Comment on “Mechanisms of synchronization and pattern formation in a lattice of pulse-coupled oscillators”

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In a recent paper, Diaz-Guilera et al. [Phys. Rev. E 57, 3820 (1998)] analyze the mechanisms of synchronization and pattern formation in a lattice of pulse-coupled oscillators. In essence, their analysis consists in the study of the stability of the fixed points of several linear return maps which are obtained from the original system by means of matrix manipulations. We show that although the model they consider is very specific and actually unable to account even for a linear phase response curve, their method does not give correct information on the original system since many of the assumptions involved are in general not correct. To clarify these aspects, several issues concerning the real dynamics are also discussed.

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I. INTRODUCTION

In Ref. [1], Diaz-Guilera et al. propose a new analytical procedure to study a lattice model of pulse-coupled oscillators in a one-dimensional ring with unidirectional coupling. This procedure, which consists in the linear stability analysis of the fixed points of some return maps obtained from the original system by means of matrix manipulations, is based on some ad hoc assumptions and in a particular form for the phase response curve (PRC). Diaz-Guilera et al. claim that their description gives complete information on the original system as well as that the results they obtain for the particular PRC they consider can also be extrapolated to a general PRC. In this comment we show that the model they consider is very specific and unable to account even for the case of a linear PRC. We also show that Diaz-Guilera et al.’s results, in addition, do not give complete information on the original system, not even for the specific PRC they consider, owing to the fact that many of the assumptions involved are not valid.

A first step toward simplifying the system and getting a linear description is to consider a linear PRC. This requirement for the PRC is often not too restrictive since one can expand the PRC in powers of the convexity of the driving hopping to grasp the behavior of the system by only considering the first two terms in the expansion, i.e., a constant term plus another proportional to the phase. Diaz-Guilera et al., in contrast, only consider the term proportional to the phase and claim that the constant term is unimportant since its only effect is to shift the threshold. This claim, which is not proved neither in Ref. [1] nor in the reference they quote, is not true. This fact can easily be seen, for instance, in the inhibitory situation if one considers the linear PRC, \( \Delta(\phi) = \alpha + \epsilon \phi \), with \(-1 - \alpha < \epsilon < -\alpha\) and \(\alpha < 0\). In this case, when an oscillator gets a pulse it can or cannot instantaneously be reset to zero depending on the value of its phase, \(\phi\). In the case addressed by Diaz-Guilera et al., when the interactions are inhibitory, an oscillator can never be reset to zero after getting a pulse if \(\epsilon > -1\) or it is always reset to zero if \(\epsilon = -1\), i.e., whether an oscillator is reset to zero or not does not depend on the value of its phase. This simple example makes it evident that the approximation of Diaz-Guilera et al. is not only unable to account even for the general linear case but also that it is highly pathological. In this sense, one should hardly expect that the results they obtain could be extrapolated to any other system with some degree of generality, as we will show.

II. NEGATIVE COUPLING

To proceed further with their analysis, Diaz-Guilera et al. assume also that the oscillators fire in a cyclic order, i.e., advancements between oscillators are not allowed. This additional assumption, which holds in the all-to-all case, enables them to construct some linear return maps for the particular PRC they are considering. These return maps are intended to describe the dynamics of the system and since they are linear, it can be done easily by only looking at their fixed points and the corresponding eigenvalues. In their analysis, however, Diaz-Guilera et al. additionally constrain the fixed points to those in which each oscillator fires just once, neglecting then other possibilities. For the case in which \(\epsilon\) is negative they found that the fixed points are stable whereas they are unstable for the case in which \(\epsilon\) is positive. After performing some simulations for a system of three oscillators they realize that in fact advancements are possible but they claim, again without proof, that advancements only matter during the transient dynamics. Thereby, they proceed with their analysis without taking into account this annoying hindrance.

In Fig. 1 we have displayed the typical time evolution for the Diaz-Guilera et al.’s PRC for the case of three oscillators when the phase of an oscillator is always reset to zero after receiving a pulse. For this coupling, Diaz-Guilera et al.’s results predict that there exists a stable fixed point in which just before an oscillator fires, the other two should have a phase equal to 0.5. Fig. 1, however, clearly illustrates that such a state is not reached. Diaz-Guilera et al. incorrectly computed
of the oscillators firing in a cyclic order except perhaps in the transient, as assumed by Diaz-Guilera et al., is in general not valid and other kind of behavior with different periodicity appears. From the methodological point of view, the previous examples illustrate that the existence of advancements cannot be disregarded.

It is worth to notice that although in the appendix Diaz-Guilera et al. conclude that for an arbitrary number of oscillators the moduli of the eigenvalues are always lower than 1 when \( \epsilon < 0 \), in fact, if the moduli are correctly computed, that result is in general not valid. For instance, in the case of three oscillators there is an eigenvalue with modulus 1 when \( \epsilon = -1 \). We performed numerical simulations for the same PRC as in previous figures and for values of the number of oscillators ranging from 3 to 1000 and in all of them we found that Diaz-Guilera et al.'s results were unable to account for the real dynamics. In the opposite situation, when we considered values for the coupling which never reset the phase to zero, we obtained results that can be compatible with Diaz-Guilera et al.'s predictions.

### III. POSITIVE COUPLING

In the case wherein the coupling is excitatory (\( \epsilon > 0 \)), by only taking into account the results concerning the stability of the previous fixed points and by assuming, again without proof, that a set of synchronized oscillators acts as a single unit that cannot be broken, Diaz-Guilera et al. concluded that eventually the whole population fires in synchrony. This procedure to state the presence of synchrony is clearly inconsistent with the assumptions they use to try to describe the real system through linear return maps. They assumed that advancements are only important during the transient dynamics. However, the whole time evolution until synchrony is reached is precisely a transient. In addition, the fact that the state corresponding to the cyclic ordering they propose for the eventual evolution is unstable does not imply that the system will go far away from those fixed points. For instance, the system might oscillate around an unstable...

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**FIG. 1.** Time evolution for a lattice of three oscillators for Diaz-Guilera et al.'s PRC with \( \epsilon = -1 \). The symbols are depicted each time any oscillator reaches the threshold.

**FIG. 2.** Time evolution for a lattice of five oscillators for Diaz-Guilera et al.'s PRC with \( \epsilon = -1 \). Filled circles, squares, diamonds, triangles up, and triangles left correspond to \( \phi_1 \), \( \phi_2 \), \( \phi_3 \), \( \phi_4 \), and \( \phi_5 \), respectively. The symbols are depicted each time that any oscillator reaches the threshold.
fixed point as occurs in the logistic map, as well as in many other systems, when period doubling appears [2].

Notice that Figs. 1 and 2 show also that, in contrast to the all-to-all case, when short-range interactions are present two synchronized oscillators can lose their mutual synchrony. This situation is a general property of the model, even for the excitatory case [3]. In general, there are not “absorbing barriers surrounding the repellers”, as quoted by Diaz-Guilera et al. In the real dynamics the situation is more complex: Two synchronized oscillators can lose their mutual synchrony but eventually they are able to recover it if some conditions for the PRC are fulfilled [3].

It is fair to say that sufficient conditions for synchronization to occur in the all-to-all case were rigorously found by Mirollo and Strogatz [4]. Based on numerical simulations, these authors also conjectured that in the excitatory case the same sufficient conditions should hold if a local coupling is considered. In this regard, despite the procedure followed by Diaz-Guilera et al. in view to show synchrony is not correct, their model meets the criteria for synchronization to occur when $\epsilon$ is positive [3]. In essence, Diaz-Guilera et al. results concerning synchronization are another numerical verification of Mirollo and Strogatz’s conjecture [4], which has been widely analyzed through numerical simulations [5] and recently rigorously proved in a model with short-range interactions [3].

IV. LINEAR COUPLING

Previously, we explained how the PRC that Diaz-Guilera et al. considered is unable to account even for a linear PRC and why one should hardly expect that the results they obtain can be extrapolated to any system with a certain degree of generality. We also showed that the results they obtain do not account neither for the specific PRC they consider when the phase is reset to zero after receiving a pulse. In order to address the issue of what happens when such pathologies are not present, we now consider a PRC including the constant term neglected by Diaz-Guilera et al. In this case, whether the phase is reset to zero or not will depend on the value of the phase. When the phase is reset to zero, i.e., $\alpha + \epsilon\phi \leq -\phi$, the PRC is effectively described by the $\epsilon = -1$ case previously studied since the resulting phase cannot be lower than zero by definition of the model. In contrast, when the phase is not reset to zero it is given by $\alpha + \epsilon\phi$.

The existence of these two effective contributions on the PRC breaks the linearity of the coupling and makes Diaz-Guilera et al.’s description inapplicable in a general case. Three examples of what happens under these circumstances are displayed in Figs. 3(a), 3(b) and 3(c). We show the time evolution of the return map corresponding to a decreasing inhibitory PRC ($\alpha < 0$ and $\epsilon < 0$) for three different values of this couple of parameters. Although Diaz-Guilera et al.’s procedure does not account even for this PRC, Fig. 3(a) looks like their predictions, i.e., the phase of each oscillator is always the same just before the first oscillator fires. However, Figs. 3(b) and 3(c) are clearly incompatible with their results since there is not an eventual cyclic firing and continuous overtakings between oscillators occur. Notice that these states are not transient since they are periodic.

In Figs. 3(d) and 3(e) we display the time evolution for an excitatory decreasing and an inhibitory increasing PRC, respectively. Although the PRC corresponding to Fig. 3(d) is positive and in Fig. 3(e) $\epsilon > 0$, the system does not synchronize. These results again do not fit Diaz-Guilera et al.’s predictions wherein for a positive PRC ($\epsilon > 0$) the system will eventually synchronize. In general, the appearance of synchronization cannot be stated by only considering the inhibitory or excitatory character of the coupling neither by only considering the derivative of the PRC. In the all-to-all case, whether the coupling is excitatory or inhibitory, synchronization appears for a positive derivative of the PRC. In general, the “absorbing barriers surrounding the repellers” called for by Diaz-Guilera et al.’s to assert the existence of synchronization do not exist and the system will not synchronize despite the interactions be excitatory or the derivative of the PRC ($\epsilon$) be positive. When short-range interactions are considered both conditions are simultaneously required [3]. Synchronization in Diaz-Guilera et al. simulations appears since the two conditions are same due to the specific PRC they consider.
V. CONCLUSIONS

To summarize, we have shown that a general, complete, and correct description of a lattice model of pulse-coupled oscillators is not possible through the method proposed by Diaz-Guilera et al. In essence, Diaz-Guilera et al.’s results are only valid when studying the linear stability of the fixed points corresponding to a cycle in which each oscillator fires exactly once per period, for a specific PRC which is proportional to the phase and for the inhibitory situation when phase resettings to zero are not allowed. The failure for their description to be applied to any other situation relies on the fact that many of the assumptions required to the development of their analysis are not valid. In particular, a PRC proportional to the phase is not equivalent to a linear PRC; advancements between oscillators can be important for the final state and not only during the transient dynamics; fixed points in which an oscillator fires several times per period are present; synchronized oscillators can lose their mutual synchrony. Diaz-Guilera et al. method can never be used to analyze the global dynamics of the system since it consists only in the study of the linear stability of some fixed points. Therefore, the appearance of synchronization, which is the most relevant case from the experimental point of view [6], can not be inferred from their results.

Our analysis makes it clear that a priori indiscriminate assumptions in view to obtain a known result are not only unjustified but can also give a misunderstanding of what is really happening. Non-linear systems do not usually behave in the way one can suspect from the intuition arising only from some simulations for a particular model. Conversely, rigorous mathematical results [3,4] are very useful to understand when a given behavior should be expected and under which conditions this behavior can or cannot be extrapolated to other systems.

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