The influence of non-Oberbeck-Boussinesq effects on rotating turbulent Rayleigh-Bénard convection

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Abstract. The influence of non-Oberbeck-Boussinesq (NOB) effects on rotating turbulent Rayleigh-Bénard convection is investigated by means of three-dimensional direct numerical simulations (DNS). For this purpose the impact of temperature dependent material properties of water with a Prandtl number of $Pr = 4.38$ is considered at a Rayleigh number of $Ra = 10^8$. Simulations with and without superimposed rotation were performed within a Rossby number range of $0.3 \leq Ro \leq 3.0$ and $Ro = \infty$. The generated flow fields are analysed with respect to deviations from the Oberbeck-Boussinesq (OB) cases.

We obtain a breakdown of the top-bottom symmetry, that is, different boundary layer thicknesses, modified mean temperature profiles including an increase of the centre temperature and asymmetric velocity flow patterns in the non-rotating case. When the Rayleigh-Bénard cell is rotated, NOB effects decrease with increasing rotation rate, but are still significant. In particular they lead to a smaller temperature gradient within the bulk. The Nusselt number $Nu$ in the non-rotating NOB cases slightly decreases, while it slightly increases in the rotating ones. However, the change in $Nu$ remains within a few percent for all cases.

1. Introduction

Turbulent flows driven by thermal convection are ubiquitous phenomena, not only in nature, as for example in the outer layer of stars, the interior of gaseous planets or the earth’s atmosphere and oceans, but also in engineering applications as crystal growth or the ventilation of buildings and aircrafts. Additionally, these flows are quite often superimposed by rotation, especially in the astrophysical and geophysical context. To get insight into this intriguing subject it is common to consider an idealisation, the so-called Rayleigh-Bénard convection (RBC), i.e. a fluid confined between a heating-plate at the bottom and a cooling-plate at the top.

A well-established method to study this classical problem from a mathematical and numerical point of view is to make use of the Oberbeck-Boussinesq (OB) approximation (Boussinesq, 1903; Oberbeck, 1879). This means, all material properties are assumed to be constant, in particular also the density, and correspondingly the fluid is incompressible. However, within the buoyancy term, the density varies linearly with temperature. This is indeed admissible for certain cases, but nonetheless the range of validity of this approach is actually quite restrictive. Deviations due to the violation of the OB assumption are commonly referred to as non-Oberbeck-Boussinesq (NOB) effects.
Gray & Giorgini (1976) provided a mathematically straightforward way to calculate the validity range of the OB approximation by the requirement that certain $\epsilon$ factors are smaller than the requested accuracy. Since at this point we are only interested in NOB effects induced by the temperature dependencies of the material properties, these are given by

$$\epsilon_1 = \frac{\alpha_m g H T_m}{c_{p,m} \Delta T}, \quad \epsilon_2 = \frac{\alpha_m g H \nu_m}{c_{p,m} \kappa_m},$$

$$\epsilon_3 = \frac{-\Delta T}{\rho_m} \frac{\partial \rho}{\partial T} \bigg|_{T_m}, \quad \epsilon_4 = \frac{-\Delta T}{c_{p,m}} \frac{\partial c_p}{\partial T} \bigg|_{T_m},$$

$$\epsilon_5 = \frac{-\Delta T}{\rho_m \nu_m} \frac{\partial (\rho \nu)}{\partial T} \bigg|_{T_m}, \quad \epsilon_6 = \frac{-\Delta T}{\Lambda_m} \frac{\partial \Lambda}{\partial T} \bigg|_{T_m},$$

$$\epsilon_7 = \frac{-\Delta T}{\alpha_m} \frac{\partial \alpha}{\partial T} \bigg|_{T_m}. \quad (1)$$

Here $H$ is the height of the cylinder, $g$ the gravitational acceleration, $\alpha$ the isobaric expansion coefficient, $\rho$ the density, $\nu$ the kinematic viscosity, $\Lambda$ the heat conductivity and $c_p$ the specific heat at constant pressure. The indices $t$, $b$ and $m$ here and in the following refer to the quantity at the top, the bottom and the arithmetic mean temperature $T_m = \frac{T_t + T_b}{2}$, respectively. That means, if $|\epsilon_1| \cdots |\epsilon_7| \leq 0.1$, a residual error of at most 10% is guaranteed. The factor $\epsilon_3$ represents the common $\alpha \Delta T \leq 0.1$ criterion which is often quoted as the sufficient one.

Here we focus on water, which temperature dependencies of its material properties can be seen in figure 1. It is already perspicuous that here neither the isobaric expansion coefficient nor the kinematic viscosity, the thermal diffusivity and the heat conductivity can be assumed to be constant with temperature if the imposed adverse temperature difference $\Delta T$ becomes too large. Indeed an accurate calculation yields that the factor $\epsilon_5$ is the demanding parameter, requiring $\Delta T < 1 \text{K}$.

2. Numerical methodology

We model turbulent Rayleigh-Bénard convection making use of a 4th order finite volume code for cylindrical domains developed by Shishkina & Wagner (2007) and performing direct numerical simulations (DNS). For the purpose of investigating NOB effects, the code has been advanced by taking temperature dependent material properties into account. That is, the viscosity $\nu$, the thermal conductivity $\Lambda$ and the density in the buoyancy term $\rho$ are described by polynomials

![Figure 1. Relative deviations of water properties $X$ from their values $X_m$ at a mean temperature of $T_m = 40^\circ\text{C}$, according to Ahlers et al. (2006); black solid line: density $\rho$; green dashed line: thermal diffusivity $\kappa$; orange short dashed line: specific heat capacity $c_p$; purple dashed dotted line: kinematic viscosity $\nu$; blue dashed triple-dotted line: expansion coefficient $\alpha$; pink dotted line: thermal conductivity $\Lambda$.](image-url)
defined at the mean temperature parameters for our simulations are essentially the Rayleigh, the Prandtl and the Rossby number as reference scales for the non-dimensional equations solved numerically. Thus, the control at the side walls is then given by

$$X = \nu, \Lambda, \rho$$

where the coefficients \( a_i \) are adopted from Ahlers et al. (2006), while the density \( \rho \), except within the buoyancy term, and the isobaric specific heat capacity \( c_p \) are set constant to their values at the arithmetic mean temperature \( T_m \). This approach is adequate for most liquids, and in particular also for water (see figure 1), and accounts for the major relevant NOB effects. Hence, the flow characteristics are obtained by solving the continuity equation for incompressible fluids (6), the Navier-Stokes equations (7) and the energy equation (8) in cylindrical coordinates \((r, \phi, z)\), including the aforementioned material functions.

$$1 \frac{1}{r} \partial_r (ru_r) + \frac{1}{r} \partial_\phi u_\phi + \partial_z u_z = 0$$

$$\rho_m \left( \frac{Du_r}{Dt} - \frac{u_\phi^2}{r} \right) = -\partial_r p + \rho_m \left( \frac{1}{r} \partial_r (rv\tau_{rr}) + \frac{1}{r} \partial_\phi (v\tau_{r\phi}) + \partial_z (v\tau_{rz}) - \frac{1}{r} v\tau_{\phi\phi} \right)$$

$$\rho_m \frac{Du_\phi}{Dt} + \frac{u_\phi u_\phi}{r} = -\partial_\phi p + \rho_m \left( \frac{1}{r^2} \partial_r (r^2 v\tau_{\phi\phi}) + \frac{1}{r} \partial_\phi (v\tau_{\phi\phi}) + \partial_z (v\tau_{\phi z}) \right)$$

$$\rho_m \frac{Du_z}{Dt} = -\partial_z p + \rho_m \left( \frac{1}{r} \partial_r (r v\tau_{zr}) + \frac{1}{r} \partial_\phi (v\tau_{z\phi}) + \partial_z (v\tau_{zz}) \right) + (\rho_m - \rho) g$$

$$\rho_m c_p,m \frac{DT}{Dt} = \frac{1}{r} \partial_r (\Lambda r \partial_r T) + \frac{1}{r^2} \partial_\phi (\Lambda \partial_\phi T) + \partial_z (\Lambda \partial_z T)$$

Here \( \tau \) is a tensor defined via the stress tensor \( \tau \) by \( \tau = \frac{\tau}{\rho m r} \) and \( \frac{Du}{Dt} \) denotes the substantial derivative.

As boundary conditions for the temperature, we impose the lateral walls being adiabatic, and the top and bottom plate being isothermal, i.e. they have a constant temperature \( T_t \) and \( T_b \), respectively. As boundary conditions for the velocity, we apply impermeability and no-slip conditions. In the case of rotation the \( \phi \)-component of the velocity at the top and bottom and at the side walls is then given by

$$u_\phi |_{z=H} = u_\phi |_{z=0} = \Omega r \quad \text{and} \quad u_\phi |_{r=R} = \Omega R,$$

with \( \Omega \) being the constant angular velocity, \( R \) the radius and \( H \) the height of the cylinder. Both times the boundary conditions are completed by setting a \( 2\pi \)-periodicity in \( \phi \)-direction.

The physical parameters radius \( R \), buoyancy velocity \( \sqrt{g \alpha_m R \Delta T} \), temperature difference \( \Delta T \) and the matching material properties at the mean temperature, i.e. \( \nu_m, \Lambda_m, \rho_m \), are used as reference scales for the non-dimensional equations solved numerically. Thus, the control parameters for our simulations are essentially the Rayleigh, the Prandtl and the Rossby number defined at the mean temperature \( T_m \),

$$Ri = \frac{\alpha_m g \Delta T H^3}{\kappa_m \nu_m}, \quad Pr = \frac{\nu_m}{\kappa_m}, \quad Ro = \frac{\sqrt{\alpha_m g \Delta T H}}{2 \Omega H}.$$  

At the moment we restrict ourselves to Rayleigh-Bénard cells with an aspect ratio of \( \Gamma = \frac{2R}{H} = 1 \).

For our performed DNS we use staggered computational meshes with nodes distributed equidistantly in \( \phi \)-direction and non-equidistantly in \( z \) and \( r \)-direction, i.e. they are clustered in the boundary layers. We perform an \textit{a priori} analysis of our meshes according to Shishkina et al. (2010). That is for water with \( Pr = 4.38 \) the global mesh size is everywhere

$$h_{\text{global}} \leq \frac{H}{Ri^{1/4} (Nu - 1)^{1/4}}$$
and to ensure that the simulations with rotation as well as with NOB effects are properly resolved, twice as many nodes $N$ as required in the pure OB case are placed in the thermal and viscous boundary layer, $\lambda_\theta$ and $\lambda_v$, 

\[
N(\lambda_\theta) > 2 \cdot (0.69 \, Nu^{1/2}) \quad (12)
\]

\[
N(\lambda_v) > 2 \cdot (0.70 \, Nu^{1/2} \, Pr^{1/3}) \quad (13)
\]

This leads to a mesh of $N_r \times N_\phi \times N_z = 192 \times 512 \times 384$ nodes for the here considered cases with $Ra = 10^8$.

3. Results and discussion

To study the influence of non-Oberbeck-Boussinesq effects on rotating Rayleigh-Bénard convection, two different series of simulations have been performed. That is, in one series rotation is applied on pure OB cases and in the other one temperature dependent materials properties are also considered, focussing on the NOB case $\Delta T = 40$ K and $T_m = 40°C$. The Rossby numbers range from $Ro = 0.3$ with rotation dominating over buoyancy, over $Ro = 1.0$ with buoyancy and rotation equally competing, to $Ro = 3.0$ where buoyancy is dominating over rotation, and eventually we also included the special case of $Ro = \infty$ corresponding to the non-rotating case. Details on varying $\Delta T$ and $T_m$ without rotation will be reported elsewhere.

3.1. Coherent flow structures

The flow organisation in two-dimensional non-rotating NOB Rayleigh-Bénard convection was already investigated by Sugiyama et al. (2009). They found a diagonal large scale circulation (LSC) and small corner flows with different strengths. However, since we are performing three-dimensional simulations we can gather information about the flow phenomenology that are suppressed in two-dimensional simulations. First of all we also obtain four smaller convection rolls in the plane perpendicular to the LSC. Under OB conditions these roll structures and the LSC are perfectly symmetrical situated within the Rayleigh-Bénard cell and are also of same strengths as can be seen in figure 2. In contrast to that, under NOB conditions the different viscosities in the cold top and hot bottom boundary layer lead to a break-up of the symmetry. The convection rolls arrange themselves in a tilted manner and are of different size. The resultant higher complexity of the flow pattern is also visible in the helicity $\omega \cdot u$, where $\omega = \nabla \times u$ is the vorticity (see figure 2).

To study the large coherent structures just as the secondary circulations induced by rotation, we looked amongst other things at the azimuthally averaged velocity components. For the NOB case of $\Delta T = 40$ K and $T_m = 40°C$ and for $Ro = \infty$, $Ro = 3.0$ and $Ro = 1.0$ such plots are shown in figure 3. The plots we obtained for the OB cases agree well with the ones reported by Kunnen (2008).

Under rotation the plumes are stretched to rather columnar vortices, also known as Ekman vortices. When $Ro$ is decreased the amount of vortices rises, while their mean radius decreases and, depending on $Ro$, they are able to penetrate more or less far into the bulk. Thus only for rather low rotation rates ($Ro \gtrsim 2.0$) the LSC keeps on being the predominant structure. The LSC vanishes completely for $Ro \lesssim 1.2$ (see Kunnen, 2008). These observations hold for the OB and the NOB cases.

The signature of the LSC, i.e. the clear separation of upward and downward velocity component $u_z$ with opposite signs in the bulk and close to the cylinder walls can be found for $Ro = \infty$ and for $Ro = 3.0$ in figure 3. For $Ro = 0.3$ the columnar vortices dominate the flow and no division between upper and lower part of the cylinder can be observed. The radial component $u_r$ has its maximum close to the top and bottom plate and an inward pointing flow
Figure 2. Slices of the temporal averaged helicity $\omega \cdot u$ in the non-rotation case. The left panels show the plane of the LSC, the right ones the plane perpendicular to it, overplotted are the velocity vectors. The top row corresponds to OB, the bottom one to NOB conditions.

in the middle of the cell, consistent with the four rolls seen in figure 2. With increasing rotation rate the outward flow is clinging closer to the plates. The azimuthal component $u_\phi$ only shows small fluctuations when there is no rotation, however, when rotation is applied the Coriolis force leads to a cyclonic motion in the center and an anticyclonic motion at the plates, close to the walls. These anticyclonic flow regions are shrinking in size with decreasing $Ro$. At $Ro = 0.3$ a secondary circulation driven by the boundary layers is found.

We note that only small deviations from the symmetry of the OB can be found in the NOB case and they are becoming less for lower Rossby numbers. Hence, the impact of temperature dependent material properties on the azimuthally averaged velocity fields can apparently be considered to be weak and under strong rotation even to be negligible in the case of water.
Figure 3. Azimuthally and temporally averaged velocity components and temperature for different Rossby numbers, $Ro = \infty$, $Ro = 3.0$ and $Ro = 0.3$ (from top to bottom), all for the NOB case of $\Delta T = 40K$. The first panels show the vertical velocity $u_z$, the second ones the radial velocity $u_r$, the third ones the azimuthal velocity $u_\phi$ and the fourth ones the temperature $T$ (from left to right).
3.2. Temperature field
On the opposite, the temperature field keeps on being sensitive to NOB effects, even though their influence is as well lower when the rotation rate is too high. The most salient feature in NOB simulations of water is the enhanced bulk temperature $T_c$ compared to the arithmetic mean temperature $T_m$ and closely related to that the modified mean temperature profiles and different boundary layer thicknesses. These asymmetries can be ascribed to the different viscosities in the cold top and hot bottom layer. That is, the lower viscosity at the warm bottom makes the plumes more prone to leave the bottom layer and they are also more mobile, i.e. faster. The cold plumes from the top show the exact opposite behaviour, i.e. they are very viscous and thus rather stuck within the cold bottom boundary layer. Furthermore the eventually emanating plumes move much slower and hence, they are much longer in contact with the ambient medium in the bulk and heat up on their way down (see also Horn et al., 2010). Ergo, the response from the system due the interplay of cold and hot plumes, is the higher center temperature and is also revealed when the Rayleigh-Bénard cell is rotated as visualised in figure 3 and 4. The only difference is that while in the OB and NOB case without rotation also the temperature fields show clear signs of a large scale circulation, the fast rotating cases simply show the traces of the Ekman vortices that are transported along approximately concentric trajectories.

![Figure 4. Cross sections of the time-averaged dimensionless temperature field $T$ at half height of the cylinder, i.e. $z = H/2$, for water with $Pr = 4.38$ and $Ra = 10^8$. The left panels show the OB case, the right ones the NOB case with $\Delta T = 40 K$, the upper panels correspond to the non-rotating the lower ones to the rotating case with $Ro = 0.3$.](image)

The mean temperature profiles on the other hand also vary in another aspect, showing a persisting temperature gradient over the height of the cell. The reason for that is provided by the fact that the Ekman vortices promote horizontal mixing when compared to vortical mixing and hence fully three dimensional turbulent mixing is less efficient. To act contrary to the
attenuated vertical motion caused by rotation, the upwards moving plumes also need a larger temperature difference between them and the adjacent bulk fluid, i.e. the rms temperature increases with $Ro$.

NOB effects seem to inhibit the lateral mixing and diminish the temperature fluctuations, leading to decreased rms, skewness and kurtosis values, as depicted in figure 5, which is consistent with the fact that the mean temperature gradient solely depends on $Ro$ and $Pr$ (Julien et al., 1996) and these two parameters certainly vary when the material properties are not constant.

![Figure 5](image)

**Figure 5.** Mean profiles of water for different Rossby numbers $Ro$ with $Ro = \infty$ corresponding to the non-rotating case. The solid lines indicate the OB, the dashed line the NOB cases. The top left panel shows the average temperature (note, that not the whole temperature range is shown), the top right one the temperature rms values, the bottom right one the skewness and the bottom left one the flatness of the temperature, all averaged in time $t$ and in every $r - \phi$ plane $A$ and plotted as function of the vertical coordinate.

### 3.3. Heat flux

Another interesting aspect of rotating Rayleigh-Bénard convection is the enhancement of the heat flux due to Ekman pumping (see e.g. Kunnen et al., 2006), the columnar vortices extract fluid from the boundary layers and thereby increase the vertical heat flux. Since they are only the prominent structures for low to moderate rotation rates, only then, of course, this enhancement can be found. Indeed, the Nusselt numbers obtained in the OB simulations, and given in table 1 with different rotation rates, show exactly this behaviour and are also in good agreement with Zhong et al. (2009) and Stevens et al. (2009), albeit at a slightly different Rayleigh and Prandtl number, respectively. When now looking at $Nu$ for the NOB case we obtain a similar phenomenological behaviour, but a definite increase of $Nu$ opposing to the decrease of $Nu$ found in the non-rotating case.
| $Ro$ | $Nu_{OB}$ | $Nu_{NOB}$ |
|------|-----------|-----------|
| $\infty$ | 32.7 | 32.0 |
| 3.0 | 32.8 | 33.9 |
| 1.0 | 33.7 | 34.6 |
| 0.3 | 37.6 | 38.1 |

4. Summary and Conclusion

In the presented study the influence of NOB and rotational effects was investigated by simulating Rayleigh-Bénard convection of water in a cylindrical cell of unity aspect ratio with different rotation rates.

Without as well as with rotation, the temperature dependency of the material properties results in asymmetric flow fields, well reflected in the increased bulk temperature, boundary layers and the convective roll structures. It is also noteworthy, that the temperature gradient maintained under rotation is diminished under NOB conditions. The deviation of the heat flux $Nu$, however, is within a range of a few percent, despite the fact that our simulations were performed with $\Delta T$ severely violating the OB validity conditions. Nonetheless, it is astonishing that $Nu$ increases when the Rayleigh-Bénard cell is rotated while it decreases if not.

Acknowledgments

The authors acknowledge support by the Deutsche Forschungsgemeinschaft (DFG) under grant SH405/2-1. Furthermore, the authors would also like to thank the Leibniz-Rechenzentrum (LRZ) in Garching for providing computational resources on the national supercomputer HLRB-II.

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