Quantum cosmology, minimal length and holography

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We study the effects of a generalized uncertainty principle on the classical and quantum cosmology of a closed Friedmann Universe filled with two fluids, namely, dust and radiation, non-interacting. More concretely, assuming the existence of a minimal length, we show that a corresponding minimal area will constitute a Dirac observable. In addition, ’t Hooft conjecture on the cosmological holographic principle is also investigated for either the case of dust or radiation being present. We describe how this holographic principle is satisfied for large values of a quantum number, n. This occurs when the entropy is computed in terms of the minimal area.

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I. INTRODUCTION

It has been pointed that our existing theories will break-down, when applied to small distances or very high-energies. In particular, the geometrical continuum beyond a certain limit will no longer be valid. This suggested the development of a scenario based on indivisible units of length. In recent years, the concept of minimal length has been described through algebraic methods, by means of a generalized uncertainty principle, which could be induced by gravitational effects, first proposed by Mead [1]. Moreover, a generalized uncertainty principle (GUP), as presented in theories such as string theory and doubly special relativity, conveys the prediction of a minimal measurable length [2]. A similar feature appears in the polymer quantization in terms of a mass scale [3]. The concept of a subsequent minimal area can, thus, be considered. This then raises the question whether it can be used in discussions about entropy. Namely, involving the holographic principle.

The holographic principle in quantum gravity was first suggested by ’t Hooft [4] and, later, extended to string theory by Susskind [5]. The most radical part of this principle proposes that the degrees of freedom of a spatial region reside not in the bulk but in the boundary. Furthermore, the number of boundary degrees of freedom per Planck area should not be larger than one. The general assumption is that the Bekenstein-Hawking area law applies universally to all cosmic or black hole horizons. Moreover, recently, in Ref. [6], it has been shown that there is an derivation for the holographic/conformal anomaly Friedmann equation. The mentioned derivation is obtained by assuming the effect of a GUP on the entropy from the apparent horizon and admitting a constraint, which relates the anomaly coefficient and the GUP parameter.

In this paper we investigate the quantum cosmology of a closed FLRW Universe, filled with a perfect fluid of radiation together with a dust fluid, non-interacting. Our aim in the herein scenario, by assuming a minimal length, is to determine whether a corresponding minimal area will be of relevance in discussing the holographic principle. The paper is organized as follows. In Section II we present the classical setting in the presence of a GUP. Section III provides the quantum cosmological description of our model. Section IV conveys how our model can be used to discuss holographic features. In Section V we summarize our results.

II. CLASSICAL MODEL WITH GUP

Let us start with the line element of homogeneous and isotropic FLRW geometry for the closed Universe

\[
ds^2 = -N^2(\eta)d\eta^2 + a^2(\eta)d\Omega^2_{(3)},
\]

where \(N(\eta)\) is the lapse function, \(a(\eta)\) is the scale factor and \(d\Omega^2_{(3)}\) is the standard line element for a unite three-sphere. The action functional corresponding to the line element (1) for gravitational and matter content as perfect fluid is given by [10]

\[
S = \frac{M^2_{Pl}}{2} \int_{\mathcal{M}} \sqrt{-g}Rd^4x + M^2_{Pl} \int_{\partial\mathcal{M}} \sqrt{g^{(3)}}Kd^3x - \int_{\mathcal{M}} \sqrt{-g}\rho d^4x = 6\pi^2 M^2_{Pl} \int \left( -\frac{a^2}{N} + Na \right) d\eta - 2\pi^2 \int Na^3 \rho d\eta,
\]
where \(M^2_{\text{Pl}} = \frac{1}{2\pi G}\) is the reduced Planck’s mass in natural units, \(M = I \times S^3\) is the spacetime manifold, \(\partial M = S^3\), \(K\) is the trace of the extrinsic curvature of the spacetime boundary and an overdot denotes differentiation with respect to \(\eta\).

If we assume radiation to have the form of a perfect fluid as \(\rho_\gamma = \frac{4}{3}p_\gamma\), and by redefining the scale factor and lapse function as

\[
\begin{align*}
  a(\eta) &= x(\eta) + \frac{M}{2\pi M^2_{\text{Pl}}} := x - x_0, \\
  N(\eta) &= 12\pi^2 M_{\text{Pl}} a(\eta) \tilde{N},
\end{align*}
\]

the total Lagrangian, constructed of gravitational and dust matter. In addition, \(N\) the total Hamiltonian as

\[
\mathcal{H} = -\frac{1}{2N} M_{\text{Pl}} \dot{x}^2 + \frac{\tilde{N}}{2} M_{\text{Pl}} \dot{\omega}^2 x^2 - \mathcal{E} \tilde{N}.
\]

where we have employed

\[
\begin{align*}
  \mathcal{E} &= \frac{M^2}{4M^2_{\text{Pl}}} + 12\pi^2 N_\gamma M_{\text{Pl}}, \\
  \omega &= 12\pi^2 M_{\text{Pl}}.
\end{align*}
\]

Besides, we introduce \(M\) and \(\mathcal{N}\gamma\) as

\[
\begin{align*}
  M &= \int_{\partial M} \sqrt{g^{(3)}} \rho_0 a^0 d^3 x, \\
  \mathcal{N}_\gamma &= \int_{\partial M} \sqrt{g^{(3)}} \rho_0 a^0 d^3 x.
\end{align*}
\]

In the above definition \(M\) denotes the total mass of the dust matter. In addition, \(\mathcal{N}_\gamma\) could be related to the total entropy of radiation as follows: the energy density \(\rho_\gamma\), the number density \(n_\gamma\), the entropy density \(s_\gamma\) and the scale factor are related to temperature via \(\rho_\gamma = \frac{\pi^2 g T^4}{360}\), \(n_\gamma = \frac{\alpha(3)}{12\pi} gT^3\), \(s_\gamma = \frac{4\pi^4}{27}\) and \(a(\eta) \sim \frac{1}{\eta}\) for radiation \([11]\).

Consequently, we can find \(\mathcal{N}_\gamma = \frac{\gamma}{3} (\frac{\pi^2 g T^4}{360})^{1/3}(S^{(\gamma)})^{4/3}\), where \(S^{(\gamma)}\) denotes the total entropy of radiation. \([22]\)

The momenta conjugate to \(x\) and the primary constraint, which are necessary to construct the Hamiltonian of the model, are given by

\[
\begin{align*}
  \Pi_x &= \frac{\partial \mathcal{E}}{\partial \dot{x}} = -\frac{\tilde{N}}{M_{\text{Pl}}} x, \\
  \Pi_{\tilde{N}} &= \frac{\partial \mathcal{E}}{\partial \dot{\tilde{N}}} = 0.
\end{align*}
\]

Therefore, the Hamiltonian corresponding to \([11]\) become

\[
\mathcal{H} = -\tilde{N} \left[ \frac{1}{2M_{\text{Pl}}} \Pi_x^2 + \frac{1}{2} M_{\text{Pl}} \dot{\omega}^2 x^2 - \mathcal{E} \right].
\]

From \([11]\), we see that the momentum conjugate to \(\tilde{N}\) vanishes, i.e. the Lagrangian of the system is singular. Therefore, we have to add it to the Hamiltonian \([8]\) and construct the total Hamiltonian as

\[
\mathcal{H}_T = -\tilde{N} \left[ \frac{1}{2M_{\text{Pl}}} \Pi_x^2 + \frac{1}{2} M_{\text{Pl}} \dot{\omega}^2 x^2 - \mathcal{E} \right] + \lambda \Pi_{\tilde{N}},
\]

where \(\lambda\) is a Lagrange multiplier.

During the evolution of the system, the primary constraint should hold; namely, we have

\[
\dot{\Pi}_{\tilde{N}} = \{ \Pi_{\tilde{N}}, \mathcal{H}_T \} \approx 0,
\]

which leads to the secondary (Hamiltonian) constraint as

\[
H := \frac{1}{2M_{\text{Pl}}} \Pi_x^2 + \frac{1}{2} M_{\text{Pl}} \omega^2 x^2 - \mathcal{E} \approx 0.
\]

We should note that a gauge-fixing condition is required for the constraint \([11]\), in which \(\tilde{N} = \text{const.}\) can be a possibility. Thus, by choosing the gauge as \(\tilde{N} = 1/\omega\), and reminding that the canonical variables satisfying the Poisson algebra \([x, \Pi_x] = 1\), we get the following Hamilton equations of motion

\[
\begin{align*}
  \dot{x} &= -\frac{1}{\omega M_{\text{Pl}}} \Pi_x, \\
  \dot{\Pi}_x &= \omega M_{\text{Pl}} x.
\end{align*}
\]

Employing the Hamiltonian constraint \([11]\), easily leads us to the well known solution for a closed Universe as

\[
\begin{align*}
  a(\eta) &= \frac{a_{\text{Max}}}{1 + \sec \phi} \left[ 1 - \sec \phi \cos(\eta + \phi) \right], \\
  a_{\text{Max}} &= \frac{M}{12\pi^2 M_{\text{Pl}}} + \left( \frac{2\pi}{M_{\text{Pl}} a_{\text{Max}}} \right)^2, \\
  \cos \phi &= \frac{\sqrt{8\pi}}{M_{\text{Pl}}} \eta,
\end{align*}
\]

where \(a_{\text{Max}}\) represents the maximum radius of the closed Universe and we have assumed that the initial singularity occurs at \(\eta = 0\).

Let us now investigate the effects, at a classical level, of a deformed Poisson algebra in the presence of a minimal length. We write \([12]\)

\[
\begin{align*}
  \{x, x\} &= \{\Pi_x, \Pi_x\} = 0, \\
  \{x, \Pi_x\} &= 1 + a^2 L_{\text{Pl}}^2 \Pi_x^2,
\end{align*}
\]

where \(a\) is a dimensionless constant and \(L_{\text{Pl}}\) denotes Planck’s length in natural units. It is normally assumed that \(a\) is of the order of unit. In this case, the deformation will contribute only in the Planck regime of the Universe and for this reason we address in the next section the quantum cosmology of the herein model.

A physical length of the order \(aL_{\text{Pl}}\) is yet unobserved so it cannot exceed the electroweak scale \([13]\), which implies \(\frac{1}{\sqrt{8\pi}} \leq a \leq 10^{17}\). As a consequence of the above deformation with Hamiltonian \([8]\), Hamilton equations become

\[
\begin{align*}
  \dot{x} &= -\frac{\tilde{N}}{M_{\text{Pl}}}(1 + a^2 L_{\text{Pl}}^2 \Pi_x^2) \Pi_x, \\
  \dot{\Pi}_x &= \tilde{N} \omega M_{\text{Pl}} (1 + a^2 L_{\text{Pl}}^2 \Pi_x^2) x.
\end{align*}
\]

To solve these equations, we change the variable \(\Pi_x\) to \(y\) as

\[
\Pi_x = \frac{1}{a L_{\text{Pl}}} \tan(\alpha L_{\text{Pl}} y),
\]

where \(\alpha\) is a dimensionless constant.
which, using Hamiltonian constraint \( |11] \) and gauge \( \tilde{N} = 1/\omega \), gives

\[
\begin{align}
\alpha(\eta) &= \frac{\alpha_{\text{Max}}}{B} \left( 1 - \frac{A \cos((1 + \phi) \eta)}{\sqrt{1 + 2\alpha^2 L_P^2 \cos^2((1 + \phi) \eta)}} \right),
\end{align}
\]

where

\[
\begin{align}
B &:= 1 + \Omega^{-1} \sqrt{1 + 2\alpha^2 L_P^2 \cos^2 \phi}, \\
A &:= \sec \phi \sqrt{1 + 2\alpha^2 L_P^2 \cos^2 \phi}, \\
\Omega &:= \sqrt{1 + 2\alpha^2 L_P^2}, \\
\cos \phi &:= \frac{M}{\sqrt{2\alpha L_P^2}} (1 + 24\alpha^2 \tilde{N})^{-1/2},
\end{align}
\]

and \( \alpha_{\text{Max}} \) is similar to the non-deformed case defined in \( |11] \). If we take the limit \( \alpha \to 0 \), we find solution \( |13] \) which shows that the canonical behavior is recovered in this limit. We immediately obtain from \( |17] \) that the Universe reaches its maximum radius at \( \eta = \frac{\pi - \phi}{\Omega} \), and it terminates in the big-crunch singularity at \( \eta = \frac{2(\pi - \phi)}{\Omega} \).

### III. QUANTUM COSMOLOGY WITH MINIMAL LENGTH

At the quantum level, the deformed Poisson algebra \( |14] \) is replaced by the following commutation relation between the phase space variables of minisuperspace

\[
[x, \Pi_x] = i(1 + \alpha^2 L_P^2 \Pi_x^2).
\]

Commutation relation \( |18] \) provides the minimal length uncertainty relation \( \text{(MLUR)} \) \( |14] \)

\[
\Delta x \geq \frac{1}{2} \left( \frac{1}{\Delta \Pi_x} + \alpha^2 L_P^2 \Delta \Pi_x \right).
\]

This MLUR implies the existence of a minimal length

\[
\Delta x_{\text{min}} = \alpha L_P,
\]

which indicates it is impossible to consider any physical state as the eigenstate of the position operator \( |15] \). On a dense domain of the Hilbert space, the position and momentum operators obeying relation \( |18] \) could be represented in momentum space as \( |16] \)

\[
\Pi_x = \Pi_x, \\
x = i(1 + \alpha^2 L_P^2 \Pi_x^2) \frac{d}{d \Pi_x}.
\]

Hence, the inner product between two arbitrary states on a dense domain will be

\[
\langle \phi | \psi \rangle = \int_{-\infty}^{\infty} \frac{d \Pi_x}{1 + \alpha^2 L_P^2 \Pi_x^2} \phi^*(\Pi_x) \psi(\Pi_x).
\]

Therefore the modified WDW equation in the presence of MLUR \( |19] \) is given by

\[
\left[ -M_P^2 \omega^2 \left( 1 + \alpha^2 L_P^2 \Pi_x^2 \frac{d}{d \Pi_x} \right)^2 + \frac{1}{2M_P^2} \Pi_x^2 \right] \psi = E \psi.
\]

Let us proceed using variable transformation \( |10] \), the above WDW equation will be changed to the trigonometric Pöch-Teller (TPT) form

\[
\frac{d^2 \psi(z)}{dz^2} + \left( \epsilon - \frac{V}{\cos^2(z)} \right) \psi(z) = 0,
\]

where \( z = \alpha L_P y \) and

\[
\begin{align}
\epsilon &= \frac{1}{(12\pi^3 + \alpha^2)}, \\
V &= \frac{1}{(12\pi^3 + \alpha^2)}.
\end{align}
\]

The normalized eigenfunctions of TPT equation are given by \( |17] \)

\[
\psi_n(z) = 2^n \Gamma(\nu) \sqrt{n!(n+1)! \alpha L_P^2} \cos^\nu(z) C_n^\nu(\sin(z)),
\]

where \( n \) is an integer, \( C_n^\nu \) is the Gegenbauer polynomial and

\[
\nu := \frac{1}{2} \left( 1 + \sqrt{1 + \frac{1}{36\pi^4 \alpha^4}} \right).
\]

Moreover, the corresponding eigenvalue of the WDW equation is given by

\[
E_n = 12\pi^2 M_P^2 \{ (n + \frac{1}{2}) \sqrt{1 + 36\pi^4 \alpha^4} + 6\pi^2 \alpha^2 (n^2 + n + \frac{1}{2}) \}.
\]

Let us now obtain the Dirac observables of the model. According to Dirac \( |18] \), the observables of a theory are those quantities which have vanishing commutators with the constraints of theory. In order to retrieve them, we start by finding the spectrum of the WDW equation \( |24] \). Considering the infinite number of bound states of Eq. \( |24] \), the underlying Lie algebra could be expected as its spectrum generating algebra. The lowering and raising operators for the WDW equation with the TPT potential \( |24] \) can be built using a factorization type method. If we define the ladder operators \( |19] \)

\[
\begin{align}
A &:= \exp(\partial_\nu)(\frac{d}{dz} + \nu \tan(z)), \\
A^\dagger &:= (-\frac{d}{dz} + \nu \tan(z)) \exp(-\partial_\nu),
\end{align}
\]

then the action of these factor operators on normalized eigenfunctions will be

\[
\begin{align}
A |\nu, n \rangle &= \sqrt{n(2\nu + 2 + n)} |\nu + 2, n - 1 \rangle, \\
A^\dagger |\nu, n \rangle &= \sqrt{(n + 1)(2\nu - 1 + n)} |\nu - 2, n + 1 \rangle.
\end{align}
\]

It can be verified that these operators together with \( \tilde{A} |\nu, n \rangle = (1/2 - \nu) |\nu, n \rangle \), obey the \( su(2) \) Lie algebra

\[
[\tilde{A}, A] = -A, [\tilde{A}, A^\dagger] = A^\dagger, [A, A^\dagger] = -2\tilde{A}.
\]

Also, based on recursion relations of Gegenbauer polynomials, we can introduce the following three operators
Consequently, the minimal surface defined as \[21\] can be written as \[20\] and according to (27), (33) and (37) is also a Dirac observable. The action of the above generators on a set of basis eigenvectors \(|\nu, n\rangle\) is given by
\[
\begin{aligned}
J_0|\nu, n\rangle &= (-j + n)|\nu, n\rangle, \\
J_-|\nu, n\rangle &= \sqrt{n(-2j + n - 1)}|\nu, n - 1\rangle, \\
J_+|\nu, n\rangle &= \sqrt{(n + 1)(2j + n)}|\nu, n + 1\rangle,
\end{aligned}
\] where \(j := -(\nu + 1)/2 < 0\) denotes the Bargmann index of dynamical group. The corresponding Casimir operator can be calculated as
\[
\begin{aligned}
J^2 := J_0(J_0 - 1) - J_+J_- , \\
J^2|\nu, n\rangle &= j(j + 1)|\nu, n\rangle,
\end{aligned}
\] with well known properties
\[34\]
\[\left[J^2, J_\pm \right] = 0, \quad \left[J^2, J_0 \right] = 0.\]
Hence, a representation of \(su(1,1)\) is determined by the Bargmann index and the eigenvectors of the Casimir and \(J_0\). In addition, we find that the Hamiltonian \[11\] could be written as \[21\]
\[
H = 72\pi^4\alpha M_{Pl}\left[ J_+ J_- - (2j + 1) J_0 \right] - 3\pi^2\alpha M_{Pl}(j + 1)(2j + 1) - \mathcal{E},
\]
from which we can conclude that the point that the Casimir operator \[31\] and \(J_0\) commute with the Hamiltonian
\[37\]
\[\left[J^2, H \right] = \left[J_0, H \right] = 0.\]
Therefore, \(J^2\) and \(J_0\) leave the physical Hilbert space invariant and we choose them as physical operators of the model as \(\{J^2, J_0, 1\}\). Moreover, Eqs. \[37\] show that \(J^2\) and \(J_0\) are Dirac observables of the cosmological model. Consequently, the minimal surface defined as \[21\]
\[
\mathcal{A}_{\text{min}} := 4\Delta S_{\text{min}}^2 = 4\alpha^2 L_{\text{Pl}}^2,
\]
in terms of a minimal surface
\[39\]
\[S_n^{(\gamma)} \simeq \left( \frac{4\pi^7g}{45} \right)^\frac{1}{3} \left( \frac{\mathcal{A}_{\text{min}}}{4G} \right)^\frac{2}{3} (n^2 + 2\nu + \nu)^{\frac{2}{3}},\]
which, according to Eqs. \[27\], \[33\] and \[37\] indicates that the entropy of is a Dirac observable. On the other hand, the expectation value of the square of scale factor could be calculated from \[21\], \[29\] and \[30\], which gives
\[40\]
\[\langle a^2 \rangle = \frac{\alpha^2 L_{\text{Pl}}^2}{n^2 + 2\nu + \nu + n - \frac{1}{2}}.
\]
In the presence of minimum length, \(\nu \simeq 1\). Hence for large values of the quantum number, \(n\), we find \(\langle a^2 \rangle \simeq \mathcal{A}_{\text{min}} n^2/8\). On the other hand, the apparent horizon of a FLRW model for the radiation case, is given by \(R_{\text{ah}} = (H^2 + 1/\alpha^2)^{-1/2} = \sqrt{\frac{2\pi^2 g}{\alpha^2}} a^2\). Inserting this result into Eq. \[39\], we find, for large values of quantum number \(n\), that
\[41\]
\[S_n^{(\gamma)} \simeq \left( \frac{2048\pi^7g}{45} \right)^\frac{1}{3} \left( \frac{\mathcal{A}_{\text{ah}}}{4G} \right)^\frac{2}{3},\]
where \(\mathcal{A}_{\text{ah}} := 4\pi (R_{\text{ah}})^2\) denotes the area of the apparent horizon. The above equation is in the form as conjectured by ’t Hooft \[4\]: assuming that the matter occupy a specific volume, then the entropy of that matter will be \(S^{(\gamma)} \propto (4G)^{-3/4} \mathcal{A}^{3/4}\) \[4\], where \(\mathcal{A}\) denotes the area of the containing volume.

We now turn to a Universe filled with only dust matter, \((\mathcal{N}_s = 0)\). We assume as probable that at the very first stages of expansion of the Universe, primordial black holes (PBH) were formed, such that these constitute the content of the dust fluid \[7\]. In this case, Eqs. \[28\] and \[31\] give us the total entropy of the PBHs: If we assume each PBH to have the same mass, \(m_{\text{bh}}\), then we have \(M = Zm_{\text{bh}}\), where \(Z\) denotes the total number of PBHs. The event horizon area of a Schwarzschild black hole is given by \(A_{\text{bh}} = 16\pi G^2 m_{\text{bh}}^2\) and consequently we obtain \(M^2 = 4\pi Z^2 A_{\text{bh}} M_{\text{Pl}}^4\), assuming each PBH to be of a Schwarzschild type. Therefore, from Eqs. \[28\] and \[31\] we can obtain the following relations between the PBH event horizon area and the minimal surface area
\[42\]
\[A_{\text{bh}} \simeq \frac{9\pi^3}{Z^2} \mathcal{A}_{\text{min}} (n^2 + 2\nu + \nu).\]
Hence, the event horizon of PBHs are Dirac observables and the local event horizon of PBHs is related to the global structure of the Universe, via the quantum number \(n\). Using the Bekenstein-Hawking formula \(S^{(bh)} = \frac{4\pi}{3G} A_{\text{bh}}\), we obtain the total entropy of PBHs, \((ZS^{(bh)})\), as
\[43\]
\[S_n^{(PBHs)} \simeq \frac{9\pi^3}{Z} A_{\text{min}} (n^2 + 2\nu + \nu).\]
For a dust dominated Universe, the apparent horizon is \(R_{\text{ah}} = \sqrt{\frac{4\pi}{3} M_{\text{Pl}}^2} a^{3/2}\). Also, the expectation value of the
The square of the scale factor is given by

$$\langle a^2 \rangle = \frac{A_{\text{min}}}{8} (n^2 + 2\nu + n - \frac{1}{2} - \frac{x_0^2}{2}).$$  \hfill (44)

Consequently, we find

$$\langle R^2_{ab} \rangle = \frac{7}{32} A_{\text{min}} (n^2 + 2\nu + n - \frac{1}{2}).$$  \hfill (45)

Therefore, for large values of quantum number \(n\) we obtain

$$S^{(\text{PBHs})} = \frac{81\pi^3}{7Z} \left( \frac{A_{\text{ah}}}{4G} \right),$$  \hfill (46)

which is in agreement with t’ Hooft holographic conjecture [4]. Note that, if we want to obtain the holographic conjecture for a Universe filled with radiation and dust, simultaneity, then we obtain

$$ZS^{(\text{PBHs})} + \left( \frac{5x^3}{4g} \right)^\frac{1}{2} S^\frac{3}{2} = 9\pi^3 \left( \frac{A_{\text{ah}}}{4G} \right) (n^2 + 2\nu + \nu),$$  \hfill (47)

which shows, it is impossible to obtain the total entropy of model in terms of minimal length.

\section{CONCLUSIONS}

In this paper, we have studied the effects of a deformed Heisenberg algebra in terms of a MLUR in a closed quantum FRLW model, whose matter is formed by a two fluid content with non-interacting radiation and dust. Quantum cosmologies with a perfect fluid matter content were investigated previously in [3]. In particular, the case of a dust and radiation dominated quantum Universe was studied in [3]. Our main result is that the extended dynamical group of the model \(su(1,1)\) admits a minimal area retrieved from a MLUR, in the form of a Dirac observable. The total entropy of PBHs or radiation are subsequently shown to be Dirac observables as well. They reasonably agree with the cosmological holographic principle, in the case of a large quantum number, \(n\). We are aware that our results are obtained within a very simple as well as restrictive setting. Nevertheless, we think they are intriguing and provide motivation for subsequent research works. Possible extensions to test the relation between a GUP, a minimal surface being subsequently obtained (constituting a Dirac observable) and cosmological holography, may include:

- Considering other perfect fluids besides radiation and dust.
- Including instead, e.g., scalar fields.
- Considering a Bianchi IX geometry.
- To explore string features by means of a broader gravitational section.

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[22] These expressions can be used, with $g = g_b + \frac{7}{8}g_f$, where $g_b$ and $g_f$ are the total number of internal degrees of freedom of all ultra-relativistic bosons and fermions, respectively.