Magnetic Instability in a Parity Invariant 2D Fermion System

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We consider the parity invariant QED$_{2+1}$ where the matter is represented as a mixture of fermions with opposite spins. It is argued that the perturbative ground state of the system is unstable with respect to the formation of magnetized ground state. Carrying out the finite temperature analysis we show that the magnetic instability disappears in the high temperature regime.

I. INTRODUCTION

Interest to the (2+1)-dimensional field theory is heated up by its successful applications to the problems of planar condensed matter systems like the quantum Hall fluid and unconventional superconductors. One of the relevant issues concerns the phenomenon of spontaneous magnetization in the systems of planar fermion matter.

Different authors [1–6] have given convincing arguments that the perturbative ground state (the one with $F_{\mu\nu} = 0$) of a 2D dense fermion matter exhibits the magnetic instability. In other terms there does exist the true ground state with the nonzero value of a proper magnetic field.

The peculiar property of planar physics is that spin is not quantized, and spin-up and spin-down states belong to the different representations of the rotation group SO(2). Remark, that the existence of magnetized ground state was demonstrated in the systems of fermions with one and the same spin orientation. Hence one can propose the simple mechanism explaining this phenomenon: matter consisting entirely of either up or down spin particles possesses nonvanishing spin density leading to a proper magnetization of the system.

In the present paper we study the double-spin model (also referred to as duplicated) where the matter represents the mixture of opposite spin fermions. Such a consideration can be motivated by the observation$^7$ that the magnetic properties of duplicated and single-spin (up or down) systems can be quite different.

In the rest part of this section we remind some principal aspects of spontaneous magnetization in the single-spin system. In Section 2 we perform the perturbative study of duplicated system and find out magnetic instability of perturbative ground state. The mechanism of instability in the double-spin model turns out to be different from that in the single-spin systems. Some necessary calculations are placed in the Appendix.

The basic object of our account is the finite temperature effective action $W[A]$ which arises after integrating out the fermion fields. In the Matsubara formalism it is given by

$$\exp \{-W[A]\} = \int D\psi^* D\psi \exp \left\{ -\int L d\tau dr \right\},$$

where $L$ is the Euclidean Lagrangian, $0 < \tau < T^{-1}$, and $T$ is the temperature.

Consider the nonrelativistic fermion matter interacting with $U(1)$ gauge field. The corresponding Euclidean Lagrangian is given by

$$L = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - ie \rho A_\tau + \psi^* (\partial_\tau + ie A_\tau) \psi +$$

$$+ \frac{1}{2m} |(\partial_k + ie A_k) \psi|^2 - \frac{\sigma e}{2m} B \psi^* \psi - \mu \psi^* \psi,$$

where $\rho$ is the neutralizing charge density, $\sigma$ describes the spin degree of freedom, magnetic field is defined as $B = \partial_2 A_1 - \partial_1 A_2$, and $\mu$ is the chemical potential.

The linear part of $W[A]$ appears as

$$W_1 = e \int \left\{ i (\langle \psi^* \psi \rangle - \rho) A_\tau - \frac{\sigma}{2m} \langle \psi^* \psi \rangle B \right\} d\tau dr,$$

where the symbol $\langle \cdots \rangle$ denotes the thermal average corresponding to $F_{\mu\nu} = 0$. 

[1–6]
From the last expression one can extract the principal conclusion concerning the magnetic properties of the single-spin system. Due to the term, linear in the magnetic field, the extremum of $W[A]$ will be realized for some $B \neq 0$. The common conclusion achieved in the finite temperature analysis of single-spin models, implies that the spontaneous magnetization persists at any temperatures. The reason is quite simple: the finite spin density ($\sigma/2\langle \psi^*\psi \rangle$) creates the proper magnetization of the system

$$M = -\frac{\delta W_1}{\delta B} = \frac{\sigma e}{2m} \langle \psi^*\psi \rangle,$$

which survives at any temperatures, provided the particle density is finite.

Concerning the $A_\tau$-term, it leads to the electric field corresponding to a local charge distribution $e\langle \psi^*\psi \rangle - e\rho$.

**II. DOUBLE-SPIN SYSTEM**

In this section we consider the duplicated model where the matter represents the mixture of opposite spin ($\pm \sigma$) fermions.

The system is assumed to be in contact with a particle reservoir which keeps the total particle density fixed and guarantees the chemical equilibrium between spin-up and spin-down subsystems. This can be realized by equalizing the corresponding chemical potentials. Then $\langle \psi_\uparrow^*\psi_\uparrow \rangle = \langle \psi_\downarrow^*\psi_\downarrow \rangle$, and the linear part of $W$ appears as

$$W_1 = ie \int [\langle \psi_\uparrow^*\psi_\uparrow \rangle + \langle \psi_\downarrow^*\psi_\downarrow \rangle - \rho] A_\tau d\tau dr,$$

where the term linear in the magnetic field is absent since the duplicated model is parity invariant. Therefore, the configuration with $B = 0$ represents the extremum of the corresponding effective action. The prior question which arises in this connection is the one concerning the stability of this extremum. This issue can be studied in terms of the two-point functions calculated within the Gaussian approximation. Considering these functions as integral operators we can introduce the corresponding eigenvalues which characterize the spectrum of gauge field fluctuations. Stability of the perturbative ground state requires the non-negative definiteness of this eigenvalue spectrum. If the spectrum contains at least one negative eigenvalue, then one is faced with an instability leading to the formation of the nontrivial ground state. In what follows we adopt this criterion and search for the negative eigenvalues of the two-point functions determining the Gaussian part. The later is given by

$$W_2 = \frac{1}{4} \int F_{\mu\nu}F_{\mu\nu} dx + \frac{e^2 m}{2} \int A_\mu(x_1)G_{\mu\nu}(x_1-x_2)A_\nu(x_2) dx_1 dx_2,$$

where $dx = d\tau dr$, and $G_{\mu\nu}(x)$ are the current correlation functions. In the Fourier representation they look as

$$G_{\tau\tau} = -\int \frac{dk'}{2\pi^2} \frac{f(\beta E_+)-f(\beta E_-)}{(kk')^2 + m^2 \xi^2} (kk'),$$

$$G_{\tau i} = G_{i\tau} = -\frac{\xi k_i}{k^2} G_{\tau\tau},$$

$$G_{ij} = \delta_{ij} \frac{\xi^2}{k^2} G_{\tau\tau} + \frac{1}{m^2} (k^2 \delta_{ij} - k_i k_j) G,$$

$$G = \int \frac{dk'}{2\pi^2} \frac{f(\beta E_+)-f(\beta E_-)}{(kk')^2 + m^2 \xi^2} (kk') \left\{ \frac{k^2}{k^4} - \frac{2}{k^4} \frac{(kk')^2}{k^4} + \frac{\sigma^2}{4} \right\},$$

1Magnetic field defined in 2+1 dimensions is a pseudoscalar. Therefore, the spin interaction leads to a parity violation in single-spin systems. Provided spin-up and spin-down fermions are interchanged under the parity transformation the duplicated model is parity invariant.
where $\xi = 2\pi T n$ with $n = 0, \pm 1, \pm 2\ldots$, and $\beta = T^{-1}$. The quantities $E_{\pm}$ and $f(z)$ are given by

$$E_{\pm} = \frac{1}{2m} \left( k' \pm \frac{k}{2} \right)^2,$$

$$f(z) = \frac{1}{1 + e^{z-\mu/T}}.$$

The induced Chern-Simons terms corresponding to the spin-up and spin-down fermions are mutually canceled out in accordance with the parity invariance.

From the definition of $E_{\pm}$ it follows that $E_{\pm} > E_{\mp}$ for $\text{sgn}(k'k) = \pm 1$, and since $f(z)$ is a decreasing function, we get

$$[f(\beta E_+) - f(\beta E_-)](kk') < 0.$$  (2)

This implies that $G_{\tau\tau} > 0$, and the corresponding contributions to $W_2$ are positive. The rest part of $W_2$ is related with pure magnetic configurations and determines the two-point function

$$\Lambda(x_1, x_2) = \frac{\delta^2 W_2}{\delta B(x_1) \delta B(x_2)},$$

which, up to a constant factor, is the susceptibility of the perturbative ground state.

Because of the translational invariance, it is convenient to analyze the eigenvalues of $\Lambda(x_1, x_2)$ in the Fourier representation. These eigenvalues can be written in the following form

$$\lambda(\xi, k, T) = 1 + \frac{e^2}{m} G(\xi, k, T),$$

(3)

where $G(\xi, k, T)$ is given by the Eq. (1).

From these expressions one can detect the possibility of the existence of negative eigenvalues leading to the magnetic instability. In fact, due to the relation (2), the $\sigma$-dependent part of $G$ is negative, and for sufficiently large values of $\sigma^2$ the quantity $G(\xi, k, T)$ will become negative at least for some $\xi$ and $k$.

In order, to expose the underlying physics, consider the dense matter of spinless fermions. In the absence of gauge fields fermions are organized in a Fermi sphere. Turning on the homogeneous magnetic field, fermions will be rearranged into the Landau levels, and the orbital diamagnetism will lead to the increase of the energy of the system. Let us now attach the spin to these fermions. Inclusion of the spin with positive (negative) $\sigma eB$ decreases (increases) the energy. Due to the chemical equilibrium between the opposite spin subsystems, the partial density of fermions with $\sigma eB > 0$ exceeds the one corresponding to $\sigma eB < 0$. Therefore, the spin effects lead to the spin paramagnetism promoting the decrease of the energy. If $\sigma^2$ is large enough, then the paramagnetism dominates over the diamagnetism and causes the overall decrease of the energy of the fermion system.

Further, for sufficiently large values of $e^2/m$, the negative values of $G$ will dominate in Eq. (3), and for some $\xi$ and $k$ we shall get $\lambda < 0$.

In the light of these arguments we can formulate two main points required for the existence of negative modes:

(i) the magnitude of $\sigma$ must be large enough to generate the instability of the fermion matter.

(ii) the value of the fraction $e^2/m$ must be large enough to guarantee that the matter instability will not be compensated by the gauge field contributions.

In the following two subsections we present more detailed analysis of the negative modes for static ($\xi = 0$) and nonstatic ($\xi \neq 0$) cases, respectively.

A. Static case

Let us first explore the point (i) concerning the negative values of $G$. As it is shown in the Appendix, the quantity $G(0, k, T)$ can be written in the following form

$$G(0, k, T) = \int_0^1 \frac{ds}{4\pi} \frac{s^2 - \sigma^2}{1 + e^{s(1-s^2)-\mu/T}},$$

(4)
where \( \zeta = k^2/8mT \) and \( k = |k| \).

Assuming that the total particle density is fixed, we express the chemical potential in terms of other parameters. The defining equation is \( \langle \psi_\uparrow^\ast \psi_\uparrow \rangle + \langle \psi_\downarrow^\ast \psi_\downarrow \rangle = n_e \), with

\[
\langle \psi_\uparrow^\ast \psi_\uparrow \rangle = \langle \psi_\downarrow^\ast \psi_\downarrow \rangle = \frac{mT}{2\pi} \ln(1 + e^{\mu/T}),
\]

and consequently

\[
\mu = T \ln \left( e^{1/\Theta} - 1 \right),
\]

where \( \Theta = mT/\pi n_e \).

Consider first the zero temperature limit. In that case the Fermi distribution function becomes steplike, and we get

\[
G(0, k, 0) = \frac{1 - 3\sigma^2}{12\pi} - \frac{1}{12\pi} \sqrt{1 - \frac{k_F^2}{k^2}} \left[ 1 - 3\sigma^2 - \frac{k_F^2}{k^2} \right] \theta(k - k_F),
\]

where the characteristic momentum \( k_F \) is defined by

\[
k_F^2 = 8m \lim_{T \to 0} \mu = 8\pi n_e.
\]

In Fig. 1(a) we present \( G(0, k, 0) \) for the different values of \( \sigma^2 \). As one can be convinced, the negative eigenvalues appear only for \( \sigma^2 > 1/3 \).

Consider now the point (ii) and trace out the condition guaranteeing that the negative values of \( G(0, k, 0) \) will lead to \( \lambda < 0 \). For \( \sigma^2 > 1/3 \) the minimal value of \( G(0, k, 0) \) is given by

\[
G(0, 0 < k < k_F, 0) = -\frac{3\sigma^2 - 1}{12\pi},
\]

and the required condition appears to be

\[
\frac{m}{e^2} < \frac{3\sigma^2 - 1}{12\pi}.
\]

In this case one obtains that \( \lambda(0, k, 0) < 0 \) for \( 0 < k < k_c \), where \( k_c \) is some critical value which is the solution to \( \lambda(0, k, 0) = 0 \). Remark, that \( k_c > k_F \).
The finite temperature behaviour of $G(0, k, T)$ is depicted in Fig. 1(b), where the case of $\sigma^2 = 1$ is represented for different values of the parameter $\Theta$. We see that $G(0, k, T)$ tends to zero when $T$ increases. Therefore, the negative modes, observed at low temperatures, disappear in the high temperature regime.

Remark, that the relation (5) can be realized only for $\sigma^2 > 1/3$ and therefore, embraces the points (i) and (ii) simultaneously. Thus, the relation (5) appears to be a sufficient condition for the existence of $\lambda(0, k, T) < 0$. Moreover, it is also a necessary one, since in the opposite case $\lambda(0, k, T)$ is positive for all $k$ and $T$. In order to check up this assertion, one can carry out some estimates. Note, that in (4) we have $0 < s < 1$. In this interval $e^{\zeta(1-s^2)} > e^{\zeta(1-s)}$ and $e^{\zeta(1-s^2)} < e^{\zeta(1-s^3)}$. Using these inequalities in (4), one gets

$$\lambda(0, k, T) > 1 + \frac{e^2}{m} \frac{1 - 3\sigma^2}{12\pi} \frac{1}{\zeta} \ln \frac{1 + e^{\mu/T}}{1 + e^{\mu/T - \zeta}}.$$  

Further, since $\zeta > 0$, we have

$$0 < \frac{1}{\zeta} \ln \frac{1 + e^{\mu/T}}{1 + e^{\mu/T - \zeta}} < 1$$

and consequently,

$$\lambda(0, k, T) > 1 + \frac{e^2}{m} \frac{1 - 3\sigma^2}{12\pi}.$$  

This is a general relation valid for all values of $k$ and $T$, and leading to $\lambda(0, k, T) > 0$ when the condition (5) is not held.

B. Nonstatic case

Integrating out the polar angle in (1), the corresponding expression can be presented in the following form

$$G(\xi, k, T) = \int_0^\infty \frac{dz}{4\pi} \left[ \frac{1 - \sigma^2 - z^2}{2} \frac{x + 4m^2\xi^2z^2}{4m^2\xi^2 + k^4(1+z)^2} \frac{1}{x} \right] (\varepsilon - 1)f(\xi z^2),$$  

where

$$\varepsilon = \sqrt{\frac{4m^2\xi^2 + k^4(1+z)^2}{4m^2\xi^2 + k^4(1-z)^2}},$$

$$x = \sqrt{\frac{1}{2} \frac{4m^2\xi^2 + k^4(1-z)^2}{4m^2\xi^2 + k^4(1+z)^2} + \frac{1}{2\varepsilon}}.$$  

In Fig. 2 we depict the behaviour of $G(\xi, k, T)$ for $\sigma^2 = 0$. In that case paramagnetism is absent and the eigenvalue spectrum is positively defined.

FIG. 2. (a) $G(\xi, k, T)$ versus $k$, $\xi = 0.1(\pi n_e/m)$, $\sigma^2 = 0$; (b) $G(\xi, k, T)$ versus $\xi$, $k = 0.8k_F$, $\sigma^2 = 0$.  

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The case $\sigma^2 = 1$ is depicted in Fig. 3, where we observe that $G$ becomes negative for some values of $\xi$ and $k$.

**FIG. 3.** (a) $G(\xi, k, T)$ versus $k$, $\xi = 0.1(\pi n_e/m)$, $\sigma^2 = 1$; (b) $G(\xi, k, T)$ versus $\xi$, $k = 0.8k_F$, $\sigma^2 = 1$.

From the expression (6) one can extract the asymptotic properties of $G(\xi, k, T)$ for $k \to \infty$ and $\xi \to \infty$. In the large-$k$ limit we obtain

\[
G \to 2 \frac{1 - \sigma^2}{2\pi\delta} \frac{mT}{k^2} \ln(1 + e^{\mu/T}) \quad \text{for} \quad \sigma^2 \neq 1
\]

\[
G \to 8 \frac{\delta - 2}{\pi\delta^2} \frac{m^2T^2}{k^4} \int_0^\infty \ln(1 + e^{-z + \mu/T}) dz \quad \text{for} \quad \sigma^2 = 1
\]

where $\delta = 1 + (2m\xi/k^2)^2$. These expressions are valid irrespectively whether the fraction $\xi/k^2$ vanishes, diverges or stays finite when $k \to \infty$.

Consider the limit $\xi \to \infty$. The case when $k$ tends to infinity together with $\xi$ has been already discussed above. Therefore, we assume that $k$ is finite and get

\[
G \to \frac{1 - \sigma^2}{4\pi} \frac{Tk^2}{m\xi^2} \ln(1 + e^{\mu/T}) + \frac{2}{\pi} \frac{T^2}{\xi^2} \int_0^\infty \ln(1 + e^{-z + \mu/T}) dz.
\]

The asymptotic relations imply that the negative values of $\lambda$ can be located in a finite region of the $(\xi, k)$-plane. Figures 2 and 3 demonstrate that the magnitude of the negative values of $G$ tend to zero as $T$ increases. Therefore, in the high temperature regime we get $\lambda > 0$.

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**IV. APPENDIX**

Here we comment on the calculation of $G(0, k, T)$. The common details are shown for its $\sigma$-dependent part:

\[
G_{\sigma\tau}(0, k, T) = -\int_0^{2\pi^2} \frac{f(\beta E_\tau) - f(\beta E_\sigma)}{kk'}.
\]
Due to the singularity, the $E_\pm$ parts are separately divergent and the integral cannot be decoupled in the corresponding way.

Expanding $f(\beta E_\pm)$ in powers of $kk'$, we use the polar variables and integrating over $k'$ get

$$G_{\tau\tau}(0,k,T) = \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \xi^n}{n!(2n+1)} f^{(n)}(\xi),$$

where $\xi = k^2/8mT$. Expand $f^{(n)}(\xi)$ and arrange the powers of $\xi$. Besides, we use $f^{(j)}(0) = (-1)^j \phi^{(j)}(\mu/T)$ with $\phi(z) = (1 + e^{-z})^{-1}$ and get

$$G_{\tau\tau}(0,k,T) = \frac{1}{2\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n \xi^n}{\Gamma(n+3/2)} \phi^{(n)}(\mu/T).$$

(A.1)

Analyze the structure of $\phi^{(n)}(z)$. From $\phi'(z) = \phi(1 - \phi)$ it follows that $\phi^{(n)}(z)$ can be presented as an $(n+1)^{th}$ order polynomial with respect to $\phi$

$$\phi^{(n)}(z) = P_{n+1}[\phi(z)].$$

(A.2)

These polynomials satisfy the recurrency relation $P_{n+1} = \phi(1 - \phi)(dP_n/d\phi)$ with $P_1[\phi] = \phi$. By the direct calculation it can be checked up that the solution to this recurrency chain appears as

$$P_{n+1}[\phi] = \sum_{k=0}^{\infty} \sum_{l=0}^{l} (-1)^l (l+1)^n C_k^l \phi^{k+1}.$$  

(A.3)

Substituting Eqs. (A.2) and (A.3) into Eq. (A.1), we use

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{\Gamma(n+\nu+1)} = \frac{2}{\Gamma(\nu)} \int_0^1 e^{-x(1-s^2)} s^{2\nu-1} ds,$$

and summing up over $l$, get

$$G_{\tau\tau}(0,k,T) = \int_0^1 \frac{dt}{\pi} e^{-\xi(1-s^2)} \sum_{k=0}^{\infty} \frac{[1 - e^{-\xi(1-s^2)}]_k}{[1 + e^{-\mu/T}]^{k+1}}.$$

The infinite sum over $k$ converges, yielding

$$G_{\tau\tau}(0,k,T) = \frac{1}{\pi} \int_0^1 \frac{ds}{1 + e^{\xi(1-s^2) - \mu/T}}.$$  

(A.4)

Performing the similar manipulations for the $\sigma$-independent part of $G$ and combining with (A.4), we get (4).

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