Nonlocal symmetry of CMA generates ASD
Ricci-flat metric with no Killing vectors

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Abstract

The complex Monge-Ampère equation (CMA) in a two-component form is treated as a bi-Hamiltonian system. We present explicitly the first nonlocal symmetry flow in the hierarchy of this system. An invariant solution of CMA with respect to this nonlocal symmetry is constructed which, being a noninvariant solution in the usual sense, does not undergo symmetry reduction in the number of independent variables. We also construct the corresponding 4-dimensional anti-self-dual (ASD) gravitational metric with either Euclidean or neutral signature. It admits no Killing vectors which is one of characteristic features of the famous gravitational instanton K3.

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1 Introduction

In his pioneer paper \([1]\), Plebański demonstrated that anti-self-dual (ASD) Ricci-flat metrics on four-dimensional complex manifolds are completely determined by a single scalar potential which satisfies his first or second heavenly equation. Such metrics are solutions to complex vacuum Einstein equations. Real four-dimensional hyper-Kähler ASD metrics

\[
ds^2 = u_{11}dz^1d\bar{z}^1 + u_{12}dz^1d\bar{z}^2 + u_{21}dz^2d\bar{z}^1 + u_{22}dz^2d\bar{z}^2 \tag{1.1}
\]

that solve the vacuum Einstein equations with either Euclidean or ultrahyperbolic signature are governed by a scalar real-valued potential \(u = u(z^1, z^2, \bar{z}^1, \bar{z}^2)\) which satisfies elliptic or hyperbolic complex Monge-Ampère equation (CMA)

\[
u_{11}u_{22} - u_{12}u_{21} = \varepsilon \tag{1.2}
\]
with \( \varepsilon = \pm 1 \) respectively. Here \( u \) is a real-valued function of the two complex variables \( z^1, z^2 \) and their conjugates \( \bar{z}^1, \bar{z}^2 \), the subscripts denoting partial derivatives with respect to these variables, e.g. \( u_{11} = \partial^2 u / \partial z^1 \partial \bar{z}^1 \) and suchlike. A modern proof of this result one can find in the books by Mason and Woodhouse [2] and Dunajski [3].

To illustrate this property, we introduce the coframe of one-forms

\[
\omega_1 = \frac{1}{\sqrt{u_{11}}} (u_{11}dz_1 + u_{21}dz_2), \quad \bar{\omega}_1 = \frac{1}{\sqrt{u_{11}}} (u_{11}d\bar{z}_1 + u_{12}d\bar{z}_2)
\]

\[
\omega_2 = \frac{1}{\sqrt{u_{11}}} dz_2, \quad \bar{\omega}_2 = \frac{1}{\sqrt{u_{11}}} d\bar{z}_2.
\] (1.3)

The metric (1.1) takes the canonical form

\[
ds^2 = \omega_1 \otimes \bar{\omega}_1 + \varepsilon \omega_2 \otimes \bar{\omega}_2
\] (1.4)

where complex Monge-Ampère equation (1.2) has been used. Equation (1.4) makes obvious the claim about the signature of the metric.

We are mostly interested in ASD Ricci-flat metrics that describe gravitational instantons which asymptotically look like a flat space, so that their curvature is concentrated in a finite region of a Riemannian space-time (see [3] and references therein). The most important gravitational instanton is \( K^3 \) which geometrically is Kummer surface [4], for which an explicit form of the metric is still unknown while many its properties and existence had been discovered and analyzed [5, 6]. A characteristic feature of the \( K^3 \) instanton is that it does not admit any Killing vectors, that is, no continuous symmetries which implies that the metric potential should be a noninvariant solution of \( CMA \) equation. As opposed to the case of invariant solutions, for noninvariant solutions of \( CMA \) there should be no symmetry reduction [7] in the number of independent variables. In this paper we achieve this goal by utilizing the invariance under an nonlocal symmetry.

The paper is organized as follows. In Section 2, we convert \( CMA \) equation into real variables and two-component form. In Section 3 we exhibit two alternative bi-Hamiltonian representations of \( CMA \) which we discovered earlier [8]. In Section 4 we explicitly construct first nonlocal flows in the hierarchy of \( CMA \) system related to each of the two bi-Hamiltonian structures. In Section 5 we formulate the invariance conditions with respect to both nonlocal symmetries appearing in each of the two alternative hierarchies. In this way, we keep in the invariance conditions the obvious discreet symmetry relating the two bi-Hamiltonian structures. Here for simplicity
we restrict ourselves to the special first nonlocal symmetry setting $\Phi = 0$ and $\chi = 0$.

In Section 6, we show in detail how the careful analysis of integrability conditions specifies various functional parameters in the invariance equations with no additional assumptions made. In Section 7, we integrate completely all the obtained equations and end up with a noninvariant solution of CMA which is a general form of the solution invariant under the first nonlocal symmetry in the hierarchy. In Section 8, we use this solution for constructing the corresponding (anti-)self-dual gravitational metrics with either Euclidean or neutral signature.

2 Real variables and 2-component form of CMA

In our earlier paper [8] we presented bi-Hamiltonian structure of the two-component version of (1.2), which by Magri’s theorem [9] proves that it is a completely integrable system in four dimensions.

We impose additional reality condition for all the objects in the theory. The transformation from complex to real variables has the form

$$t = z^1 + \bar{z}^1, \quad x = i(\bar{z}^1 - z^1), \quad y = z^2 + \bar{z}^2, \quad z = i(\bar{z}^2 - z^2).$$

Introduce the notation

$$a = \Delta(u) = u_{yy} + u_{zz}, \quad b = u_{xy} - v_z, \quad c = v_y + u_{xz}, \quad Q = \frac{b^2 + c^2 + \varepsilon}{a}$$ (2.2)

where $v = u_t$ is the second component of the unknown and $\Delta = D_y^2 + D_z^2$ is the two-dimensional Laplace operator. The definitions (2.2) imply the relations

$$a_x = b_y + c_z, \quad c_y - b_z = \Delta(v).$$ (2.3)

The CMA equation (1.2) in the real variables becomes

$$(u_{tt} + u_{xx})\Delta(u) - b^2 - c^2 - \varepsilon = 0$$

or in the two-component form

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \begin{pmatrix} v \\ Q - u_{xx} \end{pmatrix}$$ (2.4)

which we will call CMA system.
The metric (1.1) in real variables reads
\[ ds^2 = (v_t + u_{xx})(dt^2 + dx^2) + \Delta(u)(dy^2 + dz^2) - 2b(dt\, dz - dx\, dy) + 2c(dt\, dy + dx\, dz). \] (2.5)

The coframe of one-forms becomes
\[ \Omega_1 = \frac{1}{2\sqrt{v_t + u_{xx}}}[c + ib(dy + idz) + (v_t + u_{xx})(dt + idx)] \]
\[ \Omega_2 = \bar{\Omega}_1, \quad \Omega_3 = \frac{dy + idz}{2\sqrt{v_t + u_{xx}}}, \quad \Omega_4 = \bar{\Omega}_3 \] (2.6)
with the metric
\[ ds^2 = \Omega_1 \otimes \bar{\Omega}_1 + \varepsilon \Omega_2 \otimes \bar{\Omega}_2. \] (2.7)

### 3  Bi-Hamiltonian representations of CMA system

The CMA system (2.4) can be put in the Hamiltonian form
\[ \begin{pmatrix} u_t \\ v_t \end{pmatrix} = J_0 \begin{pmatrix} \delta_u H_1 \\ \delta_v H_1 \end{pmatrix} \] (3.1)
where \( \delta_u \) and \( \delta_v \) are Euler-Lagrange operators [7] with respect to \( u \) and \( v \). Here \( J_0 \) is the Hamiltonian operator
\[ J_0 = \begin{pmatrix} 0 & \frac{1}{a} \\ -\frac{1}{a} & \frac{1}{a}(cD_y + D_y c - bD_z - D_z b)\frac{1}{a} \end{pmatrix} \] (3.2)
determining the structure of Poisson bracket and \( H_1 \) is the corresponding Hamiltonian density
\[ H_1 = \frac{1}{2} [v^2 \Delta(u) - u_{xx}(u_y^2 + u_z^2)] - \varepsilon u. \] (3.3)

The first real recursion operator has the form
\[ R_1 = \begin{pmatrix} 0 & 0 \\ QD_z - cD_x & b \end{pmatrix} + \Delta^{-1} \begin{pmatrix} D_y(-aD_x + cD_z) + D_z(cD_y - bD_z) & -D_z a \\ D_x \left[ D_y(cD_y - bD_z) + D_z(aD_x - bD_y - cD_z) \right] & -D_x D_y a \end{pmatrix} \] (3.4)
where $\Delta^{-1}$ means operator multiplication, and the second recursion operator reads

$$
R_2 = \begin{pmatrix}
0 & 0 \\
bD_x - QD_y & c
\end{pmatrix} + \Delta^{-1} \begin{pmatrix}
D_y(bD_x - cD_y) + D_z(-aD_x + bD_y + cD_z) \\
D_z\left[D_y(-aD_x + bD_y + cD_z) + D_z(cD_y - bD_z)\right]
\end{pmatrix} \begin{pmatrix}
D_ya \\
-D_xD_z a
\end{pmatrix}.
$$

The two recursion operators $R_1$ and $R_2$ generate two alternative second Hamiltonian operators $J_1 = R_1J_0$ and $J^1 = R_2J_0$

$$
J_1 = R_1J_0 = \Delta^{-1} \begin{pmatrix}
D_z & -D_xD_y \\
D_xD_y & D_z^2
\end{pmatrix} + \begin{pmatrix}
0 \\
-b/a
cy(aD dx - aDz) + (D_yb - D_z a) \frac{c}{a^2} + \frac{Q^-}{2a} D_z + D_z \frac{Q^-}{2a}
\end{pmatrix}
$$

where $Q^- = (c^2 - b^2 + \varepsilon)/a$, and

$$
J^1 = R_2J_0 = \Delta^{-1} \begin{pmatrix}
D_y & D_zD_x \\
-D_xD_y & D_z^2
\end{pmatrix} + \begin{pmatrix}
0 \\
-c/a
cy (cD_z - aD_x) + (D_xc - D_z a) \frac{b}{a^2} + \frac{Q^-}{2a} D_y + D_y \frac{Q^-}{2a}
\end{pmatrix}
$$

where $Q^- = (b^2 - c^2 + \varepsilon)/a$.

The flow (3.1) can be generated by the Hamiltonian operator $J_1$ from the Hamiltonian density

$$
H_0 = zv\Delta(u) + u_x u_y
$$

so that CMA in the two-component form (3.1) is a bi-Hamiltonian system [9]

$$
\begin{pmatrix}
u_t \\
v_t
\end{pmatrix} = J_0 \begin{pmatrix}
\delta_u H_1 \\
\delta_v H_1
\end{pmatrix} = J_1 \begin{pmatrix}
\delta_u H_0 \\
\delta_v H_0
\end{pmatrix}.
$$

The same flow (3.1) can also be generated by the Hamiltonian operator $J^1$ from the Hamiltonian density

$$
H^0 = yv\Delta(u) - u_x u_z
$$

which yields another bi-Hamiltonian representation of the CMA system (3.1)

$$
\begin{pmatrix}
u_t \\
v_t
\end{pmatrix} = J_0 \begin{pmatrix}
\delta_u H_1 \\
\delta_v H_1
\end{pmatrix} = J^1 \begin{pmatrix}
\delta_u H^0 \\
\delta_v H^0
\end{pmatrix}.
$$
4 Nonlocal flows

The first nonlocal flows of the hierarchy of CMA system are generated by \( J_1 \) and \( J^1 \) acting on the vector of variational derivatives of \( H_1 \)

\[
\begin{pmatrix}
  u_{\tau_2} \\
v_{\tau_2}
\end{pmatrix} = J_1 \begin{pmatrix}
  \delta_u H_1 \\
  \delta_v H_1
\end{pmatrix}
\]

(4.1)

\[
\begin{pmatrix}
  u_{\tau_2'} \\
v_{\tau_2'}
\end{pmatrix} = J^1 \begin{pmatrix}
  \delta_u H_1 \\
  \delta_v H_1
\end{pmatrix}
\]

(4.2)

where \( \tau_2, \tau_2' \) are time variables of the flows (4.1), (4.2), respectively. Using the expressions (3.6), (3.7) and (3.3) for \( J_1 \), \( J^1 \) and \( H_1 \) we obtain explicit expressions for the flows (4.1) and (4.2)

\[
u_{\tau_2} = \Delta^{-1} \left\{ D_z (a u_{xx} - u_{xy}^2 - u_{xz}^2 - \varepsilon) - D_x D_y (a v) \right\} + u_{xy} v
\]

\[
u_{\tau_2'} = \Delta^{-1} \left\{ D_y (a u_{xx} - u_{xy}^2 - u_{xz}^2 - \varepsilon) + D_x D_z (a v) \right\} - u_{xz} v.
\]

(4.3)

Second components of these flows are time derivatives of (4.3), \( v_{\tau_2} = D_t [u_{\tau_2}], v_{\tau_2'} = D_t [u_{\tau_2'}], \) so that the flows (4.1) and (4.2) commute with the flow (2.4) of CMA system and hence they are nonlocal symmetries of the CMA system.

5 Invariance conditions with respect to nonlocal symmetries

Solutions invariant with respect to nonlocal symmetries are determined by the conditions \( u_{\tau_2} = 0 \) and \( u_{\tau_2'} = 0 \) which due to (4.3) take the explicit form

\[
D_z \left[ \Delta [u] u_{xx} - u_{xy}^2 - u_{xz}^2 \right] - D_x D_y [v \Delta [u]] + \Delta [vu_{xy}] = 0
\]

\[
D_y \left[ \Delta [u] u_{xx} - u_{xy}^2 - u_{xz}^2 \right] + D_x D_z [v \Delta [u]] - \Delta [vu_{xz}] = 0.
\]

(5.1)

Here we impose both invariance conditions (5.1) on solutions of the CMA system in order to keep the discreet symmetry \( z \mapsto y, y \mapsto -z \) between the two bi-Hamiltonian structures. Differentiating the first and second equations (5.1) with respect to \( y \) and \( z \), respectively, and taking the difference of the results yields the integrability condition

\[
\Delta \{ D_y [vu_{xy}] + D_z [vu_{xz}] - D_x [v \Delta [u]] \} = 0
\]
or, equivalently
\[
D_y[vu_{xy}] + D_z[vu_{xz}] - D_x[v\Delta[u]] = \Phi(x, y, z, t)
\]
\[
\iff v_yu_{xy} + v_zu_{xz} - v_x\Delta[u] = \Phi,
\]
where \( \Delta[\Phi] = 0 \).

On account of the relation (5.2) each of the relations (5.1) becomes
\[
v_yu_{xz} - v_zu_{xy} = \Delta[u]u_{xx} - u_{xy}^2 - u_{xz}^2 + \chi(x, y, z, t)
\]
where \( \Phi_y = \chi_z \) and \( \Phi_z = -\chi_y \) and hence \( \Delta[\Phi] = \Delta[\chi] = 0 \). Thus, we end up with the system of two equations (5.2) and (5.3) linear in derivatives of \( v \).

Solving this system algebraically for \( v_y \) and \( v_z \) and denoting \( \delta = u_{xy}^2 + u_{xz}^2 \), we obtain
\[
v_y = \frac{1}{\delta} \left\{ \Delta[u](v_xu_{xy} + u_{xx}u_{xz}) - \delta u_{xz} + \Phi u_{xy} + \chi u_{xz} \right\}
\]
\[
v_z = \frac{1}{\delta} \left\{ \Delta[u](v_xu_{xz} - u_{xx}u_{xy}) + \delta u_{xy} + \Phi u_{xz} - \chi u_{xz} \right\}.
\]

In the following for simplicity we set \( \Phi = 0, \chi = 0 \) and refer to this case as special first nonlocal symmetry. In the following it is convenient to introduce the quantity
\[
w = \frac{\delta}{\Delta[u]} - u_{xx}.
\]

Equations (5.4) become
\[
v_y = \frac{u_{xy}v_x - u_{xz}w}{w + u_{xx}}, \quad v_z = \frac{u_{xz}v_x + u_{xy}w}{w + u_{xx}}
\]
with the immediate consequences
\[
v_x = \frac{1}{\Delta[u]} (u_{xy}v_y + u_{xz}v_z), \quad w = \frac{1}{\Delta[u]} (u_{xy}v_z - u_{xz}v_y).
\]

On account of the equations (5.5) and (5.6), the real two-component form (2.4) of CMA becomes
\[
v_t = \frac{v_x^2 - u_{xx}w}{w + u_{xx}} + \frac{\varepsilon}{\Delta[u]}.
\]

Integrability condition \((v_y)_z - (v_z)_y = 0\) of equations (5.6) yields
\[
(u_{xz}w_y - u_{xy}w_z)v_x = u_{xx}(u_{xy}w_y + u_{xz}w_z) - \Delta[u](w + u_{xx})w_x.
\]
6 Further integrability conditions

It is convenient to take equation (5.9) in the form

\[(u_{xx}v_x - u_{xx}u_{xy})w_y = (u_{xy}v_x + u_{xx}u_{xx})w_z - \Delta[u](w + u_{xx})w_x.\] (6.1)

The integrability conditions \((v_t)_y = (v_y)_t\) and \((v_t)_z = (v_z)_t\) with the use of (6.1) simplify to

\[\Delta[u_y] = \frac{u_{xy}}{w + u_{xx}}\Delta[u_x], \quad \Delta[u_z] = \frac{u_{xx}}{w + u_{xx}}\Delta[u_x]\] (6.2)

with the integrability condition \((\Delta[u_y])_z - (\Delta[u_z])_y = 0\) resulting in

\[u_{xx}w_y - u_{xy}w_z = 0.\] (6.3)

On account of (6.3) equation (5.9) becomes

\[u_{xx}(u_{xy}w_y + u_{xx}w_z) - \Delta[u](w + u_{xx})w_x = 0.\] (6.4)

Equations (6.3) and (6.4) can be put in the form

\[w_y = \frac{u_{xy}}{u_{xx}}w_x, \quad w_z = \frac{u_{xx}}{u_{xx}}w_x\] (6.5)

with the integrability condition \((w_y)_z = (w_z)_y\) identically satisfied. Differentiating the definition (5.5) of \(w\) with respect to \(t\) we obtain

\[w_t = \frac{v_x}{u_{xx}}w_x.\] (6.6)

The integrability conditions \((w_t)_y = (w_y)_t\) and \((w_t)_z = (w_z)_t\) of equations (6.5) and (6.6) are identically satisfied.

Equations (6.5) are integrated by the method of characteristics with the result \(w = w(u_x, t)\) whereas (6.6) further implies \(w = w(u_x)\). The remaining equations (6.2) take the form of two first-order PDEs for \(\Delta[u]\)

\[u_{xx} + w(u_x)(\Delta[u])_y - u_{xy}(\Delta[u])_x = 0\]
\[(u_{xx} + w(u_x))(\Delta[u])_z - u_{xz}(\Delta[u])_x = 0.\] (6.7)

We start with the combination of these equations

\[u_{xz}(\Delta[u])_y - u_{xy}(\Delta[u])_z = 0\] (6.8)
which is integrated by the method of characteristics to give
\[ \Delta[u] = f(u_x, x, t), \]
Then equations (6.7) are satisfied by the solution
\[ \Delta[u] = f(\zeta, t), \quad \zeta = \omega(u_x) + x \quad \text{and} \quad \omega(u_x) = \int \frac{du_x}{w(u_x)}. \quad (6.9) \]
Equation (5.5) becomes
\[ u_{xy}^2 + u_{xz}^2 = A^2, \quad \text{where} \quad A = \sqrt{f(u_{xx} + \frac{1}{\omega'})}. \quad (6.10) \]
We rewrite (6.10) in the form
\[ u_{xy} = A \sin \theta, \quad u_{xz} = A \cos \theta \quad (6.11) \]
where we have introduced the new unknown \( \theta = \theta(x, y, z, t) \). The definition of \( A \) in (6.10) implies
\[ A_y = a \sin \theta + \frac{f}{2} \cos \theta \cdot \theta_x, \quad A_z = a \cos \theta - \frac{f}{2} \sin \theta \cdot \theta_x \]
\[ A_x = \frac{f \zeta A^3}{2f^2 \omega'} - \frac{A \omega''}{2 \omega'^2} + \frac{f}{2A} \left( \frac{\omega''}{\omega'^3} + u_{xxx} \right) \quad (6.12) \]
where
\[ a = \frac{3}{4} \left( \frac{f \zeta A^2 \omega' - f \omega''}{f \omega'^2} \right) + \frac{f^2}{4A^2} \left( \frac{\omega''}{\omega'^3} + u_{xxx} \right). \quad (6.13) \]
The integrability condition \( (u_{xy})_z - (u_{xz})_y = 0 \) of the system (6.11) has the form
\[ \sin \theta \cdot \theta_y + \cos \theta \cdot \theta_z = \frac{f}{2A} \theta_x. \quad (6.14) \]
The first equation \( \Delta[u] = f(\zeta, t) \) in (6.9) implies
\[ \Delta[u_x] = (u_{xy})_y + (u_{xz})_z = (A \sin \theta)_y + (A \cos \theta)_z = f(\omega' u_{xx} + 1) \quad (6.15) \]
which finally results in the equation
\[ \cos \theta \cdot \theta_y - \sin \theta \cdot \theta_z = \frac{f \zeta A^2 \omega' - a}{f \omega'^2} \quad (6.16) \]
We solve algebraically the system of equations (6.14) and (6.16) to obtain
\[ \theta_y = \frac{f}{2A} \sin \theta \cdot \theta_x + b \cos \theta, \quad \theta_z = \frac{f}{2A} \cos \theta \cdot \theta_x - b \sin \theta \quad (6.17) \]
where
\[ b = \frac{f \zeta}{f} A \omega' - \frac{a}{A}. \] (6.18)

Integrability condition for the system (6.17) reads
\[ (\theta_y)_z - (\theta_z)_y = b_z \cos \theta + b_y \sin \theta - \frac{f}{2A} b_x + b^2 + \frac{f^2}{4A^2} \theta_x^2 = 0 \] (6.19)
or in the explicit form
\[ \left( \frac{f^2}{2A^3} u_{xxx} - \frac{f \zeta}{f^2} A \omega' + \frac{f^2}{2A^3} \omega'' \right) + \frac{3f}{2A} \omega'' \omega'/\omega^2 = 0. \] (6.20)

Reality condition for (6.20) implies that both quadratic terms vanish separately
\[ u_{xxx} = \frac{f \zeta}{f^2} A^4 \omega' - \frac{\omega''}{\omega'^2} + \frac{3A^2 \omega''}{f} \omega'/\omega^2 \] (6.21)
and \( \theta_x = 0 \iff \theta = \theta(y,z,t) \). Using (6.21) in (6.18) for \( a \) and then in (6.13) for \( b \) we obtain
\[ b = 0, \quad a = \frac{f \zeta}{f} A^2 \omega' \] (6.22)
and (6.17) with \( \theta_x = 0 \) implies \( \theta_y = \theta_z = 0 \), so that \( \theta = \theta(t) \). Equations (6.12) become
\[ A_y = a \sin \theta, \quad A_z = a \cos \theta, \quad A_x = \frac{f \zeta}{f^2} A^3 \omega' + A^2 \omega''/\omega'^2. \] (6.23)

From the definition (6.10) of \( A \) it follows
\[ u_{xx} = \frac{A^2}{f} - \frac{1}{\omega'^2} \] (6.24)
Since \( f = f(\zeta,t) \) where \( \zeta = \omega(u_x) + x \), we have
\[ f_x = \frac{f \zeta}{f} A^2 \omega', \quad f_y = f \zeta A \sin \theta \quad f_z = f \zeta A \cos \theta \] (6.25)
and hence \( A_y/A = f_y/f, \quad A_z/A = f_z/f \), so that \( A = \alpha(x,t)f \). Then the last equation in (6.23) implies
\[ \frac{A_x}{A} = \frac{f_x}{f} + \frac{\omega''}{\omega'^2} = \frac{f_x}{f} + \frac{\alpha_x(x,t)}{\alpha} \quad \implies \left( \frac{\omega''}{\omega'^2} \right)_y = 0, \quad \left( \frac{\omega''}{\omega'^2} \right)_z = 0 \] (6.26)
and hence
\[ \left( \frac{\omega''}{\omega'} \right)' = 0 \implies \frac{1}{\omega'} = c_1 u_x + c_2 \iff \omega' = \frac{1}{c_1 u_x + c_2}. \quad (6.27) \]

From (6.26) and (6.27) it follows that
\[ \alpha(x, t) = c_3(t) e^{-c_1(t)x}, \quad A = c_3(t) e^{-c_1(t)f(\zeta, t)} \quad (6.28) \]
while (6.24) and (6.11) imply
\[ u_{xx} = c_3^2(t) e^{-2c_1 x f} - (c_1 u_x + c_2) \]
\[ u_{xy} = c_3 e^{-c_1 x f} \sin \theta, \quad u_{xz} = c_3 e^{-c_1 x f} \cos \theta \quad (6.29) \]
while (6.21) takes the form
\[ u_{xxx} = c_4^3(t) e^{-4c_1 x f} f(\mu) - 3c_3^2(t) e^{-2c_1 x f} + c_1 (c_1 u_x + c_2). \quad (6.30) \]

It is easy to check that the integrability condition \((u_{xx})_x = u_{xxx}\) of equations (6.29) and (6.30) is identically satisfied.

Expressions (5.6) and (5.8) for derivatives of \(v\) take the form
\[ v_y = \frac{e^{c_1 x}}{c_3(t)} [v_x \sin \theta - (c_1 u_x + c_2) \cos \theta] \]
\[ v_z = \frac{e^{c_1 x}}{c_3(t)} [v_x \cos \theta + (c_1 u_x + c_2) \sin \theta] \]
\[ v_t = \frac{e^{2c_1 x}}{c_3(t)} [v_x^2 + (c_1 u_x + c_2)^2] - (c_1 u_x + c_2) + \frac{\varepsilon}{f}. \quad (6.31) \]

Integrability condition \((v_t)_x = (v_x)_t\) where \(v_x\) is determined by (5.7) in the form
\[ v_x = c_3(t) e^{-c_1 x} (v_y \sin \theta + v_z \cos \theta) \]
yields the relation
\[ (c_1 + \theta'(t))(c_1 u_x + c_2) - \frac{c_3'(t)}{c_3(t)} v_x = 0. \quad (6.32) \]

If \(c_3'(t) \neq 0\), (6.32) can be solved for \(v_x\) and the integrability conditions \((v_y)_x = (v_x)_y, \quad (v_z)_x = (v_x)_z\) imply \(c_1 c_3 = 0\) and, since \(c_1 \neq 0\) (otherwise (6.32) leads to a reduction), we have \(c_3 = 0\) which contradicts to our assumption \(c_3'(t) \neq 0\).
Thus, we proceed to the opposite case of constant $c_3$ in (6.32) which implies $\theta(t) = -c_1 t + \theta_0$ with $v_x$ remaining undetermined. Then we check that all the integrability conditions $(v_y)_t = (v_t)_y$, $(v_z)_t = (v_t)_z$ and $(v_y)_z = (v_z)_y$ are identically satisfied.

Differentiating first two equations (6.31) with respect to $x$ we obtain

$$
v_{xy} = \frac{e^{c_1 x}}{c_3} (v_{xx} + c_1 v_x) \sin \theta - c_1 c_3 e^{-c_1 x} f \cos \theta
$$

$$
v_{xz} = \frac{e^{c_1 x}}{c_3} (v_{xx} + c_1 v_x) \cos \theta + c_1 c_3 e^{-c_1 x} f \sin \theta.
$$

(6.33)

Another type of integrability conditions arises from the utilization of the relation $u_t = v$. We differentiate with respect to $t$ equations (6.29). The first equation in (6.29) yields

$$
v_{xx} = -c_1 v_x + c_2^2 e^{-2c_1 x} \left( \frac{f^2 v_x}{c_1 u_x + c_2} + f_t \right)
$$

(6.34)

while the two last equations give the same results as (6.33). The same result (6.34) we obtain by differentiating with respect to $t$ the equation $\Delta[u] = f(\zeta, t)$. By using the result (6.34) in (6.33) we obtain

$$
v_{xy} = c_3 e^{-c_1 x} \left[ \left( \frac{f^2 v_x}{c_1 u_x + c_2} + f_t \right) \sin \theta - c_1 f \cos \theta \right]
$$

$$
v_{xz} = c_3 e^{-c_1 x} \left[ \left( \frac{f^2 v_x}{c_1 u_x + c_2} + f_t \right) \cos \theta + c_1 f \sin \theta \right].
$$

(6.35)

The two-component real form (5.8) of CMA equation becomes

$$
v_t = \frac{e^{2c_1 x}}{c_3^2 f} \left[ v_x^2 + (c_1 u_x + c_2)^2 \right] + \frac{\varepsilon}{f} - (c_1 u_x + c_2)
$$

(6.36)

together with its derivative with respect to $x$

$$
v_{tx} = \frac{f^2}{f} \left[ \frac{v_x^2}{c_1 u_x + c_2} - (c_1 u_x + c_2) - \frac{\varepsilon c_3^2 e^{-2c_1 x}}{c_1 u_x + c_2} \right]
$$

$$
+ 2 \frac{f^2}{f} v_x + 3 c_1 (c_1 u_x + c_2) - c_1 c_3^2 e^{-2c_1 x} f.
$$

(6.37)

Differentiating (6.37) with respect to $x$ we discover the integrability condition $(v_t)_{xx} - (v_{xx})_t = 0$ of (6.36) and (6.34) in the form

$$
f_{tt} f - 2 f_t^2 - 4c_1^2 f^2 + f_{\zeta \zeta} f - 2 f_{\zeta}^2 + 4c_1 f_{\zeta} f
$$

$$
+ \frac{\varepsilon c_3^2 e^{-2c_1 x}}{(c_1 u_x + c_2)^2} (f_{\zeta \zeta} f - 2 f_{\zeta}^2 - c_1 f_{\zeta} f) = 0.
$$

(6.38)
Since $u_x$ and $x$ in equations for $f$ are allowed only in the combination $\zeta = \ln(c_1u_x + c_2)/c_1 + x + \zeta_0$, which is the independent variable in $f(\zeta, t)$, the equation (6.38) splits into two equations

\[f_{\zeta \zeta}f - 2f_\zeta^2 - c_1f_\zeta f = 0 \tag{6.39}\]
\[f_{tt}f - 2f_t^2 - 4c_1^2f^2 + f_{\zeta \zeta}f - 2f_\zeta^2 + 4c_1f_\zeta f \tag{6.40}\]

Using (6.39) in the equation (6.40) we simplify the latter equation to the form

\[f_{tt}f - 2f_t^2 - 4c_1^2f^2 + 5c_1f_\zeta f = 0 \tag{6.41}\]

Equation (6.39) by the substitution $g = f_\zeta/f$ is reduced to the first-order separable equation

\[g_\zeta - g(g + c_1) = 0 \]

with the solution $g = c_1/(\gamma_0(t)e^{-c_1\zeta} - 1)$. The corresponding general solution for $f$ is given by the quadrature

\[\ln f = \ln f_0(t) + c_1 \int \frac{d\zeta}{\gamma_0(t)e^{-c_1\zeta} - 1}\]

or in the explicit form

\[f = \frac{f_0(t)}{\gamma_0(t) - e^{c_1\zeta}} \tag{6.42}\]

In the original variables this becomes

\[f = \frac{f_0(t)}{\gamma_0(t) - c_0e^{c_1x}(c_1u_x + c_2)} \tag{6.43}\]

Next we input the expression (6.42) for $f$ in the equation (6.41) and after cancelation of the common factor $\gamma_0 - e^{c_1\zeta}$ we obtain the result

\[(f_0''f_0 - 2f_0'^2 - 4c_1^2f_0^2)(\gamma_0 - e^{c_1\zeta}) + 2f_0f_0'\gamma_0' - f_0^2\gamma_0'' + 5c_1^2f_0^2e^{c_1\zeta} = 0 \tag{6.44}\]

where primes denote derivatives of functions depending on $t$ only. Terms with the first and zeroth powers of $e^{c_1\zeta}$ should vanish separately. The terms with $e^{c_1\zeta}$ yield the equation

\[f_0''f_0 - 2f_0'^2 - 9c_1^2f_0^2 = 0 \tag{6.45}\]

with the general solution

\[f_0 = \frac{1}{\mu(t)}, \quad \text{where} \quad \mu'' = -9c_1^2\mu \tag{6.46}\]
or, explicitly,

\[ \mu(t) = \mu_1 \cos(3c_1 t) + \mu_2 \sin(3c_1 t) \]

with arbitrary constants \( \mu_1 \) and \( \mu_2 \). The remaining equation consists of the terms in (6.44) without \( e^{c_1 \zeta} \) which with the use of (6.46) becomes

\[ \gamma_0'' + 2 \frac{\mu'}{\mu} \gamma_0' - 5c_1^2 \gamma_0 = 0. \]  
(6.47)

For the integration of (6.47) we use its two commuting Lie point symmetries

\[ X_1 = \gamma_0 \partial_{\gamma_0}, \quad X_2 = -\partial_t + \frac{\mu'}{\mu} \gamma_0 \partial_{\gamma_0}. \]  
(6.48)

Using the appropriate algorithm for the case \( G_2 Ia \) from H. Stephani’s book \( [10] \) we easily obtain the general solution

\[ \gamma_0(t) = \frac{\nu(t)}{\mu(t)}, \quad \text{where} \quad \nu'' = -4c_1^2 \nu \]  
(6.49)

or, explicitly

\[ \nu(t) = \nu_1 \cos(2c_1 t) + \nu_2 \sin(2c_1 t) \]

whereas \( \mu \) is defined in (6.46) and \( \nu_1, \nu_2 \) are arbitrary real constants.

Using our results in (6.42) we obtain the final result for \( f \)

\[ f(\zeta, t) = \frac{1}{\nu(t) - \mu(t)e^{c_1 \zeta}} \]  
(6.50)

or, alternatively, by using \( e^{c_1 \zeta_0} = c_0 \) in the definition \( \zeta = \ln(c_1 u_x + c_2)/c_1 + x + \zeta_0 \) coming from (6.39) and (6.27)

\[ f = \frac{1}{\nu(t) - \mu(t)e^{c_1 x}c_0(c_1 u_x + c_2)} \]  
(6.51)

where \( \mu(t) \) is defined by (6.46) and \( \nu(t) \) defined in (6.49). It turns out that the integrability conditions \( (v_{xt})_y = (v_{xy})_t \) and \( (v_{xt})_z = (v_{xz})_t \) of the equations (6.35) and (6.37) with \( f \) defined by (6.51) are identically satisfied without further constraints on the unknowns \( v_x, u_x, \mu \) and \( \nu \).

### 7 Integration of equations

After making sure that the integrability conditions of our equations are satisfied we may proceed to the integration of these equations. We start
with the integration of the first equation in (6.29) with respect to $x$ with
the result

$$ \frac{1}{2} c_0 \mu e^{2c_1 x} (c_1 u_x + c_2)^2 - \nu e^{c_1 x} (c_1 u_x + c_2) = c_2^3 e^{-c_1 x} - \sigma(y, z, t) $$

which can be rewritten as

$$ c_1 u_x + c_2 = \frac{1}{c_0 \mu} e^{-c_1 x} \left( \nu \pm \Lambda^{1/2} \right) \tag{7.1} $$

where

$$ \Lambda = 2 c_0 \mu \left( c_2^3 e^{-c_1 x} - \sigma(y, z, t) \right) + \nu^2 \tag{7.2} $$

with $\sigma(y, z, t)$ playing the role of the constant of integration with respect to $x$. Henceforth we are free to use either upper or lower sign in all the formulas. Using (7.1) together with the notation (7.2) in the formula (6.51) for $f$ we finally obtain

$$ f = \mp \Lambda^{-1/2}. \tag{7.3} $$

Next we utilize the results (7.1) and (7.3) in the second and third equations in (6.29), using $\Lambda_y = -2 c_0 \mu \sigma_y$ and $\Lambda_z = -2 c_0 \nu \sigma_z$ to obtain $\sigma_y = c_1 c_3 \sin \theta$ and $\sigma_z = c_1 c_3 \cos \theta$ and hence

$$ \sigma = c_1 c_3 (y \sin \theta + z \cos \theta) + \sigma_0(t) \tag{7.4} $$

which implies $\sigma_t = c_1^2 c_3 (z \sin \theta - y \cos \theta) + \sigma_0'(t)$. The result (7.3) should be used in the definition (7.2) of $\Lambda$.

Total derivative of (7.1) with respect to $t$ yields

$$ v_x = \frac{e^{-c_1 x}}{c_1 c_0 \mu} \left[ -\frac{\mu'}{\mu} \left( \nu \pm \Lambda^{1/2} \right) + \nu' \pm \frac{1}{2} \Lambda^{-1/2} \Lambda_t \right] \tag{7.5} $$

where

$$ \Lambda_t = 2 c_0 \mu' \left( c_3^2 e^{-c_1 x} - \sigma \right) - 2 c_0 \mu \sigma_t + 2 \nu \nu'. $$

Integrating (7.1) with respect to $x$ we obtain

$$ u = -\frac{\nu}{c_1 c_0 \mu} e^{-c_1 x} + \frac{1}{3 c_1^3 c_0^2 c_2^2 \mu^2} \Lambda^{3/2} - \frac{c_2}{c_1} x + \rho(y, z, t) \tag{7.6} $$

with the “constant of integration” $\rho(y, z, t)$. Differentiation of (7.6) with respect to $t$ yields the result

$$ v = u_t = -\frac{1}{c_1 c_0} \left( \frac{\mu'}{\mu} \right) e^{-c_1 x} \pm \frac{1}{c_1 c_0^2 c_2^2 \mu^2} \left( 2 \mu' \Lambda^{3/2} - \frac{1}{2} \Lambda^{1/2} \Lambda_t \right) + \rho_t. \tag{7.7} $$
Now we use the expression (7.7) to compute \( v_y \) and \( v_z \) in the first two equations (6.31) which imply the following equations

\[
\rho_{ty}(y, z, t) = \frac{\nu}{c_0} \left( \frac{\nu \sin \theta}{\mu} \right) \frac{t}{t}, \quad \rho_{tz}(y, z, t) = \frac{\nu}{c_0} \left( \frac{\nu \cos \theta}{\mu} \right) \frac{t}{t}
\]

with the final result

\[
\rho(y, z, t) = \frac{\nu}{c_0} (y \sin \theta + z \cos \theta) + \rho_0(t) + r(y, z) \tag{7.8}
\]

with arbitrary \( \rho_0(t) \) and \( r(y, z) \). Equation \( \Delta[a] = f \) implies \( \Delta[r(y, z)] = 0 \).

We differentiate \( v \) in (7.7) with respect to \( t \) and use \( v_t \) in the third equation of (6.31) which is our basic complex Monge-Ampère equation in the two-component form. This equation becomes

\[
v_t \equiv -\frac{1}{c_1 c_0} \left( \frac{\nu}{\mu} \right)^{''} e^{-c_1 x} \pm \frac{2}{3 c_1 c_0 c_3^2} \left( \frac{\mu'}{\mu} \right) \Lambda^{3/2} \pm \frac{2 \mu'}{c_1 c_0 c_3^2 \mu^3} \Lambda^{1/2} \Lambda_t \nonumber \\
\pm \frac{1}{2 c_1 c_0 c_3^2 \mu^2} \left( \frac{\mu'^2}{\mu^2} \Lambda - \frac{\mu'}{\mu} \Lambda_t + \frac{1}{4} \Lambda^{-1} \Lambda_t^2 \right) + \rho_{tt} \tag{7.9}
\]

\[
= \left[ \left( \frac{\mu'}{\mu} \right)^2 \Lambda - \frac{\mu'}{\mu} \Lambda_t + \frac{1}{4} \Lambda^{-1} \Lambda_t^2 \right] + \frac{2 \mu'}{\mu} \left( \frac{\nu'}{\mu} \right) \Lambda^{1/2} 
\pm \left( \nu' - \frac{\mu'}{\mu} \right) \Lambda^{-1/2} \Lambda_t + \left( \frac{\nu'}{\mu} \right)^2 \right] \nonumber \\
\pm \frac{1}{c_0 \mu} e^{-c_1 x} \Lambda^{1/2} - \frac{\nu}{c_0 \mu} e^{-c_1 x} \mp \varepsilon \Lambda^{1/2}. \nonumber
\]

Here we have not done an obvious cancelation of a couple of terms on both sides of the equation in order to keep the expression for \( v_t \) which we will need for the metric. A lengthy but straightforward check shows that the equation (7.9) is identically satisfied provided that arbitrary functions \( \rho_0(t) \) and \( \sigma_0(t) \) satisfy the equations

\[
\rho_0''(t) = -\frac{2}{c_1 c_0 c_3^2} \left\{ \nu \left( \gamma_0^2 + c_1^2 \gamma_0^2 \right) + \gamma_0 \left( \frac{\mu'}{\mu} \sigma_0 - \sigma_0' \right) - 2 c_1^2 \gamma_0 \sigma_0 \right\} \tag{7.10}
\]

where \( \gamma_0(t) = \nu/\mu \), as defined in (6.49), which determines \( \rho_0(t) \) by taking two quadratures and

\[
\sigma_0''(t) + c_1^2 \sigma_0(t) = -\varepsilon c_1 c_0 c_3^2 \mu. \tag{7.11}
\]

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The solution to (7.11) is determined by the elementary quadratures
\[
\sigma_0(t) = -\varepsilon c_1 c_0 c_3^2 \left[ \sin c_1 t \int \cos c_1 t \partial t - \cos c_1 t \int \sin c_1 t \partial t \right] \\
+ A \cos c_1 t + B \sin c_1 t 
\] (7.12)
where \( \mu = \mu_1 \cos 3c_1 t + \mu_2 \sin 3c_1 t. \)

8 The metric

All our equations being completely solved, we can explicitly construct the corresponding metric
\[
ds^2 = (v_t + u_{xx})(dt^2 + dx^2) \mp \Lambda^{-1/2}(dy^2 + dz^2) \\
- 2b(dt \, dz - dx \, dy) + 2c(dt \, dy + dx \, dz). 
\] (8.1)

The metric coefficients are defined as follows
\[
u = \frac{1}{c_0 \mu} \int c_3 e^{-c_1 x} \cos \left( \pm \Lambda^{1/2} + \nu \right) \\
+ e^{-2c_1 x} \Lambda^{-1/2} 
\] (8.2)
\[
u' = \frac{1}{c_0 c_3 \mu} \left[ -\frac{\mu'}{\mu} \left( \nu \pm \Lambda^{1/2} \right) + \nu' \pm \frac{1}{2} \Lambda^{-1/2} \Lambda_t \right] \cos \theta \\
+ \left[ \frac{1}{c_0 c_3 \mu} \left( \pm \Lambda^{1/2} + \nu \right) \pm c_3 e^{-c_1 x} \Lambda^{-1/2} \right] \cos \theta 
\] (8.3)
\[
u = \frac{1}{c_0 c_3 \mu} \int c_3 e^{-c_1 x} \cos \left( \pm \Lambda^{1/2} + \nu \right) \\
+ e^{-2c_1 x} \Lambda^{-1/2} 
\] (8.4)
while for \( v_t \) we can use the r.h.s. of equation (7.9). Here \( \Lambda \) is defined in (7.2) whence it follows
\[
\Lambda_t = 2c_0 \mu' \left( c_3^2 e^{-c_1 x} - \sigma \right) - 2c_0 \mu \sigma_t + 2\nu \nu' 
\]
where \( \sigma \) is defined in (7.4), \( \theta = -c_1 t + \theta_0 \) and the expression for \( \sigma_t \) is given right after the equation (7.4).

Since \( \sigma(t) \) and \( \sigma'(t) \) involve two different combinations of \( y \) and \( z \) and there is obviously no reduction in either \( x \) or \( t \), there is no symmetry reduction of the metric (8.1) in the number of independent variables and hence
this metric does not admit any Killing vectors. Thus, we have obtained the Ricci-flat (anti-)self-dual metric without Killing vectors. It is generated by a non-invariant solution of the complex Monge-Ampère equation determined solely by its invariance with respect to the special first nonlocal symmetry of $CMA$ without any additional assumptions. We see that such an invariance does not lead to a reduction in the number of independent variables in the solution, on the contrary to the invariance under Lie point symmetries. This explains our special interest to nonlocal symmetry flows in the hierarchies of bi-Hamiltonian systems of Monge-Ampère type which we constructed recently [11].

9 Conclusion

Our search for non-invariant solutions to the elliptic complex Monge-Ampère equation has been motivated by the fundamental problem of obtaining explicitly the metric of the gravitational instanton $K3$, since it will not admit any Killing vectors (continuous symmetries). Recently, we produced an example of such a metric, though not an instanton one, by combining our previous approaches to the problem and choosing at random a very particular solution to resulting equations [12].

Here we have demonstrated that a general requirement of invariance under nonlocal symmetries of $CMA$ yields solutions which are not invariant with respect to any local symmetries and therefore no symmetry reduction results in the number of independent variables. We have explicitly constructed such a solution by a meticulous analysis of all integrability conditions of the invariance equations which made it possible a complete integration of these equations with no additional assumptions made. Thus, we have obtained the most general form of the solution of $CMA$ which is invariant under the special first nonlocal symmetry in the hierarchy of $CMA$, not just a solution taken out by chance. We have also presented the corresponding ASD Ricci-flat metric without Killing vectors. It has rather a complicated form and further analysis is needed to study its properties.

This method seems to be a direct approach for obtaining noninvariant solutions of $CMA$ from the invariance under other nonlocal flows in the hierarchy. We also point out that all our constructions and results have been obtained simultaneously for elliptic and hyperbolic $CMA$ and so the corresponding metrics have either Euclidean or neutral (ultra-hyperbolic) signature, respectively.
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