The Worst Distortions of Astrometric Instruments and Orthonormal Models for Rectangular Fields of View

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ABSTRACT. The nonorthogonality of algebraic polynomials of field coordinates traditionally used to model field-dependent corrections to astrometric measurements, gives rise to subtle adverse effects. In particular, certain field-dependent perturbations in the observational data propagate into the adjusted coefficients with considerable magnification. We explain how the worst perturbation, resulting in the largest solution error, can be computed for a given nonorthogonal distortion model. An algebraic distortion model of full rank can be converted into a fully orthonormal model based on the Zernike polynomials for a circular field of view, or a basis of functions can be constructed from the original model by a variant of the Gram-Schmidt orthogonalization process for a rectangular field of view. The relative significance of orthonormal distortion terms is assessed simply by the numerical values of the corresponding coefficients. Orthonormal distortion models are easily extendable when the distribution of residuals indicate the presence of higher-order terms.

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1. INTRODUCTION

The need to exactly characterize the mapping of celestial angular coordinates onto Cartesian coordinates in the plane of a detector, expressed in suitable units, arises in many astronomical observations. Often called distortion models, such mapping models usually include the basic geometry of an optical projection, possible deviations in the focal length, optical aberrations, etc. The propagation of accidental error in traditional distortion models based on algebraic polynomials of field coordinates has been studied in depth, and the use of orthogonal functions has been advocated in the astrometric literature. However, it appears that simple algebraic polynomials are still in use for large-volume and high-precision data reduction systems. For example, McArthur et al. (2008) represent the field-dependent errors in the fine guidance sensor observations with the Hubble Space Telescope with two independent fifth-order polynomials for $x$ and $y$, a total of 16 terms in each polynomial. The distortion model is expressed through 32 coefficients, which were initially determined by ray traces on the ground, but later updated several times by means of special calibration observations of M35 while in orbit. The high-volume astrometric data reductions in the Sloan Digital Sky Survey (SDSS; Pier et al. 2003) were performed with a two-step characterization of field distortions. For each CCD, the geometric part of the coordinate transformation was separately modeled as a third-order polynomial of $y$ only, because the observations were taken in the time-delayed integration mode. This transformation was further superimposed with a 2D affine transformation. Bellini & Bedin (2009) used independent third-order algebraic polynomials for $\Delta x$ and $\Delta y$ to fit field-dependent corrections to a nominal distortion model (sometimes called astrometric transfer function), represented by a conformal transformation of field coordinates.

We investigate the propagation of systematic error and conclude that the use of nonorthogonal terms can lead to significant errors in the instrument characterization. The worst systematic perturbation in the data, producing the largest possible error in the model parameters, can be exactly computed for any particular model, which is exemplified by the Joint Milli-Arcsecond Pathfinder Survey (JMAPS) astrometric project currently under development (Hennessy et al. 2010). For the original JMAPS focal-plane model, the worst perturbation is amplified by a factor of 7 with respect to the more benign perturbations of the same norm. To avoid this potentially harmful buildup of systematic error, we suggest using orthogonal functions on the unit square (for a square detector), and we explain how an existing nonorthogonal model can be orthogonalized through the Gram-Schmidt process. Determining the significance and temporal stability of individual model terms, which are nearly independent on sufficiently dense star fields, becomes straightforward.

2. POLYNOMIAL PLATE MODELS

Traditionally, the transformation of standard coordinates $\xi$ and $\eta$ (related to the local angular coordinates on the celestial sphere through a simple gnomonic projection) into Cartesian coordinates in the detector $x$-$y$ plane is represented by a simple algebraic polynomial in powers of $x$ and $y$. The repercussions of
this choice for the propagation of accidental measurement errors into the astrometric plate reductions have been discussed in numerous articles (e.g., Eichhorn 1957; Jefferys 1963; Eichhorn & Williams 1963; de Vegt 1967). The main motivation for selecting this functional form was probably its mathematical simplicity, but also the fact that some common kinds of distortions (such as rotation, scale, and tilt) are represented by monomials in field coordinates. In the following, we will use a model currently considered for the JMAPS astrometric satellite, but the techniques developed in this article can be applied to any variety of polynomial distortion models. The JMAPS model is

\[
\begin{align*}
\xi &= a_0 x + a_1 y + a_2 + a_6 x^2 + a_7 x y + a_8 y^2 + a_{10} x r^2 \\
\eta &= a_3 x + a_4 y + a_5 + a_6 x y + a_7 y^2 + a_9 y r^2 + a_{10} y r^2,
\end{align*}
\]

with \( r \) denoting the radius, \( r = \sqrt{x^2 + y^2} \), for simplicity. This model includes the shifts in \( x \) and \( y \) (model parameters \( a_2 \) and \( a_9 \)); rotation, shear, and scale (\( a_0, a_3, a_6, \) and \( a_9 \)); tilt (\( a_3 \) and \( a_4 \)); cubic radial distortion (\( a_{10} \)); and differential radial quadratic distortion (\( a_8 \) and \( a_7 \)). It is convenient to represent this 2D model in vector and matrix forms, by stacking all the \( x \) and \( y \) measurements in a \( 2 \times n \) column vector (where \( n \) is the number of reference stars), and likewise the standard coordinates \( \xi \) and \( \eta \), and arranging the model term values in a \( 2n \times 11 \) matrix, so that the least-squares problem takes the form

\[
\begin{bmatrix}
x & y & 1 & x^2 & x y & r^2 & x r^2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & x y & y^2 & 0 & y r^2 & x & y & 1 & r^2
\end{bmatrix} \mathbf{a} = \mathbf{F} \mathbf{a}
\]

\[
= \begin{bmatrix} \xi \\ \eta \end{bmatrix},
\]

where

\[
\mathbf{a} = [a_0 \ a_1 \ a_2 \ a_6 \ a_7 \ a_8 \ a_{10} \ a_3 \ a_4 \ a_5 \ a_9]^T.
\]

The system of linear equations on a discrete set of data points (\( \xi_i, \eta_i \)) is solved by the least-squares method, yielding a solution:

\[
\mathbf{a} = (\mathbf{F}^T \mathbf{F})^{-1} \mathbf{F}^T \begin{bmatrix} \xi \\ \eta \end{bmatrix},
\]

We are concerned with the propagation of the measurement errors in the least-squares solution. Certainly, it depends on the distribution of data points in the focal plane, among other factors. Reference stars in the focal plane can be nonuniformly distributed (for example, clustered in the center of the field or in one of the corners), giving rise to large errors in some of the parameters and to enhanced covariances between distortion coefficients. In space astrometry projects, however, a single set of distortion coefficients is determined for a large collection of frames, representing a large sample of random stellar configurations and accruing a sufficiently high number density of reference stars. It is therefore justified for practical and theoretical reasons to consider an idealized sampling of the focal plane with a large and dense regular grid of data points. The inner product of two vectors in that case is replaced with its form for a Hilbert function space. For example, the inner product of the fourth and the sixth columns of matrix \( \mathbf{F} \) in equation (2) is defined as

\[
(f_4, f_6) = \int_{-1}^{1} dx \int_{-1}^{1} x^2(x^2 + y^2) dy = \frac{56}{45}.
\]

We assumed in this equation (without a loss of generality) that the detector area is square and the field coordinates are normalized to unity.

In practice, when an astrometric instrument is calibrated by its own observations, it is convenient to simplify the transformations between the angular coordinates (\( \xi, \eta \)) and the detector coordinates (\( x, y \)) to a simple form: e.g., a gnomonic projection plus a nominal scale conversion. The corrections to the nominal parameters and the higher-order terms, which may be time-dependent, are sought for as small corrections to the detector coordinates (\( \Delta x, \Delta y \)). The distortion coefficients are small enough to use the same model (such as in eq. [1]) in its differential form, by replacing \( \xi \) and \( \eta \) on the left-hand side with \( \Delta x \) and \( \Delta y \), respectively (Hiltner 1962; Green 1985). Once the small corrections to the nominal distortion model are known, both forward and backward transformations between the field coordinates can be easily performed. The JMAPS data reduction system is using this approach too, but we retain the notations of equation (1) in this article for simplicity.

3. ERROR PROPAGATION

An error in the right-hand part of equation (2) (i.e., measurement error \( \epsilon \)) propagates linearly into the least-squares solution \( \mathbf{a} \), so that

\[
\Delta \mathbf{a} = \mathbf{F}^T \epsilon,
\]

where \( \mathbf{F}^T \) is the pseudoinverse of \( \mathbf{F} \). Therefore, the error in the model parameters is defined not only by the measurement error \( \epsilon \), but also by the properties of the design matrix, i.e., the composition of distortion terms. Modeling field-dependent distortions becomes an important issue, defining the resulting astrometric accuracy to some degree. Eichhorn & Williams (1963) derived the covariances of plate constants for a series of commonly used models for the case of random uncorrelated errors \( \epsilon \). They realized that the uncertainty of field characterization in the presence of random error may be a complex, but quite predictable, function of field coordinates. The uncertainties tend to become larger and more complex for more sophisticated polynomial models with higher-order terms. They also showed that the residual field-dependent error (which they called...
systematic error) can be larger than the gain in precision for unnecessarily involved distortion models.

Correlations between some of the terms of a polynomial model are the main reason for this potentially harmful feature. A large correlation between two distortion terms should always be a concern, because the commonly used performance statistics based on postfit residuals, such as reduced $\chi^2$, do not capture the intrinsic statistical dependencies within the solution. Astrometrists rarely know their instruments well enough in advance, to the extent that a priori distortion models can be used without some experimentation or verification on real data. Even then, it is not clear where to stop in building up a distortion model, because each additional term usually results in smaller residuals, and the solution value can be deceptively large, indicating a significant term, which is actually correlated with a number of other terms.

For these reasons, the use of orthogonal functions instead of algebraic polynomials has long been advocated in the literature. Brosche et al. (1989), followed by Bienaymé (1993), explained the adverse effects of correlated terms and considered the ease of determination of a minimally sufficient model to be the main advantage of orthogonalized designs. Having determined the level of statistically significant reduction in advantage of orthogonalized designs. Having determined the level of statistically significant reduction in $\chi^2$, one can, for example, remove orthogonal terms from the model in turns, until this level is achieved with the smallest number of terms.

The effects of systematic errors have largely been unheeded in the literature. By a systematic error we understand any deterministic perturbation of the data that can be expressed as a function of field coordinates. Obviously, there is an infinite set of possible systematic errors, and analysis of their propagation appears to be intractable. Beyond the common systematic errors, such as a focal-length variation, these effects are hard to predict in complex astrometric instruments. However, instead of guessing possible systematic errors and simulating their effects, one can pose this problem: Is there a specific systematic error that propagates into the least-squares solution with the largest gain? An astonishingly simple answer is obtained through the singular-value decomposition (SVD) technique (Golub & van Loan 1996). Let

$$F = U \Sigma V^T$$  \hspace{1cm} (7)

be the SVD of the design matrix $F$. The least-squares (LS) solution is

$$\hat{a} = V \Sigma^{-1} U^T \begin{bmatrix} \xi \\ \eta \end{bmatrix}.$$  \hspace{1cm} (8)

For an overdetermined system with $k$ model terms (i.e., of rank $k$), $\Sigma^{-1}$ is a diagonal matrix with $k$ nonzero (positive) numbers in the diagonal in increasing order. Therefore, in the projection of the data vector onto the basis, $U^T \begin{bmatrix} \xi \\ \eta \end{bmatrix}$, only the first $k$ columns of the orthogonal matrix $U$ contribute to the solution. In other words, the first $k$ columns of $U$ define the basis of the subspace spanned by $F$, and the remainder is the basis of its null space.\(^1\) The LS solution is impervious to any perturbation of data in the null space. If $\epsilon$ is a systematic perturbation of the data, the squared norm of the resulting systematic error of the solution is

$$\|\delta\|^2 = \delta^T \delta = \epsilon_U^T \Sigma^{-2} \epsilon_U,$$  \hspace{1cm} (9)

where $\epsilon_U = U^T \epsilon$ is the projection of $\epsilon$ onto the basis of $F$. Recalling that $\Sigma = \text{diag}(\sigma_1, \sigma_\ldots, \sigma_k)$ and $\sigma_1 > \sigma_2 > \ldots > \sigma_k$, the maximum $\|\delta\|^2$ is achieved when $\epsilon$ is aligned with $u_k = U(:, k)$. This follows from the constrained Lagrange problem, where the Lagrange multiplier is one of the $\sigma_i^{-2}$, and among all $\epsilon$ of unit norm, the largest solution error happens when $\epsilon_U = u_k^T \epsilon$; hence, $\epsilon = u_k$.

Thus, the $k$th long singular vector, which is the least significant principal component of model $F$, is the worst systematic error among all possible systematic perturbations of unit norm.

4. THE CASE OF JMAPS

Using a dense grid of points on the unit square, the SVD of the nominal JMAPS distortion model equation (2) can be computed and the worst systematic error can be numerically derived. The result is a higher-order field distortion, shown in Fig. 1. It is similar to the scale in the center of the field of view, but instead

\(^1\)In MATLAB, the significant part of SVD is realized through the economic SVD: $[u, \text{sigma}, v] = \text{svd}(f, 0)$.  

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of growing linearly with radius, it turns to zero and reverses the sign in the outer regions. Recalling that the worst perturbation is one of the basis vectors of the subspace spanned by the columns of $F$, it can be expressed as a certain linear combination of the model functions. In this case, apparently, the differential scale terms $[x \ 0]^T$ and $[0 \ y]^T$ in combination with the radial distortion term $[x^2 \ y^2]^T$ are the dominating contributors to the worst perturbation $u_{11}$.

How does one estimate the impact of the worst systematic error? Clearly, the term “worse” or “bad” can only be meaningful in a relative, or comparative, sense. The singular values readily provide a meaningful comparison of various possible systematic errors. The most benign systematic perturbation is $u_1$, and the ratio of the smallest possible to the largest possible systematic error $\|\delta\|$ is simply $\sigma_k/\sigma_1$, which is the condition number of $F$. A very small condition number indicates an ill-conditioned problem. An inadequately chosen distortion model may have a deficient rank, in which case one or more singular values equal zero, and the solution error can be infinite. For the JMAPS model in equation (2), the condition number is 0.14652, indicating a moderately conditioned model. The worst unit-norm perturbation results in a systematic error whose norm is 7 times that of the most benign unit-norm perturbation from the subspace of $F$.

5. ORTHOGONAL DISTORTION MODELS ON THE UNIT DISK

The two-dimensional Zernike functions (also known as the circle polynomials of Zernike) provide a ready-to-use orthogonal basis of scalar functions on the unit disk. These functions are used to decompose the optical aberration function (Born & Wolf 1999), and they find other numerous applications in optics and engineering. Table A lists several low-order terms starting with $Z^0_0$, which is a constant. An expanded set of the Zernike functions and transformations of algebraic monomials is given by Mathar (2009). The Zernike functions are normalized to $\sqrt{\pi}$ with a weight $r$:

$$\int_{0}^{1} \int_{0}^{2\pi} Z_m^n Z_n^l rdrd\phi = \pi \delta_{mn} \delta_{kl},$$

and each of the algebraic monomials in equation (2) is uniquely represented by a linear combination of Zernike functions. The polynomial model is not orthogonal, because some of the terms include the same Zernike function, as can be seen in Table A. For example, the product of the sixth term (with $a_6$) and zeroth term (with $a_0$) is equal to $\pi/6$. One can replace the nonorthogonal terms in model equation (2) with their significant Zernike counterparts, arriving at

$$a = \left[ \frac{\xi}{\eta} \right].$$

This is the closest to the original equation (2) model, which is orthogonal on the unit disk. It can be easily expanded to higher orders by including more terms constructed from the Zernike functions.

6. ORTHOGONAL DISTORTION MODELS ON THE UNIT SQUARE

Instruments of optical astrometry are equipped today with CCD detectors or other electronic sensors that are rectangular in shape. It is practical to use a functional basis, which is orthogonal on a quadrangle. Without a loss of generalization, a basis on the unit square can be applied to any rectangular field of view by means of coordinate normalization. The Zernike functions described in the previous section are not relevant, because some of them are not mutually orthogonal on the unit square.

We demonstrate here how a functional basis can be constructed on the unit square starting with a model, which includes polynomials or Zernike terms. This is implemented by the Gram-Schmidt orthogonalization process in the space of 2D scalar functions. If $Y_m$ are the original (nonorthogonal) vector functions and $V_m$ are the new basis functions orthogonal on the square, the algorithm is

$$V_1 = Y_1 \quad V_2 = Y_2 - (Y_1, Y_2)/(V_1, Y_1)V_1$$

$$V_3 = Y_3 - (Y_1, Y_3)/(V_1, Y_1)V_1 - (Y_2, Y_3)/(V_2, Y_2)V_2 \cdots.$$
perfectly conditioned model. However, a small fitting error to levels that are not of practical concern. However, fields will be reasonably well sampled, reducing the impact of functions does not result in a significant reduction in solution JMAPS instead of the original model equation (1), using these JMAPS distortions in terms of orthogonal functions is

\[ ^\text{Normalized functions} \]

The unit quadrangle, i.e.,

\[ (V_m, V_n) = \int_{-1}^{+1} \int_{-1}^{+1} V_m \cdot V_n \, dx \, dy. \quad (13) \]

Normalized functions \( \hat{V}_m = V_m / \sqrt{(V_m, V_m)} \) have the additional advantage of all singular values being equal to unity for a dense and uniform set of sample points, which makes a perfectly conditioned model.

A set of orthogonal functions \( V_m (m = 1, \ldots, 12) \), derived from the original JMAPS model equation (1) is given in Table 2, along with their norms on the unit square. The updated model of JMAPS distortions in terms of orthogonal functions is

\[ [\hat{V}_1 \ \hat{V}_2 \ \cdots \ \hat{V}_t] \mathbf{a} = \begin{bmatrix} \xi \\ \eta \end{bmatrix}. \quad (14) \]

As long as the field of view is well sampled with observations of calibration stars, the design matrix in these equations is nearly orthogonal: i.e., \( F^T F \approx I_6 \).

As for the effective difference in using this solution for JMAPS instead of the original model equation (1), using these functions does not result in a significant reduction in solution residuals if the same number of terms (11 terms) are used. Our fields will be reasonably well sampled, reducing the impact of fit condition to levels that are not of practical concern. However, the utility of this solution for the JMAPS program has been that it provides a base from which to analyze our optical models, which can be expanded in a straightforward manner. As the fidelity of the JMAPS optical simulations improve, higher-order effects can be fitted, orthogonalized, and added to the solution to determine if fitting for particular effects is worthwhile. More advantages are expected in treating real calibration data, where the orthonormal terms most faithfully describe the actual state and evolution of the instrument.

7. DISCUSSION

Using the normalized vector fields based on the 2D Zernike polynomials for circular fields of view (eq. [11]) or 2D orthonormal polynomials for rectangular fields of view (eq. [14]) leads to the most stable and well-conditioned solutions for instrument calibration parameters. Any systematic error of unit norm will result in a solution error of the same magnitude. A random error of unit weight \( (e^2 = I) \) per eq. [6]) will result in an uncorrelated set of model coefficients, each having the same error expectancy, because the covariance of the LS solution is close to unity. This property of orthonormal distortion models comes in handy when we have to assess the significance of different terms in our model. On one hand, the insignificant terms, which describe certain types of field-dependent corrections that are not really present in the data, should be identified and eliminated as soon as possible, because the redundant degrees of freedom degrade the overall astrometric solution. This is especially important when the instrument is not very stable, resulting in numerous field-dependent calibration unknowns in the global adjustment. On the other hand, if the observational residuals prove too large, and their distribution suggests the presence of higher-order terms, which are not captured by the distortion model, the orthonormal basis should be built up using the generalized Gram-Schmidt algorithm (§ 6).

Orthogonal plate models are also useful when possible internal degeneracies in the instrument characterization have to be identified. A specific, but very common, case is when the focal-plane assembly includes more than one photodetector array, each being characterized by its own set of calibration parameters. In such hierarchical calibration models, lower-level parameters may be strongly correlated or completely degenerate with the higher-level parameters. For example, a shift of each CCD in the same direction by the same amount is indistinguishable from a common shift of the entire focal plane. Such double accounting of the same effect should be avoided, because it can lead to a complete solution failure or divergence of iterated solutions. With an orthonormal plate model, the degeneracies are simply quantified by projection of the additional terms onto the previously accumulated basis. If the projections turn out to be significantly nonzero, the new terms can be orthogonalized analytically, as described in this article, or numerically by the principal component method.

| Function | Square norm |
|----------|-------------|
| \( V_1 = 1 \) | \( (V_1, V_1) = 4 \) |
| \( V_2 = 0 \) | \( (V_2, V_2) = 4 \) |
| \( V_3 = x \) | \( (V_3, V_3) = 4 \) |
| \( V_4 = 0 \) | \( (V_4, V_4) = 4 \) |
| \( V_5 = y \) | \( (V_5, V_5) = 4 \) |
| \( V_6 = 0 \) | \( (V_6, V_6) = 4 \) |
| \( V_7 = \left( x^2 - \frac{1}{5} \right) \) | \( (V_7, V_7) = 4 \) |
| \( V_8 = \left( x^3 + \frac{1}{5} xy \right) \) | \( (V_8, V_8) = 4 \) |
| \( V_9 = \left[ \frac{1}{5} x^2 + y^2 - \frac{1}{5} \right] \) | \( (V_9, V_9) = \) |
| \( V_{10} = \left[ x^3 + \frac{1}{5} xy \right] \) | \( (V_{10}, V_{10}) = 27344 \) |
| \( V_{11} = \left[ \frac{1}{5} x^2 + y^2 - \frac{1}{5} \right] \) | \( (V_{11}, V_{11}) = 27344 \) |
| \( V_{12} = \left[ \frac{1}{5} x^3 - \frac{1}{5} xy \right] \) | \( (V_{12}, V_{12}) = 16 \) |
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