THE CLASSIFICATION OF REAL SINGULARITIES USING SINGULAR
PART III: UNIMODAL SINGULARITIES OF CORANK 2

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Abstract. We present a classification algorithm for isolated hypersurface singularities of corank 2 and modality 1 over the real numbers. For a singularity given by a polynomial over the rationals, the algorithm determines its right equivalence class by specifying all representatives in Arnold’s list of normal forms belonging to this class, and the corresponding values of the moduli parameter.

We discuss how to computationally realize the individual steps of the algorithm for all singularities in consideration, and give explicit examples. The algorithm is implemented in the Singular library realclassify.lib.

1. Introduction

In the ground breaking work on singularities by Arnold (1974), all isolated hypersurface singularities over the complex numbers up to modality 2 have been classified. Arnold has also given fundamental theorems on the classification of real singularities up to modality 1, which has been made explicit in Arnold et al. (1985). For his classification, Arnold considers stable equivalence of functions. Two function germs are stably equivalent if they are right equivalent after the direct addition of a non-degenerate quadratic form. Let K = R or K = C. Two function germs \( f, g \in m^2 \subset K[[x_1, \ldots, x_n]] \), where \( m = \langle x_1, \ldots, x_n \rangle \), are right equivalent if there is a \( K \)-algebra automorphism \( \phi \) of \( K[[x_1, \ldots, x_n]] \) such that \( \phi(f) = g \). Arnold presents a finite list of normal forms, each of which is a family of polynomial equations with moduli parameters such that each equivalence class contains at least one, but finitely many, elements of these families. We refer to such elements as normal form equations. Over the complex numbers, Arnold gives an algorithmic determinator which determines a normal form for a given power series.

Considering singularities over the reals, an algorithm for computing the degenerate part and the inertia index is given in Marais and Steenpaß (2015a). For singularities of modality 0, an algorithm to determine the corresponding normal form is developed in the same article.

In the case of singularities of modality 1, it turns out that an equivalence class may contain several normal form equations, as specified by Arnold. In Marais and Steenpaß (2015b), a complete classification of singularities up to corank 2 is given, in the sense that all complex and real equivalences between complex, respectively real, normal form equations are determined.

In this paper, we develop a determinator for real singularities of modality 1 and corank 2, which computes, for a given input polynomial, all normal form equations in Arnold’s list (see Table 1). In this case, the complex types correspond to real main types, which split up into real subtypes by modifying the signs of the terms in the normal form of the real main type, except in the case \( Y_{r,r} \). This complex normal form splits up into the real main types \( \bar{Y}_r \) and \( Y_{r,r} \).

In fact, we describe an algorithm which computes, for an arbitrary input polynomial \( f \in m^3 \subset \mathbb{Q}[x, y] \), the following data: all real singularity subtypes of \( f \) as well as all normal form equations in the right equivalence class of \( f \) with the respective parameter given as the unique root of a minimal polynomial over \( \mathbb{Q} \) in a specified interval. The algorithm is implemented in the library realclassify.lib for the computer algebra system Singular (Decker et al., 2015).

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This paper is structured as follows: In Section 2 we introduce the fundamental definitions and give the required background on the classification of singularities. In Section 3 we develop a general algorithm for the classification of real singularities of modality 1 and corank 2. Although the general algorithm is applicable to all cases under consideration, some steps do not have a straight-forward implementation for certain types of singularities. Therefore, in the subsequent sections, we give explicit computational realizations for all steps of the algorithm. The exceptional cases \((W_{12}, W_{13}, Z_{11}, Z_{12}, E_{12}, E_{13}, E_{14})\) follow the general algorithm in a direct way. In Section 4, as an example, we give an explicit algorithm for the case \(E_{14}\). Section 5 handles the parabolic cases \((X_9, J_{10})\). Section 6 deals with the hyperbolic cases \((X_{9+k}, J_{10+k}, Y_{r,a}, \tilde{Y}_r)\). The cases \(X_{9+k}\) and \(J_{10+k}\) follow the general algorithm in a straight-forward manner, whereas the cases \(Y_{r,a}\) and \(\tilde{Y}_r\) require some attention to detail.

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2. Definitions and Preliminary Results

In this section we give some basic definitions and results, as well as some notation that will be used throughout the paper.

We only consider classes of germs with respect to right equivalence.

**Definition 1.** Let \(\mathbb{K}\) be a field. Two power series \(f, g \in \mathbb{K}[[x_1, \ldots, x_n]]\) are called right equivalent, denoted by \(f \overset{\mathbb{K}}{\sim} g\), if there exists a \(\mathbb{K}\)-algebra automorphism \(\phi\) of \(\mathbb{K}[[x_1, \ldots, x_n]]\) such that \(\phi(f) = g\). If \(\mathbb{K} = \mathbb{R}\), we also write \(\sim\) to denote \(\mathbb{R}\).

Using this equivalence relation, Arnold [1974] gives the following formal definition of a normal form. From now on, let \(\mathbb{K}\) be either \(\mathbb{K}\) or \(\mathbb{C}\).

**Definition 2.** Let \(K \subset \mathbb{K}[[x_1, \ldots, x_n]]\) be a union of equivalence classes with respect to the relation \(\sim\). A normal form for \(K\) is given by a smooth map

\[
\Phi : B \rightarrow \mathbb{K}[x_1, \ldots, x_n] \subset \mathbb{K}[[x_1, \ldots, x_n]]
\]

of a finite dimensional \(\mathbb{K}\)-linear space \(B\) into the space of polynomials for which the following three conditions hold:

1. \(\Phi(B)\) intersects all equivalence classes of \(K\);
2. the inverse image in \(B\) of each equivalence class is finite;
3. \(\Phi^{-1}(\Phi(B) \setminus K)\) is contained in a proper hypersurface in \(B\).

We call the elements of the image of \(\Phi\) normal form equations.

**Remark 3.** There is a type \(T\) associated to each of the corank 2 real normal forms of modality 0 and 1 (see Table 1), and we denote the corresponding normal form by \(\text{NF}(T)\). Depending on whether \(T\) is a real or a complex type, we consider \(\mathbb{K} = \mathbb{R}\) or \(\mathbb{K} = \mathbb{C}\). For \(b \in \text{par}(\text{NF}(T)) := \Phi^{-1}(K)\) with \(K\) as in Definition 2 we write \(\text{NF}(T)(b) = \Phi(b)\) for the corresponding normal form equation.

We briefly introduce the concepts of weighted jets, filtrations, Newton polygons, and permissible chains. For background regarding the definitions in this section, we refer to Arnold [1974] and de Jong and Pfister [2000].

**Remark 4.** Let \(w = (c_1, \ldots, c_n) \in \mathbb{N}^n\) be a weight on the variables \((x_1, \ldots, x_n)\). We define the weighted degree on \(\text{Mon}(x_1, \ldots, x_n)\) by \(\text{w-deg}(\prod_{i=1}^n x_i^{s_i}) = \sum_{i=1}^n c_i s_i\). If the weight on all variables is 1, we call the weighted degree of a monomial \(m\) the standard degree of \(m\) and write \(\deg(m)\) instead of \(\text{w-deg}(m)\).
Remark 7. Let \( w = (w_1, \ldots, w_s) \in (\mathbb{N}^s)^* \) be a finite family of weights on the variables \((x_1, \ldots, x_n)\). For any monomial (or term \( t \)) \( m \in \mathbb{K}[x_1, \ldots, x_n] \), we define the **piecewise weight** with respect to \( w \) as

\[
\text{w-deg}(m) := \min_{i=1, \ldots, s} w_i \cdot \text{deg}(m).
\]

A polynomial is called **piecewise quasihomogeneous** of degree \( d \) with respect to \( w \) if \( \text{w-deg}(t) = d \) for any term \( t \) of \( f \). If \( s = 1 \), we call a piecewise quasihomogeneous function **quasihomogeneous**.

Definition 6. Let \( w \) be a (piecewise) weight on \( \text{Mon}(x_1, \ldots, x_n) \).

1. Let \( f = \sum_{i=0}^{\infty} f_i \) be the decomposition of \( f \in \mathbb{K}[[x_1, \ldots, x_n]] \) into weighted homogeneous parts \( f_i \) of w-degree \( i \). We denote the **weighted \( j \)-jet** of \( f \) by

\[
\text{w-jet}(f, j) := \sum_{i=0}^{j} f_i.
\]

2. A power series in \( \mathbb{K}[[x_1, \ldots, x_n]] \) has **filtration** \( d \in \mathbb{N} \) if all its monomials are of weighted degree \( d \) or higher. The power series of filtration \( d \) form a vector space

\[
E^w_d \subset \mathbb{K}[[x_1, \ldots, x_n]].
\]

3. A power series \( f \) in \( \mathbb{K}[[x_1, \ldots, x_n]] \) is called (piecewise)** semi-quasihomogeneous** if \( f \) has filtration \( d \in \mathbb{N} \) with respect to \( w \) and if the (piecewise) quasihomogeneous function \( \text{w-jet}(f, d) \) is non-degenerate, that is, if its Milnor number is finite. We call \( \text{w-jet}(f, d) \) the (piecewise) **quasihomogeneous part** of \( f \).

4. A power series \( f \in \mathbb{K}[[x_1, \ldots, x_n]] \) is weighted \( k \)-determined with respect to the weight \( w \) if

\[
f \sim \text{w-jet}(f, k) + g \quad \text{for all } g \in E^w_{k+1}.
\]

We define the weighted determinacy of \( f \) as the minimum number \( k \) such that \( f \) is \( k \)-determined.

Remark 7. (1) If for a given type \( T \), \( \text{w-jet}(\text{NF}(T)(b), j) \) is independent of \( b \in \text{par}(\text{NF}(T)) \), we denote it by \( \text{w-jet}(T, j) \).

(2) Note that \( d < d' \) implies \( E^w_{d'} \subset E^w_d \). Since the product \( E^w_d \cdot E^w_{d'} \) has filtration \( d' + d \), it follows that \( E^w_d \) is an ideal in the ring of power series.

(3) If the weight of each variable is 1, we write \( E_d \) and \( \text{jet}(f, j) \) instead of \( E^w_d \) and \( \text{w-jet}(f, j) \), respectively.

There are also similar concepts for coordinate transformations:

Definition 8. Let \( \phi \) be a \( \mathbb{K} \)-algebra automorphism of \( \mathbb{K}[[x_1, \ldots, x_n]] \) and let \( w \) be a (piecewise) weight on \( \text{Mon}(x_1, \ldots, x_n) \).

1. For \( j > 0 \) we define the **w-jet**(\( \phi, j \)), denoted by \( \phi^w_j \), to be the automorphism given by

\[
\phi^w_j (x_i) := \text{w-jet}(\phi(x_i), \text{w-deg}(x_i) + j) \quad \text{for all } i = 1, \ldots, n.
\]

If the weight of each variable is 1, that is, \( w = (1, \ldots, 1) \), we simply write \( \phi_j \) for \( \phi^w_j \).

2. \( \phi \) has filtration \( d \) if, for all \( \lambda \in \mathbb{N} \),

\[
(\phi - \text{id}) E^w_{\lambda+d} \subset E^w_{\lambda+d}.
\]

Remark 9. Note that \( \phi^w_0(x_i) = \text{jet}(\phi(x_i), 1) \) for all \( i = 1, \ldots, n \). Furthermore note that \( \phi^w_0 \) has filtration \( \leq 0 \) and that \( \phi^w_j \) has filtration \( j \) if \( j > 0 \) and \( \phi^w_{j-1} = \text{id} \).

For the next definition we restrict ourselves to the power series ring \( \mathbb{K}[[x, y]] \) in two variables.
Theorem 16. Let \( f \) be a semi-quasihomogeneous function with quasihomogeneous part \( f_0 \) of weighted \( w \)-degree \( d_w \), and let \( e_1, \ldots, e_s \) be the elements of weighted degree greater than \( d \) in a...
monomial vector space basis for the local algebra of $f_0$. Then $f$ is equivalent to a function of the form $f_0 + \sum c_k x^k$. We call $e_1, \ldots, e_r$ a system of the local algebra of $f$.

Arnold proved Theorem 10 by iteratively applying the following lemma.

**Lemma 17.** Let $f_0$ be a quasihomogeneous function of weighted $w$-degree $d_w$ and let $e_1, \ldots, e_r$ be the monomials of a given degree $d' > d_w$ in a system of the local algebra of $f_0$. Then, for every series of the form $f_0 + f_1$, where the filtration of $f_1$ is greater than $d_w$, we have

$$f_0 + f_1 \sim f_0 + f'_1,$$

where the terms in $f'_1$ of degree less than $d'$ are the same as in $f_1$, and the part of degree $d'$ can be written as $c_1 e_1 + \cdots + c_r e_r$, $c_1, \ldots, c_r \in \mathbb{R}$.

**Proof.** Let $g(x)$ denote the sum of the terms of degree $d'$ in $f_1$. There exists a decomposition of $g$ of the form

$$g(x) = \sum_i \frac{\partial f_0}{\partial x} v_i(x) + c_1 e_1 + \cdots + c_r e_r,$$

since $e_1, \ldots, e_r$ is a monomial vector space basis for the local algebra of $f_0$. Applying the transformation defined by

$$x_i \mapsto x_i - v_i(x),$$

we transform $f$ to

$$f_0(x) + (f_1(x) + (c_1 e_1(x) + \cdots + c_r e_r(x) - g(x)) + R(x),$$

where the filtration of $R$ is greater than $d'$.

**Remark 18.** A system of a local algebra is in general not unique. For his lists of quasihomogeneous normal forms of hypersurface singularities, Arnold has chosen a specific system for the local algebra in each case. In the rest of the paper, we call these systems the Arnold systems.

The following result is proved for $a \geq 4$, $b \geq 5$ and $\mathbb{K} = \mathbb{C}$ by Arnold (1974). The same proof is also valid for $a = 3$, $b \geq 7$. Moreover the proof is also valid for $\mathbb{K} = \mathbb{R}$. Hence we obtain the following more general result.

**Lemma 19.** Let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. Suppose that $f \in \mathbb{K}[[x, y]]$ is a non-degenerate power series with principal part $f_0 = x^n + \lambda x^2 y^2 + y^b$, where $0 \neq \lambda \in \mathbb{K}$ and $a, b \in \mathbb{N}$, such that $\Gamma(f_0)$ has two faces. Then

$$f \sim f_0.$$

**Remark 20.** Note that it follows from Lemma 17 and Lemma 19 that all hypersurface singularities of corank 2 are finitely weighted determined. Moreover, we explicitly obtain the weighted determinacy for every such singularity.

We proof the following result in addition to Marais and Steenpaß (2015a) Lemma 9 which has similar results for linear factorizations.

**Lemma 21.** Let $n \in \mathbb{N}$, $n \geq 2$. Suppose $f \in \mathbb{Q}[x, y]$ is homogeneous and factorizes as $g_1^n g_2$ over $\mathbb{R}$, where $g_1$ is a polynomial of degree 1 and $g_2$ is a polynomial of degree 2 that does not have a multiple root over $\mathbb{C}$. Then $f = ag_1^n g_2^2$, where $g_1$ is a polynomial of degree 1 over $\mathbb{Q}$, $g_2$ is a polynomial of degree 2 over $\mathbb{Q}$ and $a \in \mathbb{Q}$.

**Proof.** Let $f = \left(a_1 x + a_2 y\right)\left(a_3 x^2 + a_4 xy + a_5 y^2\right)$ with $a_i \in \mathbb{R}$. Since the coefficient of $x^{n+2}$ in $f$ is a rational number, it follows that

$$(x + \frac{a_2}{a_1} y)^n (x^2 + \frac{a_4}{a_3} xy + \frac{a_5}{a_3} y^2) \in \mathbb{Q}[x, y].$$

Since $\mathbb{Q}$ is a perfect field, $(x + \frac{a_2}{a_1} y)(x^2 + \frac{a_4}{a_3} xy + \frac{a_5}{a_3} y^2) \in \mathbb{Q}[x, y]$. Hence $(x + \frac{a_2}{a_1} y)^{n-1} \in \mathbb{Q}[x, y]$ which implies that $x + \frac{a_2}{a_1} y \in \mathbb{Q}[x, y]$. Therefore $f = ag_1^n g_2^2$, where $g_1 = x + \frac{a_2}{a_1} y$, $g_2 = x^2 + \frac{a_4}{a_3} xy + \frac{a_5}{a_3} y^2$ and $a = a_1^2 a_3$. \qed
Table 1. Normal forms of singularities of modality 1 and corank 2 as given in Arnold et al. (1985)

| Complex normal form | Normal forms of real subtypes | Restrictions |
|---------------------|-----------------------------|--------------|
| \( x^4 + ax^2y^2 + y^4 \) | \(+x^4 + ax^2y^2 + y^4 (X_9^+), -4 < \alpha \neq 4 \) | \( a^2 \neq +4 \) |
| \( x^3 + ax^2y^2 + xy^4 \) | \(+x^3 + ax^2y^2 + xy^4 (J_{10}^+) \) | \( a^2 \neq +4 \) |
| \( x^3 + ax^2y^2 + ay^{6+k} \) | \(+x^3 + ax^2y^2 + ay^{6+k} (J_{10+k}^+) \) | \( a \neq 0, k > 0 \) |
| \( x^4 + ax^2y^2 + ay^{4+k} \) | \(+x^4 + ax^2y^2 + ay^{4+k} (X_{9+k}^+) \) | \( a \neq 0, k > 0 \) |
| \( x^2y^2 + x^r + ay^s \) | \(+x^2y^2 + x^r + ay^s (Y_{r,s}^+) \) | \( a \neq 0, r, s > 4 \) |
| \((x^2 + y^2)^2 + ax^r \) | \(-x^2 + y^2)^2 + ax^r (Y_r^-) \) | \( a \neq 0, r > 4 \) |
| \( E_{12} \) | \( x^3 + y^5 + axy^5 \) | \( x^3 + y^5 + axy^5 \) |
| \( E_{13} \) | \( x^3 + xy^5 + ay^6 \) | \( x^3 + xy^5 + ay^6 \) |
| \( E_{14} \) | \( x^3 + y^5 + axy^6 \) | \(+x^3 + y^5 + axy^6 (E_{14}^+) \) |
| \( Z_{11} \) | \( x^3y + ay^5 + axy^4 \) | \(-x^3y + ay^5 + axy^4 \) |
| \( Z_{12} \) | \( x^3y + ax^4 + ay^5 \) | \(-x^3y + ax^4 + ay^5 \) |
| \( Z_{13} \) | \( x^3y + y^6 + axy^5 \) | \(+x^3y + y^6 + axy^5 (Z_{13}^+ \) |
| \( W_{12} \) | \( x^4 + y^6 + ax^2y^3 \) | \(+x^4 + y^6 + ax^2y^3 (W_{12}^+ \) |
| \( W_{13} \) | \( x^4 + xy^4 + ay^6 \) | \(+x^4 + xy^4 + ay^6 (W_{13}^+ \) |

\(^1\)Note that the restriction \( a^2 \neq 4 \) applies to the normal forms of the real subtypes \( X_{9}^+, X_9^-, \) and \( J_{10}^+ \) as well as to the normal forms of the complex types \( X_9 \) and \( J_{10} \) while the restriction \( a^2 \neq -4 \) applies to the normal forms of the real subtypes \( X_9^-, X_9^- \), and \( J_{10}^- \) if we allow complex parameters.
3. General Classification Algorithm

In this section we outline the general structure of an algorithm (see Algorithm 5) to determine, for a given input polynomial $f \in \mathbb{Q}[x,y]$, the real types as well as, for each type, the corresponding normal form equations to which $f$ is equivalent (see Table 1). Each parameter is given as the unique root of its minimal polynomial over $\mathbb{Q}$ in a specified interval.

Figures 1 to 4 show in the gray shaded area all monomials which can possibly occur in a polynomial $f$ of the given type $T$. The Newton polygon $\Gamma(T)$ is shown as the blue line with the non-moduli monomials of $NF(T)$ shown as blue dots. Red dots indicate monomials which are not in $Jac(f)$. The dot with the thick black circle indicates the moduli monomial in the Arnold system.

We start the classification process by classifying the input polynomial according to the complex classification in [Arnold et al., 1985]. The classifier to obtain the complex types has been implemented in the SINGULAR library classify.lib [Krüger, 1997]. As shown by [Arnold, 1974], the complex types correspond to real main types, which split up into real subtypes by modifying the signs of the terms in the normal form of the real main type, except in the case $Y_{r,r}$. This complex normal form splits up into the real main types $\tilde{Y}_r$ and $Y_{r,r}$. We can easily distinguish whether an input polynomial $f$ is of real main type $Y_{r,r}$ or $\tilde{Y}_r$, using [Marais and Steenpaß, 2015a] Proposition 8 by considering the number of real roots of $\text{jet}(f,4)$. The SINGULAR library rootsur.lib [Tobis, 2012] can be used for this task. We postpone the discussion of the case $\tilde{Y}_r$ to Section 6.2, where we will give a modification of the general algorithm applicable to this case.

Note that in all the other cases, the Newton polygon $\Gamma(T)$ of a real subtype $T$ coincides with the Newton polygon of its corresponding complex and main real type. Also note that, according to [Marais and Steenpaß, 2015b], normal form equations of different complex types, and hence of different real main types, cannot be equivalent.

Suppose $NF(T)(b)$, $b \in \text{par}(NF(T))$, is one of the normal form equations to which $f$ is equivalent. Then there exists an $\mathbb{R}$-algebra automorphism $\phi$ of $\mathbb{R}[x,y]$ such that $f = \phi(NF(T)(b))$. We write $w = w(T)$ for the piecewise weight induced by $\Gamma(T)$ and let $d_w = d(T)$ be the degree of the monomials on $\Gamma(T)$ with respect to $w$.

Our first goal is to transform $f$, and thus $\phi$, iteratively by composing $\phi$ with a suitable $\mathbb{R}$-algebra automorphism such that, after every step, we have

$$f = \phi(NF(T)(b))$$

and, after step $i$, we have

$$\text{supp}(\text{jet}(f,i)) = \text{supp}(T,i)$$

if $\Gamma(T)$ has one face, and

$$\text{supp}(w-\text{jet}(f,i),0) = \text{supp}(T,i)$$

if $\Gamma(T)$ has two faces. This implies that, after a finite number of steps, we may assume, by an appropriate choice of $\phi$, that

$$\phi_w^e(x) = c_xx \quad \text{and} \quad \phi_w^e(y) = c_yy,$$

where $w$ is the weight defined by $\Gamma(T)$. After this, in the case where $\Gamma(T)$ has two faces, Theorem 19 can be used to determine the equivalence class of $f$ by eliminating the monomials above $\Gamma(T)$.

In the case where $\Gamma(T)$ has only one face, the algorithmic proof of Theorem 16 can be used to eliminate monomials above $\Gamma(T)$ which are not in $\text{supp}(T)$, by again iteratively transforming $f$ and $\phi$ such that we have

$$f = \phi(NF(T)(b))$$

after each step, and

$$\text{supp}(w-\text{jet}(f,d_w+j_1)) = \text{supp}(w-\text{jet}(T,d_w+j_1)),$$

where $1 \leq j_1 < \cdots < j_i$, after step $i$. We may thus assume that

$$\phi_{j_1}^e(x) = c_xx \quad \text{and} \quad \phi_{j_1}^e(y) = c_yy,$$
where \(c_x, c_y \in \mathbb{R}^*\). Since the weighted determinacy \(k\) of \(f\) is finite, terms of \(w\)-degree greater than \(k\) may be discarded after each step. In particular, this process can be stopped once it reaches \(d_w + j_i > k\). Hence a normal form equation of \(f\) can be determined in finitely many steps. To summarize, the general classification algorithm involves the following main steps:

I. Determine the complex type and the corresponding Newton polygon \(\Gamma(T)\).
II. Eliminate all monomials in \(\text{supp}(f) \setminus \text{supp}(T)\) underneath or on \(\Gamma(T)\).
III. Eliminate all monomials in \(\text{supp}(f) \setminus \text{supp}(T)\) above \(\Gamma(T)\).
IV. Read off the real subtypes and the corresponding normal form equations.

Step (I) is straightforward using classify.lib. We now discuss Steps (II) to (IV) in detail:

**Step (II): Elimination of monomials underneath and on the Newton Polygon.** We first eliminate the monomials in \(\text{jet}(f, d)\), which are not in \(\text{jet}(T, d)\), where \(d\) is the maximum filtration of \(f\) with respect to the standard degree. This can be done by a linear transformation over \(\mathbb{R}\), since these terms can only be created from the monomials in \(\phi_0(x)\) and \(\phi_0(y)\). Removing these terms amounts to transforming \(\phi\) such that

\[
\phi_0(x) = c_x x, \quad c_x \in \mathbb{R},
\]

if \(\text{jet}(T, d) = x^n\) for some \(n \in \mathbb{N}\), or such that

\[
\phi_0(x) = c_x x, \quad c_x \in \mathbb{R}, \quad \text{and} \quad \phi_0(y) = c_y y, \quad c_y \in \mathbb{R},
\]

if \(\text{jet}(T, d) \neq x^n\) for any \(n \in \mathbb{N}\). For the exceptional cases and the parabolic cases \(J_{10}\) and \(J_{10+k}\), this can be done by factorizing \(\text{jet}(f, d)\) over \(\mathbb{Q}\) using [Marais and Steenpaß (2015a), Proposition 8] and [Marais and Steenpaß (2015a) Lemma 9]. Similarly, we can use Lemma 21 for the case \(X_{9+k}\). See Algorithms 1 and 2 respectively, which implement the required transformations.

**Algorithm 1**

**Input:** \(f \in \mathbb{Q}[x, y]\) of real type \(T\), where \(T\) is exceptional, or parabolic of type \(J_{10}\) or \(J_{10+k}\), with maximum filtration \(d\)

**Output:** \(h \in \mathbb{Q}[x, y]\) with \(f \sim h\) and \(\text{supp}(\text{jet}(h, d)) = \text{supp}(T, d)\)

1. : \(g := \text{jet}(f, d)\)
2. : factorize \(g\) over \(\mathbb{Q}\) as \(cf_1^\alpha f_2^\beta\), with \(f_1\) and \(f_2\) linear and non-associated, \(c \in \mathbb{Q}\), and \(\alpha > \beta \geq 0\)
3. : if \(\beta > 0\) then
4. : apply \(f_1 \mapsto x, f_2 \mapsto y\) to \(f\)
5. : else if \(f_1 \neq c'y, c' \in \mathbb{Q}\) then
6. : apply \(f_1 \mapsto x, y \mapsto y\) to \(f\)
7. : else
8. : apply \(f_1 \mapsto x, x \mapsto y\) to \(f\)
9. : return \(f\)

In the cases \(X_9\) and \(Y_{r,s}\), in general, a real field extension is required for this step. This leads to an implementational problem for the subsequent steps of the algorithm, since if we represent an algebraic number field as \(\mathbb{Q}[z]/\langle m\rangle\), with \(m \in \mathbb{Q}[z]\) irreducible, we have to determine which root of \(m\) the generator \(\pi\) corresponds to. We discuss how to handle this problem in the cases \(X_9\) and \(Y_{r,s}\) in Sections 5 and 6 respectively.

Next we eliminate the remaining monomials in \(\text{supp}(f) \setminus \text{supp}(T)\) underneath or on \(\Gamma(T)\). We consider the cases where \(\Gamma(T)\) has exactly one face and where \(\Gamma(T)\) has two faces separately.

**The Newton polygon has only one face.** Note that in the case \(X_9\), there are no monomials in \(\text{supp}(f) \setminus \text{supp}(T)\) underneath or on \(\Gamma(T)\). In the remaining cases, we have \(\text{jet}(T, d) = m_0\), where \(m_0\) is a monomial, and \(d\) is the maximum filtration of \(f\) with respect to the standard degree as before. Then \(\text{jet}(f, d) = cm_0\), with \(0 \neq c \in \mathbb{Q}\). By considering \(\frac{1}{c}f\), we may assume that \(c = 1\). Indeed, the transformations that remove the required terms of \(\frac{1}{c}f\) also remove the same terms of \(f\).
Algorithm 2

Input: \( f \in \mathbb{Q}[x,y] \) of real type \( T = X_{9+k} \)
Output: \( h \in \mathbb{Q}[x,y] \) with \( f \sim h \) and \( \text{supp} (\text{jet}(h,4)) = \text{supp}(T,4) \)
1: \( g := \text{jet}(f,4) \)
2: factorize \( g \) over \( \mathbb{Q} \) as \( cf_1^2f_2 \) with \( f_1 \) linear, \( f_2 \) quadratic, and \( c \in \mathbb{Q} \)
3: if \( f_1 \neq c'y, \ c' \in \mathbb{Q} \) then
   4: apply \( f_1 \mapsto x, \ y \mapsto y \) to \( f \)
5: else
6: apply \( f_1 \mapsto x, \ x \mapsto y \) to \( f \)
7: write \( f = a_0x^4 + a_1x^3y + a_2x^2y^2 + R, \ a_0, a_1 \in \mathbb{Q}, \ a_2 \in \mathbb{Q}^*, \ R \in E_5 \)
8: apply \( y \mapsto y - \frac{a_2}{a_0}x, \ x \mapsto x \) to \( f \)
9: return \( f \)

We first eliminate the monomials in \( f \) strictly underneath \( \Gamma(T) \) and then the monomials on \( \Gamma(T) \), which are not in \( \text{supp}(T) \).

We inductively consider jets of \( f \) of increasing standard degree, starting at degree \( d + 1 \), until all monomials strictly underneath \( \Gamma(T) \) have been removed. For each degree \( j \), the jet is transformed such that \( \text{supp}(\text{jet}(f,j)) = \text{supp}(T,j) \). While \( \text{jet}(T,j) = m_0 \), all terms of degree \( j \) in \( f \) have been created through a term in homogeneous non-linear polynomials \( v_1 \) or \( v_2 \) in
\[
\begin{align*}
\phi(x) &= \text{linear terms} + v_1 + \text{terms of higher degree than } v_1 \\
\phi(y) &= \text{linear terms} + v_2 + \text{terms of higher degree than } v_2.
\end{align*}
\]

Considering Equations (1) and (2), and taking into account that by \( c = 1 \), we may assume \( c_x = 1 \) and \( c_y = c_y = 1 \), respectively, such a transformation maps \( m_0 = x^ay^b \) in \( \text{NF}(T)(b) \) to
\[
x^ay^b + \alpha x^{a-1}y^b v_1 + \beta x^ay^{b-1}v_2 + \text{terms of higher degree}.
\]
Since \( m_0 \) has degree less than \( j \) and is the only term in \( \text{NF}(T)(b) \) with degree less than or equal to \( j \), the terms of degree \( j \) in \( f \) can be written as
\[
\frac{\partial m_0}{\partial x}v_1 + \frac{\partial m_0}{\partial y}v_2,
\]
where \( \deg(v_1) > 1 \) and \( \deg(v_2) > 1 \). We can eliminate these terms by the transformation \( x \mapsto x - v_1, \ y \mapsto y - v_2 \). This transformation possibly creates terms of degree greater than \( j \), but does not change \( \text{jet}(f,j-1) \).

As soon as the \( j \)-jet of \( \text{NF}(T)(b) \) contains more than one term, the situation becomes more complicated. For example, if \( j \) is the degree of a monomial in \( \text{supp}(T) \), then some of the monomials of degree \( j \) in \( \text{supp}(f) \setminus \text{supp}(T) \) may result from the linear terms of \( \phi(x) \) and \( \phi(y) \). However, all monomials in \( f \) underneath \( \Gamma(T) \) have already been eliminated before we reach such a case. Indeed, taking Equations (1) and (2) into account, linear terms in \( \phi(x) \) will not create any additional monomials and \( \phi(y) \) will only create additional monomials above \( \Gamma(T) \). Moreover, a case by case analysis shows that terms in \( f \) underneath \( \Gamma(T) \) also cannot result from higher order terms in \( \phi(x) \) and \( \phi(y) \):

- If \( m_0 = x^d \), then the real main type of \( f \) is one of the following: \( W_{12}, \ W_{13}, \ E_{12}, \ E_{13}, \ E_{14}, \ J_{10} \) (see Figures (1), (2), and (3)).
  Considering the normal forms of \( W_{12} \) and \( W_{13} \), we have \( \text{jet}(T,4) = x^4 \) and \( \text{jet}(T,5) \neq x^4 \). Suppose \( \text{jet}(f,4) = x^4 \). If \( f \) is of type \( W_{12} \), then \( f \) cannot have any monomials underneath \( \Gamma(T) \). Let \( f \) be of type \( W_{13} \) and suppose \( y^5 \in \text{supp}(f) \). Then it follows from Theorem 16 that \( f \) is of type \( W_{12} \) which leads to a contradiction. Hence \( y^5 \notin \text{supp}(f) \).
  If \( f \) is of type \( E_{12} \) or \( E_{13} \), then \( \text{jet}(T,5) = x^3 \) and \( \text{jet}(T,6) \neq x^3 \). After applying the algorithm described above, we may assume that \( \text{jet}(f,5) = x^3 \). If \( y^5 \in \text{supp}(f) \), then \( f \) is of type \( J_{10} \). If \( y^5 \in \text{supp}(f) \), then \( f \) is not of type \( E_{13} \) but rather of type \( E_{12} \). Hence if \( f \) is of type \( E_{12} \) or \( E_{13} \), then \( f \) has no monomials underneath \( \Gamma(T) \). Similarly, it can be
shown that if \( f \) is of type \( E_{14} \), then \( f \) has no monomials underneath \( \Gamma(T) \) after \( f \) has been transformed such that \( \text{jet}(f, 6) = x^3 \).

If \( f \) has real main type \( J_{10} \) and \( \text{jet}(f, 3) = x^3 \), the only monomials that may occur underneath \( \Gamma(T) \) are \( xy^3 \), \( y^4 \) and \( y^5 \). Suppose that \( xy^3 \in \text{supp}(f) \). Since, by Equation (1), this term cannot result from linear terms of \( \phi(x) \) or \( \phi(y) \), and since \( \text{jet}(f, 3) = x^3 \), it follows that \( xy^3 \in \text{supp}(v_1 z^2) = \text{supp}(3v_1 x^2) \). Since \( x^2 \nmid xy^3 \), this leads to a contradiction. If \( y^4 \in \text{supp}(f) \), then \( f \) is of real main type \( E_6 \) by Theorem 16 which again gives a contradiction. Similarly, if \( y^5 \in \text{supp}(f) \), then \( f \) is of real main type \( E_8 \).

- If \( m_0 \neq x^4 \), then the real main type of \( f \) is one of the following: \( Z_{11}, Z_{12}, Z_{13} \) (see Figure 3).

We show that, in these cases, there are no monomials underneath \( \Gamma(T) \) after the linear transformation that transforms \( f \) such that \( \text{jet}(f, d) = m_0 \). Suppose \( f \) is of one of the main types under consideration and that \( \text{jet}(f, d) = m_0 \). If \( f \) is of type \( Z_{11} \), there are no monomials of degree greater than \( d \) underneath \( \Gamma(T) \). Suppose \( f \) is of real main type \( Z_{12} \) or \( Z_{13} \). If \( y^5 \) or \( xy^4 \) are elements of \( \text{supp}(f) \), then, by Theorem 16, \( f \) is of real main type \( W_{12} \) or \( Z_{12} \), respectively, which leads to a contradiction.

Therefore, if \( m_0 \neq x^4 \), then after the linear transformation given in Algorithm 1, \( f \) has no monomials underneath \( \Gamma(T) \). If \( m_0 = x^4 \) and \( j \) is such that \( \text{jet}(T, j) = m_0 \), then one can write

\[
\text{jet}(f, j) = m_0 + v_1 \frac{\partial m_0}{\partial x} + 0 \frac{\partial m_0}{\partial y}.
\]

Hence, iterative application of Algorithm 1 with \( t \) a term of \( f \) of standard degree \( j \) removes all monomials underneath \( \Gamma(T) \). Note that in the algorithm we have \( m_x = m_y = 0 \), so we make only use of the case in lines [8]-[9]. The other case will be used later to eliminate monomials above the diagonal in a similar manner. Moreover note, that Algorithm 4 also works for the polynomial \( t = \text{jet}(f, j) - \text{jet}(f, j - 1) \).

After eliminating all the monomials in \( \text{supp}(f) \) strictly underneath \( \Gamma(T) \), we may assume that \( \phi^w_{-1}(x) = 0 \) and \( \phi^w_{-1}(y) = 0 \).

Next we eliminate the remaining monomials in \( \text{supp}(f) \setminus \text{supp}(T) \) on \( \Gamma(T) \), see line 10 in Algorithm 5. The only case where such monomials can occur is the \( J_{10} \) case. In this case, these monomials can theoretically be eliminated by a weighted homogeneous transformation. However, similar to the linear transformation discussed above, this transformation creates implementational difficulties, since it may require real field extensions. We will discuss how to handle this problem in Section 6.

**The Newton polygon has two faces.** In this case, all the normal forms we have to consider are of the form \( x^n + x^2 y^2 + ay^m \) with \( m, n \in \mathbb{N} \), see Figure 6.

After a linear transformation (see Algorithm 1 for the case \( J_{10+k} \), Algorithm 2 for the case \( X_{9+k} \), and Section 6 for the case \( Y_{r,s} \)), we may assume that \( \text{supp}([\text{jet}(f, d)]) = \text{supp}(T, d) \), where \( d \) is the maximum filtration of \( f \) with respect to the standard grading. We first remove all monomials \( m \) with

\[
\max \{ w_j \cdot \text{deg}(m) \mid i \} \leq d_w
\]

which are not in \( \text{supp}(T) \), see line 12 in Algorithm 5. The only cases in which such monomials may occur are the \( J_{10+k} \) cases. The monomials \( y^4 \), \( y^5 \) and \( xy^3 \) do not occur in \( f \) if it is of type \( J_{10+k} \), \( k > 0 \), since these monomials lead to a different type. Using Algorithm 5 we transform \( \phi \) by a weighted linear transformation such that \( \text{supp}(w \cdot \text{jet}(f, 6)) = \text{supp}(w \cdot \text{jet}(T, 6)) \) where \( w = (2,1) \), which means that \( f \) is in the desired form.

Similar to the case where \( \Gamma(T) \) has one face, we may assume that \( \text{coeff}(f, x^2 y^2) = 1 \) by considering \( \frac{1}{c} f \), where \( c = \text{coeff}(f, x^2 y^2) \). We now, again, consider jets of \( f \) of increasing standard degree, starting at degree \( d + 1 \). However, in this case, we only eliminate those monomials not lying in \( \text{supp}(T) \) that are strictly underneath or on \( \Gamma(T) \). Suppose we are considering the standard weighted \( j \)-jet. We first eliminate terms of the form \( c_1 x y^{j-1} \) and \( c_2 y x^{j-1} \) which lie underneath

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**THE CLASSIFICATION OF REAL SINGULARITIES USING SINGULAR, PART III**

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Algorithm 3

Input: \( f \) of type \( J_{10+k} \) with \( \text{supp}(2,1)\text{-jet}(f,5)) = \emptyset \)
Output: \( f \) with \( \text{supp}(2,1)\text{-jet}(f,6) = \text{supp}(2,1)\text{-jet}(T,6)) \)

1: write \( f \) as \( f = x^3 + cx^2y^2 + dxy^4 + ey^5 + R \) with \( c,d,e \in \mathbb{Q} \) and \( R \in \mathbb{E}_{2}^{(2,1)} \)
2: let \( q \) be the double root of \( x^3 + cx^2 + dx + e \)
3: apply \( x \mapsto x + qy^2, y \mapsto y \) to \( f \)
4: return \( f \)

or on \( \Gamma(T) \), using permissable \( x \)- and \( y \)-segments, that is, by applying the transformation

\[
x \mapsto x + \frac{c_1}{2}y^{j-3}, \quad y \mapsto y
\]
or

\[
x \mapsto x, \quad y \mapsto y + \frac{c_2}{2}x^{j-3},
\]
respectively. This process is described in Algorithm 4 with \( t = c_1xy^{j-1} \) or \( t = c_2yx^{j-1} \), respectively. Now, if \( n,m \neq j \), the monomials \( x^j \) and \( y^j \) do not occur in \( \text{supp}(f) \). Otherwise, by Lemma 19 and Theorem 16, \( f \) would be of a different type than assumed. No other monomials can occur in \( \text{supp}(f) \setminus \text{supp}(T) \) underneath or on \( \Gamma(T) \).

Algorithm 4

Input: \( f \in \mathbb{K}[x,y] \) of corank 2, modality 1 and type \( T \), and a term \( t \notin \text{supp}(T) \) of \( f \) such that either

(a) \( t \in \text{Jac}(f), |\text{supp}(T,j)| = 1, \text{supp}(\text{jet}(f,j-1)) = \text{supp}(T,j-1) \) where \( j = \deg(t) \),

(b) \( t \) is in a permissable chain, and \( \text{supp}(\text{jet}(f,d)) = \text{supp}(T,d) \) where \( d \) is the maximum filtration of \( f \), or

(c) \( t \in \text{Jac}(f), j := w(T)\text{-deg}(t) > d(T), \text{supp}(w(T)\text{-jet}(f,j-1)) = \text{supp}(w(T)\text{-jet}(T,j-1)) \), and \( j \) is smaller than the \( w(T)\text{-degree} \) of the unique element in Arnold’s system.

Output: \( g \in \mathbb{K}[x,y] \) such that \( f \sim g \) and \( \text{supp}(g) \cap \text{supp}(t) = \emptyset \)

1: let \( f_0 \) be piecewise quasihomogeneous part of \( f \) of degree \( d(T) \) w.r.t. \( w(T) \)
2: if \( t \in \text{Jac}(f) \) then
3: \((u_1,u_2) := ((1,1),(1,1)) \)
4: else
5: \((u_1,u_2) := w(T) \)
6: let \( m_x \) be the term of \( \frac{\partial f_0}{\partial x} \) of lowest \( u_2\text{-degree} \)
7: let \( m_y \) be the term of \( \frac{\partial f_0}{\partial y} \) of lowest \( u_1\text{-degree} \)
8: if \( m_x|t \) then
9: \[ \alpha : \mathbb{K}[x,y] \rightarrow \mathbb{K}[x,y] \]
\[
x \mapsto x - t/m_x
\]
\[
y \mapsto y
\]
10: else
11: \[ \alpha : \mathbb{K}[x,y] \rightarrow \mathbb{K}[x,y] \]
\[
x \mapsto x
\]
\[
y \mapsto y - t/m_y
\]
12: \( g := \alpha(f) \)
13: return \( g \)
Step (III): Elimination of monomials above the Newton polygon. We first consider the case where $\Gamma(T)$ has one face. Here we use the algorithmic method that was used in the proof of Theorem 15 to eliminate the monomials above $\Gamma(T)$ in $\text{Jac}(f)$. Let $\{m_1\}$ be the Arnold system for the singularity under consideration. Let $d'_w$ be the degree of $m_1$. Again we iteratively consider the $w$-degree $j$ part of $f$ in $\text{Jac}(f)$, for increasing $j$ with $d_w < j \leq d'_w$. This polynomial can be written as

$$\frac{\partial f_0}{\partial x} v_1 + \frac{\partial f_0}{\partial y} v_2,$$

where $f_0$ is the quasihomogeneous part of $f$ and $v_1, v_2 \in \mathbb{R}[x, y]$. After applying the transformation $x \mapsto x - v_1$, $y \mapsto y - v_2$ to $f$, the $w$-degree $j$ part has no terms of degree $j$ in $\text{Jac}(f)$. Note that for $d_w < j < d'_w$, all terms of $w$-degree $j$ are in $\text{Jac}(f)$. In all the cases we consider, each weighted diagonal above $\Gamma(T)$ of $w$-degree less than or equal to $d'_w$ will only contain one monomial in $\text{Jac}(f)$. Hence, either $v_1$ or $v_2$ are zero. We therefore can apply Algorithm 4 to remove this monomial.

The remaining $w$-degree $d'_w$ part of $f$ can be written as

$$\frac{\partial f_0}{\partial x} v_1 + \frac{\partial f_0}{\partial y} v_2 + cm_1,$$

where $v_1, v_2 \in \mathbb{R}[x, y]$ and $c \in \mathbb{R}$. Applying the transformation $x \mapsto x - v_1$, $y \mapsto y - v_2$ results in transforming the part of $f$ of weighted degree $d'_w$ to $cm_1$. A case by case analysis shows that we can always choose $v_2 = 0$. See line 22 in Algorithm 5.

Since all terms of $w$-degree greater than $d'_w$ are above the weighted determinacy, they can be deleted.

Remark 22. The question may be asked why we change the strategy, in the sense that we iterating consider the terms of increasing standard degree in the process of removing all terms underneath $\Gamma(T)$ and the terms of increasing $w$-degree above $\Gamma(T)$. The reason is that canceling terms underneath $\Gamma(T)$ can only be done using transformations of negative filtration with respect to $w$, since the terms underneath $\Gamma(T)$ have negative $w$-degree. This means that in the process of deleting monomials of a given $w$-degree, monomials of lower and higher $w$-degree may be created. Since the filtration of a transformation with respect to the standard degree cannot be negative, this is not the case when canceling terms by increasing standard degree via the above methods. However, terms above $\Gamma(T)$ are canceled using transformations with non-negative filtration with respect to $w$ via the above method and therefore we consider terms by increasing $w$-degree in this case, which leads to a much simpler process.

In the case where $\Gamma(T)$ has two faces, Theorem 19 allows us to delete all monomials above $\Gamma(T)$.

Step (IV): Scaling and reading off the desired information. Lastly we scale $f$ such that the coefficients of the non-moduli terms coincide with those in one of the real subtypes of the real main type of $f$, read off the needed information and determine all normal form equations in the equivalence class of $f$, using [Marais and Steenpaß 2015b].

Handling the case $\tilde{Y}_r$. In case $f$ is of real main type $\tilde{Y}_r$, we can follow the general algorithm (Algorithm 5) to compute its complex equivalence class. Determining the number of real roots of the initial input polynomial, the real class can then be read off using [Marais and Steenpaß 2015b, Lemma 27]. Unavoidably, a complex field extension is already needed in line 3 of the general algorithm. Again, this results in problems for the implementation of the algorithm. We discuss this case in detail in Section 8.

4. Implementing the Exceptional Cases

For the exceptional singularities (see Figures 1, 2, and 3), the general algorithm can be implemented in a straightforward manner. As an example, we consider singularities of real main type $E_{14}$, and give all details in Algorithm 6. In this case, we have $w = (8, 3)$, $d = 3$, $d_w = 24$, ...
Algorithm 5 Classification and Determination of Parameter: General Structure

Input: \( f \in \mathbb{m}^3 \subset \mathbb{Q}[x, y] \), a germ of modality 1 and corank 2

Output: the real singularity types and normal forms of \( f \) with the respective parameter given as the unique root of its minimal polynomial over \( \mathbb{Q} \) in a specified interval

I. Determine the complex type:
1: compute the complex type \( T \) of \( f \) using classify() from \(^{9}\text{Krüger (1997)}\)
2: let \( d \) be the maximum filtration of \( f \) w.r.t. the standard grading
3: apply a linear coordinate change to \( f \) such that \( \text{supp}(\text{jet}(f, d)) = \text{supp}(T, d) \)
4: \( w := (w_1, \ldots, w_n) := w(T) \)
5: \( d_w := d(T) \)
6: if \( n = 1 \) then
7: \hspace{1em} for \( j = d + 1, d + 2, \ldots \) while \( \text{supp}(f) \cap (\mathbb{R}^2 \backslash \Gamma_+(T)) \neq \emptyset \) do
8: \hspace{2em} while \( f \) has a term \( t \) of standard degree \( j \) do
9: \hspace{3em} replace \( f \) by the output of Algorithm \(^{4}\) applied to \( f \) and \( t \)
10: \hspace{2em} apply a weighted homogeneous transformation to \( f \) such that \( \text{supp}(w\text{-jet}(f, d_w)) = \text{supp}(w\text{-jet}(T, d_w)) \)
11: else
12: \hspace{1em} apply a \( w \)-weighted homogeneous transformation to \( f \) such that \( \text{supp}(w_2\text{-jet}(f, d_w)) = \text{supp}(w_2\text{-jet}(T, d_w)) \)
13: \hspace{1em} for \( j = d + 1, d + 2, \ldots \) while \( \text{supp}(f) \cap (\mathbb{R}^2 \backslash \Gamma_+(T)) \neq \emptyset \) do
14: \hspace{2em} while \( f \) has a term \( t \) of the form \( bxy^{-1} \) or \( byx^{-1} \) underneath or on \( \Gamma(T) \) do
15: \hspace{3em} replace \( f \) by the output of Algorithm \(^{4}\) applied to \( f \) and \( t \)
III. Eliminate the monomials above \( \Gamma(T) \) which are not in \( \text{supp}(T) \):
16: if \( n = 1 \) then
17: \hspace{1em} let \( d'_w \) be the \( w \)-degree of the unique element in Arnold’s system
18: \hspace{1em} for \( j = d_w + 1, \ldots, d'_w \) do
19: \hspace{2em} while \( f \) has a term \( t \in \text{Jac}(g) \) of \( w \)-degree \( j \) do
20: \hspace{3em} replace \( f \) by the output of Algorithm \(^{4}\) applied to \( f \) and \( t \)
21: \hspace{2em} \( f := w\text{-jet}(f, d'_w) \)
22: \hspace{1em} modulo \( \text{Jac}(f) \), write the sum of the terms of \( f \) above \( \Gamma(T) \) in terms of Arnold’s system
23: else
24: \hspace{1em} \( f := w\text{-jet}(f, d_w) \)
IV. Scale and read off the desired information:
25: apply a transformation of the form \( x \mapsto \pm x, \ y \mapsto \pm y \) to \( f \) such that the signs of the coefficients of the non-moduli monomials match the signs of one possible real normal form \( F \) in Table \(^{1}\)
26: read off the corresponding type of \( F \)
27: apply a transformation of the form \( x \mapsto \lambda_1 x, \ y \mapsto \lambda_2 y \) with \( \lambda_1, \lambda_2 > 0 \) in a real algebraic extension of \( \mathbb{Q} \) to transform the non-moduli terms to the terms of \( F \)
28: determine the minimal polynomial over \( \mathbb{Q} \) for the parameter in \( F \)
29: using \(^{10}\text{Marais and Steenpaß (2015b)}\), determine all possible types and corresponding parameters
30: return all types, normal forms, and parameters

\[ d'_w = 26, \ m_0 = x^3, \ m_1 = xy^6 \] and \( f_0 = x^3 + y^8 \) in the general algorithm. Algorithm \(^{6}\) corresponds directly to our implementation of the respective case in \textsc{SINGULAR}. Note that, although the scaling step in line \(^{27}\) of Algorithm \(^{5}\) has not been implemented explicitly, it has been taken
into account in Algorithm 6 when determining the minimal polynomial. We use the following notation.

**Notation 23.** Suppose \( p \in \mathbb{Q}[z] \) is a univariate polynomial over the rational numbers, and \( I \subset \mathbb{R} \) is an interval such that there is exactly one \( a \in I \) with \( p(a) = 0 \). We then denote the monic irreducible divisor of \( p \) with root \( a \) by \( m_I(p) \).

![Figure 1. Exceptional Singularities of type W](image1)

**Figure 1.** Exceptional Singularities of type W

![Figure 2. Exceptional Singularities of type E](image2)

**Figure 2.** Exceptional Singularities of type E

When denoting real subcases, we identify + with \(+1\) and − with \(-1\). For example, we use the notation \( E_{14}^{\text{sign}(b)} \) for \( E_{14}^+ \) if \( b > 0 \), and \( E_{14}^- \) if \( b < 0 \).

5. **Parabolic Singularities**

In this section, we give details on the application of the general algorithm (Algorithm 5) for the parabolic singularities. See Figure 4 for these cases.
5.1. The $J_{10}$ case. In this case we have $w = (6, 3)$ and $d_w = d_w' = 18$. As discussed in Section 3 lines 7 - 9 in the general algorithm are redundant. Moreover, by $d_w = d_w'$, Step (III) is redundant. Unfortunately we may need a real field extension in line 10 of the general algorithm. This leads to an implementational problem for the subsequent steps of the algorithm, since if we represent an algebraic number field as $\mathbb{Q}[z]/m$, with $m$ irreducible, we have to determine which root of $m$ the generator $z$ corresponds to. In this section, we will work around this problem by presenting a method to read off the required information without explicitly doing the required weighted linear transformation, see Algorithm 7.

In Marais and Steenpaß (2015b) it has been shown that in the case of real main type $J_{10}$, the equivalence class of $f$ either contains exactly one normal form equation of type $J_{10}^+$ or it contains exactly three normal form equations, two of which are of type $J_{10}^+$ and one is of type $J_{10}^-$. This shows, in particular, that the real subtype $J_{10}^-$ is redundant. The following algorithmic approach to the classification also reconfirms these findings.

Let $f \in m^3$ be a polynomial of complex type $J_{10}$ such that $f \in E_{18}^{(6,3)}$, that is, it is of the form

$$f = cx^3 + bx^2y^2 + dxy^4 + ey^6 + R, \quad R \in E_{19}^{(6,3)}.$$
Algorithm 6: Algorithm for the case $E_{14}$

**Input:** $f \in \mathbb{m}^3 \subset \mathbb{C}[x,y]$ of complex singularity type $T = E_{14}$

**Output:** all normal form equations in the right equivalence class of $f$, each specified as a tuple of the real singularity type, the normal form, the minimal polynomial of the parameter, and an interval such that the parameter is the unique root of the minimal polynomial in this interval

**II. Eliminate the monomials in $\text{supp}(f)$ underneath or on $\Gamma(T)$:**

Apply Algorithm 1:

1: $h := \text{jet}(f, 3)$
2: factorize $h$ as $h_1 \cdot c$, where $h_1$ is homogeneous of degree one and $c \in \mathbb{Q}^*$
3: if $h \neq c' y, c' \in \mathbb{Q}$ then
4: apply $h_1 \mapsto x, y \mapsto y$ to $f$
5: else
6: apply $h_1 \mapsto x, x \mapsto y$ to $f$
7: $g := \frac{1}{c} f$

Consider the terms of $g$ of standard degree $j = 4$ and apply Algorithm 4:

8: $h_1 := \text{jet}(g, 4) - \frac{x^5}{3x^2}$
9: apply $x \mapsto x - h_1, y \mapsto y$ to $g$

Consider the terms of $g$ of standard degree $j = 5$ and apply Algorithm 4:

10: $h_2 := \text{jet}(g, 5) - \frac{x^5}{3x^2}$
11: apply $x \mapsto x - h_2, y \mapsto y$ to $g$

**III. Eliminate the monomials above $\Gamma(T)$ which are not in $\text{supp}(T)$:**

12: $w := (8, 3)$

Consider the terms of $g$ of $w$-degree $j = 25$ and apply Algorithm 4:

13: $h'_1 := \frac{w \cdot \text{jet}(g, 25) - w \cdot \text{jet}(g, 24)}{3x^2}$
14: $x \mapsto x - h'_1, y \mapsto y$
15: $f := cg$

**IV. Read off the desired information:**

16: if $c < 0$ then
17: apply $x \mapsto -x, y \mapsto y$ to $f$
18: write $f$ as $f = cx^3 + by^8 + R$ with $c \in \mathbb{Q}_{>0}$, $b \in \mathbb{Q}^*$, and $R \in E_{25}^{(8,3)}$
19: $t := \text{coeff}(f, xy^6)$
20: $p := z^{12} - c^{-4} |b|^{-9} t^{12} \in \mathbb{Q}[z]$
21: if $t > 0$ then
22: return $(E_{14}^{(b)}, x^3 + \text{sign}(b) \cdot y^8 + axy^6, m_{(0,\infty)}(p), (0, \infty))$
23: else
24: return $(E_{14}^{(b)}, x^3 + \text{sign}(b) \cdot y^8 + axy^6, m_{(-\infty,0]}(p), (-\infty, 0])$

Since $f$ is weighted 18-determined, $f \sim f - R$. Hence, we may assume that

$$f = cx^3 + bx^2 y^2 + dxy^4 + cy^6.$$ 

If $c < 0$, then we apply the transformation

$$x \mapsto -x, \quad y \mapsto y.$$
Since $f$ is weighted homogeneous, a rescaling of the coordinates achieves that $c = 1$. By applying
\[ x \mapsto x - \frac{b}{3} y^2, \quad y \mapsto y, \quad (3) \]
$f$ is transformed to a polynomial of the form \[ f = x^3 + dx y^4 + ey^6. \]
Since, on the other hand, for $a', d' \in \mathbb{R}$ we have
\[ x^3 + a' x^2 y^2 \pm |d'| xy^4 \sim x^3 + a x^2 y^2 \pm xy^4, \quad a = \frac{a'}{\sqrt{|d'|}}, \quad (4) \]
via the invertible transformation
\[ \alpha' : x \mapsto x, \quad y \mapsto \frac{1}{\sqrt{|d'|}} y \]
(as applied in Step (IV) of the general algorithm), the problem is reduced to finding a transformation $\alpha$ such that $f = x^3 + dx y^4 + ey^6$ is mapped to
\[ \alpha(f) = x^3 + a' x^2 y^2 \pm |d'| xy^4, \quad a', d' \in \mathbb{R}. \quad (5) \]
The composition of $\alpha$ with the transformation in (3) and the scaling of the coordinates, as described above, is then a suitable transformation to be applied in line 10 of the general algorithm.

In the following, we describe a method to determine all polynomials as in (5) in the equivalence class of $f$. Since $\alpha$ has to be weighted homogeneous, it is of the form
\[ \alpha : x \mapsto x + cy^2, \quad y \mapsto ty \quad (6) \]
with appropriate $t, c \in \mathbb{R}, t \neq 0$. We now determine all possible values for $t$ and $c$. Applying a transformation as in (6) to $f$, we obtain
\[ \alpha(f) = x^3 + 3cx^2y^2 + (3c^2 + t^4 d)xy^4 + (c^3 + t^4 dc + et^6)y^6. \quad (7) \]
Taking $c' = \frac{c}{\sqrt{|d|}}$, we can rewrite (7) as
\[ \alpha(f) = x^3 + 3t^2 c'^2 x^2 y^2 + t^4(3c'^2 + d)xy^4 + t^6(c'^3 + dc' + c)y^6. \quad (8) \]
Clearly, for a fixed value $t \neq 0$, $c'$ is any real root of $k := s^3 + ds + c \in \mathbb{Q}[s]$. Due to the scaling of $y$ by $\alpha'$, we may assume that $t = 1$ and $c = c'$.

To determine all possible normal form equations in the equivalence class of $f$, we have to consider the following cases:

(i) $k(s)$ has one real root;
(ii) $k(s)$ has three real roots.

Denote by $c_1$, $c_2$ and $c_3$ the complex roots of $k(s)$. Then
\[ c_1 + c_2 + c_3 = 0, \quad c_1 c_2 + c_1 c_3 + c_2 c_3 = d, \quad \text{and} \quad c_1 c_2 c_3 = e. \quad (9) \]

(i) Let $c = c_1$ be the real root and let $c_2$ and $c_3$ be the complex conjugate roots of $k(s)$. It follows from (9) that $c_2 c_3 = d - c_1^2$. Since the product of two non-zero complex conjugates is positive, $d + c_1^2 > 0$, which implies $d + 3c_1^2 > 0$. Hence, considering (5), $f$ is of type $J_{10}^*$. Since the real root $c_1$ is uniquely determined, the transformation $\alpha$ as considered above and the real subtype are also uniquely determined. In particular, $a = \frac{3c_1}{\sqrt{3c_1^2 + d}}$ is uniquely determined. Define \[ a_1 := a, \quad a_2 := \frac{3c_2}{\sqrt{3c_2^2 + d}}, \quad \text{and} \quad a_3 := \frac{3c_3}{\sqrt{3c_3^2 + d}}. \quad \text{Note that} \quad da_j^2 = c_j^3(9 - 3a_j^2). \quad \text{Since} \quad c_1, c_2, \quad \text{and} \quad c_3 \quad \text{are} \quad \text{the} \quad \text{roots} \quad \text{of} \quad k(s), \quad \text{they} \quad \text{are} \quad \text{also} \quad \text{roots} \quad \text{of} \quad \text{of} \quad \hat{k}(s) := -k(s) \cdot k(-s) = s^6 + 2ds^4 + d^2s^2 - e^2. \quad \text{Multiplying} \quad \hat{k}(s) \quad \text{with} \quad (9 - 3a_j^2)^3, \quad \text{we} \quad \text{see} \quad \text{that} \quad da_j^2 \quad \text{is} \quad \text{a} \quad \text{root} \quad \text{of} \quad \hat{h}_j(w) = w^3 + 2dw^2(9 - 3a_j^2) + d^2 w(9 - 3a_j^2)^2 - e^2(9 - 3a_j^2)^3 \in \mathbb{Q}[w].
that is, \( \pm a_j, j = 1, 2, 3 \), are the roots of

\[
h(z) := (4d^3 + 27e^2)z^6 + (-36d^3 - 243e^2)z^4 + (81d^3 + 729e^2)z^2 - 729e^2 \in \mathbb{Q}[z].
\]

By \( e = c_1c_2c_3 = 0 \) and \( c_2, c_3 \neq 0 \), it follows that

\[
e = 0 \iff c_1 = 0 \iff a = 0.
\]

We show that this is the case if and only if \( a \in \mathbb{R} \) or \( a_3 \in \mathbb{R} \). Assume that, without loss of generality, \( a_2 \) is real. Then \( a_2^2 = \frac{9e^2}{36 - 729} \) is real, which implies that \( 3c_2^2 + d = \lambda c_2^2 \) for some \( \lambda \in \mathbb{R} \). Therefore \( d = \lambda' c_2^2 \) for some \( \lambda' \in \mathbb{R} \). Since \( d \in \mathbb{R} \), it follows that \( c_2^2 \in \mathbb{R} \). Because \( c_2 \in \mathbb{C} \setminus \mathbb{R} \), it follows that \( c_2 = \gamma i \) for some \( \gamma \in \mathbb{R} \), hence \( c_3 = -\gamma i \). This implies that \( a_3 \) is real, since \( d = c_2c_3 = \gamma^2 \). By \( [9] \) we obtain \( c_1 = 0 \). For the converse, note that if \( c_1 = 0 \), then \( c_2 = \gamma i \) and \( c_3 = -\gamma i \), hence \( a_2 \) and \( a_3 \) are real.

We now consider the case where \( e \neq 0 \). Using the fact that \( c_2c_3 \) is positive and \( e = c_1c_2c_3 \), it follows that

\[
\text{sign}(e) = \text{sign}(c_1) = \text{sign}(a).
\]

Hence, a minimal polynomial for \( a \) is obtained as the monic irreducible factor of \( h(x) \) with a root in \((0, \infty)\) in case \( e > 0 \), or in \((-\infty, 0)\) in case \( e < 0 \). Moreover, \( a \) is the unique root of this factor in this interval.

(ii) In the case where \( k(s) \) has three real roots, they must be pairwise different, otherwise \( k(s) \) and its derivative \( k'(s) \) have a common root, which implies that \( f \) is degenerate, since the coefficient of \( xy^2 \) in \([8]\) vanishes for \( c \) a double root of \( k(s) \). Without loss of generality, we can assume that \( c_1 < c_2 < c_3 \). Therefore \( a \) can attain the three different values

\[
a_j := \frac{3c_j}{\sqrt{|d + 3c_j^2|}}, \quad j = 1, 2, 3.
\]

In case \( c_2 = 0 \), we have \( c_3 = -c_1 = \sqrt{-d} \) with \( d < 0 \), which implies that \( a_1 = -\frac{3}{\sqrt{2}} \), \( a_2 = 0 \), and \( a_3 = \frac{3}{\sqrt{2}} \). Considering the sign of \( 3c_2^2 + d \), we obtain that \( c_1 \) and \( c_3 \) correspond to type \( J_{10}^- \) and \( c_2 \) corresponds to \( J_{10}^0 \).

Next we consider the case \( c_2 \neq 0 \). Since \( c_1 + c_2 + c_3 = 0 \), not all three roots have the same sign. Suppose \( -\text{sign}(c_1) = \text{sign}(c_2) = \text{sign}(c_3) \). Note that \( |c_1| = |c_2 + c_3| = |c_2| + |c_3| \), that is, \( |c_1| > |c_2| \) and \( |c_1| > |c_3| \). Then \( 3c_1^2 + d = 3c_1^2 + c_1c_2 + c_1c_3 + c_2c_3 > 0 \), since \( |c_1^2| > |c_1c_3| \) and \( |c_1^2| > |c_1c_2| \), and, moreover, \( c_1^2 > 0 \), \( c_1c_2 < 0 \), \( c_1c_3 < 0 \), and \( c_2c_3 > 0 \). Hence, \( c_1 \) corresponds to type \( J_{10}^- \). Furthermore

\[
3c_2^2 + d = 3c_2^2 + c_2(c_2 + c_3) + c_2c_3 = 3c_2^2 - (c_2 + c_3)(c_2 + c_3) + c_2c_3 = 3c_2^2 - (c_2^2 + c_2c_3 + c_3^2),
\]

for \( j = 2, 3 \). Since \( |c_2| < |c_3| \), we obtain that \( c_2 \) corresponds to type \( J_{10}^- \) and \( c_3 \) to \( J_{10}^+ \). The remaining case \( -\text{sign}(c_1) = -\text{sign}(c_2) = \text{sign}(c_3) \) can be treated in a similar manner, leading again to types \( J_{10}^+ \), \( J_{10}^- \), and \( J_{10}^0 \) corresponding to \( c_1, c_2 \), and \( c_3 \).

We now determine the corresponding values of the moduli parameter. We first consider the case \( J_{10}^- \). Note that

\[
-d(a_2)^2 = c_2^2(9 + 3(a_2)^2) \quad \text{and} \quad -d(ia_2)^2 = c_2^2(9 + 3(ia_2)^2), \quad j = 1, 3.
\]

By multiplying \( \tilde{k}(s) \) with \( 9 + 3(ia_2)^2 \), for \( j = 1, 3 \), and with \( 9 + 3a_2^2 \), for \( j = 2 \), it follows that the roots of

\[
h^-(z) := (-4d^3 - 27e^2)z^6 + (-36d^3 - 243e^2)z^4 + (-81d^3 - 729e^2)z^2 - 729e^2 \in \mathbb{Q}[z]
\]

are \( \pm ia_1, \pm a_2, \) and \( \pm ia_3 \). Since \( \text{sign}(c_1) = -\text{sign}(c_3) \), we have

\[
\text{sign}(a_2) = \text{sign}(c_2) = -\text{sign}(e),
\]

which determines an appropriate interval containing \( a_2 \).
For the parameters which correspond to $J_{10}^+$, we can construct, in a similar manner, the polynomial

$$h^+(z) := (4d^3 + 27e^2)z^9 + (-36d^3 - 243e^2)z^4 + (81d^3 + 729e^2)z^2 - 729e^2 \in \mathbb{Q}[z]$$

with roots $\pm a_1$, $\pm a_2$, and $\pm a_3$. Since, in these cases, either $|c_2| < |c_1|$ or $|c_2| < |c_3|$, it follows that $d < 0$. An easy calculation shows that $|c_1| < |c_3|$ if and only if $|a_3| < |a_1|$.

If $e > 0$, that is, $c_2 < 0$, we have that $|c_1| + |c_2| = |c_1 + c_2| = |c_3|$ and, hence, that $|a_3| < |a_1|$. Therefore, $a_1$ is the smallest negative real root and $a_3$ is the smallest positive real root of $h^+(z)$. Similarly, if $e < 0$, then $|a_1| < |a_3|$, hence $a_1$ is the largest negative real root, and $a_3$ the largest positive real root of $h^+(z)$.

**Remark 24.** With regard to the implementation, we use the SINGULAR library `solve.lib` [Wenk and Pohl 1999] and Sturm chains, as implemented in the SINGULAR library `rootsur.lib` [Tobis 2012], to determine intervals with rational boundaries containing the roots.

5.2. The $X_9$ case. According to Theorem 29 in Marais and Steenpaß (2015b), the real right equivalence class of a singularity of real main type $X_9$ always contains exactly two normal form equations from Arnold’s list, of possibly different real subtypes. There are four different possible cases for a given singularity of real main type $X_9$:

(A) The singularity is right equivalent to $\text{NF}(X_9^{++}) (a)$ for two different values of the parameter $a$ with $a > -2$.

(B) The singularity is right equivalent to $\text{NF}(X_9^{--}) (a)$ for two different values of the parameter $a$ with $a < 2$.

(C) The singularity is right equivalent to both $\text{NF}(X_9^{+-}) (a)$ and $\text{NF}(X_9^{-+}) (a)$ for some unique value $a \in \mathbb{R}$ of the parameter.

(D) The singularity is right equivalent to $\text{NF}(X_9^{++}) (a)$ for some value $a < -2$ of the parameter and to $\text{NF}(X_9^{--}) (a)$ for some $a > 2$.

In the following, we discuss how Algorithm 8 determines in which of these cases a given singularity falls, and which are the corresponding values of the parameter. We do not strictly follow Algorithm 8 because line 3 would, in general, require working over algebraic extensions. Due to this difficulty, especially Step (IV) needs a specific approach.

First of all, the determinacy of a given polynomial $f \in \mathbb{Q}[x, y]$ of complex singularity type $X_9$ is 4, thus it suffices to consider $\text{jet}(f, 4)$. Some calculations in linear algebra show that we can assume the coefficient of $x^4$ to be non-zero, by applying a linear coordinate transformation as in line 2 of Algorithm 8 if necessary. For convenience, we can then get rid of the term $x^2 y$ by the transformation given in line 3 such that $f$ is of the form

$$f = b x^4 + c x^2 y^2 + d x y^3 + e y^4$$

with $b, c, d, e \in \mathbb{Q}$ and $b \neq 0$.

The number of real roots of $f(x, 1) = b x^4 + c x^2 + d x + e$, which geometrically correspond to the roots of the strict transform on the exceptional divisor of the blow-up at the origin, is invariant under right equivalence and can thus be used as a first step to distinguish the four cases mentioned above. In fact, in the cases (A) and (B) the polynomial $f(x, 1)$ has no real roots, in the case (C) it has two real roots, and in the case (D) it has four real roots. Furthermore, the two cases (A) and (D) can be distinguished by the sign of the coefficient of $x^4$ because it is invariant under right equivalence if $f$ is of either one of these cases.

It remains to determine the possible values of the parameter $a$. Using the techniques from Section 5.1 in Marais and Steenpaß (2015b), one can determine, for each real subtype, a polynomial whose coefficients depend on those of $f$ and whose roots are precisely the possible values of the parameter in the normal form equations to which $f$ is complex right equivalent. In Algorithm 8 this is the polynomial $p^{(\sigma)}$ defined in line 5 with $\sigma = +1$ for the subtypes $X_9^{++}$ and $X_9^{--}$, and with $\sigma = -1$ for $X_9^{+-}$ and $X_9^{-+}$.

However, it turns out that $p^{(\sigma)}$ does not always factorize into linear factors over $\mathbb{Q}$ and that the number of its real roots within the intervals specified in the cases (A) to (D) above is larger.
Algorithm 7 Algorithm for the case $J_{10}$

Input: $f \in m^3 \subset \mathbb{Q}[x, y]$ of complex singularity type $J_{10}$

Output: all normal form equations in the right equivalence class of $f$, each specified as a tuple of the real singularity type, the normal form, the minimal polynomial of the parameter, and an interval such that the parameter is the unique root of the minimal polynomial in this interval

1: $f := \text{jet}(f, 6)$
2: if $\text{coeff}(f, x^3) = 0$ then
3: \quad apply $x \mapsto y$, $y \mapsto x$ to $f$
4: $h := \text{jet}(f, 3)$
5: factorize $h$ as $bg^3$ with $b \in \mathbb{Q}$, $b > 0$, and $g_1$ homogeneous of degree 1
6: apply $x \mapsto g_1$, $y \mapsto y$ to $f$
7: $f := \text{coeff}(f, x^3)f$
8: write $f$ as $f = x^3 + ax^2y^2 + dxy^4 + ey^6$ with $a, d, e \in \mathbb{Q}$
9: apply $x \mapsto x - \frac{a}{2}y^2$, $y \mapsto y$ to $f$
10: write $f$ as $f = x^3 + dxy^4 + ey^6$ with $d, e \in \mathbb{Q}$
11: $p^{(\sigma)} := \sigma(4d^3 + 27e^2)z^6 + (-36d^3 - 243e^2)z^4 + \sigma(81d^3 + 729e^2)z^2 - 729e^2 \in \mathbb{Q}[z]$
12: $k := s^3 + ds + e \in \mathbb{Q}[s]$
13: if $k$ has exactly one real root then
14: \quad if $e > 0$ then
15: \quad \quad \quad return $(J_{10}^{+}, x^3 + ax^2y^2 + xy^4, m_{(0, \infty)}(p^+), (0, \infty))$
16: \quad \quad \quad if $e < 0$ then
17: \quad \quad \quad return $(J_{10}^{+}, x^3 + ax^2y^2 + xy^4, m_{(-\infty, 0)}(p^+), (-\infty, 0))$
18: \quad \quad \quad if $e = 0$ then
19: \quad \quad \quad return $(J_{10}^{+}, x^3 + ax^2y^2 + xy^4, z, [0, 0])$
20: \quad else
21: \quad \quad if $k(0) = 0$ then
22: \quad \quad \quad $C_1 := (J_{10}^{+}, x^3 + ax^2y^2 + xy^4, z^2 - \frac{2}{3}, (-\infty, 0))$
23: \quad \quad \quad $C_2 := (J_{10}^{+}, x^3 + ax^2y^2 - xy^4, z, [0, 0])$
24: \quad \quad \quad $C_3 := (J_{10}^{+}, x^3 + ax^2y^2 + xy^4, z^2 - \frac{2}{3}, (0, \infty))$
25: \quad \quad else
26: \quad \quad \quad $\varepsilon := 1$
27: \quad \quad \quad do
28: \quad \quad \quad replace $\varepsilon$ by a rational number in the interval $(0, \frac{\pi}{2})$
29: \quad \quad \quad let $(z_1, \ldots, z_4) \in \mathbb{Q}^4$ be approximations of the four distinct real roots of $p^+$ in increasing order with error smaller than $\varepsilon$
30: \quad \quad \quad $(I_1, \ldots, I_4) := ((z_1 - \varepsilon, z_1 + \varepsilon), \ldots, (z_4 - \varepsilon, z_4 + \varepsilon))$
31: \quad \quad \quad while $p^+$ has more than one real root in any of the intervals $I_1, \ldots, I_4$
32: \quad \quad \quad if $e > 0$ then
33: \quad \quad \quad \quad $C_1 := (J_{10}^{+}, x^3 + ax^2y^2 - xy^4, m_{(-\infty, 0)}(p^-), (-\infty, 0))$
34: \quad \quad \quad \quad $C_2 := (J_{10}^{+}, x^3 + ax^2y^2 + xy^4, m_{I_1}(p^+), I_1)$
35: \quad \quad \quad \quad $C_3 := (J_{10}^{+}, x^3 + ax^2y^2 + xy^4, m_{I_2}(p^+), I_2)$
36: \quad \quad \quad \quad else
37: \quad \quad \quad \quad $C_1 := (J_{10}^{+}, x^3 + ax^2y^2 - xy^4, m_{I_3}(p^-), (0, \infty))$
38: \quad \quad \quad \quad $C_2 := (J_{10}^{+}, x^3 + ax^2y^2 + xy^4, m_{I_4}(p^+), I_3)$
39: \quad \quad \quad \quad $C_3 := (J_{10}^{+}, x^3 + ax^2y^2 + xy^4, m_{I_4}(p^+), I_4)$
40: \quad \quad \quad return $C_1, C_2, C_3$
than the number of admissible values for the parameter. This is due to the fact that \( p(\sigma) \) also takes into account complex transformations. In other words, for some of the real roots of \( p(\sigma) \), there is no real transformation which takes \( f \) to the respective normal form equation where the value of the parameter is that root.

We use the following method to solve this problem. Considering, again, Theorem 29 in Marais and Steenpaß (2015b), one may observe that the real roots of \( p(\sigma) \) lie in fixed disjoint intervals. In the case \([A]\), for example, the polynomial \( p(\sigma) \) has exactly one real root in each of the intervals \((-2,0)\), \((0,2)\), \((2,6)\), and \((6,\infty)\) (and two more in \((-\infty,-6)\) and \((-6,-2)\), which are not admissible values for the parameter in this case), or it has two double roots at 0 and \(6\) (and one more double root at \(-6\), which we do not consider either). According to that theorem, the two roots which are admissible values for the parameter are either those in \([0,2)\) and \((2,6]\) or those in \((-2,0)\) and \((6,\infty)\).

Using the techniques from Section 5.3 in Marais and Steenpaß (2015b), we set up, for a generic parameter \(a\), the ideal of transformations which take \(f\) to the normal form equation with this parameter. Note that a real point in the vanishing set of this ideal corresponds to a real transformation of \(f\). We can thus determine whether there exists a real transformation which maps \(f\) to a normal form equation where the parameter lies in a specific interval by applying Algorithm 12.8 from Basu et al. (2008). Continuing with case \([A]\) if there exists a real transformation for the real root of \(p(\sigma)\) in \([0,2)\), then the two admissible values of the parameter are given by this root and the one in \((2,6]\), otherwise they are given by the two roots of \(p(\sigma)\) in \((-2,0)\) and \((6,\infty)\), see lines \(5\) - \(14\) in Algorithm \(5\). The cases \([B]\), \([C]\) and \([D]\) are treated in an analogous way.

6. Hyperbolic Singularities

In this section, we give details on the application of the general algorithm (Algorithm \(5\)) for the hyperbolic singularities. See Figure 5 for these cases.

![Hyperbolic Singularities](image)

**Figure 5.** Hyperbolic Singularities

6.1. An example for the general algorithm. In Algorithm \(9\) we demonstrate the general algorithm in the cases \(X_{9+k}\). Note that the Milnor number \(\mu\) of the input polynomial \(f\) can be easily computed, using Gröbner basis techniques, and that \(k = \mu - 9\), see Arnold et al. (1985).

Also note that \(E_{5} \cap E_{d(\mu-7,2)}\) is the vector space generated by all monomials above the Newton polygon. In line \(12\) we use that if the exponent of the \(y^{k+4} = y^{\mu-5}\) term is odd, there are two
Algorithm 8 Algorithm for the case $X_9$

Input: $f \in m^3 \subset \mathbb{Q}[x, y]$ of complex singularity type $T = X_9$

Output: all normal form equations in the right equivalence class of $f$, each specified as a tuple of the real singularity type, the normal form, the minimal polynomial of the parameter, and an interval such that the parameter is the unique root of the minimal polynomial in this interval

1: $f := \text{jet}(f, 4)$
2: make sure that the coefficient of $x^4$ in $f$ is non-zero by applying either $(x, y) \mapsto (y, x)$, $(x, y) \mapsto (x, x + y), (x, y) \mapsto (x, 2x + y), \text{or} (x, y) \mapsto (x, 3x + y)$ to $f$ if necessary
3: apply $x \mapsto x - \frac{\text{coeff}(f, x^2) y}{\text{coeff}(f, x^2) y}$, $y \mapsto y$ to $f$
4: write $f$ as $f = bx^4 + cx^2y^2 + dxy^3 + ey^4$ with $b, c, d, e \in \mathbb{Q}$ and $b \neq 0$
5: $p(\sigma) := (-256b^3c^3 + 128b^2c^2e^2 - 144bc^2d^2e + 27b^2d^4 - 16bc^4e + 4bc^3d^2) \cdot z^6$
   \[+ \sigma (18432b^3c^3 + 11520b^2c^2e^2 - 5184bc^2d^2e + 972b^2d^4 + 144bc^4e + 16c^6) \cdot z^4\]
   \[+ (-331776b^3c^3 - 62208b^2c^2e^2 + 11664bc^2d^2e - 11520bc^4e + 1728bc^3d^2 - 128c^6) \cdot z^2\]
   \[+ \sigma (331776b^3c^3 - 248832b^2c^2e^2 + 46656bc^2d^2e - 18432bc^4e + 6912bc^3d^2 + 256c^6)\]
   \[\in \mathbb{Q}[z]\]
6: let $r$ be the number of real roots of $f(x, 1) = bx^4 + cx^2 + dx + e$
7: if $r = 0$ then
8: if $b > 0$ then
9: if $f \sim +x^4 + ax^2y^2 + y^4$ for some $a \in [0, 2)$ then
10: $I_1 := (0, 2), I_2 := (2, 6)$
11: else
12: \quad $I_1 := (-2, 0), I_2 := (6, 6)$
13: $C_1 := (X_9^{++}, +x^4 + ax^2y^2 + y^4, m_1(p^+), I_1)$
14: $C_2 := (X_9^{++}, +x^4 + ax^2y^2 + y^4, m_1(p^+), I_2)$
15: else
16: if $f \sim -x^4 + ax^2y^2 - y^4$ for some $a \in (-2, 0]$ then
17: $I_1 := [-6, -2), I_2 := (-2, 6)$
18: else
19: \quad $I_1 := (-\infty, -6), I_2 := (0, 2)$
20: $C_1 := (X_9^{--}, -x^4 + ax^2y^2 - y^4, m_1(p^-), I_1)$
21: $C_2 := (X_9^{--}, -x^4 + ax^2y^2 - y^4, m_1(p^-), I_2)$
22: if $r = 2$ then
23: if $f \sim +x^4 + ax^2y^2 - y^4$ for some $a \in (-\infty, 0)$ then
24: $I := (-\infty, 0)$
25: else
26: \quad $I := (0, \infty)$
27: $C_1 := (X_9^{++}, +x^4 + ax^2y^2 - y^4, m_1(p^-), I)$
28: $C_2 := (X_9^{++}, +x^4 + ax^2y^2 - y^4, m_1(p^-), I)$
29: if $r = 4$ then
30: if $f \sim +x^4 + ax^2y^2 + y^4$ for some $a \in (-\infty, -6]$ then
31: $I_1 := (-\infty, -6), I_2 := (2, 6)$
32: else
33: \quad $I_1 := (-6, -2), I_2 = (6, \infty)$
34: $C_1 := (X_9^{++}, +x^4 + ax^2y^2 + y^4, m_1(p^+), I_1)$
35: $C_2 := (X_9^{++}, +x^4 + ax^2y^2 - y^4, m_1(p^+), I_2)$
36: return $C_1, C_2$
possible normal form equations, which can be obtained from each other by the transformation 
\( x \mapsto x, \ y \mapsto -y \). In the algorithm we use the following notation.

**Notation 25.** If \( I \subset \mathbb{R} \) and \( \lambda \in \mathbb{R} \), then we define \( \lambda I \) as the set 
\[ \lambda I := \{ \lambda x \mid x \in I \} \]

The cases \( J_{10+k} \) can be handled in a similar manner.

**Algorithm 9** Algorithm for the case \( X_{9+k} \)

**Input:** \( f \in m^3 \subset \mathbb{Q}[x,y] \) of complex singularity type \( T = X_{9+k} \)

**Output:** all normal form equations in the right equivalence class of \( f \), each specified as a tuple of the real singularity type, the normal form, the minimal polynomial of the parameter, and an interval such that the parameter is the unique root of the minimal polynomial in this interval

**II. Eliminate the monomials in \( \text{supp}(f) \) underneath or on \( \Gamma(T) \)**

1. apply Algorithm 2 to \( f \)

   Use Algorithm 2 iteratively:
2. \( \mu := 9 + k \)
3. \( c := \text{coeff}(f, x^2 y^2) \)
4. for \( i = 4, \ldots, \left\lfloor \frac{\mu}{2} \right\rfloor \) do
5. \( t := \text{coeff}(f, x^2 y^2) \)
6. apply \( x \mapsto x - \frac{1}{2} y^{i-2}, \ y \mapsto y \) to \( f \)

**III. and IV. Discard the monomials above \( \Gamma(T) \) and read off the desired information:**

7. write \( f \) as \( f = b_0 x^4 + b_1 x^2 y^2 + b_2 y^{\mu-5} \) with \( b_0, b_1, b_2 \in \mathbb{Q}^* \) and \( R \in E_5 \cap E_{2 \mu-9}^{(\mu-7,2)} \)
8. \( T_R := X_{9+k}^{\text{sign}(b_0)}, \text{sign}(b_1) \)
9. \( F := \text{sign}(b_0) \cdot x^4 + \text{sign}(b_1) \cdot x^2 y^2 + ay^{\mu-5} \)
10. \( p := z^4 - b_2 \left( \frac{|b_0|}{|b_1|^2} \right)^{\mu-5} \)
11. \( I_1 := \text{sign}(b_2) \cdot (0, \infty), \ I_2 := \text{sign}(b_2) \cdot (-\infty, 0) \)
12. if \( \mu \) is odd then
13. return \( (T_R, F, m_{I_1}(p), I_1) \)
14. else
15. return \( (T_R, F, m_{I_1}(p), I_1), (T_R, F, m_{I_2}(p), I_2) \)

6.2. The cases \( Y_{r,s} \) and \( \tilde{Y}_r \). Recall from Section 3 that the complex type \( Y_{r,s} \) splits up into the real main types \( \tilde{Y}_r \) and \( Y_{r,s} \). Similar to the case \( D_4 \) treated in [Marais and Steenpaß (2015a)](#), we separate the case \( Y_{r,s} \) for the remaining cases using Proposition 8 from [Marais and Steenpaß (2015a)](#) and real root counting via Sturm chains. The 4-jet of functions of real type \( Y_{r,s} \) has four real roots (counted with multiplicity), while the 4-jet of functions of real type \( \tilde{Y}_r \) does not have any real roots. By the following lemma, if the given polynomial \( f \in \mathbb{Q}[x,y] \) is of real main type \( Y_{r,s} \) with \( r \neq s \), then the 4-jet of \( f \) always factors into linear factors over \( \mathbb{Q} \). Hence we can follow the general algorithm without any modification. The same may happen if \( f \) is of real main type \( Y_{r,r} \). However, if \( f \) is of real main type \( Y_{r,s} \) and the 4-jet of \( f \) does not factor into linear factors over \( \mathbb{Q} \), line 3 of Algorithm 4 requires a real algebraic field extension. Similarly, if \( f \) is of real main type \( Y_{r,s} \), line 3 requires a complex field extension. See the discussion after the lemma on how to handle this problem.

**Lemma 26.** Let \( f \in \mathbb{Q}[x,y] \) be of real main type \( Y_{r,s} \). If the 4-jet of \( f \) does not factorize into linear factors over \( \mathbb{Q} \), then \( r = s \), that is, \( f \) is of real main type \( Y_{r,r} \).
Proof. Let $f$ be of real main type $Y_{r,s}$ for some $r, s > 4$. Then $f$ is of the form
\[ f = (a_0x + a_1y)(b_0x + b_1y)^2 + \text{higher terms in } x \text{ and } y \]
where $(a_0, a_1), (b_0, b_1) \in \mathbb{R}^2$ are linearly independent. Hence, the strict transform of the blow-up of $f$ at the origin intersects the exceptional divisor in exactly two points.

Without loss of generality, we consider the chart where $x = 0$ is the local equation of the exceptional divisor. Then the strict transform is given by
\[ \tilde{f} = (a_0 + a_1y)^2(b_0 + b_1y)^2 + \text{terms that are divisible by } x. \]

Therefore, the intersection points of the strict transform with the exceptional divisor \( \{ x = 0 \} \) correspond to the zeros of the rational polynomial $p^2 = (a_0 + a_1y)^2(b_0 + b_1y)^2$. Since $\mathbb{Q}$ is a perfect field and $p^2 \in \mathbb{Q}[y]$, it follows that $p = (a_0 + a_1y)(b_0 + b_1y) \in \mathbb{Q}[y]$.

Now consider the standard charts where $x = 0$ and $y = 0$, respectively, define the exceptional divisor. We show that if the strict transform has an irrational point on the exceptional divisor $E$ in any of the two charts, then in both charts it has two irrational points on $E$. Indeed, if $a_1 = 0$ or $b_1 = 0$, then $p$ has only rational roots. Hence, by the assumption that the strict transform has an irrational point on $E$, we have $a_1 \neq 0$ and $b_1 \neq 0$, and at least one of the two roots $-\frac{a}{a_1}$, $-\frac{b}{b_1}$ is irrational. Since $p$ is the minimal polynomial for these roots, we conclude that both $-\frac{a}{a_1}$ and $-\frac{b}{b_1}$ are irrational.

Note that these roots correspond to the irrational points $(0, -\frac{a_0}{a_1})$ and $(0, -\frac{b_0}{b_1})$ of the strict transform on $E$. Hence, the points of the strict transform on the exceptional divisor are irrational if and only if $p$ does not have any rational root. This, in turn, is the case if and only if $\text{jet}(f, 4) = (a_0x + a_1y)^2(b_0x + b_1y)^2$ does not factorize into linear factors over $\mathbb{Q}$.

Thus, under the assumptions of the lemma, both roots of $p$ and both points of the strict transform on the exceptional divisor are irrational. Consider the blow-up of the normal form equation of $f$ at the origin. The germs at the two intersection points $q_1$ and $q_2$ of the strict transform with the exceptional divisor are right equivalent to $f_1 = \pm x^2 \pm y^{r-4}$ and $f_2 = \pm x^2 \pm y^{s-4}$. Since any right equivalence of a singularity induces right equivalences for each germ of the strict transform, the singularities of the strict transform of $f$ are also right equivalent to $f_1$ and $f_2$.

The Galois group of the quadratic extension $\mathbb{Q} \subset \mathbb{Q}[y]/p =: K$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Let $\kappa$ be the non-trivial element in this group, and let $\varphi_1$ and $\varphi_2$ be the $K$-algebra automorphisms which transform the germs at $q_1$ and $q_2$ to $f_1$ and $f_2$, respectively. Then $q_1$ and $q_2$ are conjugate via $\kappa$, and we have $\kappa^{-1} \circ \varphi_1 \circ \kappa = \varphi_2$ as $K$-algebra automorphisms because both $\tilde{f}$ and $f_1$ are invariant under $\kappa = \kappa^{-1}$. In other words, $\varphi_1$ and $\varphi_2$ can be identified with each other by the group action of $\text{Gal}(K|\mathbb{Q})$ on $\text{Aut}(K)$. Also note that the map $\kappa^{-1} \circ \varphi_1 \circ \kappa$ can be extended to an $\mathbb{R}$-algebra automorphism because its restriction to $K$ is just the identity, and note that the germ at $q_2$ is right equivalent to $f_1$ via this map. Hence $f_1 \sim f_2$ and thus $r = s$. \( \square \)

We now discuss an explicit version of Algorithm [5] which works for all rational input polynomials $f \in \mathbb{Q}[x, y]$ of complex type $Y_{r,s}$. The framework is given in Algorithm [11]. We realize Step (II) of the general algorithm in Algorithm [11] and Step (IV) in Algorithms [12] and [13] corresponding to the real main types $Y_{r,s}$ and $\tilde{Y}_r$, respectively.

Suppose $f$ is of real main type $Y_{r,s}$. Algorithm [11] determines a normal form equation of $f$ over $\mathbb{Q}$ or over a simple real algebraic extension of $\mathbb{Q}$. From this normal form equation, Algorithm [12] computes a linear or quadratic minimal polynomial for the moduli parameter. Using the original input polynomial if necessary, the algorithm then determines which of the roots of the minimal polynomial are valid moduli parameters and computes the corresponding real subtypes.

Now assume $f$ is of real main type $\tilde{Y}_r$. In this case, to determine the normal form equation, Algorithm [11] requires a simple complex, non-real algebraic extension of $\mathbb{Q}$. From this normal form equation, Algorithm [13] computes a minimal polynomial for all complex moduli parameters.
Algorithm 10 Algorithm for the cases $Y_r,s$

Input: $f \in \mathbb{m}^3 \subset \mathbb{Q}[x,y]$ of complex singularity type $T = Y_r,s$

Output: all normal form equations in the right equivalence class of $f$, each specified as a tuple of the real singularity type, the normal form, the minimal polynomial of the parameter, and an interval such that the parameter is the unique root of the minimal polynomial in this interval

1: $f_0 := f$

II. Eliminate the monomials in $\text{supp}(f)$ underneath $\Gamma(T)$
2: replace $f$ by the output of Algorithm 11 applied to $f$

III. Eliminate the monomials above $\Gamma(T)$
3: $f := w\cdot\text{jet}(f,dw)$ with $w := w(T)$ and $dw := d(T)$

IV. Read off the desired information
4: $h := \text{jet}(f_0,4)$
5: if $h(1,y)$ has at least one real root then
6: return output of Algorithm 12 applied to $f_0$ and $f$
7: else
8: return output of Algorithm 13 applied to $f_0$ and $f$

Algorithm 11 Step (II) in the general algorithm for the cases $Y_r,s$

Input: $f \in \mathbb{m}^3 \subset \mathbb{Q}[x,y]$ of complex singularity type $T = Y_r,s$

Output: $f' \in \mathbb{m}^3 \subset \mathbb{Q}[x,y]$, where $\mathbb{Q}$ is a quadratic extension field of $\mathbb{Q}$, such that $f \sim f'$ and $f'$ has no terms underneath $\Gamma(T)$

1: $h := \text{jet}(f,4)$
2: $b := \text{coeff}(f,x^2y^2)$

Implement line 3 from Algorithm 5
3: if $h$ has a linear factor over $\mathbb{Q}$ then
4: factorize $h$ as $bg_1^2g_2^2$ over $\mathbb{Q}$, where $g_1,g_2$ are homogeneous of degree 1
5: apply $g_1 \mapsto x$, $g_2 \mapsto y$ to $f$

else
7: factorize $h$ as $bg^2$ over $\mathbb{Q}$, where $g$ is homogeneous of degree 2
8: over $K$, factorize $h$ as $bg_1^2g_2^2$, where $g_1,g_2$ are homogeneous of degree 1
9: apply $g_1 \mapsto x$, $g_2 \mapsto y$ to $f$

Implement lines 13 - 15 from Algorithm 3
11: $n_1 := 4$, $n_2 := 4$
12: while $\text{coeff}(f,x^{n_1}) = 0$ or $\text{coeff}(f,y^{n_2}) = 0$ do
13: if $\text{coeff}(f,x^{n_1}) = 0$ then
14: $n_1 := n_1 + 1$
15: apply $x \mapsto x$, $y \mapsto y - \text{coeff}(f,x^{n_1}y)x^{n_1-2}$ to $f$
16: if $\text{coeff}(f,y^{n_2}) = 0$ then
17: $n_2 := n_2 + 1$
18: apply $x \mapsto x - \text{coeff}(f,xy^{n_2})y^{n_2-2}$, $y \mapsto y$ to $f$
19: return $f'$

According to Theorem 35 in Marais and Steenpaß (2015b), only two of the roots of the minimal polynomial are real, and one is the negative of the other. As shown in Section 5.4 in Marais and Steenpaß (2015b), both real roots are valid values for the parameter if $r$ is odd; if $r$ is even, then only one of the two values is valid. As above, the original input polynomial is used to determine
which of the roots of the minimal polynomial are valid moduli parameters and to compute the corresponding real subtypes.

We now give a detailed exposition of Algorithms 11 to 13. We set \( f_0 := f \) to keep track of the real equivalence class, and change \( f \) in the course of the process, possibly using real or complex transformations. For \( h := \text{jet}(f_0, 4) \), Algorithm 11 considers the following two cases:

1. \( h(1, y) \) has a linear factor over \( \mathbb{Q} \):

   This case is handled in lines 3 - 18 of Algorithm 11. It includes all singularities of real main type \( Y_{r,s} \) with \( r \neq s \) and those of type \( Y_{r,r} \) for which the 4-jet factorizes into linear factors over \( \mathbb{Q} \). We exactly follow the general algorithm.

2. \( h(1, y) \) does not have a linear factor over \( \mathbb{Q} \):

   This case is handled in lines 6 - 18 of Algorithm 11. It includes all singularities of real main type \( \tilde{Y}_r \) and those of type \( Y_{r,r} \) for which the 4-jet does not factorize into linear factors over \( \mathbb{Q} \). We first discuss lines 6 - 10. Over the reals, \( \text{jet}(f, 4) \) factorizes as \( g_1^2g_2^2 \), with \( g_1 \) and \( g_2 \) non-associate and homogeneous of degree 1. Since \( \mathbb{Q} \) is a perfect field, we have \( g_1g_2 \in \mathbb{Q}[x, y] \). Since \( g_1, g_2 \notin \mathbb{Q}[x, y] \), the polynomial \( g_1g_2(1, y) \) is irreducible. Passing to the extension field \( K = \mathbb{Q}[y]/g_1g_2(1, y) \), we apply the automorphism \( g_1 \mapsto x, g_2 \mapsto y \) to \( f \).

Lines 10 - 18 of Algorithm 11 follow the general algorithm, using transformations over \( K \) to find a polynomial which is right equivalent to \( f \) and of the form

\[
\frac{bx^2y^2 + dxr + eyy^2 + R}{b\in \mathbb{Q}, d, e \in K, \text{ and } R \in E_{n_1+1}^{(n_1-2, 1)} \cap E_{n_1+1}^{(2, n_1-2)}.}
\]

If \( f_0 \) is of real main type \( Y_{r,s} \), we now pass to Algorithm 12 which considers the following two cases:

1. \( h(1, y) \) has a linear factor over \( \mathbb{Q} \):

   This case is handled in lines 2 - 29 of Algorithm 12 which follow the general algorithm. We use Theorem 33 in Marais and Steenpaß (2015b) to determine the possible normal form equations, depending on whether \( r \) and \( s \) are even or odd, respectively.

2. \( h(1, y) \) does not have a linear factor over \( \mathbb{Q} \):

   This case is handled in lines 30 - 18 of Algorithm 12. Note that the input polynomial \( f \) is an element of \( K[x, y] \), where \( K = \mathbb{Q}[\eta] \) is a real quadratic number field, where \( \eta = \sqrt{D} \) for some positive discriminant \( D \in \mathbb{Q} \). Furthermore, note that \( f \) is of the form

\[
f = bx^2y^2 + dxr + eyy^2 \quad \text{with} \quad b \in \mathbb{Q} \quad \text{and} \quad d, e \in K,
\]

and let \( \sigma = \text{sign}(b) \).

The case where \( r \) is odd is handled in lines 33 - 67. In this case, \( f \) is right equivalent to both the types \( Y_{\sigma r}^{\sigma} \), taking into account the transformation \( x \mapsto -x \) and \( y \mapsto y \). By Theorem 33 from Marais and Steenpaß (2015b) there are only two possible values for the parameter, which are both roots of the polynomial \( z^2 - |b|^{-r}(de)^2 \). If \( a \) is a possible value for the parameter, then so is \( -a \) by the transformation \( x \mapsto -x, y \mapsto y \). On the other hand, suppose \( \varphi \) is an automorphism which transforms \( f_0 \) to a normal form equation with parameter \( \lambda_1 + \lambda_2t \), and let \( \kappa \) be the conjugation on \( K \). Then \( \kappa \circ \varphi \circ \kappa^{-1} \) is also an automorphism and transforms \( f_0 \) to the same normal form equation with the parameter replaced by \( \kappa(\lambda_1 + \lambda_2t) = \lambda_1 - \lambda_2t \). Hence, \( \lambda_1 + \lambda_2t = -(\lambda_1 - \lambda_2t) \) or \( \lambda_1 + \lambda_2t = \lambda_1 - \lambda_2t \). Thus, \( \lambda_1 = 0 \) or \( \lambda_2 = 0 \), and the square of the parameter is a rational number. Therefore, \( z^2 - |b|^{-r}(de)^2 \) is a polynomial with rational coefficients.

Consider now the case where \( r \) is even. We either have \( a = |b|^{r/2}de \) or \( a = -|b|^{r/2}de \). By Theorem 33 of Marais and Steenpaß (2015b) the parameter \( a \) is uniquely determined. Hence, by a similar argument as before, \( |b|^{r/2}a \) and thus \( de \) is invariant under conjugation, and we conclude that \( de \in \mathbb{Q} \).

The case where \( r \) is even and \( de < 0 \) is handled in lines 39 - 13. Taking the transformation \( x \mapsto y, y \mapsto x \) into account, \( f \) is both of type \( Y_{\sigma r}^{\sigma} \) with \( a = |b|^{r/2}de \), and of type \( Y_{r,\sigma}^{\sigma} \) with \( a = -|b|^{r/2}de \).
Algorithm 12 Step (IV) in the general algorithm for the real main types $Y_{r,s}$

**Input:** $f_0 \in m^3 \subset \mathbb{Q}[x,y]$ of real singularity type $Y_{r,s}$ and $f \in m^3 \subset K[x,y]$, where $K$ is a real quadratic extension of $\mathbb{Q}$, such that $f$ is of complex singularity type $T = Y_{r,s}$, $\text{supp}(f) = \text{supp}(T)$, and $f \not\in K$.

**Output:** all normal form equations in the right equivalence class of $f_0$, each specified as a tuple of the real singularity type, the normal form, the minimal polynomial of the parameter, and an interval such that the parameter is the unique root of the minimal polynomial in this interval.

1: \[ h := \text{jet}(f_0, 4) \]
2: if $h$ has a linear factor over $\mathbb{Q}$ then
3: \[ C := \text{normal form of } h \text{ as in } \text{Step (III)} \]
4: if $r$ is odd and $s$ is even then
5: \[ \text{apply } x \mapsto y, y \mapsto x \text{ to } f \]
6: write $f$ as $f = bx^2y^2 + dx^r + ey^s$ with $b, d, e \in \mathbb{Q}^*$
7: \[ p := z^{2r} - |b|^{-rs}d^{2r} + z^{2s} - |b|^{-rs}d^{2s}2r \in \mathbb{Q}[z] \]
8: \[ \sigma_1 := \text{sign}(b), \sigma_2 := \text{sign}(d), \sigma_3 := \text{sign}(e), I := (0, \infty) \]
9: if $r$ and $s$ are even then
10: \[ C_1 := (Y_{r,s}, \sigma_1x^2y^2 + \sigma_2x^r + ay^s, m_{s,r}(p), \sigma_3 \cdot I) \]
11: \[ C_2 := (Y_{s,r}, \sigma_1x^2y^2 + \sigma_3x^r + ay^s, m_{s,r}(\tilde{\rho}), \sigma_2 \cdot I) \]
12: if $r = s$ and $\sigma_2 = \sigma_3$ then
13: return $C_1$
14: else
15: return $C_1, C_2$
16: if $r$ is even and $s$ is odd then
17: \[ \text{for } i = 1, 2 \text{ do} \]
18: \[ C_i := (Y_{r,s}, \sigma_1x^2y^2 + \sigma_2x^r + ay^s, m_{s,r}(p), (-1)^i \cdot I) \]
19: \[ C_{i+2} := (Y_{s,r}, \sigma_1x^2y^2 + (-1)^ix^s + ay^r, m_{s,r}(\tilde{\rho}), \sigma_2 \cdot I) \]
20: return $C_1, C_2, C_3, C_4$
21: if $r$ and $s$ are odd then
22: \[ \text{for } i = 1, 2 \text{ do} \]
23: \[ \text{for } j = 1, 2 \text{ do} \]
24: \[ C_{2i+j-2} := (Y_{r,s}, \sigma_1x^2y^2 + (-1)^ix^r + ay^s, m_{s,r}(p), (-1)^{i+j} \cdot I) \]
25: \[ C_{2i+j+2} := (Y_{s,r}, \sigma_1x^2y^2 + (-1)^{i+j}x^s + ay^r, m_{s,r}(\tilde{\rho}), (-1)^i \cdot I) \]
26: if $r = s$ then
27: return $C_1, C_2, C_3, C_4$
28: else
29: return $C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8$
30: else
31: write $f$ as $f = bx^2y^2 + dx^r + ey^r$ with $b \in \mathbb{Q}^*$ and $d, e \in K^*$
32: \[ \sigma := \text{sign}(b), I := (0, \infty) \]
33: if $r$ is odd then
34: \[ p := z^{2r} - |b|^{-rs}(de)^2 \in \mathbb{Q}[z] \]
35: for $i = 1, 2$ do
36: \[ C_i := (Y_{r,s}, \sigma x^2y^2 + x^r + ay^r, m_{s,r}(p), (-1)^i \cdot I) \]
37: \[ C_{i+2} := (Y_{s,r}, \sigma x^2y^2 - x^r + ay^r, m_{s,r}(p), (-1)^i \cdot I) \]
38: return $C_1, C_2, C_3, C_4$
... Again, \( f \) is of the form
\[ f = bx^2y^2 + dx^r + ey^r \]
with \( b \in \mathbb{Q} \) and \( d, e \in K \).

Note that the coefficient of \( x^r \) in \( f_0 \) must be nonzero, since \( f_0 \) would be of the real main type \( Y_{r,r} \) otherwise. The real subtype of \( f_0 \) is determined by the sign \( \sigma \) of this coefficient. We now determine the possible values for the parameter \( a \).

If \( r \) is odd, then taking the transformation \( x \mapsto -x, y \mapsto y \) into account, there are exactly two possible values for the parameter, where one is the negative of the other, see Theorem 33 in Marais and Steenpaß (2015b). Using the Figure 1 from Marais and Steenpaß (2015b) and taking...
into account that $Y_{r,r}^{+\pm}$ is complex equivalent to $Y_{r,r}^{-\pm}$, see Table 7 in Marais and Steenpaß (2015b),
the complex parameter is determined up to multiplication with a 4th root of unity. Hence, the
two possible values of the parameter correspond to the unique positive and the unique negative
root of $z^8 - (de)^4 (4b^{-1})^2 r \in \mathbb{Q}[z]$.

In a similar way, if $r$ is even, the parameter $a$ is uniquely determined. Since the complex
parameter is determined up to multiplication with a 2nd root of unity, we observe that the
parameter is one of the two real roots of $z^4 - (de)^2 (4b^{-1}) r \in \mathbb{Q}[z]$. We now determine the sign of
$a$. Note that the function $(x, y) \mapsto \sigma \cdot (1 + (x^2 + y^2)^2 + |a| x')$ takes positive and negative values
on all except one line through the origin, whereas the function $(x, y) \mapsto \sigma \cdot (1 + (x^2 + y^2)^2 + |a| x')$
takes only positive or only negative values. Hence, we can determine the sign of $a$ as follows:
Let $r_1$ and $r_2$ be the number of real roots of $\sigma + f_0(x, 0)$ and $\sigma + f_0(0, y)$, respectively. If any of
the $r_i$ is positive, then $\sigma \cdot a < 0$, otherwise $\sigma \cdot a > 0$.

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