Computing Lengths of Shortest Non-Crossing Paths in Planar Graphs

Lorenzo Balzotti*  Paolo G. Franciosa†

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Abstract

Given a plane undirected graph $G$ with non-negative edge weights and a set of $k$ terminal pairs on the external face, it is shown in Takahashi et al., (Algorithmica, 16, 1996, pp. 339–357) that the lengths of $k$ non-crossing shortest paths joining the $k$ terminal pairs (if they exist) can be computed in $O(n \log n)$ worst-case time, where $n$ is the number of vertices of $G$. This technique only applies when the union $U$ of the computed shortest paths is a forest.

We show that given a plane undirected weighted graph $U$ and a set of $k$ terminal pairs on the external face, it is always possible to compute the lengths of $k$ non-crossing shortest paths joining the $k$ terminal pairs in linear worst-case time, provided that the graph $U$ is the union of $k$ shortest paths, possibly containing cycles. Moreover, each shortest path $\pi$ can be listed in $O(\ell + \ell \log \lceil \frac{k}{\ell} \rceil)$, where $\ell$ is the number of edges in $\pi$.

As a consequence, the problem of computing multi-terminal distances in a plane undirected weighted graph can always be solved in $O(n \log k)$ worst-case time.

Keywords: shortest paths, planar undirected graphs, non-crossing paths, multi-terminal distances

1 Introduction

The problem of finding shortest non-crossing paths in a plane graph (i.e., a planar graph with a fixed embedding) has its primary applications in VLSI layout, [1], where two paths are non-crossing if they do not cross each other in the chosen embedding. It also appears as a basic step in the computation of maximum flow in a planar network, [7]. The problem can be formalised as follows: given an undirected plane graph $G$ with nonnegative edge lengths and $k$ terminal pairs that lie on a specified face boundary, find $k$ shortest non-crossing paths in $G$, each connecting a terminal pair. It is assumed that terminal pairs appear in the external face so that non-crossing paths exist, this can be easily verified in linear time.

Takahashi et al. [8] proposed an algorithm for the multi-terminal shortest paths that requires $O(n \log n)$ worst-case time, where $n$ is the size of $G$. In the same article it is also analysed the case where the terminal pairs lie on two different face boundaries, and this case is reduced to the previous one with the same computational complexity. The complexity of their solution can be reduced to $O(n \log k)$ by plugging in the linear time algorithm by Henzinger et al. [4] for computing the shortest path tree in a planar graph.

The algorithm proposed in [8] first computes the union of the $k$ shortest paths, which is claimed to be a forest. The second step in [8] relies on a data structure due to Gabow and Tarjan [3] for

* Dipartimento di Scienze di Base e Applicate per l’Ingegneria, Sapienza Università di Roma, Via Antonio Scarpa, 16, 00161 Roma, Italy. lorenzo.balzotti@sbai.uniroma1.it.
† Dipartimento di Scienze Statistiche, Sapienza Università di Roma, p.le Aldo Moro 5, 00185 Roma, Italy. paolo.franciosa@uniroma1.it
efficiently solving least common ancestor (LCA) queries in a forest, in order to obtain distances between the terminal pairs in $O(n)$ worst-case time.

Actually, the union of the shortest paths may contain cycles. An instance is shown in Figure 1 such that any set of $k$ shortest paths must contain a cycle, hence the distances between terminal pairs cannot always be computed by solving LCA queries in a forest. We notice that this fact was noted first in [6].

In this article we solve this problem, showing an algorithm that given a plane graph $U$ with non-negative edge weights and $k$ terminal pairs lying on the external face computes the $k$ distances between terminal pairs in $O(|U|)$ worst-case time, provided that $U$ is the union of the shortest paths. Moreover, we also propose an algorithm for listing the sequence of edges in a shortest path $\pi$ joining a terminal pair in $O(\ell + \ell \log \left\lceil \frac{k}{\ell} \right\rceil)$, where $\ell$ is the number of edges in $\pi$. The listing algorithm can be applied to any terminal pair.

Our algorithm can replace the second step in the solution proposed in [8], obtaining an $O(n \log k)$ worst-case time algorithm that solves any possible instance of the shortest non-crossing paths problem in a plane graph.

We follow the same approach of Polischuck and Mitchell [6], that was inspired by Papadopoulou’s work [5]. Their paper solves the problem of finding $k$ shortest non-crossing paths in a polygon with $n$ vertices, where distances are defined according to the Euclidean metric. As in [6, 8], we first build the genealogy tree $T_g$, that describes the order in which terminal pairs appear on the external face. Then we find, one at a time, the shortest paths joining terminal pairs, according to a postorder visit of $T_g$.

The main novelty in our solution is the definition of shortcuts, that are portions of the boundary of a face that allow us to modify a path without increasing (and possibly decreasing) its length. We show that it is possible to establish whether a path is a shortest path by looking at the presence of shortcuts. This is the main theoretical result of this paper: indeed, while the property of being a shortest path is a global property, we can check it locally by looking at a single face at a time for the presence of shortcuts adjacent to the path, ignoring the rest of the graph. Notice that this is only possible when the input graph is the union of shortest paths, not for general plane graphs.

Using this result, we can introduce the algorithm MarkPaths. It computes an implicit representation of shortest non-crossing paths consisting in marking by a function $Mark$ all darts (for convenience, we consider each shortest path as a directed path) that appear in one or more of the directed shortest paths computed by the algorithm. This implicit representation is necessary to compute all distances between terminal pairs in $O(|U|)$ worst-case time. Function $Mark$ is also used to solve the problem of listing the edges of a single shortest path. This problem is discussed in the geometrical case, [5, 6], but is new in non-crossing paths in a plane graph.

Figure 1: in this example the union of shortest paths from $s_i$ to $t_i$, for $i = 1, 2, 3$, contains a cycle (the union is highlighted with bold edges).
Shortest paths computed by algorithm MarkPaths fulfill the single-touch property, i.e., the intersection of any pair of paths is itself a path. Single-touch property implies that shortest non-crossing single-touch paths are non-crossing for every planar embedding of \( U \), provided that a non-crossing solution exists. For this reason, we solve the problem for planar graphs and not only for plane graphs.

The paper is organised as follows: in Section 2 we give preliminary definitions and notations that will be used in the whole paper. In Section 3 we deal with shortcuts, and in Section 4 we show that our found paths are non-crossing for almost all embeddings. In Section 5 we introduce first the main algorithm MarkPaths and then algorithms for computing distances and listing edges in a path. Finally, in Section 6 there are the conclusion of this paper.

2 Preliminaries

General definitions and notations are given. Then we deal with paths and we prove some properties of a special class of non-crossing paths, strongly non-crossing paths. We define a partial order on terminal pairs, the genealogy tree, and we introduce representations, that are sets of shortest non-crossing paths. Finally, we prove some properties of non-crossing directed paths with respect to the genealogy tree.

2.1 Definitions and notations

Let \( G = (V(G), E(G)) \) be a plane graph, i.e., a planar graph with a fixed planar embedding. We denote by \( F_G \) the set of faces and by \( F_G^\circ \) its external face. When no confusion arises we use the term face to denote both the border cycle and the finite region bounded by the border cycle.

Let \( w : E(G) \to \mathbb{R}_{\geq 0} \) be a weight function on edges. The weight function is extended to a subgraph \( H \) of \( G \) so that \( w(H) = \sum_{e \in E(H)} w(e) \). W.l.o.g., we assume that \( w \) is strictly positive, indeed if an edge has zero weight, then we can delete it and join its extremal vertices. In case a zero weight cycle \( C \) exists, then all edges internal to \( C \) should have zero weight and can be contracted in a single vertex.

Even if we solve the multi-terminal shortest path problem on undirected graphs, we will make use of both undirected and directed graphs in our proofs. Unless otherwise specified, statements hold for undirected and directed graphs.

We use standard union and intersection operators on graphs, we define the empty graph as a graph without edges.

**Definition 1** Given two undirected (directed) graphs \( G = (V(G), E(G)) \) and \( H = (V(H), E(H)) \), given an edge (dart) \( e \) and a vertex \( v \) we define the following operations and relations:

- \( G \cup H = (V(G) \cup V(H), E(G) \cup E(H)) \),
- \( G \setminus H = (V(G), E(G) \setminus E(H)) \),
- \( G \cap H = (V(G) \cap V(H), E(G) \cap E(H)) \),
- \( G \subseteq H \iff V(G) \subseteq V(H) \) and \( E(G) \subseteq E(H) \),
- \( G = \emptyset \iff E(G) = \emptyset \) (\( V(G) \) can be nonempty),
- \( e \in G \iff e \in E(G) \),
- \( v \in G \iff v \in V(G) \).

We denote by \( uv \) the edge whose endpoints are \( u \) and \( v \) and we denote by \( \ ¬ → uv \) the dart from \( u \) to \( v \). For every dart \( \¬ → uv \) we define \( \text{rev}(\ ¬ → uv) = \ ¬ → vu \), \( \text{head}(\ ¬ → uv) = v \) and \( \text{tail}(\ ¬ → uv) = u \).

Let \( G = (V(G), E(G)) \) be an undirected graph, we denote by \( \overrightarrow{G} = (V(\overrightarrow{G}), D(\overrightarrow{G})) \) the directed version of \( G \) defined by \( V(\overrightarrow{G}) = V(G) \) and \( ab \in E(G) \) if and only if \( \overrightarrow{ab} \in D(\overrightarrow{G}) \). We stress that
2.2 Paths and non-crossing paths

If \( a \hat{b} \in D(G) \), then \( \overrightarrow{b a} \in D(G) \). Let \( P = (V(P), D(P)) \) be a directed graph, we denote by \( \overline{P} = (V(P), E(\overline{P})) \) the undirected version of \( P \) such that \( \overrightarrow{ab} \in D(P) \) implies \( ab \in E(\overline{P}) \), where \( w(\overrightarrow{ab}) = w(ab) \). Moreover, we say that \( P \subseteq G \) if \( P \subseteq G \).

For every vertex \( v \in V(G) \), we define the degree of \( v \) as \( \text{deg}(v) = |\{ e \in E(G) | v \text{ is an endpoint of } e \}| \); for every vertex \( x \in V_P \) we define the outgoing degree of \( x \) as \( \text{deg}^+(x) = |\{ d \in D(P) | \text{tail}(d) = x \}| \).

A graph \( p \) is a path from \( a \) to \( b \) if the set of edges in \( p \) forms a sequence \( av_1, v_1v_2, \ldots, v_{r−1}v_r, v_rb \); we also say \( p \) is an \( a-b \) path, clearly, if \( p \) is a directed path, then \( p \) starts in \( a \) and it finishes in \( b \). A path is simple if each vertex has degree at most two. A cycle is a path from \( a \) to \( a \), and it is a simple cycle if no vertex but \( a \) appears twice.

We use round brackets to denote ordered sets. For example, \( \{a, b, c\} = \{c, a, b\} \) and \( \{a, b, c\} \neq \{c, a, b\} \).

For every \( k \in \mathbb{N} \) we denote by \( [k] \) the set \( \{1, \ldots, k\} \).

Let \( U = \bigcup_{i \in [k]} P_i \) be a plane undirected graph, where \( P_i \) is a shortest path from \( s_i \) to \( t_i \) in a planar unknown graph \( G \), and the terminal pairs \( \{ (s_i, t_i) \}_{i \in [k]} \) lie on the external face \( f^\infty \) of \( U \).

We stress that we work with a fixed embedding of \( U \).

Let \( U' \) be a subgraph of \( U \), we define \( f^\infty_{U'} \), the unique face of \( U' \) such that \( f^\infty \subseteq f^\infty_{U'} \), and we call it the external face of \( U' \). Given a (possibly not simple) cycle \( C \), we define the region bounded by \( C \) the maximal subgraph of \( U \) whose external face is \( C \).

W.l.o.g., we assume that the terminal pairs are distinct, i.e., there not exist \( i, j \in [k] \) such that \( \{s_i, t_i\} = \{s_j, t_j\} \). Let \( \gamma_i \) be the undirected clockwise \( s_i-t_i \) path in \( f^\infty \), for \( i \in [k] \). By hypothesis, we assume that couples \( \{(s_i, t_i)\}_{i \in [k]} \) are well-formed, i.e., for all \( j, l \in [k] \) either \( \gamma_j \cup \gamma_l \cap \gamma_i = \emptyset \) or \( \gamma_i \cap \gamma_l = \emptyset \).

Let \( i \in [k] \), we denote by \( i \)-path an \( s_i-t_i \) path and by \( \Pi_i \) the set of shortest undirected \( i \)-paths in \( U \). If \( p \) is a simple undirected \( i \)-path, we denote by \( \overrightarrow{P} \) its directed version from \( s_i \) to \( t_i \). It is always useful to see each \( i \)-path as oriented from \( s_i \) to \( t_i \), for \( i \in [k] \), even if the path is undirected. Given an \( i \)-path \( p \), we define \( \text{Left}_p \) as the undirected left portion of \( U \) with respect to \( p \), i.e., the region bounded by the cycle formed by \( p \) and \( \gamma_i \); similarly, we define \( \text{Right}_p \) as the undirected right portion of \( U \) with respect to \( p \), i.e., the region bounded by the cycle formed by \( p \) and \( f^\infty \setminus \gamma_i \).

Given an \( i \)-path \( p \) and a \( j \)-path \( q \), we say that \( q \) is right of \( p \) if \( q \subseteq \text{Right}_p \), similarly, we say that \( q \) is left of \( p \) if \( q \subseteq \text{Left}_p \).

Let \( R \subseteq U \) and let \( p \subseteq R \) be an \( i \)-path, for some \( i \in [k] \). We say that \( p \) is the leftmost \( i \)-path in \( R \) if for each \( i \)-path \( q \subseteq R \) \( p \) is left of \( q \). Similarly, we say that \( p \) is the rightmost \( i \)-path in \( R \) if for each \( i \)-path \( q \subseteq R \) \( p \) is right of \( q \).

Let \( R \) be a subgraph of \( U \), we denote by \( \partial R \) the external face of \( R \). Moreover, we define \( R = R \setminus \partial R \).

Given an \( i \)-path \( p \), for some \( i \in [k] \), we observe that \( p = \partial \text{Left}_p \cap \partial \text{Right}_p \).

2.2 Paths and non-crossing paths

Given an \( a-b \) path \( p \) and a \( b-c \) path \( q \), we define \( p \circ q \) as the (possibly not simple) path obtained by the concatenation of \( p \) and \( q \).

Let \( p \) be a simple path and let \( a, b \in V(p) \). We denote by \( p[a, b] \) the subpath of \( p \) with extremal vertices \( a \) and \( b \).

Now we introduce an operator that allows us to replace a subpath in a path.

Definition 2 Let \( p \) be a simple \( a-b \) path, let \( u, v \in V(p) \) such that \( a, u, v, b \) appear in this order in \( p \) and let \( q \) be a \( a-v \) path. We denote by \( p \bowtie q \) the (possibly not simple) path \( p[a, u] \circ q \circ p[v, b] \).

Figure 2 shows an example of operator \( \bowtie \): in (a) there are three paths \( p, q \) and \( r \), in (b) there is \( p \bowtie q \) and in (c) there is \( p \bowtie r \).
The following definition introduces right envelopes and left envelopes, that are important for transforming paths. Figure 3 shows an example of right envelope and left envelope.

**Definition 3** Let \( p \) be an undirected \( i \)-path and let \( q \) be an undirected \( j \)-path, for \( i, j \in [k] \). Let \( V(p) \cap V(q) = (v_1, \ldots, v_r) \) ordered so that \( v_\ell \) appears before \( v_{\ell+1} \) going from \( s_i \) to \( t_i \) through \( p \), for \( \ell \in [r-1] \). If \( r \geq 2 \), for all \( \ell \in [r-1] \) we define

\[
\begin{align*}
    p \circ REq[v_\ell, v_{\ell+1}] &= \begin{cases} q[v_\ell, v_{\ell+1}] & \text{if } p \circ q[v_\ell, v_{\ell+1}] \subseteq \text{Right}_p, \\ p[v_\ell, v_{\ell+1}] & \text{otherwise.} \end{cases} \\
    p \circ LEq[v_\ell, v_{\ell+1}] &= \begin{cases} q[v_\ell, v_{\ell+1}] & \text{if } p \circ q[v_\ell, v_{\ell+1}] \subseteq \text{Left}_p, \\ p[v_\ell, v_{\ell+1}] & \text{otherwise.} \end{cases}
\end{align*}
\]

Moreover, we define \( p \) right envelope \( q \) as \( p \circ REq = p[s_i, v_1] \circ p \circ REq[v_1, v_2] \circ \ldots \circ p \circ REq[v_{r-1}, v_r] \circ p[v_r, t_i] \) and \( p \) left envelope \( q \) as \( p \circ LEq = q[s_i, v_1] \circ p \circ LEq[v_1, v_2] \circ \ldots \circ p \circ LEq[v_{r-1}, v_r] \circ p[v_r, t_i] \).

We say that two paths in a plane graph \( G \) are **non-crossing** if the curve they describe in the graph embedding do not cross each other; a formal definition of crossing/non-crossing paths can be found in [2]. We stress that this property depends on the embedding of the graph. Non-crossing paths may share vertices and/or edges or darts. We also define a class of paths that has nice properties that will be used later.

**Definition 4** Two paths \( p \) and \( q \) are **single-touch** if \( p \cap q \) is a (possibly empty) path.
Examples of non-crossing paths and single-touch paths are given in Figure 4.

Figure 4: the paths in (a) and (b) are crossing, while paths in (c), (d), (e) are non-crossing. Moreover, the paths in (a), (c) and (d) are not single-touch, while the paths in (b) and (e) are single-touch.

Our algorithms allows us to obtain non-crossing single-touch paths even if the input graph $U$ is the union of crossing and/or not single-touch paths. Let us assume that these shortest paths are found by starting from a graph $G$, thus $U$ is a subgraph of $G$. We observe that if there are more shortest $a \to b$ paths in $G$, then the shortest paths found can be not single touch. It is possible to overcome this problem by introducing small perturbations in the weight function, so that uniqueness of shortest paths is ensured. It is easy to see that with this change, the shortest paths found are always single-touch. The technique we describe in this paper does not rely on perturbation but we break ties by choosing rightmost or leftmost paths.

2.3 Strongly non-crossing paths

In this section, we prove some useful properties of a special class of non-crossing paths, called strongly non-crossing paths.

Definition 5 Let $p$ be an $i$-path and let $q$ be a $j$-path, for some $i, j \in [k]$. We say that $p$ and $q$ are strongly non-crossing if either $a \subseteq \text{Right}_p$ or $a \subseteq \text{Left}_q$, for all $a, b \in \{p, q\}$. Moreover, we say that a set of paths $P$ is strongly non-crossing if for all $p, q \in P$, $p$ and $q$ are strongly non-crossing.

Definition 5 is explained in Figure 5: in (a) there are an $i$-path $p$ and a $j$-path $q$ that are non-crossing but they are not strongly non-crossing. Figure 5(b) shows the previous paths modified by envelopes in order to obtain two strongly non-crossing paths, $p'$ and $q'$, such that $p \cup q = p' \cup q'$. We observe that if $p$ and $q$ are shortest paths, then so are $p'$ and $q'$.

Figure 5: in (a) there are two non-crossing but not strongly non-crossing paths $p$ and $q$, (b) shows two strongly non-crossing paths.

Now we give some lemmata that deal with strongly non-crossing paths.

Lemma 1 Let $P = \{p_1, \ldots, p_\ell\}$ be a set of strongly non-crossing undirected paths such that $\text{Left}_{p_1} \supseteq \text{Left}_{p_2} \supseteq \ldots \supseteq \text{Left}_{p_\ell}$ and $f \subseteq \bigcap_{i \in [\ell]} \text{Left}_{p_i}$. Then $f \cap (\bigcup_{i \in [\ell]} p_i) = f \cap p_\ell$. 

6
Proof. Let us assume by contradiction that there exists \( q \in P \) such that \((f \setminus pe) \cap q \neq \emptyset\). Let \( e \) be an edge in \((f \setminus pe) \cap q\). Being \( e \in f \setminus pe \) and being \( f \subseteq \text{Left}_{p_j} \), then \( e \in \text{Left}_{p_j} \). Thus, being \( P \) strongly non-crossing, \( q \in \text{Left}_{p_j} \). Hence \( \text{Left}_{p_j} \supseteq \text{Left}_q \), absurdum. \( \square \)

Lemma 2 Let \( f \in F \), let \( \sigma \) be a subpath of \( \partial f \) whose extremal vertices are \( u \) and \( v \). Let \( P \) be a set of strongly non-crossing undirected paths such that \( \sigma \subseteq \bigcup_{p \in P} p \). If there exists \( q \in P \) such that \( u,v \in V(q) \) and \( q \cap \sigma \neq \emptyset \), then there exists \( r \in P \) such that \( \sigma \subseteq r \).

Proof. Let \( P' \subseteq P \) be the minimal set of paths such that \( \sigma \subseteq \bigcup_{p \in P'} p \). W.l.o.g., we assume that \( f \subseteq \text{Left}_q \), and, being \( \sigma \cap q \neq \emptyset \), \( \sigma \subseteq \text{Left}_q \).

Let \( e \) be an edge in \( \sigma \setminus q \), thus \( e \in \text{Left}_q \). Let \( r \in P \) be such that \( e \in r \), then \( r \subseteq \text{Left}_q \). Let \( s_q,t_q \) be the extremal vertices of \( q \), and let \( s_r,t_r \) be the extremal vertices of \( r \). Being \( r \subseteq \text{Left}_q \), then \( s_r \) and \( t_r \) are in the clockwise path in \( f^\infty \) from \( s_q \) to \( t_q \).

Let \( S \) be the region bounded by the (not simple) face formed by \( \sigma \) and \( q[u,v] \). Being \( e \in S \) and being \( s_r,t_r \notin S \), then \( r \) has to “go in” \( S \) and “go out” from \( S \). Being \( q \) and \( r \) non-crossing and being \( \partial S \) composed by \( \sigma \) and \( q[u,v] \), then \( r \) has to go in \( S \) and to go out from \( S \) through \( \sigma \). Being \( f \) a face, then \( r \) goes in \( S \) by \( u \) and \( r \) goes out from \( S \) by \( v \) (or vice-versa).

Informally, we have proved that \( r \) “turns around” \( f \), and w.l.o.g. we assume that \( f \subseteq \text{Left}_r \) (if it is false, then it is sufficient to switch \( s_r \), with \( t_r \)).

Thus we can repeat this reasoning, i.e., if \(|P'| = \ell\), then \( P' = \{p_j\}_{j \in \ell}; p_1 = p, p_{\ell+1} = \text{Left}_{p_j} \) and \( f \subseteq \text{Left}_{p_j} \) for all \( j \in \ell \). We conclude by Lemma 1 \( \square \)

Figure 6 is useful to understand Lemma 2 in particular the necessity of the hypothesis that \( P \) has to be a set of strongly non-crossing paths. In (a) there is a face \( f \) and a subpath \( \sigma \) of \( f \) whose extremal vertices are \( u \) and \( v \). In (b) there is a set \( P = \{q,p,p'\} \) of three paths such that \( u,v \in V(q) \), \( \sigma \subseteq q \cup p \cup p' \) but there is not \( r \in \{q,p,p'\} \) such that \( \sigma \subseteq r \). Indeed Lemma 2 can not be applied because \( P \) is not a set of strongly non-crossing paths.

Figure 6: in (a) there is a face \( f \) and a subpath \( \sigma \) of \( f \) whose extremal vertices are \( u \) and \( v \). In (b) there is a set \( P = \{q,p,p'\} \) of three not strongly non-crossing paths such that their union covers \( \sigma \).

Lemma 3 Let \( P = \{p_1,\ldots,p_r\}, r \leq k \), be a set of undirected paths such that \( p_j \in \Pi_j \), for \( j \in [r] \). Then there exists \( Q = \{q_1,\ldots,q_r\} \) such that \( q_j \in \Pi_j \), for \( j \in [r] \), \( q \) and \( q' \) are non-crossing, for \( q,q' \in Q \), and \( \bigcup_{q \in Q} q = \bigcup_{p \in P} p \).

Proof. Let \( a,b \) be two paths, we define \( c_{a,b} \) as the number of times that \( a \) crosses \( b \) (clearly, \( c_{a,b} = c_{b,a} \)). Now, let us assume that in \( P \) there are at least two crossing paths, otherwise the thesis is true for \( Q = P \). Let \( p,q \in P \) be two crossing paths. Thus there exist \( u,v \in V(P) \cap V(q) \) such that \( p[u,v] \subseteq \text{Left}_q \) (or \( p[u,v] \subseteq \text{Right}_p \)), that implies \( q[u,v] \subseteq \text{Left}_p \) (or \( q[u,v] \subseteq \text{Right}_p \).
Let $R = p[u, v] \circ q[v, u]$, by above, $R$ is a simple cycle. Moreover, being $p$ and $q$ shortest paths, then $p \cap R = q \cap R = \emptyset$.

Finally, let $p' = p \times q[u, v]$ and $q' = q \times p[u, v]$, it holds that $c_{p', q'} = c_{p, q} - 2$ (two crossing paths whose pairs of extremal vertices are well-formed necessary cross an odd number of times), $p' \cup q' = p \cup q$ and $c_{r, p} + c_{r, q} = c_{r, p'} + c_{r, q'}$, for all $r \in P$, because $R$ is a simple cycle and $p \cap R = q \cap R = \emptyset$. By repeating this argument, the thesis follows. □

2.4 Genealogy tree

Given a well-formed set of couples $\{(s_i, t_i)\}_{i \in [k]}$, we define here a partial ordering as in [8] that represents the inclusion between $\gamma_i$’s. This relation intuitively corresponds to an adjacency relation between non-crossing shortest paths joining each pair.

Choose an arbitrary $i^*$ such that there are neither $s_j$ nor $t_j$, with $j \neq i^*$, walking on $f^\infty$ from $s_{i^*}$ to $t_{i^*}$ (either clockwise or counterclockwise), and let $e^*$ be an arbitrary edge on that walk. For each $j \in [k]$, we can assume that $e^* \not\subseteq \gamma_j$, indeed if it is not true, then it suffices to switch $s_j$ with $t_j$. We say that $i \prec j$ if $\gamma_i \subseteq \gamma_j$ (for all $i, j \in [k]$, either $\gamma_i \subset \gamma_j$ or $\gamma_j \subset \gamma_i$ or $\gamma_i \cap \gamma_j = \emptyset$, since terminal pairs are well-formed and distinct). Let $T_q$ be the transitive reduction of poset $([k], \prec)$.

If $i \prec j$, then we say that $i$ is an ancestor of $j$ and $j$ is a descendant of $i$; moreover, we define $[i, j] = \{l \in [k] \mid i \prec l \text{ and } l \prec j\}$.

Figure 7 (a) shows an example of well-formed terminal pairs, with a possible genealogy tree for $i^* = 1$ in Figure 7 (b). From now on, in all figures we draw $f^\infty_G$ with a solid light grey line.

![Figure 7](image)

Figure 7: in (a) an example of well-formed terminal pairs. If we choose $i^* = 1$, then we obtain the genealogy tree in (b).

2.5 Representations

We remember that $U$ is the union of shortest $i$-paths, for $i \in [k]$. That is, there exist $p_1, \ldots, p_k$ such that $p_i \in \Pi_i$, for $i \in [k]$, and $U = \bigcup_{i \in [k]} p_i$. Despite this, it is possible that there are $p'_1, \ldots, p'_k$ such that $p'_i \in \Pi_i$, for $i \in [k]$, $U = \bigcup_{i \in [k]} p'_i$ and there exists $j \in [k]$ such that $p_j \neq p'_j$. In other words, the union of shortest paths does not imply univocally all shortest paths. To overcome this ambiguity, we introduce the definition of representation.
We say that a set of undirected paths $P = \{p_1, \ldots, p_k\}$ is a representation if for all $i, j \in [k]$, $p_i \in \Pi_i$ and $p_j$ and $p_j$ are non-crossing. We stress that, being the terminal couples well-formed, we can always require non-crossing property. We denote the set of all possible representations by $\mathcal{P}$.

We underline that $U$ might be obtained by the union of crossing paths. Despite this, the following lemma ensures us that it always exists a representation $P$ formed by strongly non-crossing paths such that the union of all paths in $P$ is exactly $U$.

**Lemma 4** There exists $P \in \mathcal{P}$ such that $U = \bigcup_{p \in P} p$ and $P$ is strongly non-crossing.

**Proof.** Let $U'$ be the graph obtained from $U$ adding the vertices $s'_i, t'_i$ and the edges $s_i s'_i, t_i t'_i$, for $i \in [k]$ (the weight of the new edges are arbitrary) in order to obtain that the couples $(s'_i, t'_i)_{i \in [k]}$ are well-formed for $U'$. We want to use $U'$ to prove this lemma, indeed, a set $P'$ of non-crossing $s'_i - t'_i$ shortest paths in $U'$, for $i \in [k]$, is always strongly non-crossing (where the definitions of Right$_p$ and Left$_p$, for all $p \in P'$, are extended naturally).

Let $\Pi'_i$ be the set of $s'_i - t'_i$ shortest path in $U'$. We define $\tau : \bigcup_{i \in [k]} \Pi_i \to \bigcup_{i \in [k]} \Pi'_i$ as $\tau(p) = s'_i s_i \circ p \circ t_i t'_i$, if $p \in \Pi_i$, for $i \in [k]$. It is clear that, for all $i \in [k]$, $p \in \Pi_i$ if and only if $\tau(p) \in \Pi'_i$.

Now, let $P = \{p_1, \ldots, p_k\} \in \mathcal{P}$ be such that $P$ satisfies $U = \bigcup_{p \in P} p$ (such a representation exists because of definition of $U$). Let $P' = \{\tau(p_1), \ldots, \tau(p_k)\}$. By above, $P'$ is a set of $s'_i - t'_i$ shortest paths such that $\bigcup_{p \in P'} p = U'$, but some couples of paths can be crossing.

By Lemma 3 there exists $Q = \{q_1, \ldots, q_k\}$ obtained from $P'$, such that $\bigcup_{p \in Q} p = U'$ and $p$ and $q$ are non-crossing, for all $p, q \in Q$. Finally, let $Q' = \{\tau^{-1}(q_1), \ldots, \tau^{-1}(q_k)\}$. It holds that $Q' \in \mathcal{P}$ and $Q'$ satisfies all required properties. 

If a representation $P \in \mathcal{P}$ is such that $U = \bigcup_{p \in P} p$ and $P$ is strongly non-crossing, then we call $P$ a complete strongly non-crossing representation.

**Remark 1** From now on, we can assume that for all $i, j \in [k]$, $i \neq j$, $s_i$, $t_i$, $s_j$ and $t_j$ are distinct vertices of $U$. It is sufficient to create the graph $U'$ described in Lemma 4, embedding the new edges so that the new pairs $\{(s'_i, t'_i)\}_{i \in [k]}$ are still well-formed. Trivially, Lemma 4 still holds.

### 2.6 Properties of non-crossing directed paths

In this section we introduce two lemmata that prove two useful properties of non-crossing and/or shortest paths with respect to the genealogy tree.

**Lemma 5** Let $\pi_i$ be a shortest directed $i$-path and let $\pi_j$ be a shortest directed $j$-path, for some $i, j \in [k]$. If $j$ is not an ancestor neither a descendant of $i$ in $T_q$, then $\pi_i$ and $\pi_j$ have no common darts.

**Proof.** Being $\{(s_i, t_i)\}_{i \in [k]}$ well-formed and being $j$ not an ancestor neither a descendant of $i$ in $T_q$, then starting in $s_i$ and going clockwise on $f^\infty$ we met in order $s_i, t_i, s_j$ and $t_j$. Let us assume by contradiction that $\pi_i$ and $\pi_j$ have some common darts, and let $d$ be the dart in $\pi_i \cap \pi_j$ that appears first in $\pi_i$. Let $R$ be the region bounded by $\pi_j[s_i, \text{tail}(d)]$, $\pi_j[s_j, \text{tail}(d)]$ and the clockwise undirected $s_i - s_j$ path in $f^\infty$. It holds that $t_j \notin V(R)$ and $\text{head}(d) \in V(R) \setminus V(\partial R)$, hence $\pi_j$ has to go out from $R$ (see Figure 8 (a) shows an example of $\pi_i$, $\pi_j$ and the region $R$).

Being $\pi_i$ a shortest path, then it is a simple path. Thus, in order to go out from $R$, $\pi_j$ crosses $\pi_i$ in at least a vertex in $\pi_i[s_i, \text{tail}(d)]$. Let $x$ be the first vertex in $\pi_i[s_i, \text{tail}(d)]$ after $\text{head}(d)$ in $\pi_i$. Now it suffices to look at the cycle formed by $\pi_i[x, \text{head}(d)]$ and $\pi_j[\text{head}(d), x]$ to understand that $\pi_i$ and $\pi_j$ can be both shortest paths, absurdum (in Figure 8 (b) shows this cycle).

**Lemma 6** Let $\{\pi_i\}_{i \in [k]}$ be a set of strongly non-crossing directed paths such that $\pi_i$ is a $i$-path, for $i \in [k]$. Let $i, j \in [k]$, if $i$ is a descendant of $j$, then $\pi_i \cap \pi_j \subseteq \pi_i$, for all $l \in [i, j]$. 

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Proof. We assume that \( [i, j] \setminus \{ i, j \} \neq \emptyset \) and \( \pi_i \cap \pi_j \neq \emptyset \), otherwise there is nothing to prove. Let \( l \in [i, j] \), being \( \{(s_i, t_i)\}_{i \in [k]} \) well-formed, then starting in \( s_i \) and going clockwise on \( f^\infty \) we met in order \( s_i, s_l, s_j, t_j, t_l \) and \( t_i \).

Let \( d \) be the dart in \( \pi_i \cap \pi_j \) that first appears in \( \pi_i \) and let \( R \) be the portion bounded by \( \pi_j[s_j, \text{tail}(d)], \pi_i[s_i, \text{tail}(d)] \) and the clockwise undirected \( s_i - s_j \) path in \( f^\infty \). It holds that \( t_l \notin V(R) \setminus V(\partial R) \), hence \( \pi_l \) has to go out from \( R \) because \( s_l \in V(R) \) (in Figure 9 there is an example of the region \( R \)).

It is clear that if \( d \notin \pi_i \), then \( \{\pi_i, \pi_j, \pi_l\} \) is not a set of strongly non-crossing paths, absurdum. \( \square \)

3 Shortcuts

Now we have all the machinery necessary to introduce shortcuts, that are the main tool of our algorithms. Shortcuts allow us to check whether a path is a shortest path by only looking at faces that are adjacent to the path. This property obviously does not hold for general plane graphs, but it only holds when the input graph is known to be the union of shortest paths whose terminal pairs are well-formed.

In this way we can check a global property of a path — i.e., being a shortest path — by checking a local property — i.e., the presence of shortcuts. This is the main theoretical result of this paper.

As we have said, to compute lengths of shortest paths we follow the same procedure of [6], that solves the problem of finding \( k \) shortest (with respect to a geometric distance) non-crossing paths in a polygon with \( n \) vertices. We deal with a very different problem. First, Polischuck and Mitchell, [6], find shortest paths, while we have their union as input. Second, they work in a
geometric setting, while a plane weighted graph has a more general structure.

Another little difference with respect to [6] is that we find the shortest \( s_i - t_i \) paths according to a postorder visit of \( T_g \). While, in [6], the genealogy tree is visited by levels: first are visited the leaves, then the parents of the leaves, and so on. We stress that this change has no effect on proofs or on computational complexity, but our choice makes the implementation easier.

**Definition 6** Let \( i \in [k] \) and let \( \lambda \) be an undirected i-path. Let \( f \) be a face adjacent to \( \lambda \), i.e., \( f \cap \lambda \neq \emptyset \). Let \( u, v \in V(\lambda) \cap V(f) \). We say that a \( u - v \) subpath \( q \) of \( f \) is a shortcut of \( \lambda \) if \( w(\lambda \cap q) \leq w(\lambda) \).

**Theorem 7** Let \( i \in [k] \) and let \( \lambda \) be an undirected i-path. If there are no shortcuts of \( \lambda \), then \( \lambda \) is the unique shortest i-path.

**Proof.** First, we introduce a definition. Let \( p \) be an arbitrary i-path. We define \( h_p \) as the region bounded by the (possibly not simple) cycle formed by \( p \) and \( \lambda \). Formally, \( h_p = (\text{Left}_\lambda \cap \text{Right}_p) \cup (\text{Right}_\lambda \cap \text{Left}_p) \) and \( \partial h_p = \lambda \cup p \). We observe that if \( h_p \) has no faces, then \( p \) is equal to \( \lambda \).

Let \( P \in \mathcal{P} \) be a complete strongly non-crossing representation. Let us assume by contradiction that there exists \( \tau \in P \) such that \( \tau \in \Pi_i \) and \( \tau \neq \lambda \).

Let \( f \subseteq h_\tau \) be a face such that \( f \cap \lambda \neq \emptyset \). W.l.o.g., we assume that \( f \subseteq \text{Left}_\tau \cap \text{Right}_\lambda \) and \( \tilde{f} \subseteq \text{Left}_{\tau} \) for all \( q \in P \).

Let \( e \) be an edge in \( \tilde{f} \setminus \lambda \) and let \( p \in P \) be such that \( e \in p \), it holds that \( p \subseteq \text{Left}_\tau \) because \( \tau \) and \( p \) are strongly non-crossing. Let \( F_p = \{ f \in \mathcal{F} \mid f \subseteq h_\tau \cap \text{Left}_p \} \). We want to prove that there is a shortcut by proceeding by induction on \( |F_p| \).

In the base case there is only one face \( f \) in \( h_\tau \cap \text{Left}_p \). Let \( u, v \) be the extremal vertices of \( \partial f \setminus \lambda \), it is clear that \( u, v \in p \) (otherwise, \( |F_p| > 1 \)). By Lemma 2 there exists \( r \in P \) such that \( \partial f \setminus \lambda \subseteq r \), thus \( \partial f \setminus \lambda \) is a shortcut.

Now we prove the induction step. We suppose that the thesis is true for \( |F_p| \leq \ell \). Now we assume that \( |F_p| = \ell + 1 \). There exists \( Q \subseteq P \) such that \( \bigcup_{f \in F_p} \partial f \setminus \lambda \subseteq \bigcup_{q \in Q} q \). W.l.o.g., we can assume that \( Q \) is minimal, i.e., there is not \( Q' \subset Q \) such that \( \bigcup_{f \in F_p} \partial f \setminus \lambda \subseteq \bigcup_{q \in Q'} q \).

Let \( f \in F_\lambda \). There are two cases: either there exists \( r \subseteq Q \) such that \( f \not\subseteq \text{Left}_r \) (case (a)), or \( f \subseteq \bigcap_{q \in Q} \text{Left}_q \) (case (b)).

If case (a) happens, then the number of faces of \( F_p \) in \( \text{Left}_r \) is less or equal than \( \ell \), because \( q \) cannot cross \( p \). Thus we conclude by the induction assumption.

If case (b) occurs, let \( u, v \) be the extremal vertices of the longest subpath \( s \) of \( p \) such that \( e \in s \) and \( s \subseteq \text{Left}_\lambda \). Now the proof is identical to Lemma 2’s proof. It is sufficient to see all faces of \( F_\lambda \) as a unique face. \( \square \)

## 4 Preserving non-crossing property for almost all embeddings

In this section we deal with embeddings. We know that the non-crossing property depends on the embedding of \( U \), but we can find non-crossing paths if and only if the terminal pairs are well-formed.

The following lemma assures that single-touch property implies the strongly non-crossing property. For example, Figure 5 and 6 shows two not strongly non-crossing paths that are not single-touch.

**Lemma 8** If \( P = \{ p_i \}_{i \in [k]} \) is a set of single-touch paths, where \( p_i \) is an i-path, for \( i \in [k] \), then \( P \) is a set of strongly non-crossing paths.
Remark 2. If for an embedding of \( f \) terminal pairs well-formed in the input embedding of \( U \) we have the terminal pairs \( \{s_i, t_i\} \in [k] \) on the external face \( f \), \( \partial f \) is a simple cycle, and the terminal pairs are well-formed, then the same properties hold for all embeddings of \( U \) having \( f \) as external face.

Proof. Let us assume by contradiction that there exist \( \pi_i \) and \( \pi_j \) paths, for some \( i, j \in [k] \), such that \( \pi_i \) and \( \pi_j \) are not strongly non-crossing.

Being the extremal vertices well-formed, then either \( s_i, t_i \in \text{Left}_{\pi_i} \) or \( s_i, t_i \in \text{Right}_{\pi_i} \). W.l.o.g., we assume that \( s_i, t_i \in \text{Left}_{\pi_i} \) and at least one among \( s_i \) and \( t_i \) is in \( V(\text{Left}_{\pi_i}) \setminus V(\pi_j) \) (otherwise \( \{s_i, t_i\} = \{s_j, t_j\} \) and the single-touch property implies that \( \pi_i = \pi_j \)). W.l.o.g., let us assume that \( s_i \in V(\text{Left}_{\pi_i}) \setminus V(\pi_j) \).

Now, being \( \pi_i \) and \( \pi_j \) crossing, and being \( s_i \in V(\text{Left}_{\pi_i}) \setminus V(\pi_j) \), then there exist \( e \in \pi_i \) such that \( e \in \text{Right}_{\pi_i} \). Let \( q \) be the maximal subpath of \( \pi_i \) such that \( q \subseteq \text{Right}_{\pi_i} \) and \( e \in q \), and let \( u, v \) be the extremal vertices of \( q \). Single-touch property implies that \( \pi_i[u, v] = \pi_j[u, v] \), consequently \( e \not\in \text{Right}_{\pi_i} \), absurdum.

The following theorem explains that the single-touch property allows us to preserve the non-crossing property for all “acceptable” embeddings.

**Theorem 9** If \( \{\pi_i\}_{i \in [k]} \) is a set of simple single-touch undirected paths, where \( \pi_i \) is an \( i \)-path, for \( i \in [k] \), then \( \{\pi_i\}_{i \in [k]} \) is a set of strongly non-crossing paths for all the embeddings of \( U \) such that the terminal pairs \( \{\{s_i, t_i\}\}_{i \in [k]} \) are well-formed.

Proof. The thesis follows from Lemma 8 by observing that single-touch property does not depend on the embedding of \( U \). □

From now on, thanks to Lemma 8, we are interested only in single-touch property, being the terminal pairs well-formed in the input embedding of \( U \).

Remark 2. If for an embedding of \( U \) we have the terminal pairs \( \{\{s_i, t_i\}\}_{i \in [k]} \) on the external face \( f \), \( \partial f \) is a simple cycle, and the terminal pairs are well-formed, then the same properties hold for all embeddings of \( U \) having \( f \) as external face.

If \( \partial f \) is not a simple cycle, then we can fix the order of terminal pairs in the following way. We hang all vertices of terminal pairs in \( f \) by an edge having arbitrary weight and we add a cycle of edges with infinite weights that reaches all terminal pairs in the desired order.

5 Algorithms

We present here algorithm MarkPaths, that is a preprocessing step that is used to compute distances between terminal pairs and for listing shortest paths between terminal pairs, and we show that this algorithm has linear worst-case time complexity. Then we show that also distances between terminal pairs can be computed in linear worst-case time and that the sequence of darts in a shortest path can be listed in \( O(\ell + \ell \log(\frac{1}{\varepsilon})) \), where \( \ell \) is the number of edges in \( \pi \).

5.1 Preprocessing algorithm and its correctness

In this section it is easier to consider all graphs as directed. Thus, unless otherwise specified, every graph is directed. We use definitions and notations introduced for undirected graph by extending it naturally to the directed case. Despite this, in order to avoid possible confusions, sometimes we use first the undirected version of paths or graphs and then we return to its directed version. We stress that now all graphs are subgraphs of \( \overrightarrow{U} \). To simplify the notation, we often omit the superscript for directed graphs.

In this section we introduce algorithm MarkPaths. It gives an implicit set of shortest single-touch directed paths, that will be used to compute distances for each \( s_i - t_i \) pairs (see Theorem 17). Moreover, algorithm MarkPaths marks every dart of the output, this is essential to list a single path (see Theorem 18).
In Lemma 10, we give some preliminary results to Theorem 11 that proves the correctness of algorithm \texttt{MarkPaths} and it explains some properties of found paths.

The main idea behind algorithm \texttt{MarkPaths} is the following. We build a set of shortest i-paths \((\lambda_i)_{i \in [k]}\), by finding \(\lambda_i\) at iteration \(i\), where the terminal pairs are numbered according to a postorder visit of \(T_g\). In particular, at iteration \(i\) we find the rightmost shortest i-path in \(U_i = \bigcap_{j \in [i-1]} \text{Right}_j\), in the following way: first we set \(\lambda_i\) as the leftmost i-path in \(U_i\), then we update \(\lambda_i\) by moving right through shortcuts (the order in which shortcuts are chosen is not relevant). When \(\lambda_i\) has no more right shortcuts, then it is the rightmost shortest path in \(U_i\) by Theorem 7.

For this reason, given a directed i-path \(p\), for some \(i \in [k]\), a directed \(u - v\) path \(q\), we say that \(q\) is a directed right shortcut for \(p\) if \(s_i, u, v, t_i\) appear in this order in \(p\), \(\overline{p}\) is a shortcut for \(\overline{\overline{p}}\) and \(q \subseteq \text{Right}_p\).

Another result is that algorithm \texttt{MarkPaths} marks every dart in the union of found paths by using the function \texttt{Mark} : \(D(\bigcup_{i \in [k]} \lambda_i) \rightarrow [k]\). If a dart \(d\) is used the first time at iteration \(i\), i.e., \(d \in \lambda_i \setminus \bigcup_{j < i} \lambda_j\), then \(\text{Mark}(d) = i\). This function will be used in [5,3] to list shortest paths.

\begin{algorithm}
\textbf{Algorithm MarkPaths:}
\begin{itemize}
    \item \textbf{Input:} a plane graph \(U\), a set of well-formed terminal pairs \(\{(s_t,t_i)\}_{i \in [k]}\) on the external face, where \(U\) is the union of \(k\) shortest paths, each joining \(s_t\) and \(t_i\), for \(i \in [k]\), in an unknown undirected planar graph \(G\) with positive edge weights
    \item \textbf{Output:} an implicit linear space set of paths \(\{\lambda_1, \ldots, \lambda_k\}\), where \(\lambda_i\) is a shortest \(s_t - t_i\) path, for \(i \in [k]\),
    \item a function \texttt{Mark} associating values in \([k]\) to darts in \(\bigcup_{i \in [k]} \lambda_i\)
\end{itemize}

1. Compute \(T_g\) and renumber the terminal pairs \((s_1, t_1), \ldots, (s_k, t_k)\) according to a postorder visit of \(T_g\);
2. for \(i = 1, \ldots, k\) do
3. \hspace{0.5cm} Let \(\lambda_i\) be the directed leftmost i-path in \(U_i = \bigcap_{j \in [i-1]} \text{Right}_j\);
4. \hspace{0.5cm} while there exists \(\tau\) directed right shortcut of \(\lambda_i\) in \(U_i\) do
5. \hspace{1cm} \(\lambda_i := \lambda_i \times \tau\);
6. \hspace{1cm} for all dart \(d \in \lambda_i \setminus \bigcup_{j < i} \lambda_j\), \(\text{Mark}(d) = i\);
\end{algorithm}

\begin{lemma}
Let \(i \in [k]\), let \(\{p_j\}_{j \in [i-1]}\) be a set of directed single-touch paths such that \(p_j\) is a shortest j-path, for \(j \in [i-1]\). Let \(U_i = \bigcap_{j \in [i-1]} \text{Right}_{p_j}\). If terminal pairs are numbered according to a postorder visit of \(T_g\), then the following properties hold:
\begin{itemize}
    \item 10.(1) \(s_t, t_i \in U_i\);
    \item 10.(2) there exists a shortest i-path \(p_i\) such that \(p_i\) and \(p_j\) are single-touch, for all \(j \in [i-1]\), and \(p_i \subseteq U_i\).
\end{itemize}
\end{lemma}

\begin{proof}
Let's start with 10.(1). We suppose by contradiction that there exists \(j' < i\) such that \(s_t \in V(\text{Left}_{p_{j'}}) \setminus V(p_{j'})\) (or \(t_i \in V(\text{Right}_{p_{j'}}) \setminus V(p_{j'})\)). In particular, \(s_t \in \gamma_{j'}\), thus \(j'\) is an ancestor of \(i\), and hence \(j'\) is visited after \(i\) in every postorder visit. This implies \(j' > i\), absurdum.

Now we prove statement 10.(2). Let \(p_i\) be any shortest i-paths. If there exists \(j < i\) such that \(p_j\) and \(p_i\) are not single-touch, then let \(V(p_i) \cap V(p_j) = (v_1, \ldots, v_t)\) ordered so that \(v_t\) appears before \(v_{i+1}\) in \(p_i\). Then let \(p'_i = p_i[s_t, v_1] \circ p_j[v_1, v_t] \circ p_i[v_t, t_i]\) and \(p_j\) are single-touch and \(w(p_i) = w(p'_i)\). By repeating this reasoning, 10.(2) follows. \(\square\)
\end{proof}

\begin{theorem}
Let \(\{\lambda_i\}_{i \in [k]}\) be the set of paths computed by algorithm \texttt{MarkPaths}. Then
\begin{itemize}
    \item 11.(1) \(\lambda_i\) is the rightmost shortest i-path in \(U_i\), for \(i \in [k]\),
\end{itemize}
\end{theorem}
Now we give some definitions for \( \lambda \) we define \( C \) in \( d \). Let that \( p \) \( a \) it is clear that \( T \) in \( \lambda \) bounded by \( p \). 

5.2 Computational Complexity of algorithm MarkPaths

This section is split in three parts. In 5.2.1 we address the complexity of computing \( \lambda \) in Line 3 for all \( i \in [k] \). In 5.2.2 we show how to compute shortcuts, and, finally, in 5.2.3 we show that algorithm MarkPaths requires \( O(|U|) \) worst-case time.

From now on, let \( \{\lambda_i\}_{i \in [k]} \) be the output set of shortest paths given by algorithm MarkPaths. For \( i \in [k] \), we define \( \lambda_i \) as the initial \( \lambda_i \) computed in Line 3 and we define \( \lambda_j \), as \( \lambda_i \) after Line 5 has been executed \( j \) times, i.e., after \( j \) shortcuts have been considered. Moreover, let \( M_i \) be the number of times that algorithm MarkPaths executes Line 5 at iteration \( i \), i.e., \( \lambda_i = \lambda_{i,M_i} \). Finally, we define \( C_i = \{j \in [k] | p(j) = i \} \) the set of children of \( i \) in \( T_y \).

5.2.1 Computing \( \lambda_i \)

Now we give some definitions for \( \lambda_{i,j} \), for \( i \in [k] \) and \( j \leq M_i \). Actually, for any fixed \( i \), we are interested only in indices \( j = 0 \) and \( j = M_i \), i.e., the leftmost and the rightmost at iteration \( i \).

**Definition 7** Let \( i \in [k] \), let \( j \leq M_i \) and let \( (v_1, \ldots, v_r) \) be the sequence of vertices of \( \lambda_{i,j} \) ordered so that \( v_l \) appears before \( v_{l+1} \) in \( \lambda_{i,j} \). Let \( \Delta_{i,j} = \{x \in \text{Right}_{\lambda_{i,j}} | x \in V(\lambda_{i,j}) \text{ and } y \not\in V(\lambda_{i,j}) \} \) and let \( \overrightarrow{v_l v_{l+1}} \in \Delta_{i,j} \), for some \( l, l' \in [r] \). We say that \( \overrightarrow{v_l v_{l+1}} \) is before \( \overrightarrow{v_{l'} v_{l'+1}} \) in \( p \) if

- \( l < l' \),
- \( l = l' \) and \( \overrightarrow{v_l v_{l+1}} \) appears before \( \overrightarrow{v_{l'} v_{l'+1}} \) when darts outgoing from \( v_l \) are visited counterclockwise from \( \overrightarrow{v_l v_{l+1}} \) to \( \overrightarrow{v_{l'+1} v_{l'+2}} \) (if \( l = 1 \) or \( l = r \), for the sake of simplicity, we add two dummy darts, \( \overrightarrow{v_l v_{l+2}} \) and \( \overrightarrow{v_{l'+1} v_{l'+2}} \), in the external face).

Moreover, we define \( L_{i,j} = (d_1, d_2, \ldots, d_{|\Delta_{i,j}|}) \) as the list on \( \Delta_{i,j} \) ordered so that \( d_l \) is before \( d_{l+1} \) in \( \lambda_{i,j} \), for all \( l \in [|\Delta_{i,j}|] \).

In order to simplify the notation, for any \( i \in [k] \), we denote \( L_{i,M_i} \) by \( L_i \).

**Figure 10** shows an example of Definition 7. There is \( \lambda_i \) and darts in \( L_i \) are drawn in solid lines.

The main idea to compute \( \lambda_{i,0} \), for \( i \in [k] \), is the following: being the paths non-crossing, if a dart \( d \) is in \( \lambda_{i,0} \) and \( \lambda_j \) for some \( j < i \), then \( j \) is a descendant of \( i \) in \( T_y \); in particular \( j \) is a child of \( i \) in \( T_y \). Moreover, the darts that are in \( \lambda_{i,0} \) and \( \lambda_j \) are a subpath of \( \lambda_j \). In formula it holds that \( \lambda_{i,0} \cap \bigcup_{j < i} \lambda_j \subseteq \bigcup_{j \in C_i} \lambda_j \), thus we have to find \( a_j, b_j \), for \( j \in C_i \), such that \( \lambda_{i,0} \cap \lambda_j = \lambda_j[a_j, b_j] \).

It is clear that \( a_j, b_j \in L_j \). The following definitions and lemmata solve this problem.

**Definition 8** Let \( i \in [k] \), let \( j \in C_i \). We say that \( d \in L_j \) is internal if there exist paths \( p,q \) such that \( p \subseteq \lambda_i \) and \( \lambda_j \subseteq \lambda_i \), for some \( z, z' \in C_i \), \( p \circ q \) is a \( s_z - t_{z'} \) simple path and \( d \in R \) with \( R \) the region bounded by \( p \circ q \circ \{ \text{clockwise path in } f^{\infty} \text{ from } t_{z'} \rightarrow s_z \} \).
Figure 10: the $i$-path drawn is $\lambda_i$ and darts in $L_i$ are represented by solid lines, instead the others are drawn with dotted lines. It holds that $L_i = (d_1, d_2, d_3, d_4, d_5)$. 

**Definition 9** Let $i \in [k]$, let $j \in C_i$. We define $L_j = L_j \setminus \{d \in L_j \mid d \text{ is internal}\}$. Moreover, let $d^-_j, d^+_j$ be the first and the last element of $L_j$, respectively, and let $v^-_j = \text{tail}(d^-_j)$ and $v^+_j = \text{tail}(d^+_j)$. We impose $d^-_j, d^+_j = \emptyset$ if $L_j = \emptyset$. 

Figure 11 explains definitions 8 and 9. Let us assume that $\{j_1, j_2, j_3\} = C_i$ for some $i \in [k]$. The darts in $L_{j_1}$ are highlighted: $d_6, d_7$ and $d_8$ are internal, $d_1, d_2, d_3, d_4, d_5$ are not. Consequently, $L_{j_1} = (d_1, d_2, d_3, d_4), d^-_{j_1} = d_1$ and $d^+_{j_1} = d_4$. We stress that all darts in $L_{j_2}$ are internal.

Figure 11: darts in $L_{j_1}$ are highlighted: darts represented by solid lines are not internal, instead, darts drawn in dotted lines are internal. It holds that $L_{j_1} = (d_1, d_2, d_3, d_4, d_5)$.

**Lemma 13** Let $i \in [k]$, the following statements hold:

13.(1) let $p$ be the directed path on $U_i$ starting in $s_i$ and always turning left until a vertex in $\{t_i\} \cup \{v^-_j\}_{j \in C_i}$ is reached. Then $p$ is well defined and $p \subseteq \lambda_{i,0}$. 

Let $j \in C_i$ be such that $d^+_j \neq \emptyset$ or $d^-_j \neq \emptyset$. Then

13.(2) $d^+_j \in \lambda_{i,0}$ and $\text{rev}(d^-_j) \in \lambda_{i,0}$, 

13.(3) $d^-_j \neq \emptyset$ and $d^+_j \neq \emptyset$, 

13.(4) let $q$ be the directed path on $U_i$ starting in $d^+_j$ and always turning left until a vertex in $\{t_i\} \cup \{v^-_j\}_{j \in C_i}$ is reached. Then $q$ is well defined and $q \subseteq \lambda_{i,0}$,
13.(5) $\lambda_{i,0} \cap \lambda_j = \lambda_j[v_j^-, v_j^+]$.

**Proof.** Let $P = \{\sigma_1, \ldots, \sigma_k\} \in \mathcal{P}$ be a complete strongly non-crossing representation (we remember that $\sigma_i$ is undirected, for all $i \in [k]$), let us assume that $\sigma_j$ is a shortest $j$-path, for all $j \in [k]$. We use $P$ to prove all statements.

- First we prove [13.(1)] Let us assume by contradiction that $p$ reaches a vertex $v$ of degree one in $U_i$, $\tau \notin P$ and $v_j \notin p$, for all $j \in C_i$. Being the degree of $v$ equal to one, then $p$ can not continue to go left (we observe that this is the only case to obtain $p$ not well defined). Being $U$ the union of shortest $i$-path, for $i \in [k]$, then $v \in f^\infty$ and, in particular, there exists $j \in [k]$ such that either $v = s_j$ or $v = t_j$. If $j < i$, then $v \in U_i \cap f^\infty$, thus $p$ passes through $\partial U_i$, absurdum. Else, $j > i$, and, being the terminal pairs well-formed, then going clockwise on $f^\infty$ starting from $s_i$, we find $s_i, t_i$ and $v$ in this order. Hence $\sigma_i$ is a path left of $p$, absurdum because $p$ is the leftmost in $U_i$ and $v_j \notin p$, for all $j \in C_i$. Therefore we have proved that $p$ is well defined.

Now we prove that $p \subseteq \lambda_{i,0}$. Let us assume by contradiction that $p \not\subseteq \lambda_{i,0}$. Let $d$ be a dart in $p$ and let $j \in [k]$ be such that $\overrightarrow{d} \in \sigma_j$. Let us assume that $d \notin \sigma_j$, if not, then switch $s_j$ with $t_j$. It holds that $j \not= i$, otherwise all darts in $p$ are in $\sigma_j$, hence $p \subseteq \lambda_{i,0}$. If $j < i$, then [13.(1)] is denied for $j$. Finally, if $j > i$, then, being all terminal couples well-formed, $\sigma_j$ crosses $\lambda_{i,0}$ and $p \not\subseteq \lambda_{i,0}$ is not the leftmost $i$-path in $U_i$. Indeed, it holds that $d_i^+ \in \text{Left}_{\lambda_{i,0}}$ and $s_z, t_z \in \text{Right}_{\lambda_{i,0}}$. Thus $\lambda_{i,0}$ and $\sigma_z$ are crossing. Hence $\mathbb{P} \mathbb{L} \mathbb{E} \sigma_z$ is an $i$-path in $U_i$, left of $\lambda_{i,0}$, absurdum. Therefore we have proved [13.(2)]

- Finally, [13.(3)] and [13.(5)] are implied by [13.(2)] and [13.(4)] can be proved analogously to [13.(1)].

The following lemma and remark explain in details the output of algorithm MarkPaths. For all $i \in [k]$, we define $\text{ind}(\lambda_i) = i$.

**Lemma 14** Let $i \in [k]$. If we know $\bigcup_{j \in C_i} \mathcal{L}_j$, then

14.(1) computing $\lambda_{i,0}$ costs $O(|\lambda_{i,0} \cap \bigcup_{j \in C_i} \lambda_j| + |C_i|)$.

14.(2) $\lambda_{i,0} = p_1 \circ \lambda_j, [v_j^-, v_j^+], p_2 \circ \lambda_j, [v_j^-, v_j^+], \ldots, p_r \circ \lambda_j, [v_j^-, v_j^+]$, for some (possibly empty) paths $\{p_j\}_{j \in [r]}$ in $U_i$ such that $\bigcup_{j \in [r]} p_j^+ \cap \bigcup_{j \in C_i} \lambda_j = \emptyset$, and $\{j_1, \ldots, j_r\} = \{j \in C_i | \mathcal{L}_j \neq \emptyset\}$.

14.(3) $\lambda_i = q_1 \circ \lambda_j, [a_{i1}, b_{i1}], q_2 \circ \lambda_j, [a_{i2}, b_{i2}], \ldots, q_{r_i} \circ \lambda_j, [a_{iR_i}, b_{R_i}]$, for some (possibly empty) paths $\{q_j\}_{j \in [r_i]}$ in $U_i$ such that $\bigcup_{j \in [r_i]} q_j^+ \cap \bigcup_{j \in C_i} \lambda_j = \emptyset$, $\text{ind}(\lambda_{i1}), \ldots, \text{ind}(\lambda_{iR_i}) \subseteq \{j_1, \ldots, j_{r_i}\}$, and $a_{i1}, b_{i1} \in \{\text{tail}(d) | d \in \mathcal{L}_{\text{ind}(_i)}\}$.

14.(4) computing $L_{i,0}$ costs $O(|\lambda_{i,0} \cap \bigcup_{j \in C_i} \lambda_j| + |C_i| + |L_{i,0} \cap \bigcup_{j \in C_i} \mathcal{L}_j|)$.

Moreover, for every dart $d$ it holds that Mark(d) = $i$ if and only if $d \in q_j$ for some $j \in [R_i]$.

**Proof.** To prove [14.(1)] we write $\lambda_{i,0} = (\bigcup_{j \in C_i} \lambda_{i,0} \cap \lambda_j) \cup (\lambda_{i,0} \cap \bigcup_{j \in C_i} \lambda_j)$. It is clear that computing $\lambda_{i,0} \cap \bigcup_{j \in C_i} \lambda_j$ costs $O(|\lambda_{i,0} \cap \bigcup_{j \in C_i} \lambda_j|)$, by [14.(3)] and [13.(1)] and computing $\bigcup_{j \in C_i} \lambda_{i,0} \cap \lambda_j$ costs $|C_i|$, by Lemma [13].

- [14.(2)] follows from Lemma [13]. Moreover, [14.(3)] follows from [14.(2) and Line 5 of algorithm.
MarkPaths.
- [14.(1)] and [14.(3)] imply [14.(4)]
- The last part of the proof is implied by definition of Mark and by [14.(3)]. Indeed, \( \bigcup_{j \in C_i} \lambda_j = \emptyset \) and Mark \( (d) = i \iff d \in \lambda_i \setminus \bigcup_{t < i} \lambda_t \), finally, by Lemma 5 and Lemma 6, \( d \in \lambda_i \setminus \bigcup_{j \in C_i} \lambda_j \).

Figure 12 shows an example of [14.(2)] and [14.(3)]. Let us assume that \( C_i = \{ j_1, j_2, j_3, j_4 \} \), in [a] the shortest \( j \)-paths, for \( j \in C_i \), are drawn in solid lines. It holds that \( v_{j_1}^+ = v_{j_2}^- = \emptyset \), while \( v_{j_1}^-, v_{j_2}^-, v_{j_3}^+, v_{j_4}^+, v_{j_4}^- \) are all non empty. We stress that \( v_{j_3}^+ = v_{j_4}^- = v \). Path \( \lambda_{i,0} \) is drawn in a dotted line, and, referring to [14.(2)] \( p_1, p_2 \) and \( p_4 \) are inserted (it holds that \( p_3 = \emptyset \) because \( v_{j_3}^+ = v_{j_4}^- \)). In [b], \( \lambda_i \) is highlighted by a solid line, and, referring to [14.(3)] \( q_1, q_2 \) and \( q_3 \) are inserted.

![Figure 12](image.png)

**Figure 12:** let us assume that \( C_i = \{ j_1, j_2, j_3, j_4 \} \), in [a] the shortest \( j \)-paths, for \( j \in C_i \), are drawn in solid lines and \( \lambda_{i,0} \) is drawn in a dotted line. In [b] \( \lambda_i \) is represented by a solid lined.

**Remark 3** The output of algorithm MarkPaths is \( \bigcup_{i=1}^k \{ q_j^i \}, \text{ind}(\lambda_j^i), (a_j^i, b_j^i) \} \) \( R_{\lambda_j^i} \) and the function Mark on darts in \( \bigcup_{i \in [k]} \lambda_i \). Being \( \sum_{i=1}^k \sum_{j=1}^{R_{\lambda_j^i}} |q_j^i| \leq 2|E(U)| \) by Lemma 5 then the output has size \( O(|U|) \).

### 5.2.2 Computing shortcuts

The following lemma shows how to preprocess faces in order to efficiently check the existence of shortcuts.

**Lemma 15** Let \( i \in [k] \), let \( j \leq M_i \), let \( f \in F \) and let \( e = uv, e' = xy \) be two non adjacent edges of \( f \) such that \( e, e' \in \lambda_{i,j} \) and \( u, v, x, y \) appear in this order going from in \( \lambda_{i,j} \). Let \( \sigma \) be the undirected subpath of \( \partial f \) such that \( v, x \in V(\sigma) \) and \( u, y \notin V(\sigma) \). Let us assume that \( \sigma \notin \lambda_{i,j} \). Then \( \sigma \) is a shortcut of \( \lambda_{i,j} \) and \( \sigma \subseteq \text{Right}(\lambda_{i,j}) \).

**Proof.** Let \( P \in \mathcal{P} \) be a completely strongly non-crossing representation. Let \( Q \subseteq P \) be the minimal subset of paths of \( P \) such that \( \sigma \subseteq \bigcup_{q \in Q} q \). Let \( R = \{ q \in \mathcal{P} \setminus Q \} \). It is clear that \( \sigma \subseteq \bigcup_{r \in R} r \), \( R \) is strongly non-crossing, and \( r \) and \( \lambda_{i,j} \) are non-crossing, for all \( r \in R \). Hence \( R \cup \lambda_{i,j} \) is a set of strongly non-crossing paths such that \( \sigma \subseteq \bigcup_{r \in R} r \) and the extremal vertices of \( \sigma \) (\( v \) and \( x \)) are in \( \lambda_{i,j} \). By Lemma 2 there exists \( q \in R \) such that \( \sigma \subseteq q \). Trivially, \( q \neq \lambda_{i,j} \), and, being \( q \) obtained from \( \lambda_{i,j} \) by shortcuts, then \( w(q) \leq w(\lambda_{i,j}) \), in particular \( w(q[v,x]) \leq w(\lambda_{i,j}) \). Thus \( w(\sigma) \leq w(\lambda_{i,j}) \) proving that \( \sigma \) is a shortcut of \( \lambda_{i,j} \). Finally, it is clear that \( \sigma \subseteq \text{Right}(\lambda_{i,j}) \). □
5.2.3 Algorithm MarkPaths is linear

We can now prove that algorithm MarkPaths requires $O(|U|)$ worst-case time.

**Theorem 16** If $\lambda_i$ is expressed as in (14.3) for all $i \in [k]$, then algorithm MarkPaths costs $O(|U|)$ worst-case time.

**Proof.** We easily observe that computing Line 1 for all iterations costs $O(|U|)$ total worst-case time. Moreover, by the last part of Lemma 13, the complexity of Line 6 is included in the complexity of previous lines.

To prove the thesis, we proceed in the following way: we first deal with the complexity of computing $\lambda_{i,0}$ and $\lambda_i$, for $i \in [k]$, as a function of the complexity of finding all shortcuts and the complexity of updating all the lists $L_{i,j}$ and $L_j$, varying $i$ and $j$; after this, we show that all shortcuts in Line 5 can be found in $O(|U|)$ total worst-case time and also the updating of lists $L_i$ and $L_{i,j}$ can be made in $O(|U|)$ total worst-case time.

Let $i \in [k]$, we claim that computing $\lambda_{i,0}$, $\lambda_i$ and $L_i$, knowing $L_j$, for $j \in C_i$ costs $O(|\lambda_{i,0} \cup \bigcup_{j \in C_i} \lambda_j| + |C_i| + |\lambda_{i,0} \setminus \bigcup_{j \in C_i} L_j|)$ plus the complexity of Line 5 and the complexity of updating lists at iteration $i$. We proceed by induction.

By 13.(1) and 14.(3) the base of the induction is trivial because $C_1 = \emptyset$. To prove the inductive step, we use Lemma 14.

First, by 14.(1) and 14.(4) we can compute $\lambda_{i,0}$ and $L_{i,0}$ in $O(|\lambda_{i,0} \cup \bigcup_{j \in C_i} \lambda_j| + |C_i| + |\lambda_{i,0} \setminus \bigcup_{j \in C_i} L_j|)$. Moreover, if $\lambda_{i,0}$ has not right shortcut, then $\lambda_{i,0} = \lambda_i$. Otherwise, every times that algorithm MarkPaths executes Line 5 by finding a shortcut, then we can update $L_{i,j}$ to $L_{i,j+1}$ by modifying the part of the list $L_{i,j}$ regarding the shortcut.

Let $i \in [k]$, now we know the complexity of computing $L_j$, for $j \in C_i$, but we need $L_j$, for $j \in C_i$, to obtain $\lambda_i$; thus we have to remove internal darts. It is clear that if $V(\lambda_i) \cap V(\lambda_j) = \emptyset$, for all $j, l \in C_i$, then $L_j = L_j'$, for $j \in C_i$. Moreover, if $\lambda_j$ intersects $\lambda_i$ on vertex $v$, for some $j, l \in C_i$, then there exists a dart $d$ in $L_j$ and a dart $d'$ in $L_l$ such that $v = \text{tail}(d) = \text{tail}(d')$ and the suffix of $d$ in $L_j$ and the prefix of $d'$ in $L_l$ (or vice-versa) are internal darts. By using this observation, then all internal darts can be found and removed in $O(|U|)$ total worst-case time, because $|\bigcup_{j \in [k]} L_j| = O(|U|)$.

We claim that computing $\lambda_i$, for all $i \in [k]$, requires worst-case time $O(|U|) + \sum_{i \in [k]} O(|\lambda_{i,0} \cup \bigcup_{j \in C_i} \lambda_j| + |C_i| + |\lambda_{i,0} \setminus \bigcup_{j \in C_i} L_j|) + \{\text{the complexity of finding all shortcuts}\} + \{\text{the complexity of updating lists}\} = O(|U|) + \{\text{the complexity of finding all shortcuts}\}$, the equality is implied by 11.(2) if we represent $L_j$ and $L_j'$ by double linked lists, for all $j \in [k]$. It remains to evaluate the complexity of finding all shortcuts. Let $f \in F$. If $\lambda_{i,0} \cap f \subseteq \lambda_{i,j}$, for some $i \in [k]$ and $j \in C_i$, then there is not any right shortcut for $\lambda_{i,j}$ in $f$, otherwise 11.(1) would be denied.

Thus in the whole algorithm MarkPaths, we ask if there is a right shortcut at most $\sum_{f \in F} |f| = O(|U|)$ times and we answer every time in $O(1)$ in the following way.

We preprocess $U$ so that for every face $f \in F$ and for every vertices $u, v \in \partial f$ we can compute the weight of the clockwise path on $\partial f$ from $u$ to $v$ and the weight of the counterclockwise path on $\partial f$ from $u$ to $v$ in $O(1)$. It is sufficient to compute $w(\partial f)$ and the weight of the clockwise path on $\partial f$ from a fixed vertex $u$ to any vertex $v$ in $\partial f$. This preprocessing clearly costs $O(|V(\partial f)|)$ worst-case time, for every $f \in F$, thus it costs $O(|U|)$ worst-case time for the whole set of faces $F$.

Now we observe that, by Lemma 15 $\lambda_{i,j}$, $|\partial f|$ can be considered always connected when $i$ and $j$ grow. Hence we can establish if $w(\lambda_{i,j} \cap \partial f) \geq w(\partial f \setminus (\lambda_{i,j} \cap \partial f))$ in $O(1)$ using the above preprocessing.

The last paragraph of the previous proof shows that one can understand that the posterior visit of the genealogy tree is more convenient than the visit by levels proposed in [6].
5.3 Computing distances for all terminal pairs

By using the output of algorithm MarkPaths explained in Remark 3 we can compute distances between terminal pairs in $O(|U|)$ worst-case time, as showed in the following theorem.

**Theorem 17** Computing $w(\lambda_i)$ and $\ell_i$, for all $i \in [k]$, costs $O(|U|)$ worst-case time.

**Proof.** Let $i \in [k]$. By (14.3) it holds that $\lambda_i = \lambda_i^0 \circ \lambda_i^1[\ell_i^0, a_i^1] \circ \lambda_i^2[\ell_i^1, b_i^1] \circ \cdots \circ \lambda_i^n[\ell_i^n, b_i^n]$, for some (possibly empty) $\{q^j_i\}_{j \in [R_i]}$ paths in $U_i$ such that $\bigcup_{j \in [R_i]} q^j_i \cap \bigcup_{j \in [C_i]} \lambda_j = \emptyset$, $\{\text{ind}(\lambda_i^j), \ldots, \text{ind}(\lambda_i^n)\} \subseteq \{j \in C_i | \mathcal{L}_j \neq \emptyset\}$, and $a_i^j, b_i^j \in \{\text{tail}(d) | d \in \mathcal{L}_{\text{ind}(\lambda_i^j)}\}$. After the execution of algorithm MarkPaths, by Remark 3 we know $\bigcup_{j=1}^{k} q^j_i, \text{ind}(\lambda_i^j), (a_i^j, b_i^j)_{j=1}^{R_i}$. We compute $w(\lambda_i^j[a_i^j, b_i^j])$ by starting from $w(\lambda_i^{j_0})$ and by removing $w(\lambda_i^{j_0}[s_{\text{ind}(\lambda_i^{j_0})}, a_i^{j_0}])$ and $w(\lambda_i^{j_0}[b_i^{j_0}, \ell_{\text{ind}(\lambda_i^{j_0})}])$. Indeed $\lambda_i^j[s_{\text{ind}(\lambda_i^j)}, a_i^j]$ and $\lambda_i^j[b_i^j, \ell_{\text{ind}(\lambda_i^j)}]$ are not in $U_i$ (by (14.3), thus it suffices compute distances of $\lambda_1, \ldots, \lambda_k$ in this order to get $O(|U|)$ worst-case time. Indeed, this procedure visits the output of algorithm MarkPaths one time, and it has size $O(|U|)$.

We use the same reasoning to compute $\ell_i$, for all $i \in [k]$. \hfill \square

The following algorithm ComputeDistances implements the process described in the proof of Theorem 17. Its correctness and worst-case time are already proved in Theorem 17. For the sake of simplicity, for $i \in [k]$ and $j \in [R_i]$ we define $s_i^j = s_{\text{ind}(\lambda_i^j)}$, $t_i^j = t_{\text{ind}(\lambda_i^j)}$ and $\ell_i^j = \ell_{\text{ind}(\lambda_i^j)}$, moreover, we remember that $\lambda_i^j = \lambda_i^j[s_{\text{ind}(\lambda_i^j)}, a_i^j]$. By (14.3)

**ComputeDistances:**

**Input:** $\bigcup_{j=1}^{R_i} \{q^j_i, \text{ind}(\lambda_i^j), (a_i^j, b_i^j)\}_{j=1}^{R_i}$

**Output:** $w(\lambda_i)$, the length of $\lambda_i$,

$\ell_i$, the number of darts in $\lambda_i$, for all $i \in [k]$

1. for $i = 1, \ldots, k$

2. $w(\lambda_i) = \sum_{j=1}^{R_i} w(q^j_i) + \sum_{j=1}^{R_i} (w(\lambda_i^j) - w(\lambda_i^j[s_{\text{ind}(\lambda_i^j)}, a_i^j]) - w(\lambda_i^j[b_i^j, \ell_i^j]));$

3. $\ell_i = \sum_{j=1}^{R_i} |q^j_i| + \sum_{j=1}^{R_i} (\ell_i^j - |\lambda_i^j[s_{\text{ind}(\lambda_i^j)}, a_i^j]| - |\lambda_i^j[b_i^j, \ell_i^j]|);$

5.4 Listing paths

In this section we study the problem of listing the darts in an $i$-path, for some $i \in [k]$, after the execution of algorithm MarkPaths. We want to underline the importance of single-touch property. In Figure 13 in [m] four shortest paths are drawn (the graph is unit-weighted). We observe that the single-touch property is clearly not satisfied. In [h], instead, a single-touch version of the previous four paths is drawn; it can be obtained with algorithm MarkPaths choosing $i^* = 1$ in the genealogy tree. It is clear that the problem of listing the darts in a path in this second case is easier. We stress that in general cases, the union of a set of single-touch paths can form cycles, see Figure 14 for an example.

**Theorem 18** Let $i \in [k]$ and let $\ell_i = |\lambda_i|$. After the execution of algorithm MarkPaths, darts in $\lambda_i$ can be listed in $O(\ell_i + \ell_i \log \frac{1}{\ell_i})$ worst-case time. Moreover, darts in $\lambda_i$, for all $i \in [k]$, can be listed in $O(\sum_{i \in [k]} \ell_i)$.

**Proof.** By Line 6 after the execution of algorithm MarkPaths, every dart is marked with $i$ if and only if $d \in \lambda_i \cup \bigcup_{j<i} \lambda_j$. Now we explain how to find darts in $\lambda_i$.

Let us assume that $(d_1, \ldots, d_{\ell_i})$ are the ordered darts in $\lambda_i$. Let $v = \text{head}(d_{\ell_i-1})$, and let us assume that $\text{deg}(v) = r$ in the graph $\bigcup_{j \in [k]} \lambda_j$. We claim that if we know $d_{\ell_i-1}$, then we find $d_j$ in $O(\log r)$. First we order the outgoing darts in $v$ in clockwise order starting in $d_{\ell_i-1}$, thus let
Figure 13: (a) the union of shortest $i$-paths, for $i \in [4]$, in unit-weighted graph, every different path has different style, (b) the union of $\{\lambda_i\}_{i \in [4]}$, the output paths of algorithm MarkPaths, by assuming that $1$ is the root of the genealogy tree.

$Out_v = (g_1, \ldots, g_r)$ be this ordered set (this order is given by the embedding of the input plane graph). We observe that all darts in $Out_v$ that are in $Left_{\lambda_i}$ are in $\lambda_w$ for some $w \leq i$, thus $Mark(d) \leq i$ for all $d \in Out_v$ such that $d \in Left_{\lambda_i}$. Similarly, all darts in $Out_v$ that are $Right_{\lambda_i}$ are in $\lambda_z$ for some $z \geq i$, thus $Mark(d) \geq i$ for all $d \in Out_v$ such that $d \in Right_{\lambda_i}$. Using this observation, we have to find the unique $\ell \in [r]$ such that $Mark(g_\ell) \leq i$ and $Mark(g_{\ell+1}) > i$. This can be done in $O(\log r)$ using a binary search.

Being $\{\lambda_i\}_{i \in [k]}$ single-touch, then $\sum_{v \in V(\lambda_i)} \deg(v) \leq 2k$, where the equality holds if and only if every $\lambda_j$, for $j \neq i$, intersects on vertices $\lambda_i$ exactly two times, that is the maximum allowed by single-touch property.

Finally, being $\sum_{\ell=1}^{\ell_{\text{max}}} \log a_1^{\ell} \leq \ell \log \lceil \frac{2k}{\ell} \rceil$, the complexity of worst-case follows.

The second part of the thesis follows from $14.(3)$. Indeed, it is sufficient to find $\lambda_1, \lambda_2, \ldots, \lambda_k$ in this order using $14.(3)$. □

6 Conclusions

In this article we extend the result of Takahashi et al., [8], by computing the lengths of shortest multi-terminal paths in undirected plane graphs also in the general case when the union of shortest paths is not a forest. Moreover, we provide an algorithm for listing the sequence of edges of each path in $O(\ell + \ell \log(k/\ell))$, where $\ell$ is the number of edges in the shortest path. Our results are based on the local concept of shortcut.

All results of this paper can be easily applied in a geometrical setting, in which one deals with polygons instead of plane graphs.

We left open the following problems:

- listing a shortest path in $O(\ell)$ worst-case time;
- solving the multi-terminal shortest path problem on $k$ terminal pairs in $o(n \log k)$ in the unweighted case.

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