Realization of Shor’s Algorithm at Room Temperature

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Shor’s algorithm can find prime factors of a large number more efficiently than any known classical algorithm. Understanding the properties that gives the speedup is essential for a general and scalable construction. Here we present a realization of Shor’s algorithm, that does not need any of the simplifications presently needed in current experiments and also gives smaller systematic errors than any former experimental implementation. Our realization is based on classical pass-transistor logic, runs at room temperature, and uses the same amount of resources as a scalable quantum computer. In this paper, the focus is not on the result of the factorization, but to compare our realization with current state-of-the-art experiment, factoring 15. Our result gives further insight to the resources needed for quantum computation, aiming for a true understanding of the subject.

Shor’s algorithm is one of the few quantum algorithms that solves a computational problem with real-world applications: to efficiently find a factor q in a composite number N, which is otherwise thought to be hard. In fact, the hardness of finding the integer factorization of a composite number, is one of the most widely believed conjectures in computer science, and cryptographic applications that we all use in our daily life are built upon this.

While large high-fidelity quantum computers are still far away, several experimental realizations of Shor’s algorithm for small numbers have been presented [1–7]. These are all very impressive demonstration of quantum optimal control, but experimental realization of Shor’s algorithm with the currently available technology is demanding, and this has led to the need for vast simplifications in the algorithm. There are essentially two parameters subject to optimization with regard to circuit size, bit-depth and circuit-depth, see [8] and citations therein. Also the approximate quantum Fourier transform [9] is crucial for scalability. In this optimization procedure, one has to be careful not to over-simplify, or make explicit (or implicit) use of knowledge about the solution [10].

Shor’s algorithm for integer factorization [11] finds the order (or period) of an element a in the multiplicative group of integers modulo N. Here, the order is the smallest integer r such that $a^r = 1 \mod N$. This is sufficient information to find a factor in N. The algorithm makes use of an input-register quantum state, containing an integer x, and an output-register in which modular exponentiation $a^x \mod N$ is computed. By changing the quantum state in the input-register using the quantum Fourier transform, performing the calculation, and then inverting the transform, one can with high probability retrieve sufficient information to calculate the order, see Figure 1A. More specifically, this procedure let us sample from a probability distribution with peaks at $s/r$, where s is uniformly distributed over the integers between 0 and $r - 1$.

Ideally the peaks are completely localized to $s/r$ but in most cases there is peak broadening due to Fourier leakage, and to ensure that the measurement yields a binary fraction sufficiently close to $s/r$, the input-register needs to be at least $2n$ qubits in size, where $n$ is the size of the output-register which is large enough to perform calculations mod N. The full procedure to retrieve r is as follows:

1. Pick at random an integer $a \neq \pm 1$ modulo N. If GCD($a, N$) is a nontrivial factor of N, we have a solution.
2. Otherwise generate, setup, and run the quantum subroutine of Figure 1A to find a candidate for $s/r$.
3. Use the continued fraction expansion to retrieve r (or a factor in r when s and r has a common factor).
4. If r is even, one of GCD($a^{r/2} \pm 1, N$) may be a nontrivial factor of N. This happens with high probability.

For our example $N = 15$ the possible integers that can occur in steps 2-4 are $a \in \{2, 4, 7, 8, 11, 13\}$. One should beware of simplification such as “compilation”, where the element a is chosen deliberately to give a short period that is easy to find. It is therefore important that the element $a$ is chosen randomly [see e.g., 10]. In what follows, we have used all alternatives; this is of course only possible because of the small N used.

Some useful simplifications are allowed. The Fourier transform in Figure 1A can be exchanged for Hadamard gates, while the inverse Fourier transform can be exchanged for Hadamards followed by classically controlled single qubit rotations [12]; by advancing the measurement of the controlling qubit and using the outcome as a classical control. This decouples the 2n qubits of the input-register in the sense that the procedure of preparation, transformation, and measurement can be performed individually on each qubit. It is common to perform these single qubit procedures in sequence on one single qubit, a method known as qubit recycling, which reduces the overall bit-depth from $3n$ to $n + 1$ at the cost of circuit-depth.

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The demonstration of Monz et al [1] is the most advanced to date. They use non-Clifford group operations which is absolutely necessary to demonstrate the advantage of quantum computation [13, 14], and also refrain from “compiling” the circuitry. One simplification they do use is restricting the resolution of the input register from 2n qubits to three qubits. This is possible because all elements in the multiplicative group mod 15 has power-of-two periods, and Monz et al only verify the behavior of the exponentiation until and including the first that is equivalent to the identity map (see Figure 1C).

It has been argued [15] that the three-bit precision of Monz et al is insufficient since 2n bits are required for the algorithm to overcome Fourier leakage in general, to succeed with a bounded error rate required for scalability. However, for N = 15 there is no Fourier leakage because of the power-of-two periods, and this is clear when building the quantum gate array. Therefore, measuring more qubits will only add noise, not precision. The distribution is completely described at two bits of precision. On the other hand, this also means that the process of generating the circuitry solves the factoring problem, since it is enough to know at which point the identity emerges from exponentiation. Thus, the experiment [1] is not so much about factoring 15 through the Shor algorithm, but more a verification that the quantum circuit used behaves as expected when not using a compiled circuit.

We now present an experimental realization of Shor’s algorithm, factoring 15. Please note that the motivation for doing this is not to factor 15, but to compare our construction with the current state-of-the-art. We use basically the same algorithmic setup, but employ the semiclassical Fourier transform and not of qubit recycling, and the emerging identity operations are of course all omitted. Our setup is similar to that of Monz et al, without the qubit recycling, see Figure 1B. Similar to Monz et al we use the multiplication operators from Markov et al [8], but avoid precomputing their effect.
FIG. 3: QSL realization of Shor’s algorithm, when \( a = 8 \) so that the modular multipliers used are, from left to right, \( \times 8^2 \equiv \times 4 \pmod{15} \), and \( \times 8 \pmod{15} \).

FIG. 4: Estimated output probability distributions of the subroutine for all non-trivial elements \( a \) in the multiplicative group of integers mod 15. Each plot (distribution) is estimated from \( 10^6 \) samples.

on the initial state, see Figure 1C.

Our setup runs on quantum logical gates from an improved version of the QSL framework [16, 17] inspired by Spekkens’ model, where a qubit is simulated by two classical bits; one for the computational basis and one for the phase. A source of \(|0\rangle\) or \(|1\rangle\) initializes the computational bit to the bit value and randomizes the phase bit. A measurement returns the value of the computational bit (and randomizes the phase bit). The operation of the gates can be found in Figure 2, where Hadamard and CNOT are as in [17]. The new classically controlled \( R_2 \)-gate XORs the computational-bit onto the phase-bit and then inverts the computational-bit, if the control is 1. The Toffoli construction is much improved, and is used to construct the Fredkin gate needed for Shor’s algorithm. The physical implementation is in 2-complementary reversible pass-transistor logic, specifically using transmission gates [18]. These are constructed with currently available semiconductor technology, and operated at an ambient temperature around 300 K.

The output probability distributions are estimated from \( 10^6 \) samples for each element \( a \in \{2, 4, 7, 8, 11, 13\} \), see Figure 4. We see that the distributions for \( a = 7, 8, 13 \) are not uniform as predicted by quantum theory, but to contrast with the square statistical overlap (SSO) [19] used by Monz et al. as a fidelity measure, we get \{0.9999(1), 0.9999(1), 0.933(3), 0.984(2), 0.9999(1), 0.984(2)\} for \( a \in \{2, 4, 7, 8, 11, 13\} \) respectively (statistical errors as one standard deviation). Notably, the implementation gives the same probability (0.5) of returning a good candidate for \( r \), as the ideal quantum subroutine.

In conclusion, we have created and implemented a framework consisting of sources, gate array transformations, and measurements that can be used to run a quan-
quantum algorithm to a precision at least as good as state-of-the-art experiments factoring 15. The framework uses only classical resources, and the overhead is constant, in terms of memory and gate count (space and time). As any other physical implementation of quantum gates, our simulation suffers from systematic errors. This will result in an error propagation that suppresses the amount of useful information that we can retrieve when scaling the algorithm to larger numbers — even though there is no practical restriction for doing so. Further work is needed to reduce these systematic errors, perhaps by altering the framework or using error correcting techniques, so that the framework becomes useful for larger instances.

Most importantly, the QSL framework provides a fair comparison between quantum and classical computation. Even though it uses classical resources, it contains quantum-like degrees of freedom, and reproduces many phenomena of quantum mechanics including interference. This is enough to approach the general behavior of quantum algorithms. The remaining systematic errors point to properties of quantum gates and systems that the QSL framework does not reproduce. Knowledge of this substantially narrows down the search area for what the truly quantum resources are, and also points to the properties that are necessary for providing a quantum advantage. This is crucial information for present and future projects aiming at building a quantum computer.

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