An anatomy of a quantum adiabatic algorithm that transcends the Turing computability

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We give an update on a quantum adiabatic algorithm for the Turing noncomputable Hilbert’s tenth problem, and briefly go over some relevant issues and misleading objections to the algorithm.

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1. Introduction

Hilbert’s tenth problem\(^1\) asks for a single procedure/algorithm to systematically determine if any given Diophantine equation has some positive integer solution or not. This problem has now been proved to be Turing noncomputable, indirectly through an equivalence mapping to the noncomputable Turing halting problem\(^2\).

Nevertheless, we have proposed and argued (with more and more details provided) in a series of papers\(^3,4,5,6,7\) for an algorithm for Hilbert’s tenth problem based on quantum adiabatic computation\(^8\). Among the many valid concerns about our algorithm there are also some misleading objections which are still being spread despite their falsehood. In this short note we will give an updated overall picture of the algorithm and go through, with pointers provided for further discussions elsewhere, some of the valid concerns as well as the misleading objections to dispel the entrenched but baseless prejudice against our proposed algorithm.

2. On the quantum adiabatic algorithm

A precise statement of the algorithm can be found elsewhere\(^7\), but in a few words, given a Diophantine equation our aim is to obtain (by physical means or simulations or otherwise) the (Fock) ground state of an appropriate (and bounded from below) Hamiltonian \(H_P\) carrying the information of the input Diophantine polynomial. We will achieve that by starting with yet another easily obtained ground state of another Hamiltonian \(H_I\) and by adiabatically deforming the Hamiltonian
Essential ingredients of the algorithm are:

- **The quantum adiabatic theorem (QAT)** (for an unbounded Hamiltonian in a dimensionally infinite space) will ensure that, provided the adiabaticity and other conditions are satisfied, the initial ground state will turn into the sought-after ground state of $H_I$ with *high probability*. Relevant to this point is the paper by Tsirelson 9 criticising our algorithm when it first came out in 2001. This reference has not been published but somehow has been selectively cited by some as an evidence against our algorithm! Those citing, intentionally or not, have either ignored or missed out our reply to Tsirelson 10 which was posted only 3 days later on the same arXiv. In that reply, we clearly pointed out that Tsirelson’s arguments were simply wrong: Had they been right, the QAT, and not just our algorithm, would have been mathematically wrong for all those years!

- No level crossing is necessary to obtain the adiabaticity condition for the QAT with a *finite* rate of Hamiltonian deformation. We have employed certain mathematical theorems and a gauge-like symmetry for the class of time-dependent Hamiltonians of the algorithm to show 7 that there is indeed no level crossing.

- The identification of the final ground state is necessary because the QAT is a non-constructive theorem and does not tell us how the final probability of obtaining the ground state is approached as a function of the inverse of the rate of change of the Hamiltonian deformation. Such final probability certainly does not increase monotonically but varies for different Diophantine equations in a complicated manner. To be an algorithm for Hilbert’s tenth we need a *single and universal* criterion, applicable to any Diophantine equation, to identify the sought-after ground state. We have shown 7 that it is only the final ground state that can be obtained with a measurement probability *more than 1/2* with our algorithm. Such a probability is our identification criterion for the ground state. (The proof has now been extended to an infinite number of energy levels, not just the two levels of the ground state and the next excited state, and will appear elsewhere.) These analytical results have also been supported by numerical simulations 6.

- The probabilistic nature is inevitable for our quantum algorithm 7, either in determining the identification probability through relative frequencies in physical measurements or in extrapolating to zero-size time step in some numerical simulations 6. We will come back to this probabilistic nature in a section below.

Further modification and extension of our work has also appeared in the literature 11,12.

### 3. Exploration of the infinite in a finite time?

The recursive noncomputability of Hilbert’s tenth problem lies in the fact that a systematic substitution of positive integers (in some increasing order of magnitude, say) into a given Diophantine equation could only terminate if the equation has a solution; otherwise, the substitution would go on indefinitely without any termination.
point. (For this reason, the problem is sometimes also termed semi-computable, as we do not have a general method to determine when the equation has no solution.)

On the other hand, our quantum algorithm apparently is somehow able to “explore the infinite (of the whole domain of positive integers) in a finite time!” How can that be? And some have even used this as an indication, which is misleading, that there must be something wrong with our algorithm!

The logic behind all this is that Hilbert’s tenth problem belongs to the class of finitely refutable mathematical problems. That is, for any given Diophantine equation it only requires a substitution up to some positive integer to determine whether it has any solution or not, even in the case of no solution: if the equation has no solution within a certain finite domain of positive integers, it will not have a solution anywhere else in the whole infinite domain! The noncomputability of Hilbert’s tenth problem is precisely because we do not have any universal recursive method to determine this finite decisive domain for every Diophantine equation. In contradistinction to the recursive mathematics, quantum mechanics can give us the means to determine such finite decisive domain (in order to make some conclusion in the infinite domain) through the (energetically) ground state. The finiteness of such a domain is encoded and reflected accordingly in the finiteness of the evolution time and in the finiteness of the energy and occupation number of the final ground state. This is how the paradoxical power of our quantum algorithm can be understood. (In fact, the quantum algorithm provides an alternative proof for the finitely refutable character of Hilbert’s tenth and related problems.)

4. What about Cantor’s diagonal arguments?

Our quantum adiabatic algorithm is probabilistic in the sense that it can produce a result with a probability, which can be made arbitrarily high, of being the correct result. As such, it has a non-zero probability, even though it can be made arbitrarily small, of being incorrect! Thus, in a way, in order to compute the noncomputable or decide the undecidable with our algorithm, we will have to allow for the possibility of being wrong, even though we can reduce this chance (at a cost). It is this probabilistic nature of the algorithm that renders it outside the jurisdiction of Cantor’s diagonal arguments employed in the noncomputability proof of the Turing halting problem (and hence of Hilbert’s tenth problem). Indeed, the discoverers of noncomputability and of incompleteness in Mathematics were very much aware of this power of (probabilistic) flexibility, as reflected in their own statements.

- Gödel: “... it remains possible that there may exist (and even be empirically discovered) a theorem-proving machine which in fact is equivalent to mathematical intuition, but cannot be proved to be so, nor can be proved to yield only correct theorems of finitary number theory.”

- Turing: “... if a machine is expected to be infallible, it cannot be intelligent. There are several theorems which say almost exactly that. But these theorems say nothing about how much intelligence may be displayed if a machine makes no pretence at
infallibility."

Probabilistic computation may also be more powerful than Turing deterministic computation in yet a different way, contrary to the often misquoted statement that the two are equivalent in terms of computability.

5. Hypercomputation and quantum mechanics

5.1. A class of noncomputable linear Schrödinger equations

Pour-El and Richards have shown that, surprisingly, the solution at a finite time of the linear wave equation in 3 dimensions can be noncomputable, even with some computable initial condition! This result not only shows the limitation of Turing computability even with linear and supposedly simple differential equations like the well-known wave equations, but also implies a hypothetical physical hypercomputation: starting a controlled wave propagation with a precisely prepared initial condition and then measuring the wave configuration at a given time later to obtain (compute) the otherwise noncomputable. This would work provided that (i) the physical wave propagation so performed is governed by the mathematical wave equation; and (ii) the measurement of the wave configuration could be done with infinite precision.

Our quantum algorithm not only implies the noncomputability of the solution of the Schrödinger equation with a special class of time-dependent Hamiltonians (associated with Hilbert’s tenth problem) but also provides the procedure, physical or otherwise, for computing the Turing noncomputable, albeit in a probabilistic manner. And thanks to the flexibility of this probabilistic character, we may not require an infinitely precise measurement (but see below) – but still have to assume that the mathematics of quantum mechanics underlies any physical implementation of the algorithm, unless the algorithm is simulated on Turing machines.

5.2. (Probabilistic) hypercomputation and (quantum) randomness

Beside ours, there are also other appeals to quantum mechanics as the possible evidence and/or resource for hypercomputation. Postulated in quantum mechanics is the inherent and irreducible randomness; and true randomness is outside the Turing computability and thus belongs to the domain of hypercomputation, and so does probabilistic computation in general. Turing machines are not capable of generating truly random numbers, but only pseudo-random output with some finite and recursive algorithms. Hypercomputation beyond the Turing computability is thus not so mythical. At least everyone agrees that random number generation is a kind of hypercomputation, albeit a very special kind.

The problem is whether and how randomness can be harnessed for more interesting computation rather than just being random in itself. Each series of random numbers generated by a series of quantum measurements is different from each other and is not reproducible (being random by the very definition). What reproducible is not the outcomes but more often is the probabilities for the outcomes.
We have pointed out how some hypercomputation could be performed with the help of a (quantum) probability which is a noncomputable real number. The kind of problems that can be solved by this hypercomputation depends on the properties of that real number. Can we make direct use of a particular (irreproducible) series of random numbers generated quantum mechanically or otherwise? We hope to report these findings elsewhere.

6. Physical implementation and simulations on classical computers

Probabilistic computation may be more powerful than Turing computation provided a suitable probability measure can be defined. The systematic substitution of positive integers into a Diophantine equation, for example, does not lend itself to a definable probability measure since the cardinality of any subset used in the substitution is always finite while that of all the positive integers is not. (Recall that there is no recursive way to determine the finite decisive subsets, even when we know that the problem is finitely refutable. Were there a way, the problem would not have been Turing noncomputable.)

Our algorithm, on the other hand, possesses naturally defined probability measures through the use of quantum mechanics. In a physical implementation of the algorithm, the probability comes from the weak law of large numbers in determining some other quantum probabilities through the relative frequencies obtained from repetitive measurements. This determination does not require measurements of infinite precision. However, there have been some concerns that infinite precision is still required in physically setting up the various integer parameters in the time-dependent quantum Hamiltonians. While the issue deserves further investigations as surely any systematic errors in the Hamiltonians would be fatal, we still are not convinced that such integer parameters cannot be satisfactorily set up. In particular, we would like to understand the effects of statistical (as opposed to systematic) errors on the statistical behaviour of the spectrum of our adiabatic Hamiltonians.

However, physical implementation is not the only way to carry out the algorithm. The algorithm could also be simulated on Turing machines. Then how does probability come into such simulations? It comes in under the necessary extrapolation of the simulation time steps to zero sizes, which is essentially probabilistic. The probability measures here are different from those in the physical implementation but, on the other hand, we do not have the above problem associated with the integer parameters in our Hamiltonians.

7. Concluding remarks

We have listed and dealt with some of the concerns and objections against our quantum adiabatic algorithm for the Turing noncomputable tenth problem of Hilbert. Most of these objections (none of which actually appears in print) simply root in the belief, and no more than a belief, that there must be something wrong with the algorithm because it claims to be able to compute the very problem that has been
mathematically (and recursively) proven to be noncomputable! Closer inspection, however, shows that such a belief is simply false as all the noncomputability proof is only valid within a certain framework, outside of which the quantum algorithm operates and hence entails no contradiction with the known and proven facts.

So, what can we expect from the algorithm? Once and if implemented, it could give an answer, with any pre-determined probability, with respect to the existence of solution of any given Diophantine equation. The probability can be raised arbitrarily without bounds; but the higher it is, the higher the cost of time and resources it takes. The other caveat is that the time it takes for a successful application of the algorithm is not known beforehand but only as an end product, even though it is always finite. (This successful time is not the time we fix a priori for each run of the algorithm – each run of the algorithm always has an end point. We must then keep increasing this running time until successful.)

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