Behaviour of trajectories near a two-cycle heteroclinic network

Olga Podvigina

Institute of Earthquake Prediction Theory and Mathematical Geophysics, Moscow, Russian Federation

ABSTRACT

Heteroclinic networks and cycles are invariant sets comprised of interacting nodes connected by heteroclinic trajectories. Often the sets are not asymptotically stable but attract a positive measure set from its small neighbourhood. This property is called fragmentary asymptotic stability (f.a.s.). The definition implies that if a stable cycle is a subset of a heteroclinic network, then the entire network is stable. In general, the converse is wrong. In the examples given in the literature, the presence of spiralling due to complex eigenvalues in the linearization around an equilibrium implies switching between subcycles of the f.a.s. network, thus preventing individual cycles from being stable. We study the behaviour of trajectories near a heteroclinic network comprised of two cycles where the eigenvalues of the linearizations are real. The trajectories can be attracted to one of the cycles, or they can switch regularly or irregularly between them. To describe regular switching, we introduce the notions of an omnicycle and its trail-stability, and prove conditions for trail-stability of an omnicycle in the considered network.

1. Introduction

Heteroclinic cycles are comprised of saddle equilibria (or more complex invariant sets) cyclically connected by heteroclinic trajectories. Heteroclinic network is a connected union of at least two and finitely many heteroclinic cycles. These objects do not exist in a generic numerical system, but often emerge in a system with intrinsic constraints, such as symmetries, the existence of invariant ‘extinction subspaces’ (e.g., Lotka–Volterra system) or due to the construction (coupled cells or oscillators).

An invariant set of a dynamical system is called asymptotically stable if it attracts all trajectories from its small neighbourhood. A non-asymptotically stable set can be stable in a weaker sense and attract a positive measure set of initial conditions from the neighbourhood. In [20], this property was called fragmentary asymptotic stability (f.a.s.). Conditions for asymptotic stability or fragmentary asymptotic stability for certain types of heteroclinic cycles are known for a long time, see, e.g. [9,19,20,23,28] and references therein. The study of stability of a heteroclinic network is more difficult, because a trajectory can follow different sequences of connections along the network.
The definition of fragmentary asymptotic stability implies that if a subcycle of a heteroclinic network is f.a.s., then the whole network is f.a.s. The converse is not true. Suppose that the linearization of the flow near one of the equilibria in the network possesses complex eigenvalues and consider a trajectory in a small neighbourhood of this equilibrium. A small perturbation of the trajectory results on a substantial change of the ratio of the coordinates in the respective eigenspace for the trajectory leaving the neighbourhood. Because of this change, the perturbed trajectory follows a different path (sequence of connections) along the network. Examples of attracting networks with subcycles that are unstable due to complex eigenvalues can be found, e.g. in [1,14,16].

When all eigenvalues of linearization are real, it is more difficult to find such an example. For all networks in $\mathbb{R}^4$ studied in the literature, fragmentary asymptotic stability of a network implies that at least one of its subcycles is f.a.s. [3,5,7,15,24–26]. The same property holds true for some networks in higher-dimensional subspaces, see [10,27].

Numerical results reported in [29,30] indicate that the considered networks can be f.a.s., while all subcycles are completely unstable. However, in [29] the authors did not derive the conditions for the stability of the network, as well as for its subcycle involving six equilibria. The network considered in [30] involves subcycles with 3, 4 or 5 equilibria. The stability of these subcycles was further investigated in [8], where it was shown that the network can be asymptotically stable, while none of its subcycles are.

In this paper, we give an example of an f.a.s. network in $\mathbb{R}^6$ which is a union of two unstable cycles. The network is comprised of six equilibria connected by seven heteroclinic trajectories, and its simplicity allows a detailed study of the behaviour of nearby trajectories.

To prove the stability of the network, we introduce the notion of an omnicycle, an ordered sequence of nodes and heteroclinic connections, where repeating nodes are allowed. Omnicycles were implicitly employed in [29] to describe the behaviour of trajectories near a heteroclinic network, where the existence of an attracting omnicycle was indicated by so-called ‘regular cycling’. In [30], the expression ‘root sequence’ was used to describe the behaviour of trajectories following the connections in the prescribed order. However, the terminology introduced *ibid* was closely related to the specific structure of the considered networks, while the definition introduced in this paper is of a general type.

Omnicycles, similarly to heteroclinic cycles and network, do not exist in a generic dynamical systems, because a connection between saddles can be destroyed by a small perturbation. However, they may exist and be robust in a system where some restrictions are imposed, e.g. symmetries [17,18], natural constraints, as in population dynamics or evolutionary game theory [10,12,13] or due to prescribed patterns of interaction, as in a system of coupled cells or oscillators [2,4]. Typically, conditions for the stability of a cycle are derived by following nearby trajectories and constructing the respective transition maps. Since it is irrelevant how many times each particular node or connection is present in the sequence, the existing conditions for the stability of cycles can be straightforwardly employed for the trail-stability of omnicycles.

The network under consideration is a union of two cycles, $\mathcal{X} = \mathcal{C}_L \cup \mathcal{C}_R \subset \mathbb{R}^6$, admitted by a $\mathbb{Z}_2^6$-equivariant dynamical system
\[
\mathbf{x} = f(\mathbf{x}), \quad f : \mathbb{R}^6 \to \mathbb{R}^6, \tag{1}
\]
where $\mathbb{Z}_2^6$ is generated by symmetries reversing the sign of one of the spatial coordinates. Six equilibria involved in the network belong to coordinate axes and heteroclinic connections
belong to coordinate planes. The cycles are:

\[ C_L : (\xi_1 \to \xi_2 \to \xi_3 \to \xi_4 \to) \quad \text{and} \quad C_R : (\xi_1 \to \xi_2 \to \xi_5 \to \xi_6 \to). \]

In terminology of [20], \( \mathcal{X} \) is a type Z heteroclinic network and \( C_L \) and \( C_R \) are type Z heteroclinic cycles. The generalization of this classification to omnicycles implies that any omnicycle, which is a subset of \( \mathcal{X} \), is a type Z omnicycle. Theorems in [20] that give necessary and sufficient conditions for a type Z heteroclinic cycle to be f.a.s. are also applicable to study trail-stability of type Z heteroclinic omnicycles.

The conditions are inequalities that involve eigenvalues and eigenvectors of transition matrices, whose entries depend on eigenvalues of Jacobians \( df(\xi_j) \). For \( C_L \) and \( C_R \), the respective \( 4 \times 4 \) transitions matrices are sparse, hence allowing to calculate eigenvalues and eigenvectors. Transition matrix of the omnicycle \( C_{LR} \), where

\[ C_{LR} : (\xi_1 \to \xi_2 \to \xi_3 \to \xi_4 \to \xi_1 \to \xi_2 \to \xi_5 \to \xi_6 \to), \]

is full, therefore eigenvalues and eigenvectors cannot be found analytically in a general case. To be able to calculate them, we assume that some eigenvalues of \( df(\xi_j) \) are much larger than the others. For vanishing small eigenvalues, we calculate eigenvalues and eigenvectors of the respective sparse transition matrix. Because of the continuous dependence of the eigenvalues and eigenvectors of a matrix on its entries, the stability or instability conditions are satisfied for small non-vanishing eigenvalues of \( df(\xi_j) \) as well.

This paper is organized as follows: in Section 2, we recall basic definitions of robust heteroclinic cycles, networks and fragmentary asymptotic stability and introduce the notions of omnicycles and trail-stability. In Section 3, we discuss trail-stability of omnicycles. In Sections 4 and 5, we study analytically and numerically fragmentary asymptotic stability of the network shown in Figure 1. Namely, in Section 4 we derive conditions for the stability of the cycles \( C_L \) and \( C_R \) and show that the network can be f.a.s. while both cycles are completely unstable. In Section 5, we present results of numerical simulations of the solutions of system (1), in particular, in the cases when \( C_L \) and \( C_R \) are completely unstable,\[ \text{Figure 1. The network studied in Sections 4 and 5. The cross-sections } H_L \text{ and } H_R \text{ employed in the derivation of conditions of stability are introduced in Section 3.} \]
and $C_{LR}$ is not trail-stable. By varying the r.h.s. of (1), we obtain examples of trail-stable omnicycles, different from $C_{LR}$, or of irregular switching between $C_L$ and $C_R$.

2. Background and definitions

In this section, we introduce the notions of omnicycles and trail-stability, and recall definitions that will be used in the following sections.

2.1. Robust heteroclinic cycles, omnicycles and networks

Consider a smooth $\Gamma$-equivariant dynamical systems

$$\dot{x} = f(x), \quad f(\gamma x) = \gamma f(x), \quad \text{for all } \gamma \in \Gamma \text{ and } x \in \mathbb{R}^n,$$

where we assume that $\Gamma \subset O(n)$ is finite. For a group $\Gamma$ acting on $\mathbb{R}^n$ the isotropy group of the point $x \in \mathbb{R}^n$ is the subgroup $\Sigma_x = \{ \gamma \in \Gamma : \gamma x = x \}$, and a fixed-point subspace of a subgroup $\Sigma \subset \Gamma$ is the subspace $\text{Fix}(\Sigma) = \{ x \in \mathbb{R}^n : \sigma x = x \text{ for all } \sigma \in \Sigma \}$. (For more details about equivariant dynamical systems, see, e.g. Golubitsky and Stewart [11].)

Let $\xi_1, \ldots, \xi_m$ be hyperbolic equilibria of (2), where $\xi_j \neq \xi_k$ for any $1 \leq j, k \leq m, j \neq k$, and $\kappa_{jj+1} \neq \emptyset, \kappa_{jj+1} \subset W^u(\xi_j) \cap W^s(\xi_{j+1})$, be a set of trajectories from $\xi_j$ to $\xi_{j+1}$, where $\xi_{m+1} = \xi_1$ is assumed. A heteroclinic cycle is an invariant set $X \subset \mathbb{R}^n$ which is a union of equilibria $\{\xi_1, \ldots, \xi_m\}$ and heteroclinic connections $\{\kappa_{12}, \ldots, \kappa_{m1}\}$. A heteroclinic network is a connected union of a finite number of heteroclinic cycles.

Definition 2.1: An omnicycle is a union of equilibria $\{\xi_1, \ldots, \xi_m\}$ and heteroclinic connections, $\{\kappa_{12}, \ldots, \kappa_{m1}\}$, where $\kappa_{ij} \subset W^u(\xi_i) \cap W^s(\xi_j)$ and $\kappa_{ij} \neq \emptyset$.

I.e. the difference between a heteroclinic cycle and an omnicycle is that for the omnicycle repeating equilibria are allowed.

By analogy with a building block of a heteroclinic cycle [24], we introduce building block of an omnicycle.

Definition 2.2: Given an omnicycle $\mathcal{X} = \{\xi_1, \ldots, \xi_m; \kappa_{12}, \ldots, \kappa_{m1}\}$, we say that its subset $\{\xi_1, \ldots, \xi_i; \kappa_{12}, \ldots, \kappa_{ij} \}$ and the symmetry $\gamma \in \Gamma$ are a building block of $\mathcal{X}$ if $\gamma \xi_i = \xi_{i+j}$ and $\gamma \kappa_{i,i+1} = \kappa_{i+j,i+1}$ for any $\xi_i \in \mathcal{X}$ and $\kappa_{i,i+1} \subset \mathcal{X}$.

A heteroclinic cycle (or an omnicycle, or a network) $\mathcal{X}$ is called robust, if any connection $\kappa_{ij} \subset \mathcal{X}$ belongs to a flow-invariant subspace $P_{ij}$. In system (2), the invariance of $P_{ij}$ typically follows from its equivariance, namely that $P_{ij} = \text{Fix}(\Sigma_{ij})$ for some $\Sigma_{ij} \subset \Gamma$. By $\Delta_j$, we denote the isotropy subgroup of $\xi_j$ and $L_j = \text{Fix}(\Delta_j)$. Denote by $P_{ij}^\perp$ and by $L_j^\perp$ the orthogonal complement to $P_{ij}$ and $L_j$, respectively, in $\mathbb{R}^n$.

Given an equilibrium $\xi_j \in L_j = P_{j-1,j} \cap P_{j,j+1}$ that belongs to a robust heteroclinic cycle $\mathcal{C} : (\xi_1 \to \xi_2 \to \cdots \to \xi_m \to)$, the eigenvalues of the Jacobian $df(\xi_j)$ can be divided into radial (the associated eigenvectors belong to $L_j$), contracting (the eigenvectors belong to $P_{j-1,j} \ominus L_j$), expanding (the eigenvectors belong to $P_{j,j+1} \ominus L_j$) and transverse (the remaining ones), where $P \ominus L$ denotes the orthogonal complement to $L$ in $P$. For an omnicycle, the eigenvalues of linearizations can be similarly split into four groups. However, if in
the sequence \( \{\xi_1, \ldots, \xi_m\} \) we have that \( \xi_i = \xi_j \) then the groupings of eigenvalues of the Jacobians \( df(\xi_i) \) and \( df(\xi_j) \) can be different.

**Definition 2.3 (Adapted from [20]):** A heteroclinic network \( \mathcal{X} \) (cycle or omnicycle) is called of type \( Z \) if

- for any \( \xi_j \in \mathcal{X} \) we have that \( \dim P_{ij} = \dim P_{jk} \) for any incoming connection \( \kappa_{i,j} \) and any outgoing connection \( \kappa_{j,k} \);
- for any \( \kappa_{i,j} \subset \mathcal{X} \) the isotropy subgroup \( \Sigma_{i,j} \) decomposes \( P_{i,j} \) into one-dimensional isotropic components.

### 2.2. Stability

Denote by \( \Phi(\mathbf{x}, \tau) \) a trajectory of system (2) through the point \( \mathbf{x} \in \mathbb{R}^n \) and by \( \Phi(\mathbf{x}, (\tau_1, \tau_2)) = \bigcup_{\tau \in (\tau_1, \tau_2)} \Phi(\mathbf{x}, \tau) \) the subset of this trajectory corresponding to the time interval \( (\tau_1, \tau_2) \). For a set \( \mathcal{X} \subset \mathbb{R}^n \) and \( \varepsilon > 0 \), the \( \varepsilon \)-neighbourhood of \( \mathcal{X} \) is

\[
B_\varepsilon(\mathcal{X}) = \{ \mathbf{x} \in \mathbb{R}^n : d(\mathbf{x}, \mathcal{X}) < \varepsilon \}.
\]  

Given a compact invariant set \( \mathcal{X} \) of (2), its \( \delta \)-local basin of attraction defined as

\[
B_\delta(\mathcal{X}) = \{ \mathbf{x} \in \mathbb{R}^n : d(\Phi(\mathbf{x}, \tau), \mathcal{X}) < \delta \text{ for any } \tau \geq 0 \text{ and } \lim_{\tau \to \infty} d(\Phi(\mathbf{x}, \tau), \mathcal{X}) = 0 \}.
\]

**Definition 2.4:** An invariant set \( \mathcal{X} \) is called *asymptotically stable*, if for any \( \delta > 0 \) there exists an \( \varepsilon > 0 \) such that

\[
B_\varepsilon(\mathcal{X}) \subset B_\delta(\mathcal{X}).
\]

**Definition 2.5:** An invariant set \( \mathcal{X} \) is called *fragmentarily asymptotically stable*, if for any \( \delta > 0 \)

\[
\mu(B_\delta(\mathcal{X})) > 0.
\]

(Here \( \mu \) is the Lebesgue measure of a set in \( \mathbb{R}^n \).)

**Definition 2.6:** An invariant set \( \mathcal{X} \) is called *completely unstable*, if there exists \( \delta > 0 \) such that \( \mu(B_\delta(\mathcal{X})) = 0 \).

**Definition 2.7:** Given an omnicycle \( \mathcal{X} = \{\xi_1, \ldots, \xi_m; \kappa_{1,2}, \ldots, \kappa_{m,1}\} \), we say that a trajectory \( \Phi(\mathbf{x}, (\tau^m, \tau^m)) \) *\( \varepsilon \)-follows* this omnicycle if there exists a number \( 1 \leq j_0 \leq m \) and time instances \( \{\tau_0', \tau_0'', \tau_1', \tau_1'', \ldots, \tau_s', \tau_s''\} \), where \( \tau^m \leq \tau_0' < \tau_0'' < \tau_1' < \cdots < \tau_s' < \tau_s'' \leq \tau^m \), such that

\[
\Phi(\mathbf{x}, (\tau_0', \tau_0'')) \subset B_\varepsilon(\xi_j) \text{ and } \Phi(\mathbf{x}, (\tau_s', \tau_s'')) \subset B_\varepsilon(\kappa_{j,j+1}) \ \big( B_\varepsilon(\xi_j) \cup B_\varepsilon(\kappa_{j,j+1}) \big),
\]

where \( j = j_0 + s \text{ (mod } m) \), for all \( 0 \leq s \leq S \).

Consider a trajectory that *\( \varepsilon \)-follows* the omnicycle \( \mathcal{X} \). According to the definition, for \( \tau^m < \tau < \tau_0' \) the trajectory belongs to the \( \varepsilon \)-neighbourhood of \( \kappa_{j_0-1,j_0} \), for \( \tau_0' < \tau < \tau_0'' \) to the neighbourhood of \( \xi_{j_0} \), for \( \tau_0'' < \tau < \tau_1' \) to the neighbourhood of \( \kappa_{j_0,j_0+1} \) and so
on. That is, the trajectory stays in the $\varepsilon$-neighbourhood of the omnicycle and follows the connections in the prescribed order.

**Definition 2.8:** Given an omnicycle $\mathcal{X}$ of (2), its $\delta$-local basin of attraction is defined as

$$
B_{\delta}^{\text{omni}}(\mathcal{X}) = \{ x \in \mathbb{R}^n : \Phi(x, (0, \tau)) \delta - \text{follows} \mathcal{X} \text{ for any } \tau > 0 \text{ and } \lim_{\tau \to \infty} d(\Phi(x, \tau), \mathcal{X}) = 0 \}.
$$

**Definition 2.9:** An omnicycle $\mathcal{X}$ is called *trail-stable*, if for any $\delta > 0$

$$
\mu(B_{\delta}^{\text{omni}}(\mathcal{X})) > 0.
$$

**Theorem 2.1:** Suppose that a trail-stable omnicycle $\mathcal{X}$ is a subset of a heteroclinic network $X$. Then the network is f.a.s.

**Proof:** Trail-stability of $\mathcal{X}$ implies that $\mu(B_{\delta}^{\text{omni}}(\mathcal{X})) > 0$ for any positive $\delta$. Since $\mathcal{X} \subseteq X$ and due to definitions (2.7–2.8) we have that $B_{\delta}^{\text{omni}}(\mathcal{X}) \subseteq B_{\delta}(X)$. Therefore, $\mu(B_{\delta}(X)) > 0$ for any $\delta > 0$ and in agreement with definition 2.5 the set $X$ is f.a.s. QED

**Remark 2.1:** For an omnicycle which is a heteroclinic cycle, the notions of f.a.s. and trail-stability are equivalent, because trajectories near the cycle visit the equilibria in the prescribed order specified in the definition of the cycle. An omnicycle, which is not a cycle, can be f.a.s. according to Definition 2.5 without being trail-stable. For example, consider the network [15] comprised of two heteroclinic cycles, $(\xi_1 \rightarrow \xi_2 \rightarrow \xi_3 \rightarrow)$ and $(\xi_1 \rightarrow \xi_2 \rightarrow \xi_4 \rightarrow)$. The results *ibid* imply that the omnicycle $(\xi_1 \rightarrow \xi_2 \rightarrow \xi_3 \rightarrow \xi_1 \rightarrow \xi_2 \rightarrow \xi_4 \rightarrow)$ never is trail-stable. As a subset of the phase space, the omnicycle coincides with the network, which can be f.a.s.

**Remark 2.2:** Suppose $\mathcal{X}$ is a robust type Z omnicycle, which is not a heteroclinic cycle, in a $\Gamma$-equivariant system. Then $\mathcal{X}$ is not asymptotically stable. This follows from Theorem 3.1 in [27], since such an omnicycle must have an equilibrium with more than one outgoing connection, and therefore its unstable manifold is not contained in $\mathcal{X}$.

### 3. Stability of omnicycles

In this section, we state conditions for trail-stability of type Z omnicycles. Due to the similarity between heteroclinic cycles and omnicycles, Theorem 5 in [20], which gives the conditions for fragmentary asymptotic stability of heteroclinic cycles of type Z, is also applicable to trail-stability of omnicycles of that type. The conditions involve eigenvalues and eigenvectors of the transition matrices, which are constructed from local and global maps defined below.

#### 3.1. Local and global maps

To study the stability of an omnicycle, similarly to heteroclinic cycles and networks, the behaviour of nearby trajectories is approximated by local and global maps. By $N_\delta(\xi_j)$, we
denote the neighbourhood of \( \xi_j \) bounded by \( |a_s| = \delta \), where \( a_s \) are coordinates in the basis comprised of eigenvectors of \( df(\xi_j) \). We assume \( \delta \) to be fixed and small, so that in \( N_\delta(\xi_j) \) the equation (2) can be approximated by its linearization \( df(\xi_j) \), and \( \delta \gg \epsilon \) and \( \delta \), where \( \epsilon \) and \( \delta \) are those employed in the definitions of stability in Section 2.2.

Given an omnicycle \( \mathcal{X} = \{\xi_1, \ldots, \xi_m; \kappa_1, \ldots, \kappa_m\} \), let \((u_j, v_j, w_j, z_j)\) be local coordinates near \( \xi_j \) in the basis, where radial eigenvectors come the first (the respective coordinates are \( u_j \)), followed by the contracting and the expanding eigenvectors, the transverse eigenvectors being the last. (If repeating equilibria are present in the sequence, i.e. \( \xi_i = \xi_j \) for \( i \neq j \), then such decompositions for eigenvectors of \( df(\xi_i) \) and \( df(\xi_j) \) can be different.) Denote by \( H_{j-1,j}^{(\text{in})} \) the face of \( N_\delta(\xi_j) \) that intersects with the connection \( \kappa_{j-1,j} \) and by \( H_{j,j+1}^{(\text{out})} \) the face intersected by \( \kappa_{j,j+1} \). In the literature, \( H_{j-1,j}^{(\text{in})} \) and \( H_{j,j+1}^{(\text{out})} \) are often called cross-sections or Poincaré sections. For a trajectory close to the cycle, the local map relates coordinates of its intersection with \( H_{j-1,j}^{(\text{in})} \) to the coordinates of its intersection with \( H_{j,j+1}^{(\text{out})} \).

Let the superscripts \( \text{in} \) and \( \text{out} \) denote the coordinates in \( H_{j-1,j}^{(\text{in})} \) and \( H_{j,j+1}^{(\text{out})} \), respectively. The local map \( \phi_j : H_{j-1,j}^{(\text{in})} \rightarrow H_{j,j+1}^{(\text{out})} \) (see, e.g. [17] or [20]) is

\[
v^{\text{out}} = K_{1j}(w^{\text{in}})^{-c_j/e_j} \quad \text{and} \quad z^{\text{out}} = K_j z^{\text{in}}(w^{\text{in}})^{-t_j/c_j},
\]

where \( K_{1j} \) and \( K_j \) are constants (for a fixed \( \delta \)), and \( c_j, e_j, t_j \) are the contracting, expanding and transverse eigenvalues of \( df(\xi_j) \), respectively. The coordinates \( u^{\text{in}}, v^{\text{in}}, u^{\text{out}} \) and \( w^{\text{out}} \) are ignored, because they are irrelevant in the study of stability.

Near the connection \( \kappa_{j,j+1} \), the system (2) can be approximated by a global map (also called a connecting diffeomorphism) \( \psi_{j,j+1} : H_{j,j+1}^{(\text{out})} \rightarrow H_{j,j+1}^{(\text{in})} \), which relates the coordinates of the point where a trajectory exits \( N_\delta(\xi_j) \) to the coordinates of the point where it enters \( N_\delta(\xi_{j+1}) \). The global map is predominantly linear

\[
(w^{j+1,\text{in}}, z^{j+1,\text{in}}) = A'_j(v^{\text{out}}, z^{\text{out}}),
\]

where \( A'_j \) is an \((n_t + 1) \times (n_t + 1)\) matrix. For type Z omnicycles, the matrix \( A'_j \) is a product of a diagonal matrix \( A''_j \) and a permutation matrix \( A_j \).

Denote by \( g_j \) the superpositions of the local and global maps:

\[
g_j = \phi_j \circ \psi_{j,j-1} : H_{j}^{(\text{out})} \rightarrow H_{j+1}^{(\text{out})}.
\]

We call the set of maps \( \{g_1, \ldots, g_m\} \), \( g_j : \mathbb{R}^{n_t+1} \rightarrow \mathbb{R}^{n_t+1} \), the collection of maps associated with the omnicycle \( \mathcal{X} = \{\xi_1, \ldots, \xi_m; \kappa_1, \ldots, \kappa_m\} \). For a given a collection of maps \( \{g'_m\} \), we define superpositions

\[
G_j = g_{j-1} \circ \cdots \circ g_1 \circ g_m \circ \cdots \circ g_{j+1} \circ g_j, \quad G_j : H_{j}^{(\text{out})} \rightarrow H_{j}^{(\text{out})},
\]

which approximate the behaviour of trajectories near the cycle and are called return maps.
3.2. Transition maps and transition matrices

Let us introduce new coordinates:

\[ \eta = (\ln |w|, \ln |z_1|, \ldots, \ln |z_{n_t}|). \] (9)

Due to the smallness of \( \epsilon \), in these coordinates the maps \( g_j \) become approximately linear, \( \eta^{j+1} = M_j \eta^j \), where the basic transition matrix \( M_j \) is a product of the permutation matrix \( A_j \) that relates local coordinates near \( \xi_{j+1} \) to the ones near \( \xi_j \) and the matrix \( B_j \):

\[
B_j := \begin{pmatrix}
    b_{j,1} & 0 & 0 & \ldots & 0 \\
    b_{j,2} & 1 & 0 & \ldots & 0 \\
    b_{j,3} & 0 & 1 & \ldots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    b_{j,N} & 0 & 0 & \ldots & 1
\end{pmatrix}. \] (10)

The entries \( b_{j,l} \) of the matrix \( B_j \) depend on the eigenvalues of the linearization \( df(\xi_j) \) of (2) near \( \xi_j \) as follows:

\[ b_{j,1} = c_j/e_j \text{ and } b_{j,l+1} = -t_{j,l}/e_j, \quad 1 \leq l \leq n_t, \; 1 \leq j \leq m. \] (11)

Transition matrices of the superposition \( G_j \) defined in (8) are the products \( \mathcal{M}^{(j)} = M_{j-1} \ldots M_1 M_m \ldots M_{j+1} M_j \).

Consider the matrix \( M := \mathcal{M}^{(1)} = M_m \ldots M_1 : \mathbb{R}^N \to \mathbb{R}^N \); it is a product of the basic transition matrices \( M_j = A_j B_j \), where \( A_j \) is a permutation matrix and \( B_j \) is given by (10). We separate the coordinate vectors \( e_l, 1 \leq l \leq N \), into two groups. The first group is comprised of the vectors \( e_l \) for which there exist such \( k \) and \( j \) that \( (A_j)^k A_{j-1} e_l = e_1 \) (recall that \( A_j \) are permutation matrices), the second one incorporates the remaining vectors. Denote by \( V^{\text{sig}} \) and \( V^{\text{ins}} \), the subspaces spanned by vectors from the first and second groups, respectively (the superscripts 'ins' and 'sig' stand for significant and insignificant). In the basis where significant vectors come the first, matrix \( M \) takes the form

\[
M = \begin{pmatrix}
    M^{\text{sig}} & 0 \\
    1 & 0 & \ldots & 0 \\
    0 & 1 & \ldots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \ldots & 1
\end{pmatrix}. \] (12)

**Theorem 3.1 (Adapted Theorem 3.2 in [20]):** Let \( V^{\text{sig}} \) and \( V^{\text{ins}} \) be the subspaces defined above.

(a) The subspace \( V^{\text{ins}} \) is \( M \)-invariant and all eigenvalues associated with the eigenvectors from this subspace are one.

(b) Generically all components of eigenvectors that do not belong to \( V^{\text{ins}} \) are non-zero.
**Remark 3.1:** In the study of stability instead of the whole heteroclinic cycle or omnicycle, one can consider its building block, see Definition 2.2. In such a case, the transition matrix takes the form \( M_{\text{block}} = AM \), where \( A \) is the permutation matrix relating bases \( \{ e_l \} \) and \( \{ \gamma e_l \} \) and \( M \) has the form (12). The definition of significant and insignificant subspaces implies that the permutation is a composition of two independent permutations, a permutation of the vectors in the significant subspace and a permutation of the ones in the insignificant subspace. Therefore, the matrix \( M_{\text{block}} \) has the same form as (12), except that the lower right submatrix is the permutation matrix of the insignificant subspace. The statement (a) becomes [20] ‘The subspace \( V_{\text{ins}} \) is \( M \)-invariant and the absolute value of all eigenvalues associated with the eigenvectors from this subspace is one.’

We call **insignificant** the eigenvalues associated with eigenvectors from \( V_{\text{ins}} \), and **significant** the rest ones. Generically the absolute values of all significant eigenvalues differ from one. Below \( \lambda_{\text{max}} \neq 1 \) denotes the largest significant eigenvalue of a transition matrix \( \mathcal{M}^{(j)} \).

The proofs of the conditions for the stability of type Z heteroclinic cycles [20] involve (a) the construction of the collection of maps, associated with the cycle, (b) derivation of the necessary and sufficient conditions for (fragmentary) asymptotic stability of the fixed point 0 of the collection of maps and (c) the proof that the stability or instability of the fixed point implies the stability or instability of the cycle. As noted, the only difference between an omnicycle and a heteroclinic cycle is that in the cycle all equilibria and connections are distinct. That is, a trajectory close to a heteroclinic cycle necessarily follows the connections in the prescribed order (there is no choice), while if repeating equilibria are present, we need the definition of trail-stability which requires trajectories in \( \mathcal{B}_{\text{omni}}^\delta (X) \) to visit the equilibria and connections in this order. The parts (a) and (b) are identical for heteroclinic cycles and omnicycles. Concerning (c), by the construction of the collection of maps, the approximated trajectory visits the equilibria and connections in the order

\[ \ldots, \kappa_{j-2}, j-1, \xi_{j-1}, \kappa_{j-1}, j, \xi_{j}, \kappa_{j}, j-1, \xi_{j-1}, \kappa_{j-1}, j-2 \ldots, \]

i.e. as prescribed. Therefore, the proof of (c) can be extended from heteroclinic cycles to omnicycles and the theorem of [20] that gives necessary and sufficient conditions for a heteroclinic cycle to be f.a.s. can be generalized as follows:

**Theorem 3.2 (Adapted Theorem 5 in [20]):** Let \( M_j \) be basic transition matrices of a collection of maps \( \{ g_l^{(m)} \} \) associated with an omnicycle of type Z. Denote by \( j = j_1, \ldots, j_L \) the indices, for which \( M_j \) involves negative entries; all entries are non-negative for all remaining \( j \). Denote by \( \lambda_{\text{max}} \) the dominant eigenvalue of \( \mathcal{M}^{(j)} \) and by \( \mathbf{w}_{\text{max}} \) the associated eigenvector.

(a) If for at least one \( j = j_l + 1 \) the matrix \( \mathcal{M}^{(j)} \) does not satisfy the following conditions:
   (i) \( \lambda_{\text{max}} \) is real;
   (ii) \( \lambda_{\text{max}} > 1 \);
   (iii) \( w_{l}^{\text{max}} w_{q}^{\text{max}} > 0 \) for all \( l \) and \( q, 1 \leq l, q \leq N \).
   then the omnicycle is trail-unstable.

(b) If the matrices \( \mathcal{M}^{(j)} \) satisfy the above conditions (i) –(iii) for all \( j \) such that \( j = j_l + 1 \), then the omnicycle is trail-stable.
4. Stability: analytical results

In this section, we apply Theorem 3.2 to study dynamics near the network $X = C_L \cup C_R$ shown in Figure 1. The network exists in a $\Gamma$-equivariant system (2) in $\mathbb{R}^6$, where the group $\Gamma \cong \mathbb{Z}_2^6$ is generated by symmetries inverting the sign of one of the coordinates. The equilibria in the network belong to the coordinate axes, the connections are one-dimensional and belong to the coordinate planes, and the equilibria are stable in directions transverse to the network.

According to Theorem 3.2, the stability of a cycle or an omnicycle depends on the dominant eigenvalue and the associated eigenvectors of the respective transition matrices. Transition matrices of the cycles $C_L$ and $C_R$ have two-dimensional significant subspace, therefore we can calculate the dominant eigenvalues and the associated eigenvectors. The significant subspace of a transition matrix of the omnicycle

$$C_{LR} : (\xi_1 \rightarrow \xi_2 \rightarrow \xi_3 \rightarrow \xi_4 \rightarrow \xi_1 \rightarrow \xi_2 \rightarrow \xi_5 \rightarrow \xi_6 \rightarrow)$$

has dimension 4, therefore in a general case its eigenvalues cannot be found analytically. To show that the omnicycle can be trail-stable, we calculate the dominant eigenvalue and the associated eigenvector in the case when some eigenvalues of $df(\xi_j)$ vanish. Since the conditions for stability are non-strict inequalities, the inequalities remain true for small non-vanishing eigenvalues as well. Under our assumption that the chosen eigenvalues of $df(\xi_j)$ vanish, the conditions for stability of $C_{LR}$ are not compatible with conditions for the stability of $C_L$ and $C_R$. So, in agreement with Theorem 2.1 the network $X = C_L \cup C_R$ is f.a.s. while both subcycles are completely unstable.

4.1. Stability of cycles $C_L$ and $C_R$

Since the system under consideration is $\mathbb{Z}_2^6$-symmetric, the eigenvectors of $df(\xi_j)$ coincide with the Cartesian basis vectors $e_j$, $1 \leq j \leq 6$. In cross-sections to the connections comprising the cycle $C_L$, we take the following local bases:

$$H_{12}^{out} : \{e_3, e_4, e_5, e_6\}, \ H_{23}^{out} : \{e_4, e_1, e_5, e_6\}, \ H_{34}^{out} : \{e_1, e_2, e_5, e_6\}, \ H_{41}^{out} : \{e_2, e_3, e_5, e_6\}.$$  

(13)

In agreement with (10), the basic transition matrices are:

$$M_{412} = \begin{pmatrix} b_{132} & 1 & 0 & 0 \\ b_{142} & 0 & 0 & 0 \\ b_{152} & 0 & 1 & 0 \\ b_{162} & 0 & 0 & 1 \end{pmatrix}, \quad M_{123} = \begin{pmatrix} b_{243} & 1 & 0 & 0 \\ b_{213} & 0 & 0 & 0 \\ b_{253} & 0 & 1 & 0 \\ b_{263} & 0 & 0 & 1 \end{pmatrix},$$

$$M_{234} = \begin{pmatrix} b_{314} & 1 & 0 & 0 \\ b_{324} & 0 & 0 & 0 \\ b_{354} & 0 & 1 & 0 \\ b_{364} & 0 & 0 & 1 \end{pmatrix}, \quad M_{341} = \begin{pmatrix} b_{421} & 1 & 0 & 0 \\ b_{431} & 0 & 0 & 0 \\ b_{451} & 0 & 1 & 0 \\ b_{461} & 0 & 0 & 1 \end{pmatrix}.$$  

(14)
where
\[ b_{ijk} = -\mu_{ijk}/\mu_{ik} \]  
(15)
and \( \mu_{ij} \) is the eigenvalue of \( df((\xi)) \) associated with the eigenvector \( e_j \). Here we use the notation \( M_{ijk} \) \( (j = i + 1(\text{mod } 4), \ k = j + 1(\text{mod } 4)) \) for the basic transition matrix \( M_{ij} : H_{ij}^{out} \to H_{ij}^{out} \) introduced in Section 3.2.

The cycle \( C_L \) has only one equilibrium, \( \xi_2 \), where a transverse eigenvalue of linearization is positive. Therefore, Theorem 3.2 implies that the cycle is f.a.s. whenever the matrix \( M_L = M_123M_{412}M_341M_{234} \) satisfies the conditions (i) –(iii). Multiplying the matrices (14), we obtain that
\[ M_L = \begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & 1 & 0 \\ a_{41} & a_{42} & 0 & 1 \end{pmatrix}, \]  
(16)
where
\[
\begin{align*}
    a_{11} &= (b_{314}b_{421} + b_{324})(b_{132}b_{243} + b_{142}) + b_{314}b_{431}b_{243} \\
    a_{21} &= (b_{314}b_{421} + b_{324})b_{132}b_{213} + b_{314}b_{431}b_{213} \\
    a_{31} &= (b_{314}b_{421} + b_{324})(b_{132}b_{253} + b_{152}) + b_{314}b_{431}b_{253} \\
    a_{41} &= (b_{314}b_{421} + b_{324})(b_{132}b_{263} + b_{162}) + b_{314}b_{431}b_{263} \\
    a_{12} &= b_{421}(b_{132}b_{243} + b_{142}) + b_{431}b_{243}, \quad a_{22} = b_{421}b_{132}b_{213} + b_{431}b_{213} \\
    a_{32} &= b_{421}(b_{132}b_{253} + b_{152}) + b_{431}b_{253}, \quad a_{42} = b_{421}(b_{132}b_{263} + b_{162}) + b_{431}b_{263}.
\end{align*}
\]  
(17)
Due to the assumption that the network \( \mathcal{X} \) is stable in the transverse directions, in (17) the ratios \( b_{ijk} \) are positive, except for \( b_{253} \). Therefore, the entries of matrix (16) are positive, except possibly for \( a_{31} \) and \( a_{32} \).

The dominant eigenvalue of \( M_L \), \( \lambda_{\max} \), is the largest in absolute value eigenvalue of its upper left 2 \( \times \) 2 submatrix which we denote by \( M'_L \). The entries of \( M'_L \) are positive, implying that the discriminant of its characteristic polynomial \( F(\lambda) \) is positive. Hence, the eigenvalues of \( M'_L \) are real, which implies that \( \lambda_{\max} \) satisfies condition (i) of Theorem 3.2. Since \( F(a_{11}) = F(a_{22}) = -a_{12}a_{21} < 0 \), the dominant eigenvalue satisfies \( \lambda_{\max} > \max(a_{11}, a_{22}) > 0 \). Condition (ii) is satisfied whenever (see inequality (25) in [23])
\[
\max\left(\frac{a_{11} + a_{22}}{2}, a_{11} + a_{22} - a_{11}a_{22} + a_{12}a_{21}\right) > 1.
\]  
(18)
Let \( v = (v_1, v_2, v_3, v_4) \) be the eigenvector of \( M_L \) associated with \( \lambda_{\max} \) and \( v_1 = 1 \). The component \( v_2 \) is positive because
\[
v_2 = a_{21}v_1/(\lambda_{\max} - a_{22}).
\]  
(19)
From (16), the latter two components satisfy
\[
\begin{align*}
    v_3 &= \frac{a_{31}v_1 + a_{32}v_2}{\lambda_{\max} - 1}, \quad v_4 = \frac{a_{41}v_1 + a_{42}v_2}{\lambda_{\max} - 1}.
\end{align*}
\]  
(20)
Since \( a_{41} > 0 \) and \( a_{42} > 0 \), the latter equality implies that \( v_4 > 0 \). Substituting (19) into the first inequality in (20), we obtain that (iii) is equivalent to
\[
a_{31}(\lambda_{\text{max}} - a_{22}) + a_{32}a_{21} > 0. \tag{21}
\]
Therefore, the conditions for fragmentary asymptotic stability of the cycle \( C_L \) are (18) and (21), where \( \lambda_{\text{max}} \) is the largest eigenvalue of the upper left \( 2 \times 2 \) submatrix of (16) and the dependence of its entries on \( \mu_{ij} \) is given by (15) and (17). The permutation of coordinates \((x_1, x_2, x_3, x_4, x_5, x_6) \rightarrow (x_1, x_2, x_5, x_6, x_3, x_4)\) maps \( C_L \) to \( C_R \). Hence, the conditions for stability of \( C_R \) are obtained from those for \( C_L \) by the permutation of subscripts \((123456) \rightarrow (125634)\) of \( \mu_{ij} \) in (15) and (17).

### 4.2. Trail-stability of \( C_{LR} \)

In this section, we show that the omnicycle \( C_{LR} \) can be trail-stable. The stability of the cycle depends on the eigenvalues and eigenvectors of its \( 4 \times 4 \) transition matrices. To be able to satisfy the conditions for stability we make two assumptions. Namely, we assume that Equation (1) has an additional symmetry \((x_1, x_2, x_3, x_4, x_5, x_6) \rightarrow (x_1, x_2, x_5, x_6, x_3, x_4)\) \( (22)\) and that the eigenvalues \( \mu_{24}, \mu_{26}, \mu_{31}, \mu_{35}, \mu_{36}, \mu_{42}, \mu_{45}, \mu_{46}, \mu_{52}, \mu_{53}, \mu_{54}, \mu_{61}, \mu_{63}, \mu_{64} \) \( (23)\) are much smaller than other eigenvalues of the Jacobians \( df(\xi_j) \). The symmetry (22) implies that the following eigenvalues of the Jacobians are equal:
\[
\begin{align*}
\mu_{i,j} &= \mu_{i,j+2}, \quad \mu_{j,i} = \mu_{j+2,i}, \quad i = 1, 2, j = 3, 4 \\
\mu_{i,j} &= \mu_{i+2,j+2}, \quad \mu_{i+2,j} = \mu_{i,j+2}, \quad i = 3, 4, j = 3, 4.
\end{align*}
\]
(24)

To prove the existence of a system where \( C_{LR} \) is trail-stable, we first prove that for vanishing eigenvalues (23) it is possible to identify other eigenvalues \( \mu_{ij} \) such that the transition matrices satisfy conditions (i)–(iii) of Theorem 3.2. The conditions are not strict and eigenvalues and eigenvectors of a matrix depend continuously on its entries, therefore (i)–(iii) remain true for sufficiently small eigenvalues in (23) as well. For a given set of eigenvalues of \( df(\xi_j) \), a system that has such eigenvalues can be constructed by the procedure discussed in Section 5.1.

By Theorem 3.2, to check trail-stability of \( C_{LR} \), we should check conditions (i)–(iii) for the transition matrices \( M_{123}M_{612}M_{561}M_{256}M_{125}M_{412}M_{341}M_{234} \) and \( M_{125}M_{412}M_{341}M_{234}M_{123}M_{612}M_{561}M_{256} \) of the return maps \( H_{23}^{\text{out}} \rightarrow H_{23}^{\text{out}} \) and \( H_{25}^{\text{out}} \rightarrow H_{25}^{\text{out}} \), respectively. Here the basic transition matrix \( M_{ijk} \) approximates the superposition of the global map \( H_{ij}^{\text{out}} \rightarrow H_{ij}^{\text{in}} \) and the local map \( H_{ij}^{\text{in}} \rightarrow H_{jk}^{\text{out}} \). The existence of the symmetry (22) implies that the omnicycle is comprised of two building blocks (see Definition 2.2), therefore to study its trail-stability it is sufficient to consider the stability of the return map \( H_{23}^{\text{out}} \rightarrow H_{25}^{\text{out}} \) with the respective transition matrix \( M_{LR} = M_{125}M_{412}M_{341}M_{234} \). Let the bases in \( H_{12}^{\text{out}}, H_{34}^{\text{out}} \) and
$H_{41}^{out}$ be given by (13) and the basis in $H_{25}^{out}$ be \{e_6, e_1, e_3, e_4\}. The basic transition matrices of the cycle $C_{LR}$ in a system with the symmetry (22) and vanishing eigenvalues (23) are

\[
M_{412} = \begin{pmatrix} b_{132} & 1 & 0 & 0 \\ b_{142} & 0 & 0 & 0 \\ b_{152} & 0 & 1 & 0 \\ b_{162} & 0 & 0 & 1 \end{pmatrix}, \quad M_{234} := \begin{pmatrix} 0 & 1 & 0 & 0 \\ b_{324} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_{341} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ b_{431} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_{125} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & b_{215} & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}
\]

(25)

where $b_{ijk}$ are defined in (15).

The multiplication of the matrices implies that

\[
M_{LR} = \begin{pmatrix} \tilde{a}_{11} & 0 & 0 & 1 \\ \tilde{a}_{21} & 0 & \tilde{a}_{23} & 0 \\ 0 & \tilde{a}_{32} & -1 & 0 \\ \tilde{a}_{41} & 0 & 0 & 0 \end{pmatrix},
\]

(26)

where

\[
\tilde{a}_{11} = b_{162} b_{324}, \quad \tilde{a}_{21} = b_{152} b_{324} b_{213}, \quad \tilde{a}_{23} = b_{213}, \quad \tilde{a}_{32} = b_{413}, \quad \tilde{a}_{41} = b_{142} b_{324}.
\]

(27)

Here all $b_{ijk}$ are positive, except for $b_{152}$. Therefore, $\tilde{a}_{21} < 0$ and all the other coefficients are positive.

Upon the permutation of bases in $H_{25}^{out}$: \{e_6, e_1, e_3, e_4\} → \{e_6, e_4, e_1, e_3\}, matrix (26) takes the form

\[
M_{LR} = \begin{pmatrix} \tilde{a}_{11} & 1 & 0 & 0 \\ \tilde{a}_{41} & 0 & 0 & 0 \\ \tilde{a}_{21} & 0 & \tilde{a}_{23} & 0 \\ 0 & \tilde{a}_{32} & -1 & 0 \end{pmatrix}.
\]

(28)

The eigenvalues of this matrix are

\[
\lambda_{1,2} = \frac{\tilde{a}_{11} \pm (\tilde{a}_{11}^2 + 4\tilde{a}_{41})^{1/2}}{2}, \quad \lambda_{3,4} = \frac{-1 \pm (1 + 4\tilde{a}_{23}\tilde{a}_{32})^{1/2}}{2},
\]

(29)

where the indices 1 and 3 correspond to positive signs in front of the square roots. Here $|\lambda_1| > |\lambda_2|$ and $\lambda_1$ is positive, while $|\lambda_3| < |\lambda_4|$ and $\lambda_4$ is negative. Therefore, conditions
(i) and (ii) of Theorem 3.2 for the matrix $M_{LR}$ take the form
\[ \lambda_1 > \max(1, |\lambda_4|). \] (30)

From (28) the components of the eigenvector $v = (v_1, v_2, v_3, v_4)$ of $M_{LR}$ associated with $\lambda_1$ satisfy
\[ v_2 = \frac{v_1 \tilde{a}_{41}}{\lambda_1}, \quad v_3 = \frac{v_4 \tilde{a}_{23}}{\lambda_1}, \quad v_4 = v_1 \tilde{a}_{31} \left( \lambda_1 + 1 - \frac{\tilde{a}_{32} \tilde{a}_{23}}{\lambda_1} \right)^{-1}. \]

Therefore, the condition (iii) of Theorem 3.2 holds true whenever
\[ (\lambda_1 + 1) \lambda_1 - \tilde{a}_{32} \tilde{a}_{23} > 0. \] (31)

Finally, we show that conditions (30),(31) imply that the cycle $C_L$ is completely unstable. (Therefore, by the same arguments as applied to show trail-stability of $C_{LR}$, it remains unstable for a slightly perturbed non-vanishing eigenvalues listed in (23).) The transition matrix (16) under the assumptions of symmetry (22) and for vanishing eigenvalues (23) takes the form
\[ M_L = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & a_{32} & 1 & 0 \\ a_{41} & 0 & 0 & 1 \end{pmatrix}, \] (32)

where $a_{32} = b_{235} b_{413} < 0$. The components of the dominant eigenvector satisfy $v_3 = v_1 a_{32} / (\lambda_{\max} - 1)$. Hence, $\lambda_{\max} > 1$ implies that $v_1 v_3 < 0$, i.e. for the matrix $M_L$ the conditions (ii) and (iii) of Theorem 3.2 cannot be satisfied simultaneously.

**Remark 4.1**: To show that the cycle $C_L$ can be stable in symmetric system (22) assume that the eigenvalues
\[ \mu_{13}, \mu_{16}, \mu_{24}, \mu_{26}, \mu_{3j}, j = 1, 3, 5; \mu_{41}, \mu_{46} \] (33)
are much smaller than the other ones. Under the assumption of vanishing eigenvalues (33), the matrix (16) takes the form
\[ M_L = \begin{pmatrix} b_{324} b_{142} & 0 & 0 & 0 \\ 0 & b_{213} b_{431} & 0 & 0 \\ 0 & b_{451} - b_{431} & 1 & 0 \\ b_{324} b_{142} & 0 & 0 & 1 \end{pmatrix}. \] (34)

The conditions (i)-(iii) are satisfied if
\[ \max(b_{324} b_{142}, b_{213} b_{431}) > 1 \text{ and } b_{451} - b_{431} > 0. \]

These conditions are also satisfied for sufficiently small eigenvalues (33).

The symmetry (22) implies that the stability properties of $C_R$ are the same as the ones of $C_L$. As noted above, for sufficiently small eigenvalues (23) the trail-stability of $C_{LR}$ is not compatible with fragmentary asymptotic stability of $C_L$ (and $C_R$). We do not know if they are compatible or not without this assumption. Concerning the fragmentary asymptotic stability of the whole network, it might be related to the existence of f.a.s. subcycles, trail-stability of omnicycles (see Theorem 2.1), or due to the existence of nearby trajectories, switching irregularly between the subcycles.
5. Numerical examples

In this section, we study numerically behaviour of trajectories near the heteroclinic networks $X = \mathcal{C}_L \cup \mathcal{C}_R$ shown in Figure 1. In Section 5.1, we discuss the construction of a $\mathbb{Z}_2^6$-equivariant dynamical system, possessing such a network with prescribed eigenvalues of Jacobians $d f(\xi_j)$, $1 \leq j \leq 6$. Then, using results of Section 4, we give numerical examples of attracting $\mathcal{C}_L$ and $\mathcal{C}_{LR}$. Our simulations indicate that when none of $\mathcal{C}_L$, $\mathcal{C}_R$ or $\mathcal{C}_{LR}$ is attracting, nearby trajectories nevertheless can be attracted to the network displaying regular or irregular switching between $\mathcal{C}_L$ and $\mathcal{C}_R$. (‘Regular’ and ‘irregular’ switching are understood as regular and irregular cycling introduced in [29].)

5.1. Construction of the system

Consider the system

$$\dot{x}_i = x_i(1 + \sum_{1 \leq j \leq 6} \beta_{ij} x_j^2), \quad \text{where } \beta_{ii} = -1, \ 0 \leq i \leq 6. \quad (35)$$

By construction (35) is $\mathbb{Z}_2^6$-equivariant with the symmetry group generated by the inversions of Cartesian coordinates. Any coordinate axis, plane or a hyperplane is an invariant subspace of (35). It has equilibria $\xi_i = \pm 1$ on each of the coordinate axes, which are attractive in the radial direction.

The restriction of (35) to the coordinate plane $< e_l, e_k >$ is

$$\dot{x}_l = x_l(1 - x_l^2 + \beta_{lk} x_k^2), \ \dot{x}_k = x_k(1 - x_k^2 + \beta_{kl} x_l^2). \quad (36)$$

It is known (this follows, e.g. from the theorem in the appendix of [21]) that if both $\xi_l$ or $\xi_k$ are stable in the radial direction, one of them is unstable in the orthogonal direction and the other equilibrium is stable, then there exists a robust heteroclinic trajectory connecting these equilibria.

The eigenvalues of $d f(\xi_l)$ and $d f(\xi_k)$ in the directions of $e_k$ and $e_k$, respectively, are

$$\mu_{lk} = 1 + \beta_{kl}, \ \mu_{kl} = 1 + \beta_{lk}. \quad (37)$$

Each of $\beta_{ij}$ enters into only one expression for $\mu_{kl}$, therefore we can design system (35) for any prescribed $\mu_{kl}$ compatible with the connections shown in Figure 1. (Except for radial eigenvalues, which do not enter into the expressions for stability. These eigenvalues can be altered by modifying $\beta_{ii}$.)

5.2. Numerical results: symmetric system

Here we present results of numerical simulations (see Figure 2) of solutions to Equations (35) in the case when the system has additional symmetry (22). In Section 3, we derived conditions for the stability of heteroclinic cycles $\mathcal{C}_L$ and $\mathcal{C}_R$, and omnicycle $\mathcal{C}_{LR}$, which involve ratios $b_{ijk} = -\mu_{ij}/\mu_{jk}$. Therefore, we can choose $\mu_{lk}$ to make the cycles stable or unstable. The values of $\mu_{lk}$ employed in simulations shown in the figure are given in Table 1.
We start from stable $C_L$ and $C_{LR}$ (plates (a) and (b)) where the values of $\mu_{ik}$ given in Table 1 are taken to satisfy conditions (18),(21) and (27),(29)–(31). For all other considered variants, where $\mu_{ik}$ are taken so that both $C_L$ and $C_{LR}$ are unstable, the trajectories staying near the network display regular or irregular behaviour: they can switch between $C_L$ and

\begin{figure}[h]
\centering
\begin{subfigure}{0.49\textwidth}
\includegraphics[width=\linewidth]{figure2a}
\caption{}
\end{subfigure}
\begin{subfigure}{0.49\textwidth}
\includegraphics[width=\linewidth]{figure2b}
\caption{}
\end{subfigure}
\begin{subfigure}{0.49\textwidth}
\includegraphics[width=\linewidth]{figure2c}
\caption{}
\end{subfigure}
\begin{subfigure}{0.49\textwidth}
\includegraphics[width=\linewidth]{figure2d}
\caption{}
\end{subfigure}
\begin{subfigure}{0.49\textwidth}
\includegraphics[width=\linewidth]{figure2e}
\caption{}
\end{subfigure}
\caption{The dependence on time of $x_3$ (solid line) and $x_5$ (dashed line).}
\end{figure}
Table 1. Eigenvalues of linearizations employed in numerical simulations shown in Figure 2. We do not list \( \mu_i = -2 \) or eigenvalues that are equal to \(-0.01\).

| Plate | Eigenvalues |
|-------|-------------|
| (a)   | \( \mu_{12} = \mu_{34} = \mu_{41} = \mu_{56} = \mu_{61} = 1, \mu_{14} = \mu_{16} = -1.2 \) |
| (b)   | \( \mu_{21} = -1, \mu_{23} = \mu_{25} = 1/4, \mu_{32} = \mu_{52} = -1.5, \mu_{43} = \mu_{65} = -1.1, \mu_{45} = \mu_{63} = -1.7 \) |
| (c)   | \( \mu_{12} = \mu_{34} = \mu_{41} = \mu_{56} = \mu_{61} = 1, \mu_{14} = \mu_{16} = -1.2 \) |
| (d)   | \( \mu_{21} = -1, \mu_{23} = \mu_{25} = 1/4, \mu_{32} = \mu_{52} = -1.5, \mu_{35} = \mu_{53} = -0.4, \mu_{43} = \mu_{65} = -1.1 \) |
| (e)   | \( \mu_{21} = -1, \mu_{23} = \mu_{25} = 1/4, \mu_{32} = \mu_{52} = -1.5, \mu_{35} = \mu_{53} = -0.4, \mu_{43} = \mu_{65} = -1.2 \) |

\( C_R \) making a fixed number of turns (this number depends on parameters of (35)) around one cycle before switching to the other one, or this number can vary. Examples of such regular or irregular switching are shown in plates (c)–(e).

For \( \tau \to \infty \) the trajectories are attracted by the heteroclinic network, which is indicated by exponentially increasing subsequent time intervals between visiting \( \xi_2 \) (i.e. time intervals needed to make a turn around \( C_L \) or \( C_R \)). On the plots the horizontal axis (time) is scaled exponentially, implying that temporal behaviour is visually similar to periodic.

Remark 5.1: In simulations, the values of \( \beta_k \) are taken such that the eigenvalues of transition matrices \( M_L \) and \( M_R \) satisfy \( \lambda_{iL}^{\text{max}} > 1 \) and \( \lambda_{iR}^{\text{max}} > 1 \). We cannot prove that these inequalities imply fragmentary asymptotic stability of the network, however, numerical results indicate that this might be the case, because for all performed runs trajectories were attracted by the network.

5.3. Numerical results: non-symmetric system

In this section, we present results of numerical integration of system (35) when additional symmetry (22) is not imposed. Typical behaviour of trajectories near the network is shown in Figure 3 and the respective eigenvalues of \( df(\xi_j) \) are given in Table 2. The eigenvalues are chosen such that none of \( C_L \) or \( C_R \) is f.a.s.

A regular behaviour (examples are shown in plates (a) and (d)) can be described as a repeating pattern comprised of \( n_L \) iterations along \( C_L \) followed by \( n_R \) iterations along \( C_R \). Apparently, this indicates that the respective omnicycle, which can be labelled as \( (n_L, n_R) \), is trail-stable. Often we observe less regular behaviour (as shown in plates (b), (c) and (e)) which either indicate irregular switching or a long omnicycle. Due to the presence of numerical errors, it makes no sense to investigate the behaviour in more details: what we see in simulations may be different from what really takes place in the system.

6. Conclusion

In this paper, we studied the behaviour of trajectories near a network in \( \mathbb{R}^6 \) comprised of two heteroclinic cycles. We derived condition for f.a.s. of these heteroclinic cycles, proved that the network can be f.a.s. while both cycles are completely unstable (the stability of the network follows from the trail-stability of \( C_{LR} \) and Theorem 2.1) and presented results of
Figure 3. The dependence on time of $x_3$ (solid line) and $x_5$ (dashed line).

numerical simulations indicating that for a positive measure set of initial conditions the behaviour of nearby trajectories is irregular.

To prove the stability of the network, we introduce the notions of an omnicycle and trail-stability. (Note that the notions were implicitly used in [29,30] to study the behaviour
Table 2. Eigenvalues of linearizations employed in numerical simulations shown in Figure 3. We do not list $\mu_i = -2$ or eigenvalues that are equal to −0.01.

| Plate | Eigenvalues |
|-------|-------------|
| (a)   | $\mu_{12} = \mu_{34} = \mu_{41} = \mu_{56} = \mu_{61} = 1$, $\mu_{14} = \mu_{16} = -1.2$, $\mu_{21} = -1$, $\mu_{23} = 1/1.4$ $\mu_{25} = 1/1.5$, $\mu_{32} = \mu_{52} = -1.5$, $\mu_{43} = 1.8$, $\mu_{65} = -1.4$, $\mu_{35} = -0.4$, $\mu_{53} = -0.5$ |
| (b)   | $\mu_{12} = \mu_{34} = \mu_{41} = \mu_{56} = \mu_{61} = 1$, $\mu_{14} = \mu_{16} = -1.4$, $\mu_{21} = -1$ $\mu_{23} = 1/1.4$, $\mu_{32} = \mu_{52} = -1.5$, $\mu_{43} = -0.4$, $\mu_{43} = 2.9$, $\mu_{65} = -1.6$ |
| (c)   | $\mu_{12} = \mu_{34} = \mu_{41} = \mu_{56} = \mu_{61} = 1$, $\mu_{14} = \mu_{16} = -1.4$ $\mu_{21} = -1$, $\mu_{23} = 1/1.4$, $\mu_{32} = \mu_{52} = -1.5$, $\mu_{43} = -0.4$, $\mu_{43} = 2.9$, $\mu_{65} = -1.6$ |
| (d)   | $\mu_{12} = \mu_{34} = \mu_{41} = \mu_{56} = \mu_{61} = 1$, $\mu_{14} = \mu_{16} = -1.2$, $\mu_{21} = -1$, $\mu_{23} = 1/1.4$ $\mu_{25} = 1/1.7$, $\mu_{32} = \mu_{52} = -1.5$, $\mu_{35} = \mu_{53} = -0.4$, $\mu_{43} = -1.6$, $\mu_{65} = -1.1$ |
| (e)   | $\mu_{12} = \mu_{34} = \mu_{41} = \mu_{56} = \mu_{61} = 1$, $\mu_{14} = \mu_{16} = -1.2$, $\mu_{21} = -1$, $\mu_{23} = 1/1.4$ $\mu_{25} = 1/1.5$, $\mu_{32} = \mu_{52} = -1.5$, $\mu_{35} = \mu_{53} = -0.4$, $\mu_{43} = -1.7$, $\mu_{65} = -1.5$ |

of trajectories near a heteroclinic network.) The definition of omnicycle is similar to the one of heteroclinic cycle, except that the equilibria and heteroclinic connections are not required to be distinct. Hence, the conditions for trail-stability of an omnicycle are identical to the condition of f.a.s. of a heteroclinic cycle, for type Z objects they were proven in [20].

To prove the possibility of the existence of an attracting omnicycle, we have assumed that some eigenvalues of linearizations are much smaller than the others, which enables us to calculate eigenvalues and eigenvectors of transition matrices. The assumption of smallness of some of the eigenvalues of linearizations can be employed in other studies of stability of cycles, omnicycles or heteroclinic networks in high-dimensional spaces, where direct calculation of eigenvalues is not possible due to large dimensions of transition matrices. In particular, the smallness of eigenvalues can allow for the calculation of stability indices of heteroclinic cycles and omnicycles in dimension larger than 6. The respective transition matrices are of dimension 4 of larger. Hence, unless they are sparse, its eigenvalues and eigenvectors cannot be found analytically.

According to the definition of type Z heteroclinic network (cycle or omnicycle), the object emerges in an equivariant dynamical system. Robust heteroclinic sets also exist in systems, e.g. related to the game theory or populations dynamics, where the invariance of subspaces follows from the structure of governing equations. For such systems the notion of type Z heteroclinic cycles was generalized to quasi-simple heteroclinic cycles [9]. Similarly, the definition of type Z network or omnicycle can be generalized to quasi-simple ones. As well, the conditions for trail-stability of type Z omnicycles proven in Section 3 can be extended to quasi-simple ones.

Notes
1. As proven in [22], the dimension of $W^u(\xi_2) \cap W^s(\xi_3)$ (and of $W^u(\xi_2) \cap W^s(\xi_5)$) in the system under consideration is two. In Section 2 we write that $\kappa_{ij+1} \subseteq W^u(\xi_i) \cap W^s(\xi_{j+1})$, i.e. we can choose any subset of the intersection. We choose one-dimensional connections in order to be able to derive conditions for stability. (With multidimensional connections the derivations of conditions for stability becomes very difficult.)
2. We choose these particular eigenvalues to be small, because on the one hand this choice provides transition matrices for which we can calculate eigenvalues and eigenvectors analytically. On the other hand, it gives a dominant eigenvector with all components non-vanishing, so that we can satisfy conditions of Theorem 3.2.
3. On the figures only the coordinates $x_3$ and $x_5$ are displayed: $x_3$ close to one indicates that after $\xi_2$ a trajectory goes to $C_L$, while $x_5$ close to one corresponds to the trajectory going to $C_R$.

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