Collisionless sound of bosonic superfluids in lower dimensions

L. Salasnich

Dipartimento di Fisica e Astronomia “Galileo Galilei” and CNISM, Università di Padova, Via Marzolo 8, 35131 Padova, Italy
Istituto Nazionale di Ottica (INO) del Consiglio Nazionale delle Ricerche (CNR), via Nello Carrara 1, 50125 Sesto Fiorentino, Italy
Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Padova, Via Marzolo 8, 35131 Padova, Italy

The superfluidity of low-temperature bosons is well established in the collisional regime. In the collisionless regime, however, the presence of superfluidity is not yet fully clarified, in particular in lower spatial dimensions. Here we compare the Vlasov-Landau equation, which does not take into account the superfluid nature of the bosonic system, with the Andreev-Khalatnikov equations, which instead explicitly contain a superfluid velocity. We show that recent experimental data of the sound mode in a two-dimensional collisionless Bose gas of $^{87}$Rb atoms are in good agreement with both theories but the sound damping is better reproduced by the Andreev-Khalatnikov equations below the Berezinskii-Kosterlitz-Thouless critical temperature $T_c$, while above $T_c$, the comparison is not conclusive. For one dimensional bosonic fluids, where experimental data are not yet available, we find larger differences between the sound velocities predicted by the two transport theories and, also in this case, the existence of a superfluid velocity reduces the sound damping.

PACS numbers: 05.30.-d; 67.85.-d; 52.20.-j

Introduction. According to Landau [1], the liquid helium below the critical temperature is characterized by a superfluid component and a normal component. This idea was inspired by similar models for superconductors of London brothers [2], and for superfluids of Tisza [3]. In the standard hydrodynamic treatment of a neutral super-fluid [4-6] the normal component is supposed to be in the collisional regime. The very special case of the collisionless superfluid Helium-4, where the normal component is in the collisionless regime was analyzed by Andreev and Khalatnikov [7]. In the collisionless regime [5,8] the dimensionless parameter $\omega \tau_c$ is such that $\omega \tau_c \gg 1$, where $\tau_c$ is the collision time of quasi-particles and $\omega$ is the frequency of a generic macroscopic oscillation traveling along the fluid. Usually $\tau_c$ grows by decreasing the temperature $T$, and at extremely low temperature one expects that collisionless phenomena dominate the dynamics of superfluids and, more generally, the dynamics of quantum liquids. Indeed, the Andreev and Khalatnikov [7] collisionless approach is in full agreement with experimental measurements [9] of the sound velocity of Helium-4 for the temperature below 0.4 Kelvin. In general, depending on size and density, the system can be in the collisionless regime also far from zero temperature [4,8]. Actually, such natural systems as ionized plasmas do exist which, due to the velocity dependence of the collision frequency, become collisionless in the opposite regime of very high temperature [10].

The interest in collisionless superfluids has been renewed by a recent experiment [11], where it was measured the sound mode in a uniform quasi-2D Bose gas made of $^{87}$Rb atoms. The experimental data of the speed of sound are in good agreement with theoretical results [12,13] based on the Vlasov-Landau equation [14,15] (which is substantially equivalent to the random-phase approximation [16]) for neutral collisionless bosons. There are, however, some discrepancies between the experimental data of sound damping and the prediction of the Vlasov-Landau equation [12].

In Refs. [12,13] the superfluid nature of the system is not taken into account: the superfluid velocity $v_s(r,t)$ does not appear and the phase-space distribution $f(r,p,t)$ of particles is used instead of the phase-space distribution $f_{qp}(r,p,t)$ of quasi-particles.

In this paper we investigate the collisionless sound mode of bosonic quantum gases both in two and one spatial dimensions. We compare the Vlasov-Landau equation, which does not take into account the superfluid nature of the neutral bosonic system, with the Andreev-Khalatnikov equations [7], which instead explicitly contain a superfluid velocity. We find that the behavior of the speed of sound obtained with the two approaches is similar but the experimental data of sound damping [11] in a 2D collisionless Bose gas are closer to the theoretical predictions based on Andreev-Khalatnikov equations, below the Berezinsky-Kosterlitz-Thouless critical temperature $T_c$ [7,18]. In 1D the superfluidity is much more elusive [19], but it could be experimentally found at low temperature for finite-size systems where phase slips are inhibited [20]. For the collisionless 1D Bose gas we show that the speed of sound predicted by the two transport theories is quite different. The imaginary parts of the two complex sound velocities are instead very close each other, but also in this 1D case the presence of a superfluid velocity suppresses the sound damping.

Vlasov-Landau theory of neutral collisionless bosons. The equilibrium distribution of a weakly-interacting gas of $D$-dimensional neutral bosons, each of them with mass
where \( \mu \) is the chemical potential, fixed by the condition \( n_0 = \int dV_p f_0(p) \) with \( n_0 \) the total number density at equilibrium, \( dV_p = d^3p/(2\pi \hbar)^3 \) and \( p = |p| \). Here we assume a weakly-interacting bosonic gas with zero-range interaction of strength \( g \). Notice that, because \( n_0 \) is constant, introducing the effective chemical potential \( \tilde{\mu} = \mu - g n_0 \), \( f_0(p) \) can also be interpreted as the distribution of non-interacting bosons.

The interaction strength \( g \) appears also in the out-of-equilibrium mean-field external potential \( U_{m,f}(r,t) = g \int dV_p f(r, p, t) \), where \( f(r, p, t) \) is the out-of-equilibrium distribution function, which is driven by the following mean-field collisionless Vlasov-Landau equation

\[
\left( \frac{\partial}{\partial t} + \frac{p}{m} \cdot \nabla - \nabla U_{m,f}(r,t) \cdot \nabla_p \right) f(r, p, t) = 0
\]

where \( \nabla = (\partial_x, \partial_y, \partial_z) \) and \( \nabla_p = (\partial_{p_x}, \partial_{p_y}, \partial_{p_z}) \). Linearizing this equation around the equilibrium configuration one finds a plane-wave solution with frequency \( \omega \) and wavevector \( k \) such that \( \omega = u_0 k \), where \( u_0 \) is the speed of sound and \( k = |k| \). The determination of this complex quantity \( u_0 \) requires non trivial integrations in the complex domain [21]. For analytical and numerical details see the Supplemental Material. In general, the frequency \( \omega \) and, correspondingly, the velocity \( u_0 \) are complex numbers.

Andreev-Khalatnikov theory of neutral collisionless superfluids. Let us now consider a \( D \)-dimensional collisionless superfluid made of identical bosonic particles of mass \( m \). At thermal equilibrium the system is characterized by the total mass density \( \rho_0 = \rho_{n0} + \rho_{s0} \) where \( \rho_{n0} \) is the superfluid mass density and \( \rho_{s0} \) is the normal mass density. At fixed \( \rho_0 \) both \( \rho_{n0} \) and \( \rho_{s0} \) depend on the absolute temperature \( T \). In particular, the normal mass density \( \rho_{n0} \) can be obtained from the equilibrium distribution \( f_{qp,0}(p) \) of quasi-particles [11] as \( \rho_{n0} = -\frac{1}{g} \int dV_p p^2 \frac{df_0(p)}{dE} \) with \( p = |p| \) and

\[
f_{qp,0}(p) = \frac{1}{e^{\beta E[p,\rho_0]} - 1}
\]

where \( \beta = 1/(k_BT) \) with \( k_B \) the Boltzmann constant and \( E(p) \) is the spectrum of quasi-particles. Here we assume the Bogoliubov spectrum [22] of a weakly-interacting bosonic gas with zero-range interaction of strength \( g \), given by \( E[p, \rho_0] = \sqrt{p^2/(2m)} + (p^2/2m) + (2g/m)\rho_0 \).

Within the Andreev and Khalatnikov theory [5, 7, 8], the collisionless superfluid is characterized by three dynamical variables: the phase-space distribution of quasi-particles \( f_{qp}(r, p, t) \), the local mass density \( \rho(r, t) \) and the superfluid velocity \( \mathbf{v}_s(r, t) \). There are three coupled partial differential equations. One is the collisionless Vlasov-Landau equation for the distribution of quasi-particles

\[
\left( \frac{\partial}{\partial t} + \nabla \cdot (E[p, \rho(r, t)] + \mathbf{v}_s(r, t) \cdot \mathbf{p}) \cdot \nabla - \nabla (E[p, \rho(r, t)]) \right) f_{qp}(r, p, t) = 0
\]

and it is important to observe that in front of \( \mathbf{v}_s(r, t) \) it appears \( \rho(r, t) \). Finally, there is an equation for the superfluid velocity \( \mathbf{v}_s(r, t) \), which reads

\[
\frac{\partial \mathbf{v}_s(r, t)}{\partial t} + \nabla \left[ \frac{1}{2} \mathbf{v}_s(r, t)^2 + \frac{\mu_0[\rho(r, t)]}{m} \right]
+ \int dV_p \frac{\partial E[p, \rho(r, t)]}{\partial \rho} f_{qp}(r, p, t) = 0
\]

and \( \mu_0 \) the chemical potential of the system at zero temperature (i.e. \( T = 0 \)).

Similarly to the linearized Vlasov-Landau equation, also the linearized Andreev-Khalatnikov equations around the equilibrium configuration admit plane-wave solutions with frequency \( \omega \) and wavevector \( k \) such that \( \omega = u_0 k \) with \( u_0 \) the corresponding speed of sound. Analytical and numerical details on the derivation and solution of the linearized Andreev-Khalatnikov equations are discussed in the Supplemental Material.

In Fig. [1] we report our numerical solutions of the speed of sound \( u_0 = c_R - i c_I \) in the 2D case, with \( i = \sqrt{-1} \) the imaginary unit. Solid curves are obtained by using the Vlasov-Landau equation while dashed curves are produced by adopting the Andreev-Khalatnikov equations. In the figure there are also, as filled red circles, the experimental data of Ref. [11] obtained with a collisionless Bose gas of \(^{87}\)Rb atoms. In the figure, the quantities are plotted versus the scaled temperature \( T/T_c \), with \( T_c \) the Berezinskii-Kosterlitz-Thouless critical temperature [17, 18] predicted at thermal equilibrium for 2D interacting superfluid bosons [23, 24]. The superfluid-to-normal Kosterlitz-Thouless phase transition occurs due to the unbinding of vortex-antivortex pairs, whose number strongly increases close to the critical temperature.
FIG. 1: Results from numerical solution of the dispersion equation \( u_0 = c_R - ic_I \) vs temperature in the 2D case. Upper panel: the normalized speed of sound \( c_R/c_B \) as a function of the normalized temperature \( T/T_c \), where \( c_B = \sqrt{gn_0/m} \) is the Bogoliubov velocity, \( T_c = \frac{2\pi n_0^2 h^2}{mk_B \ln (380 h^2/(mg))} \) is the Berezinskii-Kosterlitz-Thouless critical temperature, and \( n_0 \) the 2D number density at equilibrium. Lower panel: \( Q = c_R/c_I \) quality factor of the sound damping. To compare the two transport theories with the experiment of Ref. [11] we choose \( g = 0.16 \frac{\hbar^2}{m} \). The black solid curve is the result of the Vlasov-Landau theory; the blue dashed curve is obtained using the Andreev-Khalatnikov theory. Red dots are measured data of Ref. [11].

The presence of vortices with quantized circulation is strictly related to the existence of a superfluid velocity \( \mathbf{v}_s(\mathbf{r},t) \), which must satisfy the equation \( \mathbf{v}_s(\mathbf{r},t) = (\hbar/m) \nabla \phi(\mathbf{r},t) \) with \( \phi(\mathbf{r},t) \) the angle of the phase of a complex order parameter [25]. As previously stressed, the Vlasov-Landau equation does not include a superfluid velocity but only the formation of quantized vortices. Thus, one can expect that below \( T_c \) the 2D Bose gas follows the Andreev-Khalatnikov theory and above \( T_c \) the 2D bosonic system is better described by the Vlasov-Landau equation.

In the upper panel of Fig. 1 we plot the real part of the scaled speed of sound \( c_R/c_B \), with \( c_B = \sqrt{gn_0/m} \) the Bogoliubov sound velocity. Remarkably, the experimental data (filled circles) are very well reproduced, both below and above \( T_c \), by the Vlasov-Landau equation (solid curve) but also by the Andreev-Khalatnikov equations (dashed curve). At very low temperature \( T \) the two curves of the two theories practically superimposed. In the lower panel of Fig. 1 there is instead the quality factor \( Q = c_R/c_I \) of the sound damping, namely the ratio between the real and the imaginary part of the sound velocity \( u_0 = c_R - ic_I \). For this quality factor \( Q \), the Andreev-Khalatnikov theory (dashed curve) is in much better agreement with the experimental results (filled circles) with respect to the Vlasov-Landau theory (solid curve) up to the critical temperature \( T_c \). Above the critical temperature \( T_c \) it seems that the quality factor \( Q \) can be better reproduced by the Vlasov-Landau equation but the presence of only few experimental points makes this statement questionable.

FIG. 2: Results from numerical solution of the dispersion equation \( u_0 = c_R - ic_I \) versus temperature in the 1D case. Upper panel: the normalized speed of sound \( c_R/c_B \) as a function of the normalized temperature \( T/T_B \), where \( c_B = \sqrt{gn_0/m} \) is the Bogoliubov velocity, \( T_B = \frac{2\pi n_0^2 h^2}{mk_B} \) is the degeneration temperature, and \( n_0 \) the 1D number density at equilibrium. Lower panel: \( Q = c_R/c_I \) quality factor of the sound damping. We choose \( g = 0.16 \frac{\hbar^2}{n_0 m} \). The black solid curve is the result of the Vlasov-Landau theory; the blue dashed curve is obtained using the Andreev-Khalatnikov theory.

We investigate also the 1D weakly-interacting Bose gas in the collisionless regime. Unfortunately there are
we plot the quality factor the superfluid nature of the Bose gas. In the lower panel of this slope can be experimentally use to the determine slope. Indeed, this suggests that in 1D the determination the Andreev-Khalatnikov theory predicts a much larger $u_c$ case, in 1D the real part ported in the upper panel of Fig. 2, contrary to the $T/T_0$ scaled temperature curves) and the Andreev-Khalatnikov equations (dashed obtained by solving the Vlasov-Landau equation (solid curves). The quantities are plotted as a function of the scaled temperature $T/T_B$ where $T_B = 2\pi n_0 h^2/(mk_B)$ is the temperature of Bose degeneracy, where the 1D thermal de Broglie wavelength $\lambda_T = h\sqrt{2\pi/(mk_B T)}$ becomes equal to the average distance $n_0^{-1}$ between bosons, with $n_0$ the equilibrium 1D number density. As clearly reported in the upper panel of Fig. 2 contrary to the 2D case, in 1D the real part $c_R$ of the sound velocity $v_0$ increases by increasing the temperature $T$. However, the Andreev-Khalatnikov theory predicts a much larger slope. Indeed, this suggests that in 1D the determination of this slope can be experimentally use to determine the superfluid nature of the Bose gas. In the lower panel we plot the quality factor $Q = c_s/c_I$ of the sound damping: the two theoretical curves are very close each other. This result implies that in 1D the damping is not very useful to discriminate between the two transport theories. 

Conclusions. We have analyzed the collisionless sound mode of a 2D weakly-interacting bosonic fluid, where recent experimental data are available [14], but also the collisionless sound mode of the 1D bosonic fluid, where experimental data are not yet available. We have compared two theories: the Vlasov-Landau equation versus the Andreev-Khalatnikov equations. The Andreev-Khalatnikov equations are more sophisticated because, contrary to the Vlasov-Landau equation, they also take into account the presence of a superfluid velocity. Our 2D theoretical results, also confronted with the experimental data, strongly suggest that below the critical temperature of the superfluid-to-normal transition the bosonic fluid is better described by the Andreev-Khalatnikov theory, while above the critical temperature the Vlasov-Landau theory could be more reliable. For the collisionless 1D Bose gas, our calculations show that the real part of the sound velocity grows by increasing the temperature and its slope determines the superfluid nature of the system. This prediction, as well as the reduction of sound damping due to the superfluid velocity, can be very useful for forthcoming theoretical and experimental investigations of collisionless superfluids.

ACKNOWLEDGMENTS

This work was partially supported by the University of Padova, BIRD project “Superfluid properties of Fermi gases in optical potentials”. LS thanks A. Cappellaro, K. Furutani, F. Toigo, and A. Tononi for useful suggestions. LS acknowledges F. Sattin for many fruitful discussions and valuable numerical support.
Supplemental Material
Collisionless sound of bosonic superfluids in lower dimensions

In this Supplemental Material we provide technical details useful for appreciating how we determine numerically the the sound mode from the Vlasov-Landau equation and the Andreev-Khalatnikov equations. We set \( h = m = k_B = 1 \).

**S1. Linearized Vlasov-Landau equation**

We start from the Vlasov-Landau equation of the main text. Setting

\[
 f(r, p, t) = f_0(p) + \hat{f}(p) e^{i(kr - \omega t)} \tag{S1}
\]

where the plane-wave fluctuations are supposed to be small with respect to the equilibrium quantities, we get the following linearized equation

\[
 (\omega - p \cdot k) \hat{f}(p) + g \int dV_p \hat{f}(p') k \cdot \nabla_p f_0(p) = 0 \tag{S2}
\]

From this expression one gets an implicit formula for the collisionless (zero-sound) velocity \( u_0 = \omega / k \), namely

\[
 1 - g \int dV_p \frac{\nabla_p f_0(p) \cdot n}{p \cdot n - u_0} = 0 \tag{S3}
\]

where \( n = k / k \) with \( k = |k| \).

In the Vlasov-Landau equation the quantity appears

\[
 \int d^D p \frac{\nabla_p f_0 \cdot n}{p \cdot n - u_0} \tag{S4}
\]

By chosing \( n \) parallel to x-axis, this expression simplifies to

\[
 \int d^D p \frac{\partial_{p_x} f_0}{p_x - u_0} \tag{S5}
\]

In dimension \( D = 2 \) it is straightforward to note that

\[
 \int d^2 p \frac{\partial_{p_x} f_0}{p_x - u_0} = \int dp_x \partial_{p_x} \left( \int dp_y f_0 \right) \frac{1}{p_x - u_0} \tag{S6}
\]

Thus, both in dimension one and two, ultimately one has to deal with one-dimensional integrals. The integral operator comes from an inverse Laplace transform, hence the path of integration is defined in the complex \( p \)-plane. The recipe for choosing the right path was given by Landau \[15\], and may be found in several recent references, e.g., \[10, 21\]. Here we provide just the results. The integral \( [S6] \) writes as the sum of an integral along the real axis plus a contribution coming from poles in the complex plane:

\[
 \int_0^{+\infty} dp_x \partial_{p_x} \left( \int dp_y f_0 \right) \frac{1}{p_x - u_0} + \mathcal{J} \tag{S7}
\]

If \( \text{Im}(u_0) > 0 \) then \( \mathcal{J} = 0 \). Conversely, if \( \text{Im}(u_0) < 0 \) we have

\[
\mathcal{J} = 2\pi i \partial_{p_x} f_x(p_x = u_0) \tag{S8}
\]

with

\[
 f_x(p_x) = \int dp_y p_y f_0(p_x, p_y) \tag{S9}
\]

**S2. Linearized Andreev-Khalatnikov equations**

We linearize the Andreev-Khalatnikov equations of the main text setting

\[
 f_{qp}(r, p, t) = f_{qp,0}(p) + \hat{f}_{qp}(p) e^{i(kr - \omega t)} \tag{S10}
\]

\[
 \rho(r, t) = \rho_0 + \hat{\rho} e^{i(kr - \omega t)} \tag{S11}
\]

\[
 v_s(r, t) = 0 + \hat{v}_s e^{i(kr - \omega t)} \tag{S12}
\]

where the plane-wave fluctuations are supposed to be small with respect to the equilibrium quantities. It follows that the linearized equations of motion are given by

\[
 (\omega - \nabla_p E(p) \cdot k) \hat{f}_{qp}(p) + \nabla_p E(p) \cdot k \frac{d f_{qp,0}(p)}{dE} \left( \frac{dE(p)}{\rho_0} \hat{\rho} + \hat{\rho} \cdot \hat{v}_s \right) = 0 \tag{S13}
\]

\[
 \omega \hat{\rho} - \rho_0 k \hat{v}_s - k \int dV_p \rho \hat{f}_{qp}(p) = 0 \tag{S14}
\]

\[
 -\omega \hat{v}_s + k \left( \frac{1}{\rho_0} \frac{d \hat{\rho}}{d \rho_0} + \int dV_p \hat{f}_{qp,0}(p) \frac{d^2 E(p)}{d \rho_0^2} \right) \hat{\rho} + k \int dV_p \frac{dE(p)}{d \rho_0} \hat{f}_{qp}(p) = 0 \tag{S15}
\]

where \( \rho_0 \) is the pressure at zero temperature. Equations \( [S14] \) and \( [S15] \) contain respectively the terms

\[
 \int dV_p \rho \hat{f}_{qp} \tag{S16}
\]

and

\[
 \int dV_p \hat{f}_{qp,0} \frac{d^2 E(p)}{d \rho_0^2} \tag{S17}
\]

Both terms may be computed from Eq. \( [S13] \); thus any dependence from \( \hat{f}_{qp,0} \) disappears from Eqns \( [S14] [S15] \), which become a set of two linear homogeneous equations for the two variables \( \hat{v}_s, \hat{\rho} \). The condition of vanishing determinant of the above set of linear equations yields the dispersion curve

\[
 (A - u_0)^2 - (C + c_T^2)(1 + B) = 0 \tag{S18}
\]
where, as before, \( u_0 = \omega / k \), and

\[
A = \int dV \rho \frac{\partial f_0}{\partial p} \frac{\partial E}{\partial p} \frac{1}{\partial_\rho_0 \partial_\rho - u_0} \tag{S19}
\]

\[
B = \int dV \rho p^2 \frac{\partial f_0}{\partial p} \frac{1}{\partial_\rho_0 \partial_\rho - u_0} \tag{S20}
\]

\[
C = \int dV \rho \frac{\partial f_0}{\partial p} \left( \frac{\partial E}{\partial \rho_0} \right)^2 \frac{1}{\partial_\rho_0 \partial_\rho - u_0} \tag{S21}
\]

\[
(S22)
\]

In the Andreev-Khalatnikov theory one has to deal with several integrals of the kind

\[
\int dp \frac{F(p)}{\partial p E(p) - u_0} \tag{S23}
\]

where we have dropped the \( x \) lowerscript for convenience. \( F(p) \) is one of the functions appearing in Eq. (S22). Since \( E(p) \), as defined in (7), is a nonlinear function of \( p \), the recipe of Eqns. (S7,S8) needs some modifications. Let \( \bar{p} \) be a root of the function

\[
D(p) = \partial_p E(p) - u_0 : \bar{D}(\bar{p}) = 0 \tag{S24}
\]

namely

\[
\bar{D} = 0 \tag{S25}
\]

Then, we may expand \( \bar{D}(p) \) around \( p = \bar{p} \):

\[
\bar{D} \simeq (p - \bar{p}) \partial_{\bar{p}}^2 E(\bar{p}) \tag{S26}
\]

Ultimately, therefore, the integrals (S23) are evaluated as

\[
\int dp \frac{F(p)}{\partial p E(p) - u_0} = \int_{-\infty}^{+\infty} dp \frac{F(p)}{\partial p E - u_0} + J' \tag{S27}
\]

This time we get

\[
J' = 2\pi i \frac{F(\bar{p})}{\partial_\bar{p}^2 E(\bar{p})}, \quad Im(\bar{p}) < 0 \tag{S28}
\]

It is important to stress that, to investigate the low-temperature properties of 2D Helium 4, in Refs. [5, 7, 8] a phonon-like spectrum was used. Here in 1D we employ the full Bogoliubov expression

\[
E[p, \rho(r, t)] = \sqrt{p^2 + 2 g \rho(r, t)} \tag{S29}
\]

In 2D numerical instabilities arose by using the Bogoliubov spectrum in our solution of the linearized Andreev-Khalatnikov equations, hence we adopt the Hartree approximation

\[
E_{\text{Hartree}}[p, \rho(r, t)] = \frac{p^2}{2} + g \rho(r, t) \tag{S30}
\]

for large \( T \) (i.e. for \( T/T_c \geq 0.1 \)), and the phonon-like approximation

\[
E_{\text{phonon}}[p, \rho(r, t)] = \sqrt{g \rho(r, t) p} \tag{S31}
\]

when \( T \) is small (i.e. for \( T/T_c < 0.1 \)). The final results show that the two approaches merge seamlessly.