RANDOM COVERINGS OF 1–COMPLEXES AND THE EULER CHARACTERISTIC.

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Abstract. This article presents an algebraic topology perspective on the problem of finding a complete coverage probability of a domain by a random covering. In particular we obtain a general formula for the chance that a collection of finitely many compact connected random sets placed on a 1-complex $X$ has a union equal to $X$. The result is derived under certain topological assumptions on the shape of the covering sets (the covering ought to be good), but no a priori requirements on their distribution. An upper bound for the coverage probability is also obtained as a consequence of the concentration inequality. The techniques rely on a formulation of the coverage criteria in terms of the Euler characteristic of the nerve complex associated to the random covering.

1. Introduction

We consider finite random coverings of a metric space $X$, i.e. finite collections of compact random sets: $A = \{A_i\}, i = 1, \ldots, n$, understood as measurable maps [17, p. 121]

$$A_i : (\Omega, \sigma, \mathbb{P}) \rightarrow (C(X) \cup \{\emptyset\}, \sigma_{\text{Borel}}),$$

where $(\Omega, \sigma, \mathbb{P})$ is an underlying probability space, and $C(X)$ is the set of compact subsets of $X$ topologized by the Hausdorff distance, with the associated Borel algebra $\sigma_{\text{Borel}}$. The set $C(X) \cup \{\emptyset\}$ is a disjoint union with the point $\{\emptyset\}$, which plays a role of the empty set. A typical example of an (infinite) random covering is a coverage process defined on the Euclidean space. In [17] it is simply a sequence of random sets: $A = \{\xi_1 + A_1, \xi_2 + A_2, \ldots, \xi_k + A_k, \ldots\}$, where $\{A_k\}$ is a fixed family of subsets of $\mathbb{R}^n$ called grains of the process, and $\xi = \{\xi_i\}$ random points on $\mathbb{R}^n$. In applications $A_i$’s are often round balls of a fixed radius, and $\xi$ defines a Poisson process (in which case $A$ is refereed to as a Boolean model).

Problem 1. Given a random covering $\{A_i\}, i = 1, \ldots, n$ of a metric space $X$, find a complete coverage probability: $P(X \subset |A|)$, where $|A| = \bigcup_i A_i$.

Reviewing the history of Problem 1 it was first considered by Whitworth, [36] in the basic case of a finite collection of independent identically distributed fixed length $\alpha$ arcs $\{A_i\}$ on a unit circumference circle. Much later, Stevens [31] provided a complete answer to the question of Whitworth in the form

$$P(S^1 \subset |A|) = \sum_{j=1}^{\lfloor \frac{n}{\alpha} \rfloor} (-1)^{j+1} \binom{n}{j} (1 - j\alpha)^{n-1}. \quad (1.1)$$
The Steven’s result was further improved by Siegel and Holst [30] where they allowed varying lengths for the arcs. In [14], Flatto obtained an asymptotic expression for coverage as $\alpha \to 0$. The extension of the circle problem to the 2-sphere $S^2$ was considered by Moran and Groth [28], who derived an approximation for the probability $\mathbb{P}(S^2 \subset |A|)$, and later Gilbert [16] showed the bounds

$$(1 - \lambda)^n \leq P(S^2 \subset |A|) \leq \frac{4}{3} n(n - 1) \lambda (1 - \lambda)^{n-1},$$

where $\lambda = \left(\sin \frac{\alpha}{2}\right)^2$ is the fraction of the surface of $S^2$ covered by a spherical $\alpha$–caps i.e. caps of radius $\alpha$. For $\alpha \in \left[\frac{\pi}{2}, \pi\right]$, the explicit expression for $P(S^2 \subset |A|)$ has been found by Miles [26]. Work in [7] provides explicit formulas for the complete coverage probability for $\alpha$–caps on the $m$-dimensional unit sphere $S^m$ when $\alpha \in \left[\frac{\pi}{2}, \pi\right]$, and upper bounds for $\alpha \in [0, \pi)$. The literature concerning the coverage probability in the asymptotic regimes (where the diameter of grains tends to zero) is vast and we only list a small fraction here [15, 29, 2, 27, 5]. Further, the reader may consult the recent work in [7] for a more accurate account of the history of Problem 1.

In this article we focus solely on the case of coverage for 1-dimensional domains $X$ which are homeomorphic to finite graphs (specifically, we will rely on the notion of the $\Delta$-1-complex [18]). Generally, we will make an underlying assumption of the covering $A$ to be good i.e. to satisfy (almost surely) the hypotheses of the Nerve Lemma (c.f. Section 4). In Proposition 4.7 of Section 4.2 we show that a covering $A$ of a 1-complex $X$ is always good if its elements $A_i$ are connected and small enough in diameter. The first version of our main theorem is stated below.
Theorem 1.1 (Coverage probability for compact connected 1–complexes $X$, with $\partial X = \emptyset$). Let $A = \{A_i\}, i = 1, \ldots, n$ be a random good covering of $X$ (with $\partial X = \emptyset$). Then, the range of $x = x(A)$ can be restricted to
\[ m = \chi(X) \leq x(A) \leq n = \overline{m}, \tag{1.4} \]
and the complete coverage probability equals
\[ \mathbb{P}(X \subset |A|) = \mathbb{P}(x(A) = \chi(X)) = \sum_{s \in \mathcal{C}_n : \chi(s) = \chi(X)} P_s = \sum_{s \in \mathcal{C}_n} a_s(x) p_s, \tag{1.5} \]
where $a_s(x) = \sum_{k=0}^{N} v_k(x) c_{s,k}(x), N = \overline{m} - m,$
with $p_s$ given in $[1.3],$ and
\[ v_k(x) = \frac{(-1)^k}{N!} \sum_{m < j_1 < j_2 < \ldots < j_{N-k} \leq \overline{m}} j_1 j_2 \ldots j_{N-k}, \quad v_N(x) = \frac{(-1)^N}{N!}, \tag{1.6} \]
\[ c_{s,k}(x) = \left\{ \begin{array}{ll}
\sum_{i=0}^{r_{top}(s)} \sum_{j=0}^{r_{top}(s)} (-1)^{r_{top}(s) - i - j} \binom{r_{top}(s)}{i} \binom{r_{top}(s)}{j} (i - j + r_{low}(s) - r_{low}(s))^k & \text{if } k \geq r(s), \\
0 & \text{if } k < r(s),
\end{array} \right. \]
where $r^\pm = r^\pm(s), r^\pm_{top} = r^\pm_{top}(s), r_{low}(s) = r^\pm - r^\pm_{top}(s)$ stand for a number of respectively total, top and lower: even(odd) dimensional faces of $s \in \mathcal{C}_n$, and $r(s)$ denotes a number of all faces.

If the diameter of $A_i$ is smaller than $\frac{1}{6}C$ almost surely (where $C$ is a length of the shortest cycle in $X$), then $p_s$ further simplifies as
\[ p_s = \mathbb{P}(\forall i,j \in V(s), i \neq j \{A_i \cap A_j \neq \emptyset\}), \tag{1.7} \]
where $V(s)$ is the vertex set of $s$.

An extension of the above result to the case of 1–complex $X$, with no assumptions on $\partial X$, is provided in Theorem 5.1 of Section 5. One obvious corollary of the above result is the fact that the complete coverage probability of any good random covering $\{A_i\}$ is determined by finitely many numbers.$^4$ A computational complexity of deriving $\mathbb{P}(X \subset |A|)$ via the method of Theorem 1.1 is not addressed here. However, one may expect that, due to the size of the set $\mathcal{C}_n$ (see Section 2.1), determination of $a_s(x)$ in (1.5) or the set $\{s \in \mathcal{C}_n \mid \chi(s) = \chi(X)\}$ to be double exponentially hard in $n$. Note that the coefficients $a_s(x)$ are independent of the underlying distribution vector $(p_s)$, therefore once computed for a certain size problem can be reapplied as $(p_s)$ changes. The vector $(p_s)$ can be conveniently estimated numerically (e.g. via the standard maximum likelihood estimation, c.f. [23]) but again in the simplest case of Equation (1.7) it is of exponential size: $2^{\frac{n(n+1)}{2}}$. Therefore, in practical situations one may be interested asymptotic distributions of $\chi(A)$ (as $n \rightarrow \infty$) which would lead to parametric estimators or useful bounds for $\mathbb{P}(X \subset |A|)$. Currently available results (e.g. in [21, 20]) concern sparse regimes and they are not applicable, unless we allow the diameter of random sets in $A = \{A_i\}$ to tend to 0 as $n \rightarrow \infty$ (as is the case considered

$^2$which is not obvious when considering vacancy, i.e. volume of $X - |A|$ c.f. [17]
e.g. in [15]). As far as the question of useful bounds for $\mathbb{P}(X \subset |A|)$, as a first step we derive an upper bound for the coverage probability, via the concentration inequality [3] in the following

**Theorem 1.2.** Let $A = \{A_i\}, i = 1, \ldots, n$ be a random good covering of $X$, then

$$
\mathbb{P}(X \subset |A|) \leq \exp\left(\frac{-\mu_0^2}{2n(|\chi_{rel}(X, \partial X)| + 2)^2}\right),
$$

(1.8)

where $\mu_0$ denotes the expected value of the relative Euler characteristic $\chi_{rel}(A, \partial A)$ of the random pair $(\mathcal{N}(A), \mathcal{N}(\partial A))$.

Below we outline our strategy to derive formulas of Theorem 1.1. The first equality in (1.5) is a consequence of basic topological considerations provided in Section 4, the second identity in (1.3) follows once we determine the probability distribution of $\chi(A)$ in terms of $(p_k)$. The required steps for this are summarized below for a general case of the Euler characteristic of some random complex $K = (\mathcal{C}_n, \mathcal{P}_K)$. Note that $\mathbb{P}_K$ is determined by the vector of probabilities $(P_k)$ representing atomic measures $P_k = \mathbb{P}_K(\{s\})$ of elements $s$ in $\mathcal{C}_n$ or equivalently a vector $(p_k)$ where

$$
p_k = \mathbb{P}_K(\{r \in \mathcal{C}_n \mid s \subset r\}).
$$

(1.9)

Thanks to the obvious identity: $p_k = \sum_{r \subset s} P_r$, vectors $(P_k)$ and $(p_k)$ are related by an invertible binary matrix $B$

$$(p_k) = B \cdot (P_k),
$$

(1.10)

determination of $B^{-1}$ is implicit in the following considerations.

We will think of random variables on $K$ as real coefficient polynomials in the indeterminates $e_I$, where $e_I = e_I(K)$ will be the indicator functions of faces $I$ of $\Delta_n$. Thus the polynomial ring $\mathbb{R}[e_I] = \mathbb{R}\{e_I\}$ is of the main interest. We will denote $\chi, f, Q, P, \ldots$, the algebraic expressions of elements in $\mathbb{R}[e_I]$, and $\chi(K), f(K), Q(K), P(K) \ldots$ etc., when treated as random variables on $K$. In the paragraph below we review a method to obtain general formulas for moments and distributions of discrete random variables of finite range.

Let $[m(Q), \overline{m}(Q)] \subset \mathbb{Z}$, $m(Q) \leq \overline{m}(Q)$ denote the integer range of a given $Q \in \mathbb{R}[e_I]$. Also, let $N = \overline{m}(Q) - m(Q)$ and $v_{kj}(Q)$ be coefficients of a certain inverse Vandermonde matrix (see Section 3) given explicitly as follows [12]

$$
v_{kj}(Q) = \frac{(-1)^{j+k}}{N!} \binom{N}{j} e_{N-k}(m(Q), \ldots, \overline{m}(Q)), \quad k, j = 0, \ldots, N
$$

(1.11)

where $e_{N-k}(m(Q), \ldots, \overline{m}(Q))$ denotes a value of the $(N - k - 1)$-th elementary symmetric polynomial $e_{N-k-1}$ on the range of integers $m(Q), \ldots, \overline{m}(Q)$ (with the $j$-th variable set to zero, where variables are indexed from 0 to $N$). Expansions for powers of $Q \in \mathbb{R}[e_I]$ are given as follows

$$
Q^k = \sum_{s \in \mathcal{C}_n} c_{s,k}(Q) e_s, \quad e_s = \prod_{I \in s} e_I.
$$

Further, for any $s \in \mathcal{C}_n$ we set

$$
Q(s) := Q(\{e_I = 1 \mid I \in s\}).
$$

Namely, $Q(s)$ is obtained from $Q$ by substituting $e_I = 1$ for all $I \in s$. We also denote by $Q(s)(0)$ be the constant coefficient of $Q(s)$ (i.e. the value of $Q(s)$ at 0). Moments and the probability
distribution of $Q(K)$ can be now expressed as follows
\begin{equation}
E(Q^k(K)) = \sum_{s \in \mathcal{C}_n} (Q(s)(0))^k P_s = \sum_{s \in \mathcal{C}_n} c_{s,k}(Q) p_s
\end{equation}
\begin{equation}
\mathbb{P}(Q(K) = m(Q) + j) = \sum_{s \in \mathcal{C}_n: Q(s)(0) = m(Q) + j} P_s = \sum_{s \in \mathcal{C}_n} a_{s,j}(Q) p_s, \quad \text{for} \quad j \in [0, N],
\end{equation}
where $a_{s,j}(Q) = \sum_{k=0}^N v_{kj}(Q) c_{s,k}(Q)$ (see Theorem 3.3 in Section 3.2).

Given a random complex $K$, a basic example of interest is the number of its $d$–dimensional faces
\begin{equation}
f_d = \sum_{\{I\} \in \mathcal{C}_n; |I| = d+1} e_I.
\end{equation}
and the Euler characteristic of $K$:
\begin{equation}
\chi(K) : (\mathcal{C}_n, \mathbb{P}_K) \longrightarrow \mathbb{Z},
\chi(K)(s) = \chi(s)(0)
\end{equation}
By the Euler–Poincare formula (see Equation (2.6), c.f. [18]) we have the following relation between
\begin{equation}
\chi = \sum_{d=0}^{n-1} (-1)^d f_d.
\end{equation}
Thanks to (1.12) to determine distributions of $f_d(K)$ or $\chi(K)$ in terms of $(p_s)$ it suffices to obtain expressions for the coefficients $c_{s,k}(\chi)$. Let $r^\pm = r^\pm(s)$, $r_{\text{top}}^\pm = r_{\text{top}}^\pm(s)$, $r_{\text{low}}^\pm = r_{\text{low}}^\pm(s)$ denote a number of respectively total, top and lower: even(odd) dimensional faces of $s \in \mathcal{C}_n$. Clearly,
\begin{equation}
\chi(s)(0) = r^+(s) - r^-(s).
\end{equation}
The inclusion–exclusion principle yields the following algebraic identity\(^3\) for $c_{s,k}(f_d)$,
\begin{equation}
c_{s,k}(f_d) = \sum_{1 \leq s \leq \mathcal{C}_n, 1 \subseteq \mathcal{I}} (-1)^{|s| - |\mathcal{I}|} (f_d(1)(0))^k = \sum_{i=1}^{r_{\text{top}}(s)} (-1)^{r_{\text{top}}(s) - i} \binom{r_{\text{top}}(s)}{i} i^k,
\end{equation}
and the following identity for $c_{s,k}(\chi)$ (where\(^4\) $k \geq r(\mathcal{I})$)
\begin{equation}
c_{s,k}(\chi) = \sum_{1 \leq s \leq \mathcal{C}_n, 1 \subseteq \mathcal{I}} (-1)^{|s| - |\mathcal{I}|} (\chi(1)(0))^k = \sum_{1 \leq s \leq \mathcal{C}_n, 1 \subseteq \mathcal{I}} (-1)^{|s| - |\mathcal{I}|} (r^+(1) - r^-(1))^k
\end{equation}
\begin{equation}
= \sum_{i=0}^{r_{\text{top}}(s)} \sum_{j=0}^{r_{\text{top}}(s) - i - j} (-1)^{r_{\text{top}}(s) - i - j} \binom{r_{\text{top}}(s)}{i} \binom{r_{\text{top}}(s) - j}{i} (i - j + r_{\text{low}}^+(s) - r_{\text{low}}^-(s))^k.
\end{equation}
An extension of these considerations to the case of the relative random Euler characteristic $\chi_{\text{rel}}(K, L)$ is stated in Corollary 3.4 of Section 3.

The article is organized as follows: In Section 2 we further discuss the general setup of random complexes and their associated invariants, mainly $\chi(A)$ and associated random variables. In Section 3 we derive identities in (1.12), in particular relevant formulas in the case of the random relative
\(^3\)where $c_{s,k}(f_d)$ vanish, unless $s$ is a subcomplex of the complex which is a union of $d$–dimensional faces, with some $d$–faces present
\(^4\)c_{s,k}(\chi) vanishes for $k < r$. 

RANDOM COVERINGS AND THE EULER CHARACTERISTIC 5
Euler characteristic, in Corollary 3.4. Further, in Section 4 we prove several basic topological results showing that the Euler characteristic of the nerve of a covering suffices to determine complete coverage of a 1–complex. In Proposition 4.7 we also provide a sufficient condition for the to be good (in terms of a diameter of its elements) and determine when its nerve is equivalent to an associated Vietoris-Rips complex. We collect relevant facts in Section 5 and provide the proof of Theorem 1.1 there. The upper bound for \( P(X \subset |A|) \) of Theorem 1.2 is shown in Section 6. Section 7 concludes the paper with a comparison of our formulas to the Steven’s formula (1.1). As a test case we exhibit specific computations in the case of three random arcs.

Beyond 1-complexes the techniques of algebraic topology provide coverage criteria for higher dimensional objects. For instance if an underlying space \( X \) is an \( m \)-dimensional manifold a necessary and sufficient condition for coverage is nonvanishing of the \( m \)-th(top) Betti number of the nerve \( \mathcal{N}(A) \). We aim to develop these ideas in subsequent papers.

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2. Random complexes and their topological invariants.

2.1. Random complexes. We refer the reader to [18] for background on algebraic topology. Consider \( \Delta_n \) to be a full simplex on \( n \)–vertices indexed by \( 1, \ldots, n \) (geometrically \( \Delta_n \) is the convex hull of \( n \) points given by the standard basis vectors in \( \mathbb{R}^{n+1} \)). Recall that a \( d \)-dimensional face in \( \Delta_n \) can indexed by the collection of its \( d+1 \) vertices: \( I = \{i_1, i_2, \ldots, i_{d+1}\} \), where \( 1 \leq i_1 < i_2 < \cdots < i_{d+1} \leq n \). Denote the set of all faces of \( \Delta_n \) by \( f(n) \) and particularly \( d \)-dimensional faces by \( f_d(n) \), i.e.

\[
\begin{align*}
    f(n) &= \{ I \mid I \subset 2^{\{1, \ldots, n\}} \}, \\
    f_d(n) &= \{ I \mid I \subset 2^{\{1, \ldots, n\}}, |I| = d + 1 \}\tag{2.1}
\end{align*}
\]

Next, we want to consider the set of all labeled sub-complexes \( \mathcal{C}_n \) of \( \Delta_n \) union a special point \( \{\emptyset\} \) with a role of the empty set. By a labeled sub-complex we understand a subcomplex of \( \Delta_n \) with the labeling given by vertices of \( \Delta_n \). A natural set to consider for enumerating labeled subcomplexes is the power set \( 2^{f(n)} \) of \( f(n) \) union \( \{\emptyset\} \), which we denote by \( \mathfrak{P}_n \). Here and thereafter, we slightly abuse the notation and use \( s, r, k \) for elements of both \( \mathcal{C}_n \) and \( \mathfrak{P}_n \).

Clearly, there is a surjective correspondence \( \Pi : \mathfrak{P}_n \to \mathcal{C}_n \) which to a given subset \( s \in \mathfrak{P}_n \) assigns a subcomplex \( \Pi(s) \) in \( \mathcal{C}_n \) given by the union of elements \( \{I\} \) of \( s \) treated as faces of \( \Delta_n \). \( \Pi \) is clearly not bijective, however, with certain choices we may easily build right inverses. In particular, we will be interested in two cases, which we refer to as the antichain and chain representations \( \overset{\sim}{\pi} : \mathcal{C}_n \to \mathfrak{P}_n, \overset{\sim}{\pi} : \mathcal{C}_n \to \mathfrak{P}_n \). The antichain representative \( \overset{\sim}{s} \in \mathfrak{P}_n \) of \( s \in \mathcal{C}_n \), contains only it’s top dimensional faces, i.e. for any \( \{I\}, \{J\} \in \overset{\sim}{s} : I \not\subset J \) and \( J \not\subset I \) (in terms of the partial order of \( \mathfrak{P}_n \) given by \( \subset \) ). The chain representative \( \tilde{s} \) of \( s \in \mathcal{C}_n \) is obtained from the antichain representative by adding all remaining subfaces of \( s \). Clearly, \( \tilde{s} \) is the poset associated to the subcomplex \( s \) (c.f. [33]). We denote the images of \( \mathcal{C}_n \) in \( \mathfrak{P}_n \) under \( \overset{\sim}{\pi} \) and \( \overset{\sim}{\pi} \) by \( \overset{\sim}{\pi}_n \) and \( \overset{\sim}{\pi}_n \) respectively. We also have the
associated projections
\[ \hat{\Pi} : \mathcal{P}_n \mapsto \hat{\mathcal{C}}_n, \quad \bar{\Pi} : \mathcal{P}_n \mapsto \bar{\mathcal{C}}_n. \]  
(2.2)

Note that the cardinality of \( \hat{\mathcal{C}}_n \), and therefore \( \mathcal{C}_n \) and \( \bar{\mathcal{C}}_n \), is given by the Dedekind number \( M(n) \), c.f. [22]. For any \( s \in \mathcal{C}_n \), we call the elements of \( \hat{s} \) top faces of \( s \).

- For the rest of the paper we make the convention to identify \( \bar{\mathcal{C}}_n \) with \( \mathcal{C}_n \) i.e. we think about subcomplexes as represented by chains (unless indicated otherwise).

Recall from Section 1 that a finite random complex on \( n \) vertices we understand a discrete probability space \( K = (\mathcal{C}_n, \mathbb{P}_K) \). It is easy to see that \( \mathbb{P}_K \) satisfies the following equivalent conditions for \( I' \subset I \)

(A) \( \mathbb{P}_K(I \mid (I')^c) = \mathbb{P}_K(\{s \in \mathcal{C}_n \mid \{I\} \subset s\} \mid \{r \in \mathcal{C}_n \mid \{I'\} \not\subset r\}) = 0 \),

(B) \( \mathbb{P}_K(I' \mid I) = \mathbb{P}(\{s \in \mathcal{C}_n \mid \{I'\} \in s\} \mid \{r \in \mathcal{C}_n \mid \{I\} \in r\}) = 1 \).

In short (A) says that if a subface \( I' \) of \( I \) has not occurred then \( I \) cannot occur as well, equivalently (B) says that \( I' \) occurs whenever \( I \) has occurred. We say that \( K \) is supported on a subcomplex, \( k \in \mathcal{C}_n \) if and only if for any \( \{I\} \not\subset k \) we have \( \mathbb{P}_K(\{I\} = 0 \). Given random complexes \( K \) and \( L \) on \( n \)-vertices, the joint probability space \( (K, L) := (\mathcal{C}_n \times \mathcal{C}_n, \mathbb{P}_{K,L}) \) is a random pair if and only if \( L \) is almost surely a subcomplex of \( K \), i.e. the following condition holds

(C) for every \( (s, r) \in \mathcal{C}_n \times \mathcal{C}_n \) such that \( r \not\subset s \) we have \( \mathbb{P}_{K,L}(s, r) = 0 \).

For a given \( K \) (or \( (K, L) \)) it will be convenient to consider Bernoulli random variables which are indicator functions of faces in \( K \), i.e. for \( I \in f(n) \) we define

\[ e_I : \mathcal{C}_n \mapsto \{0, 1\}, \quad e_I(s) = \begin{cases} 1, & \{I\} \in s, \\ 0, & \text{otherwise}, \end{cases} \]  
(2.3)

For any \( s \in \mathcal{C}_n \), we set \( e_s = \prod_{I \in s} e_I \) an indicator function of the subcomplex \( s \), clearly \( e_s \) takes value 1 on \( r \) if and only if \( s \subset r \). Recalling, vectors \((p_s)\) and \((P_s)\) from (1.10), we have the following expressions in terms of \( e_I \)'s:

\[ p_s = \mathbb{P}(e_s = \prod_{I \in s} e_I = 1), \]

\[ P_s = \mathbb{P}\left( \prod_{I \in f(n), I \subset s} e_I \prod_{J \in f(n), J \not\subset s} (1 - e_J) = 1 \right) \]  
(2.4)

It is clear that one may now construct an inverse of \( B \) from (1.10) by expanding the polynomial

\[ P_s = \prod_{I \in f(n), I \subset s} e_I \prod_{J \in f(n), J \not\subset s} (1 - e_J) \]  
(2.5)

in monomials \( e_s \) and taking the expected value (see Section 3.2).

Summarizing, any random complex \( K \) is fully determined by indicator functions \( \{e_I\} \) of faces and their joint distribution. In the next section \( e_I \) will serve as formal indeterminates for functions defining topological random variables on \( K \), such as the Euler characteristic. The main use of conditions (A)(B) and (C) will be to define (in Section 2.3) an appropriate polynomial ring where the random topological invariants, such as \( \chi(K) \), live.

\[ \text{because events } \{s \in \mathcal{C}_n \mid \{I\} \subset s\} \text{ and } \{r \in \mathcal{C}_n \mid \{I'\} \not\subset r\} \text{ are disjoint} \]
Remark 2.1. In general, one could consider more flexible model of a finite random complex with \((\mathcal{P}_n, \bar{P})\) as the underlying probability space. It can be thought of as a distribution on open faces of \(\Delta_n\) (i.e. interiors of faces) with an exception of the zero dimension (the vertices). In this model it is possible, for instance, for an edge to occur without its vertices (i.e. (A) can be violated). We have the following obvious result

**Lemma 2.2.** Suppose that \((\mathcal{P}_n, \bar{P})\) satisfies (A) or (B), (with \(\mathcal{P}_n\) in place of \(\mathfrak{C}_n\) and \(\bar{P}\) in place of \(P_K\)). Then \(\bar{P}\) is supported on \(\mathfrak{C}_n = \mathcal{P}_n \subset \mathcal{P}_n\), and \((\mathfrak{C}_n; [\bar{P}])\) defines a finite random complex.

**Proof.** Let \(k \in \mathcal{P}_n\), it suffices to see that if \(k \neq \bar{k}\) then \(\bar{P}(k) = 0\). Pick \(\{I\} \in \bar{k} - k\), such that there is \(\{I\} \in k\) satisfying \(I' \subset I\). Then \(k = \{s \in \mathcal{P}_n | \{I\} \in s\} \cap \{r \in \mathcal{P}_n | \{I'\} \notin r\}\) from (A)

\[\bar{P}(k) \leq \bar{P}(\{s \in \mathcal{P}_n | \{I\} \in s\} \cap \{r \in \mathcal{P}_n | \{I'\} \notin r\}) = 0. \quad \square\]

**Remark 2.3.** We may easily generalize the definition of the random complex \(K\) to the case \(n = \infty\), and thus removing dependence in \(n\) in the definition. This is done by considering all labeled subcomplexes \(\mathfrak{C}_\infty\) of the infinite simplex \(\Delta_\infty = \bigcup_n \Delta_n\), and regarding a random complex \(K\) as a probability space \((\mathfrak{C}_\infty; \bar{P}_K)\). Such random complex is finite provided the support of \(K\) is contained in \(\Delta_n\) for some \(n\).

2.2. **Topological invariants in the random setting.** Recall that, thanks to the Poincare-Euler formula [18], the Euler characteristic of a general \(n\)-complex \(K\) is given by

\[\chi(K) = \sum_{j=0}^{n} (-1)^j \dim_j(K), \quad (2.6)\]

where \(\dim_j(K)\) denotes the dimension of the real coefficient \(j\)th chain group \(C_j(K; \mathbb{R})\), and equals (in the absolute case) to the number of \(j\)-dimensional faces \(f_j(K)\) of \(K\). We will also need a relative version of \(\chi\). Given a pair \((K, L)\) where \(L\) is a subcomplex of \(K\) we have

\[\chi_{rel}(K, L) = \sum_{j=0}^{n} (-1)^j \dim_j(K, L), \quad (2.7)\]

where \(\dim_j(K, L)\) denotes the rank the \(j\)th relative chain group \(C_j(K, L; \mathbb{R}) \cong C_j(K; \mathbb{R})/C_j(L; \mathbb{R})\) (c.f. [18]). Note that we have

\[\dim_j(K, L) = f_j(K) - f_j(L). \quad (2.8)\]

Invariants \(\chi(K)\) and \(\chi_{rel}(K, L)\) can be expressed in terms of the Betti numbers \(\{\beta_j(K)\}, \{\beta_j(K, L)\}\) of the chain complexes \(C_*(K)\) and \(C_*(K, L)\), (c.f. [18]). Specifically,

\[\chi(K) = \sum_{j=0}^{n} (-1)^j \beta_j(K), \quad \chi_{rel}(K, L) = \sum_{j=0}^{n} (-1)^j \beta_j(K, L). \quad (2.9)\]

2.3. **Random polynomials.** Given a random complex \(K\) let us treat the indicator functions of faces \(\{e_I\}\) (or in a case of a random pair \(\{e_I, w_J\}\)) as formal indeterminates and consider a polynomial ring in \(e_I\) (without loss of generality we work over \(\mathbb{R}\)):

\[\mathbb{R}[\{e_I\}] := \mathbb{R}[e_{\{1\}}, \ldots, e_{\{n\}}, e_{\{1,2\}}, \ldots, e_{\{i_1,\ldots,i_k\}}, \ldots, e_{\{1,\ldots,n\}}],\]

(or \(\mathbb{R}[e_I, w_J]\) in case of random pairs). Observe that any random variable \(X\) on \(K\) is given as such polynomial, namely \(X = \sum_{s \in \mathcal{P}_n} X(s) P_s\), where \(P_s\) is given in (2.5).
Based on (2.6) we may express the random Euler characteristic $\chi$ from (1.15) as the following polynomial in $\mathbb{R}[e_I]$:

$$\chi = \sum_{I \in f(n)} (-1)^{|I|-1}e_I,$$

(2.10)

**Lemma 2.4.** Given a random complex $K = (\mathcal{C}_n, \mathbb{P}_K)$ and its collection of the indicator functions $\{e_I\}$. Consider $Q, Q' \in \mathbb{R}[e_I]$ as two representatives of the same coset in $\mathbb{R}[e_I]/\mathcal{I}$, where $\mathcal{I}$ is an ideal generated by the following relations

$$\{e_I e_J = e_J \mid \text{for all } I \subset J\},$$

(2.11)

(in particular: $e_I^2 = e_I$.) Then $Q(K) = Q'(K)$ almost surely.

**Proof.** It suffices to show that $\mathbb{P}(e_I e_J = e_J) = 1$ for any $I, J$ where $I \subset J$. We have

$$\mathbb{P}(e_I e_J = 0) = \mathbb{P}(e_J = 0, e_I = 1) + \mathbb{P}(e_J = 1, e_I = 0) + \mathbb{P}(e_J = 0, e_I = 0).$$

Thanks to (A): $\mathbb{P}(e_I = 1, e_J = 0) = 0$, thus

$$\mathbb{P}(e_I e_J = 0) = \mathbb{P}(e_J = 0, e_I = 1) + \mathbb{P}(e_J = 0, e_I = 0) = \mathbb{P}(e_J = 0),$$

and $\mathbb{P}(e_J e_I = 1) = 1 - \mathbb{P}(e_I e_J = 0) = 1 - \mathbb{P}(e_I = 0) = \mathbb{P}(e_J = 1)$. $\square$

We will further denote the quotient ring $\mathbb{R}[e_I]/\mathcal{I}$ by $\mathbb{R}_Z[e_I]$. Clearly, $\mathbb{R}_Z[e_I]$ has an additive basis of monomials indexed by the chain representatives: $s \in \mathcal{C}_n$:

$$e_s = \prod_{I \in s} e_I.$$

(2.12)

In the case of pairs $(K, L)$ we have a pair of sets of face indicator functions $\{e_I, w_J\}$ corresponding to $K$ and $L$ respectively. Then, it is relevant to consider a polynomial ring $\mathbb{R}[e_I, w_J]$ modulo relations in (2.11) and additionally (thanks to property (C)):

$$\{w_J w_I = w_I \mid \text{for all } I \subset J\},$$

$$\{w_I = w_I e_J, \mid \text{for all } J \subset I\}.$$

(2.13)

The resulting quotient ring will be denoted by $\mathbb{R}_Z[e_I, w_J]$, and the analogous statement as Lemma 2.4 is true for random variables expressed as representatives in $\mathbb{R}_Z[e_I, w_J]$. An important for us example of a polynomial in $\mathbb{R}_Z[e_I, w_J]$ is the relative Euler characteristic

$$\chi_{rel}(K, L) : (\mathcal{C}_n \times \mathcal{C}_n, \mathbb{P}_K) \rightarrow \mathbb{Z},$$

$$\chi_{rel}(s, s') = \chi_{rel}(s, s'), \text{ if } s' \subset s, \text{ i.e. the relative Euler characteristic of } (s, s')$$

$$= 0, \text{ if } s' \notin s.$$

(2.14)

Note, that thanks to (C), the set of pairs $(s, s')$ such that $s' \not\subset s$ is of measure zero in $(K, L)$ thus the value of $\chi_{rel}(K, L)$ on such pairs is irrelevant. Thanks to (2.8), the polynomial expression for $\chi_{rel}(K, L)$ is given as follows

$$\chi_{rel} = \sum_{I \in f(n)} (-1)^{|I|-1}(e_I - w_I),$$

(2.15)
Remark 2.5. One may restrict considerations to the invariant ring \( \mathbb{R}_*[e_I]^{\mathcal{S}_n} \) or \( \mathbb{R}[e_I]^{\mathcal{S}_n} \) where \( \mathcal{S}_n \) is a symmetric group. Indeed, \( \mathcal{S}_n \) acts on the vector space \( V(e_I) \) spanned by \( \{e_I \mid I \in f(n)\} \) by permuting the faces \( I \), i.e.

\[
\pi(e_I) = e_I \circ \pi = e_{\pi(I)}, \quad \pi \in \mathcal{S}_n.
\]

Recall that \( \mathbb{R}_*[e_I]^{\mathcal{S}_n} = \{ Q \in \mathbb{R}_*[e_I] \mid Q = Q \circ \pi \} \). Noether’s Theorem, cf. [32, p. 27], implies that \( \mathbb{R}_*[e_I]^{\mathcal{S}_n} \) is generated as a polynomial algebra by \( \{ e_s \mid s \in \mathcal{C}_n \} \), where

\[
e_s = \frac{1}{n!} \left( \sum_{\pi \in \mathcal{S}_n} e_{\pi(s)} \right)
\]

and \( e_{\pi(s)} = e_{\pi(I_1, I_2, \ldots, I_k)} = e_{(\pi(I_1), \pi(I_2), \ldots, \pi(I_k))} \), for \( s = \{I_1, I_2, \ldots, I_k\} \). Note that \( \chi \in \mathbb{R}_*[e_I]^{\mathcal{S}_n} \) has an expansion

\[
n! \chi = \binom{n}{1} e_{\{1\}} - \binom{n}{2} e_{\{1,2\}} + \binom{n}{3} e_{\{1,2,3\}} - \ldots + (-1)^n \binom{n}{n} e_{\{1,\ldots,n\}}.
\]

3. Moments and distributions of the random Euler characteristic.

We begin with basic review of the method of moments for the finite range discrete random variable \( \chi \), and provide a specific formulation based on the recent work in [12]. Alternatively, one could use fractional moments (see e.g. [4, p. 17]), however they are not easier to consider in the setting of the Euler characteristic.

3.1. Method of moments. First, we need basic information on the Vandermonde matrix \( \mathcal{V} \) (c.f. [25]). Given a sequence of values \( \mathbf{x} = \{x_0, x_1, \ldots, x_N\} \), \( \mathcal{V} \) is an \( (N+1) \times (N+1) \) explicitly given as follows

\[
\mathcal{V} = \mathcal{V}(\mathbf{x}) = \begin{pmatrix}
1 & x_0 & \cdots & x_0^N \\
1 & x_1 & \cdots & x_1^N \\
\vdots & \vdots & \ddots & \vdots \\
1 & x_N & \cdots & x_N^N 
\end{pmatrix}.
\]

\( \mathcal{V} \) is invertible provided the \( x_i \)'s are distinct (c.f. [25]). A closed form of \( \mathcal{V}^{-1} \) has derived in [12] in terms of the elementary symmetric polynomials. Denote by \( e_i(j)(\mathbf{x}) \) the \( i \)th–elementary symmetric polynomial in variables: \( x_0, \ldots, \hat{x}_j, \ldots, x_N \) for \( j = 0, \ldots, N \), where \( \hat{x}_j \) means that \( x_j \) is omitted, specifically

\[
e_i(j)(\mathbf{x}) = \begin{cases}
1 & \text{if } i = 0 \\
\sum_{1 \leq l_1 < l_2 < \ldots < l_i \leq N, l_k \neq j} x_{l_1}x_{l_2}\ldots x_{l_i} & \text{if } i > 0.
\end{cases}\tag{3.1}
\]

By [12, p. 647], we have

\[
\mathcal{V}(\mathbf{x})^{-1} = (v_{ki}(\mathbf{x})), \quad \text{where} \quad v_{ki}(\mathbf{x}) = (-1)^{N+k} e_{N-k}(i)(\mathbf{x}) \prod_{j=0, j \neq i}^{N} (x_i - x_j), \tag{3.2}
\]

for \( i = 0, \ldots, N \), \( k = 0, \ldots, N \). In the case \( \mathbf{x} \) is an integer interval \( [m, \ldots, \overline{m}] \), \( m, \overline{m} \in \mathbb{Z}, m \leq \overline{m} \) of size \( N = \overline{m} - m \) we obtain

\[
v_{ki}(\mathbf{x}) = v_{ki}(m, \overline{m}) = \frac{(-1)^{i+k}}{N!} \binom{N}{i} e_{N-k}(i)(m, \ldots, \overline{m}), \tag{3.3}
\]
Lemma 3.1. Let $X$ be a discrete random variable of a finite range $X = \{x_0, x_1, \ldots, x_N\}$, and let $\mu_k = \mathbb{E}(X^k)$ denote the $k$-th moment of $X$. Given the vector $\mu = (\mu_0, \ldots, \mu_N)$ we can recover the distribution of $X$ explicitly as follows

$$p_i = \mathbb{P}(X = x_i) = \sum_{k=0}^{N} v_{ki} \mu_k, \quad i = 0, \ldots, N,$$

where $v_{ki} = v_{ki}(x)$ are the Vandermonde coefficients.

Proof. By definition we have a linear system of $N$ equations

$$\mu_k = \sum_{i=0}^{N} x_i^k p_i, \quad k = 0, 1, \ldots, N.$$

In the matrix form this system of reads: $pV = \mu$ where $p = (p_0, \ldots, p_N)$, and $\mu = (\mu_0, \ldots, \mu_N)$. Since all $x_i$'s are distinct $\det(V) = \prod_{i \neq j} (x_i - x_j) \neq 0$. Thus $V$ is invertible and we have the unique solution $p = \mu V^{-1}$. Identity (3.4) is a direct consequence of (3.2). $\square$

Our goal for the next subsection is to provide expressions for distributions of polynomial random variables in $\mathbb{R}_I[e_I]$.

3.2. Distributions of random polynomials. Since the differences between $\mathbb{R}_I[e_I]$ and $\mathbb{R}_I[e_I, w_J]$ are mostly notational, we will work with the former. Recall from Section 2.3 that any representative polynomial in $\mathbb{R}[e_I]$ is a linear combination of monomials $e_k$ from (2.12)

$$Q = \sum_{k \in P_n} c_k e_k, \quad c_k \in \mathbb{R}, \quad (3.5)$$

where the constant coefficient $c_0 = c_{\emptyset}$ is indexed by the empty set. Note that if $Q \in \mathbb{R}_I[e_I]$ then, thanks to the relations in $\mathbb{R}_I[e_I, w_J]$, we may always pick expansions of $Q$ in terms of the antichain or chain representatives i.e.

$$Q = \sum_{\tilde{s} \in \tilde{C}_n} c_{\tilde{s}} e_{\tilde{s}}, \quad \text{or} \quad Q = \sum_{\tilde{s} \in \tilde{C}_n} c_{\tilde{s}} e_{\tilde{s}} = \sum_{s \in C_n} c_s e_s, \quad (3.6)$$

where in the second expansion we just applied our convention from Section 2.1 to identify elements of $C_n$ with their chain representatives. We refer to (3.6)(left) as the antichain representative and (3.6)(right) as the chain representative of $Q$ in $\mathbb{R}_I[e_I]$. From the perspective of Lemma 2.4 it is irrelevant which expansion of $Q$ we choose. Below, we outline a strategy to determine coefficients $c_k$ of (3.5) via the inclusion–exclusion principle.

Recall, the general form of the inclusion–exclusion principle [24]: Given a finite set $F$ and functions $f, g : 2^F \rightarrow \mathbb{R}$ satisfying

$$g(F) = \sum_{S: S \subseteq F} f(S), \quad (3.7)$$

we have

$$f(F) = \sum_{S: S \subseteq F} (-1)^{|F|-|S|} g(S). \quad (3.8)$$

Recall the following notation: given $Q \in \mathbb{R}[e_I]$ and $s \in \Psi_n$ define

$$Q(s) := Q(\{e_I = 1 \mid I \in s\}).$$
i.e. $Q(s)$ is a polynomial obtained from $Q$ by substituting $e_I = 1$ for all $\{I\} \in s$, and $Q(s)(0)$ its constant coefficient.

**Lemma 3.2.** Consider any representative $Q \in \mathbb{R}[e_I]$ in a general form (3.5). For any $k \in P_n$ the coefficient $c_k$ of $Q$ in the expansion (3.5) is given as follows

$$c_k(Q) = \sum_{r \in P_n, r \subset k} (-1)^{|k| - |r|} Q(r)(0). \quad (3.9)$$

In the case $Q$ is represented by the chain expansion (right) (3.6), for any $s \in C_n$, $s \neq \emptyset$ we have

$$c_s(Q) = \sum_{r \in C_n, r \subset s} (-1)^{|s| - |r|} Q(r)(0) - c_0, \quad (3.10)$$

where $c_0 = c_\emptyset = Q(0)$ is the constant term of $Q$.

**Proof.** In the inclusion–exclusion principle set $F = k$, then any subset $S \subset F$ is just a subset of faces $r$ of $k$, i.e. $r \in P_n$ and $r \subset k$. Directly from (3.5) for any $r \subset k$, we have

$$Q(r)(0) = \sum_{r' \subset r} c_{r'}$$

thus setting $g(r) = Q(r)(0)$ and $f(r) = c_r$, Equation (3.9) follows from (3.8). To obtain (3.10) consider the polynomial $\bar{Q} = Q - c_0$. If $r \subset k$ and $\bar{r} \neq \bar{\emptyset}$, then $\bar{Q}(r)(0) = 0$. Therefore, for $s \in C_n$, Equation (3.9) yields

$$c_s(Q) = \sum_{r \in P_n, r \subset s} (-1)^{|s| - |r|} \bar{Q}(r)(0) = \sum_{r \in C_n, r \subset s} (-1)^{|s| - |r|} \bar{Q}(r)(0).$$

Because $c_s(Q) = c_s(\bar{Q})$ for $s \neq \emptyset$, the identity in (3.10) follows. \qed

For a polynomial random variable $Q \in \mathbb{R}[e_I]$ in a general form (3.5), define constants

$$m(Q) = \sum_{s \in P_n} c_{-s}^-, \quad c_{-s}^- = \min\{c_s, 0\}, \quad \bar{m}(Q) = \sum_{s \in P_n} c_s^+, \quad c_s^+ = \max\{c_s, 0\}. \quad (3.11)$$

Denote the coefficients of the general expansion (3.5) of the chain representative of the $k$-th power $(Q)^k$ by $c_{s,k}(Q)$, i.e.

$$Q^k = \sum_{s \in C_n} c_{s,k}(Q) e_s. \quad (3.12)$$

We summarize efforts of this section by stating the following result which is a direct consequence of Lemma 3.1 and Lemma 3.2 and justifies the identity in (1.12) of Section 1.

---

6i.e. $r$ is not a set of chains
Theorem 3.3. Given \( Q \) as a chain representative in \( \mathbb{R}_\mathcal{I} \), suppose that the set of realizations of \( Q \) is in the integer interval \([m, \bar{m}]\), then the distribution of \( Q \) and its moments are given as follows

\[
\mu_k = \mathbb{E}(Q^k) = \sum_{s \in \mathcal{C}_n} (Q(s)(0))^k P_s = \sum_{s \in \mathcal{C}_n} c_{s,k}(Q) p_s,
\]

\[
\mathbb{P}(Q = m + j) = \sum_{s \in \mathcal{C}_n, Q(s)(0) = m + j} P_s = \sum_{s \in \mathcal{C}_n} a_{s,j}(Q) p_s, \quad j \in [0, N], \quad N = \bar{m} - m
\]

(3.13)

for \( a_{s,j}(Q) = \sum_{k=0}^N v_{kj}(Q) c_{s,k}(Q), \)

where \( v_{kj}(Q) \) were defined in (3.2). Further, \( c_0 = Q(0) \) and \( c_{0,k} = c_0^k \), and for \( s \neq \emptyset \):

\[
c_{s,k}(Q) = \sum_{r \in \mathcal{C}_n; s \subseteq r} (-1)^{|s| - |r|} (Q(x)(0) - c_0)^k.
\]

(3.14)

Proof. Since \( e_s \) are Bernoulli random variables

\[
\mu_k = \mathbb{E}(Q^k) = \sum_{s \in \mathcal{C}_n} c_{s,k}(Q) \mathbb{E}(e_s) = \sum_{s \in \mathcal{C}_n} c_{s,k}(Q) p_s,
\]

thus (3.13) is an immediate consequence of (3.4). Formula (3.14) follows from (3.10) applied to \( Q^k \) \( \square \).

3.3. Formulas for \( \chi(K), f_d(K) \) and \( \chi_{rel}(K, L) \). In this section we aim to provide a slightly more tractable formulas for the coefficients \( c_{s,k}(\cdot) \) and the integer ranges \([m(\cdot), \bar{m}(\cdot)]\) for the polynomials \( \chi(K), f_d(K) \) and \( \chi_{rel}(K, L) \), where \( K \) is a given random complex on \( n \) vertices. These formulas were partially listed as (1.16), (1.17) in Section 1. Thanks to Theorem 3.3, it will provide us with a more precise characterization of distributions for these polynomials.

We begin with the case of \( f_d(K) \). Clearly, the range of \( f_d \) is contained in between

\[
m(f_d) = 0, \quad \text{and} \quad \bar{m}(f_d) = \left( \begin{array}{c} n \\ d+1 \end{array} \right).
\]

(3.15)

For a subcomplex \( s \in \mathcal{C}_n \) and its corresponding antichain \( \hat{s} \), recall the following notation

\[
\begin{align*}
  r^+_{top} &= r^+_{top}(s) = \text{num of even dimensional faces in } \hat{s}, \\
  r^-_{top} &= r^-_{top}(s) = \text{num of odd dimensional faces in } \hat{s}, \\
  r^+_{low} &= r^+_{low}(s) = \text{num of even dimensional faces in } s - \hat{s}, \\
  r^-_{low} &= r^-_{low}(s) = \text{num of odd dimensional faces in } s - \hat{s}, \\
  r_{top} &= r_{top}(s) = r^+_{top} + r^-_{top} = |\hat{s}|, \\
  r_{low} &= r_{low}(s) = |s| - |\hat{s}|, \quad r = r(s) = r_{top} + r_{low} = |s|.
\end{align*}
\]

(3.16)

Proof of Formula (1.16). Applying (3.14) directly to \( f_d = f_d(K) \) we obtain the first identity in (1.16). For the second equation in (1.16), let \( l \in \mathcal{P}_n \) be the set of all \( d \)-faces. Since \( f_d = \sum_{l \in \mathcal{P}_n} e_l \), for any \( k \leq 1 \), Equation (3.9) implies

\[
c_k((f_d)^k) = \sum_{r \in \mathcal{P}_n; r \subseteq k} (-1)^{|k| - |r|} (f_d(r)(0))^k = \sum_{i=1}^{|k|} (-1)^{|k| - i} \binom{|k|}{i} i^k.
\]

(3.17)
Next, we turn to the random polynomial \( \chi \) which implies the identity in (1.16) via the notation of (3.16). □

If \( K \) is supported on some subcomplex \( k \in \mathcal{C}_n \), smaller than the full \( n \)-simplex, the above range can be narrowed to

\[
\bar{m}(\chi(K)) = - \sum_{0 \leq 2r+1 \leq \dim(k)} f_{2r+1}(k), \quad \bar{m}(\chi(K)) = \sum_{0 \leq 2r \leq \dim(k)} f_{2r}(k).
\]

Proof of Formula (1.17). Applying (3.14) to \( Q = \chi \) directly one obtains the following formula for the coefficients \( c_{a,k}(\chi) \)

\[
c_{a,k}(\chi) = \sum_{1 \in \mathcal{C}_n, 1 \subset \mathbf{s}} (-1)^{|s|-|l|}(\chi(1)(0))^k = \sum_{1 \in \mathcal{C}_n, 1 \subset \mathbf{s}} (-1)^{|s|-|l|}(r^+(1) - r^-(1))^k.
\]

which shows the first part of (1.17). To obtain the second part of (1.17) we can follow the same reasoning as for \( f_d \) above. However, we choose to present a different, slightly more involved, argument for the purpose of cross verification. It will use the inclusion–exclusion principle but in a different way than in Lemma [3.2] namely via the well known multinomial formula. Recall that given indeterminates \( x_1, \ldots, x_m \) we have

\[
(x_1 + x_2 + \ldots + x_m)^k = \sum_{\alpha=(\alpha_1, \alpha_2, \ldots, \alpha_m)} \binom{k}{\alpha} x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_m^{\alpha_m},
\]

where \( \binom{k}{\alpha} = \binom{k}{\alpha_1, \alpha_2, \ldots, \alpha_m}, \alpha_i \geq 0, |\alpha| = \sum \alpha_i \) and \( \alpha \) form all possible partitions of \( k \). Let \( \alpha \) have coordinates indexed by \( f(n) \) (i.e. the faces of \( \Delta_n \)). A direct application of (3.19) to (2.10) yields

\[
(\chi)^k = \sum_{\alpha=(\alpha_j), |\alpha|=k} \binom{k}{\alpha} \prod_{I \in f(n)} (-1)^{|I|-1} e_I^{\alpha_I}
\]

\[
= \sum_{\alpha=(\alpha_j), |\alpha|=k} \left((-1)^{\sum_{I \in s(\alpha_j)} (|I|-1)\alpha_I}\binom{k}{\alpha}\right) e_\mathbf{s}(\alpha),
\]

where we denoted

\[
s(\alpha) = \{I \in f(n) \mid \alpha_I > 0\}.
\]

Observe that for any \( \alpha \) and \( \alpha' \),

\[
e_\mathbf{s}(\alpha) = e_\mathbf{s}(\alpha'), \quad \text{in} \quad \mathbb{R}_T[e_I],
\]

Considering \( f_d \) as an element of \( \mathbb{R}_T[e_I] \) and choosing a chain representative for \( f_d^k \), we conclude that its coefficients \( c_{a,k}(f_d^k) \) vanish unless the corresponding antichain \( s \) consists of purely \( d \)-faces. In the latter case we obtain from (3.17)

\[
c_{a,k}(f_d^k) = c_k((f_d)^k), \quad \text{for} \quad k = s,
\]

which implies the identity in (1.16) via the notation of (3.16). □
if and only if the corresponding antichains are the same i.e. \( \hat{s}(\alpha) = \hat{s}(\alpha') \). Fix a chain representative of some complex \( s \in \mathcal{C}_n \) and let \( \hat{s} \) be the corresponding antichain. Clearly, \( \hat{s} \subset s \), consider partitions \( \alpha \) of \( k \) which are in the form \( \alpha = \beta + \gamma \) where \( \beta = (\beta_1) \), satisfies: \( \beta_1 > 0 \) for \( I \in \hat{s} \) and \( \beta_1 = 0 \) for \( I \in s - \hat{s} \), and \( \gamma = (\gamma_1) \) satisfies: \( \gamma_1 \geq 0 \) for \( I \in s - \hat{s} \) and \( \gamma_1 = 0 \) for \( I \in \hat{s} \). The following claim immediately follows

**Claim:** Given \( s \in \mathcal{C}_n \), and any partition \( \alpha \) of \( k \) indexed by \( f(n) \), we have \( \bar{\Pi}(s(\alpha)) = s \) if and only if \( \alpha \) has the above decomposition: \( \beta + \gamma \), (see (2.2) for the definition of \( \bar{\Pi} \)).

Therefore, the \( c_{s,k}(\chi) \) coefficient of the chain representative of \( (\chi)^k \) is a sum of coefficients of \( e_{s(\alpha)} \) for all \( \alpha \) in the form \( \beta + \gamma \). Applying notation (3.16), we may express it as

\[
(\chi)^k = \sum_{s \in \mathcal{C}_n} c_{s,k}(\chi) e_s,
\]

where

\[
c_{s,k}(\chi) = \begin{cases} \sum_{(\beta, \gamma) = (\beta_1, \ldots, \beta_{r_{top}}, \gamma_1, \ldots, \gamma_{r_{low}}),} (-1)^{\sum_{i=1}^{r_{top}} |I_i| - 1} \beta_i + \sum_{j=1}^{r_{low}} |J_j| - 1} \gamma_j \binom{k}{\beta, \gamma}, & \text{if } k \geq r, \\ 0, & \text{otherwise}, \end{cases}
\]

where we indexed the faces of \( \hat{s} \) in \( s \) by \( \{I_i\}, i = 1, \ldots, r_{top} \) and faces of \( s - \hat{s} \) in \( s \) by \( \{J_j\}, j = 1, \ldots, r_{low} \). To set up the inclusion–exclusion principle, note that the sum for \( c_{s,k}(\chi) \) is a part of the larger sum (where we allow \( \beta_i \geq 0 \)):

\[
\sum_{(\beta, \gamma) = (\beta_1, \ldots, \beta_{r_{top}}, \gamma_1, \ldots, \gamma_{r_{low}}),} (-1)^{\sum_{i=1}^{r_{top}} |I_i| - 1} \beta_i + \sum_{j=1}^{r_{low}} |J_j| - 1} \gamma_j \binom{k}{\beta, \gamma} = \left( \sum_{i=1}^{r_{low}} (-1)^{|I_i| - 1} \right)^{r_{top}} \binom{k}{\beta, \gamma}.
\]

We stratify the above sum with respect to number of \( \beta_i \)'s strictly greater than zero, and set up the inclusion–exclusion as follows. Let \( F = \{1, \ldots, r_{top}\} \) and define for any \( S \subset F \), functions \( f, g \) (in (3.7), (3.8)) as

\[
f(S) = \sum_{(\beta, \gamma) = (\beta_1, \ldots, \beta_{r_{top}}, \gamma_1, \ldots, \gamma_{r_{low}}), |\beta| + |\gamma| = k, |I_i| - 1, \beta_i \geq 0, \text{ if } i \in S, \beta_i = 0 \text{ if } i \notin S.}
\]

\[
g(S) = \left( \sum_{i \in S} (-1)^{|I_i| - 1} + \sum_{j=1}^{r_{low}} (-1)^{|J_j| - 1} \right)^{r_{top}}.
\]

Observe that \( \sum_{j=1}^{r_{low}} (-1)^{|J_j| - 1} = r_{low}^+ - r_{low}^- \), which yields

\[
g(S) = \left( |S^+| - |S^-| + r_{low}^+ - r_{low}^- \right)^{r_{top}}.
\]

where \( |S^+|(|S^-|) \) denotes number of even(odd) dimensional faces of \( \hat{s} \) indexed by \( S \). By (3.8) we obtain

\[
f(F) = \sum_{S: S \subset F} (-1)^{r_{top} - |S|} \left( |S^+| - |S^-| + r_{low}^+ - r_{low}^- \right)^{r_{top}}.
\]
Since there are $r^+_{\text{top}}$ even dimensional faces and $r^-_{\text{top}}$ odd dimensional faces in $\tilde{s}$, for a fixed $i \in [0, r^+_{\text{top}}]$ and $j \in [0, r^-_{\text{top}}]$ there are exactly $(r^+_{\text{top}}) (r^-_{\text{top}})$ subsets $S \subset F$ satisfying $i = |S^+|, j = |S^-|$. Thus the second part of (1.17) now follows from $f = c_{a,k}(x)$. □

As the last case of interest we consider is the relative Euler characteristic $\chi_{\text{rel}}(K, L)$ of a random pair $(K, L)$. Denoting the characteristic functions of $K$ by $\{e_I\}$ and of $L$ by $\{w_J\}$, (2.7) and (2.8) imply the following polynomial expression

$$\chi_{\text{rel}} = \sum_{d=0}^{n-1} (-1)^k \left( \sum_{I \in f_d(n)} (e_I - w_I) \right).$$

(3.24)

Analogously as in the absolute case the distribution of $(K, L)$ is determined by

$$p_{a,x} = \mathbb{P}(e_s = 1, w_r = 1) = \mathbb{P}(e_s w_r = 1).$$

(3.25)

The maximal constants for the range of $\chi_{\text{rel}}(K, L)$ are

$$m(\chi_{\text{rel}}) = m(\chi) - m(\chi), \quad \text{and} \quad m(\chi_{\text{rel}}) = m(\chi) - m(\chi).$$

(3.26)

For convenience we state the following corollary of Theorem 3.3.

**Corollary 3.4** (Distribution of $\chi_{\text{rel}}(K, L)$). Given a random pair $(K, L)$, the distribution of $\chi_{\text{rel}}(K, L)$ on $[m(\chi_{\text{rel}}), m(\chi_{\text{rel}})]$ is given as follows, for $j \in [0, N]$, $N = m(\chi_{\text{rel}}) - m(\chi_{\text{rel}})$

$$\mathbb{P}(\chi_{\text{rel}}(K, L) = m(\chi_{\text{rel}}) + j) = \sum_{(a,x) \in \mathfrak{c}_a \times \mathfrak{c}_x} a_{a,x,j}(\chi_{\text{rel}}) p_{a,x},$$

(3.27)

$$a_{a,x,j}(\chi_{\text{rel}}) = \sum_{k=0}^{N} (v_{kj}(\chi_{\text{rel}}) c_{a,x,k}(\chi_{\text{rel}}),$$

where (using the notation of (3.16))

$$\mathbb{E}(\chi_{\text{rel}}(K, L)^k) = \sum_{(a,x) \in \mathfrak{c}_a \times \mathfrak{c}_x} c_{a,x,k}(\chi_{\text{rel}}) p_{a,x},$$

(3.28)

$$c_{a,x,k}(\chi_{\text{rel}}) = \sum_{\substack{i \in [0, r^+_{\text{top}}(a)], j \in [0, r^-_{\text{top}}(a)], i' \in [0, r^+_{\text{top}}(a)], j' \in [0, r^-_{\text{top}}(a)]}} \begin{cases} (-1)^{r^+_{\text{top}}(a) + r^-_{\text{top}}(a) - i - j - i' - j'} \binom{r^+_{\text{top}}(a)}{i} \binom{r^-_{\text{top}}(a)}{j} \binom{r^+_{\text{top}}(a)}{i'} \binom{r^-_{\text{top}}(a)}{j'} & \text{for } k \geq r, \\ 0, & \text{for } k < r. \end{cases}$$

The proof is as fully analogous the previous arguments and is omitted. Note that the expression for $c_{a,x,k}(\chi_{\text{rel}})$ in (3.28) simplifies to (1.17) whenever $L = \emptyset$. 

4. Topological considerations.

Given a deterministic covering of a finite simplicial complex $X$, i.e. a collection of compact connected subsets $A = \{A_i\}$, we can define its nerve, $\mathcal{N}(A)$ as a finite complex where vertices are just elements $A_i$ of the covering and a $k$-face $I = \{i_1, \ldots, i_{k+1}\}$ belongs to $\mathcal{N}(A)$, if and only if $A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_{k+1}} \neq \emptyset$ (c.f. [34]). The following result, due to Borsuk [6], is of fundamental importance in algebraic topology.

**Lemma 4.1 (The Nerve Lemma [6]).** Let $A = \{A_i\}$ be a covering of $X$ and $\mathcal{N}(A)$ the associated nerve. If all intersections $A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_{k+1}}$, for $k > 0$ are contractible, then $\mathcal{N}(A)$ has a homotopy type of the subspace $|A| = \bigcup_i A_i$ of $X$.

If $A = \{A_i\}$ satisfies the assumption of this lemma then we call it a good covering (of $X$). Recall that a subset of $X$ is contractible if it can be deformed continuously to a point [18]. In the rest of this section we collect elementary facts from algebraic topology and show how the Euler characteristic of $\mathcal{N}(A)$ provides a criteria for a good covering $A = \{A_i\}$, to completely cover a connected 1–complex $X$.

**4.1. Coverage via the nerve.** We assume throughout that $X$ is a $\Delta$-complex (c.f. [18, p. 103]) which is a slight generalization of the notion of the simplicial complex, and in particular is more appropriate for quotient constructions. Formally, $X$ is equipped with finitely many vertices \{v_1, \ldots, v_k\} $\subset X$ and a collection of maps (edges) $s_{ij} : [0, 1] \mapsto X$ such that $s_{ij}(0) = v_i$, $s_{ij}(1) = v_j$ and $s_{ij}|_{[0,1)}$ is injective, and each point in $X - \{v_1, \ldots, v_k\}$ is the image of exactly one such $s_{ij}|_{[0,1)}$. A set $S \subset X$ is open in $X$ if and only if each $s_{ij}^{-1}(S)$ is open in $[0, 1]$. The boundary $\partial X$ of $X$ is the set of leaf vertices of $X$, when we treat $X$ as a graph (or equivalently the topological boundary of a geometric realization of $X$ via an embedding in $\mathbb{R}^3$).

**Proposition 4.2.** Suppose $X$ is a finite connected 1–complex, and let $\{A_i\}$ be a good covering of $X$. Let $|A| = \bigcup_i A_i$, denote $U = |A|$ and $V = |A|^\circ$. Then,

$$\beta_1(X) \geq \beta_1(U), \quad \beta_1(X) \geq \beta_1(U), \quad \beta_1(X) \geq \beta_1(U),$$

and

$$\chi(X) \leq \chi(U). \quad \chi(X) \leq \chi(U). \quad \chi(X) \leq \chi(U).$$

Moreover, if the inequality in (4.1) is strict then (4.2) is also strict.

**Proof.** Consider the Mayer-Vietoris sequence applied to $U$ and $V$:

$$0 \rightarrow H_1(U \cap V) \xrightarrow{\partial} H_1(U) \oplus H_1(V) \rightarrow H_1(X) \rightarrow H_0(U \cap V) \rightarrow H_0(U) \oplus H_0(V) \rightarrow H_0(X) \rightarrow 0.$$

Since $U \cap V = \partial A$ is just finitely many points, in real coefficients we have

$$0 \rightarrow \mathbb{R}^{\beta_1(U)} \oplus \mathbb{R}^{\beta_1(V)} \xrightarrow{d_1} \mathbb{R}^{\beta_1(X)} \rightarrow \ldots$$
From (2.9), \( \chi(X) = 1 - \beta_1(X) \), \( \chi(U) = \beta_0(U) - \beta_1(U) \), \( \chi(V) = \beta_0(V) - \beta_1(V) \). Since \( d_1 \) is injective we have \( \beta_1(U) + \beta_1(V) \leq \beta_1(X) \), which implies \(-\beta_1(X) + \beta_1(U) \leq 0\). This proves (4.1).

Now to prove (1.2) we have two cases to consider: \( \beta_0(U) > 1 \) and \( \beta_0(U) = 1 \). First assume \( \beta_0(U) > 1 \). We argue by contradiction. That is, suppose \( \chi(U) \leq \chi(X) \). Then \( \beta_0(U) - \beta_1(U) \leq \beta_0(X) - \beta_1(X) \) so that \( \beta_0(U) \leq \beta_1(U) + 1 - \beta_1(X) \). But \( \beta_1(A) - \beta_1(X) \leq 0 \) by the previous lemma. Therefore we obtain \( \beta_0(U) \leq 1 \) contrary to our assumption. Now assume \( \beta_0(U) = 1 \). Then \( \chi(U) = 1 - \beta_1(U) \) and \( \chi(X) = 1 - \beta_1(X) \) which yields \( \chi(U) - \chi(X) = -\beta_1(U) + \beta_1(X) \geq 0 \). Thus \( \chi(U) \geq \chi(X) \).

Suppose \( A = \{ A_i \} \) is a good covering of \( X \), and \(|A| = \bigcup_i A_i \), by the Nerve Lemma an obvious necessary condition for \( X \subset |A| \) is

\[
\chi(X) = \chi(|A|) = \chi(\mathcal{N}(A)).
\] (4.3)

Clearly, the condition is not sufficient in particular when \( \partial X \neq \emptyset \), but we have the following

**Corollary 4.3.** Let \( X \) satisfy \( \partial X = \emptyset \) then (4.3) implies \( X \subset |A| \).

**Proof.** Notice that generally \( X \) (even with \( \partial X \neq \emptyset \)) is homotopy equivalent to a bouquet of circles. If \(|A|^c \neq \emptyset \) in \( X \), then (since \(|A|^c \) is open) we pick \( p \in |A|^c \) which is not a vertex of \( X \). Then \( p \) is in the interior of one of the edges which we denote by \( e \). We may homotopy \( X \) away from the interior of \( e \) to a bouquet of \( r \) circles \( S = \bigvee^r S^1 \) in such a way that \( p \) is away from the wedge point (just collapse along the edges different from \( e \)). From Proposition (1.2)

\[
\beta_1(|A|) \leq \beta_1(\bigvee^{r-1} S^1 \vee (S^1 - \{ p \})) < \beta_1(S) = \beta_1(X).
\]

Thus \( \beta_1(|A|) < \beta_1(X) \) and therefore \( \chi(X) < \chi(|A|) \), which implies the claim.

When \( \partial X \neq \emptyset \), (4.3) is insufficient however we may adjust it by using the relative version \( \chi_{rel}(X, \partial X) \) of the Euler characteristic in (2.7) for the pair \((X, \partial X)\) reduces to

\[
\chi_{rel}(X, \partial X) = \chi(X) - \#(\partial X),
\]

where \( \#(\partial X) \) is the number of boundary points of \( X \). Because \( X \) is a special case of the cellular complex, from [18, p. 102] we have

\[
\chi_{rel}(X, \partial X) = \chi(X/\partial X),
\]

thus we can work with the quotient \( X' = X/\partial X \) which is a \( \Delta \)-1-complex ([18, p. 103]) with \( \partial X' = \emptyset \). Let \( q : X \to X' \) be the quotient projection, then the covering \( A \) of \( X \) projects to the covering \( A' \) of \( X' \). It is not true that \( A' \) is automatically a good covering of \( X' \), one may easily find examples where this is the case. However, the following fact is available (easy to see proof omitted)

**Lemma 4.4.** Given \( A = \{ A_i \} \) is a good covering of \( X \), let for every \( i \) the intersection \( A_i \cap \partial X \) be either empty or a point (in other words \( A_{\partial X} = \{ A_i \cap \partial X \} \) is a good covering of \( \partial X \)). Then the quotient covering \( A' \) of \( X' \) is also good.

Consequently, we say that \( A \) is a **good covering of the pair** \((X, \partial X)\) provided \( A \) is good for \( X \) and \( A_{\partial X} \) is good for \( \partial X \). Then by he above lemma \( A' \) is good for \( X' \) and Corollary 4.3 says that \( A' \) covers \( X' \), if and only if \( \chi(|A'|) = \chi(X') \). To obtain the condition analogous to (4.3), for the
general case, we need to express $\chi(|A'|)$ in terms of the data of $A$ and $X$. We have the following generalization of Corollary 4.3.

**Lemma 4.5.** Given a good covering $A = \{A_i\}$ of $(X, \partial X)$ let $|A| = \bigcup_i A_i$. Then $X \subset |A|$, if and only if

$$\chi_{\text{rel}}(\mathcal{N}(A), \mathcal{N}(A \partial X)) = \chi_{\text{rel}}(X, \partial X)$$

or equivalently

$$\chi(|A|) = \chi(X) - \#\{\partial X\} + \#\{|A| \cap \partial X\}. \quad (4.5)$$

**Proof.** Observe that $X \subset |A'|$ to $X \subset |A|$. Indeed, since $|A|$ is closed if $X - |A| \neq \emptyset$ then we may choose a point in $x \in X - |A|$ such that $x \notin \partial X$, since the projection $q$ is a homeomorphism on $X - \partial X$, we conclude that $q(x) \notin X' - |A'|$. Next, Equation (4.4) follows immediately from Corollary 4.3, the fact that $A$ and $A \partial X$ are good and the identities

$$\chi(|A'|) = \chi_{\text{rel}}(|A|, |A| \cap \partial X), \quad \chi(X') = \chi_{\text{rel}}(X, \partial X).$$

Now, thanks to (2.7) we compute

$$\chi_{\text{rel}}(X, \partial X) = \chi(X) - \#\{\partial X\},$$

$$\chi_{\text{rel}}(|A|, |A| \cap \partial X) = \chi(|A|) - \#\{|A| \cap \partial X\},$$

which yields (4.5). \qed}

**Remark 4.6.** Equivalently, the coverage condition for $(X, \partial X)$ can be obtained by looking at the enlarged covering $\hat{A}$, which is $A$ together with the boundary vertices: $\partial X = \{x_1, \ldots, x_{\#\{\partial X\}}\}$. Then $\hat{A}$ is good if satisfies the conditions of Lemma 4.4

$$\chi(|\hat{A}|) = \chi(|A| \cup \partial X) = \chi(|A|) + \chi(\partial X) - \chi(|A| \cap \partial X)$$

$$= \chi(|A|) + \#\{\partial X\} - \#\{|A| \cap \partial X\},$$

which together with (4.3) leads us to (4.4).

### 4.2 Coverage of $X$ by $\varepsilon$-balls. Vietoris–Rips complex.

A special case of interest (see e.g. [35] [10]) is when a connected 1–complex $X$ ought to be covered by $\varepsilon$-size neighborhoods, where $\varepsilon$ is allowed to be sufficiently small. In such cases the topology of the nerve of the covering simplifies and one may work with Vietoris–Rips complex [19], as we investigate in the following paragraphs.

First, equip $X$ with an intrinsic distance

$$d_X(x, y) = \min_{\gamma: [0, 1] \to X, \gamma(0) = x, \gamma(1) = y} \text{length}(\gamma). \quad (4.6)$$

i.e. $d_X(x, y)$ is the length of the shortest path between $x$ an $y$, which in practice is just a smallest sum of edge-lengths (and their pieces) connecting $x$ and $y$, (the lengths come from some choice of geometric realization of $X$ in $\mathbb{R}^3$). We will denote the intrinsic diameter of a subset $Y \subset X$ by $\text{diam}(Y)$. Recall that given a simplicial complex $K$ its Vietoris–Rips complex $\mathcal{R}(K)$, [19] is defined to be a maximal simplicial complex (with respect to inclusion) which has the same 1-skeleton as $K$. Practically, this means that $\mathcal{R}(K)$ is obtained by filling every $k$-clique in the graph $K^{(1)}$ with a $(k - 1)$-dimensional face, e.g. 3-cycles are filled with 2-simplices in $\mathcal{R}(K)$, etc.
We will consider a finite covering \( A = \{ A_1, \ldots, A_n \} \) of \((X, d_X)\) by closed \( \varepsilon \)-balls, possible shapes of such balls for \( \varepsilon \) sufficiently small is depicted on Figure 4.2.

Let us denote by \( \mathcal{R}(A) \) the Vietoris–Rips complex of the nerve of the cover. We record the following

**Proposition 4.7.** Suppose \( C \) is the length of the shortest cycle in the quotient complex \( X' = X/\partial X \),

(i) if \( \varepsilon < \frac{1}{6} C \) then the covering \( A \) by \( \varepsilon \)-balls in \((X, d_X)\) is a good cover.

(ii) if \( \varepsilon < \frac{1}{6} C \) then the nerve \( \mathcal{N}(A) \) of \( A \) equals \( \mathcal{R}(A) \).

**Proof.** For (i) we must show that every \( k \)-fold intersection \( A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k} \) has a homotopy type of a point. Because \( \text{diam}(A_i) < C \), \( A_i \) is a connected tree and therfore contractible, which shows the claim for \( k = 1 \). For \( k = 2 \), first suppose that a nonempty intersection \( A_i \cap A_j \) is disconnected i.e. \( \text{dim}(\overline{H}_0(A_i \cap A_j)) \geq 1 \) (where \( \overline{H}_r(\cdot) \) denotes the reduced homology groups c.f. [18]). Since \( A_i \) and \( A_j \) are connected, the reduced Mayer-Vietoris sequence for \( A_i \cap A_j \) then simplifies to

\[
0 \rightarrow \overline{H}_1(A_i \cup A_j) \rightarrow \overline{H}_0(A_i \cap A_j) \rightarrow \overline{H}_0(A_i) \oplus \overline{H}_0(A_j) \rightarrow 0,
\]

We obtain \( \overline{H}_1(A_i \cup A_j) \cong \overline{H}_0(A_i \cap A_j) \cong \mathbb{R}^k \) for some \( k \geq 1 \), which implies that \( A_i \cup A_j \) contains a nontrivial cycle. This however contradicts the fact that \( \text{diam}(A_i \cup A_j) \leq 4 \varepsilon < C \). Thus \( k \) has to vanish and \( A_i \cap A_j \) must be connected contain no cycle and therefore contractible. Now, for an induction step with respect to \( k \), it suffices to apply the previous step to \( A' = A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k} \) and \( A'' = A_{i_{k+1}} \).

Before proving (ii), recall the 1-dimensional version of Helly’s Theorem (c.f. [11]) implies that given a finite collection of intervals \( \{ C_1, C_2, \ldots, C_n \} \) on \( \mathbb{R} \), if the intersection of each pair is nonempty, i.e. \( C_i \cap C_j \neq \emptyset \), for every \( 1 \leq i, j \leq n \), then \( \bigcap_{i=1}^n C_i \neq \emptyset \).

First consider the case of 3-fold intersections, i.e. supposing that \( A_j \cap A_k \neq \emptyset \), 1 \leq k \neq j \leq 3, we aim to show that \( A_1 \cap A_2 \cap A_3 \neq \emptyset \). Observe that \( V = A_1 \cup A_2 \cup A_3 \) is connected and by the argument of (i) it must be a connected tree, i.e. contains no cycles. Let \( p_{1,2}, p_{2,3}, p_{1,3} \) be distinct points in \( V \) such that \( p_{i,j} \in A_i \cap A_j \). Note that for each pair: \( p_{i,j} \), \( p_{s,t} \) there exists a path in \( V \) connecting these points. We now consider two cases: (1) one of these paths, we denote by \( l_i \), contains all three points \( p_{i,j} \), then the collection \( \{ C_i \} \), \( C_i = l \cap A_i \), \( i = 1, 2, 3 \) satisfies the assumptions of Helly’s Theorem which implies the claim. (2) none of the paths between paris of \( p_{i,j} \)’s contain the third point. Consider two shortest paths: \( l_1 \) between \( p_{1,2} \) and \( p_{2,3} \), and \( l_2 \) between \( p_{1,2} \) and \( p_{2,3} \) then \( l_{1,2} = l_1 \cap l_2 \) is a segment between \( p_{1,2} \) and some vertex of \( v \in V \). The vertex \( v \) has to be in one of \( A_j \)’s, w.l.o.g. suppose \( v \in A_2 \) (as other cases are analogous). Then if \( v \) is also in \( A_1 \) or \( A_3 \) we can take \( p_{1,2} \) or \( p_{2,3} \) equal to \( v \) and use (1). If \( v \notin A_1 \) and \( v \notin A_3 \) then we observe
that either $A_1$ or $A_3$ is disconnected which is not the case. This concludes the proof of (ii) for the 3-fold case, the general case can be obtained by induction.

\[ \square \]

5. **Complete coverage probability.**

In this section we derive consequences results of Sections 4.1–4.2 in the random setting.

5.1. **Random coverings and the random nerve.** Suppose $A = \{A_i\}$ is a random covering of a metric space $X$, we define the nerve $N(A)$ of $A$ by defining a probability measure $P_A$ on $C_n$ via the process elucidated in Section 1 in (1.2) and (1.3). Observe that given a subspace $Y \subset X$ we obtain an induced random covering $A_Y$ from $A$:

$$A_Y = \{A_1 \cap Y, A_2 \cap Y, \ldots, A_n \cap Y\}$$

The definition of $P_A$ extends to pairs $(N(A), N(A_Y))$ in an obvious way. In particular given $(s, r) \in C_n \times C_n$, we set

$$p_{s, r} = P\{(s, 1) \in C_n \times C_n \mid s \subset k, r \subset 1\}$$

$$= P\{\forall \{i\} \in s\{A_i \neq \emptyset\}, \forall \{j\} \in r\{A_j \cap Y \neq \emptyset\}\}. \quad (5.1)$$

Clearly, $N(A)$ is a random complex, and $(N(A), N(A_Y))$ is a random pair. We say a finite random covering $\{A_i\}_{i=1,\ldots,n}$ of $X$ is *good* if and only if it is a good covering on $X$ almost surely. Further, we say a random covering $A = \{A_i\}$ of a pair $(X, \partial X)$ is good provided it is a good covering of $X$ and $A_{\partial X}$ is a good covering of $\partial X$. $|A|$ will denote the random set $\bigcup_i A_i$.

5.2. **Proof of the extended version of Theorem 1.1.** Let $\chi_{\text{rel}}(A, A_{\partial X})$ be the relative Euler characteristic of the pair $(N(A), N(A_{\partial X}))$. We may now state Theorem 1.1 for a general 1–complex $X$.

**Theorem 5.1** (Coverage probability of a 1-complex $X$ with $\partial X \neq \emptyset$). Let $A = \{A_i\}, i = 1, \ldots, n$ be a random good covering of the pair $(X, \partial X)$. Then, the range of $\chi_{\text{rel}}(A, A_{\partial X})$ can be restricted to

$$\underline{m} = \chi_{\text{rel}}(X, \partial X) \leq \chi_{\text{rel}}(A, A_{\partial X}) \leq \bar{n} = \overline{m}, \quad (5.2)$$

and the complete coverage probability equals

$$P(X \subset |A|) = P\{\chi_{\text{rel}}(A, A_{\partial X}) = \chi_{\text{rel}}(X, \partial X)\},$$

$$= \sum_{(s, r) \in C_n \times C_n} a_{s, r}(\chi_{\text{rel}}) p_{s, r}, \quad (5.3)$$

where $a_{s, r}(\chi_{\text{rel}}) = a_{s, r, 0}(\chi_{\text{rel}})$ are defined in (3.27) of Corollary 3.4, and $p_{s, r}$ in (5.1).

**Proof.** Under the given assumptions, Lemma 4.5 implies

$$P(X \subset |A|) = P\{\chi_{\text{rel}}(A, A_{\partial X}) = \chi_{\text{rel}}(X, \partial X)\}. \quad (5.4)$$

At this point the formula (3.27) of Corollary 3.4 can be applied to the random pair $(N(A), N(A_{\partial X}))$ to give an exact expression for $P\{\chi_{\text{rel}}(A, A_{\partial X}) = \chi_{\text{rel}}(X, \partial X)\}$. In this particular case the range of $\chi(A, A_{\partial X})$ is given by (5.2), where the lower bound follows from Proposition 4.2, and the upper bound corresponds to the case when elements of the covering $A$ are pairwise disjoint and containe
in $X - \partial X$, i.e. $N(A)$ is just $n$ distinct points. The formula for $p_{a,x}$ in (1.7) is a direct consequence of Proposition 4.7 (see also Remark 5.3). □

Remark 5.2. Note that $N(A_{\partial X})$ generally contains high dimensional faces and therefore the chain expansion of $\chi_{rel}^k$ in $\mathbb{R}_I[e_I, w_J]$ involves monomials in $e_a$ and $w_x$. To simplify this expansion one may observe that $N(A_{\partial X})$ has a homotopy type of finitely many points or is empty. Specifically, from 4.5 we have

$$\chi_{rel}(A_{\partial X}) = \chi(A) - \#\{A \cap \partial X\}.$$  

The random variable $\#\{A \cap \partial X\}$ (counting points in $A_{\partial X}$) can be expressed as follows:

$$\chi_{rel}(A_{\partial X}) = \sum_{i=1}^q w_{\{i\}}.$$  

where $\{1, \ldots, q\}$ label points of $\partial X$ and $\{w_{\{i\}}\}_{i=1}^q$ are the indicator functions of points in $A_{\partial X}$. Consequently, we may derive expressions for powers $\chi_{rel}^k$ as polynomials in $\mathbb{R}[e_I, w_{\{i\}}]$. These expansions of $\chi_{rel}^k$ involve products of $e_a$ and $w_{\{i\}}$ only, which may provide a different way to express $\mathbb{P}(X \subset |A|)$.

Remark 5.3. In order to be more explicit about how the computation of $p_{a,x}$ simplifies in the case the nerve $N(A)$ equals the Vietoris–Rips complex $\mathcal{R}(A)$: let us suppose $A_i$ are $\varepsilon$-radius closed balls in $X$ with random centers $\xi_i \in X$. In $\mathcal{R}(A)$ any simplex indexed by $I = \{i_1, i_2, \ldots, i_k\}$ is determined by its edges, and an edge $\{i, j\}$ in $\mathcal{R}(A)$ occurs if and only if $|\xi_i - \xi_j| \leq 2\varepsilon$ (where $| \cdot - \cdot |$ is a short notation for the distance $d_X(\cdot, \cdot)$ on $X$). For instance, we have

$$p_I = \mathbb{P}(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k} \neq \emptyset) = \mathbb{P}\left(|\xi_{i_s} - \xi_{i_t}| \leq 2\varepsilon \right\} \forall_{s,t} s \neq t)$$

Enumerate points in $\partial X$ as follows $\{x_1, x_2, \ldots, x_M\}$, $M = \#\{\partial X\}$. Now, $p_{a,x}$ given in (5.1) is just a volume of the set

$$A_{a,x} = \{(\xi_1, \ldots, \xi_n) \in X^n \mid \forall_{I \in k} \forall_{s,t \in I} |\xi_s - \xi_t| \leq 2\varepsilon, \forall_{I \in k} \exists_{1 \leq s \leq M} \forall_{i \in I} |\xi_i - x_s| \leq \varepsilon\}.$$  

which in the case $\mathbb{P} = d\xi_1 d\xi_2 \ldots d\xi_n$ (i.e. $\xi_i$’s are independent) can be computed via ordinary calculus techniques or estimated numerically. These formulas further simplify, if $\partial X = \emptyset$, but we do not attempt these computations here, except in the test case considered in Section 7.1.

6. Upper bounds for coverage probability

In this section we use the method of finite differences, c.f. [1], to estimate the complete coverage probability from above in terms of the mean of the Euler characteristic and prove Theorem 1.2. Let $\{A_i\}, i = 1, \ldots, n$ be a finite good covering of $X$, consider the following shifted version of the relative Euler characteristic $\chi_{rel}(A_{\partial X})$ of $(N(A), N(A_{\partial X}))$:

$$\chi_0 = \chi_{rel}(A_{\partial X}) - \bar{m},$$

where $\bar{m} = \chi_{rel}(X, \partial X)$. From (2.9) we obtain

$$\chi_{rel}(A_{\partial X}) = \beta_0 - \beta_1,$$

where $\beta_0 = \beta(A_{\partial X})$ stand for the random relative Betti numbers. Recall that $\{e_I, w_J\}, I, J \in f(n)$ stand for the indicator functions of faces in $(N(A), N(A_{\partial X}))$.  

[1]
We will consider a filtration by random vectors $V_i$ denoting $(e_{I(i)}, f_{J(i)})$ where $I(i), J(i) \in f(n)$ are subsets of $\{1, \ldots, i\}$. Note that $V_i$ reveals subcomplexes in $C_n$ spanned by vertices 1 through $i$. By analogy to the setting of Erdős–Rényi model [1], we set up a vertex exposure martingale, associated with $X_0$ and $\{V_i\}$ as follows:

$$Y_0 = \mu_0 = E(X_0), \quad Y_i = E(X_0 | V_i), \quad i = 1, \ldots, n, \quad (6.2)$$

Clearly, $Y_n = \chi_0$ and the sequence $\{Y_i\}$ is an instance of Doob’s martingale [1]. Recall the following variant of the Azuma-Hoeffding inequality [1, 3], for $\{Y_i\}$:

$$\mathbb{P}(Y_n - Y_0 \leq -a) \leq \exp\left(\frac{-a^2}{2 \sum_{i=1}^n c_i^2}\right) \quad (6.3)$$

where $a > 0$, and $c_i$ is a difference estimate

$$|Y_i - Y_{i-1}| \leq c_i. \quad (6.4)$$

Exposing a vertex (or a face containing it) changes $\beta_0$ by at most 1 and $\beta_1$ by at most $\beta_1(X, \partial X) = 1 - \chi_{rel}(X, \partial X)$ thus we obtain

$$|Y_i - Y_{i-1}| \leq 2 + |\chi_{rel}(X, \partial X)|.$$

Let $a = \mu_0$, then

$$\mathbb{P}(\chi_0 = 0) = \mathbb{P}(\chi_0 \leq \mu_0 - a) = \mathbb{P}(Y_n - Y_0 \leq -a).$$

Using the above estimates for $c_i$ and (6.3) yields

$$\mathbb{P}(X \subset |A|) = \mathbb{P}(\chi_0 = 0) \leq \exp\left(\frac{-\mu_0^2}{2n(|\chi_{rel}(X, \partial X)| + 2)^2}\right),$$

which completes the proof of Theorem 1.2.

7. Distribution of $\chi$ on $S^1$ and Steven’s results.

In [31] Stevens proved Formula (1.1) for the complete coverage probability by independent identically distributed $n$ random $\alpha$-length arcs on the unit circumference circle $X = S^1$. Naturally, it is a case of a random covering $A = \{A_i\}, i = 1, \ldots, n$ on $X = S^1$ we consider in this paper. Work in [31] provides more information, i.e. exact distribution of number of gaps $G = G(A) = \beta_0(X - |A|)$ in $X - |A|$. It is given as follows: let $k = \min([1/\alpha], n)$ then [31, p. 319]

$$\mathbb{P}(G(A) = j) = \binom{n}{j} \sum_{i=j}^k (-1)^{i-j} \binom{n-j}{i-j} (1 - i \alpha)^{n-1}, \quad \text{for } 0 \leq j \leq k \quad (7.1)$$

$$\mathbb{P}(G(A) = j) = 0, \quad \text{for } j > k,$$

where the coverage probability equals $\mathbb{P}(G = 0)$ (i.e. no gaps). By Proposition 4.7 the covering $A$ is good provided $\alpha < 1/2$. The random Euler characteristic of the nerve $N(A)$ (thanks to 2.9) satisfies

$$\chi(A) = \beta_0(A) - \beta_1(A)$$

where (thanks to the Nerve Lemma 4.1) the random Betti number $\beta_0$ measures the number of connected components of $A$ and $\beta_1(A)$ a number of cycles in $A$, where the later in the case of $S^1$
cannot exceed 1. If \( \beta_1(A) = 0 \) we have \( \chi(A) = \beta_0(A) \). Because any realization of \( |A| \) is just a union of arcs it follows that

\[
\chi(A) = \mathcal{G}(A),
\]

(7.2)

Thus, (7.1) is also the distribution of \( \chi(A) \). In the following subsection we compare formulas in (7.1) and those obtained from Theorem 1.1 in the case of three arcs \((n = 3)\).

7.1. Verification of the Steven’s formula in the case of three arcs. Suppose we are given three arcs \{A, B, C\} on the unit circle \( S^1 \) of length \( \alpha \) and suppose that arcs’ centers \{X, Y, Z\} are distributed uniformly and independently on \( S^1 \). In order to simplify calculations we will just compare the moments of \( \chi(A) \) obtained from Equations (1.12) and (1.17) to the moments of \( \mathcal{G} := \mathcal{G}(A) \) directly computed from the distribution in (7.1). In the relevant case of three arcs of length \( \alpha < 1 \), possible values of \( k = \min(\lfloor \frac{1}{3} \rfloor, n) \) are 2 and 3. Formula (7.1) yields the following

\[
\begin{align*}
p_0 &= \mathbb{P}(G = 0) = 1 - \left( \frac{3}{1} \right)(1 - \alpha)^2 + \left( \frac{3}{2} \right)(1 - 2\alpha)^2 - (1 - 3\alpha)^2 \\
&= (3\alpha - 1)^2 \text{ or } 0,
\end{align*}
\]

\[
\begin{align*}
p_1 &= \mathbb{P}(G = 1) = \left( \frac{3}{1} \right)(1 - \alpha)^2 - \left( \frac{2}{1} \right)(1 - 2\alpha) + (1 - 3\alpha)^2 \\
&= -3 + 18\alpha - 21\alpha^2 \text{ or } 6\alpha^2,
\end{align*}
\]

\[
\begin{align*}
p_2 &= \mathbb{P}(G = 2) = \left( \frac{3}{2} \right)(1 - 2\alpha)^2 - (1 - 3\alpha)^2 \\
&= 3(1 - 2\alpha)^2 \text{ or } 3\alpha(2 - 5\alpha),
\end{align*}
\]

\[
\begin{align*}
p_3 &= \mathbb{P}(G = 3) = 0 \quad \text{ or } (1 - 3\alpha)^2,
\end{align*}
\]

where the “unboxed” expressions are valid for \( k \geq 2 \) and “boxed” expressions for \( k = 3 \). As a result we obtain direct formulas for the moments

\[
\begin{align*}
E(G) &= p_1 + 2p_2 + 3p_3 = 3 - 6\alpha + 3\alpha^2 \\
E(G^2) &= p_1 + 4p_2 + 9p_3 = 9 - 30\alpha + 27\alpha^2 \\
E(G^3) &= p_1 + 8p_2 + 27p_3 = \begin{cases} 21 - 78\alpha + 75\alpha^2, & \text{for } k = 2, \\
27 - 114\alpha + 129\alpha^2, & \text{for } k = 3. \end{cases}
\end{align*}
\]

(7.3)

We aim to verify the above by applying the technology of Theorem 1.1. As a first step we compute the discrete probability measure defining the random nerve complex \( N(A) \). Let us denote by \( x, y, z \) the characteristic functions of vertices (i.e. \( e_{\{i\}} \)'s), then \( a, b, c \) the indicator functions of the edges (i.e. \( e_{\{i,j\}} \)'s) and \( F \) the indicator function of the face (i.e. \( e_{\{1,2,3\}} \)). With this notation, the random Euler characteristic is represented in \( \mathbb{R}_3[x, y, z, a, b, c, F] \) as

\[
\chi = x + y + z - a - b - c + F \\
= x + y + z - axy - byz - cxz + Fabcxyz.
\]

From the underlying assumptions

\[
\mathbb{P}(A = \emptyset) = \mathbb{P}(B = \emptyset) = \mathbb{P}(C = \emptyset) = 0, \quad \text{thus} \quad \mathbb{P}(x = 1) = \mathbb{P}(y = 1) = \mathbb{P}(z = 1) = 1.
\]
From (1.17) we obtain

\[
\chi^k = (x + y + z) + (-2 + 2^k)(xy + xz + yz) + (3 - 3 \cdot 2^k + 3^k)(xyz)
+ (1 - 2^k)(axy + byz + cxz) + (-1 + 2^k + 1 - 3^k)(axy + bxyz + cxyz)
+ (1 - 2^k + 3^k)(abxyz + bxyz + acxyz) + (1 - 3 \cdot 2^k - 3^k)(abxyz)
+Fabxyz,
\]

and after substituting \(x = y = z = 1\)

\[
\chi = 3 - a - b - c + abcF,
\]

\[
\chi^2 = 9 - 5(a + b + c) + 2(ab + ac + bc) + abcF,
\]

\[
\chi^3 = 27 - 19(a + b + c) + 12(ab + ac + bc) - 6abc + abcF.
\]

Next, we need distributions of the Bernoulli monomials: \(a, b, c, ab, ac, bc, abc, abcF\) which represent all possible realizations \(s \in \mathcal{C}_3\) of \(N(A)\). Recall that the arcs centers are uniformly distributed on

\[
S^1 \text{ and independent, we denote the distance on } S^1 \text{ by } |\cdot|.
\]

Due to the symmetry it suffices to compute only probabilities for \(a = axy, ab = abxyz, abc = abxyz,\) and \(F = Fabxyz\).

\[
\begin{align*}
\mathbb{P}(axy) &= \mathbb{P}(\{|X - Y| \leq \alpha\}) = \iiint_{S^1 \times S^1 \times S^1} \mathbf{1}(\{|X - Y| \leq \alpha\}) \, dX \, dY \, dZ \\
&= \iiint_{\{U \leq \alpha\}} dU \, dV \, dW = 2\alpha,
\end{align*}
\]

where in the third identity we applied \(U = X - Y, V = X, W = Z\). Analogously, we have

\[
\begin{align*}
\mathbb{P}(abxyz) &= \mathbb{P}(\{|X - Y| \leq \alpha, |X - Z| \leq \alpha\}) \\
&= \iiint_{S^1 \times S^1 \times S^1} \mathbf{1}(\{|U| \leq \alpha, |V| \leq \alpha\}) \, dU \, dV \, dW = 4\alpha^2,
\end{align*}
\]
\[ \mathbb{P}(abcxyz) = \mathbb{P}(|X - Y| \leq \alpha, |X - Z| \leq \alpha, |Z - Y| \leq \alpha) \]

\[ = \int_{S^1 \times S^1 \times S^1} \mathbf{1}(|U| \leq \alpha, |V| \leq \alpha, |U - V| \leq \alpha) \, dU \, dV \, dW \]

\[ = \begin{cases} 3\alpha^2, & \alpha \leq \frac{1}{3}, \\ 3\alpha^2 + (3\alpha - 1)^2, & \alpha \geq \frac{1}{3}, \end{cases} \]

for \( U = X - Y, V = X - Z, W = Z \). The last computation is represented on Figure 7.1 which pictures one of the faces of the torus \((S^1)^2\) as a unit square in \(U, V\) coordinates. \( \mathbb{P}(abcxyz) \) is the sum of areas outside the lines \( V = U \pm (1 - \alpha) \) and inside the lines \( V = U \pm \alpha \), i.e. the sum of the checkered pattern region and the cross pattern region. These regions have a direct geometric interpretation the checkered region corresponds to configurations of arcs which have non-empty triple intersection and the cross region of area \((3\alpha - 1)^2\) corresponds to configurations of arcs which have only pairwise intersections (these are the configurations which actually give us coverage of \(S^1\)). Note that the checkered region disappears for \( \alpha < \frac{1}{3} \) which is consistent with the fact that in this case it is impossible to cover the unit circle with three arcs.

Computing \( \mathbb{P}(F) \) directly is a bit more involved, it requires finding the volume of \( \{|X + Y - Z| \leq \frac{\alpha}{2} + |Y - X|\} \) in the torus. We omit this calculation and use the observation of the previous paragraph to conclude that \( \mathbb{P}(F) \) equals the area of the cross pattern region of Figure 7.1 and thus

\[ \mathbb{P}(xyzabcF) = 3\alpha^2. \]

Collecting the above information yields the probability distribution of the random nerve of \{A, B, C\} as follows

|      | \( P(x) = P(y) = \frac{1}{2} \) | \( P(z) = 1 \) |
|------|-------------------------------|-----------------|
|      | \( P(axy) = P(byz) = P(cxz) = 2\alpha \) | \( P(abxyz) = P(acxyz) = 2\alpha \) | \( P(abxyz) = P(acxyz) = 4\alpha^2 \) |
|      | \( P(abcxyz) = 3\alpha^2 \) | \( \text{for } \alpha \leq \frac{1}{3}, \text{ or } = 12\alpha^2 - 6\alpha + 12 \text{ if } \alpha \geq \frac{1}{3}. \) | \( P(Fabcxyz) = 3\alpha^2 \) |

As the last step we compute moments of \( \chi \), the set of possible realizations of \( \chi \) is given as \{0, 1, 2, 3\}, three first moments suffice to recover the probability distribution of \( \chi \). After substitution we obtain

\[ E(\chi) = 3 - 3P(axy) + P(xyzabcF) = 3 - 6\alpha + 3\alpha^2 \]

\[ E(\chi^2) = 9 - 5P(axy) + 2P(abxyz) + P(xyzabcF) = 9 - 30\alpha + 27\alpha^2 \]
\[ E(\chi^3) = 27 - 19\mathbb{P}(axy) + 12\mathbb{P}(abxyz) - 6\mathbb{P}(abcxyz) + \mathbb{P}(xyzabcF) \]

\begin{align*}
&= \begin{cases} 
21 - 78\alpha + 75\alpha^2, & \alpha \leq \frac{1}{3}, \\
27 - 114\alpha + 129\alpha^2, & \alpha \geq \frac{1}{3}.
\end{cases}
\end{align*}

These results are in full agreement with those obtained in (7.3).

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