The Markov approximation revisited: inconsistency of the standard quantum Brownian motion model

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Abstract

We revisit the Markov approximation necessary to derive ordinary Brownian motion from a model widely adopted in literature for this specific purpose. We show that this leads to internal inconsistencies, thereby implying that further search for a more satisfactory model is required.

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I. INTRODUCTION

According to Penrose [1], the time evolution of a quantum system should be thought of as the combination of two processes: the $U$-process, i.e., the unitary time evolution prescribed by the Schrödinger equation, and the $R$-process, namely, the genuine randomness associated to the collapse of the wave function. This perspective is very attractive, and many attempts are currently being made to set the $R$-process on the same dynamical basis as the $U$-process. This purpose is realized by some authors [2–4] by supplementing the Schrödinger equation with stochastic corrections, which are then interpreted as a manifestation of the environment influence.

As well known [5], the path-integral formalism is quite equivalent to ordinary quantum mechanics. Within this formalism the joint action of the $U$-process and the $R$-process is described by the theory of the path integral with constraints developed by Mensky [6]. We think that a possible physical meaning of the Mensky approach is made especially transparent by the interesting recent work of Presilla, Onofrio and Tambini [7–9]. These authors prove that the well known method of the influence functional introduced by Feynmann and Vernon [10] can be used to derive the Mensky path integral with constraint as an effect of the influence of the environment. This is a very interesting result which might reflect significant elements of truth [11]. The conviction that the wave-function collapse is generated by an environment and that no such systems exist in nature as isolated systems is accepted by the majority of physicists.

In spite of this general consensus, we find some element of inconsistency in this view. First, we observe that no claim of a genuine derivation of stochastic properties can be made without involving the Markov approximation. Adelman [11] built up a sort of Fokker-Planck equation equivalent to the generalized Langevin equation and this, in turn, can be derived from a dynamic picture not involving any Markov approximation [12,13], thereby generating the impression that the Markov assumption is not indispensable to the foundation of a stochastic process. However, as later proved by Fox [14], the non-Markovian Fokker-Planck equation of Adelman is not a stochastic process.

Within a quantum mechanical context the close connection between the Markov assumption and a stochastic picture is pointed out by the recent work of Kleiinter and Shabanov [15]. On the same token we think that a stochastic picture not implying the Markov approximation is not genuinely stochastic, in spite of a recent claim to the contrary [16]. The celebrated generalized Langevin equation of Mori [12,13] is an illuminating example showing that the so called stochastic force is actually a deterministic function of the initial conditions and can be considered as stochastic only if additional assumptions are made, such as that of coarse graining or of an incomplete information. On the other hand, these arbitrary assumptions might have essential effects also on the dynamics: for instance, that of abolishing slow tails of the velocity correlation function, conflicting with the exponential nature of the relaxation of macroscopic variables, a basic tenet of stochastic physics [17].

This paper is devoted to the discussion of the physical conditions necessary to establish from within a quantum-mechanical picture a standard fluctuation-dissipation process for a macroscopic variable. This property implies the exponential nature of the relaxation process of the macroscopic variable, and this is incompatible with both classical [18] and quantum mechanics [19]. The dynamical approach to the fluctuation-dissipation process [17] leads
us to conclude that there are problems even if we disregard the case where the microscopic time scale is infinite, and the Markov approximation is impossible. In the case where the Markov approximation seems to be legitimate due to the existence of a finite time scale, it has the effect of disregarding a weak but persistent slow tail of the correlation function of the macroscopic variable. Thus, the Markovian approximation turns out to be equivalent to the influence of real physical processes, the so called *spontaneous fluctuations* [10], which are in fact shown to kill these slow tails [20]. The change of perspective is radical, even if it does not affect the resulting transport equations, and throughout this letter we shall refer to it as the transition from the *subjective* to the *objective* Markov approximation.

Another way of expressing our aim is as follows. We plan to establish the intensity of the corrections to ordinary quantum mechanics necessary to make valid the Markov assumption, which is incompatible with ordinary quantum mechanics. In literature a scarce attention is devoted to the physical meaning of the error associated to the Markov approximation. To properly establish the intensity of this error we interpret the Markov approximation as a property made genuine by proper corrections to quantum mechanics. Then we make a balance on the intensity of these corrections. If they turn out to be too large, we shall consequently judge the error associated with the Markov approximation to be unacceptable.

To realize this purpose we adopt the Caldeira and Leggett model [21] and thus our departure point is close to that established by the results of Presilla, Onofrio and Tambini [7,8], who, as earlier pointed out, showed that the contraction over the irrelevant bath variables yields the Mensky path integral. Similar results have been more recently obtained by Mensky himself, adopting as a bath the internal atomic structure [22]. To trigger the transition from these subjective Markov assumptions to an objective Markov approximation we adopt the same perspective as that used in [10] to address the problem of the wave-function collapse, through the environment-induced enhancement of spontaneous fluctuations. Furthermore, rather than using the model of Ghirardi, Pearle and Rimini [23] as a source of spontaneous fluctuations, we adopt here a Mensky constraint conceived as a correction to ordinary quantum mechanics.

II. MARKOV APPROXIMATION TO FRICTION IN THE ZERO-TEMPERATURE LIMIT

Notice that the high-temperature assumption of [7,8,21] with the Mensky constraint as a source of spontaneous fluctuations would make the calculations exceedingly difficult. Thus, we do not address the calculation of fluctuations and we limit ourselves to studying dissipation in the zero-temperature limit. As proved by Caldeira and Leggett [21], in the case of harmonic baths the friction term derived from the subjective Markov approximation is independent of temperature. Thus, we make the plausible assumption that even in the presence of the Mensky constraint, the high-temperature friction is the same as that derived by us in the zero-temperature limit (see Section III)\(^1\).

\(^1\)Of course, after completing the calculation of Section III, yielding a Markov dissipation, we make the implicit assumption that the fluctuations are properly decorrelated by the Mensky constraint.
Let us consider as in [21], an oscillator of interest interacting with an environment of $N$ oscillators, with $N \gg 1$. The Lagrangians describing this system are given by the following expressions:

$$L_0(q) = \frac{1}{2}m_0\dot{q}^2 - \frac{1}{2}m_0\omega_0^2q^2$$  \hspace{1cm} (1)

$$L_B(Q_i) = \sum_{j=1}^{N} \left\{ \frac{1}{2}m_B\dot{Q}_j^2 - \frac{1}{2}m_B\omega_j^2Q_j^2 \right\}$$  \hspace{1cm} (2)

$$L_I(q, Q_j) = q \sum_{j=1}^{N} g_j Q_j.$$  \hspace{1cm} (3)

Thus the resulting quantum mechanical motion of the whole system is described by the following quantum mechanical propagator:

$$\langle q_F, Q_{F1}, ..., Q_{FN}, T|q_I, Q_{I1}, ..., Q_{IN}, 0 \rangle = \int [dq]\exp\left\{ \frac{i}{\hbar} \int_0^T dt \left[ \frac{1}{2}m_0\dot{q}^2 - \frac{1}{2}m_0\omega_0^2q^2 \right]\right\} \times \prod_{j=1}^{N} \int [dQ_j]\exp\left\{ \frac{i}{\hbar} \int_0^T dt \left[ \frac{1}{2}m_B\dot{Q}_j^2 - \frac{1}{2}m_B\omega_j^2Q_j^2 + g_jqQ_j \right]\right\}.$$  \hspace{1cm} (4)

Let us use the contracted description already adopted by Feynman and co-workers [5,9]. This means the reduced density matrix $\rho_A$ [21] for the system of interest defined by:

$$\rho_A(q_F, q_I, T) = \int dq'_F, dq'_I K(q_F, q_I, T; q'_F, q'_I, 0)\rho_A(q'_F, q'_I, 0).$$  \hspace{1cm} (5)

Here $K$ denotes the superpropagator

$$K(q_F, q_I, T; q'_F, q'_I, 0) = \int [dq_1][dq_2]\exp\left\{ \frac{i}{\hbar} [S_0(q_1) - S_0^*(q_2)] \right\} \mathcal{F}[q_1, q_2],$$  \hspace{1cm} (6)

where $S_0$ is the action related to $L_0$ and $\mathcal{F}$ is the influence functional [5,6] whose explicit expression is

$$\mathcal{F}[q_1, q_2] = \exp\left\{ -\frac{1}{\hbar} \int_0^T dt [q_1(t) - q_2(t)]I_R(t) - \frac{i}{\hbar} \int_0^T dt [q_1(t) - q_2(t)]I_I(t) \right\},$$  \hspace{1cm} (7)

with the two convolution integrals $I_R(t)$ and $I_I(t)$ given by:

$$I_R(t) = \int_0^t dt'\alpha_R(t - t')[q_1(t') - q_2(t')],$$  \hspace{1cm} (8)

$$I_I(t) = \int_0^t dt'\alpha_I(t - t')[q_1(t') + q_2(t')].$$  \hspace{1cm} (9)

The two memory kernels $\alpha_R(t-t')$ and $\alpha_I(t-t')$ present in (8) and (9) in the zero-temperature limit are given by the expressions

so as to recover, in the proper limit, an ordinary fluctuation-dissipation process.
\[ \alpha_R(t - t') = \frac{\eta}{\pi} \int_0^\Omega d\omega \cos(\omega(t - t')), \quad (10) \]
\[ \alpha_I(t - t') = -\frac{\eta}{\pi} \int_0^\Omega d\omega \sin(\omega(t - t')), \quad (11) \]

where \( \eta \) is the friction and \( \Omega \) is a cut-off frequency introduced assuming ohmic environment \[24\].

We are now in a position to discuss the problems raised by the subjective Markov approximation on the two kernels (10) and (11). First, we observe that the Markov approximation is expressed by:

\[ I(t) = \int_0^t dt' f(t')K(t - t') = \int_0^t d\tau f(t - \tau)K(\tau) \simeq f(t) \int_0^\infty d\tau K(\tau). \quad (12) \]

According to the traditional wisdom, the closer \( K(\tau) \) to \( \delta(\tau) \), the more accurate this approximation is expected to be. From within the perspective adopted in this paper, however, we have to establish a clearcut separation between the case \( \Omega = \infty \) and the case \( \Omega < \infty \), regardless of how large \( \Omega \) might be in this latter case. This is so because we know \[17\] that, in accordance with the work of Ref. \[18,19\], the relaxation process of a variable made unstable by a bath with a bounded spectrum, cannot be exactly exponential. At long times, after the completion of the exponential decay of the macroscopic variable, non-exponential tails appear \[17\]. As weak as these tails are made by increasing the value of \( \Omega \), their existence is incompatible with the claim that a fluctuation-dissipation relation has been derived.

The ideal case of \( \Omega = \infty \), on the contrary, would make the property of (12) exact, and there would be no conflict with the requirements of stochastic theory. However, we think that the recourse to the property \( \Omega = \infty \) is a sort of subterfuge and that this choice and the thermodynamical properties of crystals are at odds.

Let us discuss now in this light the memory kernels of the case under study. Let us consider first the real part of the memory kernel \( \alpha_R(t - t') \). This corresponds to fluctuations, and in accordance with the known low-temperature predictions \[25\] we expect a strong non-Markovian behavior to emerge out of it. In fact, renaming \( \tau = t - t' \), and integrating (10), we get:

\[ \alpha_R(\tau) = \frac{\eta}{\pi} \left[ \frac{\sin(\Omega\tau)}{\tau} - 2 \frac{\sin^2(\Omega\tau/2)}{\tau^2} \right]. \quad (13) \]

This is integrable, and the adoption of the Markovian approximation, which is apparently possible, would lead to the puzzling result:

\[ \int_0^\infty \alpha_R(\tau) = \frac{\eta\Omega}{2} - \frac{\eta\Omega}{2} = 0. \quad (14) \]

Actually, in accordance with \[25,26\], this is a case of infinite memory strikingly departing from the \( \delta \)-function condition which is reached with \( \Omega \to \infty \) by the dissipation part. Not even the subjective Markov approximation is admitted in this case.

Then let us consider \( \alpha_I(\tau) \). In this case, on the basis of \[21\] we expect a Markovian behavior, temperature independent, to be legitimate at a subjective level. The following calculations confirm our expectation. From Eq. (11), we get
\[ \alpha_I(\tau) = \eta \frac{d}{d\tau} \left( \frac{1 \sin \Omega \tau}{\pi \tau} \right). \]  

(15)

By using

\[ \lim_{\Omega \to \infty} \frac{1 \sin \Omega \tau}{\pi \tau} = \delta(\tau), \]

we get the expression:

\[ \alpha_I(\tau) = \eta \delta'(\tau). \]  

(17)

Therefore the convolution integral (9) becomes:

\[ I_I(t) = -\eta \delta(0)[q_1(t) + q_2(t)] + \frac{\eta}{2}[\dot{q}_1(t) + \dot{q}_2(t)]. \]  

(18)

The adoption of this approach makes it possible to evidentiate that, as earlier pointed out, the case \( \Omega = \infty \) yields an exact result. In the case \( \Omega < \infty \) the same result can be obtained using the approximation of (12):

\[ I_I(t) = \eta \int_0^t dt' \frac{1}{\pi \tau} \left( \frac{1 \sin \Omega (t - t')}{\tau - t'} \right) [q_1(t') + q_2(t')] \]

\[ \simeq -\eta \left[ \frac{1 \sin \Omega (t - t')}{\pi \tau} \right] [q_1(t') + q_2(t')] \bigg|_0^t + \frac{\eta}{\pi} [\dot{q}_1(t) + \dot{q}_2(t)] \int_0^\infty d\tau \frac{\sin \Omega \tau}{\tau} \]

\[ = -\eta \frac{\Omega}{\pi} [q_1(t) + q_2(t)] + \frac{\eta}{2}[\dot{q}_1(t) + \dot{q}_2(t)]. \]  

(19)

This result coincides with that resting on the condition \( \Omega = \infty \). However, as earlier remarked, the Markov approximation is equivalent to disregarding weak but very slow tails [17].

In both cases, the dissipative part of the influence functional is therefore written as [25,26]:

\[ F^{(diss)}[q_1, q_2] = \exp \left\{ i \frac{\hbar}{\hbar} \int_0^T dt \eta \delta(0) [q_1^2(t) - q_2^2(t)] \right\} \]

\[ - i \frac{\hbar}{\hbar} \int_0^T dt \eta \frac{1}{2} [q_1(t) - q_2(t)][\dot{q}_1(t) + \dot{q}_2(t)] \bigg\}. \]  

(20)

Notice that the divergence associated with \( \delta(0) \) can be settled either by assigning to the Lagrangian a proper counter-term [24,27] or assuming a correlated initial condition for the total system [28].

III. FROM THE SUBJECTIVE TO THE OBJECTIVE MARKOV APPROXIMATION

We want to explore here the possibility that the problems with the choice of a finite value of \( \Omega \) might be solved assuming that the bath oscillators are subjected to a Mensky
measurement process [6,7]. Adopting the perspective of Onofrio, Presilla and Tambini [7,8], these measurement processes should be traced back to the interaction between each bath oscillator and its own bath. This would not be yet satisfactory because the finite $\Omega$ problem would now affect the baths of the bath: This would be the beginning of an endless chain of baths of baths. We truncate this endless chain by assuming that the measurement process on the bath oscillator is an expression of a generalized version of quantum mechanics, an expression of spontaneous fluctuations, in the spirit of the theory of Ghirardi, Pearle and Rimini [23]. Of course, also the oscillator of interest should undergo the direct influence of this process. However, we are also making the assumption that the corrections to ordinary quantum mechanics are extremely weak, thereby implying that the direct influence of the Mensky measurement process on the oscillator of interest can be safely neglected.

On these premises we are naturally led to adopt the following quantum mechanical propagator:

$$\langle q_F, Q_1^F, ..., Q_N^F, T|q_I, Q_1^I, ..., Q_N^I, 0 \rangle_T = \int [dq] \exp \left\{ \frac{i}{\hbar} \int_0^T dt \left[ \frac{1}{2} m_0 \dot{q}^2 - \frac{1}{2} m_0 \omega_0^2 q^2 \right] \right\} \times \prod_{j=1}^N \int [dQ_j] \exp \left\{ \frac{i}{\hbar} \int_0^T dt \left[ \frac{1}{2} m_B \dot{Q}_j^2 - \frac{1}{2} m_B \omega_j^2 Q_j^2 + i\hbar k(\omega_j)(Q_j - a_j)^2 + g_j q Q_j \right] \right\},$$

(21)

where $a_j = a_j(t)$ is a function which expresses the result of the measurement process taking place on the $j$-th bath oscillator and $k(\omega_j)$ is the strength of such a process. For the sake of generality, we assume the strength of this measurement process to depend on the bath-oscillator frequency. After several trials we established that the most convenient form to adopt for our purposes is the following linear dependence:

$$k(\omega_j) = \frac{\Gamma m_B}{\hbar} \left( \omega_j - \frac{i \Gamma}{2} \right).$$

(22)

The main tenet of all theoretical derivations of fluctuation-dissipation processes [29] is that the fluctuation-dissipation processes are perceived at a contracted level of description, and that these processes are nothing but a manifestation of the interaction with a bath, whose elementary constituents are not directly observed. This makes tempting the adoption of Markov approximation, as an expression of this lack of knowledge. However, in the same way as the information approach to statistical mechanics by Jaynes [30] implies probabilistic ingredients which might be foreign to classical mechanics, the Markov approximation is equivalent to an elementary randomness, which would be foreign to quantum mechanics, if this is not supplemented by the action of $R$-processes [1]. The purpose of this letter is to establish the amount of this elementary randomness with the help of the Mensky formalism.

Note that the assumption that the bath is found in a condition of equilibrium, expressed by a canonical distribution at temperature $T$ is a very strong assumption, already implying the inclusion of thermodynamical arguments. As discussed in Section IV, we have in mind a perspective, based on chaotic dynamics, where thermodynamics can be really reduced to dynamics, and this should prevent us from adopting this strong assumption. At the same time, this assumption, based on the fact that dynamics and statistics are decoupled the ones from the others, makes it possible to adopt a harmonic thermal bath, whose regular dynamics has the effects of rendering excessively large, as we shall see, the corrections to quantum mechanics necessary to make the Markov assumption possible.
In conclusion, using again the contracted description already adopted in the previous Section, we get for the influence functional the expression:

\[
\mathcal{F}_1[q_1, q_2] = \exp \left\{ -\frac{1}{\hbar} \int_0^\infty d\omega \frac{g^2(\omega)}{2m_B\omega} \left( \frac{dN}{d\omega} \right) \left[ \int_0^T dt \int_0^t dt' q_1(t)q_1(t')e^{-i\omega(t-t')-\Gamma(t-t')} \right. \\
+ \int_0^T dt \int_0^t dt' q_2(t)q_2(t')e^{i\omega(t-t')-\Gamma(t-t')} \\
- \int_0^T dt \int_0^t dt' q_1(t)q_2(t')e^{i\omega(t-t')-\Gamma(2T-t-t')} \\
- \int_0^T dt \int_0^t dt' q_2(t)q_1(t')e^{-i\omega(t-t')-\Gamma(2T-t-t')} \right\},
\]

where we have neglected terms of order $\Gamma$ or higher related with the definition (22). Notice the presence of the two memory kernels depending on the sum of the two times $t$ and $t'$ rather than on their difference.

Eq. (23) can be rewritten under a form similar to (7):

\[
\mathcal{F}_1[q_1, q_2] = \exp \left\{ -\frac{1}{\hbar} \int_0^T dt [q_1(t)I_{R,1}^{(\Gamma)}(t) + q_2(t)I_{R,2}^{(\Gamma)}(t) - q_1(t)J_{R,1}^{(\Gamma)}(t) - q_2(t)J_{R,1}^{(\Gamma)}(t)] \\
- \frac{i}{\hbar} \int_0^T dt [q_1(t)I_{I,1}^{(\Gamma)}(t) - q_2(t)I_{I,2}^{(\Gamma)}(t) + q_1(t)J_{I,1}^{(\Gamma)}(t) - q_2(t)J_{I,1}^{(\Gamma)}(t)] \right\},
\]

where

\[
I_{\lambda,k}^{(\Gamma)}(t) = \int_0^t dt' q_k(t')\alpha_{\lambda}^{(\Gamma)}(t-t')
\]

and

\[
J_{\lambda,k}^{(\Gamma)}(t) = e^{-2\Gamma(T-t)}I_{\lambda,k}^{(\Gamma)}(t),
\]

with $k = 1, 2$ and $\lambda = R, I$. The modified kernels $\alpha_{R}^{(\Gamma)}(\tau)$ and $\alpha_{I}^{(\Gamma)}(\tau)$ are defined as follows:

\[
\alpha_{R}^{(\Gamma)}(\tau) = \alpha_{R}(\tau)e^{-\Gamma\tau} = \frac{\eta}{\pi} \int_0^\Omega d\omega \cos \omega \tau e^{-\Gamma\tau}
\]

\[
\alpha_{I}^{(\Gamma)}(\tau) = \alpha_{I}(\tau)e^{-\Gamma\tau} = -\frac{\eta}{\pi} \int_0^\Omega d\omega \sin \omega \tau e^{-\Gamma\tau},
\]

that is, neglecting terms of order $\Gamma$:

\[
\alpha_{R}^{(\Gamma)}(\tau) = \frac{\eta}{\pi} \left[ \frac{\Omega}{\tau} \sin \frac{\Omega\tau}{\tau} - 2\frac{\sin^2(\Omega\tau/2)}{\tau^2} \right] e^{-\Gamma\tau}
\]

\[
\alpha_{I}^{(\Gamma)}(\tau) = \frac{\eta}{\pi} \frac{d}{d\tau} \left( \frac{\sin \Omega\tau}{\tau} e^{-\Gamma\tau} \right).
\]

Both memory kernels decay exponentially in time, a fact that according to the earlier remarks fully legitimates the Markov behavior of the system. Let us see it in some more detail. As far as $\alpha_{R}^{(\Gamma)}$ is concerned, its scale becomes now
\[ \tau_R = \int_0^\infty \alpha_R^{(\Gamma)}(\tau) d\tau = \frac{\eta}{2\pi} \frac{\Gamma \ln \left( \frac{\Gamma^2 + \Omega^2}{\Gamma^2} \right)}{\Gamma} = \frac{\eta}{2\pi} \mathcal{O}(\Gamma), \]  

that is, it is different from zero for \( \mathcal{O}(\Gamma) \). In the limit \( \Gamma \to 0, \tau_R \to 0 \), restating the already discussed non-Markovian properties.

More interesting is the case of \( \alpha_R^{(\Gamma)} \). For \( \Gamma \) small but finite, the Markov approximation becomes now rigorous because of the presence of the \( e^{-\Gamma \tau} \) term. We get, for the convolution integral (25):

\[
I_{I,k}^{(\Gamma)}(t) = \frac{\eta}{\pi} \int_0^t d\tau \frac{d}{d\tau} \left( \frac{\sin \Omega \tau}{\tau} e^{-\Gamma \tau} \right) q_k(t - \tau)
= -\frac{\eta}{\pi} \Omega q_k(t) + \frac{\eta}{\pi} \left( \int_0^\infty d\tau \frac{\sin \Omega \tau}{\tau} e^{-\Gamma \tau} \right) \dot{q}_k(t)
= -\frac{\eta}{\pi} \Omega q_k(t) + \frac{\eta}{\pi} \text{arctan} \left( \frac{\Omega}{\Gamma} \right) \dot{q}_k(t)
\]  

Notice that we could get this result without carrying out the limit \( \Omega \to \infty \). In the limit \( \Gamma \to 0 \),

\[
I_{I,k}^{(\Gamma)}(t) \to -\frac{\eta}{\pi} \Omega q_k(t) + \frac{\eta}{2} \dot{q}_k(t) \quad (33)
\]

\[
J_{I,k}^{(\Gamma)}(t) \to -\frac{\eta}{\pi} \Omega q_k(t) + \frac{\eta}{2} \dot{q}_k(t) \quad (34)
\]

and from (24), Eq. (19) and (20) are recovered, with \( \Omega \) replacing \( \delta(0) \).

IV. INTENSITY OF THE CORRECTIONS TO ORDINARY QUANTUM MECHANICS NECESSARY TO THE OBJECTIVE MARKOV APPROXIMATION

This Section is devoted to a balance of the results obtained in this paper. To make this balance crystal clear it is convenient to warn the reader from mistaking our main purpose with that of many papers on the subject of master equation and the foundation of stochastic Schrödinger equation. To establish this difference of perspective and purposes, we find it convenient to make some comments on a representative group of papers. These are those of Refs. [31–37].

We have to point out that it is possible, in principle, to derive exact master equations to describe the time evolution of an open system [31]. However, in the specific case of the model of Caldeira and Leggett [21], which is the specific model studied in this letter, it is well known that the Markov approximation is responsible for the birth of unphysical effects. This difficulty has been addressed by different authors with different methods. For instance, Diosi [32] has recovered the Lindblad form [38] by adding two additional damping terms to the result of the Markov approximation. Munro and Gardiner [33] made the interesting observation that the unphysical effects might be the consequence of a transient process produced by the regression to equilibrium from the initial factorization of the system and bath’s density operator. However, the reduced density matrix fails being positive semidefinite in a short-time region which is probably beyond experimental observation due to the
coarse-grain time-scale approach used in its derivation. This explicit admission of resting on a coarse-graining procedure is of significant importance to stress the main aim of this letter, as we shall see in the remainder part of this Section.

We have to mention finally that the important problem of unravelling the master equation so as to build up an equivalent Schrödinger equation is now being currently extended with success to the case of non-Markov master equations [34–37].

The main purpose of this paper is totally different from that of these papers, but it is much more closely related to the conceptual issues of [1] and [23]. It is probably not so easy to appreciate these differences especially because the formal structures of the stochastic Schrödinger equation of [23] might generate the false impression that these authors, and we with them, adopt a phenomenological approach rather than a more attractive derivation from ordinary quantum mechanics.

First of all, we have to stress that our main thrust is on the connection between the unification of quantum and classical physics and the unification of mechanics and thermodynamics. The structure of the stochastic Schrödinger equation of Ghirardi, Pearle and Rimini [23] has been determined by the need of establishing a unified perspective embracing macroscopic classical physics. The spontaneous wave-function collapses are described by a correction to the ordinary Schrödinger equation, compatible with the Lindblad structure [38], but conceived as real correction to ordinary quantum mechanics rather than as expression of the influence of an environment. We are convinced that to consider these corrections as manifestations of the environment influence we should prove that the Lindblad structure [38] can be derived from the quantum mechanical picture of a system interacting with an environment with no conflict with ordinary quantum mechanics. This cannot be done because the Lindblad structure [38] implies a rigorously exponential decay. When it is used, as we did in this paper, as a seed of stochasticity, it certainly kills the long-time deviations from the exponential decay, in conflict with the well-known fact that the exponential-like decay regime is only admitted in an intermediate time region [19].

The conviction that stochastic processes are compatible with ordinary statistical mechanics is questionable from a conceptual perspective, even if it is especially attractive under the proviso of adopting the for all practical purposes point of view. We know that the prototype of stochastic processes is given by Brownian motion, and that the formal structure encompassing it within the usual statistical treatments is the ordinary Fokker-Plank equation. Therefore a convincing demonstration of the compatibility between ordinary quantum mechanics and stochastic processes would be given by a derivation of the Fokker-Planck equation with no statistical or coarse-graining assumption whatsoever. The ordinary approach to the Fokker-Planck equation is not only affected by statistical assumptions such as averaging on the initial conditions: If the Markovian assumption is not made, the resulting equation of motion cannot be identified with a bona fide Fokker-Planck equation [14].

This does not rule out, of course, the possibility that a stochastic process with infinite memory might exist. However, it can be shown [39] that in this case this stochastic process turns out to be equivalent to transport equations expressed in terms of fractional derivatives, thereby departing from conventional statistical mechanics to which we would like to limit the discussion of this final Section.

The problem with advocating our point of view is that of the experimental verification of the so-called spontaneous fluctuations [40]. Bonci et al. [40] have recently shown
that the influence of the environment might not result, in principle, in a wave-function collapse, but only in a blurring of the wave function, even if from a statistical point of view wave-function collapses turn out to be indistinguishable from decoherence processes with no collapse. A wave-function collapse is a single-system property which can be provoked by the environmental fluctuations only if these happen to be genuinely stochastic. Unfortunately, to experimentally prove this perspective it would be necessary to find cases where the effect of spontaneous fluctuations is statistically more significant than the environmental-induced fluctuations. We feel uncomfortable in accepting a view where a wave-function collapse, requiring the action of a genuine stochastic process, is actually caused by our ignorance of the microscopic details, or more in general by the limitations of the human observer. However, to change a philosophical debate into a scientific issue it would be necessary to single out experimental effects, and in this sense the results of [40] are not encouraging. In fact, the toy model adopted to study quantum jumps show that the rate of the spontaneous wave-function collapses is much lower than that of the environment-induced decoherence, thereby making them virtually invisible to statistical observation: A nice price to pay to leave quantum statistical mechanics essentially unchanged.

We can show, however, that in spite of the discouraging conditions concerning the experimental settlement of this problem, the adoption of our perspective can lead to the definition of new interesting theoretical problems, concerning the triggering action of stochastic processes, regardless the philosophical view of the investigator. A nice example of this kind is provided by Zurek and Paz [41]. These authors address indeed the problem of the foundation of the second principle of thermodynamics by means of the same theoretical arguments as those adopted by Zurek to address the intriguing problem of the wave-function collapse [42]. Although we do not share Zurek’s view, since we think it to be impossible to derive white noise from the contraction over the environmental degrees of freedom, we find attractive the ensuing picture of how the influence of this randomness, which in our perspective would be a spontaneous fluctuations, is enhanced by the chaotic properties of the system under study.

It has to be pointed out that much of the confidence of the advocates of the foundation of thermodynamics on the basis of ordinary physics rests on the identification of the correlation time $\tau_M$ with the predictability time $\tau_S$. Both times are defined in the paper of Ref. [43], which has been devoted to shed light on this intriguing issue, and where it was proved that the conventional view is reinforced by the key role of the following inequality

$$\tau_M \ll \tau_S \ll \frac{1}{\eta}.$$  \hspace{1cm} (35)

This is Eq. (24) of Ref. [43] with the oscillator friction denoted by $\eta$ to fit the notation of this letter. If this inequality is fulfilled, then the resulting macroscopic stochastic dynamics become indistinguishable from those predicted by ordinary physics supplemented by the arbitrary assumption of identifying $\tau_M$ with the predictability time $\tau_S$. Actually $\tau_M$ is the lifetime of a correlation function, and this is not necessarily identical to the time at which the system under study stops being predictable. An illuminating example is given by the superposition of many harmonic oscillations with slight different frequencies. The resulting process might be characterized by a relaxation-like behavior, but it would mistifying to identify the resulting decorrelation time with the beginning of an unpredictable regime [43]. In classical physics this unpredictability time is proved to be inversely proportional to the
Lyapunov coefficient. Apparently there might be a conflict between this interpretation and the fact that other investigators might identify this time with the inverse of the correlation time $1/\tau_M$. This apparent conflict is settled by adopting the perspective of Zurek and Paz [41]: The key action of the environmental seed of irreversibility makes it possible to identify the two times, since in the case studied by Zurek and Paz $\tau_M$ is slightly shorter than $\tau_S$. This is in perfect accordance with the point of view that we are trying to illustrate here, even if we are inclined to identify the seed of irreversibility with spontaneous fluctuations either of the form of those of Ghirardi, Pearle and Rimini, or of the Mensky constrained path integral studied in this letter.

The main aim of this paper has been that of assessing whether or not the adoption of the Mensky path integral with constraint, as an expression of correction to the ordinary quantum dynamics of the bath oscillators, makes it possible to realize the basic condition (35). This condition, fully expressed in terms of the notations used in this paper, would read

$$\eta < \Gamma < \Omega.$$  \hspace{1cm} (36)

In fact, $1/\Omega$ plays the same role as ordinary correlation time, and the parameter $\Gamma$ can be adopted to denote the rate of the Mensky process on a single oscillator. Note that the Mensky measurement process is the R-process of the perspective established by Penrose [1], and consequently the time $\tau_S$ must be identified with $1/\Gamma$, since the measurement process is by definition a genuinely stochastic process and the calculations of Section III have shown that a single bath oscillator is made unpredictable in time with the rate $\Gamma$.

Note that establishing this inequality, as simple as it is, cannot be easily done without the extended calculations of Section III, and especially without the complex process of repeated trials yielding Eq. (22). Following [10], we adopt for the dissipation parameter of the oscillator, $\eta$, the value $10^9$ sec$^{-1}$. This plausible choice and the inequality (36) set a minimum value for the parameter $\Gamma$, which turns out too large to be compatible with the Taylor series expansion adopted in our theoretical treatment. Furthermore, if we fix $\Gamma = 10^{10}$ sec$^{-1}$, $m_B = 10^{-24}$ gr, $\Omega = 10^{12}$ sec$^{-1}$, we find that the real part of $k$ is equal to the value $10^{25}$ sec$^{-1}$ cm$^{-2}$. This means that the corrections made are unacceptable from the physical as well as from the mathematical point of view. In other words, the results are not selfconsistent and imply corrections to ordinary physics too large to be seriously taken into account. The spontaneous fluctuations of Ghirardi, Pearle and Rimini [23] lead to much smaller corrections, and consequently might be regarded as being in principle an acceptable corrections. However, as established in [10], in spite of the enhancement produced by the interplay with the ordinary fluctuation-dissipation process, these fluctuations fail producing the objective wave-function collapse in a reasonably short time, thereby sharing the weakness of the Mensky constrained path integral studied in this letter.

According to the perspective here adopted, that the Markov assumption corresponds to tacitly correcting quantum mechanics, this conclusion seems to be equivalent to establishing that the Markov approximation made in literature are unacceptable. However, suggesting that all master equations based on this assumption, widely used in literature, cannot be trusted, would be a dramatic conclusion. We are convinced that the results of this letter lead to a rather different conclusion that will be illustrated at the end as the main result of this paper. Before reaching this conclusion, we want to caution the reader from mistaking
this conclusion for the discovery that the Markov assumption can break the condition that the density matrix is positive definite \[32,33\]. We cannot rule out completely the possibility that the perspective of this paper can be used to shed light on some issues raised, for instance, by Munro and Gardiner \[33\]. However, our thrust focuses on the different issue widely illustrated in this Section.

We are convinced that the results of this letter leads to a conclusion different from stating the breakdown of the master equations. The master equations, supplemented in case with the improvements established by \[32\] and \[33\], might lead to correct result, in spite of the fact that Eq. \[30\] seems to be unphysical. It has to be remarked that the approach to statistical mechanics here adopted is not genuinely dynamic: the bath of oscillators is linear and it is arbitrarily forced to stay in a canonical equilibrium condition. For this reason the conventional assumption that the bath oscillators are in a state of canonical equilibrium has to be replaced by dynamical ingredients (see, for instance, the discussion made in recent literature \[44,45\]). This means that each bath oscillator must be assigned its own bath, and this bath must be non-linear so as to result, in the classical limit, in deterministic chaos. This is expected to lead to the fulfillment of Eq. \(36\) without internal inconsistencies, thereby explaining why, after all, the linear baths of literature can be profitably used with no conflict with reality.

This is a plausible conjecture that has to be proved by a direct use of a nonlinear bath. There are already outstanding examples of dynamic approaches to quantum statistical mechanics, based on the so called quantum chaos \[46,47\]. We note that within this context it has already been noticed \[41\] that a chaotic dynamic system leads to a significant increase of the Gibbs entropy in times of the order of the inverse of the Lyapunov coefficients, provided that a seed of genuine irreversibility is introduced as an effect of environmental fluctuations. As earlier said, this fits very well the perspective established by the present paper. The dependence on these environmental fluctuations is through the logarithm of their intensity, thereby implying no significant change to ordinary quantum mechanics. If we interpret these environmental fluctuations as weak corrections to ordinary quantum mechanics, of the same type as that here studied, we might reach the satisfactory conclusion that a non arbitrary Markov assumption can be realized with extremely weak corrections to ordinary quantum mechanics. The results of this paper therefore must be interpreted as an incentive to study directly, from now on, the influence that white fluctuations have on dynamic systems which would be strongly chaotic in the classical limit.

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\(^2\)We refer here to some key equations of Ref. \[33\]. It seems to us that the short-time expansion of the density matrix terms of Eqs. (3.3) and (3.5) with the joint use of (3.11) might lead to a breakdown of the property of the density matrix to be positive definite in a time region more extended than the correlation time of Eq. (2.20). This seems to support our view that the correlation time \(\tau_M\), identified by us with the time of Eq. (2.20), cannot be regarded as the true coarse-graining time. Averages should be made on times more extended, as indicated by Eq. \(36\) of this letter.

13
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