Seshadri constants on the self-product of an elliptic curve

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Abstract

The purpose of this paper is to study Seshadri constants on the self-product $E \times E$ of an elliptic curve $E$. We provide explicit formulas for computing the Seshadri constants of all ample line bundles on the surfaces considered. As an application, we obtain a good picture of the behaviour of the Seshadri function on the nef cone.

Introduction

For an ample line bundle $L$ on a smooth projective variety $X$ over the complex numbers, the Seshadri constant of $L$ at $x \in X$ is by definition the real number

$$\varepsilon(L, x) = \sup \{ \varepsilon > 0 \mid f^*L - \varepsilon E \text{ is nef} \} ,$$

where $f : \text{Bl}_x(X) \to X$ is the blow-up of $X$ at $x$ and $E$ is the exceptional divisor over $x$ (see [2] and [17, Chapt. 5]). Seshadri constants are invariants of ample line bundles that measure their local positivity at a given point. While they were originally intended as a means to produce sections of adjoint linear series, it soon became clear that they are interesting invariants quite in their own right. It has turned out, however, that it is quite difficult to determine explicit values except in obvious cases like projective space.

There has been a considerable amount of work on Seshadri constants in recent years. One line of investigation concerns specific classes of surfaces, aiming for explicit bounds and, as far as possible, for explicit values of these subtle invariants (see for instance [6, [10, [14, [19, [21, [22]). Starting with [18], Seshadri constants have been studied quite intensively on abelian varieties (see [16, [2, [3, [13, [8]). Here, by homogeneity, the Seshadri constant $\varepsilon(L, x)$ is independent of the point $x$, so it is an invariant $\varepsilon(L)$ that is attached to every polarized abelian variety $(X, L)$. For abelian surfaces of Picard number one, the problem of finding explicit values for Seshadri constants was solved in [4, Sect. 6]. In the present paper we attack the problem from the opposite end: we consider products of elliptic curves. While the task of determining Seshadri constants on a product $E_1 \times E_2$ of two elliptic curves that are not isogenous is an immediate exercise, the behaviour of Seshadri constants on the self-product $E \times E$ of one elliptic curve turns out to be an interesting and non-trivial problem. The latter fact does perhaps not come as a surprise, as increasing the rank of the Néron-Severi group dramatically increases the choice of ample line bundles and curves that have to be taken into account in (*)..

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The problem naturally breaks up into two parts according to whether the elliptic curve has complex multiplication or not. In each case we are able to provide a complete picture.

**Theorem 1** Let $E$ be an elliptic curve without complex multiplication. On the abelian surface $X = E \times E$ denote by $F_1, F_2$ the fibers of the projections and by $\Delta$ the diagonal. Let $L = \mathcal{O}_X(b_1 F_1 + b_2 F_2 + b_3 \Delta)$ be any ample line bundle on $X$, and take a permutation $(a_1, a_2, a_3)$ of $(b_1, b_2, b_3)$ satisfying $a_1 \geq a_2 \geq a_3$.

Then $\varepsilon(L)$ is the minimum of the following finitely many numbers:

1. $a_2 + a_3$,
2. $\frac{a_2 a_1^2 + a_1 a_2^2 + a_3 (a_1 + a_2)^2}{\gcd(a_1, a_2)^2}$,
3. $\min \left\{ a_1 d^2 + a_2 c^2 + a_3 (c + d)^2 \mid c, d \in \mathbb{N} \text{ coprime}, c + d < \sqrt{2/(a_1 + a_2)} \right\}$.

As an application, we obtain in Sect. 3 a good picture of the behaviour of the Seshadri function $\varepsilon : \text{Nef}(X) \longrightarrow \mathbb{R}$, $L \mapsto \varepsilon(L)$.

We find that this function is continuous on the nef cone of $X$, and that its cross-sections are piecewise-linear (see Sect. 3 for examples).

Our second main result concerns elliptic curves with complex multiplication. We focus on those two curves that admit an automorphism $\neq \pm 1$. We prove:

**Theorem 2** Let $E_1$ be the elliptic curve admitting the automorphism $\iota : [x] \mapsto [ix]$, i.e., $E_1 = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$. On the abelian surface $X = E_1 \times E_1$ denote by $F_1, F_2$ the fibers of the projections, by $\Delta$ the diagonal, and by $\Sigma$ the graph of $\iota$. Let $L = \mathcal{O}_X(a_1 F_1 + a_2 F_2 + a_3 \Delta + a_4 \Sigma)$ be any ample line bundle on $X$. Then

$$
\varepsilon(L) = \min_{a, b, c, d \in \mathbb{Z}} \left\{ a_1(a^2 + b^2) + a_2(c^2 + d^2) + a_3((a-c)^2 + (b-d)^2) + a_4((a-d)^2 + (b+c)^2) \right\},
$$

where

$$
B \overset{\text{def}}{=} \frac{8 \max \left\{ |a_1 + a_3 + a_4|^2, |a_3|^2, |a_4|^2, |a_2 + a_3 + a_4|^2 \right\}}{a_1 a_2 + a_1 a_3 + a_1 a_4 + a_2 a_3 + a_2 a_4 + 2 a_3 a_4}.
$$

A result of similar shape holds for the elliptic curve with automorphism $[x] \mapsto [e^{\pi i/3} x]$ (see Theorem 4.9 for the precise statement).

In our opinion it is a nice feature of both Theorem 1 and Theorem 2 that they allow the quick and effective computation of Seshadri constants just by taking the minimum of finitely many numbers. Concrete examples are shown in Tables 1 and 2 in Sections 2 and 3.

This paper is organized as follows. We start in Sect. 1 by very briefly providing the necessary background on Seshadri constants as well as an auxiliary result. In Sect. 2 we study abelian surfaces $E \times E$ where $E$ does not have complex multiplication. We apply these results in Sect. 3 in order to gain insight into the behaviour of the Seshadri function on the nef cone. Abelian surfaces $E \times E$ where $E$ has complex multiplication are studied in Sect. 4. The latter case is – probably expectedly – technically harder and requires somewhat different methods.
Convention. We work throughout over the field of complex numbers.

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1 Seshadri constants

Let $L$ be an ample line bundle on a smooth projective variety $X$, and let $\epsilon(L,x)$ be the Seshadri constant of $L$ at $x$ as defined in the introduction. An alternative definition, which we will be using, is

$$\epsilon(L,x) = \inf \left\{ \frac{L \cdot C}{\text{mult}_x C} \middle| C \text{ irreducible curve passing through } x \right\}.$$

We mention that there is also a way to characterize Seshadri constants in terms of the separation of jets: One has

$$\epsilon(L,x) = \limsup_{k \to \infty} \frac{s(kL,x)}{k},$$

where $s(kL,x)$ is the maximal number of jets that the linear series $|kL|$ separates at $x$, i.e., the maximal integer $s$ such that the evaluation map

$$H^0(X, kL) \longrightarrow H^0(X, kL \otimes \mathcal{O}_X/m_x^{s+1})$$

is onto.

As a consequence of Kleiman’s theorem, one has the upper bound $\epsilon(L,x) \leq \sqrt{L^n}$, where $n = \dim(X)$. On abelian varieties, Seshadri constants enjoy the following additional properties:

- By homogeneity, the Seshadri constant $\epsilon(L,x)$ is independent of the point $x$. So it depends only on the line bundle, and we will write $\epsilon(L)$.
- One has the lower bound $\epsilon(L) \geq 1$, again as a consequence of homogeneity (see [17, Example 5.3.10]).

Consider now a smooth projective surface $X$. The following terminology turns out to be quite convenient: If $\epsilon(L,x)$ is smaller than the theoretical upper bound $\sqrt{L^2}$, then we will say that the Seshadri constant of $L$ at $x$ is submaximal. If a curve $C$ satisfies the inequality

$$\frac{L \cdot C}{\text{mult}_x C} < \sqrt{L^2}$$

at some point $x$, then we will call $C$ a submaximal curve (for $L$ at $x$). If

$$\frac{L \cdot C}{\text{mult}_x C} = \epsilon(L,x),$$

then we will say that $C$ computes the Seshadri constant of $L$ at $x$. One knows that if $\epsilon(L,x)$ is submaximal, then there must exist a curve that computes $\epsilon(L,x)$. Interestingly, by a result of Szemberg [20, Proposition 1.8] the number of submaximal curves for a given ample line bundle is bounded from above by the rank of the Néron-Severi group of $X$.

We will make use of the following lemma from [4, Sect. 5].
Lemma 1.1 Let $X$ be a smooth projective surface, $L$ an ample line bundle on $X$, $x \in X$ and $\xi > 0$. If there is a divisor $D \in |kL|$, $k \in \mathbb{N}$, such that

$$\frac{L \cdot D}{\text{mult}_x D} \leq \xi \sqrt{L^2},$$

then every irreducible curve with

$$\frac{L \cdot C}{\text{mult}_x C} < \frac{1}{\xi} \sqrt{L^2}$$

is a component of $D$.

As a somewhat surprising consequence, which has a crucial application in Sect. 4, an ample irreducible curve that is submaximal for some ample line bundle in fact computes its own Seshadri constant:

Proposition 1.2 Let $X$ be a smooth projective surface and $x \in X$. If $C$ is an irreducible ample curve that is submaximal at $x$ for some ample line bundle $L$, then $C$ computes $\varepsilon(\mathcal{O}_X(C), x)$.

Proof. From the index inequality and the assumption on $C$ we get

$$\frac{\sqrt{L^2 C^2}}{\text{mult}_x(C)} \leq \frac{L \cdot C}{\text{mult}_x C} < \sqrt{L^2},$$

and hence

$$\frac{\mathcal{O}_X(C) \cdot C}{\text{mult}_x C} < \sqrt{\mathcal{O}_X(C)^2}.$$ 

As $C$ is irreducible, Lemma 1.1 (with $\xi = 1$) implies that there cannot be any other submaximal curves for $\mathcal{O}_X(C)$ at $x$. 

Note that the proposition remains true when “submaximal” is replaced by “weakly submaximal” (meaning that $L \cdot C/\text{mult}_x(C) \leq \sqrt{L^2}$ holds instead of the strict inequality).

2 The case $E \times E$ without complex multiplication

Let $E$ be an elliptic curve without complex multiplication. The abelian surface $X = E \times E$ is then of Picard number 3, and the Néron-Severi group is generated over $\mathbb{Z}$ by the fibers $F_1, F_2$ of the projections $X \to E$ and the diagonal $\Delta$ (see [5, Sect. 2.7]).

A line bundle

$$L = \mathcal{O}_X(a_1 F_1 + a_2 F_2 + a_3 \Delta)$$

is ample if and only if its the integer coefficients $a_1, a_2, a_2$ satisfy the following inequalities:

$$a_1 + a_2 > 0, \ a_2 + a_3 > 0, \ a_3 + a_1 > 0, \ a_1 a_2 + a_2 a_3 + a_3 a_1 > 0.$$ \hspace{1cm} (2.0.1)

In fact, if $L$ is ample then its intersections with the curves $F_1, F_2, \Delta$, as well as its self-intersection must be positive, which shows that the inequalities are necessary. Conversely, if the inequalities are satisfied, then $L^2 > 0$ and the intersection of $L$ with the ample line bundle $\mathcal{O}_X(F_1 + F_2)$ is positive, which implies that $L$ is ample (see [15, 4.3.2(b)]).
Example 2.1 By way of warm-up let us consider an easy case first. Take an ample line bundle $L = O_X(a_1F_1 + a_2F_2 + a_3\Delta)$, all of whose coefficients $a_i$ are non-negative. Let $D$ be the divisor $a_1F_1 + a_2F_2 + a_3\Delta$. For any irreducible curve $C$ passing through 0 and different from $F_1, F_2, \Delta$, we have

$$L \cdot C = D \cdot C \geq \text{mult}_0 D \cdot \text{mult}_0 C \geq (a_1 + a_2 + a_3) \cdot \text{mult}_0 C,$$

and hence

$$\frac{L \cdot C}{\text{mult}_0 C} \geq a_1 + a_2 + a_3.$$

On the other hand, as $L \cdot F_1 = a_2 + a_3$, $L \cdot F_2 = a_1 + a_3$, and $L \cdot \Delta = a_1 + a_2$, we find that

$$\varepsilon(L) = \min \{a_1 + a_2, a_2 + a_3, a_3 + a_1\}.$$ 

So in this case one of the generators $F_1, F_2, \Delta$ computes $\varepsilon(L)$. Note that the argument in this example depends crucially on the fact that we know explicitly a suitable effective divisor $D$ in the linear series $|L|$. If we consider an ample line bundle like $O_X(7F_1 + 6F_2 - 3\Delta)$ instead, no suitable effective divisor is apparent, and it is therefore not so clear how its Seshadri constant can be computed. We will return to this example in 2.11.

Our purpose in this section is to determine the Seshadri constants of all ample line bundles on $X$. The first point is to prove that all Seshadri constants on $X$ are computed by elliptic curves (Theorem 2.2). Based on this result we can then carry out the computation of the Seshadri constants (Theorem 2.9).

Theorem 2.2 Let $E$ be an elliptic curve without complex multiplication, and let $X = E \times E$. For any ample line bundle $L$ on $X$, the Seshadri constant $\varepsilon(L)$ is computed by an elliptic curve.

So in particular, Seshadri constants on $X$ are always integers. For the proof of the theorem we need some preparation. To begin with, we determine all elliptic curves on $X$:

Proposition 2.3 (i) For every elliptic curve $N$ on $X$ that is not a translate of $F_1$, $F_2$ or $\Delta$ there exist coprime integers $c$ and $d$ such that one has the numerical equivalence

$$N \equiv_{\text{num}} c(c + d)F_1 + d(c + d)F_2 - cd\Delta.$$

(ii) Conversely, for every pair of coprime integers $c$ and $d$ the linear series

$$|c(c + d)F_1 + d(c + d)F_2 - cd\Delta|$$

consists of an elliptic curve.

Remarks 2.4 (i) We will denote henceforth by $N_{c,d}$ the elliptic curve specified by Proposition 2.3(ii). The curves $N_{c,d}$, along with the curves $F_1$, $F_2$, and $\Delta$, constitute then a complete system of representatives for the numerical classes of elliptic curves on $X$.

(ii) If we drop in Proposition 2.3(ii) the assumption that $c$ and $d$ be coprime, then even the curves $F_1$, $F_2$, and $\Delta$ occur among the $N_{c,d}$. Take $(c, d) = (1, 0), (0, 1)$, and $(1, -1)$ respectively. However, the system $|c(c + d)F_1 + d(c + d)F_2 - cd\Delta|$ then represents non-reduced curves $N_{c,d}$ as well: If $m$ is the greatest common divisor of $c$ and $d$, then $N_{c,d} = mN$, where $N$ is an elliptic curve. It will be useful to take this broader point of view in the proof of 2.2.
Proof. (i) Let \( N \) be an elliptic curve as in the hypothesis. We can write
\[
N \equiv_{\text{num}} a_1 F_1 + a_2 F_2 + a_3 \Delta
\]
with integers \( a_1, a_2, a_3 \). Then
\[
0 = N^2 = 2(a_1 a_2 + a_1 a_3 + a_2 a_3)
\]  
(2.4.1)
From the hypothesis that \( N \) is not numerically equivalent to any of the generators \( F_1, F_2, \Delta \), it follows that none of the coefficients \( a_i \) can be zero. In fact, if \( a_1 = 0 \), say, then (2.4.1) implies that \( a_2 = 0 \) or \( a_3 = 0 \), which gives \( N \equiv_{\text{num}} a_3 \Delta \) or \( N \equiv_{\text{num}} a_2 F_2 \) respectively, and this in turn implies that \( N \equiv_{\text{num}} \Delta \) or \( N \equiv_{\text{num}} F_2 \) (see Lemma 2.6 below). The same kind of reasoning yields \( a_1 + a_2 \neq 0 \). Equation (2.4.1) says then that
\[
-\frac{a_1 a_2}{a_1 + a_2} = a_3,
\]
hence \( a_1 + a_2 \) divides \( a_1 a_2 \). This implies by Lemma 2.5 below that there are integers \( c, d, m \) such that \( c \) and \( d \) are coprime and
\[
a_1 = mc(c + d) \quad \text{and} \quad a_2 = md(c + d).
\]
So we have
\[
N \equiv_{\text{num}} mc(c + d) F_1 + md(c + d) F_2 - mcd \Delta.
\]
As the numerical class of \( N \) is indivisible (see Lemma 2.6 below), we get \( m = 1 \).

(ii) Let \( M \) be the line bundle \( \mathcal{O}_X(c(c + d) F_1 + d(c + d) F_2 - cd \Delta) \). We find
\[
M^2 = 0 \quad \text{and} \quad M \cdot F_1 = d^2 > 0.
\]
It follows – for instance from [L, Lemma 2.4] – that \( h^0(M) > 0 \), and is is easy to see that, up to numerical equivalence, \( M \) is of the form \( \mathcal{O}_X(mN) \), where \( N \) is an elliptic curve and \( m \) a positive integer. From the equations
\[
mN \cdot F_1 = M \cdot F_1 = d^2 \quad \text{and} \quad mN \cdot F_2 = M \cdot F_2 = c^2
\]
we see then that \( m = 1 \), since \( c \) and \( d \) are coprime. \( \square \)

**Lemma 2.5** Let \( a \) and \( b \) be non-zero integers such that \( a + b \) divides \( ab \). Then there are integers \( c, d, \) and \( m \), such that \( c \) and \( d \) are coprime and
\[
a = mc(c + d), \quad b = md(c + d).
\]

**Proof.** Let \( \ell \) be the greatest common divisor of \( a \) and \( b \), and let \( c = a/\ell \) and \( d = b/\ell \). Then \( c \) and \( d \) are coprime and we have
\[
a + b = \ell(c + d) \quad \text{and} \quad ab = \ell^2 cd.
\]
From the assumption that \( a + b \) divides \( ab \) we see that \( c + d \) divides \( \ell cd \). Let \( p \) be a prime divisor of \( c + d \). Then \( p \) also divides \( \ell cd \). If \( p \) were to divide \( c \) or \( d \), then, as a prime divisor of \( c + d \), it would divide both of them. But this cannot happen, as \( c \) and \( d \) are coprime. So none of the prime divisors of \( c + d \) divides \( c \) or \( d \), and therefore \( c + d \) divides \( \ell \). Let now \( m = \ell/(c + d) \). So we obtain
\[
a = \ell c = mc(c + d)
\]
\[
b = \ell d = md(c + d)
\]
as claimed. \( \square \)
Lemma 2.6 Let $X$ be an abelian surface and let $E \subset X$ be an elliptic curve. Then the numerical class of $E$ is indivisible. In other words, if $E \equiv_{\text{num}} kD$ for some divisor $D$ and some integer $k > 0$, then $k = 1$.

Proof. Fix an ample divisor $H$. Then $H \cdot D = \frac{1}{2}H \cdot E > 0$ and $D^2 = \frac{1}{2}E^2 = 0$, which implies that $\mathcal{O}_X(D)$ is effective (see e.g. [1, Lemma 2.4]). Then $\mathcal{O}_X(D)$, being effective and of zero self-intersection, must be numerically equivalent to a positive multiple $mE'$ of an elliptic curve $E'$. So we have $E \equiv_{\text{num}} kD \equiv_{\text{num}} kmE'$. A suitable translate of $E$ is therefore contained in the linear series $|kmE'|$. But this can only happen if $k = m = 1$, because all elements of $|kmE'|$ are reducible if $km > 1$. $\square$

We turn now to the proof of Theorem 2.2. The proof draws from two sources: First, we use a classical result from the geometry of numbers in order to show that every ample line bundle admits a submaximal elliptic curve. Secondly, we apply a result from [3] in order to prove that no curve of genus $g > 1$ can be “more submaximal” than the elliptic ones.

The result from the geometry of numbers that we will need is Hermite’s classical theorem (see e.g. [7, Sect. II.3.2]):

Theorem 2.7 (Hermite) Let $Q$ be a positive definite quadratic form of two variables,

$$Q(x, y) = ax^2 + 2bxy + cy^2,$$

and let $\delta = ac - b^2$ be its determinant. Then there is a non-zero point $p \in \mathbb{Z}^2$ such that

$$Q(p) \leq \sqrt{\frac{4}{3}\delta}.$$

Proof of Theorem 2.2. (i) Let $L = \mathcal{O}_X(a_1F_1 + a_2F_2 + a_3\Delta)$ be an ample line bundle on $X$. Its intersection number with the elliptic curve $N_{c,d}$ is a quadratic from in the variables $c$ and $d$:

$$Q(c, d) \overset{\text{def}}{=} L \cdot N_{c,d} = (c \quad d) \begin{pmatrix} a_2 + a_3 & a_3 \\ a_3 & a_1 + a_3 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix}.$$

It follows from the ampleness of $L$ (using the inequalities (2.0.1)) that $Q$ is positive definite. The discriminant of $Q$ is

$$\delta = a_1a_2 + a_1a_3 + a_2a_3 = L^2/2.$$

Applying now Theorem 2.7 we find that there is a non-zero point $(c, d) \in \mathbb{Z}^2$ such that

$$Q(c, d) \leq \sqrt{\frac{4}{3}\delta}.$$

This implies that

$$L \cdot N_{c,d} \leq \sqrt{\frac{4}{3}\delta} = \sqrt{\frac{2}{3}L^2}.$$

So in any event $N_{c,d}$ is a submaximal curve for $L$. ($N_{c,d}$ is either an elliptic curve or a multiple of an elliptic curve, see Remark 2.3b). So we have

$$\varepsilon(L) \leq \sqrt{\frac{2}{3}L^2}.$$
(ii) To complete the proof we now show that there cannot be a curve of genus > 1 computing \( \varepsilon(L) \). This can be seen as follows: It is a consequence of [3, Theorem A.1(b)] – or more precisely of the proof of that theorem – that for an irreducible curve \( C \) of arithmetic genus > 1 on an abelian surface, one has

\[
\frac{L \cdot C}{\text{mult}_x C} \geq \sqrt{\frac{7}{8}} L^2.
\]

This inequality, together with (2.7.1), guarantees that one of the curves \( N_{c,d} \) computes \( \varepsilon(L) \). \( \square \)

Having established that all Seshadri constants on \( X \) are computed by elliptic curves, we are now able to provide a complete picture of the Seshadri constants of all ample line bundles. In order to formulate the result in the most compact way, it is best to keep in mind the following easy lemma.

**Lemma 2.8** Let \( L = \mathcal{O}_X(a_1 F_1 + a_2 F_2 + a_3 \Delta) \) be an ample line bundle, let \( \pi \) be a permutation of the numbers 1,2,3, and let \( L^\pi = \mathcal{O}_X(a_{\pi(1)} F_1 + a_{\pi(2)} F_2 + a_{\pi(3)} \Delta) \) be the line bundle with permuted coefficients. Then \( L^\pi \) is ample as well, and

\[
\varepsilon(L^\pi) = \varepsilon(L).
\]

**Proof.** The intersection matrix of \( (F_1, F_2, \Delta) \) is

\[
\begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix},
\]

and any permutation of the triplet \( (F_1, F_2, \Delta) \) has the same intersection matrix. This implies that \( (L^\pi)^2 = L^2 \), and, if the linear series \( |b_1 F_1 + b_2 F_2 + b_3 \Delta| \) represents an elliptic curve, then the linear series with permuted coefficients also represents an elliptic curve \( N^\pi \). This curve \( N^\pi \) satisfies

\[
L^\pi \cdot N^\pi = L \cdot N,
\]

so that if \( N \) computes \( \varepsilon(L) \), then \( N^\pi \) computes \( \varepsilon(L^\pi) \). \( \square \)

Our result can then be stated as follows, proving Theorem 1 from the introduction.

**Theorem 2.9** Let \( E \) be an elliptic curve without complex multiplication and let \( L = \mathcal{O}_X(a_1 F_1 + a_2 F_2 + a_3 \Delta) \) be any ample line bundle on the abelian surface \( X = E \times E \). Assume that

\[
a_1 \geq a_2 \geq a_3
\]

(which in view of Lemma 2.8 means no loss in generality). Then \( \varepsilon(L) \) is the minimum of the following numbers:

1. \( a_2 + a_3 \),
2. \( \frac{a_2 a_1^2 + a_1 a_2^2 + a_3(a_1 + a_2)^2}{\gcd(a_1, a_2)^2} \),
3. \( \min \left\{ a_1 d^2 + a_2 c^2 + a_3(c + d)^2 \bigg| c, d \in \mathbb{N} \text{ coprime}, c + d < \frac{1}{\sqrt{2}}(a_1 + a_2) \right\} \).
Proof. We know by Theorem 2.2 that \( \varepsilon(L) \) is in any event computed by an elliptic curve. So \( \varepsilon(L) \) is the minimal degree \( L \cdot N \), where \( N \) runs through all elliptic curves on \( X \), i.e.,

\[
\varepsilon(L) = \min(\{ L \cdot F_1, L \cdot F_2, L \cdot \Delta \} \cup \{ L \cdot N_{c,d} \mid c \text{ and } d \text{ coprime integers} \}) .
\]

Expression (1) in the statement accounts for the curves \( F_1, F_2, \) and \( \Delta \). The point now is to explicitly restrict the range of elliptic curves \( N_{c,d} \) that have to be taken into account.

Under our assumption that \( a_1 \geq a_2 \geq a_3 \) we see from the ampleness conditions 2.0.1 that \( a_1 \) and \( a_2 \) must both be positive. We now determine when the elliptic curve \( N_{c,d} \) is submaximal for \( L \), i.e., when \( L \cdot N_{c,d} < \sqrt{L^2} \) holds. In terms of coefficients this condition evaluates to the inequality

\[
(a_2 + a_3)c^2 + 2a_3cd + (a_1 + a_3)d^2 < \sqrt{2(a_1a_2 + a_1a_3 + a_2a_3)} .
\]

A calculation shows that the latter condition can equivalently be expressed as

\[
\left( a_3(c + d)^2 + \frac{(a_1d^2 + a_2c^2)(c + d)^2 - (a_1 + a_2)}{(c + d)^2} \right)^2 < \frac{1}{(c + d)^4} \left( (a_1 + a_2)^2 - 2(a_1d - a_2c)(c + d)^2 \right) .
\]

The crucial point is now that for this inequality to be satisfied – given \( a_1, a_2, a_3 \) – it is necessary to have

\[
(a_1 + a_2)^2 > 2(a_1d - a_2c)(c + d)^2 , \tag{2.9.1}
\]

and this inequality narrows down the potential submaximal curves \( N_{c,d} \) to a finite set: First, we see that \( c \) and \( d \) must be both positive or both negative, as otherwise

\[
2(a_1d - a_2c)^2(c + d)^2 \geq 2(a_1|d| - a_2|c|)^2(c + d)^2 \geq (a_1 + a_2)^2 .
\]

We may therefore assume \( c > 0 \) and \( d > 0 \). (Note that \( N_{c,d} = N_{-c,-d} \).) Furthermore, 2.9.1 implies that

\[
\frac{a_2}{a_1} = \frac{d}{c} \quad \text{or} \quad (c + d)^2 < \frac{1}{2}(a_1 + a_2)^2 .
\]

As \( c \) and \( d \) are coprime, the first case applies only to one elliptic curve, namely to

\[
N_{a_1/\gcd(a_1,a_2), a_2/\gcd(a_1,a_2)} ,
\]

which is taken account for by expression (2) of the theorem. The second case yields the range expressed in (3). \( \square \)

Remarks 2.10 (i) Theorem 2.9 shows that it is quick and easy to compute \( \varepsilon(L) \) from the coefficients \( a_1, a_2, a_3 \) of \( L \): All one needs is to take the minimum of finitely many numbers.

(ii) Note that there would be no harm if we extended the minimum in (3) over all pairs of positive integers \( c \) and \( d \) with \( c + d < \frac{1}{\sqrt{2}}(a_1 + a_2) \), whether or not they are coprime. From a computational point of view it may in fact be more efficient to do so, forgoing any coprimality tests.
Table 1: Seshadri constants of the line bundles $L = \mathcal{O}_X(a_1F_1 + a_2F_2 + a_3\Delta)$ on $X = E \times E$. The last column lists all elliptic curves $C$ such that $L \cdot C \leq \sqrt{L^2}$.

| $a_1$ | $a_2$ | $a_3$ | $L^2$ | $\sqrt{4L^2}$ | $\varepsilon(L)$ | curves computing $\varepsilon(L)$ | weakly submaximal |
|-------|-------|-------|-------|-------------|----------------|-------------------------------|-----------------|
| 3     | 2     | -1    | 2     | $\approx 1.15$ | 1              | $F_1, N_{1,1}$                  | $F_1, N_{1,1}$   |
| 3     | 3     | -1    | 6     | 2          | 2              | $F_1, F_2, N_{1,1}$              | $F_1, F_2, N_{1,1}$ |
| 4     | 3     | -1    | 10    | $\approx 2.58$ | 2              | $F_1$                          | $F_1, N_{1,1}$   |
| 5     | 3     | -1    | 14    | $\approx 3.06$ | 2              | $F_1$                          | $F_1$            |
| 5     | 4     | -2    | 4     | $\approx 1.63$ | 1              | $N_{1,1}$                      | $F_1, N_{1,1}$   |
| 7     | 4     | -2    | 12    | $\approx 2.83$ | 2              | $F_1$                          | $F_1, N_{1,1}$   |
| 7     | 6     | -3    | 6     | 2          | 1              | $N_{1,1}$                      | $N_{1,1}$        |
| 10    | 7     | -4    | 4     | $\approx 1.63$ | 1              | $N_{1,1}$                      | $N_{1,1}, N_{2,1}$ |
| 12    | 9     | -5    | 6     | 2          | 1              | $N_{1,1}$                      | $N_{1,1}$        |
| 17    | 10    | -6    | 16    | $\approx 3.27$ | 3              | $N_{1,1}, N_{2,1}$              | $F_1, N_{1,1}, N_{2,1}$ |
| 20    | 11    | -7    | 6     | 2          | 1              | $N_{2,1}$                      | $N_{2,1}$        |
| 32    | 9     | -7    | 2     | $\approx 1.15$ | 1              | $N_{3,1}, N_{4,1}$              | $N_{3,1}, N_{4,1}$ |
| 33    | 9     | -7    | 2     | 6          | 2              | $F_1, N_{3,1}, N_{4,1}$          | $F_1, N_{3,1}, N_{4,1}$ |
| 34    | 9     | -7    | 10    | $\approx 2.58$ | 2              | $F_1$                          | $F_1, N_{3,1}, N_{4,1}$ |
| 26    | 14    | -9    | 8     | $\approx 2.31$ | 1              | $N_{2,1}$                      | $N_{2,1}$        |
| 73    | 13    | -11   | 6     | 2          | 2              | $F_1, N_{5,1}, N_{6,1}$          | $F_1, N_{5,1}, N_{6,1}$ |
| 54    | 14    | -11   | 16    | $\approx 3.27$ | 3              | $F_1, N_{4,1}$                  | $F_1, N_{3,1}, N_{4,1}$ |
| 45    | 15    | -11   | 30    | $\approx 4.47$ | 4              | $F_1, N_{3,1}$                  | $F_1, N_{3,1}$   |
| 36    | 16    | -11   | 8     | $\approx 2.31$ | 1              | $N_{2,1}$                      | $N_{2,1}$        |
| 32    | 17    | -11   | 10    | $\approx 2.58$ | 1              | $N_{2,1}$                      | $N_{2,1}$        |
| 52    | 30    | -19   | 4     | $\approx 1.63$ | 1              | $N_{2,1}$                      | $N_{2,1}, N_{3,3}$ |

Theorem 2.9 allows not only to compute Seshadri constants, but it also yields all submaximal curves as the following examples illustrate. Table 1 gives further concrete examples.

**Examples 2.11** (i) Consider the ample bundle $L = \mathcal{O}_X(7F_1 + 6F_2 - 3\Delta)$ that was mentioned briefly at the end of Example 2.1. Applying Theorem 2.9 we find that $N_{1,1}$ calculates $\varepsilon(L) = 1$, and this is the only submaximal curve for $L$.

(ii) As for an example at the other extreme: The ample bundle $L = \mathcal{O}_X(33F_1 + 9F_2 - 7\Delta)$ admits three submaximal curves, $F_1, N_{3,1}, N_{4,1}$. All three of them compute $\varepsilon(L)$ in this case. This is a case where the maximal possible number of submaximal curves occurs.

### 3 The Seshadri function on the nef cone

Our purpose now is to apply the results of the previous section in order to gain insight into the behaviour of the Seshadri function on the nef cone of $E \times E$.

Consider first an arbitrary smooth projective variety $Y$. The definition of Seshadri constants extends immediately to ample (or nef) $\mathbb{Q}$-divisors, and also to ample (or nef) $\mathbb{R}$-divisors (using either definition (*) from the introduction or the alternative characterization at the beginning of Sect. 1). Further, the definition clearly extends to nef divisors. We get thus for fixed $y \in Y$ a function

$$\varepsilon_y : \text{Nef}(Y) \longrightarrow \mathbb{R}, \quad L \longmapsto \varepsilon(L, y)$$

on the nef cone of $Y$, which we will refer to as the *Seshadri function at $y$*. Considering now an abelian variety $A$, we obtain a function

$$\varepsilon : \text{Nef}(A) \longrightarrow \mathbb{R}, \quad L \longmapsto \varepsilon(L)$$
that is independent of the point. Our first observation is:

**Proposition 3.1** Let $A$ be an abelian variety. Then the Seshadri function $\varepsilon$ is concave and continuous.

Note that this result (and the subsequent proof) remains valid more generally on homogeneous varieties.

**Proof.** The concavity is immediate, as both the equality $\varepsilon(\lambda L) = \lambda \varepsilon(L)$ for $\lambda \geq 0$ and the inequality

$$\varepsilon(L + M) \geq \varepsilon(L) + \varepsilon(M)$$

follow immediately from the definition. The continuity in the interior of Nef($A$) is then a consequence of concavity. Consider then an $\mathbb{R}$-line bundle $L$ on the boundary of the nef cone. According to the Nakai criterion for $\mathbb{R}$-divisors (see [17, Theorem 2.3.18]), there is a subvariety $V \subset A$ such that $L^d \cdot V = 0$, where $d = \dim V$. Therefore, as a suitable translate of $V$ passes through any given point $x \in A$,

$$0 \leq \varepsilon(L) \leq \sqrt[k]{\frac{L^d \cdot V}{\text{mult}_x V}} = 0,$$

and hence $\varepsilon(L) = 0$. Let now $(L_n)_{n \geq 1}$ be a sequence of $\mathbb{R}$-line bundles in Nef($A$) converging to $L$. As the intersection product is continuous, we obtain

$$0 \leq \varepsilon(L_n) \leq \sqrt[k]{\frac{L_n^d \cdot V}{\text{mult}_x V}} \xrightarrow{n \to \infty} \sqrt[k]{\frac{L^d \cdot V}{\text{mult}_x V}} = 0 = \varepsilon(L),$$

hence $\varepsilon(L_n) \to \varepsilon(L)$, as claimed. □

Consider now $X = E \times E$, the self-product of an elliptic curve $E$ without complex multiplication, as in the preceding section. We wish to study the behaviour of its Seshadri function $\varepsilon : \text{Nef}(X) \to \mathbb{R}$.

Let $L = \mathcal{O}_X(a_1 F_1 + a_2 F_2 + a_3 \Delta)$ be an (integral) nef line bundle. We may assume $a_1 \geq a_2 \geq a_3$, and even $a_1 > 0$ if $L$ is not the trivial bundle. Writing then

$$L = a_1 \cdot L_{\lambda, \mu}, \quad L_{\lambda, \mu} = \mathcal{O}_X(F_1 + \lambda F_2 - \mu \Delta)$$

with $\lambda = a_2/a_1$ and $\mu = -a_3/a_1$, it is enough to determine the Seshadri constants of the bundles $L_{\lambda, \mu}$. These are nef in the range

$$\lambda \in [0, 1], \quad \mu \in ]-\infty, \frac{\lambda}{1 + \lambda}].$$

The following statements are quickly verified:

(i) For $\mu \in ]-\infty, -1]$, the curve $\Delta$ computes $\varepsilon(L_{\lambda, \mu}) = 1 + \lambda$.

(ii) For $\mu \in ]-1, 0]$, the curve $F_1$ computes $\varepsilon(L_{\lambda, \mu}) = \lambda - \mu$.

As a consequence of Theorem 2.9 we now show:

**Proposition 3.2** For fixed rational $\lambda \in [0, 1]$, the function

$$] - \infty, \frac{\lambda}{1 + \lambda} [ \to \mathbb{R}, \quad \mu \mapsto \varepsilon(L_{\lambda, \mu})$$

is a piecewise affine-linear function and has only finitely many affine-linear pieces.
Figure 1: Piecewise linear behaviour of the Seshadri function on a cross-section of the ample cone of the surface $X = E \times E$. (The point $\mu_\infty$ is the upper boundary $\frac{\lambda}{1+\lambda}$ of the nef range.)

Proof. We may assume $\lambda > 0$, so that $L_{\lambda,\mu}$ is ample. According to (the proof of) Theorem 2.9, only finitely many of the elliptic curves $N_{c,d}$ can be submaximal for any of the line bundles $L_{\lambda,\mu}$, when $\lambda$ is fixed and $\mu$ varies in the ample range $-\infty < \mu < \lambda/(1 + \lambda)$. In fact, Condition (2.9.1), which is necessary for submaximality, is equivalent to

$$(1 + \lambda)^2 > 2(d - \lambda c)^2(c + d)^2,$$

and hence it is independent of $\mu$. Denoting the potential submaximal curves by $N_1, \ldots, N_k$, the Seshadri function in the statement of the proposition is then the pointwise minimum of finitely many affine-linear functions:

$$\varepsilon(L_{\lambda,\mu}) = \min_{i=1}^k L_{\lambda,\mu} \cdot N_i.$$ (3.2.1)

At the upper boundary $\mu_\infty = \lambda/(1 + \lambda)$ of the ample range, $L_{\lambda,\mu}$ is numerically equivalent to a multiple of an elliptic curve $N_{c,d}$, and hence $\varepsilon(L_{\lambda,\mu_\infty}) = 0$. □

The behaviour of the Seshadri function described by Proposition 3.2 is displayed in Fig. 1. We now illustrate the situation by considering concrete examples.

Example 3.3 We consider the line bundles $L_{\frac{1}{n},\mu}$ for a fixed integer $n \geq 1$. The nef range for $\mu$ is then $-\infty < \mu \leq \frac{1}{n+1}$. One shows now that the only curves that matter in the minimum in (3.2.1) are $\Delta$, $F_1$, and $N_{n,1}$. (If $n = 2$, then the curve $N_{1,1}$ is also submaximal, but this curve turns out to be irrelevant when taking the minimum.) One can then determine the Seshadri function:

$$\varepsilon(L_{\frac{1}{n},\mu}) = \begin{cases} 1 + \frac{1}{n} & \text{if } \mu \leq -1 \\
^{-1} & \text{if } \mu \in \left[-\frac{n^2+n-1}{n^2(n+2)}\right] \\
1 + n - (n+1)^2 & \text{if } \mu \in \left[\frac{n^2+n-1}{n^2(n+2)}, \frac{1}{n+1}\right] \end{cases} \quad (\Delta \text{ computes } \varepsilon)$$

$$(F_1 \text{ computes } \varepsilon)$$

For other values of $\lambda$, the number of elliptic curves $N_{c,d}$ that have to be taken into account can become larger. We conclude with a somewhat more intricate example, which is intended to illustrate this point.

Example 3.4 We consider the line bundles $L_{\frac{8}{11},\mu}$. Among the curves $N_{c,d}$ the potential submaximal curves are $N_{1,1}, N_{2,1}, N_{3,2}, N_{4,3}, N_{7,5}, N_{11,8}$. By carrying out
the necessary computations one gets

\[ \varepsilon(L^{\mu}_{\mathbb{P}^1}, \mu) = \begin{cases} 
\frac{19}{11} & \text{if } \mu \leq -1 \\
\frac{8}{11} - \mu & \text{if } \mu \in \left[ -1, \frac{1}{3} \right] \\
\frac{19}{11} - 4\mu & \text{if } \mu \in \left[ \frac{1}{3}, \frac{97}{231} \right] \\
\frac{116}{11} - 25\mu & \text{if } \mu \in \left[ \frac{97}{231}, \frac{37}{88} \right] \\
\frac{227}{11} - 49\mu & \text{if } \mu \in \left[ \frac{37}{88}, \frac{1445}{3432} \right] \\
152 - 361\mu & \text{if } \mu \in \left[ \frac{1445}{3432}, \frac{8}{19} \right] 
\end{cases} \]

(\(N_{2,1}\) and \(N_{7,5}\) turn out to be irrelevant when taking the minimum.)

4 The case \(E \times E\) with complex multiplication

In this section we consider abelian surfaces \(E \times E\) where \(E\) has complex multiplication. We will focus on the elliptic curves admitting an automorphism \(\neq \pm 1\):

\[ E_1 = \mathbb{C}/\mathbb{Z} + i\mathbb{Z} \quad \text{and} \quad E_2 = \mathbb{C}/\mathbb{Z} + e^{\pi i/3}\mathbb{Z}. \]

We will study first \(E_1 \times E_1\).

4.1 Complex multiplication by \(i\)

The Néron-Severi group of \(E_1 \times E_1\) is of rank four, with generators

\[ F_1, F_2, \Delta, \Sigma, \]

where \(F_1, F_2\) are the fibers of the projections, \(\Delta\) is the diagonal, and \(\Sigma\) is the graph of the automorphism

\[ \iota : E_1 \rightarrow E_1, [x] \mapsto [ix]. \]

(see [5, Sect. 2.7]).

Note that \(\iota\) has exactly two fixed-points: \([0]\) and \([1+i/2]\). Therefore we have \(\Delta \cdot \Sigma = 2\). As for the remaining intersection numbers, we get

\[ F_1^2 = F_2^2 = \Delta^2 = \Sigma^2 = 0 \]

and

\[ F_1 \cdot F_2 = F_1 \cdot \Delta = F_2 \cdot \Delta = F_1 \cdot \Sigma = F_2 \cdot \Sigma = 1. \]

A line bundle \(L = O_X(a_1F_1 + a_2F_2 + a_3\Delta + a_4\Sigma)\) is ample if and only if its self-intersection as well as its intersection with the curves \(F_1, F_2, \Delta, \Sigma\) are positive. (This follows in the same way as indicated after (2.0.1) in the rank three case.) So \(L\) is ample if and only if

\[
\begin{align*}
    a_2 + a_3 + a_4 & > 0 \\
    a_1 + a_3 + a_4 & > 0 \\
    a_1 + a_2 + 2a_4 & > 0 \\
    a_2 + a_2 + 2a_3 & > 0 \\
    a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + 2a_3a_4 & > 0.
\end{align*}
\]

The first step in this section is to prove an analogue of Theorem 2.2 to the effect that all Seshadri constants are computed by elliptic curves. To this end we will need to know all elliptic curves on \(E_1 \times E_1\). As a parametrization of all elliptic curves as in Sect. 2 seems difficult, we will make use of the following result instead:
Lemma 4.1 (Hayashida-Nishi [12]) Let $E$ be an elliptic curve. Then for every elliptic curve $N$ on $E \times E$ there are endomorphisms $\sigma_1, \sigma_2$ of $E$ such that $N$ is a translate of the image of the map

$$E \rightarrow E \times E, \quad x \mapsto (\sigma_1(x), \sigma_2(x)).$$

Let now $N$ be an elliptic curve on $X = E_1 \times E_1$. As $\text{End}(E_1) = \mathbb{Z} + i\mathbb{Z}$, Lemma 4.1 says that there are integers $a, b, c, d$ such that $N$ is a translate of the curve

$$N_{a,b,c,d} \overset{\text{def}}{=} \{(ax + bu(x), cx + du(x)) \mid x \in E_1\}.$$

We may assume here that $a, b, c, d$ are coprime, because a common factor would just mean that the map $(\sigma_1, \sigma_2)$ is composed with a multiplication map.

We determine next the intersection numbers of $N_{a,b,c,d}$ with the generators of the Néron-Severi group. As $F_1$ and $N_{a,b,c,d}$ intersect transversely, we have

$$N_{a,b,c,d} \cdot F_1 = \#(N_{a,b,c,d} \cap F_1) = \frac{\# \{ x \in F_1 \mid ax + bu(x) = 0 \}}{\text{deg } \sigma}.$$

where $\sigma : E_1 \rightarrow N_{a,b,c,d}$ is the map $x \mapsto (ax + bu(x), cx + du(x))$.

In the next two lemmas we will evaluate the expression on the RHS of (4.1.1).

Lemma 4.2 For integers $a$ and $b$, not both of them zero, the equation

$$ax + bu(x) = 0$$

has exactly $a^2 + b^2$ solutions $x \in E_1$.

Proof. We may assume that both $a$ and $b$ are non-zero, the assertion being clear otherwise. Let $\ell = a^2 + b^2$, and consider first the case that $a$ and $b$ are coprime. Suppose that $x$ is a solution of (4.2.1). By subtracting the two equations that are obtained from (4.2.1) by multiplication with $a$ and $b$ respectively, we see that $x$ is necessarily an $\ell$-division point on $E_1$. Now, an $\ell$-division point

$$x = \left[ \frac{m}{\ell} + in/\ell \right], \quad 0 \leq m, n < \ell,$$

solves (4.2.1) if and only if $\ell$ is a divisor of both $am - bn$ and $an + bm$. Given an integer $m \in \{0, \ldots, \ell - 1\}$, there is a unique integer $n \in \{0, \ldots, \ell - 1\}$ such that these two divisibility conditions are satisfied (since $a$ and $b$ are invertible modulo $\ell$). So there are $\ell$ distinct solutions $x \in E_1$.

Taking now general $a$ and $b$, let $d = \gcd(a, b)$ and write $a = da'$, and $b = db'$. By what we have shown so far, the equation

$$a'(dx) + b'd(dx) = 0$$

admits exactly $a'^2 + b'^2$ solutions for $dx$. As multiplication by $d$ is a map of degree $d^2$, we obtain $d^2(a'^2 + b'^2) = a^2 + b^2$ solutions for $x$, and this completes the proof. □

We now determine the degree of the map $\sigma = (\sigma_1, \sigma_2) : E_1 \rightarrow N_{a,b,c,d}$. For this, and in fact for the remainder of this section we will use the abbreviation

$$D \overset{\text{def}}{=} \gcd(a^2 + b^2, c^2 + d^2, ac + bd, ad - bc).$$

Lemma 4.3 The map $\sigma$ is of degree $D$. 

Proof. We need to determine the number of elements in the kernel of \( \sigma \). So suppose that \( x \) is a point in \( E_1 \) with

\[ ax + bx = cx + dx = 0. \tag{4.3.1} \]

As in the proof of Lemma 4.2 it follows that \( x \) is both an \((a^2 + b^2)\)-division point and a \((c^2 + d^2)\)-division point. We see from the equation \( dx(ax + bx) = (-ac - bd)x \) that \( x \) is also an \((ac + bd)\)-division point, and in the same manner that it is also a \((ad - bc)\)-division point. So we infer that \( x \) is a \(D\)-division point. Conversely, a \(D\)-division point \( x \) satisfies the equations (4.3.1) if and only if the following congruences are fulfilled:

\[
\begin{align*}
am - bn &\equiv 0 \mod D \\
bn + an &\equiv 0 \mod D \\
cm - dn &\equiv 0 \mod D \\
dm + cn &\equiv 0 \mod D.
\end{align*}
\]

The proof is now completed by invoking Lemma A.1 (in the appendix), which states that this systems admits exactly \( D \) solutions. \(\square\)

The preceding lemmas now allow us to determine the required intersection numbers:

**Proposition 4.4** We have

\[
\begin{align*}
N_{a,b,c,d} \cdot F_1 &= \frac{a^2 + b^2}{D} \\
N_{a,b,c,d} \cdot F_2 &= \frac{c^2 + d^2}{D} \\
N_{a,b,c,d} \cdot \Delta &= \frac{(a - c)^2 + (b - d)^2}{D} \\
N_{a,b,c,d} \cdot \Gamma &= \frac{(a - d)^2 + (b + c)^2}{D}.
\end{align*}
\]

**Proof.** In view of lemmas 4.2 and 4.3 the first assertion follows using (4.1.1). The proof of the remaining assertions is analogous. \(\square\)

Fix now an ample line bundle \( L = \mathcal{O}_X(a_1 F_1 + a_2 F_2 + a_3 \Delta + a_4 \Sigma) \). Using Proposition 4.4 one finds

\[
L \cdot N_{a,b,c,d} = \frac{1}{D} Q(a, b, c, d),
\]

where \( Q \) is the quadratic form

\[
Q(a, b, c, d) = \begin{pmatrix}
(a_1 + a_3 + a_4) & 0 & -a_3 & -a_4 \\
0 & a_1 + a_3 + a_4 & a_4 & -a_3 \\
-a_3 & a_4 & a_2 + a_3 + a_4 & 0 \\
-a_4 & -a_3 & 0 & a_2 + a_3 + a_4
\end{pmatrix} \begin{pmatrix}
a \\ b \\ c \\ d
\end{pmatrix}.
\]

A computation shows that \( Q \) is positive definite and of discriminant

\[
\delta = (L^2/2)^2.
\]

We can now prove:

**Theorem 4.5** Let \( E_1 \) be the elliptic curve with automorphism \([x] \mapsto [ix]\), and let \( X = E_1 \times E_1 \). For any ample line bundle \( L \) on \( X \), the Seshadri constant \( \varepsilon(L) \) is computed by an elliptic curve.
In the proof we will make use of the following result from the geometry of numbers (see [11], Chapter 6).

**Theorem 4.6 (Mahler)** Let \( Q \) be a positive definite quadratic form of four variables with discriminant \( \delta \). Then there is a non-zero point \( p \in \mathbb{Z}^4 \) such that
\[
Q(p) \leq \sqrt{2}\sqrt{\delta}.
\]

**Proof of Theorem 4.6.** Let \( L = \mathcal{O}_X(a_1F_1 + a_2F_2 + a_3\Delta + a_4\Sigma) \) be an ample line bundle. We are interested in the minimum of the intersection numbers \( L \cdot N_{a,b,c,d} \) of \( L \) with all elliptic curves \( N_{a,b,c,d} \). If the g.c.d. \( D \) that is associated with \((a, b, c, d)\) in (4.2.2) is greater than one, then Lemma A.2 (in the appendix) implies that the numbers \( a, b, c, d \) may be replaced by numbers \( \overline{a}, \overline{b}, \overline{c}, \overline{d} \) such that the corresponding g.c.d. \( \overline{D} \) equals one, without altering the intersection product \( L \cdot N_{a,b,c,d} \) in the process. The upshot of this argument is that the intersection product \( L \cdot N_{a,b,c,d} \) may be minimized by taking the minimum of \( Q \).

Now, by Theorem 4.6 there are integers \( a, b, c, d, \) not all of them zero, such that
\[
L \cdot N_{a,b,c,d} \leq \sqrt{2} \left( \frac{L^2}{2} \right)^2 = \sqrt{L^2}.
\]

To complete the proof, it therefore remains to show that there cannot be a curve of genus \( > 1 \) computing \( \varepsilon(L) \). So suppose by way of contradiction that there is a submaximal curve \( C \) for \( L \) that is not elliptic. Since a non-elliptic curve on an abelian surface is automatically ample, we see from Proposition 1.2 that \( C \) is then submaximal for \( \mathcal{O}_X(C) \) as well. On the other hand, applying to \( \mathcal{O}_X(C) \) the argument that we applied to \( L \) at the beginning of the proof, we find that there is an elliptic curve \( N \) with
\[
C \cdot N \leq \sqrt{C^2}.
\]

But then, by Lemma 1.1, \( N \) would have to be a component of \( C \), and this is a contradiction. \( \square \)

Our second aim in this section is to explicitly determine the Seshadri constants for all ample line bundles on \( X \), i.e., to provide an analogue of Theorem 2.9. It seems difficult to achieve this using the same methods that we applied in Section 2. First, the increased number of variables makes it hard to derive direct estimates. Secondly, the analogue of Lemma 2.8 is not true, i.e., the generators of \( \text{NS}(X) \) may not be interchanged in arguments involving intersection numbers. For these reasons we proceed along a different path here, using a little elementary real analysis to obtain the desired bounds.

Let us fix notation for the following lemma. If \( M \) is a subset of \( \mathbb{R}^n \), then \( U_i(M) \) will denote the set of all points \((x_1, \ldots, x_n) \in \mathbb{R}^n \) such that there is an \((m_1, \ldots, m_n) \in M \) satisfying \(|x_i - m_i| \leq 1\).

**Lemma 4.7** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a partially differentiable function. Then the points, at which the restricted function \( f|_{\mathbb{Z}^n} \) is minimal, lie in the intersection
\[
\bigcap_{i=1}^n U_i \left( \left\{ x \in \mathbb{R}^n \left| \frac{\partial f}{\partial x_i}(x) = 0 \right. \right\} \right).
\]
Proof. Suppose that \( f \big|_{\mathbb{Z}^n} \) is minimal at \( m = (m_1, \ldots, m_n) \in \mathbb{Z}^n \). Then

\[
\frac{f(m) - f(m_1 - 1, m_2, \ldots, m_n)}{m_2 - m_1} \leq f(m_1 - 1, m_2, \ldots, m_n) \quad \text{and} \quad \frac{f(m) - f(m_1, m_2, \ldots, m_n)}{m_2 - m_1} \leq f(m_1, m_2, \ldots, m_n),
\]

hence the function \( t \mapsto f(t, m_2, \ldots, m_n) \) assumes a local minimum at some point \( t_1 \) of the interval \([m_1 - 1, m_1 + 1]\). The partial derivative of \( f \) vanishes then at \((t_1, m_2, \ldots, m_n)\), which just means that \( m \) is contained in the set

\[
U_1 \left( \left\{ x \in \mathbb{R}^n \left| \frac{\partial f}{\partial x_i}(x) = 0 \right. \right\} \right).
\]

The analogous statement holds for \( i = 2, \ldots, n \). \( \square \)

We are now ready to prove:

**Theorem 4.8** Let \( E_1 \) be the elliptic curve admitting the automorphism \([x] \mapsto [ix]\), i.e., \( E = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z}) \), and let \( L = \mathcal{O}_X(a_1F_1 + a_2F_2 + a_3\Delta + a_4\Sigma) \) be any ample line bundle on the abelian surface \( X = E_1 \times E_1 \). Then

\[
\varepsilon(L) = \min_{a,b,c,d \in \mathbb{Z}} \left\{ a_1(a^2 + b^2) + a_2(c^2 + d^2) + a_3((a - c)^2 + (b - d)^2) + a_4((a - d)^2 + (b + c)^2) \right\}
\]

where

\[
B \overset{\text{def}}{=} 8 \max \left\{ |a_1 + a_3 + a_4|^2, |a_3|^2, |a_4|^2, |a_2 + a_3 + a_4|^2 \right\} / a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + 2a_3a_4.
\]

As shown in Table 2 the theorem can be used to effectively compute Seshadri constants from the coefficients \( a_1, a_2, a_3, a_4 \) of the line bundle.

**Proof.** By the argument employed at the beginning of the proof of Theorem 4.5 our task is to minimize the restriction \( Q \big|_{\mathbb{Z}^4} \). According to Lemma 4.7 the points where this function is minimal lie in the intersection \( \cap_{i=1}^4 U_i \left( \{ x \in \mathbb{R}^n \left| \frac{\partial Q}{\partial x_i}(x) = 0 \right. \right\} \). We have for \( x \in \mathbb{R}^4 \)

\[
\frac{\partial Q}{\partial x_1}(x) = 2(a_1 + a_3 + a_4, 0, -a_3, -a_4) \cdot x,
\]

so the set of points in \( \mathbb{R}^4 \) whose first component has distance 1 from \( \{ \frac{\partial Q}{\partial x_1} = 0 \} \) is the union of the two affine hyperplanes

\[
\{ x \in \mathbb{R}^4 \left| (a_1 + a_3 + a_4, 0, -a_3, -a_4) \cdot x = \pm(a_1 + a_3 + a_4) \right. \}
\]

| \( a_1 \) | \( a_2 \) | \( a_3 \) | \( a_4 \) | \( L^2 \) | \( \sqrt{L^2} \) | \( \varepsilon(L) \) | curves computing \( \varepsilon(L) \) |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 14 | \( \approx 3.74 \) | 3 | \( F_1, F_2 \) |
| 1 | 1 | 0 | 0 | 2 | \( \approx 1.41 \) | 1 | \( F_1, F_2 \) |
| 2 | 1 | 0 | 0 | 4 | 2 | 1 | \( F_1 \) |
| 0 | 0 | 1 | 1 | 4 | 2 | 2 | \( F_1, F_2, \Delta, \Sigma, N_{1,1,0,1}, N_{1,0,1,1} \) |
| 1 | 0 | 1 | 1 | 8 | \( \approx 2.83 \) | 2 | \( F_1 \) |
| 1 | 1 | 1 | 0 | 6 | \( \approx 2.45 \) | 2 | \( F_1, F_2, \Delta \) |
| 2 | 2 | 1 | -1 | 4 | 2 | 2 | \( F_1, F_2, \Delta, N_{1,1,1,0}, N_{1,0,1,-1}, N_{1,0,0,-1} \) |
| 1 | 1 | 2 | 2 | 14 | \( \approx 3.74 \) | 3 | \( F_2, N_{1,1,0,1} \) |
| -1 | 1 | 2 | 2 | 10 | \( \approx 3.16 \) | 2 | \( F_2 \) |
| 4 | 4 | -1 | -1 | 4 | 2 | 2 | \( F_1, F_2, N_{1,1,0,-1}, N_{1,0,0,-1}, N_{1,0,-1,0}, N_{-1,0,0,1} \) |
| 4 | 2 | 3 | -2 | 4 | 2 | 1 | \( N_{0,1,1,1} \) |
| 8 | 5 | -1 | -2 | 10 | \( \approx 3.16 \) | 2 | \( F_1 \) |
and consequently \( U_1(\{ \frac{2Q}{a_x} = 0 \} ) \) is the set of points between these two hyperplanes.

The intersection \( \bigcap_{i=1}^{4} U_i(\{ \frac{2Q}{a_x} = 0 \} ) \) is therefore a paralleloid, whose vertices are the solutions of the sixteen equations

\[
M \cdot x = \begin{pmatrix}
\pm (a_1 + a_3 + a_4) \\
\pm (a_1 + a_3 + a_4) \\
\pm (a_2 + a_3 + a_4) \\
\pm (a_2 + a_3 + a_4)
\end{pmatrix}
\]

where \( M \) is the matrix defining \( Q \) in (4.4.1). The lengths of these vertices, and therefore of all points in the paralleloid, are bounded from above by

\[
\|x\| \leq \|M^{-1}\| \cdot \left( a_1 + a_3 + a_4 \right),
\]

and with a computation one finds that the right hand side is in turn bounded by the number \( B \) defined in the statement of the theorem.

\[ \square \]

### 4.2 Complex multiplication by \( e^{\pi i/3} \)

We now turn to the elliptic curve \( E_2 \) with automorphism \( \sigma: [x] \mapsto [e^{\pi i/3}x] \) and study the surface \( X = E_2 \times E_2 \). A result analogous to Theorem 4.8 holds in this case:

**Theorem 4.9** Let \( E_2 \) be the elliptic curve admitting the automorphism \( [x] \mapsto [e^{\pi i/3}x] \), i.e., \( E_2 = \mathbb{C}/\mathbb{Z} + e^{\pi i/3} \mathbb{Z} \), and let \( L = \mathcal{O}_X(a_1F_1 + a_2F_2 + a_5\Delta + a_4\Sigma) \) be any ample line bundle on the abelian surface \( X = E_2 \times E_2 \). Then

\[
\varepsilon(L) = \min_{a,b,c,d \in \mathbb{Z}} \left\{ a_1(a^2 + ab + b^2) + a_2(c^2 + cd + d^2) \right. \\
+ a_3((a - c)^2 + (a - c)(b - d) + (b - d)^2) \\
\left. + a_4((-a - b + d)^2 + (-a - b + d)(b + c) + (b + c)^2) \right\},
\]

where

\[
B \overset{\text{def}}{=} \frac{8 \max \left\{ 2a_1 + 2a_3 + 2a_4 \right\}^2, 2a_3 + 2a_4, |a_3 + 2a_4|^2, |a_3 - 2a_4|^2, 2a_2 + 2a_3 + 2a_4 \right\}}{3(a_1a_2 + a_1a_3 + a_1a_4 + a_2a_3 + a_2a_4 + a_3a_4)}.
\]

While the proof follows the same general strategy that we used for Theorem 4.8, it is not totally analogous. In the remainder of this section we will indicate the course of the argument, mainly emphasizing the new aspects and formulas, without repeating arguments that can be adapted from the previous case.

The automorphism \( \sigma \) has the point \([0]\) as its only fixed point. The fibers \( F_1, F_2 \), the diagonal \( \Delta \), and the graph \( \Sigma \) of \( \sigma \) generate the Néron-Severi group of \( X \), and they have the intersection numbers

\[
F_1^2 = F_2^2 = \Delta^2 = \Sigma^2 = 0
\]

and

\[
F_1 \cdot F_2 = F_1 \cdot \Delta = F_2 \cdot \Delta = F_1 \cdot \Sigma = F_2 \cdot \Sigma = \Delta \cdot \Sigma = 1.
\]
A line bundle \( L = \mathcal{O}_X(a_1 F_1 + a_2 F_2 + a_3 \Delta + a_4 \Sigma) \) is ample if and only if
\[
\begin{align*}
    a_2 + a_3 + a_4 &= L \cdot F_1 > 0, \quad a_1 + a_3 + a_4 = L \cdot F_2 > 0, \\
    a_1 + a_2 + a_4 &= L \cdot \Delta > 0, \quad a_1 + a_2 + a_3 = L \cdot \Sigma > 0, \\
    2(a_1 a_2 + a_1 a_3 + a_1 a_4 + a_2 a_3 + a_2 a_4 + a_3 a_4) &= L^2 > 0.
\end{align*}
\]
As before, the elliptic curves on \( X \) are given as the images \( N_{a,b,c,d} \) under suitable maps \( E_2 \times E_2 \to X \). For their intersection numbers one obtains
\[
\begin{align*}
    N_{a,b,c,d} \cdot F_1 &= \frac{a^2 + ab + b^2}{D} \\
    N_{a,b,c,d} \cdot F_2 &= \frac{c^2 + cd + d^2}{D} \\
    N_{a,b,c,d} \cdot \Delta &= \frac{(a - c)^2 + (a - c)(b - d) + (b - d)^2}{D} \\
    N_{a,b,c,d} \cdot \Gamma &= \frac{(-a - b + d)^2 + (-a - b + d)(b + c) + (b + c)^2}{D}
\end{align*}
\]
where one sets
\[
D \overset{\text{def}}{=} \gcd(a^2 + ab + b^2, c^2 + cd + d^2, ac + bc + bd, ad - bc).
\]
In order to see this, one proves statements similar to Lemma 4.2 and Lemma 4.3.

The next step is then to show that there is a submaximal elliptic curve for every ample line bundle on \( X \). This is accomplished by considering the quadratic form \( Q(a,b,c,d) \) given by the matrix
\[
\begin{pmatrix}
    a_1 + a_3 + a_4 & \frac{1}{2}(a_1 + a_3 + a_4) & \frac{1}{2}(-2a_3 - a_4) & \frac{1}{2}(-a_3 - 2a_4) \\
    \frac{1}{2}(a_1 + a_3 + a_4) & a_1 + a_3 + a_4 & \frac{1}{2}(-a_3 + a_4) & \frac{1}{2}(-2a_3 - a_4) \\
    \frac{1}{2}(-2a_3 - a_4) & \frac{1}{2}(-a_3 + a_4) & a_2 + a_3 + a_4 & \frac{1}{2}(a_2 + a_3 + a_4) \\
    \frac{1}{2}(-a_3 - 2a_4) & \frac{1}{2}(-2a_3 - a_4) & \frac{1}{2}(a_2 + a_3 + a_4) & a_2 + a_3 + a_4
\end{pmatrix}
\]
which governs the intersection numbers \( L \cdot N_{a,b,c,d} \). Finally, a minimization argument then leads to the estimates in Theorem 4.9. A crucial auxiliary lemma that is needed for the proof (in the same way as Lemma A.2 is required for Theorem 4.8) is stated in the appendix as Lemma A.3.

### Appendix

We state and prove here the elementary number-theoretic lemmas that are needed in the course of Sect. 4.

**Lemma A.1** Let \( a, b, c, d \) be coprime integers, and let
\[
D = \gcd(a^2 + b^2, c^2 + d^2, ac + bd, ad - bc).
\]
Then the system of congruences
\[
\begin{align*}
    am - bn &\equiv 0 \pmod{D} \quad \text{(A.1.1)} \\
    bm + an &\equiv 0 \pmod{D} \quad \text{(A.1.2)} \\
    cm - dn &\equiv 0 \pmod{D} \quad \text{(A.1.3)} \\
    dm + cn &\equiv 0 \pmod{D} \quad \text{(A.1.4)}
\end{align*}
\]
has exactly \( D \) solutions \( (m, n) \) modulo \( D \).
Lemma A.2. Let $a, b, c, d$ be coprime integers, and let

$$D = \gcd(a^2 + b^2, c^2 + d^2, ac + bd, ad - bc).$$

Then there are coprime integers $\overline{a}, \overline{b}, \overline{c}, \overline{d}$ such that

$$\gcd(\overline{a}^2 + \overline{b}^2, \overline{c}^2 + \overline{d}^2, ac + bd, ad - bc) = 1.$$
and

\[
\begin{align*}
\pi^2 + \overline{\beta}^2 &= \frac{1}{D} (a^2 + b^2), \\
\pi^2 + \overline{\gamma}^2 &= \frac{1}{D} (c^2 + d^2), \\
\overline{\alpha} \overline{c} + \overline{\beta} \overline{d} &= \frac{1}{D} (ac + bd), \\
\overline{\alpha} \overline{d} - \overline{\beta} \overline{c} &= \frac{1}{D} (ad - bc).
\end{align*}
\]

**Proof.** The idea is to consider matrices \( M = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \) over \( \mathbb{R} \) such that the two image vectors

\[
\begin{pmatrix} \pi \\ \beta \end{pmatrix} \overset{\text{def}}{=} M \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \alpha a + \beta b \\ -\beta a + \alpha b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \pi \\ \alpha \end{pmatrix} \overset{\text{def}}{=} M \cdot \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} \alpha c + \beta d \\ -\beta c + \alpha d \end{pmatrix}
\]

are integral. If \( M \) is such a matrix, then \((a^2 + b^2)\alpha = a(\alpha a + \beta b) + b(-\beta a + \alpha b) \in \mathbb{Z}\) and similarly \((c^2 + d^2)\alpha \in \mathbb{Z}, (ac + bd)\alpha \in \mathbb{Z}, \) and \((ad - bc)\alpha \in \mathbb{Z}.\) Consequently \(\alpha,\) and for the same reason \(\beta,\) are necessarily of the form \(\alpha = \frac{y}{D}, \beta = \frac{y}{D}\) for some integers \(x, y.\) The conditions (A.2.1) are then equivalent to

\[
\begin{align*}
ax + by &\equiv 0 \mod D, \\
bx - ay &\equiv 0 \mod D, \\
cx + dy &\equiv 0 \mod D, \\
dx - cy &\equiv 0 \mod D.
\end{align*}
\]

Now, the proof of Lemma [A.1] shows that this system is solvable even with a prescribed value for \(x.\) Let then \(y\) be the solution associated with \(x = 1.\) We have modulo \(D\) the equivalence \(0 \equiv x(ax + by) - y(bx - ay) = a(x^2 + y^2),\) and similarly \(0 \equiv b(x^2 + y^2),\) as well as \(0 \equiv c(x^2 + y^2)\) and \(0 \equiv d(x^2 + y^2).\) As \(a, b, c, d\) are coprime, this implies \(x^2 + y^2 \equiv 0,\) i.e., \(x^2 + y^2\) is an integer. We find

\[
\begin{align*}
\overline{\alpha}^2 + \overline{\beta}^2 &= \frac{x^2 + y^2}{D^2} (a^2 + b^2), \\
\overline{\alpha} \overline{c} + \overline{\beta} \overline{d} &= \frac{x^2 + y^2}{D^2} (ac + bd), \\
\overline{\alpha} \overline{d} - \overline{\beta} \overline{c} &= \frac{x^2 + y^2}{D^2} (ad - bc).
\end{align*}
\]

The number \(\overline{D} \overset{\text{def}}{=} \gcd(\overline{\alpha}^2 + \overline{\beta}^2, \overline{\alpha} \overline{c} + \overline{\beta} \overline{d}, \overline{\alpha} \overline{d} - \overline{\beta} \overline{c})\) satisfies

\[
\overline{D} = \frac{x^2 + y^2}{D^2} \leq \frac{1 + (D - 1)^2}{D^2} D,
\]

where the right hand side is smaller than \(D\) if \(D > 1.\) We can now repeat the argument until eventually \(\overline{D} = 1.\)

The following lemma can be proven using similar arguments. We leave the details to the reader.

**Lemma A.3** Let \(a, b, c, d\) be coprime integers, and let

\[
D = \gcd(a^2 + ab + b^2, c^2 + cd + d^2, ac + bc + bd, ad - bc).
\]

Then there are coprime integers \(\pi, \overline{\beta}, \overline{\gamma}, \overline{\delta}\) such that

\[
\gcd(\pi^2 + \overline{\beta}^2, \overline{\gamma}^2 + \overline{\delta}^2, \overline{\alpha} \overline{c} + \overline{\beta} \overline{d}, \overline{\alpha} \overline{d} - \overline{\beta} \overline{c}) = 1.
\]

and

\[
\begin{align*}
\pi^2 + \overline{\beta}^2 &= \frac{1}{D} (a^2 + ab + b^2), \\
\overline{\gamma}^2 + \overline{\delta}^2 &= \frac{1}{D} (c^2 + cd + d^2), \\
\overline{\alpha} \overline{c} + \overline{\beta} \overline{d} &= \frac{1}{D} (ac + bc + bd), \\
\overline{\alpha} \overline{d} - \overline{\beta} \overline{c} &= \frac{1}{D} (ad - bc).
\end{align*}
\]
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