Icosahedral invariants, CM points and class fields

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Abstract

$K3$ surfaces parametrized by Klein’s icosahedral invariants are studied in the author’s previous papers [N1], [N2] and [N3]. Via periods of $K3$ surfaces, icosahedral invariants can be regarded as a pair $(X, Y)$ of holomorphic functions on $\mathbb{H} \times \mathbb{H}$. In this paper, we show that the special values of our functions $X$ and $Y$ give a simple construction of class fields over quartic CM-fields. Our result implies that modular functions coming from the moduli of $K3$ surfaces are effective in algebraic number theory.

Introduction

The aim of this paper is to study class fields derived from abelian surfaces using the pair $(X, Y)$ of explicit modular functions on $\mathbb{H} \times \mathbb{H}$. Here, $X$ and $Y$ are coming from Klein’s icosahedral invariants.

To find good special functions whose special values generate class fields over algebraic number fields is a very interesting problem in number theory (cf. Hilbert’s 12-th problem). For example, special values of elliptic $j$-function generate Hilbert class fields over imaginary quadratic fields.

Let $K$ be a CM-field such that $[K : \mathbb{Q}] = 2^n$. Shimura gave a construction of a class field $k_0$ using a modular function coming from the moduli of $n$-dimensional abelian varieties $A$ such that $\text{End}_0(A) (= \text{End}(A) \otimes \mathbb{Z})$ coincides with $K$. However, Shimura’s class fields are much more difficult than the classical class fields:

(i) Generically, $k_0/K$ does not give a class field. We need the reflex $K^*$ of $K$ and $k_0/K^*$ is a class field.

(ii) $k_0/K^*$ is an unramified class field, but is not a Hilbert class field. In fact, $\text{Gal}(k_0/K^*)$ is isomorphic to $I_K/H_0$, where $H_0$ is given by (2.4). We note that the group $H_0$ is complicated.

(iii) Generically, ”the irreducibility of the class equation” does not hold (see Remark 2.1).

In this paper, we shall give a detailed study of the class fields $k_0$ when $n = 2$ and the corresponding quartic CM-field $K$ contains the real quadratic field $\mathbb{Q}(\sqrt{5})$ with the smallest discriminant as the maximal real subfield. The feature of our study is to use a pair

$$(z_1, z_2) \mapsto (X(z_1, z_2), Y(z_1, z_2))$$

of special functions coming from Klein’s icosahedral invariants. The functions $X$ and $Y$ have the explicit theta expressions of (3.9).

Here, we note that icosahedral invariants appear in several areas in mathematics. They often give remarkable and non-trivial examples (For example, see Hirzebruch [H], Brieskorn [B] or Grothendieck [G]). Especially, our study depends on the result of Hirzebruch.

We shall consider the period mapping for the appropriate family $\mathcal{F}$ of $K3$ surfaces (see (3.3)). The period mapping for $\mathcal{F}$ induces the special functions $X$ and $Y$ of (3.9). Our approach using $X$ and $Y$ of

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\( F \) has some achievements in number theory. The structure of the division algebra \( \text{End}_0(A) \) of a simple abelian surface \( A \) determines the moduli spaces which are important in number theory. In \([N1]\) and \([NS]\), \( X \) and \( Y \) give affine coordinates of the Humbert surface \( \mathcal{H}_5 \). Moreover, in \([N3]\), explicit models of Shimura curves \( S_D \) were realized as divisors in \((X, Y)\)-space.

| \( \text{End}_0(A) \) | type | Moduli Space |
|----------------------|------|--------------|
| \( \mathbb{Q} \)     | generic | Igusa 3-fold \( \mathcal{A}_2 \) |
| Real quadratic field \( \mathbb{Q}(\sqrt{\Delta}) \) | RM | Humbert surface \( \mathcal{H}_\Delta \) |
| Indefinite quaternion \( Q \)-algebra \( B_D \) | QM | Shimura curve \( S_D \) |
| Quartic CM-field \( K \) | CM | CM points |

Table 1: \( \text{End}_0(A) \) for principally polarized abelian surface \( A \) and the corresponding moduli spaces

In our study, we shall study quartic CM-fields \( K \) and the class fields \( k_0/K^* \) via \( X \) and \( Y \):

1. The moduli of our K3 surfaces \( S(X, Y) \) coincide with the moduli of principally polarized abelian surfaces \( (A, \Theta) \) such that \( \text{End}_0(A) = \mathbb{Q}(\sqrt{\Delta}) \) (Theorem 3.4).
2. The above class field \( k_0/K^* \) is generated by the special values of \( X \) and \( Y \) of \((X, Y)\) (Theorem 3.7).
3. We have the simple model of the Kummer variety (see Definition 1.9) with the minimal field of definition (Theorem 3.6).

Moreover, in Section 4, we shall give a detailed study of the class fields \( k_0/K \), where the Galois group \( \text{Gal}(K/Q) \) is cyclic. In this case, we have \( K^* = K \). We shall investigate the complicated group \( H_0 \) in detail (Theorem 4.2). Our \((X, Y)\) of \((X, Y)\) allows us to obtain explicit class fields \( k_0/K \) corresponding to \( H_0 \). In Section 5, we give examples of our results.

Summarizing the results of \([N1]\), \([N2]\), \([N3]\) and this paper, we can say that the pair of the special functions \( X \) and \( Y \) of \((X, Y)\) gives a natural extension of the classical elliptic \( j \)-function to the Hilbert modular case for the smallest discriminant (see Table 2). This suggests that modular functions derived from the moduli of K3 surfaces are very useful in number theory. Our story gives an explicit, simple and non-trivial model of the sophisticated theory of Shimura varieties.

Moreover, we note that the special values of \( X \) and \( Y \) of \((X, Y)\) can be calculated explicitly (Section 3.2 and 3.4, see also Section 5). The author believes that our results are useful for practical applications (cryptography, etc.).

| Classical Theory | Our Story |
|------------------|-----------|
| Modular Function | \( j \)-function |
| \( X \) and \( Y \) |
| Totally Imaginary Quadratic Extension \( K \) over | \( \mathbb{Q} \) |
| \( \mathbb{Q}(\sqrt{\Delta}) \) |
| Class Field | \( K(j)/K \) |
| \( K^*(X, Y)/K^* \) |

Table 2: Elliptic \( J \)-function and our \((X, Y)\)

1 Abelian varieties with complex multiplication

In this section, we survey the results of abelian varieties. For detail, see Shimura [S4].

1.1 CM-fields and CM-types

Let \( K_0 \) be a totally real number field such that \([K_0 : \mathbb{Q}] = n\). Let \( K \) be an imaginary quadratic extension of \( K_0 \). The number field \( K \) is called a CM-field. Let \( \Omega_K \) be the ring of integers of \( K \).

For an abelian variety \( A \) over \( \mathbb{C} \), let \( \text{End}(A) \) be the ring of endomorphisms of \( A \). We set \( \text{End}_0(A) = \text{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q} \).
Let $A$ be an abelian variety of $n$ dimension and $\iota : K \hookrightarrow \text{End}_0(A)$ is an embedding such that $\iota(id_K) = id_{\text{End}(A)}$. The couple $(A, \iota)$ is called an abelian variety of type $K$.

For an abelian variety $(A, \iota)$ of type $K$, let $M$ ($S$, resp.) be a rational (analytic, resp.) representation of $\text{End}_0(A)$ of degree $2n$ ($n$, resp.). We have $M = S \oplus S^\rho$, where $\rho$ is the complex conjugate. We can take $2n$ embeddings $\varphi_1, \ldots, \varphi_{2n} : K \hookrightarrow \mathbb{C}$ such that $S$ ($S^\rho$, resp.) is equivalent to the direct sum of $\varphi_1, \ldots, \varphi_n$ ($\varphi_{n+1}, \ldots, \varphi_{2n}$, resp.). We note that $\varphi_j \circ \rho = \rho \circ \varphi_j$ ($j = 1, \ldots, 2n$) holds. We call $(A, \iota)$ an abelian variety of type $(K, \{\varphi_j\}) = (K, \{\varphi_1, \ldots, \varphi_n\})$.

**Definition 1.1.** Let $K$ be a CM-field of degree $n$ and $\varphi_1, \ldots, \varphi_n : K \hookrightarrow \mathbb{C}$ be embeddings. If there exists an $n$-dimensional abelian variety $A$ and an embedding $K \hookrightarrow \text{End}_0(A)$ such that $(A, \iota)$ is of type $(K; \{\varphi_1, \ldots, \varphi_n\})$, we call $(K; \{\varphi_1, \ldots, \varphi_n\})$ a CM-type.

If $(A_1, \iota_1)$ and $(A_2, \iota_2)$ are abelian varieties of the same type, $A_1$ and $A_2$ are isogenous to each other.

### 1.2 Procedure of an abelian variety of CM-type

If a CM-type $(K, \{\varphi_1, \ldots, \varphi_n\})$ is given, we can obtain a corresponding abelian variety as follows.

For $\alpha \in K$, we set $u(\alpha) = \left(\frac{\alpha^{\varphi_1}}{\alpha^{\varphi_n}}, \ldots, \frac{\alpha^{\varphi_n}}{\alpha^{\varphi_n}}\right) \in \mathbb{C}^n$. Let $M \subset K$ be a free $\mathbb{Z}$-module of rank $2n$. Let $\alpha_1, \ldots, \alpha_{2n}$ be a system of basis of $M$. Then, the vectors $u(\alpha_1), \ldots, u(\alpha_{2n})$ span a lattice $\Lambda = \Lambda(M)$ of $\mathbb{C}^n$. Take $\zeta \in K$ such that $K = K_0(\zeta)$, $\zeta^2 \in K_0$, $-\zeta^2$ is totally positive and $\text{Im}(\zeta^{\varphi_j}) > 0$ for $j \in \{1, \ldots, n\}$. Then, we have an alternating quadratic form

$$E(z, w) = \sum_{j=1}^n \zeta^{\varphi_j}(z_j w_j - \overline{z_j} w_j)$$

for $z = (z_1, \ldots, z_n), w = (w_1, \ldots, w_n) \in \mathbb{C}^n$. We can check that

$$E(z, \sqrt{-1} w) = -\sqrt{-1} \sum_{j=1}^n \zeta^{\varphi_j}(z_j w_j + \overline{z_j} w_j)$$

is symmetric and positive non-degenerate. Letting $\rho$ be the complex conjugate, we have $\zeta^\rho = -\zeta$. So, we obtain

$$E(u(\alpha), u(\beta)) = \text{Tr}_{K/Q}(\zeta \alpha \beta^\rho)$$

for $\alpha, \beta \in K$. We can find a positive integer $\mu$ such that $\nu \zeta^\mu M^\rho \subset \mathcal{O}_K$. Hence, from (1.3), the values of $\nu E(z, w)$ on $A(M) \times A(M)$ are integers. So, together with (1.2), $\nu E(z, w)$ defines a non-degenerate Riemann form on $\mathbb{C}^n/\Lambda(M)$. Therefore, the complex torus $\mathbb{C}^n/\Lambda$ gives an n-dimensional abelian variety $A = A(M)$. For $\alpha \in K$, set $q(\alpha) = \begin{pmatrix} \alpha^{\varphi_1} & & 0 \\ & \ddots & \vdots \\ 0 & & \alpha^{\varphi_n} \end{pmatrix}$. The linear transformation given by $q(\alpha)$ defines $\iota(\alpha) \in \text{End}_0(A(M))$. So, $(A(M), \iota)$ is an abelian variety of type $(K, \{\varphi_j\})$.

The above $(A(M), \iota)$ satisfies $\iota(M) \subset \text{End}_0(A(M))$. However, we cannot assure that

$$\iota(\mathcal{O}_K) = \text{End}(A(M)) \cap \iota(K).$$

**Definition 1.2.** If $(A(M), \iota)$ satisfies (1.4), we say that $(A(M), \iota)$ is a principal abelian variety of type $(K, \{\varphi_j\})$.

**Proposition 1.1.** ([3], Section 7) The abelian variety $(A(M), \iota)$ is principal if and only if $M$ is a fractional ideal of $K$.

To consider the class fields over CM-fields, we need principal abelian varieties. Due to the above proposition, we shall study abelian varieties $(A(\mathfrak{a}), \iota)$ coming from a fractional ideal $\mathfrak{a}$ of $K$. 

3
1.3 Reflex

**Definition 1.3.** Let \((K, \{\varphi_j\})\) be a CM-type and \((A, \iota)\) be the corresponding abelian variety. If \(A\) is a simple abelian variety, we call \((K, \{\varphi_j\})\) primitive.

**Remark 1.1.** If a CM-type \((K, \{\varphi_j\})\) is not primitive, the moduli of the corresponding abelian variety can be reduced to low dimensional cases. So, in this study, we only consider primitive abelian varieties.

**Proposition 1.2.** ([3/2], Section 8) Let \((K; \{\varphi_1, \cdots, \varphi_n\})\) be a CM-type. Let \(L\) be a Galois extension of \(Q\) containing \(K\). Set \(S\) be the set of all the elements of \(\text{Gal}(L/Q)\) inducing some \(\varphi_j\) on \(K_0\). Set \(S^* = \{\sigma^{-1}\mid \sigma \in S\}\) and \(H^* = \{\gamma \in \text{Gal}(L/Q)\mid \gamma S^* = S^*\}\). There exists the subfield \(K^*\) of \(L\) corresponding to the subgroup \(H^*\) of \(\text{Gal}(L/Q)\). Let \(\{\psi_k\}\) be the set of all the embeddings \(K^* \hookrightarrow \mathbb{C}\) coming from \(S^*\). Then, \((K^*, \{\psi_k\})\) is a primitive CM-type. Moreover, \((K^*, \{\psi_k\})\) is determined only by \((K, \{\varphi_j\})\).

**Definition 1.4.** For a CM-type \((K, \{\varphi_j\})\), the primitive CM-type \((K^*, \{\psi_k\})\) in Proposition 1.2 is called the reflex of \((K, \{\varphi_j\})\).

**Remark 1.2.** For the reflex \((K^*, \{\psi_k\})\) of a CM-type \((K, \{\varphi_j\})\), we have \([K^*: Q] \leq [K: Q]\). However, if \((K; \{\varphi_j\})\) is primitive, \([K^*: Q] = [K: Q]\).

**Example 1.1.** (see [3/2], 8.4) Let \(K_0\) be a real quadratic field and \(K\) be a totally imaginary quadratic extension of \(K_0\). Namely, let \(K\) be a CM-field of degree 4. Take \(\zeta \in K\) such that \(-\zeta^2 \in K_0\) is totally positive. We can take \(\varphi_1, \varphi_2 : K \hookrightarrow \mathbb{C}\) such that \(\varphi_1 = \text{id}\) and \(\varphi_2 = \varphi\) with \(\text{Im}(\varphi) > 0\). Then, \((K, \{\varphi_1, \varphi_2\})\) is a CM-type. We have the following three cases.

(i) Suppose \(\text{Gal}(K/Q) \simeq (\mathbb{Z}/2\mathbb{Z})^2\). Letting the group \(S\) in Proposition 1.2 be \(S = \{\text{id}, \sigma\}\), we have \(\text{Gal}(K/Q) = \{\text{id}, \sigma, \rho, \sigma \rho\}\), where \(\rho\) is the complex conjugation. In this case, the CM-type \((K, \{\text{id}, \sigma\})\) is not primitive. The reflex \(K^*\) is an imaginary quadratic field.

(ii) Suppose \(\text{Gal}(K/Q) \simeq (\mathbb{Z}/4\mathbb{Z})\). Letting the group \(S\) in Proposition 1.2 be \(S = \{\text{id}, \sigma\}\), we have \(\text{Gal}(K/Q) = \{\text{id}, \sigma, \sigma^2, \rho, \sigma^3\}\), where \(\rho\) is the complex conjugation. In this case, the CM-type \((K, \{1, \sigma\})\) is primitive. The reflex of \((K, \{\text{id}, \sigma\})\) is given by \((K^*, \{\psi_k\}) = (K, \{1, \sigma\})\).

(iii) Suppose \(K\) is not normal over \(Q\). Letting \(K_0 = Q(\sqrt{\Delta}) (\Delta > 0)\), we can assume that \(-\xi^2 = x + y\sqrt{\Delta}\) \((x, y \in Q)\) is totally positive. Set \(\xi = \sqrt{-1}\sqrt{x + y\sqrt{\Delta}}\) and \(\xi^c = \sqrt{-1}\sqrt{x - y\sqrt{\Delta}}\). Since \(K\) is not normal, \(\xi^c \not\in K\). So, the Galois closure \(L\) is given by \(L = Q(\xi, \xi^c)\). Set \(\sigma : (\xi, \xi^c) \mapsto (\xi^c, -\xi)\) and \(\tau : (\xi, \xi^c) \mapsto (\xi^c, \xi)\). We can show that the Galois group \(\text{Gal}(L/Q)\) is generated by \(\sigma\) and \(\tau\). Moreover, the CM-type \((K, \{1, \varphi\})\) is primitive and its reflex is given by \((Q(\xi + \xi^c), \{1, \sigma\})\).

1.4 Polarized abelian varieties from CM-types

For a CM-type \((K, \{\varphi_j\})\) and \(a \in I_K\), we can obtain an abelian variety \((A, \iota) = (A(a), \iota)\) as in Section 1.2. According to Proposition 1.1, \((A, \iota)\) is principal. Let us consider \((A, \Theta, \iota)\), where \(\Theta\) is a polarization of \(A\) given by the Riemann form of \((\mathbb{Z}, \iota)\). We call \((A, \Theta, \iota)\) a polarized abelian variety of type \((K, \{\varphi_j\}; \zeta, a)\), or of type \((\zeta, a)\). We note that the Riemann form of \((\mathbb{Z}, \iota)\) is determined by \(\zeta\).

Let \(C_0(K)\) be the abelian group consisting of all pairs \((b, c)\), where \(b \in K_0\) is totally positive and \(c \in I_K\). Here, the multiplication is given by \(\{b_1, c_1\}\{b_2, c_2\} = \{b_1b_2, c_1c_2\}\). Set

\[
C(K) = C_0(K)/\{(xc^p, x\Omega_K)\mid x \in K^\times\}.
\] (1.5)

We denote the coset of \((b, c)\) by \((b, c)\). This group \(C(K)\) shall be used to define the class field.

Suppose \(P_1 = (A_1, \Theta_1, \iota_1)\) \((P_2 = (A_2, \Theta_2, \iota_2), \text{resp.})\) be a polarized abelian variety of type \((K, \{\varphi_j\}; \zeta_1, a_1)\) \(((K, \{\varphi_j\}; \zeta_2, a_2), \text{resp.})\). Set

\[
(b, c) = (\zeta_1^{-1}c_2, a_2^{-1}a_1).
\] (1.6)
Then, we write
\[(P_2 : P_1) = (b, c).\] (1.7)

We remark that \((A_1, \Theta_1)\) and \((A_2, \Theta_2)\) are isomorphic if and only if \((P_2 : P_1) = (1, \mathcal{O}_K)\).

### 1.5 The field of moduli for principally polarized abelian surfaces

Let \(k_1\) and \(k_2\) be number fields. Let \(\sigma : k_1 \to k_2\) be an embedding. For an algebro-geometric object \(W\) over \(k_1\), we denote by \(W^\sigma\) the image \(W\) under \(\sigma\). Then, \(W^\sigma\) is defined over \(k_2\).

Let \(k_1\) be an algebraic number fields with \(\text{char}(k_1) = 0\). Let \((A, \Theta)\) be a polarized abelian variety defined over \(k_1\). Suppose the polarization \(\Theta\) is given by a divisor \(X\) of \(A\). Letting \(\sigma : k_1 \to \mathbb{C}\) be an embedding, we have \((A^\sigma, \Theta^\sigma)\) defined over \(k_2\), where \(\Theta^\sigma\) is the polarization given by \(X^\sigma\). Take a Galois extension \(L\) of \(k_1\) over \(\mathbb{Q}\). We set
\[G_1 = \{\sigma \in \text{Gal}(L/\mathbb{Q}) | (A, \Theta) \text{ is isomorphic to } (A^\sigma, \Theta^\sigma)\}.

Let \(M\) be the subfield of \(L\) corresponding to the subgroup \(G_1\). Then, \(\sigma \in \text{Gal}(L/M)\) is identity if and only if \((A, \Theta)\) is isomorphic to \((A^{10}, \Theta^{10})\). We note that the field \(M\) is uniquely determined by \((A, \Theta)\) and independent of the choice of the field of definition \(k_1\) of \((A, \Theta)\) and the Galois closure \(L\).

**Definition 1.5.** The above field \(M\) is called the field of moduli of \((A, \Theta)\).

In the rest of this subsection, we survey the field of moduli of principally polarized abelian surfaces. Let \((A, \Theta)\) be a principally polarized abelian surface. Then, there exists a genus 2 curve \(C\) such that \(A\) is equal to the Jacobian variety \(\text{Jac}(C)\) of \(C\). Suppose \(C\) is given by
\[C : y^2 = u_0(x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5)(x - x_6).

Then, we obtain the Igusa-Clebsch invariants \(I_2, I_4, I_6\) and \(I_{10}\) of \(C\):
\[\begin{align*}
I_2 &= u_0^2 \sum (12)^2(34)^2(56)^2, \\
I_4 &= u_0^2 \sum (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2, \\
I_6 &= u_0^2 \sum (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2(14)^2(25)^2(36)^2, \\
I_{10} &= u_0^{10} \prod_{i < j} (ij)^2.
\end{align*}\] (1.8)

Here, \((jk)\) indicates \((x_j - x_k)\).

Let \(\mathcal{M}_2\) be the moduli space of genus 2 curves. Let \(\mathbb{P}(1 : 2 : 3 : 5) = \{(\zeta_1 : \zeta_2 : \zeta_3 : \zeta_5)\}\) be the weighted projective space. It is well-known that
\[\mathcal{M}_2 = \mathbb{P}(1 : 2 : 3 : 5) - \{\zeta_5 = 0\}.

In fact, \((I_2 : I_4 : I_6 : I_{10})\) gives a well-defined point of the moduli space \(\mathcal{M}_2\). We note that the moduli space \(\mathcal{M}_2\) is a Zariski open set of the moduli space \(\mathcal{M}_2\) of principally polarized abelian surfaces (\(\mathcal{M}_2\) is the complement of the divisor given by the points corresponding to the product of elliptic curves).

In [I], Igusa defined arithmetic invariants \(J_2, J_4, J_6\) and \(J_{10}\) given by
\[\begin{align*}
J_2 &= 2^{-3}I_2, \\
J_4 &= 2^{-12}I_{10}, \\
J_6 &= 2^{-5}3^{-1}(4J_2^2 - I_4), \\
J_{10} &= 2^{-6}3^{-2}(8J_2^3 - 160J_2J_4 - I_6).
\end{align*}\] (1.9)

Due to [I, 1.9], we have an isomorphism between the ring \(\mathbb{Q}[I_2, I_4, I_6, I_{10}]\) and the ring \(\mathbb{Q}[J_2, J_4, J_6, J_{10}]\).
Set
\[ m_1 = \frac{J_5^2}{J_{10}}, \quad m_2 = \frac{J_5^2 J_6}{J_{10}}, \quad m_3 = \frac{J_6^2}{J_{10}}. \] (1.10)

They are called the absolute invariants.

Then, we have the following famous result (for example, see [SI] or [FG]).

**Proposition 1.3.** (1) Let \( C_1 \) and \( C_2 \) be two curves of genus 2 with \( I_{10} \neq 0 \). Then, the curve \( C_1 \) is isomorphic to \( C_2 \) if and only if \( m_j(C_1) = m_j(C_2) \) \((j = 1, 2, 3)\).

(2) Let \((A, \Theta)\) be a principally polarized abelian surface and \( C \) be a genus 2 curve such that \( \text{Jac}(C) = A \). Then, a subfield \( M \) of the field of definition \((A, \Theta)\) gives a field of moduli of \((A, \Theta)\) if and only if the absolute invariants \( m_j(C) \) \((j = 1, 2, 3)\) generate \( M \) over \( \mathbb{Q} \).

1.6 Kummer varieties due to Weil

Let \( \text{Aut}(A, \Theta) \) be the group of automorphisms of a polarized abelian variety \((A, \Theta)\). Due to Weil [W], the quotient variety \( A/\text{Aut}(A, \Theta) \) is called the Kummer variety of \((A, \Theta)\). We note that the group \( \text{Aut}(A, \Theta) \) always contains the subgroup \( \{id_A, -id_A\} \).

**Proposition 1.4.** Let \((A, \Theta)\) be a polarized abelian variety.

(1) For \( \alpha \in \text{End}_0(A) \), let \( \alpha \mapsto \alpha^\circ \) be the Rosati involution for the polarization \( \Theta \). For \( \alpha \in \text{End}(A) \), \( \alpha \in \text{Aut}(A, \Theta) \) if and only if \( \alpha^\circ \alpha = id_A \) holds.

(2) The group \( \text{Aut}(A, \Theta) \) is a finite group.

(3) Suppose \( \Theta \) is a principal polarization. Set \( R = \text{End}(A) \). Let \( E_R \) be the group of all the units in \( R \). Let \( E_R^{tor} \) be the all the torsion elements of \( E_R \). Then, \( \text{Aut}(A, \Theta) \subset E_R^{tor} \).

**Proof.** The proofs of (1) and (2) are given in [W] or [SI]. We give a proof of (3).

Take a divisor \( X \) in the principal polarization \( \Theta \). Let \( \varphi_X : A \to \hat{A} \) be the canonical isogeny given by \( u \mapsto X_u - X \). Since \( X \) gives a principal polarization, \( \varphi_X \) is an isomorphism between \( A \) and \( \hat{A} \). Especially, \( \varphi_X^{-1} \in \text{Hom}(\hat{A}, A) \). So, for any \( \alpha \in \text{End}(A) \), \( \alpha^\circ \in \text{End}(A) = R \) holds. Hence, \( \alpha^\circ \alpha = id_A \) implies that \( \alpha \in E_R \). Moreover, since \( \text{Aut}(A, \Theta) \) is a finite group, \( \text{Aut}(A, \Theta) \) must be contained in \( E_R^{tor} \). \( \Box \)

**Definition 1.6.** ([W], see also [SI]) The quotient variety \( A/\text{Aut}(A, \Theta) \) is called the Kummer variety of \((A, \Theta)\).

**Proposition 1.5.** ([W], see also [SI]) Let \((A, \Theta)\) be a polarized abelian variety. Let \( M \) be the field of moduli of \((A, \Theta)\). Then, the field of definition of the Kummer variety of \((A, \Theta)\) contains \( M \). Moreover, there exists a model \( W \) of the Kummer variety of \((A, \Theta)\) such that the field of definition of \( W \) coincides with \( M \).

2 Class field theory from abelian varieties

In this section, for an algebraic number field \( k \) of finite degree, the class number of \( k \) is denoted by \( h_k \).

The group of fractional (principal, resp.) ideals of \( k \) is denoted by \( I_k \) (\( P_k \), resp.). Let \( K_0 \) be a totally real field of degree \( n \) and \( K \) be an imaginary quadratic extension of \( K_0 \).

2.1 Class fields

Take a cycle \( m \) of \( k \). Let \( I_k(m) \) be the group of fractional ideals of \( k \) which are prime to \( m \). Set \( P_k(m) = \{ (\alpha) | \alpha \equiv 1 \pmod{m} \} \).

Let \( k_1/k \) be a Galois extension. Let \( I_{k_1}(m) \) be the group of fractional ideals of \( k_1 \) which are prime to \( m \). The image \( N_{k_1/k}(I_{k_1}(m)) \) is a subgroup of \( I_k(m) \). Then, we have the fundamental inequality \( [I_k(m) : P_k(m) N_{k_1/k}(I_{k_1}(m))] \leq [k_1 : k] \).
**Definition 2.1.** Let $H(m)$ be a subgroup of $I_k(m)$ such that

$$P_k(m) \subset H(m) \subset I_k(m).$$

(2.1)

If a Galois extension $k_1/k_0$ satisfies $[I_k(m) : H(m)] = [k_1 : k]$ and $H(m) = P_k(m)N_{k_1/k}(I_k(m))$, then $k_1/k$ is called the class field corresponding to $H(m)$. 

For any abelian extension $k_1/k$, there exists a group $H(m)$ satisfying (2.1) for a cycle $m$ such that $k_1/k$ is the class field corresponding to $H(m)$. Conversely, for any group $H(m)$ satisfying (2.1), there exists the unique abelian extension $k_1/k$ such that $k_1/k$ gives the class field corresponding to $H(m)$.

Let $k_1/k$ be a class field corresponding to a group $H(m)$ satisfying (2.1), we have an isomorphism

$$I_k(m)/H(m) \simeq \text{Gal}(k_1/k).$$

(2.2)

The isomorphism is given by the Artin symbol $a \mapsto \left(\frac{k_1/k}{a}\right)$ (The Artin reciprocity law).

Let $k_1/k$ be an abelian extension, the greatest common factor $f(k_1/k)$ of all cycles $m$ for which the Artin reciprocity law (2.2) holds is called the conductor of $k_1/k$. A prime ideal $p$ is ramified in $k_1/k$ if and only if $p$ divides $f(k_1/k)$. Especially, if $f(k_1/k) = (1)$, then $k_1/k$ is unramified.

### 2.2 Class fields due to Shimura

Here, we give a brief survey of a class fields derived from abelian varieties. For detail, see [S4].

Let $(K, \{\varphi_j\})$ be a CM-type and $(K^*, \{\psi_k\})$ be the corresponding reflex. We have the mapping $I_{K^*} \to I_{K^*}$ given by

$$\Phi^* : a \mapsto a^{\Phi^*} = \prod_k a^{\psi_k}.$$  

(2.3)

Let $L$ be the Galois closure of $K$ over $\mathbb{Q}$. For $a \in I_{K^*}$, we can take $g(a) \in I_K$ such that $\mathcal{D}_L g(a) = \mathcal{D}_L a^{\Phi^*}$. We need to consider the subgroup $H_0$ of $I_{K^*}$ given by

$$H_0 = \{a \in I_{K^*} | g(a) \in P_K, \text{ there exists } \mu \in K \text{ such that } g(a) = (\mu), N(a) = \mu \mu' \}.$$  

(2.4)

The group $H_0$ is complicated but very important.

Let $P = (A, \Theta, \iota)$ be a polarized abelian variety of type $(K, \{\varphi_j\})$. For $\sigma \in \text{Gal}(\mathbb{C}/K^*)$, $P^\sigma = (A^\sigma, \Theta^\sigma, \iota^\sigma)$ has the same type of $(K, \{\phi_j\})$. So, as in (2.2), we obtain $(P^\sigma : P) = (b, \epsilon)$ for some totally positive $b \in K_0$ and $\epsilon \in I_K$. Here, $(b, \epsilon) = \epsilon(\epsilon)$ holds. We have a homomorphism $h : \text{Gal}(\mathbb{C}/K^*) \to \mathfrak{C}(K)$ of groups given by $\sigma \mapsto (b, \epsilon)$. The abelian variety $P$ is isomorphic to $P^\sigma$ if and only if $\sigma \in \text{Ker}(h)$. Hence, by the definition of the field of moduli $M$ of $(A, \Theta)$, $\text{Ker}(h)$ is the set of $\sigma$ which leaves the elements of $k_0 = MK^*$ invariant. So, the homomorphism $h$ induces an embedding $\tilde{h} : \text{Gal}(k_0/K^*) \to \mathfrak{C}(K)$. Thus, $\text{Gal}(k_0/K^*)$ is an abelian group.

We can take an integral ideal $m_0$ of $K^*$ such that the Aritin symbol $\sigma : I_{K^*}(m_0) \to \text{Gal}(k_0/K^*)$ given by $a \mapsto \sigma(a) = \left(\frac{k_0/K^*}{a}\right)$ induces the homomorphism $\tilde{h} \circ \sigma : I_{K^*}(m_0) \to \mathfrak{C}(K)$ with the form

$$a \mapsto (P^{\sigma(a)} : P) = (N(a), a^{\Phi^*)}.$$  

By (2.4) and (1.6), $a \in H_0 \cap I_{K^*}(m_0)$ if and only if $(N(a), a^{\Phi^*)} = (1, \mathfrak{O}_K)$. Since $\tilde{h}$ is an embedding, we have $\text{Ker}(\sigma) = H_0 \cap I_{K^*}(m_0)$. So, the Artin reciprocity law implies that $I_{K^*}(m_0)/(H_0 \cap I_{K^*}(m_0)) \simeq \text{Gal}(k_0/K^*)$. Moreover, we can prove that $H_0 \supset P_{K^*} = P_{K^*}(1)$ and the conductor of $k_0/K^*$ is $1$.

By virtue of the results in Section 2.1, the following theorem follows.

**Theorem 2.1.** ([S4], Section 15) Let $(K, \{\varphi_j\})$ a CM-type and $(K^*, \{\psi_0\})$ be the corresponding reflex. Let $H_0$ be the subgroup of $I_{K^*}$ given in (2.4). For a polarized abelian variety $P = (A, \Theta, \iota)$ of type $(K, \{\varphi_j\}; \zeta, a)$ coming from $\zeta \in K$ and $a \in I_{K^*}$, let $M$ be the field of moduli of $(A, \Theta)$. Then, $k_0 = MK^*$ is the unramified class field over $K^*$ corresponding to $H_0$. 

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2.3 Detailed properties of ideal class groups for CM-fields

The group $H_0$ of (2.4) is complicated. To study $H_0$, we need some detailed properties of CM-fields. In this subsection, we survey such properties (for detail, see [S3]). At the last of this subsection, we shall give Lemma 2.4. This lemma is useful for our study in Section 4.

Set

$$\left\{ \begin{array}{l}
I(K/K_0) = \{a \in I_K | a(a^p)^{-1} \in P_K\}, \\
I_0(K/K_0) = P_K \{a \in I_K | a^p = a\}.
\end{array} \right. \quad (2.5)$$

Let $P^+_K$ be the subgroup of $P_K$ consisting of all principal ideals generated by totally positive elements.

Let $E^+_K$ be the group of all units in $O_K$. Let $E^+_K$ be the group of all totally positive units in $O_K$. Set $E^2_K = \{u^2 | u \in E_K\}$. Then, there exists $\varepsilon \in \mathbb{Z}_{\geq 0}$ such that

$$2^\varepsilon = [E^+_K : E^2_K] = [P_K : P^+_K]. \quad (2.6)$$

We have the mapping

$$N_{K/K_0} : I_K \to I_{K_0}. \quad (2.7)$$

There exists $\delta \in \mathbb{Z}_{\geq 0}$ such that

$$2^\delta = [I_{K_0} : P^+_K N_{K/K_0}(I_K)]. \quad (2.8)$$

Set

$$\left\{ \begin{array}{l}
R_K = \{a \in I_K | N_{K/K_0}(a) \in P_K\}, \\
R^+_K = \{a \in I_K | N_{K/K_0}(a) \in P^+_K\}.
\end{array} \right. \quad (2.9)$$

The mapping $N_{K/K_0}$ of (2.7) induces an embedding

$$I_K/R_K \hookrightarrow I_{K_0}/P_{K_0}. \quad (2.10)$$

Then, we have

$$[R_K : R^+_K] = 2^{\varepsilon - \delta}. \quad (2.11)$$

Lemma 2.1. ([S3] Appendix) (1) There exist $\eta, \beta, \gamma \in \mathbb{Z}_{\geq 0}$ such that

$$2^n = [I_{K_0} \cap P_K : P_{K_0}], \quad (2.12)$$

$$2^\beta = [I(K/K_0) : I_0(K/K_0)], \quad (2.13)$$

$$2^\gamma = [I_0(K/K_0) : I_{K_0} P_K]. \quad (2.14)$$

(2) The relations

$$[I_K : I_{K_0} P_K] = 2^n \frac{h_K}{h_{K_0}}, \quad (2.15)$$

$$2^{\eta - \beta - \gamma} \frac{h_K}{h_{K_0}} = [I_K : I(K/K_0)], \quad (2.16)$$

$$\beta \leq \varepsilon, \quad (2.17)$$

$$2^\gamma = 2^{n+1-1}[N_{K/K_0}(E_K) : E^2_{K_0}]. \quad (2.18)$$

hold. Here, $E_K$ be the group consists of all units in $O_K$ and $t$ is the number of prime ideals of $K_0$ ramified in $K$.

(3) If $2^{-n}h_{K_0}$ is an odd number, then $2^{\beta + \gamma}$ is the number of elements of order 1 or 2 in the ideal class group $I_K/P_K$. 

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Let \((K, \{\varphi_j\})\) be a CM-type and \((K^*, \{\psi_k\})\) be its reflex. In fact, the mapping \(\Phi^*\) of \([S3]\) gives a mapping \(I_{K^*} \to R^+_K\). Set

\[
I(\Phi^*) = \left\{ a \in I_{K^*} | a^{\Phi^*} \in P_K \right\}.
\]  

(2.19)

The mapping \(\Phi^*\) of \([S3]\) canonically induces

\[
\tilde{\Phi}^*: I_{K^*}/I(\Phi^*) \to R^+_K/P_K.
\]  

(2.20)

Moreover, we have the following lemmas.

**Lemma 2.2.** \([S3]\) *Appendix* There exists an embedding

\[
I(\Phi^*)/H_0 \hookrightarrow E^+_K/N_{K_0/K_0}(E_K).
\]  

(2.21)

**Lemma 2.3.** \([S3]\) *Appendix* If \(K\) is an cyclic extension over \(\mathbb{Q}\), then, \(K^* = K\). Moreover,

\[
H_0 \subset I_0(K/K_0), \quad I(\Phi^*) \subset I(K/K_0)
\]  

and

\[
I_{K_0}P_K \subset H_0 \subset I(K/K_0) \subset I_K
\]  

(2.22)

hold.

Also, we have the following lemma. We shall use it in Section 4.

**Lemma 2.4.** Let \(K_0\) be a totally real number field and \(K\) be an imaginary quadratic extension of \(K_0\). Let us consider the following three conditions:

\[
\begin{aligned}
(\text{C1}) & : h_{K_0} = 1, \\
(\text{C2}) & : E^+_{K_0} = E^2_{K_0}, \\
(\text{C3}) & : K/\mathbb{Q} \text{ is cyclic.}
\end{aligned}
\]  

(2.23)

1. If \(K\) satisfies the condition (C1), then \(\eta = 0\) and \(I_K = R_K\) hold.
2. If \(K\) satisfies the conditions (C1) and (C2), then \(\varepsilon = 0, \beta = 0, I_K = R^+_K\) and \(I(\Phi^*) = H_0\) hold.
3. Suppose the ideal class group \(I_K/P_K\) is isomorphic to \((\mathbb{Z}/2\mathbb{Z})^r \oplus G_1\), where \(G_1\) does not contain any 2 torsion element. If \(K\) satisfies the conditions (C1), (C2) and (C3), then \(\gamma = r, [I(K/K_0) : P_{K_0}] = 2^r\) and \([I_K : I(K/K_0)] = \frac{h_K}{2^r}\) hold.

**Proof.** (1) Assuming (C1), we have \(\eta = 0\) and \(I_K = R_K\) from \([2.10]\) and \([2.12]\).

(2) By (C2) and \([2.16]\), we obtain \(\varepsilon = 0\). From \([2.17]\), \(\beta = 0\) holds. Moreover, \([2.8]\), \([2.11]\) and the above (1) assure \(I_K = R^+_K\). Note that \([E^+_{K_0} : E^2_{K_0}] = [E^+_{K_0} : N_{K_0/K_0}(E_K)] = [N_{K_0/K_0}(E_K) : E^2_{K_0}]\). Together with \([2.21]\), we have \(I(\Phi^*) = H_0\).

(3) By (C1) and Lemma \([2.11] (3)\), we have \(\beta + \gamma = r\). Together with the above (2), we have \(\gamma = r\). So, by \([2.13]\) and \([2.14]\), we obtain \([I(K/K_0) : P_K] = 2^r\). Therefore, \([I_K : I(K/K_0)] = \frac{h_K}{2^r}\) follows.

**Remark 2.1.** Let \((A, \Theta, i)\) be a polarized abelian variety of type \((K, \{\varphi_j\})\). If the condition

\[
[I_{K^*} : H_0] = [R^+_K : P_K][E^+_K : N_{K_0/K_0}(E_K)]
\]  

(2.24)

holds, any abelian varieties of type \((K, \{\varphi_j\})\) is given by \((A^\sigma, \Theta^\sigma, i^\sigma)\) for some \(\sigma \in \text{Gal}(k_0/K)\). Then, if \([2.24]\) holds, we say that the irreducibility of the class equation holds for the case \((K, \{\varphi_j\})\). The irreducibility of the class equation is studied by several researchers. For detail, see \([S3]\) or \([S4]\).
3 Moduli of principally polarized abelian surfaces via periods of $K3$ surfaces

We study the class fields in Section 2.2 for the case of $n = 2$ and $K_0 = \mathbb{Q}(\sqrt{5})$, that is the simplest case of complex multiplication of abelian surfaces because the discriminant of the real quadratic field is the smallest. In this section, we shall give an explicit expression of the class fields using the special values of the functions $X$ and $Y$ of $\delta$.

3.1 The family $\mathcal{F}_{CD}$ of $K3$ surfaces due to Clinger and Doran

Let $\mathcal{M}_2$ be the Siegel upper half plan of degree 2. Let $Sp(4, \mathbb{Z})$ be the symplectic group consisting of $4 \times 4$ matrices. It is well known that the moduli space $\mathcal{A}_2$ of principally polarized abelian surfaces is given by the quotient space $\mathcal{G}_2/Sp(4, \mathbb{Z})$. Here, recall that the moduli space $\mathcal{M}_2 = \text{Proj}(\mathbb{C}[I_2, I_4, I_6, I_{10}])$ of genus 2 curves is a Zariski open set of $\mathcal{A}_2$. Clinger and Doran $\text{[CD]}$ gave the family $\mathcal{F}_{CD} = \{ S_{CD}(\alpha, \beta, \gamma, \delta) \}$ of $K3$ surfaces, where $(\alpha, \beta, \gamma, \delta) \in \mathbb{P}(2 : 3 : 5 : 6) - \{ \gamma = \delta = 0 \}$. The member $S_{CD}(\alpha, \beta, \gamma, \delta)$ is given by the explicit defining equation

$$S_{CD}(\alpha, \beta, \gamma, \delta) : y^2 = x^3 + (-3\alpha t^4 - \gamma t^5)x + (t^5 - 2\beta t^6 + \delta t^7)$$

(3.1)

of an elliptic $K3$ surface. Here, the parameter space $\mathbb{P}(2 : 3 : 5 : 6) - \{ \gamma = \delta = 0 \}$ is isomorphic to $\mathcal{A}_2$. The period mapping for $\mathcal{F}_{CD}$ is studied in detail in $\text{[NS]}$. This period mapping is defined on $\mathbb{P}(2 : 3 : 5 : 6) - \{ \gamma = \delta = 0 \}$. Applying the Torelli theorem for polarized $K3$ surfaces to the family $\mathcal{F}_{CD}$, we can prove the following theorem.

Theorem 3.1. $\text{[NS], Theorem 8.2}$. The point $(I_2 : I_4 : I_6 : I_{10}) \in \mathcal{M}_2 = \mathbb{P}(1 : 2 : 3 : 5) - \{ \zeta_5 = 0 \}$ corresponds to the point $(\alpha : \beta : \gamma : \delta) \in \mathbb{P}(2 : 3 : 5 : 6) - \{ \gamma = \delta = 0 \}$ by the following birational transformation over $\mathbb{Q}$:

$$\alpha = \frac{1}{9} I_4, \quad \beta = \frac{1}{30}(-2I_2I_4 + 3I_6), \quad \gamma = 8I_{10}, \quad \delta = \frac{2}{3}I_2I_{10}.$$

(3.2)

3.2 The family $\mathcal{F}$ of $K3$ surfaces and icosahedral invariants

Let $K_0 = \mathbb{Q}(\sqrt{\Delta})$ be a real quadratic field with discriminant $\Delta$ and $\mathcal{O}_\Delta$ be the ring of integers. If an abelian surface $A$ and an embedding $\iota : K_0 \to \text{End}_0(A)$ satisfies $\iota(\mathcal{O}_\Delta) \subset \text{End}(A)$, then the pair $(A, \iota)$ is called an abelian surface with real multiplication for $\sqrt{\Delta}$. The moduli space of principally polarized abelian surfaces with real multiplication for $\sqrt{\Delta}$ is called the Humbert surface $\mathcal{H}_\Delta$. The Humbert surface $\mathcal{H}_\Delta$ is a divisor in $\mathcal{A}_2$.

In our study, we focus on the case for the smallest discriminant $\Delta = 5$. In this case, the corresponding ring of integers is $\mathcal{O}_5 = (1, \frac{1 + \sqrt{5}}{2})$. This case is studied in detail in $\text{[H], [N1], [N2], [NS]}$ and $\text{[N3]}$. In this subsection, we survey these results.

Since $\Delta = 5 = 1^2 + 2^2$, the Humbert surface $\mathcal{H}_5$ is isomorphic to the symmetric Hilbert modular surface $(\mathbb{H} \times \mathbb{H})/(PSL(2, \mathcal{O}_5), \tau)$, where $\tau$ is the involution on $\mathbb{H} \times \mathbb{H}$ given by $(z_1, z_2) \mapsto (z_2, z_1)$. A compactification $(\overline{\mathbb{H}} \times \overline{\mathbb{H}})/(PSL(2, \mathcal{O}_5), \tau)$ is isomorphic to the weighted projective plane $\mathbb{P}(1 : 3 : 5) = \text{Proj}(\mathbb{C}[\mathfrak{A}, \mathfrak{B}, \mathfrak{C}])$, where $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ are Klein’s icosahedral invariants (for detail, see Hirzebruch $\text{[H]}$). In $\text{[N1]}$, the family $\mathcal{F}$ is the family $\{ S(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}) \}$ of $K3$ surfaces, where

$$S(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}) : z^2 = x^3 - 4(4y^3 - 5\mathfrak{A}y^2)x^2 + 20\mathfrak{B}y^3x + \mathfrak{C}y^4$$

(3.3)

is studied in detail. Here, $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C}) \in \mathbb{P}(1 : 3 : 5) - \{ (1 : 0 : 0) \}$. Applying the Torelli theorem for polarized $K3$ surfaces to the family $\mathcal{F}_{CD}$ and $\mathcal{F}$, we can study the period mapping for these families in detail. Especially, we obtained the following arithmetic results.
Theorem 3.2. (1) \([\text{[NS]}\) Theorem 4.4\) The Humbert surface \(\mathcal{H}_5\) is regarded as the divisor in \(\mathbb{P}(2 : 3 : 5 : 6) = \text{Proj}(\mathbb{C}[\alpha, \beta, \gamma, \delta])\) given by the defining equation
\[
(-\alpha^2 - \beta^2 + \delta)^2 - 4\alpha(\alpha\beta - \gamma)^2 = 0. \tag{3.4}
\]

(2) \([\text{[NS]}\) Theorem 3.6\) The point \((\alpha : \beta : \gamma : \delta) \in \mathbb{P}(2 : 3 : 5 : 6)\) satisfies the equation \((3.4)\) if and only if the point \((\alpha : \beta : \gamma : \delta)\) is in the image of the embedding \(\Psi_5: \mathbb{P}(1 : 3 : 5) \to \mathbb{P}(2 : 3 : 5 : 6)\) of varieties given by
\[
(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) \mapsto (\alpha : \beta : \gamma : \delta) = (\alpha_5(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) : \beta_5(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) : \gamma_5(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) : \delta_5(\mathfrak{A} : \mathfrak{B} : \mathfrak{C})),
\]
where
\[
\begin{align*}
\alpha_5(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) &= \frac{25}{36} \mathfrak{A}^2, \\
\beta_5(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) &= \frac{1}{2} \left( -\frac{125}{108} \mathfrak{A}^3 + \frac{5}{4} \mathfrak{B} \right), \\
\gamma_5(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) &= \frac{25}{32} \mathfrak{C}, \\
\delta_5(\mathfrak{A} : \mathfrak{B} : \mathfrak{C}) &= \frac{25}{64} \mathfrak{B}^2 - \frac{5}{96} \mathfrak{A} \mathfrak{C}.
\end{align*} \tag{3.5}
\]

The multivalued period mapping \(\Phi\) for \(\mathcal{F}\) has the form \(\Phi: \mathbb{P}(1 : 3 : 5) - \{(1 : 0 : 0)\} \to \mathbb{H} \times \mathbb{H}\). The inverse correspondence of \(\Phi\) has an explicit expression by Hilbert modular functions as follows (for detail, see \([\text{[NS]}\) ). Let us consider the holomorphic mapping \(\mu_5: \mathbb{H} \times \mathbb{H} \to \mathcal{S}_2\) given by
\[
(z_1, z_2) \mapsto \frac{1}{2\sqrt{5}} \begin{pmatrix} (1 + \sqrt{5})z_1 - (1 - \sqrt{5})z_2 & \sqrt{2}(z_1 - z_2) \\
2(z_1 - z_2) & (1 + \sqrt{5})z_1 + (1 + \sqrt{5})z_2 \end{pmatrix}. \tag{3.6}
\]
In fact, \(\mu_5\) gives a Hilbert modular embedding for \(\sqrt{5}\) (see the following diagram).

\[
\begin{array}{ccc}
\mathbb{H} \times \mathbb{H} & \xrightarrow{\mu_5} & \mathcal{S}_2 \\
(\text{PSL}(2, \mathbb{O}_5), \tau) \downarrow & & \downarrow \text{Sp}(4, \mathbb{Z}) \\
\mathbb{H} \times \mathbb{H} & \xrightarrow{\mu_5} & \mathcal{S}_2
\end{array}
\]

We note that \(\mu_5\) of \((3.6)\) gives a parametrization of the surface
\[
N_5 = \left\{ \begin{pmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \end{pmatrix} \in \mathcal{S}_2 \mid \tau_1 + \tau_2 + \tau_3 = 0 \right\}. \tag{3.7}
\]
It is known that the image \(pr(N_5)\), where \(pr: \mathcal{S}_2 \to \mathcal{S}_2/\text{Sp}(4, \mathbb{Z}) = \mathcal{A}_2\) is the canonical projection, coincides with the Humbert surface \(\mathcal{H}_5\). For \(\Omega \in \mathcal{S}_2\) and \(a, b \in \{0, 1\}^2\) with \(\text{tr} ab \equiv 0 \pmod{2}\), set
\[
\vartheta(\Omega; a, b) = \sum_{g \in \mathbb{Z}^2} \exp \left( \pi \sqrt{-1} \left( (g + \frac{1}{2} a) \Omega (g + \frac{1}{2} a) + \text{tr} gb \right) \right).
\]
For \(j \in \{0, 1, \ldots, 9\}\), we set
\[
\theta_j(z_1, z_2) = \vartheta(\mu_5(z_1, z_2); a, b),
\]
where the correspondence between \(j\) and \((a, b)\) is given by Table 3. Let \(a \in \mathbb{Z}\) and \(j_1, \ldots, j_r \in \{0, \ldots, 9\}\). We set \(\theta_{j_1}^{a_1} \cdots \theta_{j_r}^{a_r} = \theta_{j_1}^{a_1} \cdots \theta_{j_r}^{a_r}\). The following \(g_2\) \((s_6, s_{10}, s_{15}, \text{resp.})\) is a symmetric Hilbert modular form of weight 2 \((6, 10, 15, \text{resp.})\) for \(\mathbb{Q}(\sqrt{5})\) (see Müller \([\text{[M]}\)):
\[
\begin{align*}
g_2 &= \theta_{0145} - \theta_{1279} - \theta_{3478} + \theta_{0268} + \theta_{3569}, \\
s_6 &= 2^{-8} \left( \theta_{012478}^2 + \theta_{012569}^2 + \theta_{034568}^2 + \theta_{236789}^2 + \theta_{134579}^2 \right), \\
s_{10} &= s_6^2 = 2^{-12} \left( \theta_{0123456789}^2 \right). \tag{3.8}
\end{align*}
\]
Theorem 3.4. ([N2] Section 2) Set \( \chi \). We say that a
\[ \chi(\nu) \]
involution \( \nu \) induces a Hodge isometry \( \ell^* \nu \). Let \( S \) be an algebraic
\( S \rightarrow S' \) be the minimal resolution, \( S \) is a K3 surface. We have the rational
quotient mapping \( \chi : S \rightarrow S' \).
Definition 3.1. We say that a K3 surface \( S \) admits a Shioda-Inose structure if there exists a symplectic
involution \( \iota \in \text{Aut}(X) \) with the rational quotient mapping \( \chi : S \rightarrow S' \) such that \( S' \) is a Kummer surface
and \( \chi_* \) induces a Hodge isometry \( \text{Tr}(S)(2) \cong \text{Tr}(S') \).

Theorem 3.3. ([N1] Theorem 4.1) Setting \((X,Y) = \left( \frac{\mathfrak{B}}{\mathfrak{A}^1}, \frac{\mathfrak{C}}{\mathfrak{A}^2} \right)\), the inverse correspondence \((z_1, z_2) \mapsto (X(z_1, z_2), Y(z_1, z_2))\) of the period mapping \( \Phi \) for \( F \) has the following theta expression
\[ X(z_1, z_2) = 2^5 \cdot 5^2 \cdot \frac{g_4(z_1, z_2)}{g_2(z_1, z_2)}, \quad Y(z_1, z_2) = 2^{10} \cdot 5^5 \cdot \frac{g_4(z_1, z_2)}{g_2(z_1, z_2)}. \] (3.9)
Moreover, \( X \) and \( Y \) give a system of generators of the field of symmetric Hilbert modular functions for \( \mathbb{Q}(\sqrt{5}) \).

3.3 Kummer surfaces coming from the Shioda-Inose structure of \( F \)
Let \( A \) be an abelian variety. The minimal resolution \( \text{Kum}(A) \) of the quotient variety \( A/\{id_A, -id_A\} \) is called the Kummer surface. We note that \( \text{Kum}(A) \) is an algebraic K3 surface.

Let \( S \) be an algebraic K3 surface. Let \( \omega \) be the unique holomorphic 2-form on \( S \) up to a constant factor. If an involution \( \nu \) on \( S \) satisfies \( \nu^* \omega = \omega \), we call \( \nu \) a symplectic involution. Set \( G = \langle \nu, \text{id} \rangle \subset \text{Aut}(S) \). Set \( S' = S/G \). Letting \( S' \rightarrow \tilde{S}' \) be the minimal resolution, \( S \) is a K3 surface. We have the rational
quotient mapping \( \chi : S \rightarrow S' \).

Theorem 3.4. ([N2] Section 2) Set \((X,Y) = \left( \frac{\mathfrak{B}}{\mathfrak{A}^1}, \frac{\mathfrak{C}}{\mathfrak{A}^2} \right)\). If a principally polarized abelian surface \((A, \Theta)\) has real multiplication for \( \sqrt{5} \), then there exists a Hodge isometry \( \text{Tr}(S)(2) \cong \text{Tr}(S') \).

The corresponding Kummer surface \( K(X,Y) \) coming from the Shioda-Inose structure with \( \chi : S(X,Y) \rightarrow K(X,Y) \) is given by the simple equation
\[ K(X,Y) : v^2 = (u^2 - 2y^5)(u - (5y^2 - 10XY + Y)). \] (3.11)

3.4 The field of moduli of the principally polarized abelian surface with real multiplication for \( \sqrt{5} \)

Theorem 3.5. Let \((A, \Theta) \) be a principally polarized abelian surface with real multiplication for \( \sqrt{5} \). Let \( S(X,Y) \) be the corresponding K3 surface under \([5,10]\). Then, the field of moduli of \((A, \Theta) \) is given by \( \mathbb{Q}(X,Y) \).

Proof. By Proposition [3.3], the field of moduli \((A, \Theta) \) is given by \( \mathbb{Q}(m_1, m_2, m_3) \). From Theorem [3.1] and \([1.9]\) we have
\[ \mathbb{Q}[J_2, J_4, J_6, J_{10}] = \mathbb{Q}[\alpha, \beta, \gamma, \delta]. \] (3.12)
Moreover, if \((A, \Theta) \) has real multiplication for \( \sqrt{5} \), by virtue of the embedding \( \Psi_5 \) over \( \mathbb{Q} \) in Theorem [3.2] we have
\[ \mathbb{Q}[\alpha, \beta, \gamma, \delta] = \mathbb{Q}[\mathfrak{A}, \mathfrak{B}, \mathfrak{C}]. \] (3.13)
Hence, from (1.10), (3.12) and (3.13), if \((A, \Theta)\) has real multiplication for \(\sqrt{5}\), we have \(\mathbb{Q}(m_1, m_2, m_3) = \mathbb{Q}(X, Y)\).

**Theorem 3.6.** Let \((A, \Theta)\) be a principally polarized abelian surface with real multiplication for \(\sqrt{5}\). Let \(S(X, Y)\) be the corresponding K3 surface via the Shioda-Inose structure \((3.10)\). For a generic \((X, Y)\), the defining equation \((3.11)\) of the Kummer surface gives an explicit model of the Kummer variety defined over the field of moduli of \((A, \Theta)\) in the sense of Definition (1.6) and Proposition (1.6).

**Proof.** For general \((X, Y)\), the corresponding principally polarized abelian surface under \((3.10)\) satisfies \(\text{End}(A) = \mathcal{O}_5\). Remark that if an units \(u\) in the ring \(\mathcal{O}_5\) is a torsion element, then \(u = 1\) or \(u = -1\). So, from Proposition (3.12) (3), we have \(\text{Aut}(A, \Theta) = \{\text{id}_A, -\text{id}_A\}\). Hence, the Kummer surface \((3.11)\) coming from the Shioda-Inose structure of \(S(X, Y)\) coincides with the Kummer variety in the sense of Definition (1.6).

Let us apply the argument in Section 1.2 to our case. Let \(K\) be an imaginary quadratic extension over \(K_0 = \mathbb{Q}(\sqrt{5})\). Let \((K, \{\text{id}, \varphi\})\) be a CM-type of \(K\). We set \(u(\alpha) = \left(\begin{array}{cc} \alpha & 0 \\ 0 & \alpha^2 \end{array}\right) \in \mathbb{C}^2\) for \(\alpha \in K\). Take \(a \in I_K\). For a system \(\{\alpha_1, \ldots, \alpha_4\}\) of the basis of \(a\), the \(2 \times 4\) matrix \(u(\alpha_1) \cdots u(\alpha_4)\) gives a lattice \(\Lambda(a)\) and we have a complex torus \(\mathbb{C}^2/\Lambda(a) = A(a)\). If the alternating form \(E\) of \((1.1)\) given by \(\zeta \in K\) gives a polarization \(\Theta\) on \(A(a)\), \((A(a), \Theta)\) is a polarized abelian surface and \((A(a), \iota)\) is principal in the sense of Definition (1.2) (see Proposition (1.1)).

**Theorem 3.7.** In the above notation, suppose \(\Theta\) is a principal polarization.

1. There exists a basis of \(\mathbb{C}^2\) such that the matrix \((u(\alpha_1) \cdots u(\alpha_4))\) is expressed in the form
   
   \[
   (u(\alpha_1)u(\alpha_2)u(\alpha_3)u(\alpha_4)) = \left(\begin{array}{cccc} \tau_1 & \tau_2 & 1 & 0 \\ \tau_2 & \tau_3 & 0 & 1 \\ 1 & 0 & \tau_1 & \tau_2 \\ 0 & 1 & \tau_2 & \tau_3 \end{array}\right) \in \mathfrak{S}_2, \quad \tau_1 = \tau_2 + \tau_3. \tag{3.14}
   
   2. Put
   
   \[
   z_1 = \frac{\tau_2 + \sqrt{5}\tau_2 + 2\tau_3}{2}, \quad z_2 = \frac{\tau_2 - \sqrt{5}\tau_2 + 2\tau_3}{2}. \tag{3.15}
   
   Then, the field of moduli of \((A, \Theta)\) is given by \(\mathbb{Q}(X(z_1^0, z_2^0), Y(z_1^0, z_2^0))\), where
   
   \[
   X(z_1^0, z_2^0) = 2^5 \cdot 5^2 \cdot g_2(z_1^0, z_2^0), \quad Y(z_1^0, z_2^0) = 2^{10} \cdot 5^5 \cdot g_2(z_1^0, z_2^0). \tag{3.16}
   
   3. Let \(K^*\) be the reflex of the CM-type \((K, \{\text{id}, \varphi\})\). Then, the field \(K^*(\mathbb{Q}(X(z_1^0, z_2^0), Y(z_1^0, z_2^0)))\) is the unramified class field over \(K^*\) for the group \(H_0\) of \((2.4)\).

**Proof.** (1) Under the assumption, \((A, \Theta)\) is a principally polarized abelian surface satisfying \(\mathcal{O}_K = \text{End}(A)\). Since \(\mathbb{Q}(\sqrt{5})\) is a subfield of \(K\), we have \(\mathcal{O}_5 \subset \text{End}(A)\). Namely, \((A, \Theta)\) gives a principally polarized abelian surface with real multiplication for \(\sqrt{5}\).

Recall that the Humbert surface \(H_5\) is given by \(\text{pr}(N_5)\), where \(N_5\) is given in \((3.7)\). This implies that there exist a basis of \(\mathbb{C}^2\) such that the \(2 \times 4\) period matrix \((u(\alpha_1)u(\alpha_2)u(\alpha_3)u(\alpha_4))\) of \(A(a)\) is represented in the form \((3.14)\).

(2) Since \((A, \Theta)\) is a principal polarized abelian surface with real multiplication for \(\sqrt{5}\), there exists a K3 surface \(S(X, Y)\) via the correspondence \((3.10)\). According to Theorem (3.5) the field of moduli of \((A, \Theta)\) is given by \(\mathbb{Q}(X, Y)\).

By virtue of the inverse correspondence of the period mapping \(\mathcal{F} = \{S(X, Y)\}\), we obtain the parameter \((X, Y)\) of \(S(X, Y)\) from the period matrix of \((3.14)\). From \(\left(\begin{array}{cc} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{array}\right) \in N_5\) of \((3.14)\), we have the corresponding \((z_1^0, z_2^0) \in \mathbb{H} \times \mathbb{H}\) given by \((3.15)\) via the modular embedding \(\mu_5\) of \((3.6)\). Using the theta expression in Theorem (3.3) we obtain the parameter \((X, Y)\) of \(S(X, Y)\) corresponding to \((A, \Theta)\).

(3) From Theorem (3.1) and the above (2), the assertion of (3) follows.
4 Cyclic quartic CM-fields

In this section, we give detailed properties of quartic CM-fields $K$ such that $K/\mathbb{Q}$ is a cyclic extension. According to Example 1.1, the corresponding abelian surfaces $A$ is primitive and the reflex $K^*$ coincides with $K$. We can investigate class fields over such a CM-field $K$ in detail.

4.1 Integral basis of cyclic quartic fields

Any cyclic quartic field over $\mathbb{Q}$ is given by

$$K = \mathbb{Q}\left(\sqrt{A(\Delta + B\sqrt{\Delta})}\right),$$

where $A, B, C, \Delta \in \mathbb{Z}$ and

$$\begin{cases} A \equiv 1 \pmod{2}, & \text{squarefree} \\ \Delta = B^2 + C^2, & (B > 0, C > 0), \text{squarefree} \\ \text{GCD}(A, \Delta) = 1. \end{cases}$$

(4.2)

Such fields are studied in detail in [HSW], [HW], [HHRWH] or [HHRW]. To state their results, let us consider the five cases (i),(ii),(iii),(iv) and (v) (see Table 4).

| Case | Conditions |
|------|-------------|
| (i)  | $\Delta \equiv 0 \pmod{2}$ |
| (ii) | $\Delta \equiv B \equiv 1 \pmod{2}$ |
| (iii)| $\Delta \equiv 1 \pmod{2}$, $B \equiv 0 \pmod{2}$, $A + B \equiv 3 \pmod{4}$ |
| (iv) | $\Delta \equiv 1 \pmod{2}$, $B \equiv 0 \pmod{2}$, $A + B \equiv 1 \pmod{4}$, $A \equiv C \pmod{4}$ |
| (v)  | $\Delta \equiv 1 \pmod{2}$, $B \equiv 0 \pmod{2}$, $A + B \equiv 1 \pmod{4}$, $A \equiv -C \pmod{4}$ |

Table 4: The five cases (i),(ii),(iii),(iv) and (v)

Proposition 4.1. ([HSW], [HW], [HHRWH], [HHRW]) Let $K$ be a cyclic quartic field of (4.1).

(1) The conductor of the field $K$ is given by

$$2^l \Delta |A|,$$

where

$$l = \begin{cases} 3 & (\text{case (i) and (ii)}), \\ 2 & (\text{case (iii)}), \\ 0 & (\text{case (iv) and (v)}). \end{cases}$$

(2) The discriminant of the field $K$ is given by

$$2^e \Delta^3 A^2,$$

where

$$e = \begin{cases} 8 & (\text{case (i)}), \\ 6 & (\text{case (ii)}), \\ 4 & (\text{case (iii)}), \\ 0 & (\text{case (iv) and (v)}). \end{cases}$$

(3) Set

$$\alpha = \sqrt{A(\Delta + B\sqrt{\Delta})}, \quad \beta = \sqrt{A(\Delta - B\sqrt{\Delta})}.$$  

A system of basis of the ring of integers $\mathcal{O}_K$ of $K$ is given by Table 5.
| Case | Basis |
|------|-------|
| (i)  | $1, \sqrt{\Delta}, \alpha, \beta$ |
| (ii) | $1, \frac{1 + \sqrt{\Delta}}{2}, \alpha, \beta$ |
| (iii)| $1, \frac{1 + \sqrt{\Delta}}{2}, \alpha + \beta, \alpha - \beta$ |
| (iv) | $1, \frac{1 + \sqrt{\Delta}}{2}, \frac{1 + \sqrt{\Delta} + \alpha + \beta}{4}, \frac{1 - \sqrt{\Delta} + \alpha - \beta}{4}$ |
| (v)  | $1, \frac{1 + \sqrt{\Delta}}{2}, \frac{1 + \sqrt{\Delta} + \alpha - \beta}{4}, \frac{1 - \sqrt{\Delta} + \alpha + \beta}{4}$ |

Table 5: The basis of the ring of integers for the cases (i),(ii),(iii),(iv) and (v)

### 4.2 Complex tori from cyclic quartic CM-fields

A cyclic quartic field $K$ of (4.1) gives a primitive CM-type (see Example 1.1), if $A < 0$. We call such fields cyclic quartic CM-fields.

In Section 4.1, we obtain a system of basis $\alpha_1, \ldots, \alpha_4$ of the ring $\mathcal{O}_K$ of integers of $K$. As Section 1.2, we obtain the lattice $\Lambda(\mathcal{O}_K)$ of $\mathbb{C}^2$ from $\alpha_1, \ldots, \alpha_4$. Then, we have a complex torus $\mathbb{C}^2/\Lambda(\mathcal{O}_K)$. If the Riemann form $E$ of (1.1) on $\Lambda(\mathcal{O}_K) \times \Lambda(\mathcal{O}_K)$ is $\mathbb{Z}$-valued, the complex torus $\mathbb{C}^2/\Lambda(\mathcal{O}_K)$ becomes a principal abelian surface of type $K$ (see Definition 1.2 and Proposition 1.1).

Recall that the Riemann form of (1.1) depends on $\zeta \in K$ such that $K = K_0(\zeta)$, $-\zeta^2 \in K_0$ is totally positive and $\text{Im}(\zeta^2) > 0$. In our study for the field of (4.1), we canonically take

$$\zeta = \frac{\sqrt{A(\Delta + B\sqrt{\Delta})}}{\kappa}, \quad (\kappa \in \mathbb{Q}). \quad (4.3)$$

**Theorem 4.1.** For a cyclic field $K$ of (4.1) and a number $\zeta$ of (4.3), the matrix $(E(\alpha_j, \alpha_k))_{j,k=1,\ldots,4}$ coming from the alternating Riemann form $E$ of (1.1) is given as Table 6.

**Proof.** Using Proposition 4.1 (3), we can prove this by a straightforward calculation. 

### 4.3 The group $H_0$ for cyclic quartic CM-fields

The group $H_0$ of (2.4) for the class fields in Section 2.2 is very complicated. So, in this subsection, we investigate the structure of $H_0$ for cyclic quartic CM-fields.

**Theorem 4.2.** Let $K$ be a quartic CM-field. Suppose the ideal class group is given by

$$I_K/P_K \simeq (\mathbb{Z}/2\mathbb{Z})^r \oplus G_1,$$

where $G_1$ does not contain any 2-torsion element. If $K$ satisfies the conditions (C1),(C2) and (C3) in Lemma 2.4 then $[I_K : H_0] = \frac{h_K}{2^r}$ holds.

**Proof.** Since $K/\mathbb{Q}$ is cyclic, we have

$$\text{Gal}(K/\mathbb{Q}) = \{id, \sigma, \sigma^2 = \rho, \sigma^3\}, \quad (4.4)$$

where $(K, \{id, \sigma\})$ gives a CM-type of $K$ (see Example 1.1). Recall that $(K, \{id, \sigma\})$ is primitive. The corresponding reflex is given by $(K^*, \{id, \sigma^{-1}\})$. The mapping $\Phi^*$ of (2.3) has the form

$$a \mapsto a^{\Phi^*} = aa^{\sigma^{-1}}. \quad (4.5)$$
Due to Lemma 2.4 (3), the group $\frac{K}{K}$ is a proper subgroup of $0$. They are subgroups of the ideal class group $I_K/P_K$.

From (2.5), we have

$$\ker(\tilde{\Phi}) = 1.$$ (4.10)

Then, $\overline{H_0}$ is a proper subgroup of $\overline{I(K/K_0)}$. We can take $s \in \mathbb{Z}$ $(0 \leq s < r)$ and $[b_1], \ldots, [b_r]$ of $\overline{H_0}$ such that

$$\overline{H_0} = T_{s+1} \oplus \cdots \oplus T_r.$$ (4.11)
and
\[ J_0 = T_1 \oplus \cdots \oplus T_s \cong I(K/K_0)/\mathcal{P}_0, \tag{4.11} \]
where \( T_j = \langle [b_j] \rangle \cong \mathbb{Z}/2\mathbb{Z} \) for \( j = 1, \ldots, r \). So, \( I(K/K_0) = J_0 \oplus \mathcal{P}_0 \) holds. For \( \sigma \) of (4.4) and \( j \in \{1, \ldots, s\} \), let us show that
\[ [b_j]^{\sigma} = [b_k] \quad (k \in \{1, \ldots, s\}, k \neq j). \tag{4.12} \]

First, from (4.9), we note that
\[ [b_j]^{\sigma} \in J_0. \tag{4.13} \]
Moreover, if \( [b_j]^{\sigma} \in T_j \), we have \( [b_j]^{\sigma} = [b_j]^{-1} \) from (4.8). Hence, by (4.9) and (4.13), it follows that \( [b_j]^{\sigma} \notin T_j \). So, (4.12) holds. We also have
\[ [b_{j_1}]^{\sigma} \neq [b_{j_2}]^{\sigma} \quad (j_1, j_2 \in \{1, \ldots, s\}, j_1 \neq j_2). \tag{4.14} \]

From (4.8) and (4.14), \( \sigma \) of (4.4) determines a permutation \( \tilde{\sigma} \) of \( \{1, \ldots, s\} \) given by \( \tilde{\sigma}(j) = k \) such that \( [b_j]^{\sigma} = [b_k] \) \( (j \neq k) \). Put \( [b] = [b_1 \cdots b_s] \in J_0 \). We remark that \( [b] \neq id_{J_0} \). Then, we have
\[ [b]^{\sigma} = [b_{\tilde{\sigma}(1)} \cdots b_{\tilde{\sigma}(s)}] = [b_1 \cdots b_s] = [b]. \]

From (4.8) and (4.9), we have \( [b] \in \mathcal{P}_0 \). This is a contradiction.

Hence, the assumption (4.10) is not true. Therefore, we have \( I(K/K_0) = H_0 \). According to Lemma 2.4 (3), we have \([I_K : H_0] = \frac{h_K}{2^n}\). \hfill \Box

**Corollary 4.1.** Let \( K \) be a quartic CM-field as in Theorem 4.2. The unramified class field \( k_0/K \) corresponding to the group \( H_0 \) of (2.4) is an extension over \( K \) of degree \( \frac{h_K}{2^n} \).

**Proof.** It is clear because we have Theorem 2.1 and Theorem 4.2 \hfill \Box

**Remark 4.1.** For a quartic CM-field \( K \) with the condition (C1), (C2) and (C3), the irreducibility of the class equation (see Remark 2.1 for the case of CM-type \( (K, \{id, \varphi\}) \) holds if and only if the ideal class group \( I_K/P_K \) does not contain any two torsion element.

### 4.4 Cyclic quartic CM-fields with the maximal real subfield \( \mathbb{Q}(\sqrt{5}) \)

Letting \( K_0 \) be a real quadratic field, the group \( E_{K_0} \) of units in \( \mathfrak{O}_{K_0} \) is given by the direct product of \( \{id, -id\} \) and \( \{\varepsilon^m | n \in \mathbb{Z}\} \). Here, \( \varepsilon_0 \) is called a fundamental unit of \( K_0 \). If \( N_{K/K_0}(\varepsilon_0) = -1 \), then the group \( E_{K_0} \) of totally positive units in \( \mathfrak{O}_{K_0} \) is given by \( \{\varepsilon_0^m | n \in \mathbb{Z}\} \) and we have \( E_{K_0} = E_{K_0}^2 \).

If the discriminant \( \Delta \) of the real quadratic field \( K_0 = \mathbb{Q}(\sqrt{\Delta}) \) is smallest (namely, \( \Delta = 5 \)), we can obtain a precise result than cases for other discriminant. The class number of \( \mathbb{Q}(\sqrt{5}) \) is equal to 1. A fundamental unit of \( \mathbb{Q}(\sqrt{5}) \) is given by \( \frac{1 - \sqrt{5}}{2} \), that satisfies \( N_{\mathbb{Q}(\sqrt{5})/\mathbb{Q}}(\frac{1 - \sqrt{5}}{2}) = -1 \). Therefore, if \( K \) is an imaginary quadratic extension over \( \mathbb{Q}(\sqrt{5}) \), then \( K \) satisfies the conditions (C1) and (C2) in Lemma 2.4. Moreover, if \( K/\mathbb{Q} \) is cyclic and \( K/\mathbb{Q}(\sqrt{5}) \) is totally imaginary quadratic, it follows that
\[ (B, C) = (1, 2) \text{ or } (2, 1) \tag{4.15} \]
(see (4.2)).

**Theorem 4.3.** Let \( K \) of (4.7) be a totally imaginary extension of \( \mathbb{Q}(\sqrt{5}) \) such that \( K/\mathbb{Q} \) is cyclic. Set \( \alpha = \sqrt{-A(N + B\sqrt{\Delta})} \). The Riemann form \( E \) of (4.1) given by \( \zeta \) in Table 7 gives a principal polarization on the abelian variety \( A(\Omega_K) \).
Table 7: The number $\zeta$ which gives a principal polarization for $\Delta = 5$

| Case | $\zeta$ of (4.3) |
|------|------------------|
| (ii) | $\alpha - 4\Delta A$ |
| (iii)| $\alpha - 2\Delta A$ |
| (iv) | $\alpha - \Delta A$ |
| (v)  | $\alpha - \Delta A$ |

Proof. If the alternating Riemann form $E$ of (1.1) is $\mathbb{Z}$-valued, the determinant of the matrix $M_j$ in Table 6 is a perfect square number. The positive root of this determinant is called the Pfaffian of $E$. If the Pfaffian of $E$ is equal to 1, then the Riemann form $E$ induces a principal polarization on the complex torus $\mathbb{C}^2/\Lambda(\mathcal{O}_K) = A(\mathcal{O}_K)$. Hence, it is sufficient to see that the number $\zeta$ in Table 7 gives the $\mathbb{Z}$-valued matrix $M_j$ in Table 6 and the determinant of $M_j$ is equal to 1.

(ii) In this case, we consider the matrix $M_2$ in Table 6. Putting $(B,C) = (1,2)$ and $\kappa = -4\Delta A$, we have

$$M_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ -1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$  

(iii) In this case, we consider the matrix $M_3$ in Table 6. Putting $(B,C) = (2,1)$ and $\kappa = -2\Delta A$, we have

$$M_3 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ -1 & -2 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix}.$$  

(iv) In this case, we consider the matrix $M_4$ in Table 6. If we consider the case (iv), putting $(B,C) = (1,2)$ and $\kappa = -\Delta A$, we have

$$M_4 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 1 \\ -1 & -2 & 0 & 1 \\ -1 & -1 & -1 & 0 \end{pmatrix}.$$  

(v) In this case, we consider the matrix $M_5$ in Table 6. If we consider the case (v), putting $(B,C) = (1,2)$ and $\kappa = -\Delta A$, we have

$$M_5 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ -1 & -1 & 0 & 1 \\ -1 & -2 & -1 & 0 \end{pmatrix}.$$

Theorem 4.3 implies that we only need principally polarized abelian surfaces to construct unramified class fields in Section 2.2 for $\Delta = 5$. Hence, Theorem 3.7 is available for every cyclic quartic CM-field $K$ for $\Delta = 5$.

Remark 4.2. The construction of class fields in Section 2.2 for $\Delta = 5$ is simpler than the cases of $\Delta > 5$. When $\Delta > 5$, to construct class fields in Section 2.2 over cyclic quartic CM-fields, we need abelian surfaces which is not principally polarized.

Hardy, Hudson, Richmann, Williams and Hiltz [HHRWH], listed imaginary cyclic quartic fields $K$ with small conductor. Especially, cyclic quartic CM-fields $K$ for $\Delta = 5$ with small conductor is listed in Table 8.
| CM-field $K$                      | Class number | Conductor |
|----------------------------------|--------------|-----------|
| $\mathbb{Q}(\sqrt{-(5 + 2\sqrt{5})})$ | 1            | 5         |
| $\mathbb{Q}(\sqrt{-(5 + \sqrt{5})})$ | 2            | 40        |
| $\mathbb{Q}(\sqrt{-3(5 + 2\sqrt{5})})$ | 4            | 60        |
| $\mathbb{Q}(\sqrt{-13(5 + 2\sqrt{5})})$ | 4            | 65        |
| $\mathbb{Q}(\sqrt{-17(5 + 2\sqrt{5})})$ | 4            | 85        |
| $\mathbb{Q}(\sqrt{-21(5 + 2\sqrt{5})})$ | 4            | 105       |
| $\mathbb{Q}(\sqrt{-3(5 + 2\sqrt{5})})$ | 4            | 120       |
| $\mathbb{Q}(\sqrt{-7(5 + 2\sqrt{5})})$ | 4            | 140       |
| $\mathbb{Q}(\sqrt{-29(5 + 2\sqrt{5})})$ | 4            | 145       |
| $\mathbb{Q}(\sqrt{-33(5 + 2\sqrt{5})})$ | 8            | 165       |
| $\mathbb{Q}(\sqrt{-37(5 + 2\sqrt{5})})$ | 10           | 185       |

Table 8: Cyclic quartic CM-fields for $\Delta = 5$ with small conductor

5 Examples

In this section, we apply our results of the icosahedral invariants $X$ and $Y$ to concrete cyclic CM-fields $K$. We construct unramified class fields $k_0$ in Section 2.2 over cyclic quartic CM-fields $K$ with maximal real subfield $\mathbb{Q}(\sqrt{5})$. According to Theorem 3.7 and Theorem 4.3, such class fields are given by the special values of $X$ and $Y$ of (3.9).

In Section 5.1, we shall give an example such that $[k_0 : K] = 1$. In Section 5.2, we shall give an example such that $[k_0 : K] = 5 > 1$.

5.1 Case $K = \mathbb{Q}(\sqrt{-(5 + \sqrt{5})})$

Let us take $K = \mathbb{Q}(\sqrt{-(5 + \sqrt{5})})$. Then, we consider the case (ii) in Table 4.

According to Theorem 4.3, taking $\zeta = \frac{\sqrt{-(5 + \sqrt{5})}}{20}$, we have the Riemann form $E$ of (1.1). Next, we take a system $\{\alpha_1, \cdots, \alpha_4\}$ of basis as in Table 5. Then, we have the lattice $\Lambda = \langle u(\alpha_1), \cdots, u(\alpha_4) \rangle$ of $\mathbb{C}^2$, where

$$u(\alpha_1) = \frac{1}{1}, \quad u(\alpha_2) = \frac{1 + \sqrt{5}}{2},$$

$$u(\alpha_3) = \frac{\sqrt{-(5 + \sqrt{5})}}{\sqrt{-(5 + \sqrt{5})}}, \quad u(\alpha_4) = \frac{-\sqrt{-(5 - \sqrt{5})}}{-\sqrt{-(5 + \sqrt{5})}}.$$<ref

Putting $\lambda_1 = u(\alpha_1), \lambda_2 = u(\alpha_2), \lambda_3 = u(\alpha_3) - u(\alpha_4)$ and $\lambda_4 = u(\alpha_4)$, we have

$$E(\lambda_j, \lambda_k) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$<ref

We set $(M_1M_2) = (\lambda_1 \lambda_2 \lambda_3 \lambda_4)$, where $M_1, M_2 \in M(2, \mathbb{C})$. So, we have $\Omega = -M_2^{-1}M_1 \in \mathfrak{S}_2$.

Set $\begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in Sp(4, \mathbb{Z})$. 19
Remark 5.1. To Theorem 3.7, using the pair of modular functions \((X, Y)\) corresponding to the group \(\mathbb{H} \times \mathbb{H}\) as in (3.19). According to Theorem (3.7) using the pair of modular functions \((X, Y)\) of (3.9), the unramified class field \(k_0\) over \(K = \mathbb{Q}(\sqrt{-37(5 + 2\sqrt{5})})\) corresponding to the group \(H_0\) of (2.1) is given by \(k_0 = K(X(z_1^0, z_2^0), Y(z_1^0, z_2^0))\).

Due to Table 8, the class number \(h_K\) of \(K = \mathbb{Q}(\sqrt{-37(5 + 2\sqrt{5})})\) is equal to 2. Therefore, \(I_K/P_K\) is isomorphic to \((\mathbb{Z}/2\mathbb{Z})\). Applying Theorem (2.2) to this case, we have \([k_0 : K] = 1\). This case does not give a non-trivial class-field over the CM-field \(K\).

Remark 5.1. In fact, Murabayashi and Umegaki \[\text{[11]}\] proved that the field of moduli of principally polarized abelian surface corresponding to \(K = \mathbb{Q}(\sqrt{-37(5 + 2\sqrt{5})})\) coincides with \(\mathbb{Q}\). So, our result does not contradict to their result.

5.2 Case \(K = \mathbb{Q}(\sqrt{-37(5 + 2\sqrt{5})})\)

Let us take \(K = \mathbb{Q}(\sqrt{-37(5 + 2\sqrt{5})})\). Then, we consider the case (v) in Table 3.

According to Theorem (4.3) taking \(\zeta = \frac{-37(5 + 2\sqrt{5})}{185}\), we have the Riemann form \(E\) of (4.1). Next, we take a system \(\{\alpha_1, \cdots, \alpha_4\}\) of basis as in Table 5. Then, we have the lattice \(\Lambda = \langle u(\alpha_1), \cdots, u(\alpha_4) \rangle\) of \(\mathbb{C}^2\), where

\[
\left\{
\begin{array}{l}
u(\alpha_1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad u(\alpha_2) = \begin{pmatrix} 1 + \sqrt{5} \\ 1 - \sqrt{5} \end{pmatrix} \\ u(\alpha_3) = \frac{1}{1 + \sqrt{5}} \begin{pmatrix} 1 + \sqrt{5} + \sqrt{-37(5 + 2\sqrt{5})} & \sqrt{-37(5 - 2\sqrt{5})} \\ 1 - \sqrt{5} + \sqrt{-37(5 - 2\sqrt{5})} & \sqrt{-37(5 + 2\sqrt{5})} \end{pmatrix} \\ u(\alpha_4) = \frac{1}{1 + \sqrt{5}} \begin{pmatrix} 1 - \sqrt{5} + \sqrt{-37(5 + 2\sqrt{5})} & \sqrt{-37(5 - 2\sqrt{5})} \\ 1 + \sqrt{5} + \sqrt{-37(5 - 2\sqrt{5})} & \sqrt{-37(5 + 2\sqrt{5})} \end{pmatrix}
\end{array}
\right.
\]

Putting \(\lambda_1 = u(\alpha_1), \lambda_2 = u(\alpha_2), \lambda_3 = 2u(\alpha_3) - u(\alpha_4)\) and \(\lambda_4 = u(\alpha_4) - u(\alpha_3) + u(\alpha_1)\), we have

\[
E(\lambda_j, \lambda_k) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.
\]

We set \((M_1M_2) = (\lambda_1 \lambda_2 \lambda_3 \lambda_4)\), where \(M_1, M_2 \in M(2, \mathbb{C})\). So, we have \(\Omega = -M_2^{-1}M_1 \in \mathfrak{S}_2\).

Set
\[
\begin{pmatrix} A_0 & B_0 \\ C_0 & D_0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \in Sp(4, \mathbb{Z}).
\]
Then, we have \(\tilde{\Omega} = (A_0 \Omega + B_0)(C_0 \Omega + D_0)^{-1} = \begin{pmatrix} \tau_1 & \tau_2 \\ \tau_2 & \tau_3 \end{pmatrix}\), where

\[
\left\{
\begin{array}{l}
\tau_1 = \sqrt{-1} \sqrt{\frac{37}{10}(5 + \sqrt{5})}, \\
\tau_2 = \frac{3}{2} - \frac{1}{2} \sqrt{-1} \sqrt{185 - \frac{74}{5}}, \\
\tau_3 = -\frac{3}{2} + \frac{1}{2} \sqrt{-1} \sqrt{481 + \frac{74}{5}}.
\end{array}
\right.
\]
This satisfies $\tau_1 - \tau_2 - \tau_3 = 0$. So, from $\tau_2$ and $\tau_3$, we have $(z_1^0, z_2^0) \in \mathbb{H} \times \mathbb{H}$ as in (3.15). According to Theorem 3.7 using the pair of modular functions $(X, Y)$ of (3.9), the unramified class field $k_0$ over $K = \mathbb{Q}(\sqrt{-37(5 + 2\sqrt{5})})$ corresponding to the group $H_0$ of (2.4) is given by $k_0 = K(X(z_1^0, z_2^0), Y(z_1^0, z_2^0))$.

From Table 8, the class number $h_K$ of $K = \mathbb{Q}(\sqrt{-37(5 + 2\sqrt{5})})$ is equal to 10. Therefore, $I_K/P_K$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/5\mathbb{Z})$. Applying Theorem 4.2 to this case, we have $[k_0 : K] = 5$. Namely, this case gives an example of an unramified class field $k_0$ corresponding to $H_0$ of (2.4) which gives a non-trivial extension $k_0/K$.

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