THE SEVERI PROBLEM FOR ABELIAN SURFACES IN THE
PRIMITIVE CASE

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Abstract. We prove that the irreducible components of primitive class Severi
varieties of general abelian surfaces are completely determined by the maximal
factorization through an isogeny of the maps from the normalized curves.

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1. INTRODUCTION

1.1. The Severi problem for K3 and abelian surfaces. Traditionally, a Severi
variety is roughly a parameter space for plane curves of fixed degree and geometric
genus. More generally, a parameter space for curves on a given projective surface
of fixed homology class and geometric genus will also be called a Severi variety.

Although the local properties of Severi varieties – properties such as (having the
expected) dimension, smoothness, facts concerning the general curves, etc. – are
reasonably well-understood for a few types of surfaces, not much is known about
the global geometry of these parameter spaces besides enumerative aspects. By a
celebrated theorem of J. Harris [18], the Severi varieties of $\mathbb{P}^2$ are irreducible – what
was previously known as the Severi conjecture or Severi problem. By extension,
proving that the Severi varieties of certain surfaces are irreducible, or identifying
their irreducible components, may be referred to as “a” Severi problem.

Severi varieties of K3 surfaces have been carefully investigated in various con-
texts, sometimes motivated by questions concerning abstract curves. The Severi
problem for general K3 surfaces is currently open even for curves in the primitive
class [10–13, 19].

Conjecture 1.1 (Strong irreducibility conjecture for K3 surfaces, primitive case).
The Severi variety of curves of genus $g \in [1, h]$ and homology dual to the hyperplane
class of a general degree $2h - 2$ projective K3 surface is irreducible.

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The “weak irreducibility conjecture” is the statement that the universal Severi variety is irreducible. However, it also includes the genus zero case, which is trivially false for individual Severi varieties.

K3 surfaces and abelian surfaces are similar in many ways and this is also true of their Severi varieties. The study of Severi varieties of abelian surfaces, specifically the analogous situation when the curve class is primitive, was initiated quite recently by Knutsen, Lelli-Chiesa and Mongardi [21]. The authors propose [21, Question 4.2] the problem analogous to Conjecture 1.1: the Severi problem for general (1, d)-polarized abelian surfaces and curves in the polarizing class.

The main purpose of this paper is to solve this problem. The Severi varieties of abelian surfaces generally fail to be irreducible due to the presence of an additional topological invariant besides degree and genus, namely the maximal isogeny through which the map from the normalization of the curve factorizes, or equivalently, the image of the induced map on first homology groups. Nevertheless, our main result states that the loci where this quantity is constant are irreducible.

1.2. The main result. Let \((A, \mathcal{L})\) be a general \((1, d)\)-polarized abelian surface. If \(D\) is a possibly singular curve on \(A\), following [21], we write \(D \in \{\mathcal{L}\}\) if \([D]\) is algebraically equivalent to \(c_1(\mathcal{L})\), as opposed to \(D \in [\mathcal{L}]\) if \([D]\) is rationally equivalent to \(c_1(\mathcal{L})\). Recall that \(p_a(|\mathcal{L}|) = d + 1, \dim \{\mathcal{L}\} = d + 1\) and \((\mathcal{L}^2) = 2d\).

Severi varieties are often defined by requiring the curves to have at worst nodal singularities. However, these issues are somewhat of a distraction for the purposes of this paper and we will work instead with a perfectly natural custom definition which avoids any unnecessary technical complications. Please see also [21, Theorem 1.3]. We say that a curve \(D \in \{\mathcal{L}\}\) is nice if it is reduced and irreducible and the map \(\bar{D} \to A\) from the normalization to the surface is unramified. In this paper, the Severi varieties \(\nabla_g(A, \{\mathcal{L}\})\) parametrize nice curves of a fixed geometric genus \(g\) in the class \(\{\mathcal{L}\}\) and deformations of such curves.

**Definition 1.2.** If \((A, \mathcal{L})\) and \((A', \mathcal{L}')\) are polarized abelian surfaces, an isogeny \(\sigma : A' \to A\) such that

\[
\sigma_*c_1(\mathcal{L}') = c_1(\mathcal{L})
\]

in \(\text{NS}(A)\) will be referred to as a copolarized isogeny \((A', \{\mathcal{L}'\}) \to (A, \{\mathcal{L}\})\).

Two copolarized isogenies will be considered identical if the underlying isogenies are identical and the polarizations are algebraically equivalent, so the set of copolarized isogenies with fixed target is discrete. We will only be concerned with copolarized isogenies where \(\mathcal{L}\) has type \((1, d)\), which implies that \(\mathcal{L}'\) has type \((1, d')\),

\[
d = d' \deg \sigma.
\]

Indeed, on one hand, since \(c_1(\mathcal{L})\) is primitive (indivisible) in \(\text{NS}(A)\), so is \(c_1(\mathcal{L}')\) in \(\text{NS}(A')\). On the other hand, by the push-pull formula,

\[
(\mathcal{L}^2)_A = (\sigma_*|\mathcal{L}'|^2)_A = (\mathcal{L}' \cdot \sigma^*\sigma_*|\mathcal{L}'|)_{A'} = (\mathcal{L}'^2)_{A'} \deg \sigma
\]

and \(d = d' \deg \sigma\) follows since \((\mathcal{L}^2) = 2d\) and \((\mathcal{L}'^2) = 2d'\).

Returning to Severi varieties, given a copolarized isogeny \((A', \{\mathcal{L}'\}) \to (A, \{\mathcal{L}\})\) and a nice curve \(A' \in \{\mathcal{L}'\}\), the image of \(D'\) in \(A\) is trivially nice as well. Conversely, for any nice curve \(D \in \{\mathcal{L}\}\) on \(A\), the map \(\bar{D} \to A\) factorizes through a maximal isogeny \(\sigma : A' \to A\) by the topological lifting property and taking \(\mathcal{L}' = \mathcal{O}_{A}(D')\) clearly makes the isogeny copolarized in the sense of Definition 1.2. Denote by
The Severi problem for abelian surfaces

\[ \mathcal{V}_g(A, \{L\}) \] the locus which parametrizes nice curves \( D \subset A \) with maximal factorization \( \sigma \) and their deformations. Formal definitions will be given in \( \S2 \).

**Theorem 1.3** (Main Theorem). Let \((A, \mathcal{L})\) be a general \((1,d)\)-polarized complex abelian surface and \( g \in [3, d+1] \) a positive integer. Then

\[ \mathcal{V}_g(A, \{L\}) = \bigcup_{\deg \sigma \leq \frac{d}{g}} \mathcal{V}_{\sigma g}(A, \{L\}) \]

is the decomposition of the Severi variety into irreducible components. The union is taken over copolarized isogenies whose degree (divides \( d \) and) doesn’t exceed \( \frac{d}{g} \).

Note that for \( g = 1 \), the Severi variety is empty, while for \( g = 2 \), specifying a map from a genus 2 curve into an abelian surface is essentially equivalent to giving an isogeny of abelian surfaces, so there is little to say for \( g < 3 \).

The theorem consists of two parts: \( \mathcal{V}_g(A, \{L\}) \) is nonempty and irreducible. Both are proved by specializing to a polarized product of elliptic curves. The proof of the former is very easy, but relies on nontrivial folklore ideas concerning the reduction of obstruction spaces by the semiregularity map and, to the author’s knowledge, only the modern formulation in [20] can be applied directly.

The focus of the paper is on the latter. The proof follows the general philosophy in [29], relying on a careful study of the intersections of the components of the degenerate Severi variety. The most obvious difference is that we will not rely on the theory of stable maps to degenerate targets [22, 23], but the reasons are a little subtle and will be explained later.

**1.3. Structure of the paper.** The paper is organized as follows:

- \( \S2 \) The technical introduction.
- \( \S3 \) We introduce discrete data which will later turn out to label some of the components of the degeneration and intersections of such components.
- \( \S4 \) We discuss a certain class of (pointed) maps from genus 2 curves to genus 1 curves. The motivation is probably not immediately obvious, but these will turn out to be pieces of certain stable maps which are vital in our analysis.
- \( \S5 \) After introducing the specialization to the split abelian surface and some preliminaries, we describe a type of “tree-like” expansion of this target surface (please see Fig. A) and gain some insight into which stable maps to the split abelian surface deform to nearby abelian surfaces.
- \( \S6 \) After some preliminaries and dealing with the non-emptiness part of the theorem, we study some intersections of the components of the degenerate moduli space (please see Fig. B), as briefly explained above, from which we deduce Theorem 1.3.

**1.4. Conventions.** We work exclusively over the field \( \mathbb{C} \) of complex numbers. We will often use “implicit conversion” from nonsingular varieties to complex analytic manifolds without any further warning.

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2. The various moduli and parameter spaces

2.1. First definitions. Let \((A, \mathcal{L})\) be a \((1, d)\)-polarized abelian surface, which we will typically, but not always, assume to be general. Let \(\beta \in H_2(A, \mathbb{Z})\) be the class Poincaré dual to \(\chi_g\).

We will use the following spaces which capture the idea of “geometric genus \(g\) curve on \(A\) in the algebraic class \(\{\mathcal{L}\}\)” in various ways:

1. \(V^\circ_g(A, \{\mathcal{L}\})\) parametrizes curves in \(\{\mathcal{L}\}\) of geometric genus \(g\) with at worst nodal singularities;
2. \(\overline{\mathcal{M}}_g(A, \beta)\) is the Kontsevich moduli stack of unmarked stable maps of genus \(g\) and class \(\beta\);
3. \(\mathcal{Y}_g(A, \beta)\) is the open substack of \(\overline{\mathcal{M}}_g(A, \beta)\) of unramified stable maps with smooth source;
4. \(\overline{\mathcal{V}}_g(A, \beta)\) is the closure of \(\mathcal{Y}_g(A, \beta)\) inside \(\overline{\mathcal{M}}_g(A, \beta)\);
5. \(\mathcal{V}_g(A, \{\mathcal{L}\})\) is the image of \(\overline{\mathcal{V}}_g(A, \beta)\) under the cycle map \(\overline{\mathcal{M}}_g(A, \beta) \to \{\mathcal{L}\}\), \([C, f] \mapsto f^*(f)\).

Occasionally, we will use some variations, such as \(\overline{\mathcal{M}}_{g,n}(A, \beta)\), the space of stable maps with \(n\) markings. An unpleasant feature of (4) is that taking closures makes deformation theory hard to control. Deformation theory will always be worked out on \(\overline{\mathcal{M}}_g(A, \beta)\) or other moduli spaces constructed by better behaved methods.

The usual deformation-obstruction theory for stable maps simplifies in the case of maps \(f : C \to A\) in \(\mathcal{Y}_g(A, \beta)\) to \(\text{Def}^1(f) = H^0(\mathcal{N}_f)\) and \(\text{Obs}(f) = H^1(\mathcal{N}_f)\), where \(\mathcal{N}_f\) is the normal sheaf of \(f\) defined by short exact sequence

\[
0 \to T_C \to f^*T_A \to \mathcal{N}_f \to 0.
\]

By linear algebra, \(\mathcal{N}_f \cong \det f^*T_A \otimes T_C^\vee \cong K_C\), so the deformation and obstruction spaces have dimension \(g\) respectively 1. In particular, the local dimension of \(\mathcal{Y}_g(A, \beta)\) at any point is at most \(g\) and \(\mathcal{Y}_g(A, \beta)\) is nonsingular if equality occurs.

In fact, it follows from well-known ideas concerning the semiregularity map [5,24] that equality occurs. In [20, Theorem 2.4], it is proved that \(\overline{\mathcal{M}}_g(A, \beta)\) admits a reduced perfect obstruction theory \(E^\bullet_{\text{red}} \to \overline{\mathcal{M}}_g(A, \beta)\) of virtual dimension \(g\), so any irreducible component of \(\overline{\mathcal{M}}_g(A, \beta)\) must have dimension at least \(g\). Alternatively and more explicitly, the restriction of \(E^\bullet_{\text{red}}\) to \(\mathcal{Y}_g(A, \beta)\) is actually concentrated in degree 0 and locally free of rank \(g\) by [7, Lemma 2, §1.4] and the desired conclusion follows from the well-known [2, Proposition 5.5].

In conclusion, \(\mathcal{Y}_g(A, \beta)\) is smooth of pure dimension \(g\) for any (not necessarily general) \((A, \mathcal{L})\). Non-emptiness for general \((A, \mathcal{L})\) follows from [21, Theorem 1.1], but we will need to return to this point since non-emptiness of the loci which are a posteriori irreducible components doesn’t.

**Definition 2.1.** An irreducible component \(\mathcal{V}\) of \(\overline{\mathcal{M}}_g(A, \beta)\) is called **flexible** (or enumeratively relevant) if the restriction of the evaluation morphism at the marked points \(\text{ev} : \overline{\mathcal{M}}_{g,n}(A, \beta) \to A^g\) to the preimage of \(\mathcal{V}\) under the forgetful morphism \(\overline{\mathcal{M}}_{g,n}(A, \beta) \to \overline{\mathcal{M}}_g(A, \beta)\) is dominant.

By construction, \(\mathcal{Y}_g(A, \beta)\) is an open Deligne-Mumford substack of \(\overline{\mathcal{M}}_g(A, \beta)\), so \(\overline{\mathcal{V}}_g(A, \beta)\) is a union of irreducible components of \(\overline{\mathcal{M}}_g(A, \beta)\).

**Proposition 2.2.** All irreducible components of \(\overline{\mathcal{V}}_g(A, \beta)\) are flexible.
Proof: Let \([C,f] \in \mathcal{V}_g(A,\beta)(\mathbb{C})\) arbitrary. Recall that the space of first order deformations of \(f\) is naturally identified with \(H^0(C,\mathcal{N}_f^\vee)\) and that \(\mathcal{N}_f = K_C\). The classical fact that \(\text{Sym}^g C \to \text{Pic}^g C\) is birational implies that a general degree \(g\) effective divisor \(D = p_1 + p_2 + \ldots + p_g\) on \(C\) has the property that the restriction map \(K_C \to K_C \otimes \mathcal{O}_D\) is an isomorphism on global sections. Indeed, \(K_C(-D)\) is a general degree \(g-2\) linear equivalence class, hence ineffective since \(\text{Sym}^{g-2} C \to \text{Pic}^{g-2} C\) is not surjective for trivial dimension reasons.

In terms of tangent spaces, the map \(T_{[C,f]} \to \mathcal{M}_g(A,\beta) \to H^0(\mathcal{N}_f \otimes \mathcal{O}_D)\) is an isomorphism. Inspecting the following commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \to & \bigoplus_{i=1}^g T_{p_i} C & \to & T_{[C,f,(p_i)]_{i=1}^g} \mathcal{M}_g(A,\beta) & \to & T_{[C,f]} \mathcal{M}_g(A,\beta) & \to & 0 \\
0 & \to & \bigoplus_{i=1}^g T_{p_i} C & \to & \bigoplus_{i=1}^g T_{(p_i)} A & \to & H^0(\mathcal{N}_f \otimes \mathcal{O}_D) & \to & 0 \\
\end{array}
\]

we see immediately that the central downward map is also an isomorphism, i.e. the derivative of the evaluation morphism \(ev : \mathcal{M}_g(A,\beta) \to A^g\) is an isomorphism at \([C,f,p_1,p_2,\ldots,p_g]\). It follows that the irreducible component of \(\mathcal{V}_g(A,\beta)\) which contains \([C,f]\) is flexible, completing the proof.

2.2. The maximal factorization through an isogeny. We digress for a moment to clarify some trivial topological remarks which will be running in the background of the arguments in the rest of the paper. Notation is completely local. Assume \(\alpha : \mathcal{A} \to T\) is a (proper) family of abelian varieties of some given dimension over some finite type base \(T\). Then the relative cohomology sheaf \(R^i \alpha_*.\mathbb{Z}_T\) is clearly locally constant and “commutes with (any) base change.” We use the (nonsensical but hopefully) suggestive notation

\[R^i \alpha_* \mathbb{Z}_T = \text{Hom}_{\mathbb{Z}_T}(R^i \alpha_*.\mathbb{Z}_T, \mathbb{Z}_T)\]

for the dual, which contains the homological information.

If \(\alpha_1 : \mathcal{A}_1 \to T\) and \(\alpha_2 : \mathcal{A}_2 \to T\) are two such families over \(T\) with a \(T\)-morphism of abelian varieties \(\varphi : \mathcal{A}_1 \to \mathcal{A}_2\), then cohomology pullback \(R^i\varphi^*\) can be regarded as a section of \(\text{Hom}_{\mathbb{Z}_T}(R^i(\alpha_2)_*.\mathbb{Z}_T, R^i(\alpha_1)_*.\mathbb{Z}_T)\) and, mildly confusingly, it is the same section as homology pushforward in \(\text{Hom}_{\mathbb{Z}_T}(R_i(\alpha_1)_*.\mathbb{Z}_{\mathcal{A}_1}, R_i(\alpha_2)_*.\mathbb{Z}_{\mathcal{A}_2})\). In particular, the kernel and cokernel are well-defined lattices on connected components. Phrased more formally – for instance – given a locally constant subsheaf \(\mathcal{L}\) of \(R_i(\alpha_2)_*.\mathbb{Z}_{\mathcal{A}_2}\), the set

\[\{t \in T(\mathbb{C}) : \text{the image of } H_i(\mathcal{L}_t, \mathbb{Z}) \to H_i(\mathcal{L}_t, \mathbb{Z}) \text{ is } \mathcal{L}_t\}\]

is open and closed in the usual topology and hence the same holds in the Zariski topology regarding its closure. What we will actually need is the following obvious corollary concerning maps from compact type curves.

Proposition 2.3. Let \(\mathcal{X} : \mathcal{C} \to T\) and \(\alpha : \mathcal{A} \to T\) be flat families of semistable curves of compact type respectively abelian varieties of a certain dimension over a scheme \(T\) and \(\varphi : \mathcal{C} \to \mathcal{A}\) a \(T\)-morphism. Then \(R^1 \alpha_*.\mathbb{Z}_T\) and \(R^1 \varphi_*.\mathbb{Z}_T\) are locally constant and behave well with respect to base change. Moreover, given any locally constant subsheaf of subgroups \(\mathcal{L}\) of \(R_1 \alpha_*.\mathbb{Z}_T\), the set

\[\{t \in T(\mathbb{C}) : \text{the image of } H_1(\mathcal{L}_t, \mathbb{Z}) \to H_1(\mathcal{L}_t, \mathbb{Z}) \text{ is } \mathcal{L}_t\}\]
is open and closed in the sense described above.

Proof. By the Albanese universal property of Jacobians, we may replace \( \mathcal{C} \) with the relative Jacobian and invoke (2). The question is local, so we may assume that \( \mathcal{C} \) has a section.

We return to the Severi varieties of the \((1, d)\)-polarized abelian surface \((A, \mathcal{L})\). We say that a stable map \( f : C \to A \) in \( \mathcal{Y}_g(A, \beta) \) has associated lattice \( \Lambda \) if \( \Lambda \) is the image of the induced pushforward on first homology groups

\[
f_* : H_1(C, \mathbb{Z}) \to H_1(A, \mathbb{Z}).
\]

Since \( \pi_1(A) \) is abelian and \( H_1(C, \mathbb{Z}) = \pi_1(C)^{ab} \), the lattice \( \Lambda \) is also the image of the induced map \( \pi_1(C) \to \pi_1(A) \) and the topological lifting property shows that the maximal isogeny \( A' \to A \) through which \( f \) factors is the isogeny

\[
A' = V/\Lambda \to V/H_1(A, \mathbb{Z}) = A,
\]

where \( V \) denotes the universal covering space of \( A \).

Definition–Lemma 2.4. We say that a rank four sublattice \( \Lambda \subset H_1(A, \mathbb{Z}) \) is compatible with \( c_1(\mathcal{L}) \) if \( \beta = \text{PD}(c_1(\mathcal{L})) \) belongs to the image of the pushforward map \( \sigma_* : H_2(A', \mathbb{Z}) \to H_2(A, \mathbb{Z}) \), where \( A' = V/\Lambda \) and \( \sigma : A' \to A \) is the induced isogeny.

Under these circumstances, there exists a line bundle \( \mathcal{L}' \in \text{Pic}(A') \) such that \( \sigma : (A', \{\mathcal{L}'\}) \to (A, \{\mathcal{L}\}) \) is a copolarized isogeny in the sense of Definition 1.2.

Proof. “PD” stands for Poincaré dual. If \( \sigma_*(\beta') = \beta \), then

\[
\text{PD}(\beta') = \frac{\sigma^*\text{PD}(\sigma_*(\beta'))}{\deg \sigma} = \frac{\sigma^*\text{PD}(\beta)}{\deg \sigma}
\]

implies that \( \beta' \) is Poincaré dual to a \((1, 1)\)-class as \( \sigma^* : H^2(A, \mathbb{C}) \to H^2(A', \mathbb{C}) \) clearly respects \((1, 1)\)-classes. Thus \( \text{PD}(\beta') = c_1(\mathcal{L}') \) for some \( \mathcal{L}' \in \text{Pic}(A') \). Moreover, (3) also shows that \( \mathcal{L}' \) is ample via the Nakai-Moishezon criterion.

Remark 2.5. Although Definition 2.4 was phrased geometrically, the fact that the notion of compatibility is actually algebraic is very transparent: it only involves a sublattice and an element of a given (canonically oriented) lattice. What we need is an algebraic description of \( \sigma_* \). On one hand, the pullback \( \sigma^* \) is just the map \( H^1(A, \mathbb{Z}) = H_1(A, \mathbb{Z})^\vee \to \Lambda^\vee \) in degree 1, respectively exterior powers of this map in arbitrary degree. On the other hand, the push-pull relation

\[
(\sigma_*(\omega) \cdot \omega)_A = (\text{PD}(\omega) \cup \sigma^*\text{PD}(\omega), [A'])
\]

completely determines \( \sigma_* \) in terms of \( \sigma^* \) and the other operations in the formula above. Recall that intersection numbers correspond to cup products under Poincaré duality and that the cup product \( H^2(A, \mathbb{Z}) \times H^2(A, \mathbb{Z}) \to H^4(A, \mathbb{Z}) \cong \mathbb{Z} \) on \( A \) is simply the wedge product for the corresponding exterior powers of \( H^1(A, \mathbb{Z}) \) – and similarly for \( A' \) (note that a full rank sublattice of a given oriented lattice has a natural “positive” orientation). This remark will be useful in §§6.1.

In conclusion, there is a one-to-one correspondence between sublattices compatible with \( c_1(\mathcal{L}) \) and copolarized isogenies into \((A, \{\mathcal{L}\})\). Assume that \( \Lambda \) is a full rank sublattice of \( H_1(A, \mathbb{Z}) \) compatible with \( c_1(\mathcal{L}) \). Continuing the list in §§2.1, Proposition 2.3 allows us to introduce the main objects studied in this paper:

(6) \( \mathcal{Y}_g(A, \beta; \Lambda) \) is the open and closed substack of \( \mathcal{Y}_g(A, \beta) \) parametrizing stable maps with associated lattice \( \Lambda \);
the realm of sublattices. We define the degree of a partition 
$\Lambda$ as the number of summands $\Lambda_i$ of $\Lambda$. Perhaps the term “partition” is not optimal as the order of the summands are considered identical. For instance, one can introduce an equivalent notion for partitions of natural numbers, two partitions which differ only in the order of the summands are considered identical. 

Moreover, the stack $\mathcal{V}_g(A, \beta; \Lambda)$ is irreducible whenever it is nonempty.

**Theorem 2.6.** If $g \geq 3$, then $\mathcal{V}_g(A, \beta; \Lambda)$ is nonempty if and only if

$$[H_1(A, Z) : \Lambda] \leq \frac{d}{g-1}.$$ 

Moreover, the stack $\mathcal{V}_g(A, \beta; \Lambda)$ is irreducible whenever it is nonempty.

It is clear that Theorem 2.6 implies Theorem 1.3. Moreover, the “only if” part of the existence statement is trivial. Indeed, by the earlier discussion, any $f : C \to A$ in $\mathcal{V}_g(A, \beta; \Lambda)$ factors through an isogeny $\sigma : A' \to A$ of degree $[H_1(A, Z) \colon \Lambda]$ as $f = \sigma \circ f'$ and the class of $f'(C) \subset A$ is of type $(1, d')$ such that $d = d' \deg \sigma$, so

$$g = p_a(C) \leq p_a([f'(C)]) = d' + 1 = \frac{d}{[H_1(A, Z) : \Lambda]} + 1,$$

which is equivalent to $[H_1(A, Z) : \Lambda] \leq \frac{d}{g-1}$.

3. **Partitions of rank two lattices**

Let $\mathbb{Z}^2$ stand for any free abelian group/lattice of rank two. A partition of the lattice $\mathbb{Z}^2$ is a collection $\Lambda_1, \Lambda_2, ..., \Lambda_k$ of full rank sublattices such that

$$\mathbb{Z}^2 = \Lambda_1 + \Lambda_2 + ... + \Lambda_k.$$

Of course, the plus sign denotes the span inside $\mathbb{Z}^2$ and not the direct sum. As in the case of partitions of natural numbers, two partitions which differ only in the order of the summands are considered identical. For instance, one can introduce an arbitrary ordering on the set of full rank sublattices and require $\Lambda_1 \geq \Lambda_2 \geq ... \geq \Lambda_k$. Perhaps the term “partition” is not optimal as $a + b = a$ occurs very frequently in the realm of sublattices. We define the degree of a partition $p$ as

$$\deg p = [\mathbb{Z}^2 : \Lambda_1] + [\mathbb{Z}^2 : \Lambda_2] + ... + [\mathbb{Z}^2 : \Lambda_k].$$

The length of the partition is $k$, the number of summands. Trivially, $\deg p \geq k$.

Let $\mathcal{P}$ be the set of all partitions of $\mathbb{Z}^2$, $\mathcal{P}_d$ the set of partitions of degree $d$, $\mathcal{P}_k$ the set of partitions of length $k$ and $\mathcal{P}_d^k = \mathcal{P}_d \cap \mathcal{P}_k$ the set of partitions of degree $d$ and length $k$. It is clear that $\mathcal{P}_d^k \neq \emptyset$ if $d \geq k \geq 2$ — for instance, we could choose all summands but one to be $\mathbb{Z}^2$. We draw an arrow between two elements of $\mathcal{P}$ pointing from $p_1 \in \mathcal{P}_k$ to $p_2 \in \mathcal{P}_{k-1}$ if $p_2$ is obtained from $p_1$ by replacing two summands $\Lambda_i$ and $\Lambda_j$ of $p_1$ with their sum $\Lambda_i + \Lambda_j$. We define an unoriented graph structure on $\mathcal{P}$ by drawing an edge between two elements $p_1, p_2 \in \mathcal{P}_k$ of the same length if there exists some $p_3 \in \mathcal{P}_{k-1}$ with arrows $p_1 \succ p_3$ and $p_2 \succ p_3$ such that the new term of $p_3$ coming from $p_1 \succ p_3$ is also the new term of $p_3$ coming from $p_2 \succ p_3$. Under these circumstances, we say that $p_1 \succ p_3 \prec p_2$ is a roof. We take the induced graph structure on all $\mathcal{P}_k$ and all $\mathcal{P}_d^k$. Equivalently, the graph on $\mathcal{P}_d^k$.
can be described as follows: there is an edge between \( p = (\Lambda_1, \ldots, \Lambda_i, \ldots, \Lambda_j, \ldots, \Lambda_k) \) and \( p' = (\Lambda_1, \ldots, \Lambda_i', \ldots, \Lambda_j', \ldots, \Lambda_k) \) if and only if \( \Lambda_i + \Lambda_j = \Lambda_i' + \Lambda_j' \) and

\[
[\mathbb{Z}^2 : \Lambda_i] + [\mathbb{Z}^2 : \Lambda_j] = [\mathbb{Z}^2 : \Lambda_i'] + [\mathbb{Z}^2 : \Lambda_j'].
\]

Nevertheless, the partition \( p_i \) above will be important later in the paper, so it was worth introducing.

**Proposition 3.1.** The graph structure on \( \mathcal{R}_d^k \) defined above is connected for all \( d \geq k \geq 2 \) or \( d = k = 1 \).

*Proof.* We proceed by induction on \( k \). The graph is complete for \( k = 2 \). Assume that \( k \geq 3 \) and that the statement holds for all \( (d', k') \) with \( k' < k \). Let \( p \in \mathcal{R}_d^k \). We say that \( p \) is equivalent to \( p' \) if it belongs to the same connected component.

The first step is to prove that \( p \) is equivalent to a partition containing \( \mathbb{Z}^2 \) among its terms. Note that any pair of summands \((\Lambda_i, \Lambda_j)\) can be replaced with any pair \((\Lambda_i + \Lambda_j, \Lambda')\), where \( \Lambda' \subseteq \Lambda_i + \Lambda_j \) has the suitable index

\[
[\Lambda_i + \Lambda_j : \Lambda'] = [\Lambda_i + \Lambda_j : \Lambda_i] + [\Lambda_i + \Lambda_j : \Lambda_j] - 1.
\]

The only way the value of the smallest index cannot be decreased is if a sublattice realizes this minimum contains all the other summands. This lattice must be \( \mathbb{Z}^2 \).

The second step is to prove that \( p \) is equivalent to a partition containing \( \mathbb{Z}^2 \) such that the remaining lattices also add up to \( \mathbb{Z}^2 \). This will complete the proof by the inductive hypothesis since we can play the game only with the remaining \( k - 1 \) lattices. Replace \( p \) with the partition found in the first step. Now any lattice can be replaced with any other lattice of the same index. It is not hard to see that given any sequence \( i_1, \ldots, i_{k-1} \) of natural numbers, there exist \( k - 1 \) lattices of these indexes which add up to \( \mathbb{Z}^2 \). For instance, we can choose the first lattice to be \( \mathbb{Z} \oplus i_1 \mathbb{Z} \), the second lattice to be \( i_2 \mathbb{Z} \oplus \mathbb{Z} \) and the others arbitrarily. \( \square \)

4. Maps from genus 2 curves to a genus 1 curve

4.1. Generalities. Let \( E \) be a smooth genus one curve, \( \Lambda \subseteq H_1(E, \mathbb{Z}) \) a sublattice of rank two and \( d \) divisible by the index of the lattice. We start by introducing notation for this section. Let \( \mathcal{M}_{2,n}(E, d) \) be the moduli stack of genus 2 degree \( d \) stable maps into \( E \) with \( n \) markings and

1. \( \mathcal{M}_{2,n}^{sm}(E, d) \subseteq \mathcal{M}_{2,n}(E, d) \) consisting of maps with smooth sources;
2. \( \mathcal{M}_{2,n}^{ct}(E, d) \subseteq \mathcal{M}_{2,n}(E, d) \) consisting of maps with sources of compact type;

By Proposition 2.3, \( \mathcal{M}_{2,n}^{ct}(E, d) \) splits as a disjoint union according to the image of the pushforward map on first homology groups:

3. \( \mathcal{M}_{2,n}^{sm}(E, d; \Lambda) \subseteq \mathcal{M}_{2,n}^{sm}(E, d) \); and
4. \( \mathcal{M}_{2,n}^{ct}(E, d; \Lambda) \subseteq \mathcal{M}_{2,n}^{ct}(E, d) \).

As usual, we will suppress the index \( n \) when \( n = 0 \). A cover is determined up to finitely many choices by its branch divisor, so \( \dim \mathcal{M}_{2,n}^{sm}(E, d; \Lambda) = 2 \). It is certainly known that the latter space is irreducible. For instance, we could invoke the general theorem of Gabai and Kazez [16] proved by topological methods which covers all possible genera of the source and target. An algebraic proof when the target has genus 1 has been given by Bujokas [9]. The case when the source has genus 2 can also be proved by basic abelian surface theory, but we will skip this.

**Theorem 4.1** (folklore). The space \( \mathcal{M}_{2}^{sm}(E, d; \Lambda) \) is a smooth nonempty irreducible Deligne-Mumford stack of dimension 2.
Smoothness is a trivial deformation theoretic calculation which we skip.

To any length two partition (§3) \( p = (\Lambda_1 + \Lambda_2 = \Lambda) \) of \( \Lambda \) and closed \( q \in E \), we can associate a stable map \( \sigma_p^q \) in \( \mathcal{M}_{2,1}^\sigma(E, d; \Lambda) \) with the following properties:

1. The source of \( \sigma_p^q \) is a chain of three curves \( E'_1 \cup L \cup E'_2 \);
2. All 3 components are smooth and \( p_a(E'_1) = p_a(E'_2) = 1 \) and \( p_a(L) = 0 \);
3. The restriction of \( \sigma_p^q \) to \( E'_i \) is the isogeny associated to \( \Lambda_i \);
4. The restriction of \( \sigma_p^q \) to \( L \) is constant with image \( \{ q \} \);
5. The sole marked point \( r \) belongs to \( L \).

Note that \( \sigma_p^q \) is stable and admits 0 or 1 nontrivial automorphisms depending on whether \( \Lambda_1 \neq \Lambda_2 \) or not.

**Lemma 4.2.** The evaluation map \( \text{ev} : \mathcal{M}_{2,1}^\sigma(E, d; \Lambda) \to E \) is smooth of local relative dimension 2 at \( \sigma_p^q \).

**Proof.** Given the action of the the group of automorphisms of \( \mathcal{M}_{2,1}^\sigma(E, d; \Lambda) \), it suffices to prove that \( \mathcal{M}_{2,1}^\sigma(E, d; \Lambda) \) is smooth of local dimension 3 at \( [\sigma_p^q] \). The tangent space to \( \mathcal{M}_{2,1}(E, d; \Lambda) \) at \([\sigma_p^q]|_{[\sigma_p^q]}\) is naturally identified with

\[
\text{Ext}^1 \left( \left[ \sigma_p^q \right]^* \Omega_E \to \Omega_D(r), \mathcal{O}_D \right),
\]

where \( D \) denotes the source of \( \sigma_p \) and the complex lives in degrees \([-1, 0]\). However, we will take an alternative route to circumvent this unpleasant calculation.

For simplicity of notation, we suppress the “\( q \)” superscript. Let \( \overline{\sigma}_p \) be the image of \( \sigma_p \) in \( \mathcal{M}_{2,1}^\sigma(E, d; \Lambda) \), i.e. the map obtained by erasing the marking and stabilizing the resulting map. Let \( D' = E'_1 \cup E'_2 \) be the source of this map. By abuse of notation, we denote \( E'_1 \cap E'_2 \) by \( r \). To complete the proof of the lemma, it suffices to prove that \( \mathcal{M}_{2,1}^\sigma(E, d; \Lambda) \) is smooth at \( [\overline{\sigma}_p] \) and that there exist first order deformations which smooth the node at \( r \). The reduction holds in light of the following general fact: locally at a node, the total space of a family of (at worst) nodal curves over a smooth base is smooth if the node is smoothed out to first order. Indeed, the condition implies that the map to the versal deformation space of a node Spec \( \mathbb{C}[t] \) is smooth at the node in question and the claim follows since the total space of the family over it Spec \( \mathbb{C}[x, y, t]/(xy - t) \) is smooth at the origin as well.

Of course, the space of first order deformations of \( \overline{\sigma}_p \) is the hyperextension group

\[
\text{Ext}^1 \left( \left[ \overline{\sigma}_p \right]^* \Omega_E \to \Omega_{D'}, \mathcal{O}_{D'} \right).
\]

Note that there is a short exact sequence

\[
0 \to \overline{\sigma}_p^* \Omega_E \to \Omega_{D'} \to \mathbb{C}_p^{(2)} \to 0,
\]

where \( \mathbb{C}_p \) denotes the skyscraper sheaf at \( r \), so the hyperextension group simplifies to \( \text{Ext}^1(\mathbb{C}_p^{(2)}, \mathcal{O}_{D'}) \). Clearly, this has dimension 2, so \( \mathcal{M}_{2,1}^\sigma(E, d; \Lambda) \) is smooth at \( [\overline{\sigma}_p] \). Moreover, the second map in the piece

\[
\text{Ext}^0(\overline{\sigma}_p^* \Omega_E, \mathcal{O}_{D'}) \to \text{Ext}^1(\mathbb{C}_p^{(2)}, \mathcal{O}_{D'}) \to \text{Ext}^1(\Omega_{D'}, \mathcal{O}_{D'})
\]

of the long exact sequence can be interpreted as follows: the elements of the second term are first order deformations of the map, which induce abstract deformations of \( D' \), contained in the third term. However, the first term is one-dimensional since \( \overline{\sigma}_p^* \Omega_E = \mathcal{O}_{D'} \), so the second map in (5) is not identically zero, i.e. the node does indeed get smoothed to first order.

\[\Box\]

**4.2. Quasi-traceless covers.** Given a map \( f : C \to E \), the pullback map on sheaf cohomology \( H^1(\mathcal{O}_E) \to H^1(\mathcal{O}_C) \) is Serre dual to the trace map

\[\text{Tr}_f : H^0(K_C) \to H^0(K_E).\]
More generally, if \( \pi : \mathcal{C} \to S \) is a flat family of semistable curves, and \( f_S : \mathcal{C} \to E \times S \) an \( S \)-morphism, we have the cohomology pullback
\[
H^1(\mathcal{O}_E) \otimes \mathcal{O}_S = R^1\text{proj}_{E \times S / S, *}(\mathcal{O}_{E \times S}) \longrightarrow R^1\pi_*\mathcal{O}_\mathcal{C}.
\]
There is a perfect pairing \( R^1\pi_*\mathcal{O}_\mathcal{C} \times \pi_\ast \omega_{\mathcal{C}/S} \to \mathcal{O}_S \), so the map above is dual to
\[
\pi_\ast \omega_{\mathcal{C}/S} \longrightarrow H^0(K_E) \otimes \mathcal{O}_S.
\]
Note that the formation of both \( R^1\pi_*\mathcal{O}_\mathcal{C} \) and \( \pi_\ast \omega_{\mathcal{C}/S} \) commutes with base changes \( S' \to S \). The bottom line is that we obtain well-defined trace maps
\[
\text{Tr} : E \longrightarrow \mathcal{O} \otimes H^0(K_E),
\]
where \( E \) and \( \mathcal{O} \) are the Hodge bundle and structure sheaf respectively over spaces such as \( \mathcal{M}_{2,n}(E, d; \Lambda) \). Recall that the Hodge bundle is the pushforward of the relative dualizing sheaf of the universal curve.

The definition below plays a very important role in the paper, but the motivation for considering these objects won’t become clear until Proposition 6.6.

**Definition 4.3.** We say that a pointed map \( f : (C, r) \to (E, q) \), \( f(r) = q \), is quasi-traceless if the unique up to scalars 1-form \( \eta \) on \( C \) such that \( \eta(r) = 0 \) is killed by the trace map \( \text{Tr}_f \).

**Remark 4.4.** The unique up to scalars nonzero 1-form on \( C \) annihilated by the trace map can be characterized as the holomorphic form which satisfies
\[
\int_C \eta \wedge f^*dz = 0,
\]
where \( z \) is the complex coordinate on the universal cover of \( E \). We can turn the question around: given a map \( f : C \to E \) from a smooth genus 2 curve, for which points \( r \in C \) is the induced marked map \( (C, r) \to (E, f(r)) \) quasi-traceless relative to \( q = f(r) \)? They are the two points where \( \eta \) vanishes.

We can construct a compactification \( \mathcal{M}_{2,1}(E, d) \) of the space of quasi-traceless covers as a degeneracy locus inside \( \mathcal{M}_{2,1}(E, d) \) as follows. Let \( \mathcal{E} \) and \( \mathcal{L} \) be the Hodge bundle and tautological cotangent line bundle over \( \mathcal{M}_{2,1}(E, d) \) respectively. For simplicity of notation, we denote the structure sheaf of the latter stack by \( \mathcal{O} \). The space \( \mathcal{M}_{2,1}(E, d) \) can be formally defined as the degeneracy locus of the map of rank 2 bundles
\[
\gamma \oplus \text{Tr} : E \longrightarrow \mathcal{L} \oplus \mathcal{O},
\]
where \( \gamma : E \to \mathcal{L} \) is evaluation at the marked point and \( \text{Tr} \) is the relative trace map. The second summand on the right hand side should be more accurately interpreted as \( \mathcal{O} \otimes H^0(K_E) \), but \( H^0(K_E) = \mathbb{C} \), so our shorthand isn’t incorrect. The notations \( \mathcal{D}^p_{2,1}(E, d; \Lambda) \) and \( \mathcal{P}^p_{2,1}(E, d; \Lambda) \) are self-explanatory.

Recall \( \sigma^p_0 (\S 4.1) \). Note that \( \gamma((\sigma^p_0)) = 0 \) because all global sections of the dualizing sheaf of the source vanish identically on \( \mathcal{L} \), so \( \sigma^p_0 \) lives in \( \mathcal{P}^p_{2,1}(E, d; \Lambda) \).

**Proposition 4.5.** The space \( \mathcal{P}^p_{2,1}(E, d; \Lambda) \) is nonempty and irreducible of dimension 2 and contains \([\sigma^p_0]\) in its closure for all partitions \( p \) and closed points \( q \).

**Proof.** A standard deformation theoretic calculation shows that \( \mathcal{M}^p_{2,1}(E, d; \Lambda) \) is smooth of dimension 3, and hence, using the \( \text{Aut}(E) \)-action again, the restricted evaluation map \( \mathcal{M}^p_{2,1}(E, d; \Lambda) \to E \) is smooth of relative dimension 2. It is not hard to see that \( \mathcal{P}^p_{2,1}(E, d; \Lambda) \to E \) has relative dimension 1 at all points. The fact that
\[ [\sigma_q^E] \] is contained in the closure of \( \mathcal{L}_{2,1}^m(E, d; \Lambda) \) is a consequence of the following easy observation: all nearby deformations of \( \sigma_q^E \) inside the fiber of \( \mathcal{L}_{2,1}^m(E, d; \Lambda) \to E \) over \( q \) in fact have smooth sources.

The least trivial claim is the irreducibility of \( \mathcal{L}_{2,1}^m(E, d; \Lambda) \). On one hand, the restriction of the forgetful map to the quasi-traceless locus
\[
\mathcal{L}_{2,1}^m(E, d; \Lambda) \to \mathcal{M}_{2}^m(E, d; \Lambda)
\]
is 2-to-1 by Remark 4.4. The target is irreducible by Theorem 4.1, so already we can see that the source can only have one or two irreducible components. To rule out the latter possibility, we analyze \( \mathcal{L}_{2,1}(E, d) \to \mathcal{M}_{2}^m(E, d; \Lambda) \) locally at \( \sigma_q^E \). It is clear that the image of \( \sigma_q^E \) – called \( \mathcal{P}_q \) in the proof of Lemma 4.2 – doesn’t have any preimage other than \( \sigma_q^E \) itself. Thus, if we prove that \( \mathcal{L}_{2,1}(E, d) \) is smooth at \( [\sigma_q^E] \), we rule out the possibility of two irreducible components and we are done.

We claim that \( \gamma([\sigma_q^E]) = 0 \), \( \text{Tr}([\sigma_q^E]) \neq 0 \) and Lemma 4.2 imply that
\[
\text{ev}_\mathcal{P} : \mathcal{L}_{2,1}(E, d) \to E
\]
is smooth of relative local dimension 1 at \( [\sigma_q^E] \). We isolate this elementary general fact in the lemma below.

**Lemma 4.6.** Let \( \mathcal{Z} \) be a smooth Deligne-Mumford stack over \( \mathbb{C} \) of dimension 2. Assume that \( \mathcal{E}, \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are locally free coherent sheaves of ranks 2, 1 and 1 respectively and that \( \varphi = \varphi_1 \oplus \varphi_2 : \mathcal{E} \to \mathcal{F}_1 \oplus \mathcal{F}_2 \) is an \( \mathcal{O}_\mathcal{Z} \)-module homomorphism. Let \( z \in \mathcal{Z}(\mathbb{C}) \) be an (infinitesimally) isolated vanishing point of \( \varphi_1 \). If \( \varphi_2(z) \neq 0 \), then \( z \) is a smooth point of the degeneracy locus of \( \varphi \) in \( \mathcal{Z} \).

**Proof.** Pulling back along an étale map \( Z \to \mathcal{Z} \) from a scheme, it is clear that we may assume that \( \mathcal{Z} \) is a scheme itself. Complex analytically (or étale) locally at \( z \), the data is isomorphic to the analogous data in the case \( \mathcal{Z} = U \subset \mathbb{A}^2, \mathcal{E} = \mathcal{O} \oplus \mathcal{O}, \mathcal{F}_1 = \mathcal{F}_2 = \mathcal{O} \) and \( z \) is the origin. Then \( \varphi \) is a \( 2 \times 2 \) matrix-valued function with entries \( a, b, c, d \) such that \( (a, b) = (0, 0) \) and \( (c, d) \neq (0, 0) \) at the origin and the degeneracy locus is \( V(ad - bc) \). We have
\[
(6) \quad \left[ \begin{array}{c} \partial_a \det \varphi \\ \partial_b \det \varphi \end{array} \right] = \left[ \begin{array}{c} [\partial_a a \quad \partial_a b] \\ [\partial_b a \quad \partial_b b] \end{array} \right] \left[ \begin{array}{c} d \\ -c \end{array} \right] + \left[ \begin{array}{c} [\partial_a c \quad \partial_a d] \\ [\partial_b c \quad \partial_b d] \end{array} \right] \left[ \begin{array}{c} -b \\ a \end{array} \right],
\]
completing the proof. \( \square \)

Applying Lemma 4.6 to an open substack of the fiber over \( q \) of the evaluation map \( \text{ev} : \mathcal{M}_{2,1}(E, d) \to E \) containing \( [\sigma_q^E] \), we deduce that \( \text{ev}_\mathcal{P} \) is smooth at \( [\sigma_q^E] \) and hence so is \( \mathcal{L}_{2,1}(E, d) \), which completes the proof of Proposition 4.5. \( \square \)

4.3. **Relation to the Atiyah ruled surface.** The rank two Atiyah bundle over \( E \), i.e. the unique rank two bundle that can be obtained as a nonsplit extension of the structure sheaf by itself \([1]\), is denoted by \( \mathcal{V} \). It is well-known and obvious that \( \mathcal{V} \) is self-dual. Let \( \varsigma : \mathbb{P}\mathcal{V} = \text{Proj} \text{Sym} \mathcal{V} \to E \) be its projectivization and \( E_\infty \subset \mathbb{P}\mathcal{V} \) the distinguished section corresponding to the unique copy of \( \mathcal{O}_E \) inside \( \mathcal{V} \).

**Lemma 4.7.** The quasiprojective surface \( \mathbb{P}\mathcal{V} \setminus E_\infty \) contains no complete curves.

**Proof.** This follows directly from \([28, \text{Lemma 2.1 and Proposition 2.2}] \). However, note that Lemma 4.7 is false in positive characteristic by \([28, \text{Proposition 2.2}] \). \( \square \)
Let \( f : (C, r) \to (E, q) \) be a map from a smooth genus 2 curve. The cup product map \( H^1(\mathcal{O}_C) \otimes H^0(\mathcal{O}_C(r)) \to H^1(\mathcal{O}_C(r)) \) can be rearranged as a \( \mathbb{C} \)-linear map

\[
H^1(\mathcal{O}_C) \xrightarrow{\mu_0} \text{Hom}(H^0(\mathcal{O}_C(r)), H^1(\mathcal{O}_C(r)))
\]

which is dual to the following rather uninteresting Petri map

\[
H^0(\mathcal{O}_C(r)) \otimes H^0(K_C(-r)) \xrightarrow{\mu_0} H^0(K_C).
\]

The image of \( \mu_0 \) consists of all 1-forms vanishing at \( r \), so \( f \) is quasi-traceless if and only if \( \text{Tr} \circ \mu_0 \equiv 0 \). Dually, \( f \) is quasi-traceless if and only if the pullback map \( H^1(\mathcal{O}_E) \to H^1(\mathcal{O}_C) \) followed by \( \mu_0^* \) is identically zero.

**Proposition 4.8.** Let \( C \) be a smooth genus 2 curve and \( r \in C(\mathbb{C}) \). A pointed cover \( f : (C, r) \to (E, q) \) is quasi-traceless if and only if there exists a lift

\[
\begin{array}{ccc}
\mathbb{P}V & \xrightarrow{\xi} & V \\
\downarrow & & \downarrow \\
C & \xrightarrow{f} & E
\end{array}
\]

of \( f \) such that \( \xi^{-1}(E_\infty) = \{r\} \) scheme-theoretically.

**Proof.** Assume that \( f \) is quasi-traceless. First, we claim that \( h^0(f^*\mathcal{V}(r)) = 2 \). The element of \( H^1(\mathcal{O}_C) = \text{Ext}^1(\mathcal{O}_C(r), \mathcal{O}_C(r)) \) defining the extension

\[
0 \to \mathcal{O}_C(r) \to f^*\mathcal{V}(r) \to \mathcal{O}_C(r) \to 0
\]

lives in the image of \( H^1(\mathcal{O}_E) \to H^1(\mathcal{O}_C) \), hence in the kernel of \( \mu_0^* \) by the remarks in the second paragraph of this subsection. Note that the map \( \mu_0^* \) can be interpreted as associating the connecting homomorphism to a given extension, which means that the connecting homomorphism is zero in our situation. Hence (7) is exact on global sections, so \( h^0(C, f^*\mathcal{V}(r)) = 2 \), as claimed.

Let \( \sigma \) be a section of \( f^*\mathcal{V}(r) \) linearly independent to the section corresponding to \( \mathcal{O}_C(r) \hookrightarrow f^*\mathcal{V}(r) \) and \( \mathcal{O}_C(D) \hookrightarrow f^*\mathcal{V}(r) \) the line bundle spanned by \( \sigma \). Then \( f^*\mathcal{V}(r-D) \) has nonzero global sections, so \( h^0(\mathcal{O}_C(r-D)) \neq 0 \) since \( f^*\mathcal{V}(r-D) \) is an extension of \( \mathcal{O}_C(r-D) \) by itself. Because \( D \) is effective, \( D = 0 \) and \( D = r \) are the only possibilities; however, the latter is ruled out by the linear independence assumption, so \( D = 0 \). Moreover, it is clear that the image of \( \sigma \) in \( \mathcal{O}_C(r) \) is nonzero, so it only has a simple zero at \( r \).

In conclusion, we get an injective map \( \mathcal{O}_C(-r) \hookrightarrow f^*\mathcal{V} \) with locally free cokernel such that the cokernel of the composition \( \mathcal{O}_C(-r) \hookrightarrow f^*\mathcal{V} \to f^*\mathcal{O}_E = \mathcal{O}_C \) is the \( (1\text{-dimensional}) \) skyscraper sheaf on \( C \) at \( r \). This morphism induces the desired lift \( \tilde{f} : C \to \mathbb{P}V \) such that \( \tilde{f}^{-1}(E_\infty) = \{r\} \) in the usual way, that is, by sending each point \( x \in C(\mathbb{C}) \) to the image line in the fiber of \( f^*\mathcal{V} \) at \( f(x) \).

The converse is proved by reversing the argument, so the details are left to the reader. Pulling back the composition \( \mathcal{O}_{\mathbb{P}V}(-E_\infty) \to \mathcal{O}_{\mathbb{P}V} \to \xi^*\mathcal{V} \) (whose cokernel is locally free, unlike the first factor) via the lift \( \tilde{f} \) gives the map \( \mathcal{O}_C(-r) \to f^*\mathcal{V} \) which implies that \( h^0(C, f^*\mathcal{V}(r)) = 2 \). Reversing the argument in the first paragraph, we deduce that \( f \) is quasi-traceless. \( \square \)
5. The specialization to a split abelian surface

5.1. Notation and preliminaries. The moduli space of \((d_1, d_2, \ldots, d_g)\)-polarized abelian varieties is denoted by \(\mathcal{A}_g(d_1, \ldots, d_g)\).

Remark 5.1. The analytic theory of abelian surfaces implies that \(\text{NS}(A) = \{L\} \mathbb{Z}\) if \((A, L)\) is very general in \(\mathcal{A}_2(1, d)\) and, more importantly, that \(\{L\}\) is indecomposable into nonzero effective classes if \((A, L)\) is general. Hence all members of \(\{L\}\) are integral under the generality assumption. Note the analogy with primitively polarized K3 surfaces and the contrast to \(\mathbb{P}^2\) polarized by \(\mathcal{O}(d)\).

For the proof of Theorem 2.6, we will consider a one-parameter family of \((1, d)\)-polarized abelian surfaces specializing to a \((1, 2)\)-polarized product of elliptic curves. We construct this family by choosing a smooth connected affine curve \(\zeta: Z \to \mathcal{A}_2(1, d)\) which intersects the image of \(\mathcal{A}_1(1) \times \mathcal{A}_1(d) \to \mathcal{A}_2(1, d)\) transversally at a point whose unique preimage is \(o_Z \in Z\). Note that we can thread such a curve \(Z\) through any point in the moduli space. All of this follows from quasi-projectivity.

Definition 5.2. A base change map is a map \(\mu: (B', o') \to (B, o)\) between pointed smooth connected quasi-projective curves such that \(\mu^{-1}(o) = \{o'\}\) set-theoretically and the restriction \(B' \setminus \{o'\} \to B \setminus \{o\}\) is étale. The local degree of \(\mu\) is the rank of \(\hat{\mathcal{O}}_{B', o'}\) as a \(\hat{\mathcal{O}}_{B, o}\)-module via the ring homomorphism \(\hat{\mathcal{O}}_{B, o} \to \hat{\mathcal{O}}_{B', o'}\) induced by the pullback on stalks \(\mathcal{O}_{B, o} \to \mathcal{O}_{B', o'}\).

Clearly, given any pointed curve \((B, o)\) as in the definition above, there exist base change maps \(\mu: (B', o') \to (B, o)\) of any local degree.

Although in principle we are only interested in the family described above, some crucial arguments will need to be carried over families obtained by a base change. For this reason, we consider base change maps \(\gamma: (B, o) \to (Z, o_Z)\) and distinguish between two situations:

\(\text{(OS)}\) \(B = Z\) and \(\gamma = \text{id}\); and
\(\text{(BC)}\) No additional assumption.

\(\text{(OS)}\) stands for “original setup” while \(\text{(BC)}\) stands for “base change.”

Unless explicitly stated otherwise, \(\text{(BC)}\) is the running assumption.

Although there are some geometric differences between these two situations – for instance, a statement analogous to [14, Proposition 1.1] holds but obviously only under \(\text{(OS)}\), it turned out that they are not relevant for our purposes. Thus this distinction is nothing more than a convenient expository trick. However, in §§6.2, we will need to return to this point and possibly replace \(Z\) with a base change in order to ensure a certain property of the base setup.

Let \(\pi: (W, \mathcal{L}) \to B\) be the family of \((1, d)\)-polarized abelian surfaces over \(B\) specializing to \(W_o = E_1 \times E_2\) over \(o \in B\). The polarization on the central fiber is \(\mathcal{L}_o = J_1 \boxtimes J_2\) with degrees \(d\) and \(1\) on \(E_1\) and \(E_2\) respectively. We may assume that \(E_1\) and \(E_2\) are not isogeneous. Let \(\pi_1\) and \(\pi_2\) be the projections of \(W_o\) to the two factors. Then \(\{\mathcal{L}_o\}\) consists of sums of \(d + 1\) mobile fibers – one of \(\pi_2\) and \(d\) of \(\pi_1\). Let \(\pi^o: W^o \to B^o\) be the restriction to \(B^o \setminus \{o\}\).

The rest of the paper is concerned with analyzing what could imprecisely be described as the degenerate Severi variety corresponding to the split abelian surface...
essentially amounts for the property of not being a perfect map. Now we define a function for the calculation.

Let \( W = \{L \in A[t]\} \) be an integral domain of characteristic 0 and \( R = A[t] \) the ring of formal power series in \( A \). Let \( (t) \subset R \) be the ideal generated by \( t \in R \). Fix \( m \in \mathbb{N} \). Consider the set

\[
W_m(A) = \left\{ p(t, x) = x^m + \sum_{j=1}^m \alpha_j(t)x^{m-j} \in R[t,x] : \alpha_j \in (t) \text{ for all } j \right\}
\]

Now we define a function

\[
h: W_m(A) \to W_m(A) \cup \{ \text{win}, \text{lose} \}
\]

as \( h = g \circ f \) with \( f \) and \( g \) defined below. First, let

\[
f(p) = \begin{cases} 
    t^{-m}p(t, tx) & \text{if } t^{-m}p(t, tx) \in R[t,x], \\
    \text{lose} & \text{otherwise.}
\end{cases}
\]

for \( p = p(t, x) \in W_m(A) \). Second, let

\[
g(p) = \begin{cases} 
    q(t, x + a) & \text{if there exists } a \in A \text{ such that } q(t, x + a) \in W_m(A), \\
    \text{win} & \text{otherwise.}
\end{cases}
\]

for \( q = q(t, x) \in R[t,x] \). Set \( g(\text{lose}) = \text{lose} \). Note that \( a \), if it exists, is unique. Indeed, if \( p(t, x) \in W_m(A) \), then \( p(t, x + a_0) \notin W_m(A) \) for all \( a_0 \in A \), \( a_0 \neq 0 \) by a simple calculation.

Lemma 5.4. Let \( p(t, x) \in W_m(A) \) which is not a perfect \( m \)-th power. Then there exists a positive integer \( \mu \) such that the sequence of iterates defined by \( p_0 = p(t^\mu, x) \) and \( p_{n+1} = h(p_n) \) eventually ends with a “win.”

Proof. Note that \( p(t, x) \) is a perfect \( m \)th power, i.e. \( p(t, x) = (x - \phi(t))^m \) for some \( \phi \in R \), if and only if

\[
\alpha_j(t) = m^{-j} \binom{m}{j} \alpha_1(t)^j, \text{ for all } j \geq 2.
\]

(8)

The first basic observation is that, if we fix the initial polynomial \( p \), then the game only becomes harder if we a posteriori extend the base ring \( B \supset A \). Indeed, first, the property of not being a perfect \( m \)th power is preserved in the larger ring \( B \) since (8) doesn’t involve the base ring in any way; second, if no \( a \in B \) with the property

Question 5.3. Which (primitive class) stable maps to \( W \) deform to nearby \( W_t \)?

The nature of the question is obviously dictated by the fact that the description of \( \{L \in A[t]\} \) is trivial as explained above, so there is an obvious trade-off between genus and repetitions among the \( d \) “utilized” fibers of \( \pi_1 \). The locus of deformable stable maps consists of many components. While the flexible components (the components which contribute to the (reduced) Gromov-Witten invariants) are trivial to describe – they consist generically of “combs” (star dual graph) with genus 1 components with the backbone mapping isomorphically onto its image and the teeth mapping as isogenies – the others are quite mysterious and the work in §5 essentially amounts to partial progress towards understanding them.

5.2. Preliminaries for semistable reduction. All notation in this subsection is independent of other notation used in the paper. Let \( A \) be an integral domain of characteristic 0 and \( R = A[t] \) the ring of formal power series in \( A \). Let \( (t) \subset R \) be the ideal generated by \( t \in R \). Fix \( m \in \mathbb{N} \). Consider the set

\[
W_m(A) = \left\{ p(t, x) = x^m + \sum_{j=1}^m \alpha_j(t)x^{m-j} \in R[t,x] : \alpha_j \in (t) \text{ for all } j \right\}
\]
required in the definition of $g$ exists, then certainly no $a \in A$ with the respective property exists (despite that $W_m(A)$ depends on the base ring) and third, as long as all choices coming from the definition of $g$ are from $A$, the value of $f$ at the next step will live in $A[t, x]$. Thus the sequence for $A$ coincides with that for $B$, but may be shorter – only by reaching “win” faster.

Thus it suffices to prove the lemma when $A = \mathbb{K}$, an algebraically closed field of characteristic zero. By the Newton-Puiseux Theorem [26, page 61], there exist Puiseux series $\phi_1, \phi_2, \ldots, \phi_m \in \mathbb{K}[[t]]$ such that

$$p(t, x) = \prod_{i=1}^{m} (x - \phi_i(t)).$$

Since $p$ is not a perfect $m$th power, not all $\phi_i$ are equal. By the definition of Puiseux series, for each $i$ there exits some natural number $\mu_i$ such that $\phi_i(t^{\mu_i}) \in \mathbb{K}(t)$, the field of Laurent series over $\mathbb{K}$.

We will show that $\mu = \text{lcm}[\mu_1, \mu_2, \ldots, \mu_m]$ satisfies the desired property. We have

$$p(t^{\mu}, x) = \prod_{i=1}^{m} (x - \psi_i(t))$$

with $\psi_i(t) \in \mathbb{K}(t)$. Actually, $p(t^{\mu}, x)$ is monic with coefficients in $\mathbb{K}[t]$ just like $p(t, x)$, hence all $\psi_i$ are integral, so, in fact, $\psi_i(t) \in \mathbb{K}[t]$. Note that $p_0 = p(t^{\mu}, x)$ is still not a perfect $m$th power and still belongs to $W_m(\mathbb{K})$.

Since $p_0$ is not a perfect $m$th power, not all $\psi_i$ are equal. Also, since $p \in W_m(\mathbb{K})$, all $\psi_i$ must be divisible by $t$. Indeed, if exactly $\alpha \geq 1$ of the $\psi_i$’s are not divisible by $t$, then the elementary function $e_{\alpha}(\psi_1, \ldots, \psi_m)$ is not divisible by $t$, contradicting the fact that $p \in W_m(\mathbb{K})$. Denote the valuation at $t$ on $\mathbb{K}[t]$ by $v_t$. Then

$$N = \min v_t(\psi_i - \psi_j) < \infty$$

and let $\psi_k - \psi_l$ be a difference for which the minimum is attained. In other words,

$$\psi_i(t) = a_1 t + a_2 t^2 + \ldots + a_N t^N + \text{h.o.t.},$$

for all $i$, but the coefficients of $t^{N+1}$ in $\psi_k$ and $\psi_l$ are distinct. Let

$$\psi_{i,n}(t) = t^{-n} (\psi_i(t) - a_1 t - \ldots - a_n t^n)$$

$$= a_{n+1} t + a_{n+2} t^2 + \ldots + a_N t^{N-n} + \text{h.o.t.}$$

for $n = 0, 1, 2, \ldots, N$ and $i = 1, 2, \ldots, m$. We claim that

$$p_n(t, x) = \prod_{i=1}^{m} (x - \psi_{i,n}(t)).$$

For $n = 0$, this is (9). Assume inductively that this holds for $n \leq N - 1$. Then

$$f(p_n)(t, x) = \prod_{i=1}^{m} \left( x - \frac{\psi_{i,n}(t)}{t} \right)$$

and hence, by (10) and the definition of $g$,

$$p_{n+1}(t, x) = \prod_{i=1}^{m} \left( x + a_{n+1} - \frac{\psi_{i,n}(t)}{t} \right) = \prod_{i=1}^{m} (x - \psi_{i,n+1}(t)).$$
completing the inductive verification. Note that (11) continues to hold for \( n = N \). This means that \( f(p_N)(0, x) \) has at least two distinct roots: the coefficients of \( t^{N+1} \) in \( \psi_k \) and \( \psi_l \), i.e. \( f(p_N)(0, x + a) \neq x^m \) for all \( a \in \mathbb{K} \) and hence \( h(p_N) = \text{“win.”} \)

Lemma 5.4 is nothing new: the \( “A = \mathbb{K}” \) part of the argument above is merely some partial progress towards 1-dimensional Fulton-MacPherson style [15] bubbling up. Indeed, the first two paragraphs operating under \( “A = \mathbb{K}” \) amount to ordering some colliding points via base change, while the last paragraph pulls the trajectories of the ordered points apart by iterated blowups. The case \( A = \mathbb{C}[X] \) we will use in \( \S \S 5.3 \) is a somewhat “fuzzier” version of this procedure with an additional short dimension.

5.3. Drawers. There is increasing evidence (arguably some of if coming from this paper) that the right technique to approach questions of the same flavor as Question 5.3 – the reader might consider a potentially different elliptic specialization – is by blowing up the elliptic fibers in the total space of the family. This idea is due to X. Chen [14]. What is new relative to both [14] and [22, 23] is a type of “tree shaped” expansion. This is what we will introduce next.

Let \( T_2 \) and \( R_2 \) be the trivial ruled surface over \( E_2 \), that is \( T_2 = \mathbb{P}^1 \times E_2 \), and the Atiyah ruled surface over \( E_2 \) (cf. \( \S \S 4.3 \)) respectively.

We return to the setup described in \( \S \S 5.1 \).

**Definition 5.5.** A drawer is a reduced but possibly reducible surface \( Z \) with simple normal crossing singularities endowed with a morphism \( \zeta : Z \to E_1 \times E_2 \) such that the following conditions are satisfied:

1. There exists a distinguished irreducible component \( W_o \) of \( Z \) such that \( \zeta \) restricted to \( W_o \) is an isomorphism.
2. All irreducible components of \( Z \) intersect transversally and there are no triple intersections, i.e. the dual simplicial complex is a graph \( G_Z \) whose set of vertices is \( V(G_Z) \) and whose set of edges is \( E(G_Z) \).
3. All irreducible components of \( Z \) other than the one in (1) are isomorphic to either \( T_2 \) or \( R_2 \). Thus for any component \( Z_v \cong Z, v \in V(G_Z) \), there is a natural projection map \( \pi_v : Z_v \to E_2 \). We require that:
   a. \( \pi_v \) is the restriction of \( \zeta \) to \( Z_v \);
   b. for any edge \( (v, w) \in E(G_Z) \), the double curve \( Z_{vw} := Z_v \cap Z_w \) is a section of \( \pi_v \) with trivial normal bundle in \( Z_v \).
4. The dual graph \( G_Z \) is a (rooted) tree such that:
   a. the vertex \( w_o \) corresponding to the abelian component \( W_o \) is considered the “root” of the tree;
   b. all components corresponding to vertices which aren’t leaves of \( G_Z \) – with the exception of \( w_o \) if it isn’t a leaf – are isomorphic to \( T_2 \).

The fibers isomorphic to \( E_2 \) of components isomorphic to \( W_o \) or \( T_2 \) (which include all double curves) are called \( E_2 \)-curves. The \( E_2 \)-curves which are double curves are called special \( E_2 \)-curves, while the others are called simple \( E_2 \)-curves.

---

[14] is concerned with rational curves on K3 surfaces, so the blown up fibers are specifically nodal, which seems a disjoint situation from the one considered here. However, quite a few analogies hold. For instance, the twisted ruled surface of [14, \( \S \S 3.1 \)] is the Atiyah ruled surface of a nodal curve in our language (\( \S \S 4.3 \)).
Less opaquey, condition 3(b) says the following: if \( v = w_o \), then \( Z_\pi \) is a fiber of \( \pi_1 \), if \( v \) corresponds to a \( T_2 \), then \( Z_\pi \) is an elliptic fiber and if \( v \) corresponds to an \( R_2 \), then \( Z_\pi \) is the distinguished section of the Atiyah surface.

**Definition 5.6.** An arboreal expansion is a commutative diagram

\[
\begin{array}{ccc}
Y & \longrightarrow & W \\
W \times_B B' & \downarrow & \downarrow \\
B' & \longrightarrow & B
\end{array}
\]

(abridged as the map \( Y \to W \times_B B' \) from now on) such that:

1. The map \( \mu : (B', o') \to (B, o) \) is a base change map.
2. The central fiber \( Y_{o'} \to W_o = E_1 \times E_2 \) of the map \( Y \to W \times_B B' \) in the diagram above is a drawer (cf. Definition 5.5).
3. If \( \text{Exc} \) is the union of all irreducible components of \( Y_{o'} \) with the exception of the distinguished one, then the map \( Y \to W \times_B B' \) restricts to an isomorphism, as in the diagram below

\[
\begin{array}{ccc}
Y & \longrightarrow & W \times_B B' \\
Y \setminus \text{Exc} & \longrightarrow & W \times_B B' \setminus \bigcup_{i=1}^{k_0} F_i
\end{array}
\]

We say that the expansion is smooth if the total space \( Y \) is smooth.

Informally, the property that an arboreal expansion is smooth means that the singularities of the drawer in the central fiber are already smoothed out at first order. Although this property is quite convenient to have, of course, one shouldn’t impose it if one desires a flexible “modular” apparatus. However, for the purposes of this paper it turns out we are free to commit the sin of imposing smoothness.

**Lemma 5.7.** If \( Y \to W \times_B B' \) is a smooth arboreal expansion and \( Z \subset Y_{o'} \) is an \( E_2 \)-curve of its central fiber, then \( N_{Z/Y} \) is either the trivial rank two bundle over \( Z \), or the rank two Atiyah bundle over \( Z \). Moreover, if \( Z \) is a special \( E_2 \)-curve, then \( N_{Z/Y} \) is necessarily trivial.

**Proof.** Let \( \Sigma \) be an irreducible component of \( Y_{o'} \) isomorphic to either \( W_0 \) or \( T_2 \) which contains \( Z \). Consider the short exact sequence for the normal bundles of the chain of inclusions \( Z \subset \Sigma \subset Y \),

\[
0 \longrightarrow N_{Z/\Sigma} \longrightarrow N_{Z/Y} \longrightarrow N_{\Sigma/Y}|_Z \longrightarrow 0.
\]

We claim that the two line bundles in the extension above are trivial. It is clear that \( N_{Z/\Sigma} \cong \mathcal{O}_Z \) since \( Z \) is a fiber of the map \( Z \to E_1 \) or \( \mathbb{P}^1 \). On the other hand,

\[
N_{\Sigma/Y} = \mathcal{O}_Y(\Sigma)|_\Sigma \cong \mathcal{O}_Y(-\Sigma^c)|_\Sigma = \mathcal{O}_\Sigma(-\Sigma \cap \Sigma^c),
\]

where \( \Sigma^c \) is the closure of the complement of \( \Sigma \) in \( Y_{o'} \), hence \( N_{\Sigma/Y}|_Z = \mathcal{O}_\Sigma(-\Sigma \cap \Sigma^c)|_Z \), which is clearly isomorphic to \( \mathcal{O}_Z \), since \( Z \) is an \( E_2 \)-curve and \( \Sigma \cap \Sigma^c \) is a union of \( E_3 \)-curves in \( \Sigma \). Note that the isomorphism \( \mathcal{O}_Y(\Sigma)|_\Sigma \cong \mathcal{O}_Y(-\Sigma^c)|_\Sigma \) used above follows from the fact that \( \mathcal{O}_Y(\Sigma) \otimes \mathcal{O}_Y(\Sigma^c) = \mathcal{O}_Y(Y_{o'}) \) is the pullback of \( \mathcal{O}_{B'}(o') \), which is certainly locally trivial.
For the second part of the lemma, simply note that if \( Z = \Sigma_1 \cap \Sigma_2 \), where \( \Sigma_1 \) and \( \Sigma_2 \) are two irreducible components of \( Y_{\sigma'} \), then we have an isomorphism \( \mathcal{N}_{Z/Y} \cong \mathcal{N}_{Z/\Sigma_1} \oplus \mathcal{N}_{Z/\Sigma_2} \cong \mathcal{O}_Z^{\oplus 2} \) by condition 3(b) in Definition 5.5.

Let \( D \subset W \) be a divisor such that \( \{D_t\}_{t \in B} \) is a family of divisors in \( \{\{L_t\}\}_{t \in B'} \), flat over \( B \). Let \( a_1 \geq a_2 \geq \ldots \geq a_{k_0} \geq 2 \) such that
\[
D_0 = S + a_1F_1 + a_2F_2 + \ldots + a_{k_0}F_{k_0} + F_{k_0+1} + \ldots + F_k,
\]
so that \( a_1 + a_2 + \ldots + a_{k_0} + k - k_0 = d \), \( S \) is a fiber of \( \pi_2 \) and all \( F_i \) are fibers of \( \pi_1 \).
Let \( q = \pi_2(S) \in E_2 \) and \( q' \neq q \) another closed point on \( E_2 \).

**Lemma 5.8.** Let \( Y \to W \times_B B' \) be an arboreal expansion and let \( D' \) be the closure of \( D' \cap D_0 \times_{B \times B'} B' \setminus B_{\sigma'} \) inside \( Y \). Assume that \( D' \) is a Cartier divisor. Then
\[
\mathcal{O}_Y(D') \otimes \mathcal{O}_Z \cong \mathcal{O}_Z(q_Z),
\]
for any \( E_2 \)-curve \( Z \subset Y_{\sigma'} \), where \( q_Z \in Z \) is the point corresponding to \( q \) under the natural isomorphism \( Z \cong E_2 \).

**Proof.** The set of \( E_2 \)-curves is naturally in bijection with the set of (closed) points of a curve \( E_1 \), which is a semistable model of \( E \). Consider the morphism \( \tilde{E}_1 \to \text{Pic}(E_2) \) given by \( z \mapsto \mathcal{O}_Y(D') \otimes \mathcal{O}_Z \) on closed points, where \( Z \) is the \( E_2 \)-curve corresponding to the point \( z \). Since Pic(\( E_2 \)) doesn’t contain any rational curves, we conclude that the map is constant and the lemma follows because if \( Z \) is a general fiber of \( \pi_1 \), then \( Z \cap D' = \{q_Z\} \).

Lemma 5.8 gives a clear picture of what the central fiber of \( D' \) can look like, as we explain below. Let \( \Sigma \) be an irreducible component of \( Y_{\sigma'} \) and let \( L_\Sigma = \mathcal{O}_Y(D') \otimes \mathcal{O}_Z \cong \mathcal{O}_Z(D' \cap \Sigma) \). If \( \Sigma \cong E_1 \times E_2 \), then \( L_\Sigma = L' \boxtimes \mathcal{O}_{E_2}(q) \) for some \( L' \in \text{Pic}(E_1) \) and \( |L_\Sigma| \) consists of divisors which are the sum of \( S \) with a divisor which is a pullback of a divisor from \( |L'| \). An analogous picture holds when \( \Sigma \cong T_2 \); \( L_\Sigma = L' \boxtimes \mathcal{O}_{E_2}(q) \) for some \( L' \in \text{Pic}(\mathbb{P}^1) \) and \( |L_\Sigma| \) consists of divisors which are the sum of the rational fiber “sitting above” \( q \) with a divisor which is a pullback of a divisor from \( |L'| \). The only nontrivial case is \( \Sigma \cong R_2 \). Then \( L_\Sigma \) restricts to \( \mathcal{O}_{E_2}(q) \) on the distinguished section \( E_{2}^{\infty} \) and all divisors in \( |L_\Sigma| \) are of the form \( C + mE_{2}^{\infty} \) with \( C \) reduced and irreducible by [28, Proposition 2.2] and also implicitly in Proposition 2.3.

The rest of this subsection is devoted to the proof of Proposition 5.9 below. As a general preliminary premise, note that Remark 5.1 implies that \( D \) is reduced. This is the only part of the argument which fails if, for instance, all abelian surfaces in the family \( W \to B \) are split.

**Proposition 5.9.** With the same notation as in Definition 5.6, there exists a smooth arboreal expansion such that if \( D' \) is the closure of \( D' \cap D_0 \times_{B \times B'} B' \setminus B_{\sigma'} \) in \( Y \), then
\[
(1) \quad D'_{\sigma'} \text{ is reduced};
(2) \quad D'_{\sigma'} \text{ doesn’t contain any special } E_2 \text{-curves};
(3) \quad \text{any irreducible component of } D'_{\sigma'} \text{ is one of the following:}
\]
Type (a) an \( E_2 \)-curve,
Type (b) the copy of \( S \) on \( W' = W \times_B B' \),
Type (c) the rational fiber sitting above \( q \) of some component of \( Y_{\sigma'} \) isomorphic to \( T_2 \).
Type (d) a curve in a component of $Y_{o'}$ isomorphic to $R_2$ which intersects the distinguished section at only one point and transversally.

Moreover, each irreducible component of $Y_{o'}$ contains precisely one component of $D'_{o'}$ of one of the types (b) – (d) above.

---

**Proof.** The proof consists of constructing the desired reduction iteratively, as a sequence of simple blow-ups and base changes. The whole procedure will be called a process and consists of the data contained in the diagram below

\[
\begin{array}{cccccc}
D & D[0] & D[1] & D[N - 1] & D[N] \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
W & Y[0] & Y[1] & \cdots & Y[N - 1] & Y[N] \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
B & B[0] & B[1] & \cdots & B[N - 1] & B[N] \\
\end{array}
\]

such that for each $n$, the diagram corresponding to the map $Y[n] \to W \times_B B[n]$ is an arboreal expansion and one the following is true of the diagram

\[
\begin{array}{c}
D[n + 1] \hookrightarrow Y[n + 1] \to B[n + 1] \\
\downarrow \\
D[n] \hookrightarrow Y[n] \to B[n]
\end{array}
\]

(A) The map $(B[n + 1], o[n + 1]) \to (B[n], o[n])$ is an isomorphism, the map $Y[n + 1] \to Y[n]$ is the blowup of a special $E_2$-curve not contained in $D[n]$ and $D[n + 1] = D[n] \times_{Y[n]} Y[n + 1]$.

(B) The map $(B[n + 1], o[n + 1]) \to (B[n], o[n])$ is an isomorphism, the map $Y[n + 1] \to Y[n]$ is the blowup of a simple $E_2$-curve contained in $D[n]$ and $D[n + 1]$ is the proper transform of $D[n]$.

(C) The map $(B[n + 1], o[n + 1]) \to (B[n], o[n])$ is a base change map, the BY square is cartesian, and, moreover, $D[n + 1] = D[n] \times_{B[n]} B[n + 1]$.  

---

**Fig. A.** An example of a drawer and a desirable $D'_{o'}$ — assuming everything is reduced.
The basic guiding principle is the following slogan:

(P0) The divisor \( D[n] \) doesn’t contain any special \( E_2 \)-curve of \( Y[n]_{o[n]} \).

A “slogan” will be a general property which will be verified inductively (sometimes implicitly) and may apply either to all steps in the process, or only to some, as the context will make clear. More slogans to follow. The process will consist of several eons:

\[
\begin{array}{cccc}
\text{CBB...BB} & \text{CAA...AABB...BB} & \text{CAA...AABB...BB} & \text{...}\text{CAA...AABB...BB} \\
\text{1st con} & \text{2nd con} & \text{3rd con} & \text{\ldots} \text{nth con}
\end{array}
\]

Let \( 0 = n_0 < n_1 < n_2 < \ldots < n_{\ell} = N \) be the eon cutoffs. We define the fat in a Weil divisor to be the sum of one less the multiplicities of its irreducible components – so a divisor is reduced if and only if it has 0 fat. Roughly, our goal is to construct the divisor to be the sum of one less the multiplicities of its irreducible components – so let \( q = \frac{1}{n} \) be the point on \( Z \) such that

\[ Z \sim O_{q^2, Y[n_k]} \equiv C[w_1, w_2, t] \]

such that \( t \) is the pullback of a coordinate at \( o[n_k] \) on \( B[n_k] \), that is,

\[
\begin{array}{ccc}
\hat{O}_{q^2, Y[n_k]} & \longrightarrow & \hat{O}_{q^2, Y[n_k]} \\
\bigl| & & \bigl| \\
C[t] & \longrightarrow & C[w_1, w_2, t]
\end{array}
\]

is a commutative diagram and, inside the central fiber, \( w_1 \) and \( w_2 \) are pullbacks of formal coordinates on \( C_1 \) and \( E_2 \) respectively – formally, the induced isomorphism \( \hat{O}_{q^2, \Sigma} \equiv C[[w_1, w_2]] \) fits in a commutative diagram of isomorphisms

\[
\begin{array}{ccc}
\hat{O}_{q^2, \Sigma} & \longrightarrow & \hat{O}_{\kappa, C_1} \otimes_C \hat{O}_{q^2, E_2} \\
\bigl| & & \bigl| \\
C[[w_1, w_2]] & \longrightarrow & C[[w_1]] \otimes_C C[[w_2]]
\end{array}
\]

in which the isomorphism on the right is the (completed, see [27, Definition 79.4.6]) tensor product of the isomorphisms \( \hat{O}_{\kappa, C_1} \equiv C[[w_1]] \) and \( \hat{O}_{q^2, E_2} \equiv C[[w_2]] \).

\(^2\)Complex analytic coordinates can be used equally well in this proof. In particular, Lemma 5.4 can still be applied as it is.
1. The initial C-move. Choose an element $F$ of $\hat{O}_{q_Z,Y[n_k]}$ which “cuts out” $D[n_k]$ formally locally at $q_Z$. Since $D[n_k]_{o[n]}$ is given by $w_1^m = 0$ locally at $q_Z$ inside $\Sigma$, it is straightforward to see that $F$ contains some monomial divisible by neither $t$ nor $w_2$. Then, by the formal Weierstrass Preparation Theorem cf. [8, Ch. 7, §3, no. 9, Prop. 6], there exists a unit $u \in \hat{O}_{q_Z,Y[n_k]}$ such that $F = up$ for some Weierstrass polynomial

$$p(t, w_1, w_2) = w_1^m + \sum_{j=1}^{m} \alpha_j(t, w_2)w_1^{m-j}$$

with $\alpha_j(t, w_2) \in (t, w_2)$, the maximal ideal of $\mathbb{C}[w_2, t]$. However, plugging in $t = 0$ and recalling that the (formally) local equation of $D[n_k]_{o[n_k]}$ inside $\Sigma$ is $w_1^m = 0$, we conclude a posteriori that actually $\alpha_j(t, w_2) \in (t)$. In other words, using the notation in §§5.2, we have $p \in W_m(\mathbb{C}[w_2])$. The variable $t$ has the same meaning as in §§5.2, while $w_1$ corresponds to $x$ and $w_2$ is simply part of the ring. Since $u$ is a unit, $D[n_k]$ is also locally cut out by $p$, not just by $F$.

Since $D[n_k]$ is reduced, $p$ is not a perfect $m$th power. Apply Lemma 5.4. Let $\mu_k$ be the natural number given by the lemma and consider the sequence of formal power series $p_{s_k}, q_{s_k}, p_{s_k+1}, q_{s_k+1}, ..., p_N, q_N$ defined by

$$p_{s_k}(t, w_1, w_2) = p(t^{\mu_k}, w_1, w_2)$$
$$q_{s_k}(t, w_1, w_2) = t^{-m}p_{s_k}(t, tw_1, w_2)$$
$$p_n(t, w_1, w_2) = q_{s_k-1}(t, w_1 + \psi_n(w_2), w_2)$$

for some $\psi_n \in \mathbb{C}[w_2]$ such that $p_n \in W_m(\mathbb{C}[w_2])$ for all $n \leq N$, yet $q_n(t, w_1 + \psi_N(w_2), w_2) \notin W_m(\mathbb{C}[w_2])$ for all $\psi_N \in \mathbb{C}[w_2]$. The choice of the initial index $s_k$ is purely a matter of notation, which will be clarified later. Choose $(B[n_k+1], o(n_k+1)) \to (B[n_k], o(n_k))$ a base change map of local degree $\mu_k$, which determines the initial (type C) move of the $n_k \to n_{k+1}$ con.

It remains to deal with the $A$- and $B$-sequences. The purpose of the $A$-sequence is to resolve the singularities of the total space created by the base change (this step is somewhat optional and also very standard), while the purpose of the $B$-sequence is to decrease the “total fat” in $D$ without violating (P0) – this is only possible thanks to the choice of $\mu_k$ above.

2. The $A$-sequence. This amounts to little more than the resolution algorithm of $A_n$ Du Val singularities, so the details will be left to the reader. For a more detailed account of arguments of this type, we refer the reader to [17, §§2.c)]. The inductive slogan is the following:

(P2) If $Z$ is a special $E_2$-curve, there exists a positive integer $\mu$ such that for any $r \in Z(\mathbb{C})$, we have

$$\hat{O}_{r,Y[n]} \cong \mathbb{C}[z_1, z_2, z_3, t]/(z_1z_2 - t^\mu)$$

as adic rings, where $t$ is the pullback of a formal coordinate at $o[n]$ on $B[n]$ and the two components of $Y[n]o[n]$ intersecting along $Z$ are given by $z_1 = 0$ and $z_2 = 0$.

(P2) holds for $n = n_k + 1$ with $\mu = \mu_k$ because it holds for $n = n_k$ with $\mu = 1$. If $Y[n + 1]$ is obtained by blowing up the special $E_2$-curve $Z \subset Y[n]$ with $\mu \geq 3$, then the exceptional divisor of the blowup consists of two irreducible components isomorphic to $T_2$. Then $Y[n + 1]$ will be smooth along the two new “lateral” special
$E_2$-curves, so it will satisfy the required conditions with $\mu = 1$. The new "central" special $E_2$-curve will satisfy the requirements above with $\mu$ replaced by $\mu - 2$ by a trivial local calculation. Finally, in the case $\mu = 2$, the blowup has only one irreducible component, which again is isomorphic to $T_2$. The details are similar to those of the previous case. In conclusion, at the end of the $A$-sequence, (P2) will hold with $\mu = 1$ at all special $E_2$-curves.

Note that the only new irreducible components acquired by $D$ are reduced rational fibers of $T_2$ components, so the amount of fat in $D$ stays constant throughout this process.

3. The $B$-sequence. Let $s_k$ be the value of the running index after the $A$-sequence. Of course, $Y[s_k] \to Y[n_k]$ is an isomorphism locally at the preimage of $q'_Z$ in $Y[s_k]$, even in the Zariski topology. The algorithm is simply to keep blowing up the $E_2$-curve $Z[s_k] \cong Z$ of multiplicity $m$ until we see something different. The key point is that the choice of $\mu_k$ will ensure that (P0) will hold until the end.

At step $n$, we blow up $Z[n] \subset Y[n]$ and obtain $Y[n+1]$ with exceptional divisor $\Sigma[n+1]$. Let $q'_Z[n]$ be the point mapping to $q'$ under the isomorphism $Z[n] \cong E_2$. Recall that $D[n+1]$ is the proper transform of $D[n]$ under the blowup. Let $C[n+1]$ be the intersection of $\Sigma[n+1]$ with the closure of its complement inside $Y[n+1]$.

(P3) Formally locally at $q'_Z[n]$, in the coordinates $(t, w_1[n], w_2)$ described below, $D[n]$ is cut out by $p_n(t, w_1[n], w_2)$ and $Z[n]$ is defined by $w_1[n] = 0$.

Note that (P3) holds for $n = s_k$ with the coordinates $t, w_1[s_k] = w_1, w_2$ introduced earlier in the proof. The coordinates $t$ and $w_2$ will be the pullbacks of the earlier $t$ and $w_2$, while $w_1[n]$ will be a new coordinate.

First, we check that $D[n+1]$ doesn’t contain $C[n+1]$, thus justifying that (P0) continues to hold. Locally at the point on $C[n+1]$ corresponding to $q'$ under the isomorphism $C[n+1] \cong E_2$, the equation of the total transform of $D[n]$ is

$$t^m + \sum_{j=1}^{m} \alpha_j(t, w_2)t^{m-j}x_1^j = 0$$

where $x_1 = t/w_1[n]$. The local equation of the proper transform is obtained by dividing (13) by the highest power of $t$ by which the right hand side is divisible. However, the assumption that $q_n$ is integral (i.e. a formal power series rather than a Laurent series) implies that this highest power is $t^m$. Hence $D[n+1]$ doesn’t contain $C[n+1]$.

We analyze two situations. Assume first that $\Sigma[n+1] \cong T_2$ and that $\Sigma[n+1] \cap D[n+1]$ consists set-theoretically only of a rational fiber (which is necessarily the fiber $F$ over $q$ of $\Sigma[n+1] \to E_2$) and an $E_2$-curve, which then ought to be denoted by $Z[n+1]$. The fat in $D[n]$ remains unchanged because

$$O_{Y[n+1]}(D[n+1]|_{\Sigma[n+1]} \cong O_{\Sigma[n+1]}(F + mZ[n+1]).$$

We have $F \cap Z[n+1] = \{q'_Z[n+1]\}$. We may think of $w_1[n]' = w_1[n]/t$ as an affine coordinate rather than a formal coordinate. If $a \in \mathbb{C}$ is the coordinate of $q'_Z[n+1]$, then $Z[n+1]$ is given formally locally at $q'_Z[n+1]$ by the equation $w_1[n]' = \psi(w_2)$ for some $\psi \in a + (w_2) \subset \mathbb{C}[w_2]$. Then $\psi = \psi_n$ a fortiori (12). Introduce the formal coordinate $w_1[n+1] = w_1[n]'-\psi_n(w_2)$. Then, locally at $q'_Z[n+1]$, $Z[n+1]$ is given by $w_1[n+1] = 0$ and $D[n+1]$ is given by $p_{n+1}(t, w_1[n+1], w_2) = 0$, thus justifying (P2) inductively.
If the assumption in the paragraph above does not hold, i.e. if either $\Sigma[n + 1] \cong R_2$, or $\Sigma[n + 1] \cong T_2$ but $D[n + 1]$ contains more than one new $E_2$-curve, then the remarks after Lemma 5.8 show that $D[n + 1]$ has strictly less fat than $D[n]$. If that is the case, we declare this step the end of the con and note that the relevant conditions which held at the beginning of the con also hold at its end. Of course, the point is that, by Lemma 5.4, this will eventually happen.

**Corollary 5.10.** Let $f : C \to W_0$ be a stable map to $W_0$ with $f_*[C] \in \{ \mathcal{L}_a \}$.

Assume that $f$ deforms to nearby fibers of $W \to B$. Then there exists a drawer $\zeta : Z \to W_0 = E_1 \times E_2$ and a map $\hat{f} : \hat{C} \to Z$ from a nodal curve $\hat{C}$ such that

1. the composition $\zeta \circ \hat{f} : \hat{C} \to W_0$ stabilizes $[3]$ to $f$,
2. any irreducible component of $\hat{C}$ satisfies one of the following relative to $\hat{f}$:
   - Type (a) it is mapped isomorphically onto an $E_2$-curve,
   - Type (b) it is mapped isomorphically onto a fiber of $\pi_1$ of the distinguished component,
   - Type (c) it is mapped isomorphically onto a rational fiber of some component of $Y_\varphi$ isomorphic to $T_2$,
   - Type (d) it is mapped birationally onto a curve in a component of $Y_\varphi$ isomorphic to $R_2$ which intersects the distinguished section at only one point and transversally,
   - Type (e) it is a contracted component.

Moreover, for each irreducible component of $Y_\varphi$ there exists precisely one irreducible component of one of the types (b) – (d) above mapping into the chosen component of $Y_\varphi$.

**Proof.** The assumption implies that, after a base change $(B', o') \to (B, o)$, we can find a (flat) family of stable maps $F' : C' \to W'$ over $B'$, where $W' = W \times_B B'$ such that $C_{o'} = C$ and $F'_{o'} = f$. Let $D' \subset W'$ be the image of $F$, which forms a flat family of divisors in the fibers of $W' \to B$. Apply Proposition 5.9 to this family of divisors and let $Y \to W''$, $W'' = W' \times_B B''$ be the resulting arboreal expansion, where $(B'', o'') \to (B', o')$ is the suitable base change. Let $D'' \subset Y$ be the family of divisors produced by Proposition 5.9.

If $C'' = C' \times_B B''$ and $F'' : C'' \to W''$ is the pullback of the family of stable maps, then we can trivially think of the restriction of $F''$ to $B'' \setminus \{ o'' \}$ as mapping into $Y$. Applying nodal reduction, after yet one more base change $(B'', o'') \to (B'', o'')$, we may extend the latter family to a flat family of maps $F''' : C''' \to Y'''$ with semistable (connected and at at worst nodal) sources, where $Y''' = Y \times_B B'''$. Choose $Z = Y'''_{o'''}, Y' = \hat{C} = C_{o'''},$ and $\hat{f} = F'''_{o'''}. Then requirement (1) in the statement of the proposition follows from the basic fact that moduli spaces of stable maps are separated – specifically, that $\mathcal{M}, (W'''' / B'''', +)$ is separated over $B'''$, where of course $W''' = W'' \times_B B'''$. Requirement (2) follows from requirement (3) from Proposition 5.9.

---

### 6. The degeneration of the moduli spaces

#### 6.1. Compatibility in the chosen family.

Before analyzing the degeneration of the family of “Severi varieties,” we need to clarify the compatibility property (Definition 2.4). We return to the setup of §§5.1. Committing the same abuse of notation as in §2, let $R_i \pi_* Z_W$ be the locally constant sheaf dual to $R^i \pi_* Z_W$. On a
technical note, Poincaré duality – which was heavily relied on in §2 – continues to hold with these definitions because \( R^1\pi_*\mathbb{Z}_W \cong \mathbb{Z}_B \). Indeed, the cup product map
\[
R^1\pi_*\mathbb{Z}_W \otimes R^{q-i}\pi_*\mathbb{Z}_W \rightarrow R^q\pi_*\mathbb{Z}_W \cong \mathbb{Z}_B
\]
induces an isomorphism \( R^1\pi_*\mathbb{Z}_W \rightarrow \text{Hom}_{\mathbb{Z}_B} (R^{q-i}\pi_*\mathbb{Z}_W, \mathbb{Z}_B) = R^{q-i}\pi_*\mathbb{Z}_W \) which restricts to Poincaré duality on each fiber. In other words, we needn’t worry about “twisting on the base,” which would have complicated notation.

Let \( H_i = H_i(W_0, \mathbb{Z}) \) and \( H^i = H^i(W_0, \mathbb{Z}) \) be the stalks at \( o \) of the local systems above. Then \( H_1 = H_1(E_1, \mathbb{Z}) \oplus H_1(E_2, \mathbb{Z}) \) and \( c_1(\mathcal{L}_o) \), seen as an integral alternating form on \( H_1 \), is given in block form by
\[
\text{Im } H_{c_1(\mathcal{L}_o)} = \begin{bmatrix} d(\cdot, \cdot)_{H_1(E_1)} & 0 \\ 0 & d(\cdot, \cdot)_{H_2(E_1)} \end{bmatrix},
\]
where \( (\cdot, \cdot)_{H_i(E_i)} \) is the topological intersection form on \( E_i \). This follows from the obvious formula \( c_1(\mathcal{L}_o) = \pi_1^1 c_1(J_1) + \pi_2^2 c_1(J_2) \).

Let \( \Lambda \) be a rank four locally free subsheaf of \( R^1\pi_*\mathbb{Z}_W \). Denote the stalk of \( \Lambda \) at \( t \in B \) by \( \Lambda_t \). We wish to know under what circumstances the sublattices \( \Lambda_t \) are compatible with the polarization if and only if \( \Lambda \) is compatible with the polarization. Thus \( \Lambda \) is compatible with the polarization if and only \( \Lambda_o \) is compatible with \( c_1(\mathcal{L}_o) \).

**Lemma 6.1.** The “subsystem” \( \Lambda \) is compatible with the polarization if and only if there exists a rank two sublattice \( \Lambda_o^3 \subseteq H_1(E_2, \mathbb{Z}) \) such that \( [H_1(E_2, \mathbb{Z}) : \Lambda_o^3] \) divides \( \Lambda \) and \( \Lambda_o = H_1(E_1, \mathbb{Z}) \oplus \Lambda_o^3 \) as sublattices of \( H_1 \).

**Proof.** We only prove the “only if” part since the “if” part is essentially trivial.

By Lemma 2.4, there exists a copolarized isogeny \( \sigma_o : (S_o', \mathcal{L}_o') \rightarrow (W_o, \mathcal{L}_o) \). Let \( D \subset S_o' \) be the preimage under \( \sigma_o \) of a fiber of \( \pi_1 : W_0 \rightarrow E_1 \). By the push-pull formula, \( (\mathcal{L}_o', D)_{\sigma'} = 1 \) and in particular \( D \) is reduced and irreducible. It is clear that \( (D^2)_{\sigma'} = 0 \). By [4, Lemma 4.6, §10], \( (S_o', \mathcal{L}_o') \) is a polarized product of elliptic curves \( E_1' \times E_2' \). Since we’re assuming that \( E_1' \) and \( E_2' \) are non-isogeneous, \( \sigma_o \) has to split as well \( \sigma_o = \sigma_o^1 \times \sigma_o^2 \), and the conclusion follows since \( \sigma_o^1 \) clearly has to be an isomorphism.

6.2. Notation and the semi-regularity map. Everything in this subsection is independent of the (OS)/(BC) distinction. However, it is implied that all moduli spaces behave in the obvious way with respect to base change. By the remarks in §6.1, let \( \beta \) be the fiberwise Poincaré dual of \( c_1(\mathcal{L}) \) and let
\[
\varpi : \mathcal{M}_g(W/B, \beta) \rightarrow B
\]
be the family of moduli stacks of class \( \beta \) genus \( g \) stable maps with no marked points. By abuse of notation, the restrictions of \( \varpi \) to various open substacks will continue to be denoted by \( \varpi \). Consider the open substacks

1. \( \mathcal{Y}_g(W/B, \beta) \) parametrizing unramified maps with sources of compact type;
2. \( \mathcal{Y}_g(W^0/B^0, \beta) \) parametrizing unramified maps with smooth sources.

Note that the central fiber of \( \mathcal{Y}_g(W^0/B^0, \beta) \) is empty. However, by Remark 5.1, the fibers of \( \mathcal{Y}_g(W^0/B^0, \beta) \) and \( \mathcal{Y}_g(W/B, \beta) \) over the generic point of \( B \) actually coincide.
We isolate the relevance of the semiregularity map for us in the proposition stated after the next lemma. This statement is surely well-known to experts.

As a basic technical prerequisite, recall [27, Lemma 36.52.26] that given an unramified stable map \( f : C \to X \), where \( X \) is a smooth target, the conormal sheaf of \( f \), which we will denote by \( \mathcal{N}_f' \), is locally free. Moreover, \( \mathcal{N}_f' \) is isomorphic to the kernel of \( f^*\Omega_X \to \Omega_C \) and hence the first order deformations and obstructions spaces \( \mathbb{E}xt^i(f^*\Omega_X \to \Omega_C, \mathcal{O}_C) \) simplify to \( H^{i-1}(\mathcal{N}_f') \), \( i = 1, 2 \). Furthermore, the normal sheaves of the restrictions of \( f \) to irreducible components are elementary modifications of the restrictions of \( \mathcal{N}_f \) and, in the special case when \( X \) is a surface, they are simply negative twists by the nodes.

**Lemma 6.2.** For any stable map \( f : C \to W_i \) in \( \mathcal{V}_g^*(W/B, \beta) \), we have \( \mathcal{N}_f \cong \omega_C \).

**Proof.** Let \( C_i, 1 \leq i \leq k \) be the irreducible components of \( C \) and \( f_i \) the restriction of \( f \) to \( C_i \). Since \( C \) is of compact type, all components \( C_i \) are smooth. Moreover, the fact that the dual graph is a tree, hence bipartite, means that there exists a function \( \epsilon : \{1, 2, ..., k\} \to \{-1, 1\} \) such that \( \epsilon(i) \neq \epsilon(j) \) whenever \( C_i \cap C_j \neq \emptyset \). Analogously to (1), we have \( \mathcal{N}_{f_i} \cong \Omega_{C_i} \). Recall that for any \( p \in C_i(\mathbb{C}) \), the fiber of \( \Omega_{C_i}(p) \) at \( p \) is canonically identified with \( \mathbb{C} \) via the residue map. Let \( P_i = C_i \cap C_i^c \) be the divisor of nodes on \( C_i \). Then \( \mathcal{N}_{f_i} |_{C_i} \cong \mathcal{N}_{f_i}(P_i) \) and hence, for any open set \( U \subset C \) with \( U_i = U \cap C_i \), we have

\[
\mathcal{N}_f(U) = \left\{ s_i \in \prod_{i=1}^k \Omega_{C_i}(P_i)(U_i) : \text{res}_p s_i = \text{res}_p s_j \text{ if } C_i \cap C_j = \{p\} \subset U \right\}
\]

\[
\omega_C(U) = \left\{ s_i \in \prod_{i=1}^k \Omega_{C_i}(P_i)(U_i) : \text{res}_p s_i = -\text{res}_p s_j \text{ if } C_i \cap C_j = \{p\} \subset U \right\},
\]

so the map \( \mathcal{N}_f \to \omega_C \) defined by \( (s_i) \mapsto ((-1)^{\epsilon(i)} s_i) \) is an isomorphism. \( \square \)

**Proposition 6.3.** The map \( \mathcal{V}_g^*(W/B, \beta) \to B \) is smooth of relative dimension \( g \).

**Proof.** By the discussion preceding Lemma 6.2, \( \text{Def}^1(f) = H^0(\mathcal{N}_f) \) and \( \text{Obs}(f) = H^1(\mathcal{N}_f) \). However, \( \mathcal{N}_f \cong \omega_C \) by Lemma 6.2 and hence these spaces have dimension \( g \) and 1 respectively. Hence the local relative dimension cannot exceed \( g \) and smoothness holds if equality occurs.

By [20, Theorem 2.4 and Remark 3.1], \( \mathcal{M}_g(W/B, \beta) \to B \) admits a relative perfect obstruction theory \( E_{\text{rel}} \to \mathcal{M}_g(W/B, \beta) / B \) of relative virtual dimension \( g \). Therefore, any irreducible component of the stack \( \mathcal{M}_g(W/B, \beta) \) has dimension at least \( g + 1 \) and the conclusion follows. Alternatively, as in §§2.1, the restriction of \( E_{\text{rel}} \) to \( \mathcal{V}_g^*(W/B, \beta) \) is concentrated in degree 0 and locally free of rank \( g \) again by [7, Lemma 2, §1.4] and the conclusion follows from [2, Proposition 7.3]. \( \square \)

Let \( \Lambda \) be a local system compatible with the polarization \( \mathcal{L} \), cf. Definition 2.4 and its extension in families in §§6.1. Recall that \( \Lambda_0 = H_1(E_1, \mathbb{Z}) \oplus \Lambda_0^2 \) by Lemma 6.1. Proposition 2.3 allows us to introduce

\[
(3) \quad \mathcal{V}_g^*(W/B, \beta; \Lambda) \subseteq \mathcal{V}_g^*(W/B, \beta) \text{ with associated lattice } \Lambda;
\]

\[
(4) \quad \mathcal{V}_g^*(W^0/B^0, \beta; \Lambda) \subseteq \mathcal{V}_g^*(W^0/B^0, \beta) \text{ with associated lattice } \Lambda;
\]

\[
(5) \quad \mathcal{V}_g(W/B, \beta; \Lambda) \text{ is the closure of } \mathcal{V}_g(W^0/B^0, \beta; \Lambda) \text{ inside } \mathcal{M}_g(W/B, \beta).
\]

We return to the comment at the end of the (OS)/(BC) paragraph in §§5.1. In short, we need a base change to ensure we can choose the lattices \( \Lambda_i \) consistently for the purpose proving Theorem 2.6. Specifically, if \( \bar{\Lambda} \) is any full rank sublattice
of \( H_1(W_\omega, \mathbb{Z}) \) for some point \( \omega \in B \), then, after a base change, we can find a local system \( \Lambda \) such that \( \tilde{\Lambda} = \Lambda_\omega \). Obviously, this is due to the fact that there are finitely many full rank sublattices of any given index. We make this assumption.

6.3. Flexible and quasi-flexible limit components. In this section, we will start analyzing the degeneration of the family of Severi varieties. We start with an overview of how this analysis will be used to deduce Theorem 2.6.

**Definition 6.4.** Let \( \varepsilon \in \{0, 1\} \). If a substack \( \mathcal{Z} \subseteq \overline{\mathcal{M}}_{g}(W_\omega, \beta) \) has the property that the evaluation morphism

\[
ev : \overline{\mathcal{M}}_{g, g-\varepsilon}(W_\omega, \beta) \to W^{g-\varepsilon}_\omega
\]

is dominant on the preimage of \( \mathcal{Z} \) under \( \overline{\mathcal{M}}_{g, g-\varepsilon}(W_\omega, \beta) \to \overline{\mathcal{M}}_{g}(W_\omega, \beta) \), it is said to be **flexible** if \( \varepsilon = 0 \), respectively **quasi-flexible** if \( \varepsilon = 1 \) and it is not flexible.

We summarize our strategy to prove Theorem 2.6 in general terms, in the hope that the strategy might be applicable to other similar problems:

(i) Describe all the flexible components of the degenerate Severi variety (in general, this also entails computing enumerative invariants);
(ii) Describe a sufficiently large number of quasi-flexible components to “connect” all the flexible components;

Of course, all of this needs to be carried out in a sufficiently well behaved compactification (cf. the hidden (log) smoothness philosophy) so that arguments similar to those in the next section are applicable. Informally, it doesn’t seem to be necessary to ever “dig deeper” than quasi-flexible components.

For any partition \( p \) of length \( g - 1 \) and degree \( d' = d/[H_1(E_2, \mathbb{Z}) : \Lambda_0] \), we say that \( \Lambda_1 + \Lambda_2 + \ldots + \Lambda_{g-1} = \Lambda_0 \) with associated lattice \( \Lambda_i \); all fibers in (5) are different.

Of course, (6) is actually forced.

**Proof of the existence part of Theorem 2.6.** The “only if” part was proved in §§2.2. We prove the “if” part. Recall the comment at the end of §§6.2. Since

\[
d' = \frac{d}{[H_1(E_2, \mathbb{Z}) : \Lambda_0]} \geq g - 1,
\]

there exists a partition \( p \) of \( \Lambda_0^2 \) of degree \( d' \) and length \( g - 1 \) by a remark in §3. Let \( f : C \to W_\omega \) be a simple stable map of type \( p \) as above. Since \( C \) is of compact type, we have

\[
H_1(C, \mathbb{Z}) = \bigoplus_{k=0}^{g-1} H_1(C_k, \mathbb{Z})
\]

\[\text{In [29], it was sufficient to deal only with a few flexible components, so the situation for } \mathbb{P}^2 \text{ is misleadingly simple in this respect.}\]
and the image of \( f_* : H_1(C, \mathbb{Z}) \to H_1(W, \mathbb{Z}) \) is \( H_1(E_1, \mathbb{Z}) \oplus \sum_{k=2}^{g} \Lambda_k = H_1(E_1, \mathbb{Z}) \oplus \Lambda_0 \oplus \Lambda_2 = \Lambda_\sigma \). Then \([f] \) belongs to \( \mathcal{M}^g(W/B, \beta; \Lambda)(\mathbb{C}) \) and Proposition 6.3 proves that it deforms to nearby fibers, completing the proof since the fibers of \( \mathcal{M}^g(W/B, \beta; \Lambda) \) and \( \mathcal{M}^g(W^0/B^0, \beta; \Lambda) \) over the generic point of \( B \) coincide.

Let \( \mathcal{M}_g(W_o, \beta; \Lambda, p) \) be the open substack of \( \mathcal{M}_g(W_o, \beta) \) parametrizing simple stable maps into \( W_o \) of type \( p \). Trivially, it is irreducible. Also, it is clear that \( \mathcal{M}_g(W_o, \beta; \Lambda, p) \) is flexible and conversely, given the description of \( \{ \mathcal{L}_o \} \), any flexible irreducible component of the central fiber of \( \mathcal{M}_g(W/B, \beta; \Lambda) \) is topologically the closure of some \( \mathcal{M}_g(W_o, \beta; \Lambda, p) \). For the converse, we ought to invoke Proposition 2.3 to ensure that \( \Lambda_1 + \Lambda_2 + \ldots + \Lambda_g = \Lambda_0^2 \).

Let \( \mathcal{V} \) be an irreducible component of \( \mathcal{M}_g(W^0/B^0, \beta; \Lambda) \) and \( \mathcal{V} \) its closure with respect to \( \mathcal{M}_g(W/B, \beta) \). Let

\[
W_B^g = \bigwedge_{g \text{ copies of } W} W
\]

be the \( g \)-fold fiber power of \( W \) over the base \( B \). By Proposition 2.2, \( \mathcal{V} \) is flexible in the sense that the evaluation map

\[
ev : \mathcal{M}_{g,B}(W/B, \beta) \to W_B^g
\]

restricted to the preimage of \( \mathcal{V} \) under \( \mathcal{M}_{g,B}(W/B, \beta) \to \mathcal{M}_{g}(W/B, \beta) \) is dominant.

Lemma 6.5. There exists a degree \( d \) length \( g-1 \) partition \( p_0 \) such that \( \mathcal{V} \) contains \( \mathcal{M}_g(W_o, \beta; \Lambda, p_0) \) topologically.

Proof. By properness, \( \ev|_{\mathcal{V}} : \mathcal{V} \to W_B^g \) is surjective, hence \( \mathcal{V} \) contains some flexible irreducible component of \( \mathcal{M}_g(W_o, \beta; \Lambda, p_0) \) and the lemma follows from the remarks in the previous two paragraphs.

We introduce a class of quasi-flexible stable maps. Again, let \( d' = d/|H_1(E_2, \mathbb{Z}) : \Lambda_0^2| \) and \( p \) a partition of length \( g-1 \) and degree \( d' \) of \( \Lambda_0^2 \) denoted

\[
\Lambda_1 + \Lambda_2 + \ldots + \Lambda_{g-1} = \Lambda_0^2.
\]

Let \( p' \) be the length \( g-2 \) partition obtained by replacing \( \Lambda_i, \Lambda_j \) with \( \Lambda_i + \Lambda_j \) for some \( i, j \leq g-1 \). In the language of §3, we choose an arrow \( p \succ p' \). We say that a stable map \( f : C \to W_o \) is quasi-simple of type \( p' \) if the following hold:

1. the source is \( C = C_0 \cup \bigcup_{k \in [g-1]\setminus\{i,j\}} C_k \cup \tilde{C} \) such that:
2. all \( C_k \) are smooth of genus 1, while \( \tilde{C} \) is smooth of genus 2;
3. the dual graph is a star centered at \( C_0 \);
4. the restriction of \( f \) to \( C_0 \) is an isomorphism onto a fiber \( E_1 \times \{ [pt] \} \);
5. the restriction of \( f \) to \( C_k, k \neq 0, i, j \), is an isogeny onto a fiber \( \{ [pt] \} \times E_2 \) with associated lattice \( \Lambda_k \);
6. the restriction of \( f \) to \( \tilde{C} \) is a degree \( d = |H_1(E_2) : \Lambda_i| + |H_1(E_2) : \Lambda_j| \) map onto a fiber \( \{ [pt] \} \times E_2 \) such that the image of the induced map on homology

\[
H_1(C, \mathbb{Z}) \to H_1(E_2, \mathbb{Z})
\]

is \( \Lambda_i + \Lambda_j \);
7. all fibers in (5) and (6) are different.

Again, there is an open substack \( \mathcal{M}_g(W_o, \beta; \Lambda, p \succ p') \) of \( \mathcal{M}_g(W_o, \beta) \) parametrizing such stable maps. The problem is that now these stable maps vary with \( g+1 \) instead of \( g \) moduli. To fix this, we will see that we need to impose one more condition on the stable map. We say that the quasi-simple map above is quasi-splendidly (quasi-simple and quasi-traceless) if it additionally satisfies
(8) the pointed restriction \((\tilde{C}, r) \to E_2\) of \(f\) is quasi-traceless \((\S \S 4.2)\), where \(r\) denotes the node of \(C\) on \(\tilde{C}\).

Unlike (1)–(7), condition (8) is closed – at least if imposed last. Let

\[
\tilde{\cC}'_g(W_o, \beta; \Lambda, p \succ p')
\]

denote the locally closed substack of \(\overline{\cC}'_g(W_o, \beta)\) parametrizing qsqt stable maps of type \(p'\). The formal construction is given below. For this, we will need the spaces \(\cM_{2,n}^0(E_2, \tilde{d}, \Lambda_1 + \Lambda_j)\) and \(\cD_{2,n}^0(E_2, \tilde{d}, \Lambda_1 + \Lambda_j)\) cf. \(\S \S 4.1, \S \S 4.2\). Consider the map

\[
\Psi: \tilde{\cC}'_g(W_o, \beta; \Lambda, p \succ p') \to \cM_{2,1}^0(E_2, \tilde{d}, \Lambda_1 + \Lambda_j)
\]

which informally associates to a quasi-simple \(f\) the once-marked map \(\tau_2 \circ f|_C\), with the marking at the former node. Note that \(\Psi\) has irreducible fibers of the same dimension. The definition of \(\tilde{\cC}'_g(W_o, \beta; \Lambda, p \succ p')\) amounts to requiring that

\[
\tilde{\cC}'_g(W_o, \beta; \Lambda, p \succ p') \xhookrightarrow{} \tilde{\cC}'_g(W_o, \beta; \Lambda, p \succ p') \xrightarrow{\Psi} \cD_{2,1}^0(E_2, \tilde{d}, \Lambda_1 + \Lambda_j)
\]

is cartesian. Finally, note that

\[
\tilde{\cC}'_g(W_o, \beta; \Lambda, p_1 \succ p_3) = \tilde{\cC}'_g(W_o, \beta; \Lambda, p_2 \succ p_3)
\]

for any roof \(p_1 \succ p_3 \prec p_2\) \((\S 3)\). In other words, \(\tilde{\cC}'_g(W_o, \beta; \Lambda, p \succ p')\) depends on \(p\) only to the extent to which one needs to remember the distinguished term of \(p'\). It is time to clarify the quasi-traceless condition.

**Proposition 6.6.** If a quasi-simple map belongs to \(\overline{\cC}'_g(W/B, \beta; \Lambda)\), then it is qsqt.

**Proof.** Let \(f\) be a stable map which belongs to \(\overline{\cC}'_g(W/B, \beta; \Lambda)\). Apply Corollary 5.10 to \(f\) and let \(\hat{f}: \tilde{C} \to Z\) be the map to a drawer \(Z\) which stabilizes to \(f\) as a map into \(W_o\). Let \(D\) be the irreducible component of \(\tilde{C}\) which corresponds to the special irreducible component of the source of \(f\). We argue by elimination that it is a type (d) component, cf. requirement (2) in the statement of Corollary 5.10. It is not of type (e) because \(\tilde{d} \neq 0\). It is not of types (a), (b) or (c) because it has genus 2. Hence it must be of type (d). Then, the analysis carried out in \(\S \S 5.3\) shows that in fact all the requirements of Lemma 4.8 are satisfied, and the conclusion follows by applying the lemma.

Finally, we state a lemma which can be regarded as a special case of Proposition 6.6 when \(\tilde{C}\) becomes reducible. We introduce one more class of stable maps. The data is the same as for simple maps: a partition \(p\) of length \(g-1\) and degree \(d'\) of \(\Lambda_0^2\), \(\Lambda_1 + \Lambda_2 + \ldots + \Lambda_{g-1} = \Lambda_0\). We say that a stable map \(f: C \to W_o\) is pseudo-simple of type \(p\) if the following conditions are satisfied

1. the source is \(C = C_0 \cup \bigcup_{k \in [g-1]} C_k\) such that:
2. all \(C_k\) are smooth curves of genus 1;
3. the dual graph is a tree such that all components are incident to \(C_0\), except \(C_1\), which is incident only to \(C_i\);
4. the restriction of \(f\) to \(C_0\) is an isomorphism onto a fiber \(E_1 \times \{[pt]\}\);
5. the restriction of \(f\) to \(C_k\), \(k \neq 0\), is an isogeny onto a fiber \(\{[pt]\} \times E_2\) with associated lattice \(\Lambda_i\);
all fibers in (5) are different except the $i$th and $j$th fibers.

**Lemma 6.7.** No pseudo-simple stable map belongs to $\overline{\mathcal{T}}_g(W/B,\beta;\Lambda)$.

**Proof.** The proof is very similar to the proof of Proposition 6.6. Assume by way of contradiction that there exists a pseudo-simple stable map $f$ which belongs to $\overline{\mathcal{T}}_g(W/B,\beta;\Lambda)$. Again, apply Corollary 5.10 to $f$ and let $\tilde{f} : \tilde{C} \to Z$ be the map to a drawer $Z$ which stabilizes to $f$ as a map into $W_\beta$. Note that, since $\tilde{f}$ stabilizes to a stable map with source of compact type, its own source must be of compact type as well, i.e. its dual graph $\tilde{G}$ needs to be a tree. We think of $\tilde{G}$ as a rooted tree, with its root at the vertex corresponding to the irreducible component of the source of $\tilde{f}$ which corresponds to $C_0$.

Let $D_i$ and $D_j$ the irreducible component of $\tilde{C}$ which corresponds to the irreducible component of the source of $f$ denoted by $C_i$ and $C_j$ in the definition of pseudo-simple given above. They are not of type (e) because the corresponding components of the source of $f$ are not contracted by $f$; they are not of type (c) since they’re not rational and they’re visibly not of type (b). Hence they’re either of type (a), or of type (c). Let $v_i$ and $v_j$ be the vertices of $\tilde{G}$ corresponding to $D_i$ and $D_j$. Then the key combinatorial observation is that $v_i$ and $v_j$ cannot be direct ancestors/descendants of each other in $\tilde{G}$ – note that this also requires Lemma 4.7.

This contradicts the fact that $\tilde{f}$ stabilizes to $f$. □

### 6.4. Connecting flexible components via quasi-flexible ones

Before stating the key ingredient of our proof, Proposition 6.10, we introduce one last piece of jargon which will be very convenient to use in the proof of the proposition.

**Definition 6.8.** We say that a stable map $f : C \to X$ “has rational contraction” if $R^1f_*\mathcal{O}_C = 0$.

Having rational contraction means that the contracted locus is a disjoint union of rational tress. Indeed, it is a straightforward exercise that

$$R^1f_*\mathcal{O}_C = \bigoplus_{x \in X} H^1(f^{-1}(x),\mathcal{O}_{f^{-1}(x)}) \otimes \mathcal{O}_X/I_x$$

and it is clear that $H^1(\mathcal{O}_{f^{-1}(x)}) = 0$ for all $x$ if and only if the geometric condition above is satisfied. This is an open condition in families of stable maps.

Returning to the family of abelian surfaces, we make the following observation.

**Lemma 6.9.** A stable map to the central fiber with rational contraction and source of compact type is not a limit of stable maps with singular sources to different nearby fibers. In other words, nodes of sources of such stable maps survive locally only inside the central fiber, at least set-theoretically.

**Proof.** If this was possible, then nearby fibers $W_t$ would have to admit stable maps $f : C \to W_t$ with singular sources of compact type and rational contraction, since having rational contraction and source of compact type are both open conditions. We claim that this is not the case. By Remark 5.1, the algebraic class $\{L_t\}$ does not split nontrivially into effective algebraic classes, so all components of $C$ but one, $C_0$, are ghost components of arithmetic genus 0. The dual graph is a tree, but it cannot have any leaves other than $C_0$ due to the previous remark and stability, so it can have only one vertex. Hence any such stable map would need to have irreducible source, hence smooth, being of compact type. □
Proposition 6.10. Let $p = \Lambda_1 + \ldots + \Lambda_{g-1}$ be a partition of $\Lambda_0^2$ of length $g-1$ and degree $d'$ and $p \succ p'$ the arrow to the length $g-2$ partition obtained replacing $\Lambda_i$ and $\Lambda_j$ with their sum, as in §3. Then $\overline{\mathcal{T}}$ contains (topologically) $\mathcal{T}_g(W_0, \beta; \Lambda, p)$ if and only if it contains $\hat{\mathcal{T}}_g(W_0, \beta; \Lambda, p \succ p')$.

Proof. Let $f$ be an arbitrary stable map into $W_0$ with the following profile:

1. the source is $C = C_0 \cup \bigcup_{k \in [g-1] \setminus \{i,j\}} C_k \cup \tilde{C}_0 \cup \tilde{C}_i \cup \tilde{C}_j$ such that:
2. $\tilde{C}_0$ is smooth rational, while all other curves are smooth of genus 1;
3. the dual graph is a tree such that all vertices are incident to $C_0$ except $\tilde{C}_i$ and $\tilde{C}_j$ which are incident to $\tilde{C}_0$;
4. the restriction of $f$ to $C_0$ is an isomorphism onto a fiber $E_1 \times \{[pt]\}$;
5. the restriction of $f$ to $C_k$ or $\tilde{C}_k$, $k \in [g-1]$, is an isogeny onto $\{[pt]\} \times E_2$ with associated lattice $\Lambda_k$;
6. $\tilde{C}_0$ is a ghost (contracted) component;
7. all fibers in (5) other than the fibers of $\tilde{C}_i$ and $\tilde{C}_j$ are different.

On one hand, $f$ is the prototypical map obtained when the $i$ and $j$ fibers of a map in $\mathcal{T}_g(W_0, \beta; \Lambda, p)$ collide, so it lies in the closure of $\mathcal{T}_g(W_0, \beta; \Lambda, p)$. The simplest way to argue this rigorously is to construct the degeneration explicitly. For instance, we can consider an isotrivial degeneration of $E_1 \times E_2$ to $E_1 \times E_2 \cup E_1 \times \mathbb{P}^1$ and construct the family of maps into the total space of this degeneration, then contract $E_1 \times \mathbb{P}^1$. The details are straightforward and left to the reader. On the other hand, $f$ lies in the closure of $\hat{\mathcal{T}}_g(W_0, \beta; \Lambda, p \succ p')$ by Proposition 4.5. Hence, regardless of which implication we're proving, it is a given that $f$ lies in $\overline{\mathcal{T}}$.

![Fig. B.](image)

Moreover, it is clear that all nearby deformations of $f$ in $\overline{\mathcal{M}}_g(W_0, \beta)$ are one of 4 possible types: they are either (a) simple, (b) quasi-simple, (c) pseudo-simple or (d) have the same profile as $f$, depending on which subset of nodes is smoothed. If we only consider deformations inside $\overline{\mathcal{T}}$, then there are further restrictions: type (b) are actually qst due to Proposition 6.6, while (c) doesn’t occur by Lemma 6.7.

Let $r_0$, $r_i$ and $r_j$ be the nodes $C_0 \cap \tilde{C}_0$, $\tilde{C}_0 \cap \tilde{C}_i$ and $\tilde{C}_0 \cap \tilde{C}_j$ respectively. The versal deformation space of a nodal singularity is

$$\text{Spec } \mathbb{C}[x, y, t]/(xy - t) \to \text{Spec } \mathbb{C}[t] := \mathbb{V} \text{ (node)}.$$ 

To avoid technical oversophistication, let $\vartheta : U \to \overline{\mathcal{M}}_g(W/B, \beta)$ be an étale map from a scheme such that $[f]$ has a unique preimage denoted by $v$. For any $\mathcal{X} \hookrightarrow \overline{\mathcal{M}}_g(W/B, \beta)$,
let \( \mathcal{Z}_{\text{ph}} = \mathcal{Z} \times_{\mathcal{Z}_g(W/B, \beta)} U \) be the pullback to \( U \). If \( \mathcal{Z} \) passes through \([f]\), let

\[
\mathcal{Z}_{\{f\}} := \text{Spec } \hat{\mathcal{O}}_{v, \mathcal{Z}_{\text{ph}}}
\]

mapping injectively into \( U \). If \( \mathcal{Z} \) doesn’t contain \([f]\), then \( \mathcal{Z}_{\{f\}} = \emptyset \) by default. Then there exist morphisms

\[
\varsigma_{\alpha} : \mathcal{M}_g(W/B, \beta)_{\{f\}} = \text{Spec } \hat{\mathcal{O}}_{v, U} \longrightarrow \mathbb{V}(r_{\alpha})
\]

for \( \alpha \in \{0, i, j\} \) such that the locus where the node \( r_{\alpha} \) survives locally is given by the vanishing of \( \varsigma_{\alpha}^* t_{\alpha} \), where \( t_{\alpha} \) is the coordinate of the formal disk \( \mathbb{V}(r_{\alpha}) \).

Denote the vanishing locus of \( \varsigma_{\alpha}^* t_{\alpha} \) by \( N_{\alpha} \). The inequality \( \dim R(x) \geq \dim R - 1 \) on dimensions of local rings implies that

\[
\dim (N_{\alpha} \cap \mathcal{F}_{\{f\}}) \geq \dim \mathcal{F}_{\{f\}} - 1 = g,
\]

for \( \alpha \in \{0, i, j\} \). Of course, equality must occur. The key point of the whole proof is that, by Lemma 6.9, \( N_{\alpha} \cap \mathcal{F}_{\{f\}} \) must be contained, at least set-theoretically, in the central fiber, since \( f \) has rational contraction and source of compact type. The central fiber of \( U \) is the preimage of the central fiber of \( \mathcal{M}_g(W/B, \beta) \).

“⇒” We apply the observations above to \( \alpha = 0 \). By inequality (15), the observations after Figure B and the fact that \( N_{0} \cap \mathcal{F}_{\{f\}} \) lives inside the central fiber, we see that \( N_{0} \cap \mathcal{F}_{\{f\}} \) must contain

\[
\mathcal{V}_g(W_o, \beta; \Lambda, p \succ p')_{\{f\}},
\]

where the “cl” superscript denotes closure and is applied before “\([f]\)” – there are simply no other deformations of \( f \) inside \( \mathcal{V}_o \) which preserve the node \( r_0 \) and have at least the required \( g \) moduli. Hence \( \mathcal{F}_{\{f\}} \) contains \( \mathcal{V}_g(W_o, \beta; \Lambda, p \succ p')_{\{f\}} \). Thus

\[
\dim \left( \mathcal{V}_g(W_o, \beta; \Lambda, p \succ p')_{\{f\}} \cap \mathcal{F} \right) = g
\]

and since \( \mathcal{V}_g(W_o, \beta; \Lambda, p \succ p') \) is irreducible by Proposition 4.5 and the remark that the map \( \Psi \) (14) has irreducible fibers of the same dimension (and in particular, so does its restriction to \( \mathcal{V}_g(W_o, \beta; \Lambda, p \succ p') \)), we conclude that the component \( \mathcal{F} \) contains \( \mathcal{V}_g(W_o, \beta; \Lambda, p \succ p') \).

“⇐” For the converse, we use analogous arguments with \( \alpha = i \). Once more, we need to keep in mind the remarks after Figure B. As above, \( N_i \cap \mathcal{F}_{\{f\}} \) must contain \( \mathcal{V}_g(W_o, \beta; \Lambda, p)_{\{f\}} \) because these are the only potential deformations of \( f \) inside \( \mathcal{V}_o \) preserving the node \( r_i \) with the required \( g \) moduli. Hence \( \mathcal{F} \) contains \( \mathcal{V}_g(W_o, \beta; \Lambda, p) \) because \( \mathcal{V}_g(W_o, \beta; \Lambda, p) \) is irreducible.

Proposition 6.10 applied to a “roof” \( p_1 \succ p_3 \prec p_2 \) gives the final result.

**Corollary 6.11.** Let \( p_1, p_2 \in \mathcal{P}^{g-1}_d \) be partitions of \( \Lambda_0^3 \) such that \([p_1p_2]\) is an edge of the graph structure on \( \mathcal{P}^{g-1}_d \) defined in §3. If \( \mathcal{F} \) contains (topologically) either \( \mathcal{V}_g(W_o, \beta; \Lambda, p_1) \) or \( \mathcal{V}_g(W_o, \beta; \Lambda, p_2) \), then it contains both of them.

Now we can prove the main part of Theorem 2.6. Note that it suffices to prove that the Deligne-Mumford stack \( \mathcal{V}_g(W_o/B^o, \beta; \Lambda) \) is irreducible under the (BC) hypothesis. We argue as follows. First, it is easy to see that there exists a base change map \((B, o) \to (Z, o_2)\) such that the map \( \varphi : \mathcal{V}_g(W_o/B^o, \beta; \Lambda) \to B^o \) admits a section. Then [25, Proposition 2.2.1] implies that the geometric fibers of \( \varphi \) are connected, hence irreducible by Proposition 6.3. Note that no generality assumption on fibers is required after the base change.
Proof of the irreducibility part of Theorem 2.6. Let \( \mathcal{V} \) be an irreducible component of \( \mathcal{V}_g(W^o/B^o, \beta; \Lambda) \) and \( \overline{\mathcal{V}} \) its closure inside \( \overline{\mathcal{M}}_g(W/B, \beta) \). Fix any partition \( \tilde{p} \) of \( \Lambda_2^\ast \) of length \( g - 1 \) and degree \( d' = d/[H_1(E_2, \mathbb{Z}) : \Lambda_2^\ast] \). By Lemma 6.5, \( \overline{\mathcal{V}} \) contains \( \mathcal{V}_g(W_o, \beta; \Lambda, p_0) \) for some \( p_0 \). By Proposition 3.1, there exists a chain

\[
p_0, p_1, \ldots, p_k = \tilde{p} \in \mathcal{P}_{d' - 1}^{g - 1}(\Lambda_2^\ast).
\]

Applying Corollary 6.11 inductively, we deduce that \( \overline{\mathcal{V}} \) contains \( \mathcal{V}_g(W_o, \beta; \Lambda, p_i) \) for all \( i \leq k \), so in particular it contains \( \mathcal{V}_g(W_o, \beta; \Lambda, \tilde{p}) \).

If \( \mathcal{V}' \) is hypothetically a different component of \( \mathcal{V}_g(W^o/B^o, \beta; \Lambda) \) with closure \( \overline{\mathcal{V}} \), then the same argument shows that \( \overline{\mathcal{V}} \) also contains \( \mathcal{V}_g(W_o, \beta; \Lambda, \tilde{p}) \) topologically. However, \( \overline{\mathcal{M}}_g(W/B, \beta) \) is smooth at all points of \( \mathcal{V}_g(W_o, \beta; \Lambda, \tilde{p}) \) as in the proof of the existence part, so \( \overline{\mathcal{V}} = \overline{\mathcal{V}} \). Thus \( \mathcal{V}_g(W^o/B^o, \beta; \Lambda) \) is irreducible, completing the proof thanks to the remark in the paragraph preceding it. \( \square \)

References

[1] M. F. Atiyah, Vector bundles over an elliptic curve, Proc. London Math. Soc. 7, 414–452 (1957)
[2] K. Behrend and B. Fantechi, The intrinsic normal cone, Invent. Math. 128, 45–88 (1997)
[3] K. Behrend and Yu. Manin, Stacks of stable maps and Gromov-Witten invariants, Duke Math. J. 85, 1–60 (1996)
[4] C. Birkenhake and H. Lange, Complex Abelian Varieties, Grundlehren math. Wiss. 302, Springer-Verlag (1992)
[5] S. Bloch, Semiregularity and De Rham cohomology, Invent. Math. 17, 51–66 (1972)
[6] J. Bryan and C. Leung, The Enumerative Geometry of K3 surfaces and Modular Forms, J. Amer. Math. Soc. 12 (2), 371–410 (2000)
[7] J. Bryan, G. Oberdieck, R. Pandharipande, and Q. Yin, Curve counting on abelian surfaces and threefolds, Alg. Geom. 5 (4), 398–463 (2018)
[8] N. Bourbaki, Commutative algebra, Hermann (1972)
[9] G. Bujokas, The Hurwitz space of covers of an elliptic curve \( E \) and the Severi variety of curves in \( E \times \mathbb{P}^1 \), preprint https://arxiv.org/abs/1409.0927 (2014)
[10] C. Ciliberto and Th. Dedieu, On universal Severi varieties of low genus K3 surfaces, Math. Z. 271, 953–960 (2012)
[11] C. Ciliberto and Th. Dedieu, On the irreducibility of Severi varieties on K3 surfaces, preprint, https://arxiv.org/abs/1809.03914 (2018)
[12] C. Ciliberto, F. Flamini, C. Galati and A. L. Knutsen, Moduli of nodal curves on K3 surfaces, Adv. Math. 309, 624–654 (2017)
[13] X. Chen, Rational curves on K3 surfaces, J. Alg. Geom. 8, 245–278 (1999)
[14] X. Chen, A simple proof that rational curves on K3 are nodal, Math. Ann. 324 (1), 71–104 (2002)
[15] W. Fulton and R. MacPherson, A compactification of configuration spaces, Ann. Math. 139, 185–225 (1994)
[16] D. Gabai and W. Kazez, The classification of maps of surfaces, Invent. Math. 90 (2), 219–242 (1987)
[17] P. Griffiths and J. Harris, On the Noether-Lefschetz Theorem and some remarks on codimension two cycles, Math. Ann. 271, 31–51 (1985)
[18] J. Harris, On the Severi problem, Invent. Math. 84 (3), 445–461 (1986)
[19] M. Kemeny, The universal Severi variety of rational curves on K3 surfaces, Bull. London Math. Soc. 45, 159–174 (2013)
[20] M. Kool and R. P. Thomas, Reduced classes and curve counting on surfaces I: theory, Alg. Geom. 1, 334–383 (2014)
[21] A. L. Knutsen, M. Lelli-Chiesa and G. Mongardi, Severi varieties and Brill-Noether theory of curves on abelian surfaces, J. Reine Angew. Math. to appear (2015)
[22] J. Li, Stable morphisms to singular schemes and relative stable morphisms, J. Diff. Geom. 57, 509–578 (2001)
[23] J. Li, A degeneration formula of GW-invariants, J. Diff. Geom. 60, 199–293 (2002)
[24] Z. Ran, *Semiregularity, obstructions and deformations of Hodge classes*, Ann. Scuola Norm.-Sci. **28**, 809–820 (1999)

[25] M. Romagny, *Composantes connexes et irréductibles en familles*, Manuscripta Math. **136**, 1–32 (2011)

[26] J. M. Ruiz, *The basic theory of power series*, Vieweg (1993)

[27] Stacks project authors, *The Stacks project*, https://stacks.math.columbia.edu/

[28] A. Zahariuc, *Elliptic surfaces and linear systems with fat points*, preprint https://arxiv.org/abs/1711.09323 (2017)

[29] A. Zahariuc, *The irreducibility of the generalized Severi varieties*, preprint https://arxiv.org/abs/1711.09324 (2017)

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