Biharmonic hypersurfaces in 4-dimensional space forms

Adina Balmus\textsuperscript{2}, Stefano Montaldo\textsuperscript{*1}, and Cezar Oniciuc \textsuperscript{2}

\textsuperscript{1} Università degli Studi di Cagliari
Dipartimento di Matematica
Via Ospedale 72
09124 Cagliari, ITALIA

\textsuperscript{2} Faculty of Mathematics, “Al.I. Cuza” University of Iasi
Bd. Carol I Nr. 11
700506 Iasi, ROMANIA

Key words  Biharmonic submanifolds, principal curvatures, isoparametric hypersurfaces

MSC (2000) 58E20

We investigate proper biharmonic hypersurfaces with at most three distinct principal curvatures in space forms.
We obtain the full classification of proper biharmonic hypersurfaces in 4-dimensional space forms.

1 Introduction

Biharmonic maps $\varphi : (M, g) \rightarrow (N, h)$ between Riemannian manifolds are critical points of the bienergy functional

$$E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 v_g,$$

where $\tau(\varphi) = \text{trace} \nabla d\varphi$ is the tension field of $\varphi$ that vanishes for harmonic maps. The Euler-Lagrange equation corresponding to $E_2$ is given by the vanishing of the bitension field

$$\tau_2(\varphi) = -J^\varphi(\tau(\varphi)) = -\Delta \tau(\varphi) - \text{trace} R^N(d\varphi, \tau(\varphi))d\varphi,$$

where $J^\varphi$ is formally the Jacobi operator of $\varphi$. The operator $J^\varphi$ is linear, thus any harmonic map is biharmonic. We call proper biharmonic the non-harmonic biharmonic maps.

In this paper we shall focus our attention on biharmonic submanifolds, i.e. on submanifolds such that the inclusion map is a biharmonic map. In this context, a proper biharmonic submanifold is a non-minimal biharmonic submanifold.

The first ambient spaces to look for proper biharmonic submanifolds are the spaces of constant sectional curvature $c$, which we shall denote by $E^n(c)$, and the first class of submanifolds to be studied is that of the hypersurfaces. The full classification of proper biharmonic hypersurfaces in $E^n(c)$, for any $n \geq 3$, is not known yet and, up to now, these are the results obtained:

- biharmonic hypersurfaces in $\mathbb{R}^n$, $n = 3, 4$, are minimal \cite{8, 10, 15};
- biharmonic surfaces in $\mathbb{H}^3$ are minimal \cite{3};
- biharmonic surfaces in $S^3$ are open parts of the hypersphere $S^2(1/\sqrt{2})$ \cite{2}.

The aim of this paper is to go further with the classification of compact proper biharmonic hypersurfaces in $\mathbb{E}^n(c)$. This study will conduce to the following classification of proper biharmonic compact hypersurfaces in $S^4$.\footnote{The first author was supported by a INdAM doctoral fellowship, ITALY. The second author was supported by PRIN-2005 (ITALY): Riemannian Metrics and Differentiable Manifolds. The third author was supported by Grant CEEX ET 5871/2006, ROMANIA. * Corresponding author: e-mail: montaldo@unica.it, Phone: +00 39 070 6758539, Fax: +00 39 070 6758504}
Theorem 2.1 (E-space form tangent components, represents the main tool in the study of proper biharmonic submanifolds in space forms. The following characterization result, obtained in [7] and [18] by splitting the bitension field in its normal and $\phi$:

**Theorem 3.5** The only proper biharmonic compact hypersurfaces of $\mathbb{S}^4$ are the hypersphere $\mathbb{S}^4(\frac{1}{\sqrt{2}})$ and the torus $\mathbb{S}^1(\frac{\pi}{2}) \times \mathbb{S}^2(\frac{\pi}{2})$.

The strategy to prove the theorem consists in proving that proper biharmonic hypersurfaces in 4-dimensional space forms have constant mean curvature. This is done by dividing the study according to the number of distinct principal curvatures.

The simplest assumption that $M$ is an umbilical hypersurface, i.e. all principal curvatures are equal, gives an immediate picture. In fact, if $M$ is a proper biharmonic umbilical hypersurface in $\mathbb{S}^{m+1}$, then it is an open part of $\mathbb{S}^n(\frac{1}{\sqrt{2}})$. Moreover, there exist no proper biharmonic umbilical hypersurfaces in $\mathbb{R}^{m+1}$ or in the hyperbolic space $\mathbb{H}^{m+1}$ (see [3]).

For biharmonic hypersurfaces with at most two distinct principal curvatures the property of having constant mean curvature was proved in [12], for the Euclidean case, and in [1] for any space form. This property proved to be the main ingredient for the following complete classification of proper biharmonic hypersurfaces with at most two distinct principal curvatures in the Euclidean sphere.

**Theorem 1.1** ([11]) Let $M^m$ be a proper biharmonic hypersurface with at most two distinct principal curvatures in $\mathbb{S}^{m+1}$. Then $M$ is an open part of $\mathbb{S}^m(\frac{1}{\sqrt{2}})$ or of $\mathbb{S}^{m+1}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^n(\frac{1}{\sqrt{2}})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.

In this paper we first prove that there exist no compact proper biharmonic hypersurfaces, of constant mean curvature, with three distinct principal curvatures in $\mathbb{S}^n$ (Theorem 3.1). Secondly, adapting a method developed by F. Defever in [9, 10], we show that biharmonic hypersurfaces of $\mathbb{E}^4(c)$ must have constant mean curvature (Theorem 3.3).

These two results, together with Theorem 1.1, give, as a consequence, the main result of the paper. For an up-to-date bibliography on biharmonic maps we refer the reader to [20].

**2 Preliminaries**

Let $\varphi : M \to \mathbb{E}^n(c)$ be the canonical inclusion of a submanifold $M$ in a constant sectional curvature $c$ manifold, $\mathbb{E}^n(c)$. The expressions assumed by the tension and bitension fields are

$$\tau(\varphi) = mH,$$

$$\tau_2(\varphi) = -m(\Delta H - mcH),$$

where $H$ denotes the mean curvature vector field of $M$ in $\mathbb{E}^n(c)$, while $\Delta$ is the rough Laplacian on $\varphi^{-1}T\mathbb{E}^n(c)$. The following characterization result, obtained in [7] and [18] by splitting the bitension field in its normal and tangent components, represents the main tool in the study of proper biharmonic submanifolds in space forms.

**Theorem 2.1** ([7, 18]) The canonical inclusion $\varphi : M^m \to \mathbb{E}^n(c)$ of a submanifold $M$ in an $n$-dimensional space form $\mathbb{E}^n(c)$ is biharmonic if and only if

$$\begin{cases} -\Delta^\perp H - \text{trace } B(\cdot, A_H \cdot) + mcH = 0, \\ 2 \text{ trace } A_{\nabla_\perp} H(\cdot) + \frac{mc}{2} \text{ grad } (|H|^2) = 0, \end{cases}$$

where $A$ denotes the Weingarten operator, $B$ the second fundamental form, $H$ the mean curvature vector field, $\nabla^\perp$ and $\Delta^\perp$ the connection and the Laplacian in the normal bundle of $M$ in $\mathbb{E}^n(c)$.

Moreover, if $M$ is a hypersurface of $\mathbb{E}^{m+1}(c)$, then $M$ is proper biharmonic if and only if

$$\begin{cases} \Delta^\perp H - (mc - |A|^2)H = 0, \\ 2A(\text{ grad } (|H|^2)) + m|H| \text{ grad } (|H|^2) = 0. \end{cases}$$

The generalized Clifford torus $\mathbb{S}^{m+1}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{n+1}(\frac{1}{\sqrt{2}})$, $m_1 + m_2 = m$, $m_1 \neq m_2$, was the first example of proper biharmonic hypersurface in $\mathbb{S}^{m+1}$ (see [14]). Then, in [2], it was proved that the only proper biharmonic hypersphere $\mathbb{S}^{m}(a)$, $a \in (0, 1)$, in $\mathbb{S}^{m+1}$, is $\mathbb{S}^m(\frac{1}{\sqrt{2}})$.

Inspired by these fundamental examples, in [3], the authors presented two methods for constructing proper biharmonic submanifolds of codimension greater than 1 in $\mathbb{S}^n$. 

Copyright line will be provided by the publisher
Theorem 2.2 (Composition property, [3]) Let \( M \) be a minimal submanifold of \( S^{n-1}(a) \subset S^n \). Then \( M \) is proper biharmonic in \( S^n \) if and only if \( a = \frac{1}{\ell} \).

Theorem 2.3 (Product composition property, [3]) Let \( M_1^{m_1} \) and \( M_2^{m_2} \) be two minimal submanifolds of \( S^{n_1}(r_1) \) and \( S^{n_2}(r_2) \), respectively, where \( n_1 + n_2 = n - 1 \), \( r_1^2 + r_2^2 = 1 \). Then \( M_1 \times M_2 \) is proper biharmonic in \( S^n \) if and only if \( r_1 = r_2 = \frac{1}{\sqrt{\ell}} \) and \( m_1 \neq m_2 \).

2.1 Isoparametric hypersurfaces

We recall that a hypersurface \( M^m \) in \( S^{m+1} \) is said to be isoparametric of type \( \ell \) if it has constant principal curvatures \( k_1 > \ldots > k_\ell \) with respective constant multiplicities \( m_1, \ldots, m_\ell \), \( m = m_1 + m_2 + \ldots + m_\ell \). It is known that the number \( \ell \) is either 1, 2, 3, 4 or 6. For \( \ell \leq 3 \) we have the following classification of compact isoparametric hypersurfaces.

If \( \ell = 1 \), then \( M \) is totally umbilical.

If \( \ell = 2 \), then \( M = S^{m_1}(r_1) \times S^{m_2}(r_2) \), \( r_1^2 + r_2^2 = 1 \) (see [19]).

If \( \ell = 3 \), then \( m_1 = m_2 = m_3 = 2^q \), \( q = 0, 1, 2, 3 \) (see [4]).

Moreover, there exists an angle \( \theta, 0 < \theta < \frac{\pi}{\ell} \), such that

\[
k_\alpha = \cot \left( \theta + \frac{(\alpha - 1)\pi}{\ell} \right), \quad \alpha = 1, \ldots, \ell. \tag{4}
\]

In the next section we shall need the following results on isoparametric hypersurfaces

Theorem 2.4 (55) A compact hypersurface \( M^m \) of constant scalar curvature \( s \) and constant mean curvature \(|H|\) in \( S^{m+1} \) is isoparametric provided it has 3 distinct principal curvatures everywhere.

Theorem 2.5 (6) Any compact hypersurface with constant scalar and mean curvature in \( S^4 \) is isoparametric.

We end this section by recalling

Proposition 2.6 (11) Let \( M^m \) be a proper biharmonic hypersurface with constant mean curvature \(|H|^2 = k \) in \( S^n \). Then \( M \) has constant scalar curvature,

\[
s = m^2(1 + k) - 2m.
\]

3 Biharmonic hypersurfaces with three distinct principal curvatures in spheres

Using the classification result on isoparametric hypersurfaces we can prove the following non-existence result for biharmonic hypersurfaces with 3 distinct principal curvatures.

Theorem 3.1 There exist no compact proper biharmonic hypersurfaces of constant mean curvature and with three distinct principal curvatures in the unit Euclidean sphere.

Proof. By Proposition 2.6, a proper biharmonic hypersurface \( M \) with constant mean curvature \(|H|^2 = k \) in \( S^{m+1} \) has constant scalar curvature. Since \( M \) is compact with 3 distinct principal curvatures and has constant mean and scalar curvature, from Theorem 2.4, it results that \( M \) is isoparametric with \( \ell = 3 \) in \( S^{m+1} \). Now, taking into account (4), there exists \( \theta \in (0, \pi/3) \) such that

\[
k_1 = \cot \theta, \quad k_2 = \cot \left( \theta + \frac{\pi}{3} \right) = \frac{k_1 - \sqrt{3}}{1 + \sqrt{3}k_1}, \quad k_3 = \cot \left( \theta + \frac{2\pi}{3} \right) = \frac{k_1 + \sqrt{3}}{1 - \sqrt{3}k_1}.
\]

Thus, from Cartan’s classification, the square of the norm of the shape operator is

\[
|A|^2 = 2^q(k_1^2 + k_2^2 + k_3^2) = 2^q \left( \frac{9k_1^4 + 45k_1^2 + 6}{1 - 3k_1^2} \right)
\]

and \( m = 3 \cdot 2^q, q = 0, 1, 2, 3 \). On the other hand, since \( M \) is biharmonic of constant mean curvature, from (3),

\[
|A|^2 = m = 3 \cdot 2^q.
\]

The last equation, together with (5), implies that \( k_1 \) is a solution of \( 3k_1^6 - 9k_1^4 + 21k_1^2 + 1 = 0 \), which is an equation with no real roots.

Copyright line will be provided by the publisher
Remark 3.2 Investigating the biharmonicity of compact isoparametric hypersurfaces in Euclidean spheres, in [16] the authors proved that the only compact proper biharmonic isoparametric hypersurfaces of $S^{m+1}$ are the hypersphere $S^m(\frac{1}{\sqrt{m}})$ and the generalized Clifford torus $S^{m_1}(\frac{1}{\sqrt{m_1}}) \times S^{m_2}(\frac{1}{\sqrt{m_2}})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.

We shall now concentrate on biharmonic hypersurfaces in $\mathbb{E}^4(c)$. Using B-Y. Chen techniques (see also the work of F. Defever et al [9, 10, 11]) we prove

Theorem 3.3 Let $M^3$ be a biharmonic hypersurface of the space form $\mathbb{E}^4(c)$. Then $M$ has constant mean curvature.

Proof. Suppose that $|H|$ is not constant on $M$. Then there exists an open subset $U$ of $M$ such that $\nabla_p |H|^2 \neq 0$, for all $p \in U$. By eventually restraining $U$ we can suppose that $|H| > 0$ on $U$, and thus $\nabla_p |H| \neq 0$, for all $p \in U$. If $U$ has at most two distinct principal curvatures, then, by Theorem 4.1 in [1], we conclude that its mean curvature is constant and we have a contradiction. Then there exists a point in $U$ with three distinct principal curvatures. This implies the existence of an open neighborhood of points with three distinct principal curvatures and we can suppose, by restraining $U$, that all its points have three distinct principal curvatures. On $U$ we can consider the unit section in the normal bundle $\eta \equiv \frac{H}{|H|}$ and denote by $f = |H|$ the mean curvature function of $U$ in $\mathbb{E}^4(c)$ and by $k_i$, $i = 1, 2, 3$, its principal curvatures w.r.t. $\eta$.

Conclusively, the hypothesis for $M$ to be proper biharmonic with at most three distinct principal curvatures in $\mathbb{E}^4(c)$ and non-constant mean curvature, implies the existence of an open connected subset $U$ of $M$, with

\[
\begin{cases}
\nabla_p f \neq 0, \\
f(p) > 0, \\
k_1(p) \neq k_2(p) \neq k_3(p) \neq k_1(p), \quad \forall p \in U.
\end{cases}
\] (6)

We shall contradict the condition $\nabla_p f \neq 0$, for all $p \in U$.

Since $M$ is proper biharmonic in $\mathbb{E}^4(c)$, from (3) we have

\[
\begin{cases}
\Delta f = (3c - |A|^2)f, \\
A(\nabla f) = -\frac{3}{2}f \nabla f.
\end{cases}
\] (7)

Consider now $X_1 = \frac{\nabla f}{|\nabla f|}$ on $U$. Then $X_1$ is a principal direction with principal curvature $k_1 = -\frac{3}{2}f$. Recall that $3f = k_1 + k_2 + k_3$, thus

\[k_2 + k_3 = \frac{9}{2}f.\] (8)

We shall use the moving frames method and denote by $X_1, X_2, X_3$ the orthonormal frame field of principal directions and by $\{\omega^a\}^3_{a=1}$ the dual frame field of $\{X_a\}^3_{a=1}$ on $U$.

Obviously,

\[X_i(f) = \langle X_i, \nabla f \rangle = |\nabla f| \langle X_i, X_1 \rangle = 0, \quad i = 2, 3,\] (9)

thus

\[\nabla f = X_1(f)X_1.\] (10)

We write

\[\nabla X_a = \omega^b_a X_b, \quad \omega^b_a \in C(T^*U).\]

From the Codazzi equations for $M$, for mutually distinct $a, b, d = 1, 2, 3$, we get

\[X_a(k_b) = (k_a - k_b)\omega^b_a(X_b)\] (11)
From (18) we have:

\[(k_0 - k_d)\omega_i^0(X_0) = (k_0 - k_d)\omega_i^0(X_0).\]

Consider now in (11), \(a = 1\) and \(b = i\) and, respectively, \(a = i\) and \(b = j\) with \(i \neq j\). We obtain

\[\omega_i^1(X_i) = \frac{X_1(k_i)}{k_i - k_1}\]

and

\[\omega_j^1(X_j) = \frac{X_1(k_j)}{k_j - k_i} \]

For \(a = i\) and \(b = 1\), as \(X_i(k_1) = 0\), (11) leads to \(\omega_i^1(X_1) = 0\) and we can write

\[\omega_i^1(X_1) = 0, \quad a = 1, 2, 3.\]

Notice that, since \(X_i(f) = 0\), then \([X_i, X_j, X_1] = 0\), thus \(\omega_i^1(X_i) = \omega_i^1(X_j)\). Now, from (12), for \(a = 1\), \(b = i\) and \(d = j\), with \(i \neq j\), we get

\[\omega_j^2(X_3) = \omega_i^2(X_1) = \omega_i^2(X_2) = 0.\]

The structure 1-forms are thus determined by the following set of relations

\[\begin{align*}
\omega_i^2(X_1) &= 0, & \omega_i^2(X_2) &= \frac{X_i(k_2)}{k_2 + k_3} = \alpha_2, & \omega_i^2(X_3) &= 0, \\
\omega_i^3(X_1) &= 0, & \omega_i^3(X_2) &= 0, & \omega_i^3(X_3) &= \frac{X_i(k_3)}{k_3 + k_2} = \alpha_3, \\
\omega_i^3(X_1) &= 0, & \omega_i^2(X_2) &= \frac{X_i(k_2)}{k_3 - k_2} = \beta_2, & \omega_i^2(X_3) &= \frac{X_i(k_3)}{k_3 - k_2} = \beta_3,
\end{align*}\]

In order to express the first condition in (7), by using (8), we compute

\[|A|^2 = k_1^2 + k_2^2 + k_3^2 = \frac{45}{2} - f^2 - 2K,\]

where \(K\) denotes the product \(k_2 k_3\). From (10) we deduce that

\[\Delta f = -\text{div} \text{grad} f = -\text{div} (X_1(f) X_1) = -X_1(X_1(f)) - X_1(f) \text{div} X_1 = -X_1(X_1(f)) + X_1(f) (\omega_i^2(X_2) + \omega_i^3(X_3)) = -X_1(X_1(f)) + X_1(f) (\alpha_2 + \alpha_3).\]

Now, by using (14) and (15), the equation \(\Delta f = (3c - |A|^2) f\) becomes

\[X_1(X_1(f)) - X_1(f)(\alpha_2 + \alpha_3) + (2K + 3c - \frac{45}{2} f^2) f = 0.\]

We also compute

\[\begin{align*}
[X_1, X_i] &= \nabla X_i X_i - \nabla X_i X_i = \langle \nabla X_i X_i, X_i \rangle X_i - \langle \nabla X_i X_i, X_i \rangle X_i \\
&= \omega_i^1(X_i) X_i = \alpha_i X_i.
\end{align*}\]

We shall now use the Gauss equation

\[\langle R^{E^4}(c) (X, Y) Z, W \rangle = \langle B(Z, X), B(Y, W) \rangle + \langle B(Z, Y), B(X, W) \rangle - \langle B(Z, Y), B(Y, W) \rangle.\]

From (18) we have:
On the other hand, by using (19) and (26), we obtain

\[
X_1(\alpha_2) = \alpha_2^2 + c - \frac{2f}{\bar{f}}k_2, \\
X_1(\alpha_3) = \alpha_3^2 + c - \frac{2f}{\bar{f}}k_3;
\]  

(19)

- for \( X = W = X_1 \) and \( Y = Z = X_i \)

\[
\begin{align*}
X_2(\alpha_3) &= \beta_3(\alpha_3 - \alpha_2), \\
X_3(\alpha_2) &= \beta_2(\alpha_3 - \alpha_2);
\end{align*}
\]

(21)

- for \( X = W = X_2 \) and \( Y = Z = X_3 \)

\[
K + c = X_2(\beta_3) - X_3(\beta_2) - \alpha_2\alpha_3 - \beta_2^2 - \beta_3^2;
\]

(20)

- for \( Y = W = X_3, X = X_2 \) and \( Z = X_1 \) and, respectively, for \( X = W = X_2, Y = X_3 \) and \( Z = X_1 \)

\[
\begin{align*}
X_1(\beta_2) &= \alpha_2\beta_2, \\
X_1(\beta_3) &= \alpha_3\beta_3.
\end{align*}
\]

(22)

Notice now that, from (9) and (17),

\[
X_i(X_1(f)) = -[X_1, X_i]f + X_1(X_i(f)) = -\alpha_\iota X_1(f) + X_1(X_i(f)) = 0
\]

and

\[
X_i(X_1(X_i(f))) = 0.
\]

(23)

(24)

Also, since \( K = \frac{(k_2 + k_3)^2 - (k_3 - k_2)^2}{4} \) we obtain

\[
\begin{align*}
X_2(K) &= -(k_3 - k_2)^2 \beta_3, \\
X_3(K) &= (k_3 - k_2)^2 \beta_2.
\end{align*}
\]

(25)

We differentiate (16) along \( X_2 \) and \( X_3 \) and use (21), (23), (24) and (25). We get

\[
\begin{align*}
X_2(\alpha_2) &= -\beta_3(\alpha_3 - \alpha_2) - \frac{2f}{X_1(f)}(k_3 - k_2)^2 \beta_3, \\
X_3(\alpha_3) &= -\beta_2(\alpha_3 - \alpha_2) + \frac{2f}{X_1(f)}(k_3 - k_2)^2 \beta_2.
\end{align*}
\]

(26)

We intend to prove that \( X_i(k_j) = 0, i, j = 2, 3 \). In order to do this we apply \([X_1, X_2] = \alpha_2 X_2\) to the quantity \( \alpha_2 \). On one hand, from (26), we get

\[
[X_1, X_2] \alpha_2 = \alpha_2 X_2(\alpha_2) = \beta_3 \left\{ -\alpha_2 \alpha_3 + \alpha_2^2 - \frac{2f}{X_1(f)}(k_3 - k_2)^2 \alpha_2 \right\}.
\]

(27)

On the other hand, by using (19) and (26), we obtain

\[
[X_1, X_2] \alpha_2 = X_1(X_2(\alpha_2)) - X_2(X_1(\alpha_2)) = \beta_3 \left\{ -2\alpha_3^2 - \alpha_2^3 + 3\alpha_2 \alpha_3 + \frac{2f}{X_1(f)} \left[ -2(k_3 - k_2)X_1(k_3 - k_2) \\
+ (k_3 - k_2)^2(2\alpha_2 - \alpha_3) \right] - 2X_1 \left( \frac{f}{X_1(f)} \right) (k_3 - k_2)^2 \right\}.
\]

(28)
We rewrite equations (19) in the form
\[ X_1 \left( \frac{f}{X_1(f)} \right) = -\frac{(\alpha_3 - \alpha_2)^2}{(k_3 - k_2)^2} + \frac{f}{X_1(f)} \left( 3\alpha_2 - \alpha_3 - 2\frac{X_1(k_3 - k_2)}{k_3 - k_2} \right). \] (29)

Moreover,
\[ X_2 \left( \frac{X_1(k_3 - k_2)}{k_3 - k_2} \right) = -\frac{1}{k_3 - k_2} [X_1, X_2](k_3 - k_2) + X_1 \left( \frac{X_2(k_3 - k_2)}{k_3 - k_2} \right) = 2(\alpha_3 - \alpha_2)\beta_3. \]

Suppose that \( \beta_3 \neq 0 \), differentiate (29) along \( X_2 \) and use (21) and (26). We get
\[ 2(\alpha_3 - \alpha_2) = -\frac{f}{X_1(f)}(k_3 - k_2)^2. \] (30)

We differentiate now (30) along \( X_2 \) and obtain
\[ \alpha_3 - \alpha_2 = -2\frac{f}{X_1(f)}(k_3 - k_2)^2, \] (31)

and since \( k_2 \neq k_3 \) the equations (30) and (31) lead to a contradiction.

Analogously, by using the symmetry of the equations in \( X_2 \) and \( X_3 \), we immediately prove that \( \beta_2 = 0 \).

We rewrite equations (19) in the form
\[ \begin{align*}
X_1(X_1(k_2)) &= \frac{34}{2} \alpha_2 X_1(f) + 2(K + c)(k_3 + \frac{3}{2}f) + (c - \frac{3}{2}fk_2)(k_2 + \frac{3}{2}f), \\
X_1(X_1(k_3)) &= \frac{34}{2} \alpha_3 X_1(f) + 2(K + c)(k_2 + \frac{3}{2}f) + (c - \frac{3}{2}fk_3)(k_3 + \frac{3}{2}f),
\end{align*} \] (32)

and by summing up we obtain
\[ X_1(X_1(f)) = \frac{7}{3} X_1(f)(\alpha_2 + \alpha_3) + f(4K + 5c - 9f^2). \] (33)

Now, by using (16) and (33) we obtain
\[ X_1(f)(\alpha_2 + \alpha_3) = f \left( -\frac{9}{2}K - 6c + \frac{189}{8}f^2 \right). \] (34)

We replace (34) in (33) and get
\[ X_1(X_1(f)) = f \left( -\frac{13}{2}K - 9c + \frac{369}{8}f^2 \right). \] (35)

In order to get another relation on \( f \) and \( K \) we first use (20), (19), (8), (13) and determine
\[ X_1(K) = -X_1(\alpha_2 \alpha_3) \] (36)
\[ = -\alpha_2 \alpha_3 + c)(\alpha_2 + \alpha_3) + \frac{3}{2}f(\alpha_2 k_3 + \alpha_3 k_2) \]
\[ = (K + 9f^2)(\alpha_2 + \alpha_3) - \frac{27}{4}f X_1(f). \]

By differentiating (34) along \( X_1 \), and by using (36), (35), (19), (34) we get
\[ X_1(f) \left( \frac{13}{2}K + 10c - 108f^2 \right) = f(\alpha_2 + \alpha_3) \left( \frac{13}{2}K + 15c - \frac{441}{4}f^2 \right). \] (37)

We multiply (37) first by \( X_1(f) \) and secondly by \( \alpha_2 + \alpha_3 \) and, by using (34), we get
\[ \begin{align*}
\left( X_1(f) \right)^2 \left( \frac{13}{2}K + 10c - 108f^2 \right) &= f^2 \left( -\frac{9}{2}K - 6c + \frac{189}{8}f^2 \right) \left( \frac{13}{2}K + 15c - \frac{441}{4}f^2 \right), \\
\left( \frac{13}{2}K + 10c - 108f^2 \right) \left( -\frac{9}{2}K - 6c + \frac{189}{8}f^2 \right) &= \left( \alpha_2 + \alpha_3 \right)^2 \left( \frac{13}{2}K + 15c - \frac{441}{4}f^2 \right),
\end{align*} \] (38)
Differentiating (34) along \( X_1 \), and using (36), (35), (19), (34), (38), we obtain
\[
27f^2(4044800c^3 - 49579440c^2f^2 + 187840944cf^4 - 254205945f^6) \\
-6(51200c^3 - 19600320c^2f^2 + 119328660cf^4 - 80969301f^6)K \\
-208(2240c^2 - 108396cf^2 - 285363f^4)K^2 \\
+2704(16c - 2277f^2)K^3 \\
+140608K^4 = 0.
\]

Consider now \( \gamma = \gamma(t) \), \( t \in I \), to be an integral curve of \( X_1 \) passing through \( p = \gamma(t_0) \). Since \( X_2(f) = X_3(f) = 0 \) and \( X_2(K) = X_3(K) = 0 \) and \( X_1(f) \neq 0 \), we can write \( t = t(f) \) in a neighborhood of \( f_0 = f(t_0) \) and thus consider \( K = K(f) \).

Notice that if \( \frac{13}{2}K + 15c - \frac{441}{4}f^2 = 0 \) or \( 10c + \frac{13}{2}K - 108f^2 = 0 \), then from (39) the function \( f \) results to be the solution of a polynomial equation of eighth degree with constant coefficients and we would get to a contradiction. Thus, from (38) we have that
\[
\left\{
\begin{array}{l}
\left( \frac{df}{df} \right)^2 = \frac{f^2(-6K - 6c + \frac{441}{4}f^2)\left( \frac{13}{2}K + 15c - \frac{441}{4}f^2 \right)}{\frac{13}{2}K + 15c - \frac{441}{4}f^2}, \\
(\alpha_2 + \alpha_3)^2 = \frac{\left( \frac{13}{2}K + 10c - 108f^2 \right)\left( \frac{13}{2}K - 6c + \frac{441}{4}f^2 \right)}{\frac{13}{2}K + 15c - \frac{441}{4}f^2}.
\end{array}
\right.
\]

We can now compute \( \frac{dK}{df} \) by using (40), (36) and (34),
\[
\frac{dK}{df} = \frac{dK}{dt} \cdot \frac{dt}{df} = \frac{(K + 9f^2)\frac{df}{df}(\alpha_2 + \alpha_3)}{(\frac{df}{df})^2} - \frac{27}{4}f = \frac{(K + 9f^2)(\frac{13}{2}K + 10c - 108f^2)}{f(\frac{13}{2}K + 15c - \frac{441}{4}f^2)} - \frac{27}{4}f.
\]

The next step consists in differentiating (39) with respect to \( f \). By substituting \( \frac{dK}{df} \) from (41) we get another polynomial equation in \( f \) and \( K \), of fifth degree in \( K \). We eliminate \( K^5 \) between this new polynomial equation and (39). The result constitutes a polynomial equation in \( f \) and \( K \), of fourth degree in \( K \). In a similar way, by using (39) and its consequences we are able to gradually eliminate \( K^4 \), \( K^3 \), \( K^2 \) and \( K \) and we are led to a polynomial equation with constant coefficients in \( f \). Thus \( f \) results to be constant and we conclude.

In [15] the authors proved that there exist no proper biharmonic hypersurfaces in \( \mathbb{R}^4 \). By using Theorem 3.3 we reobtain this result and we can extend it to the 4-dimensional hyperbolic space.

**Theorem 3.4** There exist no proper biharmonic hypersurfaces in \( \mathbb{R}^4 \) or \( \mathbb{H}^4 \).

**Proof.** Suppose that \( M^3 \) is a proper biharmonic hypersurface in \( \mathbb{R}^4 \) or \( \mathbb{H}^4 \). From Theorem 3.3, the mean curvature of \( M \) is constant, and applying Theorem 2.1 we obtain \( |A|^2 = 0 \) or \( |A|^2 = -3 \), respectively, and we conclude.

We can now state the main result of the paper.

**Theorem 3.5** The only proper biharmonic compact hypersurfaces of \( S^4 \) are the hypersphere \( S^3(\frac{1}{\sqrt{2}}) \) and the torus \( S^1(\frac{1}{\sqrt{2}}) \times S^2(\frac{1}{\sqrt{2}}) \).
Proof. Suppose that $M^3$ is a compact proper biharmonic hypersurface of $S^4$. From Theorem 3.3 it results that $M$ has constant mean curvature and, since it satisfies the hypotheses of Proposition 2.6, we conclude that it also has constant scalar curvature. We can thus apply Theorem 2.5 and it results that $M$ is isoparametric in $S^4$. From Theorem 3.1 we get that $M$ cannot be isoparametric with $\ell = 3$, and by using Theorem 1.1 we conclude the proof.

Remark 3.6 Since we have achieved the classification of proper biharmonic curves (see [3]) and compact proper biharmonic hypersurfaces, our interest is now in proper biharmonic surfaces of $S^4$. Using Theorem 2.2 and a well known result of Lawson [17], it was proved, in [3], the existence of closed orientable embedded proper biharmonic surfaces of arbitrary genus in $S^4$. All this surfaces are minimal surfaces of $S^3(\frac{1}{\sqrt{2}})$. Moreover, Theorem 2.3 cannot be applied due to the dimensions involved, thus it does not generate examples in this case. Then it is natural to propose the following

Open problem

*Are there other proper biharmonic surfaces in $S^4$, apart from the minimal surfaces of $S^3(\frac{1}{\sqrt{2}})$?*

References

[1] A. Balmuş, S. Montaldo, C. Oniciuc. Classification results for biharmonic submanifolds in spheres. *Israel J. Math.*, to appear.
[2] R. Caddeo, S. Montaldo, C. Oniciuc. Biharmonic submanifolds of $S^3$. *Int. J. Math.* 12 (2001), 867–876.
[3] R. Caddeo, S. Montaldo, C. Oniciuc. Biharmonic submanifolds in spheres. *Israel J. Math.* 130 (2002), 109–123.
[4] E. Cartan. Sur des familles remarquables d’hypersurfaces isoparamétriques dans les espaces sphériques. *Math. Z.* 45 (1939), 335–367.
[5] S. Chang. On closed hypersurfaces of constant scalar curvatures and mean curvatures in $S^{n+1}$. *Pacific J. Math.* 165 (1994), 67–76.
[6] S. Chang. A closed hypersurface with constant scalar curvature and mean curvature in $S^4$ is isoparametric. *Comm. Anal. Geom* 1 (1993), 71–100.
[7] B-Y. Chen. *Total mean curvature and submanifolds of finite type*. Series in Pure Mathematics, 1. World Scientific Publishing Co., Singapore, 1984.
[8] B-Y. Chen, S. Ishikawa. Biharmonic pseudo-Riemannian submanifolds in pseudo-Euclidean spaces. *Kyushu J. Math.* 52 (1998), 167–185.
[9] F. Defever. Hypersurfaces of $E^4$ satisfying $\Delta \vec{H} = \lambda \vec{H}$. *Mich. Math. J.* 44 (1997), 355–363.
[10] F. Defever. Hypersurfaces of $E^4$ with harmonic mean curvature vector. *Math. Nachr.* 196 (1998), 61–69.
[11] F. Defever, G. Kaimakamis, V. Papantoniou. Biharmonic hypersurfaces of the 4-dimensional semi-Euclidean space $E^4_1$. *J. Math. Anal. Appl.* 315 (2006), 276–286.
[12] I. Dimitric. Submanifolds of $E^n$ with harmonic mean curvature vector. *Bull. Inst. Math. Acad. Sinica* 20 (1992), 53–65.
[13] J. Eells, J.H. Sampson. Harmonic mappings of Riemannian manifolds. *Amer. J. Math.* 86 (1964), 109–160.
[14] G.Y. Jiang. 2-harmonic isometric immersions between Riemannian manifolds. *Chinese Ann. Math. Ser. A* 7 (1986), 130–144.
[15] T. Hasanis, T. Vlachos. Hypersurfaces in $E^4$ with harmonic mean curvature vector field. *Math. Nachr.* 172 (1995), 145–169.
[16] T. Ichiyama, J. Inoguchi, H. Urakawa. Bi-harmonic maps and bi-Yang-Mills fields, to appear.
[17] H.B. Lawson. Complete minimal surfaces in $S^3$. *Ann. of Math.* 92 (1970), 335–374.
[18] C. Oniciuc. Biharmonic maps between Riemannian manifolds. *An. Stiint. Univ. Al.I. Cuza Iasi Mat. (N.S.)* 48 (2002), 237–248.
[19] P.J. Ryan. Homogeneity and some curvature conditions for hypersurfaces. *Tohoku Math. J.* 21 (1969), 363–368.
[20] The Bibliography of Biharmonic Maps, http://beltrami.sc.unica.it/biharmonic/.