A PROOF OF FEIGIN’S CONJECTURE

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1. Introduction

The present paper is devoted to further development of semiinfinite cohomology of small quantum groups. The topic appeared first in [Ar1], where the very definition of semiinfinite cohomology \( \text{Ext}^\infty_A(\mathbb{C}, \cdot) \) was given. The setup for the definition of semiinfinite cohomology of an algebra \( A \) includes two subalgebras \( B, N \subseteq A \) and the triangular decomposition of \( A \), i.e. the vector space isomorphism \( B \otimes N \rightarrow A \) provided by the multiplication in \( A \). Fix root data \((Y, X, \ldots)\) of the finite type \((I, \cdot)\) and a positive integer number \( \ell \). The small quantum group \( \mathfrak{u}_\ell \) with the standard triangular decomposition turns out to be a very interesting object for the investigation of semiinfinite cohomology. The explanation for this lies in the following fact proved by Ginzburg and Kumar in [GK]. Consider a morphism \( B \rightarrow \mathfrak{g} \) corresponding to \((Y, X, \ldots)\).

\[ \text{Theorem:} \quad \text{Ext}^\infty_A(\mathbb{C}, \mathbb{C}) = \mathcal{F}un(N) \text{ as an associative algebra.} \]

On the other hand it is proved in [Ar2] that the algebra \( \text{Ext}^\infty_A(\mathbb{C}, \mathbb{C}) \) acts naturally on the semiinfinite cohomology of \( A \). Thus in particular \( \text{Ext}^\infty_A(\mathbb{C}, \cdot) \) becomes a quasicoherent sheaf on \( N \). It is natural to look for the answer for semiinfinite cohomology of \( \mathfrak{u}_\ell \) in terms of geometry of \( N \). B. Feigin has proposed the following conjecture. Consider the standard positive nilpotent subalgebra \( n^+ \subseteq N \subseteq \mathfrak{g} \).

**Conjecture A:** The quasicoherent sheaf on \( N \) provided by \( \text{Ext}^\infty_{\mathfrak{u}_\ell}(\mathbb{C}, \mathbb{C}) \) is equal to the sheaf of algebraic distributions on \( N \) with support on \( n^+ \subseteq N \).

Moreover note that the simply connected Lie group \( G \) with the Lie algebra equal to \( \mathfrak{g} \) acts naturally on \( N \). This action provides a structure of a \( n^+ \)-integrable \( \mathfrak{g} \)-module on the described distributions’ space. On the other hand it was shown in [Ar1] that there exists a natural \( U(\mathfrak{g}) \)-module structure on \( \text{Ext}^\infty_{\mathfrak{u}_\ell}(\mathbb{C}, \mathbb{C}) \). The \( \mathfrak{g} \)-module version of the Feigin conjecture states that the described \( \mathfrak{g} \)-modules are isomorphic. In [Ar1] and [Ar2] the conjecture was proved on the level of characters of the \( \mathfrak{g} \)-modules.

In the present paper we give a full proof of the Feigin conjecture.

1.1. Let us describe briefly the structure of the paper. In the second section we recall necessary facts concerning the usual cohomology of small quantum groups. The main facts here include the mentioned Ginzburg-Kumar theorem, the Kostant theorem describing the algebra structure on \( \mathcal{F}un(N) \) (see [2,3,4]) and a homological description of a certain degeneration \( \tilde{u}_\ell \) of the algebra \( u_\ell \) (see [2,4]). It turns out that both the usual and the semiinfinite cohomology of the algebra \( u_\ell \) with trivial coefficients can be described as associated graded objects for the corresponding functors over \( u_\ell \) with respect to certain filtrations.

In the third section we prove the \( \mathcal{F}un(N) \)-module version of the Feigin conjecture. The main arguments here are as follows. First we describe the \( \text{Ext}^\infty_{\mathfrak{u}_\ell}(\mathbb{C}, \mathbb{C}) \)-module \( \text{Ext}^\infty_{\tilde{u}_\ell}(\mathbb{C}, \mathbb{C}) \). Next, using this description, we construct a morphism of \( \mathcal{F}un(N) \)-modules

\[ \Phi : H^0(N \setminus D_1 \cup \ldots \cup D_{\ell}(R^\times), \mathcal{O}_N) \rightarrow \text{Ext}^\infty_{\tilde{u}_\ell}(\mathbb{C}, \mathbb{C}), \]

where \( D_1, \ldots, D_{\ell}(R^\times) \) denote the coordinate divisors such that \( \cap_i D_i = n^+ \). Note that the first \( \mathcal{F}un(N) \)-module contains the distributions module \( H^\infty_{n^+}(N, \mathcal{O}_N) \) as a certain quotient module.

Finally, using the connection between \( \text{Ext}^\infty_{\tilde{u}_\ell}(\mathbb{C}, \mathbb{C}) \) and \( \text{Ext}^\infty_{\tilde{u}_\ell}(\mathbb{C}, \mathbb{C}) \) we prove that the map \( \Phi \) provides a \( \mathcal{F}un(N) \)-module isomorphism

\[ H^\infty_{n^+}(N, \mathcal{O}_N) \rightarrow \text{Ext}^\infty_{\tilde{u}_\ell}(\mathbb{C}, \mathbb{C}). \]
In the fourth section we prove the $\mathfrak{g}$-module version of the Feigin conjecture. The main steps of the proof are as follows. First we construct explicitly the $\mathfrak{n}^*$-module isomorphism. Next we recall from [Ar1] that there exists a nondegenerate $\mathfrak{g}$-equivariant contragradient pairing on $\text{Ext}^\bullet_{\mathfrak{n}^*}(\mathbb{C},\mathbb{C})$. We construct its geometric analogue on $H^n_{\mathfrak{n}^*}(\mathcal{N},\mathcal{O}_\mathcal{N})$. On the other hand it is easy to verify that $H^n_{\mathfrak{n}^*}(\mathcal{N},\mathcal{O}_\mathcal{N})$ is free over the algebra $U(\mathfrak{n}^-)$. It follows that both $H^n_{\mathfrak{n}^*}(\mathcal{N},\mathcal{O}_\mathcal{N})$ and $H^n_{\mathfrak{n}^*}(\mathcal{N},\mathcal{O}_\mathcal{N})$ are co-free over $U(\mathfrak{n}^+)$. Using the constructed contragredient pairing on the $\mathfrak{g}$-module of semiinfinite cohomology we see that the latter module is $U(\mathfrak{n}^-)$-free. It follows that both $H^n_{\mathfrak{n}^*}(\mathcal{N},\mathcal{O}_\mathcal{N})$ and $H^n_{\mathfrak{n}^*}(\mathcal{N},\mathcal{O}_\mathcal{N})$ are tilting $\mathfrak{g}$-modules. Finally a beautiful result of Andersen states that a tilting $\mathfrak{g}$-module is completely determined up to an isomorphism by its character (see [A]).

In the fifth section we present a generalization of the Feigin conjecture as follows. Consider the contragradient Weyl module of $\mathfrak{g}$ corresponding to the data $(\mathcal{Y}, \mathcal{X}, \ldots)$. The algebra $U(\mathfrak{n}^-)$ is generated by the elements $\ell_{i \in I, i}$. We call the algebras $U(\mathfrak{n}^-)$ constructed an associative algebra $\mathfrak{g}$. Weyl group of $\mathfrak{g}$ is determined up to an isomorphism by its character (see [A]).

**Conjecture B:** The $\mathcal{F}un(\mathcal{N}) = \text{Ext}^\bullet_{\mathfrak{n}^*}(\mathbb{C},\mathbb{C})$-module $\text{Ext}^\bullet_{\mathfrak{n}^*}(\mathcal{N},\mathbb{D}W(\ell \lambda))$ is isomorphic to $H^n_{\mathfrak{n}^*}(\mathcal{N}, \mu_p^*\mathcal{L}(\lambda))$. 

We give a sketch of the proof of the Conjecture B. The main tool here is a certain specialization of the quantum BGG resolution into the root of unity. We call it the contragradient quasi-BGG complex. This complex has $\mathbb{D}W(\ell \lambda)$ as a zero cohomology module. We conjecture that the contragradient quasi-BGG complex is in fact quasiisomorphic to $\mathbb{D}W(\ell \lambda)$. Still even without the last assumption we manage to prove Conjecture B.

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2. SMALL QUANTUM GROUPS.

2.1. **Root data.** Fix a Cartan datum $(I, \cdot)$ of the finite type and a simply connected root datum $(\mathcal{Y}, \mathcal{X}, \ldots)$ of the type $(I, \cdot)$. Thus we have $\mathcal{Y} = \mathbb{Z}[I]$, $\mathcal{X} = \text{Hom}(\mathcal{Y}, \mathbb{Z})$, and the pairing $\langle \cdot , \cdot \rangle : \mathcal{Y} \times \mathcal{X} \to \mathbb{Z}$ coincides with the natural one (see [L3], I 1.1, I 2.2). In particular the data contain canonical embeddings $I \hookrightarrow \mathcal{Y}, i \mapsto i$ and $I \hookrightarrow \mathcal{X}, i \mapsto i' : \langle i', j \rangle := 2i \cdot j / i$. The latter map is naturally extended to an embedding $\mathcal{Y} \subset \mathcal{X}$. Denote by $\text{ht}$ the linear function on $\mathcal{X}$ defined on elements $i', i \in I$, by $\text{ht}(i') = 1$ and extended to the whole $\mathcal{X}$ by linearity. The root system (resp. the positive root system) corresponding to the data $(\mathcal{Y}, \mathcal{X}, \ldots)$ is denoted by $R$ (resp. by $R^+$), below $W$ denotes the Weyl group of $R$.

2.2. **Quantum groups at roots of 1.** Given the root datum $(\mathcal{Y}, \mathcal{X}, \ldots)$ Drinfeld and Jimbo constructed an associative algebra $\mathcal{U}$ over the field $\mathbb{Q}(v)$ of rational functions in $v$ with generators $E_i, F_i, K_i^{\pm 1}, i \in I$, and relations being the quantum analogues of the classical Serre relations in the universal enveloping algebra of the corresponding simple Lie algebra $\mathfrak{g}$. The explicit form of the relations can be found e. g. in [L3], I 3.1. We call the algebras $\mathcal{U}$ the quantum groups.

Lusztig (see [L1], V 31.1) defined a $\mathbb{Q}[v, v^{-1}]$-subalgebra $\mathcal{U}_v$ in $\mathcal{U}$ being the quantum analogue of the integral form for the universal enveloping algebra of $\mathfrak{g}$ due to Kostant. In particular the elements $E_i, F_i, K_i^{\pm 1}, i \in I$, belong to $\mathcal{U}_v$. Let $\ell$ be an odd number satisfying the conditions from [GR]. Fix a primitive $\ell$-th root of unity $\zeta$. Define a $\mathbb{C}$-algebra $\tilde{\mathcal{U}}_\ell := \mathcal{U}_v \otimes \mathbb{Q}[v, v^{-1}]\mathbb{C}$, where $v$ acts on $\mathbb{C}$ by multiplication by $\zeta$. It is known that the elements $K_i^{\ell}, i \in I$, are central in $\tilde{\mathcal{U}}_\ell$. Set $U_\ell := \tilde{\mathcal{U}}_\ell / (K_i^{\ell} - 1, i \in I)$. The algebra $U_\ell$ is generated by the elements $E_i, E_i^{(\ell)}, F_i, F_i^{(\ell)}, K_i^{\pm 1}, i \in I$. Here $E_i^{(\ell)}$ (resp. $F_i^{(\ell)}$)
denotes the $\ell$-th quantum divided power of the element $E_i$ (resp. $F_i$) specialized at the root of unity $\zeta$.

Following Lusztig we define the small quantum group $u_\ell$ at the root of unity $\zeta$ as the subalgebra in $U_\ell$ generated by all $E_i, F_i, K_i^{\pm1}, i \in I$. Denote the subalgebra in $u_\ell$ generated by $E_i, i \in I$ (resp. $F_i, i \in I, \text{resp. } K_i, i \in I)$, by $u_\ell^+$ (resp. $u_\ell^-$, resp. $u_\ell^0$). Note that the algebra $u_\ell$ is graded naturally by the abelian group $X$. Using the function $\ell$ we obtain a $Z$-grading on $u_\ell$ from this $X$-grading. In particular the subalgebra $u_\ell^+$ (resp. $u_\ell^-$) is graded by $Z_{\geq 0}$ (resp. by $Z_{< 0}$).

Below we present several well known facts about the algebra $u_\ell$ to be used later. Recall that an augmented subalgebra $B \subset A$ with the augmentation ideal $\overline{B} \subset B$ is called normal if $\overline{AB} = \overline{BA}$. If so, the space $A/\overline{B}$ becomes an algebra. It is denoted by $A//B$. Fix an augmentation on $u_\ell$ as follows: $E_i \mapsto 0, F_i \mapsto 0, K_i \mapsto 1$ for every $i \in I$. Set $\overline{u}_\ell := u_\ell/\overline{u}_\ell$.

\textbf{2.2.1. Lemma:} (see [AJS] 1.3, [2] Theorem 8.10)

(i) The multiplication in $u_\ell$ provides a vector space isomorphism $u_\ell = u_\ell^0 \otimes u_\ell^0 + u_\ell^0 \otimes u_\ell^0$; $\dim u_\ell^0 = \ell^d(R)$; the subalgebra $u_\ell^0$ is isomorphic to the group algebra of the group $(\mathbb{Z}/\mathbb{Z})^{\ell(I)}$.

(ii) The subalgebra $u_\ell^0 \otimes u_\ell^0$ (resp. $u_\ell^0 \otimes u_\ell^0$) in $u_\ell$ is denoted by $b_\ell^+$ (resp. by $b_\ell^-$). Recall that a finite dimensional algebra $A$ is called Frobenius if the latter algebra acts on the algebraic vector fields on the nilpotent cone $\overline{V}$ of $\text{algebraic vector fields on the nilpotent cone } H$ provides a grading on $g$.

\textbf{2.2.2. Lemma:} (see [AT], Lemma 2.4.5) The algebras $u_\ell^+$ and $u_\ell^-$ are Frobenius.

Consider the filtration on the algebra $u_\ell$ as follows. Let the filtration component $F^{\leq d}(u_\ell)$ be linearly generated by $X$-homogeneous monomials $u = u^0 \otimes u^0$ such that $|\text{ht}(\deg u^-)| + |\text{ht}(\deg u^+)| \leq d$. By definition set $\tilde{u}_\ell := \text{gr}^F u_\ell$. Evidently we have $\text{gr}^F b_\ell^+ = b_\ell^+,$ $\text{gr}^F b_\ell^- = b_\ell^-$, $\text{gr}^F u_\ell^0 = u_\ell^0$.

\textbf{2.2.3. Lemma:} Elements of the subalgebra $u_\ell^+ \subset \tilde{u}_\ell$ commute with elements of the subalgebra $u_\ell^- \subset \tilde{u}_\ell$.

Both the $X$- and the $Z$-grading as well as the augmentation on $\tilde{u}_\ell$ are induced by the ones on $u_\ell$. Denote the category of $X$-graded finite dimensional left $u_\ell$-modules (resp. $\tilde{u}_\ell$-modules) $M = \bigoplus_{\lambda \in X} M_\lambda$ such that $K_i$ acts on $M_\lambda$ by multiplication by the scalar $\zeta^{(i, \lambda)}$ and $E_i : M_\lambda \rightarrow M_{\lambda + i'}, F_i : M_\lambda \rightarrow M_{\lambda - i'}$ for all $i \in I$, with morphisms preserving $X$-gradings, by $u_\ell$-mod (resp. by $\tilde{u}_\ell$-mod). For $M, N \in u_\ell$-mod and $\lambda \in \ell \cdot X$ we define the shifted module $M(\lambda) \in u_\ell$-mod : $M(\lambda)_{\mu} := M_{\lambda + \mu}$ and set $\text{Hom}_{u_\ell}(M, N) := \bigoplus_{\lambda \in \ell \cdot X} \text{Hom}_{u_\ell\text{-mod}}(M(\lambda), N)$. The functor $\text{Hom}_{u_\ell}$ is defined in a similar way. Evidently the spaces $\text{Hom}_{u_\ell}(\cdot, \cdot)$ and $\text{Hom}_{\tilde{u}_\ell}(\cdot, \cdot)$ possess natural $\ell \cdot X$-gradings.

\textbf{2.3. Cohomology of small quantum groups.} Consider the $\ell \cdot X \times \mathbb{Z}$-graded algebra $\text{Ext}^*_{u_\ell}(\mathbb{C}, \mathbb{C})$. Note that by Shapiro lemma and Lemma 2.2.1 (ii) the Lie algebra $\mathfrak{g}$ acts naturally on the Ext algebra and the multiplicity in the algebra satisfies Leibnitz rule with respect to the $\mathfrak{g}$-action. In [GK] Ginzburg and Kumar obtained a nice description of the multiplication structure as well as the $\mathfrak{g}$-module structure on $\text{Ext}^*_{u_\ell}(\mathbb{C}, \mathbb{C})$ as follows.

\textbf{2.3.1. Functions on the nilpotent cone.} Let $G$ be the simply connected Lie group with the Lie algebra $\mathfrak{g}$. Then $G$ acts on $\mathfrak{g}$ by adjunction. The action preserves the set of nilpotent elements $N \subset \mathfrak{g}$ called the nilpotent cone of $\mathfrak{g}$. The action is algebraic, thus it provides a morphism of $\mathfrak{g}$ into the Lie algebra of algebraic vector fields on the nilpotent cone $\text{Vect}(N)$. The latter algebra acts on the algebraic functions $H^0(N, O_N)$. The action is $G$-integrable. Note also that the natural action of the group $\mathbb{C}^*$ provides a grading on $H^0(N, O_N)$ preserved by the $G$-action.

\textbf{2.3.2. Theorem:} (see [GK]) The algebra and $\mathfrak{g}$-module structures on $H^0(N, O_N)$ and on $\text{Ext}^*_{u_\ell}(\mathbb{C}, \mathbb{C})$ coincide. The homological grading on the latter algebra corresponds to the grading on the former one provided by the $\mathbb{C}^*$-action. The $X$-grading on $H^0(N, O_N)$ provided by the weight decomposition with respect to the action of the Cartan subalgebra in $\mathfrak{g}$ corresponds to the natural $\ell \cdot X$-grading on the space $\text{Ext}^*_{u_\ell}(\mathbb{C}, \mathbb{C})$.

We will need a more detailed description of the algebra $H^0(N, O_N)$. First the following result of Kostant shows the size of the algebra.
2.3.3. \textbf{Proposition:} (see [H]) $\text{ch}(H^0(\mathcal{N}, \mathcal{O}_\mathcal{N}), t) = \prod_{\alpha \in R^+} \frac{\alpha \cdot t}{(1 - e^{-\alpha t})(1 - e^{\alpha t})}$.

Here the indeterminate $t$ stands for the homogeneous $\mathbb{Z}$-grading, for a weight $\alpha = \sum a_i e^i$ the symbol $e^\alpha$ denotes the monomial $\prod_i e^i^{a_i}$, and $l(w)$ denotes the length of the element of the Weyl group. \hfill $\square$

In particular the space

$$H^0(\mathcal{N}, \mathcal{O}_\mathcal{N}) = \bigoplus_{\alpha \in R^+} \mathbb{C}e^\alpha \bigoplus_{\omega \in W, l(w)=1} \mathbb{C} \xi_w \bigoplus \bigoplus_{\alpha \in R^+} \mathbb{C} f^\alpha = n^{++} \bigoplus \bigoplus_{\alpha \in R^+} \mathbb{C} \xi_w \bigoplus n^{-},$$

where the $X$-grading of the element $e^\alpha$ (resp. $f^\alpha$, resp. $\xi_w$) is equal to $-\alpha$ (resp. to $\alpha$, resp. to 0).

Consider the subalgebra $H_-$ (resp. $H_+$) in $H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})$ generated by the space $n^{++}$ (resp. $n^-$). The detailed description of the algebra structure on $H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})$ is given by the following statement.

2.3.4. \textbf{Proposition:} (see [Ko], Theorem 1.5) The algebra $H_-$ (resp. $H_+$) is equal to the free commutative algebra $S^*(n^{++})$ (resp. $S^*(n^-)$).

Note that the inclusion $S^*(n^-) \hookrightarrow H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})$ (resp. $S^*(n^{++}) \hookrightarrow H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})$) corresponds to the coordinate projection $\mathcal{N} \twoheadrightarrow g \rightarrow n^-$ (resp. $\mathcal{N} \twoheadrightarrow g \rightarrow n^+$). In particular we obtain a description of the algebra $\text{Ext}_{uc}^*(\mathbb{C}, \mathbb{C})$.

2.3.5. \textbf{Corollary:} $\text{Ext}_{uc}^*(\mathbb{C}, \mathbb{C}) = S^*(n^-) \otimes H_0 \otimes S^*(n^{++})$ as a module over $S^*(n^-) \otimes S^*(n^{++})$. Here $H_0 := \bigoplus_{w \in W} \mathbb{C} \xi_w$.

Ginzburg and Kumar also proved the following statements (see [GK]).

(i) $\text{Ext}_{uc}^*(\mathbb{C}, \mathbb{C}) = \bigoplus_{w \in W} \mathbb{C} \nu_w \otimes S^*(n^-)$ as a $X \times \mathbb{Z}$-graded vector space. Here the homological grading of the element $\nu_w$, $w \in W$, is equal to $l(w)$, and its $X$-grading equals $\rho - w(\rho)$. The homological grading (resp. the $X$-grading) of the element $f^\alpha \in n^{++}$ equals 2 (resp. $\ell(\alpha)$). A similar statement holds for the Ext algebra of $u^-_c$.

(ii) $\text{Ext}_{uc}^*(\mathbb{C}, \mathbb{C})$ (resp. $\text{Ext}_{uc}^*(\mathbb{C}, \mathbb{C})$) is equal to $S^*(n^-)$ (resp. $S^*(n^{++})$) both as an associative algebra and as a $n^-$- (resp. $n^{++}$-) module.

(iii) The natural map $\text{Ext}_{uc}^*(\mathbb{C}, \mathbb{C}) \twoheadrightarrow \text{Ext}_{uc}^*(\mathbb{C}, \mathbb{C})$ (resp. $\text{Ext}_{uc}^*(\mathbb{C}, \mathbb{C}) \twoheadrightarrow \text{Ext}_{uc}^*(\mathbb{C}, \mathbb{C})$) given by the restriction functor coincides with the morphism of the algebras of functions provided by the inclusion of the affine manifolds $n^- \hookrightarrow \mathcal{N}$ (resp. $n^+ \hookrightarrow \mathcal{N}$).

We set $A_+ := \bigoplus_{w \in W} \mathbb{C} \nu_w \subset \text{Ext}_{uc}^*(\mathbb{C}, \mathbb{C})$ and $A_- := \bigoplus_{w \in W} \mathbb{C} \nu_w \subset \text{Ext}_{uc}^*(\mathbb{C}, \mathbb{C})$. Note that the restriction functor $\text{Res}_{uc}^*(\mathbb{C}, \mathbb{C})$ provides the inclusion of algebras

$$\text{Ext}_{uc}^*(\mathbb{C}, \mathbb{C}) = S^*(n^{++}) \twoheadrightarrow \text{Ext}_{uc}^*(\mathbb{C}, \mathbb{C}) \text{ (resp. } \text{Ext}_{uc}^*(\mathbb{C}, \mathbb{C}) = S^*(n^-) \twoheadrightarrow \text{Ext}_{uc}^*(\mathbb{C}, \mathbb{C}))$$

2.3.6. \textbf{Lemma:}

(i) The subspace $A_-$ (resp. $A_+$) is a subalgebra in $\text{Ext}_{uc}^*(\mathbb{C}, \mathbb{C})$ (resp. in $\text{Ext}_{uc}^*(\mathbb{C}, \mathbb{C})$).

(ii) The multiplication maps provide the vector space isomorphisms

$$\text{Ext}_{uc}^*(\mathbb{C}, \mathbb{C}) \otimes A_+ \cong \text{Ext}_{uc}^*(\mathbb{C}, \mathbb{C}) \text{ and } \text{Ext}_{uc}^*(\mathbb{C}, \mathbb{C}) \otimes A_- \cong \text{Ext}_{uc}^*(\mathbb{C}, \mathbb{C}).$$

Note that the algebra $u_{uc}^0$ acts trivially on the first factors in these decompositions.

2.3.7. \textbf{Proposition:} The algebras $A_-$ and $A_+$ are Frobenius.

\textbf{Proof:} We consider only the case of $A_-$. Recall that when calculating cohomology of the algebra $u_{uc}^0$ Ginzburg and Kumar used certain degenerations of the algebra defined by Kac, De Concini and Procesi in [DCKP] with the help of quantum PBW filtration on $u_{uc}^-$. The algebra $\text{gr}^{PBW}(u_{uc}^-)$ appears to be a certain quotient algebra of the quantum symmetric algebra with the set of generators equal enumerated by the standard basis in the space $n^-$. In particular it was proved in [GK] that $\text{Ext}_{uc}^{gr^{PBW}(u_{uc}^-)}(\mathbb{C}, \mathbb{C}) = \Lambda^*(n^-) \otimes S^*(n^-)$ as an associative algebra. Here $\Lambda^*(n^-)$ denotes the quantum exterior algebra on generators enumerated by elements of the standard basis in $n^-$. In particular this algebra is Frobenius with the trace map given by the projection on the top homological grading component (that is one dimensional).
Consider now the spectral sequence with the term $E_2$ equal to $\text{Ext}^*_{\text{gr}^F \text{BW}(u^0_\ell)}(\mathbb{C}, \mathbb{C})$ that converges to $\text{Ext}^*_{u^0_\ell}(\mathbb{C}, \mathbb{C})$. Note that for $\ell$ big enough the subalgebra $\Lambda^*_\ell(n^{-*}) \subset \text{Ext}^*_{\text{gr}^F \text{BW}(u^0_\ell)}(\mathbb{C}, \mathbb{C})$ is a DG-subalgebra. Moreover its images on the next terms of the spectral sequence are DG-subalgebras too. Now note that the subalgebra $\text{gr} A_+$ in the term $E_\infty$ of the spectral sequence comes from the described DG-subalgebra in the term $E_2$. In particular the one dimensional top homological grading component of $\Lambda^*_\ell(n^{-*})$ has its nonzero representative on the level $E_\infty$. Thus it has its nonzero representatives on all the levels $E_m, \ m \geq 2$.

Consider the algebra $\text{gr}^F A_+$ appearing in the $E_\infty$ term of the spectral sequence. Put the linear functional generating the trace map on $\text{gr}^F A_+$ equal to the projection on the one dimensional top homological grading component. We claim that the corresponding trace pairing on the algebra is self-dual with respect to the trace pairing on $\Lambda^*_\ell(n^{-*})$ appearing in the $E_2$ term of the spectral sequence. On the other hand note that the $X$-grading components of the space $\Lambda^*_\ell(n^{-*})$ with the weights $\rho - w(\rho), \ w \in W$, are one dimensional. Moreover the subspace $\bigoplus_{\rho \in W} (\Lambda^*_\ell(n^{-*}))^\rho \subset W)$ is self-dual with respect to the trace pairing on $\Lambda^*_\ell(n^{-*})$. Thus the term $E_\infty$ of the spectral sequence becomes an orthogonal direct sum and in the term $E_2$. We have proved that the algebra $\text{gr}^F A_+$ is Frobenius. Finally by Lemma 2.4.4 from [Ar1] the algebra $A_+$ is Frobenius itself. □

2.4. Cohomology of the algebra $\widetilde{u}_\ell$. Note that the algebra $\widetilde{u}_\ell$ contains the algebra $u^0_\ell \otimes u^+_\ell$ as a subalgebra, moreover, this subalgebra is normal in $\widetilde{u}_\ell$ with the quotient algebra equal to $u_\ell^0$. On the other hand, by standard arguments, we have

$$\text{Ext}^*_{u_\ell^0 \otimes u_\ell^+}(\mathbb{C}, \mathbb{C}) = \text{Ext}^*_{u_\ell^0}(\mathbb{C}, \mathbb{C}) \otimes \text{Ext}^*_{u_\ell^+}(\mathbb{C}, \mathbb{C})$$

as an associative algebra. Now recall that the algebra $u^0_\ell$ is semisimple being a group algebra of a finite group. We have proved the following statement.

2.4.1. Lemma: $\text{Ext}^*_{u_\ell^0}(\mathbb{C}, \mathbb{C}) = \left(\text{Ext}^*_{u_\ell^0}(\mathbb{C}, \mathbb{C}) \otimes \text{Ext}^*_{u_\ell^+}(\mathbb{C}, \mathbb{C})\right)^{u_\ell^0}$ as a $\ell \cdot X \times \mathbb{Z}$-graded vector space. Here $(\cdot)^{u_\ell^0}$ denotes taking $u^0_\ell$-invariants. Moreover, the restriction functor $\text{Res}_{u_\ell}^{u_\ell^0} \otimes_{u_\ell^0}$ provides an isomorphism of associative algebras $\text{Ext}^*_{u_\ell^0}(\mathbb{C}, \mathbb{C}) \cong \left(\text{Ext}^*_{u_\ell^0}(\mathbb{C}, \mathbb{C})\right)^{u_\ell^0}$. □

The following proposition is proved similarly to Theorem 3.1 from [Ar3].

2.4.2. Lemma: $\text{ch} \left(\left(\text{Ext}^*_{u_\ell^0}(\mathbb{C}, \mathbb{C}) \otimes \text{Ext}^*_{u_\ell^+}(\mathbb{C}, \mathbb{C})\right)^{u_\ell^0}, t\right) = \frac{\sum_{w \in W} e^{2\ell(w)} \prod_{\alpha \in R^+} (1 - e^{-\ell(w)g_\alpha})}{\prod_{\alpha \in R^+} (1 - e^{-\ell \cdot \rho(\alpha)})}$. Here the indeterminate $t$ stands for the homogeneous $\mathbb{Z}$-grading, for a weight $e^\alpha$ the symbol $e^\alpha$ denotes the monomial $\prod_i (e^{\alpha_i})^\alpha$. Let us denote the filtration on the space $\text{Ext}^*_{u_\ell}(\mathbb{C}, \mathbb{C})$ that corresponds to the filtration $F$ on the algebra $u_\ell$ by the same letter.

2.4.3. Corollary: $\text{gr}^F \text{Ext}^*_{u_\ell}(\mathbb{C}, \mathbb{C}) = \text{Ext}^*_{u_\ell}(\mathbb{C}, \mathbb{C})$ as a graded associative algebra. □

Summing up the previous considerations we obtain the following statement.

2.4.4. Proposition: $\text{Ext}^*_{u_\ell}(\mathbb{C}, \mathbb{C}) = S^*(n^{-*}) \otimes \text{gr}^F H_0 \otimes S^*(n^{+*})$ as an associative algebra. Here the imbedding $S(n^{-*}) \hookrightarrow \text{Ext}^*_{u_\ell}(\mathbb{C}, \mathbb{C})$ resp. the imbedding $S(n^{+*}) \hookrightarrow \text{Ext}^*_{u_\ell}(\mathbb{C}, \mathbb{C})$ is provided by the projection of algebras $\tilde{u}_\ell \rightarrow b^\ell_\alpha : u^0_\ell \rightarrow 0$ (resp. $\tilde{u}_\ell \rightarrow b^\ell_\alpha : u^0_\ell \rightarrow 0$). □

Comparing grading components in the previous equality with gradings far smaller than $\ell$ we obtain the following statement.

2.4.5. Corollary: The associative algebra $\text{gr}^F H_0 = (A_\perp \otimes A_+)u^0_\ell$. □
3. \( \text{Ext}^{\infty,\bullet}_u(\mathbb{C}, \mathbb{C}) \) as a \( \text{Ext}^{\bullet}_u(\mathbb{C}, \mathbb{C}) \)-module.

First we recall briefly the general setup for semiinfinite cohomology of the small quantum group. The definition of semiinfinite cohomology presented below is a specialization of the general one in the case of a finite dimensional graded algebra \( u = b^- \otimes u^+ \) such that \( b^- \) is nonpositively graded, and \( u^+ \) is a positively graded Frobenius algebra with \( u^+_0 = \mathbb{C} \).

3.1. Definition of semiinfinite cohomology. Consider first the semiregular \( u \)-bimodule \( S^u_+ = u \otimes u^+ \), with the right \( u \)-module structure provided by the isomorphism of left \( u^+ \)-modules \( u^+ \cong u^+ \).

Note that \( S_3.1. \ Definition \ of \ semiinfinite \ cohomology. \) Let us compare semiinfinite cohomology of the trivial \( u \)-module consisting of \( u^+ \) on the choice of resolutions and define functors \( \text{Ext}^{\bullet}_u, \text{Hom}^{\bullet}_u \) consisting of \( u^+ \)-free (resp. \( u^- \)-free) modules.

3.1.1. Definition: We set \( \text{Ext}^{\infty,\bullet}_u(M^+, N^+) := H^\bullet(\text{Hom}^\bullet_u(R^\bullet(M^+), S^u_+ \otimes_u R^\bullet(N^+))). \)

3.1.2. Lemma: (see. [Ar1] Lemma 3.4.2, Theorem 5) The spaces \( \text{Ext}^{\infty,\bullet}_u(M^+, N^+) \) do not depend on the choice of resolutions and define functors \( \text{Ext}^{\infty,\bullet}_u(\cdot, \cdot) : \text{u-mod} \times \text{u-mod} \rightarrow \text{Vect}, k \in \mathbb{Z}. \)

Below we consider semiinfinite cohomology of algebras \( \mathfrak{u}_\ell, \bar{u}_\ell, b^+_{\ell}, b^-_{\ell} \) etc. with coefficients in \( X \)-graded modules. The \( \mathbb{Z} \)-grading on such a module is obtained from the \( \mathbb{Z} \)-grading using the function \( \text{ht} : X \rightarrow \mathbb{Z} \).

Evidently the spaces \( \text{Ext}^{\infty,\bullet}_u(M^+, N^+) \) (resp. \( \text{Ext}^{\infty,\bullet}_u(\bar{M}^+, \bar{N}^+) \)) possess natural \( \ell \cdot X \)-gradings. The following statement is a direct consequence of Lemma [2.2.1](ii).

3.1.3. Lemma: Let \( M, N \in \mathfrak{u}_\ell\text{-mod} \) be restrictions of some \( U_\ell \)-modules. Then the spaces \( \text{Ext}^{\infty,\bullet}_u(M, N) \) have natural structures of \( \mathfrak{g} \)-modules, and the \( \ell \cdot X \)-gradings on them coincide with the \( X \)-gradings provided by the weight decompositions of the modules with respect to the standard Cartan subalgebra in \( \mathfrak{g} \).

In particular we consider the character of this \( \mathfrak{g} \)-module.

3.2. Semiinfinite cohomology of the trivial \( \bar{u}_\ell \)-module. The following statement sums up the main results from \([Ar1]\).

Theorem:

(i) \( \text{Ext}^{\infty,\bullet}_u(\bar{u}_\ell, \bar{u}_\ell) = \left( \text{Ext}^{\infty,\bullet}_u(\bar{u}_\ell, \bar{u}_\ell) \right)_{\bar{u}^+_0} \).

(ii) \( \text{ch} \left( \text{Ext}^{\infty,\bullet}_u(\bar{u}_\ell, \bar{u}_\ell), t \right) = t^{-\ell(R^+)} e^{-2\ell(\rho)} \sum_{\alpha \in R^+} \frac{\ell^{2\ell(\alpha)}}{(1-e^{-\ell(\alpha-1)})(1-e^{-\ell(\alpha)})^2}. \)

The right hand side of the equality is considered as an element in \( \mathbb{C}[t, t^{-1}][e^{-it}, i \in I]. \)

(iii) Let \( M \) be a filtered module over the filtered algebra \( \mathfrak{u}_\ell \) with the filtration \( F \). Then there exists a spectral sequence with the term \( E_1 = \text{Ext}^{\infty,\bullet}_u(\mathbb{C}, \text{gr} M), \) converging to \( E_\infty = \text{gr}^F \text{Ext}^{\infty,\bullet}_u(\mathbb{C}, M). \)

(iv) For the trivial \( \mathfrak{u}_\ell \)-module the described spectral sequence degenerates in the term \( E_1 \). In particular as a \( \ell \cdot X \times \mathbb{Z} \)-graded vector space \( \text{Ext}^{\infty,\bullet}_u(\mathbb{C}, \mathbb{C}) = \text{Ext}^{\infty,\bullet}_u(\mathbb{C}, \mathbb{C}). \)

Let us compare semiinfinite cohomology of the trivial \( b^+_{\ell} \)-module with the usual cohomology of \( b^+_{\ell} \) with coefficients in this module.
3.2.1. Lemma: \( \text{Ext}^\infty_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C}) = \text{Ext}^\bullet_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C}((-\ell + 1)2\rho)) \).

**Proof.** The triangular decomposition of the algebra \( \mathfrak{b}^+_\ell \) is as follows: \( \mathfrak{b}^+_\ell = u^+_\ell \otimes u^0_\ell \). Then by definition of semiinfinite cohomology, using the fact that \( u^0_\ell \) is semisimple, we see that

\[
\text{Ext}^\infty_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C}) = H^\bullet \left( \text{Hom}^\bullet_{\mathfrak{b}^+_\ell} (R^+_\ell (\mathbb{C}), S^{u^+_\ell \otimes u^0_\ell} \mathbb{C}) \right),
\]

where \( R^+_\ell (\mathbb{C}) \) denotes a concave \( u^+_\ell \)-free resolution of the trivial \( \mathfrak{b}^+_\ell \)-module. Next note that the \( \mathfrak{b}^+_\ell \)-module \( S^{u^+_\ell \otimes u^0_\ell} \mathbb{C} = \mathfrak{b}^{+\dagger} \otimes_{\mathfrak{b}^+_\ell} \mathbb{C} = \mathbb{C}((-\ell + 1)2\rho) \) Thus we have

\[
\text{Ext}^\infty_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C}) = H^\bullet \left( \text{Hom}^\bullet_{u^+_\ell} (R^+_\ell (\mathbb{C}), \mathbb{C}((-\ell + 1)2\rho)) \right)
\]

\[
= \left( H^\bullet (\text{Hom}^\bullet_{u^+_\ell} (R^+_\ell (\mathbb{C}), \mathbb{C}C((-\ell + 1)2\rho))) \right)^{u^0_\ell} = \text{Ext}^\bullet_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C}((-\ell + 1)2\rho)).
\]

Next we investigate the structure of the \( \text{Ext}^\bullet_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C}) \)-module \( \text{Ext}^\infty_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C}) \).

3.2.2. Proposition: Up to the homological grading shift by \( \mathfrak{z}(\mathbb{R}^+) \) we have

\[
\text{Ext}^\infty_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C}) = \text{Coind}^{\text{Ext}^\bullet_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C})}_{\text{Ext}^\bullet_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C})} \left( \text{Ext}^\bullet_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C}((-\ell + 1)2\rho)) \right)
\]

as a \( \text{Ext}^\infty_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C}) \)-module.

**Proof.** By Theorem \[3.2\], Lemma \[3.2.1\] and Corollary \[2.4.3\] we have

\[
\text{Ext}^\infty_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C}) = \left( \text{Ext}^\infty_{\mathfrak{b}^+_\ell \otimes u^0_\ell} (\mathbb{C}, \mathbb{C}) \right)^{u^0_\ell} = \left( \text{Tor}^{u^0_\ell}_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C} \otimes ((1 - \ell)2\rho) \otimes \text{Ext}^\bullet_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C})) \right)^{u^0_\ell}
\]

\[
= (S^\bullet(n^-) \otimes A^\bullet_+ \otimes S^\bullet(n^{++}) \otimes A^-_+ \mathbb{C} ((1 - \ell)2\rho))^{u^0_\ell}
\]

\[
= S^\bullet(n^-) \otimes S^\bullet(n^{++}) \otimes (A^\bullet_+ \mathbb{C} ((1 - \ell)2\rho) \otimes A^-_+)^{u^0_\ell}.
\]

Now recall that the algebra \( A_+ \) is Frobenius. Making the isomorphism \( A_+ = A^\bullet_+ \mathbb{C} (-2\rho) \) compatible with the \( u^0_\ell \)-action, we see that \( A_+ = A^\bullet_+ \mathbb{C} (-2\rho) \). Thus we obtain

\[
\text{Ext}^\infty_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C}) = S^\bullet(n^-) \otimes S^\bullet(n^{++}) \otimes (A^\bullet_+ \mathbb{C} (-2\rho) \otimes A^-_+)^{u^0_\ell}
\]

\[
= S^\bullet(n^-) \otimes \text{gr}^F_0 \mathbb{C} (\mathbb{C} (-2\rho)).
\]

Note that the algebra \( \text{gr} H_0 \) is Frobenius itself, thus the latter \( \text{Ext}^\infty_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C}) \)-module can be considered both as

\[
\text{Coind}^{\text{Ext}^\bullet_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C})}_{\text{Ext}^\bullet_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C})} \left( \text{Ext}^\bullet_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C}((-2\rho))) \right) \quad \text{and as} \quad \text{Ind}^{\text{Ext}^\bullet_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C})}_{\text{Ext}^\bullet_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C})} \left( \text{Tor}^{\mathfrak{b}^-_\ell}_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C}((-2\rho))) \right).
\]

Finally precise calculation of homological gradings shows that the grading on the first \( t \) module should be shifted by \(-\mathfrak{z}(\mathbb{R}^+)\), and the grading on the second one should be shifted by \( \mathfrak{z}(\mathbb{R}^+) \). Thus we have

\[
\text{Ext}^\infty_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C}) = \text{Coind}^{\text{Ext}^\bullet_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C})}_{\text{Ext}^\bullet_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C})} \left( \text{Ext}^\bullet_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C}((-\ell + 1)2\rho))) \right)
\]

\[
= \text{Ind}^{\text{Ext}^\bullet_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C})}_{\text{Ext}^\bullet_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C})} \left( \text{Tor}^{\mathfrak{b}^-_\ell}_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C}((-\ell + 1)2\rho))) \right). \quad \square
\]

3.2.3. Corollary: \( \text{Ext}^\infty_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C}) \) is both free over the algebra \( \text{Ext}^\bullet_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C}) \) and co-free over the algebra \( \text{Ext}^\bullet_{\mathfrak{b}^+_\ell} (\mathbb{C}, \mathbb{C}) \). \quad \square
3.3. Functions on \( N \) with singularities along coordinate hyperplanes. Consider the divisors on the nilpotent cone of the form \( D_\alpha := \{ n \in N | f_\alpha^*(n) = 0 \} \), where \( f_\alpha^* \) denote the standard basic linear functions on \( n^- \), \( \alpha \in R^+ \). Note that we have \( N \supset n^+ = \bigcup_{\alpha \in R^+} D_\alpha \).

Consider the open subset in the nilpotent cone of the form \( N(f_{\alpha_1}^*,...,f_{\alpha_k}^*(R^+)) := N \backslash \bigcup_{\alpha \in R^+} D_\alpha \) and the \( H^0(N, O_N) \)-module \( H^0(N(f_{\alpha_1}^*,...,f_{\alpha_k}^*(R^+)), O_{CN}). \)

It is well known how to obtain this \( H^0(N, O_N) \)-module from the free module \( H^0(N, O_N) \) via an inductive limit construction.

For \( \alpha \in R^+ \) consider the \( H^0(N, O_N) \)-module map \( \mu_\alpha : H^0(N, O_N) \rightarrow H^0(N, O_N) \) given by multiplicaton by the element \( f_\alpha^* \in n^- \).

Next we construct an inductive system of \( H^0(N, O_N) \)-modules \( I_\bullet \) enumerated by the partially ordered set \( \mathbb{Z}^R_+ \) with \( I_v = H^0(N, O_N) \) for any \( v \in \mathbb{Z}^R_+ \) and with morphisms

\[
I(\ldots, v_\alpha, \ldots) \rightarrow I(\ldots, v_\alpha + 1, \ldots)
\]
equal to \( \mu_\alpha \).

3.3.1. Lemma: The \( H^0(N, O_N) \)-module \( H^0(N(f_{\alpha_1}^*,...,f_{\alpha_k}^*(R^+)), O_N) \) is equal to \( \lim_{\rightarrow} I_\bullet \). \( \square \)

Note that the \( H^0(N, O_N) \)-module \( H^2_{n^+} (N, O_N) \) can be realised naturally as a bottom subquotient module in \( H^0(N(f_{\alpha_1}^*,...,f_{\alpha_k}^*(R^+)), O_N) \) under the filtration on the latter module by singularities of functions. More precisely, recall that \( H^0(N, O_N) = S^\bullet(n^-) \otimes H_0 \otimes S^\bullet(n^+) \) as a module over \( S(n^+) \otimes S(n^-) \). Fix the set of standard generators in the algebra \( S(n^-) : S(n^-) = \mathbb{C}[f_\alpha^*, \alpha \in R^+] \). Then by the previous Lemma \( H^0(N(f_{\alpha_1}^*,...,f_{\alpha_k}^*(R^+)), O_N) \) is equal to \( \mathbb{C}[f_\alpha^\pm 1, \alpha \in R^+] \otimes H_0 \otimes S^\bullet(n^+) \). On the other hand note that \( H^2_{n^+} (N, O_N) \) is isomorphic to \( \bigotimes_{\alpha \in R^+} \mathbb{C}[f_\alpha^\pm 1]/\mathbb{C}[f_\alpha^*] \otimes H_0 \otimes S^\bullet(n^+) \).

Our main goal here is to construct an isomorphism of \( H^0(N, O_N) \)-modules

\[
H^2_{n^+} (N, O_N) \xrightarrow{\cong} \text{Ext}^2_{\mathbb{Z}^+\bullet} (\mathbb{C}, \mathbb{C}).
\]

We start with constructing a system of \( H^0(N, O_N) \)-module morphisms \( \varphi_\bullet : I_\bullet \rightarrow \text{Ext}^2_{\mathbb{Z}^+\bullet} (\mathbb{C}, \mathbb{C}) \) that would provide a morphism from the direct limit of the inductive system to the semiinfinite cohomology module.

3.4. Construction of the morphisms \( \varphi_\bullet \). Note that since each module \( I_v, v \in \mathbb{Z}^R_+ \), is free over the algebra \( H^0(N, O_N) \) with one generating element, to construct the system of morphisms from the described inductive system to \( \text{Ext}^2_{\mathbb{Z}^+\bullet} (\mathbb{C}, \mathbb{C}) \) one has to specify the images of \( 1 \in H^0(N, O_N) \) under the morphisms \( \varphi_v, v \in \mathbb{Z}^R_+ \). Then \( \varphi_v(a) = a \cdot \varphi_v(1) \), where \( a \in H^0(N, O_N) = \text{Ext}^0 (\mathbb{C}, \mathbb{C}) \) (the latter algebra acts naturally on \( \text{Ext}^2_{\mathbb{Z}^+\bullet} (\mathbb{C}, \mathbb{C}) \)). Suppose that

\[
\varphi(\ldots, v_\alpha, \ldots) : 1 \mapsto m(\ldots, v_\alpha, \ldots) \in \text{Ext}^2_{\mathbb{Z}^+\bullet} (\mathbb{C}, \mathbb{C}) ; \quad \varphi(\ldots, v_\alpha + 1, \ldots) : 1 \mapsto m(\ldots, v_\alpha + 1, \ldots) \in \text{Ext}^2_{\mathbb{Z}^+\bullet} (\mathbb{C}, \mathbb{C}).
\]

Then evidently the only condition on the sequence of elements \( \{ m_v \}_{v \in \mathbb{Z}^R_+} \) is that

\[
f_\alpha^* \cdot m(\ldots, v_\alpha + 1, \ldots) = m(\ldots, v_\alpha, \ldots).
\]

Consider the element

\[
\nu^+_e \otimes \nu^-_e \in (A_+ \otimes A_-)^{\mathbb{Z}_0} \otimes \mathbb{C}(-2\ell \rho) = \text{gr}^F \ H_0 \otimes \mathbb{C}(-2\ell \rho) \subset \text{Ext}^2_{\mathbb{Z}^+\bullet} (\mathbb{C}, \mathbb{C}).
\]

Here we preserve notation from Section 2.3.3 for the base elements in the algebras \( A_+ \) and \( A_- \), and \( e \in W \) denotes the unity element.

Note that the corresponding \( \ell \cdot X \times \mathbb{Z} \)-grading component in the space \( \text{Ext}^2_{\mathbb{Z}^+\bullet} (\mathbb{C}, \mathbb{C}) \) is one dimensional. Since the filtration \( F \) on the space \( \text{Ext}^2_{\mathbb{Z}^+\bullet} (\mathbb{C}, \mathbb{C}) \) is well defined with respect to the \( \ell \cdot X \times \mathbb{Z} \)-grading, we can consider the described vector as an element of \( \text{Ext}^2_{\mathbb{Z}^+\bullet}(\mathbb{C}, \mathbb{C}) \).
We are starting to construct the system of morphisms $\varphi_v, v \in \mathbb{Z}_+^R$. Put

$$
\varphi_{(0,\ldots,0)} : \text{Ext}_{\mu^\bullet}(C, C)[z(R^+)] \rightarrow \text{Ext}_{\mu^\bullet}(C, C), \quad \varphi_{(0,\ldots,0)}(1) = \nu^+_e \otimes \nu^-_e.
$$

3.4.2. **Theorem:** $\text{Ext}_{\mu^\bullet}^\infty(C, C)$ is co-free over the algebra $S^\bullet(n^-) = \text{Ext}_{\nu}^\bullet(C, C)$ with the space of co-generators equal to $H_0 \otimes \mathbb{C}(-2\ell\rho) \otimes S^\bullet(n^+)$. 

**Proof.** First recall that by 2.3.3 the $\text{Ext}_{\nu}^\bullet(C, C)$-module $\text{Ext}_{\mu^\bullet}^\infty(C, C)$ is free with the space of generators equal to $S^\bullet(n^-) \otimes H_0 \otimes \mathbb{C}(-2\ell\rho)$. 

On the other hand $\text{Ext}_{\mu^\bullet}^\infty(C, C) = \text{gr} F \text{Ext}_{\mu^\bullet}^\infty(C, C)$, and this equality is an isomorphism of modules over $\text{Ext}_{\nu}^\bullet(C, C) = \text{gr} F \text{Ext}_{\nu}^\bullet(C, C)$. Fix the set of linear independent elements $\{b_p|p \in \mathbb{Z}_+^R \times W\}$ in the generators space of $\text{Ext}_{\mu^\bullet}^\infty(C, C)$ and choose the set of representatives of the elements $\{b_p|p \in \mathbb{Z}_+^R \times W\} \subset \text{Ext}_{\mu^\bullet}^\infty(C, C)$. It is easy to verify that the $\text{Ext}_{\nu}^\bullet(C, C)$-submodule in $\text{Ext}_{\mu^\bullet}^\infty(C, C)$ generated by this set is free. Moreover its character coincides with the one of $\text{Ext}_{\mu^\bullet}^\infty(C, C)$, thus the module of semi-infinite cohomology is $\text{Ext}_{\nu}^\bullet(C, C)$-free itself. Now to complete the proof of the Theorem recall the following statement from [Ar1].

Denote by $u_{\ell}$ the finite quantum group defined in the same way as $u_{\ell}$, but with $\zeta$ replaced by $\zeta^{-1}$ in the defining relations.

3.4.2. **Lemma:** (see [Ar1], Proposition 5.4.3) There exists a nondegenerate contragradient pairing

$$
\langle \ , \ \rangle : \text{Ext}_{\mu^\bullet}^\infty(C, C) \times \text{Ext}_{\mu^\bullet}^\infty(C, C) \rightarrow \mathbb{C}
$$

well defined with respect to the actions of the algebra $H^0(N, \mathcal{O}_N) = \text{Ext}_{\mu^\bullet}^\bullet(C, C) = \text{Ext}_{\mu^\bullet}(C, C)$ and of the Lie algebra $\mathfrak{g}$. 

Using this pairing we obtain the set of co-generators $\{c_p|p \in \mathbb{Z}_+^R\}$ for the co-free $\text{Ext}_{\nu}^\bullet(C, C)$-module $\text{Ext}_{\mu^\bullet}^\infty(C, C)$ enumerated by the base vectors of the space $H_0 \otimes \mathbb{C}(-2\ell\rho) \otimes S^\bullet(n^+)$. The Theorem is proved. 

Note that we can choose the cogenerators $c_p$ constructed above to be $X \times \mathbb{Z}$-homogeneous. In particular there is a unique choice of the homogeneous base vector in the space

$$
\left(\text{Ext}_{\mu^\bullet}^{\infty-\sharp(R^+)}(C, C)\right)_{-2\ell\rho}
$$

with the top filtration factor equal to $\nu^+_e \otimes \nu^-_e \in \text{Ext}_{\mu^\bullet}^{\infty-\sharp(R^+)}(C, C)$. Now consider the direct sum decomposition of the $\text{Ext}_{\nu}^\bullet(C, C)$-module

$$
\text{Ext}_{\mu^\bullet}^\infty-\sharp(R^+)^{\bullet}(C, C) = \bigoplus_{p \in \mathbb{Z}_+^R \times W} S^\bullet(n^-) \otimes c_p
$$

and the direct summand $S^\bullet(n^-) \otimes (\nu^+_e \otimes \nu^-_e)$ where $\nu^+_e \otimes \nu^-_e$ denotes the element representing $\nu^+_e \otimes \nu^-_e$. 

Fixing standard generators $f_\alpha, \alpha \in R^+$ in the space $n^-$ we obtain a linear base in $S^\bullet(n^-) \otimes (\nu^+_e \otimes \nu^-_e)$ of the form $m_v := f_{\alpha_1}^{v_1} \ldots f_{\alpha_{\ell(R^+)}}^{v_{\ell(R^+)}} \otimes (\nu^+_e \otimes \nu^-_e)$. 

3.4.3. **Corollary:** We have $f_\alpha \cdot m_{(v, 0, \ldots, 0)} = m_{(v, 0, \ldots, 0)}$.

Thus we obtain a sequence of morphisms of $\text{Ext}_{\mu^\bullet}(C, C)$-modules

$$
\varphi_v : \mathcal{I}_v = \text{Ext}_{\mu^\bullet}(C, C)[zR^+ + 2|v|] \rightarrow \text{Ext}_{\mu^\bullet}^\infty(C, C), \quad \varphi_v(1) = m_v, \quad v \in \mathbb{Z}_+^R.
$$

Here $|v| = v_1 + \ldots + v_{\ell(R^+)}$. We have constructed a morphism

$$
H^0(N(f_{\alpha_1}^{v_1} \ldots f_{\alpha_{\ell(R^+)}}^{v_{\ell(R^+)}}), \mathcal{O}_N) = \lim \mathcal{I}_v \rightarrow \text{Ext}_{\mu^\bullet}^\infty(C, C).
$$
3.5. Construction of the isomorphism \( H^\ell_\mathbb{n}^+(R^+) (\mathcal{N}, \mathcal{O}_\mathcal{N}) \rightarrow Ext^\infty_{\mathbb{C}_n} (\mathbb{C}, \mathbb{C}) \). Our nearest goal is to show that the morphism \( \lim \phi \) provides a map of \( H^0(\mathcal{N}, \mathcal{O}_\mathcal{N}) \)-modules \( H^\ell_\mathbb{n}^+(R^+) (\mathcal{N}, \mathcal{O}_\mathcal{N}) \rightarrow Ext^\infty_{\mathbb{C}_n} (\mathbb{C}, \mathbb{C}) \).

3.5.1. **Lemma:** For any \( w \in H^0(\mathcal{N}, f_{j_1}^{*}, \ldots, f_{j_n}^{*}(R^+), \mathcal{O}_\mathcal{N}) \) there exists \( v^0 = (v_1^0, \ldots, v_n^0) \) such that for any \( v = v^0 + v^1 \), \( v^1 \in \mathbb{Z}^n \), the element \( \lim \phi (f_{j_1}^{*}, \ldots, f_{j_n}^{*}, v^0) = 0 \).

**Proof.** The statement follows immediately from the restrictions on \( \ell \cdot X \times \mathbb{Z} \)-gradings of the space \( \lim \phi \).

Recall that \( H^\ell_\mathbb{n}+ (\mathcal{N}, \mathcal{O}_\mathcal{N}) \) is the largest quotient module of \( H^0(\mathcal{N}, f_{j_1}^{*}, \ldots, f_{j_n}^{*}(R^+), \mathcal{O}_\mathcal{N}) \) supported on \( \mathbb{n}^+ \subset \mathcal{N} \).

3.5.2. **Corollary:** The map \( \lim \phi \) defines a morphism \( \Phi : H^\ell_\mathbb{n}^+(R^+) (\mathcal{N}, \mathcal{O}_\mathcal{N}) \rightarrow Ext^\infty_{\mathbb{C}_n} (\mathbb{C}, \mathbb{C}) \).

In fact it follows from the previous considerations that for any finite dimensional graded \( \mathcal{U}_\ell \)-module \( M \) the quasi-coherent sheaf on \( \mathcal{N} \) corresponding to the \( H^0(\mathcal{N}, \mathcal{O}_\mathcal{N}) \)-module \( Ext^\infty_{\mathbb{C}_n} (\mathbb{C}, M) \) is supported on \( \mathbb{n}^+ \subset \mathcal{N} \).

Note that the maps \( \mu_\alpha, \varphi_v \) and \( \lim \phi \) are well defined with respect to the filtration \( F \). The maps \( \lim F \mu_\alpha : Ext^\infty_{\mathcal{U}_\ell} (\mathbb{C}, \mathbb{C}) \rightarrow Ext^\infty_{\mathcal{U}_\ell} (\mathbb{C}, \mathbb{C})[2] \) are provided by multiplication by the elements \( f_{j_1}^{*} \in Ext^\infty_{\mathcal{U}_\ell} (\mathbb{C}, \mathbb{C}) \subset Ext^\infty_{\mathcal{U}_\ell} (\mathbb{C}, \mathbb{C}) \). Moreover we have

\[
\lim F \mu_\alpha = \bigotimes_{\alpha \in R^+} C[f_{j_1}^{*} \pm 1] \otimes Ext^\infty_{\mathbb{C}_n} (\mathbb{C}, \mathbb{C})[\ell_\mathbb{n}^+](\mathbb{n}^+).
\]

Note also that the maps \( \lim F \varphi_v \) are defined by \( \lim F \varphi_v (1) = \mathcal{P}_v = f_{j_1}^{*} \cdots f_{j_n}^{*}(R^+) \), \( \ell \cdot X \times \mathbb{Z} \)-grading components. The \( \ell \cdot X \times \mathbb{Z} \)-grading components of the spaces \( \lim \mathcal{L}_\alpha, Ext^\infty_{\mathcal{U}_\ell} (\mathbb{C}, \mathbb{C}) \), \( \lim F \mathcal{L}_\alpha \) and \( Ext^\infty_{\mathcal{U}_\ell} (\mathbb{C}, \mathbb{C}) \) are finite dimensional.

Thus we can check injectivity of the map dual to \( \lim \phi \) instead of surjectivity of \( \lim \phi \). But the injectivity of a filtered map follows from the injectivity of the associated graded map.

Thus we obtain a surjective map \( \Phi : H^\ell_\mathbb{n}^+(R^+) (\mathcal{N}, \mathcal{O}_\mathcal{N}) \rightarrow Ext^\infty_{\mathbb{C}_n} (\mathbb{C}, \mathbb{C}) \).

Next recall the following statement from [A1].

3.5.4. **Proposition:** (see [A1], Theorem A.2.2)

\[
\text{ch} \left( H^\ell_\mathbb{n}^+(R^+) (\mathcal{N}, \mathcal{O}_\mathcal{N}), t^2 \right) = \text{ch} \left( Ext^\infty_{\mathbb{C}_n} (\mathbb{C}, \mathbb{C}), t \right).
\]

Here the indeterminate \( t \) in the left hand side of the equality stands for the grading by the natural action of \( \mathbb{C}^* \), in the right hand side of the equality it stands for the homological grading.

We have proved the Feigin conjecture on the level of \( H^0(\mathcal{N}, \mathcal{O}_\mathcal{N}) \)-modules.

3.5.5. **Theorem:** The morphism \( \Phi \) constructed above provides an isomorphism of the \( H^0(\mathcal{N}, \mathcal{O}_\mathcal{N}) \)-modules

\[
\Phi : H^\ell_\mathbb{n}^+(R^+) (\mathcal{N}, \mathcal{O}_\mathcal{N}) \rightarrow Ext^\infty_{\mathbb{C}_n} (\mathbb{C}, \mathbb{C}) \).
\]
4. ISOMORPHISM OF $\mathfrak{g}$-MODULES

Recall that both sides of the $H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})$-module isomorphism $\Phi$ carry natural structures of $\mathfrak{g}$-modules.

The one on $\text{Ext}^\infty_{\mathfrak{u}_\ell} \mathfrak{g}(\mathbb{C}, \mathbb{C})$ is a consequence of the fact that the algebra $u_\ell$ is a normal subalgebra in $U_\ell$ with the quotient algebra $U_\ell/\mathfrak{u}_\ell$ equal to $U(\mathfrak{g})$.

The $\mathfrak{g}$-module structure on $H^{\ell(R_+)}(\mathcal{N}, \mathcal{O}_\mathcal{N})$ comes from the natural Lie algebra inclusion $\mathfrak{g} \hookrightarrow \text{Vect}(\mathcal{N})$ and the standard fact due to Kempf (see [K]) that the Lie algebra of algebraic vector fields acts on local cohomology spaces. Below we prove that the two described $\mathfrak{g}$-module structures are isomorphic.

4.1. Isomorphism of the $\mathfrak{n}^+$-modules. Consider the standard positive nilpotent Lie subalgebra $\mathfrak{n}^+ \subset \mathfrak{g}$. We prove first that $H^{\ell(R_+)}_{\mathfrak{n}^+}(\mathcal{N}, \mathcal{O}_\mathcal{N})$ and $\text{Ext}^\infty_{\mathfrak{u}_\ell} \mathfrak{g}(\mathbb{C}, \mathbb{C})$ are isomorphic as $\mathfrak{n}^+$-modules.

Note first that the $\mathfrak{n}^+$- and $H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})$-actions on both $H^{\ell(R_+)}_{\mathfrak{n}^+}(\mathcal{N}, \mathcal{O}_\mathcal{N})$ and $\text{Ext}^\infty_{\mathfrak{u}_\ell} \mathfrak{g}(\mathbb{C}, \mathbb{C})$ satisfy the equation

$$n \cdot (a \cdot c) = a \cdot (n \cdot c) + [n, a] \cdot c,$$

where $n \in \mathfrak{n}^+, a, [n, a] \in H^0(\mathcal{N}, \mathcal{O}_\mathcal{N}), c \in H^{\ell(R_+)}_{\mathfrak{n}^+}(\mathcal{N}, \mathcal{O}_\mathcal{N})$ (resp. $c \in \text{Ext}^\infty_{\mathfrak{u}_\ell} \mathfrak{g}(\mathbb{C}, \mathbb{C})$) and $[*, *]$ denotes the natural $\mathfrak{g}$-action on $H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})$.

Consider the $S^\bullet(\mathfrak{n}^-)$-module direct sum decomposition

$$\text{Ext}^\infty_{\mathfrak{u}_\ell} \mathfrak{g}(\mathbb{C}, \mathbb{C}) = \bigoplus_{p \in \mathbb{Z}^R_+ \times W} S^\bullet(\mathfrak{n}^-) \otimes c_p$$

with $c_0 = \nu_+^e \otimes \nu_+^e \in \text{Ext}^\infty_{\mathfrak{u}_\ell} \mathfrak{g}(\mathbb{C}, \mathbb{C})$.

4.1.1. Lemma: $S^\bullet(\mathfrak{n}^-) \otimes c_0 \subset \text{Ext}^\infty_{\mathfrak{u}_\ell} \mathfrak{g}(\mathbb{C}, \mathbb{C})$ is a $\mathfrak{n}^+$-submodule.

Proof. We prove the statement of the Lemma by induction by homological grading. First $(S^\bullet(\mathfrak{n}^-) \otimes c_0)_{-\ell(R_+)} = \mathbb{C}c_0$, and the statement follows from the $X$-grading restrictions on the $\mathfrak{n}^+$-action.

Suppose the statement is proved for $S^k(\mathfrak{n}^-) \otimes c_0, k < k_0$. We prove that $S^{k_0}(\mathfrak{n}^-) \otimes c_0$ is a $\mathfrak{n}^+$-submodule in $\text{Ext}^\infty_{\mathfrak{u}_\ell} \mathfrak{g}(\mathbb{C}, \mathbb{C})$.

Let $b \in S^{k_0}(\mathfrak{n}^-)$ and $n \in \mathfrak{n}^+$. Consider the element $n \cdot b \in \text{Ext}^\infty_{\mathfrak{u}_\ell} \mathfrak{g}(\mathbb{C}, \mathbb{C})$ and its decomposition $n \cdot b = (n \cdot b)^{\mathrm{yes}} + (n \cdot b)^{\mathrm{no}}$, where $(n \cdot b)^{\mathrm{yes}} \in S^\bullet(\mathfrak{n}^-) \otimes c_0$ and $(n \cdot b)^{\mathrm{no}} \in \bigoplus_{p \in \mathbb{Z}^R_+ \times W, p \neq 0} S^\bullet(\mathfrak{n}^-) \otimes c_p$.

Then for any $\alpha \in R^+$ the element $f^*_\alpha \cdot (n \cdot b) = n \cdot (f^*_\alpha \cdot b) + [n, f^*_\alpha] \cdot b$ belongs to $S^\bullet(\mathfrak{n}^-) \otimes c_0$, since by induction hypothesis both summands lie there. Thus for any $\alpha \in R^+$ we have $f^*_\alpha \cdot (n \cdot b)^{\mathrm{no}} = 0$, i.e. the element belongs to the $S^\bullet(\mathfrak{n}^-)$-invariants subspace of $\text{Ext}^\infty_{\mathfrak{u}_\ell} \mathfrak{g}(\mathbb{C}, \mathbb{C})$. By Theorem 3.4.1 the invariants space is equal to $H_0 \otimes S^\bullet(\mathfrak{n}^-) \otimes \mathbb{C}(-2f_\rho)$. The latter subspace is situated in homological gradings greater than or equal to $-\ell(R_+)$. On the other hand the $\mathfrak{n}^+$-submodule in question is situated in homological gradings less than or equal to $-\ell(R_+)$. It follows that $(n \cdot b)^{\mathrm{no}} = 0$.

4.1.2. Corollary: The action of $\mathfrak{n}^+$ on $S^\bullet(\mathfrak{n}^-) \otimes c_0$ coincides with the one on $S^\bullet(\mathfrak{g}/\mathfrak{b}^+)$.

Proof. Identify $S^\bullet(\mathfrak{n}^-) \otimes c_0$ with $\text{Hom}_C(S^\bullet(\mathfrak{n}^-), \mathbb{C})$. For such a linear function $b$ note that $b(a) = (a \cdot b)(1)$. Note also that $S^\bullet(\mathfrak{n}^-) = S^\bullet((\mathfrak{g}/\mathfrak{b}^+)^*)$ is a $\mathfrak{n}^+$-submodule in $H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})$. Thus for $n \in \mathfrak{n}^+, b \in \text{Hom}_C(S^\bullet(\mathfrak{n}^-), \mathbb{C})$ we have

$$(n \cdot b)(a) = a \cdot (n \cdot b)(1) = n \cdot (a \cdot b)(1) - ([n, a] \cdot b)(1) = -b([n, a]).$$

Recall that the $H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})$-module $\text{Ext}^\infty_{\mathfrak{u}_\ell} \mathfrak{g}(\mathbb{C}, \mathbb{C})$ is generated by the subspace $S^\bullet(\mathfrak{n}^-) \otimes c_0$ and is in fact isomorphic to $\text{Ind}_{S^\bullet(\mathfrak{n}^-)} H^0(\mathcal{N}, \mathcal{O}_\mathcal{N}) S^\bullet(\mathfrak{n}^-) \otimes c_0$. Summing up the previous considerations we note that we have proved the following statement.

4.1.3. Theorem: Up to an isomorphism there exists only one $\mathfrak{n}^+$-module structure on the space

$$\text{Ext}^\infty_{\mathfrak{u}_\ell} \mathfrak{g}(\mathbb{C}, \mathbb{C}) = H^{\ell(R_+)}_{\mathfrak{n}^+}(\mathcal{N}, \mathcal{O}_\mathcal{N})$$

well defined with respect to the $H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})$-structure.
4.2. Tilting modules. It remains to extend the described $n^+$-module isomorphism to a $g$-module isomorphism.

Recall that a finitely generated $n^+$-locally finite $g$-module $M$ diagonalizable over $h$ is called a tilting module if it is filtered both by Verma modules and by contragradient Verma modules. H. H. Andersen has proved the following statement.

4.2.1. Theorem: (see [A]) A tilting $g$-module is uniquely determined up to an isomorphism by its character with respect to the root space decomposition for the Cartan action. 

Our goal here is to prove that both $H^2_n(R^+)(N, O_N)$ and $\text{Ext}^2_n(R^+)(\mathbb{C}, \mathbb{C})$ are tilting $g$-modules.

4.3. A nondegenerate pairing on $H^2_n(R^+)(N, O_N)$. The following construction was proposed by V. Ostrik. Instead of constructing a contragradient pairing on $H^2_n(R^+)(N, O_N)$ we obtain a bilinear pairing

$$H^2_n(R^+)(N, O_N) \times H^2_n(R^+)(N, O_N) \rightarrow \mathbb{C}$$

well defined with respect to the $g$-module structures. Consider the canonical map for the local cohomology spaces provided by the cup product

$$\langle \ , \ \rangle : H^2_n(R^+)(N, O_N) \times H^2_n(R^+)(N, O_N) \rightarrow H^2_{n-\cap n^+}(N, O_N).$$

4.3.1. Lemma:

(i) $H^2_{n-\cap n^+}(N, O_N) = H^2_0(N, O_N) = \mathbb{C}$. Here $0$ denotes the vertex of the nilpotent cone.

(ii) The map $\langle \ , \ \rangle$ provides a nondegenerate pairing. 

4.3.2. Springer-Grothendieck resolution of the nilpotent cone. To prove that $H^2_n(R^+)(N, O_N)$ is $n^-$-free recall that in [Ar1] we obtained another geometric realization of this $g$-module as follows.

Consider the simply connected Lie group $G$ with the Lie algebra equal to $g$. Choose a maximal torus $H \subset G$ providing the root decomposition of $g$ and in particular its triangular decomposition $g = n^- \oplus h \oplus n^+$. Consider the Borel subgroup $B \subset G$ with the Lie algebra $b^+ = h \oplus n^+$ and the flag variety $G/B$. The group $G$ acts on $G/B$ by left translations and the restriction of this action to $B$ is known to have finitely many orbits. These orbits are isomorphic to affine spaces and called the Schubert cells. The Bruhat decomposition of $G$ shows that the Schubert cells are enumerated by the Weyl group. Denote the orbit corresponding to the element $w \in W$ by $S_w$.

It is well known that the cotangent bundle $T^*(G/B)$ has a nice realization $T^*(G/B) = \{ (B_x, n) \mid n \in \text{Lie}(B_x) \}$, where $B_x$ denotes some Borel subgroup in $G$ and $n$ is a nilpotent element in the Lie algebra $\text{Lie}(B_x)$. The map

$$\mu : T^*(G/B) \rightarrow N, \ (B_x, n) \mapsto n,$$

is known to be a resolution of singularities of $\mathcal{N}$ called the Springer-Grothendieck resolution.

Recall the following statement from [Ar1].

4.3.3. Proposition: (see e. g. [CC], 3.1.36)

(i) $\mu^{-1}(n^+) = \bigcup_{w \in W} T^*_x(S_w(G/B))$, where $T^*_x(S_w(G/B))$ denotes the conormal bundle to $S_w$ in $G/B$.

(ii) $H^2_{n^+}(N, O_N) \xrightarrow{\sim} H^2_{\mu^{-1}(n^+)}(T^*(G/B), O_{T^*(G/B)})$ as a $g$-module. 

4.3.4. Corollary: The $g$-module $H^2_{n^+}(N, O_N)$ is free over the algebra $U(n^-)$. 

In particular it is filtered by Verma modules. Using the fact that the $g$-module is self-dual with respect to the described contragradient pairing we construct a filtration by contragradient Verma modules on $H^2_{n^+}(N, O_N)$. We have proved the following statement.

4.3.5. Proposition: The $g$-module $H^2_{n^+}(N, O_N)$ is a tilting module.
4.4. On the other hand recall that in Lemma 3.4.2 we obtained a nondegenerate $\mathfrak{g}$-equivariant contragradient pairing

$$\text{Ext}^{\tilde{\mathfrak{g}}}_{\mathfrak{u}_\ell} (\mathbb{C}, \mathbb{C}) \times \text{Ext}^{\tilde{\mathfrak{g}}}_{\mathfrak{u}_\ell} (\mathbb{C}, \mathbb{C}) \rightarrow \mathbb{C}.$$ 

It follows from Proposition 4.3.5 and Theorem 4.1.3 that the $\mathfrak{g}$-module $\text{Ext}^{\tilde{\mathfrak{g}}}_{\mathfrak{u}_\ell} (\mathbb{C}, \mathbb{C})$ is cofree over the algebra $U(\mathfrak{n}^+)$. Thus it is filtered by contragradient Verma modules. Again using the contragradient pairing on the $\mathfrak{g}$-module we produce on it a filtration with subquotients equal to direct sums of Verma modules.

We have proved the following statement.

4.4.1. Proposition: The $\mathfrak{g}$-module $\text{Ext}^{\tilde{\mathfrak{g}}}_{\mathfrak{u}_\ell} (\mathbb{C}, \mathbb{C})$ is a tilting module.

Finally we come to the main statement of the present paper.

4.4.2. Theorem: The $\mathfrak{g}$-modules $\text{Ext}^{\tilde{\mathfrak{g}}}_{\mathfrak{u}_\ell} (\mathbb{C}, \mathbb{C})$ and $H^2(\mathfrak{g}, \mathcal{O}_{\mathcal{N}})$ are isomorphic.

Proof. The statement of the Theorem follows from Theorem 4.2.1 and Propositions 4.3.5 and 4.4.1.

The Feigin conjecture is proved.

5. Semiinfinite cohomology of contragradient Weyl modules.

5.1. Fix a dominant integral weight $\lambda \in X$. Consider the module over the “big” quantum group $U_\ell$ given by

$$\mathbb{D} W(\lambda) := \left( \text{Coind}_{U^+_\ell}^{U^0_\ell} C(\lambda) \right)^{\text{fin}} \text{ (resp. by } W(\lambda) := \left( \text{Ind}_{U^0_\ell}^{U^+_\ell} C(\lambda) \right)^{\text{fin}} \right).$$

Here $B^+_\ell = U^+_\ell \otimes U^0_\ell$ (resp. $B^-_\ell = U^0_\ell \otimes U^0_\ell$) denotes the quantum positive (resp. negative) Borel subalgebra in $U_\ell$ and $(*)^{\text{fin}}$ (resp. $(*)^{\text{fin}}$) denotes the maximal finite dimensional submodule (resp. quotient module) in $(*).$ The module $\mathbb{D} W(\lambda)$ (resp. $W(\lambda)$) is called the contragradient Weyl module (resp. the Weyl module) over $U_\ell$ with the highest weight $\lambda$.

It is known that both modules $W(\lambda)$ and $\mathbb{D} W(\lambda)$ provide the natural specialization of the finite dimensional simple modules $L(\lambda)$ over the quantum group $U$ at generic values of the quantizing parameter into the root of unity $\zeta$. In particular we have

$$\text{ch}(W(\lambda)) = \text{ch}(\mathbb{D} W(\lambda)) = \sum_{w \in W} \frac{e^{w \cdot \lambda}}{\prod_{\alpha \in R^+} (1 - e^{\alpha})},$$

just like in the Weyl character formula in the semisimple Lie algebra case. Note also that $W(0) = \mathbb{D} W(0) = \mathbb{C}$.

Below we consider semiinfinite cohomology of the algebra $\mathfrak{u}_\ell$ with coefficients in the contragradient Weyl module with a $\ell$-divisible highest weight $\ell \lambda$. Ou considerations were motivated by the following result of Ginzburg and Kumar (see [GK]).

Let $p$ denote the projection $T^*(G/B) \rightarrow G/B$. Consider the linear bundle $\mathcal{L}(\lambda)$ on $G/B$ with the first Chern class equal to $\lambda \in X = H^2(G/B, \mathbb{Z})$.

5.1.1. Theorem:

(i) $\text{Ext}^{\tilde{\mathfrak{g}}}_{\mathfrak{u}_\ell}(\mathbb{C}, \mathbb{D} W(\ell \lambda)) = 0$;

(ii) $\text{Ext}^{\tilde{\mathfrak{g}}}_{\mathfrak{u}_\ell}(\mathbb{C}, \mathbb{D} W(\lambda)) = H^0(\mathcal{N}, \mu_\ell p^* \mathcal{L}(\lambda))$ as a $H^0(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$-module.

The following conjecture provides a natural semiinfinite analogue for Theorem 5.1.1.

5.1.2. Conjecture: $\text{Ext}^{\tilde{\mathfrak{g}}}_{\mathfrak{u}_\ell}(\mathbb{C}, \mathbb{D} W(\lambda)) = H^0(\mathcal{N}, \mu_\ell p^* \mathcal{L}(\lambda))$ as a $H^0(\mathcal{N}, \mathcal{O}_{\mathcal{N}})$-module. The homological grading on the left hand side of the equality corresponds to the grading by the natural action of the group $\mathbb{C}^*$ on the right hand side.
5.1.3. Corollary:
\[
\text{ch } \left( \mathbf{Ext}_{u^+_n}^{\mathbb{F}^+}(\mathbb{C}, \mathbb{D}W(\ell)), t \right)
= t^{-\ell(R^+)} \prod_{\alpha \in R^+} (1 - e^{-\ell\alpha}) \sum_{w \in W} \prod_{\alpha \in R^+, w(\alpha) \in R^+} (1 - t^2 e^{-\ell\alpha}) \prod_{\alpha \in R^+, w(\alpha) \in R^+} (1 - t^{-2} e^{-\ell\alpha}).
\]

Below we present the main steps for the proof of the conjecture. Details of the proof will be given in the forthcoming paper [Ar7].

5.2. Local cohomology with coefficients in \(\mu_*p^*\mathcal{L}(\lambda)\). For a \(H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})\)-module \(\mathcal{M}\) consider the natural pairing \(H^i_n(\mathcal{N}, \mathcal{O}_\mathcal{N}) \times H^0(\mathcal{N}, \mathcal{M}) \rightarrow H^i_n(\mathcal{N}, \mathcal{M})\). Evidently it is equivariant with respect to the \(H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})\)-action. Thus we obtain a \(H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})\)-module morphism \(s : H^i_n(\mathcal{N}, \mathcal{O}_\mathcal{N}) \otimes_{H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})} H^0(\mathcal{N}, \mathcal{M}) \rightarrow H^i_n(\mathcal{N}, \mathcal{M})\).

5.2.1. Proposition: For \(\mathcal{M} = \mu_*p^*\mathcal{L}(\lambda)\) the map \(s\) is an isomorphism.

5.2.2. Similar construction for semiinfinite cohomology. We will need some more homological algebra. Fix a graded algebra \(A\) with a subalgebra \(B \subset A\). Recall that in [V] and [Ar] the notion of a complex of graded \(A\)-modules \(K\)-semijective with respect to the subalgebra \(B\) was developed. The following statement gives an analogue of the standard technique of projective resolutions in the semiinfinite case.

5.2.3. Theorem: (see [Ar], Appendix B) Let \(SS^*_{u^+_n}(\mathbb{C})\) (resp. \(SS^*_{u^-_n}(\mathbb{C})\)) denote a \(K\)-semijective concave (resp. convex) resolution of the \(u^+_n\)-module \(\mathbb{C}\) with respect to the subalgebra \(u^-_n\) (resp. \(u^+_n\)). Then for a finite dimensional graded \(u^-_n\)-module \(M\) we have
\[
(\text{i}) \quad H^\bullet(\text{Hom}_{u^+_n}(SS^*_{u^-_n}(\mathbb{C}), SS^*_{u^-_n}(\mathcal{M}))) \cong \text{Ext}_n^\bullet(\mathbb{C}, M);
\text{(ii)} \quad H^\bullet(\text{Hom}_{u^-_n}(SS^*_{u^-_n}(\mathbb{C}), SS^*_{u^-_n}(\mathbb{C}))) \cong \text{Ext}_n^\bullet(\mathbb{C}, M).
\]

5.2.4. Corollary: The composition of morphisms provides a natural pairing
\[
\text{Ext}_{u^+_n}^{n+i}(\mathbb{C}, \mathbb{C}) \times \text{Ext}_{u^-_n}^j(\mathbb{C}, \mathbb{M}) \rightarrow \text{Ext}_{u^-_n}^{n+i+j}(\mathbb{C}, \mathbb{M}).
\]
In particular we obtain a \(\text{Ext}_{u^+_n}(\mathbb{C}, \mathbb{C})\)-module map
\[
\text{Ext}_{u^+_n}^{n+i}(\mathbb{C}, \mathbb{C}) \otimes \text{Ext}_{u^-_n}^j(\mathbb{C}, \mathbb{C}) \rightarrow \text{Ext}_{u^-_n}^{n+i+j}(\mathbb{C}, \mathbb{M}).
\]
Combining this construction with the previous considerations we obtain the following statement.

5.2.5. Proposition: There exists a natural \(H^0(\mathcal{N}, \mathcal{O}_\mathcal{N})\)-module morphism
\[
\sigma : H^2_t(R^+)(\mathcal{N}, \mu_*p^*\mathcal{L}(\lambda)) \rightarrow \text{Ext}_{u^+_n}^{n+i}(\mathbb{C}, \mathbb{D}W(\ell\lambda)).
\]

Proof. Follows from Proposition 5.2.4.

Below we show that the morphism \(\sigma\) is an isomorphism. The main tool for the demonstration of this fact is the \textit{quasi-BGG complex} providing a specialization of the classical BGG resolution for a finite dimensional simple \(U\)-module \(L(\lambda)\) into the root of unity \(\zeta\). This complex constructed below consists of direct sums of \(U\)-modules called the \textit{quasi-Verma modules}. Moreover its zero cohomology module equals \(W(\lambda)\). On the other hand we show that semiinfinite cohomology with coefficients in quasi-Verma modules has a nice geometrical interpretation.

Now we turn to the construction of the quasi-BGG complex.
5.3. Twisted quantum parabolic subalgebras in $U_Q$. Recall that Lusztig has constructed an action of the braid group $B$ corresponding to the Cartan data $(I, \cdot)$ by automorphisms of the quantum group $U_Q$ well defined with respect to the $X$-gradings. Fix a reduced expression of the maximal length element $w_0 \in W$ via the simple reflection elements:

$$w_0 = s_{i_1} \ldots s_{i_{(R+1)}}, \; i_k \in I.$$ 

Then it is known that this reduced expression provides reduced expressions for all the elements $w \in W$: $w = s_{i_{(w)}} \ldots s_{i_{(w)}}$, $i_k \in I$.

Consider the standard generators $\{T_i\}_{i \in I}$ in the braid group $B$. Lifting the reduced expressions for the elements $w$ from $W$ into $B$ we obtain the set of elements in the braid group of the form $T_w := T_{i_{(w)}} \ldots T_{i_{(w)}}$.

In particular we obtain the set of twisted Borel subalgebras $w(B^+_Q) = T_w(B^+_Q) \subset U_Q$, where $B^+_Q := U^+_Q \otimes U^0_Q$. Note that $w_0(B^+_Q) = B^+_Q = U^+_Q \otimes U^0_Q$.

Fix a subset $J \subset I$ and consider the quantum parabolic subalgebra $P_{J,Q} \subset U_Q$. By definition this subalgebra in $U_Q$ is generated over $U^0_Q$ by the elements $E_i, i \in I$, $F_j, j \in J$, and by their quantum divided powers. The previous construction provides the set of twisted quantum parabolic subalgebras $w(P_{J,Q}) := T_w(P_{J,Q})$ of the type $J$ with the twists $w \in W$.

Note that the triangular decomposition of the algebra $U_Q$ provides the set of $w(B^+_Q)$ and $w(P_{J,Q})$:

$$w(B^+_Q) = (w(B^+_Q))^\ominus \otimes U^0_Q \otimes (w(B^+_Q))^\oplus$$

and

$$w(P_{J,Q}) = (w(P_{J,Q}))^\ominus \otimes U^0_Q \otimes (w(P_{J,Q}))^\oplus,$$

where $(w(B^+_Q))^\oplus = w(B^+_Q) \cap U^-_Q$, $(w(P_{J,Q}))^\oplus = w(P_{J,Q}) \cap U^-_Q$ etc.

Specializing the quantizing parameter into the root of unity $\zeta$ we obtain in particular the subalgebras $w(B^+_Q) \subset U_\ell$, $w(P_{J,Q}) \subset U_\ell$, $w(b^+_Q) \subset U_\ell$, $w(p_{J,Q}) \subset U_\ell$ etc.

5.4. Semiinfinite induction and coinduction. From now on we will use freely the technique of associative algebra semiinfinite homology and cohomology for a graded associative algebra $A$ with two subalgebras $B, N \subset A$ equipped with a triangular decomposition $A = B \otimes N$ on the level of graded vector spaces. We will not recall the construction of these functors referring the reader to [Ar1] and [Ar2].

Let us mention only that these functors are bifunctors $D(A\text{-mod}) \times D(A^2\text{-mod}) \rightarrow D(\text{Vect})$ where the associative algebra $A^2$ is defined as follows.

Consider the semisimple $A$-module $S^N_A := A \otimes N^*$. It is proved in [Ar2] that under very weak conditions on the algebra $A$ the module $S^N_A$ is isomorphic to the $A$-module $(S^N_A)' := \text{Hom}_B(A, B)$. Thus $\text{End}_A(S^N_A) \supset N^{opp}$ and $\text{End}_A(S^N_A) \supset B^{opp}$ as subalgebras. The algebra $A^2$ is defined as the subalgebra in $\text{End}_A(S^N_A)$ generated by $B^{opp}$ and $N^{opp}$. It is proved in [Ar2] that the algebra $A^2$ has a triangular decomposition $A^2 = N^{opp} \otimes B^{opp}$ on the level of graded vector spaces. Yet for an arbitrary algebra $A$ the algebras $A^2$ and $A^{opp}$ do not coincide.

However the following statement shows that in the case of quantum groups that correspond to the root data $(Y, X, \ldots)$ of the finite type $(I, \cdot)$, the equality of $A^{opp}$ and $A^2$ holds.

5.4.1. Proposition: We have

(i) $U^2 = U^{opp}$, $U^2_Q = U^{opp}$, $U^2_Q = U^{opp}$;
(ii) $w(B^+_Q)^2 = w(B^+_Q)^{opp}$, $w(B^+_Q)^2 = w(B^+_Q)^{opp}$, $w(P_{J,Q})^2 = w(P_{J,Q})^{opp}$, $w(p_{J,Q})^2 = w(p_{J,Q})^{opp}$.

Proof. The first part is proved similarly to Lemma 9.4.1 from [Ar4]. The second one follows immediately from the first one. \[\square\]

5.4.2. Definition: Let $M^*$ be a convex complex of $w(B^+_Q)$-modules. By definition set

$$\text{S-Ind}_{w(B^+_Q)}^U(M^*) := \text{Tor}_{w(B^+_Q)}^{w(B^+_Q)}(S^U_Q \otimes M^*)$$

and

$$\text{S-Coind}_{w(B^+_Q)}^U(M^*) := \text{Ext}_{w(B^+_Q)}^{w(B^+_Q)}(S^U_Q \otimes M^*).$$

The functors $\text{S-Ind}_{w(B^+_Q)}^U(\cdot)$, $\text{S-Coind}_{w(B^+_Q)}^U(\cdot)$ etc. are defined in a similar way.
5.4.3. **Lemma:** (see [Ar4])

(i) Tor$^{\mathbb{Z}/k \mathbb{Z}}_{\mathcal{F} \mathcal{K}} (S_{U^Q_W} \cdot \cdot) = 0$ for $k \neq 0$;

(ii) Ext$^{\mathbb{Z}/k \mathbb{Z}}_{\mathcal{F} \mathcal{K}} (S_{U^Q_W} \cdot \cdot) = 0$ for $k \neq 0$;

(iii) S-Ind$^{\mathbb{Z}/k \mathbb{Z}}_{\mathcal{F} \mathcal{K}} (S_{U^Q_W} \cdot \cdot)$ define exact functors $w(B^+_W) \text{-mod} \rightarrow U_Q \text{-mod}$. □

Similar statements hold for the algebras $w(B^+_W)$, $w(P_{J,Q})$ and $w(P_{J,\ell})$.

5.5. **Quasi-Verma modules.** We define the quasi-Verma module over the algebra $U_Q$ (resp. $U_\ell$) with the highest weight $\lambda$ by

$$M^w_Q(w \cdot \lambda) := S \text{-Ind}_{w(B^+_Q)}^U (\mathcal{C}(\lambda)) \text{ (resp. } M^w_\ell(w \cdot \lambda) := S \text{-Ind}_{w(B^+_Q)}^U (\mathcal{C}(\lambda))).$$

The contragradient quasi-Verma module $\mathbb{D}M^w_Q(w \cdot \lambda)$ resp. $\mathbb{D}M^w_\ell(w \cdot \lambda)$ is defined by

$$\mathbb{D}M^w_Q(w \cdot \lambda) := S \text{-Coind}_{w(B^+_Q)}^U (\mathcal{C}(\lambda)) \text{ (resp. } \mathbb{D}M^w_\ell(w \cdot \lambda) := S \text{-Coind}_{w(B^+_Q)}^U (\mathcal{C}(\lambda))).$$

We list the main properties of quasi-Verma modules.

5.5.1. **Proposition:** (see [Ar4])

(i) Fix a dominant integral weight $\lambda \in X$. Suppose that $\xi \in \mathbb{C}^*$ is not a root of unity. Then the $U_\ell$-module $M^w_\ell(w \cdot \lambda) := M^w_Q(w \cdot \lambda) \otimes_{\mathbb{Q}[v^{-1}]} \mathbb{C}$ (resp. $M^w_\ell(w \cdot \lambda) := \mathbb{D}M^w_Q(w \cdot \lambda) \otimes_{\mathbb{Q}[v^{-1}]} \mathbb{C}$) is isomorphic to the usual Verma module $M_\ell(w \cdot \lambda)$ (resp. to the usual contragradient Verma module $\mathbb{D}M_\ell(w \cdot \lambda)$).

(ii) For any $\lambda \in X$ we have

$$\text{ch}(M^w_Q(w \cdot \lambda)) = \text{ch}(\mathbb{D}M^w_Q(w \cdot \lambda)) = \text{ch}(M^w_\ell(w \cdot \lambda)) = \text{ch}(\mathbb{D}M^w_\ell(w \cdot \lambda)) = \frac{e^{w \cdot \lambda}}{\prod_{\alpha \in R^+} (1 - e^{-\alpha})}. □$$

Thus for a dominant weight $\lambda$ one can consider $M^w_\ell(w \cdot \lambda)$ as a flat family of modules over the quantum group for various values of the quantizing parameter with the fiber at a generic $\xi \in \mathbb{C}^*$ equal to the Verma module $M_\ell(w \cdot \lambda)$.

**Remark:** Note that by definition $M^w_\ell(\lambda) = M_\ell(\lambda)$ and $\mathbb{D}M^w_\ell(\lambda) = \mathbb{D}M_\ell(\lambda)$, where $e$ denotes the unity element of the Weyl group. In particular for a dominant weight $\lambda$ we have a natural projection $M^w_\ell(\lambda) \rightarrow W(\lambda)$ and a natural inclusion $\mathbb{D}W(\lambda) \hookrightarrow \mathbb{D}M^w_\ell(\lambda)$.

5.6. **The $\mathfrak{sl}_2$ case.** Let us investigate throughly quasi-Verma modules in the case of $U_\ell = U_\ell(\mathfrak{sl}_2)$. First we find the simple subquotient modules in the module $M_\ell^w(k \ell) = M_\ell(k \ell)$.

Recall the classification of the simple objects in the category of $X$-graded $U^0_{\ell}$-semisimple $U_\ell$-modules locally finite with respect to the action of $E_i$ and $E_i^\ell$, $i \in I$, obtained by Lusztig in [L2]. In the $\mathfrak{sl}_2$ case it looks as follows. Identify the weight lattice $X$ with $\mathbb{Z}$.

**Proposition:**

(i) For $0 \leq k < \ell$ the simple $\mathfrak{u}_\ell(\mathfrak{sl}_2)$-module $L(k)$ is a restriction of a simple $U_\ell(\mathfrak{sl}_2)$-module.

(ii) Any simple $U_\ell(\mathfrak{sl}_2)$-module from the category described above is isomorphic to a module of the form $L(k) \otimes L(m \ell)$, where $0 \leq k < \ell$. Here the simple module $L(m \ell)$ is obtained from the simple $U(\mathfrak{sl}_2)$-module $L(m)$ via restricton using the map $U_\ell(\mathfrak{sl}_2) \rightarrow U(\mathfrak{sl}_2)/\mathfrak{u}_\ell(\mathfrak{sl}_2) = U(\mathfrak{sl}_2)$. □

Denote the only reflection in the Weyl group for $\mathfrak{sl}_2$ by $s$.

**Lemma:**

(i) ch $M^w_\ell(0) = \text{ch} L(0) + \text{ch} L(-2) + \text{ch} L(-2 \ell)$.

(ii) For $k > 0$ we have ch $M^w_\ell(k \ell) = \text{ch} L(k \ell) + \text{ch} L(k \ell - 2) + \text{ch} L(-k \ell - 2) + \text{ch} L(-(k + 2) \ell)$.

(iii) ch $M^w_\ell(s \cdot 0) = \text{ch} L(2) + \text{ch} L(-2 \ell)$.

(iv) For $k > 0$ we have ch $M^w_\ell(s \cdot k \ell) = + \text{ch} L(-k \ell - 2) + \text{ch} L(-(k + 2) \ell)$. □

In fact it is easy to find the filtrations on quasi-Verma modules with simple subquotients that correspond to the character equalities above.

**Lemma:**
(i) For \( k > 0 \) there exist exact sequences
\[
0 \rightarrow L(k\ell - 2) \rightarrow W(k\ell) \rightarrow L(k\ell) \rightarrow 0,
\]
\[
0 \rightarrow L(-k\ell - 2) \rightarrow M^s_k(s \cdot k\ell) \rightarrow L(-(k+2)\ell) \rightarrow 0.
\]

(ii) There exists a filtration on \( M^s_k(0) \) with subquotients as follows:
\[
gr^1 M^s_k(0) = L(0), \quad gr^2 M^s_k(0) = L(-2), \quad gr^3 M^s_k(0) = L(-2\ell).
\]

(iii) For \( k > 0 \) there exists a filtration on \( M^s_k(k\ell) \) with subquotients as follows:
\[
gr^1 M^s_k(k\ell) = L(k\ell), \quad gr^2 M^s_k(k\ell) = L(k\ell - 2),
\]
\[
gr^3 M^s_k(k\ell) = L(-(k+2)\ell), \quad gr^4 M^s_k(k\ell) = L(-k\ell - 2).
\]

Thus we obtain the following statement.

5.6.1. **Proposition:** For any \( k \geq 0 \) there exists an exact sequence of \( U = U_\ell(\mathfrak{sl}_2) \)-modules
\[
0 \rightarrow M^s_k(s \cdot k\ell) \rightarrow M^s_k(k\ell) \rightarrow W(k\ell) \rightarrow 0.
\]

**Remark:**

(i) In fact it is easy to verify that \( M^s_k(s \cdot k\ell) \) is isomorphic to the *contragradient* Verma module \( \mathcal{D}_M(s \cdot k\ell) \).

(ii) Note that if \( \xi \) is not a root of unity then the usual BGG resolution in the \( \mathfrak{sl}_2 \) case provides an exact complex
\[
0 \rightarrow M^s_k(s \cdot k\ell) \rightarrow M^s_k(k\ell) \rightarrow L(k\ell) \rightarrow 0.
\]

Thus we see that the flat family of such complexes over \( \mathbb{C}^* \setminus \{\text{roots of unity}\} \) is extended over the whole \( \mathbb{C}^* \).

5.7. **Construction of the quasi-BGG complex.** Here we extend the previous considerations to the case of the quantum group \( U_\ell \) for arbitrary root data \( (Y, X, \ldots) \) of the finite type \( (I, \cdot) \). Fix a dominant weight \( \lambda \in X \).

First we construct an inclusion \( M^w_{\ell}(w \cdot \ell) \rightarrow M^w_{\ell}(w \cdot \ell) \) for a pair of elements \( w', w \in W \) such that \( l(w') = l(w) + 1 \) and \( w' > w \) in the Bruhat order on the Weyl group. In fact we can do it explicitly only for \( w' \) and \( w \) differing by a simple reflection: \( w' = ws_i, i \in I \).

Consider the twisted quantum parabolic subalgebra \( w(\mathfrak{p}_{i,\ell}) \). Then \( w(\mathfrak{p}_{i,\ell}) \supset w(\mathfrak{b}^+_i) \) and \( w(\mathfrak{p}_{i,\ell}) \supset ws_i(\mathfrak{b}^+_i) \). Consider also the Levi quotient algebra \( w(\mathfrak{p}_{i,\ell}) \rightarrow w(\mathfrak{l}_{i,\ell}). \)

The algebra \( w(\mathfrak{l}_{i,\ell}) \) is isomorphic to \( U_\ell(\mathfrak{sl}_2) \oplus U_\ell^0(\mathfrak{sl}_2) \).

By Proposition 5.6.1 we have a natural inclusion of \( w(\mathfrak{l}_{i,\ell})\)-modules \( \text{S-Ind}_{\mathfrak{l}_w(\mathfrak{l}_{i,\ell})}^{w(\mathfrak{l}_{i,\ell})} \mathbb{C}(\ell) \rightarrow \text{S-Ind}_{\mathfrak{l}_w(\mathfrak{l}_{i,\ell})}^{w(\mathfrak{l}_{i,\ell})} \mathbb{C}(\ell). \)

**Lemma:**

(i) \( \text{S-Ind}_{w(\mathfrak{b}^+_i)}^{w(\mathfrak{p}_{i,\ell})} (\cdot) = \text{S-Ind}_{w(\mathfrak{p}_{i,\ell})}^{w(\mathfrak{p}_{i,\ell})} (\cdot) \circ \text{S-Ind}_{w(\mathfrak{b}^+_i)}^{w(\mathfrak{p}_{i,\ell})} (\cdot). \)

(ii) \( \text{S-Ind}_{ws_i(\mathfrak{b}^+_i)}^{w(\mathfrak{p}_{i,\ell})} (\cdot) = \text{S-Ind}_{w(\mathfrak{p}_{i,\ell})}^{w(\mathfrak{p}_{i,\ell})} (\cdot) \circ \text{S-Ind}_{ws_i(\mathfrak{b}^+_i)}^{w(\mathfrak{p}_{i,\ell})} (\cdot). \)

(iii) \( \text{S-Ind}_{w(\mathfrak{b}^+_i)}^{w(\mathfrak{l}_{i,\ell})} (\mathbb{C}(\ell)) = \text{S-Ind}_{w(\mathfrak{p}_{i,\ell})}^{w(\mathfrak{l}_{i,\ell})} (\mathbb{C}(\ell)) \circ \text{Res}_{w(\mathfrak{p}_{i,\ell})}^{w(\mathfrak{l}_{i,\ell})} (\ell). \)

(iv) \( \text{S-Ind}_{ws_i(\mathfrak{b}^+_i)}^{w(\mathfrak{l}_{i,\ell})} (\mathbb{C}(\ell)) = \text{S-Ind}_{w(\mathfrak{p}_{i,\ell})}^{w(\mathfrak{l}_{i,\ell})} (\mathbb{C}(\ell)) \circ \text{Res}_{w(\mathfrak{p}_{i,\ell})}^{w(\mathfrak{l}_{i,\ell})} (\ell). \)

**Corollary:** For \( w' = ws_i > w \) in the Bruhat order we have a natural inclusion of \( U_\ell \)-modules \( i^{ws_i,w}_{\ell}(w \cdot \ell) \rightarrow M^w_{\ell}(w \cdot \ell). \)

Note that in the previous considerations we never used the fact that \( U_\ell \) was a quantum group at the root of unity. Thus a similar construction provides inclusions
\[
i^{ws_i,w}_{\ell}(w\cdot\ell) : M^w_{\ell}(ws_i \cdot \ell \lambda) \hookrightarrow M^w_{\ell}(w \cdot \ell \lambda)
\]
for any \( \xi \in \mathbb{C}^* \). Recall that if \( \xi \) is not a root of unity then \( M^w_{\ell}(w \cdot \ell \lambda) \) is isomorphic to the usual Verma module \( M_{\ell}(w \cdot \ell \lambda) \). Thus the morphism \( i^{w',w}_{\ell} \) coincides with the standard inclusion of Verma modules constructed by J. Bernshtein, I.M. Gelfand and S.I. Gelfand in [BGG] that becomes a component
of the differential in the BGG resolution. In other words we see that the flat family of inclusions 
\[ i^\nu \colon M^\nu(w \cdot \ell \lambda) \hookrightarrow M^\nu(w \cdot \ell \lambda) \] 
defined over \( \mathbb{C}^* \setminus \{ \text{roots of unity} \} \) can be extended naturally 
over the whole \( \mathbb{C}^* \).

Iterating the inclusion maps we obtain a flat family of submodules 
\[ i^\nu(M^\nu(w \cdot \ell \lambda)) \subseteq M^\nu(w \cdot \ell \lambda) \] 
for \( \xi \in \mathbb{C}^* \), \( w \in W \), providing an extension of the standard lattice of Verma submodules in \( M^\nu(\ell \lambda) \) for 
\( \xi \in \mathbb{C}^* \setminus \{ \text{roots of unity} \} \).

5.7.1. **Lemma**: For a pair of elements \( w', w \in W \) such that \( l(w') = l(w) + 1 \) and \( w' > w \) in the Bruhat order we have 
\[ i^\nu(M^\nu(w' \cdot \ell \lambda)) \hookrightarrow i^\nu(M^\nu(w \cdot \ell \lambda)). \]

**Proof.** To prove the statement note that the condition \( \{ A_\xi \} \) is a submodule in \( B_\xi \) is a closed condition 
in a flat family. \( \square \)

Now using the standard combinatorics of the classical BGG resolution we obtain the following 
statement.

5.7.2. **Theorem**: There exists a complex of \( U_\ell \)-modules \( B^\bullet_\ell(\ell \lambda) \) with 
\[ B^{-k}_\ell(\ell \lambda) = \bigoplus_{w \in W, l(w) = k} M^\nu(w \cdot \ell \lambda) \] 
and with differentials provided by direct sums of the inclusions \( i^\nu(w', w) \).

5.7.3. **Definition**: We call the complex \( B^\bullet_\ell(\ell \lambda) \) the *quasi-BGG complex* for the dominant weight 
\( \ell \lambda \in X \).

Recall that we have a natural morphism \( \text{can} : M^\nu(\ell \lambda) \rightarrow W(\ell \lambda) \).

**Proposition**: \( H^0(B^\bullet_\ell(\ell \lambda)) = W(\ell \lambda) \).

**Proof.** The statement follows from the standard fact that \( W(\ell \lambda) \) is the biggest finite dimensional 
quotient module in \( M^\nu(\ell \lambda) \). \( \square \)

5.7.4. **Conjecture**: The complex \( B^\bullet_\ell(\ell \lambda) \) is quasiisomorphic to the Weyl module \( W(\ell \lambda) \).

Note that by the previous subsection the Conjecture holds in the \( U_\ell(\mathfrak{gl}_2) \) case.

Using the contragradient duality we obtain a complex \( \mathbb{D}B^\bullet_\ell(\ell \lambda) \) consisting of direct sums of con-tra
gradient quasi-Verma modules with \( H^0(\mathbb{D}B^\bullet_\ell(\ell \lambda)) = \mathbb{D}W(\ell \lambda) \). In particular we have a morphism in 
the category of complexes of \( U_\ell \)-modules \( \mathbb{D}(\text{can}) : \mathbb{D}W(\ell \lambda) \rightarrow \mathbb{D}B^\bullet_\ell(\ell \lambda) \).

5.8. **Semiinfinite cohomology with coefficients in quasi-Verma modules**. Recall the following construction that plays crucial role in considerations of Ginzburg and Kumar in [GK].

Let \( \mathfrak{B}^\bullet_\ell \)-mod\( ^\text{fin} \) (resp. \( \mathfrak{U}_\ell \)-mod\( ^\text{fin} \), resp. \( U(\mathfrak{b}^+)-\text{mod}\( ^\text{fin} \), resp. \( U(\mathfrak{g})\)-mod\( ^\text{fin} \)) be the category 
of finite dimensional \( X \)-graded modules over the corresponding algebra with the action of the Cartan 
subalgebra semisimple and well defined with respect to the \( X \)-gradings. Consider the functors:

\[
\begin{align*}
\text{Coind}_{\mathfrak{B}^\bullet_\ell} : & \quad \mathfrak{B}^\bullet_\ell \text{-mod} \rightarrow \mathfrak{U}_\ell \text{-mod}; \\
\text{Coind}_{\mathfrak{U}_\ell} : & \quad \mathfrak{U}_\ell \text{-mod} \rightarrow \mathfrak{B}^\bullet_\ell \text{-mod} \\
\text{Coind}_{U(\mathfrak{g})} : & \quad U(\mathfrak{b}^+) \text{-mod} \rightarrow U(\mathfrak{g}) \text{-mod}; \\
\text{Coind}_{U(\mathfrak{b}^+)} : & \quad U(\mathfrak{g}) \text{-mod} \rightarrow U(\mathfrak{b}^+) \text{-mod} \\
\end{align*}
\]

\[
\begin{align*}
(\cdot)^\ell \mathfrak{b}^+ : & \quad \mathfrak{B}^\bullet_\ell \text{-mod} \rightarrow U(\mathfrak{b}^+) \text{-mod} \\
(\cdot)^\ell u_\ell : & \quad \mathfrak{U}_\ell \text{-mod} \rightarrow U(\mathfrak{g}) \text{-mod} \\
\end{align*}
\]

where \( (\cdot)^\text{fin} \) denotes taking the maximal finite dimensional submodule in \( (\cdot) \) and \( (\cdot)^\ell \mathfrak{b}^+ \) (resp. \( (\cdot)^\ell u_\ell \))
denotes taking \( \mathfrak{b}^+ \)- (resp. \( u_\ell \))-invariants.

**Proposition**: (see [GK])

(i) \( (\cdot)^\ell u_\ell \circ \text{Coind}_{\mathfrak{U}_\ell} = \text{Coind}_{U(\mathfrak{g})} \circ (\cdot)^\ell \mathfrak{b}^+ \);

(ii) \( (\cdot)^\ell u_\ell \circ \left( \text{Coind}_{\mathfrak{B}^\bullet_\ell} \right)^\text{fin} = \left( \text{Coind}_{U(\mathfrak{b}^+)} \right)^\text{fin} (\cdot)^\ell \mathfrak{b}^+ \). \( \square \)
The semiinfinite analogue for the first part of the previous statement looks as follows. Fix $w \in W$. Consider the functors:

$$\text{S-Coind}_{\varpi(w_-(B^+_\ell))}^{\text{U}(\ell)} : \text{D}(w(B^+_\ell)) - \text{mod} \rightarrow \text{D}(\text{U}(\ell)) - \text{mod},$$

$$\text{S-Coind}_{\varpi(w_-(b^+_\ell))}^{\text{U}(\ell)} : \text{D}(U(w(b^+_\ell))) - \text{mod} \rightarrow \text{D}(\text{U}(\ell)) - \text{mod},$$

$$\text{Ext}^{\infty}_{\varpi(w_-(B^+_\ell))}((\mathbb{C}, \cdot)) : \text{D}(w(B^+_\ell)) - \text{mod} \rightarrow \text{D}(U(w(b^+_\ell))) - \text{mod},$$

$$\text{Ext}^{\infty}_{\varpi(w_-(B^+_\ell))}((\mathbb{C}, \cdot)) : \text{D}(w(\text{U}(\ell))) - \text{mod} \rightarrow \text{D}(\text{U}(\ell)) - \text{mod}.$$

5.8.1. **Theorem:** We have

$$\text{Ext}^{\infty}_{\varpi(w_-(B^+_\ell))}((\mathbb{C}, \cdot)) \circ \text{S-Coind}_{\varpi(w_-(B^+_\ell))}^{\text{U}(\ell)} = \text{S-Coind}_{\varpi(w_-(b^+_\ell))}^{\text{U}(\ell)} \circ \text{Ext}^{\infty}_{\varpi(w_-(B^+_\ell))}((\mathbb{C}, \cdot)).$$

**Proof.** To simplify the notations we work with semiinfinite homology and semiinfinite induction instead of semiinfinite cohomology and semiinfinite coinduction. By [A], Appendix B, every convex complex of $w(B^+_\ell)$-modules is quasisymmetric to a K-semijective convex complex. Consider a K-semijective convex complex of $w(B^+_\ell)$-modules $SS^\bullet$. Note that both semiinfinite induction functors are exact and take K-semijective complexes to K-semijective complexes. Thus we have

$$\text{Tor}^{w_-(B^+_\ell)}_{\infty + \bullet}((\mathbb{C}, \cdot)) \circ \text{S-Ind}_{\varpi(w_-(B^+_\ell))}^{\text{U}(\ell)}(SS^\bullet) = \text{Tor}^{w_-(B^+_\ell)}_{\infty + \bullet}((\mathbb{C}, \text{Tor}^{w_-(B^+_\ell)}_{\infty + \bullet}((\mathbb{C}, \mathbb{C}^\mu_{\text{U}(\ell)}),$ SS$^\bullet))$$

$$= \text{Tor}^{w_-(B^+_\ell)}_{\infty + \bullet}((\mathbb{C}, \text{S}^\mu_{\text{U}(\ell)}),$ SS$^\bullet)) = \text{Tor}^{w_-(B^+_\ell)}_{\infty + \bullet}((\mathbb{C}, \text{S}^\mu_{\text{U}(\ell)}),$ SS$^\bullet))$$

$$= \text{Tor}^{w_-(B^+_\ell)}_{\infty + \bullet}((\mathbb{C}, \text{S}^\mu_{\text{U}(\ell)}),$ SS$^\bullet)) = \text{Tor}^{w_-(B^+_\ell)}_{\infty + \bullet}((\mathbb{C}, \text{S}^\mu_{\text{U}(\ell)}),$ SS$^\bullet)).$$

Here we used the fact that the subalgebra $w(b^+_\ell) \subset w(B^+_\ell)$ is normal with the quotient algebra equal to $U(w(b^+_\ell)).$

**Corollary:**

$$\text{Ext}^{\infty}_{\varpi(w_{-\ell}(\mathbb{C}, \cdot))}(\mathbb{C}, M^w_{\ell}(w \cdot \ell)) = H^{\infty(R^\bullet)}_{T^w_{\ell}(G/B)}(T^*(G/B), \pi^* \mathcal{L}(\lambda))$$

as a module over both $U(\mathbf{g})$ and $H^0(T^*(G/B), \mathcal{O}_{T^*(G/B)}) = H^0(N, \mathcal{O}_N)$.

Recall that for the Springer-Grothendieck resolution of the nilpotent cone $\mu : T^*(G/B) \rightarrow N$ we have $\mu^{-1}(\mathcal{N}) = \bigcup_{w \in W} T^w_{\ell}(G/B)$.

**Proposition:**

(i) There exists a filtration on $H^{\infty(R^\bullet)}_{\mu^{-1}(\mathcal{N})}(T^*(G/B), \pi^* \mathcal{L}(\lambda))$ with the subquotients equal to

$$H^{\infty(R^\bullet)}_{T^w_{\ell}(G/B)}(T^*(G/B), \pi^* \mathcal{L}(\lambda)),$$

for $w \in W$.

(ii) $\text{Ext}^{\infty}_{\varpi(w_{-\ell}(\mathbb{C}, \cdot))}(\mathbb{C}, B^w_{\ell}(\ell)) = H^{\infty(R^\bullet)}_{\mu^{-1}(\mathcal{N})}(T^*(G/B), \pi^* \mathcal{L}(\lambda))$.

(iii) The morphism

$$H^{\infty(R^\bullet)}_{\mu^{-1}(\mathcal{N})}(N, \mu_* \pi^* \mathcal{L}(\lambda)) \xrightarrow{\sigma} \text{Ext}^{\infty}_{\varpi(w_{-\ell}(\mathbb{C}, \cdot))}(\mathbb{C}, \mathbb{D}W(\ell)) \xrightarrow{\tau} \text{Ext}^{\infty}_{\varpi(w_{-\ell}(\mathbb{C}, \cdot))}(\mathbb{C}, \mathbb{D}B^w_{\ell}(\ell))$$

$$\xrightarrow{\text{isom}} H^{\infty(R^\bullet)}_{\mu^{-1}(\mathcal{N})}(T^*(G/B), \pi^* \mathcal{L}(\lambda))$$

is an isomorphism.

5.8.2. **Corollary:**

(i) The map $\sigma$ is injective and the map $\tau$ is surjective.

(ii) Conjecture 5.1.2 is true if conjecture 5.7.4 is so.

However we will manage to prove Conjecture 5.1.2 without using the exactness of the quasi-BGG complex.
5.9. **Quantum twisting functor.** Here we present a construction of a kernel functor similar to the twisting functors in [Ar5] and [Ar6]. Consider the subalgebra $U^{1/2}_\ell$ in $U_\ell$ generated by the elements $E_i$, $F_i$, $F_i^{(t)}$, $i \in I$.

**Lemma:**

(i) The subalgebra in $U_\ell$ generated by the elements $F_i^{(t)}$, $i \in I$, is isomorphic to $U(n^-)$.

(ii) The algebra $U^{1/2}_\ell$ is isomorphic to $u_\ell \otimes U(n^-)$ as a vector space.

(iii) The subalgebra $u_\ell \subset U^{1/2}_\ell$ is normal with the quotient algebra equal to $U(n^-)$.

Note yet that this splitting of the projection $U^{1/2}_\ell \rightarrow U(n^-)$ does not extend to a splitting of the projection $U_\ell \rightarrow U(g)$.

Consider the restriction of the quasi-Verma module $M^w_\ell(w \cdot \ell \lambda)$ onto the subalgebra $U^{1/2}_\ell$.

**Lemma:** The $U^{1/2}_\ell$-module $M^w_\ell(w \cdot \ell \lambda)$ is isomorphic to $\text{Ind}_{U(n^-)}^{U_\ell} C(\ell(w \cdot \ell \lambda))$.

In particular $M^w_\ell(w \cdot \ell \lambda)$ is free over the subalgebra $U(n^-) \subset U_\ell$. Consider the left semiregular $U_\ell$-module $S^w_{U_\ell}$.

**Lemma:** $S^w_{U_\ell} \lla \text{Ind}_{U(n^-)}^{U_\ell} U(n^-)^\ast$ as a left $U_\ell$-module.

**Corollary:**

(i) $\left( S^w_{U_\ell} \otimes_{U_\ell} M^w_\ell(w \cdot \ell \lambda) \right)^\tau = \mathbb{D} M^{w_0 w^{-1}}_\ell(w_0 w^{-1} \cdot \ell \lambda)$.

(ii) $\left( S^w_{U_\ell} \otimes_{U_\ell} W(\ell \lambda) \right)^\tau = \mathbb{D} W(\ell \lambda)[\mathbb{Z}(R^+)]$.

(iii) $\left( S^w_{U_\ell} \otimes_{U_\ell} B^\ast_\ell(\ell \lambda) \right)^\tau = \mathbb{D} B^\ast_\ell(\ell \lambda)[\mathbb{Z}(R^+)]$. Here $(\cdot)^\tau$ denotes twisting the $U_\ell$-action on $(\cdot)$ by the Chevalley involution $\tau$.

We denote the functor $\left( S^w_{U_\ell} \otimes_{U_\ell} \right)^\tau$ by $\mathcal{S}_\ell$. Now consider the morphism of complexes of $U_\ell$-modules $can : B^\ast_\ell(\ell \lambda) \rightarrow W(\ell \lambda)$. We have a morphism in the derived category of $U_\ell$-modules $\mathcal{G}_\ell(can) : \mathbb{D} B^\ast_\ell(\ell \lambda)[\mathbb{Z}(R^+)] \rightarrow \mathbb{D} W(\ell \lambda)[\mathbb{Z}(R^+)]$. On the other hand consider the canonical map $\mathbb{D}(can) : \mathbb{D} W(\ell \lambda) \rightarrow \mathbb{D} B^\ast_\ell(\ell \lambda)$.

5.9.1. **Proposition:** For $\ell$ big enough we have $\mathcal{G}_\ell(can) \circ \mathbb{D}(can) = \text{Id}_{\mathbb{D} W(\ell \lambda)[\mathbb{Z}(R^+)]}$.

**Proof.** Note that in the definition of the functor $\mathcal{S}_\ell$ we never used the fact that we worked in the derived category of modules over the quantum group at a root of unity. In particular one can construct similar functors

$\mathcal{G}_\xi : \mathbb{D}(U_\xi \text{-mod}) \rightarrow \mathbb{D}(U_\xi \text{-mod})$

for any $\xi \in \mathbb{C}^\ast$. It is easy to verify that for $\xi$ not a root of unity we have $\mathcal{G}_\xi(can) \circ \mathbb{D}(can) = \text{Id}_{\mathbb{D} W(\ell \lambda)[\mathbb{Z}(R^+)]}$.

On the other hand note that the endomorphism spaces of the $U_\xi$-modules $L(\ell \lambda)$ (resp. $U_\ell$-modules $\mathbb{D} W(\ell \lambda)$) is one dimensional. Thus we can consider $\mathcal{G}_\xi(can) \circ \mathbb{D}(can)$ as a (nonzero) polynomial function on $\xi$. But the number of roots of this polynomial is finite.

**Corollary:** $\text{Ext}_{U^{1/2}_\ell}^{2+}(\mathbb{C}, \mathbb{D} W(\ell \lambda))$ is a direct summand in $\text{Ext}_{U^{1/2}_\ell}^{2+}(\mathbb{C}, \mathbb{D} B^\ast_\ell(\ell \lambda))$.

Comparing this statement with Corollary 5.8.2 we obtain the main result of the section.

5.9.2. **Theorem:** For $\ell$ big enough $\text{Ext}_{U^{1/2}_\ell}^{2+}(\mathbb{C}, \mathbb{D} W(\ell \lambda)) = H^{0}_{\mu^+}(\mathcal{N}, \mu^+ p^* \mathcal{L}(\lambda))$ as a $H^0(\mathcal{N}, \mathcal{O}_{\mathcal{X}})$-module.

6. **Further results and conjectures.**

In this section we present several facts without proof. We also formulate some conjectures concerning possible origin of the quasi-BGG complex.
6.1. Alternative triangular decompositions of \( u_\ell \). Note that the definition of semiinfinite cohomology starts with specifying a triangular decomposition of a graded algebra \( u \). Fix a subset \( J \subset I \).

Instead of the usual height function consider the linear map \( \text{ht}_J : X \rightarrow \mathbb{Z} \) defined on the elements \( i', i \in I \) by \( \text{ht}_J(i') = 0 \) for \( i \in J \) and \( \text{ht}_J(i') = 1 \) otherwise and extended to the whole \( X \) by linearity. Now we work in the category of complexes of \( X \)-graded \( u_\ell \)-modules satisfying conditions of concavity and convexity with respect to the \( \mathbb{Z} \)-grading obtained from the \( X \)-grading with the help of the function \( \text{ht}_J \).

Consider the triangular decomposition of the small quantum group \( u_\ell = p_J^\ell \otimes u_J^+, \) where \( p_J^\ell \) denotes the small quantum negative parabolic subalgebra in \( u_\ell \) corresponding to the subset \( J \subset I \) and \( u_J^+ \) denotes the quantum analogue of the nilpotent radical in \( \mathfrak{b}_J^+ \) defined with the help of Lusztig generators of \( U_\ell \) and \( u_\ell \) (see [1]).

Then it is known that the subalgebra \( u_J^+ \) in \( u_\ell \) is Frobenius just like \( u_J^+ \). Thus it is possible to use the general definition of semiinfinite cohomology presented in [1]. Denote the corresponding functor by \( \text{Ext}^\infty_{u_\ell,J}(*,*). \)

On the other hand consider the classical negative parabolic subalgebra \( \mathfrak{p}_J^- \subset \mathfrak{g} \) and its nilradical \( \mathfrak{n}_J^- \). Choose the standard \( X \)-homogeneous root basis \( \{ f_\alpha \} \) in the space \( \mathfrak{n}_J^- \). Consider the subset in \( N(J) \subset N \) annihilated by all the elements of the base dual to \( \{ f_\alpha \} \).

6.1.1. **Theorem:** \( \text{Ext}^\infty_{u_\ell,J}(*,\mathbb{C}) \Rightarrow H^\dim(n_J^-)(N,\mathcal{O}_N) \) as \( H^0(N,\mathcal{O}_N) \)-modules.

6.2. Contragradient Weyl modules with non-\( \ell \)-divisible highest weights. Fix a dominant weight of the form \( \ell \lambda + w \cdot 0 \), where \( \lambda \in X \) and \( w \in W \). It is known that all the dominant weights in the linkage class containing \( 0 \) look like this. Consider the contragradient Weyl module \( \mathbb{D}W(\ell \lambda + w \cdot 0) \).

The following statement generalizes Corollary 6.1.3. Its proof is similar to the proof of Conjecture 5.1.2.

6.2.1. **Theorem:**

\[
\text{ch} \left( \text{Ext}^\infty_{u_\ell,J}(*,\mathbb{C}) \otimes \mathbb{D}W(\ell \lambda + w \cdot 0), t \right)
= \frac{t^{-2(R^+)^{\perp} + 1}(w)}{1 - e^{-2tV}} \sum_{v \in W} e^{(\ell \lambda + w \cdot 0)} C_{\alpha} \left( 1 - t^2 e^{-2t \alpha} \right)^{\mathbb{R}^+}.
\]

6.3. Contragradient Weyl modules: alternative triangular decompositions. Fix the triangular decomposition of the small quantum group \( u_\ell \) like in [1]. A natural generalization of Conjecture 5.1.2 to the case of the parabolic triangular decomposition looks as follows. We keep the notations from [1].

6.3.1. **Conjecture:** \( \text{Ext}^\infty_{u_\ell,J}(*,\mathbb{D}W(\ell \lambda)) \Rightarrow H^\dim(n_J^-)(N,\mu \ast p^* L(\lambda)) \) as \( H^0(N,\mathcal{O}_N) \)-module.

6.4. Connection with affine Kac-Moody algebras. Finally we would like to say a few words about a possible explanation for the existence of quasi-Verma modules and quasi-BGG resolutions.

Suppose for simplicity that the root data \( (Y,X,\ldots) \) are untwisted, i.e. the corresponding Cartan matrix is symmetric. Consider the affine Lie algebra \( \hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}K \) corresponding to \( \mathfrak{g} \). Fix a negative level \( -h^\vee + k \), where \( k \in 1/2\mathbb{Z}_{<0} \setminus \mathbb{Z}_{<0} \) and \( h^\vee \) denotes the dual Coxeter number for chosen root data of the finite type. Consider the Kazhdan-Lusztig category \( \hat{O}_{-k} \) of \( \mathfrak{g} \otimes \mathbb{C}[t] \)-integrable finitely generated \( \hat{\mathfrak{g}} \)-modules diagonalizable with respect to the Cartan subalgebra in \( \hat{\mathfrak{g}} \) at the level \(-2h^\vee - k\). Kazhdan and Lusztig showed that the category \( \hat{O}_k \) posses a structure of a rigid tensor category with the fusion tensor product \( \otimes \). Moreover, they proved the following statement.

6.4.1. **Theorem:** (see [KL 1,2,3,4]) Let \( \ell = -2k \). Then the tensor category \( (\hat{O}_k, \otimes) \) is equivalent to the category \( (U_\ell \otimes \mathcal{M})^{\text{fin}} \) with the tensor product provided by the Hopf algebra structure on \( U_\ell \).

We denote the functor \( (\hat{O}_k, \otimes) \rightarrow \left((U_\ell \otimes \mathcal{M})^{\text{fin}}, \otimes \right) \) providing the equivalence of categories by \( \hat{\mathfrak{g}} \).

Consider the usual category \( O_k \) for \( \mathfrak{g} \) at the same level. Finkelberg constructed a functor \( \hat{\mathfrak{l}} : O_k \rightarrow \cdots \).
\( U_\ell \)-mod extending the functor \( \tilde{kl} \) (see [F]). Note that the functor \( kl \) has no chance to be an equivalence of categories because it is known not to be exact.

Fix a dominant (resp. arbitrary) weight \( \lambda \in X \). Consider now the contragradient Weyl module \( \mathbb{D}W(\lambda) = \text{Coind}_{U_{2\hbar V^-} - k}^{U_{2\hbar V}}(\hat{g}) L(\lambda) \) and the contragradient Verma module \( \mathbb{D}M(\lambda) = \text{Coind}_{U_{2\hbar V^-} - k}^{U_{2\hbar V}}(\hat{g}) \mathbb{D}M(\lambda) \) over \( \hat{g} \) at the level \(-2\hbar V - k\), where \( L(\lambda) \) (resp. \( M(\lambda) \)) denotes the simple module (resp. the contragradient Verma module) over \( g \) with the highest weight \( \lambda \). Then the usual contragradient BGG resolution of \( L(\lambda) \) provides a resolution \( \mathbb{D}B(\lambda) \) of the contragradient Weyl module \( \mathbb{D}W(\lambda) \) consisting of direct sums of contragradient Verma modules of the form \( \mathbb{D}M(w - \lambda) \), where \( w \in W \). It is known that the Kazhdan-Lusztig functor takes Weyl and contragradient Weyl modules over \( \hat{g} \) to Weyl (resp. contragradient Weyl) modules over \( U_\ell \).

6.4.2. **Conjecture:** The functor \( kl \) takes \( \mathbb{D}M(w \cdot \ell \lambda) \) to the contragradient quasi-Verma module \( \mathbb{D}M^w_{\ell}(w \cdot \ell \lambda) \). Moreover the complex \( kl(\mathbb{D}B(\ell \lambda)) \) is quasiisomorphic to \( \mathbb{D}W(\ell \lambda) \). \( \square \)

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