LOCAL ASYMPTOTIC MINIMAX ESTIMATION OF NONREGULAR PARAMETERS WITH TRANSLATION-SCALE EQUIVARIANT MAPS

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ABSTRACT. When a parameter of interest is defined to be a nondifferentiable transform of a regular parameter, the parameter does not have an influence function, rendering the existing theory of semiparametric efficient estimation inapplicable. However, when the nondifferentiable transform is a known composite map of a continuous piecewise linear map with a single kink point and a translation-scale equivariant map, this paper demonstrates that it is possible to define a notion of asymptotic optimality of an estimator as an extension of the classical local asymptotic minimax estimation. This paper establishes a local asymptotic risk bound and proposes a general method to construct a local asymptotic minimax decision.

KEY WORDS. Nonregular Parameters; Translation-Scale Equivariant Transforms; Semiparametric Efficiency; Local Asymptotic Minimax Estimation.

AMS CLASSIFICATION. 62C05, 62C20.

1. INTRODUCTION

This paper investigates the problem of optimal estimation of a parameter \( \theta \in \mathbb{R} \) which takes the following form:

\[
\theta = (f \circ g)(\beta),
\]

where \( \beta \in \mathbb{R}^d \) is a regular parameter for which a semiparametric efficiency bound is well defined, \( g \) is a translation-scale equivariant map, and \( f \) is a continuous piecewise linear map with a single kink (i.e., nondifferentiability) point.
Examples abound, including $\max\{\beta_1, \beta_2, \beta_3\}$, $\max\{\beta_1, 0\}$, $|\beta_1|$, $|\max\{\beta_1, \beta_2\}|$, etc., where $\beta = (\beta_1, \beta_2, \beta_3)$ is a regular parameter, i.e., a parameter which is differentiable in the underlying probability. Applications where such parameters arise are numerous. We give two specific examples.

**Example 1 (Maximal Average Treatment Effects):** Suppose that $X$ is an observed discrete covariate and $D \in \{0, 1, 2, \cdots, J\}$ is a treatment indicator, where $D = j$ for $j > 0$ indicates treatment by method $j$, and $D = 0$ indicates no treatment. Let us assume that the vector of potential outcomes $(Y_0, Y_1, \cdots, Y_j)$ are conditionally independent from $X$ given $D$, and that $P\{D = j|X = x\} \in (0, 1)$ for all $j = 0, 1, 2, \cdots, J$ and $x$ in the support of $X$. The researcher observes $(Y, D, X)$, where $Y = \sum_{j=0}^{J} Y_j 1\{D = j\}$, but does not observe $(Y_j)^{J}_{j=0}$. Then the average treatment effect for method $j$ for group with $X = x$ is identified by

$$\beta_j = E[Y|X = x, D = j] - E[Y|X = x, D = 0].$$

One of the examples considered by Hirano and Porter (2012) was

$$\theta = \max_{1 \leq j \leq J} \beta_j,$$

that is, the maximum treatment effect that is possible using the $J$ methods. □

**Example 2 (Bounds for Treatment Effects under Monotonicity):** Let $Y_j$ be the potential outcome variables taking values from $[K_0, K_1]$ with known constants, $K_0$ and $K_1$, and $D$ a treatment indicator as in Example 1. Suppose that $X$ is an observed discrete random variable taking values in $\{x_1, \cdots, x_M\}$, $x_1 \leq x_2 \leq \cdots \leq x_{M-1} \leq x_M$, such that $E[Y_j|X = x] \geq E[Y_j|X = x']$ whenever $x \geq x'$ for all $j = 0, 1, \cdots, J$. The parameter of interest is the conditional outcome $E[Y_j|X = x]$ for treatment method $j$. The researcher observes $(Y, D, X)$ with $Y = \sum_{j=0}^{J} Y_j 1\{D = j\}$ as before. Manski and Pepper (2000) showed that in this set-up, the conditional outcome is interval identified as follows:

$$\max_{1 \leq k \leq m} \beta_{j,k}(K_0) \leq E[Y_j|X = x_m] \leq \min_{m \leq k \leq M} \beta_{j,k}(K_1),$$

where

$$\beta_{j,k}(K) = E[Y|X = x_k, D = j]P\{D = j|X = x_k\} + K \cdot P\{D \neq j|X = x_k\}.$$ 

Then the upper bound parameter $\theta_U = \min_{m \leq k \leq M} \beta_{j,k}(K_1)$ and the lower bound parameter $\theta_L = \max_{1 \leq k \leq m} \beta_{j,k}(K_0)$ are examples of $\theta$ in (1.1). Such a bound frequently arises in economics literature (e.g. Haile and Tamer (2003) for bidders’ valuations in English auctions.) □
In contrast to the ease with which a parameter of the form in (1.1) arises in applied researches, a formal analysis of the optimal estimation problem has remained a challenging task. One might consistently estimate \( \theta \) by using plug-in estimator \( \hat{\theta} = f(g(\hat{\beta})) \), where \( \hat{\beta} \) is a \( \sqrt{n} \)-consistent estimator of \( \beta \). However, there have been concerns about the asymptotic bias that such an estimator carries, and some researchers have proposed ways to reduce the bias (Manski and Pepper (2000), Haile and Tamer (2003), Chernozhukov, Lee, and Rosen (2013)). However, Doss and Sethuraman (1989) showed that a sequence of estimators of a parameter for which there is no unbiased estimator must have variance diverging to infinity if the bias decreases to zero. Given that one cannot eliminate the bias entirely without its variance exploding, the bias reduction may do the estimator either harm or good. (See Hirano and Porter (2012) for a recent result for nondifferentiable parameters.)

Many early researches on estimation of a nonregular parameter considered a parametric model and focused on finite sample optimality properties. For example, estimation of a normal mean under bound restrictions or order restrictions has been studied, among many others, by Lovell and Prescott (1970), Casella and Strawderman (1981), Bickel (1981), Moors (1981), and more recently van Eeden and Zidek (2004). Closer to this paper are researches by Blumenthal and Cohen (1968a,b) who studied estimation of \( \max\{\beta_1, \beta_2\} \), when i.i.d. observations from a location family of symmetric distributions or normal distributions are available. On the other hand, the notion of asymptotic efficient estimation through the convolution theorem and the local asymptotic minimax theorem initiated by Hajék (1972) and Le Cam (1979) has mostly focused on regular parameters, and in many cases, resulted in regular estimators as optimal estimators. Hence the classical theory of semiparametric estimation widely known and well summarized in monographs such as Bickel, Klassen, Ritov, and Wellner (1993) and in later sections of van der Vaart and Wellner (1996) (Sections 3.10-3.11, pp. 401-422) does not directly apply to the problem of estimation of \( \theta = (f \circ g)(\beta) \). This paper attempts to fill this gap from the perspective of local asymptotic minimax estimation.

This paper finds that for the class of nonregular parameters of the form (1.1), we can extend the existing theory of local asymptotic minimax estimation and construct a reasonable class of optimal estimators that are nonregular in general and asymptotically biased. The class of optimal estimators take the form of a plug-in estimator with semiparametrically efficient estimator of \( \beta \) except that it involves an additive bias-adjustment term which can be computed using simulations.

To deal with nondifferentiability, this paper first focuses on the special case where \( f \) is an identity, and utilizes the approach of generalized convolution theorem in van der Vaart (1989) to establish the local asymptotic minimax risk bound for the parameter \( \theta \). However, such a risk bound is hard to use in our set-up where \( f \) or \( g \) is potentially asymmetric,
because the risk bound involves minimization of the risk over the distributions of “noise” in the convolution theorem. This paper proposes a local asymptotic minimax decision of a simple form:

$$g(\hat{\beta}) + \hat{c}/\sqrt{n},$$

where $\hat{\beta}$ is a semiparametrically efficient estimator of $\beta$ and $\hat{c}$ is a bias adjustment term that can be computed through simulations.

Next, extension to the case where $f$ is continuous piecewise linear with a single kink point is done. Thus, an estimator of the form

$$(1.2) \quad \hat{\theta}_{mx} \equiv f \left( g(\hat{\beta}) + \frac{\hat{c}}{\sqrt{n}} \right),$$

with appropriate bias adjustment term $\hat{c}$, is shown to be local asymptotic minimax. In several situations, the bias adjustment term $\hat{c}$ can be set to zero. In particular, when $\theta = s^\top \beta$, for some known vector $s \in \mathbb{R}^d$, so that $\theta$ is a regular parameter, the bias adjustment term can be set to be zero, and an optimal estimator in (1.2) is reduced to $s^\top \hat{\beta}$ which is a semiparametric efficient estimator of $\theta = s^\top \beta$. This confirms the continuity of this paper's approach with the standard method of semiparametric efficiency.

This paper offers results from a small sample simulation study for the case of $\theta = \max\{\beta_1, \beta_2\}$. This paper compares the method with two alternative bias reduction methods: fixed bias reduction method and a selective bias reduction method. The method of local asymptotic minimax estimation shows relatively robust performance in terms of the finite sample risk.

The next section defines the scope of the paper by introducing nondifferentiable transforms that this paper focuses on. The section also introduces regularity conditions for probabilities that identify $\beta$. Section 3 investigates optimal decisions based on the local asymptotic maximal risks. Section 4 presents and discusses Monte Carlo simulation results. All the mathematical proofs are relegated to the Appendix.

2. NONDIFFERENTIABLE TRANSFORMS OF A REGULAR PARAMETER

In this section, we present the details of the set-up in this paper. We introduce some notation. Let $\mathbb{N}$ be the collection of natural numbers. Let $1_d$ be a $d \times 1$ vector of ones with $d \geq 2$. For a vector $x \in \mathbb{R}^d$ and a scalar $c$, we simply write $x + c = x + c 1_d$, or write $x = c$ instead of $x = c 1_d$. We define $S_1 \equiv \{x \in \mathbb{R}^d : x^\top 1_d = 1\}$, where the notation $\equiv$ indicates definition. For $x \in \mathbb{R}^d$, the notation $\max(x)$ (or $\min(x)$) means the maximum (or the minimum) over the entries of the vector $x$. When $x_1, \cdots, x_n$ are scalars, we also use the notations $\max\{x_1, \cdots, x_n\}$ and $\min\{x_1, \cdots, x_n\}$ whose meanings are obvious. We let $\bar{\mathbb{R}} = [-\infty, \infty]$ and view it as a two-point compactification of $\mathbb{R}$, and let $\bar{\mathbb{R}}^d$ be the product
of its $d$ copies, so that $\mathbb{R}^d$ itself is a compactification of $\mathbb{R}^d$. (e.g. Dudley (2002), p.74.) We follow the convention to set $\infty \cdot 0 = 0$ and $(-\infty) \cdot 0 = 0$. A supremum and an infimum of a nonnegative map over an empty set are set to be 0 and $\infty$ respectively.

As for the parameter of interest $\theta$, this paper assumes that

$$\theta = (f \circ g)(\beta),$$

where $\beta \in \mathbb{R}^d$ is a regular parameter (the meaning of regularity for $\beta$ is clarified in Assumption 2 below), and $g : \mathbb{R}^d \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ satisfy the following assumptions.

**Assumption 1:** (i) The map $g : \mathbb{R}^d \to \mathbb{R}$ is Lipschitz continuous, and satisfies the following.

(a) (Translation Equivariance) For each $c \in \mathbb{R}$ and $x \in \mathbb{R}^d$, $g(x + c) = g(x) + c$.

(b) (Scale Equivariance) For each $u \geq 0$ and $x \in \mathbb{R}^d$, $g(ux) = ug(x)$.

(c) (Directional Derivatives) For each $z \in \mathbb{R}^d$ and $x \in \mathbb{R}^d$,

$$\tilde{g}(x; z) \equiv \lim_{t \downarrow 0} t^{-1} (g(x + tz) - g(x))$$

exists.

(ii) The map $f : \mathbb{R} \to \mathbb{R}$ is continuous, piecewise linear with one kink at a point (i.e., one point of nonlinearity) in $\mathbb{R}$.

We collect here the properties of the directional derivative $\tilde{g}(x; z)$ in (c) of the translation-scale equivariant and Lipschitz continuous map $g$.

**Lemma 1:** (i) For each $z \in \mathbb{R}^d$, $x \in \mathbb{R}^d$, $c \in \mathbb{R}$, and $u \geq 0$, the following properties are satisfied:

(a) $\tilde{g}(0; z) = g(z)$.

(b) $\tilde{g}(x + c; z) = \tilde{g}(x; z)$.

(c) $\tilde{g}(x; z + c) = \tilde{g}(x; z) + c$.

(d) $\tilde{g}(ux; uz) = u\tilde{g}(x; z) = \tilde{g}(x; uz)$.

(ii) For each $x \in \mathbb{R}^d$, $\tilde{g}(x; z)$ is Lipschitz continuous in $z \in \mathbb{R}^d$.

(iii) For each $x \in \mathbb{R}^d$, the convergence in the definition of the directional derivative in Assumption 1(i)(c) is uniform over $z$ in any bounded subset of $\mathbb{R}^d$.

Assumption 1 essentially defines the scope of this paper. Some examples of $g$ are as follows.

**Examples 3:** (a) $g(x) = s^T x$, where $s \in S_1$.

(b) $g(x) = \max(x)$ or $g(x) = \min(x)$. 
(c) \( g(x) = \max\{\min(x_1), x_2\} \), \( g(x) = \max(x_1) + \max(x_2) \), \( g(x) = \min(x_1) + \min(x_2) \), \( g(x) = \max(x_1) + \min(x_2) \), or \( g(x) = \max(x_1) + s^\top x \) with \( s \in S_1 \), where \( x_1 \) and \( x_2 \) are subvectors of \( x \). \( \blacksquare \)

One might ask whether the representation of parameter \( \theta \) as a composition map \( f \circ g \) of \( \beta \) in (2.1) is unique. The following lemma gives an affirmative answer.

**Lemma 2:** Suppose that \( f_1 \) and \( f_2 \) are \( \mathbb{R} \)-valued maps on \( \mathbb{R} \) that are non-constant on \( \mathbb{R} \), and \( g_1 \) and \( g_2 \) satisfy Assumption 1(i). If \( f_1 \circ g_1 = f_2 \circ g_2 \), we have

\[
f_1 = f_2 \quad \text{and} \quad g_1 = g_2.
\]

As we shall see later, the local asymptotic minimax risk bound and the optimal estimators involve the maps \( f \) and \( g \). The uniqueness result of Lemma 2 removes ambiguity that could potentially arise when \( \theta \) had multiple equivalent representations with different maps \( f \) and \( g \).

We introduce briefly conditions for probabilities that identify \( \beta \), in a manner adapted from van der Vaart (1991) and van der Vaart and Wellner (1996) (see Section 3.11, pp. 412-422.) Let \( \mathcal{P} \equiv \{P_\alpha : \alpha \in \mathcal{A}\} \) be a family of distributions on a measurable space \((\mathcal{X}, \mathcal{G})\) indexed by \( \alpha \in \mathcal{A} \), where the set \( \mathcal{A} \) is a nonempty open subset of a Euclidean space or more generally a complete metric space.

We assume that we have i.i.d. draws \( Y_1, \ldots, Y_n \) from \( P_{\alpha_0} \in \mathcal{P} \) for some \( \alpha_0 \in \mathcal{A} \), so that \( X_n \equiv (Y_1, \cdots, Y_n) \) is distributed as \( P^n_{\alpha_0} \). Let \( \mathcal{P}(P_{\alpha_0}) \) be the collection of maps \( t \to P_{\alpha_0} \) such that for some \( h \in L_2(P_{\alpha_0}) \),

\[
(2.2) \quad \int \left\{ \frac{1}{t} \left( dP_{\alpha_1}^{1/2} - dP_{\alpha_0}^{1/2} \right) - \frac{1}{2} hdP_{\alpha_0}^{1/2} \right\}^2 \to 0, \quad \text{as} \ t \to 0.
\]

When this convergence holds, we say that \( P_{\alpha_1} \) is *differentiable in quadratic mean* to \( P_{\alpha_0} \), call \( h \in L_2(P_{\alpha_0}) \) a *score function* associated with this convergence, and call the set of all such \( h \)’s a *tangent set*, denoting it by \( T(P_{\alpha_0}) \). We assume that the tangent set is a linear subspace of \( L_2(P_{\alpha_0}) \). Taking \( \langle \cdot, \cdot \rangle \) to be the usual inner product in \( L_2(P_{\alpha_0}) \), we write \( H \equiv T(P_{\alpha_0}) \) and view \( (H, \langle \cdot, \cdot \rangle) \) as a subspace of a separable Hilbert space, with \( \tilde{H} \) denoting its completion. For each \( h \in H \), \( n \in \mathbb{N} \), and \( \lambda_h \in \mathcal{A} \), let \( P_{\alpha_0 + \lambda_h/\sqrt{n}} \) be probabilities converging to \( P_{\alpha_0} \) (as in (2.2)) as \( n \to \infty \) having \( h \) as its associated score. We simply write \( P_{n,h} = P^n_{\alpha_0 + \lambda_h/\sqrt{n}} \) and consider sequences of such probabilities \( \{P_{n,h}\}_{n \geq 1} \) indexed by \( h \in H \). (See van der Vaart (1991) and van der Vaart and Wellner (1996), Section 3.11 for details.) The collection \( \mathcal{E}_n \equiv (\mathcal{X}_n, G_n, P_{n,h}; h \in H) \) constitutes a sequence of statistical experiments for \( \beta \).
Due to differentiability in quadratic mean and i.i.d. assumption, the collection $E_n$ satisfies local asymptotic normality (LAN), that is, for any $h \in H$,

$$\log \frac{dP_{n,h}}{dP_{n,0}} = \zeta_n(h) - \frac{1}{2} \langle h, h \rangle,$$

where for any $h, h' \in H$, $[\zeta_n(h), \zeta_n(h')] \xrightarrow{d} [\zeta(h), \zeta(h')]$, under $\{P_{n,0}\}$ and $\zeta(\cdot)$ is a centered Gaussian process on $H$ with covariance function $\mathbb{E}[\zeta(h_1)\zeta(h_2)] = \langle h_1, h_2 \rangle$. Note that we require here the joint convergence of $\zeta_n(h)$ and $\zeta_n(h')$ for each pair $(h, h')$. This joint convergence is used to derive a modified version of LAN (Lemma A4 in the appendix) which is used to derive the local asymptotic minimax risk. The joint convergence can be seen to hold e.g. from the proof of Lemma 3.10.11 of van der Vaart and Wellner (1996), p.406.

The LAN property reduces the decision problem to one in which an optimal decision is sought under a single Gaussian shift experiment $E = (\mathcal{X}, \mathcal{G}, P; h \in H)$, where $P_h$ is such that $\log \frac{dP_h}{dP_0} = \zeta(h) - \frac{1}{2} \langle h, h \rangle$.

The parameter $\beta$ is represented as a functional $\beta : \mathcal{P} \to \mathbb{R}^d$. From here on, we simply write for each $\alpha \in A$, $\beta_n(h) = \beta(P_{\alpha_0 + \lambda_n / \sqrt{n}})$ and regard $\beta_n(\cdot)$ as an $\mathbb{R}^d$-valued map on $H$.

**Assumption 2**: (Regular Parameter) There exists a continuous linear $\mathbb{R}^d$-valued map, $\hat{\beta}$, on $H$ such that for any $h \in H$,

$$\sqrt{n}(\beta_n(h) - \beta_n(0)) \xrightarrow{d} \hat{\beta}(h),$$

as $n \to \infty$.

Assumption 2 requires that $\beta$ be regular in the sense of van der Vaart and Wellner (1996, Section 3.11). The map $\hat{\beta}$ in Assumption 2 is associated with the semiparametric efficiency bound of $\beta$. For each $b \in \mathbb{R}^d$, $b^T \hat{\beta}(\cdot)$ defines a continuous linear functional on $H$, and hence there exists $\hat{\beta}_b^* \in \mathcal{H}$ such that $b^T \hat{\beta}(h) = \langle \hat{\beta}_b^*, h \rangle, h \in H$. Then for any $b \in \mathbb{R}^d$, $||\hat{\beta}_b^*||^2$ represents the asymptotic variance bound of the parameter $b^T \beta$. The map $\hat{\beta}_b^*$ is called an efficient influence function for $b^T \beta$ in the literature (e.g. van der Vaart (1991)). Let $e_m$ be a $d \times 1$ vector whose $m$-th entry is one and the other entries are zero, and let $\Sigma$ be a $d \times d$ matrix whose $(m, k)$-th entry is given by $\langle \hat{\beta}_{e_m}^*, \hat{\beta}_{e_k}^* \rangle$. As for $\Sigma$, we assume the following:

**Assumption 3**: $\Sigma$ is invertible.
The inverse of matrix $\Sigma$ is called the semiparametric efficiency bound for $\beta$. In particular, Assumption 3 requires that there is no redundancy among the entries of $\beta$, i.e., one entry of $\beta$ is not defined as a linear combination of the other entries.

3. Local Asymptotic Minimax Estimators

3.1. Loss Functions. For a decision $d \in \mathbb{R}$ and the object of interest $\theta \in \mathbb{R}$, we consider the following form of a loss function:

\[(3.1) \quad L(d, \theta) = \tau(|d - \theta|),\]

where $\tau : \mathbb{R} \to \mathbb{R}$ is a map that satisfies the following assumption.

**Assumption 4:** (i) $\tau(\cdot)$ is increasing and convex on $[0, \infty)$, $\tau(0) = 0$, and there exists $\bar{\tau} \in (0, \infty]$ such that $\tau^{-1}([0, y])$ is bounded in $[0, \infty)$ for all $0 < y < \bar{\tau}$.
(ii) For each $M > 0$, there exists $C_M > 0$ such that for all $x, y \in \mathbb{R}$,

\[(3.2) \quad |\tau_M(x) - \tau_M(y)| \leq C_M |x - y|,

where $\tau_M(\cdot) = \min\{\tau(\cdot), M\}$.

The smoothness condition in (3.2) is weaker than requiring $\tau$ to be Lipschitz continuous. For example, the squared loss function $\tau(x) = x^2$ satisfies this condition, but not Lipschitz continuity. While Assumption 4 is satisfied by many loss functions, it excludes the hypothesis testing type loss function $\tau(|d - \theta|) = 1\{|d - \theta| > c\}$, $c \in \mathbb{R}$. From here on, we identify $\tau$ and $\tau_M$ as their continuous extensions to $(-\infty, \infty]$.

The following lemma establishes a lower bound for the local asymptotic minimax risk when $f$ is an identity. Let for each $b \in [0, \infty)$ and $n \geq 1$,

\[H_{n,b} \equiv \{h \in H : ||\beta_n(h) - \beta_n(0)|| \leq b/\sqrt{n}\}.\]

The set $H_{n,b}$ collects those $h$'s in $H$ at which $\beta_n(h)$ lies locally around $\beta_n(0)$. (Confining our attention to $h \in H_{n,b}$ enables us to control the convergence in Assumption 2 uniformly over $h$ in $H_{n,b}$.)

**Lemma 3:** Suppose that Assumptions 1-4 hold and that $f$ is an identity. Then for any sequence of estimators $\hat{\theta}$,

\[
\sup_{b \in [0, \infty)} \liminf_{n \to \infty} \sup_{h \in H_{n,b}} E_n[\tau(|\sqrt{n}\{\hat{\theta} - g(\beta_n(h))\}|)] \\
\geq \inf_{F \in \mathcal{F}} \sup_{r \in \mathbb{R}^d} \int \mathbb{E}[\tau(\tilde{g}_0(Z + r) - \tilde{g}_0(r) + w)] dF(w),
\]
where \( \beta_0 \equiv \beta(P_{a_0}) \), \( \tilde{g}_0(r) \equiv \tilde{g}(\beta_0; r) \), \( \mathbb{E}_h \) denotes the expectation under \( P_{n,h} \), and \( \mathcal{F} \) denotes the collection of probability measures on the Borel \( \sigma \)-field of \( \mathbb{R} \).

The lower bound in Lemma 3 involves the directional derivatives \( \tilde{g}(\cdot; \cdot) \) of \( g \). Typically computation of directional derivatives is straightforward in many examples. (However, the practical procedure of optimal estimation proposed in this paper does not require an explicit computation of the directional derivatives, as we shall see after Assumption 5.)

**Examples 4**: (a) Suppose that \( g(x) = s^\top x \), \( s \in S_1 \). Then obviously, \( \tilde{g}_0(z) = s^\top z \), and the risk lower bound in Lemma 3 becomes

\[
\inf_{F \in \mathcal{F}} \int \mathbb{E} \left[ \tau(|s^\top Z + w|) \right] dF(w) \geq \mathbb{E} \left[ \tau(|s^\top Z|) \right],
\]

the last inequality following from Anderson's Lemma.

(b) Suppose that \( g(x) = \max\{x_1, x_2\} \). Then

\[
\tilde{g}_0(z) = \begin{cases} 
 z_1, & \text{if } \beta_{0,1} > \beta_{0,2} \\
 z_2, & \text{if } \beta_{0,1} < \beta_{0,2} \\
 \max\{z_1, z_2\}, & \text{if } \beta_{0,1} = \beta_{0,2},
\end{cases}
\]

where \( \beta_{0,1} \) and \( \beta_{0,2} \), and \( z_1 \) and \( z_2 \) are the first and the second entries of \( \beta_0 \) and \( z \) respectively. ■

The lower bound in Lemma 3 is obtained by using a version of a generalized convolution theorem in van der Vaart (1989) which is adapted to the current set-up. The main difficulty with using Lemma 3 is that the supremum over \( r \in S \) and the infimum over \( F \in \mathcal{F} \) do not have an explicit solution in general. Hence this paper considers simulating the lower bound in Lemma 3 by using random draws from a distribution approximating that of \( Z \).

The main obstacle in this approach is that the risk lower bound involves infimum over an infinite dimensional space \( \mathcal{F} \).

We now simplify the risk lower bound. By Jensen’s inequality,

\[
\inf_{F \in \mathcal{F}} \sup_{r \in \mathbb{R}^d} \int \mathbb{E} \left[ \tau(|\tilde{g}_0(Z + r) - \tilde{g}_0(r) + w|) \right] dF(w) \geq \inf_{F \in \mathcal{F}} \sup_{r \in \mathbb{R}^d} \mathbb{E} \left[ \tau \left( |\tilde{g}_0(Z + r) - \tilde{g}_0(r) + \int wdF(w)| \right) \right].
\]

Thus we obtain the following theorem.
Theorem 1: Suppose that Assumptions 1-4 hold and that \( f \) is an identity. Then for any sequence of estimators \( \hat{\theta} \),

\[
\sup_{b \in [0, \infty)} \liminf_{n \to \infty} \sup_{h \in H, n} \mathbb{E}_n \left[ \tau(\sqrt{n}(\hat{\theta} - g(\beta_n(h)))) \right] \geq \inf_{c \in \mathbb{R}} B(c; 1),
\]

where for \( c \in \mathbb{R} \), and any \( \alpha \geq 0 \),

\[
B(c; \alpha) \equiv \sup_{r \in \mathbb{R}^d} \mathbb{E} \left[ \tau(a|\tilde{g}_0(Z + r) - \tilde{g}_0(r) + c)| \right].
\]

The main feature of the lower bound in Theorem 1 is that it involves infimum over a single-dimensional space \( \mathbb{R} \) in its risk bound. This simpler form now makes it feasible to simulate the lower bound for the risk.

This paper proposes a method of constructing a local asymptotic minimax estimator as follows. Suppose that we are given a consistent estimator \( \hat{\Sigma} \) of \( \Sigma \) and a semiparametrically efficient estimator \( \hat{\beta} \) of \( \beta \) which satisfy the following assumptions. (See Bickel, Klaasen, Ritov, and Wellner (1993) for semiparametric efficient estimators from various models.)

Assumption 5: (i) For each \( \epsilon > 0 \), there exists \( M > 0 \) such that

\[
\limsup_{n \to \infty} \sup_{h \in H} \{ \sqrt{n}||\hat{\Sigma} - \Sigma|| > M \} < \epsilon.
\]

(ii) For each \( t \in \mathbb{R}^d \),

\[
\limsup_{n \to \infty} \sup_{h \in H} \left| \{ P_{n,h}(\sqrt{n}(\hat{\beta} - \beta_n(h)) \leq t \} - P\{Z \leq t \} \right| = 0,
\]

as \( n \to \infty \).

Assumption 5 imposes \( \sqrt{n} \)-consistency of \( \hat{\Sigma} \) and convergence in distribution of \( \sqrt{n}(\hat{\beta} - \beta_n(h)) \), both uniform over \( h \in H \). The uniform convergence can be proved through the central limit theorem uniform in \( h \in H \). Under regularity conditions, the uniform central limit theorem of a sum of i.i.d. random variables follows from a Berry-Esseen bound, as long as the third moment of the random variable is bounded uniformly in \( h \in H \).

For a fixed large \( M_1 > 0 \), we define

\[
\hat{\theta}_{mx} = g(\hat{\beta}) + \frac{c_{M_1}}{\sqrt{n}},
\]

where \( c_{M_1} \) is a bias adjustment term constructed from the simulations of the risk lower bound in Theorem 1, as we explain now. (Note that \( \hat{\theta}_{mx} \) depends on \( M_1 \) in general though the dependence is suppressed from notation.)
To simulate the risk lower bound in Theorem 1, we first draw \{\xi_i\}_{i=1}^L i.i.d. from \(N(0, I_d)\). Since \(\hat{g}_0(\cdot)\) depends on \(\beta_0\) that is unknown to the researcher, we first construct a consistent estimator of \(\hat{g}_0(\cdot)\). Take a sequence \(\varepsilon_n \to 0\) such that \(\sqrt{n}\varepsilon_n \to \infty\) as \(n \to \infty\). Examples of \(\varepsilon_n\) are \(\varepsilon_n = n^{-1/2}\) or \(\varepsilon_n = n^{-1/2} \log n\). Observe that \(\hat{g}_0(z), z \in \mathbb{R}^d\), is approximated by
\[ \varepsilon_n^{-1}(g(z_n + \beta_0) - g(\beta_0)) = g(z_n + \varepsilon_n^{-1}(\beta_0 - g(\beta_0))), \]
as \(n \to \infty\). Hence we define
\[ \hat{g}_n(z) \equiv g(z + \varepsilon_n^{-1}(\hat{\beta} - g(\hat{\beta}))). \]
Then it is not hard to see that \(\hat{g}_n(z)\) is consistent for \(\hat{g}_0(z)\). Thus, we consider the following:
for any \(a \geq 0\),
\[ \hat{B}_{M_1}(c; a) \equiv \sup_{r \in [-M_1, M_1]} \frac{1}{L} \sum_{i=1}^L \tau_{M_1}(a, \hat{g}_n(\hat{\Sigma}^{1/2}\xi_i + r) - \hat{g}_n(r) + c). \]
Then we define
\[ (3.4) \quad \hat{b}_{M_1}(a) \equiv \sup \hat{B}_{M_1}(a), \]
where, with \(\eta_{n,L} \to 0\) as \(n, L \to \infty\), \(\eta_{n,L} \varepsilon_n \sqrt{n} \to \infty\) as \(n \to \infty\) and \(\eta_{n,L} \sqrt{L} \to \infty\) as \(L \to \infty\),
\[ \hat{B}_{M_1}(a) \equiv \left\{ c \in [-M_1, M_1] : \hat{B}_{M_1}(c; a) \leq \inf_{c_1 \in [-M_1, M_1]} \hat{B}_{M_1}(c_1; a) + \eta_{n,L} \right\}. \]
The formulation of \(\hat{b}_{M_1}(a)\) in (3.4) is designed to yield an unambiguous determination of a minimizer of \(\hat{B}_{M_1}(c; a)\) (up to a small number \(\eta_{n,L}\) over \(c \in [-M_1, M_1]\), even when the minimizer of its population version \(B(c; a)\) over \(c \in [-M_1, M_1]\) turns out to be non-unique.

Now, as for the bias adjustment term \(\hat{b}_{M_1}\) in (3.3), we take \(\hat{b}_{M_1} = \hat{b}_{M_1}(1)\). The following theorem affirms that \(\hat{\theta}_{nx}\) is local asymptotic minimax for \(\theta = g(\beta)\). (For technical facility, we follow a suggestion by Strasser (1985) (p.440) and consider a truncated loss: \(\tau_M(\cdot) = \min\{\tau(\cdot), M\}\) for large \(M\).)

**Theorem 2:** Suppose that the conditions of Theorem 1 and Assumption 5 hold. Then for any \(M > 0\) and any \(M_1 \geq M\) that constitutes \(\hat{b}_{M_1}\),
\[ \sup_{b \in [0, \infty)} \limsup_{n \to \infty} \sup_{h \in \mathcal{H}_{n,b}} \mathbb{E}_h \left[ \tau_M(\ell(\sqrt{n}\{\hat{\theta}_{nx} - g(\beta_n(h))\})) \right] \leq \inf_{c \in \mathbb{R}} B(c; 1). \]

Recall that the candidate estimators considered in Theorem 1 were not restricted to plug-in estimators with an additive bias adjustment term. As standard in the literature of local asymptotic minimax estimation, the candidate estimators are any sequences of
measurable functions of observations including both regular and nonregular estimators. The main thrust of Theorem 2 is the finding that it is sufficient for local asymptotic minimax estimation to consider a plug-in estimator using a semiparametrically efficient estimator of $\beta$ with an additive bias adjustment term as in (3.3). It remains to find optimal bias adjustment, which can be done using the simulation method proposed earlier.

We now extend the result to the case where $f$ is not an identity map, but a continuous piecewise linear map with a single kink point $\bar{x} \in \mathbb{R}$. For concreteness, suppose that for all $x \in \mathbb{R}$,

$$f(x) = \begin{cases} a_1(x - \bar{x}) + f(\bar{x}), & \text{if } x \geq \bar{x} \\ a_2(x - \bar{x}) + f(\bar{x}), & \text{if } x < \bar{x} \end{cases}$$

for $a_1, a_2 \in \mathbb{R}$. Let

$$s \equiv \begin{cases} |a_1|, & \text{if } g(\beta_0) > \bar{x} \\ |a_2|, & \text{if } g(\beta_0) < \bar{x} \\ \max\{|a_1|, |a_2|\}, & \text{if } g(\beta_0) = \bar{x} \end{cases}.$$

Then the following theorem establishes the risk lower bound for the case where $f$ is not an identity map.

**Theorem 3:** Suppose that Assumptions 1-4 hold. Then for any sequence of estimators $\hat{\theta}$,

$$\sup_{b \in (0, \infty)} \liminf_{n \to \infty} \sup_{h \in H_{n,b}} \mathbb{E}_h \left[ \tau(\sqrt{n}\{\hat{\theta} - (f \circ g)(\beta_n(h))\}) \right] \geq \inf_{c \in \mathbb{R}} B(c; s).$$

The bounds in Theorems 1 and 3 involve a bias adjustment term $c^*$ that minimizes $B(c; s)$ over $c \in \mathbb{R}$. A similar bias adjustment term appears in Takagi (1994)’s local asymptotic minimax estimation result. While the bias adjustment term arises here due to asymmetric nondifferentiable map $f \circ g$ of a regular parameter, it arises in his paper due to an asymmetric loss function, and the decision problem in this paper cannot be reduced to his set-up, even if we assume a parametric family of distributions indexed by an open interval as he does in his paper.

Now let us search for a class of local asymptotic minimax estimators that achieve the lower bound in Theorem 3. Let

$$\hat{s} \equiv \begin{cases} |a_1|, & \text{if } g(\hat{\beta}) > \bar{x} + \epsilon_n \\ |a_2|, & \text{if } g(\hat{\beta}) < \bar{x} - \epsilon_n \\ \max\{|a_1|, |a_2|\}, & \text{if } \bar{x} - \epsilon_n \leq g(\hat{\beta}) \leq \bar{x} + \epsilon_n \end{cases},$$

where $\epsilon_n \to 0$ such that $\sqrt{n}\epsilon_n \to \infty$ as $n \to \infty$. It turns out that an estimator of the form:

$$\hat{\theta}_{mx} \equiv f \left( g(\hat{\beta}) + \frac{\hat{c}_{M_1}(\hat{s})}{\sqrt{n}} \right),$$

(3.5)
where \( \hat{c}_{M_1}(\hat{s}) \) is the bias-adjustment term defined in (3.4) only with \( a \) there replaced by \( \hat{s} \), is local asymptotic minimax.

**Theorem 4:** Suppose that the conditions of Theorem 3 and Assumption 5 hold. Then, for any \( M > 0 \) and any \( M_1 \geq M \),

\[
\sup_{b \in [0, \infty)} \limsup_{n \to \infty} \sup_{h \in H_{n,b}} E_h \left[ \tau_M (|\sqrt{n}(\tilde{\theta}_{mx} - (f \circ g)(\beta_n(h)))|) \right] \leq \inf_{c \in R} B(c; s).
\]

The estimator \( \tilde{\theta}_{mx} \) is in general a nonregular estimator that is asymptotically biased. When \( \tau(x) = x^k, k \geq 1 \), we have

\[
\inf_{c \in R} B(c; s) = s^k \inf_{c \in R} B(c; 1).
\]

Hence it suffices to use \( \hat{c}_{M_1}(1) \) instead of \( \hat{c}_{M_1}(\hat{s}) \) with large \( M_1 \) in this case.

When \( g(\beta) = s^\top \beta \) with \( s \in S_1 \), the risk bound in Theorem 4 becomes

\[
\inf_{c \in R} E \left[ \tau(s|\tilde{g}_0(Z) + c) \right] = E \left[ \tau(s|s^\top Z) \right],
\]

where the equality follows by Anderson’s Lemma. In this case, it suffices to set \( \hat{c}_{M_1} = 0 \), because the infimum over \( c \in R \) is achieved at \( c = 0 \). The minimax decision thus becomes simply

(3.6) \( \tilde{\theta}_{mx} = f(\hat{\beta}^\top s) \).

This has the following consequences.

**Examples 5:**
(a) When \( \theta = \beta^\top s \) for a known vector \( s \in S_1 \), \( \tilde{\theta}_{mx} = \tilde{\beta}^\top s \). Therefore, the decision in (3.6) reduces to a semiparametric efficient estimator of \( \beta^\top s \).

(b) When \( \theta = \max\{a\beta^\top s + b, 0\} \) for a known vector \( s \in S_1 \) and known constants \( a, b \in R \), \( \tilde{\theta}_{mx} = \max\{a\hat{\beta}^\top s + b, 0\} \).

(c) When \( \theta = |\beta| \) for a scalar parameter \( \beta \), \( \tilde{\theta}_{mx} = |\hat{\beta}| \). ■

The examples of (b)-(c) involve nondifferentiable transform \( f \), and hence \( \tilde{\theta}_{mx} \) as an estimator of \( \theta \) is asymptotically biased in these examples. Nevertheless, the plug-in estimator \( \tilde{\theta}_{mx} \) that does not require any bias adjustment is local asymptotic minimax. We provide another example that has the optimal bias adjustment term equal to zero. This example is motivated by Blumenthal and Cohen (1968a).

**Examples 6:** Suppose that \( \theta = \max\{\beta_1, \beta_2\} \), where \( \beta = (\beta_1, \beta_2) \in R^2 \) is a regular parameter, and the \( 2 \times 2 \) matrix \( \Sigma \) has identical diagonal entries equal to \( \sigma^2 \). (That is, \( \beta_1 \) and \( \beta_2 \) have the same semiparametric efficiency bound.) We take \( \tau(x) = x^2 \), i.e., the squared
error loss. Then from Example 4(b), the risk lower bound becomes $\sigma^2$, if $\beta_{0.1} > \beta_{0.2}$ or $\beta_{0.1} < \beta_{0.2}$, and becomes

$$\inf_{c \in \mathbb{R}} \sup_{r \geq 0} \mathbb{E} (\max\{Z_1 - r, Z_2\} - c)^2,$$

if $\beta_{0.1} = \beta_{0.2}$, where $Z_1$ and $Z_2$ denote the first and second entries of $Z$ respectively.

For each $c \in \mathbb{R}$, $\mathbb{E} (\max\{Z_1 - r, Z_2\} - c)^2$ is quasiconvex in $r \geq 0$ so that the supremum over $r \geq 0$ is achieved at $r = 0$ or $r \to \infty$. When $r = 0$, the bound becomes $\text{Var}(\max\{Z_1, Z_2\})$ and when $r \to \infty$, the bound becomes $\text{Var}(Z_2)$. By (5.10) of Moriguti (1951), we have $\text{Var}(\max\{Z_1, Z_2\}) \leq \text{Var}(Z_2)$, so that the local asymptotic risk bound becomes $\text{Var}(Z_2) = \sigma^2$ with $r = \infty$ and $c = 0$. Therefore, regardless of $\beta_{0.1} > \beta_{0.2}$, $\beta_{0.1} < \beta_{0.2}$, or $\beta_{0.1} = \beta_{0.2}$, the risk lower bound becomes $\sigma^2$ in this case. On the other hand, it is not hard to see from (A.3) of Blumenthal and Cohen (1968b) that $\tilde{\theta}_{mx} = \max\{\hat{\beta}_1, \hat{\beta}_2\}$ (without the bias adjustment term) is local asymptotic minimax. This result parallels the finding by Blumenthal and Cohen (1968a) that for squared error loss and observations of two independent random variables $X_1$ and $X_2$ from a location family of symmetric distributions, $\max\{X_1, X_2\}$ is a minimax decision. ■

4. MONTE CARLO SIMULATIONS

4.1. Simulation Designs. In the simulation study, this paper compares the finite sample risk performances of the local asymptotic minimax estimator proposed in this paper with estimators that perform bias reductions in two methods: fixed bias reduction and selective bias reduction.

In this study, we considered the following data generating process. Let $\{X_i\}_{i=1}^n$ be i.i.d. random vectors in $\mathbb{R}^2$ where $X_1 \sim N(\beta, \Sigma)$,

$$\beta = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \delta_0 / \sqrt{n} \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} 2 & 1/2 \\ 1/2 & 4 \end{bmatrix},$$

and $\delta_0$ is chosen from grid points in $[-10, 10]$. The parameters of interest are as follows:

$$\theta_1 \equiv f_1(g_1(\beta)) \text{ and } \theta_2 \equiv f_2(g_2(\beta)),$$

where

$$f_1(x) = x \text{ and } g_1(\beta) = \max\{\beta_1, \beta_2\}, \text{ and }$$

$$f_2(x) = \max\{x, 0\} \text{ and } g_2(\beta) = \beta_1.$$ 

When $\delta_0$ is close to zero, parameters $\theta_1$ and $\theta_2$ have $\beta$ close to the kink point of the nondifferentiable map. However, when $\delta_0$ is away from zero, the parameters become more like a regular parameter themselves. We take $\hat{\beta} = \frac{1}{n} \sum_{i=1}^n X_i$ as the estimator of $\beta$. 
As for the finite sample risk, we adopt the mean squared error:

\[ \mathbb{E}[(\hat{\theta}_j - \theta_j)^2], \ j = 1, 2, \]

where \( \hat{\theta}_j \) is a candidate estimator for \( \theta_j \). In the simulation study, we investigate the finite sample risk profile of decisions by varying \( \delta_0 \).

We evaluated the risk using Monte Carlo simulations. The sample size was 300. The Monte Carlo simulation number was set to be 20,000. The sequence \( \varepsilon_n \) was taken to be \( n^{-1/3} \).

We report only the results for the case of \( \theta_1 = f_1(g_1(\beta)) \). The results for the case of \( \theta_2 = f_2(g_2(\beta)) \) were similar and hence omitted.

**Figure 1.** Comparison of the Local Asymptotic Minimax Estimators with Estimators Obtained through Other Bias-Reduction Methods: \( \theta_1 = \max\{\beta_1, \beta_2\} \).
4.2. Minimax Decision and Bias Reduction. In the case of $\theta_1 \equiv \max(\beta_1, \beta_2)$, $b_F \equiv E[\max(X_{11} - \beta_1, X_{12} - \beta_2)]$ becomes the asymptotic bias of the estimator $\hat{\theta}_1 \equiv \max(\hat{\beta}_1, \hat{\beta}_2)$ when $\beta_1 = \beta_2$. One may consider the following estimator of $b_F$:

$$\hat{b}_F \equiv \frac{1}{L} \sum_{i=1}^{L} \max(\hat{\Sigma}_{1/2}^{1/2} \xi_i)$$

where $\xi_i$ is drawn i.i.d. from $N(0, I_2)$. This adjustment term $\hat{b}_F$ is fixed over different values of $\beta_2 - \beta_1$ (in large samples). Since the bias of $\max(\hat{\beta}_1, \hat{\beta}_2)$ becomes prominent only when $\beta_1$ is close to $\beta_2$, one may instead consider performing bias adjustment only when the estimated difference $|\beta_2 - \beta_1|$ is close to zero. Thus we also consider the following estimated adjustment term:

$$\hat{b}_S \equiv \left(\frac{1}{L} \sum_{i=1}^{L} \max(\hat{\Sigma}_{1/2}^{1/2} \xi_i)\right) \mathbb{1}\{|\hat{\beta}_2 - \hat{\beta}_1| < 1.7/n^{1/3}\}.$$  

We compare the following two estimators with the minimax decision $\hat{\theta}_{mx}$:

$$\hat{\theta}_F \equiv \max(\hat{\beta}_1, \hat{\beta}_2) - \hat{b}_F / \sqrt{n} \text{ and } \hat{\theta}_S \equiv \max(\hat{\beta}_1, \hat{\beta}_2) - \hat{b}_S / \sqrt{n}.$$  

We call $\hat{\theta}_F$ the estimator with fixed bias-reduction and $\hat{\theta}_S$ the estimator with selective bias-reduction. The results are reported in Figure 1.

The finite sample risks of $\hat{\theta}_F$ are better than the minimax decision $\hat{\theta}_{mx}$ only locally around $\delta_0 = 0$. The bias reduction using $\hat{b}_F$ improves the estimator's performance in this case. However, for other values of $\delta_0$, the bias reduction does more harm than good because it lowers the bias when it is better not to, due to increased variance. This is seen in the right-hand panel of Figure 1 which presents the finite sample bias of the estimators. With $\delta_0$ close to zero, the estimator with fixed bias-reduction eliminates the bias almost entirely. However, for other values of $\delta_0$, this bias correction induces negative bias, deteriorating the risk performances.

The estimator $\hat{\theta}_S$ with selective bias-reduction is designed to be hybrid between the two extremes of $\hat{\theta}_F$ and $\hat{\theta}_{mx}$. When $\beta_2 - \beta_1$ is estimated to be close to zero, the estimator performs like $\hat{\theta}_F$ and when it is away from zero, it performs like $\max(\hat{\beta}_1, \hat{\beta}_2)$. As expected, the bias of the estimator $\hat{\theta}_S$ is better than that of $\hat{\theta}_F$ while successfully eliminating nearly the entire bias when $\delta_0$ is close to zero. Nevertheless, it is remarkable that the estimator shows highly unstable finite sample risk properties overall as shown on the left panel in Figure 1. When $\delta_0$ is away from zero and around 3 to 7, the performance is worse than the other estimators. This result illuminates the fact that a reduction of bias does not always imply a better risk performance.
The minimax decision shows finite sample risks that are robust over the values of $\delta_0$. In fact, the estimated bias adjustment term $\hat{c}_M$ of the minimax decision is close to zero. This means that the estimator $\hat{\theta}_{mx}$ requires zero bias adjustment, due to the concern for its robust performance. In terms of finite sample bias, the minimax estimator suffers from a substantially positive bias as compared to the other two estimators, when $\delta_0$ is close to zero. The minimax decision tolerates this bias because by doing so, it can maintain robust performance for other cases where bias reduction is not needed. The minimax estimator is ultimately concerned with the overall risk properties, not just a bias component of the estimator, and as the left-hand panel of Figure 1 shows, it performs better than the other two estimators except when $\delta_0$ is locally around zero, or when $\beta_2 - \beta_1$ is around roughly between $-0.057$ and $0.041$.

5. Conclusion

The paper proposes local asymptotic minimax estimators for a class of nonregular parameters that are constructed by applying translation-scale equivariant transform to a regular parameter. The results are extended to the case where the nonregular parameters are transformed further by a piecewise linear map with a single kink. The local asymptotic minimax estimators take the form of a plug-in estimator with an additive bias adjustment term. The bias adjustment term can be computed by a simulation method. A small scale Monte Carlo simulation study demonstrates the robust finite sample risk properties of the local asymptotic minimax estimators, as compared to estimators based on alternative bias correction methods.

6. Appendix: Mathematical Proofs

Proof of Lemma 1: Property (a) follows immediately because $g(0) = 0$ by scale equivariance of $g$. Properties (b) and (c) are due to translation equivariance of $g$. The first equality in property (d) is due to scale equivariance of $g$, and the second equality comes from the definition of directional derivatives. Lipschitz continuity of $\tilde{g}(x; \cdot)$ on $\mathbb{R}^d$ stems from Lipschitz continuity of $g$ (e.g. see the proof of Proposition 1.1 of Clarke (1998)). Also, Lipschitz continuity of $g$ implies the uniform convergence on bounded sets, because bounded directional differentiability and directional differentiability in Assumption 1(i)(c) are equivalent when $g$ is a Lipschitz map defined on a finite dimensional space. (See Shapiro (1990), p.484.)

Proof of Lemma 2: First, suppose to the contrary that $f_1(y) \neq f_2(y)$ for some $y \in \mathbb{R}$. Then since $f_1 \circ g_1 = f_2 \circ g_2$, it is necessary that $g_1(\beta) \neq g_2(\beta)$ for some $\beta \in \mathbb{R}^d$ such that
$g_1(\beta) = y$, because $g_1(\mathbb{R}^d) = \mathbb{R}$ and $g_2(\mathbb{R}^d) = \mathbb{R}$, as we saw before. Hence

$$ (f_1 \circ g_1)(\beta) \neq (f_2 \circ g_1)(\beta).$$

(6.1)

Now observe that $f_2(g_1(\beta)) = f_2(g_2(\beta) + g_1(\beta) - g_2(\beta)) = f_2(g_2(\beta + g_1(\beta) - g_2(\beta)))$. Since $f_1 \circ g_1 = f_2 \circ g_2$, the last term is equal to

$$ f_1(g_1(\beta + g_1(\beta) - g_2(\beta))) = f_1(2g_1(\beta) - g_2(\beta)) = f_1(g_1(2\beta - g_2(\beta))) = f_2(g_2(2\beta - g_2(\beta))) = f_2(g_2(\beta)) = f_1(g_1(\beta)).$$

Therefore, we conclude that $f_2(g_1(\beta)) = f_1(g_1(\beta))$ contradicting (6.1).

Second, suppose to the contrary that $g_1(\beta) \neq g_2(\beta)$ for some $\beta \in \mathbb{R}^d$ and $f_1 = f_2$. First suppose that $g_1(\beta) > g_2(\beta)$. Fix arbitrary $a \in \mathbb{R}$ and $c \geq 0$ and let $c_\Delta = c/\Delta_{1,2}(\beta)$ and $\Delta_{1,2}(\beta) = g_1(\beta) - g_2(\beta)$. Then

$$ f_1(a + c) = f_1(a + \Delta_{1,2}(c_\Delta \beta)) = f_1(a + g_2(c_\Delta \beta) + \Delta_{1,2}(c_\Delta \beta) - g_2(c_\Delta \beta))$$

$$ = f_1(g_2(a + c_\Delta \beta + \Delta_{1,2}(c_\Delta \beta) - g_2(c_\Delta \beta)))$$

$$ = f_2(g_2(a + c_\Delta \beta + \Delta_{1,2}(c_\Delta \beta) - g_2(c_\Delta \beta)))$$

$$ = f_1(g_1(a + c_\Delta \beta + \Delta_{1,2}(c_\Delta \beta) - g_2(c_\Delta \beta)))$$

$$ = f_1(a + g_1(c_\Delta \beta - g_2(c_\Delta \beta) + c_\Delta \Delta_{1,2}(\beta))) = f_1(a + 2c).$$

The choice of $a \in \mathbb{R}$ and $c \geq 0$ are arbitrary, and hence $f_1(\cdot)$ is constant on $\mathbb{R}$, contradicting the nonconstancy condition for $f_1$.

Second, suppose that $g_1(\beta) < g_2(\beta)$. Then, fix arbitrary $a \in \mathbb{R}$ and $c \leq 0$ and let $c_\Delta = c/\Delta_{1,2}(\beta)$. Then similarly as before, we have

$$ f_1(a + c) = f_1(a + \Delta_{1,2}(c_\Delta \beta))$$

$$ = f_1(a + g_1(c_\Delta \beta - g_2(c_\Delta \beta) + c_\Delta \Delta_{1,2}(\beta))) = f_1(a + 2c),$$

because $\Delta_{1,2}(c_\Delta \beta) = c$. Therefore, again, $f_1(\cdot)$ is constant on $\mathbb{R}$, contradicting the nonconstancy condition for $f_1$. ■

We view convergence in distribution $\stackrel{d}{\to}$ in the proofs as convergence in $\bar{\mathbb{R}}^d$, so that the limit distribution is allowed to be deficient in general. Choose $\{h_i\}_{i=1}^m$ from a complete orthonormal basis $\{h_i\}_{i=1}^\infty$ of $H$. For $p \in \mathbb{R}^m$, we consider $h(p) \equiv \sum_{i=1}^m p_i h_i$, $h_i \in H$, so that $\beta_j(h(p)) = \sum_{i=1}^m \beta_j(h_i)p_i$, where $\beta_j$ is the $j$-th element of $\hat{\beta}$. Let $B$ be an $m \times d$ matrix.
such that

\[ B \equiv \begin{bmatrix} \hat{\beta}_1(h_1) & \hat{\beta}_2(h_1) & \cdots & \hat{\beta}_d(h_1) \\ \hat{\beta}_1(h_2) & \hat{\beta}_2(h_2) & \cdots & \hat{\beta}_d(h_2) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\beta}_1(h_m) & \hat{\beta}_2(h_m) & \cdots & \hat{\beta}_d(h_m) \end{bmatrix}. \]

We assume that \( m \geq d \) and \( B \) is a full column rank matrix.

We fix \( h' \in H \), and define \( a_i = (h_i, h') \) and \( a \in \mathbb{R}^n \) to be a column vector whose \( i \)-th entry is given by \( a_i \). We also define \( \zeta \equiv (\zeta(h_1), \ldots, \zeta(h_m))' \), where \( \zeta \) is the Gaussian process that appears in LAN, and with a small \( \lambda > 0 \), let \( F_\lambda(\cdot) \) be the cdf of \( N(0, (I_m - a(a^T a)^{-1}a^T)/\lambda) \). Then by design, the distribution of \( h(p) \in \mathbb{R} \), with \( p \sim F_\lambda \) concentrate on \( \{h(p) \in \mathbb{R} : \langle h(p), h' \rangle = 0\} \). Let \( Z_{\lambda,m} \in \mathbb{R}^d \) be a random vector following \( N(0, B^T(I_m + \lambda(I_m - a(a^T a)^{-1}a^T))^{-1}B) \).

Suppose that \( \hat{\theta} \in \mathbb{R} \) is a sequence of estimators such that along \( \{P_{n,h'}\}_{n \geq 1} \), with \( h \subset H \) such that \( \langle h, h' \rangle = 0 \),

\[
\begin{bmatrix} \sqrt{n}(\hat{\theta} - g(\beta_n(h + h'))) \\ \log dP_{n,h'+h}/dP_{n,h'} \end{bmatrix} \xrightarrow{d} \begin{bmatrix} V - \bar{g}_0(\hat{\beta}(h) + r) + \bar{g}_0(r) \\ \zeta(h) - \frac{1}{2}\langle h, h' \rangle \end{bmatrix},
\]

for some nonstochastic vector \( r \in \mathbb{R}^d \), where \( V \in \mathbb{R} \) is a random variable having a potentially deficient distribution independent of \( h \in H \)\(^1\). Let \( L_g^{h+h'} \) be the limiting (potentially deficient) distribution of \( \sqrt{n}(\hat{\theta} - g(\beta_n(h + h'))) \) in \( \mathbb{R}^d \) along \( \{P_{n,h+h'}\}_{n \geq 1} \) for each \( h \in H \) and \( h' \subset H \). The following lemma is an adaptation of the generalized convolution theorem in van der Vaart (1989).

**Lemma A1:** Suppose that the map \( g \) satisfies Assumption 1(i) holds. Then the following holds.

(i) For any \( \lambda > 0 \), the distribution \( \int L_g^{h(p)+h'} dF_\lambda(p) \) is equal to that of \( -\bar{g}_0(Z_{\lambda,m} + W_{\lambda,m} + r) + \bar{g}_0(r) \in \mathbb{R}, \) where \( W_{\lambda,m} \in \mathbb{R} \) is a random variable having a potentially deficient distribution independent of \( Z_{\lambda,m} \).

(ii) As \( \lambda \to 0 \) first and then \( m \to \infty \), we have \( Z_{\lambda,m} \xrightarrow{d} N(0, \Sigma) \).

**Proof:** (i) Using Assumption 1(i) and applying Le Cam’s third lemma (van der Vaart and Wellner (1996), p.404), we find that for all \( C \in \mathcal{B}(\mathbb{R}) \), the Borel \( \sigma \)-field of \( \mathbb{R} \),

\[
L_g^{h(p)+h'}(C) = E \left[ 1_C(V - \bar{g}_0(B^T p + r) + \bar{g}_0(r)) e^{p^T \zeta - \frac{1}{2}||p||^2} \right] = E \left[ 1_{(-\bar{g}_0)^{-1}(C)}(-V + B^T p + r - \bar{g}_0(r)) e^{p^T \zeta - \frac{1}{2}||p||^2} \right],
\]

\(^1\)Song (2014) on page 146 mistakenly refers to \( V \) as a "random vector" in \( \mathbb{R}^d \) when it is a random variable in \( \mathbb{R} \). A similar mistaken reference is found after the second display on page 149 of Song (2014).
where \((-\bar{g}_0)^{-1}(C) \equiv \{ x \in \mathbb{R}^d : -\bar{g}_0(x) \in C \}\). The second equality uses translation equivariance of \(\bar{g}_0\). (See Lemma 1(c).) Define

\[ \Sigma_{\lambda} \equiv (I_m + \lambda(I_m - a(a^Ta)^{-1}a^T))^{-1}. \]

Let \(N_{\lambda} : \mathbb{R}^m \to [0, 1]\) be the distribution function of \(N(0, \Sigma_{\lambda})\). From the definition of \(F_{\lambda}\), we write

\[
\int \mathcal{L}_{\bar{g}}^h(p) dF_{\lambda}(p) = (2\pi)^{-m/2} \det(\lambda(I_m - a(a^Ta)^{-1}a^T))^{-1/2} \times \int \mathbb{E}\left[ 1_{(-\bar{g}_0)^{-1}(C)}(-V + B^Tp + r - \bar{g}_0(r)) \right] e^{p^T\zeta - \frac{1}{2}p^T\Sigma_{\lambda}^{-1}p} dp.
\]

By rearranging the terms and applying change of variables, we can rewrite the integral as

\[
\int \mathbb{E}\left[ 1_{(-\bar{g}_0)^{-1}(C)}(-V + B^Tp + r - \bar{g}_0(r)) \right] e^{p^T\zeta - \frac{1}{2}p^T\Sigma_{\lambda}^{-1}p} dp
\]

\[
= \int \mathbb{E}\left[ 1_{(-\bar{g}_0)^{-1}(C)}(-V + B^T(p + \Sigma_{\lambda}\zeta) + r - \bar{g}_0(r)) \right] e^{p^T\zeta - \frac{1}{2}p^T\Sigma_{\lambda}^{-1}p} dp.
\]

Therefore, we conclude that

\[
\int \mathcal{L}_{\bar{g}}^h(p) dF_{\lambda}(p) = \int \mathbb{E}\left[ 1_{(-\bar{g}_0)^{-1}(C)}(-V + B^Tp + \Sigma_{\lambda}\zeta + r - \bar{g}_0(r)) \right] c_{\lambda}(\zeta) dN_{\lambda}(p),
\]

where \(c_{\lambda}(\zeta) \equiv e^{\frac{1}{2}\zeta^T\Sigma_{\lambda}\zeta} \cdot \det(\lambda(I_m - a(a^Ta)^{-1}a^T))^{-1/2} / \det(\Sigma_{\lambda})^{-1/2}\). When we let \(W_{\lambda,m}\) be a random variable having potentially deficient distribution \(\mathcal{W}_{\lambda,m}\) defined by

\[ \mathcal{W}_{\lambda,m}(C) \equiv \mathbb{E}\left[ 1_{(-\bar{g}_0)^{-1}(C)}(V - B^Tp) \right] c_{\lambda}(\zeta), \quad C \in \mathcal{B}(\mathbb{R}), \]

the distribution \(\int \mathcal{L}_{\bar{g}}^h dF_{\lambda}(p)\) is equal to that of \(-\bar{g}_0(Z_{\lambda,m} + W_{\lambda,m} + r - \bar{g}_0(r))\).

(ii) Since the sequence \(\{h_i\}_{i=1}^{\infty}\) is a complete orthonormal basis of \(\bar{H}\), the covariance matrix of \(Z_{\lambda,m}\) converges to \(\Sigma\) as \(\lambda \to 0\) and then \(m \to \infty\). ■

We introduce some notation. Define \(\| \cdot \|_{BL}\) on the space of Borel measurable functions on \(\mathbb{R}^d\):

\[ \|f\|_{BL} \equiv \sup_{x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|} + \sup_{x} \|f(x)\|. \]

For any two probability measures \(P\) and \(Q\) on \(\mathcal{B}(\mathbb{R}^d)\), define

\[ d_P(P, Q) \equiv \sup \left\{ \left| \int f dP - \int f dQ \right| : \|f\|_{BL} \leq 1 \right\}. \]
LEMMA A4: Suppose that for each \( n \geq 1 \), \( \{P_{n,h} : h \in H\} \) is the set of probability measures indexed by a Hilbert space \((H, \langle \cdot, \cdot \rangle)\), such that for each \( h \in H \),
\[
\log \frac{dP_{n,h}}{dP_{n,0}} = \zeta_n(h) - \frac{1}{2}\langle h, h \rangle, \quad \text{under } \{P_{n,0}\},
\]
where for each \( h, h' \in H \), \( [\zeta_n(h), \zeta_n(h')] \overset{d}{\to} [\zeta(h), \zeta(h')] \) under \( \{P_{n,0}\} \), and \( \zeta(\cdot) \) is a Gaussian process on \( H \) with covariance function \( \mathbb{E}[\zeta(h_1)\zeta(h_2)] = \langle h_1, h_2 \rangle \), \( h_1, h_2 \in H \).

Then for each \( h, h' \in H \) such that \( \langle h, h' \rangle = 0 \),
\[
\log \frac{dP_{n,h+h'}}{dP_{n,h'}} \overset{d}{\to} \zeta(h) - \frac{1}{2}\langle h, h \rangle, \quad \text{under } \{P_{n,h'}\}.
\]

PROOF: Since
\[
\log dP_{n,h+h'}/dP_{n,h'} = \log dP_{n,h+h'}/dP_{n,0} - \log dP_{n,h'}/dP_{n,0},
\]
we observe that by the condition of the lemma,
\[
\begin{bmatrix}
\log dP_{n,h+h'}/dP_{n,h'} \\
\log dP_{n,h'}/dP_{n,0}
\end{bmatrix} \overset{d}{\to} \begin{bmatrix}
\zeta(h+h') - \zeta(h') - \frac{1}{2}\langle h, h \rangle \\
\zeta(h') - \frac{1}{2}\langle h', h' \rangle
\end{bmatrix},
\]
under \( \{P_{n,0}\}_{n \geq 1} \), because \( \langle h, h' \rangle = 0 \). By Le Cam’s third lemma (van der Vaart and Wellner (1996), p.404), under\( \{P_{n,h'}\}_{n \geq 1} \),
\[
\log dP_{n,h+h'}/dP_{n,h'} \overset{d}{\to} \mathcal{L},
\]
where \( \mathcal{L} \) is a probability measure on \( B(\mathbb{R}) \) such that for any \( B \in B(\mathbb{R}) \),
\[
\mathcal{L}(B) = \mathbb{E} \left[ 1_B \left( \zeta(h+h') - \zeta(h') - \frac{1}{2}\langle h, h \rangle \right) e^{\zeta(h')-\frac{1}{2}\langle h', h' \rangle} \right] \\
= \mathbb{E} \left[ 1_B \left( \zeta(h+h') - \zeta(h') - \frac{1}{2}\langle h, h \rangle \right) \right] \\
= \mathbb{E} \left[ 1_B \left( \zeta(h) - \frac{1}{2}\langle h, h \rangle \right) \right].
\]
The second equality above follows because \( \zeta(h+h') - \zeta(h') \) and \( \zeta(h') \) are independent and \( \mathbb{E}[e^{\zeta(h')-\frac{1}{2}\langle h', h' \rangle}] = 1 \), and the third equality above follows because
\[
\zeta(h+h') - \zeta(h') \overset{d}{\to} N(0, ||h||^2) \overset{d}{=} \zeta(h).
\]
Hence we obtain (6.4). ■
PROOF OF LEMMA 3: We show that

\[
\sup_{b \in [0, \infty)} \liminf_{n \to \infty} \sup_{h \in H_{n,b}} E_h \left[ \tau(\sqrt{n} \{ \hat{\theta} - g(\beta_n(h)) \}) \right] 
\geq \sup_{\beta \in \Gamma} \left( \int E[\tau(|Z + r|) + w|] dF(w) \right),
\]

for some $F \in F$. Then the proof is complete by taking infimum over $F \in F$.

First, we choose $r \in \mathbb{R}^d$. Then we can find some $h' \in \mathbb{H}$ such that $r = \hat{\beta}(h')$. More specifically, let $q = \Sigma^{-1} r$ and define $h' = \sum_{i=1}^d \tilde{\beta}_i q_i$, where for each $i = 1, \cdots, d$, $\tilde{\beta}_i \in \overline{\mathbb{H}}$ is such that $\langle \tilde{\beta}_i, h \rangle = \mathbf{e}_i^\top \hat{\beta}(h)$ for all $h \in H$, and $q_i$ is the $i$-th entry of $q$. Then for this choice of $h'$, we can show that $r = \tilde{\beta}(h')$.

Fix $b/2 \geq ||h'|| \geq ||\hat{\beta}^*||$, where $\hat{\beta}^* = (\hat{\beta}_{e_1}^*, \cdots, \hat{\beta}_{e_m}^*)^\top$ and $\hat{\beta}_{e_m}^*$'s are as defined after Assumption 2. We note that

\[
\liminf_{n \to \infty} \sup_{h \in H_{n,b}} E_h \left[ \tau(\sqrt{n} \{ \hat{\theta} - g(\beta_n(h)) \}) \right] 
\geq \liminf_{n \to \infty} \sup_{h \in H_{n,b/2}^*} E_{h+h'} \left[ \tau(\sqrt{n} \{ \hat{\theta} - g(\beta_n(h+h')) \}) \right],
\]

where

\[H_{n,b/2}^* \equiv \{ h \in H_{n,b/2} : \langle h, h' \rangle = 0 \} .
\]

As in the proof of Theorem 3.11.5 of van der Vaart and Wellner (1996) (p.417), choose an orthonormal basis \{ $h_i$ \}_{i=1}^\infty from $\mathbb{H}$. We fix $m$ and take \{ $h_i$ \}_{i=1}^m \subset H$ and consider $h(p) = \sum_i p_i h_i$ for some $p = (p_i)_{i=1}^m \in \mathbb{R}^m$ such that $h(p) \in H$. Fix $\lambda > 0$ and let $F_\lambda(p)$ be as defined prior to Lemma A1 (with $h' \in H$ chosen previously in this proof.) Note that by design, any vector $p$ in the support of the distribution $F_\lambda$ satisfies that $\langle h(p), h' \rangle = 0$. Hence note that for fixed $M > 0$,

\[
\liminf_{n \to \infty} \sup_{h \in H_{n,b/2}^*} E_{h+h'} \left[ \tau(\sqrt{n} \{ \hat{\theta} - g(\beta_n(h+h')) \}) \right] 
\geq \liminf_{n \to \infty} \int E_{h(p)+h'} \left[ \tau_M \left( \left| \mathbf{V}_{n,h(p)+h'} \right| \right) \right] 1 \{ h(p) \in H_{n,b/2}^* \} dF_\lambda(p)
\geq \int \liminf_{n \to \infty} E_{h(p)+h'} \left[ \tau_M \left( \left| \mathbf{V}_{n,h(p)+h'} \right| \right) \right] dF_\lambda(p)
\geq -M \limsup_{n \to \infty} \int 1 \{ h(p) \notin H_{n,b/2}^* \} dF_\lambda(p),
\]

where $\mathbf{V}_{n,h} \equiv \sqrt{n} \{ \hat{\theta} - g(\beta_n(h)) \}$. The second inequality uses Fatou's lemma.
We write
\[ \sqrt{n} \{ \hat{\theta} - g(\beta_n(h+h')) \} \]
\[ = \sqrt{n} \{ \hat{\theta} - g(\beta_n(h')) \} - \sqrt{n}g(\beta_n(h+h')) + \sqrt{n}g(\beta_n(h')). \]
Then
\[ (6.8) \quad \sqrt{n}g(\beta_n(h+h')) - \sqrt{n}g(\beta_n(0)) \]
\[ = \sqrt{n} \{ g(\beta_n(h+h') - \beta_n(0) + \beta_n(0)) - g(\beta_n(0)) \} \]
\[ = \sqrt{n} \{ g(\beta(h + h')/\sqrt{n} + \beta_n(0)) - g(\beta_n(0)) \} + o(1) \]
\[ = \tilde{g}(\beta_n(0); \beta(h + h')) + o(1) = \tilde{g}(\beta_n(0); \beta(h) + r) + o(1) \]
\[ = \tilde{g}_0(\beta(h) + r) + o(1), \]
where the second to the last equality follows by the linearity of \( \hat{\beta} \) and the choice of \( r \), and the last equality follows because \( \beta_n(0) = \beta(P_{a_0}) = \beta_0 \) and by the definition of \( \tilde{g}_0(\cdot) \). Similarly,
\[ (6.9) \quad \sqrt{n}g(\beta_n'(h')) - \sqrt{n}g(\beta_n(0)) = \tilde{g}_0(r) + o(1). \]
Combining (6.8) and (6.9), we find that
\[ (6.10) \quad \sqrt{n}g(\beta_n(h+h')) - \sqrt{n}g(\beta_n(h')) \rightarrow \tilde{g}_0(\beta(h) + r) - \tilde{g}_0(r), \]
as \( n \rightarrow \infty \).

Applying Prohorov’s Theorem (in \( \bar{\mathbb{R}}^d \)), we find that for any subsequence of \( \{n\} \), there exists a further subsequence \( \{n'\} \) along which (under \( \{P_{n',h'}\} \))
\[ \sqrt{n'} \{ \hat{\theta} - g(\beta_n(h')) \} \overset{d}{\rightarrow} V, \]
where \( V \in \bar{\mathbb{R}} \) is a random variable having a potentially deficient distribution. Observe that
\[ (6.11) \quad \sqrt{n'} \{ \hat{\theta} - g(\beta_n'(h')) \} = \sqrt{n'} \{ \hat{\theta} - g(\beta_n(h')) \} \]
\[ - \{ \sqrt{n} \{ g(\beta_n(h+h') - \beta_n(h')) - \sqrt{n}g(\beta_n(h')) \} \]
\[ \overset{d}{\rightarrow} V - \tilde{g}_0(\beta(h) + r) + \tilde{g}_0(r). \]
Invoking Assumption 2, Lemma A4, and (6.11), and noting that marginal tightness implies joint tightness, we apply Prohorov’s Theorem to deduce that for any subsequence of \( \{n\} \), there exists a further subsequence \( \{n'\} \) along which \( r_{n'} \rightarrow r \equiv \beta(h') \), and (under \( P_{n',h'} \))
\[ \left[ \sqrt{n'} \{ \hat{\theta} - g(\beta_n'(h+h')) \} \right] \overset{d}{\rightarrow} \left[ V - \tilde{g}_0(\beta(h) + r) + \tilde{g}_0(r) \right] \]
where \( \sqrt{n}\{\hat{\theta} - g(\beta_n(h'))\} \to^d V \) under \( P_{n',h'} \). By Lemma A1,
\[
\liminf_{n \to \infty} \mathbb{E}_{h(p) + h'}\left[ \tau_M\left(\left| V_{n,h(p) + h'} \right|\right) \right] dF_\lambda(p) = \mathbb{E}\left[ \tau_M(\left| \bar{\theta}_0(Z_{\lambda,m} + W_{\lambda,m} + r) - \bar{\theta}_0(r) \right|) \right],
\]
where \( Z_{\lambda,m} \) is as defined prior to Lemma A1 and \( W_{\lambda,m} \in \mathbb{R} \) is a random variable having a potentially deficient distribution and independent of \( Z_{\lambda,m} \). Furthermore, by Assumption 2 (regularity of \( \beta_n(h) \)), we have for each \( p \in \mathbb{R}^m \),
\[
1\left\{ h(p) \in H_{n,b/2}^* \right\} \to 1\left\{ h(p) \in H_b^* \right\},
\]
as \( n \to \infty \), where \( H_b^* \equiv \{ h \in H : ||\hat{\beta}(h)|| \leq b, \langle h, h' \rangle = 0 \} \), and as \( b \uparrow \infty \),
\[
1\left\{ h(p) \in H_{b/2}^* \right\} \to 1\left\{ h(p) \in H_b^* \right\},
\]
where \( H_b^* \equiv \{ h \in H : \langle h, h' \rangle = 0 \} \). Therefore, since for each \( p \) in the support of \( F_\lambda \), we have \( h(p) \in H_b^* \), we send \( n \to \infty \) and \( b \uparrow \infty \), and apply the Dominated Convergence Theorem to conclude that
\[
\lim_{b \to \infty} \limsup_{n \to \infty} \int 1\left\{ h(p) \notin H_{n,b/2}^* \right\} dF_\lambda(p) = 0.
\]
Thus, we conclude from (6.7) that
\[
(6.12) \quad \lim_{b \to \infty} \liminf_{n \to \infty} \sup_{h \in H_{n,b/2}^*} \mathbb{E}_{h + h'}\left[ \tau(\sqrt{n}\{\hat{\theta} - g(\beta_n(h + h'))\}) \right] \geq \mathbb{E}\left[ \tau_M(\left| \bar{\theta}_0(Z_{\lambda,m} + W_{\lambda,m} + r) - \bar{\theta}_0(r) \right|) \right].
\]
By Lemma A1(ii), as \( \lambda \to 0 \) and then \( m \to \infty \), \( Z_{\lambda,m} \) converges in distribution to \( Z \). Since \( \{[Z_{\lambda,m}^T, W_{\lambda,m}]^T \in \mathbb{R}^{d+1} : (\lambda,m) \in (0,\infty) \times \{1,2,\ldots\} \} \) is uniformly tight in \( \mathbb{R}^{d+1} \), by Prohorov’s Theorem, for any subsequence of \( \{\lambda_k\}_{k=1}^\infty \) with \( \lambda_k \to 0 \) as \( k \to \infty \), and subsequence of \( \{m\} \), there exist further subsequences \( \{\lambda_k'\} \subset \{\lambda_k\} \) and \( \{m'\} \subset \{m\} \), such that as \( k' \to 0 \) and then \( m' \to \infty \),
\[
[Z_{\lambda_{k'},m'}^T, W_{\lambda_{k'},m'}]^T \to [Z^T, W]^T,
\]
for some random variable \( W_m \) having a potentially deficient distribution. By applying this to the right hand side of (6.12) and recalling (6.6), and noting that the choice of \( r \in \mathbb{R}^d \) was arbitrary, we conclude that
\[
(6.13) \quad \lim_{b \to \infty} \liminf_{n \to \infty} \sup_{h \in H_{n,b}} \mathbb{E}_h\left[ \tau(\sqrt{n}\{\hat{\theta} - g(\beta_n(h))\}) \right] \geq \sup_{r \in \mathbb{R}^d} \int \mathbb{E}\left[ \tau_M(\left| \bar{\theta}_0(Z + w + r) - \bar{\theta}_0(r) \right|) \right] dF(w),
\]
where \( F \) is an element of \( \mathcal{F}_s \) and \( \mathcal{F}_s \) is the collection of distributions on \( \mathcal{B}(\mathbb{R}) \).
Fix $F \in \mathcal{F}^*$. As for the last integral in (6.13), we write it as
\begin{equation}
\int E \left[ \tau_M (|\tilde{g}_0(Z + r) - \tilde{g}_0(r) + w|) \right] dF(w)
\end{equation}
\begin{align*}
&= \int E \left[ \tau_M (|\tilde{g}_0(Z + r) - \tilde{g}_0(r) + w|) 1\{w \in \bar{R} \setminus R\} \right] dF(w) \\
&\quad + \int E \left[ \tau_M (|\tilde{g}_0(Z + r) - \tilde{g}_0(r) + w|) 1\{w \in R\} \right] dF(w).
\end{align*}
Since $\tilde{g}_0(Z + r) - \tilde{g}_0(r) \in R$, for $w \in \bar{R} \setminus R$,
\begin{equation*}
\tau_M (|\tilde{g}_0(Z + r) - \tilde{g}_0(r) + w|) = \min \left\{ \sup_{x \in [0, \infty)} \tau(x), M \right\},
\end{equation*}
so that
\begin{align*}
\int E \left[ \tau_M (|\tilde{g}_0(Z + r) - \tilde{g}_0(r) + w|) 1\{w \in \bar{R} \setminus R\} \right] dF(w)
&= \min \left\{ \sup_{x \in [0, \infty)} \tau(x), M \right\} \cdot \int_{\bar{R} \setminus R} dF(w).
\end{align*}
From (6.14), we conclude that
\begin{align*}
\int E \left[ \tau_M (|\tilde{g}_0(Z + r) - \tilde{g}_0(r) + w|) \right] dF(w)
&= \min \left\{ \sup_{x \in [0, \infty)} \tau(x), M \right\} \cdot \int_{\bar{R} \setminus R} dF(w) \\
&\quad + \int E \left[ \tau_M (|\tilde{g}_0(Z + r) - \tilde{g}_0(r) + w|) 1\{w \in R\} \right] dF(w).
\end{align*}
We identify $\mathcal{F}$ as the subset of $\mathcal{F}^*$ such that for each $F \in \mathcal{F}$, $\int_{\bar{R} \setminus R} dF(w) = 0$ and $\int_R dF(w) = 1$. Since
\begin{equation*}
\tau_M (|\tilde{g}_0(Z + r) - \tilde{g}_0(r) + w|) \leq \min \left\{ \sup_{x \in [0, \infty)} \tau(x), M \right\}
\end{equation*}
everywhere, the lower bound in (6.13) remains the same if we replace $\mathcal{F}^*$ by $\mathcal{F}$. Since $\tau_M$ increases in $M$, we obtain the desired bound by sending $M \uparrow \infty$. \[\square\]

For given $M_1, a > 0$ and $c \in \mathbb{R}$, define
\begin{equation}
B_{M_1}(c; a) \equiv \sup_{r \in \mathbb{R}^2} E \left[ \tau_{M_1}(a|\tilde{g}_0(Z + r) - \tilde{g}_0(r) + c|) \right],
\end{equation}
and

$$E_{M_1}(a) \equiv \left\{ c \in [-M_1, M_1] : B_{M_1}(c; a) \leq \inf_{c_i \in [-M_1, M_1]} B_{M_1}(c_i; a) \right\}.$$ 

Let $c_{M_1}^*(a) \equiv \sup E_{M_1}(a)$. We also define

$$\tilde{g}_n(z) \equiv g(z + e_n^{-1}(\beta_0 - g(\beta_0))),$$

for $z \in \mathbb{R}^d$, and

$$\tilde{B}_{M_1}(c; a) \equiv \sup_{r \in [-M_1, M_1]^d} \frac{1}{L} \sum_{i=1}^{L} \tau_{M_1} \left( a \left| \tilde{g}_n (\Sigma_1^{1/2} \xi_i + r) - \tilde{g}_n (r) + c \right| \right),$$

$$\tilde{B}_{M_1}(c; a) \equiv \sup_{r \in [-M_1, M_1]^d} \frac{1}{L} \sum_{i=1}^{L} \tau_{M_1} \left( a \left| \tilde{g}_n (\Sigma_1^{1/2} \xi_i + r) - \tilde{g}_n (r) + c \right| \right),$$

and

$$B_{M_1}^*(c; a) \equiv \sup_{r \in [-M_1, M_1]^d} \mathbb{E} \left[ \tau_{M_1} \left( a \left| \tilde{g}_n (\Sigma_1^{1/2} \xi_i + r) - \tilde{g}_n (r) + c \right| \right) \right].$$

We also define

$$E_{M_1}^*(a) \equiv \left\{ c \in [-M_1, M_1] : B_{M_1}^*(c; a) \leq \inf_{c_i \in [-M_1, M_1]} B_{M_1}^*(c_i; a) \right\}.$$ 

**Lemma A5:** Suppose that Assumptions 1(i), 4, and 5 hold. Then as $M \to \infty$,

$$\lim_{n \to \infty} \sup_{h \in H} \left\{ \sup_{c \in [-M_1, M_1]} \left| B_{M_1}^*(c; a) - \tilde{B}_{M_1}(c; a) \right| > M \left( L^{-1/2} + n^{-1/2} e_n^{-1} \right) \right\} \to 0.$$

**Proof:** Note that

$$|\tilde{g}_n(z) - \tilde{g}_n(z)| = |g(z + e_n^{-1}(\beta_0 - g(\beta_0))) - g(z + e_n^{-1}(\beta - g(\beta)))| \leq 2 e_n^{-1} \left\| \beta_0 - \hat{\beta} \right\|,$$

by Lipschitz continuity of $g$. The last bound does not depend on $z \in \mathbb{R}^d$. Hence using Assumption 5(ii), we conclude

$$\sup_{z \in \mathbb{R}^d} |\tilde{g}_n(z) - \tilde{g}_n(z)| = O_p \left( n^{-1/2} e_n^{-1} \right),$$

where the convergence is uniform over $h \in H$. Therefore, as $M \to \infty$,

$$\lim_{n \to \infty} \sup_{h \in H} \left\{ \sup_{c \in [-M_1, M_1]} \left| \tilde{B}_{M_1}(c; a) - \tilde{B}_{M_1}(c; a) \right| > M n^{-1/2} e_n^{-1} \right\} \to 0.$$

Since $g$ is Lipschitz, there exists $C > 0$ such that for all $n \geq 1$, for any $z, w \in \mathbb{R}^d$,

$$|\tilde{g}_n(z) - \tilde{g}_n(w)| \leq C \left\| z - w \right\|.$$
Hence by Assumptions 4(ii) and 5(i), we have

\[ \lim_{n \to \infty} \sup_{h \in H} \left\{ \sup_{c \in [-M_1, M_1]} |\tilde{B}_{M_1}(c; a) - \tilde{B}_{M_1}(c; a)| > M n^{-1/2} \right\} \to 0, \]

as \( M \to \infty \).

Now we show that as \( M \to \infty \)

\[ \lim_{n \to \infty} P \left\{ \sup_{c \in [-M_1, M_1]} |B^*_{M_1}(c; a) - \tilde{B}_{M_1}(c; a)| > M (L^{-1/2} + n^{-1/2}) \right\} \to 0. \]

(Note that \( P \) above denotes the joint distribution of the simulated quantities \( \{\xi_i\}_{i=1}^L \), and hence does not depend on \( h \in H \). Thus the convergence above is trivially uniform in \( h \in H \).) First, define \( f_n(\xi; c, r) \equiv \tau_{M_1}(a | \tilde{g}_n(\xi + r) - \tilde{g}_n(r) + c|) \) and \( J_n \equiv \{ f_n(\cdot; c, r) : (c, r) \in [-M_1, M_1] \times [-M_1, M_1]^d \} \). The class \( J \) is uniformly bounded, and \( f(\xi; c, r) \) is Lipschitz continuous in \( (c, r) \in [-M_1, M_1] \times [-M_1, M_1]^d \). Using the maximal inequality (e.g. Theorems 2.14.2 (p.240) and 2.7.11 (p.164) of van der Vaart and Wellner (1996)), we find that for some \( C_{M_1} > 0 \) that depends only on \( M_1 > 0 \),

\[ \mathbb{E} \left[ \sup_{c \in [-M_1, M_1]} \left| B^*_{M_1}(c; a) - \tilde{B}_{M_1}(c; a) \right| \right] \leq C_{M_1} \left\{ L^{-1/2} + n^{-1/2} \right\}. \]

Hence the convergence in (6.16) follows. Thus the proof is complete. 

**Lemma A6: Suppose that Assumptions 1(i) and 4 hold. Then as \( n \to \infty \),

\[ \sup_{c \in [-M_1, M_1]} \left| B^*_{M_1}(c; a) - B_{M_1}(c; a) \right| \to 0. \]

**Proof:** Since \( g \) is Lipschitz continuous, the convergence

\( \tilde{g}_n(z) \to \tilde{g}_0(z) \), as \( n \to \infty \),

is uniform over \( z \) in any given bounded subset of \( \mathbb{R}^d \). (See Shapiro (1990), p.484.) Then

\[ \sup_{c \in [-M_1, M_1]} \left| B^*_{M_1}(c; a) - B_{M_1}(c; a) \right| \]

\[ \leq \sup_{c \in [-M_1, M_1]} \sup_{r \in [-M_1, M_1]^d} \mathbb{E} \left[ \tau_{M_1}(a | \tilde{g}_n(Z + r) - \tilde{g}_n(r) + c|) \right] \to 0, \]

as \( n \to \infty \), because the domains of supremums above are bounded in a finite dimensional space. 

LEMMA A7: Suppose that Assumptions 1(i), 4, and 5 hold. Then there exists $M_0$ such that for any $M_1 > M_0$, $\varepsilon > 0$, $b > 0$, and any $a > 0$,

$$\sup_{h \in H} P_{n,h} \left\{ | \hat{c}_{M_1}(a) - c^*_M(a) | > \varepsilon \right\} \to 0,$$

as $n, L \to \infty$ jointly.

PROOF: Let the Hausdorff distance between the two subsets $E_1$ and $E_2$ of $R$ be denoted by $d_H(E_1, E_2)$. First we show that

$$d_H(E^*_M(a), \hat{E}_M(a)) \to_p 0,$$  \hspace{1cm} (6.18)

as $n \to \infty$ and $L \to \infty$ uniformly over $h \in H$. For this, we use arguments in the proof of Theorem 3.1 of Chernozhukov, Hong and Tamer (2007). Fix $\varepsilon \in (0,1)$ and let $E^*_M(a) \equiv \{ x \in [-M_1, M_1] : \inf_{y \in E^*_M(a)} | x - y | \leq \varepsilon \}$. It suffices for (6.18) to show that for any $\varepsilon > 0$,

(a) $\inf_{h \in H} P_{n,h} \left\{ \sup_{c \in E^*_M(a)} \hat{B}_M(c; a) \leq \inf_{c \in [-M_1, M_1]} \hat{B}_M(c; a) + \eta_{n,L} \right\} \to 1,$

(b) $\inf_{h \in H} P_{n,h} \left\{ \sup_{c \in \hat{E}_M(a)} B^*_M(c; a) < \inf_{c \in (-M_1, M_1) \backslash E^*_M(a)} B^*_M(c; a) \right\} \to 1,$

as $n, L \to \infty$ jointly. This is because (a) implies $\inf_{h \in H} P_{n,h} \{ E^*_M(a) \subset \hat{E}_M(a) \} \to 1$ and (b) implies that $\inf_{h \in H} P_{n,h} \{ \hat{E}_M(a) \cap ([-M_1, M_1] \backslash E^*_M(a)) = \emptyset \} \to 1$ so that $\inf_{h \in H} P_{n,h} \{ \hat{E}_M(a) \subset E^*_M(a) \} \to 1$, and hence for any $\varepsilon > 0$,

$$\sup_{h \in H} P_{n,h} \left\{ d_H(E^*_M(a), \hat{E}_M(a)) > \varepsilon \right\} \to 0, \text{ as } n, L \to \infty \text{ jointly},$$

delivering (6.18).

We focus on (a). Note that

$$\sup_{c \in E^*_M(a)} \hat{B}_M(c; a) = \sup_{c \in E^*_M(a)} B^*_M(c; a) + o_p(L^{-1/2} + n^{-1/2} \varepsilon_n^{-1})$$

$$\leq \inf_{c \in [-M_1, M_1]} \hat{B}_M(c; a) + o_p(L^{-1/2} + n^{-1/2} \varepsilon_n^{-1}),$$

where the equality follows from Lemma A5, and the inequality follows by the definition of $E^*_M(a)$. From this (a) follows because $\eta_{n,L} \varepsilon_n^{\sqrt{n}} \to \infty$ as $n \to \infty$ and $\eta_{n,L} \sqrt{L} \to \infty$ as $L \to \infty$.

Now let us turn to (b). Fix $\varepsilon > 0$. Uniformly over $h \in H$,

$$\sup_{c \in \hat{E}_M(a)} B^*_M(c; a) \leq \sup_{c \in \hat{E}_M(a)} \hat{B}_M(c; a) + o_p(1)$$

$$\leq \inf_{c \in [-M_1, M_1]} \hat{B}_M(c; a) + o_p(1)$$

$$\leq \inf_{c \in [-M_1, M_1]} B^*_M(c; a) + o_p(1),$$
where the second inequality follows by the definition of $\hat{E}_{M_1}(a)$ and the third inequality is due to $\eta_{n,L} \to 0$ as $n,L \to \infty$ and Lemma A5. By the definition of $E_{M_1}^*(a)$, we have

$$0 \leq \inf_{c \in [-M_1,M_1]} B_{M_1}^*(c;a) < \inf_{c \in [-M_1,M_1] \setminus E_{M_1}^*(a)} B_{M_1}^*(c;a).$$

Hence we obtain (b). Thus we obtain (6.18).

Now we show that as $n \to \infty$,

$$(6.20) \quad d_H(E_{M_1}^*(a), E_M(a)) \to 0.$$  

Similarly as before, it suffices to note that

$$\sup_{c \in E_{M_1}(a)} B_{M_1}(c;a) \leq \inf_{c \in [-M_1,M_1]} B_{M_1}^*(c;a) + o(1) \quad \text{and} \quad \sup_{c \in E_{M_1}(a)} B_{M_1}(c;a) < \inf_{c \in [-M_1,M_1] \setminus E_{M_1}^*(a)} B_{M_1}(c;a).$$

The first inequality follows by the definition of $E_{M_1}(a)$ and Lemma A6. The second inequality follows by Lemma A6 and the definition of $E_{M_1}^*(a)$ as in (6.19). Thus we obtain (6.20). We combine (6.18) with (6.20) to conclude that

$$(6.21) \quad d_H(E_{M_1}(a), \hat{E}_{M_1}(a)) \to_p 0.$$  

For the main conclusion of the lemma, observe that $|\hat{c}_{M_1}(a) - c_{M_1}^*(a)|$ is equal to

$$|\sup \hat{E}_{M_1}(a) - \sup E_{M_1}(a)|,$$

which we can write as

$$\left| \sup_{y \in \hat{E}_{M_1}(a)} \{y - \sup E_{M_1}(a)\} \right| = \left| \sup_{y \in \hat{E}_{M_1}(a)} \inf_{x \in E_{M_1}(a)} (y - x) \right|.$$  

We can interchange the supremum and the infimum using the fact that the sets $\hat{E}_{M_1}(a)$ and $E_{M_1}(a)$ are compact sets and using a version of minimax theorem (e.g. Lemma A.3 of Puhalskii and Spokoiny (1998)). (Note that the compactness of $\hat{E}_{M_1}(a)$ and $E_{M_1}(a)$ follows from Assumption 4(ii).) Using the fact that $z = (z)_+ - (z)_-$, where $(z)_+ = \max(z,0)$ and $(z)_- = \max(-z,0)$, and applying the minimax theorem, we bound the last term by

$$\sup_{y \in \hat{E}_{M_1}(a)} \inf_{x \in E_{M_1}(a)} (y - x)_+ + \sup_{y \in \hat{E}_{M_1}(a)} \inf_{x \in E_{M_1}(a)} (y - x)_-.$$  

The sum above is bounded by $2d_H(E_{M_1}(a), \hat{E}_{M_1}(a))$. The desired result follows from (6.21).
PROOF OF THEOREM 2: Fix $M > 0$ and $\epsilon > 0$, and take large $M_1 \geq M$ such that

$$
\sup_{r \in \mathbb{R}^d} \mathbb{E} \left[ \tau_M (|\tilde{g}_0(Z + r) - \tilde{g}_0(r) + c^*_M(1)|) \right] 
\leq \sup_{r \in [-M_1, M_1]^d} \mathbb{E} \left[ \tau_M (|\tilde{g}_0(Z + r) - \tilde{g}_0(r) + c^*_M(1)|) \right] + \epsilon.
$$

(6.22)

This is possible for any choice of $\epsilon > 0$ because $\tau_M(\cdot)$ and $\tilde{g}_0(\cdot)$ are Lipschitz continuous (recall Assumption 4(ii) and Lemma 1(iii)) and bounded by $M$. Note that

$$
\sup_{h \in H_{n,b}} \mathbb{E}_h \left[ \tau_M (\sqrt{n}|\hat{\beta} - g(\beta_n(h))|) \right] 
= \sup_{h \in H_{n,b}} \mathbb{E}_h \left[ \tau_M (\sqrt{n}|g(\beta) + \hat{c}_M(1) / \sqrt{n} - g(\beta_n(h))|) \right] 
\leq \sup_{h \in H_{n,b}} \mathbb{E}_h \left[ \tau_M (|g(\sqrt{n}(\hat{\beta} - \beta_n(h)) + r_n(h)) + \hat{c}_M(1)|) \right],
$$

(6.23)

where $r_n(h) \equiv \sqrt{n}(\beta_n(h) - g(\beta_n(h)))$. Note that for each $h \in H_{n,b}$,

$$
r_n(h) = \sqrt{n}\{\beta_n(h) - \beta_n(0) + \beta_n(0)\} - \sqrt{n}g(\beta_n(h) - \beta_n(0) + \beta_n(0)) 
= \{\sqrt{n}\beta_0 + \tilde{r}_n(h) - g(\sqrt{n}\beta_0 + \tilde{r}_n(h))\},
$$

where $\tilde{r}_n(h) \equiv \sqrt{n}\{\beta_n(h) - \beta_n(0)\}$ and $\sup_{h \in H_{n,b}} ||\tilde{r}_n(h)|| \leq b||\hat{\beta}^*|| + o(1)$ by the definition of $h \in H_{n,b}$. Using Assumption 5, and using the fact that $Z$ is a continuous random vector, we find that

$$
\sup_{t \in \mathbb{R}} \sup_{r \in \mathbb{R}^d} \sup_{h \in H_{n,b}} \left| P_{n,h} \left\{ \sqrt{n}(\hat{\beta} - \beta_n(h)) + r + \hat{c}_M(1) \leq t \right\} - P \left\{ Z + r + c^*_M(1) \leq t \right\} \right| \rightarrow 0,
$$

as $n \rightarrow \infty$. Therefore,

$$
\sup_{h \in H_{n,b}} \left| \mathbb{E}_h \left[ \tau_M (|g(\sqrt{n}(\hat{\beta} - \beta_n(h)) + r_n(h)) + \hat{c}_M(1)|) \right] - \mathbb{E} \left[ \tau_M (|g(Z + r_n(h)) + c^*_M(1)|) \right] \right| \rightarrow 0,
$$

as $n \rightarrow \infty$. Let $A_M \equiv \tau^{-1}([0, M])$ which is bounded in $[0, \infty)$ by Assumption 4(i). We take $M_2 \geq M_1$ and write

$$
\mathbb{E} \left[ \tau_M (|g(Z + r_n(h)) + c^*_M(1)|) \right] 
\leq \mathbb{E} \left[ \tau_M (|g(Z + r_n(h)) + c^*_M(1)|1 \{||Z|| \leq M_2\}) \right] + MP \{||Z|| > M_2\},
$$

where the leading expectation in the second line can be rewritten as

$$
\mathbb{E} \left[ \tau_M \left( g \left( \frac{Z + \sqrt{n}\beta_0 + \tilde{r}_n(h)}{\sqrt{n}} \right) - g(\sqrt{n}\beta_0 + \tilde{r}_n(h)) + c^*_M(1) \right) \right] 1 \{||Z|| \leq M_2\}.
$$

(6.24)
Since \( g \) is Lipschitz, \( \sup_{h \in H_{n,b}} ||f_n(h)|| \leq b||\hat{\beta}^*|| + o(1) \), and the convergence of \( g(z + \sqrt{n} \beta_0) - g(\sqrt{n} \beta_0) \to \tilde{g}_0(z) \) is uniform over \( z \) in any bounded set by Lemma 1(iii), we find that the expectation in (6.24) converges to
\[
\mathbb{E}\left[ \tau_{M}(||\tilde{g}_0(Z + \hat{\beta}(h)) - \tilde{g}_0(\hat{\beta}(h)) + c^*_M(1)||1 \{||Z|| \leq M_2\}) \right],
\]
uniformly in \( h \in H_{n,b} \) as \( n \to \infty \). Thus, we conclude that
\[
\limsup_{n \to \infty} \sup_{h \in H_{n,b}} \mathbb{E}_h\left[ \tau_{M}(||\tilde{g}_0(Z + \hat{\beta}(h)) - \tilde{g}_0(\hat{\beta}(h)) + c^*_M(1)||1 \{||Z|| \leq M_2\}) \right] \leq \sup_{r \in \mathbb{R}^d} \mathbb{E}\left[ \tau_{M}(||\tilde{g}_0(Z + r) - \tilde{g}_0(r) + c^*_M(1)||1 \{||Z|| \leq M_2\}) \right] + MP \{||Z|| > M_2\}.
\]
As we send \( M_2 \to \infty \), the last sum vanishes and the leading supremum becomes
\[
\sup_{r \in \mathbb{R}^d} \mathbb{E}\left[ \tau_{M}(||\tilde{g}_0(Z + r) - \tilde{g}_0(r) + c^*_M(1)||) \right] \leq \sup_{r \in [-M_1, M_1]^d} \mathbb{E}\left[ \tau_{M}(||\tilde{g}_0(Z + r) - \tilde{g}_0(r) + c^*_M(1)||) \right] + \varepsilon,
\]
by (6.22). Since \( M_1 \geq M \), the last supremum is bounded by
\[
\sup_{r \in [-M_1, M_1]^d} \mathbb{E}\left[ \tau_{M_1}(||\tilde{g}_0(Z + r) - \tilde{g}_0(r) + c^*_M(1)||) \right] = \inf_{-M_1 \leq c \leq M_1} \sup_{r \in [-M_1, M_1]^d} \mathbb{E}\left[ \tau_{M_1}(||\tilde{g}_0(Z + r) - \tilde{g}_0(r) + c||) \right],
\]
where the equality follows by the definition of \( c^*_M(1) \). Since the choice of \( \varepsilon \) and \( M_1 \) was arbitrary and \( \mathbb{E}\left[ \tau_{M_1}(||\tilde{g}_0(Z + r) - \tilde{g}_0(r) + c||) \right] \) is uniformly continuous in \( r \in \mathbb{R}^d \), sending \( M_1 \to \infty \) (along with \( \varepsilon \downarrow 0 \)), and then sending \( M \to \infty \), we obtain the desired result. \( \blacksquare \)

**Proof of Theorem 3:** As in the proof of Lemma 3, we choose \( r \in \mathbb{R}^d \) so that for some \( h' \in H, r = \hat{\beta}(h') \). Fix \( b/2 \geq ||h'|| \cdot ||\hat{\beta}^*|| \). Define
\[
H_{n,b,1}^* \equiv \{h \in H_{n,b}^* : g(\beta_n(h + h')) \geq \bar{x}\}, \quad \text{and}
\]
\[
H_{n,b,2}^* \equiv \{h \in H_{n,b}^* : g(\beta_n(h + h')) \leq \bar{x}\},
\]
where we recall \( H_{n,b}^* \equiv \{h \in H_{n,b} : \langle h, h' \rangle = 0\} \).
First, suppose that \( g(\beta_0) > \bar{x} \). Note that

\[
\liminf_{n \to \infty} \sup_{h \in H_{n,b}} E_h \left[ \tau(\sqrt{n}(\hat{\theta} - f(g(\beta_n(h)))) \right] \\
\geq \liminf_{n \to \infty} \sup_{h \in H_{n,b}^*} E_{h+h'} \left[ \tau_M(\sqrt{n}(\hat{\theta}_1 - a_1 g(\beta_n(h + h'))) \right] \\
\geq \liminf_{n \to \infty} \sup_{h \in H_{n,b}^*} E_{h+h'} \left[ \tau_M(\sqrt{n}(\hat{\theta}_1 - a_1 g(\beta_n(h + h'))) \right],
\]

where \( \hat{\theta}_1 \equiv \hat{\theta} + a_1 \bar{x} \). Let \( \hat{V}_{n,h,1} \equiv \sqrt{n}(\hat{\theta}_1 - a_1 g(\beta_n(h))) \), \( h(p), p = (p_i)_{i=1}^m \in \mathbb{R}^m \), and \( F_\lambda(p) \) be as in the proof of Lemma 3, so that we have

\[
\liminf_{n \to \infty} \sup_{h \in H_{n,b}^*} E_{h+h'} \left[ \tau_M(\sqrt{n}(\hat{\theta}_1 - a_1 g(\beta_n(h + h'))) \right] \\
\geq \int \liminf_{n \to \infty} E_{h(p)+h'} \left[ \tau_M(\sqrt{n}(\hat{\theta}_1 - a_1 g(\beta_n(h + h'))) \right] 1 \{ h(p) \in H_{n,b,1} \} dF_\lambda(p).
\]

Let \( R_n(h) \equiv g(\beta_n(h + h')) - g(\beta_0) \), and observe that for all \( h \in H \),

\[
R_n(h) = g(\beta_n(h + h') - \beta_n(h') + \beta_n(h')) - g(\beta_n(h')) \\
= g(\beta_n(h')) - g(\beta_0) \\
= g((\hat{\beta}(h + h'))/\sqrt{n} + \beta_n(h')) - g(\beta_n(h')) \\
+ g(\beta_n(h')) - g(\beta_0) + o(1/\sqrt{n}),
\]

as \( n \to \infty \). Since the map \( g \) is Lipshitz continuous and \( \beta_n(h') = \beta_0 + O(1/\sqrt{n}) \), we deduce that for each \( h \in H \),

\[
|R_n(h)| \to 0,
\]

as \( n \to \infty \). This means that given that \( g(\beta_0) > \bar{x} \), we have

\[
1 \{ h(p) \in H_{n,b,1}^* \} \to 1 \{ h(p) \in H_b^* \},
\]

as \( n \to \infty \).

Following the same arguments as in the proofs of Lemma 3 and Theorem 1, we deduce that

\[
\sup_{h \in (0,\infty)} \liminf_{n \to \infty} \sup_{h \in H_{n,b}^*} E_h \left[ \tau_M(\sqrt{n}(\hat{\theta}_1 - a_1 g(\beta_n(h))) \right] \\
\geq \inf_{c \in \mathbb{R}} \sup_{r \in \mathbb{R}^d} E \left[ \tau_M(|a_1||\tilde{g}_0(Z + r) - \tilde{g}_0(r) + c|) \right] - \varepsilon.
\]
Second, suppose that \( g(\beta_0) < \bar{x} \). Using similar arguments, we obtain the result that

\[
\sup_{b \in (0, \infty)} \liminf_{n \to \infty} \sup_{h \in H_{n,b,1}} E_h \left[ \tau_M \left( |\sqrt{n}\{\hat{\theta}_2 - a_2 g(\beta_n(h))\}| \right) \right] \\
\geq \inf_{c \in R} \sup_{p \in R^d} E \left[ \tau_M \left( |a_2| |\tilde{g}_0(Z + \bar{r}) - \tilde{g}_0(r) + c| \right) \right] - \epsilon,
\]

where \( \hat{\theta}_1 = \hat{\theta} + a_2 \bar{x} \).

Finally, assume that \( g(\beta_0) = \bar{x} \). Then

\[
\begin{align*}
H_{n,b,1}^* &= \{ h \in H_{n,b} : R_n(h) \geq 0 \}, \\
H_{n,b,2}^* &= \{ h \in H_{n,b} : R_n(h) \leq 0 \}.
\end{align*}
\]

By (6.25), we have for each \( h \in H \), as \( n \to \infty \),

(6.26) \[
1 \left\{ h \in H_{n,b,1}^* \right\} \to 1 \left\{ h \in H_b \right\} \quad \text{and} \\
1 \left\{ h \in H_{n,b,2}^* \right\} \to 1 \left\{ h \in H_b \right\},
\]

where \( H_b \equiv \{ h \in H : ||\hat{\theta}(h)|| \leq b \} \). Note that

\[
\sup_{h \in H_{n,b}} E_h \left[ \tau_M \left( |\sqrt{n}\{\hat{\theta} - f(g(\beta_n(h)))\}| \right) \right] \geq \max_{l=1,2} \sup_{h \in H_{n,b/l}} E_h \left[ \tau_M \left( |\sqrt{n}\{\hat{\theta}_l - a_l g(\beta_n(h))\}| \right) \right].
\]

Using (6.26) and following the same arguments as before, we conclude that

\[
\sup_{b \in (0, \infty)} \liminf_{n \to \infty} \sup_{h \in H_{n,b}} E_h \left[ \tau \left( |\sqrt{n}\{\hat{\theta} - f(g(\beta_n(h)))\}| \right) \right] \\
\geq \max_{l=1,2} \inf_{c \in R} \sup_{p \in R^d} E \left[ \tau_M \left( |a_l| |\tilde{g}_0(Z + \bar{r}) - \tilde{g}_0(r) + c| \right) \right] \\
= \inf_{c \in R} \sup_{p \in R^d} E \left[ \tau_M \left( \max \{|a_1|, |a_2|\} |\tilde{g}_0(Z + \bar{r}) - \tilde{g}_0(r) + c| \right) \right],
\]

where the last equality follows because \( \tau_M \) is an increasing function. By sending \( M \uparrow \infty \), we obtain the desired result. ■

**Lemma A8:** Suppose that Assumptions 1(i) and 5 hold. Then,

\[
\inf_{h \in H} P_{n,h} \{ \hat{s} = s \} \to 1.
\]

**Proof:** The proof can be straightforwardly proceeded as the proof of Lemma A5 by dividing the proof into cases with \( g(\beta_0) > \bar{x} \), \( g(\beta_0) < \bar{x} \), and \( g(\beta_0) = \bar{x} \), and applying Assumption A5(ii). The details are omitted. ■
PROOF OF THEOREM 4: For any $M > 0$, and $b \in [0, \infty)$,
\[
\limsup_{n \to \infty} \sup_{h \in H_{n,b}} E_h \left[ \tau_M \left( \sqrt{n} \{ \tilde{\theta}_{m,1} - f(g(\beta_n(h))) \} \right) \right]
\]
\[
= \limsup_{n \to \infty} \sup_{h \in H_{n,b}} E_h \left[ \tau_M \left( \sqrt{n} \{ \tilde{\theta}_{m,1} - f(g(\beta_n(h))) \} \right) \right] 1 \{ \hat{s} = s \},
\]
by Lemma A8. We focus on the last limsup.

First, suppose that $g(\beta_0) > \tilde{x}$. Then there exists $\varepsilon > 0$, such that $g(\beta_0) > \tilde{x} + \varepsilon$. Since we have for all $h \in H_{n,b}$,
\[
||\beta_n(h) - \beta_0|| \leq b / \sqrt{n},
\]
we conclude that from some large $n$ on, for all $h \in H_{n,b}$, we have
\[
g(\beta_n(h)) \geq \tilde{x}.
\]
Hence
\[
\sup_{b \in [0, \infty)} \limsup_{n \to \infty} \sup_{h \in H_{n,b}} E_h \left[ \tau_M \left( \sqrt{n} \{ \tilde{\theta}_{m,1} - f(g(\beta_n(h))) \} \right) \right] 1 \{ \hat{s} = s \}
\]
\[
= \sup_{b \in [0, \infty)} \limsup_{n \to \infty} \sup_{h \in H_{n,b}} E_h \left[ \tau_M \left( \sqrt{n} \{ \tilde{\theta}_{m,1} - g(\beta_n(h)) \} \right) \right] 1 \{ \hat{s} = s \}
\]
\[
= \sup_{b \in [0, \infty)} \limsup_{n \to \infty} \sup_{h \in H_{n,b}} E_h \left[ \tau_M \left( \sqrt{n} \{ \tilde{\theta}_{m,1} - g(\beta_n(h)) \} \right) \right],
\]
where $\tilde{\theta}_{m,1} = \tilde{\theta}_{m,1} / a_1$. By Assumption 5, we have
\[
\sup_{h \in H_{n,b}} E_h \left[ \tau_M \left( \sqrt{n} \{ \tilde{\theta}_{m,1} - g(\beta_n(h)) \} \right) \right] \leq \sup_{h \in H_{n,b}} E_h \left[ \tau_M \left( \sqrt{n} |a_1| g(\hat{\beta}) + \hat{c}_{M_1}(s) / \sqrt{n} - g(\beta_n(h)) \right) \right].
\]
Fix $\varepsilon > 0$, choose $M_1 \geq M$, and follow the proof of Theorem 2 to find that the limsup of the last supremum is bounded by
\[
\sup_{r \in [-M_1,M_1]} E \left[ \tau_{M_1} (|a_1| \bar{g}_0(Z + r) - \bar{g}_0(r) + \tilde{c}_{M_1}^*(s)) \right] + \varepsilon.
\]
By the definition of $\tilde{c}_{M_1}^*(s)$, the last supremum is equal to
\[
\inf_{c \in [-M_1,M_1]} \sup_{r \in [-M_1,M_1]} E \left[ \tau_{M_1} (|a_1| \bar{g}_0(Z + r) - \bar{g}_0(r) + c) \right]
\]
\[
\leq \inf_{c \in [-M_1,M_1]} \sup_{r \in \mathbb{R}} E \left[ \tau (|a_1| \bar{g}_0(Z + r) - \bar{g}_0(r) + c) \right].
\]
Sending $M_1 \uparrow \infty$, we conclude that
\[
\sup_{b \in [0, \infty)} \limsup_{n \to \infty} \sup_{h \in H_{n,b}} E_h \left[ \tau_M (|\sqrt{n}\{\tilde{\theta}_{mx} - f(g(\beta_n(h)))\}|1 \{\hat{s} = s\} \right] \\
\leq \inf_{c \in \mathbb{R}^d} \sup_{0 \leq r \leq k} E [\tau(|a_1| |\tilde{g}_0(Z + r) - \tilde{g}_0(r) + c|)] .
\]

Second, suppose that $g(\beta_0) < \bar{x}$. Then we can use the same arguments as before to show the following:
\[
\sup_{b \in [0, \infty)} \limsup_{n \to \infty} \sup_{h \in H_{n,b}} E_h \left[ \tau_M (|\sqrt{n}\{\tilde{\theta}_{mx} - f(g(\beta_n(h)))\}|1 \{\hat{s} = s\} \right] \\
\leq \inf_{c \in \mathbb{R}^d} \sup_{0 \leq r \leq k} E [\tau(|a_2| |\tilde{g}_0(Z + r) - \tilde{g}_0(r) + c|)] .
\]

Finally, suppose that $g(\beta_0) = \bar{x}$. Then note that $f(\cdot) / s$ with $s = \max(|a_1|, |a_2|)$ is a contraction mapping. Hence
\[
\limsup_{n \to \infty} \sup_{h \in H_{n,b}} E_h \left[ \tau_M (|\sqrt{n}\{\tilde{\theta}_{mx} - f(g(\beta_n(h)))\}|1 \{\hat{s} = s\} \right] \\
= \limsup_{n \to \infty} \sup_{h \in H_{n,b}} E_h \left[ \tau_M (\sqrt{n}|\tilde{\theta} + \hat{c}_M(s)/\sqrt{n} - f(g(\beta_n(h)))|1 \{\hat{s} = s\} \right] \\
\leq \limsup_{n \to \infty} \sup_{h \in H_{n,b}} E_h \left[ \tau_M (\sqrt{n}|\tilde{\theta} + \hat{c}_M(s)/\sqrt{n} - g(\beta_n(h)))|1 \{\hat{s} = s\} \right] \\
= \limsup_{n \to \infty} \sup_{h \in H_{n,b}} E_h \left[ \tau_M (\sqrt{n}|\tilde{\theta} + \hat{c}_M(s)/\sqrt{n} - g(\beta_n(h)))|1 \{\hat{s} = s\} \right] ,
\]

by Lemma A8, where the inequality above is due to $f(\cdot)/s$ being a contraction mapping.

We fix $\varepsilon > 0$ and choose $M_1 \geq M$ and follow the proof of Theorem 2 to find that the
\[
\sup_{b \in [0, \infty)} \text{ of the last limsup is bounded by}
\inf_{c \in [-M_1, M_1] \in \mathbb{R}^d} \sup_{0 \leq r \leq k} E [\tau(|s| |\tilde{g}_0(Z + r) - \tilde{g}_0(r) + c|)] .
\]

By sending $M_1 \uparrow \infty$, we obtain the desired bound. ■

REFERENCES

[1] Begun, J. M., W. J. Hall, W.-M., Huang, and J. A. Wellner (1983): “Information and asymptotic efficiency in parametric-nonparametric models,” Annals of Statistics, 11, 432-452.
[2] Bickel, P. J. (1981): “Minimax estimation of the mean of a normal distribution when the parameter space is restricted,” Annals of Statistics, 9, 1301-1309.
[3] Bickel, P. J., A.J. Klaassen, Y. Ritov, and J. A. Wellner (1993): Efficient and Adaptive Estimation for Semiparametric Models, Springer Verlag, New York.
[4] Blumenthal, S. and A. Cohen (1968a): “Estimation of the larger translation parameter,” Annals of Mathematical Statistics, 39, 502-516.
[5] Blumenthal, S. and A. Cohen (1968b): “Estimation of the larger of two normal means,” Journal of the American Statistical Association, 63, 861-876.
[6] Casella G. and W. E. Strawderman (1981): “Estimating a bounded normal mean,” Annals of Statistics, 9, 870-878.
[7] Chamberlain, G. (1987): “Asymptotic efficiency in estimation with conditional moment restrictions,” Journal of Econometrics 34, 305-334.
[8] Charras, A. and C. van Eeden (1991): “Bayes and admissibility properties of estimators in truncated parameter spaces,” Canadian Journal of Statistics, 19, 121-134.
[9] Chernozhukov, V., H. Hong, and E. Tamer (2007): “Estimation and Confidence Regions for Parameter Sets in Econometric Models,” Econometrica 75, 1243-1284.
[10] Chernozhukov, V., S. Lee and A. Rosen (2013): “Intersection bounds: estimation and inference,” Econometrica 81, 667-737.
[11] Clarke, F. H. (1998): Nonsmooth Analysis and Control Theory, Springer, New York.
[12] Doss, H. and J. Sethuraman (1989): “The price of bias reduction when there is no unbiased estimate,” Annals of Statistics, 17, 440-442.
[13] Dudley, R. M. (2002): Real Analysis and Probability, Cambridge University Press, New York.
[14] Dvoretzky, A., A. Wald, and J. Wolfowitz (1951): “Elimination of randomization in certain statistical decision procedures and zero-sum two-person games,” Annals of Mathematical Statistics 22, 1-21.
[15] Haile, P. A. and E. Tamer (2003): “Inference with an incomplete model of English auctions,” Journal of Political Economy, 111, 1-51.
[16] Hájek, J. (1972): “Local asymptotic minimax and admissibility in estimation,” in L. Le Cam, J. Neyman and E. L. Scott, eds, Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, Vol 1, University of California Press, Berkeley, p.175-194.
[17] Hirano, K. and J. Porter (2012): “Impossibility results for nondifferentiable functionals,” Econometrica 80, 1769-1790.
[18] Le Cam, L. (1979): “On a theorem of J. Hájek,” in J. Jurečková, ed. Contributions to Statistics - Hájek Memorial Volume, Akademian, Prague, p.119-135.
[19] Lovell, M. C. and E. Prescott (1970): “Multiple regression with inequality constraints: pretesting bias, hypothesis testing, and efficiency,” Journal of the American Statistical Association, 65, 913-915.
[20] Mankiw, C. F. and J. Pepper (2000): “Monotone instrumental variables: with an application to the returns to schooling,” Econometrica 68, 997–1010.
[21] Milgrom, P. J. and R. J. Weber (1985): “Distributional strategies for games with incomplete information,” Mathematics of Operations Research, 10, 619-632.
[22] Moors, J. J. A. (1981): “Inadmissibility of linearly invariant estimators in truncated parameter spaces,” Journal of the American Statistical Association, 76, 910-915.
[23] Morizuti, S. (1951): “Extremal properties of extreme value distribution,” Annals of Mathematical Statistics, 22, 523-536.
[24] Puhalskii, A. and V. Spokoiny (1998): “On large-deviation efficiency in statistical inference,” Bernoulli, 4, 203-272.
[25] Shapiro, A. (1990): “On concepts of directional differentiability,” Journal of Optimization Theory and Applications 66, 477-487.
[26] Song, K. (2014): “Local Asymptotic Minimax Estimation of Nonregular Parameters with Translation-Scale Equivariant Maps,” Journal of Multivariate Analysis, 125, 136–158.
[27] Strasser, H. (1985): Mathematical Theory of Statistics, Walter de Gruyter, New York.
[28] TAKAGI, Y. (1994): “Local asymptotic minimax risk bounds for asymmetric loss functions,” *Annals of Statistics* 22, 39–48.

[29] VAN DER VAART, A. W. (1989): “On the asymptotic information bound,” *Annals of Statistics* 17, 1487-1500.

[30] VAN DER VAART, A. W. (1991): “On differentiable functionals,” *Annals of Statistics* 19, 178-204.

[31] VAN DER VAART, A. W. AND J. A. WELLNER (1996): *Weak Convergence and Empirical Processes*, Springer-Verlag, New York.

[32] VAN EEDEN, C., AND J. V. ZIDEK (2004): “Combining the data from two normal populations to estimate the mean of one when their means difference is bounded,” *Journal of Multivariate Analysis* 88, 19-46.
CORRIGENDUM TO “LOCAL ASYMPTOTIC MINIMAX ESTIMATION OF NONREGULAR PARAMETERS WITH TRANSLATION-SCALE EQUIVARIANT MAPS”: [J. MULTIVARIATE ANAL. 125 (2014) 136–158]

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First, the proof of Theorem 1 contains a gap in the equation on page 151:

\[ \min_{u \in T_K, N} \max_{r \in J_K} \int \tilde{g}_0(z + u) d\Lambda_r(z) = \min_{u \in T_K, N} \max_{r \in J_K} E\left[ \tau_M(|\tilde{g}_0(Z + r) - \tilde{g}_0(r) + u|) \right]. \]

(I thank Yoshiyasu Rai for pointing it out to me.) Theorem 1 still holds if we focus on convex loss functions, replacing Assumption 4 (i) on page 140 by the following:

**Assumption 4** (i) \( \tau(\cdot) \) is increasing and convex on \([0, \infty)\), \( \tau(0) = 0 \), and there exists \( \bar{\tau} \) such that \( \tau^{-1}([0, y]) \) is bounded in \([0, \infty)\) for all \( 0 < y < \bar{\tau} \).

Then Theorem 1 follows from Lemma 3 by Jensen’s inequality, because

\[
\inf_{F \in F} \sup_{r \in \mathbb{R}^d} \int E \left[ \tau(|\tilde{g}_0(Z + r) - \tilde{g}_0(r) + w|) \right] dF(w) \\
\geq \inf_{c \in \mathbb{R}} \sup_{r \in \mathbb{R}^d} E \left[ \tau \left( \left| \tilde{g}_0(Z + r) - \tilde{g}_0(r) + \int wdF(w) \right| \right) \right] = \inf_{c \in \mathbb{R}} B(c; 1).
\]

Second, the last equality on page 155 in the proof of Theorem 2 as follows:

\[
\sup_{r \in [-M_1, M_1]^d} \mathbb{E} \left[ \tau_{M_1}(|\tilde{g}_0(Z + r) - \tilde{g}_0(r) + c_M^*(1)|) \right] \\
= \inf_{-M_1 \leq c \leq M_1} \sup_{r \in [-M_1, M_1]^d} \mathbb{E} \left[ \tau_{M_1}(|\tilde{g}_0(Z + r) - \tilde{g}_0(r) + c|) \right]
\]

assumes that the set \( E_{M_1}(a) \) is convex, which is not guaranteed. (I thank Zheng Fang for pointing it out to me.) Note that the set \( E_{M_1}(a) \) (defined in the first display on page 152) is compact. Hence the results of the paper including Theorem 2 follow once we redefine

\[
\hat{c}_{M_1}(a) \equiv \sup \hat{E}_{M_1}(a), \text{ in (3.4) on page 141, and} \\
c_M^*(a) \equiv \sup E_{M_1}(a), \text{ in Line 3 on page 152.}
\]

Modifying the definitions using infimum in place of supremum works as well.
I apologize for any inconvenience caused by these gaps.

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