On an Eigenvalue Property of Summation-By-Parts Operators

Viktor Linders

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Abstract
Summation-By-Parts (SBP) methods provide a systematic way of constructing provably stable numerical schemes. However, many proofs of convergence and accuracy rely on the assumption that the SBP operator possesses a particular eigenvalue property. In this note, three results pertaining to this property are proven. Firstly, the eigenvalue property does not hold for all nullspace consistent SBP operators. Secondly, this issue can be addressed without affecting the accuracy of the method by adding a specially designed, arbitrarily small perturbation term to the SBP operator. Thirdly, all pseudospectral methods satisfy the eigenvalue property.

Keywords Summation-By-Parts · Eigenvalues · Nullspace consistency · Pseudospectral methods

Mathematics Subject Classification 65M12 · 65M70 · 65M99

1 Introduction
The Summation-By-Parts (SBP) methodology [12, 36] and its extensions (generalized SBP [11] and upwind SBP [26]) constitute an algebraic framework for designing provably stable discretizations of partial differential equations. SBP operators can be designed within essentially every family of spatial discretizations, including finite difference methods [19, 34], finite volume methods [28], finite element methods [1, 2], pseudospectral methods [6], as well as more specialized discretizations such as WENO [13, 38] and DRP schemes [21, 22]. Pseudospectral SBP operators are used in discontinuous Galerkin (dG) [14] and Flux Reconstruction methods [32]. With appropriate numerical fluxes, these are combined to form global SBP operators [10]. Modal and hybridized operators have be constructed to incorporate non-collocated dG methods in the SBP framework through particular similarity tranforms [9].

SBP operators can be used as time marching schemes akin to implicit Runge–Kutta methods [4, 25, 30], in which case they satisfy an assortment of desirable stability and convergence...
properties [24, 29]. All such methods share certain algebraic similarities that can be exploited to obtain stable and accurate discretizations.

This paper addresses a difficulty pertaining to an eigenvalue property that frequently arises in SBP theory. Much of what is known about the convergence of SBP methods relies on this property. It is known that there are methods that lack the property, essentially due to the existence of SBP operators with “bad” null-spaces [31, Theorem 2.1]. Here, we consider SBP operators that do not suffer from bad nullspaces; so called nullspace consistent methods.

Three new results will be presented: After introducing nullspace consistent SBP operators in Sect. 2, it is shown that not every such operator satisfies the eigenvalue property. In Sect. 3 it is established that for each SBP operator that lacks the property, another SBP operator of the same order can be found that possesses it, and that differs from the first one by an arbitrarily small perturbation. Section 4 discusses pseudospectral methods, all of which are shown to satisfy the eigenvalue property. This generalizes an earlier result on the topic [33]. A summary is given in Sect. 5.

2 SBP Operators and the Eigenvalue Property

This section introduces the notion of nullspace consistent SBP operators as well as the eigenvalue property. Let $[a, b]$ be an interval with $b > a$.

Definition 1 The matrices $D_+, D_- \in \mathbb{R}^{(n+1) \times (n+1)}$ are said to form a pair of SBP operators of order $q \geq 1$ on the interval $[a, b]$ if there exist matrices $H, S \in \mathbb{R}^{(n+1) \times (n+1)}$ and vectors $p_0, p_n, x \in \mathbb{R}^{n+1}$ such that the following relations hold:

(A) $D_\pm x^j = jx^{j-1}$, \quad $p_0^T x^j = a^j$, \quad $p_n^T x^j = b^j$, \quad $j = 0, \ldots, q$,

(B) $H = H^\top > 0$,

(C) $HD_+ + D_- H = -p_0p_0^\top + p_np_n^\top + S$, \quad $S = S^\top \geq 0$,

(D) $HD_+ + D_- H = -p_0p_0^\top + p_np_n^\top$,

(E) $x = (x_0, \ldots, x_n)^\top$, \quad $x_i \neq x_j \forall i \neq j$.

Here and elsewhere the notation $x^j$ should be understood as the elementwise exponentiation of $x$. The convention $x^0 = 1 := (1, \ldots, 1)^\top$ is used throughout. Definition 1 incorporates classical ($S = 0, p_0 \equiv e_0 := (1, 0, \ldots, 0)^\top, p_n \equiv e_n := (0, \ldots, 0, 1)^\top$), generalized ($S = 0$) and upwind SBP methods ($S \neq 0, p_0 = e_0, p_n = e_n$) as special cases.

From (A) it is seen that $D_+$ and $D_-$ approximate derivative operators and that $p_0$ and $p_n$ interpolate grid functions to the domain boundaries. Further, $H$ defines a quadrature rule [16, 23]. The role of $S$ is somewhat obfuscated in Definition 1, which warrants a comment: Subtracting (D) from (C), transposing and multiplying by $H^{-1}$ shows that $D_- = D_+ - H^{-1} S$.

From (A) and (B) it can then be deduced that

$$Sx^j = 0, \quad j = 0, \ldots, q.$$  \tag{1}

Together with positive semi-definiteness, this reveals that $S$ contributes with artificial dissipation to the operator; see e.g. [26, 27].

SBP operators are frequently used in combination with simultaneous approximation terms (SATs) that weakly impose initial [25, 30], boundary [7] or interface conditions [8]. Consider the inflow-outflow problem on $[a, b]$ described by the scalar linear advection problem

\begin{align*}
    u_t + u_x &= 0, & x \in (a, b), & t > 0, \\
    u &= f(x), & x \in (a, b), & t = 0, \\
    u &= g(t), & x = a, & t > 0.
\end{align*}  \tag{2}
Let $\|u\|^2 = \int_a^b u^2 \, dx$ denote the (squared) $L^2$-norm. Multiplying the first row of (2) by $2u$ and integrating in space leads to the identity
\begin{equation}
\frac{d}{dt} \|u\|^2 + u(b)^2 = g^2.
\end{equation}
Integrating in time reveals that $\|u\|^2$ is bounded by the data $g$ and $f$.

Applying SBP-SA T to the spatial terms in (2) results in the semi-discretization
\begin{equation}
\mathbf{u}_t + D_+ \mathbf{u} = \sigma H^{-1} \mathbf{p}_0 (g - \mathbf{p}_0^\top \mathbf{u}),
\end{equation}
where the the right-hand side term is the SAT. Here, $\sigma$ is a scalar parameter that is chosen to ensure stability. Left-multiplying (4) by $\mathbf{u}^\top H$, then adding the transpose of the result yields
\begin{equation}
\frac{d}{dt} \|u\|^2_H + \mathbf{u}^\top \left( HD_+ + D_+^\top H \right) \mathbf{u} = 2 \sigma \mathbf{u}^\top \mathbf{p}_0 (g - \mathbf{p}_0^\top \mathbf{u}),
\end{equation}
where $\|u\|^2_H = \mathbf{u}^\top H \mathbf{u}$. Using property (C) from Definition 1, then adding and subtracting $g^2$ on the right-hand side, gives after some algebraic manipulations
\begin{equation}
\frac{d}{dt} \|u\|^2_H + (\mathbf{p}_0^\top \mathbf{u})^2 + \mathbf{u}^\top S \mathbf{u} = g^2 - (g - \mathbf{p}_0^\top \mathbf{u})^2 + 2(\sigma - 1) \mathbf{p}_0^\top \mathbf{u} (g - \mathbf{p}_0^\top \mathbf{u}).
\end{equation}
Comparing terms in the semi-discrete identity (5) with the continuous case (3) it is seen how $H$ approximates an integral and how $\mathbf{p}_n$ projects a function to the right boundary. The role of $S$ as an artificial dissipation operator is here highlighted since the term $\mathbf{u}^\top S \mathbf{u} \geq 0$ by positive semi-definiteness, and by (1) vanishes for all polynomials of degree $q$ or less. The second term on the right-hand side of (5) is non-positive and small since $\mathbf{p}_0$ is a projection operator to the inflow boundary. This term, which stems from the SAT, therefore also adds artificial dissipation.

The final term on the right-hand side is in general indefinite and may therefore cause stability issues. However, the choice $\sigma = 1$ nullifies this term and is consequently the typical choice for this problem. Note that $\sigma \geq 1/2$ suffices to guarantee stability in the case of homogeneous data, $g = 0$. Other choices of $\sigma$ are considered e.g. in [3].

Remark 1 If the flow direction in (2) is reversed so that $u_t - u_x = 0$ is solved, then $D_-^\top$ is used in place of $D_+$, the boundary condition is imposed at $b$ using $\mathbf{p}_n^\top$, and a negative sign is introduced before the SAT. The same choices of $\sigma$ leads to a stable scheme.

Remark 2 If time rather than (or in addition to) space is discretized with SBP-SA T, similar conclusions are drawn about the parameter $\sigma$; see [25]. The resulting scheme is implicit. The choice $\sigma = 1$ is usually made, which has the effect of decoupling individual time steps.

Collecting terms that multiply the solution $u$, (4) may be rewritten as
\begin{equation}
\mathbf{u}_t + \bar{D}_+ \mathbf{u} \equiv \mathbf{u}_t + (D_+ + \sigma H^{-1} \mathbf{p}_0^\top) \mathbf{u} = \sigma H^{-1} \mathbf{p}_0^\top g.
\end{equation}
A necessary condition for the existence of a unique solution to (6) is that the matrix $\bar{D}_+$ is invertible. SBP operators that have this property can be characterized in terms of nullspace consistency, a concept introduced in [37]:

Definition 2 An SBP operator is said to be nullspace consistent if
\( \ker(D_+) = \text{span}(\{1\}) \).

The following is a generalization of [24, Lemma 2]:
Lemma 1 For any SBP operator $D_+$ and $\sigma \neq 0$, the matrix $\hat{D}_+ := (D_+ + \sigma H^{-1} p_0 p_0^\top)$ is invertible if and only if $D_+$ is null-space consistent.

Proof In [24, Lemma 2], this result is established in the special case when $\sigma = 1$. Let $\hat{D}_+ = D_+ + H^{-1} p_0 p_0^\top$, so that $\hat{D}_+ = \hat{D}_+$ in this special case. To prove the lemma it suffices to show that $\hat{D}_+$ is invertible if and only if $\hat{D}_+$ is, for any $\sigma \neq 0$.

Suppose first that $\hat{D}_+$ is invertible. Note that $\hat{D}_+ = \hat{D}_+ + (\sigma - 1) H^{-1} p_0 p_0^\top$ for any $\sigma$, hence $\hat{D}_+$ is a rank one perturbation of $\hat{D}_+$. By the matrix determinant lemma (see e.g. [18, Chapter 0]), the determinant of $\hat{D}_+$ can be expressed as

$$\det(\hat{D}_+) = \det(\hat{D}_+) \left(1 + (\sigma - 1) \sigma_H^{-1} H^{-1} p_0 \right).$$

Observe that if $\sigma = 0$, then $\hat{D}_+ = D_+$, which is singular since $D_+ 1 = 0$; hence $\det(D_+) = 0$. But since $\det(\hat{D}_+) \neq 0$ by assumption, it follows that $p_0^\top \sigma_H^{-1} H^{-1} p_0 = 1$. For any $\sigma \neq 0$ it must therefore hold that $\det(\hat{D}_+) = \sigma \det(\hat{D}_+)$ such that $\hat{D}_+$ is invertible.

Suppose now instead that $\hat{D}_+$ is singular. Then there is an eigenvector $w$ of $\hat{D}_+$ such that $\hat{D}_+ w = 0$. However, by [24, Lemma 1], any such eigenvector satisfies $p_0^\top w = 0$. Consequently, $\hat{D}_+ w = \hat{D}_+ w + (\sigma - 1) H^{-1} p_0^\top w = 0$ such that $\hat{D}_+$ is also singular.

In conclusion, for any $\sigma \neq 0$ it holds that $\hat{D}_+$ is invertible if and only if $\hat{D}_+$ is, and the lemma follows. \qed

Unfortunately, nullspace consistency does not follow automatically from Definition 1 [31, Theorem 2.1]. Perhaps the simplest example of an SBP operator lacking nullspace consistency is

$$D_\pm = \begin{pmatrix} -\frac{1}{2} & 0 & 1 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \quad S = 0, \quad p_0 = e_0, \quad p_n = e_n.$$

This is an SBP operator of order $q = 1$ defined on the nodes $x = (-1, 0, 1)^\top$. By inspection, $\begin{pmatrix} 0, 1, 0 \end{pmatrix}^\top \in \ker(D_\pm)$, hence nullspace consistency is not satisfied. It is easily verified that this vector also belongs to $\ker(D_+)$ for any $\sigma$, so that $\hat{D}_+$ is not invertible, as expected by Lemma 1.

An even stronger condition than nullspace consistency is frequently needed:

Definition 3 An SBP operator is said to have the eigenvalue property if each eigenvalue of $\hat{D}_+ := (D_+ + \sigma H^{-1} p_0 p_0^\top)$ has positive real part for every $\sigma > 1/2$.

Remark 3 Excluding the limiting case $\sigma = 1/2$ from Definition 3 is motivated by the fact that the eigenvalue property for any $\sigma > 1/2$ implies the eigenvalue property for all $\sigma > 1/2$; see Remark 5. The arguments presented in subsequent sections are invalid in the case $\sigma = 1/2$. The same exclusion was made e.g. in [33] for similar reasons.

Remark 4 Here and throughout, the focus is on the SBP operator $D_+$. If $D_-$ is instead used, the relevant property is that the eigenvalues of $\hat{D}_- := (D_- - \sigma H^{-1} p_0 p_0^\top)$ have negative real parts whenever $\sigma > 1/2$.

Many important results in the theory of SBP operators rely on the eigenvalue property. The following list is by no means complete.

- The proof that spatial discretizations using (classical) finite difference SBP methods converge with order $q + 1$ for hyperbolic problems and order $q + 2$ for parabolic problems assumes the eigenvalue property [35].
The proof that functional estimates from SBP methods in two or more dimensions are superconvergent requires the eigenvalue property \[15\].

SBP methods used for time discretization of partial differential equations need the eigenvalue property to ensure uniqueness of solutions \[30\]. Proofs of convergence for linear and certain nonlinear problems also rely on the property; see \[24\] for an overview.

The eigenvalue property is evidently very important and it is therefore of interest to know which SBP operators possess it. Nullspace consistency is of course necessary. This bids the question: Is nullspace consistency sufficient for the eigenvalue property? Unfortunately, this is not the case.

**Theorem 1** There are nullspace consistent SBP operators that lack the eigenvalue property.

**Proof** Consider

\[
D_+ = \frac{1}{5} \begin{pmatrix}
-5 & 4 & 2 & 0 & -2 & 1 \\
-2 & 0 & 1 & 0 & 2 & -1 \\
-1 & -1 & 0 & 2 & 0 & 0 \\
0 & 0 & -2 & 0 & 1 & 1 \\
1 & -2 & 0 & -1 & 0 & 2 \\
-1 & 2 & 0 & -2 & -4 & 5
\end{pmatrix},
\]

defined on the nodes \(x^T = (-5, -3, -1, 1, 3, 5)/2\). This is an SBP operator of order \(q = 1\) with \(S = 0\), \(p_0 = e_0\), \(p_n = e_n\) and \(H = \text{diag}(1/2, 1, 1, 1, 1/2)\). The matrix \(\tilde{D}_+ = D_+ + \sigma H^{-1}p_0p_0^T\) is invertible for any \(\sigma \neq 0\), with inverse

\[
\tilde{D}^{-1}_+ = \frac{1}{10\sigma} \begin{pmatrix}
5 -6 & -2 & 2 & 6 & -5 \\
5 -6 & -2 & 2 & 6 & -5 \\
5 -6 & -2 & 2 & 6 & -5 \\
5 -6 & -2 & 2 & 6 & -5 \\
5 -6 & -2 & 2 & 6 & -5
\end{pmatrix} + \frac{1}{25} \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 16 & 62 & 8 & -21 & 10 \\
0 & 16 & 62 & 8 & -21 & 10 \\
0 & 16 & 62 & 8 & -21 & 10 \\
0 & 16 & 62 & 8 & -21 & 10
\end{pmatrix},
\]

hence the operator is nullspace consistent by Lemma 1. However, two of the eigenvalues of \(\tilde{D}_+\) and their corresponding eigenvectors are

\[
\lambda_\pm = \pm \frac{i}{\sqrt{5}},
\]

\[
w_\pm = (0, 1, -3, 3, -1, 0)^T \pm i \sqrt{5}(0, 1, -1, -1, 1, 0)^T.
\]

The real parts of these eigenvalues are non-positive, hence the SBP operator does not have the eigenvalue property. Note that \(w_\pm\) are orthogonal to \(p_0\), hence the independence of these eigenpairs from \(\sigma\).

\[\square\]

### 3 Perturbed SBP Operators

In this section we consider an SBP operator \(D_+\) that is nullspace consistent but lack the eigenvalue property, i.e. it satisfies Definition 2 but not Definition 3. It will be shown that the eigenvalue property can be reclaimed by adding an appropriately designed perturbation \(S'\) to \(D_+\). More precisely, the eigenvalue property is enforced by constructing a new operator

\[
D'_+ = D_+ + \frac{1}{2}H^{-1}S',
\]

which adheres to Definition 1 so long as \(S'\) is symmetric positive.
semi-definite and satisfies (1). In this case, $S$ is replaced by $S + S'$ in Definition 1. The purpose of $S'$ is to push problematic eigenvalues into the right half-plane while leaving the remaining eigenvalues untouched. Furthermore, the matrix $H$ and the vectors $p_0$ and $p_n$ are not modified. To achieve this, it is first necessary to establish certain properties of the problematic eigenvalues and eigenvectors. The starting point is the following simple result, which is a slight generalization of [24, Lemma 1] and for which the proof is identical:

**Lemma 2** Consider the SBP operator $D_+$ and let $(\lambda, w)$ be an eigenpair of $\tilde{D}_+ = D_+ + \sigma H^{-1}p_0p_0^\top$, i.e. $\tilde{D}_+w = \lambda w$. For any $\sigma > 1/2$ it holds that $\text{Re}(\lambda) \geq 0$ with equality if and only if $p_0^\top w = p_n^\top w = 0$ and $Sw = 0$.

By assumption, $D_+$ is nullspace consistent but lacks the eigenvalue property. Lemma 2 therefore shows that $\tilde{D}_+$ has an imaginary eigenvalue, which is non-zero by Lemma 1. In fact, $\tilde{D}_+$ has an even number of non-zero imaginary eigenvalues since the conjugate of $\tilde{D}_+$ is also an eigenpair of $\tilde{D}_+$.

**Remark 5** The fact that any eigenvector associated with an imaginary eigenvalue of $\tilde{D}_+$ is orthogonal to $p_0$ means that this eigenpair is independent of $\sigma$. A missing eigenvalue property can thus not be fixed by modifying the SAT. Conversely, if the eigenvalue property can be established for an SBP operator using any $\sigma > 1/2$ then it holds for every $\sigma > 1/2$.

At this point, two lemmas will be established that are key to the construction of $S'$. The first pertains to normal eigenvalues; see e.g. [17, Chapter 1.6]. An eigenvalue $\lambda$ of $\tilde{D}_+$ is normal if

(a) every eigenvector of $\tilde{D}_+$ corresponding to $\lambda$ is orthogonal to every eigenvector of $\tilde{D}_+$ corresponding to each eigenvalue different from $\lambda$, and

(b) the algebraic and geometric multiplicities of $\lambda$ are equal.

**Remark 6** Herein, the inner product of two vectors $f$ and $g$ is taken to be $\langle f, g \rangle_H = f^*Hg$, where the superscript $*$ indicates conjugate transposition. Orthogonality should thus be understood with this inner product in mind.

**Lemma 3** Let $\sigma > 1/2$ and let $(\lambda, w)$ be an eigenpair of $\tilde{D}_+ \text{ with } \text{Re}(\lambda) = 0$. Then $\lambda$ is a normal eigenvalue with respect to the inner product $\langle \cdot, \cdot \rangle_H$.

**Proof** The two properties (a) and (b) must be established. For (a), let $(\mu, v)$ be any eigenpair of $\tilde{D}_+$ with $\mu \neq \lambda$ and note that $\mu \langle w, v \rangle_H = w^*H\tilde{D}_+v$. Using (C) from Definition 1 and the definition of $\tilde{D}_+$ in (6) it follows that

$$H\tilde{D}_+ = (2\sigma - 1)p_0p_0^\top + p_np_n^\top + S - \tilde{D}_+^\top H. \quad (7)$$

Thus,

$$\mu \langle w, v \rangle_H = (2\sigma - 1)(p_0^\top w)^*p_0^\top v + (p_n^\top w)^*p_n^\top v + (Sw)^*v - (\tilde{D}_+w)^*Hv$$

$$= 0 \text{ by Lemma 2}$$

$$= -\lambda w^*Hv = \lambda \langle w, v \rangle_H.$$ 

In the final equality, $\lambda = -\lambda$ has been used, which holds since $\lambda$ is imaginary. Consequently, $(\mu - \lambda) \langle w, v \rangle_H = 0$, however since $\mu \neq \lambda$ by assumption, the sought orthogonality follows.

For (b), consider the following problem: Find a symmetric positive definite matrix $X \in \mathbb{R}^{(n+1) \times (n+1)}$ such that $X\tilde{D}_+ + \tilde{D}_+^\top X \geq 0$. From (7) it is seen that $X = H$ solves
this problem. However, the existence of a solution implies that all elementary divisors of imaginary eigenvalues of $D_+$ are linear [5, Corollary 2], which is equivalent to the stated assertion on the algebraic and geometric multiplicities [20, Chapter 7].

The importance of Lemma 3 resides in the fact that it is always possible to find a set of eigenvectors of $\tilde{D}_+$ such that each eigenvector associated with an imaginary eigenvalue is orthogonal to all the others. For simple eigenvalues, this is ensured by property (a). For a repeated imaginary eigenvalue $\lambda$ with algebraic multiplicity $m$, property (b) ensures that there are $m$ linearly independent eigenvectors $v_1, \ldots, v_m$ associated with $\lambda$. Any linear combination of these is also an eigenvector associated with $\lambda$. By the Gram-Schmidt procedure it is possible to find linear combinations that result in $m$ mutually orthogonal eigenvectors $w_1, \ldots, w_m$, which by (a) are also orthogonal to all eigenvectors associated with eigenvalues different from $\lambda$.

The next lemma pertains to the orthogonality between problematic eigenvectors and grid polynomials:

**Lemma 4** Let $\sigma > 1/2$ and let $(\lambda, w)$ be an eigenpair of $\tilde{D}_+$ with $\Re(\lambda) = 0$. Then $(x^j, w)_H = 0$ for $j = 0, \ldots, q$.

**Proof** Consider first the case $j = 0$, for which we have $x^0 = 1$, and note that $\langle 1, w \rangle_H = 1^T H \tilde{D}_+ w$. From Definition 1 and (6) it follows that

$$H \tilde{D}_+ = (\sigma - 1)p_0 p_0^T + p_n p_0^T - D_+^T H.$$

Thus,

$$\langle 1, w \rangle_H = (\sigma - 1)1^T p_0 (p_0^T w) + 1^T p_n (p_n^T w) - (D_+ 1)^T H w = 0.$$

Since $D_+$ is nullspace consistent by assumption, Lemma 1 ensures that $\lambda \neq 0$. It follows that $(\langle 1, w \rangle)_H = 0$, hence the claim holds when $j = 0$.

Next, suppose that the claim holds for some $j - 1 < q$. Then, similarly,

$$\langle x^j, w \rangle_H = (\sigma - 1)(x^j)^T p_0 (p_0^T w) + (x^j)^T p_n (p_n^T w) - (D_+ x^j)^T H w = 0,$$

where the induction hypothesis implies the final equality. From nullspace consistency it follows that $(x^j, w)_H = 0$, and the claim holds by induction.

Using Lemmas 3 and 4, the main result of this section can be proven:

**Theorem 2** For any SBP operator $D_+$ of order $q$ that satisfies Definition 2 but not Definition 3, there is another SBP operator $D'_+ = D_+ + \frac{1}{2} H^{-1} S'$ of order $q$ that satisfies both definitions. Further, $D'_+$ can be constructed such that $\|D'_+ - D_+\| \leq \varepsilon$ for any $\varepsilon > 0$, where the norm $\| \cdot \|$ is arbitrary.

**Proof** Suppose that $\tilde{D}_+$ has precisely $2m$ imaginary eigenvalues, including multiplicity, and denote these $\lambda_1, \bar{\lambda}_1, \ldots, \lambda_m, \bar{\lambda}_m$. Select corresponding eigenvectors $w_1, \bar{w}_1, \ldots, w_m, \bar{w}_m$. By Lemma 3 each of these eigenvectors can be chosen so that they are orthogonal to all other eigenvectors of $\tilde{D}_+$. 

[ Springer]
Choose \( m \) positive numbers \( \epsilon_1, \ldots, \epsilon_m \) and construct the matrix
\[
S' = \sum_{k=1}^{m} \epsilon_k ((hw_k)(hw_k)^* + (h\bar{w}_k)(h\bar{w}_k)^*).
\]
Then \( S' \) is real since \((h\bar{w}_k)(h\bar{w}_k)^* = (h\bar{w}_k)(h\bar{w}_k)^*\). Further, \( S' \) is symmetric positive semi-definite.

At this point, construct \( D'_+ = D_+ + \frac{1}{2} H^{-1} S' \). Note that
\[
S' x^j = \sum_{k=1}^{m} \epsilon_k \left( (hw_k)(hw_k)^* + (h\bar{w}_k)(h\bar{w}_k)^* \right) x^j
\]
\[
= \sum_{k=1}^{m} \epsilon_k \left( (hw_k)(x^j)_H + (h\bar{w}_k)(x^j)_H \right) = 0,
\]
for each \( j = 0, \ldots, q \). Here, Lemma 4 has been used in the final equality. Consequently, \( S' \) satisfies (1) such that \( D'_+ \) is an SBP operator of order \( q \).

Consider an eigenvalue \( \lambda \) of \( \tilde{D}_+ \). If \( \text{Re}(\lambda) \neq 0 \), then pick any eigenvector \( v \) corresponding to \( \lambda \) and note that \( \tilde{D}_+ v = \lambda v \) by the orthogonality property in Lemma 3. Thus, the eigenpair \((\lambda, v)\) is unaltered by \( S' \) whenever \( \text{Re}(\lambda) \neq 0 \). However, if \( \text{Re}(\lambda) = 0 \) so that \( \lambda = \lambda_i \) for some \( i \), then pick the appropriate eigenvector \( v = w_i \) and note that
\[
\tilde{D}_+ w_i = \tilde{D}_+ w_i + \frac{1}{2} H^{-1} S' w_i
\]
\[
= \lambda_i w_i + \frac{\epsilon_i}{2} H^{-1} (hw_i)(hw_i)^* w_i
\]
\[
= \left( \lambda_i + \frac{\epsilon_i}{2} \|w_i\|_H^2 \right) w_i,
\]
where the second equality follows from the orthogonality of the eigenvectors \( w_1, w_1, \ldots, w_m, \bar{w}_m \). Thus, \( \tilde{D}_+ \) has an eigenpair \((\lambda'_i, w_i)\), where \( \text{Re}[\lambda'_i] = \frac{\epsilon_i}{2} \|w_i\|_H^2 > 0 \). By repeating this procedure for each \( \lambda_i \) and \( \bar{\lambda}_i \), every eigenvalue of \( \tilde{D}_+ \) is accounted for. Consequently, the SBP operator \( D'_+ \) has the eigenvalue property.

Finally, note that the eigenvectors \( w_k \) and \( \bar{w}_k \) may be normalized as desired and that the constants \( \epsilon_k \) can be chosen arbitrarily small. Thus, for any norm it is possible to find \( S' \) such that for any \( \epsilon \),
\[
\| D'_+ - D_+ \| = \frac{1}{2} \| H^{-1} S' \| \leq \epsilon.
\]

The implication of Theorem 2 is that, while the eigenvalue property cannot be taken for granted (due to Theorem 1), this is not a major cause for concern. In theoretical developments of SBP methods it is safe to assume the eigenvalue property, since it can always be guaranteed via addition of the artificial dissipation operator \( S' \).

Should it be desired to use \( D'_+ \) in practical computations, note that it is not necessary to construct \( S' \) explicitly. It suffices to store the eigenvectors \( w_k, k = 1, \ldots, m \) and compute matrix-vector products as
\[
D'_+ v = D_+ v + \frac{1}{2} H^{-1} S' v = D_+ v + \sum_{k=1}^{m} \epsilon_k \text{Re}[(w_k, v)_{H \bar{w}_k}].
\]
Since \( m \) is expected to be small \((m \ll n)\), storing these vectors would be cheap. Further, any desirable structure that \( D_+ \) may possess, such as a particular sparsity pattern, can still be utilized.

4 Pseudospectral Methods

An important family of SBP operators consists of the pseudospectral methods. They all satisfy Definition 1 with order \( q = n \); see [6, 11]. These operators are also used in discontinuous Galerkin and flux reconstruction methods [14, 32]. Given \( x \), the pseudospectral SBP operator \( D_+ \) is uniquely defined. To see this, note that the accuracy conditions (A) in Definition 1 imply that the elements \( d_{i,j} \) in the \( i \)th row of \( D_+ \) satisfy

\[
\begin{pmatrix}
1 & \cdots & 1 \\
x_0 & \cdots & x_n \\
\vdots & \ddots & \vdots \\
x_n^0 & \cdots & x_n^n \\
\end{pmatrix}
\begin{pmatrix}
d_{i,0} \\
d_{i,1} \\
\vdots \\
d_{i,n} \\
\end{pmatrix}
= \begin{pmatrix}
0 \\
1 \\
\vdots \\
nx_i^{n-1} \\
\end{pmatrix}.
\]

Note that \( V \) is a Vandermonde matrix and that the numbers \( x_0, \ldots, x_n \) are distinct. Thus, \( V^\top \) is invertible and \( d_{i,0}, \ldots, d_{i,n} \) are unique. The same of course holds for each row \( i = 0, \ldots, n \), hence \( D_+ \) is uniquely defined.

**Remark 7** This argument does not imply uniqueness of \( H, p_0, p_n \) and \( S \).

The goal of this section is to demonstrate that pseudospectral methods defined on arbitrary grids satisfy the eigenvalue property. In [33] it was shown that this is the case if \( H \) defines a quadrature rule that is exact for polynomials of degree \( 2n - 1 \) or higher, i.e. if \( k^\top H x^k = b^{k+1} - a^{k+1} \) for \( k = 0, \ldots, 2n - 1 \). This holds in particular if the grid \( x \) is chosen to be the Legendre-Gauss, Legendre-Gauss-Radau or Legendre-Gauss-Lobatto nodes (mapped to the interval \([a, b]\)) and \( H \) is a diagonal matrix containing the corresponding Gaussian quadrature weights. However, for general grids \( x \), the resulting quadrature rule is exact only for polynomials of degree \( n - 1 \) [11]. A different approach is thus necessary to generalize the result from [33].

The starting point here is to show that pseudospectral methods on arbitrary grids are nullspace consistent.

**Lemma 5** Every pseudospectral SBP operator is nullspace consistent.

**Proof** The columns of the Vandermonde matrix \( V \) form a basis for \( \mathbb{R}^{n+1} \). Thus, if \( D_+ v = 0 \) and \( v \) is expanded in terms of this basis as \( v = \sum_{k=0}^{n} v_k x^k \), then

\[
0 = D_+ v = D_+ \sum_{k=0}^{n} v_k x^k = \sum_{k=0}^{n} v_k D_+ x^k = \sum_{k=1}^{n} v_k k x^{k-1}.
\]

By linear independence of the basis vectors it follows that \( v_k = 0 \) for \( k = 1, \ldots, n \) and consequently that \( v = v_0 1 \). Thus, \( \text{ker}(D_+) = \text{span}(\{1\}) \) and \( D_+ \) is consequently nullspace consistent.

With Lemma 5 in place, the results from Sect. 3 can be used to establish the eigenvalue property.
Theorem 3  Every pseudospectral SBP operator has the eigenvalue property.

Proof  Suppose that there is a pseudospectral method $D_+$ that does not have the eigenvalue property. Then, by Theorem 2, another operator $D'_+$ of the same order of accuracy can be found that operates on the same grid and that is distinctly different from $D_+$. However, this violates the uniqueness of pseudospectral methods.

Alternatively, suppose that there is a pseudospectral method $D_+$ that does not have the eigenvalue property and denote by $v$ one of the eigenvectors of $\tilde{D}_+$ that corresponds to an imaginary eigenvalue. It follows from Lemma 4 that $V^\top H v = 0$. But both $V$ and $H$ are invertible, so $v = 0$ and can therefore not be an eigenvector, which contradicts its definition.

\[\square\]

5 Summary

Three results on the eigenvalues of SBP operators have been proven. Firstly, the eigenvalue property, which is essential in many important accuracy and convergence proofs, does not hold for all nullspace consistent SBP operators. Secondly, this problem can be addressed by carefully constructing an arbitrarily small artificial dissipation term that pushes each problematic eigenvalue into the right half-plane without affecting the other eigenvalues or the order of accuracy of the method. Thirdly, all pseudospectral methods satisfy the eigenvalue property. Thus, SBP operators used within the discontinuous Galerkin and flux reconstruction frameworks are free from the nuisance of problematic eigenvalues.

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