Entanglement-assisted classical information capacity of the amplitude damping channel

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Abstract

In this paper, we calculate the entanglement-assisted classical information capacity of amplitude damping channel and compare it with the particular mutual information which is considered as the entanglement-assisted classical information capacity of this channel in Ref. 6. It is shown that the difference between them is very small. In addition, we point out that using partial symmetry and concavity of mutual information derived from dense coding scheme one can simplify the calculation of entanglement-assisted classical information capacities for non-unitary-covariant quantum noisy channels.

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Entanglement-assisted classical information capacity of the quantum channel describes the maximal rate, i.e. information sent per channel usage, when we use dense coding scheme instead of simple encoding and decoding to transmit the data through the channel $\epsilon$. In the scheme, the sender, say Alice, and receiver, say Bob, share a two-qubit entangled state prior to the transmission. At first, Alice encodes information to be transmitted in an entangled state by operating her holding qubit, and then she sends the qubit through a quantum channel to Bob, finally Bob jointly measure two qubits (one is sent from Alice and the other is held by him at the beginning of this scheme) to decode the information. If no noise to be considered the scheme called dense coding, and can transmit two bit classical information by sending one qubit. However, in fact quantum noise is always exist. When we consider the effect of noise, is this scheme still superior to the traditional simple encoding and decoding scheme? If yes, then how superior is the scheme to the traditional one? Up to now, these problems could only be concretely answered by calculating the entanglement-assisted classical information capacities for some concrete quantum noisy channels. So developing

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the method of calculating the capacity is an interesting topic. This problem was first investigated by Bennett, Shor, Smolin, and Thapliyal (BSST) in [1], where the depolarizing and erasure channels in \( d \) dimensions were studied exactly. In Ref. [2] the same authors proposed a remarkable simple formula for calculating the entanglement-assisted classical information capacity in terms of the maximal mutual information between Alice and Bob, and the capacity of the amplitude damping channel was also investigated. In this paper, we shall at first recalculate the entanglement-assisted classical information capacity of amplitude damping channel. Then we shall compare it with the particular mutual information which is taken as the entanglement-assisted classical information capacity of the amplitude damping channel in Ref. 6. In addition, we shall summarize the method for calculating the entanglement-assisted classical information capacity of this kind of non-unitary-covariant channels. Let’s review BSST theorem [2] \[3\] first.

In the BSST theorem, the entanglement-assisted classical information capacity \( C(\varepsilon) \) of a quantum noisy channel \( \varepsilon : B(H) \rightarrow B(H) \) is given by

\[
C(\varepsilon) = \sup_{\rho} I(\varepsilon, \rho),
\]

where

\[
I(\varepsilon, \rho) = S(\rho) + S(\varepsilon(\rho)) - S(\varepsilon, \rho).
\]

Here, \( S(\tau) \) denotes von Neumann entropy, \( S(\varepsilon, \tau) \) denotes entropy exchange. Nielsen et al. in [4] proposed a method for calculating the entropy exchange, namely,

\[
S(\varepsilon, \rho) = S(\Omega) = -tr(\Omega \log(\Omega)),
\]

where \( \Omega_{ij} = tr\left(E_i \rho E_j^\dagger\right) \), and \( E_i \) denote Kraus operators of the channel \( \varepsilon \). The proof of this theorem was first given by [2] and then improved by Holevo in [3]. However, the calculation of the entanglement-assisted classical information capacity may still be a difficult problem for some quantum noisy channels because in order to maximize the mutual information \( I(\varepsilon, \rho) \), we must choose the state \( \rho \) in Eq.(1) over all of the possible states. Fortunately, \( I(\varepsilon, \rho) \) is a concave function so if only we can prove the in question channel being a unitary covariant channel the calculation become easy [3]. However, some quantum noisy channels are not unitary covariant, so we cannot calculate their capacities by simply replace \( \rho \) with the maximally mixed state \( 1/d \) in Eq.(2), where \( 1 \) is the unitary matrix, \( d \) is the dimension of the channel. However, for the non-unitary covariant channel the concavity and the partial symmetry of \( I(\varepsilon, \rho) \) can still be used in the calculation.

In the quantum communication, the following problems are always encountered. What are the dynamics of an atom which is spontaneously emitting a photon? How does a spin system at high temperature approach equilibrium with its environment? What is the state of a photon in an interferometer or cavity when it is subject to scattering and attenuation? Each of these processes has its own unique features, but the general behavior of all of them is well
characterized by a quantum operation known as amplitude damping. For the qubit systems, the evolvement of the amplitude damping can be modeled by amplitude damping channel. The amplitude damping channel is a non-unitary covariant channel. In the following, we shall use the concavity and the partial symmetry of $I(\varepsilon, \rho)$ investigate its entanglement-assisted classical information capacity. The Kraus operators of amplitude damping channel are

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1 - \eta} \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & \sqrt{\eta} \\ 0 & 0 \end{pmatrix}.$$  \hspace{1cm} (4)$$

Suppose the initial state $\rho \in \mathcal{H} = C^2$ is $\rho = \frac{1}{2} (I + \vec{w} \cdot \vec{\sigma})$, its eigenvalues are

$$\lambda_{1,2} = \frac{1}{2} \left( 1 \pm \sqrt{w_1^2 + w_2^2 + w_3^2} \right).$$  \hspace{1cm} (5)$$

When the initial state pass through the amplitude damping channel, it will become

$$\rho' = \varepsilon (\rho) = \frac{1}{2} \begin{pmatrix} 1 + w_3 + \eta (1 - w_3) & \sqrt{1 - \eta} (w_1 - iw_2) \\ \sqrt{1 - \eta} (w_1 + iw_2) & (1 - \eta) (1 - w_3) \end{pmatrix}.$$  \hspace{1cm} (6)$$

The eigenvalues of $\rho'$ are

$$\lambda'_{1,2} = \frac{1}{2} \pm \sqrt{(1 - \eta)^2 w_3^2 + (1 - \eta) (2\eta w_3 + w_1^2 + w_2^2) + \eta^2}. \hspace{1cm} (7)$$

By using the Kraus operators of amplitude damping channel and formula $\Omega_{ij} = tr \left( E_i \rho E_j^\dagger \right)$, we obtain

$$\Omega = \frac{1}{2} \begin{pmatrix} 2 - \eta + \eta w_3 & \sqrt{\eta} (w_1 - iw_2) \\ \sqrt{\eta} (w_1 + iw_2) & \eta (1 - w_3) \end{pmatrix}. \hspace{1cm} (8)$$

The eigenvalues of $\Omega$ are

$$\chi_{1,2} = \frac{1}{2} \pm \sqrt{(1 - \eta)^2 + 2\eta (1 - \eta) w_3 + \eta w_1^2 + \eta w_2^2 + \eta^2 w_3^2}. \hspace{1cm} (9)$$

From Eq. 8 we can obtain the mutual information as

$$I(\varepsilon, \rho) = I(\eta, w)$$

$$= -\lambda_1 \log_2 \lambda_1 - \lambda_2 \log_2 \lambda_2 - \lambda'_1 \log_2 \lambda'_1 - \lambda'_2 \log_2 \lambda'_2 + \chi_1 \log_2 \chi_1 + \chi_2 \log_2 \chi_2. \hspace{1cm} (10)$$

On one hand, from above results we have $I(\eta, w_1) = I(\eta, -w_1)$, and $I(\eta, w_2) = I(\eta, -w_2)$, so we see $I(\eta, \vec{w})$ being symmetrical on points $p(w_1, w_2 = 0, w_3)$. On the other hand, it was proven that $I(\eta, \vec{w})$ is a concave function \[\[\], so the maximum of $I(\eta, \vec{w})$ must be restricted on the points $p(w_1, w_2 = 0, w_3)$; the maximum of $I(\eta, w)$ must be included in $I'(\varepsilon, \vec{w}) := I|_{w_1, w_2 = 0} (\eta, w_3)$, namely, $C(\eta) \subset I'(\eta, \vec{w}) = I|_{w_1, w_2 = 0} (\eta, w_3)$. Further, we can calculate the capacities...
by taking a series of $w'_3$ in different $\eta$. These $w'_3$ in different $\eta$ can be obtained as follows: first we take $I'(\eta, \vec{w})$ derivative with respect to $w_3$ as
\[
\frac{dI'(\eta, \vec{w})}{dw_3} = 0,
\]
then we solve $w_3$ from Eq. (11) we can obtain a series of $w_3$, which are $w'_3$. The numerical result of $w'_3$ is shown in Table 1 and their values as a function of $\eta$ are plotted in Figure 1. The capacities $C(\eta)$, mutual information $I(\eta, w = 0)$ and the difference of $C(\eta)$ and $I(\eta, w = 0)$ are also given in the Table 1. We plot the $I(\eta, w = 0)$ and $C(\eta)$ against $\eta$ in Fig. 2. It is shown that the difference between $C(\eta)$ and $I(\eta, w = 0)$ is very small and we cannot distinguish them in the figure. In order to compare them we plot the difference $C(\eta) - I(\eta, w = 0)$ in Fig. 3.

**Fig.1**

Fig.1 $w'_3$ versus $\eta$ for the amplitude damping channel, where $w'_3$ make the mutual information $I(\eta, w_3)$ be capacities $C(\eta)$.

**Fig.2**

Fig.2 Capacity $C(\eta)$ and mutual information $I(\eta, w = 0)$ versus $\eta$ for the amplitude damping channel. Their difference being very small; we cannot distinguish them using this figure.

**Fig.3**

Fig.3 Difference of capacity $C(\eta)$ and mutual information $I(\eta, w = 0)$ versus $\eta$
for amplitude damping channel.

| η    | \(w'_3\) | \(C(\eta, w'_3)\) | \(I(\eta, w_3 = 0)\) | \(C - I\)  |
|------|-----------|-------------------|----------------------|------------|
| 0.04 | 0.2020757505 | 1.857993856 | 1.857404993 | 0.588663e-3 |
| 0.08 | 0.294514349 | 1.754220384 | 1.753086250 | 0.1134134e-2 |
| 0.12 | 0.342043493 | 1.663598636 | 1.662142602 | 0.1456034e-2 |
| 0.16 | 0.36476402 | 1.580849799 | 1.579274705 | 0.1575094e-2 |
| 0.20 | 0.36918238 | 1.503488311 | 1.501955000 | 0.1533311e-2 |
| 0.24 | 0.35871433 | 1.430055143 | 1.428611156 | 0.1373987e-2 |
| 0.28 | 0.33529523 | 1.359582064 | 1.35844378 | 0.1137686e-2 |
| 0.32 | 0.300001598 | 1.291370839 | 1.290509150 | 0.861689e-3 |
| 0.36 | 0.25341559 | 1.224884751 | 1.224304412 | 0.580339e-3 |
| 0.40 | 0.19562439 | 1.159688417 | 1.159362804 | 0.325613e-3 |
| 0.44 | 0.12643065 | 1.095410976 | 1.095283308 | 0.127668e-3 |
| 0.48 | 0.044530009 | 1.031721423 | 1.031706094 | 0.15329e-4 |
| 0.52 | -0.0848604556 | 0.9683103674 | 0.9682939063 | 0.16411e-4 |
| 0.56 | -0.156655130 | 0.9048748897 | 0.9047166920 | 0.1581977e-3 |
| 0.60 | -0.280492412 | 0.841104189 | 0.8406371958 | 0.4669891e-3 |
| 0.64 | -0.422541602 | 0.7766639116 | 0.7756955885 | 0.9683231e-3 |
| 0.68 | -0.586084818 | 0.711176746 | 0.7094908497 | 0.1685904e-2 |
| 0.72 | -0.775716652 | 0.64419545 | 0.64155562 | 0.26398237e-2 |
| 0.76 | -0.998074512 | 0.575161542 | 0.571388441 | 0.38429681e-2 |
| 0.80 | -1.263199222 | 0.503335085 | 0.4980450000 | 0.52915085e-2 |
| 0.84 | -1.587322020 | 0.4276754835 | 0.4207252951 | 0.69492884e-2 |
| 0.98 | -1.999403638 | 0.346572468 | 0.3378573979 | 0.86998489e-2 |
| 0.92 | -2.560072406 | 0.257128324 | 0.2469137502 | 0.102150822e-1 |
| 0.96 | -3.442467036 | 0.1530143199 | 0.1425950071 | 0.104193128e-1 |

Table 1 The \(w'_3\), capacities \(C(\eta)\), mutual information \(I(\eta, w = 0)\) and the difference of \(C(\eta)\) and \(I(\eta, w = 0)\) against parameter η for amplitude damping channel.

In conclusion, on the one hand, by using the concavity and the partial symmetry of \(I(\rho, \varepsilon)\) we investigate the entanglement-assisted classical information capacity of the amplitude damping channel. It is shown that the capacities \(C(\eta)\) are always a little bigger than the mutual information \(I(\eta, w = 0)\), which were taken as the entanglement-assisted classical information capacities of the amplitude damping channel in Ref. [6]. From the results we see the difference between \(C(\eta)\) and \(I(\eta, w = 0)\), namely, \(C(\eta) - I(\eta, w = 0)\), is very small for all of the parameters η. Hence, it is convenient and accurate to replace \(C(\eta)\) with \(I(\eta, w = 0)\). On the other hand, we obtained some insight into the calculation of entanglement-assisted classical information capacity for non-unitary-covariant channels. We find that the concavity and some symmetry of \(I(\rho, \varepsilon)\) for non-unitary-covariant channels can help one simplify the calculations. In particular, a unitary covariant channel corresponds to a entirely symmetrical channel whose entanglement-assisted classical information capacity can be calculated by
simply replacing $\rho$ with the maximally mixed state $1/d$ in Eq. (2).

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