Perfectly matched layer for the elastic wave equation in layered media

Kenneth Duru*  Balaje Kalyanaraman†  Siyang Wang‡

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Abstract

In this paper, we present the stability analysis of the perfectly matched layer (PML) in two-space dimensional layered elastic media. Using normal mode analysis we prove that all interface wave modes present at a planar interface of bi-material elastic solids are dissipated by the PML. Our analysis builds upon the ideas presented in [SIAM Journal on Numerical Analysis 52 (2014) 2883–2904] and extends the stability results of boundary waves (such as Rayleigh waves) on a half-plane elastic solid to interface wave modes (such as Stoneley waves) transmitted into the PML at a planar interface separating two half-plane elastic solids. Numerical experiments in two-layer and multi-layer elastic solids corroborate the theoretical analysis, and generalise the results to complex elastic media. Numerical examples using the Marmousi model demonstrates the utility of the PML and our numerical method for seismological applications.

Keywords: Elastic waves, perfectly matched layer, interface wave modes, stability, Laplace transforms, normal mode analysis

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1 Introduction

Wave motion is prevalent in many applications and has a great impact on our daily lives. Examples include the use of seismic waves [1, 9] to image natural resources in the Earth’s subsurface, to detect cracks and faults in structures, to monitor underground explosions, and to investigate strong ground motions from earthquakes.

Most wave propagation problems are formulated in large or infinite domains. However, because of limited computational resources, numerical simulations must be restricted to smaller computational domains by introducing artificial boundaries. Therefore, reliable and efficient domain truncation techniques that significantly minimise artificial reflections are important for the development of effective numerical wave solvers.

A straightforward approach to construct a domain truncation procedure is to surround the computational domain with an absorbing layer of finite thickness such that outgoing waves are absorbed. For this approach to be effective, all outgoing waves entering the layer should decay

*Mathematical Sciences Institute, Australian National University, Australia and Department of Mathematical Sciences, University of Texas at El Paso, TX, USA (kenneth.duru@anu.edu.au).
†Department of Mathematics and Mathematical Statistics, Umeå University, Umeå, Sweden (balaje.kalyanaraman@umu.se).
‡Department of Mathematics and Mathematical Statistics, Umeå University, Umeå, Sweden (siyang.wang@umu.se). Corresponding author.
without reflections, regardless of the frequency and angle of incidence. An absorbing layer with this desirable property is called a perfectly matched layer (PML) \([6, 11, 26, 13, 19, 7, 4, 5, 8]\).

The PML was first derived for electromagnetic waves in the pioneering work \([6, 11]\) but has since then been extended to other applications, for example acoustic and elastic waves \([13, 19, 7, 4, 5, 8]\). The PML has gained popularity because of its effective absorption properties, versatility, simplicity, ease of derivation and implementation using standard numerical methods. A stable PML model, when effectively implemented in a numerical solver, can yield a domain truncation scheme that ensures the convergence of the numerical solution to the solution of the unbounded problem \([5, 17, 23]\). However, the PML is also notorious for supporting fatal instabilities which can destroy the accuracy of numerical solutions. These undesirable exponentially growing modes can be present in both the PML model at the continuous level or numerical methods at the discrete level.

The stability analysis of the PML has attracted substantial attention in the literature, see for example \([13, 19, 7, 4, 5, 8]\), and \([22]\) for a recent review.

For hyperbolic PDEs, mode analysis for PML initial value problems (IVP) with constant damping and constant material properties yields a necessary geometric stability condition \([7]\). When this condition is violated, exponentially growing modes in time are present, rendering the PML model useless. In certain cases, for example the acoustic wave equation with constant coefficients, analytical solutions can be derived \([8, 12]\). In addition, energy estimates for the PML have recently been derived in physical space \([5]\) and Laplace space \([17, 23]\), which can be useful for deriving stable numerical methods. However, in general, even if the PML IVP does not support growing modes, there can still be stability issues when boundaries and material interfaces are introduced. For the extension of mode stability analysis to boundary and guided waves in homogeneous media, see \([13, 18, 19, 21]\). The stability analysis of the PML in discontinuous acoustic media was presented in \([16]\). To the best of our knowledge, the stability analysis of the PML for more general wave media such as the discontinuous or layered elastic solids has not been reported in literature.

In geophysical and seismological applications, the wave media can be composed of layers of rocks, soft and hard sediments, bedrock layers, water and possibly oil. In layered elastic media, the presence of interface wave modes such as Stoneley waves \([32, 9]\), makes the stability analysis of the PML more challenging. Numerical experiments have also reported PML instabilities and poor performance for problems with material boundaries entering into the layer and problems with strong evanescent waves \([3]\). These existing results have motivated this study to investigate where the inadequacies of the PML arise.

The main objective of this study is to analyse the stability of interface wave modes for the PML in discontinuous elastic solids. Using normal mode analysis, we prove that if the PML IVP has no temporally growing modes, then all interface wave modes present at a planar interface of bi-material elastic solids are dissipated by the PML. The analysis closely follows the steps taken in \([13]\) for boundary waves modes, but here we apply the techniques to investigate the stability of interface wave modes in the PML. Numerical experiments in two-layered isotropic and anisotropic elastic solids, and a multi-layered isotropic elastic solid corroborate the theoretical analysis. Furthermore, we present numerical examples using the Marmousi model \([29]\), extending the results to complex elastic media and demonstrating the utility of the PML and our numerical method for seismological applications.

The remainder of the paper proceeds as follows. In section 2 we present the elastic wave equation in discontinuous media, define interface conditions and discuss energy stability for the model problem. In section 3 we introduce the mode analysis for body and interface wave modes, and formulate the determinant condition that is necessary for stability. The PML model is derived in section 4. In section 5 we present the stability analysis of the PML in a piecewise constant elastic medium and formulate the main results. Numerical examples are given in section 6 corroborating...
the theoretical analysis. In section 7 we draw conclusions.

2 The elastic wave equation in discontinuous media

Consider the 2D elastic wave equation in Cartesian coordinates \((x, y) \in \Omega \subseteq \mathbb{R}^2\),

\[
\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x} T_x + \frac{\partial}{\partial y} T_y, \quad T_x = A \frac{\partial u}{\partial x} + C \frac{\partial u}{\partial y}, \quad T_y = C^T \frac{\partial u}{\partial x} + B \frac{\partial u}{\partial y},
\]

(1)

where \(u = [u_1, u_2] \in \mathbb{R}^2\) is the displacement vector, \(T_x, T_y \in \mathbb{R}^2\) are the stress vectors and \(t \geq 0\) denotes time. We set the smooth initial conditions

\[
u(x, y, 0) = f(x, y), \quad \frac{\partial u}{\partial t}(x, y, 0) = g(x, y).
\]

(2)

The medium parameters are described by the density \(\rho > 0\) and the coefficient matrices \(A, B, C\) of elastic constants. In 2D orthotropic elastic media, the elastic coefficients are described by four independent parameters \(c_{11}, c_{22}, c_{33}, c_{12}\) and the coefficient matrices are given by

\[
A = \begin{bmatrix} c_{11} & 0 \\ 0 & c_{33} \end{bmatrix}, \quad B = \begin{bmatrix} c_{33} & 0 \\ 0 & c_{22} \end{bmatrix}, \quad C = \begin{bmatrix} 0 & c_{12} \\ c_{33} & 0 \end{bmatrix},
\]

(3)

Here, the material coefficients \(c_{11}, c_{22}, c_{33}\) are always positive, but \(c_{12}\) may be negative for certain materials. In general, for stability, we require

\[
c_{11} > 0, \quad c_{22} > 0, \quad c_{33} > 0, \quad c_{11}c_{22} - c_{12}^2 > 0.
\]

(4)

For planar waves propagating along the \(x\)-direction and \(y\)-direction, the \(p\)-wave speeds and \(s\)-wave speeds are given by

\[
c_{px} := \sqrt{\frac{c_{11}}{\rho}}, \quad c_{sx} := \sqrt{\frac{c_{33}}{\rho}}, \quad c_{px} := \sqrt{\frac{c_{22}}{\rho}}, \quad c_{sy} := \sqrt{\frac{c_{33}}{\rho}}.
\]

(5)

In the case of isotropic media, the material properties can be described by only two Lamé parameters, \(\mu > 0\) and \(\lambda\), such that \(c_{11} = c_{22} = 2\mu + \lambda, c_{33} = \mu > 0, c_{12} = \lambda > -\mu\), yielding

\[
A = \begin{bmatrix} 2\mu + \lambda & 0 \\ 0 & \mu \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ \mu & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & \lambda \\ \mu & 0 \end{bmatrix},
\]

(6)

with the wave speeds

\[
c_p := \sqrt{\frac{2\mu + \lambda}{\rho}}, \quad c_s := \sqrt{\frac{\mu}{\rho}}.
\]

(7)

In isotropic media, a wave mode propagates with the same wave speed in all directions.

In general the elastic media is heterogeneous and discontinuous. At discontinuities we define the outward unit normal vector \(n = (n_x, n_y)^T \in \mathbb{R}^2\) on the interface, resolve the traction vector

\[
T = n_x T_x + n_y T_y,
\]

and we enforce the jump conditions,

\[
[T] = 0, \quad [u] = 0.
\]

(8)
Note that if the discontinuity lies on the $y$-axis ($x$-axis) we have $(n_x, n_y) = (0, 1)$ and $T = T_y$ ($n_x, n_y) = (1, 0)$ and $T = T_x$.

We introduce the strain-energy matrix

$$P = \begin{bmatrix} A & C \\ C^T & B \end{bmatrix}. \quad (9)$$

The symmetric strain-energy matrix $P$ is positive semi-definite [27]. We define the mechanical energy in the medium $\Omega$ by

$$E(t) = \frac{1}{2} \int_\Omega \left( \rho \left( \frac{\partial \mathbf{u}}{\partial t} \right)^T \left( \frac{\partial \mathbf{u}}{\partial t} \right) + \left[ \frac{\partial \mathbf{u}}{\partial x} \right]^T P \left[ \frac{\partial \mathbf{u}}{\partial y} \right] \right) dxdy. \quad (10)$$

The following theorem states a stability result for the coupled problem, (1)–(2) and (8).

**Theorem 1** Consider the elastic wave equation (1), in heterogeneous and discontinuous elastic medium defined on the whole plane $\Omega = \mathbb{R}^2$, subject to the initial condition (2), the interface condition (8) at material discontinuities, and the decay condition $|\mathbf{u}| \to 0$ at infinity $|(x, y)| \to \infty$. The total mechanical energy $E(t) \geq 0$ is defined in (10) is conserved, that is $E(t) = E(0)$ for all $t \geq 0$.

The proof of Theorem 1 is standard and can be adapted from [15]. We say that the problem, (1)–(2) and (8), is energy-stable if $E(t) \leq E(0)$ for all $t \geq 0$.

### 3. Layered elastic media and mode analysis

As we will see later in this paper, when the PML is introduced the equations will become asymmetric and the energy method may not be applicable to establishing the stability of the model. Next we will consider layered elastic media by assuming variations of media parameters along the $y$-axis only and introduce the mode analysis which will be useful in analysing the stability of the PML in layered elastic media. A similar approach was taken in [13] to analyse the stability of boundary waves modes in the PML, but here we apply the techniques to investigate the stability of interface elastic wave modes, such as Stoneley waves, in the PML.

#### 3.1 Layered elastic media

In the coming analysis we will focus on stratified elastic media with discontinuities along the $y$-axis. In particular, we will consider the 2D elastic wave equation (1) in the two half-planes $\Omega = \Omega_1 \cup \Omega_2$, with $\Omega_1 = (-\infty, \infty) \times (0, \infty)$ and $\Omega_2 = (-\infty, \infty) \times (-\infty, 0)$, and a discontinuous planar interface at $y = 0$. In each half-plane $\Omega_i$, for $i = 1, 2$, we denote the displacement field $\mathbf{u}_i$ and the piecewise constant medium parameters $\rho_i > 0, A_i, B_i, C_i$. At the material interface $y = 0$, interface conditions (8) has a more explicit form

$$B_1 \frac{\partial \mathbf{u}_1}{\partial y} + C_1^T \frac{\partial \mathbf{u}_1}{\partial x} = B_2 \frac{\partial \mathbf{u}_2}{\partial y} + C_2^T \frac{\partial \mathbf{u}_2}{\partial x}, \quad \mathbf{u}_1 = \mathbf{u}_2. \quad (11)$$

#### 3.2 Mode analysis

Theorem 1 proves energy stability of the elastic wave equation (1) in piece-wise smooth elastic media $\Omega_i$, for $i = 1, 2$, subject to the interface condition (8) or (11). However, the theorem does not
provide information about the wave modes that may exist in the medium. In this section, we use mode analysis to gain insights on the existence of possible wave modes. More precisely, we start by considering a constant-coefficient problem for the existence of body waves. After that, we analyse interface waves in media with piecewise constant material property and formulate a stability result in the framework of normal mode analysis. This mode analysis framework will be useful when proving the stability of the PML at the presence of interface wave modes.

3.2.1 Plane waves and dispersion relations

To study the existence of body wave modes, we consider the problem (1) in the whole real plane \((x, y) \in \mathbb{R}^2\) with constant medium parameters,

\[
\rho_i = \rho, \quad A_i = A, \quad B_i = B, \quad C_i = C,
\]

for \(\Omega_i, \ i = 1, 2\). In this case, there is no interface condition at \(y = 0\), and the material parameters are constant in the entire domain \(\Omega_1 \cup \Omega_2\).

Consider the wave-like solution

\[
u (x, y, t) = u_0 e^{st + i(k_x x + k_y y)}, \quad u_0 \in \mathbb{R}^2, \quad k_x, k_y, x, y \in \mathbb{R}, \quad t \geq 0, \quad i = \sqrt{-1}. \tag{12}\]

In (12), \(k = (k_x, k_y) \in \mathbb{R}^2\) is the wave vector, and \(u_0 \in \mathbb{R}^2\) is a vector of constant amplitude called the polarization vector. By inserting (12) into (1), we have the eigenvalue problem

\[
-s^2 u_0 = \mathcal{P}(k) u_0, \quad \mathcal{P}(k) = \frac{k_x^2 A + k_y^2 B + k_x k_y (C + C^T)}{\rho}. \tag{13}\]

The polarization vector \(u_0 \in \mathbb{R}^2\) is an eigenvector of the matrix \(\mathcal{P}(k)\) and \(-s^2\) is the corresponding eigenvalue. For problems that are energy conserving, the matrix \(\mathcal{P}(k)\) is symmetric positive definite for all \(k \in \mathbb{R}^2\). Thus, the eigenvectors \(u_0\) of \(\mathcal{P}(k)\) are orthogonal and the eigenvalues are real and positive, \(-s^2 > 0\).

The wave-mode (12) is a solution of the elastic wave equation (1) in the whole plane \((x, y) \in \mathbb{R}^2\) if \(s\) and \(k\) satisfy the dispersion relation

\[
F(s, k) := \det (s^2 I + \mathcal{P}(k)) = 0. \tag{14}\]

Evaluating the determinant and simplifying further, we obtain

\[
F(s, k) = s^4 + \left(\frac{(c_{11} + c_{33}) k_x^2 + (c_{22} + c_{33}) k_y^2}{\rho} + c_{11} c_{33} k_x^4 + c_{22} c_{33} k_y^4 + (c_{11} c_{22} + c_{33}^2 - (c_{33} + c_{12})^2) k_x^2 k_y^2}{\rho^2}\right) = 0. \tag{15}\]

In an isotropic medium, with \(c_{11} = c_{22} = 2\mu + \lambda, \ c_{33} = \mu > 0, \ c_{12} = \lambda > -\mu\), the dispersion relation simplifies to

\[
F(s, k) = \left(s^2 + c_p^2 |k|^2\right) \left(s^2 + c_s^2 |k|^2\right) = 0, \quad c_p = \sqrt{\frac{2\mu + \lambda}{\rho}}, \quad c_s = \sqrt{\frac{\mu}{\rho}}, \quad |k| = \sqrt{k_x^2 + k_y^2}. \tag{16}\]

Then, the eigenvalues are given by

\[
s_1^2 = c_p^2 |k|^2, \quad s_2^2 = c_s^2 |k|^2, \tag{17}\]
which correspond to the P-wave and S-wave propagating in the medium. In linear orthotropic elastic media, the eigenvalues \(-s^2\) also have closed form expressions

\[
-s_1^2 = \frac{1}{2\rho} \left( (c_{11} + c_{33})k_2^2 + (c_{22} + c_{33})k_y^2 \right) + \frac{1}{2\rho} \sqrt{\left( (c_{11} + c_{33})k_2^2 + (c_{22} + c_{33})k_y^2 \right)^2 - 4 \left( (c_{11}c_{33}k_2^2 + c_{22}c_{33}k_y^2) + (c_{11}c_{22} + c_{22}c_{33}k_2^2) + (c_{11}c_{22} + c_{22}c_{33}k_2^2) \right)},
\]

\[
-s_2^2 = \frac{1}{2\rho} \left( (c_{11} + c_{33})k_2^2 + (c_{22} + c_{33})k_y^2 \right) - \frac{1}{2\rho} \sqrt{\left( (c_{11} + c_{33})k_2^2 + (c_{22} + c_{33})k_y^2 \right)^2 - 4 \left( (c_{11}c_{33}k_2^2 + c_{22}c_{33}k_y^2) + (c_{11}c_{22} + c_{22}c_{33}k_2^2) + (c_{11}c_{22} + c_{22}c_{33}k_2^2) \right)}. \tag{18}
\]

Using the stability conditions \([4]\), it is easy to check that the two eigenvalues are strictly positive, that \(-s_j^2 > 0\) for \(j = 1, 2\). These two eigenvalues again indicate two body-wave modes, corresponding to the quasi-P waves and the quasi-S waves.

**Remark 1** The indeterminate \(s \in \mathbb{C}\) that solves the dispersion relation \([14]\) is related to the temporal frequency. Since Theorem \([7]\) holds for all stable medium parameters, the whole plane problem \([1]\) conserves energy. Thus, the real part of the roots \(s\) must be zero, that is \(s \in \mathbb{C}\) with \(\text{Re}\{s\} = 0\). Otherwise, if the roots \(s\) have non-zero real parts then the energy will grow or decay, contradicting Theorem \([7]\).

We write \(s = i\omega\), where \(\omega \in \mathbb{R}\) is called the temporal frequency, and introduce

\[
K = \left( \begin{array}{c} k_x \\ k_y \\ \|k\| \end{array} \right), \quad \text{normalized propagation direction,}
\]

\[
V_p = \left( \frac{\omega}{k_x}, \frac{\omega}{k_y} \right), \quad \text{phase velocity,}
\]

\[
S = \left( \frac{k_x}{\omega}, \frac{k_y}{\omega} \right), \quad \text{slowness vector,}
\]

\[
V_g = \left( \frac{\partial \omega}{\partial k_x}, \frac{\partial \omega}{\partial k_y} \right), \quad \text{group velocity.} \tag{19}
\]

For the Cauchy problem in a constant coefficient medium, the dispersion relation \(F(i\omega, k) = 0\) and the quantities \(K, V_p, S, V_g\), defined above give detailed description of the wave propagation properties in the medium. In addition, they determine a stability property for the corresponding PML model, which is discussed in section 5.1. In Figure \([1]\) we plot the dispersion relations of two different elastic solids, showing the slowness diagrams.

When boundaries and interfaces are present, additional boundary and interface wave modes, such as Rayleigh \([31]\) and Stoneley waves \([32, 9]\), are introduced. In the following, we consider the problem in two half-planes coupled together at a planar interface and formulate an alternative procedure to characterise the stability property of interface wave modes.

### 3.2.2 Normal mode analysis and the determinant condition

Here, we present the normal mode analysis for interface wave modes in discontinuous media, which tightly connects to the analysis of the PML in the next section. We note that a similar approach was taken in \([13]\) to investigate the stability of boundary waves in a half plane elastic solid. However, we will include the details here to ensure a self-contained narrative. To begin, we consider piecewise constant media parameters

\[
\rho_i > 0, \quad A_i, \quad B_i, \quad C_i,
\]
for \( \Omega_i, \ i = 1, 2 \), where \( A_i, B_i, C_i \) are given in (3). The material parameters are constant in each half-plane, but are discontinuous at the interface \( y = 0 \), where the equations (1) are coupled by the interface condition (8). We look for wave solutions in the form

\[
\mathbf{u}_i(x, y, t) = \phi_i(y)e^{st+ik_xx}, \quad \|\phi_i\| < \infty, \quad k_x \in \mathbb{R}, \ (x, y) \in \Omega_i, \ t \geq 0. \tag{20}
\]

The variable \( s \) is related to the stability property of the model, which is characterised in the following lemma.

**Lemma 1** The elastic wave equations (1) with piecewise constant media parameters (3) and the interface condition (8) or (11) are not stable in any sense if there are nontrivial solutions of the form (20) with \( \Re\{s\} > 0 \).

If there are nontrivial solutions of the form (20), we can always construct solutions that grow arbitrarily fast [13], which is not supported by a stable system. Now, we reformulate Lemma 1 as an algebraic condition, i.e. the so-called determinant condition in Laplace space [13, 24].

For a complex number \( z = a + ib \), we define the branch of \( \sqrt{z} \) by

\[
-\pi < \arg(a + ib) \leq \pi, \quad \arg(\sqrt{a + ib}) = \frac{1}{2} \arg(a + ib).
\]

We insert (20) in the equation (1) and the interface condition (8), and obtain

\[
s^2 \rho_i \phi_i = -k_x^2 A_i \phi_i + B_i \frac{d^2 \phi_i}{dy^2} + ik_x (C_i + C_i^T) \frac{d\phi_i}{dy}, \quad i = 1, 2, \tag{21}
\]

\[
\phi_1 = \phi_2, \quad B_1 \frac{d\phi_1}{dy} + ik_x C_i \phi_1 = B_2 \frac{d\phi_2}{dy} + ik_x C_i^T \phi_2. \tag{22}
\]

For \( \phi_i \), we seek the modal solution

\[
\phi_i = \Phi_i e^{\kappa y}, \quad \Phi_i \in \mathbb{C}^2, \quad i = 1, 2. \tag{23}
\]

Inserting the modal solution (23) in (21), we have the eigenvalue problem

\[
-s^2 \Phi_i = \mathcal{P}_i(k_x, \kappa) \Phi_i, \quad \mathcal{P}_i(k_x, \kappa) = \frac{k_x^2 A_i - \kappa^2 B_i - ik_x \kappa (C_i + C_i^T)}{\rho_i}, \quad i = 1, 2. \tag{24}
\]
The solutions satisfy the condition
\[ F_i(s, k_x, \kappa) := \det \left( s^2 I + \mathcal{P}_i(k_x, \kappa) \right) = 0. \tag{25} \]

We note that if we set \( \kappa = ik_y \) in \( F_i(s, k_x, \kappa) \), we get exactly the same dispersion relation \( (14) \) for the Cauchy problem.

For a fixed \( k_x \in \mathbb{R} \) and \( s \) with sufficiently large \( \text{Re}\{s\} > 0 \), the roots \( \kappa_i \) come in pairs and have non-vanishing real parts, \({13}\), with
\[ \kappa_{ij}^-(s, k_x), \quad \kappa_{ij}^+(s, k_x), \quad j = 1, 2. \tag{26} \]

The following lemma states an important property of the roots.

**Lemma 2** The real parts of the roots \( \kappa_{ij}^\pm \), \( i, j = 1, 2 \) in \( (26) \) do not change sign for all \( s \) with \( \text{Re}\{s\} > 0 \).

**Proof 1** We note that the roots vary continuously with \( s \). Thus, if the real part of a root \( \kappa_{ij} \) changes sign, then for some \( s \) with \( \text{Re}\{s\} > 0 \) the root is purely imaginary, \( \kappa_{ij} = ik_y \). Because of the equivalence between the dispersion relations \( (25) \) and \( (14) \) when \( \kappa_{ij} = ik_y \), a purely imaginary root corresponds to an exponentially growing mode for the Cauchy problem, which contradicts Theorem 4

We use the notation \( (26) \) to denote the roots with the stated sign convention for all \( s \) with \( \text{Re}\{s\} > 0 \). That is, the superscript \( (+) \) denotes the root with positive real part and the superscript \( (−) \) denotes the root with negative real part. Because of the condition \( \| \Phi_0 \| < \infty \), the general solution of \( (21) \) takes the form
\[ \Phi_1(y) = \delta_{11} e^{\kappa_{11} y} \Phi_{11} + \delta_{12} e^{\kappa_{12} y} \Phi_{12}, \quad \Phi_2(y) = \delta_{21} e^{\kappa_{21} y} \Phi_{21} + \delta_{22} e^{\kappa_{22} y} \Phi_{22}, \tag{27} \]
where \( \Phi_{ij}, \ i, j = 1, 2 \) are the corresponding eigenvectors. As an example, in isotropic linear elastic media, the analytical expressions of the roots and eigenvectors are
\[ \kappa_{11}^\pm = \pm \sqrt{k_x^2 + \frac{s^2}{c_{si}^2}}, \quad \kappa_{12}^\pm = \pm \sqrt{k_x^2 + \frac{s^2}{c_{pi}^2}}, \quad i = 1, 2, \]
and
\[ \Phi_{11} = \begin{bmatrix} \frac{1}{k_x} \kappa_{11}^- \\ 1 \end{bmatrix}, \quad \Phi_{12} = \begin{bmatrix} \frac{-ik_x}{k_x} \kappa_{12}^- \\ 1 \end{bmatrix}, \quad \Phi_{21} = \begin{bmatrix} \frac{i}{k_x} \kappa_{12}^+ \\ 1 \end{bmatrix}, \quad \Phi_{22} = \begin{bmatrix} \frac{-ik_x}{k_x} \kappa_{22}^+ \\ 1 \end{bmatrix}. \tag{28} \]

For orthotropic elastic media, the roots can also be expressed in closed form, but the expressions are much more complicated. We refer the reader to \( {13} \) for more details.

The coefficients \( \delta = [\delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}]^T \) are determined by inserting \( (27) \) into the interface conditions \( (22) \), yielding the following equation
\[ \mathcal{C}(s, k_x) \delta = 0, \tag{29} \]
where the \( 4 \times 4 \) interface matrix \( \mathcal{C} \) takes the form
\[ \mathcal{C}(s, k_x) = \begin{bmatrix} \Phi_{11} & \Phi_{12} \\ (\kappa_{11}^- B_1 + i k_x C_{11}^T) \Phi_{11} & (\kappa_{21}^- B_1 + i k_x C_{11}^T) \Phi_{12} \\ (\kappa_{12}^- B_2 + i k_x C_{12}^T) \Phi_{11} & (\kappa_{22}^- B_2 + i k_x C_{12}^T) \Phi_{12} \end{bmatrix}. \tag{30} \]
To ensure only trivial solutions for \( \text{Re}\{s\} > 0 \), the coefficients \( \delta \) must vanish, and thus we require the determinant condition

\[
\mathcal{F}(s, k_x) := \det (\mathcal{C}(s, k_x)) \neq 0, \quad \forall \text{Re}\{s\} > 0.
\] (31)

We will now formulate an algebraic definition of stability equivalent to Lemma 1, for the coupled problem, (1)–(2) and (8), with piecewise constant media parameters (3).

**Lemma 3** The solutions of the elastic wave equation (1) with piecewise constant media parameters (3) and the interface condition (8) are not stable in any sense if for some \( k_x \in \mathbb{R} \) and \( s \in \mathbb{C} \) with \( \text{Re}\{s\} > 0 \), we have

\[
\mathcal{F}(s, k_x) := \det (\mathcal{C}(s, k_x)) = 0.
\]

The determinant condition is defined for all \( s \) with \( \text{Re}\{s\} > 0 \). The case when \( \text{Re}\{s\} = 0 \) would correspond to time-harmonic and important interface wave modes, such as Stoneley waves [24, 25].

The energy stability in Theorem 1 states that the coupled problem, (1)–(2) and (8), with piecewise constant media parameters (3) conserves energy. Therefore, similar to Remark 1, the roots \( s \) of \( \mathcal{F}(s, k_x) = 0 \) must be zero or purely imaginary, i.e. \( s \in \mathbb{C} \) with \( \text{Re}\{s\} = 0 \). We conclude that all nontrivial and stable interface wave modes, such as Stoneley waves, that solve \( \mathcal{F}(s, k_x) = 0 \), must have purely imaginary roots, \( s = i\xi \) with \( \xi \in \mathbb{R} \). A main objective of the present work is to determine how the purely imaginary roots \( s = i\xi \) will move in the complex plane when the PML is introduced.

### 4 The perfectly matched layer

We consider the elastic wave equation (1) with the interface conditions (8). Let the Laplace transform, in time, of \( u(x, y, t) \) be defined by

\[
\mathcal{U}(x, y, s) = \int_0^\infty e^{-st} u(x, y, t) \, dt, \quad s = a + ib, \quad \text{Re}\{s\} = a > 0.
\] (32)

We consider a setup where the PML is included in the \( x \)-direction only. Without loss of generality, we assume that we are only interested in the solution in the left half-plane \( x \leq 0 \). To absorb outgoing waves, we introduce a PML outside the left half-plane and require that the material properties are invariant in \( x \) in PML.

To derive the PML model, we Laplace transform (1) in time, and obtain

\[
\rho_i s^2 \mathbf{u}_i = \frac{\partial}{\partial x} \left( A_i \frac{\partial \mathbf{u}_i}{\partial x} \right) + \frac{\partial}{\partial y} \left( B_i \frac{\partial \mathbf{u}_i}{\partial y} \right) + \frac{\partial}{\partial x} \left( C_i \frac{\partial \mathbf{u}_i}{\partial y} \right) + \frac{\partial}{\partial y} \left( C_i^T \frac{\partial \mathbf{u}_i}{\partial x} \right), \quad (x, y) \in \Omega_i, \quad \text{Re}\{s\} > 0.
\] (33)

Note that we have tacitly assumed homogeneous initial data. Next, we consider (33) in the transformed coordinate \((\tilde{x}, y)\), such that

\[
\frac{d\tilde{x}}{dx} = 1 + \frac{\sigma(x)}{\alpha + s} =: S_x.
\] (34)

Here, \( \sigma(x) \geq 0 \) is the damping function and \( \alpha \geq 0 \) is the complex frequency shift (CFS) [26]. For all \( s \in \mathbb{C} \), we have \( S_x \neq 0 \) and \( 1/S_x \neq 0 \), and the smooth complex coordinate transformation [11],

\[
\frac{\partial}{\partial x} \to \frac{1}{S_x} \frac{\partial}{\partial x}.
\] (35)
The PML model in Laplace space is

\[ s^2 \rho_i \hat{u}_i = \frac{1}{S_x} \frac{\partial}{\partial x} \left( \frac{1}{S_x} A_i \frac{\partial \hat{u}_i}{\partial x} \right) + \frac{\partial}{\partial y} \left( B_i \frac{\partial \hat{u}_i}{\partial y} \right) + \frac{1}{S_x} \frac{\partial}{\partial x} \left( C_i \frac{\partial \hat{u}_i}{\partial y} \right) + \frac{\partial}{\partial y} \left( C_i^T \frac{1}{S_x} \frac{\partial \hat{u}_i}{\partial x} \right), \quad i = 1, 2, \]  

(36)

with the transformed interface conditions

\[ \hat{u}_1 = \hat{u}_2, \quad B_1 \frac{\partial \hat{u}_1}{\partial y} + C_i^T \frac{1}{S_x} \frac{\partial \hat{u}_1}{\partial x} = B_2 \frac{\partial \hat{u}_2}{\partial y} + C_i^T \frac{1}{S_x} \frac{\partial \hat{u}_2}{\partial x}. \]  

(37)

Choosing the auxiliary variables

\[ \hat{v}_i = \frac{1}{s + \sigma + \alpha} \frac{\partial \hat{u}_i}{\partial x}, \quad \hat{w}_i = \frac{1}{s + \alpha} \frac{\partial \hat{u}_i}{\partial y}, \quad \hat{q}_i = \frac{\alpha}{s + \alpha} \hat{u}_i, \]

we invert the Laplace transformed equation (36) and obtain the PML model in physical space,

\[ \rho_i \left( \frac{\partial^2 u_i}{\partial t^2} + \sigma \frac{\partial u_i}{\partial t} - \sigma \alpha (u_i - q_i) \right) = \frac{\partial}{\partial x} \left( A_i \frac{\partial u_i}{\partial x} + C_i \frac{\partial u_i}{\partial y} - \sigma A_i v_i \right) + \frac{\partial}{\partial y} \left( B_i \frac{\partial u_i}{\partial y} + C_i^T \frac{\partial u_i}{\partial x} + \sigma B_i w_i \right), \]

\[ \frac{\partial v_i}{\partial t} = -(\sigma + \alpha) v_i + \frac{\partial u_i}{\partial x}, \]

\[ \frac{\partial w_i}{\partial t} = -\alpha w_i + \frac{\partial u_i}{\partial y}, \]

\[ \frac{\partial q_i}{\partial t} = \alpha (u_i - q_i). \]

(38)

Similarly, inverting the Laplace transformed interface conditions (37) for the PML model gives

\[ u_1 = u_2, \quad B_1 \frac{\partial u_1}{\partial y} + C_i^T \frac{\partial u_1}{\partial x} + \sigma B_1 w_1 = B_2 \frac{\partial u_2}{\partial y} + C_i^T \frac{\partial u_2}{\partial x} + \sigma B_2 w_2. \]  

(39)

In the absence of the PML, \( \sigma = 0 \), the above model problem is energy-stable in the sense of Theorem 1 for all elastic material parameters. When \( \sigma > 0 \), however, the coupled PML model (38)-(39) is asymmetric with auxiliary differential equations. Thus, a similar energy-stability cannot be established in general. To analyse the stability properties of the PML model in a piecewise constant elastic medium, we use the mode analysis discussed in Section 3 to prove that exponentially growing wave modes are not supported. In A we derive a more general model with PML included in both \( x \) and \( y \)-directions.

## 5 Stability analysis of the PML model

The stability analysis of the PML will mirror directly the mode analysis described in Section 3. We will split the analysis into two parts: plane wave analysis for the Cauchy PML problem and normal mode analysis for the interface wave modes.

### 5.1 Plane waves analysis

We now investigate the stability of body wave modes in the PML in the whole real plane \((x, y) \in \mathbb{R}^2\) with constant medium parameters. As before, we consider constant PML damping \( \sigma > 0 \) and uniformly constant coefficients medium parameters

\[ \rho_i = \rho, \quad A_i = A, \quad B_i = B, \quad C_i = C, \]
for $\Omega_i, i = 1, 2$, that is there is no discontinuity of material parameters at the interface at $y = 0$. Consider the wave-like solution

$$u(x, y, t) = u_0 e^{st + ik_xx + ky_y}, \quad u_0 \in \mathbb{R}^8, \quad k_x, k_y, x, y \in \mathbb{R}, \quad t \geq 0,$$

(40)

where $s \in \mathbb{C}$ is to be determined and relates to the stability property of the PML model.

**Lemma 4** The PML model (38) is not stable if there are nontrivial solutions $u$ of the form (40) with $\text{Re}\{s\} > 0$.

An $s$ with a positive real part, $\text{Re}\{s\} > 0$, corresponds to a plane wave solution with exponentially growing amplitude. A stable system does not admit such wave modes.

We consider the normalised wave vector $K = (k_1, k_2)$, with $\sqrt{k_1^2 + k_2^2} = 1$ and the normalised variables

$$\lambda = \frac{s}{|k|}, \quad \epsilon = \frac{\sigma}{|k|}, \quad \nu = \frac{\alpha}{|k|}, \quad S_x(\lambda, \epsilon, \nu) = 1 + \frac{\epsilon}{\lambda + \nu}.$$

Thus, if there are $\text{Re}\{\lambda\} > 0$, the PML is unstable.

We insert the plane wave solution (40) in the PML and obtain the dispersion relation

$$F_\epsilon(\lambda, K) := F(\lambda, \frac{1}{S_x(\lambda, \epsilon, \nu)} k_1, k_2) = 0,$$

(41)

where the function $F(\lambda, K)$ is defined by (15) and (16). The scaled eigenvalue $\lambda$ is a root of the complicated nonlinear dispersion relation $F_\epsilon(\lambda, K)$ for the PML and defined in (41). When the PML damping vanishes, $\epsilon = 0$ we have $S_x = 1$, and $F_0(\lambda, K) \equiv F(\lambda, K)$. As shown in Section 3.2.1, the roots of $F(\lambda, K)$ are purely imaginary and correspond to the body wave modes propagating in a homogeneous elastic medium. When the PML damping is present $\epsilon > 0$, the roots $\lambda$ can be difficult to determine. However, standard perturbation arguments yield the following well-known result [20, 7, 4].

**Theorem 2 (Necessary condition for stability)** Consider the constant coefficient PML, with $\epsilon > 0, \nu \geq 0$. Let the elastic medium be described by the phase velocity $V_p = (V_{px}, V_{py})$ and the group velocity $V_g = (V_{gx}, V_{gy})$ defined in (19). If $V_{px}V_{gx} < 0$, then at sufficiently high frequencies, $|k| \to \infty$, there are corresponding unstable wave modes with $\text{Re}\{\lambda\} > 0$.

For the elastic subdomains $\Omega_i, i = 1, 2$, we will consider only media parameters where the necessary geometric stability condition, $V_{px}V_{gx} > 0$, is satisfied and there are no growing modes for the Cauchy PML problem. In particular, it can be shown for isotropic elastic materials that body wave modes inside the PML are asymptotically stable for all frequencies [19, 13]. In many anisotropic elastic materials the geometric stability condition and the complex frequency shift $\alpha > 0$ will ensure the stability of plane wave modes for all frequencies [4]. Next, we will characterise the stability of interface wave modes in the PML.

### 5.2 Stability analysis of interface wave modes

As above, we assume constant PML damping $\sigma \geq 0$ and piecewise constant elastic media parameters with a planar interface at $y = 0$. We Laplace transform (38)-(39) in time, perform a Fourier transformation in the spatial variable $x$ of (38)-(39) and eliminate all PML auxiliary variables. We have

$$\rho_i s^2 \dddot{\mathbf{u}}_i = -k_x^2 A_i \dddot{\mathbf{u}}_i + B_i \frac{d^2 \mathbf{u}_i}{dy^2} + ik_x \left( C_i + C_i^T \right) \frac{d \mathbf{u}_i}{dy}, \quad i = 1, 2,$$

(42)
where \( \tilde{k}_x = k_x/S_x \). The Laplace-Fourier transformed interface conditions are

\[
\tilde{u}_1 = \tilde{u}_2, \quad B_1 \frac{d\tilde{u}_1}{dy} + i\tilde{k}_x C_1^T \tilde{u}_1 = B_2 \frac{d\tilde{u}_2}{dy} + i\tilde{k}_x C_2^T \tilde{u}_2, \quad y = 0. \tag{43}
\]

Note the similarity between \((42) - (43)\) and \((21) - (22)\); the only difference is that we have replaced \(k_x\) with \(\tilde{k}_x\) and \(\phi_i\) with \(\bar{u}_i\). When the PML damping vanishes \(\sigma = 0\), we have \(S_x \equiv 1\) and \(\tilde{k}_x \equiv k_x\). In this case, the PML model \((42) - (43)\) is equivalent to the original equation \((21) - (22)\), and \((42)\) is the Laplace-Fourier transformations of equation \((1)\).

We seek modal solutions to \((42)\) in the form

\[
\tilde{u}_i = \Phi_i e^{\kappa y}, \quad \Phi_i \in \mathbb{C}^2, \quad i = 1, 2. \tag{44}
\]

Substituting \((44)\) into \((42)\), we obtain

\[
\left(s^2 I + \mathcal{P}_i(\tilde{k}_x, \kappa)\right) \Phi_i = 0, \quad i = 1, 2, \tag{45}
\]

where

\[
\mathcal{P}_i(\tilde{k}_x, \kappa) = \tilde{k}_x^2 A_i - \kappa^2 B_i - i\tilde{k}_x \kappa (C_i + C_i^T), \quad i = 1, 2.
\]

The existence of nontrivial solutions to \((45)\) requires that

\[
F_i \left(s, \tilde{k}_x, \kappa\right) := \det \left(s^2 I + \mathcal{P}_i(\tilde{k}_x, \kappa)\right) = 0, \quad i = 1, 2. \tag{46}
\]

As above, we note that if we set \(\kappa = i k_y\) in \(F_i \left(s, k_x, \kappa\right)\), we get exactly the same PML dispersion relation \((41)\) for the Cauchy problem. Again, note also the close similarity between \((25)\) and \((46)\).

The roots, \(\kappa = \tilde{\kappa}_{ij}^\pm\), of the characteristic function \(F_i \left(s, \tilde{k}_x, \kappa\right)\) are

\[
\tilde{\kappa}_{ij}^- (s, k_x) = \kappa_{ij}^- (s, \tilde{k}_x), \quad \tilde{\kappa}_{ij}^+ (s, k_x) = \kappa_{ij}^+ (s, \tilde{k}_x), \quad j = 1, 2.
\]

For the proceeding analysis to directly mirror the mode analysis discussed in section 3.2.2, we will need the sign consistency between \(\text{Re}\left\{ \kappa_{ij}^\pm \right\}\) and \(\text{Re}\left\{ \tilde{\kappa}_{ij}^\pm \right\}\). That is for \(\text{Re}\{s\} > 0, \sigma \geq 0\) and \(\alpha \geq 0\) we must have

\[
\text{sign}\left(\text{Re}\left\{ \kappa_{ij}^\pm \right\}\right) = \text{sign}\left(\text{Re}\left\{ \tilde{\kappa}_{ij}^\pm \right\}\right). \tag{47}
\]

The following lemma, which uses a standard continuity argument, was first proven in [13].

**Lemma 5** If the PML Cauchy problem has no temporally growing modes, then for all \(k_x \in \mathbb{R}\) and all \(s \in \mathbb{C}\) with \(\text{Re}\{s\} > 0\) the PML characteristic equation has roots \(\tilde{\kappa}_{ij}^\pm (s, k_x)\) with

\[
\text{sign}\left(\text{Re}\left\{ \kappa_{ij}^\pm \right\}\right) = \text{sign}\left(\text{Re}\left\{ \tilde{\kappa}_{ij}^\pm \right\}\right).
\]

**Proof 2** As above, we note that the roots vary continuously with \(s\). Thus, if the real part of a root \(\tilde{\kappa}_{ij}\) changes sign, then for some \(s\) with \(\text{Re}\{s\} > 0\) the root must be purely imaginary, \(\bar{\kappa}_{ij} = i k_y\). When \(\kappa_{ij} = i k_y\) the PML dispersion relations \((41)\) for the Cauchy problem and the characteristic \((46)\) are equivalent. Therefore a purely imaginary root \(\kappa_{ij} = i k_y\) with \(\text{Re}\{s\} > 0\) corresponds to an exponentially growing mode for the Cauchy PML problem, which contradicts the assumption that the Cauchy PML problem has no growing wave modes.
Taking into account the boundedness condition, the general solution of (42) is
\[
\tilde{u}_1(y) = \delta_{11} e^{\tilde{\kappa}_{11} y} \Phi_{11} + \delta_{12} e^{\tilde{\kappa}_{12} y} \Phi_{12}, \quad \tilde{u}_2(y) = \delta_{21} e^{\tilde{\kappa}_{21} y} \Phi_{21} + \delta_{22} e^{\tilde{\kappa}_{22} y} \Phi_{22}.
\] (48)
The coefficients \( \delta = [\delta_{11}, \delta_{12}, \delta_{21}, \delta_{22}]^T \) are determined by inserting (48) into the interface conditions (22). We have the following equation
\[
\mathcal{C}(s, \tilde{k}_x) \delta = 0,
\] (49)
where
\[
\mathcal{C}(s, \tilde{k}_x) = \begin{bmatrix}
\Phi_{11} & \Phi_{12} \\
(\tilde{\kappa}_{11} B_1 + i \tilde{k}_x C_1^T) \Phi_{11} & (\tilde{\kappa}_{12} B_1 + i \tilde{k}_x C_1^T) \Phi_{12} - \Phi_{21} & -\Phi_{21} \\
(\tilde{\kappa}_{21} B_2 + i \tilde{k}_x C_2^T) \Phi_{21} & -(\tilde{\kappa}_{21} B_2 + i \tilde{k}_x C_2^T) \Phi_{22} & -\Phi_{22}
\end{bmatrix}.
\] (50)
Using the determinant condition given in Definition 3 we formulate a stability condition for the PML in a piecewise constant elastic medium.

**Lemma 6 (Stability condition)** The solution to the PML model (42) with piecewise constant material parameters (3) and interface condition (43) is not stable in any sense if for some \( k_x \in \mathbb{R} \) and \( s \in \mathbb{C} \) with \( \text{Re}\{s\} > 0 \), the determinant vanishes,
\[
\mathcal{F}(s, \tilde{k}_x) := \det \left( \mathcal{C}(s, \tilde{k}_x) \right) = 0.
\]

The roots of \( \mathcal{F}(s, \tilde{k}_x) \) are tightly connected to the roots of \( \mathcal{F}(s, k_x) \) by the homogeneous property of \( \mathcal{F} \). As a consequence, it is enough to analyse the roots of \( \mathcal{F}(s, \tilde{k}_x) \) for the stability property of the PML model. Below, we define the homogeneous property, followed by a theorem for the PML stability.

**Definition 1** Let \( f(v) \) be a function with the vector argument \( v \). If \( f(\alpha v) = \alpha^n f(v) \) for all nonzero scalar \( \alpha \neq 0 \) and some \( n \in \mathbb{Z} \), then \( f(v) \) is homogeneous of degree \( n \).

**Theorem 3** Let \( \mathcal{F}(s, k_x) \) be a homogeneous function of degree \( n \in \mathbb{Z} \). Assume that \( \mathcal{F}(s, k_x) \neq 0 \) for all \( \text{Re}\{s\} \geq 0 \) and \( k_x \in \mathbb{R} \). Let \( \tilde{k}_x = k_x / S_x \), where \( S_x \) is the PML metric (34). Then the function \( \mathcal{F}(s, \tilde{k}_x) \) has no root \( s \) with positive real part, \( \text{Re}\{s\} > 0 \).

**Proof 3** Consider the homogeneous function \( \mathcal{F}(s, k_x) \), we have
\[
\mathcal{F}(s, \tilde{k}_x) = \mathcal{F} \left( s, \frac{k_x}{S_x} \right) = \mathcal{F} \left( \frac{sS_x}{S_x}, \frac{k_x}{S_x} \right) = \left( \frac{1}{S_x} \right)^n \mathcal{F} \left( sS_x, k_x \right).
\]
Since \( S_x \neq 0 \) and \( 1/S_x \neq 0 \), we must have
\[
\mathcal{F}(s, \tilde{k}_x) = 0 \iff \mathcal{F}(\tilde{s}, k_x) = 0, \quad \tilde{s} = sS_x.
\]
Assume that \( s = a + ib \) with \( a > 0 \), we have
\[
\text{Re}\{\tilde{s}\} = \left( a + \left( \frac{a(a + \alpha) + b^2}{s + \alpha^2} \right) \sigma \right) \geq a > 0.
\]
Thus if \( \mathcal{F}(\tilde{s}, k_x) = 0 \) then \( \tilde{s} \) with \( \text{Re}\{\tilde{s}\} > 0 \) is a root. This will contradict the assumption that \( \mathcal{F}(s, k_x) \neq 0 \) for all \( \text{Re}\{s\} > 0 \). We conclude that for \( s = a + ib \) with \( a > 0 \), we must have \( \mathcal{F}(\tilde{s}, k_x) \neq 0 \) for all \( \sigma \geq 0 \) and \( \alpha \geq 0 \).

To determine the homogeneity property of \( \mathcal{F}(s, k_x) = \det(\mathcal{C}(s, k_x)) \), we may evaluate the corresponding determinant of \( \mathcal{C}(s, k_x) \). We have the following result.
Theorem 4 In piecewise constant elastic media, the determinant $\mathcal{F}(s, k_x) = \det(\mathcal{C}(s, k_x))$ given in \cite{31} is homogeneous of degree two.

Proof 4 Consider the modified boundary matrix $\mathcal{C}_1(s, k_x)$ where we have multiplied the first two rows of $\mathcal{C}(s, k_x)$ by $s \neq 0$, that is

$$
\mathcal{C}_1(s, k_x) = \begin{bmatrix}
\Phi_{11} & s\Phi_{12} & -s\Phi_{21} & -s\Phi_{22} \\
(k_1 B_1 + ik_x C_1^T)\Phi_{11} & (k_2 B_1 + ik_x C_1^T)\Phi_{12} & -(k_1 B_2 + ik_x C_2^T)\Phi_{21} & -(k_2 B_2 + ik_x C_2^T)\Phi_{22}
\end{bmatrix}.
$$

By inspection, every element of $\mathcal{C}_1(s, k_x)$ is homogeneous of degree one. Therefore the determinant $\det(\mathcal{C}_1(s, k_x))$ of the $4 \times 4$ matrix $\mathcal{C}_1(s, k_x)$, using cofactor expansion, must be homogeneous of degree four. Note that

$$
\mathcal{C}_1(s, k_x) = \mathcal{X}(s)\mathcal{C}(s, k_x), \quad \mathcal{X}(s) = \begin{bmatrix}
s & 0 & 0 & 0 \\
0 & s & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}.
$$

Using the properties of the determinant of products of matrices we have

$$
\det(\mathcal{C}_1(s, k_x)) = \det(\mathcal{X}(s))\det(\mathcal{C}(s, k_x)) = s^2 \det(\mathcal{C}(s, k_x)) = s^2 \mathcal{F}(s, k_x).
$$

Since $\det(\mathcal{C}_1(s, k_x))$ is homogeneous of degree four, therefore the determinant $\mathcal{F}(s, k_x)$ is homogeneous of degree two.

We can now state the result that shows that exponentially growing waves modes are not supported by the PML in a discontinuous elastic medium.

Theorem 5 Consider the PML \cite{42} in a discontinuous elastic medium with the interface condition \cite{43} at $y = 0$. Let $\mathcal{F}(s, k_x)$ be the homogeneous function given in \cite{31}. If $\mathcal{F}(s, k_x) \neq 0$ for all $\Re\{s\} > 0$ and $k_x \in \mathbb{R}$ and the PML Cauchy problem has no temporally growing modes, then there are no growing interface wave modes in the PML. That is $\mathcal{F}(s, k_x) \neq 0$ for all $\Re\{s\} > 0$ and $k_x \in \mathbb{R}$.

Proof 5 The proof is identical to the proof of Theorem 3 with degree of homogeneity $n = 2$.

The following theorem states that interface wave modes are dissipated by the PML.

Theorem 6 Consider the PML \cite{42} in a discontinuous elastic medium with the interface condition \cite{43} at $y = 0$. If the PML Cauchy problem has no temporally growing modes then all stable interface wave modes that solve $\mathcal{F}(s, k_x) = 0$ for all $k_x \in \mathbb{R}$ with $s = i\xi$, are dissipated by the PML.

Proof 6 It suffices to prove that $\mathcal{F}(s, k_x) = 0$ implies $\Re\{s\} \leq 0$ for all $k_x \in \mathbb{R}$, $\alpha \geq 0$ and $\sigma \geq 0$. We will split the proof into two cases, for $\alpha = 0$ and $\alpha > 0$.

Consider

$$
\mathcal{F}(s, k_x) = 0 \iff \mathcal{F}(s, k_x) = 0, \quad s S_x = \frac{\alpha + s + \sigma}{\alpha + s} s, \quad \alpha, \sigma \geq 0.
$$

Since $\mathcal{F}(s, k_x)$ has purely imaginary roots $s_0 = i\xi$, we must have

$$
\frac{\alpha + s + \sigma}{\alpha + s} s = i\xi, \quad (52)
$$
for some $\xi \in \mathbb{R}$. Thus, if $\alpha = 0$, then $s = -\sigma + i\xi$ and $\text{Re}\{s\} = -\sigma < 0$, for $\sigma > 0$.

When $\alpha > 0$, we consider

$$\frac{\alpha + s + \sigma}{\alpha + s} s = i\xi \iff s^2 + (\alpha + \sigma - i\xi)s - i\alpha\xi = 0.$$ 

If $\xi = 0$, then the roots are $s = 0$ and $s = -(\alpha + \sigma) < 0$, for $\alpha > 0$, $\sigma > 0$. Clearly the real parts of the roots are non-positive. If $\xi \neq 0$, then the roots are given by

$$s = -\frac{(\alpha + \sigma - i\xi)}{2} \pm \frac{1}{2} \sqrt{(\alpha + \sigma - i\xi)^2 + 4\alpha\xi}.$$ 

The real parts of the two roots are

$$\text{Re}\{s\} = -\frac{(\alpha + \sigma)}{2} \pm \frac{1}{2\sqrt{2}} \sqrt{(\alpha + \sigma)^2 - \xi^2 + ((\alpha + \sigma)^2 - \xi^2)^2 + 4\xi^2(\alpha - \sigma)^2}. $$

We note that the root with a negative sign has a negative real part,

$$\text{Re}\{s\} = -\frac{(\alpha + \sigma)}{2} - \frac{1}{2\sqrt{2}} \sqrt{(\alpha + \sigma)^2 - \xi^2 + ((\alpha + \sigma)^2 - \xi^2)^2 + 4\xi^2(\alpha - \sigma)^2} < 0.$$ 

For the other root with a positive sign, we have

$$\text{Re}\{s\} = -\frac{(\alpha + \sigma)}{2} + \frac{1}{2\sqrt{2}} \sqrt{(\alpha + \sigma)^2 - \xi^2 + ((\alpha + \sigma)^2 - \xi^2)^2 + 4\xi^2(\alpha - \sigma)^2}. $$

If we assume that $\text{Re}\{s\} > 0$ for $\alpha > 0$, $\sigma > 0$ and $\xi \in \mathbb{R}$, then this implies that

$$(\alpha + \sigma) < \frac{1}{\sqrt{2}} \sqrt{(\alpha + \sigma)^2 - \xi^2 + ((\alpha + \sigma)^2 - \xi^2)^2 + 4\xi^2(\alpha - \sigma)^2}. $$

Squaring both sides of the inequality gives

$$(\alpha + \sigma)^2 + \xi^2 < \sqrt{((\alpha + \sigma)^2 - \xi^2)^2 + 4\xi^2(\alpha - \sigma)^2}. $$

Squaring both sides again and simplifying further yields

$$(\alpha + \sigma)^2 < (\alpha - \sigma)^2.$$ 

This is a contradiction since $\alpha > 0$ and $\sigma > 0$. Thus, for $\alpha > 0$ and $\sigma > 0$, we must have $\text{Re}\{s\} < 0$. The roots are moved further by the PML into the stable complex plane.

### 6 Numerical Experiments

In this section, we present extensive numerical examples to verify the stability analysis performed in the previous sections and demonstrate the absorption properties of the PML model for the elastic wave equation. We will consider a sequence of numerical experiments with increasing model complexities. We will end the section by simulating elastic wave scattering using the Marmourisi model [29] defined by heterogeneous and discontinuous elastic medium in a 2D domain $\Omega \subset \mathbb{R}^2$.

For the spatial discretisation, we use the SBP finite difference operators with fourth-order accurate interior stencil [30]. The boundary conditions and material interface conditions are imposed weakly by the penalty technique [13] [14] [10] such that a discrete energy estimate is obtained when the damping vanishes. For details on the SBP discretisation and stability for the undamped problem, we refer the reader to [14] [15]. We discretise in time using the classical fourth-order accurate Runge-Kutta method with stable explicit time steps, and small enough so that the error is dominated by spatial discretisation. The Julia code to solve all the examples can be found online on a Github Repository.\footnote{Github link to the repository: \url{https://github.com/Balaje/Summation-by-parts}}
6.1 Two Layers

We consider the elastic wave equation in a two-layered medium $\Omega_1 \cup \Omega_2$, where $\Omega_1 = [0, 4\pi] \times [-4\pi, 0]$ and $\Omega_2 = [0, 4\pi] \times [-4\pi, 0]$. The material properties in each layer are either isotropic or orthotropic. For the isotropic case, we use the material properties $\rho_1 = 1.5, \mu_1 = 4.86, \lambda_1 = 4.8629$ in $\Omega_1$, and $\rho_2 = 3, \mu_2 = 27, \lambda_2 = 26.9952$ in $\Omega_2$. For the orthotropic material property, we choose $\rho_1 = 1, c_{111} = 4, c_{121} = 3.8, c_{221} = 20$ and $c_{331} = 2$ in $\Omega_1$, and the material properties in $\Omega_2$ are chosen as $\rho_2 = 0.25$ and $c_{ij2} = 4c_{ij1}$ for $i, j = 1, 2$.

For initial conditions, we set the initial displacements as the Gaussian
centered at $(x_s, y_s) = (2\pi, 1.6\pi)$ and zero initial conditions for the velocity field and all auxiliary variables. We impose the characteristic boundary conditions at the left boundary $x = 0$, the bottom boundary $y = -4\pi$, and the top boundary $y = 4\pi$. Outside the right boundary, at $x = 4\pi$, we use a PML $[4\pi, 4.4\pi] \times [-4\pi, 4\pi]$ closed by the characteristic boundary condition at the PML boundaries. Because of the PML, the boundary conditions must be modified as

\[
Z_{1y} \frac{\partial u_1}{\partial t} + B_1 \frac{\partial u_1}{\partial y} + C_1^T \frac{\partial u_1}{\partial x} + B_1 \sigma w_1 + \sigma Z_{1y}(u_1 - q_1) = 0, \quad y = 4\pi, \tag{54}
\]

\[
Z_{ix} \frac{\partial u_i}{\partial t} - A_i \frac{\partial u_i}{\partial x} - C_i \frac{\partial u_i}{\partial y} + A_i \sigma v_1 = 0, \quad x = 0, \quad i = 1, 2, \tag{55}
\]

\[
Z_{ix} \frac{\partial u_i}{\partial t} + A_i \frac{\partial u_i}{\partial x} + C_i \frac{\partial u_i}{\partial y} - A_i \sigma v_1 = 0, \quad x = 4.4\pi, \quad i = 1, 2, \tag{56}
\]

\[
Z_{2y} \frac{\partial u_2}{\partial t} - B_2 \frac{\partial u_2}{\partial y} - C_2^T \frac{\partial u_2}{\partial x} - B_2 \sigma w_2 + \sigma Z_{2y}(u_2 - q_2) = 0, \quad y = -4\pi. \tag{57}
\]

The impedance matrices $Z_{ix}$ and $Z_{iy}$ are given by

\[
Z_{ix} = \begin{bmatrix}
\rho_i c_{pxi} & 0 \\
0 & \rho_i c_{sx}\end{bmatrix}, \quad Z_{iy} = \begin{bmatrix}
\rho_i c_{sxi} & 0 \\
0 & \rho_i c_{pyi}\end{bmatrix},
\]

and the wave speeds $c_{pxi}, c_{pyi}, c_{sx}, c_{syi}$ are defined in [5], for each layer.

The PML damping function is

\[
\sigma(x) = \begin{cases}
0 & \text{if } x \leq L_x, \\
\sigma_0 \left(\frac{x - L_x}{\delta}\right)^3 & \text{if } x \geq L_x,
\end{cases} \tag{58}
\]

where the damping strength is

\[
\sigma_0 = \frac{4c_{p,\text{max}}}{2\delta} \log \left(\frac{1}{Re f}\right). \tag{59}
\]

Here, $c_{p,\text{max}} = \max(c_{p1}, c_{p2})$, $c_{p1} = \max(c_{px1}, c_{py1})$ and $c_{p2} = \max(c_{px2}, c_{py2})$ are the maximum pressure wave speeds in $\Omega_1$ and $\Omega_2$, respectively. The parameter $L_x = 4\pi$ is the length of the domain, $\delta = 0.1L_x$ is the width of the PML and $Re f = 10^{-4}$ is the relative PML modeling error. Additionally, we choose the CFS parameter $\alpha = 0.05\sigma_0$ in both subdomains.

In Figures 2 and 3 we plot the numerical solutions at three time points for the isotropic and anisotropic media, respectively. In both cases, the initial data is a Gaussian in the top layer. At $t = 1$, we observe that a wave mode propagates at the same speed in the two spatial directions in the isotropic medium but at different speeds in the anisotropic medium. At $t = 2$, the elastic
Figure 2: The solution at three time points $t = 1, 2, 3$ in a piecewise isotropic medium. The red, solid curve denotes the interface between the two layers. The region to the right of the black, dashed line is the PML.

Figure 3: Solution at three time points $t = 1, 2, 5$ in a piecewise anisotropic, othrotropic medium. The red, solid curve denotes the interface between the two layers. The region to the right of the black, dashed line is the PML.
waves have propagated into the bottom layer in the middle panel, where the effects of discontinuity are observed. In the last panel, we observe that the waves coming into the PML are effectively absorbed without reflections.

In Figure 4, we plot the $l_2$-norm, $\|u\|_H = \sqrt{\sum_{i=1}^2 u_i^T H u_i}$, of the numerical solutions in time, where $i$ is the layer index and $H$ is the discrete norm associated with the SBP operator. We observe that $\|u\|_H$ decays monotonically in both the isotropic and anisotropic media. Note that at the final time $t = 100$ the largest amplitude is about $10^{-7}$, demonstrating the numerical stability and the effectiveness of PML.

Finally, we compare the absorbing property of the PML model with the first order absorbing boundary conditions (ABC) [28]. We compute a solution in the large domain $[0, 4\pi] \times [-4\pi, 12\pi]$, which is the original domain extended three times in the positive $x$ direction, and regard the part of the solution in $[0, 3.6\pi] \times [-4\pi, 4\pi]$ as a reference solution. As before, we consider the initial displacement (53) and zero initial conditions for the velocity and the auxiliary variables. In Figure 5, we plot the PML error defined as the maximum norm of the difference between the PML solution and the reference solution, and the ABC error that is defined analogously as the maximum norm of the difference between the solution computed by using the ABC on all boundaries and the reference solution. We observe that the PML error is about two order of magnitude smaller than the ABC error in both isotropic and anisotropic media.
Next we demonstrate extension of the results to multiple elastic layers. We consider the elastic wave equation in domain $\Omega = [0, 40] \times [-40, 0]$. The medium has a four-layered structure and the material parameters are summarized in Table 1. In each layer, the material property is homogeneous and isotropic. At the interface between two adjacent layers, the material property is discontinuous, and the equations are coupled by imposing continuity of displacement and traction in the form of (39).

At time $t = 0$, we initialise the displacement fields as the Gaussian

$$u_i = e^{-5((x-x_i)^2+(y-y_i)^2)}, \quad i = 1, 2, 3, 4,$$

centered at the middle of Layer 2, that is $(x_s, y_s) = (20, -15)$.

We impose a traction free boundary condition at the top boundary $y = 0$. At the left, bottom and right boundaries, we add a PML of width $\delta = 4$ which is 10% of the computational domain in $x$. The PML in the $y$-direction introduces a new auxiliary variable in the governing equation, see Appendix A. At the boundaries of the PML ($x = \{-4, 44\}$ and $y = -44$), we impose the characteristic boundary condition. Because of the PML, the boundary conditions must be modified as

$$B_i \frac{\partial u_1}{\partial y} + C_i^T \frac{\partial u_1}{\partial x} + \sigma_x B_1 w_1 = 0, \quad y = 0,$$

$$Z_{ix} \frac{\partial u_i}{\partial t} - A_i \frac{\partial u_i}{\partial x} - C_i \frac{\partial u_i}{\partial y} + A_i \sigma_x v_1 + \sigma_y Z_{ix}(u_i - q_i) = 0, \quad x = 0, \quad i = 1, 2, 3, 4,$$

$$Z_{ix} \frac{\partial u_4}{\partial t} + A_i \frac{\partial u_4}{\partial x} + C_i \frac{\partial u_4}{\partial y} - A_i \sigma_x v_1 + \sigma_y Z_{ix}(u_i - q_i) = 0, \quad x = 44, \quad i = 1, 2, 3, 4,$$

$$Z_{4y} \frac{\partial u_4}{\partial t} - B_4 \frac{\partial u_4}{\partial y} - C_4^T \frac{\partial u_4}{\partial x} - B_4 \sigma_x w_4 + \sigma_x Z_{4y}(u_4 - q_4) = 0, \quad y = -40.$$

More precisely, on the $y$-boundaries the modified traction includes the auxiliary variable $w$. In addition, the time derivative in the characteristic boundary condition introduces a lower order term, see (63). Similarly, on the $x$-boundaries, the modified traction includes the auxiliary variable $v$.

Inside the PML of all four layers, we choose the damping functions $\sigma_x(x)$, $\sigma_y(y)$ which are cubic monomials similar to (58) and the damping strength $\sigma_0 > 0$ is given by (59) with $Re f = 10^{-4}$ the relative PML modeling error. Here, $\sigma_{p, max} = \max_i c_{pi}$ is the largest pressure wave speed $c_{pi}$ in $\Omega_i$, $i = 1, 2, 3, 4$. Additionally, we choose the CFS parameter $\alpha = 0.05\sigma_0$.

We use the same spatial and temporal discretisation parameters as in the previous numerical example. In Figure 6 we plot the solutions at three time points with the grid size $h = 0.1$. We observe that at $t = 3$, the initial Gaussian displacement field has expanded from its centre across the top three layers and the reflections at the material interfaces are clearly visible. At $t = 5$, the wave has propagated across all four layers, and has interacted with the free surface, at $y = 0$, and

| Layer | $\rho$ | $c_s$ | $c_p$ | Domain |
|-------|-------|-------|-------|--------|
| 1     | 1.5   | 1.8   | 3.118 | $[0, 40] \times [-10, 0]$ |
| 2     | 1.9   | 2.3   | 3.984 | $[0, 40] \times [-20, -10]$ |
| 3     | 2.1   | 2.7   | 4.667 | $[0, 40] \times [-30, -20]$ |
| 4     | 3     | 3     | 5.196 | $[0, 40] \times [-40, -30]$ |

Table 1: Material properties in the four layers.
Figure 6: The absolute displacement $|u|$ at three time points $t = 3, 5, 9$ with Gaussian initial data and grid size $h = 0.1$. The horizontal red lines indicate the material interfaces.

Figure 7: The solution at three time points $t = 3, 5, 9$ with single point moment source and grid size $h = 0.1$. The horizontal red lines indicate the material interfaces.

the characteristic boundary condition, at $x = -4$. The plot at $t = 9$ shows that the surface and interface waves entering the PML is effectively absorbed without reflections.

Next, we consider an example driven by seismological sources, an explosive moment tensor point source $F = gM_0 \nabla f_\delta$, as the forcing in the governing equation. The moment time function $g$ and the approximated delta function $f_\delta$ take the form

$$g = e^{-\frac{(t-0.215)^2}{0.15}} , \quad f_\delta = \frac{1}{2\pi \sqrt{s_1 s_2}} e^{-\frac{(t-20)^2}{2s_1^2} - \frac{(y+15)^2}{2s_2^2}},$$

where the parameters $s_1 = s_2 = 0.5h$ and $M_0 = 1000$. We note that the peak amplitude of $F$ is located in the middle of Layer 2, that is at $(x, y) = (20, -15)$. With zero initial data for all variables, we run the simulation with $h = 0.1$ and plot the solutions in Figure 7. We have similar observation as the case with a Gaussian initial data.

To see the stability property of the PML, we plot $\|u\|_H$ in time in Figure 8. The first plot corresponds to the case with initial Gaussian data, and the second plot corresponds to the case with the single point moment source. It is clear that the PML remains stable after a long time $t = 1000$.

6.3 Curved material interface

In this section, we consider the elastic wave equation in a two-layered medium in $(x, y) \in [0, 4\pi] \times [-4\pi, 4\pi]$ with a curved material interface shown in Figure 9 (left). The two layers are separated by a smooth Gaussian hill centered at the midpoint of the domain that is parameterised by $y =$
Figure 8: The quantity $\|u\|_H$ with $h = 0.1$ for the Gaussian initial data (left) and the single point moment source (right).

$0.8\pi e^{-10(x-2\pi)^2}$. We consider isotropic media with the material properties $\rho_1 = 1.5, \mu_1 = 4.86, \lambda_1 = 4.8629$ in $\Omega_1$, and $\rho_2 = 3, \mu_2 = 27, \lambda_2 = 26.9952$ in $\Omega_2$.

We impose the characteristic boundary conditions on the left, bottom and the top boundary. We introduce a PML $[4\pi, 4.4\pi] \times [-4\pi, 4\pi]$ with the damping function (58)–(59), closed by the characteristic boundary condition at the PML boundaries. We set the initial displacements as the Gaussian centered at $(x_s, y_s) = (2\pi, 1.6\pi)$ and zero initial conditions for the velocity and auxiliary variables.

For the spatial discretisation, we construct a curvilinear grid in each subdomain that conforms with its boundaries by the transfinite interpolation technique. The governing PML equation is then transformed to the reference domain $[0, 1]^2$ and discretised, see the details of the coordinate transformation in Appendix B, and the reference [2]. We use a uniform reference grid of size $h$ in both layers, which results in a conforming grid interface, see the coarse versions of the finite difference grids on the physical domain in Figure 9. The numerical tests were performed on a $201 \times 201$ uniform grid on the reference domain $[0, 1]^2$.

In Figure 10, we show snapshots of the solution to the PML. We clearly observe the effect of the curved interface from the solution-snapshots at $t = 1$ and $t = 3$. At $t = 3$, we observe that the waves entering the PML are damped. We also show the long-time solution at $T = 100$ and observe that the largest amplitude is about $10^{-7}$, demonstrating the numerical stability of PML with a curved material interface. In Figure 11 we plot $\|u\|_H$ in time for both isotropic and orthotropic cases. We observe that $\|u\|_H$ decays monotonically in both cases. Finally, we compare PML with the absorbing boundary condition in Figure 12. We consider curvilinear, isotropic and orthotropic layered media with an interface conforming discretisation as shown in Figure 9. For the orthotropic material property, we choose $\rho_1 = 1, c_{111} = 4, c_{121} = 3.8, c_{221} = 20$ and $c_{331} = 2$ in $\Omega_1$, and the material properties in $\Omega_2$ are chosen as $\rho_2 = 0.25$ and $c_{ij2} = 4c_{ij1}$ for $i, j = 1, 2$. We observe that the maximum error of the PML solution is about two magnitudes smaller in both the isotropic and orthotropic case.

6.4 Marmousi Model

In this section, we consider the application of the PML to a heterogeneous and discontinuous elastic solid. We use Marmousi2 [29], a large geological dataset based upon the geology from the North Quenguela Trough in the Quanza Basin of Angola. The Marmousi2 is a fully elastic
Figure 9: (Left) Geometry of the two-layered media along with the interface (shown in red, solid line) and the PML boundary (shown in black, dashed line). Coarse versions of an interface-conforming finite difference grid (right).

Figure 10: The solution at three time points $t = 1, 3, 100$ in a piecewise isotropic medium. The red, solid curve denotes the interface between the two layers. The region to the right of the black, dashed line is the PML.
Figure 11: The quantity $\|u\|_H$ with $t$ for the conforming finite difference grid for the isotropic case (left) and the orthotropic case (right).

Figure 12: The maximum error for the isotropic (left) and the orthotropic (right) curvilinear media. ABC is the absorbing boundary condition.
model which contains the density of the material and the $p$- and $s$-wave speeds. In Figure 13, we show the plots of the $p$- and $s$-wave speeds and the density in the material without the water column. The material is highly heterogeneous containing various structural elements including water, water-wet sand channels, oil-gas mixtures, salt and shale. The full computational model is defined on a rectangular domain $[0, 16.9864] \times [-3.4972, -0.44964]$ sampled on a $13601 \times 2801$ uniform grid. Since the primary goal of the section is to demonstrate the stability of the PML, to save computational cost we consider the down-sampled version of the model on $1201 \times 241$ uniform grid.

We divide the domain into two layers by introducing an interface on a naturally-occurring
discontinuity in the material. We map each layer to a reference grid with 1201 × 201 points on Layer 1 and 1201 × 41 points on Layer 2. We consider the PML damping functions σ(x), which is a cubic monomial similar to (58) with the PML width δ = 0.1L, where L = 16.9864 is the length of the domain. We consider the explosive moment tensor point sources \( F = gM_0 \nabla f_\delta \) as the forcing in the governing equation. The function g and the approximated delta function \( f_\delta \) is given by

\[
g = e^{-\frac{(t-0.215)^2}{0.15}}, \quad f_\delta = \frac{1}{2\pi s_1 s_2} \sum_{i=1}^{3} e^{-\frac{(x-x_i)^2 + (y-y_i)^2}{2s_1^2 + 2s_2^2}},
\]

where the parameters \( s_1, s_2 = 0.5\delta \), half the grid-spacing in the physical domain, and \( M_0 = 1000 \). We apply the forcing at three different locations \((x_1, y_1) = (2.54796, -1.04916), (x_2, y_2) = (8.4932, -1.04916)\) and \((x_3, y_3) = (14.43844, -1.04916)\). We assume zero initial conditions for the displacement, velocity and the auxiliary variables. As the number of unknowns is relatively large, \( \sim 3.5 \cdot 10^6 \), we solve till the final time \( T = 10 \). At the top boundary \((y = -0.44964)\), we use a traction-free boundary condition and on the left \((x = 0)\), bottom \((y = -3.4972)\) and right \((x = 16.9864)\) boundaries, we use the characteristic boundary condition.

In Figure 14, we show the snapshots of the solution at four different times. We observe the scattering of elastic waves due to the heterogeneous and discontinuous nature of the medium. At \( t = 0.5 \), we observe that the waves reach the top boundary and gets reflected due to the traction-free boundary condition. At \( t = 1.0 \), we observe the effects of the interface as the waves reach the second layer. On the right-hand side of the domain, we see that the waves near the PML gets damped. From the snapshots at \( t = 1.5 \) and \( t = 10 \), we observe that the waves generated due to the three point-sources interact with each other and then subsequently reach the PML. We observe more clearly that the waves reaching the PML gets damped. In Figure 15, we observe that \( \|u\|_H \) decays monotonically with respect to \( t \), indicating the stability of the PML.

7 Conclusion

We have analysed the stability of the PML for the elastic wave equation with piecewise constant material parameters and interface conditions at material interfaces. The elastic wave equation and the interface conditions, without the PML, satisfy an energy estimate in physical space. Alternatively, a mode analysis can also be used to prove that exponentially growing modes are not supported by the elastic wave equation subject to the interface conditions. In particular, the normal mode analysis in Laplace space for interface waves gives a boundary matrix \( \mathcal{C}(s, k_x) \) of which the determinant is a homogeneous function \( \mathcal{F}(s, k_x) \) of \((s, k_x)\) and does not have any roots \( s \) with a positive real part \( \text{Re}\{s\} > 0 \) in the complex plane. When the PML is present, the energy method is in general not applicable but the normal mode analysis can be used to investigate the existence of exponentially growing modes in the PML. The normal mode analysis when applied to the PML in a discontinuous elastic medium yields a similar boundary matrix perturbed by the PML. Our analysis shows that if the PML IVP does not support growing modes, then the PML moves the roots of the determinant \( \mathcal{F}(s, k_x) \) further into the stable complex plane. This proves that interface wave modes present at layered material interfaces in elastic solids are dissipated by the PML. Our analysis builds upon the ideas presented in [13] and extends the stability results of boundary waves (such as Rayleigh waves) on a half-plane elastic solid to interface wave modes (such as Stoneley waves) transmitted into the PML at a planar interface separating two half-plane elastic solids. We have presented numerical examples for both isotropic and anisotropic elastic solids verifying the analysis, and demonstrating that interface wave modes decay in the PML. Numerical examples
Figure 14: (Top to bottom) Snapshots of the numerical solution in a heterogeneous and discontinuous elastic solid defined by the Marmoursi model and truncated by PML model. The solutions are plotted at times $t = 0.5, 1, 1.5$ and 10.
using the Marmousi model [29] demonstrates the utility of the PML and our numerical method for seismological applications.

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**A Perfectly Matched Layers in both spatial directions**

To derive the PML model in both spatial directions, we consider the Laplace transformed elastic wave equation (33) in the transformed coordinate \((\tilde{x}, \tilde{y})\), such that

\[
\frac{d\tilde{x}}{dx} = 1 + \frac{\sigma_x(x)}{\alpha + s} =: S_x, \quad \frac{d\tilde{y}}{dy} = 1 + \frac{\sigma_y(y)}{\alpha + s} =: S_y.
\]

Here \(\sigma_x(x) \geq 0\) and \(\sigma_y(y) \geq 0\) are the damping functions and \(\alpha \geq 0\) is the complex frequency shift. By introducing a smooth coordinate transform

\[
\frac{\partial}{\partial x} \rightarrow \frac{1}{S_x} \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y} \rightarrow \frac{1}{S_y} \frac{\partial}{\partial y},
\]

the PML model in the Laplace space becomes

\[
s^2 \rho_i \tilde{\mathbf{u}}_i = \frac{1}{S_x} \frac{\partial}{\partial x} \left( \frac{1}{S_x} A_i \frac{\partial \tilde{\mathbf{u}}_i}{\partial x} \right) + \frac{1}{S_y} \frac{\partial}{\partial y} \left( \frac{1}{S_y} B_i \frac{\partial \tilde{\mathbf{u}}_i}{\partial y} \right) + \frac{1}{S_x S_y} \frac{\partial}{\partial x} \left( C_i \frac{\partial \tilde{\mathbf{u}}_i}{\partial y} \right) + \frac{1}{S_x S_y} \frac{\partial}{\partial y} \left( C_i^T \frac{\partial \tilde{\mathbf{u}}_i}{\partial x} \right),
\]

\[
i = 1, 2,
\]

along with the interface conditions

\[
\tilde{\mathbf{u}}_1 = \tilde{\mathbf{u}}_2, \quad B_1 \frac{1}{S_y} \frac{\partial \tilde{\mathbf{u}}_1}{\partial y} + C_i^T \frac{1}{S_x} \frac{\partial \tilde{\mathbf{u}}_1}{\partial x} = B_2 \frac{1}{S_y} \frac{\partial \tilde{\mathbf{u}}_2}{\partial y} + C_i^T \frac{1}{S_x} \frac{\partial \tilde{\mathbf{u}}_2}{\partial x}.
\]
Introducing the auxiliary variables

\[ \hat{v}_i = \frac{1}{\alpha + s + \sigma_x} \frac{\partial \hat{u}_i}{\partial x}, \quad \hat{w}_i = \frac{1}{\alpha + s + \sigma_y} \frac{\partial \hat{u}_i}{\partial y}, \quad \hat{q}_i = \frac{\alpha}{\alpha + s} \hat{u}_i, \quad \hat{r}_i = \frac{\alpha}{\alpha + s} (\hat{u}_i - \hat{q}_i) \]

and inverting the Laplace transformed PDE, we obtain the system

\[ \rho \frac{\partial^2 u_i}{\partial t^2} + (\sigma_x + \sigma_y) \frac{\partial u_i}{\partial t} - \alpha (\sigma_x + \sigma_y) (u_i - q_i) + \sigma_x \sigma_y (u_i - q_i - r_i) = \]

\[ \rho \frac{\partial^2 u_i}{\partial t^2} + (\sigma_x + \sigma_y) \frac{\partial u_i}{\partial t} - \alpha (\sigma_x + \sigma_y) (u_i - q_i) + \sigma_x \sigma_y (u_i - q_i - r_i) = \]

\[ \frac{\partial}{\partial x} \left( A_i \frac{\partial u_i}{\partial x} + (\sigma_y - \sigma_x) A_i v_i + C_i \frac{\partial u_i}{\partial y} \right) + \frac{\partial}{\partial y} \left( B_i \frac{\partial u_i}{\partial y} + (\sigma_x - \sigma_y) B_i w_i + C_i^T \frac{\partial u_i}{\partial x} \right), \quad (66a) \]

\[ \frac{\partial v_i}{\partial t} = \frac{\partial u_i}{\partial x} - (\alpha + \sigma_x) v_i, \quad (66b) \]

\[ \frac{\partial w_i}{\partial t} = \frac{\partial u_i}{\partial y} - (\alpha + \sigma_y) w_i, \quad (66c) \]

\[ \frac{\partial q_i}{\partial t} = \alpha (u_i - q_i), \quad (66d) \]

\[ \frac{\partial r_i}{\partial t} = \alpha (u_i - q_i - r_i). \quad (66e) \]

along with the interface conditions

\[ B_1 \frac{\partial u_1}{\partial y} + (\sigma_x - \sigma_y) B_1 w_1 + C_1^T \frac{\partial u_1}{\partial x} = B_2 \frac{\partial u_2}{\partial y} + (\sigma_x - \sigma_y) B_2 w_2 + C_2^T \frac{\partial u_2}{\partial x}. \quad (67) \]

Setting \( \sigma_y \equiv 0 \) reduces (66)–(67) to the PML equations discussed in Section 4. We note that adding the PML along the \( y \)-direction introduces (66e) coupled with (66a).

### B Elastic wave equation on curvilinear domains

Let us assume that the PML model (38) is defined on curvilinear domains. We introduce a smooth transformation \((q(x, y), r(x, y)) \leftrightarrow (x(q, r), y(q, r))\), for each block, where a point \((q, r)\) in the reference domain is mapped to a point \((x, y)\) in the physical domain. We denote by \( J \) and \( J^{-1} \), the Jacobian matrix and its inverse, defined as

\[ J = \begin{bmatrix} \frac{\partial x}{\partial q} & \frac{\partial y}{\partial q} \\ \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \end{bmatrix}, \quad J^{-1} = \begin{bmatrix} \frac{\partial q}{\partial x} & \frac{\partial r}{\partial x} \\ \frac{\partial q}{\partial y} & \frac{\partial r}{\partial y} \end{bmatrix}. \]

We apply the transformation

\[ \frac{\partial}{\partial x} = \frac{\partial}{\partial q} \left( \frac{\partial q}{\partial x} \right) + \frac{\partial}{\partial r} \left( \frac{\partial r}{\partial x} \right), \quad \frac{\partial}{\partial y} = \frac{\partial}{\partial q} \left( \frac{\partial q}{\partial y} \right) + \frac{\partial}{\partial r} \left( \frac{\partial r}{\partial y} \right), \]
and write the elastic wave equation in the reference coordinates as follows.

\[
|J| \rho_i \left( \frac{\partial^2 \mathbf{u}_i}{\partial t^2} + (\sigma_x + \sigma_y) \frac{\partial \mathbf{u}_i}{\partial q} - \sigma_x \alpha (\mathbf{u}_i - \mathbf{q}_i) + \sigma_y \alpha (\mathbf{u}_i - \mathbf{q}_i) \right) = \frac{\partial}{\partial q} \left( \hat{A}_i \frac{\partial \mathbf{u}_i}{\partial q} + \hat{C}_i \frac{\partial \mathbf{u}_i}{\partial r} + \hat{A}_i \mathbf{v}_i \right) + \frac{\partial}{\partial r} \left( \hat{B}_i \frac{\partial \mathbf{u}_i}{\partial r} + \hat{C}_i^T \frac{\partial \mathbf{u}_i}{\partial q} + \hat{B}_i \mathbf{w}_i \right),
\]

where

\[
\begin{bmatrix}
\hat{A}_i & \hat{C}_i \\
\hat{C}_i^T & \hat{B}_i 
\end{bmatrix} = |J| \begin{bmatrix} J^{-1} & 0 \\ 0 & J^{-1} \end{bmatrix} \begin{bmatrix} A_i & C_i \\ C_i^T & B_i \end{bmatrix} \begin{bmatrix} J^{-1} & 0 \\ 0 & J^{-1} \end{bmatrix}, \quad \hat{A}_i = |J|(\sigma_y - \sigma_x)J^{-1}A_i,
\]

\[
\hat{B}_i = |J|(\sigma_x - \sigma_y)J^{-1}B_i.
\]

The stress tensor \( \tau_i \) and the outward normal \( \mathbf{n} \) on the boundaries can be written as

\[
\tau_i = \begin{bmatrix} A_i \left( \frac{\partial \mathbf{u}_i}{\partial q} \frac{\partial q}{\partial x} + \frac{\partial \mathbf{u}_i}{\partial r} \frac{\partial r}{\partial x} \right) + C_i \left( \frac{\partial \mathbf{u}_i}{\partial q} \frac{\partial q}{\partial y} + \frac{\partial \mathbf{u}_i}{\partial r} \frac{\partial r}{\partial y} \right) + (\sigma_y - \sigma_x)A_i \mathbf{v}_i \\
C_i^T \left( \frac{\partial \mathbf{u}_i}{\partial q} \frac{\partial q}{\partial x} + \frac{\partial \mathbf{u}_i}{\partial r} \frac{\partial r}{\partial x} \right) + B_i \left( \frac{\partial \mathbf{u}_i}{\partial q} \frac{\partial q}{\partial y} + \frac{\partial \mathbf{u}_i}{\partial r} \frac{\partial r}{\partial y} \right) + (\sigma_x - \sigma_y)B_i \mathbf{w}_i \end{bmatrix}, \quad \mathbf{n} = \frac{J^{-1} \hat{\mathbf{n}}}{|J^{-1} \hat{\mathbf{n}}|}.
\]

where \( \hat{\mathbf{n}} \) is the corresponding outward normal in the reference domain. Let \( \mathbf{n}_{\Gamma 1} = -\mathbf{n}_{\Gamma 2} = \mathbf{n}_\Gamma \) denote the outward normal of the interface boundary \( \Gamma \) in the physical domain. The interface condition enforcing the continuity of traction is as follows

\[
\tau_1 \cdot \mathbf{n} = \tau_2 \cdot \mathbf{n}.
\]

Without loss of generality, let us assume that the interface is mapped to the side \( r = 0 \), where \( \hat{\mathbf{n}}_1 = -\hat{\mathbf{n}}_2 = [0, -1] \). Now in terms of the reference coordinates, the traction on the interface \( \Gamma \) satisfies

\[
\frac{1}{|J||J^{-1} \hat{\mathbf{n}}_1|} \left( \hat{B}_1 \frac{\partial \mathbf{u}_1}{\partial r} + \hat{C}_1^T \frac{\partial \mathbf{u}_1}{\partial q} + \hat{B}_1 \mathbf{w}_1 \right) = \frac{1}{|J||J^{-1} \hat{\mathbf{n}}_2|} \left( \hat{B}_2 \frac{\partial \mathbf{u}_2}{\partial r} + \hat{C}_2^T \frac{\partial \mathbf{u}_2}{\partial q} + \hat{B}_2 \mathbf{w}_2 \right).
\]

The displacement continuity remains unchanged and can be written as

\[
\mathbf{u}_1 = \mathbf{u}_2 \quad \text{on} \quad \Gamma.
\]

Now we have a formulation where the derivatives are in terms of the reference coordinates \( q \) and \( r \). We can now discretise the PDE using the summation-by-parts operators defined on a unit square and solve for the displacements.
References

[1] Keiiti Aki and Paul G. Richards. *Quantitative Seismology*. University Science Books, 2 edition, 2002.

[2] M. Almquist and E. M. Dunham. Elastic wave propagation in anisotropic solids using energy-stable finite differences with weakly enforced boundary and interface conditions. *J. Comput. Phys.*, page 109842, 2021.

[3] Daniel Appelö and Tim Colonius. A high-order super-grid-scale absorbing layer and its application to linear hyperbolic systems. *Journal of Computational Physics*, 228(11):4200–4217, 2009.

[4] Daniel Appelö and Gunilla Kreiss. A new absorbing layer for elastic waves. *Journal of Computational Physics*, 215(2):642–660, 2006.

[5] Daniel H. Baffet, Marcus J. Grote, Sébastien Imperiale, and Maryna Kachanovska. Energy decay and stability of a perfectly matched layer for the wave equation. *Journal of Scientific Computing*, 81(3):2237–2270, 2019.

[6] Jean-Pierre Berenger. A perfectly matched layer for the absorption of electromagnetic waves. *Journal of Computational Physics*, 114(2):185–200, 1994.

[7] E. Bécache, S. Fauqueux, and P. Joly. Stability of perfectly matched layers, group velocities and anisotropic waves. *Journal of Computational Physics*, 188(2):399–433, 2003.

[8] Eliane Bécache and Maryna Kachanovska. Stability and convergence analysis of time-domain perfectly matched layers for the wave equation in waveguides. *SIAM Journal on Numerical Analysis*, 59(4):2004–2039, 2021.

[9] L. Cagnaird. Reflection and refraction of progressive seismic waves: by L. Cagnaird, Translated from the French and revised by E. A. Flinn and C. H. Dix. McGraw-Hill Book Co. Inc. 1962. *Bulletin of the Seismological Society of America*, 53(1):217–218, 01 1963.

[10] M. H. Carpenter, D. Gottlieb, and S. Abarbanel. Time–stable boundary conditions for finite–difference schemes solving hyperbolic systems: methodology and application to high–order compact schemes. *Journal of Computational Physics*, 111:220–236, 1994.

[11] Weng Cho Chew and William H. Weedon. A 3d perfectly matched medium from modified maxwell’s equations with stretched coordinates. *Microwave and Optical Technology Letters*, 7(13):599–604, 1994.

[12] J. Diaz and P. Joly. A time domain analysis of PML models in acoustics. *Computer Methods in Applied Mechanics and Engineering*, 195(29):3820–3853, 2006.

[13] K. Duru and G. Kreiss. Boundary waves and stability of the perfectly matched layer for the two space dimensional elastic wave equation in second order form. *SIAM Journal on Numerical Analysis*, 52:2883–2904, 2014.

[14] K. Duru, G. Kreiss, and K. Mattsson. Stable and high–order accurate boundary treatments for the elastic wave equation on second–order form. *SIAM Journal on Scientific Computing*, 36:A2787–A2818, 2014.
[15] K. Duru and K. Virta. Stable and high order accurate difference methods for the elastic wave equation in discontinuous media. *Journal of Computational Physics*, 279:37–62, 2014.

[16] Kenneth Duru. A perfectly matched layer for the time-dependent wave equation in heterogeneous and layered media. *Journal of Computational Physics*, 257:757–781, 2014.

[17] Kenneth Duru, Alice-Agnes Gabriel, and Gunilla Kreiss. On energy stable discontinuous Galerkin spectral element approximations of the perfectly matched layer for the wave equation. *Computer Methods in Applied Mechanics and Engineering*, 350:898–937, 2019.

[18] Kenneth Duru, Jeremy E. Kozdon, and Gunilla Kreiss. Boundary conditions and stability of a perfectly matched layer for the elastic wave equation in first order form. *Journal of Computational Physics*, 303:372–395, 2015.

[19] Kenneth Duru and Gunilla Kreiss. On the accuracy and stability of the perfectly matched layer in transient waveguides. *Journal of Scientific Computing*, 53:642–671, 2012.

[20] Kenneth Duru and Gunilla Kreiss. A well-posed and discretely stable perfectly matched layer for elastic wave equations in second order formulation. *Communications in Computational Physics*, 11(5):1643–1672, 2012.

[21] Kenneth Duru and Gunilla Kreiss. Numerical interaction of boundary waves with perfectly matched layers in two space dimensional elastic waveguides. *Wave Motion*, 51(3):445–465, 2014.

[22] Kenneth Duru and Gunilla Kreiss. The perfectly matched layer (PML) for hyperbolic wave propagation problems: A review. *arXiv:2201.03733*, 2022.

[23] Kenneth Duru, Leonhard Rannabauer, Alice-Agnes Gabriel, Gunilla Kreiss, and Michael Bader. A stable discontinuous Galerkin method for the perfectly matched layer for elastodynamics in first order form. *Numerische Mathematik*, 146(4):729–782, 2020.

[24] B. Gustafsson, H. O. Kreiss, and J. Oliger. *Time–Dependent Problems and Difference Methods*. John Wiley & Sons, 2013.

[25] H. O. Kreiss, O. E. Ortiz, and N. A. Petersson. Initial–boundary value problems for second order systems of partial differential equations. *ESAIM–Mathematical Modelling and Numerical Analysis*, 46:559–593, 2012.

[26] M. Kuzuoglu and R. Mittra. Frequency dependence of the constitutive parameters of causal perfectly matched anisotropic absorbers. *IEEE Microwave and Guided Wave Letters*, 6(12):447–449, 1996.

[27] W. M. Lai, D. Rubin, and E. Krempl. *Introduction to Continuum Mechanics*. Butterworth-Heinemann, 1993.

[28] John Lysmer and Roger L. Kuhlemeyer. Finite dynamic model for infinite media. *Journal of the Engineering Mechanics Division*, 95(4):859–877, 1969.

[29] Gary S. Martin, Robert Wiley, and Kurt J. Marfurt. Marmousi2: An elastic upgrade for Marmousi. *The Leading Edge*, 25(2):156–166, 2006.

[30] K. Mattsson and J. Nordström. Summation by parts operators for finite difference approximations of second derivatives. *Journal Computational Physics*, 199:503–540, 2004.
[31] Lord Rayleigh. On waves propagated along the plane surface of an elastic solid. *Proceedings of the London Mathematical Society*, s1-17(1):4–11, 1885.

[32] Robert Stoneley and Henry Frederick Baker. Elastic waves at the surface of separation of two solids. *Proceedings of the Royal Society of London. Series A, Containing Papers of a Mathematical and Physical Character*, 106(738):416–428, 1924.