Unconditionally Secure Quantum Key Distribution
In Higher Dimensions

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Abstract—In search of a quantum key distribution scheme that could stand up for more drastic eavesdropping attack, I discover a prepare-and-measure scheme using $N$-dimensional quantum particles as information carriers where $N$ is a prime power. Using the Shor-Preskill-type argument, I prove that this scheme is unconditional secure against all attacks allowed by the laws of quantum physics. Incidentally, for $N = 2^n > 2$, each information carrier can be replaced by $n$ entangled qubits. And in this case, I discover an eavesdropping attack on which no unentangled-qubit-based prepare-and-measure quantum key distribution scheme known to date can generate a provably secure key. In contrast, this entangled-qubit-based scheme produces a provably secure key under the same eavesdropping attack whenever $N \geq 16$. This demonstrates the advantage of using entangled particles as information carriers to combat certain eavesdropping strategies.

Index Terms—Entanglement purification, local quantum operation, phase error correction, quantum key distribution, Shor-Preskill proof, two way classical communication, unconditional security

I. INTRODUCTION

Key distribution is the art of sharing a secret key between two cooperative players Alice and Bob in the presence of an eavesdropper Eve. If Alice and Bob distribute their key by exchanging classical messages only, Eve may at least in principle wiretap their conversations without being caught. So, given unlimited computational resources, Eve can crack the secret key. In contrast, in any attempt to distinguish between two non-orthogonal states, information gain is only possible at the expenses of disturbing the state [1]. Therefore, if Alice and Bob distribute their secret key by sending non-orthogonal quantum signals, any eavesdropping attempt will almost surely affect their signal fidelity. Consequently, a carefully designed quantum key distribution (QKD) scheme allows Alice and Bob to accurately determine the quantum channel error rate, which in turn reflects the eavesdropping rate. If the estimated quantum channel error rate is too high, Alice and Bob abort the scheme and start all over again. Otherwise, they perform certain privacy amplification procedures to distill out an almost perfectly secure key [2], [3], [4], [5], [6]. Therefore, it is conceivable that a provably secure QKD scheme exists even when Eve has unlimited computational power.

With this belief in mind, researchers proposed many QKD schemes [6]. These schemes differ in many ways such as the Hilbert space dimension of the quantum particles used, as well as the states and bases Alice and Bob prepared and measured. The first QKD scheme, commonly known as BB84, was invented by Bennett and Brassard [7]. In BB84, Alice randomly and independently prepares each qubit in one of the following four states: $|0\rangle$, $|1\rangle$ and $(|0\rangle \pm |1\rangle)/\sqrt{2}$, and sends them to Bob. Upon reception, Bob randomly and independently measures each qubit in either the $(|0\rangle, |1\rangle)$ or $(|0\rangle \pm |1\rangle)/\sqrt{2}$ bases [7]. In short, BB84 is an experimentally feasible prepare-and-measure scheme involving the transfer of unentangled qubits [6]. Later, Bruß introduced another experimentally feasible prepare-and-measure scheme known as the six-state scheme [8]. In her scheme, Alice randomly and independently prepares each qubit in one of the following six states: $|0\rangle$, $|1\rangle$, $(|0\rangle \pm |1\rangle)/\sqrt{2}$ and $(|0\rangle \pm i|1\rangle)/\sqrt{2}$, and Bob measures each of them randomly and independently in the following three bases: $(|0\rangle, |1\rangle)$, $(|0\rangle \pm |1\rangle)/\sqrt{2}$ and $(|0\rangle \pm i|1\rangle)/\sqrt{2}$. Although the six-state scheme is more complex and generates a key less efficiently, Bruß found that it tolerates higher noise level than BB84 if Eve attacks each qubit individually [8]. In addition to qubit-based schemes such as BB84 and the six-state scheme, a number of QKD schemes involving higher dimensional as well as continuous systems have been proposed [9], [10], [11], [12], [13], [14], [15], [16].

Are these QKD schemes really secure? Is it really true that the six-state scheme tolerates higher error level than BB84? The answers to these questions turn out to be highly non-trivial. Recall that the all powerful Eve may choose to attack the transmitted qubits collectively by applying a unitary operator to entangle these qubits with her quantum particles. In this situation, most of our familiar tools such as law of large numbers and classical probability theory do not apply to the resultant highly entangled non-classical state. These make rigorous cryptanalysis of BB84 and the six-state schemes extremely difficult.

In spite of these difficulties, a few air-tight security proofs against all possible eavesdropping attacks for BB84 and the six-state scheme have been discovered. Rigorous proofs for QKD schemes with better error tolerance capability are also found. After a few years of work, Mayers [4] and Biham et al. [18] eventually proved the security of BB84 against all kinds of attack allowed by the known laws of quantum physics. In particular, Mayers showed that in BB84 a provably secure key can be generated whenever the channel bit error rate is less than about 7% [4]. A precise definition of bit error rate can be found in Def. 4 in Subsection IV-A. Along a different line, Lo and Chau [3] proved the security of an entanglement-based QKD scheme that applies up to 1/3 bit error rate by means of a random hashing technique based on entanglement purification.
Their security proof is conceptually simple and appealing. Nevertheless, their scheme requires quantum computers and hence is not practical at this moment. By ingenuously combining the essence of Mayers and Lo-Chau proofs, Shor and Preskill gave a security proof of BB84 that applies up to 11.0% bit error rate \[22\]. This is a marked improvement over the 7% bit error tolerance rate in Mayers’ proof. Since then, the Shor-Preskill proof became a blueprint for the cryptanalysis of many QKD schemes. For instance, Lo [21] as well as Gottesman and Lo [22] extended it to cover the six-state QKD scheme. At the same time, the work of Gottesman and Lo also demonstrates that careful use of local quantum operation plus two way classical communication (LOCC2) increases the error tolerance rate of QKD [22]. Furthermore, they found that the six-state scheme tolerates a higher bit error rate than BB84 because the six-state scheme gives better estimates for the three Pauli error rates \[22\]. In search of a qubit-based QKD scheme that tolerates higher bit error rate, Chau recently discovered an adaptive entanglement purification procedure inspired by the technique used by Gottesman and Lo in Ref. [22]. He further gave a Shor-Preskill-based proof showing that this adaptive entanglement purification procedure allows the six-state scheme to generate a provably secure key up to a bit error rate of \((5 - \sqrt{5})/10 \approx 27.6\%\) [23], making it the most error-tolerant prepare-and-measure scheme involving unentangled qubits to date.

Unlike various qubit-based QKD schemes, a rigorous security proof against the most general type of eavesdropping attack on a QKD scheme involving higher dimensional quantum systems is lacking. Besides, the error tolerance capability for this kind of QKD schemes against the most general eavesdropping attack is virtually unexplored. In fact, almost all relevant cryptanalysis focus on individual particle attack; and they suggest that QKD schemes involving higher dimensional systems may be more error-tolerant [13], [14], [15], [17]. It is, therefore, instructive to give air-tight security proofs and analyze the error tolerance capability for this type of schemes.

In this paper, I analyze the security and error tolerance capability of a prepare-and-measure QKD scheme involving the transmission of higher dimensional quantum systems. In fact, this scheme makes use of \(N\)-dimensional quantum states prepared and measured randomly in \((N + 1)\) different bases. Because of the randomization of bases, the probabilities of certain kinds of quantum errors in the transmitted signal are correlated. This makes the error estimation effective and hence the error tolerance rate high. Nonetheless, the high error tolerance rate comes with a price, namely, that the efficiency of the scheme is lowered.

In QKD, we assume that Alice and Bob have access to two communication channels. The first one is an insecure noisy quantum channel. The other one is an unjammable noiseless authenticated classical channel in which everyone, including Eve, can listen to but cannot alter the content passing through it. We also assume that Alice and Bob have complete control over the apparatus in their own laboratories; and everything outside their laboratories except the unjammable classical channel may be manipulated by the all powerful Eve. We further make the most pessimistic assumption that Eve is capable of performing any operation in her controlled territory that is allowed by the known laws of quantum physics [5], [6].

Given an unjammable classical channel and an insecure quantum channel, a QKD scheme consists of three stages [2]. The first is the signal preparation and transmission stage where quantum signals are prepared and exchanged between Alice and Bob. The second is the signal quality test stage where a subset of the exchanged quantum signals is measured in order to estimate the eavesdropping rate in the quantum channel. The final phase is the signal privacy amplification stage where a carefully designed privacy amplification procedure is performed to distill out an almost perfectly secure key.

No QKD scheme can be 100% secure as Eve may be lucky enough to guess the preparation or measurement bases for each quantum state correctly. Hence, it is more reasonable to demand that the mutual information between Eve’s measurement results after eavesdropping and the final secret key is less than an arbitrary but fixed small positive number. Hence I adopt the following definition of security.

**Definition 1 (Based on Lo and Chau [3]):** With the above assumptions on the unlimited computational power of Eve, a QKD scheme is said to be **unconditionally secure** with security parameters \((\epsilon_p, \epsilon_f)\) provided that whenever Eve has a cheating strategy that passes the signal quality control test with probability greater than \(\epsilon_p\), the mutual information between
Eve’s measurement results after eavesdropping and the final secret key is less than $\epsilon_1$.

III. AN ENTANGLEMENT-BASED QUANTUM KEY DISTRIBUTION SCHEME

In what follows, I first explicitly construct a unitary operator $T$ which plays a pivotal role in the design of the QKD scheme in Subsection III-A. Then, I make use of the operator $T$ to construct the entanglement-based QKD scheme in Subsection III-B.

A. The Unitary Operator $T$

In the analysis of certain quantum error correcting codes, Gottesman introduced a unitary operator that cyclically permutes the $\sigma_x$, $\sigma_y$, and $\sigma_z$ errors by conjugation [24]. Later on, Lo observed that conjugation by the same operator permutes the three bases used by the six-state scheme, namely, $\{|0\rangle, |1\rangle\}, \{|(0)\pm|1\rangle\}/\sqrt{2}$ and $\{|(0)\pm|1\rangle\}/\sqrt{2}$. He further used the permuting property of this unitary operator to argue that the $\sigma_x$, $\sigma_y$, and $\sigma_z$ error rates of the transmitted quantum signals in the six-state scheme are equal [21]. This is an important step in the analysis of the error tolerance rate of signals in the six-state scheme.

To devise a highly error-tolerant higher dimensional QKD scheme, one naturally asks if it is possible to find a unitary operator $T$ that cyclically permutes as many types of single errors by conjugation in this paper when the map $T$ is clearly known to readers.

Definition 2 (Ashikhmin and Knill [25]): Suppose $a \in GF(N)$ where $N = p^n$ with $p$ being a prime. We denote the unitary operators $X_a$ and $Z_a$ acting on an $N$-dimensional Hilbert space by

$$X_a |b\rangle = |a + b\rangle$$

and

$$Z_a |b\rangle = \chi_a(b) |b\rangle \equiv \omega_p^{Tr(ab)} |b\rangle,$$

where $\chi_a$ is an additive character of the finite field $GF(N)$, $\omega_p$ is a primitive $p$th root of unity and $\text{Tr}(a) = a + a^p + a^{p^2} + \cdots + a^{p^{n-1}}$ is the absolute trace of $a \in GF(N)$. Note that, the arithmetic inside the state ket and in the exponent of $\omega_p$ is performed in the finite field $GF(N)$.

It is easy to see from Definition 2 that $\{X_a Z_b : a, b \in GF(N)\}$ spans the set of all possible linear operators for an $N$-dimensional quantum register over $\mathbb{C}$. Besides, $X_a$ and $Z_b$ follow the algebra

$$X_a X_b = X_b X_a = X_{a+b},$$

$$Z_a Z_b = Z_b Z_a = Z_{a+b}$$

and

$$Z_b X_a = \omega_p^{Tr(ab)} X_a Z_b$$

for all $a, b \in GF(N)$, where arithmetic in the subscripts is performed in $GF(N)$.

Let $T$ be a linear operator acting on an $N$-dimensional space where $N = p^n$ is a prime power. Inspired by the permuting property of the unitary operator used by Lo in the security proof of the six-state scheme [21], one naturally demands that $T^{-1} X_a Z_b T = \omega_p^{f(a,b)} X_{E(a,b)} Z_{E(a,b)}$ for all $a, b \in GF(N)$. The factor $\omega_p^{f(a,b)} \in \mathbb{C}$ satisfying $|\omega_p^{f(a,b)}| = 1$ is sometimes known as the global phase because it simply multiplies a quantum state by a phase independent of that state. In order for $T$ to cyclically permute as many $X_a Z_b$’s as possible, one may demand that

$$\begin{bmatrix} a' \\ b' \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \equiv M(T) \begin{bmatrix} a \\ b \end{bmatrix},$$

for all $a, b \in GF(N)$, where $\alpha, \beta, \gamma \in GF(N)$. I shall simply denote $M(T)$ by $M$ in this paper when the map $T$ is clearly known to readers.

The phase factor $\omega_p^{f(a,b)}$ and the matrix $M(T)$ cannot be arbitrarily chosen. To show this, I use Eqs. (3)-(6) to manipulate the expression $X_a Z_b T$. On the one hand, it equals

$$\omega_p^{f(a+c,b+d)} X_{a+c} Z_{b+d} T = \omega_p^{f(c,d)-Tr(bc)} X_a Z_b T X_{a+c} Z_{b+d} T = \omega_p^{f(a,b)+f(c,d)} \omega_p^{T(c,d)-Tr(bc)} X_a Z_b T X_{a+c} Z_{b+d} T X_{a+c} Z_{b+d} T$$

and

$$f(a,b) = \frac{1}{2} \text{Tr}(\beta[a^2 \alpha + b^2 \gamma]) + \text{Tr}(ab \beta^2) + \delta \sum_{i>j} g_i g_j [a_i a_j \alpha + b_i b_j \gamma]$$

for all $a, b \in GF(N)$. Note that in Eq. (2), $a = \sum_{i=1}^{n} a_i g_i$ and $b = \sum_{i=1}^{n} b_i g_i$ where $\{g_1, g_2, \ldots, g_n\}$ is a fixed basis of $GF(N)$ over the field $GF(p)$ and $a_i, b_i \in GF(p)$. Moreover, $\delta p^2$ in the above equation is the Kronecker delta.

Two important remarks are in place. First, when $p > 2$ and hence $N$ is odd, 2 is invertible in $GF(N)$. Consequently, global phase $\omega_p^{f(a,b)}$ may be chosen from $p$th roots of unity. Following this convention, I demand

$$f(a,b) \in \mathbb{Z}/p\mathbb{Z}$$

for any $a, b \in GF(N)$ if $2 \not| N$. (10)

In contrast, when $p = 2$ and hence $N$ is even, 2 is not invertible in $GF(N)$. In this case $f(a,b)$ may be integral or
half-integral. Consequently, $\omega_p^{f(a,b)} \in \{\pm 1, \pm i\}$. In this case, I use the convention that
\[
\omega_2^{\frac{\text{Tr}(\alpha \beta \gamma a_2^2 g_2^2)}{\sqrt{2}}} =
\begin{cases}
1 & \text{if } \text{Tr}(\alpha \beta \gamma a_2^2 g_2^2) = 0, \\
i & \text{if } \text{Tr}(\alpha \beta \gamma a_2^2 g_2^2) = 1,
\end{cases}
\]
and
\[
\omega_2^{\frac{\text{Tr}(\beta \gamma b_2^2 g_2^2)}{\sqrt{2}}} =
\begin{cases}
1 & \text{if } \text{Tr}(\beta \gamma b_2^2 g_2^2) = 0, \\
i & \text{if } \text{Tr}(\beta \gamma b_2^2 g_2^2) = 1,
\end{cases}
\]
for all $a_j, b_j \in GF(p)$, where $j = 1, 2, \ldots, n$.

The second remark concerns the reason why we have the last term in Eq. (8). Recall that the identity $\text{Tr}(a_i^2 + a_j^2)/2 + \text{Tr}(a_i a_j) = \text{Tr}[(a_i + a_j)^2]/2$ holds only for $p > 2$. In contrast, $\text{Tr}(a_i^2 + a_j^2) = \text{Tr}[(a_i + a_j)^2]$ for $p = 2$. So, I cannot use the first identity to absorb the last term in Eq. (8) into the first term when $p = 2$.

**Lemma 1:** A linear operator $T$ obeys Eqs. (7)–(12) is unitary after a proper scaling. Specifically, $T$ is unitary if and only if its operator norm satisfies $||T|| = 1$.

**Proof:** I only need to show that $||T|| = 1$ is a sufficient condition as this condition is clearly necessary. Eqs. (7)–(12) lead to $X_a Z_b T T^\dagger = \omega_p^{f(a,b)} T X_{a + b} Z_{a + b} T T^\dagger = \omega_p^{f(a,b)} T X_{a + b} Z_{a + b} T T^\dagger = \omega_p^{f(a,b)} T X_{a + b} Z_{a + b} T T^\dagger$. The same argument, $X_a Z_b T T^\dagger = T T^\dagger X_a Z_b$ for all $a, b \in GF(N)$. Since $T$ acts on a finite dimensional Hilbert space and $\{X_a Z_b : a, b \in GF(N)\}$ spans the set of all linear operators on that Hilbert space, $TT^\dagger$ and $T T^\dagger$ are constant multiples of the identity operator. Therefore, $||T|| = 1$ implies $TT^\dagger = I = T T^\dagger$. Hence, $T$ is unitary.

In order to fully utilize the error tolerance capability of an $N$-dimensional QKD scheme, $T$ must satisfy one more constraint, namely, the order of $T$ must be as large as possible. The theorem below gives us an attainable upper bound for the order of $T$.

**Theorem 1:** There exists a unitary operator $T$ satisfying the constraints Eqs. (7)–(9), the phase conventions stated in Eqs. (10)–(12) as well as the condition that $I, T, T^2, \ldots, T^N$ are distinct operators up to a global phase. (That is, for all $0 \leq i < j \leq N$ and $\theta \in \mathbb{R}$, $T^i \neq e^{i\theta} T^j$.) Furthermore, the order of $T$ up to a global phase satisfying Eqs. (7)–(12) is at most $(N + 1)$. Suppose further that $\{g_1, g_2, \ldots, g_N\}$ is a fixed basis of $GF(N)$ over $GF(p)$, then $T$ is given by
\[
T = \sum_{a, b \in GF(N)} \frac{\omega_p^{\text{Tr}(f(a,b)) - \frac{1}{2} \text{Tr}(f(a,b))} X_a Z_b \omega_p^{\frac{\text{Tr}(f(a,b)) - \frac{1}{2} \text{Tr}(f(a,b))}{\sqrt{2}}} X_a Z_b
\]
for some $\theta \in \mathbb{R}$, where
\[
\varphi_2(a, b) = \frac{\beta \gamma}{(2 - \alpha - \gamma)^2} [(\beta \gamma - 1) a_2^2 - (\gamma - 1) [(\alpha - 1)^2 + \beta^2(2\alpha - 1)] a_2 b_2 + \beta(\alpha(\gamma - 1) + \gamma - 1) b_2^2] + \delta(a, b)
\]
and
\[
\varphi_1(a, b) = \frac{1}{(2 - \alpha - \gamma)^2} [(\beta \gamma - 1) a_2^2 - (\gamma - 1) [(\alpha - 1)^2 + \beta^2(2\alpha - 1)] a_2 b_2 + \beta(\alpha(\gamma - 1) + \gamma - 1) b_2^2] + \delta(a, b)
\]
for some $\delta(a, b)$.

Note that all the arithmetic in the above two equations are performed in the finite field $GF(N)$. Besides, in Eq. (14), $a_i, b_i \in GF(p)$ are the unique solutions of the equations
\[
\sum_{i=1}^{n} \tilde{a}_i g_i = \frac{(\gamma - 1) a - \beta b}{2 - \alpha - \gamma}
\]
and
\[
\sum_{i=1}^{n} \tilde{b}_i g_i = \frac{(\alpha - 1) b - \beta a}{2 - \alpha - \gamma}.
\]

**Proof:** From Eqs. (5) and (8), I know that the order of $T$ up to a global phase is equal to the order of $M = M(T)$. Combining with Eq. (7), the characteristic equation of $M$ is $\text{Char}(M) = \lambda^2 - (\alpha + \gamma) \lambda + 1$. If $\text{Char}(M)$ is reducible in $GF(N)$, the order of $M$ and hence also the order of $T$ up to a global phase are at most $(N - 1)$. So, to construct $T$ with a larger order, I must look for $\text{Char}(M)$ that is irreducible in $GF(N)$. Nevertheless, a degree two irreducible polynomial over $GF(N)$ splits in $GF(N^2)$. Since the constant term of $\text{Char}(M)$ is 1, the roots of $\text{Char}(M) = 0$ over $GF(N^2)$ can be written as $\xi$ and $\xi^{-1}$ respectively. Since $\alpha + \gamma \in GF(N)$, I conclude that $\xi + \xi^{-1} = (\xi + \xi^{-1})^N = \xi^N + \xi^{-N}$. Therefore, $(\xi^N + \xi^{-N}) = 0$. However, $\xi \notin GF(N)$ and hence $\xi^{N+1} = 1$. In other words, the order of the irreducible polynomial $\text{Char}(M)$ and hence the order of $T$ up to a global phase both divide $(N + 1)$. More importantly, since $N \notin 1$ mod $(N + 1)$ and $N^2 \equiv 1$ mod $(N + 1)$, Theorem 3.5 in Ref. [26] assures the existence of an order $(N + 1)$ irreducible polynomial in the form $\lambda^2 + c\lambda + 1$ over $GF(N)$. (Actually, Theorem 3.5 in Ref. [26] implies that $\lambda^2 + c\lambda + 1$ is irreducible over $GF(N)$ if and only if it is equal to $(\lambda + \xi)(\lambda + \xi^{-1})$ for $\xi \in GF(N^2) \setminus GF(N)$ with $\xi^{N+1} = 1$. Hence, such irreducible polynomials can be found efficiently.)

It remains to show that there exists $T$ whose order of the corresponding characteristic polynomial $\text{Char}(M(T))$ equals $(N + 1)$. I divide the proof into two cases.

Case 1: $p = 2$ or $p = 1$ mod 4 where $N = p^n$. In this case, I simply pick $\alpha = 0$, $\gamma = -c$ and $\beta = (1/2)$. (Such a $\beta \in GF(N)$ exists because $x^2 \equiv -1$ mod $p$ is solvable when $p = 2$ or $p$ is a prime satisfying $p \equiv 1$ mod 4.) Then, it is easy to check that Eq. (7) is satisfied and hence $T$ exists.

Case 2: $p > 2$. In this case, I pick $\alpha = 1$, $\gamma = -c - 1$. In this way, $\beta^2 = -c - 2 = \xi + \xi^{-1} - 2 = (\xi - 1)(\xi^{-1} - 1) = (\xi - 1)^2 \xi^{-1} - 1$. Hence, I choose $\beta = (\xi - 1)^{-1/2} = \xi^{1/2} - \xi^{-1/2}, (\xi^{1/2})$ exists since $p$ is an odd prime and $\xi^{N+1} = 1$ so that $\xi = \kappa^{N+1}$ where $\kappa$ is a primitive element in $GF(N^2)$. Moreover, $\beta \in GF(N)$ since $(\xi^{1/2} - \xi^{-1/2})^N = \xi^{-N/2} = -\xi^{-1/2} + \xi^{1/2}$.

Now, I am ready to explicitly construct $T$. To do so, I write $T = \sum_{a, b \in GF(N)} \lambda_{ab} X_a Z_b$ for some $\lambda_{ab} \in C$. From Eq. (8),
I conclude that
\[ \Lambda_{ij} = \omega_p^{f(a, b) + \sum \delta \omega_i (\alpha \alpha + \beta \beta) [j - a \beta - b \gamma - 1] - b} \times \]
for all \( a, b, i, j \in GF(4) \). Since the order of \( T \) is greater than 1, \( M(T) = I \) is invertible. Hence, I can choose suitable \( a = (a, i, j) \) and \( b = (b, i, j) \) in Eq. (18) to relate every \( \Lambda_{ij} \) to \( \Lambda_{00} \). In this way, I conclude that every \( \Lambda_{ij} \) is proportional to \( \Lambda_{00} \). Besides, all \( |\Lambda_{ij}| \)'s are equal. Consequently, the unitarity of \( T \) implies that \( |\Lambda_{00}| = 1/N \). By explicitly substituting \( a, b \) into Eq. (18) and after a tedious but straightforward calculation, I arrive at Eqs. (13)–(17).

The explicit construction of the operator \( T \) in the above proof also shows that once the \( 2 \times 2 \) matrix \( M(T) \) and the primitive root \( \omega_p \) are fixed, \( T \) is uniquely determined up to a global phase and a convention for \( \omega_p^{(a, b)} \).

For illustration purpose, the choices of \( M(T) \)'s and hence the unitary operators \( T \)'s for \( N = 2, 3, 4 \) computed by Eqs. (18)–(17) are tabulated in Table I. Incidentally, the unitary operator \( T \) listed in Table II for \( N = 2 \) is, up to a global phase, the same as the one used by Lo in his security proof of the six-state scheme in Ref. [21].

Now, I report several important properties of \( M(T) \) and \( M(T)^k \) that will be used in the security proof of this QKD scheme in Section IV.

**Lemma 1:** Suppose the order of \( M(T) \) equals \( (N + 1) \), then \( M(T)^k \) is in the form \( aI \) for some \( a \in GF(N) \) if and only if (1) \( p > 2 \) and \( (N + 1)/2 \) \( \mid k \). In fact, if \( p > 2 \), \( M(T)^{(N+1)/2} = -I \).

Proof: Since \( \text{Char}(M(T)) = \lambda^2 + c \lambda + 1 \), \( M(T) \) can be written in the form \( P^{-1}DP \) where \( D = \text{diag}(\xi, \xi^{-1}) \) where \( \xi \in GF(N^2) \) and \( \xi^{N+1} = 1 \). Hence \( M(T)^k = aI \) if and only if \( \xi^{2k} = 1 \). If \( p = 2 \), \( \xi^{2k} = 1 \Leftrightarrow \xi^k = 1 \Leftrightarrow (N + 1) \). If \( p > 2 \), \( \xi^{2k} = 1 \Leftrightarrow \xi^k = \pm 1 \Leftrightarrow (N + 1)/2 \). Moreover, \( \xi^k = -1 \) if and only if \( k = [(N + 1)/2] \mod (N + 1) \).

**Corollary 1:** The period of the sequence \( T^{-k}X_aZ_bT^{k'} \) : \( k, k' \in [N] \) up to global phases equals \( (N + 1) \) whenever \( a, b \in GF(N) \) are not all zero. Furthermore, if \( p > 2 \), there is exactly one \( 0 \leq k \leq N \) with \( T^{-k}X_aZ_bT^k = \Lambda Z_c \) for some \( \Lambda \in \text{C} \) and \( c \in GF(N) \). If \( p > 2 \), either \( T^{-k}X_aZ_bT^{k'} \neq \Lambda Z_c \) for all \( k \) or there are two distinct \( 0 \leq k, k' \leq N \) with \( T^{-k}X_aZ_bT^{k'} = \Lambda Z_c \) and \( T^{-k'}X_aZ_bT^{k'} = N'Z_{c'} \) for \( \Lambda, N' \in \text{C} \) and \( c \neq c' \in GF(N) \).

**Proof:** Direct application of Lemma 2.

**Definition 3:** \( T \) defines an equivalent relationship for \( GF(N)^2 \) by \( (a, b) \sim (a', b') \) if there exists \( i \in N \) and \( \Lambda \in \text{C} \{0\} \) such that \( T^{-i}X_aZ_bT^i = \Lambda X_{a'}Z_{b'} \). I denote elements in the corresponding equivalent class by \( (a, b)/\sim \).

**Corollary 2:** There are \( N \) elements in the equivalent class \( GF(N)^2/\sim \). Besides, \( (a, b)/\sim = (a + 1, b)/\sim \) if \( (a, b) \neq (0, 0) \). For every \( a \in GF(N) \), there exists at most two distinct \( b, b' \in GF(N) \) such that \( (a, b) \sim (a, b') \). Furthermore, if \( p > 2 \), \( b \neq b' \) and \( c \neq 0 \), then \( (0, c) \sim (a, b) \sim (a, b') \) if and only if \( N = 3 \). If \( p = 2 \), \( (0, b) \sim (0, b') \) implies \( b = b' \). In addition, suppose that \( p = 2 \) and \( a 
eq 0 \). Then, for any \( b \in GF(N) \), there exists \( c = c(b) \) such that \( (a, c(b)) \sim (b, c) \). In summary, \( GF(N)^2/\sim \) \( \approx \{(0, a)/\sim : a \in GF(N)\} \) if \( p = 2 \). On the other hand, if \( p > 2 \), there are \( (N - 1)/2 \) elements of \( GF(N)^2/\sim \) each containing two distinct elements in the form \( (0, b) \).

**Proof:** By writing
\[ M(T) = P^{-1} \begin{bmatrix} \xi^0 & 0 \\ 0 & \xi^{-1} \end{bmatrix} P = \begin{bmatrix} \beta & \xi - \alpha \\ \xi - 1 & \alpha \end{bmatrix}^{-1} \begin{bmatrix} \xi^0 & 0 \\ 0 & \xi^{-1} \end{bmatrix} \begin{bmatrix} \beta & \xi - \alpha \\ \xi - 1 & \alpha \end{bmatrix} \]
then \( (a, b) \sim (a', b') \) if and only if there exists \( k \) such that
\[ \begin{bmatrix} \xi^{k} & 0 \\ 0 & \xi^{-k} \end{bmatrix} P \begin{bmatrix} a \\ b \end{bmatrix} = P \begin{bmatrix} a \\ b' \end{bmatrix} \]
B. An Entanglement-Based QKD Scheme

Let $N$ be a prime power and $T$ be the order $(N + 1)$ unitary operator described in Theorem 1 in Subsection IV-A. Then, the QKD scheme goes as follows.

Entanglement-based QKD Scheme A

1) Alice prepares $L \gg 1$ quantum particle pairs in the state $\sum_{i \in GF(N)} |ii\rangle/\sqrt{N}$. She applies one of the following unitary transformation to the second particle in each pair randomly and independently: $I, T, T^2, \ldots, T^N$. For every pair of particles, Alice keeps the first one and sends the second one to Bob. He acknowledges the receipt of a particle from each of the $N$ sets $S_i$ if Alice and Bob have applied $T$ to the first particle and $T^i$ to the second one respectively. Thus in the absence of noise and Eve, each pair of shared particles kept by Alice and Bob should be in the state $\sum_{i \in GF(N)} |ii\rangle/\sqrt{N}$. Note that when $N = 2$, Scheme A is a variation of the six-state scheme introduced by Chau in Ref. [23]. The key difference is that the present one does not make use of Calderbank-Shor-Steane quantum code after PEC while the former does.

2) Alice and Bob estimate the (quantum) channel error rate by sacrificing a few particle pairs. Specifically, they randomly pick $O((N + 1)^2 \log((N + 1)/\epsilon)/\delta^2 N^2)$ pairs from each of the $(N + 1)$ sets $S_i$ and measure each particle of the pair in the basis $\{|0\rangle, |1\rangle, \ldots, |N - 1\rangle\}$ basis. They publicly announce and compare their measurement results. In this way, they know the estimated channel error rate within standard deviation $\delta$ with probability at least $(1 - \epsilon)$. (Detail proof of this claim can be found in Ref. [2].) A brief outline of the proof will also be given in Subsection IV-B for handy reference.) If the channel error rate is too high, they abort the scheme and start all over again.

3) Alice and Bob perform the following privacy amplification procedure. (Readers will find out in Section IV that step below reduces errors in the form $X_a Z_b$ with $a \neq 0$ at the expense of increasing errors in the form $Z_c$ with $c \neq 0$. In contrast, step below reduces errors in the form $X_a Z_b$ with $b \neq 0$ at the expense of increasing errors in the form $X_c$ with $c \neq 0$. Most vitally, applying steps in turn is an effective way to reduce all kinds of errors.)

a) Alice and Bob apply the entanglement purification procedure by two way classical communication (LOCC2 EP) similar to the ones reported in Refs. [19], [27]. Specifically, Alice and Bob randomly group their remaining quantum particles in tetrads; and each tetrad consists of two pairs shared between Alice and Bob in Step 1. Alice randomly picks one of the two particles in her share of each tetrad as the control register and the other as the target. She applies the following unitary operation to the control and target registers:

$$|i\rangle_{\text{control}} \otimes |j\rangle_{\text{target}} \mapsto |i\rangle_{\text{control}} \otimes |j - i\rangle_{\text{target}},$$

where the subtraction is performed in the finite field $GF(N)$. Bob applies the same unitary transformation to his corresponding share of particles in the tetrad. Then, they publicly announce their measurement results of their target registers in the standard basis. They keep their control registers only when the measurement results of their corresponding target registers agree. They repeat the above LOCC2 EP procedure until there is an integer $r > 0$ such that a single application of step will bring the quantum channel error rate of the resultant particles down to less than $\epsilon^2/\ell$ for an arbitrary but fixed security parameter $\epsilon > 0$, where $r\ell$ is the number of remaining pairs they shared currently. They abort the scheme either when $r$ is greater than the number of remaining quantum pairs they possess or when they have used up all their quantum particles in this procedure.

b) They apply the majority vote phase error correction (PEC) procedure introduced by Gottesman and Lo [22]. Specifically, they randomly divide the resultant particles into sets each containing $r$ pairs of particles shared between Alice and Bob. Alice and Bob separately apply the $|1, r, r\rangle$ phase error correction procedure to their corresponding shares of $r$ particles in each set and retain their phase error corrected quantum particles. At this point, Alice and Bob show $\ell$ almost perfect pairs $\sum_{i \in GF(N)} |ii\rangle/\sqrt{N}$ with fidelity at least $(1 - \epsilon^2/\ell)$. By measuring their shared pairs in the standard basis, Alice and Bob obtain their common key. More importantly, Eve’s information on this common key is less than the security parameter $\epsilon_j$. (Proof of this claim can be found in Theorem 4 in Subsection IV-C below.)

Note that when $N = 2$, Scheme A is a variation of the six-state scheme introduced by Chau in Ref. [23]. The key difference is that the present one does not make use of Calderbank-Shor-Steane quantum code after PEC while the former does.

IV. CRYPTOANALYSIS OF THE ENTANGLEMENT-BASED QUANTUM KEY DISTRIBUTION SCHEME

In this section, I am going to report a detail unconditional security proof of Scheme A in the limit of large number of quantum particle $L$ transmitted. I will also investigate the maximum error tolerance rate for Scheme A against the most general type of eavesdropping attack allowed by the laws of quantum physics. With suitable modifications, the security proof reported here can be extended to the case of a small finite $L$. Nevertheless, working in the limit of large $L$ makes the asymptotic error tolerance rate analysis easier.

Before carrying out the cryptanalysis, I will first define various error rate measures and discuss how to fairly compare error tolerance capabilities between different QKD schemes in Subsection IV-A. Then, I will briefly explain why a reliable upper bound of the channel error can be obtained by
randomly testing only a small subset of quantum particles in step 2 of Scheme A in Subsection IV-B. Finally, I will prove the security of the privacy amplification procedure in step 3 of Scheme A and analyze its error tolerance rate in Subsection IV-C. This will complete the proof of unconditional security for entanglement-based Scheme A.

A. Fair Comparison Of Error Tolerance Capability And Various Measures Of Error Rates

Definition 4: Recall that Alice prepares \( L \) particle pairs each in the state \( \sum_{i \in GF(N)} |i\rangle/\sqrt{N} \) and randomly applies powers of \( T \) to each pair. Denote the resultant (pure) state of the pairs by \( \bigotimes_{j=1}^{L} |\phi_j\rangle \). Then, she sends one particle in each pair through an insecure quantum channel to Bob; and upon reception, Bob randomly applies powers of \( T \) to his share of the pair. The channel quantum error rate in this situation is defined as the marginal error rate of the measurement results when Alice and Bob were going to make an hypothetical measurement on the \( j \)th shared quantum particle pair in the basis \( \{ X_i Z_j \otimes 1 |\phi_j\rangle : a, b \in GF(N) \} \) for all \( j \). In other words, the channel quantum error rate equals \( 1/|L| \) times the expectation value of the cardinality of the set \( \{ j : \text{hypothetical measurement of the } j\text{th pair equals } X_a Z_b \otimes 1 |\phi_j\rangle \text{ with } (a, b) \neq (0, 0) \} \). The channel standard basis measurement error rate is defined as \( 1/|L| \) times the expectation value of the cardinality of the set \( \{ j : \text{hypothetical measurement of the } j\text{th pair equals } X_a Z_b \otimes 1 |\phi_j\rangle \text{ with } a \neq 0 \} \). The next two definitions concern only those quantum particle pairs retained by Alice and Bob in \( \bigcup_{i} S_i \). (That is, those Alice and Bob have applied \( T^j \) and \( T^{-j} \) to the second particle of the shared pair for some \( j \) respectively.) In the absence of noise and eavesdropper, all such particle pairs should be in the state \( \sum_{i \in GF(N)} |i\rangle/\sqrt{N} \). The signal quantum error rate (or quantum error rate (QER) for short) in this situation is defined as the expectation value of the proportion of particle pairs in \( \bigcup_{i} S_i \) whose measurement result in the basis \( \{ \sum_{i \in GF(N)} |i\rangle \otimes X_a Z_b |i\rangle/\sqrt{N} : a, b \in GF(N) \} \) equals \( \sum_{i \in GF(N)} |i\rangle \otimes X_a Z_b |i\rangle/\sqrt{N} \) for some \( (a, b) \neq (0, 0) \). The signal standard basis measurement error rate (or standard basis measurement error rate (SBMER) for short) is defined as the expectation value of the proportion of particle pairs in \( \bigcup_{i} S_i \) whose measurement result in the basis \( \{ \sum_{i \in GF(N)} |i\rangle \otimes X_a Z_b |i\rangle/\sqrt{N} : a, b \in GF(N) \} \) equals \( \sum_{i \in GF(N)} |i\rangle \otimes X_a Z_b |i\rangle/\sqrt{N} \) for some \( a \neq 0 \). In other words, SMBER measures the apparent error rate of the signal when Alice and Bob measure their shares of particles in the standard basis. In the special case of \( N = 2^n \), any standard basis measurement result can be bijectively mapped to a \( n \)-bit string. Thus, it makes sense to define the signal bit error rate (or bit error rate (BER) for short) as the marginal error rate of resultant \( n \)-bit string upon standard basis measurement of the signal at the end of the signal preparation and transmission stage.

Three important remarks are in place. First, SMBERs and BERs for QKD schemes using quantum particles of different dimensions as information carriers should never be compared directly. This is because the quantum communication channels used are different. In addition, the same eavesdropping strategy may lead to different error rates \([13],[14],[15],[16],[17]\). It appears that the only sensible situation to meaningfully compare the error tolerance capabilities of two QKD schemes is when the schemes are using the same quantum communication channel and are subjected to the same eavesdropping attack. Specifically, suppose Alice reversibly maps every \( p^n \)-dimensional quantum state used in Scheme A into \( n \) possibly entangled \( p \)-dimensional quantum particles and sends them through an insecure \( p \)-dimensional quantum particle communication channel to Bob. Moreover, since we assume that Alice and Bob do not have quantum storage capability, it is reasonable to regard Alice to send every packet of \( n \) possibly entangled \( p \)-dimensional quantum particles consecutively. In this way, Scheme A becomes an entangled-particle-based QKD scheme. More importantly, Eve may apply the same eavesdropping attack on the insecure \( p \)-dimensional quantum particle channel used by Alice and Bob irrespective of \( n \). In this way, I can fairly compare the error tolerance capability between two entangled-particle-based QKD schemes derived from Scheme A using \( p^n \)- and \( p^n \)-dimensional particles respectively against any eavesdropping attack on the \( p \)-dimensional quantum particle channel.

Second, the BER defined above for \( N = 2^n \) with \( n > 1 \) depends on the bijection used. Fortunately, a useful lower bound on the BER can be found amongst all bijections immediately before Eq. (46) in Subsection IV-C.

Third, since quantum errors in the form \( X_a Z_b \) with \( (a, b) \neq (0, 0) \) permute under the conjugation by powers of \( T \), the channel quantum error rate is equal to the QER of the signal. Roughly speaking, QER refers to the rate of any quantum error (phase shift and/or spin flip) occurring in the pair \( \sum_{i \in GF(N)} |i\rangle/\sqrt{N} \) shared by Alice and Bob. In contrast, due to the permutation of quantum errors by powers of \( T \), the channel standard basis measurement error rate does not equal to the SMBER in general.

B. Reliability On The Error Rate Estimation

In Scheme A, Alice and Bob keep only those particle pairs that are believed to be in the state \( \sum_{i \in GF(N)} |i\rangle/\sqrt{N} \) at the end of step 2. Then, they measure some of them in the standard basis in the signal quality control test in step 3. More importantly, since all the LOCC2 EP and PEC privacy amplification procedures in step 4 map standard basis to standard basis, we can imagine conceptually that the final standard basis measurements of their shared secret key were performed right at the beginning of step 3. In this way, any quantum eavesdropping strategy used by Eve is reduced to a classical probabilistic cheating strategy [3].

Further recall that in step 5 Alice and Bob do not care about the measurement outcome of an individual quantum register; they only care about the difference between the measurement outcome of Alice and the corresponding outcome of Bob. In
other words, they apply the projection operators

\[ P_a = \sum_{i \in GF(N)} |i, i + a\rangle \langle i, i + a| \]  

(22)
to the randomly selected quantum registers they share in the set \( S_0 \). These projection operators can be rewritten in a form involving Bell-like states as follows. Define \( |\Phi_{ab}\rangle \) to be the Bell-like state \( \sum_{i \in GF(N)} |i\rangle \otimes X_a Z_b |i\rangle /\sqrt{N} \equiv \sum_{i \in GF(N)} |i, i + a\rangle /\sqrt{N} \). Then the projection operator \( P_a \) can also be written as

\[ P_a = \sum_{i \in GF(N)} |\Phi_{ai}\rangle \langle \Phi_{ai}|. \]  

(23)

In a similar way, Alice and Bob apply the projection operators \( T^{-1} P_a T^i \) to the set \( S_i \) for all \( i \). Now, it is straightforward to check that the unitary operator \( T \) maps Bell-like states to Bell-like states. Combining with Eqs. \( 22 \) and \( 23 \), the signal quality control test in step \( 2 \) of Scheme A can be regarded as an effective random sampling test for the fidelity of the pairs as \( |\Psi_{00}\rangle \equiv \sum_{i \in GF(N)} |i\rangle /\sqrt{N} \).

At this point, classical sampling theory can be used to estimate the quantum channel error rate and hence the eavesdropping rate of the classical probabilistic cheating strategy used by Eve as well as the fidelity of the remaining pairs as \( |\Psi_{00}\rangle \).

**Lemma 3 (Adapted from Lo, Chau and Ardehali [2]):** Suppose that immediately after step \( 1 \) in Scheme A, Alice and Bob share \( L_i \) pairs of particles in the set \( S_i \), namely, those particles that are evolved under \( T \) and then \( T^{-1} \). Suppose further that Alice and Bob randomly pick \( O(\log[1/\epsilon]/\delta^2) \leq 0.01L_i \) out of the \( L_i \) pairs for testing in step \( 2 \) Define the estimated channel standard basis measurement error rate \( \hat{e}_i \) to be the portion of tested pairs whose measurement results obtained by Alice and Bob differ. Denote the channel standard basis measurement error rate for the set \( S_i \) by \( e_i \). Then, the probability that \( |e_i - \hat{e}_i| > \delta \) is of the order of \( \epsilon \) for any fixed \( \delta > 0 \).

**Proof:** Using earlier discussions in this subsection, the problem depicted in this lemma is equivalent to a classical random sampling problem without replacement whose solution follows directly from Lemma 1 in Ref. [2].

**Lemma 3** assures that by randomly choosing \( O(\log[1/\epsilon]/\delta^2) \) out of \( L_i \) pairs to test, the unbiased estimator \( \hat{e}_i \) cannot differ from the actual channel standard basis measurement error rate \( e_i \) significantly. More importantly, the number of particle pairs they need to test is independent of \( L_i \). Therefore, in the limit of large \( L_i \) and hence large \( L \), randomly testing a negligibly small portion of quantum particle pairs is sufficient for Alice and Bob to estimate with high confidence the channel standard basis measurement error rate in the set \( S_i \) [2]. In addition, the QER of the remaining untested particle pairs is the same as that of \( \bigcup_{i=0}^{N} S_i \) in the large \( L \) limit.

**Theorem 2:** Using the notation in Lemma 3, the sum

\[ \sum_{i,j \in GF(N)} \hat{e}_{ij}/N \]

is a reliable estimator of the upper bound of the QER. Specifically, the probability that the QER exceeds \( \sum_{i=0}^{N} \hat{e}_i/N + (N + 1)\delta/N \) is less than \( \epsilon(N + 1) \).

**Proof:** Recall that Eve does not know the choice of unitary operators applied by Alice and Bob in step 1 in Scheme A. Hence, in the limit of large \( L \), the \( X_a Z_b \) error rate in the set \( S_i \) is equal to that of \( T^{-1} X_a Z_b T^j \) in the set \( S_k \). Therefore, this theorem follows directly from Corollary 1 and Lemma 3.

To summarize, once the signal quality control test in step 2 of Scheme A is passed, Alice and Bob have high confidence (of at least \( (1 - \epsilon) \)) that the QER of the remaining untested particle pairs is small.

Before leaving this subsection, I would like to point out that one can estimate the QER in a more aggressive way. Specifically, Alice and Bob do not simply know whether the measurement results of each tested pair are equal, in fact they know the difference between their measurement results in each tested pair. They may exploit this extra piece of information to better estimate the probability of \( X_a Z_b \) error in the signal for each \( a, b \in GF(N) \). Such estimation helps them to devise tailor-made privacy amplification schemes that tackle the specific kind of error caused by channel noise and Eve. While this methodology will be useful in practical QKD, I shall not pursue this direction further here as the aim of this paper is the worst-case cryptanalysis in the limit of large number of quantum particle transfer \( L \).

**C. Security Of Privacy Amplification**

**Definition 5:** We denote the \( X_a Z_b \) error rate of the quantum particles shared by Alice and Bob just before step \( 3 \) in Scheme A by \( e_{a,b} \). And when there is no possibility confusion in the subscript, we shall write \( e_{ab} \) instead of \( e_{a,b} \). Similarly, we denote the \( X_a Z_b \) error rate of the resultant quantum particles shared by them after \( k \) rounds of LOCC EP by \( e_{a,b}^{\text{EP}} \) or \( e_{ab}^{\text{EP}} \). Suppose further that Alice and Bob perform PEC using the \( [r, 1, r]_N \) majority vote code after \( k \) rounds of LOCC EP. We denote the resultant \( X_a Z_b \) error rate by \( e_{a,b}^{\text{PEC}} \) or \( e_{ab}^{\text{PEC}} \).

Recall that Alice and Bob randomly and independently apply \( T^j \) and \( T^{-j} \) to each transmitted quantum register. More importantly, their choices are unknown to Eve when the quantum particle is traveling in the insecure channel. Let \( E \) be the quantum operation that Eve applies to the quantum particles in the set \( \bigcup_{i=0}^{N} S_i \). (In other words, \( E \) is a completely positive convex-linear map acting on the set of density matrices describing the quantum particle pairs to which Alice and Bob has applied \( T^j \) and \( T^{-j} \) respectively for some \( j \). Moreover, the trace of \( E \) is between 0 and 1.) After Alice and Bob publicly announced their choices of quantum operations, the quantum particle pairs in \( \bigcup_{i=0}^{N} S_i \) had equal chance of suffering from \( (\otimes (T^{-i}) E (\otimes (T^{j})) \) where \( 0 \leq i, j \leq N \). Note that the index \( j \) in the tensor product in the above expression runs over all particles pairs in \( \bigcup_{i=0}^{N} S_i \). Besides, the privacy amplification procedure in step 4 is performed irrespective to which set \( S_i \) the particle belong to. Therefore, the QER satisfies the constraints

\[ \sum_{i,j \in GF(N)} e_{ij} = 1 \]

(24)
and
\[ e_{ab} = e_{a'b'} \text{ if } (a, b) \sim (a', b'). \tag{25} \]

After knowing the initial conditions for the QER, I am going to investigate the effect of LOCC2 EP on the QER.

**Lemma 4:** In the limit of a large number of transmitted quantum registers, \( e_{ab}^{\text{EP}} \) is given by
\[ e_{ab}^{\text{EP}} = \frac{\sum_{i\in GF(N)} \left( \sum_{j\in GF(N)} e_{ij} \right)^2}{\sum_{i\in GF(N)} \left( \sum_{j\in GF(N)} e_{ij} \right)^2}. \tag{26} \]
Moreover, in this limit, \( e_{ab}^{\text{EP}} = e_{a,b}^{\text{EP}} \) for all \( a, b \in GF(N) \) and \( k \in \mathbb{N} \).

**Proof:** Suppose the control and target registers in Bob’s laboratory suffer from \( X_a Z_b \) and \( X_{a'} Z_{b'} \) errors respectively. (In contrast, those in Alice’s hand are error-free as they never pass through the insecure noisy channel.) Then after applying the unitary operation in Eq. (21), the errors in the control and target registers will become \( X_a Z_{b+b'} \) and \( X_{a'-a} Z_{b'} \) respectively.

In the limit of large number of transmitted quantum registers, the covariance between probabilities of picking any two distinct quantum register tends to zero. Besides, the covariance between probabilities of picking any two distinct pairs of quantum registers also tends to zero. Hence, in this limit, the expectation value of the \( X_a Z_b \) error rate just after applying the unitary operation in Eq. (21) can be computed by assuming that the error in every control and target register pair is independent. Moreover, the variance of the \( X_a Z_b \) error rate tends to zero in this limit.

To show that Eq. (26) is valid, let us recall that Alice and Bob keep their control registers only when the measurement results of their corresponding target registers agree. In other words, they keep the control registers only when \( a = a' \). Thus, once the control register in Bob’s laboratory is kept, it will suffer an error \( X_d Z_e \), where \( d = a \) and \( e = b+b' \). Therefore, in the limit of a large number of transmitted quantum registers, the number of quantum registers remains after \((k+1)\) rounds of LOCC2 EP is proportional to \( \sum_{i\in GF(N)} \left( \sum_{j\in GF(N)} e_{ij} \right)^2 \).

Similarly, the number of quantum registers suffering from \( \sum_{i\in GF(N)} \left( \sum_{j\in GF(N)} e_{ij} \right)^2 \) after \((k+1)\) rounds of LOCC2 EP is proportional to \( \sum_{i\in GF(N)} \left( \sum_{j\in GF(N)} e_{ij} \right)^2 \). More importantly, the two proportionality constants are the same. Therefore,
\[ e_{(k+1)}^{\text{EP}} = \frac{\sum_{i\in GF(N)} e_{ij}^{\text{EP}} \sum_{j\in GF(N)} e_{ij}^{\text{EP}}}{\sum_{i\in GF(N)} \left( \sum_{j\in GF(N)} e_{ij} \right)^2} \tag{27} \]

for all \( k \in \mathbb{N} \). Eq. (26) can then be proven by mathematical induction on \( k \). (It is easier to use mathematical induction to prove the validity of the numerator in Eq. (26) and then use Eq. (27) to determine the denominator.)

To show that \( e_{ab}^{\text{EP}} = e_{a,b}^{\text{EP}} \), I only consider the case of \( p > 2 \) since the assertion is trivially true when \( p = 2 \). From Corollary 2 and Eq. (26), we have \( e_{ab} = e_{a-b} \). Inductively, assuming the validity of the assertion for \( k \), then
\[ e_{ab}^{(k+1)} = \sum_{i\in GF(N)} e_{ij}^{(k+1)\text{EP}} = \frac{\sum_{i\in GF(N)} e_{ij}^{(k+1)\text{EP}}}{D_k} = e_{a-b}^{(k+1)\text{EP}}. \tag{28} \]

In particular, if \( e_{ab} \) satisfies
\[ e_{ab} = \begin{cases} \frac{1 - e_0}{N + 1} & \text{if } (a, b) \sim (0, 1), \\ 0 & \text{if } (a, b) \not\sim (0, 0) \text{ and } (0, 1), \end{cases} \tag{29} \]
then for \( p = 2 \),
\[ e_{00}^{\text{EP}} = \frac{(e_0 + e_1)^2 + (e_0 - e_1)^2}{2[(e_0 + e_1)^2 + (e_0 - e_1)^2]^2} \times \left[ \sum_{j\in GF(N)} e_{aj} \cos \left( \frac{2\pi \sum_{i=0}^{n-1} m_i j_i}{p} \right) \right]^{2k} \times \left[ N \sum_{i\in GF(N)} \left( \sum_{j\in GF(N)} e_{ij} \right)^2 \right]^{-1}. \tag{30} \]

To prove the remaining parts of this lemma now follow directly from Eq. (28) and Corollary 2.
In the qubit case, that is when \( N = p = 2 \), Eqs. (24) and (25) demand that \( e_{01} = e_{10} = e_{11} = (1 - e_{00})/3 \). In other words, the evolution of QER under the action of LOCC2 EP depends on a single parameter, namely, \( e_{00} \). Nevertheless, the situation is more complicated when \( N > 2 \) because \( e_{ab}^{k} \) depends on more than one parameter. Fortunately, as we shall see later on, it is possible to determine the worst case scenario for \( e_{ab} \) when the number of rounds of LOCC2 EP, \( k \), is sufficiently large when \( p = 2 \).

Lemma 5: The following two statements hold provided that either (1) \( p = 2 \) and \( e_{00} > 1/(N + 2) \) or (2) \( p > 2 \) and \( e_{00} > 2/(N + 3) \).

(a) The maximum term in the denominator of Eq. (28) is

\[
(\sum_{j \in GF(N)} e_{0j})^{2p}.
\]

(b) \( e_{00}^{k}EP > e_{0b}^{k}EP \) whenever \( b \neq 0 \).

Proof: To prove the first statement, I first consider the \( p = 2 \) case. Using Corollary 2 plus the two constraints in Eqs. (24) and (25), we have \( e_{00} > (1 - e_{00})/(N + 1) = \sum_{j \neq 0} e_{0j} \geq e_{ab} \) for all \((a, b) \neq (0, 0)\). Hence, Corollary 2 demands that \( \sum_{j \neq 0} e_{0j} \geq e_{00} > 0 \) for all \( i \neq 0 \). By the same argument, in the \( p \geq 2 \) case, \( \sum_{j \neq 0} e_{0j} \geq e_{00} - 2(1 - e_{00})/(N + 1) > 0 \) for all \( i \neq 0 \).

To prove the second statement, I express \( e_{00}^{k}EP - e_{0b}^{k}EP \) in terms of \( e_{ij}^{(k-1)EP} \) by invoking Eq. (27). The denominator of this expression is positive and the numerator is given by

\[
\sum_{e \in GF(N)} e_{0c}^{(k-1)EP} - e_{0b,c}^{(k-1)EP} = \sum_{e \in GF(N)} e_{0c}^{(k-1)EP} - e_{0b,c}^{(k-1)EP} = \frac{1}{2} \sum_{e \in GF(N)} e_{0c}^{(k-1)EP} - e_{0b,c}^{(k-1)EP} \geq (33)
\]

where I have used Lemma 4 to arrive at the second line. Therefore, \( e_{00}^{k}EP \geq e_{0b}^{k}EP \) for all \( b \). In fact, our assumption on the value of \( e_{00} \) implies \( e_{00} > e_{0b} \) for all \( b \neq 0 \). Hence from Eq. (33), statement (b) holds for \( k = 1 \). The validity of statement (b) for all \( k \in \mathbb{Z}^{+} \) can then be shown by mathematical induction on \( k \).

Theorem 3: In the limit of large number of quantum registers transmitted from Alice to Bob, the \( X_{a}Z_{b} \) error rate after PEC \( e_{ij}^{PEC} \) using \( [r, 1, r]_{N} \) majority vote code satisfies

\[
\sum_{i \neq 0} \sum_{j \in GF(N)} e_{ij}^{PEC} \leq r \sum_{i \neq 0} \sum_{j \in GF(N)} e_{ij}^{kEP}. \tag{34}
\]

Moreover, if \( p = 2 \) and \( e_{00} > 1/(N + 2) \), then

\[
\sum_{i \in GF(N)} \sum_{j \neq 0} e_{ij}^{PEC} \leq (N - 1) \left[ 1 - \frac{(e_{00} - 1 - e_{00})^{2k+1}}{2(e_{00} + 1 - e_{00})^{2k+1}} \right]^{r}, \tag{35}
\]

as \( k \to \infty \).

Proof: Recall that the error syndrome of the \([r, 1, r]_{N}\) minority vote code is

\[
\begin{bmatrix}
1 & -1 & 1 & -1 \\
1 & -1 & 1 & -1 \\
& & & \\
1 & -1 & 1 & -1
\end{bmatrix}. \tag{36}
\]

Therefore, after measuring the \((phase)\) syndrome, \( Z_{b} \) error stays on the control register while \( X_{a} \) error propagates from the control as well as all target registers to the resultant control quantum register \([29]\). Specifically, suppose the error on the \( i \)th quantum register is \( X_{a_{i}}Z_{b_{i}} \), for \( i = 1, 2, \ldots, r \). Then, after measuring the error syndrome, the resultant error in the remaining control register equals \( X_{a_{1}+\ldots+a_{r}}Z_{b_{1}} \). Consequently, upon PEC, the error in the remaining registers is \( X_{a_{1}+\ldots+a_{r}}Z_{b} \) where \( b \) is the majority of \( b_{i} \) \((i = 1, 2, \ldots, r)\). In other words, after PEC, spin flip error rates are increased by at most \( r \) times. Hence, Eq. (34) holds.

By the same argument used in Lemma 4 in the limit of large number of quantum register transfer, the rate of any kind of phase error after PEC, \( \sum_{i \in GF(N)} \sum_{j \neq 0} e_{ij}^{PEC} \), satisfies

\[
\sum_{i \in GF(N)} \sum_{j \neq 0} e_{ij}^{PEC} \leq (N - 1) \max \{ \Pr \{ \text{the number of registers suffering from error in the form } X_{a_{i}}Z_{1} \text{ is greater than or equal to those suffering from error in the form } X_{a_{i}} \text{ when drawn from a random sample of } r \text{ registers, given a fixed } e_{00} \} \}, \tag{37}
\]

where the maximum is taken over all possible probabilities with different \( e_{ab} \)'s satisfying the constraints in Eqs. (24).
and (25). I denote the sum $\sum_{i \in GF(N)} e_{ij}^{EP}$ by $e_{Z_0}^{EP}$. Then,

$$\sum_{i \in GF(N)} \sum_{j \neq 0} e_{ij}^{PEC} \leq (N-1) \max\{ \sum_{s=0}^{r} \left( (1 - e_{Z_0}^{EP} - e_{Z_1}^{EP})^{r-s} \times \right. \left. (e_{Z_0}^{EP} + e_{Z_1}^{EP})^{s} \exp \left[ -2s \left( 1 - \frac{e_{Z_0}^{EP}}{e_{Z_0}^{EP} + e_{Z_1}^{EP}} \right)^2 \right] \right) \} \leq (N-1) \max\{ 1 - (e_{Z_0}^{EP} + e_{Z_1}^{EP}) \times \left[ e^{-2(1/2 - e_{Z_1}^{EP}/(e_{Z_0}^{EP} + e_{Z_1}^{EP})^2)} \right] \} \leq (N-1) \max\{ \left( 1 - 2\epsilon(e_{Z_0}^{EP} + e_{Z_1}^{EP}) \times \left[ (1 - e_{Z_0}^{EP} + e_{Z_1}^{EP})^{2} \right] \right) \} \quad \text{(38)}$$

where $t \to 1$ as $k \to \infty$. Note that I have used Eq. (1.2.5) in Ref. [30] to arrive at the second inequality above. (Eq. (1.2.5) is applicable because Lemma 5 implies that $e_{Z_0}^{EP} > e_{Z_1}^{EP}$ for a sufficiently large $k$.)

Since $e_{00}$ satisfies $p = 2$ and $e_{00} > 1/(N+2), \text{Lemma 5}$ tells us that $e_{Z_0}^{EP}$ is the dominant term in the denominator of Eq. (28) when $k$ is sufficiently large. Thus, using Eq. (28), it is easy to check that both $e_{Z_0}^{EP}$ and $e_{Z_0}^{EP} + e_{Z_1}^{EP}$ are maximized if $e_{ab} = (1 - e_{00})/(N + 1)$ for all $(a, b) \sim (0, 1)$ subject to the following two constraints: (1) $e_{00}$ is fixed; and (2) Eqs. (24) and (25) are satisfied. Therefore, the last line of Eq. (38) is maximized if Eq. (29) holds. Consequently, Eqs. (30) and (32) imply the validity of Eq. (35).

The above theorem tells us that the effect of PEC is reducing errors in the form $X_aZ_b$ with $b \neq 0$ at the expense of possibly increasing errors in the form $X_c$ with $c \neq 0$. For this reason, powerful signal privacy amplification procedure can be constructed by suitably combining LOCC2 EP and PEC.

Now, I am going to prove the unconditional security of Scheme A.

**Theorem 4:** Let $N = p^n$ be a prime power, $\epsilon_p$, $\epsilon_2$ and $\delta$ be three arbitrarily small but fixed positive numbers. Define

$$e^{QER} = \frac{(N+1)(\sqrt{5}-2)}{1 + (N+1)(\sqrt{5}-2)} \quad \text{for } p = 2. \quad \text{(39)}$$

Then, the entanglement-based QKD Scheme A involving the transfer of $N$-dimensional quantum particles is unconditionally secure with security parameters $(\epsilon_p, \epsilon_2)$ when the number of quantum register transfer $L = L(\epsilon_p, \epsilon_2, \delta)$ is sufficiently large. Specifically, provided that Alice and Bob abort the scheme whenever the estimated QER in step 2 is greater than $(e^{QER} - \delta)$, then the secret key generated by Alice and Bob is provably secure in the $L \to \infty$ limit. In fact, if Eve uses an eavesdropping strategy with at least $\epsilon_p$ chance of passing the signal quality test stage in step 2, the mutual information between Eve’s measurement results after eavesdropping and the final secret key is less than $\epsilon_p$. In this respect, Scheme A tolerates asymptotically up to $e^{QER}$ QER.

**Proof:** Since $L \gg (N+1)^4 \log((N+1)/\epsilon_p)/\delta^2 N^2$, therefore by applying Lemma 5 and Theorem 2, I conclude that by testing $O((N+1)^3 \log((N+1)/\epsilon_p)/\delta^2 N^2)$ pairs, any eavesdropping strategy that causes a QER higher than $e^{QER}$ has less than $\epsilon_p$ chance of passing the signal quality test stage in step 2 of Scheme A. (Similarly, if the QER is less than $(e^{QER} - \delta)$, it has at least $(1 - \epsilon_p)$ chance of passing step 2. As $\delta$ can be chosen to be arbitrarily small, the signal quality test stage in step 2 of Scheme A is not overly conservative.)

Now, suppose that Alice and Bob arrive at the signal privacy amplification stage in step 3 of Scheme A. Since $L \to \infty$, the quantum particle pairs used in the signal quality test stage in step 2 do not affect the error rates $e_{ab}$’s of the remaining untested particle pairs.

First, I consider the case when $p = 2$. After applying $k$ rounds of LOCC2 EP, Alice and Bob may consider picking $r$ used in the majority vote PEC to be $e^{1/2 \sum_{i \in GF(N)} \sum_{j \neq 0} e_{ij}^{EP}}$. In the limit of $k \to \infty$, Corollaries 2 and 3 imply that in the worst case scenario, there are at most two distinct $b = b(a)$ and $b' = b'(a)$ such that $e_{ab}, e_{ab'} > 0$ for all $a \neq 0$. Hence, $r$ can be chosen to be

$$r \approx e^{\frac{\epsilon_2}{2} \left( \sum_{i \in GF(N)} \sum_{j \neq 0} e_{ij}^{EP} \right)} \ell N [2(1 - e_{00})/(N+1)]^{2k} \quad \text{(40)}$$

whenever $e_{00} > 1/(N+2)$, where $\ell$ is the number of quantum particle pairs Alice and Bob share immediately after the PEC procedure in step 3. Besides, $r \to \infty$ in the $k \to \infty$ limit. So, from Eqs. (35) and (36) in Theorem 3, the QER of the remaining quantum registers after PEC, $e_{\text{final}}$ is upper-bounded by

$$e_{\text{final}} \leq \frac{\epsilon_2}{2k} + (N-1) \exp \left[ -\frac{2\ell N [2(1 - e_{00})/(N+1)]^{2k}}{e^{\frac{\epsilon_2}{2} \left( \sum_{i \in GF(N)} \sum_{j \neq 0} e_{ij}^{EP} \right)} \ell N [2(1 - e_{00})/(N+1)]^{2k}} \right]. \quad \text{(41)}$$

In other words, $e_{\text{final}} < \epsilon_2/\ell$ provided that

$$\frac{e_{00} - 1 - e_{00}}{N+1}^2 > \frac{2(1 - e_{00})}{N+1} \left[ e_{00} + \frac{1 - e_{00}}{N+1} \right]. \quad \text{(42)}$$

This condition is satisfied if and only if

$$e_{00} > \frac{1}{1 + (N+1)(\sqrt{5}-2)}. \quad \text{(43)}$$

It is easy to verify that the constraint in Eq. (43) is consistent with the assumption that $e_{00} > 1/(N+2)$. Hence, provided that the initial QER satisfies

$$\sum_{(i,j) \neq (0,0)} e_{ij} < \frac{(N+1)(\sqrt{5}-2)}{1 + (N+1)(\sqrt{5}-2)} = e^{QER}, \quad \text{(44)}$$
the fidelity of the $\ell$ quantum particle pairs shared between Alice and Bob immediately before they perform standard basis measurement to obtain their secret key is at least $1-e_{\text{final}}^\ell > 1-\epsilon_1/\ell$. By Footnote 28 in [3], the mutual information between Eve’s final measurement result after eavesdropping and the final secret key is at most $\epsilon_f$. Thus, if Alice and Bob abort the scheme if the estimated QER in step 2 exceeds $(e_{\text{QER}} - \delta)$, the secret key generated is provably secure. More importantly, the scheme is unconditionally secure with security parameters $(\epsilon_p, \epsilon_f)$.

A few remarks are in order. First, the unconditional security of Scheme A for $p > 2$ can be proven in a similar way. However, the computation of $e_{\text{QER}}$ is getting messy as the condition for minimizing $e_{\text{QER}}$ turns out to be $N$ dependent.

Second, from Corollary 1 when $p = 2$, $GF(N)/ \sim = \{(0, b)/ \sim; b \in GF(N)\}$ and hence the ratio between QER and SBMER for any kind of eavesdropping attacks equals $(N+1): N$. In contrast, when $p > 2$, such a ratio varies between $(N+1): (N-1)$ and $1:1$. Combining these observations with Theorem 4, I conclude that the maximum tolerable SBMER for Scheme A is given by

$$e_{\text{SBMER}}^A = \begin{cases} \frac{N_{e_{\text{QER}}}^A}{N+1} & \text{if } p = 2, \\ \frac{(N-1)e_{\text{QER}}^A}{N+1} & \text{if } p > 2. \end{cases}$$

(45)

In addition, if $p = 2$, Corollary 2 implies that there is a unique $a \neq 0$ such that $(0, 1) \sim (a, b) \sim (a, b')$ for some $b \neq b'$. Hence, no matter which bijective map Alice and Bob use to convert their standard basis measurement result of an $N$-dimensional quantum particle into a $\log_2 N$-bit string, the ratio between QER and BER is at least $(N+1): (1 + 0.5N \log_2 N)/ \log_2 N$. Consequently, the maximum tolerable BER for Scheme A is given by

$$e_{\text{BER}} = e_{\text{BER}}^A \left(\frac{1}{2} + \frac{1}{N \log_2 N}\right).$$

(46)

I tabulate the tolerable SBMER and BER in Table II. However, I must emphasize once again that according to the discussions in Subsection XV.A we should not and cannot deduce the relative error tolerance capability from Table II.

Third, I study the tolerable error rate of Scheme A as a function of $N$. Table III shows that the maximum tolerable BER $e_{\text{BER}}^A$ for $N = 2$ is the same as the one obtained earlier by Chau in Ref. [23]. More importantly, $e_{\text{SBMER}}$ increases as $n$ increases.

Actually, according to Eqs. 29 and Eqs. 45, 46, the tolerable SBMER and BER tend to 100% and 50% respectively as $2^n \to \infty$. More precisely, as $n \to \infty$, the tolerable BER for Scheme A using $2^n$-level quantum particles scales as $\approx 1/2 - (3 + \sqrt{5})/2^{n+1}$.

On the other hand, the lemma below set the upper limit for the tolerable SBMER for Scheme A.

**Lemma 6:** The tolerable SBMER for Scheme A is upper-bounded by $(N-1)/(N+1)$ if $p = 2$ and $(N-1)^2/[N(N+1)]$ if $p > 2$. In fact, these bounds are set by the following interpret-and-resend strategy: Eve randomly and independently measures each $N$-dimensional particle in the insecure quantum channel in the standard basis $\{|0\rangle, |1\rangle, \ldots, |N-1\rangle\}$. Then, she records the measurement result and resends the measured particle to Bob.

**Proof:** The proof follows the idea reported in Ref. [22]. Clearly, using this intercept-and-resend strategy, no quantum correlation between Alice and Bob can survive and hence no provably secure key can be distributed. Thus, this eavesdropping strategy set the upper bound for the tolerable SBMER and BER for Scheme A. It is easy to check that the bases $\{T^i[0], T^i[1], \ldots, T^i[N-1]\}$ where $i = 0, 1, \ldots, N$ if $p = 2$ and $i = 0, 1, \ldots, (N-1)/2$ if $p > 2$ are mutually unbiased. (A proof can be found in Lemma 7 in Section XV below.) Consequently, if it turns out that the measured qubit is prepared in the standard basis, that qubit will be accepted by Scheme A as error-free. In contrast, if the measured qubit is not prepared in the standard basis, it has $(N-1)/N$ chance of being detected as erroneous. Therefore, the tolerable SBMER is upper-bounded by $N/(N+1) \times (N-1)/N = (N-1)/(N+1)$ if $p = 2$ and $(N+1)/2 - 1)/(N+1)/2 \times (N-1)/N = (N-1)^2/[N(N+1)]$ if $p > 2$.

Thus, the difference between the tolerable SBMER and its theoretical upper bound tends to zero in the limit of large $N$. So in the limit, the error tolerance capability of Scheme A approaches its maximally allowable value.

Fourth, readers may wonder why Scheme A is highly error-tolerant especially when $N$ is large. Recall that Eve does not know which particles are in set $S_i$ when the particles are transmitted from Alice to Bob. Hence, in the limit of large number of quantum particle transfer $L$, $e_{\text{err}}$ satisfies the constraints in Eqs. 24 and 25. This greatly limits the relative occurrence rates between different types of quantum errors. At this point, the LOCC2 EP becomes a powerful tool to reduce the spin errors at the expense of increasing phase errors. Furthermore, provided that the condition in Lemma 5 holds, $e_{\text{err}}^A < e_{\text{err}}^Z$ for all $b \neq 0$. In other words, the dominant kind of phase error is having no phase error at all. Thus, the majority vote PEC procedure is effective in bringing down the phase error. This is the underlying reason why Scheme A is so powerful that in the limit $N \to \infty$, $e_{\text{SBMER}} \to 1^{-}$.

Fifth, the privacy amplification performed in Scheme A is based entirely on entanglement purification and phase error correction. In fact, the key ingredient in reducing the QER used in the proof of Theorem 4 is the validity of conditions shown in Eq. 42. Nonetheless, there is no need to bring down the QER to an exponentially small number. In fact, one may...
devise an equally secure scheme by following the adaptive procedure introduced by Chau in Ref. [23]. That is to say, Alice and Bob may switch to a concatenated Calderbank-Shor-Steane quantum code when the PEC brings down the QER to about 5%. The strategy of adding an extra step of quantum error correction towards the end of the privacy amplification procedure may increase the key generation rate. This is because from the proof of Theorem 4 together with Eq. (40), I conclude that in order to bring the QER down to less than $\epsilon$ after $k$ rounds of LOCC2 EP, Alice and Bob have to choose $r$ and hence the number of quantum registers needed in PEC to be $\approx c^{2k}$ for some constant $c > 1$. In contrast, by randomizing the quantum registers, the QER after each application of the Steane’s seven quantum register code is reduced quadratically whenever the QER is less than about 5%. Consequently, Alice and Bob may increase the key generation rate by performing less rounds of LOCC2 EP, choosing $\epsilon \approx 0.01$, and finally adding a few rounds of Calderbank-Shor-Steane code quantum error correction procedure.

V. REDUCTION TO THE PREPARE-AND-MEASURE SCHEME

Finally, I apply the standard Shor and Preskill proof [20] to reduce the entanglement-based Scheme A to a provably secure prepare-and-measure scheme in this section. Let me first write down the detail procedures of Scheme B before showing its security.

Prepare-and-measure QKD Scheme B

1) Alice randomly and independently prepares $L \gg 1$ quantum particles in the standard basis. She applies one of the following unitary transformation to each particle randomly and independently: $I, T, T^2, \ldots, T^N$. Alice records the states and transformations she applied and then sends the states to Bob. He acknowledges the reception of these particles and then applies one of the following transformation to each received particles randomly and independently: $I^{-1}, T^{-1}, T^{-2}, \ldots, T^{-N}$. Now, Alice and Bob publicly reveal their unitary transformations applied to each particle. A particle is kept and is said to be in the set $S_i$ if Alice and Bob have applied $T^i$ and $T^{-i}$ to it respectively. Bob measures the particles in $S_i$ in the standard basis and records the measurement results.

2) Alice and Bob estimate the quantum channel error rate by sacrificing a few particles. Specifically, they randomly pick $O((N + 1)^2 \log((N + 1)/\epsilon)/\delta^2 N^2)$ pairs from each of the $(N + 1)$ sets $S_i$ and publicly reveal the preparation and measured states for each of them. In this way, they obtain the estimated channel error rate within standard deviation $\delta$ with probability at least $(1 - \epsilon)$. If the channel error rate is too high, they abort the scheme and start all over again.

3) Alice and Bob perform the following privacy amplification procedure.
   a) They apply the privacy amplification procedure with two way classical communication similar to the ones reported in Refs. [22], [23]. Specifically, Alice and Bob randomly group their corresponding remaining quantum particles in pairs. Suppose the $j$th particle of the $i$th pair was initially prepared in the state $|s_{ij}\rangle$. Then, Alice publicly announces the value $s_{ij} - s_{ij}' \in GF(N)$ for each pair $i$. Similarly, Bob publicly announces the value $s_{ij}' - s_{ij}''$ where $|s_{ij}'\rangle$ is the measurement result of the $j$th particle in the $i$th pair. They keep one of their corresponding registers of the pair only when their announced values the corresponding pairs agree. They repeat the above procedure until there is an integer $r > 0$ such that a single application of step b) will bring the quantum channel error rate of the resultant particles down to $\epsilon_1/\ell^2$ for a fixed security parameter $\epsilon_1 > 0$, where $\ell r$ is the number of remaining quantum particles they have. They abort the scheme either when $r$ is greater than the number of remaining quantum particles they possess or when they have used up all their quantum particles in this procedure.
   b) They apply the majority vote phase error correction procedure introduced by Gottesman and Lo [22]. Specifically, Alice and Bob randomly divide their corresponding resultant particles into sets each containing $r$ particles. They replace each set by the sum of the values prepared or measured of the $r$ particles in the set. These replaced values are bits of their final secure key string.

Theorem 5 (Based on Shor and Preskill [20]): Scheme A in Section IV and Scheme B above are equally secure. Thus, conclusions of Theorem 4 is also applicable to Scheme B.

Proof: Recall from Ref. [20] that Alice may measure all her share of quantum registers right at step II in Scheme A without affecting the security of the scheme. Besides, LOCC2 EP and PEC procedures in Scheme A simply permute the measurement basis. More importantly, the final secret key generation does not make use of the phase information of the transmitted quantum registers. Hence, the Shor-Preskill argument in Ref. [20] can be applied to Scheme A, giving us an equally secure prepare-and-measure Scheme B above.

From the discussions in Subsection IV-A, we should not and cannot compare the error tolerant capability of Scheme B that uses unentangled quantum particles of different dimensions as information carrier. Nonetheless, we may compare the error tolerant capability of the entangled-qubit-based prepare-and-measure QKD scheme derived from Scheme B against the same eavesdropping attack. Recall that in the absence of quantum storage, we may regard the transfer of a 16-dimensional quantum particle as the transfer of 4 consecutive qubits in the insecure quantum channel. Now, I consider the following eavesdropping strategy: Qubits passing through the insecure communication channel are partitioned into sets each containing 4 consecutive qubits. Eve randomly and independently measure each set in the standard basis with probability
q. Suppose q satisfies

\[ 0.8292 \approx \frac{3}{10}(5-\sqrt{5}) < q < \frac{68}{1335}(19-\sqrt{5}) \approx 0.8539. \]  

(47)

From Lemma 5 and Eq. 46, the BER caused by this eavesdropping strategy on the entangled-qubit-based prepare-and-measure QKD scheme derived from Scheme B for \( N = 2^n \) is given by \( e_{\text{BER}}^B(N) = q(N-1)(Nn + 2)/[2Nn(N+1)] \). Using Eqs. (49), (45), (47), I conclude that \( e_{\text{BER}}^B(2) > (5-\sqrt{5})/10 \). In other words, \( e_{\text{BER}}^B(2) \) is greater than tolerable BERs of all known unentangled-qubit-based prepare-and-measure QKD schemes to date. In contrast, \( e_{\text{BER}}^B(16) < 33/(19-\sqrt{5})/1424 \). Hence, from Theorem 5 together with Eqs. (49), (45) and (46), Scheme B can generate a provably secure key under this eavesdropping attack when \( N = 16 \). Actually, one may construct an eavesdropping attack that can be tolerated by the entangled-qubit-based prepare-and-measure scheme derived from Scheme B for a fixed \( N = 2^n \geq 16 \) in a similar way. (The strategy is partition the qubits into sets each containing \( n \) consecutive qubits. Eve makes standard basis measurement on each set with probability \( q \) chosen from an interval similar to the one stated in Eq. (47).) All known unentangled-qubit-based prepare-and-measure schemes to date, in contrast, cannot generate a provably secure key under the same attack.

On the other hand, suppose Eve chooses a slightly different strategy by measuring randomly and independently a qubit in each set of 4 consecutive qubits with probability \( q' = 1 - [(43 + 68\sqrt{5})/1335]^{1/4} \approx 0.3817 \) in the standard basis. Under this modified eavesdropping attack, the probability that a randomly chosen 4 consecutive qubits are not chosen equals \((1 - q')^4 \) in the limit of large number of qubit transfer. Thus, the BER induced by this attack on the entangled-qubit-based prepare-and-measure scheme derived from Scheme B for \( N = 16 \) is given by \( 1 - (1 - q')^4(N - 1)(Nn + 2)/[2Nn(N+1)] = 33/(19-\sqrt{5})/1424 \). This BER rate is just too high for the entangled-qubit-based scheme derived from Scheme B for \( N = 16 \) to handle. In contrast, the BER caused by the same eavesdropping attack for the six-state scheme equals \( q'/3 \approx 0.1272 \). This attack, therefore, can be handled easily by the unentangled-qubit-based prepare-and-measure QKD scheme introduced by Chau in Ref. [23]. To summarize, the entangled-qubit-based prepare-and-measure scheme derived from Scheme B for \( N > 2 \) is more error resilience when dealing with burst type of errors than the unentangled-qubit-based prepare-and-measure schemes.

Now, I need to point out an important remark on the number of different kinds of states Alice have to prepare in Scheme B. To distribute the key using an \( N \)-level quantum system with \( N = 2^n \), Corollary 1 tells us that \( T^k \neq I \) for all \( k = 1, 2, \ldots, N \). Therefore, \( T^i \)’s are distinct states for \( 0 \leq i < N \) and \( j \in GF(N) \). Thus, Scheme B is a \( N(N+1) \)-state scheme. In contrast, if \( N = p^n \) with \( p > 2 \), then \( T^{(N+1)/2} = -I \) by Corollary 1. Hence, in this case, upon measurement on the standard basis, Scheme B is a \( N(N+1)/2 \)-state scheme. This observation suggests that there may be rooms for improving the error tolerance rate of an prepare-and-measure QKD scheme involving \( N \)-dimensional quantum particles for an odd \( N \).

Finally, I remark that the lemma below suggests the possibility of a subtle relation between Scheme B and the so-called mutually unbiased bases.

**Lemma 7**: If \( N = 2^n \), then the bases \( \{ |k\rangle \}_{k \in GF(N)} \), \( \{ |T(k)\rangle \}_{k \in GF(N)} \), \( \{ |T^2(k)\rangle \}_{k \in GF(N)} \), \( \ldots \), \( \{ |T^{N-1}(k)\rangle \}_{k \in GF(N)} \) are mutually unbiased. While if \( N = p^n \) with \( p > 2 \), the bases \( \{ |k\rangle \}_{k \in GF(N)} \), \( \{ |T(k)\rangle \}_{k \in GF(N)} \), \( \ldots \), \( \{ |T^{(N+1)/2}(k)\rangle \}_{k \in GF(N)} \) are mutually unbiased.

**Proof**: I shall only consider the case when \( N = 2^n \). The other case can be proven in the same way. Let \( 0 \leq i < j \leq N \). I consider the equation

\[
\langle k'|T^iT^j|k\rangle = \langle 0|Z_jX_{-k'}T^{i-j}X_k|0\rangle,
\]

which holds for all \( j \in GF(N) \). Since \( 0 < i-j < N \), Corollary 1 implies that \( M(T^{i-j}) \) is in the form \[
\begin{pmatrix}
2 & a & b \\
0 & b & c
\end{pmatrix}
\]
for some \( b \neq 0 \). Therefore, applying Eqs. 5 and 48 to the right hand side of Eq. 48 gives an expression proportional to \( \langle 0|T^{i-j}X_{k\rightarrow k'+a}Z_{j\rightarrow j+b+1}|0\rangle = \langle 0|T^{i-j}|k - k' - j + b\rangle \). More importantly, the magnitude of the proportionality constant equals 1 for all \( j, k, k' \in GF(N) \). Hence, \( |\langle k'|T^i|k\rangle|^2 = |\langle k''|T^j|k''\rangle|^2 \) for all \( k, k', k'' \in GF(N) \) whenever \( 0 < i-j < N \). Hence, \( \{ |k\rangle \}_{k \in GF(N)} \), \( \{ |T(k)\rangle \}_{k \in GF(N)} \), \( \ldots \), \( \{ |T^{(N+1)/2}(k)\rangle \}_{k \in GF(N)} \) are mutually unbiased.

Since the maximum number of mutually unbiased bases equals \( (N+1) \) for any prime power \( N \) \([31], [32], [33]\), the construction in Scheme B provides a simple way to build such mutually unbiased bases for \( N = 2^n \). Perhaps one may build a more error tolerant QKD scheme using mutually unbiased bases for the case of an odd prime power \( N \).

**VI. DISCUSSIONS**

In summary, I have introduced a prepare-and-measured QKD scheme (Scheme B) and proved its unconditional security. In particular, I show that for a sufficiently large Hilbert space dimension of quantum particles \( N \) used, Scheme B generates a provably secure key close to 100% SBFMER or 50% BER. This result demonstrates the advantage of using unentangled higher dimensional quantum particles as signal carriers in QKD.

A variation to the theme is worth discussing. Suppose Alice can only send qubits. Besides, she can entangle the qubits but she cannot store them. Then, she may group \( n \) qubits together as a \( 2^n \)-dimensional system and apply Scheme B. Under this situation, Scheme B can generate a provably secure key under certain eavesdropping attack whenever \( n \geq 4 \). In contrast, no unentangled-qubit-based prepare-and-measure QKD scheme known to date can tolerate the same eavesdropping attack. Nonetheless, there exists another eavesdropping attack that Scheme B cannot tolerate unless \( N = 2 \). Recall that Scheme B is equivalent to the unentangled-qubit-based prepare-and-measure QKD scheme proposed by Chau in Ref. [23]. Therefore, the ability to create, transfer but not to store entangled qubits is advantageous in quantum cryptography using certain quantum channels with burst errors.

There is a tradeoff between the error tolerance rate and key generation efficiency, however. It is clear from the proof of
Theorem\[4\] that \( r \) and hence the number of quantum particle transfer from Alice and Bob \( L \) scales as \( 2^L \). Besides, the probability that the measurement results agree and hence the control quantum register pairs are kept in LOCC2 EP equals \( \approx 1/N \) in the worst case. As a result, while the Scheme B is highly error-tolerant, it generates a secret key with exponentially small efficiency in the worst case scenario. Fortunately, the adaptive nature of Scheme B makes sure that this scenario will not happen when the error rate of the channel is small. To improve the error tolerance rate in the case \( p > 2 \), Scheme B uses only \( N(N + 1)/2 \) different quantum states in signal transmission. It is instructive to explore such a possibility.

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REFERENCES

[1] M. A. Nielsen and I. L. Chuang, Quantum Computation And Quantum Information. Cambridge: CUP, 2000. p. 586.

[2] H.-K. Lo, H. F. Chau, and M. Ardehali, “Efficient quantum key distribution scheme and proof of its unconditional security,” 2001. (quant-ph/001056v2), to appear in J. Crypt.

[3] H.-K. Lo and H. F. Chau, “Unconditional security of quantum key distribution over arbitrarily long distances,” Science, vol. 283, pp. 2050–2056, 1999. As well as the supplementary material available at http://www.sciencemag.org/feature/data/984035.shtml.

[4] D. Mayers, “Unconditional security in quantum cryptography,” J. Assoc. Comp. Mach., vol. 48, pp. 351–406, 2001. See also his preliminary version in D. Mayers, Advances in Cryptology — Proceedings of Crypto’96 (Springer Verlag, Berlin, 1996), pp. 343–357.

[5] D. Gottesman and H.-K. Lo, “From quantum cheating to quantum security,” Phys. Today, vol. 53, no. 11, pp. 22–27, 2000. And references cited therein.

[6] N. Gisin, G. Ribordy, W. Tittel, and H. Zbinden, “Quantum cryptography,” Rev. Mod. Phys., vol. 74, pp. 145–195, 2002. And references cited therein.

[7] C. H. Bennett and G. Brassard, “Quantum cryptography: Public key distribution and coin tossing;” in Proceedings of the IEEE International Conference on Computers, Systems and Signal Processing, (New York), pp. 175–179, Bangalore, India, IEEE, 1984.

[8] D. Bruß, “Optimal eavesdropping in quantum cryptography with six states,” Phys. Rev. Lett., vol. 81, pp. 3018–3021, 1998.

[9] T. C. Ralph, “Continuous variable quantum cryptography,” Phys. Rev. A, vol. 61, pp. 013031(R):1–4, 2000.

[10] M. Hillery, “Quantum cryptography with squeezed states,” Phys. Rev. A, vol. 61, pp. 022309:1–8, 2000.

[11] D. Gottesman and J. Preskill, “Secure quantum key distribution using squeezed states,” Phys. Rev. A, vol. 63, pp. 022309:1–8, 2001.

[12] H. Bechmann-Pasquinucci and A. Peres, “Quantum cryptography with 3-state systems,” Phys. Rev. Lett., vol. 85, pp. 3313–3316, 2000.

[13] H. Bechmann-Pasquinucci and W. Tittel, “Quantum cryptography using larger alphabets,” Phys. Rev. A, vol. 61, pp. 062308:1–6, 2000.

[14] M. Bourennane, A. Karlsson, and G. Björk, “Quantum key distribution using multilevel encoding,” Phys. Rev. A, vol. 64, pp. 021306:1–5, 2001.