A Nonlinear Damage Creep Model with Time-varying Viscoelasticity Based on Fractional Theory for Rock Materials

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Abstract: This study is aimed at introducing a nonlinear damage creep model with time-varying viscoelasticity, which is based on the Caputo fractional derivative with respect to the Mittag-Leffler (M-L) and damage functions. First, the M-L function was applied to the Caputo fractional calculus to describe the nonlinear creep behavior. A series of creep experiments of sandstone were performed and the mechanical behavior of the accelerating creep was depicted in detail. Second, the experimental data of sandstone and experimental data of salt rock and mudstone achieved from previous classical studies were used to confirm the reasonability and applicability of the proposed damage creep model. Finally, the proposed damage creep model was reported to be in excellent agreement with the obtained experimental creep data, and the accuracy between predicted data and measured data is analyzed.

Keywords: Rock creep; Damage creep model; Caputo fractional calculus; Time-varying viscoelasticity; Model validation

1. Introduction
The rheology of rock has attracted considerable attention in rock engineering. Creep is the time-dependent deformation through which rock materials fail [1, 2]. However, the time of failure in the creeping stage cannot be precisely
determined [3, 4]. Thus, many studies have been conducted to this effect and considerable results have been achieved [4, 5]. Jiang et al. studied the stress measurement in a deep soft rock under creep [5]. Zhou et al performed several creep experiments for a salt rock and a creep damage model was proposed to predict the creep deformation, which was based on fractional and acoustic emission theories [6].

To describe the mechanical behavior of creep, the fractional theory was introduced to model the creep behavior of rock materials; indeed, excellent results have been achieved. A variable-order fractional creep model was proposed to describe the creep behavior of soft rock materials. The model was reported to be in excellent agreement with the data from the creep experiment of the salt rock materials [7]. Arıkoğlu et al. proposed a fractional creep model with ten parameters for describing the viscoelastic behavior of rocks [8]. Wu et al. conducted a series of creep experiments of salt rocks and introduced a fractional creep model that has been confirmed [9]. Thus, the fractional theory has been proven effective for investigating the mechanical behavior of rock materials under creep.

The damage effect in creep is another important aspect of rock failure. Rock failure is usually attributed to the damaging effect of creep [10–11]. Thus, considerable effort has been devoted to explore such a problem. Kang et al. proposed a modified nonlinear creep model for describing the creep behavior of coal under various conditions [12]. Wu et al. conducted a series of creep experiments of salt rocks. Based on the experimental data obtained, a nonlinear creep damage model was proposed to describe the nonlinear characteristics of creep [13]. Thus, the damage effect must be considered while formulating creep models.

Hence, for developing creep models, the fractional theory and damage effect should be considered. In this study, based on the Caputo fractional theory with respect to the M-L function, an improved viscoelastic Scott-Blair (S-B) model was proposed. Furthermore, a nonlinear fractional damage creep model was proposed based on a form of Norton power, which is a modified damage function, in conjunction with the improved viscoelastic S-B model. Based on the experimental data of sandstone, salt rock, and mudstone, the applicability of the proposed model was verified and confirmed to be in excellent agreement with experimental data.

2. Caputo fractional theory

**Definition 1**

Let $\lambda > 0$, $I \in (a, b)$, $f(t) \in L(I)$, $g(t) \in C(I)$, and $g'(t) \neq 0$. $g(t)$ is an increasing function for all $t \in I$. The fractional integral of $f(t)$ with respect to another function $g(t)$ is shown as follows:
\[
\int_{a}^{t} I_{a+}^{\lambda} g f(t) = \frac{1}{\Gamma(\lambda)} \int_{a}^{t} g'(t) [g(t) - g(\tau)]^{\lambda-1} f(\tau) d\tau
\]

When \( g(t) = t \) and \( g(t) = \ln(t) \), the Riemann–Liouville fractional integral and Hadamard fractional operator can be obtained, respectively. Then, if \( g(t) = t^{\alpha/\beta} \), the Erdelyi–Kober fractional integral can be obtained [14].

\textbf{Definition 2}

Let \( \text{Re}(\lambda) > 0 \), \( t > 0 \), and \( \lambda > 0 \). The Caputo fractional calculus is expressed as follows:

\[
\frac{\mathcal{C}}{C} D_{t}^{\lambda} f(t) = I_{a}^{n-\lambda} f^{(n)}(t) = \frac{1}{\Gamma(n-\lambda)} \int_{a}^{t} f^{(n)}(t) (t-\tau)^{\lambda-n+1} d\tau
\]

where \( n \geq \lambda + 1 \) and \( f(t) \) can be differentiated \( n \) times.

When \( a = 0 \) and \( n = 1 \), the abovementioned function can be derived as follows:

\[
\frac{\mathcal{C}}{C} D_{t}^{\lambda} f(t) = \frac{1}{\Gamma(1-\lambda)} \int_{0}^{t} f'(t) (t-\tau)^{\lambda-1} d\tau
\]

where \( \Gamma \) is a Gamma function.

\[
\Gamma(t) = \int_{0}^{t} e^{-\tau} \tau^{t-1} d\tau
\]

Considering Eqs. (1) and (2), the Caputo fractional derivative of \( f(t) \) compared to \( g(t) \) can be expressed as follows:

\[
\left( \frac{d}{a+ g'(t)dt} \right)^{\lambda} f(t) = I_{a+}^{n-\lambda, \alpha} \left( \frac{d}{g'(t)dt} \right)^{n} f(t)
\]

A valuable differential operator was introduced, which would aid in understanding the fractional law of a bottom derivative, i.e., \( \frac{d}{g'(t)dt} f(t) \).

By applying the well-known definition of \( f(t) = [g(t) - g(a +)]^{u-1} \) in Eq. (5) (Almeida. 2017), the new fractional equation can be expressed as follows:

\[
\left( \frac{d}{a+ g'(t)dt} \right)^{\lambda} f(t) = \frac{\Gamma(v)}{\Gamma(v-\lambda)} [g(t) - g(a +)]^{u-\lambda-1}
\]

\textbf{3. A nonlinear fractional damage creep model}

\textit{3.1 The Maxwell model}

The Maxwell model is a popular model used in describing the viscoelastic behavior of rocks. The creep constitutive model of the Maxwell model is shown as follows:

\[
\varepsilon(t) = \frac{\sigma_1}{E} + \frac{\sigma_1}{\eta} t
\]
where $\sigma_1$ is the constant stress during creep, $E$ is the elastic modulus, and $\eta$ is the viscosity coefficient. If $t = 0$, the Maxwell model is converted to a Hooke body.

However, the Maxwell model cannot predict the time effect of creep, and it is not capable of predicting the time-varying viscoelasticity in creep. Thus, the viscous element should be modified and the Maxwell model should be improved to completely describe the creep behavior.

### 3.2 The improved Scott-Blair model with time-varying viscoelasticity based on power function

In 1947, Scott-Blair et al. introduced a popular fractional viscoelastic model [13]. The constitutive equation of their model is shown as follows:

$$\sigma(t) = M \frac{d^\alpha \varepsilon(t)}{dt^\alpha}$$  \hspace{1cm} (8)

where $\sigma(t)$ is the stress, $\varepsilon(t)$ represents the strain, $\alpha$ is the fractional order, and $M$ is a parameter of material.

Wu et al. applied the Riemann–Liouville (R–L) fractional calculus to obtain the expression of the S-B model [15]. When stress is constant, the S-B model can be expressed as follows:

$$\sigma_1 = M^0_0 D^\alpha \varepsilon(t) = M \frac{\varepsilon(t)}{\Gamma(1 - \alpha) t^{-\alpha}}$$  \hspace{1cm} (9)

The time-varying property under various stress levels cannot be efficiently described using Eq. (9). Thus, Eq. (1) and (5) were applied to deduce the improved S-B model based on the Caputo fractional calculus.

$$\sigma(t) = M \frac{d^\alpha \varepsilon(t)}{dt^\alpha} = M \left( \frac{d}{g(t) dt} \right)^{\alpha} \varepsilon(t)$$  \hspace{1cm} (10)

where $\alpha$ is the fractional-order and $\alpha \in (0, 1)$. Note that $g(t)$ is an increasing function when $t > 0$.

When stress is constant, $\sigma(t) = \sigma_2$ and $n = 1$, the new creep constitutive model can be obtained as follows:

$$\varepsilon(t) = \frac{\Gamma(1 - \alpha) \sigma_2}{M} [g(t) - g(0)]^{\alpha+1}$$  \hspace{1cm} (11)

In Eq. (11), $g(t)$ is a kernel function describing the time-varying behavior. In this study, the M-L function was regarded as the kernel function, i.e., $g(t) = \frac{1}{c} E_{\alpha,\beta}(ct)$. Thus, Eq. (11) is expressed as follows:

$$\varepsilon(t) = \frac{\sigma_2}{\Gamma(1 + \alpha) M} \left( \frac{E_{\alpha,\beta}(ct) - 1}{c} \right)^{1+\alpha}$$  \hspace{1cm} (12)

On observation, when $\alpha = 0$, Eq. (12) shows the creep behavior of a solid material; however, when $\alpha = 1$, it represents the Newton fluid body.
3.3 The time-dependent damage model

It is well known that the damage behavior is along with the total creep [16]. Based on the damage theory, Tang et al. introduced a damage factor that can be expressed as follows [19]:

\[ D = 1 - e^{-\left(\frac{\varepsilon(t)}{\varepsilon_0}\right)^m} \quad (13) \]

where \( D \) is the damage factor, \( \varepsilon_0 \) is the strain, and \( m \) is a parameter related to the stress level.

Based on the Norton power law, the strain rate can be expressed as follows using Eq. (14):

\[ \frac{d\varepsilon(t)}{dt} = M_1 \left(\frac{\sigma(t)}{\sigma^*}\right)^{\gamma} \quad (14) \]

where \( \gamma \) is the exponential law, \( \frac{d\varepsilon(t)}{dt} \) is the strain rate, \( M_1 \) is the parameter of material, and \( \sigma^* \) is the unit stress.

Based on the function style of the damage factor with Eq. (13), the time-dependent damage model can be reconstituted as follows:

\[ \varepsilon(t) = M_1 \left(\frac{\sigma(t)}{1 - e^{\left(\frac{\varepsilon}{\varepsilon_0}\right)^m}}\right) t \quad (15) \]

3.4 The new nonlinear fractional damage creep model

As shown in Fig. 3, the creep process could be divided into three parts: the decaying phase (primary phase), stable phase (steady phase), and the accelerating phase (tertiary phase). This phenomenon is similar to the results in the literature [16]. According to the strain superposition principle in creep, the creep model can be expressed as follows:

\[ \varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \quad (16) \]

where \( \varepsilon \) is the total strain. When stress is constant \( \sigma = \sigma_3 \). Then, the strain as a function of time is given as follows:

\[ \varepsilon(t) = \frac{\sigma_3}{E} + \frac{\sigma_3}{\Gamma(1 + \alpha)M} \left(\frac{E_{\alpha,\beta}(ct) - 1}{c}\right)^{1+\alpha} + M_1 \left(\frac{\sigma_3}{1 - e^{\left(\frac{\varepsilon}{\varepsilon_0}\right)^m}}\right)^n t \quad (17) \]

4. Experimental preparation

4.1 Experimental device

The creep experiments were performed using five-connected rheological apparatus at the State Key Laboratory for GeoMechanics and Deep Underground Engineering, China University of Mining and Technology (Beijing). As shown in Fig. 1, this device embraces high precision for tests.
4.2 Experimental sample
The sandstone sample was obtained from the Ningtiao coal mine in Shanxi, China. As shown in Fig. 2, the samples were drilled from a sound rock. The experimental operations all conform to the demand of “Test Standard for Engineering Rocks.” The tested sandstone sample is cylindrical in shape with a height and diameter of 100 and 50 mm, respectively.

5. Experimental results and verification of the proposed model
5.1 Applicability of introduced model for rock materials
The applicability and rationality of the proposed model were verified in this study. Except for the experimental creep data of sandstone samples obtained, the experimental creep data of salt rock and mudstone samples were obtained to verify the proposed model, which was derived from previously conducted classical studies. Moreover, the formulated model was compared with the Maxwell model to determine its advantage. The fitting parameters of the formulated model are presented in Table 1. The maximum relative error of the predicted and measured data of the rock samples were all <5%, which are within reasonable ranges.
Figure 3 (a) Experimental creep data of the sandstone sample, fitting curves from the proposed model, and the Maxwell model under a stress level of 39.4 MPa. (b) Comparisons between the measured and predicted data obtained from the proposed model under a stress level of 39.4 MPa.

For verifying the proposed model, the accelerating creep stage of sandstone is considered. As shown in Fig. 3 (a), the fitting curve obtained from the proposed model is in excellent agreement with the experimental data compared to the Maxwell model. Except for the decaying and stable creep, the nonlinear behavior of the accelerating creep can be depicted by the proposed model. In Fig. 3(b), the experimental data and the predicted data from the proposed model were compared based on a standard line and the dependency of both sets of data has high accuracy. Moreover, the proposed model can excellently predict the creep data with damage when the whole stress is considered.

Fig. 4 shows the experimental creep data of the salt rock sample obtained from the study conducted by Wang [17]. The fitting curve of the proposed model has a better correlation with the experimental data than that of the Maxwell model. The relative error between the measured and predicted data falls within a reasonable range.

Considering the mechanical properties of the soft rock sample, the mechanical behavior of the mudstone sample is similar to that of the salt rock in most conditions [18]. Based on the proposed model, in Fig. 5(b), the experimental data of the mudstone sample excellently correlates with the predicted data under a constant stress level of 30 MPa.

Figure 4 (a) Experimental creep data of the salt rock sample obtained from the study conducted by Wang and fitting curves from the proposed model, and the Maxwell model under a stress level of 26 MPa; (b) Comparisons between the measured and predicted data obtained from the proposed model under a stress level of 26 MPa.
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Figure 5 (a) Experimental creep data of the mudstone sample and fitting curves from the proposed model and the Maxwell model under a stress level of 30 MPa; (b) Comparisons between the measured and predicted data obtained from the proposed model under a stress level of 30 MPa.

Finally, Table 1 are the fitting parameters of the proposed model, which were calculated by the least squares method.

| Parameters | Material | $E$ (GPa) | $M$ (GPa * $h^{\alpha}$) | $\alpha$ | $\beta$ | $M_1$ | $\varepsilon_0$ | $m$ | $n$ |
|------------|----------|-----------|----------------|---------|-------|------|----------|---|---|
|            | Sandstone | 79.88     | 3.21          | 0.62    | 1.66  | 0.112| 0.25     | 0.03 | 0.58 |
|            | Salt rock | 13.54     | 28.98         | 0.80    | 0.85  | 0.008| 0.25     | 0.19 | 0.73 |
|            | Mudstone  | 19.21     | 20.15         | 0.57    | 1.29  | 1.211| 0.03     | 0.13 | 0.04 |

6. Conclusion

1) Based on the Caputo fractional theory with respect to the M-L function and the modified damage function, a nonlinear fractional damage creep model was proposed.
2) Series of creep experiments were performed using sandstone. Based on the experimental creep data of sandstone, salt rock, and mudstone, the applicability of the introduced model was confirmed.
3) The proposed model is in excellent agreement with the experimental data and the relative error between both sets of data were deduced and analyzed. Thus, the introduced model can excellently describe the creep behavior of rock materials.

References

[1] Nopola J R., Roberts L A.: Time-dependent deformation of Pierre Shale as determined by long-duration creep tests. Houston: American Rock Mechanics Association, (2016).
[2] Mighani S., Taneja S., Sondergeld C H, et al.: Nanoindentation creep measurements on shale. San Francisco: American Rock Mechanics Association, (2015).
[3] Mauricio G.: Soft rocks in Argentina. International Journal of Mining Science and Technology 24, 883-892 (2014).
[4] Yang R., Li Y., Guo D., et al.: Failure mechanism and control technology of water-immersed roadway in high-stress and soft rock in a deep mine. Int J of Min Sci Tech 27, 245-252 (2017).
[5] Jiang J., Liu Q., Xu J.: Analytical investigation for stress measurement with the rheological stress recovery method in deep soft rock. Int J of Min Sci Tech 26, 1003-1009 (2016).
[6] Zhou H.W., Wang C.P., Han B.B., et al.: A creep constitutive model for salt rock based on fractional derivatives. Int J Rock Mech Min Sci 48, 116-121 (2011).
[7] Tang H., Wang D., Huang R., et al.: A new rock creep model based on variable-order fractional derivatives and continuum damage mechanics. Bull Eng Geol Environ 77, 1-9 (2017).
[8] A. Arikoglu.: A new fractional derivative model for linearly viscoelastic materials and parameter identification via genetic algorithms, Rheol. Acta 53, 219-233 (2014).
[9] Wu, F., Liu, J.F., Wang, J.: An improved Maxwell creep model for rock based on variable order fractional derivatives. Environ. Earth Sci 73, 6965-6971 (2015).
[10] Cao P., Youdao W., Yixian W., et al.: Study on nonlinear damage creep constitutive model for high-stress soft rock. Environ Earth Sci 75(10), 900-909 (2016).
[11] Zhao Y., Wang Y., Wang W., et al.: Modeling of non-linear rheological behavior of hard rock using triaxial rheological experiment. Int J Rock Mech Min Sci 93, 66-75 (2017).
[12] Kang J.H., Zhou F.B., Liu C., Liu. Y.K.: A fractional non-linear creep model for coal considering damage effect and experimental validation, Int. J. Non-Linear Mech 76, 20-28 (2015).
[13] Wu F., Chen J., Zou Q.: A nonlinear creep damage model for salt rock. International Journal of Damage Mechanics. 28(5), 758-771 (2018).
[14] B. Suryanto.: Predicting the creep strain of PVA-ECC at high stress levels based on the evolution of plasticity and damage. J. Adv. Concr. Techol 11, 35-48 (2013).
[15] Wu, F., Liu, J.F., Wang, J.: An improved Maxwell creep model for rock based on variable-order fractional derivatives. Environ. Earth Sci 73, 6965-6971 (2015).
[16] Cai, M.F.: Rock Mechanics and Engineering; Beijing Science Press: Beijing, China, (2006).