THE $q$-LIDSTONE SERIES INVOLVING $q$-BERNOULLI AND $q$-EULER POLYNOMIALS GENERATED BY THE THIRD JACKSON $q$-BESSEL FUNCTION

Z. MANSOUR AND M. AL-TOWAILB

Abstract. In this paper, we present $q$-Bernoulli and $q$-Euler polynomials generated by the third Jackson $q$-Bessel function to construct new types of $q$-Lidstone expansion theorem. We prove that the entire function may be expanded in terms of $q$-Lidstone polynomials which are $q$-Bernoulli polynomials and the coefficients are the even powers of the $q$-derivative $\frac{\delta_q f(z)}{\delta_q z}$ at $0$ and $1$. The other forms expand the function in $q$-Lidstone polynomials based on $q$-Euler polynomials and the coefficients contain the even and odd powers of the $q$-derivative $\frac{\delta_q f(z)}{\delta_q z}$.

1. Introduction

A Lidstone series provides a generalization of Taylor series that approximates a given function in a neighborhood of two points instead of one [11]. Recently, Ismail and Mansour [8] introduced a $q$-analog of the Lidstone expansion theorem. They proved that, under certain conditions, an entire function $f(z)$ can be expanded in the form

$$f(z) = \sum_{n=0}^{\infty} \left[ A_n(z) D_{q^{-1}}^{2n+1} f(1) - B_n(z) D_{q^{-1}}^{2n} f(0) \right],$$

where $A_n(z)$ and $B_n(z)$ are the $q$-Lidstone polynomials defined by

$$A_n(z) = \eta_{q^{-1}}^1 B_n(z) \text{ and } B_n(z) = \frac{q^{2n+1}}{[2n+1]_q!} B_{2n+1}(z/2; q).$$

Here $\eta_{q^{-1}}^y$ denotes the $q$-translation operator defined by

$$\eta_{q^{-1}}^y z^n = q^{\frac{n(n-1)}{2}} z^n (-y/z; q^{-1})_n = y^n (-z/y; q)_n,$$

and $B_n(z; q)$ is the $q$-analog of the Bernoulli polynomials which defined by the generating function

$$\frac{t E_q(zt)}{E_q(t/2) c_q(t/2)} - 1 = \sum_{n=0}^{\infty} B_n(z; q) \frac{t^n}{[n]_q!},$$

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where $E_q(z)$ and $e_q(z)$ are the $q$-exponential functions defined by

$$E_q(z) := \sum_{j=0}^{\infty} q^{j(j-1)/2} \frac{z^j}{[j]_q!}; \quad z \in \mathbb{C} \quad \text{and} \quad e_q(z) := \sum_{j=0}^{\infty} \frac{z^j}{[j]_q!}; \quad |z| < 1.$$

This paper aims to construct the $q$-Lidstone polynomials which are $q$-Bernoulli and $q$-Euler polynomials generated by the third Jackson $q$-Bessel function, and then to derive two formula of $q$-Lidstone expansion theorem. More precisely, we will prove that the entire function may be expanded in terms of $q$-Lidstone polynomials in two different forms. In the first form, the $q$-Lidstone polynomials are $q$-Bernoulli polynomials and the coefficients are the even powers of the $q$-derivative $\delta_q f(z)$ at 0 and 1. The other form expand the function in $q$-Lidstone polynomials based on $q$-Euler polynomials and the coefficients contain the even and odd powers of the $q$-derivative $\delta_q f(z)$. The publications [12, 13] are the most affiliated with this work.

This article is organized as follows: in Section 2, we state some definitions and present some background on $q$-analysis which we need in our investigations. In Section 3 and Section 4, we introduce $q$-Bernoulli and $q$-Euler polynomials generated by the third Jackson $q$-Bessel function. Section 5 contains a $q$-Lidstone expansion theorem involving $q$-Bernoulli polynomials while Section 6 contains a $q$-Lidstone series involving $q$-Euler polynomials.

2. Definitions and Preliminary results

Throughout this paper, unless otherwise is stated, $q$ is a positive number less than one and we follow the notations and terminology in [1, 6].

The symmetric $q$-difference operator $\delta_q$ is defined by

$$\delta_q f(z) = f(q^{1/2} z) - f(q^{-1/2} z),$$

(see [4, 6]) and then

$$\frac{\delta_q f(z)}{\delta_q z} := \frac{f(q^{1/2} z) - f(q^{-1/2} z)}{z(q^{1/2} - q^{-1/2})} \quad z \neq 0. \quad (2.1)$$

We use a third $q$-exponential function $exp_q(z)$ which has the following series representation

$$exp_q(z) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}}{[n]_q!} z^n; \quad z \in \mathbb{C}. \quad (2.2)$$

This function has the property $\lim_{q \to 1} exp_q(z) = e^z$ for $z \in \mathbb{C}$, and it is an entire function of $z$ of order zero (see [6]).

Remark 2.1. From the identity $[n]_{1/q}! = q^{n(1-n)/2} [n]_q!$, one can verify that

$$exp_q(z) = exp_{q^{-1}}(z); \quad z \in \mathbb{C}. \quad (2.3)$$
We consider the domain $\Omega := \{ z \in \mathbb{C} : |1 - \exp_q(z)| < 1 \}$.

**Lemma 2.2.** Let $z \in \Omega$. Then

\[
\frac{1}{\exp_q(z)} := 1 + \sum_{n=1}^{\infty} c_n z^n,
\]

where

\[
c_n = \sum_{k=1}^{n} (-1)^k \sum_{s_1 + s_2 + \ldots + s_k = n \atop s_j > 0 \ (j = 1, \ldots, k)} \frac{q^{\sum_{i=1}^{k} s_i (s_i - 1)/4}}{[s_1]_q [s_2]_q ! \ldots [s_k]_q !}.
\]

**Proof.** Observe that, for $z \in \Omega$ the function $\frac{1}{\exp_q(z)}$ can be represented as

\[
\frac{1}{\exp_q(z)} := \left[ 1 + (\exp_q(z) - 1) \right] = \sum_{k=0}^{\infty} (-1)^k \left[ \exp_q(z) - 1 \right]^k.
\]

Using the series expansion (2.2) of $\exp_q(z)$, we get

\[
\frac{1}{\exp_q(z)} = \sum_{k=0}^{\infty} (-1)^k \left( \sum_{n=1}^{\infty} q^{n(n-1)/4} \frac{z^n}{[n]_q !} \right)^k
\]

\[
= 1 + \sum_{k=1}^{\infty} (-1)^k \left( \sum_{n=1}^{\infty} q^{n(n-1)/4} \frac{z^n}{[n]_q !} \right)^k
\]

\[
= 1 + \sum_{k=1}^{\infty} (-1)^k \sum_{n=k}^{\infty} z^n \sum_{s_1 + s_2 + \ldots + s_k = n \atop s_j > 0 \ (j = 1, \ldots, k)} \frac{q^{\sum_{i=1}^{k} s_i (s_i - 1)/4}}{[s_1]_q [s_2]_q ! \ldots [s_k]_q !}.
\]

Put $a_n(k) = \sum_{s_1 + s_2 + \ldots + s_k = n \atop s_j > 0 \ (j = 1, \ldots, k)} \frac{q^{\sum_{i=1}^{k} s_i (s_i - 1)/4}}{[s_1]_q [s_2]_q ! \ldots [s_k]_q !}$. Then, the power series of $\frac{1}{\exp_q(z)}$ takes the form

\[
\frac{1}{\exp_q(z)} = 1 + \sum_{n=1}^{\infty} z^n \sum_{k=1}^{n} (-1)^k a_n(k),
\]

and then we obtain the desired result. \qed

The $q$-sine and $q$-cosine, $S_q(z)$ and $C_q(z)$, are defined by

\[
\exp_q(iz) := C_q(z) + iS_q(z),
\]

where

\[
C_q(z) := \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n-1)/2}}{[2n]_q !} z^{2n},
\]

(2.5)

\[
S_q(z) := \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)/2}}{[2n + 1]_q !} z^{2n+1}.
\]
These functions can be written in terms of the third Jackson $q$-Bessel function or (Hahn-Exton $q$-Bessel function [10]) as

$$C_q(z) := q^{-\frac{3}{8}} \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} (1 - q)^{\frac{1}{2}} J_{-\frac{3}{2}}^{(3)}(q^{-\frac{3}{4}}(1 - q)z; q^2),$$

$$S_q(z) := q^{\frac{1}{8}} \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} ((1 - q)^{\frac{1}{2}} J_{-\frac{1}{2}}^{(3)}(q^{-\frac{1}{4}}(1 - q)z; q^2),$$

and satisfy

$$\frac{\delta_q C_q(wz)}{\delta_q z} = -w S_q(wz), \quad \frac{\delta_q S_q(wz)}{\delta_q z} = w C_q(wz).$$

(see [4, 6]). Therefore,

$$\frac{\delta_q \exp_q(wz)}{\delta_q z} = w \exp_q(wz).$$

(2.7)

Note that since the third Jackson $q$-Bessel functions have only real roots and the roots are simple (see [10]), it follows that the roots of $C_q(z)$ and $S_q(z)$ are also real and simple as shown in Figure 1. Also, because $C_q(z)$ and $S_q(z)$ are respectively even and odd, the roots of these functions are symmetric. Throughout this paper we assume that $S_1$ and $C_1$ are the smallest positive zero of the functions $S_q(z)$ and $C_1$, respectively.

Here, the $q$-analog of the hyperbolic functions sinh $z$ and cosh $z$ are defined for $z \in \mathbb{C}$ by

$$\text{Sinh}_q(z) := -i S_q(iz) = \frac{\exp_q(z) - \exp_q(-z)}{2},$$

$$\text{Cosh}_q(z) := C_q(iz) = \frac{\exp_q(z) + \exp_q(-z)}{2}.$$

(2.8)
3. A \(q\)-Bernoulli polynomials generated by the third Jackson\n\(q\)-Bessel function

In this section, we use the third \(q\)-exponential function \(\exp_q(x)\) to define a \(q\)-analog of the Bernoulli polynomials which are suitable for our approach.

**Definition 3.1.** A \(q\)-Bernoulli polynomials \(\widetilde{B}_n(z; q)\) are defined by the generating function

\[
(3.1) \quad \frac{w \exp_q(z w) \exp_q(-\frac{w}{2})}{\exp_q(w/2) - \exp_q(-\frac{w}{2})} = \sum_{n=0}^{\infty} \widetilde{B}_n(z; q) \frac{w^n}{[n]_q!},
\]

and \(\widetilde{\beta}_n(q) := \widetilde{B}_n(0; q)\) are the \(q\)-Bernoulli numbers. Therefore,

\[
(3.2) \quad \frac{w \exp_q(-w/2)}{\exp_q(w/2) - \exp_q(-\frac{w}{2})} = \sum_{n=0}^{\infty} \frac{\widetilde{\beta}_n(q)}{[n]_q!} w^n.
\]

**Remark 3.2.** \(\widetilde{B}_{2n+1}(\frac{1}{2}; q) = 0\). Indeed, for \(z = \frac{1}{2}\), the left hand side of Equation (3.1) is an even function. Therefore, the odd powers of \(w\) on the left hand side vanish. Also, note that

\[
\widetilde{B}_0(z; q) = \frac{w \exp_q(z w) \exp_q(-\frac{w}{2})}{\exp_q(w/2) - \exp_q(-\frac{w}{2})} \big|_{w=0} = 1.
\]

**Proposition 3.3.** The \(q\)-Bernoulli polynomials \(\widetilde{B}_n(z; q)\) are given recursively by \(\widetilde{B}_0(z; q) = 1\), and for \(n \in \mathbb{N}\)

\[
\widetilde{B}_n(z; q) = \sum_{k=0}^{n} \binom{n}{k}_q q^{\frac{k(k-1)}{4}} \widetilde{\beta}_{n-k}(q) z^k.
\]

**Proof.** By substituting (3.2) into (3.1) and using the series representation of \(\exp_q(wz)\) we obtain

\[
\frac{w \exp_q(z w) \exp_q(-\frac{w}{2})}{\exp_q(w/2) - \exp_q(-\frac{w}{2})} = \sum_{n=0}^{\infty} \frac{\widetilde{\beta}_n(q)}{[n]_q!} \sum_{n=0}^{\infty} \frac{q^{n(n-1)/4}}{[n]_q!} (wz)^n
\]

\[
= \sum_{n=0}^{\infty} \frac{w^n}{[n]_q!} \sum_{k=0}^{n} \binom{n}{k}_q q^{\frac{k(k-1)}{4}} \tilde{\beta}_{n-k}(q) z^k.
\]

This implies

\[
(3.3) \quad \sum_{n=0}^{\infty} \frac{w^n}{[n]_q!} \sum_{k=0}^{n} \binom{n}{k}_q q^{\frac{k(k-1)}{4}} \tilde{\beta}_{n-k}(q) z^k = \sum_{n=0}^{\infty} \tilde{B}_n(z; q) \frac{w^n}{[n]_q!}.
\]

Comparing the coefficient of \(\frac{w^n}{[n]_q!}\), we obtain the required result. \(\square\)

**Proposition 3.4.** For \(n \in \mathbb{N}\) and \(z \in \mathbb{C}\), we have

\[
(3.4) \quad \tilde{B}_n(z; q) = q^{\frac{n(n-1)}{2}} \tilde{B}_n(z; 1/q),
\]

\[
(3.5) \quad \tilde{\beta}_n(q) = q^{\frac{n(n-1)}{2}} \tilde{\beta}_n(1/q).
\]
Proof. By replacing $q$ by $1/q$ on the generating function in (3.1), and then using Equation (2.3) we obtain
\[ \sum_{n=0}^{\infty} q \frac{n(n-1)}{2} \widetilde{B}_n(z; 1/q) \frac{w^n}{[n]_q!} = \sum_{n=0}^{\infty} \widetilde{B}_n(z; q) \frac{w^n}{[n]_q!}. \]
Equating the coefficients of $w^n$ yields (3.4) and substituting with $z = 0$ in (3.4) yields directly (3.5).

**Theorem 3.5.** The $q$-Bernoulli polynomials satisfy the $q$-difference equation
\[ \delta_q \widetilde{B}_n(z; q) \delta_q z = [n]_q \widetilde{B}_{n-1}(z; q) \quad (n \in \mathbb{N}). \]

**Proof.** Calculating the $q$-derivative $\delta_q$ of the two sides of (3.1) with respect to the variable $z$ and using Equation (2.7), we obtain
\[ \frac{w^2 \exp_q(zw) \exp_q(-\frac{w}{2})}{\exp_q(\frac{w}{2}) - \exp_q(-\frac{w}{2})} = \sum_{n=1}^{\infty} \delta_q \widetilde{B}_n(z; q) \frac{w^n}{[n]_q!}. \]
This implies
\[ \sum_{n=1}^{\infty} \delta_q \widetilde{B}_n(z; q) \frac{w^n}{[n]_q!} = \sum_{n=1}^{\infty} \widetilde{B}_{n-1}(z; q) \frac{w^n}{[n-1]_q!}. \]
Equating the corresponding $n$th power of $w$ in the two series of (3.7), we obtain the required result.

**Corollary 3.6.** For $k \geq 2$, we have
\[ \frac{\delta^2_q \widetilde{B}_k(z; q)}{\delta_q z^2} = [k]_q [k-1]_q \widetilde{B}_{k-2}(z; q). \]

**Proof.** It follows directly by calculating the derivative $\delta_q$ of (3.6) for even and odd index of $\widetilde{B}_k(z; q)$. \hfill \Box

**Proposition 3.7.** The $q$-Bernoulli numbers of odd index satisfy
\[ \widetilde{\beta}_1(q) = -\frac{1}{2}, \quad \widetilde{\beta}_{2n+1}(q) = 0; \quad n \in \mathbb{N}. \]

**Proof.** Observe that,
\[ \frac{w \exp_q(-\frac{w}{2})}{\exp_q(\frac{w}{2}) - \exp_q(-\frac{w}{2})} = -w + \frac{w \exp_q(-\frac{w}{2})}{\exp_q(\frac{w}{2}) - \exp_q(-\frac{w}{2})}. \]
So, we can write Equation (3.2) in the form
\[ \sum_{n=0}^{\infty} \frac{\widetilde{\beta}_n(q)}{[n]_q!} w^n = -w + \sum_{n=0}^{\infty} \frac{\widetilde{\beta}_n(q)}{[n]_q!} (-w)^n. \]
This implies
\[ \sum_{n=0}^{\infty} \left( 1 - (-1)^n \right) \tilde{\beta}_n(q) \frac{w^n}{[n]_q!} = -w. \]

Therefore, \( \tilde{\beta}_1(q) = -\frac{1}{2} \) and \( \tilde{\beta}_{2n+1}(q) = 0 \) for every \( n \in \mathbb{N} \). \( \square \)

**Theorem 3.8.** For \( z \in \mathbb{C} \) and \( n \in \mathbb{N} \), we have the identity
\[ \frac{q^{n(n-1)}}{[n]_q!} \left( \frac{1}{2} \right)^n (2q^{\frac{1-n}{2}} z; q)_n = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{1}{2k} \frac{q^{k(2k+1)}}{[2k+1]_q!} \tilde{B}_{n-2k}(z; q). \]

**Proof.** By using (3.1), we have
(3.9) \( w \exp_q(zw) \exp_q \left( \frac{-w}{2} \right) = \left[ \exp_q \left( \frac{w}{2} \right) - \exp_q \left( \frac{-w}{2} \right) \right] \sum_{n=0}^{\infty} \tilde{B}_n(z; q) \frac{w^n}{[n]_q!}. \)

Using the series representation of \( \exp_q(zw) \), we can prove that
(3.10) \( \exp_q(zw) \exp_q \left( \frac{-w}{2} \right) = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{w^n}{[n]_q!} \left( -\frac{1}{2} \right)^n (2q^{\frac{1-n}{2}} z; q)_n. \)

Substituting (3.10) into (3.9) and using (2.2), we obtain
\[ \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{w^n}{[n]_q!} (\frac{1}{2})^n (2q^{\frac{1-n}{2}} z; q)_n \]
\[ = \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} \frac{w^n}{[2n+1]_q!} (\frac{1}{2})^{2n} \sum_{n=0}^{\infty} \tilde{B}_n(z; q) \frac{w^n}{[n]_q!} \]
\[ = \sum_{n=0}^{\infty} w^n \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{1}{2k} \frac{q^{k(2k+1)}}{[2k+1]_q!} \tilde{B}_{n-2k}(z; q). \]

Comparing the coefficient of \( w^n \) we obtain the required result. \( \square \)

Note that if we substitute with \( z = 0 \) in the identity of Theorem 3.8 we get the following recurrence relation:

**Corollary 3.9.** For \( n \in \mathbb{N} \), we have
\[ \frac{q^{\frac{n(n-1)}{2}}}{[2n]_q!} (\frac{1}{2})^{2n} = \sum_{k=0}^{n} \frac{1}{2k} \frac{q^{k(2k+1)}}{[2k+1]_q!} \tilde{B}_{2n-2k}(q). \]
As a consequence of the above result, we have

\[ \tilde{\beta}_0(q) = 1, \quad \tilde{\beta}_1(q) = -\frac{1}{2}, \quad \tilde{\beta}_2(q) = \frac{(1 - q^3)q^{\frac{1}{2}} - (1 - q)q^{\frac{3}{2}}}{4(1 - q^3)}, \]

\[ \tilde{\beta}_3(q) = 0, \quad \tilde{\beta}_4 = \frac{q^3(q^3; q^2)_2 - [3]_q(q^5(1 - q)(1 - q^3)) - (1 + q)(1 - q^3)(1 - q^5)}{16(1 - q^3)^2(1 - q^5)}. \]

In the following result we prove that the function \( \text{Coth}_q(z) \) has a \( q \)-analog of Taylor series expression with only odd exponents for \( z \).

**Proposition 3.10.** Let \( w \) be a complex number such that \( 0 < \frac{|w|}{2} < C_1 \). Then

\[ \text{Coth}_q\left(\frac{w}{2}\right) = \left(\frac{w}{2}\right)^{-1} + \sum_{n=1}^{\infty} 2\tilde{\beta}_{2n}(q) \frac{w^{2n-1}}{[2n]_q!}. \]

**Proof.** By using Equation (3.2) and the identity

\[ \text{Coth}_q\left(\frac{w}{2}\right) = \frac{\exp_q\left(\frac{w}{2}\right) + \exp_q\left(-\frac{w}{2}\right)}{\exp_q\left(\frac{w}{2}\right) - \exp_q\left(-\frac{w}{2}\right)}, \]

we obtain

\[ \sum_{n=0}^{\infty} \tilde{\beta}_n(q) \frac{w^n}{[n]_q!} = w \text{Coth}_q\left(\frac{w}{2}\right) - \frac{w \exp_q\left(\frac{w}{2}\right)}{\exp_q\left(\frac{w}{2}\right) - \exp_q\left(-\frac{w}{2}\right)} = w \text{Coth}_q\left(\frac{w}{2}\right) - \sum_{n=0}^{\infty} \tilde{\beta}_n(q) \frac{(-w)^n}{[n]_q!}. \]

Therefore, \( w \text{Coth}_q\left(\frac{w}{2}\right) = 1 + \sum_{n=1}^{\infty} 2\tilde{\beta}_{2n}(q) \frac{w^{2n-1}}{[2n]_q!} \) and then the result follows.

\[ \square \]

We define the polynomials \( \tilde{A}_n(z; q) \) by the generating function

\[ \frac{\exp_q(zw)}{\exp_q\left(\frac{w}{2}\right) - \exp_q\left(-\frac{w}{2}\right)} = \sum_{n=0}^{\infty} \tilde{A}_n(z; q) \frac{w^n}{[n]_q!}. \]

**Proposition 3.11.** For \( n \in \mathbb{N} \), the \( q \)-Bernoulli polynomials \( \tilde{B}_n(z; q) \) can be represented in terms of \( \tilde{A}_n(z; q) \) as

\[ \tilde{B}_n(z; q) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right]_q \frac{(-1)^k q^{k(k+1)/2}}{n-k} \tilde{A}_{n-k}(z; q). \]
Proof. From (3.11), (2.2) and Definition 3.1 we get
\[
\sum_{n=0}^{\infty} \tilde{A}_n(z;q) \frac{w^n}{[n]_q!} = \frac{w \exp_q(zw) \exp_q(-w/2)}{\exp_q(w/2) - \exp_q(-w/2)}
\]
\[
= \sum_{n=0}^{\infty} \tilde{A}_n(z;q) \frac{w^n}{[n]_q!} \sum_{n=0}^{\infty} q^{\frac{n(n-1)}{2}} (-w)^n
\]
\[
= \sum_{n=0}^{\infty} \frac{w^n}{[n]_q!} \sum_{k=0}^{n} \left[ \frac{n}{k} \right] (-\frac{1}{2})^k q^{\frac{k(k-1)}{4}} \tilde{A}_{n-k}(z;q).
\]
Comparing the coefficient of \(\frac{w^n}{[n]_q!}\) we obtain the required result. \(\square\)

**Theorem 3.12.** Let \(z \in \mathbb{C}\). Then, the polynomials \(\tilde{A}_n(z;q)\) can be represented in terms of the \(q\)-Bernoulli polynomials \(\tilde{B}_n(z;q)\) as

\[
\tilde{A}_n(z;q) = [n]_q! \sum_{j=0}^{n-1} (-\frac{1}{2})^{j+1} \frac{\tilde{a}_j}{[n-j-1]_q!} \tilde{B}_{n-j-1}(z;q),
\]

where

\[
\tilde{a}_j = \sum_{k=0}^{j} (-1)^k \sum_{\sum_{i=1}^{k} s_i = n, \ s_i > 0} q^{\sum_{i=0}^{k} s_i(s_i+1)/4} [s_1+1]_q! [s_2+1]_q! \cdots [s_k+1]_q!.
\]

Proof. We can write the generating function of the \(q\)-polynomials \(\tilde{A}_n(z;q)\) as

\[
\frac{w \exp_q(zw)}{\exp_q(w/2) - \exp_q(-w/2)} = \frac{1}{\exp_q(w/2) - \exp_q(-w/2)} \left[ w \exp_q(zw) \exp_q(-w/2) \right].
\]
Putting \(a_n(k) = \sum_{\sum_{i=1}^{k} s_i = n, \ s_i > 0} q^{\sum_{i=0}^{k} s_i(s_i+1)/4} [s_1+1]_q! [s_2+1]_q! \cdots [s_k+1]_q!\), and then using Lemma 2.2 we obtain

\[
\sum_{n=0}^{\infty} \tilde{A}_n(z;q) \frac{w^n}{[n]_q!} = w \left( \sum_{n=0}^{\infty} (-\frac{1}{2})^{n+1} [n]_q! \frac{w^n}{[n]_q!} \sum_{k=0}^{n} (-1)^k a_n(k) \left( \sum_{n=0}^{\infty} \tilde{B}_n(z;q) \frac{w^n}{[n]_q!} \right) \right)
\]
\[
= w \sum_{n=0}^{\infty} \frac{w^n}{[n]_q!} \sum_{j=0}^{n} \left[ \frac{n}{j} \right] q^j (-\frac{1}{2})^{j+1} \sum_{k=0}^{j} a_j(k)(-1)^k \tilde{B}_{n-j}(z;q)
\]
\[
= \sum_{n=0}^{\infty} \frac{w^{n+1}[n+1]_q}{[n+1]_q!} \sum_{j=0}^{n} \left[ \frac{n}{j} \right] q^j (-\frac{1}{2})^{j+1} \sum_{k=0}^{j} a_j(k)(-1)^k \tilde{B}_{n-j}(z;q).
\]
This implies

\[
\tilde{A}_{n+1}(z; q) = [n+1]q \sum_{j=0}^{n} \left[ \begin{array}{c} n \\ j \end{array} \right] [j]_q! \left( -\frac{1}{2} \right)^{j+1} \sum_{k=0}^{j} a_j(k)(-1)^k \tilde{B}_{n-j}(z; q),
\]

and then we obtain the required result. \(\square\)

**Corollary 3.13.** For \(n \in \mathbb{N}_0\) and \(z \in \mathbb{C}\), the power series of the polynomial \(\tilde{A}_n(z; q)\) takes the form

\[
\tilde{A}_n(z; q) = \frac{z^n}{n!} \sum_{m=0}^{n-1} \tilde{c}_m(n) \frac{z^m}{[m]_q!},
\]

where

\[
(3.15) \quad \tilde{c}_m(n) = [n]_q! \left( -\frac{1}{2} \right)^{n+1} \sum_{r=n-1}^{m} \frac{(-2)^r \tilde{a}_r \beta_r(q)}{[r]_q!} \beta_{r-m}(q).
\]

**Proof.** From Theorem 3.12 and Proposition 3.3 we get

\[
\tilde{A}_n(z; q) = [n]_q! \left( -\frac{1}{2} \right)^n \sum_{r=0}^{n-1} \frac{(-2)^r \tilde{a}_r \beta_r(q)}{[r]_q!} \tilde{B}_r(z; q)
\]

\[
= [n]_q! \left( -\frac{1}{2} \right)^n \sum_{r=0}^{n-1} \frac{(-2)^r \tilde{a}_r \beta_r(q)}{[r]_q!} \sum_{m=0}^{r} \left[ \begin{array}{c} r \\ m \end{array} \right] q^{-\frac{m(m-1)}{2}} \frac{\beta_{r-m}(q)}{[r-m]_q!} z^m
\]

\[
= [n]_q! \left( -\frac{1}{2} \right)^n \sum_{m=0}^{n-1} \left( \sum_{r=n-1}^{m} q^{-\frac{m(m-1)}{2}} \frac{(-2)^r \tilde{a}_r \beta_r(q)}{[r]_q!} \beta_{r-m}(q) \right) \frac{z^m}{[m]_q!}.
\]

**Corollary 3.14.** Let \(w\) be a complex number such that \(|w| < S_1\). Then

\[
\frac{1}{\text{Sinh}_q(w)} = \sum_{n=0}^{\infty} d_n (2w)^n,
\]

where \(d_0 = 1\), \(d_n = [n+1]q \sum_{j=0}^{n} \left( -\frac{1}{2} \right)^{j+1} \frac{\tilde{a}_j}{[n-j]_q!} \beta_{n-j}(q)\) and \(\tilde{a}_j\) the constants defined in (3.14).

**Proof.** The proof follows immediately from (3.11), (3.13) and replacing \(w\) by \(2w\). \(\square\)

**Remark 3.15.** According to the definition of \(q\)-Bernoulli numbers (3.2), we have \(\beta_n(q) = \tilde{A}_n(-\frac{1}{2}; q)\). That is, for \(n \in \mathbb{N}_0\) we have

\[
\tilde{\beta}_{n+1}(q) = [n+1]q! \sum_{j=0}^{n} \left( -\frac{1}{2} \right)^{j+1} \frac{\tilde{a}_j}{[n-j]_q!} \tilde{B}_{n-j}(-\frac{1}{2}; q),
\]

where \(\tilde{a}_j\) is the constants which defined in (3.14).
4. A $q$-Euler polynomials generated by the third Jackson $q$-Bessel function

**Definition 4.1.** A $q$-Euler polynomials $\tilde{E}_n(z; q)$ are defined by the generating function

\[
2 \frac{\exp_q(zw) \exp_q(-w)}{\exp_q(w/2) + \exp_q(-w/2)} = \sum_{n=0}^{\infty} \tilde{E}_n(z; q) \frac{w^n}{[n]_q!},
\]

and the $q$-Euler numbers $\tilde{e}_n(q)$ are defined in terms of generating function

\[
\frac{2}{\exp_q(w) + \exp_q(-w)} = \sum_{n=0}^{\infty} \tilde{e}_n(q) \frac{w^n}{[n]_q!}.
\]

Clearly, $\tilde{e}_{2n+1}(q) = 0$ for all $n \in \mathbb{N}_0$. Consequently,

\[
\frac{1}{C_q(z)} = \sum_{n=0}^{\infty} (-1)^n \frac{\tilde{e}_{2n}(q)}{[2n]_q!} z^{2n}; \quad |z| < C_1,
\]

where $C_1$ is the first positive zeros of $C_q(z)$.

We use the notation $\tilde{E}_n$ to denotes the first Euler number, i.e.,

$$\tilde{E}_n := \tilde{E}_n(0; q), \quad (n \in \mathbb{N}_0).$$

**Proposition 4.2.** The $q$-Euler polynomials $\tilde{E}_n(z; q)$ are given by

$$\tilde{E}_0(z; q) = 1,$$

and for $n \in \mathbb{N}$

$$\tilde{E}_n(z; q) = \sum_{k=0}^{n} \left[ n \atop k \right]_q q^{\frac{k(k-1)}{2}} \tilde{e}_{n-k} z^k.$$

*Proof.* The proof is similar to the proof of Proposition 3.3 and is omitted. □

**Proposition 4.3.** For $n \in \mathbb{N}_0$, we have

\[
\tilde{E}_n\left(\frac{1}{2}; q\right) = \left(\frac{1}{2}\right)^n \sum_{n=0}^{\infty} \left[ n \atop n \right]_q (-1)^n q^{\frac{k(k-1)}{4}} \left( q^{\frac{1-k}{2}} ; q \right)_k \tilde{e}_{n-k}(q).
\]

*Proof.* Since

\[
\exp_q(w^2) \exp_q(-w/2) = \sum_{n=0}^{\infty} \left( -\frac{1}{2} \right)^n q^{\frac{n(n-1)}{4}} \left( q^{\frac{1-k}{2}} ; q \right)_n \frac{w^n}{[n]_q!},
\]

then, by using (1.11) and (1.2) we get

\[
\sum_{k=0}^{\infty} \tilde{E}_n\left(\frac{1}{2}; q\right) \frac{w^n}{[n]_q!} = \sum_{k=0}^{\infty} \frac{w^n}{[n]_q!} \sum_{k=0}^{n} \left( -\frac{1}{2} \right)^n (-1)^k \left[ n \atop k \right]_q q^{\frac{k(k-1)}{4}} \left( q^{\frac{1-k}{2}} ; q \right)_k \tilde{e}_{n-k}(q),
\]

which implies the result. □
Note that if $z = \frac{1}{2}$, then the left hand side of (4.1) is an even function. Hence,

\[ E_{2n+1}(\frac{1}{2}; q) = 0. \]

**Proposition 4.4.** For $n \in \mathbb{N}_0$, we have $E_{2n} = \delta_{n,0}$, where $\delta_{n,0}$ is the Kronecker’s delta.

**Proof.** Observe that

\[ 2 \exp_q(-\frac{w}{2}) + \exp_q(\frac{w}{2}) - 1 = \exp_q(-\frac{w}{2}) - \exp_q(\frac{w}{2}). \]

So, we obtain

\[ \sum_{n=0}^{\infty} \frac{E_n}{|n|q} w^n = 1 + \frac{\exp_q(-\frac{w}{2}) - \exp_q(\frac{w}{2})}{\exp_q(\frac{w}{2}) + \exp_q(-\frac{w}{2})}. \]

The right hand side of (4.7) is an odd function, therefore the even powers of $w$ on the left hand side of this equation vanish. Hence $E_0 = 1$ and $E_{2n} = 0$ for every $n \in \mathbb{N}$. \qed

The following results can be proved by the same way of Proposition 3.4, Theorem 3.5 and Theorem 3.8.

**Proposition 4.5.** For $n \in \mathbb{N}$ and $z \in \mathbb{C}$, we have

1. $E_n(z; q) = q^{\frac{n(n-1)}{2}} E_n(z; 1/q)$;
2. $\delta_q E_n(z; q) = [n]_q E_{n-1}(z; q)$.

**Theorem 4.6.** For $z \in \mathbb{C}$, we have the identities

\[ q^{\frac{n(n-1)}{2}} |n|_q \left( -\frac{1}{2} \right)^n (2q^{\frac{1-n}{2}} z)_q = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \left( \frac{1}{2} \right)^{k(2k-1)} q^{2k} \frac{E_{n-2k}(z; q)}{|n-2k|_q!}; \]

\[ q^{\frac{n(n-1)}{2}} |n|_q \left( -\frac{1}{2} \right)^n = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \left( \frac{1}{2} \right)^{k(2k-1)} q^{2k} \frac{E_{n-2k}}{|n-2k|_q!}. \]

As a consequence of Theorem 4.6 we get

\[ E_0 = 1, \quad E_1 = -\frac{1}{2}, \quad E_2 = 0, \quad E_3 = \frac{(q-1)q^{\frac{3}{2}} + (1-q^3)q^{\frac{1}{2}}}{8(1-q)}, \]

\[ E_5 = \frac{(q^2-1)q^5 + (1-q^2)(5)q^3 + (q-1)q^\frac{3}{2} [4]_q [5]_q ([3]_q q^{\frac{1}{2}} - \frac{3}{2})}{32(1-q^2)}. \]

**Proposition 4.7.** We have the identity

\[ \tanh_q(\frac{w}{2}) = \sum_{n=0}^{\infty} E_{2n+1}(\frac{w}{2}; q) = \frac{w^{2n+1}}{|2n+1|_q!}, \quad |\frac{w}{2}| < S_1. \]
Proof. The proof is similar to the proof of Proposition 3.10 and is omitted. □

Recall that the $q$-tangent and $q$-secant numbers defined by the series expansions of $\tan_qz$ and $\sec_qz$ by

\[
\tan_qz = \sum_{n=0}^{\infty} T_{2n+1}(q) \frac{z^{2n+1}}{[2n+1]_q!},
\]

\[
(4.9)
\]

\[
\sec_qz = \frac{1}{C_qu} = \sum_{n=0}^{\infty} S_{2n}(q) \frac{z^{2n}}{[2n]_q!},
\]

(for more details see [5, 7]).

Consider $S_q(z)$ and $C_q(z)$ which defined in (2.5). Then, from (4.3) and (4.8) we get

\[
T_{2n+1}(q) = (-1)^n \bar{E}_{2n+1} 2^{2n+1}, \quad S_{2n}(q) = (-1)^n \bar{e}_{2n}(q).
\]

**Theorem 4.8.** For $n \in \mathbb{N}_0$

\[
\sum_{n=0}^{\infty} \left(-1\right)^n 2^{2k} \beta_{2k}(q) \frac{T_{2n-2k+1}(q)}{[2k]_q! \ [2n-2k+1]_q!} = \delta_{n,0},
\]

where $\delta_{n,0}$ is the Kronecker’s delta.

**Proof.** From Equation (4.10), we have

\[
(4.11)
\]

\[
z \cot_q(z) = \sum_{n=0}^{\infty} \left(-1\right)^n 2^{2n} \beta_{2n}(q) \frac{z^{2n}}{[2n]_q!}.
\]

Observe that $z \tan_q(z) \cot_q(z) = z$. So, by using (4.9) and (4.11) we obtain

\[
z = \sum_{n=0}^{\infty} \left(-1\right)^n 2^{2n} \beta_{2n}(q) \frac{z^{2n}}{[2n]_q!} \sum_{n=0}^{\infty} T_{2n+1}(q) \frac{z^{2n+1}}{[2n+1]_q!}.
\]

Therefore,

\[
\sum_{n=0}^{\infty} z^{2n} \sum_{k=0}^{n} \left(-1\right)^k 2^{2k} \beta_{2k}(q) \frac{T_{2n-2k+1}(q)}{[2k]_q! \ [2n-2k+1]_q!} = 1.
\]

Comparing the coefficient of $z^{2n}$, we obtain the desired result. □

**Corollary 4.9.** Let $n \in \mathbb{N}_0$. Then, the $q$-tangent numbers $T_{2n+1}(q)$ are positive numbers.

**Proof.** From Equation (4.10), we get

\[
T_{2n+1}(q) = \sum_{k=1}^{n} \left(-1\right)^{k-1} (2k) \beta_{2k}(q) \frac{T_{2n-2k+1}(q)}{[2k]_q! \ [2n-2k+1]_q!}.
\]

Since $(-1)^{k-1} \beta_{2k} > 0$ for $k \in \mathbb{N}$ and $T_1(q) = 1 > 0$, then we can prove the result by induction on $n$ for all $n \in \mathbb{N}_0$. □
We define a sequence of polynomials $\tilde{M}_n(z; q)$ by the generating function

\[(4.12) \quad \frac{\exp_q(zw)}{\exp_q(w/2) + \exp_q(-w/2)} = \sum_{n=0}^{\infty} \tilde{M}_n(z; q) [n]_q^w.\]

Similarly to Proposition 3.11, Theorem 3.12 and Corollary 3.14, we have the following results.

**Proposition 4.10.** For $n \in \mathbb{N}$, the $q$-Euler polynomials $\tilde{E}_n(z; q)$ can be represented in terms of $\tilde{M}_n(z; q)$ as

\[(4.13) \quad \tilde{E}_n(z; q) = \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] q \left( \frac{1}{2} \right)^{k-1} \tilde{M}_{n-k}(z; q).\]

**Theorem 4.11.** For $n \in \mathbb{N}$ and $z \in \mathbb{C}$, $\tilde{M}_n(z; q)$ can be represented in terms of the $q$-Euler polynomials $\tilde{E}_n(z; q)$ as

\[(4.14) \quad \tilde{M}_n(z; q) = \frac{[n]_q!}{2} \sum_{j=0}^{n} (-1)^{j+1} \frac{\tilde{a}_j}{[n-j]_q!} \tilde{E}_{n-j}(z; q),\]

where

\[\tilde{a}_j = \sum_{k=0}^{j} (-1)^k \sum_{\substack{s_1 + s_2 + \ldots + s_k = n \\ s_i > 0, \forall i}} \frac{q^{\sum_{i=0}^{k} s_i (s_i + 1)/4}}{[s_1 + 1]_q! [s_2 + 1]_q! \ldots [s_k + 1]_q!}\]

**Corollary 4.12.** For $n \in \mathbb{N}_0$ and $z \in \mathbb{C}$, the power series of the polynomial $\tilde{M}_n(z; q)$ takes the form

\[\tilde{M}_n(z; q) = \sum_{m=0}^{n} c_m(n) \frac{z^m}{[m]_q!},\]

where

\[c_m(n) = \frac{[n]_q!}{2} \left( \frac{1}{2} \right)^{n+1} \sum_{r=n}^{m} q^{r(m-1)/4} \left( -2 \right)^r \tilde{a}_r \tilde{E}_{r-m}.\]

**Proposition 4.13.** For $z \in \mathbb{C}$, we have

\[\frac{1}{\cosh_q(w/2)} = \sum_{n=0}^{\infty} \tilde{d}_n w^n, \quad |\frac{w}{2}| < C_1,\]

where $\tilde{d}_n = \sum_{j=0}^{n} (-1)^{j+1} \frac{\tilde{a}_j}{[n-j]_q!} \tilde{E}_{n-j}$ and $\tilde{a}_j$ the constants defined in (3.14).
5. A $q$-Lidstone series involving $q$-Bernoulli polynomials

Our aim of this section is to prove that an entire function $f$ may be expanded in terms of $q$-Lidstone polynomials, where the coefficients of these polynomials are the even powers of the $q$-derivative $\delta_q f(z)$ at 0 and 1.

We begin by recalling some definitions and results from [16] which will be used in the proof of the main result.

**Definition 5.1.** Let $k$ be a non zero real number, and let $p$ be a real number with $|p| > 1$. An entire function $f$ has a $p$-exponential growth of order $k$ and a finite type, if there exist real numbers $K > 0$ and $\alpha$, such that

$$|f(z)| < Kp^{\frac{k}{2}} \left(\frac{\log|z|}{\log p}\right)^2 |z|^{\alpha},$$

or equivalently,

$$|f(z)| \leq K e^{\frac{k}{2} \log p (\log |z|)^2 + \alpha \log |z|}.$$

**Definition 5.2.** Let $k$ be a non zero real number and let $p$ be a real number, with $|p| > 1$. A formal power series expansion $\hat{f} := \sum_{n=0}^{\infty} a_n z^n$ is $p$-Gevery of order $-k$ (or of level $k$), if there exists real numbers $C, A > 0$ such that

$$|a_n| < C p^{\frac{n(n+1)}{2k}} A^n.$$

**Proposition 5.3.** Let $k$ be a non zero real number and $p$ be a real number, with $p > 1$. The following statements are equivalent.

i. The series $\hat{f} := \sum_{n=0}^{\infty} a_n z^n$ is $p$-Gevery of order $-k$;

ii. The series $\hat{f}$ is the power series expansion at the origin of an entire function $f$ having a $p$-exponential growth of order $k$ and a finite type $\alpha$, where

$$|a_n| < Ke^{-\frac{(n-\alpha)^2}{2k}}, \quad K > 0.$$

**Remark 5.4.** The series $\sum_{n=0}^{\infty} \frac{n(n-1)}{[n]_q!} z^n$ which defines the function $exp_q(z)$ is $q^{-1}$-Gevery of order $-2$. Consequently, $exp_q(z)$ has $q^{-1}$ exponential growth of order 2.

**Proposition 5.5.** Let $z$ and $w$ be complex numbers such that $|w| < S_1$. Then

$$\text{Sinh}_q(wz) \text{Csch}_q(w) = \sum_{n=0}^{\infty} \frac{2^{2n+1}}{[2n+1]_q!} \bar{A}_{2n+1}(z/2; q) w^{2n},$$

where $\bar{A}_n(z; q)$ are the $q$-polynomials defined in (5.11).
Proof. First, note that the function \( g_q(z, w) := \text{Sinh}_q(wz) \text{Csch}_q(w) \) is holomorphic for \( |w| < S_1 \). By using (2.8), we can write

\[
g_q(z, w) := \frac{\exp_q(zw) - \exp_q(-zw)}{\exp_q(w) - \exp_q(-w)}.
\]

Then, by using (3.11) we get

\[
g_q(z, w) := \frac{\exp_q(zw) - \exp_q(-zw)}{\exp_q(w) - \exp_q(-w)} = \frac{1}{2w} \sum_{n=0}^{\infty} \tilde{A}_n(z/2; q) \left( \frac{2w^n}{[n]_q} \right) - \frac{1}{2w} \sum_{n=0}^{\infty} \tilde{A}_n(z/2; q) \left( \frac{-2w^n}{[n]_q} \right).
\]

Henceforth, we will consider the notation

\[
(5.2) \quad \tilde{A}_n(z) = \frac{2^{2n+1}}{[2n+1]_q!} \tilde{A}_{2n+1}(z/2; q).
\]

So, the previous result can be restated in the following form:

\[
(5.3) \quad \frac{\exp_q(zw) - \exp_q(-zw)}{\exp_q(w) - \exp_q(-w)} = \sum_{n=0}^{\infty} \tilde{A}_n(z) w^{2n},
\]

**Corollary 5.6.** For \( n \in \mathbb{N} \), the \( q \)-polynomials \( \tilde{A}_n(z) \) satisfy the \( q \)-difference equation

\[
\frac{\delta_q^2 \tilde{A}_n(z)}{\delta_q z^2} = \tilde{A}_{n-1}(z),
\]

with the boundary conditions \( \tilde{A}_n(0) = \tilde{A}_n(1) = 0 \), and \( \tilde{A}_0(z) = z \).

Proof. By using (2.7) we obtain

\[
\frac{\delta_q^2 g(z, w)}{\delta_q z^2} = \sum_{n=0}^{\infty} \frac{\delta_q^2 \tilde{A}_n(z)}{\delta_q z^2} w^{2n} = w^2 \frac{\exp_q(zw) - \exp_q(-zw)}{\exp_q(w) - \exp_q(-w)} = \sum_{n} \tilde{A}_n(z) w^{2n+2}.
\]

Therefore, \( \frac{\delta_q^2 \tilde{A}_n(z)}{\delta_q z^2} = \tilde{A}_{n-1}(z) \) (\( n \in \mathbb{N} \)). Furthermore,

\[
\tilde{A}_0(z) = \lim_{w \to 0} \frac{\exp_q(zw) - \exp_q(-zw)}{\exp_q(w) - \exp_q(-w)} = z.
\]
Substitute with $z = 0$ and $z = 1$ in Equation (5.3), we obtain

$$\tilde{A}_n(0) = \tilde{A}_n(1) = 0$$

for all $n \in \mathbb{N}$. □

**Proposition 5.7.** Let $z$ and $w$ be complex numbers such that $|w| < S_1$. Then

$$\frac{\exp_q(zw)\exp_q(-w) - \exp_q(-zw)\exp_q(w)}{\exp_q(w) - \exp_q(-w)} = \sum_{n=0}^{\infty} \tilde{B}_n(z) w^{2n},$$

where

$$\tilde{B}_n(z) = \frac{2^{2n+1}}{[2n+1]!} \tilde{B}_{2n+1}(z/2; q).$$

**Proof.** If $z$ and $w$ are complex numbers such that $|w| < S_1$, then

$$\frac{\exp_q(zw)\exp_q(-w) - \exp_q(-zw)\exp_q(w)}{\exp_q(w) - \exp_q(-w)} = 1$$

$$= \frac{1}{2w} \left[ \frac{2w \exp_q(zw)\exp_q(-w)}{\exp_q(w) - \exp_q(-w)} \right] - \frac{1}{2w} \left[ \frac{2w \exp_q(-zw)\exp_q(w)}{\exp_q(w) - \exp_q(-w)} \right]$$

$$= \frac{1}{2w} \sum_{n=0}^{\infty} \frac{(2w)^n - (-2w)^n}{[n]!} \tilde{B}_n(z/2; q)$$

$$= \sum_{n=0}^{\infty} \frac{w^{2n}}{[2n+1]!} 2^{2n+1} \tilde{B}_{2n+1}(z/2; q).$$

□

As in Corollary 5.6 one can verify that $\tilde{B}_0(z) = z - 1$ and for $n \in \mathbb{N}$, the $q$-polynomials $\tilde{B}_n(z)$ satisfy the $q$-difference equation

$$\Delta_q^2 \tilde{B}_n(z) = \tilde{B}_{n-1}(z),$$

with the boundary conditions $\tilde{B}_n(0) = \tilde{B}_n(1) = 0$.

Now, observe that

$$\exp_q(zw) = \frac{\exp_q(zw)\exp_q(-w) - \exp_q(-zw)\exp_q(w)}{\exp_q(-w) - \exp_q(w)}$$

$$+ \exp_q(w) \frac{\exp_q(zw) - \exp_q(-zw)}{\exp_q(w) - \exp_q(-w)}.$$

So, from Proposition 5.5 and Proposition 5.7 we get immediately the following result.
Proposition 5.8. If \( z \) and \( w \) are complex numbers such that \( |w| < S_1 \), then
\[
\exp_q(zw) = \exp_q(w) \sum_{n=0}^{\infty} \tilde{A}_n(z)w^{2n} - \sum_{n=0}^{\infty} \tilde{B}_n(z)w^{2n}.
\]

In the following, we assume that \( \Psi \) is a comparison function, i.e. \( \Psi(t) = \sum_{n=0}^{\infty} \Psi_n t^n \) such that \( \Psi_n > 0 \) and \( \left( \Psi_{n+1}/\Psi_n \right) \downarrow 0 \) (see [2, 14]). We denote by \( \mathcal{R}_\Psi \) the class of all entire functions \( f \) such that, for some numbers \( \tau \),
\[
|f(re^{i\theta})| \leq M\Psi(\tau r),
\]
as \( r \to \infty \). Here, the complex variable \( z \) was written as \( z = re^{i\theta} \) to emphasize that the limit must hold in all directions \( \theta \). The infimum of numbers \( \tau \) for which (5.6) holds is the \( \Psi \)-type of the function \( f \). This type can be computed by applying Nachbin’s theorem [14] which states that a function \( f(z) = \sum_{n=0}^{\infty} f_n z^n \) is of \( \Psi \)-type \( \tau \) if and only if
\[
\tau = \limsup_{n \to \infty} \left| \frac{f_n}{\Psi_n} \right|^{\frac{1}{n}}.
\]

In [2], the authors applied Nachbin’s theorem for the generalized Borel transform
\[
F(w) = \sum_{n=0}^{\infty} \frac{f_n}{\Psi_n w^{\tau+1}},
\]
and they proved the following result.

Theorem 5.9. Let \( f(z) \) belong to the class \( \mathcal{R}_\Psi \), and let \( D(f) \) be the closed set consists of the union of the set of all singular points of \( F \) and the set of all points exterior to the domain of \( F \). Then
\[
f(z) = \frac{1}{2\pi i} \int_{\Gamma} \Psi(zw)F(w)dw
\]
where \( \Gamma \) encloses \( D(f) \).

According to the above arguments and results we will prove the main theorem.

Theorem 5.10. Let \( S_1 \) be the smallest positive zero of \( S_q(z) \). Assume that one of the following conditions hold:

(i) The function \( f(z) \) is an entire function of \( q^{-1} \)-exponential growth of order 2 and a finite type \( \alpha \), where
\[
\alpha < 2 \left( \frac{1}{4} - \frac{\log S_1}{\log q} \right).
\]

(ii) The function \( f(z) \) is an entire function of \( q^{-1} \)-exponential growth of order less than 2.
Then \( f(z) \) has a convergent \( q \)-Lidstone representation

\[
f(z) = \sum_{n=0}^{\infty} \left[ A_n(z) \frac{\delta^{2n} f(1)}{\delta_q z^{2n}} - \tilde{B}_n(z) \frac{\delta^{2n} f(0)}{\delta_q z^{2n}} \right],
\]

where \( \tilde{A}_n(z) \) is the polynomial of degree \( 2n + 1 \) defined in (5.2) and

\[
\tilde{B}_n(z) := \frac{2^{2n+1}}{(2n + 1)!} \tilde{B}_{2n+1}(z/2; q).
\]

**Proof.** We apply Theorem 5.9 when \( \Psi(z) \) chosen as \( \exp(qz) \) and

\[
\Psi_n = \frac{q^{n(n-1)/4}}{|n|_q^{1/2}}.
\]

Notice, the sequence

\[
\frac{\Psi_{n+1}}{\Psi_n} = \frac{q^{n/2}(1 - q)}{1 - q^{n+1}} = \frac{q^{n/2}}{|n+1|_q}
\]

is decreasing and vanishes at \( \infty \). By using Proposition 5.3, we have for any entire function \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) of \( q^{-1} \)-exponential growth of order \( k \) and a finite type \( \alpha \), there exists a real number \( K > 0 \) such that

\[
|a_n| \leq K q^{\frac{(n-\alpha)^2}{2k}}.
\]

According to the assumption, we have two cases:

Case 1. If \( k = 2 \), then \( |a_n| \leq K q^{\frac{(n-\alpha)^2}{4}} \). This implies (5.6) holds and \( f \in \mathcal{R}_\Psi \).

Here, the \( \Psi \)-type of the function \( f \) given by

\[
\tau := \limsup_{n \to \infty} \left| \frac{a_n}{\Psi_n} \right|^\frac{1}{n}
\]

\[
\leq \frac{q^{\frac{1}{4}-\alpha/2}}{(1 - q)} \limsup_{n \to \infty} \left( K (q; q)_n q^{\alpha^2/4} \right)^\frac{1}{n}
\]

\[
\leq q^{\frac{1}{4}-\alpha/2} < S_1.
\]

Case 2. If \( k < 2 \), then \( \tau = 0 \).

So, we can take \( D(f) \) lies in the closed disk \( |w| \leq \tau \leq q^{\frac{1}{4}-\alpha/2} < S_1 \) and take the curve \( \Gamma \) as the circle \( |w| = \tau + \epsilon < S_1, \epsilon > 0 \) which encloses \( D(f) \). Note that the inequality \( q^{\frac{1}{4}-\alpha/2} < S_1 \) satisfies the condition (5.7) on the type of the function \( f(z) \). We obtain

\[
f(z) = \frac{1}{2\pi i} \int_{\Gamma} \exp_q(zw) F(w) \, dw.
\]
Therefore,
\[
\frac{\delta_q^{2n} f(0)}{\delta_q z^{2n}} = \frac{1}{2\pi i} \int_{\Gamma} \frac{\delta_q^{2n} \exp_q(zw)|_{z=0}}{\delta_q z^{2n}} F(w) \, dw
\]
\[
= \frac{1}{2\pi i} \int_{\Gamma} w^{2n} F(w) \, dw,
\]
\[
\frac{\delta_q^{2n} f(1)}{\delta_q z^{2n}} = \frac{1}{2\pi i} \int_{\Gamma} w^{2n} \exp_q(w) F(w) \, dw
\]
Now, by using Proposition 5.8 we have
\[
f(z) = \frac{1}{2\pi i} \int_{\Gamma} \exp_q(zw) F(w) \, dw
\]
\[
= \frac{1}{2\pi i} \int_{\Gamma} \left\{ \exp_q(w) \sum_{n=0}^{\infty} \tilde{A}_n(z)w^{2n} - \sum_{n=0}^{\infty} \tilde{B}_n(z)w^{2n} \right\} F(w) \, dw
\]
\[
= \sum_{n=0}^{\infty} \left[ \tilde{A}_n(z) \frac{\delta_q^{2n} f(1)}{\delta_q z^{2n}} - \tilde{B}_n(z) \frac{\delta_q^{2n} f(0)}{\delta_q z^{2n}} \right].
\]

**Remark 5.11.** In Theorem 5.10, it is obvious if
\[
\frac{\delta_q^{2n} f(0)}{\delta_q z^{2n}} = \frac{\delta_q^{2n} f(1)}{\delta_q z^{2n}} = 0, \quad (n \in \mathbb{N})
\]
then \(f(z)\) is identically zero.

The following example shows that the sign of equality can not be admitted in (5.7).

**Example 5.12.** Consider \(f(z) = S_q(S_1z)\). Then \(f\) is an entire function of \(q^{-1}\)-exponential growth of order 2 and a finite type \(\alpha = \frac{1}{2} - 2 \frac{\log_q S_1}{\log_q q}\). By using (2.6), one can verify that
\[
\frac{\delta_q^{2n} f(0)}{\delta_q z^{2n}} = \frac{\delta_q^{2n} f(1)}{\delta_q z^{2n}} = 0, \quad (n \in \mathbb{N}).
\]
This implies the \(q\)-Lidstone expansion of \(f(z)\) vanishes identically but the function does not.

We end this section by given the \(q\)-Lidstone series of the functions \((z; q)_n\).

**Example 5.13.** Consider the functions \(g_n(z) = (z; q)_n, \quad n \in \mathbb{N}\). Then, Condition (ii) of Theorem 5.10 is satisfied. So, these polynomials have a \(q\)-Lidstone representation
\[
g_n(z) = \sum_{m=0}^{n} \left[ \tilde{A}_m(z) \frac{\delta_q^{2m} g_n(1)}{\delta_q z^{2m}} - \tilde{B}_m(z) \frac{\delta_q^{2m} g_n(0)}{\delta_q z^{2m}} \right].
\]
One can verify that
\[
\frac{\delta^2 m g_n(z)}{\delta q^{-2n}} = q^{-m(m+\frac{1}{2})} [n]_q [n-1]_q \ldots [n-2m+1]_q (q^m z; q)_{n-2m}.
\]

Therefore, \( g_n(z) \) have the convergent \( q \)-Lidstone representation
\[
g_n(z) = \sum_{m=0}^{n} q^{-m(m+\frac{1}{2})} [n]_q [n-1]_q \ldots [n-2m+1]_q \left[ (q^m z; q)_{n-2m} \tilde{A}_m(z) - \tilde{B}_m(z) \right].
\]

6. A \( q \)-Lidstone series involving \( q \)-Euler polynomials

In this section, we introduce another \( q \)-extension of Lidstone theorem. We expand the function in \( q \)-Lidstone polynomials which are \( q \)-Euler polynomials \( \tilde{E}_n(z; q) \) defined by the generating function (4.1). All the results can be studied in the same manner of the results of the previous section.

Proposition 6.1. If \( z \) and \( w \) are complex numbers such that \(|w| < C_1\), then
\[
Cosh_q(wz) Sech_q(w) = \sum_{n=0}^{\infty} \tilde{M}_n(z) w^{2n},
\]
where
\[
\tilde{M}_n(z) := \frac{2^{2n}}{[2n]_q!} \tilde{M}_{2n}(z; q),
\]
and \( \tilde{M}_n(z; q) \) are the \( q \)-polynomials defined in (4.12).

Proposition 6.2. If \( z \) and \( w \) are complex numbers such that \(|w| < C_1\), then
\[
\frac{\exp_q(zw) \exp_q(-w) - \exp_q(-zw) \exp_q(w)}{\exp_q(w) + \exp_q(-w)} = \sum_{n=0}^{\infty} \frac{w^{2n+1}}{[2n+1]_q!} 2^{2n+1} \tilde{E}_{2n+1}(z/2; q).
\]

Proposition 6.3. If \( z \) and \( w \) are complex numbers such that \(|w| < C_1\), then
\[
\exp_q(zw) = \exp_q(w) \sum_{n=0}^{\infty} \tilde{M}_n(z) w^{2n} - \sum_{n=0}^{\infty} \tilde{N}_{n+1}(z) w^{2n+1},
\]
where
\[
\tilde{N}_{n+1}(z) = \frac{2^{2n+1}}{[2n+1]_q!} \tilde{E}_{2n+1}(z/2; q).
\]

Theorem 6.4. Assume that one of the following conditions hold:
(i) The function \( f(z) \) is an entire function of \( q^{-1} \)-exponential growth of order 2 and a finite type \( \alpha \), where

\[
\alpha < 2 \left( \frac{1}{4} - \frac{\log C_1}{\log q} \right) ;
\]

(ii) The function \( f(z) \) is an entire function of \( q^{-1} \)-exponential growth of order less than 2.

Then \( f(z) \) has the convergent representation

\[
f(z) = \sum_{n=0}^{\infty} \left[ \bar{M}_n(z) \frac{\delta_q^{2n} f(1)}{\delta_q z^{2n}} - \bar{N}_{n+1}(z) \frac{\delta_q^{2n+1} f(0)}{\delta_q z^{2n+1}} \right],
\]

where \( \bar{M}_n \) is the polynomial defined in (6.2) and

\[
\bar{N}_{n+1}(z) := \frac{2^{2n+1}}{(2n+1)_q} \bar{E}_{2n+1}(z/2; q).
\]

As in Remark 5.11, the sign of equality can not be admitted in (6.5). For example, the function \( f(z) = C_q(C_1 z) \) is a function of type \( \left( \frac{1}{2} - 2 \frac{\log C_1}{\log q} \right) \) and one can verify that

\[
\frac{\delta_q^{2n} f(1)}{\delta_q z^{2n}} = 0 = \frac{\delta_q^{2n+1} f(0)}{\delta_q z^{2n+1}}.
\]

Hence, the \( q \)-Lidstone expansion of \( f(z) \) vanishes while the function does not.

7. Concluding Remarks

The \( q \)-Lidstone’s series approximates an entire function in a neighborhood of two points in terms of \( q \)-analog of Lidstone polynomials. In [8], the authors introduced these polynomials which were \( q \)-Bernoulli polynomials generated by second Jackson \( q \)-Bessel function.

In this paper, we presented \( q \)-Bernoulli and \( q \)-Euler polynomials generated by the third Jackson \( q \)-Bessel function to construct new types of \( q \)-Lidstone expansion theorem [8].

This work provides the basis for several applications that we can search in the future. Firstly, we are interested in studying the generalization of \( q \)-Lidstone’s series. The analogous problem for the classical case was studied in [17] by Whittaker. Secondly, we are interested in constructing the \( q \)-Fourier series for the \( q \)-Lidstone polynomials \( \bar{A}_n(z) \) and \( \bar{B}_n(z) \), and applying such expansions to a solution of certain \( q \)-boundary value problems as in [12] and [13].

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Z. Mansour, Department of Mathematics, Faculty of Science, Cairo University, Giza, Egypt.

Email address: zainab@sci.cu.edu.eg

M. AL-Towailb, Department of Computer Science and Engineering, King Saud University, Riyadh, KSA

Email address: mtowailb@ksu.edu.sa