POSITIVE BIDIAGONAL FACTORIZATION
OF TETRADIAGONAL HESSENBERG MATRICES

AMÍLCAR BRANQUINHO\textsuperscript{1,\textdegree}, ANA FOULQUIÉ-MORENO\textsuperscript{2,\textdegree}, AND MANUEL MAÑAS\textsuperscript{3,*}

ABSTRACT. Recently a spectral Favard theorem for bounded banded lower Hessenberg matrices that admit a positive bidiagonal factorization was presented. In this paper conditions, in terms of continued fractions, for an oscillatory tetradiagonal Hessenberg matrix to have such positive bidiagonal factorization are found. Oscillatory tetradiagonal Toeplitz matrices are taken as a case study of matrix that admits a positive bidiagonal factorization. Moreover, it is proved that oscillatory banded Hessenberg matrices are organized in rays, with the origin of the ray not having the positive bidiagonal factorization and all the interior points of the ray having such positive bidiagonal factorization.

1. INTRODUCTION

In this paper we will study for the tetradiagonal Hessenberg matrix of the form

\begin{equation}
T = \begin{bmatrix}
c_0 & 1 & 0 & \cdots & \\
& b_1 & c_1 & 1 & \\
& & a_2 & b_2 & c_2 & 1 & \\
& & & a_3 & b_3 & c_3 & 1 & \\
& & & & \ddots & \ddots & \ddots & \\
\end{bmatrix},
\end{equation}

where we assume that \(a_n > 0\), whether it is possible or not to find a positive bidiagonal factorization (PBF)

\begin{equation}
T = L_1L_2U,
\end{equation}

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\end{itemize}
with bidiagonal matrices given by

\[
L_1 = \begin{bmatrix}
1 & 0 & 0 & \cdots \\
\alpha_2 & 1 & 0 & \cdots \\
0 & \alpha_5 & 1 & \cdots \\
& & & & \ddots
\end{bmatrix}, \quad L_2 = \begin{bmatrix}
1 & 0 & 0 & \cdots \\
\alpha_3 & 1 & 0 & \cdots \\
0 & \alpha_6 & 1 & \cdots \\
& & & & \ddots
\end{bmatrix}, \quad U = \begin{bmatrix}
\alpha_1 & 1 & 0 & \cdots \\
0 & \alpha_4 & 1 & \cdots \\
0 & 0 & \alpha_7 & \cdots \\
& & & & \ddots
\end{bmatrix},
\]

with the fulfillment of the following positivity requirement

\[
\alpha_j > 0, \quad j \in \mathbb{N}.
\]

In [5] this factorization was shown to be sufficient for a Favard theorem for bounded banded Hessenberg semi-infinite matrices (with \(p + 2\) diagonals) and the existence of positive measures such that the recursion polynomials are multiple orthogonal polynomials and the Hessenberg matrix is the recursion matrix for this sequence of multiple orthogonal polynomials. Let us also mention [13] were the authors present an important application of this factorization in order to obtain stochastic bidiagonal factorizations of stochastic Hessenberg matrices.

Finite truncations of these matrices having this PBF are oscillatory matrices. In fact, we will be dealing in this paper with totally nonnegative matrices that are oscillatory matrices and, consequently, we require of some definitions and properties that we are about to present succinctly.

Totally nonnegative (TN) matrices are those with all their minors nonnegative [10, 12], and the set of nonsingular TN matrices is denoted by \(\text{InTN}\). Oscillatory matrices [12] are totally nonnegative, irreducible [14] and nonsingular. Notice that the set of oscillatory matrices is denoted by \(\text{IITN}\) (irreducible invertible totally nonnegative) in [10]. An oscillatory matrix \(A\) is equivalently defined as a totally nonnegative matrix \(A\) such that for some \(n\) we have that \(A^n\) is totally positive (all minors are positive). From Cauchy–Binet Theorem one can deduce the invariance of these sets of matrices under the usual matrix product. Thus, following [10, Theorem 1.1.2] the product of matrices in \(\text{InTN}\) is again \(\text{InTN}\) (similar statements hold for TN or oscillatory matrices).

We have the important result:

**Theorem 1** (Gantmacher–Krein Criterion). [12, Chapter 2, Theorem 10] A totally nonnegative matrix \(A\) is oscillatory if and only if it is nonsingular and the elements at the first subdiagonal and first superdiagonal are positive.

Let us discuss the connection of oscillatory matrices with the standard case, i.e. for tridiagonal semi-infinite matrices, named in the literature as Jacobi matrices.

**Definition 1.** Jacobi matrices are tridiagonal real matrices

\[
J := \begin{bmatrix}
m_0 & 1 & 0 & \cdots \\
\ell_1 & m_1 & 1 & \cdots \\
0 & \ell_2 & m_2 & 1 & \cdots \\
& & & & \ddots
\end{bmatrix},
\]
with \( \ell_j > 0, \ j = 1, 2, \ldots \) and its finite truncations, that is leading principal submatrices, are

\[
J^{[N]} := \begin{bmatrix}
m_0 & 1 & 0 & \cdots & 0 \\
\ell_1 & m_1 & 1 & \cdots & 0 \\
0 & \ell_2 & m_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \ell_N & m_N
\end{bmatrix} \in \mathbb{R}^{(N+1) \times (N+1)}, \quad \Delta_N := \det J^{[N]},
\]

further important submatrices are

\[
J^{[N,k]} := \begin{bmatrix}
m_k & 1 & 0 & \cdots & 0 \\
\ell_{k+1} & m_{k+1} & 1 & \cdots & 0 \\
0 & \ell_{k+2} & m_{k+2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & \ell_N & m_N
\end{bmatrix} \in \mathbb{R}^{(N+1-k) \times (N+1-k)}, \quad \Delta_{N,k} := \det J^{[N,k]}.
\]

Regarding oscillatory Jacobi matrices we have

**Theorem 2** (Oscillatory Jacobi matrices). [11, Chapter XIII,§9] and [12, Chapter 2,Theorem 11].

A tridiagonal matrix is oscillatory if and only if,

i) The matrix entries of the first subdiagonal and first superdiagonal are positive.

ii) All leading principal minors are positive.

If the matrix \( J \) is bounded, i.e. \( |J|_\infty < \infty \), all the possible eigenvalues of the submatrices \( J^{[N]} \) belong to the disk \( D(0, |J|_\infty) \). As all the eigenvalues are real, let us consider those that are negative, and let \( b \) be the supreme of the absolute values of all negative eigenvalues. Notice that \( b \leq |J|_\infty \).

**Theorem 3.** For \( s \geq b \) the matrix \( J_s = J + sI \) is oscillatory.

**Proof.** Take \( s \geq b \), then \( J_s \) has the eigenvalues of its leading principal submatrices \( J_s^{[N]} = J^{[N]} + sI_{N+1} \) all positive. The corresponding characteristic polynomials are \( P_{N+1}(x-s) = \det (xI_{N+1} - J_s^{[N]}) \), so that \( \det J_s^{[N]} = (-1)^{N+1}P_{N+1}(-s) \), but as \(-s\) is a lower bound for any possible zero of this monic polynomial, we have that \((-1)^{N+1}P_{N+1}(-s) > 0\). Hence, the leading principal minors of \( J_s \) are all positive and the entries on the subdiagonal a superdiagonal are positive. Thus we conclude, attending to Theorem 2, that \( J_s \) is an oscillatory matrix.

A very important consequence of this fact, i.e., that there exists a positive \( s \) such that \( J + sI \) is oscillatory is that all eigenvalues are simple, and that \( P_{N+1} \) interlaces \( P_N \) and \( P_{N+1}^{(1)} \). Indeed, the characteristic polynomial \( P_{N+1}(x-s) \) of the oscillatory matrix \( J_s^{[N]} \) interlaces the characteristic polynomials of the submatrices \( J_s^{[N]}(1) = J_s^{[N,1]} \), i.e. \( P_{N+1}^{(1)}(x-s) \), and of \( J_s^{[N]}(N+1) = J_s^{[N-1]} \), i.e. \( P_N(x-s) \). Hence, we deduce the positivity of the Christoffel coefficients from the oscillatory character \( J_s^{[N]} \). Thus, the spectral Favard theorem, see for example [22, §4.1], is a theorem for bounded Jacobi matrices which are oscillatory up to a appropriate shift. Then, all the interlacing properties follow immediately.

Let us show that for Jacobi matrices the PBF and oscillatory properties are equivalent.
Proposition 1. A Jacobi matrix is oscillatory if and only if it admits a PBF.

Proof. Let us assume that the Jacobi matrix $J_{[N,1]}$ is oscillatory. Then, the Gauss–Borel factorization of $J_{[N,1]}$, i.e.
\[
\begin{bmatrix}
m_1 & 1 & 0 & \cdots & 0 \\
\ell_2 & m_2 & 1 & \cdots & 1 \\
0 & \ell_3 & m_3 & \cdots & 1 \\
& & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \ell_N & m_N
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & \cdots & 0 \\
\beta_2 & 0 & \cdots & 0 \\
0 & \beta_4 & \cdots & 0 \\
& & \ddots & \ddots \\
0 & \cdots & 0 & \beta_{2N-2} & 1 \\
& & & & \beta_{2N-1}
\end{bmatrix},
\]
leads to $m_1 = \beta_1$ as well as $m_n = \beta_{2n-2} + \beta_{2n-1}$ and $\ell_n = \beta_{2n-2}\beta_{2n-3}$, $\beta_0 := 0$. Hence, as $\ell_n > 0$, $n \in \{2, 3, \ldots\}$, we get that $\beta_n > 0$ for $n \in \mathbb{N}$.

If the Jacobi matrix $J_{[N,1]}$ admits a PBF, we deduce that is oscillatory from the Gatmancher–Krein Criterion Theorem 1.

The structure of the paper is as follows. Section 2 is devoted to introduce truncations and continued fractions needed for the discussion. Next, in § 3 we prove, in the finite case, the existence of the PBF for tetradiagonal matrices in terms of positive finite continued fractions. Then, the extension of this PBF for the semi-infinite case is shown to happen when certain nonnegative infinite continued fraction is indeed positive. In § 4 we discuss oscillatory tetradiagonal Toeplitz–Hessenberg matrices as a case study of matrix that admits a PBF. Finally, in § 5 we show that from any given oscillatory tetradiagonal matrix we can construct tetradiagonal matrices with a PBF in several ways. Also, we will find that oscillatory matrices are organized in rays, the origin of the ray is a oscillatory matrix that don’t have a PBF and all the interior points of the ray are PBF matrices. We also show that for any PBF tetradiagonal matrix there is a retraction that is oscillatory but without a PBF and vice versa.

2. Factorization properties

We now discuss some aspects of the Gauss–Borel factorization and PBF of oscillatory tetradiagonal matrices. For more on Gauss–Borel factorization and orthogonal polynomials see for example [19] and references therein.

First, let us introduce some convenient notation for the tetradiagonal case. Let us denote by $T^{[N]} = T[\{0, 1, \ldots, N\}] \in \mathbb{R}^{(N+1) \times (N+1)}$ the $(N + 1)$-th leading principal submatrix of the banded Hessenberg matrix $T$:
\[
T^{[N]} := \begin{bmatrix}
c_0 & 1 & 0 & \cdots & 0 \\
\ell_1 & c_1 & 1 & \cdots & 1 \\
\ell_2 & b_2 & c_2 & 1 & \cdots & 1 \\
0 & \ell_3 & b_3 & c_3 & \cdots & 1 \\
& & \ddots & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \ell_N & b_N & c_N
\end{bmatrix}, \quad \delta^{[N]} := \det T^{[N]}.
\]
Further truncations are
\[
T^{[N,k]} := \begin{bmatrix}
 c_k & 1 & 0 & \cdots & 0 \\
 b_{k+1} & c_{k+1} & 1 & \cdots & 0 \\
 a_{k+2} & b_{k+2} & c_{k+2} & 1 & \cdots \\
 & a_{k+3} & b_{k+3} & c_{k+3} & \cdots \\
 & & & \ddots & \ddots \\
 & & & & a_{N-1} & b_{N-1} & c_{N-1} & 1 \\
 & & & & & a_N & b_N & c_N \\
\end{bmatrix} \in \mathbb{R}^{(N+1-k) \times (N+1-k)}, \ k \in \{0, 1, \ldots, N\},
\]

notice that \(T^{[N,N+1]} := 1\) and \(T^{[N]} = T^{[N,0]}\).

The Gauss–Borel factorization of the leading principal submatrices \(T^{[N]}\) in (7) is

(8)
\[
T^{[N]} = L^{[N]} U^{[N]}, \quad L^{[N]} := \begin{bmatrix}
1 & 0 & \cdots & 0 \\
m_1 & 1 & \cdots & 0 \\
m_2 & m_3 & \cdots & 0 \\
& m_3 & \cdots & \ddots \\
& & \ddots & \ddots & 0 \\
& & & \ddots & \ddots \\
& & & & \ell_N & m_N & 1 \\
\end{bmatrix}, \quad U^{[N]} := \begin{bmatrix}
\alpha_1 & 1 & 0 & \cdots & 0 \\
0 & \alpha_4 & \cdots & 0 \\
& 0 & \alpha_7 & \cdots & 0 \\
& & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots \\
& & & & 0 & \alpha_{3N+1} \\
\end{bmatrix}.
\]

**Proposition 2.** The Gauss–Borel factorization (8) exists if and only if all leading principal minors \(\delta^{[n]}\), \(n \in \{0, 1, \ldots, N\}\), of \(T^{[N]}\) are not zero. For \(n \in \mathbb{N}\), the following expressions for the coefficients hold

(9)
\[
\ell_{n+1} = \frac{a_n \delta^{[n-2]}}{\delta^{[n-1]}}, \quad m_n = c_n - \frac{\delta^{[n]}}{\delta^{[n-1]}}, \quad \alpha_{3n-2} = \frac{\delta^{[n-1]}}{\delta^{[n-2]}},
\]

where \(\delta^{[-1]} = 1\) and \(a_1 = 0\), and we have the following recurrence relation for the determinants

(10)
\[
\delta^{[n]} = a_n \delta^{[n-3]} - b_n \delta^{[n-2]} + c_n \delta^{[n-1]},
\]

is satisfied.

**Proof.** Notice that \(\delta^{[N]} = \det T^{[N]} = \det U^{[N]} = \alpha_1 \alpha_4 \cdots \alpha_{3N+1}\). Hence, we get \(\alpha_{3N+1} = \frac{\delta^{[N]}}{\delta^{[N-1]}}\). From the last row of the \(LU\) factorization we get
\[
a_N = \ell_N \alpha_{3N-5}, \quad b_N = \ell_N + m_N \alpha_{3N-2}, \quad c_N = m_N + \alpha_{3N+1},
\]

so that
\[
\ell_N = \frac{\delta^{[N-3]}}{\delta^{[N-2]}} a_N, \quad m_N = c_N - \frac{\delta^{[N]}}{\delta^{[N-1]}},
\]

and
\[
b_N - a_N \frac{\delta^{[N-3]}}{\delta^{[N-2]}} - \frac{\delta^{[N-1]}}{\delta^{[N-2]}} (c_N - \frac{\delta^{[N]}}{\delta^{[N-1]}}) = 0,
\]

that is we have (10). \(\square\)

Note that, in this proof, we could get (10) by expanding the determinant \(\delta^{[N]}\) along the last row.

**Proposition 3.** Let us assume that \(T^{[N]}\) given in (7) is an oscillatory matrix. Then,
\[
a_n, b_n, c_n > 0.
\]
Proof. According to the Gantmacher–Krein Criterion, Theorem 1, [12, II.7 Theorem 10], \( b_n > 0 \) for \( n \in \{1, \ldots, N\} \). Moreover, as is an invertible totally nonnegative matrix (InTN) according to [10, page 50, Chapter 2] we also need \( c_n \) to be positive.

We introduce some auxiliary submatrices that will be instrumental in the following developments.

**Definition 2 (Auxiliary submatrices).** Given the lower triangular factor \( L^{[N]} \), determined by the Gauss–Borel factorization (8), we consider its complementary submatrix, by deleting first row and last column, that we call the auxiliary Jacobi matrix, \( J^{[N,1]} = L^{[N]}(\{1\}, \{N+1\}) \in \mathbb{R}^{N \times N} \) as in (6) with \( k = 1 \). For \( k \in \{0, \ldots, N-1\} \), associated with the auxiliary Jacobi matrix \( J^{[N,1]} \) we introduce, as we did with the banded Hessenberg matrix \( T^{[N]} \), the principal submatrices \( J^{[N,k+1]} \) defined in (6). Additionally, we introduce \( T_1^{[N]} = T(\{1\}, \{N+1\}) \in \mathbb{R}^{N \times N} \) as the complementary submatrix obtained by removing the first row and last column of \( T^{[N]} \), that is

\[
T_1^{[N]} := \begin{bmatrix}
    b_1 & c_1 & 1 & 0 & \cdots & 0 \\
    a_2 & b_2 & c_2 & 1 & \cdots & 0 \\
    0 & a_3 & b_3 & c_3 & 1 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & \cdots & 0 & a_{N-1} & b_{N-1} & c_{N-1} & 1 \\
    0 & \cdots & 0 & a_N & b_N & b_N & \cdots & 0 \\
    \end{bmatrix}, \quad \delta_1^{[N]} := \det T_1^{[N]}.
\]

Further auxiliary complementary submatrices that we will consider are

\[
T_1^{[N,k]} := \begin{bmatrix}
    b_{k+1} & c_{k+1} & 1 & 0 & \cdots & 0 \\
    a_{k+2} & b_{k+2} & c_{k+2} & 1 & \cdots & 0 \\
    0 & a_{k+3} & b_{k+3} & c_{k+3} & 1 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    0 & \cdots & 0 & a_{N-1} & b_{N-1} & c_{N-1} & 1 \\
    0 & \cdots & 0 & a_N & b_N & b_N & \cdots & 0 \\
    \end{bmatrix}, \quad \delta_1^{[N,k]} := \det T_1^{[N,k]},
\]

so that \( T_1^{[N,0]} = T_1^{[N]} \) and \( \delta_1^{[N,0]} = \delta_1^{[N]} \), and the upper bidiagonal matrix

\[
U_1^{[N-1,k]} = \begin{bmatrix}
    \alpha_{3k+1} & 1 & 0 & \cdots & 0 \\
    0 & \alpha_{3k+4} & \alpha_{3k+7} & 0 & \cdots & 0 \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
    0 & \cdots & 0 & \alpha_{3N-2} & 1 & \cdots & 0 \\
\end{bmatrix}.
\]

A set of finite continued fractions will be needed in the subsequent analysis. For continued fractions we refer to the interested reader to [16, 18, 23].
**Definition 3** (Finite continued fractions). We introduce

\[ \mathcal{K}[n, k] := m_k - \frac{\ell_{k+1}}{m_{k+1}}, \quad n \in \{k + 1, k + 2, \ldots\}, \quad \mathcal{K}[k + 1, k] := m_k. \]

**Theorem 4** (Determinants and continued fractions). Let us assume that \( T^{[N]} \) as in (7) is an oscillatory matrix. Then, the following holds:

i) For the triangular factors in (8) we have \( L^{[N]}, U^{[N]} \in \text{ln} T \mathcal{N} \).

ii) For \( n \in \mathbb{N} \), the matrix entries of the triangular factors of Gauss–Borel factorization (8) of the Hessenberg matrix \( T^{[N]} \) are positive: \( \ell_2, \ldots, \ell_N, m_1, \ldots, m_N, \alpha_1, \alpha_4, \ldots, \alpha_{3N-2} > 0 \).

iii) For \( k \in \mathbb{N} \), the recurrence relation

\[ D(n + 1) = m_{k+n}D(n) - \ell_{k+n}D(n - 1), \quad n \in \mathbb{N}, \]

for the initial conditions \( D(0) = 1, D(1) = m_k \), has as solution \( D(n) = \Delta_{k+n-1,k} \), while for the initial conditions \( D(0) = 0 \) and \( D(1) = 1 \) has as solution \( \Delta_{k+n-1,k+1} \). The determinants \( \Delta_{N,k} \) were defined in (6).

iv) The ratio of consecutive determinants are bounded as follows:

\[
\frac{a_{n+2}}{b_{n+2}} < \frac{\delta[n]}{\delta[n-1]} < c_n, \quad \frac{\ell_{n+1}}{m_{n+1}} < \frac{\Delta_{n,1}}{\Delta_{n-1,1}} < m_n.
\]

v) For \( k \in \mathbb{N} \), the continued fraction given in Definition 3 is the ratio of the consecutive determinants defined in Equation (6)

\[ \mathcal{K}[n, k] = \frac{\Delta_{n,k}}{\Delta_{n,k+1}}. \]

vi) For \( k = 1 \) the determinants in (6) are positive, i.e. \( \Delta_{n,1} > 0 \).

**Proof.**

i) As \( T^{[N]} \) is TN, we know it has a Gauss–Borel factorization with totally nonnegative factors, see [10, Theorem 2.4.1].

ii) For \( n \in \{2, 3 \ldots, N\} \) we have \( a_n = \ell_n \alpha_{3n-5} \) and, consequently, as \( a_n \neq 0 \) we deduce that \( \ell_2, \ldots, \ell_N, \alpha_1, \alpha_4, \ldots, \alpha_{3N-2} > 0 \). Moreover,

\[ \left| \frac{m_n}{\ell_{n+1}} \right| \frac{1}{m_{n+1}} \geq 0, \]

and as \( \ell_{n+1} > 0 \) we deduce that \( m_nm_{n+1} \neq 0 \), and all \( m_1, \ldots, m_N > 0 \).

iii) Expand the determinant \( \Delta_{n,k} \) along the last row. One can check that the initial conditions lead to the sequence of determinants.

iv) From (9) and \( m_n > 0 \) we get \( b_n \delta[n-2] > a_n \delta[n-3], c_n \delta[n-1] > \delta[n] \) and the first inequality follows. For the second we use the proof of Proposition 1. From the factorization we get \( \Delta_{n,1} = \beta_1 \cdots \beta_{2n-1} \) and, consequently, \( \beta_{2n-1} = \frac{\Delta_{n,1}}{\Delta_{n-1,1}} \) and \( \beta_{2n-2} = m_n - \frac{\Delta_{n,1}}{\Delta_{n-1,1}} \). As \( \beta_n > 0 \),

\[ ... \]
$n \in \mathbb{N}$, we deduce that $m_n > \frac{\Delta_{n,1}}{\Delta_{n-1,1}}$. As for the oscillatory case we require $\Delta_n > 0$, the recursion relation (13), i.e., $\Delta_{n,1} = m_n \Delta_{n-1,1} - \ell_n \Delta_{n-2,1}$, implies that $m_n \Delta_{n-1,1} > \ell_n \Delta_{n-2,1}$ and the lower bound follows immediately.

v) Use the Euler–Wallis theorem for continued fractions, see for example [9, Theorem 9.2].

vi) The first two determinants are positive, then we apply induction. Let us assume that $\Delta_{n-1,1} > 0$, and that $\Delta_{n,1} = 0$. Then, for $k = 0$ Equation (13) implies that $\Delta_{n+1,1} = -\ell_{n+1} \Delta_{n-1,1} < 0$ in contradiction with the fact $\Delta_{n+1,1} \geq 0$.

\[ \Box \]

**Theorem 5** (Factorizations and oscillatory matrices). For the submatrices of $T^{[N]}$ and its determinants introduced in Definition 2 we find that:

i) The auxiliary Jacobi matrix $J^{[N,1]}$ is oscillatory.

ii) The following factorizations are fulfilled

\[ T_1^{[N]} = J^{[N,1]} U^{[N-1]}, \]  
\[ T_1^{[N,k]} = m_{k+1} E_{1,1} = J^{[N,k+1]} U^{[N-1,k]}. \]

Moreover, $\delta_1^{[N]} > 0$ and we have the following relation between determinants

\[ \Delta_{N,1} = \frac{\delta_1^{[N]}}{\delta_1^{[N-1]}}, \]
\[ \Delta_{N,k+1} = \alpha_1 \cdots \alpha_{3k-2} \frac{\delta_1^{[N,k]} - m_{k+1} \delta_1^{[N,k+1]}}{\delta_1^{[N-1]}}. \]

(Recall that $\Delta^{[N,k]} := \det J^{[N,k]}$, $\delta_1^{[N]} := \det T_1^{[N]}$ and $\delta_1^{[N,k]} := \det T_1^{[N,k]}$.)

iii) The submatrix $T_1^{[N]}$ is oscillatory.

iv) The submatrices $J^{[N,k+1]}$ and $T_1^{[N,k]}$ are oscillatory. In particular, $\Delta_{N,k+1}, \delta_1^{[N,k]} > 0$.

v) The following relations are satisfied

\[ \Delta_{N,2} \delta_1^{[N-1]} = c_0 \delta_1^{[N,1]} - a_2 \delta_1^{[N,2]} , \]
\[ \Delta_{N,1} = \frac{\delta_1^{[N]}}{c_0 \delta_1^{[N,1]} - a_2 \delta_1^{[N,2]}}. \]

vi) The recursion relation in $k$ is satisfied

\[ \Delta_{N,k+1} = m_{k+1} \Delta_{N,k+2} - \ell_{k+2} \Delta_{N,k+3}. \]

**Proof.**

i) According to Theorem 2, see [11, Chapter XIII,§9] and [12, Chapter 2,Theorem 11], the Jacobi matrix $J^{[N,1]}$ is oscillatory if and only if,

(a) The matrix entries $\ell_2, \ldots, \ell_N$ are positive.

(b) All leading principal minors $\Delta_{n,1}$ are positive.

As we have seen in previous points, both requirements are satisfied.

ii) Equations (14) and (15) follows directly from the Gauss–Borel factorization of $T^{[N]}$. Taking determinants and expanding the determinant along the first row we get (17). From Equation (14) we conclude that $\alpha_1 \alpha_2 \cdots \alpha_{3n-2} \Delta_n = \det T_1^{[n]}$. As previously said all $\alpha_1, \alpha_4, \ldots, \alpha_{3n-2} > 0$ and $\Delta_{n,1} > 0$. Therefore, $\delta_1^{[N]} \neq 0$. 

iii) Given that the matrix $T_1^{[N]}$ belongs to InTN with $a_n, b_n, c_n > 0$, see Proposition 3, the Gantmacher–Krein Criterion, Theorem 1, leads to the oscillatory character of this submatrix.

iv) If a matrix $A$ is oscillatory then so is any submatrix $A[\alpha]$ for any contiguous subset of indexes $\alpha$, see [12, Chapter 2, §7] and [10, Corollary 2.6.7]. Then, $J^{[N,k+1]} = J^{[N]}(\{k + 1, \ldots, N\})$ and $T_1^{[N,k]} = T_1^{[N]}(\{k + 1, \ldots, N\})$ are oscillatory and, consequently, $\Delta_{N,k+1} = \det J^{[N]}(\{k + 1, \ldots, N\}) > 0$ and $\delta_1^{[N,k]} = \det T_1^{[N]}(\{k + 1, \ldots, N\}) > 0$.

v) Put $k = 1$ in (17) and recall that $\alpha_1 = c_0$ and $\alpha_1 \ell_2 = a_2$. For Equation (18) use (16).

vi) Expand the determinants along the first row.

A set of convergent infinite continued fractions are important in what follows.

**Definition 4** (Infinite continued fraction and tails). We introduce the following infinite continued fraction

$$K[1] := m_1 - \frac{\ell_2}{m_2 - \frac{\ell_3}{m_3 - \cdots}}.$$  

and its tails

$$K[k + 1] := m_{k+1} - \frac{\ell_{k+2}}{m_{k+2} - \frac{\ell_{k+3}}{m_{k+3} - \cdots}}, \quad k \in \mathbb{N}.$$  

**Corollary 1.** The infinite continued fraction in (20) can be computed as the following large $N$ limit ratio

$$K[1] = \lim_{N \to \infty} \frac{\delta_1^{[N]}}{c_0 \delta_1^{[N,1]} - d_2 \delta_1^{[N,2]}}.$$  

of determinants given in (11) and (12).

**Proof.** Direct consequence of (18). \qed

Now, an important result follows regarding the behavior of these infinite continued fractions.

**Theorem 6** (Infinite continued fractions). For the infinite continued fractions $K[1]$ given in (20) we have:

i) For $k \in \mathbb{N}_0$, the sequences $\{K[n, k]\}_{n=k+1}^\infty$ of the finite continued fractions given in Definition 3 are positive and strictly decreasing.

ii) The infinite continued fraction $K[1]$ converges and is nonnegative.

iii) The tails converge and are positive, i.e. $K[k + 1] > 0$ for $k \in \mathbb{N}$.

**Proof.** i) The positivity follows at once from the positivity of $\Delta_{N,k}$. From (19) we have

$$\frac{\Delta_{N+1,k+1}}{\Delta_{N+1,k+2}} = m_{k+1} - \frac{\ell_{k+2}}{\Delta_{N+1,k+3}}, \quad \frac{\Delta_{N,k+1}}{\Delta_{N,k+2}} = m_{k+1} - \frac{\ell_{k+2}}{\Delta_{N,k+3}}.$$
As $m_k, \Delta_{N,k+1} > 0$ the inequality

\[
\frac{\Delta_{N+1,k+1}}{\Delta_{N+1,k+2}} < \frac{\Delta_{N,k+1}}{\Delta_{N,k+2}},
\]

(23) can be written

\[
m_{k+1} - \frac{\ell_{k+2}}{\Delta_{N+1,k+2}} < m_{k+1} - \frac{\ell_{k+2}}{\Delta_{N,k+2}},
\]

where we have used (22). Therefore, (23) is equivalent to the inequality

\[
\frac{\Delta_{N+1,k+2}}{\Delta_{N+1,k+3}} < \frac{\Delta_{N,k+2}}{\Delta_{N,k+3}}.
\]

Hence, if for $k = N - 2$ the inequality

\[
\frac{\Delta_{N+1,N-1}}{\Delta_{N+1,N}} < \frac{\Delta_{N,N-1}}{\Delta_{N,N}}
\]

(24) is fulfilled, the inequality (23) will hold. But,

\[
\frac{\Delta_{N+1,N-1}}{\Delta_{N+1,N}} = m_N - \frac{\ell_{N+1}}{m_{N+1}}, \quad \frac{\Delta_{N,N-1}}{\Delta_{N,N}} = m_N,
\]

and (24) satisfied.

ii) Obvious from the previous result, any positive decreasing sequence is convergent to a non-negative number.

iii) For $k \in \mathbb{N}$ we have $\mathcal{K}[k] = m_k - \frac{\ell_{k+1}}{\mathcal{K}[k+1]}$, $\ell_{k+1} > 0$, so that the convergence of $\mathcal{K}[k]$ requires $\mathcal{K}[k+1] > 0$.

$\square$

3. Positive bidiagonal factorization of tetradiagonal Hessenberg matrices

Now we discuss how the Gauss–Borel factorization can be used to find a bidiagonal factorization of the banded Hessenberg matrix. This will lead to the appearance of continued fractions in our theory.

**Lemma 1.** The factorization of any lower triangular matrix of the form

\[
L^{[N]} = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & m_1 & 1 & \cdots \\
0 & 0 & m_2 & 1 & \cdots \\
\vdots & \vdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & m_{N-1} & 1 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & m_{N}
\end{bmatrix},
\]
into bidiagonal factors, i.e.,

\[
L^{[N]} = L_1^{[N]}L_2^{[N]}, \quad L_1^{[N]} = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
\alpha_2 & 0 & \cdots & 0 \\
0 & \alpha_5 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \alpha_{3N-1}
\end{bmatrix}, \quad L_2^{[N]} = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
\alpha_3 & 0 & \cdots & 0 \\
0 & \alpha_6 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \alpha_{3N}
\end{bmatrix},
\]

is uniquely determined in terms of \(\alpha_2\), with

\[
\alpha_{3n} = m_n - \frac{\ell_n}{\ell_{n-1}}, \quad \alpha_{3n-1} = \frac{\ell_n}{\ell_{n-1}}.
\]

The factorization exists if and only if \(\alpha_{3n} \neq 0\) for \(n \in \{1, \ldots, N-1\}\).

**Proof.** The factorization (25) implies that

\[
m_n = \alpha_{3n-1} + \alpha_{3n}, \quad n \in \{1, \ldots, N\}, \quad \ell_n = \alpha_{3n-1}\alpha_{3n-3}, \quad n \in \{2, \ldots, N\}.
\]

These can be solved recursively as

\[
\alpha_3 = m_1 - \alpha_2, \quad \alpha_5 = \frac{\ell_2}{m_1 - \alpha_2}, \\
\alpha_6 = m_2 - \frac{\ell_2}{m_1 - \alpha_2}, \quad \alpha_8 = \frac{\ell_3}{m_2 - \frac{\ell_2}{m_1 - \alpha_2}}, \\
\alpha_9 = m_3 - \frac{\ell_3}{m_2 - \frac{\ell_2}{m_1 - \alpha_2}}, \quad \alpha_{10} = \frac{\ell_4}{m_3 - \frac{\ell_3}{m_2 - \frac{\ell_2}{m_1 - \alpha_2}}},
\]

and the result follows by induction. Hence, for a given \(\alpha_2\) the factorization exists if and only if \(\alpha_{3n} \neq 0\), for \(n \in \{1, \ldots, N-1\}\). \(\square\)

**Proposition 4.** For each \(\alpha_2 < \mathcal{K}[N, 1]\), with \(\mathcal{K}[N, 1]\) the finite continued fraction in Definition 3, the factorization of \(L^{[N]}\) into bidiagonal factors

\[
L^{[N]} = L_1^{[N]}L_2^{[N]}, \quad L_1^{[N]} = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
\alpha_2 & 0 & \cdots & 0 \\
0 & \alpha_5 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \alpha_{3N-1}
\end{bmatrix}, \quad L_2^{[N]} = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
\alpha_3 & 0 & \cdots & 0 \\
0 & \alpha_6 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \alpha_{3N}
\end{bmatrix},
\]

with \(\alpha_3, \alpha_5, \alpha_6, \alpha_8, \ldots, \alpha_{3n-1}, \alpha_{3n} > 0\), exists and is unique. If \(\alpha_2 \in [0, \mathcal{K}[N, 1))\) then \(L_1^{[N]}, L_2^{[N]} \in \text{InTN}\).
Proof. In the solution provided by Equation (26) we require that $\alpha_3, \alpha_5, \alpha_6, \alpha_8, \ldots, \alpha_{3N-1}, \alpha_{3N} > 0$. Let us proceed step by step, firstly if $\alpha_2 < m_1$ we see that $\alpha_3, \alpha_5 > 0$. In the next step, we get that if $\alpha_2 < m_1$ and $\alpha_2 < m_1 - \frac{\ell_2}{m_2}$ we have $\alpha_3, \alpha_5, \alpha_8 > 0$. Notice that as the sequence $\mathcal{K}[N, 1] > 0$ is decreasing $m_1 - \frac{\ell_2}{m_2} < m_1$ and only one condition is needed. Then, in the next step we conclude that what is needed for $\alpha_3, \alpha_5, \alpha_6, \alpha_8, \alpha_9, \alpha_{10} > 0$ is $\alpha_2 < m_1 - \frac{\ell_2}{m_2 - \frac{\ell_3}{m_3}}$. Finally, induction implies the result.

\[ \text{Theorem 7 (Positive bidiagonal factorization in the finite case). Let us assume that the matrix } T^{[N]} \text{ given in (7) is oscillatory. Then, each } \alpha_2 < \mathcal{K}[N, 1] \text{ determines a positive sequence } \{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \ldots, \alpha_{3N+1}\} \text{ such that the factorization} \]

\[ T^{[N]} = L_1^{[N]} L_2^{[N]} U^{[N]}, \]

with bidiagonal matrices given by

\[ L_1^{[N]} = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
\alpha_2 & 0 & \alpha_5 & \cdots & 0 \\
0 & \alpha_5 & \cdots & \alpha_{3N-1} & 1
\end{bmatrix}, \\
L_2^{[N]} = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
\alpha_3 & 0 & \alpha_6 & \cdots & 0 \\
0 & \alpha_6 & \cdots & \alpha_{3N} & 1
\end{bmatrix}, \\
U^{[N]} = \begin{bmatrix}
\alpha_1 & 1 & 0 & \cdots & 0 \\
0 & \alpha_4 & 0 & \cdots & 0 \\
0 & \cdots & \alpha_{3N+1} & 1
\end{bmatrix}, \]

is satisfied. When $\alpha_2 \in [0, \mathcal{K}[N, 1])$ each bidiagonal factor is InTN.

Proof. Consequence of ii) in Theorem 4 and Proposition 4.

\[ \text{Theorem 8 (PBF in the semi-infinite case). Let us assume that the banded Hessenberg matrix } T \text{ in (1) is oscillatory. Then, for each } \alpha_2 < \mathcal{K}[1], \text{ with } \mathcal{K}[1] \text{ the infinite continued fraction in (20), there exist a unique positive sequence } \{\alpha_1, \alpha_3, \alpha_4, \ldots\} \text{ such that the PBF (2), (3) holds. If } \alpha_2 \in [0, \mathcal{K}[1]) \text{ then } L_1, L_2, U \in \text{InTN}. \text{ Moreover, we have the following relations for the matrix entries,} \]

\[ \begin{align*}
c_n &= \alpha_{3n+1} + \alpha_{3n} + \alpha_{3n-1}, \\
b_n &= \alpha_{3n} \alpha_{3n-2} + \alpha_{3n-1} \alpha_{3n-2} + \alpha_{3n-1} \alpha_{3n-3}, \\
a_n &= \alpha_{3n-1} \alpha_{3n-3} \alpha_{3n-5}. 
\end{align*} \]

It is known that the infinite continued fraction $\mathcal{K}[1]$ in (20) could be zero, and this is an important issue in the constructions of the spectral measure representation for the banded Hessenberg matrix $T$, as $\alpha_2$ can not be taken as a positive number (cf. [5]).

Notice that in [2] this was taken for granted, and that in the hypergeometric case [3, 17] and the Jacobi-Piñeiro in the semi-band [2, 7] is also true that $\alpha_2 > 0$.

4. Oscillatory Toeplitz tetradiagonal matrices

We discuss now the uniform case that appears when

\[ a_n = a > 0, \quad b_n = b \geq 0, \quad c_n = c \geq 0. \]
That is, the Hessenberg matrix $T$ is a banded Toeplitz matrix

$$
T = \begin{bmatrix}
  c & 1 & 0 & \cdots & \\
  b & c & 1 & & \\
  a & b & c & 1 & \\
  & 0 & a & b & c \\
  & & & & \\
\end{bmatrix}.
$$

(28)

**Proposition 5** (Edrei–Schoenberg). The Toeplitz matrix (28) is oscillatory if and only if there exists $\beta_1 > \beta_2 > \beta_3 > 0$ such that

$$
a = \beta_1 \beta_2 \beta_3, \quad b = \beta_1 \beta_2 + \beta_1 \beta_3 + \beta_2 \beta_3, \quad c = \beta_1 + \beta_2 + \beta_3.
$$

(29)

**Proof.** According to the Edrei–Schoenberg Theorem, see [8, 21], the matrix $T$ is TN if and only if the generating function

$$
f(t) = 1 + ct + bt^2 + at^3
$$

can be written as

$$
f(t) = (1 + \beta_1 t)(1 + \beta_2 t)(1 + \beta_3 t), \quad \beta_1 \geq \beta_2 \geq \beta_3 \geq 0.
$$

In terms of these $\beta$'s we find (29). As $a > 0$ we must have $\beta_1 \geq \beta_2 \geq \beta_3 > 0$. Hence $b > 0$ and the Gantmacher–Krein criterion, see Theorem 1, leads to the oscillatory character of the Toeplitz matrix. \qed

Now we will show that all tetradiagonal Toeplitz matrices (28) are oscillatory if and only if admits a PBF.

**Proposition 6.** If $T$ is a oscillatory banded Toeplitz matrix as in (28) with $\beta_1 > \beta_2 > \beta_3 > 0$, then the determinants $\delta^{[n]} = \det T^{[n]}$ are explicitly given in terms of $\{\beta_1, \beta_2, \beta_3\}$ as follows:

$$
\delta^{[n]} = \frac{\beta_1^{n+2}}{\beta_1 - \beta_2} + \frac{\beta_2^{n+2}}{\beta_2 - \beta_1} + \frac{\beta_3^{n+2}}{\beta_3 - \beta_2}.
$$

(30)

**Proof.** The determinants $\delta^{[n]} = \det T^{[n]}$ are subject to the recursion relation

$$
\delta^{[n]} - c\delta^{[n-1]} + b\delta^{[n-2]} - a\delta^{[n-3]} = 0,
$$

(31)

being the initial conditions: $\delta^{[-2]} = \delta^{[-1]} = 0$ and $\delta^{[0]} = 1$. Following the theory of recursion relations, see for example [9], we consider the so called characteristic polynomial

$$
p(\lambda) := \lambda^3 - c\lambda^2 + b\lambda - a,
$$

and notice that $p(\lambda) = \lambda^3 f\left(-\frac{1}{\lambda}\right)$; hence, the characteristic roots are $\beta_1, \beta_2, \beta_3 > 0$. If the roots are distinct, i.e., simple, then the general solution to the recursion (31) will be

$$
C_1 \beta_1^n + C_2 \beta_2^n + C_3 \beta_3^n
$$

for some constants $C_1, C_2, C_3$ determined by the initial conditions:

$$
\begin{bmatrix}
  1 & 1 & 1 \\
  \beta_1 & \beta_2 & \beta_3 \\
  \beta_1^2 & \beta_2^2 & \beta_3^2 \\
\end{bmatrix}
\begin{bmatrix}
  C_1 \\
  C_2 \\
  C_3 \\
\end{bmatrix}
= 
\begin{bmatrix}
  1 \\
  0 \\
  0 \\
\end{bmatrix}.
$$
so that (30) holds.

**Corollary 2.** For $T$ as in Proposition 5, for the large $N$ ratio asymptotics of the determinants we find

\[
\lim_{N \to \infty} \frac{\delta_1[N]}{\delta_1[N-1]} = \beta_1.
\]

**Proof.** In the case of distinct characteristic roots $\beta_1 > \beta_2 > \beta_3 > 0$ is a direct consequence of Proposition 6.

When the two smaller characteristic roots coincide $\beta_2 = \beta_3$ the general solution will be

\[
C_1 \beta_1^n + (C_2 + C_3 n) \beta_2^n
\]

and the large $N$ ratio asymptotics of the determinant do not change. When the largest characteristic root is degenerate, with multiplicity two or three, then asymptotically the determinant will have a dominant term $q(n)\beta_1^n$ with a polynomial $q$ such that $\deg q = 1, 2$ in the determinants and (32) is recovered.

**Theorem 9** (Infinite continued fractions and harmonic mean). The continued fraction considered in (20) for an oscillatory tetradiagonal Toeplitz $T$ as (28) is a half of the harmonic mean of the two largest characteristic roots, i.e.

\[
\mathcal{K}[1] = \frac{\beta_1 \beta_2}{\beta_1 + \beta_2}.
\]

**Proof.** In order to compute the continued fraction $\mathcal{K}[1]$ according to (21) we only require the use of the determinants $\delta_1[N]$ of

\[
T_1[N] = \begin{bmatrix}
    b & c & 1 & 0 & \cdots & 0 \\
    a & b & c & 1 & 0 & \cdots \\
    0 & a & b & c & 1 & 0 & \cdots \\
    0 & 0 & 0 & a & b & c & 0 & \cdots \\
    0 & 0 & 0 & 0 & a & b & c & \cdots \\
\end{bmatrix} \in \mathbb{R}^{N \times N},
\]

as in this Toeplitz case $\delta_1[N,k] = \delta_1[N-k]$. These determinants are subject to the following uniform recursion relation

\[
\delta_1^[n] - b \delta_1^[n-1] + ac \delta_1^[n-2] - a^2 \delta_1^[n-3] = 0,
\]

as follows from an expansion along the last column. The initial conditions are $\delta_1[-2] = \delta_1[-1] = 0$ and $\delta_1[0] = 1$. The characteristic roots $\gamma_1,\gamma_2,\gamma_3$ are the zeros of

\[
q(t) = t^3 - bt^2 + act - a^2,
\]

and we find that $p(t) = -\frac{\beta_1}{a^3} q(t)$ or $q(t) = -\frac{\beta_1}{a^3} p(t)$. Hence, the characteristic roots are $\gamma = \frac{\beta_1}{\beta_2}$, that arranged in decreasing order can be written as follows

\[
\gamma_1 = \beta_1 \beta_2, \quad \gamma_2 = \beta_1 \beta_3, \quad \gamma_3 = \beta_3 \beta_2.
\]
Let us assume that the roots are distinct, i.e., simple, the other degenerate cases can be treated similarly. Then, the general solution to the recursion (33) will be

$$C_1 y_1^n + C_2 y_2^n + C_3 y_3^n,$$

for some constants $C_1, C_2, C_3$ determined by the initial conditions:

$$\begin{bmatrix}
1 & 1 & 1 \\
\gamma_1 & \gamma_2 & \gamma_3 \\
\gamma_1 & \gamma_2 & \gamma_3
\end{bmatrix}
\begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and, proceeding as above, we find

$$\delta^{[n]}_1 = \frac{\beta_1^{n+1}\beta_2^{n+1}}{(\beta_2 - \beta_3)(\beta_1 - \beta_3)} + \frac{\beta_1^{n+1}\beta_3^{n+1}}{(\beta_3 - \beta_2)(\beta_1 - \beta_2)} + \frac{\beta_2^{n+1}\beta_3^{n+1}}{(\beta_3 - \beta_1)(\beta_2 - \beta_1)}.$$

Hence, according to (21) with $c_0 = c$ and $a_2 = a$ we have

$$\mathcal{K}[1] = \lim_{n \to \infty} \frac{\beta_1^{n+1}\beta_2^{n+1}}{(\beta_2 - \beta_3)(\beta_1 - \beta_3)} = \frac{\beta_1^2\beta_2^2}{(\beta_1 + \beta_2 + \beta_3)\beta_1\beta_2 - \beta_1\beta_2\beta_3},$$

and after some manipulations the representation for $\mathcal{K}[1]$ follows.

**Theorem 10.** A tetradiagonal Toeplitz matrix is oscillatory if and only if admits a PBF.

### 5. Retractions and tails

We will show that from any given oscillatory tetradiagonal matrix we can construct tetradiagonal matrices with PBF in several ways. Also, we will find that oscillatory matrices are organized in rays, the origin of the ray is a oscillatory matrix that do not have a PBF and all the interior points of the ray have a PBF.

Given a TN matrix $A$, then $A + sE_{1,1}$ is also TN for $s \geq -\frac{\det A}{\det A(1)}$, see [10, Section 9.5] on retractions of TN matrices and in particular the proof of [10, Lemma 9.5.2]. When $s$ is a negative number this is known as a retraction. As we will show, for any matrix with a PBF there is a retraction that is oscillatory but with $a_2 = 0$ and vice versa. Following [10], we use the notation $E_{1,2}(s) := I + sE_{1,2}$ for the bidiagonal matrix with a nonzero contribution only possibly at the entry in the second row and first column.

**Theorem 11 (Retractions and PBF).**

i) Given an oscillatory tetradiagonal matrix $T$ as in (1) then the tetradiagonal matrix
Proof. 

i) The Jacobi matrix

\[
T_s = E_{1,2}(s)T = \begin{bmatrix}
c_0 & 1 & 0 & \\
\vdots & \ddots & \ddots & \\
b_1 + sc_0 & c_1 + s & c_2 & 1 \\
a_2 & b_2 & c_2 & 1 \\
0 & a_3 & b_3 & c_3 \\
\end{bmatrix},
\]

has a PBF for \( s > -\mathcal{K}[1] \).

ii) Given a tetraditional matrix \( T \) as in (1) that admits a PBF, i.e., with \( \mathcal{K}[1] > 0 \), then the tetraditional matrix

\[
\tilde{T} = E_{1,2}(-\mathcal{K}[1])T = \begin{bmatrix}
c_0 & 1 & 0 & \\
\vdots & \ddots & \ddots & \\
b_1 - \mathcal{K}[1]c_0 & c_1 - \mathcal{K}[1] & 1 & \\
a_2 & b_2 & c_2 & 1 \\
0 & a_3 & b_3 & c_3 \\
\end{bmatrix},
\]

is a oscillatory matrix that does not admit a PBF, i.e. \( \mathcal{K}[1] = 0 \).

Proof.

i) The Jacobi matrix \( J_s^{[N, 1]} = J^{[N, 1]} + sE_{1,1} \) is TN for \( s \geq -\frac{\Delta_{N, 1}}{\Delta_{N, 2}} \) and InTN for \( s > -\frac{\Delta_{N, 1}}{\Delta_{N, 2}} = -\mathcal{K}[N, 1] \). Thus, attending to Theorem 1, is an oscillatory matrix for \( s > -\mathcal{K}[N, 1] \). Then, the corresponding lower unitriangular matrix \( L_s^{[N]} \) that has \( J_s^{[N, 1]} \) as complementary submatrix, \( L_s^{[N]}(\{1\}, \{N + 1\}) = J_s^{[N, 1]} \) (obtained by deleting first row and last column), is InTN for \( s > -\mathcal{K}[N, 1] \). This is a consequence of [10, Lemma 3.3.4]. The continued fraction \( \mathcal{K}s[N, 1] \), corresponding to the oscillatory Jacobi matrix \( J_s^{[N, 1]} \), is \( \mathcal{K}[N, 1] + s \).

Now, let us consider the banded Hessenberg matrix by defining \( T_s^{[N]} = L_s^{[N]}U^{[N]} \), which is clearly InTN for \( s > -\mathcal{K}[N, 1] \) as its factors are. A direct computation shows that

\[
T_s^{[N]} = \begin{bmatrix}
c_0 & 1 & 0 & \\
\vdots & \ddots & \ddots & \\
b_1 + sc_0 & c_1 + s & c_2 & 1 \\
a_2 & b_2 & c_2 & 1 \\
0 & a_3 & b_3 & c_3 \\
0 & \vdots & \ddots & \\
a_{N-1} & b_{N-1} & c_{N-1} & 1 \\
\end{bmatrix}.
\]

Observe that \( m_1 = \frac{b_1}{c_0} = \mathcal{K}[2, 1] \), and recall that \( \{\mathcal{K}[n, 1]\}_{n=2}^{\infty} \) is positive and decreasing sequence and, consequently, \( b_1 + sc_0 > 0 \) for \( s > -\mathcal{K}[N, 1] \) and \( N \in \{2, 3, \ldots\} \). Therefore, using the Gantmacher–Krein Criterion, we conclude that \( T_s^{[N]} \) is oscillatory. Finally, for the large \( N \) limit with \( s > -\mathcal{K}[1] \), the matrix \( T_s \) is oscillatory and has \( \mathcal{K}[1] > 0 \).

ii) The retraction \( J^{[N, 1]} = J^{[N, 1]} - \mathcal{K}[1]E_{1,1} \) of the Jacobi matrix of \( T^{[N]} \) is oscillatory. This is a direct consequence of the fact that \( 0 < \mathcal{K}[1] < \mathcal{K}[n, 1] \), for all \( n \in \mathbb{N} \). The associated finite continued fraction is \( \mathcal{K}[N, 1] = \mathcal{K}[N, 1] - \mathcal{K}[1] \). To continue, let us consider as in
the previous discussion the unitriangular matrix \( \tilde{L}^N \) such that \( \tilde{J}^{[N,1]} = \tilde{L}^N \langle \{1\}, \{N+1\} \rangle \) is a complementary submatrix, deleting first row and last column. This triangular matrix is InTN, and the matrix \( \bar{T}^N = \bar{L}^N U^N \) is oscillatory, as well, with associated continued fraction \( \mathcal{K}[N, 1] = \mathcal{K}[N, 1] - \mathcal{K}[1] \). Thus, in the large \( N \) limit the semi-infinite banded Hessenberg matrix \( \bar{T} \) is oscillatory with \( \mathcal{K}[1] = 0 \), i.e. no PBF exists.

\( \square \)

**Corollary 3.** If the tetradiagonal matrix \( T \) given in (1) has \( \mathcal{K}[1] = 0 \), then \( T_s \) as in (34) has a PBF for \( s > 0 \).

The next result is based on the fact that the tails \( \mathcal{K}[2], \mathcal{K}[3], \ldots \) of the continued fraction \( \mathcal{K}[1] \) are positive.

**Theorem 12** (Tails and positive bidiagonal factorization). If \( T \) is an oscillatory banded Hessenberg matrix as in (1) then the matrices

\[
T^{(2)} := \begin{bmatrix}
c_1 - b_1 c_0 & 1 & 0 & \cdots \\
b_2 - \frac{a_2}{c_0} c_2 & 1 \\
a_3 & b_3 & c_3 & 1 \\
0 & a_4 & b_4 & c_4 & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix},
\]

\[
T^{(k+1)} := \begin{bmatrix}
\alpha_{3k+1} & 1 & 0 & \cdots \\
(c_{k+1} + \alpha_{3k+4}) \alpha_{3k+1} & c_{k+1} & 1 \\
\frac{a_{k+2}}{c_{k+3}} & b_{k+2} & c_{k+2} & 1 \\
0 & a_{k+3} & b_{k+3} & c_{k+3} & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix}, \quad k \in \{2, 3, \ldots \},
\]

have a PBF with associated continued fractions \( \mathcal{K}[2] \) and \( \mathcal{K}[k+1], k \in \{2, 3, \ldots \} \), respectively.

**Proof.** For \( k \in \mathbb{N} \), the tail \( \mathcal{K}[k+1] \) is the continued fraction of the Jacobi matrix

\[
J^{(k+1)} := \begin{bmatrix}
m_{k+1} & 1 & 0 & \cdots \\
\ell_{k+2} & m_{k+2} & 1 \\
0 & \ell_{k+3} & m_{k+3} & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix},
\]

that is oscillatory and a submatrix of

\[
L^{(k+1)} = \begin{bmatrix}
1 & 0 & \cdots \\
m_{k+1} & 1 \\
\ell_{k+2} & m_{k+2} & 1 \\
0 & \ell_{k+3} & m_{k+3} & 1 \\
\vdots & \vdots & \vdots & \vdots & \vdots
\end{bmatrix},
\]
with all its leading principal submatrices InTN. We introduce the upper triangular matrix

\[ U^{(k+1)} = \begin{bmatrix} \alpha_{3k+1} & 1 & 0 & \cdots & \cdots \\ 0 & \alpha_{3k+4} & 1 & \cdots \\ 0 & 0 & \alpha_{3k+7} & \cdots \\ \vdots & \vdots & \ddots & \ddots \\ \end{bmatrix}, \]

with all its leading principal submatrices InTN, and the corresponding tetradiagonal matrix \( T^{(k+1)} = L^{(k+1)} U^{(k+1)} \) is

\[ T^{(k+1)} = \begin{bmatrix} c_k - m_k & 1 & 0 & \cdots & \cdots \\ b_{k+1} - \ell_{k+1} & c_{k+1} & 1 & \cdots \\ a_{k+2} & b_{k+2} & c_{k+2} & 1 & \cdots \\ 0 & a_{k+3} & b_{k+3} & c_{k+3} & 1 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ \end{bmatrix}, \]

with \( c_k - m_k = \alpha_{3k+1} > 0 \) and \( b_{k+1} - \ell_{k+1} = m_{k+1} \alpha_{3k+1} = (c_{k+1} + \alpha_{3k+4}) \alpha_{3k+1} > 0 \) is, according to the Gantmacher–Krein Criterion, an oscillatory matrix, having its continued fraction \( \mathcal{K}^{(k+1)}[1] \) equal to the tail \( \mathcal{K}[k+1] \) that is positive. To get \( T^{(2)} \) recall that \( a_1 = c_0, m_1 \alpha_1 = b_1 \) and \( \ell_2 \alpha_1 = a_2 \). \( \square \)

We end the paper with a result that ensures, given any oscillatory tetradiagonal matrix, the finding of associated tetradiagonal matrices with a PBF.

**Theorem 13.** Let us assume that the Hessenberg matrix \( T \) in (1) is oscillatory. Then, the tetradiagonal matrices

\[ \tilde{T} := \begin{bmatrix} b_1 & 1 & 0 & \cdots & \cdots \\ a_2 c_1 & b_2 & 1 & \cdots \\ a_2 a_3 & a_3 c_2 & b_3 & 1 & \cdots \\ 0 & a_3 a_4 & a_4 c_3 & b_4 & 1 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \end{bmatrix}, \]

\[ \tilde{T}^{[k]} := \begin{bmatrix} b_{k+1} - m_{k+1} & 1 & 0 & \cdots & \cdots \\ a_{k+2} c_{k+1} & b_{k+2} & 1 & \cdots \\ a_{k+2} a_{k+3} & a_{k+3} c_{k+2} & b_{k+3} & 1 & \cdots \\ 0 & a_{k+3} a_{k+4} & a_{k+4} c_{k+3} & b_{k+4} & 1 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \end{bmatrix}, \quad k \in \mathbb{N}, \]

admit a PBF.

**Proof.** Theorem 5 ensures that the submatrix \( T^{[N]}_1 \) and its submatrices \( T^{[N,k]}_1 \) are oscillatory. Moreover, Equation (15) and the fact that \( J^{[N,k]} \) is oscillatory and \( U^{[N-1,k]} \) is InTN (notice that the product is InTN and the elements in the first superdiagonal and subdiagonal are positive, then use Gantmacher–Krein Criterion, Theorem 1) imply that the retraction \( T^{[N,k]}_1 - \ell_{k+1} E_{1,1} \) is oscillatory.
as well. These matrices are upper Hessenberg, so that transposition will transform then in lower Hessenberg

\[
\begin{bmatrix}
    b_1 & a_2 & 0 & \cdots & 0 \\
    c_1 & b_2 & a_3 & \cdots & 0 \\
    0 & c_2 & b_3 & a_4 & \cdots \\
    \vdots & \vdots & \ddots & \ddots & \ddots \\
    0 & 0 & \cdots & b_{N-1} & a_N \\
\end{bmatrix}
\]

They are not normalized to be monic on the first superdiagonal but on the second subdiagonal. However, a similarity \( T \mapsto ATA^{-1} \) transformation by the diagonal matrix

\[
A = \text{diag}(1, a_2, a_2a_3, \ldots, a_2a_3 \cdots a_N)
\]

leads to monic banded Hessenberg matrix,

\[
\tilde{T}^{[N-1]} := A(T^{[N]}_1)^T A^{-1} =
\begin{bmatrix}
    b_1 & 1 & 0 & \cdots & 0 \\
    a_2c_1 & b_2 & 1 & \cdots & 0 \\
    a_2a_3 & a_3c_2 & b_3 & 1 & \cdots \\
    0 & a_3a_4 & a_4c_3 & b_4 & 1 \\
    \vdots & \vdots & \ddots & \ddots & \ddots \\
    0 & 0 & \cdots & b_{N-1} & a_N \\
\end{bmatrix}
\]

which happens to be oscillatory. Analogously, a similarity \( T \mapsto ATA^{-1} \) transformation by the diagonal matrix \( A_k = \text{diag}(1, a_{k+2}, a_{k+2}a_{k+3}, \ldots, a_{k+2}a_{k+3} \cdots a_N) \) leads to monic banded Hessenberg matrix,

\[
\tilde{T}^{[N-1,k]} := A_k(T^{[N,k]}_1 - m_{k+1}E_{1,1})^T A_k^{-1} =
\begin{bmatrix}
    b_{k+1} - m_{k+1} & 1 & 0 & \cdots & 0 \\
    a_{k+2}c_{k+1} & b_{k+2} & 1 & \cdots & 0 \\
    a_{k+2}a_{k+3} & a_{k+3}c_{k+2} & b_{k+3} & 1 & \cdots \\
    0 & a_{k+3}a_{k+4} & a_{k+4}c_{k+3} & b_{k+4} & 1 \\
    \vdots & \vdots & \ddots & \ddots & \ddots \\
    0 & 0 & \cdots & b_{N-1}a_N & a_Nc_{N-1} \\
\end{bmatrix}
\]
retraction that happens to be oscillatory. Moreover, according to (14) and (15) these matrices admit the factorizations
\begin{equation}
\begin{aligned}
T^{[N-1]} &= \tilde{L}^{[N-1]} J^{[N]}, & L^{[N-1]} &= A (U^{[N-1]})^T A^{-1}, & J^{[N,1]} &= A (J^{[N,1]})^T A^{-1}, \\
T^{[N-1,k]} &= \tilde{L}^{[N-1,k]} J^{[N,1]}, & L^{[N-1,k]} &= A_k (U^{[N-1,k]})^T A_k^{-1}, & J^{[N,k+1]} &= A_k (J^{[N,k+1]})^T A_k^{-1},
\end{aligned}
\end{equation}

with (recalling that $\alpha_1 = c_0$)
\begin{align*}
\tilde{L}^{[N-1]} &= \begin{bmatrix}
c_0 & 0 & 0 & \cdots & 0 \\
a_2 & a_4 & 0 & \cdots & 0 \\
0 & a_3 & a_7 & 0 & \cdots \\
0 & 0 & \cdots & \cdots & a_N & a_{3N-2}
\end{bmatrix}, & J^{[N,1]} &= \begin{bmatrix}
m_1 & \ell_2 & 0 & \cdots & 0 \\
a_2 & m_2 & \ell_3 & 0 & \cdots \\
a_3 & m_3 & \ell_4 & 0 & \cdots \\
0 & \cdots & \cdots & \cdots & a_N & m_N
\end{bmatrix}, \\
\tilde{L}^{[N-1,k]} &= \begin{bmatrix}
\alpha_{3k+1} & 0 & 0 & \cdots & 0 \\
a_{k+2} & \alpha_{3k+4} & 0 & \cdots & 0 \\
0 & a_{k+3} & \alpha_{3k+7} & 0 & \cdots \\
0 & 0 & \cdots & \cdots & a_N & a_{3N-2}
\end{bmatrix}, & J^{[N,k+1]} &= \begin{bmatrix}
m_{k+1} & \ell_{k+2} & 0 & \cdots & 0 \\
a_{k+2} & m_{k+2} & \ell_{k+3} & 0 & \cdots \\
a_{k+3} & m_{k+3} & \ell_{k+4} & 0 & \cdots \\
0 & \cdots & \cdots & \cdots & a_N & m_N
\end{bmatrix}.
\end{align*}

Now, the Jacobi matrices involved in these factorizations are oscillatory and consequently admit a PBF. Hence, the existence of a PBF follows.

\[
\square
\]

\section*{Conclusions and outlook}

In \cite{5} it was shown that for bounded banded Hessenberg matrices (having $p + 2$ diagonals) with a PBF we can ensure a spectral Favard theorem \cite{22} for multiple orthogonal polynomials. This is the first extension of the spectral Favard theory to multiple orthogonality.\footnote{For an account of multiple orthogonal polynomials see \cite{20} and \cite{15} (chapter written by Van Assche), see \cite{1} for its description in terms of a Gauss–Borel factorization and its connection with integrable systems and \cite{4} for a discussion of Pearson equations and Christoffel formulas for general Christoffel/Geronimus transformations.} Therefore, it is important to discuss how the oscillatory character of a banded Hessenberg matrix is related to this PBF. We have shown that for the well known tridiagonal case, the corresponding Jacobi matrix when conveniently shifted is oscillatory, and that oscillatory and PBF coincide in this case. Next step is the tetradiagonal case. In this work we study the existence of a PBF for a generic tetradiagonal oscillatory Hessenberg matrix.

In terms of continued fractions, in the finite case we prove the existence of the bidiagonal positive factorization in the tetradiagonal scenario and for the infinite case we present a bound for the existence of the bidiagonal positive factorization. We take the oscillatory Toeplitz matrices as a case study and prove that they admit PBF. Also, it is shown that whenever a oscillatory tetradiagonal matrix is not PBF there are several ways of finding associated oscillatory matrices that have a PBF.
In [6] for the tetradiagonal case, and corresponding multiple orthogonal polynomials in the step-line with two weights, the PBF factorization is given in terms of the values of the orthogonal polynomials of type I and II at 0 and, consequently, an spectral interpretation of the Darboux transformation is given.

For the future, we will like to understand what happens with Hessenberg matrices with more diagonals. Preliminary attempts show that the role of continued fractions must be replaced by more general objects, maybe branched continued fractions. Similar questions open for the constant Toeplitz matrix, do a oscillatory pentadiagonal Toeplitz matrix always admit a PBF? What happens when the banded recursion matrix has several superdiagonals as well as subdiagonals?

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1CMUC, Departamento de Matemática, Universidade de Coimbra, 3001-454 Coimbra, Portugal. Email address: ajplb@mat.uc.pt

2CIDMA, Departamento de Matemática, Universidade de Aveiro, 3810-193 Aveiro, Portugal. Email address: foulqui@ua.pt

3Departamento de Física Teórica, Universidad Complutense de Madrid, Plaza Ciencias 1, 28040-Madrid, Spain & Instituto de Ciencias Matemáticas (ICMAT), Campus de Cantoblanco UAM, 28049-Madrid, Spain. Email address: manuel.manas@ucm.es