CERTAIN ALGEBRAIC INVARIANTS OF EDGE IDEALS OF JOIN OF GRAPHS

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Abstract. Let \( G \) be a simple graph and \( I(G) \) be its edge ideal. In this article, we study the Castelnuovo-Mumford regularity of symbolic powers of edge ideals of join of graphs. As a consequence, we prove Minh’s conjecture for wheel graphs, complete multipartite graphs and a subclass of co-chordal graphs. We obtain a class of graphs of regularity 3. By constructing graphs, we prove that multiplicity of an edge ideal is independent with depth, dimension, regularity and degree of \( h \)-polynomial. Also, we prove that depth of an edge ideal is independent with regularity and degree of \( h \)-polynomial, by constructing graphs.

1. Introduction

Let \( G \) be a simple graph on the vertex set \( V(G) = \{x_1, \ldots, x_n\} \) and edge set \( E(G) \). Set \( S = k[x_1, \ldots, x_n] \), where \( k \) is an arbitrary field. The edge ideal of \( G \), denoted by \( I(G) \), is defined by \( I(G) = (x_ix_j : \{x_i, x_j\} \in E(G)) \). From the last decade, various authors established connections between the combinatorial properties of \( G \) with algebraic properties of \( I(G) \). The \( s \)-th symbolic power of \( I(G) \) is defined as follows:

\[
I(G)^{(s)} = \bigcap_{p \in \text{Ass}(I(G))} p^s.
\]

The regularity of \( I(G) \) is defined by \( \text{reg}(I(G)) = \max\{j : \text{Tor}_j^S(I(G), k)_{i+j} \neq 0\} \). Minh stated the following conjecture for the regularity of symbolic power of edge ideal, see \([6]\):

Conjecture 1.1. Let \( G \) be a simple graph and \( I(G) \) be its edge ideal. Then \( \text{reg}(I(G)^{(s)}) = \text{reg}(I(G)^s) \) for all \( s \geq 1 \).

Gu et al. studied this conjecture for odd cycles in \([6]\). In \([11]\), Jayanthan and Kumar settled the conjecture for a class of graphs which is a clique sum of an odd cycle and some.
bipartite graphs. In [15] and [16] Fakhari studied the conjecture for chordal graphs and unicyclic graphs. In [14], the author studied the regularity of \( I(G)^* \), where \( G \) is the join of graphs. In this article we study the regularity of \( I(G)^{(s)} \), where \( G \) is the join of graphs. As a consequence, we prove the conjecture for some classes of graphs, for example wheel graphs, complete multipartite graphs and a subclass of co-chordal graphs.

In [10], Hibi et al. studied the relation between regularity, degree of \( h \)-polynomial and number of vertices. For a given pair of positive integers \((r, s)\), they constructed a graph whose edge ideal has regularity \( r \) and degree of \( h \)-polynomial \( s \). In [9], Hibi et al. constructed graph for given pair \((b, r)\) with \( b \leq r \), where \( b \) is the number of extremal Betti numbers of \( S/I(G) \) and \( r = \text{reg}(S/I(G)) \). In [4], the authors characterize the bipartite graphs whose edge ideal has regularity 3. In this article, we obtain a class of graphs \( G \) with \( \text{reg}(I(G)) = 3 \). Moreover, for a given pair of positive integers \((r, d)\) with \( r \leq d \), we construct a graph \( G \) such that \( \text{reg}(S/I(G)) = r \) and \( \dim(S/I(G)) = d \). Also we prove that for edge ideal the multiplicity has no relation with dimension, depth, regularity and degree of \( h \)-polynomial. It is known that the depth and degree of \( h \)-polynomial for a squarefree monomial ideals are bounded by the dimension. We prove that for an edge ideal the depth is independent from the degree of \( h \)-polynomial and regularity.

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2. Preliminaries

In this section, we collect all the notation and definitions which are used throughout this article. For undefined terminology, we refer the book [8] by Herzog and Hibi.

2.1. Combinatorics. Let \( G \) be a simple graph on the vertex set \( V(G) \) and the edge set \( E(G) \).

a) A graph \( H \) is called an induced subgraph of \( G \) if \( V(H) \subset V(G) \) and for \( i, j \in V(H) \), \( \{i, j\} \in E(H) \) if and only if \( \{i, j\} \in E(G) \).

b) The complement of a graph \( G \), denoted by \( G^c \), is the graph on the vertex set \( V(G) \) and edge set \( \{\{i, j\} : \{i, j\} \notin E(G)\} \).
c) A graph $G$ is chordal if there is no induced cycle of length $\geq 4$ and it is co-chordal if $G^c$ is chordal.

d) A collection $C \subset E(G)$ of disjoint edges is said to be an induced matching if the induced subgraph on the vertices of $C$ has no edge other than the edges in $C$. The maximum size of an induced matching in $G$ is called the induced matching number of $G$ and is denoted by $\nu(G)$.

e) A collection $\Delta$ of subsets of $[n]$ is called a simplicial complex if it satisfies the following:
   i) for any $i \in [n]$, $\{i\} \in \Delta$,
   ii) $F \subseteq G$ with $G \in \Delta$ implies that $F \in \Delta$.

f) An element of $\Delta$ is called a face and a maximal face with respect to inclusion is called a facet.

g) The dimension of a simplicial complex is defined as
   \[ \dim(\Delta) = \max\{|F| - 1 : F \text{ is a facet in } \Delta\}. \]

h) Let $G$ and $H$ be graphs on vertex sets $V(G) = \{x_1, \ldots, x_m\}$ and $V(H) = \{y_1, \ldots, y_n\}$, respectively. Then the join of $G$ and $H$, denoted by $G \ast H$, is a graph on the vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{\{x_i, y_j\} : 1 \leq i \leq m, 1 \leq i \leq n\}$.
   The below graph is join of a vertex and cycle $C_5$.

2.2. Hilbert Series. Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring and $I \subseteq S$ be a homogeneous ideal with $\dim(S/I) = d$. By [8] Theorem 6.1.3, the Hilbert series of $S/I$ is $H(S/I, t) = (h_0 + h_1t + \cdots + h_st^s)/(1 - t)^d$ for $h_i \in \mathbb{Z}$ with $h_s \neq 0$.

a) The polynomial $h_0 + h_1t + \cdots + h_st^s$, denoted by $h_{S/I}(t)$, is called the $h$-polynomial of $S/I$.

b) The multiplicity of $S/I$, denoted by $e(S/I)$, is $h_{S/I}(1)$. 
3. Regularity of symbolic powers of join of graphs

In this section, we study the regularity of symbolic powers of join of graphs. First, we prove a technical lemma which is useful to prove our main result of this section.

**Lemma 3.1.** Let $S = k[x_1, \ldots, x_m, y_1, \ldots, y_n]$ be a polynomial ring. Let $I \subset \langle x_1, \ldots, x_m \rangle^2$ be a squarefree monomial ideal in $k[x_1, \ldots, x_m]$ and $J \subset \langle y_1, \ldots, y_n \rangle^2$ be a squarefree monomial ideal in $k[y_1, \ldots, y_n]$. Then

$$reg\left(\left(I + \langle y_1, \ldots, y_n \rangle\right)^{(s)} + \left(J + \langle x_1, \ldots, x_m \rangle\right)^{(s)}\right) = 2s - 1.$$

**Proof.** Let $K = \left(I + \langle y_1, \ldots, y_n \rangle\right)^{(s)} + \left(J + \langle x_1, \ldots, x_m \rangle\right)^{(s)}$. Since $x_i^s, y_j^s \in K$ for all $i, j$, $K$ is an $m$-primary ideal, where $m = \langle x_1, \ldots, x_m, y_1, \ldots, y_n \rangle$. Therefore, $reg(K) = \max\{j + 1 : (S/K)_j \neq 0\}$. It is easy to see that $(S/K)_j = 0$ for all $j \geq 2s - 1$, we have $reg(K) \leq 2s - 1$. Since $I$ and $J$ are squarefree monomial ideals, $x_i^{s-1}y_j^{s-1} \not\in K$. Therefore, $x_i^{s-1}y_j^{s-1} \in (S/K)_{2s-2}$ is a nonzero element and the assertion follows.

We now fix some notation which are used throughout this section. For $1 \leq i \leq r$, set $X_i = \{x_{i1}, \ldots, x_{im_i}\}$. Let $G_1, \ldots, G_r$ be graphs on the vertex sets $X_1, \ldots, X_r$, respectively.

Also, for $1 \leq j \leq r$, set $m_j = \langle x_{j1}, \ldots, x_{jm_j} \rangle$.

**Theorem 3.2.** Let $G_1, \ldots, G_r$ be graphs on vertex sets $X_1, \ldots, X_r$, respectively. Let $G = G_1 \ast \cdots \ast G_r$ and $S = k[V(G)]$. Then

$$reg(I(G)^{(s)}) = \max\{reg\left(I(G_j)^{(i)}\right) - i + s : 1 \leq i \leq s, 1 \leq j \leq r\}.$$

**Proof.** We prove the assertion by induction on $r$. If $r = 2$, then consider the following exact sequence

$$0 \to \frac{S}{I(G)^{(s)}} \to \frac{S}{I(G_1) + m_2^{(s)}} \oplus \frac{S}{I(G_2) + m_1^{(s)}} \to \frac{S}{I(G_1) + m_2^{(s)} + I(G_2) + m_1^{(s)}} \to 0.$$

It follows from [7, Proposition 4.10] that $reg(\langle I(G_2) + m_1\rangle^{(s)}) = \max\{reg(\langle I(G_2)\rangle^{(i)}), i + s : 1 \leq i \leq s\} \geq 2s$. Similarly, $reg(\langle I(G_1) + m_2\rangle^{(s)}) \geq 2s$. By virtue of Lemma 3.1, we have $reg\left(\langle I(G_1) + m_2\rangle^{(s)} + \langle I(G_2) + m_1\rangle^{(s)}\right) = 2s - 1 < \max\{\min\{\langle I(G_1) + m_2\rangle^{(s)}, \langle I(G_2) + m_1\rangle^{(s)}\}\}$, and hence it follows from the regularity behavior on short exact sequences that

$$reg\left(I(G)^{(s)}\right) = \max\{reg\left(\langle I(G_1) + m_2\rangle^{(s)}\right), reg\left(\langle I(G_2) + m_1\rangle^{(s)}\right)\}.$$
Now, the assertion follows from \cite{7, Proposition 4.10}. Assume that \( r > 2 \) and let \( G' = G_1 * \cdots * G_{r-1} \). Note that \( G = G' * G_r \). Hence the result follows from induction hypothesis and the base case.

In \cite{14}, the author defined following classes of graphs:

\[
\mathcal{A} = \{ G \mid \text{reg} \left( (I(G))^s : u \right) \leq \text{reg} \left( I(G) \right), u \in \mathcal{G}(I(G)^s), s \geq 1 \}
\]

and

\[
\mathcal{A}_1 = \{ G \mid \text{reg} \left( (I(G))^s \right) = 2s + \text{reg}(I(G)) - 2, s \geq 1 \},
\]

where \( \mathcal{G}(I(G)^s) \) denotes the minimal generating set of \( I(G)^s \). It follows from \cite{14, Theorem 4.4, Remark 4.10} that \( \mathcal{A} \) as well as \( \mathcal{A} \cap \mathcal{A}_1 \) are closed under the operation of join of graphs. We study the regularity of symbolic power of join of graphs which belong to the class \( \mathcal{A} \cap \mathcal{A}_1 \). There are many classes of graphs which are contained in \( \mathcal{A} \cap \mathcal{A}_1 \). For example, co-chordal graphs, unmixed bipartite, bipartite \( P_6 \)-free graphs, forest graphs are contained in \( \mathcal{A} \cap \mathcal{A}_1 \), for more details see \cite{14}.

**Corollary 3.3.** Let \( G_1, \ldots, G_r \in \mathcal{A}_1 \) such that \( G_i \)'s are bipartite and \( G = G_1 * \cdots * G_r \). Then \( \text{reg} \left( (I(G))^s \right) = 2s + \text{reg}(I(G)) - 2 \). Furthermore, if \( G_i \in \mathcal{A} \) for all \( i \), then \( \text{reg} \left( (I(G))^s \right) = \text{reg}(I(G)^s) \). In particular, \( \text{reg}(I(G)^s) = 2s \), if \( G \) is a complete multipartite graph.

**Proof.** Since, for \( 1 \leq i \leq r \), \( G_i \in \mathcal{A}_1 \) and \( G_i \)'s are bipartite, we have

\[
\text{reg} \left( (I(G_i))^s \right) = \text{reg}(I(G_i)^s) < \text{reg} \left( (I(G_i))^{s+1} \right) = \text{reg} \left( (I(G_i))^{s+1} \right).
\]

Therefore, by virtue of Theorem 3.2 we get \( \text{reg} \left( (I(G))^s \right) = \max \{ \text{reg} \left( (I(G_i))^s \right) : 1 \leq i \leq r \} \). Since, for all \( 1 \leq i \leq r \), \( \text{reg}(I(G_i)^s) = 2s + \text{reg}(I(G_i)) - 2 \), it follows from \cite{13, Proposition 3.1.2} that

\[
\text{reg} \left( (I(G))^s \right) = \max \{ \text{reg} \left( (I(G_i)) \right) : 1 \leq i \leq r \} + 2s - 2 = 2s + \text{reg}(I(G)) - 2.
\]

Now, assume that each \( G_i \in \mathcal{A} \). By \cite{14, Remark 4.10}, \( \text{reg} \left( (I(G))^s \right) = \text{reg}(I(G)^s) \) which completes the proof.

In \cite{14}, the author studied the regularity of powers of edge ideals of wheel graphs. In the following corollary, we prove Minh’s conjecture for wheel graphs.
Corollary 3.4. Let $G = W_n$ be a wheel graph for $n \geq 4$. Then

$$\text{reg}(I(G)^{(s)}) = \text{reg}(I(G)^{s}) = 2s + \nu(C_n) - 1 \text{ for all } s \geq 2.$$ 

Proof. Since $G = W_n$ is a join of a vertex $v$ and a cycle graph $C_n$, by Theorem 3.2 we have $\text{reg}(I(G)^{(s)}) = \max\{\text{reg}(I(C_n)^{(i)}) - i + s : 1 \leq i \leq s\}$. It follows from [6, Theorem 5.3] that for $i \geq 2$, $\text{reg}(I(C_n)^{(i)}) = \text{reg}(I(C_n)^{i}) = 2i + \nu(C_n) - 1$, and hence our result follows from [14, Theorem 4.7].

4. Construction of Graphs

In this section, we construct graphs with different pair of algebraic invariants. Let $G$ and $H$ be graphs on vertex sets $V(G) = \{x_1, \ldots, x_m\}$ and $V(H) = \{y_1, \ldots, y_n\}$, respectively. One can see that if $C$ is a minimal vertex cover of $G \ast H$, then either $C = A \cup V(H)$ or $C = V(G) \cup B$, where $A$ and $B$ are minimal vertex cover of $G$ and $H$, respectively. Therefore, we have the following short exact sequence

$$0 \to \frac{S}{I(G \ast H)} \to \frac{S}{I(G) + m_2} \oplus \frac{S}{I(H) + m_1} \to k \to 0,$$

where $m_1 = \langle x_1, \ldots, x_m \rangle$ and $m_2 = \langle y_1, \ldots, y_n \rangle$ and $S = k[V(G \ast H)]$. Thus, by the depth lemma, $\text{depth}(S/I(G \ast H)) = 1$.

Note that, for a squarefree monomial ideal $I$, by [12, Observation 5.2], $e(S/I) = |\text{Minh}(I)|$, where $\text{Minh}(I) = \{p \in \text{Ass}(S/I) : \text{ht}(p) = \text{ht}(I)\}$. We now compute the multiplicity of join of graphs.

Theorem 4.1. Let $G$ and $H$ be graphs on the vertex sets $V(G) = \{x_1, \ldots, x_m\}$ and $V(H) = \{y_1, \ldots, y_n\}$, respectively. Suppose $S = k[V(G \ast H)]$. Then

$$e(S/I(G \ast H)) = \begin{cases} 
  e(S/I(G)) + e(S/I(H)) & \text{if } m + \text{ht}(I(H)) = n + \text{ht}(I(G)) \\
  e(S/I(G)) & \text{if } m + \text{ht}(I(H)) > n + \text{ht}(I(G)) \\
  e(S/I(H)) & \text{if } m + \text{ht}(I(H)) < n + \text{ht}(I(G)). 
\end{cases}$$

Proof. For a graph $G$, it is well known that minimal primes of $I(G)$ corresponds to minimal vertex covers of $G$. 
Note that if \( p \in \text{Minh}(I(G * H)) \), then either \( p = p_1 + m_2 \) or \( p = p_2 + m_1 \), where \( p_1 \in \text{Minh}(I(G)) \) and \( p_2 \in \text{Minh}(I(G)) \). Thus, we get

\[
|\text{Minh}(I(G * H))| = \begin{cases} 
|\text{Minh}(I(G))| + |\text{Minh}(I(H))| & \text{if } m + \text{ht}(I(H)) = n + \text{ht}(I(G)) \\
|\text{Minh}(I(G))| & \text{if } m + \text{ht}(I(H)) > n + \text{ht}(I(G)) \\
|\text{Minh}(I(H))| & \text{if } m + \text{ht}(I(H)) < n + \text{ht}(I(G)).
\end{cases}
\]

Hence, the assertion follows. \( \square \)

Now, we prove the algebraic properties of join of graphs which are used in the rest of the section.

**Proposition 4.2.** Let \( G^*l = G \cdots \cdots G \) be the join of \( l \)-copies of \( G \), \( S_G = k[V(G)] \) and \( S = k[V(G^*)] \). Then

i) \( \text{reg}(S/I(G^*)) = \text{reg}(S_G/I(G)) \),

ii) \( e(S/I(G^*)) = l \cdot e(S_G/I(G)) \),

iii) \( H(S/I(G^*), t) = l \cdot H(S_G/I(G), t) - (l - 1) \).

**Proof.** (i) and (iii) follow from [13, Proposition 3.12, Corollary 4.6], respectively. (ii) can be obtained by recursively applying Theorem 4.1. \( \square \)

Using above result, one can see that \( e(S/I(K_n)) = n \), and using (1), \( \text{depth}(S/I(K_n)) = 1 \).

**Notation 4.3.** Let \( K_n \) be the complete graph on the vertex set \( \{x_1, \ldots, x_n\} \). Then, for \( 1 \leq r \leq n \), we define a graph \( W(n, r) \) on the vertex set \( \{x_1, \ldots, x_n, y_1, \ldots, y_r\} \) with edge set \( E(K_n) \cup \{\{x_i, y_i\} : 1 \leq i \leq r\} \). The graph shown in the figure is \( W(5, 3) \).

Now, we study the algebraic invariants of \( W(n, r) \).

**Theorem 4.4.** Let \( G = W(n, r) \) and \( S = k[V(G)] \). Then we have the following:

i) \( I(G) \) has linear resolution,

ii) \( \text{pd}(S/I(G)) = n \) and \( \text{depth}(S/I(G)) = r \),

iii) if \( 1 \leq r < n \), then \( e(S/I(G)) = n - r \).

**Proof.** (i) It is easy to see that \( W(n, r) \) is a co-chordal graph. Hence, the assertion follows from [5, Theorem 1].
(ii) We proceed by induction on \( n \). Consider the following short exact sequence
\[
0 \rightarrow \frac{S}{I(G) : x_1} \xrightarrow{x_1} \frac{S}{I(G)} \rightarrow \frac{S}{I(G) + \langle x_1 \rangle} \rightarrow 0.
\]
Note that \( I(G) : x_1 = \langle x_2, \ldots, x_n, y_1 \rangle \) and \( \langle I(G), x_1 \rangle = \langle I(W(n-1, r-1)), x_1 \rangle \). Since, \( I(G) : x_1 \) is generated by a regular sequence of length \( n \), \( \text{pd}(S/\langle I(G) : x_1 \rangle) = n \) and \( \text{depth}(S/\langle I(G) : x_1 \rangle) = r \). Observe that \( y_1 \) is a regular element on \( S/\langle I(G) : x_1 \rangle \). Thus, by induction, \( \text{depth}(S/\langle I(G) + x_1 \rangle) = r \). Now, by applying the depth lemma on the short exact sequence (2), we have \( \text{depth}(S/I(G)) = r \).

(iii) We use induction on \( n \). Let \( A \) be a minimal vertex cover of \( G \). Then \( |A \cap \{x_1, \ldots, x_n\}| \geq n - 1 \). Also, \( \{x_1, \ldots, x_{n-1}\} \) is a minimal vertex cover of \( G \). Therefore, \( \text{ht}(I(G)) = n - 1 \) and hence \( \text{dim}(S/I(G)) = r + 1 \). From the proof of (ii), we have \( \text{dim}(S/\langle I(G) : x_1 \rangle) = r \) and \( \text{dim}(S/\langle I(G), x_1 \rangle) = r + 1 \). Thus, by short exact sequence (2), \( e(S/I(G)) = e(S/\langle I(G), x_1 \rangle) \). Hence, the assertion follows by induction.

Now we give an upper bound for the regularity of squarefree monomial ideals. It is known result but for sake of completeness we are proving it.

**Lemma 4.5.** Let \( I \subset S = k[x_1, \ldots, x_n] \) be a squarefree monomial ideal. Then \( \text{reg}(S/I) \leq \text{dim}(S/I) \).

**Proof.** Let \( \text{dim}(S/I(G)) = d \). Since \( I \) is a squarefree monomial ideal, there exists a simplicial complex \( \Delta \) on \( [n] \) such that the Stanley-Reisner ideal of \( \Delta \) is \( I \). It follows from [8, Section 1.5] that \( \text{dim}(\Delta) = d - 1 \). Therefore \( \tilde{H}_l(\Delta, k) = 0 \), for \( l \geq d \). By Hochster’s formula [8, Theorem 8.1.1],
\[
\beta_{i,j}(S/I) = \sum_{A \subset [n], |A| = j} \dim_k \tilde{H}_{i-1}(\Delta_A, k).
\]
Let \( \text{reg}(S/I(G)) = r \). Then there exists \( i, j \) such that \( j - i = r \) and \( \beta_{i,j}(S/I(G)) \neq 0 \). Therefore, for some subset \( A \subset [n] \), \( \dim_k \tilde{H}_{r-1}(\Delta_A, k) \neq 0 \), which implies that \( r - 1 \leq d - 1 \) and hence, \( r \leq d \). 

As a consequence, we have the following:

**Corollary 4.6.** Let \( G \) be a triangle free graph which is not a forest. Then \( \text{reg}(S/I(G^c)) = 2 \).

**Proof.** It is well known fact that \( \text{dim}(S/I(G^c)) \) is the maximum size of maximal independent set of \( G^c \) which is equal to maximum size of maximal clique in \( G \). Since, \( G \) is triangle free,
the maximum size of maximal clique is 2. Hence, by Lemma 4.5 \( \reg(S/I(G^c)) \leq 2 \). Further, by [3 Theorem 1], \( \reg(S/I(G)) = 2 \) as \( G \) is not chordal.

\[ \square \]

**Remark 4.7.** Let \( G \) and \( H \) be graphs with \( \reg(I(G)) = 3 \) and \( \reg(I(H)) \leq 3 \). Then, by [13 Proposition 3.12], \( \reg(I(G*H)) = 3 \). For example, if \( G^c \) is triangle free which is not a forest and \( H \) is a bipartite graph whose edge ideal has regularity 3 ([4 Theorem 4.1]) or co-chordal graph, then by Corollary 4.6 \( \reg(I(G*H)) = 3 \). In this way, one can construct a graph whose edge ideal has regularity 3 which is neither a bipartite graph nor a compliment of a triangle free graph.

Let \( F_n \) denote the bipartite graph on the vertex set \( \{x_1, \ldots, x_n, y_1, \ldots, y_n\} \) with edge set \( \{\{x_i, y_j\} : 1 \leq i \leq j \leq n\} \). By virtue of [8 Corollary 9.1.14], \( S/I(F_n) \) is Cohen-Macaulay. Since, \( \{x_1, \ldots, x_n\} \) is a minimal vertex cover of \( F_n \), \( \ht(I(F_n)) = \dim(S/I(F_n)) = n \). Observe that \( \mu(I(F_n)) = \binom{n+1}{2} \). Thus, it follows from [17 Theorem 4.3.7] that \( F_n \) has linear resolution. The following figure represent the graph \( F_4 \).

Now, for \( 1 \leq r \leq d \), we construct a graph \( G \) with \( \reg(S/I(G)) = r \) and \( \dim(S/I(G)) = d \).

**Theorem 4.8.** For \( 1 \leq r \leq d \), there exists a graph \( G \) such that \( \reg(S/I(G)) = r \) and \( \dim(S/I(G)) = d \).

**Proof.** Let \( G \) be a disconnected graph with \( r \) components such that \( r-1 \) components are edges and one component is \( F_{d-r+1} \). Then \( \reg(S/I(G)) = r \) and \( \dim(S/I(G)) = d \). \[ \square \]

Note that \( \{x_{2i} : 1 \leq i \leq n\} \) is a minimal vertex cover of both \( P_{2n} \) and \( P_{2n+1} \). Therefore, \( \ht(I(P_{2n})) \leq n \) and \( \ht(I(P_{2n+1})) \leq n \). Now, if possible, let \( A \) be a minimal vertex cover of \( P_{2n} \) such that \( |A| < n \). Then it is easy to see that \( |\{e : e \in E(P_{2n}) \text{ and } e \cap A \neq \emptyset\}| \leq 2n-2 \), which is a contradiction. Similarly, one can prove that \( \ht(I(P_{2n+1})) = n \). Hence, \( \dim(S/I(P_{2n})) = n \) and \( \dim(S/I(P_{2n+1})) = n + 1 \).

**Lemma 4.9.** Let \( G = P_{2n+1} \) be a path graph. Then \( e(S/I(P_{2n+1})) = 1 \).
Proof. We prove the assertion by induction on $n$. If $n = 1$, then $P_3$ has only one minimal vertex cover of minimal size. Hence, $e(S/I(P_3)) = 1$. Assume that $n > 1$ and $e(S/I(P_{2n-1})) = 1$. Now, consider the following short exact sequence,

$$0 \to \frac{S}{I(P_{2n+1}) : x_{2n+1}} \to \frac{S}{I(P_{2n+1})} \to \frac{S}{I(P_{2n}), x_{2n+1}} \to 0.$$  

Note that $I(P_{2n+1}) : x_{2n+1} = \langle I(P_{2n-1}), x_{2n} \rangle$. Observe that $\dim(S/(I(P_{2n}), x_{2n+1})) = n$, $\dim(S/(I(P_{2n+1}) : x_{2n+1})) = n + 1$ and $\dim(S/I(P_{2n+1})) = n + 1$. Hence, $e(S/I(P_{2n+1})) = e(S/(I(P_{2n-1}), x_{2n})) = 1$.  

Remark 4.10. Let $I \subseteq m^2$ be a nonzero homogeneous ideal in a polynomial ring $S$ with $H(S/I, t) = (1 + h_1 t + h_2 t^2 + \cdots + h_s t^s)/(1-t)^d$, where $\dim(S/I) = d$. Then $h_1 = \text{codim}(S/I)$. This forces that for any graph $G$, if $e(S/I(G)) = 1$, then $\deg(h_{S/I(G)}(t)) \geq 2$. Note that if $I$ is a squarefree monomial ideal, then by [3, Theorem 5.1.7], $\deg(h_{S/I}(t)) \leq \dim(S/I)$. Therefore, $e(S/I(G)) = 1$ implies that $\dim(S/I(G)) \geq 2$.

We now prove that for an edge ideal multiplicity is not bounded by algebraic invariants such as regularity, depth, dimension and degree of $h$-polynomial. In fact, we construct graphs with fixed multiplicity and one of the other invariants.

Theorem 4.11. Let $e, r, s, \delta, d$ be positive integers. Then we have the followings:

i) There exists a graph $G$ with $e(S/I(G)) = e$ and $\text{reg}(S/I(G)) = r$.

ii) There exists a graph $G$ with $e(S/I(G)) = e$ and $\deg(h_{S/I(G)}(t)) = s$, provided $e \cdot s \geq 2$.

iii) There exists a graph $G$ with $e(S/I(G)) = e$ and $\text{dim}(S/I(G)) = \delta$.

iv) There exists a graph $G$ with $e(S/I(G)) = e$ and $\text{dim}(S/I(G)) = d$, provided $e \cdot d \geq 2$.

Observe that, by Proposition 4.2, regularity and dimension remain same under self join operation but multiplicity increases. This observation is the key idea for the proof of Theorem 4.11.

Proof of Theorem 4.11. (i) If $r$ is odd, then take $H = P_{3r}$ otherwise take $H = P_{3r+1}$. Then by [1, Theorem 4.7] $\text{reg}(S_H/I(H)) = r$ and by Lemma 4.9 $e(S_H/I(H)) = 1$, where $S_H = k[V(H)]$. Now take $G = H^*e$. Then by Proposition 4.2 we have $\text{reg}(S/I(G)) = r$ and $e(S/I(G)) = e$.  


(ii) and (iv): For \( d = s = 1 \), take \( G = K_e \). Now, we assume that \( s \geq 2 \) and \( d \geq 2 \). By short exact sequence \( \ref{seq1} \) we get
\[
H \left( \frac{S_{K_{1,s}}}{I(K_{1,s})}, t \right) = \frac{1}{(1-t)^s} + \frac{1}{1-t} - 1,
\]
where \( S_{K_{1,s}} = k[V(K_{1,s})] \). Take \( G = K_{1,s}^e \). Using Proposition \ref{prop4.2}, we have
\[
H \left( \frac{S}{I(G)}, t \right) = e + \frac{e}{1-t} + \frac{e}{(1-t)^s} - (2e - 1).
\]
Thus, \( e(S/I(G)) = e \) and \( \deg(h_{S/I(G)}(t)) = s = \dim(S/I(G)) \). (iii) For any \( e \) and \( \delta \), take \( G = W(e+\delta, \delta) \). Then it follows from Theorem \ref{thm4.4} we get \( e(S/I(G)) = e \) and \( \depth(S/I(G)) = \delta \). This completes the proof.

Now, we give an upper bound for the multiplicity in terms of cover number.

**Theorem 4.12.** Let \( G \) be a graph and \( \alpha(G) \) be its cover number. Then \( e(S/I(G)) \leq 2^{\alpha(G)} \).

**Proof.** Since \( I \) is a squarefree quadratic monomial ideal, by Taylor resolution, we know that \( \beta_{0,j}^S(S/I) = 0 \) for all \( j > 2i \). Hence, using \[2, Theorem 4.6\], we get the result. \( \square \)

We now prove that for an edge ideal depth is incomparable with regularity and degree of \( h \)-polynomial. Indeed we construct graphs whose edge ideal has depth \( \delta \) and regularity \( r \). Also we show the existence of a graph whose edge ideal has depth \( \delta \) and degree of \( h \)-polynomial \( s \).

**Theorem 4.13.** Let \( \delta, r, s \) be positive integers. Then we have the followings:

i) There exists a graph \( G \) such that \( \depth(S/I(G)) = \delta \) and \( \reg(S/I(G)) = r \).

ii) There exists a graph \( G \) such that \( \depth(S/I(G)) = \delta \) and \( \deg(h_{S/I(G)}(t)) = s \).

First we prove the above theorem for \( \delta = 1 \) or \( r = 1 \) or \( s = 1 \). Then we use the following fact to complete our theorem. Let \( I \subset R = k[x_1, \ldots, x_m] \) and \( J \subset T = k[y_1, \ldots, y_n] \) be homogeneous ideals and \( S = k[x_1, \ldots, x_m, y_1, \ldots, y_n] \). Then \( \reg(S/(I+J)) = \reg(R/I) + \reg(T/J) \), \( \depth(S/(I+J)) = \depth(R/I) + \depth(T/J) \) and \( \deg(h_{S/(I+J)}(t)) = \deg(h_{R/I}(t)) + \deg(h_{T/J}(t)) \).

**Proof of Theorem 4.13** (i) **Case I.** For \( \delta = 1 \) and \( r \geq 1 \), take \( G = P_{3r}^2 \). By Proposition \ref{prop4.2} and \[11, Theorem 4.7\], \( \reg(S/I(G)) = r \) and by short exact sequence \( \ref{seq1} \), \( \depth(S/I(G)) = 1 \).

**Case II.** For \( 2 \leq \delta \leq r \), take \( G \) to be the union of \( P_{3(r-\delta+1)}^2 \) and \( \delta - 1 \) disjoint edges. Then we get \( \depth(S/I(G)) = \delta \) and \( \reg(S/I(G)) = r \).
**Case III.** For $r = 1$, consider $G = W(\delta + 1, \delta)$. By Theorem 4.4, $\text{reg}(S/I(G)) = 1$ and $\text{depth}(S/I(G)) = \delta$.

**Case IV.** For $2 \leq r < \delta$, take $G$ to be the union of $W(\delta - r + 2, \delta - r + 1)$ with $r - 1$ disjoint edges. This completes the proof of (i).

(ii) **Case I.** For $\delta = 1$, consider $G = K_{1,s}$. It follows from (1) that $\text{depth}(S/I(G)) = 1$ and by (3), $\deg(h_{S/I(G)}(t)) = s$.

**Case II.** For $2 \leq \delta \leq s$, take $G$ to be the union of $K_{1,s-\delta+1}$ and $\delta - 1$ disjoint edges. Then $\text{depth}(S/I(G)) = \delta$ and $\deg(h_{S/I(G)}(t)) = s$.

**Case III.** For $s = 1$, take $G = F_{\delta}$. Since $S/I(G)$ is Cohen-Macaulay of dimension $\delta$ and $\text{reg}(S/I(G)) = 1$, we have $\text{depth}(S/I(G)) = \delta$ and $\deg(h_{S/I(G)}(t)) = 1$.

**Case IV.** For $2 \leq s < \delta$, take $G$ to be the union of $F_{\delta-s+1}$ and $s - 1$ disjoint edges. Now the assertion follows from Case III.

□

**References**

[1] Selvi Beyarslan, Huy Tài Hà, and Tràn Nam Trung. Regularity of powers of forests and cycles. *J. Algebraic Combin.*, 42(4):1077–1095, 2015.

[2] Mats Boij and Jonas Söderberg. Betti numbers of graded modules and the multiplicity conjecture in the non-Cohen-Macaulay case. *Algebra Number Theory*, 6(3):437–454, 2012.

[3] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993.

[4] Oscar Fernández-Ramos and Philippe Gimenez. Regularity 3 in edge ideals associated to bipartite graphs. *J. Algebraic Combin.*, 39(4):919–937, 2014.

[5] Ralf Fröberg. On Stanley-Reisner rings. In *Topics in algebra, Part 2 (Warsaw, 1988)*, volume 26 of *Banach Center Publ.*, pages 57–70. PWN, Warsaw, 1990.

[6] Yan Gu, Huy Tai Ha, Jonathan L. O’Rourke, and Joseph W. Skelton. Symbolic powers of edge ideals of graphs. *arXiv e-prints*, page arXiv:1805.03428, May 2018.

[7] Huy Tai Ha, Dang Hop Nguyen, Ngo Viet Trung, and Tran Nam Trung. Symbolic powers of sums of ideals. *arXiv e-prints*, page arXiv:1702.01766, Feb 2017.

[8] Jürgen Herzog and Takayuki Hibi. *Monomial ideals*, volume 260 of *Graduate Texts in Mathematics*. Springer-Verlag London, Ltd., London, 2011.

[9] Takayuki Hibi, Kyouko Kimura, and Kazunori Matsuda. Extremal Betti numbers of edge ideals. *arXiv e-prints*, page arXiv:1810.08969, Oct 2018.

[10] Takayuki Hibi, Kazunori Matsuda, and Adam Van Tuyl. Regularity and $h$-polynomials of edge ideals. *arXiv e-prints*, page arXiv:1810.07140, Oct 2018.
[11] A. V. Jayanthan and Rajiv Kumar. Regularity of Symbolic Powers of Edge Ideals. arXiv e-prints, page arXiv:1903.05313, Mar 2019.

[12] Arvind Kumar, Rajiv Kumar, Rajib Sarkar, and S. Selvaraja. Symbolic powers of certain cover ideals of graphs. arXiv e-prints, page arXiv:1903.00178, Mar 2019.

[13] Amir Mousivand. Algebraic properties of product of graphs. Comm. Algebra, 40(11):4177–4194, 2012.

[14] S. Selvaraja. Regularity of powers of edge ideals of product of graphs. J. Algebra Appl., 17(7):1850128, 20, 2018.

[15] S. A. Seyed Fakhari. Regularity of symbolic powers of edge ideals of chordal graphs. preprint.

[16] S. A. Seyed Fakhari. Regularity of symbolic powers of edge ideals of unicyclic graphs. arXiv e-prints, page arXiv:1903.10962, Mar 2019.

[17] Rafael H. Villarreal. Monomial algebras, volume 238 of Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker, Inc., New York, 2001.

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