On functions and inverses, both positive, decreasing and convex: And Stieltjes functions

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Abstract: Any function \( f \) from \((0, \infty)\) onto \((0, \infty)\) which is decreasing and convex has an inverse \( g \) which is positive, decreasing and convex. When \( f \) has some form of generalized convexity we determine additional convexity properties inherited by \( g \). When \( f \) is positive, decreasing and \( p, q \)-convex, its inverse \( g \) is \( q, p \)-convex. Related properties which pertain when \( f \) is a Stieltjes function are developed. The results are illustrated with the Stieltjes function \( f(x) = \arctan(1/\sqrt{x})/\sqrt{x} \) via a transcendental equation.

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1. Introduction

If \( f \) is positive and (strictly) decreasing then, clearly, it has an inverse which is positive and decreasing: the inverse will be denoted by \( g \). If, in addition, \( f \) is convex, it is very easy to show that \( g \) is also convex. For ease of exposition, assume appropriate differentiability. Denote derivatives with a prime. Starting from \( f(x) = y, g(y) = x \) and \( f(g(y)) = y \), and thence \( f'g' = 1 \), a further differentiation gives

\[
(f''(g))^2 + f'g'' = 0. \tag{1.1}
\]

Using \( f' < 0 \) it follows that, if one of \( f \) or \( g \) is convex, so is the other. This result is well-known (see Mršević, 2008; Proposition 1 of Hiriart-Urruty & Martinez-Legaz, 2003), as is the fact that one can remove the differentiability assumptions (for which, see the remarks following Theorem 1).

Theorems A to D are either elementary results or those proved by other authors. Theorems numbered numerically are proved in this paper.

Theorem A. The inverse of a positive, decreasing convex function is positive, decreasing and convex.

ABOUT THE AUTHOR

All the authors are applied mathematicians with interests in a variety of problems from fluid mechanics. See google scholar. New mathematical theorems are sometimes needed to establish results in the applications. This cogent paper is an instance of this.

PUBLIC INTEREST STATEMENT

Convex functions have many applications. New facts about them are likely to be valuable in many areas. Our paper establishes new facts concerning inverse of decreasing convex functions. Our proofs are elementary. This illustrates, again, that simple results concerning classical topics remain to be discovered.
Another elementary fact is

**Theorem B.** Both the sum, \( f_0 + f_1 \), and the product, \( f_0 f_1 \), of a pair of positive, decreasing convex functions, \( f_0 \) and \( f_1 \), are positive, decreasing and convex.

The preceding theorems A and B are well-known but, prior to the authors’ investigations, straightforward developments, in which further convexity properties of \( f \) yield further properties of its inverse, do not appear to have had any systematic treatment. In this connection a small development of equation (1.1) is useful:

\[
\frac{g''}{(g')^2} = -\frac{f''}{f'} \quad \text{so} \quad \frac{gg''}{(g')^2} = -\frac{xf''}{f'}.
\]  

(1.2)

The background to this paper is that an application to a partial differential equation (pde) problem—studied in (Keady & Wiwatanapataphee, 2016, 2017, 2018)—required knowledge of convexity or concavity properties of a function \( \mu \) defined through an implicit equation. It happened that the inverse of this function, denoted \( \phi_1 \), and the closely related function \( \phi_2 \),

\[
\phi_1(z) = \frac{1}{2} \ \arctan \left( \frac{1}{z} \right), \quad \phi_2(z) = \phi_1(\sqrt{z}),
\]  

(1.3)

happen to be completely monotone and have various other easily established properties. The function \( \phi_2 \) is Stieltjes. (For definitions, see 4.) While calculations with the implicitly defined function were possible—and, indeed were made before the neater approach here was discovered—the routine calculation effort was somewhat unsatisfying. Reducing the amount of calculation (though some remains) led to the topic of this paper: determining convexity properties of the inverse of a function from those of the original function.

There are several functions for this paper.

- General results:
  - (1) We publicise results published in Appendix A of the pde paper (Keady & Wiwatanapataphee, 2018). This is because results such as Theorem 1 will have applications outside pde.
  - (2) Theorems 2, 3 and 4 form a survey of convexity properties of Stieltjes functions and their inverses. There are many open questions, some of which concern derivatives higher than the second, e.g. Question 2(ii) of §4.1.

- Results concerning \( \phi_1 \) and \( \phi_2 \) and their inverses:
  - (1) Convexity properties of \( \phi_1 \), \( \phi_2 \) and their inverses, such as are needed to establish the results in (Keady & Wiwatanapataphee, 2018, 2017), are briefly reviewed.
  - (2) Several of the properties of \( \phi_1 \) and \( \phi_2 \) proved to be easiest established by beginning by noting that they are completely monotone, and \( \phi_2 \) is Stieltjes.

Very early on in this study we obtained evidence, based on the first few derivatives of the inverses of the \( \phi \), that the following questions might be answered affirmatively.

**Question 1.** Is the inverse of \( \phi_2 \) completely monotone?

Is the inverse of \( \phi_1 \) completely monotone (and, perhaps even Stieltjes)?

These questions—and the more general ones in §4—remain unresolved.
Theorem 1 is very elementary and appears to be new (to this paper). Theorems 2 onwards will include some readers coming from the application in pde. of (Keady & Wiwatanapataphee, 2017, 2018) who are not familiar with Stieltjes functions, etc. For this reason, we have surveyed relevant definitions and simple properties in a way we hope will be useful for such readers. Because we have many questions in the paper, we highlight the outline of the paper here so readers can return to it if they need to do so.

In §2, by way of introduction to the general fact that derivatives of a function \(f\) allow one to find properties of its inverse, \(g\), we consider first derivatives. A memorable result is that, if \(f\) is a Stieltjes function not equal to a multiple of \(1/x\), the inverse of \(f\) is not a Stieltjes function.

In §3 we treat second derivatives and convexity matters. The main result from this section is Theorem 1 which establishes that if a positive, decreasing, convex function \(f\) is \((p,q)\)-convex, its inverse \(g\) is \((q,p)\)-convex. Various special cases of “convexity with respect to means” are noted.

In §4 we note convexity properties of Stieltjes functions. The Stieltjes function \(\phi_2\) is the key to neat proofs of properties of its inverse.

In §5 we present the transcendental equation which led to this study, and review the properties of \(\phi_1, \phi_2\) and their inverses \(\mu, \mu_2\).

In §6 we instance a few other situations where a function is defined through a transcendental equation and the methods of this paper might be useful.

In §7 we conclude with a short discussion and comment on the questions asked earlier in the paper.

In our application, Stieltjes functions, denoted \(S\), and/or their reciprocals, the Complete Bernstein Functions, denoted \(CBF\) arise. In view of this, and, possible future applications, we discuss these at various points in this paper, notably in §4 and subsequently. As an instance of the sort of neat similarity that sometimes occurs we note the elementary fact:

A function \(h : (0, \infty) \rightarrow \mathbb{R}\) is convex (concave), if and only if \(xh(1/x)\) is convex (concave).

This has a parallel, see (Schilling, Song, & Vondracek, 2010) p61 equation (7.3):

A function \(h : (0, \infty) \rightarrow \mathbb{R}\) is \(CBF\) if and only if \(xh(1/x)\) is \(CBF\).

A function \(s : (0, \infty) \rightarrow \mathbb{R}\) is \(S\) if and only if \(s(1/x)/x\) is \(S\).

As an example concerning the last statement, we mention the Stieltjes function \(\phi_2\) occuring in our application:

\[
\phi_2(z) := \frac{1}{\sqrt{2}} \arctan\left(\frac{1}{\sqrt{2}}z\right), \quad \text{and} \quad \frac{1}{z} \phi_2(z) = \frac{\arctan(\sqrt{z})}{\sqrt{2}}.
\]

The latter function is also a Stieltjes function, which is easily established independently. See the table in §4.3.

2. First derivatives

This section is genuinely elementary, with extremely simple calculations. Nevertheless it yields results which may not be very well known.

As mentioned earlier, an interest arising from our \(\phi_2\) being Stieltjes is what can be said about the inverses of Stieltjes functions. Completely monotone and Stieltjes functions are treated in §4, but some easy results can be stated now. The first concerns first derivatives only.

**Theorem C.** Let \(f\) be a positive decreasing (differentiable) function on \(x > 0\).

(i) Each of the functions \(f(\frac{1}{x})/x\) and \(1/(xf(x))\) is positive and decreasing iff
\[ xf''(x) + f(x) > 0 \quad \text{for } x > 0. \tag{2.1} \]

(ii) The set of positive decreasing functions satisfying inequality \((2.1)\) is closed under addition.

(iii) For any positive, decreasing \(f\) satisfying \((2.1)\), its inverse, denoted by \(g\), is positive, decreasing, and
\[ yg'(y) + g(y) > 0 \quad \text{for } y > 0. \tag{2.2} \]

Hence that each of the functions \(g\left(\frac{1}{y}\right)\) and \(1/(yg(y))\) is positive and increasing.

\textbf{Proof.} (i) follows from routine differentiation. (ii) is obvious.

(iii) Writing \(x = g(y)\) and \(y = f(x)\) and noting \(f'(x) = 1/g'(y)\), inequality \((2.1)\) rewrites to
\[ \frac{g(y)}{g'(y)} + y > 0. \]
Since \(g'(y) < 0\), inequality \((2.2)\) follows. The result in the final sentence is established with routine differentiation.

\textbf{EXAMPLE.}
\[ f(x) = \frac{1}{\sqrt{x}}, \quad xf''(x) + f(x) = \frac{1}{2\sqrt{x}}. \]
\[ g(y) = \frac{1}{y^2}, \quad yg'(y) + g(y) = -\frac{1}{y^2}. \]

\textbf{Corollary 1.} The only (positive decreasing) Stieltjes functions \(f\) whose inverses are also Stieltjes functions are the positive multiples of (the involution) \(f(x) = 1/x\).

\textbf{Proof.} One can appeal to the results that for any Stieltjes function \(f\) each of the functions \(f\left(\frac{1}{x}\right)\) and \(1/(xf(x))\) is Stieltjes. This can also be proved directly, as follows. Since, for \(t > 0\),
\[ x \frac{d}{dx} \frac{1}{x + t} + \frac{1}{x + t} = \frac{t}{(x + t)^2} > 0, \tag{2.3} \]
the function \(1/(x + t)\) satisfies inequality \((2.1)\), and, hence, using equation \((4.1)\), so does any Stieltjes function.

Any Stieltjes function is positive and decreasing. The previous paragraph ensures that Theorem C applies and so inequality \((2.1)\) is satisfied

Finally the inverse \(g\) of \(f\) does not satisfy inequality \((2.1)\) so it is not Stieltjes. \(\square\)

(As an aside here, we remark that there are many completely monotone involutions mapping \((0, \infty)\) to \((0, \infty)\), and, as in the previous Corollary, none except \(1/x\) is Stieltjes.)

As we have already noted \(\phi_2\) satisfies inequality \((2.1)\). The inverse of \(\phi_1\), denoted \(\mu\) below, also satisfies inequality \((2.1)\) (and, in so doing, shares this property with Stieltjes functions).

As a lead in to consideration of higher derivatives we remark that equation \((2.3)\) generalises in several ways, for example
\[ \frac{d^n}{dx^n} \left( \frac{x^n}{x + t} \right) = \frac{n! t^n}{(x + t)^{n+1}}. \tag{2.4} \]
and
\[ \frac{d}{dx} \left( x^n \frac{d^{n-1}}{dx^{n-1}} \left( \frac{1}{x+t} \right) \right) = \frac{(-1)^{n-1} n! x^{n-1} t}{(x+t)^{n+1}}. \] (2.5)

The \( n = 2 \) version of the latter is the \((p = -1, q = 1)\) result on a case of \((p, q)\)-concavity of Stieltjes functions. See Definition 1. Theorem 3 is a general statement of the HA-concavity of Stieltjes functions.

### 3. Second derivatives and \((p, q)\)-convexity

**Definition 1.** The function \( f \) is \((p, q)\)-convex (\((p, q)\)-concave) if and only if
\[ x \rightarrow x^{-p} f'(x) (f(x))^q \]
is increasing (decreasing). See (Baricz, 2010; Bhayo & Yin, 2014).

Special cases arise sufficiently frequently that there are other notations. There is some literature, notably (Anderson, Vamanamurthy, & Vuorinen, 2007), in connection with ‘convexity with respect to means’ and the letters A for ‘arithmetic’, G for “geometric”, and H for “harmonic” are used to label these. For example, AA-convex is ordinary convexity, AG-convex means log-convex, etc. The correspondence between the \((p, q)\) and the main examples of convexity with respect to means is given in the table.

| \( p = 1 \) | \( q = 1 \) | \( q = 0 \) | \( q = -1 \) |
|-----------|----------|---------|----------|
| AA        | AG       | AH      |          |
| GA        | GG       | GH      |          |
| HA        | HG       | HH      |          |

The set of \((p, 1)\)-convex functions is obviously closed under addition. \((1, q)\)-convexity is related to power-convexity defined and discussed below. Further facts concerning these various convexity properties are given in Appendix A of (Keady & Wiwatanapataphee, 2018).

**Definition 2.** A nonnegative function \( f \) is said to be \( q \)-th power convex if, for \( q \neq 0 \), \( q f(x)^q \) is convex, and 0-power convex if \( \log(f) \) is convex, also called log-convex, or as in (Anderson et al., 2007), AG-convex. See (Lindberg, 1982).

(When \( q < 0 \), and \( f \) is \( q \)-th power convex, then \( f(x)^q \) is concave.)

\[ f \geq 0 \text{ is } q_0 \text{-th power convex then it is } q_1 \text{-th power convex for } q_1 \geq q_0. \]

Another property, used here and again in our application in (Keady & Wiwatanapataphee, 2017, 2018), is, from p159 of (Lindberg, 1982):

If \( q \leq 1 \) then the set of positive, decreasing, convex functions which are \( q \)-th power convex is closed under addition. This set is a convex cone in appropriate function spaces.

Whether the set is closed under multiplication is less important in our present application. As an aside we recall Theorem B and mention (noting that definitions from \( \S 4 \) are needed for later items):

**Theorem D.** The product of AG-convex functions is AG-convex. The product of AG-concave functions is AG-concave.

The product of GG-convex functions is GG-convex. The product of GG-concave functions is GG-concave.
The product of two functions $f_1, f_2 \in AH$ is not, in general in $AH$, but $\sqrt{f_0 \bar{f}_1} \in AH$, i.e. $1/\sqrt{f_0 \bar{f}_1}$ is concave.

The product of completely monotone functions is completely monotone.

The product of two Stieltjes functions $f_1$ is not, in general Stieltjes, but $\sqrt{f_0 \bar{f}_1}$ is.

(In connection with the GG functions, see also (Niculescu & Persson, 2004) Lemma 2.3.4.)

In the proof of Theorem 1 we use the notation

$$D(f(x), p, q) := \frac{d}{dx} \left( (x^{1-p}f(x)(f(x))^{q-1}) \right).$$

(3.1)

Before proving the theorem we note a simple identity (which can be used in connection with $\mu$ and $\mu^2$ in §5.2):

$$\frac{D(f(x), p, q)}{D(f(x), p, 2q)} = 2.$$

**Theorem 1.** If a positive, decreasing, convex function $f$ is $(p, q)$-convex, its inverse $g$ is $(q, p)$-convex.

**Proof.** $f$ is $(p, q)$-convex iff

$$x^p f_1^{q-1} D(f(x), p, q) = x f'' + x(q-1)(f')^2 + (1-p)f > 0.$$

Setting $f'' = -g''/(g')^2, f' = 1/g$, $f(x) = y$ and $x = g(y)$ in the preceding equation gives

$$- \frac{gg''}{(g')^3} + \frac{(q-1)g}{y(g')^2} + \frac{1-p}{g} = \frac{g(y)g'' + (1-q)g' + y(p-1)(g')^2}{g^3} > 0.$$

Using that $g' < 0$ the term in parentheses in the numerator of the long expression above is positive. This is the result that $g$ is $(q, p)$-convex.

The elementary calculus proof above is appropriate here. However differentiability assumptions can be weakened. Define, for $p \neq 0$, the power-mean

$$M_p(x, y) = \left( \frac{x^p + y^p}{2} \right)^{1/p} \quad \text{and} \quad M_0(x, y) = \sqrt{xy}.$$

The definition that $f$ is $(p, q)$-convex is often given that $f(M_p(x, y)) \leq M_q(f(x), f(y))$. The proof of Theorem 1 given in Appendix A of (Keady & Wiwatanapataphee, 2018) begins with these means.

There are corresponding results for positive, decreasing, $(p, q)$-concave functions. Also, when the functions are increasing rather than decreasing, convexity of $f$ gives concavity of $g$ and vice-versa.

**EXAMPLE.** $D(1/x^p, p, q) = x^{-1-p-qa(p+aq)}$. From this the convexity properties of the positive, decreasing convex functions $1/\sqrt{x}$ and its inverse $1/y^2$ are indicated in this diagram:

| AH - vex | AG - vex | AA - vex | AH - ave | AG - vex | AA - vex |
|----------|----------|----------|--------|----------|----------|
| GH - ave | GG | GA - vex | GH - ave | GG | GA - vex |
| HH - ave | HG - ave | HA - ave | HH - ave | HG - ave | HA - vex |

$$\frac{1}{\sqrt{x}} \quad \frac{1}{y^2}$$
For our application to the functions $\phi_1$ and $\phi_2$ we have the differentiabiliy needed to apply Theorem 1. However, for other applications, we note that there are other proof techniques. With notation as in that theorem, $g$ the inverse of $f$, obviously for any invertible $\alpha$ and $\beta$, $(\beta \circ f \circ \alpha)^{-1} = \alpha^{-1} \circ g \circ \beta^{-1}$. Hence, for example, with id the identity and recip the reciprocal function taking $x$ to $1/x$,

convex, decreasing $f$ is HA $\rightarrow$ convex $\iff$ id $\circ$ f $\circ$ recip is convex, increasing

$\iff$ recip $\circ$ g $\circ$ id is concave, increasing

$\iff$ convex, decreasing $g$ is AH $\rightarrow$ convex.

A few results anticipating parts of the preceding theorem have been published. For example, concerning the inverse of a GG functions, we have the following, adapted from (Niculescu & Persson, 2004) Lemma 2.3.4, items denoted with an a are additions not explicitly in their Lemma 2.3.4.)

**Theorem E.** If a function $f$ is increasing, multiplicatively convex (GG), and one-to-one, then its inverse is multiplicatively concave (and vice versa).

(GG-a) If a function $f$ is decreasing, multiplicatively convex (GG), and one-to-one, then its inverse is multiplicatively convex.

(HH-a) If a function $f$ is increasing, HH-convex, and one-to-one, then its inverse is HH-concave (and vice versa).

If a function $f$ is decreasing, HH-convex, and one-to-one, then its inverse is HH- convex.

**Proof.** Once again, simple proofs for $C^2$ functions use the identities relating derivatives of $g$, the inverse of $f$, to those of $f$.

4. Completely monotone and Stieltjes functions

4.1. Higher derivatives. Introduction

This subsection is an aside to the main function of this paper, namely to establish generalised convexity/concavity properties of $\phi_1$ and $\phi_2$ and their inverses, for use in (Keady & Wiwatanapataphee, 2017, 2018). The subsection is included as the questions here, besides being of interest in their own right, may, if answered, provide simpler and neater routes to the properties used n (Keady & Wiwatanapataphee, 2017, 2018).

If one knows a function $f$ is completely monotone or even Stieltjes this provides information about derivatives of all orders. In the same way as we have treated first derivatives in §2 and second derivatives in §3 one may obtain an expression for the higher derivatives of the inverse, $g$, in terms of those of the original function $f$. The formulae are given in (Johnson, 2002).

There are many ways that completely monotone and Stieltjes functions either determine relations involving their derivatives or are characterised by these. Concerning third derivatives of Stieltjes functions, consider $n = 3$ in equations (2.4) and (2.5). For a Stieltjes function $f$, not only is it completely monotone, with the sign information on the derivatives of $f$, but as $f_3(x) = f(1/x)/x$ is also Stieltjes, we also have sign information from the derivatives of $f$. Concerning completely monotone functions we remark that a result concerning their Hankel determinants is stated near the middle of p167 of (Widder, 1941).
We do not investigate higher derivatives here, but believe they may be useful in answering questions like the following:

**Question 2.** (i) What conditions (if any) are needed to ensure that a Stieltjes function $f$ with range $(0, \infty)$ is GA-convex?

(ii) What conditions (if any) are needed to ensure that the inverse $g$ of a Stieltjes function $f$ with range $(0, \infty)$ which is GA-convex is such that $g$ is completely monotone?

(iii) Does $\phi_2$ satisfy these conditions?

An affirmative answer to part (iii) would answer the first part of Question 1. There are many inequalities satisfied by derivatives of Stieltjes functions, some of which are given in inequalities (2.3) to (2.5). Others are given in (Widder, 1941), etc.

Perhaps an easier question is:

**Question 3.** (i) Let $f$ be a completely monotone function mapping $(0, \infty)$ onto itself. Is the inverse of $f$ also completely monotone?

(ii) If, as we expect, not, give an example.

It is relatively easy to construct an example of a positive, decreasing, log-convex functions whose inverse is not log-convex. If considerations of second derivatives do not suffice to answer Question 3, or if higher derivatives make it easier to answer, the reference (Johnson, 2002) has the relevant formulae.

### 4.2. Completely monotone functions

A function $f : (0, \infty) \rightarrow (0, \infty)$ is called completely monotone if $f$ has derivatives of all orders and satisfies $(-1)^n f^{(n)}(x) \geq 0$ for all $x > 0$ and all nonnegative integers $n$. In particular, completely monotonic functions are decreasing and convex.

**Lemma 3.4** of (Merkle, 2002) (i) If $g \in \mathcal{C}M$ then the function $x \mapsto \exp(-g(x))$ is $\mathcal{C}M$.

(ii) If $\log f \in \mathcal{C}M$, then $f$ is $\mathcal{C}M$ (the converse is not true).

(iii) If $f \in \mathcal{C}M$ and $g$ is a positive function with a $\mathcal{C}M$ derivative (i.e. a Bernstein function), then the composition $x \mapsto f(g(x))$ is $\mathcal{C}M$.

A particular case of Lemma 3.4(iii) applies to our functions $\phi_1$ and $\phi_2$. As $\sqrt{x}$ is Bernstein, as $\phi_1 \in \mathcal{C}M$ so also $\phi_1(\sqrt{x}) = \phi_2(x) \in \mathcal{C}M$. More generally, since, for $t > 0$, $\exp(-t\sqrt{z})$ is $\mathcal{C}M$, it follows from the Laplace transform representation of $f_1(z) \in \mathcal{C}M$ that $f_1(\sqrt{z})$ is also in $\mathcal{C}M$.

Starting with a function $f_2(z)$ and obtaining properties of $f_2(z^2)$, i.e. $f_2 \circ \text{square}$, seems more difficult. Of course, $\phi_1(z) = \phi_2(z^2)$ is $\mathcal{C}M$. Starting with $\phi_1 \in \mathcal{C}M$ then forming $\phi_1(z^2)$, we note that the inverse Laplace transform of the latter can be found, and it changes sign. $\phi_1(z^2)$ is not in $\mathcal{C}M$, but seems to be log-convex. Starting from the Laplace representation of $f_2$ seems to be unhelpful in general. We remark that $\exp(-tz^2)$ is log-concave in $z$ and sums of log-concave functions are not necessarily log-concave.
4.3. Stieltjes functions

4.3.1. Definition and basic properties

Stieltjes functions \( S \) are a subclass of \( \mathcal{CM} \). A non-negative function \( f \) is called a Stieltjes function (\( f \in S \) for short) if there exists \( a_0 \geq 0, a_1 \geq 0 \) and a non-negative measure \( \mu(dt) \) on \([0, \infty)\) integrating \((z + t)^{-1}\) such that

\[
f(z) = a_0 + \frac{a_1}{z} + \int_0^\infty \frac{1}{z + t} \mu(dt).
\] (4.1)

While \( S \) is not closed under multiplication it is “logarithmically convex” in the sense that for all \( g_0, g_1 \in S \) and \( \alpha \in (0, 1) \) we have \( g_0^\alpha g_1^{1-\alpha} \in S \). (See Schilling, Song, & Vondracek, 2010 Proposition 7.10.) Various other cones of functions are treated in (Schilling et al., 2010). The (nonzero) complete Bernstein functions (\( \mathcal{CBF} \)) are the reciprocals of (nonzero) Stieltjes functions.

Some of the convexity properties of the kernel \( 1/(z + t) \) are given here:

\[
\begin{align*}
\text{AH} & \quad \Rightarrow \quad \text{AG} \quad \Rightarrow \quad \text{AA} \\
\text{GH} & \quad \Rightarrow \quad \text{GA} \quad \Rightarrow \quad \text{HG} \\
\text{HH} & \quad \Rightarrow \quad \text{HA}
\end{align*}
\]

The function \( f(x) = 1/(1 + x) \) has as inverse the function \( g \) defined on \((0, 1)\) by \( g(y) = (1/y) - 1 \). The function \( g \) is log-convex when \( y < 1/2 \) and log-/concave for \( 1/2 < y \leq 1 \). The function \( f(x) \) is GA-convex for \( x > 1 \) and GA-concave for \( 0 < x < 1 \). One observation that follows from this is if the input \( \mu(t) \) to the representation (4.1) is larger for small \( t \) than at larger \( t \) the Stieltjes function so formed is more likely to be GA-convex.

There are other characterizations of Stieltjes functions.

- Besides that of equation (4.1) there is that of iterated Laplace transforms (for which, see Schilling et al., 2010 Theorem 2.2 p12). For function \( f \) to be a Stieltjes function its inverse Laplace transform should be completely monotone. A simple consequence of this is that if \( f(z) \) is Stieltjes, then the completely monotone function \( f(z)/z \) cannot be Stieltjes. This is because the inverse Laplace transform of \( f(z)/z \) is the integral from 0 of the inverse Laplace transform of \( f(z) \), so increasing and hence not completely monotone.
- There are also characterizations in terms of Nevalinna-Pick functions (for which, see Schilling et al., 2010, p. 56).

4.3.2. Examples, focusing on functions related to that in the application in §5

Define, for \( 0 \leq a < b \leq \infty \),

\[
St(a, b, u, z) = \int_a^b \frac{u}{z + t} dt.
\]
Various examples of Stieltjes functions follow:

| St               | $f(z)$ | in terms of $\phi_2$ | invlaplace(\(I\)) |
|------------------|--------|----------------------|-------------------|
| 1                | $\text{St}(0, 1, \frac{1}{z^2}; z)$ | $\frac{1}{2} \arctan \left( \frac{1}{z} \right)$ | $\phi_2(z)$ | \(\sqrt{\exp(\pi^2 t)}\) |
| 2                | $\text{St}(0, b, \frac{1}{z^2}; z)$ | $\frac{1}{2} \arctan \left( \frac{1}{z} \right)$ | $\phi_2(z) / \sqrt{b}$ |
| 3                | $\text{St}(0, \infty, \frac{1}{z^2}; z)$ | $\frac{\pi}{2 \sqrt{z}}$ | $\phi_2(z) + \frac{1}{2} \phi_2(z)$ | $\frac{1}{2} \sqrt{t}$ |
| 4                | $\text{St}(1, \infty, \frac{1}{z^2}; z)$ | $\frac{1}{2} \arctan \left( \frac{1}{z} \right)$ | $\sqrt{\frac{\pi}{2} z - \phi_2(z)}$ | $\frac{1}{2} \sqrt{\frac{t}{z}} - F_1 \left( \frac{1}{4} z^2; z \right) - s$ |
| 5                | $\text{St}(1, 0, \frac{1}{z^2}; z)$ | $\frac{1}{2} \arctan \left( \frac{1}{z} \right)$ | $\phi_2(z)$ | $\frac{1}{2} \sqrt{\frac{t}{z}} - F_1 \left( \frac{1}{4} z^2; z \right) - s$ |
| 6                | $\text{St}(0, 1, \frac{1}{z^2}; z)$ | $1 - \arctan \left( \frac{1}{z} \right)$ | $1 - z \phi_2(z)$ | $\sqrt{\frac{t}{z}} - \pi \exp(s) \text{erfc}(\sqrt{s})$ |
| 7                | $\text{St}(0, \infty, \frac{1}{z^2}; z)$ | $\frac{\pi}{2 \sqrt{z}}$ | $1 - \arctan \left( \frac{1}{z} \right)$ | $\sqrt{\frac{t}{z}} - \pi \exp(s) \left( 1 - 2 \sqrt{s} F_1 \left( \frac{1}{4} z^2; z \right) - s \right)$ |
| 8                | $\text{St}(0, 1, \frac{1}{z^2}; z)$ | $\frac{1}{2} \arctan \left( \frac{1}{z} \right)$ | $\phi_2(z)$ | $\frac{1}{2} \sqrt{\frac{t}{z}} - F_1 \left( \frac{1}{4} z^2; z \right) - s$ |
| 9                | $\text{St}(0, 1, \frac{1}{z^2}; z)$ | $\frac{1}{2} \arctan \left( \frac{1}{z} \right)$ | $\phi_2(z)$ | $\frac{1}{2} \sqrt{\frac{t}{z}} - F_1 \left( \frac{1}{4} z^2; z \right) - s$ |
| 10               | $\text{St}(0, \infty, \delta(t - 1); z)$ | $\frac{1}{2 \sqrt{z}}$ | $\exp(-s)$ |

Some comments on the table are appropriate. The first few entries all have range $(0, \infty)$. The later entries have finite ranges (and some, e.g. item 10, are not GA-convex).

- Entry 5 for $\text{St}(1, \infty, \frac{1}{z^2}; z)$ checks against

$$\arctan \left( \frac{1}{\sqrt{2}} \right) + \arctan \left( \sqrt{2} \right) = \frac{\pi}{2}.$$ 

Also, as mentioned before, for any $f \in \mathcal{S}$, we have that $x \rightarrow f(1/x) / x$ is also in $\mathcal{S}$, checking against:

$$\arctan \left( \sqrt{2} \right) = \frac{1}{2} \phi_2 \left( \frac{1}{2} \right).$$

- Entry 9's function $\arctan \left( 1 / \sqrt{2} \right)$ can be seen to be a Stieltjes function as it is the Laplace transform of a completely monotonic function:

$$\arctan \left( \frac{1}{\sqrt{2}} \right) = \int_0^\infty \exp(-zt) \frac{\exp(-t) \text{erfi}(\sqrt{t})}{2t} \, dt.$$ 

The complete monotonicity is proved via the following steps. Beginning from the definition of $\text{erfi}$ and using $v = (t - s^2) / t$ as a change of variable, we find

$$\frac{\exp(-t) \text{erfi}(\sqrt{t})}{2t} = \frac{1}{2^{3/2}} \int_0^\infty \frac{\exp(-vt)}{\sqrt{t^2 - 1}} \, dv.$$ 

However, for $v > 0$, $\exp(-vt) / \sqrt{t}$ is the product of $\mathcal{C}_M$ functions, so $\mathcal{C}_M$, and sums and integrals of $\mathcal{C}_M$ functions are $\mathcal{C}_M$. Hence $\arctan(1 / \sqrt{2})$ is Stieltjes.

- Entry 4 follows from entry 9 on using Property (ii) from §4.3.3.
• In entry 10, \( \delta \) is the Dirac delta measure.

Entry 10 is well known to be Stieltjes and from it one notes that 
\(-z\phi_2(z)\) is Stieltjes, as

\[-z\phi_2(z) = \frac{1}{Z} \left( \phi_2 + \frac{1}{1+z} \right).

4.3.3. Various properties of Stieltjes functions

Here is a short list of some properties of the cone of Stieltjes functions:

(i) \( f \in S \setminus \{0\} \Rightarrow \frac{1}{f+1} \in S \), i.e. \( \text{recip} \circ f \circ \text{recip} \in S \)

(ii) \( f \in S \setminus \{0\} \Rightarrow \frac{1}{f\text{recip}} \in S \)

From these, \( f \in S \setminus \{0\} \Rightarrow f(1/x)/x \in S \)

(iii) \( f \in S, \lambda > 0 \Rightarrow \frac{f}{\lambda} \in S \)

(iv) \( f, g \in S \setminus \{0\} \Rightarrow f \circ \frac{1}{g}, \frac{1}{f\text{recip}} \in S \)

(v) \( f, g \in S, 0 < \alpha < 1 \Rightarrow f^\alpha \circ g^{1-\alpha} \in S \)

(vi) \( f \in S, 0 < \alpha < 1 \Rightarrow f^\alpha \in S \)

4.3.4. Stieltjes functions, AG and GA

Any completely monotone function is log-convex, i.e. AG-convex.

We have yet to check when (if always) a Stieltjes function with range \((0, \infty)\) is GA-convex. The Stieltjes function \(\phi_2\) is GA-convex. Any Stieltjes function which is the inverse of a log-convex function, e.g. a completely monotone function, is GA-convex.

4.3.5. Stieltjes functions, AH and HA

Theorem 2. For any Stieltjes function \(\phi\), \((1/\phi) \in CB.F \) so, in particular, \(1/\phi\) is concave, or, In other words, \(\phi\) is AH-convex.

Proof. This follows as Theorem 7.3 of (Schilling et al., 2010) ensures that \(1/\phi\) is a complete Bernstein function. Also any Bernstein function \(b\) is positive with \(b' \in CM\), so we have \((1/\phi)' \in CM\) is positive and decreasing. That it is decreasing is \((1/\phi)'' < 0\), i.e. \(1/\phi\) is concave. In other words, \(\phi\) is AH-convex.

Theorem 3. For any Stieltjes function \(\phi\), the function \(x \to \phi(1/x)\) is concave, or, in other words, \(\phi\) is HA-concave.

Proof. The result of Theorem 3 follows from item (i) in the above list and Theorem 2. See also (Merkle, 2003).

In connection with our later application, we remark that Theorem 3 gives that \(\phi_2(1/x)\) is concave, whereas \(\phi_1\), which is not Stieltjes, is such that \(\phi_1(1/x)\) is convex.

Stieltjes functions are simultaneously AH-convex and HA-concave. This, with the AH-HA case of Theorem 1, gives another proof of Corollary 1: the only Stieltjes functions whose inverses are Stieltjes are positive multiples of \(1/x\).
4.3.6. Stieltjes functions, GG, HG and HH

The next result is weaker than the HA-concavity of Theorem 3 which implies HG-concavity which, in turn, implies HH-concavity:

**Theorem 4.** (i) Any Stieltjes function is HG-concave.

(ii) Any Stieltjes function is HH-concave.

**Proof.** (ii) Perhaps item (ii) is the easier. There is a one-line proof:

\[ f \in S \iff \frac{1}{f'(\frac{x}{2})} \in S \Rightarrow \frac{1}{f(s(\frac{x}{2}))} \text{ is convex } \iff f \text{ is HH -- concave.} \]

See (Anderson et al., 2007) Theorem 2.4(9), with calculus proofs using 2.5(9).

(i) The one-line can be adapted:

\[ f \in S \iff \frac{1}{f'(\frac{x}{2})} \in S \Rightarrow -\log(f'(\frac{x}{2})) \text{ is convex } \iff f \text{ is HG -- concave.} \]

We remark also that while the Stieltjes function \( \phi_2 \) is necessarily HH-concave, the function \( \phi_1 \), which is merely in \( CM \), is HH-convex.

The Stieltjes function \( \phi_2 \) is GG-concave. The Stieltjes function \( 1/\log(1 + x) \) (see Schilling et al., 2010, p. 228 entry 26) is GG-convex.

5. A transcendental equation

The transcendental equation

\[ X \tan(X) = Y, \quad \text{with } Y > 0 \]

and an interest in solutions \( X \) with \( 0 < X < \pi/2 \) arises in various applications. The purpose of the remainder of this paper is to extract information on its solutions in a form that can be used in our subsequent paper (Keady & Wiwatanapataphee, 2017, 2018). Before doing this, we note that there are other applications. This transcendental equation has been widely studied, e.g. (Burniston & Siewert, 1973; Luo, Wang, & Jiurong Han, 2015; Markushin, Rosenfelder, & Schreiber, 2003). Numerical values, often used for checks, are given in Table 4.20 of (Abramowitz & Stegun, 1964).

Amongst the applications, other than ours in (Keady & Wiwatanapataphee, 2017, 2018), are (i) the energy spectrum for the one-dimensional quantum mechanical finite square well (though with \( c < 0 \) in equation (5.1), and (ii) (though again with \( c < 0 \)) zeros of the spherical Bessel function \( y_1(x) = j_2(x) \).

In the application, and notation, in (Keady & Wiwatanapataphee, 2017, 2018) the problem is given \( \beta > 0 \), how does \( \mu \) depend on \( c \) where \( \mu(c) \) solves the transcendental equation:

\[ \mu \tan(c\mu) = \frac{1}{\beta}. \quad (5.1) \]

It happens that one can re-scale variables so that there is just one independent variable \( \hat{c} \):

\[ \hat{\mu} \tan(\hat{c}\hat{\mu}) = 1, \quad \text{where } \hat{\mu} = \beta\mu, \quad \hat{c} = \frac{c}{\beta}. \quad (5.2) \]

We have an interest in the smallest positive solutions,

\[ 0 < \mu(c) < \pi/(2c), \quad 0 < \hat{\mu} < \pi/(2\hat{c}). \]
In the application in (Keady & Wiwatanapataphee, 2017, 2018) much of the effort involves obtaining results valid for $\beta/C210$—for Robin boundary conditions—where the corresponding result with $\beta=0$—for Dirichlet boundary conditions—has been available for decades.

We will, henceforth, also drop the hat notation. In the next subsection we explore the behaviour of the function $\phi_1(\mu)$, defined as in (1.3), which is inverse to $\mu(c)$, that is

$$\mu \tan(\mu \phi_1(\mu)) = 1.$$ 

Also explored are the convexity properties of $\phi_2$ where $\phi_2(z) = \phi_1(\sqrt{z})$.

### 5.1. The convexity properties of $\phi_2$ and $\phi_1$

Some properties follow from complete monotonicity, and, for $\phi_2$ others follow from it being a Stieltjes function. Yet further properties follow from calculation (the details of which are relegated to appendices of Keady & Wiwatanapataphee, 2017, 2018).

**Theorem 5.** Both $\phi_1$ and $\phi_2$ are completely monotone. Furthermore $\phi_2$ is a Stieltjes function.

**Proof.** We have

$$\phi_1(z) = \int_0^\infty \exp(-zt) \text{Si}(t) \, dt. \quad (5.3)$$

where $\text{Si}$ is the sine integral

$$\text{Si}(t) = \int_0^t \frac{\sin(\tau)}{\tau} \, d\tau. \quad (5.4)$$

Since $\text{Si}(t) > 0$ for $t > 0$, $\phi_1(z)$ is completely monotone.

The same Laplace transform representation also shows $\phi_2$ to be completely monotone as

$$\phi_2(z) = \int_0^\infty \exp(-zt) \frac{1}{Z} \sqrt{\frac{\pi}{t}} \text{erf}(\sqrt{t}) \, dt. \quad (5.5)$$

and the integrand in the expression above is positive. Furthermore

$$\sqrt{\frac{\pi}{t}} \text{erf}(\sqrt{t}) = \int_0^1 \frac{\exp(-st)}{\sqrt{s}} \, ds,$$

which is completely monotone, so by (Schilling et al., 2010) Theorem 2.2(i), $\phi_2 \in S$. That $\phi_2$ is a Stieltjes function also follows, as we have already noted in the table of examples in §4.3

$$\phi_2(z) = \frac{1}{\sqrt{z}} \arctan\left(\frac{1}{\sqrt{z}}\right) = \int_0^1 \frac{1}{z + t} \frac{dt}{2\sqrt{t}}$$

This completes the proof.

### 5.2. Properties of $\mu$ used in (2017, 2018)

Further convexity properties are given in (Keady & Wiwatanapataphee, 2018), especially Appendix B. We remark that the properties of $\mu$ are the same as in the corresponding diagram for the Stieltjes function $\phi_2$. Theorems 2 and 3 ensure that any Stieltjes function is both AH-convex and HA-concave. We have no information yet to preclude the possibility that $\mu$ is Stieltjes but, as we have no proof that it is even completely monotone, it is too early to speculate.

From the properties in (Keady & Wiwatanapataphee, 2018) one has that $\mu(2)$ is not Stieltjes. If $\mu$ were to be shown to be completely monotone, then so is its square, $\mu(2)$.
As $\mu(2) = \mu^2$ there are some obvious checks. For example, it is clear that the AG, GG- and HG-convexity properties of $\mu(2)$ and of $\mu$ must be the same. The convexity properties that differ are AH and HA.

6. Other transcendental equations involving CM functions

Denote the Lambert W function by $W$. Results concerning Stieltjes representations of $W$ are given in (Kalugin, Jeffreya, Corless, & Borwein, 2012).

The Stieltjes function $f(x) = W(1/x)$ is the solution of $g(f) := \exp(-f)/f = x$, and $g \in \text{CM}$ as

$$g(y) = \frac{\exp(-y)}{y} = \int_1^\infty \exp(-ty) dt.$$ 

The Stieltjes function $f(x) = 1/W(x)$ is the solution of $g(f) := \exp(1/f)/f = x$, and $g \in \text{CM}$ as

$$g(y) = \frac{\exp(1/y)}{y} = \int_0^\infty \text{BesselI}(0, 2\sqrt{t}) \exp(-ty) dt.$$ 

The Stieltjes function $g(y) = \frac{\tanh(y/\sqrt{y})}{y}$ has arisen in connection with one of the author’s applied mathematical interests—water waves. That $g \in \mathbb{S}$ is from Proposition 2.2.2 of (Ehrnstrom & Wahlén, 2016). Amongst the ever-growing menagerie of special functions are “generalized Lambert W” functions. The solution of $g(y) = x$ is given in (Mezo & Keady, 2016) as

$$\sqrt{y} = \frac{1}{2} W \left( \frac{2/x}{-2/x}, -\frac{1}{1} \right).$$ (6.1)

To the best of the authors’ knowledge, there has been no systematic study of complete monotonicity properties of generalized Lambert W functions.

Here is another example involving a generalized Lambert function, this time with two upper parameters. The equation to be solved, for $x$ is

$$\exp(xy) = (1 + x/a_1)(1 + x/a_2)$$ with $a_1 > 0$, $a_2 > 0$. (6.2)

When $a > 0$, $\log(1 + z/a)/z$ is a Stieltjes function for $z > 0$. Hence it is completely monotone, so convex, and log-convex. A version of the generalized Lambert function—which is a Stieltjes function for $x > 0$—arises in solving equation (6.2). One is interested in $x(y)$. Take logs

$$y = \log((1 + x/a_1)(1 + x/a_2))/x.$$ 

Now the expression on the right is a Stieltjes function of $x$, and, from this one can draw some conclusions concerning the convexity properties of $x(y)$. (This may be related to a physical problem. See (Barsan, 2016), equations (4) and (62)-(63).)

7. Conclusion

As a consequence of research associated with convexity properties of a domain functional from a partial differential equation problems (see Keady & Wiwatanapataphee, 2017, 2018) various theorems associated with convex functions and their inverses were discovered. Theorem 1 is the simplest of these.

The application involved a function $\phi_2$ which was instantly noted to be Stieltjes from which AH-convexity and HA-concavity (and consequences) follow immediately. Several open questions are posed. See §4.1. That which seems the most important for further results associated with the application of (Keady & Wiwatanapataphee, 2017, 2018) is whether the inverse of $\phi_2$ is completely monotone.
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