THE BACK REACTION OF GRAVITATIONAL PERTURBATIONS

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The back reaction of gravitational perturbations in a homogeneous background is determined by an effective energy-momentum tensor quadratic in the perturbations. We show that this nonlinear feedback effect is important in the case of long wavelength scalar perturbations in inflationary universe models. We also show how to solve an old problem concerning the gauge dependence of the effective energy-momentum tensor of perturbations.

1 Back Reaction in Chaotic Inflation

It is a well known fact that gravitational waves carry energy and momentum, and as such are themselves a source of curvature for spacetime. High-frequency gravity waves, in particular, have an equation of state of a radiation fluid \((p = \rho/3)\) and this can be used to constrain their amplitude during BBN by taking into account the effect they have on the expansion and cooling rates of the universe.

Back reaction can also be quite important in the inflationary models of the Universe evolution. In the chaotic inflation model with a massive scalar field, for example, it is usually supposed that once the inflaton \(\varphi\) drops below the self-reproduction scale \(\varphi_{sr} \sim m^{-1/2}\) (in Plank units), the dynamics of the homogeneous FRW background proceeds classically with no influence from metric perturbations created henceforth. Our calculations show, however, that the effect of back reaction may become crucial midway through the period of slow-roll inflation.

To see that, consider the effective energy-momentum tensor (EEMT) obtained after averaging quadratic terms of the perturbative expansion of Einstein’s Equations about the FRW background,

\[
\tau_{\mu\nu} = \frac{1}{16\pi} \langle (G_{\mu\nu} - 8\pi T_{\mu\nu})_{ab}\delta q^a\delta q^b \rangle ,
\]

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where \( \langle \ldots \rangle \) denotes spatial averaging and \( \delta q^a \equiv \{ \delta g_{\mu\nu}, \delta \varphi \} \) are the perturbations to the metric and matter fields, for which \( \langle \delta q^a \rangle = 0 \) but \( \langle \delta q^2 \rangle \neq 0 \). This energy-momentum tensor, of second order in the perturbations, serves as an effective source in the RHS of Einstein’s equations.

The energy density in long wavelength scalar (density) perturbations (i.e., the 0-0 component of the EEMT) is proportional to \( \langle \delta \varphi^2 \rangle \). Making use of the known spectrum of first order perturbations, it can be shown that \( \tau_{00} \) becomes comparable to the energy density of the background before the end of inflation if the initial value of the scalar field was bigger than \( \varphi_0 \sim m^{-1/3} \).

\section{Gauge Dependence of the EEMT}

Once it is established that back reaction can be relevant and that the EEMT is the proper tool for handling it, we must deal with a problem inherent to that tensor: namely, its gauge dependence.

On a given manifold, a coordinate transformation \( x \rightarrow \tilde{x} = x + \xi \) (\( \xi \) small and \( \langle \xi \rangle = 0 \)) induces a gauge transformation on the tensors of that manifold which is expressed in terms of the Lie derivative: \( q \rightarrow \tilde{q} = q - L_\xi q \). In particular, the matter and metric field perturbations are transformed by the same law, which perturbatively reads

\[ \delta \tilde{q}^a = \delta q^a - [L_\xi q_0]^a. \]  

(2)

Since the EEMT is a function of the explicit perturbations on the fixed background, it is clear by Eq. 1 that it will change accordingly, \( \tilde{\tau}_{\mu\nu} \neq \tau_{\mu\nu} \).

The question now becomes, how to calculate back reaction and be sure that it does not include spurious “gauge” effects? In order to answer this we must first clarify the origin of the mystery, i.e. why does the EEMT change while the background apparently does not?

\section{Finite Gauge Transformations}

Back reaction is a second order effect, and therefore any terms of like order should be accounted for. In particular, it must be recognized that a coordinate transformation in fact induces a gauge transformation which involve terms of all orders in perturbation theory. Rather than the simple law Eq. 2 valid only up to first order, tensor fields are transformed by the Lie operator found upon exponentiation of the Lie derivative,

\[ \tilde{q} = e^{-L_\xi} q. \]  

(3)
Now it becomes clear that, although to first order the background variables do not change under a gauge transformation, to second order they do:

\[
\tilde{q}_a^0 = q_0^a + \left\langle \frac{1}{2} \mathcal{L}_\xi \mathcal{L}_\xi q_0^a - \mathcal{L}_\xi \delta q^a \right\rangle + \mathcal{O}(\epsilon^3),
\]

where all first order terms from Eq. 3 vanish by virtue of the spatial average.

We conclude then that, although under a coordinate transformation the EEMT change, so does the background to which it is referred. In this case we would write Einstein’s equations in another frame as

\[
\langle \tilde{\Pi} \rangle = \langle e^{-L_\xi} \Pi \rangle = 0 \quad \rightarrow \quad \tilde{\Pi}_0 = -\frac{1}{2} \langle \Pi, ab \delta \tilde{q}^a \tilde{q}^b \rangle
\]

where tensor indices have been suppressed for simplicity.

Above, we have shown that the back reaction equation 5 for cosmological perturbations is covariant, that is, takes the same form in any coordinate system. An obvious question is whether a gauge invariant formulation exists. The answer is yes, and the construction is in fact analogous to one that leads to the gauge invariant theory of linear perturbations. 

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