STOCHASTIC TRANSPORT EQUATION WITH SINGULAR DRIFT

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Abstract. We prove existence, uniqueness and Sobolev regularity of weak solution of the Cauchy problem of the stochastic transport equation with drift in a large class of singular vector fields containing, in particular, the $L^d$ class, the weak $L^d$ class, as well as some vector fields that are not even in $L^2_{locl}$ for any $\varepsilon > 0$.

1. Introduction

Throughout this paper we assume $d \geq 3$. Let $(B_t)$ be a Brownian motion in $\mathbb{R}^d$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to a complete and right-continuous filtration $(\mathcal{F}_t)$. Let $\circ$ denote the Stratonovich multiplication. Set $L^p = L^p(\mathbb{R}^d) = L^p(\mathbb{R}^d, dx)$, $L^p_{locl} = L^p_{locl}(\mathbb{R}^d)$, $W^{1,p} = W^{1,p}(\mathbb{R}^d)$, $W^{1,p}_{locl} = W^{1,p}_{locl}(\mathbb{R}^d)$, $C^\infty_c = C^\infty_c(\mathbb{R}^d)$. We denote by $\| \cdot \|_{L^p \to L^q}$ the operator norm $\| \cdot \|_{L^p \to L^q}$.

The subject of this paper is the problem of existence, uniqueness and Sobolev regularity of weak solution to the Cauchy problem for the stochastic transport equation (STE)

$$du + b \cdot \nabla u dt + \sigma \nabla u \circ dB_t = 0 \quad \text{on } (0, \infty) \times \mathbb{R}^d,$$

$$u|_{t=0} = f,$$

where $u(t, x)$ is a scalar random field, $\sigma \neq 0$ is a constant, $f$ is in $L^p$ or $W^{1,p}$ and $b : \mathbb{R}^d \to \mathbb{R}^d$ is in the class of form-bounded vector fields (see definition below), a large class of singular vector fields containing, in particular, vector fields $b \in L^d$ (i.e. all components of $b$ are in $L^d$), or with $b$ in the weak $L^d$ class, as well as some vector fields $b \notin L^{2+\varepsilon}_{locl}$ for any $\varepsilon > 0$.

It is well known that the Cauchy problem for the deterministic transport equation $\partial_t u + b \cdot \nabla u = 0$ (corresponding to $\sigma = 0$ in (1)) is in general not well posed already for a bounded but discontinuous $b$. Moreover, in that case, even if the initial function $f$ is regular, one can not hope that the corresponding solution $u$ will be regular immediately after $t = 0$. This, however, changes if one adds the noise term $\sigma \nabla u \circ dB_t$, $\sigma > 0$. For the stochastic STE (1), a unique weak solution exists and is regular for some discontinuous $b$. This effect of regularization and well-posedness by noise, demonstrated by the STE, attracted considerable interest in the past few years, as a part of the more general program of establishing well-posedness by noise for SPDEs whose deterministic counterparts arising in fluid dynamics are not well-posed, see [BFGM, GM] for detailed discussions and further references.

In the fundamental paper [BFGM], the authors establish existence, uniqueness and Sobolev $W^{1,p}$ regularity (up to the initial time $t = 0$, with $p$ large) for weak solutions of (1) with time-dependent
drift \( b \) satisfying
\[
\|b(\cdot, \cdot)\| \in L^\alpha([0, \infty), L^r + L^\infty), \quad \frac{d}{r} + \frac{2}{q} \leq 1
\]
(actually, [BFGM] allows \( b = b_1 + b_1 \) with \( b_1 \) satisfying the condition above and \( b_2 \) being continuously differentiable with at most linear growth at infinity; their uniqueness result imposes additional assumptions on \( \text{div} b \)). They apply this result to study the SDE
\[
X_t = x - \int_0^t b(r, X_r)dr + \sigma(B_t - B_s),
\]
constructing, in particular, a unique, \( W^{1,p} \)-regular stochastic Lagrangian flow that solves (2) for a.e. \( x \in \mathbb{R}^d \). The STE can be viewed as the equation behind both the SDE (via path-wise interpretation of the STE and the SDE, see [BFGM]) and the parabolic equation \((\partial_t - \frac{\alpha^2}{2} \Delta + b \cdot \nabla)v = 0\) (arising from (1) upon taking expectation, i.e. \( v = \mathbb{E}[u] \), see, if needed, (16) below).

In this paper, we show that the regularity and well-posedness for (1) hold for a much larger class of drifts \( b \), at least in the time-independent case \( b = b(x) \) (see, however, Remark 2 below concerning time-dependent \( b \)).

1. Form-bounded vector fields.

**Definition 1.** A Borel vector field \( b : \mathbb{R}^d \to \mathbb{R}^d \) is said to be form-bounded with relative bound \( \delta > 0 \), written as \( b \in F_\delta \), if \( b \in L^2_{\text{loc}} \) and there exists a constant \( \lambda = \lambda_\delta \geq 0 \) such that
\[
\|b(f - \Delta)^{-\frac{1}{2}}\|_{L^2} \leq \sqrt{\delta}.
\]

The condition \( b \in F_\delta \) can be stated equivalently as the following quadratic form inequality:
\[
\|b\varphi\|^2 \leq \delta \|\nabla \varphi\|^2 + c_\delta \|\varphi\|^2, \quad \varphi \in W^{1,2},
\]
for a constant \( c_\delta (= \lambda_\delta) \). Indeed, for any \( g \in L^2 \), let \( \varphi = (\lambda - \Delta)^{-\frac{1}{2}} g \). Then
\[
\delta \|\nabla \varphi\|^2 + \lambda \delta \|\varphi\|^2 = \delta \|\nabla(\lambda - \Delta)^{-\frac{1}{2}}g\|_{L^2}^2 + \delta \|\varphi\|_{L^2}^2 = \delta \|(\lambda - \Delta)^{-\frac{1}{2}}g\|_{L^2}^2 - \delta \|\varphi\|_{L^2}^2 = \delta \|(\lambda - \Delta)^{-1}g\|_{L^2}^2.
\]

Thus if (4) holds, then \( \|b(\lambda - \Delta)^{-\frac{1}{2}}g\|_{L^2} = b\varphi\|_{L^2} \leq \delta \|g\|_{L^2} \), i.e., (3) holds. Similarly, we get that (3) implies (4).

Let us also note that
\[
b_1 \in F_\delta, b_2 \in F_\delta \quad \Rightarrow \quad b_1 + b_2 \in F_\delta, \quad \sqrt{\delta} := \sqrt{\delta_1} + \sqrt{\delta_2}.
\]

**Examples.** 1. Any vector field
\[
b \in L^d(\mathbb{R}^d, \mathbb{R}^d) + L^\infty(\mathbb{R}^d, \mathbb{R}^d)
\]
is in \( F_\delta \) for \( \delta > 0 \) that can be chosen arbitrarily small. Indeed, for any \( \varepsilon > 0 \) we can write \( b = f + h \) with \( \|f\|_d < \varepsilon, h \in L^\infty(\mathbb{R}^d, \mathbb{R}^d) \). It follows from Hölder’s inequality and the Sobolev embedding theorem that for any \( g \in L^2 \),
\[
\|b(\lambda - \Delta)^{-\frac{1}{2}}g\|_{L^2} \leq \|f\|_d \|\lambda - \Delta\|_{L^\infty}^{\frac{1}{2}} \|g\|_{L^2} + \|h\|_\infty \lambda^{\frac{1}{2}} \|g\|_{L^2} \leq c\|f\|_d \|g\|_{L^2} + \|h\|_\infty \lambda^{\frac{1}{2}} \|g\|_{L^2} \leq (c + 1)\varepsilon \|g\|_{L^2} \quad \text{for} \quad \lambda = \varepsilon^{-2} \|h\|_{L^\infty}^2.
\]

2. The class \( F_\delta \) also contains vector fields having critical-order singularities, such as
\[
b(x) = \pm \sqrt{\delta} \frac{d - 2}{2} |x|^{-2} x
\]
(by Hardy’s inequality \( \frac{(d-2)^2}{4} \|x|^{-1}\varphi\|_2^2 \leq \|\nabla \varphi\|_2^2, \varphi \in W^{1,2} \)).

3. More generally, the class \( F_\delta \) contains vector fields \( b \in L^{d,w} \) (the weak \( L^d \) space). Recall that a Borel function \( h : \mathbb{R}^d \to \mathbb{R} \) is in \( L^{d,w} \) if

\[
\|h\|_{d,w} := \sup_{s > 0} s\{x \in \mathbb{R}^d : |h(x)| > s\}^{1/d} < \infty.
\]

By the Strichartz inequality with sharp constant [KPS, Prop. 2.5, 2.6, Cor. 2.9], if \( b \in L^{d,w} \), then \( b \in F_\delta \) with

\[
\sqrt{\delta_1} = \|b(\lambda - \Delta)^{-1/2}\|_{2 \to 2} \leq \|b\|_{d,w} \Omega_{d}^{-\frac{1}{2}} \|\|x|^{-1}(\lambda - \Delta)^{-1/2}\|_{2 \to 2} \leq \|b\|_{d,w} \Omega_{d}^{-\frac{1}{2}} \frac{2}{d - 2},
\]

where \( \Omega_d = \pi^{d/2} \Gamma(d/2 + 1) \) is the volume of the unit ball in \( \mathbb{R}^d \).

We also note that if \( h \in L^2(\mathbb{R}) \), \( T : \mathbb{R}^d \to \mathbb{R} \) is a linear map, then the vector field \( b(x) = h(Tx)e \), where \( e \in \mathbb{R}^d \), is in \( F_\delta \) with appropriate \( \delta \), but \( b \) may not be in \( L^{d,w} \).

4. More generally, the class \( F_\delta \) contains vector fields in the Campanato-Morrey class and the Chang-Wilson-Wolff class, with \( \delta \) depending on the respective norms of the vector field in these classes, see [CWW].

5. We note that, for any \( \varepsilon > 0 \), there exists \( b \in F_\delta \) such that \( b \not\in L^{2+\varepsilon}_{\text{loc}} \), e.g., consider \( b(x) = (b_1(x), 0, \ldots, 0) \) with

\[
b_1(x) = C \frac{1_{B(0,1+\alpha)} - 1_{B(0,1-\alpha)}}{|x|^{-1}(\ln|\ln|x|-1|)^{\beta}}, \quad \beta > 1/2, \quad 0 < \alpha < 1.
\]

We emphasize that the condition \( b \in F_\delta \) is not a refinement of \( b \in L^d + L^\infty \) in the sense that \( F_\delta \) is not situated between \( L^d + L^\infty \) and \( L^p + L^\infty \), \( p < d \). In contrast to the elementary sub-classes of \( F_\delta \) listed above, the class \( F_\delta \) is defined in terms of the operators that, essentially, constitute the equation in (1).

There is a quantitative dependence between the value of the relative bound \( \delta \) of \( b \) and the regularity properties of solutions to the corresponding parabolic PDE, SDE and the STE. For instance, see (8) below where the admissible \( q \) depend on \( \delta \). Consider also the SDE

\[
X_t = x - \int_0^t b(X_r)dr + \sqrt{2}B_t.
\]

In [BFGM, Sect. 7], the authors show that SDE (5) with drift \( b(x) = \sqrt{d-2} |x|^{-2} x \) and starting point \( x = 0 \) does not have a weak solution if \( \delta > 4(\frac{d}{d-2})^2 \). In view of Example 2 above, this drift \( b \) belongs to \( F_\delta \). It was proved in [KiSi1] that for all \( b \in F_\delta \), \( x \in \mathbb{R}^d \), the SDE (5) possesses a weak solution as long as \( \delta < 4 \), so the counterexample of [BFGM] is sharp at least as \( d \to \infty \). Thus, the weak well-posedness of (5) is sensitive to changes in the value of the relative bound \( \delta \). In this sense, the singularities of \( b \in F_\delta \) are critical.

The dependence between the regularity properties of solutions and \( \delta \) is lost if one considers \( b \in L^d + L^\infty \) since any such \( b \) has arbitrarily small relative bound, cf. Example 1.

Let us note that the diffusion process with drift \( b(x) = c|x|^{-2} x, c \in \mathbb{R} \), with the initial point \( \not= 0 \), was also studied in [W].
Concerning the difference between classes $F_d$ and its subclass $L^d + L^\infty$, let us also note the following: if $v$ is a weak solution of the elliptic equation $(\lambda - \Delta + b \cdot \nabla)v = f$, $\lambda > 0$, $f \in C^\infty_c$ with $b \in L^d + L^\infty$ and $v \in W^{1,r}$ for $r$ large (e.g. by (9) below), then, by Hölder’s inequality,

$$\Delta v \in L^{\frac{2d}{2+d}}_{loc}.$$ 

However, for $b \in F_\delta$, one can only say that (cf. Example 5 above)

$$\Delta v \in L^{\frac{2d}{2+d}}_{loc}$$

(one can in fact show that $v \in W^{2,2}$). That is, in case $b \in F_\delta$, there are no $W^{2,p}$ estimates on the solution $v$ for $p$ large. See [KiS2] for detailed discussions of the role of $\delta$ in the regularity theory of parabolic PDEs.

Given a form-bounded vector field $b$, one can approximate it by vector fields $b_m \in C^\infty_c$ that preserve the relative bound $\delta$:

$$b_m \to b \text{ in } L^2_{loc} \text{ as } m \to \infty$$

and, for every $m = 1, 2, \ldots$,

$$\|b_m \varphi\|^2 \leq \delta \|\nabla \varphi\|^2 + 2c_\delta \|\varphi\|^2, \quad \varphi \in W^{1,2}.$$ 

(7)

For instance, one can take $b_m$ as in the next proposition which we will prove in Section 3.

**Proposition 1.** Let $1_m$ be the indicator of $\{|x| \leq m, |b(x)| \leq m\}$, $\eta_m \in C^\infty_c$ be $[0,1]$-valued functions such that $\eta_m = 1$ on $B(0,m)$, $m = 1, 2, \ldots$. Fix $0 < \gamma_m \downarrow 0$. Then the sequence

$$b_m := \frac{\delta}{(\sqrt{\delta} + \sqrt{\gamma_m})^2} \eta_m e^{\epsilon_m \Delta}(1_m b)$$

satisfies (6) and (7) provided that $\{\epsilon_m\} \downarrow 0$ sufficiently rapidly.

Another option is to take $b_m := e^{\epsilon_m \Delta} b$ with $\epsilon_m \downarrow 0$. Then $b_m \in L^\infty \cap C^\infty \cap F_\delta$ with the same $c_\delta$ as $b$. See [KiS5] for the proof.

2. **Summary of results and strategy.** The main results of the paper can be summarized as follows:

1) Construction of a “weak solution” $u$ to the Cauchy problem (1) via an approximation procedure (Theorem 1). It follows, in particular, that $u$ does not depend on the choice of a regularization of $b$ as long as it preserves the relative bound $\delta$.

2) Sobolev regularity of $u$ in $W^{1,2q}$ for $q = 1, 2, \ldots$, up to the initial time $t = 0$:

$$\sup_{t \in [0,T]} \|E|\nabla u|^{2q}\|_2 \leq C \|\nabla f\|_{4q}, \quad q = 1, 2, \ldots,$$

(8)

and so

$$\sup_{t \in J_T} E(\rho|\nabla u|^{2q}) \leq C_2 \|\nabla f\|_{4q},$$

provided that $b$ is in $F_\delta$ with $\delta$ smaller than a certain explicit constant $c = c(q)$, see Theorem 2; the proof of the a priori variant of (8) follows the approach of [BFGM].

3) Uniqueness of weak solution in an appropriate class (Theorem 3).

Estimates (8) can be viewed as stochastic parabolic counterparts of the Sobolev regularity estimates for solutions of the corresponding deterministic elliptic equation established in [KS]. More precisely,
in [KS] the authors consider the operator \( -\Delta + b \cdot \nabla \), \( b \in F_\delta \) with \( 0 < \delta < 1 \wedge \left( \frac{2}{d - 2} \right)^2 \), \( d \geq 3 \) and establish the following Sobolev regularity of solutions \( v \) to the elliptic equation \( (\mu - \Delta + b \cdot \nabla)v = f \) for \( f \in L^q \) with \( q \in [2 \vee (d - 2), 2/\sqrt{\delta}) \):

\[
\|\nabla v\|_{q; \frac{d}{d - 2}} \leq K\|f\|_q,
\]

with \( K \) depending only on \( d \), \( q \), the relative bound \( \delta \) and \( c_3 \). These estimates were used in [KS] to construct the corresponding Feller semigroup. For analogous Sobolev regularity estimates in the parabolic setting with time-dependent form-bounded \( b \), and construction of the corresponding Feller evolution family, see [Ki]. It was established in [KiS3] that, given \( b \in F_\delta \) with \( \delta < 1 \wedge \left( \frac{2}{d - 2} \right)^2 \), this Feller semigroup determines, for every starting point \( x \in \mathbb{R}^d \), a weak solution to the SDE (5). See also [KiS4] where the authors consider drifts in an even larger class although at the expense of requiring a smaller \( \delta \), and [KiM] where the authors consider SDEs with time-dependent form-bounded drifts, and prove uniqueness in a large class of weak solutions satisfying Krylov-type estimates.

The approach to studying SDEs via regularity theory of the STE, developed in [BFGM], can be combined with Theorem 2 to obtain strong existence and uniqueness for (2) with \( b \in F_\delta \) (cf. Remark 1 below), albeit potentially excluding a measure zero set of starting points \( x \in \mathbb{R}^d \). For results on strong existence and uniqueness for any \( x \in \mathbb{R}^d \), with \( b \) satisfying (in the time-independent case) \( b \in L^p + L^\infty \) with \( p \geq d \), see [Kr1, Kr2, KrR].

3. Plan of the paper. In Section 2, we state our results in detail. In Section 3 we prove Proposition 1 on regularization of form-bounded vector fields. Section 4 contains a priori variants of Theorems 1 and 2, i.e. with \( b = b_m \) and initial function \( f \in C^\infty_c \) (Propositions 2 and 3). Armed with these a priori results, in Section 5 we prove that the solutions \( u_m \) of the approximating STEs (i.e. with \( b = b_m \) and \( f \in C^\infty_c \)) constitute a Cauchy sequence \( L^\infty([0, T], L^2(\Omega, L^2_\rho)) \), where \( \rho \) is a weight vanishing at infinity; then we show that the limit of this sequence is the sought process \( u \) in Theorem 1. In Section 6 we use Theorem 1 to pass to the limit in the Sobolev regularity estimates of Proposition 3, which gives us Theorem 2. In Section 7 we adapt an argument of [BFGM] to prove uniqueness of the constructed weak solution. In contrast to [BFGM], we prove uniqueness in a smaller class of weak solutions, but without any assumptions on \( \text{div} \ b \).

Notations. Denote

\[
\langle f, g \rangle = \langle fg \rangle := \int_{\mathbb{R}^d} fg \, dx
\]

(all functions considered below are assumed to be real-valued).

Set

\[
\rho(x) \equiv \rho_{\kappa, \theta}(x) := (1 + \kappa|x|^2)^{-\theta}, \quad \kappa > 0, \quad \theta > \frac{d}{2}, \quad x \in \mathbb{R}^d.
\]

It is easily seen that

\[
|\nabla \rho(x)| \leq \theta \sqrt{\kappa} \rho(x), \quad x \in \mathbb{R}^d.
\]

Below we will be applying (10) to \( \rho \) with \( \kappa \) chosen sufficiently small.

In this, for a given vector field \( b \), the notation \( b \in X \) (e.g. \( X = L^d \)) means that all components of \( b \) belong to \( X \).
For any $p > 1$, we use $p'$ to denote its conjugate $p/(p-1)$. Let $L^p_p := L^p(\mathbb{R}^d, pdx)$. Denote by $\| \cdot \|_{p, p}$ the norm in $L^p_p$, and by $\langle \cdot, \cdot \rangle_p$ the inner product in $L^2_p$.

Set $W^{1, 2}_\rho := \{ g \in W^{1, 2}_\rho | \| g \|_{W^{1, 2}_\rho} := \| g \|_{2, \rho} + \| \nabla g \|_{2, \rho} < \infty \}$.

Define constants

$$\beta_{2q} := 1 + 4qd, \quad q = 1, 2, \ldots$$

Put $J_T := [0, T]$.

## 2. Main results

Let us first make a few preliminary remarks.

If $b \in C_{c}^{\infty}(\mathbb{R}^d, \mathbb{R}^d)$ and $f \in C_{c}^{\infty}$, then (see [Ku, Theorem 6.1.9]) there exists a unique adapted strong solution to

$$u(t) - f + \int_0^t b \cdot \nabla u ds + \sigma \int_0^t \nabla u \circ dB_s = 0 \text{ a.s.,} \quad t \in J_T,$$

given by

$$u(t) = f(\Psi_t^{-1}), \quad t \geq 0,$$

where $\Psi_t : \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ is the stochastic flow for the SDE

$$X_t = x - \int_0^t b(X_r) dr + \sigma B_t,$$

i.e. there exists $\Omega_0 \subset \Omega, \mathbb{P}(\Omega_0) = 1$, such that, for all $\omega \in \Omega_0$, $\Psi_t(\cdot, \omega)\Psi_s(\cdot, \omega) = \Psi_{t+s}(\cdot, \omega)$,

1) for every $x \in \mathbb{R}^d$, the process $t \mapsto \Psi_t(x, \omega)$ is a strong solution to (12),

2) $\Psi_t(x, \omega)$ is continuous in $(t, x)$, $\Psi_t(\cdot, \omega) : \mathbb{R}^d \to \mathbb{R}^d$ are homeomorphisms and $\Psi_t(\cdot, \omega), \Psi_t^{-1}(\cdot, \omega) \in C_{c}^{\infty}(\mathbb{R}^d, \mathbb{R}^d)$.

Applying Itô’s formula, one easily obtain that for every $\mu \geq 0$,

$$u(t) = e^{\mu t} f(\Psi_t^{-1}), \quad t \geq 0$$

solves

$$u(t) - f + \mu \int_0^t u ds + \int_0^t b \cdot \nabla u ds + \sigma \int_0^t \nabla u \circ dB_s = 0 \text{ a.s.,} \quad t \in J_T.$$

Thus, solutions of the Cauchy problems (11) and (13) differ by the factor $e^{-\mu t}$.

Below we prove well-posedness of the Cauchy problem

$$du + \mu ud t + b \cdot \nabla u dt + \sigma \nabla u \circ dB_t = 0 \quad \text{on } (0, \infty) \times \mathbb{R}^d,$$

$$u|_{t=0} = f \in L^p, \quad p \geq 2,$$

with a general form-bounded $b$ having a sufficiently small relative bound $\delta$, for a fixed $\mu \geq 0$. Note that while $u$ satisfies (19) below, $e^{\mu t} u$ satisfies the same identity without the term involving $\mu$, see Remark 4.
We rewrite the equation in (14), using the identity relating Stratonovich and Itô integrals

$$
\int_0^t \nabla u \circ dB_s = \int_0^t \nabla u dB_s - \frac{1}{2} \sum_{k=1}^d [\partial_{x_k} u, B^k]_t, \quad B_t = (B_t^k)_{k=1}^d,
$$

as

$$
du + \mu u dt + b \cdot \nabla u dt + \sigma \nabla u dB_t - \frac{\sigma^2}{2} \Delta u dt = 0.
$$

We first state our basic existence result. Recall that $b \in F_\delta$ if and only if

$$
\|b\varphi\|^2_2 \leq \delta \|\nabla \varphi\|^2_2 + c_\delta \|\varphi\|^2_2, \quad \varphi \in W^{1,2},
$$

for some constant $c_\delta \geq 0$.

**Theorem 1.** Assume that $d \geq 3$, $b \in F_\delta$ with $\sqrt{\delta} < \frac{a^2}{2}\frac{1}{\sigma^2}$. Let $T > 0$, $p \geq 2$. Provided that $\kappa$ is chosen sufficiently small, there are constants $\mu_1(\delta, c_\delta, p) \geq 0$, $C_1 = C_1(\delta, c_\delta, p) > 0$ and $C_2 = C_2(\delta, c_\delta, p, T) > 0$ such that for any $\mu \geq \mu_1(\delta, c_\delta, p)$, for every $f \in L^{2p}$ there exists a function $u \in L^\infty(J_T, L^2(\Omega, L^2_p))$ for which the following are true:

(i) 

$$
\sup_{t \in J_T} \|\mathbb{E} u^2(t)\|_p \leq \|f\|^2_{2p}, \quad \int_{J_T} \|\nabla (\mathbb{E} u^2)^{\frac{1}{2}}\|^2_2 ds \leq C_1 \|f\|^2_{2p},
$$

so, in particular, for a.e. $\omega \in \Omega$, $\nabla \int_0^T u(s, \cdot, \omega) ds \in L^2_{loc}(\mathbb{R}^d, \mathbb{R}^d)$ and hence

$$
b \cdot \nabla \int_{J_T} u(s, \cdot, \omega) ds \in L^1_{loc},
$$

and, for every test function $\varphi \in C_c^\infty$, we have a.s. for all $t \in J_T$,

$$
\langle u(t), \varphi \rangle - \langle f, \varphi \rangle + \mu \int_0^t u ds, \varphi \rangle + \langle b \cdot \nabla \int_0^t u ds, \varphi \rangle - \sigma \int_0^t u dB_s, \nabla \varphi \rangle + \frac{\sigma^2}{2} \langle \nabla \int_0^t u ds, \nabla \varphi \rangle = 0.
$$

(ii) For any sequence of smooth vector fields $b_m \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d)$, $m = 1, 2, \ldots$, that are uniformly form-bounded in the sense that $b_m \in F_\delta$ with $c_\delta$ independent of $m$, and are such that

$$
b_m \to b \text{ in } L^2_{loc}(\mathbb{R}^d, \mathbb{R}^d) \text{ as } m \to \infty,
$$

we have for any initial function $f \in C_c^\infty$,

$$
u_m(t) \to u(t) \text{ in } L^2(\Omega, L^2_p) \text{ uniformly in } t \in J_T,
$$

where $u_m$ is the unique strong solution to (14) (with $b = b_m$).

Theorem 1 is proved in Section 5.

The next theorem establishes the Sobolev regularity of $u$ up to the initial time $t = 0$. 

Theorem 2. Assume that $d \geq 3$, $b \in \mathbf{F}_\delta$ with $\sqrt{\delta} < \frac{\sigma_p}{\sigma_b}$ and $f \in W^{1,4}$. Let $\kappa$ be sufficiently small and $\mu_1(\delta, c_\delta, 2)$ be the constant in Theorem 1 with $p = 2$. For $\mu \geq \mu_1(\delta, c_\delta, 2)$, let $u$ be the process constructed in Theorem 1. There exists $\mu_2(\delta, c_\delta) \geq \mu_1(\delta, c_\delta, 2)$ such that for $\mu \geq \mu_2(\delta, c_\delta)$, the following are true:

(a) $\mathbb{E}u^2, \mathbb{E}|\nabla u|^2 \in L^\infty(J_T, L^2)$, so $u \in L^\infty(J_T, L^2(\Omega, W^{1,2}))$;

(b) for any test function $\varphi \in C_c^\infty$, the process $t \mapsto \langle u(t), \varphi \rangle$ is $(\mathcal{F}_t)$-progressively measurable and has a continuous $(\mathcal{F}_t)$-semi-martingale modification that satisfies a.s. for every $t \in J_T$,

$$
\langle u(t), \varphi \rangle - \langle f, \varphi \rangle + \mu \int_0^t \langle u, \varphi \rangle ds + \int_0^t \langle b \cdot \nabla u, \varphi \rangle ds - \sigma \int_0^t \langle u, \nabla \varphi \rangle dB_s + \frac{\sigma^2}{2} \int_0^t \langle u, \Delta \varphi \rangle ds = 0.
$$

Moreover, if $\sqrt{\delta} < \frac{\sigma_p^2}{2\sigma_b}$ for some $q = 1, 2, \ldots$, then there exist constants $\mu_2(\delta, c_\delta, q) \geq \mu_1(\delta, c_\delta, 2q)$ (with $\mu_2(\delta, c_\delta, 1)$ equal to the $\mu_2(\delta, c_\delta)$ above) and $C_1 = C_1(\delta, c_\delta, q) > 0$ such that when $\mu \geq \mu_2(\delta, c_\delta, q)$ and $f \in W^{1,4q}$, we have

$$
\sup_{0 \leq \alpha \leq 1} ||| \mathbb{E}|\nabla u|^{2q} |||_{L^{1+\frac{2q}{4q}}(J_T, L^{\frac{2q}{4q}+\frac{2q}{q-2}})} \leq C_1 || \nabla f ||_{4q}^{2q}.
$$

In particular, there exists $C_2 > 0$ such that

$$
\sup_{t \in J_T} \mathbb{E}(|\nabla u|^{2q}) \leq C_2 || \nabla f ||_{4q}^{2q}.
$$

If $2q > d$, then for a.e. $\omega \in \Omega$, $t \in J_T$, the function $x \mapsto u(t, x, \omega)$ is Hölder continuous, possibly after modification on a set of measure zero in $\mathbb{R}^d$ (in general, depending on $\omega$).

The proof of Theorem 2 is given in Section 6.

Theorem 3. Assume that $d \geq 3$, $b \in \mathbf{F}_\delta$ with $\sqrt{\delta} < \frac{\sigma_p^2}{2\sigma_b}$ and $f \in W^{1,4}$. Provided $\kappa$ is sufficiently small, there exists $\mu_3 = \mu_3(\delta, c_\delta) \geq 0$ such that for $\mu \geq \mu_3(\delta, c_\delta)$, the Cauchy problem (14) has a unique solution in the class of functions satisfying (a), (b) of Theorem 2.

We prove Theorem 3 in Section 7.

A function satisfying (a), (b) of Theorem 2 will be called a weak solution of (14). This definition of weak solution is close to [BFGM, Definition 2.13]. It should be noted however that the authors in [BFGM] prove their uniqueness result, in the time-dependent case, in a larger class of weak solutions (not requiring any differentiability, see [BFGM, Definition 3.3]) but under additional assumptions on $b$. Specialized to the time-dependent case, they assume that $b$ satisfies

$$
div b \in L^d + L^\infty
$$

in addition to $b \in L^d + L^\infty$. The latter is needed to establish (21) for solutions of the adjoint equation to the STE, i.e. the stochastic continuity equation (which allows to prove an even stronger result: the uniqueness of weak solution to the corresponding random transport equation), see [BFGM, Sect. 3].

We expect that an analogue of (23) for $b \in \mathbf{F}_\delta$ can be found with some additional effort. However, we will not address this matter in this paper. Of course, in the case $b \in \mathbf{F}_\delta$, $\text{div } b = 0$, one has (21) for solutions to the stochastic continuity equation, so one can prove the uniqueness for (14) by repeating the argument in [BFGM, Sect. 3].

The proof of the uniqueness result in Theorem 3 (see Section 7) adopts the method of [BFGM, Sect. 3].
Remark 1 (On applications to SDEs). Armed with Theorems 1 and 2, one can repeat the argument in [BFGM, Sect. 4] to prove the following result. Assuming that $b \in F_\delta$ with $\delta$ sufficiently small, there exists a stochastic Lagrangian flow for SDE (12), i.e. a measurable map $\Phi : J_T \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ such that, for a.e. $x \in \mathbb{R}^d$, the process $t \mapsto \Phi_t(x, \omega)$ is a strong solution of the SDE (12):

$$ \Phi_t(x, \omega) = x - \int_0^t b(s, \Phi_s(x, \omega))ds + \sigma B_t(\omega), \quad \text{a.s.,} \quad t \in J_T, $$

and $\Phi_t(\cdot, \cdot)$ is $\mathcal{F}_t$-progressively measurable. If also $\sqrt{\delta} < \frac{\sigma^2}{2b_2q}$, $q = 1, 2, \ldots$, then $\Phi_t(\cdot, \omega) \in W^{1,2q}_{\text{loc}}(t \in J_T)$ for a.e. $\omega \in \Omega$. Moreover, $\Phi_t$ is unique, i.e. any two such stochastic flows coincide a.s. for every $t > 0$ for a.e. $x$.

Remark 2 (STE with time-dependent $b$). The proof of Proposition 3, i.e. the a priori Sobolev regularity of solutions of the STE, carries over, without change, to time-dependent form-bounded (STE with time-dependent $b$).

Definition 2. A vector field $b \in L^2_{\text{loc}}([0, \infty) \times \mathbb{R}^d, \mathbb{R}^d)$ is said to be form-bounded with relative bound $\delta > 0$, written as $b \in \tilde{F}_\delta$, if $b \in L^2_{\text{loc}}([0, \infty) \times \mathbb{R}^d)$ and

$$ \int_0^\infty \|b(t, \cdot)\|_2^2 dt < \delta \int_0^\infty \|
abla \phi(t, \cdot)\|_2^2 dt + \int_0^\infty g(t)\|\phi(t, \cdot)\|_2^2 dt $$

for some $g = g_\delta \in L^1_{\text{loc}}([0, \infty)$, for all $\phi \in C_c^\infty([0, \infty) \times \mathbb{R}^d)$.

The class $\tilde{F}_\delta$ contains vector fields

$$ |b(\cdot, \cdot)| \in L^q([0, \infty), L^r + L^\infty), \quad \frac{d}{r} + \frac{2}{q} \leq 1 $$

(using Hölder’s inequality and the Sobolev embedding theorem). An example is $b(x) = (b_1(x), 0, \ldots, 0)$ with

$$ |b_1(t, x)|^2 \leq c_1 |x - x_0|^2 + c_2 |t - t_0|^{-1} \left(\log(e + |t - t_0|^{-1})\right)^{-1 - \varepsilon}, \quad \varepsilon > 0, \quad (t, x) \in [0, \infty) \times \mathbb{R}^d, $$

which belongs to the class $\tilde{F}_\delta$ with $\delta = c_1 (2/(d - 2))^2$ (using Hardy’s inequality).

We plan to address the regularity theory of the STE with $b \in \tilde{F}_\delta$ elsewhere (regarding weak well-posedness of the corresponding SDE (2) with $b \in \tilde{F}_\delta$, see [KiM]).

3. Proof of Proposition 1

Recall that $1_m$ is the indicator of $\{|x| \leq m, |b(x)| \leq m\}$. First, consider

$$ b_m := \eta_m e^{\varepsilon_m \Delta} (1_m b), \quad (24) $$

where $\varepsilon_m \downarrow 0$ is to be chosen later.

Let us show that, for any $\{\gamma_m\} \downarrow 0$ we can select $\{\varepsilon_m\} \downarrow 0$ so that

$$ b_m \in F_{\delta_m} \quad \text{with} \quad \delta_m = (\sqrt{\delta} + \sqrt{\gamma_m})^2 \downarrow \delta \quad \text{and} \quad c_{\delta_m} \leq 2c_\delta \text{ starting from some } m \text{ on.} $$

Since $b \in F_\delta$, there exists $\lambda \geq 0$ such that $\|b(\lambda - \Delta)^{-\frac{1}{2}}\|_{2 \to 2} \leq \sqrt{\delta}$. Then $c_\delta = \lambda \delta$. We claim that, we can select $\{\varepsilon_m\} \downarrow 0$ fast enough so that

$$ \|b_m(\lambda - \Delta)^{-\frac{1}{2}}\|_{2 \to 2} \leq \sqrt{\delta_m}. \quad (25) $$
Once this claim is proven, we will have \( c_{\delta m} = \lambda \delta_m \leq 2c_{\delta} \) starting from some \( m \) on, which implies the required. Now we prove the claim. We have

\[
 b_m = 1_m b + (b_m - 1_m b),
\]

where, clearly, \( \|1_m b\|((\lambda - \Delta)^{-\frac{1}{2}}) \|2 \|_2 \leq \sqrt{\delta} \) for every \( m \), while \( b_m - 1_m b \in L^d \). It follows from Hölder’s inequality and the Sobolev embedding theorem that for any \( g \)

\[
\|(b_m - 1_m b)(\lambda - \Delta)^{-\frac{1}{2}} g\|_2 \leq \|b_m - 1_m b\|\|\lambda - \Delta\|^{-\frac{1}{2}} \|g\|_2 \|2 \|_2 \leq c \|b_m - 1_m b\|\|g\|_2.
\]

It is easily seen that, for every \( m \), the norm \( \|b_m - 1_m b\|\) can be made smaller than \( c^{-1} \gamma_m \) by selecting \( \{\epsilon_m\} \downarrow 0 \) sufficiently rapidly. Thus

\[
\|(b_m - 1_m b)(\lambda - \Delta)^{-\frac{1}{2}}\|_2 \|_2 \leq \gamma_m,
\]

and (25) follows.

Finally, to have \( b_m \) form-bounded with the original relative bound \( \delta \), it suffices to multiply \( b_m \) in (24) by \( \frac{\delta}{\delta_m} \). (Although, to carry out the proofs of Theorems 1-3, the last step is not necessary since all our assumptions on \( \delta \) are strict inequalities.)

4. A PRIORI ESTIMATES

Assume \( b \in F_\delta \). In the remainder of this paper, we fix some \( b_m \in C_c^\infty(\mathbb{R}^d, \mathbb{R}^d) \) such that

\[
 b_m \rightarrow b \text{ in } L^2_{loc}(\mathbb{R}^d, \mathbb{R}^d) \text{ as } m \rightarrow \infty
\]

and for every \( m = 1, 2, \ldots \)

\[
\|b_m \varphi\|_2^2 \leq \delta \|\nabla \varphi\|_2^2 + c_{\delta} \|\varphi\|_2^2, \quad \varphi \in W^{1,2}
\]

with \( c_{\delta} \) independent of \( m \). We gave one example of such \( b_m \) in Proposition 1.

Let \( f \in C_c^\infty \). Let \( u_m \) be the unique strong solution to

\[
 u_m(t) - f + \mu \int_0^t u_m ds + \int_0^t b_m \cdot \nabla u ds + \sigma \int_0^t \nabla u_m \circ dB_s = 0 \text{ a.s.}, \quad t \in J_T = [0, T].
\]

Then, by [Ku, Section 6.1], for any \( p, r \geq 1 \) and any multiindex \( \alpha = (\alpha_1, \ldots, \alpha_d) \) of non-negative integers,

\[
\mathbb{E}(|D^\alpha u_m|^p) \in L^\infty(J_T \times \mathbb{R}^d)
\]

and

\[
\int_{\mathbb{R}^d}(1 + |x|^r)(\mathbb{E}|u_m|^p + \mathbb{E}|\nabla u_m|^p)dx \in L^\infty(J_T).
\]

We prove the next proposition under more general assumptions on \( \delta \) and \( p \) than in Theorem 1.

**Proposition 2.** Let \( b \in F_\delta \) with \( \sqrt{\delta} < \sigma^2 \). Let \( T > 0, p \in (p_c, \infty), p_c := (1 - \frac{\sqrt{\delta}}{\sigma^2})^{-1} \). Let \( f \in C_c^\infty \), let \( b_m \) and \( u_m \) be as above. There exist constants \( \mu(\delta, c_{\delta}, p) > 0, C_1 = C_1(\delta, c_{\delta}, p) > 0 \) and \( C_2 = C_2(\delta, c_{\delta}, p, T) > 0 \) independent of \( m \) such that for any \( \mu \geq \mu(\delta, c_{\delta}, p) \) and \( m = 1, 2, \ldots \), the following are true:

(i)

\[
\sup_{t \in J_T} \mathbb{E}|u_m|^2(t) \|_p \leq \|f\|_{2p}^2, \quad \int_{J_T} \mathbb{E}|\nabla u_m|^2 \|_2 \|_2 ds \leq C_1 \|f\|_{2p};
\]

(27)
(ii) if \( \sqrt{\delta} < \frac{\sigma^2}{2} \), then
\[
\mathbb{E}(\rho \left( \nabla \int_{J_T} u_m(s) ds \right)^2) \leq C_2\|f\|_{2p}^2.
\]  

(28)

Next, we establish a priori \( W^{1,2q} \) regularity of solutions of (26), \( q = 1, 2, \ldots \)

**Proposition 3.** Let \( b \in \mathbf{F}_\delta \) and \( f \in C_c^\infty \), let \( b_m \) and \( u_m \) be as above. For every \( q \geq 1 \), there exist constants \( \mu(\delta, c_\delta, q) \geq 0 \) and \( C = C(\delta, c_\delta, q) > 0 \) independent of \( m \) such that if \( \sqrt{\delta} < \frac{\sigma^2}{2\delta_0} \) and \( \mu \geq \mu(\delta, c_\delta, q) \), then
\[
\sup_{0 \leq \alpha \leq 1} \| \mathbb{E} \| \nabla u_m \|_{2q} \|_{L^{\frac{1}{2p}}(\mathbb{R}^n \times [0,T])} \leq C \| \nabla f \|_{4q}.
\]

Here \( L^{\frac{1}{2p}} := L^{\infty} \).

**Proof of Proposition 2.** For brevity, we write \( u \) for \( u_m \) in this proof. The identity (15) allows us to rewrite (26) as
\[
 u(t, \cdot) - f + \mu \int_0^t u ds + \int_0^t b_m \cdot \nabla u ds + \sigma \int_0^t \nabla u dB_s - \frac{\sigma^2}{2} \int_0^t \Delta u ds = 0 \quad \text{a.s.,} \quad t \in J_T.
\]

(30)

Below we will be appealing to (30).

We first prove (27). Applying Itô’s formula to \( u^2 \), we obtain, in view of (30),
\[
 u^2(t) - f^2 = -2\mu \int_0^t u^2 ds - \int_0^t b_m \cdot \nabla u^2 ds - \sigma \int_0^t \nabla u^2 dB_s + \frac{\sigma^2}{2} \int_0^t \Delta u^2 ds.
\]

Since \( t \mapsto \int_0^t \nabla u^2 dB_s \) is a martingale, \( v = \mathbb{E}u^2 \) satisfies
\[
 \partial_t v = -2\mu v - b_m \cdot \nabla v + \frac{\sigma^2}{2} \Delta v, \quad v(0) = f^2.
\]

We multiply the last equation by \( v^{p-1} \) and integrate by parts (put \( v_p := v^{\frac{p}{2}} \)),
\[
 \frac{1}{p} \partial_t \langle v_p^2 \rangle + 2\mu \langle v_p^2 \rangle + \frac{4}{pp'} \langle \|\nabla v_p\|^2 \rangle - \frac{\sigma^2}{2p} \langle b_m \cdot \nabla v_p, v_p \rangle \leq 0,
\]
so applying the quadratic inequality we have (for \( \varepsilon > 0 \))
\[
 \partial_t \langle v_p \rangle + 2\mu \langle v_p \rangle + \frac{2\sigma^2}{p'} \langle \|\nabla v_p\|^2 \rangle - 2 \left( \varepsilon \langle \|\nabla v_p\|^2 \rangle + \frac{1}{4\varepsilon'} \langle \|b_m\|_{2p'}^2 \rangle \right) \leq 0.
\]

Finally, by our assumption on \( b_m \),
\[
 \partial_t \langle v_p \rangle + 2\mu \langle v_p \rangle + \frac{2\sigma^2}{p'} \langle \|\nabla v_p\|^2 \rangle - 2 \left( \varepsilon \langle \|\nabla v_p\|^2 \rangle + \frac{\delta}{4\varepsilon'} \langle \|\nabla v_p\|^2 \rangle + \frac{c_\delta}{4\varepsilon'} \langle v_p \rangle \right) \leq 0.
\]

Taking \( \varepsilon = \frac{\delta}{2} \) in the last inequality and integrating with respect to \( t \), we obtain for \( t > 0 \)
\[
 \langle v(t)p \rangle + 2 \left( \frac{\sigma^2}{p'} - \sqrt{\delta} \right) \int_0^t \langle \|\nabla v_p\|^2 \rangle ds + \left[ 2\mu - \frac{c_\delta}{2\sqrt{\delta}} \right] \int_0^t \langle v_p \rangle ds \leq \|f^2\|_{p'},
\]
where \( \frac{\sigma^2}{p'} - \sqrt{\delta} > 0 \) since \( p > p_c \). Taking \( \mu \geq \frac{c_\delta}{4\sqrt{\delta}} \), we arrive at (27).
Now we deal with (28). Let $\mu \geq \frac{c_6}{4\sqrt{p}}$ as above. By (27),

$$\sup_{t \in J_T} \langle \rho \mathbb{E}u^2(t) \rangle \leq \|\rho\|_{p'} \sup_{t \in J_T} \|\mathbb{E}u^2(t)\|_p \leq c_1 \|f\|_{2p}^2,$$

since $\theta > \frac{d}{2}$ in the definition of $\rho$.

We multiply (30) by $\rho \int_0^t u ds$, integrate, and take expectation, to get

$$I_0 := \mathbb{E} \langle \rho \int_0^t u ds, u(t) \rangle = \mathbb{E} \langle \rho \int_0^t u ds, f \rangle - \mathbb{E} \langle \rho \int_0^t u ds, b_m \cdot \nabla \int_0^t u ds \rangle$$

$$- \sigma \mathbb{E} \langle \rho \int_0^t u ds, \int_0^t \nabla u dB_s \rangle + \frac{\sigma^2}{2} \mathbb{E} \langle \rho \int_0^t u ds, \int_0^t \Delta u ds \rangle$$

$$+ \mu \mathbb{E} \langle \rho \int_0^t u ds, \int_0^t u ds \rangle =: I_1 + I_2 + I_3 + I_4 + I_5.$$  

Set

$$U := \int_0^t u ds.$$  

By Hölder’s inequality and (31),

$$\mathbb{E} \langle \rho U^2 \rangle \leq t \mathbb{E} \langle \rho \mathbb{E}u^2 \rangle \leq t^2 c_1 \|f\|_{2p}^2.$$  

Integrating by parts in $I_4$ and using the quadratic inequality, we have

$$\frac{2}{\sigma^2} I_4 = -\mathbb{E} \langle \rho |\nabla U|^2 \rangle - \mathbb{E} \langle U \nabla \rho, \nabla U \rangle$$

$$\leq -\mathbb{E} \langle \rho |\nabla U|^2 \rangle + \alpha \mathbb{E} \langle |\nabla \rho|U^2 \rangle + \frac{1}{4\alpha} \mathbb{E} \langle |\nabla \rho| |\nabla U|^2 \rangle \quad (\alpha > 0)$$

(we are applying (10) in the last term, and (10), (33) in the middle term)

$$\leq - \left( 1 - \frac{\theta \sqrt{\kappa}}{4\alpha} \right) \mathbb{E} \langle \rho |\nabla U|^2 \rangle + \theta \sqrt{\kappa} T^2 c_1 \|f\|_{2p}^2.$$  

Substituting the last estimate into (32), we obtain

$$\frac{\sigma^2}{2} \left( 1 - \frac{\theta \sqrt{\kappa}}{4\alpha} \right) \mathbb{E} \langle \rho |\nabla U|^2 \rangle \leq \frac{\sigma^2}{2} \theta \sqrt{\kappa} T^2 \|f\|_{2p}^2 + |I_0| + |I_1| + |I_2| + |I_3| + |I_5|.$$  

We now estimate $|I_i|, i = 0, 1, 2, 3, 5$. By (31) and (33),

$$|I_0| \leq \left( \mathbb{E} \langle \rho U^2 \rangle \right)^{\frac{1}{2}} \left( \mathbb{E} \langle \rho u^2(t) \rangle \right)^{\frac{1}{2}} \leq c_2 \|f\|_{2p}^2.$$  

Similarly,

$$|I_1| \leq c_3 \|f\|_{2p}^2, \quad |I_5| \leq \mu c_4 \|f\|_{2p}^2.$$
Next, applying the quadratic inequality, we get

\[ |I_2| \leq \nu \mathbb{E} \langle \rho |b_m|^2 U^2 \rangle + \frac{1}{4\nu} \mathbb{E} \langle \rho |\nabla U|^2 \rangle \quad (\nu > 0) \]

(in the first term, we apply \( b_m \in \mathcal{F}_\delta \) with \( \varphi := \sqrt{\rho} U \))

\[ \leq \nu \delta \mathbb{E} \langle |\nabla (\sqrt{\rho} U)|^2 \rangle + c_3 \mathbb{E} \langle \rho U^2 \rangle + \frac{1}{4\nu} \mathbb{E} \langle \rho |\nabla U|^2 \rangle \]

(in the first term, we use \( (a + c)^2 \leq (1 + \epsilon)a^2 + (1 + \frac{1}{\epsilon})c^2, \epsilon > 0 \))

\[ \leq \nu \delta (1 + \epsilon) \mathbb{E} \langle \sqrt{\rho} |\nabla U|^2 \rangle + \nu \delta (1 + \frac{1}{\epsilon}) \mathbb{E} \langle |U| \sqrt{\rho} |\nabla|^2 \rangle + \nu c_3 \mathbb{E} \langle \rho U^2 \rangle + \frac{1}{4\nu} \mathbb{E} \langle \rho |\nabla U|^2 \rangle \]

(in the second term, we apply (10) and then use (33); also, we apply (33) in the last term)

\[ \leq \left( \nu \delta (1 + \epsilon) + \frac{1}{4\nu} \right) \mathbb{E} \langle \rho |\nabla U|^2 \rangle + 4 \mathbb{E} \langle \rho U^2 \rangle + 1 \mathbb{E} \langle \rho \nabla U \rangle \]

\[ + T^2 c_5 \|f\|_{2p}^2, \quad c_5 = c_5(\nu, \delta, c_3, \theta, \kappa, \epsilon). \]

In the current setting, we have \( \int_0^t \nabla u dB_s = \nabla \int_0^t u dB_s \) (see, for instance, [HN]). Thus, integrating by parts, we obtain

\[ I_3 = \sigma \mathbb{E} \langle \rho \nabla U, \int_0^t u dB_s \rangle + \sigma \mathbb{E} \langle U \nabla \rho, \int_0^t u dB_s \rangle, \]

so

\[ |I_3| \leq \sigma \left( \mathbb{E} \langle \rho |\nabla U|^2 \rangle \right)^{\frac{1}{2}} \left( \mathbb{E} \left( \int_0^t u dB_s \right)^2 \right)^{\frac{1}{2}} + \sigma \left( \mathbb{E} \langle |\nabla \rho |^2 \rangle \right)^{\frac{1}{2}} \left( \mathbb{E} \langle |U| \nabla \rho |^2 \rangle \right)^{\frac{1}{2}} \]

\[ \left( \mathbb{E} \langle |U| |\nabla \rho |^2 \rangle \right)^{\frac{1}{2}} \left( \mathbb{E} \langle \rho U^2 \rangle \right)^{\frac{1}{2}} \left( \mathbb{E} \langle \rho \nabla U \rangle \right)^{\frac{1}{2}} \]

\[ \leq \sigma \gamma \mathbb{E} \langle \rho |\nabla U|^2 \rangle + \frac{\sigma T^2 c_1}{4\gamma} \|f\|_{2p}^2 + \theta \sqrt{\kappa} \sigma T^2 c_1 \|f\|_{2p}^2 \quad (\gamma > 0). \]

Substituting the above estimates on \( |I_0|, |I_1|, |I_2|, |I_3| \) and \( |I_5| \) in (34), we obtain

\[ \left( \frac{\sigma^2}{2} - \nu \delta (1 + \epsilon) - \frac{1}{4\nu} - \sigma \gamma - \frac{\sigma^2 \theta \sqrt{\kappa}}{2} \right) \mathbb{E} \langle \rho |\nabla U|^2 \rangle \leq c_6 \|f\|_{2p}^2 \]

for an appropriate constant \( c_6 = c_6(\alpha, \gamma, \nu, \delta, \theta, \kappa, \epsilon, c_3, \mu) < \infty \). Take \( \nu = (2\sqrt{\delta})^{-1} \). Since \( \sqrt{\delta} < \frac{\sigma^2}{2} \) by assumption, we can select \( \gamma, \epsilon \) sufficiently small and \( \alpha \) sufficiently large so that

\[ \frac{\sigma^2}{2} - (\nu \delta + \frac{1}{4\nu}) - \nu \delta \epsilon - \sigma \gamma - \frac{\sigma^2 \theta \sqrt{\kappa}}{2} > 0, \]

and thus (28) follows with constant \( C_2 = c_6 \left( \frac{\sigma^2}{2} - \nu \delta (1 + \epsilon) - \frac{1}{4\nu} - \sigma \gamma - \frac{\sigma^2 \theta \sqrt{\kappa}}{2} \right)^{-1} \).
Remark 3. In Proposition 2, the interval \((p_c, \infty)\) of admissible values of \(p\) decreases to the empty set as \(\sqrt{\delta} \uparrow \sigma^2\). In fact, one can show that if \(b \in \mathbf{F}_\delta, \sqrt{\delta} < \sigma^2\) and \(b_m \in C_c^\infty\) are as above, then the limit
\[
\lim_{m \to \infty} e^{-tA_m}( \text{loc. uniformly in } t \geq 0), \quad p > p_c,
\]
where \(A_m = -\frac{\sigma^2}{2}\Delta + b \cdot \nabla, D(A_m) = W^{2,p}\), exists and determines a \(L^\infty\) contraction, quasi contraction holomorphic semigroup in \(L^p\), say, \(e^{-tA}\), see [KiS2, Theorems 4.2, 4.3]. The operator \(\Lambda\) is an appropriate operator realization of the formal operator \(-\frac{\sigma^2}{2}\Delta + b \cdot \nabla\) in \(L^p\). One can compare this result with the counterexample in [BFGM, Sect. 7] concerning the SDE
\[
X_t = -\int_t^0 b(X_s) ds + B_t, \quad b(x) = \sqrt{\delta} \frac{2}{\sigma^2}|x|^{-2}x \in \mathbf{F}_\delta,
\]
which corresponds to operator \(-\frac{1}{2}\Delta + b \cdot \nabla\), see discussion in the introduction.

Proof of Proposition 3. For any multiindex \(I\) with entries in \(\{1, \ldots, d\}\), i.e., an element of \(\{1, \ldots, d\} \times \cdots \times \{1, \ldots, d\}\), say, \(p\) times, we write \(|I| = p\). For any such multiindex \(I\) and \(l \in \{1, \ldots, d\}\), we denote by \(I - l\) the multiindex obtained from \(I\) by dropping an index of value \(l\). Let \(I - l + k\) be the multiindex \(I\) with an index of value \(l\) dropped and replaced with an index of value \(k\). It does not matter from which component the value \(l\) is dropped.

For brevity, we write \(u\) for \(u_m\) in this proof. Set
\[
w_r := \partial_x u, \quad 1 \leq r \leq d,
\]
where \(u\) is the strong solution to (26), and
\[
w_I := \prod_{r \in I} \partial_x u.
\]

Step 1. We apply Itô’s formula in Stratonovich form to \(w_I\), obtaining
\[
w_I(t) - \prod_{r \in I} \partial_x f = \sum_{r \in I} \int_0^t w_{I-r}(s) \circ dw_r(s).
\]
Next, differentiating (30) in \(x_r\) and then substituting the resulting expression for \(dw_r\) into the previous formula, we obtain
\[
w_I(t) - \prod_{r \in I} \partial_x f = -\mu \int_0^t w_I ds - \sum_{r \in I} \int_0^t w_{I-r}(b_m \cdot \nabla w_r + \partial_x b_m \cdot \nabla u) ds - \sigma \sum_{r \in I} \int_0^t w_{I-r} \nabla w_r \circ dB_s.
\]
Let \(b^k_m, k = 1, \ldots, d,\) be the components of the vector field \(b_m\). We have
\[
w_I(t) - \prod_{r \in I} \partial_x f = -\mu \int_0^t w_I ds - \sum_{r \in I} \int_0^t w_{I-r}(b_m \cdot \nabla w_r + \partial_x b_m \cdot \nabla u) ds - \sigma \int_0^t \nabla w_I \circ dB_s
\]
\[
(\text{we use } \int_0^t \nabla w_I \circ dB_s = \int_0^t \nabla w_I dB_s - \frac{1}{2} \sum_{k=1}^d [\partial_{x_k} w_I, B^k]_t)
\]
\[
= -\mu \int_0^t w_I ds - \sum_{r \in I} \int_0^t w_{I-r}(b_m \cdot \nabla w_r + \partial_x b_m \cdot \nabla u) ds - \sigma \int_0^t \nabla w_I dB_s + \frac{\sigma^2}{2} \int_0^t \Delta w_I ds
\]
\[
= -\mu \int_0^t w_I ds - \int_0^t b_m \cdot \nabla w_I ds - \sum_{r \in I} \sum_{k=1}^d \int_0^t \partial_x b_m w_{I-r+k} ds - \sigma \int_0^t \nabla w_I dB_s + \frac{\sigma^2}{2} \int_0^t \Delta w_I ds.
\]
Put
\[ v_I := E[w_I]. \]

Since \( t \mapsto \int_0^t \nabla w_I dB_s \) is a martingale, \( v_I \) satisfies
\[
v_I(t) - \prod_{r \in I} \partial_{x_r} f = -\mu \int_0^t v_I ds - \int_0^t b_m \cdot \nabla v_I ds - \sum_{r \in I} \sum_{k=1}^d \int_0^t \partial_{x_r} b_m^k v_{I-r+k} ds + \frac{\sigma^2}{2} \int_0^t \Delta v_I ds,
\]
i.e.,
\[
\partial_t v_I = -\mu v_I + \frac{\sigma^2}{2} \Delta v_I - b_m \cdot \nabla v_I - \sum_{r \in I} \sum_{k=1}^d \partial_{x_r} b_m^k v_{I-r+k}, \quad v_I(0) = \prod_{r \in I} \partial_{x_r} f. \tag{35}
\]

Step 2. We multiply the equation in (35) by \( v_I \), and integrate:
\[
\frac{1}{2} \partial_t \langle v_I^2 \rangle + \mu \langle v_I^2 \rangle + \frac{\sigma^2}{2} \langle (\nabla v_I)^2 \rangle = -\langle v_I, b_m \cdot \nabla v_I \rangle - \langle v_I, \sum_{r \in I} \sum_{k=1}^d \partial_{x_r} b_m^k v_{I-r+k} \rangle.
\]

Then, for every \( t \in J_T \),
\[
\frac{1}{2} \langle v_I^2(t) \rangle - \frac{1}{2} \langle v_I^2(0) \rangle + \mu \int_0^t v_I^2 ds + \frac{\sigma^2}{2} \int_0^t \langle (\nabla v_I)^2 \rangle ds
\]
\[
= -\int_0^t \langle v_I, b_m \cdot \nabla v_I \rangle ds - \int_0^t \langle v_I, \sum_{r \in I} \sum_{k=1}^d \partial_{x_r} b_m^k v_{I-r+k} \rangle ds =: -S_1^I - S_2^I.
\]

We estimate \( |S_1^I| \) and \( |S_2^I| \) as follows:
\[
|S_1^I| \leq \left| \int_0^t \langle v_I, b_m \cdot \nabla v_I \rangle ds \right| \leq \gamma \int_0^t \langle (\nabla v_I)^2 \rangle ds + \frac{1}{4\gamma} \int_0^t \langle v_I^2 b_m^2 \rangle ds
\]
\[
\quad \text{(we use } b_m \in F_\delta) \]
\[
\leq \left( \gamma + \frac{\delta}{4\gamma} \right) \int_0^t \langle (\nabla v_I)^2 \rangle ds + \frac{c\delta}{4\gamma} \int_0^t \langle v_I^2 \rangle.
\tag{37}
\]

Next, integrating by parts, and applying the quadratic inequality, we have
\[
|S_2^I| = \left| -\int_0^t \sum_{r \in I} \sum_{k=1}^d \left( (v_{I-r+k} \partial_{x_r} v_I + v_I \partial_{x_r} v_{I-r+k}) b_m^k \right) ds \right|
\]
\[
\leq \alpha \int_0^t \sum_{r \in I} \sum_{k=1}^d \langle (\partial_{x_r} v_I)^2 + (\partial_{x_r} v_{I-r+k})^2 \rangle ds + \frac{1}{4\alpha} \int_0^t \sum_{r \in I} \sum_{k=1}^d \langle v_{I-r+k}^2 b_m^k \rangle + \langle v_I^2 b_m^k \rangle^2 ds.
\]

Let \( q = 1, 2, \ldots \). Summing over all \( I \) with \( |I| = 2q \) and noticing that every multiindex of length \( 2q \) is counted \( 4qd \) times, we obtain
\[
\sum_I |S_2^I| \leq 4a\alpha q^2 \int_0^t \langle |\nabla v_I|^2 \rangle ds + \frac{q^2 d}{\alpha} \int_0^t \langle v_I^2 b_m^2 \rangle ds
\]
\[
\quad \text{(use } b_m \in F_\delta \text{ in the second term)}
\]
\[
\leq 4a\alpha q^2 \int_0^t \langle |\nabla v_I|^2 \rangle ds + \frac{q^2 d\delta}{\alpha} \int_0^t \langle |\nabla v_I|^2 \rangle ds + \frac{q^2 d\delta}{\alpha} \int_0^t \langle v_I^2 \rangle ds.
\]
Also, by (37), we have
\[
\sum I |S^1_I| \leq \left( \gamma + \frac{\delta}{4\gamma} \right) \sum I \int_0^t \langle |\nabla v_I|^2 \rangle \, ds + \frac{c_\delta}{4\gamma} \sum I \int_0^t \langle v_I^2 \rangle.
\]

Now, armed with the last two estimates, we sum both sides of (36) over all I with |I| = 2q to obtain
\[
\frac{1}{2} \sum I \langle v_I^2(t) \rangle + \mu \int_0^t v_I^2 \, ds + \kappa \int_0^t \sum I \langle |\nabla v_I|^2 \rangle \, ds
\leq \frac{1}{2} \sum I \langle v_I^2(0) \rangle + \left[ \frac{q d c_\delta}{\alpha} + \frac{c_\delta}{4\gamma} \right] \sum I \int_0^t \langle v_I^2 \rangle,
\]
where
\[
\kappa := \frac{\sigma^2}{2} - \gamma - \frac{\delta}{4\gamma} - 4\alpha q d - \frac{q d \delta}{4\alpha}.
\]

The maximum \( \kappa_* := \max_{\alpha, \gamma > 0} \kappa = \frac{\sigma^2}{2} - \sqrt{\delta} - 4qd\sqrt{\delta} \) is attained at
\[
\alpha = \frac{\sqrt{\delta}}{2}, \quad \gamma = \frac{\sqrt{\delta}}{2}.
\]

For this choice of \( \alpha \) and \( \gamma \), we have \( \kappa_* = \frac{\sigma^2}{2} - \beta_2 q \sqrt{\delta} \). Since \( \beta_2 q \sqrt{\delta} < \frac{\sigma^2}{2} \) by assumption, we have \( \kappa_* > 0 \) and
\[
\frac{1}{2} \sum I \langle v_I^2(t) \rangle + (\mu - \hat{\kappa}) \int_0^t v_I^2 \, ds + \kappa_* \int_0^t \sum I \langle |\nabla v_I|^2 \rangle \, ds \leq \frac{1}{2} \sum I \langle v_I^2(0) \rangle,
\]
where \( \hat{\kappa} := \frac{2q d c_\delta}{\sqrt{\delta}} + \frac{c_\delta}{2\sqrt{\delta}}. \) Thus, choosing \( \mu \geq \hat{\kappa} \), we obtain
\[
\frac{1}{2} \sup_{\tau \in [0,t]} \sum I \langle v_I^2(\tau) \rangle + \kappa_* \int_0^t \sum I \langle |\nabla v_I|^2 \rangle \, ds \leq \frac{1}{2} \sum I \langle v_I^2(0) \rangle.
\]

Step 3. Recalling that \( v_I = E[\prod_{r \in I} x_r u], v_I(0) = \prod_{r \in I} \partial_{x_r} f \), we obtain from the previous estimate:
\[
\sup_{I \in J_T} \sum_{1 \leq k \leq d} \langle \langle \mathbb{E}(\partial_{x_k} u)^{(2q)} \rangle^2 \rangle \leq c_1 \langle |\nabla f|^{2q} \rangle, \tag{38}
\]
\[
\sum_{1 \leq k \leq d} \int_0^t \langle |\nabla \mathbb{E}(\partial_{x_k} u)^{(2q)}|^2 \rangle \, ds \leq c_2 \langle |\nabla f|^{2q} \rangle, \tag{39}
\]
for appropriate positive constants \( c_1, c_2. \) By the Sobolev embedding theorem,
\[
\int_0^t \langle (\nabla \mathbb{E}|\nabla u|^{2q})^2 \rangle \, ds \geq c_3 \int_0^t \langle (\mathbb{E}|\nabla u|^{2q})^{\frac{2q}{d+2}} \rangle^\frac{d+2}{d} \, ds,
\]
so (39) yields
\[
\| \mathbb{E}|\nabla u|^{2q} \|^2_{L^2(J_T, L^{\frac{2q}{d+2}})} \leq c_4 \| \nabla f \|_{4q}^{4q},
\]
for appropriate constant \( c_4 > 0. \)

Interpolating between the last estimate, and (38), that is, \( \| \mathbb{E}|\nabla u|^{2q} \|^2_{L^\infty(J_T, L^2)} \leq c_1 \| \nabla f \|_{4q}^{4q}, \) we obtain (29). \[\square\]
5. Proof of Theorem 1

Recall that \( \| \cdot \|_{p, \rho} \) denotes the norm in \( L^p(\mathbb{R}^d, \rho dx) \), and \( \langle \cdot, \cdot \rangle_{\rho} \) the inner product in \( L^2(\mathbb{R}^d, \rho dx) \). We assume throughout this section that \( b \in \mathbf{F}_\delta \) and \( b_m, m = 1, 2, \ldots \) are as in the beginning of the previous section.

**Lemma 1.** Let \( b \in \mathbf{F}_\delta \), and let \( b_m \) be as above. Then the following are true:

(i) \( \| b \sqrt{\rho} \|_2 < \infty \).

(ii) \( \| b \sqrt{\rho} \mathbf{1}_{B^c(0,R+1)} \|_2 \downarrow 0 \) as \( R \to \infty \).

(iii) \( \langle \rho |b - b_m|^2 \rangle \to 0 \) as \( m \to \infty \).

**Proof.** (i) Using \( b \in \mathbf{F}_\delta \), and applying (10) and \( \langle \rho \rangle < \infty \), we have

\[
\| b \sqrt{\rho} \|^2 \leq \delta \| \nabla \sqrt{\rho} \|^2 + c_\delta \langle \rho \rangle < \infty.
\]

(ii) For any \( R \geq 1 \), let \( \eta_R \) be a \([0,1]\)-valued smooth function such that \( \eta_R(x) = 1 \) if \( |x| > R + 1 \);

\( \eta_R(x) = 0 \) if \( |x| \leq R \); and \( \sup_{R \geq 1} \| \nabla \eta_R \|_{\infty} \leq C \). Then

\[
\| b \sqrt{\rho} \eta_R \|^2 \leq \delta \| \sqrt{\rho} \eta_R \|^2 + c_\delta \langle \rho \eta_R^2 \rangle.
\]

We have \( \nabla [\sqrt{\rho} \eta_R] = \frac{1}{2 \sqrt{\rho}} (\nabla \rho) \eta_R + \sqrt{\rho} \nabla \eta_R =: S_1 + S_2 \). Using (10), we have

\[
\| S_1 \|^2 \leq C \langle \rho \eta_R^2 \rangle \to 0 \quad \text{as} \quad R \to \infty.
\]

Next, we use \( \sup_{R \geq 1} \| \nabla \eta_R \|_{\infty} \leq C \) to get

\[
\| S_2 \|^2 \leq C(1 + \kappa R^2)^{-\theta} \langle 1_{B(0,R+1)} - 1_{B(0,R)} \rangle = c_d C(1 + \kappa R^2)^{-\theta} R^d \to 0 \quad \text{as} \quad R \to \infty
\]

since \( \theta > \frac{d}{2} \). This completes the proof of (ii).

(iii) This is a consequence of (ii) and \( b_m \to b \) in \( L^2_{\text{loc}}(\mathbb{R}^d) \).

The proof of Lemma 1 is complete. \( \square \)

**Lemma 2.** Let \( \beta \sqrt{\delta} < \frac{\sigma^2}{2} \), \( f \in C^\infty_{c} \) and \( u_m \) be the strong solution to (26). Provided that \( \kappa > 0 \) in the definition of \( \rho \) is chosen sufficiently small, there exists \( \mu(\delta, c_\delta) \geq 0 \) such that for any \( \mu \geq \mu(\delta, c_\delta) \),

\[
\lim_{n,m \to \infty} \sup_{t \in J_T} \| \mathbb{E}[u_n(t) - u_m(t)] \|_{2, \rho} = 0.
\]

**Proof.** Set

\[
g \equiv g_{n,m} := u_n - u_m, \quad n, m = 1, 2, \ldots,
\]

then

\[
g(t) + \mu \int_0^t gds + \int_0^t b_m \cdot \nabla gds + \int_0^t (b_n - b_m) \cdot \nabla u_m ds + \sigma \int_0^t \nabla g dB_s - \frac{\sigma^2}{2} \int_0^t \Delta g ds = 0.
\]

Applying Itô's formula, we obtain

\[
g^2(t) = -2 \mu \int_0^t g^2 ds - \int_0^t b_m \cdot \nabla g^2 ds - 2 \int_0^t (g(b_n - b_m)) \cdot \nabla u_m ds - \sigma \int_0^t \nabla g^2 dB_s + \frac{\sigma^2}{2} \int_0^t \Delta g^2 ds,
\]

so denoting \( h := \mathbb{E}[g^2] \) we arrive at

\[
\partial_t h + 2 \mu h - \frac{\sigma^2}{2} \Delta h + b_m \cdot \nabla h + 2 (b_n - b_m) \cdot \mathbb{E}[\nabla u_m] = 0, \quad h(0) = 0.
\]
Multiplying this equation by $\rho h$ and integrating by parts, we obtain
\[
\frac{1}{2} \|h(t)\|_{2,\rho}^2 + 2\mu \int_0^t \|h\|_{2,\rho}^2 ds + \frac{\sigma^2}{2} \int_0^t \|\nabla h\|_{2,\rho}^2 ds + \frac{\sigma^2}{2} \int_0^t \langle (\nabla \rho), \nabla h \rangle ds \quad (40)
\]
\[+ \int_0^t \langle b_m \cdot \nabla h, h \rangle \rho ds + 2 \int_0^t \langle h(b_n - b_m) \cdot \mathbb{E}[g \nabla u_m] \rangle \rho ds = 0.
\]

Since our assumption on $\delta$ is a strict inequality, using (10) and selecting $\kappa$ sufficiently small, we can and will ignore in what follows the terms containing $\nabla \rho$.

Applying the quadratic inequality and using $b_m \in \mathbb{F}_\delta$, we obtain (cf. the proof of (27))
\[
\frac{\sigma^2}{2} \int_0^t \|\nabla h\|_{2,\rho}^2 ds + \int_0^t \langle b_m \cdot \nabla h, h \rangle \rho ds \geq \left( \frac{\sigma^2}{2} - \sqrt{\delta} \right) \frac{\mu}{4\sqrt{\delta}} \int_0^t \|h\|_{2,\rho}^2 ds,
\]
where $\frac{\sigma^2}{2} - \sqrt{\delta} > 0$ by the assumption on $\delta$.

We obtain from (40):
\[
\frac{1}{2} \sup_{\tau \in [0,t]} \|h(\tau)\|_{2,\rho}^2 + \left( \frac{\sigma^2}{2} - \sqrt{\delta} \right) \frac{\mu}{4\sqrt{\delta}} \int_0^t \|h\|_{2,\rho}^2 ds \leq 2 \int_0^t \langle h|b_n - b_m| \cdot \mathbb{E}[g \nabla u_m] \rangle \rho ds.
\]

Select $\mu \geq \frac{c_\delta}{4\sqrt{\delta}}$. Then the previous estimate yields
\[
\frac{1}{2} \sup_{\tau \in [0,t]} \|h(\tau)\|_{2,\rho}^2 \leq 2 \int_0^t \langle h|b_n - b_m| \cdot \mathbb{E}[g \nabla u_m] \rangle \rho ds,
\]
so it remains to show that
\[
\int_0^t \langle h|b_n - b_m| \cdot \mathbb{E}[g \nabla u_m] \rangle \rho ds \to 0 \quad \text{as } n, m \to \infty.
\]

We estimate
\[
\langle h|b_n - b_m| \cdot \mathbb{E}[g \nabla u_m] \rangle \rho \leq \langle |b_n - b_m| h(\mathbb{E}[g]) \frac{1}{2}(\mathbb{E}[|\nabla u_m|^2])^{\frac{1}{2}} \rangle \rho \equiv \langle |b_n - b_m| h^{\frac{3}{2}}(\mathbb{E}[|\nabla u_m|^2])^{\frac{1}{2}} \rangle \rho
\]
\[\leq \langle |b_n - b_m|^2 \rangle \rho \mathbb{E}[|\nabla u_m|^2] \frac{1}{2} \rho \leq \langle |b_n - b_m|^2 \rangle \rho \langle \mathbb{E}[|\nabla u_m|^2] \rangle \rho \frac{1}{2}
\]
\[\leq \langle |b_n - b_m|^2 \rangle \rho \left( \mathbb{E}[|\nabla u_m|^2] \right) \frac{1}{2}
\]
\[
\text{(we apply Proposition 2, and (38) with } q = 1)
\]
\[\leq c \langle |b_n - b_m|^2 \rangle \rho \|f\|^2 \left\| \nabla f \right\|_4
\]
\[
\text{(we apply Lemma 1(iii))}
\]
\[\to 0 \quad \text{as } n, m \to \infty.
\]

The proof of Lemma 2 is complete. \qed

Lemma 2 allows to prove that $\{u_m\}$ is a Cauchy sequence in $L^\infty(J_T, L^2(\Omega, L^2_\rho))$.

**Lemma 3.** Let $\beta_2 \sqrt{\delta} < \frac{\sigma^2}{2}$, $f \in C_c^\infty$ and $u_m$ be the strong solution to (26). Provided that $\kappa > 0$ in the definition of $\rho$ is chosen sufficiently small, it holds that $u_m$ converges in $L^2(\Omega, L^2_\rho)$ to a process $u$, uniformly in $t \in J_T$. 


Proof. Let \( \kappa \) be small enough and \( \mu \geq \mu(\delta, c_\delta) \). Then by Lemma 2,

\[
\sup_{t \in J_T} \mathbb{E} \| (u_n(t) - u_m(t)) \|_{2, \rho}^2 \leq \langle \rho \rangle^{\frac{1}{2}} \sup_{t \in J_T} \mathbb{E} \| u_n(t) - u_m(t) \|_{2, \rho}^2 \to 0
\]

as \( m, n \to \infty \). Thus, we can define

\[
u(t) := \lim_{m \to \infty} u_m(t) \quad \text{uniformly in } t \in J_T.
\]

The proof is complete.

We are in position to give the proof of Theorem 1.

Proof of Theorem 1. It suffices to carry out the proof for \( f \in C_c^\infty \), and then use a density argument.

It follows from the assumption \( \sqrt{\delta} < \frac{a^2}{2\beta} \) that \( p \geq 2 \) is in the interval \((p_c, \infty)\), \( p_c = \left(1 - \frac{\sqrt{\delta}}{a^2} \right)^{-1} \). (Indeed, \( p_c < 2 \) if and only if \( \sqrt{\delta} < \frac{a^2}{2\beta} \). In particular, \( p_c < 2 \) if \( \sqrt{\delta} < \frac{a^2}{2\beta} \) since \( \beta > 1 \).) Let \( \mu(\delta, c_\delta, p) \) be the constant from Proposition 2. Assume that \( \mu \geq \mu(\delta, c_\delta, p) \). Then the conclusions of Proposition 2 are valid.

We prove (i) first. We do this in two steps.

Step 1. Selecting \( \kappa \) sufficiently small so that Lemma 3 applies, we obtain that \( u_m \) converges in \( L^2(\Omega, L^2_\rho) \) to a process \( u \), uniformly in \( t \in J_T \). Thus \( u \in L^\infty(J_T, L^2_\rho(\mathbb{R}^d, L^2(\Omega))) \), and we have for all \( t \in J_T \),

\[
u_m \to u \quad \text{in } L^\infty(J_T, L^2(\Omega, L^2_\rho)),
\]

which yields

\[
\int_0^t u_m ds \to \int_0^t u ds \quad \text{in } L^2(\Omega, L^2_\rho);
\]

the latter, (28) and a standard weak compactness argument yield

\[
\nabla \int_0^t u_m ds \to \nabla \int_0^t u ds \quad \text{weakly in } L^2(\Omega, L^2(\mathbb{R}^d, \mathbb{R}^d)).
\]

Step 2. Given a test function \( \varphi \in C_c^\infty \), we multiply (26) by \( \rho \varphi \), integrate and write

\[
\langle u_m(t) - u(t), \rho \varphi \rangle + \langle u(t), \rho \varphi \rangle - \langle f, \rho \varphi \rangle = -\mu \int_0^t \langle u_m - u, \rho \varphi \rangle ds + \mu \int_0^t \langle u, \rho \varphi \rangle
\]

\[
- \langle (b_m - b) \cdot \nabla \int_0^t u_m ds, \rho \varphi \rangle - \langle b \cdot \nabla \int_0^t u_m ds, \rho \varphi \rangle
\]

\[
+ \sigma \int_0^t (u_m - u) dB_s, \nabla (\rho \varphi) \rangle + \sigma \int_0^t u dB_s, \nabla (\rho \varphi) \rangle \]

\[
- \sigma^2 \int_0^t (u_m - u) ds, \nabla (\rho \varphi) \rangle - \sigma^2 \int_0^t u ds, \nabla (\rho \varphi) \rangle.
\]

Let us now note the following. In view of (41),

\[
\mu \int_0^t \langle u_m - u, \rho \varphi \rangle ds \to 0, \quad \langle u_m(t) - u(t), \rho \varphi \rangle \equiv \langle u_m(t) - u(t), \varphi \rangle \to 0 \text{ in } L^2(\Omega).
\]
Similarly, using (43) and (10),
\[
\langle \nabla \int_0^t (u_m - u) ds, \nabla \varphi \rangle \to 0 \text{ weakly in } L^2(\Omega),
\]  
and, since \( \varphi b \in L^2_\rho \) (using that \( \varphi \) has compact support),
\[
\langle b \cdot \nabla \int_0^t u_m ds, \rho \varphi \rangle \to \langle b \cdot \nabla \int_0^t u ds, \rho \varphi \rangle \text{ weakly in } L^2(\Omega) .
\]  
By (28), \( \| \nabla \int_0^t u_m ds \|_{L^2(\Omega; L^2_\rho)} \leq c_1 \) with \( c_1 < \infty \) independent of \( m \), and \( \varphi|b_m - b_n| \to 0 \) in \( L^2_\rho \) (in fact, in \( L^2 \)). Thus
\[
\langle (b_m - b) \cdot \nabla \int_0^t u_m ds, \rho \varphi \rangle \to 0 \text{ in } L^2(\Omega).
\]  
Finally, let us show that
\[
\langle \int_0^t (u_m - u) dB_s, \nabla \varphi \rangle \to 0 \text{ in } L^2(\Omega).
\]  
Indeed, using Itô’s isometry, we have using (10)
\[
E \langle \int_0^t (u_m - u) dB_s, \nabla \varphi \rangle ^2 \leq c_2 E \langle \int_0^t (u_m - u) dB_s \|_{L^2_\rho} \rangle ^2 \langle |\varphi|^2 \rangle_{\rho} \]
\[
= c_3 \langle E \int_0^t (u_m - u)^2 ds \|_{\rho} \rangle \to 0 \quad \text{by (41)}.
\]  
The convergence (49) follows.

Thus, using (45)-(49), we can pass to the \( L^2(\Omega) \)-weak limit in (44) as \( m \to \infty \), obtaining that \( u \) satisfies (19) (with test functions \( \varphi \rho \) which, clearly, exhaust \( C_\infty^c \)).

The estimates in (17), (18) now follow from Proposition 2.

The last assertion (ii) is Lemma 3 proved above.

The proof of Theorem 1 is complete. \( \Box \)

**Remark 4.** In the notation of the proof of Theorem 1, let \( \tilde{u}_m := e^{\mu t} u_m \). Then, at least if the initial function \( f \) is sufficiently regular, \( \tilde{u}_m \) is a strong solution to
\[
d\tilde{u}_m + b_m \cdot \nabla \tilde{u}_m dt + \sigma \nabla \tilde{u}_m \circ dB_t = 0 \quad \text{on } (0, \infty) \times \mathbb{R}^d ,
\]
\[
\tilde{u}_m|_{t=0} = f
\]
(see the discussion at the beginning of Section 2). Since \( \tilde{u}_m \) and \( u_m \) differ only by factor \( e^{\mu t} \), \( t \in J_T \), the convergence results for \( u_m, \nabla u_m \) established in the proof of Theorem 1 yield the corresponding convergence results for \( \tilde{u}_m, \nabla \tilde{u}_m \). In particular, \( \tilde{u}_m \to \tilde{u} := e^{\mu t} u \) in \( L^\infty(J_T, L^2(\Omega, L^2_\rho)) \), and \( \tilde{u} \) satisfies: for a.e. \( \omega \in \Omega , \)
\[
b \cdot \nabla \int_{J_T} \tilde{u}(s, \cdot, \omega) ds \in L^1_{\text{loc}},
\]
and, for every test function \( \varphi \in C_\infty^c \), one has a.s. for all \( t \in J_T, \)
\[
\langle \tilde{u}(t), \varphi \rangle - \langle f, \varphi \rangle + \langle b \cdot \nabla \int_0^t \tilde{u} ds, \varphi \rangle - \sigma \langle \int_0^t \tilde{u} dB_s, \nabla \varphi \rangle + \frac{\sigma^2}{2} \langle \nabla \int_0^t \tilde{u} ds, \nabla \varphi \rangle = 0.
\]
6. Proof of Theorem 2

Proof of Theorem 2. Part (a) follows from Theorem 1(i). The last assertion, (21), follows from Proposition 3 and Lemma 3. So we only need to prove part (b).

Since the weak-$L^2(J_T \times \Omega)$ limit of any sequence of $(\mathcal{F}_t)$-progressively measurable processes on $J_T$ remains $(\mathcal{F}_t)$-progressively measurable and $t \mapsto \langle u_m(t), \varphi \rangle$ is $(\mathcal{F}_t)$-progressively measurable for every $m$, in view of (42), the process $t \mapsto \langle u(t), \varphi \rangle$ is $(\mathcal{F}_t)$-progressively measurable as well. The proof of (20) follows closely the proof of (19) above except that now, instead of (28), we appeal to the Sobolev regularity estimate (22) with $q = 1$.

The existence of a continuous $(\mathcal{F}_t)$-semi-martingale modification of $t \mapsto \langle u(t), \varphi \rangle$ is a consequence of the identity (20).

The proof of Theorem 2 is complete.

7. Proof of Theorem 3 (Weak uniqueness)

The fact that (14) has at least one weak solution was proved in Theorem 2. We now prove its uniqueness. We adopt the argument of [BFGM, Sect. 3]. We will need the following definitions and results. Let us fix a version of the Brownian motion $(B_t)$ having continuous trajectories $B_t(\omega)$ for every $\omega \in \Omega$.

Lemma 4. Let $b \in \mathbf{F}_\delta$ with $\sqrt{\delta} < \frac{\sigma^2}{2\beta}$ and $f \in W^{1,4}$. Let $u = u(t, x, \omega)$ be a weak solution to (14). Then for a.e. $\omega \in \Omega$,

$$\tilde{u}^\omega(t, x) := u(t, x + \sigma B_t(\omega), \omega)$$

is a weak solution to the Cauchy problem

$$\partial_t \tilde{u}^\omega + \mu \tilde{u}^\omega + \tilde{b}^\omega \cdot \nabla \tilde{u}^\omega = 0, \quad \tilde{u}^\omega|_{t=0} = f, \quad \text{where } \tilde{b}^\omega(t, x) := b(x + \sigma B_t(\omega)),$$

that is, the following are true:

1) $\tilde{u}^\omega \in L^\infty(J_T, W^{1,2}_0)$;

2) for every $\psi \in C^1(J_T, C^\infty_c)$, the function $t \mapsto \langle \tilde{u}^\omega(t), \psi(t) \rangle$ has a continuous representative, i.e. a continuous function which coincides with $t \mapsto \langle \tilde{u}^\omega(t), \psi(t) \rangle$ for a.e. $t \in J_T$;

3) for every $\psi \in C^1(J_T, C^\infty_c)$, this continuous representative of $t \mapsto \langle \tilde{u}^\omega(t), \psi(t) \rangle$ satisfies for every $t \in J_T$,

$$\langle \tilde{u}^\omega(t), \psi(t) \rangle = \langle f, \psi(0) \rangle + \mu \int_0^t \langle \tilde{u}^\omega(s), \psi(s) \rangle ds + \int_0^t \langle \tilde{u}^\omega(s), \partial_s \psi(s) \rangle ds - \int_0^t \langle \nabla \tilde{u}^\omega(s), \tilde{b}^\omega(s) \psi(s) \rangle ds.$$

The proof of Lemma 4 follows closely the proof of [BFGM, Prop.3.4] (taking into account the definition of the weak solution to (14)) and we omit the details.

Consider the terminal value problem

$$dv_m + \mu v_m dt + \nabla \cdot (b_m v_m) dt + \sigma \nabla v_m \circ dB_t = 0, \quad t \in [0, t_*], \quad v_m|_{t=t_*} = v_0 \in C^\infty_c, \quad (50)$$

where $b_m \in C^\infty_c(\mathbb{R}^d, \mathbb{R}^d)$ ($m = 1, 2, \ldots$) (since $b_m$ are bounded and smooth, we have strong existence and uniqueness for this equation).

The following is an analogue of [BFGM, Cor.3.8].
Lemma 5. \( \tilde{v}_m^\omega(t,x) := v_m(t,x + \sigma B_t(\omega)) \) satisfies, for a.e. \( \omega \in \Omega \), \( \tilde{v}_m^\omega \in C^1([0,t_*],C_c^\infty) \) and 
\[
\partial_t \tilde{v}_m^\omega + \mu \tilde{v}_m^\omega + \nabla \cdot (b_m^\omega \tilde{v}_m^\omega) = 0, \quad \tilde{v}_m^\omega(t_*,x) = v_0(x + \sigma B_{t_*}(\omega)).
\]

We will also need

Lemma 6. Let \( \sqrt{\delta} < \frac{\sigma_\omega^2}{6} \). There exist a constant \( \mu(c_\delta) \geq 0 \) and a sufficiently small \( \kappa > 0 \) (in the definition of \( \rho \)) such that 
\[
\sup_{t \in J_T} \|\rho^{-1}E[v_m^2(t)]\|_2 \leq \|\rho^{-1}v_0\|_4^2, \quad \mu \geq \mu(c_\delta), m = 1, 2, \ldots
\]
where \( v_m \) is the strong solution to (50).

Proof. Without loss of generality, we will carry out the proof for the forward equation, and will drop the subscript \( m \) from \( b_m \). Set \( w := E[v_m^2] \). Arguing as in the proof of Proposition 2, we obtain that \( w \) satisfies
\[
\partial_t w + 2\mu w - \frac{\sigma_\omega^2}{2} \Delta w - 2\nabla \cdot (bw) + b \cdot \nabla w = 0, \quad w(0) = v_0^2.
\]

We first carry out the proof for \( \rho \equiv 1 \). Multiplying the previous equation by \( w \) and integrating, we obtain
\[
\frac{1}{2} \partial_t \langle |w|^2 \rangle + 2\mu \langle |w|^2 \rangle + \frac{\sigma_\omega^2}{2} \langle |\nabla w|^2 \rangle + 3 \langle \nabla w, bw \rangle = 0.
\]

Applying the quadratic inequality and the form-boundedness condition \( b \in F_\delta \), we get that, for any \( \gamma > 0 \),
\[
\frac{1}{2} \partial_t \langle |w|^2 \rangle + (2\mu - 3\gamma c_\delta) \langle |w|^2 \rangle + \left[ \frac{\sigma_\omega^2}{2} - 3(\gamma \delta + \frac{1}{4\gamma}) \right] \langle |\nabla w|^2 \rangle \leq 0,
\]
and so, selecting \( \mu(c_\delta) := \frac{3}{2} \gamma c_\delta \) and \( \mu \geq \mu(c_\delta) \), we obtain
\[
\frac{1}{2} \langle |w(t)|^2 \rangle + \left[ \frac{\sigma_\omega^2}{2} - 3(\gamma \delta + \frac{1}{4\gamma}) \right] \int_0^t \langle |\nabla w|^2 \rangle ds \leq \frac{1}{2} \langle |w_0|^4 \rangle.
\]

Upon maximizing the coefficient in the square brackets in \( \gamma \) (thus, selecting \( \gamma = \frac{1}{2\sqrt{\delta}} \)), we obtain that the coefficient is positive since \( \sqrt{\delta} < \frac{\sigma_\omega^2}{6} \). In particular, it follows that \( \sup_{t \in J_T} \|E[v_m^2(t)]\|_2 \leq \|v_0\|_4^2 \).

In presence of \( \rho^{-1} \), we argue as above but get new terms containing \( \nabla \rho^{-1} \), which we bound appealing to the estimate
\[
|\nabla \rho^{-1}| = \left| \frac{\nabla \rho}{\rho^2} \right| \leq \theta \sqrt{\kappa} \rho^{-1} \quad \text{(by (10))},
\]
with \( \kappa \) selected sufficiently small. Note that to justify \( \|\rho^{-1}E[v_m^2(t)]\|_2 < \infty \) we can appeal to qualitative Gaussian upper bound on the heat kernel of (51).

Let us note that the assumption of the theorem \( \beta_2 \sqrt{\delta} < \frac{\sigma_\omega^2}{2} \) implies \( \sqrt{\delta} < \frac{\sigma_\omega^2}{6} \).

We are now in position to complete the proof of Theorem 3.

Proof of Theorem 3. Let \( \mu \) and \( \kappa \) be as in Lemma 6. In view of the linearity of the stochastic transport equation, it suffices to show that a weak solution \( u \) to (14) with initial condition \( u(0) = 0 \) must be identically zero for all \( t \in J_T \). In view of Lemma 4, it suffices to show that \( \tilde{u}^\omega \) corresponding to \( u \) is identically zero a.s.
Let \( v_0 \in C_c^\infty \). It follows from Lemma 5 that, for a.e. \( \omega \in \Omega \), \( \tilde{\omega}(s) \in C^1(J_T, C_c^\infty) \). Thus by Lemma 4, for a.e. \( \omega \in \Omega \) with \( \psi(s) := \tilde{\omega}(s) \), for all \( 0 < t_s \leq T \),
\[
\langle \tilde{u}^\omega(t_s), v_0(\cdot + \sigma B_t(\omega)) \rangle \\
= \mu \int_0^{t_s} \langle \tilde{u}^\omega(s), \bar{\omega}^\omega_m(s) \rangle ds + \int_0^{t_s} \langle \bar{\omega}^\omega(s), \partial_s \bar{\omega}^\omega_m(s) \rangle ds - \int_0^{t_s} \langle \nabla \bar{u}^\omega(s), \bar{b}^\omega(s) \bar{\omega}^\omega_m(s) \rangle ds \\
= \int_0^{t_s} \langle \nabla \bar{u}^\omega, (\bar{b}^\omega_m(s) - \bar{b}^\omega(s)) \bar{\omega}^\omega_m \rangle ds =: I.
\]

Step 1. Let us first show that for every \( \varepsilon \)
\[
\mathbb{E} \left| \int_0^{t_s} \langle \nabla u, (b - b_m)v_m \rangle ds \right| \to 0 \quad \text{as } m \uparrow \infty. 
\]

We have
\[
\mathbb{E} \left| \int_0^{t_s} \langle \nabla u, (b - b_m)v_m \rangle ds \right| \leq \int_0^{t_s} \langle |b - b_m| \mathbb{E}[|\nabla u|^2]^{\frac{1}{2}} \mathbb{E}[|v_m|^2]^{\frac{1}{2}} \rangle ds \leq \left( \int_0^{t_s} \langle \rho |b - b_m|^2 \rangle ds \right)^{\frac{1}{2}} \left( \int_0^{t_s} \langle (\mathbb{E}[|\nabla u|^2])^2 \rangle ds \right)^{\frac{1}{2}} \left( \int_0^{t_s} \langle \rho^{-2}(\mathbb{E}[|v_m|^2])^2 \rangle ds \right)^{\frac{1}{2}}.
\]

The first integral converges to 0 as \( m \uparrow \infty \) by Lemma 1(iii), the second integral is finite by the definition of weak solution before Theorem 3, and the third integral is bounded from above uniformly in \( m \) by \( \sqrt{t_s} \| \rho^{-1} v_0 \|_2^2 < \infty \), see Lemma 6. Thus, (53) follows.

Step 2. By Step 1, there exists a subset \( \Omega_{t_s,v_0} \subset \Omega \) of probability 1 and a sequence \( m_k \uparrow \infty \) such that for every \( \omega \in \Omega_{t_s,v_0} \),
\[
\int_0^{t_s} \langle \nabla u, (b - b_{m_k})v_{m_k} \rangle ds \to 0 \quad \text{as } m_k \uparrow \infty.
\]

Making the change of variable \( x \mapsto x + \sigma B_t(\omega) \) and using the fact that \( c_{t_s,w}^{-1} \rho(\cdot) \leq \rho(\cdot + \sigma B_t(\omega)) \leq c_{t_s,w} \rho(\cdot) \) for some constant \( c_{t_s,w} > 1 \) we obtain that for every \( \omega \in \Omega_{t_s,v_0} \),
\[
I \to 0 \quad \text{as } m_k \uparrow \infty.
\]

Fix a countable dense subset \( D \) of \( C_c^\infty(\mathbb{R}^d) \) and define
\[
\bar{\Omega} := \bigcap_{t_s \in [0,T] \cap \mathbb{Q}, v_0 \in D} \Omega_{t_s,v_0},
\]
a full measure set in \( \Omega \). Applying the diagonal argument and so passing to a subsequence of \( \{ \varepsilon_k \} \), we obtain by (52) and Step 2 that for every \( \omega \in \bar{\Omega} \), \( \bar{\omega}(t) = 0 \) for all \( t \in [0,T] \cap \mathbb{Q} \). Since \( t \mapsto \langle \tilde{u}^\omega(t), \varphi \rangle \), \( \varphi \in C_c^\infty(\mathbb{R}^d) \) is continuous, we obtain that \( \tilde{u}^\omega(t) = 0 \) for all \( t \in [0,T] \) for all \( \omega \in \bar{\Omega} \), as needed.

The proof of Theorem 3 is complete.

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