CAUCHY PROBLEM OF SEMILINEAR INHOMOGENEOUS ELLIPTIC EQUATIONS OF MATUKUMA-TYPE WITH MULTIPLE GROWTH TERMS

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ABSTRACT. We consider the structure and the stability of positive radial solutions of a semilinear inhomogeneous elliptic equation with multiple growth terms

\[ \Delta u + \sum_{i=1}^{k} K_i(|x|)u^{p_i} + \mu f(|x|) = 0, \quad x \in \mathbb{R}^n, \]

which is a generalization of Matukuma’s equation describing the dynamics of a globular cluster of stars. Equations similar to this kind have come up both in geometry and in physics, and have been a subject of extensive studies. Our result shows that any positive radial solution is stable or weakly asymptotically stable with respect to certain norm.

1. Introduction. In this paper, we consider the structure and stability of positive radial solutions of the following semilinear inhomogeneous equation

\[ \Delta u + \sum_{i=1}^{k} K_i(|x|)u^{p_i} + \mu f(|x|) = 0, \quad x \in \mathbb{R}^n, \quad (1) \]
which is the positive steady state of the following Cauchy problem
\[
\begin{cases}
    u_t = \Delta u + \sum_{i=1}^{k} K_i(|x|) u^{p_i} + \mu f(|x|), & (x, t) \in \mathbb{R}^n \times (0, \infty), \\
    u(x, 0) = \varphi(x), & x \in \mathbb{R}^n, 
\end{cases}
\]  
(2)

where \( n \geq 3, p_1 \geq p_2 \geq \cdots \geq p_k > 1, \mu > 0, 0 \leq f \leq C^1(\mathbb{R}^n \setminus \{0\}) \), \( K_i \geq 0, i = 1, 2, \cdots, k \), are locally Hölder continuous in \( \mathbb{R}^n \setminus \{0\} \), \( \varphi(x) \) is a bounded non-negative continuous function in \( \mathbb{R}^n \).

Equation (1) arises from both physics and mathematics, mainly from astrophysics. In 1930, Matukuma, an astrophysicist, proposed the mathematical model
\[
\Delta u + \frac{1}{1+|x|^2} u^p = 0, \quad x \in \mathbb{R}^3
\]
(3)
to describe the dynamics of a globular cluster of stars, where \( p > 1 \), \( u \) represents the gravitational potential (therefore \( u > 0 \)). By (3), we get \( -\frac{1}{4\pi} \Delta u = \frac{u^p}{4\pi(1+|x|^2)} \). In astrophysics, this expression represents the density and the integral \( \int_{\mathbb{R}^3} \frac{u^p}{4\pi(1+|x|^2)} \, dx \) represents the total mass of the globular cluster of stars, for details, see [28] for example. By using model (3), Matukuma’s aim was to improve the model
\[
\Delta u + \frac{1}{1+|x|^2} e^{2u} = 0, \quad x \in \mathbb{R}^3
\]
(4)
proposed earlier by Eddington in [13], where, likewise, \( u \), the expression \( -\frac{1}{4\pi} \Delta u = \frac{e^{2u}}{4\pi(1+|x|^2)} \) and the integral \( \int_{\mathbb{R}^3} \frac{e^{2u}}{4\pi(1+|x|^2)} \, dx \) represent the gravitational potential, the density and the total mass of the globular cluster of stars, respectively. For the reader’s convenience, more information on (3), (4) and models arising from astrophysics, one can refer to [13, 28, 32] and the references therein (Especially, following Matukuma’s paper [28], in the Appendix of [32], the authors gave a brief explanation of a derivation for the Eddington’s equation and the Matukuma’s equation).

For the homogeneous case, when \( k = 1 \) and \( K_1 \) is a constant, equation (1) is known as the Lane-Emden equation, sometimes it is also referred to as the Emden-Fowler equation in astrophysics. In this case, \( u \) corresponds to the density of a single star. When \( k = 1 \) and \( K_1 = c|x|^l \) with \( c \) being a positive constant and \( l > -2 \), equation (1) is called the Henon equation which is a model to describe the rotating stellar systems. For \( k = 1 \) and \( p_1 = \frac{n+2}{n-2}, n \geq 3 \), equation (1) is called the conformal scalar curvature equation in \( \mathbb{R}^n \). For a detailed overview on equation (1), we refer the readers to the survey paper [32].

Since the globular cluster has a radial symmetry, positive radial entire solutions (i.e., solutions with \( u(x) = u(|x|) > 0 \) for all \( x \in \mathbb{R}^n \)) are of particular interest. It is just for this reason, many works considered radial solutions for mathematical models arising from astrophysics. Under certain conditions, radial solutions to (1) may oscillate near zero, see [6, 11, 32, 33]. In [6, 11], the authors studied the asymptotic behavior between two consecutive zeros of solutions for their model. Moreover, the asymptotic behavior and separation properties of positive solutions similar to the equation (homogeneous and inhomogeneous) in [39] have received much more attention, see [2, 5, 7, 10, 14, 25, 26, 27, 29, 31, 34, 35, 38] for example.

For positive radial solutions of related topics, there has been a large number of valuable works, see [8, 9, 10, 14, 20, 21, 32] for example. For \( k = 1, \mu = 0 \) in (1), it seems that the first general and systematic study of the equation is due to Ni [31]. It was proved there that if \( K_1 \) grows faster than or equal to \( \rho^{(n-3)(p-1)-2} \) at
\(\infty\), then the equation possesses no positive entire solution. Existence results in the case that \(K_1\) decays faster than \(r^{-2}\) at \(\infty\) were established in [31] and slightly improved in [22, 30] later. (In fact, all these results dealt with \(K_1 = K_1(x)\), not just radial cases.) Other existence and non-existence results were also discussed in [31]. Ding and Ni [12] proved a striking result for the case \(p_1 = \frac{n+2}{n-2}\) and \(K_1\) is a perturbation of the constant 1. Recently, Kusano and Naito [23] generalized the above result. For \(K_i(r) = c_i r^{l_i}, c_i > 0, l_i > -2, i = 1, 2, \cdots, k, \mu = 0\) in (1), Ni and Yotsutani analyzed the existence and zero points of positive radial solutions to the equation by considering the relation between \(p_i\) and \(\frac{n+2l_i+2}{n-2}\) for all \(1 \leq i \leq k\) in [32]. For the same case, a systematic and delicate study of the stability of positive steady states was given by Gui, Ni and Wang in [21] and later in [20], and in [39], Yang and Zhang gave the stability and asymptotic stability of positive radial steady states to the model. In [8, 9], authors discussed the stability and instability of positive radial steady states of (1) in both of homogeneous and inhomogeneous cases. In [31], Gui, Ni and Wang gave the existence and asymptotic behavior of positive solutions for an inhomogeneous semi-linear elliptic equation were investigated. The same problem as is in [32] was investigated when \(k = 2\), and Franca studied the \(m\)-Laplace equations with two growth terms and obtained the existence and decaying rate at infinity for positive radial solutions by the Emden-Fowler transformation and dynamical system theory in [14].

Without loss of generality, in the present paper, we also consider the positive radial solutions of (1), that is, we consider the case \(K_i(0) = K_i(r), f(|x|) = f(r)\) with \(r = |x|\). An application of transformation between rectangular and polar coordinates induces that equation (1) then converts to the form

\[
 u'' + (n - 1)r^{-1}u' + \sum_{i=1}^{k} K_i(r) u^{p_i} + \mu f(r) = 0, \quad r > 0. \tag{5}
\]

For the same reasons as mentioned above, regular solutions (means that such solutions have finite limits at \(r = 0\)) are particularly interesting, which lead us to consider the Cauchy problem

\[
 \begin{cases}
 u'' + (n - 1)r^{-1}u' + \sum_{i=1}^{k} K_i(r) u^{p_i} + \mu f(r) = 0, & r > 0, \\
 u(0) = \alpha
\end{cases} \tag{6}
\]

for some \(\alpha > 0\). We use \(u_\alpha = u_\alpha(r) = u(r, \alpha)\) to denote the unique solution of (6) when it exists.

Firstly, let us introduce a collection of hypotheses on \(K_i, i = 1, 2, \cdots, k\).

(K.1) \(K_i(r) > 0\) for \(r > 0\), \(\lim_{r \to \infty} r^{-l_i} K_i(r) = k_i \in \mathbb{R}^+\) with \(l_i > -2, i = 1, 2, \cdots, k\).

(K.2) \(K_i(r) > 0\) for \(r > 0\), \(\lim_{r \to 0} r^{-l_i} K_i(r) = k_i \in \mathbb{R}^+\) with \(l_i > -2, i = 1, 2, \cdots, k\).

(K.3) \(K_i(r)\) is differentiable and \(\frac{d}{dr}(r^{-l_i} K_i(r)) \leq 0\) for \(r > 0, i = 1, 2, \cdots, k\).

Also, we introduce following notations, which will be used throughout this paper.

\[
 m_i = \frac{l_i + 2}{p_i - 1}, \quad m = \max\{m_i | i = 1, 2, \cdots, k\},
\]

\[
 M = n - 2m - 2, \quad N = (m(n - m - 2))^{\frac{1}{m-1}}, \quad P = (p_1 - 1)N^{p_1 - 1},
\]

\[
p(l_i) = \begin{cases}
 \frac{(n-2)^2 - 2(l_i + 2)(n+l_i) + 2(l_i + 2)\sqrt{(n+l_i)^2 - (n-2)^2}}{(n-2)(n-4l_i-10)} & n > 4l_i + 10, \\
 \infty & 3 \leq n \leq 4l_i + 10.
\end{cases}
\tag{7}
\]

Note that we have \(m_i > 0, M > 0\) when \(p_i > \frac{n+2l_i+2}{n-2}\) and \(l_i > -2\).
We say that $m$ is of multiplicity $l$ ($1 \leq l \leq k$) if there exist $i_1, i_2, \ldots, i_l \in \{1, 2, \ldots, k\}$ such that $m_{i_j} = m$, $j = 1, 2, \ldots, l$. In this paper, we always assume that $m$ is of multiplicity $l$. Denote $I = \{1, 2, \ldots, k\}$, $I_m = \{i_1, i_2, \ldots, i_l\}$, respectively. Then $m = m_i$ if $i \in I_m$. We will use $I, I_m$ frequently in the following.

Denote

$$Q = \sum_{i \in I_m} (p_i-1)k_{i\infty}^{p_i-1}$$

with $u_{\infty}$ satisfying the equality $\sum_{i \in I_m} k_{i\infty}^{p_i-1} = N^{p_{\infty}-1}$. Then it is easy to see that $Q \leq P$ since $p_i$ is decreasing in $i$. Consider the equation

$$\lambda^2 + M\lambda + Q = 0. \quad (8)$$

It is easy to verify that if $p_1 > p(l_1)$, then $(8)$ has two negative roots $-\lambda_2 < -\lambda_1 < 0$ satisfying $M > \lambda_2 > \lambda_1$. While when $p_1 = p(l_1)$, $(8)$ has two equal negative roots $-\lambda_2 = -\lambda_1 = -\frac{M}{2} < 0$.

Now, we state some hypotheses on $f$.

(f.1) $\lim_{r \to 0} r^{-i} f(r) = 0$ with $l_i > -2$, $i = 1, 2, \ldots, k$.

(f.2) $\lim_{r \to \infty} r^{-q} f(r) = 0$ for some constant $q > n - m - \lambda_2$, where $\lambda_2$ is defined by the equation $(8)$.

Using the technique as in [14, 25, 39], for the asymptotic behavior, separation and intersection properties of positive solutions, we have following two results, which are important in proving the stability of positive solutions in Section 5.

**Theorem 1.1.** Let $u(r)$ be a positive radial solution of (5). If $p_i > \frac{n+l_1}{n-2}$, $i = 1, 2, \ldots, k$, then

$$\lim_{r \to \infty} r^m u(r) = \begin{cases} u_{\infty} \neq 0, \text{ or } \\ 0, \end{cases}$$

where $u_{\infty}$ satisfies the equality $\sum_{i \in I_m} k_{i\infty}^{p_i-1} = N^{p_{\infty}-1}$. Moreover, if $\lim_{r \to \infty} r^m u(r) = 0$, then $\lim_{r \to \infty} r^{n-2} u(r)$ exists and is finite and positive.

It is known that the monotone property of solutions (which means whether two solutions with different initial values can intersect each other) of $(6)$ has some practical implications. So our next result is about the monotone property of solutions.

**Theorem 1.2.** Let $u_\alpha, u_\beta$ be two positive radial solutions of (5) with initial value $\alpha, \beta$ and $0 < \alpha < \beta$ respectively. Then

$$\lim_{r \to \infty} r^m u_\alpha = \lim_{r \to \infty} r^m u_\beta = u_{\infty}$$

if $p_i > \frac{n+2l_1+2}{n-2}$ and $m = m_1$. Furthermore, the following holds.

(i) $u_\alpha, u_\beta$ cannot intersect each other (specifically, $u_\alpha < u_\beta$) if $p_1 > p(l_1)$, i.e., solutions of (6) have monotone property.

(ii) $u_\alpha, u_\beta$ intersect each other infinitely many times if $p_1 < p(l_1)$ and $m$ is of multiplicity 1, i.e., solutions of (6) have no monotone property.

From [9], it is known that when $k = 1$, $\mu = 0$ and $p$ is large, then for every $\alpha > 0$, the solution of (6) is positive under the suitable hypotheses on $K_1$. But when $\mu \neq 0$, then the solution with sufficiently small initial values has finite zeros. Moreover, the authors showed that there exists a constant $\alpha_*>0$ such that every solution of
the equation has no zero for any $\alpha > \alpha^*$. We think these aspects are also true for model (6).

For some homogeneous equations with single growth term, the authors in [8], [32] and [36] showed that, for small growth exponent $p$, any two positive solutions intersect each other. Wang [36] showed that for large $p$, solutions of the corresponding problem possess monotone property for a special class of $K$, and gave explicitly the lower bound of the $p$ value. Then Gui [18] extended the result to a more general class of $K$. In [3, 4], authors studied the monotonicity of solutions with respect to the initial data and got a sharp estimate on the exponent $p$ under some general conditions imposed on $K$. In the case of homogeneous equations with multiple growth terms (specifically, $K_i = c_i r^{l_i}$ with $c_i > 0, i = 1, 2, \ldots, k$), Yang and Zhang [39] studied the stability and asymptotic stability of positive radial steady states to the problem. For the inhomogeneous equation with single growth term, the monotonicity and stability of positive radial steady states were given in [9]. It was also showed that, for small $p$, any positive solution is nonmonotonic but is monotonic for large $p$. In [38], for an inhomogeneous semi-linear bi-harmonic equation, Yang studied the optimal decay coefficient of the inhomogeneous term for existence and nonexistence, and showed that the corresponding problem had at least two types of decay solutions at infinity with assumptions on the inhomogeneous term.

The organization of this paper is as follows. Section 2 contains the statement of the main result of this paper. In Section 3, we introduce some preliminary lemmas which will be used throughout the paper. In Section 4, we give the asymptotic expansion of positive radial solutions at infinity. Some relevant aspects are extensions of those results obtained in [21, 25]. Section 5 is devoted to the stability and asymptotic stability of positive steady states of Cauchy problem (2), and is the main content of this paper.

2. **Main result.** Stable property is a very important characteristic for solutions. For the stability and instability of positive radial steady states of Cauchy problem (2) with initial function $\varphi \neq 0$, it seems that the first general result was given in [16]. In the case of $k = 1, K_1(r) = r^l$ and $\mu = 0$, for the global existence of solution $u(x,t;\varphi)$, the condition given by Fujita on $\varphi$ is that it is bounded by $\varepsilon e^{-|x|^2}$ for some small $\varepsilon$. By studying the problem in $L^p$-space in [37], Weissler found that if $\varphi$ is bounded by $\varepsilon(1 + |x|)^{-\gamma}$ for some constant $\gamma > \frac{2}{p-1}$ and $\varepsilon$ is small enough, then the global existence of $u(x,t;\varphi)$ is ensured. Lee and Ni in [24] gave a sharp condition that $\varphi$ has a decay rate of $C|x|^{-\frac{2\gamma}{p-1}}$ at $\infty$, where $C$ is a positive constant. Other recent and relevant results were also included in [34, 35]. Systematically delicate studies of the stability of positive steady states of corresponding equations were given by Deng, Li and Liu in [8], Deng, Li and Yang in [9], and Gui, Ni and Wang in [20, 21]. To describe the stability, they introduced the following norm

$$\| \psi \|_{\lambda} = \limsup_{x \in \mathbb{R}^n} |(1 + |x|)^{\lambda} \psi(x)|,$$

where $\psi(x)$ is a nonnegative and continuous function in $\mathbb{R}^n$, $\lambda$ is a real number. Here we adopt the following definition on stability and weakly asymptotical stability of positive steady states of (2) due to Gui, Ni and Wang [21].

**Definition 2.1.** A steady state $u_\alpha$ of (2) is stable with respect to the norm $\| \cdot \|_{\lambda}$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for $\| \varphi - u_\alpha \|_{\lambda} < \delta$, we have $\| u(.,t;\varphi) - u_\alpha \|_{\lambda} < \varepsilon$ for all $t > 0$: $u_\alpha$ is said to be weakly asymptotically stable.
with respect to the norm \( \| \cdot \|_\lambda \) if \( u_\alpha \) is stable with respect to norm \( \| \cdot \|_\lambda \) and there exists \( \delta > 0 \) such that, for \( \| \varphi - u_\alpha \|_\lambda < \delta \), we have \( \lim_{t \to \infty} \| u(\cdot, t; \varphi) - u_\alpha \|_{\lambda'} \to 0 \) as for all \( \lambda' < \lambda \).

In the case of \( k = 1 \), \( K_1(r) = 1 \) and \( \mu = 0 \) in (6), it was showed in [21] that steady states of (6) are stable with respect to the norm \( \| \cdot \|_{m+\lambda_1} \), and weakly asymptotically stable with respect to the norm \( \| \cdot \|_{m+\lambda_2} \), here \( m, \lambda_1, \lambda_2 \) are defined by (7) and (8). Deng, Li and Liu [8] extended the results in [21] to a more general class of \( K(r) \) for homogeneous equation. Later, For the inhomogeneous case, Deng, Li and Yang [9] showed the structure of positive solutions and got similar separation and intersection properties. Using these results, they proved that steady states are stable with respect to the norm \( \| \cdot \|_{m+\lambda_1} \), and weakly asymptotically stable with respect to the norm \( \| \cdot \|_{m+\lambda_2} \).

With the topology induced by (9), we prove the stability and weakly asymptotic stability of positive steady states of (2) for a more general class, which is an extension of results obtained in \([8, 9, 21]\). Since the nonlinear terms are more and some of them are not included in the multiplicity of \( m \), some key technique used in \([8, 9, 21]\) does not apply to equation (2). To overcome the corresponding difficulty, we especially introduce two special marks \( P \) and \( Q \) (see Section 1) and give an estimate related to \( Q \), then by constructing upper and lower solutions, we give some estimates on solutions. It should be noted that all of these depend on Theorem (2) and the asymptotic expansion of positive solutions of (6) at infinity obtained in Section 4.

The main purpose of this paper is dedicate to the investigation on the stability and weakly asymptotic stability of positive steady states of (2). Our main result is as the following.

**Theorem 2.1.** Let \( p_1 > p(l_1) \) and \( I_m = \{1, 2, \ldots, l\} \). Then any positive steady state \( u_\alpha \) of (2) is (i) stable with respect to the norm \( \| \cdot \|_{m+\lambda_1} \); (ii) weakly asymptotically stable with respect to the norm \( \| \cdot \|_{m+\lambda_2} \).

3. Preliminaries. In this section, we only give some preliminaries which will be used in the proof of Theorem 2.1. The idea was developed in \([18, 21, 30, 36]\) for the case \( k = 1, K_1 = 1 \) and \( \mu = 0 \) and extended for a general case \( f \neq 0 \) in \([19]\).

First, we introduce the following transformation, which will be used in this and latter sections. It can be proved by a straightforward calculation, thus we omit it here.

**Lemma 3.1.** Suppose that \( u(r) \) is a positive solution of (5). Let \( r = e^t, t \in \mathbb{R} \) and \( v(t) = r^q u(r) \). Then \( v \) satisfies

\[
v'' + (n - 2q - 2)v' - q(n - q - 2)v + \sum_{i \in I} K_i(e^t)e^{(q-p_i)q+2)t}v^{p_i} + \mu f(e^t)e^{(q+2)t} = 0.
\]

Taking \( q = m \), then we have

\[
v'' + Mv' - Nv^{-1}v + \sum_{i \in I} e^{-l_i t}K_i(e^t)v^{p_i} + \mu f(e^t)e^{(m+2)t} = 0
\]

where \( v = e^{mt}u(e^t) \), \( m, M \) and \( N \) are as the same as in (7).

This result can also be found in references \([9, 25, 27]\).

Now, we give two different estimates on positive solutions of (5).
Lemma 3.2. Let \( u(r) \) be a positive solution of (5). If \( m = m_1 \) and \( p_1 \geq p(l_1) \), then

(i) \( r^m u(r) \) is strictly increasing in \( r \) and \( \lim_{r \to \infty} r^m u(r) = u_\infty; \)

(ii) \( \sum_{i \in I} r^{2-m(p_i-1)} K_i(r)(r^m u(r))^{p_i-1} < N^{p_i-1}. \)

Using the similar argument as in proving Lemma 2.2 in [8] and Theorem 4.1 in [40], we can prove Lemma 3.2, and so, we omit it here.

Lemma 3.3. Let \( p_1 \geq p(l_1) \). If \( I_m = \{1, 2, \ldots, l\} \), then the solution \( u(r) \) of (5) satisfies \( \sum_{i \in I} (p_i - 1)r^{2-m(p_i-1)} K_i(r)(r^m u(r))^{p_i-1} < \sum_{i \in I_m} (p_i - 1)k_i u_\infty^{p_i-1} = Q. \)

Proof. We first have

\[
\sum_{i \in I} (p_i + 1 - 1)r^{2-m(p_i-1)} K_i(r)(r^m u(r))^{p_i-1} < (p_i + 1 - 1)N^{p_i-1}
\]

by Lemma 3.2. Since \( 2 + l_i - m(p_i - 1) = 0 \) for \( i \in I_m \), use Lemma 3.2 again to get

\[
\sum_{i \in I_m} (p_i - p_i + 1)r^{2-m(p_i-1)} K_i(r)(r^m u(r))^{p_i-1}
\]

\[
= \sum_{i \in I_m} (p_i - p_i + 1)r^{-1} K_i(r)(r^m u(r))^{p_i-1}
\]

\[
< \sum_{i \in I_m} (p_i - p_i + 1)k_i u_\infty^{p_i-1}.
\]

Considering \( p_i \geq p_i + 1 \), adding these two inequalities we get the required result and the proof is completed.

\[ \square \]

4. Asymptotic expansion of positive solutions at infinity. To prove Theorem 2.1, we need to obtain certain expansion of solutions to (5). The method used here was introduced by Gui et al. [21] and Li [25], we will use their techniques to our more general equation (5).

Suppose that \( u(r) \) is a positive solution of (5) satisfying \( \lim_{r \to \infty} r^m u(r) = u_\infty \). Let \( w(t) = r^m u(r) - u_\infty \) with \( r = e^t, t \in \mathbb{R} \). Then Lemma 3.1 induces that \( w(t) \) satisfies

\[
w'' + Mw' - N^{p_i-1}(w + u_\infty) + \sum_{i \in I} K_i(e^t)e^{(2-m(p_i-1))t}(w + u_\infty)^{p_i} + \mu f(e^t)e^{(m+2)t} = 0.
\]

Let

\[
\xi(t) = \sum_{i \in I \setminus I_m} K_i(e^t)e^{(2-m(p_i-1))t}u_\infty^{p_i} + \mu f(e^t)e^{(m+2)t}
\]

and

\[
\eta_i(t, w) = K_i(e^t)e^{(2-m(p_i-1))t}((w + u_\infty)^{p_i} - u_\infty^{p_i} - p_i u_\infty^{p_i-1}w), \quad i \in I.
\]

Then \( \eta_i \) has the expansion

\[
\eta_i(t, w) = K_i(e^t)e^{(2-m(p_i-1))t}(a_{i2}w^2 + \cdots + a_{iz}w^z + a(w^{z+1}))
\]

at \( w = 0 \) for any positive integer \( z \geq 2 \) since \( \frac{d^n \eta_i(t, w)}{dw^n} \bigg|_{w=0} = \frac{d^n \eta_i(t, w)}{dw^n} \bigg|_{w=0} = 0 \), where

\[
a_{ij} = \frac{p_i(p_i - 1) \cdots (p_i - j + 1)u_\infty^{p_i-j}}{j!}, \quad i \in I, j = 2, \ldots, z,
\]
depend only on $n, p_i, l_i, i \in I$. Especially, for $i \in I_m$, we have
\[
\eta_i(t, w) = e^{-lt}K_i(e^t)((w + u_\infty)^{p_i} - u_\infty^{p_i} - p_iu_\infty^{p_i-1}w) =: \eta_i(w),
\]
and then $\eta_i$ has the expansion
\[
\eta_i(w) = e^{-lt}K_i(e^t)(a_{i2}w^2 + \cdots + a_{iz}w^z + o(w^{z+1}))
\]
at $w = 0$ and $a_{i2} = \frac{p_i(p_i-1)}{2}u_\infty^{p_i-2} > 0$. Denote
\[
\Psi(t, w) = \xi(t) + \sum_{i \in I}(\eta_i(w) + p_iK_i(e^t)e^{(2-m(p_i-1))t}u_\infty^{p_i-1}w) - \sum_{i \in I_m}p_ie^{-lt}u_\infty^{p_i-1}w.
\]
Then a direct calculation gives that
\[
Qw + \Psi(t, w) = \sum_{i \in I_m}(p_i - 1)e^{-lt}K_i(e^t)u_\infty^{p_i-1}w
\]
\[
+ \sum_{i \not\in I_m}K_i(e^t)e^{(2-m(p_i-1))t}u_\infty^{p_i} + \mu f(e^t)e^{(m+2)t}
\]
\[
+ \sum_{i \in I}(K_i(e^t)e^{(2-m(p_i-1))t}((w + u_\infty)^{p_i} - u_\infty^{p_i} - p_iu_\infty^{p_i-1}w)
\]
\[
+ p_iK_i(e^t)e^{(2-m(p_i-1))t}u_\infty^{p_i-1}w - \sum_{i \in I_m}p_ie^{-lt}K_i(e^t)u_\infty^{p_i-1}w
\]
\[
= (\sum_{i \in I_m} - \sum_{i \not\in I_m})K_i(e^t)e^{(2-m(p_i-1))t}u_\infty^{p_i} + \mu f(e^t)e^{(m+2)t}
\]
\[
- \sum_{i \in I_m}e^{-lt}K_i(e^t)u_\infty^{p_i-1}w + \sum_{i \in I}K_i(e^t)e^{(2-m(p_i-1))t}(w + u_\infty)^{p_i}
\]
\[
- \sum_{i \in I_m}e^{-lt}K_i(e^t)u_\infty^{p_i-1}(e^{(2-m(p_i-1))t}u_\infty + w)
\]
\[
+ \sum_{i \in I}K_i(e^t)e^{(2-m(p_i-1))t}(w + u_\infty)^{p_i} + \mu f(e^t)e^{(m+2)t}
\]
\[
= -\sum_{i \in I_m}e^{-lt}K_i(e^t)u_\infty^{p_i-1}(w + u_\infty)
\]
\[
+ \sum_{i \in I}K_i(e^t)e^{(2-m(p_i-1))t}(w + u_\infty)^{p_i} + \mu f(e^t)e^{(m+2)t}
\]
\[
= -N^{p_i-1}(w + u_\infty) + \sum_{i \in I}K_i(e^t)e^{(2-m(p_i-1))t}(w + u_\infty)^{p_i}
\]
\[
+ \mu f(e^t)e^{(m+2)t},
\]
where we use $2 - m(p_i - 1) + l_i = 0, i \in I_m$ and Theorem 1.1. Therefore, (10) transforms into
\[
w'' + Mw' + Qw + \Psi(t, w) = 0
\]
with $Q = -N^{p_i-1} + \sum_{i \in I_m}p_i(-lt)u_\infty^{p_i-1}$. If $p_i > p(l_1)$, then the characteristic equation (8) of (11) has two negative roots $-\lambda_2, -\lambda_1$ satisfying $-\lambda_2 < -\lambda_1 < 0$ and $M > \lambda_2$, and so $n - m - \lambda_2 > n - m - M = m + 2$.

The condition (f.2) shows that $\lim_{r \to \infty} r^{-q}f(r)e^{(m+2)t} = 0$ since $q + m + 2 > q > n - m - \lambda_2$. Moreover, the definition of $m$ implies that $m + 2 + l_i - mp_i < 0$ for
Remark 1. For the case of $r = r(13) \text{ or } (14)$ at $t = \infty$

Theorem 4.1. $r$ are similar to those of $(3.18)$ in [8], $\gamma > 0$ (In fact, we can choose $\gamma$ only if $0 < \gamma < -(m + 2 + l_i - mp_i)$). These facts induce that there is $\gamma > 0$ such that

$$
\sum_{i \in I \setminus I_m} K_i(e^t)e^{(m+2-mp_i)t}u_{\infty} + \mu f(e^t)e^{(m+2)t} = o(e^{-\gamma t}), \ t = \infty.
$$

That is $\xi(t) = o(e^{-\gamma t}), \ t = \infty$.

Simple calculus shows that (5) can be rewritten as

$$
r^{n-1}u'(r) = - \int_0^r s^{n-1}(\sum_{i \in I} K_i(s)u^p(s) + \mu f(s))ds,
$$

then

$$
r^{m+1}u'(r) = -r^{m+2-n}\int_0^r s^{n-1}(\sum_{i \in I} K_i(s)u^p(s) + \mu f(s))ds.
$$

Thus, conditions (K.1) and (f.2) imply that $\lim_{r \to \infty} r^{m+1}u'(r)$ exists. By using $u'(t) = u'(r)r'(t) = mr^m r(t) + r^{m+1}u'(r)$ and Theorem 1.1, we know that $\lim_{t \to \infty} u'(t)$ exists and is finite.

Now, using the same argument as in [8, 25], for above $\gamma > 0$, we can obtain following results similarly (The detailed process will not be given here, one can refer to [8, 25]).

1. If $\gamma \leq \lambda_1$, then any solution $u(r)$ of (5) has the expression

$$
u(r) = \frac{1}{r^n} \sum_{i=1}^{k_2} \psi_i(r) + \sum_{j \in I, i < (k_2+\theta)/\lambda_1} a_{ij}(r) + b_1 \frac{1}{r^{m+1}} + \cdots + o\left(\frac{1}{r^{n-2+\varepsilon}}\right),
$$

at $r = \infty$ for some $\varepsilon > 0$, where $k_2, \theta, a_{ij}, b_1, I_i$ and $\psi_i$ are similar to those of (3.14) in [8].

2. If $\gamma \geq \lambda_2$, then $u(r)$ has the expression

$$
u(r) = \begin{cases} 
\frac{u_{\infty}}{r^m} + \frac{a_1}{r^{m+x_1} + \cdots + \frac{b_1}{r^{m+x_2}} + \cdots + o(\frac{1}{r^{m-2+\varepsilon}})}, \lambda_2 \neq \lambda_1, \\
\frac{u_{\infty}}{r^m} + \frac{a_1}{r^{m+x_1} + \cdots + \frac{b_1}{r^{m+x_2}} + \cdots + o(\frac{1}{r^{m-2+\varepsilon}})}, \lambda_2 = \Lambda \lambda_1
\end{cases}
$$

at $r = \infty$ for some $\varepsilon > 0$ and some positive integer $\Lambda > 1$, where $a_i, b_1$ and $d_1$ are similar to those of (3.18) in [8], $u_{\infty}$ satisfies $\sum_{i \in I_m} e^{-\lambda_1 t} K_i(r)u^p_{\infty} = N^{p_1-1}$ at $r = \infty$.

According to the analysis above, we get following conclusion.

**Theorem 4.1.** Let $p_1 > p(l_1)$. Suppose that there is a $\gamma > 0$ such that (12) holds. If $u(r)$ is a solution of (6) satisfying $\lim_{r \to \infty} r^m u(r) = u_{\infty}$, then $u(r)$ has an expression (13) or (14) at $r = \infty$.

**Remark 1.** For the case of $\lambda_1 < \gamma < \lambda_2$, we have a similar expression of $u(r)$ at $r = \infty$, and the expression constitutes with some mixed terms between $\frac{a_1}{r^{m+x_1}}$ and $\frac{b_1}{r^{m+x_2}}$, which are generated by $\frac{a_1}{r^{m+x_1}}$ and $\varphi (\log r)$. For a given solution, $\frac{a_1}{r^{m+x_1}}$ and $\frac{b_1}{r^{m+x_2}}$ are the two independent terms in the expansion at $r = \infty$. 

5. **Stability and asymptotic stability.** This section aims at the proof of Theorem 2.1, which is an extension of results obtained in [8, 21, 39]. We divide the proof into two parts. First we show that the solution of (6) is stable with the norm \( \| \cdot \|_{m+\lambda_1} \).

The following proposition shows that any positive solution of (6) is continuous in initial value in the sense of the norm \( \| \cdot \|_{m+\lambda_1} \).

**Proposition 1.** Let \( u_\beta \) and \( u_\alpha \) be two positive solutions of (6). If \( p_1 > p(l_1) \), then

\[
\lim_{\beta \to \alpha} \| u_\beta - u_\alpha \|_{m+\lambda_1} = 0.
\]

**Proof.** Let \( w = r^{m+\lambda_1}(u_\beta - u_\alpha) \). Then it is easy to show that \( w \) satisfies

\[
\begin{align*}
\frac{d^2}{dt^2} w &= \frac{d}{dt}\left(r^{m+\lambda_1}(u_\beta^\prime - u_\alpha^\prime)\right) + (m + \lambda_1)(m + \lambda_1 - 1)r^{m+\lambda_1-2}(u_\beta - u_\alpha) \\
&= r^{m+\lambda_1}(-\sum_{i \in I} K_i(r)u_\beta^{p_i} + \mu f(r)) - (\sum_{i \in I} K_i(r)u_\alpha^{p_i} + \mu f(r)))
\end{align*}
\]

Choose \( |\beta - \alpha| < \frac{\alpha}{2} \). Then we get \( |u_\beta - u_\alpha| < |u_\frac{\alpha}{2} - u_\frac{\alpha}{2}| = o(r^{- (m + \lambda_1)}) \) (or \( o(e^{- (m + \lambda_1)t}) \)) by Theorem 1.2 and Theorem 4.1.

The proofs of Lemma 2.2 in [9] and Lemma 3.3 give that

\[
K_i(r) \frac{u_\beta^{p_i} - u_\alpha^{p_i}}{u_\beta - u_\alpha} > p_i K_i(r) u_\frac{\alpha}{2}^{p_i - 1},
\]

which leads to

\[
- \sum_{i \in I} r^2 K_i(r) \frac{u_\beta^{p_i} - u_\alpha^{p_i}}{u_\beta - u_\alpha} - (m + \lambda_1)(m + \lambda_1 + 1)
\]

\[
< - \sum_{i \in I} p_i r^2 K_i(r) u_\frac{\alpha}{2}^{p_i - 1} - (m + \lambda_1)(m + \lambda_1 + 1)
\]

\[
< 0.
\]

Furthermore, the proofs of Lemma 2.2 in [9] and Lemma 3.3 also show that

\[
K_i(r) \frac{u_\beta^{p_i} - u_\alpha^{p_i}}{u_\beta - u_\alpha} < p_i K_i(r) u_\frac{\alpha}{2}^{p_i - 1},
\]

which leads to

\[
- \sum_{i \in I} r^2 K_i(r) \frac{u_\beta^{p_i} - u_\alpha^{p_i}}{u_\beta - u_\alpha} > - \sum_{i \in I} p_i r^2 K_i(r) u_\frac{\alpha}{2}^{p_i - 1}
\]

\[
= - \sum_{i \in I} p_i (r^{-i} K_i(r)) (r^{m_1} u_\frac{\alpha}{2})^{p_i - 1}.
\]
Therefore, $-\sum_{i \in I} r^2 K_i(r) \frac{u_{\beta_i}^{m_i} - u_\alpha^{m_i}}{u_\beta - u_\alpha} - (m + \lambda_1)(m + \lambda_1 + 1)$ is bounded in $(0, \infty)$ in view of (K.1) and Theorem 1.1. Denote
\[- \sum_{i \in I} r^2 K_i(r) \frac{u_{\beta_i}^{p_i} - u_\alpha^{p_i}}{u_\beta - u_\alpha} - (m + \lambda_1)(m + \lambda_1 + 1))r^{m+\lambda_1-2}(u_\beta - u_\alpha) = \tilde{w}.
\]
Then (15) changes into
\[w'' - 2(m + \lambda_1)r^{-1}w' = \tilde{w}. \tag{16}\]
Now, multiply (16) by $r^{-2(m+\lambda_1)}$ and then integrate in $(0, \infty)$ to obtain
\[w'(r) = \int_0^r r^{2(m+\lambda_1)}s^{-2(m+\lambda_1)}\tilde{w}(s)ds. \tag{17}\]
Integrating (17) in $(0, \infty)$ and exchanging the integrals order, we then have
\[w(r) = \frac{r^{1+2(m+\lambda_1)}}{1 + 2(m + \lambda_1)} \int_0^r s^{-2(m+\lambda_1)}\tilde{w}(s)ds. \tag{18}\]
For any $\varepsilon > 0$ and $0 < r_\varepsilon < r$, (18) can be rewritten as
\[w(r) = \frac{r^{1+2(m+\lambda_1)}}{1 + 2(m + \lambda_1)} \int_0^{r_\varepsilon} s^{-2(m+\lambda_1)}\tilde{w}(s)ds + \int_{r_\varepsilon}^r s^{-2(m+\lambda_1)}\tilde{w}(s)ds.
\]
For $\beta > \frac{\alpha}{2}$, there exists $r_\varepsilon$, which is independent of $\beta$ such that
\[\left| \int_{r_\varepsilon}^r s^{-2(m+\lambda_1)}\tilde{w}(s)ds \right| \leq \int_{r_\varepsilon}^\infty s^{-2(m+\lambda_1)}\tilde{w}(s)ds < \frac{(1 + 2(m + \lambda_1))\varepsilon}{2r^{1+2(m+\lambda_1)}}.\]
For such $r_\varepsilon$, take $\beta$ closing to $\alpha$ enough to get
\[\int_{r_\varepsilon}^r s^{-2(m+\lambda_1)}\tilde{w}(s)ds < \frac{(1 + 2(m + \lambda_1))\varepsilon}{2r^{1+2(m+\lambda_1)}}.\]
Therefore, we have $w(r) < \varepsilon$. This completes the proof.

Denote by $a_{1\alpha}$ the coefficient of the term $\frac{1}{r^{m+\lambda_1}}$ in (14) corresponding to the solution $u_\alpha$ of (6). The next result shows that $a_{1\alpha}$ is increasing in $\alpha$.

**Proposition 2.** Let $u_\alpha$ and $u_\beta$ be positive solutions of (6) with initial value $\alpha, \beta$ respectively and $\beta > \alpha$. If $p_1 > p(l_1)$, then $a_{1\beta} > a_{1\alpha}$.

**Proof.** By Theorem 4.1, we get
\[u_\beta - u_\alpha = \frac{a_{1\beta} - a_{1\alpha}}{r^{m+\lambda_1}} + o\left(\frac{1}{r^{m+\lambda_1}}\right)\]
at $r = \infty$. Moreover, $a_{1\beta} \geq a_{1\alpha}$ since $u_\beta > u_\alpha$ in view of Theorem 1.2. Suppose that the equality holds. Then $0 < u_\beta - u_\alpha = o\left(\frac{1}{r^{m+\lambda_1}}\right)$. Let $w = r^{m+\lambda_1}(u_\beta - u_\alpha)$. Then $w = o\left(\frac{1}{r^{m+\lambda_1}}\right) > 0, w = 0$ at $r = 0, \infty$, and $w$ satisfies
\[w'' - 2(m + \lambda_1)r^{-1}w' = \left(\sum_{i \in I} r^2 K_i(r) \frac{u_{\beta_i}^{p_i} - u_\alpha^{p_i}}{u_\beta - u_\alpha} - (m + \lambda_1)(m + \lambda_1 + 1))r^{-2}w. \]
Since
\[- \sum_{i \in I} r^2 K_i(r) \frac{u_{\beta_i}^{p_i} - u_\alpha^{p_i}}{u_\beta - u_\alpha} - (m + \lambda_1)(m + \lambda_1 + 1) < 0, \]
the maximum principle implies that $w \leq 0$ in $(0, \infty)$, which is a contradiction. The proof is completed.
Now, we need definitions of sub- and super-solutions and weak sub- and super-solutions for certain elliptic or parabolic equations (see [8]).

**Definition 5.1.** A function $u$ is said to be a super-solution of the equation
\[ \Delta u + f(x, u) = 0 \]
in an open set $\Omega \subset \mathbb{R}^n$ if $\Delta u + f(x, u) \leq 0$ in $\Omega$, and $u$ is said to be a sub-solution if $\Delta u + f(x, u) \geq 0$ in $\Omega$.

**Definition 5.2.** A function $u$ is a continuous weak super-solution of (2) if

(i) $u$ is continuous in $\Omega_T = \mathbb{R}^n \times [0, T)$ for some $T > 0$ and $u(\cdot, 0) \geq \varphi$;

(ii) $u$ satisfies
\[ \int_{\mathbb{R}^n} u(x, t) \eta(x, t) \, dx \geq \int_0^{T'} \int_{\mathbb{R}^n} (u(x, s)(\Delta \eta + \eta_t)(x, t) + \eta(x, t) f(u(x, t), t)) \, dx \, dt \]
for all $T' \in [0, T)$ and $0 \leq \eta(x, t) \in C^{2,1}(\mathbb{R}^n \times (0, T))$ with
\[ f(u(x, t), t) = \sum_{i=1}^k K_i(|x|) u^{p_i} + \mu f(|x|) \]
and supp($\eta(\cdot, t)$) being compact in $\mathbb{R}^n$ for $t \in [0, T)$. Similarly, a continuous weak sub-solution is defined by reversing the inequalities in (i) and (19).

In what follows, mainly use the techniques introduced by Wang [36], the following result can be obtained.

**Theorem 5.1.** Suppose that $K_i$, $i = 1, 2, \ldots, k$, satisfy (K.1) and (K.3) in $(R, \infty)$ for some large $R$, and $f$ satisfies (f.1). Then

(i) if $\bar{u}$ and $\underline{u}$ are bounded continuous weak super- and sub-solutions of (2), respectively, and $\bar{u} \geq \underline{u}$ on $\mathbb{R}^n \times (0, T)$, then (2) has a unique classical solution $u = u(x, t; \varphi)$ satisfies $\bar{u} \geq u \geq \underline{u}$;

(ii) if $\varphi$ is radially symmetric, so is $u(x, t; \varphi)$ in the x-variable.

(iii) if the initial value $\varphi$ in (2) is a bounded continuous super- or sub-solution of the elliptic equation (1) in $\mathbb{R}^n$, then the solution $u(x, t; \varphi)$ of (2) is strictly decreasing or increasing in $t$ as long as it exists provided that $\varphi$ is not a steady state of (2).

**Proof.** (i) Firstly, we consider the case $l_i \geq 0, i = 1, 2, \ldots, k$. Choose a set sequence $\{\Omega_\rho\}_{\rho=1}^\infty$ satisfying
\[ \Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_\rho \subset \cdots \subset \mathbb{R}^n, \cup_{\rho=1}^\infty \Omega_\rho = \mathbb{R}^n, \cup_{\rho=1}^\infty (\partial(\Omega_\rho) \cap \partial(\mathbb{R}^n)) = \partial(\mathbb{R}^n) \]
each $\partial(\Omega_\rho)$ satisfying the exterior sphere condition. Denote $\Sigma_\rho = \Omega_\rho \times (0, T)$ and $\Gamma = \{\partial(\Omega_\rho) \times (0, T)\} \cup \{\Omega_\rho \times \{0\}\}$.

Suppose that $\bar{\varphi}$ is the continuous extension of $\varphi$ in $\mathbb{R}^{n+1}$. Let
\[ \phi = \max\{\min\{\bar{\varphi}, \underline{u}\}, \underline{u}\}, \quad (x, t) \in \mathbb{R}^n \times (0, T). \]
Then $\bar{u} \geq \phi \geq \underline{u}$ holds in $\mathbb{R}^n \times (0, T)$. Now, consider the problem
\[ u_t = \Delta u + \sum_{i=1}^k K_i(|x|) u^{p_i} + \mu f(|x|), \quad (x, t) \in \Sigma_\rho, \quad u|_{r_\rho} = \phi|_{r_\rho}. \]
By the proof of Lemma 1.2 in [36] for the case of bounded region, we know that (20) has a classical solution $u_\rho = u_\rho(x, t; \varphi)$ satisfying $\bar{u} \geq u_\rho \geq \underline{u}$. Using the $L^p$ and Schauder interior estimates to $u_\rho$, for any compact subset $\Sigma \subset \mathbb{R}^n \times (0, T)$, we know
that \( \|u_p\|_{C^{2+s,1+\frac{s}{2}}(\Sigma)} < \infty \). Thus, there is a convergent subsequence of \( \{u_p\}_{p=1}^\infty \), still denoted by \( \{u_p\}_{p=1}^\infty \), such that \( \lim_{p \to \infty} u_p = u = u(x,t; \varphi) \in C^{2,1}(\mathbb{R}^n \times (0,T)) \).

Clearly, \( u \) satisfies the problem

\[
 u_t = \Delta u + \sum_{i=1}^k K_i(|x|)u^{p_i} + \mu f(|x|), \quad (x,t) \in \mathbb{R}^n \times (0,T), \quad u|_{\Gamma} = \varphi
\]

and \( \overline{u} \geq u \geq \underline{u} \) in \( \mathbb{R}^n \times (0,T) \), where \( \Gamma = \{\partial(\mathbb{R}^n) \times (0,T)\} \cup \{\mathbb{R}^n \times \{0\}\} \).

In the following, we need to verify \( u \in C(\mathbb{R}^n \times (0,T)) \) and \( u|_{\Gamma} = \varphi (= \phi|_{\Gamma}) \). We use the techniques introduced in [15]. Take \( z \in \Gamma \). Then there is \( \rho \) such that \( z \in \Omega_{\rho} \).

Find a barrier \( B_z \in C(\Sigma_{\rho}) \cap C^{2,1}(\Sigma_{\rho}) \) (the existence is justified by the regularity of \( \Omega_{\rho} \)) such that

\[
 B_z(\xi) > 0, \quad \xi \in \Sigma_{\rho}, \quad B_z(z) = 0, \quad \xi \neq z \quad \text{and} \quad (B_z)_t - \Delta B_z \geq 1 \text{ in } \Omega_{\rho}.
\]

For fixed \( \varepsilon > 0 \), denote

\[
u^+_\varepsilon = \varphi(z) + cB_z + \varepsilon, \quad \nu^-_\varepsilon = \varphi(z) - cB_z - \varepsilon
\]

with \( c \) being a selectable constant. Since \( u_p \) is uniformly bounded in \( \Sigma_{\rho} \) and \( u_l|_{\Sigma_{\rho}} = \phi|_{\Sigma_{\rho}} \) holds near \( z \) for large \( \rho \), there exists \( c_\varepsilon \) such that for each \( \rho > \rho_i \), there hold

\[
u^+_\varepsilon|_{\Sigma_{\rho}} \geq u_l|_{\Sigma_{\rho}} \geq \nu^-_\varepsilon|_{\Sigma_{\rho}}
\]

and

\[
 (\nu^+_\varepsilon)_t - \Delta \nu^+_\varepsilon \geq (\nu^-_\varepsilon)_t - \Delta \nu^-_\varepsilon.
\]

Then the maximum principle implies \( \nu^+_\varepsilon \geq \nu^-_\varepsilon \geq u_l \) for \( (x,t) \in \Sigma_{\rho} \). Thus, we have

\[
 |u(x,t) - \varphi(z)| \leq c_\varepsilon B_z(\xi) + \varepsilon
\]

for \( \xi \in \Sigma_{\rho} \) and \( \rho > \rho_i \). Now, let \( \rho \to \infty \) to get

\[
 |u(\xi) - \varphi(z)| \leq c_\varepsilon B_z(\xi) + \varepsilon, \quad \xi \in \Sigma_{\rho}.
\]

Thus, there holds

\[
 \limsup_{\xi \to z} |u(\xi) - \varphi(z)| \leq \varepsilon, \quad \xi \in \Sigma_{\rho}.
\]

Therefore, \( u \in C(\mathbb{R}^n \times (0,T)) \) and \( u|_{\Gamma} = \varphi \). A bootstrap argument then implies that \( u \) is a classical solution of (2).

In the following, we consider the case \(-2 < l_i < 0\). We mention a basic and standard fact for the classical problem

\[
 \begin{cases}
 u_t = \Delta u + u^p, \\ u(x,0) = \varphi(x), \quad \varphi(x) \geq 0, \quad x \in \mathbb{R}^n
\end{cases}
\]

with \( p > 1, \varphi(x) \in C_B(\mathbb{R}^n) = C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) due to [1]. The relative fact reads as the following: For any \( \varphi(x) \in C_B(\mathbb{R}^n) \) with \( \varphi(x) \geq 0 \), there exists \( T_\varphi > 0 \) such that this problem has a unique classical solution \( u \) in \( \mathbb{R}^n \times [0,T_\varphi) \) and \( u \) is bounded in \( \mathbb{R}^n \times (0,T') \) for any \( 0 < T' < T_\varphi \), and if \( T_\varphi < \infty \), then \( \lim_{t \to T_\varphi} \|u(\cdot,t)\|_{L^\infty(\mathbb{R}^n)} = \infty \).

This fact can be proved by the standard contraction mapping method.

Now, let \( \overline{T} = \min\{T, T_\varphi\} \) with \( T_\varphi \) corresponding to (2). It suffices to show that \( u \geq u \geq \underline{u} \) holds in \( \mathbb{R}^n \times (0,\overline{T}) \). Define

\[
 H(u) = e^{\Delta} \varphi + \int_0^t e^{(t-s)\Delta} \left( \sum_{i=1}^k K_i(|x|)u^{p_i}(|x|, s) + \mu f(|x|) \right) ds.
\]
Then similar to the proof of Theorem 2.3 in [36], we can prove that there is $T_0 = T_0(\|\varphi\|_{L^\infty(\mathbb{R}^n)}, \rho_1, \rho_2, n)$ such that the operator $H$ maps $B_{T_0}^\delta$ into itself and $H$ is contracting, where $B_{T_0}^\delta$ is a closed ball in $C_B(\mathbb{R}^n \times [0, T])$ with center $\varphi$ and radius $\delta = 3\|\varphi\|_{L^\infty(\mathbb{R}^n)}$. Let $B = \{v \in B_{T_0}^\delta | v \leq \pi\}$. Then for any $v \in B$, we have $H(v) = H(\pi) \leq \pi$, that is, $H(v) \in B$. Thus, $u \leq \pi$ holds in $\mathbb{R}^n \times [0, T_0]$ (Note that $u$ is a fixed point of $H$ in $B$). Take $\varphi = u|_{t=T_0}$. Then $u$ satisfies $u \leq \pi$ in $\mathbb{R}^n \times [T_0, 1]$ with $T_1 = T_1(\|u(x, T_0)\|_{L^\infty(\mathbb{R}^n)})$. Following this procedure, we obtain $u \leq \pi$ holds in $\mathbb{R}^n \times [T_0, T_1], \mathbb{R}^n \times [T_2, T_3], \ldots$, where $T_i = T_i(\|u(x, T_{i-1})\|_{L^\infty(\mathbb{R}^n)})$, $i \geq 2$. By deduction, we get $u \leq \pi, (x, t) \in \mathbb{R}^n \times [0, T]$.

A similar argument gives that $u \geq \underline{u}, (x, t) \in \mathbb{R}^n \times [0, T]$. As to the proof of the classicality of $u$, we will not repeat it.

(ii) If $l_i \geq 0$. Then the conclusion can be easily proved by the reflection argument as in [17], we omit the detail.

If $-2 < l_i < 0$. It only needs to verify that $u$ is radially symmetric if $\varphi$ is radially symmetric when $t \in [0, T]$ for $T > 0$ since $t$ can go up to $T_\varphi$ by a ladder argument. Suppose that we take $\varphi$ at the beginning of the iteration scheme in Theorem 2.3 [36] from which the fixed point $u$ of $H$ (defined in the proof of (1)) is obtained. Since $e^{t\Delta}$ and hence $H$ preserve the properties desired in Lemma 1.4 [36], all terms in the iteration scheme satisfy the properties what we want and hence do the fixed point $u$ of $H$ in $\mathbb{R}^n \times [0, T]$. Indeed, these includes the radial symmetry of $u$.

(iii) We suppose that $\varphi$ is a bounded continuous super-solution of the elliptic equation (1) in $\mathbb{R}^n$ but not a steady state of (2) (The sub-solution case can be proved similarly).

If $l_i \geq 0$. Then an application of Lemma 1.3 [36] implies that $\varphi \geq u$. Let

$$u_\tau(x, t) = u(x, t + \tau), v = u - u_\tau, 0 < \tau \ll 1.$$  

Then $v|_{t=0} = \varphi - u(x, \tau) \geq 0$ and

$$v_t - \Delta v = \sum_{i=1}^{k} K_i(|x|) (\varphi^{p_i} - u^{p_i}) =: C(x, t)v$$

holds in $\mathbb{R}^n \times [0, T']$, where $C(x, t) \in C^{\alpha, \frac{\alpha}{2}}(\mathbb{R}^n \times (0, \infty))$ satisfies

$$\sum_{i=1}^{k} K_i(|x|) (\bar{\varphi}^{p_i} - \bar{u}^{p_i}) \geq C(x, t)(\bar{\varphi} - \bar{u})$$

and

$$C(x, t) \leq C_0(T')(1 + |x|^2)$$

for some $C_0(T') > 0$ (see Lemma 1.3 [36]). Using Lemma 1.3 again, we know that $v \geq 0$, that is, $u$ is decreasing in $t$, $u_t \leq 0$ for $t \geq 0$. Then the maximum principle implies that $u_t < 0$, i.e., $u$ is strictly decreasing in $t$.

If $-2 < l_i < 0$. Let $u_\tau(x, t) = u(x, t + \tau), 0 < \tau < T_\varphi, (x, t) \in \mathbb{R}^n \times [0, T_\varphi - \tau)$. Then

$$u_\tau(x, t) = e^{(t+\tau)\Delta} \varphi + \int_0^{t+\tau} e^{(t+\tau-s)\Delta} \left( \sum_{i=1}^{k} K_i(|x|) u^{p_i}(|x|, s) + \mu f(|x|) \right) ds$$

$$= e^{(t+\tau)\Delta} \varphi + \int_0^{T_\varphi} e^{(t+\tau-s)\Delta} \left( \sum_{i=1}^{k} K_i(|x|) u^{p_i}(|x|, s) + \mu f(|x|) \right) ds$$

$$+ \int_0^t e^{(t-s)\Delta} \left( \sum_{i=1}^{k} K_i(|x|) u^{p_i}(|x|, s + \tau) + \mu f(|x|) \right) ds$$
\[ e^{t \Delta} u(\tau) + \int_0^t e^{(t-s)\Delta} \left( \sum_{i=1}^k K_i(|x|) u_{r+i}^p(|x|, s) + \mu f(|x|) \right) ds \]

\[ \leq e^{t \varphi} \varphi + \int_0^t e^{(t-s)\Delta} \left( \sum_{i=1}^k K_i(|x|) u_{r+i}^p(|x|, s) + \mu f(|x|) \right) ds. \]

Again, Lemma 1.3 [36] implies that \( \varphi \geq u \). Thus, \( u_\tau \) is a sub-solution of (2) in \( \mathbb{R}^n \times [0, T_\varphi - \tau) \). By (1), we know that \( u_\tau \leq u \) holds in \( \mathbb{R}^n \times [0, T_\varphi - \tau) \). So, \( u \) is strictly decreasing in \( t \). So far, the whole proof of Theorem 5.1 is completed. \( \square \)

Now using Proposition 1, Proposition 2 and Theorem 5.1, we can show that the steady state \( u_\alpha \) is stable with respect to the norm \( \| \cdot \|_{m+\lambda_1} \).

Now, we set out to prove Theorem 2.1 (i).

**Proof.** In virtue of Theorem 5.1, for any \( \varepsilon > 0 \), there exists \( \gamma \in (0, \frac{1}{2}) \) such that \( \| u_{a+\gamma} - u_\alpha \|_{m+\lambda_1} < \varepsilon \). By Theorem 1.2 and the proof of Proposition 2, we know that for such \( \varepsilon \), there exists large \( r_\varepsilon \) such that the following two inequalities hold in \( [r_\varepsilon, \infty) \).

\[ r^{m+\lambda_1}(u_{a+\gamma} - u_\alpha) = a_1(\alpha+\gamma) - a_1\alpha + o\left( \frac{1}{r^{m+\lambda_1}} \right) > \frac{a_1(\alpha+\gamma) - a_1\alpha}{2} > 0, \]

\[ r^{m+\lambda_1}(u_\alpha - u_{a-\gamma}) = a_1\alpha - a_1(\alpha-\gamma) + o\left( \frac{1}{r^{m+\lambda_1}} \right) > \frac{a_1\alpha - a_1(\alpha-\gamma)}{2} > 0. \]

Take \( \delta_1 = \min \left\{ \frac{a_1(\alpha+\gamma) - a_1\alpha}{2}, \frac{a_1\alpha - a_1(\alpha-\gamma)}{2} \right\} \). Then for \( \| \varphi - u_\alpha \|_{m+\lambda_1} < \delta_1 \), the followings hold in \( [r_\varepsilon, \infty) \).

\[ r^{m+\lambda_1}(u_{a+\gamma} - \varphi) \geq r^{m+\lambda_1}(u_{a+\gamma} - u_\alpha) - \| \varphi - u_\alpha \| > \frac{a_1(\alpha+\gamma) - a_1\alpha}{2} - \delta_1 \geq 0, \]

\[ r^{m+\lambda_1}(\varphi - u_{a-\gamma}) \geq r^{m+\lambda_1}(u_\alpha - u_{a-\gamma}) - \| \varphi - u_\alpha \| > \frac{a_1\alpha - a_1(\alpha-\gamma)}{2} - \delta_1 \geq 0. \]

Moreover, by Theorem 1.2, we have \( u_{a-\gamma} < u_\alpha < u_{a+\gamma} \) in \( [0, r_\varepsilon) \). This implies that there is \( \delta_2 > 0 \) such that \( u_{a-\gamma} < \varphi < u_{a+\gamma} \) in \( [0, r_\varepsilon) \) if \( \| \varphi - u_\alpha \|_{m+\lambda_1} < \delta_2 \).

Now, let \( \delta = \min\{\delta_1, \delta_2\} \). Then \( u_{a-\gamma} < \varphi < u_{a+\gamma} \) in \( [0, r_\varepsilon) \) if \( \| \varphi - u_\alpha \|_{m+\lambda_1} < \delta \).

Theorem 5.1 (1) shows that the solution \( u(\cdot, t; \varphi) \) of (2) satisfies \( u_{a-\gamma} < u(\cdot, t; \varphi) < u_{a+\gamma} \) in \( [0, r_\varepsilon) \), and so \( \| u(\cdot, t; \varphi) - u_\alpha \|_{m+\lambda_1} < \varepsilon \). That is, \( u_\alpha \) is stable with respect to the norm \( \| \cdot \|_{m+\lambda_1} \) by Definition 2.1. \( \square \)

In the following, we try to prove Theorem 2.1 (ii). We first give the following result on existence.

**Theorem 5.2.** Suppose that \( H_i = H_i(r) \), \( i \in I \) are radial smooth functions satisfying

\[ K_i + \nu_i H_i > 0, \quad \sum_{i \in I} \left( (1 - \frac{n(p_i - 1)}{2(p_i + 1)}) (K_i + \nu_i H_i) + \frac{r(K_i + \nu_i H_i)^p}{p_i + 1} \right) < 0 \] (21)

with \( \nu_i, i \in I \), being small positive constants. Then for each \( \alpha > 0 \), the problem

\[
\begin{cases}
    u'' + (n - 1)r^{-1}u' + \sum_{i \in I} (K_i + \nu_i H_i) u^{p_i} + \mu f = 0, & r > 0, \\
    u(0) = \alpha
\end{cases}
\] (22)

always has a positive solution \( u_\alpha \) in \( [0, \infty) \).
Proof. We prove Theorem 5.2 by contradiction. By the standard argument, we know that (22) always has a nonnegative solution \( u_\alpha \) near \( r = 0 \). If \( u_\alpha \) dose not remain entire positive in \([0, \infty)\), then there exists some \( R > 0 \) such that \( u_\alpha(R) = 0 \) and \( u_\alpha > 0 \) in \([0, R)\). By Lemma 2.29 in [21], we conclude that

\[
\int_0^R \sum_{i \in I} ((1 - \frac{n(p_i - 1)}{2(p_i + 1)}))(K_i + \nu_i H_i) + \frac{r(K_i + \nu_i H_i)'}{p_i + 1}r^{n-1}u^{p_i+1}_\alpha \, dr = \frac{(Ru'_\alpha(R))^2}{2}.
\]

Clearly, the left is negative, but the right is nonnegative. This is a contradiction. Therefore, there dose not exist \( R > 0 \) such that \( u_\alpha(R) = 0 \), and then \( u_\alpha \) is entire positive in \([0, \infty)\). \(\square\)

Remark 2. As applications, we may take \( K_i(r) = c_i r^{\ell_i} \) and \( H_i(r) = r^{\ell_i} h_i(r) \), where \( c_i, i \in I \), are positive constants, \( h_i(r), i \in I \), are smooth functions having support set in a small ball centering at the origin and decreasing in \( r \). With these assumptions, (21) is equivalent to the inequality

\[
\sum_{i \in I} ((1 - \frac{n(p_i - 1)}{2(p_i + 1)})) + \frac{r h_i'}{(p_i + 1)(c_i + h_i)} < 0.
\]

Indeed, if we take \( p_i > \frac{n+2+2\ell_i}{n-2} \), \( i \in I \), then (21) holds. In short, the condition (21) can be satisfied.

According to Theorem 5.2, if functions \( H_i, i \in I \), are non-negative and small, and \( \pm H_i \) satisfy conditions stated in Theorem 5.2, then following problems

\[
\begin{align*}
\begin{cases}
& u'' + (n - 1)r^{-1}u' + \sum_{i \in I}(K_i + \nu_i H_i)u^{p_i} + \mu f = 0, \quad r > 0, \\
& u(0) = \alpha
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
\begin{cases}
& u'' + (n - 1)r^{-1}u' + \sum_{i \in I}(K_i - \nu_i H_i)u^{p_i} + \mu f = 0, \quad r > 0, \\
& u(0) = \alpha
\end{cases}
\end{align*}
\]

both have a positive solution, denoted by \( u_+^\alpha \) and \( u_-^\alpha \), respectively. In what follows, we will use \( u_+^\alpha \) and \( u_-^\alpha \) to denote solutions of (23) corresponding to \(+H_i\) and \(-H_i\), respectively, which would be super and sub-solutions of (6).

Now we need a result (Proposition 3) derived from Lemma 2.20 in [21].

Proposition 3. Suppose that \( w_{1i} \) is a positive radial sub-solution of \( \Delta u + K_i(x)u = 0 \) in ball \( B_R \) centering at origin and \( w_{2i} \) is a radial super-solution of the same equation in \( B_R \) with \( w_{1i}(0) > 0 \). Then

\[
w_{1i}(r) \leq \frac{w_{1i}(0)}{w_{2i}(0)} w_{2i}(r)
\]

for all \( 0 \leq r \leq R \). Moreover,

\[
w_{1i}(R) \geq \frac{w_{1i}(0)}{w_{2i}(0)} w_{2i}(R)
\]

if one of functions is not a solution of the equation.

The following proposition shows that any solution of (1) decreases if we add a small positive perturbation on \( K_i \).

Proposition 4. \( u^+_\alpha \leq u_\alpha \).
Proof. It suffices to show that $u^+_\alpha < u_\gamma$ if $\alpha < \gamma$ (Since if $\alpha < \gamma$ ensures $u^+_\alpha < u_\gamma$, then for $\alpha = \gamma$, there is at most $u^+_\alpha \leq u_\gamma$). We prove this result by Proposition 3.

Suppose on the contrary that it is not true. Let $w_1 = u_\gamma - u^+_\alpha$. Then there must exist $R > 0$ such that $w_1 > 0$ in $[0, R)$, $w_1(R) = 0$, and by (1), we have

$$
\Delta w_1 + \sum_{i \in I} K_i (u^{p_i}_{\alpha} - (u^+_\alpha)^{p_i}) = \Delta w_1 + \sum_{i \in I} K_i \frac{u^{p_i}_{\alpha} - (u^+_\alpha)^{p_i}}{u_\gamma - u^+_\alpha} w_1 = \sum_{i \in I} (u^+_\alpha)^{p_i} H_i \geq 0.
$$

Let $w_2 = u_2 - u_\gamma$. Then $w_2 > 0$ in $\mathbb{R}^n$. Use (1) to get

$$
\Delta w_2 + \sum_{i \in I} K_i (u^{p_i}_{2\gamma} - (u^+_{2\gamma})^{p_i}) = \Delta w_2 + \sum_{i \in I} K_i \frac{u^{p_i}_{2\gamma} - (u^+_{2\gamma})^{p_i}}{u_\gamma - u_\gamma} w_2 = 0.
$$

These show that $w_1$ and $w_2$ are a pair of sub- and super-solutions of $\Delta u + K_i(x)u = 0$. Then Proposition 3 induces

$$
w_1(R) \geq \frac{w_1(0)}{w_2(0)} w_2(R).
$$

And so $w_1(R) > 0$. This contradicts to $w_1(R) = 0$ and the proof is finished. $\square$

Similarly, we can prove that $u_\alpha \leq u^-_\gamma$.

From Proposition 4, a comparison result can be obtained immediately.

Proposition 5. Suppose that $H_i, H^+_i, i \in I$, are small and non-negative, and $H_i \geq H^+_i$. Denote by $v^+_\alpha$ the solution of (22) by replacing $H_i$ by $H^+_i$. Then $v^+_\alpha \geq u^+_\alpha$.

Proposition 5 gives the monotonicity of solutions of (23) with small positive perturbation on $K_i$.

Combining Proposition 4, Proposition 5 with Proposition 3, and using the similar technique as in proving Proposition 4, we can prove the following result, the detailed process will be omitted here.

Proposition 6. If $\beta > \alpha > \gamma$ for $\gamma > 0$, then there exist small nonnegative functions $H^+_i, i \in I$, such that $u^+_\beta > u_\alpha > u^-_\gamma$.

Let $H_i$ be the same as in Proposition 5. Define

$$
\beta_0 = \min \{\beta > \alpha | u^+_\beta \geq u_\alpha\}, \quad \gamma_0 = \max \{\gamma < \alpha | u^-_\gamma \leq u_\alpha\}.
$$

Then $u^-_\gamma$ and $u^+_\beta$ satisfy

$$
(u^-_\gamma)'' + (n - 1)r^{-1}(u^-_\gamma)' + \sum_{i = 1}^k K_i (u^-_\gamma)^{p_i} + \mu f \\
\geq (u^-_\gamma)'' + (n - 1)r^{-1}(u^-_\gamma)' + \sum_{i = 1}^k (K_i + H_i)(u^-_\gamma)^{p_i} + \mu f \\
= 0
$$

and

$$
(u^+_\beta)'' + (n - 1)r^{-1}(u^+_\beta)' + \sum_{i = 1}^k K_i (u^+_\beta)^{p_i} + \mu f \\
\leq (u^+_\beta)'' + (n - 1)r^{-1}(u^+_\beta)' + \sum_{i = 1}^k (K_i + H_i)(u^+_\beta)^{p_i} + \mu f \\
= 0,
$$
respectively. These show that $u^{-}_\gamma$ and $u^+_{\beta}$ are a pair of sub- and super-solutions of (6). Moreover, we have $u^+_{\beta_0} > u_\alpha > u^-_{\gamma_0}$ by Proposition 6.

Denote by $a^+_{1,\beta_0}, a_{1,\alpha}$, $a_{1,\alpha}$ the coefficients of the term $\frac{1}{r^{m+\lambda_1}}$ in (14) corresponding to $u^+_{\beta_0}, u_\alpha, u^-_{\gamma_0}$. Now, we present a relationship among $a^+_{1,\beta_0}, a_{1,\alpha}$ and $a_{1,\gamma_0}$.

**Proposition 7.** Let $p_1 > p(l_1)$. Then $a^+_{1,\beta_0} = a_{1,\alpha} = a^-_{1,\gamma_0}$.

**Proof.** We only prove the left equality, and the right one can be handled similarly.

We first have $a^+_{1,\beta_0} \geq a_{1,\alpha}$. Suppose that the inequality holds. It then follows that

$$\|u^+_{\beta_0} - u_\alpha\|_{m+\lambda_1} \geq \frac{1}{2}(a^+_{1,\beta_0} - a_{1,\alpha}) > 0$$

for large $r$. Using a similar argument as in the proof of Proposition 2, we can show that

$$\lim_{\delta \to 0} \|u^+_{\beta_0,\delta} - u^+_{\beta_0}\|_{m+\lambda_1} = 0.$$

So, there exist constants $\delta_1 > 0, R > 0$ such that

$$\|u^+_{\beta_0,\delta} - u^+_{\beta_0}\|_{m+\lambda_1} < \frac{1}{4}(a^+_{1,\beta_0} - a_{1,\alpha}), \quad 0 < \delta < \delta_1,$$

and

$$r^{m+\lambda_1} |u_{\beta_0}^+ - u_\alpha| > \frac{1}{2}(a^+_{1,\beta_0} - a_{1,\alpha}), \quad r > R.$$

Therefore, for $r > R$, we have

$$r^{m+\lambda_1}(u_{\beta_0,\delta}^+ - u_\alpha) = r^{m+\lambda_1}(u_{\beta_0,\delta}^+ - u_{\beta_0}^+) + r^{m+\lambda_1}(u_{\beta_0}^+ - u_\alpha) > \frac{1}{4}(a^+_{1,\beta_0} - a_{1,\alpha}).$$

Furthermore, there exists $\delta_2 > 0$ such that $u^+_{\beta_0,\delta} > u_\alpha$ in $[0, R]$ for $0 < \delta < \delta_2$.

Take $\delta = \min\{\frac{\delta_1}{2}, \frac{\delta_2}{2}\}$. Then $u^+_{\beta_0,\delta} > u_\alpha$ holds in $[0, R]$, which contradicts to the definition of $\beta_0$ and the proof is completed. \(\square\)

Using Theorem 1.1, Proposition 2 and Proposition 7, we know that $u_\alpha$ is the unique solution of (6) between $u^-_{\gamma_0}$ and $u^+_{\beta_0}$. From this point, the following result can be obtained easily.

**Theorem 5.3.** Suppose that $p_1 > p(l_1)$. Then there exist a sequence of sub-solutions $\{u^-_{\nu}\}_{\nu=1}^{\infty}$ and a sequence of super-solutions $\{u^+_{\nu}\}_{\nu=1}^{\infty}$ of (6) with $u^-_{1} < u^-_{2} < \cdots < u_\alpha$ and $u_\alpha < \cdots < u^-_{\gamma} < u^+_{\nu}$ such that $u_\alpha$ is unique in the ordered interval $< u^-_{1}, u^+_{\nu}>: \{u^-_{\nu} < u < u^+_{\nu}, \nu = 1, 2, \cdots | u \text{ satisfies (6)}\}$. Moreover, $\{u^-_{\nu}\}_{\nu=1}^{\infty}, u_\alpha$ and $\{u^+_{\nu}\}_{\nu=1}^{\infty}$ satisfy

$$\lim_{\nu \to \infty} u^-_{\nu} = u^-_{1} = \lim_{\nu \to \infty} u^+_{\nu}.$$  

**Proof.** Take $u^-_{1} = u^-_{\gamma_0}$. Then (6) has no solution between $u^-_{1}$ and $u_\alpha$. Consider the equation

$$u'' + (n-1)r^{-1}u' + \sum_{i \in I}(K_i - \nu^{-1}H_i^+)u^{\nu_i} + \mu f = 0, \quad \nu = 1, 2, \cdots \text{ in } \mathbb{R}^n \quad (24)$$

with $H_i^+$ being the same as in Proposition 5 or Proposition 6. It is easy to see that $u_\alpha$ and $u^-_{1}$ are a pair of super- and sub-solutions of (24) for $\nu = 2$. A similar argument as in [31] induces that (24) has a radial solution, denoted by $u_2$, which satisfies $u_\alpha < u^-_{1} < u_2$. Moreover, $u_\alpha$ and $u^-_{1}$ are also a pair of super- and sub-solutions of (24) for $\nu = 3$. Generally, we find that $u_\alpha$ and $u^-_{\nu_0}$ are a pair of super- and sub-solutions of (24) if we get $u^-_{\nu_0}$ for some $\nu_0 + 1$. Using $u_\alpha$ and $u^-_{\nu_0}$, we can
We prove (25) by contradiction. Firstly, we have
\[ b \]
where \( b \) is the coefficient of \( \frac{1}{r^{m+\lambda_2}} \) corresponding to \( u^-_\nu \), \( u^+_{\nu} \), respectively. Then the following result holds.

**Remark 3.** We construct the equation (24) by using \( \nu^{-1}H^+ \) only for the technical reason. Indeed, if \( \nu^{-1}H^+ \) is replaced by \( \nu H^+ \), then one can easily check that \( u_\alpha \) remains a super-solution of (24), but \( u^-_1 \) is not a sub-solution of (24).

Denote by \( b^-_{\nu},b^+_{\nu} \) the coefficients of term \( \frac{1}{r^{m+\lambda_2}} \) corresponding to \( u^-_\nu \), \( u^+_{\nu} \), respectively. Then the following result holds.

**Proposition 8.** The sequences \( \{b^-_{\nu}\}_{\nu=1}^\infty \) and \( \{b^+_{\nu}\}_{\nu=1}^\infty \) satisfy
\[
b^-_{11} < b^-_{12} < \cdots < b^-_{1\alpha} < \cdots < b^+_{12} < b^+_{11}
\] (25)
and
\[
\lim_{\nu \to \infty} b^-_{1\nu} = b^-_{1\alpha} = \lim_{\nu \to \infty} b^+_{1\nu},
\] (26)
where \( b^-_{1\alpha} \) is the coefficient of \( r^{-(m+\lambda_2)} \) corresponding to \( u^-_{\nu} \).

**Proof.** We prove (25) by contradiction. Firstly, we have
\[
b^-_{11} \leq b^-_{12} \leq \cdots \leq b^-_{1\alpha} \leq \cdots \leq b^+_{12} \leq b^+_{11}
\]
by constructions of \( u^-_\nu \), \( u^+_{\nu} \) and Proposition 7. Suppose that \( b^-_{1(\nu+1)} = b^-_{1\nu} \) for some \( \nu \geq 1 \). Then Theorem 4.1 induces that
\[
u_{\nu+1} - u^-_\nu = o\left(\frac{1}{r^{m+\lambda_2}}\right)
\]
at \( r = \infty \) for some \( \varepsilon > 0 \). Moreover, we know that \( \Delta (u^-_{\nu+1} - u^-_\nu) < 0 \), by the argument in [[31], Theorem 3.8], there is a constant \( C > 0 \) such that
\[
u_{\nu+1} - u^-_\nu \geq C\varepsilon
\]
at \( r = \infty \). These imply that
\[
0 = \lim_{r \to \infty} \frac{u^-_{\nu+1} - u^-_\nu}{r^{m+\lambda_2}} = \lim_{r \to \infty} (u^-_{\nu+1} - u^-_\nu)r^{n-2+\varepsilon} \geq C \lim_{r \to \infty} r^\varepsilon \geq C > 0.
\]
This is impossible. So \( b^-_{1(\nu+1)} > b^-_{1\nu} \). By a similar way, we can prove \( b^+_{1(\nu+1)} < b^+_{1\nu} \).
Now, we consider (26). Let \( w_\nu^+ = r^{m+\lambda_2} (u_\nu^+ - u_\alpha) \). Then \( w_\nu^+ \) satisfies
\[
(w_\nu^+ '') - (n - 1 - 2(m + \lambda_2)) r^{-1} (w_\nu^+ ')
\]
\[
= r^{m+\lambda_2} ((u_\nu^+ '') - (n - 2 - m - \lambda_2) r^{m+\lambda_2-2}(u_\nu^+ - u_\alpha)
\]
\[
= r^{m+\lambda_2} (-\sum_{i\in I} K_i(r) u_{\nu_i}^p + \mu f(r)) - (-\sum_{i\in I} K_i(r) u_{\alpha_i}^p + \mu f(r)))
\]
\[
+(m + \lambda_2)(n - 2 - m - \lambda_2) r^{m+\lambda_2-2}(u_\nu^+ - u_\alpha) - \sum_{i\in I} \nu^{-1} r^{m+\lambda_2} H_i^+(u_\nu^+)^{p_i}
\]
\[
= (-\sum_{i\in I} r^2 K_i(r) \frac{(u_\nu^+)^{p_i} - u_{\alpha_i}^{p_i}}{u_\nu^+ - u_\alpha} + (m + \lambda_2)(n - 2 - m - \lambda_2) r^{m+\lambda_1-2}(u_\nu^+ - u_\alpha)
\]
\[
- \sum_{i\in I} \frac{k^{-1} r^{m+\lambda_2} H_i^+(u_\nu^+)^{p_i}}{u_\nu^+ - u_\alpha}
\]
\[
< (-\sum_{i\in I} r^2 K_i(r) \frac{(u_\nu^+)^{p_i} - u_{\alpha_i}^{p_i}}{u_\nu^+ - u_\alpha} + (m + \lambda_2)(n - 2 - m - \lambda_2) r^{m+\lambda_1-2}(u_\nu^+ - u_\alpha)
\]
\[
= (-\sum_{i\in I} r^2 K_i(r) \frac{(u_\nu^+)^{p_i} - u_{\alpha_i}^{p_i}}{u_\nu^+ - u_\alpha} + N^{p_i-1} + M \lambda_2 - \lambda_2^2) r^{m+\lambda_1-2}(u_\nu^+ - u_\alpha)
\]
\[
= (-\sum_{i\in I} r^2 K_i(r) \frac{(u_\nu^+)^{p_i} - u_{\alpha_i}^{p_i}}{u_\nu^+ - u_\alpha} + 2M \lambda_2 + Q + N^{p_i-1}) r^{m+\lambda_1-2}(u_\nu^+ - u_\alpha)
\]
\[
\leq (-\sum_{i\in I} p_i r^{2 K_i(r)} u_{\nu_i}^{p_i-1} + 2M \lambda_2 + Q + N^{p_i-1}) r^{m+\lambda_1-2}(u_\nu^+ - u_\alpha)
\]
\[
\leq (-\sum_{i\in I_m} p_i r^{2 K_i(r)} u_{\nu_i}^{p_i-1} + 2M \lambda_2 + Q + N^{p_i-1}) r^{m+\lambda_1-2}(u_\nu^+ - u_\alpha)
\]
\[
= (-\sum_{i\in I_m} p_i r^{2 K_i(r)} u_{\nu_i}^{p_i-1} + 2M \lambda_2 + Q + N^{p_i-1}) r^{m+\lambda_1-2}(u_\nu^+ - u_\alpha)
\]
\[
= \alpha(r^{-2-\min\{\lambda_2, \gamma\}})
\]
at \( r = \infty \), where \( \gamma \) is defined by (12). Denote
\[
(w_\nu^+ '') - (n - 1 - 2(m + \lambda_2)) r^{-1} (w_\nu^+ ') = \tilde{w}_\nu^+.
\]
Then we have
\[
|\tilde{w}_\nu^+| = o(r^{m+\lambda_2-2} + o(r^{m+\lambda_2-2}))
\]
at \( r = 0 \). It then follows that \( s \tilde{w}_\nu^+(s) \) is integrable for \( s > 0 \) since \( l_i > -2 \). Recall that \( n - 1 - 2(m + \lambda_2) = M + 1 - 2\lambda_2 > 0 \), using the similar argument as in the proof of Proposition 1, we know that \( \lim_{\nu \to \infty} u_\nu^+(r) = 0 \) holds uniformly for \( r \geq 0 \).

Therefore, \( \lim_{\nu \to \infty} b_{1\nu}^1 = b_1 \alpha \). Similarly, we can prove that \( \lim_{\nu \to \infty} b_{1\nu}^\nu = b_1 \alpha \). This finishes the proof. \( \square \)

Now, we prove Theorem 2.1 (ii).
Proof. According to Definition 2.1, to show that $u_\alpha$ is weakly asymptotically stable with respect to the norm $\| \cdot \|_{m + \lambda_2}$, we need to show that, for each $\lambda' < \lambda_2$, there exists $\delta > 0$ such that $\lim_{t \to \infty} \| u(\cdot, t; \varphi) - u_\alpha \|_{m + \lambda'} = 0$ if $\| \varphi - u_\alpha \|_{m + \lambda_2} < \delta$.

Take $\varepsilon > 0$, small. Then Theorem 5.3 and Proposition 8 induce that $\| u_{00}^+ - u_\alpha \|_{m + \lambda_2} < \varepsilon$ for some $u_{00}$ and $u_{00}^+ < \varphi < u_{00}^-$ if $\| \varphi - u_\alpha \|_{m + \lambda_2} < \delta$ for some $\delta > 0$. Using Theorem 5.1, we have $\| u(\cdot, t; \varphi) - u_\alpha \|_{m + \lambda_2} < \varepsilon$. Choose $\delta$ small enough, such that $0 < u_0^{-} < u_0^{+}$. Then it readily follows that $\lim_{t \to \infty} \| \varphi - u_\alpha \|_{m + \lambda_2} < \delta$. And then $u_0^{-} < u_\alpha < u_0^{+}$. The monotonicity of $u(\cdot, t; \varphi)$ in initial value implies that

$$u_0^{-} < u(\cdot, t; u_0^{-}) < u(\cdot, t; \varphi) < u(\cdot, t; u_0^{+}) < u_0^{+}.$$

Moreover, $u(\cdot, t; u_0^{-})$ and $u(\cdot, t; u_0^{+})$ are monotonic in $t$ by Theorem 5.1. Using the uniqueness of $u_\alpha$, $u_0^{-} < u_\alpha < u_0^{+}$, we get

$$\lim_{t \to \infty} u(\cdot, t; u_0^{-}) = u_\alpha = \lim_{t \to \infty} u(\cdot, t; u_0^{+}).$$

Now, for $\lambda' < \lambda_2$ and any $R > 0$, it follows from Proposition 7 and Proposition 8 that

$$\|(1 + r)^{m + \lambda'}(u(\cdot, t; \varphi) - u_\alpha)\|_{m + \lambda'} \leq \frac{C(1 + r)^{m + \lambda'}}{r^{m + \lambda_2}} \leq CR^{\lambda'-\lambda_2}$$

for $r \geq R$ and

$$\|(1 + r)^{m + \lambda'}(u(\cdot, t; \varphi) - u_\alpha)\|_{m + \lambda'} \leq \|(1 + R)^{m + \lambda'}(u(\cdot, t; \varphi) - u_\alpha)\|_{L^\infty(B_R)} \leq C\left(\frac{1}{R}\right)^{\lambda'-\lambda_2},$$

for $r < R$, where $C$ is a positive constant and independent of $R$, $B_R$ is a ball with radius $R$. As $t \to \infty$, we easily get

$$\lim_{t \to \infty} \| u(\cdot, t; \varphi) - u_\alpha \|_{m + \lambda'} \leq CR^{\lambda'-\lambda_2}.$$

Then it readily follows that $\lim_{t \to \infty} \| u(\cdot, t; \varphi) - u_\alpha \|_{m + \lambda'} = 0$ by using the arbitrariness of $R$. This completes the proof. \qed

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