ON THE MAZ’YA-SHAPOSHNIKOVA FORMULA IN ORLICZ-SOBOLEV SPACES ON CARNOT GROUPS

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Abstract. In this note we study the limit as $s \downarrow 0$ of fractional Orlicz-Sobolev seminorms in Carnot groups. This closes the study started in [10].

1. Introduction

Let $\mathbb{G}$ be a Carnot group of step $k \geq 1$. Let $s \in (0, 1)$, $p \geq 1$ and $\varphi$ be a Young function (see section 2 for precise definitions), we consider the fractional Orlicz-Sobolev space $W^{s, \varphi}_0(\mathbb{G})$ defined as the closure of $C_0^\infty(\mathbb{G})$ with respect to the fractional Orlicz seminorm

$$\left( \iint_{\mathbb{G} \times \mathbb{G}} \varphi \left( \frac{|u(x) - u(y)|}{\|y^{-1} \cdot x\|^s_{\mathbb{G}}} \right) \frac{dx \, dy}{\|y^{-1} \cdot x\|^q_{\mathbb{G}}} \right)^{1/p}.$$

The aim of this note is to provide an answer to the following question: given $u \in \bigcup_{s \in (0, 1)} W^{s, \varphi}_0(\mathbb{G})$, does the following limit

$$\lim_{s \downarrow 0} s \iint_{\mathbb{G} \times \mathbb{G}} \varphi \left( \frac{|u(x) - u(y)|}{\|y^{-1} \cdot x\|^s_{\mathbb{G}}} \right) \frac{dx \, dy}{\|y^{-1} \cdot x\|^q_{\mathbb{G}}}$$

exist? In [10], the authors provided lower bounds for the liminf and upper bounds for the limsup as $s \downarrow 0$. In particular, we want to stress that those bounds were in general not the same, except for the trivial case $\varphi(t) := t^p$ (which boils down to classical fractional Sobolev spaces), but they were always expressed in terms of the same Young function $\varphi$. Before stating our result, let us spend a few words concerning the history behind the question addressed in the present paper. The first result dates back to [23] in 2002, when Maz’ya and Shaposhnikova proved that whenever $u \in \bigcup_{s \in (0, 1)} W^{s, p}_0(\mathbb{R}^n)$, then

$$\lim_{s \downarrow 0} s \iint_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy = 2 \frac{|S|^{n-1}}{p} \int_{\mathbb{R}^n} |u|^p \, dx,$$

where $|S|^{n-1}$ denotes the measure of the unit sphere. In the statement, $W^{s, p}_0(\mathbb{R}^n)$ denotes the closure of $C_0^\infty(\mathbb{R}^n)$ with respect to the fractional Gagliardo seminorm

$$[u]_{s, p} := \left( \iint_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \right)^{1/p}.$$
The result in [23] was the natural complement to the one considered by Bourgain, Brezis and Mironescu, see [3], where they proved that for every \( u \in W^{1,p}(\mathbb{R}^n) \) it holds that
\[
\lim_{s \uparrow 1} (1-s) \int_{\mathbb{R}^{2n}} \frac{|u(x) - u(y)|^p}{|x - y|^{n + sp}} \, dx \, dy = K(n, p) \int_{\mathbb{R}^n} |\nabla u|^p \, dx,
\]
where the constant \( K(n, p) \) is defined as
\[
K(n, p) := \frac{1}{p} \int_{\mathbb{S}^{n-1}} |\omega \cdot h|^p \, dH^{n-1}(h),
\]
where \( \mathbb{S}^{n-1} \subset \mathbb{R}^n \) denotes the unit sphere, \( H^{n-1} \) is the \((n-1)\)-dimensional Hausdorff measure and \( \omega \) is an arbitrary unit vector of \( \mathbb{R}^n \).

Passing over the fundamental contributions recalled above, the literature concerning asymptotic behaviors, for both \( s \uparrow 1 \) and \( s \downarrow 0 \), of fractional norms, perimeters and nonlocal functionals associated to image processing, has grown up covering several interesting variants and generalizations, see e.g. [3, 6, 15, 18, 28, 1, 32, 29, 4, 22, 24, 25, 26, 7, 30, 13, 11, 9] and the references therein.

We want now to go back to our original problem. As already recalled, a partial answer to our question is contained in [10]. Recently, in [1], the same question has been positively answered in the Euclidean case. In particular, it has been proved that the limit exists and, roughly speaking, equals a quantity which depends on a different Young function \( \varphi \), which is closely related to \( \varphi \) and whose dependence on \( \varphi \) can be made explicit.

The Euclidean case treated in [1] provides the unique example of Carnot groups of step 1, so it becomes pretty natural to wonder whether the technique adopted in [1] can be adapted to the more general framework of Carnot groups of step \( k > 1 \).

The answer is positive when dealing with Young functions \( \varphi \) satisfying the following condition: we assume that there exist constants \( 1 \leq p^- \leq p^+ \) such that
\[
\tag{L} p^- \leq \frac{t \varphi'(t)}{\varphi(t)} \leq p^+ \quad \text{for every } t > 0.
\]

Moreover, for every Young function \( \varphi \), we define the function \( \overline{\varphi} : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \) as
\[
\overline{\varphi}(t) := \int_0^t \varphi(\tau) \frac{d\tau}{\tau}, \quad t \in \mathbb{R}_0^+.
\]

With this at hand, we can state our result, which is the content of the following

**Theorem 1.1.** Let \( \varphi \) be a Young function satisfying (L). If \( u \in \bigcup_{s \in (0,1)} W^{s,\overline{\varphi}}(G) \), then
\[
\lim \int_{G \times G} \varphi \left( \frac{y^{-1} \cdot (x - u(y))}{\|y^{-1} \cdot x\|_G^{Q}} \right) \, dx \, dy = 2QC_b \int_G \varphi(|u|) \, dx,
\]
where \( Q \) is the homogeneous dimension of \( G \) and \( C_b \) stands for the measure of the unit ball.

Let us spend a few comments on Theorem 1.1. First, it is known, see e.g. [27, Chapter 4]), that condition (L) is equivalent to the so called \( \Delta_2 \)-condition (see Definition 2.2) which is the standing assumption in [1]. Second, when \( \varphi(t) = t^p \), it holds that \( \overline{\varphi}(t) = \frac{p}{p} \), and therefore we recover [10, Corollary 1.1] in the case of fractional Sobolev spaces.

The proof of Theorem 1.1 follows the path tracked in [1], which in turn is based
on the original argument used in [24]. We notice that as a byproduct of the lower bound of the liminf (see Lemma 3.1), we are able to recover a Hardy-type inequality for small enough $s$ (see (3.6)), akin to the one proved in [2]. We also stress that, coherently with our previous results in [10], the proof of Theorem 1.1 actually works if the homogeneous norm there considered satisfies the classical triangular inequality

\[ \|y\|_G - \|x\|_G \leq \|y - x\|_G \leq \|x\|_G + \|y\|_G. \]

We stress that this does not always hold true for a generic homogeneous norm. We refer to Section 2 for more details.

Finally, we want to recall that it is necessary for the Young function to satisfy the $\Delta_2$-condition for Theorem 1.1 to hold. We refer to [1, Theorem 1.2] for an example.

As a last remark, we hereby recall that the study of properties of fractional Orlicz-Sobolev spaces, even considering just the Euclidean case, is becoming more and more popular. See e.g. [16, 17, 1, 2, 14, 31] and the references therein.

The structure of the paper is the following: in section 2 we collect the basic notions and results concerning Young functions and Carnot groups, while in section 3 we prove Theorem 1.1.

2. Preliminaries

In this section we introduce definitions and notations and we recall few results needed in section 3.

2.1. Young functions. We recall now the basic notions concerning Young functions. We refer the interested reader to the book [27] for a comprehensive introduction to the subject.

**Definition 2.1.** Let $\phi: \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a function satisfying the following conditions:

(i) $\phi(0) = 0$, and $\phi(t) > 0$ for any $t > 0$;

(ii) $\phi$ is non-decreasing;

(iii) $\phi$ is right-continuous and $\lim_{t \to +\infty} \phi(t) = +\infty$.

We call Young function the real valued function $\varphi: \mathbb{R}_0^+ \to \mathbb{R}_0^+$ given by

\[ \varphi(t) = \int_0^t \phi(s) \, ds. \]

We stress that thanks to (i) – (iii), every Young function $\varphi$ is continuous, locally Lipschitz continuous, strictly increasing and convex on $\mathbb{R}_0^+$. Moreover, it holds that $\varphi(0) = 0$ and that $\varphi$ is superlinear at zero and at infinity. We can also assume without loss of generality that $\varphi(1) = 1$. Finally, it is well defined the inverse $\varphi^{-1}: \mathbb{R}_0^+ \to \mathbb{R}_0^+$ of $\varphi$. It follows from our previous considerations that $\varphi^{-1}$ is continuous, concave, strictly increasing, $\varphi^{-1}(0) = 0$ and $\varphi^{-1}(1) = 1$.

We recall that we will work with Young functions satisfying the standing assumption provided by [14], which we can now reformulate in terms of $\phi$: we assume there exist $1 \leq p^- \leq p^+$ such that

\[ p^- \leq \frac{t \phi(t)}{\varphi(t)} \leq p^+ \quad \text{for every } t > 0. \]
As a direct consequence, we notice that
\[
\begin{align*}
(\varphi_1) & \quad s^\varphi \varphi(t) \leq \varphi(st) \leq s^\varphi \varphi(t), \\
(\varphi_2) & \quad \varphi(s + t) \leq \frac{2^{p^+}}{2}(\varphi(s) + \varphi(t))
\end{align*}
\]
for any \(s, t \in \mathbb{R}_0^+\), where \(s^\varphi := \max\{s^{p^-}, s^{p^+}\}\) and \(s^\varphi := \min\{s^{p^-}, s^{p^+}\}\).

Clearly, the most trivial example of a Young function is \(\varphi(t) = t^p\) for a certain \(p \geq 1\), and in this case we obviously have \(p^- = p = p^+\). A less trivial instance of Young function is provided by logarithmic perturbation of powers, namely a function \(\varphi\) such that \(\varphi'(t) = t^a \log(b + ct)\), for assigned \(a, b, c > 0\); in this case we have that \(p^- = 1 + a\) and \(p^+ = 2 + a\).

As already remarked in the Introduction, condition \(\mathbb{L}\) is equivalent to the so called \(\Delta_2\)-condition which we recall right now.

**Definition 2.2.** Let \(\varphi\) be a Young function. We say that \(\varphi\) satisfies the \(\Delta_2\)-condition if there exists a positive constant \(C > 2\) such that
\[
\varphi(2t) \leq C \varphi(t) \quad \text{for every } t \in \mathbb{R}_0^+.
\]

Keeping in mind the definition \(\mathbb{L}\) of the Young function \(\varphi\), we end this recap noticing that the monotonicity of \(\varphi(t)\) and \(\varphi'(t)\) implies that \(\varphi\) is bounded. Moreover, \(\varphi\) and \(\varphi\) are equivalent Young functions, in the sense that
\[
\varphi\left(\frac{t}{2}\right) \leq \varphi(t) \leq \varphi(t) \quad \text{for every } t \in \mathbb{R}_0^+.
\]

2.2. *Carnot groups.* We will now recall the basic notions concerning Carnot groups. We refer to the monograph [8] for a detailed and comprehensive introduction to the subject.

A Carnot group \(G = (\mathbb{R}^n, \cdot)\) of step \(k\) is a connected, simply connected and nilpotent Lie group, whose Lie algebra \(\mathfrak{g}\) admits a stratification of step \(k\). This means that there exist \(k\) linear subspaces \(V_1, \ldots, V_k\) such that
\[
\mathfrak{g} = V_1 \oplus \cdots \oplus V_k, \quad [V_1, V_i] = V_{i+1}, \quad V_k \neq \{0\}, \quad V_i = \{0\} \quad \text{if } i > k,
\]
where
\[
[V_i, V_j] := \text{span}\{[X, Y] : X \in V_i, Y \in V_j\}.
\]
The Carnot group \(G\) and its Lie algebra \(\mathfrak{g}\) can be identified through the exponential mapping. This identification allows to write any element \(x\) of the group in exponential coordinates \((x_1, \ldots, x_n)\).

The group law is in general non-abelian (except for the Euclidean case \(k = 1\)) and can be written in coordinates by means of the Hausdorff-Campbell formula. Every Carnot group \(G\) can be also endowed with a family of *dilations* \(\delta_\lambda\) with \(\lambda \in \mathbb{R}^+\). These are automorphisms of the group \(\delta_\lambda : G \to G\) given by
\[
\delta_\lambda(x_1, \ldots, x_n) := (\lambda^{d_1} x_1, \ldots, \lambda^{d_n} x_n),
\]
where \(d_j \in \mathbb{N}\) for every \(j = 1, \ldots, n\) and \(1 = d_1 = \ldots = d_m < d_{m+1} \leq \ldots \leq d_n\) for \(m := \dim(V_1)\). As a consequence, the Haar measure on \(G\) coincides with the \(n\)-dimensional Lebesgue measure \(\mathcal{L}^n\) of \(\mathbb{R}^n\), see e.g. [8 Proposition 1.3.21].

Every Carnot group has a homogeneous dimension \(Q\) defined as \(Q := \sum_{i=1}^k i \dim(V_i)\).
This number corresponds to the Hausdorff dimension of $G$ (with respect to an appropriate sub–Riemannian distance, see below). This is generally greater than (or equal to) the topological dimension of $G$ and it coincides with it only when $G$ is the Euclidean group $(\mathbb{R}^n, +)$, which is the only Abelian Carnot group.

Carnot groups are also naturally endowed with sub–Riemannian distances. One of the most known example of such metrics is provided by the so called Carnot-Carathéodory distance $d_{cc}$, see e.g. [8, Definition 5.2.2], which is a path–metric resembling the classical Riemannian distance. In our case, we will work with metrics induced by homogeneous norms.

**Definition 2.3.** A homogeneous norm $\| \cdot \|_G: G \to \mathbb{R}^+_0$ is a continuous function with the following properties:

(i) $\|x\|_G = 0$ if and only if $x = 0$ for every $x \in G$;

(ii) $\|x^{-1}\|_G = \|x\|_G$ for every $x \in G$;

(iii) $\|\delta_\lambda x\|_G = \lambda \|x\|_G$ for every $\lambda \in \mathbb{R}^+$ and for every $x \in G$.

An example of homogeneous norm is provided by the Korányi norm, see for instance [12]. We remind that, given any homogeneous norm, it is possible to define a left–invariant homogeneous distance as follows:

$$d(x, y) := \|y^{-1} \cdot x\|_G \quad \text{for every } x, y \in G.$$ 

In this paper we will work with homogeneous norms $\| \cdot \|_G$ that need to satisfy another assumption:

(iv) the validity of the classical triangular inequality

$$\|y\|_G - \|x\|_G \leq \|y^{-1} \cdot x\|_G \leq \|x\|_G + \|y\|_G.$$ 

An example of such kind of norm, whose induced distance is equivalent to the well-known Carnot-Carathéodory distance, is given in [20, 21].

In what follows, we will write $B(x, \varepsilon)$ to denote the ball of center $x \in G$ and radius $\varepsilon > 0$ with respect to an assigned distance $d$. As it will become clear in section 3, in order to prove Theorem 1.1 we will need to be able to compute integrals of radial functions. To this aim, in light of [11, Proposition 1.13], we have the following useful result to compute integrals of radial functions. See [10, Proposition 2.3] for details.

**Proposition 2.4.** Let $f \in L^1(\mathbb{R}^+)$ and $R > 0$. Then

$$\int_{B(y, R)} f(\|y^{-1} \cdot x\|_G) \, dx = \int_{B(0, R)} f(\|x\|_G) \, dx = QC_B \int_0^R r^{Q-1} f(r) \, dr$$

and

$$\int_{G \setminus B(y, R)} f(\|y^{-1} \cdot x\|_G) \, dx = \int_{G \setminus B(0, R)} f(\|x\|_G) \, dx = QC_B \int_R^{+\infty} r^{Q-1} f(r) \, dr.$$ 

We close this section providing the relevant definitions concerning the functional spaces appearing in Theorem 1.1.
Definition 2.5. Let \( G \) be a Carnot group, let \( \varphi \) be a Young function. We define the Orlicz-Lebesgue space \( L^{\varphi}(G) \) as
\[
L^{\varphi}(G) := \{ u : G \to \mathbb{R} \text{ measurables such that } \Phi_{\varphi}(u) < \infty \},
\]
where
\[
\Phi_{\varphi}(u) := \int_G \varphi(|u(x)|) \, dx.
\]
Moreover, for \( s \in (0, 1) \) we define the fractional Orlicz-Sobolev space \( W^{s,\varphi}(G) \) as
\[
W^{s,\varphi}(G) := \{ u \in L^{\varphi}(G) \text{ such that } \Phi_{s,\varphi}(u) < \infty \},
\]
where
\[
\Phi_{s,\varphi}(u) := \iint_{G \times G} \varphi\left( \frac{|u(x) - u(y)|}{\|y^{-1} \cdot x\|_G^s} \right) \frac{dy \, dx}{\|y^{-1} \cdot x\|_G^s}.
\]

These spaces become Banach spaces when endowed with the so-called Luxemburg norms, which are defined as follows:
\[
\|u\|_{s,\varphi} := \inf \left\{ \lambda > 0 : \Phi_{s,\varphi}\left( \frac{u}{\lambda} \right) \leq 1 \right\},
\]
where
\[
\|u\|_{s,\varphi} := \|u\|_{\varphi} + [u]_{s,\varphi},
\]
is the \((s, \varphi)\)-Gagliardo seminorm.

To close this section, we recall the following

Theorem 2.6 (Theorem 2.2, [10]). Let \( \varphi \) be a Young function such that both \( \varphi \) and its Legendre transform satisfy the \( \Delta_2 \)-condition. Then, for each \( s \in (0, 1) \), the spaces \( W^{s,\varphi}(G) \) are reflexive and separable Banach spaces. Moreover, \( C_c^\infty(G) \) is dense in \( W^{s,\varphi}(G) \).

3. Proof of Theorem 1.1

In this section we prove Theorem 1.1 combining two estimates for the liminf and the limsup respectively.

Lemma 3.1 (Liminf estimate). For any \( u \in \bigcup_{s \in (0, 1)} W^{s,\varphi}_0(G) \) it holds that
\[
\liminf_{s \to 0} s\Phi_{s,\varphi}(u) \geq 2QC_\varphi \Phi_{\varphi}(u).
\]

Proof. For every \( \varepsilon > 0 \) we define
\[
\Phi_{s,\varepsilon} := \frac{1}{1 + \varepsilon} \int_G \left( \int_G \varphi\left( \left( 1 + \varepsilon \right) \frac{|u(x) - u(y)|}{\|y^{-1} \cdot x\|_G^s} \right) \frac{dy}{\|y^{-1} \cdot x\|_G^s} \right) \, dx
\]
and
\[
I_{s,1,\varepsilon} := \frac{1}{1 + \varepsilon} \int_G \left( \int_{\|y^{-1} \cdot x\|_G > 2 \|x\|_G} \varphi\left( \left( 1 + \varepsilon \right) \frac{|u(x) - u(y)|}{\|y^{-1} \cdot x\|_G^s} \right) \frac{dy}{\|y^{-1} \cdot x\|_G^s} \right) \, dx.
\]
By a change of variables and Fubini’s Theorem, we get
\[
I_{s,1}^\varepsilon = \frac{1}{1 + \varepsilon} \int_G \left( \int_{\|x^{-1} \cdot y\|_G > 2\|y\|_G} \varphi \left( (1 + \varepsilon) \frac{|u(y) - u(x)|}{\|x^{-1} \cdot y\|_G^s} \right) \frac{dx}{\|x^{-1} \cdot y\|_G^s} \right) dy
\]
\[
= \frac{1}{1 + \varepsilon} \int_G \left( \int_{\|x^{-1} \cdot y\|_G > 2\|y\|_G} \varphi \left( (1 + \varepsilon) \frac{|u(x) - u(y)|}{\|y^{-1} \cdot x\|_G^s} \right) \frac{dy}{\|y^{-1} \cdot x\|_G^s} \right) dx.
\]
Let us notice that \( \{\|y^{-1} \cdot x\|_G > 2\|x\|_G\} \cap \{\|y^{-1} \cdot x\|_G > 2\|y\|_G\} = \emptyset \). Indeed, if this were not the case, the validity of both \( \|y^{-1} \cdot x\|_G > 2\|x\|_G \) and \( \|y^{-1} \cdot x\|_G > 2\|y\|_G \), combined with the triangular inequality, would imply that
\[
\|y^{-1} \cdot x\|_G \leq \|x\|_G + \|y\|_G < 2\|y^{-1} \cdot x\|_G + \frac{1}{2}\|y^{-1} \cdot x\|_G = \|y^{-1} \cdot x\|_G,
\]
which is a contradiction. Therefore,
\[
(3.2)
\]
\[
\Phi_{s,1}^\varepsilon \geq \frac{1}{1 + \varepsilon} \int_G \left( \int_{\|y^{-1} \cdot x\|_G > 2\|y\|_G} \varphi \left( (1 + \varepsilon) \frac{|u(x) - u(y)|}{\|y^{-1} \cdot x\|_G^s} \right) \frac{dy}{\|y^{-1} \cdot x\|_G^s} \right) dx
\]
\[
+ \frac{1}{1 + \varepsilon} \int_G \left( \int_{\|y^{-1} \cdot x\|_G > 2\|y\|_G} \varphi \left( (1 + \varepsilon) \frac{|u(x) - u(y)|}{\|y^{-1} \cdot x\|_G^s} \right) \frac{dy}{\|y^{-1} \cdot x\|_G^s} \right) dx
\]
\[
= \frac{2}{1 + \varepsilon} \int_G \left( \int_{\|y^{-1} \cdot x\|_G > 2\|y\|_G} \varphi \left( (1 + \varepsilon) \frac{|u(x) - u(y)|}{\|y^{-1} \cdot x\|_G^s} \right) \frac{dy}{\|y^{-1} \cdot x\|_G^s} \right) dx
\]
\[
= 2 I_{s,1}^\varepsilon.
\]
We now define
\[
I_{s,2}^\varepsilon := \frac{\varepsilon}{1 + \varepsilon} \int_G \left( \int_{\|y^{-1} \cdot x\|_G > 2\|y\|_G} \varphi \left( \frac{1 + \varepsilon}{\varepsilon} \frac{|u(y)|}{\|y^{-1} \cdot x\|_G^s} \right) \frac{dy}{\|y^{-1} \cdot x\|_G^s} \right) dx.
\]
In order to complete the proof, it would be enough to prove both
\[
(3.3)
\]
\[
I_{s,1}^\varepsilon + I_{s,2}^\varepsilon \geq \frac{QC_b}{s} \int_G \varphi \left( \frac{|u(x)|}{2\|x\|_G^s} \right) dx
\]
and
\[
(3.4)
\]
\[
I_{s,2}^\varepsilon \leq \frac{\varepsilon}{1 + \varepsilon} C_b \left( \frac{2(1 + \varepsilon)\beta}{\varepsilon} \right)^3 \int \varphi \left( \frac{|u(y)|}{2\|y\|_G^s} \right) dy.
\]
Indeed, assume for the moment that \( (3.3) \) and \( (3.4) \) hold true, combined with \( (3.2) \), they imply that
\[
s \Phi_{s,1}^\varepsilon \geq 2s(I_{s,1}^\varepsilon + I_{s,2}^\varepsilon) - 2sI_{s,2}^\varepsilon
\]
\[
\geq 2QC_b \int_G \varphi \left( \frac{|u(x)|}{2\|x\|_G^s} \right) dx \left[ 1 - \frac{s\varepsilon}{Q(1 + \varepsilon)} \left( \frac{2(1 + \varepsilon)\beta}{\varepsilon} \right)^3 \right].
\]
We stress that, exploiting \( (2.1) \) and \( (2.1) \), we can even get a Hardy-type inequality for small enough \( s \in (0, 1) \) in the following sense. Fixed \( \varepsilon > 0 \), from \( (3.5) \) we find that
\[
(3.6)
\]
\[
\int_G \varphi \left( \frac{|u(x)|}{2\|x\|_G^s} \right) dx \leq KS_{\varepsilon} \Phi_{s,1}^\varepsilon \leq K (1 + \varepsilon)^{\beta - 1} \Phi_{s,1}^\varepsilon (u) < \infty,
\]
where

\[ K = K(s, \varepsilon) := \left[ 2QCb \left[ 1 - \frac{s\varepsilon}{Q(1 + \varepsilon)} \left( \frac{2(1 + \varepsilon)_{\varepsilon}}{\varepsilon} \right)^{\frac{1}{\varepsilon}} \right] \right]^{-1} > 0 \]

whenever \( s \in (0, 1) \) is small enough in dependence on \( \varepsilon \). Moreover, we notice that

\[ K \to \frac{1}{2QCb} \quad \text{as} \quad s \downarrow 0. \]

Thanks to (3.6), we can argue as in [1], proving (3.1) by applying Fatou’s Lemma and then exploiting the arbitrariness of \( \varepsilon > 0 \). We are therefore left with the proof of (3.3) and (3.4).

We start with (3.3). By the convexity and the monotonicity of \( \varphi \), it follows that

\[ \varphi \left( \frac{|u(x)|}{\|y^{-1} \cdot x\|_G} \right) \leq \frac{\varepsilon}{1 + \varepsilon} \varphi \left( \frac{1 + \varepsilon}{\varepsilon} \frac{|u(y)|}{\|y^{-1} \cdot x\|_G} \right) + \frac{1}{1 + \varepsilon} \varphi \left( (1 + \varepsilon) \frac{|u(x) - u(y)|}{\|y^{-1} \cdot x\|_G} \right). \]

Therefore, once we define

\[ I_{s,1} := \int_G \int_{\{\|y^{-1} \cdot x\|_G > 2\|x\|_G\}} \varphi \left( \frac{|u(x)|}{\|y^{-1} \cdot x\|_G} \right) \frac{dy}{\|y^{-1} \cdot x\|_G} dx, \]

we easily get

\[ I_{s,1} \leq I_{s,1}^\varepsilon + I_{s,2}^\varepsilon. \]

Moreover, by Proposition 2.4 and a change of variables we get

\[ I_{s,1} = QCb \int_G \left( \int_{2\|x\|_G}^{+\infty} \varphi \left( \frac{|u(x)|}{r^{\frac{s}{\varepsilon}}} \right) \frac{dr}{r} \right) dx = QCb \int_G \left( \int_{0}^{\frac{|u(x)|}{2\|x\|_G}} \varphi (\tau) d\tau \right) dx \]

\[ = QCb \int_G \varphi \left( \frac{|u(x)|}{2\|x\|_G^{\frac{s}{\varepsilon}}} \right) dx, \]

where \( \varphi \) is as (1.1). This proves (3.3).

Let us now prove (3.4). Notice that \( \|y^{-1} \cdot x\|_G \geq 2\|x\|_G \) implies

\[ \frac{2}{3} \|y\|_G \leq \|y^{-1} \cdot x\|_G \leq 2\|y\|_G, \]

\[ \|\delta_{\frac{1}{3}} y \cdot x\|_G \leq \frac{2}{3} \|y\|_G. \]

Indeed, (3.7) trivially follows from the triangular inequality, while (3.8) follows from the r.h.s. of (3.7) replacing \( y \) with \( \delta_{\frac{1}{3}} y \). Moreover, we remind that (3.8) is equivalent to write \( x \in B((\delta_{\frac{1}{3}} y)^{-1}, \frac{4}{3}\|y\|_G). \)
Now, by using (3.7), Fubini’s theorem, (3.8), Proposition 2.4 and the monotonicity of \( \varphi \), we get

\[
I_{s,2}^s \leq \frac{\varepsilon}{1 + \varepsilon} \left( \frac{3}{2} \right)^Q \int_G \left( \int_{\|y - x\| \geq 2\|x\|} \varphi \left( \frac{1 + \varepsilon}{\varepsilon} \left( \frac{3}{2} \right)^s \frac{|u(y)|}{\|y\| G} \right) \, dy \right) \, dx
\]

\[
= \frac{\varepsilon}{1 + \varepsilon} \left( \frac{3}{2} \right)^Q \int_G \left( \int_{\|y - x\| \geq 2\|x\|} dx \right) \varphi \left( \frac{1 + \varepsilon}{\varepsilon} \left( \frac{3}{2} \right)^s \frac{|u(y)|}{\|y\| G} \right) \, dy \frac{Q}{y}
\]

\[
\leq \frac{\varepsilon}{1 + \varepsilon} \left( \frac{3}{2} \right)^Q \int_G \left( \int_{B((\delta \frac{1}{2} y - \delta \frac{1}{2} y - x))} dx \right) \varphi \left( \frac{1 + \varepsilon}{\varepsilon} \left( \frac{3}{2} \right)^s \frac{|u(y)|}{\|y\| G} \right) \, dy \frac{Q}{y}
\]

\[
= \frac{\varepsilon}{1 + \varepsilon} \left( \frac{3}{2} \right)^Q \int_G \left( \frac{QC_b}{\|y\| G} - \int_0 Q \, dy \right) \varphi \left( \frac{1 + \varepsilon}{\varepsilon} \left( \frac{3}{2} \right)^s \frac{|u(y)|}{\|y\| G} \right) \, dy \frac{Q}{y}
\]

Moreover, by (21) and (21), we have

\[
\int_G \varphi \left( \frac{1 + \varepsilon}{\varepsilon} \left( \frac{3}{2} \right)^s \frac{|u(y)|}{\|y\| G} \right) \, dy \leq \left( \frac{2(1 + \varepsilon)}{\varepsilon} \right)^{2s} \int_G \varphi \left( \frac{|u(y)|}{2s |y| G} \right) \, dy,
\]

and now (3.4) easily follows. This closes the proof. \( \square \)

We can now move to the upper estimate for the limsup.

**Lemma 3.2** (Limsup estimate). For any \( u \in \bigcup_{s \in (0,1)} W^{s,\varphi}_0(G) \) it holds that

\[
\limsup_{s \downarrow 0} s \Phi_{s,\varphi}(u) \leq 2QC_b \Phi_\varphi(u).
\]

**Proof.** Let us first notice that, for any \( \alpha > 0 \), in light of Fubini’s Theorem and by a change of variables, it holds that

\[
\int_G \left( \int_{\|y\| \leq \alpha \|x\|} \varphi \left( \frac{|u(x) - u(y)|}{\|y - x\| G} \right) \, dy \right) \, dx
\]

\[
\leq \int_G \left( \int_{\|y\| \leq \alpha \|x\|} \varphi \left( \frac{|u(x) - u(y)|}{\|y - x\| Q G} \right) \, dy \right) \, dx
\]

(3.9)

\[
= \int_G \left( \int_{\|y\| \leq \alpha \|x\|} \varphi \left( \frac{|u(x) - u(y)|}{\|y - x\| Q G} \right) \, dy \right) \, dx
\]

Then, by (3.9) we can split the expression of \( \Phi_{s,\varphi}(u) \) integrating over the regions \( G \times \{ \|y\| \geq \|x\| \} \) and \( G \times \{ \|y\| < \|x\| \} \), obtaining

\[
\Phi_{s,\varphi}(u) = 2 \int_G \left( \int_{\|y\| \leq \alpha \|x\|} \varphi \left( \frac{|u(x) - u(y)|}{\|y - x\| Q G} \right) \, dy \right) \, dx.
\]
To this aim, we first notice that, by the triangular inequality,

\[
(3.12)
\]

\[
(3.13)
\]

which in turn gives that

\[
(3.10)
\]

\[
(3.11)
\]

\[
(3.12)
\]

\[
(3.13)
\]

\[
(3.14)
\]

From this, together with the convexity and monotonicity of \( \varphi \), we get that for any \( \varepsilon > 0 \), it holds

\[
J_1 \leq \frac{QC_b}{s} \int_G \varphi \left( (1 + \varepsilon) \frac{|u(x)|}{\|x\|_G^s} \right) dx < +\infty,
\]

and

\[
J_2 \leq C_b \int_G \varphi \left( \frac{1 + \varepsilon}{\varepsilon} \frac{|u(x)|}{\|x\|_G^s} \right) dx < +\infty.
\]

To this aim, we first notice that, by the triangular inequality, \( \|y\|_G \geq 2\|x\|_G \) implies that

\[
\|y^{-1} \cdot x\|_G \geq \|y\|_G - \|x\|_G \geq \|x\|_G
\]

and

\[
\|y^{-1} \cdot x\|_G \geq \|y\|_G - \|x\|_G \geq \frac{1}{2}\|y\|_G
\]

which in turn gives that

\[
\{ \|y\|_G \geq 2\|x\|_G \} \subset \{ \|y^{-1} \cdot x\|_G \geq \|x\|_G \},
\]

(3.13)

\[
\|y^{-1} \cdot x\|_G \geq \frac{1}{2}\|y\|_G.
\]

Then, by (3.13) and Proposition 2.4,

\[
J_1 \leq \frac{QC_b}{s} \int_G \left( \int_{\{\|y\|_G \geq 2\|x\|_G \}} \varphi \left( (1 + \varepsilon) \frac{|u(x)|}{\|y^{-1} \cdot x\|_G^s} \right) \frac{dy}{\|y^{-1} \cdot x\|_G^s} \right) dx
\]

\[
= QC_b \int_G \left( \int_{\|x\|_G}^{+\infty} \varphi \left( (1 + \varepsilon) \frac{|u(x)|}{\|y^{-1} \cdot x\|_G^s} \right) \frac{dr}{r} \right) dx
\]

\[
= QC_b \int_G \left( \int_0^{(1+\varepsilon)\|x\|_G} \varphi(r) \frac{dr}{r} \right) dx = \frac{QC_b}{s} \int_G \left( (1 + \varepsilon) \frac{|u(x)|}{\|x\|_G^s} \right) dx.
\]
Moreover, since \((1 + \varepsilon)2^s > 1\), by \((2.1)\) and \((3.3)\), we get that
\[
\int_G \varphi \left( (1 + \varepsilon) \frac{|u(x)|}{\|x\|_G} \right) \, dx \leq \left[ (1 + \varepsilon)2^s \right]^p \int_G \varphi \left( \frac{|u(x)|}{2^s \|x\|_G} \right) \, dx
\]
\[
\leq \left[ (1 + \varepsilon)2^s \right]^p K(1 + \varepsilon)^{p-1} \Phi_{s, \varphi}(u)
\]
\[
= 2^{sp} K(1 + \varepsilon)^{p-1} \Phi_{s, \varphi}(u) < \infty.
\]
This proves \((3.11)\).

Now, by \((3.14)\), Fubini’s Theorem and the monotonicity of \(\varphi\), we also get
\[
J_2 \leq 2^Q \int_G \left( \int_{\|x\|_G \leq \frac{1}{2} \|y\|_G} \varphi \left( \frac{1 + \varepsilon}{2} \frac{|u(y)|}{\|y\|_G^s} \right) \, dy \int_0^1 x^{Q-1} \, dx \right)
\]
\[
= 2^Q Q B \int_G \left( \int_{\|x\|_G \leq \frac{1}{2} \|y\|_G} \varphi \left( \frac{1 + \varepsilon}{2} \frac{|u(y)|}{\|y\|_G^s} \right) \, dy \right)
\]
\[
= C_b \int_G \varphi \left( \frac{1 + \varepsilon}{2} \frac{|u(y)|}{\|y\|_G^s} \right) \, dy.
\]
The finiteness of the right hand side can be proved as follows: recalling \((2.1)\), \((2.1)\) and \((3.3)\), we find that
\[
\int_G \varphi \left( \frac{1 + \varepsilon}{2} \frac{|u(x)|}{\|x\|_G^s} \right) \, dx = \int_G \varphi \left( \frac{1 + \varepsilon}{2} \frac{|u(x)|}{\|x\|_G^s} \right) \, dx
\]
\[
\leq \left[ \frac{1 + \varepsilon}{2} \right]^{p-1} \int_G \varphi \left( \frac{|u(x)|}{\|x\|_G^s} \right) \, dx
\]
\[
\leq \left[ \frac{1 + \varepsilon}{2} \right]^{p-1} K(1 + \varepsilon)^{p-1} \Phi_{s, \varphi}(u) < \infty.
\]
This proves \((3.12)\).

Looking back at \((3.11)\), we only need to provide a suitable upper estimate from above for \(J_3\). We claim that, fixed \(N > 3\), there exists \(\overline{s} \in (0, 1)\) such that
\[
(3.15) \quad J_3 \leq \int_G \left( \int_A \varphi \left( N^{\overline{s}-s} \frac{|u(x) - u(y)|}{\|y^{-1} \cdot x\|_G} \right) \, dy \right) \, dx + \frac{\varepsilon}{s}
\]
for any \(s \in (0, \overline{s})\), where \(A := \{ \|x\|_G \leq \|y\|_G < 2\|x\|_G, \|y^{-1} \cdot x\|_G \leq N \}\).

In order to prove \((3.15)\), we first notice that \(J_3\) can be written as
\[
J_3 = (i) + (ii),
\]
where
\[
(i) := \int_G \left( \int_A \varphi \left( \frac{|u(x) - u(y)|}{\|y^{-1} \cdot x\|_G} \right) \, dy \right) \, dx
\]
\[
(ii) := \int_G \left( \int_B \varphi \left( \frac{|u(x) - u(y)|}{\|y^{-1} \cdot x\|_G} \right) \, dy \right) \, dx
\]
where \(B := \{ \|x\|_G \leq \|y\|_G < 2\|x\|_G, \|y^{-1} \cdot x\|_G > N \}\).
Since \( u \in \bigcup_{s \in (0,1)} W^{s,\varphi}_0(G) \), there exists \( \overline{s} \in (0,1) \) such that \( u \in W^{\overline{s},\varphi}_0(G) \). Let now \( s < \overline{s} \). Then

\[
(i) = \int_G \left( \int_A \varphi \left( \frac{\|y^{-1} \cdot x\|_G^{\overline{s}}} {\|\varphi\|_G} \frac{|u(x) - u(y)|}{\|y^{-1} \cdot x\|_G^{\overline{s}}} \right) \frac{dy}{\|y^{-1} \cdot x\|_G^{\overline{s}}} \right) dx
\]

(3.16)

\[
\leq \int_G \left( \int_A \varphi \left( \frac{\|y^{-1} \cdot x\|_G^{\overline{s}}} {\|\varphi\|_G} \frac{|u(x) - u(y)|}{\|y^{-1} \cdot x\|_G^{\overline{s}}} \right) \frac{dy}{\|y^{-1} \cdot x\|_G^{\overline{s}}} \right) dx.
\]

Moreover, since

\[
B \subset \left\{ \|x\|_G > \frac{N}{3}, \|y^{-1} \cdot x\|_G > N \right\},
\]

by convexity of \( \varphi \), Fubini’s Theorem, a change of variables and \( (3.16) \), we get that

\[
(ii) \leq \frac{1}{2} \int_G \left( \int_B \varphi \left( \frac{2|u(x)|}{\|y^{-1} \cdot x\|_G^{\overline{s}}} \right) \frac{dy}{\|y^{-1} \cdot x\|_G^{\overline{s}}} \right) dx
\]

\[
+ \frac{1}{2} \int_G \left( \int_B \varphi \left( \frac{2|u(x)|}{\|y^{-1} \cdot x\|_G^{\overline{s}}} \right) \frac{dy}{\|y^{-1} \cdot x\|_G^{\overline{s}}} \right) dx
\]

\[
= \int_G \left( \int_B \varphi \left( \frac{2|u(x)|}{\|y^{-1} \cdot x\|_G^{\overline{s}}} \right) \frac{dy}{\|y^{-1} \cdot x\|_G^{\overline{s}}} \right) dx
\]

\[
\leq \int_{\{|x| \geq \frac{N}{3}\}} \left( \int_{\{|y^{-1} \cdot x| > N\}} \varphi \left( \frac{2|u(x)|}{\|y^{-1} \cdot x\|_G^{\overline{s}}} \right) \frac{dy}{\|y^{-1} \cdot x\|_G^{\overline{s}}} \right) dx
\]

\[
= QC_b \int_{\{|x| \geq \frac{N}{3}\}} \left( \int_{N}^{+\infty} \varphi \left( \frac{2|u(x)|}{r^s} \right) \frac{dr}{r} \right) dx
\]

\[
= QC_b \int_{\{|x| \geq \frac{N}{3}\}} \left( \int_{0}^{2u(x)} \varphi(\tau) \frac{d\tau}{\tau} \right) dx
\]

\[
= QC_b \int_{\{|x| \geq \frac{N}{3}\}} \varphi \left( \frac{2|u(x)|}{N^s} \right) dx
\]

\[
\leq QC_b \int_{\{|x| \geq \frac{N}{3}\}} \varphi(2|u(x)|) dx < \frac{\varepsilon}{s},
\]

for \( N \) sufficiently large. From this, \( (3.15) \) easily follows.

In order to justify the passage to the limsup, we can argue as in \([1]\) once again.

In this way, by the arbitrariness of \( \varepsilon \), by gathering \((3.10), (3.11), (3.12), (3.15)\) and by using Fatou’s Lemma, we close the proof.

**Proof of Theorem 1.1.** Combining the lower bounds obtained in Lemma 3.1 with the upper bounds provided by Lemma 3.2 we easily get (1.2). \(\square\)

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