] \leq E \left[ e^{-\gamma V} |x|^\gamma \pi_1 \right] = E \left[ e^{-\gamma V} |x|^\gamma \pi \right]
$$

$$
\leq H(|x|^\gamma) E \left[ \int_0^{\nu_1} e^{-\gamma V} ds \right] < \infty
$$

in virtue of the first property in (13).

Case where $\gamma > 1$. Now we shall split integrals using the elementary inequality

$$(|x| + |y|)^\gamma \leq 2^{\gamma-1}(|x|^\gamma + |y|^\gamma).$$

Because of the second property in (13), we need to consider only the integral with respect to the jump component of $P$. Note that $e^{-V}|x| \pi_1 \leq \infty$. Then

$$
E \left[ (e^{-V} |x| \pi_1)^\gamma \right] \leq 2^{\gamma-1} \left( E \left[ e^{-V} |x| (\pi - \tilde{\pi}) \pi_1 \right] \right) + \left( E \left[ (e^{-V} |x| \pi_1)^\gamma \right] \right).
$$

Due to the first property in (13)

$$
E \left[ (e^{-V} |x| \pi_1)^\gamma \right] \leq (H(|x|))^{\gamma} E \left[ \left( \int_0^{\nu_1} e^{-V} ds \right)^\gamma \right] < \infty.
$$

Let $I_* := e^{-V} |x| (\pi - \tilde{\pi}) \pi_1$. According to the Novikov inequality with $\alpha = 1$ the moment of the order $\gamma > 1$ of the random variable $I_*^{\gamma} := \sup_{s \leq t} |I_*|$ admits the bound

$$
E \left[ I_*^{\gamma} \right] \leq C_{\gamma,1} \left( E \left[ (e^{-V} |x| \pi_1)^\gamma \right] + E \left[ (e^{-V} |x| \pi_1)^\gamma \right] \right)
$$

$$
\leq C_{\gamma,1} \left[ \left( \int_0^{\nu_1} e^{-V} ds \right)^\gamma \right] + C_{\gamma,1} \left[ \int_0^{\nu_1} e^{-\gamma V} ds \right]
$$

See [10] and a discussion in [11].
where \( C'_{n,1} := C_{n,1}(H(|x|)) \gamma < \infty \), \( C''_{n,1} := C_{n,1} H(|x|) \gamma < \infty \) due to our assumption. The both integrals in the right-hand side, as we proved, are finite. \( \square \)

5 Study of the process \( Y \)

**Lemma 5** The process \( Y \) has the following properties:

(i) \( Y_t \) converges almost surely as \( t \to \infty \) to a finite random variable \( Y_\infty \).

(ii) \( Y_\infty = Q_1 + M_1 Y_{1,\infty} \) where \( Y_{1,\infty} \) is a random variable independent on \((Q_1, M_1)\) and having the same law as \( Y_i \).

(iii) \( Y_\infty \) is unbounded from above.

**Proof.** (i) Take \( p \in [0, \gamma \wedge 1] \). Then \( r := E\left[M_1^p\right] < 1 \) and, Lemma 4 \( E[|Q_i|^p] < \infty \). It follows that \( E[|Y_{n+1} - Y_n|^p] = E[M_1^p \ldots M_1^p Q_{n+1}^p] = r^nE[|Q_1|^p] \) and, therefore,

\[
E\left[\sum_{n \geq 0} |Y_{n+1} - Y_n|^p\right] \leq \sum_{n \geq 0} E[|Y_{n+1} - Y_n|^p] < \infty.
\]

Thus, \( \sum_n |Y_{n+1} - Y_n| < \infty \) a.s. implying that \( Y_{n} \) converges a.s. to some finite random variable we shall denote \( Y_\infty \).

Let \( Y_n^\star := \sup_{s \leq t} Y_s \). Then

\[
E[Y_{n+1}^\star] \leq e^p E\left[\left(\int_0^{V_1} e^{-V_s} ds\right)^p\right] + (H(|x|))^{p-1} E\left[\left(\int_0^{V_1} e^{-p V_s} ds\right)^p\right] < \infty.
\]

Put

\[
\Delta_n = \sup_{v \in [V_{n+1}, V_{n+2}]} \left| \int_{V_n}^v e^{-V_s} dP_s \right|.
\]

Then

\[
E[\Delta_n^p] = E\left[M_1^p \ldots M_1^p \sup_{v \in [V_n, V_{n+1}]} \left| \int_{V_n}^v e^{-(V_s - V_n)} dP_s \right|^p\right] \leq r^n E[|Y_{n+1}^\star|]
\]

and, therefore, for any \( \varepsilon > 0 \)

\[
\sum_{n \geq 0} P[\Delta_n \geq \varepsilon] \leq e^{-p} \sum_{n \geq 0} E[\Delta_n^p] < \infty.
\]

By the Borel–Cantelli lemma for all \( \omega \) except a null-set \( \Delta_n(\omega) \leq \varepsilon \) for all \( n \geq n(\omega) \). This implies that \( Y_t \) converges a.s. to the same limit as the sequence \( Y_{n} \).

(ii) Rewriting (12) in the form

\[
Y_{n+1} = Q_1 + M_1 (Q_2 + M_2 Q_3 + \ldots + M_{n-1} Q_n)
\]

and observing that the sequence of random variables in the parentheses converges almost surely to a random variable with same law as \( Y_\infty \) and independent on \((Q_1, M_1)\) we get the needed assertion.

(iii) In virtue of (ii) it is sufficient to check that the set \( \{Q_1 \geq N, M_1 \leq 1/N\} \) is non-null whatever is \( N \geq 1 \).
Recall that
\[ Q_1 = -e^{-V} x \pi_{v_1} - c \int_0^{v_1} e^{-V} ds, \]
where \( dV_s = \sigma_0 \, dW_s + (1/2) \sigma_0^2 \beta_0 \, ds. \)

We consider several cases.

1) \( c < 0. \) Using conditioning with respect to \( \theta \) we may argue as \( \theta \) would be deterministic, i.e. assuming that \( V \) is a process with a deterministic switching of parameters and \( v_1 \) is just a number, say, \( t > 0. \) On the set \( \{ T_1 > v_1 \} \) we have \( Q_1 = -c \int_0^{v_1} e^{-V} ds. \) Since \( T_1 \) is independent on \( W \) and the set \( \{ T_1 > v_1 \} \) we need to check only that the set
\[ B_N(t) := \left\{ -c \int_0^t e^{-V} ds \geq N, e^{-V} \leq 1/N \right\} \]
is non-null. In the case where \( \theta \) has no jumps on \([0, t]\) the process \( V_s = \sigma_0 W_t + (1/2) \sigma_0^2 \delta_0 t \) we get the latter property using conditioning with respect to \( W_t = x. \) Indeed, the conditional distribution of \((W_s)_{s \leq t}\) given \( W_t = x \) is the same as the (unconditional) distribution of the Brownian bridge \( B^x = (B^x_s)_{s \leq t} \) ending at \( t \) at the value \( x. \) The latter is a continuous Gaussian process. This implies that the conditional distribution of the integral involved in \( B_N(t) \) is unbound from above. Integrating over a suitable set with respect to the distribution of \( W_t \) shows that \( B_N(t) \) is non-null. In the case of several jumps at the moments \( t_1, \ldots t_k \) we can show that the integral over the interval \([0, t]\) has unbounded conditional distribution given \((W_{t_1}, W_{t_2} - W_{t_1}, \ldots, W_{t_k} - W_{t_{k-1}}) = (x_{t_1}, x_{t_2}, \ldots x_{t_k+1}) \) and conclude by integrating with respect to the distribution of the increments of \( W \) over a set \([x, \infty[k+1] \) for sufficiently large \( x \in \mathbb{R}_. \)

2) \( c \geq 0. \) Put \( \sigma^* := \max_j \sigma_j, \kappa^* := \max_j (1/2) \sigma_j^2 \beta_j, \kappa_* := \min_j (1/2) \sigma_j^2 \beta_j. \) Let \( \delta > 0 \) and let \( r_N := (\sigma^* \delta + \ln N)/\kappa_. \) The set
\[ A_N := \{ |W_s| < \delta, \forall s \leq r_N + 1 \} \cap \{ r_N \leq v_1 \leq r_N + 1 \} \]
is non-null. On this set for all \( s \in [0, v_1] \) we have the bounds
\[ -\sigma^* \delta + \kappa_* r_N \leq V_s \leq \sigma^* \delta + \kappa^* (r_N + 1) \]
implying that
\[ M_1 = e^{-V_{v_1}} \leq e^{\sigma^* \delta - \kappa_* r_N} = 1/N \]
and
\[ c \int_0^{v_1} e^{-V} ds \leq (e/N)(r_N + 1) =: C_N. \]

Since \( P \) is not an increasing process, \( P([\epsilon, \infty, 0]) > 0. \) Hence, the set
\[ \left\{ e^{-\sigma^* \delta - \kappa^* (r_N + 1)} |x| I_{\{x < 0\}} \ast \pi_{r_N} \geq C_N + N, x I_{\{x > 0\}} \ast \pi_{r_N} = 0 \right\} \]
is non-null and its intersection with \( A_N \) is also non-null. But this intersection is a subset of the set \( \{ Q_1 \geq N, M_1 \leq 1/N \}. \) \( \square \)
6 Bounds for the ruin probability

Lemma 6 For every \( u > 0 \)

\[
\bar{G}_i(u) \leq \Psi_i(u) = \frac{G_i(u)}{E[\bar{G}_{\theta_{u,i}}(0)|\tau_{u,i} < \infty]} \leq \frac{\bar{G}_i(u)}{\min_j \bar{G}_j(0)},
\]

where \( \bar{G}_i(u) := P[Y_{\infty}^i > u] \).

Proof. Let \( \tau \) be an arbitrary stopping time with respect to the filtration \( \mathcal{F}_t^{P,R,\theta} \). As the finite limit \( Y_{\infty}^i \) exists, the random variable

\[
Y_{\tau,\infty}^i := \begin{cases} 
- \lim_{N \to \infty} \int_{\tau,\tau + N} e^{-(V_s - V_\tau)dP_s}, & \tau < \infty, \\
0, & \tau = \infty,
\end{cases}
\]

is well defined. On the set \( \{ \tau < \infty \} \)

\[
Y_{\tau,\infty}^i = e^{V_\tau}(Y_{\infty}^i - Y_{\tau}^i) = X_{\tau}^u + e^{V_\tau}(Y_{\infty}^i - u).
\]

Let \( \xi \) be an \( \mathcal{F}_t^{P,R,\theta} \)-measurable random variable. Note that the conditional distribution of \( Y_{\tau,\infty}^i \) given \( (\tau, \xi, \theta_\tau) = (t, x, j) \in \mathbb{R}_+ \times \mathbb{R} \times \{0, 1, K-1\} \) is the same as the distribution of \( Y_{\infty,j}^j \). It follows that

\[
P[Y_{\tau,\infty}^i > \xi, \tau < \infty, \theta_\tau = j] = E[\bar{G}_j(\xi) I_{\{\tau < \infty, \theta_\tau = j\}}].
\]

Thus, if \( P[\tau < \infty] > 0 \), then

\[
P[Y_{\tau,\infty}^i > \xi, \tau < \infty] = E[\bar{G}_{\theta_\tau}(\xi) | \tau < \infty] P[\tau < \infty].
\]

Noting that \( \Psi_i(u) := P[\tau_{u,i} < \infty] \geq P[Y_{\infty}^i > u] = \bar{G}_i(u) > 0 \), we deduce from here using (15) that

\[
\bar{G}_i(u) = P[Y_{\infty}^i > u, \tau_{u,i} < \infty] = P[Y_{\tau_{u,i},\infty}^i > X_{\tau_{u,i}}^u, \tau_{u,i} < \infty]
= E[\bar{G}_{\theta_{u,i}}(X_{\tau_{u,i}}^u) | \tau_{u,i} < \infty] P[\tau_{u,i} < \infty]
\]

implying the equality in (15). Also,

\[
E[\bar{G}_{\theta_{u,i}}(0)|\tau_{u,i} < \infty] = \sum_{j=0}^{K-1} \bar{G}_j(0) P[\theta_{u,i} = j|\tau_{u,i} < \infty] \leq \min_j \bar{G}_j(0)
\]

implying the result. \( \square \)
7 Ruin with probability one

Assuming that \( \beta^* := \max_j \beta_j \) is strictly negative, we give a sufficient condition under which the ruin is imminent.

**Theorem 3** Suppose that \( \beta^* < 0, \ II([\beta^* - \varepsilon, -\varepsilon]) > 0 \) for all \( \varepsilon > 0 \), and there exists \( \delta \in [0, |\beta^*| \wedge 1] \) for which \( II(|x|^\delta) < \infty \). Then \( \Psi_u = 1 \) for any \( u > 0 \) and \( i \).

Proof. Put \( \tilde{X}_n = X_n^1 := X_{n+1} \). Note that (3) implies that the sequence \( \tilde{X}_n \) satisfies the difference equation

\[
\tilde{X}_n = A_n \tilde{X}_{n-1} + B_n, \quad n \geq 1, \quad \tilde{X}_0 = u, \quad (17)
\]

where \( A_n := M_n^{-1} := e^{V_n - V_{n-1}} \) and

\[
B_n := -M_n^{-1} Q_n = \int_{[V_{n-1}, V_n]} e^{V_n - V_s} dP_s.
\]

According to Corollary 6.2 in [14], \( \inf_n \tilde{X}_n < 0 \) a.s. if the ratio \( B_1/A_1 \) is unbounded from below and there is \( \delta \in [0, 1] \) such that \( E[A_n^\delta] < 1 \) and \( E[|B_1|^\delta] < \infty \). By our assumption the event that on a fixed finite interval the process \( P \) has arbitrary many downward jumps of the size larger than \( \varepsilon \) and has no jumps upward is of strictly positive probability. Due to independence of \( P \) and \((W, \theta)\) this implies that \( -Q_1 = B_1/A_1 \) is unbounded from below.

Noting that

\[
f_j(\delta) := \frac{\lambda_j}{\lambda_j + (1/2)\sigma_j^2(\delta|\beta_j - \delta|)} < 1, \quad (18)
\]

we get that

\[
E[A_n^\delta] = E[e^{\delta V_n}] = E\left[ e^{\delta V_{\tau_1}} \prod_{i=2}^{\infty} e^{\delta(V_{\tau_i} - V_{\tau_{i-1}})} \right] = E \left[ \sum_{k=1}^{\infty} f_0(\delta)f_{\theta_1}(\delta) \ldots f_{\theta_{k-1}}(\delta) \right] < 1.
\]

Finally, the property \( E[|B_1|^\delta] < \infty \) can be proved by the same arguments as in the proof of Lemma 3 with \( \gamma \) and \( V \) replaced by \( \delta \) and \( V_{s+1} - V_s \) and the reference to [13] replaced by the reference to [19] in Lemma 7 below. \( \square \)

**Lemma 7** Suppose that \( \beta^* < 0 \). Then for any \( \delta \in [0, |\beta^*|] \)

\[
E \left[ \int_0^{V_1} e^{\delta(V_{s+1} - V_s)} ds \right] < \infty, \quad E \left[ \left( \int_0^{V_1} e^{\delta(V_{s+1} - V_s)} ds \right)^\delta \right] < \infty. \quad (19)
\]

Proof. The arguments are very similar to that of Lemma 3 and we only sketch them. The only new feature is that we need to consider processes of the form \( (V_T - V_s)_{s \in [0, T]} \) rather than \( (V_s)_{s \in [0, T]} \). The crucial observation is that the process \( (W_T - W_s)_{s \in [0, T]} \) in the reversed time \( s' := T - s \) is a Wiener process.

First, observe that

\[
E \left[ \int_0^{V_1} e^{\delta(V_{s+1} - V_s)} ds \right] = E \left[ \sum_{k=1}^{\infty} e^{\delta(V_{s_k+1} - V_{s_k})} \int_{s_{k-1}}^{s_k} e^{\delta(V_{s_k} - V_s)} ds \right]
\]

Given a trajectory of \( \theta \), the exponential and the integral in each summand are conditionally independent and their conditional expectations admit explicit expressions. For the integral it
is $1/\lambda \theta_{k-1} f_{\theta_{k-1}}(\delta)$ where $f_j$ in (18). Note that for $\delta \in [0, \beta^*]$ the conditional expectation of the integral is dominated by $1/\lambda$, implying that

$$
\mathbb{E} \left[ \int_0^{\tau_1} e^{\delta (V_{\tau_1} - V_s)} ds \right] \leq \frac{1}{\lambda^*} \mathbb{E} \left[ \sum_{k=1}^{\infty} e^{\delta (V_{\tau_k} - V_{\tau_{k-1}})} \right]
$$

$$
= \frac{1}{\lambda^*} \mathbb{E} \left[ \prod_{n=1}^{\infty} e^{\delta (V_{\tau_{n+1}} - V_{\tau_n})} \right] = \frac{1}{\lambda^*} \mathbb{E} \left[ \sum_{k=1}^{\infty} \prod_{n=k}^{\infty} f_{\theta_n}(\delta) \right].
$$

Due to the choice of $\beta$, we have that $\tilde{f}^* := \max_j f_j(\delta) < 1$ and, therefore,

$$
\mathbb{E} \left[ \sum_{k=1}^{\infty} \prod_{n=k}^{\infty} f_{\theta_n}(\delta) \right] \leq \mathbb{E} \left[ \sum_{k=1}^{\infty} (\tilde{f}^*)^{\infty-k} \leq \sum_{k=1}^{\infty} (\tilde{f}^*)^k < \infty.
$$

The first property in (19) is proven.

Let $\tau$ be an exponential random variable with parameter $\lambda > 0$. For any $\delta \in [0, \tilde{r}]$, where

$$
\tilde{r} := \sqrt{2\lambda/\sigma^2 + \beta^2/4 + |\beta|/2},
$$

we have, according to (1.1.1) in Ch. 2 of the reference book [4], that

$$
\tilde{C}(\delta, \lambda, \beta, \sigma) := \mathbb{E} \left[ e^{\delta \sigma \sup_{s \leq \tau} W_s^{(\sigma \beta/2)}} \right] = \frac{\tilde{r}}{\tilde{r} - \delta} < \infty.
$$

We get (as in Corollary [1]) that for all $k \geq 1$

$$
\mathbb{E} \left[ \left( \int_{\tau_{k-1}}^{\tau_k} e^{\delta (V_{\tau_k} - V_s)} ds \right) \right] \leq \tilde{C}^*(\delta),
$$

with some constant $\tilde{C}^*(\delta) < \infty$, and we complete the proof of the second property in (19) as in Lemma [3].

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References

1. Albrecher, H., Gerber, H., Yang, H.: A direct approach to the discounted penalty function. North American Actuarial Journal, 14, 4, 420–434 (2010)
2. Anderson, W.J.: Continuous-time Markov chains. An Applications-Oriented Approach. Springer, Berlin (1991)
3. Asmussen, S., Albrecher, H.: Ruin Probabilities. World Scientific, Singapore (2010)
4. Borodin, A.N., Salminen, P.: Handbook of Brownian Motion – Facts and Formulae. 2nd Edition, Basel–Boston–Berlin, Springer Basel AG, Birkhäuser (2002)
5. Buraczewski, D., Damek, E.: A simple proof of heavy tail estimates for affine type Lipschitz recursions. Stochastic processes and their applications, 127, 657 – 668 (2017)
6. Di Masi, G.B., Kabanov, Yu.M., Runggaldier, W.J.: Mean-square hedging of options on a stock with Markov volatilities. Theory Probab. Appl., 39, 1, 172–182 (1994)
7. Ellanskaya, A., Kabanov, Yu.: On ruin probabilities with risky investments in a stock with stochastic volatility. Extremes, DOI 10.1007/s10687-021-00420-8
8. Feller, W.: An Introduction to Probability Theory and Its Applications, Volume 1, 3rd Edition, Wiley (1991)
9. Frolova, A., Kabanov, Yu., Pergamenshchikov, S.: In the insurance business risky investments are dangerous. Finance and Stochastics, 6, 2, 227 – 235 (2002)
10. Goldie, C.M.: Implicit renewal theory and tails of solutions of random equations. Ann. Appl. Probab., 1, 1, 126-166 (1991)
11. Grandell, J.: Aspects of Risk Theory. Springer, Berlin (1990)
12. Guivarc’h, Y., Le Page, E.: On the homogeneity at infinity of the stationary probability for affine random walk. In: Bhattacharya S., Das T., Ghosh A., Shah R. (eds). “Recent trends in ergodic theory and dynamical systems”. Contemporary Mathematics AMS, 119 – 130 (2015)
13. Kabanov, Yu., Pergamenshchikov, S.: In the insurance business risky investments are dangerous: the case of negative risk sums. Finance and Stochastics, 20, 2, 355 – 379 (2016)
14. Kabanov, Yu., Pergamenshchikov, S.: Ruin probabilities for a Lévy-driven generalized Ornstein–Uhlenbeck process. Finance and Stochastics, 24, 1, 39 – 69 (2020)
15. Kabanov, Yu., Pukhlyakov, N.: Ruin probabilities with investments: smoothness, IDE and ODE, asymptotic behavior. Preprint (2020). https://arxiv.org/abs/2011.07828
16. Novikov, A.A.: On discontinuous martingales, Theory of Probability and its Applications, 20, 11 – 26 (1975)
17. Paulsen, J.: Risk theory in stochastic economic environment. Stochastic processes and their applications, 46, 327 – 361 (1993)
18. Paulsen, J.: Sharp conditions for certain ruin in a risk process with stochastic return on investments, Stoch. Process. Appl., 75, 135 – 148 (1998)
19. Paulsen, J.: On Cramér-like asymptotics for risk processes with stochastic return on investments. Annals of Applied Probability, 12, 1247 – 1260 (2002)
20. Paulsen, J., Gjessing, H. K.: Ruin theory with stochastic return on investments. Advances in Applied Probability, 29, 965 – 985 (1997)
21. Pergamenshchikov, S., Zeitouni, O.: Ruin probability in the presence of risky investments. Stoch. Process. Appl., 116, 267–278 (2006). Erratum to: “Ruin probability in the presence of risky investments”. Stoch. Process. Appl., 119, 305 – 306 (2009)