ON $A_1^2$ RESTRICTIONS OF WEYL ARRANGEMENTS

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ABSTRACT. Let $\mathcal{A}$ be a Weyl arrangement in an $\ell$-dimensional Euclidean space. The freeness of restrictions of $\mathcal{A}$ was first settled by a case-by-case method by Orlik-Terao (1993), and later by a uniform argument by Douglass (1999). Prior to this, Orlik-Solomon (1983) had completely determined the exponents of these arrangements by exhaustion. A classical result due to Orlik-Solomon-Terao (1986), asserts that the exponents of any $A_1$ restriction, i.e., the restriction of $\mathcal{A}$ to a hyperplane, are given by $\{m_1, \ldots, m_{\ell-1}\}$, where $\exp(\mathcal{A}) = \{m_1, \ldots, m_{\ell}\}$ with $m_1 \leq \cdots \leq m_{\ell}$. As a next step after Orlik-Solomon-Terao towards understanding the exponents of the restrictions, we will investigate the $A_1^2$ restrictions, i.e., the restrictions of $\mathcal{A}$ to subspaces of the type $A_1^2$. In this paper, we give a combinatorial description of the exponents of the $A_1^2$ restrictions and describe bases for the modules of derivations in terms of the classical notion of related roots by Kostant (1955).

1. INTRODUCTION

Assume that $V = \mathbb{R}^\ell$ with the standard inner product $(\cdot, \cdot)$. Denote by $\Phi$ an irreducible (crystallographic) root system in $V$ and by $\Phi^+$ a positive system of $\Phi$. Let $\mathcal{A}$ be the Weyl arrangement of $\Phi^+$. Denote by $L(\mathcal{A})$ the intersection poset of $\mathcal{A}$. For each $X \in L(\mathcal{A})$, we write $\mathcal{A}^X$ for the restriction of $\mathcal{A}$ to $X$. Set $L_p(\mathcal{A}) := \{X \in L(\mathcal{A}) \mid \text{codim}(X) = p\}$ for $0 \leq p \leq \ell$. Let $W$ be the Weyl group of $\Phi$ and let $m_1, \ldots, m_{\ell}$ with $m_1 \leq \cdots \leq m_{\ell}$ be the exponents of $W$.

Notation 1.1. If $X \in L_p(\mathcal{A})$, then $\Phi_X := \Phi \cap X^\perp$ is a root system of rank $p$. A positive system of $\Phi_X$ is taken to be $\Phi_X^+ := \Phi^+ \cap \Phi_X$. Let $\Delta_X$ be the base of $\Phi_X$ associated with $\Phi_X^+$.

Definition 1.2. A subspace $X \in L(\mathcal{A})$ is said to be of type $T$ (or $T$ for short) if $\Phi_X$ is a root system of type $T$. In this case, the restriction $\mathcal{A}^X$ is said to be of type $T$ (or $T$).

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Weyl arrangements are important examples of free arrangements. In other words, the module $D(A)$ of $A$-derivations is a free module. Furthermore, the exponents of $A$ are the same as the exponents of $W$, i.e., $\exp(A) = \{m_1, \ldots, m_\ell\}$ (e.g., [Sai93]). It is shown by Orlik-Solomon [OS83], using the classification of finite reflection groups, that the characteristic polynomial of the restriction $A^X$ ($X \in L(A)$) of an arbitrary Weyl arrangement $A$ is fully factored. Orlik-Terao [OT93] proved a stronger statement that $D(A^X)$ is free by a case-by-case study. Soon afterwards, Douglass [Dou99] gave a uniform proof for the freeness using the representation theory of Lie groups.

We are interested in studying the exponents of $A^X$ and bases for $D(A^X)$. For a general $X$, it is not an easy task. The only known general result due to Orlik-Solomon-Terao [OST86], asserts that for each $X \in A$, i.e., $X$ is of type $A_1$, $\exp(A^X) = \{m_1, \ldots, m_{\ell-1}\}$ and a basis for $D(A^X)$ consists of the restrictions to $X$ of the corresponding basic derivations. In this paper, we consider a “next” general class of restrictions, that is when $X$ is of type $A_2$. We prove that $\exp(A^X)$ is obtained from $\exp(A)$ by removing either the two largest exponents, or the largest and the middle exponents, depending upon a combinatorial condition on $X$. Furthermore, similar to the result of Orlik-Solomon-Terao, our method produces an explicit basis for $D(A^X)$ in each case. The main combinatorial ingredient in our description is the following concept defined by Kostant:

Definition 1.3 ([Kos55]). Two non-proportional roots $\beta_1, \beta_2$ are said to be related if

(a) $(\beta_1, \beta_2) = 0$,
(b) for any $\gamma \in \Phi \setminus \{\pm \beta_1, \pm \beta_2\}$, $(\gamma, \beta_i) = 0$ implies $(\gamma, \beta_{3-i}) = 0$ for all $i \in \{1, 2\}$.

In this case, we call the set $\{\beta_1, \beta_2\}$ relatedly orthogonal (RO), and the subspace $X = H_{\beta_1} \cap H_{\beta_2}$ is said to be RO.

Remark 1.4. The relatedly orthogonal sets presumably first appeared in [Kos55], wherein Kostant required $\beta_1, \beta_2$ to have the same length, and allowed a root to be related to itself and its negative. Green called the relatedly orthogonal sets strongly orthogonal and defined the strongly orthogonality in a more general setting [Gre13, Definition 4.4.1]. It should be noted that the notion of strongly orthogonal sets is probably more well-known with the definition that neither sum nor difference of the two roots is a root. For every $\beta \in \Phi$, set $\beta^\perp := \{\alpha \in \Phi \mid (\alpha, \beta) = 0\}$. Condition (b) in Definition 1.3 can be written symbolically as (b’) $\beta^\perp \setminus \{\pm \beta_2\} = \beta^\perp \setminus \{\pm \beta_1\}$.

Let $h$ be the Coxeter number of $W$. For $\phi \in D(A)$, let $\phi^X$ be the restriction of $\phi$ to $X$. We now formulate our main results.
Theorem 1.5. Assume that \( \ell \geq 3 \). If \( X \in L(A) \) is of type \( A_1^2 \), then \( A^X \) is free with
\[
\exp(A^X) = \begin{cases} 
\exp(A) \setminus \{h/2, m_\ell\} & \text{if } X \text{ is RO}, \\
\exp(A) \setminus \{m_{\ell-1}, m_\ell\} & \text{otherwise}.
\end{cases}
\]

Theorem 1.6. Assume that \( \ell \geq 3 \).

(i) Suppose that \( X \) is \( A_1^2 \) and not RO. Let \( \{\varphi_1, \ldots, \varphi_\ell\} \) be a basis for \( D(A) \) with \( \deg \varphi_j = m_j \) (\( 1 \leq j \leq \ell \)). Then \( \{\varphi_1^X, \ldots, \varphi_{\ell-2}^X\} \) is a basis for \( D(A^X) \).

(ii) Suppose that \( X \) is both \( A_1^2 \) and RO. Then \( \Phi \) must be of type \( D_\ell \) with \( \ell \geq 3 \). Furthermore, a basis for \( D(A^X) \) is given by \( \{\theta_1^X, \ldots, \theta_{\ell-2}^X\} \), where
\[
\theta_i := \sum_{k=1}^{\ell} x_k^{2i-1} \partial_k \quad (1 \leq i \leq \ell - 2).
\]

Theorem 1.5 gave a little extra information: when \( X \) is RO, half of the Coxeter number is an exponent of \( W \) (hence it lies in the “middle” of the exponent sequence in the increasing order). We emphasize that Theorem 1.5 can be readily verified by looking at the numerical results in [OS83]. It is interesting to search for a proof, free of case-by-case considerations. In this paper, we provide a conceptual proof with a minimal use of classifications of root systems (we will use up to the classification of rank-4 root systems, and such classifications will be needed only in Subsection 5.2). Theorem 1.6 seems to be, however, less straightforward even if one relies on the classification. In comparison with the classical result of Orlik-Solomon-Terao, Theorem 1.6(ii) gives a bit more flexible construction of basis for \( D(A^X) \). Namely, a wanted basis is obtained by taking the restriction to \( X \) of any basis for \( D(A) \), without the need of basic derivations. Nevertheless, the remaining part of the basis construction (Theorem 1.6(ii)) can not avoid the classification. It may happen that there are more than one derivations in a basis for \( D(A) \) having the same degree. Hence some additional computation is required to examine which derivation vanishes after taking the restriction to \( X \) (see Remark 2.10).

Beyond the \( A_1^2 \) restrictions, computations on the restrictions in higher codimensions or of irreducible types are more complicated. We leave them for future research.

The rest of this paper is organized as follows. In §2, we first review preliminary results on free arrangements and their exponents, and a recent result (Combinatorial Deletion) relating the freeness to combinatorics of arrangements (Theorem 2.8). We also prove an important result in the paper, a construction of a basis for \( D(A^H) \) (\( H \in A \)) from a basis for \( D(A) \) when
\(\mathcal{A}\) and \(\mathcal{A} \setminus \{H\}\) are both free (Theorem 2.9). We then review the background information on root systems, Weyl groups, and Weyl arrangements. Based on Combinatorial Deletion Theorem, we provide a slightly different proof for the classical result of Orlik-Solomon-Terao (Remark 2.15). In §3, we first gather some numerical facts on \(A_1^2\) and RO sets (Remark 3.5). Then we provide a proof for the freeness part of Theorem 1.5, i.e., the freeness of \(A_1^2\) restrictions (Theorem 3.9). The proof is different to (and more direct than) the proofs presented in [OT93] and [Dou99]. Finally, we present a proof of Theorem 1.6 (Theorem 3.10 and Example 3.11). In §4, we give an evaluation to the cardinality of every \(A_1^2\) restriction (Proposition 4.8). This evaluation can be expressed in terms of the local and global second smallest exponents of the Weyl groups (Remark 4.9). In §5, we complete the proof of Theorem 1.5 by providing a proof for the exponent part (Theorem 5.1). The proof contains two halves which we present the proof for each half in Theorems 5.3 and 5.31. We close the paper by giving two results about local-global inequalities on the second smallest exponents and the largest coefficients of the highest roots (Corollary 5.32), and the Coxeter number of any irreducible component of the subsystem orthogonal to the highest root in the simply-laced cases (Corollary 5.33).

2. Preliminaries

2.1. Free arrangements and their exponents. For basic concepts and results of free arrangements, we refer the reader to [OT92].

Let \(\mathbb{K}\) be a field and let \(V := \mathbb{K}^\ell\). A hyperplane in \(V\) is a subspace of codimension 1 of \(V\). An arrangement is a finite set of hyperplanes in \(V\). We choose a basis \(\{x_1, \ldots, x_\ell\}\) for \(V^*\) and let \(S := \mathbb{K}[x_1, \ldots, x_\ell]\). Fix an arrangement \(\mathcal{A}\) in \(V\). The defining polynomial \(Q(\mathcal{A})\) of \(\mathcal{A}\) is defined by

\[
Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H \in S,
\]

where \(\alpha_H = a_1x_1 + \cdots + a_\ell x_\ell \in V^* \setminus \{0\}, a_i \in \mathbb{K}\) and \(H = \ker \alpha_H\). The number of hyperplanes in \(\mathcal{A}\) is denoted by \(|\mathcal{A}|\). It is easy to see that \(Q(\mathcal{A})\) is a homogeneous polynomial in \(S\) and \(\deg Q(\mathcal{A}) = |\mathcal{A}|\).

The intersection poset of \(\mathcal{A}\), denoted by \(L(\mathcal{A})\), is defined to be

\[
L(\mathcal{A}) := \{\cap_{H \in B} H \mid B \subseteq \mathcal{A}\},
\]

where the partial order is given by reverse inclusion. We agree that \(V \in L(\mathcal{A})\) is the unique minimal element. For each \(X \in L(\mathcal{A})\), we define the localization of \(\mathcal{A}\) on \(X\) by

\[
\mathcal{A}_X := \{K \in \mathcal{A} \mid X \subseteq K\},
\]
and define the restriction $A^X$ of $A$ to $X$ by
$$A^X := \{ K \cap X \mid K \in A \setminus A_X \}.$$ The Möbius function $\mu : L(A) \to \mathbb{Z}$ is formulated by
$$\mu(V) := 1, \quad \mu(X) := -\sum_{X \subsetneq Y \subseteq V} \mu(Y).$$

The characteristic polynomial $\chi(A, t)$ of $A$ is defined by
$$\chi(A, t) := \sum_{X \in L(A)} \mu(X) t^{\dim X}.$$ A derivation of $S$ over $K$ is a linear map $\phi : S \to S$ such that for all $f, g \in S$, $\phi(fg) = f \phi(g) + g \phi(f)$. Let $\text{Der}(S)$ denote the set of derivations of $S$ over $K$. Then $\text{Der}(S)$ is a free $S$-module with a basis $\{\partial_1, \ldots, \partial_\ell\}$, where $\partial_i := \partial/\partial x_i$ for $1 \leq i \leq \ell$. Define an $S$-submodule of $\text{Der}(S)$, called the module of $A$-derivations, by
$$D(A) := \{ \phi \in \text{Der}(S) \mid \phi(Q) \in QS \}.$$ A non-zero element $\phi = f_1 \partial_1 + \cdots + f_\ell \partial_\ell \in \text{Der}(S)$ is homogeneous of degree $b$ if each non-zero polynomial $f_i \in S$ for $1 \leq i \leq \ell$ is homogeneous of degree $b$. We then write $\deg \phi = b$. The arrangement $A$ is called free if $D(A)$ is a free $S$-module. If $A$ is free, then $D(A)$ admits a basis $\{\phi_1, \ldots, \phi_\ell\}$ consisting of homogeneous derivations ([OT92, Proposition 4.18]). Such a basis is called a homogeneous basis. Although homogeneous basis need not be unique, the degrees of elements of a basis are unique (with multiplicity but neglecting the order) depending only on $A$ ([OT92, Proposition A.24]). In this case, we call $\deg \phi_1, \ldots, \deg \phi_\ell$ the exponents of $A$, store them in a multiset denoted by $\exp(A)$ and write
$$\exp(A) = \{ \deg \phi_1, \ldots, \deg \phi_\ell \}.$$ Interestingly, when an arrangement is free, the exponents turn out to be the roots of the characteristic polynomial due to a result of Terao.

**Theorem 2.1** (Factorization). If $A$ is free with $\exp(A) = \{d_1, \ldots, d_\ell\}$, then
$$\chi(A, t) = \prod_{i=1}^\ell (t - d_i).$$

**Proof.** See [Ter81] or [OT92, Theorem 4.137]. \qed

For $\phi_1, \ldots, \phi_\ell \in D(A)$, we define the $(\ell \times \ell)$-matrix $M(\phi_1, \ldots, \phi_\ell)$ as the matrix with $(i, j)$th entry $\phi_j(x_i)$. In general, it is difficult to determine whether a given arrangement is free or not. However, using the following criterion, we can verify that a candidate for a basis is actually a basis.
Theorem 2.2 (Saito’s criterion). Let $\phi_1, \ldots, \phi_\ell \in D(A)$. Then $\{\phi_1, \ldots, \phi_\ell\}$ forms a basis for $D(A)$ if and only if
$$\det M(\phi_1, \ldots, \phi_\ell) = cQ(A) \quad (c \neq 0).$$
In particular, if $\phi_1, \ldots, \phi_\ell$ are all homogeneous, then $\{\phi_1, \ldots, \phi_\ell\}$ forms a basis for $D(A)$ if and only if the following two conditions are satisfied:
(i) $\phi_1, \ldots, \phi_\ell$ are independent over $S$,
(ii) $\sum_{i=1}^\ell \deg \phi_i = |A|$.

Proof. See [OT92, Theorems 4.19 and 4.23].

In addition to the Saito’s criterion, we have a way to check if a set of derivations is part of a homogeneous basis, and sometimes a full basis. First, the notation $\{d_1, \ldots, d_\ell\} \leq$ indicates $d_1 \leq \cdots \leq d_\ell$.

Theorem 2.3. Let $A$ be a free arrangement with $\exp(A) = \{d_1, \ldots, d_\ell\} \leq$. If $\phi_1, \ldots, \phi_k \in D(A)$ satisfy for $1 \leq i \leq k$,
(i) $\deg \phi_i = d_i$,
(ii) $\phi_i \notin S\phi_1 + \cdots + S\phi_{i-1}$,
then $\phi_1, \ldots, \phi_k$ may be extended to a basis for $D(A)$.

Proof. See [OT92, Theorem 4.42].

Definition 2.4. For $X \in L(A)$, let $I = I(X) := \sum_{H \in A_X} \alpha_H S$ and $\overline{S} := S/I$. For $\phi \in D(A)$, define $\phi^X \in \text{Der}(\overline{S})$ by $\phi^X(f + I) = \phi(f) + I$. We call $\phi^X$ the restriction of $\phi$ to $X$.

Proposition 2.5. If $\phi \in D(A)$, then $\phi^X \in D(A^X)$. If $\phi^X \neq 0$, then $\deg \phi^X = \deg \phi$.

Proof. See [OST86, Lemma 2.12].

Fix $H \in A$, denote $A' := A \setminus \{H\}$ and $A'' := A^H$. We call $(A, A', A'')$ the triple with respect to the hyperplane $H \in A$.

Proposition 2.6. Define $h : D(A') \rightarrow D(A)$ by $h(\phi) = \alpha_H \phi$ and $q : D(A) \rightarrow D(A'')$ by $q(\phi) = \phi^H$. The sequence
$$0 \rightarrow D(A') \xrightarrow{h} D(A) \xrightarrow{q} D(A'')$$
is exact.

Proof. See [OT92, Proposition 4.45].

Then the freeness of any two of the triple, under a certain condition on their exponents, implies the freeness of the third.

Theorem 2.7 (Addition-Deletion). Let $A$ be a non-empty arrangement and let $H \in A$. Then two of the following imply the third:
(1) $A$ is free with $\exp(A) = \{d_1, \ldots, d_\ell, d_\ell\}$.
(2) $A'$ is free with $\exp(A') = \{d_1, \ldots, d_{\ell-1}, d_\ell - 1\}$.
(3) $A''$ is free with $\exp(A'') = \{d_1, \ldots, d_{\ell-1}\}$.
Moreover, all the three hold true if $A$ and $A'$ are both free.

Proof. See [Ter80] or [OT92, Theorems 4.46 and 4.51].

The Addition, Deletion Theorems above are rather “algebraic” as they rely on the conditions concerning the exponents. Recently, Abe found “combinatorial” versions for these theorems [Abe], [Abe18], [Abe19]. In this paper, we focus on the Deletion theorem.

Theorem 2.8 (Combinatorial Deletion). Let $A$ be a free arrangement and $H \in A$. Then $A'$ is free if and only if $|A_X| - |A_X^H|$ is a root of $\chi(A_X, t)$ for all $X \in L(A^H)$.

Proof. See [Abe, Theorem 8.2].

When $A$ and $A'$ are both free, one may construct a basis for $D(A'')$ from a basis for $D(A)$. Although the following theorem is probably well-known among experts, we give a detailed proof for the sake of completeness.

Theorem 2.9. Let $A$ be a non-empty free arrangement and $\exp(A) = \{d_1, \ldots, d_\ell\}$. Assume further that $A'$ is also free. Let $\{\varphi_1, \ldots, \varphi_\ell\}$ be a basis for $D(A)$ with $\deg \varphi_i = d_j$ for $1 \leq j \leq \ell$. Then there exists some $p$ with $1 \leq p \leq \ell$ such that $\{\varphi_1^H, \ldots, \varphi_\ell^H\} \setminus \{\varphi_p^H\}$ forms a basis for $D(A'')$.

Proof. By Theorem 2.7, $A''$ is also free and we may write $\exp(A'') = \exp(A) \setminus \{d_k\}$ for some $k$ with $1 \leq k \leq \ell$. Denote $\tau_i := \varphi_i$ $(1 \leq i \leq k-1)$, and $\tau_j := \varphi_{j+1}^H$ $(k \leq j \leq \ell - 1)$.

If $\tau_i^H \notin S\tau_i^H + \cdots + S\tau_{i-1}^H$ for all $1 \leq i \leq \ell - 1$, then by Theorem 2.3, $\{\tau_1^H, \ldots, \tau_{\ell-1}^H\} = \{\varphi_1^H, \ldots, \varphi_{\ell-1}^H\} \setminus \{\varphi_p^H\}$ forms a basis for $D(A''')$.

If not, there exists some $p$, $1 \leq p \leq \ell - 1$, such that $\tau_p^H \notin S\tau_1^H + \cdots + S\tau_{p-1}^H$. By Proposition 2.6,

$$\tau_p = f_1 \tau_1 + \cdots + f_{p-1} \tau_{p-1} + \alpha_H \tau,$$

where $f_i \in S$ $(1 \leq i \leq p - 1)$, $\tau \in D(A')$, $\deg \tau = d_p - 1$. By Theorem 2.2 (Saito’s criterion), $\{\tau_1, \ldots, \tau_p, \alpha_H \tau, \tau_{p+1}, \ldots, \tau_{\ell-1}, \varphi_k\}$ is a basis for $D(A')$. By Theorem 2.7, $\exp(A'') = \exp(A) \setminus \{d_p\}$. It means that $d_k = d_p$. Note that if $d_k$ appears only once in $\exp(A)$, then we obtain a contradiction here and the proof is completed. Now suppose $d_k$ appears at least twice. Again by Theorem 2.2, $\{\tau_1, \ldots, \tau_p, \alpha_H \tau, \tau_{p+1}, \ldots, \tau_{\ell-1}, \varphi_k\}$ is a basis for $D(A)$. Since the map $q : D(A) \to D(A'')$ is surjective ([OT92, Proposition 4.57]), $D(A'')$ is generated by $\{\tau_1^H, \ldots, \tau_p^H, \tau_{p+1}^H, \ldots, \tau_{\ell-1}^H, \varphi_k^H\}$. This set is the same as $\{\varphi_1^H, \ldots, \varphi_p^H\} \setminus \{\varphi_p^H\}$ which indeed forms a basis for $D(A'')$ by [OT92, Proposition A.3]. It completes the proof. □
Remark 2.10. If $d_k$ appears only once in $\exp(A)$, then Theorem 2.9 gives an explicit basis for $D(A'')$. However, if $d_k$ appears at least twice, Theorem 2.9 may not be sufficient to derive an explicit basis for $D(A'')$. It requires some additional computation to examine which derivation vanishes after taking the restriction to $H$. This observation will be useful to construct an explicit basis for $D(A^X)$ when $X$ is $A_1^2$ and RO (Example 3.11).

2.2. Root systems, Weyl groups and Weyl arrangements. Our standard reference for root systems and their Weyl groups is [Bou68].

Let $V := \mathbb{R}^\ell$ with the standard inner product $(\cdot, \cdot)$. Let $\Phi$ be an irreducible (crystallographic) root system spanning $V$. The rank of $\Phi$, denoted by $\text{rank}(\Phi)$, is defined to be $\dim(V)$. We fix a positive system $\Phi^+$ of $\Phi$. We write $\Delta := \{\alpha_1, \ldots, \alpha_\ell\}$ for the simple system (base) of $\Phi$ associated with $\Phi^+$. For $\alpha = \sum_{i=1}^\ell d_i \alpha_i \in \Phi^+$, the height of $\alpha$ is defined by $\text{ht}(\alpha) := \sum_{i=1}^\ell d_i$. There are four classical types: $A_\ell (\ell \geq 1)$, $B_\ell (\ell \geq 2)$, $C_\ell (\ell \geq 3)$, $D_\ell (\ell \geq 4)$ and five exceptional types: $E_6$, $E_7$, $E_8$, $F_4$, $G_2$. We write $\Phi = T$ if the root system $\Phi$ is of type $T$, otherwise we write $\Phi \neq T$.

A reflection in $V$ with respect to a vector $\alpha \in V \setminus \{0\}$ is a mapping $s_\alpha : V \to V$ defined by $s_\alpha(x) := x - 2\frac{(x, \alpha)}{(\alpha, \alpha)}\alpha$. The Weyl group $W := W(\Phi)$ of $\Phi$ is a group generated by the set $\{s_\alpha \mid \alpha \in \Phi\}$. An element of the form $c = s_\alpha_1 \ldots s_\alpha_\ell \in W$ is called a Coxeter element. Since all Coxeter elements are conjugate (e.g., [Bou68, Chapter V, §6.1, Proposition 1]), they have the same order, characteristic polynomial and eigenvalues. The order $h := h(W)$ of Coxeter elements is called the Coxeter number of $W$. For a fixed Coxeter element $c \in W$, if its eigenvalues are of the form $\exp(2\pi \sqrt{-1} m_1/h), \ldots, \exp(2\pi \sqrt{-1} m_\ell/h)$ with $0 < m_1 \leq \cdots \leq m_\ell < h$, then the integers $m_1, \ldots, m_\ell$ are called the exponents of $W$ (or of $\Phi$).

Theorem 2.11. For any irreducible root system $\Phi$ of rank $\ell$,

(i) $m_j + m_{\ell+1-j} = h$ for $1 \leq j \leq \ell$,
(ii) $m_1 + m_2 + \cdots + m_\ell = \ell h/2$,
(iii) $1 = m_1 < m_2 \leq \cdots \leq m_{\ell-1} < m_\ell = h - 1$,
(iv) $h = 2|\Phi^+|/\ell$,
(v) $h = \text{ht}(\theta) + 1$, where $\theta$ is the highest root of $\Phi$.
(vi) $h = 2\sum_{\mu \in \Phi^+}(\gamma, \mu)^2$, for arbitrary $\gamma \in \Phi^+$. Here $\hat{x} := x/(x, x)$.

Proof. See, e.g., [Bou68, Chapter V, §6.2 and Chapter VI, §1.11].

Let $\Theta^{(r)} \subseteq \Phi^+$ be the set consisting of positive roots of height $r$, i.e., $\Theta^{(r)} = \{\alpha \in \Phi^+ \mid \text{ht}(\alpha) = r\}$. The height distribution of $\Phi^+$ is defined as a multiset of positive integers:

$$\{t_1, \ldots, t_r, \ldots, t_{h-1}\},$$
where \( t_r := |\Theta^{(r)}| \). The dual partition \( \mathcal{DP}(\Phi^+) \) of the height distribution of \( \Phi^+ \) is given by a multiset of non-negative integers:

\[
\mathcal{DP}(\Phi^+) := \{(0)^{\ell-t_1}, (1)^{t_1-t_2}, \ldots, (h-2)^{t_{h-2}-t_{h-3}}, (h-1)^{t_{h-1}}\},
\]

where notation \((a)^b\) means the integer \(a\) appears exactly \(b\) times.

**Theorem 2.12.** The exponents of \( W \) are given by \( \mathcal{DP}(\Phi^+) \).

**Proof.** See, e.g., [Ste59], [Kos59], [Mac72], [ABC+16]. \(\square\)

Let \( S^W \) denote the ring of \( W \)-invariant polynomials. Let \( \mathcal{F} = \{f_1, \ldots, f_\ell\} \) be a set of basic invariants with \( \deg f_1 \leq \cdots \leq \deg f_\ell \). Then \( S^W = \mathbb{R}[f_1, \ldots, f_\ell] \) and \( m_i = \deg f_i - 1 \) \((1 \leq i \leq \ell)\). Let \( \mathcal{D} = \{\theta_{f_1}, \ldots, \theta_{f_\ell}\} \) be the set of basic derivations associated to \( \mathcal{F} \) (see, e.g., [OST86, Definition 2.4] and [OT92, Definition 6.50]). The Weyl arrangement of \( \Phi^+ \) is defined by

\[
\mathcal{A} = \mathcal{A}(\Phi^+) := \{(\mathbb{R} \alpha)^\perp | \alpha \in \Phi^+\}.
\]

**Theorem 2.13.** \( \mathcal{A} \) is free with \( \exp(\mathcal{A}) = \{m_1, \ldots, m_\ell\} \) and \( \{\theta_{f_1}, \ldots, \theta_{f_\ell}\} \) is a basis for \( D(\mathcal{A}) \).

**Proof.** See, e.g., [Sai93] and [OT92, Theorem 6.60]. \(\square\)

Recall from Proposition 2.6 the map \( q : D(\mathcal{A}) \to D(\mathcal{A}') \) defined by \( q(\phi) = \phi^H \).

**Theorem 2.14.** If \( H \in \mathcal{A} \), then \( \mathcal{A}^H \) is free with \( \exp(\mathcal{A}^H) = \{m_1, \ldots, m_{\ell-1}\} \). Furthermore, \( \{\theta_{f_1}^H, \ldots, \theta_{f_{\ell-1}}^H\} \) is a basis for \( D(\mathcal{A}^H) \).

**Proof.** See [OST86, Theorem 1.12]. \(\square\)

**Remark 2.15.** We can give a slightly different proof of Theorem 2.14. By [OST86, Theorem 3.7], \( |\mathcal{A}| - |\mathcal{A}^H| = m_\ell \). Thus \( |\mathcal{A}| - |\mathcal{A}^H| \) is a root of \( \chi(\mathcal{A}, t) \) by Theorems 2.13 and 2.1. By Theorem 2.8, \( \mathcal{A}' \) is free (see also Theorem 3.9 for a similar and more detailed explanation). Thus \( \mathcal{A}^H \) is free with \( \exp(\mathcal{A}^H) = \{m_1, \ldots, m_{\ell-1}\} \) which follows from Theorem 2.7. A basis for \( D(\mathcal{A}^H) \) can be constructed a bit more flexibly, without the need to introduce the basic derivations. Namely, let \( \{\varphi_1, \ldots, \varphi_\ell\} \) be any basis for \( D(\mathcal{A}) \) with \( \deg \varphi_j = m_j \) for \( 1 \leq j \leq \ell \). Note that \( m_\ell \) appears exactly once in \( \exp(\mathcal{A}) \). Then by Theorem 2.9, \( \{\varphi_{f_1}^H, \ldots, \varphi_{f_{\ell-1}}^H\} \) is a basis for \( D(\mathcal{A}') \).

A subset \( \Gamma \subseteq \Phi \) is called a (root) subsystem if it is a root system in \( \text{span}_R(\Gamma) \subseteq V \). For any \( J \subseteq \Delta \), set \( \Phi(J) := \Phi \cap \text{span}(J) \). Let \( W(J) \) be the group generated by \( \{s_\delta | \delta \in J\} \). By [Car72, Proposition 2.5.1], \( \Phi(J) \) is a subsystem of \( \Phi \), \( J \) is a base of \( \Phi(J) \) and the Weyl group of \( \Phi(J) \) is \( W(J) \). For any subset \( \Gamma \subseteq \Phi \) and \( w \in W \), denote \( w\Gamma := \{w(\alpha) | \alpha \in \Gamma\} \subseteq \Phi \). A parabolic subsystem is any subsystem of the form \( w\Phi(J) \), likewise, a
parabolic subgroup of $W$ is any subgroup of the form $wW(J)w^{-1}$, where $J \subseteq \Delta$ and $w \in W$.

**Theorem 2.16.** For $\Gamma \subseteq \Phi$, $\Phi \cap \text{span}(\Gamma)$ is a parabolic subsystem of $\Phi$ and its Weyl group is a parabolic subgroup of $W$.

**Proof.** See, e.g., [Kra94, Lemma 3.2.3], [HRT97, Proposition 2.6]. □

Recall the notation of $\Phi_X$, $\Phi_X^+$, $\Delta_X$ for $X \in L(A)$ from Notation 1.1. Note that if $X \in L_p(A)$, then $\Phi_X$ is a parabolic subsystem of rank $p$ (Theorem 2.16). For each $X \in L(A)$, define the fixer of $X$ by

$$W_X := \{w \in W \mid w(x) = x \text{ for all } x \in X\}.$$ 

**Proposition 2.17.** If $X \in L(A)$, then

(i) $W_X$ is the Weyl group of $\Phi_X$. Consequently, $W_X$ is a parabolic subgroup of $W$.

(ii) $A_X$ is the Weyl arrangement of $\Phi_X^+$.

**Proof.** (i) The first statement follows from [Car72, Proposition 2.5.5]. The second statement follows from Theorem 2.16. (ii) follows from (i) and [Bou68, Chapter V, §3.3, Proposition 2]. □

**Definition 2.18.** Two subsets $\Phi_1, \Phi_2 \subseteq \Phi$ (resp., two subspaces $X_1, X_2 \in L(A)$) lie in the same $W$-orbit if there exists $w \in W$ such that $\Phi_1 = w\Phi_2$ (resp., $X_1 = wX_2$). Two subgroups $W_1, W_2$ of $W$ are $W$-conjugate if there exists $w \in W$ such that $W_1 = w^{-1}W_2w$.

**Lemma 2.19.** Let $X_1, X_2$ be subspaces in $L(A)$. The following statements are equivalent:

(i) $\Phi_{X_1}$ and $\Phi_{X_2}$ lie in the same $W$-orbit.

(ii) $\Delta_{X_1}$ and $\Delta_{X_2}$ lie in the same $W$-orbit.

(iii) $X_1$ and $X_2$ lie in the same $W$-orbit.

(iv) $W_{X_1}$ and $W_{X_2}$ are $W$-conjugate.

Consequently, if any one of the statements above holds, then $|A^{X_1}| = |A^{X_2}|$.

**Proof.** The equivalence of the statements follows from [OS83, Lemmas (3.4), (3.5)] (see also [Kan01, Chapter VIII, 27-3, Proposition B]). The consequence is straightforward. □

3. $A_2^1$ RESTRICTIONS: FREENESS, EXPONENTS AND BASIS

**Definition 3.1.** A set $\{\beta_1, \beta_2\} \subseteq \Phi$ with $\beta_1 \neq \pm \beta_2$ is called an $A_2^1$ set if it spans a subsystem of type $A_2^1$, i.e., $\Phi \cap \text{span}\{\beta_1, \beta_2\} = \{\pm \beta_1, \pm \beta_2\}$.

**Lemma 3.2.**
(i) For any $\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha \in \Phi$, the set of $\alpha \in \Delta$ such that $c_\alpha \neq 0$ forms a non-empty connected induced subgraph of the Dynkin graph of $\Phi$.
(ii) If $G$ is a non-empty connected subgraph of the Dynkin graph, then $\sum_{\alpha \in G} \alpha \in \Phi$.
(iii) If $\{\beta_1, \beta_2\} \subseteq \Delta$ and $(\beta_1, \beta_2) = 0$, then $\{\beta_1, \beta_2\}$ is an $A_1^2$ set.

**Proof.** Proofs of (i) and (ii) can be found in [Bou68, Chapter VI, §1.6, Corollary 3 of Proposition 19]. (iii) is an easy consequence of (i).

Let $T(A_1^2)$ (resp., $T(RO)$) be the set consisting of $A_1^2$ (resp., RO) sets.

**Proposition 3.3.**
(i) If $\{\beta_1, \beta_2\}$ is $A_1^2$ (resp., RO), then $w\{\beta_1, \beta_2\}$ is $A_1^2$ (resp., RO) for all $w \in W$.
(ii) $T(A_1^2) = T(\Delta)$, where $T(\Delta) := \{ w\{\alpha_i, \alpha_j\} \mid \{\alpha_i, \alpha_j\} \subseteq \Delta, (\alpha_i, \alpha_j) = 0, w \in W \}$.

**Proof.** (i) is straightforward. (ii) follows from (i), Theorem 2.16 and Lemma 3.2(ii).

**Remark 3.4.** By Proposition 3.3 and Definition 1.3, $T(A_1^2) \neq \emptyset$ (resp., $T(RO) \neq \emptyset$) only when $\dim(V) \geq 3$.

**Remark 3.5.** Now we collect some numerical facts about $T(A_1^2)$ and $T(RO)$. By a direct check on all root systems, $T(RO) \cap T(A_1^2) \neq \emptyset$ if and only if $\Phi = D_\ell$ with $\ell \geq 3 (D_3 = A_3)$. In general, $T(RO) \setminus T(A_1^2) \neq \emptyset$, for example when $\Phi = B_\ell (\ell \geq 3), \{\epsilon_1 - \epsilon_2, \epsilon_1 + \epsilon_2\}$ is RO but spans a subsystem of type $B_2$ (notation in [Bou68, Plate II]). There is only one orbit of $A_1^2$ sets, with the following exceptions:

(i) when $\Phi = D_4$, $T(RO) = T(A_1^2)$, and there are three different orbits,
(ii) when $\Phi = D_\ell (\ell \geq 5), T(RO) \subsetneq T(A_1^2)$, and there are two different orbits: $T(RO)$ and $T(A_1^2) \setminus T(RO)$.
(iii) when $\Phi \in \{B_\ell, C_\ell\} (\ell \geq 4), T(RO) \cap T(A_1^2) = \emptyset$, and there are two different orbits: $T(\Delta^c) := \{ \{\alpha, \beta\} \in T(A_1^2) \mid ||\alpha|| = ||\beta|| \}$, and $T(A_1^2) \setminus T(\Delta^c)$.

Let us recall some general facts on hyperplane arrangements.

**Lemma 3.6.**
(i) If $X, Y \in L(A)$, then $(A^X)^{X \cap Y} = A^{X \cap Y}$. Similarly, $(\phi^X)^{X \cap Y} = \phi^{X \cap Y}$ for any $\phi \in D(A)$.
(ii) If $H \in A$ and $X \in L(A^H)$, then $(A^H)_X = (A_X)^H$. We will use the notation $A^H_X$ to denote these arrangements.

**Proof.** Straightforward.
Lemma 3.7. Let $(A_i, V_i)$ be irreducible arrangements $(1 \leq i \leq n)$. Let \( A = A_1 \times \cdots \times A_n \) and \( V = \bigoplus_{i=1}^n V_i \). Then

(i) \( \chi(A, t) = \prod_{i=1}^n \chi(A_i, t) \),

(ii) for \( H = V_1 \oplus \cdots \oplus V_{k-1} \oplus H_k \oplus V_{k+1} \oplus \cdots \oplus V_n \in A \) (\( H_k \in A_k \)),

\[ A^H = A_1 \times \cdots \times A_{k-1} \times A_k^{H_k} \times A_{k+1} \times \cdots \times A_n. \]

Proof. (i) is a well-known fact. (ii) follows by definition of restrictions. \( \square \)

Note that \( X \in L(A) \) is of type \( A_1^2 \) if and only if \( \Delta_X \) is an \( A_2^2 \) set. The freeness of every restriction is settled by a case-by-case study in [OT93], and by a uniform method in [Dou99]. In Theorem 3.9 below, we provide a different and more direct proof for the freeness of \( A_1^2 \) restrictions by using the Addition-Deletion Theorem (Theorem 2.7) and Combinatorial Deletion Theorem (Theorem 2.8). We also need the following result, which we prove it in Theorem 5.1.

Proposition 3.8. If \( \{\alpha, \beta\} \) is an \( A_1^2 \) set, then \( |A^{H_\alpha \cap H_\beta}| = |A^{H_\alpha \cap H_\beta}| \) is a root of \( \chi(A^{H_\alpha}, t) = \prod_{i=1}^n (t - m_i) \).

Theorem 3.9. If \( \{\alpha, \beta\} \) is an \( A_1^2 \) set, then the following statements are equivalent:

(1) \( A^{H_\alpha \cap H_\beta} \) is free and \( \exp(A^{H_\alpha \cap H_\beta}) = \exp(A) \setminus \{m_i, m_\ell\} \) for some \( i \) with \( 1 \leq i \leq \ell - 1 \).

(2) \( |A^{H_\alpha}| - |A^{H_\alpha \cap H_\beta}| = m_i. \)

Proof. It suffices to prove (2) \( \Rightarrow \) (1). By Theorem 2.14, \( A^{H_\alpha} \) is free with \( \exp(A^{H_\alpha}) = \{m_1, \ldots, m_{\ell-1}\} \). By Lemma 3.6, \( A^{H_\alpha \cap H_\beta} = (A^{H_\alpha})^{H_\alpha \cap H_\beta} \). If we can prove that \( A^{H_\alpha} \setminus \{H_\alpha \cap H_\beta\} \) is free, then Theorem 2.7 and Condition (2) guarantee that \( A^{H_\alpha \cap H_\beta} \) is free with \( \exp(A^{H_\alpha \cap H_\beta}) = \exp(A) \setminus \{m_i, m_\ell\} \) for some \( 1 \leq i \leq \ell - 1 \).

To show the freeness of \( A^{H_\alpha} \setminus \{H_\alpha \cap H_\beta\} \), we use Theorem 2.8. We need to show that \( |A^{H_\alpha}_X| - |A^{H_\alpha \cap H_\beta}_X| \) is a root of \( \chi(A^{H_\alpha}_X, t) \) for all \( X \in L(A^{H_\alpha \cap H_\beta}) \). It is clearly true by Proposition 3.8 provided that \( A_X \) is irreducible. If \( A_X \) is reducible, write \( A_X = A_1 \times \cdots \times A_n \) where each \( A_i \) is irreducible. By Lemma 3.7, \( |A^{H_\alpha}_X| - |A^{H_\alpha \cap H_\beta}_X| \) either equals \( |A^{H_k}_X| - |A^{H_{k'} \cap H_{k'}}_X| \) where \( 1 \leq k \leq n, H_k, H_k' \in A_k, H_k \cap H_k' \) is \( A_1^2 \) with respect to \( A_k \), or equals \( |A_j| - |A^{H_j}_X| \) for some \( j \neq k \). In either case, \( |A^{H_\alpha}_X| - |A^{H_\alpha \cap H_\beta}_X| \) is a root of \( \chi(A^{H_\alpha}_X, t) = \chi(A_1, t) \cdots \chi(A_k, t) \cdots \chi(A_{k'}, t) \cdots \chi(A_n, t). \) \( \square \)

Now we give an explicit basis for \( D(A^X) \) for a given \( A_1^2 \) subspace \( X \).

Theorem 3.10. Assume that \( X = H_1 \cap X_2 \) is \( A_1^2 \) but not RO. Let \( \{\varphi_1, \ldots, \varphi_\ell\} \) be a basis for \( D(A) \) with \( \deg \varphi_j = m_j \) (1 \( \leq j \leq \ell \)). Then \( \{\varphi_1^X, \ldots, \varphi_{\ell-2}^X\} \) is a basis for \( D(A^X) \).
Proof. The statement is checked by a case-by-case method when $\ell \leq 4$. As we will see in Corollary 5.16, if $\ell \geq 5$, then $m_{\ell-1}$ appears exactly once in $\exp(A)$. By Remark 2.15, $A^{H_1}$ is free and $\{\varphi_{H_1}^{1}, \ldots, \varphi_{H_1}^{\ell-1}\}$ is a basis for $D(A^{H_1})$. We will also see in Theorem 5.1, that $|A^{H_1}| - |A^X| = m_{\ell-1}$. By Proof of Theorem 3.9, $A^{H_1} \setminus \{X\}$ is also free. Theorem 2.9 completes the proof.

Example 3.11. Assume that $X \in L(A)$ is both $A_2^1$ and RO. By Remark 3.5, $\Phi = D_{\ell}$ with $\ell \geq 3$. Suppose that

$$Q = \prod_{1 \leq i < j \leq \ell} (x_i - x_j) \prod_{1 \leq i < j \leq \ell} (x_i + x_j),$$

where $\{x_1, \ldots, x_\ell\}$ is an orthonormal basis for $V^*$. Let $H_1 = \ker(x_1 + x_2)$, $H_2 = \ker(x_1 - x_2)$, and $X = H_1 \cap X_2$. Then $X$ is $A_2^1$ and RO. Define

$$\theta_i := \sum_{k=1}^{\ell} x_k^{2i-1} \partial_k \quad (1 \leq i \leq \ell - 1),$$

$$\eta := \sum_{k=1}^{\ell} \frac{x_1 \ldots x_\ell}{x_k} \partial_k.$$

Then it is known that $\theta_1, \ldots, \theta_{\ell-1}, \eta$ form a basis for $D(A)$. Let

$$\varphi := \left( \prod_{k=3}^{\ell} (x_1^2 - x_k^2) \right) \partial_1 + \left( \prod_{k=3}^{\ell} (x_2^2 - x_k^2) \right) \partial_2.$$

Then it is not hard to verify that $\varphi \in D(A \setminus \{H_1\})$ and thus $(x_1 + x_2)\varphi \in D(A)$. By Saito’s criterion, we may show that $\theta_1, \ldots, \theta_{\ell-2}, \eta, (x_1 + x_2)\varphi$ also form a basis for $D(A)$, and $\theta_1, \ldots, \theta_{\ell-2}, \eta, \varphi$ form a basis for $D(A \setminus \{H_1\})$. Therefore, $\{\theta_1^{H_1}, \ldots, \theta_{\ell-2}^{H_1}, \eta^{H_1}\}$ is a basis for $D(A^{H_1})$. This basis may have two elements having the same degree, for example, $\deg \theta_{\ell/2} = \ell - 1$ when $\ell$ is an even number. However, it is easy to check that for every case $\eta^X = 0$. Hence, $\{\theta_1^X, \ldots, \theta_{\ell-2}^X\}$ is a basis for $D(A^X)$ and $\exp(A^X) = \{1, 3, 5, \ldots, 2\ell - 5\}$ as predicted in Theorem 2.9. This is also consistent with the fact that the $A^X$ above is exactly the Weyl arrangement of type $B_{\ell-2}$.

Proof of Theorem 1.6. It follows from Theorem 3.10 and Example 3.11.

4. Enumerate the cardinalities of $A_2^1$ restrictions

In this section, we will express the cardinality $|A^X|$ where $X$ is of type $A_2^1$ in terms of the Coxeter number $h$ and sum of inner products of positive roots (Proposition 4.8). To do that, for an $A_2^1$ set $\{\beta_1, \beta_2\} \subseteq \Phi$, we work
with rank-3 root subsystems of $\Phi$ that contain $\{\beta_1, \beta_2\}$. Recall the notation $\beta^\perp = \{\alpha \in \Phi \mid (\alpha, \beta) = 0\}$ for $\beta \in \Phi$.

**Definition 4.1.** For an $A_1^2$ set $\{\beta_1, \beta_2\} \subseteq \Phi$, define

$$N_0 = N_0(\{\beta_1, \beta_2\}) := \{\Psi \subseteq \Phi \mid \Psi \text{ is irreducible of rank } 3, \{\beta_1, \beta_2\} \subseteq \Psi\},$$

and for each $i \in \{1, 2\}$,

$$M_{\beta_{3-i}}(\beta_i) := \left\{ \Lambda \subseteq \beta_i^\perp \bigg| \Lambda \text{ is irreducible of rank } 2, \beta_i \in \Lambda, \Phi \cap \text{span}(\{\beta_{3-i}\} \cup \Lambda) \text{ is reducible of rank } 3 \right\},$$

and

$$N_{\beta_{3-i}}(\beta_i) := \{\Psi \subseteq \Phi \mid \Psi = \{\pm\beta_{3-i}\} \times \Lambda, \Lambda \in M_{\beta_{3-i}}(\beta_i)\}.$$

**Proposition 4.2.** $N_0$ is not empty.

**Proof.** For $\beta_1, \beta_2 \in \Phi$, there exists $\delta \in \Phi$ such that $(\delta, \beta_1) \neq 0$ and $(\delta, \beta_2) \neq 0$, e.g., see [HRT97, Lemma 2.10]. Thus $\Phi \cap \text{span}\{\beta_1, \beta_2, \delta\} \in N_0$. $\square$

In the remainder of this section, we assume that $X = H_{\beta_1} \cap H_{\beta_2}$ is an $A_1^2$ subspace with $\beta_1, \beta_2 \in \Phi^+$. If $Y \in \mathcal{A}^X$, then $\Phi_Y$ is a subsystem of rank 3 and contains $\Delta_X = \{\beta_1, \beta_2\}$ (Notation 1.1).

**Proposition 4.3.** If $X \in L_2(\mathcal{A})$ is an $A_1^2$ subspace, then

$$N_0 = \{\Phi_Y \mid Y \in \mathcal{A}^X, \Phi_Y \text{ is irreducible}\},$$

and for each $i \in \{1, 2\}$,

$$N_{\beta_{3-i}}(\beta_i) = \{\Phi_Y \mid Y \in \mathcal{A}^X, \Phi_Y = \{\pm\beta_{3-i}\} \times \Xi_Y \text{ and } \Xi_Y \text{ is irreducible of rank } 2\}.$$

**Proof.** We only give a proof for the first equality. The others follow by a similar method. Let $\Psi \in N_0$. There exists $\delta \in \Phi^+$ such that $\Psi = \Phi \cap \text{span}\{\beta_1, \beta_2, \delta\}$. Thus $\Psi = \Phi \cap Y^\perp$ where $Y := X \cap H_3$. Therefore $\Psi = \Phi_Y$ with $Y \in \mathcal{A}^X$. $\square$

**Lemma 4.4.** Let $X \in L(\mathcal{A})$.

(i) $\mathcal{A} = \bigcup_{Y \in \mathcal{A}^X} \mathcal{A}_Y$.

(ii) If $Y, Y' \in \mathcal{A}^X$ and $Y \neq Y'$, then $\mathcal{A}_Y \cap \mathcal{A}_{Y'} = \mathcal{A}_X$.

(iii) $\mathcal{A} \setminus \mathcal{A}_X = \bigcup_{Y \in \mathcal{A}^X} (\mathcal{A}_Y \setminus \mathcal{A}_X)$ (disjoint).

**Proof.** (i) is straightforward. For (ii), $\mathcal{A}_X \subseteq \mathcal{A}_Y \cap \mathcal{A}_{Y'}$ since $Y, Y' \subseteq X$. Arguing on the dimensions, and using the fact that $\dim(X) - \dim(Y) = 1$ yield $X = Y + Y'$. Thus if $H \in \mathcal{A}_Y \cap \mathcal{A}_{Y'}$, then $X \subseteq H$, i.e., $H \in \mathcal{A}_X$. (iii) follows automatically from (i) and (ii). $\square$

**Corollary 4.5.** Set $\mathcal{N} := N_0 \bigcup N_{\beta_2}(\beta_1) \bigcup N_{\beta_1}(\beta_2) \bigcup N_3$, where

$$N_3 = N_3(\{\beta_1, \beta_2\}) := \{\Psi \subseteq \Phi \mid \Psi = \{\pm\beta_1\} \times \{\pm\beta_2\} \times \{\pm\gamma\}, \gamma \in \Phi^+\}.$$

For $\Psi \in \mathcal{N}$, set $\Psi^+ := \Phi^+ \cap \Psi$. Then

$$\Phi^+ = \{\beta_1, \beta_2\} \bigcup \bigcup_{\Psi \in \mathcal{N}} (\Psi^+ \setminus \{\beta_1, \beta_2\}).$$
Proof. It follows from Proposition 2.17(ii), Proposition 4.3 and Lemma 4.4.

\begin{proposition}
For each \( i \in \{1, 2\} \), set
\[
\mathcal{K}_0 := \sum_{\Psi \in \mathcal{N}_0} \sum_{\delta \in \Psi^+ \setminus \{\beta_i\}} (\beta_i, \delta)^2,
\]
\[
\mathcal{K}_{\beta_{3-i}}(\beta_i) := \sum_{\Lambda \in \mathcal{M}_{\beta_{3-i}}(\beta_i)} \sum_{\delta \in \Lambda^+ \setminus \{\beta_i\}} (\beta_i, \delta)^2.
\]
Then \( 2(\mathcal{K}_0 + \mathcal{K}_{\beta_{3-i}}(\beta_i) + 1) = h \). In particular, \( \mathcal{K}_{\beta_2}(\beta_1) = \mathcal{K}_{\beta_1}(\beta_2) \).

\end{proposition}

Proof. By Corollary 4.5,
\[
2(\mathcal{K}_0 + \mathcal{K}_{\beta_{3-i}}(\beta_i) + 1) = 2 \sum_{\delta \in \Phi^+} (\beta_i, \delta)^2,
\]
which is equal to \( h \) by Theorem 2.11(vi).

Proof. It follows from items (iv) and (vi) of Theorem 2.11.

\begin{proposition}
If \( X = H_{\beta_1} \cap H_{\beta_2} \) is of type \( A_2 \), then for each \( i \in \{1, 2\} \)
\[
|\mathcal{A}^{H_{\beta_i}}| - |\mathcal{A}^X| = h/2 + \mathcal{K}_{\beta_{3-i}}(\beta_i).
\]

\end{proposition}

Proof. The proof is similar in spirit to the proof of [OST86, Proposition 3.6]. By Corollary 4.5,
\[
|\mathcal{A}| - 2 = \sum_{\Psi \in \mathcal{N}} (|\Psi^+| - 2).
\]
It is not hard to see that \( |\mathcal{A}^X| = |\mathcal{N}| \) (via the bijection \( Y \mapsto \Phi_Y \)). By Proposition 4.7,
\[
|\mathcal{A}| - |\mathcal{A}^X| = \sum_{\Psi \in \mathcal{N}} (|\Psi^+| - 3) + 2
\]
\[
= 3\mathcal{K}_0 + 2\mathcal{K}_{\beta_2}(\beta_1) + 2\mathcal{K}_{\beta_1}(\beta_2) + 2.
\]
By Theorem 2.14 and Proposition 4.6,
\[
|\mathcal{A}^{H_{\beta_i}}| - |\mathcal{A}^X| = |\mathcal{A}| - |\mathcal{A}^X| - m_\ell
\]
\[
= h/2 + \mathcal{K}_{\beta_{3-i}}(\beta_i).
\]
Remark 4.9.

(i) The conclusion of Proposition 4.8 can also be written as

\[(4.1) \quad |A_{H_\beta}^i| - |A_X| = m_\ell - K_0.\]

(ii) For each \(\Psi \in \mathcal{N}_0\), denote by \(h(\Psi)\) the Coxeter number of \(\Psi\) and write \(m_1(\Psi) \leq m_2(\Psi) \leq m_3(\Psi)\) for the exponents of \(\Psi\). In fact, \(h(\Psi) = 2m_2(\Psi)\) since \(\text{rank}(\Psi) = 3\). Thus

\[(4.2) \quad K_0 = \sum_{\Psi \in \mathcal{N}_0} (m_2(\Psi) - 1).\]

In particular, if \(\ell = 3\), then \(\mathcal{N}_0 = \{\Phi\}\) and \(K_0 = m_2 - 1\). In this case,

\[|A_{H_\beta}^i| - |A_X| = m_2 = h/2.\]

5. Proof of Theorem 1.5

Theorem 1.5 follows from Theorem 3.9 and Theorem 5.1 below.

**Theorem 5.1.** If \(\{\beta_1, \beta_2\} \subseteq \Phi^+\) is \(A_2^1\), then for each \(i \in \{1, 2\}\)

\[|A_{H_\beta}^i| - |A_{H_\beta_1 \cap H_\beta_2}| = \begin{cases} h/2 & \text{if } \{\beta_1, \beta_2\} \text{ is RO}, \\ m_\ell-1 & \text{if } \{\beta_1, \beta_2\} \text{ is not RO}. \end{cases}\]

Proof of Theorem 5.1 will be divided into two parts, which we present the proof for each part in Theorems 5.3 and 5.31.

5.1. First half of Proof of Theorem 5.1.

**Lemma 5.2.** Let \(C = (c_{ij})\) with \(c_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)\) be the Cartan matrix of \(\Phi\). The roots of the characteristic polynomial of \(C\) are \(2 + 2 \cos(m_i \pi/h)\) \((1 \leq i \leq \ell)\).

**Proof.** See, e.g., [BLM89, Theorem 2]. \(\Box\)

We denote by \(\mathcal{D}(\Phi)\) the Dynkin graph and by \(\tilde{\mathcal{D}}(\Phi)\) the extended Dynkin graph of \(\Phi\).

**Theorem 5.3.** Assume that there exists a set \(\{\beta_1, \beta_2\} \subseteq \Phi^+\) such that \(\{\beta_1, \beta_2\}\) is both \(A_2^1\) and RO. Then \(h/2\) is an exponent of \(W\). Moreover, for each \(i \in \{1, 2\}\),

\[|A_{H_\beta}^i| - |A_{H_\beta_1 \cap H_\beta_2}| = h/2.\]

**Proof.** By Lemma 5.2, it suffices to prove that the characteristic polynomial of the Cartan matrix \(C\) admits 2 as a root. By Proposition 3.3, we may assume that \(\{\beta_1, \beta_2\} \subseteq \Delta\). Since the Dynkin graph \(\mathcal{D}(\Phi)\) of \(\Phi\) is a tree, there is a unique path in \(\mathcal{D}(\Phi)\) which admits \(\beta_1\) and \(\beta_2\) as endpoints. By the definition of RO sets, this path contains exactly one more vertex of \(\mathcal{D}(\Phi)\), say \(\beta_3\) with \((\beta_1, \beta_3) \neq 0\) and \((\beta_2, \beta_3) \neq 0\). Also, the other vertices of \(\mathcal{D}(\Phi)\),
if any, connect to the path only at \( \beta_3 \). The Cartan matrix has the following form:

\[
C = \begin{bmatrix}
2 & 0 & c_{13} & 0 & \cdots & 0 \\
0 & 2 & c_{23} & 0 & \cdots & 0 \\
c_{13} & c_{23} & 2 & 0 & \cdots & 0 \\
0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & 0 & \cdots & \cdots & \cdots & 2 \\
0 & 0 & \cdots & \cdots & \cdots & 2
\end{bmatrix}.
\]

By applying the Laplace’s formula, it is easily seen that \( \det(xI_\ell - C) \) is divisible by \( x - 2 \).

Assume that \( \mathcal{M}_{\beta_2}(\beta_1) \neq \emptyset \) and let \( \Lambda \in \mathcal{M}_{\beta_2}(\beta_1) \) (notation in Definition 4.1). Since \( \{\beta_1, \beta_2\} \) is RO, the fact that \( \Lambda \subseteq \beta_2^\perp \) implies that \( (\beta_1, \alpha) = 0 \) for all \( \alpha \in \Lambda \setminus \{\pm \beta_1\} \). This contradicts the irreducibility of \( \Lambda \). Thus \( K_{\beta_2}(\beta_1) = 0 \). Proposition 4.8 completes the proof.

5.2. Second half of Proof of Theorem 5.1. In this section, we need the classification of irreducible root systems of rank at most 4 to make some proofs work. The classifications of rank 3 and 4 root systems will be announced before use, while that of rank 2 root systems (has been and) will be used without announcing explicitly.

The first key ingredient for the proof of Theorem 5.31 is Theorem 5.22. It asserts that Theorem 5.31 is true for a special class of \( A_1^2 \) sets, and we can describe the class by the height function without the RO condition. Before we can get to the proof, some preparatory results are in order.

For \( \alpha \in V, \beta \in V \setminus \{0\} \), denote \( \langle \alpha, \beta \rangle := \frac{2(\alpha, \beta)}{(\beta, \beta)} \). Let \( \theta := \sum_{i=1}^{\ell} c_{\alpha_i} \alpha_i \) be the highest root of \( \Phi \), and we call \( c_{\alpha_i} \in \mathbb{Z}_{>0} \) the coefficient of \( \theta \) at the simple root \( \alpha_i \). Denote by \( c_{\max} := \max \{c_{\alpha_i} | 1 \leq i \leq \ell \} \) the largest coefficient.

**Proposition 5.4.** Let \( \Phi \) be an irreducible root system in \( \mathbb{R}^\ell \). Let \( \theta \) be the highest root of \( \Phi \), and denote \( \lambda_0 := -\theta, c_{\lambda_0} := 1 \). Suppose that the elements of a fixed base \( \Delta = \{\lambda_1, \ldots, \lambda_\ell\} \) are labeled so that \( \Lambda := \{\lambda_0, \lambda_1, \ldots, \lambda_q\} \) is a set of minimal cardinality such that \( c_{\max} = c_{\lambda_q} \) and \( (\lambda_s, \lambda_{s+1}) < 0 \) for \( 0 \leq s \leq q - 1 \).

(i) Then \( c_{\lambda_s} = s + 1 \) for \( 0 \leq s \leq q \) and \( |\Lambda| = c_{\max} \).

(ii) Assume that \( c_{\max} \geq 2 \). Then \( (\lambda_0, \lambda_1, \ldots, \lambda_{q-1}) \) is a simple chain of \( \overline{D}(\Phi) \) connected to the other vertices only at \( \lambda_{q-1} \).

**Proof.** See, e.g., [MT11, Lemma B.27, Appendix B] and [Tra17, Proposition 3.1].

**Remark 5.5.** Proposition 5.4 is closely related to lemma of the string which was originally formulated and proved in terms of coroots in [Ste75, Lemma
1.5]. The proof of [MT11, Lemma B.27, Appendix B] contains a small error, that was resolved in [Tra17, Proposition 3.1].

**Corollary 5.6.**

(i) If $c_{\text{max}} = 1$, then all roots of $\Phi$ have the same length. In addition, if $\ell \geq 2$, then $D(\Phi)$ is a simple chain and $-\theta$ is connected only to two terminal vertices of $D(\Phi)$.

(ii) If $c_{\text{max}} \geq 2$, then $-\theta$ is connected only to one vertex $\lambda$ of $D(\Phi)$ with $\{\theta, \lambda\} \in \{1, 2\}$ and $c_\lambda = 2$. In particular, if $\lambda_i \in \Delta$ with $c_\lambda_i = 1$ is connected to $\lambda_j \in \Delta$ with $c_\lambda_j \geq 2$, then $\lambda_i$ must be a terminal vertex of $D(\Phi)$ and $\langle \lambda_j, \lambda_i \rangle = -1$.

**Proof.** The second statement of (ii) follows from the equation $\xi = \max$. The other statements can be found in [Tra17, Proposition 3.1 and Remark 3.2]. □

**Corollary 5.7.** Assume that $c_{\text{max}} \geq 2$. Either $\langle \lambda_{q-1}, \lambda_q \rangle \in \{-2, -3\}$ or $\lambda_q$ is a ramification point of $D(\Phi)$.

**Proof.** See [Tra17, Corollary 3.3]. □

**Definition 5.8.** Define $U := \{\theta_i \in \Phi^+ \mid \text{ht}(\theta_i) > m_{\ell-1}\}$, and set $m := |U|$. By Theorems 2.12 and 2.11(i), (iii), we can write $m = m_\ell - m_{\ell-1} = m_2 - 1 > 0$. Suppose that the elements of $U$ are labeled so that $\theta_1$ denotes the highest root, and $\xi_i = \theta_i - \theta_{i+1} \in \Delta$ for $1 \leq i \leq m - 1$. We also adopt a convention $\xi_0 := -\theta_1$. Set $\Xi := \{\xi_i \mid 0 \leq i \leq m - 1\}$. Note that $\Xi$ is a multiset, not necessarily a set. For a finite multiset $S = \{(a_1)^{b_1}, \ldots, (a_n)^{b_n}\}$, we write $\mathcal{B}$ for the base set of $S$, i.e., $\mathcal{B} = \{a_1, \ldots, a_n\}$. Let us call the case “there is an integer $t$ such that $1 \leq t \leq m - 1$ and $\langle \theta_t, \xi_t \rangle = 3$” Case 1, and its negation Case 2.

**Proposition 5.9.**

(i) If Case 1 occurs, then $t = m - 2$ and $\Xi = \{\xi_0, \xi_1, \ldots, (\xi_{m-2})^2\}$ with $\xi_i \neq \xi_j$ for $0 \leq i < j \leq m - 2$. As a result, $|\Xi| = m_2 - 2$.

(ii) If Case 2 occurs, then $\Xi = \{\xi_0, \xi_1, \ldots, \xi_{m-1}\}$ with $\xi_i \neq \xi_j$ for $0 \leq i < j \leq m - 1$. As a result, $|\Xi| = m_2 - 1$.

**Proof.** See [Tra17, Propositions 3.9 and 3.10]. □

**Theorem 5.10.** With the notations we have seen from Proposition 5.4 to Proposition 5.9, we have that $q = m - 1$ and $\lambda_i = \xi_i$ for all $1 \leq i \leq q$. In particular, $\Xi = \Lambda$ (as sets). Moreover, if Case 1 occurs, then $m_2 = c_{\text{max}} + 2$; and if Case 2 occurs, then $m_2 = c_{\text{max}} + 1$.

**Proof.** See [Tra17, Theorem 4.1]. □
Corollary 5.11.
(i) If Case 1 occurs, then $\theta_i - \theta_j \in \Phi^+$ for $1 \leq i < j \leq m$, $\{i, j\} \neq \{m - 2, m\}$, and $\theta_{m-2} - \theta_m \in 2\Delta$.
(ii) If Case 2 occurs, then $\theta_i - \theta_j \in \Phi^+$ for $1 \leq i < j \leq m$.

Proof. See [Tra17, Corollary 3.11]. □

Corollary 5.12. The following statements are equivalent: (i) Case 1 occurs, (ii) $\Phi = G_2$, (iii) $c_{\text{max}} = m_2 - 2$.

Proof. See [Tra17, Theorem 4.2]. □

Recall the notation $\Theta^{(r)} = \{\alpha \in \Phi^+ \mid \text{ht}(\alpha) = r\}$.

Proposition 5.13. There is always a long root in $\Theta^{(m_{\ell - 1})}$.

Proof. We may assume that $c_{\text{max}} \geq 2$. If not, all roots of $\Phi$ have the same length. The assertion is trivial. If Case 1 occurs, then by Corollary 5.12, $\Phi = G_2$. The assertion is also trivial.

Now we can assume that Case 2 occurs. By Proposition 5.9, $\xi_i \neq \xi_j$ for $0 \leq i < j \leq m - 1$. If $\langle \xi_{m-2}, \xi_{m-1} \rangle = -2$, then $\langle \theta_{m-1}, \xi_{m-1} \rangle = 2$. Thus $\theta_{m-1} - 2\xi_{m-1} \in \Theta^{(m_{\ell - 1})}$, that is a long root. We are left with the case $\langle \xi_{m-2}, \xi_{m-1} \rangle = -1$. By Corollary 5.7, $\xi_{m-1}$ is connected to at least two vertices of $D(\Phi)$ apart from $\xi_{m-2}$, say $\mu_1, \ldots, \mu_k$ ($k \geq 2$). We claim that there exists $\mu_i$ such that $\langle \xi_{m-1}, \mu_i \rangle = -1$. Proof of the claim when $m = 2$ (i.e., $c_{\text{max}} = 2$) and $m \geq 3$ uses very similar technique (the case $m = 2$ is actually Lemma 5.23(2)). We only give a proof when $m \geq 3$. From $\langle \theta, \xi_{m-1} \rangle = 0$ (it equals 1 if $m = 2$), $c_{\xi_{m-1}} = c_{\text{max}}$, $c_{\xi_{m-2}} = c_{\text{max}} - 1$, and $\xi_{m-1}$ is a long root, we have $c_{\text{max}} + 1 - \sum_{i=1}^k c_{\mu_i} = 0$. Suppose to the contrary that $\langle \xi_{m-1}, \mu_i \rangle \leq -2$ for all $1 \leq i \leq k$. From $0 = \langle \theta, \mu_i \rangle \leq 2c_{\mu_i} + c_{\text{max}} \langle \xi_{m-1}, \mu_i \rangle$, we obtain $c_{\mu_i} = c_{\text{max}}$. Thus, $c_{\text{max}} + 1 - k c_{\text{max}} = 0$, a contradiction. So we can choose $\mu_i$ so that $\langle \xi_{m-1}, \mu_i \rangle = -1$. Therefore, $\langle \theta_m, \mu_i \rangle = -\langle \xi_{m-1}, \mu_i \rangle = 1$ and $\theta_m - \mu_i \in \Theta^{(m_{\ell - 1})}$, that is a long root. □

Lemma 5.14. Suppose $\beta_1, \beta_2, \beta_3 \in \Phi$ with $\beta_1 + \beta_2 + \beta_3 \in \Phi$ and $\beta_i + \beta_j \neq 0$ for $i \neq j$. Then at least two of the three partial sums $\beta_i + \beta_j$ belong to $\Phi$.

Proof. See, e.g., [LN04, §11, Lemma 11.10]. □

Proposition 5.15. If $\gamma \in \Theta^{(m_{\ell - 1})}$, then $\theta_i - \gamma \in k\Phi^+$ with $k \in \{1, 2, 3\}$ for every $1 \leq i \leq m$.

Proof. To avoid the triviality, we assume that Case 2 occurs and $1 \leq i \leq m - 1$. Denote $\mu := \theta_m - \gamma \in \Delta$.

Assume that $\mu = \xi_{m-1}$. Then $\theta_{m-1} - \gamma = 2\xi_{m-1} \in 2\Delta$. We have $\langle \theta_{m-1}, \xi_{m-1} \rangle = \langle \gamma, \xi_{m-1} \rangle + 4$. It follows that $\langle \theta_{m-1}, \xi_{m-1} \rangle = 2$. Fix
i with \(1 \leq i \leq m - 2\), and set \(\alpha := \theta_i - \theta_{m-1} \in \Phi^+\) (by Corollary 5.11). Since \(\theta_i = \theta_1 - (\xi_1 + \cdots + \xi_{i-1})\), we have \(\langle \theta_i, \xi_{m-1} \rangle = 0\). Thus \(\langle \alpha, \xi_{m-1} \rangle = -\langle \theta_m - 1, \xi_{m-1} \rangle = -2\). Then \(\theta_i - \gamma = \alpha + 2\xi_{m-1} \in \Phi^+\).

Assume that \(\mu \neq \xi_{m-1}\). Then \(\theta_m - 1 = \gamma + \mu + \xi_{m-1}\). By Lemma 5.14, \(\mu + \xi_{m-1} \in \Phi^+\) since \(\gamma + \xi_{m-1} \notin \Phi^+\). Thus \(\mu\) and \(\xi_{m-1}\) are adjacent on \(D(\Phi)\). If \(\mu \neq \xi_{m-2}\) (of course, \(\mu \neq \xi_i\) for all \(1 \leq i \leq m - 3\) since \(D(\Phi)\) is a tree), then by Lemma 3.2(ii), \(\theta_i - \gamma = \xi_i + \cdots + \xi_{m-1} + \mu \in \Phi^+\) for each \(1 \leq i \leq m - 1\). If \(\mu = \xi_{m-2}\), then \(\theta_{m-2} = \gamma + \xi_{m-1} + 2\xi_{m-2}\). Thus \(\langle \theta_{m-2}, \xi_{m-2} \rangle = \langle \gamma, \xi_{m-2} \rangle + \langle \xi_{m-1}, \xi_{m-2} \rangle + 4\). Using the fact that \(\xi_{m-2}\) is a long root, we obtain a contradiction since the left-hand side is at most 1 while the right-hand side is at least 2.

**Corollary 5.16.** If \(\ell \geq 5\), then \(m_{\ell-2} < m_{\ell-1}\).

**Proof.** By Theorem 2.12, it suffices to prove that there are exactly two roots of height \(m_{\ell-1}\), i.e., \(|\Theta^{(m_{\ell-1})}| = 2\). To avoid the triviality, we assume that Case 2 occurs and \(c_{\text{max}} \geq 2\). Proofs for the cases \(\langle \xi_{m-2}, \xi_{m-1} \rangle = -2\) and \(\langle \xi_{m-2}, \xi_{m-1} \rangle = -1\) are very similar, and we only give a proof for the latter (slightly harder) case. Suppose to the contrary that \(\Theta^{(m_{\ell-1})} = \{\gamma_1, \ldots, \gamma_k\}\) with \(k \geq 3\). By Proposition 5.15, \(\theta_{m-1} - \gamma_i = \xi_{m-1} + \mu_i \in \Phi^+\), where \(\mu_i \in \Delta\). Thus \(\mu_i\) is adjacent to \(\xi_{m-1}\) on \(D(\Phi)\). By the same argument as the one used in the end of Proof of Proposition 5.15, \(\mu_i \neq \xi_{m-2}\) for all \(1 \leq i \leq k\). If \(m \geq 3\), then the same argument as in Proof of Proposition 5.13 gives \(c_{\text{max}} + 1 = \sum_{i=1}^{k} c_{\mu_i}\). From \(0 = \langle \theta, \mu_i \rangle \leq 2(c_{\mu_i} - c_{\text{max}})\), we obtain \(c_{\mu_i} \geq c_{\text{max}}/2\) for all \(1 \leq i \leq k\). This forces \(k = 3\). But it implies that \(c_{\text{max}} \leq 2\), i.e., \(m \leq 2\), a contradiction. Now consider \(m = 2\). A similar argument as above shows that \(k = 3\) and \(c_{\mu_1} = c_{\mu_2} = c_{\mu_3} = 1\). The second statement of Corollary 5.6(ii) implies that \(c_{\mu_i}\) must be all terminal. Thus \(\ell = 4\), a contradiction.

**Lemma 5.17.** If \(m_2 = h/2\), then \(\ell \leq 4\). More specifically, when \(\ell = 4\), \(m_2 = h/2\) if and only if \(\Phi = D_4\).

**Proof.** If \(m_2 = c_{\text{max}} + 2\), then by Corollary 5.12, \(\Phi = G_2\). However, \(m_2 > h/2\) by a direct check. Now assume that \(m_2 = c_{\text{max}} + 1\). Recall from Proposition 5.4 that the coefficients of \(\theta\) at elements of \(\Lambda = \{\lambda_0, \lambda_1, \ldots, \lambda_q\}\) form an arithmetic progression, starting with \(c_{\lambda_0} = 1\) and ending with \(c_{\lambda_q} = c_{\text{max}} = q + 1\). From \(\sum_{\lambda \in \Lambda} c_{\lambda} + \sum_{\lambda \in \Delta \setminus \Lambda} c_{\lambda} = h\), we have

\[
c_{\text{max}}(c_{\text{max}} + 1)/2 + \ell - (c_{\text{max}} - 1) \leq 2c_{\text{max}} + 2.
\]

Thus \(\ell \leq (-c_{\text{max}}^2 + 5c_{\text{max}} + 2)/2\). Therefore, \(\ell \leq 4\). The second statement is clear from the classification of irreducible root systems of rank 4. \(\square\)
Remark 5.18. The first statement of Lemma 5.17 is an easy consequence of the well-known fact that every exponent of $\Phi$ appears at most twice. A uniform proof of this fact is probably well-known among experts.

Unless otherwise stated, we assume that $\ell \geq 3$ in the remainder of this subsection.

Lemma 5.19. If $\{\gamma_1, \gamma_2\} \subseteq \Theta^{(r)}$ with $\gamma_1 \neq \gamma_2$ and $r \geq \lfloor m_\ell/2 \rfloor + 1$ (floor function), then $\{\gamma_1, \gamma_2\}$ is $A_{1}^{2}$. This assertion is true, in particular, if $r = m_\ell - 1$.

Proof. It is immediate because $\gamma_1 + \gamma_2 \notin \Phi$ and $\gamma_i - \gamma_j \notin \sum_{\alpha \in \Delta} \mathbb{Z}_{\geq 0} \alpha$ for $i \neq j$. □

Definition 5.20. Let $\{\gamma_1, \gamma_2\}$ be an $A_{1}^{2}$ set. Let $\Psi \in \mathcal{N}_0 = \mathcal{N}_0(\{\gamma_1, \gamma_2\})$ (Definition 4.1 and Corollary 4.5). For any $\alpha = \sum_{\mu \in \Delta(\Psi)} c_\mu \mu \in \Psi^+$ ($\Delta(\Psi)$ is the base of $\Psi$ associated with $\Psi^+$), its local height in $\Psi$ (following [ABC+16, Section 4]), denoted by $ht_\Psi(\alpha)$, is defined by $ht_\Psi(\alpha) := \sum_{\mu \in \Delta(\Psi)} c_\mu$.

Lemma 5.21. With the notations and assumptions in Lemma 5.19, Definition 5.20 and Remark 4.9, we have $ht_\Psi(\gamma_i) = m_2(\Psi)$ for $i \in \{1, 2\}$.

Proof. This follows from a direct verification on all rank-3 irreducible root systems ($A_3, B_3, C_3$). We will check only a (most) non-obvious case when $\Psi = C_3$, and $\gamma_1 = \mu_1 + \mu_2, \gamma_2 = 2\mu_2 + \mu_3$ (see Table 1). Set $\alpha := 2\mu_1 + 2\mu_2 + \mu_3$. Since $\alpha = 2\gamma_1 + \mu_3 \in \Psi^+ \subseteq \Phi^+$, we have $ht(\alpha) > 2ht(\gamma_1) > m_\ell$, which is a contradiction. □

Height

| $\nu$ | $2\mu_1 + 2\mu_2 + \mu_3$ |
|------|--------------------------|
| 5    | $\mu_1 + 2\mu_2 + \mu_3$ |
| 4    | $\mu_1 + \mu_2 + \mu_3$  |
| 3    | $2\mu_2 + \mu_3$          |
| 2    | $\mu_1 + \mu_2 + \mu_3$  |
| 1    | $\mu_1 + \mu_2 + \mu_3$  |

Table 1. $\Psi^+$ when $\Psi = C_3$.

Theorem 5.22. If $\gamma_1, \gamma_2 \in \Theta^{(m_\ell - 1)}$ with $\gamma_1 \neq \gamma_2$, then for each $i \in \{1, 2\}$

$$|A^{H_{\gamma_i}}| - |A^{H_{\gamma_1} \cap H_{\gamma_2}}| = m_{\ell - 1}.\]

Proof. By Remark 4.9, Theorem 5.22 is proved once we prove

$$\sum_{\Psi \in \mathcal{N}_0} (m_2(\Psi) - 1) = m_2 - 1.\]
Recall the notation \( \mathcal{U} = \{ \theta_j \in \Phi^+ \mid \text{ht}(\theta_j) > m_{\ell-1} \} \), and \( |\mathcal{U}| = m_2 - 1 \). For each \( \Psi \in \mathcal{N}_0 \), set \( \mathcal{U}_\Psi := \{ \mu \in \Psi^+ \mid \text{ht}_\Psi(\mu) > \text{ht}_\Psi(\gamma_i) \} \) (by Lemma 5.21, this definition does not depend on the index \( i \)). Moreover, \( |\mathcal{U}_\Psi| = m_2(\Psi) - 1 \). Since \( \mathcal{U}_\Psi \cap \mathcal{U}_{\Psi'} = \emptyset \) for \( \Psi \neq \Psi' \), we have

\[
\left| \bigcup_{\Psi \in \mathcal{N}_0} \mathcal{U}_\Psi \right| = \sum_{\Psi \in \mathcal{N}_0} |\mathcal{U}_\Psi| = \sum_{\Psi \in \mathcal{N}_0} (m_2(\Psi) - 1).
\]

Equality (5.1) will be proved once we prove \( \mathcal{U} = \bigcup_{\Psi \in \mathcal{N}_0} \mathcal{U}_\Psi \). For any \( \mu \in \mathcal{U}_\Psi \),

\[
\mu - \gamma_i = \sum_{\mu \in \Delta(\Psi)} \mathbb{Z}_{\geq 0} \mu \subseteq \sum_{\alpha \in \Delta} \mathbb{Z}_{\geq 0} \alpha \quad (i \in \{1, 2\}).
\]

Thus \( \text{ht}(\mu) > \text{ht}(\gamma_i) = m_{\ell-1} \) hence \( \mu \in \mathcal{U} \). Therefore \( \mathcal{U} \supseteq \bigcup_{\Psi \in \mathcal{N}_0} \mathcal{U}_\Psi \). To prove the inclusion it suffices to prove that for every \( \theta_j \in \mathcal{U} \), the subsystem \( \Gamma := \Phi \cap \text{span}\{\theta_j, \gamma_1, \gamma_2\} \in \mathcal{N}_0 \). Obviously, \( \Gamma \) is of rank 3 since it contains the \( A_1^2 \) set \( \{\gamma_1, \gamma_2\} \). The irreducibility of \( \Gamma \) follows from Proposition 5.15.

The second key ingredient for the proof of Theorem 5.31 is Corollary 5.28. One of the main reasons that the exponents are given by two different formulas is that the \( A_1^2 \) sets may form more than one orbits under the action of \( W \) on \( \Phi \) (Remark 3.5). In fact, the \( A_1^2 \) sets described in Theorem 5.22 are not enough to form all possible orbits. Thus if we focus on non-RO \( A_1^2 \) sets and there are two of them (which may form different orbits), we wish to find a relation between these sets. Corollary 5.28 indicates such a relation.

The problem of classifying orbits is known to be related to properties of \( \theta^\perp \), the subsystem orthogonal to the highest root \( \theta \). The idea is to first investigate the “local” picture of the extended Dynkin graph at the highest root and the simple root adjacent to it, then find connection with RO properties.

**Lemma 5.23.** Assume that \( \ell \geq 2 \) and \( c_{\text{max}} \geq 2 \). Let \( \lambda \) be the unique simple root connected to \(-\theta \). Denote by \( \gamma_1, \ldots, \gamma_k \) (\( k \geq 1 \)) the simple roots connected to \( \lambda \). Then there are the following possibilities:

1. If \( \langle \theta, \lambda \rangle = 2 \), then \( k = 1, c_{\gamma_1} \in \{1, 2\} \).
2. If \( \langle \theta, \lambda \rangle = 1 \) (in particular, \( \lambda \) is long), then either (2a) \( k = 3, c_{\gamma_1} = c_{\gamma_2} = c_{\gamma_3} = 1 \) (i.e., \( \Phi = D_4 \)), or (2b) \( k = 2, c_{\gamma_1} = 2, c_{\gamma_2} = 1 \) (\( \gamma_2 \) is terminal and long), or (2c) \( k = 1, c_{\gamma_1} = 3 \).

**Proof.** We have \( \langle \theta, \lambda \rangle = 2c_{\lambda} + \sum_{i=1}^k c_{\gamma_i} \langle \gamma_i, \lambda \rangle \leq 4 - \sum_{i=1}^k c_{\gamma_i} \). Since \( \langle \theta, \lambda \rangle \geq 1 \), we have \( \sum_{i=1}^k c_{\gamma_i} \in \{1, 2, 3\} \). Then we can list all possibilities and rule out impossibilities. For example, if \( \langle \theta, \lambda \rangle = 1 \) and \( \sum_{i=1}^k c_{\gamma_i} = 2 \), then either (i) \( k = 1, c_{\gamma_1} = 2 \), or (ii) \( k = 2, c_{\gamma_1} = c_{\gamma_2} = 1 \). For (i), it follows
that \(1 = \langle \theta, \lambda \rangle = 4 + 2 \langle \gamma_1, \lambda \rangle\), which is a contradiction. For (ii), the second statement of Corollary 5.6(ii) implies that \(\gamma_1, \gamma_2\) must be all terminal. Thus \(\ell = 3\). However, such root system does not exist by the classification of irreducible root systems of rank 3. Similarly, to conclude that \(\Phi = D_4\) in (2a), we need the classification of irreducible root systems of rank 4. \(\square\)

It is known that \(\theta^\perp\) is the standard parabolic subsystem of \(\Phi\) generated by \(\{\alpha \in \Delta \mid (\alpha, \theta) = 0\}\). Also, \(\theta^\perp\) may be reducible and decomposed into irreducible, mutually orthogonal components.

**Corollary 5.24.** If \(\theta^\perp\) is reducible, then either Possibility (2a) or (2b) occurs.

*Proof.* It follows immediately from Lemma 5.23 by noting that \(k \geq 2\). \(\square\)

**Corollary 5.25.** When \(\ell = 4\), there exists a set that is both \(A_1^2\) and RO if and only if \(\Phi = D_4\). Moreover, if \(\Phi = D_4\), then every \(A_1^2\) set is RO.

*Proof.* Use the classification of irreducible root systems of rank 4. \(\square\)

**Proposition 5.26.** Assume that \(\ell \geq 4\). If \(\theta^\perp\) is reducible and there exists an \(A_1^2\) set that is not RO, then Possibility (2b) in Lemma 5.23 occurs. In particular, \(\theta^\perp = \{\pm \gamma_2\} \times \Omega\) for a long simple root \(\gamma_2\) and \(\Omega\) is irreducible with \(\operatorname{rank}(\Omega) \geq 2\).

*Proof.* If \(c_{\text{max}} = 1\), then Corollary 5.6(i) implies that \(\theta^\perp\) is irreducible and of rank at least 2, a contradiction. Now consider \(c_{\text{max}} \geq 2\). By Corollary 5.24, either Possibility (2a) or (2b) occurs. However, Corollary 5.25 ensures that Possibility (2a) can not occur because a non-RO \(A_1^2\) set exists. Thus Possibility (2b) must occur. Then \(\theta^\perp = \{\pm \gamma_2\} \times \Omega\) where \(\Omega\) is irreducible with \(\operatorname{rank}(\Omega) \geq 2\). \(\square\)

**Proposition 5.27.** Assume that \(\ell \geq 4\). Suppose that \(\{\theta, \alpha\}\) is an \(A_1^2\) set with \(\alpha \in \Delta\). If \(\{\pm \alpha\}\) is a component of \(\theta^\perp\), then \(\{\theta, \alpha\}\) is RO.

*Proof.* We may assume that \(c_{\text{max}} \geq 2\). If not, Corollary 5.6(i) implies that \(\theta^\perp\) is irreducible and of rank at least 2, a contradiction. Now \(\neg \theta\) connects only to one vertex of \(\mathcal{D}(\Phi)\), \(\theta^\perp\) must be reducible; otherwise, \(\theta^\perp = \{\pm \alpha\}\), a contradiction. Thus \(\{\beta, \alpha\}\) is \(A_1^2\) for every \(\beta \in \theta^\perp \setminus \{\pm \alpha\}\). With the notations in Definition 4.1 and Proposition 4.6, we have \(\mathcal{M}_\theta(\alpha) = \emptyset\) and \(\mathcal{K}_\theta(\alpha) = 0\). Suppose to the contrary that \(\{\theta, \alpha\}\) is not RO. Note that \(\theta^\perp \setminus \{\pm \alpha\} \subseteq \alpha^\perp \setminus \{\pm \theta\}\) since \(\{\pm \alpha\}\) is a component of \(\theta^\perp\). Then there exists \(\beta \in \alpha^\perp \setminus \{\pm \theta\}\) such that \(\langle \beta, \theta \rangle \neq 0\). Note that \(\Gamma := \Phi \cap \langle \beta, \theta, \alpha \rangle\) is a subsystem of rank 3 since it contains the \(A_1^2\) set \(\{\theta, \alpha\}\). \(\Gamma\) must be irreducible, otherwise, \(\Gamma \in \mathcal{M}_\theta(\theta)\) and \(\mathcal{K}_\theta(\theta) \neq 0\) which contradicts Proposition 4.6. Using the facts that \(\{\theta, \alpha\}\) is \(A_1^2\) in \(\Gamma\), \(\alpha^\perp \cap \Gamma\) is an irreducible subsystem of rank 2 of \(\Gamma\) (as it contains \(\beta\) and \(\theta\)), and the classification
of irreducible rank 3 root systems, we conclude that $\Gamma = B_3$ and $\alpha$ is the unique short simple root of $\Gamma$. Since $\theta^\perp$ is reducible, either Possibility (2a) or (2b) occurs by Corollary 5.24. Thus Possibility (2b) must occur since $\Phi$ contains the subsystem $\Gamma$ of type $B_3$. Since $\{\pm \alpha\}$ is a component of $\theta^\perp$ and $\alpha$ is short, with the notation in Possibility (2b), $\Delta = \{\gamma_2, \lambda, \alpha\}$. Thus $\ell = 3$, a contradiction. 

Corollary 5.28. Assume that $\ell \geq 4$. If there are two $A_2^2$ sets $\{\theta, \lambda_1\}$, $\{\theta, \lambda_2\}$ that are both non-RO, then $\lambda_1$ and $\lambda_2$ lie in the same irreducible component $\Omega$ of $\theta^\perp$ with $\text{rank}(\Omega) \geq 2$.

Proof. The statement is trivial if $\theta^\perp$ is irreducible. If $\theta^\perp$ is reducible, then the statement follows from Propositions 5.26 and 5.27. \qed

The final key ingredient for the proof of Theorem 5.31 is Proposition 5.29.

Proposition 5.29. If there exists an $A_2^2$ set which is not RO, then the set $S := \{\{\gamma_1, \gamma_2\} \subseteq \Theta^{(m_\ell-1)} \mid \text{at least one of } \gamma_1, \gamma_2 \text{ is a long root}\}$ contains a non-RO set.

Proof. Note that $S \neq \emptyset$ since $\Theta^{(m_\ell-1)}$ always contains a long root (Proposition 5.13). The case $\ell = 3$ is checked directly by the classification. Assume that $\ell \geq 4$ and suppose to the contrary that every element in $S$ is RO. We can take $\{\gamma_1, \gamma_2\} \in S$ and assume that $\gamma_1$ is long. By Theorems 5.22 and 5.3, $m_\ell-1 = h/2$. By Lemma 5.17, $\ell = 4$. By Corollary 5.25, $\Phi = D_4$, and all $A_2^2$ sets must be RO. This contradicts the Proposition’s assumption. \qed

Lemma 5.30. If $\{\beta_1, \beta_2\} \subseteq \Phi^+$ contains a long root and $\langle \beta_1, \beta_2 \rangle = 0$, then $\{\beta_1, \beta_2\}$ lies in the same $W$-orbit with $\{\theta, \mu\}$ for some $\mu \in \Delta$.

Proof. This is well-known. There is an irreducible component $\Psi \subseteq \theta^\perp$ such that $\{\beta_1, \beta_2\}$ lies in the same $W$-orbit with $\{\theta, \gamma\}$ for some $\gamma \in \Psi$. There exist $\mu \in \Delta \cap \Psi$ and $w \in W(\Psi)$ such that $\gamma = w(\mu)$ and this $w$ fixes $\theta$, i.e., $\theta = w(\theta)$. Thus $\{\beta_1, \beta_2\}$ lies in the same $W$-orbit with $\{\theta, \mu\}$. \qed

Now we are ready to prove the main result of this subsection.

Theorem 5.31. If $\{\beta_1, \beta_2\} \subseteq \Phi^+$ is $A_2^2$ but not RO, then for each $i \in \{1, 2\}$

$$|A^{H_{\beta_i}}| - |A^X| = m_\ell-1.$$ 

Proof. It suffices to prove Theorem 5.31 under the condition that the $A_2^2$ set $\{\beta_1, \beta_2\}$ contains a long root. Otherwise, we would consider the dual root system $\Phi^\vee$ where short roots become long roots. By Lemma 5.30, $\{\beta_1, \beta_2\}$ lies in the same $W$-orbit with $\{\theta, \lambda_1\}$ for some $\lambda_1 \in \Delta$. We may also assume that $\ell \geq 4$ since the case $\ell = 3$ is done in Remark 4.9. Note also
that Theorem 5.31 is proved once we prove that $K_0(\lambda_1) = m_{\ell - 1} - h/2$ (notation in Proposition 4.6).

By Proposition 5.29, we can find a non-RO set $\{\gamma_1, \gamma_2\} \subseteq \Theta^{(m_{\ell - 1})}$ where $\gamma_1$ is a long root. Again by Lemma 5.30, $\{\gamma_1, \gamma_2\}$ lies in the same $W$-orbit with $\{\theta, \lambda_2\}$ for some $\lambda_2 \in \Delta$. Corollary 5.28 implies that $\lambda_1$ and $\lambda_2$ lie in the same irreducible component $\Omega$ of $\theta^\perp$ with $\text{rank}(\Omega) \geq 2$.

We already know from Theorem 5.22 that Theorem 5.31 is proved for $\{\theta, \lambda_2\}$, i.e., $K_0(\lambda_2) = m_{\ell - 1} - h/2$. So we want to prove that $K_0(\lambda_1) = K_0(\lambda_2)$. This formula is proved if $\|\lambda_1\| = \|\lambda_2\|$. Now consider $\|\lambda_1\| \neq \|\lambda_2\|$. Note that at most two root lengths occur in $\Omega$, they are $\|\lambda_1\|$ and $\|\lambda_2\|$. Then the fact that $\{\theta, \lambda_1\}$ and $\{\theta, \lambda_2\}$ are both $A_1^2$ implies that $\{\theta, \beta\}$ is $A_1^2$ for all $\beta \in \Omega$. For every $\beta \in \Omega$, with the notations in Definition 4.1 applying to the $A_1^2$ set $\{\theta, \beta\}$, we have

$$M_\theta(\beta) = \left\{ \Lambda \subseteq \Omega \mid \text{$\Lambda$ is irreducible of rank 2, $\beta \in \Lambda$,} \right\} \cup \left\{ \Phi \cap \text{span}(\{\theta\} \cup \Lambda) \text{ is reducible of rank 3.} \right\}$$

Indeed, if $\Lambda \subseteq \theta^\perp$, then $\Lambda \subseteq \Omega$; otherwise, $\Lambda$ is an $A_1^2$ set, a contradiction. We further make the following definition

$$M_\theta'(\beta) := \left\{ \Lambda \subseteq \Omega \mid \text{$\Lambda$ is irreducible of rank 2, $\beta \in \Lambda$,} \right\} \cup \left\{ \Phi \cap \text{span}(\{\theta\} \cup \Lambda) \text{ is irreducible of rank 3.} \right\}$$

Using Proposition 4.6 and Theorem 2.11(vi), we compute

$$2K_\theta(\beta) = 2 \sum_{\Lambda \in M_\theta(\beta)} \sum_{\delta \in \Lambda^+ \setminus \{\beta\}} \left(\tilde{\beta}, \tilde{\delta}\right)^2$$

$$= h(\Omega) - 2 - 2 \sum_{\Lambda \in M_\theta'(\beta)} \sum_{\delta \in \Lambda^+ \setminus \{\beta\}} \left(\tilde{\beta}, \tilde{\delta}\right)^2.$$

We claim that $M_\theta'(\beta) = \emptyset$ for every $\beta \in \Omega$. Suppose not and let $\Lambda \in M_\theta'(\beta)$. Note that $\Gamma := \Phi \cap \text{span}(\{\pm \theta\} \cup \Lambda)$ admits $\theta$ as the highest root in its positive system $\Phi^+ \cap \text{span}(\{\pm \theta\} \cup \Lambda)$. By a direct check on all possible irreducible rank 3 root systems and using the fact that $\{\theta, \beta\}$ is $A_1^2$ for all $\beta \in \Lambda \subseteq \Gamma \cap \Omega$, we obtain a contradiction. It follows, in particular, that $M_\theta'(\lambda_1) = M_\theta'(\lambda_2) = \emptyset$. By the computation above, $K_\theta(\lambda_1) = K_\theta(\lambda_2)$, which equals $h(\Omega)/2 - 1$. This completes the proof. \(\square\)

5.3. Some corollaries.

**Corollary 5.32** (Local-global inequalities). Assume that $\ell \geq 3$. Assume that a set $\{\beta_1, \beta_2\} \subseteq \Phi$ is $A_1^2$. Recall the notation $N_0 = N_0(\{\beta_1, \beta_2\})$ in Definition 4.1. For each $\Psi \in N_0$, denote by $c_{\text{max}}(\Psi)$ the largest coefficient of the highest root of the subsystem $\Psi$. Then

(a) $\sum_{\Psi \in N_0} (m_2(\Psi) - 1) \geq m_2 - 1,$
(b) $\sum_{\Psi \in \mathcal{N}_0} c_{\text{max}}(\Psi) \geq c_{\text{max}}$.

The equality in (a) (resp., (b)) occurs if and only if either (i) $\ell \leq 4$, or (ii) $\ell \geq 5$ and $\{\beta_1, \beta_2\}$ is not RO.

Proof. Since $\ell \geq 3$, $m_2 = c_{\text{max}} + 1$ by Theorem 5.10 and Corollary 5.12. Thus (a) and (b) are essentially equivalent. The left-hand sides of these inequalities are equal to $K_0$ by Remark 4.9. By Theorem 5.1 and Remark 4.9,

$$K_0 = \begin{cases} h/2 - 1 & \text{if } \{\beta_1, \beta_2\} \text{ is RO}, \\ m_2 - 1 & \text{if } \{\beta_1, \beta_2\} \text{ is not RO}. \end{cases}$$

The inequalities are proved. If $\ell = 3$, the equalities always occur since $h/2 = m_2$. If $\ell = 4$, we need only care about the case $\{\beta_1, \beta_2\}$ is both $A_1^2$ and RO. This condition forces $\Phi = D_4$ by Corollary 5.25. Again, we have $h/2 = m_2$. So the equalities always occur if $\ell \leq 4$. If $\ell \geq 5$, by Lemma 5.17, $h/2 > m_2$. Thus the equalities occur if $\{\beta_1, \beta_2\}$ is not RO. \(\square\)

Corollary 5.33. Assume that $\ell \geq 3$. Let $\Omega$ be an irreducible component of $\theta^\perp$. If $\Phi$ is simply-laced, then the Coxeter number of $\Omega$ is given by

$$h(\Omega) = \begin{cases} 2 & \text{if } \text{rank}(\Omega) = 1, \\ h - 2m_2 + 2 & \text{if } \text{rank}(\Omega) \geq 2. \end{cases}$$

Proof. We need only give a proof for the second line. Note that by Lemma 5.23, there exists at most one irreducible component $\Omega$ of $\theta^\perp$ satisfying $\text{rank}(\Omega) \geq 2$. For every $\beta \in \Omega$, $\{\theta, \beta\}$ is $A_1^2$ since $\Phi$ is simply-laced. Moreover, $\{\theta, \beta\}$ is not RO by the reason of rank. With the notations in Proof of Theorem 5.31, $\mathcal{M}'_\theta(\beta) = \emptyset$. It completes the proof. \(\square\)

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