Quantitative Quermassintegral Inequalities for Nearly Spherical Sets

Caroline VanBlargan, Yi Wang

Abstract
In this paper, we establish quantitative Alexandrov-Fenchel inequalities for quermassintegrals on nearly spherical sets. In particular, we bound the \((k, m)\)-isoperimetric deficit from below by the Frankel asymmetry. We also find a lower bound on the \((k, m)\)-isoperimetric deficit using the spherical deviation.

1 Introduction
In this paper, we study stability in the quermassintegral inequalities for nearly spherical sets, which is largely motivated by work done on the stability in the classical isoperimetric inequality. In particular, we are motivated by analysis done on the isoperimetric deficit \(\delta(\Omega)\) of a domain \(\Omega \subseteq \mathbb{R}^{n+1}\), defined as

\[
\delta(\Omega) := \frac{P(\Omega) - P(\bar{B}_\Omega)}{P(\bar{B}_\Omega)}.
\]

Here \(|\bar{B}_\Omega|\) is the volume of \(B_\Omega\), \(B_\Omega\) is a ball such that \(|\Omega| = |\bar{B}_\Omega|\), and \(P(\cdot)\) gives the perimeter of a set. The classical isoperimetric inequality is equivalent to \(\delta(\Omega) \geq 0\), with equality if and only if \(\Omega\) is a ball. There has been a lot of work studying quantitative isoperimetric inequalities inspired by the Bonnesen type inequalities, which was named by Osserman in [17]. This was based off work by Bonnesen, where he studied inequalities in the form

\[
L^2 - 4\pi A \geq \lambda(C).
\]

In this setting, \(L\) and \(A\) represent the length and area enclosed by a simple closed curve \(C\) in \(\mathbb{R}^2\). Moreover, \(\lambda(C)\) satisfies three conditions:

1. \(\lambda(C) \geq 0\).
2. \(\lambda(C) = 0\) precisely when \(C\) is a circle.
3. \(\lambda(C)\) measures geometrically how close \(C\) is to a circle.

Fuglede worked to expand these results to higher dimensions in [6] and [7], where they proved a quantitative isoperimetric inequality for nearly spherical sets. They used this result to study the stability for convex domains using the spherical deviation.

**Definition 1.1.** For a domain \(\Omega \subseteq \mathbb{R}^{n+1}\), set \(\tilde{\Omega} := \frac{\text{Vol}(B)}{\text{Vol}(\Omega)}(\Omega - \text{bar}(\Omega))\), where \(\text{bar}(\Omega)\) is the barycenter of \(\Omega\) and \(B\) is the unit ball. The spherical deviation of \(\Omega\), \(d(\Omega)\), is defined as

\[
d(\Omega) := d_H(\tilde{\Omega}, B),
\]

where \(d_H(\cdot, \cdot)\) gives the Hausdorff distance between two sets.

Specifically, for a convex domain \(\Omega\), they established an inequality in the form \(d(\Omega) \leq f(\delta(\Omega))\), for some function \(f\).

To establish a quantitative isoperimetric inequality for more general domains, the Fraenkel asymmetry, \(\alpha(\Omega)\), is a well-studied quantity used as a lower bound.

**Definition 1.2.** Suppose \(\Omega \subseteq \mathbb{R}^{n+1}\). The Fraenkel asymmetry of \(\Omega\) is denoted by \(\alpha(\Omega)\), where

\[
\alpha(\Omega) := \inf \left\{ \frac{|\Omega \Delta (x + B_\Omega)|}{|B_\Omega|}, x \in \mathbb{R}^{n+1} \right\}.
\]

\(B_\Omega\) denotes the ball centered at the origin with the same volume as \(\Omega\), and \(\Delta\) denotes the symmetric difference between two sets.
Using the Fraenkel asymmetry in the study of stability brings us to the \textit{quantitative isoperimetric inequality}, which asks if there is a fixed $C(n) > 0$ such that all Borel sets $\Omega \subseteq \mathbb{R}^{n+1}$ with finite measure satisfy the inequality
\begin{equation}
\delta(\Omega) \geq C(n)\alpha^n(\Omega),
\end{equation}
for some exponent $m$. The quantitative isoperimetric inequality $\delta(\Omega) \geq C\alpha(\Omega)$ was shown for Steiner symmetrical sets in $[15]$ by Hall, Hayman, and Weitsman. Later, by using results on the Steiner symmetrical, Hall showed $[5]$ in $[14]$, but with a suboptimal exponent of $m = 4$.

Finally, in $[8]$, Fusco, Maggi, and Pratelli showed, by using symmetrizations of $\Omega$, that $[5]$ holds with optimal exponent $m = 2$ for Borel sets $\Omega \subseteq \mathbb{R}^{n+1}$ of finite measure. Figalli, Maggi, and Pratelli in $[5]$ proved this optimal result in the more general setting of the anisotropic perimeter.

\textbf{Theorem 1.1} ($[5, 8]$). Suppose $n \geq 1$. Then for any Borel set $\Omega \subseteq \mathbb{R}^{n+1}$ of finite measure,
\begin{equation}
\delta(\Omega) \geq C(n)\alpha^2(\Omega),
\end{equation}
where $C(n) > 0$ depends only on $n$.

This theorem was proved in $[8]$ by reducing the problem to $n$-symmetric sets and then using the method of Steiner symmetry. To prove the result in $[5]$, the authors did not use symmetrization arguments as done previously. Instead, they applied the Brenier map to employ methods in mass transportation theory. For further reading on the quantitative isoperimetric inequality see $[9]$ and $[16]$.

In this paper, we will be working with quantities motivated by the theory of mixed volumes in convex geometry. For a convex body $\Omega \subseteq \mathbb{R}^n$, the $k$-th quermassintegral of $\Omega$ is the mixed volume
\begin{equation}
W_k(\Omega) := V(\Omega, ..., \Omega, B, ..., B),
\end{equation}
where $\Omega$ appears in the first $n + 1 - k$ entries and $B$, which is the unit ball in $\mathbb{R}^{n+1}$, appears in the last $k$ entries. The famous Steiner formula states that the volume of $\Omega + tB$ is a polynomial in $t$. In particular,
\begin{equation}
\text{Vol}(\Omega + tB) = \sum_{k=0}^{n+1} \binom{n+1}{k} W_k(\Omega)t^k.
\end{equation}

Next, denoting $\omega_n$ as the volume of the unit $m$-ball, we set
\begin{equation}
V_k(\Omega) := \frac{\omega_k}{\omega_{n+1}} W_{n+1-k}(\Omega).
\end{equation}

Note that $V_{n+1}(A) = \text{Vol}(\Omega)$ and $V_n(\Omega) = \frac{\omega_{n+1}}{(n+1)!}\text{Area}(\partial \Omega)$. We obtain, as a consequence of the Alexandrov-Fenchel inequalities, the \textit{quermassintegral inequalities}
\begin{equation}
\left( \frac{V_{k+1}(\Omega)}{V_{k+1}(B)} \right)^\frac{n+1}{k+1} \leq \left( \frac{V_k(\Omega)}{V_k(B)} \right)^\frac{1}{k}.
\end{equation}

When $k = n$, the inequality in $[10]$ is simply the classical isoperimetric inequality. For convex domains and $k \geq 1$, quermassintegrals have the useful integral formula
\begin{equation}
V_{n+1-k} = \frac{(n + 1 - k)! (k - 1)!}{(n + 1)!} \frac{\omega_{n+1-k}}{\omega_{n+1}} \int_M \sigma_{k-1}(L)d\mu,
\end{equation}
where $M := \partial \Omega$, $L$ is the second fundamental form of $M$, and $\sigma_k(L)$ is the $k$-th \textit{mean curvature} of $M$. The $k$-th mean curvature is the $k$-th elementary symmetric polynomial of the principal curvatures. The inequalities in $[10]$ equivalently state that for convex domains
\begin{equation}
\left( \int_M \sigma_{k-1}(L)d\mu \right)^\frac{n+1}{k+1} \leq C(n, k) \left( \int_M \sigma_k(L)d\mu \right)^\frac{1}{k},
\end{equation}
where $C(n, k)$ is the constant that gives equality in the case where $M$ is a sphere.

Much of the previous work to establish $[12]$ relies heavily on working with convex domains. There has been work extending $[12]$ to a class of nonconvex domains known as $k$-convex domains, where $\sigma_j(L) \geq 0$ for $1 \leq j \leq k$. Guan and Li showed in $[13]$ that $[12]$ holds in $k$-convex, starshaped domains. To prove
the inequalities, they used a normalization of the flow \( X_t = \frac{\sigma_k(L)}{\sigma_{k+1}(L)} \nu \), which was studied by Gerhardt in [12] and Urbas in [18].

In [3], Chang and Wang were able to show (12) without the requirement of a starshaped domain, but with the added assumption of having \((k+1)\)-convexity instead of just \(k\)-convexity, and the constant \(C(n,k)\) is non-optimal. They proved this using optimal transport methods. See also [1], [2], and [19].

Our study of stability in the quermassintegral inequalities is inspired by work done by Fuglede in [7] and by Cicalese and Leonardi in [4].

**Definition 1.3.** Suppose \( M = \{(1 + u(x))x : x \in \partial B\} \), where \( \partial B \) is the unit sphere in \( \mathbb{R}^{n+1} \) and \( u : \partial B \to (-1, \infty) \) is a smooth function on the unit sphere. \( M \) is referred to as a nearly spherical set when we have suitable, small bounds on \(|u|, |\nabla u|, \) and \(|D^2u|\).

Remark. Nearly spherical sets in [6] and [4] only require bounds on \(|u| \) and \(|\nabla u|\). However, since we will be working with curvature terms, we will require small bounds on \(|D^2u|\) as well.

In [4], Cicalese and Leonardi introduced a new method to show (6) for all Borel sets of finite measure. In this paper, we aim to prove a version of (13) as it applies to the quermassintegral inequalities.

First, we define \( I_k(\Omega) \) by integrating the \( k\)-th mean curvature of nearly spherical sets.

\[
I_k(\Omega) := \int_M \sigma_k(L) \ dA.
\]

Also, for \( k = -1 \) we define

\[
I_{-1}(\Omega) := \text{Vol}(\Omega).
\]

We are now able to define a natural generalization of the isoperimetric deficit for quermassintegrals.

**Definition 1.4.** For \(-1 < k \leq n \) and \(-1 \leq m < k\), the \((k,m)\)-isoperimetric deficit is denoted by \( \delta_{k,m}(\Omega) \), where

\[
\delta_{k,m}(\Omega) := \frac{I_k(\Omega) - I_k(B_{\Omega,m})}{I_k(B_{\Omega,m})}.
\]

Here \( B_{\Omega,m} \) is the ball centered at the origin where \( I_m(B_{\Omega,m}) = I_m(\Omega) \).

In Section 3, we compute an explicit formula for the \( k\)-th mean curvature of nearly spherical sets.

Then in Section 4, we add in the assumption that \(|u|_{W^{2,\infty}} < \epsilon \) to expand out \( I_k(\Omega) \). This brings us to our first main theorem.

**Theorem 1.2.** Suppose \( \Omega \in \{(1 + u(x))x : x \in B\} \subseteq \mathbb{R}^{n+1} \), where \( u \in C^3(\partial B) \), \( \text{Vol}(\Omega) = \text{Vol}(B) \), and \( \text{bar}(\Omega) = 0 \). For all \( \eta > 0 \), there exists \( \epsilon > 0 \) such that if \(|u|_{W^{2,\infty}} < \epsilon \), then

\[
\delta_{k,j}(\Omega) \geq \left( \frac{(n-k)(k+1)}{2n(n+1)^2} - \eta \right) \alpha^2(\Omega).
\]
In Section 5, we prove a similar theorem, but we assume that $I_j(\Omega) = I_j(B)$ instead of $\text{Vol}(\Omega) = \text{Vol}(B)$.

**Theorem 1.3.** Fix $0 \leq j < k$. Suppose $\Omega = \{(1 + u(\frac{x}{|x|}))x : x \in B\} \subseteq \mathbb{R}^{n+1}$, where $u \in C^3(\partial B)$, $I_j(\Omega) = I_j(B)$, and $\text{bar}(\Omega) = 0$. For all $\eta > 0$, there exists $\epsilon > 0$ such that if $||u||_{W^{2,\infty}} < \epsilon$, then

$$
\delta_{k,j}(\Omega) \geq \left( \frac{n(n-k)(k-j)}{4(n+1)^2} - \eta \right) \alpha^2(\Omega). \tag{19}
$$

We remark that for sufficiently small $||u||_{W^{2,\infty}}$, $\Omega$ is a convex domain. Then, we already know from the result of Guan and Li, which assumes $\Omega$ is $k$-convex and starshaped, that $\delta_{k,j}(\Omega) \geq 0$. So, we are establishing a quantitative isoperimetric inequality in this case.

In Section 6, we prove the following theorem.

**Theorem 1.4.** Suppose $\Omega = \{(1 + u(\frac{x}{|x|}))x : x \in B\} \subseteq \Omega$, where $u \in C^3(\partial B)$, $\text{Vol}(\Omega) = \text{Vol}(B)$, and $\text{bar}(\Omega) = 0$. There exists an $\eta > 0$ so if $||u||_{W^{2,\infty}} < \eta$, then

$$
||u||_{L^{\infty}}^n \leq \begin{cases} 
C\delta_{k-1}^{1/2}(\Omega) & n = 1 \\
C\delta_{k-1}(\Omega) \log \frac{A}{A_{k-1}(\Omega)} & n = 2 \\
C\delta_{k-1}(\Omega) & n \geq 3,
\end{cases}
$$

where $A, C > 0$ depend only on $n$.

Theorem 1.4 shows that the $(k-1)$-deficit gives a control on $||u||_{L^{\infty}}$, which is equivalent to the spherical deviation $d(\Omega)$. The proof of this theorem follows closely to Fuglede’s in [7], where they proved this theorem when $k = 0$ (for the classical isoperimetric deficit). In [10], Fusco, Gelli, and Pisante expanded this stability result for domains where they impose a uniform cone condition on the boundary. Studying the control on $||u||_{L^{\infty}}$ gives a stronger result than just the Fraenkel asymmetry, although there must be some regularity condition imposed for it to hold. In [7], bounding $||u||_{L^{\infty}}$ by the classical isoperimetric deficit for nearly spherical domains was a key result to establish stability results for convex domains (without the the assumption that the domain is nearly spherical). We hope Theorem 1.4 will be useful for establishing a stability result with the spherical deviation for less restrictive $k$-convex domains.

## 2 Preliminaries

### 2.1 The $k$-th mean curvature

For $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n$, we denote $\sigma_k(\lambda)$ as the $k$-th elementary symmetric polynomial of $\lambda_1, ..., \lambda_n)$. That is, for $1 \leq k \leq n$,

$$
\sigma_k(\lambda) = \sum_{i_1 < i_2 < \cdots < i_k} \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_k}, \tag{20}
$$

and

$$
\sigma_0(\lambda) = 1. \tag{21}
$$

This leads to a natural generalization of the mean curvature of a surface.

**Definition 2.1.** Suppose $\Omega$ is a smooth, bounded domain in $\mathbb{R}^{n+1}$. For $x \in M := \partial \Omega$, the $k$-th mean curvature of $M$ at $x$ is $\sigma_k(\lambda)$, where $\lambda = (\lambda_1(x), ..., \lambda_n(x))$ are the principal curvatures of $M$ at $x$.

Observe that in this definition, $\sigma_1(\lambda)$ is the mean curvature and $\sigma_n(\lambda)$ is the Gaussian curvature. When $\{\lambda_1, ..., \lambda_n\}$ are the eigenvalues of a matrix $A = \{A^i_j\}$, we denote $\sigma_k(A) = \sigma_k(\lambda)$, which can be equivalently calculated as

$$
\sigma_k(A) = \frac{1}{k!} \det_{i_1 < i_2 < \cdots < i_k \in n} A^{i_1}_{j_1} \cdots A^{i_k}_{j_k}, \tag{22}
$$

using the Einstein convention to sum over repeated indices.

So, if $L$ is the second fundamental form of $M$, we can use this expression for $\sigma_k(L)$ to compute the $k$-th mean curvature of $M$. Throughout this paper, we will be working with family of surfaces where, for $0 < j \leq k, \sigma_j(L) \geq 0$ at each point. Such surfaces are called $k$-convex.
Definition 2.2. Let $\Omega$ be a domain in $\mathbb{R}^{n+1}$. Then the hypersurface $M := \partial \Omega$ is said to be strictly $k$-convex if the principal curvatures $\lambda = (\lambda_1, ..., \lambda_n)$ lie in the Garding cone $\Gamma_k^+$, which is defined as

$$\Gamma_k^+ := \{ \lambda \subseteq \mathbb{R}^n : \sigma_j(\lambda) > 0, 1 \leq j \leq k \}.$$  

Note that $n$-convexity is the same as normal convexity. A useful operator related to $\sigma_k$ is the Newton transformation tensor $[T_k]_{ij}^l$.

Definition 2.3. The Newton transformation tensor, $[T_k]_{ij}^l$, of $n \times n$ matrices $\{A_1, ..., A_k\}$ is defined as

$$[T_k]_{ij}^l(A_1, ..., A_k) := \frac{1}{k!} \delta_{i_1, ..., i_k}^{j_1, ..., j_k} (A_1)^{i_1}_{j_1} \cdots (A_k)^{i_k}_{j_k},$$  

When $A_1 = A_2 = ... = A_k = A$, we denote $[T_k]_{ij}^l(A) = [T_k]_{ij}^l(A, ..., A)$. A related operator is $\Sigma_k$, which the polarization of $\sigma_k$.

Definition 2.4. Suppose $\{A_1, ..., A_k\}$ is a collection of $n \times n$ matrices. We denote

$$\Sigma_k(A_1, ..., A_k) := (A_1)_{ij}^l [T_{k-1}]_{ij}^l (A_2, ..., A_k)$$

$$= \frac{1}{(k-1)!} \delta_{i_1, ..., i_k}^{j_1, ..., j_k} (A_1)^{i_1}_{j_1} \cdots (A_k)^{i_k}_{j_k}. \quad (25)$$

Two useful identities are

$$\sigma_k(A) = \frac{1}{k} \Sigma_k(A, ..., A) = \frac{1}{k} A_1^l [T_{k-1}]_{ij}^l (A), \quad (26)$$

and

$$\frac{\partial \sigma_k(A)}{\partial A_j^l} = \frac{1}{k} [T_{k-1}]_{ij}^l (A). \quad (27)$$

We will also use the identity

$$A_1^l [T_{m}]_{ij}^l (A) = \delta_{i}^{j} \sigma_{m+1}(A) - [T_{m+1}]_{s}^l (A). \quad (28)$$

2.2 The $(k, m)$-isoperimetric deficit

In this paper, we look at a more general notion of the isoperimetric deficit as it pertains to the $k$-th mean curvature. We consider a bounded domain $\Omega \subset \mathbb{R}^{n+1}$ where $M := \partial \Omega$ is a smooth hypersurface. First, we define $I_k(\Omega)$ by integrating the $k$-th mean curvature of $M$. That is,

$$I_k(\Omega) := \int_{M} \sigma_k(L) d\mu. \quad (29)$$

The definition extends to $k = -1$ so that

$$I_{-1}(\Omega) := \text{Vol}(\Omega). \quad (30)$$

Furthermore, because $\sigma_0(L) = 1$,

$$I_0(\Omega) = \text{Area}(M). \quad (31)$$

As seen from the identity $I_1(\Omega)$ is equal to a quermassintegral of $\Omega$ up to a constant. We aim to study the $(k, m)$-deficit, $\delta_{k, m}(\Omega)$, from Definition 1.4. Note that $\delta_{0, -1}(\Omega)$ is the classical isoperimetric deficit from $[1.4]$.

For $k \geq 0$, $I_k(B) = (\nu_k) Area(\partial B)$. If we scale $\Omega$ by a fixed $r > 0$, so $rE = \{ rx : x \in \Omega \}$, then $I_k(rE) = r^{n-k} I_k(\Omega)$. It follows that

$$I_m(r\Omega) = I_m(rB_{\Omega, m}). \quad (32)$$

Therefore, $\delta_{k, m}(\Omega)$ is invariant under scaling. It is also invariant under translation.
Furthermore, if \( r \) is the radius of \( B_{\Omega,m} \), then \( I_m(\Omega) = \binom{n}{m} r^{n-m} \text{Area}(\partial B) \), giving
\[
r = \left( \frac{I_m(\Omega)}{\binom{n}{m} \text{Area}(\partial B)} \right)^{\frac{1}{n-m}}.
\]

Then,
\[
I_k(B_{\Omega,m}) = \left( \frac{I_m(\Omega)}{\binom{n}{m} \text{Area}(\partial B)} \right)^{\frac{n-k}{n-m}} \binom{n}{k} \text{Area}(\partial B)^{\frac{k-m}{n-m}} I_m(\Omega)^{\frac{n-k}{n-m}}.
\]

In particular, when setting \( m = k - 1 \),
\[
\delta_{k-1}(\Omega) = \frac{I_k(\Omega)}{I_{k-1}(B)} = \frac{I_k(\Omega)}{I_{k-1}(\Omega)^{\frac{n-k}{n-m}}} - 1.
\]

Thus, the inequality \( \delta_{k-1}(\Omega) \geq 0 \) is equivalent to the quermassintegral inequalities in [12].

Our goal is to look at the quantitative isoperimetric inequality in the setting of the \( k \)-th mean curvature, which we refer to as the quantitative \((k,m)\)-isoperimetric inequality. That is, we aim to compare \( \delta_{k,m}(\Omega) \) to the Fraenkel asymmetry of \( \Omega \), \( \alpha(\Omega) \), which measures how close \( \Omega \) is to a ball (see Definition [12]).

The Fraenkel asymmetry of a set is invariant under scalings and translations. Therefore, when studying the quantitative \((k,m)\)-isoperimetric inequality, i.e. when there is a fixed \( C > 0 \) so that
\[
\delta_{k,m}(\Omega) \geq C\alpha^2(\Omega),
\]

we only need to consider sets where \( I_m(\Omega) = I_m(B) \) and \( \text{bar}(\Omega) = 0 \).

### 2.3 Nearly spherical sets

The focus of this paper is to establish the \((k,m)\)-isoperimetric inequality for nearly spherical sets. Our approach is inspired by Cicalese and Leonardi’s work in the classical quantitative isoperimetric inequality for nearly spherical sets in [3]. That is, we consider a smooth, bounded domain \( \Omega \) that is starshaped with respect to the origin, which is enclosed by \( M := \partial \Omega \). We write \( M = \{(1 + u(x))x : x \in \partial B\} \), where \( u : \partial B \to \mathbb{R} \) is a smooth function. The set \( M \) is referred to as a nearly spherical set when there is a suitable, small bound on \(|u||_{W^{2,\infty}}\). In this section, we establish some useful formulas for nearly spherical sets.

We write \( \mathbb{R}^{n+1} \) in spherical coordinates with the tangent basis \( \{\partial/\partial \theta^1, \partial/\partial \theta^2, \ldots, \partial/\partial \theta^n\} \). Denoting \( s_{ij} \) as the metric on the sphere, we have \( <\partial/\partial \theta^i, \partial/\partial \theta^j> = 0 \), \( <\partial/\partial r, \partial/\partial \theta^j> = 1 \), and \( <\partial/\partial \theta^i, \partial/\partial \theta^j> = r^2 s_{ij} \). Set \( u_i = \partial u/\partial \theta^i \). Then, \( \{e_i\} \) forms a tangent basis of \( M \) where
\[
e_i = \frac{\partial}{\partial \theta^i} + u_i \frac{\partial}{\partial r}.
\]

We find,
\[
N = \frac{-\sum_{i=1}^n s^{ij} u_j \frac{\partial}{\partial \theta^i} + (1 + u)^2 \frac{\partial}{\partial r}}{(1 + u)\sqrt{|\nabla u|^2 + (1 + u)^2}},
\]

where \( N \) is the the outward unit normal on \( M \), and the norm \(|\nabla u|\) is taken with respect to the standard metric on \( \partial B \). We compute the metric \( g_{ij} \) on \( M \) as
\[
g_{ij} = <e_i, e_j> = (1 + u)^2 s_{ij} + u_i u_j,
\]

where \(<\cdot,\cdot>\) is the standard Euclidean inner product on \( \mathbb{R}^{n+1} \). Setting \( g^{ij} \) to be the inverse of \( g_{ij} \), we have
\[ g^{ij} = \frac{s^{ij}}{(1+u)^2} - 1 \frac{u_k u_l s^{kl}s^{ij}}{(1+u)^2 |\nabla u|^2 + (1+u)^2}. \]  

(40)

We denote \( h_{ij} \) as the second fundamental form on \( M \). That is, \( h_{ij} = -\langle \nabla_n e_i, e_j \rangle \), and we form the shape operator \( h^i_j \) by

\[ h^i_j = g^{ik} h_{kj}. \]  

(41)

We now explicitly calculate \( h^i_j \). First, note

\[
\left( \nabla \frac{\partial}{\partial r_j} \right) = \frac{1}{2} r^2 s^{kl} \left( \partial_k (r^2 s_{il}) + \partial_l (r^2 s_{ik}) - \partial_i (r^2 s_{jl}) \right) = \frac{1}{2} s^{kl} \left( \partial_i s_{kl} + \partial_j s_{ik} - \partial_l s_{ij} \right) = \Gamma^k_{ij},
\]

(42)

(43)

(44)

where \( \Gamma^k_{ij} \) refers to the Christoffel symbol on \( S^n \), and

\[
\left( \nabla \frac{\partial}{\partial r_j} \right)^{\nabla} = -r s_{ij}.
\]

(45)

We thus obtain

- \( \nabla \frac{\partial}{\partial r_j} \frac{\partial}{\partial r_i} = \Gamma^k_{ij} \frac{\partial}{\partial r_i} - r s_{ij} \frac{\partial}{\partial r_k} \).

Similarly,

- \( \nabla \frac{\partial}{\partial r_i} \frac{\partial}{\partial r_j} = \nabla \frac{\partial}{\partial r_i} \frac{\partial}{\partial r_j} = \frac{1}{r} \frac{\partial}{\partial r_i} \).

- \( \nabla \frac{\partial}{\partial r_i} = 0 \).

Then,

\[
\nabla e_i e_j = \nabla \frac{\partial}{\partial r_j} + u_i \frac{\partial}{\partial r} \left( \frac{\partial}{\partial r_j} + u_j \frac{\partial}{\partial r} \right) \\
= \nabla \frac{\partial}{\partial r_j} + u_j \nabla \frac{\partial}{\partial r} + u_i \nabla \frac{\partial}{\partial r_j} + u_i \nabla \frac{\partial}{\partial r} - 1 (1+u) s_{ij} \frac{\partial}{\partial r} + (1+u) \frac{\partial}{\partial r} + u_i \frac{\partial}{\partial r_j} + u_i \frac{\partial}{\partial r} - 1 (1+u) s_{ij} \frac{\partial}{\partial r} + (1+u) \frac{\partial}{\partial r}. \]

(46)

(47)

So, our expression for \( h_{ij} \) becomes:

\[
- \left( -s^{pq} u_p \frac{\partial}{\partial r} + (1+u)^2 \frac{\partial}{\partial r}, \Gamma^k_{ij} \frac{\partial}{\partial r_k} - (1+u) s_{ij} \frac{\partial}{\partial r} + 1 (1+u) \frac{\partial}{\partial r} + u_i \frac{\partial}{\partial r} + u_i \frac{\partial}{\partial r} + \frac{\partial^2}{\partial \theta_i \partial \theta_j} u \right) \frac{\partial}{\partial r}.
\]

(48)

Thus,

\[
\frac{1}{\sqrt{|\nabla u|^2 + (1+u)^2}} \left( (1+u) u_k \Gamma^k_{ij} + (1+u)^2 s_{ij} + 2 u_i u_j - (1+u) \left( \frac{\partial^2}{\partial \theta_i \partial \theta_j} u \right) \right) \]

(49)

\[
= \frac{1}{\sqrt{|\nabla u|^2 + (1+u)^2}} \left( 2 u_i u_j + (1+u)^2 s_{ij} - (1+u) u_{ij} \right),
\]

where \( u_{ij} \) denotes the Hessian of \( u \) on \( S^n \). Set

\[
D := \sqrt{|\nabla u|^2 + (1+u)^2}.
\]

(50)
Then,
\[ h_j^i = g^{ik} h_{kj} \]
\[ = \frac{s^{ik}}{(1 + u)^2} - \frac{1}{(1 + u)^2} \frac{u_m u_i s^{mi} s^{lk}}{D^2} \frac{1}{D} \left( 2 u_k u_j + (1 + u)^2 s_{kj} - (1 + u) u_{kj} \right) \]
\[ = \frac{2 u^i u_j}{(1 + u)^2 D} + \frac{\delta^i_j}{D} - \frac{u^j_i}{(1 + u) D} \frac{2 u^i u_j |\nabla u|^2}{(1 + u)^2 D^3} - \frac{u^i u_j}{D^3} + \frac{u^i u_j l}{(1 + u) D^3}. \]  

We observe that
\[ \frac{2 u^i u_j}{(1 + u)^2 D} - \frac{2 u^i u_j |\nabla u|^2}{(1 + u)^2 D^3} = \frac{u^i u_j}{D^3}, \]
which yields
\[ h_j^i = \frac{\delta^i_j}{D} - \frac{u^j_i}{(1 + u) D} + \frac{u^i u_j l}{(1 + u) D^3}. \]

Next, note
\[ \sqrt{\text{det} \ g_{ij}} = (1 + u)^{n} \sqrt{\frac{|\nabla u|^2}{(1 + u)^2} + 1}. \]
Therefore,
\[ \text{Area}(M) = \int_{\partial B} (1 + u)^{n} \sqrt{\frac{|\nabla u|^2}{(1 + u)^2} + 1} \ dA. \]

We list a few more relevant formulae below:
\[ |\Omega| = \frac{1}{n + 1} \int_{\partial B} (1 + u)^{n+1} dA, \]
\[ |\Omega \Delta B| = \sum_{k=1}^{n+1} \int_{\partial B} \frac{1}{n + 1} \binom{n + 1}{k} |u|^k dA, \]
\[ \text{bar}(\Omega) = \frac{1}{\text{Area}(\partial B)} \int_{\partial B} (1 + u)^{n+2} x dA. \]

Finally, we consider how to compute \( \nabla_j [T_m^l](D^2 u) \) for nearly spherical sets. This is particularly useful when applying integration by parts on \( I_k(\Omega) \) in Section 4. See [3] for a similar computation. We compute
\[ \nabla_j [T_m^l](D^2 u) = \frac{1}{m!} \nabla_j \delta^{j_1 j_2 \ldots j_m}_{i_1 i_2 \ldots i_m} u_{j_1}^{i_1} u_{j_2}^{i_2} \ldots u_{j_m}^{i_m} \]
\[ = \frac{m}{m!} \delta^{j_1 j_2 \ldots j_m}_{i_1 i_2 \ldots i_m} (\nabla_j u_{j_1}^{i_1}) u_{j_2}^{i_2} \ldots u_{j_m}^{i_m}. \]

Note that
\[ \delta^{j_1 j_2 \ldots j_m}_{i_1 i_2 \ldots i_m} (\nabla_j u_{j_1}^{i_1}) u_{j_2}^{i_2} \ldots u_{j_m}^{i_m} = -\delta^{j_1 j_2 \ldots j_m}_{i_1 i_2 \ldots i_m} (\nabla_j u_{j_1}^{i_1}) u_{j_2}^{i_2} \ldots u_{j_m}^{i_m}. \]
We obtain
\[ \nabla_j [T_m^l](D^2 u) = \frac{1}{2(m - 1)!} \delta^{j_1 j_2 \ldots j_m}_{i_1 i_2 \ldots i_m} (\nabla_j u_{j_1}^{i_1} - \nabla_{j_1} u_{j_1}^{i_1}) u_{j_2}^{i_2} \ldots u_{j_m}^{i_m} \]
\[ = \frac{1}{2(m - 1)!} \delta^{j_1 j_2 \ldots j_{m-1}}_{i_1 i_2 \ldots i_{m-1}} (u_{p} R_{p j}^{l}) u_{j_1}^{i_1} \ldots u_{j_{m-1}}^{i_{m-1}}, \]
where \( R_{p j}^{l} \) is the curvature tensor on \( \Omega \). On \( S^n \), we know by the Gauss equation,
\[ R_{p j}^{l} = h_{p l}^i h_{j k}^i - h_{p l}^i h_{j k}^k = \delta_{p j}^{l} - \delta_{p l}^{j}. \]
\[ (62) \]
Therefore,
\[
\nabla_j [T_{m-1}]^j_i (D^2 u) = \frac{1}{(m-1)!} \frac{1}{2} u_{p} (\delta^p i \delta_j^i - \delta^{p} j \delta_i^j) \delta^j_{i_1 \ldots i_{m-1}} u_{j_1} \ldots u_{j_{m-1}} \\
= \frac{1}{(m-1)!} \frac{1}{2} \left( u_{a} \delta^a j_{i_1 \ldots i_{m-1}} u_{j_1} \ldots u_{j_{m-1}} - u_{j} \delta^j i_{i_1 \ldots i_{m-1}} u_{i_1} \ldots u_{i_{m-1}} \right) \\
= \frac{1}{(m-1)!} \left( u_{j} \delta^j i_{i_1 \ldots i_{m-1}} u_{j_1} \ldots u_{j_{m-1}} \right) \\
= -(n-m) u_j [T_{m-1}]^j_i (D^2 u). \tag{63}
\]

3 Computation of \( \sigma_k(L) \) for nearly spherical sets

Our goal is to control a lower bound on the \((k,m)\)-isoperimetric deficit \( \delta_{k,m}(\Omega) \), where \( M := \partial \Omega \) is a nearly spherical set. In this section, we focus on calculating \( \sigma_k(h_j^i) \), where

\[
\sigma_k(h_j^i) = \frac{1}{k!} \delta_{i_1 \ldots i_k} \left( \frac{-u_{i_1}}{D(1 + u)} + \frac{\delta_{i_1}^j}{D} + \frac{u_{i_1} u_{j_1}}{D^2(1 + u)} \right) \\
\ldots \left( \frac{-u_{i_k}}{D(1 + u)} + \frac{\delta_{i_k}^j}{D} + \frac{u_{i_k} u_{j_k}}{D^2(1 + u)} \right). \tag{64}
\]

In particular, the mean curvature on \( M \) is given by

\[
\sigma_1(h_j^i) = \delta_{i_1}^j \left( \frac{-u_{i_1}}{D(1 + u)} + \frac{\delta_{i_1}^j}{D} + \frac{u_{i_1} u_{j_1}}{D^2(1 + u)} \right) \\
= -\frac{\Delta u}{D(1 + u)} + \frac{n}{D} + \nabla u \frac{2}{D^3}. \tag{65}
\]

Computing \( \sigma_k(h_j^i) \) for any \( k > 0 \) requires a bit more work, as we show in the following lemma.

**Lemma 3.1.** Suppose \( \Omega \subseteq \mathbb{R}^{n+1} \) where \( M = \{(1 + u(x))x : x \in \partial B\} \) and \( u \in C^2(\partial B) \). Then

\[
\sigma_k(h_j^i) = \frac{1}{((1 + u)^2 + |\nabla u|^2)^{\frac{k+2}{2}}} \sum_{m=0}^{k} \frac{(-1)^m (n-m)}{(1 + u)^m} \left( (1 + u)^2 \sigma_m(D^2 u) + \frac{n + k - 2m}{n - m} u_i u_j [T_m]_{ij}^i(D^2 u) \right). \tag{66}
\]

**Proof.** In this proof we set

\[
D := \sqrt{(1 + u)^2 + |\nabla u|^2}. \tag{67}
\]

We expand out each term of \( \sigma_k(h_j^i) \) in [64]. Many of the terms in the expansion of this sum turn out to be zero. We compute the terms in the following cases:

1. \( m \geq 0 \) instances of \( \frac{-u_i^j}{D(1 + u)} \), and the rest are in the form \( \frac{\delta_{i_1}^j}{D} \).

First, consider the sum of all terms where \( \frac{-u_i^j}{D(1 + u)} \) occurs in the first \( m \) terms. This equals

\[
\frac{1}{k!} \delta_{i_1 \ldots i_k} \left( \frac{-u_{i_1}}{D(1 + u)} \ldots \frac{-u_{i_m}}{D(1 + u)} \right) \left( \frac{\delta_{i_1}^j}{D} \ldots \frac{\delta_{i_m}^j}{D} \right) \\
= \frac{1}{k!} \frac{(-1)^m}{D^k(1 + u)^m} \delta_{i_1 \ldots i_m} \frac{u_{i_1}}{D} \ldots \frac{u_{i_m}}{D} \left( \begin{array}{c} n - m \\ k - m \end{array} \right) \left( \begin{array}{c} n - m \\ k - m \end{array} \right) ! \\
= \frac{(-1)^m (n-m)}{(k-m)} \frac{\sigma_m(D^2 u)}{D^k(1 + u)^m}. \tag{68}
\]

However, to account for any permutation of the ordering of the terms above, we multiply [68] by \( \frac{n-m}{k-m} \).

So, the sum of all terms in this case is:

\[
\frac{(-1)^m (n-m)}{D^k(1 + u)^m} \sigma_m(D^2 u). \tag{69}
\]
2. One instance of \( \frac{u_i u_j}{D^2(1+u)^m} \) and \( m \geq 1 \) instances of \( \frac{-u_i}{D(1+u)^2} \).

The sum of these terms is equal to:

\[
\frac{(-1)^m k \cdot (k-1) \cdot \ldots \cdot (1)\delta_{i_1} \ldots \delta_{i_k} u_{i_1} u_{i_2} \ldots u_{i_{m+1}}}{k!} \frac{u_{i_3}}{D^3} \frac{u_{i_4}}{D(1+u)} \ldots \frac{u_{i_{m+1}}}{D(1+u)} \frac{\delta_{j_{m+2}} \ldots \delta_{j_k}}{D}\]

\[
= \frac{(-1)^m (k-1)!}{(k-1)!} \frac{1}{D^{k+2}(1+u)^{m+1}} \delta_{i_1} \ldots \delta_{i_k} u_{i_1} u_{i_2} \ldots u_{i_{m+1}} (n - (m+1)) (k - (m+1))!
\]

\[
= \left( n - (m+1) \right) \frac{(-1)^m}{k - (m+1)} \frac{u_i u_j [T_m]_i^j (D^2 u)}{D^{k+2}(1+u)^{m+1}}.
\]

(70)

3. One instance of \( \frac{u_i u_j}{D^3(1+u)} \) and \( m \geq 1 \) instances of \( \frac{u}{D(1+u)^2} \), or one instance of \( \frac{u_i u_j}{D^2(1+u)} \) and one instance of \( \frac{u}{D(1+u)^2} \).

In this case, the sum of all the terms is:

\[
\frac{(-1)^m (k-1)!}{k!} \frac{1}{D^{k+2}(1+u)^{m+1}} \delta_{i_1} \ldots \delta_{i_k} u_{i_1} u_{i_2} \ldots u_{i_{m+1}} (n - (m+1)) (k - (m+1))!
\]

\[
= \left( n - (m+1) \right) \frac{(-1)^m}{k - (m+1)} \frac{u_i u_j [T_m]_i^j (D^2 u)}{D^{k+2}(1+u)^{m+1}}.
\]

(71)

4. When there are either two instances of \( \frac{u_i u_j}{D^2(1+u)} \), two instances of \( \frac{u_i u_j}{D(1+u)^2} \), or one instance of \( \frac{u_i u_j}{D(1+u)^2} \) and one instance of \( \frac{u}{D(1+u)^2} \).

In this case, we apply the following Lemma [3,2] to conclude the sum of all these terms is zero.

Next, we simplify the expression for \( \sigma_k(h_j^i) \) by noting the identity

\[
u_j^i [T_m]_i^j (D^2 u) = \delta_i^j \sigma_{m+1}(D^2 u) - [T_{m+1}]_i^j (D^2 u).
\]

We compute

\[
\sigma_k(h_j^i) = \sum_{m=0}^{k-1} \left( \frac{n-m}{k-m} \right) \frac{(-1)^m \sigma_m(D^2 u)}{D^{k+2}(1+u)^{m+1}} + \sum_{m=0}^{k-1} \left( n - (m+1) \right) \frac{(-1)^m u_i u_j [T_m]_i^j (D^2 u)}{D^{k+2}(1+u)^{m+1}}
\]

\[
+ \sum_{m=0}^{k-1} \left( n - (m+1) \right) \frac{(-1)^m u_i u_j [T_m+1]_i^j (D^2 u) + \nabla u_i^2 \sigma_{m+1}(D^2 u)}{D^{k+2}(1+u)^{m+1}}
\]

\[
= \sum_{m=0}^{k} \frac{[1+u]^2 + |\nabla u|^2 (n-m)\sigma_m(D^2 u)}{D^{k+2}(1+u)^m} + \sum_{m=0}^{k-1} \left( n - (m+1) \right) \frac{(-1)^m u_i u_j [T_m]_i^j (D^2 u)}{D^{k+2}(1+u)^m}
\]

\[
+ \sum_{m=0}^{k} \frac{n-m}{k-m} \frac{(-1)^m u_i u_j [T_m]_i^j (D^2 u) - |\nabla u|^2 \sigma_m(D^2 u)}{D^{k+2}(1+u)^m}
\]

\[
= \frac{1}{D^{k+2}} \sum_{m=0}^{k} \frac{(-1)^m (n-m)}{(1+u)^m} \left( (1+u)^2 \sigma_m(D^2 u) + \frac{n-k-2m}{n-m} u_i u_j [T_m]_i^j (D^2 u) \right).
\]

(73)

Now we give a quick proof of the lemma that was used in case 4 in Lemma [3,1].
Lemma 3.2. Suppose $2 \leq k \leq n$ where $M_1, ..., M_{k-2}$ are $n \times n$ matrices, and $w_1, w_2, v$ are $n$-dimensional vectors. Then,

$$\Sigma_k(w_1v^T, w_2v^T, M_1, M_2, M_3, ..., M_{k-2}) = 0.$$  

(74)

Proof. We compute,

$$\Sigma_k(w_1v^T, w_2v^T, M_1, M_2, ..., M_{k-2}) = \frac{1}{(k-1)!} \delta_{i_1j_2...j_k}^i \delta_{i_1j_2...j_k}^i w_1^{i_1} v_{j_1} w_2^{i_2} v_{j_2} M_1^{i_3} \cdots M_{k-2}^{i_k}$$

$$= \frac{1}{(k-1)!} \delta_{i_1j_2...j_k}^i \delta_{i_1j_2...j_k}^i w_1^{i_1} v_{j_1} w_2^{i_2} v_{j_2} M_1^{i_3} \cdots M_{k-2}^{i_k}$$

$$= -\frac{1}{(k-1)!} \delta_{i_1j_2...j_k}^i \delta_{i_1j_2...j_k}^i w_1^{i_1} v_{j_1} w_2^{i_2} v_{j_2} M_1^{i_3} \cdots M_{k-2}^{i_k}$$

$$= -\Sigma_k(w_1v^T, w_2v^T, M_1, M_2, ..., M_{k-2}).$$  

(75)

Since $\Sigma_k(w_1v^T, w_2v^T, M_1, M_2, ..., M_{k-2}) = -\Sigma_k(w_1v^T, w_2v^T, M_1, M_2, ..., M_{k-2})$, we conclude

$$\Sigma_k(w_1v^T, w_2v^T, M_1, M_2, M_3, ..., M_{k-2}) = 0.$$  

(76)

□

4 $I_k(\Omega)$ for nearly spherical sets

We continue to look at $I_k(\Omega)$ for starshaped domains, but now we add in the additional assumption that $||u||_{W^{2,\infty}} < \epsilon$, making $M := \partial \Omega$ a nearly spherical set as described in Definition 1.3. Note,

$$\int_M \sigma(h^j)^2 d\mu = \int_{\partial B} \sigma(h^j)(1 + u)^n \sqrt{1 + \left|\nabla u\right|^2 / (1 + u)^2} dA.$$  

(77)

Using our formula for $\sigma(h^j)$ in Lemma 3.1, we have the following expression for $\int_M \sigma(h^j) dp$:

$$\int_{\partial B} \sum_{m=0}^{k} \frac{(-1)^m \delta_{k-m}^m (1 + u)^{n-m-1}}{(1 + u^2 + |\nabla u|^2)^{1/2}} \left(1 + u^2\right)^2 \sigma_m(D^2 u) + \frac{n + k - 2m}{n-m} \nabla u_j [T_m]_{ij} (D^2 u) dA.$$  

(78)

Later, in the main theorems, our analysis deals mainly with lower order terms in $O(||u||)$ and $O(||\nabla u||)$. In the following lemma, we expand out $(1 + u^2 + |\nabla u|^2)^{1/2}$ in the integral using its Taylor expansion, and we are able to group all of the higher order terms in $O(\epsilon)||u||_{L^2}^2 + O(\epsilon)||\nabla u||_{L^2}^2$.

Lemma 4.1. Suppose $u \in C^1(\partial B)$ and for sufficiently small $\epsilon > 0$ that $||u||_{L^\infty}, ||\nabla u||_{L^\infty} < \epsilon$. Then,

$$\frac{1}{(|\nabla u|^2 + (1 + u^2)^{1/2})^{1/2}} = 1 - mu + \frac{m(m+1)}{2} u^2 - \frac{m}{2} |\nabla u|^2 + O(\epsilon)u^2 + O(\epsilon)|\nabla u|^2.$$  

(79)

Proof. First, we will expand out $(|\nabla u|^2 + (1 + u^2)^{1/2})^{1/2}$, which we rewrite as $(1 + 2u + u^2 + |\nabla u|^2)^{-1/2}$. Setting $f(x) = (1 + x)^{-1/2}$, we obtain

$$f^n(0) = \frac{(-1)^n \prod_{m=1}^{n}(2m-1)}{2^n}.$$  

(80)

In the radius of convergence for its Taylor expansion, $f(x) = \sum_{n=0}^{\infty} c_n x^n$, where $c_n = \frac{(-1)^n \prod_{m=1}^{n}(2m-1)}{n!2^n}$ for $n \geq 1$ and $c_0 = 1$. Note that

$$|c_n| = \frac{\prod_{m=1}^{n}(2m-1)}{n!2^n} = \frac{(2n)!}{(n!)^2 4^n} \leq 1.$$  

(81)

Then, for small $|u|$ and $|\nabla u|$, and setting $g(u) := 2u + u^2 + |\nabla u|^2$,

$$\frac{1}{(|\nabla u|^2 + (1 + u^2)^{1/2})^{1/2}} = 1 - \frac{1}{2} (g(u)) + \frac{3}{8} (g(u))^2 + \sum_{n=3}^{\infty} c_n (g(u))^n.$$  

(82)
Furthermore, for $|g(u)| \leq \frac{1}{2}$,

$$\left| \sum_{n=3}^{\infty} c_n(g(u))^n \right| \leq \sum_{n=3}^{\infty} |g(u)|^n = \frac{|g(u)|^3}{1-g(u)} \leq 2|2u + u^2 + |\nabla u|^2|^3 = O(\epsilon)u^2 + O(\epsilon)|\nabla u|^2.$$  

(83)

Therefore,

$$\frac{1}{(|\nabla u|^2 + (1+u)^2)^{\frac{m}{2}}} = 1 - \frac{1}{2}(2u + u^2 + |\nabla u|^2)^2 + \frac{3}{8}(2u + u^2 + |\nabla u|^2)^2 + \sum_{n=3}^{\infty} c_n(g(u))^n$$

$$= 1 - u + u^2 - \frac{1}{2}|\nabla u|^2 + O(\epsilon)u^2 + O(\epsilon)|\nabla u|^2.$$  

(84)

We conclude,

$$\frac{1}{(|\nabla u|^2 + (1+u)^2)^{\frac{m}{2}}} = \left(1 - u + u^2 - \frac{1}{2}|\nabla u|^2 + O(\epsilon)u^2 + O(\epsilon)|\nabla u|^2 \right)^m$$

$$= \sum_{j=0}^{m} \binom{m}{j} (-u + u^2 - \frac{1}{2}|\nabla u|^2)^j + O(\epsilon)u^2 + O(\epsilon)|\nabla u|^2$$

$$= 1 - mu + \frac{m(m+1)}{2} u^2 - \frac{m}{2} |\nabla u|^2 + O(\epsilon)u^2 + O(\epsilon)|\nabla u|^2.$$  

(85)

Using the expansion in Lemma 4.1, we further expand $\int_M \sigma_k(h_j^p) \, d\mu$. We perform integration by parts on many of the terms to convert them to include $|\nabla u|^2$. The new format of the integral will be useful later on when we find a lower bound involving the Frénet asymmetry.

**Lemma 4.2.** Suppose $\Omega = \{(1 + u(\frac{1}{\sqrt{m}})) : x \in B \} \subseteq \mathbb{R}^{n+1}$ and $u \in C^2(\partial B)$. If $||u||_{W^{2, \infty}} < \epsilon$, then

$$\int_M \sigma_k(h_j^p) \, d\mu = \int_{\partial B} \binom{n}{k} + \binom{n}{k}(n-k)u + \binom{n}{k}(n-k)(n-k-1)u^2$$

$$+ \sum_{m=0}^{k} \frac{(-1)^m}{k-m} \frac{(n-m)(k+1)}{2(m+1)(n-m)} |\nabla u|^2 \sigma_m(D^2u) \, dA + O(\epsilon)||\nabla u||_{L^2}^2 + O(\epsilon)||u||_{L^2}^2.$$  

(86)

**Proof.** Applying the Taylor expansion in Lemma 4.1 we find

$$\int_M \sigma_k(h_j^p) \, d\mu = \sum_{m=0}^{k} \frac{(-1)^m}{k-m} \frac{(n-m)(k+1)}{2(m+1)(n-m)} |\nabla u|^2 \sigma_m(D^2u) + \frac{(n-m-k)(n-m-k-1)}{2} u^2$$

$$- \frac{k+1}{2} |\nabla u|^2 \sigma_m(D^2u) + \frac{(n-k-2m)}{n-m} u^i u^j [T_m]_1^j \sigma_m(D^2u) dA + O(\epsilon)||u||_{L^2}^2 + O(\epsilon)||\nabla u||_{L^2}^2.$$  

(87)

Next, recall from the preliminaries that

$$\nabla_j [T_m]_1^j(D^2u) = -(n-m) u_j [T_m-1]_1^j(D^2u),$$

(88)

and

$$\sigma_m(D^2u) = \frac{1}{m} u_j [T_m]_1^j(D^2u).$$

(89)

Using these identities, we rewrite many of the terms using integration by parts. First, for $m \geq 1$

$$\int_{\partial B} |\nabla u|^2 \sigma_m(D^2u) \, dA = \frac{1}{m} \int_{\partial B} |\nabla u|^2 u_j [T_m-1]_1^j(D^2u) \, dA$$

$$= -\frac{1}{m} \int_{\partial B} u^j u_j u^j [T_m-1]_1^j(D^2u) + u^i |\nabla u|^2 \nabla_j [T_m-1]_1^j(D^2u) \, dA$$

$$= -\frac{2}{m} \int_{\partial B} u^j u^i u_j u^j [T_m-1]_1^j(D^2u) dA + O(\epsilon)||\nabla u||_{L^2}^2$$

$$= \frac{2}{m} \int_{\partial B} u^j u_j [T_m]_1^j (D^2u) - |\nabla u|^2 \sigma_m(D^2u) dA + O(\epsilon)||\nabla u||_{L^2}^2.$$  

(90)
Therefore,
\[ \int_{\partial B} u^i u_j [T_{m}]_{ij} (D^2 u) dA = \frac{m+2}{2} \int_{\partial B} |\nabla u|^2 \sigma_m (D^2 u) dA + O(\epsilon)||\nabla u||_{L^2}^2. \]  
(91)

Next, we integrate each \( \sigma_m (D^2 u) \) term. For \( m \geq 2 \), we have
\[ \int_{\partial B} \sigma_m (D^2 u) dA = \frac{1}{m} \int_{\partial B} u^i u_j [T_{m-1}]_{ij} (D^2 u) dA \]
\[ = \frac{1}{m} \int_{\partial B} u^i \nabla_j [T_{m-1}]_{ij} (D^2 u) dA \]
\[ = \frac{n-m+1}{m} \int_{\partial B} u^i u_j [T_{m-2}]_{ij} (D^2 u) dA. \]  
(92)

By (91),
\[ \int_{\partial B} \sigma_m (D^2 u) dA = \frac{n-m+1}{2} \int_{\partial B} |\nabla u|^2 \sigma_{m-2} (D^2 u) dA + O(\epsilon)||\nabla u||_{L^2}^2. \]  
(93)

Also, for \( m = 0 \) and \( 1 \),
\[ \int_{\partial B} \sigma_0 (D^2 u) dA = \int_{\partial B} 1 dA \text{ and } \int_{\partial B} \sigma_1 (D^2 u) dA = 0. \]  
(94)

Similarly, for \( m \geq 1 \),
\[ \int_{\partial B} u \sigma_m (D^2 u) dA = \frac{1}{m} \int_{\partial B} u u_j u_i [T_{m-1}]_{ij} (D^2 u) dA \]
\[ = \frac{1}{m} \int_{\partial B} u^i u_j [T_{m-1}]_{ij} (D^2 u) + uu_i \nabla_j [T_{m-1}]_{ij} (D^2 u) dA \]
\[ = \frac{1}{m} \int_{\partial B} u^i u_j [T_{m-1}]_{ij} (D^2 u) + O(\epsilon)||\nabla u||_{L^2}^2. \]  
(95)

By (91),
\[ \int_{\partial B} u \sigma_m (D^2 u) dA = \frac{-(m+1)}{2m} \int_{\partial B} |\nabla u|^2 \sigma_{m-1} (D^2 u) dA + O(\epsilon)||\nabla u||_{L^2}^2. \]  
(96)

And,
\[ \int_{\partial B} u \sigma_0 (D^2 u) dA = \int_{\partial B} u dA. \]  
(97)

Lastly, for \( m \geq 1 \),
\[ \int_{\partial B} u^2 \sigma_m (D^2 u) dA = \frac{1}{m} \int_{\partial B} u^2 u_j [T_{m-1}]_{ij} (D^2 u) dA \]
\[ = \frac{1}{m} \int_{\partial B} u^i \left( 2u_j u_i [T_{m-1}]_{ij} (D^2 u) + u^2 \nabla_j [T_{m-1}]_{ij} (D^2 u) \right) dA \]
\[ = O(\epsilon)||\nabla u||_{L^2}^2. \]  
(98)

And,
\[ \int_{\partial B} u^2 \sigma_0 (D^2 u) dA = \int_{\partial B} u^2 dA. \]  
(99)

All together, we find
\[ \int_{M} \sigma_k (h^i_j) d\mu = \int_{\partial B} \left( \begin{array}{c} n \\ k \end{array} \right) + \left( \begin{array}{c} n \\ k \end{array} \right) (n-k) u + \left( \begin{array}{c} n \\ k \end{array} \right) \frac{(n-k)(n-k-1)}{2} u^2 \\
+ \sum_{m=0}^{k} (-1)^m \frac{(n-m)(n-k)(k+1)}{k-m} 2(m+1)(n-m) |\nabla u|^2 \sigma_m (D^2 u) dA + O(\epsilon)||\nabla u||_{L^2}^2 + O(\epsilon)||u||_{L^2}^2. \]  
(100)
Next we turn our attention to the \((k, -1)\)-isoperimetric deficit. Recall from Section 2.2
\[
\delta_{k,-1}(\Omega) = \frac{I_k(\Omega) - I_k(B_{\Omega,-1})}{I_k(B_{\Omega,-1})}
\]
(101)
where \(B_{\Omega,-1}\) is the ball centered at the origin satisfying \(\text{Vol}(B_{\Omega,-1}) = \text{Vol}(\Omega)\). In particular, if \(\Omega\) is normalized so that \(\text{Vol}(\Omega) = \text{Vol}(B)\), then
\[
\delta_{k,-1}(\Omega) = \frac{I_k(\Omega) - I_k(B)}{I_k(B)}.
\]
(102)
In the next proposition, with the additional assumption that the barycenter of \(\Omega\) is at the origin, we are able to bound \(I_k(\Omega) - I_k(B)\) below by terms involving \(\|u\|_{L^{1,2}}\). In order to get our main theorem bounding \(\delta_{k,-1}(\Omega)\) below by \(\alpha^2(\Omega)\) (see Section 2.2), we only need the term with \(\|u\|_{L^2}^2\) in the lower bound. However, we form a stronger statement that also includes \(\|\nabla u\|_{L^2}^2\) in the lower bound.

**Proposition 4.3.** Suppose \(\Omega = \{(1 + u(x)/|x|)x : x \in B\} \subseteq \mathbb{R}^{n+1}\) where \(u \in C^3(\partial B)\), \(\text{Vol}(\Omega) = \text{Vol}(B)\), and \(\text{bar}(\Omega) = 0\). Additionally, assume for sufficiently small \(\epsilon > 0\) that \(\|u\|_{W^{1,\infty}} < \epsilon\). Then,
\[
I_k(\Omega) - I_k(B) \geq \frac{n}{2k}
\]
\[
\left(1 + O(\epsilon)\right)\|u\|_{L^2}^2 + \left(\frac{1}{2} + O(\epsilon)\right)\|\nabla u\|_{L^2}^2.
\]
(103)

**Proof.** From Lemma 4.2
\[
I_k(\Omega) - I_k(B) = \int_{\partial B} \left(\frac{n}{2k} - \frac{n-k}{k}\right) u + \frac{(n-k)(n-k-1)}{2}\frac{n}{k} u^2
\]
\[
+ \sum_{m=0}^{k} (-1)^m \frac{n-m}{k-m} \frac{(n-k)(k+1)}{2(m+1)(n-m)} \|\nabla u\|_{L^2}^2 \sigma_m(D^2 u) dA + O(\epsilon)\|\nabla u\|_{L^2}^2 + O(\epsilon)\|u\|_{L^2}^2.
\]
(104)
Using the assumption that \(\text{Vol}(\Omega) = \text{Vol}(B)\), we have from formula (56) for the volume that
\[
\int_{\partial B} u dA = \int_{\partial B} \frac{-n}{2} u^2 dA + O(\epsilon)\|u\|_{L^2}^2.
\]
(105)
Substituting this expression into (104) yields
\[
I_k(\Omega) - I_k(B) = \int_{\partial B} \left(\frac{n}{2k} - \frac{n-k}{k}\right) |\nabla u|^2 - \frac{(n-k)(k+1)}{2} u^2
\]
\[
+ \sum_{m=1}^{k} (-1)^m \frac{n-m}{k-m} \frac{(n-k)(k+1)}{2(m+1)(n-m)} \|\nabla u\|_{L^2}^2 \sigma_m(D^2 u) dA + O(\epsilon)\|\nabla u\|_{L^2}^2 + O(\epsilon)\|u\|_{L^2}^2.
\]
(106)
Then, using the assumptions that \(\text{Vol}(\Omega) = \text{Vol}(B)\) and \(\text{bar}(\Omega) = 0\), Cicalaese and Leonardi showed (see Lemma 4.2 in [4]), by writing \(u\) in terms of its spherical harmonics basis, that
\[
\|\nabla u\|_{L^2}^2 \geq 2(n+1)\|u\|_{L^2}^2 + O(\epsilon)\|u\|_{L^2}^2.
\]
(107)
Finally, by applying the inequality (107) to (106), we find that \(I_k(\Omega) - I_k(B)\) is bounded below by the following expression:
\[
\frac{n}{k} \frac{(n-k)(k+1)}{2n} \left(\frac{1}{2} \|\nabla u\|_{L^2}^2 + (n+1)\|u\|_{L^2}^2 + n\|u\|_{L^2}^2\right) + O(\epsilon)\|\nabla u\|_{L^2}^2 + O(\epsilon)\|u\|_{L^2}^2.
\]
(108)
Now, with the lower bound on \(I_k(\Omega) - I_k(B)\) being controlled by \(\|u\|_{L^2}^2\), we are equipped to show one of our main results. Observe, as shown in [4], that for \(\|u\|_{L^\infty} < \epsilon\), Hölder’s inequality yields
\[
\frac{|\Omega\Delta B|}{|B|} = \frac{1}{|B|} \left(\|u\|_{L^1} + \sum_{k=2}^{n+1} \int_{\partial B} \frac{1}{n+1} \left(\frac{n+1}{k}\right) \|u\| dA\right)
\]
\[
\leq \frac{1}{|B|} \left(\text{Area}(\partial B)^{1/2}\|u\|_{L^2} + \sum_{k=2}^{n+1} \int_{\partial B} \frac{1}{n+1} \left(\frac{n+1}{k}\right) \|u\| dA\right).
\]
(109)
Therefore, when \( \text{Vol}(\Omega) = \text{Vol}(B) \),
\[
\alpha^2(\Omega) \leq \frac{\|\Omega \Delta B\|^2}{|B|^2} \leq \frac{\text{Area}(\partial B)}{|B|^2} \|u\|_{L^2}^2 + O(\epsilon)\|u\|_{L^2}^2 = \frac{(n + 1)^2}{\text{Area}(\partial B)} \|u\|_{L^2}^2 + O(\epsilon)\|u\|_{L^2}^2.
\]

Now we are ready to prove Theorem \ref{thm:quantitative-isoperimetric-inequality}.

**Theorem 1.2.** Suppose \( \Omega = \{(1 + u(x/|x|))x : x \in B\} \subseteq \mathbb{R}^{n+1} \), where \( u \in C^3(\partial B) \), \( \text{Vol}(\Omega) = \text{Vol}(B) \), and \( \text{bar}(\Omega) = 0 \). For all \( \eta > 0 \), there exists \( \epsilon > 0 \) such that if \( \|u\|_{W^{2,\infty}} < \epsilon \), then
\[
\delta_{k,j}(\Omega) \geq \left( \frac{(n - k)(k + 1)}{2(n + 1)^2} - \eta \right) \alpha^2(\Omega).
\]

**Proof.** Suppose \( \|u\|_{W^{2,\infty}} < \epsilon \). Since \( \text{Vol}(\Omega) = \text{Vol}(B) \), the definition for the \((k,-1)\)-isoperimetric deficit becomes
\[
\delta_{k,-1}(\Omega) = \frac{I_k(\Omega) - I_k(B)}{I_k(B)} = \frac{I_k(\Omega) - I_k(B)}{(n_k)\text{Area}(\partial B)}.
\]

From Proposition \ref{prop:isoperimetric-deficit}, we have
\[
\delta_{k,-1}(\Omega) \geq \left( \frac{(n - k)(k + 1)}{2n\text{Area}(\partial B)} \right) \|u\|_{L^2}^2 + O(\epsilon)\|u\|_{L^2}^2.
\]

Next, as noted in \ref{thm:quantitative-isoperimetric-inequality},
\[
\alpha^2(\Omega) \leq \frac{\|\Omega \Delta B\|^2}{|B|^2} \leq \frac{(n + 1)^2}{\text{Area}(\partial B)} \|u\|_{L^2}^2 + O(\epsilon)\|u\|_{L^2}^2.
\]

It follows that
\[
\|u\|_{L^2}^2 \geq \frac{\text{Area}(\partial B)}{(n + 1)^2} \alpha^2(\Omega) + O(\epsilon)\alpha^2(\Omega).
\]

Therefore,
\[
\delta_{k,-1}(\Omega) \geq \left( \frac{(n - k)(k + 1)}{2n\text{Area}(\partial B)} \right) \left( 1 + O(\epsilon) \right)\|u\|_{L^2}^2
\]
\[
\geq \left( \frac{(n - k)(k + 1)}{2n(n + 1)^2} \right) \alpha^2(\Omega) + O(\epsilon)\alpha^2(\Omega).
\]

\( \square \)

## 5 Quantitative isoperimetric inequality for \( \delta_{k,j}(\Omega) \) when \( j \geq 0 \)

In this section we extend the result from Theorem \ref{thm:quantitative-isoperimetric-inequality} to \( \delta_{k,j}(\Omega) \) for \( 0 \leq j < k \). The proof turns out to be quite similar to the case for \( \delta_{k,-1}(\Omega) \) in the previous section. The expression for \( I_k(\Omega) \) will contain the quantity \( C \int_{\partial B} |\nabla u|^2 - nu^2 dA \) under the assumption \( I_j(\Omega) = I_j(B) \), which we bound in the same manner as in Proposition \ref{prop:isoperimetric-deficit}.

We begin with the following proposition.

**Proposition 5.1.** Fix \( j \) where \( 0 \leq j < k \). Suppose \( \Omega = \{(1 + u(x/|x|))x : x \in B\} \subseteq \mathbb{R}^{n+1} \), where \( u \in C^3(\partial B) \), \( I_j(\Omega) = I_j(B) \), and \( \text{bar}(\Omega) = 0 \). Assume for sufficiently small \( \epsilon > 0 \) that \( \|u\|_{W^{2,\infty}} < \epsilon \). Then,
\[
I_k(\Omega) - I_k(B) \geq \frac{n}{k} \left( \frac{n - k}{2n} \right) \left( (1 + O(\epsilon)) \|u\|_{L^2}^2 + \left( \frac{1}{2} + O(\epsilon) \right) \|\nabla u\|_{L^2}^2 \right).
\]

**Proof.** First, for any \( s \geq 0 \), we have from Lemma \ref{lem:isoperimetric-deficit} that
\[
I_s(\Omega) - I_s(B) = \int_{\partial B} \left( \frac{n}{s} \right) (n - s)u + \left( \frac{n}{s} \right) \left( (n - s)(n - s - 1) \right) u^2
\]
\[
+ \sum_{m=0}^{\infty} (-1)^m \left( \frac{n - m}{s - m} \right) \frac{(n - s)(s + 1)}{2(m + 1)(n - m)} \|\nabla u\|^2 \sigma_m(D^2 u) dA + O(\epsilon) \|\nabla u\|_{L^2}^2 + O(\epsilon)\|u\|_{L^2}^2.
\]
Therefore, if \( I_j(\Omega) = I_j(B) \),
\[
\int_{\partial B} u \, dA = - \int_{\partial B} \frac{n-j-1}{2} u^2 + \frac{j+1}{2n} |\nabla u|^2 + \left( \frac{1}{n} \right) \sum_{m=1}^{k} (-1)^m \left( \frac{n-m}{j-m} \right) \frac{j+1}{2(m+1)(n-m)} \sigma_m(D^2 u) \, dA \\
+ O(\epsilon)||\nabla u||_{L^2}^2 + O(\epsilon)||u||_{L^2}^2.
\] (119)
Substituting this expression in (118) for \( s = k \) yields
\[
I_k(\Omega) - I_k(B) = \left( \frac{n}{k} \right) \frac{(n-k)(k-j)}{2n} \int_{\partial B} |\nabla u|^2 - nu^2 + \left( \sum_{m=1}^{k} d_m \sigma_m(D^2 u) \right) |\nabla u|^2 \, dA \\
+ O(\epsilon)||\nabla u||_{L^2}^2 + O(\epsilon)||u||_{L^2}^2,
\] (120)
where each \( d_m \) is coefficient for \( \sigma_m(D^2 u) |\nabla u|^2 \). Using the assumptions that \( I_j(\Omega) = I_j(B) \) and \( \text{bar}(\Omega) = 0 \), we show in the following lemma that
\[
||\nabla u||_{L^2}^2 \geq 2(n+1)||u||_{L^2}^2 + O(\epsilon^2)||u||_{L^2}^2 + O(\epsilon)||\nabla u||_{L^2}^2.
\] (121)
Therefore,
\[
I_k(\Omega) - I_k(B) \geq \left( \frac{n}{k} \right) \frac{(n-k)(k-j)}{4n} ||\nabla u||_{L^2}^2 + \left( \frac{n}{k} \right) \frac{(n-k)(k-j)}{2n} ||u||_{L^2}^2 + O(\epsilon)||\nabla u||_{L^2}^2 + O(\epsilon)||u||_{L^2}^2
\] (122)
We now prove the lower bound on \( ||\nabla u||_{L^2}^2 \) used in the previous lemma. The proof closely resembles that in [4] by Cicala and Leonardi in their work with the classical quantitative isoperimetric inequality (see also [6] and [7] by Fuglede).

**Lemma 5.2.** Suppose \( \Omega = \{(1 + u(x))x : x \in B \} \subseteq \mathbb{R}^{n+1} \), with \( u \in C^3(\partial B) \), \( \text{bar}(\Omega) = 0 \), and \( I_j(\Omega) = I_j(B) \) for a fixed \( j \) where \( 0 \leq j \leq n \). It holds that
\[
||\nabla u||_{L^2}^2 \geq 2(n+1)||u||_{L^2}^2 + O(\epsilon^2)||u||_{L^2}^2 + O(\epsilon)||\nabla u||_{L^2}^2.
\] (123)

**Proof.** We write
\[
u = \sum_{k=0}^{\infty} a_k Y_k,
\] (124)
where \( \{Y_k\} \) are spherical harmonics which form an orthonormal basis for \( L^2(\partial B) \). Since \( Y_0 = 1 \) we have
\[
a_0 = <u, 1 >_{L^2(\partial B)} = \int_{\partial B} u \, dA.
\] (125)
Additionally, using the assumption \( I_j(\Omega) = I_j(B) \), we have from (119) that
\[
\int_{\partial B} u \, dA = \int_{\partial B} \frac{n-j-1}{2} u^2 + \frac{j+1}{2n} |\nabla u|^2 + \left( \frac{1}{n} \right) \sum_{m=1}^{k} (-1)^m \left( \frac{n-m}{j-m} \right) \frac{j+1}{2(m+1)(n-m)} \sigma_m(D^2 u) \, dA \\
+ O(\epsilon)||\nabla u||_{L^2}^2 + O(\epsilon)||u||_{L^2}^2.
\] (126)
This further implies that \( \int_{\partial B} u \, dA = O(\epsilon^2) \). Hence,
\[
a_0^2 = O(\epsilon^2)||u||_{L^2}^2 + O(\epsilon^2)||\nabla u||_{L^2}^2.
\] (127)
As shown in [4], combining \( \bar{\text{bar}}(\Omega) = 0 \) and \( \int_{\partial B} Y_1 \, dA = 0 \) gives

\[
\int_{\partial B} ((1 + u)^{n+2} - 1)Y_1 \, dA = 0.
\]

(128)

So,

\[
\int_{\partial B} uY_1 \, dA = \sum_{k=2}^{n+2} \binom{n+2}{k} \int_{\partial B} u^k Y_1 \, dA = O(||u||_{L^2}^2).
\]

(129)

Therefore,

\[
a_1^2 = O(\epsilon^2)||u||_{L^2}^2.
\]

(130)

Next, we consider the corresponding eigenvalue of the spherical harmonic \( Y_k \), which is explicitly given by \( \lambda_k = -k(k + n - 1) \). Noting that \( |\lambda_k| \geq 2(n+1) \) when \( k \geq 2 \), we compute

\[
||\nabla u||_{L^2}^2 = \sum_{k=1}^\infty |\lambda_k| a_k^2 \\
= \sum_{k=2}^\infty |\lambda_k| a_k^2 + na_1^2 \\
\geq 2(n+1) \sum_{k=2}^\infty a_k^2 + na_1^2 \\
= 2(n+1) \sum_{k=0}^\infty a_k^2 - 2(n+1)a_0^2 - (n+2)a_1^2 \\
= 2(n+1)||u||_{L^2}^2 + O(\epsilon^2)||u||_{L^2}^2 + O(\epsilon^2)||\nabla u||_{L^2}^2.
\]

(131)

Next, we aim to use Proposition 5.1 to bound the \((k, j)\)-isoperimetric deficit below by the Frankel asymmetry \( \alpha(\Omega) \). When \( j = -1 \) (when the volume is preserved), estimating \( \alpha(\Omega) \) was reduced to being bounded above by \( |\Omega| \Delta B|_{\Omega}^2 \), which was bounded by \( ||u||_{L^2}^2 \) (up to a constant). When the \( \Vol(\Omega) \neq \Vol(B) \), estimating this quantity is a bit more difficult, and we show in the next theorem that we can bound it above by \( ||\nabla u||_{L^2}^2 \).

**Lemma 5.3.** Suppose \( M := \partial \Omega \) is a nearly spherical set, then

\[
\frac{|\Omega \Delta B_{\Omega}|^2}{|B_{\Omega}|^2} \leq \frac{(n+1)^2}{n^2 \text{Area}(\partial B)} ||\nabla u||_{L^2}^2 + O(\epsilon)||\nabla u||_{L^2}^2,
\]

(132)

where \( ||u||_{W^{1,\infty}} < \epsilon \).

**Remark.** Note that we do not need to assume \( ||D^2 u||_{L^\infty} < \epsilon \) in this lemma.

**Proof.** Recall the formula

\[
|\Omega| = \frac{1}{n+1} \int_{\partial B} (1 + u)^{n+1} \, dA = |B| + \sum_{k=1}^{n+1} \frac{n+1}{k+1} \int_{\partial B} u^k \, dA.
\]

(133)
And, if $r$ is the radius of $B_{\Omega}$, then $|\Omega| = |B_{\Omega}| = r^{n+1}|B|$. Hence, $r^{n+1} = |\Omega| / |B|$. We compute,

\[
\frac{|\Omega \Delta B_{\Omega}|}{|B_{\Omega}|} = \frac{1}{n+1} \frac{1}{|B_{\Omega}|} \int_{\partial B} \left| (1+u)^{n+1} - r^{n+1} \right| dA \\
= \frac{1}{n+1} \frac{1}{|B_{\Omega}|} \int_{\partial B} \left| (1+u)^{n+1} - \frac{|B| + \sum_{k=1}^{n+1} \int_{\partial B} u^k dA}{|B|} \right| dA \\
= \frac{1}{n+1} \frac{1}{|B_{\Omega}|} \int_{\partial B} \left| \sum_{k=1}^{n+1} \left( \begin{array}{c} n+1 \\ k \end{array} \right) u^k - \frac{1}{\operatorname{Area}(\partial B)} \left( \begin{array}{c} n+1 \\ k \end{array} \right) \int_{\partial B} u^k dA \right| dA \\
= \frac{1}{n+1} \frac{1}{|B_{\Omega}|} \int_{\partial B} \sum_{k=1}^{n+1} \left( \begin{array}{c} n+1 \\ k \end{array} \right) \left( u^k - \operatorname{Avg}(u^k) \right) dA \\
\leq \frac{1}{n+1} \frac{1}{|B_{\Omega}|} \sum_{k=1}^{n+1} \left( \begin{array}{c} n+1 \\ k \end{array} \right) \|u^k - \operatorname{Avg}(u^k)\|_{L^1}, \quad (134)
\]

where $\operatorname{Avg}(u^k)$ denotes the average value of $u^k$ on $\partial B$. Then, by applying H"older’s inequality and the Poincaré inequality, we continue to bound

\[
\frac{|\Omega \Delta B_{\Omega}|}{|B_{\Omega}|} \leq \frac{1}{n+1} \frac{\operatorname{Area}(\partial B)^{1/2}}{|B_{\Omega}|} \sum_{k=1}^{n+1} \left( \begin{array}{c} n+1 \\ k \end{array} \right) \|u^k - \operatorname{Avg}(u^k)\|_{L^2} \\
\leq \frac{1}{n(n+1)} \frac{\operatorname{Area}(\partial B)^{1/2}}{|B_{\Omega}|} \sum_{k=1}^{n+1} \left( \begin{array}{c} n+1 \\ k \end{array} \right) \|\nabla(u^k)\|_{L^2} \\
\leq \frac{\operatorname{Area}(\partial B)^{1/2}}{n|B_{\Omega}|} \sum_{k=1}^{n+1} \left( \begin{array}{c} n \\ k-1 \end{array} \right) \|u^{k-1}\|_{L^\infty} \|\nabla u\|_{L^2}. \quad (135)
\]

Therefore, noting that $\|u^{k-1}\|_{L^\infty} = O(\epsilon)$ for $k \geq 2$,

\[
\frac{|\Omega \Delta B_{\Omega}|^2}{|B_{\Omega}|^2} \leq \frac{1}{n^2} \frac{\operatorname{Area}(\partial B)}{|B_{\Omega}|^2} \|\nabla u\|_{L^2}^2 + O(\epsilon) \|\nabla u\|_{L^2}^2 \\
= \frac{(n+1)^2}{n^2 \operatorname{Area}(\partial B)} \frac{|B|^2}{|B_{\Omega}|^2} \|\nabla u\|_{L^2}^2 + O(\epsilon) \|\nabla u\|_{L^2}^2. \quad (136)
\]

Then, because $\frac{|B|^2}{|B_{\Omega}|^2} = 1 + O(\epsilon)$

\[
\frac{|\Omega \Delta B_{\Omega}|^2}{|B_{\Omega}|^2} \leq \frac{(n+1)^2}{n^2 \operatorname{Area}(\partial B)} \|\nabla u\|_{L^2}^2 + O(\epsilon) \|\nabla u\|_{L^2}^2. \quad (137)
\]

We now prove the main theorem of this section, where we obtain a quantitative isoperimetric inequality for the $(k,j)$-isoperimetric deficit. Recall from Section 2.2 that normalizing $\Omega$ such that $I_j(\Omega) = I_j(B)$ yields

\[
\delta_{k,j}(\Omega) = \frac{I_k(\Omega) - I_k(B)}{I_k(B)}. \quad (138)
\]

**Theorem 1.3** Fix $0 \leq j < k$. Suppose $\Omega = \{ (1 + u(x)) : x \in B \} \subset \mathbb{R}^{n+1}$, where $u \in C^3(\partial B)$, $I_j(\Omega) = I_j(B)$, and $\bar{u}(\Omega) = 0$. For all $\eta > 0$, there exists $\epsilon > 0$ such that if $\|u\|_{W^{2,\infty}} < \epsilon$, then

\[
\delta_{k,j}(\Omega) \geq \left( \frac{n(n-k)(k-j)}{4(n+1)^2} - \eta \right) \alpha^2(\Omega). \quad (139)
\]
Proof. Suppose \( ||u||_{W^{2,\infty}} < \epsilon \). Applying Lemma 5.3

\[
\alpha^2(\Omega) \leq \frac{|\Omega \Delta B|^2}{|B|^2} \leq \left( \frac{(n+1)^2}{n^2 \text{Area}(\partial B)} + O(\epsilon) \right) ||\nabla u||^2.
\] (140)

Thus

\[
||\nabla u||^2_{L^2} \geq \left( \frac{n^2}{(n+1)^2} \text{Area}(\partial B) + O(\epsilon) \right) \alpha^2(\Omega).
\] (141)

Additionally, since \( I_j(\Omega) = I_j(B) \),

\[
\delta_{k,j}(\Omega) = \frac{I_k(\Omega) - I_k(B)}{I_k(B)} = \frac{I_k(\Omega) - I_k(B)}\left( \frac{\epsilon}{n} \right) \text{Area}(\partial B).
\] (142)

Therefore, applying Proposition 5.1 when \( ||u||_{W^{2,\infty}} < \epsilon \), we have that

\[
\delta_{k,j}(\Omega) \geq \left( \frac{(n-k)(k-j)}{4n \text{Area}(\partial B)} + O(\epsilon) \right) ||\nabla u||^2_{L^2}
\]
\[
\geq \left( \frac{(n-k)(k-j)}{4n \text{Area}(\partial B)} + O(\epsilon) \right) \left( \frac{n^2}{(n+1)^2} \text{Area}(\partial B) + O(\epsilon) \right) \alpha^2(\Omega)
\]
\[
= \left( \frac{n(n-k)(k-j)}{4(n+1)^2} + O(\epsilon) \right) \alpha^2(\Omega).
\] (143)

\[\square\]

6 Bounds on \( ||u||_{L^\infty} \)

Following the argument of Fuglede in [7] for stability of the classical isoperimetric inequality, we control \( ||u||_{L^\infty} \) using the \((k,-1)\)-isoperimetric deficit. Because we consider \( \Omega \) when \( \text{Vol}(\Omega) = \text{Vol}(B) \) and \( \text{bar}(\Omega) = 0 \), \( ||u||_{L^\infty} \) is simply the spherial deviation \( d(\Omega) \) from Definition 1.3. First, we state a lemma from [7].

Lemma 6.1. (Fuglede [7], Lemma 1.4) Suppose \( w : \partial B \to \mathbb{R} \) is a Lipschitz function where \( \int_{\partial B} wdA = 0 \). Then

\[
||w||_{L^\infty}^n \leq \begin{cases} 
\pi ||\nabla w||_{L^1} & n = 1 \\
\frac{\pi}{4} ||\nabla w||_{L^2}^2 & n = 2 \\
\frac{C||\nabla w||_{L^2}^2 ||w||_{L^\infty}^{n-2}}{n-2} & n \geq 3,
\end{cases}
\]

where \( C > 0 \) depends only on \( n \).

This lemma is useful because the assumption that \( \text{Vol}(\Omega) = \text{Vol}(B) \) is equivalently stated expressed as:

\[
\int_{\partial B} (1 + u)^{n+1} - 1 dA = 0.
\] (144)

So, we set \( w = \frac{1}{n+1}((1 + u)^{n+1} - 1) \) and apply Lemma 6.1 to \( w \).

Theorem 1.4. Suppose \( \Omega = \{(1 + u(x)) x : x \in B \} \subseteq \mathbb{R}^{n+1} \), where \( u \in C^3(\partial B) \), \( \text{Vol}(\Omega) = \text{Vol}(B) \), and \( \text{bar}(\Omega) = 0 \). There exists an \( \eta > 0 \) so if \( ||u||_{W^{2,\infty}} < \eta \), then

\[
||u||_{L^\infty}^n \leq \begin{cases}
C_{k,-1}^\delta(\Omega) & n = 1 \\
C_{k,-1}^\delta(\Omega) \log A & n = 2 \\
C_{k,-1}^\delta(\Omega) & n \geq 3
\end{cases}
\]

where \( A, C > 0 \) depend only on \( n, k \).
Proof. As noted above, setting \( w = \frac{1}{n+1} \left((1 + u)^{n+1} - 1\right) \) gives
\[
\int_{B} w \, dA = 0. \tag{145}
\]
Therefore, Lemma \ref{lemma6.1} applies to \( w \). Moreover, as shown in \cite{7}, there exists an \( \eta > 0 \) such that when \( \|u\|_{W^{2,\infty}} < \beta \), then
\[
(1 - O(\eta)) |w| \leq |w| \leq (1 + O(\eta)) |u|, \tag{146}
\]
and
\[
(1 - O(\eta)) |\nabla u| \leq |\nabla w| \leq (1 + O(\eta)) |\nabla u|. \tag{147}
\]
First we suppose \( n \geq 3 \). We will then prove the theorem for \( n = 2 \), and \( n = 1 \) follows similarly. Applying Lemma \ref{lemma6.1} there is a constant \( C > 0 \) (possibly changing from line to line) where
\[
\|\nabla u\|_{L^2}^2 \geq C \|\nabla w\|_{L^2}^2 \geq \frac{C \|u\|_{L^\infty}^2}{(C(n)) \|\nabla w\|_{L^\infty}^2} \geq C \|u\|_{L^\infty}^2. \tag{148}
\]
By Proposition \ref{prop4.3} for small enough \( \|D^2 u\|_{L^\infty} \),
\[
\delta_{k,-1}(\Omega) \geq C \|\nabla u\|_{L^2}^2, \tag{149}
\]
which together with \eqref{148} gives the statement of the theorem for \( n \geq 3 \).

Next suppose \( n = 2 \). Then, there is a \( M > 0 \) such that
\[
\|\nabla u\|_{L^2}^2 \log \frac{M \|\nabla u\|_{L^\infty}^2}{\|\nabla u\|_{L^2}^2} \geq C \|\nabla w\|_{L^2}^2 \log \frac{8e \|\nabla w\|_{L^\infty}^2}{\|\nabla w\|_{L^2}^2} \geq C \|u\|_{L^\infty}^2 \geq C \|u\|_{L^\infty}^2. \tag{150}
\]
Furthermore,
\[
\|\nabla u\|_{L^2}^2 \log \frac{M \|\nabla u\|_{L^\infty}^2}{\|\nabla u\|_{L^2}^2} \leq C_1 \delta_{k,-1}(\Omega) \log \frac{C_2}{\|\nabla u\|_{L^2}^2} \leq C_1 \delta_{k,-1}(\Omega) \log \frac{C_2}{\delta_{k,-1}(\Omega)}. \tag{151}
\]
The last line follows from the observation in \eqref{106}, where for sufficiently small \( \|u\|_{W^{2,\infty}} \) we have \( \delta_{k,-1} \leq C(n,k) \|\nabla u\|_{L^2}^2 \) for some positive constant \( C(n,k) > 0 \). Combining \eqref{150} and \eqref{151} concludes the statement of the theorem for \( n = 2 \). The proof for \( n = 1 \) follows similarly. \qed

References

\begin{enumerate}
\item Sun-Yung A. Chang and Yi Wang, \textit{Some higher order isoperimetric inequalities via the method of optimal transport}, Int. Math. Res. Not. IMRN \textbf{24} (2014), 6619–6644. MR3291634
\item Sun-Yung Alice Chang and Yi Wang, \textit{On Aleksandrov-Fenchel inequalities for k-convex domains}, Milan J. Math. \textbf{79} (2011), no. 1, 13–38. MR2831436
\item Marco Cicalese and Gian Paolo Leonardi, \textit{A selection principle for the sharp quantitative isoperimetric inequality}, Arch. Ration. Mech. Anal. \textbf{206} (2012), no. 2, 617–643. MR2880529
\item A. Figalli, F. Maggi, and A. Pratelli, \textit{A mass transportation approach to quantitative isoperimetric inequalities}, Invent. Math. \textbf{182} (2010), no. 1, 167–211. MR2672283
\item Bent Fuglede, \textit{Stability in the isoperimetric problem}, Bull. London Math. Soc. \textbf{18} (1986), no. 6, 599–605. MR859955
\item Nicola Fusco, \textit{The quantitative isoperimetric inequality and related topics}, Bull. Math. Sci. \textbf{5} (2015), no. 3, 517–607. MR3401715
\end{enumerate}
[10] Nicola Fusco, Maria Stella Gelli, and Giovanni Pisante, *On a Bonnesen type inequality involving the spherical deviation*, J. Math. Pures Appl. (9) **98** (2012), no. 6, 616–632. MR2994695
[11] Nicola Fusco and Vesa Julin, *A strong form of the quantitative isoperimetric inequality*, Calc. Var. Partial Differential Equations **50** (2014), no. 3-4, 925–937. MR3216859
[12] Claus Gerhardt, *Flow of nonconvex hypersurfaces into spheres*, J. Differential Geom. **32** (1990), no. 1, 299–314. MR1064876
[13] Pengfei Guan and Junfang Li, *The quermassintegral inequalities for k-convex starshaped domains*, Adv. Math. **221** (2009), no. 5, 1725–1732. MR2522434
[14] R. R. Hall, *A quantitative isoperimetric inequality in n-dimensional space*, J. Reine Angew. Math. **428** (1992), 161–176. MR1166511
[15] R. R. Hall, W. K. Hayman, and A. W. Weitsman, *On asymmetry and capacity*, J. Analyse Math. **56** (1991), 87–123. MR1166511
[16] F. Maggi, *Some methods for studying stability in isoperimetric type problems*, Bull. Amer. Math. Soc. (N.S.) **45** (2008), no. 3, 367–408. MR2402947
[17] Robert Osserman, *Bonnesen-style isoperimetric inequalities*, Amer. Math. Monthly **86** (1979), no. 1, 1–29. MR519520
[18] John I. E. Urbas, *On the expansion of starshaped hypersurfaces by symmetric functions of their principal curvatures*, Math. Z. **205** (1990), no. 3, 355–372. MR1082861
[19] Yi Wang, *Michael-Simon inequalities for k-th mean curvatures*, Calc. Var. Partial Differential Equations **51** (2014), no. 1-2, 117–138. MR3247383