Surface Patches with Rounded Corners

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Abstract

We analyze surface patches with a corner that is rounded in the sense that the partial derivatives at that point are antiparallel. Sufficient conditions for $G^1$ smoothness are given, which, up to a certain degenerate case, are also necessary. Further, we investigate curvature integrability and present examples.

1 Introduction

Surface parametrizations of form $x : [a, b] \times [c, d] \rightarrow \mathbb{R}^3$, such as Bézier patches or NURBS surfaces, are frequently used in geometric modeling. The regularity of $x$ in the sense that $\det Dx \neq 0$ is a standard assumption to guarantee the geometric smoothness of the trace of $x$, and also, most analytic tools from differential geometry rely on that assumption. However, regularity implies that the shape of the parametrized surface is necessarily four-sided. Moreover, the parametrically smooth contact of $n \neq 4$ such patches sharing a vertex in a composite model is impossible. Methods to overcome these restrictions include trimming [MH18] and the concept of geometric continuity [Pet02]. Another approach is based on deliberately dropping regularity at isolated spots of the surface. This increases flexibility, but special care has to be taken that the resulting surfaces are geometrically smooth of order $G^k$ in a vicinity of the singularity in the sense that, locally, there exists a regular reparametrization of class $C^k$.

There exist different types of patches with singularities: First, certain partial derivatives of $x$ can be set to zero at a corner of the domain. In [Rei97] and [BR97], conditions for $C^1$- and $C^2$-smoothness of such parametrizations are derived, see also [SLL11]. These constructions are useful for the parametrically smooth contact of $n \neq 4$ patches meeting at a point.

Second, whole edges of the domain can be requested to collapse to single points in the image of $x$, see for instance [YHL14]. This facilitates the representation of three- and also two-sided shapes. In [ST21], an extraction matrix is utilized to construct $C^1$-smooth splines on such shapes, allowing smooth single-patch parametrizations of ellipsoids.

Third, the first partial derivatives of $x$ at certain corners of the domain may be antiparallel. This means that the edges sharing such corners are mapped to curves meeting
at straight angles. Again, this construction admits the representation of three- or two-sided shapes. But it is equally possible to parametrize, for instance, a hemispherical shape by a single map \( x : [0, 1]^2 \to \mathbb{R}^3 \), see Section 3.1.

Surface patches with such rounded corners appear for instance in watertight Boolean operations presented in [UMC+19]. In this approach, turning points of trimming curves are utilized to define a layout of surface patches for constructing non-trimmed watertight boundary representations of volumes in \( \mathbb{R}^3 \). In the resulting model, a turning point becomes a corner of the surfaces obtained, and if the initial trimming curve is smooth at this point, this corner is prone to be a rounded one. Other applications can be found in isogeometric analysis, where patches with rounded corners may be convenient to parametrize the physical domain, see for instance [TJ12]. The latter reference addresses aspects of Sobolev regularity, while a specific analysis of geometric smoothness seems to be unknown in the literature. This paper aims at filling that gap.

In Section 2, we define surface patches with rounded corners and provide sufficient conditions for \( C^1 \)-smoothness. These conditions are shown to be almost necessary in the sense that only cases with degeneracies of higher order are left undecided. Further, we show that the principal curvatures of such \( C^1 \)-patches are square-integrable. In Section 3, we present two examples featuring the use of surface patches with rounded corners in applications.

## 2 Analysis of rounded corners

To simplify notation, we consider surface patches with domain \([0, H]^2\) and analyze their behavior in the vicinity of the vertex \((0, 0)\). The generalization to an arbitrary corner of an arbitrary rectangle is straightforward.

**Definition 2.1** Let

\[
  x : [0, H]^2 \to \mathbb{R}^3, \quad x(u, v) = \sum_{j+k=0}^{2} \frac{w^j v^k}{j!k!} \xi_{j,k} + O((u+v)^3),
\]

be a three times differentiable surface patch. It has a rounded corner at \((u, v) = (0, 0)\) if the following conditions are satisfied:

- **Antiparallelism.** There exists a unit vector \( t \in \mathbb{R}^3 \) and factors \( \mu, \lambda > 0 \) such that
  \[
  \xi_{1,0} = \lambda t, \quad \xi_{0,1} = -\mu t.
  \]

- **Coplanarity.** The three vectors
  \[
  r := \mu \xi_{2,0} + \lambda \xi_{1,1}, \quad s := \lambda \xi_{0,2} + \mu \xi_{1,1},
  \]
  and \( t \) are linearly dependent.
- **Onesidedness.** The three vectors satisfy

\[ \langle \mathbf{t} \times \mathbf{r}, \mathbf{t} \times \mathbf{s} \rangle > 0. \]

Throughout, we will assume that the size $H$ of the domain is chosen so small that properties of the surface $\mathbf{x}$ away from the origin can be discarded. Furthermore, we define

\[ h := \max\{u, v\}. \]

Antiparallelism of the partial derivatives $\xi_{1,0}, \xi_{0,1}$ causes a loss of regularity of $\mathbf{x}$ at the origin. This means that geometric smoothness of the trace cannot be taken for granted at that point, despite the smoothness of the parametrization. However, we are going to demonstrate that coplanarity together with onesidedness guarantees that the patch $\mathbf{x}$ is $G^1$, meaning that there exists a regular $C^1$-parametrization of the trace of $\mathbf{x}$ near the rounded corner. We will also show that the $G^1$-property is lost if the vectors $\mathbf{r}, \mathbf{s}, \mathbf{t}$ are linearly independent, or if the quadruple product is negative, leaving only the particular case $\langle \mathbf{t} \times \mathbf{r}, \mathbf{t} \times \mathbf{s} \rangle = 0$ undecided.

Onesidedness implies that the limit normal

\[ \mathbf{n} := \frac{\mathbf{t} \times \mathbf{r}}{\|\mathbf{t} \times \mathbf{r}\|} \]

is well defined. Together, $\mathbf{t}$ and the cross vector

\[ \mathbf{c} := \mathbf{n} \times \mathbf{t} \]

span the limit tangent space

\[ \mathbf{T} := \{\alpha \mathbf{t} + \beta \mathbf{c} : (\alpha, \beta) \in \mathbb{R}^2\}, \]

and we observe that $\mathbf{r}, \mathbf{s} \in \mathbf{T}$. Denoting the cross components of $\mathbf{r}$ and $\mathbf{s}$ by

\[ \varrho := \langle \mathbf{c}, \mathbf{r} \rangle, \quad \sigma := \langle \mathbf{c}, \mathbf{s} \rangle, \]

respectively, we have

\[ \mathbf{t} \times \mathbf{r} = \varrho \mathbf{n} \quad \text{and} \quad \mathbf{t} \times \mathbf{s} = \sigma \mathbf{n}. \quad (2) \]

Onesidedness yields $\langle \mathbf{t} \times \mathbf{r}, \mathbf{t} \times \mathbf{s} \rangle = \varrho \sigma > 0$ so that

\[ \varrho = \|\mathbf{t} \times \mathbf{r}\| > 0 \quad \text{and} \quad \sigma > 0. \quad (3) \]

Geometrically speaking, positivity of the cross components means that the vectors $\mathbf{r}$ and $\mathbf{s}$ lie on the same side of the vector $\mathbf{t}$ within the plane $\mathbf{T}$, which accounts for the name of the third property in Definition 2.1.

The following theorem shows that $\mathbf{n}$ is in fact the limit of normal vectors

\[ \nu := \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|} \]

at the rounded corner.
Theorem 2.2 (Normal continuity.) The surface patch $\mathbf{x}$ with a rounded corner according to Definition 2.1 is normal continuous at $(0,0)$ with

$$\mathbf{n} = \lim_{(u,v) \to (0,0)} \nu(u,v).$$

Proof. The partial derivatives of $\mathbf{x}$ are

$$\mathbf{x}_u = \lambda \mathbf{t} + u \xi_{2,0} + v \xi_{1,1} + O(h^2), \quad \mathbf{x}_v = -\mu \mathbf{t} + v \xi_{0,2} + u \xi_{1,1} + O(h^2). \quad (4)$$

Their cross product is

$$\mathbf{x}_u \times \mathbf{x}_v = \mathbf{t} \times (u \mathbf{r} + v \mathbf{s}) + O(h^2) = \mathbf{t} \times \mathbf{r} + v \mathbf{t} \times \mathbf{s} + O(h^2) \quad (5)$$

$$= (\varrho u + \sigma v) \mathbf{n} + O(h^2), \quad (6)$$

where we used (2) to derive the last equality. By positivity of the cross components according to (3), the reciprocal of the first factor in (6) is bounded by

$$0 < \frac{1}{\varrho u + \sigma v} \leq \frac{1}{\min\{\varrho, \sigma\} h} = O(1/h), \quad (u, v) \in [0, H]^2 \setminus \{(0,0)\}. \quad (7)$$

Hence,

$$\frac{1}{\|\mathbf{x}_u \times \mathbf{x}_v\|} = \frac{1}{(\varrho u + \sigma v) + O(h^2)} = \frac{1}{(\varrho u + \sigma v)(1 + O(h))} = O(1/h),$$

and convergence of normal vectors follows from

$$\nu = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|} = \frac{(\varrho u + \sigma v)\mathbf{n} + O(h^n)}{(\varrho u + \sigma v)(1 + O(h))} = \mathbf{n} + O(h).$$

The next theorem clarifies that coplanarity and onesidedness are essential for normal continuity when antiparallelism is assumed.

Theorem 2.3 (Normal discontinuity.) Let $\mathbf{x}$ be a surface patch with antiparallel partial derivatives $\xi_{1,0}, \xi_{0,1}$ at the origin, as in Definition 2.1. If the vectors $\mathbf{r}, \mathbf{s}, \mathbf{t}$ are linearly independent, or if

$$\langle \mathbf{t} \times \mathbf{r}, \mathbf{t} \times \mathbf{s} \rangle < 0,$$

then $\mathbf{x}$ is not normal continuous.

Proof. First, let us assume that the vectors $\mathbf{r}, \mathbf{s}, \mathbf{t}$ are linearly independent. Then also the vectors $\mathbf{n}_1 := \mathbf{t} \times \mathbf{r}$ and $\mathbf{n}_2 := \mathbf{t} \times \mathbf{s}$ are linearly independent since $\mathbf{n}_1 \times \mathbf{n}_2 = \det[\mathbf{r}, \mathbf{s}, \mathbf{t}] \mathbf{t} \neq 0$. According to (5), we have

$$\mathbf{x}_u \times \mathbf{x}_v = u \mathbf{t} \times \mathbf{r} + v \mathbf{t} \times \mathbf{s} + O(h^2) = u \mathbf{n}_1 + v \mathbf{n}_2 + O(h^2).$$
Comparing

\[ \nu(u, 0) = \frac{u\mathbf{n}_1 + O(u^2)}{u\|\mathbf{n}_1\|(1 + O(u))} = \frac{\mathbf{n}_1}{\|\mathbf{n}_1\|} + O(u) \]

and

\[ \nu(0, v) = \frac{v\mathbf{n}_2 + O(v^2)}{v\|\mathbf{n}_2\|(1 + O(v))} = \frac{\mathbf{n}_2}{\|\mathbf{n}_2\|} + O(v) \]

shows that \( \nu \) does not have a unique limit at \((0, 0)\).

Second, let us assume that the vectors \( \mathbf{r}, \mathbf{s}, \mathbf{t} \) are coplanar, and that \( \langle \mathbf{t} \times \mathbf{r}, \mathbf{t} \times \mathbf{s} \rangle < 0 \).

Then we can follow the proof of the preceding theorem up to (6),

\[ x_u \times x_v = (\varrho u + \sigma v) \mathbf{n} + O(h^2), \]

but now, the factors \( \varrho \) and \( \sigma \) have opposite sign,

\[ \langle \mathbf{t} \times \mathbf{r}, \mathbf{t} \times \mathbf{s} \rangle = \varrho \sigma < 0. \]

As before, comparing

\[ \nu(u, 0) = \frac{\varrho u \mathbf{n} + O(u^2)}{|\varrho|u\|\mathbf{n}\|(1 + O(u))} = \mathbf{n} + O(u) \]

and

\[ \nu(0, v) = \frac{\sigma v \mathbf{n} + O(v^2)}{|\sigma|v\|\mathbf{n}\|(1 + O(v))} = -\mathbf{n} + O(v) \]

shows that \( \nu \) does not have a limit at \((0, 0)\). \( \Box \)

The only case of surface patches with antiparallel partial derivatives \( \xi_{1,0}, \xi_{0,1} \) not covered by Theorems 2.2 and 2.3 is that of coplanar vectors \( \mathbf{r}, \mathbf{s}, \mathbf{t} \) with \( \langle \mathbf{t} \times \mathbf{r}, \mathbf{t} \times \mathbf{s} \rangle = 0 \), which represents a degeneracy of higher order.

Normal continuity is a relatively weak notion of smoothness since it does not imply that the trace of the given parametrization is a smooth manifold. In particular, local self-intersections cannot be excluded. As an example, consider the surface

\[ \mathbf{x}(u, v) = \begin{bmatrix} u^7 - 21u^5v^2 + 35u^3v^4 - 7uv^6 \\ v^7 - 21v^5u^2 + 35v^3u^4 - 7vu^6 \\ u^{10} + v^{10} \end{bmatrix}, \quad [u, v] \in [0, 1]^2, \]

see Figure 1 (left). It is easily verified by inspection that

\[ \lim_{(u,v) \to (0,0)} \nu(u, v) = (0, 0, 1)^t, \]

but the projection of \( \mathbf{x} \) into the \( xy \)-plane is not injective, see Figure 1 (right). The following result states that this cannot happen near rounded corners if the conditions of Definition 2.1 are satisfied.
Theorem 2.4 (Single-sheetedness.) Let $\mathbf{x}$ be a surface patch with a rounded corner according to Definition 2.1. When restricted to a sufficiently small neighborhood $[0, H]^2$ of the origin, the orthogonal projection of $\mathbf{x} - \mathbf{x}_{0,0}$ to the limit tangent space $\mathbf{T}$, corresponding to the map

$$\Pi : [0, H]^2 \to \mathbf{T}, \quad \Pi(\eta) := (\text{Id} - \mathbf{n}\mathbf{n}^t)(\mathbf{x}(\eta) - \mathbf{x}_{0,0}),$$

is injective.

**Proof.** Assume that $\Pi(\eta_0) = \Pi(\eta_1)$ for $\eta_0, \eta_1 \in [0, H]^2$, and let

$$\delta := M^{-1}(\eta_1 - \eta_0), \quad M := \begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix}.$$  

Then the scalar function

$$g(t) := \delta^t [\mathbf{t} \quad \mathbf{c}]^t \Pi(\eta_0 + M\delta t)$$

has equal values at $t = 0$ and $t = 1$,

$$g(1) - g(0) = \delta^t [\mathbf{t} \quad \mathbf{c}]^t (\Pi(\eta_1) - \Pi(\eta_0)) = 0.$$  

By the mean value theorem, there exists $\tau \in (0, 1)$ such that

$$g'(\tau) = \delta^t [\mathbf{t} \quad \mathbf{c}]^t (\text{Id} - \mathbf{n}\mathbf{n}^t) D\mathbf{x}(u, v) M\delta = \delta^t ([\mathbf{t} \quad \mathbf{c}]^t D\mathbf{x}(u, v) M) \delta = 0,$$

where $(u, v) = \eta_0 + M\delta \tau \in [0, H]^2 \setminus \{(0, 0)\}$ by convexity of the domain. Here, we used that, by definition, both $\mathbf{t}$ and $\mathbf{c}$ are perpendicular to $\mathbf{n}$. Denoting the matrix

$$\lambda = x_3, \quad \mu = x_4,$$
in parentheses by $J$, we obtain $g'(\tau) = \delta^t J \delta$. With $Dx = [x_u \ x_v]$, a short computation yields

$$J := \begin{bmatrix} \lambda^2 + \nu^2 + O(h) & O(h) \\ O(h) & \rho u + \sigma v + O(h^2) \end{bmatrix}.$$  

We symmetrize the quadratic form by setting $J_s := (J + J^t)/2$ and use (7) again to obtain $g'(\tau) = \delta^t J_s \delta$ with

$$J_s := \begin{bmatrix} (\lambda^2 + \nu^2)(1 + O(h)) & O(h) \\ O(h) & (\rho u + \sigma v)(1 + O(h)) \end{bmatrix}.$$  

Recalling $0 < h = \max\{u, v\} \leq H$,

$$\det J_s = (\lambda^2 + \nu^2)(\rho u + \sigma v)(1 + O(h))$$

$$\text{trace } J_s = (\lambda^2 + \nu^2)(1 + O(h))$$

shows that $J_s$ is positive definite provided that $H$ is sufficiently small. Eventually, $g'(\tau) = \delta^t J_s \delta = 0$ implies $\delta = 0$ and $\eta_0 = \eta_1$, showing that $\Pi$ is injective. 

Together, normal continuity and single-sheetedness imply that the parametrization of the patch $x$ as a graph over the limit tangent space is $C^1$, see [PR08, Theorem 2.13]. We state this result as

**Corollary 2.5 (C$^1$-regularity.)** A surface patch $x$ with a rounded corner according to Definition 2.1 possesses a regular $C^1$-parametrization in a neighborhood of that corner.

The $G^1$-property of surface patches with a rounded corner is a prerequisite for many design applications. However, also the asymptotic behavior of the principal curvatures $\kappa_1, \kappa_2$ is significant. In particular, square integrability is requested when such patches shall be used for the Ritz-Galerkin simulation of 4th order PDEs, like thin shell equations. The following theorem settles this issue.

**Theorem 2.6 (Curvature integrability.)** In a neighborhood of the rounded corner, the principal curvatures $\kappa_{1,2}$ of a surface patch $x$ according to Definition 2.1 are almost in $L^3$ in the sense that

$$\int_{[0,H]^{2}} |\kappa_{1,2}|^p \ d\mu < \infty \text{ for any } p \in [1, 3),$$

where $d\mu$ denotes the surface element of $x$ and $H > 0$ is chosen sufficiently small.

**Proof.** The first fundamental form of $x$ is

$$G := \begin{bmatrix} \langle x_u, x_u \rangle & \langle x_u, x_v \rangle \\ \langle x_v, x_u \rangle & \langle x_v, x_v \rangle \end{bmatrix} = \begin{bmatrix} \lambda^2 & -\lambda \mu \\ -\lambda \mu & \mu^2 \end{bmatrix} + O(h).$$
By (6) and (7), its inverse is given by
\[
G^{-1} = \frac{1}{\|x_u \times x_v\|^2} \begin{bmatrix} \mu^2 & \lambda \mu \\ \lambda \mu & \lambda^2 \end{bmatrix} + O(1/h).
\]

With the second fundamental form
\[
B := \begin{bmatrix} \langle x_{uu}, v \rangle & \langle x_{uv}, v \rangle \\ \langle x_{uv}, v \rangle & \langle x_{vv}, v \rangle \end{bmatrix} = \begin{bmatrix} \langle \xi_{2,0}, n \rangle & \langle \xi_{1,1}, n \rangle \\ \langle \xi_{1,1}, n \rangle & \langle \xi_{0,2}, n \rangle \end{bmatrix} + O(h),
\]

we obtain the shape operator
\[
S := G^{-1}B = \frac{1}{\|x_u \times x_v\|^2} \begin{bmatrix} \mu(r, n) & \mu(s, n) \\ \lambda(r, n) & \lambda(s, n) \end{bmatrix} + O(1/h) = O(1/h).
\]

Its eigenvalues are the principal curvatures \( \kappa_{1,2} \), which are of the same order of magnitude,
\[
\kappa_{1,2} = O(1/h).
\]

The surface element is \( d\mu = \|x_u \times x_v\| dudv \), where \( \|x_u \times x_v\| = O(h) \). Hence, there exists a constant \( c \) such that \( |\kappa_{1/2}|^p \|x_u \times x_v\| \leq ch^{1-p} \), and we obtain, using \( h = \max\{u, v\} \),
\[
\int_{[0,H]^2} |\kappa_{1,2}|^p d\mu \leq c \int_0^H \int_0^H h^{1-p} dudv = \frac{2cH^{3-p}}{3-p} < \infty
\]
for \( p < 3 \), as claimed. \( \square \)

In applications, surface patches are often given in B-spline format. The following theorem specifies conditions for control points that are equivalent to Definition 2.1. These conditions take the simplest form when the boundary knots have maximal multiplicity so that we focus on that case. In particular, Bézier patches are covered.

**Theorem 2.7** Denote by \( b^1_j \) and \( b^2_k \) the B-splines of degrees \( n_1, n_2 \geq 2 \) with knots
\[
T_1 = [0, \ldots, 0, \tau_1^1, \tau_2^1, \ldots, \tau_{N_1}^1], \quad T_2 = [0, \ldots, 0, \tau_1^2, \tau_2^2, \ldots, \tau_{N_2}^2],
\]
respectively, where \( \tau_1^1, \tau_2^2 > 0 \). The spline surface
\[
x(u, v) = \sum_{j=0}^{N_1-1} \sum_{k=0}^{N_2-1} b^1_j(u)b^2_k(v)\ p_{j,k}
\]
with control points \( p_{j,k} \in \mathbb{R}^3 \) has a rounded corner at \((0, 0)\) according to Definition 2.1 if the following conditions are satisfied:
• There exist weights \( \alpha_1, \alpha_2 \in (0, 1) \) with \( \alpha_1 + \alpha_2 = 1 \) such that
  \[ p_{0,0} = \alpha_1 p_{1,0} + \alpha_2 p_{0,1}. \] (8)

• The three vectors
  \[ r^* := (n_1 - 1)\tau_1^1 \alpha_1 (p_{2,0} - p_{0,0}) + n_1\tau_2^1 \alpha_2 (p_{1,1} - p_{0,0}) \]
  \[ s^* := (n_2 - 1)\tau_1^2 \alpha_2 (p_{0,2} - p_{0,0}) + n_2\tau_2^2 \alpha_1 (p_{1,1} - p_{0,0}) \]
  \[ t^* := p_{1,0} - p_{0,1} \]

  are linearly dependent.

• We have
  \[ \langle t^* \times r^*, t^* \times s^* \rangle > 0. \] (10)

**Proof.** In the following, \( c_1, \ldots, c_5 \) denote real factors, the specific values of which are irrelevant. The partial derivatives of \( x \) are given by

\[
\begin{align*}
\xi_{0,0} &= p_{0,0} \\
\xi_{1,0} &= \frac{n_1}{\tau_1}(p_{1,0} - p_{0,0}) \\
\xi_{0,1} &= \frac{n_2}{\tau_1}(p_{0,1} - p_{0,0}) \\
\xi_{2,0} &= \frac{n_1(n_1 - 1)}{\tau_1^1\tau_2^1}(p_{2,0} - p_{0,0}) + c_1t^* \\
\xi_{1,1} &= \frac{n_1n_2}{\tau_1^2\tau_2^1}(p_{1,1} - p_{0,0}) + c_2t^* \\
\xi_{0,2} &= \frac{n_2(n_2 - 1)}{\tau_1^2\tau_2^2}(p_{0,2} - p_{0,0}) + c_3t^*
\end{align*}
\]

First, we observe that \( t^* \neq 0 \) because of (10). Further, by (8), we have \( p_{1,0} - p_{0,0} = \alpha_2 t^* \) and \( p_{0,1} - p_{0,0} = \alpha_1 t^* \). Hence, the condition of antiparallelism is satisfied with

\[ t = t^*/\|t^*\|, \quad \lambda = n_1\alpha_2\|t^*\|/\tau_1^1, \quad \mu = n_2\alpha_1\|t^*\|/\tau_1^2. \]

Second, we find

\[ r = \frac{n_1n_2\|t^*\|}{(\tau_1^1)^2\tau_2^1\tau_2^1} r^* + c_4t^*, \quad s = \frac{n_1n_2\|t^*\|}{(\tau_2^1)^2\tau_1^1\tau_2^1} s^* + c_5t^* \]

and conclude that \( r, s, t \) are linearly dependent if so are \( r^*, s^*, t^* \), thus establishing coplanarity. Third, positivity follows from

\[ \langle t \times r, t \times s \rangle = \frac{n_1^2n_2^2}{(\tau_1^1\tau_2^1)^3\tau_1^1\tau_2^2} \langle t^* \times r^*, t^* \times s^* \rangle > 0. \]

\[ \square \]
3 Experimental results

In this section, we apply the rounded corner constraints to B-spline models. First, the impact of these conditions on the approximation of a hemisphere, its normal, and curvature is investigated. Then, they are utilized to improve the representation of a boat fender model.

3.1 Approximation of a hemisphere

The first example considers the approximation of a hemisphere with radius $r = 1$ by a spline surface with four rounded corners. Figure 2 illustrates the reference surface $y$ and details its parametrization. Based on $y$, we construct single-patch B-spline surfaces $x : [-1, 1]^2 \rightarrow \mathbb{R}^3$ of bi-degree $(n, n)$ with knot spacing $h = 2^{-\ell}$ for various values of $n$ and $\ell$. The following schemes are employed:

- standard: approximation by conventional $L^2$-projection
- constrained: $L^2$-projection including the rounded corner constraints (RCC)

In both cases, the $L^2$-projection is performed in two steps: first, the boundary control points are fitted in the $xy$-plane, and subsequently, the inner control points are computed. This procedure yields better visual comparability of coarse discretization. For the constrained case, we set $\alpha_1 = \alpha_2 = 1/2$ and use the known limit normal $n$ of each rounded corner to specify the corresponding orientation of the limit tangent space $T$. Lagrange multipliers are used to enforce these conditions together with the antiparallelism and coplanarity constraints, i.e., (8) and (9). After the construction, we check for onesidedness (10). In our experiments, this condition was never violated.

Parametrization:

$$y(u, v) = \begin{bmatrix} 2us_2s_3 \\ 2vs_2s_3 \\ 2s_3 - 1 \end{bmatrix}, \ [u, v] \in [-1, 1]$$

with

$$s_1 = (1 - u^2)(1 - v^2)$$

$$s_2 = \left(1 - s_4^2\right)^2 \sqrt{u^2 + v^2 + s_1}^{-1}$$

$$s_3 = \left(u^2s_2^2 + v^2s_2^2 + 1\right)^{-1}$$

Figure 2: Parametrization of the hemisphere over the domain $[-1, 1]^2$. 
The implementation is first validated by a convergence study of the approximation error. Therefore, the maximal error of different B-splines with various degrees and element numbers are summarized in Figure 3(a). Note that both schemes obtain optimal convergence rates. In fact, the graphs of the standard and the constrained approach are almost identical, indicating that the effects of the constraints on the approximation power are marginal.

Let us now focus on geometric aspects of the approximation process. First, we investigate the error in representing normal vectors, which is measured by the angle between the reference normal and that of the approximation. Figure 3(b) shows the maximal deviations of normal vectors, again for different degrees and both schemes. It is worth noting that we never evaluate directly in a rounded corner, where the normal may be undefined. Obeying the constraints for rounded corners yields convergence of normals at rates growing with the chosen degree, while standard approximation performs significantly worse. Those issues of the standard scheme are induced by the loss of regularity near rounded corners. In Figure 4, the error in representing normal vectors is plotted along a diagonal emanating from a rounded corner of a B-spline surface \( x(u, v) \) with \( n = 3 \) and \( \ell = 3 \). To be precise, the errors are evaluated at \( u = v = \alpha \) with \( \alpha \in [10^{-7}, 0.1] \). Note that with the proposed RCC, the error in the normal vector goes to zero, as \( \alpha \to 0 \).
Figure 4: Error distribution along a diagonal emitting from a rounded corner for a B-spline surface with degree \( n = 3 \) and \( \ell = 3 \).

Finally, we demonstrate the impact of rounded corners on curvature. Therefore, Figure 5 shows the reflection lines of each approximation scheme for surfaces \( x \) with \( n = 2 \) and \( \ell = 2 \). Note the irregularities close to the rounded corner in the standard approach, which vanish in the constrained case.

Figure 5: Reflection lines of a B-spline with \( n = 2 \) and \( \ell = 2 \). The white lines are the \( C^1 \) isolines of the surfaces and the close-ups show the reflections around a rounded corner.
3.2 Watertight boat fender model

The following example utilizes the constrained $L^2$-projection investigated in Section 3.1 in the context of a modeling process. In particular, we consider a “watertight” boat fender B-spline model, i.e., the boundary representation has no trimmed patches, and all splines surfaces are connected by explicit continuity conditions. The initial model is constructed by watertight Boolean operations detailed, in [UMC+19]. These operations connect intersecting surfaces in a non-trimmed $C^0$-continuous manner. At the same time, this construction may introduce rounded corners in the spline model. Figure 6 illustrates the initial model of the boat fender. Note that the close-up shows four rendering defects. There the model possesses rounded corners.

Figure 7 outlines how the rounded corner constraints can be used to improve watertight models:

(a) detect all rounded corners and the adjacent surfaces,

(b) employ the constrained approximation scheme described in Section 3.1 and

(c) update the adjacent surfaces to maintain a watertight representation.

Here, adjacent surfaces refer to surfaces that are connected to another surface’s rounded corner. Their update is necessary since the constrained approximation scheme affects the control points along the shared surface edge (cf. red and green control points in Figure 7(c)). The resulting model is shown in Figure 8.
Figure 7: Implementation of rounded corner constraints (RCC): (a) detection of a surface patch with rounded corners, (b) adjustment of the control points of this patch by constrained $L^2$-projection, and (c) update of the control points of the surface patches adjacent to the rounded corner. The orange reflection lines indicate the impact of the constraints. Initial control points are shown in red, while the final ones are shown in green. In (c), they are plotted on top of each other for better comparison.

Figure 8: Final watertight model of the boat fender with rounded corner constraints
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