Homotopy transfer for QFT on non-compact manifold with boundary: a case study

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Abstract

In this work we report a homological perturbation calculation to construct effective theories of topological quantum mechanics on $\mathbb{R}_{\geq 0}$. Such calculation can be regarded as a generalization of Feynman graph computation. The resulting effective theories fit into derived BV algebra structure, which generalizes BV quantization. Besides, our construction may serve as the simplest example of a process called “boundary transfer”, which may help study bulk-boundary correspondence.

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1 Introduction

Based on Batalin-Vilkovisky (BV) formalism [BV81], Kevin Costello has developed a framework [Cos11] for perturbative quantum field theory (QFT), written in the language of homological algebra. Once a QFT is defined in this way, there will be a set of cochain complexes of observables encoding the gauge symmetry and interaction of this theory. Elements in this set are labeled by renormalization scales \( t \in \mathbb{R}_{>0} \) (if we use heat kernel or fake heat kernel to perform renormalization), while observables at different scales are connected by homotopic renormalization. We may call such data a “renormalized QFT”.

\[
\text{renormalized QFT} \xrightarrow{\text{homotopy transfer}} \ \text{"effective theory"} \xleftarrow{\text{"effective factorization algebra"}} \ \text{"factorization algebra"}
\]

We are interested in two aspects of a given renormalized QFT, schematically depicted above.

The horizontal direction of (1.1) is to ask what are the “physical observables”. The answer is an “effective observable complex” quasi-isomorphic to the “renormalized observable complexes”, while does not depend on the renormalization scale. (The scale should be regarded as an auxiliary parameter to make the theory well-defined.) The information of interaction on this effective observable complex is obtained by homotopy transfer from the renormalized observable complexes. We may say this effective complex defines an effective theory.

Remark 1.0.1 Our terminology differs from that in [Cos11]. The “renormalized” here corresponds to “effective” there. Roughly speaking, we use “effective” to refer to structures “on the cohomology” (or “on the physical objects”). See also Remark 2.4.2 for a concrete comparison.

As for the vertical direction of (1.1), the question is what structure on the observables reflects the fact that QFT is local on the spacetime manifold? In [CG16, CG21], Kevin Costello and Owen Gwilliam developed a formalism to construct a factorization algebra of observable complexes from a renormalized QFT. It is this “renormalized factorization algebra” that encodes spacetime locality. Naturally, we hope to remove the scales from its data by homotopy transfer, and construct an “effective factorization algebra”. If we achieve this goal for various QFT’s, the effective factorization algebras are expected to exhibit structures such as canonical quanization, vertex operator algebras, bulk-boundary correspondence and functorial formulation of QFT.

The process of homotopy transfer mentioned above is a calculation using homological perturbation theory. To obtain the effective theory of a QFT on a closed manifold, this calculation implies Feynman graph formulae, and the resulting effective theory will fit into BV formalism. (See [DJP19] for a formal explanation of this point using finite dimensional toy model.) However, in order to obtain the effective factorization algebra, we also need to figure out effective theories of QFT’s living on non-compact manifolds. In this case, the argument leading to Feynman graph formulae fails, and the resulting effective theories can exceed the scope of BV formalism. We will find that they fit into a structure called derived BV algebra [Kra99].
Moreover, to disclose bulk-boundary correspondences using effective factorization algebras, we have to figure out effective theories of QFT’s on manifolds with boundary. But with the presence of boundary, renormalization has not been systematically developed yet in general. If the spacetime is $\mathbb{H}^n$ equipped with the Euclidean metric, discussion of heat kernel renormalization can be found in [Alb16]. Later, Eugene Rabinovich formulated the renormalized theories and factorization algebras for field theories which are “topological normal to the boundary” [Rab21]. Particularly, topological quantum mechanics (TQM) on $\mathbb{R}_{\geq 0}$ can be constructed in the current sense.

### 1.1 Main results

In this work, we focus on the calculation for effective theories of TQM on $\mathbb{R}_{\geq 0}$. Since $\mathbb{R}_{\geq 0}$ is a non-compact manifold with boundary, this simple model carries double difficulties described in the last two paragraphs. Eventually, we construct an effective observable complex in Theorem 4.1.1. The differential of this complex and the projection from the renormalized observable complex to this effective complex have concrete formulae given in (4.5), (4.6), respectively. They are simplified from those initial formulae in homological perturbation lemma, just like Feynman graph formulae arise from homological perturbation calculation (reviewed in Section 2.4). As expected, the effective observable complex is independent of renormalization scales in the renormalized theory.

Using Theorem 4.1.1 we give three examples of such effective theories.

Example 4.1.1 is essentially a free theory, and reproduces known result in [Rab21, Theorem 3.4.3]. This is the simplest example showing how degenerate field theories arise from field theories on manifold with boundary (we refer to [BY16] for more discussion on this point). Example 4.1.2 deals with BF theory, and refines a conclusion in [Rab21, Theorem 5.0.2].

As for Example 4.1.3 we specifically design it to explicitly show that BV structure is not enough to describe such effective theory. A suitable candidate structure is the derived BV algebra structure [Kra99].

We emphasize two aspects of our calculation. The former corresponds to the spacetime being non-compact, and the latter corresponds to the presence of spacetime boundary.

### Derived BV algebras

Example 4.1.3 motivates us to discuss derived BV algebras. We base our discussion on [Ban20]. For the case we consider, Proposition 5.0.3 concludes that the renormalized observable complexes and the specific homotopies (renormalization) between them are objects and morphisms in the category of derived BV algebras, respectively. Moreover, Proposition 5.0.8 concludes that both the renormalized and effective observable complexes, together with the quasi-isomorphisms between them also lie in the category of derived BV algebras.

### Boundary transfer

Suppose there is a renormalized QFT on a manifold $X$ with boundary $\partial X$, we now sketch a process to obtain a factorization algebra on $\partial X$.

The renormalized QFT should give rise to a renormalized factorization algebra $\text{Obs}_T$ living on (small) tubular neighborhood $T \simeq [0, \varepsilon) \times \partial X$ of $\partial X$. Then, we can pushforward $\text{Obs}_T$ to $\partial X$ via the
projection $[0, \varepsilon) \times \partial X \to \partial X$, and find its effective version by homotopy transfer. We call this process a “boundary transfer”, and hope that it can help study various bulk-boundary correspondences.

Guided by this consideration, we make a “quasi-calculation” for BF theory on $\mathbb{R}_{>0} \times \mathbb{R}$ associated to a unimodular Lie algebra $\mathfrak{g}$. Instead of rigorously defining the renormalized theory, we treat it as a TQM and apply Theorem 4.1.1 ignoring singularity in the data. For the so-called B boundary condition and A boundary condition, we obtain two “effective observable complexes” (4.11) and (4.12). Then, they are identified with the Chevalley-Eilenberg complex $C_\bullet(\Omega^\bullet(\mathbb{R}) \otimes \mathfrak{g})$ and the Chevalley-Eilenberg algebra $\text{CE}(\Omega^\bullet(\mathbb{R}) \otimes \mathfrak{g})$, respectively (after taking $\hbar = 1$). This result echos the observation (see a review [PW21] and references therein) that Koszul duality between certain algebras can appear in QFT’s with proper setting.

1.2 Future directions

There are many questions to study in the future, we list several of them here.

First, we set up Definition 3.0.1 for the renormalized theory, but have not provided a direct criterion to determine whether a given $I^\partial$ satisfies the definition. For TQM on $S^1$, such a direct criterion has been worked out. (See [GLL17] Theorem 3.10 or Theorem 3.22. A proof of the algebraic index theorem can be obtained by studying the effective theory there, see also [GLX21].) The criterion for our case should be a modification of that one, taking into account that $\rho(I^\partial)$ may have boundary anomaly. Besides, Definition 3.0.1 excludes boundary terms (i.e. functionals supported on the boundary) in the interaction functional, but they are relevant to bulk-boundary correspondence and should be considered in principle.

Second, the argument leading to Theorem 4.1.1 relies on the facts that the spacetime is 1-dimensional, and the theory is topological. We hope to find ways to simplify homological perturbation calculations for general configurations. It could be formulated using Costello’s framework. In the 2D case, another possible approach is to use a geometric renormalization method of regularized integral introduced in [LZ21] (see also [GL21]).

Third, we recognize that the renormalized and effective observable complexes and their quasi-isomorphisms lie in the category of derived BV algebras, but we have not touched consequences of this fact yet. Just like BV algebras, derived BV algebras have quantum master equations associated to them. The solutions of these equations behave well under derived BV algebra morphisms. We leave these considerations for later study.

1.3 Organization of the paper

The paper is organized as follows.

In Section 2 we briefly introduce homological perturbation theory and BV formalism. As a warm-up, we use them to construct effective theories of free QFT’s (on arbitrary manifolds) and interactive QFT’s on closed manifolds. We perform the free theory calculation without using Hodge decomposition, hence the result has a more general setting than that in [DJP19], and can be used in Section 3. In the interactive case we review the way Feynman graph formulae arise from homotopy transfer. Moreover, these two warm-up examples exhibit all essential homological perturbation calculations in later sections, including the one in Proposition 5.0.4 which simplifies [Ban20] Proposition 1.27].
In Section 3 we review relevant constructions in [Rab21], and set up our renormalized theories. Their effective theories are obtained in Section 4.

In Section 5 we review the definition and homotopy transfer of derived BV algebras based on [Ban20], then apply them to our constructions in previous sections.

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Convention

• Let $V$ be a $\mathbb{Z}$-graded $k$-vector space. We use $V_m$ to denote its degree $m$ component. Given homogeneous element $a \in V_m$, we let $|a| = m$ be its degree.
  
  $V[n]$ denotes the degree shifting of $V$ such that $V[n]_m = V_{n+m}$.
  
  $V^*$ denotes its linear dual such that $V^*_m = \text{Hom}_k(V_{-m}, k)$. Our base field $k$ will mainly be $\mathbb{R}$.
  
  $\text{Sym}^m(V)$ and $\wedge^m(V)$ denote the $m$-th power graded symmetric product and graded skew-symmetric product respectively. We also denote
  \[
  \text{Sym}(V) := \bigoplus_{m \geq 0} \text{Sym}^m(V), \quad \widehat{\text{Sym}}(V) := \prod_{m \geq 0} \text{Sym}^m(V).
  \]
  The latter is a graded symmetric algebra with the former being its subalgebra. We will omit the multiplication mark for this product in expressions (unless confusion occurs).
  
  $V[[\hbar]], V((\hbar))$ denote formal power series and Laurent series respectively in a variable $\hbar$ valued in $V$.

• We use the Einstein summation convention throughout this work.

• We use $(\pm)_{\text{Kos}}$ to represent the sign factors decided by Koszul sign rule. We always assume this rule in dealing with graded objects.

  Example: let $j$ be a homogeneous linear map on $V$, then $j^*$ denotes the induced linear map on $V^*$: for $\forall f \in V^*, a \in V$ being homogeneous,
  \[
  j^* f(a) := (\pm)_{\text{Kos}} f(j(a)) \quad \text{with } (\pm)_{\text{Kos}} = (-1)^{|j| |f|} \text{ here}.
  \]

  Example: let $f, g, h \in V^*$ be homogeneous elements, then $f \otimes g \otimes h \in (V^*)^{\otimes 3}$ is regarded as an element in $(V^{\otimes 3})^*$: for $\forall a, b, c \in V$ being homogeneous,
  \[
  (f \otimes g \otimes h)(a \otimes b \otimes c) := (\pm)_{\text{Kos}} f(a) g(b) h(c) \quad \text{with } (\pm)_{\text{Kos}} = (-1)^{|b||a|+|a||b|+|g||a|} \text{ here}.
  \]

  Example: let $(A, \cdot)$ be a graded algebra, then $[\cdot, \cdot]$ means the graded commutator, i.e, for homogeneous elements $a, b$,
  \[
  [a, b] := a \cdot b - (\pm)_{\text{Kos}} b \cdot a \quad \text{with } (\pm)_{\text{Kos}} = (-1)^{|a||b|} \text{ here}.
  \]
• We fix an embedding of vector spaces \( \text{Sym}^m(V) \hookrightarrow V^\otimes m \) by

\[
a_1a_2\cdots a_m \mapsto \sum_{\sigma\in S_m} (\pm)_{\text{Kos}} a_{\sigma(1)} \otimes a_{\sigma(2)} \otimes \cdots \otimes a_{\sigma(m)},
\]

where \( S_m \) denotes the symmetric group. Accordingly, any \( f_1f_2\cdots f_m \in \text{Sym}^m(V^*) \) is regarded as an element in \( (\text{Sym}^m(V))^* \): for \( \forall a_1a_2\cdots a_m \in \text{Sym}^m(V) \),

\[
f_1f_2\cdots f_m(a_1a_2\cdots a_m) = m! \sum_{\sigma\in S_m} (\pm)_{\text{Kos}} f_1(a_{\sigma(1)})f_2(a_{\sigma(2)})\cdots f_m(a_{\sigma(m)}).
\]

• We call \((V, d)\) a cochain complex if \( d \) is a degree 1 map on the graded vector space \( V \) such that \( d^2 = 0 \). Such \( d \) is called a differential. A cochain map \( f : (V, d) \mapsto (W, b) \) is a degree 0 map from \( V \) to \( W \) such that \( bf = fd \).

• Given a manifold \( X \), we denote the space of real smooth forms by

\[
\Omega^\bullet(X) = \bigoplus_k \Omega^k(X)
\]

where \( \Omega^k(X) \) is the subspace of \( k \)-forms, lying at degree \( k \).

• Now that we have mentioned differential forms, definitely we will work with infinite dimensional functional spaces that carry natural topologies. The above notions for \( V \) will be generalized as follows. We refer the reader to [Tre06] or [Cos11, Appendix 2] for further details. Besides, [Rab21, Appendix A] contains specialized discussion for sections of vector bundles with boundary conditions.

  – All topological vector spaces we consider will be nuclear and we still use \( \otimes \) to denote the completed projective tensor product. For example, given two manifolds \( X, Y \), we have a canonical isomorphism

\[
\mathcal{C}^\infty(X) \otimes \mathcal{C}^\infty(Y) = \mathcal{C}^\infty(X \times Y).
\]

  – In the involved categories, dual space is defined to be the continuous linear dual, equipped with the topology of uniform convergence of bounded subsets. We still use \((–)^*\) to denote taking such duals.

2 Algebraic Preliminaries

In this section, we briefly introduce homological perturbation theory and BV formalism. As a warm-up, we use them to schematically construct perturbative effective theories of free QFT’s (on arbitrary manifolds) and interactive QFT’s on closed manifolds.

2.1 Homological perturbation theory

A special deformation retract (SDR for short) is the following data:

\[
\begin{align*}
\quad & i \\
(N, b) & \equiv (M, d), K
\end{align*}
\]

(2.1)
where \((N, b)\) and \((M, d)\) are cochain complexes, and \(i, p\) are cochain maps between them. \(K\) is a degree \(-1\) map on \(M\), such that

\[
pi = 1, \quad ip = 1 + dK + Kd, \quad pK = 0, \quad Ki = 0, \quad K^2 = 0.
\]

Consider a perturbation \(\delta\) to the differential on \(M\):

\[
d_1 := d + \delta, \quad d_1^2 = 0
\]

we say \(\delta\) is a small perturbation if \((1 - \delta K)\) is invertible. For example, if \(\sum_{n=0}^{+\infty}(\delta K)^n\) is well defined on \(M\), then \(\delta\) is small. Note that \((1 - \delta K)\) being invertible implies \((1 - K\delta)\) is also invertible, because we can verify that

\[
(1 - K\delta)^{-1} = 1 + K(1 - \delta K)^{-1}\delta.
\]

**Lemma 2.1.1 “Homological Perturbation”**

Given an SDR as \((2.1)\) and a small perturbation \(\delta\), denote \(A := (1 - \delta K)^{-1}\delta\), the following data is also an SDR:

\[
i_1
\]

\[
(N, b_1) \cong (M, d_1 = d + \delta), K_1
\]

\[
p_1
\]

where

\[
b_1 = b + pAi, \quad i_1 = i + KAi, \quad p_1 = p + pAK, \quad K_1 = K + KAK.
\]

We can also write

\[
\begin{align*}
i_1 & = (1 - K\delta)^{-1}i, & p_1 & = p(1 - \delta K)^{-1}, & K_1 & = (1 - K\delta)^{-1}K = K(1 - \delta K)^{-1}, \\
b_1 & = b + p_1\delta i = b + p\delta i_1 = p_1d_1i = pd_1i_1.
\end{align*}
\]

We refer the reader to [Cra04] and references therein for proof and further discussion of this lemma.

**Proposition 2.1.1** In the settings of Lemma 2.1.1, the following three statements are equivalent:

A. \(p\delta K = 0\)

B. \(p_1 = p\)

C. \(p\delta = p\delta i\)

**Proposition 2.1.2** In the settings of Lemma 2.1.1, the following three statements are equivalent:

A. \(K\delta i = 0\)

B. \(i_1 = i\)

C. \(i_1 \delta i = \delta i\)
These two propositions can be checked using the formulae in Lemma 2.1.1. We leave it as an exercise.

**Lemma 2.1.2 “Associativity of Homological Perturbation”**

Given initial data written as (2.1), suppose there are two small perturbations to \(d\):

\[
d_1 := d + \delta_1, \quad d_2 := d + (\delta_1 + \delta_2),
\]

where the former induces an SDR written as (2.2). Then, \(\delta_2\) must be small with respect to (2.2), because we can verify

\[
(1 - \delta_2 K_1)^{-1} = (1 - \delta_1 K)(1 - (\delta_1 + \delta_2)K)^{-1}.
\]

Denote

\[
A_2 := (1 - (\delta_1 + \delta_2)K)^{-1}(\delta_1 + \delta_2), \quad A'_2 := (1 - \delta_2 K_1)^{-1}\delta_2.
\]

For the perturbed differential \(d_2\), we have the “one-step perturbation” to (2.1):

\[
\begin{align*}
(N, b_2) &\equiv (M, d_2), \quad K_2 \\
p_2 &
\end{align*}
\]

where

\[
b_2 = b + pA_2i, \quad i_2 = i + KA_2i, \quad p_2 = p + pA_2K, \quad K_2 = K + KA_2K.
\]

We also have the “two-step perturbation” to (2.1):

\[
\begin{align*}
(N, b'_2) &\equiv (M, d_2), \quad K'_2 \\
p'_2 &
\end{align*}
\]

where

\[
b'_2 = b_1 + p_1 A'_2 i_1, \quad i'_2 = i_1 + K_1 A'_2 i_1, \quad p'_2 = p_1 + p_1 A'_2 K_1, \quad K'_2 = K_1 + K_1 A'_2 K_1.
\]

Then the conclusion is that these two perturbed SDR’s actually coincide.

The proof of this lemma is not hard and we leave it as an exercise.

**Lemma 2.1.3 “Dual Construction”**

Given data written as (2.1), take the duals, then the following is an SDR:

\[
\begin{align*}
p^* &
\end{align*}
\]

The verification is straightforward and we skip it. This taking dual operation commutes with homological perturbation.
Lemma 2.1.4 “Symmetric Tensor Power Construction” (See e.g., [Ber14, Section 5].) Given data written as (2.1), the following is an SDR:

\[(\text{Sym}(N), b_{\text{der}}) \equiv (\text{Sym}(M), d_{\text{der}}), K_{\text{sym}}\] (2.3)

where \(i_{\text{sym}}, p_{\text{sym}}\) are algebraic maps extended from \(i, p\):

\[i_{\text{sym}} = \sum_{n \geq 0} i^{\otimes n}, \quad p_{\text{sym}} = \sum_{n \geq 0} p^{\otimes n},\]

and \(b_{\text{der}}, d_{\text{der}}\) are derivations extended from \(b, d\) using Leibniz rule:

\[b_{\text{der}} = \sum_{n \geq 1} n^{-1} \sum_{m=0}^{n-1} 1^{\otimes m} \otimes b \otimes 1^{\otimes (n-m-1)}, \quad d_{\text{der}} = \sum_{n \geq 1} n^{-1} \sum_{m=0}^{n-1} 1^{\otimes m} \otimes d \otimes 1^{\otimes (n-m-1)},\]

and

\[K_{\text{sym}} = \sum_{n \geq 1} \frac{1}{n!} \sum_{\sigma \in S_n} \sigma^{-1} \left( \sum_{m=0}^{n-1} 1^{\otimes m} K \otimes \pi^{\otimes (n-m-1)} \right) \sigma = qK_{\text{der}} = K_{\text{der}} q,\]

with \(\pi := ip, \sigma \in S_n\) permuting the tensor factors of \(\text{Sym}^n(M)\), \(K_{\text{der}}\) being the derivation extended from \(K\), and

\[q := \sum_{n \geq 1} \sum_{\epsilon \in \{0,1\}^n} \frac{|\epsilon|!(n-1-|\epsilon|)!}{n!} \pi^{\epsilon_1} \otimes \pi^{\epsilon_2} \otimes \cdots \otimes \pi^{\epsilon_n}, \quad \text{with } |\epsilon| = \epsilon_1 + \cdots + \epsilon_n.\]

The above statement remains unchanged if we replace \(\text{Sym}(N)\) and \(\text{Sym}(M)\) by \(\hat{\text{Sym}}(N)\) and \(\hat{\text{Sym}}(M)\), respectively.

It is direct to see that taking symmetric tensor power commutes with taking dual. For the sake of brevity, in the rest of this paper we will just use \(b, d, i, p\) to denote \(b_{\text{der}}, d_{\text{der}}, i_{\text{sym}}, p_{\text{sym}}\) in symmetric tensor power constructions if there is no ambiguity.

**Perturbation by conjugation**

Given an SDR as (2.1), consider a conjugation on \(M\)

\[U \quad (M, d) \equiv (M, UdU^{-1}) \quad U^{-1}\]

Denote \(d_U := UdU^{-1}\). If \((d_U - d)\) is a small perturbation with respect to the initial data (2.1), we then write the perturbed SDR as

\[(N, b'_U) \equiv (M, d_U), K'_U \quad p'_U\]

with

\[b'_U = \sum_{n \geq 1} \sum_{m=0}^{n-1} 1^{\otimes m} \otimes b \otimes 1^{\otimes (n-m-1)} - \sum_{n \geq 1} \sum_{m=0}^{n-1} 1^{\otimes m} \otimes d \otimes 1^{\otimes (n-m-1)},\]

and

\[p'_U = \sum_{n \geq 1} \sum_{m=0}^{n-1} 1^{\otimes m} \otimes p \otimes 1^{\otimes (n-m-1)} - \sum_{n \geq 1} \sum_{m=0}^{n-1} 1^{\otimes m} \otimes p \otimes 1^{\otimes (n-m-1)},\]

and

\[K'_U = \sum_{n \geq 1} \frac{1}{n!} \sum_{\sigma \in S_n} \sigma^{-1} \left( \sum_{m=0}^{n-1} 1^{\otimes m} K \otimes \pi^{\otimes (n-m-1)} \right) \sigma - \sum_{n \geq 1} \frac{1}{n!} \sum_{\sigma \in S_n} \sigma^{-1} \left( \sum_{m=0}^{n-1} 1^{\otimes m} K \otimes \pi^{\otimes (n-m-1)} \right) \sigma qK_{\text{der}} = K_{\text{der}} q,\]

with \(\pi := ip, \sigma \in S_n\) permuting the tensor factors of \(\text{Sym}^n(M)\), \(K_{\text{der}}\) being the derivation extended from \(K\), and

\[q := \sum_{n \geq 1} \sum_{\epsilon \in \{0,1\}^n} \frac{|\epsilon|!(n-1-|\epsilon|)!}{n!} \pi^{\epsilon_1} \otimes \pi^{\epsilon_2} \otimes \cdots \otimes \pi^{\epsilon_n}, \quad \text{with } |\epsilon| = \epsilon_1 + \cdots + \epsilon_n.\]
Proposition 2.1.3 In the above setting, the following statements are equivalent:

A. There is a conjugation on $N$

\[
\begin{align*}
W \\
(N, b) & \Leftrightarrow (N, WbW^{-1}) \\
W^{-1}
\end{align*}
\]

such that

\[
b'_U = WbW^{-1} \quad \text{and} \quad p'_U = WpU^{-1}
\]

B. The invertible map $U$ satisfies

\[
pU^{-1}K = 0, \quad \text{and} \quad (pU^{-1}i)^{-1} \text{ exists on } N.
\]

If [A] (hence also [B]) holds, then there must be

\[
W = (pU^{-1}i)^{-1}.
\]

Proof Assume statement [A] holds. By the formula for perturbed projection map in Lemma 2.1.1, it is easy to observe that $p'_UK = 0$ and $p'_Ui = 1$. So $pU^{-1}K = W^{-1}p'_UK = 0$, $W(pU^{-1}) = p'_Ui = 1$. This means statement [B] holds, and $W = (pU^{-1}i)^{-1}$.

Assume statement [B] holds. Then $pU^{-1}i'_U = pU^{-1}i$. So,

\[
p'_U - (pU^{-1}i)^{-1}pU^{-1} = (pU^{-1}i)^{-1}(pU^{-1}i)p'_U - (pU^{-1}i)^{-1}pU^{-1}
\]

\[
= (pU^{-1}i)^{-1}pU^{-1}i'_U p'_U - (pU^{-1}i)^{-1}pU^{-1}
\]

\[
= (pU^{-1}i)^{-1}pU^{-1}(i'_U p'_U - 1)
\]

\[
= (pU^{-1}i)^{-1}pU^{-1}(d_U K'_U + K'_U d_U)
\]

\[
= (pU^{-1}i)^{-1}pU^{-1}d_U K'_U
\]

\[
= (pU^{-1}i)^{-1}pdU^{-1} K'_U
\]

\[
= (pU^{-1}i)^{-1}bpU^{-1} K'_U
\]

\[
= 0.
\]

So, $b'_U = p'_U d_U i = (pU^{-1}i)^{-1}pU^{-1}U dU^{-1}i = (pU^{-1}i)^{-1}b(pU^{-1}i)$. This means statement [A] holds. \qed

Proposition 2.1.4 In the same setting for Proposition 2.1.3, the following statements are equivalent:

A. There is a conjugation on $N$

\[
\begin{align*}
W \\
(N, b) & \Leftrightarrow (N, WbW^{-1}) \\
W^{-1}
\end{align*}
\]

such that

\[
b'_U = WbW^{-1} \quad \text{and} \quad i'_U = UiW^{-1}
\]

B. The invertible map $U$ satisfies

\[
KUi = 0, \quad \text{and} \quad (pU^{-1}i)^{-1} \text{ exists on } N.
\]
If $A$ (hence also $B$) holds, then there must be

$$W = pU i.$$ 

The proof is left as an exercise.

### 2.2 Batalin-Vilkovisky algebras and the quantum master equation

**Definition 2.2.1** A differential Batalin-Vilkovisky (BV) algebra is a triple $(A, Q, \Delta)$ where

- $A$ is a $\mathbb{Z}$-graded commutative associative unital algebra. Assume the base field is $\mathbb{R}$.
- $Q : A \to A$ is a derivation of degree 1 such that $Q^2 = 0$.
- $\Delta : A \to A$ is a linear operator of degree 1 such that $\Delta^2 = 0$, and $[Q, \Delta] = Q\Delta + \Delta Q = 0$.
- $\Delta$ is a “second-order” operator w.r.t. the product of $A$. Precisely, define the binary operator $\{−, −\} : A \otimes A \to A$ as:

$$\{a, b\} := \Delta(ab) - (\Delta a)b - (-1)^{|a|}a\Delta b, \quad \text{for } \forall a, b \in A.$$ 

Then for $\forall a \in A$, $\{−, −\}$ is a derivation of degree $(|a| + 1)$: for $\forall b, c \in A$

$$\{a, bc\} = \{a, b\}c + (\pm)_{\text{Kos}} b\{a, c\}, \quad \text{with } (\pm)_{\text{Kos}} = (-1)^{|b||a|+|b|} \text{ here.}$$

We call $\Delta$ the BV operator. $\{−, −\}$ is called the BV bracket, which measures the failure of $\Delta$ being a derivation. The above description implies the following properties:

- $\{a, b\} = (-1)^{|a||b|}\{b, a\}$.
- $\Delta\{a, b\} = -\{\Delta a, b\} - (-1)^{|a|}\{a, \Delta b\}$.
- $Q\{a, b\} = -\{Qa, b\} - (-1)^{|a|}\{a, Qb\}$.
- $\{a, \{b, c\}\} = (-1)^{|a|+1}\{\{a, b\}, c\} + (-1)^{|a|+1)(|b|+1)}\{b, \{a, c\}\}$.

Let $\hbar$ be a formal variable of degree 0 (representing the quantum parameter), we can extend the above $Q, \Delta$ to $\mathbb{R}[[\hbar]]$-linear operators on $A[[\hbar]]$. Then, $(A, Q, \Delta)$ being a differential BV algebra implies $Q + \hbar\Delta$ is a differential on $A[[\hbar]]$. There is a systematic way to twist (i.e., perturb) this differential, sketched in the following.

**Definition 2.2.2** Let $(A, Q, \Delta)$ be a differential BV algebra. A degree 0 element $I \in A[[\hbar]]$ is said to satisfy quantum master equation (QME) if

$$QI + \hbar\Delta I + \frac{1}{2}\{I, I\} = 0. \quad (2.4)$$

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It is direct to check that (2.4) implies 
\[(Q + \hbar \Delta + \{I, -\})^2 = 0 \text{ on } \mathcal{A}[[\hbar]].\]

The “second-order” property of \(\Delta\) allows us to write down a formally equivalent equation of QME:
\[(Q + \hbar \Delta)e^{I/\hbar} = 0,\]
and it implies this formal conjugation relation of operators on \(\mathcal{A}[[\hbar]]\):
\[Q + \hbar \Delta + \{I, -\} = e^{-I/\hbar}(Q + \hbar \Delta)e^{I/\hbar}.\]

Although \(e^{I/\hbar}\) is not defined on \(\mathcal{A}[[\hbar]]\), we can still make sense of the above two formulae without worrying about the powers of \(\hbar^{-1}\).

For later convenience, we give an example of differential BV algebra here. In Lemma 2.1.4 we have seen how to construct a differential graded commutative algebra from a cochain complex
\[(M, d) \rightsquigarrow (\hat{\text{Sym}}(M), d).\]  

(2.5)

**Definition 2.2.3** We call a linear operator \(G\) on \(\hat{\text{Sym}}(M)\) a “2-to-0 operator”, if

- \(G(\text{Sym}^{\leq 1}(M)) = 0\), and \(G(\text{Sym}^2(M)) \subset \text{Sym}^0(M)\).
- For \(n \geq 2\), \(\forall m_1 m_2 \cdots m_n \in \text{Sym}^n(M)\),
  \[G(m_1 m_2 \cdots m_n) = \sum_{i<j} (\pm)_{\text{Kos}} G(m_i m_j) m_1 \cdots \hat{m}_i \cdots \hat{m}_j \cdots m_n.\]

**Remark** In short, a 2-to-0 operator can be regarded as a second-order differential operator with constant coefficients.

It is easy to verify the following propositions:

**Proposition 2.2.1** Any two 2-to-0 operators \(G_1, G_2\) commute: \([G_1, G_2] = 0\). This means that any 2-to-0 operator of degree 1 is a differential.

**Proposition 2.2.2** For \((\hat{\text{Sym}}(M), d)\) in (2.5) and a 2-to-0 operator \(G\), \([G, d]\) is also a 2-to-0 operator, decided by
\[[G, d](m_1 m_2) = Gd(m_1 m_2) \quad \text{for } \forall m_1, m_2 \in M.\]

**Proposition 2.2.3** For \((\hat{\text{Sym}}(M), d)\) in (2.5), let \(\Delta\) be a 2-to-0 operator of degree 1 such that
\[\Delta d(m_1 m_2) = \Delta((dm_1) m_2 + (\pm)_{\text{Kos}} m_1(dm_2)) = 0 \quad \text{for } \forall m_1, m_2 \in M,\]
then \((\hat{\text{Sym}}(M), d, \Delta)\) is a differential BV algebra.

We refer to the review [L] for geometric descriptions of BV formalism and its relations to quantum field theory.
2.3 Application: effective theory of a free QFT

In Costello’s framework, for each QFT we construct a family of differential BV algebras which are all equivalent in the sense of homotopic renormalization. Since renormalization specified for our main case will be discussed in later sections, we ignore this complexity here (i.e., we only refer to structures at a fixed scale $t$ in the renormalized theory).

So, attached to each free QFT, there is a differential BV algebra $(\hat{\text{Sym}}(M), d, \Delta)$ as described in Proposition 2.2.3. The meaning of $M$ is “the set of classical linear observables”, and usually $(M, d)$ is the dual of the field space for the theory. Then, there is a cochain complex of quantum observables

$$(\hat{\text{Sym}}(M)[[\hbar]], d + \hbar \Delta). \tag{2.6}$$

Roughly speaking, an effective theory is a “smaller” and possibly “more invariant” cochain complex quasi-isomorphic to (2.6). In our scope, it is constructed using homological perturbation theory, explained in the following.

We start from an SDR written as (2.1)

$$i(N, b) \mapsto (M, d, K) \tag{2.1}$$

with $N$ regarded as “the effective classical linear observables”. Then, by symmetric tensor power construction (Lemma 2.1.4), there is an SDR

$$i(\hat{\text{Sym}}(N)[[\hbar]], b) \mapsto (\hat{\text{Sym}}(M)[[\hbar]], d, K^{sym}) \tag{2.7}$$

where we have extended the maps linearly over $\mathbb{R}[[\hbar]]$. It is direct to see that $\sum_{n=0}^{+\infty}(\hbar \Delta K^{sym})^n$ is well defined on $\hat{\text{Sym}}(M)[[\hbar]]$, so $\hbar \Delta$ is a small perturbation to this SDR.

Proposition 2.3.1 In above settings, we can write down the perturbation result of (2.7) by $\hbar \Delta$ as follows:

$$i_h(\hat{\text{Sym}}(N)[[\hbar]], b_h) \Rightarrow (\hat{\text{Sym}}(M)[[\hbar]], d + \hbar \Delta), K_h \tag{2.8}$$

where

$$i_h = i, \quad b_h = b + \hbar p \Delta i, \quad p_h = pe^{b(\Delta K^{sym})_2}$$

with $(\Delta K^{sym})_2$ being the 2-to-0 operator decided by

$$(\Delta K^{sym})_2(m_1m_2) = \Delta K^{sym}(m_1m_2) = \frac{1}{2}(K \otimes (1 + ip) + (1 + ip) \otimes K)(m_1m_2) \quad \text{for } \forall m_1, m_2 \in M.$$
Proof By associativity of perturbation theory (Lemma 2.1.2), we can calculate the resulting SDR by “two-step perturbation”. First, define another 2-to-0 operator \( \Delta^{-}\infty \):

\[
\Delta^{-}\infty (m_1m_2) := \Delta(m_1m_2) + [\Delta K^{\text{sym}}, d](m_1m_2) = \Delta(m_1m_2) + K^{\text{sym}}d(m_1m_2) = (i + K^{\text{sym}}d)(m_1m_2) = (ip - dK^{\text{sym}})(m_1m_2) = \Delta (ip)(m_1m_2).
\]

It is direct to verify that \((\widehat{\text{Sym}}(M), d, \Delta^{-}\infty)\) is also a differential BV algebra described by Proposition 2.2.3. By construction we have

\[
\Delta^{-}\infty i = i(p \Delta i), \quad p \Delta^{-}\infty = (p \Delta i)p.
\]

So if we use \( \hbar \Delta^{-}\infty \) to perturb \( (2.7) \), the statement A in Proposition 2.1.1 and statement A in Proposition 2.1.2 are both valid, hence the perturbed result is:

\[
(\widehat{\text{Sym}}(N)[[\hbar]], b, \hbar p \Delta i) \leftarrow (\widehat{\text{Sym}}(M)[[\hbar]], d + \hbar \Delta^{-}\infty), K^{-}\infty
\]

where we do not touch the concrete formula for \( K^{-}\infty \), only keep in mind that \( K^{-}\infty \) can be written as \( K^{\text{sym}}T \) or \( T'K^{\text{sym}} \) with some invertible operator \( T, T' \).

Now, consider a conjugation on the RHS of \( (2.9) \) defined by \( U = e^{-\hbar(\Delta K^{\text{sym}})^2} \). By Proposition 2.2.1 and Proposition 2.2.2, it is direct to see

\[
U(d + \hbar \Delta^{-}\infty)U^{-1} = (d + \hbar \Delta^{-}\infty) - \hbar((\Delta K^{\text{sym}})_2, (d + \hbar \Delta^{-}\infty)) + \frac{1}{2!}\hbar^2[(\Delta K^{\text{sym}})_2, ((\Delta K^{\text{sym}})_2, (d + \hbar \Delta^{-}\infty))] - \ldots = d + \hbar \Delta^{-}\infty - \hbar((\Delta K^{\text{sym}})_2, d] = d + \hbar \Delta.
\]

This conjugation indeed defines a small perturbation to \( (2.8) \), and after this second perturbation the differential on \( \widehat{\text{Sym}}(M)[[\hbar]] \) becomes exactly the one on the RHS of \( (2.8) \).

Since \( Ki = 0 \), we have

\[
U i = U^{-1} i = i.
\]

By similar reason for Proposition 2.2.2, we can find that \( [(\Delta K^{\text{sym}})_2, K^{\text{der}}] = 0 \), so \( pU^{-1}K^{\text{der}} = 0 \). Hence

\[
pU^{-1}K^{-}\infty = 0, \quad K^{-}\infty Ui = 0, \quad pU i = pU^{-1} i = 1,
\]

which means that the statement B in Proposition 2.1.3 and statement B in Proposition 2.1.4 are both valid. So after this perturbation by conjugation, \( (2.9) \) will exactly give rise to \( (2.8) \). \( \square \)

In this way, we obtain an effective theory which also corresponds to a differential BV algebra \((\widehat{\text{Sym}}(N), b, p \Delta i)\). The injection and projection maps between the effective quantum observable complex and the renormalized quantum observable complex are figured out. We should pay attention to the operator \( (\Delta K^{\text{sym}})_2 \), it corresponds to “contracting with propagator” in physicists’ language.
2.4 Application: effective theory of an interactive QFT on closed manifold

In Costello’s framework, an interactive QFT which is “in the neighborhood” of a free QFT \((\hat{\text{Sym}}(M), d, \Delta)\) will give rise to a quantum observable complex

\[
(\hat{\text{Sym}}(M)[[\hbar]], d + \hbar \Delta + \delta^{\text{int}}) \tag{2.10}
\]

with \(\delta^{\text{int}}\) being a \(\mathbb{R}[[\hbar]]\)-linear derivation on \(\hat{\text{Sym}}(M)[[\hbar]]\). Particularly, if the theory is constructed on a closed manifold, then the observable complex is expected to have the form

\[
(\hat{\text{Sym}}(M)[[\hbar]], d + \hbar \Delta + \{I, -\}) \tag{2.11}
\]

where \(I \in \hat{\text{Sym}}(M)[[\hbar]]\) is a degree 0 element satisfying the QME for \((\hat{\text{Sym}}(M), d, \Delta)\). We call \(I\) the “action functional” encoding the interaction. More precisely, we require

\[
I \in \hat{\text{Sym}}(M)[[\hbar]]^+ \tag{2.12}
\]

where

\[
\hat{\text{Sym}}(M)[[\hbar]]^+ := \left( \prod_{n \geq 3} \text{Sym}^n(M) + \hbar \hat{\text{Sym}}(M)[[\hbar]] \right) \subset \hat{\text{Sym}}(M)[[\hbar]]. \tag{2.13}
\]

This implies that, given the SDR \((2.7), \sum_{n=0}^{+\infty}((\hbar \Delta + \{I, -\})K_{\text{sym}})^n\) is well defined on \(\hat{\text{Sym}}(M)[[\hbar]]\). (This point can be checked by analyzing the \(\hbar\)-grading and symmetric tensor power grading, which is left as an exercise.) So \(\hbar \Delta + \{I, -\}\) is a small perturbation to \((2.7)\). To express the perturbation result we need to introduce another notation.

**Definition 2.4.1** Let \(G\) be a 2-to-0 operator of degree 0 on \(\hat{\text{Sym}}(M)\). We define the **first homotopic renormalization group (HRG) operator**

\[
W(G, -) : \hat{\text{Sym}}(M)[[\hbar]]^+ \mapsto \hat{\text{Sym}}(M)[[\hbar]]^+
\]

by this formal formula

\[
W(G, I) := \hbar \log(e^{\hbar G}e^I) \quad \text{for} \quad I \in \hat{\text{Sym}}(M)[[\hbar]]^+.
\]

The real content of this formula is a summation over connected Feynman graph expansion. We refer to [Cost11, Chapter 2] for details of this definition. (Note that the operator \(G\) here corresponds to “contracting with propagator” in the reference, so their notation differs a little from ours.)

Similarly, we define the **second HRG operator**

\[
W(G, -, -) : \hat{\text{Sym}}(M)[[\hbar]]^+ \times \hat{\text{Sym}}(M)[[\hbar]] \mapsto \hat{\text{Sym}}(M)[[\hbar]]
\]

by this formal formula

\[
W(G, I, f) := e^{-W(G,I)/\hbar}e^{\hbar G}(e^{I/h}f) \quad \text{for} \quad I \in \hat{\text{Sym}}(M)[[\hbar]]^+, f \in \hat{\text{Sym}}(M)[[\hbar]].
\]

Actually this is also a summation over connected Feynman graphs, with the restriction that each graph should contain exactly one vertex representing \(f\).
We impose another condition (only within the current subsection) between the BV operator $\Delta$ and the map $K$ in (2.1):
\[
\Delta((Km_1)m_2) = (-1)^{|m_1|}\Delta(m_1 Km_2) \quad \text{for } \forall m_1, m_2 \in M.
\] (2.14)
This implies
\[
(\Delta K_{\text{sym}})_2(m_1 Km_2) = 0 \quad \text{for } \forall m_1, m_2 \in M,
\] (2.15)
and
\[
(\Delta K_{\text{sym}})_2(m_1 im_2) = 0 \quad \text{for } \forall m_1, m_2 \in M.
\] (2.16)

**Remark 2.4.1** If the SDR (2.1) for classical linear observables comes from a Hodge decomposition of field space on closed manifold, (2.14) will be automatically satisfied. It is convenient to illustrate this point schematically by considering finite dimensional field space. We refer to [DJP19, Section 3.2] for Hodge decomposition of dg (degree $-1$) symplectic vector space and how it induces an SDR.

**Proposition 2.4.1** In above settings, we can write down the perturbation result of (2.7) by $\hbar \Delta + \{I, -\}$ as follows:
\[
(\widehat{\text{Sym}}(N)[[\hbar]], b_{\hbar}^{\text{int}}) \xlongleftarrow{\text{b}_{\hbar}^{\text{int}} \text{_{pert}}} (\widehat{\text{Sym}}(M)[[\hbar]], d + h\Delta + \{I, -\}), K_h^{\text{int}}
\] (2.17)
where
\[
\begin{align*}
\beta_{\hbar}^{\text{int}} &= b + h\beta_{\Delta i} + \{I_{\text{eff}}, -\}_{p\Delta i} \\
\gamma_{\hbar}^{\text{int}} &= p\mathcal{W}(\Delta K_{\text{sym}})_2, I, -,
\end{align*}
\]
with $\{-, -, \}_{p\Delta i}$ being the BV bracket on $\widehat{\text{Sym}}(N)$ induced by $p\Delta i$, and
\[
I_{\text{eff}} := p\mathcal{W}(\Delta K_{\text{sym}})_2, I) \in \widehat{\text{Sym}}(N)[[\hbar]]^+.
\]

**Proof** By associativity of perturbation theory (Lemma 2.1.2), we can use $\{I, -\}$ to perturb (2.8):
\[
(\widehat{\text{Sym}}(N)[[\hbar]], b + h\beta_{\Delta i}) \xlongleftarrow{\text{b}_{\hbar}^{\text{int}} \text{_{pert}}} (\widehat{\text{Sym}}(M)[[\hbar]], d + h\Delta, K_h)
\]
where
\[
p_h = pe^{h(\Delta K_{\text{sym}})_2}
\]
to find out the desired result. (Lemma 2.1.2 contains the fact that this perturbation is also small.)

Recall we have a formal conjugation
\[
d + h\Delta + \{I, -\} = e^{-I/\hbar}(d + h\Delta)e^{I/\hbar}
\]
Although $U := e^{-I/\hbar}$ is not a well-defined operator on $\widehat{\text{Sym}}(M)[[\hbar]]$, we can still go through the following formal calculation:
\[
p_h U^{-1} K_{\text{sym}} = pe^{h(\Delta K_{\text{sym}})_2} e^{I/\hbar} K_{\text{sym}} = 0.
\]
This is a consequence of (2.15). Also by (2.16) we have

\[ p\hbar U^{-1}i = pe^{h(\Delta K_{\text{sym}})2}e^{I/h}i \]
\[ = (pe^{h(\Delta K_{\text{sym}})2}e^{I/h}) \]
\[ = (pe^{W((\Delta K_{\text{sym}})2),I)/h}) \]
\[ = e^{I_{\text{eff}}/h}. \]

So, formally we have verified statement B in Proposition 2.1.3

\[ p\hbar U^{-1}K_h = 0, \quad (p\hbar U^{-1}i)^{-1} = e^{-I_{\text{eff}}/h}, \]

and we can formally write down the perturbed projection predicted by Proposition 2.1.3

\[(p\hbar U^{-1}i)^{-1}p\hbar U^{-1} = e^{-I_{\text{eff}}/h}pe^{h(\Delta K_{\text{sym}})2}e^{I/h} \]
\[ = (pe^{-W((\Delta K_{\text{sym}})2),I)/h})pe^{h(\Delta K_{\text{sym}})2}e^{I/h} \]
\[ = pe^{-W((\Delta K_{\text{sym}})2),I)/h}e^{h(\Delta K_{\text{sym}})2}e^{I/h} \]
\[ = pW((\Delta K_{\text{sym}})2,I,--) \]

which is exactly \( p^\text{int}_h \).

Since (formally) we have

\[ (b + \hbar p\Delta i)e^{I_{\text{eff}}/h} = (b + \hbar p\Delta i)pe^{I/h} = p_h(d + \hbar \Delta)e^{I/h} = 0, \]

we can formally write down the perturbed differential on \( \tilde{\text{Sym}}(N) [[\hbar]] \) predicted by Proposition 2.1.3

\[ e^{-I_{\text{eff}}/h}b_h e^{I_{\text{eff}}/h} = b + \hbar p\Delta i + \{ I_{\text{eff}}, - \} p\Delta i \]

which is exactly \( b^\text{int}_h \).

In the current context, all the above formal arguments make (the proof of) Proposition 2.1.3 really works, because we can handle the intermediate calculations by involving \( \prod_{n \geq 0} (\text{Sym}^n(M)((\hbar))) \) and \( \prod_{n \geq 0} (\text{Sym}^n(N)((\hbar))) \). This completes the proof.

So we obtain an effective interactive theory, encoded by an effective action functional \( I_{\text{eff}} \) which is a solution to QME of the effective free theory \( (\tilde{\text{Sym}}(N), b, p\Delta i) \). We also obtain a concrete formula for the projection map \( p^\text{int}_h \) from the observable complex of the renormalized theory to that of the effective theory. Feynman graph calculations are packaged in the formulae for \( I_{\text{eff}} \) and \( p^\text{int}_h \), supporting our language as a substitute for physicists’ (perturbative) path integral story.

Remark 2.4.2 As mentioned in the introduction, our terminology “effective” differs from that in [Cos11]. The effective action \( I_{\text{eff}} \) here is the “restriction of scale \( \infty \) effective action to harmonic fields” there (see [Cos11, Proposition 10.7.2] for details).

3 TQM on \( \mathbb{R}_{\geq 0} \): the Renormalized Theory

We have seen that, for a free QFT or an interactive QFT on closed manifold, the effective theory can be described using the same kind of algebraic structure of the renormalized theory. For an interactive
QFT on non-compact manifold, we can still construct an observable complex that looks like (2.11), but now

\[ I \notin \hat{\text{Sym}}(M)[[\hbar]], \quad (3.1) \]
i.e., the action functional is not an observable (because integration might be divergent on a non-compact manifold). Actually this is a very common case, because a QFT on a closed manifold can be restricted to any open subset of this manifold (see [Cos11, Section 2.14] and [CG21, Section 8.7] for details).

On a manifold \( X \) with boundary \( \partial X \), if there is a QFT constructed using our current formulation, we can then restrict it to a (small) tubular neighborhood \( T \simeq [0, \varepsilon) \times \partial X \) of the boundary \( \partial X \). Then, an effective observable complex of this QFT on \( T \) might be regarded as a system on \( \partial X \). Because of (3.1), if we calculate this complex by homological perturbation, the argument leading to Proposition 2.4.1 fails, and the result can exceed the scope of BV formalism. Certainly we are interested in such results.

With the presence of spacetime boundary, renormalization has not been systematically developed yet in general. If the spacetime is \( \mathbb{H}^n \) equipped with the Euclidean metric, discussion of heat kernel renormalization can be found in [Alb16]. Later, Eugene Rabinovich formulated the renormalized theories and factorization algebras for field theories which are “topological normal to the boundary” [Rab21]. If we restrict such a theory on \( X \) to a tubular neighborhood of \( \partial X \), the result will be equivalent to a QFT on \( \mathbb{R}_{\geq 0} \times \partial X \). If \( \partial X \) is a point, this is a topological quantum mechanics, which is the main case we study in this work.

In this section, relevant constructions will be extracted from [Rab21], making up the renormalized theories. Effective theories will be calculated in the next section.

**Content of field space**

Let \( (V, Q^0) \) be a cochain complex of finite dimensional vector space. By Leibniz rule, the differential \( Q^0 \) induces differentials on various tensors of \( V, V^* \), still denoted by \( Q^0 \). Let

\[ \omega^0 \in \Lambda^2(V^*), \quad \text{s.t.} \quad Q^0 \omega^0 = 0 \quad (3.2) \]

be a degree 0 symplectic pairing on \( V \) compatible with \( Q^0 \). Let \( L, L' \) be two Lagrangian subspaces of \( V \) satisfying

\[ V = L \oplus L', \quad Q^0(L) \subseteq L, \quad Q^0(L') \subseteq L'. \quad (3.3) \]

For \( v \in V \), the map \( v \mapsto \omega^0(v, -) \) induces a vector space isomorphism \( V \simeq V^* \), hence also a vector space isomorphism \( \Lambda^2 V \simeq \Lambda^2(V^*) \). Let

\[ K^0 \in \Lambda^2(V) \]

be the image of \( \omega^0 \) under \( \Lambda^2(V^*) \simeq \Lambda^2(V) \). If we regard \( K^0 \) as an element in

\[ V^\otimes 2 = (L \otimes L) \oplus (L' \otimes L') \oplus (L \otimes L') \oplus (L' \otimes L), \]

we can write

\[ K^0 = K^0_0 + K^0_+, \quad K^0_0 \in L \otimes L', K^0_+ \in L' \otimes L, \quad \text{s.t.} \quad \sigma K^0_- = -K^0_+, \quad (3.4) \]
where $\sigma$ permutes the two factors of $V \otimes^2$. By (3.2),

$$Q^0 K_0^0 = Q^0 K_+^0 = 0. \quad (3.5)$$

We fix the spacetime manifold to be $\mathbb{R}_{\geq 0}$. Let $\iota^* : \Omega^\bullet(\mathbb{R}_{\geq 0}) \otimes V \mapsto V$ denote the pullback of $V$-valued forms induced by the inclusion map $\iota : \{0\} \hookrightarrow \mathbb{R}_{\geq 0}$. The field space of our theory is

$$\mathcal{E}_L := \{ f \in \Omega^\bullet(\mathbb{R}_{\geq 0}) \otimes V | \iota^* f \in L \}. \quad (3.6)$$

Note that we have chosen a boundary condition in above definition. The set of classical linear observables is the dual space $\mathcal{E}_L^*$ (we have mentioned the meaning of dual in the convention part), and

$$\mathcal{O}(\mathcal{E}_L) := \hat{\text{Sym}}(\mathcal{E}_L^*)$$

is the set of all classical observables. By construction we have a surjection

$$\hat{\text{Sym}}((\Omega^\bullet(\mathbb{R}_{\geq 0}) \otimes V)^*) \mapsto \mathcal{O}(\mathcal{E}_L).$$

There is a differential $Q := d + Q^\partial$ on $\mathcal{E}_L$, where $d$ is the de Rham differential on $\Omega^\bullet(\mathbb{R}_{\geq 0})$. $Q$ induces differentials on $\mathcal{E}_L^*$ and $\mathcal{O}(\mathcal{E}_L)$, still denoted by $Q$.

There is a subcomplex of $(\mathcal{E}_L, Q)$

$$(\mathcal{E}_{L,c} := \{ f \in \mathcal{E}_L | f \text{ is compactly supported} \}, Q). \quad (3.7)$$

$\omega^\partial$ can be $\Omega^\bullet(\mathbb{R}_{\geq 0})$-linearly extended to a map $\mathcal{E}_L \times \mathcal{E}_L \mapsto \Omega^\bullet(\mathbb{R}_{\geq 0})$. (The wedge product on $\Omega^\bullet(\mathbb{R}_{\geq 0})$ is implicitly used.) Then, we have a degree $-1$ pairing:

$$\int_{\mathbb{R}_{\geq 0}} \omega^\partial : \mathcal{E}_{L,c} \times \mathcal{E}_L \mapsto \mathbb{R} \quad (f, g) \mapsto \int_{\mathbb{R}_{\geq 0}} \omega^\partial(f, g).$$

It induces an embedding of cochain complex

$$(\mathcal{E}_{L,c}[1], -Q) \hookrightarrow (\mathcal{E}_L^*, Q) \quad \eta f \mapsto \int_{\mathbb{R}_{\geq 0}} \omega^\partial(f, -), \quad (3.8)$$

where $\eta$ is a formal variable of degree $-1$, and we use $\eta f$ to represent the element in $\mathcal{E}_{L,c}[1]$ corresponding to $f \in \mathcal{E}_{L,c}$. This embedding is actually a quasi-isomorphism (see [Rab21] Appendix A.3 for details). So, we have a set of “smeared observables” quasi-isomorphic to $\mathcal{O}(\mathcal{E}_L)$:

$$\mathcal{O}_{\text{sm}}(\mathcal{E}_L) := \hat{\text{Sym}}(\mathcal{E}_{L,c}[1]).$$

Let $\Delta_0$ be the 2-to-0 operator on $\mathcal{O}_{\text{sm}}(\mathcal{E}_L)$ decided by

$$\Delta_0(\eta f_1 \eta f_2) = \int_{\mathbb{R}_{\geq 0}} \omega^\partial((-1)^{|f_1|} f_1, f_2).$$

Then it is easy to verify:

**Proposition 3.0.1** $(\mathcal{O}_{\text{sm}}(\mathcal{E}_L), -Q, \Delta_0)$ is a differential BV algebra.
Free quantum observable complex

In order to incorporate interaction, we have to define BV structure directly on \( O(\mathcal{E}_L) \). For simplicity, we will not mention the “doubling trick” method in [Rab21 Chapter 4]. Expressions for the BV operator and propagator here should be equivalent to those in [Rab21 Chapter 5.1].

We fix a metric on \( \mathbb{R}_{>0} \) by \( \langle \partial_x, \partial_x \rangle = 1 \), where \( x \) is the coordinate. Let \( d^{GF} \) denotes the Hodge dual to the de Rham operator induced by this metric:

\[
d^{GF}(f \, dx) = -\partial_x f \quad \text{for} \; f \in \Omega^0(\mathbb{R}_{>0}).
\]

The heat kernels, being smooth functions on \( \mathbb{R}_{>0} \times (\mathbb{R}_{>0} \times \mathbb{R}_{>0}) \), are the following:

\[
H_D(t, x, y) = \frac{1}{\sqrt{4\pi t}} \left( e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}} \right), \quad H_N(t, x, y) = \frac{1}{\sqrt{4\pi t}} \left( e^{-\frac{(x-y)^2}{4t}} + e^{-\frac{(x+y)^2}{4t}} \right),
\]

where the indice “\( D \)” means Dirichlet boundary condition, and indice “\( N \)” means Neumann boundary condition. Let \( H_t := H_D(t, x, y) \, dy - H_N(t, x, y) \, dx \), then it is direct to verify

\[
(1 \otimes d + d \otimes 1) \, H_t = 0 \tag{3.9}
\]

\[
-\frac{1}{2} (1 \otimes d + d \otimes 1) \left( d^{GF} \otimes 1 + 1 \otimes d^{GF} \right) H_t = \partial_t H_t.
\]

So, for \( \forall \varepsilon, \Lambda \in \mathbb{R}_{>0} \),

\[
H_\Lambda = H_\varepsilon - \frac{1}{2} (1 \otimes d + d \otimes 1) \left( d^{GF} \otimes 1 + 1 \otimes d^{GF} \right) \int_\varepsilon^\Lambda dt \, H_t. \tag{3.10}
\]

Let \( \phi \in C^\infty(\mathbb{R}) \) be a compactly supported even function which evaluates to 1 in a neighborhood of \( \{0\} \). Define

\[
\widetilde{H}_t := \frac{1}{\sqrt{4\pi t}} \left( \phi(x-y)e^{-\frac{(x-y)^2}{4t}}(dy - dx) - \phi(x+y)e^{-\frac{(x+y)^2}{4t}}(dy + dx) \right),
\]

then

\[
\widetilde{H}_t - H_t = \frac{1}{\sqrt{4\pi t}} \left( (\phi(x-y) - 1)e^{-\frac{(x-y)^2}{4t}}(dy - dx) - (\phi(x+y) - 1)e^{-\frac{(x+y)^2}{4t}}(dy + dx) \right)
\]

can be extended to a smooth form on \( \mathbb{R}_{>0} \times (\mathbb{R}_{>0} \times \mathbb{R}_{>0}) \), vanishing at \( t = 0 \). For \( t > 0 \), define the BV kernel to be

\[
K_t := -\frac{1}{2} \left( H_t - \frac{1}{2} (1 \otimes d + d \otimes 1) \left( d^{GF} \otimes 1 + 1 \otimes d^{GF} \right) \int_0^t ds (\widetilde{H}_s - H_s) \right) \otimes K^0_+ + \frac{1}{2} \sigma \left( H_t - \frac{1}{2} (1 \otimes d + d \otimes 1) \left( d^{GF} \otimes 1 + 1 \otimes d^{GF} \right) \int_0^t ds (\widetilde{H}_s - H_s) \right) \otimes K^0_- \tag{3.11}
\]

where the \( \sigma \) here permutes variables \( x \) and \( y \), \( K^0_+, K^0_- \) are defined in (3.4). By a little bit calculus, we can verify that \( K_t \in \text{Sym}^2(\mathcal{E}_L) \), if the tensor factors are properly rearranged. Due to (3.5), (3.9), \( K_t \) is \( Q \)-closed. Define the propagator to be

\[
P(\varepsilon, \Lambda) := \left( -\frac{1}{4} \left( d^{GF} \otimes 1 + 1 \otimes d^{GF} \right) \int_\varepsilon^\Lambda dt \, \widetilde{H}_t \right) \otimes K^0_+ + \sigma \left( \frac{1}{4} \left( d^{GF} \otimes 1 + 1 \otimes d^{GF} \right) \int_\varepsilon^\Lambda dt \, \widetilde{H}_t \right) \otimes K^0_- \tag{3.12}
\]
where \( \varepsilon, \Lambda \in \mathbb{R}_{>0} \). Similarly, \( P(\varepsilon, \Lambda) \in \text{Sym}^2(\mathcal{E}_L) \). By (3.10), we have

\[
K_\Lambda = K_\varepsilon + (1 \otimes d + d \otimes 1) P(\varepsilon, \Lambda).
\] (3.13)

For \( \forall E \in \text{Sym}^2(\mathcal{E}_L) \), let \( \partial_E \) denote the 2-to-0 operator on \( \mathcal{O}(\mathcal{E}_L) \) decided by

\[
\partial_E(a_1 a_2) = (-1)^{|E||a_1| + |a_2|} a_1 a_2(E) \quad \text{for } a_1, a_2 \in \mathcal{E}_L^*.
\]

It is direct to verify:

**Proposition 3.0.2** Denote \( \partial_{K_t} \) by \( \Delta_t \), then,

- \( (\mathcal{O}(\mathcal{E}_L), Q, \Delta_t) \) is a differential BV algebra for \( \forall t > 0 \).
- The embedding \( \mathcal{O}_e(\mathcal{E}_L) \hookrightarrow \mathcal{O}(\mathcal{E}_L) \) induced by (3.5) makes \( (\mathcal{O}_e(\mathcal{E}_L), -Q, \Delta_t) \) also a differential BV algebra. Moreover, the \( t \to 0 \) limit of \( \Delta_t \) exists on \( \mathcal{O}_e(\mathcal{E}_L) \), and equals to \( \Delta_0 \) in Proposition 3.0.1.
- For \( \forall \varepsilon, \Lambda > 0 \), \( \Delta_\Lambda = \Delta_\varepsilon + [\partial_{P(\varepsilon, \Lambda)}, Q] \). Equivalently, there is such a conjugation of cochain complexes

\[
\begin{align*}
\epsilon^{b \partial_{P(\varepsilon, \Lambda)}} (\mathcal{O}(\mathcal{E}_L)[[\hbar]], Q + \hbar \Delta_\varepsilon) &\quad \equiv \quad \epsilon^{-b \partial_{P(\varepsilon, \Lambda)}} (\mathcal{O}(\mathcal{E}_L)[[\hbar]], Q + \hbar \Delta_\Lambda).
\end{align*}
\]

**Remark 3.0.1** Regarded as forms on \( \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \), \( K_t \) and \( P(\varepsilon, \Lambda) \) have proper supports. Namely, each of the projection maps \( \pi_1, \pi_2 : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) is proper when restricted to \( \text{supp}(K_t) \) or \( \text{supp}(P(\varepsilon, \Lambda)) \). For \( P(\varepsilon, \Lambda) \) this is by construction, because \( \text{supp}(\tilde{H}_t) \) is bounded by a proper neighborhood of the diagonal, uniformly for all \( t \). For \( K_t \), if we take \( \varepsilon \to 0 \) in (3.13):

\[
K_t = K_0 + (1 \otimes d + d \otimes 1) P(0, t),
\]

we obtain a relation in the space of distributions. Since \( K_0, P(0, t) \) have proper supports, so does \( K_t \).

**Interaction**

We have introduced the functional space \( \mathcal{O}(\mathcal{E}_L) \), and obtained a family of free observable complexes \( \{ (\mathcal{O}(\mathcal{E}_L)[[\hbar]], Q + \hbar \Delta_t) \mid t \in \mathbb{R}_{>0} \} \). We can also write down functionals on the space \( \mathcal{E}_{L,c} \):

\[
\mathcal{O}(\mathcal{E}_{L,c}) := \overline{\text{Sym}}(\mathcal{E}_{L,c}^*).
\]

Then, there is a natural inclusion \( \mathcal{O}(\mathcal{E}_L) \hookrightarrow \mathcal{O}(\mathcal{E}_{L,c}) \). A functional \( f \in \text{Sym}^n(\mathcal{E}_{L,c}^*) \) with \( n > 0 \) belongs to \( \mathcal{O}(\mathcal{E}_L) \) if \( f \) has compact support. Here we regard the input of \( f \) as forms on \( \mathbb{R}_{\geq 0}^n \). The support of \( f \) means \( \mathbb{R}_{\geq 0}^n \setminus \mathcal{U} \), where \( \mathcal{U} \) is the union of all open sets \( U \subset \mathbb{R}_{\geq 0}^n \) satisfying \( f(e) = 0 \) whenever \( e \) has support on a compact subset of \( U \). For \( n > 0 \), define

\[
\mathcal{O}^n_p(\mathcal{E}_{L,c}) := \{ f \in \text{Sym}^n(\mathcal{E}_{L,c}) \mid \text{supp}(f) \text{ is proper} \},
\]

where \( \text{supp}(f) \) is proper if each of the \( n \) projection maps \( \mathbb{R}_{\geq 0}^n \to \mathbb{R}_{\geq 0} \) is proper when restricted to \( \text{supp}(f) \). Since a compact support is also proper, \( \text{Sym}^n(\mathcal{E}_{L,c}^*) \subset \mathcal{O}^n_p(\mathcal{E}_{L,c}) \). \( \mathcal{O}(\mathcal{E}_{L,c}) \) has a subspace

\[
\mathcal{O}^n_{p>0}(\mathcal{E}_{L,c}) := \prod_{n>0} \mathcal{O}^n_p(\mathcal{E}_{L,c})
\]

which is not a subring: for \( f_1, f_2 \in \mathcal{O}^n_{p>0}(\mathcal{E}_{L,c}) \), \( f_1 f_2 \in \mathcal{O}(\mathcal{E}_{L,c}) \) may not have proper support.
Remark 3.0.2  Due to Remark 3.0.1, we have the following facts:

- \( \Delta_t, \partial_{P(\epsilon,\Lambda)} \) are well defined on \( \mathcal{O}_P^\geq(\mathcal{E}_{L,c}) \).

- The BV bracket induced by \( \Delta_t \):
  \[
  \{-,-\}_t : \mathcal{O}(\mathcal{E}_L) \otimes \mathcal{O}(\mathcal{E}_L) \mapsto \mathcal{O}(\mathcal{E}_L)
  \]
  can be extended to a map \( \{-,-\}_t : \mathcal{O}_P^\geq(\mathcal{E}_{L,c}) \otimes \mathcal{O}(\mathcal{E}_L) \mapsto \mathcal{O}(\mathcal{E}_L) \). This is a derivation with respect to \( \mathcal{O}(\mathcal{E}_L) \). We can further extend \( \{-,-\}_t \) to
  \[
  \{-,-\}_t : \mathcal{O}_P^\geq(\mathcal{E}_{L,c}) \otimes \mathcal{O}_P^\geq(\mathcal{E}_{L,c}) \mapsto \mathcal{O}_P^\geq(\mathcal{E}_{L,c}),
  \]
  which makes \( \mathcal{O}_P^\geq(\mathcal{E}_{L,c})[-1] \) a graded Lie algebra.

(The reasoning behind these facts is similar to the proof of [Cos11, Chapter 2, Lemma 14.5.1].)

So, although \( \mathcal{O}_P^\geq(\mathcal{E}_{L,c}) \) is not an graded algebra, the QME “at scale \( t \)"
\[
QI + \hbar \Delta_t I + \frac{1}{2} \{I, I\}_t = 0
\]
is well defined for degree 0 elements in \( \mathcal{O}_P^\geq(\mathcal{E}_{L,c})[[\hbar]] \). If \( I \) is a solution, \( Q + \hbar \Delta_t + \{I, -\}_t \) will be a differential on \( \mathcal{O}(\mathcal{E}_L)[[\hbar]] \), and we can formally write
\[
Q + \hbar \Delta_t + \{I, -\}_t = e^{-I/\hbar}(Q + \hbar \Delta_t)e^{I/\hbar}.
\]

By definition (3.6), we have a decomposition for the field space
\[
\mathcal{E}_L = (\Omega^\bullet(\mathbb{R}_\geq 0) \otimes L) \oplus \{f \in \Omega^\bullet(\mathbb{R}_\geq 0) \otimes L' | \iota^* f = 0 \}.
\]
Let \( \mathcal{L}, \mathcal{L}' \) denote the first and the second component, respectively. Define
\[
\mathcal{O}_P(\mathcal{E}_{L,c})[[\hbar]]^+ := \left( \mathcal{O}_P(\mathcal{L}_c)^2 + \prod_{n \geq 3} \mathcal{O}_P(\mathcal{E}_{L,c}) + \hbar \mathcal{O}_P^\geq(\mathcal{E}_{L,c})[[\hbar]] \right) \subset \mathcal{O}_P^\geq(\mathcal{E}_{L,c})[[\hbar]],
\]
where \( \mathcal{O}_P(\mathcal{L}_c) \subset \mathcal{O}_P^\geq(\mathcal{E}_{L,c}) \) are functionals only depending on \( \mathcal{L}' \). Such a quadratic component at \( \hbar^0 \)-order is not included in (2.13) to ensure well-defined perturbation, but the presence here will cause no problem. The Definition-Lemma 4.4.6 in [Rab21] tells us these facts:

- The first HRG operator is well defined:
  \[
  \mathcal{W}(\partial_{P(\epsilon,\Lambda)}, -) : \mathcal{O}_P(\mathcal{E}_{L,c})[[\hbar]]^+ \mapsto \mathcal{O}_P(\mathcal{E}_{L,c})[[\hbar]]^+,
  \]
  formally, for \( I \in \mathcal{O}_P(\mathcal{E}_{L,c})[[\hbar]]^+ \),
  \[
  \mathcal{W}(\partial_{P(\epsilon,\Lambda)}, I) = \hbar \log(e^{\hbar \partial_{P(\epsilon,\Lambda)}} e^{I/\hbar}) - \hbar \log(e^{\hbar \partial_{P(\epsilon,\Lambda)}} e^{I/\hbar}))(0),
  \]
  where the second term is the (ill-defined) constant part of the first term.

This operator has Feynman graph summation formula modified from that in Definition 2.4.1 by discarding graphs without external leg.
The second HRG operator is well defined:

$$\mathcal{W}(\partial_{P(\varepsilon, \Lambda)}, - , -) : \mathcal{O}_P(\mathcal{E}_{L,c})[[\hbar]]^+ \times \mathcal{O}(\mathcal{E}_L)[[\hbar]] \mapsto \mathcal{O}(\mathcal{E}_L)[[\hbar]],$$

formally, for $I \in \mathcal{O}_P(\mathcal{E}_{L,c})[[\hbar]]^+$,

$$\mathcal{W}(\partial_{P(\varepsilon, \Lambda)}, I, -) = (e^{\hbar \partial_{P(\varepsilon, \Lambda)}} e^{I/\hbar})^{-1} e^{\hbar \partial_{P(\varepsilon, \Lambda)}} e^{I/\hbar}(-).$$

(3.17)

Its Feynman graph summation formula is the same with that in Definition 2.3.1.

The HRG operators satisfy certain associativity. For $\Lambda_1, \Lambda_2, \Lambda_3 > 0, I \in \mathcal{O}_P(\mathcal{E}_{L,c})[[\hbar]]^+$,

$$\mathcal{W}(\partial_{P(\Lambda_1, \Lambda_3)}, I) = \mathcal{W}(\partial_{P(\Lambda_2, \Lambda_3)}, \mathcal{W}(\partial_{P(\Lambda_1, \Lambda_2)}, I)),$$

$$\mathcal{W}(\partial_{P(\Lambda_1, \Lambda_3)}, I, -) = \mathcal{W}(\partial_{P(\Lambda_2, \Lambda_3)}, \mathcal{W}(\partial_{P(\Lambda_1, \Lambda_2)}, I), \mathcal{W}(\partial_{P(\Lambda_1, \Lambda_2)}, I, -)).$$

(3.18) (3.19)

Lemma 3.0.1 Given a degree 0 functional $I_\varepsilon \in \mathcal{O}_P(\mathcal{E}_{L,c})[[\hbar]]^+$ satisfying the QME at a certain scale $\varepsilon > 0$:

$$Q I_\varepsilon + \hbar \Delta_\varepsilon I_\varepsilon + \frac{1}{2} \{I_\varepsilon, I_\varepsilon\}_\varepsilon = 0,$$

let $I_\Lambda$ denote $\mathcal{W}(\partial_{P(\varepsilon, \Lambda)}, I_\varepsilon)$ for another scale $\Lambda > 0$. Then, $I_\Lambda$ satisfies the QME at scale $\Lambda$. In addition, we have a conjugation of cochain complexes

$$\mathcal{W}(\partial_{P(\varepsilon, \Lambda)}, I_\varepsilon, -) = (\mathcal{O}(\mathcal{E}_L)[[\hbar]], Q + \hbar \Delta_\varepsilon + \{I_\varepsilon, -\}_\varepsilon) \quad = \quad (\mathcal{O}(\mathcal{E}_L)[[\hbar]], Q + \hbar \Delta_\Lambda + \{I_\Lambda, -\}_\Lambda).$$

(3.20)

The above properties are parallel with those for HRG operators in [Cos11, CG21], and can be proved similarly using the formal formulae (3.14), (3.16) and (3.17). While using these formal expressions, we need to involve $\prod_{n \geq 0} (\text{Sym}^n(\mathcal{E}_{L,c}^*))(\hbar)$ and discard ill-defined constant factors to make sense of intermediate calculations. We leave the details to the interested reader.

Given a function $f \in \text{Sym}^n(V^*)$ with $n > 0$, we can extend it $\Omega^*(\mathbb{R}_{\geq 0})$-linearly and then integrate over $\mathbb{R}_{\geq 0}$, thus obtain a functional in $\text{Sym}^n(\mathcal{E}_{L,c}^*)$. Denote this functional by $\rho(f)$, it is supported on the diagonal of $\mathbb{R}_{\geq 0}^n$. In this way we have a degree $-1$ map:

$$\rho : \text{Sym}^2((L')^*) \oplus \prod_{n \geq 3} \text{Sym}^n(V^*) \oplus \hbar \prod_{n > 0} \text{Sym}^n(V^*)[[\hbar]] \mapsto \mathcal{O}_P(\mathcal{E}_{L,c})[[\hbar]]^+$$

(3.21)

Now, we are ready to define the concrete theory to study.

Definition 3.0.1 A UV finite topological quantum mechanics (TQM) on $\mathbb{R}_{\geq 0}$ with only bulk interaction consists of the following:

- a family of differential BV algebras $\{(\mathcal{O}(\mathcal{E}_L), Q, \Delta_t)| t \in \mathbb{R}_{\geq 0}\}$ stated in Proposition 3.0.2.
- a “scale 0” action functional $\rho(I^0) \in \mathcal{O}_P(\mathcal{E}_{L,c})[[\hbar]]^+$, where $I^0 \in \widehat{\text{Sym}}(V^*)[[\hbar]]$ has degree 1, $\rho$ is the map in (3.21).
They are required to satisfy these conditions:

- (UV finiteness) For a given scale \( t > 0 \) (hence for all scales), the following limit exists:

\[
I_t := \lim_{\varepsilon \to 0} \mathcal{W}(\partial P(\varepsilon, t), \rho(I^0)).
\]

- This limit \( I_t \) is a solution to the QME at scale \( t \).

**Remark 3.0.3** For simplicity, we impose the UV finiteness by hand. Detailed analysis may disclose that this condition is automatically satisfied, as in the case of TQM on \( S^1 \) (see [GLL17]). We leave this consideration for later work.

**Example 3.0.1** Given \( (V = L \oplus L', Q^0, \omega^0) \) as in (3.2) and (3.3), let \( Q^{rel} : L' \mapsto L \) be a degree 1 map, such that

\[
Q^{rel}Q^0 + Q^0Q^{rel} = 0, \quad \text{and} \quad \omega^0(Q^{rel}a, b) + (-1)^{|a|}\omega^0(a, Q^{rel}b) = 0
\]

for \( a, b \in V \). Consider the degree \(-1\) function

\[
I^0 : V \otimes V \mapsto \mathbb{R}, \quad I^0(a, b) := -\omega^0(Q^{rel}a, b) \quad \text{for} \ a, b \in V,
\]

it is direct to verify that actually \( I^0 \in \text{Sym}^2((L')^*) \subset (V^*)^\otimes 2 \).

Recall we have a decomposition \( E_L = L \oplus L' \) in (3.15). The operators \( \partial P(\varepsilon, t), \Delta_t, \{-, -\}_t \) have a common feature: they always pair a linear functional factor depending on \( L \) with a linear functional factor depending on \( L' \). The functional

\[
\rho(I^0) = -\int_{\mathbb{R}_{>0}} \omega^0(Q^{rel}, -, -) \in \mathcal{O}_P^2(L'_\varepsilon)
\]

only depends on \( L' \), hence satisfies

\[
\partial P(\varepsilon, t)\rho(I^0) = 0, \quad \Delta_t\rho(I^0) = 0, \quad \{\rho(I^0), \rho(I^0)\}_t = 0
\]

for \( \forall \varepsilon, t > 0 \). It is also easy to verify

\[
d\rho(I^0) = Q^0\rho(I^0) = 0.
\]

So, \( I_t := \lim_{\varepsilon \to 0} \mathcal{W}(\partial P(\varepsilon, t), \rho(I^0)) = \rho(I^0) \), and \( I_t \) satisfies the QME at scale \( t \), thus defines a theory in our sense.

**Example 3.0.2** “BF theory with B boundary condition”

Let \( g \) be a Lie algebra with basis \( \{t^a\}_{a=1}^\ell, [t^a, t^b] = f_{bc}^{ab}t^c \). We impose the unimodular condition

\[
f^{cb}_{c} = 0. \quad \text{(3.22)}
\]

(We have used the Einstein summation convention.) Let

\[
L := (g[1])^* = (g^*)[-1], \quad L' := g[1].
\]
For $\beta \in g^*$, we use $\epsilon \beta$ to denote the element in $L$ corresponding to $\beta$, where $\epsilon$ is a formal variable of degree 1; similarly for $\alpha \in g$ we have $\eta \alpha \in L'$ where $\eta$ is a formal variable of degree $-1$. There is a degree 0 symplectic pairing $\omega^0$ on $V = L \oplus L'$, decided by
\[
\omega^0(\epsilon t_a, \eta t_b) := \delta_a^b
\]
where $\{t_a\}_{a=1}^\ell$ is the basis of $g^*$ dual to $\{t^a\}_{a=1}^\ell$. Then, $(V, Q^0 := 0, \omega^0)$ satisfies (3.2) and (3.3).

We define $I^0 \in \text{Sym}^3(V^*) \subset (V \otimes \mathbb{R})^*$ to be the function decided by
\[
I^0(\epsilon \beta, \eta \alpha_1, \eta \alpha_2) := -\beta([\alpha_1, \alpha_2]), \quad \text{for } \epsilon \beta \in L, \eta \alpha_1, \eta \alpha_2 \in L'.
\]
Let $\{B^a\}_{a=1}^\ell$ be the basis of $L^*$ and $\{A_a\}_{a=1}^\ell$ be the basis of $(L')^*$ so that
\[
B^a(\epsilon t_b) = A_b(\eta t^a) = \delta^a_b.
\]
Then we can write
\[
I^0 = \frac{1}{2} f_{abc} B^c A_a A_b.
\]
So, $\rho(I^0) = \frac{1}{2} \int_{\mathbb{R}_{\geq 0}} f_{abc} B^c \wedge A_a \wedge A_b$ is the interaction action functional of BF theory on $\mathbb{R}_{\geq 0}$. The boundary condition we choose is the so-called B condition. This data indeed fits into Definition 3.0.1 and we refer to [Rab21, Section 5.2] for the proof.

Example 3.0.3 Let $\mu_1, \mu_2, \nu_1, \nu_2$ be formal variables of degree 0, 1, 0, $-1$, respectively. Define
\[
L := \mathbb{R}\mu_1 \oplus \mathbb{R}\mu_2, \quad L' := \mathbb{R}\nu_1 \oplus \mathbb{R}\nu_2.
\]
There is a degree 0 symplectic pairing $\omega^0$ on $V := L \oplus L'$, decided by
\[
\omega^0(\mu_i, \nu_j) = \delta_{ij}, \quad \omega^0(\nu_i, \nu_j) = \omega^0(\mu_i, \mu_j) = 0, \quad i, j \in \{1, 2\}.
\]
Denote the dual space of $L, L'$ by
\[
L^* = \mathbb{R}q_1 \oplus \mathbb{R}q_2, \quad (L')^* := \mathbb{R}p_1 \oplus \mathbb{R}p_2
\]
with $q_i(\mu_j) = p_i(\nu_j) = \delta_{ij}$ for $i, j \in \{1, 2\}$. Then, $(V, Q^0 := 0, \omega^0)$ satisfies (3.2) and (3.3). Let
\[
I^0 := -p_1^2 p_2,
\]
then $\rho(I^0) = -\int_{\mathbb{R}_{\geq 0}} p_1 \wedge p_1 \wedge p_2$ is a degree 0 functional depending only on $L'$. For the same reason as in Example 3.0.1
\[
I_t := \lim_{\epsilon \to 0} \mathcal{W}(\partial P(\epsilon, t), \rho(I^0)) = \rho(I^0)
\]
satisfies the QME at scale $t$. So we have defined a theory.
4 TQM on $\mathbb{R}_{\geq 0}$: the Effective Theory

4.1 Homological perturbation construction

Given a TQM as in Definition 3.0.1, we have a family of quantum observable complexes

$$\{(\mathcal{O}(\mathcal{E}_L)[[\hbar]], Q + \hbar \Delta_t + \{I_t, -\})| t \in \mathbb{R}_{>0}\}$$

(4.1)

with $I_t = \lim_{\varepsilon \to 0} W(\partial P(\varepsilon, t), \rho(I^0))$ satisfying the QME at scale $t$. These cochain complexes at different scales are related by conjugations (3.20). To construct an effective observable complex by homological perturbation, we need to start with an SDR whose RHS can be perturbed to elements of (4.1).

**Lemma 4.1.1** For the field space $(\mathcal{E}_L = \{f \in \Omega^*(\mathbb{R}_{\geq 0}) \otimes V| \iota^* f \in \mathcal{L}\}, Q = d + Q^\partial)$, we have an SDR:

$$\tau(L, Q^\partial) \impliedby (\mathcal{E}_L, Q), \iota^*$$

(4.2)

where $\tau, \iota^*$ both have images in $\Omega^0(\mathbb{R}_{\geq 0}) \otimes V$: for $l \in L, f \in \mathcal{E}_L, x \in \mathbb{R}_{\geq 0}$,

$$\tau(l)(x) = l, \quad (\iota^*(f))(x) = - \int_0^x f.$$

The check is left to the reader as an exercise.

We denote the dual construction (Lemma 2.1.3) of (4.2) by

$$i(L^*, Q^\partial) \impliedby (\mathcal{E}_L^*, Q), K.$$

It further induces an SDR by symmetric tensor power construction:

$$i(O(L)[[\hbar]], Q^\partial) \impliedby (\mathcal{O}(\mathcal{E}_L)[[\hbar]], Q), K^\text{sym},$$

(4.3)

where $\mathcal{O}(L) := \hat{\text{Sym}}(L^*)$.

The homological degree of an element in $\mathcal{O}(\mathcal{E}_L)[[\hbar]]$ is the sum of two parts, the first comes from the grading of $\Omega^0(\mathbb{R}_{\geq 0})$, and the second comes from the grading of $V$. In the following we will call the first part the “de Rham degree”. Similarly, we can define the de Rham degree of an operator on $\mathcal{O}(\mathcal{E}_L)[[\hbar]]$. Denote this degree by $|\alpha|_{dR}$ for $\alpha$ being a functional or an operator. It is easy to see that

- elements in $\mathcal{O}(\mathcal{E}_L)[[\hbar]]$ have nonpositive de Rham degree;
- for $f \in \mathcal{O}(L)[[\hbar]], |i(f)|_{dR} = 0$, elements in $p^{-1}(f)$ also have de Rham degree 0;
- $|Q^\partial|_{dR} = 0, |d|_{dR} = 1, |K^\text{sym}|_{dR} = -1$;
- $|\Delta|_{dR} = |\{-, -\}|_{dR} = 1$;
• $|\partial_P(e,\Lambda)|_{dr} = 0$;

• $|\rho(f)|_{dr} = -1$ for $f \in \text{Sym}^n(V^*)$ with $n > 0$. ($\rho$ is the map in (3.21).)

Now, pick a scale $t > 0$, we use

$$h\Delta_t + \delta_t^{\text{int}} := h\Delta_t + \{I_t, -\}_t$$

to perturb (4.3), whose RHS will be changed to $(O(E_L)[[h]], Q + h\Delta_t + \{I_t, -\}_t)$. By analyzing the $h$-grading and symmetric tensor power grading, we can observe that $\sum_{n=0}^{+\infty}((h\Delta_t + \delta_t^{\text{int}})K^{\text{sym}})^n$ is well defined on $O(E_L)[[h]]$. So, $h\Delta_t + \delta_t^{\text{int}}$ is a small perturbation. (This is similar to the implication following (2.13), although here $I_t$ can contain terms in $O^2(P_L,c)$.)

**Theorem 4.1.1** Let $\{(O(E_L), Q, \Delta_t)| t \in \mathbb{R}_{>0}\}$ and $\rho(I^0) \in \mathcal{O}_P(E_L,c)[[h]]^+$ encode a TQM on $\mathbb{R}_{\geq 0}$ as in Definition 3.0.1. At any scale $t > 0$, we have an SDR manifesting an effective theory for this TQM. Concretely, this SDR is the result of using $h\Delta_t + \delta_t^{\text{int}}$ to perturb (4.3), written as follows:

$$O(L)[[h]], Q^0 + b_t^{\text{int}} \rightleftharpoons O(E_L)[[h]], Q + h\Delta_t + \delta_t^{\text{int}}, K_t^{\text{int}}$$

where

$$b_t^{\text{int}} = pe^{h(\Delta_t K^{\text{sym}})}_{2}\delta_t^{\text{int}} i,$$

$$p_t = pe^{h(\Delta_t K^{\text{sym}})}_{2}.$$ (4.5)

Moreover, for any two different scales $\varepsilon, \Lambda > 0$,

$$p_\varepsilon = p_\Lambda W(\partial_P(\varepsilon,\Lambda), I_\varepsilon, -)$$

$$b_\varepsilon^{\text{int}} = b_\Lambda^{\text{int}}.$$ (4.6)

Namely, we obtain an effective observable complex $(O(L)[[h]], Q^0 + b^{\text{int}})$ independent of the scale we pick to construct it.

**Proof** By associativity of perturbation theory (Lemma 2.1.2), we do the calculation in two steps. First, we use only $h\Delta_t$ to perturb (4.3). By Proposition 2.3.1 the result can be written as

$$i$$

$$O(L)[[h]], Q^0 \rightleftharpoons O(E_L)[[h]], Q + h\Delta_t, K.$$ (4.7)

Note that the differential on $O(L)[[h]]$ is not perturbed, because $\Delta_t i = 0$ here. Then, we use $\delta_t^{\text{int}}$ to perturb (4.7). By definition,

$$\delta_t^{\text{int}} = \{I_t, -\}_t = \lim_{\varepsilon \to 0} \{W(\partial_P(\varepsilon,t), \rho(I^0)), -\}_t.$$ (4.8)

Since

$$|\rho(I^0)|_{dr} = -1, \quad |\partial_P(e,t)|_{dr} = 0,$$
it is direct to conclude from (3.16) that, terms in $W(\partial P_{(\varepsilon,\Lambda)}, \rho(I^0))$ will have de Rham degree at most $-1$. Besides, $|\cdot|_{dR} = 1$, so given $f \in O(\mathcal{E}_L)[[\hbar]]$ with specific de Rham degree $|f|_{dR}$, terms in $\delta^\text{int}_t(f)$ will have de Rham degree at most $|f|_{dR}$. By

$$|K^\text{sym}|_{dR} = -1, \quad |\Delta_t|_{dR} = 1,$$

we have $|(\Delta_t K^\text{sym})^2|_{dR} = 0$. Recall that $p(f) = 0$ for $f \in O(\mathcal{E}_L)[[\hbar]]$ with specific $|f|_{dR} < 0$, so

$$pe^{h(\Delta_t K^\text{sym})^2} \delta^\text{int}_t K^\text{sym} = 0.$$

This implies $pe^{h(\Delta_t K^\text{sym})^2} \delta^\text{int}_t \hbar = 0$. Namely, perturbation $\delta^\text{int}_t$ to (4.7) validates the statement $[A]$ in Proposition 2.1.1, hence gives rise to exactly the formulae (4.5) and (4.6).

By the same counting as above, given $f \in O(\mathcal{E}_L)[[\hbar]]$ with specific $|f|_{dR}$, (3.17) implies

$$W(\partial P_{(\varepsilon,\Lambda)}, I, f) = e^{h\partial P_{(\varepsilon,\Lambda)}} f + \ldots,$$

where terms in “...” have de Rham degree less than $|f|_{dR}$. So,

$$p_{\Lambda} W(\partial P_{(\varepsilon,\Lambda)}, I, -) = pe^{h(\Delta_{\Lambda} K^\text{sym})_2} e^{h\partial P_{(\varepsilon,\Lambda)}} = pe^{h((\Delta_{\Lambda} K^\text{sym})_2 + \partial P_{(\varepsilon,\Lambda)})}.$$

Note that $\Delta_{\Lambda} = \Delta_{\varepsilon} + [\partial P_{(\varepsilon,\Lambda)}, Q] = \Delta_{\varepsilon} + [\partial P_{(\varepsilon,\Lambda)}, d]$, we have

$$(\Delta_{\Lambda} K^\text{sym})^2 + \partial P_{(\varepsilon,\Lambda)} = (\Delta_{\varepsilon} K^\text{sym})^2 + ([\partial P_{(\varepsilon,\Lambda)}, d] K^\text{sym})^2 + \partial P_{(\varepsilon,\Lambda)}.$$

It is easy to verify that $\partial P_{(\varepsilon,\Lambda)} t = 0$, and for $f_1, f_2 \in \mathcal{E}_L^*$, $\partial P_{(\varepsilon,\Lambda)}(f_1 K f_2) = 0$ (by de Rham degree reason). So

$$((\partial P_{(\varepsilon,\Lambda)}) d] K^\text{sym})^2 + \partial P_{(\varepsilon,\Lambda)}(f_1 f_2) = (\partial P_{(\varepsilon,\Lambda)} d K^\text{sym} + \partial P_{(\varepsilon,\Lambda)})(f_1 f_2) = \partial P_{(\varepsilon,\Lambda)}(ip - K^\text{sym} d)(f_1 f_2) = 0,$$

which means $([\partial P_{(\varepsilon,\Lambda)}, d] K^\text{sym})^2 + \partial P_{(\varepsilon,\Lambda)} = 0$ because this is a 2-to-0 operator. Hence we have proved $p_{\Lambda} W(\partial P_{(\varepsilon,\Lambda)}, I, -) = p_{\varepsilon}$. As a consequence,

$$Q^\partial + b^\text{int}_\varepsilon = p_{\varepsilon}(Q + \hbar \Delta_{\varepsilon} + \delta^\text{int}_t f^\text{int}_\varepsilon) = p_{\Lambda}(Q + \hbar \Delta_{\Lambda} + \delta^\text{int}_t f^\text{int}_\Lambda) = (Q^\partial + b^\text{int}_\Lambda) p_{\varepsilon} W(\partial P_{(\varepsilon,\Lambda)}, I, -) = (Q^\partial + b^\text{int}_\Lambda) p_{\varepsilon} f^\text{int}_\varepsilon.$$

So we have finished. \(\square\)
Remark 4.1.1 In the proof we have seen that, if we had set $Q^0 = 0$ in the beginning, before turning on the interaction we would obtain a complex (the LHS of (4.7)) with 0 differential. Then, the interaction could give rise to a nonzero $b^{\text{int}}$ on it (see Example 4.1.3). This is different from the case in Section 2.4. There, the interaction induces a formal conjugation on the effective observable complex, so will preserve a zero differential.

We can actually simplify formula (4.3) a little. Formally we have
$$
\hat{\partial}_{\text{int}}^0 = \{I_t, -\} = h e^{-I_t/h} \{e^{I_t/h}, -\} = h e^{-I_t/h} \lim_{\varepsilon \to 0} \left\{ e^{\hbar D_P(x,t) \rho(I^0)/h}, -\right\}.
$$
So, by de Rham degree argument,
$$
b^{\text{int}} = pe^{\hbar(\Delta t K^{sym})^2} \hat{\partial}_{\text{int}}^0 \hat{i} = \lim_{\varepsilon \to 0} pe^{\hbar(\Delta t K^{sym})^2} \left\{ e^{\hbar D_P(x,t) \rho(I^0)}, -\right\} \hat{i}.
$$
(4.8)

For later convenience, we rewrite the BV kernel $K_t$ (defined in (3.11)) as
$$
K_t(x, y) = (H_t^{\text{e}}(x, y) \text{ dx} + H_t^{\text{pe}}(x, y) \text{ dy}) \otimes K^0_+ + \sigma(H_t^{\text{e}}(x, y) \text{ dx} + H_t^{\text{pe}}(x, y) \text{ dy}) \otimes \sigma K^0_+
$$
with
$$
H_t^{\text{e}}(x, y) = -\frac{1}{2} H_N(t, x, y) - \frac{1}{4} \partial_x \left( \left( d^{\text{GF}} \otimes 1 + 1 \otimes d^{\text{GF}} \right) \int_0^t ds(\tilde{H_s} - H_s) \right)
$$
$$
H_t^{\text{pe}}(x, y) = -\frac{1}{2} H_D(t, x, y) - \frac{1}{4} \partial_y \left( \left( d^{\text{GF}} \otimes 1 + 1 \otimes d^{\text{GF}} \right) \int_0^t ds(\tilde{H_s} - H_s) \right).
$$
Now we will work out the explicit formula of $b^{\text{int}}$ for the examples in the last section.

Example 4.1.1 For the theory given in Example 3.0.1 we have
$$
\rho(I^0) = -\int \omega(\rho(I^0)) = -\int \omega(\rho(I^0)),
$$
so $b^{\text{int}} = pe^{\hbar(\Delta t K^{sym})^2} \left\{ \rho(I^0), -\right\} \hat{i}.$

By construction (recall $\hat{E}_L = \mathcal{L} \oplus \mathcal{L}^r$, $p^{-1}(\mathcal{O}(L)) \subset \hat{\text{Sym}}(\mathcal{L}^r)$, $i(\mathcal{O}(L)) \subset \hat{\text{Sym}}(\mathcal{L}^r)$).

Just like $\partial_P(x,t), \Delta t$ and $\{-, -\}_t$, the operator $(\Delta t K^{sym})^2$ also pairs a linear functional factor depending on $\mathcal{L}$ with a linear functional factor depending on $\mathcal{L}^r$. So we have
$$
b^{\text{int}} = pe^{\hbar(\Delta t K^{sym})^2} \left\{ \rho(I^0), -\right\} \hat{i} = \hat{h} p(\Delta t K^{sym})^2 \left\{ \rho(I^0), -\right\} \hat{i} = \hat{h} p((\Delta t K^{sym})^2, \{\rho(I^0), -\}) \hat{i}.
$$
By Proposition 2.2.2, this is a 2-to-0 operator, decided by
$$
b^{\text{int}}(f_1 f_2) = \hat{h} \Delta t((K \{\rho(I^0), i f_1\} t)(i f_2) + (i f_1)(K \{\rho(I^0), i f_2\} t))
$$
$$
= (-1)^{|f_1|+|f_2|} \hat{h} \int_{R^0} \text{ dx } H_t^{\text{e}}(x, 0) \int_0^\infty \text{ dy } H_t^{\text{pe}}(y, 0) f_1 f_2 ((\rho(I^0) \otimes 1) K^0_+)
$$
$$
= (-1)^{|f_1|+|f_2|} \hat{h} \left( \int_0^\infty \text{ dx } H_t^{\text{e}}(x, 0) \right)^2 f_1 f_2 ((\rho(I^0) \otimes 1) K^0_+)
$$
$$
= (-1)^{|f_1|+|f_2|} \frac{\hat{h}}{2} \left( \int_0^\infty \text{ dx } H_N(t, x, 0) \right)^2 f_1 f_2 ((\rho(I^0) \otimes 1) K^0_+)
$$
$$
= \frac{(-1)^{|f_1|+|f_2|} \hat{h}}{2} f_1 f_2 ((\rho(I^0) \otimes 1) K^0_+),
$$
29
with \( f_1, f_2 \in L^* \).

So, we obtain an effective theory described by a differential BV algebra \((O(L), Q^\partial, b^{\text{int}}/\hbar)\). The effective BV operator \( b^{\text{int}}/\hbar \) can be degenerate (as a pairing on \( L^* \)) for certain choices of \( Q^{\text{rel}} \). This example reproduces the simplest case (i.e., the boundary manifold is a point here) of the second relation in [Rab21, Theorem 3.4.3]. It has been observed that degenerate classical field theories can arise from classical field theories on manifold with boundary [BY16]. We hope our homotopy transfer method can extend this study to quantum level in the future.

**Example 4.1.2** For BF theory with B boundary condition (Example 3.0.2), by the unimodular condition \( f_{cb} = 0 \) and \( K_2^a = -\eta^a \otimes \epsilon_t, K_2^a = -\epsilon_t \otimes \eta^a \), it is direct to see

\[
\partial_{P(\varepsilon, t)} \rho(I^\partial) = \frac{1}{2} \partial_{P(\varepsilon, t)} \int_{\mathbb{R}^0} f_{cb} B^c \wedge A_a \wedge A_b = 0. \tag{4.9}
\]

So \( e^{\hbar \partial_{P(\varepsilon, t)}} \rho(I^\partial) = \rho(I^\partial) \). Using the same argument in Example 4.1.1, we have

\[
b^{\text{int}} = \rho(\Delta^2_{1K^{\text{sym}}}) \{ \rho(I^\partial), - \} i = \hbar \rho(\Delta^2_{1K^{\text{sym}}}) \{ \rho(I^\partial), - \} i.
\]

By the same reason for (1.3), \( (\Delta_t K^{\text{sym}})_2 \{ \rho(I^0), i(f) \} t = 0 \) for \( f \in L^* \). We conclude that

- for \( n \geq 2 \), \( \forall f_1, f_2, \ldots, f_n \in \text{Sym}^n(L^*) \),
  \[ b^{\text{int}}(f_1 f_2 \cdots f_n) = \sum_{i<j} (\pm)_{K_{\text{sym}}} b^{\text{int}}(f_i f_j) f_1 \cdots \hat{f}_i \cdots \hat{f}_j \cdots f_n. \]

- \( b^{\text{int}}(\text{Sym} <1(L^*)[[\hbar]]) = 0 \), and \( b^{\text{int}}(\text{Sym}^2(L^*)) \subset \hbar L^* \). For \( f_1, f_2 \in L^* \),
  \[
b^{\text{int}}(f_1 f_2) = 4\hbar f_{cb} B^c \int_{\mathbb{R}^0} dx H^0_t(x, 0) \int_0^x dy H^0_t(y, 0) \partial_{B^a} \partial_{B^b}(f_1 f_2)
  = \frac{\hbar}{2} f_{cb} B^c \partial_{B^a} \partial_{B^b}(f_1 f_2)
\]
  where \( \partial_{B^a} \) is the derivation on \( O(L) \) decided by \( \partial_{B^a}(B^b) = \delta^b_a \).

The isomorphism \( L^* \simeq \mathfrak{g}[1] \) identifies \( (O(L)[[\hbar]], b^{\text{int}}) \) with \( (\text{Sym}(\mathfrak{g}[1])[\hbar], \hbar d_{[-,-]} \) ), where \( d_{[-,-]} \) is the differential of the Chevalley-Eilenberg complex

\[
C_\bullet(\mathfrak{g}) := (\text{Sym}(\mathfrak{g}[1]), d_{[-,-]}).
\]

So, we have refined the first conclusion in [Rab21, Theorem 5.0.2] by preserving the \( \hbar \)-grading structure.

**Example 4.1.3** For the theory given in Example 3.0.3, using the same argument in Example 4.1.1 we have

\[
b^{\text{int}} = \lim_{\varepsilon \to 0} pe^{\hbar(\Delta^2_{1K^{\text{sym}}})_2} \{ e^{\hbar \partial_{P(\varepsilon, t)}} \rho(I^\partial), - \} i = \frac{\hbar}{2} p(\Delta^2_{tK^{\text{sym}}})_2 \{ \rho(I^\partial), - \} i
\]

with \( \rho(I^0) = - \int_{\mathbb{R}^0} p_1 \wedge p_1 \wedge p_2 \). This is a “3-to-0 operator” on \( O(L)[[\hbar]] \), in the sense that

- \( b^{\text{int}}(\text{Sym} <2(L^*)[[\hbar]]) = 0 \), and \( b^{\text{int}}(\text{Sym}^3(L^*)) \subset \mathbb{R} \hbar^2 \).
For \( n \geq 3 \), \( \forall f_1, f_2, \ldots, f_n \in \text{Sym}^n(L^*) \),
\[
\hat{b}^{\text{int}}(f_1 f_2 \cdots f_n) = \sum_{i<j<k} (\pm)_{\text{Kos}} \hat{b}^{\text{int}}(f_i f_j f_k) f_1 \cdots \hat{f}_i \cdots \hat{f}_j \cdots \hat{f}_k \cdots f_n.
\]
And for \( f_1, f_2, f_3 \in L^* \),
\[
\hat{b}^{\text{int}}(f_1 f_2 f_3) = 24\hbar^2 \int_{\mathbb{R}^3} dx \, H_{10}^i(x,0) \int_0^x dy \, H_{10}^j(y,0) \int_0^x dz \, H_{10}^k(z,0) \partial_{q_1} \partial_{q_2} \partial_{q_2} (f_1 f_2 f_3)
\]
where \( \partial_{q_i} \) is the derivation on \( \mathcal{O}(L) \) decided by \( \partial_{q_i}(q_j) = \delta_{ij} \) for \( i, j \in \{1, 2\} \). Obviously, \( \hat{b}^{\text{int}}/\hbar^2 \) is not a BV operator on \( \mathcal{O}(L) \).

### 4.2 A quasi-example: 2D BF theory

In [Rab21], field theory topological normal to the boundary can be defined on a higher-dimensional spacetime \( X \), inducing QFT on \( \mathbb{R}_{\geq 0} \times \partial X \). We can ask whether Theorem 4.1 can be extended to the case when \( \partial X \) is not a point. Now, to perform renormalization, the propagator will contain “\( \mathbb{R}_{\geq 0} \) direction” terms and “\( \partial X \) direction” terms. This fact unfortunately invalidates the de Rham degree argument in the proof of Theorem 4.1.1. So, in the following we do not define the theory rigorously, but instead, make a trial regarding the theory on \( \mathbb{R}_{\geq 0} \times \partial X \) as a TQM on \( \mathbb{R}_{\geq 0} \) (with singularity in the data). In this way we will study BF theory on \( \mathbb{R}_{\geq 0} \times \mathbb{R} \), and see whether we can obtain some meaningful “effective theory” by applying Theorem 4.1.1 anyway.

#### B boundary condition

Let \( g \) be the unimodular Lie algebra mentioned in Example 3.0.2. For BF theory on \( \mathbb{R}_{\geq 0} \times \mathbb{R} \) with B boundary condition, we define
\[
L := \Omega^1(\mathbb{R}) \otimes g^*, \quad L' := \Omega^1(\mathbb{R}) \otimes g[1], \quad V := L \oplus L'.
\]
Let \( V_c \) denote the set of compactly supported forms in \( V \). There is a degree 0 nondegenerate antisymmetric pairing \( \omega^{\partial} \) on \( V_c \), decided by
\[
\omega^{\partial}(\varphi_1 \otimes t_a, \varphi_2 \otimes \eta t^b) := \delta^b_a \int_{\mathbb{R}} \varphi_1 \wedge \varphi_2
\]
with \( \varphi_1, \varphi_2 \in \Omega^1(\mathbb{R}) \) having compact supports and \( \eta \) being formal variable of degree \(-1\). Let \( Q^{\partial} \) be the de Rham operator on \( \Omega^1(\mathbb{R}) \), then, \( Q^{\partial} \omega^{\partial} = 0 \). Now,
\[
\mathcal{E}_L = \{ f \in \Omega^1(\mathbb{R}_{\geq 0}) \otimes V | \iota^{\ast} f \in L \},
\]
with \( \iota^{\ast} \) induced by \( \iota : \mathbb{R} \hookrightarrow \mathbb{R}_{\geq 0} \times \mathbb{R} \). \( \mathcal{O}(\mathcal{E}_L) \) consists of functionals compactly supported on \( \mathbb{R}_{\geq 0} \times \mathbb{R} \). We can write
\[
K^0_+ = \delta(x' - y')(dy' - dx') \otimes \eta t^a \otimes t_a, \quad K^0_- = -\sigma K^0_+
\]
where \((x', y')\) are coordinates on \(\mathbb{R} \times \mathbb{R}\). Then, we formally define \(K_t\) and \(P(\varepsilon, \Lambda)\) as in \((3.11)\) and \((3.12)\). Let \(d\) still denote the de Rham operator on \(\Omega^\bullet(\mathbb{R}_{\geq 0})\). The data
\[
(O(E_L), Q = d + Q^\partial, \Delta_t = \partial_{K_t})
\]
is ill-defined, but if we restrict \(O(E_L)\) to its subspace of smooth distributions, it will become a differential BV algebra. Let \(\{B^a\}_{a=1}^t\) be the basis of \((\mathfrak{g}^*)^\bullet\) dual to \(\{t_a\}_{a=1}^t\), \(\{A_a\}_{a=1}^t\) be the basis of \((\mathfrak{g}[1])^\bullet\) such that \(A_a(\eta^b) = \delta^b_a\), and \(I^\partial := \frac{1}{2} \int_\mathbb{R} f_c^{ab} B^c \wedge A_a \wedge A_b\). Then we pretend that
\[
\rho(I^\partial) = \frac{1}{2} \int_{\mathbb{R}_{\geq 0}} \int_\mathbb{R} f_c^{ab} B^c \wedge A_a \wedge A_b,
\]
together with \((O(E_L), Q, \Delta_t)\), defines an interactive theory in the sense of Definition 3.1.1. The proof of Theorem 4.1.1 and arguments in Example 4.1.2 formally hold, allowing us to tentatively write down an effective observable complex \((O(L)[[h]] = O(\Omega^\bullet(\mathbb{R}) \otimes \mathfrak{g}^*)[[h]], Q^\partial + b^{\text{int}})\), where

- for \(n \geq 2\), \(\forall f_1 f_2 \cdots f_n \in \text{Sym}^n(L^\bullet)\),
  \[
b^{\text{int}}(f_1 f_2 \cdots f_n) = \sum_{i<j} (\pm)_{\text{Koszul}} b^{\text{int}}(f_i f_j) f_1 \cdots \hat{f}_i \cdots \hat{f}_j \cdots f_n,
\]

- \(b^{\text{int}}(\text{Sym} \leq 1(L^\bullet)[[h]]) = 0\), and \(b^{\text{int}}(\text{Sym}^2(L^\bullet)) \subset \h L^\bullet\). For \(f_1, f_2 \in L^\bullet\), suppose
  \[
f_j = \int_\mathbb{R} f_{ja} \wedge B^a \quad \text{for } j = 1, 2
\]
  with each \(\int_\mathbb{R} f_{ja} \wedge (-)\) representing a functional on \(\Omega^\bullet(\mathbb{R})\), then
  \[
b^{\text{int}}(f_1 f_2) = h(-1)^{|f_{1a}|+1} f_c^{ab} \int_\mathbb{R} f_{1a} \wedge f_{2b} \wedge B^c.
\]

This is still ill-defined, because we cannot multiply two distributions. Recall that we have an embedding
\[
(\Omega^\bullet(\mathbb{R})[1], -d_{\mathbb{R}}) \hookrightarrow ((\Omega^\bullet(\mathbb{R}))^\bullet, Q^\partial) \quad \text{\eta} f \mapsto \int_\mathbb{R} f \wedge (-),
\]
with \((\Omega^\bullet(\mathbb{R}), d_{\mathbb{R}})\) being the compactly supported de Rham complex on \(\mathbb{R}\) and \(\eta\) being a degree \(-1\) formal variable. By Atiyah-Bott lemma, this is a continuous quasi-isomorphism (we refer to \[CG16,\] Appendix D for an explanation). \(1.10\) and the identification \((\mathfrak{g}^*)^\bullet \simeq \mathfrak{g}\) induce a quasi-isomorphism
\[
(\widehat{\text{Sym}}(\Omega^\bullet(\mathbb{R})[1] \otimes \mathfrak{g}), -d_{\mathbb{R}}) \hookrightarrow (O(L), Q^\partial).
\]
Then, \(-d_{\mathbb{R}} + b^{\text{int}}\) is a well-defined differential on \(\widehat{\text{Sym}}(\Omega^\bullet(\mathbb{R})[1] \otimes \mathfrak{g})[[h]]\). For \(f_1, f_2 \in \Omega^\bullet(\mathbb{R}) \otimes \mathfrak{g}\),
\[
b^{\text{int}}((\eta f_1)(\eta f_2)) = h(-1)^{|f_1|+1} \eta [f_1, f_2],
\]
where we have recognized \(\Omega^\bullet(\mathbb{R}) \otimes \mathfrak{g}\) to be a dg Lie algebra, with bracket still denoted by \([-, -]\). In this way, we can identify
\[
(\widehat{\text{Sym}}(\Omega^\bullet(\mathbb{R})[1] \otimes \mathfrak{g})[[h]], -d_{\mathbb{R}} + b^{\text{int}})
\]
with the Chevalley-Eilenberg complex \(C^\bullet(\Omega^\bullet(\mathbb{R}) \otimes \mathfrak{g})\) (after taking \(h = 1\)).

**Remark 4.2.1** If we take the boundary manifold to be an open subset \(U \subset \mathbb{R}\), the above quasi-calculation will give rise to \(C^\bullet(\Omega^\bullet(U) \otimes \mathfrak{g})\). This assignment actually leads to the enveloping factorization algebra \(U(\Omega^\bullet(U) \otimes \mathfrak{g})\) on \(\mathbb{R}\) (also called factorization envelope, see \[CG16,\] Section 3.6]). The cohomological factorization algebra of \(U(\Omega^\bullet \otimes \mathfrak{g})\) is the universal enveloping algebra \(U \mathfrak{g}\), regarded as a factorization algebra on \(\mathbb{R}\) (see \[CG16,\] Section 3.4)].
A boundary condition

For the A boundary condition, we modify the defining data in the above case as follows:

\[ L = \Omega^*(\mathbb{R}) \otimes g[1], \quad L' = \Omega^*(\mathbb{R}) \otimes g^*, \quad K^0_+ = \delta(x' - y')(dy' - dx') \otimes t_a \otimes \eta t^a. \]

Then, \( \mathcal{O}(L)[[\hbar]] = \mathcal{O}(\Omega^*(\mathbb{R}) \otimes g[1])[\hbar] \), and we can formally derive

\[ b^\text{int} = pe^{b(\Delta t^K)^{\text{sym}}/2}\{\rho(I^0),-\}t_i = p\{\rho(I^0),-\}t_i. \]

So, we obtain a cochain complex

\[ (\mathcal{O}(\Omega^*(\mathbb{R}) \otimes g[1])[\hbar], d_R + b^\text{int}), \quad (4.12) \]

where \( b^\text{int} \) is a derivation decided by

\[ b^\text{int} \left( \int f^a \wedge A_a \right) = \frac{(-1)^{|f|}}{2} \int f^a c \int f^c \wedge A_a \wedge A_b. \]

Thus we can identify \( (\mathcal{O}(\Omega^*(\mathbb{R}) \otimes g[1])[\hbar], d_R + b^\text{int}) \) with \( \text{CE}(\Omega^*(\mathbb{R}) \otimes \mathfrak{g}) \), the Chevalley-Eilenberg algebra of the dg Lie algebra \( \Omega^*(\mathbb{R}) \otimes \mathfrak{g} \).

Remark 4.2.2 If we apply the constructions in [Rab21, Section 4.7] to BF theory on \( \mathbb{R} \times \mathbb{R} \), we will obtain a factorization algebra of observables living on \( \mathbb{R} \times \mathbb{R} \), denoted by \( \text{Obs}^g_{BF,B} \) or \( \text{Obs}^g_{BF,A} \) for the B or A boundary condition case. Then, the projection \( \pi: \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R} \) will induce pushforward factorization algebras \( \pi_* (\text{Obs}^g_{BF,B}) \) and \( \pi_* (\text{Obs}^g_{BF,A}) \) on \( \mathbb{R} \). If we figured out a rigorous version of the definitions and calculations presented in the current subsection, we would confirm a weak equivalence between \( \pi_* (\text{Obs}^g_{BF,B}) \big|_{\hbar=1} \) and \( U_g \), and a weak equivalence between \( \pi_* (\text{Obs}^g_{BF,A}) \big|_{\hbar=1} \) and \( \text{CE}(\mathfrak{g}) \). Expectations for this relation can be found in the literature (see e.g., [GRW20, Remark 16]).

5 Effective Theory and Derived BV Algebras

If we use BV formalism to construct a renormalized QFT, the differential on the observable complex \( (2.10) \) will always consist of a BV operator (“second-order”) and a derivation (“first-order”). However, we have seen in Example 4.1.3 that the differential on the effective observables can be a “higher-order” operator. In order to describe it, we resort to a generalization of BV algebra called “derived BV algebra”, introduced by Olga Kravchenko [Kra99] (the terminology there is “commutative BV∞ algebra”). In [Ban20], Ruggero Bandiera showed that derived BV algebra structures can be transferred along certain kind of SDR’s via homological perturbation theory. We will review this result (also simplify it a little) in this section, and conclude that constructions in previous sections fit into derived BV algebra structure (see Proposition 5.0.3 and Proposition 5.0.8).

Cumulants and Koszul brackets

In this subsection we collect basics to prepare for discussions of derived BV algebras. We refer to [Ban20] and references therein for details.
Definition 5.0.1 Let \( A, B \) be graded commutative unital algebras over field \( k \), \( f : A \to B \) be a degree 0 linear map such that \( f(1_A) = 1_B \). The cumulants \( \{ \kappa(f)_n : A^{\otimes n} \to B \mid n \in \mathbb{Z}_+ \} \) are defined by the following formula:

\[
\kappa(f)_n(a_1, a_2, \ldots, a_n) := \sum_{\sigma \in S(m_1, \ldots, m_j) \mid m_1 + \cdots + m_j = n} (-1)^{j-1} \sum_{j, m_1, \ldots, m_j > 1} \sigma \frac{1}{j!} \left( \sum_{1 \leq i < j} f(a_{\sigma(i)} \cdots a_{\sigma(m_1)}) \cdots f(a_{\sigma(n-m_j+1)} \cdots a_{\sigma(n)}) \right) \tag{5.1}
\]

where \( a_1, a_2, \ldots, a_n \in A \), and

\[
S(m_1, \ldots, m_j) := \{ \sigma \in S_n \mid \sigma(1) < \cdots < \sigma(m_1), \ldots, \sigma(n-m_j+1) < \cdots < \sigma(n) \} \tag{5.2}
\]

can be regarded as the set of ordered \((m_1, \ldots, m_j)\)-partitions of \( \{1, 2, \ldots, n\} \). Actually \( S(m_1, \ldots, m_j) \) is defined for \( m_1, \ldots, m_j \geq 0 \), although in (5.1) we only sum over \( \sigma \in S(m_1, \ldots, m_j) \) with \( m_1, \ldots, m_j \geq 1 \).

By definition, \( \kappa(f)_n \) is symmetric in the inputs, and \( \kappa(f)_1 = f \). Roughly speaking, the cumulants of \( f \) for \( n \geq 2 \) measure the deviation of \( f \) from being a morphism of algebras. For example, it is direct to see

\[
\kappa(f)_2(a_1, a_2) = f(a_1 a_2) - f(a_1)f(a_2).
\]

If \( f \) is a morphism of algebras, then \( \kappa(f)_n = 0 \) for all \( n \geq 2 \).

Definition 5.0.2 Let \( A \) be a graded commutative unital algebra over field \( k \), \( D : A \to A \) be a linear map such that \( D(1_A) = 0 \). The Koszul brackets \( \{ \mathcal{K}(D)_n : A^{\otimes n} \to A \mid n \in \mathbb{Z}_+ \} \) are defined by the following formula:

\[
\mathcal{K}(D)_n(a_1, a_2, \ldots, a_n) := \sum_{m=1}^{n} \sum_{\sigma \in S(m, n-m)} (-1)^{n-m} D(a_{\sigma(1)} \cdots a_{\sigma(m)}) a_{\sigma(m+1)} \cdots a_{\sigma(n)}
\]

for \( a_1, a_2, \ldots, a_n \in A \).

By definition, \( \mathcal{K}(D)_n \) is symmetric in the inputs, and \( \mathcal{K}(D)_1 = D \). Roughly speaking, the Koszul brackets of \( D \) for \( n \geq 2 \) measure the deviation of \( D \) from being a derivation of graded algebras. For example, it is direct to see

\[
\mathcal{K}(D)_2(a_1, a_2) = D(a_1 a_2) - D(a_1)a_2 - (-1)^{|D||a_1|} a_1 D(a_2).
\]

Remark 5.0.1 For \( n \geq 1 \), let \( d_1, d_2, \ldots, d_n \) be derivations on \( A \). Then, their composition \( D := d_1 d_2 \cdots d_n \) is an “\( n \)-th-order” operator, satisfying the following properties:

- \( \mathcal{K}(D)_m = 0 \) for \( \forall m > n \);
- \( \mathcal{K}(D)_n \) is a “multi-derivation”, i.e., it is a derivation with respect to each input argument.

In Definition 2.2.1, we require the BV operator \( \Delta \) to be “second-order”, so \( \mathcal{K}(\Delta)_{m>2} = 0 \). The BV bracket \( \{ -, - \} \) is exactly \( \mathcal{K}(\Delta)_2 \).
Besides defining \(\{\kappa(f)_n\}\) and \(\{\mathcal{K}(D)_n\}\) by concrete formulae as above, we can also generate these maps from \(f\) and \(D\) in a systematic way. This method involves the cofree cocommutative coalgebra cogenerated by \(\mathcal{A},\mathcal{B}\).

Given a graded vector space \(V\) over \(k\), we can construct a coalgebra \((\text{Sym}(V), \Delta, \varepsilon)\), where

\[
\Delta(v_1 v_2 \cdots v_n) := \sum_{m=0}^{n} \sum_{\sigma \in S(m,n-m)} (\pm)^{Kos} (v_{\sigma(1)} v_{\sigma(2)} \cdots v_{\sigma(m)}) \otimes (v_{\sigma(m+1)} v_{\sigma(m+2)} \cdots v_{\sigma(n)})
\]

for \(v_1, v_2, \ldots, v_n \in V\), and \(\varepsilon\) is the projection \(\text{Sym}(V) \rightarrow \text{Sym}^0(V)\).

There is a correspondence:

\[
\{\text{coalggebra morphism} \quad F : \text{Sym}(V) \rightarrow \text{Sym}(W)\} \cong \{\text{linear maps} (f_1, \ldots, f_n, \ldots), \quad f_n \in \text{Hom}_k(\text{Sym}^n(V), W)\}.
\] (5.3)

The bijections between them are given by:

\[
F(v_1 \cdots v_n) = \sum_{j,m_1, \ldots, m_j \geq 1} \frac{1}{j!} \sum_{m_1 + \cdots + m_j = n} (\pm)^{Kos} f_{m_1}(v_{\sigma(1)} \cdots v_{\sigma(m_1)}) \cdots f_{m_j}(v_{\sigma(n-m_j+1)} \cdots v_{\sigma(n)}),
\]

\[
f_n = p_1 F i_n
\]

where \(i_n, p_n\) are the natural injection and projection between \(\text{Sym}^n(-)\) and \(\text{Sym}(-)\). The linear maps \((f_1, \ldots, f_n, \ldots)\) are called the Taylor coefficients of \(F\). In a similar manner, we have another correspondence:

\[
\{\text{coderivation} Q on \, \text{Sym}(V)\} \cong \{\text{linear maps} (q_0, q_1, \ldots, q_n, \ldots), \quad q_n \in \text{Hom}_k(\text{Sym}^n(V), V)\}
\] (5.4)

with the bijections between them given by:

\[
Q(v_1 \cdots v_n) = \sum_{m=0}^{n} \sum_{\sigma \in S(m,n-m)} (\pm)^{Kos} q_m(v_{\sigma(1)} \cdots v_{\sigma(m)}) v_{\sigma(m+1)} \cdots v_{\sigma(n)},
\]

\[
q_n = p_1 Q i_n.
\]

Now, \((q_0, q_1, \ldots, q_n, \ldots)\) are called the Taylor coefficients of \(Q\).

**Remark** To avoid confusion in the rest of this paper, for an algebra \(\mathcal{A}\), we will use \(S(\mathcal{A})\) to denote \(\text{Sym}(\mathcal{A})\), and use \(\odot\) to denote the symmetric tensor product. So, \(\text{Sym}^n(\mathcal{A})\) will be written as \(\mathcal{A} \odot^n\). The multiplication mark of \(\mathcal{A}\) itself will still be omitted.

Let \(\mathcal{A}, \mathcal{B}\) be graded commutative unital algebras. For \(t \in k, t \neq 0\), we have a coalgebra automorphism

\[
E_t^\mathcal{A} : S(\mathcal{A}) \rightarrow S(\mathcal{A}), \quad p_1 E_t^\mathcal{A} i_n(a_1 \odot a_2 \odot \cdots \odot a_n) = \frac{1}{t^{n-1}} a_1 a_2 \cdots a_n. \quad (5.5)
\]

It is direct to verify that the inverse of \(E_t^\mathcal{A}\) is

\[
L_t^\mathcal{A} : S(\mathcal{A}) \rightarrow S(\mathcal{A}), \quad p_1 L_t^\mathcal{A} i_n(a_1 \odot a_2 \odot \cdots \odot a_n) = \frac{(-1)^{n-1} (n-1)!}{t^{n-1}} a_1 a_2 \cdots a_n. \quad (5.6)
\]

Given a degree 0 linear map \(f : \mathcal{A} \rightarrow \mathcal{B}\) satisfying \(f(1_\mathcal{A}) = 1_\mathcal{B}\), we define a coalgebra morphism

\[
S(f) : S(\mathcal{A}) \rightarrow S(\mathcal{B}), \quad \text{its Taylor coefficients are} \, \,(f, 0, \ldots, 0, \ldots).
\]
Proposition 5.0.1 In above settings, we define a coalgebra morphism
\[ \kappa_t(f) : S(A) \mapsto S(B), \quad \kappa_t(f) := L_t^B S(f) E_t^A. \] (5.7)
Then, the cumulants in Definition 5.0.1 can be obtained from \( \kappa_t(f) \):
\[ \kappa(f)_n(a_1, a_2, \ldots, a_n) = t^{n-1} p_1 \kappa_t(f) 1_n(a_1 \odot a_2 \odot \cdots \odot a_n). \]

Given a linear map \( D : A \mapsto A \) satisfying \( D(1_A) = 0 \), we define a coderivation
\[ \overline{D} : S(A) \mapsto S(A), \quad \text{its Taylor coefficients are } (0, D, 0, \ldots, 0, \ldots). \]

Proposition 5.0.2 In above settings, we define a coderivation
\[ \mathcal{K}_t(D) : S(A) \mapsto S(A), \quad \mathcal{K}_t(D) := L_t^A \overline{D} E_t^A. \] (5.8)
Then, \( p_1 \mathcal{K}_t(D) 1_0 = 0 \), and the Koszul brackets in Definition 5.0.2 can be obtained from \( \mathcal{K}_t(D) \):
\[ \mathcal{K}(D)_n(a_1, a_2, \ldots, a_n) = t^{n-1} p_1 \mathcal{K}_t(D) 1_n(a_1 \odot a_2 \odot \cdots \odot a_n). \]

These two propositions can be verified by direct computation. Moreover, \( \kappa_t \) and \( \mathcal{K}_t \) satisfy the following relations:

- For a degree 0 map \( D : A \mapsto A \) satisfying \( D(1_A) = 0 \), suppose the map \( e^D \) is well defined on \( A \). Then \( e^D(1_A) = 1_A \), and \( e^{\mathcal{K}_t(D)} \) is a well-defined automorphism on \( S(A) \), satisfying
  \[ e^{\mathcal{K}_t(D)} = \kappa_t(e^D). \] (5.9)

- Let \( f : A \mapsto B, g : B \mapsto C \) be degree 0 linear maps such that \( f(1_A) = 1_B, g(1_B) = 1_C \), then
  \[ \kappa_t(gf) = \kappa_t(g) \kappa_t(f). \] (5.10)

Derived BV algebras

Now we are ready to introduce the derived BV algebras.

Definition 5.0.3 Let \( A \) be a graded commutative unital algebra over \( k \), \( \hbar \) be a formal variable with \( |\hbar| \) being an even integer. Let \( D \) be a \( k[[\hbar]] \)-linear operator on \( A[[\hbar]] \), \( |D| = 1, D(1_A) = 0 \). Then, \( (A[[\hbar]], D) \) is called a degree \( (1 - |\hbar|) \) derived BV algebra if the following conditions are satisfied:

- \( D^2 = 0 \);

- if we \( k((\hbar)) \)-linearly extend \( D \) to an operator on \( A((\hbar)) \), we can define \( \mathcal{K}_\hbar(D) \) on \( S(A)((\hbar)) \) as in (5.8), then \( \mathcal{K}_\hbar(D) \) preserves \( S(A)[[\hbar]] \subset S(A)((\hbar)) \).
Remark 5.0.2 By Proposition~5.0.2 the second condition is equivalent to
\[
\mathcal{K}(D)_n(a_1, \ldots, a_n) \equiv 0 \pmod{\hbar^{n-1}} \quad \text{for all } n \geq 2, \text{ and } a_1, \ldots, a_n \in \mathcal{A}.
\] (5.11)

Hence Definition 5.0.3 is equivalent to [Ban20, Definition 2.1]. If we expand $D$ as
\[
D = \sum_{n=0}^{+\infty} \hbar^n D_n, \quad D_n \in \text{Hom}_k(\mathcal{A}, \mathcal{A}),
\]
then (5.11) is equivalent to $\mathcal{K}(D_n)_{n+2} = 0$ for all $n \geq 0$.

In this paper we will take $|\hbar| = 0$ if the degree is not explicitly specified.

Remark 5.0.3 The observable complex (2.10) is born to be a derived BV algebra. Besides, the effective observable complexes in Proposition 2.3.1, 2.4.1, and those $(\mathcal{O}(L)[[\hbar]], Q^{\partial + \hbar^{\text{int}}})$’s in Example 4.1.1, 4.1.2, 4.1.3, and the quasi-calculation results (4.11), (4.12) are all derived BV algebras.

Definition 5.0.4 Given a pair of degree $(1 - |\hbar|)$ derived BV algebras $(\mathcal{A}[[\hbar]], D), (\mathcal{B}[[\hbar]], D')$, a morphism between them is a degree $0$ $k[[\hbar]]$-linear map $f : \mathcal{A}[[\hbar]] \mapsto \mathcal{B}[[\hbar]], f(1_A) = 1_B$, satisfying:

- $fD = D'f$;
- if we $k((\hbar))$-linearly extend $f$ to $f : \mathcal{A}((\hbar)) \mapsto \mathcal{B}((\hbar))$, we can define
  \[
  \kappa_{\hbar}(f) : S(\mathcal{A})((\hbar)) \mapsto S(\mathcal{B})((\hbar))
  \]
  as in (5.7), then $\kappa_{\hbar}(f)(S(\mathcal{A})[[\hbar]]) \subset S(\mathcal{B})[[\hbar]]$.

By the relation (5.10), compositions of such morphisms are well defined, so we obtain a category of derived BV algebras.

Remark 5.0.4 By Proposition 5.0.4 the second condition is equivalent to
\[
\kappa(f)_n(a_1, a_2, \ldots, a_n) \equiv 0 \pmod{\hbar^{n-1}} \quad \text{for all } n \geq 2, \text{ and } a_1, \ldots, a_n \in \mathcal{A}.
\]

Hence Definition 5.0.4 is equivalent to [Ban20, Definition 2.7].

Remark 5.0.5 The map $i_{\hbar} = i$ in (2.8) is a morphism of algebras, hence $\kappa(i)_n = 0$ for $n \geq 2$. So, $i_{\hbar}$ is a morphism of derived BV algebras. As for $p_{\hbar} = pe^{h(\Delta K^{\text{sym}})2}$ in (2.8), by (5.9) and (5.10) we have
\[
\kappa_{\hbar}(p_{\hbar}) = \kappa_{\hbar}(p)\kappa_{\hbar}(e^{h(\Delta K^{\text{sym}})2}) = \kappa_{\hbar}(p)e^{\kappa_{\hbar}(h(\Delta K^{\text{sym}})2)}.
\]

Hence $p_{\hbar}$ is also a morphism of derived BV algebras.

Proposition 5.0.3 The second HRG operators $\mathcal{W}(\partial_{P(\varepsilon, \Lambda)}, I_{\varepsilon}, -)$ and $\mathcal{W}(\partial_{P(\Lambda, \varepsilon)}, I_{\Lambda}, -)$ in (3.20) are morphisms of derived BV algebras.
Recall that $P_G$ denote $\{0,\}$. Now we give an explanation.

In Remark 5.0.3 we find that the effective observable complexes in the examples are all derived BV algebras. Homotopy transfer for derived BV algebras

We have

$$\kappa_h(W(\partial_{P(\varepsilon, \Lambda)}), I_\varepsilon, -)) = \kappa_h(W(\partial_{P(\Lambda, \Lambda^+)}), I_\Lambda, -)) \kappa_h(W(\partial_{P(\varepsilon, \Lambda)}), I_\varepsilon, -)).$$

Recall that $P(\Lambda, \Lambda) = 0$ and $W(0, I, -) = \text{id}_{O(\mathcal{E}_L)[[\hbar]]}$, we have

$$\left( \frac{\partial}{\partial N} S(W(\partial_{P(\Lambda, \Lambda')}, I_\Lambda, -)) \right)_{N'=\Lambda} = \left( \frac{\partial}{\partial N} W(\partial_{P(\Lambda, \Lambda')}, I_\Lambda, -) \right)_{N'=\Lambda}.$$

So,

$$\frac{\partial}{\partial \Lambda}\kappa_h(W(\partial_{P(\varepsilon, \Lambda)}), I_\varepsilon, -)) = \left( \frac{\partial}{\partial N} S(W(\partial_{P(\Lambda, \Lambda')}, I_\Lambda, -)) \right)_{N'=\Lambda} E_h^{O(\mathcal{E}_L)[[\hbar]]} \kappa_h(W(\partial_{P(\varepsilon, \Lambda)}), I_\varepsilon, -)) = \kappa_h(W(\partial_{P(\varepsilon, \Lambda)}), I_\varepsilon, -)).$$

Define $G_\Lambda := (\frac{\partial}{\partial N} \partial_{P(\Lambda, \Lambda')})_{N'=\Lambda}$, it is a 2-to-0 operator on $O(\mathcal{E}_L)$. Similar to Remark 3.0.2, we use $\{-, -\}_{G_\Lambda}$ to denote $\mathcal{K}(G_\Lambda)_2 : O(\mathcal{E}_L)^{\otimes 2} \mapsto O(\mathcal{E}_L)$, and extend it to

$$\{-, -\}_{G_\Lambda} : O_P^{=0}(\mathcal{E}_{L,c})[[\hbar]] \otimes O(\mathcal{E}_L)[[\hbar]] \mapsto O(\mathcal{E}_L)[[\hbar]],$$

which is a derivation with respect to $O(\mathcal{E}_L)[[\hbar]]$. It is direct to check that

$$\left( \frac{\partial}{\partial N} W(\partial_{P(\Lambda, \Lambda')}, I_\Lambda, -) \right)_{N'=\Lambda} = \hbar G_\Lambda + \{I_\Lambda, -\}_{G_\Lambda}.$$

Thus $\kappa_h(W(\partial_{P(\varepsilon, \Lambda)}), I_\varepsilon, -))$ satisfies a first order linear differential equation

$$\frac{\partial}{\partial \Lambda}\kappa_h(W(\partial_{P(\varepsilon, \Lambda)}), I_\varepsilon, -)) = \kappa_h(hG_\Lambda + \{I_\Lambda, -\}_{G_\Lambda}) \kappa_h(W(\partial_{P(\varepsilon, \Lambda)}), I_\varepsilon, -)).$$

Since $\kappa_h(hG_\Lambda + \{I_\Lambda, -\}_{G_\Lambda})$ and $\kappa_h(W(\partial_{P(\varepsilon, \Lambda)}), I_\varepsilon, -))$ preserve $S(O(\mathcal{E}_L))[[\hbar]]$, we conclude that for $\forall \Lambda > 0$, $\kappa_h(W(\partial_{P(\varepsilon, \Lambda)}), I_\varepsilon, -))$ preserves $S(O(\mathcal{E}_L))[[\hbar]]$. □

**Homotopy transfer for derived BV algebras**

In Remark 5.0.3 we find that the effective observable complexes in the examples are all derived BV algebras. Now we give an explanation.

Let $\mathcal{A}, \mathcal{B}$ be graded commutative unital algebras over field $k$, suppose there is an SDR of $k[[\hbar]]$-cochain complexes

$$i \quad (\mathcal{B}[[\hbar]], D') = (\mathcal{A}[[\hbar]], D), \bar{\mathcal{R}},$$

where $(\mathcal{A}[[\hbar]], D)$ is a derived BV algebra, $i(1_B) = 1_A, p(1_A) = 1_B$, and for $\forall n \geq 1, b_1, b_2, \ldots, b_n \in \mathcal{B},$

$$p(i(b_1)i(b_2) \cdots i(b_n)) = b_1b_2 \cdots b_n.$$ (5.13)
By symmetric tensor power construction (Lemma 2.1.4) and $k((h))$-linear extension, (5.12) leads to

$$S(i) \quad (S(B)((h)), \overline{D}) \equiv (S(A)((h)), \overline{D}), R^{sym}.$$  \hspace{1cm} (5.14)

(It is easy to check that $i^{sym} = S(i), p^{sym} = S(p), (D')^{der} = \overline{D}^{der}$ and $D^{der} = D$, if there is any confusion here.) There is a conjugation on the RHS of (5.14):

$$L_h^A \quad (S(A)((h)), \overline{D}) \equiv (S(A)((h)), K_h(D)).$$  \hspace{1cm} (5.15)

It is direct to check that $(K_h(D) - \overline{D})$ is a small perturbation to (5.14). So, we can write down a perturbed SDR:

$$S(i)_{EL} \quad (S(B)((h)), (D')_{EL}) \equiv (S(A)((h)), K_h(D)), (R^{sym})_{EL}.$$  \hspace{1cm} (5.16)

$(A[[h]], D)$ being a derived BV algebra implies the perturbation $(K_h(D) - \overline{D})$ preserves $A[[h]]$. Hence the perturbed maps automatically satisfy

$$(D')_{EL}(S(B)[[h]]) \subset S(B)[[h]], \quad (R^{sym})_{EL}(S(A)[[h]]) \subset S(A)[[h]],$$

$$S(i)_{EL}(S(B)[[h]]) \subset S(A)[[h]], \quad S(p)_{EL}(S(A)[[h]]) \subset S(B)[[h]].$$

**Proposition 5.0.4** In above settings, we have the following facts:

I. If for $\forall n \geq 1, a_1, a_2, \ldots, a_n \in A, p, R$ further satisfy

$$\sum_{m=1}^{n} (\pm)_{Kos} p(a_1 \cdots R(a_m) \cdots a_n) = 0,$$  \hspace{1cm} (5.17)

then,

$$(D')_{EL} = K_h(D'), \quad S(p)_{EL} = \kappa_h(p).$$  \hspace{1cm} (5.18)

So now $(B[[h]], D')$ is a derived BV algebra, and $p$ is a morphism of derived BV algebras.

II. If for $\forall n \geq 1, b_1, b_2, \ldots, b_n \in B, i, R$ further satisfy

$$R(i(b_1)i(b_2) \cdots i(b_n)) = 0,$$  \hspace{1cm} (5.19)

then,

$$(D')_{EL} = K_h(D'), \quad S(i)_{EL} = \kappa_h(i).$$  \hspace{1cm} (5.20)

So now $(B[[h]], D')$ is a derived BV algebra, and $i$ is a morphism of derived BV algebras.
**Proof** For the perturbation to (5.13) induced by conjugation (5.15), it is direct to check that (5.13) and (5.17) imply the statement B of Proposition 2.1.3. Similarly, (5.13) and (5.19) imply the statement B of Proposition 2.1.4. Then, these two propositions will lead to (5.18) and (5.20), respectively. Details are left as an exercise. □

**Definition 5.0.5** Let $A, B$ be graded commutative unital algebras, suppose there is an SDR of cochain complexes

$$i$$

$$(B, D_B) \cong (A, D_A), \mathcal{R},$$

where $D_B(1_B) = 0, D_A(1_A) = 0$. We call this SDR a **semifull algebra contraction** if the following identities are satisfied for all $a_1, a_2 \in A, b_1, b_2 \in B$:

$$\mathcal{R}(\mathcal{R}(a_1)i(a_2)) = \mathcal{R}(\mathcal{R}(a_1)i(b_2)) = \mathcal{R}(i(b_1)i(i(b_2))) = \mathcal{R}(1_A) = 0$$

$$\mathcal{P}(\mathcal{R}(a_1)i(a_2)) = \mathcal{P}(\mathcal{R}(a_1)i(b_2)) = 0, \quad \mathcal{P}(i(b_1)i(b_2)) = b_1b_2, \quad \mathcal{P}(1_A) = 1_B.$$

Note that these imply $i(1_B) = i\mathcal{P}(1_A) = (1 + \mathcal{R}D_A + D_A\mathcal{R})(1_A) = 1_A$.

**Proposition 5.0.5** Given a semifull algebra contraction as (5.21), the following identities are satisfied for $\forall n \geq 1, b_1, b_2, \ldots, b_n \in B$:

$$\mathcal{P}(i(b_1)i(b_2) \cdots i(b_n)) = b_1b_2 \cdots b_n, \quad \mathcal{R}(i(b_1)i(b_2) \cdots i(b_n)) = 0.$$

**Proof** Hint: use $i(b_1) \cdots i(b_n) = (i\mathcal{P} - \mathcal{R}D_A - D_A\mathcal{R})(i(b_1) \cdots i(b_n))$ and induction on $n$. □

**Proposition 5.0.6** Given a semifull algebra contraction as (5.21), if we perturb it by a small perturbation $\delta_A$ satisfying $\delta_A(1_A) = 0$, the resulting perturbed SDR will also be a semifull algebra contraction.

**Proof** By direct check. □

**Remark 5.0.6** The result (2.3) of symmetric tensor power construction is a semifull algebra contraction.

So, we have the following conclusion:

**Proposition 5.0.7** Let $A, B$ be graded commutative unital algebras over field $k$, suppose there is a semifull algebra contraction of $k[[h]]$-algebras

$$i$$

$$(B[[h]], D_B) \cong (A[[h]], D_A), \mathcal{R},$$

where $(A[[h]], D_A)$ is a derived BV algebra, $i(1_B) = 1_A, p(1_A) = 1_B$. Then, $(B[[h]], D_B)$ is also a derived BV algebra, and the injection $i$ is a morphism of derived BV algebras.

If we perturb (5.22) by a small perturbation $\delta_A$ such that $(A[[h]], D_A + \delta_A)$ is still a derived BV algebra, then the perturbed differential on $B[[h]]$ endows $B[[h]]$ a derived BV algebra structure, and the perturbed injection is still a morphism of derived BV algebras.
Finally, we come to our constructions in previous sections:

**Proposition 5.0.8** The SDR’s (2.17) and (4.4) are homotopies between derived BV algebras, i.e., the injections and projections are morphisms of derived BV algebras.

**Proof** By the previous proposition we know the injections in (2.17) and (4.4) are morphisms of derived BV algebras. As for the projections, (2.15) implies the maps in (2.17) satisfy

\[ p_h^\text{int}(a_1K_h^\text{int}(a_2)) = 0 \quad \text{for } \forall a_1, a_2 \in \hat{\text{Sym}}(M)[[\hbar]]. \]

Similarly (by de Rham degree reason) the maps in (4.4) satisfy

\[ p_t(a_1K_t^\text{int}(a_2)) = 0 \quad \text{for } \forall a_1, a_2 \in \mathcal{O}(E_L)[[\hbar]]. \]

Hence both (2.17) and (4.4) satisfy the condition (5.17). By Proposition 5.0.4, the projections in (2.17) and (4.4) are also morphisms of derived BV algebras. \( \square \)

**Remark 5.0.7** As commutative algebras, the observable complexes in this paper are quite special: they all have the form of \( \hat{\text{Sym}}(V)[[\hbar]] \) for some \( V \). Inspired by [Ban20, Section 3], if \( (\hat{\text{Sym}}(V)[[\hbar]], D) \) is a derived BV algebra, we may call it a “dual IBL\(_\infty\) algebra”. We might modify Theorem 3.6 in [Ban20] and use it to prove Proposition 5.0.8 without referring to the condition (5.17). We leave this consideration for later work.

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