Entropy of conditional tomographic probability distributions for classical and quantum systems

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Abstract. The possibility to describe hybrid systems containing classical and quantum subsystems by means of conditional tomographic probability distributions (tomograms) is discussed. Tomographic Shannon and Rényi entropies associated with the tomograms are studied, and new tomographic uncertainty relations are obtained.

1. Introduction
The possibility to describe the hybrid systems containing both classical and quantum subsystems was considered, for example, in [1–5]. The problems presented in the process of considering combined systems are related to a difference in the description of classical and quantum states. The classical states are associated with the probability density on the phase space. The quantum states are associated either with the wave function or with the density matrix [6,7].

Recently the probability representation of quantum states was introduced using the tomographic approach [8] (see also the review [9]). Within the framework of this approach, the states in quantum mechanics are identified with the probability density of the particle position (called the tomogram) measured in a rotated and scaled reference frame in the phase space. It is important that the states in classical mechanics can also be identified with tomograms [10,11].

In view of the mentioned above, the difference in the concept of classical and quantum states in the tomographic approach is reduced to the difference of the sets where the classical and quantum tomograms are located. This circumstance was used to make an attempt to describe a hybrid system consisted of the classical and quantum subsystems by a joint tomographic probability distribution depending on the positions of the classical and quantum particles. Also it was introduced that the tomograms can be interpreted as conditional probability distributions [12,13]. For example, the optical tomogram $w(X, \theta)$, being dependent on a random homodyne quadrature $X$ and the local oscillator phase $\theta$ [14,15], which is a measurable parameter (see, e.g., [16,17]), can be interpreted as the conditional probability density $w(X, \theta) = w(X \mid \theta)$.

In view of Bayesian formula, the conditional probability distribution is connected with a joint probability distribution $P(X, \theta)$ of the both random variables.
The aim of our work is to study the possibility to associate the states of hybrid systems consisted of classical and quantum subsystems with the conditional probability densities and joint probability distributions related to the densities by the Bias formula. Using the tomographic approach and present optical and symplectic tomograms of classical and quantum systems. We give our conclusions and prospectives in section 6.

This paper is organized as follows.

In section 2, we review the tomographic approach and present optical and symplectic tomograms of classical and quantum systems. In section 3, we consider the tomograms of hybrid systems. In section 4, we study the tomographic entropies. In section 5, we discuss the properties of possible kinetic equations for the tomograms of the hybrid systems. We give our conclusions and prospectives in section 6.

2. Tomograms of classical and quantum systems

Following [13] we review in this section the properties of tomograms.

By definition, the optical tomogram $w(\vec{X} \mid \vec{\theta})$ of a classical system with $N$ degrees of freedom, i.e., $\vec{X} = (X_1, X_2, \ldots, X_N)$ and $\vec{\theta} = (\theta_1, \theta_2, \ldots, \theta_N)$, where $-\infty < X_j \leq \infty$ and $0 \leq \theta_j \leq 2\pi$, is the joint conditional probability density of random positions $X_j$, having in mind that the measurements of the positions are made in the set of reference frames in the system phase space with rotated initial axes $q_j \rightarrow X_j = q_j \cos \theta_j + p_j \sin \theta_j$ and $p_j \rightarrow P_j = -q_j \sin \theta_j + p_j \cos \theta_j$.

The set of $N$ angles $\theta_j$ determines a new set of reference frames rotated in the phase space.

The nonnegative symplectic tomogram $M(\vec{X} \mid \vec{\mu}, \vec{\nu})$ of a classical system is the joint conditional probability density of random positions $X_j$ measured in the canonically transformed reference frame in the system phase space, having in mind that new scales in the space and time for each degree of freedom are introduced, i.e., $q_j \rightarrow \mu_j q_j$, $t \rightarrow \nu_j^{-1} t$, and the measured positions are $X_j = \mu_j q_j + \nu_j p_j$. The scaling parameters satisfy the condition $-\infty < \mu_j, \nu_j \leq \infty$.

For the case $\mu_j = \cos \theta_j$ and $\nu_j = \sin \theta_j$, the symplectic tomogram provides the optical tomogram, i.e., $M(\vec{X} \mid \cos \theta, \sin \theta) = w(\vec{X} \mid \vec{\theta})$, where $\sin \theta = (\sin \theta_1, \sin \theta_2, \ldots, \sin \theta_N)$ and $\cos \theta = (\cos \theta_1, \cos \theta_2, \ldots, \cos \theta_N)$.

The joint probability density $f(q, p) \geq 0$ of $N$ positions and $N$ momenta is determined by the symplectic tomogram due to the Fourier transform

$$f(q, p) = \int M(\vec{X} \mid \vec{\mu}, \vec{\nu}) \left\{ \prod_{k=1}^{N} \exp[i(X_k - \mu_k q_k - \nu_k p_k)] \frac{dq_k dp_k}{4\pi^2} \right\}.$$

Since $f(q, p) \geq 0$, only such symplectic tomograms $M(\vec{X} \mid \vec{\mu}, \vec{\nu})$ are admissible for describing the classical system states for which Fourier integral (1) yields a nonnegative function.

The symplectic tomogram can be found in view of the inverse Radon transform

$$M(\vec{X} \mid \vec{\mu}, \vec{\nu}) = \int f(q, p) \left\{ \prod_{k=1}^{N} \delta(X_k - \mu_k q_k - \nu_k p_k) dq_k dp_k \right\}.$$

Both optical and symplectic tomograms are normalized functions

$$\int M(\vec{X} \mid \vec{\mu}, \vec{\nu}) d\vec{X} = \int w(\vec{X} \mid \vec{\theta}) d\vec{X} = 1.$$
The conditional probability densities can be associated with joint probability densities via Bayesian formula. For example, one has the optical tomogram

\[ w(\bar{X} | \bar{\theta}) = \tilde{w}(\bar{X}, \bar{\theta}) \left\{ \int \tilde{w}(\bar{X}, \bar{\theta}) \, d\bar{X} \right\}^{-1} \]

(4)
in terms of a joint probability density \( \tilde{w}(\bar{X}, \bar{\theta}) \) of random positions \( X_j \) and rotations \( \theta_j \).

An analogous relation takes place for symplectic tomograms

\[ M(\bar{X} | \bar{\mu}, \bar{\nu}) = \tilde{M}(\bar{X}, \bar{\mu}, \bar{\nu}) \left\{ \int \tilde{M}(\bar{X}, \bar{\mu}, \bar{\nu}) \, d\bar{X} \right\}^{-1}, \]

(5)
where \( \tilde{M}(\bar{X}, \bar{\mu}, \bar{\nu}) \) is a joint probability density of random positions \( X_j \) and scaling parameters \( \mu_j \) and \( \nu_j \).

For a given joint probability density, there exists the conditional probability density determined without an ambiguity, but for a given conditional probability density there exist many joint probability densities. For example, by choosing an arbitrary marginal probability distribution in the denominator of (4) as

\[ P(\bar{\theta}) = \int \tilde{w}(\bar{X}, \bar{\theta}) \, d\bar{X}, \]

(6)
we have

\[ \tilde{w}(\bar{X}, \bar{\theta}) = w(\bar{X} | \bar{\theta}) P(\bar{\theta}). \]

(7)
Analogously, for symplectic tomogram we have

\[ \tilde{w}(\bar{X}, \bar{\mu}, \bar{\nu}) = M(\bar{X} | \bar{\mu}, \bar{\nu}) \Pi(\bar{\mu}, \bar{\nu}), \]

(8)
where \( \Pi(\bar{\mu}, \bar{\nu}) \) is an arbitrary joint probability density of random variables \( \mu_j \) and \( \nu_j \).

There exists the other tomographic construction called the center-of-mass tomography (see, e.g., [9]). The center-of-mass tomogram \( C(X | \bar{\mu}, \bar{\nu}) \) depends only on one random position \( X \) interpreted as an analog of the system center-of-mass coordinate. Thus, this tomogram is the probability density of the coordinate \( X \) which is considered in a reference frame in the phase space with rescaled time and positions for all degrees of freedom. Then the conventional probability density \( f(\bar{q}, \bar{p}) \) is given by the Fourier integral

\[ f(\bar{q}, \bar{p}) = (2\pi)^{-2N} \int C(X | \bar{\mu}, \bar{\nu}) \exp[i(X - \bar{\mu}\bar{q} - \bar{\nu}\bar{p})] \, dX \, d\bar{\mu} \, d\bar{\nu}. \]

(9)
The inverse Radon transform determines the center-of-mass tomogram

\[ C(X | \bar{\mu}, \bar{\nu}) = \int f(\bar{q}, \bar{p}) \delta(X - \bar{\mu}\bar{q} - \bar{\nu}\bar{p}) \, dq \, dp. \]

(10)
The tomogram is nonnegative and normalized conditional probability density

\[ \int C(X | \bar{\mu}, \bar{\nu}) \, dX = 1. \]
The joint probability density $\tilde{C}(X, \vec{\mu}, \vec{\nu})$ can be reconstructed

$$\tilde{C}(X, \vec{\mu}, \vec{\nu}) = C(X \mid \vec{\mu}, \vec{\nu})\Pi(\vec{\mu}, \vec{\nu}),$$

(12)

where $\Pi(\vec{\mu}, \vec{\nu})$ is an arbitrary joint probability density of random variables $\mu_j$ and $\nu_j$.

For classical systems, the probability densities $C(X \mid \vec{\mu}, \vec{\nu})$ are admissible as the state tomograms only if the Fourier integral given by (9) provides a nonnegative function.

For quantum states, one has the same tomograms given in terms of the density operators $\hat{\rho}$. For example, the optical tomogram reads

$$w(\vec{X} \mid \vec{\theta}) = \text{Tr} \hat{\rho} \prod_{j=1}^{N} \delta(X_j\hat{1} - \hat{q}_j \cos \theta_j - \hat{p}_j \sin \theta_j).$$

(13)

The symplectic tomogram is given by an analogous formula

$$M(\vec{X} \mid \vec{\mu}, \vec{\nu}) = \text{Tr} \hat{\rho} \prod_{j=1}^{N} \delta(X_j\hat{1} - \mu_j \hat{q}_j - \nu_j \hat{p}_j).$$

(14)

The center-of-mass tomogram is

$$C(X \mid \vec{\mu}, \vec{\nu}) = \text{Tr} \hat{\rho} \delta(X\hat{1} - \vec{\mu}\hat{q} - \vec{\nu}\hat{p}).$$

(15)

The density operator $\hat{\rho}$ can be found by means of quantum Radon transform. For example, the symplectic tomogram provides the reconstruction of the density operator as follows:

$$\hat{\rho} = (2\pi)^{-2N} \int M(\vec{X} \mid \vec{\mu}, \vec{\nu}) \left\{ \prod_{k=1}^{N} \exp[i(X_k\hat{1} - \mu_k \hat{q}_k - \nu_k \hat{p}_k)] \right\} dX d\mu d\nu.$$  

(16)

For the center-of-mass tomography, one has the density operator in the form

$$\hat{\rho} = (2\pi)^{-2N} \int C(X \mid \vec{\mu}, \vec{\nu}) \exp[i(X\hat{1} - \vec{\mu}\hat{q} - \vec{\nu}\hat{p})] dX d\mu d\nu.$$  

(17)

Quantum states are associated with nonnegative density operators. Thus, the probability densities $M(\vec{X} \mid \vec{\mu}, \vec{\nu})$ and $C(X \mid \vec{\mu}, \vec{\nu})$ are admissible as tomograms of quantum states only if integrals (16) and (17) yield the Hermitian operators with nonnegative eigenvalues.

In the quantum case, the joint tomographic probability distributions can be constructed in view of the same relations (8) and (12) used for classical systems. It is worth pointing out that the nonnegativity conditions for the Fourier integrals of the tomograms of classical and quantum systems are different. There can exist the probability distributions satisfying the both conditions, either only classical or only quantum condition, as well as neither classical nor quantum one.

In spite of the fact that, in the tomographic picture of classical and quantum mechanics, the states (both classical and quantum) are associated with the same tomographic probability densities like optical, symplectic, and center-of-mass tomograms, sets of the tomograms are different. The sets are determined by the nonnegativity conditions of the corresponding integrals reconstructing the probability density in the classical domain and the density operator in the quantum domain. This is just the essence of the difference in the classical and quantum cases in the context of tomographic description of the states.
3. Hybrid systems

In [19, 20], it was suggested to describe hybrid systems consisted of classical and quantum subsystems by tomographic probability distributions. Below we develop this proposal on an example of the center-of-mass tomogram.

We consider a classical subsystem with $N_1$ degrees of freedom and a quantum subsystem with $N_2$ degrees of freedom. With the hybrid-system state, we associate the cluster center-of-mass tomogram $C(X, Y \mid \vec{\mu}, \vec{\nu}, \vec{m}, \vec{n})$, where $\vec{\mu} = (\mu_1, \mu_2, \ldots, \mu_{N_1})$, $\vec{\nu} = (\nu_1, \nu_2, \ldots, \nu_{N_1})$, $\vec{m} = (m_1, m_2, \ldots, m_{N_2})$, and $\vec{n} = (n_1, n_2, \ldots, n_{N_2})$. The cluster center-of-mass tomogram $C(X, Y \mid \vec{\mu}, \vec{\nu}, \vec{m}, \vec{n})$ is a nonnegative normalized distribution function

$$\int C(X, Y \mid \vec{\mu}, \vec{\nu}, \vec{m}, \vec{n}) \, dX \, dY = 1.$$  

(18)

The state of the classical subsystem is associated with the marginal center-of-mass tomogram

$$C_{cl}(X \mid \vec{\mu}, \vec{\nu}) = \int C(X, Y \mid \vec{\mu}, \vec{\nu}, \vec{m}, \vec{n}) \, dY,$$  

(19)

and the state of the quantum subsystem is associated with the following marginal center-of-mass tomogram

$$C_q(Y \mid \vec{m}, \vec{n}) = \int C(X, Y \mid \vec{\mu}, \vec{\nu}, \vec{m}, \vec{n}) \, dX.$$  

(20)

The tomogram of the hybrid-system state must satisfy the conditions

$$f(\vec{q}, \vec{p}) = (2\pi)^{-2N_1} \int C(X, Y \mid \vec{\mu}, \vec{\nu}, \vec{m}, \vec{n}) \exp[i(X - \vec{\mu} \vec{q} - \vec{\nu} \vec{p})] \, dX \, dY \, d\vec{\mu} \, d\vec{\nu} \geq 0,$$  

(21)

$$\hat{\rho} = (2\pi)^{-2N_2} \int C(X, Y \mid \vec{\mu}, \vec{\nu}, \vec{m}, \vec{n}) \exp[i(Y - \vec{m} \vec{q} - \vec{n} \vec{p})] \, dX \, dY \, d\vec{m} \, d\vec{n} \geq 0,$$  

(22)

where $\vec{\mu} \vec{q} = \sum_{k=1}^{N_1} \mu_k q_k$, $\vec{\nu} \vec{p} = \sum_{k=1}^{N_1} \nu_k p_k$, $\vec{m} \vec{q} = \sum_{j=1}^{N_2} m_j \hat{q}_j$, and $\vec{n} \vec{p} = \sum_{j=1}^{N_2} n_j \hat{p}_j$, with $\hat{q}_j$ and $\hat{p}_j$ being the position and momenta operators of quantum degrees of freedom. Analogous conditions can be written for optical and symplectic tomograms [20]. Conditions (21) and (22) correspond to the nonnegativity of the probability density $f(\vec{q}, \vec{p})$ on the phase space of the classical subsystem state and the nonnegativity of the density operator $\hat{\rho}$ of the quantum subsystem state.

The simplest case of the hybrid-system-state tomogram is the tomographic probability density without correlations, i.e.,

$$C(X, Y \mid \vec{\mu}, \vec{\nu}, \vec{m}, \vec{n}) = C_{cl}(X \mid \vec{\mu}, \vec{\nu}) C_q(Y \mid \vec{m}, \vec{n}).$$  

(23)

The case of correlated classical quantum states can be described by tomograms which are convex sums of uncorrelated tomograms, e.g.,

$$C(X, Y \mid \vec{\mu}, \vec{\nu}, \vec{m}, \vec{n}) = \sum_k P(k) C_{cl}^{(k)}(X \mid \vec{\mu}, \vec{\nu}) C_q^{(k)}(Y \mid \vec{m}, \vec{n}),$$  

(24)

where $P(k) \geq 0$ is the probability distribution satisfying the normalization condition $\sum_k P(k) = 1$, and the index $k$ can be either discrete or continuous one. The form of tomogram (24) corresponds to separable probability distributions.
Also other kinds of the probability distributions can exist, which cannot be presented in the form (24), but can be presented in the form of entangled probability distributions like

\[ C(X, Y \mid \vec{\mu}, \vec{\nu}, \vec{m}, \vec{n}) = (1 + \delta) \sum_k P(k)C^{(k)}_c(X \mid \vec{\mu}, \vec{\nu})C^{(k)}_q(Y \mid \vec{m}, \vec{n}) \]
\[-\delta \sum_j P(j)C^{(j)}_c(X \mid \vec{\mu}, \vec{\nu})C^{(j)}_q(Y \mid \vec{m}, \vec{n}), \tag{25}\]

where \( \delta > 0 \), and \( P(k) \) and \( P(j) \) are normalized probability distributions of the corresponding indices \( k \) and \( j \).

Analogous arguments provide the possibility to write a structure of the optical tomogram \( w(\vec{X}, \vec{Y}, \vec{\theta}, \vec{\varphi}) \) of a hybrid-system state with \( N_1 \) classical degrees of freedom \( \vec{X} \) and \( N_2 \) quantum degrees of freedom \( \vec{Y} \), i.e.,

\[ w(\vec{X}, \vec{Y} \mid \vec{\theta}, \vec{\varphi}) = (1 + \delta) \sum_k P(k)w^{(k)}_c(\vec{X} \mid \vec{\theta})w^{(k)}_q(\vec{Y} \mid \vec{\varphi}) \]
\[-\delta \sum_j P(j)w^{(j)}_c(\vec{X} \mid \vec{\theta})w^{(j)}_q(\vec{Y} \mid \vec{\varphi}). \tag{26}\]

In (25) and (26), at \( \delta = 0 \), one has a separable form of the tomographic conditional probability distributions of the hybrid-system states.

For the symplectic tomogram, the corresponding formula reads

\[ M(\vec{X}, \vec{Y} \mid \vec{\mu}, \vec{\nu}, \vec{m}, \vec{n}) = (1 + \delta) \sum_k P(k)M^{(k)}_c(\vec{X} \mid \vec{\mu}, \vec{\nu})M^{(k)}_q(\vec{Y} \mid \vec{m}, \vec{n}) \]
\[-\delta \sum_j P(j)M^{(j)}_c(\vec{X} \mid \vec{\mu}, \vec{\nu})M^{(j)}_q(\vec{Y} \mid \vec{m}, \vec{n}). \tag{27}\]

4. Admissible dynamics of the hybrid-system tomograms

Now we discuss a possible dynamics of tomograms of the hybrid-system states starting from expression (24) for the center-of-mass tomogram. Since the tomogram is the probability distribution, its dynamics can be described as a map \( \Phi(t) \) of the probability distribution onto a probability distribution of the same kind. Thus, we determine some admissible maps given by the formula

\[ \Phi(t)C(X, Y \mid \vec{\mu}, \vec{\nu}, \vec{m}, \vec{n}) = C(X, Y, t \mid \vec{\mu}, \vec{\nu}, \vec{m}, \vec{n}) \]
\[= \sum_k P(k)C^{(k)}_c(X, t \mid \vec{\mu}, \vec{\nu})C^{(k)}_q(Y, t \mid \vec{m}, \vec{n}). \tag{28}\]

The above dynamics is coded by the dynamics of the probability distribution \( P(k) \) as well as by the time dependence of the classical subsystem-state tomograms \( C^{(k)}_c(X, t \mid \vec{\mu}, \vec{\nu}) \) and quantum subsystem-state tomograms \( C^{(k)}_q(Y, t \mid \vec{m}, \vec{n}) \). If the map \( \Phi(t) \) is factorized in the form of direct product of the classical and quantum maps of the tomographic probability distributions \( \Phi(t) = \Phi_c(t) \otimes \Phi_q(t) \) and the probability distributions \( P(k) \) and \( P(j) \) do not vary with time, the kinetic equation for the tomogram can be considered as a combination of independent kinetic equations for classical and quantum subsystems. In this case, for the
factorized initial states the classical and quantum subsystems evolve according to the classical and quantum evolution equations. Thus, the zero initial correlations of classical and quantum degrees of freedom remain to be zero at any time moment. The map \( \Phi_{cl}(t) \) acts only on the classical degrees of freedom of the hybrid system while the map \( \Phi_{q}(t) \) acts only on the quantum degrees of freedom of the hybrid system. This means that the maps preserve the conditions of tomogram admissibility for any time moment.

The map \( \Phi(t) \) can be constructed as a convex sum of the direct product maps

\[
\Phi(t) = \sum_k \Pi(k,t) \Phi_{cl}^{(k)}(t) \otimes \Phi_{q}^{(k)}(t),
\]

where \( \Pi(k,t) \geq 0 \) and \( \sum_k \Pi(k,t) = 1 \). In this case, due to the dynamics, the initial uncorrelated state of the hybrid system starts to have nonzero correlations of classical and quantum observables.

The set of all admissible maps of the hybrid-system states has a structure of the semigroup. The tomograms of the hybrid-system states belong to the domain in the simplex of all probability distributions. This domain is determined for the center-of-mass tomograms by conditions (21) and (22). The admissible dynamics of the hybrid-system states is determined by the invariance semigroup of the domain in the simplex.

We consider a simple example of the dynamics of a system with one classical \( X \) and one quantum \( Y \) degrees of freedom. Let the initial state to be described by a factorized symplectic tomogram

\[
M(X,Y | \mu, \nu, m, n) = [\pi(m^2 + n^2)]^{-1/2} \delta(X - \mu q_0 - \nu p_0) \exp[-Y^2/(m^2 + n^2)].
\]

(30)

The initial state tomogram (30) corresponds to the Gaussian state of the quantum degree of freedom with equal dispersions of position and momentum \( \sigma_{qq} = \sigma_{pp} = 1/2 \) and the classical system with position \( q_0 \) and momentum \( p_0 \).

Let the dynamics to be described by the map

\[
\Phi(t) = \alpha 1 \otimes \Phi_{q}(t) + (1 - \alpha) \Phi_{cl}(t) \times 1, \quad 0 \leq \alpha \leq 1,
\]

(31)

where \( \Phi_{q}(t) \) provides the free-motion map of the quantum degree of freedom and \( \Phi_{cl}(t) \) provides the free-motion map of the classical system. After the action of dynamical map (31), the tomogram of the hybrid-system state (30) becomes

\[
M(X,Y,t | \mu, \nu, m, n) = \Phi(t)M(X,Y | \mu, \nu, m, n)
\]

\[
= \alpha \delta(X - \mu q_0 - \nu p_0)[2\pi \sigma_{YY}(t)]^{-1/2} \exp[-Y^2/2\sigma_{YY}(t)]
\]

\[
+ (1 - \alpha) \delta(X - \mu(q_0 + pt) - \nu p_0)[\pi(m^2 + n^2)]^{-1/2} \exp[-Y^2/(m^2 + n^2)].
\]

(32)

The dispersion parameter \( \sigma_{YY}(t) \) reads

\[
\sigma_{YY}(t) = [(m^2 + n^2)/2] + mnt + m^2 t^2 / 2.
\]

(33)

The constant parameter \( \alpha \) can be replaced by the time-dependent probability \( \alpha \to \cos^2 \Omega t \). In this case, the classical–quantum correlations are modeled by the map \( \Phi(t) \), which provides the variation in time of the probability distribution of uncorrelated products in the convex sum of such products.

Other examples of admissible dynamics of hybrid-system tomograms can be constructed. In this way, the classical subsystem can follow the dynamics given by kinetic equations.
with dissipation, and the quantum subsystem can follow the dynamics provided by Gorini–Kossakowski–Sudarshan–Lindblad equations [21, 22]. The time dependence of the probability distribution \( P_k(t) \) can be modeled by an arbitrary stochastic map (both Markovian or non-Markovian). The physical meaning and possible applications of the discussed dynamical models of the hybrid-system states need extra study.

5. Tomographic entropy of hybrid systems

The optical, symplectic, and center-of-mass tomograms under discussion are standard probability distributions. For classical and quantum systems, the Shannon and Rényi tomographic entropies and their specific properties were reviewed, e.g., in [18]. Some new inequalities for the tomographic entropies were checked in experiments [17] with homodyne detection of photon states where the optical tomograms are measured.

The tomographic entropy can also be introduced for hybrid systems. For example, the Rényi entropy associated with the center-of-mass tomogram of the hybrid-system state is defined as follows:

\[
R_C(q) = \frac{1}{1 - q} \ln \left\{ \int \left[ C(X, Y \mid \bar{\mu}, \bar{\nu}, \bar{m}, \bar{n}) \right]^q dX dY \right\}.
\] (34)

The limit \( q \to 1 \) provides the Shannon tomographic entropy for the hybrid-system state \( H_C = R_C(1) \).

An analogous tomographic Rényi entropy can be associated with the optical tomogram of the hybrid-system state

\[
R_w(q) = \frac{1}{1 - q} \ln \left\{ \int \left[ w(X, Y \mid \bar{\theta}, \bar{\varphi}) \right]^q d\vec{X} d\vec{Y} \right\}.
\] (35)

The Shannon tomographic entropy associated with the optical tomogram of the hybrid-system state \( H_w = R_w(1) \).

The symplectic tomographic Rényi entropy is associated with the symplectic tomogram of the hybrid-system state \( R_M(q) \)

\[
R_M(q) = \frac{1}{1 - q} \ln \left\{ \int \left[ M(X, Y \mid \bar{\mu}, \bar{\nu}, \bar{m}, \bar{n}) \right]^q d\vec{X} d\vec{Y} \right\}.
\] (36)

The corresponding Shannon entropy reads \( H_M = R_M(1) \).

All three entropies depend on the parameters \( \bar{\mu}, \bar{\nu}, \bar{m}, \bar{n} \) and \( \bar{\theta}, \bar{\varphi} \), respectively.

It is obvious that the introduced tomographic entropies for the hybrid-system states must satisfy some inequalities. We present here a new inequality for the optical tomographic entropy of the hybrid-system state; it reads

\[
- \int \left[ \int w(\vec{X}, \vec{Y} \mid \bar{\theta}, \bar{\varphi}) d\vec{X} \right] \ln \left( \int w(\vec{X}, \vec{Y} \mid \bar{\theta}, \bar{\varphi}) d\vec{X} \right) d\vec{Y} \\
- \int \left[ \int w(\vec{X}, \vec{Y} \mid \bar{\theta}, \bar{\varphi}') d\vec{X} \right] \ln \left( \int w(\vec{X}, \vec{Y} \mid \bar{\theta}, \bar{\varphi}') d\vec{X} \right) d\vec{Y} \geq N_2 \ln \pi e,
\]

\( \bar{\varphi}' = (\varphi_1 + \pi/2, \varphi_2 + \pi/2, \ldots, \varphi_{N_2} + \pi/2) \).

(37)

The bound \( N_2 \ln \pi e \) in this inequality corresponds to quantum fluctuations of the quantum degrees of freedom. It is worth pointing out that there is no such a bound for fluctuations of the classical subsystem.
6. Conclusions
To conclude, we point out the main results of our work.

We showed the possibility to describe the states of hybrid systems consisted of classical and quantum subsystems by tomographic probability distributions. We discussed three equivalent variants of the tomographic probability distributions: optical tomograms, symplectic tomograms, and center-of-mass tomograms. We demonstrated that these tomograms are admissible for describing the states of hybrid systems if they satisfy specific constrains. These constrains have the form of inequalities for the marginal tomographic probability densities. These inequalities determine a domain in the simplex where the hybrid-system-state tomograms are located. We modeled the dynamics of the hybrid-system states by stochastic maps which connect the initial tomogram with the tomogram at any future time moment. We considered the example of such dynamics related to the free motion of classical and quantum subsystems in an explicit form. We introduced the tomographic entropies associated with the hybrid-system-state tomograms. For the optical tomogram of the hybrid-system state, we obtained the new inequality extending the entropic uncertainty relations known for quantum systems. We will study the other examples of dynamical maps of the tomographic probability distributions and their connections with other models of the hybrid systems and their evolution suggested by Elze et al. [2, 3, 23–25] in the future publication.

Acknowledgments
This study was supported by the Russian Foundation for Basic Research under Projects Nos. 11-02-00456 and 10-02-00312. The authors are grateful to the Organizers of the Sixth International Workshop DICE2012 Spacetime–Matter–Quantum Mechanics from the Planck Scale to Emergent Phenomena (Castello Pasquini/Castiglioncello, Tuscany, Italy, September 17–21, 2012) and especially Prof. Thomas Elze for invitation and kind hospitality.

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