Approximate Bayesian inference with queueing networks and coupled jump processes

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Abstract

Queueing networks are systems of theoretical interest that give rise to complex families of stochastic processes, and find widespread use in the performance evaluation of interconnected resources. Yet, despite their importance within applications, and in comparison to their counterpart stochastic models in genetics or mathematical biology, there exist few relevant approaches for transient inference and uncertainty quantification tasks in these systems. This is a consequence of strong computational impediments and distinctive properties of the Markov jump processes induced by queueing networks. In this paper, we offer a comprehensive overview of the inferential challenge and its comparison to analogue tasks within related mathematical domains. We then discuss a model augmentation over an approximating network system, and present a flexible and scalable variational Bayesian framework, which is targeted at general-form open and closed queueing systems, with varied service disciplines and priorities. The inferential procedure is finally validated in a couple of uncertainty quantification tasks for network service rates.

Keywords: Queueing networks, Bayesian variational inference, mean-field methods, Markov jump process, non-homogeneous counting process

1 Introduction

Queueing networks (QNs) are systems of theoretical and practical interest in the design of computing applications (Kleinrock, 1976), and in the optimization of business processes arising in factories, shops, offices or hospitals (Buzacott and Shanthikumar, 1993; Koole and Mandelbaum, 2002; Osorio and Bierlaire, 2009). They are formed by interconnected resources routing and processing jobs, and their behaviour often gives rise to complex families of stochastic processes. In applications, they provide the means to assess modifications, diagnose performance and evaluate robustness in multiple service infrastructures.

The quantitative basis for the evaluation and optimal design of a networked system is a set of estimates for the service requirements in the resources, and there exist several inferential approaches designed for this purpose. However, the task is challenging as the observations often provide little indirect information (Sutton and Jordan, 2011). The starting point for a
A complete inferential study is a set of measurements such as queue lengths, visit counts or response times, along with a likelihood function for network process trajectories. Unfortunately, there exist strong computational hurdles in the design of an efficient framework that can integrate over the uncertainty (Perez et al., 2017). As a consequence, real-world solutions are mostly centred around steady-state metrics (Kraft et al., 2009), end-to-end measurements (Liu et al., 2006) or fluid approximations (Hayden et al., 2012), and rely on complete data. Typically, the inferential task is addressed by means of strict product-form model assumptions (cf. Baskett et al. (1975)), which are inapplicable to a complete transient analysis. This implies that the effects of system uncertainty in the predictive outcomes are not well understood, and the motivation for a Bayesian approach is clear (cf. Armero and Bayarri (1994)). For a review on the evaluation of commonly used inference procedures in queueing theory, we refer the reader to Spinner et al. (2015). In this paper, we offer a formal presentation to the inferential and uncertainty quantification problem with QNs. Focusing on the underlying Markov jump processes (MJPs), we review the properties and complexity posed by varied system configurations, and we discuss the relation to analogue statistical tasks in domains such as genetics or mathematical biology. Within our main technical contributions, we propose to address the problem by means of an augmented process that leads to a flexible approximate Bayesian framework, which may be targeted at general-form open and closed QNs. The resulting inference procedure is suitable for single or multi-class Markovian systems with either finite or infinite processors, multiple types of service disciplines and probabilistic routings.

Bayesian inferential procedures for the estimation of service requirements in QNs have recently been explored in both open (Sutton and Jordan, 2011) and closed (Wang et al., 2016) systems. Additionally, a flexible approach to uncertainty quantification in open systems has been reported in Perez et al. (2017). These methods vary in their scope and whether they carry inference by means of transient analyses (i.e. conditional transition rates and a Markov family of transition probabilities) or steady-state metrics (e.g. product-form equilibrium distributions). In common, they each focus on a reduced type of system and, most importantly, employ Markov Chain Monte Carlo sampling procedures that suffer scalability problems due to long mixing times and strong correlation properties across variables. In its simplest transient form, the work above further relates to inferential tasks with jump processes common in the applied sciences, such as in mathematical biology (Hobolth and Stordal, 2009) or genetics (Golightly and Wilkinson, 2015), to name a few. In all cases, intensive numerical procedures are still inefficient, hard to implement and often only applicable to a small class of problems, despite recent advances in computational approaches (Rao and Teh, 2013). The major impediments are posed by complex structured models, along with event synchronization and countably-infinite support spaces (Bobbio et al., 2008; Rao and Teh, 2013; Georgoulas et al., 2017). In the specific case of QNs, job routings, queue loads, priorities and the existence of feedback loops create strong and characteristic temporal dependency structures in the underlying stochastic dynamics (Sutton and Jordan, 2011). This, along with jump synchronization or coupling properties in the resources (due to instantaneous routings) differentiates the problem from alternative application domains, and requires the design of purpose-specific inferential tools.

In this work, the reader will find a probabilistic hierarchical formulation of queueing systems, which enables the understanding of uncertainty quantification challenges presented by classes of open and closed networks. The paper further describes a Bayesian approximate inferential framework that overcomes synchronization problems and scales to complex, structured and coupled stochastic jump processes. The methods derived build on existing variational mean-field theory for continuous jump systems (Opper and Sanguinetti, 2008; Cohn et al., 2010), and in order to overcome their computational limitations, we discuss an
alternate optimization procedure with slack variables and inequality constraints. This allows us to study the behaviour of network models by means of a suitably-defined augmented family of independent inhomogeneous counting processes, which model interactions among the resources. We offer evidence of the procedure providing a flexible and scalable transient approach for the estimation of resource demands. For this purpose, we present worked examples addressing various inference and uncertainty quantification tasks for service diagnosis and system evaluation.

The rest of the paper is organised as follows. In Section 2 we provide a Bayesian hierarchical formulation of a queueing system, we further introduce the inferential problem and we overview a running case for clarity in the presentation. Section 3 introduces an approximating network model and offers a summary of the main results to be presented later in the paper. Sections 4 and 5 include the main contributions in our work; these discuss the treatment of the network system by means of interactions in network resources, and further present the results, proofs and technical details that contribute to the final inferential algorithm. In Section 6 we guide the reader through some examples and in Section 7 we conclude the paper with a discussion.

2 Queueing systems and jump processes

In the following, we employ shorthand notation for densities, base measures and distributions whenever these are clear from the context. From here on, let \((\Omega, \mathcal{F})\) denote a measurable space with the regular conditional probability property, supporting the various rates, trajectories and observations. A general form queueing network comprises some \(M \in \mathbb{N}\) service stations along with a set of job classes \(C\). The stations are connected by a network topology that governs the underlying routing mechanisms; when a job is serviced in a station, it can either queue for service at a different node, or depart the network. Such topology is often defined as a set of routing probability matrices \(\{P^c\}_{c \in C}\), with elements \(p^c_{i,j}\) that denote the probability for a class \(c \in C\) job to immediately transit to queueing station \(j\) after service completion in station \(i\), for all \(0 \leq i, j \leq M\). In open queueing systems, the index 0 is used as a virtual external node that represents the source and destination of job arrivals and departures to, and from, the network. In closed systems, this index may either not exist, or instead refer to a delay server that routes departing jobs back into the network. Also, it holds that \(\sum_{j=0}^{M} p^c_{i,j} = 1\), for all \(0 \leq i \leq M, c \in C\).

In this work, we address time-homogeneous Markovian systems that are parametrized by exponential inter-arrival and service times, with non-negative rates \(\mu = \{\mu^c_i \in \mathbb{R}_+ : 0 \leq i \leq M, c \in C\}\), which may vary across service stations and job classes. The servers in the network stations may have finite or infinite processors, and service disciplines can differ across a range of processor sharing (PS) policies, first-come first-served (FCFS) and variations including last-come first-served (LCFS) or random order (RO) nodes. In some cases, FCFS processors may require shared processing times across the various job classes (cf. Baskett et al. (1975)). For simplicity and ease of notation, class switching, service priorities or queue-length dependent service rates are not discussed in detail, however, these follow naturally and we later present some examples of such instances. Under standard exponential service assumptions, the underlying system behaviour is described by an MJP \(X = (X_t)_{t \geq 0}\) with values defined in a measurable space \((S, \mathcal{P}(S))\). Here, \(S\) denotes a countable set of feasible states in the network, usually infinite in open or mixed systems and finite within closed ones; \(\mathcal{P}(S)\) denotes the power set of \(S\). We allow for \(S\) to support vectors of integers that represent job counts across the various
class types and service nodes, and denote by $X_t^{i,c}$ the number of class $c$ jobs in station $i > 0$ at time $t \geq 0$. Note that here we ignore the loads in delay nodes ($i = 0$) within closed systems, since these are uniquely determined given the number of jobs in the remaining stations. The infinitesimal generator matrix $Q$ of $X$ is such that

$$\mathbb{P}(X_{t+dt} = x'|X_t = x) = \mathbbm{1}(x = x') + Q_{x,x'} dt + o(dt)$$

for all $x, x' \in S$. This can be an infinite matrix, it is generally sparse and its entries describe rates for transitions across states in $S$. Rows in $Q$ must sum to 0 so that $Q_{x,x'} \geq 0$ for all $x \neq x'$, and $Q_x := Q_{x,x} = -\sum_{x' \in S, x \neq x'} Q_{x,x'}$.

Hence, jumps in the process $X$ are caused by jobs being routed through nodes in the underlying network model. We often say that a state $x \in S$ is accessible from $x \in S$, and write $x \xrightarrow{ij,c} x'$ for its corresponding jump, if $x'$ may be reached from $x$ by means of a class-$c$ job transition between the stations $i$ and $j$, in the direction $i \rightarrow j$. We further denote

$$T = \{(i,j,c) \in \{0,\ldots,M\}^2 \times C : p_{i,j}^c > 0\}$$

for the finite set of all feasible job transitions in the system, and we remark that the generator $Q$ of $X$ is populated by some positive real-valued rates $\lambda = \{\lambda_{\eta} \in \mathbb{R}_+ : \eta \in T\}$ that define the intensities for these job routings, with $\lambda_{i,j,c} = \mu_i^c \cdot p_{i,j}^c$ for all $(i,j,c) \in T$.

In Figure 1 we observe diagrams that illustrate this notation in an open single-class network. On the left, we see 3 stations with different rates, disciplines and server counts. The topology $P$ is such that $|T| = 5$ and $p_{0,1} = 1 - p_{0,2} \in (0,1)$, $p_{1,3} = p_{2,3} = p_{3,0} = 1$ ($p_{i,j} = 0$ otherwise). On the right, we find the corresponding job transition rates across the 4 pairs of connected nodes. In this single-class example, $X$ monitors counts across the stations s.t. $X_t = (X^{1,c}_t, X^{2,c}_t, X^{3,c}_t) \in S$ for all $t \geq 0$; also, the generator $Q$ is an infinite matrix with $Q_{x,x'} = \lambda_{i,j} \cdot (K_i \wedge x_i)$ for all pairs $x, x' \in S$ with associated transition $x \xrightarrow{ij} x'$, where $K_i, x_i \in \mathbb{N}_0$ denote the number of processors and the queue-length within station $i \geq 0$. We finally have $K_1 = 1, K_2 = \infty$ and $K_3 = 2$; at the virtual node, it holds $K_0 \wedge x_0 = 1$ always. Thus, note that transition rates in $X$ further depend on the network loads, and resemble kinetic laws within chemical reaction models [Georgoulas et al., 2017].

![Figure 1: Left, open bottleneck network with 3 service stations. Shaded circles indicate servers, queueing areas are pictured as empty rectangles. The box is a probabilistic junction for the routing of arrivals. Right, job transition intensities across network nodes.](image)

### 2.1 A hierarchical formulation of queueing systems

In order to capture model uncertainty and reflect ignorance regarding network parameters, we employ a Bayesian multilevel formulation. Here, rates in $\lambda$ have a prior distribution (or image) $\mathbb{P}_{\lambda} \equiv \lambda_* \mathbb{P}$ under a reference measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$. We assume this to admit a density...
This is assumed to be a continuous-time Markov chains parametrized by a collection of hyper-parameters, and analoge modelling choices for the infinitesimal rates associated with fixed values in $\lambda$.

Let $2.2$ Network observations and problem statement

appropriate framework for approximate transient analyses, parameter inference and reverse modelling choice is not suitable for traditional probabilistic studies of queueing systems by means of balance equations, due to parameter uncertainty; however, we will show that it offers an appropriate framework for approximate transient analyses, parameter inference and reverse uncertainty quantification tasks.

for all $B \in \Sigma_X$ (see Appendix $A$ for details). In this case,

$$f_X|\lambda(t, x) = \pi(x_0) e^{Q_x(T-t)} \prod_{i=1}^I Q_{x_{i-1},x_i} e^{Q_{t_{i-1},x_{i-1}}},$$

for every pair of ordered times $t = \{0, t_1, \ldots, t_I\}$ in $[0, T]$ and states $x = \{x_0, \ldots, x_I\}$. Here, $\pi(\cdot)$ denotes an arbitrary distribution over initial states, and $Q \equiv Q(\lambda)$ is the matrix of infinitesimal rates associated with fixed values in $\lambda$. The queueing network model is thus fully parametrized by a collection of hyper-parameters, and analogue modelling choices for the Bayesian study of continuous-time Markov chains (CTMCs) or MJPs can be found in Huisken et al. (2002), Baele et al. (2010) or Zhao et al. (2016), to name a few. We finally note that this modelling choice is not suitable for traditional probabilistic studies of queueing systems by means of balance equations, due to parameter uncertainty; however, we will show that it offers an appropriate framework for approximate transient analyses, parameter inference and reverse uncertainty quantification tasks.

2.2 Network observations and problem statement

Let $T > 0$ denote some arbitrary terminal time and $x_0 \in S$ an initial state in $X$. For simplicity, this is assumed to be a $0$-valued vector, where no jobs populate the system. Now, let $0 \leq t_1 < \cdots < t_K \leq T$ denote some fixed network monitoring times along with observation variables $\{O_k \in \mathcal{O}, k = 1, \ldots, K\}$, for some arbitrary support set $\mathcal{O}$, such that

$$P\left(\prod_{k=1}^K O_k^{-1}(o_k)|X\right) = \prod_{k=1}^K P(O_k^{-1}(o_k)|X) = \prod_{k=1}^K f_{O_k|X_k}(o_k)$$

for any sequence of elements $o_1, \ldots, o_K$ where $o_k$ denotes the time-$t_k$ network observation across all nodes. Hence, any two observations are mutually independent if conditioned on their network states. Also, the term $f_{O_k|X_k}$ stands for a conditional mass function assigned to the measurements across the $M$ nodes, and defined w.r.t a counting measure $\mu_O$. In real applications, network monitoring is often distributed across the different stations, hence, this term will further factor across network components. In this paper it is assumed that $f_{O_k|X} > 0$ (everywhere) for all $x \in S$, however, deterministic observations such as queue lengths can be easily approximated by means of regularised indicator functions; we discuss such examples within Section $6$.

Additionally, extensions to continuous settings are straightforward.

Now, let $P(A|o_1, \ldots, o_K), A \in \mathcal{F}, (o_1, \ldots, o_K) \in \mathcal{O}^K$ denote the regular conditional probability across global events and observations. Our interest lies in its induced distribution over the intensity rates (which we denote $P_{\lambda|o_1,\ldots,o_K}$, often referred to as the posterior. This
posterior exists and admits a density carried by its corresponding prior (see Appendix A), moreover, the transformation is proportional to a weighted product of network paths, and defined by the Radon-Nikodym derivative

\[
\frac{dP_{\lambda|o_1,\ldots,o_K}}{dP_{\lambda}} = \frac{\int_X \prod_{k=1}^K P(O_k^{-1}(o_k)|t,x) f_{X|\lambda}(t,x) \mu_X(dt,dx)}{P(O_1^{-1}(o_1) \cap \ldots \cap O_K^{-1}(o_K))},
\]

which corresponds to Bayes’ equation. There, the denominator denotes a normalising constant that integrates over trajectories and rates. This transformation will often induce a density representation \(f_{\lambda|o_1,\ldots,o_K}\) for the posterior distribution w.r.t a suitable (Lebesgue) base. In these cases, we may think of the above derivative as a Likelihood-ratio. However, this ratio poses a tractability problem, that is, the integral over trajectories cannot be computed analytically and must be approximated. This is a common problem in inferential tasks with jump processes (cf. Hobolth and Stone (2009); Rao and Teh (2013)), and proposed solutions often rely on intensive MCMC procedures that iterate between trajectories and parameters; including direct sampling, rejection sampling or uniformization-based methods. Yet, algorithms are still hard to implement, computationally demanding or only applicable to reduced classes of problems. In the case of queueing networks and similar coupled (or synchronized) models, the strong temporal dependencies in the stochastic trajectories \(X\) impose hard coupling properties amongst rates and paths (Sutton and Jordan, 2011). In practice, this limits the applicability of state of the art MCMC solutions to the simplest types of networks (Perez et al., 2017).

In the following, we formally describe the inherent complexity of jump processes induced by networks of queues, and build on the mean-field methods presented in Opper and Sanguinetti (2008) and references therein, in order to describe a variational Bayesian framework that scales to complex systems and offers an approximation to the induced rate densities under the posterior measure. This is however not straightforward; first, we must address coupled jump systems where state changes (a job arrival or departure) in each network resource are typically synchronized to an arrival or departure in another node. This prevents the use of simplifying independence assumptions previously presented in Opper and Sanguinetti (2008); Cohn et al. (2010). Second, we ought to account for uncertainty within the prior distribution of the infinitesimal rates, and inspect prior conjugacy, as opposed to exercising simpler point calibrations of the parameter values. Third, we must address a functional optimization task by imposing inequality constraints, and ensuring that previously reported computational bottlenecks are not reproduced in our inferential algorithm. Next, we present and explore an augmented model of inhomogeneous counting processes for interactions in the resources. The reader will notice that while we draw motivation from queueing systems, the method is relevant for the study of a broader family of jump processes, with applications in stochastic epidemics, gene regulatory networks and many biological phenomena.

2.3 Closed network example

For clarity in the presentation and notation, and in order to ease the introduction of later formulae, we first discuss a toy closed network example as shown in Figure 2. This running case will further facilitate the discussion of our results and experiments in Section 6. The network includes one FCFS service station, with \(K_1 = 1\) processing unit, along with a delay node, together processing a population of \(N\) jobs cyclically in a closed loop. All jobs belong to the same class and have equal service rates, we denote by \(\mu_1\) the job processing rate within the service station. On completion, a job proceeds to the delay node where it awaits for an exponentially distributed time before being routed back to the queue. We use \(\mu_0\) to denote the
delay rate; and note that the arrival rate to the queue is directly proportional to the number of jobs at the delay.

![Figure 2: Closed queueing network with a single FCFS service station and a delay.](image)

Both nodes are independent and $\mu_0$ is fixed in order to ensure model identifiability within the service station. In this instance, the network topology is deterministic and trivial, and the evolution of $X = (X_t)_{t \geq 0}$ monitors the total number of jobs within the service station, with $X_0 = 0$. The generator $Q$ of $X$ is finite and s.t.

$$Q = \begin{bmatrix} 0 & N\mu_0 & 0 & \cdots & 0 & 0 \\ 1 & -N\mu_0 + \mu_0 - \mu_1 & (N-1)\mu_0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ N-1 & 0 & 0 & \cdots & \mu_1 - (\mu_0 + \mu_1) & \mu_0 \\ N & 0 & 0 & \cdots & 0 & -\mu_1 \end{bmatrix},$$

where row and column labels denote the number of jobs in the queueing node. Since $\mu_0$ is fixed, our interest lies in inferring $\lambda \equiv \lambda_{1,0} = \mu_1 \cdot p_{1,0} = \mu_1$. We further assign to this rate a prior distribution $P_\lambda \equiv \lambda_\beta P$ with density $f_\lambda$ such that its hyperparameters fix some reasonably uninformative prior knowledge on the system. We monitor the delay node and FCFS service station at fixed and equally spaced times $t_1 < \cdots < t_K$ in an interval $[0,T]$. Here, variables $O_k$ are supported on $O = \{0, \ldots, N\}^2$ and the observation model factors across the network components s.t. $f_{O_k}(o) = f_{O_k|N=0}(o_0) \cdot f_{O_k|X}(o_1)$ with $f_{O_k|X}(o) = \frac{1}{N} + \mathbb{I}(o=x) \cdot (1 - \frac{N-1}{N} \epsilon)$ and

$$P(O_k^{-1}(o)|X) = \begin{cases} (1-\epsilon)^2 & o_0 = N - X_{t_k}, o_1 = X_{t_k}, \\ (\epsilon/N)^2 & o_0 \neq N - X_{t_k}, o_1 \neq X_{t_k}, \\ (1-\epsilon) \cdot \epsilon/N & \text{otherwise}. \end{cases}$$

for $\epsilon > 0$ and all $k = 1, \ldots, K$. This accounts for some $\%100 \cdot \epsilon$ faulty measurements, also, we note that a system with discrete observations is approximated as $\epsilon \to 0$. Now, fix $N = 50$, $\mu_0 = 0.1$, $\epsilon = 0.2$, $T = 100$, $K = 50$ and assume there exist some sample observations $o_1, \ldots, o_K$ from a model realization in this closed network. These can be easily produced from (4) given a trajectory $(X_t)_{t \in [0,T]}$. In order to produce the trajectory from (1), given the service rates, we may employ Gillespie’s algorithm [Gillespie, 1977] or faster uniformization based alternatives [Rao and Tel, 2013].

Finally, assume that $f_\lambda$ denotes a Gamma density function. In this case, the transformation $dP_{\lambda|o_1,\ldots,o_K}/dP_\lambda$ is such that, conditioned on $o_1, \ldots, o_K$, the distribution over $\lambda$ admits a density carried by a Lebesgue measure $\mu_\lambda$, so that $dP_{\lambda|o_1,\ldots,o_K} = f_{\lambda|o_1,\ldots,o_K} \cdot d\mu_\lambda$. However, the transformation is analytically intractable and $f_{\lambda|o_1,\ldots,o_K}$ can only be approximated. In Figure 3 we observe some approximate posterior densities, superimposed to alternative prior densities. The real rate generating the trajectory $X$ and its observations is $\lambda = 3$, and we note slight variations in the approximate posterior caused by different prior choices in
In this example, numerical MCMC procedures can offer similar approximations. However, network deployments are generally complex within applications, and include multiple stations, delay nodes or queues with varied specifications and unobservable routing mechanisms; also, they process greater counts of jobs. There, inference is intractable by means of existing simulation-based inferential procedures (Perez et al., 2017), and we offer a longer discussion within Section 6. In the following, we present results that allow for the approximation of posterior densities in a wide class of closed and open systems.

3 Overview of main results

To begin with, we describe an approximating network model that will later enable the inferential task, and continue by offering an overview of our main results to follow in the paper. Under the natural measure \( P \) tied to the infinitesimal generator \( Q \), an underlying MJP \( X \) as introduced in Section 2 is supported in a set \( S \) of feasible vectors of integers, which is often just \( S = \mathbb{N}_0^{|C| \times M} \). Here, we assume the existence of an approximating measure \( \tilde{P} \) on an augmented space of network paths \( \tilde{X} \), such that we further assign a mass to network loads with negative values. Rates for transitions across the states are induced by a generator \( \tilde{Q} \) with

\[
\tilde{Q}_{x,x'} = \delta + Q_{x,x'}, \quad \delta > 0
\]

whenever \( x \xrightarrow{i,j,c} x' \) is such that \( (i, j, c) \in T \), and \( \tilde{Q}_{x,x'} = Q_{x,x'} = 0 \) otherwise. Hence, intensities for job transitions between nodes \( i \) and \( j \) are strictly positive whenever \( p_{i,j}^c > 0 \), for any class and regardless of the network loads. In the event of a \( t \)-time class-\( c \) job departure from a station \( i \) when \( X_i^t \leq 0 \), then we assume this job to be virtually generated and a unit will be subtracted from the state vector at the corresponding index, in order to represent the fact and preserve the global population count. For values of \( \delta \) small enough, the \( \tilde{P} \)-density assigned to trajectories outside of \( X \) is negligible. Note that a density \( \tilde{f} \) in (1) with generator \( \tilde{Q} \) in (5) is such that, for any network path \( (t, x) \in \tilde{X} \setminus X \), it holds \( \tilde{f}_{X \setminus X}(t, x) \leq \prod_{i=1}^{|X|} \tilde{Q}_{x_i-1, x_i} = O(\delta^r) \) as \( \delta \to 0 \), for some \( r \in \{1, \ldots, I\} \). Thus, \( X, \tilde{P}(X) = 1 - \int_{\tilde{X} \setminus X} \tilde{f}_{X \setminus X} \tilde{d}\tilde{\mu}_X \xrightarrow{\delta \to 0} 1 \), where \( \tilde{\mu} \) denotes an appropriately augmented base measure, and the limiting system dynamics under \( \tilde{P} \) will offer a perfect approximation to the original network model. Within the rest of the paper,

- In Section 4, we discuss a counting process over job transitions in the augmented network with generator \( \tilde{Q} \) in (5); we ease the use of simplifying independence assumptions
and introduce an alternative absolutely continuous mean-field measure \(\tilde{\pi}\). In Lemma 1, we derive a lower bound to the equivalent log-likelihood for the network observations.

- Propositions 1 and 2 within Section 5 inspect the structure of \(\tilde{\pi}\) that approximates the regular conditional probability and corresponding likelihood-ratio in \(\tilde{\pi}\). Corollaries 1.1 and 2.1 focus on real-world network monitoring scenarios and the posterior rate density \(d\pi_{\lambda_{d+1},\ldots,\lambda_d}\) by looking at conjugacy properties and limiting behaviour as \(\delta \to 0\).

- Finally, Subsection 5.1 describes an iterative algorithm to obtain approximate posterior distributions across the various service rates \(\mu\) and routing probabilities in \(P\) within the original queueing network.

## 4 A counting process over job transitions

A network system as introduced in Section 2 further gives rise to a multivariate Markov counting process \(Y = (Y_t)_{t \geq 0}\) on \((\Omega, F)\), where each indexed \(Y_t = (Y^n_t)_{\eta \in \mathcal{T}} \in S^\mathcal{T}\) accounts for job transitions across all classes in \(\mathcal{T}\) up to a time \(t \geq 0\). That is, each \(Y^n_t\) denotes the cumulative count in \(Y^n_t\) of transitions \(x \xrightarrow{\eta} x'\) in \(X^n\), with \(x, x' \in S\), and \(Y^n_0 = 0\) for all \(\eta \in \mathcal{T}\).

At a basic level, these are simply non-decreasing counting processes for job transitions in the directions defined within \(\mathcal{T}\). We further note that \(|\mathcal{T}|\) is often small, as underlying network topologies impose strict routing mechanisms. The support set \(S^\mathcal{T}\) for the counting process is determined by the connectivity structure amongst the stations. Under the approximating measure \(\tilde{\pi}\), it holds \(S^\mathcal{T} = \mathbb{N}_{0, |\mathcal{T}|}\), since job transitions may occur regardless of network loads.

Now, let

\[
\mathcal{T}_{i,c}^- = \{\eta \in \mathcal{T} : \eta_2 = i, \eta_3 = c\} \quad \text{and} \quad \mathcal{T}_{i,c}^+ = \{\eta \in \mathcal{T} : \eta_1 = i, \eta_2 = c\}
\]

denote the subsets of \(\mathcal{T}\) that include class \(c \in \mathcal{C}\) job transitions to, and from, the network node \(i \in \{0, \ldots, M\}\), respectively. Also, recall that \(X^{i,c}\) denotes the number of class \(c\) jobs in station \(i \geq 0\) at time \(t \geq 0\), then

\[
X^{i,c}_t = \sum_{\eta \in \mathcal{T}_{i,c}^-} Y^n_t - \sum_{\eta \in \mathcal{T}_{i,c}^+} Y^n_t \quad (6)
\]

for all \(t \geq 0\), assuming initially empty networked systems. We note that for all \(\eta = (i, j, c) \in \mathcal{T}\) it holds \(\eta \in \mathcal{T}_{j,c}^-\) and \(\eta \in \mathcal{T}_{i,c}^+\). Thus, paths in \(X^n\) and \(Y^n\) differ in that the former is coupled, i.e. a job transition in the direction \(\eta = (i, j, c)\) is relevant to (and thus is synchronized across) a pair of marginal processes \((X^{i,c}_t)_{t \geq 0}, (X^{j,c}_t)_{t \geq 0}\); in the latter, this is only relevant to the indexed process \((Y^n_t)_{t \geq 0}\).

In view of (6), we further denote \(x_{i,c} = \sum_{\eta \in \mathcal{T}_{i,c}^-} y_\eta - \sum_{\eta \in \mathcal{T}_{i,c}^+} y_\eta\) to the class-\(c\) queue-length in station \(i \geq 0\) for any \(y \in S^\mathcal{T}\). Then, the \(\tilde{\pi}\)-associated infinitesimal generator matrix \(\Xi\) of \(Y^n\) is such that

\[
\Xi_{y,y',\eta} = \delta + \lambda_\eta \cdot [\Upsilon(y, \eta_1, \eta_3) \wedge 0]
\]

with a station load

\[
\Upsilon(y, i, c) = x_{i,c} \cdot \left(\frac{K_i}{\sum_{c' \in \mathcal{C}} x_{i,c'}} \wedge 1\right)
\]

for all jumps \(y \xrightarrow{\eta} y', \eta = (i, j, c)\), where the origin station \(i > 0\) has PS discipline (here we have set \(0/0 = 0\), and

\[
\Upsilon(y, i, c) = K_i \wedge x_{i,c}
\]
in stations $i > 0$ with FCFS policy within single-class networks. We further have $\Xi_{y,y'} = \delta + \lambda_{0,j,c}$ for arrivals from virtual nodes (in open networks) and $\Xi_{y,y'} = \delta + \lambda_{0,j,c} \cdot (N + \sum_{\eta \in T^+} y_{\eta} - \sum_{\eta \in T^+} y_{\eta})$ for arrivals from delays, where $N$ denotes the job population in a closed system. Finally, $\Xi_y := \Xi_{y,y} = -\sum_{y' \in S^Y, y \neq y'} \Xi_{y,y'}$.

### 4.1 A mean field decomposition and lower bound

The likelihood for observation events in (2) readily transfers to counts $Y$ by means of (6), we thus may write $f_{O|Y=\eta} (o_k) \equiv f_{O|X=\eta} (o_k)$. Under the measure $\mathbb{P}$ network states can have negative values, the likelihood is undefined in such instances. Now, note that piecewise $S^Y$-valued trajectories also represent elements $(t, y)$ in a space $(Y, \Sigma_Y)$, similar to network paths in $X$. Let $f_{Y|\lambda, o_1, \ldots, o_K}$ be a density function, w.r.t. some base measure $\mu_Y$, where for all $B \in \Sigma_Y$ it holds

$$
P(Y^{-1}(B)|\lambda, o_1, \ldots, o_K) = \int_B f_{Y|\lambda, o_1, \ldots, o_K} \, d\mu_Y.$$

It may be shown by properties of conditional distributions that, conditioned on observations, $Y$ is a non-homogeneous semi-Markov process with hazard functions

$$
\Lambda_{y,y'}(t) = \Xi_{y,y'} \cdot \frac{P(\bigcap_{t_i \geq t} O_{\eta} = y')}{P(\bigcap_{t_i > t} O_{\eta} = y')} \quad \text{for} \quad y' \neq y,
$$

and $\Lambda_y(t) = -\sum_{y' \neq y} \Lambda_{y,y'}(t)$, s.t.

$$
f_{Y|\lambda, o_1, \ldots, o_K}(t, y) = \pi(y_0) e^{\int_0^t \Lambda_{y,y}(u) \, du} \prod_{t_i \geq t} \Lambda_{y_{i-1}, y_i} (t_i) e^{\int_{t_i}^{t_i+1} \Lambda_{y_{i-1}, y_i}(u) \, du}.
$$

Here, $\Xi \equiv \Xi(\lambda)$ denotes the generator matrix associated with fixed values in $\lambda$. For a deeper look at conditional jump processes we refer the reader to [Serfozo (1972), Daley and Vere-Jones (2002)]. This conditional counting process is of key importance, however, the structure of rates in (2) poses a trivial analytical impediment. In our approximating effort, we assume the existence of an alternative measure $Q$ on $(\Omega, \mathcal{F})$. Under this measure, network trajectories in $X$ are subject to a mean-field decomposition across interacting pairwise nodes, that is, the $Q$-law of $Y$ is that of a family of $|T|$ independent non-homogeneous Poisson counting processes with state-dependent intensity functions $\nu^\eta = (\nu_t^\eta(\cdot))_{t \geq 0}$, for all $\eta \in T$. Here,

- Intensity rates for jumps $y \xrightarrow{\nu_t^\eta} y'$, $t \geq 0$, are independent of $\lambda$, change over time, and are given by $\nu_t^\eta(y_\eta)$.

- Holding rates in $Y$ evolve according to $|\nu_t(Y_\eta)|$, with $\nu_t(Y_\eta) = -\sum_{\eta \in T} \nu_t^\eta(Y_\eta)$.

- The state probability of the multivariate process $Y$ factors across the job transition directions, s.t.

$$
Q(Y_t = y) = \prod_{\eta \in T} Q(Y_\eta = y_\eta)
$$

for every $y \in S^Y$.

In order to ensure computational tractability within the forthcoming procedures, the intensity functions $\nu$ must be bounded from above by some arbitrary constant, s.t. $\nu_t^\eta(y_\eta) \leq \bar{\nu}$ for all $t > 0, \eta \in T$ and $y \in S^Y$. Furthermore, transition rates in $\lambda$ are assumed mutually independent under $Q$, and admit undetermined densities $dQ_{\lambda, \eta}$, $\eta \in T$, that must integrate to 1 on $(\mathbb{R}_+, B(\mathbb{R}_+))$. We will later observe that this still induces dependence structures across
the services rates and routing probabilities in the original network model. We finally note that \( Q \) and \( \tilde{P} \) are equivalent on \( \mathcal{F} \), as both assign a positive measure to every marginally-increasing sequence of \( N^\| t \| \)-valued counts.

**Lemma 1** (Mean-field lower bound). Denote \( O = \bigcap_{k=1}^K O^{-1}_k(o_k) \) and let \( \tilde{P} \) and \( Q \) be the probability measures on \((\Omega, \mathcal{F})\), as defined above. Recall notation \( \Xi_{y,y'} = \Xi_{y,\eta} \) for jumps \( y \xrightarrow{\eta} y' \) with direction \( \eta \), then

\[
\log \tilde{P}(O) \geq \sum_{k=1}^K \mathbb{E}_{Y_k,\lambda}^Q \left[ \log f_O|Y_k(o_k) \right] - \mathbb{E}_{Y,\lambda}^Q \left[ \log \frac{dQ_{\lambda}}{dP_{\lambda}} \right] - \int_0^T \mathbb{E}_{Y,\lambda}^Q \left[ \sum_{\eta \in T} \nu^\eta(Y_t^\eta) \log \frac{\nu^\eta(Y_t^\eta)}{\Xi_{Y_t,\eta}} - \Xi_{Y_t} + \nu_t(Y_t) \right] dt
\]

(10)

offers a lower bound on the \( \tilde{P} \)-probability of retrieved observation events.

**Proof.** Note that

\[
\log \tilde{P}(O) = \log \int_{Y \times \mathbb{R}_+^{|\mathcal{T}|}} \mathbb{P}(O|Y) d(Y,\lambda), \tilde{P} = \log \mathbb{E}_{Y,\lambda}^Q \left[ \mathbb{P}(O|Y) \frac{d(Y,\lambda), \tilde{P}}{d(Y,\lambda), Q} \right]
\]

where we use Jensen’s inequality for finite measures. This is known as a variational mean-field lower bound on the log-likelihood, and

\[
\mathbb{E}_{Y,\lambda}^Q \left[ \log \mathbb{P}(O|Y) \right] = \mathbb{E}_{Y}^Q \left[ \log \prod_{k=1}^K f_O|Y_k(o_k) \right] = \sum_{k=1}^K \mathbb{E}_{Y_k,\lambda}^Q \left[ \log f_O|Y_k(o_k) \right]
\]

(11)

follows directly from (2). The negative part in (11) is the Kullback-Leibler (KL) divergence between image measures of \( Q \) and \( \tilde{P} \). By noting that these share base measures, and \( Y,\lambda \) are independent under \( Q \), it holds

\[
\mathbb{E}_{Y,\lambda}^Q \left[ \log \frac{d(Y,\lambda), Q}{d(Y,\lambda), \tilde{P}} \right] = \mathbb{E}_{\lambda}^Q \left[ \log \frac{g_{\lambda}}{f_{\lambda|Y}} \right] + \mathbb{E}_{Y}^Q \left[ \log \frac{\Xi_{Y}}{\mathbb{Q}_Y} \right],
\]

(12)

where \( g_{\lambda} \) and \( f_{Y|\lambda} \) denote the \( Y \) -trajectory densities associated with rates \( \nu^\eta \) and \( \Xi(\lambda) \), respectively. The last term in (12) is a \( Q \)-average of the KL divergence on \( Y \), where the mean is taken across the infinitesimal transition rates. For a fixed starting \( Y_0 \in \mathcal{S}^Y \), the inner expectation is shown in [Opper and Sanguinetti (2008)](OpperSanguinetti2008) to take the equivalent form

\[
\mathbb{E}^Q_Y \left[ \log \frac{g_{\lambda}}{f_{Y|\lambda}} \right] = \lim_{R \to \infty} \frac{1}{R} \sum_{r=0}^{R-1} \mathbb{E}^Q_{Y} \left[ \sum_{y \in \mathcal{S}^Y} Q(Y_{T(r+1)} = y|Y_T^\eta) \log \frac{Q(Y_{T(r+1)} = y|Y_T^\eta)}{\mathbb{Q}(Y_{T(r+1)} = y|Y_T^\eta, \lambda)} \right]
\]

Note that within an infinitesimal time interval a jump in \( Y \) may only happen in one direction within \( \mathcal{T} \). With this in mind, we retrieve the limit of a Riemann sum in the interval \([0, T]\), i.e.

\[
\mathbb{E}_{Y,\lambda}^Q \left[ \log \frac{g_{\lambda}}{f_{Y|\lambda}} \right] = \lim_{R \to \infty} \frac{T}{R} \sum_{r=0}^{R-1} \mathbb{E}^Q_{Y} \left[ \sum_{\eta \in \mathcal{T}} \nu^\eta(Y_t^\eta) \log \frac{\nu^\eta(Y_t^\eta)}{\Xi_{Y_t,\eta}} + R T \log \frac{1 + 1}{1 + \frac{1}{R} T} \right]
\]

(11)

\[
= \int_0^T \mathbb{E}_{Y,\lambda}^Q \left[ \sum_{\eta \in \mathcal{T}} \nu^\eta(Y_t^\eta) \log \frac{\nu^\eta(Y_t^\eta)}{\Xi_{Y_t,\eta}} - \Xi_{Y_t} + \nu_t(Y_t) \right] dt,
\]

and

\[
\mathbb{E}_{Y,\lambda}^Q \left[ \log \frac{d(Y,\lambda), Q}{d(Y,\lambda), \tilde{P}} \right] = \mathbb{E}_{\lambda}^Q \left[ \log \frac{dQ_{\lambda}}{dP_{\lambda}} \right] + \int_0^T \mathbb{E}_{Y,\lambda}^Q \left[ \sum_{\eta \in \mathcal{T}} \nu^\eta(Y_t^\eta) \log \frac{\nu^\eta(Y_t^\eta)}{\Xi_{Y_t,\eta}} - \Xi_{Y_t} + \nu_t(Y_t) \right] dt
\]

completes the proof. \( \square \)
Thus, the lower bound in (10) depends on both the latent variables $Y$ and $\lambda$, accounting for the various counts and rates. On a basic level, this is built by 3 distinguishable components, that is, the expected log-observations, the Kullback-Leibler divergence across service rate densities, and a $Q$-weighted divergence across hazard functions and rates, further integrated along the entire network trajectory.

5 A functional representation

The above bound includes the prior rates density $d\bar{P}_\lambda$ along with the $\bar{P}$-generator $\Xi$ for the approximating network system with negative loads. In addition, we can find the unknown $\bar{Q}$ distribution for the time-indexed random variables $Y_t$, along with undetermined hazard rates $\nu$ and densities for infinitesimal rates in $\lambda$. Hence, by maximising this bound, we may derive properties on $\bar{Q}$ that allow for the construction of an approximating distribution to $d\bar{P}_\lambda|_{\bar{O}_1,\ldots,\bar{O}_K}$ and the corresponding likelihood-ratio in $\bar{Q}$. In this Section, we begin by generalizing work in [Upper and Sanguinetti 2008] and present results that accommodate parameter uncertainty in transition rates and impose computational restrictions in the resulting iterative system of equations. Later, we move on to inspect posterior rate densities and conjugacy properties as $\delta \to 0$.

**Proposition 1.** Let $d\bar{Q}_\lambda$ be some valid joint density assigned to the instantaneous rates $\lambda$ under the approximating mean-field measure $\bar{Q}$. Also, define $Y^\lambda_{t,\eta} = \{Y^{\eta'}_t : \eta' \in \mathcal{T}\setminus\{\eta\}\}$. Then, the $\bar{Q}$-dynamics of $\bar{Y}$ that optimize the lower bound (10) may be parametrized by a system of equations, so that the intensity functions $\nu^\eta(y)$ may be given by

$$\nu^\eta(y) = \frac{\nu^\eta_0(y) + \int_{\mathbb{T}} e^{\nu^\eta_{\eta'}_y,\lambda} \left[ \log E_{Y^{\eta',\eta}_t | Y^{\eta'}_t = y} - \frac{1}{Q(Y^{\eta'}_t = y)} \right] - k^\eta(y)}{e^{k^\eta(y) / Q(Y^{\eta'}_t = y)}}$$  \hspace{1cm} (13)

for all $t \in [0, \tau], \eta \in \mathcal{T}$ and $y \in \mathbb{N}$, with $k^\eta(y) \geq 0$ and

$$\frac{dr^\eta_k(t)}{dt} = r^\eta_k(t) \mathbb{E}^\bar{Q}_{Y_k | \lambda} \left[ E_{Y^{\eta'}_t | Y^{\eta'}_t = y} - 1 + \frac{k^\eta(y)}{Q(Y^{\eta'}_t = y)} \right] r^\eta_k(t) + 1 \frac{e^{k^\eta(y) / Q(Y^{\eta'}_t = y)}}{e^{k^\eta(y) / Q(Y^{\eta'}_t = y)}}$$  \hspace{1cm} (14)

whenever $t \neq t_k$, $k = 1, \ldots, K$, and

$$\lim_{t \to t_k^{-}} r^\eta_k(t) = r^\eta_k(t) \exp \left( e^{\nu^\eta_{\eta'}_y,\lambda} \left[ \log f_{O} | Y_k (\nu_k) | Y^{\eta'}_t = y \right] \right)$$  \hspace{1cm} (15)

at network observation times. In addition, $k^\eta(y) (\nu^\eta(y) - \bar{\nu}) = 0$.

**Proof.** We identify a stationary point to the Lagrangian associated with this constrained optimization problem, where optimization is w.r.t. the jump rates and the finite dimensional distributions of $Y$, subject to $\nu^\eta(y) \leq \bar{\nu}$ and the master equation

$$\frac{dQ(Y^{\eta'}_t = y)}{dt} = \nu^\eta_0(y - 1) \cdot Q(Y^{\eta'}_t = y - 1) - \nu^\eta(y) \cdot Q(Y^{\eta'}_t = y)$$  \hspace{1cm} (16)

for $y \geq 1$, with $dQ(Y^{\eta'}_t = 0) = -\nu^\eta_0(0)Q(Y^{\eta'}_t = 0)dt$. Denote by $\phi^\eta(y) = Q(Y^{\eta'}_t = y)$ the functional representing the marginal $Q$-probability of the state $Y_t$ in the direction of $\eta$, for all $y \in \mathbb{N}$. In view of (10), the object function may be expressed as the functional

$$\Phi[\phi, \nu, l] = C + \sum_{k=1}^{K} \mathbb{E}^Q_{Y_k} \left[ \log f_{O} | Y_k (\nu_k) \right] - \int_{0}^{T} \sum_{\eta \in \mathcal{T}} \mathbb{E}^Q_{Y_t} \left[ \Psi | Y^{\eta'}_t, \phi^\eta_{\eta'}_t (Y^{\eta'}_t, \nu^\eta_0(Y^{\eta'}_t), \nu^\eta_0(Y^{\eta'}_t)) \right] dt$$

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then the complementary slackness conditions imply $k^n = (k^n_t)_{t \geq 0}$ and $k^n = (k^n_t(\cdot))_{t \geq 0}$ are multiplier functions that ensure (16) and the complementary inequality on rates are satisfied. Above, the term $C$ includes the remainder bits in the lower bound in (10) that are independent of the finite dimensional distributions of $Y$ under $Q$. Hence, we obtain the following functional derivatives

$$
\frac{\delta \Phi}{\delta \phi^n_t(y)} = \sum_{k=1}^K \delta(t - k) \mathbb{E}^{Q_{Y_t^n, \lambda}}_{Y_t^n}[\log f_{0|Y_t^n}(\omega_k)|Y_t^n = y] - \mathbb{E}^{Q_{Y_t^n, \lambda}}_{Y_t^n}[\log \mathbb{E}_{Y_t^n}|Y_t^n = y] - \frac{d \varphi^n_t(y)}{dt} - \nu^n_t(y) \left( \log \nu^n_t(y) - \mathbb{E}^{Q_{Y_t^n, \lambda}}_{Y_t^n}[\log \mathbb{E}_{Y_t^n}|Y_t^n = y] - 1 - \varphi^n_t(y) + \varphi^n_t(y + 1) \right),
$$

and

$$
\frac{\delta \Phi}{\delta \nu^n_t(y)} = -\varphi^n_t(y) \left( \log \varphi^n_t(y) - \mathbb{E}^{Q_{Y_t^n, \lambda}}_{Y_t^n}[\log \mathbb{E}_{Y_t^n}|Y_t^n = y] + \varphi^n_t(y + 1) - \varphi^n_t(y) - \varphi^n_t(y) \right),
$$

for all $t \in [0, T], \eta \in T$ and $y \in \mathbb{N}_0$, to be complemented by the slackness conditions $r^n_t(y)(\nu^n_t(y) - \nu) = 0, \nu^n_t(y) \geq 0$ and $\nu^n_t(y) \leq \nu$. By letting $l^n_t(y) = -\log \varphi^n_t(y)$ and setting the above expressions to 0, we obtain

$$
\frac{d \varphi^n_t(y)}{dt} = \varphi^n_t(y) \cdot \left( \mathbb{E}^{Q_{Y_t^n, \lambda}}_{Y_t^n}[\log f_{0|Y_t^n}(\omega_k)|Y_t^n = y] - \sum_{k=1}^K \delta(t - k) \mathbb{E}^{Q_{Y_t^n, \lambda}}_{Y_t^n}[\log f_{0|Y_t^n}(\omega_k)|Y_t^n = y] - \frac{\varphi^n_t(y)}{\varphi^n_t(y)} \cdot \mathbb{E}^{Q_{Y_t^n, \lambda}}_{Y_t^n}[\log \mathbb{E}_{Y_t^n}|Y_t^n = y] - \varphi^n_t(y) \right)
$$

and

$$
\nu^n_t(y) = \frac{\varphi^n_t(y + 1)}{\varphi^n_t(y)} \cdot \mathbb{E}^{Q_{Y_t^n, \lambda}}_{Y_t^n}[\log \mathbb{E}_{Y_t^n}|Y_t^n = y] - \varphi^n_t(y) \cdot \frac{\varphi^n_t(y + 1)}{\varphi^n_t(y + 1)}. \mathbb{E}^{Q_{Y_t^n, \lambda}}_{Y_t^n}[\log \mathbb{E}_{Y_t^n}|Y_t^n = y] - \varphi^n_t(y) \cdot \frac{\varphi^n_t(y + 1)}{\varphi^n_t(y + 1)}.
$$

Observe above that, for fixed values of $\phi$ and $r$, if

$$
\frac{\varphi^n_t(y + 1)}{\varphi^n_t(y)} \cdot \mathbb{E}^{Q_{Y_t^n, \lambda}}_{Y_t^n}[\log \mathbb{E}_{Y_t^n}|Y_t^n = y] < \nu
$$

then the complementary slackness conditions imply $k^n_t(y) = 0$; otherwise, $\nu^n_t(y) = \nu$ and

$$
k^n_t(y) = \mathbb{E}^{Q_{Y_t^n, \lambda}}_{Y_t^n}[\log \mathbb{E}_{Y_t^n}|Y_t^n = y] - \log \frac{\varphi^n_t(y + 1)}{\varphi^n_t(y)} \geq 0,
$$

yielding a valid system of equations, leading to (13)-(15) and concluding the proof. \hfill \Box

**Corollary 1.1 (Distributed network monitoring).** Assume that network observations are distributed and independent across the stations, so that

$$
f_{0|Y_t^n}(\omega_k) = \prod_{i=1}^M f_{0|Y_t^n, \eta \in T_i}(\omega_k)
$$

for some conditional mass function $f_{0|Y_t^n, \eta \in T_i}$, where $T_i = (\cup_{c \in C} T_{i,c}) \cup (\cup_{c \in C} T_{i,c}^+)$ is the set of job transitions relevant to network activity in station $i > 0$, and $t_k$ denotes the time $t_k$ observations
across classes in the station. Further assume that $\hat{\nu} = \infty$, so that there exists no bound on intensity rates $\nu^\eta_t(y)$ under $\mathbb{Q}$. Then, the system of equations in Proposition 1 reduces to

$$\nu^\eta_t(y) = \frac{r^\eta_t(y+1)}{r^\eta_t(y)} e^{\sum_{k \in \mathcal{A}} \mathbb{E}^\eta_{Y_t, \lambda} \{ \log \mathbb{E}^{\mathcal{Y}_{t,n|Y^\eta_t}}_{\lambda|Y^\eta_t} - \mathbb{E}^{\mathcal{Y}_{t,n|Y^\eta_t}}_{\lambda|Y^\eta_t} \}}$$

for all $t \in [0, T]$, $\eta \in \mathcal{T}$ and $y \in \mathbb{N}_0$, with

$$\frac{dr^\eta_t(y)}{dt} = r^\eta_t(y) \mathbb{E}^\eta_{Y_t, \lambda} \{ \Xi_{Y_t, \eta} | Y^\eta_t = y \} - r^\eta_t(y+1) e^{\sum_{k \in \mathcal{A}} \mathbb{E}^\eta_{Y_t, \lambda} \{ \log \mathbb{E}^{\mathcal{Y}_{t,n|Y^\eta_t}}_{\lambda|Y^\eta_t} - \mathbb{E}^{\mathcal{Y}_{t,n|Y^\eta_t}}_{\lambda|Y^\eta_t} \}}$$

whenever $t \neq t_k$, $k = 1, \ldots, K$, and

$$\lim_{t \to t_k^-} r^\eta_t(y) = r^\eta_{t_k}(y) e^{\sum_{k \in \mathcal{A}} \mathbb{E}^\eta_{Y_{t_k}, \lambda} \{ \log \mathbb{E}^{\mathcal{Y}_{t,n|Y^\eta_{t_k}}}_{\lambda|Y^\eta_{t_k}} - \mathbb{E}^{\mathcal{Y}_{t,n|Y^\eta_{t_k}}}_{\lambda|Y^\eta_{t_k}} \}}$$

accounting for observations at origin and departure nodes in $\eta \in \mathcal{T}$.

Here, we have obtained a system of equations with iterated dependencies given a distribution $\mathbb{Q}_\lambda$. The hazard rate in each counting process depends on the network state probability across the indexed times; complementarily, these state probabilities may be updated independently by means of the master equation (10). In Corollary 1, we further notice that by simplifying the network observation model, and easing restrictions on rates under the approximating measure $\mathbb{Q}$ we retrieve an analogue result to that previously presented in Opper and Sanguinetti (2008), Cohn et al. (2010). However, this is reportedly problematic and can cause a computational bottleneck when reconstructing the jump rates $\nu^\eta_t(y)$, as these may approach infinity at observation times. Next, we derive the main result on the infinitesimal transition rates.

**Proposition 2.** Let densities for the infinitesimal rates $\lambda$ be defined w.r.t to a (Lebesgue) product base measure $\mu_\lambda$, so that $d\mathbb{Q}_\lambda = g_\lambda d\mu_\lambda$ with $g_\lambda = \prod_{\eta \in \mathcal{T}} g^\eta_\lambda$ and marginal densities $g^\eta_\lambda = d\mathbb{Q}_\lambda / d\mu_\lambda$. Also, let $\nu^\eta_t(y)$, $\eta \in \mathcal{T}$, be some (independent) intensity functions assigned to $Y$ under the approximating mean-field measure $\mathbb{Q}$. Finally, define $\lambda, \eta = \{ \lambda^\eta : \eta^\prime \in \mathcal{T} \setminus \{ \eta \} \}$ and recall definitions for network station loads $\mathcal{Y}$ in (7) and (8). Then, as $\delta \to 0$ in (9), the distribution $\mathbb{Q}_\lambda$ that optimizes the lower bound (10) is such that

$$g^\eta_\lambda(z) \propto e^{-\sum_{\eta \in \mathcal{T}} \mathbb{E}^\eta_{\mathcal{Y}_{t}, \lambda} \{ \log f_{\mathcal{Y}_{t}|Y^\delta} \} - \int_{0}^{T} \mathbb{E}^\eta_{\mathcal{Y}_{t}, \lambda} \{ \nu^\eta_\lambda(Y^\delta_t) \} dt}$$

up to a normalizing constant, for $z \in \mathbb{R}_+$ and every $\eta \in \mathcal{T}$.

**Proof.** We again identify a stationary point to the Lagrangian associated with a constrained optimization problem, w.r.t to arbitrary (positive) densities $g^\eta_\lambda$ with $\int_{\mathbb{R}_+} g^\eta_\lambda d\mu_\lambda = 1$. Since $\mathbb{P}_\lambda$ and $\mathbb{Q}_\lambda$ share base measures, the object function can be written as

$$\Phi[g] = C - \sum_{\eta \in \mathcal{T}} \mathbb{E}^\eta_{\mathcal{Y}_{t}, \lambda} \{ \Psi[\lambda, g^\eta_\lambda] \} - \sum_{\eta \in \mathcal{T}} \mathbb{E}^\eta_{\mathcal{Y}_{t}, \lambda} \{ \nu^\eta_\lambda(Y^\delta_t) \} dt$$

where $\{l_\eta\} \eta \in \mathcal{T}$ are non-functional Lagrange multipliers, and

$$\Psi[\lambda, g^\eta_\lambda] = \log g^\eta_\lambda - \frac{1}{|\mathcal{T}|} \mathbb{E}^\eta_{\mathcal{Y}_{t}, \lambda} \{ \log f_{\mathcal{Y}_{t}} \} + \int_{0}^{T} \mathbb{E}^\eta_{\mathcal{Y}_{t}, \lambda} \{ \nu^\eta_\lambda(Y^\delta_t) \} \log \frac{\nu^\eta_\lambda(Y^\delta_t)}{\Xi_{Y^\delta_t, \eta}} + \Xi_{Y^\delta_t, \eta} - \nu^\eta_\lambda(Y^\delta_t) dt$$

The term $C$ includes the remainder bits in the lower bound in (10) that are independent of the rates $\lambda$. It follows that

$$\frac{\delta \Phi}{\delta g^\eta_\lambda} = \mathbb{E}^\eta_{\mathcal{Y}_{t}, \lambda} \{ \log f_{\mathcal{Y}_{t}} \} - \log(g^\eta_\lambda) - 1 - l_\eta - \int_{0}^{T} \mathbb{E}^\eta_{\mathcal{Y}_{t}, \lambda} \{ \nu^\eta_\lambda(Y^\delta_t) \} \log \frac{\nu^\eta_\lambda(Y^\delta_t)}{\Xi_{Y^\delta_t, \eta}} + \Xi_{Y^\delta_t, \eta} - \nu^\eta_\lambda(Y^\delta_t) dt$$

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in its support set \( \mathbb{R}_+ \), for all \( \eta \in \mathcal{T} \). By equating the above to 0, considering constraints and analysing the relevant terms up to proportionality, we note that

\[
g_\lambda^\eta \propto \exp \left( \mathbb{E}_{\lambda,\eta}^Q [\log f_\lambda(\eta)] + \int_0^T \mathbb{E}_{Y_t}^Q [\nu_t^\eta(Y_t^\eta) \log \Xi_{Y_t,\eta} - \Xi_{Y_t,\eta}] \, dt \right),
\]

so that

\[
g_\lambda^\eta(z) \propto e^{-\lambda^\eta \cdot z} - \int_0^T \mathbb{E}_{Y_t}^Q [\nu_t^\eta(Y_t^\eta) \log(\delta + z \cdot \Xi(Y_t,\eta,\eta_3))] \, dt
\]

and

\[
g_\lambda^\eta(z) \propto e^{-\lambda^\eta \cdot z} - \int_0^T \mathbb{E}_{Y_t}^Q [\nu_t^\eta(Y_t^\eta)] \, dt - \int_0^T \mathbb{E}_{Y_t}^Q \nu_t^\eta(Y_t^\eta) \, dt
\]

as \( \delta \to 0 \), for \( z \in \mathbb{R}_+ \) and every \( \eta \in \mathcal{T} \).

**Corollary 2.1 (Conjugate prior).** Assume that the prior density on \( \lambda \) also factors across the individual rates, s.t \( d\mu_\lambda = \prod_{\eta \in \mathcal{T}} f_\lambda(\eta) \, d\mu_\lambda \), where \( f_\lambda(\eta) \) for \( \eta \in \mathcal{T} \) denote Gamma density functions with shape \( \alpha_\eta \) and rate \( \beta_\eta \). Then, as \( \delta \to 0 \) in \( 5 \), these are conjugate priors and

\[
\lambda_\eta \sim \Gamma \left( \alpha_\eta + \int_0^T \mathbb{E}_{Y_t}^Q [\nu_t^\eta(Y_t^\eta)] \, dt, \beta_\eta + \int_0^T \mathbb{E}_{Y_t}^Q [\nu_t^\eta(Y_t^\eta)] \, dt \right)
\]

for all \( \eta \in \mathcal{T} \).

Hence, as the network model with negative loads offers a better approximation of its original counterpart, we may numerically approximate posterior distributions across the infinitesimal rates in \( \lambda \), under the mean-field measure \( \mathcal{Q} \). In the special case with independent Gamma prior densities, this is an easily interpretable posterior where the shape and rate parameters depend, respectively, on the integrated expected jump intensities and the integrated expected station loads.

### 5.1 Inference on service rates and routing topology

The above results suggest an iterative inferential procedure by means of coordinate ascent. Here, we iteratively update the values of the various rates, functions and Lagrange multipliers while evaluating, and assessing convergence, in the lower bound \( (10) \) to the log-likelihood. This is a standard approach in variational inference when looking for a (local) maxima (Blei et al., 2017), and the problem is known to be convex.

Maximising the bound by calibrating the measure \( \mathcal{Q} \) will yield an approximation to the regular conditional probability of events in \( \mathcal{F} \) under \( \tilde{\mathbb{P}}_\lambda \), conditioned on the observations. Projected over the rates \( \lambda_\eta \) and noting that base measures in prior and posterior must agree, this yields an approximation to the posterior rate density and the likelihood ratio in \( 3 \). This projected densities are valid in order to approximate the posterior distributions of the service rates \( \mu \) and routing probabilities \( \mathcal{P} \) in the original queueing network system. The final iterative procedure is described here.

- First, input network observations in \( 2 \) and assign a (conjugate) Gamma prior \( d\mu_\lambda \) across job transition intensities \( \lambda_\eta \), with shape parameters \( \alpha_\eta \) and a (shared) rate parameter \( \beta_\eta = \beta, \eta \in \mathcal{T} \).
- Define a discretization grid of the time interval \([0, T]\), and operate through interpolation within points in the grid.
• Set an arbitrary density \( dQ_\lambda \). Fix \( \kappa_i^n(y) = 0, \rho_i^n(y) = 1 \) and input (valid) arbitrary starting values \( Q(Y_t^n = y) \), for all \( t \in [0, T] \), \( \eta \in T \) and \( y \in \mathbb{N}_0 \).

• Iterate until convergence:
  - In each direction \( \eta \in T \), numerically compute \( \rho_i^n(y) \) for every \( t \in [0, T] \) and \( y \in \mathbb{N}_0 \), by means of (13)-(15). Then, update intensity and slack functions \( \nu_i^n(y), \kappa_i^n(y) \) with (13), so that \( \kappa_i^n(y) = 0 \) if
    \[
    \frac{\rho_i^n(y + 1)}{\rho_i^n(y)} \cdot e^{\mathbb{E}_\lambda T_i^n | \log \Xi Y_i, \eta | Y_t^n = y} < \bar{\nu},
    \]
    and
    \[
    \nu_i^n(y) = \bar{\nu}, \quad \kappa_i^n(y) = Q(Y_t^n = y) \cdot \left[ \mathbb{E}^Q_{Y_i, \eta_\lambda} | \log \Xi Y_i, \eta | Y_t^n = y \right] - \log \bar{\nu} \cdot \rho_i^n(y) - 1
    \]
    otherwise. Renew transient state probabilities in \( Y \) by means of the master equation (16), for all \( t \in [0, T] \).
  - Derive expected jump intensities \( \mathbb{E}^Q_{Y_i, \eta_n}[\nu_i^n(Y_t^n)] \) and station loads \( \mathbb{E}^Q_{Y_i, \eta_\lambda}[\Upsilon(Y_t, \eta, \eta_3) \mid 0] \), for all directions \( \eta \in T \) and times \( t \in [0, T] \). Update \( Q \)-densities for rates s.t.
    \[
    \lambda \eta \sim \Gamma \left( \alpha + \int_0^T \mathbb{E}^Q_{Y_i, \eta_n}[\nu_i^n(Y_t^n)] dt, \beta + \int_0^T \mathbb{E}^Q_{Y_i, \eta_\lambda}[\Upsilon(Y_t, \eta_1, \eta_3) \mid 0] dt \right),
    \]
    for all \( \eta \in T \). The likelihood ratio in (3) can be computed at this stage.
  - Evaluate the lower bound (10), given the current densities and infinitesimal rates under the approximating measure \( Q \). Assess variation in the bound across iterations and establish convergence.

• Finally, infer the structure of the various service rates and routing probabilities in the queueing network system.
  - Note that \( \mathbb{E}^Q_{Y_i}[\Upsilon(Y_t, \eta_1, \eta_3)] \) remains the same across directions \( \eta \in T \) with shared origin station. Since \( \mu_i = \sum_{\eta \in T_{i,c}} \lambda_\eta \) it holds
    \[
    \mu_i \sim \Gamma \left( |T_{i,c}| \cdot \alpha_\eta + \sum_{\eta \in T_{i,c}} \int_0^T \mathbb{E}^Q_{Y_i, \eta_n}[\nu_i^n(Y_t^n)] dt, \beta + \int_0^T \mathbb{E}^Q_{Y_i, \eta_\lambda}[\Upsilon(Y_t, \eta_1, \eta_3) \mid 0] dt \right),
    \]
    for all \( 0 \leq i \leq M, c \in C \).
  - Retrieve distributions for routing probabilities by noting that \( \rho^{i,j}_c = \lambda_{i,j,c}/\mu_i^n \) for all \( 0 \leq i, j \leq M, c \in C \). This suggests a Dirichlet distribution.

6 Examples

The following examples treat open and closed network models, and discuss applications of the previously presented results.\(^1\) These are introduced in increasing order of complexity.

\(^1\)Source code to these examples can be found at [github.com/IkerPerez/variationalQueues].
6.1 Single class closed network

First, we work through the single-class closed system previously introduced in Subsection 2.3 and pictured in Figure 2. We analyse simulated data by assigning a conjugate Gamma prior to \( \lambda \equiv \mu_1 \), so that \( \lambda \sim \Gamma(\alpha, \beta) \) under the reference measure \( \mathbb{P} \). In this instance, \( \alpha = 5 \) and \( \beta = 2 \). Recall that \( \mu_0 \) is fixed to ensure model identifiability, and \( X_t \in \{0, \ldots, N\} \) denotes the number of jobs in the service station at any time \( t \geq 0 \). For later reference, the stationary distribution of the system is given by

\[
\pi_\mathbb{P}(x|\lambda) = \lim_{t \to \infty} \mathbb{P}(X_t = x|\lambda) = \frac{(\frac{\mu_0}{\lambda})^x}{\sum_{x=0}^{N} (\frac{\mu_0}{\lambda})^x},
\]

for any fixed value \( \lambda \), so that, assuming the observations are sufficiently spaced, and that the system has reached stationarity, it holds that

\[
P\left( \bigcap_{k=1}^{K} O_k^{-1}(\mathbf{o}_k)|\lambda \right) \approx \prod_{k=1}^{K} \prod_{x=0}^{N} f_{\mathcal{O}_k}(\mathbf{o}_k) \pi_\mathbb{P}(x|\lambda) = \frac{\prod_{k=1}^{K} \sum_{x=0}^{N} f_{\mathcal{O}_k}(\mathbf{o}_k)(\frac{\mu_0}{\lambda})^x}{\sum_{x=0}^{N} (\frac{\mu_0}{\lambda})^x},
\]

(17)

where \( f_{\mathcal{O}_k}(\mathbf{o}_k) \) is as defined in \( \mathcal{O} \). Parameter choices and data remain as previously described, so that \( N = 50, \mu_0 = 0.1, \epsilon = 0.2, T = 100 \) and \( K = 50 \). Note that here \( \mathcal{T} = 2 \), and the process \( Y = (Y_t^{0,1}, Y_t^{1,0})_{t \geq 0} \) monitors transitions between the delay and service station, in both the directions \( 0 \to 1 \) and \( 1 \to 0 \). The lower bound to the log-likelihood in (10) reduces to

\[
\log \overline{\mathbb{P}}(\mathcal{O}) \geq \sum_{k=1}^{K} E_{\mathbf{y}_k}^\mathbb{Q} \left[ \log \mathbb{P}(O_k^{-1}(\mathbf{o}_k)|Y_{t_k}^{0,1} - Y_{t_k}^{1,0}) \right] - E_{\lambda}^\mathbb{Q} \left[ \log \frac{\delta \lambda}{\delta \|Y\| \cdot Y \cdot Y} \right] + \int_0^T E_{\mathbf{y}_t, \lambda}^\mathbb{Q} \left[ \Psi[Y_t, \nu_t, \lambda] \right] dt
\]

(18)

with

\[
\Psi[Y_t, \nu_t, \lambda] = \nu_t^{1,0}(Y_t^{1,0}) + \nu_t^{0,1}(Y_t^{0,1}) - 2 \delta - \lambda \cdot \mathbb{I}(Y_{t_k}^{0,1} - Y_{t_k}^{1,0} > 0) - \mu_0 \cdot \mathbb{I}(N + Y_{t_k}^{1,0} - Y_{t_k}^{0,1})
\]

\[
- \nu_t^{1,0}(Y_t^{1,0}) \log \frac{\nu_t^{1,0}(Y_t^{1,0})}{\delta + \lambda \cdot \mathbb{I}(Y_{t_k}^{0,1} - Y_{t_k}^{1,0} > 0)} - \nu_t^{0,1}(Y_t^{0,1}) \log \frac{\nu_t^{0,1}(Y_t^{0,1})}{\delta + \mu_0 \cdot \mathbb{I}(N + Y_{t_k}^{1,0} - Y_{t_k}^{0,1})}
\]

s.t. it contains only two hazard functions in the approximating measure \( \mathbb{Q} \), namely \( \nu_t^{0,1} \) and \( \nu_t^{1,0} \). In (18), we again notice that the lower bound is dominated by 3 distinguishable components, i.e. (i) the expected log-observations, (ii) the Kullback-Leibler divergence across the service rate density, and (iii) a weighted \( \mathbb{P} \)-to-\( \mathbb{Q} \) divergence in the expected path likelihood, further integrated along the entire network trajectory. Also, differential equations for functionals in (13) reduce to

\[
\frac{dr_t^{0,1}}{dt} = r_t^{0,1}(y) \left( \delta + \mu_0 \cdot \mathbb{E}_{Y_t^{1,0}}^\mathbb{Q} \left[ (Y_t^{1,0} - y) \vee 0 \right] \right) - \frac{1 + k_t^{0,1}(y)/\mathbb{Q}(Y_{t_k}^{1,0} = y)}{e^{k_t^{0,1}(y)/\mathbb{Q}(Y_{t_k}^{0,1} = y)}} r_t^{0,1}(y + 1)e^{k_t^{0,1}(y)/\mathbb{Q}(Y_{t_k}^{1,0} = y)} \left[ \log \left( \frac{\delta + \mu_0 \cdot (Y_t^{1,0} - y) \vee 0}{\mathbb{Q}(Y_{t_k}^{0,1} = y)} \right) \right],
\]

and

\[
\frac{dr_t^{1,0}}{dt} = r_t^{1,0}(y) \left( \delta + \mathbb{E}_{Y_t^{0,1}}^\mathbb{Q} \left[ (Y_t^{0,1} - y) \right] \right) - \frac{1 + k_t^{1,0}(y)/\mathbb{Q}(Y_{t_k}^{1,0} = y)}{e^{k_t^{1,0}(y)/\mathbb{Q}(Y_{t_k}^{0,1} = y)}} r_t^{1,0}(y + 1)e^{k_t^{1,0}(y)/\mathbb{Q}(Y_{t_k}^{0,1} = y)} \left[ \log \left( \frac{\delta + \lambda \cdot (Y_t^{0,1} - y)}{\mathbb{Q}(Y_{t_k}^{1,0} = y)} \right) \right].
\]

In Figure 3 (left) we observe the evolution of the lower bound (18) during the iterative inferential procedure, for a sufficiently small and negligible value of \( \delta \). There, we notice that
the procedure has converged to a (local) optima within approximately 13 iterations. On the right hand side of the Figure, we further observe summary statistics (mean and standard deviation) for $\lambda$ under the approximating measure $Q$. In addition, in Figure 5 (left) we find the $P$-prior density for $\lambda$ with the $Q$-posterior superimposed; along with them, the dotted line represents a posterior density obtained through Metropolis-Hastings Markov chain Monte Carlo, by means of strong stationarity assumptions leading to the likelihood function shown in 17. On the right hand side we observe the network observations on both the service station and delay node. Delay node observations are displayed by subtracting their value from the job population $N$ (thus representing a second measurement on the service station). Whenever both observations match, these are displayed with a large-sized dot. Along with it, we find a mean-average network trajectory and 95% credible interval for job counts on the service station $X_t = (Y_0^{0,1} - Y_t^{1,0})_{t \geq 0}$, under the approximating measure $Q$. Noticeably, the average mean-field trajectory flows through the most informative observations (thick dots), and the credible interval widens-up to account for some faulty measurements within either network node. Finally, in this example the topology is obvious and we note that the estimate on $\lambda$ is equivalent to that of the service rate $\mu_1$. 

Figure 4: Left, evolution of lower bound to log-likelihood during the inferential procedure. Right, evolution of mean and standard deviation values for $\lambda$ under $Q$. 

Figure 5: Left, prior and posterior densities for $\lambda$; along with MCMC density estimate (dotted). The black dot on the horizontal axis represents the original value in the network simulation. Right, network observations along with mean-average network trajectory and 95% credible interval for job counts in the service station.
6.2 Multi-class parallel tandems with bottleneck and service priorities

In our main example we analyse an open multi-class queueing network as pictured in Figure 6. In this network, there exists two classes \( c = 1, 2 \) of jobs that simultaneously transit the system. The first class consists of high priority jobs with low arrival and service intensity rates. The second class includes low priority jobs with high arrival and service rates. Once a job enters the system, a probabilistic routing junction (pictured as a square within the Figure) sends this job through either a PS or priority-FCFS tandem; later, it will be serviced within an infinite node before leaving the network. In the top processor-sharing tandem, each station has 5 processing units; these will fraction their working capacity as seen in (7), in order simultaneously service all jobs regardless of their class and priority level, however, service rates will differ depending on the job class. On the contrary, the bottom tandem includes two FCFS stations with a single processing unit and priority scheduling. Within these nodes, low priority jobs are only serviced if each station is fully empty of any high priority jobs; consequently, station loads in (8) are rewritten s.t.

\[
\Upsilon(y, i, 1) = 1 \land x_{i,1} \quad \text{and} \quad \Upsilon(y, i, 2) = (1 \land x_{i,2}) \cdot I(x_{i,1} < 1),
\]

at stations \( i \in \{2, 4\} \) and for any \( y \in S^Y \), where we recall

\[
x_{i,c} = \sum_{\eta \in T_{i,c}^-} y_\eta - \sum_{\eta \in T_{i,c}^+} y_\eta
\]

and thus \( x_{2,c} = y_{0,2,c} - y_{2,4,c}, x_{4,c} = y_{2,4,c} - y_{4,5,c} \) for \( c = 1, 2 \). Due to the presence of service priorities, the ordering of jobs within the queue is irrelevant (this is also the case with random order disciplines); hence, our inferential framework allows for different service rates assigned to jobs in each class. Finally, the last service node includes an infinite amount of processing units, and processing rates also differ depending on the job class.

![Figure 6: Open queueing network with one routing junction (pictured as a square) and 5 service stations with varied disciplines and processing rates.](image)

We analyse synthetic data created during a time interval \([0, T]\) \( (T = 100)\), with arrival intensities \( \mu_0^c = 0.5, \mu_2^c = 3 \), routing probabilities \( p_{0,i}^c = 0.5, i, c \in \{1, 2\} \) and service rates as shown in Table 1. We collect a reduced set of noiseless and equally spaced observations with \( K = 50 \); these are essentially snapshots of the full system state across its service stations and job classes, s.t. \( O = N^{10} \) and the observation density in (2) is defined with

\[
f_{O|x}(o) = \prod_{i=1}^{5} \prod_{c=1}^{2} I(x_{i,c} = o_{i,c})
\]

for \( x \in S \), where \( o_{i,c} \) is an indexed observation in the element \( o \) denoting the class-\( c \) queue length at station \( i > 0 \). Within the inferential procedure, this observation likelihood must

\[\text{Source code for the data simulation process may be found at github.com/IkerPerez/variationalQueues.}\]
be approximated with some regularized variant similar to (4), while taking $\epsilon \to 0$. Next, we assign conjugate Gamma priors to the various service intensities; in order to ensure identifiability in the problem, arrival rates and routing probabilities are fixed and we focus this inferential task on the various service stations. Hence $\lambda \equiv \{\mu_c^i : c = 1, 2 \text{ and } i = 1, \ldots, 5\}$, and we set $\lambda_\eta \sim \Gamma(1, 0.3)$ under the reference measure $\mathbb{P}$, for all $\eta \in \mathcal{T}$.

In the following, we omit the cumbersome mathematical details related to this complex model formulation, and we focus on discussing prior choices across queue-lengths, calibration of the algorithm in Subsection 5.1 and results following the inferential procedure. First, we note that transient inference in systems with priorities is specially challenging, due to the strong dependencies this generates on the queue lengths across the nodes and classes; here, simulation-based inferential methods relying on MCMC techniques do not scale (cf. Sutton and Jordan, 2011; Perez et al., 2017). Additionally, there exist no analytic product-form distributions that can enable approximate inferential methods under assumptions of system stationarity, as also discussed in the first example in this section. As a consequence, we evaluate our algorithm on the basis of its ability to carry inference on previously unaddressed systems.

### 6.2.1 Algorithm calibration

Prior choices in the system state $Y$ must initially accommodate a strictly positive, albeit not necessarily large, likelihood for low-priority jobs to be serviced at any point in time. Here, we achieve this by means of assigning Poisson process priors to task transition counts in $Y$; that is, we first run the master equation (16) with some user-specified constant intensity rates. This creates monotone mean average queue lengths in the service nodes, and we ensure they flow aligned to the network observations in every instance.

Also, the presence of strong temporal dependencies will often trigger the approximating rates $\nu$ in (13) to become unreasonably large, ultimately deeming the algorithm computationally unfeasible. This is a phenomena also observed in Opper and Sanguinetti (2008) or Cohn et al. (2010), within the context of simpler stochastic dynamics. To ensure computational tractability, we exploit the capping functionals $k$ as in introduced in Proposition 1, and set a global rate cap of $\bar{\nu} = 50$. Furthermore, we run the differential equations for $r$ in (15) in log-form. Specific details can be found within the aforementioned source code.

### 6.2.2 Results

Within the plots in Figure 7 with the exception of the bottom right one, we observe 95% credible intervals for the queue length processes $X_i^c$ over time, across the various service stations and job classes. There, intervals in dark gray colour relate to high priority jobs, and their corresponding queue length observations are represented by black circles. This information is superimposed over its analogue for low priority jobs, where intervals are coloured in light gray and observations represented by small diamonds. These interval approximations ignore small positive densities that are sometimes assigned to negative queue lengths. Note that this is a consequence of employing counts across job transitions in $Y$ as a basis for inference on $X$, however, we recall this is a necessity in order to overcome the coupling challenges described in Sections 2 and 4. Overall, we note that the mean-field flow captures well the collected observations, with some few exceptions in the nodes with priority scheduling; hence, it offers a good basis to build approximate estimates for parameters and the likelihood ratio (3).

Additionally, the bottom right plot in Figure 7 shows an overview of the expected jump
intensity $E_{Y|\eta}^{Q}[\nu_{t}^{\eta}(Y_{t}^{\eta})]$ and station load $E_{Y|\eta}^{Q}[Y(\eta, \eta_{1}, \eta_{3}) \lor 0]$ in the direction $\eta = (0, 1, 1)$ at times $t \in [0, T]$. The sharp peaks in the intensities come at observations times, and ensure the process density transits through the observations. Finally, we notice that the expected station load differs from the estimate of the high-priority queue-length in node 1, as this process combines and weights the queue-length across the two priorities according to (7).

Figure 7: 95% credible intervals for queue lengths across the service stations. Dark (light) gray corresponds to high (low) priority jobs. Also, expected jump intensity and station load in the direction $\eta = (0, 1, 1)$.

Next, we find in Table 1 summary statistics for the posterior service rates under the approximating mean-field measure $Q$. There, we observe how the proposed framework allows for us to gain a good overview of the system properties and variability in the processing speed across the various stations. Noticeably, there exists a few significant deviations from real values, within the posterior estimates for high priority service rates in $PS$ nodes. This is likely due to a combination of sampling variance, high model complexity and the limitations of such approximate variational procedures for transient analyses of stochastic processes.

7 Discussion

In this paper, we have offered a probabilistic overview of the properties in the inferential problem with systems of networked queues, and we have presented a flexible approximate
Bayesian framework targeted at general-form open and closed QNs, capable of overcoming the challenges posed by coupling properties inherent in such complex and structured stochastic processes. The resulting procedure is suitable for single or multiple class Markovian systems with either finite or infinite processors and varied service disciplines. To achieve this goal, we have built on existing variational mean-field theory (see Opper and Sanguinetti, 2008; Cohn et al., 2010), and discussed an alternate optimization procedure with slack variables and inequality constraints that can address computational limitations within existing techniques. Notably, results within this paper contribute to existing Bayesian statistical literature in Sutton and Jordan (2011); Wang et al. (2016); Perez et al. (2017), and first allow for the study of the latent stochastic behaviour across complex mixed network models, by means of an augmented process for interactions in the resources.

Furthermore, even though the proposed framework relies on an approximated network model as a basis for inference (which ensures the absolute continuity across base measures), and while it further analyses queue-lengths by means of augmented job transitions in the resources, we have shown we can reliably capture the finite-dimensional posterior distributions of the various marginal stochastic processes, and offer a good overview of the network structure and likely flow of workload. This is important as it can enable the evaluation and uncertainty quantification tasks in several networked systems found in many application domains, where full data observations may be hard to retrieve. Currently, existing state-of-the-art alternatives rely on strong assumptions leading to stationary analyses of such systems, or use alternate MCMC procedures that reportedly find limitations due to existing computational constrains (Sutton and Jordan, 2011; Perez et al., 2017).

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Let \((\Omega, \mathcal{F})\) be a measurable space with the regular conditional probability property; also, let \(0 \leq t_1 < \cdots < t_K \leq T\) be some fixed observation times, with \(T > 0\). In a standard queueing network with \(M\) stations, \(\Omega\) may denote a product set supporting instantaneous rates, trajectories and observations, and \(\mathcal{F}\) the corresponding product \(\sigma\)-algebra. The space of rates and observations will consist of trivial Borel algebras and power sets, so that \(\lambda\) is an \((\mathbb{R}_+^n, B(\mathbb{R}_+^n))\)-valued random variable of rates in the infinitesimal generator matrix \(Q\) of \(X\), where \(n \in \mathbb{N}\) denotes an arbitrary number determined by the network topology. In addition, \(\{O_k : k = 1, \ldots, K\}\) corresponds to random measurement variables for the network monitoring activity, each defined on \((\mathcal{O}, \mathcal{P}(\mathcal{O}))\), where \(\mathcal{O}\) denotes an arbitrary countable support set for observations in every service station. A network trajectory \(X = (X_t)_{0 \leq t \leq T}\) is an \((\mathcal{S}, \mathcal{P}(\mathcal{S}))\)-valued stochastic process with a countably infinite support set \(\mathcal{S}\). Note that this is a piecewise deterministic jump-process, so that \(X = (t, x)\) is formed by a sequence of transition times \(t\) along with states \(x\). Every pair \((t, x)\) can be further defined as a random variable on a measurable space \((\mathcal{X}, \Sigma_X)\), with \(\mathcal{X} = \bigcup_{i=0}^{\infty}([0, T] \times \mathcal{S})^i\) and the corresponding union \(\sigma\)-algebra \(\Sigma_X\). This space can support all finite \(\mathcal{S}\)-valued trajectories and allows the assignment of a dominating base measure \(\mu_X\) w.r.t which define a trajectory density. For details, we refer the reader to Daley and Vere-Jones (2007).

Let \(\mathbb{P}\) be a reference probability measure on \((\Omega, \mathcal{F})\). For all \(A \in B(\mathbb{R}_+^n)\), we write

\[
\mathbb{P}(\lambda^{-1}(A)) = \mathbb{P}_\lambda(A) = \int_A f_\lambda(a) \mu_{\mathbb{R}_+^n}(da),
\]

where \(f_\lambda\) denotes the joint density function of \(n\) independent Gamma distributed variables. Hence, we assume that the distribution of instantaneous rates under \(\mathbb{P}\) admits a density carried by a (Lebesgue) measure \(\mu_{\mathbb{R}_+^n}\). Next, let \(\kappa_1 : \mathcal{F} \times \mathbb{R}_+^n \rightarrow [0, 1]\) be a regular conditional probability; i.e. a Markov kernel that defines a probability measure on \(\mathcal{F}\) for all \(\lambda \in \mathbb{R}_+^n\), with

\[
\mathbb{P}(B \cap \lambda^{-1}(A)) = \int_A \kappa_1(B, a) f_\lambda(a) \mu_{\mathbb{R}_+^n}(da)
\]
for \( A \in B(\mathbb{R}_+^n) \) and \( B \in \mathcal{F} \). By definition, \( \kappa_1(B, a) = P(B \mid \lambda = a) \) and most importantly

\[
\kappa_1(X^{-1}(C), a) = \int_C f_{X \mid \lambda = a}(t, x) \mu_X(dt, dx)
\]

for all \( C \in \Sigma_X \) (note this often poses an intractable integral). The conditional density \( f_{X \mid \lambda = a} \) is such that for every \( I \in \mathbb{N} \) and pair of ordered times \( t = \{0, t_1, \ldots, t_I\} \) in \([0, T]\) and states \( x = \{x_0, \ldots, x_I\} \) in \( S \) we have

\[
f_{X \mid \lambda = a}(t, x) = \pi(x_0) e^{Q^\top(t - t_{I-1})} \prod_{i=1}^I Q_{x_{i-1}, x_i} e^{Q^{x_{i-1}}(t_i - t_{i-1})},
\]

where \( Q \equiv Q(a) \) is the matrix of infinitesimal transition rates in \( X \) associated to values in \( a \). Finally, network observations are assumed to be discrete events, independent of transition rates given a trajectory. Thus, there exists a kernel \( \kappa_2 : \mathcal{F} \times (X \times \mathbb{R}_+^n) \to [0, 1] \) s.t.

\[
P(O_k \in D \mid X = (t, x), \lambda = a) = \kappa_2(O_k^{-1}(D), (t, x), a) = \sum_{d \in D} f_{O_k \mid (t, x)}(d) \mu_O(d)
\]

for all \( k = 1, \ldots, K \) and \( D \in \mathcal{P}(O) \). Here, \( f_{O_k \mid (t, x)} \) defines an arbitrary probability mass function on \( O \) carried by a counting measure; in our applications, each observation only depends on the state of the system at the observation time, so the above expression could be further simplified.

Under the above model construction, the support over infinitesimal rates is a standard Borel space and the existence of a posterior distribution is guaranteed (cf. Orbanz and Teh (2010)). Also, measures induced by the kernel \( \kappa_2 \) are \( \sigma \)-finite and such that \( \kappa_2(\cdot, (t, x), a) << \mu_O \), for every \( ((t, x), a) \in X \times \mathbb{R}_+^n \). The posterior is thus carried by its corresponding prior and defined by means of the Radon-Nikodym derivative

\[
\frac{dP_{\lambda | O_1 = o_1, \ldots, O_K = o_K}}{dP_\lambda}(a) = \frac{\int f_{X \mid \lambda = a}(t, x) \prod_{k=1}^K f_{O_k \mid (t, x)}(o_k) \mu_X(dt, dx)}{\int f_X \prod_{k=1}^K f_{O_k \mid (t, x)}(o_k) f_{X \mid \lambda = a}(t, x) \mu_X(dt, dx) f_\lambda(a) \mu_{\mathbb{R}_+^n}(da)},
\]

where we employ the shorthand notation \( dP_{\lambda | .}(a) = P_\lambda(da|.) \).