Thermal model at RHIC, part II: elliptic flow and HBT radii

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Abstract. We continue the analysis of the preceding talk with a discussion of the elliptic flow and the Hanbury-Brown–Twiss pion correlation radii. It is shown that the thermal model can be extended to describe these phenomena. The description of the elliptic flow involves an appropriate deformation of the freeze-out hyper-surface and flow velocity. The obtained results agree reasonably with the data for soft (< 2 GeV) transverse momenta. For the pionic HBT correlation radii the experimental feature that $R_{\text{out}}/R_{\text{side}} \approx 1$ is naturally obtained. The reproduction of individual $R_{\text{side}}$ and $R_{\text{out}}$ can be achieved with the inclusion of the excluded volume corrections, which effectively increase the radii by $\sim 30\%$.

INTRODUCTION

In the preceding talk [1], from now on referred to as (I), it has been shown that the thermal approach is successful in the description of the particle ratios and $p_\perp$-spectra at RHIC. Here we continue our investigation, studying azimuthal asymmetry of the spectra and the pionic Hanbury-Brown–Twiss correlation radii.

ELLIPTIC FLOW

When the nuclei collide at non-zero impact parameter, $b \neq 0$, the momentum distribution of the produced particles carries azimuthal asymmetry. In general, at mid-rapidity ($y = 0$) we may write the following Fourier decomposition in the azimuthal angle $\phi$, measured from the reaction plane:

$$\left. \frac{dN}{d^2p_\perp dy} \right|_{y=0} = \frac{dN}{2\pi p_\perp dp_\perp dy} \left|_{y=0} \right. \left( 1 + 2v_2 \cos 2\phi + 2v_4 \cos 4\phi + \ldots \right). \quad (1)$$

The sines are absent due to the symmetry condition $\phi \rightarrow -\phi$, which is simply the reflexion with respect to the reaction plane, while the coefficients of cosines with odd multiples of $\phi$ vanish for the case of symmetric nuclei and at $y = 0$, when the symmetry

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\[ \phi \rightarrow \pi - \phi \] holds. The elliptic-flow coefficient, \( v_2 \), can therefore be computed as

\[ v_2 = \frac{\int_0^{2\pi} \frac{dN}{d^2p_\perp dy}|_{y=0} \cos 2\phi \, d\phi}{\int_0^{2\pi} \frac{dN}{d^2p_\perp dy}|_{y=0} \, d\phi}. \] (2)

The value of \( v_2 \) is an important signature of the physics occurring in heavy-ion collisions. Most importantly, its non-vanishing value indicates that the production mechanism is not a simple composition of nucleon-nucleon collisions, since in that case the asymmetry of production in each such collision would average out practically to zero. Thus, interactions with other particles (rescattering, asymmetric collective flow, ...) are necessary to generate non-vanishing \( v_2 \). In hydrodynamical approaches the elliptic flow has been analyzed in many papers, see e.g. [2, 3, 4, 5, 6]). The coefficient \( v_2 \) depends on the impact parameter, \( b \), on the transverse momentum \( p_\perp \), as well as, obviously, on the type of the considered particle. All these dependences are measured at RHIC. The impact parameter, \( b \), is traded for the experimentally more useful centrality parameter, \( c \), which to a very good accuracy is given by [7]

\[ c \simeq \frac{b^2}{(2R)^2}. \] (3)

There are two basic empirical facts from RHIC which we will use in our approach. Firstly, \( v_2 \) is positive [8, 9, 10, 11, 12, 13, 14, 15, 16, 17], which means that the collective flow is faster in the reaction plane than out-of-plane. Secondly, the measurement [18] of the azimuthal dependence of the \( R \) side pion HBT radius shows, that the shape of the system at freeze-out is elongated out of the reaction plane. We will now use these two facts in our choice of the parameterization of the hypersurface at freeze-out. A natural extension of Eq. (I.6) is to introduce the azimuthal shape asymmetry,

\[ r_x = \rho_{\text{max}}\sqrt{1 - \varepsilon \cos \phi}, \]
\[ r_y = \rho_{\text{max}}\sqrt{1 + \varepsilon \sin \phi}, \] (4)

with \( r_z \) and \( t \) kept as in the symmetric case of Eq. (I.6) of paper (I). Our convention is that \( r_x \) lies in the reaction plane, and \( r_y \) is perpendicular to the reaction plane. For positive \( \varepsilon \) this produces elongation out of the reaction plane, as seen in the experiment. The four-velocity of Eq. (I.3) is modified as follows:

\[ u_x = \frac{1}{N} r_x \sqrt{1 + \delta \cos \phi}, \]
\[ u_y = \frac{1}{N} r_y \sqrt{1 - \delta \sin \phi}, \]
\[ u_z = \frac{1}{N} r_z, \]
\[ u_t = \frac{1}{N} t. \] (5)
FIGURE 1. The model fit of the pion, kaon, and proton spectra to the PHENIX data for $\sqrt{s_{NN}} = 130$ GeV \cite{15} at three centrality bins, arranged top to bottom. Negative (positive) hadron are shown in the left (right) side. The optimum values of the size parameters $\tau$ and $\rho_{\text{max}}$ are given for each centrality bin.

The normalization $N$ is such that $u\mu u\mu = 1$. Positive $\delta$ means faster flow in the reaction plane, which corresponds to positive $v_2$. Certainly, the choice (4,5) is by no means unique, but it grasps the essential empirical features.

The modified expansion model has four geometric parameters: $\tau$, $\rho_{\text{max}}$, $\epsilon$ and $\delta$. These parameters depend on the centrality parameter, $c$. Fortunately, the effect of $\epsilon$ and $\delta$ on the $\phi$-averaged spectra, \[ dN/(2\pi p_\perp dp_\perp dy) \big|_{y=0}, \] is negligible and enters at the level of a few percent. Thus we may first fit $\tau$ and $\rho_{\text{max}}$ to the $\phi$-averaged $p_\perp$-spectra at various centrality parameters, assuming for the moment vanishing $\epsilon$ and $\delta$. The result is shown in Fig. 1. We note that the fit works as good as for the most-central case presented in (1). The optimum values of parameters are collected in Table 1 where they are also compared to the minimum-bias fit and the joint fit to the most central PHENIX \cite{15} and STAR \cite{19} data. In fact, the qualitative dependence of $\tau$ and $\rho_{\text{max}}$ on $c$ is as expected: the larger $c$, i.e. the more peripheral collision, the smaller values of the size parameters. Figure 2 visualizes this dependence. In Table 1 we also list the ratio of $\rho_{\text{max}}/\tau$, and the maximum and average values of the flow parameter, $\beta$. Interestingly, these quantities
FIGURE 2. The dependence of the size parameters $\tau$ and $\rho_{\text{max}}$ on the centrality parameter $c$, as fitted to the PHENIX data on $\phi$-integrated $p_{\perp}$-spectra at $\sqrt{s_{NN}} = 130$ GeV [15].

depend very weakly on $c$. They are defined as follows:

$$\beta_{\perp}^{\text{max}} = \frac{\rho_{\text{max}}}{\sqrt{\tau^2 + \rho_{\text{max}}^2}},$$

$$\langle \beta_{\perp} \rangle = \frac{\int_0^{\rho_{\text{max}}} rdr \frac{r}{\sqrt{\tau^2 + r^2}}}{\int_0^{\rho_{\text{max}}} rdr}. \quad (6)$$

Ideally, the dependence of the $\varepsilon$ parameter on $c$ should come from the measurement of $R_{\text{side}}$ at various centralities. Then the model evaluation of this quantity would allow to fit independently $\varepsilon(c)$ to the data. We hope to be able to proceed in such a manner in the future. Unfortunately, no necessary experimental results are available at the moment. In this circumstance we take a reasonable theoretical estimate for $\varepsilon(c)$ based on Ref. [4], which leads to $\varepsilon = 0.1, 0.21,$ and $0.35$ in the centrality bins $0 - 15\%$, $15 - 30\%$,

| TABLE 1. | Comparison of the size parameters obtained by fitting particle spectra at various values of centrality parameter $c$. Their ratio, as well as the maximum and average value of the flow parameter, $\bar{\beta}_{\perp}$, are also given. |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| $c$ [%] | PHENIX @130GeV | PHENIX+STAR @130GeV |
| $\tau$ [fm] | $\rho_{\text{max}}$ [fm] | $\rho_{\text{max}}/\tau$ | $\beta_{\perp}^{\text{max}}$ | $\langle \beta_{\perp} \rangle$ |
| min. bias | 0-5 | 15-30 | 60-92 | 0-5/0-6 |
| 5.6 | 8.2 | 6.3 | 2.3 | 7.7 |
| 4.5 | 6.9 | 5.3 | 2.0 | 6.7 |
| 0.81 | 0.84 | 0.84 | 0.87 | 0.87 |
| 0.62 | 0.64 | 0.64 | 0.66 | 0.66 |
| 0.46 | 0.47 | 0.47 | 0.48 | 0.48 |
and 30 – 60%, respectively. Finally, the parameter $\delta$ is obtained by fitting the model predictions to the $v_2$ measurements. Our approach includes, as described in detail in (I), the decays of resonances. The calculation is straightforward and very similar to the one discussed in (I), although takes a much longer computer time due to a lower degree of symmetry. The technicalities will be presented elsewhere. The simplified results for $v_2$ presented here include all resonances up to $m_\Delta = 1.232$ GeV, and do not take into account three-body decays.

Figure 3 shows the result of our calculation for three different centrality bins. The elliptic flow coefficient grows with the momentum. In continues to grow for large momenta, where saturation is seen in the experiment, however the thermal model cannot be trusted at momenta larger than about, say, 2 GeV, where hard dynamics is important. We observe that the effects of resonance decays, large in both the numerator and denominator of Eq. (2), cancel to a large degree in the ratio, and the net effect in $v_2$ is small. The dependence of $v_2$ on centrality for various transverse-momentum bins is show in

**FIGURE 3.** Dependence of $v_2$ (accumulated from all particle species) on the perpendicular momentum, $p_\perp$, for three centrality bins: 0 – 15% (bottom), 15 – 30% (middle), and 30 – 60% (top). Thick (thin) lines correspond to the calculation with (without) resonance decays. Experimental points come from the PHENIX collaboration at $\sqrt{s_{NN}} = 130$ GeV [17]. The taken values for the shape-asymmetry parameter $\varepsilon$ are, from the lowest to highest centrality bin, 0.1, 0.21, and 0.35, while the fitted values of the flow-asymmetry parameter $\delta$ are 0.145, 0.34, and 0.35, respectively.

Fig. 4 We note that our calculation works well except large $c$ and $p_\perp$, i.e. except for high-momentum particles from most peripheral collisions.

In Fig. 5 we compare the model predictions for $v_2$ integrated over $c$ for identified particles: pions, kaons, and protons. The general characteristics, with lighter particles having larger $v_2$, are reproduced. The agreement is not satisfactory only for the case of protons at lowest centrality, where the data is compatible with zero.

To summarize this part we note that

1. Elliptic flow can be introduced in the thermal approach by suitably modifying the freeze-out hypersurface.
2. The shape-deformation parameter, $\varepsilon(c)$, should and hopefully will be taken independently from future data on the azimuthal asymmetry of the $R_{\text{side}}$ HBT correlation radius at various centralities.

3. The velocity-asymmetry parameter, $\delta(c)$, can be fitted to reproduce $v_2$. Predictions for identified-particle $v_2$ follow and are reasonable.

4. The $p_\perp$-dependence shows a monotonic growth, which agrees with the data at lower values of $p_\perp$, but certainly fails to produce saturation at large momenta, where the thermal approach is not applicable.

5. Resonance decays do not have a very large effect on $v_2$.

**HBT RADIIs**

Now we pass to the description of the pionic Hanbury-Brown–Twiss correlation radii (for a review of the problem see, e.g., [21]). The studied object is the two-particle correlation function for identical particles, in the present case $\pi^+\pi^+$ or $\pi^-\pi^-$. It is given by

$$C(\vec{q}, \vec{P}) = \frac{\left\{n_{\vec{p}_1} n_{\vec{p}_2}\right\}}{\left\{n_{\vec{p}_1}\right\}\left\{n_{\vec{p}_2}\right\}},$$  

(7)
where \{\} denotes averaging over events, \(p_1\) and \(p_2\) are the momenta of the pions, \(\vec{q} = \vec{p}_2 - \vec{p}_1\), and \(\vec{P} = \vec{p}_1 + \vec{p}_2\). We use the Bertch-Pratt parameterization [21, 22, 23],

\[
C(\vec{q}, \vec{P}) = 1 + \lambda e^{- \left( q_{\text{out}}^2 R_{\text{out}}^2 + q_{\text{side}}^2 R_{\text{side}}^2 + q_{\text{long}}^2 R_{\text{long}}^2 + 2 q_{\text{out}} q_{\text{long}} R_{\text{out}} R_{\text{long}} \right)}.
\]

First, let us briefly recall the experimental highlights. Two facts came as a great surprise with the RHIC data. First, the \(R_{\text{side}}\) and \(R_{\text{out}}\) radii practically do not depend on the collision energy [8, 19], and acquire similar values from AGS to RHIC, despite the increase of the energy by almost two orders of magnitude. Secondly, the ratio of \(R_{\text{out}}\) to \(R_{\text{side}}\), which can be interpreted as a measure of the duration time of the freeze-out, is close or even less than one. This is in contradiction to anticipations from numerous hydrodynamic simulations, which had predicted \(R_{\text{out}}/R_{\text{side}}\) significantly larger than one.
Our model evaluation of the correlation function is performed according to the formalism of Ref. [24], with a technical approximation of neglecting the finite life-time of resonances. In that case

\[ C(\vec{q}, \vec{P}) = 1 + \frac{\left| \int d\Sigma(x) \cdot u(x) e^{iq \cdot x} S(P \cdot u(x)) \right|^2}{\int d\Sigma \cdot u S((P + \frac{q}{2}) \cdot u(x)) \int d\Sigma \cdot u S((P - \frac{q}{2}) \cdot u(x))}, \]

(9)

where the source function is

\[ S(p \cdot u) = \frac{1}{(2\pi)^3} e^{-(p-u-\mu)/T} + \text{contribution from resonances}. \]

(10)

As discussed in Ref. [25], our model values for the geometric parameters \( \tau \) and \( \rho_{\text{max}} \) are low, of the order of the size of the colliding nuclei. As a result, the values of the \( R_{\text{side}} \) and \( R_{\text{out}} \) HBT radii obtained with the procedure described above are about 30\% too low compared to the experiment. The problem can be alleviated with the inclusion of the excluded-volume (Van der Waals) corrections. Such effects have been realized to be important since the early thermal studies of the particle production in relativistic heavy-ion collisions [26, 27, 28, 29], where they led to a significant dilution of the system.

In the case of the Boltzmann statistics, which is a very good approximation in our case [30], the excluded volume corrections bring in a factor [31]

\[ e^{-P \nu_i/T} \frac{1}{1 + \sum_j \nu_j e^{-P \nu_j/T} n_j}, \]

(11)

into the phase-space integrals, where \( P \) is the pressure, \( \nu_i = 4 \frac{4}{3} \pi r_i^3 \) is the excluded volume for the particle of species \( i \), and \( n_i \) is the density of particles of species \( i \). The pressure can be calculated self-consistently from the equation

\[ P = \sum_i P_i^0(T, \mu_i - P \nu_i/T) = \sum_i P_i^0(T, \mu_i) e^{-P \nu_i/T}, \]

(12)

where \( P_i^0 \) denotes the partial pressure of the ideal gas of hadrons of species \( i \). With the simplest assumption that the excluded volumes for all particles are equal, \( r_i = r \), the excluded-volume correction manifests itself as a common scale factor, which we may denote by \( S^{-3} \). The Frye-Cooper formula can then be written in the form [31, 32]

\[ \frac{dN_i}{d^2p_\perp dy} = \tau^3 \int_{-\infty}^{+\infty} d\alpha_\perp \int_0^{P_{\text{max}}/\tau} \sinh \alpha_\perp d \left( \sinh \alpha_\perp \right) \times \int_0^{2\pi} d\xi \ p \cdot u S^{-3} f_i(p \cdot u), \]

(13)

where \( p \cdot u = m_\perp \cosh \alpha_\parallel \cosh \alpha_\perp - p_\perp \cos \xi \sinh \alpha_\perp \). As can be immediately seen from this expression, the presence of the factor \( S^{-3} \) in Eq. (13) may be compensated by rescaling \( \rho \) and \( \tau \) by the factor \( S \). That way the system becomes more dilute and larger in such a way, that the particle multiplicities and the spectra are left intact.
FIGURE 6. Model predictions for the pionic $R_{\text{side}}$ and $R_{\text{out}}$ HBT correlation radii (top two panels), and their ratio (bottom panel), confronted with the PHENIX data $Au + Au$ data at $\sqrt{s_{NN}} = 130$ GeV and average centrality 10%. The quantity $k_{\perp}$ is the total momentum of the pion pair.

With our values of the thermodynamic parameters $\sum_i P_i^0(T, \mu_i) = 80$ MeV/fm$^3$, and we find $S = 1.3$ with $r = 0.6$ fm. Such a value of the excluded volume is compatible with values typically obtained in other calculations. The increase of the size parameters by 30% is what we need to bring the values of $R_{\text{side}}$ and $R_{\text{out}}$ up to the experimental ball park.

Our results are shown in Fig. 6. We note that the agreement with data is very reasonable. In particular, the ratio of $R_{\text{out}}/R_{\text{side}}$ is close to one, and drops below one at larger values of the pair momentum $k_{\perp}$. The plots of $R_{\text{side}}$ and $R_{\text{out}}$ include the excluded-volume correction factor $S = 1.3$. We observe that these radii decrease with the pair momentum $k_{\perp}$, although somewhat slower than indicated by the data. We note that the radius $R_{\text{long}}$ cannot be reliably evaluated in our model. This is due to the assumption of the boost invariance, cf. Eq. (1.2), which leads to too large values of $R_{\text{long}}$. 
CONCLUSION

To conclude, we list the main results of our approach:

1. The thermal model works for the particle ratios, see (I).
2. Supplied with expansion, it works for the $p_\perp$-spectra, see (I). The complete treatment of resonances is essential, and the assumption of single freeze-out leads to good predictions. Moreover, the strange particles, including $\Omega$, are described properly with no need for extra parameters.
3. Supplied with azimuthal asymmetry, the model can be used to describe the elliptic flow at moderate transverse momenta, up to $p_\perp \sim 2$ GeV. The fitted values of the velocity asymmetry parameter, $\delta$, are reasonable.
4. Supplied with the excluded-volume corrections, the model works also for the HBT radii $R_{\text{side}}$ and $R_{\text{out}}$. In particular, the ratio of $R_{\text{out}}/R_{\text{side}}$ is close to 1.
5. The description is efficient, involving two thermal parameters $T$ and $\mu_B$, two size parameters $\tau$ and $\rho_{\text{max}}$, and in the case of azimuthal asymmetry, two deformation parameters, $\varepsilon$ and $\delta$.
6. Finally, we note that the model also works for the case of RHIC at $\sqrt{s_{\text{NN}}} = 200$ GeV A as well as for SPS at $\sqrt{s_{\text{NN}}} = 17$ GeV [33, 34].

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