POWER SERIES SOLUTION OF A NONLINEAR SCHRÖDINGER EQUATION

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Abstract. A slightly modified variant of the cubic periodic one-dimensional nonlinear Schrödinger equation is shown to be well-posed, in a relatively weak sense, in certain function spaces wider than \(L^2\). Solutions are constructed as sums of infinite series of multilinear operators applied to initial data; no fixed point argument or energy inequality are used.

1. Introduction

1.1. The NLS Cauchy problem. The Cauchy problem for the one-dimensional periodic cubic nonlinear Schrödinger equation is

\[
\begin{align*}
  iu_t + u_{xx} + \omega |u|^2 u &= 0 \\
  u(0, x) &= u_0(x)
\end{align*}
\]

where \(x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}\), \(t \in \mathbb{R}\), and the parameter \(\omega\) equals \(\pm 1\). Bourgain [2] has shown this problem to be wellposed in the Sobolev space \(H^s\) for all \(s \geq 0\), in the sense of uniformly continuous dependence on the initial datum. In \(H^0\) it is wellposed globally in time, and as is typical in this subject, the uniqueness aspect of wellposedness is formulated in a certain auxiliary space more restricted than \(C^0([0, T], H^s(\mathbb{T}))\), in which existence is also established. For \(s < 0\) it is illposed in the sense of uniformly continuous dependence [3], and is illposed in stronger senses [5] as well.

The objectives of this paper are twofold. Firstly, we seek to establish the existence of solutions for wider classes of initial data than \(H^0\). Secondly, we aim to develop an alternative method of solution.

The spaces of initial data considered here are the spaces \(H^{s,p}\) for \(s \geq 0\) and \(p \in [1, \infty]\), defined as follows:

**Definition 1.1.** \(H^{s,p}(\mathbb{T}) = \{ f \in D(\mathbb{T}) : \langle \cdot \rangle^s \hat{f}(\cdot) \in \ell^p \} \).

Here \(D(\mathbb{T})\) is the usual space of distributions, and \(H^{s,p}\) is equipped with the norm \(\|f\|_{H^{s,p}} = \|\hat{f}\|_{\ell^p(\mathbb{Z})} = \left( \sum_{n \in \mathbb{Z}} \langle n \rangle^{ps} |\hat{f}(n)|^p \right)^{1/p} \). We write \(H^p = H^{0,p}\), and are mainly interested in these spaces since, for \(p > 2\), they are larger function spaces than the borderline Sobolev space \(H^0\) in which \(\text{(NLS)}\) is already known to be wellposed.

1.2. Motivations. At least four concrete considerations motivate analysis of the Cauchy problem in these particular function spaces. Firstly, \(H^p\) scales like \(H^{s(p)}\) where \(s(p) = -\frac{1}{2} + \frac{1}{p} \downarrow -\frac{1}{2}\) as \(p \uparrow \infty\), thus spanning the gap between the optimal exponent \(s = 0\) for Sobolev space wellposedness, and the scaling exponent \(-\frac{1}{2}\).

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A second motivation is the work of Kappeler and Topalov [2], [10], who showed via an inverse scattering analysis that the periodic KdV and mKdV equations are wellposed for wider ranges of Sobolev spaces $H^s$ than had previously been known. It is reasonable to seek a corresponding improvement for (NLS), but this problem has been shown to be illposed in strong senses in $H^s$ for all $s < 0$ [2]. Christ and Erdogan have investigated in unpublished work the relatively simple “action variable” portion of the inverse scattering theory relevant to (NLS), and have found that for any distribution in $H^p(T)$ with small norm, the sequence of gap lengths for the associated Dirac operator belongs to $\ell^p$ and has comparable norm.\footnote{Having slightly better than bounded Fourier coefficients seems to be a minimal condition for the applicability of this machinery, since the eigenvalues for the free periodic Dirac system are equally spaced, and gap lengths for perturbations are to leading order proportional to absolute values of Fourier coefficients of the perturbing potential.}

Thus $H^p$ for $2 < p < \infty$ may be a natural setting for inverse scattering theory for the Dirac operator relevant to the periodic cubic nonlinear Schrödinger equation.

A related third motivation is the goal of developing an alternative approach to the results of Kappeler and Topalov, independent of inverse scattering theory. NLS seems to be technically simpler than mKdV or KdV, so it may be a reasonable starting point. Fourthly, Grünrock [7] has proved wellposedness for the cubic nonlinear Schrödinger equation in spaces analogous to $H^{s,p}$, with $T$ replaced by $\mathbb{R}$, and for other PDE in these function spaces, as well.

1.3. **Modified equation.** In order for the Cauchy problem to make any sense in $H^p$ for $p > 2$ it seems to be essential to modify the differential equation. We consider

\[(NLS^*) \quad \begin{cases} iu_t + u_{xx} + \omega(|u|^2 - 2\mu(|u|^2))u = 0 \\ u(0, x) = u_0(x) \end{cases}\]

where

\[(1.1) \quad \mu(|f|^2) = (2\pi)^{-1} \int_T |f(x)|^2 \, dx\]

equals the mean value of the absolute value squared of $f$. In (NLS), $\mu(|u|^2)$ is shorthand for $\mu(|u(t, \cdot)|^2) = \|u(t, \cdot)|^2_{L^2}$, which is independent of $t$ for all sufficiently smooth solutions; modifying the equation in this way merely introduces a unimodular scalar factor $e^{2i\mu t}$, where $\mu = \mu(|u_0|^2)$. For parameters $p, s$ such that $H^{s,p}$ is not embedded in $H^0$, $\mu(|u|^2)$ is not defined for typical $u_0 \in H^{s,p}$, but of course the same goes for the function $|u_0(x)|^2$, and we will nonetheless prove that the equation makes reasonable sense for such initial data.

The coefficient 2 in front of $\mu(|u|^2)$ is the unique one for which solutions depend continuously on initial data in $H^p$ for $p > 2$.

1.4. **Conclusions.** Our main result is as follows. Recall that there exists a unique mapping $u_0 \mapsto Su_0(t, x)$, defined for $u_0 \in C^\infty$, which for all sufficiently large $s$ extends to a uniformly continuous mapping from $H^s(T)$ to $C^0([0, \infty), H^s(T)) \cap C^1([0, \infty), H^{s-2}(T))$, such that $Su_0$ is a solution of the modified Cauchy problem (NLS). $C^\infty(T)$ is of course a dense subset of $H^{s,p}$ for any $p \in [1, \infty]$.

**Theorem 1.1.** For any $p \in [1, \infty)$, any $s \geq 0$, and any $R < \infty$, there exists $\tau > 0$ for which the solution mapping $S$ extends by continuity to a uniformly continuous mapping from the ball centered at 0 of radius $R$ in $H^{s,p}(T)$ to $C^0([0, \tau], H^{s,p}(T))$.\footnote{Having slightly better than bounded Fourier coefficients seems to be a minimal condition for the applicability of this machinery, since the eigenvalues for the free periodic Dirac system are equally spaced, and gap lengths for perturbations are to leading order proportional to absolute values of Fourier coefficients of the perturbing potential.}
For the unmodified equation this has the following consequence. Denote by \( H_0^c = H_0^0(\mathbb{T}) \) the set of all \( f \in H^0 \) such that \( \|f\|_{L^2} = c \). Denote by \( S' u_0 \) the usual solution \([\text{NLS}]\) of the unmodified Cauchy problem \([\text{NLS}]\) with initial datum \( u_0 \), for \( u_0 \in H^0 \).

**Corollary 1.2.** Let \( p \in [1, \infty) \) and \( s \geq 0 \). For any \( R < \infty \) there exists \( \tau > 0 \) such that for any finite constant \( c > 0 \), the mapping \( H_0^c \ni u_0 \mapsto S' u_0 \) is uniformly continuous as a mapping from \( H_0^c \) intersected with the ball centered at \( 0 \) of radius \( R \) in \( H^{s,p} \), equipped with the \( H^{s,p} \) norm, to \( C^0([0, \tau], H^{s,p}(\mathbb{T})) \).

The unpublished result of the author and Erdogan shows that for initial data in \( L^2 \), for which the solution is known to exist globally in time, \( \|u(t)\|_{H^p} \leq C\|u_0\|_{H^p} \) uniformly for all \( t \in [0, \infty) \), provided that \( \|u_0\|_{H^p} \) is sufficiently small. This result, once published, will combine with Theorem \([\text{NLS}]\) to yield global wellposedness for sufficiently small data.

The following result quantifies the relation between the nonlinear evolution \([\text{NLS}]\) and the corresponding linear Cauchy problem

\[
\begin{cases}
iv_t + v_{xx} = 0 \\
v(0, x) = u_0(x).
\end{cases}
\]

**Proposition 1.3.** Let \( R < \infty \) and \( p \in [1, \infty) \). Let \( q > p/3 \) also satisfy \( q \geq 1 \). Then there exist \( \tau, \varepsilon > 0 \) and \( C < \infty \) such that for any initial datum \( u_0 \) satisfying \( \|u_0\|_{H^p} \leq R \), the solutions \( u = Su_0 \) of \([\text{NLS}]\) and \( v \) of \((1.2)\) satisfy

\[
\|u(t, \cdot) - v(t, \cdot)\|_{H^q} \leq C \tau^\varepsilon \text{ for all } t \in [0, \tau].
\]

Here \( u \) the solution defined by approximating \( u_0 \) by elements of \( C^\infty \) and passing to the limit. Thus for \( p > 1 \) the nonlinear terms are in a sense smoother than the linear evolution.

Our next result indicates that the function \( u(t, x) \) defined by the limiting procedure of Theorem \([\text{NLS}]\) is a solution of the differential equation in a more natural sense than merely being a limit of smooth solutions. Define Fourier truncation operators \( T_N \), acting on \( H^{s,p}(\mathbb{T}) \), by \( T_N f(n) = 0 \) for all \( |n| > N \), and \( = \hat{f}(n) \) whenever \( |n| \leq N \). \( T_N \) acts also on functions \( v(t, x) \) by acting on \( v(t, \cdot) \) for each time \( t \) separately. We denote by \( S(u_0) \) the limiting function whose existence, for nonsmooth \( u_0 \), is established by Theorem \([\text{NLS}]\).

**Proposition 1.4.** Let \( p \in [1, \infty) \), \( s \geq 0 \), and \( u_0 \in H^{s,p} \). Write \( u = S(u_0) \). Then for any \( R < \infty \) there exists \( \tau > 0 \) such that whenever \( \|u_0\|_{H^{s,p}} \leq R \), \( u(u(t, x) = (|u|^2 - 2\mu(|u|^2))u \) exists in the sense that

\[
\lim_{N \to \infty} \mathcal{N}(T_N u)(t, x) \text{ exists in the sense of distributions in } C^0((0, \tau), \mathcal{D}'(\mathbb{T})).
\]

Moreover if \( \mathcal{N}(u) \) is interpreted as this limit, then \( u = S(u_0) \) satisfies \([\text{NLS}]\) in the sense of distributions in \((0, \tau) \times \mathbb{T}\).

More generally, the same holds for any sequence of Fourier multipliers of the form \( \widehat{T_N f}(n) = m_\nu(n) \hat{f}(n) \) where each sequence \( m_\nu \) is finitely supported, \( \sup_\nu \|m_\nu\|_\infty < \infty \), and \( m_\nu(n) \to 1 \) as \( \nu \to \infty \) for each \( n \in \mathbb{Z} \); the limit is of course independent of the sequence \( (m_\nu) \). Making sense of the nonlinearity via this limiting procedure is connected with general theories of multiplication of distributions \([\text{NLS}], [6]\), but the existence here of the limit over all sequences \( (m_\nu) \) gives \( u \) stronger claim to the title of solution than in the general theory.

Unlike the fixed point method, our proof yields no uniqueness statement corresponding to these existence results. But this failing is unavoidable; for all \( p > 2 \), solutions of the Cauchy problem in the class \( C^0([0, \tau], H^p) \), in the sense of Proposition \([\text{NLS}]\) are not unique \([1]\).
1.5. Method. Define the partial Fourier transform
\begin{equation}
\hat{u}(t, n) = (2\pi)^{-1} \int_T e^{-inx} u(t, x) \, dx.
\end{equation}

Our approach is to regard the partial differential equation as an infinite coupled nonlinear system of ordinary differential equations for these Fourier coefficients, to express the solution as a power series in the initial datum
\begin{equation}
\hat{u}(t, n) = \sum_{k=0}^{\infty} \hat{A}_k(t)(\hat{u}_0, \cdots, \hat{u}_0)
\end{equation}
where each $\hat{A}_k(t)$ is a bounded multilinear operator to a product of $k$ copies of $H^{s,p}$ to $H^{s,p}$, to show that the individual terms $\hat{A}_k(t)(\hat{u}_0, \cdots, \hat{u}_0)$ are well-defined, and to show that the formal series converges absolutely in $C_0(\mathbb{R}, H^{s,p})$ to a solution in the sense of (1.4). The case $s \geq 0$ follows from a very small modification of the analysis for $s = 0$, so we discuss primarily $s = 0$, indicating the necessary modifications for $s > 0$ at the end of the paper.

The analysis is quite elementary, much of the paper being devoted to setting up the definitions and notation required to describe the operators $\hat{A}_k(t)$. A single number theoretic fact enters the discussion: the number of factorizations of an integer $n$ as a product of two integer factors is $O(n^\delta)$ as $n \to \infty$, for all $\delta > 0$; this same fact was used in a more sophisticated way by Bourgain [2].

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2. A SYSTEM OF COUPLED ORDINARY DIFFERENTIAL EQUATIONS

2.1. General discussion. Define
\begin{equation}
\sigma(j, k, l, n) = n^2 - j^2 + k^2 - l^2.
\end{equation}
It factors as
\begin{equation}
\sigma(j, k, l, n) = 2(n-j)(n-l) = 2(k-l)(k-j) \text{ provided that } j-k+l = n.
\end{equation}

Written in terms of Fourier coefficients $\hat{u}_n(t) = \hat{u}(t, n)$, the equation $iu_t + u_{xx} + \omega(|u|^2 - 2\mu(|u|^2)) u = 0$ becomes
\begin{equation}
d\hat{u}_n/dt - n^2 \hat{u}_n + \omega \sum_{j-k+l=n} \hat{u}_j \hat{u}_k \hat{u}_l - 2\omega \sum_m |\hat{u}_m|^2 \hat{u}_n = 0.
\end{equation}
Here the first summation is taken over all $(j, k, l) \in \mathbb{Z}^3$ satisfying the indicated identity, and the second over all $m \in \mathbb{Z}$. Substituting
\begin{equation}
a_n(t) = e^{in^2t} \hat{u}(t, n),
\end{equation}
becomes
\begin{equation}
d\bar{a}_n/dt = i\omega \sum_{j-k+l=n} a_j a_k a_l e^{i\sigma(j,k,l,n)t} - i\omega |a_n|^2 a_n.
\end{equation}

\footnote{Throughout the discussion we allow multilinear operators to be either conjugate linear or linear in each of their arguments, independently.}
where the notation $\sum_{j-k+l=n}^*$ means that the sum is taken over all $(j, k, l) \in \mathbb{Z}^3$ for which neither $j = n$ nor $l = n$. This notational convention will be used throughout the discussion. The effect of the term $-2 \omega \mu \|u\|^2 u$ in the modified differential equation $\text{(NLS)}^2$ is to cancel out a term $2i\omega \sum_m |a_m|^2 a_n$, which would otherwise appear on the right-hand side of $\text{(2.5)}$.

Reformulated as an integral equation, $\text{(2.5)}$ becomes

\begin{equation}
(2.6) \quad a_n(t) = a_n(0) + i\omega \sum_{j-k+l=n}^* a_j(s) a_k(s) a_l(s) e^{i\sigma(j,k,l,n)s} ds - i\omega \int_0^t |a_n(s)|^2 a_n(s) ds.
\end{equation}

However, in deriving $\text{(2.6)}$ from $\text{(2.5)}$, we have interchanged the integral over $[0, t]$ with the summation over $j, k, l$ without any justification. $\text{(2.6)}$ is fully equivalent to $\text{(2.7)}$

$$
\hat{u}(t, n) = \hat{u}_0(n) - in^2 \int_0^t \hat{u}(s, n) ds + i\omega \sum_{j-k+l=n}^* \hat{u}(s, j) \hat{u}(s, k) \hat{u}(s, l) ds - i\omega \int_0^t |\hat{u}(s, n)|^2 \hat{u}(s, n) ds.
$$

Substituting for $a_j, a_k, a_l$ in the right-hand side of $\text{(2.6)}$ by means of the equation itself yields

\begin{equation}
(2.8) \quad a_n(t) = a_n(0) + i\omega \sum_{j-k+l=n}^* a_j(0) a_k(0) a_l(0) \int_0^t e^{i\sigma(j,k,l,n)s} ds - i\omega |a_n(0)|^2 a_n(0) \int_0^t 1 ds

+ \text{additional terms}.
\end{equation}

$$
= a_n(0) \left(1 - i\omega t |a_n(0)|^2 \right) + \frac{1}{2} \omega \sum_{j-k+l=n}^* \frac{a_j(0) a_k(0) a_l(0)}{(n-j)(n-l)} (e^{i(n^2-j^2+k^2-l^2)t} - 1)

+ \text{additional terms}.
$$

We recognize $1 - i\omega t |a_n(0)|^2$ as a Taylor polynomial for $\exp(-i|a_n(0)|^2 t)$, but for our purposes it will not be necessary to exploit this by recombining terms, and in particular we will not exploit the coefficient $i$ which makes this exponential unimodular.

2.2. A sample term. One representative additional term is

\begin{equation}
(2.9) \quad (i\omega)^4 \sum_{j_1-j_2+j_3=n}^* \sum_{m_1^1-m_1^2+m_1^3=j_1}^* \sum_{m_2^1-m_2^2+m_2^3=j_2}^* \sum_{m_3^1-m_3^2+m_3^3=j_3}^*

\int_{0 \leq r_1, r_2, r_3 \leq s \leq t} a_{m_1^1}(r_1) a_{m_2^2}(r_2) a_{m_3^3}(r_3) a_{m_1^1}(r_1) a_{m_2^2}(r_2) a_{m_3^3}(r_3) a_{m_1^1}(r_1) a_{m_2^2}(r_2) a_{m_3^3}(r_3)

e^{i\sigma(j_1, j_2, j_3, n)s} e^{i\sigma(m_1^1, m_2^2, m_3^3, j_1)} e^{-i\sigma(m_1^1, m_2^2, m_3^3, j_2)} e^{i\sigma(m_1^1, m_2^2, m_3^3, j_3)} dr_1 dr_2 dr_3 ds.
\end{equation}

Substituting via $\text{(2.6)}$ for each coefficient $a$ yields a main term

\begin{equation}
(2.10) \quad (i\omega)^4 \sum_{(m_k^i)_{1 \leq i, k \leq 3} \subseteq \mathbb{Z}^3}^* \mathcal{I}(t, (m_k^i)_{1 \leq i, k \leq 3}) \prod_{i,j=1}^3 a_{m_j^i}^*(0)

+ \text{higher-degree terms, where the superscript }^* \text{ indicates here that the sum is taken over only certain } (m_k^i)_{1 \leq i, k \leq 3} \subseteq \mathbb{Z}^3 \text{ (more precisely, over most of a copy of } \mathbb{Z}^8 \text{ affinely embedded}
\end{equation}
in \( \mathbb{Z}^9 \), \( a_{m_j}^+(0) = a_{m_j}(0) \) if \( i + j \) is even and \( = \overline{a_{m_j}(0)} \) if \( i + j \) is odd, and

\[
\mathcal{I}(t, (m_k^i)_{1 \leq i, k \leq 3}) = \int_{0 \leq r_1, r_2, r_3 \leq s \leq t} e^{i\theta(t, s, r_1, r_2, r_3, (m^i_j)_{1 \leq i, j \leq 3})} dr_1 dr_2 dr_3 ds,
\]

with

\[
\theta(t, s, r_1, r_2, r_3, (m^i_j)_{1 \leq i, j \leq 3}) = \sigma(j_1, j_2, j_3, n)s + \sum_{i=1}^{3} (-1)^{i+1}\sigma(m^i_1, m^i_2, m^i_3, j_i)r_i;
\]

here \( j_1, j_2, j_3, n \) are defined as functions of \( (m^i_j) \) by the equations governing the sums in (2.9). Continuing in this way yields formally an infinite expansion for the sequence \( (a_n(t))_{n \in \mathbb{Z}} \) in terms of multilinear expressions in the initial datum \( (a_n(0)) \). This expansion is doubly infinite; the single (and relatively simple) term (2.10) is for instance an infinite sum over most of a copy of \( \mathbb{Z}^3 \) for each \( n \).

The discussion up to this point has been purely formal, with no justification of convergence. In the next section we will begin to describe the terms in this expansion systematically.

3. Trees

On a formal level \( a(t) = (a_n(t))_{n \in \mathbb{Z}} \) equals an infinite sum \( \sum_{k=1}^{\infty} A_k(t)(a(0), a(0), a(0), \ldots) \) where each \( A_k(t) \) is a sum of finitely many multilinear operators, each of degree \( k \). We now describe a class of trees which will be used both to name, and to analyze, these multilinear operators.

Definition 3.1. A tree \( T \) is a finite partially ordered set with the following properties:

(1) Whenever \( v_1, v_2, v_3, v_4 \in T \) and \( v_1 \leq v_2 \leq v_3 \leq v_4 \), then either \( v_2 \leq v_3 \) or \( v_3 \leq v_2 \).

(2) There exists a unique element \( v_0 \in T \) satisfying \( v_0 \geq v \) for all \( v \in T \).

(3) Each \( v \in T \) has either three children, or no children; \( w \) is said to be a child of \( v \) if \( w < v \) and if there exists no \( u \in T \) satisfying \( w < u < v \).

(4) For each \( v \in T \) there is given an element of \( \{\pm 1\} \), denoted \( \pm v \).

Definition 3.2. Elements of \( T \) are called nodes. A terminal node is one with zero children. The maximal element of \( T \) is called its root node. For any \( u \in T \), \( T_u = \{ v \in T : v \leq u \} \) is a tree, with root node \( u \). \( T^\infty \) denotes the set of all terminal nodes of \( T \), while \( T^0 = T \setminus T^\infty \) denotes the set \( T^0 \) of all non-terminal nodes. The three children of any \( v \in T^0 \) are denoted by \( (v, 1), (v, 2), (v, 3) \).

The number \( |T| \) of nodes of a tree is of the form \( 1 + 3k \) for some nonnegative integer \( k \). The number of terminal nodes is then

\[
|T^\infty| = 1 + 2k = \frac{2}{3}|T| + \frac{1}{3}.
\]

Definition 3.3. An ornamented tree is a tree \( T \) together with the following additional structure:

(1) Associated to each node \( v \in T \) is copy of \( \mathbb{Z} \), indexed by the variable \( j_v \).

(2) There is given a partition of the set of all non-terminal nodes of \( T \) into two disjoint classes, called simple nodes and general nodes. Terminal nodes are neither simple nor general.
Definition 3.7. A weathered ornamented tree \((T, T')\) is an ornamented tree \(T\) together with a subset \(T' \subset T^0\) and the collection

\[ \mathcal{J}(T, T') = \{ j \in \mathcal{J}(T) : v \in T \text{ is frozen if and only if } v \in T'. \} \]

4. Multilinear Operators Associated to Trees

Definition 4.1. Let \(T\) be any tree, and let \(t \in \mathbb{R}\). The associated tree coefficients are

\[ I_T(t,j) = \int_{\mathbb{R}(T,t)} \prod_{u \in T^0} e^{\pm u \sigma_u(j) t_u} dt_u \]
where
\[ R(T, t) = \{ (t_u)_{u \in T^0} : 0 \leq t_u \leq t \text{ whenever } u \leq u' \}. \]

The following upper bounds for the coefficients \( I_T(t, j) \) are the only information concerning them that will be used in the analysis.

**Lemma 4.1.** Let \( T \) be any tree, and let \( j \in J(T) \). Then for all \( t \in [0, 1] \),
\[ |I_T(t, j)| \leq t^{T^0} \]
and
\[ |I_T(t, j)| \leq 2^{|T|} \prod_{(u, v) \in T^0} (\rho_w(j))^{-1}. \]

The notation \( \langle x \rangle \) means \( (1 + |x|^2)^{1/2} \). The sum here is taken over all of the \( 3^{|T^0|} \) possible choices of \( \varepsilon_{u, v} \in \{ 0, 1, -1 \} \); these choices in turn determine the quantities \( \rho_w \). This lemma will be proved in \( \S 6 \).

**Definition 4.2.** Let \( T \) be any ornamented tree. The tree operator \( S_T(t) \) associated to \( T \) is for each \( t \in \mathbb{R} \) the multilinear operator that maps \( (x_v)_{v \in T^\infty} \), where each \( x_v \) is a sequence of complex numbers, to the sequence of complex numbers
\[ S_T(t)(x_v)_{v \in T^\infty}(n) = \sum_{j \in J(T) : j_{T^0} = n} I_T(t, j) \prod_{w \in T^\infty} x_w(j_w). \]

5. Formalities

With all these definitions and notations in place, we can finally formulate the conclusion of the discussion in \( \S 4 \) proofs will be supplied later.

**Proposition 5.1.** The recursive procedure indicated in \( \S 4 \) yields a formal expansion
\[ a(t) = \sum_{k=1}^\infty A_k(t)(a(0), a(0), \ldots), \]
where each \( A_k(t) \) is a multilinear operator of the form
\[ A_k(t) = \sum_{|T| = 3k+1} c_T S_T(t) \]
where the scalars \( c_T \in \mathbb{C} \) satisfy \( |c_T| \leq C^k \) for some finite constant \( C \). The sum in \( [4.2] \) is taken over all ornamented trees \( T \) of the indicated cardinalities. There exists a finite positive constant \( c_0 \) such that whenever \( a(0) \in \ell^1 \), the multiply infinite series \( \sum_k A_k(t)(a(0), \ldots) \) converges absolutely to a function in \( C^0([0, \tau], \ell^1) \) provided that \( \tau \|a(0)\|_{\ell^1} \leq c_0 \).

By this last statement we mean that \( \sum_{j \in J(T)} |I_T(t, j)| \prod_{w \in T^\infty} |a(0)(j_w)| \) converges absolutely for each ornamented tree \( T \), and that if its sum is denoted by \( S_T^\ast(a(0), a(0), \ldots)(t) \) then the resulting series \( \sum_{k=1}^\infty \sum_{|T| = 3k+1} |c_T S_T^\ast(a(0), a(0), \ldots)(t)| \) likewise converges. All but the last sentence follows from the discussion in \( \S 4 \) and the definitions in \( \S 5 \) section:treeops.

The operators \( S_T \) and coefficients \( c_T \) were defined so that the following holds automatically.
Lemma 5.2. There exists $c > 0$ with the following property. Let $\hat{u}_0$ be any numerical sequence and define $a(0)(n) = \hat{u}_0(n)$. Suppose that the infinite series defining $S_T^a((0), a(0), \cdots)(t)$ converges absolutely and uniformly for all $t \in [0, \tau]$ and that its sum is $O(e^{\epsilon T})$, uniformly for every ornamented tree $T$. Define a $(t)$ to be the sequence $\sum_{k=1}^{\infty} A_k(t)(0), a(0), \cdots)$. Then a satisfies the integral equation (2.6) for $t \in [0, \tau]$. Moreover the function $u(t, x)$ defined by $\hat{u}(t, n) = e^{-in^2t}a(t, n)$ is a solution of the modified Cauchy problem (NLS) in the corresponding sense (2.7).

The main estimate in our analysis is as follows.

Proposition 5.3. Let $p \in (1, \infty)$. Then for any exponent $q > \frac{p}{p-1}$ satisfying also $q \geq 1$, there exist $\epsilon > 0$ and $C < \infty$ such that for all ornamented trees $T$ and all sequences $x_v \in \ell^1$,

$$\|S_T(t)(x_v)_{v \in T}\|_{\ell^q} \leq (C t^\epsilon)^{|T|^{\infty}} \prod_{v \in T^{\infty}} \|x_v\|_{\ell^q}.$$ (5.3)

Propositions 5.1 and 5.3 and Lemma 5.2 will be proved in subsequent sections. Together, they give:

Corollary 5.4. Let $p \in [1, \infty)$. For any $R < \infty$ there exists $\tau > 0$ such that the solution mapping $u_0 \mapsto u(t, \cdot)$ for the modified Cauchy problem (NLS), initially defined for all sufficiently smooth $u_0$, extends by uniform continuity to a real analytic mapping from $\{u_0 \in \mathcal{H}^p : \|u_0\|_{\mathcal{H}^p} \leq R\}$ to $C^0([0, \tau], \mathcal{H}^p(T))$.

We emphasize that analytic dependence on $t$ is not asserted.

6. Tree coefficient bound

Proof of Lemma 5.1. The first conclusion of the lemma holds simply because $|I_T(t, j)| \leq |R(T, t)|$. The proof of the main conclusion (12) proceeds recursively. In step 1 we integrate with respect to $t_v$ for certain nodes $v$, holding fixed all other coordinates $t_w$ in the integral defining $R(T, t)$. Specifically, we hold fixed the coordinate $t_v$ whenever at least one child $v$ is terminal. We also fix $t_v$ for every simple node $v$ having only terminal children. The former coordinates $t_v$, and underlying nodes $v$, are said to be temporarily fixed; the latter coordinates and nodes are said to be permanently fixed. No other coordinates are fixed at this step.

When $|T| = 1$ there is nothing to prove. Otherwise there must always exist at least one node, all of whose children are terminal. If there exists such a node which is also general, then at least coordinate $t_v$ is not fixed. The subset, or slice, of $R(T, t)$ defined by setting each of the fixed coordinates equal to some constant is either empty, or takes the product form $\times_u$ not fixed $[0, t_{uw}]$, where $u^*$ denotes the parent of $u$. Integrating over this slice with respect to all of the non-fixed coordinates thus yields

$$\prod_{u \neq} e^{\pm iu_\omega \sigma_u t_u} \prod_{u} \int_0^{t_{uw}} e^{\pm iu_\omega \sigma_u t_u} dt_u,$$

where the first product is taken over all fixed $w \in T^0$, and the second over all remaining non-fixed $u \in T^0$.

None of the quantities $\sigma_u$ can vanish in step 1, since a general node having only terminal children can never be exceptional, by (3.5). Therefore the preceding expression equals

$$\prod_{u} e^{\pm iu_\omega \sigma_u t_u} \prod_{u} (\pm iu_\omega \sigma_u)^{-1} (e^{\pm iu_\omega \sigma_u t_{uw}} - 1).$$
This may be expanded as a sum of $2^N$ terms, where $N$ is the number of non-fixed nodes in $T^0$. Each of these terms has the form

$$\pm \prod_w e^{\pm w i\omega \sigma_w t_w} \prod_u (i\omega \sigma_u)^{-1} e^{w i\omega \sigma_u t_u}$$

for some numbers $\varepsilon_u \in \{0, 1, -1\}$.

The other possibility in step 1 is that $|T| > 1$, but every nonterminal node that has only terminal children is simple. In that case all coordinates $t_v$ are fixed at step 1, no integration is performed, and we move on to step 2.

We now carry out step 2. If a node $v$ was permanently fixed at step 1 then it remains fixed for all subsequent steps; we never integrate with respect to $t_v$. More generally, any node that is permanently fixed at any step remains fixed through all subsequent steps. On the other hand, once we’ve integrated with respect to some $t_w$, then the node $w$ is also removed from further consideration. All other nodes remain active, including those temporarily fixed in step 1. Denote by $T_1$ the set of all nodes that are active after the completion of step 1.

$T_1$ is itself a tree. There is an associated subset $R_{T_1}$ of $\{(t_w : w \in T_1)\}$, defined by the inequalities $0 \leq t_w \leq t$ whenever $w \leq w'$, and also by $t_u \leq t_w$ if $u \leq w$ and $u$ was permanently fixed in step 1. To each node $w \in T_1$ is associated a modified phase $\sigma_w^{(2)}$, defined to be $\sigma_w + \sum_i \varepsilon_{(w,i)}(w,u)\sigma_{(w,i)}$, where the sum is taken over all $i \in \{1, 2, 3\}$ such that we integrated with respect to $t_{(w,i)}$ in the first step.

A node $w$ is permanently fixed at the second step if $w$ is terminal in $T_1$ and satisfies $\sigma_w^{(2)} = 0$. A node $w \in T_1$ is temporarily fixed at the second step if $w$ is not terminal in $T_1$. We now integrate $\prod_{w \in T_1} e^{\pm i\omega \sigma_w^{(2)}(t_w)}$ over $R_{T_1}$ with respect to $t_u$ for all $u \in T_1$ that are neither temporarily nor permanently fixed. As in step 1, this integral has a product structure, and $2^{N_2}$ terms are obtained, where $N_2$ is the number of variables with respect to which we integrate.

In step 3 we consider the tree $T_2$ consisting of all $w \in T_1$ that were temporarily fixed in step 2. Associated to $T_2$ is a set $R_{T_2}$, and associated to each node $v \in T_2$ is a modified phase $\sigma_v^{(3)} = \sigma_v^{(2)} + \sum_i \varepsilon_{(v,i)}(v,u)\sigma_{(v,i)}$, the sum being taken over all $i \in \{1, 2, 3\}$ such that $(v,i)$ was not fixed in step 2. A node $v \in T_2$ is then permanently fixed if it is terminal in $T_2$ and $\sigma_v^{(3)} = 0$. $v \in T_2$ is temporarily fixed if it is not terminal in $T_2$. We then integrate with respect to $t_v$ for all $v \in T_2$ that are neither temporarily nor permanently fixed.

This procedure terminates after finitely many steps, when for each node $v \in T^0$, either $v$ has become permanently fixed, or we have integrated with respect to $t_v$. This yields a sum of at most $2^{2|T^0|}$ terms. Each term arises from some particular choice of the parameters $\varepsilon_{u,i}$, and is expressed as an integral with respect to $t_v$ for all nodes $v \in T^0$ that were permanently fixed at some step; the vector $(t_v)$ indexed by all such $v$ varies over a subset of $[0, t]^M$ where $M$ is the number of such $v$. At step $n$, each integration with respect to some $t_u$ yields a factor of $(\sigma_u^{(n)})^{-1}$, multiplied by some unimodular factor; recall that $\sigma_u^{(n)} \neq 0$, since otherwise $u$ would have been permanently fixed.

Thus for each term we obtain an upper bound of $\prod_u |\rho_u|^{-1}$, where the product is taken over all nonexceptional nodes $u$; this bound must still be integrated with respect to all $t_w$ where $w$ ranges over all the exceptional nodes. Each such coordinate $t_w$ is restricted to
Thus we obtain a total bound

\begin{equation}
|I(t,j)| \leq \sum_{\{e_{u,i}\}} t^M \prod_{w \in T^0} |\rho_w(j)|^{-1}
\end{equation}

where for each \((e_{u,i})\), \(M = M((e_{u,i}))\) is the number of exceptional nodes encountered in this procedure, that is, the number of permanently fixed nodes, and where for each \((e_{u,i})\), \(\prod_{w \in T^0} w\) denotes the product over all nonexceptional nodes \(w \in T^0\) that are nonexceptional with respect to \((e_{u,i})\).

\[\square\]

7. A simple \(\ell^1\) bound

This section is devoted to a preliminary bound for simplified multilinear operators. For any tree \(T\) and any sequences \(y_v \in \ell^1\), define

\begin{equation}
\mathcal{S}_T(y_v)_{v \in T^\infty}(n) = \sum_{j: \|j\|_V = n} \prod_{u \in T^\infty} y_u(j_u).
\end{equation}

The notation \(\sum_{j: \|j\|_V = n}^*\) indicates that the sum is taken over all indices \(j \in \mathbb{Z}^T\) that satisfy (3.4) as well as \(j_v = n\); the restrictions (3.3) and (3.6) are not imposed here.

**Lemma 7.1.** For any tree \(T\) and any sequences \(\{y_v : v \in T^\infty\}\)

\begin{equation}
\|\mathcal{S}_T(y_v)_{v \in T^\infty}\|_{\ell^1} \leq \prod_{w \in T^\infty} \|y_w\|_{\ell^1},
\end{equation}

with equality when all \(y_v(j_v)\) are nonnegative.

**Proof.** Recall that for some nonnegative integer \(k\), \(|T| = 3k + 1\), \(|T^\infty| = 2k + 1\), and \(|T^0| = k\). Consider the set \(B \subset T\) whose elements are \(v_0\) together with all \((v,i)\) such that \(v \in T^0\) and \(i \in \{1,3\}\). Thus \(|B| = 1 + 2k = |T^\infty|\). Define

\begin{equation}
k_{v,i} = j_v - j_{(v,i)} \text{ for } v \in T^0 \text{ and } i \in \{1,3\}.
\end{equation}

Consider the \(\mathbb{Z}\)-linear mapping \(L\) from \(\mathbb{Z}^{T^\infty}\) to \(\mathbb{Z}^B\) defined so that \(L(j)\) has coordinates \(j_{v_0}\) and all \(k_{v,i}\).

\(j_v\) and \(j_{(v,i)}\) are well-defined linear functionals of \(j \in \mathbb{Z}^{T^\infty}\), because given the quantities \(j_w\) for all \(w \in T^\infty\), \(j_v\) can be recovered for all other \(v \in T\) via the relations (3.3), by ascending induction on \(v\). We claim that \(L(j)\) is invertible. Indeed, from the quantities \(j_{v_0}\) and all \(j_v - j_{(v,i)}\) with \(v \in T^0\) and \(i \in \{1,3\}\), \(j_u\) can be recovered for all \(u \in T\) by descending induction on \(u\), using again (3.4) at each stage. For instance, at the initial step, \(j_{(v_0,i)} = j_{v_0} + k_{v_0,i}\) for \(i = 1,3\), and then \(j_{(v_0,2)}\) can be recovered via (3.4). Thus \(L(j)\) is injective, hence invertible.

By descending induction on nodes it follows in the same way from (3.4) that \(j = (j_w)_{w \in T^\infty}\) satisfies a certain linear relation of the form

\begin{equation}
\sum_{w \in T^\infty} \pm_w j_w
\end{equation}

where each coefficient \(\pm_w\) equals \(\pm 1\). By the conclusion of the preceding paragraph, this can be the only relation to which \((j_w)_{w \in T^\infty}\) is subject; the sum defining \(\mathcal{S}_T(y_v)_{v \in T^\infty}(j_{v_0})\) is taken over all \(j\) satisfying this relation. Therefore \(\sum_{j_{v_0}} \mathcal{S}_T(j_{v_0})\) equals the summation over all \(w \in T^\infty\) and all \(j_w \in \mathbb{Z}\), without restriction, of \(\prod_{w \in T^\infty} y_w(j_w)\). The lemma follows. \[\square\]
Corollary 7.2. For any ornamented tree, the sum defining \( \tilde{S}_T(y_v)_{v \in T^\infty}(n) \) converges absolutely for all \( n \in \mathbb{Z} \) whenever all \( y_v \in \ell^1 \), and the resulting sequence satisfies
\[
\| \tilde{S}_T(y_v)_{v \in T^\infty} \|_{\ell^1} \leq \prod_{v \in T^\infty} \| y_v \|_{\ell^1}.
\]

There is no bound for \( \tilde{S}_T \) in terms of the quantities \( \| y_v \|_{\ell^p} \) for \( p > 1 \). It is the additional factors \( (\rho_u)^{-1} \) in the second tree coefficient bound (4.4), reflecting the dispersive character of the partial differential equation, which make possible estimates in terms of weaker \( \ell^p \) norms.

Proof of Proposition 5.1. Consider any tree \( T \) and associated function
\[
(7.6) \quad \int_{R(T,t)} \sum_{j \in \mathcal{J}(T)} \prod_{v \in T^0} e^{\pm i \sigma_v t_u} \prod_{u \in T^\infty} y_u(t_u, j_u) \, dt_u
\]
for \( 0 \leq t \leq \tau \), with \( t_v \equiv t \), under the assumption that for each \( u \in T^\infty \), \( y = y_u \) belongs to \( C^0([0, \tau], \ell^1) \). Here for each \( t_u \), \( y_u(t_u) \) is the sequence whose components are \( y_u(t_u, j_u) \), \( j_u \in \mathbb{Z} \).

Let each node \( v \in T^\infty \) be designated as either finished or unfinished. Assume that for each finished node, \( y_u(t_u, j_u) \) is independent of \( t_u \), while for each unfinished node, either the sequence-valued function \( t_u \mapsto y_u \) satisfies the integral equation
\[
(7.7) \quad y_u(t, n) = y_u(0, n) - i \omega \int_0^t |y_u(s, n)|^2 y_u(s, n) \, ds
\]
\[
+ i \omega \sum_{j-k+l=n} \int_0^t y_u(s, j) y_u(s, k) y_u(s, l) e^{i \sigma(j, k, l, n) s} \, ds,
\]
or its complex conjugate satisfies this same equation.

The \( C^0(\ell^1) \) hypothesis guarantees that if we substitute the right-hand side of (7.7) for \( y_u(t_u, j_u) \) in (7.6) for each unfinished node, then an absolutely convergent integral and sum are obtained. Thus we may interchange the outer integral with the sums. What results is a finite linear combination of expressions of the same character as (7.6), each associated to a larger tree \( T' \supset T \). At most \( 3^{|T^\infty|} \) such expressions are obtained, and each is multiplied by a unimodular numerical coefficient. Each non-terminal node of \( T \) is a non-terminal node of \( T' \), each finished node of \( T^\infty \) remains a terminal node of \( T' \), and each unfinished node of \( T^\infty \) becomes a non-terminal node of \( T' \), each of whose three children may independently be either finished or unfinished. An unfinished node \( u \) of \( T \) gives rise either to a simple node or a general node of \( T' \), depending on which of the two trilinear terms of (7.7) is substituted for \( y_u \) in (7.6).

This discussion justifies the formal derivation of the expansion in §2. It follows by recursion that for any solution \( u(t, x) \) of the modified Cauchy problem [NLS] in \( C^0([0, \tau], H^s) \) for sufficiently large \( s \), the associated coefficients \( a_n(t) = e^{i n^2 t} \tilde{u}(t, n) \) are given by the absolutely convergent infinite power series (5.1), (5.2) for all sufficiently small \( t \). \( \square \)

8. Tree sum majorants

Definition 8.1. Let \( T \) be an ornamented tree. The tree sum majorant associated to \( T \) is the multilinear operator
\[
(8.1) \quad S_T(y_w)_{w \in T^\infty} = \sum_{j \in \mathcal{J}(T) : j_v = n} \prod_{u \in T^0} (\rho_u(j))^{-1} \prod_{w \in T^\infty} y_w(j_w).
\]
Here \( t \geq 0 \) and \( S_T \) is initially defined when all \( y_w \in \ell^1 \), in order to ensure absolute convergence of the sum.

**Definition 8.2.** Let \((T, T')\) be a weathered ornamented tree. The associated tree sum majorant is the multilinear operator

\[
S_{(T, T')}(y_w)_{w \in T^\infty}(n) = \sum_{j \in J(T, T') \colon j_{n_0} = n} \prod_{u \in T^0} \langle \rho_u(j) \rangle^{-1} \prod_{w \in T^\infty} y_w(j_w).
\]

Thus

\[
S_T = \sum_{T' \subset T^0} S_{(T, T')},
\]

the sum being taken over all subsets \( T' \subset T^0 \). The total number of such subsets is \( 2^{|T^0|} \leq 2|T| \leq 2^{3|T^\infty|/2} = C|T^\infty| \).

Let \((T, T')\) be a weathered ornamented tree. We seek an upper bound for the associated tree sum operator \( S_{(T, T')} \). The factors \( \langle \rho_v \rangle^{-1} \) in the definition of \( S_{(T, T')} \) are favorable when \(|\rho_v|\) is large; frozen nodes are those for which \(|\rho_v|\) is relatively small, and hence these require special attention.

Denote by \( \Gamma = (\gamma_u)_{u \in T'} \) any element of \( 2^{T'} \). Let

\[
J(T, T', \Gamma) = \{ j \in J(T, T') : \rho_u(j) = \gamma_u \ \text{for all} \ u \in T' \}.
\]

\( T' \) is the set of all frozen nodes, so by its definition we have

\[
|\gamma_u| = |\rho_u(j)| \leq c_0 |\sigma_u(j)|^{1-\delta} \ \forall u \in T'
\]

with the shorthand notation \( \sigma_u(j) = \sigma(j_{(u, 1)}, j_{(u, 2)}, j_{(u, 3)}; j_u) \) introduced earlier. In the remainder of the discussion, we always assume tacitly that \( \Gamma \) satisfies \((8.5)\).

This leads to a further decomposition

\[
S_{(T, T')}(y_v)_{v \in T^\infty}(n) = \sum_{\Gamma} \sum_{j \in J(T, T', \Gamma) : j_{n_0} = n} \prod_{u \in T^0} \langle \rho_u(j) \rangle^{-1} \prod_{w \in T^\infty} y_w(j_w)
\]

\[
\leq C|T| \sum_{N_v} \prod_{v \in T'} 2^{-N_v} \sum_{M_v} \prod_{u \in T^0 \setminus T'} 2^{-(1-\delta)M_u} \sum_{\Gamma} \sum_{j \in J(T, T', \Gamma) : j_{n_0} = n} \prod_{w \in T^\infty} y_w(j_w)
\]

where \( N_v = (N_v)_{v \in T'} \) and \( M_v = (M_v)_{u \in T^0 \setminus T'} \). The notation in the last line means that the first two sums are taken over all nonnegative integers \( N_v, M_u \) as \( v \) ranges over \( T' \) and \( u \) over \( T^0 \setminus T' \); the third sum is taken over all \( \Gamma \) such that

\[
\langle \gamma_v \rangle \in [2^{N_v}, 2^{1+N_v}) \ \text{for all} \ v \in T';
\]

and the sum with respect to \( j \) is taken over all \( j \) satisfying the additional restrictions

\[
|\sigma_j(j_{(u, 1)}, j_{(u, 2)}, j_{(u, 3)}; j_u)| \sim 2^{M_u} \ \text{for all} \ u \in T^0 \setminus T'
\]

\[
\rho_v(j) = \gamma_v \ \text{for all} \ v \in T'.
\]

Thus there is an upper bound \( 2^{N_v} \leq C_{c_0} |\sigma_v(j)|^{1-\delta} \) for all \( v \in T' \).

For any \( v \in T' \) and any parameter \( \gamma_v \), for any \( j \in J(T, T', \Gamma) \), \( \sigma_v(j) = \gamma_v - \sum_{i=1}^3 \varepsilon_{v,i} \rho_{v,i}(j) \) where \( \rho_{v,i} = \rho_{v,i}(j) \) depends only on \( \{j_w - j_{(w,i)} : w < v, i \in \{1, 2, 3\} \} \). Since the quantity \( \sigma_v \) on the left-hand side equals \( 2(j_v - j_{(v,1)})(j_v - j_{(v,3)}) \), for any \( \{j_w - j_{(w,l)} : w < v, l \in \{1, 2, 3\} \} \) and any \( \gamma_v \) there are at most \( C_{\delta_1} |\gamma_v - \sum_{i=1}^3 \varepsilon_{v,i} \rho_{v,i}|^{\delta_1} \) ordered pairs \( (j_v - j_{(v,1)}, j_v - j_{(v,3)}) \) satisfying \((8.9)\). Here \( \delta_1 \) is an arbitrarily small constant, to be chosen later.
For any frozen node \( v \in T' \), \( |\gamma_v| \) is small relative to \( \sum_{i=1}^3 |\rho_{(v,i)}|^{1-\delta} \), provided that \( c_0 \) is taken to be small in the definition of a frozen node. Therefore we can choose for each combination of parameters \( N, M \) a family \( \mathcal{F} = \mathcal{F}_{N,M} \) of vector-valued functions \( F = (f_{v,i} : v \in T', i \in \{1,3\}) \) of cardinality at most \( C_{\delta_1}|T|^2 \prod_{v \in T'} 2^{\max_i N_{(v,i)} \delta_1} \) such that for any \( \Gamma \) satisfying (S.11) and any \( j \in J(T,T',\Gamma) \), there exists \( F \in \mathcal{F}_{N,M} \) such that for each \( v \in T' \) and each \( i \in \{1,3\} \),

\[
k_{v,i} = j_v - j_{(v,i)} = f_{v,i}(\gamma_v, (k_{w,i} : w < v)).
\]

Thus for all nonnegative sequences \( y_w \) and all \( n \in \mathbb{Z} \),

\[
|S_{(T,T')}(y_w)_{w \in T^\infty}(n)| \leq C|T| \sum_{N,M} |N|^2 (1-\delta)|M| \sum_{F \in \mathcal{F}_{N,M}} |S_{(T,T',N,M,\Gamma,F)}(y_w)_{w \in T^\infty}(n)|
\]

where \( |N| = \sum_v N_v \), \( |M| = \sum_u M_u \), and

\[
S_{(T,T',N,M,\Gamma,F)}(y_w)_{w \in T^\infty}(n) = \sum_{j \in J(T,T',\Gamma); j v_0 = n} \prod_{w \in T^\infty} y_w(j_w).
\]

In (S.11), the second summation is taken over all \( \Gamma = (\gamma_u)_{u \in T'} \) satisfying both (S.17) and (S.18). In (S.12), the sum is taken over all \( j \in J(T,T',\Gamma) \) satisfying \( j v_0 = n \), (S.39), and the additional restriction (S.10). We have finally arrived at our basic building blocks, the multilinear operators \( S_{(T,T',N,M,\Gamma,F)} \).

**Lemma 8.1.** Let \( p \in [1, \infty) \) and \( \delta_1 > 0 \). Then for every exponent \( q \) satisfying \( q \geq 1 \) and \( q > p/|T^\infty| \), there exists \( C < \infty \) such that for every \( T, T', N, M, \Gamma, F \) and for every sequence \( y_v \),

\[
\|S_{(T,T',N,M,\Gamma,F)}(y_w)_{w \in T^\infty}\|_{\ell_q} \leq C|T|2^{(1+\delta_1)|M|} \prod_{v \in T^\infty} \|y_v\|_{\ell_p}.
\]

**Proof.** As was shown in the proof of Lemma 8.1, each quantity \( j_v \) in the summation defining \( S_{(T,T',N,M,\Gamma,F)}(y_w)_{w \in T^\infty}(j v_0) \) can be expressed as a function, depending on \( \Gamma, F \), of \( j v_0 \) together with all \( k_{w,i} = j_w - j_{(w,i)} \), where \( w \) varies over the set \( T^0 \setminus T' \) of all nodes that are neither frozen nor terminal, and \( i \) varies over \( \{1,3\} \). More precisely, \( j_v \) equals \( j v_0 + g_v \), where \( g_v \) is some function of all these \( k_{w,i} \).

\[
\prod_{w \in T^\infty} y_w(j_v) \text{ can thus be rewritten as } \prod_{w \in T^\infty} y_w(j_v + g_v).
\]

If every \( k_{w,i} \) is held fixed, then as a function of \( j v_0 \), this product belongs to \( \ell^q \) for \( q = p/|T^\infty| \) with bound \( \prod_{v \in T^\infty} \|y_v\|_{\ell_p} \), by Hölder’s inequality.

The total number of terms in the sum defining \( S_{(T,T',N,M,\Gamma,F)} \) is the total possible number of vectors \( (k_{w,i}) \) where \( w \) ranges over \( T^0 \setminus T' \) and \( i \) over \( \{1,3\} \). The number of such pairs for a given \( w \) is \( \leq C_{\delta_1} 2^{(1+\delta_1)M} \), since \( 2|k_{w,1} k_{w,3}| = |\sigma_{w}(j)| \leq 2^{M+1} \). Thus in all there are at most \( C_{\delta_1} 2^{(1+\delta_1)|M|} \) terms. Minkowski’s inequality thus gives the stated bound. \( \square \)

(S.13) is a satisfactory bound, but it must be summed over all \( F \in \mathcal{F}_{N,M} \). An upper bound for the number of such functions \( F \) is, roughly speaking, \( C_{\delta_1}|T| \) times the product over all \( w \in T' \) of \( \max_i |\rho_{(w,i)}|^{\delta_1} \). However, this does not quite make sense since \( \max_i |\rho_{(w,i)}| \) is a function of \( j \), which we wish to allow to vary while \( F \in \mathcal{F}_{N,M} \) remains fixed. Thus a correct upper bound is

\[
|\mathcal{F}_{N,M}| \leq C_{\delta_1}|T| \prod_{w \in T'} 2^{\max_i K_{(w,i)} \delta_1}
\]
where \( K_u = N_u \) for \( u \in T' \) and \( K_u = M_u \) for \( u \in T_0 \setminus T' \), and the maximum is taken over \( i \in \{1, 3\} \).

A difficulty now appears. For each \( v \in T' \) we have a compensating factor of \( (\gamma_v (j))^{-1} \sim 2^{-N_v} \), but there is no upper bound whatsoever for the ratio \( \max_i |\rho_{(v,i)}|^{\delta_1} / |\gamma_v| \). Thus the factor gained for a given \( v \in T' \) cannot compensate for the factor lost for that same node. However in aggregate the factors gained compensate for those lost, as will now be shown.

**Lemma 8.2.** For any \( \varepsilon > 0 \) there exists \( C_\varepsilon < \infty \) such that uniformly for all \( T, T', N, M \),

\[
|F_{N,M}| \leq C_\varepsilon^{T[2\varepsilon]}{|M|}.
\]

**Proof.** If the constant \( c_0 \) in the definition (8.17) of a frozen node is chosen to be sufficiently small, then any frozen node \( u \) has a child \( (u, i) \) such that \( |\rho_u| \leq \frac{1}{2} |\rho_{(u,i)}|^{1-\delta} \). Consider any chain \( v = u_h \geq u_{h-1} \geq \cdots \geq u_1 \) of nodes such that \( u_k \) is not frozen for each \( 1 \leq k < h \) \( (u_k \) is the parent of \( u_{k+1} \) for each \( k \) \). Consider any chain \( v = u_h \geq u_{h-1} \geq \cdots \geq u_1 \) of nodes such that \( u_k \) is the \( (k-1) \)-th generation ancestor of \( u_1 \). Since \( u_k \) is frozen for all \( k > 1 \), \( u_1 \) is either not frozen or is terminal, and \( |\rho_k| \leq \frac{1}{2} |\rho_{k-1}|^{1-\delta} \). Then \( |\rho_k| \leq 2^{1-k} |\rho_{(u_1)}| \). Hence \( 2^{K_u} \leq 2 C_\varepsilon K_{(u_1,i)} \).

If \( u_1 \) is terminal then \( \rho_{(u_1)} = 0 \) by definition, whence the inequality \( |\rho_k| \leq 2^{1-k} |\rho_{(u_1)}| \) forces \( \rho_k = 0 \) for all \( k \), as well. This means that \( 2^{K_u} \leq 2 C_\varepsilon \).

In particular, this holds for \( u_k = v \), so the factor \( 2^{K_v} \) will be harmless in our estimates. We say that a node \( v \) is negligible if there exists such a chain, with \( v = u_h \) for some \( h \geq 1 \).

For each nonnegligible frozen node \( v \), choose one such chain with \( u_h = v \), thus uniquely specifying \( h \) and \( u_1 \) as functions of \( v \); then write \( u_1 = D(v) \). Given \( u_1 \) and \( h \), there can be at most one \( v \) such that \( u_1 = D(v) \) and \( v \) is the \( h \)-th generation ancestor of \( u_1 \), simply because any node has at most one \( h \)-th generation ancestor. Now taking the product only over nonnegligible nodes \( v \in T' \) on the left-hand side,

\[
\prod_{v \in T'} 2^{K_v} \leq \prod_{w \in T_0 \setminus T'} \prod_{h=1}^\infty 2^{(1-\delta)^h - 1} M_w = \prod_{w \in T_0 \setminus T'} 2^{M_w 1/\delta},
\]

since each factor \( 2^{K_v} \) in the first product is majorized by \( 2^{(1-\delta)^h - 1} M_w \) in the second product, where \( w = D(v) \) and \( v \) is the \( h \)-th generation ancestor of \( w \). This is not so for negligible nodes, but they contribute at most \( C[|\Gamma|] \) to the left-hand side so the conclusion remains valid for the full product. Thus by choosing \( \delta_1 \) so that \( \delta_1 / \delta = \varepsilon \), since \( |F_{N,M}| \leq C_\varepsilon^{T[2\varepsilon]} \prod_{v \in T'} 2^{K_v} \rho \), we obtain \( |F_{N,M}| \leq C_\varepsilon^{T[2\varepsilon]} \prod_{v \in T_0 \setminus T'} 2^{M_w} = C_\varepsilon^{T[2\varepsilon]} |M| \).

**Conclusion of proof of Proposition 5.3.** Combining the preceding two lemmas gives

\[
\sum_{F \in F_{N,M}} \|S_{T,T',N,M,T'}(y_F)\|_{L^\infty} \leq C_\varepsilon^{T[2(1+\varepsilon)]}|M| \prod_{v \in T_0 \setminus T'} \|y_v\|_{L^p}
\]

for arbitrarily small \( \varepsilon > 0 \), provided \( p \geq \max(1, |\Gamma|) \). Since \( |\Gamma| \leq C[|T|]|2^{N_o}| \), it follows that

\[
\sum_{F \in F_{N,M}} \sum_{v \in T_0 \setminus T'} \|S_{T,T',N,M,T'}(y_F)\|_{L^\infty} \leq C_\varepsilon^{T[2^N_o]} \|y_v\|_{L^p}.
\]

On the other hand, Lemma 7.1 gives a uniform \( \|L^1 \) norm bound of \( C[|T|] \prod_{v \in T_0 \setminus T'} \|y_v\|_{L^1} \) for the summation over all \( j \). Thus if \( q > P_{\Gamma}^{-1} \) and \( q \geq 1 \) we may interpolate to find that there

\[\text{The exponent } 1-\delta < 1 \text{ in the definition (8.17) of a frozen node was introduced solely in order to produce a summable series of exponents } (1-\delta)^h.\]
exists $\eta > 0$ depending on $q - \frac{2}{p}$ but not on $\delta$ such that
\begin{equation}
\sum_{F} \sum_{r} \|S_{T,T',N,M,T,F}(y_{v})\|_{\ell^{p}_{v}} \leq C_{\eta}^{2(1-\eta)}\|N\|^{(1-\eta)} \prod_{v \in T^{\infty}} \|y_{v}\|_{\ell^{p}}.
\end{equation}
Taking into account the factors $2^{-|N|}2^{-|1-\eta|M}$ in (8.11), summing over $N, M$ as well as over all subsets $T' \subset T^{0}$ yields a convergent series and completes the proof of Proposition 5.3. \hfill \Box

9. Loose ends

We may reinterpret the sum of our power series \eqref{5.1}, \eqref{5.2} as a function via the relation $\hat{u}(t,n) = e^{in^{2}t}a_{n}(t)$ with $a(0)$ defined by $\hat{u}(0)(n) = a_{n}(0)$, and will do so consistently without further comment, abusing notation mildly by writing $u(t,x) = S(t)u_{0}(x)$.

**Lemma 9.1.** Let $p \in [1, \infty)$. For any $R > 0$ there exists $\tau > 0$ such that for any $u_{0} \in \mathcal{H}^{p}$ with norm $\leq R$, the element $u(t,x) \in C^{\infty}([0,\tau], \mathcal{H}^{p})$ defined by \eqref{5.1}, \eqref{5.2} is a limit, in $C^{\infty}([0,\tau], \mathcal{H}^{p})$, norm, of smooth solutions of \eqref{NLS}.

**Proof.** All of our estimates apply also in the spaces $\mathcal{H}^{s,p}$ defined by the condition that $\langle (n^{2})^{\frac{3}{2}}f(n) \rangle_{n \in \mathbb{Z}} \in \ell^{p}$, provided that $1 \leq p \leq \infty$ and $s > 0$. This follows from the proof given for $s = 0$ above, for the effect of working in $\mathcal{H}^{s,p}$ is to introduce a factor of $\prod_{v \in T^{0}} (\frac{(j_{v})^{s}}{\prod_{j=1}^{\infty}(j_{v,j}^{s})^{s}})$ in the definition of the tree operator. The relation \eqref{3.4} ensures that $\max_{j} |j_{v,j}| \geq \frac{1}{4} |j_{v}|$, whence $\prod_{j_{v,j}} (\frac{(j_{v,j})^{s}}{(j_{v}^{s})^{s}}) \lesssim 1$, so the estimates for $s = 0$ apply directly to all $s > 0$. More generally, if $\mathcal{H}^{s,p}$ is equipped with the norm
\[ \|f\|_{\mathcal{H}^{s,p}} = \|(1 + |\cdot|^{2s})^{1/2} \mathcal{F}(\cdot)\|_{\ell^{p}} \]
then all estimates hold uniformly in $\varepsilon \in [0,1]$ and $s \geq 0$.

Given $s$ and any initial datum $u_{0}$ satisfying $\|u_{0}\|_{\mathcal{H}^{p}} \leq R$ with the additional property that $\hat{u}_{0}(n) = 0$ for all $|n| > N$, we may choose $\varepsilon > 0$ so that $\|u_{0}\|_{\mathcal{H}^{s,p}} \leq 2R; \varepsilon$ depends on $N$ but not on $R$. Thus the infinite series converges absolutely and uniformly in $C^{\infty}([0,\tau], H^{s-\frac{1}{2}, \frac{1}{p}})$ if $p \geq 2$ and in $C^{\infty}([0,\tau], H^{s})$ if $p \leq 2$, where $\tau$ depends only on $R$, not on $s$. By Lemma 5.2, the series sums to a solution of \eqref{NLS} in the sense \eqref{2.7}; but since the sum is very smooth as a function of $x$ (that is, its Fourier coefficients decay rapidly) this implies that it is a solution in the classical sense. Given an arbitrary $u_{0}$ satisfying $\|u_{0}\|_{\mathcal{H}^{p}} \leq R$, we can thus approximate it by such special initial data to conclude that $S(t)u_{0}$ is indeed a limit, in $C^{\infty}([0,\tau], \mathcal{H}^{p})$, of smooth solutions. \hfill \Box

**Proof of Proposition 5.4**. Let $u_{0} \in \mathcal{H}^{p}$ be given, let $u(t,x) = S(t)u_{0}) \in C^{0}([0,\tau], \mathcal{H}^{p})$. We aim to prove that the nonlinear term $\omega|u|^{2}u$ has an intrinsic meaning as $\lim_{N \to \infty} \omega|T_{N}u|^{2}T_{N}u$ in the sense of distributions in $C^{0}(0,\tau) \times \mathbb{T}$. Forming $T_{N}S(t)(u_{0})$ is of course not the same thing as forming $S(t)(T_{N}u_{0})$.

Define $a_{n}(t) = e^{in^{2}t} \hat{u}(t,n)$. Denote also by $T_{N}$ the operator that maps a sequence-valued function $(b_{n}(t))$ to $(T_{N}b_{n}(t))$ where $T_{N}b_{n} = b_{n}$ if $|n| \leq N$, and $= 0$ otherwise. It suffices to prove that
\begin{equation}
\int_{0}^{t} \sum_{j-k+l=n} T_{N}a_{j}(s)T_{N}a_{k}(s)T_{N}a_{l}(s)e^{ij(s,k,l,n)s} ds - \int_{0}^{t} |T_{N}a_{n}(s)|^{2}T_{N}a_{n}(s) ds
\end{equation}

converges in $\ell^p$ norm as $N \to \infty$, uniformly for all $t \in [0, \tau]$, to 
$$
\sum_{j-k+l=n}^* \int_0^t a_j(s)a_k(s)a_l(s)e^{i\sigma(j,k,l,n)s} \, ds - \int_0^t |a_n(s)|^2 a_n(s) \, ds.
$$

Convergence in the distribution sense follows easily from this by expressing any sufficiently smooth function of the time $t$ as a superposition of characteristic functions of intervals $[0, t]$.

Now in the term $\int_0^t \sum_{j-k+l=n} T_na_j(s)T_na_k(s)T_na_l(s)e^{i\sigma(j,k,l,n)s} \, ds$, the integral may be interchanged with the sum since the truncation operators restrict the summation to finitely many terms. Expanding $a_j, a_k, a_l$ out as infinite series of free operators applied to $a(0)$, we obtain finally an infinite series of the general form $\sum_{k=1}^\infty B_k(t)(a(0), \ldots, a(0))$ where $B_k(t)$ is a finite linear combination of $O(C^k)$ tree sum operators, with coefficients $O(C^k)$, applied to $a(0)$ just as before, with the sole change that the extra restriction $|j(v_0,i)| \leq N$ for $i \in \{1, 2, 3\}$ for indices corresponding to children of the root node is placed on $j$ in the summation defining $S_T$ for each tree $T$.

Since we have shown that all bounds hold for the sums of the absolute values of the terms in the tree sum, it follows immediately that this trilinear term converges as $N \to \infty$. Convergence for the other nonlinear term is of course trivial. Likewise it is trivial that $(T_Nu)_t \to u_t$ and $(T_Nu)_{xx} \to u_{xx}$, by linearity.

This reasoning shows that the limit of each term equals the sum of a convergent power series, taking values in $C^0([0, \tau], \mathcal{H}^p)$, in $u_0$.

Given $R > 0$, there exists $\tau > 0$ for which we have shown that for any $a(0) \in \ell^p$ satisfying $\|a(0)\|_{\ell^p} \leq R$, our power series expansion defines $a(t) \in C^0([0, \tau], \ell^p)$, as an $\ell^p$-valued analytic function of $a(0)$. Moreover for any $t \in [0, \tau]$, both cubic terms in the integral equation (2.6) are well-defined as limits obtained by replacing $a(s)$ by $T_Na(s)$, evaluating the resulting cubic expressions, and passing to the limit $N \to \infty$.

**Lemma 9.2.** Whenever $\|a(0)\|_{\ell^p} \leq R$, the function $a(t) \in C^0([0, \tau], \ell^p)$ defined as the sum of the power series expansion (5.1) satisfies the integral equation (2.6) when the nonlinear terms in (2.6) are defined by the limiting procedure described in the preceding paragraph.

**Proof.** This follows by combining Lemma 5.2 with the result just proved. □

**Proof of Proposition 1.5.** Let $u_0 \in \mathcal{H}^p$. If $u = Su_0$, and if $v$ is the solution of the Cauchy problem (NLS) for the modified linear Schrödinger equation with initial datum $u_0$, then $u_0 - v$ is expressed as $\sum_{k=1}^\infty B_k(t)(u_0, \ldots, u_0)$ where the $n$-th Fourier coefficient of $B_k(t)(u_0, \ldots)(t)$ equals $e^{-in^2t} A_k(t)(a(0), \ldots)$ with $a_n(0) = \hat{u_0}(n)$. According to Proposition 5.3, $\|A_k(t)(a(0), \ldots)\|_{\ell^q} = O(t^{k\frac{p}{3}}a(0)\|_{\ell^p}^q)$ whenever $q > \frac{p}{3}$ and $q \geq 1$. Summing over $k$ yields the conclusion. □

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