OUTCOME SPACE ALGORITHM FOR GENERALIZED MULTIPLICATIVE PROBLEMS AND OPTIMIZATION OVER THE EFFICIENT SET

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Abstract. In this paper, an algorithm of the branch and bound type in outcome space is proposed for solving a global optimization problem that includes, as a special case, generalized multiplicative problems. As an application, we solve the problem of optimizing over the efficient set of a bicriteria concave maximization problem. Preliminary computational experiments show that this algorithm works well for problems where the dimensions of the decision space can be fairly large.

1. Introduction. The problem of central interest in this paper is given by

\[ \max \Phi(x) = \varphi(f(x)) \text{ s.t. } x \in X, \]  

(GP)

where \( f(x) = (f_1(x), f_2(x)) \), \( f_1, f_2 \) are two concave functions defined on \( \mathbb{R}^n \), \( X \subset \mathbb{R}^n \) is a nonempty, compact convex set, and \( \varphi(y_1, y_2) \) is continuous and increasing in the sense of

\[ \varphi(y^1) > \varphi(y^2) \text{ whenever } y^1 \geq y^2, y^1 \neq y^2. \]

An application of Problem (GP) follows from the observation that the Generalized Concave Multiplicative Programming problem

\[ \max_{x \in X} \left( g(x) = g_0(x) + \prod_{i=1}^{k} g_i(x) \right), \]  

(GCMP)

where \( k \geq 2 \) and \( X \) is as in Problem (GP), the functions \( g_i \) are concave on \( \mathbb{R}^n \) \((i = 0, 1, \ldots, k)\) such that

\[ \forall x \in X \quad g_i(x) > 0, \quad i = 1, 2, \ldots, k, \]  

(1)

is a special case of Problem (GP). Indeed, in Problem (GP), let \( f_1(x) = g_0(x) \) and

\[ f_2(x) = \left( \prod_{i=1}^{k} g_i(x) \right)^{1/k}. \]

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From Proposition 2.7 in [20], since \( g_i, i = 1, \ldots, k \) are concave positive functions on the convex set \( X \), this implies that the function \( f_2(x) \) is also concave on \( X \). Let \( \varphi(y_1, y_2) = y_1 + y_2^k \). It is easily seen that \( \varphi \) is a continuous and increasing function and Problem (GP) is identical to Problem (GCMP). Apparently, the problem in literature most closely related to Problem (GCMP) is the Generalized Convex Multiplicative Programming problem studied in [8]. This problem is obtained from (GCMP) by minimizing (rather than maximizing) \( g(x) \) over \( X \) under the same assumptions as for Problem (GP), except that \( g_i, i = 0, 1, \ldots, k \) are assumed to be convex, rather than concave, functions on \( \mathbb{R}^n \). In the case \( k = 2 \), Problem (GCMP) can be seen as a special case of the problems considered in [1], [4] and [16].

As usual, the outcome set \( Y \) for Problem (GP) is defined by

\[
Y := \{ y \in \mathbb{R}^2 | y = f(x) \text{ for some } x \in X \}.
\]

A direct reformulation of problem (GP) as an outcome space problem is given by

\[
\max \varphi(y) \text{ s.t. } y \in Y. \tag{OP_1}
\]

By the definition, it is simple to prove that if \( y^* \) is a global optimal solution to Problem (OP_1) then any \( x^* \in X \) such that \( f(x^*) \geq y^* \) is a global optimal solution to Problem (GP).

To globally solve Problem (GP), we present an algorithm of branch and bound type in the outcome space \( \mathbb{R}^2 \) for solving a problem (OP_2) that is equivalent to Problem (OP_1). The proposed algorithm is established on the basis of the relationship between the global optimal solution to Problem (OP_1) and the efficient set of the outcome set \( Y \). As an application, we solve the problem

\[
\max \Phi(x) = \varphi(f(x)) \text{ s.t. } x \in X_E, \tag{P_{X_E}}
\]

where \( X_E \) is the efficient set of the following bicriteria programming problem

\[
\text{Vmax } f(x) = (f_1(x), f_2(x))^T \text{ s.t. } x \in X, \tag{BOP}
\]

with \( \varphi, f_1, f_2 \) and \( X \) are as in Problem (GP). Problem (P_{X_E}) could happen in certain common situations and has attracted the attention from many researchers (see e.g. [2, 3, 5, 6, 7, 9, 10, 11, 14, 18, 19, 21]). See [6] for a detailed discussion about Problem (P_{X_E}).

The paper organized as follows. In the Section 2, Problem (OP_1) is reformulated as a nonconvex optimization problem (OP_2) in the outcome space \( \mathbb{R}^2 \). Section 3 presents the basic operations which are used for the establishment of our algorithm of branch and bound type in Section 4. Section 5 shows that an optimal solution to Problem (P_{X_E}) can be obtained by solving Problem (OP_2). In the last section, preliminary computational experiments are reported and show that our algorithm works well for problems where the dimensions of decision space can be fairly large.

2. Reformulation in the outcome space. Let \( a^1, a^2 \in \mathbb{R}^2 \). As a matter of notation, we write \( a^1 \geq a^2 \) to mean that \( a^1_i \geq a^2_i \) for \( i = 1, 2 \), and write \( a^1 \gg a^2 \) to mean that \( a^1_i > a^2_i \) for \( i = 1, 2 \).

Let \( A \) be a nonempty subset of \( \mathbb{R}^2 \). We denote the interior of \( A \) by \( \text{int}A \), the boundary of \( A \) by \( \partial A \). A point \( a^0 \in A \) is said to be an efficient point of \( A \) if there is no point \( a \in A \) such that \( a \geq a^0 \) and \( a \neq a^0 \), i.e., \( (a^0 + \mathbb{R}^2_+) \cap A = \{a^0\} \). Analogously, a point \( a^* \in A \) is said to be a weakly efficient point of \( A \) if there is no point \( a \in A \) such that \( a \gg a^* \), i.e., \( (a^0 + \text{int}\mathbb{R}^2_+) \cap A = \emptyset \). Denote by \( E(A) \) and
\( WE(A) \) the set of all efficient points of \( A \) and the set of all weakly efficient points of \( A \), respectively. By the definition,
\[
E(A) \subseteq WE(A).
\]

The property of the optimal solutions to Problem (OP\(_1\)) is stated by the following proposition.

**Proposition 1.** Any global optimal solution to Problem (OP\(_1\)) must belong to the efficient set \( E(Y) \).

**Proof.** Let \( y^0 \) be a global optimal solution to Problem (OP\(_1\)). Assume the contrary that \( y^0 \not\in E(Y) \). By the definition, there is \( y^* \in Y \) such that \( y^* \geq y^0 \) and \( y^* \neq y^0 \). Since \( \varphi(y) \) is increasing, we have \( \varphi(y^*) > \varphi(y^0) \). This contradicts the hypothesis that \( y^0 \in \operatorname{Argmax}\{ \varphi(y) | y \in Y \} \) and the proof is complete. \( \square \)

Now, consider the set \( Z \) given by
\[
Z = Y - \mathbb{R}^2_+ = \{ z \in \mathbb{R}^2 | y \geq z \text{ for some } y \in Y \}.
\]
It is clear that \( Z \) is a nonempty, full-dimensional closed convex set. Below is some interesting properties of \( Z \) (see [13, 17, 22]) which will be useful in the sequel.

**Proposition 2.**

i) \( E(Z) \) is homeomorphic to a nonempty closed interval in \( \mathbb{R} \);

ii) \( E(Z) = E(Y) \).

Combining Proposition 1 and Proposition 2(ii), to globally solve Problem (OP\(_1\)), the algorithm will instead solve the problem
\[
\max \varphi(z) \text{ s.t. } z \in E(Z).
\]  
\text{(OP\(_2\))}

Recall that a point \( z^I = (z^I_1, z^I_2) \in \mathbb{R}^2 \) is called the ideal point of the set \( Z \) if
\[
z^I_i = \max\{ z_i \mid z \in Z \}, \text{ } i = 1, 2.
\]
Notice that the ideal point \( z^I \) need not belong to \( Z \). If \( z^I \in Z \) then \( E(Z) = \{ z^I \} \) and \( z^I \) is the unique optimal solution for Problem (OP\(_2\)). We therefore assume henceforth that \( z^I \not\in Z \). In this case, by Proposition 2(i), the efficient set \( E(Z) \) is a connected curve on the boundary of \( Z \) with the starting efficient point \( z^{\text{start}} \) and the end efficient point \( z^{\text{end}} \) (see Fig. 1). By the geometry of the efficient set \( E(Z) \subset \mathbb{R}^2 \), we can see that the efficient extreme point \( z^{\text{start}} \) is the unique optimal solution of the problem
\[
\max\{ z_1 | z \in Z, \text{ } z_2 = z^I_2 \} \tag{P\(_1\))
\]
and the efficient extreme point \( z^{\text{end}} \) is the unique optimal solution of the problem
\[
\max\{ z_2 | z \in Z, \text{ } z_1 = z^I_1 \}. \tag{P\(_2\))
\]
By definition, it is easily seen that for each \( i = 1, 2 \), Problem (P\(_i\)) has the explicit formulation as follows
\[
\max \begin{cases} z_i \\ \text{s.t.} \quad z - f(x) \leq 0 \\ x \in X \\ z_k = z^I_k, \end{cases} \tag{IP\(_i\))}
\]
where \( k \in \{1, 2\} \setminus \{i\} \).
Remark 1. Let $z^* \in (z^I - \mathbb{R}_2^n) \setminus Z$. By the geometry, if $p^*$ is the projection of $z^*$ on $Z$ then $p^*$ is the efficient point of $Z$. It is easily seen that $p^*$ is an optimal solution of the convex programming problem
\[
\min \|z^* - z\| \quad \text{s.t.} \quad z \in Z,
\] (Pro($z^*$))
which has the explicit formulation as follows
\[
\begin{align*}
\min & \quad \|z^* - z\| \\
\text{s.t.} & \quad z - f(x) \leq 0 \\
& \quad x \in X.
\end{align*}
\]
Let
\[
d^* = \frac{z^* - p^*}{\sum_{i=1}^2 (z^*_i - p^*_i)}.
\]
According to [15], we find that $d^*$ is the normal vector of the supporting hyperplane of $Z$ at $p^*$.

In the remain part of this paper, an algorithm of a branch and bound type will be established to solve Problem (OP$_2$).

3. Basic operations.

3.1. Upper bound for subproblem. Let $z^\ell, z^r \in E(Z)$ satisfy
\[
z^\ell_1 > z_1^r \quad \text{and} \quad z^\ell_2 > z^r_2, \tag{2}
\]
and $\mathcal{E} \subseteq E(Z)$ be the unique curve connecting $z^\ell$ and $z^r$. Let $d^\ell$ and $d^r$ be two normal vectors associated with $z^\ell$ and $z^r$, respectively. It is clear that if $z^\ell = z^{\text{start}}$ and $z^r = z^{\text{end}}$ then $\mathcal{E} \equiv E(Z)$ and $d^\ell = (0,1)^T, d^r = (1,0)^T$. Otherwise, we have $\mathcal{E} \subset E(Z)$. In the section, we present a procedure for computing an upper bound $\alpha = \alpha(\mathcal{E})$ for the subproblem
\[
\max \varphi(z) \quad \text{s.t.} \quad z \in \mathcal{E}. \tag{SP($\mathcal{E}$)}
\]
There are following three cases:
Case 1. Two vectors $d^\ell$ and $d^r$ are dependent (See Fig. 2(a)). For this case, it is plain that $E$ is the line segment joining the two points $z^\ell$ and $z^r$ and denoted by $[z^\ell, z^r]$. By the definition, each $z \in [z^\ell, z^r]$ can be formed by
\[ z = z^r + t(z^\ell - z^r), \quad 0 \leq t \leq 1. \]

Let
\[ \psi(t) = \varphi(z) = \varphi(z^r + t(z^\ell - z^r)). \]
Then Problem (SP($E$)) becomes the following one-variable optimization problem
\[ \max \psi(t) \text{ s.t. } t \in [0, 1]. \] (P$_{\text{sub}}^1$)
Let $t^{opt}$ denote an optimal solution of this problem. Then $\hat{z} = z^r + t^{opt}(z^\ell - z^r) \in E(Z)$ is a feasible solution of Problem (OP$_2$), and can be used for improving the lower bound of Problem (OP$_2$). Moreover, $\alpha = \alpha(E) = \varphi(\hat{z})$ is the exactly upper bound of Problem (SP($E$)).

Case 2. Two vectors $d^\ell$ and $d^r$ are independent and the linear system
\[ \begin{cases} \langle d^\ell, z \rangle = \langle d^\ell, z^\ell \rangle \\ \langle d^r, z \rangle = \langle d^r, z^r \rangle \end{cases} \] (3)
has the solution $z^* \in \{z^\ell, z^r\}$ (See Fig. 2(b)). For this case, it is easily seen that $E = [z^\ell, z^r]$ and solving Problem (SP($E$)) is carried out as Case 1.

Case 3. Two vectors $d^\ell$ and $d^r$ are independent and the linear system (3) has the solution $z^* \notin \{z^\ell, z^r\}$ (See Fig. 3). Then $z^* \in (z^\ell \mathbb{R}_+^2 \setminus Z$. Let $S$ be a 2-simplex with three vertices $z^\ell, z^r$ and $z^*$. It is clear that
\[ E \subset S. \] (4)
Let $\hat{z}$ be an optimal solution of the following relaxation problem
\[ \max \varphi(z) \text{ s.t. } z \in S. \] (RP($E$))
Then $\alpha = \alpha(E) = \varphi(\hat{z})$ is an upper bound of Problem (SP($E$)).
Remark 2. By Proposition 1, any global optimal solution to Problem (RP(\mathcal{E})) must belong to the efficient set \( E(S) \) of the simplex \( S \). By the geometry, it is easy to check that the weakly efficient set of \( S \) is

\[
\text{WE}(S) = [z^\ell, z^*] \cup [z^*, z^r] \supseteq E(S).
\]

By the definition, each \( z \in [z^\ell, z^*] \cup [z^*, z^r] \) can be formed by

\[
z = \begin{cases} 
z^* + t(z^* - z^\ell), & -1 \leq t \leq 0 \\
z^* + t(z^r - z^*), & 0 \leq t \leq 1.
\end{cases}
\]

Let

\[
\phi(t) = \varphi(z) = \begin{cases} 
\varphi(z^* + t(z^* - z^\ell)), & -1 \leq t \leq 0 \\
\varphi(z^* + t(z^r - z^*)), & 0 \leq t \leq 1.
\end{cases}
\]

Then Problem (RP(\mathcal{E})) becomes the following one-variable optimization problem

\[
\max \phi(t) \text{ s.t. } t \in [-1, 1]. \tag{P^2_{\text{sub}}}
\]

Let \( t^{opt} \) denote an optimal solution of Problem (P^2_{\text{sub}}). If \( t^{opt} > 0 \) then set \( \z = z^* + t^{opt}(z^r - z^*) \), otherwise set \( \z = z^* + t^{opt}(z^* - z^\ell) \). Then \( \alpha = \alpha(\mathcal{E}) = \varphi(\z) \) is an upper bound for Problem (SP(\mathcal{E})).

The procedure for computing the upper bound for subproblem (SP(\mathcal{E})) can be described as follows.

**Procedure Solve (RP(\mathcal{E}))**

- **Input**: Two efficient points \( z^\ell, z^r \) satisfying (2) and two normal vectors \( d^\ell, d^r \) respectively.
- **Output**: The optimal solution \( \z \) to Problem (RP(\mathcal{E})) and an index \( id \). If Problem (RP(\mathcal{E})) belongs to the third case then \( id = 1 \) and \( \z \in (z^I - \mathbb{R}^2_+) \setminus Z \). Otherwise, \( id = 0 \) and \( \z \in E(Z) \).

**Step 1.** If \( d^\ell = t d^r \) with \( t > 0 \) Then go to Step 3.

**Else** Find the solution \( z^* \) of the linear system (3).

**Step 2.** If \( z^* \in \{z^\ell, z^r\} \) Then go to Step 3.

**Else** Set \( id = 1 \). Solve Problem (P^2_{\text{sub}}) to obtain the optimal solution \( t^{opt} \).
If \( t_{\text{opt}} > 0 \) Then Let \( \hat{z} = z^* + t_{\text{opt}}(z^r - z^*) \)
Else Let \( \hat{z} = z^* + t_{\text{opt}}(z^s - z^f) \).

**Step 3.** Set \( id = 0 \). Solve Problem \( (P^1_{\text{sub}}) \) to obtain the optimal solution \( t_{\text{opt}} \).

Let \( \hat{z} = z^r + t_{\text{opt}}(z^s - z^r) \).

3.2. **Partitioning and branching.** Let \( Q \) be a subset of \( \mathbb{R}^2 \). Let us recall that a collection \( \{Q_j, j \in J\} \) of subsets of \( Q \), where \( J \) is a finite set of indices, is called a partition of \( Q \) if

\[
Q = \bigcup_{j \in J} Q_j \quad \text{and} \quad Q_i \cap Q_j = \partial Q_i \cap \partial Q_j \quad \forall i, j \in J, \ i \neq j,
\]

where \( \partial Q_i \) denotes the boundary of \( Q_i \).

The initial partition \( \mathcal{D}^0 \) of \( E(Z) \) consists of simply \( E_0 = E(Z) \), i.e.,

\[
\mathcal{D}^0 = \{E_0\}.
\]

At the beginning of Iteration \( k \) of the algorithm, we have available a partition \( \mathcal{D}^k \) of the efficient set \( E(Z) \). Denote \( R_k \) the set consisting of the elements of the partition \( \mathcal{D}^k \) that cannot yet be excluded due to not containing a global optimal solution for Problem \( (OP_2) \). As usual, we find \( \mathcal{E}_k \in R_k \) that is an unique efficient curve connecting \( z^I_k \in E(Z) \) and \( z^R_k \in E(Z) \) the such that

\[
\alpha(\mathcal{E}_k) = \max\{\alpha(\mathcal{E}) \mid \mathcal{E} \in R_k\},
\]

where \( \alpha(\mathcal{E}) \) is the upper bound for Problem \( (SP(\mathcal{E})) \). Then, \( \alpha_k = \alpha(\mathcal{E}_k) \) is the upper bound for Problem \( (OP_2) \). Let \( \hat{z}^k \) be the optimal solution for Problem \( (RP(\mathcal{E})) \) with \( \mathcal{E} = \mathcal{E}_k \). By the construction, Problem \( (RP(\mathcal{E})) \) with \( \mathcal{E} = \mathcal{E}_k \) belongs to the third case, i.e. \( id = 1 \) and \( \hat{z}^k \in (z^I - R^2_2) \setminus Z \).

![Figure 4](image-url)

By Remark 1, the optimal solution \( \hat{p}^k \) to Problem \( (Pro(z^*)) \) with \( z^* = \hat{z}^k \) is an efficient point of \( Z \) and \( d^k = (\hat{z}^k - \hat{p}^k)/\sum_{i=1}^2(\hat{z}^k_i - \hat{p}^k_i) \gg 0 \) is a normal vector of the supporting hyperplane with \( Z \) at \( \hat{p}^k \) (see Fig. 4). Then the point \( \hat{p}^k \) can be used for improving the lower bound of Problem \( (OP_2) \). Let \( \mathcal{E}_{k_1} \subset E(Z) \) be the efficient
curve connecting $z^{k_1}$ and $p^k$, and $E_{k_2} \subset E(Z)$ be the efficient curve connecting $p^k$ and $z^{r_k}$. Then

$$E_k = E_{k_1} \cup E_{k_2}$$

and the point $p^k$ is called to be a bisection point for $E_k$. Let

$$D^{k+1} := (D^k \setminus E_k) \cup \{E_{k_1}, E_{k_2}\}.$$ 

Then $D^{k+1}$ is a partition of $E(Z)$ that refiner than partition $D^k$.

### 4. The algorithm

Let $\varepsilon > 0$ be a given sufficiently small number and $z^{opt} \in E(Z)$. It is clear that $\varphi(z^{opt})$ is a lower bound of Problem (OP$_2$). The point $z^{opt} \in E(Z)$ is said to be an $\varepsilon$-optimal solution to Problem (OP$_2$) if there is an upper bound $\alpha^*$ of Problem (OP$_2$) such that $|\alpha^* - \varphi(z^{opt})| \leq \varepsilon(|\varphi(z^{opt})| + 1)$.

Using notations and basic operations given in the Section 3, in this section the branch and bound algorithm in outcome space for solving Problem (OP$_2$) may be stated as follows.

**Algorithm OS**

**Initialization step.**

(0.1) Solve problem (IP$_i$), $i = 1, 2$, to find the starting point $z^{start}$ and the end point $z^{end}$ of the efficient set $E(Z)$.

(0.2) Set $\beta_0 = \max\{\varphi(z^{start}), \varphi(z^{end})\}$ (the currently best lower bound) and take $\tilde{z}^0$ (currently best feasible solution) from $\{z^{start}, z^{end}\}$ such that $\beta_0 = \varphi(\tilde{z}^0)$.

(0.3) Solve Problem (RP($E$)) with $E = E_0$, where $E_0 = E(Z)$ is the efficient curve connecting the two points $z^\ell = z^{start}$ and $z^r = z^{end}$, to obtain the optimal solution $\tilde{z}^0$. Set $\alpha(\tilde{E}_0) = \varphi(\tilde{z}^0)$. (current best upper bound)

(0.4) Set $k := 0$ and $R_0 = \{E_0\}$.

Iteration $k, k = 0, 1, 2, \ldots$. See Step 1 through Step 4 below.

**Step 1. (Selecting)**

(1.1) Find $E_k \in R_k$ such that $\alpha(E_k) = \max\{\alpha(E) \mid E \in R_k\}$, where $E_k$ is the efficient curve connecting two efficient points $z^{k_1}$ and $z^{r_k}$.

(1.2) Let $\alpha_k := \alpha(E_k)$. (current best upper bound)

**Step 2. (Branching)**

(2.1) Solve the problem (Pro($z^*$)) with $z^* = \tilde{z}^k$ to obtain the projection $\tilde{p}^k \in E(Z)$. Then $\tilde{d}^k = (\tilde{z}^k - \tilde{p}^k) / \sum_{i=1}^2 (\tilde{z}^k - \tilde{p}^k_i)$ is a normal vector associated with $\tilde{p}^k$.

(2.2) Update $\beta_{k+1} = \max\{\beta_k, \varphi(\tilde{p}^k)\}$ (the currently best lower bound) and take $\tilde{z}^{k+1}$ (currently best feasible solution) from $\{\tilde{z}^k, \tilde{p}^k\}$ such that $\beta_{k+1} = \varphi(\tilde{z}^{k+1})$.

(2.3) Set $R_{k+1} = (R_k \setminus E_k) \cup \{E_{k_1}, E_{k_2}\}$, where $E_{k_1}$ is the efficient curve connecting two efficient points $z^{k_1}$ and $p^k$, $E_{k_2}$ is the efficient curve connecting two efficient points $p^k$ and $z^{r_k}$.

**Step 3. (Updating and Rejecting)**

(3.1) For each $i \in \{1, 2\}$, solve problem (RP($E$)) with $E = E_{k_i}$, to obtain the index $I_{d_i}$ and the optimal solution $\tilde{z}^{k_i}$.

Set $\alpha(E_{k_i}) = \varphi(\tilde{z}^{k_i})$ (the upper bound for Problem (SP($E$)) with $E = E_{k_i}$).

If $I_{d_i} = 0$ and $\beta_{k+1} < \varphi(\tilde{z}^{k_i})$, then set $\tilde{z}^{k+1} = \tilde{z}^{k_i}$ (currently best feasible solution).
Set
\[ R_{k+1} = \{ E \in R_{k+1} \mid \alpha(E) - \beta_{k+1} > \varepsilon(\beta_{k+1} + 1) \} \]

**Step 4. (Stop Criterion)** If \( R_{k+1} = \emptyset \), then terminate the algorithm: \( z^{opt} = \hat{z}^{k+1} \)
is an \( \varepsilon \)-optimal solution. Otherwise, Set \( k := k + 1 \) and go to Step 1.

**Remark 3.** By construction, when the algorithm terminates at Step 4, we receive an \( \varepsilon \)-optimal solution \( z^{opt} \) to Problem (OP2). Then a feasible solution \( x^{opt} \) to Problem (GP) that satisfies \( f(x^{opt}) \geq z^{opt} \) is an approximate optimal solution to Problem (GP). It is easy to see that the point \( x^{opt} \) can be obtained by solving a convex programming problem with a linear objective function over the nonempty compact convex feasible set \( \{ f(x) \geq z^{opt}, x \in X \} \).

**Proposition 3.** Algorithm OS can be infinite only if \( \varepsilon = 0 \) and in that case the sequence \( \{ \hat{z}^k \} \) has a cluster point that solves Problem (OP2) globally.

**Proof.** Suppose that Algorithm OS is not finite. Since the sequence \( \{ \hat{z}^k \} \) belonging to the compact set \( [(z^I - \mathbb{R}^2_+) \setminus Z] \cup E(Z) \) (Remark 1), by taking subsequence if necessary, we may assume that there exists \( z^* \in [(z^I - \mathbb{R}^2_+) \setminus Z] \cup E(Z) \) such that \( \hat{z}^k \to z^* \). Recall that \( \hat{p}^k \in E(Z) \) is the projection of \( \hat{z}^k \) at Step 2.1. By continuity of projections, there exists \( p^* \in E(Z) \) satisfying that \( \hat{p}^k \to p^* \). Now let us show that \( z^* \equiv p^* \). On the contrary, suppose that \( z^* \neq p^* \). Then, for \( k \) sufficiently large, one has
\[
\langle z^k - \hat{p}^k, \hat{z}^{k+1} - \hat{p}^k \rangle \geq \| z^* - p^* \| > 0,
\]
which means that
\[
\hat{z}^{k+1} \notin \{ z^I - \mathbb{R}^2_+ \mid \langle z^k - \hat{p}^k, z - \hat{p}^k \rangle \leq 0 \},
\]
contrary to the selection of \( \hat{z}^{k+1} \) at Step 1.1 and Step 3.1.

From Step 2.2 and Step 3.1 of the algorithm, we have \( \beta_{k+1} \geq \varphi(\hat{p}^k) \) and \( \alpha(E_k) = \varphi(\hat{z}^k) \). Since the function \( \varphi \) is continuous, this implies that \( \lim_{k \to \infty} \beta_{k+1} \geq \varphi(p^*) \) and \( \lim_{k \to \infty} \alpha(E_k) = \varphi(z^*) \). Moreover, for all \( k \geq 0 \), we have \( \alpha(E_k) - \beta_{k+1} > \varepsilon \) because \( E_k \in R_{k+1} \). It follows that
\[
\varepsilon \leq \lim_{k \to \infty} (\alpha(E_k) - \beta_{k+1}) \leq \varphi(z^*) - \varphi(p^*) = 0.
\]
Since \( \varepsilon \) is a non-negative real number, we conclude \( \varepsilon = 0 \).

Suppose that the algorithm generates the sequence \( \{ \hat{z}^k \} \) belonging to the compact set \( E(Z) \) (Proposition 2) and it has a cluster point \( \tilde{z} \in E(Z) \). There is no loss of generality in assuming that \( \{ \hat{z}^k \} \) is the subset of the above sequence \( \{ \hat{p}^k \} \). Since \( \hat{p}^k \to z^* \), we have \( \tilde{z} \equiv z^* \). Since \( \varphi \) is continuous, \( \varphi(z^*) = \lim_{k \to \infty} \varphi(\hat{z}^k) = \lim_{k \to \infty} \alpha(E_k) \) is an upper bound of Problem (OP2). This means that \( z^* \) is an optimal solution of Problem (OP2). The proof is complete.

**5. Application to optimization over the efficient set.** Consider the problem
\[
\max \Phi(x) = \varphi(f(x)) \quad \text{s.t. } x \in X_E, \quad (P_{X_E})
\]
where \( X_E \) is the efficient set of the following bicriteria programming problem
\[
\max f(x) = (f_1(x), f_2(x))^T \quad \text{s.t. } x \in X, \quad (BOP)
\]
with \( \varphi \), \( f_1 \), \( f_2 \) and \( X \) are as in Problem (GP). Recall that a point \( x^0 \in X \) is called an efficient solution of problem (BOP), if there is no point \( \bar{x} \in X \) such that \( f(\bar{x}) \geq f(x^0) \) and \( f(\bar{x}) \neq f(x^0) \). It means that \( x^0 \in X_E \) if \( y^0 = f(x^0) \in E(Y) \).
Proposition 4. Problem \((P_{X_E})\) is equivalent to problem \((OP_2)\) in the following sense: If \(z^*\) is a global optimal solution of problem \((OP_2)\), then any \(x^* \in X\) such that \(f(x^*) \geq z^*\) is a global optimal solution of Problem \((P_{X_E})\).

Proof. Suppose that \(z^*\) is a global optimal solution of problem \((OP_2)\), i.e. \(\varphi(z^*) \geq \varphi(z)\) for all \(z \in E(Z)\). From Proposition 2(ii) and (5), we have \(E(Z) = Y_E\). This implies that
\[
\varphi(z^*) \geq \varphi(z) \quad \forall \, z \in Y_E.
\]

For any \(x^* \in X\) such that \(f(x^*) \geq z^*\), combining the monotone of \(\varphi(z)\) and the definition of \(Y_E\) gives
\[
\varphi(f(x^*)) \geq \varphi(f(x)) \quad \forall \, x \in X_E.
\]
Therefore, \(x^*\) must be a global optimal solution for Problem \((P_{X_E})\). \(\square\)

By Proposition 4, we can see that the optimal solution \(x^*\) to Problem \((P_{X_E})\) can be obtained by finding the optimal solution \(z^*\) to Problem \((OP_2)\) and optimizing a linear objective function over the nonempty compact convex set \(\{f(x) \geq z^*, x \in X\}\).

6. Illustrative example and computational experiments. Now we consider some examples to illustrate the algorithm.

Example 1. Consider the following problem with the tolerance \(\varepsilon = 10^{-6}\).
\[
\begin{align*}
\max \quad & (x_1 - x_2 + 4) + (5 - 0.25x_1^2)(4 - 0.125x_2^2) \\
\text{s.t.} \quad & 5x_1 - 8x_2 \geq -24 \\
& 5x_1 + 8x_2 \leq 44 \\
& 6x_1 - 3x_2 \leq 15 \\
& 4x_1 + 5x_2 \geq 10 \\
& x_1 \geq 0.
\end{align*}
\]

(Ex1)

It is easily to see that Problem \((Ex1)\) has the form of Problem \((GCMP)\) with \(g_0(x) = x_1 - x_2 + 4, \, g_1(x) = 5 - 0.25x_1^2\) and \(g_2(x) = 4 - 0.125x_2^2\). It is easily to check that \(g_0, g_1, g_2\) is concave. We transform this problem to the form of Problem \((GP)\) with \(\varphi(y_1, y_2) = y_1 + y_2^2, \, f_1(x) = g_0(x)\) and \(f_2(x) = (g_1(x)g_2(x))^{1/2}\).

Initialization step. By solving two convex problems \((IP_i), i = 1, 2\), we obtain the optimal solutions
\[
z^{\text{start}} = (6.5000, 3.7081), \quad z^{\text{end}} = (3.3266, 4.2717).
\]
Step 1. Since $\varphi(\begin{smallmatrix} z^{start} \end{smallmatrix}) = 20.2500 < \varphi(\begin{smallmatrix} z^{end} \end{smallmatrix}) = 21.5739$, set the currently best lower bound $\beta_0 = \varphi(\begin{smallmatrix} z^{end} \end{smallmatrix}) = 21.5739$ and the currently best feasible solution $\tilde{z}^0 = z^{end}$.

Let $E_0$ be the efficient curve connecting the points $z^{start}$ and $z^{end}$. Solve Problem $(RP(E))$ with $E = E_0$ to obtain the optimal solution $\tilde{z}^0 = (6.5000, 4.2717)$. Set the currently best upper bound $\alpha(E_0) = \varphi(\tilde{z}^0) = 24.7471$.

Set $k := 0$, $R_0 = \{E_0\}$ and go to Step 1.

**Iteration $k = 0$.**

**Step 1.** Since $R_0 = \{E_0\}$, where $E_0$ is the efficient curve connecting $z^{start}$ and $z^{end}$, we select $E_0$ for branching at Step 2. Set $\alpha_0 = \alpha(E_0) = 24.7471$.

**Step 2.** By solving Problem $(Pro(z^*))$ with $z^* = \tilde{z}^0$, we obtain the point $\hat{p}^0 = (6.3248, 3.7715)$ and the normal vector $\hat{d}^0 = (0.2591, 0.7409)$.

Since $\beta_0 > \varphi(\hat{p}^0) = 16.4354$, we update the currently best lower bound $\beta_1 = \beta_0$ and $\tilde{z}^1 = \tilde{z}^0$.

Set $R_1 = (R_0 \setminus E_0) \cup \{E_{01}, E_{02}\} = \{E_{01}, E_{02}\}$, where $E_{01}$ is the efficient curve connecting two efficient points $z^{start}$ and $\hat{p}^0$, $E_{02}$ is the efficient curve connecting two efficient points $\hat{p}^0$ and $z^{end}$.

**Step 3.** By solving Problem $(RP(E))$ with $E = E_{01}$ and $E = E_{02}$, we obtain the indices $Id_1 = 1, Id_2 = 1$ and the optimal solutions $\tilde{z}^{01} = (6.3249, 3.7715), \tilde{z}^{02} = (4.8948, 4.2717)$, respectively. Set $\alpha(E_{01}) = \varphi(\tilde{z}^{01}) = 20.5492$ and $\alpha(E_{02}) = \varphi(\tilde{z}^{02}) = 23.1421$. Set $R_1 = \{E \in R_1 \mid \alpha(E) - \beta_1 > \varepsilon(|\beta_1| + 1)\} = \{E_{02}\}$.

**Step 4.** Since $R_1 \neq \emptyset$, set $k := 1$ and go to Step 1.

After 5 iteration steps, the algorithm generates the currently lower bound $\beta_5 = 22.1252$, the set of remaining partitions $R_5 = \{E_5\}$ where $E_5$ is the efficient curve connecting $z^{start} = (4.4931, 1.9911)$ and $z^{end} = (4.3009, 4.2209)$, and the optimal solution $\tilde{z}^5 = (4.4078, 4.2097)$ to Problem $(RP(E))$ with $E = E_5$.

**Iteration $k = 5$.**

**Step 1.** Since $R_5 = \{E_5\}$, we select $E_5$ for branching at Step 2. Set $\alpha_5 = \alpha(E_5) = 22.1298$.

**Step 2.** By solving Problem $(Pro(z^*))$ with $z^* = \tilde{z}^5$, we obtain the point $\hat{p}^5 = (4.4077, 4.2088)$ and the normal vector $\hat{d}^5 = (0.1042, 0.8958)$.

Since $\beta_5 > \varphi(\hat{p}^5) = 22.1220$, we update the currently best lower bound $\beta_6 = 22.1252$ and $\tilde{z}^6 = \tilde{z}^5$.

Set $R_6 = (R_5 \setminus E_5) \cup \{E_{61}, E_{62}\} = \{E_{61}, E_{62}\}$, where $E_{61}$ is the efficient curve connecting two efficient points $z^{start}$ and $\hat{p}^5$, $E_{62}$ is the efficient curve connecting two efficient points $\hat{p}^5$ and $z^{end}$.

**Step 3.** By solving Problem $(RP(E))$ with $E = E_{61}$ and $E = E_{62}$, we obtain the indices $Id_1 = 1, Id_2 = 1$ and the optimal solutions $\tilde{z}^{61} = (4.4931, 1.9911), \tilde{z}^{62} = (4.4077, 4.2088)$, respectively. Set $\alpha(E_{61}) = \varphi(\tilde{z}^{61}) = 22.1252$ and $\alpha(E_{62}) = \varphi(\tilde{z}^{62}) = 22.1220$. Set $R_6 = \{E \in R_6 \mid \alpha(E) - \beta_6 > \varepsilon(|\beta_6| + 1)\} = \emptyset$.

**Step 4.** Since $R_6 = \emptyset$, then terminate the algorithm. The $\varepsilon$-optimal solution to Problem $(OP_2)$ is $\tilde{z}^6 = (4.4931, 1.9911)$. Then the approximate optimal solution to Problem $(GCMP)$ is $\tilde{z}^6 = (1.3850, 0.8920)$ and the $\varepsilon$-optimal value of Problem $(GCMP)$ is $\varphi(\tilde{z}_6) = 22.1252$. The computational result is shown in Table 1.

**Example 2.** Consider the problem proposed in [1] and [12].

$$
\begin{align*}
\text{max} \quad & (3x_1 - 4x_2 + 15) + (x_1 + 2x_2 - 1.5)(2x_1 - x_2 + 4) \\
& + (x_1 - 2x_2 + 8.5)(2x_1 + x_2 - 1)
\end{align*}
$$

(Ex2)
Initialization step. Let $\alpha, \beta$ are concave and positive over the feasible region of Problem (Ex2).

Step 1. Since $\beta_{k+1} = 0$, we update the currently best lower bound $\beta_0 = \varphi(z^{\text{start}}) = 156.5000$ and the currently best feasible solution $\hat{z}^0 = z^{\text{start}}$.

Set $\mathcal{E}_0$ be the efficient curve connecting the points $z^{\text{start}}$ and $z^{\text{end}}$. Solve Problem (RP(\mathcal{E})) with $\mathcal{E} = \mathcal{E}_0$ to obtain the optimal solution $\hat{z}^0 = (9.0830, 8.8046)$. Set the currently best upper bound $\alpha(\mathcal{E}_0) = \varphi(\hat{z}^0) = 158.1874$.

Set $k := 0$, $\mathcal{R}_0 = \{\mathcal{E}_0\}$ and go to Step 1.

Iteration $k = 0$.

Step 1. Since $\mathcal{R}_0 = \{\mathcal{E}_0\}$, where $\mathcal{E}_0$ is the efficient curve connecting $z^{\ell_0} = z^{\text{start}}$ and $z^{r_0} = z^{\text{end}}$. Set $\alpha_0 = \alpha(\mathcal{E}_0) = 1158.1874$.

Step 2. By solving Problem (Pro(z*)) with $z^* = \hat{z}^0$, we obtain the point $\hat{p}^0 = (9.0641, 8.7121)$ and the normal vectors $d^0 = (0.1694, 0.8306)$.

Since $\beta_0 > \varphi(\hat{p}^0) = 156.2253$, we update the currently best lower bound $\beta_1 = \beta_0$ and $z^1 = \hat{z}^1$.

Set $\mathcal{R}_1 = (\mathcal{R}_0 \setminus \mathcal{E}_0) \cup \{\mathcal{E}_{0_1}, \mathcal{E}_{0_2}\} = \{\mathcal{E}_{0_1}, \mathcal{E}_{0_2}\}$, where $\mathcal{E}_{0_1}$ is the efficient curve connecting two efficient points $z^{\text{start}}$ and $\hat{p}^0$, $\mathcal{E}_{0_2}$ is the efficient curve connecting two efficient points $\hat{p}^0$ and $z^{\text{end}}$.

Step 3. By solving Problem (RP(\mathcal{E})) with $\mathcal{E} = \mathcal{E}_{0_1}$ and $\mathcal{E} = \mathcal{E}_{0_2}$, we obtain the indices $I_{d_1} = 1, I_{d_2} = 1$ and the optimal solutions $\hat{z}^{0_1} = (9.0830, 8.7083), \hat{z}^{0_2} = (9.0641, 8.7121)$, respectively. Set $\alpha(\mathcal{E}_{0_1}) = \varphi(\hat{z}^{0_1}) = 156.5007$ and $\alpha(\mathcal{E}_{0_2}) = \varphi(\hat{z}^{0_2}) = 156.2252$. Set $\mathcal{R}_1 = \{\mathcal{E} \in \mathcal{R}_1 \mid \alpha(\mathcal{E}) - \beta_1 > \varepsilon(|\beta_1| + 1)\} = \{\mathcal{E}_{0_1}\}$.

Step 4. Since $\mathcal{R}_1 \neq \emptyset$, set $k := 1$ and go to Step 1.

| $k$ | $z^k$   | $\hat{p}^k$ | $\alpha_k$ | $\beta_{k+1}$ |
|-----|--------|-------------|------------|---------------|
| 0   | (6.5000, 4.2717) | (6.3248, 3.7715) | (0.2591, 0.7409) | 24.7471 | 21.5739 |
| 1   | (4.8948, 4.2717) | (4.8732, 4.1433) | (0.1440, 0.8560) | 23.1421 | 22.0401 |
| 2   | (4.1101, 4.2717) | (4.1074, 4.2392) | (0.0767, 0.9233) | 22.3574 | 22.0783 |
| 3   | (4.4941, 4.2071) | (4.4931, 4.1991) | (0.1114, 0.8886) | 22.1936 | 22.1252 |
| 4   | (4.3012, 4.2231) | (4.3009, 4.2209) | (0.0944, 0.9056) | 22.1358 | 22.1252 |
| 5   | (4.4078, 4.2097) | (4.4077, 4.2088) | (0.1042, 0.8958) | 22.1298 | 22.1252 |

Table 1. The computational result for Example 1.
Iteration $k = 1$.

Step 1. Since $\mathcal{R}_1 = \{\mathcal{E}_1\}$, where $\mathcal{E}_1$ is the efficient curve connecting $z^\ell_1 = (9.0830, 8.7083)$ and $z^r_1 = (9.0641, 8.7121)$. Set $\alpha_1 = \alpha(\mathcal{E}_1) = 156.5007$.

Step 2. By solving Problem (Pro$(z^*)$) with $z^* = \hat{z}^1$ where $\hat{z}^1 = (9.0830, 8.7083)$ is the optimal solution for Problem (RP$(\mathcal{E})$) with $\mathcal{E} = \mathcal{E}_1$ found in previous iteration step, we obtain the point $\hat{p}^1 = (9.0818, 8.7074)$ and the normal vector $\hat{d}^1 = (0.5742, 0.4258)$.

Since $\beta_1 > \varphi(\hat{p}^1) = 156.4646$, we update the currently best lower bound $\beta_2 = \beta_1$ and $\hat{z}^2 = \hat{z}^1$. Set $\mathcal{R}_2 = (\mathcal{R}_1 \setminus \mathcal{E}_1) \cup \{\mathcal{E}_{2_1}, \mathcal{E}_{2_2}\} = \{\mathcal{E}_{2_1}, \mathcal{E}_{2_2}\}$, where $\mathcal{E}_{2_1}$ is the efficient curve connecting two efficient points $z^\ell_1$ and $\hat{p}^1$, $\mathcal{E}_{2_2}$ is the efficient curve connecting two efficient points $\hat{p}^1$ and $z^r_1$.

Step 3. By solving Problem (RP$(\mathcal{E})$) with $\mathcal{E} = \mathcal{E}_{2_1}$ and $\mathcal{E} = \mathcal{E}_{2_2}$, we obtain the indices $Id_1 = 1, Id_2 = 1$ and the optimal solutions $\hat{z}^{1_1} = (9.0830, 8.7082), \hat{z}^{1_2} = (9.0809, 8.7087)$, respectively. Set $\alpha(\mathcal{E}_{1_1}) = \varphi(\hat{z}^{1_1}) = 156.5000$ and $\alpha(\mathcal{E}_{1_2}) = \varphi(\hat{z}^{1_2}) = 156.4704$. Set $\mathcal{R}_3 = \{\mathcal{E} \in \mathcal{R}_2 \mid \alpha(\mathcal{E}) - \beta_2 > \varepsilon(|\beta_2| + 1)\} = \emptyset$.

Step 4. Since $\mathcal{R}_1 = \emptyset$, then terminate the algorithm. The $\varepsilon$–optimal solution to Problem (OP$_2$) is $\hat{z}^2 = (9.0830, 8.7082)$.

Then the optimal solution to Problem (GP) is $\hat{x}^2 = (4.0000, 3.0000)$ and the $\varepsilon$–optimal value of Problem (GP) is $\varphi(\hat{x}^2) = 156.5000$. This computational result is the same as one in [1]. Furthermore, our algorithm terminated after only 2 iterations while the algorithm in [1] terminated after 5 iterations.

Example 3. Consider the Problem (P$_{X,k}$) (Example 5.1 in [6]), with

$$\varphi(y_1, y_2) = y_1 y_2 + 0.1 \sum_{i=1}^{2} y_i^2 - \frac{1}{1 + y_i}$$

and $f(x) = Cx$ and $X = \{x \in \mathbb{R}^3 \mid Ax \leq b, x \geq 0\}$ where

$$A = \begin{bmatrix}
0.455018 & 0.877264 & 0.657945 \\
-0.513731 & -0.216323 & 0.959626 \\
-0.046754 & 0.155691 & -0.538654 \\
0.668292 & -0.463549 & 0.056380 \\
-0.950449 & 0.806154 & -0.269284 \\
-0.160535 & -0.066810 & -0.351285
\end{bmatrix},$$

$$b = \begin{bmatrix}
3.938022 \\
0.104009 \\
-0.125144 \\
1.532750 \\
0.000000 \\
0.000000
\end{bmatrix},$$

$$C = \begin{bmatrix}
0.950449 & -0.806154 & 0.269284 \\
0.160535 & 0.066810 & 0.351280
\end{bmatrix}.$$
Since \( \varphi(z_{start}) = 2.5029 < \varphi(z_{end}) = 3.7476 \), set the currently best lower bound \( \beta_0 = \varphi(z_{end}) = 3.7476 \) and the currently best feasible solution \( \hat{z}^0 = z_{end} \).

Set \( \mathcal{E}_0 \) be the efficient curve connecting the points \( z_{start} \) and \( z_{end} \). Solve Problem (RP(\( \mathcal{E} \))) with \( \mathcal{E} = \mathcal{E}_0 \) to obtain the optimal solution \( \hat{z}^0 = (2.4216, 1.3043) \). Set the currently best upper bound \( \alpha(\mathcal{E}_0) = \varphi(\hat{z}^0) = 3.8424 \).

**Iteration** \( k = 0 \).

**Step 1.** Since \( \mathcal{R}_0 = \{\mathcal{E}_0\} \), where \( \mathcal{E}_0 \) is the efficient curve connecting \( z^0 = z_{start} \) and \( z^{\alpha_0} = z_{end} \). Set \( \alpha_0 = \alpha(\mathcal{E}_0) = 3.8424 \).

**Step 2.** By solving Problem \((\text{Pro}(z^*))\) with \( z^* = \hat{z}^0 \), we obtain the point \( \hat{p}^0 = (2.3693, 1.2988) \) and the normal vector \( \hat{d}^0 = (0.9046, 0.0954) \).

Since \( \beta_0 > \varphi(\hat{p}^0) = 3.7340 \), we update the currently best lower bound \( \beta_1 = \beta_0 = 3.7476 \) and \( \hat{z}^1 = \hat{z}^0 \).

Set \( \mathcal{R}_1 = (\mathcal{R}_0 \setminus \mathcal{E}_0) \cup \{\mathcal{E}_{01}, \mathcal{E}_{02}\} = \{\mathcal{E}_{01}, \mathcal{E}_{02}\} \), where \( \mathcal{E}_{01} \) is the efficient curve connecting two efficient points \( z_{start} \) and \( \hat{p}^0 \), \( \mathcal{E}_{02} \) is the efficient curve connecting two efficient points \( \hat{p}^0 \) and \( z_{end} \).

**Step 3.** By solving Problem (RP(\( \mathcal{E} \))) with \( \mathcal{E} = \mathcal{E}_{01} \) and \( \mathcal{E} = \mathcal{E}_{02} \), we obtain the indices \( I_d_1 = 1, I_d_2 = 1 \) and the optimal solutions \( \hat{z}_{01}^1 = (2.3693, 1.2987) \), \( \hat{z}_{02}^1 = (2.3687, 1.3043) \), respectively. Set \( \alpha(\mathcal{E}_{01}) = \varphi(\hat{z}_{01}^1) = 3.7339 \) and \( \alpha(\mathcal{E}_{02}) = \varphi(\hat{z}_{02}^1) = 3.7476 \). Set \( \mathcal{R}_1 = \{\mathcal{E} \in \mathcal{R}_1 \mid \alpha(\mathcal{E}) - \beta_1 > \epsilon(|\beta_1| + 1)\} = \emptyset \).

**Step 4.** Since \( \mathcal{R}_1 \neq \emptyset \), then terminate the algorithm. The \( \epsilon \)-optimal solution of Problem (OP2) is \( \hat{z}^0 = (2.3687, 1.3043) \). Then the approximate optimal solution to Problem (P_{X,p}) is \( \hat{x}^0 = (3.0631, 1.3593, 2.0546) \) and the \( \epsilon \)-optimal value is \( \varphi(f(\hat{x}^0)) = 3.7476 \), which are the same as the results in [6].

The following test is implemented by codes written in Matlab2009a and performed on a laptop HP Pavilion 1.8Ghz, RAM 2GB.

**Example 4.** Problem (GCMP) with randomly generated values are given in two types (see [8]):

**Type I:**

\[
\begin{align*}
\text{max} & \quad \langle \alpha^0, x \rangle + \prod_{i=1}^{k} (\langle \alpha^i, x \rangle + \beta_i) \\
\text{s.t.} & \quad Ax \leq b \\
& \quad x \geq 0,
\end{align*}
\]

where \( \beta_i = 1 - \min_{x \in X} \langle \alpha^i, x \rangle \) for \( i = 1, \ldots, k \).

**Type II:**

\[
\begin{align*}
\text{max} & \quad (\alpha^0 x + x^T D^0 x) + \prod_{i=1}^{k} (\alpha^i x + x^T D^i x + \beta_i) \\
\text{s.t.} & \quad x_1 + \sum_{j=2}^{k} \frac{j-1}{j} x_j^{1.5} \leq 1000
\end{align*}
\]
\[
\left(-2 + \sum_{j=1}^{k} \frac{x_j}{j}\right)^2 \leq 100 \quad \text{Ax} \leq b \quad \text{x} \geq 0,
\]

where \( \beta_i = 1 - \min_{x \in X} (\alpha^i x + x^T D^i x) \) for \( i = 1, \ldots, k \).

The parameters are defined as follows:

- \( \alpha^i \), for all \( i = 0, 1, \ldots, k \) are randomly generated vectors with all components belonging to \([0, 1]\).
- \( A = (a_{ij}) \in \mathbb{R}^{m \times n} \) is a randomly generated matrix with elements belonging to \([-1, 1]\).
- \( b = (b_1, b_2, \ldots, b_m)^T \) is randomly generated vector such that

\[
b_i = \sum_{j=1}^{n} a_{ij} + 2b_0, \quad i = 1, 2, \ldots, m,
\]

with \( b_0 \) being a randomly generated real in \([0, 1]\).
- \( D^i = (d_{ij}) \in \mathbb{R}^{n \times n} \) are diagonal matrices with diagonal elements \( d_{ii} \) randomly generated in \([-1, 0]\).

| \( k \) | \( m \) | \( n \) | Problems in Type I #Iter Time | Problems in Type II #Iter Time |
|------|------|------|-----------------------------|-----------------------------|
| 2    | 80   | 100  | 4.2 13.4307                  | 6.7 18.9782                  |
| 2    | 100  | 100  | 4.5 16.4012                  | 8.1 32.3003                  |
| 2    | 100  | 120  | 5.1 20.3567                  | 8.9 41.7732                  |
| 2    | 120  | 120  | 5.5 26.2341                  | 9.2 50.6674                  |
| 3    | 80   | 100  | 5.1 16.0467                  | 8.7 22.4429                  |
| 3    | 100  | 100  | 5.4 14.3470                  | 8.4 27.9831                  |
| 3    | 100  | 120  | 6.4 28.2346                  | 10.5 58.7864                 |
| 3    | 120  | 120  | 6.8 31.6772                  | 11.2 48.9017                 |

Table 2. The computational result for Example 4.

For \( \varepsilon = 0.005 \), the set of 10 problems associated with each set of parameters \( k, m, n \), are solved. We obtain the numerical results in Table 2. In this table, the number of iterations and the computing time in seconds are presented by \#Iter and Time, respectively. We also compare two types of problems in linear case (Type I) and convex case (Type II). The computational experiments show that the algorithm works well for problems with the fairly large dimension of the decision space.

7. Conclusion. A global optimization approach for maximizing the composite function of an increasing function and a concave vector function was proposed in this paper. By using some transformations, the original problem was reformulated in the outcome space as a problem over the efficient curve, which can be solved by partitioning and branching. The algorithm utilizes the interesting properties of
the efficient curve and solves efficiently the relaxation problems by optimizing one-variable functions. As an application, we also solved the Generalized Multiplicative Concave Programming and the problem of optimizing over the efficient set of a bicriteria concave maximization problem. The proposed algorithm can be implemented easily by the available optimization packages and the numerical experiments have presented its efficiency.

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