ON UPPER BOUNDS FOR THE FIRST $\ell^2$-BETTI NUMBER

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Abstract. This article presents a method for proving upper bounds for the first $\ell^2$-Betti number of groups using only the geometry of the Cayley graph. As an application we prove that Burnside groups of large prime exponent have vanishing first $\ell^2$-Betti number.

Our approach extends to generalizations of $\ell^2$-Betti numbers, that are defined using characters. We illustrate this flexibility by generalizing results of Thom-Peterson on $q$-normal subgroups to this setting.

Over the last 30 years the $\ell^2$-Betti numbers have become a major tool in the investigation of infinite groups. The purpose of this article is to explore the first $\ell^2$-Betti number of groups using only the geometry of the Cayley graph. Our method is based on Pichot’s observation [11, Proposition 2] that the first $\ell^2$-Betti number can be expressed with the rate of relations in the Cayley graph. It follows from an elementary identity (see Lemma 1.1) that explicit cycles in the Cayley graph give rise to upper bounds for the first $\ell^2$-Betti number. Surprisingly, these elementary bounds can be used to prove new results.

Theorem 0.1. Let $p$ be a prime and let $G$ be a torsion group of exponent $p$. Then $b_1^{(2)}(G) \leq 2p - 2$.

Using a theorem of Gaboriau this implies a vanishing result for the first $\ell^2$-Betti number of Burnside groups $B(m,p)$ of exponent $p$.

Corollary 0.2. Let $p$ be a prime number. If $p$ is sufficiently large, then $b_1^{(2)}(B(m,p)) = 0$.

On the other hand, suppose that $b_1^{(2)}(B(m,p)) \neq 0$ for some prime $p$. Then Theorem 0.1 offers a simple solution of the restricted Burnside problem for $m$-generated $p$-groups using the multiplication formula for $\ell^2$-Betti numbers of finite index subgroups.

Our method can neatly be adapted to character-theoretic generalizations of the first $\ell^2$-Betti number. We recall that every character $\psi$ (see [8, Def. 2.5]) of the group $G$, gives rise to a $\psi$-Betti number $b_1^\psi(G)$; see [8] or Section II. The ordinary Betti numbers and the $\ell^2$-Betti numbers are special cases of this construction. However, it is difficult to calculate or bound $\psi$-Betti numbers under general assumptions of $\psi$.
We extend Pichot’s observation to the general setting and we use our method to generalize a result of Thom-Peterson [10, Theorem 5.6] to $\psi$-Betti numbers; see Corollary 3.3. Even for $\ell^2$-Betti numbers our argument contains a new proof of their result. This provides a convenient way to bound (and sometimes calculate) $\psi$-Betti numbers in some generality. We illustrate this by proving a vanishing result for certain $\psi$-Betti numbers of right-angled groups; see Theorem 3.6.

In Section 1 we discuss basic results on $\psi$-Betti numbers and we introduce our main method. In Section 2 we apply it in the case of $p$-torsion groups. Section 3 is concerned with $q$-normality and presents applications to $\psi$-Betti numbers.

1. Betti numbers and the Cayley graph

The following simple result is essential for our approach.

**Lemma 1.1.** Let $\mathcal{H}$ be a Hilbert space and let $W \subseteq \mathcal{H}$ be a subspace. Let $P: \mathcal{H} \to W$ denote the orthogonal projection onto the closure of $W$. Then for all $v \in \mathcal{H}$

$$\langle Pv, v \rangle = \sup_{w \in W} \frac{|\langle w, v \rangle|^2}{\langle w, w \rangle}$$

where the supremum is taken over all non-zero elements of $W$ (and is defined to be 0 if $W = 0$).

**Proof.** For $v = 0$ the assertion is obvious. We may assume that $\|v\| = 1$. For all $w \in W$, we note that

$$|\langle w, v \rangle|^2 = |\langle Pw, v \rangle|^2 = |\langle w, P v \rangle|^2 \leq \|w\|^2 \|Pv\|^2 = \langle Pv, v \rangle \|w\|^2.$$

If $w \neq 0$ we obtain

$$\frac{|\langle w, v \rangle|^2}{\langle w, w \rangle} \leq \langle Pv, v \rangle.$$ 

In particular, the proof is complete if $Pv = 0$.

For the converse we assume $Pv \neq 0$. Let $\varepsilon \in (0, 1)$. Since $W$ is dense in $\overline{W}$, there is $w \in W$ with $\|Pv - w\| < \varepsilon \|Pv\|^2$ and we deduce

$$|\langle w, v \rangle| \geq \langle Pv, v \rangle - |\langle w - Pv, v \rangle| \geq (1 - \varepsilon) \langle Pv, v \rangle.$$

In addition, we note that $\|w\| = \|w - Pv + Pv\| \leq (1 + \varepsilon) \|Pv\|$ and so

$$\frac{|\langle w, v \rangle|^2}{\langle w, w \rangle} \geq \frac{(1 - \varepsilon)^2}{(1 + \varepsilon)^2} \langle Pv, v \rangle.$$

The assertion follows as $\varepsilon$ can be arbitrarily close to 0. \qed

Let $G$ be a group. A character of $G$ is a function $\psi: G \to \mathbb{C}$ of positive type, which is constant on conjugacy classes of $G$ and satisfies $\psi(1_G) = 1$; see [8, Def. 2.5]. Let $\text{Ch}(G)$ denote the space of all characters of $G$. Every character $\psi \in \text{Ch}(G)$ gives rise to a semi-definite $G$-invariant inner product $\langle g, h \rangle_\psi = \psi(h^{-1}g)$ on the group ring $\mathbb{C}[G]$. Passing to the completion provides us with a tracial Hilbert $G$-bimodule $\ell^2(G)$; see [8, Def. 2.1]. Using the GNS construction, this provides a tracial von Neumann algebra and a notion of dimension, which can be used to define the $\psi$-Betti numbers $b^\psi_k(G)$.
Lemma 1.3. \( \partial \) The image of \( \ell \) \( \sum \) Since the associated free resolution of \( C \) Proof. Let \( G \) and assertion (a) follows from Lemma 1.1. Let \( G \) \( k \) \( \psi \) is where \( \psi \) \( | \langle \rangle \rangle = 1 \) for all \( g \) gives rise to the ordinary rational Betti numbers of \( G \) since \( \ell^\psi(G) \cong \mathbb{C} \).

**Definition 1.2.** Let \( G \) be a group and let \( \psi \in \text{Ch}(G) \). A subgroup \( K \leq G \) is \( \psi \)-regular, if \( \psi|_K \) is the regular character on \( K \), i.e. \( \psi(k) = 0 \) for all \( k \in K \setminus \{1\} \).

Here we are mainly interested in the first Betti numbers \( b^\psi_1(G) \). It will however be useful and instructive to initially consider the 0-th Betti number. Let \( J_G \) denote the augmentation ideal in \( \mathbb{C}[G] \), i.e. the set of elements \( w = \sum_{g \in G} w_g g \) which satisfy \( \sum_{g \in G} w_g = 0 \).

**Lemma 1.3.** Let \( G \) be a group and let \( \psi \in \text{Ch}(G) \) be a character.

(a) \( b^\psi_0(G) = 1 - \sup_{w \in J_G} \langle \psi|_G, w \rangle \) where the supremum is taken over all non-zero elements of \( J_G \).

(b) If \( G = \bigcup_{i \in I} G_i \) is a directed union of subgroups \( G_i \), then \( \lim_{i \in I} b^\psi_0(G_i) = b^\psi_0(G) \).

(c) \( b^\psi_0(G) \leq \frac{1}{|K|} \) for every \( \psi \)-regular subgroup \( K \leq G \).

**Remark 1.4.** It is well-known that \( b^\psi_0(G) = \frac{1}{|G|} \); see [9, Thm. 1.35 (8)].

**Proof.** Let \( S \) be a generating set for \( G \). We consider the initial segment of the associated free resolution of \( \mathbb{C} \):

\[ \mathbb{C}[G]^S \xrightarrow{\partial_1} \mathbb{C}[G] \longrightarrow \mathbb{C}. \]

The image of \( \partial_1 \) is the augmentation ideal. We take the tensor product with \( \ell^\psi(G) \) and deduce that

\[ b^\psi_0(G) = 1 - \dim_{\psi}(J_G). \]

where \( J_G \) denotes the closure of the image of the augmentation ideal in \( \ell^\psi(G) \). Let \( P \colon \ell^\psi(G) \to J_G \) denote the orthogonal projection. By definition

\[ \dim_{\psi}(J_G) = \langle P(1), 1 \rangle_{\psi} \]

and assertion (a) follows from Lemma [1]. Let \( G = \bigcup G_i \) be a direct union of subgroups, then \( J_G = J_G \cup J_G_i \) and (b) follows immediately from (a).

Let \( K \leq G \) be a \( \psi \)-regular subgroup. Let \( T \subseteq K \setminus \{1\} \) be a finite subset. Then

\[ w = |T| \cdot 1_G - \sum_{k \in T} k \in J_G. \]

Since \( K \) is \( \psi \)-regular, the elements of \( K \) are orthonormal and we deduce

\[ \frac{|\langle w, 1 \rangle_{\psi}|^2}{\langle w, w \rangle_{\psi}} = \frac{|T|^2}{|T|^2 + |T|} = \frac{|T|}{|T| + 1}. \]

Now (a) implies \( b^\psi_0(K) \leq 1 - \frac{|T|}{|T| + 1} = \frac{1}{|T| + 1} \). Statement (c) follows by taking \( T = K \setminus \{1\} \) if \( K \) is finite respectively letting \( |T| \) tend to \( \infty \) otherwise. \( \square \)
We would like to apply the same ideas to the first $\psi$-Betti number $b_1^\psi(G)$. However, up to now we only have a definition of $b_1^\psi(G)$ for all finitely generated groups $G$. We also require a definition for groups which are not finitely generated. This could be done using Lück’s generalized dimension function (discussed in [9 §6.1, 6.2]), but this is not convenient for our purposes and for simplicity we work with the following variation.

**Definition 1.5.** Let $G$ be a group and let $\psi \in \text{Ch}(G)$. Then

$$\bar{b}_1^\psi(G) := \liminf_{H \leq G} b_1^\psi(H)$$

where the limit is taken over the directed system of all finitely generated subgroups $H \leq G$.

**Remark 1.6.** For a finitely generated group $\bar{b}_1^\psi(G) = b_1^\psi(G)$. In general however, $\bar{b}_1^\psi(G)$ can be strictly larger than the properly defined value of the first $\psi$-Betti number. It is easy to see this for the ordinary Betti numbers. For instance, it follows from the methods developed in [5] that $\langle (x_i)_{i \in \mathbb{Z}} \mid x_i x_{i+1} x_i^{-1} = x_{i+1}^2 \rangle$, is a perfect and locally indicable group, i.e., the ordinary rational Betti number of every finitely generated subgroup is $\geq 1$.

For the classical $\ell^2$-Betti number the inequality $b_1^{(2)}(G) \leq \bar{b}_1^{(2)}(G)$ follows from the argument given in the proof of [9 Theorem 7.2 (3)].

For later reference we state the following observation.

**Lemma 1.7.** Let $G$ be a group and let $\psi \in \text{Ch}(G)$. If $G = \bigcup_{i \in I} G_i$ is a directed union of subgroups $G_i$, then

$$\bar{b}_1^\psi(G) \leq \liminf_{i \in I} \bar{b}_1^\psi(G_i)$$

**Proof.** Let $\varepsilon > 0$. There is a finitely generated subgroup $H_0 \leq G$ such that $b_1^\psi(H) \geq \bar{b}_1^\psi(G) - \varepsilon$ for all finitely generated subgroups $H$ that contain $H_0$. Since $H_0$ is finitely generated, there is $i \in I$ such that $H_0 \subseteq G_i$. Thus for all $j \geq i$ we have $\bar{b}_1^\psi(G_j) \geq \bar{b}_1^\psi(G_i) - \varepsilon$. \[
\]

Assume that $G$ is finitely generated and that $S$ is a finite generating set. The Cayley graph $\text{Cay}(G,S)$ is the directed graph with vertex set $G$ and edges

$$E_{G,S} = \{(g,gs) \mid g \in G, s \in S\}.$$ 

The edge $(1_G,s)$ will be denoted by $s$. The Cayley graph is equipped with a left action of $G$. Let $\mathbb{C}[E_{G,S}]$ be the vector space with basis $E_{G,S}$ and let $\partial : \mathbb{C}[E_{G,S}] \to \mathbb{C}[G]$ denote the boundary map. A finite cycle in $\text{Cay}(G,S)$ is an element $z \in \mathbb{C}[E_{G,S}]$ with $\partial(z) = 0$. Let $Z_{G,S}$ denote the space of finite cycles. If $\psi \in \text{Ch}(G)$ is a character, then the semi-definite inner product $\langle \cdot, \cdot \rangle_\psi$ extends to a $G$-invariant semi-definite inner product on $\mathbb{C}[E_{G,S}]$ such that the edges $\{s \mid s \in S\}$ are orthonormal; this means

$$\langle (g, gs), (h, ht) \rangle_\psi = \begin{cases} 0 & \text{if } s \neq t \\ \psi(h^{-1}g) & \text{if } s = t \end{cases}.$$ 

The following extends [11 Prop. 2] and is our main tool.
Lemma 1.8. Let $G$ be a group and let $S$ be a finite generating set. Let $\psi \in \text{Ch}(G)$ be a character. Then

$$b_1^\psi(G) = |S| - 1 + b_0^\psi(G) - \sum_{s \in S} \sup_{z \in Z_{G,S}} |\langle z, \bar{s} \rangle_\psi|^2$$

where the suprema are taken over all non-zero elements of $Z_{G,S}$.

Proof. The finite generating set $S$ provides us with a presentation $G \cong F/R$ of $G$ where $F$ is the free group over $S$ and $R$ is the subgroup of relations. We consider the initial segment of the associated free resolution of $\mathbb{C}$:

$$\mathbb{C}[G] \xrightarrow{\partial_2} \mathbb{C}[G]^S \xrightarrow{\partial_1} \mathbb{C}[G] \rightarrow \mathbb{C}.$$ Tensoring with $\mathbb{C}[G]$ gives

$$\mathbb{C}[G]^R \xrightarrow{\partial_2} \mathbb{C}[G]^S \xrightarrow{\partial_1} \mathbb{C}[G] \rightarrow \mathbb{C}.$$

The middle term is naturally isomorphic to the completion of $\mathbb{C}[E_{G,S}]$ with respect to $\langle \cdot, \cdot \rangle_\psi$. The image of $\partial_2$ is the closure of $Z_{G,S}$. The $\psi$-dimension of the closure of the image of $\partial_1$ is $1 - b_0^\psi(G)$. We deduce that

$$b_1^\psi(G) = |S| - (1 - b_0^\psi(G)) - \text{dim}_\psi(Z_{G,S}).$$

Let $P: \mathbb{C}(G)^S \rightarrow Z_{G,S}$ denote the orthogonal projection. By definition

$$\text{dim}_\psi(Z_{G,S}) = \sum_{s \in S} \langle P\bar{s}, \bar{s} \rangle_\psi$$

and the result follows from Lemma 1.1. \qed

Remark 1.9. It seems surprising that the value on the right hand side is independent from the chosen set of generators. This is a consequence of the homotopy invariance of the $\psi$-Betti numbers, which can be proven using the standard argument; e.g. \cite[Thm. 3.18]{7} or \cite{9}.

2. Torsion Groups

In view of Remark 1.6 the following result implies Theorem 0.1

Theorem 2.1. Let $p$ be a prime. Let $G$ be a torsion group of exponent $p$. Then $\overline{b}_1^{(2)}(G) \leq 2p - 2$.

Proof. We may assume that $G$ is infinite (and $\overline{b}_0^{(2)} = 0$), otherwise $\overline{b}_1^{(2)}(G) = 0$ and there is nothing to show. By the definition of $\overline{b}_1^{(2)}(G)$ (see Remark 1.6), we may assume that $G$ is finitely generated.

We choose a minimal generating set $S$ of $G$ and denote the number of elements by $N = |S|$. Since all elements of $G$ have prime order, all pairwise distinct elements $a, b, c \in S$ satisfy

$$\langle ac \rangle \cap \langle ab \rangle = \{1\}$$

Suppose for a contradiction that there are three distinct elements $a, b, c \in S$ with $\langle ac \rangle \cap \langle ab \rangle \neq \{1\}$, then these cyclic groups of prime order coincide and

$$ac = (ab)^k$$

for some $k \in \mathbb{N}$, i.e., $c \in \langle a, b \rangle$ which contradicts the minimality of $S$. 

For all \(a \in S\) we have \(N - 1\) relations
\[(ab)^p\]
of length \(2p\) for all \(b \neq a\) in \(S\). By condition (1), the only common edge in the Cayley graph is the first edge \(\bar{a}\) from 1 to \(a\). Summing up these cycles, we obtain a cycle \(z_a\) in \(\text{Cay}(G, S)\) with
\[
\frac{\langle z_a, \bar{a} \rangle^2}{\langle z_a, z_a \rangle} = \frac{(N - 1)^2}{(N - 1)^2 + (N - 1)(2p - 1)} = \frac{1}{1 + \frac{2p - 1}{N - 1}}.
\]
We deduce from Lemma 1.8 that
\[
b_{1}^{(2)}(G) \leq N - 1 - \sum_{a \in S} \frac{\langle z_a, \bar{a} \rangle^2}{\langle z_a, z_a \rangle} = N - 1 - \frac{N}{1 + \frac{2p - 1}{N - 1}} = \frac{2p - 2}{1 + \frac{2p - 1}{N - 1}} \leq 2p - 2.
\]

\[\square\]

**Theorem 2.2.** Let \(p\) be a prime number and let \(G\) be a countable torsion group of exponent \(p\). If \(G\) has an infinite normal subgroup \(N\) of infinite index, then
\[b_{1}^{(2)}(G) = 0.\]

**Proof.** By Theorem 2.1 and Remark 1.6 we have \(b_{1}^{(2)}(N) \leq b_{1}^{(2)}(N) \leq 2p - 2\). By Gaboriau’s Theorem [4] Thm. 6.8, this implies \(b_{1}^{(2)}(G) = 0\). \[\square\]

**Proof of Corollary 0.2** Recall that \(B(m, p)\) denotes the Burnside group of exponent \(p\) and rank \(m\). Since \(B(1, p)\) is finite, we have \(b_{1}^{(2)}(B(1, p)) = 0\).

Assume \(m \geq 2\). For sufficiently large \(p\), the main result of [6] implies that \(B(m, p)\) contains a \(Q\)-subgroup \(H\) which is isomorphic to \(B(\infty, p)\). A \(Q\)-subgroup has the property that the normal closure \(\langle K \rangle^{B(m, p)}\) in \(B(m, p)\) of any normal subgroup \(K \trianglelefteq H\) intersects \(H\) exactly in \(K\).

Take a projection from \(B(\infty, p)\) onto \(B(\infty, p)\) with an infinite kernel \(K\). Then the normal closure \(\langle K \rangle^{B(m, p)}\) is an infinite normal subgroup of \(B(m, p)\) of infinite index. Now Theorem 2.2 implies the result. \[\square\]

**Remark 2.3.**

1. Ivanov [6] quantifies sufficiently large as \(p > 10^{78}\).

2. One can also deduce \(b_{1}^{(2)}(B(m, p)) = 0\) for \(m \geq 3\) under the assumption that \(B(2, p)\) is infinite\(^1\) using the normal subgroup \(N = \ker(B(m, p) \to B(m - 1, p))\).

   Indeed, let \(x_1, x_2, \ldots, x_m\) be a free generating set of \(B(m, p)\) such that \(N\) is the normal closure of \(x_1\) in \(B(m, p)\). Since \(\langle x_1, x_2 \rangle \subseteq N\langle x_2 \rangle\) and \(\langle x_1, x_2 \rangle \cong B(p, 2)\) is infinite, we deduce that \(N\) is infinite. Moreover, \(N\) has infinite index, since \(B(m, p)/N \cong B(m - 1, p)\).

3. We expect that \(b_{1}^{(2)}(B(m, p)) = 0\) for all \(p, m\). On the other hand, if \(b_{1}^{(2)}(B(m, p)) > 0\) holds for some \(m\) and \(p\), then this offers a simple solution to the restricted Burnside problem for \(m\)-generated \(p\)-groups.

\(^1\)According to Adian [2] the Burnside groups \(B(2, p)\) are infinite for all \(p > 100\).
More precisely, every finite index normal subgroup \( N \leq B(m, p) \) satisfies
\[
|B(m, p) : N| \cdot b_1^{(2)}(B(m, p)) = b_1^{(2)}(N) \leq 2p - 2
\]
by Theorem 2.1 and this inequality imposes an upper bound on the index of \( N \).

3. \( q \)-NORMALITY AND APPLICATIONS

Lemma 3.1. Let \( G = \langle H, a \rangle \) be a group and let \( \psi \in \text{Ch}(G) \) be a character. Assume that \( aHa^{-1} \cap H \) contains a \( \psi \)-regular subgroup of order \( n \in \mathbb{N} \cup \{\infty\} \). Then
\[
\bar{b}_1^\psi(G) - b_0^\psi(G) \leq \bar{b}_1^\psi(H) + \frac{3 + 2\Re(\psi(a))}{n + 2 + 2\Re(\psi(a))}
\]
In particular
\[
\bar{b}_1^\psi(G) \leq \bar{b}_1^\psi(H).
\]
for \( n = \infty \).

Proof. Without loss of generality we assume that \( a \notin H \). We denote the \( \psi \)-regular subgroup of order \( n \) in \( H \cap aHa^{-1} \) by \( K \). For a finite subset \( S \subseteq H \), we define \( H_S = \langle S \rangle \). Since \( K \) is \( \psi \)-regular, we obtain \( b_0^\psi(H) \leq b_0^\psi(G) \leq \frac{1}{n} \) and \( b_0^\psi(H_S) \leq |K \cap H_S|^{-1} \) by Remark 1.8.

For \( n = \infty \), let \( S \subseteq H \) be any finite subset and denote by \( h_1, h_2, \ldots, h_k \) the pairwise distinct elements of \( S \cap aSa^{-1} \cap K \). If \( n < \infty \), we choose \( S \subseteq H \) such that
\[
S \cap aSa^{-1} = K \setminus \{1\} = \{h_1, h_2, \ldots, h_k\}
\]
In both situations we define \( S' = S \cup \{a\} \) and \( G' = \langle S' \rangle \). Lemma 1.8 implies
\[
b_1^\psi(G') - b_0^\psi(G') = |S'| - 1 - \sum_{s \in S'} \sup_{z \in Z_{G', S'}} \frac{|\langle z, s \rangle^\psi|^2}{\langle z, z \rangle^\psi}
\]
\[
\leq |S'| + 1 - 1 - \sum_{s \in S} \sup_{z \in Z_{H_S, S}} \frac{|\langle z, s \rangle^\psi|^2}{\langle z, z \rangle^\psi} - \sup_{z \in Z_{G', S'}} \frac{|\langle z, a \rangle^\psi|^2}{\langle z, z \rangle^\psi}
\]
\[
\leq b_1^{(2)}(H_S) + 1 - \sup_{z \in Z_{G', S'}} \frac{|\langle z, a \rangle^\psi|^2}{\langle z, z \rangle^\psi}
\]
(2)

To obtain a lower bound for \( \sup_{z \in Z_{G', S'}} \frac{|\langle z, a \rangle^\psi|^2}{\langle z, z \rangle^\psi} \), we consider the Cayley graph \( \text{Cay}(G', S') \) of \( G' \) and exhibit a suitable cycle \( z \). Each relation \( ah_i a^{-1}(ah_i a^{-1})^{-1} \) provides a cycle \( z_i \) of length 4 in \( \text{Cay}(G', S') \), i.e.,
\[
\rho_i = (1, a) + (a, ah_i) - (ah_i a^{-1}, ah_i) - (1, ah_i a^{-1}).
\]
Note that the cycles \( z_i \) touch exactly four vertices, since \( h_i \neq 1 \) and \( a \notin H \).

In addition, the cycles \( z_i \) have no common edges, except for \( a \). We define
\[
z = \sum_{i=1}^k z_i
to be the sum of these cycles. Since \( \psi(ah_i a^{-1}) = \psi(h_i) = 0 \) holds for all \( i \leq k \), we deduce
\[
\langle z, a \rangle^\psi = k\langle a, a \rangle^\psi - \sum_{i=1}^k \langle (ah_i a^{-1}, ah_i), a \rangle^\psi = k - \sum_{i=1}^k \psi(ah_i a^{-1}) = k
\]
We note further (using again that $K$ is $\psi$-regular) that for given $i,j \leq k$ the edges $\neq \bar{a}$ in $z_i, z_j$ are orthogonal unless $h_i = ah_ja^{-1}$ or $ah_i a^{-1} = h_j$. Each of these cases occurs at most once for every $i$ and then $\langle (a, ah_i), (1, ah_j a^{-1}) \rangle_{\psi} = \psi(a)$ and $\langle (1, ah_i a^{-1}), (a, ah_j) \rangle_{\psi} = \psi(\bar{a})$ respectively. We deduce

$$\langle z, z \rangle_{\psi} = \sum_{i,j} \langle z_i, z_j \rangle_{\psi} \leq \sum_{i=j}^{4k} + k^2 - k + k(\psi(a) + \psi(\bar{a}))$$

$$\leq k^2 + (3 + 2\text{Re}(\psi(a)))k$$

and conclude

$$\frac{\langle z, \bar{a} \rangle_{\psi}^2}{\langle z, z \rangle_{\psi}} \geq \frac{1}{1 + \frac{3 + 2\text{Re}(\psi(a))}{k}}.$$}

Finally, we use this cycle in combination with inequality (2) to obtain

$$b_1^\psi(G') - b_0^\psi(G') \leq b_1^{(2)}(H_S) + 1 - \sup_{z \in Z_{G', S'}} \frac{|\langle z, \bar{a} \rangle_{\psi}|^2}{\langle z, z \rangle_{\psi}}$$

$$\leq b_1^{(2)}(H_S) + 1 - \frac{1}{1 + \frac{3 + 2\text{Re}(\psi(a))}{k}}$$

$$= b_1^{(2)}(H_S) + \frac{3 + 2\text{Re}(\psi(a))}{k + 3 + 2\text{Re}(\psi(a))} \quad \underset{k \to \infty}{\longrightarrow} b_1^{(2)}(H_S)$$

For $n = \infty$ we can make $k$ arbitrary large. In the case $n < \infty$ we have $k = n - 1$ by construction. We note that every finitely generated subgroup of $G$ is contained in a group of the form $G'$. The result follows from Lemma 1.7 and Lemma 1.3 (b).

If $n = \infty$, then $G$ contains an infinite $\psi$-regular subgroup and $b_0^\psi(G) = 0$ by Lemma 1.3 (c).

In the spirit of Popa [12] and Thom-Peterson [10] we introduce the following notion.

**Definition 3.2.** Let $G$ be a group and let $\psi \in \text{Ch}(G)$. A subgroup $H \leq G$ is $q$-$\psi$-normal, if there is a set $A \subseteq G$ such that $G = \langle H \cup A \rangle$ and $H \cap aH a^{-1}$ contains an infinite $\psi$-regular subgroup for all $a \in A$.

A subgroup $H \leq G$ is weakly $q$-$\psi$-normal, if there is an ordinal number $\alpha$ and an increasing chain of subgroups $H_0 = H$ to $H_\alpha = G$ such that $\bigcup_{\beta < \gamma} H_\beta$ is $q$-$\psi$-normal in $H_\gamma$ for all $\gamma \leq \alpha$.

Based on Lemma 3.1 we obtain the following analog of [10, Theorem 5.6].

**Corollary 3.3.** Let $G$ be a group and let $\psi \in \text{Ch}(G)$. If $H \leq G$ is a weakly $q$-$\psi$-normal subgroup, then

$$\bar{b}_1^\psi(G) \leq \bar{b}_1^\psi(H).$$

**Proof.** Assume that $H$ is $q$-$\psi$-normal. Then $G = \langle H \cup A \rangle$ and $H \cap aH a^{-1}$ contains an infinite $\psi$-regular subgroup for all $a \in A$. If $A$ is finite, then the assertion follows inductively from Lemma 3.1 Assume that $A$ is infinite. For every finite subset $B \subseteq A$, we define $G_B = \langle H \cup B \rangle$. Then $G = \bigcup_{B \subseteq A} G_B$ and Lemma 1.7 implies

$$\bar{b}_1^\psi(G) \leq \liminf_{B \subseteq A} \bar{b}_1^\psi(G_B) \leq \bar{b}_1^\psi(H).$$
The result for weakly $q$-$\psi$-normal subgroups follows by transfinite induction using Lemma 1.7. □

**Corollary 3.4.** Let $G$ be a group and let $\psi \in \text{Ch}(G)$.

1. If $G$ is an HNN-extension of $H$ with associated subgroups $A, B$ and $A$ contains an infinite $\psi$-regular subgroup, then we have $\overline{b}_1^\psi(G) \leq \overline{b}_1^\psi(H)$.
2. If $G = A *_C B$ is an amalgamated product such that $C$ is $q$-$\psi$-normal in $B$. Then we have $\overline{b}_1^\psi(G) \leq \overline{b}_1^\psi(A)$.
3. If $G$ contains an infinite normal amenable $\psi$-regular subgroup, then $\overline{b}_1^\psi(G) = 0$.

**Proof.** (1): follows immediately from Lemma 3.1.

(2): The assumptions imply that $A$ is $q$-$\psi$-normal in $G = A *_C B$ and the assertion follows from Corollary 3.3.

(3): The infinite normal amenable subgroup $N$ is $q$-$\psi$-normal in $G$ and $\overline{b}_1^\psi(N) = \overline{b}_1^{(2)}(N) = 0$ by [3, Thm. 0.2]. □

We illustrate the helpfulness of $q$-normality with an application to right-angled groups. This notion was put forward in [1, Definition 1].

**Definition 3.5** (right-angled groups). A group $G$ is right-angled, if it is the quotient of a right-angled Artin group $A_\Gamma$ with a finite connected graph $\Gamma = (\mathcal{I}, \mathcal{E})$ such that the image of every generator $\sigma_i$ ($i \in \mathcal{I}$) has infinite order in $G$.

The image of the generating set of $A_\Gamma$ will be called a right-angled set of generators.

**Theorem 3.6.** Let $G$ be a right-angled group and let $S = \{s_i \mid i \in \mathcal{I}\}$ be a right-angled set of generators. If $\psi \in \text{Ch}(G)$ is such that the cyclic subgroup $\langle s_i \rangle$ is $\psi$-regular for every $i \in \mathcal{I}$. Then we have $\overline{b}_1^\psi(G) = 0$.

**Proof.** Our proof will be by induction over the number $n = |\mathcal{I}| \in \mathbb{N}$ of generators. For the base of induction we note that $b_1^{(2)}(\mathbb{Z}) = 0$. We assume for the induction step w.l.o.g. $S = \{s_1, s_2, \ldots, s_n, s_{n+1}\}$ such that $s_n$ commutes with $s_{n+1}$ and such that $G' = \langle s_1, \ldots, s_n \rangle$ is a right-angled group with $\overline{b}_1^{(2)}(G') = 0$. We claim that $G'$ is $q$-$\psi$-normal. Indeed, set $H = \langle s_n \rangle$ and $a = s_{n+1}$, $a^{-1}G'a \cap G' \supseteq H$ and $H$ is $\psi$-regular by assumption. Now the result follows from Lemma 3.1. □

Using this calculation and the approximation methods from [8] one can control the growth of Betti numbers in right-angled Artin groups with respect to normal chains with non-trivial intersection.

**Corollary 3.7.** Let $A_\Gamma$ be a right-angled Artin group for a finite connected graph $\Gamma$ with generating set $\{\sigma_i \mid i \in \mathcal{I}\}$. Let $N_1 \supseteq N_2 \supseteq \ldots$ be a descending chain of finite index normal subgroups in $A_\Gamma$. If the order $\text{ord}_{A_\Gamma/N_n}(<\sigma_i>)$ in the finite factors $A_\Gamma/N_n$ is unbounded for each generator $\sigma_i$, then

$$\lim_{n \to \infty} \frac{b_1(N_n)}{|A_\Gamma : N_n|} = 0.$$
Proof. Let $\psi_n$ be the character of the permutation action of $A_\Gamma$ on $A_\Gamma/N_n$. Since the sequence of normal subgroups is descending, the sequence $\psi_n$ converges in $\text{Ch}(A_\Gamma)$ to a character $\psi$. Since each character $\psi_n$ is sofic and $A_\Gamma$ is finitely presented, it follows from [8, Theorem 3.5] that

$$b_1^\psi(A_\Gamma) = \lim_{n \to \infty} b_1^{\psi_n}(A_\Gamma) = \lim_{n \to \infty} \frac{b_1(N_n)}{|A_\Gamma : N_n|}$$

If the order $\text{ord}_{A_\Gamma/N_n}(\sigma_i)$ tends to infinity, $\psi_n(\sigma_i^k)$ vanishes for all $k \neq 0$ and all large $n$, i.e., $\langle \sigma_i \rangle$ is a $\psi$-regular subgroup. Theorem 3.6 implies that $b_1^\psi(A_\Gamma) = 0$ and this completes the proof. □

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