Small Gaps in the Spectrum of Tori: Asymptotic Formulae

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Received: 29 August 2021 / Accepted: 22 April 2023
Published online: 9 August 2023 – © The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2023

Abstract: We establish an asymptotic formula, uniformly down to the Planck scale, for the number of small gaps between the first \( N \) eigenvalues of the Laplacian on almost all flat tori and also on almost all rectangular flat tori.

1. Introduction

What is the distribution of values at integer arguments of a randomly chosen positive binary quadratic form? This number theoretic question has the following well-known dynamical interpretation. If \( \Lambda \subseteq \mathbb{R}^2 \) is a lattice of rank 2 and \( \Lambda^* \) denotes the dual lattice, then the numbers \( 4\pi^2 \|\omega\|^2, \omega \in \Lambda^* \), are the eigenvalues of the Laplacian on \( \mathbb{R}^2 / \Lambda \) which are given by the values of a positive binary quadratic form at integer arguments. The equations of motions of a free particle moving through \( \Lambda \) are integrable, thus according to the conjectures of Berry–Tabor [BT] the energy levels of the corresponding quantized system (i.e., \( 4\pi \|\omega\|^2, \omega \in \Lambda^* \)) should behave like a sequence of points coming from a Poisson process, at least for generic \( \Lambda \).

The first to investigate this phenomenon systematically in the case of quadratic forms was Sarnak [Sa] who showed that for almost all (in a Lebesgue sense) quadratic forms the pair correlation is Poissonian. To state this more precisely, let us fix some notation. Given \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \) with \( 4\alpha_1\alpha_3 > \alpha_2^2 \) and \( \alpha_1 > 0 \), let \( q_\alpha(m, n) = \alpha_1m^2 + \alpha_2mn + \alpha_3n^2 \) denote the corresponding positive binary quadratic form. Without loss of generality we may assume that \( q_\alpha \) is reduced, so that we can restrict \( \alpha \in \mathcal{D} \) with

\[
\mathcal{D} = \{ (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3 \mid 0 \leq \alpha_2 \leq \alpha_1 \leq \alpha_3 \}.
\]
The precise measure that we choose on $\mathcal{D}$ is relatively unimportant, but it is most natural to choose the $\text{GL}_2(\mathbb{R})$-invariant hyperbolic measure

$$d^*_{\text{hyp}} \alpha = \frac{d\alpha}{\pi^3 D(\alpha)^3} = \frac{d\alpha_1 d\alpha_2 d\alpha_3}{(4\alpha_1\alpha_3 - \alpha_2^2)^{3/2}}, \quad D(\alpha) = \frac{1}{\pi} \sqrt{4\alpha_1\alpha_3 - \alpha_2^2}.$$ 

Each quadratic form $q_\alpha$ has the automorphism $(m, n) \mapsto (-m, -n)$, so each positive eigenvalue occurs with multiplicity at least 2. It is therefore natural to desymmetrize the spectrum and consider only the values $q_\alpha(m, n)$ with $m > 0$ or $m = 0$ and $n > 0$. We denote by $0 < \Lambda_1 \leq \Lambda_2 \leq \cdots$ the ordered set of values

$$\frac{q_\alpha(m, n)}{D(\alpha)},$$

where $m > 0$ or $m = 0$ and $n > 0$. We usually suppress the dependence on $\alpha$ of the numbers $\Lambda_j$. Asymptotically the average spacing between the $\Lambda_j$ is one, thus we think of $\Lambda_j$ as the properly rescaled multi-set of eigenvalues of the Laplacian on a suitable lattice. For an interval $I \subseteq \mathbb{R}$ and $N \geq 1$ we write

$$P(\alpha, N, I) = \frac{1}{N} \# \{(j, k) \mid \Lambda_j, \Lambda_k \leq N, j \neq k, \Lambda_j - \Lambda_k \in I\}$$

and denote by $\mu(I)$ the length of $I$. Sarnak [Sa, Theorem 1] shows that almost all $\alpha$ satisfy

$$P(\alpha, N, I) \sim \mu(I)$$

for any fixed interval $I$ as $N \to \infty$. That this holds even for all diophantine $\alpha$ was proved in [EMM, Theorem 1.7]. It is clear that it cannot hold for all $\alpha$, for instance it is clearly wrong for all integral forms $q_\alpha$ and any nonempty $I$ not containing zero of size strictly less than $D(\alpha)^{-1}$.

Sarnak’s result is an effective asymptotic formula that comes with an unspecified power saving in the error term, so it is clear that one can shrink the interval $I$ a little bit with $N$. Our first result shows that the asymptotic formula remains true for almost all $\alpha$ even for $\mu(I)$ as small as $N^{-1+\varepsilon}$. This is, up to the value of $\varepsilon$, the smallest scale at which we expect that gaps exist at all; it corresponds to the Planck scale. In this sense Theorem 1 is best possible.

**Theorem 1.** Let $\eta > 0$ be given. There exists a subset $\mathcal{E} \subseteq \mathcal{D}$ of measure zero with the following property: for all $\alpha \in \mathcal{D} \setminus \mathcal{E}$ we have

$$P(\alpha, N, [0, \Delta]) = (1 + o(1))\Delta, \quad N \to \infty,$$

uniformly in

$$N^{-1+\eta} \leq \Delta \leq N^{-\eta}.$$  

The restriction $\Delta \leq N^{-\eta}$ could easily be removed. We included it for convenience to streamline the argument as the main interest is certainly the case of small $\Delta$.

Our next result concerns the thin subset of rectangular tori where $\alpha_2 = 0$, i.e., the value distribution of diagonal quadratic forms. It was shown in [BBRR, Theorem 1.2] that almost all rectangular tori have pairs of eigenvalues of size at most $N$ with difference at most $N^{-1+\varepsilon}$. Here we upgrade the mere existence of small gaps to an asymptotic formula for its cardinality.
Diagonal forms have four symmetries generated by \((m, n) \mapsto (m, -n)\) and \((m, n) \mapsto (-m, n)\), and we denote by \(0 < \Lambda_1 \leq \Lambda_2 \leq \cdots\) the ordered sequence of values of

\[
\frac{\pi}{4\sqrt{\alpha_1\alpha_3}} q_\alpha(m, n), \quad m > 0, \quad n \geq 0
\]

and accordingly define \(P(\alpha, N, I)\) as in (1.1). Let \(\mathcal{R}\) be the set of \((\alpha_1, \alpha_3) \in \mathbb{R}_+^2\) which is naturally equipped with the measure \(d\alpha_1 d\alpha_3\).

**Theorem 2.** Let \(\eta > 0\) be given. There exists a subset \(\mathcal{F} \subseteq \mathcal{R}\) of measure zero with the following property: for all \(\alpha \in \mathcal{R}\setminus\mathcal{F}\) we have

\[
P(\alpha, N, [0, \Delta]) = (1 + o(1))\Delta, \quad N \to \infty,
\]

uniformly in

\[
N^{-1+\eta} \leq \Delta \ll 1. \tag{1.4}
\]

The proofs of Theorems 1 and 2 use a variety of techniques. Both of them start with Fourier analysis and transform the problem at hand into a diophantine question. The arithmetic part of the proof of Theorem 1 is mainly based on lattice point arguments and the geometry of numbers. Theorem 2 uses more advanced machinery. The desired asymptotic formula follows without much difficulty from the Generalized Riemann Hypothesis, or even from a Lindelöf-type bound for the 8th moment of the Riemann zeta function on the half-line. In [BBRR] the use of GRH was avoided by introducing an additional bilinear structure (and hence a second set of variables) along with the best known bounds for the Riemann zeta function close to the one-line. This comes at the price of losing density in the asymptotic formula and returns only a lower bound for \(P(\alpha, N, [0, \Delta])\). Therefore we need a new idea. The plan is to restrict the second set of variables to primes and employ ideas from [MR] together with an analysis of numbers without small and large prime factors.

The problem of determining the smallest \(\Delta\) for which

\[
\liminf_{N \to \infty} P(\alpha, N, [0, \Delta]) > 0 \quad \text{almost surely in } \alpha
\]

has attracted some attention, in the context of Theorem 2. Recently Aistleitner et al. [AEM] showed that for almost \(\alpha\) there exist gaps that are at most \((\log N)^{2c}/N\) with \(c = 1 - \frac{1+\log\log 2}{\log 2} \approx 0.086\) the Erdős–Tenenbaum–Ford constant. In this direction we note that with more effort Theorem 2 can be shown to still hold for \(\Delta = (\log N)^A/N\) and \(A > 0\) some large fixed constant.

2. Proof of Theorem 1

Since \(\mathcal{D}\) can be covered by countably many compact sets, it suffices to consider a compact subset \(\mathcal{D}_0 \subseteq \mathcal{D}\) and show that almost all \(\alpha \in \mathcal{D}_0\) satisfy (1.2). Next we observe that \(P(\alpha, N, I) = P(\lambda\alpha, N, I)\) for every \(\lambda > 0\), so we can de-homogenize by setting \(\alpha_2 = 1\), as the set of \(\alpha \in \mathcal{D}_0\) with \(\alpha_2 = 0\) has measure 0. From now on we write

\[
q_\alpha(m, n) = \alpha_1 m^2 + mn + \alpha_3 n^2
\]
where $\alpha = (\alpha_1, 1, \alpha_3)$ is contained in a compact domain $D^*$ inside (a neighbourhood of)

$$
\{(\alpha_1, 1, \alpha_3) \in \mathbb{R} \times \{1\} \times \mathbb{R} \mid 1 \leq \alpha_1 \leq \alpha_3\}.
$$

(2.1)

Correspondingly we write

$$
d^*_\text{hyp}\alpha = \frac{d\alpha_1 d\alpha_3}{(4\alpha_1\alpha_3 - 1)^{3/2}} = \frac{d\alpha}{\pi^3 D(\alpha)^3}
$$

where now $D(\alpha) = D((\alpha_1, 1, \alpha_3)) = \pi^{-1}(4\alpha_1\alpha_3 - 1)^{1/2}$. Let $V, W$ be fixed smooth, real-valued functions with compact support in $(-\infty, \infty)$ respectively. For $M, T \geq 1$ we define

$$
G_\alpha(M, T) := \frac{1}{4} \sum_{x_1, x_2, y_1, y_2 \in \mathbb{Z}} W(T \cdot q_\alpha(x_1, y_1) - q_\alpha(x_2, y_2)) V\left(\frac{q_\alpha(x_1, y_1)}{M^2 D(\alpha)}\right).
$$

As the argument of $V$ is non-negative, we define

$$
V^*(x) = \delta_{x \geq 0} V(x).
$$

(2.2)

As a precursor to Theorem 1 we show the following proposition.

**Proposition 3.** Fix $\eta > 0$ and suppose that

$$
M^\eta \leq T \leq M^{2-\eta}.
$$

(2.3)

Then for all $\varepsilon > 0$ we have

$$
\int_{D^*} \left| G_\alpha(M, T) - \hat{V}^*(0) \hat{W}(0) \frac{M^2}{T} \right|^2 d^*_\text{hyp}\alpha \ll \frac{M^{4+\frac{1}{1+\varepsilon}}}{T^2} + \frac{M^{2+\varepsilon}}{T}.
$$

Here and in the following we denote by $\hat{f}$ the Fourier transform of $f$. We postpone the proof of Proposition 3 to Sect. 3 and complete the proof of Theorem 1. For $0 < \delta < 1$ and $M, T$ as in (2.3) let $S_\delta^+(M, T^{-1})$ be the set of $\alpha$ such that

$$
\#\{(j, k) \mid \Lambda_j, \Lambda_k \leq M^2, j \neq k, 0 \leq \Lambda_j - \Lambda_k \leq T^{-1}\} \geq (1 + \delta) \frac{M^2}{T}.
$$

We specialize $V, W$ to be smooth, non-negative functions such that $V, W \geq 1$ on $[0, 1]$ and $1 \leq \hat{V}^*(0), \hat{W}(0) \leq 1 + \delta/3$. Then $S_\delta^+(M, T^{-1})$ is contained in the set of $\alpha$ such that

$$
G_\alpha(M, T) \geq (1 + \delta) \frac{M^2}{T},
$$

so that

$$
\mu_{\text{hyp}}(S_\delta^+(M, T^{-1})) \ll \delta^{-2} M^{-\eta/2}.
$$

if (2.3) holds and $\eta \leq 1/23$. In the same way we can bound the measure of the set $S_\delta^-(M, T^{-1})$ of $\alpha$ such that

$$
\#\{(j, k) \mid \Lambda_j, \Lambda_k \leq M^2, j \neq k, 0 \leq \Lambda_j - \Lambda_k \leq T^{-1}\} \leq (1 - \delta) \frac{M^2}{T}.
$$
Hence if \( S_\delta(N, \Delta) \) is the set of \( \alpha \) such that
\[
\left| \{ (j, k) \mid \Lambda_j, \Lambda_k \leq N, j \neq k, 0 \leq \Lambda_j - \Lambda_k \leq \Delta \} - N\Delta \right| \geq \delta N\Delta,
\]
we conclude that
\[
\mu_{\text{hyp}}(S_\delta(N, \Delta)) \ll \delta^{-2} M^{-\eta/2} = \delta^{-2} N^{-\eta/4}
\]
uniformly in the region (1.3). Now let \( S_\delta \) be the set of \( \alpha \) such that there exists a pair of sequences \( N_j, \Delta_j \) with \( N_j \to \infty \) and \( (N_j, \Delta_j) \) satisfying (1.3) such that \( \alpha \in S_\delta(N_j, \Delta_j) \) for all \( j \). For approximation purposes we now consider the special sequences \( N_m^s = (1 + \delta^2)^m \), and \( \Delta_n^s = (1 + \delta^2)^{-n} \), where \( m, n \in \mathbb{N} \). If \( \delta \) is sufficiently small, then for each \( j \) there exists a pair \( m, n \) such that \( |N_j - N_m^s| \ll \delta^2 N_j \) and \( |\Delta_j - \Delta_n^s| \ll \delta^2 \Delta_j \) and so \( S_\delta(N_j, \Delta_j) \subseteq S_{\delta/2}(N_m^s, \Delta_n^s) \). Here we have necessarily \( n \ll \log N_m^s \). We conclude that
\[
S^{s}_{\delta}(N_j, \Delta_j) \subseteq \bigcup_{n \ll \log N_m^s} S_{\delta/2}(N_m^s, \Delta_n^s) =: S_{\delta/2}(N_m^s),
\]
and hence \( S_\delta \subseteq \lim \sup_{m} S_{\delta/2}(N_m^s) \). As
\[
\mu_{\text{hyp}}(S_{\delta/2}(N_m^s)) \ll (N_m^s)^{-\eta/4} \log N_m^s = (1 + \delta^2)^{-\eta m/4} \log(1 + \delta^2)^m
\]
we conclude from the Borel–Cantelli lemma that \( \mu_{\text{hyp}}(S_\delta) = 0 \). Therefore, the set
\[
\bigcup_n S_{1/n}
\]
has measure zero, and Theorem 1 follows.

3. Proof of Proposition 3

Let \( F \) be a smooth non-negative function with compact support in small neighbourhood of \( \mathcal{D}^x \) as described in (2.1). We write \( \| F \| := \int F(\alpha) d_{\text{hyp}}^{s} \alpha \) and recall (2.3). It suffices to estimate
\[
\int_{\mathbb{R}^2} F(\alpha) \left| \mathcal{G}_\alpha(M, T) - \widehat{V}^s(0) \widehat{W}(0) \frac{M^2}{T} \right|^2 d_{\text{hyp}}^{s} \alpha \ll_{\varepsilon} \frac{M^{4-\frac{1}{1+\varepsilon}}}{T^2} + \frac{M^{2+\varepsilon}}{T}.
\]
Opening the square, the proposition follows from the following two estimates
\[
\mathcal{I}_1(M, T) = \int_{\mathbb{R}^2} F(\alpha) \mathcal{G}_\alpha(M, T) d_{\text{hyp}}^{s} \alpha = \| F \| \left( \frac{M^2}{T} + O\left( \frac{M^{3/2+\varepsilon}}{T} \right) \right)
\]  
(3.1)

and
\[
\mathcal{I}_2(M, T) = \int_{\mathbb{R}^2} F(\alpha)|\mathcal{G}_\alpha(M, T)|^2 d_{\text{hyp}}^{s} \alpha
\]
\[
= \| F \|^2 \left( \frac{M^2}{T^2} + O\left( \frac{M^{3/2+\varepsilon}}{T} \right) \right).
\]  
(3.2)

Before we proceed, we remark that without changing \( \mathcal{G}_\alpha(M, T) \) we can (and will) insert a redundant function \( \psi(x/M, y/M) \) into the sum \( (x = (x_1, x_2), y = (y_1, y_2)) \), where \( \psi \) a smooth function that is constantly 1 on some sufficiently large box in \( \mathbb{R}^4 \) (depending on the support of \( F \)).
3.1. Proof of (3.1). Fourier-inverting $V$ and $W$ we obtain

$$I_1(M,T) = \frac{1}{4} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} F(\alpha) \sum_{(x_1,y_1) \neq (x_2,y_2)} \psi\left( \frac{x}{M}, \frac{y}{M} \right) \hat{V}(z) \hat{W}(u) e\left( Tu \cdot \frac{q_\alpha(x_1, y_1) - q_\alpha(x_2, y_2)}{D(\alpha)} \right) du dz \frac{d\alpha_1 d\alpha_3}{\pi^3 D(\alpha)^3}.$$ 

Changing variables $z \leftarrow z/D(\alpha)$, $u \leftarrow u/D(\alpha)$ and integrating over $\alpha$ we obtain

$$I_1(M,T) = \frac{1}{4} \int_{\mathbb{R}^2} \sum_{(x_1, y_1) \neq (x_2, y_2)} \psi\left( \frac{x}{M}, \frac{y}{M} \right) e\left( Tu(x_1 y_1 - x_2 y_2) + z \frac{x_1 y_1}{M^2} \right)\right) du dz$$

where

$$G(\alpha_1, \alpha_3; z, u) = \frac{F(\alpha)}{\pi^3 D(\alpha)} \hat{\hat{V}}(z D(\alpha)) \hat{\hat{W}}(u D(\alpha))$$

and the Fourier transform is taken with respect to the first two variables. Note that $\hat{G}$ is a Schwartz-class function in all variables. We change variables

$$a_1 = y_1 - y_2, \quad a_2 = y_1 + y_2, \quad b_1 = x_1 - x_2, \quad b_2 = x_1 + x_2, \quad (3.3)$$

so that

$$a_1 \equiv a_2 \text{ (mod 2),} \quad b_1 \equiv b_2 \text{ (mod 2),} \quad (a_1, b_1) \neq (0, 0) \neq (a_2, b_2). \quad (3.4)$$

With $a = (a_1, a_2)$, $b = (b_1, b_2)$ we obtain

$$I_1(M,T) = \frac{1}{4} \sum_{a,b}^* \tilde{\psi}\left( \frac{a}{M}, \frac{b}{M} \right) \mathcal{H}(a,b) \quad (3.5)$$

where the star indicates that the $a, b$-sum is subject to (3.4),

$$\tilde{\psi}(a, b) = \psi\left( \frac{b_1 + b_2}{2}, \frac{b_2 - b_1}{2}, \frac{a_1 + a_2}{2}, \frac{a_2 - a_1}{2} \right)$$

and

$$\mathcal{H}(a, b) = \int_{\mathbb{R}^2} e\left( \frac{1}{2} Tu(a_1 b_2 + a_2 b_1) + \frac{z}{4M^2} (a_1 + a_2)(b_1 + b_2) \right)\right) \hat{G}\left( Tu b_1 b_2 + \frac{z}{4M^2} (b_1 + b_2)^2, Tu a_1 a_2 + \frac{z}{4M^2} (a_1 + a_2)^2; z, u \right) du dz.$$ \quad (3.6)$$

Let

$$C_{\text{max}} := \max(|a_1|, |a_2|, |b_1|, |b_2|), \quad C_{\text{min}} := \min(|a_1|, |a_2|, |b_1|, |b_2|), \quad P = \max(|a_1 a_2|, |b_1 b_2|, |a_1 b_2 + a_2 b_1|).$$
By (3.4) we have $C_{\text{max}}$, $P \neq 0$. Integrating by parts with respect to $u$ we find
\[
\mathcal{H}(\mathbf{a}, \mathbf{b}) \ll_A \frac{1}{T(|a_1a_2| + |b_1b_2|)} + \frac{1}{TP + 1} \left(1 + \frac{|a_1b_2 + a_2b_1|}{|a_1a_2| + |b_1b_2| + 1/T}\right)^{-A}
\]
(3.7)

for any $A > 0$. (If $\mathbf{a}, \mathbf{b}$ are integral vectors satisfying (3.4), we have $TP + 1 \asymp TP$, but for arbitrary arguments it is important to keep the extra +1). A similar argument shows that we can restrict the $u$-integral in (3.6) to
\[
TuP \ll M^\varepsilon
\]
(3.8)

up to a negligible error.

For some $1 < C < M$ to be determined later we first estimate the contribution of tuples $(\mathbf{a}, \mathbf{b})$ with $C_{\text{min}} \leq C$ to (3.5). Assume without loss of generality that $C_{\text{min}} = |a_1|$. If $a_1 = 0$, we distinguish the cases $b_2 \neq 0$ (in which case $b_1b_2 \neq 0$ by (3.4)) and $b_2 = 0$ (in which case $\mathcal{H}(\mathbf{a}, \mathbf{b})$ is negligible by (3.7)). It is then easy to see that we get a contribution of
\[
\ll \frac{M(\log M)^2}{T}
\]
to (3.5). If $|a_1| > 0$, then by (3.7) we obtain a contribution
\[
\sum_{0 \neq |a_1| \leq C} \sum_{0 \neq a_2, b_1, b_2 \ll M} \frac{1}{T|b_1b_2|} \ll \frac{M^{1+\varepsilon}C}{T}.
\]

We now attach a smooth cut-off function $\Psi(\mathbf{a}/C, \mathbf{b}/C)$ to the $(\mathbf{a}, \mathbf{b})$-sum in (3.5) where $\Psi$ is supported on $(-\infty, -1/2] \cup [1/2, \infty)$ in each variable and constantly 1 on $(-\infty, -1] \cup [1, \infty)$ in each variable. By the above remarks we thus obtain
\[
\mathcal{I}_1(M, T) = \frac{1}{4} \int_{\mathbb{R}^4} \Psi \left(\frac{\mathbf{a}}{C}, \frac{\mathbf{b}}{C}\right) \tilde{\psi} \left(\frac{\mathbf{a}}{M}, \frac{\mathbf{b}}{M}\right) \mathcal{H}(\mathbf{a}, \mathbf{b}) d(\mathbf{a}, \mathbf{b}) + O \left(\frac{M^{1+\varepsilon}C}{T}\right).
\]

Recalling (3.8) and (3.7) and taking the gradient with respect to $\mathbf{a}$ and $\mathbf{b}$, we conclude
\[
\| \nabla \left(\Psi \left(\frac{\mathbf{a}}{C}, \frac{\mathbf{b}}{C}\right) \tilde{\psi} \left(\frac{\mathbf{a}}{M}, \frac{\mathbf{b}}{M}\right) \mathcal{H}(\mathbf{a}, \mathbf{b}) \right) \| \ll \int \frac{1}{Tu \ll M\varepsilon/TP} \left(Tu|C_{\text{max}} + \frac{C_{\text{max}}}{M^2}\right) du + \frac{1}{CTP}
\ll \frac{M^\varepsilon C_{\text{max}}}{P} \frac{1}{TP} + \frac{1}{CTP} \ll \frac{M^\varepsilon}{CTP}.
\]

By the Euler–MacLaurin formula we obtain altogether
\[
\mathcal{I}_1(M, T) = \frac{1}{4} \int_{\mathbb{R}^4} \Psi \left(\frac{\mathbf{a}}{C}, \frac{\mathbf{b}}{C}\right) \tilde{\psi} \left(\frac{\mathbf{a}}{M}, \frac{\mathbf{b}}{M}\right) \mathcal{H}(\mathbf{a}, \mathbf{b}) d(\mathbf{a}, \mathbf{b}) + O \left(\frac{M^{2+\varepsilon}}{CT} + \frac{M^{1+\varepsilon}C}{T}\right).
\]

We can now remove $\Psi(\mathbf{a}/C, \mathbf{b}/C)$ in the same way as we introduced it at the cost of an error
\[
\ll \int_{|a_1| \leq C} \int_{|a_1| \ll a_2, b_1, b_2 \ll M} \frac{1}{T|b_1b_2| + 1} d(\mathbf{a}, \mathbf{b}) \ll \frac{M^{1+\varepsilon}C}{T}.
\]
by \( (3.7) \). Choosing \( C = M^{1/2} \), we obtain
\[
\mathcal{I}_1(M, T) = \frac{1}{4} \cdot \frac{1}{4} \int_{\mathbb{R}^4} \tilde{\psi} \left( \frac{a}{M}, \frac{b}{M} \right) \mathcal{H}(a, b) d(a, b) + O \left( \frac{M^{3/2 + \varepsilon}}{T} \right). \tag{3.9}
\]
At this point we revert all transformations. We change variables \((a, b)\) back to \((x, y)\) using \((3.3)\), undo the Fourier inversion with respect to \( \alpha \), the linear changes of variables of \((z, u)\), and the Fourier inversion with respect to \(u, z\). In this way the main term in (3.9) equals
\[
\frac{1}{4} \int_{\mathbb{R}^2} \int_{\mathbb{R}^4} F(\alpha) \psi \left( \frac{x}{M}, \frac{y}{M} \right)
W \left( T \cdot \frac{q\alpha(x_1, y_1) - q\alpha(x_2, y_2)}{D(\alpha)} \right) V \left( \frac{q\alpha(x_1, y_1)}{M^2 D(\alpha)} \right) dxdy d_{\text{hyp}} \alpha.
\]
We drop \( \psi \) because it is redundant. Let
\[
x' = x + \frac{y}{2\alpha_1}, \quad x'' = \frac{x'\alpha_1^{1/2}}{(4\alpha_1 \alpha_3 - 1)^{1/4}}, \quad y' = \frac{y}{4\alpha_1 \alpha_3 - 1}^{1/4}.
\]
Then
\[
\frac{q\alpha(x, y)}{D(\alpha)} = \pi \frac{\alpha_1(x')^2 + (4\alpha_1 \alpha_3 - 1) y'^2 / (4\alpha_1)}{(4\alpha_1 \alpha_3 - 1)^{1/2}} = \pi ((x'')^2 + (y'')^2).
\]
Changing variables, we obtain
\[
\|F\| \int_{\mathbb{R}^4} W \left( T \pi (x_1^2 + y_1^2 - x_2^2 - y_2^2) \right) V \left( \pi \frac{(x_1^2 + y_1^2)}{M^2} \right) dxdy.
\]
Changing to polar coordinates and recalling (2.2), we get
\[
\|F\| \int_{[0, \infty)^2} W \left( T(r_1 - r_2) \right) V \left( \frac{r_1}{M^2} \right) dr_2 dr_1
\]
\[
= \|F\| \frac{M^2}{T} \int_0^\infty \int_{-\infty}^{\infty} W(r_2) V(r_1) dr_2 dr_1
= \|F\| \frac{M^2}{T} \left( \tilde{W}(0) \tilde{V}^*(0) + O \left( \frac{1}{M^2 T} \right) \right). \tag{3.9}
\]
Together with (3.9) this establishes (3.1).

3.2. Proof of (3.2). This follows to some extent the analysis in the proof of [ABR, Proposition 5]. By Fourier inversion we have (again with \( x = (x_1, \ldots, x_4), \ y = (y_1, \ldots, y_4) \))
\[
\mathcal{I}_2(M, T) = \frac{1}{16} \int_{\mathbb{R}^2} F(\alpha) \sum_{(x_1, y_1) \neq (x_2, y_2), (x_3, y_3) \neq (x_4, y_4)} \psi \left( \frac{x}{M}, \frac{y}{M} \right) \int_{\mathbb{R}^4} \tilde{V}(z_1) \tilde{V}(z_2) \tilde{W}(u) \tilde{W}(v)
\]
\[
eq e \left( T u \frac{q\alpha(x_1, y_1) - q\alpha(x_2, y_2)}{D(\alpha)} \right) e \left( T v \frac{q\alpha(x_3, y_3) - q\alpha(x_4, y_4)}{D(\alpha)} \right) e \left( \frac{z_1 q\alpha(x_1, y_1) - z_2 q\alpha(x_3, y_3)}{M^2 D(\alpha)} \right) d(u, v, z_1, z_2) \frac{d\alpha}{\pi^3 D(\alpha)^3},
\]
where $\psi$ is a similar redundant function as before. We change variables $z_j \leftrightarrow z_j/D(\alpha)$, $u \leftrightarrow u/D(\alpha)$, $v \leftrightarrow v/D(\alpha)$, integrate over $\alpha$ and introduce the variables

\[ a_1 = y_1 - y_2, \quad a_2 = y_1 + y_2, \quad a_3 = x_3 - x_4, \quad a_4 = x_3 + x_4, \]
\[ b_1 = x_1 - x_2, \quad b_2 = x_1 + x_2, \quad b_3 = y_3 - y_4, \quad b_4 = y_3 + y_4, \]

obtaining (with $a = (a_1, \ldots, a_4)$, $b = (b_1, \ldots, b_4)$)

\[ \mathcal{I}_2(M, T) = \frac{1}{16} \sum_{a,b}^\prime \tilde{\psi} \left( \frac{a}{M}, \frac{b}{M} \right) \mathcal{H}(a, b) \]  

(3.10)

where $\sum^\prime$ indicates the conditions

\[ a_1 \equiv a_2 (\text{mod } 2), \quad b_1 \equiv b_2 (\text{mod } 2), \quad a_3 \equiv a_4 (\text{mod } 2), \quad b_3 \equiv b_4 (\text{mod } 2) \]  

(3.11)

as well as

\[ (a_1, b_1) \neq (0, 0) \neq (a_2, b_2), \quad (a_3, b_3) \neq (0, 0) \neq (a_4, b_4). \]  

(3.12)

Moreover,

\[ \tilde{\psi}(a, b) := \psi \left( \frac{b_1 + b_2}{2}, \frac{b_2 - b_1}{2}, \frac{a_3 + a_4}{2}, \frac{a_4 - a_3}{2}, \right. \]
\[ \left. \frac{a_1 + a_2}{2}, \frac{a_2 - a_1}{2}, \frac{b_3 + b_4}{2}, \frac{b_4 - b_3}{2} \right) = \psi(x, y) \]

and $\mathcal{H}(a, b)$ is defined by

\[ \int_{\mathbb{R}^4} e \left( \frac{Tu}{2} (a_1 b_2 + a_2 b_1) + \frac{Tv}{2} (a_3 b_4 + a_4 b_3) \right. \]
\[ + \frac{z_1 (a_1 + a_2) (b_1 + b_2) - z_2 (a_3 + a_4) (b_3 + b_4)}{4M^2} \]
\[ \left. \tilde{G} \left( Tu b_1 b_2 + Tv a_3 a_4 + \frac{z_1 (b_1 + b_2)^2 - z_2 (a_3 + a_4)^2}{4M^2} \right) \right) \]
\[ T u a_1 a_2 + T v b_3 b_4 + \frac{z_1 (a_1 + a_2)^2 - z_2 (b_3 + b_4)^2}{4M^2} \right) d(u, v, z_1, z_2) \]

with

\[ G(\alpha_1, \alpha_3; z_1, z_2, u, v) = \pi^{-3} D(\alpha) F(\alpha) \tilde{V}(z_1 D(\alpha)) \tilde{V}(z_2 D(\alpha)) \tilde{W}(u D(\alpha)) \tilde{W}(v D(\alpha)) \]

and the Fourier transform $\tilde{G}$ of $G$ is taken with respect to the first two variables $\alpha_1, \alpha_3$. We have

\[ \mathcal{D} \tilde{G}(U, V; z_1, z_2, u, v) \]
\[ \ll_{A, \mathcal{D}} \left( (1 + |U|)(1 + |V|)(1 + |z_1|)(1 + |z_2|)(1 + |u|)(1 + |v|) \right)^{-A} \]  

(3.13)
for all $A > 0$ and any differential operator $\mathcal{D}$ with constant coefficients. Put

$$P = \max(|a_1 a_2|, |a_3 a_4|, |b_1 b_2|, |b_3 b_4|),$$

$$\Delta = a_1 a_2 a_3 a_4 - b_1 b_2 b_3 b_4,$$

$$\Delta_1 = a_1 a_2 b_3 a_4 + a_1 a_2 a_3 b_4 - a_1 b_2 b_3 b_4 - a_1 a_2 b_3 a_4,$$

$$\Delta_2 = a_1 b_2 a_3 a_4 + b_1 a_2 a_3 a_4 - b_1 b_2 a_3 a_4 - b_1 b_2 b_3 a_4.$$ 

We see immediately that the contribution of $P = 0$ to (3.10) is negligible by (3.12), since in this case $a_1 a_2 = b_1 b_2 = a_3 a_4 = b_3 b_4 = 0$, but $a_1 b_2 + a_2 b_1 \neq 0 \neq a_3 b_4 + a_4 b_3$, so that repeated partial integration in $u$ or $v$ saves as many powers of $T$ as we wish (and $T \geq M^0$ by (2.3)). From now on we restrict to $P \neq 0$. If $\Delta \neq 0$, we change variables

$$u = \frac{a_3 a_4 V - b_3 b_4 U}{T \Delta}, \quad v = \frac{a_1 a_2 U - b_1 b_2 V}{T \Delta}$$

to see that $\mathcal{H}(a, b)$ equals

$$\frac{1}{T^2 |\Delta|} \int_{\mathbb{R}^4} e\left(\frac{U \Delta_1}{2 \Delta} + \frac{V \Delta_2}{2 \Delta} + \frac{z_1 (a_1 + a_2) (b_1 + b_2) - z_2 (a_3 + a_4) (b_3 + b_4)}{4 M^2}\right)\tilde{G}\left(U + \frac{z_1 (b_1 + b_2)^2 - z_2 (a_3 + a_4)^2}{4 M^2}, V + \frac{z_1 (a_1 + a_2)^2 - z_2 (b_3 + b_4)^2}{4 M^2}\right)\frac{z_1, z_2, a_3 a_4 V - b_3 b_4 U, a_1 a_2 U - b_1 b_2 V}{\Delta T} \, d(U, V, z_1, z_2).$$

(3.14) By (3.13) and repeated integration by parts we conclude

$$\mathcal{H}(a, b) \ll_A \frac{1}{T^2 (|\Delta| + P/T)} \left(1 + \frac{|\Delta_1|}{|\Delta| + P/T}\right) \left(1 + \frac{|\Delta_2|}{|\Delta| + P/T}\right)^{-A}$$

(3.15) for any $A \geq 0$ and all $a, b \ll M$ satisfying $P \neq 0$. This remains true for $\Delta = 0$, in which case we change variables

$$ua_1 a_2 + vb_3 b_4 = U$$

so that $ub_1 b_2 + va_3 a_4 = Ub_1 b_2 / a_1 a_2$ and apply the same argument, cf. also [ABR, (2.13)]. We also conclude that

$$\|\nabla \mathcal{H}(a, b)\| \ll \frac{1}{T^2 |\Delta|} \left(\frac{1}{M} + \frac{M^3}{\Delta}\right).$$

(3.16) We finally observe the trivial bound

$$\mathcal{H}(a, b) \ll 1,$$

(3.17) valid for all real $a, b$ (even in the case $P = 0$). Fix $0 < \delta < 1/2$. We claim that the error from dropping terms in $\mathcal{I}_2(M, T)$ in (3.10) with $\Delta \ll M^{4-\delta}$ is small, more precisely (recall (3.15))

$$\#\{(a, b) \text{ satisfying (3.12) } | \ a, b \ll M, \Delta \ll M^{4-\delta}, \Delta_1, \Delta_2 \ll M^\delta (|\Delta| + P/T)\} \ll M^\delta \left(\frac{M^{4-\delta/5}}{T^2} + \frac{M^2}{T}\right).$$

(3.18)
This is the analogue of [ABR, Proposition 5]. We postpone the proof of (3.18) to the next subsection. Let $\phi$ be a smooth function with support on $[1/2, \infty]$ that is 1 on $[1, \infty]$, and write

$$\Phi(a, b) := \tilde{\psi} \left( \frac{a}{M}, \frac{b}{M} \right) \phi \left( \frac{|\Delta|}{M^{4-\delta}} \right),$$

so that

$$I_2(M, T) = \frac{1}{16} \sum_{a,b}' \Phi(a, b) H(a, b) + O \left( M^\delta \left( \frac{M^{4-\delta/5}}{T^2} + \frac{M^2}{T} \right) \right).$$

Note that the condition (3.12) is now void. From (3.15) and (3.16) we conclude

$$\| \nabla \Phi(a, b) H(a, b) \| \ll \frac{1}{T^2 |\Delta|} \left( \frac{1}{M^{1-\delta}} + \frac{M^3}{\Delta} \right) \ll \frac{1}{T^2 M^{5-2\delta}}.$$

By the Euler–MacLaurin formula we conclude

$$\sum_{a,b}' \Phi(a, b) H(a, b) = \frac{1}{16} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \Phi(a, b) H(a, b) da \, db + O \left( \frac{M^{3+2\delta}}{T^2} \right),$$

where the factor 1/16 comes from the congruences (3.11). We now re-insert the contribution of the terms $\Delta \ll M^{4-\delta}$ into the integral by dropping the cut-off function $\phi(|\Delta|/M^{4-\delta})$, and we claim that this introduces an error of at most

$$\int_{\Delta \ll M^{4-\delta}} \tilde{\psi} \left( \frac{a}{M}, \frac{b}{M} \right) H(a, b) d(a, b) \ll \frac{M^{4-\delta/5+\varepsilon}}{T^2}, \quad (3.19)$$

which is already present. Again we postpone the proof and revert all steps as in the previous subsection and in the end of [ABR, Section 2], namely the change of variables $(x, y) \mapsto (a, b)$, the integration over $\alpha$ and the Fourier inversions. In this way we finally obtain that $I_2(M, T)$ equals

$$\frac{1}{16} \int_{\mathbb{R}^2} F(\alpha) \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} \psi \left( \frac{x}{M}, \frac{y}{M} \right) W \left( \frac{q_\alpha(x_1, y_1) - q_\alpha(x_2, y_2)}{D(\alpha)} \right) W \left( \frac{q_\alpha(x_3, y_3) - q_\alpha(x_4, y_4)}{D(\alpha)} \right) V \left( \frac{q_\alpha(x_1, y_1)}{M^2 D(\alpha)} \right) V \left( \frac{q_\alpha(x_3, y_3)}{M^2 D(\alpha)} \right) dx \, dy \, d_{\text{hyp}}^* \alpha$$

$$+ O \left( M^\delta \left( \frac{M^{4-\delta/5}}{T^2} + \frac{M^2}{T} + \frac{M^{3+2\delta}}{T^2} \right) \right).$$

Here we can drop the function $\psi(x/M, y/M)$ because it is redundant. We choose $\delta = 5/11$. By the same change of variables as in the previous subsection we obtain

$$I_2(M, T) = \|F\| \int_{\mathbb{R}^8} W \left( T \pi \left( \frac{x_1^2 + y_1^2}{M^2} - x_2^2 - y_2^2 \right) \right) W \left( T \pi \left( \frac{x_3^2 + y_3^2}{M^2} - x_4^2 - y_4^2 \right) \right) V \left( \frac{\pi \left( x_1^2 + y_1^2 \right)}{M^2} \right) V \left( \frac{\pi \left( x_3^2 + y_3^2 \right)}{M^2} \right) dx \, dy + O \left( \frac{M^{43/11 + \varepsilon}}{T^2} + \frac{M^{2+\varepsilon}}{T} \right).$$
The main term equals

\[
\|F\| \frac{M^4}{T^2} \int_0^\infty \int_{-\infty}^{M^2r_1} \int_0^\infty \int_{-\infty}^{M^2r_3} W(r_2)V(r_1)W(r_4)V(r_3)dr_4
dr_3 dr_2 dr_1
= \|F\| \frac{M^4}{T^2} (\widehat{W}(0)\widehat{V}(0) + O\left(\frac{1}{TM^2}\right))^2,
\]
and (3.2) follows. It remains to prove (3.18) and (3.19) to which the following two subsections are devoted.

3.3. Proof of (3.18). We use the notation \(X \lesssim Y\) to mean \(X \ll M^\delta Y\). We put all variables into dyadic intervals and suppose that

\[
A_1 \leq |a_1| \leq 2A_1, \ldots, A_4 \leq |a_4| \leq 2A_4, B_1 \leq |b_1| \leq 2B_1, \ldots, B_4 \leq |b_4| \leq 2B_4
\]

with \(0 \leq A_1, \ldots, B_4 \ll M\). We also assume

\[
D \leq |\Delta| \leq 2D \ll M^{4-\delta}. \tag{3.20}
\]

For \(A = (A_1, \ldots, A_4), B = (B_1, \ldots, B_4)\) we now count the number \(N'(A, B, D)\) of 8-tuples \((a, b)\) subject to these size conditions as well as (3.12) and

\[
\Delta_1, \Delta_2 \ll |\Delta| + P/T. \tag{3.21}
\]

We start with some degenerate cases and denote by \(N_0(A, B, D)\) the contribution of \(A_1 \cdots A_4 B_1 \cdots B_4 D = 0\).

Let us first assume that some of the \(a\)-variables vanish, but none of the \(b\)-variables. Then \(D \neq 0\), and by a divisor bound, this contribution is \(\ll DM^3\). If some \(a\)-variable vanishes, say \(a_1\), and also some \(b\)-variable vanishes (which cannot be \(b_1\)), say \(b_2\), then \(|b_1a_2| \geq 1\), \(\Delta = 0\), \(P = \max(A_3A_4, B_3B_4)\), \(\Delta_1 = b_1a_2b_3b_4\), \(\Delta_2 = b_1a_2a_3a_4\), which is impossible by (2.3) and (3.21).

So from now on we will assume \(a_1 \cdots a_4 b_1 \cdots b_4 \neq 0\). Let us next assume \(D = 0\), i.e., \(a_1 \cdots a_4 = b_1 \cdots b_4\). By a divisor bound this contributes \(\ll PM^2\). So from now on we assume \(D \neq 0\), and we have shown

\[
N_0(A, B, D) \ll DM^3 + PM^2. \tag{3.22}
\]

Let \(N'(A, B, D)\) denote the contribution with \(A_1 \cdots A_4 B_1 \cdots B_4 D \neq 0\) and let us write \(\min(A_1, \ldots, A_4, B_1, \ldots, B_4) = M^{1-\eta}\), say, with \(0 < \eta < 1\). We have trivially

\[
N'(A, B, D) \ll D \min(A_1A_2A_3A_4, B_1B_2B_3B_4) \ll DM^{4-\eta}.
\]

We consider another degenerate case, namely \(a_2a_4 = b_2b_4\). In this case let \(d = a_1a_3 - b_1b_3 = \Delta/(b_2b_4) \neq 0\). We have \(\ll DM^2\) choices for \((d, b_2, b_4, a_1, a_3)\), and then \(b_1, b_3\) are determined by a divisor argument. This can be absorbed in the existing count (3.22), so that from now we assume \(a_2a_4 \neq b_2b_4\) and similarly \(a_2a_3 \neq b_2b_3\).

Substituting \(b_1 = (a_1a_2a_3a_4 - \Delta)/(b_2b_3b_4)\) into the definition of \(\Delta_1\), we obtain

\[
\Delta_1 = -\frac{a_1}{b_2}(a_2a_3 - b_2b_3)(a_2a_4 - b_2b_4) + \frac{a_2}{b_2}\Delta
\]
so
\[(a_2a_3 - b_2b_3)(a_2a_4 - b_2b_4) \leq \frac{B_2}{A_1}(D + \frac{P}{T}) + \frac{A_2}{A_1} \ll \left( D + \frac{P}{T} \right) M^\eta. \quad (3.23)\]

By a standard lattice point argument [Sa, pp. 200–201] the number of such (non-zero) 6-tuples is
\[\lesssim (D + P/T) M^{2+\eta}. \]

Having fixed \(a_2, a_3, a_4, b_2, b_3, b_4\), we are left with pairs \((a_1, b_1)\) satisfying
\[a_1 = \frac{b_1b_2b_3b_4}{a_2a_3a_4} + O\left( \frac{D}{A_2A_3A_4} \right) \]

so that we obtain in total
\[N(A, B, D) \lesssim DM^3 + PM^2 + \min\left( DM^{4-\eta}, \left( D + \frac{P}{T} \right) M^{3+\eta}\left( \frac{D}{M^{3-3\eta}} + 1 \right) \right) \]
\[\leq DM^3 + PM^2 + \left( D + \frac{P}{T} \right) (M^{7/2} + D^{1/5} M^{16/5}) \]
\[\ll \left( D + \frac{P}{T} \right) (M^{4-\delta/5} + M^2 T) \]

by (3.20) (and since \(\delta < 1/2\)). The quantity on the left hand side of (3.18) is
\[\lesssim \max_{A,B,D} \frac{N(A, B, D)}{T^2(D + P/T)} \]

and so (3.18) follows.

3.4. Proof of (3.19). Here we must estimate
\[\int_{\Delta \leq M^{4-\delta}} \tilde{\psi}\left( \frac{a}{M}, \frac{b}{M} \right) \mathcal{H}(a, b) d(a, b) \]

based on the bounds (3.15) and (3.17). This is the (much simpler) continuous analogue of the previous subsection. We put all variables into dyadic intervals \(A_j \leq |a_j| \leq 2A_j\), \(B_j \leq |b_j| \leq 2B_j\), \(D \leq |\Delta| \leq 2D\), \(\Delta_1, \Delta_2 \leq \Delta + P/T\). The numbers \(A_j, B_j, D\) run through logarithmically many positive and negative powers of 2 and are bounded by \(M^{-100} \leq A_j, B_j \ll M, M^{-100} \leq D \ll M^{4-\delta}\). (If one of these is \(\ll M^{-100}\), the coarsest trivial estimates suffice - this is the only point where (3.17) is needed.) We call the corresponding set \(S(A, B, D)\) and as before we write \(\min(A_1, \ldots, A_4, B_1, \ldots B_4) = M^{1-\eta}\). We have the trivial bound
\[\text{vol}(S(A, B, D)) \ll DM^{4-\eta+\epsilon} \]

On the other hand, as in (3.23) we see that the integration condition implies
\[(a_2a_3 - b_2b_3)(a_2a_4 - b_2b_4) \leq \frac{B_2}{A_1}(D + \frac{P}{T}) + \frac{A_2}{A_1} \ll \left( D + \frac{P}{T} \right) M^\eta \]
An easy computation shows that the volume of such 6-tuples is \( \ll (D + P/T)M^{2+\eta+\varepsilon} \), so that we obtain the alternative bound
\[
\text{vol}(S(A, B, D)) \ll \left(D + \frac{P}{T}\right)M^{2+\eta+\varepsilon} \cdot M \cdot \frac{D}{M^{3-3\eta}} \ll \left(D + \frac{P}{T}\right)M^{4+4\eta-\delta+\varepsilon}.
\]
Combining the two bounds we obtain
\[
\text{vol}(S(A, B, D)) \ll (D + P/T)M^{4-\delta/5+\varepsilon}
\]
and (3.19) follows from this and (3.15).

4. Proof of Theorem 2

As in the proof of Theorem 1 we can specialize \( \alpha_3 = 1 \), and we can restrict ourselves to a compact set of \( \alpha \in \mathcal{R}_0 \subseteq \mathbb{R}_{>0} \). Thus our quadratic forms have the shape \( q_\alpha(m, n) = am^2 + n^2, m > 0, n \geq 0 \) with \( \alpha \asymp 1 \). By the same argument as in the proof of Theorem 1 it suffices to show that the measure of \( \alpha \in \mathcal{R}_0 \) such that
\[
\left| \#(\Lambda_i, \Lambda_j \leq N \mid 0 \leq \Lambda_j - \Lambda_i \leq \Delta, i \neq j) - N\Delta \right| > \delta N\Delta
\]
is \( \ll_{\delta, \eta} 1/(\log N)^{2.5} \) (the exponent has to be larger than 2 for the Borel–Cantelli argument to work), uniformly in the range (1.4). We define
\[
t_1 = m_1 - m_2, \quad t_2 = m_1 + m_2, \quad t_3 = n_2 - n_1, \quad t_4 = n_2 + n_1,
\]
so that \( \#(\Lambda_i, \Lambda_j \leq N \mid 0 \leq \Lambda_j - \Lambda_i \leq \Delta, i \neq j) \) equals the cardinality of all quadruples \( (t_1, t_2, t_3, t_4) \in \mathcal{T}_\alpha(N, \Delta) \) where \( \mathcal{T}_\alpha(N, \Delta) \) is defined by
\[
(t_1, t_3) \neq (0, 0), \quad t_1 \equiv t_2 \pmod{2}, \quad t_3 \equiv t_4 \pmod{2}, \quad t_2 > |t_1|, \quad t_4 \geq |t_3|,
\]
\[
\alpha\left(\frac{t_1 \pm t_2}{2}\right)^2 + \left(\frac{t_3 \mp t_4}{2}\right)^2 \leq \frac{4\sqrt{\alpha}N}{\pi}, \quad 0 \leq \alpha t_1 t_2 - t_3 t_4 \leq \frac{4\sqrt{\alpha}\Delta}{\pi}.
\]
For \( \rho > 0 \) let
\[
S_\rho(N) = \left\{ n \in \mathbb{N} \mid n \text{ has a prime divisor in } \exp((\log N)^{\rho}), \exp((\log N)^{1-\rho}) \right\}.
\]
Let \( \mathcal{T}_\alpha^{\rho, A}(N, \Delta) \) be the set of \( (t_1, \ldots, t_4) \in \mathcal{T}_\alpha(N, \Delta) \) such that
- at least one of \( t_1, t_2, t_3 \) is in \( S_\rho(N) \) (we make no assumption on \( t_4 \)) and
- all \( t_j \) satisfy \( |t_j| \geq N^{1/2}(\log N)^{-A} \).

We claim that the contribution of \( (t_1, \ldots, t_4) \in \mathcal{T}_\alpha(N, \Delta) \setminus \mathcal{T}_\alpha^{\rho, A}(N, \Delta) \) is negligible.

**Proposition 4.** Let \( A > 10 \) and \( \rho < 1/20 \). Then the measure of all \( \alpha \in \mathcal{R}_0 \) such that
\[
\#(\mathcal{T}_\alpha(N, \Delta) \setminus \mathcal{T}_\alpha^{\rho, A}(N, \Delta)) \geq \frac{1}{2} \Delta N
\]
is \( \ll (\log N)^{-2.5} \). Moreover, the measure of \( \alpha \in \mathcal{R}_0 \) such that
\[
\#\{(t_1, \ldots, t_4) \in \mathcal{T}_\alpha(N, \Delta) \mid (t_1 - t_2)(t_3 - t_4)(\alpha t_1 t_2 - t_3 t_4) = 0\} \geq \frac{1}{10} \Delta N
\]
is \( \ll N^{-1/3} \).
We postpone the proof to Sect. 5. By the second part of the lemma we may freely insert or remove cases where \( t_1 = t_2 \) or \( t_3 = t_4 \) (i.e., \( m_2n_1 = 0 \)) or \( \alpha t_1 t_2 = t_3 t_4 \). This observation is needed for the following symmetry arguments.

We now focus on \( T^{\rho, A}_\alpha (N, \Delta) \). Clearly \( t_1 \) and \( t_3 \) are of the same sign, and by symmetry we may assume \( t_1 > 0 \) at the cost of replacing the condition \( 0 \leq \alpha t_1 t_2 - t_3 t_4 \leq 4\sqrt{\alpha} \Delta / \pi \) with

\[
|\alpha t_1 t_2 - t_3 t_4| \leq 4\sqrt{\alpha} \Delta / \pi.
\]

Again by symmetry we may drop at the cost of a factor \( 1/4 \) the conditions \( t_2 > |t_1| \) and \( t_4 \geq |t_3| \) (i.e., we may swap \( t_1 \leftrightarrow t_2, t_3 \leftrightarrow t_4 \)) where we also observe that the first part of Proposition 4 holds in the exact same way if \( T^{\rho, A}_\alpha (N, \Delta) \) is defined for one of \( t_1, t_2, t_4 \) being in \( S_\rho (N) \). Thus we see that it suffices to analyse one quarter times the cardinality of \( T^{\rho, A}_\alpha (N, \Delta)^* \) which we define to be the set of \((t_1, \ldots, t_4)\) such that

1. \( N^{1/2} (\log N)^{-A} < t_i \) for all \( i \in \{1, \ldots, 4\} \).
2. \( t_1 \equiv t_2 \) (mod 2) and \( t_3 \equiv t_4 \) (mod 2)
3. \( |\alpha t_1 t_2 - t_3 t_4| \leq (4/\pi) \sqrt{\alpha} \Delta 
4. \) at least one of \( t_1, t_2, t_3 \) belongs to \( S_\rho (N) \),
5. and

\[
\alpha \left( \frac{t_1 + t_2}{2} \right)^2 + \left( \frac{t_3 + t_4}{2} \right)^2 \leq \frac{4\sqrt{\alpha} N}{\pi}.
\]

For the smaller count we only consider tuples \((D_1, \ldots, D_4)\) satisfying (4.3) such that the boxes lie completely inside the region

\[
\alpha \left( \frac{t_1 + t_2}{2} \right)^2 + \left( \frac{t_3 + t_4}{2} \right)^2 \leq \frac{4\sqrt{\alpha} N}{\pi}, \quad t_1, t_2, t_3, t_4 \geq 0
\] (4.4)

We call the collection of such quadruples \( \mathcal{D}_- \).

For \( 0 \leq \theta \leq 1 \) we note that

\[
\log \left( \frac{t_3 t_4}{t_1 t_2} + \theta \frac{4\sqrt{\alpha} \Delta}{t_1 t_2} \right) = \log \left( \frac{t_3 t_4}{t_1 t_2} \right) + \theta \frac{4\sqrt{\alpha} \Delta}{t_1 t_2} \Delta + O \left( \frac{\theta^2 \Delta^2}{D_3^2 D_4^2} \right).
\]

Therefore we sharpen the inequality

\[
|\alpha t_1 t_2 - t_3 t_4| \leq \frac{4\sqrt{\alpha} \Delta}{\pi}
\] (4.5)

to

\[
\left| \log \alpha - \log \left( \frac{t_3 t_4}{t_1 t_2} \right) \right| \leq (1 - 3\delta') \frac{4\sqrt{\alpha} \Delta}{\pi D_3 D_4},
\]
and we detect this with a smooth weight function
\[ W_-(\log \alpha - \log \frac{t_3 t_4}{t_1 t_2} \frac{\pi D_3 D_4}{4\sqrt{\alpha \Delta}}) \]
where \( W_- \) is constantly 1 on \([-1 + 4\delta', 1 - 4\delta']\) and vanishes outside \([-1 + 3\delta', 1 - 3\delta']\).

For the larger count we consider boxes satisfying (4.3) that intersect the part of (4.4) where \( t_i \geq N^{1/2}(\log N)^{-A} \), calling this larger collection of quadruples \( D_+ \). We relax (4.5) to
\[
\left| \log \alpha - \log \frac{t_3 t_4}{t_1 t_2} \right| \leq (1 + \delta') \frac{4\sqrt{\alpha \Delta}}{\pi D_3 D_4}
\]
and detect it with a factor
\[ W_+(\log \alpha - \log \frac{t_3 t_4}{t_1 t_2} \frac{\pi D_3 D_4}{4\sqrt{\alpha \Delta}}) \]
where \( W_+ \) is constantly 1 on \([-1 - \delta', 1 + \delta']\) and vanishes outside \([-1 - 2\delta', 1 + 2\delta']\). Eventually we will choose \( \delta' = c\delta \) for some sufficiently small \( c \). For notational simplicity we write
\[ D_j' = D_j(1 + \delta'). \]

We summarize that
\[
\frac{1}{4} \# T_{\alpha, A}^\rho(N, \Delta)^* \leq \frac{1}{4} \sum_{(D_1, \ldots, D_4) \in D_+} \sum_{t_i \in (D_i, D_i')} \sum_{\text{some } t_1, t_2, t_3 \in \mathcal{S}_\rho(N) \atop 2|t_1 - t_2, t_3 - t_4} W_+(\log \alpha - \log \frac{t_3 t_4}{t_1 t_2} \frac{\pi D_3 D_4}{4\sqrt{\alpha \Delta}}) \]
\[
= \frac{1}{4} \sum_{(D_1, \ldots, D_4) \in D_+} \frac{4\sqrt{\alpha \Delta}}{\pi D_3 D_4} \int_\mathbb{R} \hat{W}_+(\frac{4\sqrt{\alpha \Delta} y}{\pi D_3 D_4}) \sum_{t_i \in (D_i, D_i')} \sum_{\text{some } t_1, t_2, t_3 \in \mathcal{S}_\rho(N) \atop 2|t_1 - t_2, t_3 - t_4} (\alpha t_1 t_2)^{2\pi i y} dy,
\]
and we have a similar lower bound where the subscripts + are replaced with the subscripts −. We consider only the + case, the other one being identical. We extract the main term from small values of \( y \) in the integral. Let \( V \) be a smooth non-negative function that is 1 on \([-1, 1]\) and vanishes for \(|x| > 2\). Let \( B > 3A \), define
\[
Y := \frac{Y'D_3 D_4}{N}, \quad Y' = (\log N)^B
\]
and decompose the previous \( y \)-integral as \( I_1(\alpha) + I_2(\alpha) \) where \( I_1(\alpha) \) contains the factor \( V(y/Y) \) and \( I_2(\alpha) \) contains the factor \( 1 - V(y/Y) \). We claim

**Proposition 5.** We have
\[
\frac{1}{4} \sum_{(D_1, \ldots, D_4) \in D_+} \frac{4\sqrt{\alpha \Delta}}{\pi D_3 D_4} I_1(\alpha) = (1 + O(\delta')) \Delta N
\]
where the implied constant depends only on \( \rho, A, B \) and \( \mathcal{R}_0 \).
Proposition 6. For suitable choices of $A$, $B$ we have

$$
\int_{R_0} \left| \sum_{(D_1, \ldots, D_4) \in \mathbb{D}_+} \sqrt{2} \frac{d\alpha}{\alpha} I_2(\alpha) \right|^2 \ll \Delta^2 N^2 (\log N)^{-50}
$$

where the implied constant depends only on $\rho$ and $R_0$.

We postpone the proofs to Sects. 6 and 7 respectively. From these two propositions we obtain that the measure of $\alpha \in R_0$ such that

$$
\# T^\rho A(N, \Delta) - \Delta N \gg \delta' \Delta N
$$

is $O((\log N)^{-50})$. We choose $\delta' = \delta c$ where $c$ is sufficiently small in terms of the implied constant in Propositon 5. Together with Proposition 4 we see that the measure of $\alpha \in R_0$ satisfying (4.1) is indeed $\ll (\log N)^{-2.5}$ which completes the proof.

5. Proof of Proposition 4

We recall that $T_\alpha(N, \Delta) \setminus T^\rho A(N, \Delta)$ is contained in the set of all $(t_1, \ldots, t_4)$ such that

$$(t_1, t_3) \neq (0, 0), \quad t_2 > |t_1|, \quad t_4 \geq |t_3|, \quad \alpha \left( \frac{t_1 \pm t_2}{2} \right)^2 + \left( \frac{t_3 \mp t_4}{2} \right)^2 \ll N, \quad 0 \leq \alpha t_1 t_2 - t_3 t_4 \ll \Delta$$

(5.1)

and none of $t_1, t_2, t_3$ is in $S_\rho(N)$ or at least one $t_j$ satisfies $|t_j| \leq N^{1/2} (\log N)^{-A}$.

Let us first consider the contribution $T_1(\alpha)$ of those quadruples, where at least one $t_i$, say $t_3$, satisfies $|t_3| \leq N^{1/2} (\log N)^{-A}$. We observe directly that the contribution of $t_1 t_2 t_3 t_4 = 0$ is bounded by $O(N^{1/2})$ if $\Delta \gg 1$ and vanishes otherwise, so it is $\ll N^{1/2} \Delta$. We assume from now on that all $t_j$ are non-zero, and without loss of generality positive. We localize each $t_i$ in a dyadic interval of the shape $(D_i, 4D_i)$; there are at most $O((\log N)^4)$ such boxes, and we must have

$$
D_1 D_2 \asymp D_3 D_4 \ll N (\log N)^{-A}
$$

(5.2)

and moreover

$$
\left| \log \alpha + \log \frac{t_1 t_2}{t_3 t_4} \right| \ll \frac{\Delta}{D_3 D_4}.
$$

If $W$ denotes a suitable non-negative smooth compactly supported function, we conclude that

$$
T_1(\alpha) \ll \sum_{D_1, D_2, D_3, D_4} \sum_{t_i \in (D_i, 4D_i]} W \left( \frac{D_3 D_4}{\Delta} \left( \log \alpha + \log \frac{t_1 t_2}{t_3 t_4} \right) \right) + N^{1/2} \Delta
$$

where the outer sum runs over powers of 2 subject to (5.2). Therefore the measure of $\alpha \in R_0$ such that $T_1(\alpha) > \Delta N/2$ is at most

$$
\int_{R_0} \frac{1}{\Delta N} \sum_{D_1, D_2, D_3, D_4} \sum_{t_i \in (D_i, 4D_i]} W \left( \frac{D_3 D_4}{\Delta} \left( \log \alpha + \log \frac{t_1 t_2}{t_3 t_4} \right) \right) \frac{d\alpha}{\alpha}.
$$
We write $\beta = \log \alpha$, insert the weight $e^{-\beta^2/2}$ and extend the integration to all of $\mathbb{R}$. By Fourier inversion we bound the previous display by

$$
\sum_{D_1, D_2, D_3, D_4} \frac{1}{\Delta N} \frac{\Delta}{D_3 D_4} \int_{\mathbb{R}} e^{-u^2/2} \hat{W}\left(\frac{\Delta u}{D_3 D_4}\right) F(u, D_1) F(u, D_2) F(-u, D_3) F(-u, D_4) du
$$

where

$$
F(u, D) = \sum_{D < t \leq 4D} \frac{1}{t^2 \pi i u}.
$$

By trivial estimates and (5.2), the above is

$$
\ll \sum_{D_1, D_2, D_3, D_4} \frac{D_1 D_2}{N} \ll (\log N)^{4-A}.
$$

Let now $T_2(\alpha)$ denote the set of all quadruples satisfying (5.1) with $|t_j| \geq N^{1/2}$ $(\log N)^{-A}$, but none of $t_1, t_2, t_3$ is in $S_\rho(N)$. Again we localize each $t_i$ in a dyadic interval of the shape $(D_i, 4D_i]$; by our current assumptions, there are at most $O_{A}(\log \log N)$ such boxes. By the same argument as before, the measure of $\alpha \in R_0$ such that $T_2(\alpha) > \Delta N/2$ is at most

$$
\sum_{D_1, D_2, D_3, D_4} \frac{1}{\Delta N} \frac{\Delta}{D_3 D_4} \int_{\mathbb{R}} e^{-u^2/2} \hat{W}\left(\frac{\Delta u}{D_3 D_4}\right) F_{\rho, N}(u, D_1) F_{\rho, N}(u, D_2) F_{\rho, N}(-u, D_3) F(-u, D_4) du
$$

where

$$
F_{\rho, N}(u, D) = \sum_{D < t \leq 4D} \frac{1}{t^2 \pi i u}.
$$

At this point we invoke [We, Corollary 1.2(iii)] with $y = \exp((\log N)^{1-\rho})$, $z = \exp((\log N)^{\rho})$, $u \asymp (\log N)^{\rho}$, $r = (\log N)^{2\rho-1}$ in the form

$$
\sum_{n \leq D} 1 = D P_N + O\left(D \exp(-((\log N)^{\rho/2}))\right)
$$

where

$$
P_N = \prod_{\exp((\log N)^{\rho}) < p \leq \exp((\log N)^{1-\rho})} \left(1 - \frac{1}{p}\right) \ll (\log N)^{2\rho-1},
$$

uniformly in $N^{1/3} \ll D \ll N^{1/2}$, say. By trivial bounds, (5.3) is at most

$$
\ll (\log \log N)^{4}(\log N)^{6\rho-3},
$$

and the proof of the first part is complete.
The second part of the proposition is easy to see. Obviously the condition $\alpha t_1 t_2 - t_3 t_4 = 0$ can only hold for $\alpha \in \mathbb{Q} \cap \mathcal{R}_0$, which has measure 0. Suppose now $t_1 = t_2 = t$, say.

By the same argument as before it suffices to estimate

$$
\int_{\mathcal{R}_0} \frac{1}{\Delta N} \sum_{0 \neq t, t_3, t_4 \ll N^{1/2}} 1 \, d\alpha \ll \sum_{0 \neq t \ll N^{1/2}} \int_{\mathcal{R}_0} \frac{1}{\Delta N} \sum_{r \ll N} \tau(r) \, d\alpha
\ll \frac{N^\varepsilon}{\Delta N} \sum_{0 \neq t \ll N^{1/2}} \text{meas}(\{\alpha \in \mathcal{R}_0 \mid \text{dist}(\alpha t^2, \mathbb{Z}) \ll \Delta\}) \ll N^{-1/2+\varepsilon}.
$$

In the case $t_3 = t_4 = t$, say, we argue in the same way after dividing the inequality $\alpha t_1 t_2 - t^3 \ll \Delta$ by $\alpha$. This completes the proof.

### 6. Proof of Proposition 5

The aim of this section is an asymptotic evaluation of

$$
\frac{1}{4} \sum_{(D_1, \ldots, D_4) \in D_+} \frac{4 \sqrt{\alpha} \Delta}{\pi D_3 D_4} \int_{\mathbb{R}} \hat{W}_+(\frac{4 \sqrt{\alpha} \Delta y}{\pi D_3 D_4}) V\left(\frac{y}{Y}\right) \sum_{t_i \in [D_1, D_2]} \sum_{s \in \mathcal{S}_\rho(N)_{2|t_1-t_2-t_3-t_4}} (\alpha t_1 t_2 \hat{t}_3 t_4)^{2\pi iy} dy
$$

with $Y$ as in (4.7). By a Taylor argument we have

$$
\hat{W}_+(\frac{4 \sqrt{\alpha} \Delta y}{\pi D_3 D_4}) = \hat{W}_+(0) + O_Y\left(\frac{\Delta Y}{D_3 D_4}\right).
$$

(The definition of $W_\pm$ depends on $\delta'$.) By inclusion–exclusion we can detect the condition that some $t_i \in \mathcal{S}_\rho(N)$ by an alternating sum of terms where a certain subset of the $t_i$ is in $\mathcal{S}_\rho(N)$ while the complement is unrestricted. All of these terms are handled in the same way, so for notational simplicity let us focus on the case $t_1 \in \mathcal{S}_\rho(N)$, $t_2$, $t_3$, $t_4$ unrestricted. We recall the notation $D' = D(1+\delta')$ and define

$$
P \sum_{D < t < D'} \frac{1}{t^{2\pi iy}} \quad \text{and} \quad G_{\rho,N}(y, D) = \sum_{D < t \leq D'} \frac{1}{t^{2\pi iy}}.
$$

By partial summation we have

$$
G(y, D) = G^*(1 - 2\pi iy, D', D) + O(Y \log D), \quad G^*(z, D_1, D_2) = \frac{D_1 z - D_2 \hat{z}}{z}
$$

and using (5.4), we also have

$$
G_{\rho,N}(y, D) = (1 - P_N) G^*(1 - 2\pi iy, D', D) + O\left(Y (\log D) D \exp\left(-\left(\log N\right)^{\rho/2}\right)\right)
$$

for $y \ll Y$. Let

$$
\Phi_{\rho,N}(y, D_1, D_2) = G_{\rho,N}(-y, D_1) G(-y, D_2), \quad \Phi(y, D_3, D_4) = G(y, D_3) G(y, D_4).
$$
Detecting the congruence condition modulo 2, we have
\[
\sum_{t_i \in \{D_1, D_2\}} \left( \frac{t_1 t_2}{t_3 t_4} \right)^{2\pi i y} = \left( \Phi_{\rho, N}(y, D_1, D_2) - \frac{\Phi_{\rho, N}(y, \frac{1}{2} D_1, D_2) + \Phi_{\rho, N}(y, D_1, \frac{1}{2} D_2)}{2 - 2\pi iy} + \frac{2\Phi_{\rho, N}(y, \frac{1}{2} D_1, \frac{1}{2} D_2)}{4 - 2\pi iy} \right)
\]
\[
\left( \Phi(y, D_3, D_4) - \frac{\Phi(y, \frac{1}{2} D_3, D_4) + \Phi(y, D_3, \frac{1}{2} D_4)}{2^2\pi iy} + \frac{2\Phi(y, \frac{1}{2} D_3, \frac{1}{2} D_4)}{4^2\pi iy} \right).
\]
(6.3)
Substituting all of this, we recast the portion of (6.1) with \( t_1 \in S_{\rho}(N) \), \( t_2, t_3 \) unrestricted as
\[
\frac{1}{4} \sum_{(D_1, \ldots, D_4) \in \mathcal{D}_+} \frac{4 \sqrt{\alpha \Delta}}{\pi D_3 D_4} \hat{W}_+(0)(1 - P_N)
\]
\[
\int_{\mathbb{R}} V\left( \frac{y}{Y} \right) dt_1^{2\pi iy} \prod_{j=1}^{2} G^*(1 + 2\pi iy, D_j', D_j) \prod_{j=3}^{4} G^*(1 - 2\pi iy, D_j', D_j) dy
\]
\[+ O(Y^2 \Delta N \exp(- (\log N)^{\rho/3})). \]
(6.4)
Define
\[ H(t, D_1, D_2) = \delta_{\log D_2 < t \leq \log D_1} e^{t}. \]
Then
\[ \hat{H}(y, D_1, D_2) = \int_{\mathbb{R}} H(t, D_1, D_2) e(ty) dt = G^*(1 + 2\pi iy, D_1, D_2). \]
Thus the main term in (6.4) becomes
\[
\frac{1}{4} \sum_{(D_1, \ldots, D_4) \in \mathcal{D}_+} \frac{\sqrt{\alpha \Delta}}{\pi D_3 D_4} \hat{W}_+(0)(1 - P_N) \int_{\mathbb{R}} V\left( \frac{y}{Y} \right) \int_{\log D_1}^{\log D_1'} \int_{\log D_2}^{\log D_2'} \int_{\log D_3}^{\log D_3'}
\]
\[
\int_{\log D_4}^{\log D_4'} e^{t_1 + t_2 + t_3 + t_4} e((\log \alpha + t_1 + t_2 - t_3 - t_4) y) dt_1 dt_2 dt_3 dt_4 dy.
\]
Changing variables and recalling (4.7), this equals
\[
\frac{1}{4} \sum_{(D_1, \ldots, D_4) \in \mathcal{D}_+} \frac{\sqrt{\alpha \Delta Y'}}{\pi N} \hat{W}_+(0)(1 - P_N) \int_{D_1}^{D_1'} \int_{D_2}^{D_2'} \int_{D_3}^{D_3'} \int_{D_4}^{D_4'} \hat{V}
\]
(6.5)
By the rapid decay of \( \hat{V} \) we can restrict to \( \log \alpha t_1 t_2 / t_3 t_4 \ll Y^{-3/4} \) at the cost of a total error \( \Delta N Y^{-100} \). By a Taylor argument we then have
\[
Y \log \frac{\alpha t_1 t_2}{t_3 t_4} = \frac{Y}{t_3 t_4} (\alpha t_1 t_2 - t_3 t_4) + O(Y^{-1/2})
\]
\[
= \frac{Y'}{N} \frac{D_3 D_4}{t_3 t_4} (\alpha t_1 t_2 - t_3 t_4) + O(Y^{-1/2})
\]
\[
= (1 + O(\delta')) \frac{Y'}{N} (\alpha t_1 t_2 - t_3 t_4) + O(Y^{-1/2}).
\]
Using also $\hat{W}_+(0)(1 - P_N) = 2 + O(\delta')$ and defining

$$D_+ = \bigcup_{(D_1, \ldots, D_4) \in D_+} \times [D_i, D'_i],$$

we recast the main term (6.5) as

$$\frac{(1 + O(\delta'))\sqrt{\alpha} \Delta Y'}{2\pi N} \int_{D_+} \hat{V} \left( \frac{Y'}{N}(\alpha t_1 t_2 - t_3 t_4) \right) dt.$$

We now add to the integration domain $D_+$ those points of (4.4) with $\min(t_1, \ldots, t_4) \leq N^{1/2}(\log N)^{-A}$. It is easy to see (e.g., by putting the variables into dyadic boxes) that this infers an error of at most $O(\Delta N^{1/2}(\log N)^{-A})$. Next we replace the integration domain with the exact region (4.4), the error of which can be absorbed in the existing $(1 + O(\delta'))$-term. By symmetry we can assume $t_2 \geq t_1$ and $t_4 \geq t_3$ after multiplying by 4, and we drop the condition $t_1 \geq 0$ at the cost of dividing by 2 and note that for negative $t_1$ we automatically have $t_3 \leq 0$ up to a negligible error. Finally we reverse the change of variables (4.2) (the Jacobian infers a factor 4) getting

$$4\sqrt{\alpha}/\pi \frac{(1 + O(\delta'))\Delta Y'}{N} \int_{cN} \hat{V} \left( \frac{Y'}{N}(\alpha n_1^2 + m_1^2 - (\alpha n_2^2 + m_2^2)) \right) dn_1 dn_2 dm_1 dm_2.$$

where the integration is taken over

$$an_j^2 + m_j^2 \leq cN, \quad n_1, n_2, m_1, m_2 \geq 0$$

where $c = 4\sqrt{\alpha}/\pi$. Changing variables, this equals

$$\frac{1}{c} (1 + O(\delta')) \Delta \frac{Y'}{N} \int_0^{cN} \int_0^{cN} \hat{V} \left( \frac{Y'}{N} (r_1 - r_2) \right) dr_2 dr_1$$

$$= \frac{1}{c} (1 + O(\delta')) \Delta \frac{Y'}{N} \int_0^{cN} \int_{-r_1}^{cN-r_1} \hat{V} \left( - \frac{Y'}{N} r_2 \right) dr_2 dr_1.$$

The portion $r_1 \leq cN(Y')^{-3/4}$ and $r_1 \geq cN - cN(Y')^{-3/4}$ is negligible, in the remaining part we can extend the $r_2$-integral to all of $\mathbb{R}$ by the rapid decay of $\hat{V}$ and finally obtain

$$\frac{1}{c} (1 + O(\delta')) \Delta \int_{cN(Y')^{-3/4}}^{cN-cN(Y')^{-3/4}} V(0) dr_1 = (1 + O(\delta')) \Delta N$$

as desired.

### 7. Proof of Proposition 6

We will need the following Plancherel type lemma, which is a simple application of Schur’s test.
Lemma 1. Let $F$ be any measurable function, $V : \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{C}$ a compactly supported Schwartz function and $T > 0$. Let

$$I(\alpha) = \int_{\mathbb{R}} V\left(\alpha, \frac{y}{T}\right) F(y) e^{iy} \, dy.$$ 

Then there exists a compactly supported Schwartz function $H : \mathbb{R} \to \mathbb{R}_{\geq 0}$ depending only on $V$ such that

$$\int_{\mathbb{R}} |I(\alpha)|^2 \, d\alpha \leq \int_{\mathbb{R}} |F(y)|^2 H\left(\frac{y}{T}\right) \, dy.$$ 

Proof. We have

$$\int_{\mathbb{R}} |I(\alpha)|^2 \, d\alpha = \int_{\mathbb{R}^2} F(y) \overline{F(y')} G\left(\frac{y}{T}, \frac{y'}{T}\right) \, dy \, dy'.$$

where

$$G(y, y') = \int_{\mathbb{R}} e^{i(T(y-y')} V(\alpha, y) \overline{V(\alpha, y')} \, d\alpha.$$ 

Integrating by parts twice, we can easily find a suitable Schwartz function $H$ (depending only on $V$) such that

$$|G(y, y')| \leq \frac{H(y)}{1 + (T|y - y'|)^2}.$$ 

We conclude

$$\int_{\mathbb{R}} |I(\alpha)|^2 \, d\alpha \leq \frac{1}{2} \int_{\mathbb{R}^2} \left(|F(y)|^2 + |F(y')|^2\right) \left|G\left(\frac{y}{T}, \frac{y'}{T}\right)\right| \, dy \, dy' \ll \int_{\mathbb{R}} |F(y)|^2 H\left(\frac{y}{T}\right) \, dy$$

as claimed. \hfill \Box

We now start the proof. By Cauchy–Schwarz we have

$$\int_{\mathcal{R}_0} \left| \sum_{(D_1, \ldots, D_4) \in \mathcal{D}_+} \frac{\sqrt{\alpha} \Delta}{\pi D_3 D_4} I_2(\alpha) \right|^2 \, \frac{d\alpha}{\alpha} \ll \sup_{D_i} \frac{\Delta^2 \left(\log \log N\right)^8}{D_3 D_4^2} \int_{\mathcal{R}_0} |I_2(\alpha)|^2 \, d\alpha.$$ 

We majorize the characteristic function of $\mathcal{R}_0$ by a fixed smooth function (with support bounded away from zero and infinity) and recall the definition of $I_2(\alpha)$ as the portion of the $y$-integral in (4.6) restricted to $y \geq Y$ (in a smooth way). Applying the previous lemma, we have

$$\int_{\mathcal{R}_0} |I_2(\alpha)|^2 \, d\alpha \ll \int_{|y| \geq Y} W^*\left(\frac{\Delta y}{D_3 D_4}\right) \sum_{\substack{t_i \in (D_i, D_i'] \\text{some } t_1, t_2, t_3, t_4 \in \mathcal{S}_p(N) \\frac{2|t_1-t_2, t_3-t_4|}{2|t_1-t_2, t_3-t_4|}} \left(\frac{t_1 t_2}{t_3 t_4}\right)^{2\pi y} \, dy$$

for some non-negative Schwartz function $W^*$. 
Again we treat the case \( t_1 \in S_p(N) \) and \( t_2, t_3, t_4 \) unrestricted, the other cases being similar, and we note that \( t_4 \) is unrestricted in all cases. We can detect the parity conditions in the same way as in (6.3) which replaces potentially some \( D_i \) by \( \frac{1}{2} D_i \). Thus we have reduced the problem to bounding,

\[
\sup_{D_i} \frac{\Delta^2 (\log N)^4}{D_3^2 D_4^2} \int_{|y| \geq Y} W^* \left( \frac{\Delta y}{D_3 D_4} \right) |G_{\rho, N}(y, D_1) G(y, D_2) G(y, D_3) G(y, D_4)|^2 dy
\]

(7.1)

using the notation (6.2). We now manipulate \( G_{\rho, N}(y, D_1) \) similarly as in [MR, Lemma 12]. Let \( I = (\exp((\log N)\rho)), \exp((\log N)^{1-\rho}) \) and \( \omega_I(m) \) be the number of prime divisors of \( m \) in \( I \). We have

\[
G_{\rho, N}(y, D) = \sum_{\substack{D \leq n \leq D' \atop n \in S_p(N)}} \frac{1}{n^{2\pi i y}} = \sum_{p \in I} \sum_{D/p < m \leq D'/p} \frac{\left( \omega_I(m) + \delta_{(p,m)=1} \right)^{-1}}{(pm)^{2\pi i y}}
\]

\[
= \sum_{p \in I} \sum_{D/p < m \leq D'/p} \frac{\omega_I(m)^{-1}}{(pm)^{2\pi i y}} - \sum_{p \in I} \sum_{D/p < m \leq D'/p} \frac{\left( \omega_I(m) + 1 \right)^{-1}}{(pm)^{2\pi i y}}
\]

(this is often called Ramaré’s identity). We split the first \( p \)-sum into \( O(\kappa^{-1} \log N) \) intervals of the shape \( P < p \leq P(1+\kappa) \) for

\[
\kappa = (\log N)^{-C}
\]

and \( \exp((\log N)\rho) \leq P \leq \exp((\log N)^{1-\rho}) \). We argue as in [MR, Lemma 12] and write

\[
\sum_{p \in I} \sum_{D/p < m \leq D'/p} \frac{\left( \omega_I(m) + 1 \right)^{-1}}{(pm)^{2\pi i y}}
\]

\[
= \sum_{P < p \leq P(1+\kappa)} \sum_{p \in I} \sum_{D/(P(1+\kappa)) < m \leq D'/P} \frac{\left( \omega_I(m) + 1 \right)^{-1}}{m^{2\pi i y}}
\]

\[
= \sum_{P < p \leq P(1+\kappa)} \sum_{p \in I} \sum_{D/P < m \leq D'/P} \frac{\left( \omega_I(m) + 1 \right)^{-1}}{m^{2\pi i y}} + \sum_{m \in J} \frac{d_m}{m^{2\pi i y}}
\]

for certain \( |d_m| \leq 1 \), where \( J = [D/(1+\kappa), D(1+\kappa)] \cup [D', D'(1+\kappa)] \) and \( P \) runs over a sequence of the type \( P_0(1+\kappa)^j \). We substitute this back into (7.1) getting
For the second term in (7.2) we apply (7.3) directly getting the bound

\[ |Q_P(y)R_P(y, D_1)G(y, D_2)G(y, D_3)G(y, D_4)|^2 dy \]

\[ + \sup_{D_i} \frac{\Delta^2 (\log N)^e}{D_3^2 D_4^2} \int_{\mathbb{R}} W^*(\frac{\Delta y}{D_3 D_4}) |V(y, D_1)G(y, D_2)G(y, D_3)G(y, D_4)|^2 dy \]

\[ + \sup_{D_i} \frac{\Delta^2 (\log N)^e}{D_3^2 D_4^2} \int_{\mathbb{R}} W^*(\frac{\Delta y}{D_3 D_4}) |U(y, D_1)G(y, D_2)G(y, D_3)G(y, D_4)|^2 dy \]

(7.2)

where

\[ Q_P(y) = \sum_{\substack{p < p' \leq P(1+\epsilon) \text{ or } p \not\in I}} \frac{1}{p^{2\pi i y}}, \quad R_P(y, D) = \sum_{D/P < m \leq D'/P} \frac{(\omega_I(m) + 1)^{-1}}{m^{2\pi i y}}, \]

\[ V(y, D) = \sum_{m \in J} \frac{d_m}{m^{2\pi i y}}, \]

\[ U(y, D) = \sum_{p \in I} \sum_{D/p < m \leq D'/p} \frac{\omega_I(m)^{-1}}{(pm)^{2\pi i y}} - \sum_{p \in I} \sum_{D/p < m \leq D'/p} \frac{(\omega_I(m) + 1)^{-1}}{(pm)^{2\pi i y}}. \]

We estimate the three terms in (7.2) separately and recall the standard mean value estimate [IK, Theorem 9.1] for Dirichlet polynomials

\[ \left( \int_{-T}^T \left| \sum_{n \leq X} \frac{a_n}{n^{it}} \right|^2 dt \right) \ll (T + X) \sum_{n \leq X} |a_n|^2. \]

(7.3)

For the second term in (7.2) we apply (7.3) directly getting the bound

\[ \sup_{D_i} \frac{\Delta^2 (\log N)^e}{D_3^2 D_4^2} \left( \frac{D_3 D_4}{\Delta} + D_1 D_2 D_3 D_4 \right) \]

\[ \sum_{m_1 \in J} \sum_{D_2 < m_2 \leq D'_2} \sum_{D_3 < m_2 \leq D'_3} \sum_{D_4 < m_2 \leq D'_4} \tau_4(m_1 m_2 m_3 m_4) \]

\[ \ll \sup_{D_i} \frac{\Delta^2 (\log N)^e}{D_3^2 D_4^2} \left( \frac{D_3 D_4}{\Delta} + D_1 D_2 D_3 D_4 \right) (D_1 D_2 D_3 D_4)^{1/2} \left( \sum_{n \leq X} \tau_4(n) \tau_4(n)^2 \right)^{1/2} \]

\[ \ll (\Delta N + \Delta^2 N^2) (\log N)^{32-C/2} \ll \Delta^2 N^2 (\log N)^{32-C/2} \]

(7.4)

where we used Cauchy–Schwarz, \( \sum_{n \leq X} \tau_4(n)^3 \ll X (\log X)^{63} \) and the assumption \( \Delta \geq N^{-1+\eta} \).
For the third term in (7.2) we proceed similarly, applying (7.3) directly. This gives the bound
\[
\sup_{D_i} \frac{\Delta^2 (\log N)^\varepsilon}{D_3 D_4^2} \left( \frac{D_3 D_4}{\Delta} + D_1 D_2 D_3 D_4 \right) \sum_{p \in I} \sum_{D_2 < m_2 \leq D_2'} \tau_4(m_1 p^2 m_2 m_3 m_4)
\]
\[\ll \sup_{D_i} \frac{\Delta^2 (\log N)^\varepsilon}{D_3 D_4^2} \left( \frac{D_3 D_4}{\Delta} + D_1 D_2 D_3 D_4 \right) \exp(-\log N)^{\rho/2})
\]
\[\ll (\Delta N + \Delta^2 N^2) \exp(-\log N)^{\rho/4}) \ll \Delta^2 N^2 \exp(-\log N)^{\rho/4}). \quad (7.5)
\]

In the first term in (7.2) we split the \(y\)-integral into two parts. For those \(y\) with \(Q_P(y) \leq P (\log N)^{-\Delta} \) we use (7.3) to bound their contribution by
\[
\sup_{D_i} \frac{\Delta^2 (\log N)^\varepsilon (k-1) \log N} {D_3 D_4^2} . \left( \frac{D_3 D_4}{\Delta} + \frac{D_1 D_2 D_3 D_4}{P} \right) \sum_{n \ll D_1 D_2 D_3 D_4/P} \tau_4(n)^2
\]
\[\ll (\Delta N P + \Delta^2 N^2) (\log N)^{2C+18-2D} \ll \Delta^2 N^2 (\log N)^{2C+18-2D} \quad (7.6)
\]

since \(\Delta \gg N^{-1+\eta} \) and \(P \leq N^{\eta/10}\).

Now we treat the integral over the remaining \(y\) where \(Q_P(y) \geq P (\log N)^{-\Delta}. \) Here we can discretize the integral and estimate it by a sum over certain points of distance at most 1. From [MR, Lemma 8] with \(T = N^2, V = (\log N)^D \) and \(\exp(\log N)^{\rho}) \leq P \leq \exp((\log N)^{1-\rho}) \) we conclude that the number of such points is at most \(\ll \exp((\log N)^{1-\rho/2}). \) This gives the bound
\[
\sup_{D_i} \frac{\Delta^2 (\log N)^\varepsilon (k-1) \log N} {D_3 D_4^2} \sum_{P} \sum_{j} |Q_P(y_j) R_P(y_j, D_1) G(y_j, D_2) G(y_j, D_3) G(y_j, D_4)|^2 \quad (7.7)
\]

where \(P \) runs over \(O(k^{-1} \log N) \) numbers and \(j \) runs over \(O(N^\varepsilon) \) numbers with \(Y \leq |y_j| \ll N^2. \) We recall from (4.7) that \(Y \gg (\log N)^{B-2A}. \) A standard application of Perron’s formula and the convexity bound for the Riemann zeta function shows
\[
G(y, D) \ll \frac{D \log N}{T} (T + |y|)^\varepsilon + \frac{D}{1 + |y|} + D^{1/2} (T + |y|)^{1/4+\varepsilon}
\]
for any parameter \(T > 1, \) so that under the present conditions \(G(y_j, D_4) \ll D_4 Y^{-1}. \) For the remaining sum over \(j \) we use the discrete mean value theorem [MR, Lemma 9] (which is [IK, Theorem 9.6]) and bound (7.7) by
\[
\sup_{D_i} \frac{\Delta^2 (\log N)^\varepsilon (k-1) \log N} {D_3 D_4^2} \frac{D_4^2}{Y^2} (D_1 D_2 D_3 + \sqrt{D_1 D_2 N^\varepsilon}) \sum_{n \ll D_1 D_2 D_3} \tau_4(n)^2 \ll \Delta^2 N^2 (\log N)^{18-2B+4A+2C}. \quad (7.8)
\]

Combining (7.4)–(7.6) and (7.8) and choosing \(A = 20, B = 400, C = 200, D = 300, \) we complete the proof.
Acknowledgement  We would like to thank the referee for a very careful reading of the manuscript.

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Communicated by S. Dyatlov