The O(3) sigma model in two spatial dimensions admits topological (Bogomol’nyi) lower bound on its energy. This paper proposes a lattice version of this system which maintains the Bogomol’nyi bound and allows the explicit construction of static solitons on the lattice. Numerical simulations show that these lattice solitons are unstable under small perturbations; in fact, their size changes linearly with time.

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1. Introduction.

The nonlinear O(3) sigma model in (2+1) dimensions is a popular model in theoretical physics; the static system is integrable and of Bogomol’nyi type (all minimal energy solutions can be obtained by solving the Bogomol’nyi equations). As a result, one can explicitly write down soliton solutions of arbitrary degree in term of rational functions [1]; but the model is scale invariant and therefore, its solitons have no fixed size and so their stability is a central question. Under small perturbations they shrink towards infinitely tall spikes of zero width or may spread out, with this expansion continuing indefinitely. That this indeed happens is confirmed by numerical experiments [2, 3]. General time-dependent solutions cannot be constructed explicitly, and so it is natural to investigate numerical evolution techniques which discretize the partial differential equations.

Given a continuum field theory, there are many different lattice systems which reduce to it in the continuum limit. In systems where there are topological configurations (instantons, monopoles, etc) one often has a Bogomol’nyi bound which is related to the stability of the topological objects in question. If the bound is maintained on the lattice, the topological objects will be well-behaved even when their size is comparable to the lattice spacing. Lattice versions of these systems are important for purposes of numerical computations but they have, generally, not preserved the Bogomol’nyi bound. Few years ago, Leese [4] discretized the (unmodified) O(3) sigma model in (2 + 1) dimensions. He imposed radial symmetry, made the radial coordinate $r$ discrete and found a reduced lattice system with Bogomol’nyi bound. But, although the topological lower bound can be attained, the minimum-energy configurations are not explicit. On the other hand, Ward [5] described a lattice version of this model with Bogomol’nyi bound, without any symmetry constraint. In this general case, however, the lower bound cannot be attained.

The purpose of this paper is to present a lattice version of the O(3) sigma model in two space dimensions, in which the Bogomol’nyi bound is maintained. The primary aim is not to simulate the continuum system, but rather to define an alternative lattice system with similar properties but more convenient to study numerically. Following Leese, only field configurations for which the energy density (and not necessarily the fields) is radially symmetric will be considered here, so that in effect one obtains a one-dimensional system and therefore, the construction of the discrete Bogomol’nyi equations is less complicated.
Solutions of these equations (which were obtained analytically) are then used as the basis for a numerical study of soliton stability. In fact we study the shrinking of lattice solitons and try to estimate analytically the rate of this shrinking for a soliton and 2-soliton configuration. It was shown [3] that a collision of two solitons can induce their shrinking, therefore we can then follow one of these solitons and study its behaviour. However, due to the soliton movement it is difficult to be very quantitative about the soliton shrinking. For this reason it is convenient the restriction to axial symmetry. It is worth remarking that if radial symmetry is imposed on the fields [4] then the model is integrable, and an inverse scattering transformation exist. But this restriction requires that the obtained solutions have topological charge zero.

The rest of this paper is arranged as follows. In the next section we describe initially, the familiar continuum O(3) sigma model in (2 + 1) dimensions then reparametrize the fields in order to impose radial symmetry and, finally, discretize the model. In section 3 we study the dynamics of the 2-soliton configuration at low shrinking velocities using the slow-motion approximation, make approximate analytic predictions of its behaviour, and compare these with numerical results, while in section 4 we investigate the properties of the lattice O(3) solitons numerically.

2. The Lattice O(3) Sigma Model.

Let us begin with a brief review of the continuum O(3) sigma model in two space dimensions. The field $\phi$ is a unit 3-vector field on $\mathbb{R}^2$ (i.e. a smooth function from $\mathbb{R}^2$ to the target space $S^2$), with the boundary condition $\phi \rightarrow \phi_0$ as $r \rightarrow \infty$ in $\mathbb{R}^2$. Here $\phi_0$ is some fixed point on the image sphere $S^2$. Hence there are distinct topological sectors classified by an integer $k$ (topological charge), which represents the number of times $\mathbb{R}^2$ is wrapped around $S^2$. Roughly speaking, $k$ is the number of solitons. The potential energy of the field is $E_p = (8\pi)^{-1} \int [(\partial_x \phi)^2 + (\partial_y \phi)^2] \, dx \, dy$ and the appropriate Bogomol’nyi argument gives the bound $E_p \geq |k|$. There are fields which attain this lower bound (such minimum-energy fields will be called solitons in what follows). Since $E_p$ is invariant under the scaling transformation $\phi(x^i) \mapsto \phi(\lambda x^i)$ these configurations are metastable rather than stable (their size is not fixed).

From now on we will restrict attention to fields which are invariant under simultaneous rotations and reflections in space and target space. Thus we assume that $\phi = (\phi^\alpha, \phi^3)$
with $\alpha = 1, 2$ is of the hedgehog form
\[\phi^\alpha = \sin g(r, t) k^\alpha, \quad \phi^3 = \cos g(r, t),\] (1)
characterized by its topological charge $k$, defining the unit vector $k^\alpha = (\cos k\theta, \sin k\theta)$ in (\ref{eq:1}) in terms of the azimuthal angle $\theta$; and by the real (profile) function $g$ of the polar coordinates and $t$ which satisfies certain boundary conditions. The corresponding potential energy of the field (\ref{eq:1}) is
\[E_p = \frac{1}{4} \int_0^\infty (r g'^2 + \frac{k^2}{r} \sin^2 g) \, dr,\] (2)
where $g' = dg/dr$. This is normalized so that a static configuration has $k$ energy. The boundary conditions are $g(0, t) = \pi$, in order to ensure a unique definition of $\phi$ at the origin and $g(r, t) \to 0$ as $r \to \infty$, so that $E_p$ converges.

The standard Bogomol’nyi argument \[8\] is
\[0 \leq \frac{1}{4} \int_0^\infty (\sqrt{r} g' + \frac{k}{\sqrt{r}} \sin g)^2 \, dr\]
\[= E_p - \frac{k}{2} \int_0^\infty \partial_r (\cos g) \, dr\]
\[= E_p - k.\] (3)
So the energy $E_p$ is bounded below by $k$; and $E_p$ equals $k$ if and only if $g' = -k \sin g/r$, the solution of which is the static $k$-soliton configuration
\[g(r) = 2 \arctan \left( \frac{a}{r^k} \right),\] (4)
located at the origin, with $a$ being a positive real constant which determines the soliton size. If $a$ is large, the soliton configuration is flat and broad; while if $a$ is small, it is tall but narrow. In fact, the height of the configuration (maximum of the energy density) is proportional to $a^{-2/k}$; while its radius (width) is proportional to $a^{1/k}$. Notice that, for $k = 0$ the field is constant and the energy density is zero everywhere; while for $k = 1$ the configuration looks like a lump peaked at the origin; and for $k > 1$ it is a ring centered at the origin. In what follows, we will assume that in all cases $k > 0$, since taking $k = 0$ does not test the ability of the model to handle nontrivial topologies.

So far all we have done is to re-express the $k$-soliton solution in terms of a real field $g$, which is a function of the polar radius $r$. It will now been seen how this description is useful in constructing discrete analogues of the Bogomol’nyi equations.
From now on, \( r \) becomes a discrete variable, with lattice spacing \( h \). So the real-valued field \( g(r, t) \) depends on the continuous variable \( t \), and the discrete variable \( r = nh \) \((n \in \mathbb{Z}, n \geq 0)\). The subscript \(+\) denotes forward shift, i.e. \( g_+(r, t) = g(r + h, t) = g((n + 1)h, t) \); and so the forward difference is given by \( \Delta g = (g_+ - g)/h \). The question of how best to incorporate topological ideas into a lattice formulation has been the subject of much discussion, especially in lattice gauge theories contexts (cf. [9]). One approach, following Speight and Ward [10], is to begin with the same function \( \cos g \) as appears in (3) and reconstruct the inequality, i.e.

\[
k(\Delta \cos g) = -D_n F_n,
\]

where \( D_n \to \sqrt{r} g' \) and \( F_n \to k \sin g/\sqrt{r} \) in the continuum limit \( h \to 0 \). The formula \( k(\Delta \cos g) = -2k/h \sin \left( \frac{g_+ - g}{2} \right) \sin \left( \frac{g_+ + g}{2} \right) \) suggests the choices

\[
D_n = \frac{2f(h)\sqrt{n}}{h} \sin \left( \frac{g_+ - g}{2} \right),
\]

\[
F_n = \frac{k}{f(h)\sqrt{hn}} \sin \left( \frac{g_+ + g}{2} \right), \quad n > 0,
\]

where \( f(h) \) is an arbitrary function of the lattice spacing with constraints \( f(h) \to 1 \) as \( h \to 0 \) and \( f(h) > \sqrt{k}/2 \) (see below). The origin must be treated in a special way since (5) are undefined when \( n = 0 \). One possibility is to arrange that \( D_0 + F_0 = 0 \) identically. So choose

\[
D_0 \equiv -F_0 = \sqrt{\frac{2k}{h}} \cos \left( \frac{g(h, t)}{2} \right).
\]

The potential energy of the lattice O(3) sigma model field is defined to be

\[
E_p = \frac{h}{4} \sum_{n=0}^{\infty} \left( D_n^2 + F_n^2 \right)
= k \cos^2 \left( \frac{g(h, t)}{2} \right) + \sum_{n=1}^{\infty} \left[ f^2 n \sin^2 \left( \frac{g_+ - g}{2} \right) + \frac{k^2}{4f^2 n} \sin^2 \left( \frac{g_+ + g}{2} \right) \right].
\]

As in the continuum case, it follows that \( E_p \) is bounded below by \( k \); and the minimum is attained if and only if \( D_n + F_n = 0 \). For models with different lattice spacing the effect of \( f(h) \) is to decrease the importance of the \( \sin^2 g \) term in the energy density, although the total energy is still the same as in the continuum \((k \text{ in our units})\).

The kinetic energy can be defined by the simple choice

\[
E_k = \frac{h^2}{4} \sum_{n=1}^{\infty} n g_n^2,
\]
where \( \dot{g} = dg/dt \). The boundary condition on \( g \) is that it should tend to zero at spatial infinity; this guarantees finite energy. For such fields, the total energy \( E_t = E_p + E_k \) is bounded below by \( k \); and this lower bound is attained if and only if \( \dot{g} = 0 \), and \( D_n + F_n = 0 \) for \( n > 0 \). [Recall that \( D_0 + F_0 = 0 \) identically.]

This latter condition, i.e. \( D_n + F_n = 0 \), is called the Bogomol’nyi equation. It is a first-order difference equation, whose solutions (for the aforementioned boundary conditions) minimize the potential energy, and therefore, are also static solutions of the Euler-Lagrange equations

\[
\sum_{n=1}^{\infty} n \dot{g}(nh, t) = -\frac{2}{h^2} \frac{\partial E_p}{\partial g},
\]

since \( \partial E_p/\partial g = 0 \) at a minimum. So using the discrete Bogomol’nyi equations one gets first-order equations whose solutions are also static solutions of the second-order equations of motion. Moreover, these solutions have energy which is at its topological minimum value.

The Bogomol’nyi equation \( D_n + F_n = 0 \), may also be written as

\[
\tan \frac{g_n}{2} = \frac{2f^2 n - k}{2f^2 n + k} \tan \frac{g_n}{2}, \quad n > 0,
\]

from which one sees that the function \( f(h) \) should be greater than \( \sqrt{k/2} \) in order the profile function to be monotonic. So for \( k = 1, 2 \) the form \( f(h) = 1 + h \) will work for all \( h \); while for \( k \geq 3 \) we need a condition on \( h \), i.e. \( h > \sqrt{k/2} - 1 \).

The solution of (11) can be written down explicitly, i.e.

\[
g(nh) = \begin{cases} 
\pi, & n = 0, \\
2 \arctan(z_1 Z_n), & n > 0,
\end{cases}
\]

where

\[
Z_n = \frac{\Gamma(n - k/2f^2) \Gamma(1 + k/2f^2)}{\Gamma(n + k/2f^2) \Gamma(1 - k/2f^2)},
\]

and \( z_1 \) is an arbitrary positive constant which specifies, as in the continuum model, the soliton size. This is a static lattice \( k \)-soliton solution located at the origin; which corresponds to a minimum of the energy in the \( k \) sector and thus, it is stable under perturbations which remain in that sector. From now on, we will concentrate on the \( k = 1, 2 \) sectors and therefore, we will take on the first one the function \( f(h) \) to be constant and equal to unity, for any \( h \); while on the second one to be \( f(h) = 1 + h \), for small \( h \).
It would be nice to have a lattice analogue of the configuration width $a^{1/k}$, which appeared in (12). One possibility is to set

$$a_n = (nh)^k \tan \frac{g(nh)}{2},$$

and then to define $a = \lim_{n \to \infty} a_n$, provided this limit exists. Indeed, $a$ is proportional to $z_1 h^k$. If $E_p = \sum_{n=0}^{\infty} E_{pn}$, the energy density at the origin is $E_{p0} = k/(1 + z_1^2)$; therefore, $E_{p0}$ is close to the Bogomol’nyi bound as $z_1 \to 0$. A diagram illustrating the profiles of the function $g(nh)$ and the energy densities profiles are represented in figure 1 for $k = 1$ and $k = 2$ with $z_1 = 15$ and $h = 0.19$.

The situation we wish to study is that of an isolated perturbed static $k$-soliton configuration and investigate the effects of the perturbation. As it costs them no energy to shrink or expand they can shrink to almost a zero width (radius) configuration in the energy density plot. Since the soliton configuration is described by a few points on a lattice it is difficult to decide what is meant by its width and how to calculate it. The lattice analogue will be a field configuration with $g(0, t) = \pi$ (due to the boundary conditions) and $g(nh, t) = 0$, for $n > 0$. In fact, this corresponds to a spike soliton (of almost zero width) in the continuum and thus, our interest lies in the study of the time dependence of the shrinking of a $k$-soliton configuration.

Since, there is no explicit solution in this case, one has to resort to approximation, or to numerical solutions of the equations of motion (12), namely

$$\ddot{g} = \frac{1}{h^2} [k \sin g(h, t) + f^2 \sin (g(2h, t) - g(h, t))] - \frac{k^2}{4f^2 h^2} \sin (g(2h, t) + g(h, t)), \quad n = 1,$$

$$n\ddot{g} = \frac{f^2}{h^2} [n \sin (g_+ - g) - (n - 1) \sin (g - g_-)] - \frac{k^2}{4f^2 h^2} \left[ \frac{\sin (g_+ + g)}{n} + \frac{\sin (g + g_-)}{n - 1} \right], \quad n > 1. \quad (15)$$

3. The Slow-Motion Approximation.

There is a fundamental difference between the cases $k = 1$ and $k > 1$, which becomes apparent when one considers the slow-motion approximation, originally proposed in connection with monopole scattering [11]. In this scheme one assumes that the field $g$ is a static solution, but slightly perturbed. More precisely, since the energy is conserved, and due to the existence of the Bogomol’nyi bound, we may assume that a $k$-soliton dynamics
is obtained by restricting \( g \) to have the form of (12), with \( z_1 \) now becoming a dynamical variable \( z_1(t) \). So the number of degrees of freedom is reduced from infinite to one. These static solutions form a manifold, which is equipped with a natural metric coming from the kinetic energy, and the evolution is given by the resulting geodesics. Since every configuration of the form (12) has the same potential energy, the kinetic energy may be taken as the Lagrangian; thus, the corresponding Euler-Lagrange equations are precisely the geodesic equations associated with the aforementioned metric. This approximation is a good one if the speeds are small (if \( E_k \) is small compared to \( E_p = k \)).

For \( k = 1 \), the requirement of finite kinetic energy means that \( z_1 \) should be independent of \( t \) at spatial infinity, so ruling out the slow-motion approximation. In other words, taking \( z_1 \) to be a function only of \( t \) leads to a divergent kinetic energy. But when \( k > 1 \) there are sufficient powers of \( n \) in the denominator of (12) to keep the energy finite. For this case the slow-motion approximation has been considered in [3, 12] in order to study the dynamics of \( \mathbb{CP}^1 \) lumps. However, they have not looked at the speed of the shrinking in any detailed. Let us concentrate on the \( k = 2 \) topological sector, where two solitons are sitting on top of each other at the origin, forming a ring structure. The Lagrangian is

\[
L = E_k - E_p
\]

\[
= l(z_1) \dot{z}_1^2 - 2,
\]

where

\[
l(z_1) = \hbar^2 \sum_{n=1}^{\infty} \frac{nZ_n^2}{(1 + z_1^2 Z_n^2)^2}.
\]

The Euler-Lagrange equation of the system is

\[
2l(z_1) \ddot{z}_1 + l'(z_1) \dot{z}_1^2 = 0 \quad \Rightarrow \quad \frac{d}{dt} \left( l(z_1) \dot{z}_1^2 \right) = 0.
\]

which may be reduced to quadratures:

\[
v t = \int_{c}^{z_1(t)} \sqrt{l(\tilde{z}_1)} \frac{d\tilde{z}_1}{l(c)}
\]

\[
\equiv \Lambda_h(z_1),
\]

where \( z_1(0) = c, \dot{z}_1(0) = v \). Recall that, \( z_1 \) determines the configuration size which evolves with \( t \); more precisely, \( \sqrt{\chi c h} \) is the initial width of the configuration and, \( v \) is the initial rate of change of the configuration width in each lattice site per unit time. In fact, \( v < 0 \) corresponds to an initial contraction and \( v > 0 \) to an initial expansion.
The function $\Lambda_h(z_1)$ decreasing or increasing depending on the value of $z_1(t)$, which corresponds to contraction or expansion of the configuration. It is easily inverted to give the time variation of the configuration size, i.e.

$$z_1(t) = \Lambda_h^{-1}(vt).$$

(20)

Recall that, the lattice analogue of the configuration width is proportional to $\sqrt{z_1}h$. In fact, the time taken for the configuration to shrink from the initial width to zero, i.e. to become a spike, is

$$t_c = \frac{\Lambda_h(0)}{v}.$$  

(21)

We believe that this analytical approximation is accurate for small $|v|$.

The accuracy of the approximation has been tested numerically using a fully-explicit fourth-order Runge-Kutta algorithm with fixed time step 0.0053. The initial condition was a static 2-soliton profile whose width we perturbed to shrink with initial velocity 0.1 lattice site per unit time $(v = -0.1h)$. Simulations of duration 2985 time units were performed for $h = 0.01$. Inspection of the rate of change of $z_1(t) = \tan(g(h,t)/2)$ reveals close agreement with $\dot{z}_1(t)$ calculated from (20) (see figure 2).

4. Dynamics of the Lattice O(3) Solitons.

Since the slow-motion approximation is expected to fail at high velocities (except for small $h$), we incorporate the notion of lattice solitons in a full numerical evolution scheme. Throughout the simulations, the extensive use of the difference equations (15) have not revealed any instabilities (the total energy is conserved).

The lattice formulation necessarily has a spatial boundary at $n = n_{\text{max}},$ say. Hence, the quantities that we are going to use in order to study the soliton dynamics will be calculated within some radius ($n_{\text{max}}$). Moreover, the infinite sums on the energies will be truncated. In fact, for $k = 1$ the finiteness of the grid imposes an artificial cutoff which provides a finiteness in the energy. On the boundary though, the fields are taken to be fixed in time, i.e.

$$g((n_{\text{max}} + 1)h,t) = g((n_{\text{max}} + 1)h,0),$$

(22)

since if one attempts to apply boundary conditions which allow the field to change with time at arbitrary distances, then the total energy of the system for $k = 1$, grows rapidly
and without bound. One may also, choose absorbing boundary conditions or may place the boundary far enough from the configuration (i.e., no radiation effects). But, in this scheme, the choice of the boundary conditions has no impact on the rate of shrinking.

Moving on to the question of initial data, there are clearly many different types of perturbation which we could apply to the configuration, the only restriction being that we do not perturb the field close to the boundary. Since the evolution equations (15) are second order the initial data must specify the field values \( g(nh, t) \) and its time derivatives \( \dot{g}(nh, t) \) at \( t = 0 \). So the field configuration at \( t = 0 \) is taken to be the static lattice one (12) but slightly perturbed, i.e.

\[
\dot{g}(nh, t) \bigg|_{t=0} = \frac{2\nu Z_n}{1 + z_1^2 Z_n^2}.
\]  

Physically the picture is this: there is a continuous interpolation between the inner region where \( \nu \) is the amplitude of the perturbation (as in the slow-motion approximation) and the outer one where there is no perturbation at all. This class of perturbation reveals all the qualitative types of behaviour that can occur.

So we have a \( k \)-soliton configuration whose centre remains fixed, but whose radius decreases to a minimum (close to zero) and then increases again. More precisely, the initial perturbation (for \( \nu < 0 \)) tends to shrink the configuration, while large burst of radiation travel outwards at the speed of light (see figure 3), together with a residual motion in the central region occupied by the soliton. When the radiation reaches the boundary, is reflected back, reabsorbed by the configuration which expands and then another pulse is emitted a short time later; and so the process repeats. The data were produced by the aforementioned Runge-Kutta algorithm for \( \nu = -0.1 \) and \( z_1 = 1 \), on a lattice of unit spacing \( (h = 1) \) in the \( k = 1 \) sector.

We are interested in the speed of the shrinking in detail. In order to analyze the results of the numerical simulations we look at the dynamical quantity \( g(h, t) \). Since we are on a lattice, the soliton will be highly localized (a spike) when the profile function at the first site \( (n = 1) \) and consequently, at all others \( (n > 1) \) will be zero. Then, the soliton configuration occupies essentially only one lattice site, while the slope of the curve \( g(h, t) \) correspond to the power law of shrinking.

In figure 4 we present the time dependence of the field \( g(h, t) \) for a single soliton and a
2-soliton configuration. The results are derived from a relatively small mesh \( n_{max} = 200 \) (in fact, they do not change for larger mesh sizes), for \( \nu = -0.1 \) and \( z_1 = 1 \). In the single soliton case, we make the simple choice \( f(h) = 1 \) whereas \( h = 1 \); while in the 2-soliton case, we take \( f(h) = 1 + h \) with \( h = 0.01 \). (Note that, figure 4(b) corresponds to figure 2.) The field configuration saturates the Bogomol’nyi bound throughout the numerical evolution and due to that in the \( k = 1 \) sector the lattice spacing is comparable to the soliton size without compromising its behaviour. As it is clear for the graphs the curves are nearly straight, confirming the power law for the rate of shrinking. The linear curve in figure 4(b) is due to the fact that the lattice spacing is small compare to the size of the topological soliton and thus, the model is closer to the continuum one. Let us conclude with the observation that our results are consistent with the ones obtained by studying the continuum O(3) sigma model (cf. [3]).

5. Conclusions.

We have studied the time evolution of the lattice O(3) sigma solitons which were allowed to shrink. We have used two different methods to analyze numerically the soliton shrinking. The first method is based on the slow-motion approximation and leads to an ordinary differential equation. The second one consists of integrating numerically the (1+1)-dimensional semi-discrete equations. From all the numerical studies we have performed we conclude that a single soliton and a 2-soliton configuration of the O(3) lattice sigma model does shrink to a highly localized (spike) lattice soliton, linearly with time. If one is close to the continuum limit, in a sense that the lattice spacing is small compared to the size of the topological solitons, then there may not be much difference between various lattice versions of the continuum system. An advantage of the lattice model described in this paper is that the lattice spacing can be relatively large without compromising the soliton dynamics.

Let us conclude by stressing once again that the slow-motion approximation works very well.

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Figure 1: (a) Profiles of $g$ for charges $k = 1, 2$. (b) Profiles of the energy densities $E_{pn}/(2\pi nh)$ for the charges in (a). The $k = 2$ energy density is ring shaped.
Figure 2: The time variation of $|\ddot{z}_1|$ in the analytic slow-motion approximation (solid line), and also for the numerical evolution (dotted line).
Figure 3: Radiation emitted by the 1-soliton solution.
Figure 4: The variation of $g(h, t)$ over the range (a) $0 \leq t \leq 12.6$ for a slowly shrinking 1-soliton lump and (b) $0 \leq t \leq 16$ for a slowly shrinking 2-soliton ring, for the numerical evolution.