Nonlinear systems’ equilibrium points: branching, blow-up and stability

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Abstract. This article considers the nonlinear dynamic model formulated as the system of differential and operator equations. This system is assumed to enjoy an equilibrium point. The Cauchy problem with the initial condition with respect to one of the desired functions is formulated. The second function controls the corresponding nonlinear dynamic process. The sufficient conditions of the global classical solution’s existence and stabilisation at infinity to the equilibrium point are formulated. The solution can be constructed by the method of successive approximations. If the conditions of the main theorem are not satisfied, then several solutions may exist. Some solutions can blow-up in a finite time, while others stabilise to an equilibrium point. The special case of considered dynamic models are differential-algebraic equations which model various nonlinear phenomena in circuit analysis, power systems, chemical processes and many other processes.

Let us consider the system

\[ A \frac{dx}{dt} = F(x, u), \]
\[ 0 = G(x, u). \]

Here the linear operator \( A : D \subset X \to E \) has bounded inverse, the nonlinear operators \( F : X + U \to E, G : X + U \to U \) are continuous in the neighbourhoods \( ||x||_X \leq r_1, ||u||_U \leq r_2 \) of real Banach spaces \( X, U; E \) is a linear real normed space. The following operators decompositions are assumed

\[ F(x, u) = F(0, 0) + A_1 x + A_2 u + R(x, u); \]
\[ ||R(x, u)|| = o(||x|| + ||u||); \]
\[ G(x, u) = G(0, 0) + \sum_{k=1}^{n} d^k(G(0, 0); (x, u)) + r(x, u); \]
\[ ||r(x, u)||_U = o(||x|| + ||u||)^n). \]

Here the derivatives and Fréchet differentials calculated in point \((0, 0)\) as follows

\[ A_1 := \frac{\partial F(0, 0)}{\partial x}; \quad A_2 := \frac{\partial F(0, 0)}{\partial u}; \quad A_3 := \frac{\partial G(0, 0)}{\partial x}; \quad A_4 := \frac{\partial G(0, 0)}{\partial u}; \]
\[ d^k (\mathcal{G}(0,0);(x,u)) = \sum_{i+j=k} C_k^i \frac{\partial^k \mathcal{G}(0,0)}{\partial x^i \partial u^j} \bigg|_{x=x_0,u=u_0} x^i u^j; \]

\[ \frac{\partial^k \mathcal{G}(x,u)}{\partial x^i \partial u^j} : X + \cdots + X + U + \cdots + U \rightarrow U. \]

It is assumed that point \((0,0)\) satisfies operator equations \(\mathcal{F}(x,u) = 0, \mathcal{G}(x,u) = 0\). Therefore, \((0,0)\) is the equilibrium point of system (1), (2). The special cases of system (1), (2) with initial condition

\[ x(0) = \Delta \]

are considered in [1–4]. Here \(\Delta\) is element from neighbourhood of equilibrium point. Solution \(x(t), u(t)\) on semiaxis \([0, +\infty)\) is constructed such as \(\lim_{t \to +\infty} (||x(t)|| + ||u(t)||) = 0\). Functions \(u(t)\) are selected such as solution \(x(t), u(t)\) is stabilised to the equilibrium point \((0,0)\) as \(t \to +\infty\). Such problem is important for solution of various automatic control problem. It is also important for nonlinear dynamic systems’ mathematical modelling using “input” – “output” approach [7,8], when \(u\) is “input”, and \(y = h(x,u)\) is “output”. In case of closed loop the output must satisfy the given criteria in the form of equation \(\mathcal{G}(y) = 0\). Some results of this paper were announced in article [9].

In works [1, 2], [3] only models (1), (2) with ordinary differential equations have been considered. Calculations were performed to demonstrate the problem complexity caused by stability analysis and possible blow-up of the solution.

Constructed theory enables the unified consideration of “input-output” models involving both differential and integral-differential equations. Let \(\mathcal{F}(0,0) = 0, \mathcal{G}(0,0) = 0\).

**Definition 0.1** We call the ball \(||x|| \leq r\) as basin of attraction of equilibrium point \((0,0)\) of system (1), (2) such as for arbitrary \(\Delta\) from this ball there exist solutions \(x : [0, +\infty) \to X, u : [0, +\infty) \to U\) with initial condition \(x(0) = \Delta\) stabilising to zero on the positive semiaxis.

**Definition 0.2** If basin of attraction of an equilibrium point is nonempty \((r > 0)\), then stationary solution \((0,0)\) of system (1), (2) is called asymptotically stable.

The objective of this work is to construct the sufficient condition of non-emptiness of the basin of attraction of equilibrium points, proof of the existence and uniqueness theorem, the construction of sufficient conditions for the nonempty basin of attraction of the equilibrium point, the proof of the existence and uniqueness theorem for the solution (1), (2) with the initial condition \(x(0) = \Delta\); the development of the method of successive approximations of the solution of the Cauchy problem on semiaxis \([0, +\infty)\). The sufficient conditions are formulated for the Cauchy problem’s solution branching for the system (1), (2) with stability analysis of individual branches of this solution.

The main part of this work concentrated on the case when the linear operators \(A\) and \(A_4\) have bounded inverse. Spectrum of linear bounded operator acting from \(X\) to \(X\)

\[ \mathcal{M} = A^{-1}(A_1 - A_2A_4^{-1}A_3), \]

will be used. It is assumed that \(\text{Re}\ \lambda \leq -l < 0\) for \(\lambda \in \sigma(\mathcal{M})\). The Existence and Uniqueness Theorem on the semiaxis \([0, \infty)\) is proved, as well as the asymptotic stability of the Cauchy problem for sufficiently small norm \(||x(0)||\). It is shown that the solution branching of the Cauchy problem can occur. Some of the branches extend to the whole semi-axis \([0, +\infty)\) and they stabilize to zero as \(t \to +\infty\), and others can collapse (go to infinity). The illustrative examples are given.
1. Reduction of a non-linear system in the neighbourhood of an equilibrium point to a single differential equation

Let \( G(x, u) = A_3x + A_1u + r(x, u) \), where \( ||r(x, u)|| = o(||x|| + ||u||) \). The Implicit Mapping Theorem allows us to prove the following Lemmas.

**Lemma 1.1** Let operator \( A_3 \) has bounded inverse, here \( A_4 = \frac{\partial G(0,0)}{\partial u} \) is Fréchet derivative. Then for arbitrary ball \( S_1 : ||x|| \leq r_1 \) exists ball \( S_2 : ||u|| \leq r_2 \) such as for any \( x \in S_1 \) equation (2) enjoy unique continuous solution \( u(x) \) in ball \( S_2 \). In this case the following asymptotic representation of the solution is valid

\[
u = -A_4^{-1}A_3x + o(||x||),\]

as \( ||x|| \to 0 \), here \( A_3 = \frac{\partial G(0,0)}{\partial x} \) is Fréchet derivative.

From Lemma 2.1 it follows

**Lemma 1.2** Exists neighbourhood \( ||x|| \leq r, ||u|| \leq r_2 \), such as system (1), (2) can be uniquely reduced to the following differential equation

\[
A \frac{dx}{dt} = f(x).
\]

Here mapping \( f : S(0, r) \subset X \to E_2 := F(x, u(x)) \) is defined as follows

\[
f(x) = A_1x - A_2A_4^{-1}A_3x + L(x).
\]

Nonlinear mapping \( L : X \to E \) satisfies the estimate \( ||L(x)|| = o(||x||) \).

After determination of \( x(t) \) one may find the approximate \( u(t) \). Indeed, let \( x(t) \) is solution to differential equation (3) constructed in Lemma 1.2. Then function \( u(t) \) is constructed by following asymptotics \( u(t) \sim -A_4^{-1}A_3x(t) \) as \( ||x|| \to 0 \) using Lemma 1.1.

2. An a priori estimate of the Cauchy problem’s solution

Let us consider the system (1), (2) with initial Cauchy condition \( x(0) = \Delta \). Based on the Lemmas 1.1 and 1.2 and the Gronwall-Bellman inequality the following Lemma can be proved.

**Lemma 2.1** Let \( x : [0, +\infty) \to X \) be solution to Cauchy problem for system (1), (2), where \( ||\Delta|| \) in the initial condition, is sufficiently small. Let \( Re \lambda \leq -l < 0 \) for all \( \lambda \in \sigma(M) \). Then there are \( C \geq 1 \) and \( \varepsilon \in (0, l) \), such as \( ||\exp(Mt)||_{C(X \to X)} \leq Ce^{-lt} \) and \( ||x(t)||_X \leq C||\Delta||e^{(\varepsilon-l)t} \) for \( t \in [0, +\infty) \).

A priori assessment of the solution justifies the continuation of a local continuous solution of the Cauchy problem to the entire interval \( [0, +\infty) \), see the following Theorem 3.1.

3. Existence, uniqueness, and asymptotic stability

**Theorem 3.1** Let \( (0, 0) \) be equilibrium point of system (1), (2). Let \( Re \lambda \leq -l < 0 \) for all \( \lambda \in \sigma(M) \), ||\Delta|| is sufficiently small. Then system (1), (2) with condition \( x(0) = \Delta \) enjoys unique solution \( x : [0, +\infty) \to X, u : [0, +\infty) \to U \). Moreover, \( \lim_{t \to +\infty} (||x(t)|| + ||u(t)||) = 0 \).

**Proof.** By virtue of Lemmas 1.1 and 1.2 and the obvious validity of Picard’s theorem for equation (3), the Cauchy problem (1), (2), \( x(t_0) = x_0 \) has a unique local solution for \( \forall t_0 \in [0, \infty) \). Therefore, the set of values of the arguments \( t \), for which the local solution continuously extends, is open in any interval \( [t_0, +\infty) \). Since the solution of the Cauchy problem for sufficiently small \( ||x(0)|| \) on the basis of Lemma 2.1, satisfies the a priori estimate \( x(t) < \delta \) for \( \forall t \in (0, +\infty) \) and
does not reach \( \delta \), then the set of values of the argument \( t \), on which the solution can be continued, will be closed. Therefore, on the basis of known facts about the method of continuation with respect to a parameter, the solution continuously extends to the entire interval \([0, +\infty)\). In view of above proved Lemmas the desired functions \( x(t), u(t) \) stabilise as \( t \to +\infty \) to point \((0, 0)\) for sufficiently small initial value \( \Delta \). Theorem 3.1 is proved.

Example 3.1

\[
\begin{aligned}
\frac{\partial x(t,z)}{\partial t} &= -x(t,z) + x^3(t,z) + u^2(t,z), \\
x(0,z) &= \Delta(z), \quad t \in [0, +\infty), \quad z \in [0,1], \\
u(t,z) + \int_0^1 zs u(t,s) \, ds + x^2(t,z) + u^2(t,z) &= 0,
\end{aligned}
\]

\(|\Delta(z)| \leq \varepsilon, \varepsilon \) is sufficiently small. Here \( A = A_1 = I, A_3 = 0, X = E = U = \mathcal{C}[0,1] \). Operator \( A_4 = I + \int_0^1 z s\hat{\cdot}\, ds \) has bounded inverse \( A_4^{-1} = I - \frac{2}{3} \int_0^1 z s\hat{\cdot}\, ds, M = -I \). If \(|x(t,z)|\) is sufficiently small then sequence \( \{u_n(t,z)\} \), where \( u_n(t,z) = -x^2(t,z) - u_{n-1}(t,z) + \frac{2}{3} \int_0^1 z \{x^3(t,s) + u_{n-1}^2(t,s)\} \, ds, u_0(t,z) = 0, \) converges and enables the algorithm for construction of solution \( u(t,z) \) of integral equation as function of \( x(t,z) \). Substituting this solution into a differential equation, we reduce the problem to the following differential equation:

\[
\frac{\partial x(t,z)}{\partial t} = -x(t,z) + O(||x||^2), \quad x(0,z) = \Delta(z). \quad \text{Here } ||u|| = O(||x||^2).
\]

Therefore, this model, consisting of differential and integral equations satisfies conditions of the Theorem 3.1 and on the semiaxis \([0, \infty)\) enjoy unique continuous solution \( x(t,z), u(t,z), \) stabilising as \( t \to +\infty \) to the equilibrium point \((0,0)\) if \( \max_{0 \leq z \leq 1} |\Delta(z)| \) is sufficiently small.

4. On the construction of a solution of a nonlinear system by the successive approximations method

Under the conditions of Theorem 3.1, the desired solution \( x(t), u(t) \) of system (1), (2) with condition \( x(0) = \Delta \) can be constructed without prior system’s reduction to one differential equation. Indeed, we introduce two sequences \( \{x_n(t)\}, \{u_n(t)\} \) with conditions \( x_n(0) = \Delta, n = 0, 1, \ldots, \) where \(||\Delta||\) is sufficiently small. Let \( u_0 = 0, \) and \(||\Delta||\) is sufficiently small, \( x_n(t) \) is solution to Cauchy problem \( A_4 \frac{dx}{dt} = \mathcal{F}(x_n, u_{n-1}), x_n(0) = \Delta, n = 1, 2, \ldots \). Obviously solution \( x_n(t) \) exists and unique for \( t \geq 0 \) due to Theorem 3.1.

Next, let us construct functions \( u_n \) using the iterations \( u_n = u_{n-1} + w_n, \) where \( u_0 = 0. \) Due to invertibility of the operator \( A_4 \), functions \( w_n \) can be found from the linear equation \( A_4 w_n + \mathcal{G}(x_n, u_{n-1}) = 0, n = 1, 2, \ldots \). Then, under Theorem 3.1 assumptions \( lim_{n \to \infty} x_n(t) = x(t), \) \( lim_{n \to \infty} u_n(t) = u(t), \) \( lim_{n \to \infty} (||x(t)|| + ||u(t)||) = 0. \) It is essential to require small \(||\Delta||\) otherwise solution to nonlinear differential equation may blow-up in the point \( t^* \) (refer to examples below).

Example 4.1 Let us consider the system

\[
\begin{aligned}
\frac{dx(t)}{dt} &= -\frac{x(t)}{2} - u(t) + x^2(t), \\
0 &= 2u(t) - x(t) + 2u(t)\sin u(t) - x(t)\sin u(t).
\end{aligned}
\]

with initial condition \( x(0) = \Delta, 0 \leq t < +\infty. \) The replacement \( u(t) = \frac{x(t)}{2} \) will reduce this system to Cauchy problem \( \dot{x}(t) = -x(t) + x^2(t), \) \( x(0) = \Delta. \) It is easy to verify that the latter model has an exact solution \( x(t) = \frac{\Delta}{\Delta - 1} + \frac{\Delta}{x - \Delta + \Delta}. \) Let us demonstrate that point \( t^* = \ln \frac{\Delta}{\Delta - 1} \) may appear to be blow-up of the constructed solution. We consider the following 4 cases.

Case 1. If \( \Delta \in (0, 1), \) then blow-up point \( t^* \) is complex, solution is continuous for \( t \in (0, +\infty) \) and stabilising to the equilibrium point \( x = 0 \) as \( t \to +\infty \) (ref. Fig. 1).
Case 2. If $\Delta \in (-\infty, 0)$, then blow-up point is negative, and on semiaxis $[0, +\infty)$ solution is continuous and stabilising to the equilibrium point $x = 0$ as $t \to +\infty$ (ref. Fig. 2).

Case 3. If $1 < \Delta < \infty$, then solution blows-up for $t^{*} = \ln \frac{\Delta}{\Delta - 1}$, where $\frac{\Delta}{\Delta - 1} > 0$. If $t > t^{*}$ then solution is continuous and also stabilising to the equilibrium point $x = 0$ as $t \to +\infty$ (ref. Fig. 3).

Case 4. For $\Delta = 0$ and $\Delta = 1$ we get the stationary solutions.

Based on the above, the following conclusion can be drawn. In Example 2 for $\Delta \in (-\infty, 1)$ exists the unique solution to the Cauchy problem as $t \geq 0$, this solution stabilising to the equilibrium point as $t \to +\infty$. It is to be mentioned that for $\Delta > 1$ the Cauchy problem’s solution will blow-up on the finite time $\ln \frac{\Delta}{\Delta - 1}$.

Remark 1 The absence of real equilibrium points can generate solutions with a countable set of blow-up points.

Example 4.2

\[
\left\{ \begin{array}{l}
\frac{dx}{dt} = \alpha x + \beta u + u^2 + x^3, \\
 x(0) = \Delta, \\
 \alpha x^2 + 2\beta xu + u^2 = 0, 0 \leq t < \infty.
\end{array} \right.
\]
Here \((0, 0)\) is the equilibrium point. Under assumption \(u = cx\), where \(c\) is const, we get the following quadratic equation \(c^2 + 2bc + a = 0\). Then \(u\) is double-valued \(u_1, u_2 = x(t)(-b \pm \sqrt{b^2 - a})\).

Let \(a < b^2\). Let us substitute the determined values of \(u\) into the differential equation. Then the problem of determining the function \(x(t)\) is reduced to the solution of two Cauchy problems

\[
\begin{align*}
\begin{cases}
\frac{dx}{dt} = (\alpha + \beta(-b \pm \sqrt{b^2 - a})x \pm (-b \pm \sqrt{b^2 - a})^2 x^2 \pm x^3, \\
x(0) = \Delta.
\end{cases}
\end{align*}
\]

Let \(\alpha + \beta(-b - \sqrt{b^2 - a}) < 0\). Then exists branch \(x_-(t)\) for small \(|\Delta|\) for \(t \geq 0\) and stabilising to zero as \(t \to +\infty\).

5. Possible generalizations

In this work until now only the autonomous systems were considered. This can be relaxed. For example, if we have system

\[
\begin{align*}
\begin{cases}
\mathcal{A}_1 \frac{dx}{dt} = (\mathcal{A}_1 + \mathcal{A}_1(t))x(t) + (\mathcal{A}_2 + \mathcal{A}_2(t))u(t) + \mathcal{R}(x, u, t), \\
0 = \mathcal{A}_3 x + \mathcal{A}_4 u + \mathcal{r}(x, u, t),
\end{cases}
\end{align*}
\]

where \(\mathcal{A}_1(t) \to 0, \mathcal{A}_2(t) \to 0\) as \(t \to +\infty\), \(||\mathcal{R}(x, u, t)|| = o(||x|| + ||u||)\) and \(||\mathcal{r}(x, u, t)|| = o(||x|| + ||u||)\) for \(||x|| + ||u|| \to 0\) coversges uniformly \(t \geq 0\), then results of Theorem 3.1 remains correct. In the theory of systems (1)–(2), the most difficult case is the case of irreversible Fréchet derivative \(\frac{d}{du}G(x, u)\) at the equilibrium point, and therefore the Implicit Operator Theorem is not fulfilled for the map \(G(x, u) = 0\). This case needs the results of the state-of-the-art analytic theory of branching solutions of nonlinear operator equations and weakly regular equations are considered in [11–19] and other. It is also interesting to consider the systems (1)–(2) with a discontinuity in the equilibrium points’ neighbourhood, when the stability condition in the first approximation is not satisfied, and more advanced methods must be used. For example, methods related to Lyapunov functions construction can be employed to evaluate the location of potential blow-up points using method of convex majorants of L.V. Kantorovich used in our works [8, 10, 20]. In this case, when one construct the algorithms for analysing stability and construct the estimates of the regions of attraction of the equilibrium points of the power systems, it is necessary to use the methods based on the Lyapunov vector-functions theory. Finally, it is interesting to consider the system (1)–(2) with equilibrium points for an irreversible operator.
In this case, the standard Cauchy problem can have no classic solutions and it is necessary to introduce other initial conditions. If the irreversible operator $A$ admits a finite-length skeleton decomposition, then new correct initial conditions for the problem (1)–(2) can be formulated using the results of the monograph [14] and article [21].

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