STABILITY ESTIMATES IN INVERSE PROBLEMS FOR THE SCHRÖDINGER AND WAVE EQUATIONS WITH TRAPPING

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Dedicated to the memory of Slava Kurylev

Abstract. For a class of Riemannian manifolds with boundary that includes all negatively curved manifolds with strictly convex boundary, we establish Hölder type stability estimates in the geometric inverse problem of determining the electric potential or the conformal factor from the Dirichlet-to-Neumann map associated with the Schrödinger equation and the wave equation. The novelty in this result lies in the fact that we allow some geodesics to be trapped inside the manifold and have infinite length.

1. Introduction

In this article we study a geometric inverse problem associated with the anisotropic Schrödinger equation and the wave equation on a compact Riemannian manifold \((M, g)\) with boundary \(\partial M\).

Let \(\Delta_g\) be the non-negative Laplace-Beltrami operator associated with the metric \(g\), we consider two initial-value-problems. First, we consider the Schrödinger equation for finite time of propagation and with Dirichlet conditions:

\[
\begin{aligned}
(i \partial_t - \Delta_g + q(x))u(t, x) &= 0, & \text{in } (t, x) &\in I \times M, \\
u(0, \cdot) &= 0, & \text{in } x &\in M, \\
u(t, x) &= f(t, x), & \text{on } (t, x) &\in I \times \partial M,
\end{aligned}
\]

where \(I = (0, T)\) for \(T > 0\) fixed. Secondly, we consider the wave equation for infinite time of propagation and with Dirichlet conditions:

\[
\begin{aligned}
(\partial_t^2 + \Delta_g + q(x))u(t, x) &= 0, & \text{in } (t, x) &\in I \times M, \\
u(0, \cdot) &= 0, & \partial_t u(0, \cdot) &= 0, & \text{in } x &\in M, \\
u(t, x) &= f(t, x), & \text{on } (t, x) &\in I \times \partial M,
\end{aligned}
\]

where \(I = (0, T)\) and \(T\) can be equal to \(+\infty\).

We aim at studying the problem of the stable recovery of the potential \(q\), or alternatively conformal factor in a conformal class of a metric \(g\), from the Dirichlet-to-Neumann map (DN map in short) is the operator defined for each \(T < \infty\)

\[
\begin{align*}
\Lambda^S_{g,q} &: H^1_0([0, T) \times \partial M) \to L^2((0, T) \times \partial M), & \Lambda^S_{g,q}f &= -\partial_n u^S|_{(0,T)\times\partial M}, \\
\Lambda^W_{g,q} &: H^1_0([0, T) \times \partial M) \to L^2((0, T) \times \partial M), & \Lambda^W_{g,q}f &= -\partial_n u^W|_{(0,T)\times\partial M},
\end{align*}
\]
where $u^S$ solves (1.1) and $u^W$ solves (1.2), and $\partial_u$ is the unit inward normal derivative at $\partial M$. Here $H^1_0([0,T] \times \partial M)$ denotes the closed subspace of functions in $H^1([0,T] \times \partial M)$ vanishing at $t = 0$. For the wave equation, we shall need to consider the case $T = \infty$, and we will show that there is $\nu_0 \geq 0$ depending only on $\|q\|_{L^\infty}$ so that for all $\nu \geq \nu_0$

$$
\Lambda_{g,q}^W : e^{\nu t}H_0^1(\mathbb{R}_+ \times \partial M) \to e^{\nu t}L^2(\mathbb{R}_+ \times \partial M)
$$

is bounded.

By a stability estimate, we mean that there is a constant $C > 0$, possibly depending on some a priori bound on $\|q_j\|_{H^s(M)}$ for some $s \geq 0$ such that an estimate of the following form holds

$$
\|q_1 - q_2\|_{L^2(M)} \leq CF(\|\Lambda_{g,q_1}^{S/W} - \Lambda_{g,q_2}^{S/W}\|_{*,\nu}),
$$

where the used norm for the Schrödinger/wave DN map are respectively

$$
\| \cdot \|_{s,\nu} = \| \cdot \|_{H^s(1 \times \partial M) \to L^2(1 \times \partial M)}, \quad \| \cdot \|_{*,\nu} = \| \cdot \|_{e^{\nu t}H_0^1(1 \times \partial M) \to e^{\nu t}L^2(1 \times \partial M)}
$$

and $F$ is a continuous function satisfying $F(0) = 0$; we shall write simply $\| \cdot \|_s$ for the wave case when $I = [0,T]$ with $T < \infty$, and $\nu = 0$. We say that the stability is of Hölder type if $F(x) = x^\beta$ for some $\beta > 0$, it is said of log-type if $F(x) = \log(1/x)^{-\beta}$ for some $\beta > 0$. More generally, one can ask if there is a stability for the problem of recovering the metric, i.e.

$$
\|q_1 - \psi^*q_2\|_{L^2(M)} \leq CF(\|\Lambda_{g_1,0}^{S/W} - \Lambda_{g_2,0}^{S/W}\|_{*})
$$

for some diffeomorphism $\psi$ (depending on $q_1, g_2$). Here we have used the $L^2$ norm on $M$ to measure $q_1 - q_2$, but one could also ask the same question for Sobolev or Hölder norms. Assuming a priori bounds on $q$ in some large enough Sobolev spaces $H^s(M)$ allows to deduce (by interpolation) bounds on $\|q_1 - q_2\|_{H^s}$ for $s < s_0$ if one has bounds on $\|q_1 - q_2\|_{L^2}$ (and similary for $q_1 - q_2$).

The problem of determination of the metric $g$ or the potential $q$ from $\Lambda_{g,q}^W$ was solved in general by Belishev-Kurylyev [BK92] (see also [KKL01]) but the stability estimates in the general setting appeared only recently in the work of Burago-Ivanov-Lassas-Lu [BILL20] and are of log-log type (i.e. $F(x) = |\log|\log x||^{-\beta}$) for the case with no potential. When $g = g_{\text{eucl}}$ is the Euclidean metric on a domain $M \subset \mathbb{R}^n$, a Hölder type stability was proved by Sun [Sun90] and Alessandrini-Sun-Sylvester [AZS90] for the determination of the potential $q$ from $\Lambda_{g_{\text{eucl}},q}^W$. In the case of non-Euclidean metrics, but close to the Euclidean metric on a ball in $\mathbb{R}^n$, Stefanov-Uhlmann [SU98] obtained Hölder estimates for the metric recovery (with no potential involved), and they extended this result in [SU05] to Riemannian metrics close to a simple metric $g_0$ with injective X-ray transform on symmetric 2-tensors. Such simple metrics are dense among simple metrics. We recall that simple metrics are Riemannian metrics with no conjugate points on a ball $B$ in $\mathbb{R}^n$ with strictly convex boundary, in particular all geodesics in $B$ for such a metric have finite length with endpoints on the boundary $\partial B$. If $g_0$ is a fixed simple metric, Bellassoued and Dos Santos Ferreira [BDSF10, BDSF11] proved Hölder stability of the inverse problem for both $\|\Lambda_{g_0,q_1}^S - \Lambda_{g_0,q_2}^S\|_{*}$ and $\|\Lambda_{g_0,q_1}^W - \Lambda_{g_0,q_2}^W\|_{*}$. When $g$ is close to a fixed simple metric $g_0$ with
injective X-ray transform on 2-tensors, Montalto [Mon14] extended the previous result
to the recovery of the pair \((g, q)\) (and a magnetic potential term in addition) in a Hölder stable way. For non-simple metrics, we are aware of only two results showing strong stability: the first by Bao-Zhang [BZ14] who prove for a non-trapping metric \(g = c(x)^2 g_{\text{eucl}}\), conformal to the Euclidean metric, and satisfying certain assumptions on their conjugate points, that if \(\|\Lambda^W_{c^2 g_{\text{eucl}}} - \Lambda^W_{\tilde{c}^2 g_{\text{eucl}}}\|_*\) is small enough then the conformal factors agree \(c = \tilde{c}\); the second by Stefanov-Uhlmann-Vasy [SUV16] is of the same kind but under the assumption that \(g_{\text{eucl}}\) is replaced by a metric \(g_0\) so that the manifold \((M, g_0)\) can be foliated by strictly convex hypersurfaces. In all these results, the time interval \(I = (0, T)\) can be taken with \(T > 0\) finite but large enough for the wave case (depending on the diameter of the domain), while for the Schrödinger case it can be taken finite and small using infinite speed of propagation.

All these mentioned results where Hölder stability results hold assume no trapped geodesic rays for the Riemannian manifold \((M, g)\), i.e. geodesics staying inside the interior \(M^o\) of \(M\) for infinite time. Existence of trapped geodesics means that some regions of the phase space are not accessible from the boundary by geodesic rays, and some waves can possibly stay (microlocally) trapped for a long time near these trapped rays, so that a part of the information can not be read off microlocally from the DN map at the boundary. It is thus an open question to understand how stable is the recovery of the coefficients of the wave equation or the Schrödinger equation when the metric is not simple. The difficulty to obtain such Hölder estimates lies in the fact that one usually reduces the inverse problem for the DN map to some X-ray tomography problem using wave packets or WKB solutions of the wave/Schrödinger equations that concentrate near single geodesics going from a point of the boundary to another point. It is likely that under general assumptions, no Hölder stability estimates hold but log stability estimates do; we mention the recent work of Koch-Rüland-Salo [KRS] about this question. Our purpose in this work is to address this stability question in a family of cases where the trapped set is sufficiently filamentary, the typical example being that of a non-simply connected Riemannian metric with negative curvature and strictly convex boundary.

Our main geometric assumptions are the hyperbolicity of the trapped set for the geodesic flow and the absence of conjugate points. We notice that these two assumptions are satisfied if \((M, g)\) is negatively curved. Let us recall the precise definition of hyperbolic trapped set. Let \(\varphi_t : SM \to SM\) be the geodesic flow for \(t \in \mathbb{R}\), where \(SM = \{(x, v) \in TM : |v|_{g(x)} = 1\}\) is the unit tangent bundle. We call, for every \(z = (x, v) \in SM\), the escape time of \(SM\) in positive (+) and negative (−) times,

\[
\tau_+(z) := \sup \{t \geq 0 \mid \forall s < t, \varphi_s(z) \in SM^o \in [0, +\infty], \\
\tau_-(z) := \inf \{t \leq 0 \mid \forall s > t, \varphi_s(z) \in SM^o \in [-\infty, 0].
\]

In other words, \(\tau_\pm(z)\) is the time needed for the geodesic \((\varphi_\pm t(z))_{t \geq 0}\) to reach \(\partial_\pm SM \cup \partial_0 SM\). The incoming (−) and outgoing (+) tails in \(SM\) are defined by

\[
\Gamma_\mp = \{z \in SM \mid \tau_\pm(z) = \pm \infty\},
\]
and the trapped set for the flow on $SM$ is the set $K := \Gamma_+ \cap \Gamma_-$. If $\partial M$ is strictly convex for $(M, g)$ (i.e. the second fundamental form of $\partial M$ is positive), the trapped set $K$ is a compact flow-invariant subset of the interior $SM^\circ$ of $SM$. We say that the trapped set $K \subset SM$ is a hyperbolic set if there exists $C > 0$ and $\nu > 0$ so that, there is a continuous flow-invariant splitting over $K$

$$T_K(SM) = \mathbb{R}X \oplus E_u \oplus E_s, \quad (1.3)$$

where $X$ is the geodesic vector field on $SM$, and $E_s, E_u$ are vector subspaces satisfying for all $z \in K$

$$\|d\phi_t(z)w\| \leq Ce^{-\nu t}\|w\|, \quad \forall t > 0, \; \forall w \in E_s(z), \quad (1.4)$$

$$\|d\phi_t(z)w\| \leq Ce^{-\nu|t|}\|w\|, \quad \forall t < 0, \; \forall w \in E_u(z), \quad (1.5)$$

with respect to any fixed metric on $SM$. The notion of conjugate points can be defined as follows. If $\pi_0 : SM \to M$ is the projection and $\nu := \ker d\pi_0 \subset T(SM)$ is the vertical bundle of the fibration, we say that there is no conjugate point if $d\phi_t(\nu) \cap \nu = \{0\}$ for all $t \neq 0$, where $\{0\}$ denotes the 0-section of $T(SM)$.

1.1. The case of the Schrödinger equation. The DN map associated with (1.1) is continuous [BDSF10, Thm. 1] as an operator from $H^1((0, T) \times \partial M)$ to $L^2((0, T) \times \partial M)$. Our first goal is to obtain a Hölder stability estimate of the form

$$\|q_1 - q_2\|_{L^2(M)} \leq C\|\Lambda_{g,q_1} - \Lambda_{g,q_2}\|_s^\beta, \quad (1.6)$$

for some $\beta > 0$ for the Schrödinger equation on a bounded time interval $(0, T)$. Here we assume that $q_1$ and $q_2$ belong to the family of admissible electrical potentials

$$Q(N_0) := \{q \in W^{1,\infty}(M) \mid \|q\|_{W^{1,\infty}(M)} \leq N_0\}, \quad (1.7)$$

with $N_0 > 0$ fixed, and that $q_1$ and $q_2$ coincide on the boundary $\partial M$. It is known that the estimate (1.6) holds on simple manifolds [BDSF10] with $\beta = 1/8$. Our aim is to extend this result to the case of hyperbolic trapped set of the geodesic flow and no conjugate points.

Our first result gives the stable determination of the potential $q$ from the DN map.

**Theorem 1.** Let $(M, g)$ be a compact Riemannian manifold of dimension $d \geq 2$ with strictly convex boundary. Let $T, N_0 > 0$ fixed. Assume that the trapped set $K$ is hyperbolic and there are no conjugate points. Then, there exists a constant $C = C(M, g, T, N_0) > 0$ such that, for any $q_1, q_2 \in Q(N_0)$ with $q_1 = q_2$ on $\partial M$,

$$\|q_1 - q_2\|_{L^2(M)} \leq C\|\Lambda^S_{g,q_1} - \Lambda^S_{g,q_2}\|_s^\beta, \quad (1.8)$$

for some $\beta > 0$ depending only on $(M, g)$.

We notice from our proof that the constant $\beta$ can be expressed in terms of the volume entropy and dynamical quantities on the geodesic flow of $(M, g)$, more precisely the pressure of the unstable jacobian of the geodesic flow on the trapped set and the maximal expansion rate of the flow.
In order to obtain a stability estimate for the conformal factor of the metric, we consider the family of admissible conformal factors given by
\[ \mathcal{C}(N_0, k, \epsilon) := \{ c \in C^\infty(M) \mid c > 0 \text{ in } M, \|1 - c\|_{C^0(M)} \leq \epsilon, \|c\|_{C^k(M)} \leq N_0 \}. \] (1.9)
Our second result gives the stable determination of the conformal factor.

**Theorem 2.** Let \((M, g)\) be a compact Riemannian manifold of dimension \(d \geq 2\) with strictly convex boundary, hyperbolic trapped set \(K\) and no conjugate points. Let \(T, N_0 > 0\) be fixed. Then, there exist \(k \geq 1\) depending only on \(\dim(M)\), \(\epsilon > 0\) depending on \((M, g, N_0)\) and a constant \(C = C(M, g, T, N_0) > 0\) such that, for any \(c \in \mathcal{C}(N_0, k, \epsilon)\) with \(c = 1\) near \(\partial M\),
\[ \|1 - c\|_{L^2(M)} \leq C\|\Lambda_{g,0}^S - \Lambda_{cg,0}^S\|_{\ast}^\beta, \] (1.10)
for some \(\beta > 0\) depending only on \((M, g)\).

As far as we know, these two results are the first Hölder stability results for the Schrödinger equation when the principal symbol of the operator has trapped bicharacteristic rays.

### 1.2. The case of the wave equation.

The DN map associated with (1.2) with \(I = (0, T)\) is bounded as an operator from \(H^1_0((0, T) \times \partial M)\) to \(L^2((0, T) \times \partial M)\) (see [LM72a, LM72b]). In the case \(I = (0, \infty)\), it is necessary to introduce an exponential weight in the time as \(T \to +\infty\) to obtain boundedness of the DN map. For our result, due to the fact that some geodesics have infinite length (those that are trapped), we need to consider the wave equation for all positive time.

For \(k, \ell \in \mathbb{N}_0\), let \(\nu > 0\). We define the weighted Sobolev space \(e^{\nu t}H^k(I; H^\ell(M))\) as the space of functions \(f \in H^k(I; H^\ell(M))\), with finite norm
\[ \|f\|_{e^{\nu t}H^k(I; H^\ell(M))} := \sum_{j=0}^k \left( \int_0^\infty e^{-2\nu t} \|\partial_t^j f(t, \cdot)\|_{H^\ell(M)}^2 dt \right)^{\frac{1}{2}}. \]

In particular, we denote \(e^{\nu t}H^k(I \times M) := e^{\nu t}H^k(I; H^k(M))\). Similarly we define the weighted Sobolev spaces \(e^{\nu t}H^k(I; H^\ell(\partial M))\) on the boundary \(\partial M\), and denote \(e^{\nu t}H^k(I \times \partial M) := e^{\nu t}H^k(I; H^k(\partial M))\).

The DN map associated with (1.2) is continuous from \(e^{\nu t}H^1_0(I \times \partial M)\) to \(e^{\nu t}L^2(I \times \partial M)\) for every \(\nu \geq \nu_0\): this follows from [CP82, Thm 6.10 and Thm. 7.1] and can be checked that \(\nu_0 \geq 0\) depends only on \(\|q\|_{L^\infty}\), as we show in Lemma 4.3 and the comment that follows. We denote:
\[ \|\Lambda_{g,q}^W\|_{\ast, \nu} := \|\Lambda_{g,q}^W\|_{\mathcal{L}(e^{\nu t}H^1_0(I \times \partial M), e^{\nu t}L^2(I \times \partial M))}. \] (1.11)

We next state our main result on the stable determination of the electric potential from the DN map.

**Theorem 3.** Let \((M, g)\) be a compact Riemannian manifold of dimension \(n \geq 2\) with strictly convex boundary, hyperbolic trapped set and no conjugate points. Let \(N_0 > 0\) be
fixed. There is $\nu_0$ depending only on $N_0$ such that for every $\nu > \nu_0$, there exists $C > 0$ such that, for any $q_1, q_2 \in \mathcal{Q}(N_0)$ with $q_1 = q_2$ on $\partial M$,

$$\|q_1 - q_2\|_{L^2(M)} \leq C\|\Lambda^W_{q,q_1} - \Lambda^W_{q,q_2}\|_{*,\nu},$$

(1.12)

for some $\beta > 0$ depending only on $(M, g)$ and $\nu$.

We finally state our main result on the stable recovery of the conformal factor.

**Theorem 4.** Let $(M, g)$ be a compact Riemannian manifold of dimension $d \geq 2$ with strictly convex boundary, hyperbolic trapped set and no conjugate points. Let $N_0 > 0$ be fixed. Then, there exist $\nu_0 > 0$, $k \geq 1$ depending only on $d$, and $\epsilon > 0$ depending on $(M, g, N_0)$, such that for all $\nu > \nu_0$, there is $C$ depending on $(M, g, N_0, \nu)$ so that, for any $c \in \mathcal{C}(N_0, k, \epsilon)$ with $c = 1$ near $\partial M$,

$$\|1 - c\|_{L^2(M)} \leq C\|\Lambda^W_{g,0} - \Lambda^W_{c,g,0}\|_{*,\nu},$$

(1.13)

for some $\beta > 0$ depending only on $(M, g)$ and $\nu$.

1.3. **Method of proof.** To obtain the stability results, we use the general method of [SU05, BDSF10, BDSF11] of reducing the problem to some estimate on X-ray transform of $q_1 - q_2$. We however need to perform several important modifications due to trapping. Ultimately we rely on some results of the second author [Gui17] on the injectivity and stability estimates of the X-ray transform for the class of manifold under study, but it is not a simple reduction to that problem, as we now explain. We first follow the well known route of constructing WKB solutions $u$ of the Schrödinger/wave equation concentrating on each geodesic $\gamma$ of length less or equal to $T_0 > 0$ with endpoints on the boundary. We use the universal covering of $M$ to construct $u$ since $M$ is not assumed simply connected. We can then bound the integral of $q_1 - q_2$ along these geodesics by a constant times $\|\Lambda^S_{g,0} - \Lambda^S_{q,0}\|_{*,\nu}$. The non-simple metric assumption complicates that step compared to the simple metric case, due to the fact that geodesics self intersect. In the Schrödinger equation, using the infinite speed of propagation, we can take $T_0$ as large as we want by taking WKB solutions with frequencies $\lambda \gg T_0/T$, while for the wave we need to know the DN map on time $[0, \infty)$ to be able to let $T_0$ be arbitrarily large. We then use some estimate on the volume of the set of geodesics staying in $M^\circ$ for time $\leq T_0$: this volume decays exponentially in $T_0$. We deduce that the transform $I^*_0 I_0(q_1 - q_2)$ of $q := q_1 - q_2$ can be controled in $L^2$ by

$$C e^{-C_0 T_0} \|\Lambda^S_{q,q_1} - \Lambda^S_{q,q_2}\|_{*,\nu}^{1/4} \|q\|_{W^{1,\infty}}^{1/2} + C e^{-\epsilon T_0} \|q\|_{L^\infty}$$

(1.14)

for some $C_0 > 0, C > 0, \epsilon > 0$ independent of $T_0$. Here $I_0 : L^{\infty}(M) \to L^2_{loc}(\partial SM \setminus \Gamma_-)$ is the X-ray transform defined by

$$I_0 q(z) := \int_0^{\pi_+(z)} q(\pi_0(\varphi_t(z))) dt,$$

that extends continuously to $L^{\infty}(M) \to L^2(\partial SM)$ by [Gui17]; here $\pi_0 : SM \to M$ is the projection on the base. For simple metrics, it is well-known (see [PU05]) that the normal operator $\Pi_0 := I_0^* I_0$ is an elliptic pseudo-differential operator of order $-1$ thus
satisfying \( \| \Pi_0 f \|_{H^s} \geq C_s \| f \|_{H^{s-1}} \) for all \( s \geq 0 \) and \( C_s > 0 \) depending on \( s \). In [Gui17], using anisotropic Sobolev spaces an Fredholm theory for vector fields generating Axiom A flows [DG16], it is shown that the same properties hold on \( \Pi_0 \) for metrics with no conjugate points and hyperbolic trapping. We can then bound the norm of \( \| q_1 - q_2 \| \) by a constant times some norm \( \| \Pi_0 (q_1 - q_2) \| \), which in turn is bounded by (1.14). Taking \( T_0 \) large enough (depending on \( \| \Lambda^{S/W}_{g,q_1} - \Lambda^{S/W}_{g,q_2} \|_{s, \nu} \)) and using interpolation estimates, we can then show that the second term of (1.14) can be absorbed into the first term, and we obtain the desired stability bound. The case of the recovery of the conformal factor is using a similar type of arguments.

We make a final comment about the assumption \( q_1 = q_2 \) on \( \partial M \) (resp. \( c = 1 \) near \( \partial M \)): this assumption could be removed by standard arguments provided the potentials \( q_i \) (resp. for \( c \)) have uniform bounds in \( C^k(M) \) for \( k \) large enough. Since this amounts to construct geometrical optics solutions concentrated on very short geodesics almost tangent to \( \partial M \), the proof is basically the same as in [SU05, Section 3] and [Mon14, Theorem 2] in the case of the wave equation, and a slight variation in the case of Schrödinger equation. We write this short argument in an Appendix, where \( q_j \in C^4(M) \) (resp. \( q_j \in C^8(M) \)) is sufficient for the wave (resp. Schrödinger) equation.

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**Notations:** In what follows, we shall use the notational convention of writing \( C > 0 \) for constants appearing in upper/lower bounds, where this constant may change from line to line, and we shall indicate its dependence on the parameters of our problem when this is important.

### 2. Geometric setting and dynamical properties of the geodesic flow

In this section we recall, for \((M, g)\) a Riemannian manifold with strictly convex boundary, some notions about the geometry of the unit tangent bundle \( SM := \{(x, v) \in TM \mid g_x(v, v) = 1\} \) and the dynamics of the geodesic flow on \( SM \). Let 

\[
\pi_0 : SM \to M, \quad \pi_0(x, v) = x,
\]

be the natural projection on the base. We will denote by \( X \) the geodesic vector field on \( SM \) defined by \( Xf(x, v) = \partial_t f(\gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t))|_{t=0} \) where \( \gamma_{(x,v)}(t) \) is the unit speed geodesic with initial condition \( (\gamma_{(x,v)}(0), \dot{\gamma}_{(x,v)}(0)) = (x, v) \). We will denote by \( \varphi_t(x, v) = (\gamma_{(x,v)}(t), \dot{\gamma}_{(x,v)}(t)) \) the geodesic flow, which in turn is the flow of the vector field \( X \).
The incoming (−) and outgoing (+) boundaries of the unit tangent bundle of $M$ are defined by
\[
\partial_{\pm} SM := \{(x, v) \in SM \mid x \in \partial M, \mp g_x(v, n) > 0\},
\]
where $n$ is the inward pointing unit normal vector field to $\partial M$. For any $(x, v) \in SM$, define the forward and backward escaping time
\[
\tau_+(x, v) = \sup\{t \geq 0 \mid \varphi_t(x, v) \in \partial SM \text{ or } \forall s \in (0, t), \varphi_s(x, v) \in SM^o \} \in [0, +\infty],
\]
\[
\tau_-(x, v) := -\tau_+(x, -v) \in [-\infty, 0],
\]
which satisfies $X\tau_+ = -1$ in $SM$ with $\tau_+|_{\partial_+ SM} = 0$. For $(x, v) \in \partial_- SM$, the geodesic $\gamma(x, v)$ with initial point $x$ and tangent vector $v$ either has infinite length (i.e. $\tau_+(x, v) = +\infty$) or it intersects $\partial M$ at a boundary point $x' \in \partial M$ with tangent vector $v'$ with $(x', v') \in \partial_+ SM$.

The incoming (−) and outgoing (+) tails in $SM$ are defined by
\[
\Gamma_{\pm} = \{z \in SM \mid \tau_\pm(z) = \pm\infty\},
\]
and the trapped set for the flow on $SM$ is the set
\[
K := \Gamma_+ \cap \Gamma_-.
\]
It is a compact subset of $SM^o$ that is flow invariant ([Gui17]). We define the subset $T_+(t) \subset SM$ given by the points $(x, v) \in SM$ for which the orbit of the geodesic flow issued from $(x, v)$ remains in $SM$ after time $t$:
\[
T_+(t) := \{(x, v) \in SM \mid \tau_+(x, v) \geq t\}.
\]
We define the non-escaping mass function $V(t)$ as
\[
V(t) := \text{Vol}(T_+(t)),
\]
where Vol is the volume with respect to the Liouville measure $\mu$ on $SM$. Let us also denote
\[
T_{+\partial SM}(t) := T_+(t) \cap \partial_- SM.
\]
The escape rate $Q \leq 0$ measures the exponential rate of decay of $V(t)$. It is given by:
\[
Q := \limsup_{t \to +\infty} \frac{1}{t} \log V(t).
\]
By [Gui17, Prop. 2.4], if the trapped set $K$ is hyperbolic, then $Q = \text{Pr}(-J_u)$ is the topological pressure of (minus) the unstable Jacobian $J_u := \partial_t \det(\text{d}\varphi_t|_{E_u})|_{t=0}$ of the geodesic flow on the trapped set $K$, and it satisfies
\[
Q = \text{Pr}(-J_u) < 0.
\]
Let $d\mu_n$ be the measure on $\partial SM$ defined by
\[
d\mu_n(x, v) := |g_x(v, n)|_t^*|d\mu(x, v)|,
where \(|d\mu|\) is the Liouville density, and \(\iota : \partial SM \to SM\) is the inclusion map. When \(\text{Vol}(\Gamma_- \cup \Gamma_+) = 0\), then \(\text{Vol}_{\partial SM}(\Gamma_\pm \cap \partial_\pm SM) = 0\) and one can use Santaló's formula ([Gui17, Section 2.5]) to integrate functions in \(SM\): for all \(f \in L^1(SM)\):

\[
\int_{SM} f d\mu = \int_{\partial_- SM \setminus \Gamma_-} \int_0^{\tau_+(x,v)} f \circ \varphi_t(x,v) dt \, d\mu_n(x,v). \tag{2.3}
\]

It is convenient to view \((M, g)\) as a strictly convex region of a larger smooth manifold \((M_\varepsilon, g_\varepsilon)\) with strictly convex boundary so that each geodesic in \(M_\varepsilon \setminus M\) has finite length with endpoints on \(\partial M_\varepsilon \cup \partial M\). The existence of such extension is proved in [Gui17, Sect. 2.1 and Lemma 2.3]. Moreover, if \((M, g)\) has hyperbolic trapped set and no conjugate points, one can choose \((M_\varepsilon, g_\varepsilon)\) with the same properties as \((M, g)\), as is shown in [Gui17, Lemma 2.3]. The vector field \(X\) and the flow \(\varphi_t\) are extended in \(SM_\varepsilon\) and we define the function \(\tau_\pm\) on \(SM_\varepsilon\) just as we did for \(\tau_\pm\) on \(SM\). The trapped set of the flow in \(SM_\varepsilon\) is still \(K \subset SM^0\), the incoming tail \(\Gamma_\varepsilon^\pm\) on \(SM_\varepsilon\) is \(\Gamma_\varepsilon^\pm = \bigcup_{t \geq 0} \varphi_{\pm t}(\Gamma_\pm) \cap SM_\varepsilon\) and \(\Gamma_\varepsilon^\pm \cap SM = \Gamma_\pm\).

### 3. The X-ray Transform

In this section we recall from [Gui17] the main properties of the X-ray transform acting on functions in our geometric setting. Let \((M, g)\) be a smooth compact Riemannian manifold with strictly convex boundary and \(M_\varepsilon\) a small extension with the same property.

The X-ray transform \(I\) is defined as the map:

\[
I : C_c^\infty(SM \setminus (\Gamma_- \cup \Gamma_+)) \to C_c^\infty(\partial_- SM \setminus \Gamma_-), \quad If(x, v) := \int_0^{\tau_+(x,v)} f \circ \varphi_t(x,v) dt.
\]

The X-ray transform can be extended to more general spaces. If \(\text{Vol}(K) = 0\), then Santaló's formula implies that the operator \(I\) extends as a bounded operator

\[
I : L^1(SM) \to L^1(\partial_- SM; d\mu_n).
\]

When moreover the escape rate \(Q\) of (2.2) satisfies \(Q < 0\) then, by [Gui17, Lemma 5.1], one has that

\[
\forall p > 2, \quad I : L^p(SM) \to L^2(\partial_- SM, d\mu_n). \tag{3.1}
\]

For our purposes to extend the results of [BDSF10], it is more convenient to deal with the X-ray transform acting on functions in \(C^\infty(M)\). The projection \(\pi_0 : SM_\varepsilon \to M_\varepsilon\) on the base induces a pullback map

\[
\pi_0^* : C_c^\infty(M_\varepsilon^0) \to C^\infty(SM_\varepsilon), \quad \pi_0^* f := f \circ \pi_0,
\]

and a pushforward map \(\pi_{0*}\) defined by duality:

\[
\pi_{0*} : \mathcal{D}'(SM_\varepsilon) \to \mathcal{D}'(M_\varepsilon^0), \quad \langle \pi_{0*} u, f \rangle := \langle u, \pi_0^* f \rangle.
\]

When acting on \(L^1\) functions, the pushforward \(\pi_{0*}\) acts as

\[
\pi_{0*} f(x) := \int_{SM} f(x, v) d\omega_x(v).
\]
where $d\omega_x$ is the measure on $S_x M$ induced by $g$. The pullback by $\pi_0$ gives a bounded operator $\pi^*_0 : L^p(M) \to L^p(SM)$ for all $p \in (1, \infty)$. We define the X-ray transform on functions by

$$I_0 = I\pi^*_0.$$ 

If $Q < 0$ then $I_0$ extends as a bounded operator

$$I_0 := I\pi^*_0 : L^p(M) \to L^2(\partial SM, d\mu_n),$$

(3.2)

for any $p > 2$. The adjoint $I^*_0 : L^2(\partial SM, d\mu_n) \to L^{p'}(M)$ is bounded for $1/p' + 1/p = 1$, and it is given precisely by $I^*_0 = \pi_0 I^*$. The operator $\Pi_0$ is defined as the bounded self-adjoint operator

$$\Pi_0 : I^*_0 I_0 = \pi_0 I^* I\pi^*_0 : L^p(M) \to L^{p'}(M), \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad p > 2.$$ 

Similarly, we define the extended X-ray transform $I^*_0$ associated with $M_e$, and

$$\Pi^*_0 = I^*_0 I^*_0 : L^p(M) \to L^{p'}(M), \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad p > 2.$$ 

Lemma 3.1. [Gui17, Prop. 5.7] Assume that $(M, g)$ has strictly convex boundary, no conjugate points and hyperbolic trapped set and let $(M_e, g_e)$ be an extension with the same properties, and with same trapped set. The operator $\Pi_0$, resp. $\Pi^*_0$, is an injective elliptic pseudo-differential operator of order $-1$ in $M^c$, resp. $M^c_e$, with principal symbol $\sigma(\Pi^*_0) (x, \xi) = C_d |\xi|^{-1}$ for some constant $C_d > 0$ depending only on $d = \dim M$. For each $k \in \mathbb{Z}$ and each compact subset $\Omega \subset M^c_e$ with smooth boundary, there exists $C_1, C_2 > 0$ such that for all $f \in C^\infty_c(\Omega)$

$$C_1 \|f\|_{H^k(M_e)} \leq \|\Pi^*_0 f\|_{H^{k+1}(M_e)} \leq C_2 \|f\|_{H^k(M_e)}.$$ 

(3.4)

Moreover, a direct calculation yields for $z \not\in \Gamma^e_+ \cup \Gamma^e_-$

$$I^*(I^*_0 f)(z) = \int_{\tau^+_e(z)}^{\tau^+_e(z)} \pi^*_0 f(\varphi_\ell(z))d\ell, \quad I^*_0(I_0 f)(z) = \int_{\tau^-_e(z)}^{\tau^-_e(z)} \pi^*_0 f(\varphi_\ell(z))d\ell$$

and thus if $f \in L^p(M_e)$ satisfies $\text{supp } f \subset M$, we have $I^*(I^*_0 f) = I^*(I_0 f)$ on $SM \setminus (\Gamma^e_+ \cup \Gamma^e_-)$. In particular this implies that

$$f \in W^s_{comp}(M^c) \implies \Pi^*_0 f \in W^{s+1, p}_{loc}(M^c) \quad (3.5)$$

Since pseudo-differential operators of order $-1$ map $W^{s, p}_{comp}(M^c)$ continuously to $W^{s+1, p}_{loc}(M^c)$ for all $(s, p) \in \mathbb{R} \times (1, \infty)$ (see [Tay91, Thm. 0.11.A]), $(3.5)$ implies that

$$f \in W^s_{loc}(M) \implies \Pi_0 f \in W^{s+1, p}(M), \quad f \in W^s_{loc}(M_e) \implies \Pi^*_0 f \in W^{s+1, p}(M_e)$$ 

(3.6)

where $W^{s, p}(M)$ denotes the Sobolev space (with $s$ derivatives in $L^p$) on the manifold with boundary $M$, $W^s_{loc}(M)$ is the closure of $C^\infty(M^c)$ for the $W^s_{loc}(M)$ topology, and similarly on $M_e$. 

\[\]
4. Geometrical optics solutions

We will assume along this section that \((M, g)\) is a smooth compact Riemannian manifold with boundary such that

- The boundary \(\partial M\) is strictly convex,
- The metric \(g\) has no pairs of conjugate points,
- The trapped set \(K\) is hyperbolic.

We shall take \((M_e, g_e)\) an extension of \((M, g)\) with the same properties and for notational simplicity we will write \(g\) instead of \(g_e\) for the extended metric on \(M_e\).

4.1. Geometrical optics for the Schrödinger equation. In this section we generalize the geometrical optics solutions given in [BDSF10, Sect. 4] for simple manifolds to our geometric setting. Since the map \(\exp^{-1}_x(M) \to M\), with \(x \in M\), is no longer a diffeomorphism, but the exponential map behaves well on the universal cover of \(M\), we then make the construction in the universal cover of \(M\), periodize it with respect to the fundamental group \(\pi_1(M)\), and then project it down to \(M\).

We first recall the following:

**Lemma 4.1.** [BDSF10, Lemma 3.2 and eq (3.5)] Let \(T > 0\) and \(q \in L^\infty(M)\). If \(F \in W^{1,1}([0, T]; L^2(M))\) such that \(F(0, \cdot) \equiv 0\), then the unique solution \(v\) to

\[
\begin{cases}
(i\partial_t - \Delta_g + q(x))v(t, x) = F(t, x) & \text{in } (0, T) \times M, \\
v(0, x) = 0 & \text{in } M, \\
v(t, x) = 0 & \text{on } (0, T) \times \partial M,
\end{cases}
\]

satisfies

\[
v \in C^1([0, T]; L^2(M)) \cap C([0, T]; H^2(M) \cap H^1_0(M)).
\]

In addition, there is a \(C > 0\) depending on \((M, g), T\) and \(\|q\|_{L^\infty}\) such that for any \(\eta > 0\) small and \(t \in [0, T]\)

\[
\|v(t, \cdot)\|_{L^2(M)} \leq C \int_0^T \|F(s, \cdot)\|_{L^2(M)} \, ds
\]

\[
\|v(t, \cdot)\|_{H^1_0(M)} \leq C(\eta \|\partial_t F\|_{L^1([0, T]; L^2(M))} + \eta^{-1} \|F\|_{L^1([0, T]; L^2(M))}^2).
\]

Let us consider extensions \(M \Subset M_e \Subset M_{ee}\) of the manifold \(M\) and extend the metric \(g\) smoothly in a way that \((M_e, g_e)\) has the same properties as \((M, g)\). The potentials \(q_1, q_2\) may also be extended to \(M_{ee}\) and their \(W^{1,\infty}(M)\) norms may be bounded by \(N_0\). Since \(q_1\) and \(q_2\) coincide on the boundary, their extension outside \(M\) can be taken so that \(q_1 = q_2\) in \(M_{ee} \setminus M\).

We first recall the construction, following [BDSF10, Sect. 4], of a geometric optics solution for simple manifolds and we will explain how to extend it to our setting. If \((M, g)\) is a simple manifold, a geometric optics solution \(u \in C^1([0, T]; L^2(M)) \cap C([0, T]; H^2(M))\) for the Schrödinger equation

\[
(i\partial_t - \Delta_g + q(x))u = 0, \quad \text{in } (0, T) \times M, \\
u(0, x) = 0,
\]

\[
(4.3)
\]
can be constructed in terms of a function $\psi \in C^2(M)$ satisfying the eikonal equation

$$|\nabla^g \psi(x)|_g = 1, \quad \forall x \in M,$$

and a function $a \in H^1(\mathbb{R}; H^2(M))$ solving the transport equation

$$\frac{\partial a}{\partial t} + da(\nabla^g \psi) - \frac{1}{2}(\Delta_g \psi_y)a = 0, \quad \forall t \in \mathbb{R}, \quad x \in M,$$

with $a(t, x) = 0, \quad \forall x \in M, \quad$ and $t \leq 0, \text{ or } t \geq T_0,$

for some $T_0$ sufficiently large (which in the simple case is taken to satisfy $T_0 > 1 + \text{Diam}(M_e)$ where $\text{Diam}(M_e)$ is the $g$-diameter of $M_e$), and $da$ is the exterior derivative of $a$. More precisely if $\partial M_e$ is chosen close enough to $\partial M$ so that $(M_e, g)$ is a simple manifold, one can define, for any fixed $y \in \partial M_e$,

$$\psi(x) = \psi_y(x) := d_g(y, x), \quad x \in M_e.$$

Using geodesic polar coordinates we can write each $x \in M_e$ as

$$x = \exp_y(r(x)v(x)), \quad r(x) = d_g(y, x), \quad v(x) \in S_y M_e$$

where $\exp_y : TM_e \to M_e$ denotes the exponential map at $y$ for the metric $g$. One defines a solution to the transport equation $a \in H^1(\mathbb{R}; H^2(M))$ in polar coordinates as

$$a(t, x) := \alpha^{-1/4}(r(x), v(x))\phi(t - r(x))b(y, v(x)),$$

where $\alpha = \alpha(r, v) = |\det(g_{ij}(r, v))|$ denotes the square of the volume element in geodesic polar coordinates, $\phi \in C^\infty_c(\mathbb{R})$ satisfies $\text{supp} \phi \subset (0, \varepsilon_0)$ for $\varepsilon_0 > 0$ small, and $b \in H^2(\partial_S M_e)$ is a fixed initial data.

A geometrical optics solution $u \in C^1([0, T]; L^2(M)) \cap C([0, T]; H^2(M))$ for (4.3) is then defined by $u(t, x) = G_\lambda(t, x) + v_\lambda(t, x)$, where

$$G_\lambda(t, x) := a(2\lambda t, x)e^{i\lambda(\psi_y(x) - \lambda t)},$$

and the remainder $v_\lambda$ satisfies ([BDSF10, Lemma 4.1]):

$$v_\lambda(t, x) = 0, \quad \forall (t, x) \in (0, T) \times \partial M,$$

$$v_\lambda(0, x) = 0.$$

Moreover, there exists $C > 0$ such that, for all $\lambda \geq T_0/2T$,

$$\|v_\lambda(t, \cdot)\|_{H^k(M)} \leq C\lambda^{k-1}\|a\|_{H^1([0, T_0]; H^2(M))}, \quad k = 0, 1.$$

The constant $C$ depends only on $T$ and $(M, g)$. One can also construct a geometrical optics solution $u(t, x)$ if the initial condition $u(0, x) = 0$ is replaced by the final condition $u(T, x) = 0$ provided $\lambda \geq T_0/2T$; in this case $v_\lambda$ satisfies $v_\lambda(T, x) = 0$.

In our setting, that is, assuming that the trapped set $K$ is hyperbolic and that $g$ has no conjugate points, the construction is a bit more subtle, since the exponential map $\exp_y : \exp_y^{-1}(M) \to M$ is no longer a diffeomorphism. We work on the universal cover $\tilde{M}$.
of \( M \), which is a non-compact manifold with boundary (the boundary has infinitely many connected components), whose interior is diffeomorphic to a ball. One has

\[
M = \tilde{M}/\pi_1(M),
\]

where the fundamental group \( \pi_1(M) \) is identified with the group of deck transformations on \( \tilde{M} \), that is, the set of homeomorphisms \( f : \tilde{M} \to \tilde{M} \) such that \( \pi \circ f = \pi \), with the composition, where \( \pi : \tilde{M} \to M \) is the covering map. The metric \( g \) lifts to a smooth metric \( \tilde{g} \) on \( \tilde{M} \) satisfying \( \gamma^*\tilde{g} = \tilde{g} \) for all \( \gamma \in \pi_1(M) \). More generally, we denote by \( \tilde{\cdot} \) the lifted objects to the universal cover. If \((M, g)\) is assumed to have no pair of conjugate points, \( \tilde{y} \) does not have pairs of conjugate points. Thus, for each \( y \in \tilde{M} \) the exponential map

\[
\exp_y : U_y \subset T\tilde{M} \to \tilde{M}
\]

is a diffeomorphism for some simply connected set \( U_y \). Similarly, we define the universal cover \( \tilde{M}_e \) of \( M_e \), and note that \( \pi_1(M) = \pi_1(M_e) \) so that each deck transformation \( \gamma \) of \( \tilde{M} \) extends naturally as a deck transformation on \( \tilde{M}_e \). Let us fix \( y \in \partial M_e \) and we lift this point to \( \tilde{y} \in \partial \tilde{M}_e \). We can choose a fundamental domain \( F \subset \tilde{M} \), so that \( M = F/\pi_1(M) \) via identification of the points of the boundary of \( F \) by the action of the elements of \( \pi_1(M) \). Note that \( F \) has two types of boundary components, the boundary components \( \partial F \) in the interior \( \tilde{M}^0 \) of \( \tilde{M} \) which are identified by elements of \( \pi_1(M) \), and the boundary components \( F \cap \pi^{-1}(\partial M) \). Similarly, we choose a fundamental domain \( F_e \) for \( \pi_1(M_e) \simeq \pi_1(M) \) in \( \tilde{M}_e \) extending \( F \) and denote by \( \partial_e F \) the interior boundary of \( F_e \). We can freely assume that \( \tilde{y} \in F_e \) does not belong to the closure of \( \partial_e F \). Recall the definition of the volume entropy of \( M \)

\[
h(M, g) := \limsup_{R \to \infty} \frac{1}{R} \log \text{Vol}_g(B_g(\tilde{y}; R)) \tag{4.5}
\]

where \( B_g(\tilde{y}; R) \subset \tilde{M}_g \) is the \( \tilde{g} \)-geodesic ball centered at \( \tilde{y} \) of radius \( R \). Since \( M \) is compact, one has \( h(M, g) < \infty \) (for instance by Bishop-Gromov comparison theorem) and \( h(M, g) \) is not depending on the choice of \( \tilde{y} \).

We define on \( \tilde{M} \) the solution to the “lifted” eikonal equation \( |\nabla_{\tilde{g}} \psi_{\tilde{y}}| = 1 \) given by

\[
\psi_{\tilde{y}}(\tilde{x}) := d_{\tilde{g}}(\tilde{y}, \tilde{x}), \quad \tilde{x} \in \tilde{M},
\]

where \( d_{\tilde{g}} \) denotes the distance associated to the lifted metric \( \tilde{g} \) on \( \tilde{M} \). Notice that \( \psi_{\tilde{y}} \) is well defined and smooth outside \( \tilde{x} = \tilde{y} \) since \( \tilde{g} \) has no conjugate points (for any \( \tilde{x} \in \tilde{M} \) there is a unique geodesic joining \( \tilde{y} \) with \( \tilde{x} \) and realizing \( d_{\tilde{g}}(\tilde{y}, \tilde{x}) \)). Let

\[
\partial_- S_{\tilde{y}} \tilde{M}_e := \{ v \in S_{\tilde{y}} \tilde{M}_e | \langle v, \nu(\tilde{y}) \rangle > 0 \}.
\]

Using geodesic polar coordinates on \( \tilde{M} \) we can write each \( x \in \tilde{M}_e \)

\[
x = \exp_{\tilde{y}}(r(x)v(x)), \quad r(x) = \psi_{\tilde{y}}(x), \quad v(x) \in S_{\tilde{y}} \tilde{M}_e.
\]

Fix \( T_0 > 1 + \text{Diam}(M_e) \). For any given \( b \in H^2(\partial_- S M_e) \) with \( b|_{\tau_{\tilde{y}}^{\text{gsm}}(T_0)} = 0 \) (recall definition (2.1)), we denote by \( \tilde{b} \) its lift to \( \partial_- S F_e \). Let \( \phi \in C_0^\infty(\mathbb{R}) \) with \( \text{supp} \phi \subset (0, \varepsilon_0), \varepsilon_0 > 0 \) small,
we define
\[ a(t, x) := \alpha^{-1/4}(r(x), v(x))\phi(t - r(x))\tilde{b}(\tilde{y}, v(x)), \] (4.6)
where \( \alpha(r, v) = |\det(g_{ij}(r, v))| \) denotes the square of the volume element of \( \tilde{M}_e \) in geodesic polar coordinates. We introduce the norm \( \|a\|_* \) on functions on \([0, T_0] \times \tilde{M} \) given by
\[ \|a\|_* := \|a\|_{H^1([0, T_0]; H^2(\tilde{M}))}. \]

For any \( \lambda > 0 \), we set
\[ G_\lambda(t, x) := \sum_{\gamma \in \pi_1(M)} a(2\lambda t, \gamma(x))e^{i\lambda(\psi_\theta(\gamma(x)) - \lambda t)}, \quad t \in (0, T), \quad x \in \tilde{M}_e, \] (4.7)
where \( \gamma(x) \) denotes the lift of the point \( \pi(x) \in M_e \) to the fundamental domain \( \gamma(F_e) \). Notice that this definition does not depend on the choice of the lift \( \tilde{y} \) but on the base point \( y \), and since \( \text{supp}(a(2\lambda t, \cdot)) \) is contained in a fixed compact set of \( \tilde{M}_e \) for \( t \in [0, T] \), the sum in \( \gamma \in \pi_1(M) \) is locally finite. Notice also that the condition \( T_0 > 1 + \text{Diam}(M_e) \) together with \( \tilde{b}|_{T_0^{\partial M}} = 0 \) ensures that \( G_\lambda(t, x) \) vanishes on \((0, T) \times \partial M \) provided that \( 2\lambda T \geq T_0 \), since the solution to the transport equation crosses the whole manifold \( M_e \) in time \( T \). Moreover, as \( G_\lambda(t, \gamma x) = G_\lambda(t, x) \), the function \( G_\lambda \) descends to \( M_e \) (and will also be denoted \( G_\lambda \) downstairs), and satisfies \( G_\lambda \in H^1([0, T]; H^2(M_e)) \).

**Lemma 4.2.** Let \( q \in L^\infty(M) \). For \( T_0 > 0 \) and \( T > 0 \), the Schrödinger equation
\[ (i\partial_t - \Delta_g + q(x))u = 0, \quad \text{in} \ (0, T) \times M, \]
\[ u(0, \cdot) = 0, \quad \text{in} \ M, \]
has a solution of the form
\[ u(t, x) = G_\lambda(t, x) + v_\lambda(t, x), \]
with \( G_\lambda \) given by (4.7), such that
\[ u \in C^1([0, T]; L^2(M)) \cap C([0, T]; H^2(M)), \]
where \( v_\lambda(t, x) \) satisfies, for \( \lambda \geq \frac{T_0}{2T} \),
\[ v_\lambda(t, x) = 0, \quad \forall (t, x) \in (0, T) \times \partial M, \]
\[ v_\lambda(0, x) = 0, \quad x \in M. \]
Furthermore, for each \( \epsilon > 0 \) there exists \( C > 0 \) depending only on \( (\epsilon, M, g, \|q\|_{L^\infty}, T) \) but not on \( T_0, y \), such that, for any \( \lambda \geq \frac{T_0}{2T} \) the following estimate holds true:
\[ \|v_\lambda(t, \cdot)\|_{H^k(M)} \leq C e^{(h+\epsilon)T_0} \lambda^{k-1} \|a\|_*, \quad k = 0, 1, \] (4.8)
where \( h = h(M, g) \) denotes the volume entropy of \((M, g)\). The result remains valid after replacing the initial condition \( u(0, \cdot) = 0 \) by the final condition \( u(T, \cdot) = 0 \).

**Proof.** We prove the Lemma with initial condition \( u(0, \cdot) = 0 \), the proof for \( u(T, \cdot) = 0 \) being analogous. As in [BDSF10, Proof of Lemma 4.1], we consider for \( x \in M_e \)
\[ k(t, x) = -\sum_{\gamma \in \pi_1(M)} (i\partial_t - \Delta_\theta + \tilde{q}) \left( a(2\lambda t, \gamma(\tilde{x}))e^{i\lambda(\psi_\theta(\gamma(\tilde{x})) - \lambda t)} \right), \]
where \( \tilde{x} \in \tilde{M} \) is a lift of \( x \). Let \( v_\lambda \) be the solution, given by Lemma 4.1, to the homogeneous boundary value problem

\[
\begin{aligned}
(i \partial_t - \Delta_g + q)v_\lambda(t, x) &= k(t, x) & \text{in } (0, T) \times M, \\
v_\lambda(0, x) &= 0, & \text{in } M, \\
v_\lambda(t, x) &= 0 & \text{on } (0, T) \times \partial M.
\end{aligned}
\]

We shall show that \( v_\lambda \) satisfies the estimate (4.8). A computation gives

\[
-k(t, x) = \sum_{\gamma \in \pi_1(M)} e^{i\lambda(\psi_g(\gamma(\tilde{x}))) - \lambda t} (\Delta_g + \tilde{q}(\gamma(\tilde{x}))) a(2\lambda t, \gamma(\tilde{x}))
\]

\[
+ 2i\lambda \sum_{\gamma \in \pi_1(M)} e^{i\lambda(\psi_g(\gamma(\tilde{x}))) - \lambda t} \left( \partial_t a + da(\nabla_{\tilde{g}} \psi_g) - \frac{a}{2} \Delta_{\tilde{g}} \psi_g \right) (2\lambda t, \gamma(\tilde{x}))
\]

\[
+ \lambda^2 \sum_{\gamma \in \pi_1(M)} a(2\lambda t, \gamma(\tilde{x})) e^{i\lambda(\psi_g(\gamma(\tilde{x}))) - \lambda t} \left( 1 - |\nabla_{\tilde{g}} \psi_g|_{\tilde{g}}^2 \right).
\]

Using that \( a \) solves the transport equation, that \( \psi_g \) solves the eikonal equation and that \( \Delta_{\tilde{g}} \) commute with \( \gamma^* \) (since \( \gamma \) are isometries of \( \tilde{g} \)), we obtain

\[
-k(t, x) = \sum_{\gamma \in \pi_1(M)} e^{i\lambda(\psi_g(\gamma(\tilde{x}))) - \lambda t} (\Delta_g a + \tilde{q} a) (2\lambda t, \gamma(\tilde{x}))
\]

\[
=: \sum_{\gamma \in \pi_1(M)} e^{i\lambda(\psi_g(\gamma(\tilde{x}))) - \lambda t} k_0(2\lambda t, \gamma(\tilde{x})).
\]

Notice that \( k_0 \in H^1_0([0, T]; L^2(M)) \) for \( \lambda \geq T_0 / 2T \) and \( k_0(s, \cdot)|_{\tilde{M}} = 0 \) for \( s > T_0 \). Then, using Lemma 4.1 we get

\[
v_\lambda \in C^1([0, T]; L^2(M)) \cap C([0, T]; H^2(M) \cap H^1_0(M)).
\]

Moreover, using (4.1), there exists \( C > 0 \) depending on \( (M, g), T > 0 \), and \( ||q||_{L^\infty} \) such that

\[
||v_\lambda(t, \cdot)||_{L^2(M)} \leq C \sum_{\gamma \in \pi_1(M)} \int_0^T ||k_0(2\lambda t, \cdot)||_{L^2(\gamma(\tilde{F}))} dt \leq \frac{C}{\lambda} \sum_{\gamma \in \pi_1(M)} \int_0^{T_0} ||k_0(s, \cdot)||_{L^2(\gamma(\tilde{F}))} ds
\]

\[
\leq \frac{C(1 + ||q||_{L^\infty}) T_0^{1/2} N(T_0)}{\lambda} ||a||_*,
\]

where \( N(T_0) \) denotes the number of fundamental domains which intersects the geodesic ball \( B(\tilde{y}, T_0) \) of center \( \tilde{y} \) and radius \( T_0 \) in \( \tilde{M} \), that is:

\[
N(T_0) = \# \{ \gamma \in \pi_1(M) \mid \exists x \in F, \; d_{\tilde{g}}(\gamma(x), \tilde{y}) \leq T_0 \}
\]

\[
\leq \# \{ \gamma \in \pi_1(M) \mid \max_{x \in F} d_{\tilde{g}}(\gamma(x), \tilde{y}) \leq T_0 + \text{Diam}(M) \}.
\]

Clearly, \( N(T_0) \text{Vol}(M) \leq \text{Vol}(B(\tilde{y}, T_0 + \text{Diam}(M))) \), and therefore for each \( \epsilon > 0 \), there is \( C > 0 \) (depending on \( \text{Diam}(M) \) and \( \text{Vol}(M) \)) such that for all \( T_0 > 0 \) large enough

\[
N(T_0) \leq Ce^{(h+\epsilon)T_0}.
\]
where \( h = h(M, g) \) is the volume entropy of \((M, g)\) defined in (4.5). We notice that the constants \( C > 0 \) above can be chosen independently of \( y \) and that \( h \) is in fact not depending on \( y \).

Finally, by Lemma 4.1, there is \( C > 0 \) (depending on \((M, g, T, \|q\|_{L^\infty})\)) such that for each \( \eta > 0 \)

\[
\| \nabla^g v_\lambda(t, \cdot) \|_{L^2(M)} \leq C \eta \sum_{\gamma \in \pi_1(M)} \int_0^T \left( \lambda^2 \| k_0(2\lambda t, \cdot) \|_{L^2(\gamma(\mathcal{F}))} + \lambda \| \partial_t k_0(2\lambda t, \cdot) \|_{L^2(\gamma(\mathcal{F}))} \right) dt \\
+ C \eta^{-1} \sum_{\gamma \in \pi_1(M)} \int_0^T \| k_0(2\lambda t, \cdot) \|_{L^2(\gamma(\mathcal{F}))} dt.
\]

Choosing \( \eta = \lambda^{-1} \), for each \( \epsilon > 0 \) there is \( C > 0 \) such that for all \( \lambda \geq T_0/2T \)

\[
\| \nabla^g v_\lambda(t, \cdot) \|_{L^2(M)} \leq C \sum_{\gamma \in \pi_1(M)} \left( \int_0^{T_0} \| k_0(s, \cdot) \|_{L^2(\gamma(\mathcal{F}))} ds + \int_0^{T_0} \| \partial_t k_0(s, \cdot) \|_{L^2(\gamma(\mathcal{F}))} ds \right) \\
\leq C e^{(b+\epsilon)T_0} \| a \|_*.
\]

\( \square \)

4.2. Geometrical optics for the wave equation. In this section we give the construction of geometric optics solutions for the wave equation with hyperbolic trapped set. First we use the following

**Lemma 4.3.** Let \( q \in L^\infty(M) \). There are constants \( C > 0 \) depending only on \((M, g, \|q\|_{L^\infty})\) and \( v_0 \geq 0 \) depending only on \( \|q\|_{L^\infty} \), such that for all \( 0 < t \leq T \), and all \( F \in L^2((0, T) \times M) \), there is a unique solution \( v \) to

\[
\begin{cases}
\left( \partial_t^2 + \Delta_g + q(x) \right) v(t, x) = F(t, x) & \text{in } [0, T] \times M, \\
v(0, x) = 0, \quad \partial_t v(0, x) = 0 & \text{in } M, \\
v(t, x) = 0 & \text{on } [0, T] \times \partial M,
\end{cases}
\]

in \( C^1([0, T]; L^2(M)) \cap C^0([0, T]; H^1_0(M)) \) satisfying for each \( \nu \geq v_0 \)

\[
\| v(t, \cdot) \|^2_{L^2(M)} + \| \partial_t v(t, \cdot) \|^2_{L^2(M)} + \| \nabla^g v(t, \cdot) \|^2_{L^2(M)} \leq C \int_0^t e^{\nu(t-s)} \| F(s, \cdot) \|^2_{L^2(M)} ds,
\]

\( \forall \nu \geq v_0 \)

\[
\| e^{-\frac{\nu}{2} t} \partial_t v \|^2_{L^2((0,T)\times\partial M)} \leq C \| e^{-\frac{\nu}{2} t} F \|^2_{L^2((0,T)\times M)}
\]

where \( C \) depends only on \( \| q \|_{L^\infty} \) and \( \nu \).

There is \( C > 0 \) as above such that for each \( f \in H^1([0, T] \times \partial M) \) with \( f(0, \cdot) = \partial_t f(0, \cdot) = 0 \), there is a unique solution \( u \in C^1([0, T]; L^2(M)) \cap C^0([0, T]; H^1_0(M)) \) such that

\[
\begin{cases}
\left( \partial_t^2 + \Delta_g + q(x) \right) u(t, x) = 0 & \text{in } [0, T] \times M, \\
u(0, x) = 0, \quad \partial_t u(0, x) = 0 & \text{in } M, \\
u(t, x) = f(t, x) & \text{on } [0, T] \times \partial M,
\end{cases}
\]

and \( u \) satisfies,

\[
\| e^{-\nu t/2} \partial_t u \|^2_{L^2([0,T]\times\partial M)} \leq C \| e^{-\nu t/2} f \|^2_{H^1([0,T]\times\partial M)}
\]
As a consequence, the operator $\Lambda^W_q : e^{i\nu t/2} H^1_0(\mathbb{R}_+ \times \partial M) \to e^{i\nu t/2} L^2(\mathbb{R}_+ \times \partial M)$ is bounded with norm depending only on $(M, g)$, $\|q\|_{L^\infty}$ and $\nu$.

**Proof.** The uniqueness and existence is done in [LM72a, Chapter 3, Section 8 and 9] and based on energy estimates. Here we want a uniform estimate in time involving the exponent $\nu$, in particular for what concerns its dependence in $q$. Let $\nu$ be a solution to (4.9), then, for any $\nu \geq 0$, $v_\nu(t, x) = e^{-\nu t/2} v(t, x)$ satisfies the damped-wave equation

$$
\left\{
\begin{array}{ll}
(\partial_t^2 + \Delta_g + \frac{\nu^2}{4} + q(x) + \nu \partial_t) v_\nu(t, x) = F_\nu(t, x) & \text{in } [0, T] \times M, \\
v_\nu(0, x) = 0, & \text{on } [0, T] \times \partial M, \\
\partial_t v_\nu(0, x) = 0 & \text{in } M,
\end{array}
\right.
$$

(4.12)

where $F_\nu(t, x) = e^{-\nu t/2} F(t, x)$. Similarly, let $u$ be the solution to (4.11), then $u_\nu(t, x) = e^{-\nu t/2} u(t, x)$ solves the boundary-value problem:

$$
\left\{
\begin{array}{ll}
(\partial_t^2 + \Delta_g + \frac{\nu^2}{4} + q(x) + \nu \partial_t) u_\nu(t, x) = 0 & \text{in } [0, T] \times M, \\
u_\nu(0, x) = 0, & \text{on } [0, T] \times \partial M, \\
\partial_t u_\nu(0, x) = 0 & \text{in } M,
\end{array}
\right.
$$

(4.13)

where $f_\nu(t, x) = e^{-\nu t/2} f(t, x)$. First, multiply equation (4.12) by $\bar{v}_\nu$ and integrate in $[0, t] \times M$ to get

$$
- \int_0^t \int_M |\partial_s v_\nu|^2 dv_g ds + \int_0^t \int_M |\nabla g v_\nu|^2 dv_g ds + \int_0^t \int_M \left(\frac{\nu^2}{4} + q\right)|v_\nu|^2 dv_g ds
$$

$$
= -\int_M \partial_t v_\nu \bar{v}_\nu dv_g - \nu \int_0^t \int_M \partial_s v_\nu \bar{v}_\nu dv_g ds + \int_0^t \int_M F_\nu \bar{v}_\nu dv_g ds.
$$

Then:

$$
\int_0^t \int_M |\nabla g v_\nu|^2 dv_g ds + \frac{\nu^2}{4} \int_0^t \int_M |v_\nu|^2 dv_g ds
$$

$$
\leq \frac{1}{2} \int_M |\partial_t v_\nu(t)|^2 dv_g + \frac{1}{2} \int_M |v_\nu(t)|^2 dv_g + \frac{\nu}{2} \int_0^t \int_M (|\partial_s v_\nu|^2 + |v_\nu|^2) dv_g ds
$$

$$
+ (\|q\|_{L^\infty} + \frac{1}{2}) \int_0^t \int_M |v_\nu|^2 dv_g ds + \frac{1}{2} \int_0^t \int_M |F_\nu|^2 dv_g ds
$$

and we get, for $C_\nu = \frac{\nu^2}{4} - \frac{\nu}{2} - \|q\|_{L^\infty} - \frac{1}{2}$,

$$
\int_0^t \|\nabla g v_\nu(s)\|_{L^2(M)}^2 ds + C_\nu \int_0^t \|v_\nu(s)\|_{L^2(M)}^2 ds
$$

$$
\leq \frac{1}{2} \int_0^t |\partial_t v_\nu(t)|_{L^2(M)}^2 + \frac{1}{2} \|v_\nu(t)\|_{L^2(M)}^2
$$

$$
+ \frac{\nu}{2} \int_0^t \|\partial_s v_\nu\|_{L^2(M)}^2 + \frac{1}{2} \int_0^t \|F_\nu(s)\|_{L^2(M)}^2.
$$

(4.14)
We next multiply equation (4.12) by $\partial_t \bar{v}_\nu$, integrate in $[0, t] \times M$, take the real part and integrate by parts to obtain

$$
\frac{\nu^2}{4} \|v_\nu(t)\|_{L^2(M)}^2 + \|\partial_t v_\nu(t)\|_{L^2(M)}^2 + \|\nabla^g v_\nu(t)\|_{L^2(M)}^2 \\
\leq 2 \int_0^t (\|F_\nu(s)\|_{L^2(M)} \|\partial_s v(s)\|_{L^2(M)}) ds + \|q\|_{L^\infty} \int_0^t \|v_\nu(s)\|_{L^2(M)}^2 ds \\
+ (\|q\|_{L^\infty} - \nu) \int_0^t \|\partial_s v_\nu(s)\|_{L^2(M)}^2 ds \\
\leq \int_0^t \|F_\nu(s)\|_{L^2(M)}^2 ds + \|q\|_{L^\infty} \int_0^t \|v_\nu(s)\|_{L^2(M)}^2 ds + (1 + \|q\|_{L^\infty} - \nu) \int_0^t \|\partial_s v_\nu(s)\|_{L^2(M)}^2 ds.
$$

Defining $\Phi_{v_\nu}(t) := \|\partial_t v_\nu(s, \cdot)\|_{L^2(M)}^2 + \|v_\nu(t)\|_{L^2(M)}^2 + \|\nabla^g v_\nu(t)\|_{L^2(M)}^2$, using (4.14) and taking $\nu > 2$ large enough depending only on $\|q\|_{L^\infty}$, we obtain:

$$
\frac{1}{2} \Phi_{v_\nu}(t) \leq \frac{3}{2} \int_0^t \|F_\nu(s, \cdot)\|_{L^2(M)}^2 ds + (\|q\|_{L^\infty} - C_{v}) \int_0^t \|v_\nu(s)\|_{L^2(M)}^2 ds \\
+ (1 + \|q\|_{L^\infty} - \nu) \int_0^t \|\partial_s v_\nu(s)\|_{L^2(M)}^2 ds - \int_0^t \|\nabla^g v_\nu(s)\|_{L^2(M)}^2 ds.
$$

Take $\nu$ large enough so that $C_{v} > \|q\|_{L^\infty} > 1$ and $\nu > 2 + \|q\|_{L^\infty}$, we then get

$$
\Phi_{v_\nu}(t) + \int_0^t \Phi_{v_\nu}(s) ds \leq \frac{3}{2} \int_0^t \|F_\nu(s)\|_{L^2(M)}^2 ds.
$$

Since $\Phi_{v_\nu}(t) \geq \frac{2}{\nu^2 + 1} e^{-\nu t} \|\partial_t v\|_{L^2(M)}^2 + \frac{1}{2} e^{-\nu t} \|v(t)\|_{L^2(M)}^2$, this shows in particular the first estimate of (4.10). Next, let $N$ be a smooth vector field equal to the inward normal vector field on $\partial M$. Multiplying equation (4.12) by $\langle N, \nabla^g \bar{v} \rangle = d\bar{v}_\nu(N)$, one has:

$$
\int_0^t \int_M F_\nu(s, \cdot) d\bar{v}_\nu(N) dv_g ds \\
= \int_0^t \int_M \partial_s^2 v_\nu d\bar{v}_\nu(N) dv_g ds + \int_0^t \int_M \Delta_g v_\nu d\bar{v}_\nu(N) dv_g ds + \int_0^t \int_M (\frac{L^2}{4} + q) v_\nu d\bar{v}_\nu(N) dv_g ds \\
+ \nu \int_0^t \int_M \partial_s v_\nu d\bar{v}_\nu(N) dv_g ds \\
= I_1 + I_2 + I_3 + I_4.
$$

Using integration by parts in $s$

$$
2\text{Re}(I_1 + I_4) = 2\text{Re}(\langle \partial_t v_\nu, d\bar{v}_\nu(N) \rangle_{L^2(M)}) - \int_0^t \langle N, (\nabla^g |\partial_s v_\nu|^2) \rangle_{L^2(M)} ds \\
+ 2\nu \text{Re}(\int_0^t \langle \partial_s v_\nu, dv_\nu(N) \rangle_{L^2(M)} ds)$$
which gives, after integrating by parts in \( x \in M \) the second term, using \( \partial_s v = 0 \) on \((0, t) \times \partial M \) and the estimates (4.15) (for some \( C_{\nu, ||q||L^\infty} > 0 \) depending only on \((M, g), \nu, ||q||L^\infty) \)

\[
|2\text{Re}(I_1 + I_4)| \leq C_{\nu, ||q||L^\infty} \|F_{\nu}\|_{L^2((0, t) \times M)}^2.
\]

For the \( I_2 \) term, one gets by integration by parts

\[
2\text{Re}(I_2) = 2 \int_0^t \int_{\partial M} |\partial_h v_\nu|^2 dv_{\partial M} ds + 2\text{Re} \int_0^t \langle \nabla^g v_\nu, \nabla^g (d_{\nu}(N)) \rangle_{L^2(M)} ds
\]

and an easy computation shows

\[
\langle \nabla^g v_\nu, \nabla^g (d_{\nu}(N)) \rangle_{L^2(M)} = \int_M \left( (\nabla^g N \nabla^g v_\nu, \nabla^g \tilde{v}_\nu) + \frac{1}{2} \text{div}(|\nabla^g v_\nu|^2 N) - \frac{1}{2} |\nabla^g v_\nu|^2 \text{div}(N) \right) dv_g
\]

\[
= \int_M \left( (\nabla^g N \nabla^g v_\nu, \nabla^g \tilde{v}_\nu) - \frac{1}{2} |\nabla^g v_\nu|^2 \text{div}(N) \right) dv_g - \frac{1}{2} \int_{\partial M} |\nabla^g v_\nu|^2 dv_{\partial M}
\]

thus, using \(|\nabla^g v_\nu|^2 = |\partial_h v_\nu|^2\) (since \( v_\nu = 0 \) on \( \partial M \)), we obtain, by (4.15),

\[
|2\text{Re}(I_2) - \int_0^t \int_{\partial M} |\partial_h v_\nu|^2 dv_{\partial M} ds| \leq C_{\nu, ||q||L^\infty} \|F_{\nu}\|_{L^2((0, t), L^2(M))}^2.
\]

Finally, (4.15) directly gives \(|I_3| \leq C_{\nu, ||q||L^\infty} \|F_{\nu}\|_{L^2((0, t) \times M)}^2 \) and we conclude that

\[
\int_0^t \int_{\partial M} |\partial_h v_\nu|^2 dv_{\partial M} ds \leq C_{\nu, ||q||L^\infty} \|F_{\nu}\|_{L^2((0, t), L^2(M))}^2.
\]

(4.16)

The same results apply for solutions of the equation (4.12) on \([0, T] \times M\) with \( \nu \) replaced by \( -\nu \) and with boundary condition \( v_{-\nu}(T) = \partial_t v_{-\nu}(T) = 0 \).

Notice that if \( u_\nu \) solves (4.13) then for any \( F_{-\nu} \in L^2((0, T) \times M) \), one has, by duality with (4.12),

\[
\ell(F_{-\nu}) := \int_0^T \int_M u_\nu(t, x) F_{-\nu}(t, x) dv_g dt = \int_0^T \int_{\partial M} f_\nu(t, x) \partial_h v_{-\nu}(t, x) dv_{\partial M} dt,
\]

where \( v_{-\nu} \) is the solution to (4.12) with \( \nu \) replaced by \( -\nu \) and with boundary condition \( v_{-\nu}(T) = \partial_t v_{-\nu}(T) = 0 \) rather than \( v_\nu(0) = \partial_t v_\nu(0) = 0 \). By (4.16), there is \( C > 0 \) depending only on \((M, g)\) so that

\[
|\ell(F_{-\nu})| \leq C \|f_\nu\|_{L^2((0, T) \times \partial M)} \|F_{-\nu}\|_{L^2((0, T) \times M)}.
\]

Therefore

\[
\|u_\nu\|_{L^2((0, T) \times M)} \leq C \|f_\nu\|_{L^2((0, T) \times \partial M)}.
\]

Moreover, if \( f_\nu \in H^1((0, T) \times M) \), then \( w_\nu = \partial_t u_\nu \) solves the equation

\[
\begin{cases}
(\partial_t^2 + \Delta_g + \frac{\nu^2}{4} q + \nu \partial_t) w_\nu(t, x) = 0, & \text{in } [0, T] \times M \\
w_\nu(0, x) = \partial_t u_\nu(0, x) = 0, & \text{in } M \\
w_\nu(x, t) = \partial_t f_\nu(t, x), & \text{on } \partial M.
\end{cases}
\]
This implies that \( w_\nu = \partial_t u_\nu \in L^2((0, T) \times M) \) by the uniqueness result [LM72a, Chapter 3, Section 8 and 9] and with bound \( \|w_\nu\|_{L^2((0,T) \times M)} \leq C \|\partial_t f_\nu\|_{L^2((0,T) \times \partial M)} \). Note that 
\[
\partial_t w_\nu = \partial_t^2 u_\nu = - (\Delta_g + \frac{\nu^2}{4} + q + \nu \partial_t) u_\nu \in L^2((0,T); H^{-2}(M)) \text{ and}
\]
\[
\|\partial_t w_\nu\|_{L^2((0,T); H^{-2}(M))} \leq C_{\nu,\|q\|_{L^\infty}} \|f_\nu\|_{H^1((0,T) \times \partial M)}
\]
for some constant \( C_{\nu,\|q\|_{L^\infty}} \) depending only on \( \|q\|_{L^\infty}, \nu \) and \((M,g)\). On the other hand, 
\[
\partial_t^2 w_\nu = - (\Delta_g + \frac{\nu^2}{4} + q + \nu \partial_t) w_\nu \in L^2((0,T); H^{-2}(M)) \text{ with norm}
\]
\[
\|\partial_t w_\nu\|_{L^2((0,T); H^{-2}(M))} \leq C_{\nu,\|q\|_{L^\infty}} \|f_\nu\|_{H^1((0,T) \times \partial M)}
\]
for some constant as above, thus by interpolation with \( w_\nu \in L^2((0,T); L^2(M)) \), we also have \( \partial_t w_\nu \in L^2((0,T); H^{-1}(M)) \) ([LM72a, Proposition 2.2]) with bound 
\[
\|\partial_t w_\nu\|_{L^2((0,T); H^{-1}(M))} \leq C_{\nu,\|q\|_{L^\infty}} \|f_\nu\|_{H^1((0,T) \times \partial M)}.
\]
In particular, we obtain:
\[
\|u_\nu\|_{L^2((0,T) \times M)} + \|\partial_t u_\nu\|_{L^2((0,T) \times M)} + \|\partial_t^2 u_\nu\|_{L^2((0,T); H^{-1}(M))} \leq C_{\nu,\|q\|_{L^\infty}} \|f_\nu\|_{H^1((0,T) \times \partial M)}.
\]

We next observe that
\[
\begin{align*}
\Delta_g u_\nu &= - \partial_t^2 u_\nu - \left(\frac{\nu^2}{4} + q\right) u_\nu - \nu \partial_t u_\nu =: \psi_\nu, \quad \text{in } M, \\
u_\nu &= f_\nu, \quad \text{on } \partial M.
\end{align*}
\]
Then, by elliptic regularity and since \( \partial_t^2 u_\nu, \partial_t u_\nu \in L^2((0,T); H^{-1}(M)) \), there is \( C > 0 \) depending only on \((M,g)\) so that
\[
\|u_\nu(t, \cdot)\|_{H^1(M)} \leq C \left( \|f_\nu(t, \cdot)\|_{H^1(\partial M)} + \|\psi_\nu(t, \cdot)\|_{H^{-1}(M)} \right).
\]
Therefore:
\[
\|u_\nu\|_{L^2((0,T) \times M)} + \|\partial_t u_\nu\|_{L^2((0,T) \times M)} + \|\nabla^g u_\nu\|_{L^2((0,T) \times M)} \leq C_{\nu,\|q\|_{L^\infty}} \|f_\nu\|_{H^1((0,T) \times \partial M)}. \tag{4.17}
\]

Next, multiplying equation (4.13) by \( \langle N, \bar{u}_\nu \rangle = d\bar{u}(N) \) and applying the same reasoning as we did above for \( u_\nu \) and equation (4.12), it is direct to obtain the bound
\[
\int_0^T \int_{\partial M} |\partial_t u_\nu(t)|^2 \nu_{\partial M} dt \leq C_{\nu,\|q\|_{L^\infty}} \|f_\nu\|_{H^1((0,T) \times \partial M)}^2. \tag{4.18}
\]
This concludes the proof. \( \square \)

Given \( T > 0 \), let \( b \in H^2(\partial SM) \) with \( b|_{\partial^+ SM(T)} = 0 \). Let \( a(t, \bar{x}) \) be a solution to the lifted transport equation defined by (4.6). We extend \( a \) to \( \mathbb{R}_+ \times \tilde{M} \) by zero and denote
\[
\|a\|_* := \|a\|_{e^t W^{1,1}(\mathbb{R}_+; H^2(\tilde{M}))} + \|a\|_{e^t W^{1,1}(\mathbb{R}_+; L^2(\tilde{M}))}. \tag{4.19}
\]
For \( y \in \partial M_e \), we use the function \( \psi_y = d_y(\bar{y}, \cdot) \) on \( \tilde{M}_e \) defined above and define, for any \( \lambda > 0 \),
\[
G_\lambda(t, x) := \sum_{\gamma \in \pi_1(\tilde{M})} a(t, \gamma(x)) e^{i\lambda \psi_y(\gamma(x))}. \tag{4.20}
\]
The sum is locally finite and this function is \( \pi_1(M) \) invariant, and thus descends to \( M_e \).
Lemma 4.4. Let \( q \in L^\infty(M) \) and \( \nu_0 \geq 0 \) be defined in Lemma 4.3. For any \( \lambda > 0 \) and \( T > 0 \) the equation

\[
\begin{aligned}
&(\partial_t^2 + \Delta_g + q(x))u = 0, \quad \text{in } \mathbb{R}_+ \times M, \\
u(0, x) = \partial_t u(0, x) = 0, \quad x \in M,
\end{aligned}
\]

has a solution of the form

\[
u(t, x) = G_\lambda(t, x) + v_\lambda(t, x),
\]

such that, for every \( \nu > \nu_0 \),

\[
u \in e^{\nu t}H^1(\mathbb{R}_+; L^2(M)) \cap e^{\nu t}L^2(\mathbb{R}_+; H^1(M)),
\]

and where \( v_\lambda(t, x) \) satisfies

\[
\begin{aligned}
v_\lambda(t, x) &= 0, \quad \forall (t, x) \in \mathbb{R}_+ \times \partial M, \\
v_\lambda(0, x) &= 0, \quad \partial_t v_\lambda(0, x) = 0, \quad x \in M,
\end{aligned}
\]

and for all \( \epsilon > 0 \), there is \( C > 0 \) depending only on \( M, \nu, \epsilon \) so that

\[
\lambda \|v_\lambda\|_{e^{\nu t}L^2(\mathbb{R}_+ \times M)} + \|\partial_t v_\lambda\|_{e^{\nu t}L^2(\mathbb{R}_+ \times M)} + \|\nabla_x v_\lambda\|_{e^{\nu t}L^2(\mathbb{R}_+ \times M)} \leq Ce^{(h+\epsilon)T}\|a\|_*,
\]

(4.21)

where \( h = h(M, g) \) denotes the volume entropy. The result remains valid if one replaces the initial condition \( u(0, x) = \partial_t u(0, x) = 0 \) by the final condition \( u(t, x) = \partial_t u(t, x) = 0 \) for \( t \geq T \). In this case, \( v_\lambda \) satisfies

\[
v_\lambda(t, x) = 0, \quad \partial_t v_\lambda(t, x) = 0, \quad t \geq T, \quad x \in M.
\]

Proof. We prove the Lemma with initial condition \( u(0, \cdot) = \partial_t u(0, \cdot) = 0 \). The proof for \( u(T, \cdot) = \partial_t u(T, \cdot) = 0 \) is analogous. As in [BDSF11, Proof of Lemma 4.1], we consider

\[
k(t, x) = - \sum_{\gamma \in \pi_1(M)} (\partial_t^2 + \Delta_g + \bar{q}) (a(t, \gamma(x)) e^{i\lambda(\psi_g(\gamma(x)) - t)})
\]

Let \( v_\lambda \) be the solution, given by Lemma 4.3, to the boundary value problem

\[
\begin{aligned}
&(\partial_t^2 + \Delta_g + q) v_\lambda(t, x) = k(t, x) \quad \text{in } \mathbb{R}_+ \times M, \\
v_\lambda(0, x) = \partial_t v_\lambda(0, x) = 0, \quad \text{in } M, \\
v_\lambda(t, x) = 0 \quad \text{on } \mathbb{R}_+ \times \partial M.
\end{aligned}
\]

(4.22)

To prove the claim it is sufficient to show that \( v_\lambda \) satisfies the estimate (4.21). A computation yields

\[
-k(t, x) = \sum_{\gamma \in \pi_1(M)} e^{i\lambda(\psi_g(\gamma(x)) - t)} (\partial_t^2 + \Delta_g + \bar{q}) a(t, \gamma(x))
\]

\[-2i\lambda \sum_{\gamma \in \pi_1(M)} e^{i\lambda(\psi_g(\gamma(x)) - t)} \left( \partial_t a + da(\nabla_{\tilde{g}} \psi_{\tilde{g}}) - \frac{a}{2} \Delta_{\tilde{g}} \psi_{\tilde{g}} \right) (t, \gamma(x))
\]

\[-\lambda^2 \sum_{\gamma \in \pi_1(M)} a(t, \gamma(x)) e^{i\lambda(\psi_g(\gamma(x)) - t)} (1 - |\nabla_{\tilde{g}} \psi_{\tilde{g}}|^2).
\]
Using that $a$ solves the transport equation and $\psi_\bar{y}$ solves the eikonal equation, we obtain

$$-k(t, x) = \sum_{\gamma \in \pi_1(M)} e^{i\lambda(\psi_{\bar{y}}(\gamma(x)) - t)}((\partial_t^2 + \Delta_{\bar{y}} + \bar{q})a)(t, \gamma(x))$$

$$= \sum_{\gamma \in \pi_1(M)} e^{i\lambda(\psi_{\bar{y}}(\gamma(x)) - t)}k_0(t, \gamma(x)).$$

Notice that $k_0 \in H^1_0([0, T]; L^2(\tilde{M}))$. Extending $k_0$ to $I$ by zero, we obtain

$$\|k_0\|_{e^{\nu t}L^1(I; L^2(\tilde{M}))} + \|\partial_t k_0\|_{e^{\nu t}L^1(I; L^2(\tilde{M}))} \leq C\|a\|_*.$$

Moreover, the function

$$w_\lambda(t, x) := \int_0^t v_\lambda(s, x)ds$$

solves the mixed hyperbolic problem (4.22) with right-hand side $k_1(t, x) = \int_0^t k(s, x)ds$. For $t \geq T$, this is equal to

$$k_1(t, x) = \frac{1}{i\lambda} \sum_{\gamma \in \pi_1(M)} \int_0^t k_0(s, \gamma(x))\partial_s(e^{i\lambda(\psi_{\bar{y}}(\gamma(x)) - s)})ds$$

$$= -\frac{1}{i\lambda} \sum_{\gamma \in \pi_1(M)} \int_0^t \partial_s k_0(s, \gamma(x))e^{i\lambda(\psi_{\bar{y}}(\gamma(x)) - s)}ds.$$

Then, by Lemma 4.3, we obtain for $\nu_0$ defined in Lemma 4.3

$$\|v_\lambda(t, \cdot)\|_{L^2(M)} = \|\partial_t w_\lambda(t, \cdot)\|_{L^2(M)} \leq C \int_0^t e^{\nu_0(t-s)}\|k_1(s, \cdot)\|_{L^2(M)}ds$$

$$\leq \frac{C}{\lambda} \sum_{\gamma \in \pi_1(M)} \int_0^t e^{\nu_0(t-s)}\int_s^a \|\partial_r k_0(r, \cdot)\|_{L^2(\gamma(F))}drds$$

$$\leq \frac{CN(T)}{\lambda} \int_0^t e^{\nu_0(t-s)}\int_0^s \|\partial_r k_0(r, \cdot)\|_{L^2(\tilde{M})}drds.$$
Therefore, using that $N(T) \leq C e^{(h+\epsilon)T}$ and the Minkowski integral inequality, we obtain for $\nu > \nu_0$
\[
\|v_\lambda\|_{e^{\nu t}L^2(I \times M)} \leq \frac{C e^{(h+\epsilon)T}}{\lambda} \left( \int_0^\infty \left( \int_0^t \int_s^\infty e^{-(\nu-\nu_0)t} e^{-\nu_0 s} \|\partial_r k_0(r, \cdot)\|_{L^2(\tilde{M})} dr ds \right)^2 dt \right)^{\frac{1}{2}}
\leq \frac{C e^{(h+\epsilon)T}}{\lambda} \int_0^\infty \int_r^\infty \left( \int_s^\infty e^{-2(\nu-\nu_0)(t-s)} e^{-2\nu_0 s} \|\partial_r k_0(r, \cdot)\|_{L^2(\tilde{M})} dt \right)^{\frac{1}{2}} ds dr
\leq \frac{C e^{(h+\epsilon)T}}{\lambda} \int_0^\infty e^{-\nu t} \|\partial_r k_0(r, \cdot)\|_{L^2(\tilde{M})} dr
\leq \frac{C e^{(h+\epsilon)T}}{\lambda} \|a\|_*\]
for some constants $C$ independent of $(T, \lambda)$ (but depending on $\nu - \nu_0$). On the other hand, using again Lemma 4.3 for $v_\lambda$, we obtain
\[
\|\partial_t v_\lambda(t, \cdot)\|_{L^2(M)} + \|\nabla^q v_\lambda(t, \cdot)\|_{L^2(M)} \leq C \sum_{\gamma \in \pi_1(M)} \int_0^t e^{\nu(t-s)} \|k_0(s, \cdot)\|_{L^2(\gamma(F))} ds
\leq C e^{(h+\epsilon)T} \int_0^t e^{\nu(t-s)} \|k_0(s, \cdot)\|_{L^2(\tilde{M})} ds.
\]
Repeating the previous argument, we get:
\[
\|\partial_t v_\lambda\|_{e^{\nu t}L^2(\mathbb{R} \times M)} + \|\nabla^q v_\lambda(t, \cdot)\|_{e^{\nu t}L^2(\mathbb{R} \times M)} \leq C e^{(h+\epsilon)T} \|a\|_*.
\]

5. Stable determination of the electrical potential for the Schrödinger equation

In this section we prove Theorem 1 about the stability estimate for the Schrödinger equation. The main idea relies in taking geometric optics solutions with initial data $b \in H^2(\partial_+ SM_e)$ with $b|_{\mathcal{T}^e_{\partial SM}(T_0)} = 0$ where $T_0 > 0$ is taken large. These solutions are concentrating on geodesics with endpoints on the boundary and with length at most $T_0$; when $T_0 \to \infty$, these geodesics will cover a set of full measure in $M$. Using these solutions, one can control the $L^2$ to norm of $\Pi_0^e q$ by the difference of the DN maps for $q_1$ and $q_2$, with a remainder term coming from $L^2$ estimates of the X-ray transform of $q$ and $\Pi_0^e q$ on $\mathcal{T}^e_{\partial SM}(T_0)$; here $\Pi_0^e = I_0^e I_0^e$ is the normal operator associated to the X-ray transform on $M_e$ introduced in (3.3). As we shall see, using the estimate (3.4), this is enough to obtain our stability estimate since, due to hyperbolicity of the trapped set, the volume of $\mathcal{T}_+(T_0)$ decays exponentially as $T_0 \to \infty$ and this remainder term can be absorbed in the stability estimate for $T_0$ sufficiently large.
5.1. Preliminary estimates. We first reformulate Lemmas 5.1 and 5.2 from [BDSF10] in our setting. Let \( q = q_1 - q_2 \) extended to \( M_e \) by \( q = 0 \) in \( M_e \setminus M \). Recall that \( \Lambda_g^S \) is the Dirichlet-to-Neumann map associated with the Schrödinger equation (1.1).

**Lemma 5.1.** Let \( q_1, q_2 \in W^{1,\infty}(M) \) with \( q_1|_{\partial M} = q_2|_{\partial M} \) and set \( q := q_1 - q_2 \). There exists \( C > 0 \) depending only on \( (g, g, \|q\|_{W^{1,\infty}}) \) and \( C_0 > 0 \) depending only on \( (\partial_g \psi_y) \) satisfying the transport equation (4.4) on \( \tilde{M} \) (for the same solution \( \psi_y \) to the eikonal equation \( |\nabla_{\tilde{g}} \psi_y| = 1 \)) with condition \( a_j|_{\partial_{-}S \tilde{M}} = \pi^* b_j \) for \( b_1, b_2 \in H^2(\partial_{-}SM) \) satisfying \( b_1|_{\Gamma^{\partial_{-}SM}(T_0)} = b_2|_{\Gamma^{\partial_{-}SM}(T_0)} = 0 \), the following estimate holds true:

\[
\left| \int_0^T \int_M \tilde{q}(x) a_1(2\lambda t, x) a_2(2\lambda t, x) \, dv_{\tilde{g}}(x) \, dt \right| \leq C e^{C_0 T_0} (\lambda^{-2} + \|\Lambda_g^S - \Lambda_{g, q_1}^S\|_{\ast}) \|a_1\|_{\ast} \|a_2\|_{\ast},
\]

for any \( \lambda \geq \frac{T_0}{2T} \), where \( \tilde{q} = \pi^* q \in W^{1,\infty}(\tilde{M}) \) is the lift of \( q \).

**Proof.** By Lemma 4.2, one can construct \( a_2, \psi_{\tilde{g}} \), such that for \( \lambda \geq T_0/2T \) and \( G_{2, \lambda} \) defined as in (4.7), the solution

\[
u_2(t, x) = G_{2, \lambda}(t, x) + v_{2, \lambda}(t, x)
\]
to the Schrödinger equation corresponding to the potential \( q_2 \),

\[
\begin{cases}
(i\partial_t - \Delta_g + q_2)u(t, x) = 0, \quad \text{in } (0, T) \times M, \\
u(0, \cdot) = 0, \quad \text{in } M,
\end{cases}
\]
satisfies \( v_{2, \lambda}(t, x) = 0 \) for all \( (t, x) \in (0, T) \times \partial M \), and

\[
\lambda \|v_{2, \lambda}(t, \cdot)\|_{L^2(M)} + \|\nabla v_{2, \lambda}(t, \cdot)\|_{L^2(M)} \leq C e^{(h+c)T_0} \|\tilde{a}_2\|_{\ast}. \tag{5.1}
\]

Moreover, \( u_2 \in C^1([0, T]; L^2(M)) \cap C([0, T]; H^2(M)) \). We next denote by \( f_\lambda \) the restriction of \( G_{2, \lambda} \) on \( (0, T] \times \partial M \)

\[
f_\lambda(t, x) := G_{2, \lambda}(t, x) = \sum_{\gamma \in \pi_1(M)} a_2(2\lambda t, \gamma(x)) e^{i\lambda(\psi(\gamma(x)) - \lambda t)}.
\]

Let \( v \) be the solution to the non-homogeneous boundary value problem

\[
\begin{cases}
(i\partial_t - \Delta_g + q_1)v(t, x) = 0, \quad (t, x) \in (0, T) \times M, \\
v(0, x) = 0, \quad x \in M, \\
v(t, x) = u_2(t, x) = f_\lambda(t, x), \quad (t, x) \in (0, T) \times \partial M,
\end{cases}
\]

and denote \( w = v - u_2 \). Notice that \( w \) solves the following homogeneous boundary value problem for the Schrödinger equation:

\[
\begin{cases}
(i\partial_t - \Delta_g + q_1)w(t, x) = q(x)u_2(t, x), \quad (t, x) \in (0, T) \times M, \\
w(0, x) = 0, \quad x \in M, \\
w(t, x) = 0, \quad (t, x) \in (0, T) \times \partial M.
\end{cases}
\]

Since \( q(x)u_2 \in W^{1,1}([0, T]; L^2(M)) \) with \( u_2(0, \cdot) \equiv 0 \), by Lemma 4.1 we obtain that \( w \in C^1([0, T]; L^2(M)) \cap C([0, T]; H^2(M) \cap H^1_0(M)) \).
On the other hand, using Lemma 4.2, we construct a special solution \( u_1 \in C^1([0, T]; L^2(M)) \cap C([0, T]; H^2(M)) \) to the backward Schrödinger equation
\[
\begin{cases}
(i\partial_t - \Delta_g + q_1(x))u_1(t, x) = 0, \quad (t, x) \in (0, T) \times M, \\
u_1(T, x) = 0, \quad x \in M,
\end{cases}
\]
having the special form
\[
u_1(t, x) = \sum_{\gamma \in \pi_1(M)} a_1(2\lambda t, \gamma(x))e^{i\lambda(\psi_g(\gamma(x)) - \lambda t)} + \nu_{1,\lambda}(t, x),
\]
which corresponds to the electric potential \( \tilde{q}_1 \), where \( \nu_{1,\lambda} \) vanishes on \( (0, T) \times \partial M \), satisfies \( \nu_{1,\lambda}(T, \cdot) = 0 \), and
\[
\forall t \in [0, T], \quad \lambda\|v_{1,\lambda}(t, \cdot)\|_{L^2(M)} + \|\nabla^g v_{1,\lambda}(t, \cdot)\|_{L^2(M)} \leq Ce^{(h+\varepsilon)T_0}\|a_1\|_*. \tag{5.2}
\]
By integration by parts and Green’s formula, we obtain
\[
\int_0^T \int_M (i\partial_t - \Delta_g + q_1)w\nu_1 \, dv_g \, dt = \int_0^T \int_M qu_2\tilde{\nu}_1 \, dv_g \, dt = -\int_0^T \int_{\partial M} \partial_n w\tilde{\nu}_1 \, dv_{g|\partial M} \, dt.
\]
We then obtain
\[
\int_0^T \int_M qu_2\tilde{\nu}_1 \, dv_g \, dt = -\int_0^T \int_{\partial M} (\Lambda^S_{g, q_1} - \Lambda^S_{g, q_2})(f, \xi)(t, x)\tilde{\nu}_1(t, x) \, dv_{g|\partial M}(x) \, dt,
\]
where the boundary data \( g_\lambda \) is given by
\[
g_\lambda(t, x) := \sum_{\gamma \in \pi_1(M)} a_1(2\lambda t, \gamma(x))e^{i\lambda(\psi_g(\gamma(x)) - \lambda t)}, \quad (t, x) \in (0, T) \times \partial M.
\]
Using the definition of \( u_1 \) and \( u_2 \), we get
\[
\sum_{\gamma_1, \gamma_2 \in \pi_1(M)} \int_0^T \int_{\mathcal{F}} \tilde{q}(x)a_2(2\lambda t, \gamma_1(x))\overline{a_1(2\lambda t, \gamma_2(x))}e^{i\lambda(\psi_g(\gamma_1(x)) - \psi_g(\gamma_2(x)))} \, dv_g(x) \, dt
\]
\[
= -\int_0^T \int_{\partial M} g_\lambda(\Lambda^S_{g, q_1} - \Lambda^S_{g, q_2})(f, \xi) \, dv_{g|\partial M} \, dt - \int_0^T \int_M qv_{2,\lambda}\tilde{\nu}_{1,\lambda} \, dv_g \, dt
\]
\[
- \int_0^T \int_M qf\tilde{\nu}_{1,\lambda} \, dv_g \, dt - \int_0^T \int_M qv_{2,\lambda}\tilde{\nu}_1 \, dv_g \, dt.
\]
By (5.2), there is \( C > 0 \) depending on \( ||q||_{L^\infty}, T \) and \( \varepsilon > 0 \) so that
\[
\left| \int_0^T \int_M qf\tilde{\nu}_{1,\lambda} \, dv_g \, dt \right| \leq ||q||_{L^\infty} \sum_{\gamma \in \pi_1(M)} \int_0^T \|a_2(2\lambda t, \cdot)\|_{L^2(\gamma(x))} \|v_{1,\lambda}(t, \cdot)\|_{L^2(M)} \, dt
\]
\[
\leq Ce^{2(h+\varepsilon)T_0} \lambda^{-2}\|a_2\|_*\|a_1\|_*.
\]
where \( \| \cdot \|_* = \| \cdot \|_{H^1([0,T_0]; H^2(\tilde{M}))} \) as before, and the second \( \lambda^{-1} \) comes from the change of variables in \( t \). With the same argument, using (5.1) and (5.2),
\[
\left| \int_0^T \int_M qv_{2,\lambda}\tilde{\nu}_1 \, dv_g \, dt \right| + \left| \int_0^T \int_M qv_{2,\lambda}\tilde{\nu}_{1,\lambda} \, dv_g \, dt \right| \leq Ce^{2(h+\varepsilon)T_0} \lambda^{-2}\|a_2\|_*\|a_1\|_*.
Finally, using the trace theorem, we get
\[
\left| \int_0^T \int_{\partial M} ((\Lambda_{g,q_1}^S - \Lambda_{g,q_2}^S) f_\lambda) \bar{\gamma}_\lambda \, dv_{g|_{\partial M}} \, dt \right| \leq C \| \Lambda_{g,q_1}^S - \Lambda_{g,q_2}^S \|_* \| f_\lambda \|_{H^1((0,T) \times \partial M)} \| g_\lambda \|_{L^2((0,T) \times \partial M)}
\]
\[
\leq C e^{\varepsilon T_0} \| a_1 \|_* \| a_2 \|_* \| \Lambda_{g,q_1}^S - \Lambda_{g,q_2}^S \|_* .
\]
The diagonal term gives, using that \( \gamma_1, \gamma_2 \) are isometries of \( \tilde{g} \) and preserve \( dv_{\tilde{g}} \),
\[
\sum_{\gamma \in \pi_1(M)} \int_0^T \int_F \tilde{q}(x) a_2(2\lambda t, \gamma(x)) a_1(2\lambda t, \gamma(x)) \, dv_{\tilde{g}}(x) \, dt
\]
\[
= \int_0^T \int_M \tilde{q}(x) a_2(2\lambda t, x) a_1(2\lambda t, x) \, dv_{\tilde{g}}(x) \, dt.
\]
The last point consists in bounding the off-diagonal terms
\[
\left| \sum_{\gamma_1 \neq \gamma_2 \in \pi_1(M)} \int_0^T \int_F \tilde{q}(x) a_2(2\lambda t, \gamma_1(x)) a_1(2\lambda t, \gamma_2(x)) e^{i\lambda \psi(x)} e^{i\lambda \psi(y)} dv_{\tilde{g}}(x) \, dt \right|
\]
\[
= \left| \sum_{\gamma \neq Id \in \pi_1(M)} \int_0^T \int_M \tilde{q}(x) a_2(2\lambda t, x) a_1(2\lambda t, x) e^{i\lambda \psi(x)} e^{i\lambda \psi(y)} dv_{\tilde{g}}(x) \, dt \right| .
\]
We will apply non stationary phase: to that aim, we need to bound below the norm
\[
|\nabla \tilde{g}(\psi(x)) - \psi(\gamma(x))| = |\nabla \tilde{g} d\psi(\cdot, \tilde{y}) - (d\gamma)^{-1} \nabla \tilde{g} d\psi(\cdot, \tilde{y}) \circ \gamma|.
\]
This corresponds to bounding below the distance between two vectors \( v_1 = d\pi \nabla \tilde{g} \psi_1(x) \) and \( v_2 = d\pi (d\gamma)^{-1} (\nabla \tilde{g} \psi_2)(\gamma(x)) \in T_y M_\gamma \) tangent to two geodesics \( \alpha_1 \) and \( \alpha_2 \) of length \( \leq T_0 \) in \( M_\gamma \), starting at \( y \in \partial M_\gamma \). Then there is a homotopy \( h : [0,1] \times M_\gamma \to M_\gamma \) such that \( h(0, t) = \alpha_1(t) \) and \( h(1, t) = \alpha_2(t) \). However by a standard estimate on flows of smooth vector fields, there is \( C_0 > 0 \) independent of \( y, x \in M_\gamma \) \( (C_0 \) depends on the \( C^1 \) norm of the geodesic vector field \( X_{g_0}) \) such that
\[
d_{g}(\alpha_2([0,1]), \alpha_1([0,1])) \leq |v_1 - v_2|_{\bar{g}} e^{C_0 T_0} .
\]
We then deduce that \( |v_1 - v_2|_{\bar{g}} \geq \delta \) and therefore we can apply one integration by parts in the second line of (5.3) and the usual change of coordinates \( t = s/\lambda \) to obtain
\[
\left| \sum_{\gamma \in \pi_1(M) \setminus \{ Id \}} \int_0^T \int_M \tilde{q}(x) a_2(2\lambda t, x) a_1(2\lambda t, \gamma(x)) e^{i\lambda \psi(x) - \psi(y(x))} dv_{\tilde{g}}(x) \, dt \right|
\]
\[
\leq \lambda^{-2} C e^{(2\varepsilon - C_0 T_0)} \| a_2 \|_* \| a_1 \|_*
\]
where \( C > 0 \) now depends on the \( \| \cdot \|_{W^{1,\infty}(\bar{M})} \) norm instead of \( \| \cdot \|_{L^\infty} \).
Notice that $C_0 > 0$ above depends only the maximal expansion rate of the flow defined as the smallest constant $\theta > 0$ such that for each $\epsilon > 0$ there is $C > 0$ such that for all $t$ large enough

$$\|d\varphi_t\| \leq C e^{(\theta+\epsilon)t}.$$ (5.4)

Next, we show the following Lemma which is key to relate the X-ray transform of $q$ to the DN map of the Schrödinger equation.

**Lemma 5.2.** Let $q_1, q_2 \in W^{1,\infty}(M)$ with $q_1|_{\partial M} = q_2|_{\partial M}$ and set $q := q_1 - q_2$. There is $C_0 > 0$ depending only on $(M, g)$ and $C > 0$ depending on $(M, g, \|q_1\|_{W^{1,\infty}}, \|q_2\|_{W^{1,\infty}})$ such that for any $T_0 > 0$ and any $b \in H^2(\partial SM_e)$ such that $b|_{\tau^+_T(SM_e(T_0))} = 0$, the following estimate

$$\left| \int_{\partial \tilde{g}M_e} \int_0^{\tau^+_T(y,v)} q(\exp_y(sv))b(y,v)\mu(y,v)ds \, d\omega_y(v) \right| \leq C e^{C_0 T_0}\|\Lambda_{g,q_1}^S - \Lambda_{g,q_2}^S\|_2^{1/2}\|b(\cdot, \cdot)\|_{H^2(\partial \tilde{g}M_e)},$$

holds uniformly for any $y \in \partial M_e$, where $\mu(y,v) = g(n_y,v)$, with $n_y$ the inward unit normal of $\partial M_e$ at $y$.

**Proof.** We take two solutions $a_1, a_2$ to the transport equation on the universal cover $\tilde{M}_e$ defined as before by

$$a_1(t,x) = \alpha^{-1/4}\phi(t-r(x))\tilde{b}(\tilde{y}(t,x)), \quad a_2(t,x) = \alpha^{-1/4}\phi(t-r(x))\tilde{\mu}(\tilde{y}(t,x))\chi_{T_0}(\tau^+_T(y,v(x)))$$

where $r(x) = d_{\tilde{y}}(x,\tilde{y})$, $\exp_{\tilde{y}}(r(x)v(x)) = x$, $\chi_{T_0} \in C_0^\infty(\mathbb{R})$ is supported in $[0, T_0 + 1]$ and equal to 1 in $[0, T_0]$ and $\tilde{\mu}, \tilde{b}$ are lifts of $\mu, b$ to $SM_e$ as before. Here we have used the natural identification $S_{\tilde{g}}M_e \simeq S_gM_e$ to define $\tau^+_T(y,v(x))$. We write using geodesic polar coordinates $x = \exp_{\tilde{y}}(rv)$ with $v \in \partial \tilde{g}M_e$

$$\int_0^{T} \int_{\tilde{M}} \tilde{q}(x)a_1(2\lambda t, x)a_2(2\lambda t, x) \, dv_g(x)dt$$

$$= \int_0^{T} \int_{\tilde{g}M_e} \int_0^{\tau^+_T(y,v)} \tilde{q}(r,v) a_1(2\lambda t, r,v) a_2(2\lambda t, r,v) \alpha^{1/2} dr d\omega_{\tilde{y}}(v) dt$$

$$= \int_0^{T} \int_{\tilde{g}M_e} \int_0^{\tau^+_T(y,v)} q(\exp_{y}(rv))\phi^2(2\lambda t - r)b(y,v)\mu(y,v)dr d\omega_y(v) dt$$

$$= \frac{1}{2}\lambda \int_0^{2\lambda T} \int_{\tilde{g}M_e} \int_0^{\tau^+_T(y,v)} q(\exp_{y}(rv))\phi^2(t - r)b(y,v)\mu(y,\theta)dr d\omega_y(v) dt.$$ 

By Lemma 5.1, we obtain the bound (using that $r \leq T_0 \leq 2\lambda T$ by our assumption)

$$\left| \int_0^{\infty} \int_{\tilde{g}M_e} \int_0^{\tau^+_T(y,v)} q(\exp_{y}(rv))\phi^2(t - r)b(y,v)\mu(y,v)dr d\omega_y(v) dt \right| \leq C e^{C_0 T_0}\left(\lambda^{-1} + \lambda\|\Lambda_{g,q_1}^S - \Lambda_{g,q_2}^S\|_2\|\phi\|_{L^2(\mathbb{R})}\|b(\cdot, \cdot)\|_{H^2(S_m^M)}\right).$$
Moreover, using the properties of the function $\phi$, we also have
\[
\int_0^\infty \int_{\partial S_p M_\varepsilon} \int_0^{\tau_+^{(y,v)}} q(\exp_y(rv)) \phi(\gamma - r) b(y,v) \mu(y,v) \, dr \, d\omega_y(v) dt \\
= \left( \int_{-\infty}^{\infty} \phi^2(t) dt \right) \int_{\partial S_p M_\varepsilon} \int_0^{\tau_+^{(y,v)}} q(\exp_y(rv)) b(y,v) \mu(y,v) \, dr \, d\omega_y(v).
\]
Finally, to prove the Lemma it suffices to take
\[
\lambda = \frac{T_0}{2T} \left( \frac{2\delta(N_0)}{\| \Lambda_{g,q_1}^S - \Lambda_{g,q_2}^S \|_*} \right)^{1/2},
\]
where $\delta(N_0) := \sup_{q \in Q(N_0)} \| \Lambda_{g,q}^S \|_*$ is finite by [BDSF10, Thm. 1].

5.2. Proof of the stability estimate. Using Lemma 5.2 we have, for any $y \in \partial M_\varepsilon$ and $b \in H^2(\partial S_p M_\varepsilon)$ such that $b|_{\tau^{ASM}(T_0)} = 0$,
\[
\left| \int_{\partial S_p M_\varepsilon} \int_0^{\tau_+^{(y,v)}} q(\exp_y(sw)) ds b(y,v) \mu(y,v) \, d\omega_y(v) \right| \\
\leq C e^{C_0 T_0} \| \Lambda_{g,q_1}^S - \Lambda_{g,q_2}^S \|_*^{1/2} \| b(y, \cdot) \|_{H^2(\partial S_p M_\varepsilon)},
\]
where $C, C_0$ are uniform in $y, T_0$. Now we take bump function $\chi_{T_0} \in C^\infty_c(\mathbb{R})$ supported in the interval $[0,T_0]$ and equal to 1 on $[0, T_0 - 1]$, for $T_0 \gg 1$, and set
\[
b(y,v) := \chi_{T_0}(\tau_+^{(y,v)}) I_0^c(\Pi_0^c q)(y,v). \tag{5.5}\]
Since $\Pi_0^c$ is a pseudo-differential operator of order $-1$ on $M_\varepsilon$ ([Gui17, Prop. 5.7]) and $q = 0$ in $M_\varepsilon \setminus M$, $\Pi_0^c q \in W^{2,p}(M_\varepsilon)$ for all $p < \infty$. Integrating with respect to $y \in \partial M_\varepsilon$, we obtain
\[
\left| \int_{\partial S_p M_\varepsilon} I_0^c(q)(y,v) I_0^c(\Pi_0^c q)(y,v) d\mu_n(y,v) \right| \\
\leq C e^{C_0 T_0} \| \Lambda_{g,q_1}^S - \Lambda_{g,q_2}^S \|_*^{1/2} \| b \|_{H^2(\partial S_p M_\varepsilon)} + \left| \int_{\tau^{ASM}(T_0)} I_0^c(q)(y,v) I_0^c(\Pi_0^c q)(y,v) d\mu_n(y,v) \right|.
\tag{5.6}\]
Moreover, we can write
\[
I_0^c(\Pi_0^c q)(y,v) = \int_0^{\tau_+^{(y,v)}} \pi_0^c(\Pi_0^c q) \circ \varphi_t(y,v) dt, \quad (y,v) \in \partial S_p M_\varepsilon \setminus \Gamma_. \tag{5.7}\]
By Cauchy-Schwarz inequality,
\[
\left| \int_{\tau^{ASM}(T_0)} I_0^c(q)(y,\theta) I_0^c(\Pi_0^c q)(y,\theta) d\mu_n(y,\theta) \right| \\
\leq \left( \int_{\tau^{ASM}(T_0)} |I_0^c(q)(y,\theta)|^2 d\mu_n(y,\theta) \right)^{1/2} \left( \int_{\tau^{ASM}(T_0)} |I_0^c(\Pi_0^c q)(y,\theta)|^2 d\mu_n(y,\theta) \right)^{1/2}.
\]
Since \( q \in W^{1,\infty}(M) \subset H^1(M) \), then by [Gui17, Prop. 5.7], \( \Pi_0^* q \in H^2(M) \). By Sobolev embedding theorem, we also have that
\[
\pi_0^* q, \quad \pi_0^* \Pi_0^* q \in L^p(SM),
\]
for some \( p > 2 \). Let us now give an estimate on the \( L^2 \)-norm of the X-ray transform of an \( L^p \)-function in \( T_+^{\partial SM}(T_0) \):

**Lemma 5.3.** Let \( Q < 0 \) be the escape rate defined by (2.2). Let \( p \in (2, \infty) \), then there exists \( C = C(Q, p, \dim M) > 0 \) such that for all \( f \in L^p(M) \) and \( T_0 \gg 1 \) large,
\[
\int_{T_+^{\partial SM}(T_0)} |I_0^e f(y, v)|^2 d\mu_n(y, v) \leq C e^{2Q T_0} \| f \|_{L^p(M)}^2.
\]

**Proof.** We follow the argument in [Gui17, Lemma 5.1]. Using Hölder inequality, with \( \frac{1}{p} + \frac{1}{p'} = 1 \) and \( \frac{p}{p'} \geq \frac{p-1}{2} > 1 \) and Santaló’s formula we have:
\[
\| I_0^e f \|_{L^2(T_+^{\partial SM}(T_0))}^2 = \int_{T_+^{\partial SM}(T_0)} \left| \int_0^{\tau_+^e(y,v)} \pi_0^* f(\varphi_t(y,v)) dt \right|^2 d\mu_n
\leq \int_{T_+^{\partial SM}(T_0)} \left( \int_0^{\tau_+^e(y,v)} |\pi_0^* f(\varphi_t(y,v))| dt \right)^{2p} \tau_+^e(y,v)^{2p'} d\mu_n
\leq \left( \int_{T_+^{\partial SM}(T_0)} \int_0^{\tau_+^e(y,v)} |\pi_0^* f(\varphi_t(y,v))| dt d\mu_n \right)^{2p} \| \tau_+^e \|_{L^{2p'/p'}(T_+^{\partial SM}(T_0))}^{2p/p'}
\leq C \| f \|_{L^p(M)}^2 \left( \int_{T_0-L-1}^\infty (t + L + 1)^{2p-1} V(t) dt \right)^{\frac{1}{2}}
\leq C \| f \|_{L^p(M)}^2 e^{2Q T_0/2}
\]
with \( C \) depending only on \( (Q, p, M) \) and \( L \) some fixed constant satisfying \( \tau_+ + L \geq \tau_+^e \) (thus depending only on \( M \)). In the third inequality, we have used [Gui17, eq. (4.13)] which states that for \( L > 0 \) as above and for all \( T \gg L \)
\[
\int_{\partial SM} 1_{[T, \infty)}(\tau_+^e) d\mu_v \leq 2 V(T - L - 1)
\]
and then the Cavalieri principle gives (by definition of \( V(t) \))
\[
\int_{T_+^{\partial SM}(T_0)} (\tau_+^e(x, v))^{2p'/p'} d\mu_v(x, v) \leq C \int_{T_0-L-1}^\infty (t + L + 1)^{2p-1} V(t) dt.
\]

Notice that as \( p \to +\infty \) we have \( r \to 1 \) and \( C(Q, p, \dim M) \) can be taken uniform in \( p \).

**Lemma 5.4.** There is \( C > 0 \) and \( \theta > 0 \) such that for all \( b \in H^2(\partial SM_e) \) given by (5.5),
\[
\| b \|_{H^2(\partial SM)} \leq C e^{\theta T_0} \| q \|_{W^{1,\infty}(M)},
\]

**Proof.** First, by the implicit function theorem, \( \tau_+^e : \partial SM_e \setminus T_+^{\partial SM}(T_0) \to \mathbb{R}_+ \) is a smooth function. We shall compute its \( C^2 \)-norm. Let \( \rho \) be a boundary defining function of \( M_e \) so
that \(|d\rho|_{\eta} = 1\) near \(\partial M_e\) and \(d(\pi_0^e \rho)(X) = -g(v, n)\) at \(\partial_i SM_e\). The function \(\tau_+^e\) is defined by the implicit equation

\[
\pi_0^e \rho(\varphi_{\tau_+^e(x,v)}(x, v)) = 0.
\]

Therefore, denoting \(S(x, v) := \varphi_{\tau_+^e(x,v)}(x, v)\) one has on \(\partial_- SM_e \setminus \Gamma_\partial\)

\[
d\tau_+^e(x, v) = \frac{d(\pi_0^e \rho)_{S(x, v)} \cdot d\varphi_{\tau_+^e(x,v)}(x, v)}{g(S(x, v), n)}.
\] (5.8)

From standard estimates on flows of autonomous \(C^2\)-vector fields, there is \(C, \theta > 0\) depending on \(|X|_{C^2}\) such that for all \(t \in \mathbb{R}\) for which the flow is defined,

\[
\|\varphi_t\|_{C^2} \leq Ce^{\theta|t|}.
\]

Using this, the fact that \(g(S(x, v), n) > c_0 > 0\) for some \(c_0\) if \(\tau_+(x, v) > 1\), we see from the expression (5.8) and its derivative that there is \(C > 0, \theta > 0\) independent of \(T_0\) such that

\[
\sup_{(x, v) \in \partial \tau_+(T_0), \tau_+(x, v) > 1} \|\nabla \tau_+^e(x, v)\| + \|\nabla^2 \tau_+^e(x, v)\| \leq Ce^{2\theta T_0},
\] (5.9)

where \(\nabla\) is any fixed Riemannian connection on \(\partial SM_e\) (for example that given by Sasaki metric). First, by (3.2) and Lemma 3.1, for each \(p > 2\) we have (using Sobolev embedding)

\[
\|b\|_{L^2(\partial SM_e)} \leq C\|\Pi_0^e q\|_{L^p(M_e)} \leq C\|q\|_{L^2(M)}.
\] (5.10)

Next we compute \(db\). Let \(f := \Pi_0^e q\). Since \(\text{supp}(q) \subset M\), one can use (3.6) to deduce that \(f \in W^{2,p}(M_e)\) for all \(p < \infty\) and that \(f \in C^\infty(M_e \setminus M)\). For \(z \in \partial SM_e\)

\[
\begin{align*}
db(z) &= d\tau_+^e(z)\chi'_{T_0}(\tau_+^e(z))(I_0^e f)(z) + \chi_{T_0}(\tau_+^e(z))\left(d\tau_+^e(z)\pi_0^e f(S(z)) + \int_0^{\tau_+^e(z)}d(\pi_0^e f).d\varphi_t(z)dt\right) \\
&\quad + 2\chi'_{T_0}(\tau_+^e(z))d\tau_+^e(z) \otimes \left(d\tau_+^e(z)\pi_0^e f(S(z)) + \int_0^{\tau_+^e(z)}d(\pi_0^e f).d\varphi_t(z)dt\right) \\
&\quad + \chi_{T_0}(\tau_+^e(z))\left(\nabla^2 \tau_+^e(z)\pi_0^e f(S(z)) + 2d\tau_+^e(z) \otimes (d(\pi_0^e f).dS(z))\right) \\
&\quad + \chi_{T_0}(\tau_+^e(z))\int_0^{\tau_+^e(z)}\nabla(\varphi^*_t d(\pi_0^e f))dt.
\end{align*}
\]

and therefore by (5.9) and (3.2), there is \(C > 0, \theta > 0\) independent of \(T_0, q\) such that

\[
\|db\|_{L^2} \leq CT_0 e^{\theta T_0}\|q\|_{W^{1,\infty}(M)}.
\] (5.11)

Finally, we compute another derivative of \(b\), and write

\[
\begin{align*}
\nabla^2 b(z) &= \nabla^2 \tau_+^e(z)\chi'_{T_0}(\tau_+^e(z))(I_0^e f)(z) + (d\tau_+^e \otimes d\tau_+^e(z)\chi'_{T_0}(\tau_+^e(z))(I_0^e f)(z) \\
&\quad + 2\chi'_{T_0}(\tau_+^e(z))d\tau_+^e(z) \otimes (d\tau_+^e(z)\pi_0^e f(S(z)) + \int_0^{\tau_+^e(z)}d(\pi_0^e f).d\varphi_t(z)dt) \\
&\quad + \chi_{T_0}(\tau_+^e(z))\left(\nabla^2 \tau_+^e(z)\pi_0^e f(S(z)) + 2d\tau_+^e(z) \otimes (d(\pi_0^e f).dS(z))\right) \\
&\quad + \chi_{T_0}(\tau_+^e(z))\int_0^{\tau_+^e(z)}\nabla(\varphi^*_t d(\pi_0^e f))dt.
\end{align*}
\]

Since \(\pi_0^e f\) is smooth near \(\partial SM_e\) and since the bounds (5.9) hold and \(I_0^e f \in L^2\), we obtain that there is \(C > 0, \theta > 0\) independent of \(T_0, q\) such that

\[
\|\nabla^2 b\|_{L^2} \leq CT_0 e^{2\theta T_0}\|q\|_{W^{1,\infty}(M)}.
\] (5.12)

Combining (5.12), (5.11) and (5.10), we get the desired result.\qed
Proof of Theorem 1. By (5.6), Lemma 5.3, Lemma 5.4, and using that \( \| \Pi_0^p q \|_{L^p(M_c)} \leq C\| q \|_{L^\infty(M)} \) for each \( p \in (2, \infty) \), we have that there is \( C_0 > 0 \) depending only on \((M, g)\) and \( C \) depending on \((M, g, \| q \|_{W^{1,\infty}})\) such that
\[
\int_{M_c} |\Pi_0^p(q)|^2 \, dv_g \leq Ce^{C_0 T_0} \| \Lambda_{g, q_1}^S - \Lambda_{g, q_2}^S \|_{*}^{1/2} + Ce^{\frac{Q}{2} T_0}.
\] (5.13)
Then, defining \( \alpha := e^{-\frac{Q}{2} T_0} \) and \( m := -\frac{2C_0}{Q} \), we deduce from (5.13) that
\[
\| \Pi_0^p q \|_{L^2(M_c)}^2 \leq C \left( \alpha^m \| \Lambda_{g, q_1}^S - \Lambda_{g, q_2}^S \|_{*}^{1/2} + \alpha^{-1} \right).
\]
We next take \( T_0 \) sufficiently large so that \( \alpha = \| \Lambda_{g, q_1}^S - \Lambda_{g, q_2}^S \|_{*}^{1/2} \). With this choice, we obtain
\[
\| \Pi_0^p q \|_{L^2(M_c)}^2 \leq C \| \Lambda_{g, q_1}^S - \Lambda_{g, q_2}^S \|_{*}^{1/2}.
\] (5.14)
This holds in the regime \( T_0 = \frac{2}{Q} \log \left( \| \Lambda_{g, q_1}^S - \Lambda_{g, q_2}^S \|_{*}^{1/2} \right) \).

Finally, by (3.4), there is \( C > 0 \) depending only on \((M, g)\) such that
\[
\| q \|_{L^2(M_c)} \leq C\| \Pi_0^p q \|_{H^1(M_c)}, \quad \| \Pi_0^p q \|_{H^2(M_c)} \leq C\| q \|_{H^1(M)}.
\] (5.15)
Using this, an interpolation estimate, (5.14), we obtain
\[
\| \Pi_0^p q \|_{H^1(M_c)}^2 \leq C\| \Pi_0^p q \|_{L^2(M_c)} \| \Pi_0^p q \|_{H^2(M_c)} \leq C \| \Lambda_{g, q_1}^S - \Lambda_{g, q_2}^S \|_{*}^{1/2} \| q \|_{H^1(M)}.
\]
Finally, by the first bound of (5.15), we conclude that for \( q_1, q_2 \in Q(N_0) \)
\[
\| q \|_{L^2(M)}^2 \leq C \| \Lambda_{g, q_1}^S - \Lambda_{g, q_2}^S \|_{*}^{1/2} N_0.
\]
This concludes the proof.

\[\square\]

6. Stable determination of the electrical potential for the wave equation

In this section we shall prove Theorem 3.

6.1. Preliminary estimates. We start with a Lemma very similar to Lemma 5.1 but now in the context of the wave equation. Its proof follows the lines of that of Lemma 5.1.

Lemma 6.1. Let \( q_1, q_2 \in W^{1,\infty}(M) \) with \( q_1|_{\partial M} = q_2|_{\partial M} \) and set \( q := q_1 - q_2 \). There exist \( C > 0 \) depending only on \((M, g, \| q \|_{W^{1,\infty}(M)})\), \( \nu > 0 \) depending only on \( \| q \|_{L^\infty} \) and \( C_0 > 0 \) depending only on \((M, g, \nu)\) such that for any \( T > 0, \lambda > 1 \), and for \( a_1, a_2 \in H^1_0((0, T], H^2(\widetilde{M})) \) the functions constructed in (4.6) with function \( b \) given respectively by \( b_1, b_2 \in H^2(\partial_{\Delta} SM) \) satisfying \( b_1|_{T^{\Delta}_{T^+}(T)} = b_2|_{T^{\Delta}_{T^+}(T)} = 0 \), the following estimate holds true:
\[
\left| \int_0^T \int_M \bar{q}(x)a_1(t,x)a_2(t,x) \, dv_g(x) \, dt \right| \leq Ce^{C_0 T} (\lambda^{-1} + \lambda \| \Lambda_{g, q_1}^W - \Lambda_{g, q_2}^W \|_{*, \nu}) \| a_1 \|_{*} \| a_2 \|_{*}.
\]
where \( \tilde{q} \) is the lift of \( q \) to \( \tilde{M} \).

**Proof.** We shall proceed as for the Schrödinger equation. By Lemma 4.4, let \( \lambda > 0 \), there exist \( a_2, \psi_\gamma \) as in (4.6), such that the for \( G_{2,\lambda}(t, x) \) given by (4.20) with \( a = a_2 \), the solution
\[
u_2(t, x) = G_{2,\lambda}(t, x) + v_{2,\lambda}(t, x)
\]
to the wave equation corresponding to the potential \( q_2 \)
\[
\begin{aligned}
&\left( \partial_t^2 + \Delta_g + q_2(x) \right) u(t, x) = 0, \quad \text{in } I \times M, \\
u(0, \cdot) = 0, \quad \partial_t u(0, \cdot) = 0, \quad \text{in } M,
\end{aligned}
\]
satisfies \( v_{2,\lambda}(t, x) = 0 \) for all \((t, x) \in (0, T) \times \partial M\), and (for \( \epsilon > 0 \) small)
\[
\lambda \|v_{2,\lambda}\|_{e^{\epsilon t}L^2(I \times M)} + \|\nabla^g v_{2,\lambda}\|_{e^{\epsilon t}L^2(I \times M)} \leq C e^{(h+\epsilon)T} \|a_2\|_*.
\]  
(6.1)

We next denote by \( f_\lambda \) the restriction of \( G_{2,\lambda} \) to \([0, T] \times \partial M\)
\[
f_\lambda(t, x) := G_{2,\lambda}(t, x) = \sum_{\gamma \in \pi_1(M)} a_2(t, \gamma(x)) e^{i\lambda (\psi_\gamma(\gamma(x)) - t)}.
\]

Let \( v \) be the solution to the boundary value problem
\[
\begin{aligned}
&\left( \partial_t^2 + \Delta_g + q_1 \right) v(t, x) = 0, \quad (t, x) \in I \times M, \\
v(0, \cdot) = 0, \quad \partial_t v(0, x) = 0, \quad x \in M, \\
v(t, x) = u_2(t, x) := f_\lambda(t, x), \quad (t, x) \in I \times \partial M,
\end{aligned}
\]
and denote \( w = v - u_2 \). Notice that \( w \) solves the following homogeneous boundary value problem for the wave equation:
\[
\begin{aligned}
&\left( \partial_t^2 + \Delta_g + q_1 \right) w(t, x) = q(x) u_2(t, x), \quad (t, x) \in I \times M, \\
w(0, \cdot) = 0, \quad \partial_t w(0, x) = 0, \quad x \in M, \\
w(t, x) = 0, \quad (t, x) \in I \times \partial M.
\end{aligned}
\]

Since \( q(x) u_2 \in C([0, T]; L^2(M)) \) with \( u_2(0, \cdot) \equiv 0 \), by Lemma 4.3, we obtain that
\[
w \in C^1(I; L^2(M)) \cap C(I; H^1_0(M)).
\]

On the other hand, we construct a special solution
\[
u_1 \in e^{\epsilon t}H^1(I; L^2(M)) \cap e^{\epsilon t}L^2(I; H^1(M))
\]
to the backward wave equation
\[
\begin{aligned}
&\left( \partial_t^2 + \Delta_g + \tilde{q}_1(x) \right) u_1(t, x) = 0, \quad (t, x) \in (0, T) \times M, \\
u_1(T, x) = 0, \quad \partial_t u_1(T, x) = 0, \quad x \in M,
\end{aligned}
\]
having the special form
\[
u_1(t, x) = \sum_{\gamma \in \pi_1(M)} a_1(t, \gamma(x)) e^{i\lambda (\psi_\gamma(\gamma(x)) - t)} + v_{1,\lambda}(t, x),
\]
which corresponds to the electric potential \( q_1 \), where \( v_{1,\lambda} \) satisfies \( v_{1,\lambda}(T, \cdot) = \partial_t v_{1,\lambda}(T, \cdot) = 0 \), and for each \( \epsilon > 0 \) there is \( C \) independent of \( T, \lambda \) such that
\[
\lambda \|v_{1,\lambda}\|_{e^{\epsilon t}L^2(I \times M)} + \|\nabla^g v_{1,\lambda}\|_{e^{\epsilon t}L^2(I \times M)} \leq C e^{(h+\epsilon)T} \|a_1\|_*.
\]  
(6.2)
By integration by parts and Green’s formula, we obtain
\[
\int_0^T \int_M (\partial_t^2 + \Delta_g + q_1) w \overline{\eta}_1 \, dv_g \, dt = \int_0^\infty \int_M q u_2 \overline{\eta}_1 \, dv_g \, dt
\]
\[
= - \int_0^T \int_{\partial M} \partial_n w \overline{\eta}_1 \, dv_{g|\partial M} \, dt.
\]
We therefore obtain
\[
\int_0^T q u_2 \overline{\eta}_1 \, dv_g \, dt = \int_0^T \int_{\partial M} (\Lambda_{g,q_1}^W - \Lambda_{g,q_2}^W) (f_\lambda) g_\lambda \, dv_{g|\partial M} \, dt,
\]
where the boundary data \(g_\lambda\) is given by
\[
g_\lambda(t, x) := \sum_{\gamma \in \pi_1(M)} a_1(t, \gamma(x)) e^{i\lambda(\psi_\gamma(\gamma(x)) - t)}, \quad (t, x) \in (0, T) \times \partial M.
\]
Using the definition of \(u_1\) and \(u_2\), we get
\[
\sum_{\gamma, \gamma_2 \in \pi_1(M)} \int_0^T \int_{\mathcal{F}} \tilde{q}(x) a_2(t, \gamma_1(x)) \overline{a_1(t, \gamma_2(x))} \, dv_\tilde{q}(x) \, dt
\]
\[
= \int_0^T \int_{\partial M} \overline{g_\lambda}(\Lambda_{g,q_1}^W - \Lambda_{g,q_2}^W) f_\lambda \, dv_{g|\partial M} \, dt - \int_0^T \int_M q u_2, \lambda \overline{\eta}_1, \lambda \, dv_g \, dt
\]
\[
- \sum_{\gamma \in \pi_1(M)} \int_0^T \int_{\mathcal{F}} q e^{i\lambda(\psi_\gamma(\gamma(\cdot)) - t)} a_2(t, \gamma(\cdot)) \overline{\eta}_1, \lambda \, dv_\tilde{q} \, dt
\]
\[
- \sum_{\gamma \in \pi_1(M)} \int_0^T \int_{\mathcal{F}} q u_2, \lambda e^{-i\lambda(\psi_\gamma(\gamma(\cdot)) - t)} \overline{a_1(t, \gamma(\cdot))} \, dv_\tilde{q} \, dt.
\]
By (6.1) and (6.2), for each \(\epsilon > 0\) small, there is \(C > 0\) depending on \(g, \|q\|_{L^\infty}, \epsilon\) such that for all \(T, \lambda\)
\[
\sum_{\gamma \in \pi_1(M)} \left| \int_0^T \int_{\mathcal{F}} q e^{i\lambda(\psi_\gamma - t)} a_2(t, \gamma(\cdot)) \overline{\eta}_1, \lambda \, dv_\tilde{q} \, dt \right| + \left| \int_0^T \int_{\mathcal{F}} q e^{i\lambda(\psi_\gamma - t)} a_1(t, \gamma(\cdot)) \overline{\eta}_2, \lambda \, dv_\tilde{q} \, dt \right|
\]
\[
\leq C \sum_{\gamma \in \pi_1(M)} \int_0^T \|a_2(t, \cdot)\|_{L^2(\mathcal{F})} \|v_1, \lambda(t, \cdot)\|_{L^2(M)} \, dt
\]
\[
\leq C e^{2(h + \nu + \epsilon)T} \lambda^{-1} \|a_2\|_* \|a_1\|_*.
\]
and \(\int_0^T \int_M q u_2, \lambda \overline{\eta}_1, \lambda \, dv_g \, dt \leq C e^{2(h + \nu + \epsilon)T} \lambda^{-2} \|a_1\|_* \|a_2\|_*\). We also have, using that \(\forall \epsilon > 0, \|f_\lambda\|_{H^1([0,T] \times \partial M)} \leq C \epsilon e^{2(h + \epsilon)T} \|a_2\|_*\) and \(\|g_\lambda\|_{L^2([0,T] \times \partial M)} \leq C \epsilon e^{2(h + \epsilon)T} \|a_1\|_*\) for some \(C > 0\) depending only on the metric \(g\) and \(\epsilon\),
\[
\left| \int_0^T \int_{\partial M} \Lambda_{g,q_1}^W - \Lambda_{g,q_2}^W (f_\lambda) g_\lambda \, dv_{g|\partial M} \, dt \right| \leq C e^{2(h + \nu + \epsilon)T} \lambda \|a_1\|_* \|a_2\|_* \|\Lambda_{g,q_1}^W - \Lambda_{g,q_2}^W\|_{*,\nu}.
\]
Finally, using that
\[ \sum_{\gamma \in \pi_1(M)} \int_0^T \int_F \tilde{q}(x)a_2(t, \gamma(x))a_1(t, \gamma(x)) \, dv_g(x) \, dt = \int_0^T \int_M \tilde{q}(x)a_2(t, x)a_1(t, x) \, dv_g(x) \, dt \]
and bounding the off-diagonal terms as in (5.3), the Lemma holds. \hfill \square

We then obtain the following Lemma comparable to Lemma 5.2:

**Lemma 6.2.** There exist \( C > 0 \) depending only on \((M, g, \|q_1\|_{W^{1, \infty}}, \|q_2\|_{W^{1, \infty}})\), \( \nu \) depending on \((\|q_1\|_{L\infty}, \|q_2\|_{L\infty})\) and \( C_0 > 0 \) depending on \((M, g, \nu)\) such that, for any \( T > 1, b \in H^2(\partial_{-}SM_e) \) such that \( b|\tau_{\partial M}^{T} = 0 \), one has
\[
\left| \int_{\partial_{-}S_{y}M_e} \int_0^{\tau_{\partial M}^{y}(v)} q(\exp_y(sv))b(y, v)\mu(y, v) \, ds \, d\omega_y(v) \right| \leq C e^{C_0T} \|\Lambda_{g, q_1}^W - \Lambda_{g, q_2}^W\|_{*}^{1/2} \|b(y, \cdot)\|_{H^2(\partial_{-}S_{y}M_e)},
\]
holds uniformly for any \( y \in \partial M_e \), where \( \mu(y, v) = g(n_y, v) \).

**Proof.** Take \( a_1, a_2 \) solutions to the transport equation on the universal cover \( \widetilde{M} \) as before. Then as in the proof of Lemma 5.2, we obtain
\[
\int_0^T \int_M \tilde{q}a_1a_2 \, dv_g \, dt = \int_0^T \int_{\partial_{-}S_{y}M_e} \int_0^{\tau_{\partial M}^{y}(v)} q(\exp_y(sv))\phi^2(t-r)b(y, v)\mu(y, v) \, dr \, d\omega_y(v) \, dt
\]
\[ = \|\phi\|_{L^2(\mathbb{R})} \int_{\partial_{-}S_{y}M_e} \int_{0}^{\tau_{\partial M}^{y}(v)} q(\exp_y(sv))b(y, v)\mu(y, v) \, dr \, d\omega_y(v). \]
Combining this with Lemma 6.1 and choosing \( \lambda = \|\Lambda_{g, q_1}^W - \Lambda_{g, q_2}^W\|_{*}^{-1/2} \) yields the desired result. \hfill \square

6.2. **Proof of the stability estimate.** Using Lemma 6.2 there is \( C > 0 \) such that for any \( y \in \partial M \) and \( b \in H^2(\partial_{-}SM_e) \) such that \( b|\tau_{\partial M}^{T} = 0 \),
\[
\left| \int_{\partial_{-}S_{y}M_e} \int_0^{\tau_{\partial M}^{y}(v)} I_0^q(y, v)b(y, v) \, d\mu_{\nu}(y, v) \right| \leq C e^{C_0T} \|\Lambda_{g, q_1}^W - \Lambda_{g, q_2}^W\|_{*}^{1/2} \|b(y, \cdot)\|_{H^2(\partial_{-}S_{y}M_e)}.
\]
Now we take a bump function \( \chi_T \in C_e(\mathbb{R}) \) as in Section 5.2 and choose \( b \) by (5.5) with \( T \) instead of \( T_0 \).

**Proof of Theorem 3.** The proof is then almost the same as the proof of Theorem 1, we then just briefly describe it. From (6.3), the same as (5.6) holds with \( \Lambda_{g, q_1}^s - \Lambda_{g, q_2}^s \), then using Lemmas 5.3 and 5.4 with \( \|\Pi_0^q\|_{L^p(M_e)} \leq \|q\|_{L^\infty(M)} \), we obtain
\[
\left| \int_{M_e} \|\Pi_0^q\|^2 \, dv_g \right| \leq C e^{C_0T} \|\Lambda_{g, q_1}^W - \Lambda_{g, q_2}^W\|_{*}^{1/2} \|q\|_{W^{1, \infty}(M)} + C e^{Q_T} \|q\|_{L^\infty(M)}^2
\]
The last part of the proof is exactly the same as for Theorem 1 by choosing $T$ so that $e^{-QT/2} = \|\Lambda_{g,q_1}^W - \Lambda_{g,q_2}^W\|_{*,*}$, with $m = -2C_0/Q$. \hfill \Box

7. Stable determination of the conformal factor for the Schrödinger equation

In this section we prove Theorem 2. The proof follows the argument of [BDSF10, Thm. 3], and we indicate the main modifications. Let $c \in \mathcal{C}(N_0, k, \epsilon)$, be such that $c = 1$ near the boundary $\partial M$, we denote

$$
\rho_0(x) := 1 - c(x), \quad \rho_1(x) := e^{d/2}(x) - 1, \quad \rho_2(x) := e^{d/2-1}(x) - 1, \quad \rho(x) := \rho_2(x) - \rho_1(x) = e^{d/2-1}(x)(1 - c(x)),
$$

(7.1)

where recall that $d = \dim(M)$. We modify the construction of geometric optics solutions [BDSF10, Sect. 6.1] in our geometrical setting similarly to what was discussed in previous sections, using the universal cover $\tilde{M}$. We consider two solutions $\tilde{\psi}_1, \tilde{\psi}_2$, respectively to the lifted eikonal equations $|\nabla_{\tilde{g}}\tilde{\psi}_1| = 1$ and $|\nabla_{\tilde{g}}\tilde{\psi}_2| = 1$, a solution $a_2$ to the lifted transport equation

$$
\partial_t a_2 + da_2(\nabla_{\tilde{g}}\tilde{\psi}_1) - \frac{a_2}{2} \Delta_{\tilde{g}} \tilde{\psi}_1 = 0,
$$

(7.2)

given in geodesical polar coordinates with respect to $g$, as in (4.6), by

$$
a_2(t, x) = \alpha^{-1/4} \phi(t - r(x)) b(\tilde{y}, v(x)), \quad \tilde{y} \in \partial SM_\epsilon,
$$

for some $b \in H^2(\partial_+ SM)$ satisfying that $b|_{T^1_0SM(T_0)} = 0$, and $\phi \in C^\infty_0(\mathbb{R})$ with $\supp \phi \subset (0, \varepsilon_0)$, $\varepsilon_0 > 0$ small, and a solution $a_3$ to the lifted transport equation

$$
\partial_t a_3 + da_3(\nabla_{\tilde{g}}\tilde{\psi}_2) - \frac{a_3}{2} \Delta_{\tilde{g}} \tilde{\psi}_2 = -\frac{1}{2i} a_2(t, x) \left(1 - \frac{1}{c}\right) e^{i\lambda(\tilde{\psi}_1 - \tilde{\psi}_2)}
$$

which satisfies (analogously to [BDSF10, (6.10)]) the bound

$$
\|a_3\|_* \leq C\lambda^2 \|1 - c\|_{C^2(M)} \|a_2\|_*,
$$

(7.3)

for some $C > 0$ depending on $(M, g)$ (it does not depend on $T_0$, notice that the derivatives of the difference between $\tilde{\psi}_1$ and $\tilde{\psi}_2$ can be bounded in terms of derivatives of $c$), where $\| \cdot \|_* = \| \cdot \|_{H^1([0, T_0]; H^2(\tilde{M}))}$. Then, [BDSF10, Lemma 6.2] becomes in our setting:

**Lemma 7.1.** The equation

$$
\begin{aligned}
(i\partial_t - \Delta_{cg}) u &= 0, \quad \text{in} \ (0, T) \times M, \\
u(0, x) &= 0, \quad \text{in} \ M,
\end{aligned}
$$

has a solution of the form

$$u_2(t, x) = \sum_{\gamma \in \pi_1(M)} \left(\frac{1}{\lambda} a_2(2\lambda t, \gamma(x)) e^{i\lambda(\tilde{\psi}_1(\gamma(x)) - \lambda t)} + a_3(2\lambda t, \gamma(x)) e^{i\lambda(\tilde{\psi}_2(\gamma(x)) - \lambda t)}\right) + v_{2,\lambda}(t, x),$$
which satisfies, for \( \lambda \geq T_0/2T \),
\[
\lambda \| v_{2,\lambda}(t, \cdot) \|_{L^2(M)} + \| \nabla^g v_{2,\lambda}(t, \cdot) \|_{L^2(M)} + \lambda^{-1} \| \partial_t v_{2,\lambda}(t, \cdot) \|_{L^2(M)} \\
\leq C e^{C_0 T_0} (\lambda^2 \| \rho_0 \|_{C^2(M)} + \lambda^{-1}) \| a_2 \|_\ast .
\]
for \( C > 0 \) depending on \((M, g, N_0)\) and \( C_0 > 0 \) depending on \((M, g)\).

The constant \( C_0 \) above can be written in terms of the max of the volume entropies of the metrics \( cg \) for \(|c - 1|c^{1}(M) \leq 1/2\) (this can be bounded by a uniform constant times the volume entropy of \((M, g)\)).

Moreover, reasoning as in previous Sections, [BDSF10, Lemma 6.3] becomes in our setting:

**Lemma 7.2.** There exist constants \( C > 0 \) depending on \((M, g, N_0)\) and \( C_0 > 0 \) depending on \((M, g)\) such that, for any \( a_1, a_2 \in H^1([0, T_0]; H^2(\tilde{M}))\) solving (7.2) associated to \( b_1, b_2 \in H^2(\partial_- S M_e) \) with \( [\rho_0]_{\partial^+ S M_e(T_0)} = b_2 |_{\partial^+ S M_e(T_0)} = 0 \), the following estimate holds:

\[
\left| \sum_{\gamma \in \pi_1(M)} \int_0^T \int_M (x,\gamma(x))a_1(2\lambda t, \gamma(x))a_2(2\lambda t, \gamma(x)) \, d\nu_g(x) \, dt \right| \\
\leq C \lambda^{-1} e^{C_0 T_0} (\| \rho_0 \|_{C^1(M)} (\lambda^{-1} + \lambda^3 \| \rho_0 \|_{C^2(M)}) + \lambda \| \Lambda^S_g - \Lambda^S_{cg} \|_\ast) \| a_1 \|_\ast \| a_2 \|_\ast .
\]

for any \( \lambda \geq T_0/2T \).

Finally, [BDSF10, Lemma 6.4] becomes in our setting.

**Lemma 7.3.** There exists \( C > 0 \) and \( C_0 > 0 \) as in previous Lemma such that, for any \( b \in H^2(\partial_- S M_e) \) with \( b |_{\partial^+ S M_e(T_0)} = 0 \) the following estimate

\[
\left| \int_{\partial_- S M_e} I^e_0(\rho)(y, v) b(y, v) d\mu_\ast(y, v) \right| \\
\leq C e^{C_0 T_0} \left( (\lambda^{-1} + \lambda^3 \| \rho_0 \|_{C^2(M)}) \| \rho_0 \|_{C^1(M)} + \lambda \| \Lambda^S_g - \Lambda^S_{cg} \|_\ast \right) \| b \|_{H^2(\partial_- S M)}
\]

holds for any \( \lambda \geq T_0/2T \). Here \( I^e_0 \) is the X-ray transform for \( g \) on functions on \( M_e \).

**Proof of Theorem 2.** We take \( b \) as in (5.5) with \( q \) replaced by \( \rho \). By Lemma 7.3, Lemma 5.3, and Lemma 5.4, there is \( C > 0, C_0 > 0 \) depending on \((M, g, N_0, \epsilon)\) such that

\[
\| \Pi^\ast_0 \rho \|_{L^2(M_e)}^2 \\
\leq C e^{C_0 T_0} \left( (\lambda^{-1} + \lambda^3 \| \rho_0 \|_{C^2(M)}) \| \rho_0 \|_{C^1(M)} + \lambda \| \Lambda^S_g - \Lambda^S_{cg} \|_\ast \right) \| \rho \|_{W^{1,p}(M)} + C e^{C_0 T_0} \| \rho \|_{L^p(M)}^2
\]

for some \( p > 2 \). By interpolation,

\[
\| \Pi^\ast_0 \rho \|_{H^2(M_e)}^2 \leq C \| \Pi^\ast_0 \rho \|_{L^2(M_e)} \| \Pi^\ast_0 \rho \|_{H^2(M_e)}
\]

\[
\leq C e^{C_0 T_0} \left( (\lambda^{-1} + \lambda^3 \| \rho_0 \|_{C^2(M)}) \| \rho_0 \|_{C^1(M)} + \lambda \| \Lambda^S_g - \Lambda^S_{cg} \|_\ast \right) \| \rho \|_{W^{1,p}(M)} \| \rho \|_{H^1(M)}^{1/2} \| \rho \|_{L^p(M)} \| \rho \|_{H^1(M)}^{1/2} \\
+ C e^{C_0 T_0} \| \rho \|_{L^p(M)} \| \rho \|_{H^1(M)}. 
\]
We use (3.4) to deduce the bound
\[
\|\rho\|^2_{L^2(M)} \leq C e^{C_0 T_0/2} (\lambda^{-1} + \lambda^3 \|\rho_0\|_{C^2(M)} \frac{1}{2} \|\rho_0\|_{C^1(M)} \|\rho\|_{W^{1,\infty}(M)} \|\rho\|_{H^1(M)} + C e^{C_0 T_0} \lambda^2 \|\rho_0\|_{C^1(M)} \|\Lambda^S_g - \Lambda^S_{cg}\|^2_{\ast} + C e^{C_0 T_0} \|\rho\|_{L^p(M)} \|\rho\|_{H^1(M)}.
\]
Taking
\[
\lambda = \frac{T_0}{2T} \cdot \left(\frac{\|\rho_0\|_{C^2(M)}}{2N_0}\right)^{-\frac{1}{4}},
\]
we obtain for \(T_0 > 0\) large
\[
\|\rho\|^2_{L^2(M)} \leq C e^{C_0 T_0} (\|\rho_0\|^2_{C^2(M)} + \|\rho_0\|_{C^1(M)} \|\Lambda^S_g - \Lambda^S_{cg}\|^2_{\ast}) + C e^{C_0 T_0} \|\rho\|_{L^p(M)} \|\rho\|_{H^1(M)} \leq C e^{C_0 T_0} (\|\rho_0\|^2_{C^2(M)} + \|\Lambda^S_g - \Lambda^S_{cg}\|^2_{\ast}) + C e^{C_0 T_0} \|\rho\|_{L^p(M)} \|\rho\|_{H^1(M)},
\]
where \(C\) depends on \(N_0, \ell = \frac{1}{16}(1 - \frac{2}{k})\), and we have used the interpolation estimate
\[
\|\rho_0\|_{C^2(M)} \leq C \|\rho_0\|^{1-\frac{2}{k}}_{C^2(M)} \leq C e^{1-\frac{2}{k}}.
\]
By interpolation, choosing \(k > \max(s, s')\) large enough, we have for some \(\delta, \delta' > 0\)
\[
\|\rho_0\|_{C^2(M)} \leq C \|\rho_0\|_{H^\frac{33+\delta}{4} + 2+\delta(M)} \leq C \|\rho_0\|_{L^2(M)}^{32/33+\delta} \|\rho_0\|_{H^1(M)}^{1/33-\delta} \leq C \|\rho_0\|_{L^2(M)}^{32/33+\delta}, \quad (7.4)
\]
\[
\|\rho\|_{H^1(M)} \leq \|\rho\|_{L^2(M)}^{1-\delta'} \|\rho\|_{H^{\delta'}(M)}^{\delta'} \leq C \|\rho\|_{L^2(M)}^{1-\delta'}, \quad (7.5)
\]
\[
\|\rho\|_{L^p(M)} \leq \|\rho\|_{L^2(M)}^{1-\delta'} \|\rho\|_{L^{p'}(M)}^{\delta'} \leq C \|\rho\|_{L^2(M)}^{1-\delta'}, \quad (7.6)
\]
where \(C > 0\) depends on \(N_0\). Thus, using that \(C^{-1} \|\rho_0\|_{L^2} \leq \|\rho\|_{L^2} \leq C \|\rho_0\|_{L^2}\), we see that there is \(C > 0\) depending on \((M, g, N_0, k)\) and \(C_0 > 0\) depending on \((M, g)\) such that
\[
\|\rho_0\|^2_{L^2(M)} \leq C e^{C_0 T_0} \|\rho_0\|^2_{L^2(M)} + C e^{C_0 T_0} \|\Lambda^S_g - \Lambda^S_{cg}\|^2_{\ast} + C e^{C_0 T_0} \|\rho_0\|_{L^2(M)}^{2-2\delta'}. \quad (7.7)
\]
We finally take \(T_0\) sufficiently large so that
\[
C e^{C_0 T_0} < \frac{1}{2} \|\rho_0\|_{L^2}^{2\delta'}.
\]
This allows us to absorb the third term of the right-hand-side in the left-hand side:
\[
\|\rho_0\|^2_{L^2(M)} \leq C e^{\ell} \|\rho_0\|^2_{L^2(M)} + C \|\rho_0\|^2_{L^2(M)} \|\Lambda^S_g - \Lambda^S_{cg}\|^2_{\ast},
\]
for \(m = C_0/4|\Omega| > 0\). Choosing \(\delta' > 2m\delta'\) and taking \(\epsilon\) sufficiently small, we can absorb the first term of the right-hand-side into the left-hand-side. Therefore, there exists \(\beta > 0\) depending on \((M, g)\) and \(C > 0\) depending on \((M, g, N_0, k, \epsilon)\) such that
\[
\|\rho_0\|_{L^2(M)} \leq C \|\Lambda^S_g - \Lambda^S_{cg}\|_{\ast}^\beta,
\]
and the proof is complete. \(\Box\)
8. Stable determination of the conformal factor for the wave equation

In this section we sketch the proof of Theorem 4. We just indicate the modifications with respect to the proof of [BDSF11, Thm. 2]. We will assume that the conformal factor satisfies $c \in \mathcal{C}(N_0, k, \epsilon)$ be such that $c = 1$ near the boundary $\partial M$, and we use the notation (7.1). First, [BDSF11, Lemma 6.2] becomes in our setting:

**Lemma 8.1.** The equation

\[
\begin{cases}
(\partial_t^2 + \Delta_{cg})u = 0, & \text{in } (0, T) \times M, \\
u(0, x) = 0, & \text{in } M,
\end{cases}
\]

has a solution of the form

\[
u_2(t, x) = \sum_{\gamma \in \pi_1(M)} \left( \frac{1}{\lambda} \alpha_2(t, \gamma(x)) e^{i\lambda(\psi_1(\gamma(x))-t)} + \alpha_3(t, \gamma(x); \lambda) e^{i\lambda(\psi_2(\gamma(x))-t)} \right) + \nu_\lambda(t, x),
\]

such that there is $\nu_0 > 0$ (given by Lemma 4.3) so that for all $\nu > \nu_0$, there is $C > 0$ depending only on $(M, g, N_0, \nu)$, and $C_0 > 0$ depending only on $(M, g, \nu)$ so that $\forall \lambda > 1$

\[
\lambda \|v_2, \lambda\|_{L^2(\mathbb{R}^+ \times M)} + \|\nabla^g v_2, \lambda\|_{L^2(\mathbb{R}^+ \times M)} + \lambda^{-1} \|\partial_t v_2, \lambda\|_{L^2(\mathbb{R}^+ \times M)} \leq Ce^{C_0T} (\|\rho_0\|_{C^2(M)} \lambda^2 + \lambda^{-1}) \|a_2\|_*.
\]

Moreover, [BDSF11, Lemma 6.3] is replaced by:

**Lemma 8.2.** There exist constants $C_0 > 0$ depending on $(M, g, \nu)$ and $C > 0$ depending on $(M, g, N_0, \nu)$ such that, for any $a_1, a_2 \in H^1([0, T]; H^2(M))$ solving (7.2) associated to $b_1, b_2 \in H^2(\partial_- S M_e)$ with $b_1|_{T^0 S M(T)} = b_2|_{T^0 S M(T)} = 0$, the following estimate holds:

\[
\left| \sum_{\gamma \in \pi_1(M)} \int_0^T \int_M \rho(x) a_1(t, \gamma(x)) a_2(t, \gamma(x)) \, dv_x(x) \, dt \right| \\
\leq Ce^{C_0T} \|\rho_0\|_{C^1(M)} (\lambda^{-1} + \lambda^3 \|\rho_0\|_{C^2(M)}) \|a_1\|_* \|a_2\|_* + Ce^{C_0T} \lambda \|\Lambda_{gW} - \Lambda_{cgW}\|_{*, \nu} \|a_1\|_* \|a_2\|_*,
\]

for all $\lambda > 1$.

Finally, [BDSF11, Lemma 6.4] is replaced by:

**Lemma 8.3.** There exists $C > 0$ depending on $(M, g, N_0, \nu)$ and $C_0 > 0$ depending on $(M, g, \nu)$ such that, for any $b \in H^2(\partial_- S M_e)$ with $b|_{T^0 S M(T)} = 0$ the following estimate

\[
\left| \int_{\partial_- S M_e} I_0^e(\rho)(y, v) b(y, v) \, d\mu_\rho(y, v) \right| \\
\leq Ce^{C_0T} \left( (\lambda^{-1} + \lambda^3 \|\rho_0\|_{C^2(M)}) \|\rho_0\|_{C^1(M)} + \lambda \|\Lambda_{gW} - \Lambda_{cgW}\|_{*, \nu} \right) \|b\|_{H^2(\partial_- S M)}
\]

holds for all $\lambda > 1$. 
Proof of Theorem 4. We take $b$ as in (5.5) with $q$ replaced by $\rho$. By Lemma 7.3, Lemma 5.3, and Lemma 5.4, there is $C > 0$ depending on $(M, g, N_0, \nu)$ and $C_0 > 0$ depending on $(M, g)$ such that

$$
\|\Pi_0^e \rho\|_{L^2(M)}^2 \\
\leq C e^{C_0 T \left( (\lambda^{-1} + \lambda^3 \|\rho_0\|_{C^2(M)}) \|\rho_0\|_{C^1(M)} + \lambda \|\Lambda_g^W - \Lambda_{cg}^W\|_{\ast, \nu} \right)} \|\rho\|_{W^{1, p}(M)} + C e^{\frac{Q}{4} T} \|\rho\|_{L^p(M)}^2
$$

for some $p > 2$. By interpolation,

$$
\|\Pi_0^e \rho\|_{H^1(M)}^2 \\
\leq C (\|\Pi_0^e \rho\|_{L^2(M)} \|\Pi_0^e \rho\|_{H^2(M)}) \\
\leq C e^{\frac{C_0 T}{2} \left( (\lambda^{-1} + \lambda^3 \|\rho_0\|_{C^2(M)}) \|\rho_0\|_{C^1(M)} + \lambda \|\Lambda_g^W - \Lambda_{cg}^W\|_{\ast, \nu} \right)} \|\rho\|_{C^1(M)}^{\frac{3}{2}} \\
+ C e^{\frac{Q}{4} T} \|\rho\|_{L^p(M)} \|\rho\|_{H^1(M)}.
$$

We use (3.4) to deduce the bound

$$
\|\rho\|_{L^2(M)}^2 \leq C e^{\frac{C_0 T}{2} \left( (\lambda^{-1} + \lambda^3 \|\rho_0\|_{C^2(M)}) \|\rho_0\|_{C^1(M)} \right)} \|\rho\|_{C^1(M)}^{\frac{3}{2}} \\
+ C e^{\frac{Q}{4} T} \|\rho\|_{L^p(M)} \|\rho\|_{H^1(M)}.
$$

Taking $\lambda = \|\rho_0\|_{C^2(M)}^{-\frac{1}{6}}$, we obtain for $T > 0$ large:

$$
\|\rho\|_{L^2(M)}^2 \leq C e^{C_0 T} \|\rho_0\|_{C^2(M)}^{\frac{17}{6}} + C e^{C_0 T} \|\rho_0\|_{C^1(M)} \|\Lambda_g^W - \Lambda_{cg}^W\|_{\ast, \nu} + C e^{\frac{Q}{4} T} \|\rho\|_{L^p(M)} \|\rho\|_{H^1(M)},
$$

where $\ell = \frac{1}{16} \left( 1 - \frac{2}{k} \right)$. Choosing $k > \max(s, s')$ large enough as in the previous section, we have for some $\delta, \delta' > 0$ small the interpolation estimates (7.4), (7.5), and (7.6). Thus we get that there is $C > 0$ depending on $(M, g, N_0, k)$ and $C_0 > 0$ depending on $(M, g)$ such that

$$
\|\rho_0\|_{L^2(M)}^2 \leq C e^{C_0 T} \ell^{\delta} \|\rho_0\|_{L^2(M)}^{2 + \delta} + C e^{C_0 T} \|\Lambda_g^W - \Lambda_{cg}^W\|_{\ast, \nu} + C e^{\frac{Q}{4} T} \|\rho_0\|_{L^2(M)}^{2 - 2\delta'}.
$$

We finally take $T$ sufficiently large so that

$$
C e^{\frac{Q}{4} T} < \frac{1}{2} \|\rho_0\|_{L^2(M)}^{2\delta'}.
$$

This allows us to absorb the third term of the right-hand-side in the left-hand side:

$$
\|\rho_0\|_{L^2(M)}^2 \leq C e^{\ell} \|\rho_0\|_{L^2(M)}^{2 - 2m\delta'} \|\rho_0\|_{L^2(M)}^{2 + \delta} + C \|\rho_0\|_{L^2(M)}^{-2m\delta'} \|\Lambda_g^W - \Lambda_{cg}^W\|_{\ast, \nu},
$$

for $m = C_0/4 |Q| > 0$. Choosing $\delta > 2m\delta'$ and taking $\epsilon$ sufficiently small (depending only on $\|1 - c\|_{L^2}$), we can absorb the first term of the right-hand-side into the left-hand-side. Therefore, there exists $\beta > 0$ depending on $(M, g, \nu)$ and $C > 0$ depending on $(M, g, N_0, k, \epsilon)$ such that

$$
\|\rho_0\|_{L^2(M)} \leq C \|\Lambda_g^W - \Lambda_{cg}^W\|_{\ast, \nu}^{-\beta},
$$

and the proof is complete. \qed
9. APPENDIX

In this appendix, we discuss how to remove the boundary condition $q_1 = q_2$ on $\partial M$ in Theorem 1 and 3, at the cost of assuming more regularity on $q_j$.

9.1. Boundary stability: the wave case. We follow the argument of [Mon14, Section 3]. Assuming that $q_1, q_2 \in C^4(M)$, one can construct two geometrical optics solutions $(\partial_t^2 + \Delta_g + q_j)u_j = 0$ concentrated on a small geodesic close to a boundary point $x_0 \in \partial M$

$$u_j = e^{i\lambda(t-x)}(a_0 + \lambda^{-1}a_1^j + \lambda^{-2}a_2^j) + v_j =: u_j^0 + v_j,$$

where $\psi$ is a suitable solution to the eikonal equation $|\nabla_g\psi| = 1$ with $\psi|_{U \cap \partial M} = x'.\omega$ for some $\omega \in T_{x_0}\partial M \simeq \mathbb{R}^{n-1}$ satisfying $|\omega'| < 1$ (but close to 1), $a_0$ solves a transport equation involving only the metric and some derivatives of $\psi$, $a_0|_{\partial M} = \chi \in C_\infty((0,T') \times \partial M; \mathbb{R}^+)$ is some cutoff function equal to 1 near $(t_0, x_0)$, and $a_1^j, a_2^j$ solve the transport equation

$$i\left(2\partial_t + 2\nabla^g\psi - \Delta_g \psi\right)a_k^j = -(\partial_t^2 + \Delta_g + q_j)a_k^j, \quad a_k^j|_{\partial M} = 0. \quad (9.1)$$

By Lemma 4.3 one can construct the remainder term $v_j$ so that $\|\partial_n v_j\|_{C^0([0,T'] \times \partial M)} = \mathcal{O}(\lambda^{-2})$ for some small $T' > 0$, with uniform dependence wrt $\|q_j\|_{C^4(M)}$ and $v_j|_{(0,T') \times \partial M} = 0$. Using that

$$\Lambda_{g,q_j}(u_j|_{\partial M}) = e^{i\lambda(t-x)}(i\lambda(\partial_n \psi)a_0 - \partial_n a_0 - \lambda^{-1}\partial_n a_1) + \mathcal{O}(\lambda^{-2}),$$

that $\|u_j^0\|_{H^1([0,T'] \times \partial M)} \leq C\lambda$, and choosing $\lambda \sim \|\Lambda_{g,q_1}^W - \Lambda_{g,q_2}^W\|^{-1/3}_*\{1\}$, one gets the bound

$$\|\partial_n a_1^j - \partial_n a_2^j\|_{L^2([0,T'] \times \partial M)} \leq C\|\Lambda_{g,q_1}^W - \Lambda_{g,q_2}^W\|^{1/3}_*.$$

This implies $2|\partial_n a_1^j - \partial_n a_2^j| = |(q_1 - q_2)\chi|$ by restricting (9.1) to $(0,T') \times \partial M$. Therefore, there is an open neighborhood $V \subset U \cap \partial M$ of $x_0$ of uniform size and $C > 0$ uniform in $\|q_j\|_{C^4}$ such that

$$\|q_1 - q_2\|_{L^2(V)} \leq C\|\Lambda_{g,q_1}^W - \Lambda_{g,q_2}^W\|^{1/3}_*.$$

Using interpolation and Sobolev embeddings, we get $\|q_1 - q_2\|_{C^1(\partial M)} \leq C\|\Lambda_{g,q_1}^W - \Lambda_{g,q_2}^W\|^{\mu}_*$ for some $\mu > 0$. We finally replace $q_2$ by $\tilde{q}_2 = q_2 + \theta(\|\Lambda_{g,q_1}^W - \Lambda_{g,q_2}^W\|^{\mu/2}_d g(\cdot, \partial M))(q_1 - q_2)$ in Theorem 3 for some $\theta \in C_\infty((0,1))$ equal to 1 near 0 so that $\tilde{q}_2 = q_1$ on $\partial M$.

9.2. Boundary stability: the Schrödinger case. We use a similar argument for the Schrödinger equation. Assume that $q_j \in C^5(M)$. We construct a geometric optic solution of the form

$$u_j(t, x) = e^{i\lambda(t-x)}(\sum_{k=0}^4 a_k^j(2\lambda t, x)\lambda^{-k}) + v_j(t, x) =: u_j^0(t, x) + v_j(t, x),$$

where $a_0^j(s, x)$ solves the same transport equation as for the wave equation (thus not depending on $q_j$) with $a_0^j|_{\partial M} = \chi$ and $a_k^j$ for $k \geq 1$ solves

$$i\left(2\partial_t + 2\nabla^g\psi - \Delta_g \psi\right)a_k^j(t, x) = (\Delta_g - q_j)a_k^j(t, x), \quad a_k^j|_{\partial M} = 0. \quad (9.2)$$
and the remainder can be estimated by \( \|\partial_{\lambda}v_j\|_{L^2((0,T)\times\partial M)} = O(\lambda^{-3}) \) using [BDSF10, Lemma 3.2] and the fact that \( \|\lambda^{-4}e^{i\lambda(\psi-\lambda t)}((\Delta g - q_j)a_j^2)\|_{H^1((0,T);L^2(M))} = O(\lambda^{-3}) \) (we loose \( \lambda^2 \) from the derivative but gain one \( \lambda^{-1} \) from the change of variable \( t \mapsto t/\lambda \) in the \( dt \) integral as \( a_k^2 \) are supported in time interval of size \( O(\lambda^{-1}) \)). We then obtain
\[
A_{g,q_1}^S(u_j|\partial M) = e^{i\lambda(\psi-\lambda t)} \left( i\lambda(\partial_{x_n}\psi)a_0 - \partial_{x_n}a_0 - \lambda^{-1}\partial_{x_n}a_1 \right) (2\lambda t, x) + O_{L^2}(\lambda^{-3})
\]
Proceeding as for the wave equation, we deduce that
\[
\|\partial_{x_n}a_1^1 - \partial_{x_n}a_1^2\|_{L^2((0,T)\times\partial M)} \leq C\|A_{g,q_1}^W - A_{g,q_2}^W\|_{L^2((0,T)\times\partial M)}^{1/3}
\]
and the end of the proof is the same as for the wave equation.

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