Automorphism-Invariant Non-Singular Rings and Modules

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Abstract. A ring $A$ is a right automorphism-invariant right non-singular ring if and only if $A = S \times T$, where $S$ a right injective regular ring and $T$ is a strongly regular ring which contains all invertible elements of its maximal right ring of quotients. Over a ring $A$, each direct sum of automorphism-invariant non-singular right modules is an automorphism-invariant module if and only if the factor ring of the ring $A$ with respect to its right Goldie radical is a semiprime right Goldie ring.

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Key words: automorphism-invariant ring, automorphism-invariant module, injective module, quasi-injective module

1. Introduction and preliminaries

All rings are assumed to be associative and with zero identity element; all modules are unitary. A module $M$ is said to be automorphism-invariant if $M$ is invariant under any automorphism of its injective hull. In [2] Dickson and Fuller studied automorphism-invariant modules, when the underlying ring is a finite-dimensional algebra over a field with more than two elements. In [3, Theorem 16] Er, Singh and Srivastava proved that a module $M$ is automorphism-invariant if and only if $M$ is a pseudo-injective module, i.e., for any submodule $X$ of $M$, every monomorphism $X \to M$ can be extended to an endomorphism of the module $M$. Pseudo-injective modules were studied in several papers; e.g., see [8], [15], [16]. Automorphism-invariant modules were studied in several papers; e.g., see [1], [3], [5], [11], [13], [16], [17], [18], [19], [20].

A ring $A$ is said to be regular if every its principal right (left) ideal is generated by an idempotent. A ring $A$ is said to be strongly regular if every its principal right (left) ideal is generated by a central idempotent. A module $X$ is said to be injective relative to the module $Y$ or $Y$-injective if for any submodule $Y_1$ of $Y$, every homomorphism $Y_1 \to X$ can be extended to a homomorphism $Y \to X$. A module is said to be injective if it is injective with respect to any module. A module is said to be square-free if it does not contain a direct sum of two non-zero isomorphic submodules. A submodule $Y'$ of the module $X$ is said to be essential in $X$ if $Y \cap Z \neq 0$ for any non-zero submodule $Z$ of $X$. A submodule $Y$ of the module $X$ is said to be closed in $X$ if $Y = Y'$ for every submodule $Y'$ of $X$ which is an essential extension of the module $Y$. We denote by Sing $X$ the singular submodule of the right $A$-module $X$, i.e., Sing $X$ is a fully invariant submodule of $X$ which consists of all elements $x \in X$ such that $r(x)$ is an essential right ideal of the ring $A$. A module $X$ is said to be non-singular if Sing $X = 0$.

Remark 1.1. In [3] Theorem 7, Theorem 8, Example 9] Er, Singh and Srivastava proved the following results.

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(1) If $A$ is a right non-singular, right automorphism-invariant ring, then $A = S \times T$, where the ring $S$ is right injective, the module $T_P$ is square-free, any sum of closed right ideals of $T$ is a two-sided ideal which is an automorphism-invariant right $T$-module, and for any prime ideal $P$ of $T$ which is not essential in $T_P$, the factor ring $T/P$ is a division ring.

(2) If $A$ is a right non-singular, right automorphism-invariant prime ring, then the ring $A$ is right injective.

(3) Let $F$ be the field of order 2, $S$ the direct product of a countable set of copies of $F$, and $A = \{(f_n)_{n=1}^\infty \in S : \text{almost all } f_n \text{ are equal to some } a \in F\}$. Then $A$ is a commutative automorphism-invariant regular ring, but it is not an injective $A$-module.

In connection to Remark 1.1, we will prove Theorem 1.2 which is the first main result of the given paper.

**Theorem 1.2.** For a ring $A$, the following conditions are equivalent.

1) $A$ is a right automorphism-invariant right non-singular ring.

2) $A$ is a right automorphism-invariant regular ring.

3) $A = S \times T$, where $S$ is a right injective regular ring and $T$ is a strongly regular ring which contains all invertible elements of its maximal right ring of quotients.

**Remark 1.3.** A module $X$ is said to be quasi-injective if $X$ is injective relative to $X$, i.e., for any submodule $X_1$ of $X$, every homomorphism $X_1 \to X$ can be extended to an endomorphism of the module $X$. Every quasi-injective module is an automorphism-invariant module, since the module $X$ is quasi-injective if and only if $X$ is invariant under any endomorphism of its injective hull; e.g., see [10, Theorem 6.74]. Every finite cyclic group is a quasi-injective non-injective module over the ring $\mathbb{Z}$ of integers.

For a module $X$, we denote by $G(X)$ the intersection of all submodules $Y$ of the module $X$ such that the factor module $X/Y$ is non-singular. The submodule $G(X)$ is a fully invariant submodule of $X$; it is called the Goldie radical of the module $X$.

**Remark 1.4.** In [9, Theorem 3.8] Kutami and Oshiro proved that any direct sum of non-singular quasi-injective right modules over the ring $A$ is quasi-injective if and only if $A/G(A_A)$ is a semiprime right Goldie ring.

In connection to Remark 1.4, we will prove Theorem 1.5 which is the second main result of the given paper.

**Theorem 1.5.** For a ring $A$ with right Goldie radical $G(A_A)$, the following conditions are equivalent.

1) $A/G(A_A)$ is a semiprime right Goldie ring.

2) Any direct sum of automorphism-invariant non-singular right $A$-modules is an automorphism-invariant module.

3) Any direct sum of automorphism-invariant non-singular right $A$-modules is an injective module.
The proof of Theorem 1.2 and Theorem 1.5 is decomposed into a series of assertions, some of which are of independent interest.

We give some necessary definitions. A module \( X \) is said to be singular if \( X = \text{Sing} X \). A module \( X \) is a Goldie-radical module if \( X = G(X) \). The relation \( G(X) = 0 \) is equivalent to the property that the module \( M \) is non-singular. In the paper, we use well-known properties of \( \text{Sing} X, G(X), \) non-singular modules and maximal right rings of quotients; e.g., see [2, Chapter 2], [10, Section 7] and [12, Section 3.3]. A module \( Q \) is called an injective hull of the module \( M \) if \( Q \) is an injective module and \( M \) is an essential submodule of the module \( Q \). A module \( M \) is called a CS module if every its closed submodule is a direct summand of the module \( M \). A module \( M \) is said to be uniform if the intersection of any two non-zero submodules of the module \( M \) is not equal to zero. A module \( M \) is said to be finite-dimensional if \( M \) does not contain an infinite direct sum of non-zero submodules.

A ring \( A \) is called a right Goldie ring if \( A \) is a right finite-dimensional ring with the maximum condition on right annihilators. A ring \( A \) is said to be reduced if \( A \) does not have non-zero nilpotent elements. A ring without non-zero nilpotent ideals is said to be semiprime ring. A ring \( A \) is said to be right strongly semiprime if any its ideal, which is an essential right ideal, contains a finite subset with zero right annihilator. A ring is said to be right strongly prime if every its non-zero ideal contains a finite subset with zero right annihilator.

Remark 1.6. Every right strongly semiprime ring is a right non-singular semiprime ring. It is clear that every right strongly prime ring is right strongly semiprime. The direct product of two finite fields is a finite commutative strongly semiprime ring which is not strongly prime. The direct product of a countable number of fields is an example of a commutative semiprime non-singular ring which is not strongly semiprime. All finite direct products of rings without zero-divisors and all finite direct products of simple rings are right and left strongly semiprime rings.

Remark 1.7. If \( A \) is a semiprime right Goldie ring, then it is well known that every essential right ideal of the ring \( A \) contains a non-zero-divisor. Therefore, all semiprime right Goldie rings are right strongly semiprime. In particular, all right Noetherian semiprime rings are right strongly semiprime.

2. Automorphism-Invariant Nonsingular Rings

Lemma 2.1 [12, Section 3.3]. Let \( A \) be a right non-singular ring with maximal right ring of quotients \( Q \). Then \( Q \) is an injective right regular ring and \( Q \) can be naturally identified with the ring \( \text{End} Q_A \) and \( Q_A \) is an injective hull of the module \( A_Q \).

Lemma 2.2. If \( A \) is a right non-singular ring with maximal right ring of quotients \( Q \), then \( A \) is a right automorphism-invariant ring if and only if \( A \) contains all invertible elements of the ring \( Q \).

Lemma 2.2 follows from Lemma 2.1.

Lemma 2.3 [13, Chapter 12, 5.1–5.4]. Let \( A \) be a reduced ring with
maximal right ring of quotients Q. Then ring A is right and left non-singular. If each of the closed right ideals of the ring A is an ideal, then A is a reduced ring and Q is a right and left injective strongly regular ring.

**Lemma 2.4.** Let A be a right non-singular ring in which all closed right ideals are ideals. Then the ring A is reduced.

**Proof.** Let a be an element of A with $a^2 = 0$. There exists a closed right ideal B of A such that $B \cap aA = 0$ and $B + aA$ is an essential right ideal of A. By assumption, the closed right ideal B is an ideal. Therefore, $aB = 0$, since $aB$ is contained in the intersection of $aA$ and B. Then $a(B + aA) = 0$ and $B + aA$ is an essential right ideal. Since A is right non-singular, $a = 0$. □

**Lemma 2.5.** If A is a right automorphism-invariant right non-singular ring, then $A = S \times T$, where S is a right injective regular ring and T is a strongly regular ring which contains all invertible elements of its maximal right ring of quotients.

**Proof.** By Remark 1.1(1) and Lemma 2.1, $A = S \times T$, where S is a right injective regular ring, T is a right automorphism-invariant right non-singular ring, and any closed right ideal of T is an ideal. By Lemma 2.4, T is a reduced ring. Let Q be the maximal right ring of quotients of the ring T. By Lemma 2.3, T is a reduced ring and Q is a right and left injective strongly regular ring. To prove that T is a strongly regular ring, it is sufficient to prove that an arbitrary element t of the ring T is the product of a central idempotent and an invertible element. Since t is an element of the strongly regular ring Q, we have that $t = eu$, where e is a central idempotent of the ring Q and u is an invertible element of the ring Q. By Lemma 2.2, T contains all invertible elements of the ring Q. Therefore, $u \in T$. Then $e = tu^{-1} \in T$ and every element of the ring T is the product of a central idempotent and an invertible element. □

**Remark 2.6.** The completion of the proof of Theorem 1.2. In Theorem 1.2, the implication 1) $\Rightarrow$ 3) follows from Lemma 2.5, the implication 3) $\Rightarrow$ 2) follows from the property that the direct product of regular rings S and T is a regular ring, and the implication 2) $\Rightarrow$ 1) follows from the property that every regular ring is right and left non-singular.

**Corollary 2.7.** If A is a right automorphism-invariant right non-singular indecomposable ring, then A is a right injective ring; see Remark 1.1(2).

Corollary 2.7 follows from Theorem 1.2 and the property that every strongly regular indecomposable ring is a division ring and, consequently, a right injective ring.

**Corollary 2.8.** Let A be a right automorphism-invariant right non-singular ring which does not contain an infinite set of non-zero central orthogonal idempotents. Then A is a right injective ring.

Corollary 2.8 follows from Corollary 2.7 and the property that every ring, which does not contain an infinite set of non-zero central orthogonal idempotents, is a finite direct product of indecomposable rings.

3. Automorphism-Invariant Non-singular Modules

**Lemma 3.1.** Let A be a ring and X a right A-module which is not an essential extension of a singular module. Then there exists a non-zero right
ideal $B$ of the ring $A$ such that the module $B_A$ is isomorphic to a submodule of the module $X$.

Proof. Since the module $X$ is not an essential extension of a singular module, there exists an element $x \in X$ such that $xA$ is a non-zero non-singular module. Since $xA \cong A_A/r(x)$ and the module $xA$ is non-singular, the right ideal $r(x)$ is not an essential. Therefore, there exists a non-zero right ideal $B$ with $B \cap r(x) = 0$. In addition, there exists an epimorphism $f: A_A \to xA$ with kernel $r(x)$. Since $B \cap \ker f = 0$, we have that $f$ induces the monomorphism $g: B \to xA$. Therefore, $xA$ contains the non-zero submodule $g(B)$ which is isomorphic to the module $B_A$. □

Lemma 3.2. Let $A$ be a ring, $G = G(A_A)$ the right Goldie radical of the ring $A$, $h: A \to A/G$ the natural ring epimorphism and $X$ a non-singular non-zero right $A$-module.

1) If $B$ is a essential right ideal of the ring $A$, then $h(B)$ is an essential right ideal of the ring $h(A)$.

2) If $B$ is a right ideal of the ring $A$ such that $G \subseteq B$ and $h(B)$ is an essential right ideal of the ring $h(A)$, then $B$ is an essential right ideal of the ring $A$.

3) For any right $A$-module $M$, the module $MG$ is contained in the Goldie radical of $M$.

4) $XG = 0$ and the natural $h(A)$-module $X$ is non-singular. In addition, if $Y$ is an arbitrary non-singular right $A$-module, then $YG = 0$ and the $h(A)$-module homomorphisms $Y \to X$ coincide with the $A$-module homomorphisms $Y \to X$. Therefore, $X$ is an $Y$-injective $A$-module if and only if $X$ is an $Y$-injective $h(A)$-module. The essential submodules of the $h(A)$-module $X$ coincide with the essential submodules of the $A$-module $X$.

5) $X$ is an injective $h(A)$-module if and only if $X$ is an injective $A$-module.

6) $X_{h(A)}$ is a uniform module (resp., an essential extension of a direct sum of uniform modules) if and only if $X_A$ is a uniform module (resp., an essential extension of a direct sum of uniform modules).

7) $X_A$ is an essential extension of a direct sum of modules each of them is isomorphic to some non-zero right ideal of the ring $A$.

8) If the ring $A$ is right finite-dimensional, then $X_A$ is an essential extension of a direct sum of modules each of them is isomorphic to some non-zero uniform right ideal of the ring $A$.

9) If the ring $h(A)$ is right finite-dimensional, then $X_{h(A)}$ is an essential extension of a direct sum of modules each of them is isomorphic to some non-zero uniform right ideal of the ring $h(A)$.

Proof. 1. Let us assume that $h(B)$ is not an essential right ideal of the ring $h(A)$. Then there exists a right ideal $C$ of the ring $A$ such that $C$ properly
contains \( G \) and \( h(B) \cap h(C) = h(0) \). Since \( h(B) \cap h(C) = h(0) \), we have that \( B \cap C \subseteq G \). Since \( C \) properly contains the closed right ideal \( G \), we have that \( C_A \) contains a non-zero submodule \( D \) with \( D \cap G = 0 \). Since \( B \) is an essential right ideal, \( B \cap D \neq 0 \) and \( (B \cap D) \cap G = 0 \). Then \( h(0) \neq h(B \cap D) \leq h(B) \cap h(C) = h(0) \). This is a contradiction.

2. Let us assume that \( B \) is not an essential right ideal of the ring \( A \). Then \( B \cap C = 0 \) for some non-zero right ideal \( C \) of the ring \( A \) and \( G \cap C \subseteq B \cap C = 0 \). Therefore, \( h(C) \neq h(0) \). Since \( h(B) \) is an essential right ideal of the ring \( h(A) \), we have that \( h(B) \cap h(C) \neq h(0) \). Let \( h(0) \neq h(b) = h(c) \in h(B) \cap h(C) \), where \( b \in B \) and \( c \in C \). Then \( c - b \in G \subseteq B \). Therefore, \( c \in B \cap C = 0 \) and \( h(c) = h(0) \). This is a contradiction.

3. For any element \( m \in M \), the module \( mG_A \) is a Goldie-radical module, since \( mG_A \) is a homomorphic image of the Goldie-radical module \( G \). Therefore, \( mG = M \), since it contains at least one maximal element \( K \).

4. By 3, \( XG = 0 \). Let us assume that \( x \in X \) and \( xh(B) = 0 \) for some essential right ideal \( h(B) \), where \( B = h^{-1}(h(B)) \) is the complete pre-image of \( h(B) \) in the ring \( A \). By 2, \( B \) is an essential right ideal of the ring \( A \). Then \( xB = 0 \) and \( x \in \text{Sing} X = 0 \). Therefore, \( X \) is a non-singular \( h(A) \)-module. The remaining part of 4 is directly verified.

5. Let \( R \) be one of the rings \( A \), \( h(A) \) and \( M \) a right \( R \)-module. By Lemma 1(4), the module \( M \) is injective if and only if \( M \) is injective relative to the module \( R_R \). Now the assertion follows from 4.

6. The assertion follows from 4.

7. Let \( \mathcal{M} \) be the set of all submodules of the module \( X \) which are direct sums of modules each of them is isomorphic to a non-zero right ideal of the ring \( A \). The set \( \mathcal{M} \) is not empty by Lemma 3.1. There exists a partial order in \( \mathcal{M} \) such that for any \( M, M' \in \mathcal{M} \), the relation \( M \not\leq M' \) is equivalent to the property that \( M' = M \oplus N \) for some \( N \in \mathcal{M} \). By the Zorn lemma, the set \( \mathcal{M} \) contains at least one maximal element \( K \).

Let us assume that \( K \) is not an essential submodule of the module \( X \). Then there exists a non-zero submodule \( L \) of the non-singular module \( X \) with \( K \cap L = 0 \). By Lemma 3.1, there exists a non-zero right ideal \( B \) of the ring \( A \) such that the module \( B_A \) is isomorphic to some submodule \( L' \) of the module \( L \). This contradicts to the property that \( K \) is a maximal element of the set \( \mathcal{M} \).

8. Since the ring \( A \) is right finite-dimensional, every non-zero right ideal of the ring \( A \) is an essential extension of a finite direct sum of non-zero uniform right ideals. Now the assertion follows from 7.

9. The assertion follows from 6 and 8. \( \square \)

Remark 3.3. Let \( M \) be an automorphism-invariant non-singular module. In [3] Theorem 3, Theorem 6(ii)] Er, Singh and Srivastava proved that \( M = X \oplus Y \), where \( X \) is a quasi-injective non-singular module, \( Y \) is an automorphism-invariant non-singular square-free module, the modules \( X \) and \( Y \) are injective relative to each other, \( \text{Hom}(X,Y) = 0 = \text{Hom}(Y,X) \) and \( \text{Hom}(D_1, D_2) = 0 \) for any two submodules \( D_1, D_2 \) of the module \( Y \) with \( D_1 \cap D_2 = 0 \). In addition, for any set \( \{K_i | i \in I\} \) of closed submodules of \( Y \), the submodule \( \sum_{i \in I} K_i \) is
Remark 3.4. Let $M$ be a direct sum of CS modules $M_i$, $i \in I$. In [11] Corollary 15 Lee and Zhou proved that $M$ is a quasi-injective module if and only if $M$ is an automorphism-invariant module.

Remark 3.5. Let $A$ be a ring with right Goldie radical $G(A_A)$. In [9] Theorem 3.4 Kutami and Oshiro proved that the factor ring $A/G(A_A)$ is a right strongly semiprime ring if and only if every non-singular quasi-injective right $A$-module is injective.

In [9] Theorem 3.8 Kutami and Oshiro proved that the factor ring $A/G(A_A)$ is a semiprime right Goldie ring if and only if every direct sum of non-singular quasi-injective right $A$-modules is quasi-injective.

Lemma 3.6. Let $A$ be a ring with right Goldie radical $G(A_A)$ and $M$ an automorphism-invariant non-singular right $A$-module which is an essential extension of a direct sum of uniform modules.

1) $M$ is an essential extension of some quasi-injective non-singular module $K$ which is direct sum of uniform modules closed in $M$.

2) If the factor ring $A/G(A_A)$ is a right strongly semiprime ring, then $M$ is an injective module.

Proof. 1. By Remark 3.3, $M = X \oplus Y$, where $X$ is a quasi-injective module, $Y$ is an automorphism-invariant square-free module. Therefore, we can assume that $M$ is an automorphism-invariant square-free module. Since $M$ is an essential extension direct sum of uniform submodules, $M$ is an essential extension of some module $K$ which is the direct sum of uniform closed submodules $K_i$ of $M$, $i \in I$. By Remark 3.3, $K$ is an automorphism-invariant module. Since every uniform module is a CS module, $K$ is a quasi-injective module by Remark 3.4.

2. By 1, $M$ is an essential extension of some quasi-injective non-singular module $K$. By Remark 3.5, $K$ is an injective essential submodule of the module $M$. Therefore, $K$ is an essential direct summand of the module $M$. Then $M = K$ and $M$ is an injective module.

Lemma 3.7. Let $A$ be a ring with right Goldie radical $G(A_A)$ and $M$ an automorphism-invariant non-singular right $A$-module. If the factor ring $A/G(A_A)$ is a semiprime right Goldie ring, then $M$ is an injective module.

Proof. By Lemma 3.2(9), $M$ is an essential extension of a direct sum of uniform modules. In addition, the semiprime right Goldie ring $A/G(A_A)$ is a right strongly semiprime ring [6]. By Lemma 3.6(2), $M$ is an injective module.

Remark 3.8. The completion of the proof of Theorem 1.5. In Theorem 1.5, the implications $3) \Rightarrow 2) \Rightarrow 1)$ are obvious.

1) $\Rightarrow 3)$. Let $M$ be the direct sum of automorphism-invariant non-singular right $A$-modules $M_i$, $i \in I$. By Lemma 3.7, each of the modules $M_i$ is injective. By Remark 3.5, $M$ is a quasi-injective module. By Remark 3.5, $M$ is an injective module.
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