Emergence of Time from Dimensional Reduction in Noncommutative Geometry

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Abstract

By considering a new form of dimensional reduction for noncommutative field theory, we show that the signature of spacetime may be changed. In particular, it is demonstrated that a temporal dimension can emerge from a purely Euclidean geometry. We suggest that this mechanism may hint at the origin of time in the fundamental theory of quantum gravity.
1 Introduction

In general, given an action with the kinetic term

\[ S = \int h^{\mu\nu} \partial_\mu \psi \partial_\nu \psi \]  

for a real scalar field \( \psi \), one can read off the metric \( h_{\mu\nu} \) of the underlying space directly by taking the inverse of the matrix \( (h^{\mu\nu}) \). If Lorentzian, the metric yields a light cone, determines the field propagation and gives rises to the concept of macrocausality. Performing a canonical quantization, one may obtain from the vanishing of the commutator of observables a microcausality condition. It is well known that ordinary local interaction in quantum field theory cannot modify the lightcone. It is also well known that while conventional dimensional reduction can change a Minkowskian spacetime to a Euclidean one by a simple reduction on time, to generate a time dimension from a purely Euclidean space is impossible.

The situation is drastically different for field theory in noncommutative space (for reviews see [1–3]). Recently, it has been shown that the microcausality of noncommutative field theory is generally modified from a lightcone to a lightwedge [4,5]. This modification is due to the highly nonlocal nature of the noncommutative interaction. In this letter we demonstrate another effect of noncommutative geometry on the lightcone. We will show that, by performing a more general form of dimensional reduction, one can generate a time dimension from a purely Euclidean noncommutative space.

The change of signature in noncommutative geometry hints at a novel possibility to explain the origin of time. In our framework, time is not fundamental but a concept emergent from space. We emphasize that noncommutativity plays a crucial role for our mechanism to work. It has been argued by DeWitt that quantum gravity has an uncertainty principle which prevents one from measuring positions to better than Planck length accuracy [6]. It is natural to expect that noncommutative geometry plays an important role in this realm. Therefore, one may speculate the emergence of time to be an intrinsic property of quantum gravity. Our results suggest a fundamental “timeless” formulation of quantum gravity. Even more, it has been proposed that both space and time are not fundamental and will have to be replaced by something more sophisticated [7]. It is likely that the fundamental theory of quantum gravity is “pointless”, meaning that both space and time are secondary, derived or effective notions.\(^1\) In this letter, we provide a simple mechanism towards a realization of this speculation.

\(^1\)A step in this direction is provided by the IKKT matrix model [8] where it has been argued that a spacetime continuum is generated dynamically. However a Minkowski signature is assumed there.
2 Dimensional reduction in the commutative case

Let us start with a review of the commutative case. Consider a commutative gauge theory with the action

\[ S_{D+1} = \int \text{tr} \left[ \left( \hat{D}_\mu \hat{\phi} \right)^2 + \frac{1}{2} \hat{F}_{\mu\nu}^2 \right] , \]  

where

\[ \hat{D}_\mu \hat{\phi} = \partial_\mu \hat{\phi} + \left[ \hat{A}_\mu, \hat{\phi} \right] , \quad \hat{F}_{\mu\nu} = \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + \left[ \hat{A}_\mu, \hat{A}_\nu \right] , \]

and the fields \( \hat{\phi} \) and \( \hat{A}_\mu \) are \( N \times N \) real matrices. The action is \( \text{GL}(N, \mathbb{R}) \) invariant under the gauge transformation

\[ \hat{\phi} \rightarrow g^{-1} \hat{\phi} g \quad \text{and} \quad \hat{A}_\mu \rightarrow g^{-1} \hat{A}_\mu g + g^{-1} \partial_\mu g \quad \text{for} \quad g \in \text{GL}(N, \mathbb{R}) . \]

We will be interested in the Euclidean space with metric \( g_{\mu\nu} = \delta_{\mu\nu} \). Let the coordinates be

\[ (x^\mu) = (x^0, x^1, \cdots, x^{D-1}, z) = (\{x^i\}, z) =: (\bar{x}, z) , \]

and \( z \) is the coordinate we will take to reduce the theory.

Usually the dimensional reduction takes the form that all the fields are declared independent of \( z \):

\[ \hat{\phi} = \phi(\bar{x}) \quad \text{and} \quad \hat{A}_\mu = A_\mu(\bar{x}) . \]

This leads to the action \(^2\)

\[ S_D = \int \text{tr} \left[ (D_i \phi)^2 + (D_i \lambda)^2 + [\phi, \lambda]^2 + \frac{1}{2} F_{ij}^2 \right] , \]

where

\[ \lambda := A_z(\bar{x}) , \quad F_{ij} := \partial_i A_j - \partial_j A_i + [A_i, A_j] , \quad D_i := \partial_i + [A_i, \cdot] . \]

Let us consider a more general reduction of the “twisted” form

\[ \hat{\phi} = U(\bar{x}, z) \phi(\bar{x}) U^{-1}(\bar{x}, z) \quad \text{and} \quad \hat{A}_\mu = U(\bar{x}, z) A_\mu(\bar{x}) U^{-1}(\bar{x}, z) \]

for \( U(\bar{x}, z) \in \text{GL}(N, \mathbb{R}) \). This is gauge equivalent to

\[ \hat{\phi} = \phi(\bar{x}) \quad \text{and} \quad \hat{A}_\mu = A_\mu(\bar{x}) + \partial_\mu U U^{-1}(\bar{x}, z) , \]

which is just a reduction in the presence of a vacuum background gauge field

\[ \hat{A}_\mu^{(B)} = \partial_\mu U U^{-1} . \]

\(^2\)We drop the unimportant normalization \( \int dz \).
The choice of $U$ is not arbitrary. To see this, note that our ansatz (9) gives
\begin{align}
U^{-1}\hat{F}_{ij}U &= F_{ij} + [U^{-1}\partial_i U, A_j] - [U^{-1}\partial_j U, A_i] \quad , \quad (12) \\
U^{-1}\hat{F}_{iz}U &= D_i \lambda + [U^{-1}\partial_i U, \lambda] - [U^{-1}\partial_z U, A_i] \quad , \quad (13) \\
U^{-1}\hat{D}_i\tilde{\phi}U &= D_i \phi + [U^{-1}\partial_i U, \phi] \quad , \quad (14) \\
U^{-1}\hat{D}_z\tilde{\phi}U &= [\lambda, \phi] + [U^{-1}\partial_z U, \phi] \quad . \quad (15)
\end{align}

To be a consistent reduction, the action should be independent of $z$ when these expressions are substituted. To achieve this, we want (12)–(15) to be independent of $z$. Without making any special assumption on the fields $\phi$ and $A_{\mu}$, a sufficient condition is
\begin{equation}
U^{-1}\partial_\mu U = h_\mu(\vec{x}, z) + k_\mu(\vec{x}) \quad \text{with} \quad h_\mu \text{ being central} \quad . \quad (16)
\end{equation}

For example, one may take $U = U(\vec{x})$. In this case, the background gauge field (11) is $z$-independent. As a result of (16), it is easy to verify that the reduced action is given by (7) plus a couple of additional terms that arise due to the background gauge configuration.

To obtain the propagator, gauge fixing is necessary to make sure that the quadratic part of (2) or (7) is invertible. A convenient choice of the gauge-fixing condition is
\begin{equation}
\partial_\mu \hat{A}^\mu = 0 \quad . \quad (17)
\end{equation}

Including the gauge-fixing term $S_{\text{g.f.}} = \int \text{tr}(\partial_\mu \hat{A}^\mu)^2$, this leads to the gauge-fixed action,
\begin{equation}
S'_{D+1} := S_{D+1} + S_{\text{g.f.}} = \int \text{tr} \left[ -\hat{\phi} \hat{\Delta} \hat{\phi} - \delta^{\mu\nu} \hat{A}_\mu \hat{\Delta} \hat{A}_\nu + \cdots \right] \quad \text{with} \quad \hat{\Delta} := \partial_\mu \partial^\mu \quad , \quad (18)
\end{equation}

where we wrote out only the kinetic terms. Note that
\begin{equation}
U^{-1}\partial_\mu \hat{A}^\mu U = \partial_i A^i + [U^{-1}\partial_i U, A^i] + [U^{-1}\partial_z U, \lambda] = \partial_i A^i + [k_i, A^i] + [k_z, \lambda] \quad , \quad (19)
\end{equation}

and so the gauge-fixing condition reduces to a lower-dimensional one provided that (16) is satisfied. The reduced gauge-fixed action takes the form
\begin{equation}
S'_D = \int \text{tr} \left[ -\phi \Delta \phi - A_i \Delta A_i - \lambda \Delta \lambda + \cdots \right] \quad , \quad \text{where} \quad \Delta := \partial_i \partial_i \quad . \quad (20)
\end{equation}

It is clear that the metric gets reduced as follows,
\begin{equation}
\delta_{\mu\nu} \to \delta_{ij} \quad , \quad (21)
\end{equation}

and remains Euclidean. This conclusion will not be affected by using another choice of gauge fixing.
3 Noncommutative reduction and generation of time

We are interested in dimensional reduction in a noncommutative Euclidean space. Note that one cannot take the fields $\hat{\phi}$ and $\hat{A}_\mu$ to be real any more since, in the presence of the star product, this is not compatible with the gauge transformation (11). In fact, the gauge group must now be extended to $\text{GL}_\star(1,\mathbb{C})$, and the action $S_{D+1}$ in (2) ceases to be real. This does not bother us as the latter only serves to formulate the extremum principle for the equations of motion.

Normally one would like to associate the action with quantum amplitudes via the phase $e^{iS}$. However, a space without time does not provide an arena for standard quantum mechanics, and one should not expect the action to do more than generating the equations of motion. After reduction, we will arrive at a lower-dimensional world with time. Therefore, the reduced action should be real in order to accommodate quantum processes. For the original Euclidean theory, one may consider a path integral defined by $Z := \int [D\hat{\phi}] e^{i\text{Re}S_{D+1}}$. Upon dimensional reduction, this partition function gives rise to the standard path integral for the lower dimensional Minkowskian theory. It would be interesting to understand further the physical properties of $Z$.

Now back to the dimensional reduction. We assume that there is at least one coordinate which is commuting, say $z$, and we will reduce on this coordinate. For simplicity let us take $N = 1$ and consider the case of $D = 2$, i.e. $(x^\mu) = (t, x, z) = (\vec{x}, z)$ with Euclidean metric. The generalization to the nonabelian case and to higher dimensions is straightforward. The noncommutativity is given by

\[ [t, x] = i\theta \quad \text{and } z \text{ being central} \quad . \]

Since $\text{GL}_\star(1,\mathbb{C})$ is infinite-dimensional, the condition (16) becomes much more restrictive. In particular, in order to be central, $h_\mu$ must not depend on $t$ or $x$, enforcing $^3$

\[ U^{-1}\partial_\mu U = h_\mu(z) + k_\mu(\vec{x}) \quad . \]

We will now show that these equations essentially fix the form of $U$ to be

\[ U = U_0 := e^{z f(t, x)} \quad , \quad \text{where } f(t, x) = \alpha t + \beta x \quad \text{and } \alpha, \beta \text{ are constants} \quad . \]

The proof is straightforward. First, from the consistency of the $\mu=i$ equations, one easily see that $k_i$ has to be a pure gauge, i.e.

\[ k_i = W^{-1}\partial_i W \quad \text{for some } W(\vec{x}) \quad . \]

Second, the $\mu=i$ equations are compatible with the $\mu=z$ equation only if

\[ \partial_i h_i = c_i = \partial_i k_z + [k_i, k_z] \quad \text{for constants } c_i \quad , \]

$^3$The star product and star exponential are understood below.
which implies that

\[ h_i = c_i z + d_i \quad \text{and} \quad k_z = W^{-1}(c_i x^i + d) W \]  

(27)

with further constants \( d_i \) and \( d \). The function \( h_z \) is unconstrained. It follows that the corresponding \( U \) factorizes as

\[ U = V(z) U_0(z, \vec{x}) W(\vec{x}) \quad \text{with} \quad U_0 = e^{c_i x^i + d_i x^i + d z} , \]  

(28)

so that we have

\[ h_i = U_0^{-1} \partial_i U_0 \quad \text{and} \quad h_z = V^{-1} \partial_z V \]  

(29)

Clearly, the factors \( V \) and \( W \) can be absorbed into redefinitions of \( \phi, A_i, \lambda \) and \( \hat{\phi}, \hat{A}_i \), respectively. For the same reason we can drop the constants \( d_i \) and \( d \). Therefore, the essential part of \( U \) is the one that entangles the \( z \) and \( x^i \) dependence, as claimed in (24).

The corresponding background gauge field is

\[ \hat{A}_i^{(B)} = \alpha z \quad , \quad \hat{A}_x^{(B)} = \beta z \quad , \quad \hat{A}_z^{(B)} = \alpha t + \beta x \]  

(30)

and the gauge-fixing condition (19) reads

\[ \partial_i A_i + [\hat{A}_z^{(B)}, \lambda] = 0 \]  

(31)

So far, we have shown that for the noncommutative gauge theory (2), the twisted reduction defined in (9) is essentially determined by (24). It corresponds to having a linear gauge potential background (30). Therefore, we may call our reduction a “linear background reduction”.

For the \( U \) of (24), we have

\[ U^{-1} \hat{F}_{ij} U = F_{ij} \]  

(32)

\[ U^{-1} \hat{F}_{iz} U = D_i \lambda - [f, A_i] \]  

(33)

\[ U^{-1} \hat{D}_i \hat{\phi} U = D_i \phi \]  

(34)

\[ U^{-1} \hat{D}_z \hat{\phi} U = [\lambda, \phi] + [f, \phi] \]  

(35)

\[ U^{-1} \partial_\mu \hat{A}_\mu U = \partial_i A_i + [f, \lambda] \]  

(36)

Hence the gauge-fixed reduced action is given by

\[ S_2 = \int \left\{ (\partial_i \phi)^2 + [f, \phi]^2 + (\partial_i \lambda)^2 + [f, \lambda]^2 + F_{tx}^2 + (\partial_i A_i)^2 + [f, A_i]^2 + [f, A_x]^2 \right. \]

\[ + 2 \partial_i \phi [A_i, \phi] + 2 \partial_i \lambda [A_i, \lambda] - 2 [A_i, \lambda][f, A_i] + 2 [f, \phi][\lambda, \phi] \]

\[ \left. + [A_i, \phi]^2 + [A_i, \lambda]^2 + [\phi, \lambda]^2 \right\} \]  

(37)

The first line represents the kinetic terms, while the second and third lines are the interactions. At this stage we further restrict our field configurations by imposing reality conditions on the reduced fields, namely

\[ \phi^\dagger = \phi \quad , \quad \lambda^\dagger = \lambda \quad \text{and} \quad A_i^\dagger = -A_i \]  

(38)
This renders the action real and reduces the gauge group to $U_\star(1)$.

Summarizing, our linear background reduction (39) takes the form

$$\hat{\psi} = e^{zf} \psi e^{-zf} \quad \text{with} \quad f = \alpha t + \beta x \quad \text{for} \quad \hat{\psi} = \hat{\phi}, i \hat{A}_\mu \quad \text{real} .$$  \hspace{1cm} (39)

Note that the $z$-independence of the reduced fields $\psi$ implies the linear derivative constraint

$$\partial_z \hat{\psi} = [f, \hat{\psi}] = i \theta (\alpha \partial_x - \beta \partial_t) \hat{\psi} ,$$  \hspace{1cm} (40)

where we have used

$$[t, ] = i \theta \partial_x \quad \text{and} \quad [x, ] = -i \theta \partial_t .$$  \hspace{1cm} (41)

Likewise, the reality condition (38) lifts to

$$\hat{\psi}^\dagger = e^{-z(f+f^\dagger)} \hat{\psi} e^{z(f+f^\dagger)}$$  \hspace{1cm} (42)

for the three-dimensional fields. It is noteworthy that the two conditions (40) and (42) are consistent.

Now substituting the relation

$$\partial_z \hat{\psi} = U[f, \psi] U^{-1}$$  \hspace{1cm} (43)

in the action (37), its kinetic terms read

$$-\phi \tilde{\Delta} \phi - A_i \tilde{\Delta} A_i - \lambda \tilde{\Delta} \lambda ,$$  \hspace{1cm} (44)

where the kinetic operator

$$\tilde{\Delta} := h^{ij} \partial_i \partial_j$$  \hspace{1cm} (45)

contains the (inverse) metric

$$\begin{pmatrix} h^{tt} & h^{tx} \\ h^{xt} & h^{xx} \end{pmatrix} = \begin{pmatrix} 1 - \beta^2 \theta^2 & \alpha \beta \theta^2 \\ \alpha \beta \theta^2 & 1 - \alpha^2 \theta^2 \end{pmatrix} .$$  \hspace{1cm} (46)

Thus the reduction of the metric now reads

$$\delta_{\mu\nu} \rightarrow h_{ij} .$$  \hspace{1cm} (47)

Note that the signature of the metric depends on the values of $\alpha$ and $\beta$. For the metric to be real, it is necessary that $\alpha$ and $\beta$ be real or purely imaginary. In particular, the metric is Minkowskian if

$$(\alpha^2 + \beta^2) \theta^2 > 1 .$$  \hspace{1cm} (48)

Therefore a temporal direction can be generated in the lower dimension if the parameters of our reduction (39) obey (48)! It is clear that (48) can never be satisfied in the commutative limit. In fact, for $\theta \rightarrow 0$ our reduction (39) goes back to the naive reduction (6).
We note that unlike in ordinary Minkowskian gauge theory where the time component of the gauge field has a kinetic term of the wrong sign, there is no such problem in our case, see (44). The difference is due to the appearance of gauge-fixing term \((\partial_iA_i)^2\) in (37). In ordinary gauge theory, this gauge-fixing term is not acceptable since it is not compatible with the Lorentz symmetry \(SO(1,1)\). In the noncommutative case, however, the Lorentz symmetry breakdown renders this term admissible.

We remark that our reduction (39) resembles the one used in [9]. There the form of the reduction was fixed by requiring integrability, and it implied a change of signature from Lorentzian to Euclidean. This kind of signature change is typical for conventional dimensional reductions. Here we find that, by performing a more general dimensional reduction, one can do the opposite: changing of signature from Euclidean to Lorentzian. This is a novel phenomenon.

One may think that a constraint of the form (40),

\[
(\partial_z + ia \partial_x + ib \partial_t) \hat{\psi}(t,x,z) = 0 \quad \text{with} \quad a, b \in \mathbb{R},
\]

(49)
can be imposed in a commutative theory for achieving the same change of signature without invoking noncommutative geometry. However, it is easy to see that this is not the case: The complex nature of the constraint renders its resolution in terms of real commuting fields impossible. From the above analysis, we see that what noncommutative geometry effectively does for us is to allow for a satisfactory implementation of the constraint (40). This brings about the desired change of signature and would not be possible without noncommutative geometry at work.

The two key assumptions which enable this mechanism are, first, the reduction along some commutative direction in a noncommutative geometry and, second, the linear background reduction (39). While the first ingredient is essential, the second one is open to generalization. The linear background reduction (39) arose from the choice (16), which is the simplest one. It is conceivable that more general forms of \(U\) in the twisted reduction ansatz (9) work as well. Another obvious generalization consists in adding further (commutative or noncommutative) dimensions. Here, we have considered only the simplest case of \(D=2\), which we think is generic. The existence of commutative coordinates in a Moyal-deformed geometry requires either a special choice of the noncommutativity matrix \((\theta^{\mu\nu})\) or an odd dimensionality of our space. In string theory, such a situation is realized on D-branes in the presence of a suitably chosen \(B\)-field background.

Our lower-dimensional theory features a noncommutative time variable. In terms of the Moyal star product, the equation of motion contains an infinite number of time derivatives. The quantization of such theories is an unresolved problem (see for example [10–14] for related discussions). Yet we think that time-space noncommutative theories can make sense quantum mechanically. A proper understanding of their quantum properties might in fact help us to better grasp the nature of spacetime in quantum gravity.
Our reduction is parameterized by $\alpha$ and $\beta$ which describe the background on which the reduction takes place. Just as for the case of an open string in a background NS-NS $B$-field, where the closed string metric and the background $B$-field are free and any value of the noncommutativity parameters $\theta^{\mu\nu}$ is allowed, the background parameters $\alpha$ and $\beta$ can be dialed freely in our setup. Different choices give rise to lower-dimensional worlds with different “spacetime” metrics, including the untwisted reduction as a special case. To decide which values for $\alpha$ and $\beta$ are physically preferred, an underlying “microscopical” theory is desired, which generates the reduction via some compactification mechanism and lifts the flat directions in our moduli space of reductions. This is beyond our reach at the moment. Clearly, it will be very interesting to obtain a dynamical/statistical understanding of the origin of these parameters and the emergence of time.

One of the most exciting prospects in the AdS/CFT proposal \cite{15} is the possibility to understanding properties of spacetime in terms of the dual gauge theory dynamics. It will be tantalizing to understand how a change of spacetime signature manifests itself on the gauge theory side in the context of AdS/CFT. It will also be very interesting to investigate further the phenomenological implications of models built on this kind of dimensional reduction. We leave these fascinating issues for further investigation.

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