Lattice Points in Large Borel Sets and Successive Minima

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Abstract. Let $B$ be a Borel set in $\mathbb{E}^d$ with volume $V(B) = \infty$. It is shown that almost all lattices $L$ in $\mathbb{E}^d$ contain infinitely many pairwise disjoint $d$-tuples, that is sets of $d$ linearly independent points in $B$. A consequence of this result is the following: let $S$ be a star body in $\mathbb{E}^d$ with $V(S) = \infty$. Then for almost all lattices $L$ in $\mathbb{E}^d$ the successive minima $\lambda_1(S, L), \ldots, \lambda_d(S, L)$ of $S$ with respect to $L$ are 0. A corresponding result holds for most lattices in the Baire category sense. A tool for the latter result is the semi-continuity of the successive minima.

Key words. Borel sets, star bodies, lattices, successive minima, measure, Baire category, semicontinuity.

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1 Introduction and Statement of Results

A lattice $L$ in Euclidean $d$-space $\mathbb{E}^d$ is the system of all integer linear combinations of $d$ linearly independent vectors in $\mathbb{E}^d$. These vectors form a basis of $L$ and the absolute value of their determinant is the determinant $d(L)$ of $L$. $d(L)$ is independent of the particular choice of a basis. To each lattice $L$ we let correspond all $d \times d$ matrices the column vectors of which form a basis of $L$. Identify each such matrix with a point in $\mathbb{E}^{d^2}$. There is a Borel set $\mathcal{F}$ in $\mathbb{E}^{d^2}$ consisting of such matrices, which has infinite Lebesgue measure and such that to each lattice $L$ corresponds precisely one matrix in $\mathcal{F}$. Thus there is a one-to-one correspondence between the space $\mathcal{L}$ of all lattices in $\mathbb{E}^d$ and $\mathcal{F}$. The Lebesgue measure on $\mathcal{F}$ then yields a measure $\nu$ on $\mathcal{L}$.

Results of Rogers [7] (for $d \geq 3$) and Schmidt [8] (for $d = 2$) show that for a Borel set $B$ in $\mathbb{E}^d$ with Lebesgue measure $V(B) = \infty$, $\nu$-almost all lattices $L$ contain infinitely many primitive points in $B$, where a point $l \in L$ is primitive if it is different from the origin $o$ and on the line-segment $[o, l]$ there are no points of $L$, except $o$ and $l$. A refinement of this result is as follows.

Theorem 1. Let $B$ be a Borel set in $\mathbb{E}^d$ with $V(B) = \infty$. Then for $\nu$-almost every lattice $L \in \mathcal{L}$, the set $B$ contains infinitely many, pairwise disjoint $d$-tuples of linearly independent primitive points of $L$.

Tools for the proof are measure theoretic results of Rogers [7] and Schmidt [8] and a result of Yao and Yao [10] from applied computational geometry on dissection of sets in $\mathbb{E}^d$.

A star body $S$ in $\mathbb{E}^d$ is a closed set with $o$ in its interior such that each ray with endpoint $o$ meets the boundary of $S$ in at most one point. Equivalently,
\( S = \{ x : f(x) \leq 1 \} \), where \( f : \mathbb{E}^d \to \mathbb{R} \) is a distance function, i.e. it is non-negative, continuous and positively homogeneous of degree 1. The successive minima of \( S \) or \( f \) with respect to a lattice \( L \) are defined as follows:

\[
\lambda_i(S,L) = \lambda_i(f,L) = \inf \{ \lambda > 0 : \lambda S \cap L \text{ contains } i \text{ linearly independent vectors} \} = \inf \{ \max \{ f(l_1), \ldots, f(l_i) \} : l_1, \ldots, l_i \in L \text{ linearly independent} \}
\]

for \( i = 1, \ldots, d \). Clearly,

\[
(1) \quad 0 \leq \lambda_1(S,L) \leq \cdots \leq \lambda_d(S,L) \leq \infty.
\]

Successive minima play an important role in the geometry of numbers, algebraic number theory, Diophantine approximation and computational geometry, see e.g. [3, 2, 9, 5, 1]. For a surprising relation to Nevanlinna’s value distribution theory see [4].

Let \( \mathcal{L} \) be endowed with its natural topology, see [3]. Then \( \mathcal{L} \) is locally compact by Mahler’s compactness theorem. Thus a version of the Baire category theorem implies that \( \mathcal{L} \) is Baire. That is, any meager set has dense complement, where a set is meager or of first Baire category, it is a countable union of nowhere dense sets, see [6].

**Theorem 2.** Let \( S \) be a star body in \( \mathbb{E}^d \) with \( V(S) = \infty \). Then \( \lambda_1(S,L) = \cdots = \lambda_d(S,L) = 0 \) for

(i) \( \nu \)-almost all lattices \( L \) in \( \mathcal{L} \) and for

(ii) all lattices \( L \) in \( \mathcal{L} \), with a meager set of exceptions.

Tools for the proof are Theorem 1 and a semi-continuity result for successive minima which may be described as follows:

Let \( (S_n) \) be a sequence of star bodies and \( (f_n) \) the corresponding sequence of distance functions. Then \( (S_n) \) converges to a star body \( S \) with corresponding distance function \( f \) if the sequence \( (f_n) \) converges uniformly to \( f \) on the solid unit ball \( \{ x : \| x \| \leq 1 \} \) of \( \mathbb{E}^d \). A sequence \( (L_n) \) of lattices converges to a lattice \( L \), if there are bases \( \{ b_{n1}, \ldots, b_{nd} \} \) of \( L_n \) and \( \{ b_1, \ldots, b_d \} \) of \( L \) such that \( b_{n1} \to b_1, \ldots, b_{nd} \to b_d \). This notion of convergence induces the topology on \( \mathcal{L} \).

**Lemma.** Let \( (S_n) \) be a sequence of star bodies and \( (L_n) \) a sequence of lattices in \( \mathbb{E}^d \), converging to a star body \( S \) and a lattice \( L \), respectively. Then,

(i) \( \limsup_{n \to \infty} \lambda_i(S_n,L_n) \leq \lambda_i(S,L) \), for \( i = 1, \ldots, d \), and

(ii) if \( S \) is bounded, then \( \lim_{n \to \infty} \lambda_i(S_n,L_n) \) exists and is equal to \( \lambda_i(S,L) \), for \( i = 1, \ldots, d \).

To see that \( \lambda_i \) is not continuous, let \( S \) be a star body with \( V(S) = \infty \) such that there is a lattice \( L \) which has only 0 in common with the interior of \( S \), for example the star body \( \{ x : |x_1 \cdots x_d| \leq 1 \} \), see [3], p. 28. Then \( 1 \leq \lambda_i(S,L) < \infty \), while by Theorem 2 there is a sequence \( (L_n) \) of lattices such that \( L_n \to L \) with \( \lambda_i(S,L_n) = 0 \) for all \( n \).
2 Proof of Theorem 1

A result of Yao and Yao [10] says that any mass distribution in $\mathbb{E}^d$ with positive, continuous density which tends rapidly to 0 as $\|x\| \to \infty$, and of total mass $V$, can be dissected into $2^d$ disjoint Borel parts, each of mass $2^{-d}V$ and such that no hyperplane meets all these $2^d$ masses. We need the following version of this result:

(2) Let $A \subset \mathbb{E}^d$ be a bounded Borel set with volume $V(A) > V > 0$. Then $A$ contains $2^d$ pairwise disjoint Borel subsets, each of volume $2^{-d}V$ and such that no $(d-1)$-dimensional subspace of $\mathbb{E}^d$ meets each of these $2^d$ sets.

To see (2), choose a compact set $C \subset A$ with $V(C) > V$. This is possible by the inner regularity of Lebesgue measure. Next, choose a continuous function $g : \mathbb{E}^d \to \mathbb{R}^+$ such that

$$g \geq \chi_C, \int_{\mathbb{E}^d} (g - \chi_C) \, dx < 2^{-d}(V(C) - V), \quad g(x) \to 0 \text{ rapidly as } \|x\| \to \infty,$$

where $\chi_C$ is the characteristic function of $C$. This is possible by the outer regularity of Lebesgue measure and Urysohn’s lemma. Let $F_i, i = 1, \ldots, 2^d$, be a dissection of $\mathbb{E}^d$ for the density $g$ as described by Yao and Yao such that

$$\int_{F_i} g \, dx = 2^{-d} \int_{\mathbb{E}^d} g \, dx \geq 2^{-d}V(C).$$

Then there is no $(d-1)$-dimensional subspace of $\mathbb{E}^d$ which meets each of the sets $C \cap F_i$, and for the respective volumes of these sets we have the following estimate:

$$V(C \cap F_i) = \int_{F_i} \chi_C \, dx = \int_{F_i} g \, dx - \int_{F_i} (g - \chi_C) \, dx \geq \int_{F_i} g \, dx - \int_{\mathbb{E}^d} (g - \chi_C) \, dx \geq 2^{-d}V(C) - 2^{-d}(V(C) - V) = 2^{-d}V.$$

This concludes the proof of (2).

For the proof of Theorem 2 assume first that $d \geq 3$. The following result is an immediate consequence of a result of Rogers [7], p. 286:

(3) Let $k = 1, 2, \ldots$, and $A$ a Borel set in $\mathbb{E}^d$ with $0 < V(A) < \infty$. Then the function $\#^*(A \cap \cdot) : \mathcal{L} \to \{0, 1, \ldots\}$, which counts the number of primitive points of $L$ in $A$, is Borel measurable and

$$\int_{\mathcal{L}(k)} \left( \frac{\#^*(A \cap L)}{\zeta(d)} - \frac{V(A)}{\zeta(d)} \right)^2 \, d\nu(L) \leq \alpha V(A).$$

Here $\mathcal{L}(k) = \{L \in \mathcal{L} : d(L) \leq k\}$, $\zeta(\cdot)$ denotes the Riemann zeta-function, and $\alpha > 0$ is a constant depending on $k$ and $d$. 
The main step of the proof is to show the following proposition:

(4) Let \( k = 1, 2, \ldots \) Then for \( \nu \)-almost every lattice \( L \in \mathcal{L}(k) \) the set \( B \) contains infinitely many pairwise disjoint \( d \)-tuples of linearly independent points of \( L \).

To prove this, let \( 0 = \varrho_0 < \varrho_1 < \ldots \) be such that

\[
V(B_n) > 2^d \zeta(d)n, \text{ where } B_n = \{ x \in B : \varrho_{n-1} < \| x \| \leq \varrho_n \}.
\]

By (2),

(5) for \( n = 1, 2, \ldots \), there are \( 2^d \) pairwise disjoint Borel sets \( B_{ni} \), \( i = 1, \ldots, 2^d \), of \( B_n \) such that \( V(B_{ni}) = \zeta(d)n \) and no \((d-1)\)-dimensional subspaces of \( \mathbb{E}^d \) meets each set \( B_{ni} \).

Consequently (3) implies that

(6) \[
\int_{\mathcal{L}(k)} (\#^*(B_{ni} \cap L) - n)^2 d\nu(L) \leq \alpha \zeta(d)n.
\]

By (5) and (3) the sets \( \mathcal{L}_{ni} = \{ L \in \mathcal{L}(k) : \#^*(B_{ni} \cap L) = 0 \}, i = 1, \ldots, 2^d \), are Borel. It thus follows from (6) that \( n^2 \nu(\mathcal{L}_{ni}) \leq \alpha \zeta(d)n \), or

(7) \[
\nu(\mathcal{L}_{ni}) \leq \frac{\alpha \zeta(d)}{n}.
\]

The set \( \mathcal{L}_n = \mathcal{L}_{n1} \cup \cdots \cup \mathcal{L}_{n2^d} \) is Borel and consists of all lattices \( L \in \mathcal{L}(k) \) such that at least one of the sets \( B_{ni} \) contains no primitive point of \( L \). Hence \( \mathcal{L}(k) \setminus \mathcal{L}_n \) is the set of all lattices \( L \in \mathcal{L}(k) \) such that each set \( B_{ni} \) contains a primitive point of \( L \). Hence (5) shows that

(8) \[
\text{for any lattice } L \in \mathcal{L}(k) \setminus \mathcal{L}_n, \text{ the set } B_n \text{ contains a } d\text{-tuple of linearly independent points of } L.
\]

By (7),

(9) \[
\nu(\mathcal{L}_n) \leq \frac{\alpha 2^d \zeta(d)}{n}.
\]

By definition the sets \( B_n \) are pairwise disjoint subsets of \( B \). Hence (8) implies that

\[
\{ L \in \mathcal{L}(k) : B \text{ contains infinitely many pairwise disjoint } d\text{-tuples of linearly independent primitive points of } L \} \\
\supset \{ L \in \mathcal{L}(k) : \text{for infinitely many } n, \text{ the set } B_n \text{ contains a } d\text{-tuple of linearly independent primitive points of } L \} \\
\supset \{ L \in \mathcal{L}(k) : \text{for infinitely many } n \text{ the lattice } L \text{ is not contained in } \mathcal{L}_n \} \\
= \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} (\mathcal{L}(k) \setminus \mathcal{L}_n) = \mathcal{L}(k) \setminus \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \mathcal{L}_n.
\]
Since by (9),
\[ \nu \left( \bigcap_{n=m}^{\infty} \mathcal{L}_n \right) = 0 \text{ for } m = 1, 2, \ldots, \text{ and thus } \nu \left( \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \mathcal{L}_n \right) = 0, \]
the proof of (4) is complete.

Since \( \mathcal{L} = \bigcup \mathcal{L}(k) \), Theorem 1 for \( d \geq 3 \) is an immediate consequence of (4).

Assume now that \( d = 2 \) and let \( \xi \) be the measure on \( \mathcal{L} \) used by Schmidt [8], p. 525. A result of Schmidt [8], p. 526/7, shows that (3) continues to hold, but with the following weaker inequality:
\[ \int_{\mathcal{L}(k)} \left( \#^* (A \cap L) - \frac{V(A)}{\zeta(2)} \right)^2 d\xi(L) \leq \beta V(A) \log_2 V(A). \]

Using this, we see that in the case \( d = 2 \) the proof is, in essence, the same as the above proof for \( d \geq 3 \). Finally, note that the sets of measure 0 with respect to \( \xi \) and \( \nu \) coincide. \( \square \)

3 Proof of the Lemma

Let \( f_n \) and \( f \) be the distance functions of \( S_n \) and \( S \), respectively. Since distance functions are positively homogeneous of degree 1, and \( f_n \to f \) uniformly for \( \|x\| \leq 1 \), we have that \( f_n \to f \) uniformly on each bounded set in \( \mathbb{E}^d \). This yields the following statement:

(10) Let \( l_n, l \in \mathbb{E}^d \) be such that \( l_n \to l \). Then \( f_n(l_n) \to f(l) \).

The following claims are simple consequences of the convergence \( L_n \to L \), see [3], p. 178/9:

(11) Given \( l \in L \), there are \( l_n \in L \) such that \( l_n \to l \).

(12) If \( l_n \in L_n \) and \( l \in \mathbb{E}^d \) such that \( l_n \to l \), then \( l \in L \).

(i): Let \( \varepsilon > 0 \). By the definition of successive minima one can show that there are linearly independent lattice points \( l_1, \ldots, l_d \in L \) such that

(13) \( \max \{ f(l_1), \ldots, f(l_d) \} \leq \lambda_i(S, L) + \varepsilon \).

By (11) we may choose points \( l_{nj} \in L_n, j = 1, \ldots, d \), such that

(14) \( l_{nj} \to l_j \).

Since \( l_1, \ldots, l_d \) are linearly independent, it follows that

(15) \( l_{n1}, \ldots, l_{nd} \in L_n \) are also linearly independent for all sufficiently large \( n \).
Hence, by the definition of $\lambda_i$ together with (15), (14), (10), and (13) we obtain that

$$
\lambda_i(f_n, L_n) \leq \max\{f_n(l_{n1}), \ldots, f_n(l_{ni})\} \leq \max\{f(l_1), \ldots, f(l_{ni})\} + \varepsilon
$$

$$
\leq \lambda_i(f, L) + 2\varepsilon \text{ for all sufficiently large } n,
$$

Since $\varepsilon > 0$ was arbitrary, this concludes the proof of claim (i).

(ii): Let $0 < \varepsilon < 1$. Since $f_n \to f$ uniformly on $\{x : \|x\| = 1\}$ and $f(x) > 0$ for $\|x\| = 1$ by the boundedness of $S$, and $f_n, f$ all are continuous and positively homogeneous of degree 1, there is a constant $\alpha > 0$ such that

$$
(16) \quad \alpha\|x\| \leq (1 - \varepsilon)f(x) \leq f_n(x) \text{ for all } x \in \mathbb{E}^d \text{ if } n \text{ is sufficiently large.}
$$

For such $n$ we have that $f(x) > 0$ for $x \neq 0$, hence $S_n$ is bounded. The definition of $\lambda_i$ then yields that

$$
(17) \quad \text{for all sufficiently large } n, \text{ there are linearly independent points } l_{n1}, \ldots, l_{nd}
$$

in $L_n$, such that $\lambda_i(f_n, L_n) = \max\{f_n(l_{n1}), \ldots, f_n(l_{ni})\}$ for $i = 1, \ldots, d$.

(16), (17), (1) and (i) together imply that

$$
(18) \quad \|l_{ni}\| \leq \frac{1}{\alpha}f_n(l_{ni}) \leq \frac{1}{\alpha}\lambda_i(f_n, L_n) \leq \frac{1}{\alpha}\lambda_d(f_n, L_n)
$$

$$
\leq \frac{1}{\alpha}\lambda_d(f, L) + \varepsilon \text{ for all sufficiently large } n.
$$

For all sufficiently large $n$, the vectors $l_{n1}, \ldots, l_{nd}$ are linearly independent by (17). Consequently, $|\det(l_{n1}, \ldots, l_{nd})|$ is an integer multiple of $d(L)$. By assumption, $L_n \to L$. Hence $d(L_n) \to d(L)$. Combining this, it follows that

$$
(19) \quad |\det(l_{n1}, \ldots, l_{nd})| \geq d(L_n) \geq (1 - \varepsilon)d(L) \text{ for all sufficiently large } n.
$$

By (18), all the sequences $(l_{n1}), \ldots, (l_{nd})$ are bounded. Fix an index $i = 1, \ldots, d$. By considering a suitable subsequence of 1, 2, $\ldots$, and re-numbering, if necessary, we may suppose that

$$
(20) \quad \liminf_{n \to \infty} \lambda_i(f_n, L_n) \text{ is the same as for the original sequence,}
$$

and $l_{n1} \to l_1, \ldots, l_{nd} \to l_d$, say. By (10), (12) and (19) the latter implies that

$$
f_n(l_{n1}) \to f(l_1), \ldots, f_n(l_{nd}) \to f(l_d), l_1, \ldots, l_d \in L,
$$

$$
|\det(l_1, \ldots, l_d)| \geq (1 - \varepsilon)d(L) > 0.
$$

In particular, $l_1, \ldots, l_d$ are linearly independent. Using (17) and the definition of $\lambda_i$, it follows that

$$
\lambda_i(f_n, L_n) = \max\{f_n(l_{n1}), \ldots, f_n(l_{ni})\} \to \max\{f(l_1), \ldots, f(l_i)\} \geq \lambda_i(f, L).
$$

This together with (20) and (i) finally yields (ii). □
4 Proof of Theorem 2

(i): Apply Theorem 1 with $B = \frac{1}{k} S, k = 1, 2, \ldots$, to see that for $\nu$-almost all lattices $\varepsilon S$ contains a $d$-tuple of linearly independent primitive points of $L$ for any $\varepsilon > 0$. Hence $\lambda_d(S, L) = 0$ for $\nu$-almost all lattices $L$. In conjunction with (1), this completes the proof of claim (i).

(ii): Let $M_n = \{ L \in \mathcal{L} : \lambda_d(S, L) \geq \frac{1}{n} \}, n = 1, 2, \ldots$. Since $\lambda_d(S, \cdot)$ is upper semi-continuous by the Lemma, $M_n$ is closed. If the interior of $M_n$ is non-empty, then $\nu(M_n) > 0$ by the definitions of $\nu$ and the topology on $\mathcal{L}$, in contradiction to (i). Hence $M_n$ has empty interior. Being closed, $M_n$ is nowhere dense in $\mathcal{L}$. Hence

$$\bigcup_{n=1}^{\infty} M_n = \{ L \in \mathcal{L} : \lambda_d(S, L) > 0 \}$$

is meager.

Now note (1) to conclude the proof of claim (ii). □

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