STACK-NUMBER IS NOT BOUNDED BY QUEUE-NUMBER

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We describe a family of graphs with queue-number at most 4 but unbounded stack-number. This resolves open problems of Heath, Leighton and Rosenberg (1992) and Blankenship and Oporowski (1999).

1. Introduction

Stacks and queues are fundamental data structures in computer science, but which is more powerful? In 1992, Heath, Leighton and Rosenberg [28,29] introduced an approach for answering this question by defining the graph parameters stack-number and queue-number (defined below), which respectively measure the power of stacks and queues for representing graphs. The following fundamental questions, implicit in [28,29], were made explicit by Dujmović and Wood [21]1:

- Is stack-number bounded by queue-number?
- Is queue-number bounded by stack-number?

If stack-number is bounded by queue-number but queue-number is not bounded by stack-number, then stacks would be considered to be more pow-

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1 A graph parameter is a function α such that α(G) ∈ ℝ for every graph G and such that α(G1) = α(G2) for all isomorphic graphs G1 and G2. A graph parameter α is bounded by a graph parameter β if there exists a function f such that α(G) ≤ f(β(G)) for every graph G.
erful than queues. Similarly, if the converse holds, then queues would be considered to be more powerful than stacks. Despite extensive research on stack- and queue-numbers, these questions have remained unsolved.

We now formally define stack- and queue-number. Let $G$ be a graph and let $\prec$ be a total order on $V(G)$. Two disjoint edges $vw, xy \in E(G)$ with $v \prec w$ and $x \prec y$ cross with respect to $\prec$ if $v \prec x \prec w \prec y$ or $x \prec v \prec y \prec w$, and nest with respect to $\prec$ if $v \prec x \prec y \prec w$ or $x \prec v \prec w \prec y$. Consider a function $\phi : E(G) \to \{1, \ldots, k\}$ for some $k \in \mathbb{N}$. Then $(\prec, \phi)$ is a $k$-stack layout of $G$ if $vw$ and $xy$ do not cross for all edges $vw, xy \in E(G)$ with $\phi(vw) = \phi(xy)$. Similarly, $(\prec, \phi)$ is a $k$-queue layout of $G$ if $vw$ and $xy$ do not nest for all edges $vw, xy \in E(G)$ with $\phi(vw) = \phi(xy)$. See Figure 1 for examples. The smallest integer $s$ for which $G$ has an $s$-stack layout is called the stack-number of $G$, denoted $\text{sn}(G)$. The smallest integer $q$ for which $G$ has a $q$-queue layout is called the queue-number of $G$, denoted $\text{qsn}(G)$.

Given a $k$-stack layout $(\prec, \phi)$ of a graph $G$, for each $i \in \{1, \ldots, k\}$, the set $\phi^{-1}(i)$ behaves like a stack, in the sense that each edge $vw \in \phi^{-1}(i)$ with $v \prec w$ corresponds to an element in a sequence of stack operations, such that if we traverse the vertices in the order of $\prec$, then $vw$ is pushed onto the stack at $v$ and popped off the stack at $w$. Similarly, each set $\phi^{-1}(i)$ in a queue layout behaves like a queue. In this way, the stack-number and queue-number respectively measure the power of stacks and queues to represent graphs.

Note that stack layouts are equivalent to book embeddings (first defined by Ollmann [34] in 1973), and stack-number is also known as page-number, book-thickness or fixed outer-thickness. Stack and queue layouts have other applications including computational complexity [10,11,19,26], RNA folding [27], graph drawing in two [1,2,39] and three dimen-

![Figure 1. A 2-queue layout and a 2-stack layout of the triangulated grid graph $H_4$ defined below. Edges drawn above the vertices are assigned to the first queue/stack and edges drawn below the vertices are assigned to the second queue/stack](image-url)
Theorem 1.1. For every $s \in \mathbb{N}$ there exists a graph $G$ with $qsn(G) \leq 4$ and $sn(G) > s$.

This demonstrates that stacks are not more powerful than queues for representing graphs.

### Cartesian products

As illustrated in Figure 2, the graph $G$ in Theorem 1.1 is the cartesian product $^2$ $S_b \square H_n$ for sufficiently large $b$ and $n$, where $S_b$ is the star graph with root $r$ and $b$ leaves, and $H_n$ is the dual of the hexagonal grid, defined by

$$V(H_n) := \{1, \ldots, n\}^2$$
$$E(H_n) := \{(x, y)(x + 1, y) : x \in \{1, \ldots, n - 1\}, y \in \{1, \ldots, n\}\}$$
$$\cup \{(x, y)(x, y + 1) : x \in \{1, \ldots, n\}, y \in \{1, \ldots, n - 1\}\}$$
$$\cup \{(x, y)(x + 1, y + 1) : x, y \in \{1, \ldots, n - 1\}\}.$$

We prove the following:

For graphs $G_1$ and $G_2$, the cartesian product $G_1 \square G_2$ is the graph with vertex set $\{(v_1, v_2) : v_1 \in V(G_1), v_2 \in V(G_2)\}$, where $(v_1, v_2)(w_1, w_2) \in E(G_1 \square G_2)$ if $v_1 = w_1$ and $v_2w_2 \in E(G_2)$, or $v_1w_1 \in E(G_1)$ and $v_2 = w_2$. The strong product $G_1 \boxtimes G_2$ is the graph obtained from $G_1 \square G_2$ by adding the edge $(v_1, v_2)(w_1, w_2)$ whenever $v_1w_1 \in E(G_1)$ and $v_2w_2 \in E(G_2)$. Note that Pupyrev [35] independently suggested using graph products to show that stack-number is not bounded by queue-number.
Theorem 1.2. For every \( s \in \mathbb{N} \), if \( b \) and \( n \) are sufficiently large compared to \( s \), then

\[
\text{sn}(S_b \Box H_n) > s.
\]

We now show that \( \text{qsn}(S_b \Box H_n) \leq 4 \), which with Theorem 1.2 implies Theorem 1.1. We need the following definition due to Wood [41]. A queue layout \((\varphi, \prec)\) is strict if for every vertex \( u \in V(G) \) and for all neighbours \( v, w \in N_G(u) \), if \( u \prec v \prec w \) or \( v \prec w \prec u \), then \( \varphi(uv) \neq \varphi(uw) \). Let \( \text{qsn}(G) \) be the minimum integer \( k \) such that \( G \) has a strict \( k \)-queue layout. To see that \( \text{qsn}(H_n) \leq 3 \), order the vertices row-by-row and then left-to-right within a row, with vertical edges in one queue, horizontal edges in one queue, and diagonal edges in another queue. Wood [41] proved that for all graphs \( G_1 \) and \( G_2 \),

\[
\text{qsn}(G_1 \Box G_2) \leq \text{qsn}(G_1) + \text{qsn}(G_2).
\]

Of course, \( S_b \) has a 1-queue layout (since no two edges are nested for any vertex-ordering). Thus \( \text{qsn}(S_b \Box H_n) \leq 4 \).

Bernhart and Kainen [4] implicitly proved a result similar to (1) for stack layouts. Let \( \text{dsn}(G) \) be the minimum integer \( k \) such that \( G \) has a \( k \)-stack layout \((\prec, \varphi)\) where \( \varphi \) is a proper edge-colouring of \( G \); that is, \( \varphi(vx) \neq \varphi(vy) \) for any two edges \( vx, vy \in E(G) \) with a common endpoint. Then for every
The key difference between (1) and (2) is that $G_2$ is assumed to be bipartite in (2). Theorem 1.2 says that this assumption is essential, since it is easily seen that $(dsn(H_n))_{n \in \mathbb{N}}$ is bounded, but the stack number of $(S_b \Box H_n)_{b,n \in \mathbb{N}}$ is unbounded by Theorem 1.2. We choose $H_n$ in Theorem 1.2 since it satisfies the Hex Lemma (Lemma 2.4 below), which quantifies the intuition that $H_n$ is far from being bipartite (while still having bounded queue-number and bounded maximum degree so that (1) is applicable).

**Subdivisions**

A noteworthy consequence of Theorem 1.1 is that it resolves a conjecture of Blankenship and Oporowski [6]. A graph $G'$ is a subdivision of a graph $G$ if $G'$ can be obtained from $G$ by replacing the edges $vw$ of $G$ by internally disjoint paths $P_{vw}$ with endpoints $v$ and $w$. If each $P_{vw}$ has exactly $k$ internal vertices, then $G'$ is the $k$-subdivision of $G$. If each $P_{vw}$ has at most $k$ internal vertices, then $G'$ is a $(\leq k)$-subdivision of $G$. Blankenship and Oporowski [6] conjectured that the stack-number of $(\leq k)$-subdivisions ($k$ fixed) is not much less than the stack-number of the original graph. More precisely:

**Conjecture 1.3 ([6]).** There exists a function $f$ such that for every graph $G$ and integer $k$, if $G'$ is any $(\leq k)$-subdivision of $G$, then $sn(G) \leq f(sn(G'),k)$.

Dujmović and Wood [21] established a connection between this conjecture and the question of whether stack-number is bounded by queue-number. In particular, they showed that if Conjecture 1.3 was true, then stack-number would be bounded by queue-number. Since Theorem 1.1 shows that stack-number is not bounded by queue-number, Conjecture 1.3 is false. The proof of Dujmović and Wood [21] is based on the following key lemma: every graph $G$ has a 3-stack subdivision with $1 + 2\lceil \log_2 qsn(G) \rceil$ division vertices per edge. Applying this result to the graph $G = S_b \Box H_n$ in Theorem 1.1, the 5-subdivision of $S_b \Box H_n$ has a 3-stack layout. If Conjecture 1.3 was true, then $sn(S_b \Box H_n)$ would be at most $f(3,5)$, contradicting Theorem 1.1.

**Is queue-number bounded by stack-number?**

It remains open whether queues are more powerful than stacks; that is, whether queue-number is bounded by stack-number. Several results are
known about this problem. Heath et al. [28] showed that every 1-stack graph has a 2-queue layout. Dujmović et al. [14] showed that planar graphs have bounded queue-number. (Note that graph products also feature heavily in this proof.) Since 2-stack graphs are planar, this implies that 2-stack graphs have bounded queue-number. It is open whether 3-stack graphs have bounded queue-number. In fact, the case of three stacks is as hard as the general question. Dujmović and Wood [21] proved that queue-number is bounded by stack-number if and only if 3-stack graphs have bounded queue-number. Moreover, if this is true, then queue-number is bounded by a polynomial function of stack-number.

2. Proof of Theorem 1.2

We now turn to the proof of our main result, the lower bound on $sn(G)$, where $G := S_b □ H_n$. Consider a hypothetical $s$-stack layout $(ϕ,≺)$ of $G$, where $n$ and $b$ are chosen sufficiently large compared to $s$ as detailed below. We begin with three lemmas that, for sufficiently large $b$, provide a large sub-star $S_d$ of $S_b$ for which the induced stack layout of $S_d □ H_n$ is highly structured.

For each node $v$ of $S_b$, define $π_v$ as the permutation of $\{1, \ldots, n\}^2$ in which $(x_1, y_1)$ appears before $(x_2, y_2)$ if and only if $(v, (x_1, y_1)) ≺ (v, (x_2, y_2))$. The following lemma is an immediate consequence of the Pigeonhole Principle:

**Lemma 2.1.** There exists a permutation $π$ of $\{1, \ldots, n\}^2$ and a set $L_1$ of leaves of $S_b$ of size $a ≥ b/(n^2)!$ such that $π_v = π$ for each $v ∈ L_1$.

For each leaf $v$ in $L_1$, let $ϕ_v$ be the edge colouring of $H_n$ defined by $ϕ_v(xy) := ϕ((v, x)(v, y))$ for each $xy ∈ E(H_n)$. Since $H_n$ has maximum degree 6 and is not 6-regular, it has fewer than $3n^2$ edges. Therefore, there are fewer than $s^{3n^2}$ edge colourings of $H_n$ using $s$ colours. Another application of the Pigeonhole Principle proves the following:

**Lemma 2.2.** There exists a subset $L_2 ⊆ L_1$ of size $c ≥ a/s^{3n^2}$ and an edge colouring $ϕ: E(H_n) → \{1, \ldots, s\}$ such that $ϕ_v = ϕ$ for each $v ∈ L_2$.

Let $S_c$ be the subgraph of $S_b$ induced by $L_2 ∪ \{r\}$. The preceding two lemmas ensure that, for distinct leaves $v$ and $w$ of $S_c$, the stack layouts of the isomorphic graphs $G[\{(v, p): p ∈ V(H_n)\}]$ and $G[\{(w, p): p ∈ V(H_n)\}]$ are identical. The next lemma is a statement about the relationship between the stack layouts of $G[\{(v, p): v ∈ V(S_c)\}]$ and $G[\{(v, q): v ∈ V(S_c)\}]$ for distinct $p, q ∈ V(H_n)$. It does not assert that these two layouts are identical but it does state that they fall into one of two categories.
Lemma 2.3. There exists a sequence \( u_1, \ldots, u_d \in L_2 \) of length \( d \geq c^{1/2n^2 - 1} \) such that, for each \( p \in V(H_n) \), either \((u_1, p) < (u_2, p) < \cdots < (u_d, p)\) or \((u_1, p) > (u_2, p) > \cdots > (u_d, p)\).

Proof. Let \( p_1, \ldots, p_{n^2} \) denote the vertices of \( H_n \) in any order. Begin with the sequence \( V_1 := v_{1,1}, \ldots, v_{1,c} \) that contains all \( c \) elements of \( L_2 \) ordered so that \((v_{1,1}, p_1) < \cdots < (v_{1,c}, p_1)\). For each \( i \in \{2, \ldots, n^2\} \), the Erdős–Szekeres Theorem \([24]\) implies that \( V_{i-1} \) contains a subsequence \( V_i := v_{i,1}, \ldots, v_{i,|V_i|} \) of length \(|V_i| \geq \sqrt{|V_{i-1}|}\) such that \((v_{i,1}, p_i) < \cdots < (v_{i,|V_i|}, p_i)\) or \((v_{i,1}, p_i) > \cdots > (v_{i,|V_i|}, p_i)\). It is straightforward to verify by induction on \( i \) that \(|V_i| \geq c^{1/2i - 1}\) resulting in a final sequence \( V_{n^2} \) of length at least \( c^{1/2n^2 - 1} \). 

For the rest of the proof we work with the star \( S_d \) whose leaves are \( u_1, \ldots, u_d \) described in Lemma 2.3. Consider the improper colouring of \( H_n \) obtained by colouring each vertex \( p \in V(H_n) \) red if \((u_1, p) < \cdots < (u_d, p)\) and colouring \( p \) blue if \((u_1, p) > \cdots > (u_d, p)\). We need the following famous Hex Lemma \([25]\).

Lemma 2.4 ([25]). Every vertex 2-colouring of \( H_n \) contains a monochromatic path on \( n \) vertices.

Apply Lemma 2.4 with the above-defined colouring of \( H_n \). We obtain a path subgraph \( P := (p_1, \ldots, p_n) \) of \( H_n \) that, without loss of generality, consists entirely of red vertices; thus \((u_1, p_1) < \cdots < (u_d, p_j)\) for each \( j \in \{1, \ldots, n\} \). Let \( X \) be the subgraph \( S_d \square P \) of \( G \).

Lemma 2.5. \( X \) contains a set of at least \( \min\{[d/2^n], [n/2]\} \) pairwise crossing edges with respect to \(<\).

Proof. Extend the total order \(<\) to a partial order over subsets of \( V(G) \), where for all \( V, W \subseteq V(G) \), we have \( V < W \) if and only if \( v < w \) for each \( v \in V \) and each \( w \in W \). We abuse notation slightly by using \(<\) to compare elements of \( V(G) \) and subsets of \( V(G) \) so that, for \( v \in V(G) \) and \( V \subseteq V(G) \), \( v < V \) denotes \( \{v\} < V \). We will define sets \( A_1 \supseteq \cdots \supseteq A_n \) of leaves of \( S_d \) so that each \( A_i \) satisfies the following conditions:

(C1) \( A_i \) contains \( d_i \geq d/2^{i-1} \) leaves of \( S_d \).
(C2) Each leaf \( v \in A_i \) defines an \( i \)-element vertex set \( Z_{i,v} := \{(v, p_j) : j \in \{1, \ldots, i\}\} \). For any distinct \( v, w \in A_i \), the sets \( Z_{i,v} \) and \( Z_{i,w} \) are separated with respect to \(<\); that is, \( Z_{i,v} < Z_{i,w} \) or \( Z_{i,v} > Z_{i,w} \).

Before defining \( A_1, \ldots, A_n \) we first show how the existence of the set \( A_n \) implies the lemma. To avoid triple-subscripts, let \( d' := d_n \geq d/2^{n-1} \). By (C2),
the set $A_n$ defines vertex sets $Z_{n,v_1} \prec \cdots \prec Z_{n,v_{d'}}$ (see Figure 3). The root $r$ of $S_b$ is adjacent to each of $v_1, \ldots, v_{d'}$ in $S_d$. Thus, for each $j \in \{1, \ldots, n\}$ and each $i \in \{1, \ldots, d'\}$, the edge $(r, p_j)(v_i, p_j)$ is in $X$. Hence, $(r, p_j)$ is adjacent to an element of each of $Z_{n,v_1}, \ldots, Z_{n,v_{d'}}$.

Since $Z_{n,v_1}, \ldots, Z_{n,v_{d'}}$ are separated with respect to $\prec$, if we imagine identifying the vertices in each set $Z_{n,v_i}$, this situation looks like a complete bipartite graph $K_{n,d'}$ with the root vertices $L := \{(r, p_j): j \in \{1, \ldots, n\}\}$ in one part and the groups $R := Z_{n,v_1} \cup \cdots \cup Z_{n,v_{d'}}$ in the other part. Any linear ordering of $K_{n,d'}$ has a large set of pairwise crossing edges. So, intuitively, the induced subgraph $X[L \cup R]$ should also have a large set of pairwise crossing edges.

We formalize this idea as follows: Label the vertices in $L$ as $r_1, \ldots, r_n$ so that $r_1 \prec \cdots \prec r_n$. Then at least one of the following two cases applies (see Figure 4):

1. $Z_{n,[d'/2]} \prec r_{[n/2]}$ in which case the graph between $r_{[n/2]}, \ldots, r_n$ and $Z_{n,1}, \ldots, Z_{n,[d'/2]}$ has a set of at least $\min\{[d'/2], [n/2]\}$ pairwise-crossing edges.

2. $r_{[n/2]} \prec Z_{[d'/2]+1}, \ldots, Z_{d'}$ in which case the graph between $r_1, \ldots, r_{[n/2]}$ and $Z_{[d'/2]+1}, \ldots, Z_{d'}$ has a set of $\min\{[d'/2], [n/2]\}$ pairwise-crossing edges.

Since, by (C1), $d' \geq d/2^{n-1}$, either case results in a set of pairwise-crossing edges of size at least $\min\{[d/2^n], [n/2]\}$, as claimed.

It remains to define the sets $A_1 \supseteq \cdots \supseteq A_n$ that satisfy (C1) and (C2). Let $A_1$ be the set of all the leaves of $S_d$. For each $i \in \{2, \ldots, n\}$, assuming that $A_{i-1}$ is already defined, the set $A_i$ is defined as follows: For brevity, let $m := |A_{i-1}|$. Let $Z_1, \ldots, Z_m$ denote the sets $Z_{i-1, v}$ for each $v \in A_{i-1}$ ordered...
so that $Z_1 < \cdots < Z_m$. By Property (C2), this is always possible. Label the vertices of $A_{i-1}$ as $v_1, \ldots, v_m$ so that $(v_1, p_{i-1}) \prec \cdots \prec (v_m, p_{i-1})$. (This is equivalent to naming them so that $(v_j, p_{i-1}) \in Z_j$ for each $j \in \{1, \ldots, m\}$.) Define the set $A_i := \{v_{2k+1}: k \in \{0, \ldots, \lceil (m-1)/2 \rceil \} \} = \{v_j \in A_{i-1}: j \text{ is odd}\}$. This completes the definition of $A_1, \ldots, A_n$.

We now verify that $A_i$ satisfies (C1) and (C2) for each $i \in \{1, \ldots, n\}$. We do this by induction on $i$. The base case $i = 1$ is trivial, so now assume that $i \in \{2, \ldots, n\}$. To see that $A_i$ satisfies (C1) observe that $|A_i| = \lceil |A_{i-1}|/2 \rceil \geq |A_{i-1}|/2 \geq d/2^{i-1}$, where the final inequality follows by applying the inductive hypothesis $|A_{i-1}| \geq d/2^{i-2}$. Now it remains to show that $A_i$ satisfies (C2). Again, let $m := |A_{i-1}|$.

Recall that, for each $v \in A_{i-1}$, the edge $e_v := (v, p_{i-1})(v, p_i)$ is in $X$. We have the following properties:

(P1) By Lemma 2.2, $\varphi(e_v) = \emptyset(p_{i-1}p_i)$ for each $v \in A_{i-1}$.
(P2) Since $p_{i-1}$ and $p_i$ are both red, for each $v, w \in A_{i-1}$, we have $(v, p_{i-1}) \prec (w, p_{i-1})$ if and only if $(v, p_i) \prec (w, p_i)$.
(P3) By Lemma 2.1, $(v, p_{i-1}) \prec (v, p_i)$ for every $v \in A_{i-1}$ or $(v, p_{i-1}) \succ (v, p_i)$ for every $v \in A_{i-1}$.

We claim that these three conditions imply that the vertex sets $\{(v, p_{i-1}): v \in A_{i-1}\}$ and $\{(v, p_i): v \in A_{i-1}\}$ interleave perfectly with respect to $\prec$. More precisely:

**Claim 1.** $(v_1, p_{i-1}+t) \prec (v_1, p_{i-1}) \prec (v_2, p_{i-1}+t) \prec (v_2, p_{i-1}) \cdots \prec (v_m, p_{i-1}+t) \prec (v_m, p_{i-1})$ for some $t \in \{0, 1\}$.

**Proof of Claim 1.** By (P3) we may assume, without loss of generality, that $(v, p_{i-1}) \prec (v, p_i)$ for each $v \in A_{i-1}$, in which case we are trying to prove the claim for $t = 0$. Therefore, it is sufficient to show that $(v_j, p_i) \prec (v_{j+1}, p_{i-1})$ for each $j \in \{1, \ldots, m-1\}$. For the sake of contradiction, suppose
(v_j, p_i) > (v_{j+1}, p_{i-1}) for some \( j \in \{1, \ldots, m - 1\} \). By the labelling of \( A_{i-1} \), \((v_j, p_i) \prec (v_{j+1}, p_{i-1})\) so, by (P2), \((v_j, p_i) \prec (v_{j+1}, p_i)\). Therefore,

\[
(v_j, p_i - 1) \prec (v_{j+1}, p_i - 1) \prec (v_j, p_i) \prec (v_{j+1}, p_i).
\]

Therefore, the edges \( e_{v_j} = (v_j, p_i - 1)(v_j, p_i) \) and \( e_{v_{j+1}} = (v_{j+1}, p_i - 1)(v_{j+1}, p_i) \) cross with respect to \( \prec \). But this is a contradiction since, by (P1), \( \varphi(e_{v_j}) = \varphi(e_{v_{j+1}}) = \phi(p_{i-1}p_i) \). This contradiction completes the proof of Claim 1.

We now complete the proof that \( A_i \) satisfies (C2). Apply Claim 1 and assume without loss of generality that \( t = 0 \), so that

\[
(v_1, p_i - 1) \prec (v_1, p_i) \prec (v_2, p_i - 1) \prec (v_2, p_i) \cdots \prec (v_m, p_i - 1) \prec (v_m, p_i).
\]

For each \( j \in \{1, \ldots, m - 2\} \), we have \((v_{j+1}, p_i - 1) \in Z_{j+1} \prec Z_{j+2} \), so \((v_{j+1}, p_i - 1) \prec (v_{j+1}, p_i) \prec Z_{j+2} \). Therefore \( Z_j \cup \{(v_j, p_i)\} \prec Z_{j+2} \). By a symmetric argument, \( Z_j \cup \{(v_j, p_i)\} \prec Z_{j-2} \) for each \( j \in \{3, \ldots, m\} \). Finally, since \((v_{j+2}, p_i) \) for each odd \( i \in \{1, \ldots, m\} \), we have \( Z_j \cup \{(v_j, p_i)\} \prec Z_{j+2} \cup \{(v_{j+2}, p_i)\} \) for each odd \( j \in \{1, \ldots, m-2\} \). Thus \( A_i \) satisfies (C2) since the sets \( Z_1 \cup \{(v_1, p_i)\}, Z_3 \cup \{(v_3, p_i)\}, \ldots, Z_2 \cup \{(v_{(m-1)/2} + 1, p_i)\} \) are precisely the sets \( Z_{i, 1}, \ldots, Z_{i, d} \) determined by our choice of \( A_i \).

**Proof of Theorem 1.2.** Let \( G := S_b \square H_n \), where \( n := 2s + 1 \) and \( b := (n^2)!s^{3n^2}((s + 1)2^n)^{2n^2 - 1} \). Suppose that \( G \) has an \( s \)-stack layout \((\varphi, \prec)\). In particular, there are no \( s + 1 \) pairwise crossing edges in \( G \) with respect to \( \prec \). By Lemmas 2.1 to 2.3, we have \( a \geq b/(n^2)! = s^{3n^2}((s + 1)2^n)^{2n^2 - 1} \) and \( c \leq a/s^{3n^2} \geq ((s + 1)2^n)^{2n^2 - 1} \) and \( d \geq c^{1/2n^2 - 1} \geq (s + 1)2^n \). By Lemma 2.5, the graph \( X \), which is a subgraph of \( G \), contains \( \min\{[d/2^n], [n/2]\} = s + 1 \) pairwise crossing edges with respect to \( \prec \). This contradiction shows that \( \text{sn}(G) > s \).

**3. Reflections**

We now mention some further consequences and open problems that arise from our main result.

Nešetřil, Ossona de Mendez and Wood [33] proved that graph classes with bounded stack-number or bounded queue-number have bounded expansion; see [32] for background on bounded expansion classes. The converse is not true, since cubic graphs (for example) have bounded expansion, unbounded stack-number [31] and unbounded queue-number [42]. However, prior to the present work it was open whether graph classes with polynomial expansion have bounded stack-number or bounded queue-number. It
follows from the work of Dvořák, Huynh, Joret, Liu and Wood [23, Theorem 19] that \((S_b \square H_n)_{b,n \in \mathbb{N}}\) has polynomial expansion. So Theorem 1.2 implies there is a class of graphs with polynomial expansion and with unbounded stack-number. It remains open whether graph classes with polynomial expansion have bounded queue-number. See [14,17] for several examples of graph classes with polynomial expansion and bounded queue-number.

Our main result also resolves a question of Bonnet, Geniet, Kim, Thomassé and Watrigant [7] concerning sparse twin-width; see [7,8,9] for the definition and background on (sparse) twin-width. Bonnet et al. [7] proved that graphs with bounded stack-number have bounded sparse twin-width, and they write that they “believe that the inclusion is strict”; that is, there exists a class of graphs with bounded sparse twin-width and unbounded stack-number. Theorem 1.2 confirms this intuition, since the class of all subgraphs of \((S_b \square H_n)_{b,n \in \mathbb{N}}\) has bounded sparse twin-width (since Bonnet et al. [7] showed that any hereditary class of graphs with bounded queue-number has bounded sparse twin-width). It remains open whether bounded sparse twin-width coincides with bounded queue-number.

Finally, we mention some more open problems:

- Recall that every 1-queue graph has a 2-stack layout [28] and we proved that there are 4-queue graphs with unbounded stack-number. The following questions remain open: Do 2-queue graphs have bounded stack-number? Do 3-queue graphs have bounded stack-number?
- Since \(H_n \subseteq P \boxtimes P\) where \(P\) is the \(n\)-vertex path, Theorem 1.1 implies that \(\text{sn}(S \boxtimes P \boxtimes P)\) is unbounded for stars \(S\) and paths \(P\). It is easily seen that \(\text{sn}(S \boxtimes P)\) is bounded [35]. The following question naturally arises (independently asked by Pupyrev [35]): Is \(\text{sn}(T \boxtimes P)\) bounded for all trees \(T\) and paths \(P\)? We conjecture the answer is “no”.

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