The resonance spectrum of the cusp map in the space of analytic functions

I. Antoniou$^{1,2}$, S. A. Shkarin$^{2,3,5}$ and E. Yarevsky$^{2,4}$

$^1$Department of Mathematics, Aristotle University of Thessaloniki, 54006, Greece
$^2$International Solvay Institutes for Physics and Chemistry, Campus Plaine ULB C.P.231, Bd du Triomphe, Brussels 1050, Belgium
$^3$Moscow State University, Dept. of Mathematics and Mechanics, Vorobjovy Gory, Moscow, 119899, Russia
$^4$Laboratory of Complex Systems Theory, Institute for Physics, St. Petersburg State University, Uljanovskaya 1, Petrodvoretz, St. Petersburg 198904, Russia
$^5$Department of Mathematics, Wuppertal University, 42119, Wuppertal, Gauss str. 20, Germany

e-mail: eiarevsk@ulb.ac.be shkarin@math.uni-wuppertal.de

Abstract

We prove that the Frobenius–Perron operator $U$ of the cusp map $F : [-1,1] \rightarrow [-1,1], F(x) = 1 - 2\sqrt{|x|}$ (which is an approximation of the Poincaré section of the Lorenz attractor) has no analytic eigenfunctions corresponding to eigenvalues different from 0 and 1. We also prove that for any $q \in (0,1)$ the spectrum of $U$ in the Hardy space in the disk $\{ z \in \mathbb{C} : |z-q| < 1+q \}$ is the union of the segment $[0,1]$ and some finite or countably infinite set of isolated eigenvalues of finite multiplicity.
1 Introduction

The so-called cusp map \[^{1}\]
\[
F : [-1, 1] \rightarrow [-1, 1], \quad F(x) = 1 - 2\sqrt{|x|}
\]
is an approximation of the Poincaré section of the Lorenz attractor \[^{2, 3}\]. This map is ergodic \[^{4}\]. The unique absolutely continuous invariant probability measure \(\mu\) has density \(\rho(x) = (1 - x)/2\) \[^{1}\].

The Frobenius–Perron operator (F.P.O.) \(U\) of \(F\) is the adjoint of the Koopman operator \(V\) \[^{5}\] in the Hilbert space \(L_2([-1, 1], \mu)\):
\[
Vf(x) = f(F(x)),
\]
\[
Uf(x) = \frac{1}{2} \left(1 - \frac{(1 - x)^2}{4}\right) f \left(\frac{(1 - x)^2}{4}\right) + \frac{1}{2} \left(1 + \frac{(1 - x)^2}{4}\right) f \left(-\frac{(1 - x)^2}{4}\right). \tag{1}
\]
The spectral analysis of the F.P.O. in different function spaces is useful for the probabilistic approach to non-linear dynamics. The spectrum of the F.P.O. known also as resonance spectrum gives estimates on the decay of correlation functions, see, e.g. \[^{5, 6, 7}\]. The spectral decomposition of the Koopman and Frobenius–Perron operators acquires meaning in locally convex topological spaces and allows for probabilistic prediction \[^{8, 9, 10, 11}\].

The cusp map is not expanding \[^{5}\]. In \[^{12, 13, 14}\], the following family of maps depending analytically on the parameter \(\varepsilon \in (0, 1/2]\) is introduced and studied:
\[
F_\varepsilon : [-1, 1] \rightarrow [-1, 1], \quad F_\varepsilon(x) = \frac{1 - \sqrt{1 - 4\varepsilon (1 - \varepsilon - 2|x|)}}{2\varepsilon}, \quad \text{for } \varepsilon \in (0, 1/2]. \tag{2}
\]
This family consists of piecewise analytic expanding maps and has the cusp map as the limit case for \(\varepsilon = 1/2\). For any map \[^{1}\], the spectrum of the F.P.O. in the space of \(C^\infty\) functions consists of a sequence of eigenvalues of finite multiplicity converging to 0, and the corresponding eigenfunctions are analytic \[^{15, 16}\]. The divergence of the eigenfunctions as the maps (2) approach the cusp map has been observed numerically \[^{13, 14, 16}\]. These numerical results indicate that the F.P.O. \[^{1}\] has no analytic eigenfunctions corresponding to eigenvalues different from 0 or 1. In the present paper we give an analytic proof of this fact. Our result confirms the reliability of the numerical works \[^{13, 14, 16}\].

The spectral properties of the F.P.O. of piecewise analytic maps with one neutral fixed point (i.e. fixed point with derivative equal to 1; this is the point \(x = -1\) for the cusp map) have been addressed by Rugh \[^{17}\]. For each map satisfying certain properties, Rugh has constructed a map-dependent Banach space of functions, analytic everywhere except at the neutral fixed point. The spectrum of the Frobenius–Perron operator in this Banach space is the union of the segment \([0, 1]\) and some isolated eigenvalues of finite multiplicity. However, Rugh’s results do not specify the spectrum of the F.P.O in the space of everywhere analytic functions. Moreover, the cusp map does not satisfy the properties of the class of maps considered by Rugh as a result of the cusp singularity. Nevertheless, we prove in this paper that for any \(q \in (0, 1)\) the spectrum of the F.P.O. \[^{1}\]
in the usual Hilbert Hardy space in the disk \( \{ z \in \mathbb{C} : |z - q| < 1 + q \} \) is the union of the segment \([0, 1]\) and a finite or countable set of isolated eigenvalues of finite multiplicity.

The paper is organized as follows. In Section 2, we summarize in Theorems 1–4 our results about spectra of the operator \( U \), and derive Theorems 1 and 2. We prove auxiliary lemmas in Section 3, while the main proofs are presented in Sections 4 and 5.

## 2 The spectrum of the F.P.O.

We shall use the following notations. \( \mathcal{A} \) is the space of real-analytic functions \( f : [-1, 1] \rightarrow \mathbb{C} \) and \( \mathcal{E} \) is the space of entire functions of one complex variable (endowed with their natural topologies \( \mathbb{E} \)). \( \mathcal{P} \) is the space of polynomials of one complex variable. As usual in algebra, we assume that the degree of zero polynomial is \(-1\).

The spectral structure of the F.P.O. in the spaces \( \mathcal{A} \) and \( \mathcal{E} \) is given by the following two theorems.

**Theorem 1.**
The spectra of both operators \( U_\mathcal{E} = U \mid _\mathcal{E} : \mathcal{E} \rightarrow \mathcal{E} \) and \( U_\mathcal{A} = U \mid _\mathcal{A} : \mathcal{A} \rightarrow \mathcal{A} \) coincide with the whole complex plane.

**Theorem 2.**
I. The point spectra of both operators \( U_\mathcal{E} \) and \( U_\mathcal{A} \) coincide with the two-point set \( \{ 0, 1 \} \).
The eigenvalue 1 is simple and the eigenvalue 0 has infinite multiplicity for both operators.
II. Let \( f \in \mathcal{A} \). Then \( U f \equiv 0 \) if and only if \( f(x) \equiv (1 + x)g(x) \) for all \( x \in [-1, 1] \), where \( g \in \mathcal{A} \) is odd.
III. Let \( \lambda \in \mathbb{C} \), \( f \in \mathcal{A} \), \( n \in \mathbb{N} \) and \( (U - \lambda I)^n f = 0 \) (where \( I \) is the identity operator).

(i) If \( \lambda \notin \{ 0, 1 \} \) then \( f = 0 \).

(ii) If \( \lambda = 1 \) then \( f \) is constant.

As we have already mentioned, Rugh’s theorem \( ^{17} \) cannot be applied for the cusp map. Moreover, one cannot use the technique of Rugh’s proof to prove a similar theorem for the cusp map. Indeed, the key element of Rugh’s proof is a holomorphic function \( \varphi \), defined in an open simply connected set, containing all points of the segment except the neutral fixed point such that \( \varphi \circ \xi \circ \varphi^{-1}(z) \equiv z + 1 \), where \( \xi \) is a branch of the inverse map containing the neutral fixed point \( (\xi(z) = -\frac{(1 - z)^2}{4} \) for the cusp map). Such a map \( \varphi \) obviously does not exist if \( \xi' \) vanishes somewhere on the segment, which is the case for the cusp map since \( \xi'(1) = 0 \). Nevertheless, using similar ideas but quite different technique, we prove that \( U \) has the spectral structure as in Rugh’s theorem in appropriate Hardy spaces. The definitions of the Hardy spaces \( H^2 \) in the unit disk and in the upper half-plane \( \{ z : \text{Im} \, z \geq 0 \} \) can be found e.g. in \( ^{19} \), Chapters 3 and 8. These spaces are separable Hilbert spaces.

**Theorem 3.**
Let \( q \in (0, 1) \) and \( X \) be the Hardy space \( H^2 \) in the disk \( \{ z \in \mathbb{C} : |z - q| < 1 + q \} \). Then \( U \) is a bounded operator on the Banach space \( X \) and the spectrum of \( U \mid_\mathcal{X} : X \rightarrow X \) is the union of the segment \([0, 1]\) and a finite or countable set of isolated eigenvalues of finite multiplicity.
The proof of Theorem 3 is given in Section 5. We do not consider Hardy spaces $H^p$, $p \neq 2$, although the Theorem 3 can be generalized for such spaces. The Hardy spaces in any other disk or half-plane are the results of appropriate linear change of variables applied to the Hardy space in the unit disk or the upper half-plane.

For the sake of completeness we formulate here the following theorem proved in [16].

**Theorem 4.**
The spectrum of the operator $U|_X : X \to X$, where $X$ is either $L_p([-1, 1], \mu)$ $(1 \leq p \leq +\infty)$ or $C^k[-1, 1]$ $(k = 0, 1, \ldots, \infty)$, is the closed unit disk. The point spectrum of $U|_X$ is the set $\{z \in \mathbb{C} : |z| < 1\} \cup \{1\}$. The eigenspace corresponding to the eigenvalue 1 is the one dimensional space of constants. The eigenspace corresponding to the eigenvalues $\{z : |z| < 1\}$ is infinite dimensional.

For the proof of Theorems 1 and 2 we need the following

**Proposition.**
I. Let $\lambda \in \mathbb{C} \setminus \{0\}$, $g \in \mathcal{E}$, $f \in \mathcal{A}$ and

\[ Uf(x) = \lambda f(x) + g(x), \tag{3} \]

for all $x \in [-1, 1]$. Then $f \in \mathcal{E}$.
II. If additionally $g \in \mathcal{P}$, then $f \in \mathcal{P}$ and $\deg f \leq \deg g + 1$.
III. If $g \in \mathcal{P}$ and $\deg g = 4k + 1$, $k = 0, 1, \ldots$ then the functional equation (3) has no solutions in $\mathcal{A}$.

The proof of the Proposition is given in Section 4.

**Proof of Theorem 1.** The number 0 is an eigenvalue of $U_\mathcal{A}$ and $U_\mathcal{E}$. For example $Uf = 0$ if $f(x) = (1 + x)x$. Therefore, the number 0 belongs to the spectra of these operators. Let $\lambda \in \mathbb{C} \setminus \{0\}$. According to the Proposition, the functional equation (3) has in the space $\mathcal{P}$ no solutions for $g$ with degree of the form $4k + 1$. Thus, the functional equation $Uf(x) = \lambda f(x) + x$ has no analytic solutions. Therefore the function $g(x) = x$ does not belong to the image of the operator $U_\mathcal{A} - \lambda I$ (and to the smaller image of $U_\mathcal{E} - \lambda I$) for any $\lambda \in \mathbb{C} \setminus \{0\}$. Hence, operators $U_\mathcal{A} - \lambda I$ and $U_\mathcal{E} - \lambda I$ are non-invertible. Therefore the spectra of $U_\mathcal{A}$ and $U_\mathcal{E}$ coincide with the whole complex plane.

**Proof of Theorem 2.** Let $z \in \mathbb{C} \setminus \{0\}$, $c \in \mathbb{C}$ and $f \in \mathcal{A}$. The Proposition implies that

\[ \text{if } Uf - zf \equiv c \text{ then } f \text{ is a constant.} \tag{4} \]

Let $\lambda \in \mathbb{C}$, $\lambda \neq 0$, $f \in \mathcal{A}$ and $(U - \lambda I)^n f = 0$. Relation (3) implies consequently that the functions $h_1 = (U - \lambda I)^{n-1} f$, $h_2 = (U - \lambda I)^{n-2} f$, $\ldots$, $h_{k-1} = (U - \lambda I) f$, $h_k = f$ are constants. In particular, $f \equiv a$ for some $a \in \mathbb{C}$. Since $U1 = 1$, we have that $0 = (U - \lambda I)^n f = (1 - \lambda)^n \cdot a$. Therefore either $\lambda = 1$ or $f = 0$. This proves part III of the theorem.

Let now $f \in \mathcal{A}$. According to (3), the equality $Uf = 0$ can be written in the form:

\[ \left(1 - \frac{(1 - x)^2}{4}\right) f \left(\frac{(1-x)^2}{4}\right) + \left(1 + \frac{(1-x)^2}{4}\right) f \left(-\frac{(1-x)^2}{4}\right) = 0, \tag{5} \]
$x \in [-1, 1]$. For $x = -1$ the equality (3) implies that $f(-1) = 0$. Therefore $f(x) \equiv (1 + x)g(x)$ for some $g \in \mathcal{A}$. Substituting $(1 + x)g(x)$ instead of $f(x)$ into (4), we obtain

$$
(1 - \frac{(1 - x)^4}{16}) \left( g \left( \frac{(1 - x)^2}{4} \right) + g \left( -\frac{(1 - x)^2}{4} \right) \right) = 0 \tag{6}
$$

for $x \in [-1, 1]$. Denoting $y = (1 - x)^2/4$, we arrive to $g(y) = g(-y)$ for $y \in [0, 1)$. Hence, $g$ is odd.

Suppose now that $f(x) \equiv (1 + x)g(x)$ for odd $g \in \mathcal{A}$. This implies the validity of (3) and therefore (5), which is equivalent to $Uf = 0$. Part II of the theorem is proved. Part I follows immediately from parts II and III. $lacksquare$

## 3 Analytic continuation of eigenfunctions

First, we introduce some notations, which we shall use in Sections 3 and 4 without additional comments. Let $\lambda$, $g$, and $f$ be as in the Proposition. For $z \in \mathbb{C}$ and $r \in (0, +\infty)$ we denote

$$D(z, r) = \{ w \in \mathbb{C} : |w - z| < r \}, \quad \overline{D}(z, r) = \{ w \in \mathbb{C} : |w - z| \leq r \}.$$

By $\mathbb{C}_-$ we denote the set $\mathbb{C} \setminus (-\infty, 0)$. We reserve the symbol $\sqrt{z}$ for the “positive” branch of the square root on the set $\mathbb{C}_-$, i.e. $\sqrt{z} = \sqrt{r}e^{i\phi/2}$, where $z = re^{i\phi}$, $-\pi < \phi < \pi$.

For an infinite connected subset $A \subseteq \mathbb{C}$ we say that a function $\varphi : A \to \mathbb{C}$ is analytic if $\varphi$ admits an analytic extension to some open set, containing $A$. In particular, $\mathcal{A}$ is the space of all functions analytic on $[-1, 1]$. For a connected set $A \subseteq \mathbb{C}$ and a subset $B \subseteq A$, having at least one limit point in $A$, we say that a function $\varphi : B \to \mathbb{C}$ is analytic on $A$ if $\varphi$ admits a (unique) analytic extension on $A$. We shall also denote by $\varphi$ the extension.

We shall show that the function $f$ admits an analytic extension to the disk $D(0, 2 + \sqrt{3})$ in several steps.

**Lemma 1.**

Let $A$ and $B$ be subsets of $\mathbb{C}$, $\varphi_1 : A \to \mathbb{C}$ and $\varphi_2 : B \to \mathbb{C}$ be analytic functions, $M \subseteq A \cap B$ be a set having at least one limit point in $A \cap B$. Let also the set $A \cap B$ be connected and $\varphi_1|_M \equiv \varphi_2|_M$. Then the function

$$\varphi : A \cup B \to \mathbb{C}, \quad \varphi(z) = \begin{cases} 
\varphi_1(z) & \text{if } z \in A; \\
\varphi_2(z) & \text{if } z \in B
\end{cases}$$

is well defined and analytic on $A \cup B$.

**Proof.** Since $M$ has a limit point in the connected set $A \cap B$, then according to the uniqueness theorem $\left[ \overline{\varphi_1}_{A \cap B} \equiv \varphi_2|_{A \cap B} \right]$. Therefore $\varphi$ is well defined. Analyticity of $\varphi$ follows from analyticity of $\varphi_1$ and $\varphi_2$. $lacksquare$

**Lemma 2.**

The analyticity of $f$ on $\overline{D}(0, c)$ ($c \geq 1$) implies the analyticity of $f$ on $\overline{D}(0, c) \cup \overline{D}(1, 2\sqrt{c})$. 

5
Proof. Let $f$ be analytic on $D(0, c)$. Consider $h : D(1, 2\sqrt{c}) \rightarrow \mathbb{C}$, $h(z) = (Uf(z) - g(z))/\lambda$. Clearly $h$ is well-defined and analytic. Since (3) is valid on $[-1, 1]$, we have that $h(z) = f(z)$ for $z \in [-1, 1]$. Lemma 1 implies that the function

$$q(y) = \begin{cases} f(y) & \text{if } y \in D(0, c); \\ h(y) & \text{if } z \in D(1, 2\sqrt{c}) \end{cases}$$

is well defined and analytic on $D(0, c) \cup D(1, 2\sqrt{c})$. This is the desired analytic extension of $f$. ■

Lemma 3.
The function $f$ is analytic on the closed disk $D(1, 2)$ and the functional equation (3) is valid for all $x \in D(1, 2)$.

Proof. It suffices to show that $f$ is analytic on $D(1, 2)$ (the validity of the functional equation (3) for all $x \in D(1, 2)$ follows then from the uniqueness theorem [20]). For this goal it suffices to show that $f$ is analytic on $D(0, 1)$. Analyticity on $D(1, 2)$ then follows from Lemma 2. For $a \in (0, 1]$ let

$$K_a = \{z \in D(0, 1) : |\text{Im } z| < a\}, \quad K_a^c = \{z \in D(0, 1) : |\text{Im } z| \leq a\}.$$

Suppose that $f$ is not analytic on $D(0, 1)$. Let us denote

$$a = \sup\{b \in (0, 1) : f \text{ is analytic on } K_b\}.$$

Then $a \in (0, 1]$, $f$ is analytic on $K_a$ and $f$ is not analytic on $K_a^c$.

Let $z \in K_a^c$, $x = \Re z$, $y = \Im z$ and $w = (1 - z^2)/4$. Since $|z| \leq 1$, we have that $|w| \leq (1 + |z|)^2/4 \leq 1$. Since $x^2 + y^2 \leq 1$ and $|y| \leq a$ we have that $|\Im w| = |(x - 1)y|/2 \leq a|x - 1|/2$. Therefore $|\Im w| \leq a$. Moreover $|\Im w| = a$ if and only if $x = -1$ and $|y| = a$. But then $|z| = \sqrt{1 + a^2} > 1$. Hence, $|\Im w| < a$. Thus, we have shown that

$$\pm(1 - z^2)/4 \in K_a \quad \text{for any } z \in K_a^c. \quad (7)$$

Formulas (7) and (1) imply that the function $h(z) = (Uf(z) - g(z))/\lambda$ is well-defined and analytic on $K_a$. The equation (3) implies that $f(x) = h(x)$ for $x \in [-1, 1]$. Therefore $h$ is an analytic continuation of $f$, i.e. $f$ is analytic on $K_a$. This contradiction completes the proof. ■

Lemma 4.
I. Let $c \in [2, +\infty)$, $z \in D(-1, c) \setminus D(1, c)$. Then $1 - 2\sqrt{-z} \in D(-1, c)$.
II. Let $a \in [3, +\infty)$, $z \in D(0, a)$, $\Re z \geq 0$. Then $w, u \in D(0, a)$, where $w = 2\sqrt{z} - 1$ and $u = z - w$.

Proof. I. According to the maximum principle [20], it suffices to show that

$$\sqrt{z} \in D(1, c/2) \quad \text{for any} \quad z \in \partial M, \quad (8)$$

where $\partial M$ is the boundary of the set $M = \{z \in D(1, c) : \Re z \geq 0\}$. Clearly (8) is equivalent to

$$\sqrt{z} \in D(1, c/2) \quad \text{for any} \quad z \in \Gamma_1 = \{it : t \in [-\sqrt{c^2 - 1}, \sqrt{c^2 - 1}]\}; \quad (9)$$

$$\sqrt{z} \in D(1, c/2) \quad \text{for any} \quad z \in \Gamma_2 = \{z \in \mathbb{C} : |z - 1| = c, \ \Re z \geq 0\}. \quad (10)$$

II. Let $a \in [3, +\infty)$, $z \in D(0, a)$, $\Re z \geq 0$. Then $w, u \in D(0, a)$, where $w = 2\sqrt{z} - 1$ and $u = z - w$.

Proof. I. According to the maximum principle [20], it suffices to show that

$$\sqrt{z} \in D(1, c/2) \quad \text{for any} \quad z \in \partial M, \quad (8)$$

where $\partial M$ is the boundary of the set $M = \{z \in D(1, c) : \Re z \geq 0\}$. Clearly (8) is equivalent to

$$\sqrt{z} \in D(1, c/2) \quad \text{for any} \quad z \in \Gamma_1 = \{it : t \in [-\sqrt{c^2 - 1}, \sqrt{c^2 - 1}]\}; \quad (9)$$

$$\sqrt{z} \in D(1, c/2) \quad \text{for any} \quad z \in \Gamma_2 = \{z \in \mathbb{C} : |z - 1| = c, \ \Re z \geq 0\}. \quad (10)$$
Parameterizing $z \in \Gamma_2$ by polar coordinates $z = re^{i\varphi}$, we obtain that (11) and (12) are equivalent to
\[
\max\{t + 1 - \sqrt{2t} : t \in [0, \sqrt{c^2 - 1}]\} \leq c^2/4; \tag{11}
\]
\[
r + 1 \leq 2\sqrt{r} \cos(\varphi/2) + c^2/4 \quad \text{if} \quad r^2 - 2r \cos \varphi = c^2 - 1, \quad \varphi \in [-\pi/2, \pi/2], \tag{12}
\]
respectively. Since $r^2 - 2r \cos \varphi = c^2 - 1$, inequality from (12) is equivalent to
\[
c^4/16 + c^2(\sqrt{r} \cos(\varphi/2) - 1) \geq 0.
\]
As $r$ and $\cos(\varphi/2)$ are decreasing with respect to $\varphi \in [0, \pi/2]$ and the function $t + 1 - \sqrt{2t}$ on the segment $[0, \sqrt{c^2 - 1}]$ takes the maximal value for $t = \sqrt{c^2 - 1}$, we obtain that inequalities (11) and (12) are respectively equivalent to
\[
4a^4 - 8a^2 + 8a - 3 \geq 0 \quad \text{for} \quad a \in [(3/4)^{1/4}, \infty); \quad \tag{13}
\]
\[
4a^4 + 16a - 15 \geq 0 \quad \text{for} \quad a \in [(3/4)^{1/4}, \infty). \quad \tag{14}
\]
Since $4a^4 - 8a^2 + 8a - 3 = (2a^2 + 2a - 3)(2a^2 + 2a + 1)$ and $4a^4 + 16a - 15 = (2a^2 + 2a - 3)(2a^2 - 2a + 5)$, the number $(\sqrt{7} - 1)/2 < (3/4)^{1/4}$ is the maximal real zero of both polynomials $4a^4 - 8a^2 + 8a - 3$ and $4a^4 + 16a - 15$. This proves (13) and (14), which imply (8).

II. We have to prove that $|w| < a$ and $|u| < a$. According to the maximum principle, it suffices to verify this for $z \in \Gamma$, where $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 = \{it : t \in [-a, a]\}$ and $\Gamma_2 = \{ae^{i\varphi} : \varphi \in [-\pi/2, \pi/2]\}$.

For $z \in \Gamma_1$ we have
\[
|w|^2 = 4|t| - 2\sqrt{2|t|} + 1, \quad |u|^2 = t^2 - 2\sqrt{2|t|^{3/2}} + 4|t| - 2\sqrt{2|t|^{1/2}} + 1.
\]
For $z \in \Gamma_2$ we have
\[
|w|^2 = 4a + 1 - 4\sqrt{a} \cos(\varphi/2),
|u|^2 = a^2 - 4a^{3/2} \cos(\varphi/2) + 4a + 2a \cos \varphi - 4a^{1/2} \cos(\varphi/2) + 1.
\]

Differentiating these functions with respect to $t$ and $\varphi$, we find that both $|u|^2$ and $|w|^2$ for $z \in \Gamma_1$ are maximal when $t = a$ and that both $|u|^2$ and $|w|^2$ for $z \in \Gamma_2$ are maximal when $\varphi = \pm \pi/2$. Thus, in any case
\[
|w|^2 \leq 4a - 2\sqrt{2a} + 1 \quad \text{and} \quad |u|^2 \leq a^2 - 2\sqrt{2a^{3/2}} + 4a - 2\sqrt{2a^{1/2}} + 1. \tag{15}
\]
Hence, it suffices to verify that
\[
a^2 - 4a + 2\sqrt{2a} - 1 > 0 \quad \text{and} \quad 2\sqrt{2a^{3/2}} - 4a + 2\sqrt{2a^{1/2}} - 1 > 0 \tag{16}
\]
for $a \geq 3$. Both functions from (16) are increasing for $a \geq 3$. Hence, we should only prove (16) for $a = 3$, which is a simple arithmetic exercise. $\blacksquare$

**Lemma 5.**
Let $c \in [2, +\infty)$. Then the analyticity of $f$ on $\overline{D}(1, c)$ implies the analyticity of $f$ on $\overline{D}(1, c) \cup D(-1, c)$.  

7
Proof. Let $f$ be analytic on $\overline{D}(1,c)$. Pick $\varepsilon > 0$ such that $f$ is analytic on $\overline{D}(1,c+\varepsilon)$ and let

$$\begin{align*}
S_0 &= \overline{D}(1,c+\varepsilon) \cap (\overline{D}(1,c) \cup \overline{D}(-1,c)), \\
S_{n+1} &= S_n \cup \{z \in \overline{D}(-1,c) : 1 - 2\sqrt{-z} \in S_n\} \\
&= S_n \cup \{-(1-w)^2/4 : w \in S_n\} \cap \overline{D}(-1,c)).
\end{align*}$$

(17)

Evidently $f$ is analytic on $S_0$. It is easy to see that $S_n, n = 0, 1, 2 \ldots$, is an increasing sequence of subsets of $\overline{D}(1,c) \cup \overline{D}(-1,c)$.

First, we shall show that analyticity of $f$ on $S_n$ implies analyticity of $f$ on $S_{n+1}$. Let $f$ be analytic on $S_n$. Consider the function $h : A_n \to \mathbb{C}$

$$h(z) = \frac{1+z}{z-1}f(-z) + \frac{2\lambda}{1-z}f(1-2\sqrt{-z}) + \frac{2}{1-z}g(1-2\sqrt{-z}),$$

(18)

where $A_n = \{z \in \overline{D}(-1,c) : z \notin [0, +\infty), 1 - 2\sqrt{-z} \in S_n\}$. Clearly $h$ is well defined and analytic. Moreover, from (3) and the definition of $h$ it follows that $h\big|_{(-1,0)} = f\big|_{(-1,0)}$.

Lemma 1 implies that the function

$$q : S_{n+1} = A_n \cup S_n \to \mathbb{C}, \quad q(z) = \begin{cases} f(z) & \text{if } z \in S_n; \\ h(z) & \text{if } z \in A_n \end{cases}$$

is well defined and is an analytic extension of $f$ to $S_{n+1}$. Therefore $f$ is analytic on $\bigcup_{n=0}^{\infty} S_n$.

It remains to show that

$$\overline{D}(1,c) \cup \overline{D}(-1,c) = \bigcup_{n=0}^{\infty} S_n.$$  

(19)

From the definition of $S_n$, the point $z \in \overline{D}(-1,c) \setminus D(1,c+\varepsilon)$ belongs to $\bigcup_{n=0}^{\infty} S_n$, if and only if there exists $m \in \mathbb{N}$ such that $z_j \in \overline{D}(-1,c) \setminus D(1,c+\varepsilon)$ for $0 \leq j \leq m$ and $z_{m+1} \in S_0$, where

$$z_0 = z, \quad z_{j+1} = 1 - 2\sqrt{-z_j}.$$  

(20)

Suppose that there exists $z \in \overline{D}(-1,c) \setminus D(1,c+\varepsilon)$ such that $z_n \in \overline{D}(-1,c)$ for all $n \in \mathbb{N}$. Let $K$ be the closure of the set $\{z_n : n = 0, 1, \ldots\}$. Then $K$ is a closed subset of the compact set $A = \overline{D}(-1,c) \setminus D(1,c+\varepsilon)$, and $\varphi(K) \subseteq K$, where $\varphi(z) = 1 - 2\sqrt{-z}$. Since $|\varphi'(u)| < 1$ for any $u \in A$, $\varphi|_K : K \to K$ is a contraction. According to the contraction map theorem, there exists a fixed point of the map $\varphi|_K$. But the unique solution (in $\mathbb{C} \setminus [1, +\infty)$ of the equation $\varphi(w) = w$ is $w = -1 \notin A$. This contradiction shows that for any $z \in \overline{D}(-1,c) \setminus D(1,c+\varepsilon)$ there exists the first positive integer $m$ for which $z_m \notin \overline{D}(-1,c) \setminus D(1,c+\varepsilon)$. Then $z_{m-1} \in \overline{D}(-1,c) \setminus D(1,c+\varepsilon)$. According to Lemma 4 $z_m = 1 - 2\sqrt{-z_m} \in \overline{D}(-1,c) \cap D(1,c+\varepsilon) \subseteq S_0$. According to the description of the set $\bigcup_{n=0}^{\infty} S_n$, we have that $z \in \bigcup_{n=0}^{\infty} S_n$. This implies (19).
\textbf{Lemma 6.}

The function \( f \) is analytic on the disk \( D(0, 2 + \sqrt{3}) \).

\textbf{Proof.} Let \( c_0 = 1, c_{n+1} = \sqrt{4c_n - 1} \) for \( n = 1, 2, \ldots \). This sequence strictly increases and converges to \( 2 + \sqrt{3} \). From Lemma 6 it follows that \( f \) is analytic on \( \overline{D}(0, c_0) \). According to Lemmas 2 and 5 analyticity of \( f \) on \( \overline{D}(1, 2\sqrt{c_n}) \cup \overline{D}(-1, 2\sqrt{c_n}) \supset \overline{D}(0, c_{n+1}) \). Therefore \( f \) is analytic on \( \overline{D}(0, c_n) \) for any \( n = 0, 1, \ldots \). Hence, \( f \) is analytic on the set \( \bigcup_{n=0}^{\infty} \overline{D}(0, c_n) = D(0, 2 + \sqrt{3}) \). \( \blacksquare \)

4 Proof of the Proposition

I. Without loss of generality we can assume that \( g(-1) = 0 \) and \( f(-1) = 0 \). If this is not the case, we can achieve these conditions just by adding suitable constants to \( f \) and \( g \). Therefore \( f(x) = \varphi(x)(1 + x), g(x) = \psi(x)(1 + x) \) and \( \varphi \in \mathcal{E}, \varphi \in \mathcal{A} \). Moreover analyticity of \( f \) on a connected set \( A \supset [-1, 1] \) implies analyticity of \( \varphi \) on the same set \( A \). Let

\[
\begin{align*}
    f_0(x) &= (\varphi(x) + \varphi(-x))/2, \\
    g_0(x) &= (\psi(x) + \psi(-x))/2,
\end{align*}
\]

\( \lambda f(x) = f_1(x) + (1 + x)(g_0(x) + g_1(x)). \) (21)

Then \( g_0, g_1 \in \mathcal{E}, f_0, f_1 \in \mathcal{A} \). The analyticity of \( f \) on a connected symmetric (with respect to 0) set \( A \supset [-1, 1] \) implies the analyticity of \( f_0, f_1 \) on the same set. Thus, according to Lemma 6 \( f_0 \) and \( f_1 \) are analytic on \( D(0, 2 + \sqrt{3}) \).

Evidently \( f_0, g_0 \) are even, \( f_1, g_1 \) are odd and

\[
    f(x) = (1 + x)(f_0(x) + f_1(x)), \quad g(x) = (1 + x)(g_0(x) + g_1(x)). \quad \text{(22)}
\]

From (21) and (22) it follows that

\[
    \left(1 - \frac{(1 - x)^4}{16}\right) f_0 \left(\frac{(1 - x)^2}{4}\right) = \lambda(1 + x)(f_0(x) + f_1(x)) + (1 + x)(g_0(x) + g_1(x))
\]

for any \( x \in [-1, 1] \). Dividing by \( (1 + x) \) we obtain

\[
    \frac{1}{16}(15 - 11x + 5x^2 - x^3) f_0 \left(\frac{(1 - x)^2}{4}\right) = \lambda(f_0(x) + f_1(x)) + g_0(x) + g_1(x). \quad \text{(23)}
\]

Adding (23) for \( x \) with (23) for \( -x \) we obtain that for any \( x \in [-1, 1] \),

\[
    \frac{1}{32}(15 - 11x + 5x^2 - x^3) f_0 \left(\frac{(1 - x)^2}{4}\right) + \\
    + \frac{1}{32}(15 + 11x + 5x^2 + x^3) f_0 \left(\frac{(1 + x)^2}{4}\right) = \lambda f_0(x) + g_0(x). \quad \text{(24)}
\]

Let us prove that \( f_0 \in \mathcal{E} \). Suppose that \( f_0 \notin \mathcal{E} \). Since \( f_0 \) is analytic on \( D(0, 2 + \sqrt{3}) \), there exists \( a \in [2 + \sqrt{3}, +\infty) \) such that \( f_0 \) is analytic on \( D(0, a) \) and is not analytic on \( \overline{D}(0, a) \). Since \( f_0 \) is even, we have that \( f_0 \) is not analytic on the set \( B = \{ z \in \overline{D}(0, a) : \)
\( z \neq 0, \Re z \geq 0 \). According to Lemma 4 \( x = 2\sqrt{y} - 1 \in D(0, a) \) and \( y - x \in D(0, a) \) for any \( y \in B \). Since \( 15 + 11x + 5x^2 + x^3 \neq 0 \) for \( y \in B \), the function

\[
h(y) = \frac{(x^3 - 5x^2 + 11x - 15)f_0(y - x) + 32\lambda f_0(x) + 32g_0(x)}{15 + 11x + 5x^2 + x^3},
\]

where \( x = x(y) = 2\sqrt{y} - 1 \), is well defined and analytic on \( B \). On the other hand, (24) and the definition of \( h \) imply that \( h(x) = f(x) \) for all \( x \in (0, 1) \). The uniqueness theorem implies that \( h \) is an analytic extension of \( f_0 \) from \( (0, 1] \) to \( B \), which does not exist. This contradiction proves that \( f_0 \in \mathcal{E} \).

From (23) it follows that

\[
f_1(x) = \frac{1}{16\lambda}(15 - 11x + 5x^2 - x^3)f_0 \left( \frac{(1 - x)^2}{4} \right) - f_0(x) - \frac{1}{\lambda}(g_0(x) + g_1(x)).
\]  

(25)

Therefore \( f_1 \in \mathcal{E} \). Formula (22) implies that \( f \) is entire and Part I is proved.

II. Let \( g \in \mathcal{P} \). We have to prove that \( f \in \mathcal{P} \). According to (21) \( g_0, g_1 \in \mathcal{P} \). Let \( k = \deg g_0 \) if \( g_0 \neq 0 \) and \( k = 0 \) if \( g_0 \equiv 0 \). Let us show that \( f_0 \in \mathcal{P} \) and \( \deg f_0 \leq k/2 - 1 \).

The functional equation (24) can be rewritten in the form

\[
f_0 \left( \frac{(1 + x)^2}{4} \right) = \frac{32\lambda f_0(x)}{x^3 + 5x^2 + 11x + 15} + \frac{32g_0(x)}{x^3 + 5x^2 + 11x + 15} + \frac{x^3 - 5x^2 + 11x - 15}{x^3 + 5x^2 + 11x + 15}f_0 \left( \frac{(1 - x)^2}{4} \right).
\]  

(26)

Since \( f_0 \) is even

\[
M(R) = \max_{|x|=R}|f_0(x)| = \max_{|x|=R, \Re x \geq 0}|f_0(x)|.
\]  

(27)

Let \( y \in \mathbb{C}, |y| = R, \Re y \geq 0, x = 2\sqrt{y} - 1, w = y - x \). Then \( |x| = 2\sqrt{R} + O(1) \) and

\[
|w| = |y| \left| \frac{1 - y}{x} \right| = R \left| 1 - \frac{2}{\sqrt{y}} + \frac{1}{y} \right| = R \left| 1 - \frac{2}{\sqrt{y}} \right| + O(1).
\]

All \( O \)-symbols are considered here for \( R \to \infty \). The number \( |1 - (2/\sqrt{y})| \) for \( |y| = R, \Re y \geq 0 \) is maximal for \( y = \pm Ri \). Therefore

\[
|w| \leq R \left| 1 - \frac{\sqrt{2}(1 + i)}{\sqrt{R}} \right| + O(1) = R - \sqrt{2R} + O(1)
\]

and

\[
|w| < R - \sqrt{R}, \quad |x| = 2\sqrt{R} + O(1) < R - \sqrt{R}
\]  

(28)

for sufficiently large \( R \). Formula (27) implies that

\[
f_0(y) = \frac{32\lambda f_0(x)}{x^3 + 5x^2 + 11x + 15} + \frac{32g_0(x)}{x^3 + 5x^2 + 11x + 15} + \frac{x^3 - 5x^2 + 11x - 15}{x^3 + 5x^2 + 11x + 15}f_0(w).
\]  

(29)
Note that
\[ \left| \frac{x^3 - 5x^2 + 11x - 15}{x^3 + 5x^2 + 11x + 15} \right| = \left| 1 - \frac{10}{x} + O\left(\frac{1}{x^2}\right) \right| = \left| 1 - \frac{5}{\sqrt{y}} \right| + O(R^{-1}). \]

The number \(|1 - (5/\sqrt{y})|\) for \(|y| = R, \text{Re}y \geq 0\) is maximal for \(y = \pm Ri\). Therefore
\[ \left| \frac{x^3 - 5x^2 + 11x - 15}{x^3 + 5x^2 + 11x + 15} \right| \leq 1 - \frac{5}{\sqrt{2R}} + O(R^{-1}). \quad (30) \]

Formulas (29), (27), (28) and (30) imply that
\[ M(R) \leq M(R - \sqrt{R}) \left(1 - \frac{5}{\sqrt{2R}} + O(R^{-1})\right) + O(R^{k-3/2}). \quad (31) \]

If \(f_0\) is a polynomial of degree at most \(k/2 - 1\), we have proved the statement. Otherwise \(R^{k/2} = O(M(R))\). Hence \(R^{(k-3)/2} = O(M(R - \sqrt{R})/R)\), and
\[ M(R) \leq M(R - \sqrt{R}) \left(1 - \frac{5}{\sqrt{2R}} + O(R^{-1})\right). \]

Therefore \(M(R) \leq M(R - c\sqrt{R})\) for sufficiently large \(R\). Hence \(M(R) = O(1)\) and \(f_0\) is a constant according to the Liouville theorem (20). Hence \(M(R)\) is constant. Formula (31) implies then that \(M(R) \equiv 0\). Therefore \(f_0 \equiv 0\) and \(k = 0\) according to (26). Thus, anyway \(f_0 \in \mathcal{P}\) and \(\deg f_0 \leq k/2 - 1\).

From (23) we find that \(f_1 \in \mathcal{P}\) and

\[ \deg f_1 \leq \max\{k + 1, \deg g_1\}. \]

Then using (22) we find that \(f \in \mathcal{P}\) and \(\deg f \leq \deg g + 1\).

III. Suppose that \(g \in \mathcal{P}\) and \(f \in \mathcal{A}\) satisfies (3). According to Part II of the Proposition

\[ f(x) = \sum_{k=0}^{n} a_k x^k, \quad a_n \neq 0. \]

If \(n\) is odd then according to (1) \(\deg Uf = 2n + 2\) and \(\deg g = \deg (Uf - \lambda f) = 2n + 2\) is even. If \(n\) is even and \(n \neq \deg Uf\), then \(\deg g = \max\{\deg f, \deg Uf\}\) is even. It remains to consider the case \(\deg Uf = n\). Since \(\deg Uf\) is always a multiple of 4, we have that \(n = 4m, m = 0, 1, \ldots\) According to (1)

\[ Uf(x) = \sum_{l=1}^{2m} (a_{2l} - a_{2l-1}) 2^{-4l} (1 - x)^{4l} + a_0. \quad (32) \]

Since \(\deg Uf = 4m\) we have that \(a_{2l} = a_{2l-1}\) if \(m \leq l \leq 2m\) and \(a_{2m} \neq a_{2m-1}\). Substituting (32) into (3) and taking these relations into account, we obtain that \(\deg g = 4m\) or \(\deg g = 4m - 1\). In any case \(\deg g\) does not have form \(4j + 1\).
5 Proof of Theorem 3

We start with three lemmas.

Lemma 7. Let \( q \in (0, 1) \) and \( z \in \mathbb{D}(q, 1 + q) \). Then \( (1 - z)^2 / 4 \in \mathbb{D}(q, 1 + q) \).

Proof. We have to prove that

\[
|z - 1|^2 - 4q < 4 + 4q. \tag{33}
\]

Since \( z \in \mathbb{D}(q, 1 + q) \), we find that \( z = q + (1 + q)u \), where \( |u| \leq 1 \). Obviously (33) is equivalent to

\[
|1 - 6q + q^2 - 2(1 - q^2)u + (1 + 2q + q^2)u^2| < 4 + 4q. \tag{34}
\]

If \( 1 - 6q + q^2 \geq 0 \), we have

\[
|1 - 6q + q^2 - 2(1 - q^2)u + (1 + 2q + q^2)u^2| \leq 1 - 6q + q^2 + 2(1 - q^2) + (1 + 2q + q^2) = 4 - 4q < 4 + 4q.
\]

If \( 1 - 6q + q^2 < 0 \), we have

\[
|1 - 6q + q^2 - 2(1 - q^2)u + (1 + 2q + q^2)u^2| \leq -1 + 6q - q^2 + 2(1 - q^2) + (1 + 2q + q^2) = 4 + 4q - 2(1 - q)^2 < 4 + 4q.
\]

Lemma 8.

Let \( \mathcal{H} \) be the Hardy space \( H^2 \) in the upper half-plane \( \Pi = \{ z \in \mathbb{C} : \text{Re} \, z > \alpha \}, \alpha \in \mathbb{R} \), \( \nu, \varphi : \Pi \to \mathbb{C} \) be bounded analytic functions and \( c, \varepsilon \in (0, +\infty) \) be such that

\[
\text{Re} \, \varphi(z) \geq \varepsilon \text{ for all } z \in \Pi; \tag{35}
\]

\[
\int_{-\infty}^{+\infty} |\nu(\alpha + is)|^2 \, ds \leq A < \infty; \quad \int_{-\infty}^{+\infty} |\varphi(\alpha + is) - c|^2 \, ds \leq A < \infty. \tag{36}
\]

Then the operator \( S : \mathcal{H} \to \mathcal{H}, Si(z) = (1 + \nu(z))f(z + \varphi(z)) \) is the sum of two operators \( A \) and \( B \), where \( A \) is bounded, self-adjoint and has purely absolutely continuous spectrum \([0, 1]\) and \( B \) is a Hilbert–Schmidt operator.

Proof. Without loss of generality we assume \( \alpha = 0 \). According to the Paley–Wiener theorem ([19], Chapter 8), the Laplace transform

\[
Lg(z) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-zt}g(t) \, dt
\]

is a unitary operator from \( L_2[0, +\infty) \) onto \( \mathcal{H} \).

Let \( D_+ \) be the subspace of \( L_2[0, +\infty) \), consisting of infinitely differentiable functions with compact support lying in \((0, +\infty)\). Consider the operator \( \tilde{S} = L^{-1}SL : L_2[0, +\infty) \to L_2[0, +\infty) \). Using the standard formula for the inverse Laplace transform we obtain that

\[
\tilde{S}g(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{0}^{+\infty} g(\tau)e^{is(t-\tau)}e^{-\varphi(is)\tau}(1 + \nu(is)) \, d\tau \, ds
\]
for any $g \in D_+$. Therefore

$$\tilde{S}g(t) = g(t)e^{-ct} + \int_0^{+\infty} g(\tau)K(\tau, t) d\tau, \quad (37)$$

where

$$K(\tau, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{is(t-\tau)} \left[ e^{-\varphi(is)\tau}(1 + \nu(is)) - e^{c\tau} \right] ds.$$ 

The existence of the last integral (in the principle value sense) for any $\tau$ and almost all $t$, follows from the Plancherel theorem, since formula (36) implies square integrability, with respect to $s$, of the function $e^{-ist} \left[ e^{-\varphi(is)\tau}(1 + \nu(is)) - e^{c\tau} \right]$.

According to the Parseval identity and using formulas (35) and (36), we obtain

$$\int_{-\infty}^{+\infty} |K(\tau, t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| e^{-\varphi(is)\tau}(1 + \nu(is)) - e^{c\tau} \right|^2 ds \leq \frac{e^{-2\tau\varepsilon}}{\pi} \int_{-\infty}^{+\infty} \left( |e^{-(\varphi(is)-\varepsilon)\tau} - e^{(c+\varepsilon)\tau}|^2 + |e^{-(\varphi(is)-\varepsilon)\tau}\nu(is)|^2 \right) ds \leq \frac{Ae^{-2\tau\varepsilon}(\tau^2 + 1)}{\pi}.$$ 

According to Fubini’s theorem

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} |K(\tau, t)|^2 dt d\tau \leq \frac{A}{\pi} \int_0^{+\infty} e^{-2\tau\varepsilon}(\tau^2 + 1) d\tau < +\infty.$$ 

Therefore, $K \in L_2([0, +\infty)^2)$. Since $D_+$ is dense in $L_2[0, +\infty)$, formula (37) is valid for any $g \in L_2[0, +\infty)$. Therefore $\tilde{S} = \tilde{A} + \tilde{B}$, where

$$\tilde{A}g(t) = g(t)e^{-ct} \quad \text{and} \quad \tilde{B}g(t) = \int_0^{+\infty} g(\tau)K(\tau, t) d\tau,$$

the operator $\tilde{A}$ is bounded, selfadjoint and has purely absolutely continuous spectrum $[0, 1]$, and the operator $\tilde{B}$ is a Hilbert–Schmidt operator. It remains to note that operators $S$ and $\tilde{S}$ are unitarily equivalent. ■

The following lemma is a consequence of Lemma 5.2 of Chapter 1 in the book [21].

**Lemma 9.**

Let $X$ be a Banach space, $A$ be a bounded linear operator on $X$ with simply connected spectrum $\sigma(A) = K$ without isolated points and $B$ be a compact linear operator on $X$.  

13
Then the spectrum of $A + B$ is the union of $K$ and a finite or countably infinite set of isolated eigenvalues of finite multiplicity.

Now we can prove Theorem 3. Evidently $U = U^0 + U^1$, where

$$U^0 f(x) = \frac{1}{2} \left( 1 + \frac{(1 - x)^2}{4} \right) f \left( \frac{1 - x^2}{4} \right),$$

$$U^1 f(x) = \frac{1}{2} \left( 1 - \frac{(1 - x)^2}{4} \right) f \left( \frac{(1 - x)^2}{4} \right).$$

According to Lemma 7, there exists a $r > 1 + q$ such that for any $f \in X$ the function $U^1 f$ admits an analytic extension to $D(q,r)$. Let $X_r$ be the Hardy space in the disk $D(q,r)$. Then formula (39) defines a bounded linear operator $U^1_X$ from $X$ to $X$. The operator $U^1_X : X \to X$ is the superposition of $U^1_r$ and the identity embedding $J$ of $X_r$ into $X$. Since the operator $J$ is nuclear ($J$ has the s-numbers $(1 + q)^n r^{-n}$), we find that $U^1_X = U^1_r|_X : X \to X$ is also nuclear and therefore compact.

Let $H$ be the Hardy space in the upper half-plane $\{ z \in \mathbb{C} : \text{Re } z > (1 + q)^{-1}\}$. One can easily verify that the operator $M : X \to H$,

$$M f(z) = \frac{1}{z} f \left( \frac{2}{z - 1} \right),$$

is unitary up to a multiplication on a positive constant. Then the operators $U^0_X$ and $W^0 = MU^0 M^{-1} : H \to H$ are unitarily equivalent.

From the definitions of (39), (40) we obtain that

$$M^{-1} f(z) = \frac{2}{z + 1} g \left( \frac{2}{z + 1} \right) \quad \text{and} \quad W^0 f(z) = (1 + \nu(z)) f(z + \varphi(z)),$$

where

$$\nu(z) = \frac{1 - z}{2z^2 - z} \quad \text{and} \quad \varphi(z) = \frac{1}{2} + \frac{1}{4z - 2}.$$

One can easily verify that $\varphi$ and $\nu$ satisfy all conditions of Lemma 8 with $\varepsilon = c = 1/2$. According to Lemma 8, $W^0$ is a sum of a self-adjoint operator with the purely absolutely continuous spectrum $[0, 1]$ and a Hilbert–Schmidt operator. Since $U^0_X$ and $W^0$ are unitarily equivalent and $U^1_X$ is nuclear, we have that $U_X$ is a sum of a self-adjoint operator with purely absolutely continuous spectrum $[0, 1]$ and a Hilbert–Schmidt operator. It remains to apply Lemma 9. ■

6 Concluding remarks

1. So far there exist very few results on the spectral properties of the F.P.O. of the maps with parabolic neutral fixed points. We would like to point out the result of H. Rugh [17], who considered the F.P.O. of piecewise analytic maps, which are expanding everywhere except one parabolic fixed point. Namely, he constructed a specific map-dependent Banach space of analytic functions, where the spectrum of the F.P.O. consists of the segment $[0,1]$ and some isolated normal eigenvalues. This space is in fact the image of
$L_1[0, +\infty)$ with respect to some map-dependent integral transformation similar to the Laplace transform.

The cusp map does not satisfy the conditions of Rugh’s theorem because of the cusp-shaped singularity. Nevertheless, we proved that the F.P.O. of the cusp map has similar spectral properties in the Hardy spaces $H^2$ in the disks $D(q, 1 + q), 0 < q < 1$. We also conjecture that the spectrum of the F.P.O. $U$ of the cusp map in the Hardy spaces $H^2$ in the disks $D(q, 1 + q), 0 < q < 1$ is precisely the segment $[0, 1]$, i.e., the set of isolated eigenvalues of $U$ is empty. Note that the functions of these Hardy spaces as well as the functions of Rugh’s spaces are analytic in all points of the segment except at the parabolic fixed point ($x = -1$ in the case of the cusp map). However, we should notice that the spectrum of the F.P.O. of a map $S$ in spaces of analytic functions with singularity at a fixed point of $S$ may differ considerably from the spectrum in spaces of everywhere analytic functions. We have proved that this is precisely the case for the cusp map.

2. The theory of the point spectrum for the maps has been recently developed in terms of locally convex topological vector spaces [9]. For different classes of observables the same evolution law may have different resonances i.e. different rates of approach to equilibrium. However, once the class of observables is chosen, the resonance structure is unique [9, 10]. In terms of the assumptions of [9], the admissible point spectra for a given map are described. Here we see that for the cusp map, we have continuous spectra in Hardy spaces. This type of spectra were not addressed in [9].

3. We would like also to notice that Theorems 1 and 2 remain valid if one replaces the F.P.O. (1) of the cusp map by some positive transfer operator [6] of the cusp map, for example:

$$\tilde{U}f(x) = \frac{1}{2} f \left( \frac{(1 - x)^2}{4} \right) + \frac{1}{2} f \left( -\frac{(1 - x)^2}{4} \right).$$

Of course, Theorems 1 and 2 do not remain valid for all positive transfer operators in the class considered in [6]. For example, let us consider the operator

$$Wf(x) = \left( \frac{1}{2} - \beta + \frac{a(x)}{4} \right) f(a(x)) + \left( \frac{1}{2} + \beta - \frac{a(x)}{4} \right) f(-a(x)),$$

where $a(x) = (1 - x)^2/4$. For the real parameter $\beta \in [-1/4, 1/2]$, this is a positive transfer operator for the cusp map, and $W1 = 1$. On the other hand, the function

$$f(x) = x - \frac{x^2}{2} + \frac{\beta}{2(1 - \beta)}$$

is the eigenfunction of $W$ corresponding to the eigenvalue $\beta$: $Wf = \beta f$. Hence Theorems 1 and 2 are not valid for the operator $W$.

4. There are few questions which remain open for the cusp map. First, the question about the asymptotics of the autocorrelation function. As the eigenvalues of the F.P.O. of the family (2) tend to unity when $\varepsilon \to 1/2$, one can expect non-exponential decrease of the autocorrelation function. The estimations in [14] show that the autocorrelation function $C(n)$ decreases as $1/n$, when $n \to \infty$. However, this conjecture has not yet been analytically proven. Another question addresses the choice of the space of analytic
functions where the spectrum of the F.P.O. is naturally defined by the dynamics of the map.

Acknowledgments. We would like to thank Profs. Ilya Prigogine and Victor Sadovnichy for helpful discussions. We would like also to thank the referee who motivated us to formulate and prove Theorem 3 which added value to the paper. This work enjoyed the financial support of the European Commission, project IST-2000-26016 IMCOMP, and the Belgian Government through the Interuniversity Attraction Poles. S. A. Shkarin is supported by the Alexander von Humboldt foundation.

References

[1] P. C. Hemmer, J. Phys. A17, L247 (1984).
[2] E. Ott, Rev. Mod. Phys. 53, 655 (1981).
[3] W. Tucker, C. R. Acad. Sci. Paris 328, 1197 (1999).
[4] I. P. Cornfeld, S. V. Fomin and Ya. G. Sinai, Ergodic Theory, Springer-Verlag, New York, 1982.
[5] A. Lasota and M. Mackey, Probabilistic Properties of Deterministic Systems, Cambridge University Press, Cambridge, 1985.
[6] V. Baladi, Positive Transfer Operators and Decay of Correlations, World Scientific, 2000.
[7] T. Bedford, M. Keane and C. Series, Ergodic theory, symbolic dynamics, and hyperbolic spaces, Oxford University Press, 1991.
[8] I. Antoniou and Bi Qiao, Phys. Lett. A215, 280 (1996).
[9] I. Antoniou and S. Shkarin, in Generalized functions, operator theory, and dynamical systems, eds. I. Antoniou and G. Lumer, (Chapman & Hall/CRC Research Notes in Mathematics 399, London) p. 171 (1999).
[10] I. Antoniou, V. Sadovnichii and S. Shkarin, Phys. Lett. A258, 237 (1999).
[11] O.F. Bandtlow, I. Antoniou and Z. Suchanecki, Computers Math. Appl. 34, 95 (1997).
[12] G. Győrgyi and P. Szépfalussy, Z. Phys. B55, 179 (1984).
[13] Z. Kaufmann, H. Lustfeld and J. Bene, Phys. Rev. 1 E53, 1416 (1996).
[14] H. Lustfeld and P. Szépfalussy, Correlation functions on the border of transient chaos, Phys. Rev. E, 1996, vol. 53, pp 5882–5889
[15] D. Ruelle, Comm. Math. Phys. 125, 239 (1989).
[16] I. Antoniou, S. A. Shkarin and E. Yarevsky, *Resonances of the cusp family*, submitted to J.Phys.A.

[17] H. H. Rugh, Invent. Math. **135**, 1 (1999).

[18] A. Robertson and V. Robertson, *Topological Vector Spaces*, Cambridge University Press, Cambridge, 1964.

[19] K. Hoffman, *Banach spaces of analytic functions*, Dover, New York, 1988.

[20] E. C. Titchmarsh, *The theory of functions*, Oxford University Press, Oxford, 1984.

[21] I. Gohberg and M. Krein, *Introduction to the theory of linear nonselfadjoint operators*, Translations of Math. Monographs, **18**, Amer. Math. Soc., Providence, R.I., 1969.