On kinematical constraints in the hadrogenesis conjecture for the baryon resonance spectrum

Yonggoo Heo\textsuperscript{a} and Matthias F.M. Lutz\textsuperscript{b}

GSI Helmholtzzentrum für Schwerionenforschung GmbH, Planck Str. 1, 64291 Darmstadt, Germany

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Abstract. We consider the reaction dynamics of bosons with negative parity and spin 0 or 1 and fermions with positive parity and spin $\frac{1}{2}$ or $\frac{3}{2}$. Such systems are of central importance for the computation of the baryon resonance spectrum in the hadrogenesis conjecture. Based on a chiral Lagrangian the coupled-channel partial-wave scattering amplitudes have to be computed. We study the generic properties of such amplitudes. A decomposition of the various scattering amplitudes into suitable sets of invariant functions expected to satisfy Mandelstam’s dispersion-integral representation is presented. Sets are identified that are free from kinematical constraints and that can be computed efficiently in terms of a novel projection algebra. From such a representation one can deduce the analytic structure of the partial-wave amplitudes. The helicity and the conventional angular-momentum partial-wave amplitudes are kinematically constrained at the Kibble conditions. Therefore an application of a dispersion-integral representation is prohibitively cumbersome. We derive covariant partial-wave amplitudes that are free from kinematical constraints at the Kibble conditions. They correspond to specific polynomials in the 4-momenta and Dirac matrices that solve the various Bethe-Salpeter equations in the presence of short-range interactions analytically.

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1 Introduction

The light vector mesons play a crucial role in the hadrogenesis conjecture \cite{1,2,3,4,5,6,8,9,10}. Together with the Goldstone bosons they are identified to be the quasi-fundamental hadronic degrees of freedom that are expected to be responsible for the formation of the meson spectrum. If supplemented with the baryon-octet and decuplet ground states the baryon resonance spectrum is conjectured to be generated by coupled-channel dynamics. For instance it was shown that the leading chiral interaction of Goldstone bosons with the light vector mesons generates an axial-vector meson spectrum that is quite close to the empirical spectrum \cite{4,9}. Similarly s- and d-wave baryon resonances were generated by the leading chiral interaction of the Goldstone bosons with the baryon octet and decuplet states \cite{11,2,3,13}.

Extensions of such computations to systems involving intermediate states of a vector meson and a baryon, as suggested by the hadrogenesis conjecture, are formidable challenges. Though it is well known how to incorporate more massive degrees of freedom into the chiral Lagrangian, it is not so clear how to organize systematic applications. For instance, the low-energy interaction of vector mesons with the baryons is characterized by various unknown two-body counter terms. This is analogous to the low-energy interaction of two nucleons \cite{14,15,16,17}. Here chiral dynamics does not predict the dominant structure of the coupled-channel interaction. This motivated a phenomenological study that parameterized a set of quasi-local coupled-channel interactions and adjusted their strength to empirical data from pion- and photon-nucleon scattering \cite{4}. By construction all long-range forces from t- and u-channel exchange processes are assumed to be integrated out and therefore the approach is applicable in the resonance region only. The number of adjusted parameter, about 50, were used to describe about 2000 data points. Various baryon resonances were shown to be dynamically generated by s-wave channels with a vector meson. Clearly, a more systematic and predictive approach would be highly desirable \cite{15}. The challenge is the consistent treatment of both long- and short-range forces from t- and u-channel exchange processes. Only then it may be possible to establish a reliable link of a chiral Lagrangian to the resonance spectrum.

The key issue is the identification of an optimal set of degrees of freedom in combination with the construction of power counting rules. A novel counting scheme for the chiral Lagrangian which includes the nonet of light vector mesons in the tensor field representation was explored in \cite{9,10}. It is based on the hadrogenesis conjecture and large-$N_c$ considerations \cite{4}. The counting scheme...
would be a consequence of an additional mass gap of QCD in the chiral limit, that may arise if the number of colors increases. How to systematically include the baryon octet and decuplet states into this Lagrangian remains an open issue.

In this work we prepare the ground for systematic coupled-channel computations involving active vector meson degrees of freedom. Based on a chiral Lagrangian coupled-channel partial-wave amplitudes need to be established in a controlled approximation. Any conceivable scheme should obey the coupled-channel unitarity condition and generate amplitudes, which analytic structures are consistent with the constraints set by micro-causality. Following [19 20 21 22 23 17] we use the concept of a generalized potential. A partial-wave scattering amplitude $T^J(\sqrt{s})$ is decomposed into two contributions

$$T^J(\sqrt{s}) = U^J(\sqrt{s}) + \int \frac{dw}{\pi} \frac{T^J(w) \rho^J(w) T^{J'}(w)}{w - \sqrt{s} + i\epsilon}. \tag{1}$$

By definition the generalized potential $U^J(\sqrt{s})$ receives contributions with left-hand cuts only. All right-hand cuts from the $s$-channel unitarity condition are generated by the second term in $U$. Given an approximated generalized potential one can solve for the partial-wave scattering amplitude $T^J(\sqrt{s})$. A systematic unitarization scheme based on [19] was developed recently in [19 20 21 22 23 17]. It is applicable in the presence of long- and short-range forces and therefore suitable for coupled-channel studies with active vector meson degrees of freedom.

Once intermediate states with a vector meson are considered there are almost always long-range forces implied by $t$- or $u$-channel exchange processes that lead to non-trivial left-hand branch points in the partial-wave amplitudes. The left- and right-hand branch cuts almost always overlap and therefore any algebraic or separable approach has to be rejected. Let us be specific and exemplify our statement: consider the $\omega$-meson nucleon scattering process. The partial-wave amplitudes have a left-hand branch point at 1351 MeV that is caused by the $s$-channel nucleon exchange process according to the Landau condition [24]. The branch point is right from the pion-nucleon threshold and therefore constitutes an example for an overlap of left- and right-hand cuts. If the on-shell reduction scheme of [2] or any other separable scheme would be applied to the system with two channels $\pi N$ and $\omega N$, the partial-wave scattering amplitudes develop necessarily unphysical left-hand branch points [19 22]. This holds at any finite truncation of the interaction kernel: all pion-nucleon partial-wave amplitudes would have an unphysical branch point at 1351 MeV. The resulting partial-wave amplitudes are analytic functions, however, they have a cut structure that is inconsistent with the constraints set by micro-causality.

Though it is straightforward to introduce partial-wave scattering amplitudes in the helicity formalism of Jacob and Wick [25], it is a nontrivial task to derive transformations that lead to amplitudes that are kinematically unconstrained. [26 27 28 29 30 31 19 17] A kinematical constraint of a partial-wave amplitude requires a boundary condition on the non-linear integral equations (1) that complicates the search of its solutions significantly. Therefore it is useful to find transformations of the helicity partial-wave amplitudes to covariant partial-wave amplitudes that do not require boundary conditions in (1). In a previous work one of the authors studied the scattering of $0^-$ and $1^-$ particles [32] and fermion-antifermion annihilation processes with $\frac{1}{2}^+$ and $\frac{3}{2}^+$ particles [33].

So far most reactions involving two-body states with $0^-$ or $1^-$ particles and $\frac{1}{2}^+$ or $\frac{3}{2}^+$ particles have not been dealt with. It is the purpose of the present work to derive the covariant partial-wave amplitudes for the latter reactions. The technique applied in this work has been used previously in studies of two-body scattering systems with photons, pions and nucleons. [26 31 32 33 34 35 36 37 38 39 40 41 42 43 44]. Since the applications of the unitarization scheme [19 20 21 22 23 17] requires a detailed study of the analytic structure of any contribution to the generalized potential, it is instrumental to generate from a given chiral Lagrangian analytic expressions for such driving terms. In an initial step we decompose the scattering amplitude into invariant functions that are free of kinematical constraints [26 27 28 29 30 31 32 33 34]. Such amplitudes are expected to satisfy a Mandelstam's dispersion-integral representation [35 36]. Like in the previous work [32] we will derive a projection algebra that allows the analytic derivation of contributions to the invariant amplitudes from a given chiral Lagrangian by means of a computer algebra program. In a second step the non-trivial transformation from the helicity to the covariant partial-wave amplitudes is derived. As a side product of such an analysis we generate specific polynomials in the 4-momenta and Dirac matrices that solve the various Bethe-Salpeter equations in the presence of short range interactions analytically. This generalizes and systematizes the results of previous works [32 30 37].

The work is organized as follows. Section 2 introduces the conventions used for the kinematics of the various two-body reactions. The scattering amplitudes are decomposed into sets of invariant amplitudes free of kinematical constraints. In the following section the helicity partial-wave amplitudes are constructed within the given convention. The transformation to partial-wave amplitudes free of kinematical constraints are derived, discussed and presented. Section 4 offers a short summary.

### 2 On-shell scattering amplitudes

We consider two-body reactions of a boson with $J^P = 0^-, 1^-$ and a fermion with $J^P = \frac{1}{2}^+, \frac{3}{2}^+$ where we use the conventions introduced in [33 32] for the kinematics and wave functions. All derivations will be completely generic. A two-body reaction is characterized by the three Mandelstam variables $s$, $t$ and $u$ with

$$s + t + u = m^2 + M^2 + \bar{m}^2 + \bar{M}^2, \tag{2}$$

with the initial and final masses $m, M$ and $\bar{m}, \bar{M}$ respectively. In the center-of-momentum frame the 4-momenta...
q and $\bar{q}$ of the incoming and outgoing boson and those of the fermion, $p$ and $\bar{p}$ are determined by the scattering angle $\theta$ and the magnitudes of the initial and final three-momenta $q_{cm}$ and $\bar{q}_{cm}$. From [33][32] we recall the further useful notations

$$
\begin{align*}
\gamma^{\mu} &= q^{\mu} + p^{\mu} = q^{\mu} + \bar{p}^{\mu}, \\
\gamma^{\mu} &= \frac{1}{2}(q^{\mu} - p^{\mu}), \\
\bar{\gamma}^{\mu} &= \frac{1}{2}(q^{\mu} - \bar{p}^{\mu}), \\
\bar{\gamma}^{\mu} &= \frac{1}{2}(q^{\mu} - q^{\mu}) = \frac{1}{2}\bar{q}^{\mu} - q^{\mu}w_{\mu}, \\
\bar{\gamma}^{\mu} &= \frac{1}{2}(\bar{q}^{\mu} + q^{\mu}) = \frac{1}{2}\bar{q}^{\mu} + q^{\mu}w_{\mu}, \\
r \cdot r &= -\bar{q}_{cm}^{2} = \bar{r} \cdot r = -\bar{q}_{cm}^{2} = \cos \theta, \\
r \cdot r &= -q_{cm}^{2}, \\
r \cdot w &= 0 = w \cdot r, \\
q \cdot w &= s, \\
q \cdot w &= s, \\
q \cdot w &= s, \\
q \cdot w &= s, \\
q \cdot w &= s,
\end{align*}
$$

where the two 4-vectors $r_{\mu}$ and $\bar{r}_{\mu}$ are orthogonal to $w_{\mu}$.

The on-shell production and scattering amplitudes are defined in terms of plane-wave matrix elements of the scattering operator where we do not make explicit internal degrees of freedom like isospin or strangeness quantum numbers for simplicity. We decompose the scattering amplitudes into sets of invariant functions. The merit of the decomposition lies in the transparent analytic properties of the functions $F_{\pm}(s,t)$, which are expected to satisfy Mandelstam’s dispersion-integral representation [35][34]. For reactions involving non-zero spin particles it is not straightforward to identify such amplitudes.

We begin with the elastic scattering of a pseudoscalar boson off a spin-one-half fermion

$$
T_{0 \uparrow \rightarrow 0 \downarrow} F_{\mp}^{\pm}(\sqrt{s},t) (P_{\mp})_{0 \uparrow \rightarrow 0 \downarrow} + F_{\mp}^{-}(\sqrt{s},t) (P_{\mp})_{0 \uparrow \rightarrow 0 \downarrow},
$$

$$
\langle P_{\mp}\rangle_{0 \uparrow \rightarrow 0 \downarrow} = \bar{u}(\bar{p},\lambda_{p}) P_{\mp} u(p,\lambda_{p}),
$$

where the convention for the baryon wave functions $u(p,\lambda_{p})$ and $\bar{u}(\bar{p},\lambda_{p})$ with the helicity projections $\lambda_{p}$ and $\lambda_{p}$ is taken from [33][34]. In [31] we use the projection matrices $P_{\mp}$ with $P_{\pm} P_{\mp} = 0$ and

$$
P_{\pm} = \frac{1}{2\sqrt{s}} \left( \gamma^{\pm} \pm \gamma \right), \\
P_{\pm} P_{\pm} = P_{\pm}.
$$

The reaction is characterized by two scalar function. It is well studied in the literature (see e.g. [33][34][35]). The number of invariant amplitudes follows readily from the number of on-shell independent Dirac matrices. Further Dirac structures like $\not{p}$ or $\not{\bar{p}}$ are redundant

$$
\langle \not{p} P_{\pm}\rangle_{0 \uparrow \rightarrow 0 \downarrow} = M (P_{\pm})_{0 \uparrow \rightarrow 0 \downarrow}, \\
\langle P_{\pm} \not{p}\rangle_{0 \uparrow \rightarrow 0 \downarrow} = M (P_{\pm})_{0 \uparrow \rightarrow 0 \downarrow}.
$$

Owing to the MacDowell symmetry [33] it holds

$$
F_{1}^{-}(\sqrt{s},t) = F_{1}^{+}(\sqrt{s},t).
$$

While the functions $F_{1}^{\pm}$ depend on $\sqrt{s}$ and $t$ the particular combinations

$$
F_{1}(s,t) = F_{1}^{+}(\sqrt{s},t) + F_{1}^{-}(\sqrt{s},t),
$$

$$
F_{2}(s,t) = \sqrt{s} (F_{1}^{+}(\sqrt{s},t) - F_{1}^{-}(\sqrt{s},t)),
$$

depend on $s$ and $t$ and do satisfy Mandelstam’s dispersion-integral representation [35][34]. This implies that the functions $F_{1}(s,t)$, unlike the functions $F_{1}^{\pm}(\sqrt{s},t)$, do not have a square-root branch point at $s = 0$.

The invariant amplitudes $F_{1}^{\pm}$ can be derived by means of the following projection algebra

$$
\frac{1}{2} \text{tr}(P_{\pm} A Q_{A} A) = 1, \\
\frac{1}{2} \text{tr}(P_{\pm} A Q_{A} A) = 0,
$$

$$
A = \bar{p} + M,
$$

$$
Q_{\pm} = \frac{s}{i\delta} (r \cdot r) P_{\mp} - E_{\pm} E_{\pm} P_{\pm},
$$

$$
E_{\pm} = \frac{\sqrt{s}}{2} (1 - \delta) \pm M, \\
\delta = \frac{m^{2} - M^{2}}{s},
$$

$$
v^{2} = s ((r \cdot r)^{2} - r^{2} r^{2}) = -s q_{cm}^{2} q_{cm}^{2} \sin^{2} \theta.
$$

A slightly more complicated process involves one vector particle in the final state

$$
T_{0 \uparrow \rightarrow 1 \downarrow} F^{n}_{n} (\sqrt{s},t) (T^{(n)}_{\pm})_{0 \uparrow \rightarrow 1 \downarrow},
$$

$$
\langle T^{(n)}_{\pm}\rangle_{0 \uparrow \rightarrow 1 \downarrow} = \epsilon^{\mu}(q,\lambda_{q}) \bar{u}(\bar{p},\lambda_{p}) T^{(n)}_{\pm} u(p,\lambda_{p}),
$$

$$
T^{(1)}_{\pm} = \bar{\gamma}_{\mu} P_{\pm} i \gamma_{5}, \\
T^{(2)}_{\pm} = w_{\mu} P_{\pm} i \gamma_{5},
$$

where we use a notation analogous to the one introduced in [33]. The sign convention for the vector wave functions $\epsilon_{\mu}(q,\lambda_{q})$ and $\epsilon_{\mu}(q,\lambda_{q})$ are given in [32]. In [31] we use [5] and

$$
\bar{\gamma}_{\mu} = \bar{\gamma}_{\mu} - \frac{1}{s} \gamma_{\mu} w_{\mu}, \\
P_{\pm} \bar{\gamma}_{\mu} = \gamma_{\mu} P_{\pm}.
$$

For notational simplicity we do not introduce different notations for the invariant amplitudes $F_{1}^{\pm}(\sqrt{s},t)$ in the two reactions [31][32].

The number of invariant on-shell amplitudes $F_{n}^{\pm}$ with $n = 1, 2, 3$ follows from the number of helicity amplitudes. Since we are assuming parity conservation the total number of independent helicity amplitudes is

$$
\frac{1}{2} (2 S_{q} + 1) (2 S_{p} + 1) (2 S_{q} + 1) (2 S_{p} + 1),
$$

where $S_{q}, S_{p}$ and $S_{q}, S_{p}$ are the spins of the initial and final particles.

Since the invariant amplitudes are supposed to be free of kinematical constraints the tensor structures should involve the minimal number of momenta. There is one structure, $\gamma_{\mu} i \gamma_{5}$ which does not involve any momentum. Owing to the transversality of the spin-one wave functions with

$$
\epsilon_{\mu}(q,\lambda_{q}) k^{\mu} = 2 \epsilon_{\mu}(q,\lambda_{q}) w^{\mu},
$$

$$
\epsilon_{\mu}(q,\lambda_{q}) k^{\mu} = 2 \epsilon_{\mu}(q,\lambda_{q}) w^{\mu}.
$$
there are three structures with one momentum involved, \( \gamma_\beta \not\equiv i \gamma_\alpha \), \( i \gamma_\alpha T_\mu \) and \( i \gamma_\beta w_\mu \). Those 3 structures are part of our basis. It is left to identify the remaining two tensors, which involve necessarily two momenta. Our basis suggests the two structures \( \not\! \! \! \! \! \not \! \! \! \! \! \! \gamma_\alpha r_\mu \) and \( \not\! \! \! \! \! \! \not \! \! \! \! \! \! \gamma_\beta w_\mu \), but there are three more candidates built from the tensor \( \epsilon_{\mu \rho \sigma \beta} \gamma^\mu \) contracted with two momenta. It is left to show that for on-shell conditions the latter three tensors can be decomposed into our basis without generating kinematical singularities. Indeed, it holds

\[
\begin{align*}
\sqrt{s} \langle \epsilon_{\mu \rho \sigma \beta} \gamma^\mu P_{\pm} \epsilon^\rho \epsilon^\sigma \rangle_{0 \rightarrow 1} &= \langle \pm \sqrt{s} E_{\pm} T^{(3)}_{\mp \mu} \\
&= - \sqrt{s} E_{\pm} E_{\pm} T^{(1)}_{\pm \mu} + \sqrt{s} (\vec{r} \cdot r) T^{(1)}_{\mp \mu} \\
&= (\vec{M} + \sqrt{s}) E_{\pm} T^{(2)}_{\pm \mu} \mp (\vec{r} \cdot r) T^{(2)}_{\mp \mu} \langle 1 \rightarrow 1 \rangle ,
\end{align*}
\]

whence

\[
\begin{align*}
1 \sqrt{s} \langle \epsilon_{\mu \rho \sigma \beta} \gamma^\mu P_{\pm} \epsilon^\rho \epsilon^\sigma \rangle_{0 \rightarrow 1} &= \langle - E_{\pm} T^{(1)}_{\mp \mu} \\
&\pm T^{(3)}_{\mp \mu} \langle 1 \rightarrow 1 \rangle ,
\end{align*}
\]

It is obvious that any tensor involving three or more momenta has a regular on-shell decomposition into our basis. For instance, consider the two additional structures \( v_\mu P_{\pm} \) with

\[
v_\mu = \epsilon_{\mu \sigma \tau \beta} \vec{k}^\alpha \vec{w}^\tau \vec{k}^\beta .
\]

The on-shell identity

\[
\begin{align*}
\frac{1}{\sqrt{s}} \langle v_\mu P_{\pm} \rangle_0 \rightarrow 1 \frac{1}{2} &= \langle \pm \vec{E}_{\pm} \vec{E}_{\pm} T^{(1)}_{\pm \mu} + (\vec{r} \cdot r) T^{(1)}_{\mp \mu} \\
&- \frac{1}{2} (\delta + 1) E_{\pm} T^{(2)}_{\pm \mu} - \vec{E}_{\pm} T^{(3)}_{\mp \mu} \langle 1 \rightarrow 1 \rangle ,
\end{align*}
\]

shows that such structures are linear dependent of the six tensors \( T^{(n)}_{\pm \mu} \), introduced in (11). We conclude that the particular choice (11) leads to invariant amplitudes \( F^{(n)}_\mu(\sqrt{s}, t) \) that are free of kinematical constraints and manifest the MacDowell relations with

\[
F^{(n)}_{\mu}(\pm \sqrt{s}, t) = F^{(n)}_{\mu}(- \sqrt{s}, t) .
\]

It is useful to derive a suitable projection algebra analogous to the one displayed in (9). We find

\[
\begin{align*}
\frac{1}{2} tr (T^{(n)}_{\alpha \beta}) A Q^{(n)}_{\alpha \beta} &= \delta_{n k} Q^{(n)}_{\mu \nu} = 0 , \\
Q^{(0)}_{\pm 1} &= \pm \frac{\sqrt{s}}{v^2} P_{\pm} v^\mu , \\
Q^{(1)}_{\pm 2} &= - i \gamma_5 R_{\pm} w_\mu - \frac{1}{2} (1 + \delta) \sqrt{\frac{s}{v^2}} E_{\pm} Q^\mu v^\mu , \\
Q^{(2)}_{\pm 3} &= - i \gamma_5 R_{\pm} r_\mu - \sqrt{\frac{s}{v^2}} E_{\pm} Q^\mu v^\mu ,
\end{align*}
\]

with

\[
R_{\pm} = \frac{s}{v^2} \left( \vec{E}_{\pm} \vec{E}_{\pm} P_{\pm} - (\vec{r} \cdot r) P_{\pm} \right) .
\]

Following (32) the 4-vectors \( r_\mu, w_\mu \) and \( w_\mu \) are suitable linear combinations of \( \vec{r}, r \) and \( w \) as to have the convenient properties

\[
\begin{align*}
R_{\pm} \cdot r &= 1 , \\
R_{\pm} \cdot \vec{r} &= 0 , \\
w_{\pm} \cdot w &= 1 , \\
w_{\pm} \cdot r &= 0 , \\
w_{\pm} \cdot \vec{r} &= 0 , \\
w_{\pm} \cdot \vec{r} &= 0 .
\end{align*}
\]

The index \( \mu \) of a vector indicates whether it is orthogonal to both 4 momenta. We recall the explicit form of the auxiliary vectors \( r_\mu, w_\mu \) and \( w_\mu \) from (32). Given three 4-vectors \( a_\mu, b_\mu \) and \( c_\mu \) we introduce a vector, \( a_{bc} = a_c b_\mu \), as follows

\[
\begin{align*}
\hat{a}_{bc} &= a_{bc} - a_{\mu} \frac{c^\mu}{c^2} , \\
\hat{a}_{bc} c_\mu &= 0 , \\
\hat{a}_{bc} c^\mu &= 0
\end{align*}
\]

In the notation of (23) the desired vectors are identified with

\[
\begin{align*}
\hat{r}_\mu &= \frac{r_\mu}{r^2} \frac{\hat{r}_\mu}{\hat{r}^2} , \\
\hat{w}_\mu &= \frac{w_\mu}{w^2} \frac{\hat{w}_\mu}{\hat{w}^2} , \\
\hat{w}_\mu &= \frac{w_\mu}{w^2} \frac{\hat{w}_\mu}{\hat{w}^2} .
\end{align*}
\]

where we introduced the additional vectors \( \hat{r}_\mu, \hat{w}_\mu \) that will turn useful below. In the derivation of (19) the following expressions are useful

\[
\begin{align*}
v^2 &= s \left( (\vec{r} \cdot r) - \vec{v}^2 \right) , \\
v^2 &= s (r_1 \cdot r_1) - s \vec{v}^2 , \\
s (w_1 \cdot w) &= \frac{1}{2} (\delta + 1) (\vec{r} \cdot \vec{r}) , \\
s (w_1 \cdot \vec{w}) &= \frac{1}{2} (\delta + 1) (\vec{r} \cdot \vec{r}) , \\
v^2 &= \frac{1}{2} (\delta + 1) (\vec{r} \cdot \vec{r}) , \\
v^2 &= \frac{1}{2} (\delta + 1) (\vec{w} \cdot \vec{w})
\end{align*}
\]

We turn to the scattering of spin-one bosons off spin-one-half fermions. The scattering amplitude takes the generic form

\[
\begin{align*}
T^{(n)}_{\pm \mu} &= \langle \gamma_\mu P_{\pm} \rangle , \\
\langle \gamma_\mu P_{\pm} \rangle &= c^{\mu}(q, \lambda_\mu) u(p, \lambda_\mu) \\
&\times T^{(n)}_{\pm \mu}(p, \lambda_\mu) e^{\mu}(q, \lambda_\mu) ,
\end{align*}
\]

with

\[
\begin{align*}
T^{(1)}_{\pm \mu} &= \gamma_\mu P_{\pm} , \\
T^{(2)}_{\pm \mu} &= \gamma_\mu P_{\pm} \gamma_\mu , \\
T^{(3)}_{\pm \mu} &= \gamma_\mu P_{\pm} w_\mu , \\
T^{(4)}_{\pm \mu} &= \gamma_\mu P_{\pm} w_\mu , \\
T^{(5)}_{\pm \mu} &= \gamma_\mu P_{\pm} \gamma_\mu , \\
T^{(6)}_{\pm \mu} &= \gamma_\mu P_{\pm} w_\mu , \\
T^{(7)}_{\pm \mu} &= \gamma_\mu P_{\pm} \gamma_\mu , \\
T^{(8)}_{\pm \mu} &= \gamma_\mu P_{\pm} w_\mu ,
\end{align*}
\]
where we use
\[ \hat{g}_{\mu\nu} = g_{\mu\nu} - \frac{w_{\mu} w_{\nu}}{s}. \] (26)

The invariant amplitudes \( F_n^\pm(\sqrt{s}, t) \) satisfy the MacDowell relations. At first there appear many possible tensor structures. Besides the 18 structures introduced in (23) there are for instance the tensors \( v_\mu P_\pm v_\mu, r_\mu P_\pm \bar{r}_\mu \) and \( \epsilon_{\mu\nu\delta} r^\nu r^\delta \bar{P}_{\pm 7} \). They can be decomposed into basis tensors with regular coefficients. As an example we display the identity
\[ v_\mu v_\mu = v^2 \left[ g_{\mu\nu} - (w_\mu \cdot \bar{w}_\nu) w_\mu w_\nu - (w_\mu \cdot \bar{r}_\nu) w_\mu \bar{r}_\mu - (r_\mu \cdot \bar{w}_\nu) r_\mu w_\nu - (r_\mu \cdot \bar{r}_\nu) r_\mu \bar{r}_\mu \right]. \] (27)

A derivation of the invariant amplitudes \( F_n^\pm(\sqrt{s}, t) \) for a given physical system turns more and more tedious as the spins of the involved particles and therewith the number of invariant amplitudes increases. For the considered case there are 18 amplitudes to be determined and without a systematicatized approach this appears prohibitively cumbersome. Fortunately it is possible to establish a projection algebra analogous to the one displayed in (23) also for more complicated cases. By means of computer algebra programs it is then straightforward to determine the amplitudes. We find
\[ \frac{1}{2} \text{tr} \left( T_n^{(\mu)} A Q_{\pm,k}^{\mu} \right) = \delta_{nk} \delta_{ab}, \] (28)
with \( q_0 Q_{\pm,k}^{\mu} = 0, \quad q_\mu Q_{\pm,k}^{\mu} = 0, \quad Q_{\pm,k}^{\mu} = \frac{1}{\sqrt{v^2}} \sqrt{v^2} Q_{\pm,k}^{\mu} \),
\[ Q_{\pm,1}^{\mu} = \frac{1}{\sqrt{v^2}} \sqrt{v^2} Q_{\pm,1}^{\mu} \]
\[ Q_{\pm,2}^{\mu} = \frac{1}{\sqrt{v^2}} \sqrt{v^2} Q_{\pm,2}^{\mu} \]
\[ Q_{\pm,3}^{\mu} = \frac{1}{\sqrt{v^2}} \sqrt{v^2} Q_{\pm,3}^{\mu} \]
\[ Q_{\pm,4}^{\mu} = \frac{1}{\sqrt{v^2}} \sqrt{v^2} Q_{\pm,4}^{\mu} \]
\[ Q_{\pm,5}^{\mu} = \frac{1}{\sqrt{v^2}} \sqrt{v^2} Q_{\pm,5}^{\mu} \]
\[ Q_{\pm,6}^{\mu} = \frac{1}{\sqrt{v^2}} \sqrt{v^2} Q_{\pm,6}^{\mu} \]
\[ Q_{\pm,7}^{\mu} = Q_{\pm} \left( (w_\mu \cdot \bar{r}_\delta) - (w_\mu \cdot \bar{r}_\delta) \right) \frac{1}{\sqrt{v^2}} \sqrt{v^2} Q_{\pm}^{\mu} \]
\[ \pm (r_\mu \cdot r_\nu) E_+ s \sqrt{v^2} Q_{\pm,4}^{\mu} \]
\[ \mp (r_\mu \cdot r_\nu) E_+ s \sqrt{v^2} Q_{\pm,4}^{\mu} \]
\[ Q_{\pm,8}^{\mu} = Q_{\pm} \left( (r_\mu \cdot \bar{r}_\delta) - (r_\mu \cdot \bar{r}_\delta) \right) \frac{1}{\sqrt{v^2}} \sqrt{v^2} Q_{\pm}^{\mu} \]
\[ \pm (r_\mu \cdot r_\nu) E_+ s \sqrt{v^2} Q_{\pm,6}^{\mu} \]
\[ \mp (r_\mu \cdot r_\nu) E_+ s \sqrt{v^2} Q_{\pm,6}^{\mu} \]
We turn to the reactions involving the spin-three-half fermions. The simplest reaction is the production process
\[ T_0 \rightarrow 0 (k, k, w) = \sum_{\pm,n} F_n^\pm(\sqrt{s}, t) (T_n^{(\mu)} A) \]
\[ \langle T_n^{(\mu)} A \rangle = \bar{u}^\mu (p, \lambda_\rho) T_n^{(\mu)} u(p, \lambda_\rho), \] (29)
where we refer to (23) for the convention used for the spin-three-half wave function \( u(p, \lambda_\rho) \). For the associated projection algebra we derive
\[ \frac{1}{2} \text{tr} \left( T_n^{(\mu)} A Q_{\pm,k}^{\mu} \right) = \delta_{nk} \delta_{ab}, \] with \( A Q_{\pm,k}^{\mu} A \gamma_\rho = 0, \bar{p}^\mu Q_{\pm,k}^{\mu} = 0, \)
\[ Q_{\pm,1}^{\mu} = \frac{1}{\sqrt{v^2}} \sqrt{v^2} Q_{\pm,1}^{\mu} \]
\[ Q_{\pm,2}^{\mu} = \frac{1}{\sqrt{v^2}} \sqrt{v^2} Q_{\pm,2}^{\mu} \]
\[ Q_{\pm,3}^{\mu} = \frac{1}{\sqrt{v^2}} \sqrt{v^2} Q_{\pm,3}^{\mu} \]
\[ Q_{\pm,4}^{\mu} = \frac{1}{\sqrt{v^2}} \sqrt{v^2} Q_{\pm,4}^{\mu} \]
\[ Q_{\pm,5}^{\mu} = \frac{1}{\sqrt{v^2}} \sqrt{v^2} Q_{\pm,5}^{\mu} \]
\[ Q_{\pm,6}^{\mu} = \frac{1}{\sqrt{v^2}} \sqrt{v^2} Q_{\pm,6}^{\mu} \]
where \( P_{\pm,k}^{\mu} \) are designed to satisfy on-shell conditions as follows:
\[ A P_{\pm,k}^{\mu} A \gamma_\rho = 0, \quad \bar{p}^\mu P_{\pm,k}^{\mu} = 0, \]
\[ P_{\pm,1}^{\mu} = w^\mu \gamma_5 P_\pm \! - \! v^\mu ((r_\mu \cdot r_\nu) E_+ s \sqrt{v^2}, P_\pm \! + \! M E_\pm P_\pm)/v^2, \]
\[ P_{\pm,2}^{\mu} = r^\mu \gamma_5 P_\pm \! - \! v^\mu (\sqrt{s} E_\pm P_\pm)/v^2. \]

There are left further reactions involving at least one spin-three-half particle. We derived complete lists of regular tensors and their associated projection algebras
\[ T_{1/2} \rightarrow 0 (k, k, w) = \sum_{\pm,n} F_n^\pm(\sqrt{s}, t) (T_n^{(\mu)} A) \]
\[ \langle T_n^{(\mu)} A \rangle = \bar{u}^\mu (p, \lambda_\rho) T_n^{(\mu)} u(p, \lambda_\rho), \]
\[
\begin{align*}
T_{(1)}^{(0)} &= \tilde{q}_\mu P_\mu, & T_{(2)}^{(0)} &= w_\mu P_\mu \tilde{q}_\mu, \\
T_{(3)}^{(0)} &= r_\mu P_\mu \tilde{q}_\mu, & T_{(4)}^{(0)} &= w_\mu P_\mu \tilde{q}_\mu, \\
T_{(5)}^{(0)} &= r_\mu P_\mu, & T_{(6)}^{(0)} &= w_\mu P_\mu, \\
\frac{1}{2} \text{tr} (T_{a, \mu}^{(n)} A Q_{\mu \nu}^{(2) A}) &= \delta_{n k} \delta_{ab}, \\
\text{with} \quad \tilde{p}_\mu Q_{\pm, k}^{\mu} &= 0 = \Lambda Q_{\pm, k}^{\mu} A \tilde{\gamma}_\nu, \\
T_{0, \mu}^{(n)} &+ \frac{1}{2} (k, k, w) = \sum_{\pm, n} F_{\mu}^{(n)} (\sqrt{s}, t) (T^{(n)}_{\mu, \mu, \mu})^{\mu \nu} \frac{1}{2} \rightarrow \frac{1}{4} \\
(T^{(n)}_{\mu, \mu, \mu})^{\mu \nu} &= \epsilon^{\mu} (\bar{q}, \lambda_\mu) \bar{u}^\nu (\bar{p}, \lambda_\nu) T^{(n)}_{\mu, \mu, \mu} u^\nu (p, \lambda_\nu), \\
T_{(1)}^{(1)} &= \tilde{g}_{\mu} P_\mu, & T_{(2)}^{(1)} &= p_\mu P_\mu \tilde{g}_\mu, \\
T_{(3)}^{(1)} &= r_\mu P_\mu \tilde{g}_\mu, & T_{(4)}^{(1)} &= w_\mu P_\mu \tilde{g}_\mu, \\
T_{(5)}^{(1)} &= r_\mu P_\mu, & T_{(6)}^{(1)} &= w_\mu P_\mu, \\
\frac{1}{2} \text{tr} (T_{a, \mu}^{(n)} A Q_{\mu \nu}^{(2) k} \Lambda) &= \delta_{n k} \delta_{ab}, \\
\text{with} \quad \tilde{p}_\mu Q_{\pm, k}^{\mu} &= 0 = \Lambda Q_{\pm, k}^{\mu} A \tilde{\gamma}_\nu, \\
T_{0, \mu}^{(n)} &+ \frac{1}{2} (k, k, w) = \sum_{\pm, n} F_{\mu}^{(n)} (\sqrt{s}, t) (T^{(n)}_{\mu, \mu, \mu})^{\mu \nu} \frac{1}{2} \rightarrow \frac{1}{4} \\
(T^{(n)}_{\mu, \mu, \mu})^{\mu \nu} &= \epsilon^{\mu} (\bar{q}, \lambda_\mu) \bar{u}^\nu (\bar{p}, \lambda_\nu) T^{(n)}_{\mu, \mu, \mu} u^\nu (p, \lambda_\nu), \\
T_{(1)}^{(1)} &= \tilde{g}_{\mu} P_\mu, & T_{(2)}^{(1)} &= p_\mu P_\mu \tilde{g}_\mu, \\
T_{(3)}^{(1)} &= r_\mu P_\mu \tilde{g}_\mu, & T_{(4)}^{(1)} &= w_\mu P_\mu \tilde{g}_\mu, \\
T_{(5)}^{(1)} &= r_\mu P_\mu, & T_{(6)}^{(1)} &= w_\mu P_\mu, \\
\frac{1}{2} \text{tr} (T_{a, \mu}^{(n)} A Q_{\mu \nu}^{(2) k} \Lambda) &= \delta_{n k} \delta_{ab}, \\
\text{with} \quad \tilde{p}_\mu Q_{\pm, k}^{\mu} &= 0 = \Lambda Q_{\pm, k}^{\mu} A \tilde{\gamma}_\nu, \\
T_{0, \mu}^{(n)} &+ \frac{1}{2} (k, k, w) = \sum_{\pm, n} F_{\mu}^{(n)} (\sqrt{s}, t) (T^{(n)}_{\mu, \mu, \mu})^{\mu \nu} \frac{1}{2} \rightarrow \frac{1}{4} \\
(T^{(n)}_{\mu, \mu, \mu})^{\mu \nu} &= \epsilon^{\mu} (\bar{q}, \lambda_\mu) \bar{u}^\nu (\bar{p}, \lambda_\nu) T^{(n)}_{\mu, \mu, \mu} u^\nu (p, \lambda_\nu), \\
T_{(1)}^{(1)} &= \tilde{g}_{\mu} P_\mu, & T_{(2)}^{(1)} &= p_\mu P_\mu \tilde{g}_\mu, \\
\end{align*}
\]
Since the expressions for the projection algebras are increasingly tedious we refrain from detailing all of them in the main text. In [Appendix A] explicit expressions for the Q's can be found for all reactions except for the most tedious case.

We summarize that all amplitudes satisfy the MacDowell relations $F^a_n(+\sqrt{s}, t) = F^a_n(-\sqrt{s}, t)$ and the even amplitude combinations
\[
\sqrt{s} \left( F^a_n(\sqrt{s}, t) - F^a_n(-\sqrt{s}, t) \right)
\]
as introduced in this section are truly uncorrelated and satisfy Mandelstam's dispersion-integral representation [35, 44].

### 3 Partial-wave decomposition

The scattering operator, $T$, is decomposed into partial-wave amplitudes characterized by the total angular momentum $J$. Following the seminal work of Jacob and Wick [25] we consider helicity projections $\lambda_q, \lambda_p$ and $\lambda_q, \lambda_p$ of the scattering matrix, where we apply the helicity wave functions in the convention as introduced in [33, 32]. We write
\[
\langle \lambda_q \lambda_p | T | \lambda_q \lambda_p \rangle = \sum_J (2J + 1) \langle \lambda_q \lambda_p | T_J | \lambda_q \lambda_p \rangle d^{(J)}_{\lambda \lambda}(\theta),
\]

\[
d^{(J)}_{\lambda \lambda}(\theta) = (-)^{\lambda - \bar{\lambda}} d^{(J)}_{\lambda \bar{\lambda}}(\theta)
\]

\[
\langle \lambda_q \lambda_p | T_J | \lambda_q \lambda_p \rangle = \int_{-1}^{1} \frac{d \cos \theta}{2} \langle \lambda_q \lambda_p | T | \lambda_q \lambda_p \rangle d^{(J)}_{\lambda \lambda}(\theta),
\]

with $\lambda = \lambda_q - \lambda_p$ and $\bar{\lambda} = \lambda_q + \lambda_p$. In [33] we recall some general properties of the Wigner rotation functions, $d^{(J)}_{\lambda \lambda}(\theta)$. The phase conventions assumed in this work imply the parity relations
\[
(-\lambda_q, -\lambda_p) | T | -\lambda_q, -\lambda_p \rangle = (-)^{\Delta} \langle \lambda_q, \lambda_p | T | \lambda_q, \lambda_p \rangle,
\]
with $\Delta = S_q - S_p - S_q + S_p + \lambda - \bar{\lambda}$. (39)

The two parity sectors with $P = \pm 1$ are decoupled introducing parity eigenstates of good total angular momentum $J$, formed in terms of the helicity states [25]. Following [33] we introduce the angular momentum projection, $|\lambda_q, \lambda_p \rangle_J$, of the helicity state $|\lambda_q, \lambda_p \rangle$. We write
\[
|\lambda_q, \lambda_p \rangle_J, \quad \text{with} \quad T |\lambda_q, \lambda_p \rangle_J = T_J |\lambda_q, \lambda_p \rangle.
\]

We introduce states, $|n_{\pm}, J \rangle$, that are eigenstates of the total angular momentum operator $J$ and the parity operator $P$. The following state convention
\[
|2_{\pm}, J \rangle_{1_{\pm}} = \frac{1}{\sqrt{2}} \left| \left| 1 + 1, +\frac{1}{2} \right| J \mp 1, -1, \frac{1}{2} \right| J \rangle,
\]
\[
|3_{\pm}, J \rangle_{1_{\pm}} = \frac{1}{\sqrt{2}} \left| \left| 1, 0, +\frac{1}{2} \right| J \mp 1, -1, +\frac{1}{2} \right| J \rangle,
\]
\[
|1_{\pm}, J \rangle_{0_{\pm}} = \frac{1}{\sqrt{2}} \left| \left| 0, -\frac{1}{2} \right| J \mp 0, +\frac{1}{2} \right| J \rangle,
\]
\[
|2_{\pm}, J \rangle_{0_{\pm}} = \frac{1}{\sqrt{2}} \left| \left| 1, 0, -\frac{1}{2} \right| J \mp 0, +\frac{3}{2} \right| J \rangle,
\]
\[
|1_{\pm}, J \rangle_{1_{\pm}} = \frac{1}{\sqrt{2}} \left| \left| 1, 0, -\frac{1}{2} \right| J \mp 0, +\frac{1}{2} \right| J \rangle,
\]
\[
|2_{\pm}, J \rangle_{1_{\pm}} = \frac{1}{\sqrt{2}} \left| \left| 1 + 1, +\frac{1}{2} \right| J \mp 1, -1, +\frac{1}{2} \right| J \rangle.
\]

will be used, where we omit the sector index $S_q S_p$ on the right-hand sides for notational convenience. It holds
\[
P |n_{\pm}, J \rangle_{S_q S_p} = \pm (-1)^{J + \frac{1}{2}} |n_{\pm}, J \rangle_{S_q S_p},
\]
for all considered systems of this work. The helicity partial-wave amplitudes, $t^{J}_{\pm, ab}$, that carry good angular momentum $J$ and good parity $P$ are defined by
\[
t^{J}_{\pm, ab} = \langle a_{\pm}, J | T | b_{\pm}, J \rangle,
\]

where $a$ and $b$ label the states in the convention [11]. For sufficiently large $s$ the unitarity condition takes the simple form
\[
\imath |t^{J}_{\pm}|^{-1} = -\frac{q_{cm}}{4\pi} M_a \delta_{ab},
\]
where $M_a$ is the baryon mass in the channel $a$. According to [3] the momentum $q_{cm}$ is the on-shell value of $\sqrt{-t}$ in the channel $a$.

Helicity partial-wave amplitudes are correlated at specific kinematical conditions, see for example the review [11]. This is seen once the amplitudes $t^{J}_{\pm, ab}(\sqrt{s})$ are expressed in terms of the invariant functions $F^{J}_{\pm}(\sqrt{s})$. In contrast covariant partial-wave amplitudes $T^{J}_{\pm}(\sqrt{s})$ are free of kinematical constraints and can therefore be used efficiently in partial-wave dispersion relations. They are associated with covariant states and covariant projector polynomials which diagonalize the Bethe-Salpeter two-body scattering equation for local interactions [24, 31, 33, 32]. We introduce
\[
T^{J}_{\pm}(\sqrt{s}) = \sqrt{\frac{4 M M s}{E_{\pm} E_{\pm} (q_{cm} q_{cm})}} J^{\pm}_{\frac{1}{2}}
\]
Table 1. Non-vanishing coefficients $a^{J+k}_{\pm n}$ and $b^{J+k}_{\pm n}$ as introduced in (53) for the $0^{-\frac{1}{2}+} \rightarrow 0^{-\frac{3}{2}+}$ reaction.

$$\times \left[ C^J_{\pm}(\sqrt{s}) \right]^T T^J_{\pm}(\sqrt{s}) C^J_{\pm}(\sqrt{s}),$$

with real triangular matrices $C^J_{\pm}(\sqrt{s})$ and $C^J'_{\pm}(\sqrt{s})$ characterizing the transformation for the initial and final states from the helicity basis to the new covariant basis. We generate a convention in which the covariant partial-wave amplitudes $T^J_{\pm}(\sqrt{s})$ satisfy the MacDowell relations

$$T^J_{\pm}(\sqrt{s}) = T^J_{\pm}(\sqrt{s}) - (\tilde{q}_{cm} q_{cm})^2 \sqrt{2 J - 1} \sqrt{2 J + 3} E_{\pm} E_{\mp}$$

for all considered reactions. The transformation (45) implies a change in the phase-space distribution

$$\rho^J_{\pm}(\sqrt{s}) = -\tilde{\Im}[T^J_{\pm}(\sqrt{s})]^{-1}$$

for the off-diagonal elements. Our reference to $J$ and $P$ for notational simplicity. In order to gain insight into the non-linear integral equation we derive the asymptotic behavior of the partial-wave amplitude. Assuming at first that the phase-space matrix $\rho_{ab} \sim \delta_{ab}$ is diagonal the unitarity condition (19) implies the particular identity

$$|\Re T_{aa}|^2 \leq \begin{cases} \frac{1}{\rho_{aa}} & \text{if } \Im T_{aa} \geq 0, \\ \rho_{aa} & \text{if } \Im T_{aa} < 0 \end{cases}$$

From (50) it follows that

$$|\Re T_{aa}| < \frac{1}{|\rho_{aa}|}, \quad |\Im T_{aa}| < \frac{1}{|\rho_{aa}|},$$

the diagonal elements of the covariant partial-wave amplitudes are asymptotically bounded for large $\sqrt{s}$. This holds if the phase-space matrix $\rho_{aa}$ is bounded asymptotically by a non-vanishing constant. An identical conclusion may be drawn from (53) for the off-diagonal elements. Our conclusions can also be proven for a general triangular transformation matrix $C^J_{\pm}(\sqrt{s})$ that is asymptotically bounded and may lead to a non-diagonal phase-space matrix.

In a first step we reproduce the previous results of [2]. The transformation matrix for the $0^{-\frac{1}{2}+}$ state is

$$C^J_{\pm,0^+}(\sqrt{s}) = 1.$$
The important merit of (53, 54) is the absence of kinematic constraints, with the possible exception at $q_{cm} \neq 0$. A singularity at $\bar{p}$ has all properties we insist on a projector to have. Negative powers are prohibited since they would destroy the defining property that if partial-wave expanded via (38) it must be assumed $|J| \leq |m|$. The projectors as given in (55) are minimal in the sense that any possible alternative has properly multiplied by positive powers of $q_{cm}/q_{cm}$. The Bethe-Salpeter equation ill-defined. A projector polynomial $P_J(\cos \theta)$ contributes exclusively to a single partial-wave amplitude with $T_J^\ell(\sqrt{s}) = 1$.

A projector is defined for off-shell conditions with $q^2 \neq m^2, p^2 \neq M^2$ and $q^2 \neq \bar{m}^2, \bar{p}^2 \neq M^2$ and is independent on any mass parameter. This is because only then a projector structure generates an analytic solution of its associated Bethe-Salpeter equation, where a particular renormalization program built on dimensional regularization must be assumed. The projectors as given in (55) are minimal in the sense that any possible alternative has necessarily a higher mass dimension. A projector appropriately multiplied by positive powers of $q^2, \gamma \cdot p$ or $\bar{q}^2, \gamma \cdot \bar{p}$ has all properties we insist on a projector to have. Negative powers are prohibited since they would destroy the property that the projector has to solve the Bethe-Salpeter equation. A singular behaviour at for instance $q^2 = 0$ or $\gamma \cdot p = 0$ renders the Bethe-Salpeter equation ill-defined.

Table 2. Non-vanishing coefficients $a_{n}^{\ell+k}$ and $b_{n}^{\ell+k}$ as introduced in (55) for the $0-1^+ \rightarrow 1-1^+$ reaction.

| $k$ $n$ | $a_{n}^{\ell+k}$ | $b_{n}^{\ell+k}$ |
|---|---|---|
| $+\frac{3}{2}$ | $\frac{1}{3} \frac{q_{cm}}{q_{cm}} E_{\bar{p}} E_{\bar{q}} (\sqrt{s} E_{\bar{p}} - \sqrt{s} E_{\bar{q}}) / (2J + 1)$ | $\frac{1}{2} \frac{q_{cm}^2}{q_{cm}} ((2J - 1) q_{cm}^2 - (2J) q_{cm}^2 / \sqrt{s})$ |
| $+\frac{1}{2}$ | $E_{\bar{p}} (2 \bar{q} - E_{\bar{q}})$ | $-\frac{1}{2} (2J - 1)$ |
| $+\frac{1}{2}$ | $\pm \sqrt{E_{\bar{p}} E_{\bar{q}}}$ | $\mp s \sqrt{s}$ |
| $-\frac{1}{2}$ | $\mp \sqrt{s} E_{\bar{p}} + \sqrt{s}$ | $\pm (2J - 1)$ |
| $k$ $n$ | $a_{n}^{\ell+k}$ | $b_{n}^{\ell+k}$ |
| $+\frac{3}{2}$ | $-\frac{1}{3} \frac{q_{cm} q_{cm}}{s} \sqrt{s} E_{\bar{p}} E_{\bar{q}} / (2J + 2)$ | $+\frac{1}{2} \frac{q_{cm} q_{cm}}{s} \sqrt{s} E_{\bar{p}} E_{\bar{q}} / (2J + 2)$ |
| $+\frac{1}{2}$ | $\mp \sqrt{E_{\bar{p}} E_{\bar{q}}}$ | $\pm \sqrt{s}$ |
| $-\frac{1}{2}$ | $-\sqrt{s} E_{\bar{p}} E_{\bar{q}} / (2J + 2)$ | $\pm \sqrt{s}$ |

with

$$A_{\pm n}^{J}(\sqrt{s}) = \left( \frac{s}{q_{cm}} \right)^{J} \times \int_{-1}^{1} \frac{d \cos \theta}{2} F_{n}^{\pm}(\sqrt{s}, t) P_{J}(\cos \theta). \quad (54)$$

The important merit of (53, 54) is the absence of kinematical constraints, with the possible exception at $s = 0$. A singularity at $q_{cm} = 0$ is not realized due to the properties of the Legendre polynomials $P_{J}(\cos \theta)$. We recover the projector polynomials first derived in [2]. In a notation using (12) and the building blocks introduced in [3,4] we write

$$\pm \sqrt{s} Y_{\pm}^{J}(\bar{r}, \bar{w}) = Y_{J+1}^{(1)}(\bar{r}, \bar{w}) P_{\pm}^{\ell} + \frac{1}{2} Y_{J+1}^{(1)}(\bar{r}, \bar{w}) (\bar{r} \cdot \bar{\gamma}) P_{\pm}(\gamma \cdot r), \quad (55)$$

with the generic polynomials

$$Y_{n}^{(k)}(\bar{r}, \bar{w}) = \left( \frac{q_{cm}}{s} \right)^{n-k} \left( \frac{d}{d \cos \theta} \right)^{k} P_{J}(\cos \theta), \quad (56)$$

regular at $q_{cm} = 0$. A projector polynomial $Y_{n}^{(k)}$ has the defining property that if partial-wave expanded via (58) it

$$\cos \theta = -\frac{\bar{r} \cdot r}{q_{cm}} = -\frac{\bar{r} \cdot r}{\sqrt{\bar{r}^{2} \bar{r}^{2}}}, \quad (56)$$

with $\gamma \cdot p = 0$ renders the Bethe-Salpeter equation ill-defined.
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As emphasized before the dimension of a projector must not be altered by a multiplication with any mass parameter. The only other available scale \( \sqrt{s} \) can not be used, since that would alter the asymptotic property \([18]\).

We proceed with the more complicated \( J = \frac{1}{2} \) and \( J = \frac{3}{2} \) states. According to \([11]\) for a given \( J \) and \( P \) there are two, three and six possible partial-wave states respectively. For the first two the transformation matrices take the form

\[
C^J_{\pm, \frac{1}{2}}(\sqrt{s}) = \begin{pmatrix}
0 & M \sqrt{s} \\
\sqrt{\frac{2J+1}{2}}(2E\pm M) \sqrt{s} & M \\
\pm E_{\frac{3}{2}} & \pm E_{\frac{1}{2}}
\end{pmatrix}, \quad (57)
\]

\[
C^J_{\pm, \frac{3}{2}}(\sqrt{s}) = \begin{pmatrix}
0 & 0 \\
\pm M \sqrt{s} & 0 \\
\pm E_{\frac{1}{2}} & \pm E_{\frac{3}{2}}
\end{pmatrix}, \quad (58)
\]

with \( J = n + 1/2 \), \( \pm M \) and \( \omega = \sqrt{s} \). The energies \( E_{\frac{1}{2}} \) were introduced already in \([10]\). For the most complicated case the non-zero elements of the transformation matrix are given in Tab. \([5]\). The transformation matrices are not unitary and therefore the associated phase-space matrices \( \rho^J_{\pm}(\sqrt{s}) \) are not diagonal. In contrast to the helicity states the phase-space matrix in the covariant states does have off-diagonal elements. The quest for the elimination of kinematical constraints leads necessarily to off-diagonal elements. Note that the condition of minimal projectors leads possibly to distinct dimensions of the different columns in a transformation matrix. The asymptotic property of the phase-space matrix \([18]\) is readily confirmed using the explicit form \([57]\) and Tab. \([5]\) To the best knowledge of the authors the transformation matrices \( C^J_{\pm, \frac{1}{2}}(\sqrt{s}) \) presented here are novel and were not derived in the literature before.

In order to verify the acclaimed properties of the transformation matrices it suffices to consider the \( 0 \frac{1}{2} \rightarrow 0 \frac{1}{2} \), \( 0 \frac{3}{2} \rightarrow 1 \frac{1}{2} \) and \( 0 \frac{3}{2} \rightarrow 1 \frac{3}{2} \) reactions. Nevertheless, we cross check our results against all considered two-body reactions. Our derivations rely on extensive use of computer algebra programs that indeed verify our claims. While we obtained explicit expressions for all of such reactions,
which are useful in numerical applications, the results are too tedious to be shown fully here.

The decomposition of the covariant partial-wave amplitudes in terms of the invariant amplitudes \( F_n^{\pm} \) should not lead to a singular behavior. Following previous works \([33, 32]\) we introduce the convenient notation

\[
T_n^J(\sqrt{s}) = \sum_{k,n} a_{k,n}^{J+}(\sqrt{s}) A_{k,n}^{J+}(\sqrt{s}) + \sum_{k,n} b_{k,n}^{J+}(\sqrt{s}) A_{k,n}^{J+}(\sqrt{s}),
\]

where we introduced the functions \( A_{k,n}^{J+}(\sqrt{s}) \) already in \([54]\). The coefficient functions \( a_{k,n}^{J+}(\sqrt{s}) \) and \( b_{k,n}^{J+}(\sqrt{s}) \) for \( 0^+ \to 0^+ \) and \( 0^± \to 0^± \) are shown in Tab. 1 and Tab. 2.

We turn to the associated projector polynomials. They do not depend on any mass parameter and are regular in particular at \( r^2 = 0, \bar{r}^2 = 0 \) and \( \bar{r} \cdot r = 0 \). Since they should have minimal dimension they may come with different dimensions. Their derivation involves repeated use of the on-shell identities

\[
\bar{u}(\bar{p}) (\bar{r} \cdot \gamma) P_\pm = \pm E_\mp \bar{u}(\bar{p}) P_\pm, \quad P_\pm (\bar{\gamma} \cdot r) u(p) = \pm E_\mp u(p),
\]

(59)

We recall that if a projector \( \mathcal{Y}_{J,\pm,ab}(\bar{r}, r, w) \) is partial-wave expanded via \([38]\) it contributes exclusively to a single partial-wave amplitude with \( T_3^{J,\pm,ab}(\sqrt{s}) = 1 \). Since the explicit expressions are rather lengthy not all of them are shown here. We provide explicit results for the two production cases \( 0^± \to 0^±, 1^± \) The projector polynomials are expressed in terms of the tensor structures introduced in \([29, 11]\), where the two sectors are discriminated by the use of specific Lorentz indices. We find

\[
\mathcal{Y}_{n_{\pm,11}(\bar{r}, r, w)} = \pm \frac{3}{2\sqrt{s/2}} Y_{n_{11}}^{(1)} T_{11,\pm,\gamma} + \frac{3}{2\sqrt{s/2}} Y_{n_{11}}^{(1)} (\bar{r} \cdot \gamma) T_{11,\pm,\gamma},
\]

(60)
4 Summary

In this work we studied the generic properties of two-body reactions that are of central importance for the computation of the resonance spectrum of baryons in the hadrogens conjecture. We have constructed covariant partial-wave amplitudes for two-body reactions with $J^P = 0^+, 1^-$ and $J^P = \frac{3}{2}^+, \frac{5}{2}^+$ particles which are free from kinematical constraints. Those covariant partial-wave amplitudes are conveniently used in partial-wave dispersion relations. Explicit transformations from the conventional helicity states to the covariant states were derived and presented in this work. It was illustrated that the covariant partial-wave amplitudes are associated with projector polynomials that generate analytic solutions of the Bethe-Salpeter equation in the presence of short-range interactions. In an initial step we identified complete sets of invariant functions that parameterize the considered reaction amplitudes and are expected to satisfy Mandelstam's dispersion-integral representation. A convenient projection algebra was constructed that is instrumental in the derivation of the invariant amplitudes by means of computer algebra codes.

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Appendix A

In this Appendix we specify the projectors introduced in (31)–(35). The various sectors are discriminated by the use of specific Lorentz indices. While the presence of the index $\mu$ and $\bar{\mu}$ signals a spin-one particle, the indices $\nu$ and $\bar{\nu}$ a spin-three-half particle in the initial and final state respectively. For the (32) case

\[
Q_{\pm,1}^{\nu} = \pm \frac{s}{v^2} \left[ (\bar{r} \cdot r) P_{\pm,1}^{\nu} - \bar{E}_+ E_+ P_{\pm,1}^{\nu} \right],
\]

\[
Q_{\pm,2}^{\nu} = \pm \frac{1}{E_\pm} \left[ (\bar{r} \cdot r) Q_{\pm,3}^{\nu} + \bar{E}_+ E_+ Q_{\pm,3}^{\nu} \right],
\]

\[
Q_{\pm,3}^{\nu} = \pm \frac{s}{v^2} \left[ \sqrt{s} (\bar{r} \cdot r) P_{\pm,1}^{\nu} - \bar{E}_+ E_+ P_{\pm,1}^{\nu} \right],
\]

\[
Q_{\pm,4}^{\nu} = \pm \frac{s}{v^2} \left[ (\bar{r} \cdot r) Q_{\pm,5}^{\nu} + \bar{E}_+ E_+ Q_{\pm,5}^{\nu} \right],
\]

\[
Q_{\pm,5}^{\nu} = \pm \frac{s}{v^2} \left[ (\bar{r} \cdot r) P_{\pm,1}^{\nu} - \bar{E}_+ E_+ P_{\pm,1}^{\nu} \right],
\]

\[
Q_{\pm,6}^{\nu} = \pm \frac{s}{v^2} \left[ (\bar{r} \cdot r) Q_{\pm,3}^{\nu} - \bar{E}_+ E_+ Q_{\pm,3}^{\nu} \right],
\]

\[
Q_{\pm,7}^{\nu} = \pm \frac{s}{v^2} \left[ (\bar{r} \cdot r) P_{\pm,1}^{\nu} - \bar{E}_+ E_+ P_{\pm,1}^{\nu} \right],
\]

\[
Q_{\pm,8}^{\nu} = \pm \frac{s}{v^2} \left[ (\bar{r} \cdot r) Q_{\pm,3}^{\nu} + \bar{E}_+ E_+ Q_{\pm,3}^{\nu} \right],
\]

where

\[
\bar{p}_\nu P_{\pm,1}^{\nu} \equiv 0 = A P_{\pm,1}^{\nu} \bar{A}_{\gamma_\nu}, \quad \bar{q}_\mu P_{\pm,2}^{\nu} \equiv 0, \quad P_{\pm,1}^{\nu} = [r_1^\nu i_\gamma_\bar{\nu}(\sqrt{s}/v^2) \bar{E}_+ P_{\pm,1}^{\nu}] v^\mu/v^2.
\]
\[ P_{\pm,2}^{\mu} = [r_{\mu}^0 P_{\pm} + v_{\mu}^0 (\sqrt{s}/v^2) \hat{E}_{\pm} i \gamma_5 P_{\pm}] r_{\mu}^0, \]
\[ P_{\pm,3}^{\mu} = [r_{\mu}^0 P_{\pm} + v_{\mu}^0 (\sqrt{s}/v^2) \hat{E}_{\pm} i \gamma_5 P_{\pm}] w_{\mu}, \]
\[ P_{\pm,4}^{\mu} = [w_{\mu}^0 i \gamma_5 P_{\pm} - v_{\mu}^0 (\hat{r}_{\mu} \cdot \hat{P}_{\pm} + M \hat{E}_{\pm} P_{\pm})/v^2] \sqrt{v^2}, \]
\[ P_{\pm,5}^{\mu} = [w_{\mu}^0 P_{\pm} + v_{\mu}^0 i \gamma_5 ((\hat{r}_{\mu} \cdot \hat{P}_{\pm} + M \hat{E}_{\pm} P_{\pm})/v^2) r_{\mu}^0, \]
\[ P_{\pm,6}^{\mu} = [w_{\mu}^0 P_{\pm} + v_{\mu}^0 i \gamma_5 ((\hat{r}_{\mu} \cdot \hat{P}_{\pm} + M \hat{E}_{\pm} P_{\pm})/v^2) w_{\mu}^0. \]

For the \textbf{33} case:
\[
Q_{\mu,1}^{\mu} = \frac{1}{v^2} [\hat{r}_{\mu} \cdot \hat{P}_{\pm,1} - \hat{E}_{\pm} E_{\pm} P_{\pm,1}^{\mu}] , \]
\[
Q_{\mu,2}^{\mu} = \frac{1}{v^2} [\hat{r}_{\mu} \cdot \hat{P}_{\pm,2} - \hat{E}_{\pm} E_{\pm} P_{\pm,2}^{\mu}] , \]
\[
Q_{\mu,3}^{\mu} = \frac{1}{v^2} [\hat{r}_{\mu} \cdot \hat{P}_{\pm,3} - \hat{E}_{\pm} E_{\pm} P_{\pm,3}^{\mu}] , \]
\[
Q_{\mu,4}^{\mu} = \frac{1}{v^2} [\hat{r}_{\mu} \cdot \hat{P}_{\pm,4} - \hat{E}_{\pm} E_{\pm} P_{\pm,4}^{\mu}] . \]

where
\[
\bar{p}_\mu P_{\pm,1}^{\mu} = 0 = A P_{\pm,1}^{\mu} \hat{A} \gamma_0, \quad \nu_0 P_{\pm,1}^{\mu} = 0 = \nu_0 A P_{\pm,1}^{\mu} \hat{A}, \]
\[
P_{\pm,2}^{\mu} = \bar{r}_{\mu}^0 \gamma_5 P_{\pm} + \nu_{\mu}^0 (s/v^2) \hat{E}_{\pm} E_{\pm} P_{\pm}/v^2 \]
\[ - \bar{r}_{\mu}^0 \nu_{\mu}^0 (\sqrt{s}/v^2) \hat{E}_{\pm} i \gamma_5 P_{\pm}, \]
\[ P_{\pm,3}^{\mu} = \bar{r}_{\mu}^0 \nu_{\mu}^0 (\sqrt{s}/v^2) \hat{E}_{\pm} i \gamma_5 P_{\pm} \]
\[ - \bar{r}_{\mu}^0 \gamma_5 P_{\pm} + \nu_{\mu}^0 \gamma_5 P_{\pm} + M \hat{E}_{\pm} P_{\pm}/v^2 \]
\[ + \nu_{\mu}^0 \gamma_5 (\hat{s}/v^2) \hat{E}_{\pm} ((\hat{r}_{\mu} \cdot \hat{P}_{\pm} + M \hat{E}_{\pm} P_{\pm})/v^2), \]
\[ P_{\pm,4}^{\mu} = \bar{r}_{\mu}^0 \nu_{\mu}^0 (\sqrt{s}/v^2) \hat{E}_{\pm} ((\hat{r}_{\mu} \cdot \hat{P}_{\pm} + M \hat{E}_{\pm} P_{\pm})/v^2), \]
\[ + \bar{r}_{\mu}^0 \gamma_5 (\hat{s}/v^2) \hat{E}_{\pm} ((\hat{r}_{\mu} \cdot \hat{P}_{\pm} + M \hat{E}_{\pm} P_{\pm})/v^2), \]
\[ + \nu_{\mu}^0 \gamma_5 (\hat{s}/v^2) \hat{E}_{\pm} ((\hat{r}_{\mu} \cdot \hat{P}_{\pm} + M \hat{E}_{\pm} P_{\pm})/v^2), \]
\[ + \nu_{\mu}^0 \gamma_5 (\hat{s}/v^2) \hat{E}_{\pm} ((\hat{r}_{\mu} \cdot \hat{P}_{\pm} + M \hat{E}_{\pm} P_{\pm})/v^2), \]
\[ - \frac{1}{v^2} (\delta - 1) \nu_{\mu}^0 \gamma_5 (\hat{r}_{\mu} \cdot \hat{P}_{\pm} + M \hat{E}_{\pm} P_{\pm})/v^2, \]
\[ + \frac{1}{v^2} (\delta - 1) \nu_{\mu}^0 \gamma_5 (\hat{r}_{\mu} \cdot \hat{P}_{\pm} + M \hat{E}_{\pm} P_{\pm})/v^2, \]
\[ + \frac{1}{v^2} (\delta - 1) \nu_{\mu}^0 \gamma_5 (\hat{r}_{\mu} \cdot \hat{P}_{\pm} + M \hat{E}_{\pm} P_{\pm})/v^2, \]
\[ + \frac{1}{v^2} (\delta - 1) \nu_{\mu}^0 \gamma_5 (\hat{r}_{\mu} \cdot \hat{P}_{\pm} + M \hat{E}_{\pm} P_{\pm})/v^2. \]

For the \textbf{33} case:
\[ Q_{\mu,1}^{\mu} = 3(\delta - 1) \frac{v}{v^2} (\hat{r}_{\mu} \cdot \hat{E}_{\pm}) = \frac{1}{v^2} [\hat{r}_{\mu} \cdot \hat{P}_{\pm,1} - \hat{E}_{\pm} E_{\pm} P_{\pm,1}^{\mu}] , \]
\[ s \hat{E}_{\pm} E_{\pm} [\hat{r}_{\mu} \cdot \hat{P}_{\pm,1}^{\mu} - E_{\pm} E_{\pm} P_{\pm,1}^{\mu}] , \]
\[ + \sqrt{s} E_{\pm} [\hat{r}_{\mu} \cdot \hat{P}_{\pm,2}^{\mu} - E_{\pm} E_{\pm} P_{\pm,2}^{\mu}] . \]
\[
Q_{\mu\nu}^{\alpha,\beta} = \pm \sqrt{\frac{s}{M}} \left[ \tilde{P}_{\mu\nu}^{\alpha,\beta} - \sqrt{E_\pm} \tilde{P}_{\mu\nu}^{\alpha,\beta,\pm,16} \right] \\
\pm \frac{1}{2} \left( \delta - 1 \right) \left( \delta + 1 \right) \frac{s}{v^2} \left( \bar{r} \cdot r \right) \left[ \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,6} + \bar{E}_\pm + \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,6} \right] \\
\pm \frac{1}{2} \left( \delta + 1 \right) \frac{s}{v^2} E_\pm \left[ \left( \bar{r} \cdot r \right) \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,1} + \bar{E}_\pm E_\pm + \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,1} \right], \\
Q_{\mu\nu,4}^{\alpha,\beta} = \sqrt{\frac{s}{M}} \left[ \left( \bar{r} \cdot r \right) \bar{P}_{\mu\nu,4}^{\alpha,\beta} - \bar{E}_\pm \bar{E}_\pm + \tilde{P}_{\mu\nu,4}^{\alpha,\beta} \right] \\
+ \frac{1}{2} \left( \delta + 1 \right) \frac{s}{v^2} \left( \bar{r} \cdot r \right) \left[ \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,8} + \bar{E}_\pm \bar{E}_\pm + \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,8} \right] \\
\pm \frac{1}{2} \left( \delta + 1 \right) \frac{s}{v^2} \left( \bar{r} \cdot r \right) \left[ \bar{Q}_{\mu\nu}^{\alpha,\beta,\pm,2} + \bar{E}_\pm E_\pm \bar{Q}_{\mu\nu}^{\alpha,\beta,\pm,2} \right] \\
\pm \frac{1}{2} \left( \delta + 1 \right) \frac{s}{v^2} E_\pm \left[ \left( \bar{r} \cdot r \right) \bar{Q}_{\mu\nu}^{\alpha,\beta,\pm,12} + \bar{E}_\pm E_\pm \bar{Q}_{\mu\nu}^{\alpha,\beta,\pm,12} \right]. \\
\]

\[
Q_{\mu\nu,5}^{\alpha,\beta} = \sqrt{\frac{s}{M}} \left[ \left( \bar{r} \cdot r \right) \bar{P}_{\mu\nu,5}^{\alpha,\beta} - \bar{E}_\pm E_\pm + \tilde{P}_{\mu\nu,5}^{\alpha,\beta} \right] \\
+ \frac{1}{2} \left( \delta + 1 \right) \frac{s}{v^2} \left( \bar{r} \cdot r \right) \left[ \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,7} + \bar{E}_\pm E_\pm + \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,7} \right] \\
\pm \frac{1}{2} \left( \delta + 1 \right) \frac{s}{v^2} \left( \bar{r} \cdot r \right) \left[ \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,2} + \bar{E}_\pm E_\pm \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,2} \right] \\
\pm \frac{1}{2} \left( \delta + 1 \right) \frac{s}{v^2} E_\pm \left[ \left( \bar{r} \cdot r \right) \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,11} + \bar{E}_\pm E_\pm \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,11} \right]. \\
\]

\[
Q_{\mu\nu,6}^{\alpha,\beta} = \frac{1}{2} \left( \delta - 1 \right) \frac{1}{(r \cdot r)} \left[ \left( \bar{r} \cdot r \right) \bar{P}_{\mu\nu,6}^{\alpha,\beta} - \bar{E}_\pm E_\pm + \tilde{P}_{\mu\nu,6}^{\alpha,\beta} \right] \\
+ \frac{1}{2} \left( \delta - 1 \right) \frac{1}{(r \cdot r)} E_\pm \left[ \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,7} + \bar{E}_\pm E_\pm + \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,7} \right] \\
\pm \frac{1}{2} \left( \delta + 1 \right) \frac{1}{(r \cdot r)} \left[ \left( \bar{r} \cdot r \right) \bar{Q}_{\mu\nu}^{\alpha,\beta,\pm,9} + \bar{E}_\pm E_\pm \bar{Q}_{\mu\nu}^{\alpha,\beta,\pm,9} \right] \\
\pm \frac{1}{2} \left( \delta + 1 \right) \frac{1}{(r \cdot r)} E_\pm \left[ \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,8} + \bar{E}_\pm E_\pm \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,8} \right]. \\
\]

\[
Q_{\mu\nu,7}^{\alpha,\beta} = \sqrt{s} \left[ \left( \bar{r} \cdot r \right) \bar{P}_{\mu\nu,7}^{\alpha,\beta} - \bar{E}_\pm E_\pm + \tilde{P}_{\mu\nu,7}^{\alpha,\beta} \right] \\
\pm \frac{1}{2} \left( \delta - 1 \right) \frac{s}{v^2} \left( \bar{r} \cdot r \right) \left[ \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,7} + \bar{E}_\pm E_\pm + \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,7} \right] \\
\pm \frac{1}{2} \left( \delta + 1 \right) \frac{s}{v^2} E_\pm \left[ \left( \bar{r} \cdot r \right) \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,1} + \bar{E}_\pm E_\pm \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,1} \right]. \\
\]

\[
Q_{\mu\nu,8}^{\alpha,\beta} = \sqrt{s} \left[ \left( \bar{r} \cdot r \right) \bar{P}_{\mu\nu,8}^{\alpha,\beta} - \bar{E}_\pm E_\pm + \tilde{P}_{\mu\nu,8}^{\alpha,\beta} \right] \\
\pm \frac{1}{2} \left( \delta + 1 \right) \frac{s}{v^2} E_\pm \left[ \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,8} + \bar{E}_\pm E_\pm + \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,8} \right] \\
\pm \frac{1}{2} \left( \delta - 1 \right) \frac{s}{v^2} \left( \bar{r} \cdot r \right) \left[ \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,7} + \bar{E}_\pm E_\pm \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,7} \right] \\
\pm \frac{1}{2} \left( \delta - 1 \right) \frac{s}{v^2} \left( \bar{r} \cdot r \right) \left[ \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,1} + \bar{E}_\pm E_\pm \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,1} \right]. \\
\]

\[
Q_{\mu\nu,9}^{\alpha,\beta} = \pm \sqrt{s} \left[ \left( \bar{r} \cdot r \right) \bar{P}_{\mu\nu,9}^{\alpha,\beta} - \bar{E}_\pm E_\pm + \tilde{P}_{\mu\nu,9}^{\alpha,\beta} \right] \\
\pm \frac{1}{2} \left( \delta - 1 \right) \frac{s}{v^2} E_\pm \left[ \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,9} + \bar{E}_\pm E_\pm + \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,9} \right] \\
\pm \frac{1}{2} \left( \delta + 1 \right) \frac{s}{v^2} \left( \bar{r} \cdot r \right) \left[ \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,7} + \bar{E}_\pm E_\pm \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,7} \right] \\
\pm \frac{1}{2} \left( \delta - 1 \right) \frac{s}{v^2} \left( \bar{r} \cdot r \right) \left[ \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,1} + \bar{E}_\pm E_\pm \tilde{Q}_{\mu\nu}^{\alpha,\beta,\pm,1} \right]. \\
\]
(r · r) E_{\mp} \left[ (\bar{r} \cdot r) Q^{\alpha \beta}_{\mp,1} + E_{\pm} Q^{\alpha \beta}_{\pm,1} \right]
- (r \cdot r) E_{\pm} \left[ (\bar{r} \cdot r) Q^{\alpha \beta}_{\pm,2} + \bar{E}_{\pm} E_{\pm} Q^{\alpha \beta}_{\pm,2} \right]
+ \frac{1}{2} (\delta + 1) (\delta - 1) \frac{s}{v^2} (r \cdot r) Q^{\alpha \beta}_{\pm,3}
- \frac{1}{2} (\delta + 1) (\delta + 1) \frac{s}{v^2} (r \cdot r) Q^{\alpha \beta}_{\pm,4}
- \frac{1}{2} (\delta + 1) (\delta - 1) \frac{s}{v^2} (r \cdot r) Q^{\alpha \beta}_{\pm,5}
\pm \frac{1}{2} (\delta + 1) \frac{s}{v^2} E_{\mp} \left[ (\bar{r} \cdot r) Q^{\alpha \beta}_{\mp,13} + E_{\pm} E_{\pm} Q^{\alpha \beta}_{\pm,13} \right]
\pm \frac{1}{2} (\delta + 1) \frac{s}{v^2} E_{\mp} \left[ (\bar{r} \cdot r) Q^{\alpha \beta}_{\mp,10} + E_{\pm} E_{\pm} Q^{\alpha \beta}_{\pm,10} \right],
Q^{\alpha \beta}_{\pm,16} = \frac{s}{v^2} \left[ (\bar{r} \cdot r) P^{\alpha \beta}_{\pm,12} - E_{\mp} E_{\mp} P^{\alpha \beta}_{\mp,12} \right]
+ \frac{1}{2} (\delta - 1) \frac{s}{v^2} (r \cdot r) \left[ (\bar{r} \cdot r) Q^{\alpha \beta}_{\mp,5} - E_{\mp} E_{\mp} Q^{\alpha \beta}_{\pm,5} \right]
\pm \frac{1}{2} (\delta + 1) \frac{s}{v^2} E_{\mp} \left[ (\bar{r} \cdot r) Q^{\alpha \beta}_{\mp,3} + E_{\pm} E_{\pm} Q^{\alpha \beta}_{\pm,3} \right]
\pm \frac{1}{2} (\delta + 1) \frac{s}{v^2} E_{\mp} \left[ (\bar{r} \cdot r) Q^{\alpha \beta}_{\mp,10} + E_{\pm} E_{\pm} Q^{\alpha \beta}_{\pm,10} \right],
Q^{\alpha \beta}_{\pm,17} = \frac{s}{v^2} \left[ (\bar{r} \cdot r) P^{\alpha \beta}_{\pm,15} - E_{\mp} E_{\mp} P^{\alpha \beta}_{\mp,15} \right]
- \frac{1}{2} (\delta + 1) (\delta - 1) \frac{s}{v^2} \left[ 2 (\bar{r} \cdot r) Q^{\alpha \beta}_{\mp,8} - E_{\mp} E_{\mp} Q^{\alpha \beta}_{\pm,8} \right]
\pm \frac{8}{v^2} E_{\mp} \left[ (\bar{r} \cdot r) ((\bar{r} \cdot r) Q^{\alpha \beta}_{\mp,1} + E_{\pm} E_{\pm} Q^{\alpha \beta}_{\pm,1}) \right]
+ E_{\mp} E_{\mp} ((\bar{r} \cdot r) Q^{\alpha \beta}_{\mp,2} + E_{\pm} E_{\pm} Q^{\alpha \beta}_{\pm,2}) \right]
+ \frac{1}{2} (\delta + 1) \frac{s}{v^2} \left[ (\bar{r} \cdot r) Q^{\alpha \beta}_{\pm,11} + E_{\pm} E_{\pm} Q^{\alpha \beta}_{\pm,11} \right]
- \frac{1}{2} (\delta + 1) \frac{s}{v^2} \left[ (\bar{r} \cdot r) Q^{\alpha \beta}_{\pm,12} + E_{\pm} E_{\pm} Q^{\alpha \beta}_{\pm,12} \right]
+ \frac{1}{2} (\delta + 1) \frac{s}{v^2} \left[ (\bar{r} \cdot r) Q^{\alpha \beta}_{\pm,15} + E_{\pm} E_{\pm} Q^{\alpha \beta}_{\pm,15} \right],
Q^{\alpha \beta}_{\pm,18} = \frac{s}{v^2} \left[ (\bar{r} \cdot r) P^{\alpha \beta}_{\pm,14} - E_{\mp} E_{\mp} P^{\alpha \beta}_{\mp,14} \right]
- \frac{1}{2} (\delta + 1) \frac{s}{v^2} \left[ (\bar{r} \cdot r) (Q^{\alpha \beta}_{\mp,8} + Q^{\alpha \beta}_{\pm,8}) \right]
\pm \frac{8}{v^2} E_{\mp} \left[ (\bar{r} \cdot r) ((\bar{r} \cdot r) Q^{\alpha \beta}_{\mp,1} + E_{\pm} E_{\pm} Q^{\alpha \beta}_{\pm,1}) \right]
+ E_{\mp} E_{\mp} ((\bar{r} \cdot r) Q^{\alpha \beta}_{\mp,2} + E_{\pm} E_{\pm} Q^{\alpha \beta}_{\pm,2}) \right]
+ \frac{1}{2} (\delta + 1) \frac{s}{v^2} \left[ (\bar{r} \cdot r) Q^{\alpha \beta}_{\pm,11} + E_{\pm} E_{\pm} Q^{\alpha \beta}_{\pm,11} \right]
+ \frac{1}{2} (\delta + 1) \frac{s}{v^2} \left[ (\bar{r} \cdot r) Q^{\alpha \beta}_{\pm,12} + E_{\pm} E_{\pm} Q^{\alpha \beta}_{\pm,12} \right],

where
q_\mu P^{\alpha \beta}_{\pm,\mu} = 0, \quad p_\mu P^{\alpha \beta}_{\pm,\mu} = 0 \quad \bar{p}_\mu P^{\alpha \beta}_{\pm,\mu} = 0 = A P^{\alpha \beta}_{\pm,\lambda} A \gamma_\nu

\text{For the } \frac{5}{5} \text{ case}
Q^{\alpha \beta}_{\pm,1} = - (\delta - 1) \frac{s}{v^2} \sqrt{\frac{5}{M}} (r \cdot r) E_{\pm} \left[ (\bar{r} \cdot r) P^{\alpha \beta}_{\pm,1} - E_{\mp} E_{\mp} P^{\alpha \beta}_{\mp,1} \right]
- \sqrt{\frac{5}{M}} E_{\mp} \left[ (\bar{r} \cdot r) P^{\alpha \beta}_{\mp,1} - E_{\mp} E_{\mp} P^{\alpha \beta}_{\mp,1} \right]
- \frac{s}{v^2} \left( \frac{5}{M} \right) (r \cdot r) E_{\pm} \left[ (\bar{r} \cdot r) P^{\alpha \beta}_{\pm,1} - E_{\mp} E_{\mp} P^{\alpha \beta}_{\mp,1} \right]
- \sqrt{\frac{5}{M}} E_{\mp} \left[ (\bar{r} \cdot r) P^{\alpha \beta}_{\mp,1} - E_{\mp} E_{\mp} P^{\alpha \beta}_{\mp,1} \right]
+ \frac{1}{2} (\delta - 1) \frac{s}{v^2} \left( \frac{5}{M} \right) E_{\pm} E_{\mp} P^{\alpha \beta}_{\pm,1} \pm \sqrt{\frac{5}{M}} (r \cdot r) P^{\alpha \beta}_{\pm,1} + P^{\alpha \beta}_{\pm,1}
Q^{\alpha \beta}_{\pm,2} = - \frac{s}{v^2} \left( \frac{5}{M} \right) E_{\pm} E_{\mp} P^{\alpha \beta}_{\pm,1} \pm \sqrt{\frac{5}{M}} (r \cdot r) P^{\alpha \beta}_{\pm,1} + P^{\alpha \beta}_{\pm,1} + P^{\alpha \beta}_{\pm,1
\[ E_{\pm} \left[ (\mathbf{r} \cdot \mathbf{Q})_{\pm} - E_{\pm} E_{\mp} \right] + \gamma (\bar{\gamma} + 1) Q_{\pm}^\mu \\
\]
\[ Q_{\pm,3} = -2 \frac{s}{v} E_{\pm} E_{\pm} \left( (\mathbf{r} \cdot \mathbf{Q})_{\pm} - E_{\pm} E_{\mp} \right) + P_{\pm,5}^\mu \\
+ \frac{s}{v^2} E_{\pm} \left[ ((\mathbf{r} \cdot \mathbf{Q})_{\pm} + E_{\pm} E_{\mp}) \right] \\
\pm E_{\pm} \left[ (\mathbf{r} \cdot \mathbf{Q})_{\pm} - E_{\pm} E_{\mp} \right] , \\
\]
\[ Q_{\pm,4} = \pm \frac{\sqrt{s}}{M} \left[ P_{\pm,5,8}^\mu - \sqrt{s} E_{\mp} P_{\pm}^\mu \right] + \frac{1}{\sqrt{s}} Q_{\pm}^{\mu,4}, \\
\]
\[ Q_{\pm,5} = \pm \sqrt{s} P_{\pm,5}^\mu + \frac{1}{s} Q_{\pm}^{\mu,4} + 1 \frac{1}{\sqrt{s}} Q_{\pm}^{\mu,5}, \\
\]
\[ Q_{\pm,6} = \pm \sqrt{s} P_{\pm,5}^\mu + \frac{1}{s} Q_{\pm}^{\mu,4} + 1 \frac{1}{\sqrt{s}} Q_{\pm}^{\mu,5}, \\
\]
\[ Q_{\pm,7} = \pm \sqrt{s} P_{\pm,5}^\mu + \frac{1}{s} Q_{\pm}^{\mu,4} + 1 \frac{1}{\sqrt{s}} Q_{\pm}^{\mu,5}, \\
\]
where
\[ q_0 P_{\pm,5}^\mu = 0, \hspace{1cm} p_0 P_{\pm,5}^\mu = 0 = A P_{\pm,5}^\mu A, \]
\[ P_{\pm,1}^\mu = \gamma^3 \left[ r_{\pm r} \mathbf{P}_3 + v^\nu v^\nu (s/v^2) \mathbf{E}_\pm \mathbf{E}_\pm /v^2 \\
- r_{\pm r} v^\nu (s/v^2) \mathbf{E}_\pm \mathbf{E}_\pm i \gamma_5 \mathbf{P}_3 + v^\nu (s/v^2) \mathbf{E}_\pm \mathbf{E}_\pm \gamma_5 \mathbf{P}_3 /v^2 \right], \\
\]
\[ P_{\pm,2}^\mu = \gamma^3 \left[ r_{\pm r} v^\nu (s/v^2) \mathbf{E}_\pm \mathbf{E}_\pm i \gamma_5 \mathbf{P}_3 + v^\nu (s/v^2) \mathbf{E}_\pm \mathbf{E}_\pm \gamma_5 \mathbf{P}_3 /v^2 \right], \\
\]
\[ P_{\pm,3}^\mu = \gamma^3 \left[ w_{\pm r} v^\nu (s/v^2) \mathbf{E}_\pm \mathbf{E}_\pm i \gamma_5 \mathbf{P}_3 + v^\nu (s/v^2) \mathbf{E}_\pm \mathbf{E}_\pm \gamma_5 \mathbf{P}_3 /v^2 \right], \\
\]
\[ P_{\pm,4}^\mu = \gamma^3 \left[ w_{\pm r} v^\nu (s/v^2) \mathbf{E}_\pm \mathbf{E}_\pm i \gamma_5 \mathbf{P}_3 + v^\nu (s/v^2) \mathbf{E}_\pm \mathbf{E}_\pm \gamma_5 \mathbf{P}_3 /v^2 \right], \\
\]
\[ P_{\pm,5}^\mu = \gamma^3 \left[ r_{\pm r} v^\nu (s/v^2) \mathbf{E}_\pm \mathbf{E}_\pm i \gamma_5 \mathbf{P}_3 + v^\nu (s/v^2) \mathbf{E}_\pm \mathbf{E}_\pm \gamma_5 \mathbf{P}_3 /v^2 \right]. \\
\]
\[ P_{E}^{\pm,9} = \frac{w}{v} \sum \left[ \frac{w}{v} \overline{w} \frac{1}{v} \gamma_{5} P_{\pm} + v \overline{w} \nu^{\prime} \left( \sqrt{s}/v^{2} \right) \overline{E}_{\pm} P_{\mp} \right] \begin{array}{c} \pm \frac{1}{\sqrt{2}} \left( \delta - 1 \right) \left( \sqrt{s}/v^{2} \right) M \overline{E}_{\pm} \gamma_{5} \left( \overline{E}_{\mp} P_{\pm} - E_{\mp} \overline{E}_{\pm} P_{\pm} \right) \\
+ \frac{1}{\sqrt{2}} \left( \delta - 1 \right) \left( \sqrt{s}/v^{2} \right) M \overline{E}_{\pm} \gamma_{5} \left( \overline{E}_{\mp} P_{\pm} - E_{\mp} \overline{E}_{\pm} P_{\pm} \right) \end{array}, \]

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