On graphs with three or four distinct normalized Laplacian eigenvalues

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Abstract In this paper, we characterize all connected graphs with exactly three distinct normalized Laplacian eigenvalues of which one is equal to 1, determine all connected bipartite graphs with at least one vertex of degree 1 having exactly four distinct normalized Laplacian eigenvalues, and find all unicyclic graphs with three or four distinct normalized Laplacian eigenvalues.

Keywords: Normalized Laplacian eigenvalue; Bipartite graph; Symmetric BIBD; Hadamard matrix; Unicyclic graph.

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1 Introduction

Let $G$ be a simple undirected graph on $n$ vertices with $m$ edges. The adjacency matrix $A = (a_{uv})$ of $G$ is the $n \times n$ matrix with rows and columns indexed by the vertices, where $a_{uv} = 1$ if $u$ is adjacent to $v$, and 0 otherwise. Let $D = \text{diag}(d_1, d_2, \ldots, d_n)$ denote the diagonal degree matrix of $G$. The well-known Laplacian matrix of $G$ is defined as $L = D - A$. The normalized Laplacian matrix ($\mathcal{L}$-matrix for short) of $G$ is the $n \times n$ matrix $\mathcal{L} = (\ell_{uv})$ with

\[
\ell_{uv} = \begin{cases} 
1 & \text{if } u = v, \ d_u \neq 0, \\
-1/\sqrt{d_ud_v} & \text{if } u \text{ is adjacent to } v, \\
0 & \text{otherwise.}
\end{cases}
\]

Clearly, if $G$ has no isolated vertices then $\mathcal{L} = D^{-\frac{1}{2}}LD^{-\frac{1}{2}} = I - D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$. Since $\mathcal{L}$ is positive semidefinite, all its eigenvalues are nonnegative. These eigenvalues are also called the normalized Laplacian eigenvalues ($\mathcal{L}$-eigenvalues for short) of $G$. We denote by $\lambda_1 > \lambda_2 > \cdots > \lambda_t$ all the distinct $\mathcal{L}$-eigenvalues of $G$ with multiplicities $m_1, m_2, \ldots, m_t$ ($\sum_{i=1}^t m_i = n$), respectively. All these $\mathcal{L}$-eigenvalues together with their multiplicities are called the normalized Laplacian spectrum ($\mathcal{L}$-spectrum for short) of $G$ denoted by

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Spec\(_L(G) = \{[\lambda_1]^{m_1}, [\lambda_2]^{m_2}, \ldots, [\lambda_t]^{m_t}\}\). With respect to adjacency matrix, the \textit{adjacency spectrum} of \(G\) can be similarly defined and denoted by \(\text{Spec}_A(G)\).

A connected graph on \(n\) vertices with \(m\) edges is called a \(k\)-\textit{cyclic graph} if \(k = m - n + 1\). In particular, the notions \textit{tree} and \textit{unicyclic graph} are respectively defined as the \(k\)-cyclic graph with \(k = 0\) and \(k = 1\).

Throughout this paper, we denote the \textit{neighborhood} of a vertex \(v \in V(G)\) by \(N_G(v)\), the disjoint union of graphs \(G\) and \(H\) by \(G \cup H\), the complete graph on \(n\) vertices by \(K_n\) and the complete multipartite graph with \(s\) parts of sizes \(n_1, \ldots, n_s\) by \(K_{n_1, \ldots, n_s}\). Also, the \(n \times n\) identity matrix, the \(n \times 1\) all-ones vector and the \(n \times n\) all-ones matrix will be denoted by \(I_n\), \(j_n\) and \(J_n\), respectively.

Connected graphs with few distinct eigenvalues have been investigated frequently for several graph matrices over the past two decades, such as the adjacency matrix [3, 5, 7, 9, 10, 13, 14, 16–19, 21, 24, 25, 27, 29, 32], the Laplacian matrix [15, 26, 35], the signless Laplacian matrix [1], the Seidel matrix [31], and the universal adjacency matrix [22]. One of the reason is that such graphs in general have pretty combinatorial properties and a rich structure [17]. With regard to normalized Laplacian matrix, Cavers [6] characterized all connected graphs with at least one vertex of degree 1 having three distinct \(L\)-eigenvalues, Van Dam et al. [20] gave all connected triangle-free graphs (particularly, bipartite graphs) with three distinct \(L\)-eigenvalues, and Braga et al. [2] determined all trees with four or five distinct \(L\)-eigenvalues.

In this paper, we characterize all connected graphs with exactly three distinct \(L\)-eigenvalues of which one \(L\)-eigenvalue is 1, determine all connected bipartite graphs with at least one vertex of degree 1 that have exactly four distinct \(L\)-eigenvalues, and find all unicyclic graphs with three or four distinct \(L\)-eigenvalues.

## 2 Main tools

First of all, we recall some basic results about \(L\)-eigenvalues.

\textbf{Lemma 2.1.} (See [8, 20].) Let \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n (n \geq 2)\) be all the \(L\)-eigenvalues of \(G\). Then \(G\) has the following properties.

(i) \(\lambda_n = 0\),

(ii) \(\sum \lambda_i \leq n\) with equality holding if and only if \(G\) has no isolated vertices,

(iii) \(\lambda_{n-1} \leq \frac{n}{n-1}\) with equality holding if and only if \(G\) is a complete graph on \(n\) vertices,

(iv) \(\lambda_{n-1} \leq 1\) if \(G\) is non-complete,

(v) \(\lambda_1 \geq \frac{n}{n-1}\) if \(G\) has no isolated vertices,

(vi) \(\lambda_{n-1} > 0\) if \(G\) is connected. If \(\lambda_{n-i+1} = 0\) and \(\lambda_{n-i} \neq 0\), then \(G\) has exactly \(i\) connected components,

(vii) The \(L\)-spectrum of \(G\) is the union of the \(L\)-spectra of its connected components,

(viii) \(\lambda_i \leq 2\) for all \(i\), with \(\lambda_1 = 2\) if and only if some connected component of \(G\) is a non-trivial bipartite graph,

(ix) \(G\) is bipartite if and only if \(2 - \lambda_i\) is an \(L\)-eigenvalue of \(G\) for each \(i\).
Two \( n \times n \) real symmetric matrices \( M \) and \( N \) are said to be congruent if there exists an invertible matrix \( S \in \mathbb{R}^{n \times n} \) such that \( S^T MS = N \). The well-known Sylvester’s law of inertia states that two congruent real symmetric matrices have the same numbers of positive, negative, and zero eigenvalues. As Lemma 2.1 (iv) suggests that \( \lambda_{n-1} \leq 1 \) if \( G \) is not a complete graph, we now characterize all the connected graphs attaining this bound.

**Corollary 2.1.** Let \( G \) be a connected non-complete graph, and \( \lambda_{n-1} \) the second least \( \mathcal{L} \)-eigenvalue of \( G \). Then \( \lambda_{n-1} \leq 1 \) with equality holding if and only if \( G \) is a complete multipartite graph.

**Proof.** Suppose \( A \) and \( D \) are the adjacency matrix and diagonal degree matrix of \( G \), respectively. Let \( A^* = D^{-\frac{1}{2}} A D^{-\frac{1}{2}} \). The \( \mathcal{L} \)-matrix of \( G \) is \( \mathcal{L} = I - A^* \). Therefore, \( \lambda_{n-1} = 1 \) if and only if the second largest eigenvalue of \( A^* \) is equal to 0, which is the case if and only if the second largest eigenvalue of \( A \) is equal to 0 because \( A^* \) and \( A \) have the same number of positive, negative, and zero eigenvalues due to \( A^* \) and \( A \) are congruent. It is well known that a connected graph has 0 as its second largest (adjacency) eigenvalue if and only if it is a complete multipartite graph (except the complete graph). The result follows. \( \square \)

The following lemma suggests that some special \( \mathcal{L} \)-eigenvalues are related to some local properties of graphs.

**Lemma 2.2.** Let \( G \) be a graph on \( n \) vertices, and let \( X = \{v_1, v_2, \ldots, v_p\} \) \( (p \geq 2) \) be a set of vertices such that \( N_G(v_1) \setminus X = N_G(v_2) \setminus X = \cdots = N_G(v_p) \setminus X = \{u_1, u_2, \ldots, u_q\} \) \( (q \geq 1) \).

We have

(i) if \( X \) is an independent set of \( G \), then 1 is an \( \mathcal{L} \)-eigenvalue of \( G \) with multiplicities at least \( p - 1 \);

(ii) if \( X \) is a clique of \( G \), then \( \frac{p+q}{p+q-1} \) is an \( \mathcal{L} \)-eigenvalue of \( G \) with multiplicities at least \( p - 1 \).

**Proof.** For any fixed \( i \) \((1 \leq i \leq p-1)\), let \( x_i \in \mathbb{R}^n \) be the vector defined by \( x_i(v_i) = 1 \), \( x_i(v_p) = -1 \), and \( x_i(v) = 0 \) for each \( v \notin \{v_i, v_p\} \) in \( G \). To prove (i), it suffices to verify that each \( x_i \) is an eigenvector of \( G \) corresponding to the \( \mathcal{L} \)-eigenvalue 1. In fact, under the assumption of (i), we have \( N_G(v_1) = N_G(v_2) = \cdots = N_G(v_p) = \{u_1, u_2, \ldots, u_q\} \) since \( X \) is independent set. Recall that \( \mathcal{L} = I - D^{-\frac{1}{2}} A D^{-\frac{1}{2}} \), the following equalities conform that \( \mathcal{L} x_i = 1 \cdot x_i \):

\[
\begin{align*}
    x_i(v_j) - \sum_{v \sim v_j} \frac{1}{\sqrt{d_v d_{v_j}}} x_i(v) &= x_i(v_j) - \sum_{l=1}^{q} \frac{1}{\sqrt{d_{u_l} d_{v_j}}} x_i(u_l) = 1 \cdot x_i(v_j) \text{ for } 1 \leq j \leq p, \\
    x_i(u_j) - \sum_{v \sim u_j} \frac{1}{\sqrt{d_v d_{u_j}}} x_i(v) &= 0 - \left( \frac{1}{\sqrt{d_{v_p} d_{u_j}}} - \frac{1}{\sqrt{d_{v_j} d_{u_j}}} \right) = 1 \cdot x_i(u_j) \text{ for } 1 \leq j \leq q, \\
    x_i(u) - \sum_{v \sim u} \frac{1}{\sqrt{d_v d_u}} x_i(v) &= 0 = 1 \cdot x_i(u) \text{ for } u \notin \{v_1, \ldots, v_p, u_1, \ldots, u_q\}.
\end{align*}
\]

Hence 1 is an \( \mathcal{L} \)-eigenvalue of \( G \) with multiplicities at least \( p - 1 \).

Under the assumption of (ii), one can similarly verify that the same \( x_i \in \mathbb{R}^n \) defined above is also an eigenvector of \( G \) with respect to the \( \mathcal{L} \)-eigenvalue \( \frac{p+q}{p+q-1} \) for \( i = 1, 2, \ldots, p-1 \). \( \square \)
By Lemma 2.1 (ix) and Lemma 2.2 (i), we obtain the following corollary immediately.

**Corollary 2.2.** Let $G$ be a bipartite graph. If there are two vertices in $G$ share the same neighborhood, then the number of distinct $\mathcal{L}$-eigenvalues of $G$ must be odd.

Let $M$ be a real symmetric matrix whose all distinct eigenvalues are $\lambda_1, \ldots, \lambda_s$. Then $M$ has the spectral decomposition $M = \lambda_1 P_1 + \cdots + \lambda_s P_s$, where $P_i = x_1 x_1^T + \cdots + x_r x_r^T$ if the eigenspace $\mathcal{E}(\lambda_i)$ has $\{x_1, \ldots, x_r\}$ as an orthonormal basis. It is well known that for any real polynomial $f(x)$, we have $f(M) = f(\lambda_1)P_1 + \cdots + f(\lambda_s)P_s$.

Note that for any graph $G$, we have $\mathcal{L} \cdot \mathcal{D}^{\frac{1}{2}} = 0 \cdot \mathcal{D}^{\frac{1}{2}}$. and so $\mathcal{D}^{\frac{1}{2}}$ is an eigenvector with respect to the $\mathcal{L}$-eigenvalue 0 of $G$. Moreover, if $G$ is connected, from Lemma 2.1 (vi) we know that 0 is a simple $\mathcal{L}$-eigenvalue of $G$. Keeping these facts in mind, one can easily deduce the following lemma by using the method of spectral decomposition.

**Lemma 2.3.** (See [6].) Let $G$ be a connected graph on $n \geq 3$ vertices with $m$ edges and fix $2 \leq s \leq n$. Then $G$ has exactly $s$ distinct $\mathcal{L}$-eigenvalues, namely $\lambda_1, \lambda_2, \ldots, \lambda_{s-1}$ and 0, if and only if there exists $s - 1$ distinct nonzero numbers $\lambda_1, \lambda_2, \ldots, \lambda_{s-1}$ such that

$$
\prod_{i=1}^{s-1}(\mathcal{L} - \lambda_i I) = (\mathcal{L} - \alpha I)\left(\mathcal{L} - \beta I\right) = (-1)^{s-1}\left(\prod_{i=1}^{s-1} \lambda_i\right)\mathcal{D}^{\frac{1}{2}} J \mathcal{D}^{\frac{1}{2}},
$$

(1)

where $D$ is the diagonal degree matrix of $G$.

By Lemma 2.3, we have the following result.

**Lemma 2.4.** Let $G$ be a connected graph with $m$ edges and $\text{Spec}_G(\mathcal{L}) = \{[\alpha]^m, [\beta]^m, [0]^1\}$ $(2 \geq \alpha > \beta > 0)$. Then $\beta \leq 1$, and one of the following holds:

(i) $\beta = 1$, and each pair of non-adjacent vertices in $G$ have the same neighbors;
(ii) $\beta < 1$, and $\frac{2m(\alpha - 1)(\beta - 1)}{\alpha \beta} \geq d_u - d_v \geq \frac{2m(\alpha - 1)(\beta - 1)}{\alpha \beta}$ holds for any two non-adjacent vertices $u, v$ in $G$.

**Proof.** By the assumption, $\text{Spec}_G(\mathcal{L}) = \{[\alpha]^m, [\beta]^m, [0]^1\}$, where $2 \geq \alpha > \beta > 0$. By Lemma 2.3, we have

$$
(\mathcal{L} - \alpha I)(\mathcal{L} - \beta I) = \frac{\alpha \beta}{2m} \mathcal{D}^{\frac{1}{2}} J \mathcal{D}^{\frac{1}{2}}.
$$

(2)

Since $\mathcal{L} = \mathcal{I} - \mathcal{D}^{-\frac{1}{2}} \mathcal{A} \mathcal{D}^{-\frac{1}{2}}$, (2) can be written as

$$
(\mathcal{D}^{-\frac{1}{2}} \mathcal{A} \mathcal{D}^{-\frac{1}{2}})^2 + (\alpha + \beta - 2)\mathcal{D}^{-\frac{1}{2}} \mathcal{A} \mathcal{D}^{-\frac{1}{2}} + (\alpha - 1)(\beta - 1) \mathcal{I} = \frac{\alpha \beta}{2m} \mathcal{D}^{\frac{1}{2}} J \mathcal{D}^{\frac{1}{2}}.
$$

(3)

By considering the $(u, u)$-entry and $(u, v)$-entry $(u \neq v)$ at both sides of (3), we obtain that

$$
\sum_{w \sim u} \frac{1}{d_w} = \frac{\alpha \beta}{2m} d_u^2 - (\alpha - 1)(\beta - 1)d_u,
$$

(4)

and

$$
\sum_{w \sim u} \frac{1}{d_w} = \begin{cases} 
- (\alpha + \beta - 2) + \frac{\alpha \beta}{2m} d_u d_v & \text{if } u \sim v, \\
\frac{\alpha \beta}{2m} d_u d_v & \text{if } u \neq v.
\end{cases}
$$

(5)
Let \( u, v \) be a pair of non-adjacent vertices in \( G \). Then from (4) and (5) we deduce that

\[
\begin{align*}
\frac{\alpha \beta}{2m} d^2_u - (\alpha - 1)(\beta - 1)d_u &= \sum_{w \sim u} \frac{1}{d_w} \geq \sum_{w \sim v} \frac{1}{d_w} = \frac{\alpha \beta}{2m} d^2_v, \\
\frac{\alpha \beta}{2m} d^2_v - (\alpha - 1)(\beta - 1)d_v &= \sum_{w \sim v} \frac{1}{d_w} \geq \sum_{w \sim u} \frac{1}{d_w} = \frac{\alpha \beta}{2m} d^2_u. 
\end{align*}
\]

This implies that

\[
-\frac{2m(\alpha - 1)(\beta - 1)}{\alpha \beta} \geq d_u - d_v \geq -\frac{2m(\alpha - 1)(\beta - 1)}{\alpha \beta}. 
\]

Since \( \alpha > 1 \) by Lemma 2.1 (v), we must have \( \beta \leq 1 \) by (7), which is consistent with Lemma 2.1 (iv). Particularly, if \( \beta = 1 \), from (7) we may conclude that \( d_u = d_v \), and moreover, \( u \) and \( v \) must share the same neighborhood in \( G \) by (6).

We complete the proof. \( \square \)

**Remark 1.** It is worth mentioning that Lemma 2.4 (i) can be also deduced from Corollary 2.1.

At the end of this section, we characterize all connected graphs with exactly three distinct \( L \)-eigenvalues of which one is equal to 1.

**Theorem 2.1.** Let \( G \) be a connected graph of order \( n \geq 3 \). Then \( G \) has exactly three distinct \( L \)-eigenvalues of which one \( L \)-eigenvalue is 1 if and only if \( G \cong K_{s,n-s} \) with \( 1 \leq s \leq n-1 \), or \( G \cong K_{n_1,n_2,\ldots,n_r} \) with \( n_1 = n_2 = \cdots = n_r = \frac{n}{r} \) and \( 3 \leq r \leq n-1 \).

**Proof.** By the assumption, we can suppose that \( \text{Spec}_L(G) = \{[\alpha]^{m_1}, [1]^{m_2}, [0]^1\} \) (\( \alpha > 1 \)). By Lemma 2.4 (i), each pair of non-adjacent vertices in \( G \) share the same neighborhood. For any \( u \in V(G) \), let \( [u] \) denote the set of vertices which are not adjacent to \( u \) in \( G \). Then each vertex in \( [u] \) has the same neighborhood as \( u \). Therefore, \( [u] \) forms an independent set of \( G \) and all vertices in \( [u] \) share the same neighborhood \( V(G) \setminus [u] \). Hence, by the arbitrariness of \( u \), we may conclude that \( G \) is a complete multipartite graph, say \( G = K_{n_1,n_2,\ldots,n_r} \), where \( n_1 + n_2 + \cdots + n_r = n \), and \( 2 \leq r \leq n-1 \) because \( G \) is connected and cannot be a complete graph (which has only two distinct \( L \)-eigenvalues). If \( r = 2 \), then \( G \) is a complete bipartite graph and our result follows. Now assume that \( 3 \leq r \leq n-1 \). Let \( V_i \) (\( 1 \leq i \leq r \)) be the \( i \)-th part of \( V(G) \) with \( |V_i| = n_i \). If \( r = 3 \), for any three vertices \( u \in V_1, v \in V_2 \) and \( w \in V_3 \), we have \( u \sim v, u \sim w \) and \( v \sim w \). By applying (5) to these three pairs of adjacent vertices, we get

\[
\begin{align*}
\frac{n_3}{n-n_3} &= 1 - \alpha + \frac{\alpha}{2m}(n_1 - n_1)(n-n_2) \\
\frac{n_2}{n-n_2} &= 1 - \alpha + \frac{\alpha}{2m}(n_1 - n_1)(n-n_3) \Rightarrow \begin{cases} n_3 - n_2 = (1-\alpha)(n_2 - n_3) \\ n_2 - n_1 = (1-\alpha)(n_1 - n_2) \end{cases} \\
\frac{n_1}{n-n_1} &= 1 - \alpha + \frac{\alpha}{2m}(n-n_2)(n-n_3)
\end{align*}
\]

which gives that \( n_1 = n_2 = n_3 \) because \( \alpha \neq 2 \) due to \( G \) is not a bipartite graph. For \( 4 \leq r \leq n-1 \), we have to deal with it by the way of contradiction since we cannot obtain
a similar symmetric relation as in (8). Assume that not all \( n_i \)'s are equal, say \( n_1 \neq n_2 \). Taking \( u_1 \in V_1, u_2 \in V_2 \) and \( u_i \in V_i (3 \leq i \leq r) \), then \( u_1 \sim u_i \) and \( u_2 \sim u_i \). By applying (5) to these two pairs of adjacent vertices, we get

\[
\begin{align*}
\sum_{i \neq 1, j} \frac{n_i}{n - n_i} &= 1 - \frac{\alpha}{2m} (n - n_1)(n - n_i), \\
\sum_{i \neq 2, j} \frac{n_i}{n - n_i} &= 1 - \frac{\alpha}{2m} (n - n_2)(n - n_i) \Rightarrow \frac{n(n_1 - n_2)}{(n - n_1)(n - n_2)} = \frac{\alpha}{2m} (n_1 - n_2)(n - n_i). \quad (9)
\end{align*}
\]

Since \( n_1 \neq n_2 \), from (9) we have

\[
\frac{\alpha}{2m} = \frac{n}{(n - n_1)(n - n_2)(n - n_i)}. \quad (10)
\]

Thus we obtain \( n_3 = \cdots = n_r \) by the arbitrariness of \( i \). Consequently, we see that \( n_1 \neq n_3 \) or \( n_2 \neq n_3 \), say \( n_1 \neq n_3 \). By exchanging the roles of \( n_2 \) and \( n_3 \), similarly as above arguments, we deduce that \( n_2 = n_4 = \cdots = n_r \). Therefore, we have \( n_2 = n_3 = \cdots = n_r \). Furthermore, putting \( u = u_1 \in V_1 \) in (4), we obtain

\[
\frac{\alpha}{2m} (n - n_1)^2 = \frac{1}{n - n_2} \cdot (n - n_1), \text{ i.e.,}
\]

\[
\frac{\alpha}{2m} = \frac{1}{(n - n_1)(n - n_2)}. \quad (11)
\]

Combining (10) and (11), we deduce that \( n_i = 0 \) for \( 3 \leq i \leq r \), which is impossible. Therefore, we have \( n_1 = n_2 = \cdots = n_r = \frac{n}{2} \), and our result follows.

Conversely, by simple computation we obtain that

\[
\begin{align*}
\text{Spec}_L(K_{s,n-s}) &= \{[2]^1, [1]^{n-2}, [0]^1\}, \\
\text{Spec}_L(K_{\frac{n}{2}, \frac{n}{2}}) &= \{[r-1]^{r-1}, [1]^{n-r}, [0]^1\},
\end{align*}
\]

(12)

where \( 1 \leq s \leq n - 1 \) and \( 3 \leq r \leq n - 1 \). It follows our result. \( \square \)

**Corollary 2.3.** Let \( G \) be a connected graph of order \( n \geq 3 \). Then \( G \) has \( L \)-spectrum \( \text{Spec}_L(G) = \{[\alpha]^1, [\beta]^{\nu-2}, [0]^1]\} (\alpha > \beta > 0) \) if and only if \( \alpha = 2 \) and \( \beta = 1 \) if and only if \( G \) is a complete bipartite graph.

**Proof.** Suppose that \( \text{Spec}_L(G) = \{[\alpha]^1, [\beta]^{\nu-2}, [0]^1]\} \). By considering the trace of \( L \), we have \( \alpha + (n-2)\beta = n \), implying that \( \beta = \frac{n-\alpha}{n-2} \geq 1 \) due to \( \alpha \leq 2 \) by Lemma 2.1 (viii). Again by Lemma 2.1 (iv), we get \( \beta = 1 \), and so \( \alpha = 2 \). Thus \( G \) is a complete bipartite graph by Theorem 2.1. Conversely, the \( L \)-spectrum of a complete bipartite graph \( G \) is of the form \( \text{Spec}_L(G) = \{[\alpha]^1, [\beta]^{\nu-2}, [0]^1]\} \) by (12), as required. \( \square \)

**Remark 2.** Note that Corollary 2.3 has been obtained by Van Dam and Omidi [20]. In fact, they have determined all connected graphs with three distinct \( L \)-eigenvalues of which two are simple.

## 3 Bipartite graphs with four distinct \( L \)-eigenvalues

In this section, we focus on connected bipartite graphs with four distinct \( L \)-eigenvalues, and determine all such graphs with at least one vertex of degree 1. First of all, we need some concepts and results coming from combinatorial design theory for later use.
A balanced incomplete block design (BIBD for short) is a pair \((V, \mathcal{B})\) where \(V\) is a \(v\)-set and \(\mathcal{B}\) is a collection of \(b\) \(k\)-subsets (blocks) of \(V\) such that each element of \(V\) is contained in exactly \(r\) blocks, and each pair of elements of \(V\) is simultaneously contained in \(\lambda\) blocks (see [12]). The integers \((v, b, r, k, \lambda)\) are called the parameters of the BIBD \((V, \mathcal{B})\). The complement of \((V, \mathcal{B})\) is \((V, \overline{\mathcal{B}})\), where \(\overline{\mathcal{B}} = \{V \setminus B : B \in \mathcal{B}\}\). Clearly, \((V, \overline{\mathcal{B}})\) is a BIBD with parameters \((v, b, b - r, v - k, b - 2r + \lambda)\). In the case \(r = k\) (and then \(v = b\)) the BIBD \((V, \mathcal{B})\) is called symmetric with parameters \((v, k, \lambda)\). In particular, the complement of a symmetric BIBD with parameters \((v, k, \lambda)\) is also a symmetric BIBD, which has parameters \((v, v - k, v - 2k + \lambda)\).

Let \((V, \mathcal{B})\) be a BIBD with parameters \((v, b, r, k, \lambda)\). The incidence matrix of \((V, \mathcal{B})\) is a \(v \times b\) matrix \(C = (c_{ij})\), in which \(c_{ij} = 1\) when the \(i\)-th element \(v_i\) of \(V\) occurs in the \(j\)-th block \(B_j\) of \(\mathcal{B}\) and \(c_{ij} = 0\) otherwise. The incidence graph of \((V, \mathcal{B})\) is the bipartite graph on \(b + v\) vertices with the bipartition \(V \cup \mathcal{B}\) in which \(v_i \in V\) and \(B_j \in \mathcal{B}\) are adjacent if and only if \(v_i \in B_j\). As shown in [12] (pp. 165–167), the incidence graph has adjacency spectrum \([\sqrt{kr}]^1, [\sqrt{r - \lambda}]^{v - 1}, [0]^{b - v}, [-\sqrt{r - \lambda}]^{-1}, [-\sqrt{kr}]^1\). In particular, if \((V, \mathcal{B})\) is symmetric, the incidence graph is a \(k\)-regular bipartite graph with adjacency spectrum

\[
[[k]^1, [\sqrt{k - \lambda}]^{v - 1}, [-\sqrt{k - \lambda}]^{-1}, [-k]^1].
\]

Conversely, a connected regular bipartite graph with four distinct (adjacency) eigenvalues is well known that a Hadamard matrix of order \(4t\) exists only if \(n = 1, 2\) or \(4t\), where \(t\) is a positive integer [30]. Multiplying any row (column) of a Hadamard matrix by \(-1\), or permuting rows (columns) of a Hadamard matrix, the result is also a Hadamard matrix. Two Hadamard matrices are said to be equivalent if one can be obtained from the other by a sequence of these operations. It is easy to see that every Hadamard matrix is equivalent to a Hadamard matrix that has every element of its first row and column \(+1\), which is called a normalized Hadamard matrix. Clearly, in a normalized Hadamard matrix of order \(4t\), every row (column) except the first contains \(+1\) and \(-1\) exactly \(2t\) times each, and further, \(+1\) (resp. \(-1\)) in any row (column) except the first overlap with \(+1\) (resp. \(-1\)) in each other row (column) except the first exactly \(t\) times each.

Assume that there exists a Hadamard matrix \(H\) of order \(4t\). Without loss of generality, suppose that \(H\) is normalized. Remove the first row and column of \(H\) and replace every \(-1\) in the resulting matrix by a zero. The final \((4t - 1) \times (4t - 1)\) matrix \(C\) can be viewed as the incidence matrix of a symmetric BIBD with parameters \((4t - 1, 2t - 1, t - 1)\) by above arguments. Conversely, given a symmetric BIBD with parameters \((4t - 1, 2t - 1, t - 1)\), a Hadamard matrix could be constructed by reversing above process. For this reason, a symmetric BIBD with parameters \((4t - 1, 2t - 1, t - 1)\) is called a Hadamard design of dimension \(t\). Therefore, a Hadamard matrix corresponds to a Hadamard design naturally.

Now we begin to consider connected bipartite graphs with four distinct \(L\)-eigenvalues. Suppose that \(G\) is a connected bipartite graph on \(n\) vertices \(m\) edges with the bipartition \(V(G) = V_1 \cup V_2\), where \(|V_i| = n_i\) for \(i = 1, 2\), and \(n_1 \leq n_2\). Then the adjacency matrix \(A\) and the diagonal degree matrix \(D\) of \(G\) can be respectively written as

\[
A = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} V_1 \quad \text{and} \quad D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}.
\]
where $D_i$ corresponds to $V_i$ for $i = 1, 2$. The $\mathcal{L}$-matrix of $G$ is of the form

\[
\mathcal{L} = I - D^{-\frac{1}{2}}AD^{-\frac{1}{2}} = I - \begin{pmatrix}
0 & D_1^{-\frac{1}{2}}BD_2^{-\frac{1}{2}} \\
D_2^{-\frac{1}{2}}B^TD_1^{-\frac{1}{2}} & 0
\end{pmatrix} = I - \begin{pmatrix}
0 & B^* \\
B^T & 0
\end{pmatrix} = I - A^*,
\]

where $B^* = D_1^{-\frac{1}{2}}BD_2^{-\frac{1}{2}}$ and $A^* = \begin{pmatrix}
0 & B^* \\
B^T & 0
\end{pmatrix}$. Then $A^{*2} = \begin{pmatrix}
B^*B^T & 0 \\
0 & B^TB^*
\end{pmatrix}$ and $|\xi - A^{*2}| = |\xi I_{n_1} - B^*B^T| \cdot |\xi I_{n_2} - B^TB^*| = \xi^{n_2-n_1}|\xi I_{n_1} - B^*B^T|^2$. Assume that $\xi_1, \ldots, \xi_{n_1}$ are all the eigenvalues of $B^*B^T$. Note that if $\xi$ is an eigenvalue of $B^*B^T$ then $\pm \sqrt{\xi}$ must be eigenvalues of $A^*$. Then all the eigenvalues of $A^*$ are given by $\pm \sqrt{\xi_1}, \ldots, \pm \sqrt{\xi_{n_1}}$, and $(n_2 - n_1)$'s 0. Therefore, we have Spec $\mathcal{L}(G) = \{[1 \pm \sqrt{\xi_1}], \ldots, [1 \pm \sqrt{\xi_{n_1}}], [0]^2\}$, which implies that $n_1 = n_2$ if $G$ has an even number of distinct $\mathcal{L}$-eigenvalues.

Now we are in a position to prove the main result of this section.

**Theorem 3.1.** Let $G$ a connected bipartite graph with at least one vertex of degree 1. Then $G$ has exactly four distinct $\mathcal{L}$-eigenvalues if and only if $G$ is the graph obtained from $G' \cup K_2$ by joining one vertex of $K_2$ to one part of $G'$, where $G' = K_2$, or $G'$ is the incidence graph of the complement of a Hadamard design.

**Proof.** Let $G$ be a connected bipartite graph on $n$ vertices $m$ edges with at least one vertex of degree 1. Suppose that $V(G) = V_1 \cup V_2$ is the bipartition of $G$, where $|V_i| = n_i$ for $i = 1, 2$. If $G$ has four distinct $\mathcal{L}$-eigenvalues, we must have $n_1 = n_2 = \frac{n}{2}$ by above arguments, and furthermore, we can assume that Spec $\mathcal{L}(G) = \{[2], [2 - \alpha]^{\frac{n_2}{2}}, [\alpha]^{\frac{n_2}{2}}, [0]^1\}$ ($0 < \alpha < 1$) by Lemma 2.1. Set

\[
x_1 = \frac{1}{\sqrt{2m}}D_1^jo_n = \frac{1}{\sqrt{2m}}\begin{pmatrix}D_1^0j_1^2 \\
D_2^0j_2^2\end{pmatrix} \text{ and } x_2 = \frac{1}{\sqrt{2m}}\begin{pmatrix}D_1^0j_z^2 \\
-D_2^0j_z^2\end{pmatrix},
\]

one can easily verify that $x_1$ and $x_2$ are the eigenvectors of $G$ corresponding to the $\mathcal{L}$-eigenvalues 0 and 2, respectively. Let $f(x) = (x - \alpha)(x - (2 - \alpha))$. By using the spectral decomposition of $f(\mathcal{L})$, we obtain

\[
f(\mathcal{L}) = (\mathcal{L} - \alpha I)(\mathcal{L} - (2 - \alpha)I) = \alpha(2 - \alpha)(x_1x_1^T + x_2x_2^T),
\]

that is,

\[
\left(D^{-\frac{1}{2}}AD^{-\frac{1}{2}}\right)^2 - (1 - \alpha)^2I = \frac{\alpha(2 - \alpha)}{m}\begin{pmatrix}D_1^0j_z^2 & 0 \\
0 & D_2^0j_z^2\end{pmatrix}.
\]

By considering the $(u, u)$-entry and $(u, v)$-entry ($u, v \in V_i$ and $u \neq v$) at both sides of (14), we have

\[
\sum_{w \sim u} \frac{1}{d_w} = (1 - \alpha)^2d_u + \frac{\alpha(2 - \alpha)}{m}d_u^2,
\]

and

\[
\sum_{w \sim u} \frac{1}{d_w} = \frac{\alpha(2 - \alpha)}{m} d_u d_v \text{ for } u, v \in V_i, \text{ where } i = 1, 2.
\]
Suppose that $u_0$ is a vertex of $G$ with degree 1. Assume that $u_0 \in V_1$, and let $v_0 \in V_2$ be the unique neighbor of $u_0$. For any $u \in V_1 \setminus \{u_0\}$, from (16) we know that

$$\sum_{w \sim u} \frac{1}{d_w} = \frac{\alpha(2-\alpha)}{m} d_u d_{u_0} > 0,$$

(17)

implying that $u \sim v_0$, and $v_0$ is the unique common neighbor of $u$ and $u_0$ due to $d_{u_0} = 1$. Thus $v_0$ is adjacent to all vertices in $V_1$ by the arbitrariness of $u$, that is, $d_{v_0} = \frac{n}{2}$, and furthermore, from (16) we have

$$\frac{2}{n} = \frac{1}{d_{v_0}} = \sum_{w \sim v_0} \frac{1}{d_w} = \frac{\alpha(2-\alpha)}{m} d_u = \frac{1 - (1 - \alpha)^2}{m} d_u$$

for any $u \in V_1 \setminus \{u_0\}$.

(18)

In addition, putting $u = u_0$ in (15) and noting that $v_0$ is the unique neighbor of $u_0$, we obtain

$$\frac{2}{n} = (1 - \alpha)^2 + \frac{\alpha(2-\alpha)}{m},$$

i.e.,

$$1 - \alpha)^2 = \frac{2m - n}{n(m - 1)}.$$

(19)

Combining (18) and (19), we deduce that $d_u = \frac{2m-2}{n^2}$ for any $u \in V_1 \setminus \{u_0\}$. For any $v \in V_2 \setminus \{v_0\}$, we see that all neighbors of $v$ share the same degree $\frac{2m-2}{n^2}$. Then from (15) we get

$$\frac{n - 2}{2m - 2} \cdot d_v = \sum_{w \sim v} \frac{1}{d_w} = (1 - \alpha)^2 d_v + \frac{1 - (1 - \alpha)^2}{m} d_v.$$  

(20)

Combining (19) and (20), one can easily deduce that $d_v = \frac{n^2 - 4m}{2n^2}$. Since the sum of all degrees of vertices in $V_2$ is equal to the number of edges of $G$, we have $\frac{4}{2} + \frac{n^2 - 4m}{2n^2} \cdot (\frac{n}{2} - 1) = m$, which implies that $m = \frac{n^2 + 2n}{8}$. Therefore, we get $\alpha = 1 - \sqrt{\frac{2}{n+2}}$ by (19), and $d_u = \frac{w+4}{4}$ for any $u \in V_1 \setminus \{u_0\}$, $d_v = \frac{n}{4}$ for any $v \in V_2 \setminus \{v_0\}$. If $n = 4$, it is obvious that $G = P_4$, which has $\mathcal{L}$-spectrum $\text{Spec}_L(P_4) = \{1, 1.5, 0.5, 0\}$, as required. If $n > 4$, then $n \geq 6$ due to $n$ is even, and so $|V_1 \setminus u_0| = |V_2 \setminus v_0| \geq 2$. Then, for any two vertices $u_1, u_2 \in V_1 \setminus \{u_0\}$, from (16) we obtain

$$\frac{2}{n} + |N_G(u_1) \cap N_G(u_2)| \cdot \frac{4}{n} = \sum_{w \sim u_1} \frac{1}{d_w} = \frac{\alpha(2-\alpha)}{m} d_{u_1} d_{u_2} = \frac{n + 4}{2n},$$

which gives that $|N_G(u_1) \cap N_G(u_2)| = \frac{n}{8}$. Similarly, for $v_1, v_2 \in V_2 \setminus \{v_0\}$, we have

$$|N_G(v_1) \cap N_G(v_2)| \cdot \frac{4}{n + 4} = \sum_{w \sim v_1} \frac{1}{d_w} = \frac{\alpha(2-\alpha)}{m} d_{v_1} d_{v_2} = \frac{n}{2(n + 4)},$$

so $|N_G(v_1) \cap N_G(v_2)| = \frac{n}{8}$. Set $V'_1 = V_1 \setminus \{u_0\}$, $V'_2 = V_2 \setminus \{v_0\}$ and $G' = G[V'_1 \cup V'_2]$, the induced subgraph of $G$ on $V'_1 \cup V'_2$. Then $G$ is just the graph obtained from $G' \cup K_2$ by joining one vertex of $K_2$ to all the vertices in $V'_1$ of $G'$. By above arguments, we see that $G'$ is a $\frac{n}{4}$-regular bipartite graph (with the bipartition $V(G') = V'_1 \cup V'_2$) on $n - 2$ vertices.
in which each pair of vertices in \( V'_1 \) (resp. \( V'_2 \)) have \( \frac{4}{5} \) common neighbors in \( V'_2 \) (resp. \( V'_1 \)). Therefore, we claim that \( G' \) is the incidence graph of a symmetric BIBD with parameters \((4t - 1, 2t, t)\) if we put \( n = 8t \). Clearly, such a symmetric BIBD is the complement of a symmetric BIBD with parameters \((4t - 1, 2t - 1, t - 1)\), which is known as the Hadamard design of dimension \( t \).

Conversely, if \( G' = K_2 \), by the assumption, we have \( G = P_4 \), which has exactly four distinct \( L \)-eigenvalues. Now assume that \( G' \) is the incidence graph (with the bipartition \( V(G') = V'_1 \cup V'_2 \)) of the complement of a Hadamard design of dimension \( t \). In other words, \( G' \) is the incidence graph of a symmetric BIBD with parameters \((4t - 1, 2t, t)\). Recall that \( G \) is the graph obtained from \( G' \cup K_2 \) \((V(K_2) = \{u_0, v_0\})\) by joining the vertex \( v_0 \) of \( K_2 \) to all the vertices in \( V'_1 \) of \( G' \). We will show that \( G \) has exactly four distinct \( L \)-eigenvalues.

Suppose \( A(G') = \begin{pmatrix} 0 & B' \\ B'^T & 0 \end{pmatrix} \). Then the \( L \)-matrix of \( G \) can be written as

\[
L = I_{8t} - \begin{pmatrix} 0 & \sqrt{1/4t} & \sqrt{1/(8t^2 + 4t)}j_{4t-1} & 0 \\ \sqrt{1/4t} & 0 & 0 & \sqrt{1/(4t^2 + 2t)}B' \\ \sqrt{1/(8t^2 + 4t)}j_{4t-1} & 0 & 0 & \sqrt{1/(4t^2 + 2t)}B'^T \\ 0 & \sqrt{1/(4t^2 + 2t)}B & 0 & 0 \end{pmatrix}
\]

By the arguments at the beginning of this section and (13), we know that \( G' \) is a \( 2t \)-regular bipartite graph on \( 8t - 2 \) vertices with adjacency spectrum

\[
\text{Spec}_{A}(G') = \left\{ [2t]^1, \left[ \sqrt{t} \right]^{4t-2}, \left[ -\sqrt{t} \right]^{4t-2}, [-2t]^1 \right\}.
\]

Note that the vectors \( y_0 = j_{8t-2} - (j^T_{4t-1}, j^T_{4t-1}) \) and \( y'_0 = (j^T_{4t-1}, -j^T_{4t-1}) \) are the eigenvectors of \( G' \) with respect to the (adjacency) eigenvalues \( 2t \) and \(-2t \), respectively. Suppose that \( y_1 = (z^T_1, w^T_1) \) and \( y'_1 = (z^T_1, -w^T_1) \) \((1 \leq i \leq 4t - 2)\) are all the orthonormal eigenvectors of \( G' \) with respect to the (adjacency) eigenvalues \( \sqrt{t} \) and \(-\sqrt{t} \), respectively. Then \( B'w_i = \sqrt{t}z_i \) and \( B'^Tz_i = \sqrt{t}w_i \) for \( 1 \leq i \leq 4t - 2 \). Also, for each \( i \), we have \( z^T_1j_{4t-1} = 0 \) and \( w^T_1j_{4t-1} = 0 \) due to \( y^T_1y_0 = 0 \) and \( y'^T_1y'_0 = 0 \).

Taking \( x_0 = D^Tj_{8t} = (1, 2\sqrt{t}, \sqrt{2t + 1}j^T_{4t-1}, \sqrt{2t}j^T_{4t-1}) \) and \( x'_0 = (1, -2\sqrt{t}, \sqrt{2t + 1}j^T_{4t-1}, -\sqrt{2t}j^T_{4t-1}) \), one can easily verify that \( \mathcal{L}x_0 = 0 \cdot x_0 \) and \( \mathcal{L}x'_0 = 2 \cdot x'_0 \) due to \( B'j_{4t-1} = B'^Tj_{4t-1} = 2t \cdot j_{4t-1} \). Furthermore, suppose \( x_i = (0, 0, z^T_I, w^T_I) \), \( x'_i = (0, 0, z^T_I, -w^T_I) \) for \( 1 \leq i \leq 4t - 2 \), and \( x_{4t-1} = (1, \sqrt{2t + 1}, \sqrt{2t + 1}j^T_{4t-1}, 1) \). Then the \( L \)-matrix of \( G \) can be written as

\[
\begin{pmatrix}
\sqrt{1/4t} & \sqrt{1/(8t^2 + 4t)}j_{4t-1} & 0 & \sqrt{1/(4t^2 + 2t)}B' \\
\sqrt{1/4t} & 0 & 0 & \sqrt{1/(4t^2 + 2t)}B'^T \\
\sqrt{1/(8t^2 + 4t)}j_{4t-1} & 0 & 0 & \sqrt{1/(4t^2 + 2t)}B & 0 \\
0 & \sqrt{1/(4t^2 + 2t)}B & 0 & 0 \\
\end{pmatrix}
\]

Since \( z^T_Ij_{4t-1} = 0 \), \( w^T_Ij_{4t-1} = 0 \), and \( B'^Tj_{4t-1} = B'^Tj_{4t-1} = 2t \cdot j_{4t-1} \), one can also verify that \( \mathcal{L}x_i = (1 - \sqrt{1/(4t + 2)})x_i \), \( \mathcal{L}x'_i = (1 + \sqrt{1/(4t + 2)})x'_i \) holds for each \( 1 \leq i \leq 4t - 1 \). Since \( x_1, x'_1, \ldots, x_{4t-1}, x'_{4t-1} \) are pairwise orthogonal, we conclude that both \( 1 - \sqrt{1/(4t + 2)} \) and \( 1 + \sqrt{1/(4t + 2)} \) are the \( L \)-eigenvalues of \( G \) with multiplicities at least \( 4t - 1 \). As \((4t - 1) \cdot 2 + 2 = 8t\), which equals to order of \( G \), we have

\[
\text{Spec}_{L}(G) = \left\{ [2]^1, \left[ 1 + \sqrt{1/(4t + 2)} \right]^{4t-2}, \left[ 1 - \sqrt{1/(4t + 2)} \right]^{4t-2}, [0]^1 \right\},
\]

and our result follows.

Let \( \mathcal{G} \) denote the set of connected bipartite graphs with at least one vertex of degree 1 having four distinct \( L \)-eigenvalues. According to Theorem 3.1, each graph (except \( P_4 \))
in $\mathcal{G}$ is of order $n = 8t$ for some positive integer $t$ and corresponds to a Hadamard design of dimension $t$, or equivalently, a Hadamard matrix of order $4t$. In the following, we list some examples on constructing Hadamard matrices.

The Kronecker product $M \otimes N$ of matrices $M = (m_{ij})_{a \times b}$ and $N = (n_{ij})_{c \times d}$ is the $ac \times bd$ matrix obtained from $M$ by replacing each element $m_{ij}$ with the block $m_{ij}N$.

**Example 1.** (Sylvester’s Construction, see [34]) Assume that $H_1$ and $H_2$ are two Hadamard matrices of order $m$ and $n$, respectively. Then $H_1H_1^T = ml_m$ and $H_2H_2^T = nl_n$. Therefore, $(H_1 \otimes H_2)(H_1 \otimes H_2)^T = (H_1 \otimes H_2)(H_1^T \otimes H_2^T) = (H_1H_1^T) \otimes (H_2H_2^T) = ml_m \otimes nl_n = mnI_{mn}$, which implies that $H_1 \otimes H_2$ is a Hadamard matrix of order $mn$. Let $H$ be the Hadamard matrix of order $2$ given by

$$H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$ 

Putting $H_1 = H$ and $H_i = H \otimes H_{i-1}$ for $i \geq 2$. We see that $H_i (i \geq 2)$ is also a Hadamard matrix, which has order $2^i = 4 \cdot 2^{i-2}$. Therefore, by Theorem 3.1, there exists a connected bipartite graph belonging to $\mathcal{G}$ of order $8 \cdot 2^{i-2} = 2^{i+1}$ for each $i \geq 2$.

**Example 2.** (Paley’s Constructions, see [28], or [23], Theorems 3.2–3.3) Firstly, let $p^\alpha$ be a prime power such that $p^\alpha \equiv 3 \pmod{4}$, and let $a_0, a_1, \ldots, a_{p^\alpha-1}$ be all the elements of the finite field $GF(p^\alpha)$. Suppose that $C = (c_{ij})$ is the matrix of order $p^\alpha$ defined as follows:

$$c_{ij} = \begin{cases} 0 & \text{if } a_i = a_j, \\ 1 & \text{if } a_i - a_j = a^2 \text{ for some } a \in GF(p^\alpha) \setminus \{0\}, \\ -1 & \text{if } a_i - a_j \text{ is not a square in } GF(p^\alpha). \end{cases}$$

Putting

$$S_1 = \begin{pmatrix} 0 & \mathbf{j}^T \\ -\mathbf{j} & C \end{pmatrix} \quad \text{and} \quad H_1 = I + S_1.$$

Then $H_1$ is a Hadamard matrix of order $p^\alpha + 1 (= 4t_1)$. Next, let $p^\beta$ be a prime power such that $p^\beta \equiv 1 \pmod{4}$. A matrix $C$ could be constructed as above. Putting

$$S_2 = \begin{pmatrix} 0 & \mathbf{j}^T \\ -\mathbf{j} & C \end{pmatrix} \quad \text{and} \quad H_2 = S_2 \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} + I \otimes \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix},$$

$H_2$ is a Hadamard matrix of order $2(p^\beta + 1) (= 4t_2)$. Therefore, by Theorem 3.1, there exists a connected bipartite graph belonging to $\mathcal{G}$ of order $2(p^\alpha + 1) = 8t_1$ when $p^\alpha \equiv 3 \pmod{4}$, and of order $4(p^\beta + 1) = 8t_2$ when $p^\beta \equiv 1 \pmod{4}$.

In addition, Wallis in [33] proved that, if $q$ is an odd natural number, there exists a Hadamard matrix of order $2^t q$ for each natural number $s \geq \lceil 2 \log_2(q - 3) \rceil$. Therefore, given any odd natural number $q$, there exists a connected bipartite graph belonging to $\mathcal{G}$ of order $2^{s+1} q$ for each $s \geq \lceil 2 \log_2(q - 3) \rceil$. For more techniques on the construction of Hadamard matrices or Hadamard designs, we refer the reader to [11, 23, 30].
unicyclic graph. Then $G$ has three distinct $L$-eigenvalues.

The following lemma shows that the diameter of a connected graph is less than the number of distinct $L$-eigenvalues.

**Lemma 4.1.** (See [8].) Let $G$ be a connected graph with diameter $d$. If $G$ has $s$ distinct $L$-eigenvalues, then $d \leq s - 1$.

According to Lemma 4.1, connected graphs with at most four distinct $L$-eigenvalues have diameter at most three. In Fig. 1, we list all unicyclic graphs whose diameter are at most three for later use. The following theorem determines all unicyclic graphs with three distinct $L$-eigenvalues.

**Theorem 4.1.** Let $G$ be a unicyclic graph. Then $G$ has three distinct $L$-eigenvalues if and only if $G = C_4$ or $G = C_5$.

**Proof.** Assume that $G$ has three distinct $L$-eigenvalues. By Lemma 4.1, the diameter of $G$ is exactly two because it cannot be a complete graph, and so $G$ must be one of the following graphs: $U_2(a)$, $U_7 = C_4$, $U_{10} = C_5$ (see Fig. 1). First suppose that $G = U_2(a)$. If $a = 1$, then $\text{Spec}_L(U_2(1)) = \{[1.7287], [1.5000], [0.7713], [0]\}$, and so $G \neq U_2(1)$. If $a \geq 2$, by Lemma 2.2, $G$ has 1 as its $L$-eigenvalue. By Theorem 2.1, $G$ must be a complete bipartite graph or a regular complete multipartite graph, both are impossible. Furthermore, by simple computation we know that $\text{Spec}_L(C_4) = \{[2], [1]^2, [0]\}$ and

Fig. 1: Unicyclic graphs with diameter at most three ($a, b, c \geq 1$).
Let $G$ be a connected graph with four distinct $L$-eigenvalues. By Lemma 2.1, we may assume that $\text{Spec}_L(G) = \{[\alpha]^m, [\beta]^m, [\gamma]^m, [0]^1\}$, where $2 \geq \alpha > \beta > \gamma > 0$. According to Lemma 2.3, we have

$$ (L - \alpha I)(L - \beta I)(L - \gamma I) = -\frac{\alpha \beta \gamma}{2m} D^\frac{1}{2} J D^\frac{1}{2}, $$

where $D$ is the diagonal degree matrix of $G$. By considering the $(u, u)$-entry at both sides of (21), we obtain that

$$ \frac{1}{d_v} + \frac{1}{d_w} + (\alpha + \beta + \gamma - 3) \frac{1}{d_v} + (\alpha - 1)(\beta - 1)(\gamma - 1)d_v = \frac{\alpha \beta \gamma}{2m} d^2_u. \tag{22} $$

By using (22) and Lemma 2.2, we now determine all unicyclic graphs with four distinct $L$-eigenvalues.

**Theorem 4.2.** Let $G$ be a unicyclic graph. Then $G$ has four distinct $L$-eigenvalues if and only if $G = C_6$, $G = C_7$, $G = U_2(1)$ or $G = U_4(1, 1, 1)$.

**Proof.** Let $G$ be a unicyclic graph with four distinct $L$-eigenvalues. Suppose $\text{Spec}_L(G) = \{[\alpha]^m, [\beta]^m, [\gamma]^m, [0]^1\}$, where $2 \geq \alpha > \beta > \gamma > 1$. Then the diameter of $G$ is equal to 2 or 3 due to $G$ cannot be a complete graph. Thus $G$ must be one of the graphs (excluding $U_1 = K_3$) shown in Fig. 1. If $G \in \{U_7, U_{10}, U_{13}, U_{14}\}$, then we have $G = U_{13} = C_6$ or $G = U_{14} = C_7$ by simple computation. To prove the result, it suffices to consider the remain cases.

First suppose that $G = U_2(a)$. If $a \geq 2$, then $u_1$ and $u_2$ (see Fig. 1) have the same neighborhood, and thus 1 is an $L$-eigenvalue of $G$ by Lemma 2.2 (i). Then $\beta = 1$ or $\gamma = 1$ by Lemma 2.1 (v). Without loss of generality, we assume that $\gamma = 1$. Putting $u = u_1$, $u = u_0$ and $u = v_0$ (see Fig. 1) in (22), respectively, we obtain the following three equalities:

$$ \begin{cases} 
(\alpha + \beta - 2) \cdot \frac{1}{a + 2} = \frac{\alpha \beta}{2m}, \\
\frac{1}{2} \cdot (\alpha + \beta - 2)(a + \frac{1}{2}) = \frac{\alpha \beta}{2m} (a + 2)^2, \\
\frac{1}{2} \cdot (a + 2)^2 + (\alpha + \beta - 2)(\frac{1}{a + 2} + \frac{1}{2}) = \frac{\alpha \beta}{2m} \cdot 2^2.
\end{cases} $$

By simple computation we obtain $a = 0$, which is contrary to $a \geq 2$. If $a = 1$, then $G = U_2(1)$ with $\text{Spec}_L(U_2(1)) = \{[1.7287], [1.5000], [0.7713], [0]\}$, as required.

Suppose $G = U_3(a, b)$. If $a \geq 2$ or $b \geq 2$, then 1 is an $L$-eigenvalue of $G$. As above, assume that $\gamma = 1$. Putting $u = u_1$, $u = v_1$, $u = u_0$ and $u = w_0$ in (22) one by one, we
obtain the following four equalities:

\[
\begin{align*}
(\alpha + \beta - 2) \cdot \frac{1}{a + 2} &= \frac{\alpha \beta}{2m}, \\
(\alpha + \beta - 2) \cdot \frac{1}{b + 2} &= \frac{\alpha \beta}{2m}, \\
\frac{1}{2(b + 2)} \cdot 2 + (\alpha + \beta - 2)(a + 1) + \frac{1}{2} &= \frac{\alpha \beta}{2m}(a + 2)^2, \\
(\alpha + \beta - 2)(\frac{1}{a + 2} + \frac{1}{b + 2}) &= \frac{\alpha \beta}{2m} \cdot 2^2,
\end{align*}
\]

from which one can easily deduce that \(a = b = 0\), a contradiction. If \(a = b = 1\), then \(\text{Spec}_L(G) = \text{Spec}_L(U_3(1, 1)) = \{[1.7676], [1.6667], [1], [0.5657], [0]\}, \) contrary to our assumption.

Now suppose that \(U = U_4(a, b, c)\). If \(\max\{a, b, c\} \geq 2\), as above, putting \(u = u_1, u = v_1\) and \(u = w_1\) in (22) one by one, we obtain

\[
\begin{align*}
(\alpha + \beta - 2) \cdot \frac{1}{a + 2} &= \frac{\alpha \beta}{2m}, \\
(\alpha + \beta - 2) \cdot \frac{1}{b + 2} &= \frac{\alpha \beta}{2m}, \\
(\alpha + \beta - 2) \cdot \frac{1}{c + 2} &= \frac{\alpha \beta}{2m},
\end{align*}
\]

which implies that \(a = b = c\), and so \(G = U_4(a, a, a)\). Then it suffices to consider the \(L\)-spectrum of the graph \(U_4(a, a, a)\), where \(a \geq 1\). In fact, the \(L\)-polynomial of \(U_4(a, a, a)\) is equal to

\[
P_L(U_4(a, a, a)) = |U - L|
\]

\[
\begin{align*}
(x - 1)I_a &\quad 0 &\quad 0 &\quad \frac{1}{\sqrt{a+2}J_a} &\quad 0 &\quad 0 \\
0 &\quad (x - 1)I_a &\quad 0 &\quad 0 &\quad \frac{1}{\sqrt{a+2}J_a} &\quad 0 \\
0 &\quad 0 &\quad (x - 1)I_a &\quad 0 &\quad 0 &\quad \frac{1}{\sqrt{a+2}J_a} \\
\frac{1}{\sqrt{a+2}J_a}^T &\quad 0 &\quad 0 &\quad x - 1 &\quad \frac{1}{a+2} &\quad \frac{1}{a+2} \\
0 &\quad \frac{1}{\sqrt{a+2}J_a}^T &\quad 0 &\quad \frac{1}{a+2} &\quad x - 1 &\quad \frac{1}{a+2} \\
0 &\quad 0 &\quad \frac{1}{\sqrt{a+2}J_a}^T &\quad \frac{1}{a+2} &\quad \frac{1}{a+2} &\quad x - 1
\end{align*}
\]

\[
= \frac{1}{(a + 2)^3} x(x - 1)^{3a - 3} p_1(x)(p_2(x))^2,
\]

where \(p_1(x) = (a + 2)x - 2(a + 1)\) and \(p_2(x) = (a + 2)x^2 - (2a + 5)x + 3\). It is easy to verify that \(1\) cannot be a root of \(p_1(x)\) or \(p_2(x)\) due to \(a \neq 0\). Also, \(p_1(x)\) and \(p_2(x)\) cannot share
the same root because $a \neq 0$. Furthermore, the roots of $p_2(x)$ must be distinct because the discriminant $(2a + 5)^2 - 4 \cdot (a + 2) = 4a^2 + 8a + 1 > 0$ due to $a \geq 1$. Therefore, $G = U_4(a, a, a)$ has four distinct $L$-eigenvalues if and only if $a = 1$.

Next suppose that $U = U_5(a)$. If $a \geq 2$, as above, putting $u = u_1$ and $u = u_0$ in (22), we have

$$\begin{cases}
(\alpha + \beta - 2) \cdot \frac{1}{a + 1} = \frac{\alpha \beta}{2m} \\
(\alpha + \beta - 2) \cdot (a + \frac{1}{3}) = \frac{\alpha \beta}{2m} \cdot (a + 1)^2,
\end{cases}$$

from which one can deduce that $a + 1 = a + \frac{1}{2}$, a contradiction. If $a = 1$, then $\text{Spec}_L(G) = \text{Spec}_L(U_5(1)) = \{1.8566, [1.5000], [1.2975], [0.3459], [0]\}$, a contradiction.

Suppose $G = U_6(a, b)$. If $a \geq 2$ or $b \geq 2$, as above, putting $u = u_1$ and $u = u_0$ in (22) in turn, we get

$$\begin{cases}
(\alpha + \beta - 2) \cdot \frac{1}{a + 1} = \frac{\alpha \beta}{2m} \\
(\alpha + \beta - 2) \cdot (a + \frac{1}{b + 3}) = \frac{\alpha \beta}{2m} \cdot (a + 1)^2,
\end{cases}$$

implying that $b = -2$, contrary to $b \geq 1$. If $a = b = 1$, then $\text{Spec}_L(G) = \text{Spec}_L(U_6(1, 1)) = \{1.8762, [1.5000]^2, [0.7838], [0.3400], [0]\}$, which is impossible.

Now suppose that $G \in \{U_8(a), U_9(a, b)\}$. Then $G$ is a bipartite graph. We claim that $a = b = 1$, since otherwise $G$ cannot have four distinct $L$-eigenvalues by Corollary 2.2. Thus $G = U_8(1)$ or $G = U_9(1, 1)$, which are also impossible due to $\text{Spec}_L(U_8(1)) = \{2, [1.4082], [1], [0.5918], [0]\}$ and $\text{Spec}_L(U_9(1, 1)) = \{2, [1.5000], [1.3333], [0.6667], [0.5000], [0]\}$.

Suppose $G = U_{11}(a)$. If $a \geq 2$, putting the $u = u_1$ and $u = u_0$ in (22), we have

$$\begin{cases}
(\alpha + \beta - 2) \cdot \frac{1}{a + 2} = \frac{\alpha \beta}{2m} \\
(\alpha + \beta - 2) \cdot (a + \frac{1}{2} \cdot 2) = \frac{\alpha \beta}{2m} \cdot (a + 2)^2,
\end{cases}$$

Thus we deduce that $a + 2 = a + 1$, a contradiction. If $a = 1$, then $\text{Spec}_L(G) = \text{Spec}_L(U_{11}(1)) = \{1.8691, [1.8090], [1.1759], [0.6910], [0.4550], [0]\}$, contrary to our assumption.

Suppose $G = U_{12}(a, b)$. If $a \geq 2$ or $b \geq 2$, putting $u = u_1$ and $u = u_0$ in (22), we obtain the following three equalities:

$$\begin{cases}
(\alpha + \beta - 2) \cdot \frac{1}{a + 2} = \frac{\alpha \beta}{2m} \\
(\alpha + \beta - 2) \cdot (a + \frac{1}{2} \cdot \frac{1}{b + 2}) = \frac{\alpha \beta}{2m} \cdot (a + 2)^2,
\end{cases}$$

which implies that $b = -\frac{4}{3}$, a contradiction. Then $a = b = 1$, then $\text{Spec}_L(G) = \text{Spec}_L(U_{12}(1, 1)) = \{1.8931, [1.8259], [1.3766], [1], [0.4642], [0.4402], [0]\}$, a contradiction.

We complete the proof. □
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