WELL-POSEDNESS IN CRITICAL SPACES FOR A MULTIDIMENSIONAL COMPRESSIBLE VISCOUS LIQUID-GAS TWO-PHASE FLOW MODEL

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Abstract. This paper is dedicated to the study of the Cauchy problem for a compressible viscous liquid-gas two-phase flow model in $\mathbb{R}^N (N \geq 2)$. We concentrate on the critical Besov spaces based on the $L^p$ setting. We improve the range of Lebesgue exponent $p$, for which the system is locally well-posed, compared to [22]. Applying Lagrangian coordinates is the key to our statements, as it enables us to obtain the result by means of Banach fixed point theorem.

1. Introduction. We address the well-posedness of the compressible liquid-gas two-phase flow model, which satisfies:

\[
\begin{align*}
\partial_t m + \text{div}(mu) &= 0, \\
\partial_t n + \text{div}(nu) &= 0, \\
\partial_t (mu) + \text{div}(mu \otimes u) + \nabla P(m, n) &= \mu \Delta u + (\mu + \lambda)\nabla \text{div} u, \\
(m, n, u)|_{t=0} &= (m_0, n_0, u_0)(x),
\end{align*}
\]

where $m = \alpha_l \rho_l$ and $n = \alpha_g \rho_g$ denote liquid mass and gas mass, respectively. The unknown variables $\alpha_l, \alpha_g \in [0, 1]$ denote liquid and gas volume fractions, respectively, satisfying the fundamental relation: $\alpha_l + \alpha_g = 1$. Furthermore, the other unknown variables $\rho_l$ and $\rho_g$ denote liquid and gas densities, respectively, satisfying.

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equations of state: 
\[ \rho_l = \rho_{l,0} + (P - P_{l,0})/a_l^2, \quad \rho_g = P/a_g^2, \] 
where \( a_l \) and \( a_g \) are sonic speeds, respectively, in liquid and gas; and \( P_{l,0} \) and \( \rho_{l,0} \) are, respectively, the reference pressure and density given as constants. \( \mathbf{u} \) denotes mixed velocity of liquid and gas, and \( P \) is common pressure for both phases, which satisfies
\[
P(m, n) = C_0 \left( -b(m, n) + \sqrt{b^2(m, n) + c(m, n)} \right)^2
\]
with \( b(m, n) = k_0 - m - a_0 n, \ c(m, n) = 4k_0a_0n, \ C_0 = a_l^2/2, \ k_0 = \rho_{l,0} - P_{l,0}/a_l^2 > 0 \) and \( a_0 = a_g^2/a_l^2 \). \( \mu \) and \( \lambda \) are the viscosity constants satisfying
\[ \mu > 0, \quad 2\mu + N\lambda \geq 0. \]

For the more detailed explanations about the above model, one can refer to [5, 17, 21, 23].

There are many results about the viscous liquid-gas two-fluid flow model. For the model (1) in 1D, where the liquid is incompressible and the gas is polytropic, the global well-posedness of weak solution to the free boundary value problem was studied in [11, 13, 25, 26]. Specifically, when both of the two fluids are compressible, one could refer to [12]. In [23], Yao, Zhang and Zhu obtained the existence of global weak solutions to the 2D model when the initial energy is small and there is no initial vacuum, which can be seen as a generalization of the results in [12] from 1D to multi-dimension. Later on, in a bounded domain, Wen, Yao, Zhang and Zhu [21, 24] proved the blow-up criterion in terms of the upper bound of the liquid mass for local strong solution to the 3D (or 2D) viscous liquid-gas two-phase flow model, in both cases when there is initial vacuum and no initial vacuum, respectively. In addition, the first author and collaborators [6] considered global well-posedness of the classical solutions for the 3D model when the initial data are only small in the energy-norm.

Let us mention that all of the above results were performed in the framework of Sobolev spaces. Inspired by [7] for the compressible Navier-Stokes equations, it is natural to study the system (1) in critical Besov spaces. We observe that system (1) is invariant by the transformation
\[
\tilde{m} = m(l^2t, lx), \quad \tilde{n} = n(l^2t, lx), \quad \tilde{\mathbf{u}} = \mathbf{l}u(l^2t, lx)
\]
up to a change of the pressure law \( \tilde{P} = l^2P \). A critical space is a space in which the norm is invariant under the scaling
\[
(\tilde{e}, \tilde{f}, \tilde{g})(x) = (e(lx), f(lx), l g(lx)).
\]
Recently, Hao and Li [14] studied the Cauchy problem (1) in Besov spaces based on the \( L^2 \) framework for all multi-dimensions. On one hand, they obtained the existence and uniqueness of the global strong solution for system (1) provided that the initial data are close to a constant equilibrium state. On the other hand, they also proved local well-posedness under the condition that initial data are slightly more regular.

Very recently, motivated by the papers [2, 3], Xu and Yuan [22] obtained the local well-posedness for large data in critical Besov spaces based on the general \( L^p \) framework, which is a generalization of \( p = 2 \) in [14]. Moreover, compared with the local well-posedness result in [14], the one in [22] dose not need slightly more regular for initial data.

The purpose of this paper is to further improve the range of the Lebesgue exponent \( p \) in [22], for which system (1) is locally well-posed under the same conditions for initial data as in [22]. More precisely, we relax the restriction on the Lebesgue
exponent $p$ in Besov spaces from $(1,N]$ to $(1,2N)$, which means that initial velocities in critical Besov spaces with negative indices generate a unique local solution.

Our main ideas come from the recent work dedicated to the compressible barotropic flow [8] and from the paper [9] concerning incompressible inhomogeneous fluids. Under the Lagrangian coordinates, the authors in [8, 9] studied the corresponding local well-posedness problem in the critical Besov spaces setting. In addition, using this method, Chikami and Danchin [4] considered the full compressible Navier-Stokes system in critical Besov spaces. They improved the range of Lebesgue exponent $p$ for which the system is locally well-posed, compared to [10]. Let us emphasize that this approach has already been successfully applied in the case of smooth data (see e.g. [15, 16, 19, 20]).

As pointed out in [4, 8, 9], the motivation behind introducing Lagrangian coordinates is to effectively eliminate the hyperbolic part of the system, provided that the system of liquid mass and gas mass becomes explicitly solvable once the flow of the velocity field has been determined. Meanwhile, the velocity equation in Lagrangian coordinates remains of parabolic type (at least for small enough time), and the Banach fixed point theorem turns out to be applicable for obtaining the existence and uniqueness of the solution in the same class of spaces as in the Eulerian framework.

We organize the rest of the paper as follows. In the next section, we introduce the compressible viscous liquid-gas two-phase flow model in Lagrangian coordinates and present our main results. In Section 3, we list some results about the Besov spaces and Lagrangian coordinates that may be found in [1, 8, 9]. Section 4 is devoted to the proofs of Theorem 2.1 and Theorem 2.2 by Banach fixed point theorem.

Throughout the paper, $C$ stands for a generic constant and we sometimes write $A \lesssim B$ as an equivalent to $A \leq CB$. For $X$ a Banach space, $p \in [1, +\infty]$ and $T > 0$, the notation $L_p^r(0,T;X)$ or $L_p^r(X)$ designates the set of measurable functions $f : [0,T] \to X$ with $t \mapsto \|f(t)\|_X$ in $L_p^r(0,T)$, endowed with the norm $\|f\|_{L_p^r(X)} := \|f\|\|X\|_{L_p^r(0,T)}$.

We agree that $C([0,T];X)$ denotes the set of continuous functions from $[0,T]$ to $X$.

2. Main results. Before introducing the Lagrangian system corresponding to (1), let us list some notational conventions.

For a $C^1$ function $F : \mathbb{R}^N \to \mathbb{R}^N \times \mathbb{R}^M$, we define $\text{div } F : \mathbb{R}^N \to \mathbb{R}^M$ by $(\text{div } F)^j := \sum_i \partial_i F_{ij}, \quad 1 \leq j \leq M.$

For $N \times N$ matrices $A = (a_{ij})_{1 \leq i,j \leq N}$ and $B = (b_{ij})_{1 \leq i,j \leq N}$, we define the trace product $A : B$ by $A : B = \text{tr} AB = \sum_{ij} a_{ij} b_{ji}$.

We denote by $\text{adj } (A)$ the adjugate matrix of $A$, i.e., the transpose of the cofactor matrix of $A$. Given some matrix $A$, we define the “twisted” deformation tensor and divergence operator (acting on vector fields $z$) by the formulae

$D_A(z) := \frac{1}{2} (Dz \cdot A + A^T \cdot \nabla z)$ and $\text{div } A z := A^T : \nabla z = Dz : A$.

Let $X$ be the flow associated to the vector-field $u$, i.e., the solution to

$X(t,y) = y + \int_0^t u(\tau, X(\tau,y))d\tau.$ (4)
Denote
\[
\overline{m}(t, y) := m(t, X(t, y)), \quad \overline{n}(t, y) := n(t, X(t, y)) \quad \text{and} \quad \overline{u}(t, y) := u(t, X(t, y))
\]
with \((m, n, u)\) a solution of (1). Setting \(J := \det DX\) and \(A := (D_y X)^{-1}\), using the chain rule and Proposition 6, one has that \((\overline{m}, \overline{n}, \overline{u})\) satisfies
\[
\begin{align*}
\partial_t (J\overline{m}) &= 0, \\
\partial_t (J\overline{n}) &= 0, \\
\partial_t \overline{u} - m_0^{-1} \text{div} \left[ \text{adj} (DX) \left( 2\mu D_A(\overline{u}) + \lambda \text{div} A \overline{u}I - P(\overline{m}, \overline{n})I \right) \right] &= 0,
\end{align*}
\]
with \((6)\) and \((\nabla f, \phi) = 1\) if \(\phi \neq 0\). The homogeneous frequency localization operator \(\hat{\Delta}_j\) and \(\hat{S}_j\) are defined by
\[
\hat{\Delta}_j u = \varphi(2^{-j}D) u, \quad \hat{S}_j u = \sum_{k \leq j-1} \hat{\Delta}_k u \quad \text{for} \quad j \in \mathbb{Z}.
\]
Let us denote the space \(\mathcal{Y}'(\mathbb{R}^N)\) by the quotient space of \(S'(\mathbb{R}^N)/\mathcal{P}\) with the polynomials space \(\mathcal{P}\). The formal equality \(u = \sum_{k \in \mathbb{Z}} \hat{\Delta}_k u\) holds true for \(u \in \mathcal{Y}'(\mathbb{R}^N)\) and is called the homogeneous Littlewood-Paley decomposition.

We then define, for \(s \in \mathbb{R}, 1 \leq p, r \leq +\infty\), the homogeneous Besov space
\[
\dot{B}_{p,r}^s = \{ f \in \mathcal{Y}'(\mathbb{R}^N) : \| f \|_{\dot{B}_{p,r}^s} < +\infty \},
\]
where
\[
\| f \|_{\dot{B}_{p,r}^s} := \| 2^{ks} \| \Delta_k f \|_{L_p} \|_{L_r}.
\]
Let \(\hat{\eta} \) and \(\hat{n} \) be two constants satisfying \(\hat{\eta} > 0\) and \(\hat{n} \geq 0\). We shall obtain the existence and uniqueness of a local-in-time solution \((\overline{m}, \overline{n}, \overline{u})\) for system \((5)\), with \(d := (\overline{m} - \hat{m})/\hat{m}, \quad \overline{e} := \overline{n} - \hat{n} \) in \(C([0, T]; \dot{B}_{p,1}^{\hat{N}})\) and \(\overline{u} \) in the space
\[
E_p(T) := \left\{ v \in C([0, T]; \dot{B}_{p,1}^{\frac{N}{p}-1}), \quad \partial_t v, \nabla^2 v \in L^1(0, T; \dot{B}_{p,1}^{\frac{N}{p}-1}) \right\}
\]
endowed with the norm
\[
\| v \|_{E_p(T)} := \left( \| v \|_{L_p^\infty(\dot{B}_{p,1}^{\frac{N}{p}-1})} + \| \partial_t v, \nabla^2 v \|_{L_p^1(\dot{B}_{p,1}^{\frac{N}{p}-1})} \right).
\]
Let us now state our main result.

**Theorem 2.1.** Let \(1 < p < 2N\) and \(N \geq 2\). Let \(\hat{\eta} > 0\) and \(\hat{n} \geq 0\) be two constants. Assume that the initial data satisfy
\[
(d_0, e_0, u_0) := \left( \frac{m_0 - \hat{m}}{\hat{m}}, n_0 - \hat{n}, u_0 \right) \in \dot{B}_{p,1}^{\hat{N}} \times \dot{B}_{p,1}^{\frac{N}{p}} \times (\dot{B}_{p,1}^{\frac{N}{p}-1})^N \quad \text{and} \quad \inf_{x \in \mathbb{R}^N} m_0(x) > 0.
\]
Then system (5) admits a unique local solution \((\bar{m}, \tilde{n}, \tilde{u})\) with \(\tilde{m}\) bounded away from zero, \(\check{d} := (\bar{m} - \check{n})/\check{m}, \tilde{m} := \bar{m} - \tilde{n}\) in \(C([0,T]; \dot{B}^N_{p,1})\) and \(\tilde{u}\) in the space \(E_p(T)\).

Moreover, the flow map \((d_0, \check{c}_0, \lambda_0) \mapsto (\check{d}, \check{c}, \lambda)\) is locally Lipschitz continuous from \((\dot{B}^N_{p,1})^2 \times \dot{B}^N_{p,1}^{-1}\) to \((C([0,T]; \dot{B}^N_{p,1}))^2 \times E_p(T)\).

In Eulerian coordinates, Theorem 2.1 means:

**Theorem 2.2.** Under the same assumptions as in Theorem 2.1 with \(1 < p < 2N\) and \(N \geq 2\), system (1) has a unique local solution \((m, n, u)\) with \(u \in E_p(T)\), \((m - \check{n}, n - \check{n}) \in (C([0,T]; \dot{B}^N_{p,1}))^2\) and \(m\) bounded away from 0.

**Remark 1.** In [22], the authors obtained the local existence and uniqueness of the solution to system (1) in critical Besov spaces based on the \(L^p\) framework. Here, Theorem 2.2 improves the range of \(p\) from \((1, N]\) to \((1, 2N)\) under the same conditions as in [22]. In addition, since \(u_0 \in (\dot{B}^N_{p,1})^N\) and \(1 < p < 2N\) by means of Theorem 2.2, we infer that initial velocities in critical Besov spaces with negative indices produce a unique local solution.

3. Preliminaries. In this section, we present some technical results that have been used repeatedly in the next content.

3.1. Linear parabolic systems with rough coefficients. The proof of the main results in this paper is based on the estimates that have been established recently in [8] for the following Lamé system with nonsmooth coefficients:

\[
\partial_t u - 2a \operatorname{div} (\mu D(u)) - b \nabla (\lambda \operatorname{div} u) = f \tag{8}
\]

(8)

Here both \(u\) and \(f\) are valued in \(\mathbb{R}^N\) when the following uniform ellipticity condition is satisfied:

\[
\alpha := \min \left\{ \inf_{(t,x) \in [0,T] \times \mathbb{R}^N} \alpha \mu(t, x), \inf_{(t,x) \in [0,T] \times \mathbb{R}^N} (2a \mu + b \lambda)(t, x) \right\} > 0. \tag{9}
\]

**Proposition 1.** (see [8]) Let \(a, b, \lambda\) and \(\mu\) be bounded functions satisfying (9).

Assume that \(a \check{\mu}, b \check{\lambda}, \lambda \check{\mu}\) and \(\lambda \check{\lambda}\) are in \(L^\infty (0, T; \dot{B}^N_{p,1}^{-1})\) for some \(1 < p < +\infty\), and that there exist some constants \(\check{a}, \check{b}, \check{\lambda}, \check{\mu}\) and \(\check{\mu}\) satisfying

\[
2a \check{\mu} + b \check{\lambda} > 0 \quad \text{and} \quad a \check{\mu}, b \check{\lambda}, \lambda \check{\mu}, \lambda \check{\lambda} > 0,
\]

and such that \(a - \check{a}, b - \check{b}, \check{\lambda} - \check{\lambda}\) and \(\mu - \check{\mu}\) are in \(C([0,T]; \dot{B}^N_{p,1}^{-1})\). Finally, suppose that

\[
\lim_{q \to +\infty} \left\| (\text{Id} - \hat{S}_q)(a \nabla \mu, b \nabla \lambda, \mu \nabla a, \lambda \nabla b) \right\|_{L^p(T; \dot{B}^N_{p,1}^{-1})} = 0.
\]

Then for any data \(u_0 \in \dot{B}^N_{p,1}^{-1}\) and \(f \in L^1 (0, T; \dot{B}^N_{p,1}^{-1})\), system (8) admits a unique solution \(u \in C([0,T]; \dot{B}^N_{p,1}^{-1})\) with \(\nabla u \in L^1 (0, T; \dot{B}^N_{p,1})\).

Furthermore, there exist two constants \(\eta\) and \(C\) such that if \(q\) is so large as to satisfy

\[
\min \left\{ \inf_{(t,x) \in [0,T] \times \mathbb{R}^N} \hat{S}_q(a \mu)(t,x), \inf_{(t,x) \in [0,T] \times \mathbb{R}^N} \hat{S}_q(2a \mu + b \lambda)(t,x) \right\} \geq \frac{\alpha}{2},
\]

\[
\left\| (\text{Id} - \hat{S}_q)(a \nabla \mu, b \nabla \lambda, \mu \nabla a, \lambda \nabla b) \right\|_{L^p(T; \dot{B}^N_{p,1}^{-1})} \leq \eta \alpha.
\]

(10)
then we have for all \( t \in [0, T] \),
\[
\|u\|_{L^\infty_t(B_{p,1}^s)} + \alpha\|u\|_{L^1_t(B_{p,1}^{\infty})} \\
\leq C\left(\|u_0\|_{B_{p,1}^s} + \|f\|_{L^1_t(B_{p,1}^{\infty})}\right) \exp\left(\frac{C}{\alpha} \int_0^t \|\dot{S}_t(a\nabla \mu, b\nabla \lambda, \mu \nabla a, \lambda \nabla b)\|_{B_{p,1}^{N}} d\tau\right).
\]

3.2. Estimates for product and composition. For the proofs of the following propositions, one can see [1, 7, 18], and Appendix of [8, 9].

Proposition 2. Let \( \nu \geq 0 \) and \( -\min\{\frac{N}{p}, \frac{N}{p'}\} < \sigma \leq \frac{N}{p} - \nu \). The following product law holds:
\[
\|uv\|_{B_{p,1}^{s}} \leq C\|u\|_{B_{p,1}^{s}}\|v\|_{B_{p,1}^{s+\nu}}.
\]

Proposition 3. Let \( s > 0, p \in [1, \infty] \) and \( a \in \dot{B}_{p,1}^{s} \cap L^\infty \). Let \( F \in W^{1,2}_1(\mathbb{R}^N) \) such that \( F(0) = 0 \). Then \( F(a) \in \dot{B}_{p,1}^{s} \) and there exists a constant \( C = C(s, p, N, F, \|a\|_{L^\infty}) \) such that
\[
\|F(a)\|_{\dot{B}_{p,1}^{s}} \leq C\|a\|_{\dot{B}_{p,1}^{s}}.
\]

Proposition 4. If \( a_1 \) and \( a_2 \) belong to \( \dot{B}_{2,1}^{N} \), \( (a_2 - a_1) \in \dot{B}_{2,1}^{s} \) for \( s \in (-\frac{N}{p}, \frac{N}{p}) \) and \( G \in W^{1,3,\infty}_1(\mathbb{R}^N) \) satisfies \( G'(0) = 0 \), then \( G(a_2) - G(a_1) \) belongs to \( \dot{B}_{2,1}^{s} \) and there exists a function of two variables \( C \) depending only on \( s, N \) and \( G \), and such that
\[
\|G(a_2) - G(a_1)\|_{\dot{B}_{2,1}^{s}} \leq C(\|a_1\|_{L^\infty}, \|a_2\|_{L^\infty})\left(\|a_1\|_{\dot{B}_{2,1}^{s}} + \|a_2\|_{\dot{B}_{2,1}^{s}}\right)\|a_2 - a_1\|_{\dot{B}_{2,1}^{s}}.
\]

3.3. Lagrangian coordinates. Let \( X \) be a \( C^1 \)-diffeomorphism over \( \mathbb{R}^N \). For a vector-valued function \( H: \mathbb{R}^N \to \mathbb{R}^M \), denote \( \overline{H}(y) := H(x) \) with \( x = X(y) \). The chain rule states that
\[
D_y\overline{H}(y) = D_xH(X(y)) \cdot D_yX(y) \quad \text{and} \quad \nabla_y\overline{H}(y) = \nabla_yX(y) \cdot \nabla_xH(X(y)).
\]

Hence, setting \( A(y) = (D_yX(y))^{-1} = D_xX^{-1}(X(y)) \), we have
\[
D_xH(X(y)) = D_y\overline{H}(y) \cdot A(y) \quad \text{and} \quad \nabla_xH(X(y)) = A^T(y) \cdot \nabla_y\overline{H}(y).
\]

Proposition 5. (See [8, 9]) Let \( X \) be a globally bi-Lipschitz diffeomorphism of \( \mathbb{R}^N \) and \( (s, p) \) with \( 1 \leq p < \infty \) and \(-\frac{N}{p'} < s \leq \frac{N}{p} \). Then \( a \mapsto a \circ X \) is a self-map over \( \dot{B}_{p,1}^{s} \) in the following cases:

(1) \( s \in (0, 1) \),
(2) \( s \in (-1, 0) \) and \( J_{X^{-1}} \) is in the multiplier space \( \mathcal{M}(\dot{B}_{p,1}^{-s}) \),
(3) \( s \geq 1 \) and \( (DX - \operatorname{Id}) \in \dot{B}_{p,1}^{N} \).

Here, the multiplier norm \( \mathcal{M}(\dot{B}_{p,1}^{s}) \) for \( \dot{B}_{p,1}^{s} \), is defined by
\[
\|f\|_{\mathcal{M}(\dot{B}_{p,1}^{s})} := \sup \|\psi f\|_{\dot{B}_{p,1}^{s}}.
\]

The supremum is taken over those functions \( \psi \) in \( \dot{B}_{p,1}^{s} \) with norm 1.

Proposition 6. (See [8, 9]) Let \( K \) be a \( C^1 \)-scalar function over \( \mathbb{R}^N \) and \( H \) be a \( C^1 \)-vector field. If \( X \) is a \( C^1 \)-diffeomorphism such that \( J := \det(D_yX) > 0 \), then
\[
\nabla_xK = J^{-1} \text{div}(\text{adj}(D_yX)K),
\]
\[
\overline{\text{div}}_x \overline{H} = J^{-1} \text{div}_y (\text{adj} (D_y X) \overline{H}),
\]

where \(\text{adj} (D_y X)\) is the adjugate matrix of \(D_y X\).

From Proposition 6, we can obtain the following relations:

\[
\Delta_x \mathbf{u} = J^{-1} \text{div}_y (\text{adj} (D_y X) \nabla_x \mathbf{u}) = J^{-1} \text{div}_y (\text{adj} (D_y X) A^T \nabla_y \mathbf{u}), \quad (14)
\]

\[
\nabla_x \text{div}_x \mathbf{u} = J^{-1} \text{div}_y (\text{adj} (D_y X) \text{div}_x \mathbf{u}) = J^{-1} \text{div}_y (\text{adj} (D_y X) A^T : \nabla_y \mathbf{u}), \quad (15)
\]

\[
\nabla_x \overline{P} = J^{-1} \text{div}_y (\text{adj} (D_y X) \overline{P}). \quad (16)
\]

### 3.4. Estimates of flow.

In the following, we recall the flow estimates that have been proved in [8, 9].

**Proposition 7.** Let \(p \in [1, +\infty)\) and \(\mathbf{v} \in E_p(T)\). There exists a positive constant \(c\) (independent of \(T\)) such that if

\[
\int_0^T \| D\mathbf{v} \|_{B^1_{p,1}} \ dt \leq c \quad (17)
\]

then for all \(t \in [0, T]\), we have

\[
\| \text{Id} - \text{adj} (D_X (t)) \|_{B^1_{p,1}} \lesssim \| D\mathbf{v} \|_{L^1(\dot{B}^1_{p,1})},
\]

\[
\| \text{Id} - A_v(t) \|_{B^1_{p,1}} \lesssim \| D\mathbf{v} \|_{L^1(\dot{B}^1_{p,1})},
\]

and

\[
\| J_v^{\pm 1}(t) - 1 \|_{B^1_{p,1}} \lesssim \| D\mathbf{v} \|_{L^1(\dot{B}^1_{p,1})}.
\]

Furthermore, if \(\mathbf{w}\) is a vector field such that \(D\mathbf{w} \in L^1(0, T; \dot{B}^0_{p,1})\) then

\[
\| (\text{adj} (D_X (t)) D_{A_v}(\mathbf{w}) - D(\mathbf{w})) (t) \|_{B^0_{p,1}} \lesssim \| D\mathbf{v} \|_{L^1(\dot{B}^1_{p,1})} \| D\mathbf{w} \|_{L^1(\dot{B}^1_{p,1})},
\]

and

\[
\| (\text{adj} (D_X (t)) \text{div}_{A_v}(\mathbf{w}) - \text{div}\mathbf{w} \text{Id}) (t) \|_{B^0_{p,1}} \lesssim \| D\mathbf{v} \|_{L^1(\dot{B}^1_{p,1})} \| D\mathbf{w} \|_{L^1(\dot{B}^1_{p,1})}.
\]

**Proposition 8.** Let \(\mathbf{v}_1\) and \(\mathbf{v}_2\) be two vector-fields satisfying (17), and \(\delta \mathbf{v} := \mathbf{v}_2 - \mathbf{v}_1\). Then for all \(p \in [1, +\infty)\) and all \(t \in [0, T]\), we have

\[
\| A_{v_2}(t) - A_{v_1}(t) \|_{B^1_{p,1}} \lesssim \| D\delta \mathbf{v} \|_{L^1(\dot{B}^1_{p,1})},
\]

\[
\| \text{adj} (D_X v_2(t)) - \text{adj} (D_X v_1(t)) \|_{B^0_{p,1}} \lesssim \| D\delta \mathbf{v} \|_{L^1(\dot{B}^1_{p,1})},
\]

and

\[
\| J^{\pm 1}_{v_2}(t) - J^{\pm 1}_{v_1}(t) \|_{B^1_{p,1}} \lesssim \| D\delta \mathbf{v} \|_{L^1(\dot{B}^1_{p,1})}.
\]
4. **Proof of the main theorem.** In this section, with the help of Proposition 1, we give the proofs of Theorem 2.1 and Theorem 2.2 through Banach fixed theorem. From now on, to simplify the notation, we drop the bars of the Lagrangian coordinates. Denoting

\[ L_{m_0} u := \partial_t u - m_0^{-1} \text{div} \left( 2\mu D(u) + \lambda \text{div} u \text{Id} \right), \]

system (5) thus writes

\[ L_{m_0} u = 2\mu m_0^{-1} \text{div} I_1(u, u) + \lambda m_0^{-1} \text{div} I_2(u, u) - m_0^{-1} \text{div} I_3(u), \]

with

\[
\begin{align*}
I_1(v, w) &:= \text{adj} (DX_v) D_{A_v} (w) - D(w), \\
I_2(v, w) &:= \text{adj} (DX_v) \text{div}_{A_v} w - \text{div} w \text{Id}, \\
I_3(v) &:= \text{adj} (DX_v) P(J_v^{-1} m_0, J_v^{-1} n_0),
\end{align*}
\]

where we have used that \( m = J_v^{-1} m_0 \) and \( n = J_v^{-1} n_0 \).

In order to solve (5) locally, it suffices to show that the map

\[ \Phi : v \mapsto u \]

with \( u \) the solution to

\[ L_{m_0} u = 2\mu m_0^{-1} \text{div} I_1(v, v) + \lambda m_0^{-1} \text{div} I_2(v, v) - m_0^{-1} \text{div} I_3(v) \]

has a fixed point in \( E_p(T) \) for small enough \( T \).

Let \( u_L \) be the solution to the linear system corresponding to (18) with \( m_0 \equiv 1 \), i.e.,

\[ L_1 u_L = 0, \quad u_L |_{t=0} = u_0. \]

### 4.1. Proof of Theorem 2.1.** We claim that the Banach fixed point theorem applies to the map \( \Phi \) defined in (21) in some closed ball \( \overline{B}_{E_p(T)}(u_L, R) \) with suitably small \( T \) and \( R \). Denoting \( \tilde{u} := u - u_L \), we obtain that \( \tilde{u} \) satisfy

\[
\begin{align*}
L_{m_0} \tilde{u} &= (L_1 - L_{m_0}) u_L + m_0^{-1} \text{div} (2\mu I_1(v, v) + \lambda I_2(v, v) - I_3(v)), \\
\tilde{u} |_{t=0} &= 0.
\end{align*}
\]

Since (7) is fulfilled, we set

\[
\alpha := \min \left\{ \inf_{x \in \mathbb{R}^N} \frac{\mu}{m_0}, \inf_{x \in \mathbb{R}^N} \left( \frac{2\mu}{m_0} + \frac{\lambda}{m_0} \right) \right\}.
\]

Because the space \( \dot{B}^s_{p,1} \) embeds in the set of bounded continuous functions, then there exists some \( q \in \mathbb{Z} \) so that

\[
\min \left\{ \inf_{x \in \mathbb{R}^N} \dot{S}_q \left( \frac{\mu}{m_0} \right), \inf_{x \in \mathbb{R}^N} \dot{S}_q \left( \frac{2\mu}{m_0} + \frac{\lambda}{m_0} \right) \right\} \geq \frac{\alpha}{2},
\]

and

\[
\left\| (\text{Id} - \dot{S}_q) \left( \frac{\mu}{m_0} \nabla m_0, \frac{\lambda}{m_0} \nabla m_0 \right) \right\|_{\dot{B}^s_{p,1}} \leq \eta \alpha.
\]

In order to prove Theorem 2.1, due to Proposition 1, it suffices to study the right hand side of (23).**

**First step.** Stability of the ball \( \overline{B}_{E_p(T)}(u_L, R) \) for small enough \( R \) and \( T \).
In what follows, we suppose that for a small enough $c$, it holds that
\[
\|Dv\|_{L^1_t(B_{p,1}^{\frac{N}{2}})} \leq c. \tag{24}
\]

Proposition 1 and the definition of the multiplier space $\mathcal{M}(\hat{B}_{p,1}^{\frac{N}{2}-1})$ (see (13)) yield that
\[
\|\tilde{u}\|_{E_{p}(T)} \leq C e^{C_{m_0,q} T} \left[\|(L_1 - L_{m_0})u_L\|_{L^1_t(B_{p,1}^{\frac{N}{2}-1})} + \|m_0^{-1}\|_{\mathcal{M}(B_{p,1}^{\frac{N}{2}-1})} \left(\|I_1(v, v)\|_{L^1_t(B_{p,1}^{\frac{N}{2}})} + \|I_2(v, v)\|_{L^1_t(B_{p,1}^{\frac{N}{2}})} + \|I_3(v)\|_{L^1_t(B_{p,1}^{\frac{N}{2}})}\right)^{\frac{1}{p'}}\right] \tag{25}
\]

for some constant $C_{m_0,q}$ depending only on $m_0$ and $q$. Owing to the definition of $\mathcal{M}(\hat{B}_{p,1}^{\frac{N}{2}-1})$ and product estimates, we have the fact that $m_0^{-1}$ belongs to $\mathcal{M}(B_{p,1}^{\frac{N}{2}-1})$ and
\[
\|m_0^{-1}g\|_{B_{p,1}^{\frac{N}{2}-1}} = \left\| \frac{1}{m} \left(1 - \frac{d_0}{1 + d_0}\right)g \right\|_{B_{p,1}^{\frac{N}{2}-1}} \leq \frac{1}{m} \left(1 + \|d_0\|_{B_{2,1}^{\frac{N}{2}}}ight) \|g\|_{B_{p,1}^{\frac{N}{2}-1}}.
\]

In order to deal with the first term of the r.h.s. of (25), it follows from
\[
(L_1 - L_{m_0})u_L = (m_0^{-1} - 1)\text{div} \left(2\mu D(u_L) + \lambda \text{div} u_L \text{Id}\right)
\]
and composition inequalities (11) and (12) that
\[
\|(L_1 - L_{m_0})u_L\|_{L^1_t(B_{p,1}^{\frac{N}{2}-1})} \lesssim (1 + \|d_0\|_{B_{2,1}^{\frac{N}{2}}}) \|Dv\|_{L^1_t(B_{p,1}^{\frac{N}{2}})}.
\]

Terms $I_1$ and $I_2$ have been bounded in [8] as follows:
\[
\|I_1(v, w)\|_{L^1_t(B_{p,1}^{\frac{N}{2}})} + \|I_2(v, w)\|_{L^1_t(B_{p,1}^{\frac{N}{2}})} \lesssim \|Dv\|_{L^1_t(B_{p,1}^{\frac{N}{2}})} \|Dw\|_{L^1_t(B_{p,1}^{\frac{N}{2}})}.
\]

Next, we focus on the estimate of $I_3(v)$. Recall that the pressure is given by
\[
P(m, n) = C_0 \left(-b(m, n) + \sqrt{b^2(m, n) + c(m, n)}\right)
\]
\[
= C_0 \left(-k_0 + m + a_0 n + \sqrt{(k_0 - m - a_0 n)^2 + 4k_0 a_0 n}\right).
\]

Then, $I_3(v)$ is written as
\[
I_3(v) = \text{adj}(DX_v)P(J_v^{-1}m_0, J_v^{-1}n_0)
\]
\[
= C_0 \text{adj}(DX_v)\left(-k_0 + J_v^{-1}m_0 + a_0 J_v^{-1}n_0\right) + C_0 \text{adj}(DX_v)B(J_v^{-1}m_0, J_v^{-1}n_0)
\]
\[
:= K_1 + K_2, \tag{26}
\]

where
\[
B(J_v^{-1}m_0, J_v^{-1}n_0) := \sqrt{(k_0 - J_v^{-1}m_0 - a_0 J_v^{-1}n_0)^2 + 4k_0 a_0 J_v^{-1}n_0}. \tag{27}
\]
Attention is now focused on bounding $K_1$ and $K_2$. Owing to Proposition 7, (24) and composition inequalities (11) and (12), it follows that

$$
\|K_1\|_{L^1_T(L^\infty_x(B_{p,1}^0))} \lesssim T \left(1 + \|Dv\|_{L^1_T(L^\infty_x(B_{p,1}^0))} \right) \left(1 + \|(d_0,e_0)\|_{B_{p,1}^0} \right),
$$

and

$$
\|K_2\|_{L^1_T(L^\infty_x(B_{p,1}^0))} \lesssim \|\text{adj} (DX_v) (B(J_v^{-1}m_0, J_v^{-1}n_0) - B(0,0))\|_{L^1_T(B_{p,1}^0)}
+ \|B(0,0)\text{adj} (DX_v)\|_{L^1_T(B_{p,1}^0)}
\lesssim \|\text{adj} (DX_v) (B(J_v^{-1}m_0, J_v^{-1}n_0) - B(J_v^{-1}m_0, 0))\|_{L^1_T(B_{p,1}^0)}
+ \|\text{adj} (DX_v) (B(0,0) - B(0,0))\|_{L^1_T(B_{p,1}^0)}
+ \|B(0,0)\text{adj} (DX_v)\|_{L^1_T(B_{p,1}^0)}
\lesssim T \left(1 + \|Dv\|_{L^1_T(B_{p,1}^0)} \right) \left(1 + \|(d_0,e_0)\|_{B_{p,1}^0} \right).
$$

Thus one arrives at

$$
\|I_3(v)\|_{L^1_T(L^\infty_x(B_{p,1}^0))} \lesssim T \left(1 + \|Dv\|_{L^1_T(L^\infty_x(B_{p,1}^0))} \right) \left(1 + \|(d_0,e_0)\|_{B_{p,1}^0} \right).
$$

Putting all the above information together, we conclude that

$$
\|\bar{u}\|_{E_p(T)} \leq Ce^{C_{m_0,q}T} \left(1 + \|(d_0,e_0)\|_{B_{p,1}^0} \right)^2
\times \left[\|Du_L\|_{L^1_T(B_{p,1}^0)} + \|Dv\|_{L^1_T(B_{p,1}^0)}^2 + T \left(1 + \|Dv\|_{L^1_T(B_{p,1}^0)} \right) \right].
$$

Since $v$ belongs to the ball $\overline{B}_{E_p(T)}(u_L,R)$, decomposing $v$ into $\bar{v} + u_L$ and keeping in mind that $v$ satisfies (24), we have

$$
\|\bar{u}\|_{E_p(T)} \leq Ce^{C_{m_0,q}T} \left(1 + \|(d_0,e_0)\|_{B_{p,1}^0} \right)^2
\times \left(\|Du_L\|_{L^1_T(B_{p,1}^0)} + \|Du_L\|_{L^1_T(B_{p,1}^0)}^2 + R^2 + T \right).
$$

Therefore, if we first choose $R$ so that for a small enough constant $\eta$,

$$
2C \left(1 + \|(d_0,e_0)\|_{B_{p,1}^0} \right)^2 R \leq \eta
$$

and then take $T$ so that

$$
C_{m_0,q}T \leq \log 2, \quad T \leq R^2, \quad \|Du_L\|_{L^1_T(B_{p,1}^0)} \leq R, \quad \|Du_L\|_{L^1_T(B_{p,1}^0)} \leq R^2,
$$

(29)
then we may conclude that $\Phi$ maps $\overline{B}_{E_p(T)}(u_L, R)$ into itself.

**Second step.** Contraction estimates.

We consider two vector-fields $v^1$ and $v^2$ in $\overline{B}_{E_p(T)}(u_L, R)$ and set $u^1 := \Phi(v^1)$ and $u^2 := \Phi(v^2)$. Let $\delta u := u^2 - u^1$ and $\delta v := v^2 - v^1$. To simplify the notation, we set $X_i := X_{\psi_i}$, $A_i := A_{\psi_i}$ and $J_i := J_{\psi_i}$.

To prove that $\Phi$ is contractive, it is mainly a matter of applying Proposition 1 to

$$L_{m_0} \delta u = 2\mu m_0^{-1} \text{div} \left( I_1(v^2, v^2) - I_1(v^1, v^1) \right)
+ \lambda m_0^{-1} \text{div} \left( I_2(v^2, v^2) - I_2(v^1, v^1) \right) - m_0^{-1} \text{div} \left( I_3(v^2) - I_3(v^1) \right).$$

Then one gets, provided that $C_{m_0,q} T \leq \log 2$,

$$\|\delta u\|_{E_p(T)} \leq C \left( 1 + \|d_0\|_{\tilde{B}_{p,1}^\infty} \right) \left( \|I_1(v^2, v^2) - I_1(v^1, v^1)\|_{L^1_p(B_{p,1}^\infty)}
+ \|I_2(v^2, v^2) - I_2(v^1, v^1)\|_{L^1_p(B_{p,1}^\infty)} + \|I_3(v^2) - I_3(v^1)\|_{L^1_p(B_{p,1}^\infty)} \right).$$

The first two terms of the r.h.s. of (30) have been estimated in [8], then we list as follows:

$$\|I_j(v^2, v^2) - I_j(v^1, v^1)\|_{L^1_p(B_{p,1}^\infty)} \leq C m_0 \|(Dv^1, Dv^2)\|_{L^1_p(B_{p,1}^\infty)} \|D\delta v\|_{L^1_p(B_{p,1}^\infty)}$$

for $j = 1, 2$. As for the last term of (30), a straightforward calculation based on Proposition 8 ensures that

$$\|I_3(v^2) - I_3(v^1)\|_{L^1_p(B_{p,1}^\infty)} \leq C \left( 1 + \|d_0\|_{\tilde{B}_{p,1}^\infty} \right) T \|D\delta v\|_{L^1_p(B_{p,1}^\infty)}.$$

Indeed, recall that

$$I_3(v) = \text{adj}(DX_\psi) P(J^{-1}_{\psi}m_0, J^{-1}_{\psi}n_0),$$

therefore,

$$I_3(v^2) - I_3(v^1) = (\text{adj}(DX_2) - \text{adj}(DX_1)) P(J^{-1}_{2}m_0, J^{-1}_{2}n_0)
+ \text{adj}(DX_1) \left[ P(J^{-1}_{2}m_0, J^{-1}_{2}n_0) - P(J^{-1}_{1}m_0, J^{-1}_{1}n_0) \right]
:= H_1 + H_2.$$ To bound $H_1$, the way to deal with $P(J^{-1}_{2}m_0, J^{-1}_{2}n_0)$ is in accordance with (26).

Then it follows from Proposition 8 that

$$\|H_1\|_{L^1_p(B_{p,1}^\infty)} \leq T \|D\delta v\|_{L^1_p(B_{p,1}^\infty)} \left( 1 + \|d_0\|_{\tilde{B}_{p,1}^\infty} \right).$$

In order to deal with $H_2$, let us first notice that

$$P(J^{-1}_{2}m_0, J^{-1}_{2}n_0) - P(J^{-1}_{1}m_0, J^{-1}_{1}n_0)
= (J^{-1} - J^{-1}_1)m_0 + a_0(J^{-1}_2 - J^{-1}_1)n_0$$
Applying Proposition 8, composition inequalities (11) and (12) again yields
\[
\|H_2\|_{L^p(B_{p,1}^\infty)} \leq CT\|D\delta v\|_{L^p(B_{p,1}^\infty)} \left(1 + \| (d_0, e_0) \|_{B_{p,1}^\infty} \right).
\]
Hence
\[
\|J_3(v^2) - I_3(v^1)\|_{L^p(B_{p,1}^\infty)} \leq C \left(1 + \| (d_0, e_0) \|_{B_{p,1}^\infty} \right) T\|D\delta v\|_{L^p(B_{p,1}^\infty)}.
\]
Finally, we get
\[
\|\delta u\|_{E_p(T)} \leq C \left(1 + \| (d_0, e_0) \|_{B_{p,1}^\infty} \right)^2 \left(T + \| (Dv^1, Dv^2)\|_{L^p(B_{p,1}^\infty)} \right)\|D\delta v\|_{L^p(B_{p,1}^\infty)}.
\]
Given that \(v^1\) and \(v^2\) are in \(\overline{B}_{E_p(T)}(u_L, R)\) with suitably small \(T\) and \(R\), we end up with
\[
\|\delta u\|_{E_p(T)} \leq \frac{1}{2}\|\delta v\|_{E_p(T)}.
\]
One can thus conclude that \(\Phi\) admits a unique fixed point in \(\overline{B}_{E_p(T)}(u_L, R)\).

**Third step.** Regularity of liquid mass and gas mass.

 Granted with the above velocity field \(u\) in \(E_p(T)\), we set \(m := J_u^{-1}m_0\) and \(n := J_u^{-1}n_0\). By construction, the triplet \((m, n, u)\) satisfies system (5). In order to prove that \(d := (m - \hat{m})/\hat{m}\) and \(e := n - \hat{n}\) are in \(C([0, T]; \hat{B}_{p,1}^\infty)\), we use the fact that
\[
d = (J_u^{-1} - 1)d_0 + d_0 + (J_u^{-1} - 1)\quad \text{and} \quad e = (J_u^{-1} - 1)e_0 + e_0 + (J_u^{-1} - 1).
\]
Given Proposition 7 and the fact that \(Du \in L^1(0, T; \hat{B}_{p,1}^\infty)\), it is clear that \((J_u^{-1} - 1)\) belongs to \(C([0, T]; \hat{B}_{p,1}^\infty)\). Hence both \(d\) and \(e\) belong to \(C([0, T]; \hat{B}_{p,1}^\infty)\) too. Because \(\hat{B}_{p,1}^\infty\) is continuously embedded in \(L^\infty\), condition \(\inf_T m > 0\) is fulfilled on \([0, T]\) if needed.

**Last step.** Uniqueness and continuity of the flow map.

Consider two triplets \((m_0^1, n_0^1, u_0^1)\) and \((m_0^2, n_0^2, u_0^2)\) of data satisfying the assumptions of Theorem 2.1. Let \((m^1, n^1, u^1)\) and \((m^2, n^2, u^2)\) be two solutions in \(E_p(T)\) corresponding to those data. Setting \(\delta u := u^2 - u^1\), we thus get
\[
L_{m_0^1}^1\delta u = (L_{m_0^1} - L_{m_0^2})u^2 + (L_{m_0^1}^2 u^2 - L_{m_0^1} u^1) := (L_{m_0^1} - L_{m_0^2})u^2 + G_1 + G_2 + G_3,
\]
where
\[
G_1 := (m_0^2)^{-1}\text{div}\left[2\mu(I_1(u^2, u^2) - I_1(u^1, u^1)) + \lambda(I_2(u^2, u^2) - I_2(u^1, u^1)) \right],
\]
\[
G_2 := ((m_0^2)^{-1} - (m_0^1)^{-1})\text{div}\left[2\mu I_1(u^1, u^1) + \lambda I_2(u^1, u^1) - I_3^2(u^1)\right],
\]
\[
G_3 := -(m_0^1)^{-1}\text{div}\left[I_3^2(u^1) - I_3^3(u^1)\right] \quad .
\]
Here, \( I_1 \) and \( I_2 \) correspond to the quantities that have been defined previously in (20), and
\[
I_1^1(v) := \text{adj}(DX_v) P(J_{v^{-1}} m_0^1, J_{v^{-1}} n_0^1)
\]
and
\[
I_1^2(v) := \text{adj}(DX_v) P(J_{v^{-1}} m_0^2, J_{v^{-1}} n_0^2).
\]
Applying Proposition 1 to (31), we can estimate each term on the r.h.s. of (31), exactly as in the estimates of the second step. Here, for the first three terms on the r.h.s. of (31), we only list the estimates as follows (see also [8]):
\[
\| (L_{m_0^1} - L_{m_0^2}) u \|_{L_T^1(B_{p_1}^{\frac{N}{2}-1})} \leq C_{m_0^1,m_0^2} \| \delta d_0 \|_{B_{p_1}^{\frac{N}{2}}} \| Du \|_{L_T^1(B_{p_1}^{\frac{N}{2}})},
\]
\[
\| G_1 \|_{L_T^1(B_{p_1}^{\frac{N}{2}-1})} \leq C_{m_0^1,m_0^2} \left( T + \| (Du^1, Du^2) \|_{L_T^1(B_{p_1}^{\frac{N}{2}})} \right) \| \delta u \|_{L_T^1(B_{p_1}^{\frac{N}{2}})}.
\]
\[
\| G_2 \|_{L_T^1(B_{p_1}^{\frac{N}{2}-1})} \leq C_{m_0^1,m_0^2} \left( T + \| Du^1 \|_{L_T^1(B_{p_1}^{\frac{N}{2}})} \right) \| \delta m_0 \|_{L_T^1(B_{p_1}^{\frac{N}{2}})}.
\]
To deal with term \( G_3 \), we first recall that
\[
I_3^1(u^1) - I_3^1(u^1) = \text{adj}(DX_{u^1}) \left[ P(J_{u^1}^{-1} m_0^2, J_{u^1}^{-1} n_0^2) - P(J_{u^1}^{-1} m_0^1, J_{u^1}^{-1} n_0^1) \right]
\]
\[
= C_0 \text{adj}(DX_{u^1}) \left[ J_{u^1}^{-1} (m_0^2 - m_0^1) + a_0 J_{u^1}^{-1} (n_0^2 - n_0^1) \right]
\]
\[
+ C_0 \text{adj}(DX_{u^1}) \left[ B(J_{u^1}^{-1} m_0^2, J_{u^1}^{-1} n_0^2) - B(J_{u^1}^{-1} m_0^1, J_{u^1}^{-1} n_0^1) \right]
\]
\[
+ C_0 \text{adj}(DX_{u^1}) \left[ B(J_{u^1}^{-1} m_0^2, J_{u^1}^{-1} n_0^1) - B(J_{u^1}^{-1} m_0^1, J_{u^1}^{-1} n_0^1) \right].
\]
Then, combining composition, flow and product estimates yields
\[
\| G_3 \|_{L_T^1(B_{p_1}^{\frac{N}{2}-1})} \leq C_{m_0^1,m_0^2} T \| \delta m_0, \delta n_0 \|_{B_{p_1}^{\frac{N}{2}}} \left( 1 + \| Du^1 \|_{L_T^1(B_{p_1}^{\frac{N}{2}})} \right).
\]
Collecting all the above estimates, we conclude that for \( t \leq T \),
\[
\| \delta u \|_{E_p(t)} \leq C_{m_0^1,m_0^2} \left[ T + \| (Du^1, Du^2) \|_{L_T^1(B_{p_1}^{\frac{N}{2}})} \right] \| \delta u \|_{L_T^1(B_{p_1}^{\frac{N}{2}})}
\]
\[
+ \| \delta u_0 \|_{B_{p_1}^{\frac{N}{2}-1}} + \| (\delta m_0, \delta n_0) \|_{B_{p_1}^{\frac{N}{2}}} \left( T + \| (Du^1, Du^2) \|_{L_T^1(B_{p_1}^{\frac{N}{2}})} \right).
\]
As in [8], the constant \( C_{m_0^1,m_0^2} \) depends only on \( m_0^1 \) through its norm, for the integer \( q \) used in Proposition 1 corresponds to \( m_0^1 \) only. So if \( \delta m_0 \) is small enough then the above inequality implies that
\[
\| \delta u \|_{E_p(t)} \leq C_{m_0^1} \left[ T + \| Du^1 \|_{L_T^1(B_{p_1}^{\frac{N}{2}})} + \| \delta u \|_{E_p(t)} \right] \| \delta u \|_{E_p(t)}
\]
\[
+ \| \delta u_0 \|_{B_{p_1}^{\frac{N}{2}-1}} + \| (\delta m_0, \delta n_0) \|_{B_{p_1}^{\frac{N}{2}}} \left( T + \| Du^1 \|_{L_T^1(B_{p_1}^{\frac{N}{2}})} + \| \delta u \|_{E_p(t)} \right).
\]
Since \(\|Du^1\|_{L^1_t(B^{n+1}_p)} \to 0\) as \(t \to 0\), a bootstrap argument yields that, for small enough \(t, \delta u_0\) and \((\delta n_0, \delta n_0)\),

\[
\|\delta u\|_{E^p(t)} \leq 2C_m \left( \|\delta u_0\|_{B^{n+1}_p} + \|\delta n_0\|_{\dot{B}^{n+1}_p} \right).
\]

Concerning the liquid mass and the gas mass, we have

\[
\delta d = J^{-1}_u \delta d_0 + (J^{-1}_u - J^{-1}_u)(a_0^2 + 1),
\]

\[
\delta e = J^{-1}_u \delta e_0 + (J^{-1}_u - J^{-1}_u)(c_0 + 1).
\]

Hence, for all \(t \in [0, T]\),

\[
\|\delta d, \delta e\|_{B^{n+1}_p} \leq C \left( 1 + \|Du^1\|_{L^1(B^{n+1}_p)} \right) \|\delta d_0, \delta e_0\|_{B^{n+1}_p} + C \left( 1 + \|d_0^2, e_0^2\|_{B^{n+1}_p} \right) \|D\delta u\|_{L^1(B^{n+1}_p)} \leq C \left( 1 + \|Du^1\|_{L^1(B^{n+1}_p)} \right) \left( \|\delta u_0\|_{B^{n+1}_p} + \|\delta d_0, \delta e_0\|_{B^{n+1}_p} \right),
\]

here, in the last line, we have used (32).

Therefore, we may conclude to both uniqueness and continuity of the flow map on a small time interval. Note that in the case where both initial data coincide, then we may iterate the proof and obtain the uniqueness on the whole time interval \([0, T]\).

4.2. Proof of Theorem 2.2. Granted with the results in Theorem 2.1, now we revert back the solution in the Lagrangian coordinates to that in the Eulerian coordinates. Theorem 2.2 is a corollary of the following proposition which states the equivalence of the systems (1) and (5) in our functional framework.

**Proposition 9.** Let \(1 < p < 2N\) with \(N \geq 2\). Assume that \((m, n, u)\) with \((m - \hat{m}, n - \hat{n}) \in [C([0, T]; \dot{B}^{N+1}_p)]^2\) and \(u \in E_p(T)\) is a solution to (1) such that

\[
\int_0^T \|\nabla u\|_{\dot{B}^{N+1}_p} \leq c.
\]

Let \(X\) be the flow of \(u\) defined in (4). Then the triplet \((\overline{m}, \overline{n}, \overline{u}) := (m \circ X, n \circ X, u \circ X)\) belongs to the same functional space as \((m, n, u)\) and satisfies (5).

Conversely, if \((\overline{m} - \hat{m}, \overline{n} - \hat{n}, \overline{u})\) belongs to \([C([0, T]; \dot{B}^{N+1}_p)]^2 \times E_p(T)\) and \((\overline{m}, \overline{n}, \overline{u})\) satisfies (5) and, for a small enough constant \(c\),

\[
\int_0^T \|\nabla u\|_{\dot{B}^{N+1}_p} \leq c,
\]

then the map \(X\) defined in (6) is a \(C^1\) (and in fact locally \(\dot{B}^{N+1}_p\)) diffeomorphism over \(\mathbb{R}^N\) and the triplet \((m, n, u) := (\overline{m} \circ X^{-1}, \overline{n} \circ X^{-1}, \overline{u} \circ X^{-1})\) satisfies (1) and has the same regularity as \((\overline{m}, \overline{n}, \overline{u})\).

**Proof.** For a solution \((m, n, u)\) to (1) with the above properties, the definition of \(X\) implies that \(DX - Id\) is in \([C([0, T]; \dot{B}^{N+1}_p)]\). Furthermore, Proposition 5 ensures that \((\overline{m}, \overline{n}, \overline{u}) := (m \circ X, n \circ X, u \circ X)\) belongs to the same functional space as \((m, n, u)\), and Proposition 7 along with (33) ensures that \(A - Id, \ adj(AX) - Id\) and \(J^{\perp} - 1\)
are in $C([0, T]; \dot{B}^N_{p, 1})$. Hence, whenever $1 < p < 2N$, the product estimates for Besov spaces enable us to use the relations (14), (15) and (16). Therefore, $(\bar{m}, \bar{n}, \bar{u})$ fulfills (5).

Conversely, given that some solution $(\bar{m}, \bar{n}, \bar{u})$ to system (5) with
\[
(\bar{m} - \hat{m}, \bar{n} - \hat{n}, \bar{u}) \in \left[ C([0, T]; B^N_{p, 1}) \right]^2 \times E_p(T),
\]
then one may show that, under condition (34), the flow $X(t, \cdot)$ of $\bar{u}$ defined by (6) is a $C^1$ diffeomorphism over $\mathbb{R}^N$ (see the appendix of [9]), and satisfies $DX - Id \in C([0, T]; \dot{B}^N_{p, 1})$. Hence, in the Eulerian coordinates, one may construct
\[
(m, n, u) := (\bar{m} \circ X^{-1}, \bar{n} \circ X^{-1}, \bar{u} \circ X^{-1}).
\]
As above, the relations (14),(15) and (16) hold whenever $1 < p < 2N$. Then $(m, n, u)$ is a solution to system (1) and Proposition 5 ensures that $(m, n, u)$ has the desired regularity. \qed

**Proof of Theorem 2.2.** Given that data $(m_0, n_0, u_0)$ with $m_0$ bounded away from 0, $(m_0 - \hat{m}) \in \dot{B}^N_{p, 1}$, $(n_0 - \hat{n}) \in \dot{B}^N_{p, 1}$ and $u_0 \in \dot{B}^{-1}_{p, 1}$, then Theorem 2.1 ensures a local solution $(m, n, u)$ to system (5) with $(m - \hat{m}), (n - \hat{n}) \in C([0, T]; B^N_{p, 1})$ and $u$ in the space $E_p(T)$. If $T$ is small enough then (34) is satisfied, so Proposition 9 provides that $(m, n, u) := (m \circ X^{-1}, n \circ X^{-1}, u \circ X^{-1})$ is a solution of (1) in the desired functional space.

To prove uniqueness, we consider two solutions $(m^1, n^1, u^1)$ and $(m^2, n^2, u^2)$ corresponding to the same data $(m_0, n_0, u_0)$, and execute the Lagrangian change of variable (pertaining to the flow of $u^1$ and $u^2$ respectively). The obtained triplets $(\bar{m}^1, \bar{n}^1, \bar{u}^1)$ and $(\bar{m}^2, \bar{n}^2, \bar{u}^2)$ are in $(\hat{m} + C([0, T]; B_{p, 1}^N)) \times (\hat{n} + C([0, T]; B_{p, 1}^N)) \times E_p(T)$ and both satisfy system (5) with the same data $(m_0, n_0, u_0)$. Hence, the uniqueness part of Theorem 2.1 implies that $(m^1, n^1, u^1) = (m^2, n^2, u^2)$ on $[0, T]$. \qed

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