New duality transformation in two-dimensional non-linear sigma models

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Abstract
A new T-duality transformation is found in two-dimensional non-linear sigma models. Abelian and non-Abelian T-dualities are special cases of this construction.
1 Introduction

The duality transformations, known as T-duality, have proven to be a crucial step in the study of string theory. These transformations stipulate that string propagation in some background could be equally described by another, a priori, completely different background. Therefore, T-duality transformations are a useful tool in the programme of classifying equivalent compactifications of string theory. These transformations are constructed at the level of the two-dimensional non-linear sigma model underlying the propagation of strings in non-trivial backgrounds. There are, however, no systematic criteria for finding these duality transformations.

The first T-duality transformations (abelian T-duality) were constructed for non-linear sigma models possessing Abelian isometries [1]. This study was later generalised to theories with non-Abelian isometries (non-Abelian T-duality) [2, 3]. It turned out, however, that the presence of isometries is not an essential ingredient in building T-duality transformations. This is certainly what happens in the case of Poisson-Lie T-duality [4]. These last transformations do not rely on isometries and are indeed symmetries of the string effective action [5].

The idea behind Poisson-Lie T-duality resides in the fact that if two theories are dual then the equations of motion in one theory become Bianchi identities in the other (see [6] for various developments of this subject). In this letter, we exploited this observation to construct new T-duality transformations. These are obvious generalisations of Abelian and non-Abelian T-dualities.

In order to clarify our approach, let us start by briefly revisiting Abelian and non-Abelian T-dualities. We consider the two-dimensional non-linear sigma model as represented by the action

$$S [x, y] = \int dzd\bar{z} \left\{ Q_{ij} (x) \partial x^i \partial x^j + G_{ia} (x) \partial x^i \bar{A}^a + \bar{G}_{ia} (x) \bar{\partial} x^i A^a + P_{ab} (x) A^a \bar{A}^b \right\}.$$  (1.1)

The target space coordinates are the “spectators” $x^i$ (labelled by $i, j, k, ...$) and the “active” coordinates $y^a$ (labelled by $a, b, c, d, e, f, g, ...$). The most important feature of these sigma models is that the $y^a$ coordinates appear only through the one form

$$\left\{ \begin{array}{lcl}
A^a &=& c^a_0 (y) \partial y^b \\
\bar{A}^a &=& c^a_0 (y) \bar{\partial} y^b
\end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{lcl}
\partial y^a &=& E^a_b (y) A^b \\
\bar{\partial} y^a &=& E^a_b (y) \bar{A}^b
\end{array} \right.$$  (1.2)

Here $c^a_0$ are vielbeins satisfying the Cartan-Maurer relation $\partial a e^c_b - \partial b e^c_a + f^c_{bd} e^d_a e^b_b = 0$, where $f^c_{bd}$ are the structure constants of a Lie algebra $\mathcal{G}$. The inverses of these vielbeins are denoted $E^a_b$ and satisfy the commutation relations $E^c_a \partial E^a_b - E^c_a \bar{\partial} E^a_b - f^c_{ab} E^d_b = 0$.

The next ingredient in the construction of the dual theory is the integrability condition (the Bianchi identity) associated with the definition (1.2). Namely, $\partial \bar{\partial} y^a - \bar{\partial} \partial y^a = 0$. When expressed in terms of $A^a$ and $\bar{A}^a$ it reads

$$E^a_b \left( \partial \bar{A}^b - \bar{\partial} A^b + f^b_{ca} A^c \bar{A}^d \right) = 0.$$  (1.3)

The crucial point about this Bianchi identity is that there is no explicit dependence on the coordinates $y^a$ in the expression between brackets.

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$^3$The world-sheet is for simplicity taken to be flat and we are using complex coordinates $z$ and $\bar{z}$. 
The Bianchi identity allows one to formulate the non-linear sigma model (1.1) in a first order formalism through the action \[ S_1 \left[ x, A, \bar{A}, \chi \right] = \int dzd\bar{z} \left\{ Q_{ij} (x) \partial x^i \partial x^j + G_{ia} (x) \partial x^i \bar{A}^a + G_{ia} (x) \bar{\partial} x^i A^a \right. \\
+ P_{ab} (x) A^a \bar{A}^b + \chi_b \left( \partial \bar{A}^b - \bar{\partial} A^b + f_{cd}^b A^c \bar{A}^d \right) \right\} . \] This last action is equivalent to (1.1) as the equations of motion for the Lagrange multiplier \( \chi_a \) is simply the Bianchi identity (1.3) whose solution is the one form (1.2). Replacing this solution in the first order action leads to our starting theory (1.1). The dual theory is obtained by keeping the Lagrange multiplier \( \chi_a \) and eliminating, instead, the independent fields \( A^a \) and \( \bar{A}^a \) through their equations of motion. The resulting two-dimensional non-linear sigma model (the dual model) is described by the action \[ \bar{S} [x, \chi] = \int dzd\bar{z} \left\{ \left[ Q_{ij} - (M^{-1})^{ab} G_{ia} \bar{G}_{jb} \right] \partial x^i \bar{\partial} x^j - (M^{-1})^{ab} G_{ia} \partial x^i \bar{\partial} \chi_b \right. \\
+ \left( M^{-1} \right)^{ba} \bar{G}_{ia} \partial x^i \partial \chi_b + \left( M^{-1} \right)^{ab} \partial \chi_a \bar{\partial} \chi_b \right\} , \] where the matrix \( M_{ab} \) is defined as \[ M_{ab} = P_{ab} + \chi_c f_{ab}^c . \] In the dual theory the field \( \chi_a \) plays the role of the field \( y^a \). It is worth mentioning that the two non-linear sigma models lead to the same string effective action as shown in [3]. This construction will be generalised below in an obvious manner.

2 The construction

The important point in the above construction is the existence of the Bianchi identity related to the definition (1.2). One notices, however, that \( A^a \) and \( \bar{A}^a \) are functions of the coordinates \( y^a \) only. We therefore take the new one-form to be defined by

\[
\begin{align*}
\left\{ \begin{array}{l}
\partial y^a = \alpha^a (x, y) A^b + \beta^a (x, y) \partial x^i \\
\bar{\partial} y^a = \gamma^a (x, y) \bar{A}^b + \omega^a (x, y) \bar{\partial} x^i
\end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l}
A^a = (\alpha^{-1})^a_b \left[ \partial y^b - \beta^b \partial x^i \right] \\
\bar{A}^a = (\gamma^{-1})^a_b \left[ \bar{\partial} y^b - \omega^b \bar{\partial} x^i \right]
\end{array} \right.
\] \tag{2.1}
\]

In terms of this one-form, the Bianchi identity corresponding to \( \partial \bar{\partial} y^a - \bar{\partial} \partial y^a = 0 \) takes the form

\[
\gamma^a_b \left( \partial \bar{A}^b - \theta^b_c \partial A^c + \varphi^b_{cd} A^c \bar{A}^d + \mu^b_{ie} \partial x^i A^c + \nu^b_{ie} \partial x^i \bar{A}^c + \xi^b_{ij} \partial x^i \bar{\partial} x^j + \rho^b_{ie} \bar{\partial} x^j \right) = 0 . \tag{2.2}
\]

The different tensors appearing in this equation are defined as

\[
\begin{align*}
\theta^a_b &= \left( \gamma^{-1} \right)^a_e \alpha^e_b , \\
\varphi^a_{ec} &= \left( \gamma^{-1} \right)^a_e \left[ \alpha^d_e \partial d \gamma^c - \gamma^d_e \partial e \alpha^c \right] , \\
\mu^a_{ib} &= - \left( \gamma^{-1} \right)^a_e \left[ \partial_i \alpha^e_b + \omega^e_i \partial e \alpha^c_b - \alpha^e_i \partial c \omega^e_i \right] , \\
\nu^a_{ib} &= \left( \gamma^{-1} \right)^a_e \left[ \partial_i \gamma^e_b + \beta^e_i \partial e \gamma^c_b - \gamma^e_b \partial c \beta^e_i \right] , \\
\xi^a_{ij} &= \left( \gamma^{-1} \right)^a_e \left[ \partial_i \omega^e_j - \partial_j \beta^e_i + \beta^e_i \partial_0 \omega^e_j - \omega^e_j \partial_0 \beta^e_i \right] , \\
\rho^a_{i} &= \left( \gamma^{-1} \right)^a_e \left[ \omega^e_i - \beta^e_i \right] ,
\end{align*}
\tag{2.3}
\]
where \( \partial_a \) and \( \partial_i \) are, respectively, the derivatives with respect to \( y^a \) and \( x^i \).

Just as in the case of ordinary duality, we require that no explicit dependence on the coordinates \( y^a \) is left in the expression between brackets of the Bianchi identity (2.2). That is, we demand that \( \theta^a_b = \theta^a_b (x) \), \( \varphi^a_b (x) = \varphi^a_b (x) \), \( \mu^a_b = \mu^a_b (x) \), \( \nu^a_b = \nu^a_b (x) \), \( \xi^a_{ij} = \xi^a_{ij} (x) \) and \( \rho^a_i = \rho^a_i (x) \). Hence our task is to find the tensors \( \alpha^a_b \), \( \gamma^a_b \), \( \omega^a_i \) and \( \beta^a_i \) such that this property holds.

We choose to express \( \alpha^a_b (x, y) \) and \( \omega^a_i (x, y) \) in terms of \( \gamma^a_b (x, y) \) and \( \beta^a_i (x, y) \). The first equation (2.3) and the sixth equation (2.8) of this set are algebraic and allow the determination of \( \alpha^a_b \) and \( \omega^a_i \). These are given by

\[
\alpha^a_b (x, y) = \gamma^c_a \theta^a_b ,
\omega^a_i (x, y) = \beta^a_i + \gamma^a_b \rho^b_i .
\]

(2.9)

(2.10)

Replacing \( \alpha^a_b \) by its expression in equation (2.4) leads to

\[
\gamma^d_a \theta^d_c - \gamma^d_b \theta^d_a = t^d_{bc} \gamma^c_d ,
\]

where the new tensor \( t^d_{bc} (x) \) is defined as

\[
\theta^d_{bc} = \varphi^c_{ab} .
\]

(2.12)

Using (2.9), (2.10), (2.6) and (2.11) in equation (2.5) we obtain

\[
\mu^a_b = - \partial_i \theta^a_b - \nu^a_c \theta^c_b - t^a_{cd} \theta^d_i \theta^d_b .
\]

(2.13)

This last equation gives simply \( \mu^a_b \) in terms of the other tensors. Finally, using (2.10) and (2.6) in equation (2.7) yields two relations: The first is

\[
\partial_i \beta^a_j - \partial_j \beta^a_i + \beta^b_i \partial_b \beta^a_j - \beta^b_j \partial_b \beta^a_i = \frac{1}{2} \gamma^a_b \left( \xi^b_{ij} - \xi^b_j - \partial_i \rho^b_j + \partial_j \rho^b_i - \nu^b_i \rho^b_j + \nu^b_j \rho^b_i \right) .
\]

(2.14)

The second relation is

\[
\frac{1}{2} \gamma^a_b \left( \xi^b_{ij} + \xi^b_{ji} - \partial_i \rho^b_j - \partial_j \rho^b_i - \nu^b_i \rho^b_j + \nu^b_j \rho^b_i \right) = 0 .
\]

(2.15)

This last equation determines the symmetric part of \( \xi^a_{ij} \) in terms of the other tensors.

The constraints on the two tensors \( \gamma^a_b \) and \( \beta^a_i \) are therefore equations (2.11), (2.6) and (2.14). These equations have a better interpretation in terms of the following differential operators

\[
T_a = \gamma^b_a (x, y) \partial_b ,
J_i = \beta^b_i (x, y) \partial_b .
\]

(2.16)

Indeed, equations (2.11), (2.6) and (2.14) become respectively

\[
[T_a, T_b] = t^c_{ab} T_c ,
[\nabla_i, T_a] = \nu^b_a T_b ,
[\nabla_i, \nabla_j] = \Lambda^a_{ij} T_a ,
\]

(2.17)

(2.18)

(2.19)
where we have introduced the new notation
\[
\nabla_i = \partial_i + J_i \quad ,
\]
\[
\Lambda^a_{ij} = \frac{1}{2} \left( \xi^a_{ij} - \xi^a_{ji} - \partial_i \rho^a_j + \partial_j \rho^a_i - \nu^a_{ic} \rho^c_j + \nu^a_{jc} \rho^c_i \right) . \quad (2.20)
\]

Of course, the above commutation relations have to satisfy certain consistency relations. These are given by
\[
t^d_{ab} t^e_{de} + t^d_{ca} t^e_{db} + t^d_{bc} t^e_{da} = 0 \quad , \quad (2.21)
\]
\[
t^d_{ab} t^e_{cd} - t^d_{ad} t^e_{ib} + t^d_{bd} t^e_{ia} = -\partial_i t^e_{ab} \quad , \quad (2.22)
\]
\[
\partial_i \nu^a_{jb} - \partial_j \nu^a_{ib} + \nu^a_{ic} \nu^c_{jb} - \nu^a_{ic} \nu^c_{ib} = -t^a_{bd} \Lambda^d_{ij} \quad , \quad (2.23)
\]
\[
(\partial_i \Lambda^a_{jk} + \nu^a_{ib} \Lambda^b_{jk}) + (\partial_k \Lambda^a_{ij} + \nu^a_{kb} \Lambda^b_{ij}) + (\partial_j \Lambda^a_{ki} + \nu^a_{jb} \Lambda^b_{ki}) = 0 \quad . \quad (2.24)
\]

These Jacobi identities do not require any further integrability conditions. Furthermore, the last Jacobi identity is automatically satisfied once the first three are obeyed. It is worth mentioning that if the Jacobi identities are fulfilled then the covariant derivative
\[
(\mathcal{D}_i)^b = \delta^b_0 \partial_i - \varepsilon^b_i t^a_{ca} + \nu^b_{ia} \quad , \quad (2.25)
\]
has vanishing curvature. Namely,
\[
(\mathcal{D}_i)^b (\mathcal{D}_j)^c - (\mathcal{D}_j)^b (\mathcal{D}_i)^c = 0 \quad . \quad (2.26)
\]

This observation together with the above commutation relations give a geometrical interpretation to our setting.

To summarise the problem so far, we are looking for two tensors \( \gamma^b_a (x, y) \) and \( \beta^a_i (x, y) \) from which we can construct the three tensors \( t^a_{bc} (x) \), \( \Lambda^a_{ij} (x) \) and \( \nu^a_{ib} (x) \). These last tensors are subject to the Jacobi identities (2.21)-(2.24).

### 3 Solutions

By examining the first commutation relation (2.17) and the first Jacobi identity (2.21), one realises that the tensor \( t^a_{bc} \) must be related by a similarity transformation to the structure constants of a Lie algebra. Namely, one must have
\[
t^a_{bc} (x) = s^d_b (x) s^c_e (x) S^a_d (x) f^a_{de} \quad , \quad (3.1)
\]
where \( S^a_b \) is the inverse of \( s^a_b \) and \( f^a_{bc} \) are the structure constants of a Lie algebra \( \mathcal{G} \), whose generators we denote \( \lambda_a \), such that
\[
[\lambda_a, \lambda_b] = f^c_{ab} \lambda_c . \quad (3.2)
\]

We deduce, from (2.11), that \( \gamma^b_a \) must be given by
\[
\gamma^b_a (x, y) = s^c_b (x) E^a_c (y) \quad . \quad (3.3)
\]
Here $E^a_b(y)$ are the inverses of the vielbeins $e^a_b(y)$ defined by

$$e^a_b \lambda_a = g^{-1} \partial_b g \ ,$$

where $g(y)$ is a Lie group element corresponding to the Lie algebra $\mathcal{G}$.

Next, the expression for $\nu^a_b$ in (2.10) can be cast into

$$\left( \nu^a_b + s^d_b \partial_b s^c_a \right) S^b_a s^a_c = \left( s^d_b \varepsilon^c_e \right) f^e_d - E^d_a \partial_d (s^a_c \varepsilon^c_i) \ .$$

(3.5)

The right-hand-side of this equation must be independent of the coordinate $y^a$. A possible solution to this problem is found by taking

$$\varepsilon^c_i (x, y) = S^c_b (x) \left[ \nu^b_i (y) \kappa^a_i (x) + C^b_i (x) \right] \ ,$$

where $\nu^b_i (y)$ is defined as

$$\nu^b_i (y) = g^{-1} \lambda_b g \ (3.7)$$

and satisfies the following crucial properties

$$E^d_c \partial_d \nu^e_b - \nu^b_c f^e_b = 0 \ , \ \nu^b_c \nu^d_c f^e_b - \nu^e_d f^d_a = 0 \ .$$

(3.8)

This solution leads to the following final expression for the tensor $\nu^a_b (x)$

$$\nu^a_b (x) = -s^a_b \left( \partial_b S^a_c - f^a_d C^d_i S^c_g \right) \ .$$

(3.9)

What remains to be determined now is the tensor $\xi^c_{ij} (x)$. Equation (2.20) leads to

$$\Lambda^c_{ij} (x) = \partial_i \varepsilon^c_j - \partial_j \varepsilon^c_i - t^c_{ab} \varepsilon^b_i \varepsilon^c_j + \nu^c_i \varepsilon^c_j - \nu^c_j \varepsilon^c_i \ .$$

(3.10)

Replacing for $\varepsilon^c_i$, $t^c_{ab}$ and $\nu^a_b$ and using the properties of $\nu^b_i$ one gets

$$\Lambda^c_{ij} (x) = \nu^b_i (y) S^c_b \left[ \partial_i \kappa^a_j - \partial_j \kappa^a_i - f^a_d \kappa^d_i \kappa^a_j \right] + S^c_b \left[ \partial_j C^a_i - \partial_j C^a_i + f^a_d C^d_i C^c_j \right] \ .$$

(3.11)

Hence, the right-hand-side is independent of $y^a$ only and only if

$$\partial_i \kappa^a_j - \partial_j \kappa^a_i - f^a_d \kappa^d_i \kappa^a_j = 0 \ .$$

(3.12)

An interesting solution to this condition is given by

$$\kappa^a_i (x) \lambda_a = \partial_i h h^{-1} \ ,$$

(3.13)

where $h(x)$ is an element of a Lie group built on the Lie algebra $\mathcal{G}$ but with parameters $x^i$. Of course, this solution requires that the range of the indices $\{a, b, c, \ldots\}$ is the same as that of a subset of the indices $\{i, j, k, \ldots\}$. If the Lie algebra $\mathcal{G}$ is Abelian then $\nu^a_b = \delta^a_b$ and the zero curvature condition (3.12) is not necessary.

Finally, the expression of the tensor $\xi^c_{ij} (x)$ is given by

$$\xi^c_{ij} (x) = S^c_a \left[ \partial_i C^a_j - \partial_j C^a_i + f^a_d C^d_i C^c_j \right] + \partial_i \rho^j - s^c_a \left( \partial_i S^c_a - f^a_d C^b_i C^c_g \right) \rho^d_j \ .$$

(3.14)

It is easy to check that all the Jacobi identities are satisfied by the solution presented here.
4 The dual models

The original two-dimensional non-linear sigma model is still given by an action of the form \( I \) but with \( A^a \) and \( \bar{A}^a \) as expressed by

\[
A^a = S^a_\delta \left( \theta^{-1} \right)^a \left[ e^d_{\delta} \partial y^b \left( v^d_{e} \kappa^e_{i} + C^d_{i} \right) \partial x^i \right] ,
\]

\[
\bar{A}^a = \bar{S}^a_\delta \left[ e^d_{\delta} \partial y^b \left( v^d_{e} \kappa^e_{i} + C^d_{i} + s^d_{e} \kappa^e_{i} \right) \bar{\partial} x^i \right] .
\]

(4.1)

We now make use of the Bianchi identity corresponding to this gauge fields to write the first order action

\[
I_1 \left[ x, A, \bar{A}, \chi \right] = \int d\nu dz \left\{ Q_{ij} \left( x \right) \partial x^i \bar{\partial} x^j + G_{ia} \left( x \right) \partial x^i A^a + G_{ia} \left( x \right) \bar{\partial} x^i A^a 
+ P_{ab} \left( x \right) A^a \bar{A}^b + \chi_b \left[ \partial \bar{A}^b - \theta^b_{c} \left( x \right) \partial A^c + \varphi^b_{cd} \left( x \right) A^c \bar{A}^d + \mu^b_{ic} \left( x \right) \partial x^i A^c 
+ \nu^b_{ic} \left( x \right) \partial x^i \bar{A}^c \right] \right\} .
\]

(4.2)

The original action is obtained by substituting for \( A^a \) and \( \bar{A}^a \). On the other hand, the dual theory is obtained by keeping the Lagrange multiplier \( \chi_a \) and integrating out the new tensors \( \chi_a \). It is, however, possible to give a simple form to the two dual models.

In order to compare our construction with ordinary non-Abelian duality, we introduce the new one-form defined by\[4\]

\[
B^a = s^a_b \theta^b_c \partial x^i + C^a_b \partial x^i = e^a_b \left( y \right) \partial y^b - v^a_b \left( y \right) \kappa^b_{i} \left( x \right) \partial x^i,
\]

\[
\bar{B}^a = s^a_b \bar{A}^b + \left( C^a_{i} + s^a_b \rho^a_{b} \right) \partial x^i = e^a_b \left( y \right) \partial y^b - v^a_b \left( y \right) \kappa^b_{i} \left( x \right) \bar{\partial} x^i.
\]

(4.3)

It is then easy express the Bianchi identity associated to \( A^a \) and \( \bar{A}^a \) in terms of the new gauge fields \( B^a \) and \( \bar{B}^a \). This takes the familiar form

\[
E^b_b \left( \partial \bar{B}^b - \bar{\partial} B^b + f^b_{cd} B^c \bar{B}^d \right) = 0 .
\]

(4.4)

Indeed, using group elements, \( B^a \) and \( \bar{B}^a \) are such that

\[
B^a \lambda_a = l^{-1} \partial l , \quad \bar{B}^a \lambda_a = l^{-1} \bar{\partial} l ,
\]

(4.5)

where \( l \left( x, y \right) = h^{-1} \left( x \right) g \left( y \right) \). This remark explains the zero curvature condition encountered in \( \{ 2, 3 \} \).

In terms of the gauge fields \( B^a \) and \( \bar{B}^a \), the above first order action takes the form

\[
I_1 \left[ x, B, \bar{B}, \chi \right] = \int d\nu dz \left\{ T_{ij} \left( x \right) \partial x^i \bar{\partial} x^j + H_{ia} \left( x \right) \partial x^i \bar{B}^a + \bar{H}_{ia} \left( x \right) \bar{\partial} x^i B^a 
+ F_{ab} \left( x \right) B^a \bar{B}^b + \chi_b \left( \partial \bar{B}^b - \bar{\partial} B^b + f^b_{cd} B^c \bar{B}^d \right) \right\} .
\]

(4.6)

Here, the new backgrounds \( T_{ij} \), \( H_{ia} \), \( \bar{H}_{ia} \) and \( F_{ab} \) are functions of the old tensors \( Q_{ij} \), \( G_{ia} \), \( \bar{G}_{ia} \) and \( P_{ab} \). The original sigma model is then obtained by solving the constraints imposed by the Lagrange multiplier \( \chi_a \). This amounts to replacing, in the first order action, \( B^a \) and \( \bar{B}^a \) by their expressions in \( \{ 2, 3 \} \). We obtain the original theory

\[
I \left[ x, y \right] = \int d\nu dz \left\{ \left( T_{ij} + F_{ab} v^a_{c} v^b_{d} \bar{\kappa}^c_{ij} \kappa^d_{jk} - H_{ia} v^a_{b} \kappa^b_{ij} - \bar{H}_{ja} v^a_{b} \kappa^b_{ij} \right) \partial x^i \bar{\partial} x^j 
+ \left( H_{ia} - F_{ab} v^a_{c} \kappa^c_{ij} \right) e^a_d \partial x^i \bar{\partial} y^d + \left( \bar{H}_{ia} - F_{ab} v^a_{c} \kappa^c_{ij} \right) e^a_d \bar{\partial} x^i \partial y^d 
+F_{ab} e^a_{d} e^b_{d} \partial y^c \bar{\partial} y^d \right\} .
\]

(4.7)

\(^4\)I am grateful to Ian jack for suggesting this to me.
Notice that all the dependance on the $y^a$ coordinates is in $e^a_b(y)$ and $v^a_b(y)$. It is clear that if $\kappa^a_i$ is set to zero then one recovers the setting for ordinary non-Abelian duality ($f_{bc}^a \neq 0$) or Abelian duality ($f_{bc}^a = 0$).

The dual non-linear sigma model is found by eliminating, through their equations of motion, the gauge fields $B^a$ and $\bar{B}^a$ from the first order action. This leads to a dual theory described by

$$
\tilde{I}[x, \chi] = \int dz \bar{z} \left\{ \left[ T_{ij} - (M^{-1})^{ab} H_{ia} \bar{H}_{jb} \right] \partial x^i \bar{\partial} x^j - (M^{-1})^{ab} H_{ia} \partial x^i \bar{\partial} \chi_b \\
+ (M^{-1})^{ba} \bar{H}_{ia} \bar{\partial} x^i \partial \chi_b + (M^{-1})^{ab} \partial \chi_a \bar{\partial} \chi_b \right\},
$$

where $M_{ab} = F_{ab} + \chi_c f^c_{ab}$.

5 Discussion

The construction associated to Abelian and non-Abelian T-duality has been generalised to find new T-duality transformations. The analyses has been carried out at the level of the classical non-linear sigma model action. It relies on the integrability condition (Bianchi identity) associated to some one-form $A^a = A^a dz + \bar{A}^a d\bar{z}$. The duality procedure can be carried out if the dependence of the original action on some fields $y^a$ appears through this one-form only. One can then trade the coordinates $y^a$ for the one-form $A^a$ at the price of introducing a Lagrange multiplier. The equations of motion for this Lagrange multiplier yield the original theory. The gauge fields $A^a$ and $\bar{A}^a$ appear quadratically in the action and without derivatives. Their equations of motion lead to a dual sigm model in which the Lagrange multiplier replaces the coordinates $y^a$.

A natural question in this context is whether this new T-duality remains a symmetry at the quantum level (see [9] and references within for a quantum treatment of Abelian and non-Abelian T-dualities). In other words, we would like to know if the string effective action corresponding to the backgrounds of the model (4.7) is equivalent to that coming from the backgrounds of the action (4.8). This study requires a knowledge of the transformation properties of the dilaton field. If the two-dimensional theory (4.7) possesses a dilaton background in the form

$$
- \frac{1}{4} \int d^2 \sigma \sqrt{\eta} R^{(2)}(x) \Phi (x) ,
$$

where $\eta$ is the metric on the world-sheet and $R^{(2)}$ is its Ricci scalar curvature. In analogy with Abelian and non-Abelian T-duality [10], we conjecture that the dilaton in the dual theory is given by

$$
\Phi = \Phi - \frac{1}{2} \ln \det (M) .
$$

The study of the string effective action under these new T-duality transformation is worth exploring.

Finally, we should mention that a duality transformation based on the gauging of the most general sigma model having a chiral symmetry ($H_L \times H_R$) was found in ref.[11] and some of its applications were explored in [12]. There, different kinds of gauging (a generalisation of Abelian axial and vector gauging) were used to construct the duality transformations. This duality symmetry reduces to ordinary Abelian duality if the chiral
group is Abelian. The relation between this “axial-vector” duality symmetry and non-Abelian duality (or our new duality) remains undetermined.

Acknowledgments: I would like to thank Janos Balog, Peter Forgács, Ian Jack and Max Niedermaier for useful discussions.

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