General Series Solutions for Stresses and Displacements in an Inner-fixed Ring

Yongshu Jiao\textsuperscript{a}, Shuo Liu\textsuperscript{b}, Dexuan Qi\textsuperscript{c}

School of Mechanical Engineering, Hebei University of Technology
Tianjin, 300401, P. R. China

*Corresponding Author

\textsuperscript{a}E-mail: ysjiao@hebut.edu.cn \textsuperscript{b}E-mail: liushuo11235@163.com \textsuperscript{c}E-mail: dxqi@tju.edu.cn

Abstract. The general series solution approach is provided to get the stress and displacement fields in the inner-fixed ring. After choosing an Airy stress function in series form, stresses are expressed by infinite coefficients. Displacements are obtained by integrating the geometric equations. For an inner-fixed ring, the arbitrary loads acting on outer edge are extended into two sets of Fourier series. The zero displacement boundary conditions on inner surface are utilized. Then the stress (and displacement) coefficients are expressed by loading coefficients. A numerical example shows the validity of this approach.

1. Introduction
The circular ring is probably the most commonly treated geometry, since it is the transverse cross-section of tubes widely used in many kinds of construction. Many researchers have investigated the stresses in elastic rings. The ring subjected to equal and opposite concentrated forces diametrically applied to the outside boundary was studied since 1901 by Timoshenko and Goodier (1951)\textsuperscript{[1]}. By means of complex analysis, Jaeger and Hoskins (1966) determined the full-field stress in a circular ring on which the loads were uniformly distributed radial pressure along two opposite arcs instead of concentrated line loads\textsuperscript{[2]}. Since then, this subject is studied continually\textsuperscript{[3]}/\textsuperscript{[4]}. Recently, Tokovyy, Hung, and Ma (2010) investigated a thin annular disk under the external compression uniformly distributed on two opposite arcs\textsuperscript{[5]}. They presented stresses in a convenient form for analysis via direct integration method. Stresses field, principal stresses, maximum shearing stress, and displacements field were obtained and verified by experimental testing. Serati, Alehossein, and Williams (2012) analyzed a circular hollow disc subjected to partially distributed radial tractions\textsuperscript{[6]}. Radial loads were distributed on internal and external boundaries and tangential loads at the boundaries were zero. Stresses and displacements were determined via Michell solution. Kourkoulis and Markides (2014) used Muskhelishvili's complex potentials technique to study the ring under parabolic distribution of radial pressure\textsuperscript{[7]}. Kourkoulis, Markides, and Pasiou (2015) showed a combined study in analytic and experiment in a ring\textsuperscript{[8]}. The main objective of the present paper is to derive the general series solutions (GSS) for stresses and displacements in an elastic ring based on the fundamental theory of elasticity and Michell-Fourier series technique. Combining the GSS for stresses and displacements derived in this paper, we can calculate the stresses and displacements in elastic rings with both loading boundary conditions and displacement boundary conditions. As an application, an inner-fixed elastic ring arbitrarily loaded on the outer surface is considered.
2. GSS for Stresses and Displacements

Consider a plane stress problem in a homogeneous, linear elastic and isotropic material. For a multi-connected body like an elastic ring, we take the following function as the stress function

\[
\varphi = C_{01}r^2 + C_{03}lr + D_{03}r + C_{14}r^3 \cos \theta + D_{14}r^3 \sin \theta + [C_{11}r^3 + C_{12}r^{-1} + 0.5(\nu - 1)D_{14}l]r \cos \theta + [D_{11}r^3 + D_{12}r^{-1} + 0.5(\nu - 1)C_{14}l]r \sin \theta + \sum_{m=2}^{\infty} (C_{m1}r^{m+2} + C_{m2}r^m + C_{m3}r^{-m+2} + C_{m4}r^{-m}) \cos m\theta + \sum_{m=2}^{\infty} (D_{m1}r^{m+2} + D_{m2}r^m + D_{m3}r^{-m+2} + D_{m4}r^{-m}) \sin m\theta
\]  

(2.1)

where \( \nu \) is the Poisson's ratio of the ring. \( C_{mn}, D_{mn} (m = 0, ..., \infty, n = 1, 2, 3, 4) \) are stress coefficients to be determined by the boundary conditions of the ring. Substituting this stress function into the stress expressions in elasticity, we have the radial normal stress

\[
\sigma_r = 2C_{01} + C_{03}r^{-2} + [2C_{11}r - 2C_{12}r^{-3} + 0.5(3 + \nu)D_{14}r^{-1}]\cos \theta + [2D_{11}r - 2D_{12}r^{-3} - 0.5(3 + \nu)C_{14}r^{-1}]\sin \theta + \sum_{m=2}^{\infty} [(1 + m)(2 - m)C_{m1}r^{m+2} + m(1 - m)C_{m2}r^{m-2} + (1 - m)(2 + m)C_{m3}r^{-m+2} + (1 + m)C_{m4}r^{-m-2} + (1 - m)(2 + m)D_{m1}r^{m+2} + m(1 - m)D_{m2}r^{m-2} + (1 - m)(2 + m)D_{m3}r^{-m+2} + (1 + m)D_{m4}r^{-m-2}] \sin m\theta,
\]

(2.2)

the circumferential normal stress

\[
\sigma_\theta = 2C_{01} - C_{03}r^{-2} + [6C_{11}r + 2C_{12}r^{-3} - 0.5(1 - \nu)D_{14}r^{-1}]\cos \theta + [6D_{11}r + 2D_{12}r^{-3} + 0.5(1 - \nu)C_{14}r^{-1}]\sin \theta + \sum_{m=2}^{\infty} [(1 + m)(2 - m)C_{m1}r^{m+2} + m(m - 1)C_{m2}r^{m-2} + (1 - m)(2 + m)C_{m3}r^{-m+2} + (1 + m)C_{m4}r^{-m-2} + (1 - m)(2 + m)D_{m1}r^{m+2} + m(m - 1)D_{m2}r^{m-2} + (1 - m)(2 + m)D_{m3}r^{-m+2} + (1 + m)D_{m4}r^{-m-2}] \cos m\theta,
\]

(2.3)

and the shearing stress

\[
\tau_{r\theta} = D_{03}r^{-2} + [-2D_{11}r + 2D_{12}r^{-3} - 0.5(1 - \nu)C_{14}r^{-1}]\cos \theta + [2C_{11}r - 2C_{12}r^{-3} - 0.5(1 - \nu)D_{14}r^{-1}]\sin \theta + \sum_{m=2}^{\infty} [m(-1 + m)D_{m1}r^{m+2} + (m - 1)mD_{m2}r^{m-2} + (m - 1)(m + 1)D_{m3}r^{-m+2} + (m + 1)mD_{m4}r^{-m-2}] \cos m\theta + \sum_{m=2}^{\infty} [m(1 + m)C_{m1}r^{m+2} - (m - 1)mC_{m2}r^{m-2} - (m - 1)(m + 1)C_{m3}r^{-m+2} + (m + 1)mC_{m4}r^{-m-2}] \sin m\theta.
\]

(2.4)

It can be easily verified that these stresses satisfy not only the equilibrium equations, but also the strain compatibility equation expressed by stresses.

In order to obtain the displacements in an elastic ring, the geometric equations and generalized Hooke's law in elasticity must be employed. In this way, we have the radial and circumferential displacements as follows

\[
Eu_r = \int (\sigma_r - \nu \sigma_\theta) dr + f(\theta),
\]

(2.5)

where \( E \) is the elastic modulus of the material, \( f(\theta) \) is an undefined function of \( \theta \) only, \( g(r) \) an unknown function of \( r \) only. Substituting the stresses \( \sigma_r, \sigma_\theta \) in (2.2)-(2.4) into the expressions (2.5) and making the integrations, we can get the radial displacement \( u_r \) and circumferential displacement \( u_\theta \). Then bringing these displacements into geometric equations, we can get the shearing strain \( \gamma_{r\theta} \) with the unknown integration functions \( f(\theta) \) and \( g(r) \).

In order to determine the unknown functions \( f(\theta) \) and \( g(r) \), we can express the shearing strain through the shearing stress \( \tau_{r\theta} \) in (2.4). The comparison of these two expressions of shearing strain demands that

\[
f(\theta) = H_1 \cos \theta + H_2 \sin \theta, \quad g(r) = H_3 r - (1 + \nu)D_{03}r^{-1} - k,
\]

(2.6)

where \( H_1 \) and \( H_2 \) represent the rigid translational displacements along \( x \) and \( y \) directions, \( H_3 \) the rigid rotational angle. \( k \) is a constant which will be disappeared in the displacements. Taking the functions \( f(\theta) \) and \( g(r) \) into the displacements in (2.5), we have the radial and circumferential displacements as follows,
$$Eu_r = 2(1 - \nu)C_{00}r - (1 + \nu)C_{03}r^{-1} + [H_1 + (1 - 3\nu)C_{11}r^2 + (1 + \nu)C_{12}r^{-2} + 0.5(3 + 2\nu - \nu^2)D_{14}\ln r]\cos\theta$$
$$+ [H_2 + (1 - 3\nu)D_{14}r^2 + (1 + \nu)D_{12}r^{-2} - 0.5(3 + 2\nu - \nu^2)C_{14}\ln r]\sin\theta$$
$$+ \sum_{m=2}^{\infty} \left[(2 - m - 2\nu - m\nu)D_{m1}r^{m+1} - m(1 + \nu)D_{m2}r^{-m-1}\cos\theta \right.$$ 
$$+ (2 + m - 2\nu + m\nu)C_{m3}r^{m+1} + m(1 + \nu)C_{m4}r^{-m-1}\cos\theta \left.] \ln r \right|_{14}\cos\theta$$
$$E\theta = -(1 + \nu)D_{03}r^{-1} + H_3r + (H_2 - (5 + \nu)D_{11}r^2 - (1 + \nu)D_{12}r^{-2} - 0.5[(1 + \nu)^2 + (3 + 2\nu - \nu^2)\ln r]D_{14}\sin\theta$$
$$+ \sum_{m=2}^{\infty} \left[- (4 + m + m\nu)D_{m1}r^{m+1} - m(1 + \nu)D_{m2}r^{-m-1} \right.$$ 
$$+ (4 - m - m\nu)D_{m3}r^{-m+1} - m(1 + \nu)D_{m4}r^{-m-1}\cos\theta \right.$$ 
$$+ \sum_{m=2}^{\infty} \left[(4 + m + m\nu)C_{m3}r^{m+1} + m(1 + \nu)C_{m4}r^{-m-1} \right.$$ 
$$- (4 - m - m\nu)C_{m3}r^{-m+1} + m(1 + \nu)C_{m4}r^{m+1}\sin\theta \right.$$ \]

(2.7)

3. The Boundary Conditions
Suppose that the normal and tangential loads acting on the outer surface are $p_0(\theta)$ and $q_0(\theta)$, respectively. Expand these loads into Fourier series forms, such as

$$p_0(\theta) = A_{30}/2 + \sum_{m=1}^{\infty} (A_{3m}\cos m\theta + B_{3m}\sin m\theta),$$
$$q_0(\theta) = A_{40}/2 + \sum_{m=1}^{\infty} (A_{4m}\cos m\theta + B_{4m}\sin m\theta),$$

(3.1)

where $A_{3m}, B_{3m}$ ($n = 3, 4, m = 0, \infty$) are loading coefficients to be determined by the usual technique of Fourier series expansion. The stress boundary conditions on this surface are

$$\sigma_r(b, \theta) = -p_0(\theta), \tau_{r\theta}(b, \theta) = q_0(\theta).$$

(3.2)

where $b$ is the outer radius of the ring.

Suppose that the ring is fixed on the inner surface, the displacement boundary conditions on this edge will be

$$u_r(a, \theta) = 0, u_\theta(a, \theta) = 0,$$

(3.3)

where $a$ is the inner radius of the ring.

4. Determination of Stress Coefficients
When we take the radial normal stress in (2.2) and the shearing stress in (2.4) into the stress boundary conditions in (3.2) and bring the radial displacement in (2.7) and the circumferential displacement in (2.8) into the displacement boundary conditions in (3.3), four equations about the stress coefficients $C_{nm}$, $D_{nm}$ ($m = 0, \ldots, \infty, n = 1, 2, 3, 4$) and loading coefficients $A_{nm}, B_{nm}$ ($n = 3, 4, m = 0, \infty$) are formulated. By comparing similar terms, these equations provide the following stress coefficients expressed by loading coefficients

$$C_{00} = \frac{b^2(1+\nu)A_{30}}{4[a^2(1+\nu)+b^2(1+\nu)]}, C_{03} = \frac{a^2b^2(1+\nu)A_{30}}{2[a^2(1+\nu)+b^2(1+\nu)]}, D_{03} = \frac{b^2A_{40}}{2},$$

(4.1)

$$C_{11} = \frac{b(1+\nu)[b^2(1+\nu)+a^2(1+\nu)]A_{11} + [a^2(1+\nu)-b^2(3+\nu)]B_{11}}{8[a^2(1+\nu)+b^2(1+\nu)]},$$

(4.2)

$$C_{12} = \frac{a^2b^2[3(2+4\nu^2)+2(1+\nu)]A_{11} + [a^2(1+\nu)-b^2(3+\nu)]B_{11}}{8[a^2(1+\nu)+b^2(1+\nu)]},$$

(4.3)

$$D_{11} = \frac{b(1+\nu)[b^2(1+\nu)+a^2(1+\nu)]B_{11} - [a^2(1+\nu)-b^2(3+\nu)]A_{11}}{8[a^2(1+\nu)+b^2(1+\nu)]},$$

(4.4)

$$D_{12} = \frac{a^2b^2[3(2+4\nu^2)+2(1+\nu)]B_{11} + [a^2(1+\nu)-b^2(3+\nu)]A_{11}}{8[a^2(1+\nu)+b^2(1+\nu)]}.$$
uniformly distributed loads in different sections on the outer surface, as shown in Fig. 1 (a) and (b).

Now, all stress coefficients \( k_{\alpha}h \), \( 1 \leq \alpha \leq 3 \) have been related to the loading coefficients.

An example investigates stresses and displacements in an elastic ring subjected to non-rigid body displacements are as follows

\[
\Delta = a^2b^4+2m^2(1+v)^2 - 2a^2+b^2(m^2+1)(1+v)^2 + a^4(2m+1+v)^2,
\]

\[
k_{11}(a,b) = a^2b^4(3+v) + a^2b^2(1+m)(1+v) - a^4b^2(1+m)(1+v),
\]

\[
k_{12}(a,b) = a^2b^4(3-v) + a^2b^2(1+m)(1+v) + a^4b^2(1+m)(1+v),
\]

\[
k_{21}(a,b) = a^2b^4(1-m)m(1+v)^2 + b^2(1-m)m(1+v)^2 + a^4b^2(1+m)(1+v)^2 + a^4b^2(1+v)^2,
\]

\[
k_{22}(a,b) = a^2b^4(2-m-m^2)(1+v)^2 + b^2(1-m)m(1+v)^2 + a^4b^2(1+m)(1+v)^2 + a^4b^2(1+v)^2,
\]

\[
k_{31}(a,b) = a^2b^4(3-v) + a^2b^2(1-m)(1+v) + b^2m(1+m)(1+v),
\]

\[
k_{32}(a,b) = a^2b^4(3-v) + b^2(1-m)m(1+v) + a^4b^2(1+m)(1+v),
\]

\[
k_{41}(a,b) = -a^2b^4m(1+m)(1+v)^2 + a^4b^2m(3-2v^2) + a^4b^2[8-8v+m^2(1+v)^2],
\]

\[
k_{42}(a,b) = b^2(1-m)^2(1+v)^2 + a^2m^2(2+m)^2(1-m)^2 + a^2b^2(2+m)^2(1-m)^2.
\]

The integration constants \( H_i (i = 1, 2, 3) \) related to the rigid body displacements are as follows

\[
H_1 = \frac{-b(1+v)[h_1(a,b)A_{11} + h_2(a,b)A_{41}]}{b[a^4(v-3) - b^4(1+v)]}, \quad H_2 = \frac{b(1+v)[h_2(a,b)A_{12} + h_1(a,b)B_{11}]}{b[a^4(v-3) - b^4(1+v)]}, \quad H_3 = \frac{(1+v)b^4a_{40}}{2a^2}.
\]

where

\[
h_1(a,b) = b^4(1+v)^2 + a^4(1-2v-3v^2) + 2a^2b^2(v^2-1) + 2[a^4(v-3)^2 + b^4(3+2v-v^2)]loga,
\]

\[
h_2(a,b) = b^4(1+v)^2 + a^4(1-2v-3v^2) + 2a^2b^2(3+4v+v^2) + 2[a^4(v-3)^2 + b^4(3+2v-v^2)]loga.
\]

Now, all stress coefficients \( C_m, D_m \) \( m = 0, ..., \infty, n = 1,2,3,4 \) as well as the integration constants \( H_i (i = 1, 2, 3) \) have been related to the loading coefficients.

5. An Numerical Example

The following example investigates stresses and displacements in an elastic ring subjected to non-uniformly distributed loads in different sections on the outer surface, as shown in Fig. 1 (a) and (b).
Figure 1: Loads in example ring. (a) Normal loads; (b) Tangential loads.

Figure 2: Comparisons of contours of displacements. (a) Radial displacement by GSS; (b) Radial displacement by FEM; (c) Circumferential displacement by GSS; (d) Circumferential displacement by FEM.
Figure 3: Comparisons of contours of stresses. (a) Radial stress by GSS; (b) Radial stress by FEM; (c) Circumferential stress by GSS; (d) Circumferential stress by FEM; (e) Shearing stress by GSS; (f) Shearing stress by FEM.

The outer radius of the ring is 100 mm, and inner radius 50 mm. The modulus of elasticity of the material is taken as 206 GPa. The Poisson’s ratio is assumed to be 0.3. The loading coefficients $A_3m, B_3m, A_4m, B_4m$ ($m = 0, \ldots, \infty$) in (3.1) for this problem can be determined by the usual technique of Fourier series expansion. The procedure is omitted for the limitation of space. After the loading coefficients are determined, the stress coefficients $C_{mn}, D_{mn}$ and the integration constants $H_i (i = 1, 2, 3)$ can be calculated according to (4.1)-(4.22). Now we are ready to evaluate the stresses and displacements at any point.
according to the expressions (2.2)-(2.4) and (2.7)-(2.8), respectively. The contour maps of displacements are shown in Fig. 2 (a), (c), and those of stresses are presented in Fig. 3 (a), (c) and (e).

In order to support the validity of our GSS for stresses and displacements, the same problem was also modeled in ANSYS using the finite element method (FEM). The corresponding displacement contour maps via ANSYS are presented in Fig. 2 (b), (d), and stress contour maps are depicted in Fig. 3 (b), (d) and (f). It can be seen that the patterns of corresponding maps appear highly similar. The stress and displacement values at the same (or nearby) positions are very close with little difference.

6. Conclusions
Based on the theory of elasticity and Fourier series approach, the GSS for stresses and displacements in an elastic ring is provided. As an application, an inner-fixed ring arbitrarily loaded on outer edge is considered. The stress (and displacement) coefficients and the integration constants related to rigid-body displacements are expressed by loading coefficients. The validity of these solutions is verified through a numerical example. The stresses and displacements in the example ring are calculated, showing great consistency with those from ANSYS.

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