Cancellation ideals of a ring extension

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Communicated by E. I. Zelmanov

Abstract. We study properties of cancellation ideals of ring extensions. Let $R \subseteq S$ be a ring extension. A nonzero $S$-regular ideal $I$ of $R$ is called a (quasi)-cancellation ideal of the ring extension $R \subseteq S$ if whenever $IB = IC$ for two $S$-regular (finitely generated) $R$-submodules $B$ and $C$ of $S$, then $B = C$. We show that a finitely generated ideal $I$ is a cancellation ideal of the ring extension $R \subseteq S$ if and only if $I$ is $S$-invertible.

1. Introduction and background

Throughout this article, we assume that all rings are commutative with identity. The notion of cancellation ideal for a ring has been studied in [1] and [2]. An ideal $I$ of a ring $R$ is called cancellation ideal if whenever $IB = IC$ for two ideals $B$ and $C$ of $R$, then $B = C$ [2]. A finitely generated ideal is a cancellation ideal if and only if for each maximal ideal $M$ of $R$, $IM$ is a regular principal ideal of $RM$ [1, Theorem 1]. D.D Anderson and D.F Anderson used the notion of cancellation ideal to characterize Prüfer domain. A ring $R$ is a Prüfer domain if and only if every finitely generated nonzero ideal of $R$ is a cancellation ideal[1, Theorem 6]. In this paper, we study the notion of cancellation ideal for ring extensions; which is a generalization of the notion of cancellation ideal for rings. Let $R \subseteq S$ be a ring extension, and let $A$ be an $R$-submodule of $S$. The $R$-submodule $A$ is said to be $S$-regular if $AS = S$[5, Definition 1, p. 84]. For two $R$-submodules $E$, $F$ of $S$, denote by $[E : F]$ the set of all $x \in S$ such that $xF \subseteq E$. 

2020 MSC: 13A15, 13A18, 13B02.

Key words and phrases: ring extension, cancellation ideal, pullback diagram.
An $R$-submodule $A$ of $S$ is said to be $S$-invertible, if there exists an $R$-submodule $B$ of $S$ such that $AB = R[\mathbf{5}, \text{Definition 3, p 90}]$. In this case, we write $B = A^{-1}$, and $A^{-1} = [R : A] = \{x \in S : xA \subseteq R\}$ $[\mathbf{5}, \text{Remark 1.10, p. 90}]$. For the $R$-submodule $A$ of $S$, and for a multiplicative subset $\tau$ of $R$, we denote by $A[\tau]$ the set of all $x \in S$ such that $tx \in A$ for some $t \in \tau$. If $p$ is a prime ideal of $R$, and $\tau = R \setminus p$, then $A[p]$ denotes the set of all $x \in S$ such that $tx \in A$ for some $t \in \tau$. The set $A[\tau]$ is called the saturation of $A$ by $\tau$. Properties of the saturation of a submodule are studied in $[\mathbf{5}, \text{p. 18}]$ and $[\mathbf{6}]$.

An $S$-regular ideal $I$ of $R$ is called (quasi)-cancellation ideal of the ring extension $R \subseteq S$ if whenever $IB = IC$ for two $S$-regular (finitely generated) $R$-submodules $B$ and $C$ of $S$, then $B = C$. In section 2, we study properties of (quasi)-cancellation ideals of ring extensions. In Proposition 2.4, we prove that a finitely generated $S$-regular ideal $I$ of $R$ is a cancellation ideal if and only if it is a quasi-cancellation ideal. In Theorem 2.12, we show that for an $S$-regular finitely generated ideal $I$ of $R$, the followings are equivalent:

1. $I$ is a cancellation ideal of the ring extension $R \subseteq S$.
2. $I$ is an $S$-invertible ideal of $R$.
3. $IR[X]$ is a cancellation ideal of the ring extension $R[X] \subseteq S[X]$.

**Remark 1.1.** Let $R \subseteq S$ be a ring extension, and let $A, B$ be two $R$-submodules of $S$. Then $A = B$ if and only if $A[m] = B[m]$ for each maximal ideal $m$ of $R$. In fact, if $A = B$, then it clear that $A[m] = B[m]$ for each maximal ideal $m$ of $R$. Conversely, if $A[m] = B[m]$ for each $m \in M$, where $M$ is the set of all maximal ideals of $R$, then by $[\mathbf{5}, \text{Remark 5.5, p. 50}]$, we have $A = \bigcap_{m \in M} A[m] = \bigcap_{m \in M} B[m] = B$.

Let $R \subseteq S$ and $L \subseteq T$ be two ring extensions, and consider the following commutative diagram

$$
\begin{array}{ccc}
R & \xrightarrow{\alpha} & L \\
\downarrow & & \downarrow \\
S & \xrightarrow{\Psi} & T
\end{array}
$$

where ker $\Psi$ is an ideal of $R$, $\Psi : S \rightarrow T$ is surjective, the restriction $\alpha : R \rightarrow L$ of $\Psi$ is also surjective and the vertical mappings are inclusions. When ker $\Psi$ is a maximal ideal of $S$, the previous commutative diagram is called a pullback diagram a type $\Box$. Pullback diagrams of type $\Box$ are studied by S. Gabelli and E. Houston in $[\mathbf{4}]$. 
Lemma 1.2. Consider the above pullback diagram of type □. If \( A, B \) are two \( S \)-regular ideals of \( R \) such that \( \Psi(A) = \Psi(B) \), then \( A = B \).

Proof. Let \( A, B \) be two \( S \)-regular ideals of \( R \) such that \( \Psi(A) = \Psi(B) \). By [7, Remark 1.1], we have \( \ker \Psi \subseteq A \) and \( \ker \Psi \subseteq B \). Let \( a \in A \). Then there exists \( b \in B \) such that \( \Psi(a) = \Psi(b) \). Hence \( a - b \in \ker \Psi \subseteq B \). Thus \( a \in B \). This shows that \( A \subseteq B \). With the same argument, \( B \subseteq A \). Thus \( A = B \). □

2. Cancellation ideals of ring extensions

In this section, we define and study properties of cancellation ideals of ring extensions.

Definition 2.1. Let \( R \subseteq S \) be a ring extension. A nonzero \( S \)-regular ideal \( I \) of \( R \) is called a (quasi)-cancellation ideal of the ring extension \( R \subseteq S \) if whenever \( IB = IC \) for two \( S \)-regular (finitely generated) \( R \)-submodules \( B \) and \( C \) of \( S \), then \( B = C \).

The following proposition studies cancellation ideals in pullback diagram of type □. In this article, the Jacobson radical of a ring is denoted \( \text{Jac}(R) \).

Proposition 2.2. Suppose that the following diagram

\[
\begin{array}{ccc}
R & \longrightarrow & L \\
\downarrow & & \downarrow \\
S & \longrightarrow & T
\end{array}
\]

is a pullback diagram of type □ such that \( \ker \Psi \subseteq \text{Jac}(R) \). Then an \( S \)-regular ideal \( I \) of \( R \) is a cancellation ideal of the extension \( R \subseteq S \) if and only if \( \Psi(I) \) is a cancellation ideal of the extension \( L \subseteq T \).

Proof. Suppose that \( I \) is a cancellation ideal of the extension \( R \subseteq S \). Since \( IS = S \), we have \( \Psi(I)\Psi(S) = \Psi(S) \). It follows that \( \Psi(I)T = T \). Hence \( \Psi(I) \) is a \( T \)-regular ideal of \( L \). Let \( E \) and \( F \) be two \( T \)-regular \( L \)-submodules of \( T \) such that \( \Psi(I)E = \Psi(I)F \). Let \( B = \Psi^{-1}(E) \) and \( C = \Psi^{-1}(F) \). Then by [7, Lemma 2.8(1)] \( B \) and \( C \) are two \( S \)-regular ideals of \( R \). Furthermore, \( E = \Psi(B) \) and \( F = \Psi(C) \) since \( \Psi \) is surjective. It follows from the equality \( \Psi(I)E = \Psi(I)F \) that \( \Psi(I)\Psi(B) = \Psi(I)\Psi(C) \). Hence \( \Psi(IB) = \Psi(IC) \). Furthermore, \( (IB)S = IS = S \) and \( (IC)S = IS = S \). Therefore, by
Lemma 1.2, we have $IB = IC$. Hence $B = C$ since $I$ is a cancellation ideal of the extension $R \subseteq S$. It follows that $E = \Psi(B) = \Psi(C) = F$. This shows that $\Psi(I)$ is a cancellation ideal of the extension $L \subseteq T$.

Conversely, suppose that $\Psi(I)$ is a cancellation ideal of the extension $L \subseteq T$. Let $B$ and $C$ be two $S$-regular $R$-submodules of $S$ such that $IB = IC$. Then $\Psi(I)\Psi(B) = \Psi(I)\Psi(C)$. Since $BS = S$, we have $\Psi(B)T = T$. Hence $\Psi(B)$ is a $T$-regular ideal of $L$. With the same argument, $\Psi(C)$ is a $T$-regular ideal of $L$. It follows that $\Psi(B) = \Psi(C)$ since $\Psi(I)$ is a cancellation ideal of the extension $L \subseteq T$. Therefore, by Lemma 1.2, we have $B = C$. This shows that $I$ is a cancellation ideal of the extension $R \subseteq S$.

In the next proposition, we give a characterization of a cancellation ideal of a ring extension. This result is an analogue of [3, Proposition 2.1, p. 10] in the case of cancellation ideal of a ring.

**Proposition 2.3.** Let $R \subseteq S$ be a ring extension, and let $I$ be an $S$-regular ideal of $R$. The following statements are equivalent.

1. $I$ is a (quasi)-cancellation ideal of the ring extension $R \subseteq S$.
2. $[IJ : I] = J$ for any $S$-regular (finitely generated) $R$-submodule $J$ of $S$.
3. If $IJ \subseteq IK$ for two $S$-regular (finitely generated) $R$-submodules $J$ and $K$ of $S$, then $J \subseteq K$.

**Proof.** (1) $\Rightarrow$ (2) Suppose that $I$ is a cancellation ideal of the extension $R \subseteq S$, and let $J$ be an $S$-regular $R$-submodule of $S$. The containment $J \subseteq [IJ : I]$ is always true. Let $x \in [IJ : I]$. Then $xI \subseteq IJ$. It follows that $(x, J)I \subseteq IJ$, where $(x, J)$ is the $R$-submodule of $S$ generated by $x$ and $J$. Therefore, $(x, J)I = IJ$ since the containment $IJ \subseteq (x, J)I$ is always true. Furthermore, $(x, J)$ is an $S$-regular $R$-submodule of $S$ since $J \subseteq (x, J)$. It follows from the definition of a cancellation ideal that $(x, J) = J$. This shows that $x \in J$, and thus $[IJ : I] \subseteq J$. Therefore $[IJ : I] = J$.

(2) $\Rightarrow$ (3) Suppose that the statement (2) is true. Let $J$ and $K$ be two $S$-regular $R$-submodules of $S$. Then by (2), we have $[IK : I] = K$. If $IJ \subseteq IK$, then $J \subseteq [IK : I] = K$.

(3) $\Rightarrow$ (1) This implication is obvious. 

**Proposition 2.4.** Let $R \subseteq S$ be a ring extension, and let $I$ be a finitely generated $S$-regular ideal of $R$. Then $I$ is a cancellation ideal of $R \subseteq S$ if and only if $I$ is a quasi-cancellation ideal of $R \subseteq S$. 


Proof. Let \( I \) be a finitely generated \( S \)-regular ideal of \( R \). If \( I \) is a cancellation ideal of the extension \( R \subseteq S \), then obviously \( I \) is a quasi-cancellation ideal of the extension \( R \subseteq S \). Conversely, suppose that \( I \) is a quasi-cancellation ideal of the extension \( R \subseteq S \). Let \( a_1, \ldots, a_n \in R \) be a set of generators of \( I \). Let \( B, C \) be two \( S \)-regular \( R \)-submodules of \( S \) such that \( IB \subseteq IC \). Let \( b \in B \). Then \( bI \subseteq IC \). So, for \( 1 \leq i \leq n \), we have \( ba_i = \sum_{j=1}^{k} a_j c_{ij} \) with \( c_{ij} \in C \) for \( 1 \leq j \leq k \). Furthermore, since \( CS = S \), there exist \( u_1, \ldots, u_\ell \in C \) and \( s_1, \ldots, s_\ell \in S \) such that \( u_1 s_1 + \cdots + u_\ell s_\ell = 1 \).

Let \( C' \) be the \( R \)-submodule of \( S \) generated by the elements of the set \( \{u_1, \ldots, u_n, c_{ij} : 1 \leq i \leq n, 1 \leq j \leq k\} \). Let \( B_0 \) be the \( R \)-submodule of \( S \) generated by \( b \). Then \( B_0 + (u_1, \ldots, u_n)R \subseteq C' \). It follows from the equivalence (1) \( \iff \) (3) of Proposition 2.3 that \( B_0 + (u_1, \ldots, u_n)R \subseteq C' \) since \( B_0 + (u_1, \ldots, u_n) \) and \( C' \) are finitely generated \( S \)-regular ideal of \( S \). Therefore, \( b \in C' \subseteq C \). Hence \( B \subseteq C \) since \( b \) was arbitrary chosen in \( B \).

This shows that \( I \) is a cancellation ideal of the extension \( R \subseteq S \). \( \square \)

Lemma 2.5. Let \( R \subseteq S \) be a ring extension, and let \( u_1, \ldots, u_\ell \in S \).

Define the sets \( E = (u_1, \ldots, u_\ell)R_\mathfrak{p} \) and \( A = (u_1, \ldots, u_\ell)R \), where \( \mathfrak{p} \) is a prime ideal of \( R \). For any ideal \( I \) of \( R \), we have:

1. \( (AI)_\mathfrak{p} = (EI)_\mathfrak{p} \). In particular, \( A_\mathfrak{p} = E_\mathfrak{p} \).
2. \( (EI)_\mathfrak{p} = (EI_\mathfrak{p})_\mathfrak{p} \).

Proof. (1) First, observe that \( AI \subseteq EI \). So \( (AI)_\mathfrak{p} \subseteq (EI)_\mathfrak{p} \). Let \( x \in (EI)_\mathfrak{p} \). Then there exists \( t \in R \setminus \mathfrak{p} \) such that \( tx \in EI \). Therefore, \( tx = \sum_{i=1}^{n} e_ix_i \) for some \( e_i \in E \) and \( x_i \in I \), \( 1 \leq i \leq n \). For each \( 1 \leq i \leq n \), write \( e_i = \sum_{j=1}^{\ell} u_j y_{ij} \) with \( y_{ij} \in R_\mathfrak{p} \) for \( 1 \leq j \leq \ell \). Let \( s_{ij} \in R \setminus \mathfrak{p} \) such that \( s_{ij}y_{ij} \in R \), \( s_i = \prod_{j=1}^{\ell} s_{ij} \) and \( s = \prod_{i=1}^{n} e_i \). It follows that \( (st)x = \sum_{i=1}^{\ell} (se_i)x_i \in AI \). Thus \( x \in (AI)_\mathfrak{p} \) since \( st \in R \setminus \mathfrak{p} \). This shows that \( (EI)_\mathfrak{p} \subseteq (AI)_\mathfrak{p} \). Hence \( (AI)_\mathfrak{p} = (EI)_\mathfrak{p} \).

In particular, if we take \( I = R \), then we get \( A_\mathfrak{p} = E_\mathfrak{p} \).

(2) The containment \( (EI)_\mathfrak{p} \subseteq (EI_\mathfrak{p})_\mathfrak{p} \) is clear since \( EI \subseteq EI_\mathfrak{p} \). Let \( x \in (EI_\mathfrak{p})_\mathfrak{p} \). Then \( tx \in EI_\mathfrak{p} \) for some \( t \in R \setminus \mathfrak{p} \). Thus \( tx = \sum_{i=1}^{k} v_i y_i \) with \( v_i \in E \) and \( y_i \in I_\mathfrak{p} \) for \( 1 \leq i \leq k \). Let \( s_i \in R \setminus \mathfrak{p} \) such that \( s_i y_i \in I \), and let \( s = \prod_{i=1}^{k} s_i \). Then \( (st)x = \sum_{i=1}^{k} v_i (s y_i) \in EI \). It follows that \( x \in (EI)_\mathfrak{p} \). Therefore, \( (EI)_\mathfrak{p} = (EI_\mathfrak{p})_\mathfrak{p} \) \( \square \)

Theorem 2.6. Let \( R \subseteq S \) be a ring extension, and let \( I \) be a finitely generated \( S \)-regular ideal of \( R \). The following statements are equivalent.

1. \( I \) is a quasi-cancellation ideal of the extension \( R \subseteq S \).
(2) For each prime ideal \( p \) of \( R \), and for each \( S \)-regular finitely generated \( R[p] \)-submodule \( E \) of \( S \), we have \([((EI)[p] : I[p]] = E[p]\).

Proof. (1) \( \Rightarrow \) (2) Suppose that \( I \) is a quasi-cancellation ideal of the extension \( R \subseteq S \), and let \( p \) be a prime ideal of \( R \). Let \( E \) be a finitely generated \( S \)-regular \( R[p] \)-submodule of \( S \). Then \( E = (u_1, \ldots , u_\ell) R[p] \) for some elements \( u_1, \ldots , u_\ell \) of \( S \). Let \( A \) be the \( R \)-submodule of \( S \) generated by \( u_1, \ldots , u_\ell \). Then by Proposition 2.3 and Proposition 2.4, we have \([AI : I] = A\). It follows from [6, Proposition 2.1(4)] that \([((AI)_p : I_p[p]] = A[p]\). Hence by Lemma 2.5, we have \([((EI)_p[p] : I[p]] = [(EI)_p : I[p]] = [(AI)_p : I[p]] = A[p] = E[p] \).

(2) \( \Rightarrow \) (1) Suppose that the statement (2) is true. Let \( A \) be an \( S \)-regular finitely generated \( R \)-submodule of \( S \), and let \( p \) be a prime ideal of \( R \). Let \( E = AR[p] \). Then by Lemma 2.5, we have \((AI)_p = (EI)_p[p] \) and \( A[p] = E[p] \).

So, by hypothesis we have \( A[p] = E[p] = [(EI)_p : I[p]] = [(AI)_p : I[p]] \). But by [6, Proposition 2.1(4)], we have \([((AI)_p : I[p]] = [(AI) : I[p]] \). Therefore, \( A[p] = [(AI) : I[p]] \) for each prime ideal \( p \) of \( R \). It follows from Remark 1.1 that \([AI : I] = A\). Therefore, by the equivalence (1) \( \Leftrightarrow \) (2) of Proposition 2.3, \( I \) is a quasi-cancellation ideal of the extension \( R \subseteq S \). \( \square \)

In their book [5], Knebusch and Zhang defined the notion of Prüfer extension using valuation ring [5, Definition 1, p. 46]. Several characterizations of a Prüfer extension are given in [5, Theorem 5.2, p. 47]. For the purpose of this work, we will use the following: a ring extension \( R \subseteq S \) is called Prüfer extension if \( R \) is integrally closed in \( S \) and \( R[\alpha] = R[\alpha^n] \) for any \( \alpha \in S \) and any \( n \in \mathbb{N} \).

Lemma 2.7. [5, Theorem 1.13, p. 91] If a ring extension \( R \subseteq S \) is a Prüfer extension, then every finitely generated \( S \)-regular \( R \)-submodule of \( S \) is \( S \)-invertible.

Proposition 2.8. Let \( R \subseteq S \) be a ring extension, and let \( I \) be an \( S \)-regular ideal of \( R \).

(1) If \( I \) is a cancellation ideal of the extension \( R \subseteq S \), then \([I : I] = R\).

(2) If the extension \( R \subseteq S \) is Prüfer, then the converse of statement (1) is also true (i.e. in a Prüfer extension \( R \subseteq S \), if \( I \) is an \( S \)-regular ideal satisfying \([I : I] = R\), then \( I \) is a quasi-cancellation ideal).

Proof. (1) The proof follows directly from the equivalence (1) \( \Leftrightarrow \) (2) of Theorem 2.3. It suffices to take \( J = R \).

(2) Suppose that the extension \( R \subseteq S \) is Prüfer, and let \( I \) be an \( S \)-regular ideal of \( R \) such that \([I : I] = R\). Let \( A \) be an \( S \)-regular finitely
generated $R$-submodule of $S$. Then by Lemma 2.7, $A$ is $S$-invertible. We show that $A[I : I] = [AI : I]$. Let $x \in [AI : I]$. Then $xI \subseteq AI$. Hence $xA^{-1} \subseteq I$. Thus $xA^{-1} \subseteq [I : I]$. It follows that $x \in A[I : I]$. On the other hand, let $y = \sum_{i=1}^{k}a_iv_i \in A[I : I]$ with $a_i \in A$ and $v_i \in [I : I]$ for $1 \leq i \leq k$. Then $v_i I \subseteq I$. Hence $a_iv_i I \subseteq AI$. Therefore, $a_iv_i \in [AI : I]$. So $y = \sum_{i=1}^{k}a_iv_i \in [AI : I]$. This shows that $[AI : I] = A[I : I]$. Hence $[AI : I] = A[I : I] = AR = A$. Hence, by the equivalence $(1) \iff (2)$ of Proposition 2.3, $I$ is a quasi-cancellation ideal of the extension $R \subseteq S$. 

Let $R \subseteq S$ be a ring extension. A nonzero $S$-regular ideal $I$ of $R$ is called an $m$-canonical ideal of the extension $R \subseteq S$ if $[I : [I : J]] = J$ for all $S$-regular ideal $J$ of $R$. Properties of $m$-canonical ideals of a ring extension are studied in [7].

**Corollary 2.9.** Any $m$-canonical ideal of a Prüfer extension is a quasi-cancellation ideal.

**Proof.** If $I$ is an $m$-canonical ideal of a Prüfer extension $R \subseteq S$, then by [7, Proposition 2.3], we have $[I : I] = R$. It follows from Proposition 2.8(2) that $I$ is a quasi-cancellation ideal of the extension $R \subseteq S$. 

**Lemma 2.10.** Let $R \subseteq S$ be a ring extension, and let $I$ be an $S$-regular ideal of $R$ which is a cancellation ideal of $R \subseteq S$. If $I = (x, y) + A$, where $A$ is an ideal of $R$ containing $mI$ for some maximal ideal $m$ of $R$, then $I = (x) + A$ or $I = (y) + A$.

**Proof.** Let $J = (x^2 + y^2, xy, xA, yA, A^2)R$. Then $IJ = I^3$. Observe that $I^2$ is $S$-regular since $I^2S = I(IS) = IS = S$. Also, from the equality $IJ = I^3$ we have $(IJ)S = I^3S = I(IS) = IS = S$. So $JS = S$. This shows that $J$ is an $S$-regular ideal of $R$. It follows from the equation $IJ = I^3$ and the fact that $I$ is a cancellation ideal of the extension $R \subseteq S$ that $J = I^2$. Thus $x^2 = t(x^2 + y^2) + \text{terms from } (xy, xA, yA, A^2)$, with $t \in R$. Suppose that $t \in m$. Then $x^2 \in (y^2, xy, xA, yA, A^2)$, since $tx \in mI \in A$. Let $K = (y) + A$. Then $I^2 = IK$. Furthermore, from the equality $IK = I^2$, we have $K(IS) = I^2S$. Hence $KS = S$. Therefore, $K$ is an $S$-regular ideal of $S$. It follows that $I = K$ since $I$ is a cancellation ideal of the extension $R \subseteq S$. The rest of the proof is similar to the proof of [2, Lemma].

**Proposition 2.11.** Let $R \subseteq S$ be a ring extension, and let $I$ be a nonzero $S$-regular ideal of $R$. If $I$ is a cancellation ideal of the extension $R \subseteq S$, then for each maximal ideal $m$ of $R$, there exists $a \in R$ such that $I_{[m]} = (a)_{[m]}$. 

Theorem 2.12. Let $R \subseteq S$ be a ring extension, and let $I$ be a nonzero finitely generated $S$-regular ideal of $R$. The following statements are equivalent.

1. $I$ is a cancellation ideal of the extension $R \subseteq S$.
2. $I$ is a quasi-cancellation ideal of the extension $R \subseteq S$.
3. $I$ is an $S$-invertible ideal of $R$.
4. $IR[X]$ is a cancellation ideal of the extension $R[X] \subseteq S[X]$.

Proof. The equivalence (1) $\Leftrightarrow$ (2) is the result of Theorem 2.4.

(1) $\Rightarrow$ (3) Suppose that $I$ is a cancellation ideal of the extension $R \subseteq S$, and let $m$ be a maximal ideal of $R$. By the previous proposition, $I[m] = (a)[m]$ for some $a \in R$. It follows that $(I[m])_{m[m]} = ((a)[m])_{m[m]}$. But by [5, Lemma 2.9(b), p. 28], we have $I_m = (I[m])_{m[m]}$ and $(a)_m = ((a)[m])_{m[m]}$. Hence $I_m = (a)_m$. This shows that $I$ is locally principal. It follows from [5, Proposition 2.3, p. 97] that $I$ is $S$-invertible.

(3) $\Rightarrow$ (4) Suppose that $I$ is an $S$-invertible ideal of the extension $R \subseteq S$. First, note that $(IR[X])(S[X]) = S[X]$ since $IS = S$. Hence $IR[X]$ is an $S[X]$-regular ideal of $R[X]$. Let $J$ be the $R$-submodule of $S$ such that $IJ = R$. Then $(IR[X])(JR[X]) = R[X]$. This shows that $IR[X]$ is an $S[X]$-invertible ideal of $R[X]$. It follows from the equivalence (1) $\Leftrightarrow$ (3) that $IR[X]$ is a cancellation ideal of the extension $R[X] \subseteq S[X]$.

(4) $\Rightarrow$ (1) Suppose that $IR[X]$ is a cancellation ideal of the extension $R[X] \subseteq S[X]$. Let $J$ be an $S$-regular ideal of $R$. Then by the equivalence (1) $\Leftrightarrow$ (2) of Proposition 2.3, we have $[(IR[X])(JR[X]) : IR[X]] = JR[X]$.

We show that $[IJ : I] = J$. First, note that the containment $J \subseteq [IJ : I]$ is always true. Let $u \in [IJ : I]$. Then $uI \subseteq IJ$. Therefore, $uIR[X] \subseteq (IJ)R[X] \subseteq (IR[X])(JR[X])$. Hence $u \in [(IR[X])(JR[X]) : IR[X]] = JR[X]$. It follows that $u \in JR[X] \cap S = J$. This shows that $[IJ : I] \subseteq J$. Hence $[IJ : I] = J$. It follows from the equivalence (1) $\Leftrightarrow$ (2) of Proposition 2.3 that $I$ is a cancellation ideal of the extension $R \subseteq S$. □
Corollary 2.13. Let $R \subseteq S$ be a ring extension, and let $I$ be a finitely generated $S$-regular ideal of $R$. If $I$ is a cancellation ideal of the extension $R \subseteq S$, then $I_{[m]}$ is a cancellation ideal of the extension $R_{[m]} \subseteq S$ for each maximal ideal $m$ of $R$.

Proof. Let $I$ be a finitely generated $S$-regular ideal of $R$, and let $m$ be a maximal ideal of $R$. Suppose that $I$ is a cancellation ideal of the extension $R \subseteq S$. Then by the previous theorem, $I$ is $S$-invertible. Let $J$ be an $R$-submodule of $S$ such that $IJ = R$. Then $I_{[m]}J_{[m]} \subseteq (IJ)_{[m]} \subseteq R_{[m]}$. Furthermore, since $IS = S$, there exist $x_i \in I$ and $y_i \in J$, $1 \leq i \leq \ell$, such that $1 = \sum_{i=1}^{\ell} x_i y_i$. Let $u \in R_{[m]}$. There exists $t \in R \setminus m$ such that $tu \in R$ and $u = \sum_{i=1}^{\ell} (ux_i)y_i$. But for $1 \leq i \leq \ell$, $t(ux_i) = (tu)x_i \in I$ since $tu \in R$ and $x_i \in I$. It follows that $ux_i \in I_{[m]}$. Therefore, $u = \sum_{i=1}^{\ell} (ux_i)y_i \in I_{[m]}J \subseteq I_{[m]}J_{[m]}$. This shows that $R_{[m]} \subseteq I_{[m]}J_{[m]}$. Thus $I_{[m]}J_{[m]} = R_{[m]}$. Hence $I_{[m]}$ is an $S$-invertible $R_{[m]}$-submodule of $S$. It follows that $I_{[m]}$ is a cancellation ideal of the extension $R_{[m]} \subseteq S$, since an invertible ideal of ring extension is always a cancellation ideal. \qed

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Received by the editors: 26.07.2019
and in final form 30.10.2020.