Nonlinear model of source of a elastic field.

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A general description of the long-range elastic interaction is proposed. The far-field type of the interaction is determined by the way of symmetry breaking of the distribution of the elastic field produced by the topological defect as isolated inclusions. Every topological defect can be present as the source of the elastic field and can be described in the terms of this field. At the short distance the source can be described as nonlinear object which present charge of linear theory of elastic field at the far distance. In this article the nonlinear models of source of a elastic field was proposed.

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In the general case we have the ground state of the continuum which can always be described in terms of the elastic field. In the ground state, this field has some definite value which does not depend on the point of the elastic continuum. The different fluctuation of elastic field which mean value is zero present the ground state too. The elastic field can be characterized by various geometrical objects, i.e., a scalar if this field describes the phase transition in a condensed matter, a vector potential in electrodynamics, a second-rank tensor in the general relativity etc. To consider the elastic field that determines this continuum, we have to describe possible deformations of the distribution of this field. The basic concept is that in any system with broken continuous symmetry exist states in which the elastic variables describe distortions of spatial configurations of the ground state. These distortions arise if the continuous symmetry of the elastic-field distribution is broken in a local area. The first way to break the continuous symmetry is to introduce a foreign inclusion in the elastic matter. Then the long-range interaction between inclusions is determined by the symmetry of the deformation of the elastic field. These deformations are produced by the inclusions and interaction can be expressed in terms of the characteristics of this inclusion. In a system with broken continuous symmetry also can exist defects in the elastic-field distribution, i.e., the topological defects. Each topological defect has some core region, where the elastic-field distribution is strongly destroyed, and a far-field region where the elastic variable slowly changes in space. Both inclusions and topological defects are foreign to the elastic field and can not be described in terms of this field. Nevertheless they must have characteristics which influence on the ground state of the elastic continuum. This characteristics determine the value and character of the deformation. Then the amount value of various elastic deformations can be associated with the effective interaction between inclusions. In other words, the presence of the field deformation immediately leads to the interaction. Such interaction did not need the carrier.

Let us consider how the deformation of the elastic-field distribution can produce interaction of additional inclusions or topological singularities. We start with the description of a scalar field with spatially uniform ground state. We have to describe probable deformations of the distribution of a scalar elastic field $\varphi(\vec{r})$. With an additional inclusion being introduced in the elastic matter, we can write the action of this system in the form

$$S = \frac{1}{8\pi} \int \left\{ \left( \nabla \varphi(\vec{r}) \right)^2 - 2 \sum_i f_i \varphi(\vec{r}_i) \right\} d\vec{r}$$

(1)

where the first term describes the deformation energy of the elastic field $\varphi$ and the second term is responsible for the effect of this inclusion, located at the point $\vec{r}_i$, on the elastic field. Note that the inclusion must possess properties which influence on the elastic continuum. The first way to describe the interaction of inclusions is to obtain the change of the deformation energy produced by this additional inclusions. In the Fourier presentation

$$\varphi(\vec{k}) = \frac{1}{(2\pi)^3} \int d\vec{r} \varphi(\vec{r}) \exp(-i\vec{k}\cdot\vec{r})$$

we can rewrite the action in the $\vec{k}$ space in the form

$$S = \frac{1}{8\pi} \left( \frac{1}{(2\pi)^3} \right)^3 \int d\vec{k} \left\{ k^2 \varphi^2(\vec{k}) - 2 \sum_i f_i \varphi(\vec{k}) \exp(i\vec{k}\cdot\vec{r}_i) \right\}$$

(2)

In order to find probable configurations of the field $\varphi$ we have to solve Euler-Lagrange (EL) equations with minimum of action with regard to the boundary conditions on the inclusion. This equation is given by

$$k^2 \varphi(\vec{k}) = 4\pi \sum_i f_i \exp(-i\vec{k}\cdot\vec{r}_i)$$

(3)

which corresponds equation $\Delta \varphi = -4\pi \sum_i f_i$ in the real space. The solution of this equation yields the field distribution in the form

$$\varphi(\vec{k}) = -4\pi \sum_i f_i \frac{\exp(-i\vec{k}\cdot\vec{r}_i)}{\vec{k}^2}$$

(4)

Taking this distribution of the field in action again, we can calculate the elastic-field deformation energy produced by two inclusions:

$$U_{ij} = \frac{4\pi}{2} \left( \frac{1}{(2\pi)^3} \right)^3 \sum_{i,j} f_i f_j \int d\vec{k} \frac{\kappa}{\vec{k}^2} \exp(-i\vec{k}\cdot(\vec{r}_i - \vec{r}_j))$$

(5)
Having integrated over \( \mathbf{k} \) we obtain the interaction energy of two inclusions in the standard form

\[
U_{ij} = \sum_{i,j} \frac{f_i f_j}{r_i - r_j}
\] (6)

If we assume that the scalar field is the electrostatic potential and \( f \) is the charge, we obtain the energy of Coulomb-like charge interaction through the deformation of the electric field. In order to take into account the charge dispersion in the matter we have to describe the distribution of this field in the area of dispersed charges. In this case, in this local area the symmetry of the elastic-field distribution is different and \( \varphi(\mathbf{r}_i) \simeq \varphi(\mathbf{r}) + (\mathbf{p}_i \nabla) \varphi(\mathbf{r}) \), where \( \mathbf{p}_i \) is the distance to a single charge and, having introduced the dipole moment \( d_i = \sum_i \mathbf{b}_i \rho_i \), we can find the dipole-dipole interaction in standard form. The symmetry of distribution of elastic field which produced by two inclusions is different as symmetry of deformation of elastic field each inclusion. Two-particle system selects the solution which corresponds to the position and properties of these inclusions.

Let us find this solution for a dynamical electromagnetic field. The action for electrodynamics may be written in the standard form, i.e.,

\[
S = -\frac{1}{16\pi c} \int \left\{ F_{ij} F^{ij} + \frac{16\pi}{c} A_i j^i \right\} d\Omega
\] (7)

where the Maxwell stress tensor \( F_{ij} = \frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} \) is determined by the vector-potential \( \mathbf{A} \), and \( j \) is the current of charges. From the minimum of action, we obtain the field equations:

\[
\frac{\partial F_{ij}}{\partial x_j} = -\frac{4\pi}{c} j_i
\] (8)

For the gauge condition \( \frac{\partial A_\mu}{\partial x_\mu} = 0 \) in the four-dimensional Euclidean space, the equation became wave equation for Fourier-transformed vector potential \( A_i(k, \omega) \) with right term \( j_i \), whose solution can write as \( A_i(k, \omega) = \frac{4\pi i j_i \exp(-i(\mathbf{k} \cdot \mathbf{v})t)}{k^2 - (\frac{\mathbf{k} \cdot \mathbf{v}}{c})^2} \). The substitution of this solution into the expression for the action gives the interaction energy of different currents in the standard form, i.e.,

\[
U_{ij} = \int d^4 \mathbf{q}_i G^{ij}(q) j_i(q)
\] (9)

where \( G(q) \) is the Green function. For example using this expression one can describe the interaction of two charges which move with the velocity \( v \). In this case the charge changes by the law \( e \delta(\mathbf{r} - v t) \). In accordance with the result [1], the Fourier component of the vector potential can be written as

\[
\varphi_k = \frac{4\pi e}{k^2 - (\frac{\mathbf{k} \cdot \mathbf{v}}{c})^2} \exp(-i(\mathbf{k} \cdot \mathbf{v})t)
\] (10)

and

\[
A_k = \frac{4\pi e \exp(-i(\mathbf{k} \cdot \mathbf{v})t)}{c k^2 - (\frac{\mathbf{k} \cdot \mathbf{v}}{c})^2}
\] (11)

Having substituted this vector potential in the expression for the interaction energy we find that in the case \( v = 0 \) we have the previous result. In the other case when \( j_0 = e(t) \delta(\mathbf{r}) \) yields the interaction energy in the form:

\[
U(\mathbf{r}, \mathbf{r}', t, t') = \frac{e^2 \delta(e(t - t') - (\mathbf{r} - \mathbf{r}'))}{|\mathbf{r} - \mathbf{r}'|}
\] (12)

that reproduces the standard resultant interaction of variable charges.

Same result we can obtain if describe the gravitation field and the appearance of the interaction of masses which produce the change of the geometry of space. The action of the gravitation field and distributed matter is given by the standard expression

\[
S = \frac{c}{16\pi G} \int R \sqrt{-g} d\Omega + \frac{1}{2c} \int T \sqrt{-g} d\Omega
\] (13)

where \( R \) is the curvature, \( T \) is the compression of the energy-momentum tensor with the metric tensor \( g_{\mu\nu} \). \( \Omega \) is the space-time volume, and \( G \) gravitational constant. Minimization of this action yields the Einstein equation

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^2} T_{\mu\nu}
\] (14)

For a distributed matter, the energy-momentum tensor can be written as \( T_{00} = -mc^2 \) and the field equation reduces to \( R_{00} = -\frac{4\pi G}{c^2} T_{00} \). In the linear approximation we have \( \Gamma_{00} \simeq -\frac{1}{2} g_{\mu\nu} \frac{\partial g^{00}}{\partial x^\mu} = \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial x^2} \) that have as result

\[
R_{00} = -\frac{1}{c^2} \frac{\partial^2 \varphi}{\partial x^2} = -\frac{1}{c^2} \Delta \varphi.
\]

Thus we obtain the Poisson equation \( \Delta \varphi = 4\pi Gm \). We substitute the solution of this equation in the expression for the action and thus obtain the interaction energy of the distributed matter in the form of the standard Newton law

\[
U = -\frac{Gm_1 m_2}{\mathbf{r}_i - \mathbf{r}_j}
\] (15)

This interaction energy has attractive nature by virtue of the specifics of gravitation filed which is described by the second-rank tensor.

In Refs. [5], [6] the field equation in the empty space was written as

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0
\] (16)

according to Einstein’s statement that the geometry can not be mixed with matter. Solution of this equation for a continuum distribution of matter does not exist. However, there exists a solution with a singular point in the distribution of matter. In this presentation we can obtain the same equation for the gravitational field. The
field equation determines the law of motion in terms of the integral of motion of the surface that surrounds this singularity. If we start from action
\[ S = \frac{\rho^2}{16\pi G} \int R \sqrt{-g} d\Omega \tag{17} \]
for free gravitation field we can obtain the energy-impulse tensor as variational parameter which describe peculiarity in geometry \[7\].

We illustrate this approach on presentation of charge in electrodynamics. We start from the action for free electrodynamics field in the standard form, i.e.,
\[ S = -\frac{1}{16\pi c} \int F_{ik} F^{ij} d\Omega \tag{18} \]
The variation of this action is follow \[1\]
\[ \delta S = -\frac{1}{4\pi c} \int \left\{ \frac{\partial F_{ik}}{\partial x_k} \delta A_i + \frac{\partial}{\partial x_k} (F_{ik} \delta A_i) \right\} d\Omega \tag{19} \]
At the standard approach the last term can be neglected because it can be reduced to the surface integral which disappear in a reason that disappear potential on the surface. But, if we have peculiarity in elastic field distribution it is not correct. This surface integral is not zero on the surface of defects and on the surface of the area where exist the electrodynamic field. One can obtain this non zero term for the defects. For \( x_k = x_0 = ct \) the general presentation take the form
\[ -\frac{1}{4\pi c} \int \left\{ \frac{\partial}{\partial x_k} (F_{ik} \delta A_i) \right\} d\Omega = -\frac{1}{4\pi c} \int \{ \nabla \varphi \delta A \} dV \tag{20} \]
This term can be rewritten in the other form in spherical coordinates
\[ -\frac{1}{4\pi c} \int \{ \nabla \varphi \delta A \} r^2 dr d\cos \theta d\phi = -\frac{1}{c} \int \{ \nabla \varphi \delta A \} r^2 dr \tag{21} \]
If we take in to account that the solution for defect \( \nabla \varphi = \frac{\delta}{\delta r} \), we can present the last presentation in the form
\[ -\frac{1}{c} \int \{ \nabla \varphi \delta A \} r^2 dr = -\frac{1}{c} \int \{ k\delta A \} dr \tag{22} \]
or
\[ -\frac{1}{c} \int k\delta A \frac{dr}{d( ct)} d(ct) = -\frac{1}{c} \int k \frac{\varphi}{c} \delta A d( ct) = -\frac{1}{c} \int \{ \delta A \} d( ct) \tag{23} \]
For \( x_k = r \) we can obtain the other stream of energy through the surface
\[ -\frac{1}{c} \int \{ \nabla \varphi \delta \varphi \} r^2 d\varphi = -\frac{1}{c} \int k\delta \varphi d\varphi \tag{24} \]
Combining both obtaining part we can write in four dimensional the additional part of the action for the electrodynamic field
\[ -\frac{1}{c} \int j \delta A dV d\varphi \tag{25} \]
After this we must solve equation for elastic field
\[ \frac{\partial F_{ij}}{\partial x_j} = j_i \tag{26} \]
The charge in electrodynamics, as well as the energy-momentum in the general relativity, is foreign with respect to the field and cannot be described in terms of either potentials or the geometry. For a system with broken continuous symmetry, we can consider a class of defects in the distribution of elastic field that are called the topological defects. A topological defect can play the role similar to a particle which changes the elastic field. In the general case, each topological defect has a core region, where the elastic-field distribution is strongly destroyed, and a far-field region where the elastic variable slowly varies in space. The boundary conditions are then determined by the conditions on the core of the defect. In this approach we can find the interaction energy of topological defects. We can start from action (1) without additional force \( f \). The equation of minimum action in this case is given by the Euler-Lagrange equation
\[ \Delta \varphi(\vec{r}) = 0 \tag{27} \]
This equation has many solutions. The first solution, \( \varphi = 0 \) or \( \varphi = \text{const} \), is trivial and describes the ground state. The particular solution of this equation, compatible to the existence of a topological singularity, may be written as \( \varphi_n = (m r^n + k r^{-(n+1)}) Y_{ij}(\theta, \phi) \), and in the case \( n = 0 \) we have a spherically symmetric solution \( \varphi = \frac{1}{r} \) where \( k \) may be interpreted as the magnitude of the topological charge. This solution can describe the singularity in the topological behavior of the scalar field. As a result, this solution has infinite eigenvalue, however the interaction energy of such topological charges is finite. A superposition of two solutions for the scalar field, \( \varphi(\vec{r}) = \varphi_1(\vec{r}) + \varphi_2(\vec{r}) \), which produce two topological charges and determine the interaction energy given by \( U_{\text{int}} \equiv E(\varphi_1(\vec{r}) + \varphi_2(\vec{r})) - E(\varphi_1(\vec{r})) - E(\varphi_2(\vec{r})) \) yields
\[ U_{ij} = \frac{2}{8\pi} \int_0^\infty (\nabla \varphi^1)(\nabla \varphi^2) d\vec{r} = \frac{k_k j_j}{d} \tag{28} \]
where \( d \) is the distance between the topological charges. This formula implies that two topological defects with like signs in the elastic scalar field repel according to the Coulomb law. In a more rigorous approach \[12\] the interaction energy of two defects in the two-dimensional have the same law.
In order to understand that can present this topological defects return to action of distribution of scalar elastic field Eq.1. We would like to have the linear theory outside the topological defects. But general theory can be nonlinearity. The nonlinearity must play role only inside of the defect. For the defects description we can not use the standard theory with characteristic \( f_i \) as function of local elastic field. In this case we can present \( f_i = a \varphi - \frac{b}{2} \varphi^3 \) when \( r \leq R \), and where \( R \) is size of core of defect. In this
case inside the defect the equation can be present in the form
\[ \triangle \varphi - a\varphi + b\varphi^3 = 0 \]  
(29)
The solution of this equation is \( \varphi = (\frac{2a}{b}) \text{sech}(\sqrt{a} r) \) and it have noncorrect asymptotical behaviour. The standard approach of nonlinearity is not good to present particle as topological defect. In more realistic situation we can take the characteristic of defect \( f_i \) as function of gradient of local elastic field. Outside the defect we must have the previous result and can present the characteristic of defect \( f_i \) in the form \( f = f - \sum_n a_n (\nabla \varphi)^n \). This series must have only odd \( n \). This correspond to the symmetry when states \( \varphi \) and \( -\varphi \) are equal. For this case the different equation for the action minimization can be obtained
\[ \nabla(\nabla \varphi) - \sum_n a_n \nabla \{ (\nabla \varphi)^{n-1} \} \varphi + \sum_n a_n (1-n)(\nabla \varphi)^n + f = 0 \]  
(30)
In the case \( n = 1 \) we have the simple equation \( \triangle \varphi = -f \). In the case \( n = 3 \) we have the nonlinear equation
\[ \nabla(\nabla \varphi)(1 - 6a_3 \nabla \varphi) - 2a_3 (\nabla \varphi)^3 + f = 0 \]  
(31)
which have exactly solution \( \varphi = kr^{\frac{3}{2}} \) with \( k = \frac{1}{a} \) when the nonlinearity is greater than linear part. It is correct for the distance \( r \leq r_0 = k^2 f^{-\frac{4}{3}} \). Outside this distance we have previous behavior of the solution of the nonlinear equation and standard presentation of the behaviour of the potential. Other way to describe defect can be realize if we take the characteristic of defect \( f_i \) in the form \( f_i = a\theta(r - r_0)\varphi \) where \( \theta \) - theta function, which is 1 at \( r < r_0 \) and 0 at \( r \geq r_0 \) where \( r_0 \) - size of the core. The action of the scalar elastic field take the form:
\[ S = \frac{1}{8\pi} \int \left\{ \left( \nabla \varphi(\mathbf{r}) \right)^2 + a\theta(r - r_0)\varphi^2 \right\} d\mathbf{r} \]  
(32)
The minimum of action realized on the equation
\[ \Delta \varphi \{ 1 + a\theta \varphi^2 \} + a\theta(\nabla \varphi)^2 \varphi^2 + 2a\delta(r_o)(\nabla \varphi)^2 \varphi^2 = 0 \]  
(33)
In the area \( r \geq r_o \) we have equation
\[ \Delta \varphi + 2a\delta(r_o)(\nabla \varphi)^2 r = 0 \]  
(34)
that is similar the Poisson equation with charge
\[ 2a\delta(r_o)(\nabla \varphi)^2 r = r_0 \]  
(35)
In the area inside the core we have the equation
\[ \Delta \varphi \{ 1 + a\theta \varphi^2 \} + a\theta(\nabla \varphi)^2 \varphi^2 = 0 \]  
(36)
which have the first integral \( (\nabla \varphi)^2 \{ 1 + a\varphi^2 \} \). The solution of this equation with such first integral can be written in the form \( a\varphi \sqrt{1 + a^2 \varphi^2} + \ln \left| a\varphi + \sqrt{1 + a^2 \varphi^2} \right| = 2ac \) which have the next asymptotic behaviour: if \( a\varphi < 1 \) \( \varphi \approx cr \) and vise versa if \( a\varphi > 1 \) \( \varphi \approx (\frac{2a}{b}) \text{sech}(\sqrt{a} r) \). The charge of particle tend to zero in the center of the core of defect. Outside the core we have the normal behavior of the elastic field. Nonlinearity in terms of the gradient of elastic field can solve problem of source of the one.

In that way we can present the characteristic of defect or charge as nonlinearity formation which outside can be consider as defect in linear theory of elastic interaction. Every action determine elastic deformation which produce the topological defect. They can be only topological singularities in the distribution of the elastic field. Interaction of these singularities is governed by the deformation of the elastic field. The presentation of the particle as topological defect has two preferences. First of all, in this case we have the law of conservation of the topological charge and this not dependent on the nature of elastic field. Second, we can estimate the size of the particle as the core of this defect. The structure of this core is a structure of particle. Every core must have the energy \( mc^2 \), where \( m \) is mass of particle or core of the defect. In the process of arise of two particles must appears two topological defects with different signs. In this case arise the energy of interaction between them. We can observe two particle only in the case when the distance between them will be larger two sizes of core. In the condition of equality of energy \( 2mc^2 = \frac{e^2}{4\pi} \) we obtain that the size of core of size of particle \( R = \frac{e^2}{4mc^2} \). If \( f \) is electric charge we obtained the size of electron, as elementary particle in the form
\[ R = \frac{e^2}{4mc^2} = \frac{e^2\hbar}{4mc^2} = \frac{1}{4}\alpha\lambda \]  
(37)
\( \alpha = \frac{e^2}{4\pi} \) is constant of electric interaction and \( \lambda = \frac{mc}{\hbar} \) is Compton length. The size of electron is the size of core of topological defect in electrostatic field and this size is smaller as Compton length in \( \sim 450 \) time. In the case of gravitation field
\[ R .. \frac{1}{4} \frac{m^2}{m_p^2} \lambda \]  
(38)
where \( m_p = \left( \frac{\hbar^2}{m^2} \right) \) is Plank mass. The elementary particle in this presentation has not structure.

Thus we obtain a general approach that makes possible to explain well-known interaction modes for different cases by considering various types of local symmetry breaking. The local continuum symmetry breaking present the topological defect in elastic field. The interaction between two defect have same form as interaction between inclusions. The defect in distribution of elastic field produce nonlinearity of behaviour of the elastic field. Nonlinearity of elastic field is motivation of appearance of the charge in electrodynamic and location of mass in general relativity.
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