Approximate Controllability for Linear Stochastic Differential Equations in Infinite Dimensions

D. Goreac

Laboratoire de Mathématiques, Unité CNRS UMR 6285, Université de Bretagne Occidentale, 6, av. Victor LeGorgeu, B.P. 809, 29200 Brest cedex, France
e-mail: Dan.Goreac@univ-brest.fr
Tel. 02.98.01.72.45, Fax. 02.98.01.67.90

AMS Classification. 60H10, 60H15

Abstract The objective of the paper is to investigate the approximate controllability property of a linear stochastic control system with values in a separable real Hilbert space. In a first step we prove the existence and uniqueness for the solution of the dual linear backward stochastic differential equation. This equation has the particularity that in addition to an unbounded operator acting on the Y-component of the solution there is still another one acting on the Z-component. With the help of this dual equation we then deduce the duality between approximate controllability and observability. Finally, under the assumption that the unbounded operator acting on the state process of the forward equation is an infinitesimal generator of an exponentially stable semigroup, we show that the generalized Hautus test provides a necessary condition for the approximate controllability. The paper generalizes former results by Buckdahn, Quincampoix and Tessitore (2006) and Goreac (2007) from the finite dimensional to the infinite dimensional case.

1 Preliminaries

This paper is concerned with the study of approximate controllability of an infinite dimensional stochastic equation with multiplicative noise

\[
\begin{cases}
    dX_t^{x,u} = (AX_t^{x,u} + Bu_t) dt + CX_t^{x,u} dW_t, \\
    X_0 = x \in H,
\end{cases}
\]

(1)
where \( u \) is a \( U \)-valued stochastic control process, and the state space \( H \) as well as the control state space \( U \) are separable real Hilbert spaces. We say that the above equation enjoys the approximate controllability property if, for any initial data \( x \in H \), and all finite time horizon \( T > 0 \), one can find a control process \( u \) which keeps the solution \( X_{T}^{x,u} \) arbitrarily close to a given square integrable final condition.

For deterministic control systems with finite dimensional state space \( \mathbb{C}^{n} \), controllability is completely characterized by the well-known Kalman condition. Often, it is convenient to study the observability of the adjoint system rather than the controllability of the initial system. Indeed, whenever dealing with a deterministic control system

\[
\begin{align*}
    dX_{t}^{x,u} &= (AX_{t}^{x,u} + Bu_{t}) \, dt, \\
    X_{0} &= x \in \mathbb{C}^{n},
\end{align*}
\]  \( (2) \)

controllability is equivalent to the observability of the dual system

\[
\begin{align*}
    dY_{t}^{y} &= -A^{*}Y_{t}^{y} \, dt, \\
    O_{t}^{y} &= B^{*}Y_{T}^{y}, \\
    Y_{0}^{y} &= y.
\end{align*}
\]  \( (3) \)

A very powerful tool for this approach is the Hautus test. According to this test, observability of \( (3) \) (and, thus, controllability for \( (2) \)) is equivalent to

\[
\text{rank} \left[ sI - A^{*} \right] = n, \text{ for all } s \in \mathbb{C}.
\]

In the case of separable Hilbert state space, whenever \( A \) generates an exponentially stable semigroup, Russell and Weiss \( [20] \) have obtained a necessary condition for observability which generalizes the Hautus criterion. They have also conjectured that this condition is even sufficient. Jacob and Zwart \( [14] \) proved that the above conjecture holds true for the class of diagonal systems satisfying the strong stability condition whenever the output space is finite dimensional. Similar arguments allow to obtain in \( [13] \) a characterization of approximate controllability of a deterministic controlled system with 1-dimensional input.

In the stochastic framework, Kalman-type characterizations of approximate controllability have been obtained, for the finite-dimensional case, by Buckdahn, Quincampoix and Tessitore \( [3] \) when the noise term is not controlled, and by Goreac \( [11] \) when the control is allowed to act on the noise. The method they use relies on the duality between approximate controllability and approximate observability for the dual equation. Riccati algebraic arguments allow to obtain in \( [3] \) and \( [11] \) an invariance criterion for the approximate controllability of the initial system.

In the case of controlled stochastic systems with infinite-dimensional state space, we cite Barbu, Răşcanu, Tessitore \( [1] \), Fernandez-Cara, Garrido-Atienza, Real \( [8] \), and Sirbu, Tessitore \( [21] \). In \( [21] \), the authors characterize the property of (null) controllability with the help of singular Riccati equations. They also provide a Riccati characterization using the duality approach.
In this paper, we prove the duality between approximate controllability for the forward system and some approximate observability for the dual system, and we use this approach to show that the generalized Hautus test is a necessary condition for approximate controllability whenever $A$ is the generator of an exponentially stable semigroup.

The paper is organized as follows: In the first section we introduce the standard notations and assumptions which will be used in what follows. After, in the second section, we investigate the existence and the uniqueness of the mild solution of the following backward stochastic differential equation which is associated as dual equation to the controlled system (1):

\[
\begin{cases}
    dY_t = -(A^*Y_t + C^*Z_t) \, dt + Z_t dW_t, \\
    Y_T = \xi \in L^2(\Omega, \mathcal{F}_T, P; H).
\end{cases}
\]

We emphasize that the drift term in our dual backward equation contains not only the unbounded operator $A^*$ acting on $Y$ but also the unbounded operator $C^*$ that acts on $Z$. To overcome the difficulties related with, we make a joint dissipativity hypothesis which corresponds, in the case of general heat equations, to the usual joint ellipticity condition. Under these minimal assumptions we are able to prove the existence and the uniqueness. Moreover, we provide a duality result between approximate controllability for the forward equation and the approximate observability of the dual system. The third section proves that, whenever $A$ generates an exponentially stable semigroup, the Russell and Weiss generalization of the Hautus test is a necessary condition for approximate controllability of stochastic systems. Finally, we discuss as example the general heat equation.

2 Introduction

Let us begin by introducing some basic notations and standard assumptions. The spaces $(H, \langle \cdot, \cdot \rangle_H)$, $(U, \langle \cdot, \cdot \rangle_U)$, $(\Xi, \langle \cdot, \cdot \rangle_\Xi)$ are separable real Hilbert spaces. We let $L(\Xi, H)$ denote the space of all bounded $H$-valued linear operators on $\Xi$, and $L_2(\Xi, H)$ be the subspace of Hilbert-Schmidt operators. Both spaces are endowed with the usual norms. Moreover, we consider a linear dissipative operator $A : D(A) \subset H \longrightarrow H$ which generates a $C_0$-semigroup of linear operators $(e^{tA})_{t \geq 0}$, a linear bounded operator $B \in L(U, H)$ and a linear operator $C : H \longrightarrow L(\Xi, H)$ such that, for all $t > 0$,

\begin{align*}
    &a) \ e^{tA}C \in L(H; L_2(\Xi, H)), \\
    &b) \ |e^{tA}C|_{L(H; L_2(\Xi, H))} \leq Lt^{-\gamma},
\end{align*}

for some constants $\gamma \in \left[0, \frac{1}{2}\right)$ and $L > 0$.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space endowed with a filtration $(\mathcal{F}_t)_{t \geq 0}$ which is supposed to satisfy the usual assumptions of completeness and right-continuity. We denote by $W$ a cylindrical $(\mathcal{F}_t)$–Wiener
process that takes its values in \( \Xi \). Finally, we let \( \mathcal{U} \) denote the space of all \((\mathcal{F}_t)\)-progressively measurable processes \( u : \mathbb{R}_+ \times \Omega \rightarrow \mathcal{U} \) such that

\[
E \left[ \int_0^T |u_t|^2 \, dt \right] < \infty, \quad \text{for all } T > 0.
\]

The aim of this paper is to give an easy and verifiable criterion for approximate controllability for the following linear stochastic differential equation

\[
\begin{aligned}
\left\{
\begin{array}{l}
\frac{dX_{t}^{x,u}}{dt} = (AX_{t}^{x,u} + Bu_t) \, dt + CX_{t}^{x,u} \, dW_t, \ t \geq 0. \\
X_0 = x \in H.
\end{array}
\right.
\end{aligned}
\]

(4)

Given an admissible control process \( u \in \mathcal{U} \), an \((\mathcal{F}_t)\)-progressively measurable process \( X_{x,u} \) with \( E \left[ \sup_{s \in [0,T]} |X_{s}^{x,u}|^2 \right] < \infty, \quad \text{for all } T > 0, \)

is a mild solution of (4) if, for all \( t > 0, \)

\[
X_t = e^{tA} x + \int_0^t e^{sA} Bu_s \, ds + \int_0^t e^{sA} CX_s \, dW_s,
\]

(5)

\( P\)-a.s. Under the standard assumptions given above, there exists a unique mild solution of (4). For further results on mild solutions, the reader is referred to Da Prato, Zabczyk [5], and Fuhrman, Tessitore [9].

3 The dual equation

Let us now consider the following backward stochastic differential equation

\[
\begin{aligned}
\left\{
\begin{array}{l}
\frac{dY_{t}}{dt} = - (A^*Y_t + C^*Z_t) \, dt + Z_t \, dW_t, \\
Y_T = \xi \in L^2 (\Omega, \mathcal{F}_T, P; H).
\end{array}
\right.
\end{aligned}
\]

(6)

Since \( C : H \rightarrow L(\Xi, H) \), also \( Ce^{tA} : H \rightarrow L(\Xi, H) \), for all \( t \geq 0 \). Let us assume that, for all \( t > 0, \) all the values of \( Ce^{tA} \) are in \( L_2 (\Xi; H) \),

\[
Ce^{tA} : H \rightarrow L_2 (\Xi; H).
\]

Then, of course, the linear operator \( (Ce^{tA})^* \) maps \( L_2 (\Xi; H) \) into \( H \) and we can introduce the notion of a mild solution for equation (6). A mild solution of (6) is a couple \((Y,Z)\) of progressively measurable processes with values in \( H \), respectively \( L_2 (\Xi; H) \), such that

\[
\begin{aligned}
(Y, Z) \in C \left( [0,T] ; L^2 (\Omega; H) \right) \times L^2 \left( [0,T] \times \Omega; L_2 (\Xi; H) \right), \\
\sup_{t \in [0,T]} E \left[ |Y_t|^2 \right] + E \left[ \int_0^T |Z_t|^2 \, dt \right] < \infty, \\
\int_0^T \left( (Ce^{(s-t)A})^* Z_s \right) ds < \infty, \ P - a.s.,
\end{aligned}
\]

\[
Y_t = e^{(T-t)A} \xi + \int_t^T (Ce^{(s-t)A})^* Z_s \, ds - \int_t^T e^{(s-t)A} Z_s \, dW_s, \ t \in [0,T].
\]
If $C$ is a bounded linear operator, then it has been shown in Confortola [4, Th. 2.2] that (6) admits a unique mild solution. Let us suppose that

(A1) The operator $C$ may be written as sum of two linear operators $C_1$, $C_2$

$$C = C_1 + C_2,$$

satisfying the following properties:

1) $C_2$ is a bounded operator from $H$ to $L_2(\Xi; H)$,

2) for all $t > 0$, $C_1 e^{tA} \in L(H; L_2(\Xi; H))$. Moreover, we suppose that there exist some $\gamma \in [0, \frac{1}{2})$ and some positive constant $L > 0$ such that

$$|C_1 e^{tA}|_{L(H; L_2(\Xi; H))} \leq Lt^{-\gamma},$$

for all $t > 0$.

3) There exists some constant $a > \frac{1}{2}$ such that

$$A + a (C_1 e^{\delta A})^* (C_1 e^{\delta A})$$

is dissipative.

Remark 1 If $A$ is a self-adjoint, dissipative operator which generates a contraction semigroup, then (A.2) is obviously satisfied.

Moreover, if we suppose that $C_1$ takes its values in $L_2(\Xi; H)$, then we may replace (A1) 3) by

3') there exists some constant $a > \frac{1}{2}$ such that

$$A + a C_1^* C_1 e^{\delta A}$$

is dissipative.

Indeed, in this case $e^{2\delta A}$ is a bounded operator which commutes with the self-adjoint positive operator $-A$ and also with its square root $\sqrt{-A}$. Thus, for all $x \in D(A)$,

$$\langle e^{2\delta A}(-A)x, x \rangle = \left| e^{\delta A} \sqrt{-A} x \right|^2 \leq \left| \sqrt{-A} x \right|^2 = \langle (-A)x, x \rangle.$$

It follows that $A - e^{\delta A}Ae^{\delta A^*}$ is dissipative. Therefore, also $A + ae^{\delta A^*}C_1^* C_1 e^{\delta A}$ is dissipative.

We now can state the main result of this section.

**Theorem 1** Under the assumptions (A1) and (A2), there exists a unique mild solution of the backward linear stochastic differential equation (6). Moreover, this solution satisfies

$$\sup_{t \in [0, T]} E \left[ |Y_t|^2 \right] + E \left[ \int_0^T |Z_s|^2 \, ds \right] \leq k E \left[ |\xi|^2 \right],$$

(7)

where $k > 0$ is some constant that doesn't depend on the particular choice of $\xi$ but only on the operators $A, C$ and the time horizon $T$. 
Remark 2.1. The existence and uniqueness of the solution for equation (6) has been studied by Tessitore [22] for the case in which $A$ generates an analytic semigroup of contractions of negative type; the Brownian motion was supposed to be finite-dimensional. His main assumption, the joint dissipativity condition, was justified by its necessity for the "well-posedness" and coercivity of the forward system. The approach is fundamentally different from ours and relies on duality methods. However, let us point out that the author obtains, for his analytic case, stronger space regularity properties for the solution of the BSDE.

2. Ma, Yong [17] treated a particular linear, degenerate BSDE. Their method relies on a parabolicity assumption and a priori estimates that allowed the authors to get the well-posedness of the problem, the existence, the uniqueness as well as regularity properties. Later the same technique was used by Hu, Ma, Yong [12] for further extensions.

Proof (of Theorem 1). We begin by proving the existence: The main difficulty to prove the existence and the uniqueness for a BSDE in infinite dimensions with unbounded linear operators consists in the fact that Itô’s formula can’t be applied directly to this equation because it is defined only in the mild sense. To overcome this difficulty, we have to reduce the problem with the help of two different approximations to BSDEs that allow the application of Itô’s formula. We first approximate our original BSDE by the following one:

$$\begin{cases} dY^\delta_t = -A^*Y^\delta_t dt - (C_1 e^{\delta A})^* Z^\delta_t dt - C_2^* Z^\delta_t dt + Z^\delta_t dW_t, \\ Y^\delta_T = \xi \in L^2(\Omega, F_T, P; H) \end{cases}$$

(8)

For this approximating equation we know that, due to the results of Confortola [3], there exists a unique mild solution $(Y^\delta, Z^\delta)$ for every $\delta > 0$.

In a first step we prove that

Step 1. There is a positive constant $k$ independent of $\delta > 0$ and $\xi$ such that

$$\sup_{t \in [0,T]} E \left[ |Y^\delta_t|^2 \right] + E \left[ \int_0^T |Z^\delta_s|^2 ds \right] \leq k E |\xi|^2. \quad (9)$$

Indeed, we introduce the Yosida approximation of the dissipative operator $A^*$, $A^*_n = n(nI - A^*)^{-1}A^* = J_n^*A^*$, and we consider the following approximating BSDE:

$$\begin{cases} dY^{n,\delta}_t = -A^*_n Y^{n,\delta}_t dt - J_n^* (C_1 e^{\delta A})^* Z^{n,\delta}_t dt - C_2^* Z^{n,\delta}_t dt + Z^{n,\delta}_t dW_t, \\ Y^{n,\delta}_T = \xi \in L^2(\Omega, F_T, P; H). \end{cases}$$

It is well known that the above equation admits a unique solution $(Y^{n,\delta}, Z^{n,\delta})$. Let $\alpha < 2a$ and $\beta > 0$ be such that $\frac{1}{\alpha} + \frac{1}{\beta} < 1$. Then, by applying Itô’s
formula to $|Y^{n,\delta}|^2$ we obtain

$$E|\xi|^2 = E \left[ |Y^{n,\delta}|^2 \right] - 2E \left[ \int_0^T \left( A_n^* Y^{n,\delta}, Y^{n,\delta} \right) \right]$$

$$- 2E \left[ \int_t^T \left( J_n^* \left( C_1 e^{\delta A} \right)^* Z^{n,\delta}_s, Y^{n,\delta}_s \right) \right]$$

$$- 2E \left[ \int_t^T \left( C_2^* Z^{n,\delta}_s, Y^{n,\delta}_s \right) \right] + E \left[ \int_t^T \left| Z^{n,\delta}_s \right|^2 ds \right]$$

$$\geq E \left[ |Y^{n,\delta}|^2 \right] + \left( 1 - \frac{1}{\alpha} - \frac{1}{\beta} \right) E \left[ \int_t^T \left| Z^{n,\delta}_s \right|^2 ds \right]$$

$$- 2E \left[ \int_t^T \left( A_n^* + \frac{\alpha}{2} J_n^* \left( C_1 e^{\delta A} \right)^* \left( C_1 e^{\delta A} \right) J_n \right) Y^{n,\delta}_s, Y^{n,\delta}_s \right]$$

$$- \beta |C_2|^2 E \left[ \int_t^T \left| Y^{n,\delta}_s \right|^2 ds \right], \quad (10)$$

On the other hand, with the help of assumption (A.2) we can prove that

$$A_n^* + \frac{\alpha}{2} J_n^* \left( C_1 e^{\delta A} \right)^* \left( C_1 e^{\delta A} \right) J_n$$

$$= -n^{-1} A_n^* A_n + J_n^* \left( A^* + \frac{\alpha}{2} \left( C_1 e^{\delta A} \right)^* \left( C_1 e^{\delta A} \right) \right) J_n$$

is a dissipative operator. It then follows from (10) that

$$E \left[ |Y^{n,\delta}|^2 \right] + \left( 1 - \frac{1}{\alpha} - \frac{1}{\beta} \right) E \left[ \int_t^T \left| Z^{n,\delta}_s \right|^2 ds \right]$$

$$\leq E |\xi|^2 + \beta |C_2|^2 E \left[ \int_t^T \left| Y^{n,\delta}_s \right|^2 ds \right],$$

and Gronwall’s inequality yields

$$\sup_{t \in [0,T]} E \left[ |Y^{n,\delta}_t|^2 \right] + E \left[ \int_0^T \left| Z^{n,\delta}_s \right|^2 ds \right] \leq kE |\xi|^2, \quad (11)$$

Notice that the constant $k$ here is independent of $n \geq 1, \delta > 0$ and of $\xi$; it denotes a generic constant whose value can change from line to line. From the above estimate we can conclude that there is a subsequence, still denoted $(Y^{n,\delta}, Z^{n,\delta})_n$, such that $Y^{n,\delta} \rightharpoonup Y^\delta$ weakly * in $L^\infty([0,T]; L^2(\Omega; H))$ and $Z^{n,\delta} \rightharpoonup Z^\delta$ weakly in $L^2(\Omega \times [0,T]; L_2(\Xi; H))$. It can be easily proved the limit $(Y^\delta, Z^\delta)$ is the unique mild solution of (8). This allows to consider for $Y^\delta$ its version in $C([0,T]; L^2(\Omega; H))$. Finally, from Mazur’s theorem we obtain that $(Y^\delta, Z^\delta)$ satisfies the estimate announced in step 1.
In preparation of the next step we observe that, since \((Y^\delta, Z^\delta)_{\delta > 0}\) is bounded in \(L^\infty ([0, T]; L^2(\Omega; H)) \times L^2 (\Omega \times [0, T]; L_2(\Xi; H))\), we get the existence of some subsequence, again denoted by \((Y^\delta, Z^\delta)_{\delta > 0}\), such that \(Y^\delta \to Y\) weak * in \(L^\infty ([0, T]; L^2(\Omega; H))\) and \(Z^\delta \to Z\) weakly in \(L^2 (\Omega \times [0, T]; L_2(\Xi; H))\), as \(\delta \to 0\).

We want to prove that the couple \((Y, Z)\) obtained above is a mild solution of our BSDE:

\[
Y_t = e^{(T-t)A^\ast} \xi + \int_t^T \left( C_1 e^{(s-t)A} \right)^* Z_s ds \\
+ \int_t^T e^{(s-t)A^*} C_2^* Z_s ds - \int_t^T e^{(s-t)A^*} Z_s dW_s. \tag{12}
\]

For this we notice that, since \((Y^\delta, Z^\delta)\) is a mild solution of (8), we have

\[
Y^\delta_t = e^{(T-t)A^\ast} \xi + \int_t^T e^{(s-t)A^*} \left( C_1 e^{\delta A} \right)^* Z^\delta_s ds \\
+ \int_t^T e^{(s-t)A^*} C_2^* Z^\delta_s ds - \int_t^T e^{(s-t)A^*} Z^\delta_s dW_s, \tag{13}
\]

and we show the following:

**Step 2** The process

\[
M_1^{1, \delta} = \int_t^T e^{(s-t)A^*} \left( C_1 e^{\delta A} \right)^* Z^\delta_s ds, \ t \in [0, T],
\]

belongs to \(L^\infty ([0, T]; L^2(\Omega; H))\) and converges weakly * in \(L^\infty ([0, T]; L^2(\Omega; H))\) to \(M^1 = \left( \int_t^T \left( C_1 e^{(s-t)A} \right)^* Z_s ds \right) e_{t \in [0, T]}\).

Indeed, by using that

\[
e^{\delta^\ast A^*} \left( C_1 e^{\delta A} \right)^* = \left( C_1 e^{(\delta + \delta')A} \right)^*,
\]

for all \(\delta, \delta' > 0\), we have

\[
E \left[ \left| \int_t^T e^{(s-t)A^*} \left( C_1 e^{\delta A} \right)^* Z^\delta_s ds \right|^2 \right] \\
\leq E \left[ \left( \int_t^T e^{\delta A^*} \left( C_1 e^{(s-t)A} \right)^* Z^\delta_s ds \right)^2 \right] \\
\leq kE \left[ \int_t^T (s-t)^{-2\gamma} ds \int_t^T |Z^\delta_s|^2 ds \right] \\
\leq kE |\xi|^2,
\]
which implies that \( \{M^{1,\delta}, \delta > 0\} \subset L^\infty([0,T]; L^2(\Omega; H)) \) is bounded. Moreover, for all \( \phi \in L^2(\Omega; H) \) and \( t \in [0,T] \),

\[
E\left[\langle M_t^{1,\delta}, \phi \rangle\right] = E\left[\int_t^T \left\langle (C_1 e^{t(s-t)A})^* Z_s^\delta, (e^{\delta A^*} - I) \phi \right\rangle ds\right]
+ E\left[\int_t^T \left\langle (C_1 e^{t(s-t)A})^* Z_s^\delta, \phi \right\rangle ds\right] =: I_1^\delta + I_2^\delta, \tag{14}
\]

where

\[
I_1^\delta = E\left[\int_t^T \left\langle (C_1 e^{t(s-t)A})^* Z_s^\delta, (e^{\delta A^*} - I) \phi \right\rangle ds\right]
\leq E\left[\left| (e^{\delta A^*} - I) \phi \right| \int_t^T \left| (C_1 e^{t(s-t)A})^* Z_s^\delta \right| ds\right]
\leq \left( E\left[\int_t^T (s-t)^{-2\gamma} ds \int_t^T |Z_s^\delta|^2 ds\right]\right)^{\frac{1}{2}} \left( E\left[\left| (e^{\delta A^*} - I) \phi \right|^2\right]\right)^{\frac{1}{2}}
\leq k (E|\xi|^2)^{\frac{1}{2}} \left( E\left[\left| (e^{\delta A^*} - I) \phi \right|^2\right]\right)^{\frac{1}{2}}.
\]

Consequently, due to the dominated convergence theorem,

\( I_1^\delta \to 0 \) as \( \delta \to 0 \).

For the second term we have

\[
I_2^\delta = E\left[\int_t^T \left\langle (C_1 e^{t(s-t)A})^* Z_s^\delta, \phi \right\rangle ds\right] = E\left[\int_t^T \left\langle Z_s^\delta, (C_1 e^{t(s-t)A}) \phi \right\rangle ds\right],
\]

and since \( (C_1 e^{t(s-t)A}) \phi \in L^2(\Omega \times [0,T]; L_2(\Xi; H)) \), it follows from the weak convergence of \( Z_s^\delta \) to \( Z_s \) that

\[
I_2^\delta = E\left[\int_t^T \left\langle (C_1 e^{t(s-t)A})^* Z_s^\delta, \phi \right\rangle ds\right] \to E\left[\int_t^T \left\langle (C_1 e^{t(s-t)A})^* Z_s, \phi \right\rangle ds\right],
\]

and from (14) we then get

\[
E\left[\langle M_t^{1,\delta}, \phi \rangle\right] \to E\left[\langle M_t^1, \phi \rangle\right] \text{ as } \delta \to 0.
\]

In order to prove that \( M^{1,\delta} \) converges in the weak * topology on \( L^\infty([0,T]; L^2(\Omega; H)) \) to \( M^1 \), we consider \( \Phi \in L^1([0,T]; L^2(\Omega; H)) \), and use the fact that, for all \( t \in [0,T] \) for which \( \Phi_t \in L^2(\Omega; H) \), the previous convergence holds with \( \Phi_t \) at the place of \( \phi \). We then apply a dominated convergence argument and get the statement of step 2.
Step 3. The couple \((Y, Z)\) is a solution of the BSDE

\[
Y_t = e^{(T-t)A^*} \xi + \int_t^T \left( C_1 e^{(s-t)A} \right)^* Z_s ds \\
+ \int_t^T e^{(s-t)A^*} C_2^* Z_s ds - \int_t^T e^{(s-t)A^*} Z_s dW_s.
\] (15)

Moreover,

\[
\sup_{t \in [0,T]} E \left[ |Y_t|^2 \right] + E \left[ \int_0^T |Z_s|^2 ds \right] \leq k E |\xi|^2.
\] (16)

To prove the above statement we write \(Y_t^\delta, t \in [0, T]\), as

\[
Y_t^\delta = e^{(T-t)A^*} \xi + M_t^{1, \delta} + M_t^{2, \delta} + M_t^{3, \delta}.
\]

While we have already studied the convergence of \(M_t^{1, \delta}\) in the preceding step, it is an immediate consequence of the boundedness of the operator \(C_2\) that \(M_t^{2, \delta} = \int_t^T e^{(s-t)A^*} C_2^* Z_s^\delta ds\) converges weakly * in \(L^\infty ([0, T]; L^2(\Omega; H))\) to \(M_t^2 = \int_t^T e^{(s-t)A^*} C_2^* Z_s ds\).

For the noise term \(M_t^{3, \delta} = \int_t^T e^{(s-t)A^*} Z_s^\delta dW_s\) we notice that since \(Z^\delta\) converges weakly in \(L^2(\Omega \times [0, T]; L^2(\mathbb{Z}; H))\) to \(Z\), \(e^{(t-s)A^*} Z^\delta\) also converges weakly to \(e^{(t-s)A^*} Z\). We apply the martingale representation theorem to get that \(\int_t^T e^{(s-t)A^*} Z_s^\delta dW_s\) converges weakly in \(L^2(\Omega; H)\) to \(\int_t^T e^{(s-t)A^*} Z_s dW_s\). Using, as before, the dominated convergence, we get that

\[
N_t^\delta = \int_t^T e^{(s-t)A^*} Z_s^\delta dW_s\converges in the weak* topology on \(L^\infty ([0, T]; L^2(\Omega; H))\) to \(N_t = \int_t^T e^{(s-t)A^*} Z_s dW_s\).
\]

We now pass to the \(L^\infty ([0, T]; L^2(\Omega; H))\) weak * limit in the approximating mild equation (13). This yields the statement of step 3, with the only difference, that for the BSDE which has been got by a weak limit, we only know for the moment that this equation is satisfied \(dtdP\)-a.e. To obtain that the BSDE is satisfied by \((Y, Z)\) for all time points of the interval \([0, T]\), \(P\)-a.s., we need the following auxiliary statement:

**Lemma 1** The process

\[
\Phi_t = e^{(T-t)A^*} \xi + (C_1 e^{(r-t)A})^* Z_r dr + \int_t^T e^{(r-t)A^*} C_2^* Z_r dr \\
- \int_t^T e^{(r-t)A^*} Z_r dW_r, t \in [0, T], is mean-square continuous.
\]

**Proof** We return to the proof of our theorem. The proof of the lemma will be given afterwards.

The above result allows to conclude the proof of step 3. Indeed, the above lemma guarantees the existence of a version of the solution \((Y, Z)\) in
The only solution \((Y, Z)\) of the BSDE

\[
\begin{cases}
    dY_t = -A^* Y_t dt - C^* Z_t dt + Z_t dW_t, \\
    \dot{Y}_T = 0.
\end{cases}
\]

is the trivial one: \((Y, Z) = (0, 0)\).

To prove this, we have to transform the BSDE into an equation which allows to apply Itô’s formula. For this reason we put, for all \(t\)

\[
J_n = \left( J_n^* e^{\delta A^*} A^* Y \right).
\]

and we observe that the such introduced process \(\widetilde{Y}\) satisfies the following backward equation:

\[
\begin{cases}
    d\widetilde{Y}_t = -A^* \widetilde{Y}_t dt - J_n^* (C_1 e^{\delta A^*})^* Z_t dt - J_n^* e^{\delta A^*} C_2 Z_t dt + J_n^* e^{\delta A^*} Z_t dW_t, \\
    \dot{\widetilde{Y}}_T = 0.
\end{cases}
\]

To this equation we can apply Itô’s formula (Indeed, notice that \(A^* \widetilde{Y} = (J_n^* e^{\delta A^*} A^*) Y\), where the operator \(J_n^* e^{\delta A^*} A^*\) is bounded). This yields:

\[
0 = E \left[ \left| J_n^* e^{\delta A^*} Y_t \right|^2 \right] - 2E \left[ \int_t^T \langle A^* \widetilde{Y}_s, \dot{\widetilde{Y}}_s \rangle \, ds \right]
- 2E \left[ \int_t^T \langle J_n^* (C_1 e^{\delta A^*})^* Z_s, \dot{\widetilde{Y}}_s \rangle \, ds \right]
- 2E \left[ \int_t^T \langle J_n^* e^{\delta A^*} C_2 Z_s, \dot{\widetilde{Y}}_s \rangle \, ds \right] + E \left[ \int_t^T |J_n^* e^{\delta A^*} Z_s|^2 \, ds \right]
\geq E \left[ \left| J_n^* e^{\delta A^*} Y_t \right|^2 \right] - 2E \left[ \int_t^T \left( A^* + \frac{\alpha}{2} J_n^* (C_1 e^{\delta A^*})^* (C_1 e^{\delta A^*}) J_n \right) \widetilde{Y}_s, \dot{\widetilde{Y}}_s \right] \, ds
- \beta |C_2|^2 E \left[ \int_t^T |Y_s|^2 \, ds \right] + E \left[ \int_t^T |J_n^* e^{\delta A^*} Z_s|^2 \, ds \right]
- \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) E \left[ \int_t^T |Z_s|^2 \, ds \right],
\]

(17)

To be able to go ahead with the above estimate we need the dissipativity of the operator \(A^* + \frac{\alpha}{2} J_n^* (C_1 e^{\delta A^*})^* (C_1 e^{\delta A^*}) J_n\).

For this end we notice that

\[
(nI - A^*) A^* (nI - A) - n^2 A^* = -n A^* A^* - n A^* A + A^* A^* A
\]
and apply this relation to the operator \((nI - A)^{-1}\). To the relation we then apply \((nI - A^*)^{-1}\). So we obtain the following equality:

\[
A^* - J^*_n A^* J_n = -n^{-1} J^*_n (A^*)^2 J_n - n^{-1} J^*_n A^* A J_n + n^{-2} J^*_n A^* A^* A J_n,
\]

which proves that the operator \(A^* - J^*_n A^* J_n\) is dissipative. It now follows easily that also the operator

\[
A^* + \frac{\alpha}{2} J^*_n \left( C_1 e^{\delta A} \right)^* \left( C_1 e^{\delta A} \right) J_n
\]

is dissipative if the parameters \(\alpha, \beta\) are chosen as in \(\text{(10)}\).

This dissipativity allows to go ahead in \(\text{(17)}\) and to conclude that

\[
E \left[ |J^*_n e^{\delta A^*} Y_t|^2 \right] + E \left[ \int_t^T |J^*_n e^{\delta A^*} Z_s|^2 ds \right] \\
\leq \beta |C_2|^2 E \left[ \int_t^T |Y_s|^2 ds \right] + \left( \frac{1}{\alpha} + \frac{1}{\beta} \right) E \left[ \int_t^T |Z_s|^2 ds \right].
\]

Recall that \((Y,Z) \in L^2 (\Omega \times [0,T]; H \times L^2(\Xi;H))\). Thus, letting \(n \to \infty\) and then \(\delta \to 0\) in the above estimate, we get

\[
E \left[ |Y_t|^2 \right] + \left( 1 - \frac{1}{\alpha} - \frac{1}{\beta} \right) E \left[ \int_t^T |Z_s|^2 ds \right] \leq k\beta |C_2|^2 E \left[ \int_t^T |Y_s|^2 ds \right].
\]

Finally, we take the supremum over \(t \in [0,T]\) and apply Gronwall’s inequality. Thus we obtain

\[
\sup_{t \in [0,T]} E \left[ |Y_t|^2 \right] + E \left[ \int_0^T |Z_s|^2 ds \right] = 0,
\]

and the claimed uniqueness follows as immediate consequence. ■

In order to really complete the proof of the theorem we still have to give the proof of Lemma 1.
Proof (of Lemma 1) A standard estimate for the process $\Phi$ defined in Lemma 1 gives the following for all $s, t \geq 0$:

\[
E \left[ |\Phi_t - \Phi_s|^2 \right] \leq k \left( E \left[ \left( e^{(t-s)|A^* - I|} \Phi_{t\vee s} \right)^2 \right] 
+ E \left[ \int_{s \land t}^{s \lor t} \left( C_1 e^{(r-s)A} \right)^* Z_r \right]^2 \right) 
+ E \left[ \int_{s \land t}^{s \lor t} e^{(r-s)A^* C_2 Z_r} \right]^2 
+ E \left[ \int_{s \land t}^{s \lor t} e^{(r-t)A^* Z_r} \right]^2 dr \right) 
\leq k \left( E \left[ \left( e^{(t-s)|A^* - I|} \Phi_{t\vee s} \right)^2 \right] 
+ (1 + |t - s|^{1-2\gamma}) E \left[ \int_{s \land t}^{s \lor t} |Z_r|^2 dr \right] \right).
\tag{18}
\]

Here $k$ denotes a generic constant that is independent of $s, t \in [0, T]$ and can change from line to line.

Since $Z \in L^2 (\Omega \times [0, T] ; L_2 (\Xi ; H))$, it is a direct consequence of the dominated convergence theorem that

\[
\lim_{s \to t} E \left[ \int_{s \land t}^{s \lor t} |Z_r|^2 dr \right] = 0.
\]

It remains to show that also $E \left[ \left( e^{(t-s)|A^* - I|} \Phi_{t\vee s} \right)^2 \right]$ converges to zero, as $s \to t$. We first consider this limit for $t > s \uparrow t$. In this case

\[
E \left[ \left( e^{(t-s)|A^* - I|} \Phi_{t\vee s} \right)^2 \right] = E \left[ \left( e^{(t-s)|A^* - I|} \Phi_t \right)^2 \right],
\]

and the wished convergence follows from the dominated convergence theorem.
Let us now study the case in which \( t < s \leq t \). For this end we notice that, for all \( s \geq t \),

\[
E \left[ \left| \left( e^{[t-s]A^*} - I \right) \Phi_{t \vee s} \right|^2 \right] \\
\leq c \left( E \left[ \left| \left( e^{(s-t)A^*} - I \right) e^{(T-s)A^*} \xi \right|^2 \right] \\
+ E \left[ \int_s^T \left( e^{(s-t)A^*} - I \right) \left( C_1 e^{(r-s)A^*} \right)^* Z_r \, dr \right]^2 \right] \\
+ E \left[ \int_s^T \left( e^{(s-t)A^*} - I \right) e^{(r-s)A^*} C_2 Z_r \, dr \right]^2 \right] \\
+ E \left[ \int_s^T \left( e^{(s-t)A^*} - I \right) e^{(r-s)A^*} Z_r \, dW_r \right]^2 \right) \\
= I_1(s) + I_2(s) + I_3(s) + I_4(s).
\] (19)

For the first term we get from the dominated convergence theorem that

\[
I_1(s) \leq k E \left[ \left| \left( e^{(s-t)A^*} - I \right) \xi \right|^2 \right] \to 0 \text{ as } s \downarrow t.
\]

Next,

\[
I_2(s) \leq \left( \int_s^T (r-s)^{-2\gamma} \, dr \right) \times \\
\times E \int_t^T I_{[s,T]}(r) \left| (r-s)^\gamma \left( e^{(s-t)A^*} - I \right) \left( C_1 e^{(r-s)A^*} \right)^* Z_r \right|^2 \, dr.
\] (20)

We let \( t < r \leq T \) and choose an arbitrary \( s_0 \in [t, r[ \). Then, for all \( t < s < s_0 \),

\[
\left| \left( e^{(s-t)A^*} - I \right) \left( C_1 e^{(r-s)A^*} \right)^* Z_r \right| \\
= \left| \left( e^{(s-t)A^*} - I \right) e^{(s_0-s)A^*} \left( \left( C_1 e^{(r-s_0)A^*} \right)^* Z_r \right) \right| \\
\leq k \left| \left( e^{(s-t)A^*} - I \right) \left( C_1 e^{(r-s_0)A^*} \right)^* Z_r \right|.
\]

Obviously, the latter expression converges to 0 as \( s \downarrow t \). Consequently

\[
I_{[s,T]}(r) \left| (r-s)^\gamma \left( e^{(s-t)A^*} - I \right) e^{(r-s)A^*} C_2 Z_r \right|^2 \to 0, \text{ for all } r > t,
\]

and from the dominated convergence theorem it follows that

\[
I_2(s) \to 0 \text{ as } s \downarrow t.
\]
A similar argument yields $I_3(s) \to 0$ as $s \downarrow t$. Finally, for the last term, we have
\[
I_4(s) \leq E \left[ \int_s^T \left( (e^{(s-t)A^*} - I) e^{(r-s)A^*} Z_r \right)^2 \, dr \right]
\leq E \left[ \int_s^T \left( (e^{(s-t)A^*} - I) Z_r \right)^2 \, dr \right],
\]
and, again by the dominated convergence theorem,
\[
I_4(s) \to 0 \text{ as } s \downarrow t.
\]
Therefore, returning to (19) we get
\[
\lim_{s \downarrow t} E \left[ \left( e^{|t-s|A^*} - I \right) \Phi_{t\vee s} \right] = 0.
\]
This concludes the proof of our lemma.

After having studied the existence and unique for the BSDE adjoint to our forward stochastic control problem we are able now to characterize their duality. For the sake of simplicity, we shall assume from now on that $C_1$ takes its values in $L_2(\Xi;H)$.

**Proposition 1** Let $X^{x,u}$ be the unique mild solution of (7) associated to an admissible control $u$, and let $(Y,Z)$ be the unique mild solution of (6). Then the following duality relation holds true
\[
E \left[ \langle X_T^{x,u}, Y_T \rangle \right] = E \left[ \langle x, Y_0 \rangle \right] + E \left[ \int_0^T \langle Bu_s, Y_s \rangle \, ds \right]. \tag{21}
\]

**Proof** For the proof of the duality relation we have the same difficulty as in the proof of Theorem 1: we can’t apply Itô’s formula directly to our forward SDE and our BSDE in infinite dimensions. This is why we consider the following approximating equations
\[
\begin{aligned}
\{dX^{n,\delta}_t\} &= \left( A_n X^{n,\delta}_t + Bu_t \right) \, dt + \left( C_1 e^{\delta A} J_n + C_2 \right) X^{n,\delta}_t \, dW_t, \\
X^{n}_0 &= x \in H,
\end{aligned}
\]
and
\[
\begin{aligned}
\{dY^{n,\delta}_t\} &= - \left( A_n^* Y^{n,\delta}_t + J_n^* e^{\delta A^*} C_1^* Z^{n,\delta}_t + C_2^* Z^{n,\delta}_t \right) \, dt + Z^{n,\delta}_t \, dW_t, \\
Y^{n,\delta}_T &= \xi \in L^2(\Omega,\mathcal{F}_T,\mathbb{P};H).
\end{aligned}
\]
Recall that $A_n^* = n(nI - A^*)^{-1}A^* = J_n^* A^*$. To the above approximating equations we now can apply Itô’s formula, and we get
\[
E \left[ Y^{n,\delta}_t X^{n,\delta}_s \right] = E \left[ Y^{n,\delta}_t X^{n,\delta}_t \right] + E \left[ \int_s^t \langle Bu_r, Y^{n,\delta}_r \rangle \, dr \right], \tag{22}
\]
for all $0 \leq t < s \leq T$. Moreover, standard SDE and BSDE estimates show that there exists some positive constant $k$ (not depending on $\delta$ and $n$), such that

$$E \left[ \sup_{t \in [0,T]} \left| X_{t}^{n,\delta} \right|^2 \right] \leq k \left( 1 + |x|^2 \right)$$

and

$$\sup_{t \in [0,T]} E \left[ \left| Y_{t}^{n,\delta} \right|^2 \right] + E \left[ \int_{0}^{T} \left| Z_{s}^{n,\delta} \right|^2 ds \right] \leq k E \left[ \left| \xi \right|^2 \right].$$

It follows that there exists some subsequence, still denoted $(X_{n,\delta}, Y_{n,\delta}, Z_{n,\delta})$, which converges weakly to some limit $(X', Y', Z)$ in

$L^2(\Omega \times [0,T]; P \otimes dt; H \times H) \times L^2(\Omega \times [0,T]; P \otimes dt; L^2(\Xi; H))$ as $n \to \infty$, $\delta \downarrow 0$. We denote by $X$ the continuous version of $X'$; it is the unique mild solution of equation (4). Moreover, we let $Y$ be the $dt \otimes dP$-version of $Y'$, which belongs to $C \left( [0,T]; L^2(\Xi; H) \right)$, and is, together with the process $Z$, the unique mild solution of (5). Moreover, from the above estimates satisfied by $(X_{n,\delta}, Y_{n,\delta}, Z_{n,\delta})$ we get with the help of Mazur’s theorem estimate (7) and

$$E \left[ \sup_{t \in [0,T]} \left| X_{t} \right|^2 \right] \leq k \left( 1 + |x|^2 \right).$$

Moreover, if we take the weak limit as $n \to \infty$ and $\delta \downarrow 0$ in (22) we get

$$E \left\langle Y_{s}', X_{t}' \right\rangle = E \left\langle Y_{t}', X_{s}' \right\rangle + E \left[ \int_{t}^{s} \langle Bu_{r}, Y_{r}' \rangle dr \right], \quad dt \otimes dP \text{-a.e.}, \ 0 \leq t < s \leq T.$$

Consequently,

$$E \left\langle Y_{s}, X_{s} \right\rangle = E \left\langle Y_{t}, X_{t} \right\rangle + E \left[ \int_{t}^{s} \langle Bu_{r}, Y_{r} \rangle dr \right], \quad \text{for } 0 \leq t < s \leq T.$$

Finally, by taking $s = T$ and $t = 0$, we get the assertion. The proof is complete.

The connection between equation (6) and the approximate controllability of (4) is given by the following result that generalizes those of the finite dimensional case.

**Proposition 2** (i) The linear stochastic equation (4) is approximately controllable if and only if, for every finite time horizon $T > 0$, any solution of the dual equation (6) that satisfies $B^*Y_s = 0$ $dP$-a.s., for all $0 \leq s \leq T$, necessarily vanishes $ds \otimes dP$ – a.s., i.e. $Y_s = 0$ $dP$-a.s., for all $0 \leq s \leq T$.

(ii) The linear stochastic equation (4) is approximately null-controllable if and only if, for all finite time horizon $T > 0$, any solution of the dual equation (6) satisfying $B^*Y_s = 0$ $dP$-a.s., for all $0 \leq s \leq T$, is such that $Y_0 = 0$ $dP$-a.e.
Proof For any arbitrarily fixed time horizon \( T > 0 \) we get from the previous proposition that

\[
E[\langle X^x_T, Y_T \rangle] = E[\langle x, 0 \rangle] + E \left[ \int_0^T (Bu_s, Y_s) \, ds \right].
\] (23)

We introduce the linear operator \( M : U \rightarrow L^2(\Omega, \mathcal{F}_T, P; H) \) which associates to every admissible control \( u \) the mild solution of (1) starting from \( x = 0 \):

\[
M(u) = X^0_T = \int_0^T e^{sA}Bu_s \, ds + \int_0^T e^{sA}CX_{x,0}u \, dW_s.
\]

Obviously, the approximate controllability (at time \( T \)) for (1) is equivalent to the condition that \( M \) has an image space dense in \( L^2(\Omega, \mathcal{F}_T, P; H) \). This allows to deduce from (23) the form of the dual operator of \( M \),

\[
M^\ast \xi = B^\ast Y.
\]

On the other hand, since the density of the value domain of the bounded linear operator \( M \in L(L^2(\Omega, \mathcal{F}_T, P; H)) \) is equivalent with the condition that the kernel of its adjoint operator \( M^\ast \) is trivial, we obtain from the above relation the first assertion.

For the proof of the second assertion we introduce the operator \( L : H \rightarrow L^2(\Omega, \mathcal{F}, P; H) \) which associates to each initial state \( x \in H \) the mild solution of (1) corresponding to the control \( u \equiv 0 \):

\[
L(x) = e^{tA}x + \int_0^T e^{sA}CX_{x,0} \, dW_s.
\]

From the relation \( X^x_T = L(x) + M(u) \) we deduce easily that the approximate null-controllability of \( X \) is equivalent to the condition that \( \overline{\text{Im}}(L) \subset \overline{\text{Im}}(M) \) (\( \overline{\text{Im}}(L), \overline{\text{Im}}(M) \) are the closures of the image spaces of \( L \) and \( M \), resp.) and hence also to the following condition:

\[
\text{Ker}(M^\ast) \subset \text{Ker}(L^\ast).
\]

On the other hand, from (23) we get \( L^\ast \xi = Y_0 \). This relation together with \( M^\ast \xi = B^\ast Y = 0 \) allow now to see the equivalence between the approximate null-controllability of \( X \) and the condition given in the second assertion. ■

In what follows we will need the notion of the backward viability kernel introduced by Buckdahn, Quincampoix, Răşcanu [2].

**Definition 1** Let \( K \) be a nonempty, convex, closed subset of \( H \).

(i) A continuous stochastic process \( \{Y_t, \, t \in [0, T]\} \) is called viable in \( K \) if and only if \( Y_t \in K, \, P\text{-a.s.}, \, \text{for all} \, t \in [0, T] \).

(ii) We say that the set \( K \) enjoys the backward stochastic viability property at time \( T \) with respect to \( \{\} \) if for every \( K \)-valued terminal condition \( \eta \in L^2(\Omega, \mathcal{F}_T, P; K) \), the solution \( \{Y^\eta_t, \, t \in [0, T]\} \) of (1) is viable in \( K \).

(iii) The largest closed, convex subset of \( K \) enjoying the backward stochastic viability property is called the backward stochastic viability kernel of \( K \).
The notion of the stochastic viability kernel allows to reformulate the criterion for the approximate controllability, stated in Proposition 2:

**Proposition 3** The linear stochastic equation (4) is approximately controllable if and only if, for every finite time horizon \( T > 0 \), the backward stochastic viability kernel of \( \text{Ker} \ B^* = \{ y \in H : B^* y = 0 \} \) at time \( T \) with respect to (6) is the trivial subspace \( \{ 0 \} \).

**Remark 3** In the finite dimensional case, the backward equation (6) may be interpreted as a forward controlled equation. Therefore, instead of studying the backward viability kernel, one may choose to investigate approximate controllability with the help of the (forward) viability kernel. Riccati methods are well adapted to control problems and allow nice characterizations of the (forward) viability kernel. The authors of [3] use these methods and show that approximate controllability of (4) is equivalent to the following invariance condition:

The largest \((A^*; C^*)\)-strictly invariant linear subspace of \( \text{Ker} B^* \) is \( \{ 0 \} \).

We recall that a linear subspace \( V \subset \mathbb{R}^n \) is said to be \((A^*; C^*)\)-strictly invariant if \( A^* V \subset \text{Span}\{ V; C^* V \} = \{ \lambda v + \mu w : v \in V, w \in C^* V \} \).

If \( H \) is infinite dimensional, and \( A \) is a generator of a strongly continuous group, similar arguments apply.

**Remark 4** Let us suppose that the Brownian motion \( W \) is 1-dimensional, \( B \in \mathcal{L}(H) \), and \( C \) is a linear (possibly unbounded) operator on \( H \) such that \( A^* B^* = B^* A^* \) and \( B^* C^* = C^* B^* \). Then (4) is approximately controllable if and only if the image space \( \text{Im}(B) \) is dense in \( H \).

Indeed, let us notice that if \( (Y, Z) \) is the mild solution of (6) and satisfies (7), then

\[
Y_t = e^{(T-t)A^*} \xi + \int_t^T e^{(s-t)A^*} C^* Z_s \, ds - \int_t^T e^{(s-t)A^*} Z_s \, dW_s,
\]

and, from the commutativity of \( B^* \) with \( A^* \) and with \( C^* \),

\[
B^* Y_t = e^{(T-t)A^*} B^* \xi + \int_t^T e^{(s-t)A^*} B^* C^* Z_s \, ds - \int_t^T e^{(s-t)A^*} B^* Z_s \, dW_s.
\]

Thus, \( B^* Y_t \) is the unique mild solution of the following BSDE:

\[
\begin{align*}
\tilde{d} Y_t &= -A^* \tilde{Y}_t dt - C^* \tilde{Z}_t dt + \tilde{Z}_t dW_t, \\
\tilde{Y}_T &= B^* \xi.
\end{align*}
\]

Obviously, \( \tilde{Y} = 0 \) if and only if \( B^* \xi = 0 \) \( P\)-a.s.. Thus, from Proposition 2 it follows that Eq. (4) is approximately controllable if, for all \( \xi \in L^2(\Omega, \mathcal{F}_T; P; H) \), the relation \( B^* \xi = 0, P-a.s. \), implies that \( \xi = 0, P-a.s. \). This is, of course, equivalent with the density of the image space \( \text{Im}(B) \) in \( H \).
4 A necessary condition for approximate controllability

We have seen that approximate controllability for the forward controlled equation (4) is equivalent to the following (approximate) observability condition on the dual equation (6):

\[ B^*Y_t = 0, \ dP - a.s., \ \text{for all} \ t \in [0, T], \ \implies Y_T = 0, \ dP - a.s. \]  \quad (24)

In the deterministic case, Russell and Weiss [20] generalized the Hautus test of observability for infinite dimensional equations with an operator \( A \) that is supposed to generate an exponentially stable semigroup. In what follows we assume besides (A1) and (A2) the following additional condition:

(A3) The linear operator \( A \) generates an exponentially stable, strongly continuous semigroup of operators.

Under the assumptions (A1)-(A3) we can prove the following statement:

Proposition 4 A necessary condition for the approximate controllability of (4) is that, for every \( y \in D(A^*) \) and every \( \alpha < 0 \),

\[ |B^*y| + |(A^* - \alpha I)y| > 0, \ \text{whenever} \ y \neq 0. \]  \quad (N1)

Proof In order to prove the claim, let us first notice that \( H_1 = D(A) \) endowed with the norm \( |h|_1 = |(A^* - \alpha I)h|_H \) is a Hilbert space. It is well known that, under the above assumptions, the family of norms indexed by \( \alpha < 0 \) are equivalent with the usual graph norm on \( H_1 \). For every \( y \in D(A^*) \) we let \((Y^y, Z^y)\) denote the unique mild solution in \( H_1 \) of the BSDE

\[
\begin{cases}
    dY^y_t = -A^*Y^y_t dt - C^*Z^y_t dt + Z^y_t dW_t, \\
    Y^y_T = y.
\end{cases}
\]

Since all data of this BSDE is deterministic it is immediate that \( Y^y \) is deterministic and \( Z^y = 0 \). In particular, we see that \( Y^y_t = e^{(T-t)A^*}y \) is a classical solution (in \( H_1 \)) of

\[
\begin{cases}
    dY^y_t = -\alpha Y^y_t dt - e^{(T-t)A^*}(A^* - \alpha I)y dt, \\
    Y^y_T = y.
\end{cases}
\]

and the function \( B^*Y^y \) is a classical solution of the following equation:

\[
\begin{cases}
    d(B^*Y^y_t) = -\alpha (B^*Y^y_t) dt - B^*e^{(T-t)A^*}(A^* - \alpha I)y dt \\
    B^*Y^y_T = B^*y.
\end{cases}
\]

It follows easily from this equation that \( B^*Y^y_t = 0 \), for all \( t \in [0, T] \), if and only if

\[
\begin{cases}
    B^*y = 0, \\
    B^*e^{(T-t)A^*}(A^* - \alpha I)y = 0, \ \text{for all} \ t \in [0, T].
\end{cases}
\]

Consequently, the condition (24) gives the following necessary condition for the approximate controllability of (4):

\[ "B^*Y^y_t = 0, \ \text{for all} \ t \in [0, T], \ \implies y = 0." \]
Obviously, the two latter conditions allow to conclude that
\[
\begin{cases}
B^*y = 0, \\
B^*e^{tA^*}(A^* - \alpha I)y = 0, \text{ for all } t \in [0, T],
\end{cases}
\] implies \( y = 0 \), \( \text{(25)} \)
and the estimate
\[
|B^*e^{tA^*}(A^* - \alpha I)y| \leq k|(A^* - \alpha I)y|,
\]
in combination with \( \text{(25)} \) allows to complete the proof. \( \blacksquare \)

**Remark 5**

Jacob, Partington \([13]\) studied the approximate controllability for a deterministic system. They supposed

\((JP)\) \( A \) is an infinitesimal generator of an exponentially stable, strongly continuous semigroup which possesses a sequence of normalized eigenvectors \( \{e_i\} \) corresponding to the eigenvalues \( \{\lambda_i\} \) such that \( \sup_i \lambda_i < 0 \). Moreover, they considered the case of a 1-dimensional input space, i.e. \( B \in L(\mathbb{R}; H) \).

In this particular case, the necessary and sufficient condition for approximate controllability of the deterministic system
\[
\begin{cases}
\frac{dX}{dt} = (AX + Bu) dt, \\
X_0 = x \in H,
\end{cases}
\]
found by the authors, says that for all \( y \in H_1 \) and all \( \alpha < 0 \),
\[
|B^*y|^2 + |(A^* - \alpha I)y|^2 > 0 \text{ whenever } y \neq 0.
\]

**Remark 6**

For the case in which \( H \) is \( n \)-dimensional Euclidean space (stochastic) approximate controllability was studied by Buckdahn, Quincampoix, Tesiitore \([3]\) and Goreac \([11]\). The equivalent condition for approximate controllability reads

The largest \((A^*; C^*)\) -strictly invariant subspace of \( \text{Ker } B^* \) is \( \{0\} \). \( \text{(26)} \)

Let us suppose that, for the framework studied by these authors, there exists a bounded linear operator \( D \in L(U) \) such that \( B^*C^* = DB^* \). Then we get that \( \text{Ker } B^* \) is \( C^* \)-invariant, and thus \( \text{(26)} \) can be written as follows:

The largest \( A^* \)-invariant subspace of \( \text{Ker } B^* \) is \( \{0\} \). \( \text{(27)} \)

Moreover, under the assumptions of Jacob, Partington \([13]\) \((JP)\), it is obvious that \( \text{(N1)} \) is equivalent to \( \text{(27)} \). Indeed, if \( \text{(N1)} \) holds true, then
\[
\begin{cases}
B^*e_i \neq 0, \text{ for all } 1 \leq i \leq n, \\
\lambda_i \neq \lambda_j, \text{ for all } 1 \leq i, j \leq n, \ i \neq j.
\end{cases}
\]
(see Jacob, Partington \([13]\), Theorem 4.1). Let \( V \) denote the largest \( A^* \)-invariant subspace of \( \text{Ker } B^* \), and let us suppose that there exists some linear combination \( v = \sum_{k=1}^m v_k e_k \) such that \( v \in V \), where \( m \leq \)
\( n, i_k \in \{1, 2, \ldots, n\} \) and \( v_{ik} \neq 0 \), for all \( 1 \leq k \leq m \). Then, for all \( j \geq m - 1 \), \( \sum_{k=1}^{m} \lambda_i^j v_{ik} e_k \in V \). Thus, since

\[
\det \left[ \lambda_i^j v_{ik} \right]_{k,j} = \prod_{1 \leq k \leq m} v_{ik} \prod_{1 \leq k < j \leq m} (\lambda_i^j - \lambda_i^k) \neq 0,
\]

we get that \( \text{span} \{e_{ik}, 1 \leq k \leq m\} \subset V \).

It follows that \( V = \text{span} \{e_{ik}, 1 \leq k \leq N\} \), for some \( N \leq n \). But then \( B^* e_{ik} = 0 \), and this contradicts our assumption and we have that \( V = \{0\} \).

For the converse, if \( [27] \) holds true and \( y \in H_1 \) such that

\[
|B^* y|^2 + |(A^* - \alpha I) y|^2 = 0 \quad \text{for some} \quad \alpha < 0,
\]

then \( V = \text{span} \{y\} \) is \( A^* \)-invariant and included in \( \text{Ker} B^* \). It follows that \( y = 0 \), and we get \( [N] \). This latter argument applies also when \( H \) has infinite dimension.

Let us now make the following assumptions:

**B** \( W \) is supposed to be a 1-dimensional Brownian motion, the control state space \( U \) is a bounded closed subspace of some separable real Hilbert space \( V \), \( B \in \mathcal{L}(V; H) \), \( A \) is a self adjoint operator which generates a semigroup of contractions on \( H \), and the operator \( C \) admits a decomposition

\[
C = C_1 + C_2,
\]

of two linear operators \( C_1, C_2 \) which are supposed to have the following properties:

1) \( C_2 \) is a bounded operator from \( H \) to \( H \);
2) for all \( t > 0 \), \( e^{tA} \), \( e^{tA} C_1 \in \mathcal{L}(H) \). Moreover, we suppose that there exist some \( \gamma \in [0, \frac{1}{2}) \) and some positive constant \( L > 0 \) such that

\[
|C_1 e^{tA}|_{\mathcal{L}(H)} + |e^{tA} C_1|_{\mathcal{L}(H)} \leq L t^{-\gamma},
\]

for all \( t > 0 \).

3) There exists some constant \( a > \frac{1}{2} \) such that

\[
A + a C_1^* C_1 \text{ is dissipative}.
\]

We recall the following

**Definition 2** Let \( A \) be the generator of a \( C_0 \)-semigroup on the Hilbert space \( H \) and \( C \) is a linear operator on \( H \). We say that \( C \) is a class-\( P \) perturbation of \( A \) if \( C \) is closed,

\[
D(C) \supset \cup_{t>0} e^{tA}(H) \quad \text{and} \quad \int_0^1 |C e^{tA}| dt < \infty.
\]
Obviously, under the above assumptions, the operator $C$ is a class-$P$
perturbation of $A$. It follows that $A + \lambda C$ is the generator of a $C_0$-semigroup
$(e^{t(A+\lambda C)})_{t \geq 0}$ for all $\lambda \in \mathbb{R}$ (cf. Davies \cite{Davies} Theorem 3.5).

For the study of the main result of this section we will need the following estimates:

**Lemma 2** Under our standard assumptions we have that, for some constant $k$,

$$
\left| C_1 e^{t(A+\lambda C)} \right|_{\mathcal{L}(H)} + \left| e^{t(A+\lambda C)} C_1 \right|_{\mathcal{L}(H)} \leq k \left( t^{-\gamma} + 1 \right),
$$

for all $t \in [0, T]$.

**Proof** From the theory of general perturbation of generators it follows that

$$
e^{t(A+\lambda C)}x = e^{tA}x + \lambda \int_0^t e^{(t-s)A} C_1 e^{s(A+\lambda C)} x ds + \lambda \int_0^t e^{(t-s)A} C_2 e^{s(A+\lambda C)} x ds,
$$

for all $x \in H$. Then, by applying on both sides of the above relation the bounded operator $C_2$, we get the following norm estimate:

$$
\left| C_1 e^{t(A+\lambda C)} x \right| \leq t^{-\gamma} \left| x \right| + \lambda \int_0^t (t - s)^{-\gamma} \left| C_1 e^{s(A+\lambda C)} x \right| ds + k \int_0^t (t - s)^{-\gamma} \left| x \right| ds,
$$

for all $t \in [0, T]$. Here $k$ denotes again a generic constant which can depend on $\lambda$ and $T$. Thus, applying Cauchy-Schwarz inequality yields

$$
\left| C_1 e^{t(A+\lambda C)} x \right|^2 \leq k \left( (t^{-2\gamma} + t^{2-2\gamma}) \left| x \right|^2 + t^{1-2\gamma} \int_0^t \left| C_1 e^{s(A+\lambda C)} x \right|^2 ds \right) \leq k \left( (t^{-2\gamma} + 1) \left| x \right|^2 + \int_0^t \left| C_1 e^{s(A+\lambda C)} x \right|^2 ds \right),
$$

and from Gronwall’s inequality we finally get

$$
\left| C_1 e^{t(A+\lambda C)} x \right|^2 \leq k \left( t^{-\gamma} + 1 \right)^2 \left| x \right|^2.
$$

It follows that $C_1 e^{t(A+\lambda C)} \in \mathcal{L}(H)$ and

$$
\left| C_1 e^{t(A+\lambda C)} \right|_{\mathcal{L}(H)} \leq k \left( t^{-\gamma} + 1 \right),
$$

for all $t \in [0, T]$. Using a similar argument we can prove that $e^{t(A+\lambda C)} C_1 \in \mathcal{L}(H)$ and

$$
\left| e^{t(A+\lambda C)} C_1 \right|_{\mathcal{L}(H)} \leq k \left( t^{-\gamma} + 1 \right),
$$

for all $t \in [0, T]$. □
To establish the main result of this section we shall further introduce
the following set standing for the joint dissipativity condition on \( A, C \):
\[
\Lambda = \left\{ \lambda \in \mathbb{R} : \exists a > \frac{1}{2} \text{ such that } A + \lambda C_1 + aC_1^*C_1 \text{ is dissipative} \right\}.
\]

Remark 7.1. If \( C \in \mathcal{L}(H) \) is a bounded operator, then \( \Lambda = \mathbb{R} \).
2. \( \Lambda \) contains at least the origin \( \{0\} \).
3. If \( C_1 \) is dissipative and the assumption (B) holds true, then \( \mathbb{R}_+ \subseteq \Lambda \).

We now can state our main result of this section.

**Theorem 2** Under assumption (B), a necessary condition for the approximate controllability of \((\mathcal{A})\) is
\[
|B^*y| + |(A^* + \lambda C^* - \alpha I)y| > 0, \text{ for all } y \neq 0, \text{ and all } (\lambda, \alpha) \in \Lambda \times \mathbb{R}_-.
\]
(28)

The above necessary condition is an immediate consequence of Proposition 4 and a \( \lambda \)-wise application of the following result:

**Theorem 3** If \((\mathcal{A})\) is approximately controllable, then the system
\[
\begin{cases}
    dX_t = ((A + \lambda C)X_t + Bu_t)dt + (C + \lambda I)X_t dW_t, \\
    X_0 = x \in H,
\end{cases}
\]
(29)
which is governed by the control process \( v \in L^2_F([0,T];V) \) is also approximately controllable.

**Proof** Step 1. Approximation of \((\mathcal{A})\) by an equation with bounded operators admitting the application of Itô’s formula.

For all \( u \in L^1_F([0,T];U) \), we denote by \( X^{x,u}_{n,\delta} \) the unique mild solution of the controlled forward equation
\[
\begin{cases}
    dX^{x,u}_{n,\delta}(t) = A_nX^{x,u}_{n,\delta}(t)dt + Bu(t)dt + J_{n\lambda}e^{\delta A}Ce^{\delta A}J_nX^{x,u}_{n,\delta}(t)dW_t, \\
    X^{x,u}_{n,\delta}(0) = x \in H,
\end{cases}
\]
where \( J_n = (I - n^{-1}A)^{-1} \) and \( A_n = J_nA \). This approximation of the operators \( A \) (by \( A_n \)) and \( C \) (by \( J_{n\lambda}e^{\delta A}Ce^{\delta A}J_n \)) explains by the same difficulties we have already met in the proof of Theorem 1. Our special choice of the approximation allows to conserve the joint dissipativity condition also for the approximating operators and allows now to apply Itô’s formula.

Let \( \mathcal{E}(\lambda W) \) denote the Doléan-Dade exponential of \( \lambda W \), i.e., \( \mathcal{E}(\lambda W)_t := e^{\lambda W_t - \frac{\lambda^2}{2}t}, \ t \in [0,T] \). Then, from Itô’s formula applied to \( \mathcal{E}(\lambda W)_t X^{x,u}_{n,\delta}(t) \) it follows that
\[
\begin{cases}
    d \left( \mathcal{E}(\lambda W)_t X^{x,u}_{n,\delta}(t) \right) = (A_n + \lambda J_{n\lambda}e^{\delta A}Ce^{\delta A}J_n) \left( \mathcal{E}(\lambda W)_t X^{x,u}_{n,\delta}(t) \right) dt \\
    + B \left( \mathcal{E}(\lambda W)_t u(t) \right) dt \\
    + (J_{n\lambda}e^{\delta A}Ce^{\delta A}J_n + \lambda I) \left( \mathcal{E}(\lambda W)_t X^{x,u}_{n,\delta}(t) \right) dW_t,
\end{cases}
\]
\( X^{x,u}_{n,\delta}(0) = x \in H. \)
After the above application of Itô’s formula we would like to take the limit as \( n \to +\infty \) and then as \( \delta \downarrow 0 \) in order to get an equation which coincides with that we would get if we applied formally Itô’s formula to \( \mathcal{E}(\lambda W)_t X^{x,u}(t) \), where \( X^{x,u} \) denotes the unique mild solution of (4). For taking these limits we need the following result whose proof will be given later.

**Proposition 5** Under the assumptions on Theorem 2 and with the notations introduced above we have that, for all \( x \in H \),

\[
\lim_{n} \sup_{0 \leq t \leq T} \left| e^{t(A_n + \lambda J_n e^{\delta A} C e^{\delta A} J_n)} x - e^{t(A + \lambda e^{\delta A} C e^{\delta A})} x \right| = 0, \quad \delta > 0, \quad (30)
\]

and

\[
\lim_{\delta} \sup_{0 \leq t \leq T} \left| e^{t(A + \lambda e^{\delta A} C e^{\delta A})} x - e^{t(A + \lambda C)} x \right| = 0. \quad (31)
\]

We continue the **Proof** of our theorem. With the help of the above proposition we are now able to prove

**Step 2.** Let \( X^{x,u} \) denote the unique mild solution of (4). Then the process \( \mathcal{E}(\lambda W)_t X^{x,u}(\cdot) \) is the unique mild solution of (29). Moreover,

\[
\sup_{0 \leq t \leq T} E \left[ \mathcal{E}(\lambda W)_t X^{x,u}(t) \right] \leq c_p \left( 1 + |x|^p \right). \quad (32)
\]

For proving this statement we first notice that from standard estimates, for all \( p > 2 \),

\[
E \left[ \sup_{0 \leq t \leq T} \left| \mathcal{E}(\lambda W)_t X^{x,u}_{n,\delta}(t) \right|^p \right] \leq c_p \left( 1 + |x|^p \right), \quad \text{and}
\]

\[
E \left[ \sup_{0 \leq t \leq T} \left| X^{x,u}_{n,\delta}(t) \right|^p \right] \leq c_p \left( 1 + |x|^p \right);
\]

cp denotes a generic constant independent of \( n, \delta \) and \( u \in L^0_F([0,T];U) \). Then, for any \( \delta > 0 \), there exists a subsequence of

\[
\left( \mathcal{E}(\lambda W), X^{x,u}_{n,\delta}(\cdot), X^{x,u}(\cdot) \right)_n,
\]

still denoted by \( \left( \mathcal{E}(\lambda W), X^{x,u}_{n,\delta}(\cdot), X^{x,u}(\cdot) \right)_n \), which converges in the weak topology on

\[
L^p([0,T] \times \Omega; H) \times L^{2p}([0,T] \times \Omega; H)
\]

to some limit \( (X^{x}_\delta(\cdot), X^{x}_\delta(\cdot)) \). With the help of Proposition 5 we can show that \( X^{x}_\delta \) is a unique mild solution of

\[
\begin{cases}
  dX^{x}_\delta(t) = (A + \lambda e^{\delta A} C e^{\delta A}) X^{x}_\delta(t)dt \\
  + B \left( \mathcal{E}(\lambda W)_t u(t) \right) dt + \left( e^{\delta A} C e^{\delta A} + \lambda I \right) X^{x}_\delta(t)dW_t,
\end{cases}
\]

\( X^{x}_\delta(0) = x \in H, \)
and $X''_\delta$ is a mild solution of
\[
\begin{cases}
    dX''_\delta(t) = (AX''_\delta(t) + Bu) dt + e^{\delta A t} Ce^{\delta A X''_\delta(t)} dW_t, \\
    X''_\delta(0) = x \in H.
\end{cases}
\]  
(33)

On the other hand, it follows from the general theory of SDEs in infinite dimensions that these mild solutions are unique and that
\[
\sup_{0 \leq t \leq T} E \left[ |X'_\delta(t)|^p \right] \leq c_p \left( 1 + |x|^p \right).
\]  
(34)

Moreover, taking into account that
\[
E(\lambda W) \cdot \zeta(\cdot) \in L^{\frac{p}{p-1}} ([0, T] \times \Omega; H), \text{ for all } \zeta \in L^{\frac{p}{p-1}} ([0, T] \times \Omega; H),
\]
we get
\[
E \left[ \int_0^T \langle X'_\delta(t), \zeta(t) \rangle dt \right] = \lim_n E \left[ \int_0^T \left\langle E(\lambda W)_t X'^{x,u}_{n,\delta}(t), \zeta(t) \right\rangle dt \right]
\]
\[
= \lim_n E \left[ \int_0^T \left\langle X'^{x,u}_{n,\delta}(t), E(\lambda W)_t \zeta(t) \right\rangle dt \right]
\]
\[
= E \left[ \int_0^T \left\langle E(\lambda W)_t X'^{x,u}_\delta(t), \zeta(t) \right\rangle dt \right].
\]

This relation allows to identify the processes $X'_\delta(\cdot)$ and $E(\lambda W) X'^{x,u}_\delta(\cdot)$ as elements of $L^p([0, T] \times \Omega; H)$. Moreover, if $X'^{x,u}_{n,\delta}$ denotes the continuous version of $X'^{x,u}_\delta$ and $\tilde{X}'^{x,u}_\delta$ the continuous version of $X'_\delta$, we have
\[
\tilde{X}'^{x,u}_\delta(t) = E(\lambda W)_t X'^{x,u}_\delta(t), \text{ dP-a.s, for all } t \in [0, T],
\]
and inequality (34) takes the form
\[
\sup_{0 \leq t \leq T} E \left[ |E(\lambda W)_t X'^{x,u}_\delta(t)|^p \right] \leq c_p \left( 1 + |x|^p \right).
\]

By repeating the argument for letting $\delta \to 0$ we get the result stated in step 2.

After having related equation (4) with equation (29) we can prove now the theorem in its proper sense.

Step 3. Conclusion.

If $\xi \in L^2(\Omega, F_T, P; H)$, then, for every $\varepsilon > 0$ there exists some $\xi^\varepsilon \in L^\infty(\Omega, F_T, P; H)$ such that
\[
E \left[ |\xi^\varepsilon - \xi|^2 \right] \leq \varepsilon.
\]

It follows from (32) that the family
\[
\left\{ |E(\lambda W)_T X^{x,u}(T) - \xi|^2, \ u \in L^p_T([0, T]; U) \right\}
\]
is uniformly integrable. Consequently, there exists $M_\varepsilon > 0$ such that
\[
E \left[ \| e(\lambda W)_T X^{x,u} (T) - \xi \|^2 1\{ e(\lambda W)_T > M_\varepsilon \} \right] \leq \varepsilon,
\]
for all $u \in L^0_F([0,T]; U)$. If the equation (4) is approximately controllable, then there exists $u_\varepsilon \in L^0_F([0,T]; U)$ such that
\[
E \left[ \left| X^{x,u_\varepsilon} - \xi e(\lambda W)_T^{-1} \right|^2 \right] \leq \frac{\varepsilon}{M_\varepsilon^2},
\]
and we get
\[
E \left[ \| e(\lambda W)_T X^{x,u_\varepsilon} - \xi \|^2 \right] \leq M_\varepsilon^2 E \left[ \left| X^{x,u_\varepsilon} - \xi e(\lambda W)_T^{-1} \right|^2 \right] + E \left[ \| e(\lambda W)_T X^{x,u_\varepsilon} (T) - \xi \|^2 1\{ e(\lambda W)_T > M_\varepsilon \} \right] \leq 2\varepsilon.
\]
Therefore, also (29) is approximately controllable. The proof of our theorem is now complete.

However, the proof of Proposition 5 still remains open:

Proof (of Proposition 5). Due to the definition of the approximation of the operators $A$ and $C$ given in step 1 of the proof of the above theorem we have for all $x \in D \left( A + \lambda e^{\delta A^*} C e^{\delta A} \right)$,
\[
\lim_n \left( A_n + \lambda J_n^* e^{\delta A^*} C e^{\delta A} J_n \right) x = \left( A + \lambda e^{\delta A^*} C e^{\delta A} \right) x.
\]
(35)

For all $n$, the operator $A_n + \lambda J_n^* e^{\delta A^*} C e^{\delta A} J_n$ is bounded. Therefore, it generates a $C_0$-semigroup $\left( e^{t \left( A_n + \lambda J_n^* e^{\delta A^*} C e^{\delta A} J_n \right)} \right)_t$ and the application $t \mapsto \left| e^{t \left( A_n + \lambda J_n^* e^{\delta A^*} C e^{\delta A} J_n \right)} \right|_x$ is continuous. From the general theory of perturbation of generators, we have
\[
e^{t \left( A_n + \lambda J_n^* e^{\delta A^*} C e^{\delta A} J_n \right)} x = e^{t A_n} x
\]
\[
+ \lambda \int_0^t e^{(t-s)A_n} J_n^* e^{\delta A^*} C_1 e^{\delta A} J_n e^{s\left( A_n + \lambda J_n^* e^{\delta A^*} C e^{\delta A} J_n \right)} xds
\]
\[
+ \lambda \int_0^t e^{(t-s)A_n} J_n^* e^{\delta A^*} C_2 e^{\delta A} J_n e^{s\left( A_n + \lambda J_n^* e^{\delta A^*} C e^{\delta A} J_n \right)} xds.
\]
It follows that, for $n$ great enough
\[
\left| e^{t \left( A_n + \lambda J_n^* e^{\delta A^*} C e^{\delta A} J_n \right)} \right| \leq 1 + \lambda \int_0^t \left( \delta^{-\gamma} + k \right) \left| e^{s\left( A_n + \lambda J_n^* e^{\delta A^*} C e^{\delta A} J_n \right)} \right| ds,
\]
where $k > 0$ is a generic constant (which may depend on $\delta$ but not on $n$), and Gronwall’s inequality yields

\[
|e^{t(A_n+\lambda J_n^*Ce^J_n)}| \leq e^{kt},
\]

(36)

for all $t > 0$, and all $n \in \mathbb{N}$. Then, from (35) and (36) we get (cf. Davies [7] Th. 3.17) that (30) holds true, for all $\delta > 0$ and all $x \in D(A)$.

To prove the second assertion, we notice that

\[
e^{t(A+\lambda e^{J^*}Ce^J)}x = e^{tA}x + \int_0^t e^{(t-s)A}\lambda e^{J^*}C_1e^{J}e^{s(A+\lambda e^{J^*}Ce^J)}xds
\]

\[
+ \int_0^t e^{(t-s)A}\lambda e^{J^*}C_2e^{J}e^{s(A+\lambda e^{J^*}Ce^J)}xds,
\]

for all $x \in H$. Then, recalling that $A$ is self adjoint, we obtain

\[
|e^{t(A+\lambda e^{J^*}Ce^J)}x| \leq |x| + \lambda \int_0^t |e^{s(A+\lambda e^{J^*}Ce^J)}x| (t-s)^{-\gamma} ds
\]

\[
+ k \int_0^t |e^{s(A+\lambda e^{J^*}Ce^J)}x| ds
\]

(37)

$k$ is again a generic constant independent of $\delta$). Thus, with the notation

\[
f(t) = |e^{t(A+\lambda e^{J^*}Ce^J)}x|,
\]

the latter estimate takes the form

\[
f(t) \leq |x| + \lambda \int_0^t f(s)(t-s)^{-\gamma} ds + k \int_0^t f(s)ds.
\]

Then, by Cauchy-Schwarz inequality,

\[
f(t) \leq |x| + k \left( \frac{1-\frac{2\gamma}{1-2\gamma}}{\sqrt{1-2\gamma}} + t^{\frac{1}{2}} \right) \left( \int_0^t f^2(s)ds \right)^{\frac{1}{2}}
\]

(37)

\[
\leq |x| + k \left( T^{\frac{1}{2}} \lor 1 \right) \left( \int_0^t f^2(s)ds \right)^{\frac{1}{2}},
\]

(38)

and, consequently,

\[
f^2(t) \leq 2 \left( |x|^2 + k (T \lor 1) \int_0^t f^2(s)ds \right).
\]

To the latter estimate we apply Gronwall’s inequality and take the square root after. This yields

\[
f(t) \leq \sqrt{2} |x| e^{k(T \lor 1)t}.
\]
Therefore, from the definition of \( f(t) \) it follows that
\[
\sup_{\delta > 0} \left| e^{\left( A + \lambda e^{\delta A} C e^{\delta A} \right)} \right| \leq M e^{c t}, \tag{39}
\]
for all \( t \leq T \), where \( M \) and \( c \) are positive constants that are independent of \( \delta > 0 \). On the other hand, for all \( x \in \mathcal{D}(A + \lambda C) \) we have
\[
\lim_{\delta \to 0} \left( A + \lambda e^{\delta A} C e^{\delta A} \right) x = (A + \lambda C) x. \tag{40}
\]
The second assertion follows (cf. Davies [7] Th. 3.17).

In the following we discuss two examples to illustrate the results of this section.

**Example 1**

Given a regular domain \( \mathcal{O} \subset \mathbb{R}^N \) we consider the following stochastic partial differential equation
\[
\begin{align*}
\{ &d_t X^u(t, x) = \sum_{i,j=1}^N \partial_i (a_{i,j}(x) \partial_j X^u(t, x)) \, dt + u(t) b(x) dt \\
&+ \sum_{i=1}^N c_i(x) \partial_i X^u(t, x) \, dW_t, \\
&X^u(t, x) = 0, \quad \forall (t, x) \in [0, T] \times \partial \mathcal{O}, \\
&X^u(0, x) = \xi(x), \quad \forall x \in \mathcal{O},
\end{align*}
\tag{41}
\]
where \( u \) is an admissible control process taking its values in \( \mathbb{R} \). We suppose that \( a(x) = (a_{i,j}(x)) \sigma(x) \sigma^*(x) \) for some \( C^\infty_{c, b} \) matrix \( \sigma \) of \( N \times N \)-type, \( c = (c_1, \ldots, c_N) \in C^\infty_{c, b} (\mathcal{O} ; \mathbb{R}^N) \), \( b \in H^1(\mathcal{O}) \) and \( \xi \in L^2(\Omega, \mathcal{F}_T, P; L^2(\mathcal{O})) \). Moreover, we suppose that the couple of coefficients \((a, c)\) satisfies the standard ellipticity condition
\[
\sum_{i,j=1}^N (a_{i,j}(x) - \alpha c_i(x) c_j(x)) \lambda_i \lambda_j \geq 0, \tag{42}
\]
for some \( \alpha > \frac{1}{2} \) and for all \( \lambda \in \mathbb{R}^N \). Then, if we put
\[
\begin{align*}
H &= L^2(\mathcal{O}), \\
\mathcal{D}(A) &= H^2(\mathcal{O}) \cap H^1_0(\mathcal{O}), \quad A \zeta = \sum_{i,j=1}^N \partial_i (a_{i,j}(x) \partial_j \zeta(x)), \\
\mathcal{D}(C) &= H^1(\mathcal{O}), \quad C \zeta = c \cdot \nabla \zeta,
\end{align*}
\]
we get that
\[
\mathcal{D}(C^*) = H^1(\mathcal{O}), \quad C^* \zeta = -c \cdot \nabla \zeta - \xi \sum_{i=1}^N \partial_i c_i.
\]
The ellipticity condition (42) insures that the dual backward stochastic partial differential equation
\[
\begin{aligned}
d_t Y(t, x) &= -\left(\sum_{i,j=1}^N \partial_i \left(a_{i,j}(x) \partial_j Y(t, x)\right)\right) dt + \left(\sum_{i=1}^N c_i(x) \partial_i Z(t, x)\right) dt \\
&\quad + \left(\sum_{i=1}^N \partial_i c_i(x) Z(t, x)\right) dt + Z(t, x) dW_t, \\
Y(t, x) &= Z(t, x) = 0, \forall (t, x) \in [0, T] \times \partial \Omega, \\
Y(T, x) &= \eta(x), \forall x \in \Omega,
\end{aligned}
\]  
has a unique mild solution. Thus we know that the approximate controllability of (41) is equivalent to the approximate observability of (43).

From (N1) it follows that, if (41) is approximately controllable and if \(\zeta_n(x)\) is a complete orthonormal base consisting of eigenvectors for \(A\), then every coefficient of \(b\) in this base must be non null.

**Remark 8** The problem of controllability for the deterministic version of (41) has been treated by Carleman estimates method in Fursikov, Imanuvilov [10].

The condition (N2) is non trivially more general then (N1) as proven by the following

**Example 2** We consider the following equation
\[
\begin{aligned}
d_t X^u(t, x) &= \Delta X^u(t, x) dt + u(t) b(x) dt \\
&\quad + \left(2 \sin(\pi x) \int_0^1 X^u(t, y) \sin(\pi y) dy\right) dW_t, \\
X^u(t, 0) &= X^u(t, 1) = 0, \forall t \in [0, T], \\
X^u(0, x) &= \xi(x), \forall x \in (0, 1),
\end{aligned}
\]  
where \(u\) is an admissible real-valued bounded control process and \(b \in L^2(0, 1)\). This equation can be expressed as an infinite dimensional linear equation. For this we put
\[
H = L^2(0, 1), \quad D(A) = H^2(0, 1) \cap H_0^1(0, 1),
\]
\[
A\zeta = \Delta \zeta, \quad \text{for all} \ \zeta \in D(A),
\]
\[
C\zeta(\cdot) = 2 \sin(\pi) \int_0^1 \zeta(y) \sin(\pi y) dy, \quad \text{for all} \ \zeta \in H.
\]
Obviously \(C\) is a self-adjoint bounded linear operator on \(H\). Furthermore, suppose that
\[
b_n = \sqrt{2} \int_0^1 b(y) \sin(\pi y) dy \neq 0,
\]
for all \(n \geq 1\). Then (N1) is obviously satisfied. However, if we choose \(\lambda = -3\pi^2, \alpha = -4\pi^2\) and \(\zeta(\cdot) = \frac{b_1}{b_2} \sin(\pi \cdot) + \sqrt{2} \sin(2\pi \cdot)\), we have
\[
|A^* + \lambda C^* - \alpha I| \zeta|^2 + |B^* \zeta|^2 = 0.
\]
It follows that (N2) is not satisfied which implies that the equation (44) cannot be approximately controllable.
References

1. V. Barbu, A. Răşcanu, G. Tessitore (2003), Carleman estimates and Controllability of stochastic heat equations with multiplicative noise, Appl. Math. Optim. 47:97–120, pp.98-120.
2. Buckdahn, R., Quincampoix, M., Răşcanu, A. (2000), Viability property for a backward stochastic differential equation and applications to partial differential equations, Probab. Theory Relat. Fields 116, No.4, pp. 485-504.
3. Buckdahn, R., Quincampoix, M., Tessitore, G. (2006), A Characterization of Approximately Controllable Linear Stochastic Differential Equations, Stochastic Partial Differential Equations and Applications, G. Da Prato and L. Tubaro Eds Series of Lecture Notes in pure and appl. Math., Chapman & Hall Vol.245, pp. 253-260.
4. Confortola, F. (2004), Dissipative backward stochastic differential equations in infinite dimensions.
5. Da Prato, G., Zabczyk, J. (1992), Stochastic equations in infinite dimensions, Cambridge University Press, Cambridge.
6. Da Prato, G., Zabczyk, J. (1996), Ergodicity for infinite-dimensional systems. London Mathematical Society Lecture Note Series, 229, Cambridge University Press, Cambridge.
7. Davies, E. B. (1980), One-parameter semigroups, London Mathematical Society Monographs, 15. Academic Press, Inc., London-New York.
8. Fernández-Cara, E., Garrido-Atienza, M. J. Real J. (1999), On the approximate controllability of a stochastic parabolic equation with a multiplicative noise, C. R. Acad. Sci. Paris, t. 328, Serie I, pp. 675-680.
9. Fuhrman, M., Tessitore, G. (2002), Nonlinear Kolmogorov equations in infinite dimensional spaces: the backward stochastic differential equations approach and applications to optimal control, Ann. Probab. 30, pp. 1397-1465.
10. Fursikov, A., Imanuvilov, O. (1996), Controllability of evolution equations, vol. 34. Seoul National University.
11. Goreac, D. (2007), Approximate Controllability for Linear Stochastic Differential Equations with Control Acting on the Noise, Applied Analysis and Differential Equations, Iaşi, România 4 - 9 September 2006, World Scientific Publishing, pp. 153-164.
12. Hu, Y., Ma, J., Yong, J. (2002), On semi-linear degenerate backward stochastic partial differential equations, Probability Theory and Related Fields, vol. 123, no. 3, pp. 381–411.
13. Jacob, B., Partington, J., R. (2006), On controllability of diagonal systems with one-dimensional input space, Systems and Control Letters 55, pp. 321 – 328.
14. Jacob, B., Zwart, H. (2001). Exact observability of diagonal systems with a finitedimensional output operator, Systems Control Lett. 43 101–109.
15. Liu, Y., Peng, S. (2002), Infinite horizon backward stochastic differential equation and exponential convergence index assignment of stochastic control systems, Automatica, 38, pp. 1417-1423.
16. Ma, J., Yong, J. (1997), Adapted solution of a degenerate backward SPDE, with applications, Stochastic Processes and Their Applications, vol. 70, no. 1, pp. 59–84.
17. Ma, J., Yong, J. (1999), On linear, degenerate backward stochastic partial differential equations, Probability Theory and Related Fields, vol. 113, no. 2, pp. 135–170.
18. Pardoux, E., Peng, S.G. (1990), *Adapted solutions of a backward stochastic differential equation*, Systems and Control Letters, 14, pp. 55-61.

19. Peng, S.G. (1994), *Backward Stochastic Differential Equation and Exact Controllability of Stochastic Control Systems*, Progr. Natur. Sci. vol. 4, No. 3, pp. 274-284.

20. Russell, D.L., Weiss, G. (1994), *A general necessary condition for exact observability*, SIAM J. Control Optim. 32 (1), pp. 1–23.

21. Sirbu, M, Tessitore, G. (2001), *Null controllability of an infinite dimensional SDE with state and control-dependent noise*, Systems and Control Letters, 44, pp. 385-394.

22. Tessitore, G. (1996), *Existence, uniqueness and space regularity of the adapted solutions of a backward SPDE*, Stochastic Analysis and Applications, vol. 14, no. 4, pp. 461–486.