FROBENIUS KERNELS OF ALGEBRAIC SUPERGROUPS AND
STEINBERG’S TENSOR PRODUCT THEOREM

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Dedicated to Professor Akira Masuoka on the occasion of his 60th birthday.

Abstract. For a split quasireductive supergroup $\mathcal{G}$ defined over a field, we study structure and representation of Frobenius kernels $\mathcal{G}_r$ of $\mathcal{G}$ and we give a necessary and sufficient condition for $\mathcal{G}_r$ to be unimodular in terms of the root system of $\mathcal{G}$. We also establish Steinberg’s tensor product theorem for $\mathcal{G}$ under some natural assumptions.

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1. Introduction

Structure and representation of algebraic group schemes (especially, connected and split reductive groups) over a field have been well studied (see [J, Mi] for example) and provide applications in many areas such as combinatorics or number theory. Over an algebraically closed field \( k \) of characteristic zero, the Lie algebra \( \text{Lie}(G) \) of a connected and split reductive group (scheme) \( G \) strongly reflects many properties of \( G \) (see [Ho]) and becomes a fundamental tool for studying representations of \( G \). For example, it is known that there exists a category equivalence between the category of left \( G \)-modules and the category of locally finite left \( \text{Lie}(G) \)-modules, where \( T \) denotes a split maximal torus of \( G \). Here, we say that a \( \text{Lie}(G) \)-module \( M \) is a \( \text{Lie}(G) \)-module if the restricted \( \text{Lie}(T) \)-module structure on \( M \) arises from some \( T \)-module structure on it. In particular, we can show that for a dominant weight \( \lambda \), the simple \( \text{Lie}(G) \)-module \( \text{ind}_G^\lambda(k^\lambda) \) of the one-dimensional \( T \)-module \( k^\lambda \) of weight \( \lambda \), where \( B \) denotes a fixed Borel subgroup of \( G \). The character of \( \text{ind}_G^\lambda(k^\lambda) \) is explicitly given by Weyl’s character formula.

On the other hand, over a field \( k \) of positive characteristic, the situation is more complicated, since the simple left \( \text{Lie}(G) \)-module \( \text{ind}_G^\lambda(k^\lambda) \) may be a proper submodule of \( \text{ind}_G^\lambda(k^\lambda) \) in general. In [T1], Takeuchi studied the hyperalgebra \( \text{hy}(G) \) of \( G \) which is a natural refinement of the universal enveloping algebra \( \mathcal{U}(\text{Lie}(G)) \) of \( \text{Lie}(G) \). Note that, \( \text{hy}(G) \) is isomorphic to \( \mathcal{U}(\text{Lie}(G)) \) as (cocommutative) Hopf algebras if \( \text{char}(k) = 0 \). By Hopf-algebraic method, as in the Lie algebra case, he showed \( \text{hy}(G) \) strongly reflects many properties of \( G \) (see [T1, T2, T3]). There also holds a category equivalence between the category of left \( \text{Lie}(G) \)-modules and the category of locally finite left \( \text{hy}(G) \)-modules (see [J] Part II, Chapter I) for example). Over a perfect field \( k \) of positive characteristic \( p \), for each positive integer \( r \), the kernel \( G_r \) of the \( r \)-th iterated Frobenius morphism \( \text{Fr}^r : G \to G \), called the \( r \)-th Frobenius kernel of \( G \), is a fundamental and powerful tool for studying \( G \). By definition, we have an ascending chain \( G_1 \subset G_2 \subset \cdots \subset G \) of normal subgroup of \( G \) and \( \text{hy}(G) = \lim_{\leftarrow} \text{hy}(G_r) \). Moreover, it is known that all Frobenius kernels \( G_r \) are unimodular, that is, there exists non-zero two-sided integral for \( G_r \), see Definition 4.2

Using the categorical equivalence of modules mentioned above, we can show Steinberg’s tensor product theorem ([J] Part II, Corollary 3.17) which states that as a left \( \text{Lie}(G) \)-module, the simple left \( \text{Lie}(G) \)-module \( L(\lambda) \) decomposes into some tensor products of \( \text{Fr}^r \)-twisted simple left \( \text{Lie}(G) \)-modules \( L(\lambda_r)^{[r]} \) as such

\[
L(\lambda) \cong L(\lambda_0) \otimes L(\lambda_1)^{[1]} \otimes L(\lambda_2)^{[2]} \otimes \cdots \otimes L(\lambda_m)^{[m]}
\]

along the “\( p \)-adic expansion” \( \lambda = \lambda_0 + p\lambda_1 + p^2\lambda_2 + \cdots + p^m\lambda_m \) of \( \lambda \), where \( \lambda_r \)’s are \( p \)-restricted weights for \( G \) (see Definition 5.18). In particular, the character of \( L(\lambda) \) can be calculated by the product of the character of \( L(\lambda_r)^{[r]} \). Note that, if we
write the character of a $G$-module $M$ as $\sum \dim(M^\lambda) e^\lambda$, then the character of $\text{Fr}^r$-twisted $G$-module $M^{[r]}$ is given by $\sum \dim(M^\lambda) e^{r \lambda}$. Therefore, the decomposition tells us that to study a simple left $G$-module, it is enough to consider simple left $G$-modules with $p$-restricted weights.

In recent years, supergeometries and superalgebras have attracted much attention. The word “super” is a synonym of “graded by the group $\mathbb{Z}_2$ of order two” (see Section 5.3). The symmetric tensor category of vector spaces is generalized by the category of superspaces (i.e., $\mathbb{Z}_2$-graded vector spaces) with the familiar tensor product and supersymmetry. The classification of finite dimensional simple Lie superalgebras over an algebraically closed field of characteristic zero was done by Kac [Ka]. Since then, many authors have studied the corresponding algebraic supergroup ([Kost] Kosz [P] [BrK] [Ma1] [Z1] [CCF] [CFZ] [FG] for example). Here, an algebraic supergroup $G$ is a representable functor from the category of commutative superalgebras to the category of groups; the representing object $O(G)$ is a finitely generated commutative Hopf superalgebra. In this paper, as the super-analogue of the connected and split reductive groups, we study an algebraic supergroup $G$ whose “even part” $G_{ev}$ is a connected and split reductive group, called a split quasireductive supergroup ([Sh1] [Sh2], see also [SG] [GT]) over a field. The class of split quasireductive supergroup has many important algebraic supergroups, for example, the general linear supergroups $GL(m|n)$, the queer supergroups $Q(n)$, the periplectic supergroups $P(n)$, Chevalley supergroups of classical type (including special linear supergroups $SL(m|n)$ and ortho-symplectic supergroups $SpO(m|n)$) due to Fiorese and Gavarini [FG], etc. As in the non super-situation, if the base field is of characteristic zero, then representation theory of a split quasireductive supergroup $G$ is essentially the same as the Lie superalgebra $\text{Lie}(G)$ of $G$.

In this paper, we are interested in modular representation theory of split quasireductive supergroup, that is, the case when the characteristic of the base field $k$ is positive. As in the non super-situation, we can define Frobenius kernels $G_r$ of a split quasireductive supergroup $G$ and these are also powerful tool for studying $G$. For example, using Frobenius kernels of $GL(m|n)$, Zubkov and Marko [ZM] provided the linkage principle and described blocks of $GL(m|n)$. In this paper, we give a necessary and sufficient condition for $G_r$ to be unimodular in terms of the root system of $G$. Thus, in contrast to the non super-situation, there exists a non unimodular $G_r$ (see Example 4.17).

Recently, it is shown by several authors that Steinberg’s tensor product theorem holds for $GL(m|n)$ [Ku], $Q(n)$ [BrK] and $SpO(m|n)$ [SW]. See also, [CSW] for (simply connected) Chevalley supergroups of type $D(2;\zeta)$, $G(3)$ and $F(3;1)$. For a split quasireductive supergroup $G$ in general, it has been shown in [MS1] that there exists a category equivalence between the category of left $G$-supermodules and the category of locally finite left $hy(G)$-$T$-supermodules. Moreover, all simple left $G$-supermodules have been systematically constructed in [Sh1]. Therefore, it is natural to ask whether Steinberg’s tensor product theorem holds for $G$, in general. To answer this question, one encounters the following two difficulties:

1. not all simple left $G$-supermodules are absolutely simple, that is, there exists a simple left $G$-supermodule which is no longer simple after base change to some field extension of $k$ (see Definition 5.3);

2. the root system of $G$ is ill-behaved (see Example 5.4 for example) without suitable extra conditions on the “odd part” of $G$. 


Note that, in the non super-situation, (1) never happens.

In this paper, we prove that these difficulties (1) and (2) can be overcome by attaching appropriate natural conditions. We first show that if the root system $\Delta$ of $G_{\text{ev}}$ does not contain the unit $0$ of the character group $X(T)$ of a fixed split maximal torus of $G_{\text{ev}}$, then all simple left $G_{\text{ev}}$-supermodules are absolutely simple (Proposition 5.6). Therefore, to resolve (1), we assume that (1)' the base field $k$ is algebraically closed if $0 \in \Delta$. To resolve (2), we also assume that (2)' the root system $\Delta$ of $G$ has a special base (i.e., an existence of even/odd “simple roots” of $\Delta$), see Definition 5.14 for the detail. We note that typical examples of split quasireductive supergroups (such as $GL(m|n), Q(n), P(n)$ or Chevalley supergroups) satisfy the conditions both (1)' and (2)'. Under these natural assumptions (1)' and (2)', we establish Steinberg’s tensor product theorem for $G$ (Corollary 5.26); the result includes those by [BrKl, Ku, SW, CSW].

**Organization of this paper.** This paper is organized as follows: In Section 2 we review some basic definitions and results for Hopf superalgebras and algebraic supergroups defined over a field. The Lie superalgebra $\text{Lie}(G)$ and the superhyperalgebra $\text{hy}(G)$ of an algebraic supergroup $G$ are reviewed in Section 2.3.

In Section 3 we define the notion of split quasireductive supergroups which is the main object of study in this paper. Since the even $G_{\text{ev}}$ part of a split quasireductive supergroup $G$ is a connected and split reductive group (scheme) by definition, we fix a split maximal torus $T$ of $G_{\text{ev}}$. Thus, inside of the character group $X(T)$ of $T$, we can define the root system $\Delta$ of $G$ with respect to $T$ (Section 3.2) which also has a parity $\Delta = \Delta_0 \cup \Delta_1$. Over a perfect field, we study structures of Frobenius kernels $G_r$ of $G$ in Section 3.4. In particular, we describe a basis of the Hopf superalgebra $O(G_r)$ (see (3.2)) and establish the PBW theorem for the super-hyperalgebra $\text{hy}(G_r)$ of $G_r$ (Theorem 3.11).

In Section 4 we discuss the unimodularity of Frobenius kernels of a split quasireductive supergroup over a perfect field. First, we review basic definitions and results for left/right (co)integrals of Hopf superalgebras (Section 4.1). A left (resp. right) integral for an algebraic supergroup $G$ is defined to be a left (resp. right) cointegral on the corresponding Hopf superalgebra $O(G)$. We say that $G$ is unimodular if there exists a two-sided (i.e., left and right) integral for $G$. In [MSS], it has been shown that $G$ has a left (resp. right) integral if and only if its even part $G_{\text{ev}}$ does. Thus, by Sullivan’s result [Su], if the characteristic of the base field is zero, then it follows that algebraic supergroup $G$ has a left (or right) integral if and only if $G$ is quasireductive. Over a field of characteristic zero, we give a necessary and sufficient condition for a split quasireductive supergroup $G$ to be unimodular in terms of its root system $\Delta$ (Theorem 4.5 and Corollary 4.6). It is known that being unimodular is equivalent to that the distinguished group-like element is trivial (cf [Rad, Chapter 10]). In Section 4.3 we investigate properties of the distinguished group-like element of a finite normal super-subgroup of an algebraic supergroup, in general. In Section 4.4 we study unimodularity of Frobenius kernels $G_r$ of a split quasireductive supergroup $G$ defined over a perfect field. Note that, $G_r$ always has an integral, since $G_r$ is finite (i.e., $O(G_r)$ is finite-dimensional). Using the result [ZM, Proposition 6.11] by Zubkov and Marko, we get an explicit description of the distinguished group-like element of $G_r$, and hence we give a necessary and sufficient condition for all $G_r$ to be unimodular in terms of $\Delta$ (Theorem 4.15 and Corollary 4.16).
In Section 5, we establish Steinberg’s tensor product theorem for a split quasireductive supergroup \( \mathcal{G} \) under some natural assumptions. In Section 5.1, we review construction of simple \( \mathcal{G} \)-supermodules \( L(\lambda) (\lambda \in X(T)^p) \) given in [Sh1]. In the super-situation, not all of simple \( \mathcal{G} \)-supermodules are absolutely simple (see Example 5.7). We show that if \( \triangle \) does not contain the unit \( \mathbf{0} \) of \( X(T) \), then all simple \( \mathcal{G} \)-supermodules are absolutely simple (Proposition 5.9). In Section 5.2, we construct simple \( \mathcal{G}_r \)-supermodules \( L_r(\lambda) (\lambda \in X(T)) \) and show that \( L_r(\lambda) \) coincides with the \( \mathcal{G}_r \)-top of the “highest weight module” \( M_r(\lambda) \) of weight \( \lambda \) (Proposition 5.13). In Section 5.3, since the root system \( \triangle \) of \( \mathcal{G} \) is somewhat ill-behaved, we introduce the notion of a special base of \( \triangle \) (see Definition 5.14). We see that typical examples of split quasireductive supergroups have bases of its root systems (Example 5.15). The rest of Section 5.3, we assume that \( \triangle \) has a special base. The set of all \( p^r \)-restricted weights for \( \mathcal{G} \) is denoted by \( X_r(T)^p \), where \( p \) is the characteristic of the base field (see Definition 5.18). Then we show that the simple \( \mathcal{G} \)-supermodule \( L(\lambda) \) of highest weight \( \lambda \in X_r(T)^p \) coincides with \( \hy(\mathcal{G}_r) \rightarrow L(\lambda)^h \), where \( L(\lambda)^h \) is the \( \lambda \)-weight space of \( L(\lambda) \) (Lemma 5.20). Because of the existence of a non absolutely simple \( \mathcal{G} \)-supermodule, in Section 5.4 we assume that the base field is algebraically closed if \( \mathbf{0} \in \triangle \). This assumption is essentially needed to prove Proposition 5.23 which states \( L(\lambda) \) is isomorphic to \( L_r(\lambda) \) as \( \mathcal{G}_r \)-supermodules (see Remark 5.24).

Using Proposition 5.23 as in the non super-situation, we can establish Steinberg’s tensor product theorem for \( \mathcal{G} \) (Theorem 5.25 and Corollary 5.26).

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2. Preliminaries

Throughout this paper, \( \kappa \) denotes a fixed base field of characteristic different from 2. The unadorned \( \otimes \) is the tensor product over \( \kappa \). In this section, we fix notations and collect some known results for Hopf superalgebras and supergroups.

2.1. Hopf superalgebras. Let \( \mathbb{Z}_2 = \{ \mathbf{0}, \mathbf{1} \} \) be the additive group of order two. The group algebra \( \kappa \mathbb{Z}_2 \) of \( \mathbb{Z}_2 \) over \( \kappa \) has a unique Hopf algebra structure and a right \( \kappa \mathbb{Z}_2 \)-comodule is naturally regarded as \( \mathbb{Z}_2 \)-graded vector space. The category \( \mathcal{C} \) of right \( \kappa \mathbb{Z}_2 \)-comodules forms a monoidal category by the tensor product \( \otimes \) over \( \kappa \). Namely, the unit object is \( \kappa = \kappa \otimes \mathbf{0} \) and \( (V \otimes W)_\varepsilon = \bigoplus_{a,b \in \mathbb{Z}_2, a+b=\varepsilon} V_a \otimes W_b \) (\( \varepsilon \in \mathbb{Z}_2 \)) for right \( \kappa \mathbb{Z}_2 \)-comodules \( V \) and \( W \). For a homogeneous element \( 0 \neq v \in V_0 \cup V_1 \) of \( V \in \mathcal{C} \), we let \( |v| \) denote the degree of \( v \), called the parity of \( v \). We say that \( V \) is purely even if \( V = V_0 \). For simplicity, when we use the symbol \( |v| \), we always assume that \( v \) is homogeneous. The following supersymmetry ensures that the category \( \mathcal{C} \) is symmetric:

\[ e_{V,W} : V \otimes W \rightarrow W \otimes V; \quad v \otimes w \mapsto (-1)^{|v||w|} w \otimes v. \]

An object of \( \mathcal{C} \) is called a superspace.

For a superspace \( V \), we define \( IV \in \mathcal{C} \) by letting \( (IV)_\varepsilon = V_{\varepsilon+1} \) for \( \varepsilon \in \mathbb{Z}_2 \). For simplicity, we put \( I^0V := V \) and \( I^1V := IV \). We denote by \( \Hom_\kappa(V,W) \) the set of all parity preserving morphism from \( V \) to \( W \) in \( \mathcal{C} \). We define a superspace \( \Hom_\kappa(V,W) \) by \( \Hom_\kappa(V,W)_\varepsilon := \Hom_\kappa(IV,V,W) \) for \( \varepsilon \in \mathbb{Z}_2 \). As usual, we set
For positive integers $m$ and $n$, the set of all matrices of size $m \times n$ whose entries are in $k$ is denoted by $\text{Mat}_{m,n}(k)$. Then we can regard $\text{Mat}_{m,n}(k) := \text{Mat}_{m+n,m+n}(k)$ as a superspace by letting

$$
\begin{align*}
\text{Mat}_{m,n}(k) &:= \left\{ \begin{pmatrix} x_{00} & O \\ O & x_{11} \end{pmatrix} \mid x_{00} \in \text{Mat}_{m,m}(k), x_{11} \in \text{Mat}_{n,n}(k) \right\}, \\
\text{Mat}_{m,n}(k) &:= \left\{ \begin{pmatrix} O & x_{01} \\ x_{10} & O \end{pmatrix} \mid x_{01} \in \text{Mat}_{m,n}(k), x_{10} \in \text{Mat}_{n,m}(k) \right\}.
\end{align*}
$$

The usual matrix multiplication makes $\text{Mat}_{m,n}(k)$ into a superalgebra. For a finite dimensional superspace $V$, we can identify $\text{End}_k(V)$ (resp. $\text{End}_k(V)$) with $\text{Mat}_{m,n}(k)$ (resp. $\text{Mat}_{m,n}(k)$), where $m = \dim(V_0)$ and $n = \dim(V_1)$.

For a supercoalgebra $C = (C, \Delta_C, \varepsilon_C)$, the Heyneman-Sweedler notation, such as $\Delta_C(c) = \sum_c c(1) \otimes c(2)$ and

$$
\sum_c \Delta_C(c(1)) \otimes c(2) = \sum_c c(1) \otimes c(2) \otimes c(3) = \sum_c c(1) \otimes \Delta_C(c(2))
$$

is used to express the comultiplication $\Delta_C : C \to C \otimes C$ of $c \in C$. Note that, $\varepsilon_C(c) = 0$ for all $c \in C_1$. For a right $C$-supermodule $V$ with the structure map $\rho_V : V \to V \otimes C$, we also use the Heyneman-Sweedler notation to express the coaction, such as $\rho_V(v) = \sum v_{(0)} \otimes v_{(1)}$ for $v \in V$. For right $C$-supercomodules $V$ and $W$, the set of all parity preserving left $C$-supermodule map form $V$ to $W$ is denoted by $\text{Hom}^C(V,W)$, and define a superspace $\text{Hom}^C(V,W) : \text{Hom}^C(V,W) \to \text{Hom}^C(V,W)$ for $c \in \mathbb{Z}_2$.

Let $H$ be a Hopf superalgebra. By definition, we get

$$
\Delta_H(ab) = \sum_{a,b} (-1)^{|a(2)|\cdot|b(1)|} a(1) \otimes a(2) b(2) \otimes b(2)
$$

for $a, b \in H$, where $\Delta_H$ is the comultiplication of $H$. In this paper, the antipode of $H$ is denoted by $S_H : H \to H$.

Example 2.2. For a vector space $V$, the exterior algebra $H = \wedge(V)$ of $V$ over $k$ naturally becomes a commutative superalgebra. Moreover, $H$ forms a Hopf superalgebra by letting: $\Delta_H(v) = v \otimes 1 + 1 \otimes v$, $\varepsilon_H(v) = 0$ and $S_H(v) = -v$ for $v \in V$. Note that, $H$ is cocommutative.

As in the non super-situation, if a Hopf superalgebra $H$ is commutative or cocommutative, then the antipode $S_H : H \to H$ of $H$ satisfies $S_H^2 = id_H$. In particular, $S_H$ is bijective.

Definition 2.3. Let $H$ be a Hopf superalgebra. A non-zero element $g$ of $H$ is called a group-like elements of $H$ if it satisfies $g \in H_0$ and $\Delta_H(g) = g \otimes g$. The set of all group-like elements of $H$ is denoted by $g.l.(H)$.

For $g, h \in g.l.(H)$, we see that $gh \in g.l.(H)$, $\varepsilon_H(g) = 1$ and $gS_H(g) = 1_H = S_H(g)g$ (in particular, $g^{-1} = S_H(g) \in g.l.(H)$), where $\varepsilon_H$ (resp. $1_H$) is the counit (resp. unit element) of $H$. Thus, $g.l.(H)$ forms an abstract group.
2.2. Algebraic supergroups. An affine supergroup scheme (supergroup, for short) over \( k \) is a representable functor \( \mathcal{G} \) from the category of commutative superalgebras to the category of groups. By Yoneda lemma, the representing object \( \mathcal{O}(\mathcal{G}) \) of \( \mathcal{G} \) forms a commutative Hopf superalgebra. A supergroup \( \mathcal{G} \) is said to be algebraic (resp. finite) if \( \mathcal{O}(\mathcal{G}) \) is finitely generated as a superalgebra (resp. finite-dimensional).

For a supergroup \( \mathcal{G} \), we define its even part \( \mathcal{G}_{ev} \) as the restricted functor of \( \mathcal{G} \) from the category of commutative superalgebras to the category of groups. If we set \( A := \mathcal{O}(\mathcal{G}) \), then \( \mathcal{G}_{ev} \) is an (ordinary) affine group scheme represented by the quotient Hopf algebra \( \overline{A} := A/(A_1) \), where \( (A_1) \) is the super-ideal of \( A \) generated by the odd part \( A_1 \) of \( A \). We denote \( \overline{\pi} \) the image of \( a \in A \) by the canonical quotient map \( A \to \overline{A} \). If \( \mathcal{G} \) is algebraic, then so is \( \mathcal{G}_{ev} \). An algebraic supergroup is said to be connected if its even part is connected, see [Ma2, Definition 8].

**Example 2.4.** We list some basic example of algebraic supergroups. In the following, \( R \) denotes a commutative superalgebra.

1. For positive integers \( m \) and \( n \), we define the supergroup \( \mathcal{G}L(m|n) \), called the general linear supergroup, by

\[
\mathcal{G}L(m|n)(R) := \left\{ \begin{pmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{pmatrix} \middle| g_{00} \in \text{GL}_m(R_0), g_{01} \in \text{Mat}_{m,n}(R_1), g_{10} \in \text{Mat}_{n,m}(R_1), g_{11} \in \text{GL}_n(R_0) \right\},
\]

where \( \text{GL}_m \) (resp. \( \text{Mat}_{m,n}(R_1) \)) denotes the general linear group scheme of size \( m \) (resp. the set of all \( m \times n \) matrices whose entries are in \( R_1 \)). It is known that \( \mathcal{G}L(m|n) \) is algebraic and its even part \( \mathcal{G}L(m|n)_{ev} \) is isomorphic to \( \text{GL}_m \times \text{GL}_n \), see [BrKj], [MZ], [Z1] for example.

2. For a positive integer \( n \), the following \( Q(n) \), called the queer supergroup, is a closed super-subgroup supergroup of \( \mathcal{G}L(n|n) \).

\[
Q(n)(R) := \left\{ \begin{pmatrix} g_{00} & g_{01} \\ -g_{01} & -g_{00} \end{pmatrix} \middle| g_{00} \in \text{GL}_n(R_0), g_{01} \in \text{Mat}_{n,n}(R_1) \right\}.
\]

The even part \( Q(n)_{ev} \) of \( Q(n) \) is isomorphic to \( GL_n \). In [Br], [BrKl], modular representation theory of this supergroup is well-studied.

3. Let \( \wedge(z) \) denote the exterior superalgebra of a one-dimensional vector space \( kz \) (see Example 2.2). The corresponding algebraic supergroup of \( \wedge(z) \) is denoted by \( G^-_\wedge \), called the one-dimensional odd unipotent supergroup, see [GZ], [MZ]. By definition, we have \( G^-_\wedge(R) = R_1 \).

Let \( \mathcal{G} \) be a supergroup with representing object \( \mathcal{O}(\mathcal{G}) \). By a left \( \mathcal{G} \)-supermodule we mean a right \( \mathcal{O}(\mathcal{G}) \)-supermodule. A homomorphism of left \( \mathcal{G} \)-supermodules is just a right \( \mathcal{O}(\mathcal{G}) \)-supermodule map. For left \( \mathcal{G} \)-supermodules \( V \) and \( W \), we set \( \mathcal{G} \text{Hom}(V, W) := \text{Hom}_{\mathcal{O}(\mathcal{G})}(V, W) \) and \( \mathcal{G} \text{Hom}(V, W) := \text{Hom}_{\mathcal{O}(\mathcal{G})}(V, W) \).

A non-zero left \( \mathcal{G} \)-supermodule \( L \) is said to be simple if \( L \) has no non-trivial \( \mathcal{O}(\mathcal{G}) \)-super-submodule. The parity change \( \Pi \) acts on the set of isomorphism classes of simple left \( \mathcal{G} \)-supermodules \( \text{Simple}(\mathcal{G}) \) as a permutation of order two. We let \( \text{Simple}_{\Pi}(\mathcal{G}) \) denote the set of \( \Pi \)-orbits in \( \text{Simple}(\mathcal{G}) \).

2.3. Lie superalgebras and super-hyperalgebras. Let \( \mathcal{G} \) be an algebraic supergroup. Set \( m_{\mathcal{G}} := \ker(\varepsilon_{\mathcal{O}(\mathcal{G})}) \), called the augmentation super-ideal of \( \mathcal{O}(\mathcal{G}) \), where \( \varepsilon_{\mathcal{O}(\mathcal{G})} : \mathcal{O}(\mathcal{G}) \to k \) is the counit of \( \mathcal{O}(\mathcal{G}) \). Set \( \text{Lie}(\mathcal{G}) := (m_{\mathcal{G}}/m_{\mathcal{G}}^2)^\ast \). This
naturally forms a Lie superalgebra (see [MS1 Proposition 4.2]), which we call the Lie superalgebra of $G$. Since $G$ is algebraic, $\text{Lie}(G)$ is finite-dimensional. The even part $\text{Lie}(G)_0$ of $\text{Lie}(G)$ can be identified with the (ordinary) Lie algebra $\text{Lie}(G_{ev})$ of $G_{ev}$.

**Example 2.5.** First, note that, $\text{Mat}_{m|n}(k)$ forms a Lie superalgebra with Lie superbracket $[X,Y] = XY - (-1)^{|X||Y|}YX$ for $X, Y \in \text{Mat}_{m|n}(k)$.

1. The Lie superalgebra of the general linear supergroup $GL(m|n)$ is isomorphic to $\text{gl}(m|n) := \text{Mat}_{m|n}(k)$.

2. As a Lie super-subalgebra of $\text{gl}(n|n)$, the Lie superalgebra of the queer supergroup $Q(n)$ is isomorphic to

$$q(n) := \left\{ \begin{pmatrix} x_{00} & x_{01} \\ x_{01} & x_{00} \end{pmatrix} \right| x_{00}, x_{01} \in \text{Mat}_{n,n}(k) \right\}.$$ 

This Lie superalgebra $q(n)$ is the so-called queer superalgebra. ■

For any positive integer $n$, we regard $(O(G)/m^0_G)^\ast$ as a super-subspace of $O(G)^\ast$ through the dual of the canonical quotient map $O(G) \to O(G)/m^0_G$. As a subsuperspace of $O(G)^\ast$, we set

$$\text{hy}(G) := \lim_{n \to 1} (O(G)/m^0_G)^\ast.$$ 

This $\text{hy}(G)$ forms a super-subalgebra of $O(G)^\ast$. We call it the super-hyperalgebra of $G$ (it is sometimes called the super-distribution algebra $\text{Dist}(G)$ of $G$). By definition, we see that $\text{hy}(G) = O(G)^\ast$ if $G$ is finite. Since $O(G)/m^0_G$ is finite-dimensional for any positive integer $n$, one sees that $\text{hy}(G)$ has a structure of a cocommutative Hopf superalgebra such that the restriction

$$\langle \ , \ \rangle : \text{hy}(G) \times O(G) \to k$$

of the canonical pairing $O(G)^\ast \times O(G) \to k$ is a Hopf pairing, see [MS1 Lemma 5.1]. If $G$ is connected, then the pairing induces an injection $O(G) \hookrightarrow \text{hy}(G)^\ast$. In particular, the unit element of $\text{hy}(G)$ is given by the restriction of the counit $\varepsilon_{O(G)} : O(G) \to k$ of $O(G)$.

Masuoka showed the following $\otimes$-split type theorem for $O(G)$ and $\text{hy}(G)$, see [Ma1 Theorem 4.5] for detail (see also [Ma2 Proposition 22]).

**Theorem 2.6.** For an algebraic supergroup $G$, there exists a counit (resp. unit) preserving isomorphism

$$O(G) \cong O(G_{ev}) \otimes \bigwedge (\text{Lie}(G)^\ast) \quad (\text{resp. } \text{hy}(G) \cong \text{hy}(G_{ev}) \otimes \bigwedge (\text{Lie}(G)^\ast))$$

of (left $O(G_{ev})$-comodule) superalgebras (resp. (left $\text{hy}(G_{ev})$)-module) supercoalgebras.

In the following, let $S : \text{hy}(G) \to \text{hy}(G)$ denote the antipode of $\text{hy}(G)$ for simplicity. Note that, $S$ is the restriction of the dual $S_{O(G)}$ of the antipode $S_{O(G)} : O(G) \to O(G)$ of $O(G)$. We set

$$[u, w] := \sum_{u,w} (-1)^{|u(2)||w(1)|} u(1)w(1)S(u(2))S(w(2)), \quad u, w \in \text{hy}(G),$$

called the super-bracket of $\text{hy}(G)$. An element $X \in \text{hy}(G)$ is said to be primitive if the comultiplication of $X$ is given by $X \otimes 1 + 1 \otimes X$, where $1$ denotes the unit.
element of $\text{hy}(\mathcal{G})$. For primitive elements $X, Y$ of $\text{hy}(\mathcal{G})$, we have $[X, Y] = XY - (-1)^{|X||Y|}YX$. If we regard $\text{Lie}(\mathcal{G})$ as a super-subspace of $\text{hy}(\mathcal{G})$, then this shows that $\text{Lie}(\mathcal{G})$ coincides with the set of all primitive elements in $\text{hy}(\mathcal{G})$.

For a left $\mathcal{G}$-supermodule $V$, we regard $V$ as a left $\text{hy}(\mathcal{G})$-supermodule by letting

$$(2.2) \quad u \mapsto v := \sum_v (-1)^{|v(0)||u|} v(0) \langle u, v(1) \rangle,$$

where $u \in \text{hy}(\mathcal{G})$, $v \in V$. Suppose that $V$ is finite-dimensional. Then the dual superspace $V^*$ of $V$ forms a right $\mathcal{O}(\mathcal{G})$-supercomodule by using the antipode of $\mathcal{O}(\mathcal{G})$. The induced left $\text{hy}(\mathcal{G})$-supermodule structure on $V^*$ satisfies the following equation.

$$(2.3) \quad (u \cdot f)(v) = (-1)^{|u||f|} f(\mathcal{S}(u) \mapsto v),$$

where $v \in V$, $f \in V^*$ and $u \in \text{hy}(\mathcal{G})$.

2.4. Normal super-subgroups. Let $\mathcal{G}$ be an algebraic supergroup, in general. Set $A := \mathcal{O}(\mathcal{G})$. A subfunctor $\mathcal{N}$ of $\mathcal{G}$ is called a closed super-subgroup if $\mathcal{N}$ is affine and the corresponding Hopf superalgebra $\mathcal{O}(\mathcal{N})$ is isomorphic to the quotient $A/I$ for some Hopf super-ideal $I$ of $A$. This $\mathcal{N}$ is said to be normal if (as an abstract group) $\mathcal{N}(R)$ is a normal subgroup of $\mathcal{G}(R)$ for all commutative superalgebra $R$. The condition is equivalent to saying that the canonical quotient map $A \rightarrow A/I$ is conormal (see [Ma1, Definition 5.7]), that is, $\text{coad}_A(I) \subset A \otimes I$. Here, $\text{coad}_A$ denotes the (left) coadjoint coaction on $A$ given by

$$\text{coad}_A : A \rightarrow A \otimes A; \quad a \mapsto \sum_a (-1)^{|a(2)||a(3)|} a(1)\mathcal{S}(a(3)) \otimes a(2),$$

where $\mathcal{S}$ is the antipode of $A$. By definition, the even part $\mathcal{N}_{ev}$ of a normal super-subgroup $\mathcal{N}$ of $\mathcal{G}$ is a normal subgroup of $\mathcal{G}_{ev}$.

As the dual notion of (left) coadjoint coaction on $A$, we define

$$(2.4) \quad u \triangleright w := \sum_u (-1)^{|w||u(2)|} u(1) w \mathcal{S}(u(2)), \quad u, w \in \text{hy}(\mathcal{G}),$$

called the (left) adjoint action on $\text{hy}(\mathcal{G})$, where $\mathcal{S}$ is the antipode of $\text{hy}(\mathcal{G})$. Note that, the super-bracket (2.1) can be rewritten as $[u, w] = \sum_w (u \triangleright w) \mathcal{S}(w)$. 

Lemma 2.7. For $u, u', w, w' \in \text{hy}(\mathcal{G})$, we have (1) $(uu') \triangleright w = u \triangleright (u' \triangleright w)$, (2) $1 \triangleright w = w$, (3) $u \triangleright (ww') = \sum_u (-1)^{|w||u(2)|} u(1) \triangleright w(u(2) \triangleright w')$ and (4) $u \triangleright 1 = \varepsilon(u) = u(1)$. Here, $\varepsilon$ (resp. 1) denotes the counit (resp. unit) of $\text{hy}(\mathcal{G})$.

Proof. It is straightforward to check (1), (2) and (4). The following direct computation shows (3):

$$\sum_u (-1)^{|u(2)||w|} u(1) \triangleright w(u(2) \triangleright w')$$

$$= \sum_u (-1)^{|u(4)||w|+|u(2)||w|+|u(4)|} u(1) w \mathcal{S}(u(2)) u(3) w' \mathcal{S}(u(4))$$

$$= \sum_u (-1)^{|u(2)||w|+|u(2)||w'|} u(1) w w' \mathcal{S}(u(2)) = u \triangleright (ww').$$

The second equation follows from the fact that $\varepsilon(x) = 0$ for $|x| = 1$ in general. \hfill $\square$
Let $\mathcal{N}$ be a normal super-subgroup of $\mathcal{G}$. By definition, the left coadjoint coaction on $\mathcal{O}(\mathcal{G})$ induces a left $\mathcal{O}(\mathcal{G})$-supermodule structure on $\mathfrak{m}_N/\mathfrak{m}_N^2$, where $\mathfrak{m}_N$ is the augmentation super-ideal of $\mathcal{O}(\mathcal{N})$. Since $\mathfrak{m}_N/\mathfrak{m}_N^2$ is finite-dimensional, its linear dual $\mathfrak{g}(\mathcal{N})$ has a left $\mathcal{G}$-supermodule structure. Thus by \cite{Ma2, Proposition 5.5(2)}, we get a left $\mathfrak{g}(\mathcal{G})$-supermodule structure on $\mathfrak{g}(\mathcal{G})$.

We regard $\mathfrak{g}(\mathcal{N})$ as a super-subspace of $\mathfrak{g}(\mathcal{G})$ by the inclusion $\mathfrak{g}(\mathcal{N}) \subseteq \mathfrak{g}(\mathcal{G})$. Then one sees that the action of $\mathfrak{g}(\mathcal{N})$ on $\mathfrak{g}(\mathcal{G})$ defined above is given by the adjoint action $\triangleright$, see \cite{Ma2, Proposition 5.5(2)}. In particular, by restricting the action of $\mathfrak{g}(\mathcal{G})$ to $\mathfrak{g}(\mathcal{N})$, we see that $\mathfrak{g}(\mathcal{N})$ is a Lie super-ideal of $\mathfrak{g}(\mathcal{G})$, that is, $\{X, N\} (= X \triangleright N) \in \mathfrak{g}(\mathcal{N})$ for all $X \in \mathfrak{g}(\mathcal{G})$ and $N \in \mathfrak{g}(\mathcal{N})$.

A Hopf super-subalgebra $H$ of $\mathfrak{g}(\mathcal{G})$ is said to be normal (see \cite[Theorem 3.10]{Ma1}) if $H$ is $\mathfrak{g}(\mathcal{G})$-stable under the adjoint action $\triangleright$, that is, $u \triangleright h \in H$ for all $u \in \mathfrak{g}(\mathcal{G})$ and $h \in H$.

**Proposition 2.8.** If $\mathcal{N}$ is a normal super-subgroup of $\mathcal{G}$, then $\mathfrak{g}(\mathcal{N}) \subseteq \mathfrak{g}(\mathcal{G})$ is normal. In particular, $\mathfrak{g}(\mathcal{N})$ is closed under super-bracket of $\mathfrak{g}(\mathcal{G})$, that is, $[u, x] \in \mathfrak{g}(\mathcal{N})$ for all $u \in \mathfrak{g}(\mathcal{G})$ and $x \in \mathfrak{g}(\mathcal{N})$.

**Proof.** By \cite[Proposition 5.5(2)\cite{Ma2}], $\mathfrak{g}(\mathcal{N})$ is normal if and only if the following four conditions are satisfied: (i) $\mathfrak{g}(\mathcal{N})_1 \subseteq \mathfrak{g}(\mathcal{G})_1$ is normal, (ii) $\mathfrak{g}(\mathcal{N})_1$ is $\mathfrak{g}(\mathcal{G})$-stable under the adjoint action $\triangleright$, (iii) $[\mathfrak{g}(\mathcal{N})_1, \mathfrak{g}(\mathcal{G})_1] \subseteq \mathfrak{g}(\mathcal{N})_1$ and (iv) $X \triangleleft u - \varepsilon(u)X \in \mathfrak{g}(\mathcal{N})_1$ for all $X \in \mathfrak{g}(\mathcal{G})_1$, $u \in \mathfrak{g}(\mathcal{N})_1$, where $X \triangleleft u := \sum_u S(u_{(1)})UX_{(2)}$ is the right adjoint action of $\mathfrak{g}(\mathcal{N})_1$ on $\mathfrak{g}(\mathcal{G})_1$.

Since $\mathfrak{g}(\mathcal{N})_1$ is a normal subgroup of $\mathfrak{g}(\mathcal{G})_1$, the condition (i) is clear by \cite[Corollary 3.4.15]{Ma1}. By the construction, $\mathfrak{g}(\mathcal{N})_1$ is $\mathfrak{g}(\mathcal{G})$-stable, and hence the condition (ii) follows. Since $\mathfrak{g}(\mathcal{N})$ is a Lie super-ideal of $\mathfrak{g}(\mathcal{G})$, the condition (iii) is trivial. Note that, in our case, the value of the counit $\varepsilon(u)$ is zero unless $u \in k$ = \{ $c \in \mathfrak{g}(\mathcal{G})$ | $c \in k$ \}. Thus, to show the condition (iv), it is enough to show that $X \triangleleft u \in \mathfrak{g}(\mathcal{N})_1$ for all $X \in \mathfrak{g}(\mathcal{G})_1$ and $u \in \mathfrak{g}(\mathcal{N})_1$. Since $\mathfrak{g}(\mathcal{N})_1$ is cocommutative, we have $S^2 = id$ and

$$X \triangleleft S(u) = \sum_u u_{(2)}X S(u_{(1)}) = \sum_u u_{(1)}XS(u_{(2)}) = u \triangleright X.$$ 

On the other hand, by the construction, $\mathfrak{g}(\mathcal{G})_1$ is $\mathfrak{g}(\mathcal{G})$-stable, and hence $\mathfrak{g}(\mathcal{N})_1$-stable. In particular, $\mathfrak{g}(\mathcal{G})_1$ is $\mathfrak{g}(\mathcal{N})_1$-stable under the adjoint action $\triangleright$. Thus, the condition (iv) easily follows from the above formula. \hfill $\Box$

### 2.5. Characters

Let $G_m := GL_1$ denote the one dimensional multiplicative group (scheme). A character of a supergroup $\mathcal{G}$ is a group homomorphism from $\mathcal{G}$ to $G_m$. The set of all characters

$$X(\mathcal{G}) := \text{Hom}(\mathcal{G}, G_m)$$

of $\mathcal{G}$, called the character group of $\mathcal{G}$, naturally forms an abstract group. For $X \in X(\mathcal{G})$, we have a group homomorphism $\chi : \mathcal{O}(\mathcal{G}) \to G_m(\mathcal{O}(\mathcal{G}))$, and hence we have a Hopf algebra homomorphism $\chi(\text{id}_{\mathcal{O}(\mathcal{G})}) : \mathcal{O}(G_m) \to \mathcal{O}(\mathcal{G})$ by the Yoneda lemma. If we realize $\mathcal{O}(G_m)$ as the Laurent polynomial algebra $k[X^{\pm 1}]$ in the variable $X$ with coefficients in $k$, then it is easy to see that $\chi(\text{id}_{\mathcal{O}(\mathcal{G})})X \in \mathcal{O}(\mathcal{G})$ is a group-like element. In this way, we have an isomorphism $X(\mathcal{G}) \cong g.l.(\mathcal{O}(\mathcal{G}))$ of abstract groups.

For each $X \in X(\mathcal{G})$, we get the one-dimensional left $\mathcal{G}$-supermodule $k^X$ so that $k^X = k$ as a purely even superspace and the right $\mathcal{O}(\mathcal{G})$-supermodule structure...
is given by
\[ k^\chi \to k^\chi \otimes \mathcal{O}(G); \quad v \mapsto \chi \cdot v. \]
In other words, \( g.v = \chi(g)v \) for all commutative superalgebra \( R \) and \( g \in G(R), v \in k^\chi \). If there is no confusion, we sometimes simply denote \( k^\chi \) by \( \chi \). In this way, we get a one-to-one correspondence between \( X(G) \cong g.l.(\mathcal{O}(G)) \) and the set of all equivalence classes of one-dimensional (simple) left \( G \)-supermodules under the parity change \( \Pi \).

Lemma 2.9. The map \( X(G) \to X(G_{ev}); \) \( \chi \mapsto \chi|_{\mathfrak{a}_{ev}} \) is injective, where \( \chi|_{\mathfrak{a}_{ev}} \) denotes the restriction of \( \chi \) : \( G \to G_{m} \) to \( G_{ev} \).

Proof. Set \( A := \mathcal{O}(G), \Gamma := g.l.(A) \) and \( \Gamma' := g.l.(\overline{A}) \). Recall that, \( \overline{A} = A/(A_1) = O(G_{ev}) \). Then the group algebra \( k\Gamma \) is a Hopf sub-superalgebra of \( A \). By [Ma1 Proposition 4.6(3)], the inclusion \( k\Gamma \subset A \) induces an injection \( k\Gamma \to \overline{A} \). On the other hand, the quotient map \( A \to \overline{A} \) induces a Hopf algebra homomorphism \( k\Gamma \to k\Gamma' \). Since \( k\Gamma = k\Gamma' \), we see that \( k\Gamma \to k\Gamma' \); \( a \mapsto \overline{a} \) is injective. This proves the claim. \( \square \)

Example 2.10. We consider the case \( G = GL(m|n) \). Recall that \( G_{ev} = GL_{m} \times GL_{n} \). Let \( R \) be a fixed superalgebra. For
\[ g = \begin{pmatrix} g_{00} & g_{01} \\ g_{10} & g_{11} \end{pmatrix} \in GL(m|n)(R), \]
we set \( \det_{0}(g) := \det(g_{00}), \det_{1}(g) := \det(g_{11}) \) and
\[ \text{Ber}(g) = \det(g_{00} - g_{01}g_{11}^{-1}g_{10})\det(g_{11})^{-1}. \]
This \( \text{Ber}(g) \) is called the Berezinian of \( g \). Then it is easy to see that \( \det_{e} \) and \( \text{Ber} \) are in \( X(G) \) for \( e \in \mathbb{Z}_{2} \). Note that, \( \text{Ber}|_{\mathfrak{a}_{ev}} = (\det_{0}|_{\mathfrak{a}_{ev}}) \cdot (\det_{1}|_{\mathfrak{a}_{ev}})^{-1} \). In [Z2 Lemma 13.5], Zubkov showed that the character group \( X(G) \) of \( G = GL(m|n) \) is generated by \( \{\text{Ber}, \det_{1}^{\bar{p}}\} \), where \( \bar{p} := \text{char}(k)(\neq 2) \). \( \blacksquare \)

3. Split Quasireductive Supergroups

3.1. Split quasireductive supergroups. Recall that, a split and connected reductive \( Z \)-group \( G_{Z} \) is a connected algebraic group (scheme) over \( Z \) having a split maximal torus \( T_{Z} \) such that the pair \( (G_{Z}, T_{Z}) \) corresponds to a root datum (cf. [SGA3]). See also [J Part II, Chapter 1] and [Mi §5.2], for example. It is known that \( O(G_{Z}) \) is free as a \( Z \)-module and \( G_{Z} \) is infinitesimally flat.

Definition 3.1 [Shi1 Definition 3.1]. An algebraic supergroup \( G_{Z} \) defined over \( Z \) is said to be split quasireductive if its even part of \( G_{Z} \) is a split and connected reductive group over \( Z \) and the odd part of \( m_{G_{Z}}/m_{G_{Z}}^{2} \) is finitely generated and free as a \( Z \)-module. Here, \( m_{G_{Z}} \) denotes the augmented ideal of \( O(G_{Z}) \).

Note that in [Shi2, Shi3], a split quasireductive supergroup is simply called a quasireductive supergroup.

In the following, we fix a split quasireductive supergroup \( G_{Z} \) over \( Z \) and a split maximal torus \( T_{Z} \) of \( (G_{Z})_{ev} \). Let \( G \) (resp. \( T \)) denote the base change of \( G_{Z} \) (resp. \( T_{Z} \)) to our base field \( k \), that is, \( O(G) := O(G_{Z}) \otimes_{Z} k \). By definition, \( G \) is connected. We can identify \( X(T) \) with \( Z' \) (\( \ell \) is the rank of \( G_{ev} \)) and we often write its group low additively with unit element \( 0 \).

Example 3.2. We list some basic examples of split quasireductive supergroups.
(1) General linear supergroups $GL(m|n)$.
(2) Queer supergroups $Q(n)$.
(3) Chevalley supergroups of classical type, see [FG]. For example, special linear supergroups $SL(m|n)$ and ortho-symplectic supergroups $SpO(m|n)$.
(4) Periplectic supergroups $P(n)$ with $n \geq 2$, see [Shi1]. For a superalgebra $R$, the supergroup is given by $P(n)(R) := \{g \in GL(n|R) | \alpha g J_n g = J_n\}$. Here, we used the following notations.

$$\begin{pmatrix} g_{00} & g_{10} \\ g_{10} & g_{11} \end{pmatrix} := \begin{pmatrix} t g_{00} \\ -g_{00} \end{pmatrix}, \quad J_n := \begin{pmatrix} O & I_n \\ I_n & O \end{pmatrix},$$

where $t g_{00}$ denotes the matrix transpose of $g_{00}$ and $I_n$ denotes the identity matrix of size $n$. One sees that $P(n)_{ev} \simeq GL_n$.

As we have seen in Section 2.3 for a left $G$-supermodule $V$, we get a left $hy(G)$-supermodule structure on $V$. It is easy to see that $V$ is locally finite and has a $T$-weight decomposition, and hence $V$ becomes a locally finite left $hy(G)$-$T$-supermodule. Here, we say that a left $hy(G)$-supermodule $V$ is left $hy(G)$-$T$-supermodule if the restricted $hy(T)$-supermodule structure on $V$ arises from some $T$-supermodule structure on it. In this way, we get a functor from the category of left $G$-supermodules to the category of locally finite left $hy(G)$-$T$-supermodules.

**Theorem 3.3** ([Mi, Theorem 5.8]). The functor discussed above gives an equivalence between the category of left $G$-supermodules and the category of locally finite left $hy(G)$-$T$-supermodules.

### 3.2. Root systems

Let $g = g_0 \oplus g_1$ be the Lie superalgebra $Lie(G)$ of $G$. As we have seen in Section 2.4 (for $N = G$), the left coadjoint action of $O(G)$ induces the adjoint action of $G$ on $g$. Restricting the action to $T$, the Lie superalgebra $g$ forms a left $T$-supermodule. Since $T$ is a diagonalizable group scheme, $g$ decomposes into weight superspaces as follows:

$$g = \bigoplus_{\alpha \in X(T)} g^\alpha = (\bigoplus_{\alpha \in X(T)} g_0^\alpha) \oplus (\bigoplus_{\gamma \in X(T)} g_1^\gamma),$$

where $g^\alpha$ denotes the $\alpha$-weight super-subspace of $g$. By [I] Part I, 7.14, we get

$$g^\alpha = \{X \in g | u \triangleright X = \alpha(u)X \text{ for all } u \in hy(T)\},$$

where $\triangleright$ is the adjoint action (2.4). Here, we regard $X(T)$ as a subset of $O(T)$. Let $h := g^0$ be the $0$-weight super-subspace of $g$ which forms a Lie super-subalgebra of $g$. Note that, the even part $g_0$ of $g$ coincides with the Lie algebra $Lie(G_{ev})$. By definition, we see that $h_0 = Lie(T)$. For $\epsilon \in \mathbb{Z}_2$, we set $\nabla_{\epsilon} := \{\alpha \in X(T) | g_0^\alpha \neq 0\} \setminus \{0\}$ and

$$\nabla := \begin{cases} \nabla_{\epsilon} \cup \nabla_{\bar{\epsilon}} & \text{if } h_\bar{\epsilon} = 0, \\ \nabla_0 \cup \nabla_{\epsilon} \cup \{0\} & \text{otherwise.} \end{cases}$$

We call $\nabla$ the root system of $G$ with respect to $T$. Note that, $\nabla_0$ is the root system of $G$ with respect to $T$ in the usual sense. Moreover, the quadruple $(X(T), \nabla_{\epsilon}, X(T)^{\vee}, \nabla_{\bar{\epsilon}}^{\vee})$ forms a root datum of the pair $(G_{ev}, T)$, see [Mi, Appendix C]. Let $\lambda_1, \ldots, \lambda_\ell$ denote a basis of $X(T) \cong \mathbb{Z}_2^\ell = \bigoplus_{i=1}^\ell \mathbb{Z}\lambda_i$, where $\ell$ is the rank of $G_{ev}$.

**Example 3.4.** Here we list some examples of root systems.
(1) If $G = \mathfrak{GL}(m|n)$ with the standard maximal torus $T$ of $G_{ev} = \mathfrak{GL}_m \times \mathfrak{GL}_n$ (i.e., the subgroup of $G_{ev}$ consisting all diagonal matrices), then $X(T) \cong \bigoplus_{i=1}^{m+n} \mathbb{Z} \lambda_i$ and $\Delta = \Delta_0 \cup \Delta_1 = \{ \lambda_i - \lambda_j \mid 1 \leq i \neq j \leq m + n \}$ with $\Delta_0 = \{ \lambda_i - \lambda_j \mid 1 \leq i \neq j \leq m \} \cup \{ \lambda_i - \lambda_j \mid m + 1 \leq i \neq j \leq m + n \}$. 

(2) If $G = \mathfrak{Q}(n)$ with the standard maximal torus $T$ of $G_{ev} = \mathfrak{GL}_n$, then $X(T) \cong \bigoplus_{i=1}^{n} \mathbb{Z} \lambda_i$ and $\Delta = \{ \lambda_i - \lambda_j \mid 1 \leq i \neq j \leq n \} \cup \{ 0 \}$ with $\Delta_0 = \Delta_1$.

(3) If $G = \mathfrak{P}(n)$ with the standard maximal torus $T$ of $G_{ev} = \mathfrak{GL}_n$, then $X(T) \cong \bigoplus_{i=1}^{n} \mathbb{Z} \lambda_i$ and $\Delta_0 = \{ \lambda_i - \lambda_j \mid 1 \leq i \neq j \leq n \}, \Delta_1 = \{ \pm (\lambda_i + \lambda_j), 2\lambda_i \mid 1 \leq i < j \leq n, 1 \leq t \leq n \}$ with $\Delta = \Delta_0 \cup \Delta_1$.

(4) Let $(X, R, X^\vee, R^\vee)$ be a root datum, and let $F$ be a corresponding connected and split reductive group (defined over $k$) with split maximal torus $T$. Take group-like elements $g_1, \ldots , g_n \in g.l.(\mathcal{O}(F))$. By slightly modifying the algebraic supergroup $G_{a,x}$, we consider in [MZ2] Section 4, we consider the semidirect product

$$F^{(g_1, \ldots , g_n)} := F \ltimes (G_a^\vee)^n$$

such that $F^{(g_1, \ldots , g_n)}(R) = F(R) \times R^n_1$ as sets and the multiplication is

$$(f, (x_i)_{1 \leq i \leq n}).(k, (y_i)_{1 \leq i \leq n}) := (f k_i(k (g_i)x_i + y_i)_{1 \leq i \leq n})$$

for $f, k \in F(R)$ and $(x_i)_{1 \leq i \leq n}, (y_i)_{1 \leq i \leq n} \in R^n_1$, where $R$ is a commutative superalgebra. By definition, $(F^{(g_1, \ldots , g_n)})_{ev} = F$ and $F^{(g_1, \ldots , g_n)}$ forms a split quasireductive supergroup. The even part $\Delta_0$ of the root system $\Delta$ of $F^{(g_1, \ldots , g_n)}$ with respect to $T$ is just $R$. Since $g.l.(\mathcal{O}(F)) \twoheadrightarrow g.l.(\mathcal{O}(T)) = X$, we shall write $\chi_i := g_i|T$ for all $1 \leq i \leq n$. Then the odd part of $\Delta_1$ of $\Delta$ is given by $\{-\chi_1, \ldots , -\chi_n\}$. 

For each $\epsilon \in \mathbb{Z}_2$, we set $\ell_\epsilon := \text{dim}(h_\epsilon)$. Note that, $\ell_0$ coincides with the rank $\ell$ of $G_{ev}$. In [Shi1] Theorem 3.11, Poincaré-Birkhoff-Witt (PBW) theorem for $hy(G)$ has been established. It states that we can take a homogeneous basis

$$\{ X_\alpha \in \mathfrak{g}^\vee \mid \alpha \in \Delta_0 \} \cup \{ H_\ell \in \mathfrak{h}_0 \mid 1 \leq \ell \leq \ell_0 \}$$

$$\cup \{ Y_{(\gamma,j)} \in \mathfrak{g}^\vee_1 \mid \gamma \in \Delta_1, 1 \leq j \leq \text{dim}(\mathfrak{g}^\vee_1) \} \cup \{ K_\ell \in \mathfrak{h}_1 \mid 1 \leq \ell \leq \ell_1 \}$$

of $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ so that the set of all products of factors of the following type (taken in any fixed total order) forms a basis of $hy(G)$:

$$H_i^{(m_i)}, \quad X_\alpha^{(n_\alpha)}, \quad K_\ell^{(t_\ell)}, \quad Y_{(\gamma,j)}^{(e_{\gamma,j})}$$

with $n_\alpha, m_i \in \mathbb{Z}_{\geq 0}, \alpha \in \Delta_0, 1 \leq i \leq \ell_0, \gamma \in \Delta_1, 1 \leq j \leq \text{dim}(\mathfrak{g}^\vee_1), 1 \leq t \leq \ell_1$ and $\epsilon_{\ell}, e_{(\gamma,j)} \in \{ 0,1 \}$. See also Theorem 2.3. Here, we used the symbol of the "divided powers" $X_\alpha^{(n_\alpha)}$ and $H_i^{(m_i)}$ for $X_\alpha$ and $H_i$. For more detail, see [Shi1] §3.4. In the following, to simplify the notation, we write $Y_{\gamma,1} := Y_{(\gamma,1)}$ if $\text{dim}(\mathfrak{g}^\vee_1) = 1$ for $\gamma \in \Delta_1$.

One sees that $hy(G)$ is a cocommutative supercoalgebra of Birkhoff-Witt type (for the non super-situation, see [T2] Section 3.3.5). In particular, if we denote the comultiplication of $hy(G)$ by $\Delta$, then we have

$$\Delta(X_\alpha^{(n_\alpha)}) = \sum_{i+j=n} X_\alpha^{(i)} \otimes X_\alpha^{(j)} \quad \text{and} \quad X_\alpha^{(n_\alpha)} X_\alpha^{(m_\alpha)} = \binom{n + m}{n} X_\alpha^{(m+n)}$$

for $n, m \in \mathbb{N} \cup \{ 0 \}$ and $\alpha \in \Delta_0$. Here, $\binom{m+n}{n}$ denotes the binomial coefficient.
3.3. **Characters.** It is known that $\mathfrak{g}_{\text{ev}}$ is generated by the split maximal torus $T$ and the $\alpha$-root subgroups $U_\alpha$ of $\mathfrak{g}_{\text{ev}}$ for all $\alpha \in \Delta$, see [31] Theorem 21.11] for example. Since each $U_\alpha$ is isomorphic to the one-dimensional additive group (scheme) $G_\alpha$, we see that $X(U_\alpha) \cong \text{g.l.}(\mathcal{O}(G_\alpha))$ is trivial, and hence any character of $\mathfrak{g}_{\text{ev}}$ is trivial on $U_\alpha$. In particular, the map $X(\mathfrak{g}_{\text{ev}}) \to X(T); \chi \mapsto \chi|_T$ is injective.

**Remark 3.5.** More precisely, it is known (see [31 Part II, 1.18]) that $X(\mathfrak{g}_{\text{ev}}) \to X_0(T) := \{ \lambda \in X(T) \mid \langle \lambda, \alpha^\vee \rangle = 0 \text{ for all } \alpha \in \Delta \}; \chi \mapsto \chi|_T$ gives an isomorphism, where $\alpha^\vee \in X(T)^\vee = \text{Hom}(G_m, T)$ denotes the dual root corresponding to $\alpha$ and $\langle \cdot, \cdot \rangle$ denotes the perfect pairing $X(T) \times X(T)^\vee \to \mathbb{Z}$. $\blacksquare$

**Lemma 3.6.** The map $X(\mathfrak{g}) \to X(T); \chi \mapsto \chi|_T$ is injective. More precisely, $X(\mathfrak{g}) \to X_0(T); \chi \mapsto \chi|_T$ is injective.

*Proof.* By Lemma 2.9 and Remark 3.5 the claim follows immediately. $\square$

**Example 3.7.** We determine the character group $X(\mathbb{Q}(n))$ of the queer supergroup $\mathbb{Q}(n)$. One easily sees that $X_0(\mathbb{Q}) = \{ m(\lambda_1 + \cdots + \lambda_n) \mid m \in \mathbb{Z} \}$. Since $\det_3$ is a non-trivial character, this shows that $X(\mathbb{Q}(n)) = \{ \det^n_m \mid m \in \mathbb{Z} \}$ by Lemma 3.6. Note that, the Berezinian $\text{Ber}$ is trivial on $\mathbb{Q}(n)$. $\blacksquare$

3.4. **Frobenius kernels.** In this subsection, we suppose that $k$ is a perfect field of characteristic $p > 2$ and fix a positive integer $r$. Let $\mathfrak{g}$ be an algebraic supergroup over $k$, in general.

For a commutative superalgebra $R$, we define a commutative superalgebra $R^{(r)}$ so that $R^{(r)} = R$ as a super-ring and the scalar multiplication is given by $c.a = c^p a$ for all $c \in k$ and $a \in R$. We define a supergroup $\mathfrak{g}^{(r)}$ so that $\mathfrak{g}^{(r)}(R) := \mathfrak{g}(R^{(r)})$, and define a morphism $\text{Fr}^r : \mathfrak{g} \to \mathfrak{g}^{(r)}$ of supergroups, called the $r$-th Frobenius morphism, as follows:

$$\text{Fr}^r(R) : \mathfrak{g}(R) \to \mathfrak{g}^{(r)}(R); \quad g \mapsto (\mathcal{O}(\mathfrak{g}) \to R^{(r)}; \cdot \mapsto g(a^p)).$$

The kernel of the morphism $\text{Fr}^r$ is called the $r$-th Frobenius kernel of $\mathfrak{g}$ which we denote by $\mathfrak{g}_r$.

It is easy to see that $\mathfrak{g}_r$ is represented by the quotient Hopf superalgebra $\mathcal{O}(\mathfrak{g})/m^r_{\mathfrak{g}}$ of $\mathcal{O}(\mathfrak{g})$, where $m_{\mathfrak{g}}$ is the augmentation super-ideal of $\mathcal{O}(\mathfrak{g})$. Since $a^2 = 0$ for all $a \in \mathcal{O}(\mathfrak{g})_1$, we see that for a commutative superalgebra $R$ and $g \in \mathfrak{g}(R)$, the map $g' := (\text{Fr}^r(R))(g) : \mathcal{O}(\mathfrak{g}) \to R^{(r)}$ factors through the canonical quotient $\mathcal{O}(\mathfrak{g}) \to \mathcal{O}((\mathfrak{g}_{\text{ev}})_r)$, and hence we may identify $g' \in \mathfrak{g}_{\text{ev}}(R^{(r)}) = \mathfrak{g}_{\text{ev}}^{(r)}(R)$.

Thus, we may and do assume that $\text{Fr}^r : \mathfrak{g} \to \mathfrak{g}_{\text{ev}}^{(r)}$. By Theorem 2.6 for $\mathfrak{g}_r$, Masuoka [Ma3] showed the following result:

**Proposition 3.8.** We have $(\mathfrak{g}_{\text{ev}})_r = (\mathfrak{g}_r)_{\text{ev}}$ and $\text{Lie}(\mathfrak{g}_r)_1 = \text{Lie}(\mathfrak{g})_1$. In particular, there exists a count preserving isomorphism $\mathcal{O}(\mathfrak{g}_r) \cong \mathcal{O}((\mathfrak{g}_{\text{ev}})_r) \otimes \Lambda(\text{Lie}(\mathfrak{g})_1^*)$ of (left $\mathcal{O}((\mathfrak{g}_{\text{ev}})_r)$-comodule) superalgebras.

Therefore, $\text{Lie}(\mathfrak{g}_r) = \text{Lie}(\mathfrak{g})$ and $\mathfrak{g}_r$ is infinitesimal, that is, $\mathfrak{g}_r$ is finite and the augmentation super-ideal $m_{\mathfrak{g}_r}$ of $\mathcal{O}(\mathfrak{g}_r)$ is nilpotent. In particular, $\mathfrak{g}_r$ is a finite normal super-subgroup of $\mathfrak{g}$, and hence by $\mathfrak{g}(\mathfrak{g}_r) = \mathcal{O}(\mathfrak{g}_r)^*$.

Let $V$ be a left $\mathfrak{g}_{\text{ev}}$-module. We regard $V$ as a superspace by letting $V_0 = V$ and $V_1 = 0$. Using the $r$-th Frobenius morphism $\text{Fr}^r : \mathfrak{g} \to \mathfrak{g}_{\text{ev}}^{(r)}$, we may consider
forms a basis of \( O \). Of for each \( u \) is a morphism of superspaces, since

\[
\text{(3.2)} \quad M \rightarrow M \otimes O(\mathfrak{g}_{\epsilon}); \quad m \mapsto \sum_{m(0)} m_{(0)} \otimes m_{(1)}^{r-1}.
\]

By definition, we have \( (M[-r])^{|r|} = M \) as a \( \mathfrak{g} \)-supermodule.

**Example 3.9.** Let \( M \) be a left \( \mathfrak{g} \)-supermodule. For the \( \mathfrak{g}_{\epsilon} \)-fixed point supersubspace \( M_{\mathfrak{g}_{\epsilon}} \) of \( M \), we can consider \( (M_{\mathfrak{g}_{\epsilon}})^{|r|} \). We naturally regard \( M \) as a left \( \mathfrak{g}_{\epsilon} \)-supermodule via the inclusion \( \mathfrak{g}_{\epsilon} \subset \mathfrak{g} \). For a finite dimensional left \( \mathfrak{g} \)-supermodule \( M' \), we can make \( \mathfrak{g}_{\epsilon} \cdot \text{Hom}(M', M) \) into a left \( \mathfrak{g} \)-supermodule by the conjugate action. As a left \( \text{hy}(\mathfrak{g}) \)-supermodule, the induced action is given by

\[
(u.f)(v) := \sum_{u} (-1)^{|f||u_{(2)}|} u_{(1)} f(S(u_{(2)})v),
\]

where \( f \in \mathfrak{g}_{\epsilon} \cdot \text{Hom}(M', M) \), \( u \in \text{hy}(\mathfrak{g}) \) and \( v \in M' \). Here, \( S \) denotes the antipode of \( \text{hy}(\mathfrak{g}) \). Since \( M_{\mathfrak{g}_{\epsilon}} \) can be identified with \( \mathfrak{g}_{\epsilon} \cdot \text{Hom}(k, M) \), we can also consider \( \mathfrak{g}_{\epsilon} \cdot \text{Hom}(M', M)_{\bar{0}} \). Note that, the “evaluation map”

\[
\varphi : \mathfrak{g}_{\epsilon} \cdot \text{Hom}(M', M) \otimes M' \rightarrow M; \quad f \otimes v \mapsto f(v)
\]

is a morphism of superspaces, since \( \mathfrak{g}_{\epsilon} \cdot \text{Hom}(M', M) = \mathfrak{g}_{\epsilon} \cdot \text{Hom}(M', M)_{\bar{0}} \) consists of parity preserving morphisms. Moreover, we get

\[
\varphi(u.(f \otimes v)) = \sum_{u} (u_{(1)}.f)(u_{(2)}v) = \sum_{u} u_{(1)} f(S(u_{(2)})u_{(3)}v) = u \varphi(f \otimes v)
\]

for each \( u \in \text{hy}(\mathfrak{g}) \), \( f \in \mathfrak{g}_{\epsilon} \cdot \text{Hom}(M', M) \) and \( v \in M' \). This shows that \( \varphi \) is actually a \( \mathfrak{g} \)-supermodule homomorphism. \( \blacksquare \)

Again, we suppose that \( \mathfrak{g} \) is split quasi-reductive and set \( \mathfrak{g} := \text{Lie}(\mathfrak{g}) \). Let \( r \) be a fixed positive integer. Set \( n_{\epsilon} := \dim(\mathfrak{g}_{\epsilon}) \) for \( \epsilon \in \mathbb{Z}_{2} \). Suppose that \( f_{1}, \ldots, f_{n_{\epsilon}} \in \mathfrak{m}_{\mathfrak{g}} \) forms a basis of \( \mathfrak{g}_{\bar{0}} = \text{Lie}(\mathfrak{g}_{\bar{0}}) \) and \( f_{n_{\epsilon}+1}, \ldots, f_{n_{\epsilon}+n_{1}} \in \mathfrak{m}_{\mathfrak{g}} \) forms a basis of \( \mathfrak{g}_{1} \). Since \( \mathfrak{g}_{\bar{0}} \) is reduced, the set \( \{ f_{1}^{a_{1}} \cdots f_{n_{\epsilon}}^{a_{n_{\epsilon}}} \mid 0 \leq a_{1}, \ldots, a_{n_{\epsilon}} \leq p^{r} - 1 \} \) forms a basis of \( \mathcal{O}((\mathfrak{g}_{\epsilon})_{\bar{r}}) \), see [J Part I, 9.6]. Thus, by Proposition 3.8, the set

\[
(3.2) \quad \left\{ f_{1}^{a_{1}} \cdots f_{n_{\epsilon}}^{a_{n_{\epsilon}}} \cdot f_{n_{\epsilon}+1}^{\epsilon_{1}} \cdots f_{n_{\epsilon}+n_{1}}^{\epsilon_{n_{1}}} \mid 0 \leq a_{1}, \ldots, a_{n_{\epsilon}} \leq p^{r} - 1, \epsilon_{1}, \ldots, \epsilon_{n_{1}} \in \{0, 1\} \right\}
\]

forms a basis of \( \mathcal{O}(\mathfrak{g}_{\epsilon}) \). In particular, we have

\[
\dim(\mathcal{O}(\mathfrak{g}_{\epsilon})) = p^{n_{\epsilon}} \cdot 2^{n_{1}}.
\]
Example 3.10. Recall that, $G^-_a$ is the one-dimensional odd unipotent supergroup with $O(G^-_a) = k[z]/(z^2)$, see Example 2.4[3]. Then for a commutative superalgebra $R$, we have
\[
\text{Fr}^r(R) : G^-_a(R) \longrightarrow (G^-_a)^{(r)}(R); \quad \begin{array}{c}
\phi \mapsto (z \mapsto z^{p^r} \mapsto g(z^{p^r})).
\end{array}
\]
Since $p > 2$ and $z^2 = 0$, we conclude that the $r$-th Frobenius kernel $(G^-_a)^r$ of $G^-_a$ coincides with $G^-_a$. For the supergroup $F^{[g_1, \ldots, g_n]} = F \ltimes (G^-_a)^n$ defined in Example 3.10, if the group-like elements $g_1, \ldots, g_n$ are trivial, then $F^{[1, \ldots, 1]} = F \ltimes (G^-_a)^n$ and $(F^{[1, \ldots, 1]})_r = F_r \ltimes (G^-_a)^n$, where $F_r$ denotes the $r$-th Frobenius kernel of $F$.

In the following, we regard $\text{hy}(G_r)$ as a Hopf super-subalgebra of $\text{hy}(G)$ via the inclusion $G_r \subset G$. The following is a PBW type theorem for the $r$-th Frobenius kernel $G_r$ of $G$.

Theorem 3.11. For any total order on the homogeneous basis of $g = g_0 \oplus g_1$, the set of all products of factors of type
\[
H_i^{(m_i)}, \quad X_{\alpha}^{(n_\alpha)}, \quad K_i^{(t_i)}, \quad Y_{(\gamma,j)}^{(e,\bar{\gamma},e)}
\]
\begin{itemize}
\item $(0 \leq n_\alpha, m_i \leq p^r - 1, \alpha \in \Delta_\delta, 1 \leq i \leq \ell_0, \bar{\gamma} \in \Delta_\delta, 1 \leq j \leq \dim(g_1^1), 1 \leq t \leq \ell_1$ and $e_i, e(\gamma, j) \in \{0, 1\}$), taken in $\text{hy}(G)$ with respect to the order, form a basis of $\text{hy}(G_r).
\end{itemize}

Proof. By Theorem 2.6 for $\text{hy}(G_r)$ and Proposition 3.8 we have an isomorphism $\text{hy}(G_r) \cong \text{hy}((G_{ev})_r) \otimes \bigwedge(g_1)$ of (left $\text{hy}((G_{ev})_r)$-module) supercoalgebras. On the other hand, since $G_{ev}$ is split reductive, the set of all products (taken in the fixed order) of factors of type $H_i^{(m_i)}, X_{\alpha}^{(n_\alpha)}$ ($0 \leq n_\alpha, m_i \leq p^r - 1, \alpha \in \Delta_\delta, 1 \leq i \leq \ell_0$) form a basis of $\text{hy}((G_{ev})_r)$, see [J] Part II, Lemma 3.3]. The proof is done.

In particular, it follows that $\text{hy}(G)$ is generated by $\text{hy}(G_{ev})$ and $\text{hy}(G_r)$ as a superalgebra.

4. Unimodularity of Algebraic Supergroups

In this section, we discuss the unimodularity of Frobenius kernels of split quasireductive supergroups.

4.1. (Co)integrals on Hopf superalgebras. Let $H$ be a Hopf superalgebra with unit $1_H$ and counit $\varepsilon_H$, in general. A left cointegral on $H$ is an element $\phi \in H^*$ satisfying
\[
\phi \ast f = f(1_H)\phi
\]
for all $f \in H^*$. Here, $\phi \ast \phi : H \rightarrow k$ denotes the convolution product of $f$ and $\phi$, that is, $(\phi \ast \phi)(h) = \sum h(-1)^{|h(1)|} |\phi| f(h(1)) \phi(h(2))$ for $h \in H$. In other words, a left cointegral is an element in the space $\int_H^L := \text{Hom}^L(H, k)$, where $k$ is regarded as a trivial left $H$-supercomodule. The notion of a right cointegral on $H$ and the symbol $\int_H^R$ are defined analogously. Using the bosonization technique (see [MZ1] Section 10 for example), we have the following:

Proposition 4.1 ([MSS Corollary 3.2]). Both of $\dim(\int_H^L)$ and $\dim(\int_H^R)$ are less than or equal to 1, that is, a non-zero left or right cointegral on $H$ is unique up to scalar multiplication if it exists. Moreover, such an element is homogeneous.
Definition 4.2. We say that $H$ is unimodular if $\int^H_t = \int^H_s \neq 0$, that is, there exists a non-zero two-sided (i.e., left and right) cointegral on $H$.

Suppose that $H$ is finite-dimensional. An element $t \in H$ is called a left (resp. right) integral in $H$ if it satisfies $ht = \varepsilon_H(h)t$ (resp. $th = \varepsilon_H(h)t$) for all $h \in H$. The space of all left (resp. right) integrals in $H$ is denoted by $\mathcal{I}^H_t$ (resp. $\mathcal{I}^H_H$).

In general, it is known that any finite dimensional Hopf algebra has both non-zero left/right integral. By this fact and the dual result of \cite[Proposition 3.1]{MSS}, we have $\dim(\mathcal{I}_H^H) = \dim(\mathcal{I}_H^H) = 1$ and $S_{\mathcal{H}}(\mathcal{I}_H^H) = \mathcal{I}_H^H$, where $S_{\mathcal{H}} : H \rightarrow H$ is the antipode of $H$. As in the non super-situation (see \cite[Chapter 10]{Rad}), one easily sees that the following holds:

Proposition 4.3. There uniquely exists $\alpha_H \in \mathfrak{g}l(H^*)$ such that $\theta = (\alpha_H, h)t$ for all $h \in H$ and $t \in \mathcal{I}^H_H$.

The element $\alpha_H$ is the so-called distinguished group-like element for $H$.

4.2. Integrals for supergroups. Let $\mathcal{G}$ be an algebraic supergroup, in general. We say that $\mathcal{G}$ has a left (resp. right) integral for $\mathcal{G}$ if there exists a non-zero left (resp. right) cointegral on $O(\mathcal{G})$. Also, we say that $\mathcal{G}$ is unimodular if $O(\mathcal{G})$ is unimodular (see Definition 4.2).

Theorem 4.4 (\cite[Theorem 3.7]{MSS}). $\mathcal{G}$ has a left (resp. right) integral if and only if $\mathcal{G}_{ev}$ does.

Assume for a moment that $\text{char}(k) = 0$. Let $F$ be an algebraic group over $k$. Then by Sullivan’s theorem (\cite{Su}), $F$ has a left (or right) integral if and only if $F$ is linearly reductive. In particular, in this case, $F$ is automatically unimodular. However, in our super-situation, the existence of an integral does not imply its unimodularity (see Theorem 4.5 below).

By Theorem 4.4 (and Sullivan’s theorem again), we note that for a connected and algebraic supergroup $\mathcal{G}$ defined over a field of characteristic zero, $\mathcal{G}$ has a left (or right) integral if and only if $\mathcal{G}$ is split quasi-reductive.

Theorem 4.5. Assume that $\text{char}(k) = 0$ and $\mathcal{G}$ is a split quasi-reductive supergroup. Then $\mathcal{G}$ is unimodular if and only if $\sum_{\gamma \in \Delta_1} \dim(\mathfrak{g}^1_{\gamma}) = 0$ in $X(T)$.

Proof. Let $\text{ad}^\prime : \mathfrak{g}_0 \rightarrow \text{End}(\mathfrak{g}_1)$ be the restriction of the adjoint representation of $\mathfrak{g}$. Then by \cite[Proposition 3.16]{MSS}, we know that $\mathcal{G}$ is unimodular if and only if the algebra map $\chi_\mathcal{G} : U(\mathfrak{g}_0) \rightarrow k$ defined by the following is trivial:

$$\chi_\mathcal{G}(X) = \text{tr}(\text{ad}^\prime(X)) \quad \text{for all } X \in \mathfrak{g}_0,$$

where the universal enveloping algebra $U(\mathfrak{g}_0)$ of $\mathfrak{g}_0$. Since $\text{hy}(\mathcal{G}_{ev}) = U(\mathfrak{g}_0)$ and $O(\mathcal{G}_{ev}) \subset \text{hy}(\mathcal{G}_{ev})^*$ (by the connectedness assumption on $\mathcal{G}_{ev}$), we may regard $\chi_\mathcal{G}$ with a character of $\mathcal{G}_{ev}$. Thus, we see that $\chi_\mathcal{G}$ is trivial if and only if the restriction $\chi_\mathcal{G}|_T$ to the split maximal torus $T$ is trivial by Lemma 3.6. Since the $T$-weight superspace decomposition of $\mathfrak{g}_1$ is given as $\mathfrak{g}_1 = \mathfrak{h}_1 \oplus \bigoplus_{\gamma \in \Delta_1} \mathfrak{g}^1_{\gamma}$ with $\mathfrak{h}_1 = \mathfrak{g}_0^0$, we can compute

$$\chi_\mathcal{G}(t) = \dim(\mathfrak{h}_1)0 + \sum_{\gamma \in \Delta_1} \dim(\mathfrak{g}^1_{\gamma})\gamma(t) = \sum_{\gamma \in \Delta_1} \dim(\mathfrak{g}^1_{\gamma})\gamma(t)$$

for all $t \in T(R)$, where $R$ is a commutative algebra. Thus we are done. \qed
Corollary 4.6. Assume that \( \text{char}(k) = 0 \). Then \( GL(m|n), Q(n) \) and Chevalley supergroups of classical type are unimodular.

Proof. As in Section 5.1 and Example 5.15 in these cases, we can define an “order” on \( \Delta_1 \) satisfying the following properties:
\[
\Delta_1 = \Delta^+_1 \cup \Delta^-_1 \quad \text{(disjoint union)}, \quad \Delta^+_1 = -\Delta^-_1,
\]
and \( \dim(g_1^\gamma) = \dim(g^-_1^-) \) for all \( \gamma \in \Delta^+_1 \).

Thus, we have \( \sum_{\gamma \in \Delta_1} \dim(g_1^\gamma) \gamma = 0 \). By Theorem 4.5 we are done. \( \square \)

Note that, in the above proof, the “order” can be found for such supergroups without assuming that the base field \( k \) is of characteristic zero.

Example 4.7. Assume that \( \text{char}(k) = 0 \).

1. Suppose that \( G = P(n) \) with \( n \geq 2 \). Then by Example 3.4, we have \( \sum_{\gamma \in \Delta_1} \dim(g_1^\gamma) \gamma = 2 \sum_{t=1}^n \lambda_t \neq 0 \). Thus, \( P(n) \) is non-unimodular.

2. We consider the following closed supergroup \( G \) of \( GL(3|3) \).

\[
G(R) := \{ \begin{pmatrix} h & 0 & x & 0 & 0 & 0 \\
0 & 1 & 0 & a & 0 & b \\
y & 0 & k & 0 & 0 & 0 \\
0 & 0 & 0 & h & 0 & x \\
a & 0 & b & 0 & 1 & 0 \\
0 & 0 & 0 & y & 0 & k \end{pmatrix} \} \in GL(3|3)(R),
\]

where \( R \) is a superalgebra. Since \( G_{ev} \cong GL_2 \), this is split quasireductive. If we take \( T \) as diagonal matrices in \( G \), then root system of \( G \) with respect to \( T \) is given by \( \Delta_0 = \{ \pm(\lambda_1 - \lambda_3) \} \) and \( \Delta_1 = \{ -\lambda_1, -\lambda_3 \} \). Thus, \( \sum_{\gamma \in \Delta_1} \dim(g_1^\gamma) \gamma = -\lambda_1 - \lambda_3 \neq 0 \), and hence this \( G \) is non-unimodular.

3. For the split quasireductive supergroup \( F^{(g_1, \ldots, g_n)} = F \ltimes (G^-_n)^n \) discussed in Example 3.4, we have seen that \( \Delta_1 = \{ -\chi_1, \ldots, -\chi_n \} \). Set \( m := \#\Delta_1 \).

If we write \( \Delta_1 = \{ -\chi_1, \ldots, -\chi_m \} \) and set
\[
d_j := \#\{ \chi \in \Delta_1 \mid \chi = \chi_i \} = \dim(g^{-\chi_i}_1),
\]
then \( F^{(g_1, \ldots, g_n)} \) is unimodular if and only if \( \sum_{j=1}^m d_j \chi_j = 0 \). \( \blacksquare \)

4.3. Integrals for finite normal super-subgroups. Again, we suppose \( k \) is a field of characteristic different from 2. Let \( G \) be an algebraic supergroup over \( k \), and let \( N \) be a finite and normal super-subgroup of \( G \). Set \( A := O(G) \) and \( B := O(N) \) for simplicity. For \( a \in A \), we denote by \( \pi^B \in B \) the image of \( a \) via the canonical Hopf quotient map \( A \to B \) corresponding to the inclusion \( N \subset G \).

Since \( N \) is normal, the left adjoint action \( \text{Ad} \) of \( G \) on \( N \) makes \( B \) into a Hopf superalgebra object in the category of left \( A \)-supermodules. Explicitly, \( \text{coad}_B : B \to A \otimes B; \quad \pi^B \mapsto \sum_a (-1)^{|a(2)| |a(3)|} a(1) S_A(a(3)) \otimes \overline{a(2)}^B \),

where \( S_A \) is the antipode of \( A \). Taking the linear dual, \( B^* \) forms a Hopf superalgebra object in the category of right \( A \)-supermodules with the dual supercomodule structure map \( \text{coad}_B^* : B^* \to B^* \otimes A \) of \( \text{coad}_B \).

Since \( B \) is finite-dimensional, the space \( T_B^* \) of left integrals in \( B^* \) is one-dimensional, see Section 4.1. In the following, we take and fix a \( k \)-base \( \phi \) of the space.
\[ T^L_{B^*} \text{ of left integrals in } B^*, \text{ that is, } T^L_{B^*} = \mathbb{k} \phi. \] Note that, \( \phi \) is homogeneous, that is, purely even or odd.

**Lemma 4.8.** The space \( T^L_{B^*} \) forms an Aスーパー-subcomodule of \( B^* \). In particular, there uniquely exists \( \chi \in \mathfrak{l}(A) \) such that \( \text{coad}_B^A(\phi) = \phi \otimes \chi \).

**Proof.** We denote by \( \Phi : G \to \text{Aut}(B^*) \) the left \( G \)-supermodule structure map on \( B^* \) corresponding to \( \text{coad}_B^A : B^* \to B^* \otimes A \). To prove the claim we show that \( T^L_{B^*} \) is stable under the action of \( g := \Phi_R(g)(-)^{\epsilon} \) for all commutative superalgebra \( R \) and \( g \in \mathcal{G}(R) \), where \( \Phi_R : \mathcal{G}(R) \to \text{Aut}_R(B^* \otimes R) \) and \( \text{Aut}_R(B^* \otimes R) := \text{End}_R(B^* \otimes R)^{\epsilon} \).

We fix \( f \in B^* \). Since \( B^* \) is a Hopf superalgebra object in the category of right \( A \)-supermodules, we have

\[ (f \otimes 1_R)^{\epsilon} (\phi \otimes 1_R) = g^{-1}(f \otimes 1_R) \ast (\phi \otimes 1_R), \]

where \( 1_R \) is the unit element of \( R \). On the other hand, since \( \phi \) is a left integral in \( B^* \), we have

\[ g^{-1}(f \otimes 1_R) \ast (\phi \otimes 1_R) = \varepsilon_{B^* \otimes A}(g^{-1}(f \otimes 1_R)) (\phi \otimes 1_R). \]

By definition, we get \( \varepsilon_{B^* \otimes A}(g^{-1}(f \otimes 1_R)) = \varepsilon_{B^*}(f) \otimes 1_R \). Thus, we conclude that \( g(\phi \otimes 1_R) \in T^L_{B^*} \). \( \square \)

We may identify the dual superspace \( B^{**} \) of \( B^* \) with \( B \), since \( B \) is finite-dimensional. Through this identification, there uniquely exists \( \alpha_{B^*} \in \mathfrak{l}(B) \) (i.e., the distinguished group-like element) such that

\[ \phi \ast f = \langle f, \alpha_{B^*} \rangle \phi \quad \text{for all } f \in B^*, \]

see Proposition 4.3. Note that, \( \alpha_{B^*} \) is an element of the even part \( B_0 \) of \( B \).

The left \( A^* \)-supermodule structure on \( B^* \) induced from \( \text{coad}_B^A : B^* \to B^* \otimes A \) is given by

\[ h \to f = \sum f(k) \otimes (g \phi) \]

where we write \( \text{coad}_B^A(f) = \sum f(k) \otimes (g \phi) \). By restricting the action to \( B^* (\subset A^*) \), we get the adjoint action \( k \to f = \sum k^{-1}(f) \otimes (g \phi) \) for all \( k, f \in B^* \), where \( S_{B^*} \) is the antipode of \( B^* \).

**Proposition 4.9.** \( \overline{\chi}^B \) coincides with the inverse \( (\alpha_{B^*})^{-1} \) of the distinguished group-like element \( \alpha_{B^*} \in \mathfrak{l}(B) \) of \( B^* \).

**Proof.** We fix \( k \in B^* \). Since \( \phi \in T^L_{B^*} \), is purely even/odd and \( \alpha_{B^*} \in B_0 \), we have

\[ k \to \phi = \sum_k (-1)^{|k|} |k_1| \phi \ast S_{B^*}(k_2) \]

where we write \( \text{coad}_B^A(f) = \sum f(k) \otimes (g \phi) \). By restricting the action to \( B^* (\subset A^*) \), we get the adjoint action \( k \to f = \sum (-1)^{|f|} k \phi \ast S_{B^*}(k_2) \) for all \( k, f \in B^* \).

On the other hand, we calculate the action \( k \to \phi \) directly. Since we know \( \text{coad}_B^A(\phi) = \phi \otimes \chi \) by Lemma 4.3, we get

\[ k \to \phi = (-1)^{|k|} \langle k, \overline{\chi}^B \rangle \phi = \langle k, \overline{\chi}^B \rangle \phi. \]
Theorem 4.10. The restriction $\chi|_{\mathbb{N}}$ is trivial if and only if $\mathbb{N}$ is unimodular. In particular, $\mathbb{N}$ is unimodular if $\chi$ is trivial.

Remark 4.11. In the non super-situation, Theorem 4.10 tells us that for a connected and split reductive group $F$, any finite and normal subgroup $K$ of $F$ is unimodular. In particular, all Frobenius kernels of $F$ are unimodular. We give a proof of this fact. The adjoint action $\text{Ad} : F \to \text{Aut}(K)$: $f \mapsto (k \mapsto kf^{-1})$ factors through the quotient $F/Z(F)$, where $Z(F)$ is the center of $F$. Thus, the corresponding coaction $O(K)^* \to O(K)^* \otimes O(F)$ factors through $O(K)^* \otimes O(F/Z(F))$:

$$O(K)^* \xrightarrow{\text{factors}} O(K)^* \otimes O(F) \xrightarrow{f} O(K)^* \otimes O(F/Z(F)).$$

Note that, we regard $O(F/Z(F))$ as a Hopf subalgebra of $O(F)$ via the canonical quotient $F \to F/Z(F)$. Thus, the group-like element $\chi$ is in $O(F/Z(F))$. On the other hand, since $F$ is connected and reductive, the quotient $F/Z(F)$ coincides with its derived group, see [Mi] Chapter 21 for example. Thus, there is no non-trivial group-like element in $O(F/Z(F))$, and hence $\chi$ must be trivial. Then by Proposition 4.11 $K$ is unimodular.

However, in our super-situation, the proof in Remark 4.11 does not work. One of the reasons is that $(G/Z(G))_{\text{ev}} = G_{\text{ev}}/Z(G)_{\text{ev}}$ (by Masuoka and Zubkov [MZ1]) is not isomorphic to $G_{\text{ev}}/Z(G_{\text{ev}})$, in general. For example, if we take $G = GL(m|n)$, then $Z(G_{\text{ev}}) \cong Z(GL_m \times GL_n) \cong G_m \times G_n$, while $Z(G) = Z(G)_{\text{ev}} \cong G_m$.

4.4. Unimodularity of Frobenius kernels. We suppose that the base field $\mathbb{k}$ is a perfect field of characteristic $p > 2$.

In [ZM Corollary 7.2], it is proved that all Frobenius kernels of the general linear supergroup $GL(m|n)$ are unimodular. In this subsection, we give a necessary and sufficient condition for Frobenius kernels of a split quasireductive supergroup to be unimodular in terms of the root system of it.

Let $G$ be a split quasireductive supergroup, and let $r$ be a positive integer. Set $\mathfrak{g} := \text{Lie}(G)$. Since the $r$-th Frobenius kernel $G_r$ of $G$ is finite and normal, there uniquely exists $\chi_r \in g.l.(O(G)) \cong X(G)$ such that $\text{coad}_{O(G_r)}(\phi_{G_r}) = \phi_{G_r} \otimes \chi_r$ by Lemma 4.8. Here, $\phi_{G_r}$ is a fixed non-zero left integral for $G_r$. As a super-analogue of [H, Part I, Proposition 9.7], Zubkov and Marko [ZM] explicitly determined the value of $\chi_r$ as follows.

Proposition 4.12 ([ZM Proposition 6.11]). Let $R$ be a commutative superalgebra. For each $g \in G(R)$,

$$\chi_r(g) = \text{Ber(Ad(g)})^{p^r-1} \cdot \text{det}_l(\text{Ad(g)})^{p^r}.$$  

Here, the left adjoint action $\text{Ad}(g)$ on $\mathfrak{g}$ is regarded as an element of $\text{Mat}_{\dim(\mathfrak{g})|\dim(\mathfrak{g})}(R)$ with respect to the fixed basis given in [3.2].
Set $T_r := T \cap (G_{ev})_r$. Note that, $T_r$ is the $r$-th Frobenius kernel of $T$. The following is a version of Lemma 3.6.

**Lemma 4.13.** The map $X(G_r) \to X(T_r)$; $\chi \mapsto \chi|_{T_r}$ is injective.

**Proof.** For each $t \in T_r$, let $(U_\alpha)_r$ denote the $r$-th Frobenius kernel of the $\alpha$-root subgroup $U_\alpha$ of $G_{ev}$. Since $U_\alpha \cong G_3$, one sees that the corresponding Hopf algebra of $(U_\alpha)_r$ is isomorphic to the quotient $k[X_\alpha]/(X_\alpha^{p^r})$ of the polynomial algebra $k[X_\alpha]$. Thus, the character group of $(U_\alpha)_r$ is trivial. Since $(G_{ev})_r$ is generated by $(U_\alpha)_r$ and $T_r$, the map $X((G_{ev})_r) \to X(T_r)$; $\chi \mapsto \chi|_{T_r}$ is injective. Then by Lemma 2.9 and Proposition 3.8 we are done. \[\square\]

Recall that $X(T) \cong \mathbb{Z}^\ell = \bigoplus_{i=1}^\ell \mathbb{Z}\lambda_i$. We shall write down the odd roots by the basis. For each $\gamma \in \Delta_1$, there uniquely exits $n(\gamma)_1, \ldots, n(\gamma)_\ell \in \mathbb{Z}$ such that

$$\gamma = \sum_{i=1}^\ell n(\gamma)_i \lambda_i.$$ 

Using this notation, we have the following result:

**Proposition 4.14.** The $r$-th Frobenius kernel $G_r$ of $G$ is unimodular if and only if $\sum_{\gamma \in \Delta_1} \dim(g_1^\gamma) n(\gamma)_i \in p^r \mathbb{Z}$ for all $1 \leq i \leq \ell$.

**Proof.** By Theorem 4.10, we have $G_r$ is unimodular if and only if the restriction $\chi_{r|G_r}$ is trivial. On the other hand, by Lemma 4.13 the restriction $\chi_{r|G_r}$ is trivial if and only if $\chi_{r|T_r}$ is trivial.

Let $R$ be a commutative algebra. By the explicit description of $\chi_r$ (Proposition 4.12), for each $t \in T_r(R)$

$$\chi_r(t) = \det_{\ell_0}(\text{Ad}(t))^{p^r-1} \cdot \det_{\ell_1}(\text{Ad}(t))$$

$$= \prod_{\alpha \in \Delta_0} \alpha(t)^{p^r-1} \cdot \prod_{\gamma \in \Delta_1} \gamma(t)^{\dim(g_1^\gamma)} = \prod_{\gamma \in \Delta_1} \gamma(t)^{\dim(g_1^\gamma)}.$$ 

Here, the last equation follows from $\sum_{\alpha \in \Delta_0} \alpha = 0$ in $X(T)$.

Recall that, the identification $X(T) \cong \mathbb{Z}^\ell$ is induced from the fixed isomorphism $T \cong G_m^\ell$. Since $T_r \cong \mu_{p^r}^\ell$, we have $X(T_r) \cong (\mathbb{Z}/p^r\mathbb{Z})^\ell$ through this identification. For each $1 \leq i \leq \ell$, we get

$$t = (1, \ldots, t_i, \ldots, 1) \in T_r(R) \cong \mu_{p^r}^\ell(R) \implies \chi_r(t) = \prod_{\gamma \in \Delta_1} t_i^{\dim(g_1^\gamma)n(\gamma)_i}.$$ 

Thus, $\chi_{r|T_r}$ is trivial if and only if $\sum_{\gamma \in \Delta_1} \dim(g_1^\gamma)n(\gamma)_i \in \mathbb{Z}$ is divided by $p^r$ for each $1 \leq i \leq \ell$. This proves the claim. \[\square\]

**Theorem 4.15.** The following conditions are equivalent:

1. For all positive integer $r$, the $r$-th Frobenius kernel $G_r$ of $G$ is unimodular.
2. $\sum_{\gamma \in \Delta_1} \dim(g_1^\gamma) = 0$ in $X(T)$.

**Proof.** By Proposition 4.14 it follows that the condition (1) holds if and only if $\sum_{\gamma \in \Delta_1} \dim(g_1^\gamma)n(\gamma)_i = 0$ for all $1 \leq i \leq \ell$. The last condition is obviously equivalent to (2). The proof is done. \[\square\]

**Corollary 4.16.** Let $G$ be one of $GL(m|n)$, $Q(n)$ or a Chevalley supergroup of classical type. For any positive integer $r$, the $r$-th Frobenius kernel $G_r$ of $G$ is unimodular.
Proof. As in the proof of Corollary 4.6 for each \( \mathcal{G} \), we can find a decomposition \( \Delta_1 = \Delta_1^+ \cup \Delta_1^- \) satisfying the condition (4.1). Thus by Theorem 4.15 we are done. \( \square \)

**Example 4.17.** As we have seen in Example 4.7 the \( r \)-th Frobenius kernels of the periplectic supergroup \( \mathcal{P}(u) \) and the supergroup \( \mathcal{G} \) defined in Example 4.7(2) are non-unimodular. For the supergroup \( \mathcal{F}^{[g_1,\ldots,g_n]} = F \ltimes (G_{2n})_r \) given in Example 3.4[3], the \( r \)-th Frobenius kernel \( (\mathcal{F}^{[g_1,\ldots,g_n]})_r \) is unimodular if and only if \( \sum_{j=1}^{m_1} d_j \chi_{ij} = 0 \), see Example 4.7(3). \( \square \)

### 5. Steinberg’s Tensor Product Theorem

We fix a split quasi-reductive supergroup \( \mathcal{G} \) with a split maximal torus \( T \) of \( \mathcal{G}_{ev} \). Set \( \mathfrak{g} := \text{Lie}(\mathcal{G}) \) and \( \mathfrak{h} := \mathfrak{g}^0 \) as before. In this section, we establish Steinberg’s tensor product theorem for \( \mathcal{G} \) under natural assumptions.

#### 5.1. Simple \( \mathcal{G} \)-supermodules.** In [Shi1], we defined some special closed subgroups of \( \mathcal{G} \) and constructed all simple left \( \mathcal{G} \)-supermodules. In the following, we briefly review the construction.

First of all, we constructed a closed super-subgroup \( \mathcal{T} \) of \( \mathcal{G} \) such that \( \mathcal{T}_{ev} = T \) with \( \text{Lie}(\mathcal{T}) = \mathfrak{h} \). We fix a group homomorphism \( \Upsilon : Z \Delta \to \mathbb{R} \) with \( \Upsilon(\Delta \setminus \{0\}) \subseteq \mathbb{R} \setminus \{0\} \) to define an “order” on \( \Delta \) as follows:

\[
\Delta^\pm := \{ \alpha \in \Delta \setminus \{0\} \mid \pm \Upsilon(\alpha) > 0 \}, \quad \Delta_\epsilon^\pm := \Delta_\epsilon \cap \Delta^\pm \quad (\epsilon \in \mathbb{Z}_2).
\]

Along this order, we can construct a closed super-subgroup \( \mathcal{B}^+ \) (resp. \( \mathcal{B} \)) of \( \mathcal{G} \), called the Borel super-subgroup of \( \mathcal{G} \), such that \( \mathcal{B}_{ev}^+ \) (resp. \( \mathcal{B}_{ev} \)) is a positive (resp. negative) Borel subgroup of \( \mathcal{G}_{ev} \) with respect to \( \Delta_0^+ \) (resp. \( \Delta_0^- \)) satisfying

\[
\text{Lie}(\mathcal{B}^+) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}^\alpha \quad \text{(resp. } \text{Lie}(\mathcal{B}) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^-} \mathfrak{g}^\alpha \text{)}.
\]

Also, we can construct a closed super-subgroup \( \mathcal{U}^+ \) of \( \mathcal{G} \) such that \( \mathcal{U}_{ev}^+ \) is a unipotent subgroup of \( \mathcal{G}_{ev} \) and \( \text{Lie}(\mathcal{U}^+) = \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}^\alpha \). One sees that \( \mathcal{U}^+ \) is unipotent, that is, for any left \( \mathcal{U}^+ \)-supermodule, its \( \mathcal{U}^+ \)-invariant super-subspace is non-zero, see [ZU]. (In other words, the corresponding Hopf superalgebra \( \mathcal{O}(\mathcal{U}^+) \) is irreducible, see also [Ma2 Definition 4(1)]). Moreover, we have \( \mathcal{B}^+ \cong T \ltimes \mathcal{U}^+ \).

Analogously, we can find \( \mathcal{U} \subset \mathcal{G} \) such that \( \mathcal{B} \cong T \ltimes \mathcal{U} \).

Using Clifford superalgebra theory, we can find a simple left \( \mathcal{T} \)-supermodule \( u(\lambda) \) for each \( \lambda \in X(\mathcal{T}) \). Moreover, the map \( X(\mathcal{T}) \to \text{Simple}_T(\mathcal{T}) \); \( \lambda \mapsto u(\lambda) \) is bijective. As left \( \mathcal{T} \)-modules, this \( u(\lambda) \) is isomorphic to a (finite) copy of the one-dimensional left \( \mathcal{T} \)-module \( k^\lambda \). Thus, if \( \mathcal{T} = T \) (i.e., \( 0 \notin \Delta \)), then \( u(\lambda) \) is just \( k^\lambda \).

**Lemma 5.1.** We have \( X(\mathcal{T}) \cong X(\mathcal{T}) \).

*Proof.* For each \( \lambda \in X(\mathcal{T}) \), there exists \( n_\lambda > 0 \) such that \( u(\lambda) \cong (k^\lambda)^{\oplus n_\lambda} \) as left \( \mathcal{T} \)-modules. Since \( X(\mathcal{T}) \) is identified with the set of all equivalence classes of one-dimensional left \( \mathcal{T} \)-supermodules under the parity change \( \Pi \), we conclude that \( X(\mathcal{T}) \) is naturally identified with \( X(\mathcal{T}) \). This proves the claim. \( \square \)
Since $\mathcal{T}$ is a closed super-subgroup of $\mathcal{B}$, we may regard $u(\lambda)$ as a left $\mathcal{B}$-supermodule (i.e., a right $\mathcal{O}(\mathcal{B})$-supercomodule), which we denote by the same symbol. For each $\lambda \in X(T)$, we get a left $\mathcal{G}$-supermodule
\[ H^0(\lambda) := \text{ind}_{\mathcal{B}}^\mathcal{G}(u(\lambda)) = u(\lambda) \square \mathcal{O}(\mathcal{B}) \mathcal{O}(\mathcal{G}), \]
where $\square \mathcal{O}(\mathcal{B})$ denotes the cotensor product over $\mathcal{O}(\mathcal{B})$ and $\mathcal{O}(\mathcal{G})$ is naturally regarded as a left $\mathcal{O}(\mathcal{B})$-supercomodule via $\mathcal{B} \subset \mathcal{G}$.

Set
\[ X(T)^\flat := \{ \lambda \in X(T) \mid H^0(\lambda) \neq 0 \}. \]
For each $\lambda \in X(T)^\flat$, we can show that $H^0(\lambda)$ has a unique simple left $\mathcal{G}$-supermodule $L(\lambda)$.

**Theorem 5.2** ([Shi 1, Theorem 4.12 and Proposition 4.15]). The map $X(T)^\flat \rightarrow \text{Simple}_1(\mathcal{G}) \ni \lambda \mapsto L(\lambda)$ is bijective. Moreover, $\lambda$ is a “highest” $T$-weight of $L(\lambda)$, in the sense that the $\lambda$-weight superspace $L(\lambda)^\lambda$ is isomorphic to $u(\lambda)$ as left $\mathcal{T}$-supermodules and the action of $L(\lambda)^\lambda$ trivial.

**Definition 5.3.** A simple left $\mathcal{G}$-supermodule $L$ is said to be *absolutely simple* if $L \otimes k'$ is a simple left $\mathcal{G}_k'$-supermodule for all field extensions $k'$ of $k$. Here, $\mathcal{G}_k'$ denotes the base change of $\mathcal{G}$ to $k'$.

The following is a corollary of Theorem 5.2

**Corollary 5.4.** Let $\lambda \in X(T)^\flat$. If $L(\lambda)$ is absolutely simple, then $L(\lambda) \otimes k' \cong L_{k'}(\lambda)$ as left $\mathcal{G}_{k'}$-supermodules for all field extensions $k'$ of $k$. Here, $L_{k'}(\lambda)$ denotes a simple left $\mathcal{G}_{k'}$-supermodule of highest weight $\lambda$.

**Proposition 5.5.** For a field extension $k'$ of $k$, we have $X(T_{k'})^\flat \subset X(T)^\flat$. If $0 \notin \Delta$ (or equivalently, $\mathcal{T} = T$), then $X(T_{k'})^\flat = X(T)^\flat$.

**Proof.** Since the induction functor $\text{ind}_{\mathcal{B}}^\mathcal{G}(\_)$ commutes with all field extensions,
\[ H^0(\lambda) \otimes k' \cong (u(\lambda) \otimes k') \square \mathcal{O}(\mathcal{B}_{k'}) \mathcal{O}(\mathcal{G}_{k'}) \supset u_{k'}(\lambda) \square \mathcal{O}(\mathcal{B}_{k'}) \mathcal{O}(\mathcal{G}_{k'}) =: H_{k'}^0(\lambda), \]
Here, $u_{k'}(\lambda)$ denotes a unique simple left $\mathcal{T}_{k'}$-supermodule with weight $\lambda$. Thus, $H_{k'}^0(\lambda) \neq 0$ implies $H^0(\lambda) \neq 0$. If $0 \notin \Delta$, then $u(\lambda) = k^\lambda$, and hence $H_{k'}^0(\lambda) \cong H^0(\lambda)$.

**Proposition 5.6.** If $0 \notin \Delta$ (or equivalently, $\mathcal{T} = T$), then all simple left $\mathcal{G}$-supermodules are absolutely simple. In particular, for a field extension $k'$ of $k$ and a left $\mathcal{G}$-supermodule $V$, we have (1) $\text{soc}_{\mathcal{G}}(V) \otimes k' \cong \text{soc}_{\mathcal{G}_{k'}}(V \otimes k')$; and (2) $V$ is $\mathcal{G}$-semisimple if and only if $V \otimes k'$ is $\mathcal{G}_{k'}$-semisimple.

**Proof.** Let $\lambda \in X(T)^\flat$. We note that $L(\lambda)^\lambda \cong u(\lambda) = k^\lambda$ and $L(\lambda) \not\cong \Pi L(\lambda)$. By Frobenius reciprocity (see [Shi 1, Section A.3]), we get
\[ \mathcal{G}\text{Hom}(L(\lambda), H^0(\lambda)) \cong \mathcal{G}\text{Hom}(L(\lambda), k^\lambda) \subset \mathcal{G}\text{Hom}(L(\lambda)^\lambda, k^\lambda) \cong k. \]
Since $\text{id}_{L(\lambda)} \in \mathcal{G}\text{End}(L(\lambda))$, we can conclude that $\mathcal{G}\text{End}(L(\lambda)) = k$, and hence $\mathcal{G}\text{End}(L(\lambda)) = k$ by Theorem 5.3. Let $\rho : \text{hy}(\mathcal{G}) \rightarrow \mathcal{G}\text{End}(L(\lambda))$ denote the $\text{hy}(\mathcal{G})$-supermodule structure map of $L(\lambda)$. Then by Jacobson density theorem for superalgebras, the above argument implies that $\rho$ is surjective.

Let $k'$ be a filed extension of $k$. Since $\mathcal{G}_k$ is infinitesimally flat and $\rho$ is surjective, the $\text{hy}(\mathcal{G}_{k'})$-supermodule structure map
\[ \rho \otimes k' : \text{hy}(\mathcal{G}_{k'}) \cong \text{hy}(\mathcal{G}) \otimes k' \rightarrow \mathcal{G}\text{End}_{k'}(L(\lambda) \otimes k') \cong \mathcal{G}\text{End}_{k'}(L(\lambda)) \otimes k', \]
of $L(\lambda) \otimes k'$ is also surjective. In general, it is easy to see that $L(\lambda) \otimes k'$ is a simple left $\mathfrak{End}_k(L(\lambda) \otimes k')$-supermodule. Therefore, we conclude that $L(\lambda) \otimes k'$ is a simple left $\mathfrak{g}_y$-supermodule. By Theorem 3.3 (for $\mathfrak{g}_y$), $L(\lambda) \otimes k'$ is a simple left $\mathfrak{g}_{ev}$-supermodule.

By counting multiplicity of simple super-submodules inside of $V$, the claim (1) easily follows. The claim (2) is just a consequence of (1). □

In the non super-situation, it is known that all left simple $\mathfrak{g}_{ev}$-modules are absolutely simple, see [1] Part II, Corollary 2.9 (and [MI] Section 22.4). However, the following example shows that this phenomenon is no longer true for the super-situation when $\emptyset \in \Delta$ (or equivalently, $\mathbb{T} \neq T$):

Example 5.7. Suppose that our base field $k$ satisfies $-1 \notin (k^\times)^2$, that is, $k$ does not contain $x$ such that $x^2 = -1$. Let $\mathfrak{g}$ be the queer supergroup $Q(2)$ over $k$. Take $T$ to be the standard maximal torus of $\mathfrak{g}_{ev} \cong \text{GL}_2$ and identify $X(T)$ with $\mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2$ as before. By construction ([ShI Section 4.1]), the simple left $\mathfrak{T}$-supermodule $u(\lambda)$ is a unique simple supermodule over the Clifford superalgebra $\mathfrak{Cl}(\mathfrak{h}_1, b^\lambda)$ of $\mathfrak{h}_1 = \text{Lie}(\mathfrak{T})_1$ with the symmetric bilinear form

$$b^\lambda : \mathfrak{h}_1 \times \mathfrak{h}_1 \longrightarrow k; \ (x, y) \longmapsto \lambda(|x, y|).$$

Let $\lambda = \lambda_1 - 2\lambda_2 \in X(T)$. For $Q(n)$, by [BrK] Theorem 6.11, we know that

$$X(T^p) = \left\{ \sum_{i=1}^n c_i \lambda_i \in \bigoplus_{i=1}^n \mathbb{Z}\lambda_i \mid c_1 \geq \cdots \geq c_n \text{ and } [c_i - c_{i+1}] = p \mid c_i \right\},$$

where $\overline{k}$ denotes the algebraic closure of $k$ and $p := \text{char}(k)$. Thus, by Proposition 5.5 we have $\lambda \in X(T^p) \subset X(T^p)$. It is easy to see that $\mathfrak{Cl}(\mathfrak{h}_1, b^\lambda)$ is isomorphic to the quaternion superalgebra $\left( -\frac{1}{k}, -1 \right) : = k\langle 1, j \rangle / (i^2 + 1, j^2 + 1, ij + ji)$ with $|1| = |j| = 1$ over $k$, and hence $u(\lambda)$ is a 4-dimensional vector space over $k$.

On the other hand, since the base change of $\mathfrak{Cl}(\mathfrak{h}_1, b^\lambda)$ to the filed $k' := k[X] / (X^2 + 1)$ is isomorphic to the matrix superalgebra $\text{Mat}_{1|1}(k')$, its simple supermodule is a 2-dimensional vector space over $k'$, which we denote by $u_{k'}(\lambda)$. By Theorem 5.2 we have

$$(L(\lambda) \otimes k')^\lambda \cong u(\lambda) \otimes k' \supseteq u_{k'}(\lambda) \cong L_{k'}(\lambda)^\lambda.$$

Thus, we conclude that $L(\lambda) \otimes k' \supseteq L_{k'}(\lambda)$. □

5.2. Simple $\mathfrak{g}_r$-supermodules. Throughout the rest of the paper, we suppose that $k$ is a perfect field of characteristic $p > 2$.

In the following, we fix a positive integer $r$. As in Section 3.4, the $r$-th Frobenius kernel $\mathbb{B}^+_r$ (resp. $\mathbb{B}_r$) of $\mathbb{B}^+$ (resp. $\mathbb{B}$) are infinitesimal and normal.

Recall that, for each $\lambda \in X(T)$, we have regarded the simple left $\mathfrak{T}$-supermodule $u(\lambda)$ as a left $\mathbb{B}$-supermodule. We also regard $u(\lambda)$ as a left $\mathbb{B}_r$-supermodule (resp. $\mathfrak{T}_r$-supermodule) via the inclusion $\mathbb{B}_r \subset \mathbb{B}$ (resp. $\mathfrak{T}_r \subset \mathfrak{T}$), which we again denote by the same symbol. By Proposition 5.8 we have $(\mathfrak{T}_r)^{ev} = T_r$, and hence we get the following result:

Lemma 5.8. For $\lambda, \mu \in X(T)$, we have $u(\lambda + pr^\mu) \cong u(\lambda)$ as left $\mathbb{B}_r$-supermodules.

By definition, we get the short exact sequence $0 \to p^r X(T) \to X(T) \to X(T_r) \to 0$, where $X(T) \to X(T_r)$ is the restriction map induced from $T_r \subset T$. Thus, by Lemma 5.8 for each $\lambda \in X(T_r)$, we can define a left $\mathbb{B}_r$-supermodule structure on $u(\lambda)$ in an obvious way.
**Proposition 5.9.** For each \( \lambda \in X(T_r) \), the induced left \( \mathcal{G}_r \)-supermodule \( \text{ind}_{\mathcal{B}_r}^{\mathcal{G}_r}(u(\lambda)) \) of \( u(\lambda) \) has a unique simple left \( \mathcal{G}_r \)-super-submodule \( L_r(\lambda) \). Moreover, the map \( X(T_r) \to \text{Simple}_{\mathcal{G}_r}(\mathcal{G}_r) ; \lambda \mapsto L_r(\lambda) \) is bijective.

**Proof.** Since the superalgebra map \( \mathcal{O}(\mathcal{G}_r) \to \mathcal{O}(\mathcal{B}_r) \otimes \mathcal{O}(\mathcal{B}_r) \) induced from the multiplication on \( \mathcal{G}_r \) is injective, one easily sees that the same argument as in \[\text{Shi1, Theorem 4.12}\] works for the quadruple \((\mathcal{G}_r, \mathcal{B}_r, \mathcal{B}_r, T_r)\). Thus, to prove the claim, it is enough to show that \( X(T_r) = \{ \lambda \in X(T_r) \mid \text{ind}_{\mathcal{B}_r}^{\mathcal{G}_r}(u(\lambda)) \neq 0 \} \).

It is easy to see that the multiplication on \( \mathcal{G}_r \) induces an isomorphism \( \mathcal{B}_r \times \mathcal{B}_r \to \mathcal{G}_r \) of superschemes, where \( \mathcal{B}_r \) is the \( r \)-th Frobenius kernel of \( \mathcal{U}^+ \). Since the isomorphism is compatible with the left \( \mathcal{B}_r \)-multiplication, we get an isomorphism \( \mathcal{O}(\mathcal{G}_r) \cong \mathcal{O}(\mathcal{B}_r) \otimes \mathcal{O}(\mathcal{U}_r^+) \) of left \( \mathcal{O}(\mathcal{B}_r) \)-supercomodules. Thus, we have
\[
\text{ind}_{\mathcal{B}_r}^{\mathcal{G}_r}(u(\lambda)) = u(\lambda) \square_{\mathcal{O}(\mathcal{B}_r)} \mathcal{O}(\mathcal{G}_r) \cong u(\lambda) \otimes \mathcal{O}(\mathcal{U}_r^+) 
\]
for each \( \lambda \in X(T_r) \). Thus, we are done. \[\square\]

The proof of Proposition 5.9 shows that the dimension of \( \text{ind}_{\mathcal{B}_r}^{\mathcal{G}_r}(N) \) is given by \( p^r \# \Delta_0^+ \cdot 2^{n_1^+} \cdot \dim(N) \) for each left \( \mathcal{B}_r \)-supermodule \( N \), where \( n_1^+ := \sum_{\gamma \in \Delta_1^-} \dim(g_1^+) \).

Since \( X(T_r) \cong X(T)/p^rX(T) \), we get the following result:

**Proposition 5.10.** For all \( \lambda, \mu \in X(T) \), we have \( L_r(\lambda + p^r\mu) \cong L_r(\lambda) \) as left \( \mathcal{G}_r \)-supermodules.

We get the following result, whose proof is similar to that of Proposition 5.6.

**Proposition 5.11.** If \( 0 \notin \Delta \) (or equivalently, \( T = T \)), then all simple left \( \mathcal{G}_r \)-supermodules are absolutely simple. In particular, for a field extension \( k' \) of \( k \) and a left \( \mathcal{G}_r \)-supermodule \( V \), we have (1) \( \text{soc}_{\mathcal{G}_r}(V) \otimes k' \cong \text{soc}_{\mathcal{G}_r}(V) \otimes k' \); and (2) \( V \) is \( \mathcal{G}_r \)-semisimple if and only if \( V \otimes k' \) is \( \mathcal{G}_r \)-semisimple.

Recall that, in Section 1.4, we have used the character \( \chi_r \in X(\mathcal{G}) \) to discuss the unimodularity of \( \mathcal{G}_r \). Since \( \mathcal{B}_r^+ \) is infinitesimal and normal, by Lemma 4.8, we also find a unique character \( \psi_r \in g.l.(\mathcal{O}(\mathcal{B}_r^+)) \) of \( \mathcal{O}(\mathcal{B}_r^+) \) such that \( \text{coad}_{\mathcal{O}(\mathcal{B}_r^+)}(\phi_{\mathcal{B}_r^+}) = \phi_{\mathcal{B}_r^+} \otimes \psi_r \), where \( \text{coad}_{\mathcal{O}(\mathcal{B}_r^+)} : \mathcal{O}(\mathcal{B}_r^+) \to \mathcal{O}(\mathcal{B}_r^+) \otimes \mathcal{O}(\mathcal{B}_r^+) \) is the induced right \( \mathcal{O}(\mathcal{B}_r^+) \)-coaction on \( \mathcal{O}(\mathcal{B}_r^+) \). In the following, we let \( \epsilon_r \) denote the sum of the parity of the integral \( \phi_{\mathcal{G}_r} \) and \( \phi_{\mathcal{B}_r^+} \), see Proposition 1.1.

We set
\[
\delta_r := \chi_r \mid_{\mathcal{B}_r^+} \cdot \psi_r^{-1}.
\]
as an element of \( X(\mathcal{B}_r^+) \cong g.l.(\mathcal{O}(\mathcal{B}_r^+)) \). Since \( -\Delta_0^- = \Delta_0^+ \) and \( \Delta_1 = \Delta_1^+ \sqcup \Delta_1^- \), we have
\[
\delta_r \mid_T = -(p^r - 1) \sum_{\alpha \in \Delta_0^+} \alpha + \sum_{\gamma \in \Delta_1^-} \dim(g_1^+) \gamma.
\]
Here, we write the group low of \( X(T) \) additively. In particular, \( \delta_r \mid_T = \sum_{\alpha \in \Delta_0^+} \alpha + \sum_{\gamma \in \Delta_1^-} \dim(g_1^+) \gamma \). In the following, we simply write \( \delta_r \mid_{\mathcal{B}_r^+} \) by \( \delta_r \).

For a left \( \mathcal{B}_r^+ \)-supermodule \( N \), we set
\[
\text{coind}_{\mathcal{B}_r^+}^{\mathcal{G}_r}(N) := \text{hy}(\mathcal{G}_r) \otimes_{\text{hy}(\mathcal{B}_r^+)} N.
\]
We fix a non-zero composition \( \sum \mathcal{H}_\gamma(BZ) \). The induced morphism is an isomorphism. □

For any \( u \) of \( M \), the argument above shows that \( u \) is simple. By Shih Proposition 4.15 (for \( \mathcal{G}_r \)), the \( \lambda \)-weight superspace \( L_r(\lambda) \) is isomorphic to \( u(\lambda) \) as \( \mathcal{T}_r \)-supermodules. Moreover, it was also shown that the action of \( U_r^+ \) on \( L_r(\lambda) \) is trivial, that is, \( \lambda \) is a “highest” \( T \)-weight of \( L_r(\lambda) \). Thus, we have

\[
\mathcal{G}_r \text{ Hom}(M_r(\lambda), L_r(\lambda)) \cong B^+_\mathcal{G}_r \text{ Hom}(u(\lambda), L_r(\lambda)) \neq 0.
\]

We fix a non-zero \( \mathcal{G}_r \)-endomorphism \( M_r(\lambda) \to L_r(\lambda) \), which is surjective since \( L_r(\lambda) \) is simple. By the definition of the radical, this morphism must factor through the quotient \( \text{top}_{\mathcal{G}_r}(M_r(\lambda)) = M_r(\lambda)/\text{rad}_{\mathcal{G}_r}(M_r(\lambda)) \). Since the quotient is simple, the induced morphism is an isomorphism.

\[\]

Proposition 5.12 ([MZ Proposition 13 and Lemma 14]). Let \( N \) be a left \( B^+_r \)-supermodule. Then there is an isomorphism \( \text{coind}_{\mathcal{G}_r}^B(N) \cong \Pi^{\ast} \text{ind}_{\mathcal{G}_r}^B(N \otimes \delta_r) \) of left \( \mathcal{G}_r \)-supermodules. If \( N \) is finite, then \( \text{coind}_{\mathcal{G}_r}^B(N)^* \cong \text{ind}_{\mathcal{G}_r}^B(N^*) \).

For each \( \lambda \in X(T) \), we set

\[ M_r(\lambda) := \text{coind}_{\mathcal{G}_r}^B(u(\lambda)). \]

Note that, \( M_r(\lambda) \) is finite-dimensional. Let \( \text{rad}_{\mathcal{G}_r}(M_r(\lambda)) \) denote the \( \mathcal{G}_r \)-radical of \( M_r(\lambda) \), that is, the intersection of all maximal \( \mathcal{G}_r \)-super-submodules of \( M_r(\lambda) \). Set \( \text{top}_{\mathcal{G}_r}(M_r(\lambda)) := M_r(\lambda)/\text{rad}_{\mathcal{G}_r}(M_r(\lambda)) \).

Proposition 5.13. For each \( \lambda \in X(T) \), \( \text{top}_{\mathcal{G}_r}(M_r(\lambda)) \) is isomorphic to \( L_r(\lambda) \) as left \( \mathcal{G}_r \)-supermodules.

Proof. First, we show that \( \text{top}_{\mathcal{G}_r}(M_r(\lambda)) \) is simple. To show this, we note that for any \( \lambda \in X(T) \), \( M_r(\lambda) \) has a unique simple \( \mathcal{G}_r \)-super-submodule. Indeed, by Proposition 5.12 we get

\[ M_r(\lambda) \cong \Pi^{\ast} \text{ind}_{\mathcal{G}_r}^B(u(\lambda) \otimes \delta_r) \cong \Pi^{\ast} \text{ind}_{\mathcal{G}_r}^B(u(\lambda + \delta_r)). \]

Then by mimicking the proof given in Proposition 5.9 this proves the claim. The dual of \( \text{top}_{\mathcal{G}_r}(M_r(\lambda)) \) can be naturally regarded as a \( \mathcal{G}_r \)-super-submodule of \( M_r(\lambda)^* \). Since \( M_r(\lambda) \) is finite-dimensional, \( \text{top}_{\mathcal{G}_r}(M_r(\lambda)) \) is semisimple as a left \( \mathcal{G}_r \)-supermodule. Thus, to prove that \( M_r(\lambda) \) is simple, it is enough to show that \( M_r(\lambda)^* \) has a unique simple \( \mathcal{G}_r \)-super-submodule. By Proposition 5.12 and \( u(\lambda)^* = u(-\lambda) \), we have

\[ M_r(\lambda)^* \cong \text{ind}_{\mathcal{G}_r}^B(u(-\lambda)) \cong \Pi^{\ast} \text{coind}_{\mathcal{G}_r}^B(u(-\lambda - \delta_r)) = \Pi^{\ast} M_r(-\lambda - \delta_r). \]

Thus, the argument above shows that \( \text{top}_{\mathcal{G}_r}(M_r(\lambda)) \) is simple.

By Shih Proposition 4.15 (for \( \mathcal{G}_r \)), the \( \lambda \)-weight superspace \( L_r(\lambda)^\lambda \) of \( L_r(\lambda) \) is isomorphic to \( u(\lambda) \) as \( \mathcal{T}_r \)-supermodules. Moreover, it was also shown that the action of \( U_r^+ \) on \( L_r(\lambda)^\lambda \) is trivial, that is, \( \lambda \) is a “highest” \( T \)-weight of \( L_r(\lambda) \). Thus, we have

\[ \mathcal{G}_r \text{ Hom}(M_r(\lambda), L_r(\lambda)) \cong B^+_\mathcal{G}_r \text{ Hom}(u(\lambda), L_r(\lambda)) \neq 0. \]
5.3. Bases of odd roots. Recall that, \( \triangle \) is the root system of \( \mathfrak{g} \) with respect to \( T \), and the quadruple \( (X(T), \triangle, X(T)^\vee, \triangle_0^\vee) \) is the root datum of the pair \((\mathfrak{g}_{ev}, T)\). Let \( \Psi_0 \) be the base of \( \triangle_0 \) with respect to \( \Upsilon \), in other words, \( \Psi_0 \) is the set of all simple roots in \( \triangle_0^\vee \) (see [3] Chapter 21d).

By [3] Part I, Proposition 7.19 and Remark(2) (see also [26] Theorem 2.1), as an algebra, \( hy(\mathfrak{g}_{ev}) \) is generated by \( hy(T) \) and \( hy(U_{\pm \alpha}) \) for \( \alpha \in \Psi_0 \), where \( U_\alpha \) is the \( \alpha \)-root subgroup of \( \mathfrak{g}_{ev} \). Note that, \( hy(U_\alpha) = \bigoplus_{n=0}^{\infty} kX_\alpha^{(n)} \) and \( hy(U_\alpha^\vee) \) is generated by \( \{X_{\alpha^{(n)}} | \alpha \in \Psi_0, n_\alpha \in \mathbb{N}\} \) as an algebra. By “SL theory”, we get the following commutator formula (see [Hi] Section 26 for example):

\[
X_\alpha^{(m)}X_\beta^{(n)} = \sum_{i=0}^{\min\{m,n\}} X_{\alpha^{(n-i)}}^{-1}H_{\alpha}^{-m-n+2i}X_{\alpha^{(m-i)}}
\]

for all \( m, n \in \mathbb{N} \cup \{0\} \) and \( \alpha \in \triangle_0 \), where \( H_\alpha := [X_\alpha, X_{-\alpha}] \).

In general, the root system of a split quasireductive supergroup is ill-behaved (see Example 3.3(1) for example). For this reason, we shall deal with a split quasireductive supergroup having a good “simple roots”, as follows:

**Definition 5.14.** A subset \( \Psi_1 \) of \( \triangle_1^+ \) is called a special base of \( \triangle \) if it satisfies the following three conditions:

1. For each \( \gamma \in \triangle_1^+ \), the odd part of the \( \gamma \)-weight super-subspace \( \mathfrak{g}^\gamma_1 \) of \( \mathfrak{g} \) is contained in the Lie super-subalgebra of \( \mathfrak{g} \) generated by \( \{Y_{(\gamma,j)} \in \mathfrak{g}_1^\gamma | \gamma \in \Psi_1, 1 \leq j \leq \dim(\mathfrak{g}_1^\gamma)\} \).
2. For all \( \alpha \in \Psi_0 \) and \( \gamma \in \Psi_1 \) with \( \alpha \neq \gamma \), we have \( \gamma - \alpha \notin \triangle \).
3. If \( \Psi_0 \cap \Psi_1 \neq \emptyset \), then \( \dim(\mathfrak{g}_{1,0}^{\alpha \gamma}) = 1 \) for all \( \alpha \in \Psi_0 \cap \Psi_1 \).

In this case, we say that the pair \( (\Psi_0, \Psi_1) \) is a special base of \( \triangle \). To clarify, we shall say that \( \Psi_0 \) is an even base of \( \triangle \).

Note that, \( \Psi_0 \) and \( \Psi_1 \) (if it exists) do depend on the choice of \( \Upsilon : \mathbb{Z}\triangle \to \mathbb{R} \).

**Example 5.15.** We use the notations in Example 3.3. In the following, we shall extend the domain \( \mathbb{Z}\triangle \) of \( \Upsilon \) to \( X(T) \) just for simplicity.

1. For the general linear supergroup \( \mathfrak{gl}(m|n) \), it is natural to take \( \Upsilon(\lambda_i) := -i \) for each \( 1 \leq i \leq m+n \). Then an even base of \( \triangle \) is given as \( \Psi_0 = \{\lambda_1 - \lambda_{i+1} | 1 \leq i \leq m-1 \text{ or } m+1 \leq i \leq m+n-1\} \) and \( \Psi_1 = \{\lambda_m - \lambda_{m+1}\} \) is a special odd base of \( \triangle \).
2. For the queer supergroup \( \mathfrak{q}(n) \), we define \( \Upsilon(\lambda_i) := -i \) for each \( 1 \leq i \leq n \). Then \( \Psi_0 = \{\lambda_i - \lambda_{i+1} | 1 \leq i \leq n-1\} \) is an even base of \( \triangle \) and \( \Psi_1 := \Psi_0 \) is a special odd base of \( \triangle \). Note that, \( \dim(\mathfrak{q}(n)^\alpha) = 1 \) for all \( \alpha \in \triangle \setminus \{0\} \).
3. For the periplectic supergroup \( \mathfrak{p}(n) \), we define \( \Upsilon(\lambda_i) := n-i+1 \) for each \( 1 \leq i \leq n \). Then \( \Psi_0 = \{\lambda_i - \lambda_{i+1} | 1 \leq i \leq n-1\} \) is an even base of \( \triangle \) and \( \Psi_1 = \{2\lambda_n\} \) is a special odd base of \( \triangle \).
4. For a Chevalley supergroup \( \mathfrak{g} \) of classical type, Fiorese and Gavarini find a special base of the root system of \( \text{Lie}(\mathfrak{g}) \) ([28] Section 3.3, see also [29] Theorem 5.35]).
5. We consider the algebraic supergroup \( \mathfrak{f}^{(g_1,\ldots,g_n)} = F \ltimes (G_{\mathbb{Z}})^n \) given in Example 3.3(1). Suppose that \( F = \mathfrak{gl}_n \) with standard split maximal torus \( T \) and \( \Upsilon(\lambda_i) := -i \) for each \( 1 \leq i \leq n \). Then \( \triangle_0^+ = \{\pm(\lambda_i - \lambda_j) | 1 \leq i < j \leq n\} \), \( \triangle_1 = \triangle_1^+ = \{-\chi_i | 1 \leq i \leq n\} \) and \( \triangle_1^- = \emptyset \). Since
Suppose that Lemma 5.17.

Remark 5.16. We give some remarks on special odd bases of root systems.

1. We explain the notion of special odd bases of $\Delta$ depends on the choice of $\Upsilon$.

Suppose that $G = P(2)$ and $\Upsilon(\lambda_i) = -i$ for each $1 \leq i \leq n$. Then one sees that $\Delta^+ = \{\lambda_1 - \lambda_2, -(\lambda_1 + \lambda_2)\}$ and $\Delta^- = \{-(\lambda_1 - \lambda_2), \lambda_1 + \lambda_2, 2\lambda_1, 2\lambda_2\}$. The even base is $\Psi_0 = \Delta_0^+ = \{\lambda_1 - \lambda_2\}$. If we let $\Psi_1 := \Delta_1^+ = \{-(\lambda_1 + \lambda_2)\}$, then obviously this satisfies Definition 5.13(2). However, one easily sees that $\Psi_1$ does not satisfy Definition 5.13(1). Thus, in this case, $\Delta$ does not have a special odd base.

2. If $\Psi_1$ is a special odd base of $\Delta$, then by Definition 5.13(1), we get

$$\Delta^+ = \Delta_0^+ \cup \Delta_1^+ \subset Z_{\geq 0}\Psi_0 + Z_{\geq 0}\Psi_1,$$

where $Z_{\geq 0} := \mathbb{N} \cup \{0\}$ and $Z_{\geq 0}\Psi_\epsilon := \{\sum_i c_i \alpha_i \mid c_i \in Z_{\geq 0}, \alpha_i \in \Psi_\epsilon\} (\epsilon \in \mathbb{Z}_2)$. However, since the dimension of an odd root space of $g$ may be greater than one (see Example 4.7[3]), the converse does not hold in general.

Lemma 5.17. Suppose that $\Delta$ has a special base $(\Psi_0, \Psi_1)$. Let $\lambda \in X(T)^\flat$ and $\alpha \in \Psi_0 \setminus \Psi_1$. If $n \geq \langle \lambda, \alpha^\vee \rangle + 1$, then $X_{\beta}^{(n)} \to w = 0$ for all $w \in \hy(U^\flat)$. By Definition 5.13(1), this is equivalent to saying that the following two conditions are satisfied:

1. $X_{\beta}^{(m)} \to w = 0$ for all $\beta \in \Psi_0$ and $m \in \mathbb{N}$.
2. $Y_{(\gamma,j)} \to w = 0$ for all $\gamma \in \Psi_1$ and $1 \leq j \leq \dim(g^\gamma)$.

As in the non super-situation, the condition (i) is clear. However, for convenience for the reader, we shall give a proof. If $\alpha \neq \beta$, then it is known that $U_{\alpha}$ commutes with $U_{\beta}$, and hence $\hy(U_{\alpha})$ commutes with $\hy(U_{\beta})$, see [13, Proposition 2.3]. Since $\nu^\lambda$ is a “maximal” vector, we have $X_\beta^{(m)} \to w = X_{-\alpha}^{(n)} \to X_\gamma^{(m)} \to \nu^\lambda = 0$. Suppose that $\alpha = \beta$. By the commutator formula (5.3), we may assume that $n \geq m$ and get

$$X_\alpha^{(m)} \to w = X_{-\alpha}^{(n-m)} \to \left(\frac{\lambda(H_{\alpha}) + m - n}{m}\right)\nu^\lambda.$$ 

Since $n$ is supposed to be greater than $\lambda(H_{\alpha}) = \langle \lambda, \alpha^\vee \rangle$, we have $X_\alpha^{(m)} \to w = 0$.

Next, we show the condition (ii). Since we have assumed that $\alpha \notin \Psi_1$, we especially get $\alpha \neq \gamma$. Then by Definition 5.13(2), we get $[Y_{(\gamma,j)}, X_{-\alpha}] = 0$, and hence $Y_{(\gamma,j)}X_{-\alpha} = X_{-\alpha}Y_{(\gamma,j)}$. Thus, we get $Y_{(\gamma,j)} \to w = (X_{-\alpha}Y_{(\gamma,j)}) \to \nu^\lambda = X_{-\alpha} \to (Y_{(\gamma,j)} \to \nu^\lambda) = 0$. The proof is done.

If $\Psi_0 \cap \Psi_1 \neq \emptyset$, then we put $K_{\alpha} := [X_{\alpha}, Y_{-\alpha}] (\in \mathfrak{j}_1)$ for each $\alpha \in \Psi_0 \cap \Psi_1$. For the notation $Y_{-\alpha}$, see the end of Section 5.2.

Definition 5.18. Suppose that $\Delta$ has a special base $(\Psi_0, \Psi_1)$. An element $\lambda \in X(T)^\flat$ is called the $p^r$-restricted weight for $G$ if it satisfies the following conditions for all $\alpha \in \Psi_0$:

1. For the case when $\alpha \notin \Psi_1$. Then $\langle \lambda, \alpha^\vee \rangle \leq p^r - 1$. 

Remark 5.19. Suppose that \( L/BZ \) of \( X(1) \), \( \text{BrKl} \), respectively. For \( \mathcal{G} = \mathfrak{SpO}(m|n) \), the above \( X_r(T)^b \) is denoted by \( X_r(T) \) in \([SW]\).

Let \( V \) be a left \( \mathcal{G}_\mathfrak{r} \)-supermodule. Recall that the induced action of \( u \in \text{hy}(\mathcal{G}_\mathfrak{r}) \) on \( v \in V \) is denoted by \( u \rightarrow v \), see (2.2). For simplicity, we set \( \text{hy}(\mathcal{G}_\mathfrak{r}) \rightarrow V := \{ u \rightarrow v \mid u \in \text{hy}(\mathcal{G}_\mathfrak{r}), v \in V \} \). The next is a key-lemma in this paper whose proof is essentially based on the proof of \([BrKl]\) Lemma 9.8:

Lemma 5.20. Suppose that \( \triangle \) has a special base \( (\Psi_0, \Psi_1) \). For each \( \lambda \in X_r(T)^b \), \( \text{hy}(\mathcal{G}_\mathfrak{r}) \rightarrow L(\lambda)^\mathfrak{r} \) forms a left \( \mathcal{G} \)-supermodule. In particular, \( L(\lambda) = \text{hy}(\mathcal{G}_\mathfrak{r}) \rightarrow L(\lambda)^\mathfrak{r} \).

Proof. First of all, we note that the second claim follows from the first one and the simplicity of \( L(\lambda) \). By Theorem 5.3, it is enough to show that \( M := \text{hy}(\mathcal{G}_\mathfrak{r}) \rightarrow L(\lambda)^\mathfrak{r} \) is \( \text{hy}(\mathcal{G}) \)-invariant. Since \( \text{hy}(\mathcal{G}) \) is generated by \( \text{hy}(\mathcal{G}_\mathfrak{ev}) \) and \( \text{hy}(\mathcal{G}_\mathfrak{r}) \), we shall see that \( M \) is \( \text{hy}(\mathcal{G}_\mathfrak{ev}) \)-invariant. For \( x \in \text{hy}(\mathcal{G}) \) and \( u \in \text{hy}(\mathcal{G}_\mathfrak{r}) \), we get
\[
xu = \sum_{x,u} (-1)^{|u_1||x_2|+|u_1||x_3|} x(u_1)u_{(2)}S(x_{(2)})S(u_{(3)})u_3x_3
\]
where \( [\ , \ ] \) denotes the super-bracket (2.1). Since \( \mathcal{G}_\mathfrak{r} \) is a normal super-subgroup of \( \mathcal{G} \), we have \( [x_{(1)}, u_{(1)}] \in \text{hy}(\mathcal{G}_\mathfrak{r}) \) by Proposition 2.8. Thus by (5.1), we see that \( M = \text{hy}(\mathcal{G}_\mathfrak{ev}) \)-stable if and only if \( x \rightarrow L(\lambda)^\mathfrak{r} \subset M \) for all \( x \in \text{hy}(\mathcal{U}_\mathfrak{ev}) \), since \( \mathcal{U}_\mathfrak{ev} \) trivially acts on \( L(\lambda)^\mathfrak{r} \). Moreover, by Theorem 5.11, it is enough to show that
\[
X_\alpha^{(n)} \rightarrow \nu^\lambda \in M \quad \text{for all} \quad \alpha \in \Psi_0, n \geq p^r \quad \text{and} \quad \nu^\lambda \in L(\lambda)^\mathfrak{r}.
\]
If \( \alpha \in \Psi_0 \setminus \Psi_1 \), then \( X_\alpha^{(n)} \rightarrow \nu^\lambda = 0 \) by Lemma 5.17 and Definition 5.18(1).

Thus, in the following, we suppose that \( \alpha \in \Psi_0 \cap \Psi_1 \). For simplicity, we write \( n = p^r + m \) for some \( m \in \mathbb{N} \cup \{ 0 \} \). If \( p \nmid \lambda([K_\alpha, K_\alpha]) \), then we put \( c := \lambda([K_\alpha, K_\alpha])^{-1} \in \mathbb{N} \). Otherwise, we put \( c := 1 \). Set \( K := 2cK_\alpha \). We note that, \( K_\alpha K = c[K_\alpha, K_\alpha] \in \mathfrak{h}_0 \). First, we show that
\[
X_\alpha^{(p^r+m)} \rightarrow \nu^\lambda = X_\alpha^{(p^r+m-1)}Y_\alpha K \rightarrow \nu^\lambda.
\]
To show this, we suppose that \( w := (X_\alpha^{(p^r+m)} - X_\alpha^{(p^r+m-1)}Y_\alpha K) \rightarrow \nu^\lambda \) is non-zero. If \( x \rightarrow w = 0 \) for all \( x \in \text{hy}(\mathcal{U}_\mathfrak{r}) \), then \( \text{hy}(\mathcal{G}) \rightarrow w \) forms a proper super-submodule of \( L(\lambda) \), a contradiction. Thus, there exists a PBW monomial \( x \in \text{hy}(\mathcal{U}_\mathfrak{r}) \) such that \( x \rightarrow w \neq 0 \) and \( x \rightarrow w \in L(\lambda)^\mathfrak{r} \). By comparing weights and by Definition 5.14, such \( x \) must be of the form (i) \( x = X_\alpha^{(p^r+m)} \) or (ii) \( x = Y_\alpha X_\alpha^{(p^r+m-1)} \). For the case (i), by the commutator formula (5.3), we have
\[
x \rightarrow w = \left(\frac{\lambda(H_\alpha)}{p^r+m}\right)\nu^\lambda = \left(\frac{\lambda(H_\alpha)}{p^r+m}-1\right)\nu^\lambda
\]
By comparing weights and by Definition 5.14, such \( x \) must be of the form (i) \( x = X_\alpha^{(p^r+m)} \) or (ii) \( x = Y_\alpha X_\alpha^{(p^r+m-1)} \). For the case (i), by the commutator formula (5.3), we have
\[
x \rightarrow w = \left(\frac{\lambda(H_\alpha)}{p^r+m}-1\right)\nu^\lambda = \left(\frac{\lambda(H_\alpha)}{p^r+m}-\lambda(H_\alpha)\right)\nu^\lambda.
\]
By Definition 5.18, we get \( x \to w = 0 \) for all \( m \). Also, for the case (ii), we have
\[
x \to w = Y_\alpha(X_{\alpha} \left( \frac{\lambda(H_\alpha) - 1}{p^r + m - 1} - \frac{\lambda(H_\alpha) - \alpha(H_\alpha)}{p^r + m - 1} \right) Y_{\alpha}K
- X_{\alpha} \left( \frac{\lambda(H_\alpha) - 2}{p^r + m - 2} \right) X_{\alpha} Y_{\alpha}K \to v^\lambda
= \left( \lambda(H_\alpha) - 1 \right) \frac{\lambda(H_\alpha) - 2}{p^r + m - 2} c\lambda([K, K_\alpha])[Y_{\alpha}, X_{\alpha}] \to v^\lambda.
\]
The second equation follows from \( \alpha(H_\alpha) = \langle \alpha, \alpha^\vee \rangle = 2 \). Thus, by the same reason as (i), we get \( x \to w = 0 \) for all \( m \). This is a contradiction, and hence \( w = 0 \). This proves the equation (5.6).

Finally, we show (5.3) by induction on \( m \). If \( m = 0 \), then (5.5) implies that \( X_{\alpha}^{(p^r)} \to v^\lambda = X_{\alpha}^{(p^r - 1)} Y_{\alpha}K \to v^\lambda \). The right hand side actually belongs to \( M \) (see Theorem 3.11). Suppose that \( m \geq 1 \). Then by (5.3) and the argument at the beginning of the proof, we get
\[
X_{\alpha}^{(p^r + m)} \to v^\lambda = \sum_u \sum_{i+j=p^r+m-1} [u(1), X_{\alpha}^{(j)}] u(2) X_{\alpha}^{(j)} \to v^\lambda
\]
where \( u := Y_{\alpha}K \in \hy(G_r) \). Here, we have used (5.1). Thus, by the induction hypothesis, we get \( X_{\alpha}^{(p^r + m)} \to v^\lambda \in M \). This completes the proof. □

5.4. Steinberg’s tensor product theorem. Throughout the rest of the paper, we assume that the root system \( \Delta \) of \( G \) (with respect to \( T \)) has a special base \( (\Psi_0, \Psi_1) \), see Definition 5.14.

As we have seen in Section 5.1, not all simple supermodules are absolutely simple, in general (see Proposition 5.6 and Example 5.7). Thus, we also assume the following condition on our base field \( k \).

Assumption 5.21. If \( 0 \in \Delta \) (or equivalently, \( T \neq T \)), then we assume that the base field \( k \) is algebraically closed.

We naturally regard a left \( G \)-supermodule as a left \( G_r \)-supermodule via the inclusion \( G_r \subset G \) as before. The following proof is due to Brundan and Kleshchev [BrK] Lemma 9.6:

Lemma 5.22. Any simple left \( G \)-supermodule is semisimple as a left \( G_r \)-supermodule.

Proof. If \( 0 \notin \Delta \), then to prove the claim, we may assume that \( k \) is algebraically closed by Propositions 5.6 and 5.11. Otherwise, by Assumption 5.21 \( k \) is supposed to be algebraically closed.

Let \( L \) be a simple left \( G \)-supermodule. Since \( L \neq 0 \), we get \( \soc_{G_r}(L) \neq 0 \) (see [Sh1] Lemma A.3)). Thus, we fix a simple left \( G_r \)-supermodule \( S \) of \( L \). Note that, the \( k \)-valued points \( G_r(\mathbb{A}, \mathfrak{m}) \) coincides with \( (G_r)_r(k) \) by Proposition 5.8. For each \( g \in (G_r)_r(k) \), \( g.S := \{ g.v \in L \mid v \in S \} \) becomes a simple left \( (G_r)_r \)-submodule of \( L \), since \( (G_r)_r \) is a normal subgroup of \( G_r \). Thus, \( M := \sum_{g \in (G_r)_r(k)} g.S \) forms a semisimple left \( (G_r)_r \)-submodule of \( L \). In particular, \( M \) is a left \( \hy((G_r)_r) \)-submodule of \( L \) by Theorem 3.3.

On the other hand, it is obvious that \( M \) is a left \( G_r \)-submodule of \( L \). Since \( k \) is algebraically closed and \( G_r \) is reduced, \( M \) is actually a left \( G_r \)-submodule of \( L \), see [H] Part I, Section 2.8). Thus, again by Theorem 3.3 \( M \) is also a locally finite left \( \hy(G_r) \)-submodule of \( L \). By Theorem 3.11 as a superalgebra, \( \hy(G_r) \)
is generated by \( \text{hy}(\mathcal{G}_r) \) and \( \text{hy}(\mathcal{G}_r) \). This shows that \( M \) is actually a locally finite left \( \text{hy}(\mathcal{G})_T \)-supermodule, and hence a left \( \mathcal{G}_r \)-supermodule again by Theorem 5.23. Since \( L \) is simple, we get \( L = M \). The proof is done. \( \square \)

**Proposition 5.23.** For \( \lambda \in X_r(T)^{\flat} \), we have \( \text{hy}(\mathcal{G}_r) \hookrightarrow L(\lambda)^{\flat} \cong L_r(\lambda) \) as left \( \mathcal{G}_r \)-supermodules. In particular, \( L(\lambda) \cong L_r(\lambda) \) as left \( \mathcal{G}_r \)-supermodules.

**Proof.** Since \( L(\lambda)^{\flat} \cong u(\lambda) \) as left \( B_r^{\flat} \)-supermodules, we get
\[
0 \neq \text{hy}(u(\lambda), L(\lambda)) \cong \mathfrak{g}_r \cdot \text{Hom}(u(\lambda), L(\lambda)).
\]
Thus, the following is a non-zero surjective homomorphism of \( \mathcal{G}_r \)-supermodules:
\[
\varphi : M_r(\lambda) \twoheadrightarrow \text{hy}(\mathcal{G}_r) \rightarrow L(\lambda)^{\flat}; \quad u \otimes_{\text{hy}(B_r^{\flat})} v \mapsto u \rightarrow v,
\]
where \( u \in \text{hy}(\mathcal{G}_r) \), \( v \in u(\lambda) \). Since the quotient \( M_r(\lambda)/\text{Ker}(\varphi) \) is semisimple \( \mathcal{G}_r \)-supermodule by Lemmas 5.20 and 5.22, the radical of \( M_r(\lambda) \) is contained in the kernel of \( \varphi \). This shows that there exists a surjective homomorphism
\[
\text{top}_{\mathfrak{g}_r}(M_r(\lambda)) \rightarrow M_r(\lambda)/\text{Ker}(\varphi) \cong \text{hy}(\mathcal{G}_r) \rightarrow L(\lambda)^{\flat},
\]
of left \( \mathcal{G}_r \)-supermodules. This map is actually bijective, since \( L_r(\lambda) \cong \text{top}_{\mathfrak{g}_r}(M_r(\lambda)) \) by Proposition 5.18. Also, by Lemma 5.20, we get \( \text{hy}(\mathcal{G}_r) \rightarrow L(\lambda)^{\flat} = L(\lambda) \). The proof is done. \( \square \)

**Remark 5.24.** If \( 0 \in \Delta \) and \( k \) is not algebraically closed, which is the same as in Example 5.7, then neither Lemma 5.22 nor Proposition 5.23 fail in general. To see this, we shall consider the queer supergroup \( \mathcal{G} = \mathbb{Q}(2) \). Suppose that \( p = \text{char}(k) = 3 \). Set \( \alpha := \lambda_1 - \lambda_2 \in \Delta \) and \( \lambda := \lambda_1 - 2\lambda_2 \in X(T)^{\flat} \). Since \( \langle \lambda, \alpha^{\vee} \rangle = 3 \), we have \( \lambda \in X_{r=2}(T)^{\flat} \). As in Example 5.7, we may identify \( L(\lambda)^{\flat} = u(\lambda) \) with the 4-dimensional super-subalgebra \( \text{hy}(T)^{\flat} \) of \( \text{hy}(T) \) generated by \( K_1 \) and \( K_2 \). By Lemma 5.17, one sees that \( \text{hy}(\mathcal{G}_r)X^{-4}(K_1 - K_2 - 4) \otimes_{\text{hy}(B_r^{\flat})} L(\lambda)^{\flat} \) is a non-zero proper super-submodule of \( M_r(\lambda) \), and hence \( M_r(\lambda) \) is not simple. On the other hand, by the PBW theorem for \( \mathcal{G}_r \) (Theorem 3.11) and Lemma 5.20, one sees that
\[
\varphi : M_r(\lambda) \twoheadrightarrow \text{hy}(\mathcal{G}_r) \rightarrow L(\lambda)^{\flat} = L(\lambda); \quad u \otimes_{\text{hy}(B_r^{\flat})} v \mapsto u \rightarrow v,
\]
is injective, and hence \( \varphi \) is bijective. If we suppose that \( L(\lambda) \) is semisimple as a left \( \mathcal{G}_r \)-supermodule (cf. Lemma 5.22) or \( L(\lambda) \cong L_r(\lambda) \) as left \( \mathcal{G}_r \)-supermodules (cf. Proposition 5.23), then this shows that \( M_r(\lambda) \cong L_r(\lambda) \). In particular, \( M_r(\lambda) \) is a simple left \( \mathcal{G}_r \)-supermodule, a contradiction. Thus, our Assumption 5.24 is actually needed.

For \( \lambda \in X(T) \), set \( H^0_{\text{ev}}(\lambda) := \text{ind}_{\mathfrak{B}_{\text{ev}}}(\mathfrak{k}^{\lambda}) \). Then it is known that
\[
X(T)^+ := \{ \lambda \in X(T) \mid H^0_{\text{ev}}(\lambda) \neq 0 \} = \{ \lambda \in X(T) \mid \forall \alpha \in \Delta^+_0, \langle \lambda, \alpha^{\vee} \rangle \geq 0 \}
\]
and the following map is bijective, see [1] Part II, Chapter 2:
\[
X(T)^+ \rightarrow \text{Simple}(\mathcal{G}_{\text{ev}}); \quad \lambda \mapsto L_{\text{ev}}(\lambda) := \text{soc}_{\mathcal{G}_{\text{ev}}}(H^0_{\text{ev}}(\lambda)).
\]
In [Sh11], Proposition 4.18, it is shown that \( X(T)^{\flat} \subset X(T)^+ \). Thus, for each \( \lambda \in X(T)^{\flat} \), we may consider \( L_{\text{ev}}(\lambda) \).

**Theorem 5.25.** Let \( \lambda \in X(T)^{\flat} \). Suppose that there exists \( \lambda' \in X_r(T)^{\flat} \) and \( \mu \in X(T)^{\flat} \) such that \( \lambda = \lambda' + p^{\prime} \mu \). Then there exists an isomorphism \( L(\lambda) \cong L(\lambda') \otimes L_{\text{ev}}(\mu)[r] \) of left \( \mathcal{G} \)-supermodules.
Lemma 5.22, \( L \)

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\( H \) is a left \( G \)-supermodule, this \( \varphi \) is surjective. By Lemma 5.22, \( L(\lambda) \) is a simple left \( G \)-supermodule. Thus, for any \( \lambda \in X(\mathfrak{g}) \), we see that \( \mathfrak{g} \)-module of “highest” weight \( \lambda \) is actually an isomorphism.

To complete the proof, we shall show \( H = L(\mathfrak{g})[\mu] \). Since \( H \) is finite-dimensional, we have \( H = (H^{-r})^{|r|} \). Thus, it is enough to see that \( H^{-r} = L(\mathfrak{g}) \). By the isomorphism \( H \otimes L(\mathfrak{g}) \cong L(\mathfrak{g}) \) of left \( G \)-supermodules, we see that \( H \) is a simple left \( G \)-supermodule of “highest” weight \( \mu \). In particular, \( H^{-r} \) is a simple left \( \mathfrak{g} \)-module of “highest” weight \( \mu \), and hence \( H^{-r} \) must be isomorphic to \( L(\mathfrak{g}) \mu \). The proof is done.

By Theorem 5.24, we can establish Steinberg’s tensor product theorem for \( G \):

\textbf{Corollary 5.26.} Let \( \lambda \in X(\mathfrak{g}) \). If we write \( \lambda = \lambda_0 + p\lambda_1 + \cdots + p^m\lambda_m \) for some \( \lambda_0, \lambda_1, \ldots, \lambda_m \in X_1(T)^{\mathbb{Z}} \), then there exists an isomorphism

\[ L(\lambda) \cong L(\lambda_0) \otimes L_{\text{ev}}(\lambda_1)^{[1]} \otimes \cdots \otimes L_{\text{ev}}(\lambda_m)^{[m]} \]

of left \( G \)-supermodules.

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