Formality of the homotopy calculus algebra of Hochschild (co)chains

Vasiliy Dolgushev, Dmitry Tamarkin, and Boris Tsygan

To Mikhail Olshanetsky on the occasion of his 70th birthday.

Abstract

The Kontsevich-Soibelman solution of the cyclic version of Deligne’s conjecture and the formality of the operad of little discs on a cylinder provide us with a natural homotopy calculus structure on the pair \((C^\bullet(A), C_\bullet(A))\) “Hochschild cochains + Hochschild chains” of an associative algebra \(A\). We show that for an arbitrary smooth algebraic variety \(X\) over a field \(K\) of characteristic zero the sheaf \((C^\bullet(\mathcal{O}_X), C_\bullet(\mathcal{O}_X))\) of homotopy calculi is formal. This result was announced in paper [29] by the second and the third author.

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1 Introduction

The standard Cartan calculus on polyvector fields and exterior forms can be naturally extended to the Hochschild cohomology $\text{HH}^\bullet(A, A)$ and the Hochschild homology $\text{HH}_\bullet(A, A)$ of an arbitrary associative algebra $A$ [11, 24]. This calculus is induced by simple operations on Hochschild (co)chains, and the identities of this algebraic structure hold for these operations up to homotopy.

The Kontsevich-Soibelman proof of the cyclic version of Deligne’s conjecture [23] and the formality of the operad of little discs on a cylinder[1] imply that this nice collection of the operations on the pair $(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A))$ “(normalized) Hochschild cochains + (normalized) Hochschild chains” can be extended to an $\infty$- or homotopy calculus structure.

This homotopy calculus structure on the pair $(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A))$ is a natural generalization of the homotopy Gerstenhaber algebra structure on the cochains $C^\bullet_{\text{norm}}(A)$. In paper [13] we proved the formality of this homotopy Gerstenhaber algebra on $C^\bullet_{\text{norm}}(A)$ for an arbitrary regular commutative algebra $A$ over a field $\mathbb{K}$ of characteristic zero. In this paper we extend this result to the homotopy calculus algebra on the pair $(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A))$.

As well as in [13] we also consider the situation when the algebra $A$ is replaced by the structure sheaf $\mathcal{O}_X$ of a smooth algebraic variety $X$ over the field $\mathbb{K}$. More precisely, we consider the homotopy calculus algebra on the pair $(C^\bullet_{\text{norm}}(\mathcal{O}_X), C^\bullet_{\text{norm}}(\mathcal{O}_X))$ where $C^\bullet_{\text{norm}}(\mathcal{O}_X)$ and $C^\bullet_{\text{norm}}(\mathcal{O}_X)$ is, respectively, the sheaf of (normalized) Hochschild cochains and the sheaf of (normalized) Hochschild chains of $\mathcal{O}_X$. In this paper we show that the sheaf of homotopy calculi $(C^\bullet_{\text{norm}}(\mathcal{O}_X), C^\bullet_{\text{norm}}(\mathcal{O}_X))$ is formal.

If $A$ is an associative algebra (with unit), the pair $(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A))$ is also equipped with an algebraic structure defined by a degree $-1$ Lie bracket on $C^\bullet_{\text{norm}}(A)$, a degree $-1$ Lie module structure on $C^\bullet_{\text{norm}}(A)$ over $C^\bullet_{\text{norm}}(A)$, and Connes’ operator on $C^\bullet_{\text{norm}}(A)$ which is compatible with the Lie module structure. In the paper we refer to such algebra structures as $\Lambda\text{Lie}^+_\delta$-algebra. (See Definition 4.)

In paper [31] the third author conjectured that if $A$ is a regular commutative algebra then this $\Lambda\text{Lie}^+_\delta$-algebra structure on $(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A))$ is formal. This conjecture was proved in [33] (at least in the case $\mathbb{R} \subset \mathbb{K}$) by Willwacher who used the constructions of B. Shoikhet [25] and the first author [12].

In general $Ho(\Lambda\text{Lie}^+_\delta)$-part of the homotopy calculus structure on $(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A))$ derived from [23] may not coincide with the $\Lambda\text{Lie}^+_\delta$-algebra on the pair $(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A))$. However, we show that this homotopy calculus algebra on $(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A))$ is quasi-isomorphic to another homotopy calculus algebra on $(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A))$ whose $Ho(\Lambda\text{Lie}^+_\delta)$-part is the ordinary $\Lambda\text{Lie}^+_\delta$-algebra given by the above Lie bracket on $C^\bullet_{\text{norm}}(A)$, the Lie algebra module on $C^\bullet_{\text{norm}}(A)$ over $C^\bullet_{\text{norm}}(A)$ and Connes’ operator on $C^\bullet_{\text{norm}}(A)$. In this sense, the formality of the homotopy calculus algebra on $(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A))$ is a generalization of Willwacher’s cyclic formality theorem [33].

The organization of the paper is as follows. In Section 2 we fix the notation and recall required results about (co)operads and (co)algebras. Section 3 is devoted to $\infty$- or homotopy versions for the algebras over the operads $\text{calc}$, $e_2$, and $\text{Lie}^+_\delta$. In Section 4 we recall the

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[1] See Proposition 11.3.3 on page 50 in [23].
Kontsevich-Soibelman operad and the operad $\text{Cyl}$ of little discs on a cylinder. We show that the homology operad $H_{-\bullet}(\text{Cyl}, K)$ of $\text{Cyl}$ with the reversed grading is the operad of calculi. Finally we recall required results from [23] and prove a useful property of the Kontsevich-Soibelman operad. Section 5 is devoted to properties of the homotopy calculus algebra on the pair $(C_{\text{norm}}^\bullet(A), C_{\text{norm}}^\bullet(A))$. In Section 6 we formulate and prove the main result of this paper. (See Theorem 5 on page 49.) In the concluding section we discussion applications and generalizations of Theorem 5. We also discuss recent articles related to our main result.

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2 Preliminaries

2.1 (Co)operads and (co)algebras

Most of the notation and conventions for (co)operads and their (co)algebras are borrowed from [13].

Depending on a context our underlying symmetric monoidal category is either the category of graded vector spaces, or the category of chain complexes, or the category of compactly generated topological spaces, or the category of sets. By suspension $sV$ of a graded vector space (or a chain complex) $V$ we mean $\varepsilon \otimes V$, where $\varepsilon$ is a one-dimensional vector space placed in degree $+1$. For a vector $v \in V$ we denote by $|v|$ its degree. The symmetric group of permutations of $n$ letters is denoted by $S_n$. The underlying field $K$ has characteristic zero.

For an operad $O$ we denote by $\text{Alg}_O$ the category of algebras over the operad $O$. Dually, for a cooperad $C$ we denote by $\text{Coalg}_C$ the category of nilpotent\footnote{For the definition of nilpotent coalgebra see section 2.4.1 in [19].} coalgebras over the cooperad $C$. By corestriction we mean the canonical map

$$\rho_V : F_C(V) \rightarrow V \quad (2.1)$$

from the free coalgebra $F_C(V)$ to the vector space of its cogenerators $V$. We often omit the subscript in the notation $\rho_V$ for the corestriction.

For a polynomial functor $P$ we denote by $T(P)$ (resp. $T^*(P)$) the free operad (resp. the free cooperad) (co)generated by $P$. The notation $\bullet$ is reserved for the monoidal product of the polynomial functors. Thus, if $P$ and $Q$ are polynomial functors then

$$P \bullet Q(n) = \bigoplus_{k_1 + \cdots + k_m = n} P(m) \otimes_{S_m} (Q(k_1) \otimes \cdots \otimes Q(k_m)) \quad (2.2)$$

This formula can be easily generalized to the colored polynomial functors.
By “suspension” of a (co)operad $O$ of graded vector spaces (or chain complexes) we mean the (co)operad $\Lambda(O)$ whose $m$-th vector space is

$$\Lambda(O)(m) = s^{1-m}O(m) \otimes \text{sgn}_m,$$

where $\text{sgn}_m$ is the sign representation of the symmetric group $S_m$.

For a commutative algebra $B$ and a $B$-module $V$ we denote by $S_B(V)$ the symmetric algebra of $V$ over $B$. $S_B^m(V)$ stands for the $m$-th component of this algebra. If $B = \mathbb{K}$ then $B$ is omitted from the notation. The abbreviation “DGLA” stands for differential graded Lie algebra.

We denote by $*$ the polynomial functor

$$* (n) = \begin{cases} \mathbb{K}, & \text{if } n = 1, \\ 0, & \text{otherwise}. \end{cases}$$

This functor carries the unique structure of the operad (resp. the cooperad) such that $*$ is the initial (resp. the terminal) object in the category of operads (resp. cooperads) of graded vector spaces or chain complexes. There is an obvious generalization of $*$ to the category of sets and to the category of topological spaces. However, we will need $*$ only for linear (co)operads, i.e. the (co)operads in the category of graded vector spaces or the category of chain complexes.

All the linear operads (resp. linear cooperads), we consider, are equipped with an augmentation (resp. coaugmentation). In other words, for every operad $O$ we will have a chosen morphism of operads:

$$\tau : O \rightarrow \ast.$$

Dually for every cooperad $C$ we will have a chosen morphism of cooperads

$$\kappa : \ast \rightarrow C.$$

We are going to deal with 2-colored (co)operads. Throughout the paper we label the two colors of all 2-colored (co)operads by $c$ and $a$. For example, the notation $\text{Lie}^+$ is reserved for the 2-colored operad which governs the pairs “Lie algebra $V$ and a Lie algebra module $W$ over $V$.” Vectors of the Lie algebra $V$ are colored by $c$ and vectors of the module $W$ are colored by $a$.

For a linear 2-colored operad $O$ we will denote by $O^c(n,k)$ (resp. $O^a(n,k)$) the vector space of operations producing a vector with the color $c$ (resp. $a$) from $n$ vectors with the color $c$ and $k$ vectors with the color $a$. We use the same notation for the linear 2-colored cooperads and for topological 2-colored operads.

The polynomial functor $*$ has the obvious generalization to the category of linear 2-colored (co)operads:

$$*^c(n,k) = \begin{cases} \mathbb{K}, & \text{if } (n,k) = (1,0), \\ 0, & \text{otherwise}. \end{cases}$$

$$*^a(n,k) = \begin{cases} \mathbb{K}, & \text{if } (n,k) = (0,1), \\ 0, & \text{otherwise}. \end{cases}$$
For a linear operad $O$ we denote by $\text{Bar}(O)$ its bar construction. Dually, for a linear cooperad $C$ we denote by $\text{Cobar}(C)$ its cobar construction.

We recall that, as a cooperad of graded vector spaces, $\text{Bar}(O)$ is freely generated by the polynomial functor $s^{-1}O$, where $O$ is the kernel of the augmentation (2.5). Dually, as an operad of graded vector spaces, $\text{Cobar}(C)$ is freely generated by the polynomial functor $sC$, where $C$ is cokernel of the coaugmentation (2.6). The differential $\partial_{\text{Bar}}$ on the operad $\text{Bar}(O)$ is defined using the multiplication of the operad $O$ and the differential $\partial_{\text{Cobar}}$ on the cooperad $\text{Cobar}(C)$ is defined using the comultiplication of the cooperad $C$. See Chapter 3 in [15] or Section 2 in [17] for details.

For a quadratic operad $O$ there is a natural sub-cooperad $O^\vee$ of $\text{Bar}(O)$ which satisfies the property:

$$\partial_{\text{Bar}}|_{O^\vee} = 0.$$ The details of the construction of $O^\vee$ can be found in Section 5.2 in [15]. Following [18] we call $O^\vee$ the Koszul dual cooperad of $O$.

For a linear operad $O$ (resp. linear cooperad $C$) and a vector space $V$ we denote by $F_O(V)$ (resp. by $F_C(V)$) the free algebra (resp. free coalgebra) over the operad $O$ (resp. cooperad $C$). For a linear 2-colored (co)operad $O$ the functor $F_O$ splits according to the colors $c$ and $a$ as

$$F_O(V, W) = F_O(V, W)_c \oplus F_O(V, W)_a,$$

where

$$F_O(V, W)_c = \bigoplus_{n,k} O^c(n, k) \otimes S_n \times S_k V^\otimes n \otimes W^\otimes k,$$

and

$$F_O(V, W)_a = \bigoplus_{n,k} O^a(n, k) \otimes S_n \times S_k V^\otimes n \otimes W^\otimes k.$$ We need to recall some facts about algebras over the operad $\text{Cobar}(C)$ for a coaugmented cooperad $C$.

Since $\text{Cobar}(C)$ is freely generated by the suspension $sC$ of the cokernel $C$ of the coaugmentation (2.6) a $\text{Cobar}(C)$-algebra structure on a chain complex $V$ is uniquely determined by the restriction of the multiplication map

$$\mu : F_{\text{Cobar}(C)}(V) \rightarrow V$$

to the subspace

$$F_sC(V) \subset F_{\text{Cobar}(C)}(V).$$

In other words, a $\text{Cobar}(C)$-algebra structure on $V$ is uniquely determined by a degree 1 map from $F(C)(V)$ to $V$.

It turns out that the maps from $F(C)(V)$ to $V$ have a elegant description in terms of coderivations of the free coalgebra $F_C(V)$. To recall this description we introduce the Lie subalgebra

$$\text{Coder}'(F_C(V)) = \{ Q \in \text{Coder}(F_C(V)) \mid Q|_V = 0 \}, \quad (2.8)$$

$^3F_O$ is called the Schur functor.
where $\mathcal{V}$ is considered as a subspace of $\mathcal{C}(1) \otimes \mathcal{V}$ via the coaugmentation (2.6). In other words, the elements of $\text{Coder}'(\mathbb{F}_C(\mathcal{V}))$ are coderivations of the free coalgebra $\mathbb{F}_C(\mathcal{V})$ which can be factored through the projection

$$\mathbb{F}_C(\mathcal{V}) \rightarrow \mathbb{F}_C(\mathcal{V}).$$

It is not hard to see that the subspace (2.8) is closed under the commutator and the differentials coming from $\mathcal{C}$ and $\mathcal{V}$. Thus the graded vector space $\text{Coder}'(\mathbb{F}_C(\mathcal{V}))$ is in fact a DGLA.

Let us recall from [17] the following proposition

**Proposition 1 (Proposition 2.14 [17])** For a coaugmented cooperad $\mathcal{C}$ the composition with the corestriction (2.1) $\rho_{\mathcal{V}} : \mathbb{F}_C(\mathcal{V}) \rightarrow \mathcal{V}$ induces an isomorphism of graded vector spaces

$$\text{Coder}'(\mathbb{F}_C(\mathcal{V})) \cong \text{Hom}(\mathbb{F}_C(\mathcal{V}), \mathcal{V}).$$

where, as above, $\overline{\mathcal{C}}$ is the cokernel of the coaugmentation (2.6) of $\mathcal{C}$.

Due to this proposition a $\text{Cobar}(\mathcal{C})$-algebra structure on a chain complex $\mathcal{V}$ is uniquely determined by a degree 1 coderivation

$$Q \in \text{Coder}'(\mathbb{F}_C(\mathcal{V})).$$

(2.10)

According to Proposition 2.15 from [17] the compatibility of the $\text{Cobar}(\mathcal{C})$-algebra structure on $\mathcal{V}$ with the total differential on $\text{Cobar}(\mathcal{C})$ and the differential on $\mathcal{V}$ is equivalent to the Maurer-Cartan equation for the corresponding derivation (2.10):

$$[\partial^{\mathcal{C}} + \partial^{\mathcal{V}}, Q] + \frac{1}{2}[Q, Q] = 0,$$

(2.11)

where $\partial^{\mathcal{C}}$ is the differential on $\mathbb{F}_C(\mathcal{V})$ induced by the one on the cooperad $\mathcal{C}$ and $\partial^{\mathcal{V}}$ comes from that on $\mathcal{V}$.

In other words,

**Proposition 2 (Proposition 2.15, [17])** There is a natural bijection between the Maurer-Cartan elements of the DGLA $\text{Coder}'(\mathbb{F}_C(\mathcal{V}))$ and and the $\text{Cobar}(\mathcal{C})$-algebra structures on $\mathcal{V}$.

If we have a map

$$\mu : \mathcal{C}_1 \rightarrow \mathcal{C}_2$$

(2.12)

of coaugmented cooperads then the corresponding map between the cobar constructions

$$\text{Cobar}(\mu) : \text{Cobar}(\mathcal{C}_1) \rightarrow \text{Cobar}(\mathcal{C}_2)$$

allows us to pull $\text{Cobar}(\mathcal{C}_2)$-algebra structure on $\mathcal{V}$ to a $\text{Cobar}(\mathcal{C}_1)$-algebra on $\mathcal{V}$.

We claim that

**Proposition 3** If $Q_1$ is a Maurer-Cartan element of the DGLA $\text{Coder}'(\mathbb{F}_{C_1}(\mathcal{V}))$ corresponding to a $\text{Cobar}(\mathcal{C}_1)$-algebra structure on $\mathcal{V}$ and $Q_2$ is a Maurer-Cartan element of the DGLA $\text{Coder}'(\mathbb{F}_{C_2}(\mathcal{V}))$ corresponding to a $\text{Cobar}(\mathcal{C}_2)$-algebra structure on $\mathcal{V}$ then

$$\rho_{\mathcal{V}} \circ Q_1 = \rho_{\mathcal{V}} \circ Q_2 \circ \mathbb{F}(\mu),$$

(2.13)

where the map

$$\mathbb{F}(\mu) : \mathbb{F}_{\overline{\mathcal{C}}_1}(\mathcal{V}) \rightarrow \mathbb{F}_{\overline{\mathcal{C}}_2}(\mathcal{V})$$

is induced by (2.12).
Proof. Let
\[ \nu_2 : F_{\text{Cobar}(C_2)}(V) \to V \]
be the \textit{Cobar}(C_2)-algebra structure on \( V \). Then the \textit{Cobar}(C_1)-algebra structure on \( V \)
\[ \nu_1 : F_{\text{Cobar}(C_1)}(V) \to V \]
is obtained by composing the map \( \nu_2 \) with the map
\[ F(\text{Cobar}(\mu)) : F_{\text{Cobar}(C_1)}(V) \to F_{\text{Cobar}(C_2)}(V). \]
It is not hard to see that the restriction of \( \nu_1 \) to the subspace \( F_{s\mathcal{C}_1}(V) \) coincides with the composition of the maps
\[ F_{s\mathcal{C}_1}(V) \xrightarrow{F(\mu)} F_{s\mathcal{C}_2}(V) \]
and
\[ \nu_2_{|F_{s\mathcal{C}_2}(V)} : F_{s\mathcal{C}_2}(V) \to V. \]
Thus the proposition follows from the equation
\[ \nu_i_{|F_{s\mathcal{C}_i}(V)} = \rho_V \circ Q \circ \sigma, \]
where \( \rho_V \) is the corestriction \textcircled{2.11} and \( \sigma \) is the suspension isomorphism
\[ \sigma : F_{s\mathcal{C}_i}(V) \to F_{\mathcal{C}_i}(V), \]
and \( i = 1, 2. \) \( \square \)

We will freely use Propositions \textcircled{1} \textcircled{2} and \textcircled{3} for colored cooperads.

We remark that all 2-colored (co)operads, we consider, satisfy the following property: \textit{an argument with the color a can enter an operation at most once. If an argument with this color enters an operation then the resulting color is also a. Otherwise the resulting color is c}. In other words, for every \( n \)
\[ O^c(n, k) = O^a(n, k) = \{0\} , \quad \forall \ k > 1 , \]
\[ O^a(n, 0) = \{0\} , \quad O^c(n, 1) = \{0\} \]
for the (co)operads of graded vector spaces or chain complexes and
\[ O^c(n, k) = O^a(n, k) = \emptyset , \quad \forall \ k > 1 , \]
\[ O^a(n, 0) = \emptyset , \quad O^c(n, 1) = \emptyset \]
for the (co)operads of topological spaces or sets.

It is not hard to see that bar and cobar constructions the (co)operads of graded vector spaces or chain complexes preserve property \textcircled{2.14}.

Let us recall that

\textbf{Definition 1 (M. Gerstenhaber, [16])} A graded vector space \( V \) is a Gerstenhaber algebra if it is equipped with a commutative and associative product \( \wedge \) of degree 0 and a Lie bracket \([ , , ]\) of degree \(-1\). These operations have to be compatible in the sense of the following Leibniz rule
\[ [a, b \wedge c] = [a, b] \wedge c + (-1)^{(|a|+1)|b|}b \wedge [a, c], \]
where \( a, b, c \) are homogeneous vectors of \( V \).
Definition 2 A precalculus is a pair of a Gerstenhaber algebra \((V, \wedge, [\cdot, \cdot])\) and a graded vector space \(W\) together with

- a module structure \(i_\bullet : V \otimes W \to W\) of the graded commutative algebra \(V\) on \(W\),
- an action \(l_\bullet : s^{-1}V \otimes W \to W\) of the Lie algebra \(s^{-1}V\) on \(W\) which are compatible in the sense of the following equations

\[
i_a l_b - (-1)^{|a|(|b|+1)} l_b i_a = i_{[a,b]}, \tag{2.17}
\]

and

\[
l_a \wedge b = l_a i_b + (-1)^{|a|} i_a l_b. \tag{2.18}
\]

Furthermore,

Definition 3 A calculus is a precalculus \((V, W, [\cdot, \cdot], \wedge, i_\bullet, l_\bullet)\) with a degree \(-1\) unary operation \(\delta\) on \(W\) such that

\[
\delta i_a - (-1)^{|a|} i_a \delta = l_a, \tag{2.19}
\]

and

\[
\delta^2 = 0. \tag{2.20}
\]

We call \(l\) and \(i\) the Lie derivative and the contraction, respectively.

We will use the following list of (co)operads:

- \textbf{Lie} (resp. \textbf{coLie}) is the operad of Lie algebras (resp. the cooperad of Lie coalgebras),
- \textbf{comm} (resp. \textbf{cocomm}) is the operad of commutative (associative) algebras (resp. the operad of cocommutative coassociative coalgebras),
- \(e_2\) denotes the operad of Gerstenhaber algebras, (see Definition 1),
- \(KS\) denotes the operad of M. Kontsevich and Y. Soibelman. This operad\(^5\) is described in sections 11.1, 11.2 and 11.3 of [23],
- \(\textbf{Lie}^+\) (resp. \textbf{coLie}^+) denotes the 2-colored operad of pairs “Lie algebra + its module” (resp. the 2-colored cooperad of pairs “Lie coalgebra + its comodule”),
- \(\textbf{comm}^+\) (resp. \textbf{cocomm}^+) denotes the 2-colored operad of pairs “commutative algebra + its module” (resp. the 2-colored cooperad of pairs “cocommutative coalgebra + its comodule”),
- \(\textbf{pcalc}\) denotes the 2-colored operad of precalculi, (see Definition 2),
- \(\textbf{calc}\) denotes the 2-colored operad of calculi, (see Definition 3),
- \(\textbf{assoc}\) is the non-symmetric operad of sets controlling unital monoids; each set \(\text{assoc}(n), n \geq 0\), is a point.

\(^4\)Although \(\delta^2 = 0\), the operation \(\delta\) is never considered as a part of the differential on \(W\).

\(^5\)In [23] this operad is denoted by \(P\).
It is not hard to show that for the vector space of the free calculus algebra generated by
the pair \((\mathcal{V}, \mathcal{W})\) we have
\[
\mathbb{F}_{\text{calc}}(\mathcal{V}, \mathcal{W}) \cong \mathbb{F}_{\text{comm}}^+(\mathbb{F}_{\text{ALie}}(\mathcal{V}, \mathcal{W} \oplus s^{-1} \mathcal{W})).
\] (2.21)

In other words, for the color components we have the isomorphisms of graded vector spaces:
\[
\mathbb{F}_{\text{calc}}(\mathcal{V}, \mathcal{W})_c \cong \mathbb{F}_{\text{comm}}(\mathbb{F}_{\text{ALie}}(\mathcal{V})),
\] (2.22)

and
\[
\mathbb{F}_{\text{calc}}(\mathcal{V}, \mathcal{W})_a \cong \mathbb{F}_{\text{comm}}^+(\mathbb{F}_{\text{ALie}}(\mathcal{V}), \mathbb{F}_{\text{ALie}}^+(\mathcal{V}, \mathcal{W} \oplus s^{-1} \mathcal{W})).
\] (2.23)

### 2.2 Hochschild (co)chain complexes

For an associative algebra \(A\)
\[
C^\bullet(A) = \text{Hom}(A^{\otimes \bullet}, A)
\]
denotes the Hochschild cochain complex and
\[
C_\bullet(A) = A \otimes A^{\otimes (-\bullet)}
\]
stands for the Hochschild chain complex of \(A\) with the reversed grading.

For the normalized versions of these complexes we reserve the notation:
\[
C^\bullet_{\text{norm}}(A) = \{P \in \text{Hom}(A^{\otimes \bullet}, A) \mid P(\ldots, 1, \ldots) = 0\}
\]
and
\[
C^\bullet_{\text{norm}}(A) = A \otimes (A/\mathbb{K}1)^{\otimes (-\bullet)}.
\]

- The notation \(\partial^{Hoch}\) is reserved both for the Hochschild coboundary operator on \(C^\bullet_{\text{norm}}(A)\) and Hochschild boundary operator on \(C^\bullet_{\text{norm}}(A)\)

\[
(\partial^{Hoch} P)(a_0, a_1, \ldots) = a_0 P(a_1, a_2, \ldots, a_k) - P(a_0 a_1, a_2, \ldots, a_k) + P(a_0, a_1 a_2, \ldots, a_k) - \ldots
\]

\[
+ (-1)^k P(a_0, \ldots, a_{k-2}, a_{k-1} a_k) + (-1)^{k+1} P(a_0, \ldots, a_{k-2}, a_{k-1}) a_k
\]

\[
\partial^{Hoch}(a_0, a_1, \ldots, a_m) = (a_0 a_1, a_2, \ldots, a_m) - (a_0, a_1 a_2, a_3, \ldots, a_m) + \cdots +
\]

\[
(-1)^{m-1}(a_0, \ldots, a_{m-2}, a_{m-1} a_m) + (-1)^m (a_m a_0, a_1, a_2, \ldots, a_{m-1}),
\]

\[a_i \in A, \quad P \in C^k_{\text{norm}}(A)\).

- The notation \(\cup\) is reserved for the cup-product on \(C^\bullet_{\text{norm}}(A)\)

\[
P_1 \cup P_2(a_1, a_2, \ldots, a_{k_1+k_2}) = P_1(a_1, \ldots, a_{k_1}) P_2(a_{k_1+1}, \ldots, a_{k_1+k_2}),
\] (2.24)

\[P_i \in C^k_{\text{norm}}(A)\).
• $[\cdot, \cdot]_G$ stands for the Gerstenhaber bracket on $C^\bullet_{\text{norm}}(A)$

$$[Q_1, Q_2]_G = \sum_{i=0}^{k_1} (-1)^i k_2 Q_1(a_0, \ldots, Q(a_i, \ldots, a_{i+k_2}), \ldots, a_{k_1+k_2}) - (-1)^{k_1 k_2} (1 \leftrightarrow 2),$$

$$Q_i \in C^{k_i+1}_{\text{norm}}(A).$$

• $I_P(c)$ is the contraction of a Hochschild cochain $P \in C^k_{\text{norm}}(A)$ with a Hochschild chain $c = (a_0, a_1, \ldots, a_m)$

$$I_P(a_0, a_1, \ldots, a_m) = \begin{cases} (a_0 P(a_1, \ldots, a_k), a_{k+1}, \ldots, a_m), & \text{if } m \geq k, \\ 0, & \text{otherwise}. \end{cases}$$

• $L_Q(c)$ denotes the Lie derivative of a Hochschild chain $c = (a_0, a_1, \ldots, a_m)$ along a Hochschild cochain $Q \in C^{k+1}_{\text{norm}}(A)$

$$L_Q(a_0, a_1, \ldots, a_m) = \sum_{i=0}^{m-k} (-1)^{k_i}(a_0, \ldots, Q(a_i, \ldots, a_{i+k}), \ldots, a_m) +$$

$$\sum_{j=m-k}^{m-1} (-1)^{m(j+1)}(Q(a_{j+1}, \ldots, a_m, a_0, \ldots, a_{k+j-m}), a_{k+j+1-m}, \ldots, a_j).$$

• $B : C^\bullet_{\text{norm}}(A) \to C_{\text{norm}-1}(A)$ denotes Connes’ operator

$$B(a_0, a_1, \ldots, a_m) = \sum_{i=0}^{m} (-1)^{m_i} (1, a_i, \ldots, a_m, a_0, a_1, \ldots, a_{i-1}).$$

The notation $HH^\bullet(A)$ (resp. $HH_\bullet(A)$) is used for the Hochschild cohomology (resp. homology groups) of $A$ with coefficients in $A$

$$HH^\bullet(A) = H^\bullet(C^\bullet_{\text{norm}}(A), \partial_{\text{Hoch}}),$$

$$HH_\bullet(A) = H^\bullet(C^\bullet_{\text{norm}}(A), \partial_{\text{Hoch}}).$$

To describe algebraic structures on pairs $(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A))$ and $(HH^\bullet(A), HH_\bullet(A))$ we use the language of operads. Thus, the Gerstenhaber bracket $[\cdot, \cdot]_G$ equips the cochain complex $C^\bullet_{\text{norm}}(A)$ with an algebra structure over the operad $\Lambda \text{Lie}$ and the Lie derivative (2.27) equips the pair $(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A))$ with the algebra structure over the operad $\Lambda \text{Lie}^+$. In order to add Connes’ operator (2.28) into this operadic picture we give one more definition
Definition 4 We say that the pair of graded vector spaces \((V,W)\) is an algebra over the operad \(\text{Lie}_\delta^+\) if \(V\) is a Lie algebra, \(W\) is a module over \(V\) and \(W\) is equipped with a degree \(-1\) unary operation \(\delta\) satisfying the equations
\[
\delta^2 = 0, \quad (2.29)
\]
and
\[
[\delta, l_a] = 0, \quad \forall \ a \in V, \quad (2.30)
\]
where \(l\) is the action of \(V\) on \(W\).

Adding Connes’ operator \(B\) into the picture we may say that the pair \((C^\text{norm}(A), C^\text{norm}(A))\) is a \(\text{ALie}_\delta^+\)-algebra.

The operations \((2.24), (2.25), (2.26), (2.27)\) and \((2.28)\) are closed with respect to the (co)boundary operator \(\partial^{\text{Hoch}}\).

According to \([16]\) the operations \(\cup (2.24)\) and \([,]_G (2.25)\) induce on \(HH^\bullet(A)\) the structure of a Gerstenhaber algebra. Furthermore, it is known \([11]\) that the operations \((2.24), (2.25), (2.26), (2.27)\) and \((2.28)\) induce on the pair \((HH^\bullet(A), HH_\bullet(A))\) the structure of the calculus algebra.

3 The operads \(\text{Ho}(\text{calc}), \text{Ho}(e_2), \text{and Ho}(\text{Lie}_\delta^+)\)

In this section we describe the homotopy versions for the algebras over the operads \(\text{calc}, e_2, \) and \(\text{Lie}_\delta^+\).

3.1 Description of the operads \(\text{Ho}(\text{calc})\) and \(\text{Ho}(e_2)\)

To describe the homotopy version of \(\text{calc}\)-algebras we use the canonical cofibrant resolution \(\text{Cobar}(\text{Bar}(\text{calc}))\). In other words, we set
\[
\text{Ho}(\text{calc}) = \text{Cobar}(\text{Bar}(\text{calc})). \quad (3.1)
\]

The cooperad \(\text{Bar}(\text{calc})\) will be used throughout the paper. For this reason we reserve a short-hand notation
\[
\mathbf{B} = \text{Bar}(\text{calc}) \quad (3.2)
\]
for this cooperad.

Recall that, as a cooperad of graded vector spaces, \(\mathbf{B} = \text{Bar}(\text{calc})\) is freely generated by the polynomial functor \(s^{-1}\text{calc}\), where \(\text{calc}\) is the kernel of the augmentation.

We represent elements of the free coalgebra \(\mathbb{F}_\mathbf{B}(\mathcal{V}, \mathcal{W})\) and elements of the cooperad \(\mathbf{B}\) graphically. Thus Figures \([1, 2]\) represent the simplest elements of \(\mathbb{F}_\mathbf{B}(\mathcal{V}, \mathcal{W})_\gamma\) with \(\gamma_1\) and \(\gamma_2\) being vectors in \(\mathcal{V}\). Figures \([3, 4]\) show the simplest elements of \(\mathbb{F}_\mathbf{B}(\mathcal{V}, \mathcal{W})_c\) with \(\gamma \in \mathcal{V}\) and \(c \in \mathcal{W}\). Figures \([5, 6]\) represent simple elements of \(\mathbf{B}^a(0, 1)\). The dashed line in figures \([3, 4, 5]\) and \(6\) is used to label the arguments of the color \(a\) and the solid line is used to label the arguments of the color \(c\).

Using this graphical notation we may perform simple computations in the coalgebra \(\mathbb{F}_\mathbf{B}(\mathcal{V}, \mathcal{W})\). For example, using equation \((2.19)\), we present on Figure \([7]\) a simple computation with the bar differential \(\partial^{\text{Bar}}\). Here \(\gamma \in \mathcal{V}\) and \(c \in \mathcal{W}\).
Figure 1: The product $\land \in \text{calc}^c(2,0)$ is used

Figure 2: The bracket $[,] \in \text{calc}^c(2,0)$ is used

Figure 3: The commutative module structure $i \in \text{calc}^a(1,1)$ is used

Figure 4: The Lie algebra module structure $l \in \text{calc}^a(1,1)$ is used
For the operad $e_2$ we use a resolution which is simpler than the canonical one $Cobar(\text{Bar}(e_2))$. More precisely, as in $[13]$, we set

$$\text{Ho } (e_2) = Cobar(e_2^\vee).$$

(3.3)

Due to koszulity of the operad $e_2$ the inclusions

$$\iota_{e_2} : e_2^\vee \hookrightarrow \text{Bar}(e_2)$$

(3.4)

and

$$Cobar(\iota_{e_2}) : Cobar(e_2^\vee) \hookrightarrow Cobar(\text{Bar}(e_2))$$

(3.5)

are quasi-isomorphisms of cooperads and operads, respectively. It is the second quasi-isomorphism (3.5) which allows us to replace the canonical resolution $Cobar(\text{Bar}(e_2))$ by (3.3).

To get a more tractable description of algebras over the operads $\text{Ho } (\text{calc})$ and $\text{Ho } (e_2)$ we introduce the following DGLAs

$$\text{Coder}'(\mathbb{F}_{e_2^\vee}(\mathcal{V})) = \{Q \in \text{Coder}(\mathbb{F}_{e_2^\vee}(\mathcal{V})) \mid Q\big|_{\mathcal{V}} = 0\},$$

(3.6)

$$\text{Coder}'(\mathbb{F}_{B}(\mathcal{V}, \mathcal{W})) = \{Q \in \text{Coder}(\mathbb{F}_{B}(\mathcal{V}, \mathcal{W})) \mid Q\big|_{\mathcal{V}\oplus\mathcal{W}} = 0\},$$

(3.7)

where $\text{Coder}(\mathbb{F}_{e_2^\vee}(\mathcal{V}))$ (resp. $\text{Coder}(\mathbb{F}_{B}(\mathcal{V}, \mathcal{W}))$) is the DGLA of coderivations of the free coalgebra $\mathbb{F}_{e_2^\vee}(\mathcal{V})$ (resp. the free coalgebra $\mathbb{F}_{B}(\mathcal{V}, \mathcal{W})$). Furthermore, $\mathcal{V}$ (resp. $\mathcal{V}\oplus\mathcal{W}$) is considered as a subspace of $\mathbb{F}_{e_2^\vee}(\mathcal{V})$ (resp. $\mathbb{F}_{B}(\mathcal{V}, \mathcal{W})$) via the corresponding coaugmentation.

According to Proposition $[2]$ the $\text{Ho } (e_2)$-algebra structures on $\mathcal{V}$ are in bijection with the Maurer-Cartan elements of the DGLA $\text{Coder}'(\mathbb{F}_{e_2^\vee}(\mathcal{V}))$. Similarly, the $\text{Ho } (\text{calc})$-algebra
Figure 7: A simple computation with $\partial Bar$
structures on the pair \((\mathcal{V}, \mathcal{W})\) are in bijection with the Maurer-Cartan elements of the DGLA \(\text{Coder}^{\prime}(\mathbb{F}_B(\mathcal{V}, \mathcal{W}))\). Moreover, due to Proposition 1 the Maurer-Cartan element \(Q\) of the DGLA \((3.6)\) (resp. the DGLA \((3.7)\)) is uniquely determined by its composition \(\rho_V \circ Q\) (resp. \(\rho_{V,W} \circ Q\)) with the corestriction \(\rho_V : \mathbb{F}_{e_2^\vee}(\mathcal{V}) \to \mathcal{V}\) (resp. the corestriction \(\rho_{V,W} : \mathbb{F}_B(\mathcal{V}, \mathcal{W}) \to \mathcal{V} \oplus \mathcal{W}\)).

The vector space of the free coalgebra \(\mathbb{F}_B(\mathcal{V}, \mathcal{W})\) splits according to the two colors \((c, a)\) as
\[
\mathbb{F}_B(\mathcal{V}, \mathcal{W}) = \mathbb{F}_B(\mathcal{V}, \mathcal{W})_c \oplus \mathbb{F}_B(\mathcal{V}, \mathcal{W})_a ,
\]
where
\[
\mathbb{F}_B(\mathcal{V}, \mathcal{W})_c = \mathbb{F}_{\text{Bar}}(e_2)(\mathcal{V}) .
\]

Thus for every \(\text{Ho}(\text{calc})\)-algebra \((\mathcal{V}, \mathcal{W})\) the graded vector space \(\mathcal{V}\) is an algebra over the operad \(\text{Cobar}(\text{Bar}(e_2))\). Using this algebra structure over \(\text{Cobar}(\text{Bar}(e_2))\) and the embedding \((3.5)\) we get a \(\text{Ho}(e_2)\)-algebra structure on \(\mathcal{V}\).

To describe the relationship between these algebras we denote by \(Q_{V,W}\) the Maurer-Cartan element of the DGLA \(\text{Coder}^{\prime}(\mathbb{F}_B(\mathcal{V}, \mathcal{W}))\) corresponding to the \(\text{Ho}(\text{calc})\)-algebra structure on \((\mathcal{V}, \mathcal{W})\). Next, we denote by \(Q_V\) the Maurer-Cartan element of the DGLA \(\text{Coder}^{\prime}(\mathbb{F}_{e_2^\vee}(\mathcal{V}))\) corresponding to the \(\text{Ho}(e_2)\)-algebra structure on \(\mathcal{V}\).

Proposition 3 implies that
\[
\rho_V \circ Q_V = \rho_{V,W} \circ Q_{V,W} \circ \mathbb{F}(\iota_{e_2}) ,
\]
where \(\iota_{e_2}\) is the embedding \((3.4)\) and
\[
Q_{V,W} = Q_{V,W} \big|_{\mathbb{F}_B(\mathcal{V}, \mathcal{W})_c} .
\]

Due to Proposition 1 the coderivation \(Q_V\) (resp. the coderivation \(Q_{V,W}\)) is uniquely determined by the composition \(\rho_V \circ Q_V\) (resp. \(\rho_{V,W} \circ Q_{V,W}\)). Thus equation \((3.10)\) indeed describes the relationship between the \(\text{Ho}(\text{calc})\)-algebra structure on \((\mathcal{V}, \mathcal{W})\) and the \(\text{Ho}(e_2)\)-algebra structure on \(\mathcal{V}\).

**Remark.** The vector space of operations of the cooperad \(\mathbf{B}\) with no arguments having color \(c\) is
\[
\mathbf{B}^c(0, 1) = \mathbb{K}[u] ,
\]
where \(u\) is an auxiliary variable of degree \(-2\). The monomial \(u^m\) corresponds to the element of \(\mathbf{B}^c(0, 1)\) which is drawn on Figure 6 (See page 13).

### 3.2 Description of the operad \(\text{Ho}(\text{Lie}_\delta^+)\)

The canonical cofibrant resolution \(\text{Cobar}(\text{Bar}(\text{Lie}_\delta^+))\) can be simplified. In this subsection we construct a sub-cooperad \((\text{Lie}_\delta^+)\vee\) of \(\text{Bar}(\text{Lie}_\delta^+)\) such that the embedding of operads
\[
\text{Cobar}((\text{Lie}_\delta^+)\vee) \hookrightarrow \text{Cobar}(\text{Bar}(\text{Lie}_\delta^+))
\]
is a quasi-isomorphism. This construction goes along the lines of \([15], [18]\). (See also Definition 3.2.1 in \([19]\).) It allows us to set
\[
\text{Ho}(\text{Lie}_\delta^+) = \text{Cobar}((\text{Lie}_\delta^+)\vee) .
\]
Let us first recall that algebras over the operad \( \mathbf{Lie}^+_\delta \) are pairs \((\mathcal{V}, \mathcal{W})\) where \( \mathcal{V} \) is a Lie algebra \( \mathcal{W} \) is a Lie algebra module over \( \mathcal{V} \) and \( \mathcal{W} \) is equipped with degree \(-1\) unary operation \( \delta \) which satisfies the identities

\[
\delta^2 = 0, \tag{3.12}
\]

and

\[
\delta l_a - (-1)^{|a|} l_a \delta = 0, \tag{3.13}
\]

where \( l : \mathcal{V} \otimes \mathcal{W} \to \mathcal{W} \) is the action of \( \mathcal{V} \) on \( \mathcal{W} \).

Thus the operad \( \mathbf{Lie}^+_\delta \) is generated by the elementary operations \([, , ], l \) and \( \delta \), where \([, , ]\) denotes the Lie bracket. These operations are subject to the homogeneous quadratic relations: the Jacobi identity for the Lie bracket \([, , ]\), and the compatibility equation between \( l \) and \([, , ]\)

\[
l_a l_b - (-1)^{|a||b|} l_a l_b = l_{[a,b]} \tag{3.14}
\]

and, finally, equations (3.12) and (3.13).

To construct the cooperad \( (\mathbf{Lie}^+_\delta)^\vee \) we introduce the polynomial functor \( S \) spanned linearly by the elementary operations \([, , ], l \), and \( \delta \) of the operad \( \mathbf{Lie}^+_\delta \).

We also introduce the linear span \( R \) of the homogeneous quadratic relations of \( \mathbf{Lie}^+_\delta \) between the elementary operations.

Next, we consider the free cooperad \( T^*(s^{-1}S) \) generated by the polynomial functor \( s^{-1}S \). The cooperad \( T^*(s^{-1}S) \) may be viewed as a sub-cooperad of \( \text{Bar}(\mathbf{Lie}^+_\delta) \) if we forget about the differential \( \partial_{\text{Bar}} \).

Let us remark that, the cooperad \( T^*(s^{-1}S) \) is equipped with the natural grading

\[
T^*(s^{-1}S) = \bigoplus_{m=0}^\infty T_m^*(s^{-1}S), \quad T_0^*(s^{-1}S) = *, \tag{3.15}
\]

where * is the terminal object \((2.7)\) in the category of 2-colored cooperads and \( T_m^*(s^{-1}S) \) consists of the elements of degree \( m \) in the elementary operations. Thus, since the relations between the elementary operations are quadratic, \( s^{-2}R \) is a subspace of \( T_2^*(s^{-1}S) \).

First, we construct the cooperad \( (\mathbf{Lie}^+_\delta)^\vee \) as a sub-cooperad of \( T^*(s^{-1}S) \) and then we will show that \( (\mathbf{Lie}^+_\delta)^\vee \) belongs to the kernel of the bar differential \( \partial_{\text{Bar}} \).

We construct \( (\mathbf{Lie}^+_\delta)^\vee \) by induction on the degree \( m \) in (3.15). The base of the induction is given by the equations

\[
(\mathbf{Lie}^+_\delta)^\vee \cap T_0^*(s^{-1}S) \oplus T_1^*(s^{-1}S) = T_0^*(s^{-1}S) \oplus T_1^*(s^{-1}S), \tag{3.16}
\]

\[
(\mathbf{Lie}^+_\delta)^\vee \cap T_2^*(s^{-1}S) = s^{-2}R, \tag{3.17}
\]

and the step is given by the condition: a vector \( v \in T_m^*(s^{-1}S) \) belongs to \( (\mathbf{Lie}^+_\delta)^\vee \) provided

\[
\bar{\Delta}(v) \in (\mathbf{Lie}^+_\delta)^\vee \bullet (\mathbf{Lie}^+_\delta)^\vee.
\]

Here \( \bar{\Delta} \) is the coproduct:

\[
\bar{\Delta} : T^*(s^{-1}S) \to T^*(s^{-1}S) \bullet T^*(s^{-1}S),
\]

and

\[
\bar{\Delta}(v) = \Delta(v) - v \otimes (1 \otimes \cdots \otimes 1) - 1 \otimes (v \otimes 1 \otimes \cdots \otimes 1) - 1 \otimes (1 \otimes v \otimes 1 \cdots \otimes 1) - \ldots
\]
−1 ⊗ (1 ⊗ ⋮ ⊗ 1 ⊗ v).

By construction \((\text{Lie}^+_h)^\vee\) is a sub-cooperad of \(T^*(s^{-1}S)\).

Equation (3.17) imply immediately that

\[ \partial \text{Bar} \ v = 0, \quad \forall \ v \in (\text{Lie}^+_h)^\vee \cap T^*_2(s^{-1}S). \]

Then the compatibility of \(\partial \text{Bar}\) with the coproduct \(\Delta:\)

\[ \Delta \partial \text{Bar} = (\partial \text{Bar} \otimes (1 \otimes \cdots \otimes 1) + 1 \otimes (\partial \text{Bar} \otimes 1 \otimes \cdots \otimes 1) + \ldots) \Delta \]

and the inductive definition of \((\text{Lie}^+_h)^\vee\) imply that

\[ \partial \text{Bar} \ v = 0, \quad \forall \ v \in (\text{Lie}^+_h)^\vee. \] (3.18)

Thus \((\text{Lie}^+_h)^\vee\) belongs to the kernel of the bar differential \(\partial \text{Bar}\) in \(\text{Bar}(\text{Lie}^+_h)\).

The following proposition gives us a description of the coalgebras over the cooperad \((\text{Lie}^+_h)^\vee\)

**Proposition 4** A pair \((\mathcal{V}, \mathcal{W})\) of graded vector spaces forms a coalgebra over the cooperad \((\text{Lie}^+_h)^\vee\) if \((\mathcal{V}, \mathcal{W})\) is a coalgebra over the cooperad \(\Lambda \text{cocomm}^+\) and \(\mathcal{W}\) is equipped with a degree 2 endomorphism

\[ \delta^\vee : \mathcal{W} \to \mathcal{W} \]

satisfying the equation

\[ l^\vee \circ \delta^\vee = (1 \otimes \delta^\vee)l^\vee, \]

where \(l^\vee\) is the coaction of \(\mathcal{V}\) on \(\mathcal{W}\)

\[ l^\vee : \mathcal{W} \to s^{-1}(\mathcal{V} \otimes \mathcal{W}). \]

**Proof.** Let us consider the restricted dual vector space

\[ [T^*(s^{-1}S)]^* = \text{Hom}_{\text{restr}}(T^*(s^{-1}S), \mathbb{K}) \] (3.19)

of the free cooperad \(T^*(s^{-1}S)\) with respect to the grading (3.15). It is not hard to see that

\[ [T^*(s^{-1}S)]^* = T(sS^*) \]

is the free operad \(T(sS^*)\) generated by the suspension \(sS^*\) of the linear dual \(S^*\) of the polynomial functor \(S\).

From the construction of \((\text{Lie}^+_h)^\vee\) it follows that the restricted dual \([(\text{Lie}^+_h)^\vee]^*\) of the cooperad \((\text{Lie}^+_h)^\vee\) is the quotient of the free operad \(T(sS^*)\) with respect to the ideal generated by the polynomial functor of dual relations

\[ R^* = \{ r \in \text{Hom}(T^*_2(s^{-1}S), \mathbb{K}) , \quad | r \big|_R = 0 \}. \] (3.20)

Let \{[,], l^*, \delta^*\} be the basis of \(S^*\) which is dual to the basis \{[,], l, \delta\} of \(S\).

Dualizing the Jacobi relation for [,] and the compatibility (3.14) of \(l\) with [,] we see that the operation \(s[,]^*\) satisfies the axioms of an associative commutative product and the operation \(sl^*\) satisfies the axiom of a module over an associative and commutative algebra.
Dualizing the relation (3.13) we see that \( s l^* \) and \( s \delta^* \) are compatible in the sense of the following relation

\[
s \delta^* s l^* = s l^*(1 \otimes s \delta^*).
\]

(3.21)

Finally the presence of the relation (3.12) implies that we should not impose any additional condition on \( s \delta^* \) besides (3.21).

Thus a pair \((\tilde{V}, \tilde{W})\) is an algebra over the operad \([([\text{Lie}^+_\delta]^\vee)^\vee])\) if \((\tilde{V}, \tilde{W})\) is a \(\Lambda^{-1}\text{comm}^+\)-algebra and \(\tilde{W}\) is equipped with a degree 2 endomorphism \(s \delta^*\) which is compatible with the action of \(\tilde{V}\) on \(\tilde{W}\) in the sense of (3.21).

Taking the dual partner of an algebra over the operad \([([\text{Lie}^+_\delta]^\vee)]^\vee\) we get the statement of the proposition. □

Proposition 4 implies that a free coalgebra over the cooperad \([([\text{Lie}^+_\delta]+\delta^\vee)]^\vee\) generated by a pair \((V, W)\) is

\[
F_{([\text{Lie}^+_\delta]^\vee)^\vee}(V, W) = F_{\Lambda\text{cocomm}^+}(V, W[[u]]),
\]

(3.22)

where \(u\) is an auxiliary variable of degree \(-2\).

We claim that

**Proposition 5** The operad \(\text{Lie}^+_\delta\) is Koszul. In other words the embedding

\[
\text{Cobar}((\text{Lie}^+_\delta)^\vee) \rightarrow \text{Cobar}(\text{Bar}(\text{Lie}^+_\delta))
\]

is a quasi-isomorphism of operads.

**Proof.** The criterion of Ginzburg and Kapranov [18] (theorem 4.2.5) reduces this question to computation of the homology of a free \(\text{Lie}^+_\delta\)-algebra. More precisely, we need to show that for every pair \((\mathcal{V}, \mathcal{W})\) of vector spaces the complex

\[
F_{([\text{Lie}^+_\delta]^\vee)^\vee} \circ F_{\text{Lie}^+_\delta}(\mathcal{V}, \mathcal{W})
\]

(3.23)

has nontrivial cohomology only in degree 0.

Here the differential on the complex (3.23) is defined along the lines of [17] using the twisting cochain of the pair \((\text{Lie}^+_\delta, ([\text{Lie}^+_\delta]^\vee)^\vee))\). (See Section 2.4 in [17] for more details.)

If we split the complex (3.23) according to the colors \(c\) and \(a\) and use equation (3.22) then we get two complexes:

\[
F_{([\text{Lie}^+_\delta]^\vee)^\vee} \circ F_{\text{Lie}^+_\delta}(\mathcal{V}, \mathcal{W})_c = F_{\Lambda\text{cocomm}} \circ F_{\text{Lie}}(\mathcal{V}),
\]

(3.24)

and

\[
F_{([\text{Lie}^+_\delta]^\vee)^\vee} \circ F_{\text{Lie}^+_\delta}(\mathcal{V}, \mathcal{W})_a = F_{\Lambda\text{cocomm}^+}(F_{\text{Lie}}(\mathcal{V}), T(\mathcal{V}) \otimes (\mathcal{W} \oplus \delta \mathcal{W})[[u]])_a,
\]

(3.25)

where \(T(\mathcal{V})\) denotes the tensor algebra of \(\mathcal{V}\), \(\delta\) is the unary operation of \(\text{Lie}^+_\delta\) and \(u\) is an auxiliary variable of degree \(-2\).

The first complex is exactly the Harrison complex of the free Lie algebra generated by \(\mathcal{V}\) and it is known that this complex has nontrivial cohomology only in degree 0.

The second complex is the tensor product of the Harrison complex of the free module generated by \(\mathcal{W}\) over the free Lie algebra \(F_{\text{Lie}}(\mathcal{V})\) and the De Rham complex

\[
(\mathbb{K}[[u]] \oplus \delta \mathbb{K}[[u]], \delta \frac{\partial}{\partial u})
\]
of the algebra $K[[u]]$. Thus the second complex also has nontrivial cohomology only in degree 0. □

This Proposition implies immediately that the embedding

$$Cobar(\Lambda(\text{Lie}^+_{\delta})) \hookrightarrow Cobar(\text{Bar}(\Lambda\text{Lie}^+_{\delta}))$$

is a quasi-isomorphism of operads. Thus we may set

$$\text{Ho}(\Lambda\text{Lie}^+_{\delta}) = Cobar(\Lambda(\text{Lie}^+_{\delta}))^\vee. \quad (3.26)$$

We would also like to remark that equation (3.22) implies that

$$F_{\Lambda(\text{Lie}^+_{\delta})^\vee}(V, W) = F_{\Lambda^2\text{cocomm}^+}(V, W[[u]]), \quad (3.27)$$

where $u$ is an auxiliary variable of degree $-2$.

4 The Kontsevich-Soibelman operad and the operad of little discs on a cylinder

4.1 The Kontsevich-Soibelman operad KS

Let us describe the auxiliary operad $\mathcal{H}$ (of sets) of “natural”\textsuperscript{6} operations on the pair

$$(C^*(A), C_*(A)).$$

This operad is going to have a countable set of colors

$$\Xi = \mathbb{Z}^+ \sqcup \mathbb{Z}^-, \quad (4.1)$$

where $\mathbb{Z}^+$ (resp. $\mathbb{Z}^-$) denotes the set of nonnegative (resp. nonpositive) integers.

The numbers from the set $\mathbb{Z}^+$ label the degrees of the Hochschild cochains and the numbers from the set $\mathbb{Z}^-$ label the degrees of Hochschild chains.

Using $\mathcal{H}$ we construct the DG operad $\mathbf{KS}$ of Kontsevich and Soibelman. The latter operad\textsuperscript{7} is described in sections 11.1, 11.2 and 11.3 of [23].

For the $\Xi$-colored operad $\mathcal{H}$ we only allow the operations in which a chain may enter as an argument at most once. If a chain enters then the result of the operation is also a chain. Otherwise the result is a cochain. We denote the set of operations producing a cochain from $n$ cochains by $\mathcal{H}(n, 0)$. The set of operations producing a chain from $n$ cochains and 1 chain is denoted by $\mathcal{H}(n, 1)$.

$\mathcal{H}(n, 0)$ is the set of equivalence classes of rooted\textsuperscript{8} planar trees $T$ with marked vertices. The equivalence relation is the finest one in which two such trees are equivalent if one of them can be obtained from the other by either:

- the contraction of an edge with unmarked ends or

\textsuperscript{6}We are not sure if these operations are natural in the sense of category theory.

\textsuperscript{7}In [23] this operad is denoted by $P$.

\textsuperscript{8}Recall that a tree called rooted if if its root vertex has valency 1.
removing an unmarked vertex with only one edge originating from it and joining the
two edges adjacent to this vertex into one edge.

If a marked vertex is internal then it is reserved for a cochain which enters as an argument
of the operation. The number of the incoming edges of such vertex is the degree of the
Corresponding cochain. If a marked vertex is terminal then it is reserved either for a cochain
of degree 0 or for an argument of the cochain produced by the operation.

The unmarked vertices (both internal and terminal) are reserved for the operations of
the non-symmetric operad assoc which controls unital monoids. For example, an unmarked
terminal vertex is reserved for unit of $A$, an unmarked vertex of valency 2 is reserved for
the identity transformation on $A$, and an unmarked vertex of valency 3 is reserved for the
associative product on $A$.

The root vertex is special. Since our trees are rooted this vertex has always valency 1.
It is always marked and reserved for the outcome of the cochain produced by the operation
corresponding to the tree.

The tree on figure 8 represents an operation which produces the 2-cochain:

$$a_1 \otimes a_2 \rightarrow Q(a_1, a_2, 1) P$$

from a degree 0 cochain $P$ and a degree 3 cochain $Q$. Marked vertices in this figure are
labeled by small circles. The unmarked terminal vertex corresponds to the insertion of the
unit into $Q(a_1, a_2, 1)$. The unmarked 3-valent vertex gives the product of $P$ and $Q(a_1, a_2, 1)$.

Let us denote by $\mathcal{H}_{n \epsilon}^m(n, 1)$ the set of operations producing a chain in $C_{-m, \epsilon}(A)$ from $n$
cochains and a chain in $C_{-m, \epsilon}(A)$.

$\mathcal{H}_{n \epsilon}^m(n, 1)$ is described using forests of rooted trees drawn on the standard cylinder

$$\Sigma = S^1 \times [0, 1]$$

and subject to the following conditions:
1. every tree of the forest has its root vertex on the boundary $S^1 \times \{0\}$;

2. all vertices of the forest lying on the boundary of the cylinder are marked:
   - the vertices lying on the boundary $S^1 \times \{1\}$ are marked by integers $0, 1, \ldots, m_a$ in the counterclockwise order; these vertices are reserved for the components of the chain which enters as an argument,
   - the roots are marked by integers $0, 1, \ldots, m_r$ in the same counterclockwise order; they are reserved for components of the resulting chain,

3. all other marked vertices of the forest lie on the lateral surface $S^1 \times (0, 1)$ of the cylinder and there are exactly $n$ such marked vertices.

On the set of these forests we introduce the finest equivalence relation in which two such forests are equivalent if one of them can be obtained from the other by either:

- isotopy, or
- the contraction of an edge with unmarked ends, or
- removing an unmarked vertex with only one edge originating from it and joining the two edges adjacent to this vertex into one edge.

$\mathcal{H}_{m_r}^{m_a}(n, 1)$ is the set of the corresponding equivalence classes.

As we see from the conditions, all unmarked vertices lie on the lateral surface $S^1 \times (0, 1)$ of the cylinder. As above, these vertices are reserved for operations of $\text{assoc}$. The marked vertices lying on the lateral surface $S^1 \times (0, 1)$ are reserved for cochains.

We allow forests with no marked vertices lying on the lateral surface $S^1 \times (0, 1)$. Such forests represent operations which produce a chain from a chain.

Figure 9 gives an example of an operation of $\mathcal{H}(2, 1)$ which produces the chain

$$(b_0, b_1, b_2, b_3) = (P a_3, Q(a_0, 1, a_1), 1, a_2)$$

from a degree 0 cochain $P$, a degree 3 cochain $Q$ and a degree $-3$ chain $(a_0, a_1, a_2, a_3)$. Marked vertices in this figure are labeled by small circles. The unmarked 3-valent vertex gives the product of $P$ and $a_3$, the two unmarked terminal vertices give units of $A$ and the unmarked 2-valent vertex gives the identity operation on $A$. The vertices lying on the boundary $S^1 \times \{1\}$ are marked by the components of the chain $(a_0, a_1, a_2, a_3)$ and the roots are marked by the components of the chain $(4.3)$.

It is clear how the operad $\mathcal{H}$ acts on the pair $(C^\bullet(A), C_\bullet(A))$. From this action it is also clear how to compose the operations. For example, the composition of operations from $\mathcal{H}(n_1, 1)$ and $\mathcal{H}(n_2, 1)$ corresponds to putting one cylinder on the top of the other matching the roots of the first cylinder with the vertices lying on the upper circle of the second cylinder, and then shrinking the resulting cylinder to the required height.

Recall that the operad $\mathcal{H}$ is colored by degrees of the cochains and degrees of the chains. It is not hard to see that $\mathcal{H}(n, 0)$ is a cosimplicial set with respect to the degree of the resulting cochain and a polysimplicial set with respect to the degrees of the cochains entering as arguments.
Figure 9: An operation produces which the chain (4.3)
Similarly, $\mathcal{H}(n, 1)$ is a cosimplicial set with respect to the degree of the chain entering as an argument and a polysimplicial set with respect to the degrees of the cochains entering as arguments and the degree of the resulting chain.

These poly-simplicial/cosimplicial structure is compatible with the compositions and we get

**Definition 5** The DG operad $\mathbf{KS}$ is the realization of the operad $\mathcal{H}$ in the category of chain complexes.

It follows from the construction that $\mathbf{KS}$ is a 2-colored operad which acts on the pair $(C_{\text{norm}}(A), C_{\text{norm}}(A))$.

It is not hard to see that the operations $\cup$ (2.24), $[,]_G$ (2.25), $I$ (2.26), $L$ (2.27) and $B$ (2.28) come from the action of the operad $\mathbf{KS}$.

**Remark 1.** The operad $\mathbf{KS}$ with its action on $(C_{\text{norm}}(A), C_{\text{norm}}(A))$ was introduced by Kontsevich and Soibelman in \[23\] in the case when $A$ is an $A_\infty$-algebra. Here we recall the construction of $\mathbf{KS}$ in the case when $A$ is simply an associative algebra. It is this assumption on $A$ which allows us to utilize the natural cosimplicial/simplicial structure on $(C_{\text{norm}}(A), C_{\text{norm}}(A))$.

**Remark 2.** If we restrict ourselves to the subspace of operations of $\mathbf{KS}$ which do not involve chains then we get the minimal operad of Kontsevich and Soibelman described in [22].

### 4.2 The operad of little discs on a cylinder

A “topological partner” of $\mathbf{KS}$ is the operad $\text{Cyl}$ of discs on a cylinder [23], [29]. As well as the operad of Kontsevich and Soibelman $\text{Cyl}$ is a 2-colored operad satisfying the property (2.15).

The spaces $\text{Cyl}^\ell(n, 0)$, $n \geq 1$ are the spaces of the little disc operad.

To introduce the space $\text{Cyl}^a(n, 1)$ for $n \geq 1$ we consider cylinders $S^1 \times [a, c]$ for $a, c \in \mathbb{R}$, $a < c$ with the natural flat metric and define the topological space $\widetilde{\text{Cyl}}_n$.

A point of the space $\text{Cyl}_n$ is a cylinder $S^1 \times [a, c]$ together with a configuration of $n \geq 1$ discs on the lateral surface $S^1 \times (a, c)$ and a position of two points $b$ and $t$ lying on the boundaries $S^1 \times a$ and $S^1 \times c$, respectively. The topology on the space $\widetilde{\text{Cyl}}_n$ is defined in the obvious way using the flat metric on the cylinder.

The space $\text{Cyl}_n$ is equipped with a free action of the group $S^1 \times \mathbb{R}$. The subgroup $S^1 \subset S^1 \times \mathbb{R}$ simultaneously rotates all the cylinders and the subgroup $\mathbb{R} \subset S^1 \times \mathbb{R}$ acts by parallel shifts

$$S^1 \times [a, c] \to S^1 \times [a + l, c + l], \quad l \in \mathbb{R}.$$  

The space $\text{Cyl}^a(n, 1)$ for $n \geq 1$ of the operad $\text{Cyl}$ is the quotient

$$\text{Cyl}^a(n, 1) = \widetilde{\text{Cyl}}_n/S^1 \times \mathbb{R}.$$  \hspace{1cm} (4.4)

\[9\] See sections 11.1, 11.2, and 11.3 in [23].
The space $\text{Cyl}^a(0,1)$ is the space of configurations of two (possibly coinciding) points $b$ and $t$ on the circle $S^1$ considered modulo rotations. Although it is obvious that $\text{Cyl}^a(0,1)$ is homeomorphic to the circle $S^1$ we still define $\text{Cyl}^a(0,1)$ using the configuration space in order to better visualize the operations of the operad.

The insertions of the type

$$\text{Cyl}^c(n,0) \times \text{Cyl}^c(m,0) \to \text{Cyl}^c(n + m - 1,0)$$

are defined in the same as for the operad of little squares. The operations of the type

$$\text{Cyl}^c(n,0) \times \text{Cyl}^a(m,1) \to \text{Cyl}^a(n + m - 1,1)$$

are insertions of the configuration of little discs of $\text{Cyl}^c(n,0)$ in a little disc on the lateral surface of a cylinder. Finally the operations of the type

$$\text{Cyl}^a(n,1) \times \text{Cyl}^a(m,1) \to \text{Cyl}^a(n + m,1)$$

correspond to putting the first cylinder under the second one while the second cylinder is rotated in such a way that the point $b$ of the second cylinder coincides with the point $t$ of the first cylinder. The composition involving degenerate configurations of $\text{Cyl}^a(0,1)$ are defined in the obvious way.

To describe the operad of homology groups of $\text{Cyl}$ we will need some results about the configuration spaces of distinct points on the punctured plane $\mathbb{R}^2 \setminus \{0\}$.

Let us denote by $\text{Conf}_n(\mathbb{R}^2 \setminus \{0\})$ the configuration space of $n$ distinct points on the punctured plane $\mathbb{R}^2 \setminus \{0\}$ and consider the following projections

$$p_k : \text{Conf}_n(\mathbb{R}^2 \setminus \{0\}) \to \mathbb{R}^2 \setminus \{0\},$$

$$p_k(x_1, \ldots, x_n) = x_k. \quad (4.5)$$

Due to E. Fadell and L. Neuwirth [14] we have

**Theorem 1 (Theorem 1, [14])** For every $k = 1, 2, \ldots, n$ the map $p_k$ is a locally trivial fibration.

Using the ideas of E. Fadell and L. Neuwirth [14] we show that

**Proposition 6** The map

$$p : \text{Conf}_n(\mathbb{R}^2 \setminus \{0\}) \to \text{Conf}_{n-1}(\mathbb{R}^2 \setminus \{0\}),$$

$$p(x_1, x_2, \ldots, x_n) = (x_2, \ldots, x_n). \quad (4.6)$$

is a locally trivial fibration. Furthermore, the fiber $F_n$ of $p$ is

$$F_n = \mathbb{R}^2 \setminus \{0, q_2, \ldots, q_n\}, \quad (4.7)$$

where $q_2, \ldots, q_n$ are $n - 1$ distinct points of the punctured plane $\mathbb{R}^2 \setminus \{0\}$. 

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Proof. For the open unit disc $D_1$ on $\mathbb{R}^2$ centered at the origin there exists a continuous map

$$\theta : D_1 \times \bar{D}_1 \to \bar{D}_1$$

satisfying the following properties:
- for all $x \in D_1$ the map $\theta(x, \cdot) : \bar{D}_1 \to \bar{D}_1$ is a homeomorphism having $\partial \bar{D}_1$ fixed.
- for all $x \in D_1$ we have $\theta(x, x) = 0$.

For distinct points $q_2, \ldots, q_n$ on the punctured plane $\mathbb{R}^2 \setminus \{0\}$ we choose open discs

$$D_{q_2}, D_{q_3}, \ldots, D_{q_n},$$

which are centered at $q_2, q_3, \ldots, q_n$, respectively. Each disc $\bar{D}_{q_i}$ avoids the origin $0$ and for each $i \neq j$

$$\bar{D}_{q_i} \cap \bar{D}_{q_j} = \emptyset.$$

Let us denote respectively by $r_2, \ldots, r_n$ the radii of the discs (4.9) and let $U_b$ be the following neighborhood of $\text{Conf}_{n-1}(\mathbb{R}^2 \setminus \{0\})$:

$$U_b = \{(x_2, \ldots, x_n) \mid x_2 \in D_{q_2}, \ x_3 \in D_{q_3}, \ldots, x_n \in D_{q_n}\}.$$ (4.10)

The desired homeomorphism $h$ from $p^{-1}(U_b) \to F_n \times U_b$ is given by the formula:

$$h(x_1, x_2, \ldots, x_n) = \begin{cases} (x_1, x_2, \ldots, x_n), & \text{if } x_1 \notin \bigcup_{j=2}^n D_{q_j}, \\ (q_j + r_j \theta\left(\frac{x_j - q_j}{r_j}, \frac{x_1 - q_j}{r_j}\right), x_2, \ldots, x_n), & \text{if } x_1 \in \bar{D}_{q_j}, \end{cases}$$ (4.11)

where $r_j$ is the radius of the $j$-th disc $D_{q_j}$.

Since the open subsets of the form $U_b$ (4.10) cover $\text{Conf}_{n-1}(\mathbb{R}^2 \setminus \{0\})$ the map $p$ (4.6) is indeed a locally trivial fibration. $\square$

Fiber $F_n$ of $p$ (4.6) is homotopy equivalent to the wedge sum $\vee^n S^1$ of $n$ circles. Hence, the homology groups of $F_n$ (4.7) are

$$H_*(F_n, \mathbb{K}) = \begin{cases} \mathbb{K}, & \text{if } \bullet = 0, \\ \mathbb{K}^n, & \text{if } \bullet = 1, \\ 0, & \text{otherwise}. \end{cases}$$ (4.12)

Let us show that

**Proposition 7** The fundamental group $\pi_1(\text{Conf}_{n-1}(\mathbb{R}^2 \setminus \{0\}))$ acts trivially on the homology groups $H_*(F_n, \mathbb{K})$ of $F_n$.

**Proof.** It is obvious that we only need to consider the action of $\pi_1(\text{Conf}_{n-1}(\mathbb{R}^2 \setminus \{0\}))$ on $H_1(F_n, \mathbb{K})$.

To get the cycles representing elements of a basis for $H_1(F_n, \mathbb{K})$ we choose closed discs

$$D_0, D_{q_2}, D_{q_3}, \ldots, D_{q_n},$$

which are centered at $0, q_2, q_3, \ldots, q_n$, respectively. The discs (4.13) are chosen in such a way that their closures

$$\bar{D}_0, \bar{D}_{q_2}, \bar{D}_{q_3}, \ldots, \bar{D}_{q_n}$$
are pairwise disjoint.

The boundaries of the discs (4.13) are cycles representing the elements of a basis for \( H_1(F_n, \mathbb{K}) \).

Let us identify \( \mathbb{R}^2 \) with the complex plane \( \mathbb{C} \) and consider the following loop in \( \text{Conf}_{n-1}(\mathbb{R}^2 \setminus \{0\}) \)

\[
f(t) = (e^{2\pi it}q_2, e^{2\pi it}q_3, \ldots, e^{2\pi it}q_n) : [0, 1] \to \text{Conf}_{n-1}(\mathbb{R}^2 \setminus \{0\}),
\]

(4.14)

where \((q_2, q_3, \ldots, q_n)\) is a fixed collection of the distinct points of \( \mathbb{R}^2 \setminus \{0\} \).

The loop (4.14) lifts to the following loop in \( \text{Conf}_n(\mathbb{R}^2 \setminus \{0\}) \)

\[
\tilde{f}(t) = (e^{2\pi it}x_1, e^{2\pi it}q_2, e^{2\pi it}q_3, \ldots, e^{2\pi it}q_n) : [0, 1] \to \text{Conf}_n(\mathbb{R}^2 \setminus \{0\}).
\]

(4.15)

As we go around the loop (4.13) the point \( x_1 \) of the fiber \( F_n \) returns to its original position. Thus the element \([f] \in \pi_1(\text{Conf}_{n-1}(\mathbb{R}^2 \setminus \{0\}))\) represented by the loop \( f \) (4.14) acts trivially on \( H_\bullet(F_n, \mathbb{K}) \).

Let now \( g \) be an arbitrary loop in \( \text{Conf}_{n-1}(\mathbb{R}^2 \setminus \{0\}) \).

To find the action of the homotopy class \([g]\) on \( H_\bullet(F_n, \mathbb{K}) \) we need to lift the map

\[
\gamma(y, t) = g(t) : F_n \times [0, 1] \to \text{Conf}_{n-1}(\mathbb{R}^2 \setminus \{0\})
\]

(4.16)

to a map

\[
\tilde{\gamma}(y, t) : F_n \times [0, 1] \to \text{Conf}_n(\mathbb{R}^2 \setminus \{0\})
\]

(4.17)

which makes the diagram

\[
\begin{array}{ccc}
F_n \times \{0\} & \hookrightarrow & \text{Conf}_n(\mathbb{R}^2 \setminus \{0\}) \\
\downarrow & \searrow & \downarrow \gamma \\
F_n \times [0, 1] & \to & \text{Conf}_{n-1}(\mathbb{R}^2 \setminus \{0\})
\end{array}
\]

(4.18)

commutative.

To construct the lift \( \tilde{\gamma} \) we divide the segment \([0, 1]\) into small enough subsegments \([t_i, t_{i+1}]\) satisfying the property

\[
g([t_i, t_{i+1}]) \subset V,
\]

(4.19)

where \( V \) is an open subset of \( \text{Conf}_{n-1}(\mathbb{R}^2 \setminus \{0\}) \) of the form (4.10). Then each individual lift

\[
\tilde{\gamma}\big|_{F_n \times [t_i, t_{i+1}]} : F_n \times [t_i, t_{i+1}] \to \text{Conf}_n(\mathbb{R}^2 \setminus \{0\})
\]

(4.20)

can be constructed using the trivialization (4.11).

With this construction in mind we consider the compositions \( p_k \circ g \) of \( g \) with the projections \( p_k \) (4.5) for \( k \in \{2, \ldots, n\} \). Since the image \( p_k \circ g([0, 1]) \) of the segment \([0, 1]\) is a compact subset in \( \mathbb{R}^2 \setminus \{0\} \) we may choose the disc \( D_0 \) in such a way that the closure \( \overline{D_0} \) avoids the points of the images \( p_k \circ g([0, 1]) \) for all \( k \in \{2, \ldots, n\} \). Therefore the lift \( \tilde{\gamma} \) (4.17) can be chosen in such a way that

\[
\tilde{\gamma}(y, t) = y, \quad \forall \ y \in \overline{D_0} \subset F_n.
\]

Hence the action of the homotopy class \([g]\) on the homology class represented by the boundary of the disc \( D_0 \) is trivial.
Let us examine the loop \( p_k \circ g \) in \( \mathbb{R}^2 \setminus \{0\} \) more closely. Since the fundamental group of \( \mathbb{R}^2 \setminus \{0\} \) is generated by the homotopy class of the loop

\[
 l(t) = e^{2\pi i t} q_k : [0, 1] \to \mathbb{R}^2 \setminus \{0\}
\]

around the origin there exists an integer \( N \in \mathbb{Z} \) such that the loop \( p_k \circ (g \ast f^N) \) is null-homotopic. Here \( f \) is the loop \((1.14)\) in \(\text{Conf}_{n-1}(\mathbb{R}^2 \setminus \{0\})\).

But the class of \( f \) acts trivially on the homology of the fiber \( F_n \). Therefore, without loss of generality, we may assume that the composition \( p_k \circ g \) is null-homotopic.

Thus, due to Theorem 1 of Fadell and Neuwirth we may further assume that \( p_k \circ g \) is a constant map

\[
 p_k \circ g(t) \equiv q_k. \tag{4.21}
\]

In other words, the \( k \)-th point \( x_k \) does not move as we go along the loop \( g \).

Since for each \( j \neq k \) the image \( p_j \circ g([0, 1]) \) of the segment \([0, 1]\) is a compact subset in \( \mathbb{R}^2 \setminus \{0\} \) we may choose the disc \( D_{q_k} \) in such a way that the closure \( \overline{D}_{q_k} \) avoids the points of the images \( p_j \circ g([0, 1]) \) for all \( j \neq k \).

Thus, using the partition of the segment \([0, 1]\) satisfying the property \((4.19)\) and constructing the lift \((4.17)\) using the trivializations of the form \((4.11)\) we see that the lift \( \tilde{\gamma} \) can be chosen in such a way that

\[
 \tilde{\gamma}(y, t) = y, \quad \forall \quad y \in \overline{D}_{q_k} \subset F_n.
\]

Therefore the action of the homotopy class \([g]\) on the homology class represented by the boundary of the disc \( D_{q_k} \) is trivial.

The proposition is proved. \( \square \)

Generalizing the result of F. R. Cohen \([9]\) we get the following

**Theorem 2** The homology operad \( H_{-\bullet}(\text{Cyl}, \mathbb{K}) \) of \( \text{Cyl} \) with the reversed grading is the operad \( \text{calc} \) of calculi.

**Proof.** Since the operad \( \text{Cyl} \) has two colors algebras over \( H_{-\bullet}(\text{Cyl}, \mathbb{K}) \) are pairs of graded vector spaces \((\mathcal{V}, \mathcal{W})\).

The components \( \text{Cyl}^\ell(n, 0) \) form the operad of little discs. Thus, due to Theorem 1.2 \(\text{in} \ [9] \) \( \mathcal{V} \) is a Gerstenhaber algebra.

The space \( \text{Cyl}^\ell(0, 0) \) is empty, the space \( \text{Cyl}^\ell(1, 0) \) is a point and the space \( \text{Cyl}^\ell(n, 0) \) for \( n > 1 \) is homotopy equivalent to the space \( \text{Conf}_n(\mathbb{R}^2) \) of configurations of \( n \) distinct points on \( \mathbb{R}^2 \). The map

\[
 \text{Cyl}^\ell(n, 0) \xrightarrow{\sim} \text{Conf}_n(\mathbb{R}^2) \tag{4.22}
\]

which establishes the equivalence associates with a configuration of disjoint discs the configuration of their centers.

The space \( \text{Conf}_2(\mathbb{R}^2) \) is, in turn, homotopy equivalent to \( S^1 \). Thus the generator of \( H_0(S^1) \) represents the commutative product on \( \mathcal{V} \) and the generator of \( H_1(S^1) \) represents the bracket on \( \mathcal{V} \). This bracket has degree \(-1\) because we use the reversed grading on the homology groups.

\(^{10}\)We only need the operations which survive in characteristic zero.
The operations without inputs of color \( c \) correspond to homology classes of the space \( \text{Cyl}^a(0, 1) \). Since this space is homeomorphic to the circle \( S^1 \) we have

\[
H_\bullet(\text{Cyl}^a(0, 1), \mathbb{K}) = \begin{cases} 
\mathbb{K}, & \text{if } \bullet = 0, 1, \\
0, & \text{otherwise}.
\end{cases}
\]

The generator of \( H_0(\text{Cyl}^a(0, 1)) \) corresponds to the identity transformation of \( \mathcal{W} \) and the generator \( \delta \) of \( H_1(\text{Cyl}^a(0, 1)) \) corresponds to a unary operation on \( \mathcal{W} \). The operation \( \delta \) has degree \(-1\) because we use the reversed grading on the homology groups.

The identity

\[
\delta^2 = 0
\]

follows immediately from the fact that

\[
H_2(\text{Cyl}^a(0, 1), \mathbb{K}) = 0.
\]

Let us construct a homotopy equivalence between the space \( \text{Cyl}^a(n, 1) \) for \( n \geq 1 \) and the space \( \text{Conf}_n(\mathbb{R}^2 \setminus 0) \times S^1 \), where \( \text{Conf}_n(\mathbb{R}^2 \setminus 0) \) is the configuration space of \( n \) distinct points on the punctured plane \( \mathbb{R}^2 \setminus 0 \).

First, we kill the rotation symmetry by fixing the position of the point \( t \) on the upper boundary \( S^1 \times \{ c \} \) of the cylinder \( S^1 \times [a, c] \). Second, we kill translation symmetry by setting \( c = 0 \).

Next we assign to each configuration of discs on the lateral surface \( S^1 \times (a, 0) \) the configuration of centers of the discs. In this way we get a homotopy equivalence between the space \( \text{Cyl}^a(n, 1) \) and the space

\[
\text{Conf}_n(S^1 \times (a, 0)) \times S^1,
\]

where the points on the factor \( S^1 \) correspond to positions of the point \( b \) on \( S^1 \times \{ a \} \).

Finally, using the map

\[
\chi : S^1 \times (a, 0) \to \mathbb{R}^2 \setminus 0, \\
\chi(\varphi, y) = \left( \frac{y}{a-y} \cos(\varphi), \frac{y}{a-y} \sin(\varphi) \right)
\]

we get the desired homotopy equivalence

\[
\text{Cyl}^a(n, 1) \simeq \text{Conf}_n(\mathbb{R}^2 \setminus 0) \times S^1.
\]

Let us consider the homology groups of \( \text{Cyl}^a(1, 1) \) in more details. The space \( \text{Cyl}^a(1, 1) \) is homotopy equivalent to

\[
\mathbb{R}^2 \setminus 0 \times S^1
\]

and the latter space is, in turn, homotopy equivalent to \( S^1 \times S^1 \). Thus

\[
\text{Cyl}^a(1, 1) \simeq S^1 \times S^1.
\]
Therefore
\[ H_{\ast}(\text{Cyl}^a(1,1), \mathbb{K}) = \mathbb{K} \oplus s^{-1} \mathbb{K}^2 \oplus s^{-2} \mathbb{K}. \]

To identify the cycles representing the homology classes we parameterize the circle \( S^1 \) by the angle variable \( \varphi \in [0, 2\pi] \) and the torus \( S^1 \times S^1 \) by the pair of angle variables \( \varphi_1, \varphi_2 \in [0, 2\pi] \).

The zeroth homology space \( H_0(\text{Cyl}^a(1,1), \mathbb{K}) \) is one-dimensional and its generator corresponds to the contraction \( i \) of elements of \( \mathcal{V} \) with elements of \( \mathcal{W} \). The second homology space \( H_2(\text{Cyl}^a(1,1), \mathbb{K}) \) is also one-dimensional. Its generator corresponds to the composition \( \delta i \delta \).

The cycle
\[ \varphi \rightarrow (0, \varphi) : S^1 \hookrightarrow S^1 \times S^1 \quad (4.27) \]
represents the homology class corresponding to the composition \( i \delta \). In order to get this cycle in \( \text{Cyl}^a(1,1) \) we need to revolve the point \( b \) on the lower boundary about the vertical axis as it is shown on Figure 10.

The composition \( \delta i \) is, in turn, represented by the diagonal
\[ \varphi \rightarrow (\varphi, \varphi) : S^1 \rightarrow S^1 \times S^1 \quad (4.28) \]
of the torus. To get this cycle we need to revolve simultaneously the disc and the point \( b \) about the vertical axis as it is shown on Figure 11.

The homology classes \( \delta i \) and \( i \delta \) form a basis of \( H_1(\text{Cyl}^a(1,1), \mathbb{K}) \).

We would like to remark that the homology class represented by the cycle
\[ \varphi \rightarrow (\varphi, 0) : S^1 \hookrightarrow S^1 \times S^1 \quad (4.29) \]
equals to the combination
\[ \delta i - i \delta. \]
Indeed it is easy to see that the cycles (4.27), (4.28), and (4.29) form the boundary of the following 2-simplex in $S^1 \times S^1$

$$\{(\varphi_1, \varphi_2) \mid \varphi_1 \leq \varphi_2 \} \subset S^1 \times S^1.$$ 

Thus the cycle (4.29) represents the homology class corresponding to the Lie derivative $l$. To get this cycle in $\text{Cyl}^a(1, 1)$ we need to revolve the disc about the vertical axis as it is shown on Figure 12.

In general, for $n \geq 1$ the homology of the space $\text{Cyl}^a(n, 1)$ can be computed with the help of the homological version of Lemma 6.2 from [9]. Due to this lemma we have

$$H_\bullet(\text{Conf}_n(\mathbb{R}^2 \setminus 0), \mathbb{K}) = \bigotimes_{j=1}^n H_\bullet(\vee^j S^1, \mathbb{K}).$$  \hspace{1cm} (4.30)

Using the homotopy equivalence (4.25) and the Künneth formula, we deduce that

$$H_\bullet(\text{Cyl}^a(n, 1), \mathbb{K}) = \bigotimes_{j=1}^n H_\bullet(\vee^j S^1, \mathbb{K}) \otimes H_\bullet(S^1, \mathbb{K}).$$ \hspace{1cm} (4.31)

Let us show that the operad $H_{-\bullet}(\text{Cyl}, \mathbb{K})$ is generated by operations of $H_{-\bullet}(\text{Cyl}^c(2, 0), \mathbb{K})$, $H_{-\bullet}(\text{Cyl}^a(0, 1))$, and $H_{-\bullet}(\text{Cyl}^a(1, 1), \mathbb{K})$.

Since the case of the operad of little discs was already considered by F. Cohen [9] we should only consider the operations of $H_{-\bullet}(\text{Cyl}^a(n, 1), \mathbb{K})$.

Due to homotopy equivalence (4.25), the homology classes of $\text{Cyl}^a(n, 1)$ are of the forms

$$\alpha \otimes 1,$$ \hspace{1cm} (4.32)
Figure 12: How to get the cycle in Cyl^a(1, 1) representing the operation \( l \)

and

\[
\alpha \otimes \phi , \quad (4.33)
\]

where \( \alpha \in H_{-\bullet}(\text{Conf}_n(\mathbb{R}^2 \setminus 0), K) \), 1 is the generator of \( H_0(S^1, K) \) and \( \phi \) is the generator of \( H_0(S^1, K) \).

It is obvious that homology classes of the form (4.33) are obtained by composing the homology classes of the form (4.32) with the generator \( \delta \) of \( H_1(\text{Cyl}^a(0, 1), K) \).

To analyze the homology classes (4.32) we consider the Serre spectral sequence corresponding to the fibration (4.6). Due to Proposition 7 the \( E^2 \) term of the sequence is

\[
E^2_{p,q} = H_p(\text{Conf}_{n-1}(\mathbb{R}^2 \setminus 0), K) \otimes H_q(F_n, K), \quad (4.34)
\]

where \( F_n \) is the fiber (4.7) of \( p \).

Since \( F_n \) is homotopy equivalent to the wedge \( \vee^n S^1 \) of \( n \) circles equation (4.30) implies that the vector spaces

\[
\bigoplus_p E^2_{p,\bullet-p}
\]

and \( H_\bullet(\text{Conf}_n(\mathbb{R}^2 \setminus 0), K) \) have the same dimension. Thus, using the fact that spectral sequence corresponding to the fibration (4.6) converges to \( H_\bullet(\text{Conf}_n(\mathbb{R}^2 \setminus 0), K) \), we deduce that this spectral sequence degenerates at \( E_2 \) and

\[
H_\bullet(\text{Conf}_n(\mathbb{R}^2 \setminus 0), K) = \bigoplus_{p+q=\bullet} H_p(\text{Conf}_{n-1}(\mathbb{R}^2 \setminus 0), K) \otimes H_q(F_n, K) \quad (4.35)
\]

Using equation (4.12) we reduce this expression further to

\[
H_\bullet(\text{Conf}_n(\mathbb{R}^2 \setminus 0), K) =
\]
\[ H_\bullet(\text{Conf}_{n-1}(\mathbb{R}^2 \setminus 0), \mathbb{K}) \otimes H_0(F_n, \mathbb{K}) \oplus H_{\bullet-1}(\text{Conf}_{n-1}(\mathbb{R}^2 \setminus 0), \mathbb{K}) \otimes H_1(F_n, \mathbb{K}). \tag{4.36} \]

Let \( v_1, v_2, \ldots, v_n \in V \) and \( w \in W \) be the arguments of an operation corresponding to a homology class in (4.36). It is not hard to see that the generator of \( H_0(F_n, \mathbb{K}) = \mathbb{K} \) corresponds to the contraction \( i \) with \( v_1 \) and the generators of \( H_1(F_n, \mathbb{K}) = \mathbb{K}^n \) correspond to the brackets \([v_1, v_j]\) for \( j \in \{2, 3, \ldots, n\} \) and the Lie derivative \( l_{v_1} \) along \( v_1 \).

Thus the homology classes of \( H_{-\bullet}(\text{Cyl}^a(n, 1), \mathbb{K}) \) are all produced by the operadic compositions of the classes in \( H_{-\bullet}(\text{Cyl}^a(n - 1, 1), \mathbb{K}) \), \( H_{-\bullet}(\text{Cyl}^a(1, 1), \mathbb{K}) \), and \( H_{-\bullet}(\text{Cyl}^c(2, 0), \mathbb{K}) \).

This inductive argument allows us to conclude that the operad \( H_{-\bullet}(\text{Cyl}, \mathbb{K}) \) is generated by the classes

\[ i \in H_0(\text{Cyl}^a(1, 1), \mathbb{K}), \quad \delta \in H_0(\text{Cyl}^0(0, 1), \mathbb{K}), \quad \wedge \in H_0(\text{Cyl}^c(2, 0), \mathbb{K}), \quad \[, \] \in H_1(\text{Cyl}^c(2, 0), \mathbb{K}). \tag{4.37} \]

Using equation (2.23) and the fact \([21]\) that \( \dim \text{Lie}(n) = (n - 1)! \) it is not hard to show that \( \text{calc}^a(n, 1) \cong \text{calc}^a(n - 1, 1) \otimes (\mathbb{K} \oplus s^{-1}\mathbb{K}^n) \)

as graded vector spaces.

On the other hand, equation (4.31) gives us the same isomorphism

\[ H_{-\bullet}(\text{Cyl}^a(n, 1), \mathbb{K}) \cong H_{-\bullet}(\text{Cyl}^a(1, 1), \mathbb{K}) \otimes (\mathbb{K} \oplus s^{-1}\mathbb{K}^n) \]

of graded vector spaces for \( H_{-\bullet}(\text{Cyl}, \mathbb{K}) \).

Therefore, the dimensions of graded components of \( H_{-\bullet}(\text{Cyl}^a(n, 1), \mathbb{K}) \) and \( \text{calc}^a(n, 1) \) are equal.

Thus, in order the complete the proof of the theorem, we need to show the operations (4.37) satisfy the identities of calculus algebra (see Definition 3).

The identities of the Gerstenhaber algebra were already checked in \([9]\). The identity (2.19) was checked above. On Figure 13 we show how to check the identity

\[ [i, l] = i[l, ] \tag{4.38} \]

The remaining identities can be checked in the similar way.

Theorem 2 is proved. \( \Box \)

4.3 Required results from \([23]\)

We will need

**Theorem 3 (M. Kontsevich, Y. Soibelman, \([23]\))** The operad \( \text{KS} \) is quasi-isomorphic to the operad of singular chains of the topological operad \( \text{Cyl} \). The homology operad \( H_{-\bullet}(\text{KS}, \mathbb{K}) \) is generated by the classes of the operations \( \cup (2.24) \), \( [, ]_G (2.25) \), \( I (2.26) \), \( L (2.27) \) and \( B (2.28) \).

Furthermore, (See Proposition 11.3.3 on page 50 in \([23]\))

\[ (^{11}\text{This picture is very reminiscent of the consideration of Hochschild-Serre spectral sequence in the proof of Proposition 4.1 in } [28].) \]

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Figure 13: How to check identity (4.38)
Proposition 8 The operad of singular chains of the topological operad $Cyl$ is formal.

Combining these two statements with Theorem [2] we get the following corollary.

Corollary 1 The pair $(C_{\text{norm}}(A), C_{\text{norm}}^*(A))$ is a homotopy calculus algebra. The operations of this algebra structure are expressed in terms of operations of $\mathcal{H}$. The induced calculus structure on the pair $(HH^*(A), HH_*(A))$ coincides with the one in [11].

4.4 A useful property of the operad KS

For the 2-colored operad $KS$ of chain complexes we have the following

Proposition 9 The elements of $KS^c(k, 0)$ have degrees

$$\deg \geq 1 - k$$

and the elements of $KS^a(k, 1)$ have degrees

$$\deg \geq -1 - k.$$ (4.40)

Proof. Let us start with $KS^c(k, 0)$ for $k = 1$. Operations of $KS^c(1, 0)$ produce a Hochschild cochain from a Hochschild cochain. In order to prove that these operations have nonnegative degrees we need to show that the operations from $\mathcal{H}(1, 0)$ which lower the number the number of arguments of the cochain do not contribute to the realization of $\mathcal{H}$.

Let $T$ be a tree representing an operation from $\mathcal{H}(1, 0)$ and let $v_a$ be the vertex adjacent to the root. If $v_a$ is marked then it is marked by the only cochain $P$ which enters as an argument. All other marked vertices are necessarily terminal and they are reserved for the arguments of the cochain produced by the operation.

In order to lower the number of arguments, we need to insert the unit into the cochain $P$. The insertion of the unit is a degeneracy of the simplicial structure on $\mathcal{H}(1, 0)$. Hence all operations from $\mathcal{H}(1, 0)$ which lower the degree of the cochain do not contribute to the realization.

If $v_a$ is unmarked then, starting with the marked vertex $P$ reserved for the cochain $P$, we can form the proper maximal subtree with $v_P$ being the vertex adjacent to the root. In order to contribute to the realization the operation corresponding to this subtree has to have a nonnegative degree. Hence, so does the operation corresponding to the whole tree.

We proved (4.39) for $k = 1$.

Let us take it as a base of the induction and assume that (4.39) is proved for all $m < k$.

We consider a tree $T$ which represents an operation from $\mathcal{H}(k, 0)$ and denote the vertex adjacent to the root of $T$ by $v_a$. Let us consider the case when the vertex $v_a$ is marked. Say, $v_a$ is reserved for a cochain $P_1$ of degree $q_1$.

Then the tree $T$ has exactly $q_1$ maximal proper subtrees whose root vertex is $v_a$. We denote these subtrees by $T_1, T_2, \ldots, T_{q_1}$. The number $q_1$ splits into the sum

$$q_1 = p_n + p_y,$$ (4.41)

where $p_n$ is the number of the subtrees with no vertices reserved for cochains and $p_y$ is the number of the subtrees in which at least one vertex is reserved for a cochain.
Let $P$ denote the cochain produced by the operation in question and let $r$ by the number of arguments of this cochain. We will find an estimate for $r$ using the inductive assumption.

Every subtree with no vertices reserved for cochains has to give at least one argument for the cochain $P$. Otherwise, we have to insert the unit as an argument of the cochain $P_1$. In this case the operation in question is a obtained from another operation by degeneracy. Therefore this operation would not contribute to the realization of $\mathcal{H}$.

If $T_j$ is a subtree with exactly $k_j$ vertices reserved for cochains then, obviously, $k_j < k$. Hence, applying the assumption of the induction, we get that the number of arguments of $P$ coming from $T_j$ is greater or equal

$$(1 - k_j) + q_j,$$

where $q_j$ is the total degree of all cochains entering as arguments of the operation corresponding to the subtree $T_j$.

Thus the number $r$ of the arguments of the cochain $P$ produced by the operation in question can be estimated by

$$r \geq p_n + \sum_{j=1}^{p_y} (1 - k_j + q_j).$$

(4.42)

This inequality can be rewritten as

$$r - (p_n + p_y + \sum_{j=1}^{p_y} q_j) \geq -\sum_{j=1}^{p_y} k_j.$$

Due to equation (4.41) the sum

$$p_n + p_y + \sum_{j=1}^{p_y} q_j$$

is the total degree of all cochains entering as arguments of the operation. Furthermore, since the vertex $v_a$ is reserved for one of the cochains

$$\sum_{j=1}^{p_y} k_j = k - 1$$

and the desired inequality (4.39) is proved in this case.

Let us now consider the case when the vertex $v_a$ is unmarked.

The valency of the vertex $v_a$ has to be at least 2. Otherwise the operation will have the empty set of arguments. If the valency of this vertex is 2 then we remove it using the equivalence transformation.

Thus, without loss of generality, we may assume that the vertex $v_a$ has at least 2 incoming edges. Let us denote by $s$ the number of these incoming edges and let $T_1, \ldots, T_s$ be the maximal proper subtrees of $T$ whose root vertex is $v_a$.

If the vertices reserved for cochains belong to only one of these subtrees $T_i$ then excising the subtrees $T_j$ for $j \neq i$ we get another operation whose degree is less or equal the degree of the original operation. Since in the new tree the unmarked vertex $v_a$ has the valency 2 we may remove this vertex by the corresponding equivalence transformation.
If in this modified tree the vertex adjacent to the root is marked then we deduce the desired inequality to the case considered above.

Otherwise, we should only consider the case when the vertex \( v_a \) is unmarked, its valency is at least 3 and each maximal proper subtree \( T_j \) of \( T \) with root vertex \( v_a \) has at least one vertex reserved for a cochain.

Let \( s \) be the number of the maximal proper subtrees of \( T \) whose root vertex is \( v_a \). Since the number of these subtrees is greater or equal than 2 therefore every subtree \( T_j \) represents an operation with the number of arguments \( k_j < k \).

Let \( \deg(T_j) \) be the degree of the operation corresponding to the \( j \)-th subtree \( T_j \). Applying the assumption of the induction we get the inequality

\[
\deg(T_j) \geq 1 - k_j .
\] (4.43)

It is clear that the degree \( \deg(T) \) of the operation corresponding to the tree \( T \) is the sum of degrees of the operations corresponding to the subtrees \( T_1, \ldots, T_s \). Therefore

\[
\deg(T) \geq \sum_{i=1}^{s} (1 - k_i) .
\]

On the other hand \( \sum_{i=1}^{s} k_i = k \). Therefore,

\[
\deg(T) \geq s - k
\]

and inequality (4.39) follows from the fact that \( s \geq 2 \).

To prove the second inequality (4.40) we denote by \( \mathcal{H}_{m_r}^{m_a}(k, 1) \) the set of operations producing a chain

\[
(b_0, b_1, \ldots, b_{m_r})
\] (4.44)
in \( C_{m_r}(A) \) from \( k \) cochains and a chain

\[
(c_0, c_1, \ldots, c_{m_a})
\] (4.45)
in \( C_{m_a}(A) \).

Let \( F \) be a forest on the cylinder (4.2) representing an operation from \( \mathcal{H}_{m_r}^{m_a}(k, 1) \) which contribute to the realization of \( \mathcal{H}(k, 1) \). Our purpose is to prove the inequality

\[
-m_r \geq -m_a + q - 1 - k ,
\] (4.46)

where \( q \) is the total degree of all \( k \) cochains of the operation.

This inequality is equivalent to

\[
m_a \geq m_r + q - 1 - k .
\] (4.47)

By construction the forest \( F \) has exactly \( m_r \) trees. Let us denote these trees by \( S_1, \ldots, S_{m_r} \), \( T_1, \ldots, T_{m_y} \) where the trees \( S_1, \ldots, S_{m_r} \) have no vertices reserved for cochains and each tree \( T_i \) has at least one vertex reserved for a cochain. Obviously,

\[
m_r = m_r^n + m_r^y .
\] (4.48)
The roots of the trees $S_1, \ldots , S_{m^y}$, $T_1, \ldots , T_{m^y}$ are marked by components of the chain (4.44). If the root of the tree $S_i$ is marked by the component $b_j$ of the for $j \neq 0$ then $S_i$ has to have at least one terminal vertex marked by a component of the chain (4.45). Otherwise we have to insert the unit as the $j$-th component of (4.44) for $j \neq 0$. In this case the operation in question is a composition of the another operation and a degeneracy. Hence, this operation would not contribute to the realization of $\mathcal{H}(k,1)$.

Let us denote by $m_i$ the number of the terminal vertices of the tree $T_i$ marked by components of the chain (4.45). If the tree $T_i$ has exactly $k_i$ vertices reserved for the cochains then $T_i$ represents an operation from $\mathcal{H}(k_i,0)$. Furthermore, if the operation corresponding to the forest $F$ contributes to the realization then so does the operation corresponding to the tree $T_i$. Hence, the number $m_i$ can be estimated using the inequality (4.39)

$$m_i \geq q_i + 1 - k_i,$$

where $q_i$ is the total degree of all cochains of the operation corresponding to the tree $T_i$.

Thus we get the following inequality for $m_a$

$$m_a \geq (m^n_r - 1) + \sum_{i=1}^{m^y_r} (q_i + 1 - k_i), \quad (4.49)$$

where the first term $(m^n_r - 1)$ in the right hand side comes from estimate of the number of the marked terminal vertices of the trees $S_1, \ldots , S_{m^y}$.

Inequality (4.49) can be rewritten as

$$m_a \geq m^n_r + m^y_r + q - 1 - k, \quad (4.50)$$

where $q$ is the total degree of all $k$ cochains of the operation in question.

Due to equation (4.48) inequality (4.50) coincides exactly with the desired inequality (4.47).

The proposition is proved. □

5 The homotopy calculus on the pair $(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A))$.

Since the operad $\Lambda\text{Lie}_+^\delta$ is a suboperad of $\text{calc}$, Corollary 1 implies that the pair

$$(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A))$$

carries a natural $\text{Ho}(\Lambda\text{Lie}_+^\delta)$-algebra structure. In this section we show that the homotopy calculus on $(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A))$ can be modified in such a way that its $\text{Ho}(\Lambda\text{Lie}_+^\delta)$-algebra part becomes the $\Lambda\text{Lie}_+^\delta$-algebra structure given by the operations $[ , ]_G$ (2.25), $L$ (2.27), and $B$ (2.28).

Due to Proposition 2 a homotopy calculus structure on the pair $(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A))$ is a Maurer-Cartan element

$$Q \in \text{Coder}'(\mathbb{F}_B(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A))). \quad (5.1)$$

In other words, $Q$ is a degree 1 coderivation of the coalgebra $\mathbb{F}_B(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A))$ satisfying the condition

$$Q|_{C^\bullet_{\text{norm}}(A) \oplus C^\bullet_{\text{norm}}(A)} = 0,$$
and the equation
$$[\partial^{\text{Bar}} + \partial^{\text{Hoch}}, Q] + Q \circ Q = 0,$$
where $C^\bullet_{\text{norm}}(A) \oplus C^\bullet_{\text{norm}}(A)$ is considered as subspace of $\mathbb{F}_B(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A))$ via coaugmentation and $\partial^{\text{Hoch}}$ is the differential coming from the Hochschild (co)boundary operator on $(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A))$.

It is convenient to reserve a notation for the Lie algebra Coder’$(\mathbb{F}_B(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A)))$
$$\mathbb{L} = \text{Coder’}(\mathbb{F}_B(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A))).$$

Proposition 1 implies that the coderivation $Q$ is uniquely determined by its composition with the corestriction $\rho : \mathbb{F}_B(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A)) \to C^\bullet_{\text{norm}}(A) \oplus C^\bullet_{\text{norm}}(A)$
$$q = \rho \circ Q : \mathbb{F}_B(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A)) \to C^\bullet_{\text{norm}}(A) \oplus C^\bullet_{\text{norm}}(A),$$
while equation (5.2) is equivalent to
$$[\partial^{\text{Hoch}}, q] + q \circ \partial^{\text{Bar}} + q \circ Q = 0. \quad (5.5)$$

The coalgebra $\mathbb{F}_B(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A))$ is equipped with a natural increasing filtration\textsuperscript{12}
$$C^\bullet_{\text{norm}}(A) \oplus C^\bullet_{\text{norm}}(A)[u] = \mathcal{F}^1 \mathbb{F}_B(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A)) \subset \mathcal{F}^2 \mathbb{F}_B(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A)) \subset \ldots$$
$$\mathcal{F}^m \mathbb{F}_B(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A)) = \bigoplus_{k \leq m} B^k(k, 0) \otimes S_k (C^\bullet_{\text{norm}}(A))^\otimes \otimes k,$$
$$\mathcal{F}^m \mathbb{F}_B(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A)) \{ k+1 \leq m \} = \bigoplus_{k \leq m} B^a(k, 1) \otimes S_k (C^\bullet_{\text{norm}}(A))^\otimes \otimes k \otimes C^\bullet(A), \quad (5.6)$$
where $u$ is an auxiliary variable of degree $-2$.

Using (5.6) we endow the Lie algebra (5.3) with the following decreasing filtration
$$\mathbb{L} = \mathcal{F}^0 \mathbb{L} \supset \mathcal{F}^1 \mathbb{L} \supset \mathcal{F}^2 \mathbb{L} \supset \ldots$$
$$\mathcal{F}^m \mathbb{L} = \{ Y \in \mathbb{L} \mid Y_{\mathcal{F}^m \mathbb{F}_B(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A))} = 0 \}. \quad (5.7)$$
Since
$$\mathbb{L} = \lim_{m} \mathbb{L}/\mathcal{F}^m \mathbb{L}$$
the Lie subalgebra $\mathcal{F}^1 \mathbb{L}^0$ is pronilpotent.

Therefore $\mathcal{F}^1 \mathbb{L}^0$ integrates to a pronilpotent group
$$\mathbb{G} = \exp(\mathcal{F}^1 \mathbb{L}^0) \quad (5.8)$$
which acts on the Maurer-Cartan elements of $\mathbb{L}$. This action is defined by the formula:
$$\exp(Y) Q = \exp([ , Y]) Q + f([ , Y]) [\partial^{\text{Bar}} + \partial^{\text{Hoch}}, Y],$$
where $f$ is the power series of the function
$$f(x) = \frac{e^x - 1}{x}.$$\textsuperscript{12}See Equation (3.11).
at the point \( x = 0 \).

Let \( Q \) be a Maurer-Cartan element of the DGLA \( \mathbb{L} \) which corresponds to the homotopy calculus structure on the pair \( (C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A)) \) which comes from the action of the operad \( \text{KS} \).

For every \( Y \in \mathcal{F}^1 \mathbb{L} \) the homotopy calculus on the pair \( (C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A)) \) corresponding to the Maurer-Cartan element \( \exp(Y) Q \) is quasi-isomorphic to the homotopy calculus corresponding the original Maurer-Cartan element \( Q \). Indeed, the desired \( \text{Ho} (\text{calc}) \)-quasi-isomorphism is expressed in terms of \( Y \) as

\[
\exp([\cdot, Y]) : (\mathbb{F}(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A)), Q) \rightarrow (\mathbb{F}(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A)), \exp(Y) Q) . \tag{5.10}
\]

Thus we get a family of mutually quasi-isomorphic homotopy calculus structures on \( (C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A)) \). Let us denote this family by \( S_{\text{calc}} \).

We claim that

**Theorem 4** The family \( S_{\text{calc}} \) contains a homotopy calculus structure whose \( \text{Ho}(\Lambda \text{Lie}_3^+)-\)algebra part is the \( \Lambda \text{Lie}_3^+ \)-algebra structure given by the operations \( [\cdot, \cdot]_G \tag{2.25} \), \( L \tag{2.27} \), and \( B \tag{2.28} \).

**Proof.** According to Proposition\[\text{2}\] and equation \( \text{3.26} \) the \( \text{Ho}(\Lambda \text{Lie}_3^+) \) is given by a Maurer-Cartan element \( M \) of the DGLA

\[
\text{Coder}'(\mathbb{F}(\Lambda \text{Lie}_3^+)^\vee(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A))) . \tag{5.11}
\]

Due to Proposition\[\text{1}\] this Maurer-Cartan element is, in turn, uniquely determined by the composition with the corestriction \( \rho \) onto \( C^\bullet_{\text{norm}}(A) \oplus C^\bullet_{\text{norm}}(A) \)

\[
m = \rho \circ M : \mathbb{F}(\Lambda \text{Lie}_3^+)^\vee(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A)) \rightarrow C^\bullet_{\text{norm}}(A) \oplus C^\bullet_{\text{norm}}(A) . \tag{5.12}
\]

Finally, Proposition\[\text{3}\] shows that the map \( m \) is related to the map \( q \tag{5.4} \) by the equation

\[
m = q \left| \mathbb{F}(\Lambda \text{Lie}_3^+)^\vee(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A)) \right. , \tag{5.13}
\]

where \( \Lambda(\text{Lie}_3^+)^\vee \) is considered as a sub-cooperad of \( B = \text{Bar}(\text{calc}) \) via the chain of embeddings:

\[
\Lambda(\text{Lie}_3^+)^\vee \hookrightarrow \text{Bar}(\Lambda \text{Lie}_3^+) \hookrightarrow \text{Bar}(\text{calc}) .
\]

The \( \text{Ho}(\Lambda \text{Lie}) \)-algebra structure on \( C^\bullet_{\text{norm}}(A) \) is induced by the \( \text{Ho}(\text{e}_2) \)-algebra structure which, in turn, comes from the action of the minimal operad of Kontsevich and Soibelman\[\text{22} \] on \( C^\bullet_{\text{norm}}(A) \). It was proved in \[\text{13} \] (see Theorem\[\text{2}\]) that this \( \text{Ho}(\text{Lie}) \)-algebra is in fact a genuine Lie algebra given by \( [\cdot, \cdot]_G \).

Thus it remains to take care about the operations involving a chain.

Due to \( \text{3.27} \) all the operations of the \( \text{Ho}(\Lambda \text{Lie}_3^+) \)-algebra involving a chain are combined into a single degree 1 map

\[
m^a = m \left| \mathbb{F}(\Lambda \text{co}\text{comm}^+(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A)[[u]]) \right. , \tag{5.14}
\]

\[
\rightarrow C^\bullet_{\text{norm}}(A) .
\]
such that
\[ m^a \big|_{C_{\text{norm}}(A)} = 0. \] (5.15)

The latter equation follows from the fact that the coderivation \( M \) belongs to the DGLA (5.11).

In other words, we have the infinite collection of operations:
\[ m^a_{k,n} : S^k(C_{\text{norm}}(A)) \otimes C^\bullet_n(A) \to C^\bullet_n(A) \] (5.16)
of degrees \( 1 - 2k - 2n \), where \( S^k(C_{\text{norm}}(A)) \) is the \( k \)-th component of the symmetric algebra \( S(C_{\text{norm}}(A)) \) of \( C_{\text{norm}}(A) \).

Due to Proposition 9 the operation \( m^a_{k,n} \) vanish if \( k + 2n > 2 \). Furthermore, due to equation (5.15) we have \( m^a_{0,0} = 0 \). Thus we need to analyze only there operations: one unary operation
\[ m^a_{0,1} : C^\bullet_{\text{norm}}(A) \to C^\bullet_{\text{norm}}(A) \] (5.17)
of degree \(-1\), one binary operation
\[ m^a_{1,0} : C^\bullet_{\text{norm}}(A) \otimes C^\bullet_{\text{norm}}(A) \to C^\bullet_{\text{norm}}(A) \] (5.18)
of degree \(-1\) and one ternary operation
\[ m^a_{2,0} : S^2(C_{\text{norm}}(A)) \otimes C^\bullet_{\text{norm}}(A) \to C^\bullet_{\text{norm}}(A) \] (5.19)
of degree \(-3\).

Theorems 2 and 3 imply that the operation (5.17) differs from Connes’ operator \( B \) by an exact operation. Namely,
\[ m^a_{0,1}(c) = B(c) + \partial^{Hoch} \beta(c) - \beta(\partial^{Hoch} c), \]
where \( \beta \) is an operation in \( KS \)
\[ \beta : C^\bullet_{\text{norm}}(A) \to C^\bullet_{\text{norm}}(A) \]
of degree \(-2\).

Using Proposition 9 we deduce that \( \beta \) is zero. Hence \( m^a_{0,1} = B \).

Due to Proposition 3 the operation (5.18) is expressed in terms of \( q \) (5.4) as
\[ m^a_{1,0}(P,c) = q(b_1), \] (5.20)
where \( P \in C^\bullet_{\text{norm}}(A) \), \( c \in C^\bullet_{\text{norm}}(A) \), and the element \( b_1 \in F_B(C^\bullet_{\text{norm}}(A), C^\bullet_{\text{norm}}(A)) \) is depicted on Figure 14.

Theorems 2 and 3 imply that \( m^a_{1,0} \) differs from the action \( L \) of cochains on chains by an exact operation. In other words,
\[ m^a_{1,0}(P,c) = -(1)^{|P|} L_P c + \partial^{Hoch} \psi(P,c) - \psi(\partial^{Hoch} P,c) - (1)^{|P|} \psi(P,\partial^{Hoch} c), \] (5.21)
where \(|P|\) is the degree of \( P \) and
\[ \psi : C^\bullet_{\text{norm}}(A) \otimes C^\bullet_{\text{norm}}(A) \to C^\bullet_{\text{norm}}(A) \]
is an operation in $\text{KS}^a(1,1)$ of degree $-2$.

We remark that $\psi$ may be considered as a map
\[
\psi : \Lambda(\text{Lie}^+)(1,1) \otimes C_{\text{norm}}^\bullet(A) \otimes C_{\text{norm}}^\bullet(A) \to C_{\text{norm}}^\bullet(A) \quad (5.22)
\]
of degree 0. Our purpose is to extend $\psi$ “by zero” to the whole vector space of the coalgebra $\mathcal{F}_B(C_{\text{norm}}^\bullet(A), C_{\text{norm}}^\bullet(A))$. This extension depends on the choice of basis in $\text{calc}^a(1,1)$.

We choose the basis
\[
\{l, i, i\delta, l\delta\} \quad (5.23)
\]
and extend $\psi$ to $\text{B}^a(1,1) \otimes C_{\text{norm}}^\bullet(A) \otimes C_{\text{norm}}^\bullet(A)$
\[
\psi : \text{B}^a(1,1) \otimes C_{\text{norm}}^\bullet(A) \otimes C_{\text{norm}}^\bullet(A) \to C_{\text{norm}}^\bullet(A), \quad (5.24)
\]
as
\[
\psi(b_1) = \psi(P,c), \quad \psi(b_2) = \psi(b_3) = \psi(b_4) = 0, \\
\psi(b_\lambda) = 0,
\]
where the elements
\[
b_1, b_2, b_3, b_4, b_\lambda \in \text{B}^a(1,1) \otimes C_{\text{norm}}^\bullet(A) \otimes C_{\text{norm}}^\bullet(A)
\]
are depicted on Figures 14, 15, 16. $\lambda$ is an arbitrary element of the basis (5.23), and $P \in$
$b_\lambda = \lambda$

Figure 16: The number of $\delta$'s is $\geq 1$
Next we extend $\psi$ by zero to the whole vector space of the coalgebra

$$F_B(C_{\text{norm}}(A), C_{\text{norm}}(A)) = \bigoplus_n B^c(n, 0) \otimes S_n (C_{\text{norm}}(A))^\otimes n \oplus \bigoplus_n B^a(n, 1) \otimes S_n (C_{\text{norm}}(A))^\otimes n \otimes C_{\text{norm}}(A).$$ (5.25)

Then according to Proposition 1 the equation

$$\rho \circ \Psi = \psi$$

defines a derivation $\Psi$ of the coalgebra (5.25). The derivation $\Psi$ has degree 0 since so does the map $\psi$. Furthermore, it is obvious that $\Psi \in F^{11}$.

Applying the element $\exp(-\Psi)$ of the group $G$ (5.8) to the Maurer-Cartan element $Q$ (5.1) we adjust the component $m_{2,0}^a$ by killing this additional exact term $\partial^{\text{Hoch}} \psi(P, c) - \psi(\partial^{\text{Hoch}} P, c) - (-1)^{|P|} \psi(P, \partial^{\text{Hoch}} c)$ in (5.21). In doing this we do not change the unary operations because $\Psi \in F^{11}$.

Thus we are left with only one non-vanishing operation (5.19).

The Maurer-Cartan equation (5.5) implies that $m_{2,0}^a$ should be closed with respect to the differential $\partial^{\text{Hoch}}$.

Since the degree of $m_{2,0}^a$ is $-3$, using Theorems 2 and 3, we deduce that up to $\partial^{\text{Hoch}}$-exact terms the operation $m_{2,0}^a$ is made of the following “building blocks”:

$$L_{[P_1, P_2]_G} B c, \quad L_{P_1} L_{P_2} B c, \quad L_{P_2} L_{P_1} B c,$$

where $P_1, P_2 \in C_{\text{norm}}(A)$ and $c \in C_{\text{norm}}(A)$.

Using the symmetry in the arguments $P_1, P_2$ and the compatibility with $\partial^{\text{Hoch}}$ it is not hard to show that (up to $\partial^{\text{Hoch}}$-exact terms) the most general expression for $m_{2,0}^a$ is

$$m_{2,0}^a(P_1, P_2, c) = (-1)^{|P_1|\mu} L_{[P_1, P_2]_G} B c,$$ (5.26)

where $\mu \in K$, $P_1, P_2 \in C_{\text{norm}}(A)$ and $c \in C_{\text{norm}}(A)$.

If necessary, we apply the above trick with the action (5.9) of the group (5.8) to modify $Q$ (5.1) so that equation (5.26) indeed holds.

To kill $m_{2,0}^a$ we will introduce the map

$$y : B^a(1, 1) \otimes C_{\text{norm}}(A) \otimes C_{\text{norm}}(A) \to C_{\text{norm}}(A).$$ (5.27)

This map is defined by the equations

$$y(b_1) = -\mu L_{P} B c, \quad y(b_2) = y(b_3) = y(b_4) = 0,$$

$$y(b_5) = 0,$$

where the elements

$$b_1, b_2, b_3, b_4, b_5 \in B^a(1, 1) \otimes C_{\text{norm}}(A) \otimes C_{\text{norm}}(A)$$

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are depicted on Figures 14, 15, 16. λ is an arbitrary element of the basis (5.23) in calc^a(1, 1) and

\[ P \in C_{\text{norm}}^*(A), \quad c \in C_{\text{norm}}^*(A). \]

Then we extend y by zero to the whole vector space of the coalgebra (5.25). It is not hard to see that y is of degree 0.

According to Proposition 4 the equation

\[ \rho \circ Y = y \]

defines a degree 0 coderivation Y of the coalgebra (5.25). Furthermore,

\[ [\partial^{\text{Hoch}}, Y] = 0, \] (5.28)

and

\[ Y \in \mathcal{F}^1 \mathbb{L}. \] (5.29)

Applying the element \( \exp(Y) \) of the group \( \mathbb{G} \) (5.8) to the Maurer-Cartan element \( Q \) (5.1) we get another homotopy calculus structure on \((C_{\text{norm}}^*(A), C^*_{\text{norm}}(A))\). This homotopy calculus structure is determined by the Maurer-Cartan element \( \exp(Y) Q \).

Let us denote by \( \tilde{m}^{a}_{0,1}, \tilde{m}^{a}_{1,0}, \) and \( \tilde{m}^{a}_{2,0} \) the operations (5.17), (5.18), (5.19) of the \( \text{Ho}(\Lambda \text{Lie}_\delta^+) \)-algebra corresponding to the new homotopy calculus \( \exp(Y) Q \). Since \( Y \in \mathcal{F}^1 \mathbb{L} \) the unary operation cannot change. Thus

\[ \tilde{m}^{a}_{0,1} = m^{a}_{0,1} = B. \]

For the binary operation we have

\[ \tilde{m}^{a}_{1,0}(P, c) = \rho \circ \exp(Y) Q(b_1), \] (5.30)

where \( \rho \) is the projection from \( F_B(C_{\text{norm}}^*(A), C^*_{\text{norm}}(A)) \) onto \( C_{\text{norm}}^*(A) \oplus C^*_{\text{norm}}(A) \) and the element \( b_1 \in F_B(C_{\text{norm}}^*(A), C^*_{\text{norm}}(A)) \) is depicted on Figure 14.

Using (5.9), (5.28), and (5.29) we simplify equation (5.30) as follows

\[ \tilde{m}^{a}_{1,0}(P, c) = m^{a}_{1,0}(P, c) + [(q \circ Y - y \circ Q) + y \circ \partial^{\text{Bar}}] b_1. \]

It is obvious that \( \partial^{\text{Bar}} b_1 = 0 \). Furthermore, it is not hard to see that \( q \circ Y(b_1) = 0 \) and \( y \circ Q(b_1) = 0 \). Thus the binary operation (5.18) is also unchanged.

For the ternary operation \( \tilde{m}^{a}_{2,0} \) we have

\[ \tilde{m}^{a}_{2,0}(P_1, P_2, c) = \rho \circ \exp(Y) Q(b) = \\
\]

\[ = m^{a}_{2,0}(P_1, P_2, c) + (q \circ Y - y \circ Q)(b) + y \circ \partial^{\text{Bar}}(b) - y \circ \partial^{\text{Bar}} \circ Y(b) + \frac{1}{2} y \circ Y \circ \partial^{\text{Bar}}(b), \]

where the element \( b \in F_B(C_{\text{norm}}^*(A), C^*_{\text{norm}}(A)) \) is depicted on Figure 17 and, as above, we used (5.28) and (5.29).

It is obvious that \( \partial^{\text{Bar}}(b) = 0 \) and it is not hard to check that \( \partial^{\text{Bar}} \circ Y(b) = 0 \). Furthermore a direct computation shows that

\[ (q \circ Y - y \circ Q)(b) = -(-1)^{|P_1|} \mu L_{[P_1, P_2]} c, B c. \]

Thus

\[ \tilde{m}^{a}_{2,0} = 0 \] (5.31)

and Theorem 4 is proved. □
Figure 17: Here $P_1, P_2 \in C_{\text{norm}}(A)$ and $c \in C_{\text{norm}}(A)$
6 Formality theorem

6.1 Enveloping algebra of a Gerstenhaber algebra

Let \((\mathcal{V}, \mathcal{W})\) be a pair of graded vector spaces. For our purposes we will need to consider \textit{calc}-algebras on \((\mathcal{V}, \mathcal{W})\) with a fixed Gerstenhaber algebra structure \((\mathcal{V}, \wedge, [\ , \ ])\) on \(\mathcal{V}\). For such \textit{calc}-algebras we call \(\mathcal{W}\) a \textit{calc}-module over \((\mathcal{V}, \wedge, [\ , \ ])\).

The category of \textit{calc}-modules over \(\mathcal{V}\) is equivalent to a category of ordinary modules over the enveloping algebra \([30]\) of the Gerstenhaber algebra \(\mathcal{V}\). In this section we recall the construction of this enveloping algebra and describe its properties.

Let us start with the following definition:

\textbf{Definition 6} For a Gerstenhaber algebra \((\mathcal{V}, \wedge, [\ , \ ])\) we define an associative algebra \(Y_0(\mathcal{V})\) which is generated by two sets of symbols

\[ l_v, i_v \quad v \in \mathcal{V} \]  

(6.1)

of degrees

\[ |i_v| = |v|, \quad |l_v| = |v| - 1. \]  

(6.2)

These symbols are \(\mathbb{K}\)-linear in \(v\) and they are subject to the following relations

\[ i_{v_1 \cdot v_2} = i_{v_1} i_{v_2}, \quad [i_{v_1}, l_{v_2}] = i_{[v_1, v_2]}, \]  

\[ l_{v_1 \cdot v_2} = l_{v_1} i_{v_2} + (-1)^{|v_1|} i_{v_1} l_{v_2}, \quad [l_{v_1}, l_{v_2}] = l_{[v_1, v_2]}. \]  

(6.3)

Furthermore,

\textbf{Definition 7 ([30])} If \((\mathcal{V}, \cdot, [\ , \ ])\) is a Gerstenhaber algebra then the associative algebra \(Y(\mathcal{V})\) is generated by symbols \((6.4)\) and an element \(\delta\) of degree \(-1\). The symbols \((6.4)\) are \(\mathbb{K}\)-linear in \(v\) and they are subject to the following relations

\[ \delta^2 = 0, \quad [\delta, i_v] = l_v, \]  

\[ i_{v_1 \cdot v_2} = i_{v_1} i_{v_2}, \quad [i_{v_1}, l_{v_2}] = i_{[v_1, v_2]}. \]  

(6.4)

It is obvious that the category of \textit{calc}-modules over \(\mathcal{V}\) is equivalent to the category of ordinary modules over the associative algebra \(Y(\mathcal{V})\).

Let us give the following natural definition

\textbf{Definition 8} A DG commutative algebra \(\mathcal{V}\) is called regular if the module \(\Omega^1(\mathcal{V})\) of its Kähler differentials is cofibrant.

\textbf{Remark.} Notice that if \(\mathcal{V}\) is a commutative algebra concentrated in degree 0 then the above condition of regularity means that the \(\mathcal{V}\)-module \(\Omega^1(\mathcal{V})\) is projective.

We claim that

\textbf{Proposition 10} Let \(\mathcal{V}\) be a DG Gerstenhaber algebra and \(\mathcal{R} \to \mathcal{V}\) be its cofibrant resolution. If the corresponding DG commutative algebra \(\mathcal{V}\) is regular then the induced map

\[ Y(\mathcal{R}) \to Y(\mathcal{V}) \]  

(6.5)

is a quasi-isomorphism of DG associative algebras.
Proof. Due to the obvious equality of chain complexes
\[ \mathcal{Y}(\mathcal{V}) = \mathcal{Y}_0(\mathcal{V}) \oplus \mathcal{Y}_0(\mathcal{V}) \delta \]  
(6.6)
it suffices to show that the map
\[ \mathcal{Y}_0(\mathcal{R}) \rightarrow \mathcal{Y}_0(\mathcal{V}) \]  
(6.7)
is a quasi-isomorphism.

For this purpose we introduce the following Lie-Rinehart algebra structure on the pair \((\mathcal{V}, \Omega^1(\mathcal{V}))\), where \(\Omega^1(\mathcal{V})\) is the module of Kähler differentials of the DG commutative algebra \(\mathcal{V}\). The Lie bracket \(\{\ ,\ \}\) on \(\Omega^1(\mathcal{V})\) and the action \(l\) of \(\Omega^1(\mathcal{V})\) on \(\mathcal{V}\) are defined in terms of the Lie bracket on \(\mathcal{V}\) as follows
\[
\{a_1 da_2, b_1 db_2\} = (-1)^{|a_2|+1}a_1 [a_2, b_1] db_2 + (-1)^{|a_2|+|b_1|}a_1 b_1 d([a_2, b_2]) + \]
\[
(-1)^{|a_1|+|a_2|+|b_1|+|b_2|} b_1 [b_2, a_1] da_2 ,
\]
(6.8)
and
\[
l_{a_1 da_2}(b) = (-1)^{|a_2|+1}a_1 [a_2, b] .
\]
(6.9)
The identities of a Gerstenhaber algebra imply that equations (6.8) and (6.9) indeed define a Lie-Rinehart algebra on the pair \((\mathcal{V}, \Omega^1(\mathcal{V}))\).

Next we remark that for every (DG) Gerstenhaber algebra \(\mathcal{V}\) the associative algebra \(\mathcal{Y}_0(\mathcal{V})\) is nothing but the enveloping algebra of the Lie-Rinehart algebra \((\mathcal{V}, \Omega^1(\mathcal{V}))\). Indeed, the required isomorphism is defined on generators as
\[
a \mapsto i_a , \quad db \mapsto l_b , \quad a, b \in \mathcal{V} .
\]

Then the PBW-filtration on \(\mathcal{Y}_0(\mathcal{V})\) is
\[
\mathcal{V} \cong \mathcal{F}^0 \mathcal{Y}_0(\mathcal{V}) \subset \mathcal{F}^1 \mathcal{Y}_0(\mathcal{V}) \subset \mathcal{F}^2 \mathcal{Y}_0(\mathcal{V}) \subset \ldots
\]  
(6.10)
where
\[ \mathcal{F}^k \mathcal{Y}_0(\mathcal{V}) \]
is spanned by monomials in which the number of symbols \(l_v\), \(v \in \mathcal{V}\) is less or equal to \(k\).

Since the DG commutative algebra \(\mathcal{V}\) is regular we can apply the PBW-theorem to the Lie-Rinehart algebra \((\mathcal{V}, \Omega^1(\mathcal{V}))\). Using this theorem (see Theorem 3.1 in [24]) we conclude that the associated graded algebra is isomorphic to the symmetric algebra \(S_{\mathcal{V}}(\Omega^1(\mathcal{V}))\)
\[
\bigoplus_k \mathcal{F}^k \mathcal{Y}_0(\mathcal{V}) / \mathcal{F}^{k-1} \mathcal{Y}_0(\mathcal{V}) \cong S_{\mathcal{V}}(\Omega^1(\mathcal{V})) .
\]  
(6.11)

Since \(\mathcal{R}\) is a cofibrant resolution of \(\mathcal{V}\) the same argument with PBW theorem from [24] works for \(\mathcal{Y}_0(\mathcal{R})\). The map (6.7) is obviously compatible with the filtrations (6.10) on \(\mathcal{Y}_0(\mathcal{R})\) and \(\mathcal{Y}_0(\mathcal{V})\). Furthermore, these filtrations are cocomplete
\[
\mathcal{Y}_0(\mathcal{V}) = \text{colim}_k \mathcal{F}^k \mathcal{Y}_0(\mathcal{V}) , \quad \mathcal{Y}_0(\mathcal{R}) = \text{colim}_k \mathcal{F}^k \mathcal{Y}_0(\mathcal{R}) .
\]

Hence, in order to prove that the map (6.7) is a quasi-isomorphism, we need to show that so is the map
\[ S_{\mathcal{R}}(\Omega^1(\mathcal{R})) \rightarrow S_{\mathcal{V}}(\Omega^1(\mathcal{V})) .
\]  
(6.12)
This statement follows from the regularity of \(\mathcal{V}\). Thus the proposition is proved. \(\square\)
6.2 Sheaves of Hochschild (co)chains on an algebraic variety

Let \( X \) be a smooth algebraic variety over \( \mathbb{K} \) with the structure sheaf \( \mathcal{O}_X \). We denote by \( V_X^\bullet \) the sheaf of polyvector fields and by \( \Omega_X^{-\bullet} \) be the sheaf of exterior forms with reversed grading. \( \mathcal{D}_X \) denotes the sheaf of differential operators on \( X \) and \( \mathcal{D}_{\Omega_X^{-\bullet}} \) denotes the sheaf of differential operators on the sheaf of (graded) commutative algebras \( \Omega_X^{-\bullet} \).

In the affine case \( X = \text{Spec}(A) \) we will use the short-hand notation for the corresponding modules of global sections \( V_X^\bullet(A) = \Gamma(X, V_X^\bullet) \), \( \Omega_X^{-\bullet}(A) = \Gamma(X, \Omega_X^{-\bullet}) \), \( \mathcal{D}(A) = \Gamma(X, \mathcal{D}_X) \), and finally \( \mathcal{D}(\Omega_X^{-\bullet}(A)) = \Gamma(X, \mathcal{D}_{\Omega_X^{-\bullet}}) \).

The pair \((V_X^\bullet, \Omega_X^{-\bullet})\) is a calculus algebra with respect to the operations: the exterior product \( \wedge \) on \( V_X^\bullet \), the Schouten-Nijenhuis bracket \( [\cdot, \cdot]_{SN} \) on \( V_X^\bullet \), the contraction \( \mathcal{I} \) of a form with a polyvector, the Lie derivative \( \mathcal{L} \) of a form along a polyvector, and finally the de Rham differential \( d \) on the exterior forms.

Using the contraction \( \mathcal{I} \) we define the natural \( \mathcal{O}_X \)-linear pairing \( \langle , \rangle \) between the sheaves \( V_X^\bullet \) and \( \Omega_X^{-\bullet} \):

\[
\langle \gamma, \eta \rangle : V_X^\bullet \otimes_{\mathcal{O}_X} \Omega_X^{-\bullet} \to \mathcal{O}_X \\
\langle \gamma, \eta \rangle = \begin{cases} 
\mathcal{I}_\gamma \eta, & \text{if } |\eta| = -|\gamma|, \\
0, & \text{otherwise}. 
\end{cases}
\]

(6.13)

Here \( \gamma \) and \( \eta \) are local sections of \( V_X^\bullet \) and \( \Omega_X^{-\bullet} \), respectively.

An appropriate version of Hochschild cochain complex for the structure sheaf \( \mathcal{O}_X \) is the sheaf of polydifferential operators \([27], [34]\). We will denote this sheaf by \( C^\bullet(\mathcal{O}_X) \). For example \( C^0(\mathcal{O}_X) \) is the structure sheaf \( \mathcal{O}_X \) and \( C^1(\mathcal{O}_X) \) is the sheaf \( \mathcal{D}_X \) of differential operators on \( X \).

Let us also denote by \( C^\bullet_{\text{norm}}(\mathcal{O}_X) \) the sheaf of normalized polydifferential operators. These are the polydifferential operators satisfying the property

\[
P(\ldots, 1, \ldots) = 0.
\]

Similarly an appropriate version of Hochschild chain complex for the structure sheaf \( \mathcal{O}_X \) is the sheaf of polyjets \([31]\):

\[
C_\bullet(\mathcal{O}_X) = \mathcal{H}om_{\mathcal{O}_X}(C^{-\bullet}(\mathcal{O}_X), \mathcal{O}_X),
\]

(6.14)

where \( \mathcal{H}om \) denotes the sheaf-Hom and \( C^\bullet(\mathcal{O}_X) \) is considered with its natural left \( \mathcal{O}_X \)-module structure. For example \( C_0(\mathcal{O}_X) \) is the structure sheaf \( \mathcal{O}_X \) and \( C_{-1}(\mathcal{O}_X) \) is the sheaf of \( \infty \)-jets.

There are natural analogs of the degenerate Hochschild chains

\[
c = (c_0, \ldots, 1, \ldots)
\]

and these degenerate chains form a subsheaf \( C^\bullet_{\text{degen}}(\mathcal{O}_X) \) of \( C_\bullet(\mathcal{O}_X) \). Furthermore the subsheaf \( C^\bullet_{\text{degen}}(\mathcal{O}_X) \) is closed with respect to the Hochschild boundary operator \( \partial^{\text{Hoch}} \).

We define the sheaf \( C^\bullet_{\text{norm}}(\mathcal{O}_X) \) of normalized Hochschild chains as the quotient sheaf \( C_\bullet(\mathcal{O}_X)/C^\bullet_{\text{degen}}(\mathcal{O}_X) \). It is not hard to show that

\[
C^\bullet_{\text{norm}}(\mathcal{O}_X) = \mathcal{H}om_{\mathcal{O}_X}(C^{-\bullet}_{\text{norm}}(\mathcal{O}_X), \mathcal{O}_X).
\]

(6.15)
As well as for Hochschild complexes of an associative algebra the inclusion
\[ C_\text{norm}^\bullet(\mathcal{O}_X) \hookrightarrow C^\bullet(\mathcal{O}_X) \]
and the projection
\[ C^\bullet(\mathcal{O}_X) \to C_\text{norm}^\bullet(\mathcal{O}_X) \]
are quasi-isomorphisms of complexes of sheaves. Furthermore, the action of the Kontsevich-Soibelman operad $\mathcal{KS}$ on the pair of sheaves $(C_\text{norm}^\bullet(\mathcal{O}_X), C^\bullet(\mathcal{O}_X))$ is well-defined\(^{13}\).

Thus the pair $(C_\text{norm}^\bullet(\mathcal{O}_X), C^\bullet(\mathcal{O}_X))$ is a sheaf of homotopy calculi.

Let us recall that the embedding
\[ \varrho_c : V^\bullet_X \hookrightarrow C_\text{norm}^\bullet(\mathcal{O}_X) \quad (6.16) \]
is called the Hochschild-Kostant-Rosenberg map. It is known \(^{20}\) that $\varrho_V$ is a quasi-isomorphism of complexes of sheaves where the sheaf $V^\bullet_X$ is considered with the zero differential.

The corresponding quasi-isomorphism for Hochschild chains
\[ \varrho_a : C^\bullet(\mathcal{O}_X) \to \Omega_X^{-\bullet} \quad (6.17) \]
is called the Connes-Hochschild-Kostant-Rosenberg map. This map is defined by the equation
\[ \langle \gamma, \varrho_a(c) \rangle = c(\varrho_c(\gamma)) , \quad (6.18) \]
where $c$ is a local section of $C_{-m}^\text{norm}(\mathcal{O}_X)$, $\gamma$ is a local section of $V_X^m$ and the pairing $\langle \cdot, \cdot \rangle$ is defined in (6.13).

It is known \(^{10}\) that the maps $\varrho_c$ and $\varrho_a$ are compatible with the operations of the Cartan calculus on the pair $(V^\bullet_X, \Omega_X^{-\bullet})$ and the operations $\cup$ (2.24), $[\cdot, \cdot]_G$ (2.25), $I$ (2.26), $L$ (2.27) and $B$ (2.28) on the pair $(C_\text{norm}^\bullet(\mathcal{O}_X), C^\bullet(\mathcal{O}_X))$ up to homotopy. We upgrade this observation to the following theorem.

**Theorem 5** If $X$ is a smooth algebraic variety over a field $\mathbb{K}$ of characteristic zero then the sheaf $(C_\text{norm}^\bullet(\mathcal{O}_X), C^\bullet(\mathcal{O}_X))$ of homotopy calculi is quasi-isomorphic to the sheaf $(V^\bullet_X, \Omega_X^{-\bullet})$ of calculi.

The proof of this theorem is given in Subsection 6.4.

### 6.3 Morita equivalence

In this subsection we will show that the sheaf of algebras $\mathcal{Y}(V^\bullet_X)$ is Morita equivalent to the sheaf $\mathcal{D}_X[\vartheta]/(\vartheta^2)$ where $\vartheta$ is an auxiliary variable of degree $-1$ commuting with all the differential operators.

First we remark that $\mathcal{Y}_0(V^\bullet_X)$-module structure on the sheaf $\Omega_X^{-\bullet}$ gives us a natural map
\[ \mathcal{Y}_0(V^\bullet_X) \to \mathcal{D}_{\Omega_X^{-\bullet}} \quad (6.19) \]
between the sheaves of associative algebras. We claim that

\(^{13}\)Obvious extensions of the operations on Hochschild chains to the operations on polyjets is discussed in details in [2].
Proposition 11 The map \((6.19)\) is an isomorphism of sheaves of associative algebras.

Proof. We need to show that \((6.19)\) gives us an isomorphism on stalks
\[
\mathcal{Y}_0(V_X^x) \rightarrow (\mathcal{D}_{\Omega^x})_x
\]
for every point \(x \in X\).

Thus, since \(X\) is smooth, it suffices to show that the map
\[
\mathcal{Y}_0(V^\bullet(A)) \rightarrow \mathcal{D}_{\Omega^\bullet(A)} \tag{6.20}
\]
is an isomorphism for every local regular (commutative) algebra \(A\) over \(\mathbb{K}\).

It is easy to see that the associative algebra \(\mathcal{Y}_0(V^\bullet(A))\) is generated by symbols:
\[
i_a, i_v, l_a, l_v, \tag{6.21}
\]
where \(a \in A\) and \(v \in V^1(A)\).

Under the map \((6.20)\) the symbols go to
\[
i_a \rightarrow I_a, \quad i_v \rightarrow I_v, \quad l_a \rightarrow L_a, \quad l_v \rightarrow L_v.
\]
Since the images of the symbols \((6.21)\) satisfy the same relations therefore the map \((6.20)\) is injective.

To show that \((6.20)\) is surjective we remark that the algebra \(\mathcal{D}_{\Omega^\bullet(A)}\) is generated by differential operators of the form
\[
a \cdot, \quad db \cdot, \quad a, b \in A
\]
and derivations \(\text{Der}_K(\Omega^\bullet(A))\) of \(\Omega^\bullet(A)\).

Since \(a \cdot = I_a\) and \(db \cdot = L_b\) it remains to show that every derivation \(W \in \text{Der}_K(\Omega^\bullet(A))\) belongs to the image of the map \((6.20)\).

The regularity of \(A\) implies that the \(A\)-modules \(\Omega^1(A)\) and \(\Omega^\bullet(A)\) are free. More precisely, if \(x_1, \ldots, x_n\) is a regular system of parameters in \(A\) then the \(A\)-module \(\Omega^1(A)\) is freely generated by the 1-forms
\[
dx^i, \quad i = 1, 2, \ldots, n, \tag{6.22}
\]
and the \(A\)-module \(\Omega^\bullet(A)\) is freely generated by the forms
\[
dx^{i_1} dx^{i_2} \ldots dx^{i_k}, \quad 1 \leq i_1 < i_2 < \cdots < i_k \leq n. \tag{6.23}
\]

Dually the \(A\)-module \(V^1(A) = \text{Der}_K(A)\) is freely generated by
\[
e_1, e_2, \ldots, e_n, \tag{6.24}
\]
where \(e_i\) is a derivation of \(A\) defined by the equation
\[
I_{e_i}(dx^j) = \delta^j_i.
\]

Since the \(A\)-module \(\Omega^\bullet(A)\) is freely generated by the forms \((6.23)\) therefore every derivation \(W \in \text{Der}_K(\Omega^\bullet(A))\) is uniquely determined by its values on the elements of \(A\) and the 1-forms \((6.22)\). In general we have
\[
W(a) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} W_{i_1 \ldots i_k}(a) dx^{i_1} dx^{i_2} \ldots dx^{i_k}
\]
and

\[ W(dx^i) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} W_{i_1 \ldots i_k}^i dx^{i_1} dx^{i_2} \cdots dx^{i_k}, \]

where \( W_{i_1 \ldots i_k} \) are derivations of \( A \) over \( \mathbb{K} \) and \( W_{i_1 \ldots i_k}^i \in A \).

Let \( W_1 \) be the following derivation of \( \Omega^{-\bullet}(A) \):

\[ W_1 = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} dx^{i_1} dx^{i_2} \cdots dx^{i_k} \mathcal{L}_{W_{i_1 \ldots i_k}}. \]

It is obvious that the difference \( W - W_1 \) is \( A \)-linear. Hence \( W_2 = W - W_1 \) is uniquely determined by its values on 1-forms (6.22):

\[ W_2(dx^i) = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} \tilde{W}_{i_1 \ldots i_k}^i dx^{i_1} dx^{i_2} \cdots dx^{i_k}, \]

where \( \tilde{W}_{i_1 \ldots i_k}^i \in A \). Thus the derivation \( W \) can be rewritten as

\[ W = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} dx^{i_1} dx^{i_2} \cdots dx^{i_k} \mathcal{L}_{W_{i_1 \ldots i_k}} + \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq n} dx^{i_1} dx^{i_2} \cdots dx^{i_k} \mathcal{I}_{\tilde{W}_{i_1 \ldots i_k}}, \]

where \( \tilde{W}_{i_1 \ldots i_k} \) is the derivation of \( A \) defined by

\[ \tilde{W}_{i_1 \ldots i_k} = \sum_i \tilde{W}_{i_1 \ldots i_k}^i e_i. \]

This shows that the map (6.20) is surjective. Hence so is the map (6.19). \( \square \)

Using the pairing (6.13) we introduce the following map of sheaves of \( \mathcal{O}_X \)-bimodules:

\[ r : \Omega^{-\bullet} \otimes \mathcal{O}_X \mathcal{D}_X \otimes \mathcal{O}_X V_X^{-\bullet} \rightarrow \mathcal{D}_{\Omega^{-\bullet}}, \]

(6.25)

\[ r(\eta, D, \gamma)(\varphi) = \eta D(\gamma, \varphi), \]

where \( \eta \) and \( \varphi \) are local sections of \( \Omega_X^{-\bullet} \), \( D \) is a local section of \( \mathcal{D}_X \), \( \gamma \) is a local section of \( V_X^{\bullet} \), and \( \mathcal{D}_X \) is considered as sheaf of bimodules over \( \mathcal{O}_X \).

We claim that

**Proposition 12** The map \( r \) in (6.25) is an isomorphism of sheaves of \( \mathcal{O}_X \)-bimodules.

**Proof.** Again, it suffices to prove that the map \( r \) gives an isomorphism on stalks

\[ (\Omega_X^{-\bullet} \otimes \mathcal{O}_X \mathcal{D}_X \otimes \mathcal{O}_X V_X^{\bullet})_x \rightarrow (\mathcal{D}_{\Omega_X^{-\bullet}})_x \]

for every point \( x \in X \).

Hence, due to smoothness of \( X \), we need to show that the map

\[ r_A : \Omega^{-\bullet}(A) \otimes A \mathcal{D}(A) \otimes A V^{\bullet}(A) \rightarrow \mathcal{D}_{\Omega^{-\bullet}(A)} \]

(6.26)

is an isomorphism for every local regular (commutative) algebra over \( \mathbb{K} \).
Since $\Omega^{-\bullet}(A)$ and $V^{\bullet}(A)$ are free modules over $A$ the injectivity of $r_A$ follows easily from the injectivity of the restriction
\[
r_A|_{\mathcal{D}(A)} : \mathcal{D}(A) \to \mathcal{D}_{\Omega^{-\bullet}(A)}.
\]

Due to Proposition 11 the algebra $\mathcal{D}_{\Omega^{-\bullet}(A)}$ is generated by differential operators of the form
\[
\mathcal{I}_a = a \cdot, \quad \mathcal{L}_a = da \cdot, \quad a \in A, \quad (6.27)
\]
\[
\mathcal{I}_v, \quad \mathcal{L}_v, \quad v \in \text{Der}_K(A). \quad (6.28)
\]
Thus, to show that $r_A$ is surjective it suffices to prove that the operators (6.27) and (6.28) belong to the image of $r_A$.

If we choose a regular system of parameters $x_1, \ldots, x_n$ in $A$ then using the generators (6.23) and (6.24) we may rewrite the operators $\mathcal{I}_a = a \cdot, \mathcal{L}_a = da \cdot$ as follows:
\[
a \cdot \varphi = \sum_{k;1 \leq i_1 < i_2 < \ldots < i_k \leq n} dx^{i_1} dx^{i_2} \ldots dx^{i_k} \langle e_{i_k} \wedge \ldots \wedge e_{i_2} \wedge e_{i_1}, \varphi \rangle,
\]
and
\[
da \cdot \varphi = \sum_{k;1 \leq i_1 < i_2 < \ldots < i_k \leq n} da dx^{i_1} dx^{i_2} \ldots dx^{i_k} \langle e_{i_k} \wedge \ldots \wedge e_{i_2} \wedge e_{i_1}, \varphi \rangle.
\]
Thus the operators (6.27) belong to the image of $r_A$.

Every derivation $v \in \text{Der}_K(A)$ can be uniquely written as
\[
v = \sum_i v^i e_i,
\]
where $v^i \in A$.

Using this decomposition we rewrite the operators $\mathcal{I}_v$ and $\mathcal{L}_v$ as
\[
\mathcal{I}_v \varphi = \sum_{k;1 \leq i_1 < i_2 < \ldots < i_k \leq n} \sum_{s=1}^k (-1)^{s-1} v^i_s dx^{i_1} \ldots \hat{dx}^{i_s} \ldots dx^{i_k} \langle e_{i_k} \wedge \ldots \wedge e_{i_2} \wedge e_{i_1}, \varphi \rangle,
\]
\[
\mathcal{L}_v \varphi = \sum_{k;1 \leq i_1 < i_2 < \ldots < i_k \leq n} dx^{i_1} dx^{i_2} \ldots dx^{i_k} v \langle e_{i_k} \wedge \ldots \wedge e_{i_2} \wedge e_{i_1}, \varphi \rangle + \quad (6.29)
\]
\[
+ \sum_{k;1 \leq i_1 < i_2 < \ldots < i_{k+1} \leq n} (-1)^{s-t} e_{i_{t+1}}(v^i_s) dx^{i_1} \ldots \hat{dx}^{i_s} \ldots dx^{i_{k+1}} \langle e_{i_{k+1}} \wedge \ldots \wedge e_{i_t} \wedge \ldots \wedge e_{i_1}, \varphi \rangle,
\]
where the symbol $\hat{}$ over $dx^i$ (resp. $e_i$) means that the 1-form $dx^i$ (resp. the vector $e_i$) is omitted. The vector $v$ in the right hand side of (6.29) is considered as a differential operator on $A$.

Thus the operators (6.28) also belong to the image of $r_A$. This concludes the proof of the proposition. □

We remark that the sheaf
\[
\mathcal{P} = \Omega^{-\bullet}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \quad (6.30)
\]

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has the natural left $\mathcal{D}_{\Omega^+_X}$-module structure and the natural right $\mathcal{D}_X$-module structure.

Similarly, the sheaf
\[ Q = \mathcal{D}_X \otimes_{\mathcal{O}_X} V_X^* \]  
(6.31)
has the natural left $\mathcal{D}_X$-module structure and a natural right $\mathcal{D}_{\Omega^+_X}$-module structure.

Proposition 12 implies that
\[ P \otimes_{\mathcal{D}_X} Q \cong \mathcal{D}_{\Omega^+_X}. \]  
(6.32)
Furthermore, it is obvious that
\[ Q \otimes_{\mathcal{D}_{\Omega^+_X}} P \cong \mathcal{D}_X. \]  
(6.33)
Thus we arrive at the following statement

**Proposition 13** The sheaves $\mathcal{P}$ (6.30) and $\mathcal{Q}$ (6.31) establish a Morita equivalence between the sheaves of associative algebras $\mathcal{D}_{\Omega^+_X}$ and $\mathcal{D}_X$. □

Combining this statement with Proposition 11 we conclude that the sheaves $\mathcal{Y}_0(V_X^*)$ and $\mathcal{D}_X$ are Morita equivalent.

In order to get the sheaf of algebras $\mathcal{Y}(V_X^*)$ from $\mathcal{Y}_0(V_X^*)$ we need to tensor $\mathcal{Y}_0(V_X^*)$ with the constant sheaf
\[ \mathbb{K}[\delta]/(\delta^2), \quad \deg(\delta) = -1 \]
and impose the equation
\[ [\delta, P] = \hat{\delta}(P), \]  
(6.34)
where $\hat{\delta}$ is the derivation of $\mathcal{Y}_0(V_X^*)$ defined by
\[ \hat{\delta}(i_\gamma) = l_\gamma, \quad \hat{\delta}(l_\gamma) = 0, \]  
(6.35)
P is a local section of $\mathcal{Y}_0(V_X^*)$ and $\gamma$ is a local section of $V_X^*$.

On the other hand, for every local section $P$ of $\mathcal{Y}_0(V_X^*)$ we have
\[ \hat{\delta}(P) = [d, P], \]
where $d$ is the de Rham differential.

The de Rham differential $d$ is a global section of the sheaf $\mathcal{D}_{\Omega^+_X}$. Hence, due to Proposition 11 $d$ is a global section of $\mathcal{Y}_0(V_X^*)$.

Thus, switching from $\delta$ to
\[ \vartheta = \delta - d \]  
(6.36)
we get the following isomorphism of the sheaves of algebras
\[ \mathcal{Y}(V_X^*) \cong \mathcal{Y}_0(V_X^*)[\vartheta]/(\vartheta^2), \]  
(6.37)
where $\vartheta$ has degree $-1$ and
\[ [\vartheta, P] = 0 \]
for every local section $P$ of $\mathcal{Y}_0(V_X^*)$.

Combining this observation with Proposition 13 we arrive at the following statement.
Let $P$ and $Q$ be the sheaves defined in (6.30) and (6.31), respectively. The sheaves $P[d]/(d^2)$, $Q[d]/(d^2)$ establish a Morita equivalence between the sheaf of associative algebras $\mathcal{Y}(V^*_X)$ and the sheaf $D_X[d]/(d^2)$, where $d$ has degree $-1$ and $[d, D] = 0$ for every local section $D$ of $D_X$. □

6.4 Proof of Theorem 5

Let us recall that, due to Proposition 2, a homotopy calculus structure on the pair $(C^\bullet_{\text{norm}}(\mathcal{O}_X), C^\bullet_{\text{norm}}(\mathcal{O}_X))$ is a Maurer-Cartan element $Q$ of the DGLA $\text{Coder}^\prime(F_B(C^\bullet_{\text{norm}}(\mathcal{O}_X), C^\bullet_{\text{norm}}(\mathcal{O}_X)))$, (6.38)

where $B$ is as above the cooperad $\text{Bar(calc)}$ and the DGLA (6.38) consists of the coderivations of $F_B(C^\bullet_{\text{norm}}(\mathcal{O}_X), C^\bullet_{\text{norm}}(\mathcal{O}_X))$ satisfying the condition

$$Q \bigg|_{C^\bullet_{\text{norm}}(\mathcal{O}_X) \oplus C^\bullet_{\text{norm}}(\mathcal{O}_X)} = 0.$$  

The codifferential on the sheaf of coalgebras $F_B(C^\bullet_{\text{norm}}(\mathcal{O}_X), C^\bullet_{\text{norm}}(\mathcal{O}_X))$ is the sum

$$\partial = \partial^{\text{Bar}} + \partial^{\text{Hoch}},$$  

(6.39)

where $\partial^{\text{Bar}}$ comes from the bar differential on $B = \text{Bar(calc)}$ and $\partial^{\text{Hoch}}$ comes from the Hochschild (co)boundary operators on $C^\bullet_{\text{norm}}(\mathcal{O}_X)$ and $C^\bullet_{\text{norm}}(\mathcal{O}_X)$.

Using the Maurer-Cartan element $Q$ we shift the codifferential $\partial$ by $[Q, \cdot]$ and get the new codifferential on $F_B(C^\bullet_{\text{norm}}(\mathcal{O}_X), C^\bullet_{\text{norm}}(\mathcal{O}_X))$:

$$\partial^Q = \partial^{\text{Bar}} + \partial^{\text{Hoch}} + [Q, \cdot].$$  

(6.40)

Let us denote the resulting sheaf of DG B-coalgebras by $C_Q$:

$$C_Q = \left(F_B(C^\bullet_{\text{norm}}(\mathcal{O}_X), C^\bullet_{\text{norm}}(\mathcal{O}_X)), \partial^Q \right).$$  

(6.41)

We use $C_Q$ to get the canonical free resolution $\mathcal{R}$ of the sheaf $(C^\bullet_{\text{norm}}(\mathcal{O}_X), C^\bullet_{\text{norm}}(\mathcal{O}_X))$ of homotopy calculi. As a sheaf of calculi,

$$\mathcal{R} = F_{\text{calc}}(C_Q)$$  

(6.42)

---

14 The construction of this free resolution is known in topology as the rectification [1]. We describe this construction in more details in [13] (See Proposition 3 therein).
and the differential on $\mathcal{R}$ is
\[
\partial^{\mathcal{R}} = \partial^{\mathcal{W}} + \partial^{\mathcal{Q}},
\] (6.43)
where $\partial^{\mathcal{Q}}$ comes from the differential on $\mathcal{C}_Q$ and $\partial^{\mathcal{W}}$ is defined using the twisting cochain between operad $\mathsf{calc}$ and cooperad $B = \text{Bar}(\mathsf{calc})$. (See Section 2.3 in [17] on twisting cochain and the construction of the differential $\partial^{\mathcal{W}}$ for algebras over an abstract operad.)

The sheaf of DG $\mathsf{calc}$-algebras $\mathcal{R}$ splits according to the colors $(c,a)$ as
\[
\mathcal{R} = \mathcal{R}_c \oplus \mathcal{R}_a,
\]
where
\[
\mathcal{R}_a = F \mathsf{calc} \left( F \text{Bar}(\mathsf{calc})(C^\bullet_{\text{norm}}(\mathcal{O}_X), C^\bullet_{\text{norm}}(\mathcal{O}_X)) \right)_{\mathfrak{a}},
\] (6.44)
and
\[
\mathcal{R}_c = F_{e_2} \circ F \text{Bar}(e_2)(C^\bullet_{\text{norm}}(\mathcal{O}_X)).
\] (6.45)
Thus the sheaf $\mathcal{R}_c$ with the differential
\[
\left. \partial^{\mathcal{R}} \right|_{\mathcal{R}_c}
\]
is a free resolution of the sheaf $C^\bullet_{\text{norm}}(\mathcal{O}_X)$ of homotopy Gerstenhaber algebras. This resolution can be simplified. More precisely, we may consider the subsheaf
\[
F_{e_2} \circ F_{e_2}^{-\vee}(C^\bullet_{\text{norm}}(\mathcal{O}_X)) \subset \mathcal{R}_c
\] (6.46)
with the differential obtained by restricting the one on (6.45). Then, using the fact that the inclusion $\iota_{e_2}$ (3.3) is a quasi-isomorphism of cooperads one can show that the inclusion
\[
F_{e_2}(\iota_{e_2}) : F_{e_2} \circ F_{e_2}^{-\vee}(C^\bullet_{\text{norm}}(\mathcal{O}_X)) \hookrightarrow F_{e_2} \circ F \text{Bar}(e_2)(C^\bullet_{\text{norm}}(\mathcal{O}_X))
\] (6.47)
is a quasi-isomorphism of sheaves of DG Gerstenhaber algebras.

Thus the sheaf (6.46) is also a free resolution of the sheaf $C^\bullet_{\text{norm}}(\mathcal{O}_X)$ of homotopy Gerstenhaber algebras.

We denote the differential on the sheaf (6.46) by $\partial^{\mathcal{R}}_c$ and reserve the notation $\mathcal{R}(C^\bullet_{\text{norm}}(\mathcal{O}_X))$ for this resolution
\[
\mathcal{R}(C^\bullet_{\text{norm}}(\mathcal{O}_X)) = (F_{e_2} \circ F_{e_2}^{-\vee}(C^\bullet_{\text{norm}}(\mathcal{O}_X)), \partial^{\mathcal{R}}_c).
\] (6.48)

The quasi-isomorphism (6.47) provides the sheaf $\mathcal{R}_a$ (6.44) with a (DG) $\mathsf{calc}$-module structure over the sheaf $\mathcal{R}(C^\bullet_{\text{norm}}(\mathcal{O}_X))$ (6.48). Thus, in order to prove Theorem 5, we need to show that the sheaf $(\mathcal{R}(C^\bullet_{\text{norm}}(\mathcal{O}_X)), \mathcal{R}_a)$ of calculi is quasi-isomorphic to the sheaf $(V^\bullet_X, \Omega^\bullet_X)$.

In paper [13] we constructed a chain of quasi-isomorphisms of sheaves of DG Gerstenhaber algebras which connects the sheaf $\mathcal{R}(C^\bullet_{\text{norm}}(\mathcal{O}_X))$ to the sheaf $V^\bullet_X$.

In this construction we use the sheaf of $e_2^{-\vee}$-coalgebras:
\[
\Xi_X = F_{\Lambda^2\text{cocomm}} \circ F_{\Lambda\text{colie}^+}(\mathcal{O}_X, V^1_X),
\] (6.49)
where $\text{colie}^+$ is the cooperad which governs pairs “a Lie coalgebra + its comodule.”
We also use the canonical free resolution

\[(F_{e_2} \circ F_{e_2}(V_X^\bullet), \partial^R)\]  

(6.50)

of the sheaf of Gerstenhaber algebras \(V_X^\bullet\). Here the differential \(\partial^R\) on the sheaf (6.50) comes from the twisting cochain [17] of the pair \((e_2, e_2^\vee)\).

It is obvious that \(\Xi_X\) is a subsheaf of

\(F_{e_2^\vee}(\mathcal{C}^\bullet_{\text{norm}}(O_X))\).

and a subsheaf of

\(F_{e_2^\vee}(V_X^\bullet)\).

Due to this observation we have two inclusions of the sheaves of free Gerstenhaber algebras

\[\sigma_1 : F_{e_2}(\Xi_X) \hookrightarrow \mathcal{R}(\mathcal{C}^\bullet_{\text{norm}}(O_X)), \]

(6.51)

and

\[\sigma_2 : F_{e_2}(\Xi_X) \hookrightarrow F_{e_2} \circ F_{e_2^\vee}(V_X^\bullet). \]

(6.52)

It was shown in [13] that the sheaf \(F_{e_2^\vee}(\Xi_X)\) is closed both with respect to the differential \(\partial^R\) on \(\mathcal{R}(\mathcal{C}^\bullet_{\text{norm}}(O_X))\) and the differential \(\partial^R_{\mathcal{C}}\) on the sheaf (6.50). Furthermore, the restriction of the differential \(\partial^R_{\mathcal{C}}\) to \(F_{e_2}(\Xi_X)\) coincides with the restriction of the differential \(\partial^R\). In other words, the sheaf of free Gerstenhaber algebras \(F_{e_2}(\Xi_X)\) is equipped with a canonical differential.

Composing \(\sigma_2\) (6.52) with the quasi-isomorphism

\[F_{e_2} \circ F_{e_2^\vee}(V_X^\bullet) \sim V_X^\bullet\]

we arrive at the following pair of maps of sheaves of Gerstenhaber algebras:

\[V_X^\bullet \xleftarrow{\lambda} F_{e_2}(\Xi_X) \xrightarrow{\sigma_1} \mathcal{R}(\mathcal{C}^\bullet_{\text{norm}}(O_X)). \]

(6.53)

In [13] it was shown that both \(\lambda\) and \(\sigma_1\) are quasi-isomorphisms of complexes of sheaves.

The quasi-isomorphism \(\sigma_1\) in (6.53) provides the sheaf \(\mathcal{R}_a\) (6.44) with a (DG) calc-module structure over the sheaf \(F_{e_2}(\Xi_X)\). Thus, in order to prove Theorem 5 we need to show that the sheaf of calculi \((F_{e_2}(\Xi_X), \mathcal{R}_a)\) is quasi-isomorphic to the sheaf \((V_X^\bullet, \Omega_X^\bullet)\).

For this purpose we introduce the bar resolution of the sheaf \(\mathcal{R}_a\) of \(\mathcal{Y}(F_{e_2}(\Xi_X))\)-modules:

\[BR_a = \bigoplus_{k \geq 1} s^{1-k} \mathcal{Y}(F_{e_2}(\Xi_X))^\otimes k \otimes \mathcal{R}_a. \]

(6.54)

The map \(\lambda\) in (6.53) induces the following map of sheaves of associative algebras

\[\mathcal{Y}(\lambda) : \mathcal{Y}(F_{e_2}(\Xi_X)) \rightarrow \mathcal{Y}(V_X^\bullet). \]

(6.55)

Considering the map \(\mathcal{Y}(\lambda)\) on the level of stalks at a point \(x \in X\) we get the map of associative algebras

\[\mathcal{Y}(F_{e_2}(\Xi(A))) \rightarrow \mathcal{Y}(V^\bullet(A)), \]

(6.56)

where \(A\) is the local algebra at the point \(x\), \(\Xi(A) = \Xi_{\text{Spec}(A)}\) and \(V^\bullet(A) = V^\bullet_{\text{Spec}(A)}\).
Since the variety $X$ is smooth the local algebra $A$ and hence the graded commutative algebra $V^\bullet (A)$ is regular. Furthermore, the Gerstenhaber algebra $\mathbb{F}_{e_2}(\Xi(A))$ is a free resolution of $V^\bullet (A)$. Thus, due to Proposition 10, the map (6.56) a quasi-isomorphism. Hence (6.55) is a quasi-isomorphism of complexes of sheaves.

Recall that $\mathcal{B}R_a$ is the free resolution (6.54) of the sheaf $\mathcal{R}_a$ of $\mathcal{Y}(\mathbb{F}_{e_2}(\Xi_X))$-modules. Therefore, applying the functor $\otimes \mathcal{Y}(\mathbb{F}_{e_2}(\Xi_X)) \mathcal{B}R_a$ to the quasi-isomorphism (6.55) we get the quasi-isomorphism of sheaves of $\mathcal{Y}(\mathbb{F}_{e_2}(\Xi_X))$-modules:

$$\mathcal{B}R_a \sim \mathcal{Y}(V_X^\bullet) \otimes \mathcal{Y}(\mathbb{F}_{e_2}(\Xi_X)) \mathcal{B}R_a.$$ 

Thus we need to show that the sheaf of $(graded)$ commutative algebras $\Omega_X^{-\bullet}$ can be realized as a subsheaf of $\mathcal{Y}(V_X^\bullet)$. Indeed,

$$\Omega_X^{-\bullet} = \mathcal{Y}_0(\mathcal{O}_X) \subset \mathcal{Y}(V_X^\bullet).$$ 

Using the global section $1 \in \Gamma(X, C^0_{\text{norm}}(\mathcal{O}_X))$ we introduce the following global section

$$E = 1_{\mathcal{Y}(V_X^\bullet)} \otimes 1_{\mathcal{Y}(\mathbb{F}_{e_2}(\Xi_X))} \otimes 1 \in \Gamma(X, \mathcal{Y}(V_X^\bullet) \otimes \mathcal{Y}(\mathbb{F}_{e_2}(\Xi_X)) \mathcal{B}R_a)$$ 

of the sheaf $\mathcal{Y}(V_X^\bullet) \otimes \mathcal{Y}(\mathbb{F}_{e_2}(\Xi_X)) \mathcal{B}R_a$. Here $1 \in \Gamma(X, C^0_{\text{norm}}(\mathcal{O}_X))$ is also considered as a global section of the sheaf

$$\mathcal{R}_a = \mathbb{F}_{\text{calc}}\left(\mathbb{F}_B(C^\bullet_{\text{norm}}(\mathcal{O}_X), C^\bullet_{\text{norm}}(\mathcal{O}_X))\right)_a,$$

via the unit of the operad $\text{calc}$ and the coaugmentation of the cooperad $B$.

It is obvious that $E$ is closed with respect to the total differential on

$$\Gamma(X, \mathcal{Y}(V_X^\bullet) \otimes \mathcal{Y}(\mathbb{F}_{e_2}(\Xi_X)) \mathcal{B}R_a).$$

Furthermore, for every point $x \in X$ the cohomology class of the germ $E_x$ corresponds to the cohomology class of the germ $1_x$.

Using the cycle $E$ and equation (6.58) we define the following map of sheaves

$$\nu : \Omega_X^{-\bullet} \rightarrow \mathcal{Y}(V_X^\bullet) \otimes \mathcal{Y}(\mathbb{F}_{e_2}(\Xi_X)) \mathcal{B}R_a,$$

$$\nu(a \, db_1 \, db_2 \ldots \, db_m) = (i_a \, l_{b_1} \, l_{b_2} \ldots \, l_{b_m} : E),$$ 

where $a, b_1, b_2, \ldots, b_m$ are local sections of the structure sheaf $\mathcal{O}_X$.

It is obvious that $\nu$ is compatible with $\Omega_X^{-\bullet}$-module structures.

We claim that

**Proposition 15** The map $\nu$ (6.60) is a quasi-isomorphism of complexes of sheaves.

**Proof.** Indeed, let us consider the corresponding map of stalks

$$\nu_x : (\Omega_X^{-\bullet})_x \rightarrow \left(\mathcal{Y}(V_X^\bullet) \otimes \mathcal{Y}(\mathbb{F}_{e_2}(\Xi_X)) \mathcal{B}R_a\right)_x.$$
at a point \( x \in X \). Let \( a_1, b_2, \ldots, b_m \) be germs of functions on \( X \) at \( x \).

Since the cohomology class of the germ \( E_x \) corresponds to the cohomology class of the germ \( 1_x \in C^0(\mathcal{O}_X)_x \) therefore the cohomology class of the element

\[
(i_a b_1 l_2 \ldots l_m, \ E_x)
\]
corresponds to the cohomology class of the Hochschild cycle

\[
\sum_{\sigma \in S_m} (-1)^{|\sigma|} (a, b_{\sigma(1)}, b_{\sigma(2)}, \ldots, b_{\sigma(m)}) = L_a L_{b_1} L_{b_2} \ldots L_{b_m} 1_x \in C^m_\text{norm}(A).
\]

Furthermore, under Connes-Hochschild-Kostant-Rosenberg map \((6.17)\) the chain \((6.61)\) goes at a point \( x \in \mathcal{Y}(\mathcal{O}_X) \) due to the isomorphism \((6.25)\) the sheaf \( D_{\Omega_X^\bullet} \) is isomorphic to the sheaf \( \Omega_X \).

Let us also recall that \( D_{\Omega_X^\bullet} \) is the sheaf of differential operators on exterior forms and \( \partial \) is an auxiliary variable of degree \(-1\) which commutes with local sections of \( D_{\Omega_X^\bullet} \). Let us also recall that due to the isomorphism \((6.25)\) the sheaf \( \Omega_X \) has a natural structure of left \( D_{\Omega_X^\bullet} \)-module.

It is not hard to see that for every sheaf \( \mathcal{M} \) of \( D_{\Omega_X^\bullet} \)-modules we have the natural isomorphism of sheaves of \( \mathcal{O}_X \)-modules

\[
\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{V}_X^{\bullet}[\partial]/(\partial^2) \otimes_{D_{\Omega_X^\bullet}[\partial]/(\partial^2)} \mathcal{M} \cong \mathcal{O}_X \otimes_{\Omega_X^\bullet} \mathcal{M},
\]

where the \( \Omega_X^\bullet \)-module structure on \( \mathcal{O}_X \) is given by the equation

\[
f \eta = \langle \eta, f \rangle, \quad \eta \in \Gamma(U, \Omega_X^\bullet), \quad f \in \Gamma(U, \mathcal{O}_X),
\]

and the pairing \( \langle \cdot, \cdot \rangle \) is defined in \((6.13)\).

Having in mind Proposition \((6.14)\) we apply the functor

\[
\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{V}_X^{\bullet}[\partial]/(\partial^2) \otimes_{D_{\Omega_X^\bullet}[\partial]/(\partial^2)}
\]
to the map \( \nu \) \((6.60)\) and get the following quasi-isomorphism of sheaves of \( \mathcal{O}_X \)-modules

\[
\bar{\nu} : \mathcal{O}_X \sim \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{V}_X^{\bullet}[\partial]/(\partial^2) \otimes_{\mathcal{Y}(\mathcal{F}_g(\mathcal{E}_X)))} \mathcal{B}\mathcal{R}_a,
\]

where the right \( \mathcal{Y}(\mathcal{F}_g(\mathcal{E}_X))) \)-module structure on \( \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{V}_X^{\bullet}[\partial]/(\partial^2) \) is obtained from the right \( \mathcal{Y}(\mathcal{V}_X^\bullet) \)-module structure via the map \( \lambda \) in \((6.53)\).

It is not hard to see that the \( \mathcal{D}_X \)-module structure on

\[
\mathcal{O}_X \cong \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{V}_X^{\bullet}[\partial]/(\partial^2) \otimes_{D_{\Omega_X^\bullet}[\partial]/(\partial^2)} \Omega_X^{-\bullet}
\]
is the standard one. Furthermore, the element \( \partial \) acts on the sections of the sheaf \( \mathcal{O}_X \) in \((6.64)\) by zero simply because \( \partial \) has degree \(-1\) and \( \mathcal{O}_X \) is concentrated in the single degree 0.

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Let us denote the target of the map $\tilde{\nu}$ (6.63) by $G$:

$$G = D_X \otimes_{O_X} V^*_X[\mathfrak{d}]/(\mathfrak{d}^2) \otimes_{\mathcal{Y}(\mathfrak{F}_{\mathfrak{H}_2}(\mathcal{E}))} B\mathcal{R}_a. \quad (6.65)$$

We also denote by $\partial^G$ the total differential on this sheaf.

For the next proposition we will need the Cech resolution $\tilde{\mathcal{C}}^\bullet(G)$ of the sheaf of $D_X[\mathfrak{d}]/(\mathfrak{d}^2)$-modules $G$ in the category of sheaves.

**Proposition 16** The map (6.63) extends to an $A_\infty$ quasi-isomorphism $\Upsilon$ from the sheaf of $D_X[\mathfrak{d}]/(\mathfrak{d}^2)$-modules $O_X$ to the the Cech resolution $\tilde{\mathcal{C}}^\bullet(G)$ of the sheaf $G$ (6.65).

**Proof.** First, let us prove that the map $\tilde{\nu}$ (6.63) is compatible with the action of the sheaf $D_X[\mathfrak{d}]/(\mathfrak{d}^2)$ up to homotopy on the level of stalks.

Let $A$ be the stalk $(O_X)_x$ of the structure sheaf $O_X$ at a point $x$. Since $X$ is smooth, $A$ is a local regular algebra.

Let, as above, $x_1, \ldots, x_n$ be a regular system of parameters in $A$. The module $\Omega^1(A)$ of Kähler differentials is freely generated by the 1-forms (6.22) and the module of derivations $\text{Der}(A)$ is freely generated by (6.24).

Since the algebra $D(A)[\mathfrak{d}]/(\mathfrak{d}^2)$ is generated by $A$, derivations of $A$ and the element $\mathfrak{d}$ it suffices to show that the action of the element $\mathfrak{d}$ and a derivation $v$ of $A$ sends the element

$$1_{D(A) \otimes_{O_X} V^*(A)[\mathfrak{d}]/(\mathfrak{d}^2)} \otimes_{\mathcal{Y}(V^*(A))} E_x$$

(6.66)

to cohomologically trivial elements of the chain complex

$$G_x = D(A) \otimes_{O_X} V^*(A)[\mathfrak{d}]/(\mathfrak{d}^2) \otimes_{\mathcal{Y}(V^*(A))} \mathcal{Y}(V^*(A)) \otimes_{\mathcal{Y}(\mathfrak{F}_{\mathfrak{H}_2}(\mathcal{E}))} (B\mathcal{R}_a)_x. \quad (6.67)$$

The isomorphism

$$\mathcal{Y}(V^*(A)) \cong \Omega^{-\bullet}(A) \otimes_A D(A) \otimes_A V^*(A)[\mathfrak{d}]/(\mathfrak{d}^2)$$

allows us to consider $\mathfrak{d}$ and the derivation $v \in \text{Der}(A)$ as elements of the algebra $\mathcal{Y}(V^*(A))$. Thus we need to show that the cocycles

$$v E_x$$

and

$$\mathfrak{d} E_x$$

are cohomologically trivial.

Since the map $\tilde{\nu}$ (6.63) is a quasi-isomorphism of complexes of sheaves, the cohomology of the complex (6.67) is concentrated in the degree 0. Hence, the cocycle $\mathfrak{d} E_x$ is a coboundary because it has degree $-1$.

Next, using the generators (6.22) and (6.24) we rewrite the cocycle $v E_x$ as

$$v E_x = \frac{1}{n!} \prod_{k=1}^n \left( k - \sum_{j=1}^n l_{x_j} e_j \right) E_x, \quad (6.68)$$

where the element

$$\frac{1}{n!} \prod_{k=1}^n \left( k - \sum_{j=1}^n l_{x_j} e_j \right) \in D(\Omega^{-\bullet}(A))$$
operates as a projection on the degree 0 forms.

Since the cohomology class of $E_x$ corresponds to the cohomology class of $1_x = 1 \in \mathcal{C}_0^{\text{norm}}(A)$ the cohomology class of the element (6.68) corresponds to the class of

$$\frac{1}{n!} L_v \prod_{k=1}^{n} \left( k - \sum_{j=1}^{n} L_{x_j} I_{e_j} \right) 1_x \in \mathcal{C}_0^{\text{norm}}(A).$$

It is easy to see that

$$\frac{1}{n!} L_v \prod_{k=1}^{n} \left( k - \sum_{j=1}^{n} L_{x_j} I_{e_j} \right) 1_x = 0.$$

Thus the map $\tilde{\nu}$ (6.63) is indeed compatible with the action of the sheaf $\mathcal{D}_X[\mathfrak{d}]/(\mathfrak{d}^2)$ up to homotopy on the level of stalks.

An $A_\infty$ morphism from the sheaf of $\mathcal{D}_X[\mathfrak{d}]/(\mathfrak{d}^2)$-modules $\mathcal{O}_X$ to the sheaf $\mathcal{C}^\bullet(\mathcal{G})$ is the degree 0 element

$$\Upsilon \in \bigoplus_{k \geq 0} s^k \text{Hom} \left( (\mathcal{D}_X[\mathfrak{d}]/(\mathfrak{d}^2))^\otimes k \otimes \mathcal{O}_X, \mathcal{C}^\bullet(\mathcal{G}) \right)$$

satisfying the cocycle condition

$$(\partial^{\mathfrak{d}} + \partial + \mathcal{D}^{Hoch}) \Upsilon = 0,$$

where $\partial^{\mathfrak{d}}$ is the differential on the sheaf (6.65), $\partial$ is the Cech differential and $\mathcal{D}^{Hoch}$ is the Hochschild coboundary operator of the sheaf $\mathcal{D}_X[\mathfrak{d}]/(\mathfrak{d}^2)$ with values in the sheaf of bimodules $\mathcal{H}om(\mathcal{O}_x, \mathcal{C}^\bullet(\mathcal{G})).$

In other words, $\Upsilon$ can be defined by the infinite collection of maps

$$\Upsilon_k \in \text{Hom} \left( (\mathcal{D}_X[\mathfrak{d}]/(\mathfrak{d}^2))^\otimes k \otimes \mathcal{O}_X, \mathcal{C}^\bullet(\mathcal{G}) \right), \quad k = 0, 1, 2 \ldots$$

such that $\Upsilon_k$ has degree $-k$ and all the maps satisfy the equations

$$(\partial^{\mathfrak{d}} + \partial) \Upsilon_{k+1} + \mathcal{D}^{Hoch} \Upsilon_k = 0.$$

Our purpose is to show that there is exists an $A_\infty$-morphism $\Upsilon$ with

$$\Upsilon_0 = \tilde{\nu}.$$

Let us show that there exists $\Upsilon_1$ satisfying the equation

$$(\partial^{\mathfrak{d}} + \partial) \Upsilon_1 + \mathcal{D}^{Hoch} \Upsilon_0 = 0.$$

We will find $\Upsilon_1$ by induction in degrees of the Cech complex. In general,

$$\Upsilon_1 = \sum_{q=0}^{\infty} \Upsilon_1^q,$$

where

$$\Upsilon_1^q \in \text{Hom} \left( \mathcal{D}_X[\mathfrak{d}]/(\mathfrak{d}^2) \otimes \mathcal{O}_X, \mathcal{C}^q(\mathcal{G}^{1-q}) \right)$$

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and the equation \((6.73)\) is equivalent to
\[
\partial^G \Upsilon^0_1 + D^{Hoch} \Upsilon_0 = 0.
\]
\[(6.76)\]
\[
\partial^G \Upsilon^q_1 + \tilde{\partial} \Upsilon^q_1 = 0, \quad q = 0, 1, 2, \ldots.
\]
\[(6.77)\]

If we set \(\Upsilon_0 = \tilde{\nu}\) then there exists a map \(\Upsilon^0_1\) satisfying equation \((6.76)\) because \(\tilde{\nu}\) is compatible with the action of \(D_X[\mathfrak{d}]/(\partial^2)\) up to homotopy on the level of stalks.

Due to equation \((6.76)\) the element \(\tilde{\partial} \Upsilon^0_1\) is closed with respect to \(\partial^G\)
\[
\partial^G (\tilde{\partial} \Upsilon^0_1) = 0.
\]

But
\[
\tilde{\partial} \Upsilon^0_1 \in \text{Hom} \left( D_X[\mathfrak{d}]/(\partial^2) \otimes \mathcal{O}_X , \mathcal{C}^1(G^{-1}) \right).
\]

Thus, using the fact that the cohomology of the stalk \(G_x\) is concentrated in degree 0 we conclude that there exists the next map \(\Upsilon^1_1\) in \((6.75)\) satisfying the equation
\[
\partial^G \Upsilon^1_1 + \tilde{\partial} \Upsilon^0_1 = 0.
\]

This is the base of the induction.

Let us now assume that for \(m > 0\) there exists the collection of maps \(\Upsilon^q_1\) \((6.75)\) for \(q < m\) satisfying equation \((6.77)\) for \(q < m - 1\). Then, due to equation \((6.77)\) for \(q = m - 2\) the map \(\tilde{\partial} \Upsilon^{m-1}_1\) is closed with respect to the differential \(\partial^G\):
\[
\partial^G (\tilde{\partial} \Upsilon^{m-1}_1) = 0.
\]

But
\[
\tilde{\partial} \Upsilon^{m-1}_1 \in \text{Hom} \left( D_X[\mathfrak{d}]/(\partial^2) \otimes \mathcal{O}_X , \mathcal{C}^m(G^{-m}) \right).
\]

Thus, using the fact that the cohomology of the stalk \(G_x\) is concentrated in degree 0 we conclude that there exists the next map \(\Upsilon^m_1\) in \((6.75)\) satisfying equation \((6.77)\) for \(q = m - 1\).

We proved the existence of the map \(\Upsilon_1\) in \((6.71)\) satisfying equation \((6.72)\) for \(k = 0\).

Now we proceed by induction on \(k\) in \((6.71)\) and \((6.72)\).

Let us assume that \(\Upsilon_k\) \((6.71)\) are constructed for \(k < m\) and equation \((6.72)\) holds for \(k < m - 1\). Then equation \((6.72)\) for \(k = m - 2\) implies that the element
\[
D^{Hoch} \Upsilon_{m-1} \in \text{Hom} \left( (D_X[\mathfrak{d}]/(\partial^2))^{\otimes (m-1)} \otimes \mathcal{O}_X , \mathcal{C}^{\bullet}(\mathcal{G}) \right)
\]
\[(6.78)\]
is closed with respect to the differential \(\partial^G + \tilde{\partial}\).

Since the sheaf \(\mathcal{C}^{\bullet}(\mathcal{G})\) is acyclic with respect to the functor of global sections the map \(\tilde{\nu}\) \((6.63)\) induces the quasi-isomorphism between the chain complex
\[
\text{Hom} \left( (D_X[\mathfrak{d}]/(\partial^2))^{\otimes (m-1)} \otimes \mathcal{O}_X , \mathcal{C}^{\bullet}(\mathcal{G}) \right)
\]
\[(6.79)\]
and the chain complex
\[
\text{Hom} \left( (D_X[\mathfrak{d}]/(\partial^2))^{\otimes (m-1)} \otimes \mathcal{O}_X , \mathcal{C}^{\bullet}(\mathcal{O}_X) \right).
\]
\[(6.80)\]
It is obvious that the cohomology of the latter complex is concentrated only in non-negative degrees.

On the other hand the cocycle (6.78) has the negative degree $-m + 1$. Hence there exists the next map $\Upsilon_m$ satisfying equation (6.72) for $k = m - 1$.

Proposition 16 is proved. □

Thus the sheaf (6.64) of $\mathcal{D}_X[\partial]/(\partial^2)$-modules is quasi-isomorphic to the sheaf

$$\mathcal{G} = \mathcal{D}_X \otimes_{\mathcal{O}_X} V_X^*[\partial]/(\partial^2) \otimes_{\mathcal{Y}(\mathcal{R}_a)} \mathcal{B}\mathcal{R}_a.$$ Combining this observation with Proposition 14 we see that the sheaves

$$\Omega_{X}^{\bullet}$$

and

$$\mathcal{Y}(V_X^\bullet) \otimes_{\mathcal{Y}(\mathcal{R}_a)} \mathcal{B}\mathcal{R}_a$$

are quasi-isomorphic as sheaves of $\mathcal{Y}(V_X^\bullet)$-modules.

Theorem 5 is proved. □

7 Applications and generalizations

Let, as above, $X$ be a smooth algebraic variety over a field $\mathbb{K}$ of characteristic zero. The homotopy calculus algebra on the pair $(C_{\text{norm}}^\bullet(X), C_{\text{norm}}^\bullet(X))$ gives us a comm$^+$-module structure on the pair $(V_X^\bullet, \Omega_X^\bullet)$. Theorem 3 implies that this comm$^+$-module structure on $(V_X^\bullet, \Omega_X^\bullet)$ is given by the $\wedge$-product of polyvector fields and contraction of polyvectors with forms.

According to [27] and [31] the Hochschild cohomology $HH^\bullet(X)$ of the variety $X$ is the hypercohomology of the sheaf $C_{\text{norm}}^\bullet(\mathcal{O}_X)$:

$$HH^\bullet(X) = \mathbb{H}^\bullet(C_{\text{norm}}^\bullet(\mathcal{O}_X)).$$ Furthermore, according to [8], the Hochschild homology $HH_\bullet(X)$ of the variety $X$ is the hypercohomology of the sheaf $C_{\text{norm}}^\bullet(\mathcal{O}_X)$

$$HH_\bullet(X) = \mathbb{H}^\bullet(C_{\text{norm}}^\bullet(\mathcal{O}_X)).$$

Thus, using Theorem 5 we get the following generalization of Corollary 2 from [13]

**Corollary 2** For every smooth algebraic variety $X$ over a field $\mathbb{K}$ of characteristic zero the comm$^+$-algebras

$$(H^\bullet(X, V_X^\bullet), H^\bullet(X, \Omega_X^\bullet))$$

and

$$(HH^\bullet(X), HH_\bullet(X))$$

are isomorphic. □
This statement is the existence part of Caldararu’s conjecture [7] on the Hochschild structure of a smooth algebraic variety. The cohomological part of this conjecture was proved in [4]. As far as we know, D. Calaque, C. Rossi, and M. Van den Bergh are currently writing an article [3] with a proof of homological part of Caldararu’s conjecture.

Combining Theorem 4 with Theorem 5 we deduce the statement of cyclic formality conjecture (see Conjecture 3.3.2 in [31]) from [31] for an arbitrary smooth algebraic variety over a field $\mathbb{K}$ of characteristic zero:

**Corollary 3 (T. Willwacher, [33])** If $X$ is a smooth algebraic variety a field $\mathbb{K}$ of characteristic zero then the sheaf of $\Lambda^+$-algebras $(C^\bullet(O_X), C^\bullet(O_X))$ is formal.

**Remark.** Strictly speaking the methods used by T. Willwacher in [33] require an additional assumption $\mathbb{R} \subset \mathbb{K}$. Theorems 4 and 5 allow us to remove the assumption $\mathbb{R} \subset \mathbb{K}$ from the statement of Corollary 3.

The proof of Theorem 5 can be easily modified for the following two cases:

- $X$ is complex manifold with $O_X$ being the sheaf of holomorphic functions,
- $X$ is a real manifold with $O_X$ being the sheaf of $C^\infty$ functions.

Thus we get the following obvious modification of Theorem 5

**Theorem 6** If $X$ is a complex manifold (resp. real manifold) with $O_X$ being the sheaf of holomorphic functions (resp. the sheaf of $C^\infty$ real functions) then the sheaf

$$(C^\bullet_{\text{norm}}(O_X), C^\bullet_{\text{norm}}(O_X))$$

of homotopy calculi is quasi-isomorphic to the sheaf $(V^\bullet_X, \Omega^{-\bullet}_X)$ of calculi.

For $C^\infty$ real case we also get the following statement

**Corollary 4** If $X$ is a real manifold with $O_X$ being the sheaf of $C^\infty$ functions then the homotopy calculus algebra

$$\left(\Gamma(X, C^\bullet_{\text{norm}}(O_X)), \Gamma(X, C^\bullet_{\text{norm}}(O_X))\right)$$

is quasi-isomorphic to the calculus algebra

$$\left(\Gamma(X, V^\bullet_X), \Gamma(X, \Omega_X^{-\bullet})\right).$$

**Proof.** In the $C^\infty$ real case the chain of quasi-isomorphisms connecting the sheaves

$$(C^\bullet_{\text{norm}}(O_X), C^\bullet_{\text{norm}}(O_X))$$

and

$$(V^\bullet_X, \Omega_X^{-\bullet})$$

consists of soft sheaves. Hence, applying the functor $\Gamma(X, )$ of global sections we get the desired result. □

We would like to mention recent papers [5] and [8]. In paper [8] A. Cattaneo and G. Felder consider the DG Lie algebra module $CC^\bullet_{\text{neg}}(O_X)$ of negative cyclic chains over the
DGLA $\mathcal{C}^\ast(\mathcal{O}_X)$ of Hochschild cochains on a $C^\infty$ real manifold equipped with a volume form. Using an interesting modification of the Poisson sigma model A. Cattaneo and G. Felder construct a curious $L_\infty$ morphism (not a quasi-isomorphism!) from this DG Lie algebra module to a DG Lie algebra module modeled on polyvector fields using the volume form. A. Cattaneo and G. Felder also apply this result to a construction of a specific trace on the deformation quantization algebra of a unimodular Poisson manifold. Although this trace can be constructed using the formality quasi-isomorphism for Hochschild chains [25], [33], the relation of the $L_\infty$ morphism of A. Cattaneo and G. Felder to the formality quasi-isomorphism is a mystery.

Paper [5] is devoted to the proof of Kontsevich’s cyclic formality conjecture for cochains formulated in paper [26]. We suspect that the statement of this conjecture may be related to Theorem [7] and Corollary [8] via the Van den Bergh duality [32] between Hochschild cohomology and Hochschild homology.

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**Department of Mathematics, University of California at Riverside,**
**900 Big Springs Drive,**
**Riverside, CA 92521, USA**

*E-mail address:* vald@math.ucr.edu

**Mathematics Department, Northwestern University,**
**2033 Sheridan Rd.,**
**Evanston, IL 60208, USA**

*E-mail addresses:* tamarkin@math.northwestern.edu, tsygan@math.northwestern.edu