The Commutators of Bochner–Riesz Operators for Hermite Operator

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Abstract
In this paper, we study the $L^p$-boundedness of the commutator $[b, S^\delta_R(H)](f) = bS^\delta_R(H)f - S^\delta_R(H)(bf)$ of a BMO function $b$ and the Bochner–Riesz means $S^\delta_R(H)$ for Hermite operator $H = -\Delta + |x|^2$ on $\mathbb{R}^n$, $n \geq 2$. We show that if $\delta > \delta(q) = n(1/q - 1/2) - 1/2$, the commutator $[b, S^\delta_R(H)]$ is bounded on $L^p(\mathbb{R}^n)$ whenever $q < p < q'$ and $1 \leq q \leq 2n/(n + 2)$.

Keywords Commutators · BMO · Bochner–Riesz means · Hermite operator · Spectral multipliers

Mathematics Subject Classification 42B15 · 42B20 · 47F10.

1 Introduction

We begin with recalling the Bochner–Riesz means on $\mathbb{R}^n$ which are defined by, for $\delta \geq 0$ and $R > 0$,

$$\hat{S^\delta_R f}(\xi) = \left(1 - \frac{|\xi|^2}{R^2}\right)\delta \hat{f}(\xi), \quad \forall \xi \in \mathbb{R}^n.$$
Here \((x)_+ = \max\{0, x\}\) for \(x \in \mathbb{R}\) and \(\hat{f}\) denotes the Fourier transform of \(f\). The commutator \([b, S^\delta_R]\) is defined by

\[
[b, S^\delta_R](f) = bS^\delta_R f - S^\delta_R (bf).
\]

Let \(b \in \text{BMO}(\mathbb{R}^n)\). If \(\delta \geq (n-1)/2\), the commutator \([b, S^\delta_R]\) is bounded on \(L^p(\mathbb{R}^n)\) for all \(1 < p < \infty\) ([10, p.129]); On the space \(\mathbb{R}^2\), if \(0 < \delta < 1/2\), Hu and Lu [6] proved that the commutator \([b, S^\delta_R]\) is bounded on \(L^p(\mathbb{R}^2)\) if \(4/(3 + 2\delta) < p < 4/(1 - 2\delta)\), which is, showed by Lu and Xia [9], also necessary. On the space \(\mathbb{R}^n, n \geq 3\), if \(0 < \delta < (n-1)/2\), Lu and Xia [9] showed that the commutator \([b, S^\delta_R]\) is bounded on \(L^p(\mathbb{R}^n)\) with \(p > 1\), then \(\delta > \delta(p) = \max\{n|1/p - 1/2| - 1/2, 0\}\). Conversely, if \((n-1)/(2n+2) < \delta < (n-1)/2\) Hu and Lu [6] showed that the commutator \([b, S^\delta_R]\) is bounded on \(L^p(\mathbb{R}^n)\) provided \(\delta > \delta(p)\). For other works about the commutator \([b, S^\delta_R]\) of Bochner–Riesz operators, see [1, 6, 9, 10, 12, 13] and the references therein.

In this paper, we are concerned the \(L^p\)-boundedness of commutators of a BMO function \(b\) and the Bochner–Riesz means for the Hermite operator \(H\) on \(\mathbb{R}^n\) for \(n \geq 2\), which is defined by

\[
H = -\Delta + |x|^2 = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + |x|^2, \quad x = (x_1, \cdots, x_n).
\]

The operator \(H\) is non-negative and self-adjoint with respect to the Lebesgue measure on \(\mathbb{R}^n\). Let \(\Phi_\mu\) be eigenfunctions for the Hermite operator with eigenvalue \((2|\mu| + n)\) and \(\{\Phi_\mu\}_{\mu \in \mathbb{N}_0^n}\) form a complete orthonormal system in \(L^2(\mathbb{R}^n)\). Thus, for every \(f \in L^2(\mathbb{R}^n)\), we have the Hermite expansion

\[
f(x) = \sum_{\mu} \langle f, \Phi_\mu \rangle \Phi_\mu(x) = \sum_{k=0}^{\infty} \sum_{|\mu| = k} \langle f, \Phi_\mu \rangle \Phi_\mu(x) =: \sum_{k=0}^{\infty} P_k f(x).
\]

For \(R > 0\), the Bochner–Riesz means for \(H\) of order \(\delta \geq 0\) is defined by

\[
S^\delta_R(H) f(x) := \left(I - \frac{H}{R^2}\right)_+^\delta := \sum_{k=0}^{\infty} \left(1 - \frac{2k + n}{R^2}\right)_+^\delta P_k f(x).
\]

On the space \(\mathbb{R}\), Thangavelu [14] showed that \(S^\delta_R(H)\) is uniformly bounded on \(L^p(\mathbb{R})\) for \(1 \leq p \leq \infty\) provided \(\delta > 1/6\). If \(0 < \delta < 1/6\), \(S^\delta_R(H)\) is uniformly bounded on \(L^p(\mathbb{R})\) if and only if \(4/(6\delta + 3) < p < 4/(1 - 6\delta)\). On the space \(\mathbb{R}^n, n \geq 2\), if \(\delta > (n-1)/2\), Thangavelu [15] showed that \(S^\delta_R(H)\) is uniformly bounded on \(L^p(\mathbb{R}^n)\) for \(1 \leq p \leq \infty\). Especially for the case \(p = 1\), \(\delta > (n-1)/2\) is also the necessary condition for the \(L^1(\mathbb{R}^n)\)-boundedness of \(S^\delta_R(H)\). When \(0 \leq \delta \leq (n-1)/2\) and \(p \neq 2\), it was conjectured (see [16, p.259]) that \(S^\delta_R(H)\) is bounded on \(L^p(\mathbb{R}^n)\) uniformly in
\[ \delta > \delta(p) = \max \left\{ n \left| \frac{1}{p} - \frac{1}{2} \right| - \frac{1}{2}, 0 \right\}. \]

Thangavelu proved the necessary part that if \( S^\delta_R(H) \) is \( L^p(\mathbb{R}^n) \) uniformly bounded, then \( \delta > \delta(p) \). Karadzhov [7] proved the sufficiency of the conjecture is true when \( p \) is in the range of \([1, 2n/(n+2)] \cup [2n/(n-2), \infty]\).

Let \( b \in \text{BMO}(\mathbb{R}^n) \). The commutator \([b, S^\delta_R(H)]\) is defined by \([b, S^\delta_R(H)](f) = bS^\delta_R(H)f - S^\delta_R(H)(bf)\). The aim of this paper is to prove the following result.

**Theorem 1.1** Let \( H \) be the Hermite operator defined in (1.1) on \( \mathbb{R}^n \), \( n \geq 2 \). Let \( 1 \leq p \leq 2n/(n+2) \) and \( \delta > \delta(p) \), then for all \( b \in \text{BMO}(\mathbb{R}^n) \)

\[
\sup_{R > 0} \left\| [b, S^\delta_R(H)] \right\|_{q \to q} \leq C \| b \|_{\text{BMO}},
\]

for all \( p < q < p' \).

We would like to mention that in [4], Tian, Ward and the first-named author obtained the \( L^q(\mathbb{R}^n) \)-boundedness of the commutator \([b, S^\delta_R(L)]\) of a BMO function \( b \) and the Bochner–Riesz means \( S^\delta_R(L) \) for a class of elliptic self-adjoint operators \( L \) on \( \mathbb{R}^n \), where \( L \) satisfies the finite speed of propagation property for the wave equation (see Sect. 2 below) and spectral measure estimate: For some \( 1 \leq p < 2 \),

\[
\| dE_{\sqrt{T}}(\lambda) \|_{p \to p'} \leq C \lambda^{n \left( \frac{1}{p} - \frac{1}{p'} \right) - 1}, \quad \lambda > 0. \tag{1.4}
\]

However, such estimate (1.4) is not available for the Hermite operator \( H = -\Delta + |x|^2 \) on \( \mathbb{R}^n \), and an argument in [4] does not work Theorem 1.1 very well. The proof of Theorem 1.1 will be given in Sect. 3 by using three non-trivial facts. The first is the restriction-type estimate due to Karadzhov [7]: for \( 1 \leq p \leq 2n/(n+2) \) and \( n \geq 2 \),

\[
\| \chi_{[k,k+1]}(H) \|_{p \to 2} = \| P_k \|_{p \to 2} \leq C k^{\frac{n}{2} \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{1}{2}}, \quad \forall k \in \mathbb{N}. \tag{1.5}
\]

The discrete restriction-type condition (1.5) is weaker than the restriction-type condition (1.4). To compensate for this difference, when proving the \( L^p \)-boundedness for the commutator \([b, S^\delta_R(H)]\) in our Theorem 1.1, we need a priori estimate that for any \( \varepsilon > 0 \) and \( 1 \leq r \leq 2 \),

\[
\| (I + H)^{-\frac{n}{2} \left( \frac{1}{r} - \frac{1}{2} \right) - \varepsilon} \|_{2 \to r} \leq C_\varepsilon. \tag{1.6}
\]

This is an important observation in our paper. Further, the fact that the eigenvalue of the Hermite operator is bigger than 1 is useful in the proof of Theorem 1.1.
2 Preliminaries

Throughout this paper, for \(1 \leq p \leq \infty\), we write the \(L^p\) norm of a function \(f\) by \(\|f\|_p\) and write \(\|T\|_{p \to q}\) for the operator norm of \(T\) if \(T\) is a bounded linear operator from \(L^p(\mathbb{R}^n)\) to \(L^q(\mathbb{R}^n)\). Given a subset \(E \subseteq X\), we denote the characteristic function of \(E\) by \(\chi_E\). For a given function \(F : \mathbb{R} \to \mathbb{C}\) and \(R > 0\), we define the function \(\delta_R F : \mathbb{R} \to \mathbb{C}\) by putting \(\delta_R F(x) = F(Rx)\).

**Lemma 2.1** Let \(H\) be the Hermite operator defined in (1.1). For any \(\varepsilon > 0\) and \(1 \leq r \leq 2\), there exists a constant \(C = C_{r, \varepsilon}\) such that

\[
\|(I + H)^{-\frac{\alpha}{2}(1 - \frac{1}{p}) - \varepsilon}\|_{2 \to r} \leq C. \tag{2.1}
\]

**Proof** The proof of (2.1) is proved in [5, Lemma 7.9] in the case \(r = 1\). For general \(1 \leq r \leq 2\), we refer to [2, p.3823 (6.5)]. \(\square\)

**Lemma 2.2** Let \(H\) be the Hermite operator defined in (1.1) on \(\mathbb{R}^n\), \(n \geq 2\). Then for \(1 \leq p \leq 2n/(n + 2)\),

\[
\|\chi_{[k,k+1]}(H)\|_{p \to 2} = \|P_k\|_{p \to 2} \leq Ck^{\frac{2}{p} - \frac{3}{2}} = Ck^{(\delta(p) - 1/2)/2}, \forall k \in \mathbb{N}. \tag{2.2}
\]

**Proof** For the proof of this lemma, we refer the reader to [7]. See also [2, 8, 16]. \(\square\)

Define

\[\mathcal{D}_t := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : |x - y| \leq t\}.\]

Given an operator \(T\) from \(L^p\) to \(L^q\), we denote

\[\text{supp} K_T \subseteq \mathcal{D}_t,\]

if \((Tf_1, f_2) = 0\) whenever \(f_i\) has support \(B(x_i, t_i), i = 1, 2\) and \(t_1 + t_2 + t < |x_1 - x_2|\). If \(T\) is an integral operator, then \(\text{supp} K_T \subseteq \mathcal{D}_t\) coincides with \(K_T(x, y) = 0, \forall (x, y) \notin \mathcal{D}_t\). We say that an operator \(L\) satisfies finite speed propagation property; it means that

\[\text{supp} K_{\cos(t\sqrt{L})} \subseteq \mathcal{D}_t, \forall t > 0. \tag{FS}\]

**Lemma 2.3** Let \(F\) be an even bounded Borel function and \(\hat{F} \in L^1(\mathbb{R})\) with \(\text{supp} \hat{F} \subseteq [-t, t]\). Then we have that \(K_{F(\sqrt{H})} \subseteq \mathcal{D}_t\).

**Proof** Since the heat kernel \(p_t(x, y)\) of \(e^{-tH}\) satisfies Gaussian upper bound, i.e.,

\[p_t(x, y) \leq (4\pi t)^{-n/2} \exp\left(-\frac{|x-y|^2}{4t}\right),\]

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it follows from (see, for example, [11, Theorem 2]) that \( H \) satisfies finite speed propagation property (FS). By Fourier inversion, for any even function \( F \),

\[
F(\sqrt{H}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{F}(t) \cos(t\sqrt{H})dt.
\]

It’s known from [3, Lemma I.1] with \( \text{supp} \hat{F} \subseteq [-t, t] \), we have that \( K_{F(\sqrt{H})} \subseteq D_t. \)

\[\square\]

3 Proof of Theorem 1.1

We start with the following lemma.

\textbf{Lemma 3.1} Suppose that \( T \) is a linear map and \( T \) is bounded from \( L^p \) to \( L^s \) for some \( 1 \leq p < s < \infty \). Assume also that \( \text{supp} K_T \subseteq D_\rho \) for some \( \rho > 0 \). Assume that function \( b \in \text{BMO} \). Then given a number \( q \) with \( 1 \leq p < q < s \), there exists a constant \( C = C_{p,q,s} \) such that

\[
\| [b, T] \|_{q \rightarrow q} \leq C \| b \|_{\text{BMO}} \rho^{n\left(\frac{1}{p} - \frac{1}{2}\right)} \| T \|_{p \rightarrow s}.
\]

\textbf{Proof} For the proof, we refer the reader to [4, Lemma 3.2]. \[\square\]

To prove Theorem 1.1, we will show the following result, which gives Theorem 1.1 as a straightforward consequence of Theorem 3.2. Indeed, we take \( F(\lambda) = (1 - \lambda^2)^\delta \) and \( t = R^{-1} \).

\textbf{Theorem 3.2} Let \( H \) be the Hermite operator defined in (1.1) on \( \mathbb{R}^n \), \( n \geq 2 \), \( 1 \leq p \leq 2n/(n+2) \) and \( s > n/(1/p - 1/2) \). Suppose \( b \in \text{BMO}(\mathbb{R}^n) \). Then for any even Borel function such that \( \text{supp} F \subseteq [-1, 1] \) and \( F \in \mathcal{W}_2^s(\mathbb{R}) \), the commutator \([b, F(\sqrt{H})]\) is bounded on \( L^q(\mathbb{R}^n) \) for all \( p < q < p' \) uniformly in \( t \). In addition,

\[
\sup_{t>0} \| [b, F(t\sqrt{H})] \|_{q \rightarrow q} \leq C \| b \|_{\text{BMO}} \| F \|_{W_2^s}. \tag{3.1}
\]

Before we start the proof of Theorem 3.2, let us show that Theorem 1.1 is a straightforward consequence of Theorem 3.2. Indeed, we take \( F(\lambda) = (1 - \lambda^2)^\delta \) and \( t = R^{-1} \). Note that \( F \in W_2^s \) if and only if \( \delta > s - 1/2 \). Hence, it follows from Theorem 3.2 that for all \( \delta > \delta(p) = n/(1/p - 1/2) - 1/2 \) and \( p < q < p' \),

\[
\| [b, F(t\sqrt{H})] \|_{q \rightarrow q} \leq C \| b \|_{\text{BMO}} \| F \|_{W_2^s} \leq C \| b \|_{\text{BMO}},
\]

with a bound \( C \) independent of \( t \). Then we have

\[
\sup_{R>0} \| [b, S_R^\delta (H)] \|_{q \rightarrow q} \leq C \| b \|_{\text{BMO}}.
\]
This completes the proof of Theorem 1.1.

The proof of Theorem 3.2 is inspired by [3–5]. To prove Theorem 3.2, we fix an even function \( \eta \in C^\infty_c(\mathbb{R}^n) \) supported in \( \{1/4 \leq |u| \leq 1\} \) such that \( \sum_{\ell \in \mathbb{Z}} \eta(2^{-\ell}u) = 1, \forall u \neq 0 \). Set \( \eta_0(u) = 1 - \sum_{\ell \geq 1} \eta(2^{-\ell}u) \). Write \( \eta^\ell (u) = \eta(2^{-\ell}u) \) for \( \ell \geq 1 \). Define

\[
F^{(\ell)}(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \eta^\ell(u) \hat{F}(u)e^{i\lambda u} du, \quad \ell \geq 0.
\]  

(3.2)

Then, we have

\[
F(t\sqrt{H}) = \sum_{\ell=0}^{\infty} F^{(\ell)}(t\sqrt{H}).
\]  

(3.3)

Noting that \( \text{supp} \ F \subseteq [-1, 1] \) and the eigenvalue of the Hermite operator is not less than 1, we can always assume that \( 0 < t \leq 1 \) in estimate (3.1). Let \( N = 50([t^{-1}] + 1) (N \approx t^{-1}) \). We pick up an even function \( \xi \in C^\infty_c([-1, 1]) \) with

\[
\hat{\xi}^{(0)}(0) = 1, \quad \hat{\xi}^{(1)}(0) = \cdots = \hat{\xi}^{(\kappa)}(0) = 0,
\]  

(3.4)

where \( \kappa \) is a large enough integer that will be chosen later. Let \( \xi_N(\lambda) = N\xi(N\lambda) \). Then

\[
[b, F(t\sqrt{H})] = [b, \delta_t F(\sqrt{H})]
\]

\[
\quad = [b, (\xi_N \ast \delta_t F)(\sqrt{H})] + [b, (\delta_t F - \xi_N \ast \delta_t F)(\sqrt{H})].
\]

It is clear that to prove Theorem 3.2, it is sufficient to show the following two lemmas 3.3 and 3.4.

**Lemma 3.3** Suppose \( b \in \text{BMO}(\mathbb{R}^n) \). Let \( 1 \leq p \leq 2n/(n+2) \), \( s > n(1/p - 1/2) \) and \( p < q < p' \). Then for any even Borel function such that \( \text{supp} F \subseteq [-1, 1] \) and \( F \in W^s_q(\mathbb{R}) \),

\[
\sup_{t > 0} \left\| [b, (\xi_N \ast \delta_t F)(\sqrt{H})] \right\|_{q \rightarrow q} \leq C \|b\|_{\text{BMO}} \|F\|_{W^s_q}.
\]  

(3.5)

**Lemma 3.4** Suppose \( b \in \text{BMO}(\mathbb{R}^n) \). Let \( 1 \leq p \leq 2n/(n+2) \), \( s > n(1/p - 1/2) \) and \( p < q < p' \). Then for any even Borel function such that \( \text{supp} F \subseteq [-1, 1] \) and \( F \in W^s_q(\mathbb{R}) \),

\[
\sup_{t > 0} \left\| [b, (\delta_t F - \xi_N \ast \delta_t F)(\sqrt{H})] \right\|_{q \rightarrow q} \leq C \|b\|_{\text{BMO}} \|F\|_{W^s_q}.
\]  

(3.6)
3.1 Proof of Lemma 3.3

First, we note that

\[ \operatorname{supp} (\hat{\xi}_N \ast \delta_t F^{(\ell)}) \subseteq [\sqrt{-2^{\ell+1}t}, 2^{\ell+1}t]. \]

By Lemma 2.3, we have that \( K_{\hat{\xi}_N \ast \delta_t F^{(\ell)}} \subseteq D_{2^{\ell+1}t} \). We then apply Lemma 3.1 to see that for \( p < q < 2 \),

\[
\|[b, (\hat{\xi}_N \ast \delta_t F^{(\ell)})(\sqrt{H})]\|_{q \to q} \leq C\|b\|_{\text{BMO}} \sum_{\ell=0}^{\infty} \| (\hat{\xi}_N \ast \delta_t F^{(\ell)})(\sqrt{H})\|_{p \to 2} (2^\ell t)^{n(\frac{1}{p} - \frac{1}{2})},
\]

and we will estimate each term \( \| \xi_N \ast \delta_t F^{(\ell)}(\sqrt{H})\|_{p \to 2} \) in the sequel. To do it, we pick up a function \( \psi \in C_0^\infty(\mathbb{R}) \) with support \([-4, 4]\) and \( \psi = 1 \) in \([-2, 2]\) to write

\[
(\xi_N \ast \delta_t F^{(\ell)})(\sqrt{H}) = \psi(t\sqrt{H})(\xi_N \ast \delta_t F^{(\ell)})(\sqrt{H}) + (1 - \psi)(t\sqrt{H})(\xi_N \ast \delta_t F^{(\ell)})(\sqrt{H}).
\]

(3.8)

Recall that \( 1 \leq p \leq 2n/(n + 2), n \geq 2 \), we have \( \delta(p) = n(1/p - 1/2) - 1/2 \). It follows by Lemma 2.2 that

\[
\| \psi(t\sqrt{H})(\xi_N \ast \delta_t F^{(\ell)})(\sqrt{H}) f \|_2^2 = \sum_{k \geq 0} \psi^2(t\sqrt{2k+n})(\xi_N \ast \delta_t F^{(\ell)})(\sqrt{2k+n}) \| P_k f \|_2^2 \leq C \sum_{k \geq 0} k^{\delta(p)-1/2} \psi^2(t\sqrt{2k+n})(\xi_N \ast \delta_t F^{(\ell)})(\sqrt{2k+n}) \| f \|_p^2.
\]

(3.9)

Noting \( \operatorname{supp} \psi \subseteq [-4, 4], \operatorname{supp} \xi \subseteq [-1, 1], t \approx N^{-1} \) and \( \delta(p) \geq 1/2 \), we have

\[
\| \psi(t\sqrt{H})(\xi_N \ast \delta_t F^{(\ell)})(\sqrt{H})\|_{p \to 2}^2 \leq C \sum_{t\sqrt{2k+n} \leq 4} k^{\delta(p)-1/2} (\xi_N \ast \delta_t F^{(\ell)})(\sqrt{2k+n}) \leq Ct^{-2(\delta(p)-1/2)} \sum_{t\sqrt{2k+n} \leq 4} (\xi_N \ast \delta_t F^{(\ell)})(\sqrt{2k+n}) \leq C\|\xi\|_2^2 t^{-2(\delta(p)-1/2)} \sum_{t\sqrt{2k+n} \leq 4} \int_{N\sqrt{2k+n}-1}^{N\sqrt{2k+n}+1} |F^{(\ell)}(tN^{-1}y)|^2 dy \leq C t^{-2(\delta(p)-1/2)} \int_{\mathbb{R}} |F^{(\ell)}(tN^{-1}y)|^2 dy \approx Ct^{-2n(\frac{1}{p} - \frac{1}{2})} \| F^{(\ell)} \|_2^2,
\]

(3.10)

where in the last inequality we use the fact that the sets \( \left\{ [N\sqrt{2k+n}-1, N\sqrt{2k+n}+1] \right\}_k \) are disjoint whenever \( t\sqrt{2k+n} \leq 4 \).
Now we consider the term $(1 - \psi)(t_\sqrt{H})(\xi_N \ast \delta_{t}F^{(\ell)})(\sqrt{H})$. We apply Lemma 2.2 to obtain

\[
\| (1 - \psi)(t_\sqrt{H})(\xi_N \ast \delta_{t}F^{(\ell)})(\sqrt{H}) \|_{p \to 2}^2 \\
\leq C \sum_{k \geq 0} k^{\delta(p)-1/2}(1 - \psi)^2(t_\sqrt{2k+n})(\xi_N \ast \delta_{t}F^{(\ell)})^2(t_\sqrt{2k+n}) \\
\leq C \sum_{t_\sqrt{2k+n} \geq 2} k^{\delta(p)-1/2}(\xi_N \ast \delta_{t}F^{(\ell)})^2(t_\sqrt{2k+n}).
\]

(3.11)

Since $\text{supp} \xi \in [-1, 1]$, $\text{supp} (1 - \psi) \in [2, \infty)$, $\text{supp} F \in [-1, 1]$, $\tilde{\eta}_{\ell}$ is a Schwartz function and $0 < tN^{-1} \leq 1/2$, we have that for any $M > 0$

\[
|(1 - \psi)(t\lambda)(\xi_N \ast \delta_{t}F^{(\ell)})(\lambda)| \leq \| \xi \|_1 |(1 - \psi)(t\lambda)| \| X_{[t\lambda-tN^{-1},t\lambda+tN^{-1}]} F \ast \tilde{\eta}_{\ell} \|_{L^\infty} \\
\leq C_M \| X_{|t\lambda| \geq 2} \int_{-1}^{1} |F(y)| 2^\ell \left(1 + 2^\ell |t\lambda| - y\right) - M dy \\
\leq C_M \| X_{|t\lambda| \geq 2} 2^{-2(M-1)\ell} |t\lambda|^{-M} F_2.
\]

(3.12)

Hence,

\[
\text{RHS of (3.11)} \leq C 2^{-2(M-1)\ell} \| F_2 \|_2^2 \sum_{t_\sqrt{2k+n} \geq 2} k^{\delta(p)-1/2}(t_\sqrt{2k+n})^{-2M} \\
\leq C 2^{-2(M-1)\ell} t^{-2n(\frac{1}{p}-\frac{1}{2})} \| F_2 \|_2.
\]

(3.13)

Recall that $s > n(1/p - 1/2)$, $W^2_s(\mathbb{R}) \subseteq B^{2,1}_{n(\frac{1}{p}-\frac{1}{2})}(\mathbb{R})$, which in combination with estimates (3.10) and (3.13) implies that

\[
\text{RHS of (3.7)} \leq C_M \| b \|_{\text{BMO}} \sum_{\ell \geq 0} \left(2^{\ell n(\frac{1}{p}-\frac{1}{2})} \| F^{(\ell)} \|_2 + 2^{-2(M-1)\ell} \| F \|_2 \right) \\
\leq C \| b \|_{\text{BMO}} \left( \| F \|_{B^{2,1}_{n(\frac{1}{p}-\frac{1}{2})}} + \| F \|_2 \right) \leq C \| b \|_{\text{BMO}} \| F \|_{W^2_s}.
\]

By duality and interpolation argument, we see that for $p < q < p'$ and $s > n(1/p - 1/2)$,

\[
\| [b, (\xi_N \ast \delta_{t}F)(\sqrt{H})] \|_{q \to q} \leq C \| b \|_{\text{BMO}} \| F \|_{W^2_s}.
\]

This completes the proof of Lemma 3.3.

\[\Box\]

3.2 Proof of Lemma 3.4

To prove Lemma 3.4, we need the following result.
Lemma 3.5  Let $1 \leq p \leq 2n/(n + 2)$. Suppose $\theta \in C^\infty_c([-8, 8])$. With the notations as in Lemma 3.4, there exists a constant $C$ independent of $t$ such that

$$\|\theta(2^{-j}t\sqrt{H})(\delta_t F^{(\ell)} - \xi_N * \delta_t F^{(\ell)})(\sqrt{H})\|_{p \to 2} \leq \begin{cases} C\|\theta\|_{\infty} F \|_{W^2_p} 2^{-j/2}N^{-2s}(2jN)^{n(\frac{1}{p} - \frac{1}{2})}(2^\ell N - 2)^{\kappa + 1 - s}, & \text{if } 2^\ell tN^{-1} \leq 1; \\
C\|\theta\|_{\infty} F \|_{W^2_p} 2^{-j/2}N^{-2s}(2jN)^{n(\frac{1}{p} - \frac{1}{2})}(2^\ell N - 2)^{1/2 - s}, & \text{if } 2^\ell tN^{-1} > 1 \end{cases}$$

(3.14)

for all $j, \ell \in \mathbb{N}$, where $\kappa$ is the constant in estimate (3.4).

The proof of Lemma 3.5 will be given later.

Now let us apply Lemma 3.5 to prove Lemma 3.4. We pick up functions $\psi \in C^\infty_c([-4, 4])$ and $\phi \in C^\infty_c([2, 8])$ such that $\psi(\lambda) + \sum_{j \geq 0} \phi(2^{-j}\lambda) = 1$ for all $\lambda > 0$. Let $p < r < 2$. For any $0 < \varepsilon < 1/2$, we consider $r_1$ such that $r < r_1 \leq 2$ and $n(1/p - 1/r_1) < \varepsilon/2$. Then we apply Lemma 3.1 with (3.3) to obtain

$$\|[b, (\delta_t F - \xi_N * \delta_t F)(\sqrt{H})]\|_{r \to r} \leq C\|b\|_{BMO} \sum_{\ell = 0}^{\infty} \|((\delta_t F^{(\ell)} - \xi_N * \delta_t F^{(\ell)})(\sqrt{H})\|_{p \to r_1}(2^\ell t)^{n(\frac{1}{p} - \frac{1}{r_1})}$$

(3.15)

and

$$\|((\delta_t F^{(\ell)} - \xi_N * \delta_t F^{(\ell)})(\sqrt{H})\|_{p \to r_1} \leq \|\psi(t\sqrt{H})(\delta_t F^{(\ell)} - \xi_N * \delta_t F^{(\ell)})(\sqrt{H})\|_{p \to r_1}$$

$$+ \sum_{j \geq 0} \|\phi(2^{-j}t\sqrt{H})(\delta_t F^{(\ell)} - \xi_N * \delta_t F^{(\ell)})(\sqrt{H})\|_{p \to r_1}$$

$$=: E_1(\ell, t) + \sum_{j \geq 0} E_2(j, \ell, t).$$

(3.16)

In the following, we set $\gamma = n(1/r_1 - 1/2)$. For the term $E_1(\ell, t)$, we note that $\text{sup supp } \psi \subseteq [-4, 4]$ and $N \approx t^{-1} > 1$. This gives

$$\sup_{\lambda} |\psi(t\lambda)(1 + |\lambda|^2)^{(\gamma + \varepsilon)/2}| \leq Ct^{-(\gamma + \varepsilon)} \leq CN^{\gamma + \varepsilon}.$$

By Lemma 2.1 and Lemma 3.5,

$$E_1(\ell, t) \leq \|\psi(t\sqrt{H})(I + H)^{(\gamma + \varepsilon)/2}(\delta_t F^{(\ell)} - \xi_N * \delta_t F^{(\ell)})$$

$$\|\psi(t\sqrt{H})(I + H)^{-(\gamma + \varepsilon)/2}\|_{2 \to r_1}$$

$$\leq C\|\psi(t\sqrt{H})(I + H)^{(\gamma + \varepsilon)/2}(\delta_t F^{(\ell)} - \xi_N * \delta_t F^{(\ell)})(\sqrt{H})\|_{p \to 2}$$
\[
\leq \begin{cases} 
C \| F \|_{W^s_2} N^{n(\frac{1}{p} - \frac{1}{2}) + \gamma - 2s + \varepsilon} (2^\ell N^{-2})^{\kappa + 1 - s}, & \text{if } 2^\ell tN^{-1} \leq 1; \\
C \| F \|_{W^s_2} N^{n(\frac{1}{p} - \frac{1}{2}) + \gamma - 2s + \varepsilon} (2^\ell N^{-2})^{1/2 - s}, & \text{if } 2^\ell tN^{-1} \geq 1.
\end{cases}
\]

This, in combination with the fact that \( s > n(1/p - 1/2) \geq 1, n(1/p - 1/r_1) < \varepsilon/2 \) and our selection of \( \kappa \) such that \( \kappa \geq s \), yields

\[
\sum_{\ell=0}^\infty \| b \|_{BMO} \| \psi(t\sqrt{\mathcal{H}})(\delta_t F^{(\ell)} - \xi_N * \delta_t F^{(\ell)})(\sqrt{\mathcal{H}}) \|_{p \rightarrow r_1} (2^\ell t)^{n(\frac{1}{p} - \frac{1}{r_1})} \
\leq C \| b \|_{BMO} \| F \|_{W^s_2} N^{2n(\frac{1}{p} - \frac{1}{2}) - 2s + \varepsilon} \
\times \left( \sum_{2^\ell tN^{-1} < 1} 2^\ell n(\frac{1}{p} - \frac{1}{2}) (2^\ell N^{-2})^{\kappa + 1 - s} + \sum_{2^\ell tN^{-1} \geq 1} 2^\ell n(\frac{1}{p} - \frac{1}{2}) (2^\ell N^{-2})^{1/2 - s} \right) \
\leq C \| b \|_{BMO} \| F \|_{W^s_2} N^{2n(\frac{1}{p} - \frac{1}{2}) - 2s + \varepsilon}.
\] (3.17)

Now we estimate the terms \( E_2(j, \ell, t) \) for \( j \geq 0, \ell \geq 0 \). Since \( \text{supp } \phi \subseteq [2, 8] \subseteq [-8, 8] \) and \( N \approx t^{-1} > 1 \), we have that \( \text{supp} \phi(2^{-j} t \lambda)(1 + |\lambda|^2)^{(\gamma + \varepsilon)/2} \leq C(2^jN)^{\gamma + \varepsilon} \). By Lemmas 2.1 and 3.5,

\[
E_2(j, \ell, t) \leq C \| \phi(2^{-j} t \sqrt{\mathcal{H}})(J + H)^{(\gamma + \varepsilon)/2} (\delta_t F^{(\ell)} - \xi_N * \delta_t F^{(\ell)})(\sqrt{\mathcal{H}}) \|_{p \rightarrow 2} \
\leq \begin{cases} 
C \| F \|_{W^s_2} 2^{-j/2} N^{-2s} (2^j N)^{n(\frac{1}{p} - \frac{1}{2}) + \gamma + \varepsilon} (2^\ell N^{-2})^{\kappa + 1 - s}, & \text{if } 2^\ell tN^{-1} \leq 1; \\
C \| F \|_{W^s_2} 2^{-j/2} N^{-2s} (2^j N)^{n(\frac{1}{p} - \frac{1}{2}) + \gamma + \varepsilon} (2^\ell N^{-2})^{1/2 - s}, & \text{if } 2^\ell tN^{-1} \geq 1.
\end{cases}
\] (3.18)

On another hand, we also have that

\[
E_2(j, \ell, t) \leq C \left( \sum_{k \geq 0} k^{\delta(p)-1/2} (2k + n)^{\gamma + \varepsilon} \phi^2(2^{-j} t \sqrt{2k + n})(\delta_t F^{(\ell)} - \xi_N * \delta_t F^{(\ell)})(\sqrt{2k + n}) \right)^{1/2} \
\leq C 2^{-M\ell} 2^{-(M+1)j} \| F \|_1 \left( \sum_{2^j \leq t \sqrt{2k + n} \leq 2^j} (2k + n)^{\delta(p)-1/2 + \gamma + \varepsilon} \right)^{1/2} \
\leq C 2^{-M\ell} 2^{-(M+1)j} (2^j N)^{n(\frac{1}{p} - \frac{1}{2}) + \gamma + \varepsilon} \| F \|_2,
\] (3.19)

where in the second inequality we use the inequality that

\[
|\phi(2^{-j} t \lambda)| \left( |\delta_t F^{(\ell)}(\lambda)| + |\xi_N * \delta_t F^{(\ell)}(\lambda)| \right) \leq C_M 2^{-M\ell} 2^{-(M+1)j} \| F \|_2,
\]

which can be obtained by a similar discussion of estimate (3.12).

Next we set \( \alpha := n(1/p - 1/2) + \gamma + \varepsilon, \beta := n(1/p - 1/r_1) \) and \( \nu = (\alpha - 1/2)/(M + 1/2) \) with \( M > \alpha - 1 \). Recall that \( 1 \leq p \leq 2n/(n + 2) \), so \( \alpha > 1 \) and
0 < \nu < 1. When $2^\ell N^{-1} t \leq 1$, we choose $j_1$ such that
\[(2^{j_1} N)^\alpha 2^{-j_1/2} N^{-2s} (2^\ell N^{-2})^{\kappa+1-s} = (2^{j_1} N)^\alpha 2^{-(M+1)j_1} 2^{-M\ell}.\]
This, in combination with estimates (3.18) and (3.19), yields
\[
\sum_{j\geq 0} E_2(j, \ell, t) \leq C_M \|F\|_{W^2_{p}} \left( \sum_{j\geq j_1} 2^{-M\ell} (2^{j} N)^\alpha 2^{-(M+1)j} \right.
\[ + \sum_{j\leq j_1} 2^{-j/2} (2^{j} N)^\alpha N^{-2s} (2^\ell N^{-2})^{\kappa+1-s} \right)
\leq C_M \|F\|_{W^2_{p}} 2^{-M\ell} N^{\alpha-2s(1-\nu)} (2^\ell N^{-2})^{(1-\nu)(\kappa+1-s)}. \tag{3.20}\]
When $2^\ell N^{-1} t \geq 1$, we choose $j_2$ such that
\[(2^{j_2} N)^\alpha 2^{-j_2/2} N^{-2s} (2^\ell N^{-2})^{1/2-s} = (2^{j_2} N)^\alpha 2^{-(M+1)j_2} 2^{-M\ell}.\]
It follows by estimates (3.18) and (3.19) that
\[
\sum_{j\geq 0} E_2(j, \ell, t) \leq C_M \|F\|_{W^2_{p}} \left( \sum_{j\geq j_2} 2^{-M\ell} (2^{j} N)^\alpha 2^{-(M+1)j} \right.
\[ + \sum_{j\leq j_2} 2^{-j/2} (2^{j} N)^\alpha N^{-2s} (2^\ell N^{-2})^{1/2-s} \right)
\leq C_M \|F\|_{W^2_{p}} 2^{-M\ell} N^{\alpha-2s(1-\nu)} (2^\ell N^{-2})^{(1-\nu)(1/2-s)}. \tag{3.21}\]
Combining estimates (3.16), (3.20) and (3.21), we use the facts that $s > n(1/p - 1/2) \geq 1$, $\kappa \geq s$ and $N \approx t^{-1}$ to see that
\[
\sum_{\ell \geq 0} \sum_{j \geq 0} \|b\|_{BMO(2^\ell t)^\beta} \|\phi(2^{-j} t \sqrt{H}) (\delta_t F^{(\ell)} - \xi_N * \delta_t F^{(\ell)})(\sqrt{H})\|_{p \to r_1}
\leq C_M \|b\|_{BMO} \|F\|_{W^2_p} N^{\alpha-2s(1-\nu)}
\times \left( \sum_{2^\ell N^{-1} t \leq 1} 2^{-M\ell} (2^\ell N^{-2})^{(1-\nu)(\kappa+1-s)} (2^\ell t)^\beta \right.
\[ + \sum_{2^\ell N^{-1} t \geq 1} 2^{-M\ell} (2^\ell N^{-2})^{(1-\nu)(1/2-s)} (2^\ell t)^\beta \right)
\leq C_M \|b\|_{BMO} \|F\|_{W^2_p} N^{\alpha+\beta-2M\nu-2s(1-\nu)}
\times \left( \sum_{2^\ell N^{-1} t \leq 1} (2^\ell N^{-2})^{-M\nu+(1-\nu)(\kappa+1-s)+\beta} \right.
\[ + \sum_{2^\ell N^{-1} t \geq 1} (2^\ell N^{-2})^{-M\nu+(1-\nu)(1/2-s)+\beta} \right)
\[ \begin{align*}
\leq C_M \|b\|_{BMO} \|F\|_{W^2_2 N^{\alpha+\beta-2s-2(M-s)\nu}}, \\
\leq C \|b\|_{BMO} \|F\|_{W^2_2},
\end{align*} \]

(3.22)

where we take \( M \) and \( \kappa \) such that \( M \geq \max\{s, 2\alpha\} \) and \( \kappa \geq 2\alpha + s - 2 > s \). Recall that \( \alpha > 1, \beta < \varepsilon/2 < 1/4 \) and \( 0 < \nu < 1 \). When \( s > \max((\alpha + \beta)/2, 1/2) \), i.e., \( s > n(1/p - 1/2) + \varepsilon/2 \), we have

\[ \begin{align*}
\alpha + \beta - 2s - 2(M-s)\nu < 0, \\
-M\nu + (1-\nu)(\kappa + 1-s) + \beta > 0, \\
-M\nu + (1-\nu)(1/2-s) + \beta < 0.
\end{align*} \]

(3.23)

So estimate (3.22) holds.

Combining the estimates (3.15), (3.16), (3.17), (3.22), we obtain that for any \( s > n(1/p - 1/2) + \varepsilon \) and any \( r \) such that \( 0 < 1/p - 1/r < \varepsilon/2 \),

\[ \| [b, (\delta_t F - \xi_N * \delta_t F)(\sqrt{H})] \|_{r \to r} \leq C_{\varepsilon} \|b\|_{BMO} \|F\|_{W^2_2}. \]

By duality and interpolation argument, we have proved estimate (3.6) of Lemma 3.4 for \( p < q < p' \) and \( s > n(1/p - 1/2) + \varepsilon \). Hence, we obtain the proof of Lemma 3.4 provided Lemma 3.5 is proved.

Finally, let us prove Lemma 3.5.

**Proof of Lemma 3.5** It follows from the Hermite expansion, Lemma 2.2 and \( \text{supp} \theta \subseteq [-8, 8] \) that

\[ \begin{align*}
\| \theta(2^{-j} t \sqrt{H})(\delta_t F^{(\ell)}) - \xi_N * \delta_t F^{(\ell)}(\sqrt{H}) \|_{p \to 2}
\leq C \left( \sum_{k \geq 0} k^{\delta(p)-1/2} \frac{\theta^2(2^{-j} t \sqrt{2k+n})(\delta_t F^{(\ell)})^2(\sqrt{2k+n})}{t^{\sqrt{2k+n} \leq 2^{j+3}}} \right)^{1/2},
\leq C \|\theta\|_{\infty} (2^{-j} t^{-1})^{\delta(p)-1/2} \left( \sum_{t^{\sqrt{2k+n} \leq 2^{j+3}}} (\delta_t F^{(\ell)})^2(\sqrt{2k+n}) \right)^{1/2}.
\end{align*} \]

(3.24)

Let \( 0 < \mu \leq 1 \), we first define two functions \( \hat{H}(\lambda) \) and \( \hat{\Omega}^{(\ell)}_{\mu,j} \) by

\[ \hat{H}(\lambda) = |2^j \lambda|^s 2^j N t^{-1} \hat{F}(2^j N t^{-1} \lambda), \]
\[ \hat{\Omega}^{(\ell)}_{\mu,j}(\lambda) = |2^j \lambda|^{-s} (1 - \hat{\xi}(2^j \lambda)) \eta(2^j \mu^{-1} \lambda). \]

Write \( \hat{\xi}_{2^{-j}}(\lambda) = 2^{-j} \hat{\xi}(2^{-j} \lambda) \). Observe that

\[ (\delta_t F^{(\ell)} - \xi_N * \delta_t F^{(\ell)})(\sqrt{2k+n}) = (\delta_t(2^j N)^{-1} F^{(\ell)} - \hat{\xi}_{2^{-j}} * \delta_t(2^j N)^{-1} F^{(\ell)})(2^j N \sqrt{2k+n}). \]
\[ H * \Omega_{tN-1, j}^{(f)}(2^j N \sqrt{2k + n}). \] (3.25)

We write \( \Omega_{\mu, j}^{(f)} \) as \( \sum_{m \in \mathbb{Z}} \Omega_{\mu, j}^{(f), m}(\lambda) \), where

\[
\Omega_{\mu, j}^{(f), m}(\lambda) := \begin{cases} \Omega_{\mu, j}^{(f)}(\lambda) \chi_{[m-1, m+1]}(\lambda), & m \text{ is odd;} \\ 0, & m \text{ is even.} \end{cases}
\]

From Hölder’s inequality and Minkowski’s inequality, we have

\[
\left( \sum_{t \sqrt{2k+n} \leq 2^{j+3}} |H * \Omega_{tN-1, j}^{(f)}(2^j N \sqrt{2k + n})|^2 \right)^{1/2} \\
\leq \left( \sum_{t \sqrt{2k+n} \leq 2^{j+3}} \left( \sum_{m \in \mathbb{Z}} \| \Omega_{tN-1, j}^{(f), m} \|_2 \left( \int_{2^j N \sqrt{2k+n} - m}^{2^j N \sqrt{2k+n} - m+1} |H(y)|^2 dy \right)^{1/2} \right)^2 \right)^{1/2} \\
\leq \sum_{m \in \mathbb{Z}} \| \Omega_{tN-1, j}^{(f), m} \|_2 \left( \sum_{t \sqrt{2k+n} \leq 2^{j+3}} \int_{2^j N \sqrt{2k+n} - m}^{2^j N \sqrt{2k+n} - m+1} |H(y)|^2 dy \right)^{1/2} \\
\leq \sum_{m \in \mathbb{Z}} \| \Omega_{tN-1, j}^{(f), m} \|_2 \| H \|_2, \] (3.26)

where we use the fact that the sets \( \{[2^j N \sqrt{2k+n} - m - 1, 2^j N \sqrt{2k+n} - m + 1] \}_k \) are disjoint whenever \( t \sqrt{2k+n} \leq 2^{j+3} \) in the last inequality. Indeed, when \( t \sqrt{2k+n} \leq 2^{j+3} \), we have

\[
\sqrt{2k+n} + 2 + \sqrt{2k+n} \leq 2^{j+4} t^{-1} \leq 2^j N,
\]

which is equivalent to

\[
2^j N \sqrt{2k+n} - m + 1 \leq 2^j N \sqrt{2k+2+n} - m - 1.
\]

By the definition of homogeneous Sobolev spaces and \( t \approx N^{-1} \), we have

\[
\| H \|_2 \leq 2^{j^s} \| \delta_{t(2^j N)^{-1}} F \|_{\dot{W}^{2s}} = 2^{j/2} (tN^{-1})^{s-1/2} \| F \|_{\dot{W}^{2s}} \leq C 2^{j/2} N^{-2s+1} \| F \|_{\dot{W}^{2s}},
\]

which in combination with estimates (3.25)-(3.26) implies that

\[
\text{RHS of (3.24)} \leq C \| \theta \|_{\infty} \| F \|_{\dot{W}^{2s}} 2^{-j/2} (2^j N)^{n(1/p - 1)} N^{-2s} \sum_m \| \Omega_{tN-1, j}^{(f), m} \|_2. \] (3.27)
Hence, the proof of estimate (3.14) reduces to show that for any $\ell, j \in \mathbb{N}$ and any $0 < \mu \leq 1$,
\[
\sum_{m \in \mathbb{Z}} \| \Omega_{\mu,j,m}^{(\ell)} \|_2 \leq \begin{cases} 
C(2^\ell \mu)^{\kappa+1-s}, & \text{if } 2^\ell \mu \leq 1; \\
C(2^\ell \mu)^{1/2-s}, & \text{if } 2^\ell \mu > 1,
\end{cases}
\] (3.28)
since the desired estimate (3.14) follows easily by setting $\mu = tN^{-1}$ and taking estimate (3.28) into estimate (3.27).

Let us prove estimate (3.28). We write $\Omega_{\mu,j,m}^{(\ell)}(\lambda) := 2^{-j} \xi_{\mu}^{(\ell)}(2^{-j} \lambda)$, where
\[
\hat{\xi}_{\mu}^{(\ell)}(a) := (1 - \hat{\xi}(a)) \eta(\mu^{-1}a)|a|^{-s}.
\]
We will show that for any $0 < \mu \leq 1$,
\[
|\xi_{\mu}^{(0)}(\lambda)| \leq C(1 + |\mu\lambda|)^{-(\kappa+2-s)} \mu^{\kappa+2-s},
\] (3.29)
and for every $\ell \in \mathbb{N}^+$ and $M \geq 0$,
\[
|\xi_{\mu}^{(\ell)}(\lambda)| \leq \begin{cases} 
C_M (2^\ell \mu)^{\kappa+2-s}(1 + |2^\ell \mu\lambda|)^{-M}, & \text{if } 2^\ell \mu \leq 1; \\
C_M (2^\ell \mu)^{1-s}(1 + |2^\ell \mu\lambda|)^{-M}, & \text{if } 2^\ell \mu > 1.
\end{cases}
\] (3.30)

Recall that $\hat{\xi}$ and $\eta_0$ are even functions, we see that
\[
\xi_{\mu}^{(0)}(\lambda) = \int_0^\infty (1 - \hat{\xi}(a)) \eta_0(\mu^{-1}a)|a|^{-s} (e^{ia\lambda} + e^{-ia\lambda}) da
\]
\[
= \sum_{j \leq 0} \int_0^\infty (1 - \hat{\xi}(a)) \eta(2^{-j} \mu^{-1}a)|a|^{-s} (e^{ia\lambda} + e^{-ia\lambda}) da
\]
\[
= \sum_{j \leq 0} (2^j \mu)^{1-s} \int_0^\infty (1 - \hat{\xi}(2^j \mu a)) \eta(a)a^{-s} \left(e^{i2^j \mu\lambda a} + e^{-i2^j \mu\lambda a}\right) da.
\]

For any $\alpha \in \mathbb{N}$, from Taylor’s expansion of $\hat{\xi}$ at original point and the compact support of $\eta$, we have $|\partial^\alpha \left( (1 - \hat{\xi}(2^j \mu a)) \eta(a)a^{-s} \right) | \leq C_\alpha \chi_{|a:1/4 \leq |a| \leq 1|}(a)(2^j \mu)^{\kappa+1}$. Then for any $M \geq 0$, $|\xi_{\mu}^{(0)}(\lambda)| \leq C_M \sum_{j \leq 0} (2^j \mu)^{\kappa+2-s}(1 + |2^j \mu\lambda|)^{-M}$. From these estimates, we obtain that for any $0 < \mu \leq 1$
\[
|\xi_{\mu}^{(0)}(\lambda)| \leq \begin{cases} 
C|\lambda|^{-(\kappa+2-s)}, & |\lambda,\mu| \geq 1; \\
C\mu^{\kappa+2-s}, & |\lambda,\mu| \leq 1,
\end{cases}
\]
which gives estimate (3.29).
Now we consider the case $\ell \geq 1$. Recall that $\hat{\xi}$ and $\eta_\ell$ are even functions. By integration by parts, we have

$$
|\xi_\mu^{(\ell)}(\lambda)| = |\int_{\mathbb{R}} \left(1 - \hat{\xi}(a)\right) \eta(2^{-\ell} \mu^{-1}a)|a|^{-s}e^{ia\lambda}da|
$$

$$
= |\int_{0}^{\infty} \left(1 - \hat{\xi}(a)\right) \eta(2^{-\ell} \mu^{-1}a)a^{-s}(e^{ia\lambda} + e^{-ia\lambda})da|
$$

$$
\leq C|\lambda|^{-s} \int_{0}^{\infty} |\partial_\alpha\left((1 - \hat{\xi}(a))\eta(2^{-\ell} \mu^{-1}a)a^{-s}\right)| |da|.
$$

If $2^\ell \mu \leq 1$, for any $\alpha \in \mathbb{N}$, from supp $\eta \subseteq \{\tau : 1/4 \leq |\tau| \leq 1\}$, Taylor’s expansion of $\hat{\xi}$ at the original point and $\hat{\xi}^{(0)}(0) = 1$, $\hat{\xi}^{(1)}(0) = \cdots = \hat{\xi}^{(\kappa)}(0) = 0$, we have

$$
|\partial_\alpha\left((1 - \hat{\xi}(a))\eta(2^{-\ell} \mu^{-1}a)a^{-s}\right)| \leq C_{\alpha} \chi_{[\alpha:2^{\ell-2}\mu \leq |\alpha| \leq 2^\ell \mu]}(a)(2^\ell \mu)^{\kappa + 1 - s - \alpha}.
$$

If $2^\ell \mu > 1$, for any $\alpha \in \mathbb{N}$, from supp $\eta \subseteq \{\tau : 1/4 \leq |\tau| \leq 1\}$ and $\xi \in C_c^\infty(\mathbb{R})$, we have

$$
|\partial_\alpha\left((1 - \hat{\xi}(a))\eta(2^{-\ell} \mu^{-1}a)a^{-s}\right)| \leq C_{\alpha} \chi_{[\alpha:2^{\ell-2}\mu \leq |\alpha| \leq 2^\ell \mu]}(a)(2^\ell \mu)^{-s - \alpha}.
$$

Consequently, for any $\ell \in \mathbb{N}^+$, $M \in \mathbb{N}$ and any $0 < \mu \leq 1$,

$$
|\xi_\mu^{(\ell)}(\lambda)| \leq \begin{cases} 
C_M (2^\ell \mu)^{\kappa + 2 - s} (1 + |2^\ell \mu \lambda|)^{-M}, & \text{if } 2^\ell \mu \leq 1; \\
C_M (2^\ell \mu)^{1-s} (1 + |2^\ell \mu \lambda|)^{-M}, & \text{if } 2^\ell \mu > 1,
\end{cases}
$$

which gives estimate (3.30).

Next we return to apply estimates (3.29) and (3.30) to prove estimate (3.28). Recall that $\Omega_\mu^{(\ell)}(x) := 2^{-j} \xi_\mu^{(\ell)}(2^{-j} x)$, which is an even function. It suffices to consider the positive odd integer $m$ since $\Omega_\mu^{(\ell), m} = 0$ when $m$ is an even integer. In the following, we assume that $\ell \in \mathbb{N}^+$ and consider two cases: $2^\ell \mu \leq 1$ and $2^\ell \mu > 1$.

**Case 1** $2^\ell \mu \leq 1$.

In this case, it follows from estimate (3.30) that for $m \in \mathbb{N}^+$ and for every $M > 1$,

$$
\|\Omega_\mu^{(\ell), m}\|_2 = \left(\int_{m-1}^{m+1} |\Omega_\mu^{(\ell)}(x)|^2 dx\right)^{1/2} = 2^{-j} \left(\int_{m-1}^{m+1} |\xi_\mu^{(\ell)}(2^{-j} x)|^2 dx\right)^{1/2}
$$

$$
\leq C_M 2^{-j} (2^\ell \mu)^{\kappa + 2 - s} \left(\int_{m-1}^{m+1} (1 + 2^{-j} \mu x)^{-2M} dx\right)^{1/2}
$$

$$
\leq C_M 2^{-j} (2^\ell \mu)^{\kappa + 2 - s} (1 + 2^{-j} \mu (m - 1))^{-M}.
$$
Hence, for any \( j \in \mathbb{N} \) and \( 0 < \mu \leq 1 \),

\[
\sum_{m \geq 1} \|\Omega_{\mu,j}^{(\ell),m}\|_2 \leq \sum_{m:m \geq 1, 2^\ell \mu(m-1) \leq 1} \|\Omega_{\mu,j}^{(\ell),m}\|_2 + \sum_{m:m \geq 1, 2^\ell \mu(m-1) > 1} \|\Omega_{\mu,j}^{(\ell),m}\|_2
\]

\[
\leq C_M 2^{-j} (2^\ell \mu)^{x+2-s} \left( \sum_{m:m \geq 1, 2^\ell \mu(m-1) \leq 1} 1 + \sum_{m:m \geq 1, 2^\ell \mu(m-1) > 1} (2^\ell \mu(m-1))^{-M} \right)
\]

\[
\leq C 2^{-j} (2^\ell \mu)^{x+2-s} \left( \# \{ m \in \mathbb{N}^+ : 1 \leq m \leq 2^j (2^\ell \mu)^{-1} + 1 \} + 2^j (2^\ell \mu)^{-1} \right)
\]

\[
\leq C (2^\ell \mu)^{x+1-s},
\]

where in the last inequality we use the fact that when \( 2^\ell \mu \leq 1, j \geq 0 \),

\[
\# \{ m \in \mathbb{N}^+ : 1 \leq m \leq 2^j (2^\ell \mu)^{-1} + 1 \} \leq 2^j (2^\ell \mu)^{-1}.
\]

**Case 2** \( 2^\ell \mu > 1 \).

In this case, it follows from estimate (3.30) that for \( m = 1 \) and for every \( M > 1 \),

\[
\|\Omega_{\mu,j}^{(\ell),1}\|_2 = \left( \int_0^2 |\Omega_{\mu,j}^{(\ell),1}(x)|^2 dx \right)^{1/2} = 2^{-j} \left( \int_0^2 |\xi_{\mu}^{(\ell)}(2^{-j} x)|^2 dx \right)^{1/2}
\]

\[
\leq C_M 2^{-j} (2^\ell \mu)^{1-s} \left( \int_0^2 (1 + 2^{\ell-j} \mu x)^{-2M} dx \right)^{1/2}
\]

\[
\leq C_M 2^{-j} (2^\ell \mu)^{1-s} (2^{\ell-j} \mu)^{-1/2} \left( \int_0^\infty (1 + y)^{-2M} dy \right)^{1/2}
\]

\[
\leq C 2^{-j/2} (2^\ell \mu)^{1/2-s}.
\]

If \( m \geq 3 \), then for any \( M > 1 \),

\[
\|\Omega_{\mu,j}^{(\ell),m}\|_2 = \left( \int_{m-1}^{m+1} |\Omega_{\mu,j}^{(\ell),m}(x)|^2 dx \right)^{1/2} = 2^{-j} \left( \int_{m-1}^{m+1} |\xi_{\mu}^{(\ell)}(2^{-j} x)|^2 dx \right)^{1/2}
\]

\[
\leq C_M 2^{-j} (2^\ell \mu)^{1-s} \left( \int_{m-1}^{m+1} (1 + 2^{\ell-j} \mu x)^{-2M} dx \right)^{1/2}
\]

\[
\leq C_M 2^{-j} (2^\ell \mu)^{1-s} \left( 1 + 2^{\ell-j} \mu(m-1) \right)^{-M}.
\]

Therefore, for any \( j \in \mathbb{N} \) and \( 0 < \mu \leq 1 \),

\[
\sum_{m \geq 1} \|\Omega_{\mu,j}^{(\ell),m}\|_2 \leq \|\Omega_{\mu,j}^{(\ell),1}\|_2 + \sum_{m:m \geq 3, 2^\ell \mu(m-1) \leq 1} \|\Omega_{\mu,j}^{(\ell),m}\|_2
\]

\[
+ \sum_{m:m \geq 3, 2^\ell \mu(m-1) > 1} \|\Omega_{\mu,j}^{(\ell),m}\|_2
\]

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\begin{align*}
&\leq C_M 2^{-j} (2^j \mu)^{1-s} \left( 2^j (2^j \mu)^{-\frac{1}{2}} + \left( \sum_{m,n \geq 3, \atop 2^{j-l} \mu (m-1) \leq 1} 1 \right) \right. \\
&\quad + \left. \sum_{m,n \geq 3, \atop 2^{j-l} \mu (m-1) > 1} (2^{l-j} \mu (m-1))^{-M} \right) \\
&\leq C 2^{-j} (2^j \mu)^{1-s} \left( 2^j (2^j \mu)^{-\frac{1}{2}} + \# \{m \in \mathbb{N}^+: 3 \leq m \leq 2^j (2^j \mu)^{-1} + 1 \} \right. \\
&\quad + \left. 2^{j} (2^j \mu)^{-1} \right) \\
&\leq C 2^{-j} (2^j \mu)^{1-s} \left( 2^j (2^j \mu)^{-\frac{1}{2}} + 3 \times 2^{j} (2^j \mu)^{-1} \right) \leq C (2^j \mu)^{\frac{1}{2}-s},
\end{align*}

where we use the inequality that

\[ \# \{m \in \mathbb{N}^+: 3 \leq m \leq 2^j (2^j \mu)^{-1} + 1 \} \leq \max \{0, 2^j (2^j \mu)^{-1} - 1 \} \leq 2^{j+1} (2^j \mu)^{-1}. \]

The proof of \( \ell = 0 \) is similar to that of Case 1 and even more simple, and we skip the detail here. Combining the above discussion, we obtain estimate (3.28). This finishes the proof of Lemma 3.5, and then the proof of Lemma 3.4 is complete. \( \square \)

4 Extensions

In the previous section, we proved Theorem 1.1 with the potential \( V = |x|^2 \). However, the form of this potential does not play a crucial role in the proof. Here we consider instead the operators \( H_V = -\Delta + V \) on \( L^2(\mathbb{R}^n) \) for \( n \geq 2 \), where \( V \) is a positive potential with the following conditions:

\[ V \sim |x|^2, \quad |\nabla V| \sim |x|, \quad |\partial_x^2 V| \leq 1. \tag{4.1} \]

Under the assumption (4.1), \( H_V \) is a non-negative self-adjoint operator on \( L^2(\mathbb{R}^n) \). Such an operator admits a spectral resolution

\[ H_V = \int_0^\infty \lambda dE_{H_V}(\lambda). \]

Now, the Bochner–Riesz means of order \( \delta \geq 0 \) for operator \( H_V \) can be defined by

\[ S_R^\delta (H_V)f := \int_0^R^2 \left( 1 - \frac{\lambda}{R^2} \right)^\delta dE_{H_V}(\lambda) f, \quad f \in L^2(\mathbb{R}^n). \tag{4.2} \]

Then, similar to the Hermite operator \( H \), the commutator \( [b, S_R^\delta (H_V)] \) is bounded on \( L^q(\mathbb{R}^n) \) for \( p < q < p' \) uniformly in \( R > 0 \) for \( 1 \leq p \leq 2n/(n+2) \) and \( \delta > \delta(p) \), as we now show.
**Theorem 4.1** Suppose the potential $V$ satisfies (4.1). Let $n \geq 2$, $1 \leq p \leq 2n/(n + 2)$ and $\delta > \delta(p)$, then for all $b \in \text{BMO}(\mathbb{R}^n)$

$$\sup_{R > 0} \left\| [b, S^\delta_R(H_V)] \right\|_{q \to q} \leq C\|b\|_{\text{BMO}}$$

for all $p < q < p'$.

**Proof** It follows from [8, Theorem 4] that for all $1 \leq p \leq 2n/(n + 2)$ and $\lambda \geq 0$

$$\|E_{H_V}[\lambda^2, \lambda^2 + 1]\|_{p \to 2} \leq C(1 + \lambda)^{n(\frac{1}{p} - \frac{1}{2}) - 1}.$$  \hspace{1cm} (4.3)

With Lemma 2.2 and the spectral projection estimate (4.3), the argument in the proof of Theorem 3.2 also establishes Theorem 4.1. \hfill \Box

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**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest regarding the work reported in this paper.

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