Notes on

RISK THEORY

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1 Introduction

Risk theory is the part of insurance mathematics that is concerned with stochastic models for the flow of payments in an insurance business. The purpose of an insurance is in general to level out fluctuations in the cost for the policyholder and to replace the often strongly varying cost with a more predictable flow of payments. To achieve this, a large group of risks – a “collective” – is created in which the costs of an individual member can be highly stochastic, but where the total cost is levelled out as a consequence of the law of large numbers.

In these lecture notes we will describe some basic natural models for “risk processes” and derive various types of asymptotic laws for the fluctuations in the amount of loss. We will also investigate how the fluctuations depend on variables such as reserve capital, premium amount, reinsurance arrangements, size of the collective and distribution of the included variables. Models for both life and property insurance will be considered.

One can distinguish between two different types of risks: the insurance risk and the uncertainty concerning the future returns from the collected reserve capital. These notes will mainly be concerned with the former type of risk, which is generally better known from a statistical point of view because it changes slower over time so that observed losses can be expected to be relevant in predicting future losses. Also, an important difference between the risk types is that uncertainty in for instance the development of the interest can not be levelled out in the same way as the first type of risk, since it can not be decomposed as a sum of many contributions, obeying the law of large numbers. However it is of interest to model the influence of both risk types and indeed the substantial development of finance mathematics during the last years has resulted in several models for financial risks. In recent research these models are combined with models from traditional risk theory in an interesting way, and new types of contracts are being analyzed.

Risk theory as a branch of probability has a long tradition, particularly within Swedish insurance research. Some of the models that we will be interested in were formulated already in the beginning of the 20th century in works by Filip Lundberg and Harald Cramér, and the theory of ruin probabilities that we will consider was developed in the 1930-50’s by Cramér, Esscher, Segerdahl and Arfwedson among others. This research inspired the development of the theory for stochastic processes, and during the 1960-80’s it has turned out that many problems in queuing theory, storage theory and risk theory are closely related and can be solved by the same methods. This has resulted in several simplifications of the theory in that technically complicated analytical methods have been replaced by probabilistic techniques which are more intuitive. In these notes we will, as far as possible, use these probabilistic methods.
2 Stochastic models for the total amount of loss during a fixed period

In risk theory there are two basic models for the amount of loss in an insurance collective: the individual model and the collective model. Both these models are described in this section. We also derive approximations for tail probabilities for the distribution of the total amount of loss.

2.1 The individual risk model

In this model we consider a (large) number of individual policies - for instance we can think of whole life assurances - that are in effect during, let’s say, one financial year. For each of the policies there is a (small) probability \( p_i \) that a loss occurs, and a probability \( q_i = 1 - p_i \) that no loss occurs. If a loss occurs the amount \( x_i \) is payed to the policyholder, where \( x_i \) is specified in the agreement. The losses are assumed to be independent. Let \( \{ M_i \} \) be independent Bernoulli-variables with \( P(M_i = 1) = 1 - P(M_i = 0) = p_i \). Then the individual amount of loss can be written as \( x_i M_i \) and the total loss is given by \( X := \sum_i x_i M_i \).

Since the total loss is a sum of independent random variables, it is natural to define its distribution via the generating function \( E[e^{\xi X}] \), which is the product of the individual generating functions, that is,

\[
E[e^{\xi X}] = \prod_i E[e^{\xi x_i M_i}]
= \prod_i (q_i + p_i e^{\xi x_i}).
\]

The mean and variance of the individual losses are \( E[x_i M_i] = x_i p_i \) and \( \text{Var}(x_i M_i) = x_i^2 p_i q_i \), implying that \( E[X] = \sum_i x_i p_i \) and \( \text{Var}(X) = \sum_i x_i^2 p_i q_i \). Now, since \( X \) is a sum of independent random variables, a natural approach might be to approximate its distribution with a normal distribution with these parameters, that is, one could believe that

\[
P \left( \frac{X - E[X]}{\sqrt{\text{Var}(X)}} > x \right) \approx 1 - \Phi(x).
\]

However, this approximation often turns out to be quite poor because of the fact that \( p_i \) is typically very small so that rather few losses occur even when the number of policies is large. In such a situation it is more natural to approximate the distribution of \( X \) with a so called compound Poisson distribution, which is constructed as follows: Let \( \{ N_i \} \) be independent Poisson distributed variables with \( E[N_i] = \lambda_i \), that is,

\[
P(N_i = n) = \frac{\lambda_i^n}{n!} e^{-\lambda_i}.
\]
Pick $\lambda_i$ so that $P(N_i = 0) = q_i$ and put
\[
M_i = \begin{cases} 
0 & \text{if } N_i = 0, \\
1 & \text{if } N_i \geq 1
\end{cases}
\]
Then $P(M_i = 0) = 1 - P(M_i = 1) = q_i$ and hence $M_i$ has the right distribution. Moreover, when $p_i$ is small, $M_i = N_i$ with large probability. To see this, note that
\[
P(M_i \neq N_i) = P(N_i \geq 2) = 1 - P(N_i = 0) - P(N_i = 1) = 1 - e^{-\lambda_i} - \lambda_i e^{-\lambda_i} \\
\approx 1 - (1 - \lambda_i + \lambda_i^2/2) - \lambda_i(1 - \lambda_i) = \lambda_i^2/2.
\]
By the choice of $\lambda_i$, we have $1 - p_i = e^{-\lambda_i}$, and, since $e^{-\lambda_i} \approx 1 - \lambda_i$, it follows that $p_i \approx \lambda_i$. Hence $P(M_i \neq N_i) \approx p_i^2/2$ when $p_i$ is small. In this situation it is natural to approximate $X$ with $S := \sum_i x_i N_i$. This quantity has a compound Poisson distribution and, since
\[
P(M_i \neq N_i \text{ for some } i) \leq \sum_i P(M_i \neq N_i) = O\left( \sum_i p_i^2 \right),
\]
the approximation is good if $\sum_i p_i^2$ is small.

Just as the distribution of $X$, the distribution of $S$ can be defined via its generating function. Remember that the generating function for a Poisson distributed variable is given by
\[
E[e^{\xi N_i}] = \sum_{n=0}^{\infty} e^{\xi n} \frac{\lambda^n}{n!} e^{-\lambda_i} = \exp\left\{ \lambda_i \left( e^\xi - 1 \right) \right\}.
\]
Since $\{N_i\}$ are independent, we have
\[
E[e^{\xi S}] = \prod_i E[e^{\xi x_i N_i}] = \prod_i \exp \left\{ \lambda_i \left( e^{\xi x_i} - 1 \right) \right\} = \exp \left\{ \sum_i \lambda_i \left( e^{\xi x_i} - 1 \right) \right\}.
\]
Introduce the notation $g(\xi) = \sum_i \lambda_i \left( e^{\xi x_i} - 1 \right)$. We then have $E[e^{\xi S}] = e^{g(\xi)}$ or, equivalently, $g(\xi) = \log E\left( e^{\xi S} \right)$. In what follows we will derive approximations
for the distribution of \( S \) and thereby hopefully also for the distribution of \( X \).
In this context it is worth noting that \( M_i \leq N_i \) for all \( i \) and hence \( X \leq S \) if \( x_i > 0 \) for all \( i \). This implies that \( P(X > x) \leq P(S > x) \) and thus, if we can find an upper bound for \( P(S > x) \), then this bound is valid also for \( P(X > x) \).

The function \( g(\xi) \) can be expressed in a slightly different way using the so called *risk mass distribution*, \( F(dx) \). Let \( \lambda = \sum_i \lambda_i \) and construct \( F(dx) \) by placing the mass \( \lambda_i/\lambda \) at the point \( x_i \) on the x-axis, \( i = 1, 2, 3 \ldots \), as demonstrated in Figure 1. We then have

\[
g(\xi) = \lambda \int_0^{\infty} (e^{\xi x} - 1) F(dx) \quad \text{and} \quad \mathbb{E}[e^{\xi S}] = e^{g(\xi)}. \tag{1}
\]

More generally we can consider \( g(\xi) \) and \( S \) defined in this way with an arbitrary probability distribution \( F(dx) \) and some constant \( \lambda < \infty \). The distribution of \( S \) is then called a compound Poisson distribution and the following proposition gives a fundamental characterization for \( S \).

**Proposition 2.1** Let \( \{X_k\} \) be independent random variables with distribution \( F(dx) \) and let \( N \) be a Poisson distributed variable, independent of \( \{X_k\} \), with \( \mathbb{E}[N] = \lambda \). Define \( S = \sum_{k=1}^N X_k \). Then \( S \) has a compound Poisson distribution defined by (1).

**Proof:** Let \( f(\xi) \) be the generating function of the distribution \( F(dx) \), that is,

\[
f(\xi) = \mathbb{E}[e^{\xi X_k}] = \int_0^{\infty} e^{\xi x} F(dx).
\]

For each fixed \( n \), the sum \( S_n := \sum_{k=1}^n X_k \) has distribution \( F^{n*}(dx) \) – the convolution of \( F \) with itself \( n \) times – with generating function \( \mathbb{E}[e^{\xi S_n}] = f^n(\xi) \). Hence, \( P(S \in dx|N = n) = F^{n*}(dx) \), and, summing over the possible values of \( N \), we obtain

\[
P(S \in dx) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} e^{-\lambda} F^{n*}(dx). \tag{2}
\]
The corresponding generating function is

\[
E[e^{\xi S}] = \sum_{n=0}^{\infty} E[e^{\xi S}|N = n] P(N = n)
= \sum_{n=0}^{\infty} f^n(\xi) \frac{\lambda^n}{n!} e^{-\lambda}
= e^{\lambda(\lambda(\xi) - 1)}.
\]

With \(g(\xi)\) defined as in (1), we have \(\lambda(\lambda(\xi) - 1) = g(\xi)\) and hence \(E[e^{\xi S}] = e^{g(\xi)}\), as desired.

2.2 The collective risk model

In the individual risk model for a portfolio of whole life assurances, the collective is changed over time as more and more policyholders die. However, for moderate times and large collectives this effect can often be neglected. A natural approximation then is to consider a collective that is stationary in time in the sense that \(\lambda\) and \(F(dx)\) are constant and the number of losses in a time interval of length \(t\) is Poisson distributed with expected value \(\lambda t\), the number of losses in disjoint time intervals being independent. Below we give a description of the total loss process \(S(t)\) in the interval \((0, t]\) motivated by this observation.

Assume that the losses occur at time points \(T_1, T_2, \ldots\) that constitute a Poisson process in time, that is, the increments \(Y_k := T_k - T_{k-1}\) are independent and exponentially distributed with density \(\lambda e^{-\lambda y} dy\). At each time of loss \(T_k\), an amount of damage \(X_k > 0\) is generated. The variables \(\{X_k\}\) are assumed to be independent with distribution \(F(dx)\) and the total loss in \((0, t]\) is given by \(S(t) := \sum_{T_k \in (0, t]} X_k\). As illustrated in Figure 2, the process \(S(t)\) is a step function with jumps of height \(X_k\) at the times \(T_k\).

To specify the distribution of \(S(t)\), let \(N(t)\) denote the number of losses in the interval \((0, t]\). We then have \(S(t) = \sum_{k=1}^{N(t)} X_k\). The process \(\{N(t)\}_{t>0}\) is a Poisson process with independent increments in disjoint intervals and hence the increments of \(S(t)\) — that is, the sums of the amounts of loss in disjoint intervals — are also independent. Furthermore, since

\[
P(N(t) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t},
\]

by proceeding as in the derivation of (2), we obtain

\[
P(S(t) \in dx) = \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda t} F^n(dx).
\]

This means that, just like \(S\) in the previous subsection, \(S(t)\) has a compound Poisson distribution and hence its generating function is given by
where, as before, $f(\xi)$ is the generating function of the distribution $F(dx)$ and $g(\xi) = \lambda \int_0^\infty (e^{\xi x} - 1) F(dx)$.

The above formulas define the collective risk model, which will be thoroughly studied in the following. The model can be used to describe both a life assurance business and a property insurance business. The total loss process $\{S(t)\}$ has independent stationary increments with a compound Poisson distribution defined by $\lambda$ and $F(dx)$ and the expected value and variance of $S(t)$ can be obtained by differentiating the generating function. Introducing the notation $\mu = \int_0^\infty x F(dx)$ and $\nu = \int_0^\infty x^2 F(dx)$, we get

$$E[S(t)] = tg'(0) = t\lambda \int_0^\infty x F(dx) = t\lambda \mu$$

and

$$\text{Var}(S(t)) = tg''(0) = t\lambda \int_0^\infty x^2 F(dx) = t\lambda \nu.$$
This so called Panjer-recursion is easy to implement numerically and is widely used. To describe it, assume for simplicity that \( t = 1 \) and write \( S(1) = S \). Also, let \( f_x := P(X_k = x) \) (\( x = 1, 2, 3, \ldots \)) and \( g_y := P(S = y) \) (\( y = 0, 1, 2, \ldots \)). Here the probabilities \( \{f_x\} \) are assumed to be known and we want to calculate \( \{g_y\} \).

To this end, introduce the generating functions

\[
\varphi(s) := \sum_{x=1}^{\infty} s^x f_x \quad \text{and} \quad \gamma(s) := \sum_{y=0}^{\infty} s^y g_y.
\]

Since \( f(\xi) = E \left[ e^{\xi X_k} \right] = \sum_x e^{\xi x} f_x \), we have \( f(\xi) = \varphi(e^\xi) \). Now let \( \xi \) and \( s \) be related in that \( s = e^\xi \). Then \( f(\xi) = \varphi(s) \) and, since \( \gamma(s) = E \left[ e^{\lambda S} \right] = e^{\lambda f(\xi) - 1} \), we get

\[
\gamma(s) = e^{\lambda (\varphi(s) - 1)}.
\]

Differentiating this relation we obtain \( \gamma'(s) = \lambda \varphi'(s) \gamma(s) \) or, more explicitly,

\[
\gamma'(s) = \lambda \sum_{x=1}^{\infty} x f_x s^{x-1} \sum_{y=0}^{\infty} g_y s^y
\]

\[
= \lambda \sum_{x=1}^{\infty} \sum_{y=0}^{\infty} x f_x g_y s^{x+y-1}
\]

\[
= \lambda \sum_{n=1}^{\infty} s^{n-1} \sum_{x=1}^{n} x f_x g_{n-x}.
\]

But we also have \( \gamma'(s) = \sum_{n=1}^{\infty} n g_n s^{n-1} \). Equating these two expressions for \( \gamma'(s) \) yields

\[
g_n = \lambda \sum_{x=1}^{n} x f_x g_{n-x}, \quad n = 1, 2, 3, \ldots
\]

The probability \( g_0 \) is determined by noting that \( g_0 = \gamma(0) = e^{\lambda \varphi(0) - 1} = e^{-\lambda} \), where the last equality follows since \( \varphi(0) = 0 \). Given \( g_0 \), the probabilities \( \{g_n\}_{n \geq 1} \) are then successively obtained from the equations (3). We get

\[
g_1 = \lambda f_1 g_0
\]

\[
g_2 = \lambda (f_1 g_1 + 2 f_2 g_0)/2
\]

\[
\vdots
\]

\[
g_n = \lambda (f_1 g_{n-1} + 2 f_2 g_{n-2} + \ldots + n f_n g_0)/n.
\]

As described above, an important quantity is \( G_m := P(S > m) \), \( m \geq 0 \). Noting that \( G_m = \sum_{n=1}^{\infty} g_n \), the probabilities \( \{G_n\} \) can be calculated together with \( \{g_n\} \) using the formula \( G_m = G_{m-1} - g_m \), with \( G_{-1} = 1 \). Finally we remark that, if the \( X_k \)’s are not integer-valued, they can be approximated by some suitable discretization and the Panjer-recursion can then be applied to this distribution.
2.4 Approximations of $P(S(t) > tx)$

In this section we derive two useful approximations of $P(S(t) > tx)$. They both involve the generating function $g(\xi)$ and are fairly easy to calculate when this function is known.

2.4.1 Chernoff bound

The first approximation is based on an inequality, Chernoff’s inequality, that is used in many statistical contexts. To derive it, introduce the notation $F(t, dx) = P(S(t) \in dx)$, fix $\xi \geq 0$, and note that

$$e^{t\xi} = \mathbb{E}[e^{\xi S(t)}] = \int_{0}^{\infty} e^{\xi x} F(t, dx) \geq e^{t\xi} \int_{tx}^{\infty} F(t, dy) = e^{t\xi} P(S(t) \geq tx).$$

Consequently we have $P(S(t) \geq tx) \leq e^{-t(x\xi - g(\xi))}$ for all $\xi \geq 0$. Clearly the best upper bound is obtained if $\xi \geq 0$ is picked so that $x\xi - g(\xi)$ is maximized. Define

$$h(x) = \max_{\xi} \{x\xi - g(\xi)\}$$

and write $\xi_x$ for the maximizing $\xi$-value. We then have

$$P(S(t) \geq tx) \leq e^{-th(x)} \quad \text{if } \xi_x \geq 0.$$ 

Analogously, it can be seen that

$$P(S(t) \leq tx) \leq e^{-th(x)} \quad \text{if } \xi_x \leq 0.$$ 

The function $h(x)$ will play an important role in what follows, and we need to study its properties a bit closer. To this end, first consider the function $g(\xi) = \lambda \int_{0}^{\infty} (e^{\xi x} - 1) F(dx)$. We will assume that $g(\xi) < \infty$ for $\xi < \xi$, where $\xi > 0$, that $g(\xi) \to \infty$ as $\xi \to \xi$, and also that $g'(\xi) \to \infty$ as $\xi \to \xi$. Since $g'(\xi) = \lambda \int_{0}^{\infty} x e^{\xi x} F(dx)$ and $g''(\xi) = \lambda \int_{0}^{\infty} x^2 e^{\xi x} F(dx)$ are both positive, the derivative $g'(\xi)$ increases monotonically from 0 to $\infty$ as $\xi$ increases from $-\infty$ to $\xi$. Hence $g(\xi)$ is strictly convex and increases from $-\lambda$ to $\infty$ for these $\xi$-values, see Figure 3(a).

Now consider the function $h(x)$. The maximizing value $\xi_x$ must satisfy $g'(\xi_x) = x$ and, since $g'(x)$ is strictly increasing and continuous, for each $x$ this equation has exactly one solution $\xi_x \leq \xi$. Furthermore, the fact that $g'(\xi)$ is strictly
increasing also implies that \( \xi_x \geq 0 \) if and only if \( x = g'(\xi_x) \geq g'(0) = \lambda \mu \).

Hence Chernoff’s inequalities tells us that

(i) \( P(S(t) \geq tx) \leq e^{-th(x)} \) if \( x \geq \lambda \mu ; \)

(ii) \( P(S(t) \leq tx) \leq e^{-th(x)} \) if \( x \leq \lambda \mu \).

A picture of the geometrical construction of the function \( h(x) \) is shown in Figure 3(b). Consider the problem of finding a tangent \(-h + x\xi \), with given slope \( x > 0 \), to the curve \( g(\xi) \). The tangent point \( \xi_x \) satisfies \( g'(\xi_x) = x \) and \( h = h(x) \) is determined so that \(-h + x\xi_x = g(\xi_x) \), that is, we have \( h(x) = x\xi_x - g(\xi_x) \). As can be seen in the figure, \( h(x) \geq 0 \) for all \( x > 0 \) and \( h(x) = 0 \) when \( \xi_x = 0 \). The geometrical construction can be thought of as if a line \(-h + x\xi \), with \( x \) fixed, is pushed upwards towards the curve \( g(\xi) \) until a point is found where the line coincides with the tangent of the curve. This means that we are looking for the smallest value of \( h \) such that \(-h + x\xi \leq g(\xi) \) for all \( \xi \), that is, such that \( h \geq x\xi - g(\xi) \) for all \( \xi \). Hence the critical value is \( h(x) = \max_{\xi} \{x\xi - g(\xi)\} \).

The derivative of \( h(x) \) is

\[
 h'(x) = \frac{d}{dx}(x\xi_x - g(\xi_x)) \\
= \xi_x + \frac{d\xi_x}{dx}(x - g'(\xi_x)) \\
= \xi_x,
\]

where the last equality follows because \( x = g'(\xi_x) \). The relation \( x = g'(\xi_x) \) between \( x \) and \( \xi_x \) is 1-1 and differentiable. We have \( \frac{d\xi_x}{dx} = g''(\xi_x) \), and, since \( g''(\xi) > 0 \), it follows that \( \frac{d\xi_x}{dx} = 1/g''(\xi_x) \). Using this, we get

\[
 h''(x) = \frac{d\xi_x}{dx} = \frac{1}{g''(\xi_x)} > 0,
\]

which means that \( h(x) \) is also strictly convex. Remembering that \( h(x) \geq 0 \) for all \( x \), and \( h(\lambda \mu) = 0 \), we can draw \( h \) as in Figure 3(c).

The relation between \( g(\xi) \) and \( h(x) \) can be inverted. For \( \xi = \xi_x \), we have \( g(\xi) = x\xi - h(x) \) and \( \xi = h'(x) \). This implies that \( g(\xi) \) is given by the formula \( g(\xi) = \max_x \{x\xi - h(x)\} \), which is analogous to the formula for \( h(x) \). In the theory for convex functions this relation is well-known and \( g(\xi) \) and \( h(x) \) are said to be each others Legendre transforms.

Since \( h(x) > 0 \) for \( x \neq \lambda \mu \), Chernoff’s inequalities tells us that, when \( x > \lambda \mu \) is fixed, the probability of the event \( \{S(t)/t \geq x\} \) decays exponentially as \( t \to \infty \). Such exponential estimates are common in the theory of large deviations. Here “deviations” refer to deviations from the mean and “large” refers to the fact that the deviations are large compared to the deviations treated by the central limit theorem, where \( x = \lambda \mu + y/\sqrt{t} \) as \( t \to \infty \). The function \( h(x) \) that measures
Figure 3: The functions $g(\xi)$ and $h(x)$. 

(a) $g(\xi)$

(b) Construction of $h(x)$.

(c) $h(x)$
the decay of the deviation probability is a fundamental object. It is called the entropy function of the distribution $F(dx)$. Let us give some examples of how it is calculated for different distributions.

1. The exponential distribution, $F(dx) = e^{-x}dx$: For $\xi < 1$, we have

$$g(\xi) = \lambda \int_0^{\infty} (e^{\xi x} - 1)e^{-x}dx$$
$$= \lambda \left( \frac{1}{1 - \xi} - 1 \right)$$
$$= \frac{\lambda \xi}{1 - \xi}.$$ 

This yields $g'(\xi) = \lambda/(1 - \xi)^2$ and the equation $x = g'(\xi)$ hence becomes $1 - \xi = \sqrt{\lambda/x}$. Thus

$$h(x) = x\xi - g(\xi)$$
$$= x \left( 1 - \sqrt{\frac{\lambda}{x}} \right) - \lambda \left( \sqrt{\frac{x}{\lambda}} - 1 \right)$$
$$= x - 2\sqrt{\lambda x} + \lambda$$
$$= \lambda \left( \sqrt{\frac{x}{\lambda}} - 1 \right)^2.$$

2. The one-point distribution $F(dx) = \delta(x - 1)dx$ gives $g(\xi) = \lambda(e^{\xi} - 1)$ and $g'(\xi) = \lambda e^{\xi}$. Putting $x = g'(\xi)$, we get $\xi = \log(x/\lambda)$ and hence

$$h(x) = x \log \left( \frac{x}{\lambda} \right) - \lambda \left( \frac{x}{\lambda} - 1 \right)$$
$$= \lambda \left[ \frac{x}{\lambda} \log \left( \frac{x}{\lambda} \right) - \frac{x}{\lambda} + 1 \right].$$

3. The Gamma distribution with $\mu = a$, $F(dx) = \gamma_a(x)dx$, gives $g(\xi) = \lambda((1 - \xi)^{-a} - 1)$ and $g'(\xi) = \lambda a(1 - \xi)^{-(a+1)}$. The relation $x = g'(\xi)$ implies that

$$(1 - \xi) = \left( \frac{\lambda a}{x} \right)^{1/(1+a)}$$

and hence

$$h(x) = x \left( 1 - \left( \frac{\lambda a}{x} \right)^{1/(a+1)} \right) - \lambda \left( \left( \frac{x}{\lambda a} \right)^{a/(a+1)} - 1 \right)$$
$$= \lambda \left[ \frac{x}{\lambda} - \left( \frac{x}{\lambda} \right)^{a/(a+1)} \left( a^{1/(a+1)} + a^{-a/(a+1)} \right) + 1 \right]$$.
2.4.2 Esscher’s approximation

The second approximation of \( P(S(t) > tx) \) is the so called *Esscher-approximation*, which is an asymptotic formula, valid as \( t \to \infty \). It states that

\[
P(S(t) > tx) \approx \frac{C}{\sqrt{t}} e^{-th(x)} \quad \text{as } t \to \infty
\]

in the sense that the quotient between the left hand side and the right hand side tends to 1. Here \( C > 0 \) is a constant, and the correction factor \( C/\sqrt{t} \) gives a more precise estimate of the exponential decay derived in the previous section.

We will see that in many cases this formula gives a good approximation also for moderate values of \( t \) and that it is easy to calculate numerically if the function \( g(\xi) \) is available.

Since the process \( \{S(t)\} \) has independent increments, it obeys the central limit theorem, that is, \( (S(t) - t\lambda\mu)/\sqrt{t\lambda\nu} \) is approximately normally distributed as \( t \to \infty \). This means that, with \( x = \lambda\mu + y\sqrt{\lambda\nu/t} \), we have that

\[
P(S(t) > tx) \to 1 - \Phi(y) \quad \text{as } t \to \infty,
\]

where \( \Phi(y) \) denotes the standard normal distribution function. The central limit theorem hence gives an approximation for “normal” deviations – that is, deviations of the form \( y\sqrt{\lambda\nu/t} \) from the mean \( \lambda\mu \). However, if we want to study “large” deviations, with \( x > \lambda\mu \) fixed as \( t \to \infty \), then this approximation is not sufficient. Below we will see that this problem can be circumvented by modifying the distribution \( F(dx) \) – and thereby also the distribution of \( S(t) \) – so that it becomes centered at the value \( tx \) that we are interested in. The central limit theorem can then be applied to the transformed distribution to get an approximation that can be used also for the original distribution close to the value \( tx \).

The modification of the distribution \( F(dx) \) that we will use is called the *Esscher-transform*. It is obtained by introducing a distribution that is proportional to \( e^{ax} \) with respect to \( F(dx) \), where \( a \) is a parameter that can be chosen freely. To be more precise, we embed \( F(dx) \) in an exponential family by defining

\[
F_a(dx) = \frac{e^{ax}}{f(a)} F(dx),
\]

where \( f(a) = \int_0^\infty e^{ax} F(dx) \). For \( a < \xi \) we have \( f(a) < \infty \) and hence \( F_a(dx) \) is a probability distribution. Now let \( F_a(dx) \) be the modified distribution of \( \{X_k\} \). It defines a different distribution of \( S_n = \sum_1^n X_k \). Write \( P_a(\cdot) \) for the modified probabilities and \( E_a[\cdot] \) for the corresponding means. Furthermore, let \( f_a(\xi) := E_a[e^{\xi X_1}] \) denote the generating function of \( F_a(dx) \). We then have

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\[ f_a(\xi) = \int_0^\infty e^{\xi x} f_a(dx) \]
\[ = \int_0^\infty e^{\xi x} \frac{e^{ax}}{f(a)} F(dx) \]
\[ = \frac{f(\xi + a)}{f(a)} . \]  \hspace{1cm} (4)

Since the \(X_k\)'s are independent also under the measure \(P_a\), the distribution of \(S_n\) under this measure is given by \(F_n^* (dx) - \) the convolution of \(F_n\) with itself \(n\) times. Hence \(E_a[e^{\xi S_n}] = \int_0^\infty e^{\xi x} F_n^*(dx)\). But, using (4), we also have

\[ E_a[e^{\xi S_n}] = \frac{f^n(\xi + a)}{f^n(a)} \]
\[ = \frac{E[e^{(a+\xi)S_n}]}{f^n(a)} \]
\[ = \int_0^\infty e^{\xi x} \frac{e^{ax}}{f^n(a)} F^n^*(dx) . \]

Thus

\[ F_n^*(dx) = \frac{e^{ax}}{f^n(a)} F^n^*(dx) , \]

that is, \(F_n^*\) is the Esscher-transform of \(F_n^*\). This means that the original distribution of \(S_n\) can be expressed in terms of the modified one via the relation \(F_n^*(dx) = f^n(a)e^{-ax}F_n^*(dx)\). Hence, if we can approximate \(F_n^*(dx)\) for some choice of \(a\), we can also approximate \(F_n^*(dx)\) via this relation. The reason for picking an exponential density for \(\{X_k\}\) is that this is the only case when the transformed distribution of \(S_n\) is obtained by applying the same transform to the original distribution of \(S_n\).

Now let us make an analogous transformation of \(S(t)\). Write \(P_a(S(t) \in dx) = F_a(t, dx)\) and define

\[ F_a(t, dx) = e^{ax-tg(a)} F(t, dx) . \]

Remembering that \(E[e^{\xi S(t)}] = e^{tg(\xi)}\), we then have

\[ \int_0^\infty F_a(t, dx) = e^{-tg(a)} \int_0^\infty e^{ax} F(t, dx) \]
\[ = e^{-tg(a)} \mathbb{E}[e^{aS(t)}] \]
\[ = 1 \]

so that \(F_a(t, dx)\) is indeed a probability distribution. The generating function is given by
\[ E_a \left[ e^{\xi S(t)} \right] = \int_0^\infty e^{\xi x} e^{ax - tg(a)} F(t, dx) \]
\[ = E_a \left[ e^{(a+\xi)S(t) - tg(a)} \right] \]
\[ = e^{tg(a+\xi) - g(a)} . \]

Hence we have \( E_a \left[ e^{\xi S(t)} \right] = e^{tg_a(\xi)} \), where \( g_a(\xi) = g(a+\xi) - g(a) \). This means that \( S(t) \) is still a compound Poisson process, since

\[ g_a(\xi) = \lambda \int_0^\infty \left( e^{(a+\xi)x} - e^{ax} \right) F(dx) \]
\[ = \lambda f(a) \int_0^\infty \left( e^{\xi x} - 1 \right) F_a(dx). \]

We thus have the important relation that, under the measure \( P_a \), \( S(t) \) has a compound Poisson distribution with \( \lambda_a = \lambda f(a) \), jump distribution \( F_a(dx) \) and generating function \( g_a(\xi) = g(a+\xi) - g(a) \). The last equation immediately gives us the mean and variance. We have

\[ E_a[S(t)] = tg'_a(0) = tg'(a) \]

and

\[ \text{Var}_a(S(t)) = tg''_a(0) = tg''(a). \]

Just as for \( S_a \), we have a simple expression for \( F(t, dx) \) in terms of \( F_a(t, dx) \), namely

\[ F(t, dx) = e^{tg(a) - ax} F_a(t, dx). \]

We will now see how this expression can be used to study large deviations for \( S(t) \). Consider the probability \( P(S(t) \geq tx) \) with \( x > \lambda \mu \). Center \( P_a \) by choosing \( a \) such that \( E_a[S(t)] = tx \), that is, such that \( g'(a) = x \). We have previously seen that this equation has a strictly positive unique solution if \( x > \lambda \mu = g'(0) \).

The central limit theorem can now be used to approximate the distribution of \( S(t) \) under the measure \( P_a \) near its mean \( tx \): Put \( S(t) = tg'(a) + Y \). Then, as \( t \to \infty \), the distribution of \( Y \) is approximately normal with mean 0 and variance \( \sigma^2 = tg''(a) \). Furthermore,

\[ P(S(t) \geq tx) = \int_{tx}^\infty F(t, dy) \]
\[ = \int_{tx}^\infty e^{tg(a) - ay} F_a(t, dy) \]
\[ = e^{tg(a)} E_a \left[ e^{-a(tg'(a) + Y)}, Y \geq 0 \right] \]
\[ = e^{t(g(a) - ag'(a))} E_a \left[ e^{-aY}, Y \geq 0 \right] . \]
In the previous section we saw that, when \( x = g'(a) \), we have \( ag'(a) - g(a) = h(x) \) and hence we arrive at the fundamental formula

\[
P(S(t) \geq tx) = e^{-th(x)}E_{a}[e^{-aY}, Y \geq 0].
\]

Now, if \( Y \) has a density, the normal approximation for \( Y \) implies that

\[
E_a(e^{-aY}, Y \geq 0) \approx \int_{0}^{\infty} e^{-ay} \phi \left( \frac{y}{\sigma} \right) \frac{dy}{\sigma} = \int_{0}^{\infty} e^{-a\sigma y} \phi(y) dy,
\]

where \( \phi(y) = e^{-y^2/2}/\sqrt{2\pi} \) denotes the normal density. In the literature,

\[
E(s) = \int_{0}^{\infty} e^{-sy} \phi(y) dy = e^{s^2/2} \int_{0}^{\infty} e^{-(s+y)^2/2} dy/\sqrt{2\pi} = e^{s^2/2}(1 - \Phi(s))
\]

is referred to as the Esscher function and, in terms of this function we have now derived Esscher’s approximation formula, which states that

\[
P(S(t) \geq tx) \approx e^{-th(x)}E(a\sigma) \quad \text{as } t \to \infty,
\]

where \( a > 0 \) is determined by the relation \( x = g'(a) \) and \( \sigma = \sqrt{tg''(a)} \). The formula is only valid if \( F(dx) \) has a density, but later we will see that there is a similar approximation if \( F(dx) \) has a discrete distribution.

As \( t \to \infty \), the same holds for \( s \), and from the definition of \( E(s) \) we see that, for large \( s \), the exponential function is quickly damped as \( y \) grows so that only values near \( y = 0 \) are essential. Near \( y = 0 \), we have \( \phi(y) \approx (1 - y^2/2)/\sqrt{2\pi} \) and hence

\[
E(s) \approx \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-sy} \left( 1 - \frac{y^2}{2} \right) dy = \frac{1}{\sqrt{2\pi}} \left( \frac{1}{s} - \frac{1}{s^2} \right).
\]

Thus we have the more explicit formula

\[
P(S(t) \geq tx) \approx \frac{e^{-th(x)}}{\sqrt{2\pi a\sqrt{tg''(a)}}}, \quad x = g'(a), \quad (5)
\]

which is also referred to as Esscher’s approximation. The formula is reasonably easy to implement numerically provided that it is possible to compute the function \( g(a) \) and its derivatives. If \( x = g'(a) \), \( h(x) = ag'(a) - g(a) \) and \( \sigma = \sqrt{tg''(a)} \)
are computed for sufficiently many values of \( a > 0 \), the Esscher approximations can also be computed and thereby we have an approximation for sufficiently many values of \( x \). This method gives an approximation that is good enough for all distributions that occur in practice.

Even if the condition that \( F(dx) \) has a density is not fulfilled, it is possible to derive an analogous approximation formula when \( F(dx) \) is a discrete distribution such that \( X_k \) takes values on the form \( nd \), for some constant \( d \) and \( n = 0, 1, 2, \ldots \). To do this, note that if \( X_k \in \{ nd; n = 0, 1, 2, \ldots \} \), the same thing holds for \( S(t) \). The normal approximation for \( Y \) becomes

\[
P(Y = y) \approx \varphi \left( \frac{y}{\sigma} \right) \frac{d}{\sigma} \quad \text{for } y = n \cdot d.
\]

and hence

\[
E \left[ e^{-aY}, Y \geq 0 \right] \approx \sum_{n=0}^{\infty} e^{-adn} \varphi \left( \frac{nd}{\sigma} \right) \frac{d}{\sigma}.
\]

In this case it is natural to introduce the discrete Esscher function

\[ E(s, b) = \sum_{n=0}^{\infty} e^{-sn} \varphi(nb)b. \]

The Esscher approximation then becomes

\[
P(S(t) \geq tx) \approx e^{-th(x)} E \left( ad, \frac{d}{\sigma} \right).
\]

As \( b \to 0 \) we have

\[
E(s, b) \approx \sum_{n=0}^{\infty} e^{-sn} \varphi(0)b = \frac{b}{\sqrt{2\pi(1-e^{-s})}},
\]

that is,

\[
E \left( ad, \frac{d}{\sigma} \right) \approx \frac{1}{\sqrt{2\pi \sigma}} \frac{d}{1-e^{-ad}} \quad \text{as } d \to \infty.
\]

Hence, in the discrete case we have the modified Esscher approximation

\[
P(S(t) \geq tx) \approx \frac{e^{-th(x)}}{\sqrt{2\pi A(d)\sqrt{tg''(a)}}}
\]

where \( A(d) = (1-e^{-ad})/d \) and \( x = g'(a) < \lambda\mu \) with \( a < 0 \). As \( d \to 0 \) we see that \( A(d) \to a \) and hence the formula is consistent with (5).

The Esscher approximation holds analogously for \( P(S(t) \leq tx) \) when \( x = g'(a) < \lambda\mu \) with \( a < 0 \). More generally, it holds for any probability \( P(S(t) \in I) \),
where $I$ is an interval $[z, y]$ with $\lambda \mu < z < y$ or $[y, z]$ with $y < z < \lambda \mu$. In both cases, $a$ should be chosen so that 

$$g'(a) = x,$$

where $x$ is the point in $I$ where $h(x)$ is as small as possible, that is, the exponent is always given by $\min_{x \in I} h(x)$.

The general formula is

$$P(S(t)/t \in I) \approx \frac{C}{\sqrt{t}} e^{-t \min_{x \in I} h(x)}$$

for some constant $C > 0$. This type of estimate is common in the more general theory for large deviations that has been developed during the last decades inspired by the pioneering work of Esscher from the 1930’s.

3 Theory of ruin probabilities

So far we have studied the total loss $S(t)$ without taking the flow of premiums in time into account, that is, we have only considered $S(t)$ at a fixed time $t$. In this case it is relevant to study $P(S(t) \geq tx)$ as we did in the previous section. The number $tx$ should be thought of as the capital available at time $t$ – that is, the sum of the capital at $t = 0$ and the amount of premiums that is paid in the interval $(0, t)$ – and we want to make sure that this capital is large enough to make the probability reasonably small. In such a setting we do not take the possibility that a deficit might arise before time $t$ into account.

To study the course of events in time we need to describe the flow of premiums. This might also be stochastic, but here we will restrict ourselves to the simplest setting, where the premiums constitute a constant continuous inflow so that the total premium paid in the interval $(0, t)$ is $ct$. If the capital at time $t = 0$ is $u$, the surplus at time $t$ is then given by $u + ct - S(t)$ (for simplicity we disregard income from interest). In the following we will study the so called ruin probability, that is, the probability that the surplus is negative at some time point during the planning period $(0, t)$, where $t = \infty$ is also a possibility. In particular, we will see how this probability depends on the parameters $u$, $c$, $\lambda$ and $F(dx)$.

3.1 The total loss process

Let us introduce the net amount of loss $U(t) := S(t) - ct$. This is a stochastic process with upward jumps of height $\{X_k\}$ at times $\{T_k\}$, just as $S(t)$, and in between these times the process decreases at rate $-c$; see Figure 1. Our main object of interest is the time of ruin, denoted by $T(u)$ and defined as the first time when $U(t) > u$. As we can see in Figure 1 if $u \geq 0$, the ruin occurs at the first time $T_k$ such that $S_k - cT_k > u$, that is, the ruin does not occur in between two loss occasions, which means that in general a non-zero deficit arises at time $T(u)$. We will also study $T(-u)$, which is the first time when $U(t) \leq -u$. Since the heights of the jumps are strictly positive, this occurs in between the jump occasions so that, unlike what holds for $T(u)$, we have $U(t) = -u$ at time
$t = T(-u)$. This will turn out to be a useful fact. Now assume that $u \geq 0$ and define the ruin probabilities as

\[
    r(u, t) = P(T(u) \leq t), \quad \text{for } t < \infty;
\]
\[
    r(u) = P(T(u) < \infty),
\]

and, analogously,

\[
    r(-u, t) = P(T(-u) \leq t), \quad \text{for } t < \infty;
\]
\[
    r(-u) = P(T(-u) < \infty).
\]

We also define $T(\pm u) = \infty$ if the passage to $\pm u$ never occurs, which, as we will see, happens with positive probability.

Classical risk theory has to a large extent been concerned with finding equations for $r(u)$ and $r(u, t)$ and, on the basis of these equations, deriving approximations analogous to the ones derived in the previous section for the distribution of $S(t)$. In the following we will treat these problems, using more probabilistic methods than the traditional ones. This often leads to a better understanding of why the approximations are valid and also to many simplifications of the derivations.

Before moving on to the mathematical treatment, we remark that $T(u)$ is of course the natural ruin time when we have a positive “risk sum”, that is, when the loss amounts $X_k$ are positive and the premium inflow has rate $c > 0$. This is the natural model for property insurance and whole life assurance. However, we can also apply the model to life assurance with negative risk sum. In this case we have a continuous outflow of payments $ct$ and $S(t)$ represents the accumulated inflow of profits made at the times of the deaths. The ruin occurs when $ct - S(t) \geq u$ for the first time, that is, at time $T(-u)$. Hence $T(-u)$ also has a natural interpretation and $r(-u, t)$ and $r(-u)$ are the ruin probabilities in this case.
3.2 Basic formulas for the ruin probabilities

We begin by deriving a clever formula for the ruin probability when \( u = 0 \). The formula will turn out to be useful also in finding expressions for the ruin probabilities when \( u \neq 0 \), as has been shown by Lajos Takács. First consider the event \( A_t := \{ T(0) > t \} \) that the time to ruin exceeds \( t \) and note that

\[
A_t = \{ S(t') \leq ct' \text{ for all } t' \in (0, t) \} = \{ S_k \leq cT_k \text{ for } k = 1, 2, \ldots, N(t) \},
\]

see Figure 5 for an illustration. The following lemma gives a simple formula for the probability of \( A_t \) given that \( S(t) = x \), \( 0 \leq x \leq ct \).

**Lemma 3.1** We have

\[
P(A_t|S(t) = x) = \left( 1 - \frac{x}{ct} \right)_+,\]

where

\[
\left( 1 - \frac{x}{ct} \right)_+ = \begin{cases} 
1 - \frac{x}{ct} & \text{if } 0 \leq x \leq ct, \\
0 & \text{if } x > ct.
\end{cases}
\]

**Proof:** We will use induction over \( N(t) = n \) to show the slightly stronger statement that

\[
P(A_t|S(t) = x, N(t) = n) = \left( 1 - \frac{x}{ct} \right)_+.
\]

To this end, first consider the case \( n = 0 \). Then \( S(t) = 0 \), so that only \( x = 0 \) has to be considered, and the event \( A_t \) occurs with probability 1. Hence (6) is true for \( n = 0 \). For \( n = 1 \), the event \( A_t \) occurs if and only if \( T_1 \geq x/c \). The conditional density for \( T_1 \) given that \( N(t) = 1 \) is
\[
f_1(z) \, dz = \frac{P(N(0, z) = 0, N(z, z + dz) = 1, N(z + dz, t) = 0)}{P(N(0, t) = 1)}
= \frac{e^{-\lambda z} \lambda dz e^{-\lambda(t-z)}}{\lambda t e^{-\lambda t}}
= \frac{dz}{t}, \quad 0 \leq z \leq t,
\]
that is, a uniform distribution on \((0, t)\). Hence
\[
P\left( T_1 \geq \frac{x}{c} \left| S(t) = x, N(t) = 1 \right. \right) = \left( 1 - \frac{x}{ct} \right)_+, \quad \text{and so } (\text{iii}) \text{ is true also for } n = 1.
\]
Now assume that (\text{iii}) holds for \(N(t) \leq n - 1\) and consider the case \(S(t) = x, N(t) = n\). Given that \(N(t) = n\), the time \(T_n\) has the conditional density \(f_n(z)\) given by
\[
f_n(z) \, dz = \frac{P(N(0, z) = n - 1, N(z, z + dz) = 1, N(z + dz, t) = 0)}{P(N(0, t) = n)}
= \frac{(\lambda z)^{n-1} e^{-\lambda z} \lambda dz e^{-\lambda(t-z)}/(n-1)!}{(\lambda t)^n e^{-\lambda t}/n!}
= n \left( \frac{z}{t} \right)^{n-1} \frac{dz}{t}, \quad 0 \leq z \leq t.
\]
If we fix \(T_n = z\) and \(S(T_{n-1}) = y\), where \(0 \leq y \leq x \leq cz \leq ct\), it follows from the induction assumption that the conditional probability for \(A_t\) is the same as for \(A_z\), that is, \(1 - y/cz\). Integrating over \(y\) with the conditional distribution of \(S(T_{n-1})\) given \(S(T_n)\) yields
\[
P(A_t|S(t) = x, N(t) = n, T_n = z) = \mathbb{E} \left[ 1 - \frac{S(T_{n-1})}{cz} \right| S(T_n) = x].
\]
By symmetry we have
\[
\mathbb{E}[S(T_{n-1})|S(T_n)] = (n - 1)\mathbb{E}[X_k|S(T_n)] = \frac{n-1}{n} S(T_n),
\]
and hence
\[
P(A_t|S(t) = x, N(t) = n, T_n = z) = \left( 1 - \frac{(n-1)x}{ncz} \right).
\]
Integrating over \(z \in (x/c, t)\) with the density \(f_n(z)\) we finally get
\[ P(A_t|S(t) = x, N(t) = n) = \int_{x/c}^t \left( 1 - \frac{(n-1)x}{ncz} \right) \frac{(z/n)^{n-1}}{t} \frac{dz}{t} = \int_{x/c}^t n \left( \frac{z}{t} \right)^{n-1} \frac{dz}{t} - (n-1) \frac{x^{n-2}}{ct^n} dz \]

\[ = 1 - \left( \frac{x}{ct} \right)^n - \frac{x}{ct} + \left( \frac{x}{ct} \right)^n = 1 - \frac{x}{ct}. \]

The formula (6) now follows by induction. Since the right hand side does not involve \( n \), the conditioning on \( N(t) = n \) can be removed without affecting the formula and hence the lemma is proved.

Multiplying the probability in Lemma 3.1 with \( P(S(t) \in dx) = F(t, dx) \) gives the joint probability

\[ P(A_t, S(t) \in dx) = \left( 1 - \frac{x}{ct} \right) F(t, dx). \]

Now fix \( S(t) = x \) such that \( U(t) = S(t) - ct = x - ct \leq 0 \) and write \( x - ct = -u \). We then have

\[ P(A_t|U(t) = -u) = \left( \frac{u}{ct} \right) \]

and

\[ P(A_t, U(t) \in -du) = \left( \frac{u}{ct} \right) F(t, ct - du), \]

that is,

\[ P(U(t) \in -du, U(t') \leq 0 \text{ for } t' \in (0, t)) = \left( \frac{u}{ct} \right) F(t, ct - du); \quad (7) \]

see Figure 6(a). Integrating this over \( x \in (0, ct) \) we obtain the non-ruin probability \( \bar{r}(0, t) := 1 - r(0, t) \) for the initial capital \( u = 0 \), that is,

\[ \bar{r}(0, t) = \int_0^{ct} \left( 1 - \frac{x}{ct} \right) F(t, dx). \quad (8) \]

In the following sections we will see how these formulas can be used to determine the ruin probabilities when \( u \neq 0 \).

### 3.2.1 The distribution of \( T(-u) \)

Let us first derive a formula for \( P(T(-u) \in dt) \). A typical trajectory with \( T(-u) \in dt \) has \( U(t') > -u \) for \( t' < T(-u) \) and \( U(t') = -u \) for \( t' = T(-u) \); see Figure 6(b). If we turn this picture upside down and move the origin to the crossing point, we see that the trajectory is transformed into the trajectory in Figure 6(a). Hence, if this transformation does not change the distribution
of the process, the probability that $T(-u) \in dt$ should be the same as the probability of the event in (7), that is,

$$P(T(-u) \in dt) = \left(\frac{u}{ct}\right) F(t, ct - du)$$

and, since $-du = cdt$, we have

$$P(T(-u) \in dt) = \left(\frac{u}{ct}\right) F(t, cdt - u) \quad \text{for } ct \geq u > 0.$$ 

This is an explicit formula for the the distribution of $T(-u)$ and we have for instance that

$$r(-u, t) = P(T(-u) \leq t) = \int_{u/c}^{t} \left(\frac{u}{cs}\right) F(s, cds - u).$$

To understand that the transformed process has the same distribution as $U(t)$ we can write it as $\hat{U}(\hat{t}) = -u - U(t - \hat{t}), 0 \leq \hat{t} \leq t$. The process $\hat{U}(\hat{t})$ has jumps at the time points $\hat{T}_k = t - T_k$, which constitute a Poisson process, and the jumps are $\hat{X}_k = X_k$, which are independent with distribution $F(dx)$. Between the jumps, $\hat{U}(\hat{t})$ is changed at rate $-c$. Hence $\hat{U}(\hat{t})$ is a process with the same distribution as $U(t)$ and initial value $\hat{U}(0) = 0$. The jumps occur in a different order, but this does not affect the distribution. The process $\hat{U}(\hat{t})$ is illustrated in Figure 6(c).

The ruin probability with $t = \infty$ is

$$r(-u) = \int_{u/c}^{\infty} \left(\frac{u}{cs}\right) F(s, cds - u),$$

or, with $x = cs - u$,

$$r(-u) = \int_{0}^{\infty} \left(\frac{u}{x + u}\right) F\left(\frac{x + u}{c}, dx\right).$$

We will mainly consider the case when $c > \lambda\mu$ so that $E[U(t)] = -(c - \lambda\mu)t < 0$. By the law of large numbers,

$$\frac{U(t)}{t} \to -(c - \lambda\mu) < 0 \quad \text{a.s. as } t \to \infty.$$ 

This implies that, with probability 1, the barrier $-u$ is hit sooner or later, that is, $r(-u) = 1$, or, equivalently,

$$\int_{0}^{\infty} \left(\frac{u}{x + u}\right) F\left(\frac{x + u}{c}, dx\right) = 1 \quad \text{if } c > \lambda\mu. \quad (9)$$

This relation will prove to be important in what follows.
Figure 6: Transformation of $U(t)$. 
3.2.2 The distribution of $T(u)$

We will now derive an explicit formula for $r(u, t) = P(T(u) \leq t)$ by using the previous results and conditioning on the value of $U(t)$. Trivially

$$r(u, t) = P(T(u) \leq t, U(t) > u) + P(T(u) \leq t, U(t) \leq u).$$

If $U(t) > u$, we know for sure that $T(u) \leq t$, and hence

$$P(T(u) \leq t, U(t) > u) = P(U(t) > u) = \int_{x=u+ct}^{\infty} F(t, dx).$$

If $T(u) \leq t$ and $U(t) \leq u$ the trajectory for $U(t)$ has to cross the level $u$ one or more times between $T(u)$ and $t$; see Figure 7. Let $s$ be the value of the last time this occurs. The probability for such an outcome is $P(U(s) \in du)P(E)$, where $E$ denotes the event to go from $u$ at $s$ to $u - dy$ at $t$ without exceeding $u$ between $s$ and $t$. The first factor equals $F(s, du+cs) = F(s, u+cds)$. By Lemma 3.1 the last factor equals $(y/c(t-s))F(t-s, c(t-s) - dy)$ and, integrating over $y \geq 0$, we get $\bar{r}(0, t-s)$, see (3). Combining all this yields

$$r(u, t) = \int_{u+ct}^{\infty} F(t, dx) + \int_{0}^{t} F(s, u+cds)\bar{r}(0, t-s).$$

This formula is called Seals’s formula and, if $F(t, dx)$ is known, it can be used to calculate $r(u, t)$. As $u$ and $t$ becomes large, it can also be used to derive an asymptotic formula using the Esscher-approximation of $F(t, dx)$, but the calculations become cumbersome.
3.2.3 The ruin probability $r(u)$

We will now derive a useful formula for $r(u) = P(T(u) < \infty)$. A conceivable method for studying $r(u)$ would be to let $t \to \infty$ in Seal’s formula for $r(u, t)$. However, we will see that it is possible to obtain an interesting formula via a more direct analysis, where the process $U(t)$ is divided into successive up-crossings, $\{U_k\}$; see Figure 8. These upcrossings are defined as follows: Initially, $U(0) = 0$. With probability $r := r(0)$, we have $U(t) > 0$ for some $t$, and with probability $1 - r$, we have $U(t) \leq 0$ for all $t$. In the first case, define $U_1$ to be the value of $U(t)$ just after it has exceeded 0 for the first time, that is, $U_1 = U(T(0))$ if $T(0) < \infty$. From this point $U(t)$ goes on for $t \geq T(0)$, and the process $U(T(0) + t) - U_1$ has the same distribution as $U(t)$ and is independent of $U_1$. Define $U_2$ as the first up-crossing in this process, and so on. In each step, there is a probability $1 - r$ that no more up-crossing occurs, and the successive $U_k$:s become independent and identically distributed.

Below we will see that the $U_k$:s have a density $k(u)$ that is easy to write down and that

$$r = \begin{cases} \frac{\lambda}{c} & \text{if } c > \lambda\mu, \\ 1 & \text{if } c < \lambda\mu. \end{cases}$$

Let $M$ denote the number of upcrossings and define $\bar{U} = \sum_1^M U_k$. When $r < 1$, $M$ has a geometric distribution with

$$P(M = m) = (1 - r)r^m, \quad m = 0, 1, 2, \ldots.$$  

This means that $M$ is finite with probability one and hence we can write $\bar{U} = \max_{t \geq 0} U(t)$. The ruin probability then becomes $r(u) = P(\bar{U} > u)$ and this probability can easily be expressed in terms of $r$ and $k(u)$. It turns out, namely, that $\bar{U}$ has a compound geometrical density,

$$l(u) = (1 - r) \sum_{m=0}^\infty r^m k^{m+1}(u), \quad (10)$$
where \( k^{m*}(u) \) denotes the convolution of \( k(u) \) with itself \( m \) times (compare with the compound Poisson distribution characterized in Proposition 2.1). This can be seen by noting that, with probability \((1-r)r^m\), we have \( M = m \) and the density of \( \bar{U} \) then becomes \( k^{m*}(u) \). Summing over the possible values of \( m \), we get \( 10 \). The formula for \( r(u) \) becomes

\[
r(u) = \int_u^\infty l(y)dy, \quad u > 0.
\]

This formula is useful, since both \( k(u) \) and \( r(u) \) can easily be calculated. To find expressions for \( k(u) \) and \( r(u) \), consider the first upcrossing \( U := U_1 \). Write \(-V\) for the value of \( U(t) \) just before this up-crossing and let \( W \) denote the time when the up-crossing occurs; see Figure 9. By Lemma 3.1, the joint distribution of \( (U, V, W) \) is

\[
P(U \in du, V \in dv, W \in dw) = P(U(w) \in -dv \text{ and } U(t) \leq 0 \text{ for } t < w) \cdot P(\text{a loss occurs in } dw) \cdot P(\text{the amount of loss } \in v + du).
\]

The distribution of \( (U,V) \) is obtained by integrating over \( w \). Assuming that \( F \) has a density \( F' \) and making the substitution \( x = cw - v \), we get

\[
P(U \in du, V \in dv) = \int_{w \geq v/c} \frac{v}{cw} F'(w, cw - v) \lambda dw dv F'(u + v) du
\]

\[
= \frac{\lambda}{c} F'(u + v) dv du \quad \text{ when } c \geq \lambda \mu,
\]
where the last equality follows from (9). Hence the pair \((U, V)\) has density 
\[
\lambda/c \cdot F'(u + v)\, du\, dv
\]
for \(u, v \geq 0\). Integrating this over \(v\) gives

\[
P(U \in du) = \frac{\lambda}{c} \int_0^\infty F'(u + v)\, dv
\]

Since

\[
\int_0^\infty (1 - F(u))\, du = \int_0^\infty uF'(u)\, du = \mu,
\]

we normalize by \(\mu\), to get

\[
\frac{\lambda \mu}{c} \cdot \frac{1 - F(u)}{\mu}\, du = rk(u)\, du,
\]

where \(r = \lambda \mu/c < 1\) and \(k(u) = (1 - F(u))/\mu\), with \(\int_0^\infty k(u)\, du = 1\).

To summarize, we see that \(P(U > 0) = r = \lambda \mu/c\), and that the conditional density of \(U\) given that \(U > 0\), is given by \(k(u) = (1 - F(u))/\mu\). Also, the ruin probability is

\[
r(u) = \int_u^\infty l(y)\, dy, \quad u > 0,
\]

where \(l(u)\) is specified in (10). In risk theory, this formula is called Cramér’s formula, and in queuing theory – where it solves a similar problem – it is referred to as Pollaczek-Khinchin’s formula.

The formula for \(l(u)\) gives rise to a corresponding equation for the generating functions

\[
\kappa(\xi) = \int_0^\infty e^{\xi u}k(u)\, du
\]

and

\[
\lambda(\xi) = \int_0^\infty e^{\xi u}l(u)\, du,
\]

which are defined at least for \(\xi \leq 0\). The equation for \(l(u)\) corresponds to the equation

\[
\lambda(\xi) = (1 - r) \sum_{m=0}^\infty r^m \kappa^m(\xi)
\]

\[
= \frac{1 - r}{1 - r\kappa(\xi)}.
\]

This equation can be used to compute \(l(u)\) by inverting the generating function.
Example. Let \( k(u) = e^{-u} \). Then

\[
\kappa(\xi) = \int_0^\infty e^{\xi u} e^{-u} \, du = \frac{1}{1 - \xi},
\]

which implies that

\[
\lambda(\xi) = \frac{1 - r}{1 - r/(1 - \xi)} = (1 - r) + \frac{r(1 - r)}{1 - r - \xi}.
\]

This is the generating function of \( l(u) = (1 - r) \delta(u) + r(1 - r)e^{-(1-r)u} \). The density \( l(u) \) can be computed in a similar way when \( \kappa(\xi) \) is a rational function of \( \xi \).

3.2.4 Panjer-approximation of \( r(u) \)

In the following two sections, two different approximations of \( r(u) \) will be derived. The first one is a kind of Panjer-recursion and the second one is an asymptotic formula as \( u \to \infty \) similar to the Esscher approximation.

As for the Panjer-recursion, consider first two discrete distributions \( \{k_n\}_1^\infty \) and \( \{l_n\}_0^\infty \), where \( \{k_n\} \) is known and \( \{l_n\} \) is “compound geometric”, that is,

\[
l_n = (1 - r) \sum_{m=0}^\infty r^m k_n^{m*}, \quad n = 0, 1, 2, \ldots.
\]

We will now see that \( \{l_n\} \) can be computed by aid of a recursive formula of Panjer-type. Convolving the equation \( l = (1 - r) \sum_0^\infty r^m k^{m*} \) with \( rk \) yields

\[
rk * l = (1 - r) \sum_{m=0}^\infty r^{m+1} k^{(m+1)*} = (1 - r) \sum_{m=1}^\infty r^m k^{m*} = l - (1 - r) \delta,
\]

where

\[
\delta_n = k_0^{0*} = \begin{cases} 
1 & \text{if } n = 0, \\
0 & \text{if } n > 0.
\end{cases}
\]
Hence we have the renewal equation \( l = (1 - r)\delta + rk + l \), or, more explicitly,

\[
  l_n = (1 - r)\delta_n + r \sum_{m=1}^{n} k_m l_{n-m}.
\]

The probabilities \( \{l_n\} \) can successively be computed for \( n = 0, 1, 2, \ldots \). We obtain

\[
  l_0 = 1 - r,
  l_1 = rk_1 l_0,
  l_2 = r(k_1 l_1 + k_2 l_0),
  \vdots
  l_n = r(k_1 l_{n-1} + \ldots + k_n l_0).
\]

If \( \{k_n\} \) is known, this recursion is easy to implement. Also, the probabilities \( r_n = \sum_n l_m \) can be computed parallel to \( l_n \).

The equations for \( l(u) \) and \( r(u) \) are similar, but, just as \( k(u) \), they are the densities of continuous random variables. However, if we make a suitable discrete approximation of \( k(u) \), we can calculate the corresponding approximations of \( l(u) \) and \( r(u) \) by the above method.

The relation between \( \{k_n\} \) and \( \{l_n\} \) can also be expressed in terms of the generating functions \( \hat{k}(s) = \sum_1^{\infty} k_n s^n \) and \( \hat{l}(s) = \sum_0^{\infty} l_n s^n \). We have

\[
  \hat{l}(s) = (1 - r) \sum_{m=0}^{\infty} r^m \hat{k}^m(s) = \frac{1 - r}{1 - rk(s)}.
\]

If, for instance, \( \hat{k}(s) \) is a rational function of \( s \), the function \( \hat{l}(s) \) is also rational and \( \{l_n\} \) can be obtained by partial fraction expansion.

**Example.** Let \( k_n = (1 - p)p^{n-1}, n = 1, 2, \ldots \) This gives

\[
  \hat{k}(s) = (1 - p) \sum_{n=1}^{\infty} s^n p^{n-1} = \frac{(1 - p)s}{1 - ps},
\]

that is,
\[ 1 - r\hat{k}(s) = 1 - \frac{r(1-p)s}{1-ps} \]
\[ = \frac{1 - qs}{1-ps}, \]

where \( q = p + r(1-p) \). Hence

\[ \hat{l}(s) = (1-r)\frac{1-ps}{1-qe} \]
\[ = (1-r)\left(1 + \frac{(q-p)s}{1-qe}\right) \]
\[ = (1-r)\left(1 + r(1-p)\sum_{1}^{\infty} s^n q^{n-1}\right). \]

This yields
\[
\begin{cases}
  l_0 = (1-r) \\
  l_n = (1-r)r(1-p)q^{n-1}, \ n \geq 1.
\end{cases}
\]

A natural method for finding a discrete approximation to the density \( k(x) \) can be obtained as follows: Approximate first the distribution \( F(x) \) by a discrete distribution with masses \( f_n \) for \( x = nd \) and \( n = 1, 2, \ldots \) and put \( F_n = \sum_{1}^{n} f_m \).

For this distribution \( F(x) \) is piecewise constant: \( F(x) = F_n \) and \( 1 - F(x) = 1 - F_n = \sum_{n+1}^{\infty} f_m \) for \( nd \leq x < (n+1)d \), and then \( \mu = d \sum_{0}^{\infty} (1-F_n) \). The density \( k(x) = (1-F(x))/\mu \) can then be approximated by a discrete distribution having masses \( k_n = \int_{(n-1)d}^{nd} k(x)dx = d(1-F_{n-1})/\mu = (d/\mu)\sum_{n}^{\infty} f_m \) for \( x = nd \) and \( n = 1, \ldots \). This distribution will have total mass one and is located at positive \( x \)-values.

3.2.5 Cramér-Lundberg’s approximation of \( r(u) \)

We will now derive a more explicit approximation formula for \( r(u) \). It is an asymptotic formula valid as \( u \to \infty \) and, as we will see, it is closely related to the Esscher approximation.

First recall from Section 3.2.3 that
\[ r(u) = \int_{u}^{\infty} l(y)dy \quad \text{for} \ u > 0, \]
where \( l(y) = (1-r)\sum_{0}^{\infty} r^m k^{m*}(y) \), \( r = \lambda \mu / c \) and \( k(u) = (1-F(u))/\mu \). Here \( r = P(U > 0) \), where \( U \) denotes the size of an up-crossing, and \( k(u) \) is the conditional density of \( U \) given that \( U > 0 \). To get an approximation of \( l(y) \) when \( y \) is large, we need an approximation of \( k^{m*}(y) \) as \( y \to \infty \). Since \( r^m \) damps large \( m \)-values in the formula for \( l(y) \) we only have to consider moderate
values of $m$. The desired approximation is obtained by introducing a modified density $k_a(y)$ as in Section 2.4.2, and choosing $a$ suitably. We have

$$k_a(y) = \frac{e^{ay}k(y)}{\kappa(a)},$$

where $\kappa$ is the generating function of the density $k$ (see (11)), and below we will see that, just as $g(a)$, $\kappa(a) < \infty$ for $a < \xi$. As in Section 2.4.2, we get

$$k_m^*(y) = \frac{e^{ay}k_m^*(y)}{\kappa_m(a)},$$

so that

$$k_m^*(y) = e^{-ay} \kappa_m(a) k_m^*(y)$$

and hence

$$l(y) = e^{-ay}(1 - r) \sum_{m=0}^{\infty} r^m \kappa_m(a) k_m^*(y).$$

Choosing $a$ such that $r\kappa(a) = 1$ yields

$$l(y) = e^{-ay}(1 - r) \sum_{m=0}^{\infty} k_m^*(y).$$

As $y \to \infty$, this expression can be approximated using the so called renewal theorem, which is an important result in renewal theory. It states that, as $y \to \infty$, the sum $\sum_{m=0}^{\infty} k_m^*(y)$ can be approximated by a uniform density with intensity $1/m_a$, where

$$m_a = \int_0^{\infty} yk_a(y)dy = \int_0^{\infty} ye^{ay}k(y)dy = \frac{\kappa'(a)}{\kappa(a)}.$$

Substituting this approximation in the formula for $r(u)$ gives

$$r(u) \approx \int_u^{\infty} e^{-ay}(1 - r) \frac{dy}{m_a} = e^{-au}(1 - r) \int_0^{\infty} e^{-ax} dx = \frac{(1 - r) e^{-au}}{am_a}.$$ 

If $a$ is known, this is a simple exponential approximation.
The equation for $a$, $r\kappa(a) = 1$, can be expressed more explicitly in terms of $g(a)$, by noting that

$$
\kappa(a) = \int_0^\infty e^{ax} \left( \frac{1 - F(x)}{\mu} \right) dx
$$

$$
= \frac{1}{a\mu} \int_0^\infty (e^{ax} - 1) F(dx)
$$

$$
= \frac{g(a)}{a\lambda\mu},
$$

where the second equality is obtained by partial integration. The equation for $a$ hence becomes $g(a)/a\lambda\mu = 1/r = c/\lambda\mu$, that is, $g(a) = ca$. Recall from Section 2.4.1 that $g(\xi)$ is strictly convex with $g(0) = 0$ and $g'(0) = \lambda\mu$. We are looking for the intersection with a line $c\xi$ with slope $c$; see Figure 10. For $c > g'(0) = \lambda\mu$, there is a strictly positive root, which is denoted by $R$ and referred to as the **Lundberg exponent**. For $c < \lambda\mu$, the root is negative.

To find an expression for the constant $C := (1 - r)/am_a$ in the formula for $r(u)$, note that

$$
m_a = \frac{\kappa'(a)}{\kappa(a)}
$$

$$
= [\log \kappa(a)]'
$$

$$
= [\log g(a)]' - [\log a]'
$$

$$
= \frac{g'(a)}{g(a)} - \frac{1}{a}
$$

$$
= \frac{g'(a) - c}{ca}.
$$

This yields
To sum up, we have deduced that \( r(u) \approx Ce^{-Ru} \), where \( R \) is the positive root of the equation \( g(a) = ca \) and \( C = (c - g'(0))/(g'(R) - c) \). Here “\( \approx \)” means that the quotient between the right hand and the left hand side tends to 1 as \( u \to \infty \). A natural way of using these formulas for the design of a system is to start by choosing \( c \) so that \( r \) has a suitable value close enough to one, and then finding the corresponding values of \( R \) and \( C \). Then \( u \) can easily be found so that \( Ce^{-Ru} \) has a value considered to be small enough to be safe.

**Example.** Approximate calculation of \( R \) and \( C \) when \( r = \lambda \mu/c \) is close to one.

The equation \( g'(R) = c \) can be expressed in terms of the Taylor expansion of \( g(R) \) as follows:

\[
g(R) = \lambda \sum_{k=1}^{\infty} \frac{\mu_k R^k}{k!}
\]

where \( \mu_k \) is the \( k \)-th moment of the claims distribution \( F \). (\( \mu_1 = \mu \) and \( \mu_2 = \nu \)).

In terms of it the equation for \( R \) is hence

\[
\lambda(\mu + \mu_2 R/2 + \mu_3 R^2/6 + \cdots) = c
\]

or

\[
\lambda(\mu_2 R/2 + \mu_3 R^2/6 + \cdots) = \lambda \mu(1/r - 1),
\]

and we see that \( r \approx 1 \) corresponds to \( \rho \equiv (1/r - 1) \approx 0 \) and hence to \( R \approx 0 \). To first order in \( \rho \) we hence have \( \mu_2 R_1/2 = \mu_3 \rho \) and \( R_1 = (2\mu_1/\mu_2)\rho \). To second order in \( \rho \) we then have

\[
R_2 = R_1 - (2/\mu_2)(\mu_3/6)(2\mu_1/\mu_2)^2 \rho^2 = (2\mu_1/\mu_2)\rho - (4/3)(\mu_3\mu_1^2/\mu_2^2)\rho^2
\]

etc. The corresponding values of \( C \) can be obtained from the relation

\[
C = \frac{(g(R)/R - \lambda \mu)/(g'(R) - g(R)/R)}{(\sum_{k=2}^{\infty} \mu_k R^{k-1}/k)!/\sum_{k=2}^{\infty} \mu_k R^{k-1}(k - 1)/k!)}
\]

\[
= \frac{(\sum_{k=2}^{\infty} \mu_k R^{k-2}/k)!/\sum_{k=2}^{\infty} \mu_k R^{k-2}(k - 1)/k!}{}
\]

\[
= (\mu_2/2) + (\mu_3/6)R + (\mu_4/24)R^2 + \cdots
\]

\[
= (\mu_2/2) + (\mu_3/3)R + (\mu_4/8)R^2 + \cdots
\]

To first order in \( \rho \) we hence have

\[
C_1 = ((\mu_2/2) + (\mu_3/6)R_1)/((\mu_2/2) + (\mu_3/3)R_1),
\]

and we get quite explicit expressions in terms of the moments \( \mu_k \) and \( \rho \).
Example. Assume that $F'(x)$ is a weighted sum of exponential densities, that is,

$$F'(x) = \sum_{i=1}^{n} a_i b_i e^{-b_i x},$$

where $a_i > 0$, $\sum_{i=1}^{n} a_i = 1$ and $0 < b_1 < b_2 < \ldots < b_n$. Then $r(u)$ can be calculated fairly explicitly via the generating functions $\kappa(\xi)$ and $\lambda(\xi)$, defined in (11) and (12) respectively, and we will be able to see how the approximation $Ce^{-R\mu}$ arises. First recall that $\kappa(\xi) = g(\xi)/\xi \lambda \mu$ and $\lambda(\xi) = (1-r)/(1-r\kappa(\xi))$.

The generating function, $f(\xi)$, of $F'(x)$ is

$$f(\xi) = \sum_{i=1}^{n} a_i \cdot \frac{b_i}{b_i - \xi}$$

and we obtain

$$g(\xi) = \lambda(f(\xi) - 1) = \lambda \sum_{i=1}^{n} \frac{a_i \xi}{b_i - \xi}$$

(13)

and $\mu = \sum_{i=1}^{n} a_i / b_i$. Hence

$$\kappa(\xi) = \frac{1}{\mu} \sum_{i=1}^{n} \frac{a_i \xi}{b_i - \xi},$$

that is, $\kappa(\xi)$ is a rational function of $\xi$, where the denominator is of degree $n$, and $\kappa(\xi) \to 0$ as $|\xi| \to \infty$. The poles of $\lambda(\xi)$ – that is, the zeroes of its denominator – are the roots of the equation $1 - r\kappa(\xi) = 0$. If the root $\xi = 0$ is ignored, this equation can be rewritten as $1 = rg(\xi)/\lambda \mu \xi$, that is, $g(\xi) = c\xi$. Using the relation (13), the equation becomes

$$\lambda \sum_{i=1}^{n} \frac{a_i}{b_i - \xi} = c.$$

For $\xi = 0$, the left hand side equals $\lambda \mu$ which is strictly smaller than $c$. A graph of the expression on the left hand side as a function of $\xi \geq 0$ is displayed in Figure 11. We see that there are $n$ real roots $R_1, \ldots R_n$, with $0 < R_1 < b_1 < R_2 < b_2 < \ldots < R_n < b_n$. Hence, the partial fraction expansion of $\lambda(\xi)$ is

$$\lambda(\xi) = (1-r) + \sum_{i=1}^{n} \frac{C_i R_i}{R_i - \xi},$$

where the coefficients $C_i R_i$ are determined by the formula
Figure 11: Graphical picture of the roots \{R_i\}.

\[ C_i R_i = \lim_{\xi \to R_i} (R_i - \xi) \lambda(\xi) \]
\[ = \lim_{\xi \to R_i} \frac{(R_i - \xi)(1 - r)}{1 - rg(\xi)/\xi \mu} \]
\[ = \lim_{\xi \to R_i} \frac{(R_i - \xi)(1 - r)c \xi}{c \xi - g(\xi)}. \]

Near \( \xi = R_i \), we have for the denominator, that

\[ g(\xi) - c \xi \approx g(R_i) - cR_i + (\xi - R_i)(g'(R_i) - c) \]
\[ = (\xi - R_i)(g'(R_i) - c), \]

since \( g(R_i) = cR_i \). Hence \( C_i = c(1 - r)/(g'(R_i) - c) \) and, using the formula for \( \lambda(\xi) \), we obtain

\[ l(x) = (1 - r)\delta(x) + \sum_{i=1}^{n} C_i R_i e^{-R_i x} \]

and

\[ r(u) = \int_{u}^{\infty} l(x) dx \]
\[ = \sum_{i=1}^{n} C_i e^{-R_i u} \text{ for } u > 0. \]
This is an elegant generalization of Cramér-Lundberg’s formula, which is obtained when only the contribution from $R_1 = R$ is included. Since $R_1 < b_1 < R_2 < \ldots < R_n$, we see that the first term $Ce^{-Ru}$ dominates, as expected.

3.2.6 An alternative derivation of Cramér’s formula for $r(u)$

In the derivation of the formula for $r(u)$, the process $U(t)$ was divided into successive upcrossings. This gave a natural probabilistic interpretation of the quantities $r$ and $k(y)$. The traditional method for determining $r(u)$ is to derive an integral equation, that is well-known in renewal theory, and to show that its solution is given by Cramér’s formula. Although it does not provide the same insight concerning the probabilistic structure of the solution, this method has the advantage of being more direct. Also, it can be generalized to the case when $c$ depends on the value of $U(t)$, which is indeed a natural extension. For the sake of completeness, we describe also this analytic derivation.

We are looking for an equation for $r(u)$ as a function of $u$, that is based on an analysis of what can happen in a small interval $(0, h)$ just after $t = 0$. Such equations are common in the more general theory for Markov processes and are referred to as backward equations. There are basically two possible scenarios that can occur in the interval $(0, h)$:

1. With probability $e^{-\lambda h} \approx 1 - \lambda h$ no loss occurs. At time $h$ we then have $U(h) = -ch$ and ruin has not yet occurred. Looking ahead from $h$, the ruin probability is $r(u + ch)$, since the surplus has increased by $ch$ in the interval $(0, h)$.

2. With probability $\approx \lambda h$ a loss occurs in $(0, h)$. Let $x$ denote the amount of loss. If $x > u$, ruin occurs immediately. If $x \leq u$, we have $U(h) \approx x$ and ruin has not yet occurred. Looking ahead from $h$, the ruin probability is $r(u - x)$, since the surplus has decreased by $x$ in the interval $(0, h)$.

(3.) With probability $o(h^2)$, more than one loss occur in $(0, h)$. As $h \to 0$, this possibility can be excluded.

Combining this gives

$$r(u) = (1 - \lambda h)r(u + ch) + \lambda h \int_0^u r(u - x)F(dx) + \lambda h \int_u^\infty F(dx) + o(h^2)$$

as $h \to 0$. If we assume that $r'(u)$ exists, we get the equation

$$cr'(u) - \lambda r(u) + \lambda \int_0^u r(u - x)F(dx) + \lambda \tilde{F}(u) = 0,$$

where $\tilde{F}(u) = 1 - F(u)$. We will solve this equation with the boundary condition that $r(u) \to 0$ as $u \to \infty$ if $c > \lambda \mu$. A difficulty is that both $r(u)$ and $r'(u)$ are included in the equation. However, it turns out that $r(u)$ can be eliminated by
partial integration of the third term. Using the fact that \( d\bar{F}(x) = -F(dx) \), we get
\[
\int_0^u r(u - x)F(dx) = r(u) - r(0)\bar{F}(u) - \int_0^u r'(u - x)\bar{F}(x)dx.
\]
Substituting this in the above equation yields
\[
er'(u) = \lambda \int_0^u r'(u - x)\bar{F}(x)dx + \lambda (r(0) - 1)\bar{F}(u),
\]
Here \( r(0) \) is a constant that can be determined from the boundary condition. This is an equation that involves only \( r'(u) \). To solve it, introduce \( r = \lambda \mu/c \) and \( k(u) = (1 - F(u))/\mu = \bar{F}(u)/\mu \). The equation then becomes
\[
r'(u) = r\int_0^u r'(u - x)k(x)dx + r(r(0) - 1)k(u).
\]
This is a renewal equation that, by a convolution operation, can be written as
\[
r'(u) = r(k * r')(u) + r(r(0) - 1)k(u).
\]
This equation can be solved by an iteration that converges when \( r < 1 \), that is, when \( c > \lambda \mu \). We have
\[
r'(u) = r(r(0) - 1)\left( \sum_{m=0}^{\infty} r^m k^{m*} \right) * k(u)
= (r(0) - 1) \sum_{m=1}^{\infty} r^m k^{m*}(u).
\]
According to the boundary condition, \( \int_0^\infty r'(u) = r(\infty) - r(0) = -r(0) \), and, since \( \int_0^\infty k^{m*}(u)du = 1 \) for all \( m \), we have
\[
\int_0^\infty \sum_{m=1}^{\infty} r^m k^{m*}(u)du = \sum_{m=1}^{\infty} r^m = \frac{r}{1 - r}.
\]
Using these relations, we obtain
\[
-r(0) = (r(0) - 1) \cdot r/(1 - r), \quad \text{that is,} \quad r(0) = r.
\]
Hence, since \( l(u) = (1 - r) \sum_0^\infty r^m k^{m*}(u) \) and \( k^{0*}(u) = \delta(u) = 0 \) when \( u > 0 \), we get
\[
r'(u) = - r(1 - r) \sum_{m=1}^{\infty} r^m k^{m*}(u)
= -l(u) \quad \text{for} \quad u > 0,
\]
By integration it follows that \( r(u) = \int_0^u l(y)dy \). This is the same formula that was derived in 3.2.3.
3.2.7 Approximation of \( r(u,t) \)

So far we have been concerned with \( r(u) = P(T(u) < \infty) \). However, it is also interesting to study when ruin occurs if \( T(u) < \infty \). In the following we will prove a law of large numbers for \( T(u) \), that states that, if \( T(u) < \infty \), then with large probability \( T(u) \approx ut \) as \( t \to \infty \), where \( t \) is given by a formula that includes the Lundberg exponent \( R \). We will show that exponential inequalities, analogous to the ones for \( \Pr(S(t) \geq tx) \), hold for \( \Pr(T(u) \leq ut) \) when \( t < \bar{t} \) and for \( \Pr(ut \leq T(u) < \infty) \) when \( t > \bar{t} \). The exponent can be expressed in terms of the function \( h(x) \).

In the derivation of Chernoff’s inequality, \( P(S(t) \geq tx) \leq e^{-th(x)} \) for \( x > \lambda \mu \), in Section 2.4.1, we started with the relation \( E[e^{\xi S(t)}] = e^{t g(\xi)} \) and picked a suitable value of \( \xi \), depending on \( x \). This relation can be written as \( E[e^{\xi S(t)} - t g(\xi)] = 1 \) for all \( t \). We will first show that, for some \( \xi \)-values, this relation holds also for the stochastic times \( T(u) \) and \( T(-u) \) so that, for instance,

\[
E \left[e^{\xi S(T(u)) - T(u) g(\xi)} , T(u) < \infty\right] = 1. \tag{14}
\]

Starting from this equation, which is called Wald’s identity, we will derive inequalities for \( T(u) \) analogous to the Chernoff bounds.

Proof of Wald’s identity:

Since the process \( S(t) \) has independent increments, for \( 0 < s < t \), we have that \( S(t) - S(s) \) is independent of all events \( A_s \) and stochastic variables that concern the values of the process up to time \( s \). This implies that

\[
E \left[e^{\xi S(t) - t g(\xi)} , A_s \right] = E \left[e^{\xi (S(t) - S(s)) - (t-s) g(\xi)} \cdot e^{\xi S(s) - s g(\xi)} , A_s \right] \\
= E \left[e^{\xi (S(t) - S(s)) - (t-s) g(\xi)} \right] \cdot E \left[e^{\xi S(s) - s g(\xi)} , A_s \right] \\
= E \left[e^{\xi S(s) - s g(\xi)} , A_s \right],
\]

where the last equality comes from the fact that \( S(t) - S(s) \) has the same distribution as \( S(t-s) \) and so its generating function is \( (t-s) g(\xi) \). Now let \( A_s = \{T(u) \in (s-ds, s]\} \). Note that when the values of the process up to time \( s \) are given, we can decide if \( A_s \) has occurred or not. We get

\[
E \left[e^{\xi S(t) - t g(\xi)} , T(u) \in ds \right] = E \left[e^{\xi S(s) - t g(\xi)} , T(u) \in ds \right] \\
\approx E \left[e^{\xi S(T(u)) - T(u) g(\xi)} , T(u) \in ds \right].
\]

In the last equality we have used the facts that, since \( S(t) \) is right-continuous at the jump points, the difference between \( S(T(u)) \) and \( S(s) \) is at most \( cds \) when \( s - T(u) \leq ds \), and that, with large probability, at most one jump occurs in \( ds \). Integrating the above relation for \( s \in (0,t] \) yields

\[
E \left[e^{\xi S(t) - t g(\xi)} , T(u) \leq t \right] = E \left[e^{\xi S(T(u)) - T(u) g(\xi)} , T(u) \leq t \right].
\]

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Recalling the definition of the Esscher-transformed distribution of $S(t)$ in Section 2.4.2, we see that the left hand side can be written as $P_\xi(T(u) \leq t)$, where $P_\xi(\cdot)$ is the transformed measure. To establish Wald’s identity we have to show that this tends to 1 as $t \to \infty$. To this end, remember that $S(t)$ is still a compound Poisson process under the measure $P_\xi$, but the mean is changed to $E_\xi[S(t)] = tg'(\xi)$. Hence $E_\xi[U(t)] = t(g'(\xi) - c)$. If $\xi$ is chosen so that this is strictly positive – that is, so that $g'(\xi) > c$ – then, by the law of large numbers, $U(t) \to \infty$ with $P_\xi$-probability 1 and it follows that $P_\xi(T(u) < \infty) = 1$. Since $g'(\xi)$ is an increasing function of $\xi$, the condition that $g'(\xi) > c$ is fulfilled for $\xi > \xi_c$, where $\xi_c$ satisfies $g'(\xi_c) = c$.

To summarize, we have showed that (14) holds for $\xi > \xi_c$, where $\xi_c$ is defined via the relation $g'(\xi_c) = c$. Analogously it can be shown for $T(-u)$ that $P_\xi(T(-u) < \infty) = 1$ if $\xi < \xi_c$, since then the drift is strictly negative and $P_\xi(T(-u) < \infty) = 1$.

From the picture of the definition of the Lundberg exponent $R$ in Figure 10 it can be seen that $0 < \xi_c < R$ if $c > \lambda \mu$ and $R < \xi_c < 0$ if $c < \lambda \mu$. We will now see how Wald’s identity can be used to study $T(u)$ for $c > \lambda \mu$. First remember that $T(u)$ is defined as the first time when $U(t) > u$, that is, the first time when $S(t) > u + ct$. Together with Wald’s identity this yields that, for $\xi > \xi_c > 0$,

$$1 \geq E\left[e^{\xi S(T(-u)) - T(-u)g(\xi)}, T(-u) < \infty\right],$$

that is,

$$e^{-\xi u} \geq E\left[e^{(c\xi - g(\xi))T(u)}, T(u) < \infty\right].$$

In particular, for $\xi = R$ we get

$$P(T(u) < \infty) = r(u) \leq e^{-Ru},$$

which is referred to as Lundberg’s inequality. This inequality holds for all $u > 0$ and the exponent is the same as in the asymptotic approximation in Section 3.2.5.

The above inequality can be used to estimate $P(T(u) \leq ut)$ (compare with the Chernoff bound from Section 2.4.1). If $c\xi - g(\xi) \leq 0$, when $T(u) \leq ut$, we have $(c\xi - g(\xi))T(u) \geq (c\xi - g(\xi))ut$ so that

$$e^{-u\xi} \geq E\left[e^{(c\xi - g(\xi))T(u)}, T(u) \leq ut\right] \geq e^{(c\xi - g(\xi))ut} P(T(u) \leq ut),$$

and hence
\[ P(T(u) \leq ut) \leq e^{-u\xi - ut(c\xi - g(\xi))} = e^{-ut((c+1/t)\xi - g(\xi))}. \]

To get the best possible estimate we want to minimize the exponent. This is done by picking \( \xi \) such that \( g'(\xi) = c + 1/t \) and, recalling the definition of \( h(x) \) from Section 2.4.1, we get

\[ P(T(u) \leq ut) \leq e^{-uth(c + 1/t)}. \]

The above calculations are valid under the assumption that \( c\xi - g(\xi) \leq 0 \), that is, \( R \leq \xi \), where \( \xi \) is defined by \( g'(\xi) = c + 1/t \). Since \( g'(\xi) \) is strictly increasing, the condition that \( \xi \geq R \) is equivalent to \( g'(\xi) \geq g'(R) \), that is, \( 1/t \geq g'(R) - c \). If we introduce \( \bar{t} \), defined by the relation \( 1/\bar{t} = g'(R) - c \), we see that the above estimate holds for \( t \leq \bar{t} \).

Analogously, if we pick \( \xi \) such that \( \xi_c < \xi \) and \( c\xi - g(\xi) \geq 0 \) we can estimate \( P(ut < T(u) < \infty) \). We get

\[ P(ut < T(u) < \infty) \leq e^{-ut((c+1/t)\xi - g(\xi))}. \]

If \( g'(\xi) = c + 1/t \) and \( \xi_c < \xi \leq R \), that is, if \( c = g'(\xi_c) < c + 1/t \leq g'(R) \), or, equivalently, \( 0 < 1/t \leq 1/\bar{t} \), that is, \( t \geq \bar{t} \), then it follows that

\[ P(ut < T(u) < \infty) \leq e^{-uth(c+1/t)}. \]

For \( \xi = R \) we get \( t = \bar{t} \) and the exponent then becomes \( \bar{t}h(c + 1/t) = \bar{t}((c + 1/\bar{t})R - g(R)) = R \), since \( cR = g(R) \).

To summarize, we have shown that there is a time \( \bar{t} \), defined by the relation \( 1/\bar{t} = g'(R) - c \), such that the deviations from \( ut \bar{t} \) can be estimated by

\[ P(T(u) \leq ut) \leq e^{-uth(c + 1/t)} \quad \text{for } t \leq \bar{t}, \]

and

\[ P(ut \leq T(u) < \infty) \leq e^{-uth(c + 1/t)} \quad \text{for } t \geq \bar{t}, \]

and \( \bar{t}h(c + 1/\bar{t}) = R \) for \( t = \bar{t} \).

We will soon see that the exponent \( H(t) := th(c + 1/t) \) is a strictly convex function of \( t \) with \( \min_t H(t) = H(\bar{t}) = R \). This fact makes it possible to study \( T(u) \) when \( T(u) < \infty \). Assume for example that \( t < \bar{t} \). By Cramér’s approximation, as \( u \to \infty \) we then have

\[ P(T(u) \leq ut | T(u) < \infty) = \frac{P(T(u) \leq ut)}{r(u)} \leq \frac{e^{-uH(t)}}{r(u)} \approx \frac{e^{-u(H(t) - R)}}{C}. \]
This tends to 0 exponentially fast, since \( H(t) > R \) for \( t < \bar{t} \). Analogously it can be seen that \( P(ut \leq T(u) < \infty | T(u) < \infty) \) tends to 0 exponentially fast when \( t > \bar{t} \) and \( u \to \infty \). This has the following important interpretation: When \( t > \bar{t} \), the ruin probability \( r(u, ut) \) can be approximated by \( r(u) \approx Ce^{-Ru} \), and when \( t < \bar{t} \), we have \( r(u, ut) \ll r(u) \), since \( r(u, ut) \leq e^{-uH(t)} \) and \( r(u) \approx Ce^{-Ru} \) with \( H(t) > R \).

**Proof of the convexity of \( H(t) \):**

We have \( H(t) = t((c + 1/t)\xi - g(\xi)) = t(c\xi - g(\xi)) + \xi \), with \( g'(\xi) = c + 1/t \). When \( t \) varies, \( \xi \) varies as well, and we get

\[
dH = (c\xi - g(\xi))dt + (tc - tg'(\xi) + 1)d\xi
\]

Hence \( H'(t) = \frac{dH}{dt} = c\xi - g(\xi) \) and

\[
H''(t) = (c - g'(\xi))\frac{d\xi}{dt}
\]

\[
= -\frac{1}{t} \cdot \frac{d\xi}{dt}
\]

From the equation for \( \xi \) it follows that \( -dt/t^2 = g''(\xi)d\xi \), that is,

\[
\frac{d\xi}{dt} = -\frac{1}{t^2}g''(\xi) < 0.
\]

Thus \( H''(t) > 0 \) and we have showed that \( H(t) \) is strictly convex. The function \( H(t) \) attains its smallest value when \( H'(t) = c\xi - g(\xi) = 0 \), that is, when \( \xi = R \) and \( t = \bar{t} \). As described above, we then have \( H(\bar{t}) = \bar{t}(cR - g(R)) + R = R \). \( \Box \)

### 3.2.8 Approximation of \( r(-u, t) \)

We will now show that the above estimates of \( r(u, t) \) also hold for \( r(-u, t) = P(T(-u) \leq t) \) with small modifications when \( c < \lambda\mu \). As we have seen, in this case we have \( R < \xi_c < 0 \) and it follows from Wald’s identity that

\[
E \left[ e^{\xi S(T(-u)) - T(-u)g(\xi)} \mid T(-u) < \infty \right] = 1
\]

for \( \xi < \xi_c \). An interesting difference as compared to the previous case is that at the time of ruin we now have \( S(T(-u)) = -u + cT(-u) \). This means that

\[
E \left[ e^{-u\xi + (c\xi - g(\xi))T(-u)} \mid T(-u) < \infty \right] = 1.
\]

This is an equation for the generating function of \( T(-u) \): Put \( w = c\xi - g(\xi) \) for \( \xi < \xi_c \). Since \( \frac{d\xi}{dw} = c - g'(\xi) > 0 \) so that \( \xi := R(w) \) is uniquely determined, this is a 1-1 relation. Hence

\[
E \left[ e^{wT(-u)} \mid T(-u) < \infty \right] = e^{uR(w)}.
\]

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For \( w = 0 \) we have \( R(0) = R < 0 \) which gives the exact relation

\[
P(T(-u) < \infty) = e^{Ru} = e^{-|R|u},
\]

where \( P(T(-u) < \infty) = r(-u) \).

As before we can also estimate \( P(T(-u) \leq ut) \). If \( c\xi - g(\xi) \leq 0 \) we have

\[
e^{u\xi} \geq E \left[ e^{(c\xi - g(\xi))T(-u)}, T(-u) < ut \right] 
\geq e^{(c\xi - g(\xi))ut} P(T(-u) \leq ut)
\]

so that

\[
P(T(-u) \leq ut) \leq e^{-ut(c\xi - g(\xi)) + u\xi} = e^{-ut(c-1/t)\xi + g(\xi)} = e^{-uth(c-1/t)}
\]

if \( \xi \) is chosen such that \( g'(\xi) = c - 1/t \). This is possible if \( g(\xi) \geq c\xi \), that is, if \( \xi \leq R < 0 \) so that \( g'(\xi) = c - 1/t \leq g'(R) \). Putting \( 1/t = c - g'(R) \) gives the condition \( 1/t \leq 1/t \), that is, \( t \leq \bar{t} \). Analogously we obtain

\[
P(ut \leq T(-u) < \infty) \leq e^{-uth(c-1/t)} \quad \text{for } t \geq \bar{t}.
\]

To summarize, we have the formulas \( 1/\bar{t} = c - g'(R) \), \( H(t) = th(c-1/t) \) and \( r(-u) = e^{-|R|u} \). Furthermore,

\[
P(T(-u) \leq ut | T(-u) < \infty) = \frac{r(-u, ut)}{r(-u)} \leq e^{-u(H(t)+R)} \quad \text{for } t \leq \bar{t},
\]

and

\[
P(ut \leq T(-u) < \infty | T(-u) < \infty) = \frac{r(-u) - r(-u, ut)}{r(-u)} \leq e^{-u(H(t)+R)} \quad \text{for } t \geq \bar{t}.
\]

Since \( H(t) \) is strictly convex with \( H(t) \geq H(\bar{t}) = -R > 0 \), we can hence localize \( T(-u) \) well near \( ut \).
3.2.9 An interpretation of the modified distribution $P_R(S(t) \in dx)$

The Esscher transformed distribution $P_R(S(t) \in dx)$ is defined by

$$P_R(S(t) \in dx) = e^{Rx - tg(R)} F(t, dx)$$

and we have seen that

$$E_R\left[e^{\xi S(t)}\right] = E_R\left[e^{(\xi+R)S(t)-tg(R)}\right] = e^{t(g(\xi+R)-g(R))}.$$ 

Under this measure, we have $E_R[S(t)] = tg'(R)$ and $E_R[U(t)] = t(g'(R) - c) > 0$ when $c > \lambda\mu$. Hence, by the law of large numbers, $P_R(t) < \infty = 1$. Also, by the same theorem, since $E_R[U(u\xi)] = ut(g'(R) - c) = u$, we should expect that $T(u) \approx u\xi$ under the measure $P_R$ as $u \to \infty$. As we have just seen, given that $T(u) < \infty$, we have that $T(u) \approx u\xi$ as $u \to \infty$. Hence, it seems as if the measure $P_R$ gives an approximate description of the conditional distribution of the process $S(t)$ given that $T(u) < \infty$ as $u \to \infty$.

Because of the Markov property, this probability equals

$$P(S(t) \in dx|T(u) < \infty) = \frac{P(S(t) \in dx, T(u) < \infty)}{P(T(u) < \infty)} \approx \frac{P(S(t) \in dx, t < T(u) < \infty)}{r(u)}.$$ 

since, if $S(t) = x$ we have $U(t) = x - ct$ and so the surplus at time $t$ is $u - x + ct$. As $u \to \infty$ with $t$ fixed, we have $r(u) \approx Ce^{-Ru}$, implying that

$$\frac{r(u-x+ct)}{r(u)} \to e^{Rx-Rct}.$$ 

Hence the conditional distribution of $S(t)$ converges to $F(t, dx)e^{Rx-Rct}$ and, since $g(R) = Rc$,

$$F(t, dx)e^{Rx-Rct} = F(t, dx)e^{Rx-tg(R)} = P_R(S(t) \in dx).$$
A corresponding result holds for $T(-u)$ when $c < \lambda u$.

Using the distribution $P_R$ a fairly intuitive proof of the central limit theorem for the quantity $(T(u) - u\bar{t})/\sqrt{u}$ can be formulated as follows: If we invert the relation between $P$ and $P_R$ we see that

$$P(T(u) \leq u\bar{t} + t\sqrt{u}) = E_R \left[ e^{-R S(T(u)) + T(u) g(R)}, T(u) \leq u\bar{t} + t\sqrt{u} \right].$$

Furthermore, $U(T(u)) = S(T(u)) - cT(u) = u + Z$, where $Z$ is the overshoot over $u$ at the passage at $T(u)$. The overshoot $Z$ is bounded when $u$ is large and approximately independent of $T(u)$. Hence, because $g(R) = cR$ the exponent in this expression can be written $-R(cT(u) + u + Z) + cRT(u) = -Ru - RZ$, so that

$$P(T(u) \leq u\bar{t} + t\sqrt{u}) \approx e^{-Ru} E_R \left[ e^{-RZ} \right] P_R(T(u) \leq u\bar{t} + t\sqrt{u}).$$

The last probability can be estimated using the fact that, under the modified measure $P_R$, the process $U(t) = S(t) - ct$ has positive drift $E_R[U(t)] = t(g'(R) - c) = t/\bar{t}$ and variance $\text{Var}(U(t)) = \text{Var}(S(t)) = t\sigma^2(R) = t\sigma^2$. We can now estimate $T(u)$ as follows. The law of large numbers tells us that $U(t)/t \to 1/\bar{t}$ when $t \to \infty$ and, since $T(u) \to \infty$ as $u \to \infty$, we have $U(T(u))/T(u) \to 1/\bar{t}$ as $u \to \infty$. But, since $U(T(u)) = u + Z$ with $Z$ bounded, this implies that $u/T(u) \to 1/\bar{t}$, that is, $T(u)/u \to \bar{t}$. We can now use the central limit theorem for $U(t)$, which tells us that the quantity

$$X := \frac{U(t) - t/\bar{t}}{\sigma \sqrt{t}}$$

has an approximately $N(0,1)$ distribution when $t \to \infty$. Using this for $t = T(u)$ we get

$$X = \frac{U(T(u)) - T(u)/\bar{t}}{\sigma \sqrt{T(u)}} \approx \frac{\bar{t}U(T(u)) - T(u)}{\sigma \bar{t}^{3/2} \sqrt{u}},$$

and, since $U(T(u)) = u + Z$, this can be written

$$X = \frac{\bar{t}u - T(u) + \bar{t}Z}{\sigma \bar{t}^{3/2} \sqrt{u}}.$$

Since $Z$ remains bounded, $Z/\sqrt{u}$ can be neglected when $u \to \infty$ and we finally get the Gaussian approximation

$$\frac{T(u) - u\bar{t}}{\sqrt{u}} \approx -\sigma \bar{t}^{3/2} X$$
under the measure $P_R$ and hence
\[
P_R \left( \frac{T(u) - u\bar{t}}{\sqrt{u}} \leq \sigma \bar{t}^{3/2} x \right) \approx \Phi(x).
\]
Finally we get the corresponding formula for the measure $P$,
\[
P \left( \frac{T(u) - u\bar{t}}{\sqrt{u}} \leq \sigma \bar{t}^{3/2} x \right) \approx C_R e^{-Ru} \Phi(x)
\]
with $C_R = \lim_{u \to \infty} E_R[e^{-RZ}]$. The value of the constant $C_R$ can be deduced from the Cramér-Lundberg approximation $r(u) = P(T(u) < \infty) \approx C e^{-Ru}$ (see Section 3.2.5). If we let $x \to \infty$ we see that $C_R = C = (c - g'(0))/(g'(R) - c)$. The asymptotic variance of $T(u)$ is hence $u\bar{t}^3 \sigma^2 = ug''(R)/(g'(R) - c)^3$.

3.2.10 An interesting property of a composite system

Let us collect the approximate formulas for $r(u)$ and $T(u)$ as follows. The exponent $R$ and the time $\bar{t}$ are determined by $c = g(R)/R$ and $\bar{t} = 1/(g'(R) - c)$. The approximate time of ruin is $\bar{T} = u\bar{t} = u/(g'(R) - c)$ and, if we define $C = (c - g'(0))/(g'(R) - c)$, then $r(u) \approx r(u, t) \approx Ce^{-Ru}$ for $t > \bar{T}$. This means that, if our planning horizon is $\bar{T}$, then the probability of ruin, $r(u)$, is a reasonable approximation for the finite time ruin probability $r(u, t)$ if $t > \bar{T}$. If ruin happens it takes place for $T(u) \approx \bar{T}$.

Let us now consider a system consisting of two independent pieces so that $S(t) = S_1(t) + S_2(t)$ with $S_1(t)$ and $S_2(t)$ independent, and hence $g(\xi) = g_1(\xi) + g_2(\xi)$. It is interesting to compare the quantities of the pieces to those of the total system. It they have the same $R$, we get
\[
c = \frac{g(R)}{R} = \frac{g_1(R)}{R} + \frac{g_2(R)}{R} = c_1 + c_2,
\]
and, if they have the same $\bar{T}$, we obtain
\[
u = \bar{T}(g'(R) - c) = \bar{T}(g'_1(R) - c_1 + g'_2(R) - c_2) = u_1 + u_2.
\]
If we use these $c_i$ and $u_i$, we get
\[
r(u) \approx Ce^{-Ru} \\
= Ce^{-Ru_1}e^{-Ru_2} \\
\approx C_1e^{-Ru_1}C_2e^{-Ru_2} \\
\approx r_1(u_1)r_2(u_2),
\]
since, from the fact that

\[ C = \frac{c - g'_1(0) + c_2 - g'_2(0)}{g'_1(R) - c_1 + g'_2(R) - c_2}, \]

it follows that \( C_1 \leq C_1 C_2 / C \leq C_2 \) if \( C_1 \leq C_2 \), that is, the constants are comparable.

There is hence a natural decomposition of \( c \) and \( u \) into \( c_1 + c_2 \) and \( u_1 + u_2 \), so that if we have a common \( \bar{T} \) and \( R \), then \( r(u) \approx r_1(u_1) r_2(u_2) \), which is the probability that both systems are ruined. None of the systems is so to speak unnecessarily safe compared to the other. This is also an example of decentralized planning: In order to calculate \( r(u) \) the central actuary only has to give the values of \( R \) and \( \bar{T} \) to the local actuaries who can then calculate \( r_1(u) \) and \( r_2(u) \) and return them, and \( r(u) \approx r_1(u) r_2(u) \).
4 Summary of the formulas

In this section we give a concise summary of the formulas that have been derived in the notes.

The individual risk model

\(X = \text{total amount of loss} = \sum_i x_i M_i\), where \(\{M_i\}\) are Bernoulli variables with \(P(M_i = 1) = p_i = 1-q_i\).

Moments: \(E[X] = \sum_i x_i p_i\)
\(\text{Var}(X) = \sum_i x_i^2 p_i q_i\)

Generating function: \(E\left[e^{\xi X}\right] = \prod_i (q_i + p_i e^{\xi x_i})\)

Compound Poisson approximation:
\(X \approx S = \sum_i x_i N_i\), where \(\{N_i\}\) are Poisson variables with \(e^{-\lambda_i} = q_i\)

Generating function: \(E\left[e^{\xi S}\right] = \exp \left\{ \sum_i \lambda_i (e^{\xi x_i} - 1) \right\} = e^{g(\xi)}\), where \(g(\xi) = \sum_i \lambda_i (e^{\xi x_i} - 1)\)

The collective risk model

\(S(t) = \text{total amount of loss in } (0, t)\)
\(N(t) = \text{number of accidents in } (0, t)\)
\(X_i = \text{the losses in the accidents}\)

\(S(t) = \sum_1^{N(t)} X_i\)
\(\{N(t)\}\) is a Poisson process with \(E[N(t)] = \lambda t\).
\(\{X_i\}\) are i.i.d. with distribution \(F(dx)\), \(E[X_i] = \mu\), \(E[X_i^2] = \nu\).

The distribution of \(S(t)\) is \(F(t, dx)\).

Moments: \(E[S(t)] = tg'(0) = t\lambda\mu\)
\(\text{Var}(S(t)) = t g''(0) = t\lambda\nu\).

Panjer-recursion for the density of \(S(t)\)
Assume that \(X_i\) have a discrete distribution with \(P(X_i = nd) = f_n\). Then \(P(S(t) = md) = g_m\) are given by the recursion
\[
\begin{align*}
m g_m &= \lambda t \sum_{n=1}^{\infty} n f_n g_{m-n}, \quad m = 1, 2, \ldots \\
g_0 &= e^{-\lambda t}.
\end{align*}
\]

Approximations of \(P(S(t) > tx)\)

Entropy function: \(h(x) = \max_{\xi} \{ x\xi - g(\xi) \} \)
\(= x\xi_x - g(\xi_x)\), with \(\xi_x\) defined by \(g'(\xi_x) = x\).

The functions \(g(\xi)\) and \(h(x)\) are convex.
We have 
\[ g(\xi) = \max_x \{ \xi x - h(x) \} \]
\[ = \xi x_\xi - h(x_\xi), \text{ with } x_\xi \text{ defined by } h(x_\xi) = \xi. \]

The functions \( x = g'(\xi) \) and \( \xi = h'(x) \) are inverses of each other.

Chernoff’s bound:
\[
\begin{cases} 
  P(S(t) \geq tx) \leq e^{-th(x)} & \text{if } x \geq \lambda \mu; \\
  P(S(t) \leq tx) \leq e^{-th(x)} & \text{if } x \leq \lambda \mu.
\end{cases}
\]

Esscher’s approximation:

The Esscher transform of \( F(dx) \) is \( F_a(dx) = e^{ax} F(dx) / f(a) \) with \( f(a) = \int_0^\infty e^{ax} F(dx) \).

\[
E_a[e^{\xi X}] = f(\xi + a) / f(a)
\]
\[
E_a[e^{\xi S(t)}] = e^{t(g(\xi + a) - g(a))} = e^{tg_a(\xi)}
\]

The transform of \( F(t, dx) \) is \( P_a(S(t) \in dx) = F_a(t, dx) = e^{ax - tg(a)} F(t, dx) \).

Moments:
\[
E_a[S(t)] = tg'(a)
\]
\[
\text{Var}_a(S(t)) = tg''(a)
\]

Esscher’s approximation tells us that
\[
P(S(t) \geq tx) \approx \frac{e^{-th(x)}}{\sqrt{2\pi a \sqrt{tg''(a)}}}
\]
with \( x = g'(a) \geq \lambda \mu = g'(0), a \geq 0 \). This is valid for a continuous distribution.

For a discrete distribution with span \( d \), the factor \( a \) is changed into \( A(d) = (1 - e^{-ad}) / d \).

Ruin probabilities

\( U(t) = S(t) - ct = \text{net amount of loss in } (0, t) \)
\( u = \text{initial capital} \)
\( T(u) = \min \{ t; \ U(t) > u \} = \text{time of ruin} \)
\( T(-u) = \min \{ t; \ U(t) = -u \} \)
\( r(\pm u, t) = P(T(\pm u) \leq t) = \text{ruin probabilities in finite time} \)
\( r(\pm u) = P(T(\pm u) < \infty) = \text{ruin probabilities in infinite time} \)

For \( u = 0 \), we have the explicit formula
\[
P(T(0) > t, S(t) \in dx) = \left( 1 - \frac{x}{ct} \right)_+ F(t, dx)
\]
and hence
\[
P(T(0) > t) = 1 - r(0, t) = \int_0^{ct} \left( 1 - \frac{x}{ct} \right) F(t, dx).
\]
The distribution of \( T(-u) \)
For \( ct \geq u > 0 \) we have \( P(T(-u) \in dt) = (u/ct)F(t, cdt - u) \). Hence
\[
r(-u, t) = \int_{u/c}^{\infty} \frac{u}{cs} F(s, cds - u)
\]
and
\[
r(-u) = \int_{u/c}^{t} \frac{u}{cs} F(s, cds - u) = \int_{0}^{\infty} \left(\frac{u}{x + u}\right) F\left(\frac{x + u}{c}, dx\right) \text{ with } x = cs - u.
\]
If \( c > \lambda\mu \), we have \( r(-u) = 1 \).

The distribution of \( T(u) \)
Seal's formula:
\[
r(u, t) = \int_{u/c}^{\infty} F(t, dx) + \int_{0}^{t} F(s, u + cds)\bar{r}(0, t - s)
\]
where \( \bar{r}(0, t - s) = 1 - r(0, t - s) \).

Cramér's formula for \( r(u) \)
The upcrossings \( \{U_k\} \) are i.i.d. with \( r := P(U_1 > 0) = \lambda\mu/c \) if \( c > \lambda\mu \), and \( U_1 \) has the conditional density \( k(u) = (1 - F(u))/\mu \), given that \( U_1 > 0 \).
The density of \( \bar{U} = \max_{t \geq 0} U(t) \) is \( l(u) = (1 - r) \sum_{n} r^n k^\ast(u) \) and we have
\[
r(u) = P(\bar{U} > u) = \int_{u}^{\infty} l(y)dy.
\]

Panjer-approximation of \( r(u) \)
Approximate the density \( k(u) \) by a discrete one with masses \( \{k_n\} \) for \( u = nd, n = 1, 2, \ldots \). Then the corresponding approximation \( \{l_n\} \) for \( l(u), u = nd \), can be calculated by the iteration
\[
\begin{align*}
  l_n &= r(k_1l_{n-1} + \ldots + k_nl_0) \quad \text{for } n \geq 1; \\
  l_0 &= 1 - r.
\end{align*}
\]
The ruin probability \( r(u) \) is approximated by \( r_n = \sum_{n} l_m \) for \( u = nd \).

Cramér-Lundberg’s approximation of \( r(u) \)
For \( c > \lambda\mu \), let \( R \) be the positive root of the equation \( g(R) = cR \) and define \( C = (c - g'(0))/(g'(R) - c) \). Then
\[
r(u) \approx Ce^{-R}u \quad \text{as } u \to \infty.
\]
Similarly, for $c < \lambda \mu$, we have $r(-u) = e^{R_0u}$, where $R$ is the negative root of $g(R) = cR$.

**Approximation of $r(u,t)$**

For $c > \lambda \mu$, define $\bar{T} = u\bar{t} = u/(g'(R) - c)$. Then $T(u) \approx \bar{T}$ if $T(u) < \infty$. More accurately, if $H(t) = th(c + 1/t)$, then

$$P(T(u) \leq ut | T(u) < \infty) \leq C^{-1} e^{-u(H(t)-R)}$$

for $t < \bar{t}$ and

$$P(ut \leq T(u) < \infty | T(u) < \infty) \leq C^{-1} e^{-u(H(t)-R)}$$

for $t > \bar{t}$

and $R = \min_t H(t) = H(\bar{t})$. Similarly, for $c < \lambda \mu$, if we define $\bar{T} = u\bar{t} = u/(c - g'(R))$ and $H(t) = th(c - 1/t)$, we have

$$P(T(-u) \leq ut | T(-u) < \infty) \leq e^{-u(H(t)+R)}$$

for $t < \bar{t}$ and

$$P(ut \leq T(-u) < \infty | T(-u) < \infty) \leq e^{-u(H(t)+R)}$$

for $t > \bar{t}$

and $-R = \min_t H(t) = H(\bar{t})$.

**Interpretation of the transformed distribution of $S(t)$**

When $c > \lambda \mu$, the transformed distribution

$$F_R(t, dx) = P_R(S(t) \in dx) = e^{Rx-tg'(R)} F(t, dx)$$

is equal to $\lim_{u \to \infty} P(S(t) \in dx | T(u) < \infty)$ and the corresponding result holds for $T(-u)$ when $c < \lambda \mu$. Hence we have

$$E[S(t) | T(u) < \infty] \to E_R[S(t)] = tg'(R) \quad \text{as } u \to \infty.$$

This explains the formula for $\bar{T}$, because $E_R[U(t)] = t(g'(R) - c)$ so $\bar{T}$ is that value of $t$ for which this is equal to $u$.

**The central limit theorem for $T(u)$**

When $u \to \infty$ we have

$$P \left( \frac{T(u) - u\bar{t}}{\sqrt{u}} \leq \alpha \bar{t}^{3/2} x \right) \approx Ce^{-Ru} \Phi(x),$$

with $C = (c - g'(0))/(g'(R) - c)$, $\bar{t} = 1/(g'(R) - c)$ and $\sigma^2 = g''(R)$. 


5 Notes and references

The notes on risk theory by Harald Cramér from 1930 [3] still form a very readable introduction to the subject. The idea of using Lemma 3.1 – the so called ballot theorem – to derive the formulas for ruin probabilities is developed by Lajos Takács in [6]. Hopefully our treatment is more understandable. The use of tools from large deviation theory to derive asymptotic estimates is developed by the author in [5]. An alternative way of studying $T(u)$, which allows a central limit theorem to be proved is developed by Bengt von Bahr in [2]. A modern and comprehensive treatment of the theory of ruin probabilities is given in [1] and [4].

[1] Asmussen, S: Ruin probabilities, World Scientific 2000.

[2] von Bahr, B: Ruin probabilities expressed in terms of ladder height distributions, Scand. Actuarial J. 1974, 190-204.

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[4] Klugman, S, Panjer H, Willmot, G: Loss Models, 2 ed., Wiley 2004.

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[6] Takács, L: Combinatorial methods in the theory of stochastic processes, Wiley 1967.