The conformal frame freedom in theories of gravitation

Éanna É. Flanagan

Center for Radiophysics and Space Research, Cornell University, Ithaca, NY 14853-5001.

It has frequently been claimed in the literature that the classical physical predictions of scalar tensor theories of gravity depend on the conformal frame in which the theory is formulated. We argue that this claim is false, and that all classical physical predictions are conformal-frame invariants.

We also respond to criticisms by Vollick [10], in which this issue arises, of our recent analysis of the Palatini form of 1/R gravity.

I. INTRODUCTION

The observed recent acceleration of the Universe’s expansion [1, 2] has prompted suggestions that the acceleration can be explained by a modification of general relativity [3, 4, 5, 6, 7]. In particular Vollick [8] suggested that the gravitational action, where

\[ T \]

\[ \sigma \]

\[ \gamma \]

\[ \eta \]

\[ \nabla \]

\[ \Phi \]

\[ \gamma_{AB}(\Phi^C) \]

\[ V(\Phi^A) \]

\[ S = \int d^4x \left[ -\frac{1}{2} \gamma_{AB}(\Phi^C) \eta^{\mu\nu} \nabla_\mu \Phi^A \nabla_\nu \Phi^B - V(\Phi^A) \right]. \]

(2.1)

This theory can be expressed in different forms using different choices of coordinates \( \Phi^A \) on the manifold \( M \). The choice of coordinates is analogous to the labeling of left-handed and right-handed in the previous example: it is an arbitrary convention. Different choices of coordinates yield theories that are mathematically equivalent.

Once again, there are two points of view on whether or not different choices of coordinates in Eq. (2.1) give from the following point of view. We can use specific objects exterior to the theory to define right-handed and left-handed (for example chiral organic molecules, whose chirality depends on an accident of history). With respect to this standard, the theory \( T_1 \) is correct while the theory \( T_2 \) disagrees with experiments.

There is a second point of view, on the other hand, according to which the two theories are physically equivalent. The difference between \( T_1 \) and \( T_2 \) amounts to a trivial relabeling of what constitutes left-handed and right-handed, the specification of which is an arbitrary convention. If we were to encounter a specification of a model of particle physics from an alien civilization, we would not know that civilization’s conventions for right-handed and left-handed, and we would naturally say that the model was correct if there were some choice of convention for which the theory agreed with experiment. More generally, in this point of view, two theories are physically equivalent if any arbitrary conventions used in the interpretation of the theory can be chosen to make the two theories agree.

As a second example, consider a general nonlinear \( \sigma \)-model as a classical field theory in Minkowski spacetime. Given an \( N \)-dimensional manifold \( M \) with coordinates \( \Phi^A = (\Phi^1, \ldots, \Phi^N) \), a Riemannian metric \( \gamma_{AB}(\Phi^C) \) and a potential function \( V \) on \( M \), the action is

Vollick [10] refers to the idea that two different physical theories can be mathematically equivalent without being physically equivalent. There are two contexts in which this idea makes sense. In order to discuss those contexts, it is useful to consider some specific examples of physical theories.

As a first example, let \( T_1 \) be the standard model of particle physics, and let \( T_2 \) be the standard model with the roles of “left-handed” and “right-handed” interchanged. Then \( T_1 \) and \( T_2 \) will differ due to parity violation in the weak interaction. Are these two theories equivalent? Clearly they are mathematically equivalent, since the states of \( T_1 \) are in one-to-one correspondence with the states of \( T_2 \). They are not physically equivalent, however.

1 Of course if one takes the dynamical field of the theory to be the mapping \( \Phi \) from Minkowski spacetime to \( M \), then a choice of coordinates is not needed to define the action (2.1). This is the standard differential-geometric point of view. Here, in order to obtain a theory that ostensibly depends on the choice of coordinates, we define the dynamical fields of the theory to be the coordinate representations \( \Phi^A \) of the map \( \Phi \).
rise to physically equivalent theories. In the first point of view, an observer could in principle define a set of coordinates \( \Phi^A \) without using the properties of these fields encoded in the action \( \mathcal{L} \), but instead using specific physical objects in the observer’s vicinity constructed out of the fields. For example, the observer might come across several boson stars constructed out of different combinations of the fields, and use those stars to establish a coordinate system. If the observer insists on using such a definition of the coordinates \( \Phi^A \), then only one of the actions of form \( \mathcal{L} \) will be correct, and the other ones (corresponding to other choices of coordinates) will disagree with observations. This situation is analogous to using chiral organic molecules to establish a convention for right-handedness, and concluding that only one of the two theories \( T_1 \) and \( T_2 \) discussed above is correct.

By contrast, the second point of view disallows the use of fixed conventions, like choices of coordinates, that are exterior to the specification of the theory. Instead, one regards such conventions as mutable and adjustable. From this point of view, all choices of coordinates on the field manifold \( M \) give rise to physically equivalent theories.

It is clear from this discussion that the second point of view – in which arbitrary conventions are taken to be mutable and not fixed when comparing two theories – is the more conventional. For example, most physicists would immediately assert that different choices of coordinates in the action \( \mathcal{L} \) give rise to physically equivalent theories. Our papers \([9, 11]\) implicitly adopted this point of view, and this definition of “physically equivalent.” Nevertheless, the first point of view is equally logical and consistent, albeit laden with the baggage of a fixed, arbitrary choice of conventions.

Note that in the second point of view, a theory agrees with experiment if there exists some choice of conventions for which the predictions of the theory agree with observations. Conversely, a theory can be falsified only by showing that for all choices of conventions, there is a disagreement with experiment. In particular, there is a danger of a naive analysis that attempts to rule out a theory by showing disagreement with experiment for only one choice of convention. It is precisely this type of fallacy which Vollick \([8]\) claims occurs in our analysis \([9]\). We will discuss this crucial point in detail below and explain why the fallacy does not in fact occur.

So far, we have discussed one context in which two mathematically equivalent theories can be regarded as physically inequivalent, namely when there are arbitrary conventions that are fixed and that are independent of the specification of the theory. A second context arises when one gives an incomplete specification of a physical theory. In particular, this arises if one specifies a theory which constitutes one sector of a larger theory, and if interactions in the larger theory determine some conventions used in the interpretation of the smaller theory. For example, source-free electromagnetism is mathematically equivalent to a dual theory in which the roles of the electric and magnetic fields have been interchanged \([12]\). This equivalence is not physical however, since once one enlarges the theory to include couplings to charged fields there are electric charges but no magnetic monopoles.

The conclusion we draw from this discussion is that if two theories are mathematically equivalent, then they will always be physically equivalent as long as (i) any arbitrary conventions arising in the interpretation of the theory are regarded as adjustable, not fixed; and (ii) the theory is complete and contains all the degrees of freedom that are involved in measurements related to the theory.

### B. Scalar-tensor theories of gravitation: classical considerations

We now turn to a discussion of scalar-tensor theories of gravitation \([13]\). In this section we will treat these theories as classical field theories, neglecting quantum effects. In the following section we will discuss the extent to which the conclusions of this section need to be modified by quantum mechanical considerations.

We start by reviewing some well-known properties of scalar-tensor theories. The action for such theories can be written as

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} A(\Phi) R - \frac{1}{2} B(\Phi)(\nabla \Phi)^2 - V(\Phi) \right] + S_m[\tilde{g}_{\mu\nu}, \psi_m].
\]  

(2.2)

Here the first term in the action depends on a metric \( \tilde{g}_{\mu\nu} \) and a scalar field \( \Phi \); \( R \) is the Ricci scalar of \( g_{\mu\nu} \) and \( \kappa^2 = 8\pi G \). We use the sign conventions of Ref. \([14]\). The second term is the matter action \( S_m[\tilde{g}_{\mu\nu}, \psi_m] \), which is some functional of the matter fields \( \psi_m \) and of the metric \( \tilde{g}_{\mu\nu} \).

This action can be taken to be the action of the standard model of particle physics. The theory \( \mathcal{L} \) depends on four freely specifiable functions of \( \Phi \): \( A(\Phi), B(\Phi), \alpha(\Phi) \) and the potential \( V(\Phi) \).

As is well known, the form \( \mathcal{L} \) of the theory is preserved under a group of field redefinitions that contains two functional degrees of freedom. Specifically, if one defines a new metric \( \tilde{g}_{\mu\nu} \) and a new scalar field \( \Phi \) via

\[
\Phi = f(\Phi)
\]

(2.4)

\[
\tilde{g}_{\mu\nu} = e^{2\gamma(\Phi)} g_{\mu\nu}
\]

(2.5)

for some functions \( f \) and \( \gamma \) with \( f' > 0 \), then the action \( \mathcal{L} \) can be rewritten up to a boundary term as

\[
S = \int d^4x \sqrt{-\tilde{g}} \left[ \frac{1}{2\kappa^2} \tilde{A}(\tilde{\Phi}) \tilde{R} - \frac{1}{2} \tilde{B}(\tilde{\Phi})(\nabla \tilde{\Phi})^2 - \tilde{V}(\tilde{\Phi}) \right] + S_m[\tilde{g}_{\mu\nu}, \psi_m].
\]

(2.6)
Here the transformed functions $\tilde{A}(\Phi), \tilde{B}(\Phi), \tilde{V}(\Phi), \tilde{\alpha}(\Phi)$ are given by

\begin{align}
\tilde{\alpha}(\Phi) &= \alpha[f(\Phi)] + \gamma(\Phi), \\
\tilde{V}(\Phi) &= e^{4\gamma(\Phi)}V[f(\Phi)], \\
\tilde{A}(\Phi) &= e^{2\gamma(\Phi)}A[f(\Phi)], \\
\tilde{B}(\Phi) &= e^{2\gamma(\Phi)}\left\{ f'(\Phi)^2B[f(\Phi)] - \frac{6}{\kappa^2}f'(\Phi)\gamma'(\Phi)A'[f(\Phi)] \\
&\quad - \frac{6}{\kappa^2}\gamma'(\Phi)^2A[f(\Phi)] \right\}.
\end{align}

(2.7a) \hspace{1cm} (2.7b) \hspace{1cm} (2.7c) \hspace{1cm} (2.7d)

For many theories the transformation group (2.4) – (2.5) can be used to obtain canonical representations of the theory, characterized by two free functions instead of four. Some of these representations (or choices of “conformal frame”) are

- The Jordan frame, which is characterized by $\alpha = 0, B = 1$. The free functions in this frame are $A(\Phi)$ and $V(\Phi)$. Freely falling objects built from the matter fields $\psi_m$ follow geodesics of the Jordan-frame metric.

- The Einstein frame, which is characterized by $A = 1, B = 1$. The free functions in this frame are $\alpha(\Phi)$ and $V(\Phi)$.

- A frame which does not have a standard name that is characterized by $B = 0, A(\Phi) = \Phi$. The free functions in this frame are $\alpha(\Phi)$ and $V(\Phi)$.

Suppose now that we are given an arbitrary scalar-tensor theory, specified by the functions $A(\Phi), B(\Phi), V(\Phi)$ and $\alpha(\Phi)$. Then there is no guarantee that the theory has a well defined classical dynamics, i.e. a well-posed initial value formulation \[15\]. For example, in the theory $B = 1, V = \alpha = 0, A = 1 - \kappa^2\Phi^2/6$ of a conformally coupled scalar field, there is a class of solutions of the equations of motion with no matter given by $\Phi = \sqrt{6}/\kappa$, $g_{\mu\nu}$ = any metric with $R = 0$. Therefore for this theory the initial value formulation is ill-posed; local uniqueness of solutions fails. However, it is often possible to obtain a sensible classical theory by passing to a subspace of the phase space defined by restricting the allowed values of $\Phi$ in an initial data set to an open interval of the form $(\Phi_1, \Phi_2)$ \[2\]. We will restrict attention to theories obtained by passing to subspaces of this form which are in one-to-one correspondence with the entire phase space of an Einstein-frame description \[3\]. This assumption implies that all the theories we consider will automatically have well-posed initial value formulations, since general theorems \[15\] guarantee this for Einstein-frame theories. It also implies that any two theories related by transformations of the form (2.4) – (2.5) are mathematically equivalent, so that it is natural to call such theories different conformal-frame representations of the same theory. Equivalent assumptions are more or less implicit in many discussions of scalar tensor theories in the literature.

We now discuss whether or not different conformal-frame representations of a theory are physically equivalent. One fairly trivial context in which an apparent physical inequivalence arises is when the theory is incompletely specified, when one specifies only the first term in the action (2.2) and not the second (matter) term. This arises if one adopts a convention in which it is implicit that the metric which is being used is the Jordan-frame metric, or equivalently in which $\alpha(\Phi) = 0$. In this context there is no conformal-frame freedom, since it is clearly inconsistent to perform a conformal transformation on the first term, thereby transforming the functions $A(\Phi), B(\Phi)$ and $V(\Phi)$, without keeping track of the transformation of the function $\alpha(\Phi)$ in the second term. Thus, while one can express the action of the theory [the first term in Eq. (2.2)] in different conformal frames, only one of those frames is physically correct, as pointed out by Brans \[16\].

Putting aside this special case, turn now to the more complete framework where the entire action (2.2) is specified, including the four functions $A(\Phi), B(\Phi), V(\Phi)$ and $\alpha(\Phi)$. Are different conformal-frame representations of the theory physically equivalent? The action (2.2) is complete and contains all the relevant degrees of freedom. Also the representations are mathematically equivalent because of our assumption discussed above. It therefore follows from the general discussion of Sec. \[11\] above that the different conformal-frame representations are physically equivalent, as long as one does not treat as fixed any convention that arises in the interpretation of the theory. The relevant convention in this case is the interpretation of the meaning of the metric that appears in the theory. For example, suppose one measures the distance to the moon using lunar laser ranging and using an atomic clock to determine the light travel time. In general relativity, the computation corresponding to this measurement is straightforward: the atomic clock measures proper time associated with the metric. In a general scalar-tensor theory and in a general conformal frame, however, the time on the clock will not be the proper time associated with the metric. Nevertheless one

\[\int_{\Phi_1}^{\Phi_2} \sqrt{F(\Phi)d\Phi} \to \infty\]

as $\Phi_1 \to \Phi_1$ from above and $\Phi_2 \to \Phi_2$ from below.

\[\int_{\Phi_1}^{\Phi_2} \sqrt{F(\Phi)d\Phi} \to \infty\]

Some of the confusion in the literature discussed below apparently arises from a misinterpretation of the context of Brans’s argument.

---

2 Here we do not exclude $\Phi_1 = -\infty$ or $\Phi_2 = \infty$. In the cases we consider below the condition $\Phi_1 < \Phi < \Phi_2$ will be preserved under dynamical evolution.

3 This assumption is equivalent to the conditions $A > 0$ and $F > 0$ on the interval $(\Phi_1, \Phi_2)$, where $F = B/A + 3(A')^2/(2\kappa^2 A^2)$, and $\int_{\Phi_1}^{\Phi_2} \sqrt{F(\Phi)d\Phi} \to \infty$ as $\Phi_1 \to \Phi_1$ from above and $\Phi_2 \to \Phi_2$ from below.
can still compute from the action $\mathcal{L}$, the time as measured by the clock, and the same result will be obtained in all conformal frames. Namely, it will be the proper time associated with the metric \( g_{\mu\nu} \), which from Eqs. (2.2) and (2.3) is a conformal-frame invariant. Different conformal frames will be physically inequivalent only if one insists on interpreting the metric in the theory as the metric which is measured by atomic clocks.

However, it has repeatedly been claimed in the literature that only one choice of conformal frame is correct or “physical”, and that as a consequence different conformal-frame representations of a theory are not physically equivalent. 5 Several different criteria have been suggested to determine the “correct” or “physical” conformal frame, including local positivity of energy \( \mathcal{F}_{\mu\nu} \mu\nu \) and the existence of a stable ground state \( \mathcal{F}_{\mu\nu} \mu\nu \) and \( \mathcal{F}_{\mu\nu} \mu\nu \) and \( \mathcal{F}_{\mu\nu} \mu\nu \) and \( \mathcal{F}_{\mu\nu} \mu\nu \) and \( \mathcal{F}_{\mu\nu} \mu\nu \). Such efforts to determine the “correct” choice of conformal frame are misguided, at least in the realm of classical physics. They are analogous to attempting to determine the “correct” choice of radial coordinate in the Schwarzschild spacetime. In that context, there is of course no correct radial coordinate, since all physical observables are coordinate-invariants. In a similar way, all observable quantities in scalar-tensor theories are conformal-frame invariants. For example, the measured 4-momentum of an isolated object is given by the Bondi 4-momentum of the frame-invariant metric \( \mathcal{L} \).

We have given a general, abstract argument in favor of the physical equivalence of different conformal frames. That abstract argument is supported by several explicit computations in the literature, where observables have been computed in different conformal frames, and where consistent results have been obtained. For example, one of the key observables of extended inflation models is the spectrum of scalar perturbations. This spectrum can be computed in the Einstein frame or in the Jordan frame; the results are identical. 6 Another example is the careful computation by Armendáriz-Picon of the observed ratio between the frequency of a quasar absorption line and the corresponding atomic transition frequency as measured in the laboratory. That computation is carried out in a theoretical context (more general than the context considered here) which allows time variations of the fine structure constant. Armendáriz-Picon computes the ratio in both the Einstein and Jordan frames, and obtains identical results.

Vollick 10 claims that different physical predictions are obtained in the Einstein and Jordan frames, and cites as evidence several papers 21, 23, 24, 25 that compute such differing predictions. The first of these papers 24 claims that that in the Einstein-frame version of Brans-Dicke theory, scalar gravitational waves produce longitudinal accelerations in gravitational wave detectors, and that no such longitudinal accelerations are present in the Jordan-frame version of the theory. In fact the prediction of both frames is that there is no measurable longitudinal acceleration. 7 The remaining papers 21, 23, 25 show that the weak energy condition is violated in one frame but not in the other. While this is true, there is no physical observable whose predicted value in all conformal frames is the sign of \( G_{\mu\nu} u^\mu u^\nu \) for timelike vectors \( u^\mu \). Thus there is no measurable inconsistency.

Another aspect of this subject is the equivalence between the choice of a conformal frame and the choice of physical units, first clearly enunciated by Dicke. 34 When we change from one system of units to another, the ratio between the original unit of length and the new unit of length is normally a constant, independent of space and time. However, if the operational procedure used to define the unit of length is changed, then the ratio between the old and new units can vary with space and time. 8 For example, suppose we define units of length and time by taking the speed of light to be unity and by taking the unit of time to be determined by some atomic transition frequency (as in the current SI definition of the second). Measurements of the geometry of spacetime in these units yield the Jordan-frame metric. However, we can instead define a system of units as follows. Suppose that we have a nonspinning black hole. We can in principle take this to be a “standard” black hole (like the original platinum-iridium standard meter), and create other nonspinning black holes of the same size. Using these black holes we can operationally define a unit of time to be the inverse of the frequency of their fundamental quasinormal mode of vibration. If we define the speed of light to be unity, and measure the geometry of spacetime in these units, the result is the Einstein-frame metric. Thus, the choice of conformal frame is no more than a choice of physical units, just a human convention.

---

5 A source of confusion is that some authors use the phrase “physical frame” as a synonym for Jordan frame, whereas others use it in the sense discussed here.

6 The first of these criteria, the local positivity of \( G_{\mu\nu} u^\mu u^\nu \) for timelike vectors \( u^\mu \), does pick out a unique conformal frame \( \mathcal{F}_{\mu\nu} \mu\nu \). The second criterion does not, since the existence and stability of a suitable ground state is a conformal-frame invariant.

7 The error of Ref. 24 was to assume that the measured acceleration in gravitational-wave detectors corresponds to the relative acceleration of two freely falling particles computed in the Einstein-frame metric, when in fact it is their relative acceleration computed in the metric \( \mathcal{L} \) which is what is measured.

8 This is related to the fact that a statement like “Newton’s constant of gravitation is changing with time” is meaningless, while the statement “Newton’s constant of gravitation in SI units is changing with time” does make sense. See Duff 31 for a lucid discussion of this issue.
C. Scalar-tensor theories of gravitation: quantum mechanical considerations

In classical physics, we have argued that scalar tensor theories can be formulated in any conformal frame without affecting the physical predictions of the theory. In quantum physics the situation is not as clear cut because (i) theories which are classically equivalent can be quantum mechanically inequivalent, and (ii) there are the usual difficulties in describing quantum gravitational degrees of freedom. The status of the conformal-frame freedom depends to some extent on the level of generality one aspires to, i.e., on the theoretical framework or context in which one is working.

One possible context is a semiclassical approximation in which the matter fields $\psi_m$ in the action (2.2) are treated quantum mechanically, but the fields $g_{\mu\nu}$ and $\Phi$ are treated classically. Here the arguments of the last section continue to apply, and identical predictions are obtained from all conformal frames. However, the arguments in Ref. 9 used a quantum mechanical treatment of the field $\Phi$, so this context is not sufficiently general for our purposes.

A second context is a semiclassical approximation in which the matter fields $\psi_m$ and the scalar field $\Phi$ are treated quantum mechanically, but the metric $g_{\mu\nu}$ is treated classically. Here it is well-known that quantization does not commute with changing conformal frames [18, 32], so that by starting in different conformal frames one obtains theories that are mathematically and physically inequivalent. On the other hand, one would expect to find such conformal-frame dependence from an approximation that freezes the quantum fluctuations in a conformal-frame-dependent combination of the scalar and tensor degrees of freedom. There is no evidence in this context that the frame dependence seen is not an artifact of the approximation being used, or that frame invariance is not present at a more fundamental level in a full quantum theory.

A third context is full quantum gravity. Here it is certainly conceivable that applying a quantization procedure to different conformal-frame representations of a theory will yield inequivalent quantum theories\(^9\). In fact, computations in two dimensional dilaton gravity theories indicate that such inequivalence does occur\(^\dagger\).

\(^9\) See for example the analysis of Ref. 35, where the two quantum theories obtained are not obviously equivalent, although the black hole entropies computed in the two different frames agree.

\(^\dagger\) In this paper we restrict attention to theories for which there exist some conformal frame in which $A = B = 1$, as discussed in footnote 3 above. However, one could instead consider the class of scalar-tensor theories which are exactly equivalent to general relativity. For these theories $\alpha(\Phi) = \lambda(\Phi)$, $V(\Phi) = 0$, $\mathcal{A}(\Phi) = \exp[2\lambda(\Phi)]$, and $\mathcal{B}(\Phi) = -6\exp[2\lambda(\Phi)]\lambda'(\Phi)^2/\kappa^2$ for some function $\lambda(\Phi)$. In Ref. 37 it is shown that these theories are quantum mechanically equivalent to general relativity.

III. PALATINI FORM OF 1/R GRAVITY

In this section we discuss the specific criticisms of Ref. 9. We start by briefly reviewing the argument of Ref. 9.

A. Review of derivation

In the theory of gravity suggested in Ref. 8, the dynamical variables are a metric $\bar{g}_{\mu\nu}$, a symmetric connection $\bar{\nabla}_\mu$, and the matter fields $\psi_m$. The action is

$$S[\bar{g}_{\mu\nu}, \bar{\nabla}_\mu, \psi_m] = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\bar{g}} f(\bar{R}) + S_m[\bar{g}_{\mu\nu}, \psi_m],$$

(3.1)

where $\bar{R}$ is the Ricci scalar of the connection $\bar{\nabla}_\mu$ and $f(\bar{R}) = \bar{R} - \mu^2/\bar{R}$. In the variational principle the metric and connection are treated as independent variables according to the Palatini prescription. In Ref. 8 we derived another description of the theory by (i) making field redefinitions; (ii) setting to zero some degrees of freedom which vanish classically on-shell; (iii) adding a new scalar field which vanishes classically on-shell; and (iv) discarding a boundary term. The last three steps are valid classically, and also quantum mechanically if one is only interested in computing tree-level scattering cross sections as we do below. The resulting action depends
on a metric $g_{\mu\nu}$ and a scalar field $\Phi$ and is given by\(^{11}\)
\[
\bar{S}[g_{\mu\nu}, \Phi, \psi_\mu] = \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa^2} - V(\Phi) \right] + S_{\text{m}}[e^{2\alpha(\Phi)} g_{\mu\nu}, \psi_\mu], \tag{3.2}
\]
where the potential $V(\Phi)$ and coupling function $\alpha(\Phi)$ are given by Eqs. (9) and (11) of Ref. \(^{8}\).

A key point about the action \((3.2)\) is that the scalar field is not an independent dynamical variable, but can be eliminated from the action. One then obtains a theory consisting of general relativity coupled to a modified matter action. Taking the matter action to be the Dirac matter action. Taking the matter action to be the Dirac
\[
\bar{S}[g_{\mu\nu}, \Psi_e] = \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa^2} - \Lambda + i\bar{\Psi}_e \gamma^\mu \nabla_\mu \Psi_e - m_e \bar{\Psi}_e \Psi_e - \frac{3\sqrt{3}}{16m_\ast^3} (i\bar{\Psi}_e \gamma^\mu \nabla_\mu \Psi_e)^2 - \frac{1}{\sqrt{3}} m_e^2 (\bar{\Psi}_e \Psi_e)^2 + \sqrt{\frac{3}{4}} \frac{m_e}{m_\ast^3} (i\bar{\Psi}_e \gamma^\mu \nabla_\mu \Psi_e)(\bar{\Psi}_e \Psi_e) + \ldots \right], \tag{3.3}
\]
where $\kappa = \sqrt{4/3k}$, $m_\ast = \sqrt{\mu/k}$ and $\Lambda = \mu^2/(\sqrt{3}k^2)$ is the induced cosmological constant. The last three terms in Eq. \((3.3)\) are characterized by the energy scale $m_\ast$ which is of order $10^{-3}$ eV. Since this energy scale is so small, one expects the action \((3.3)\) to be in conflict with atomic physics and particle physics experiments, for example electron-electron scattering. This is discussed in more detail in Sec. III C below.

**B. Objections to derivation**

Vollick \(^{10}\) gives several objections to the above derivation:

1. **“Failure to properly identify the physical frame”**

As we have argued in Secs. II B and II C, we disagree that one needs to identify a preferred “physical” conformal frame in order to define the theory\(^{12}\).

2. **“Failure to ... add the minimally coupled matter Lagrangian in [the physical] frame”**

The choice of frame in which to add the minimally coupled matter Lagrangian is a part of the specification of the theory of gravity, as discussed in Sec. III A. That specification was given in Refs. II B via the action \((3.1)\). Vollick \(^{10}\) did not make any choices in this regard; it simply analyzed the given theory of gravity.

3. **“The physical inequivalence of the Jordan and Einstein frames ... has been shown by many authors”**

This issue was discussed in Secs. II B and II C. One additional source of confusion is the following. Vollick \(^{10}\) gives an example of two conformally related metrics and notes that they are not physically equivalent. However, in order to determine whether two different situations are physically equivalent, one needs to specify not only the metric, but also the scalar field and the four functions that define the theory of gravity. For example, consider the following three different situations:

a. The metric and scalar field are $g_{\mu\nu}$ and $\Phi$, and the theory is defined by the functions $A(\Phi)$, $B(\Phi)$, $V(\Phi)$ and $\alpha(\Phi)$.

b. The metric and scalar field are $\bar{g}_{\mu\nu}$ and $\bar{\Phi}$, related to $g_{\mu\nu}$ and $\Phi$ by Eqs. (2.4) - (2.5), and the theory is defined by the same functions $A$, $B$, $V$ and $\alpha$ as before.

c. The metric and scalar field are $\bar{g}_{\mu\nu}$ and $\bar{\Phi}$, and the theory is defined by the functions $A$, $B$, $V$ and $\bar{\alpha}$ given by Eqs. (2.4d).

Here a. and b. are not physically equivalent, as noted by Vollick. However a. and c. are physically equivalent, and it is this which underlies the equivalence between the Einstein and Jordan frames.

4. **“One cannot add a minimally coupled Lagrangian in one frame and then transform to another frame ... all predictions of the theory must be calculated in [the physical] frame”**

We disagree with these assertions. As we have argued in Secs. II B and II C all physical observables are conformal-frame-invariants, and so can be computed in any convenient frame. This is true classically, and also quantum mechanically for tree-level scattering amplitudes. These assertions would only be true if one insisted on associating particular operational meanings to the dynamical fields appearing in the Lagrangian. In the point of view we adopt here [the second of the two points of view discussed in Sec. III A], we remain agnostic as to the physical interpretation of the dynamical fields \(^{29}\), and
as a consequence we can perform computations in any
conformal frame.

However, the spirit of this objection can be translated
into our point of view. It can be summarized as “In the
action (3.3), how do we know that the fields \( g_{\mu\nu} \) and \( \Psi_e \)
correspond to the graviton and electron that we actu-
ally measure? How do we know that there is not some
nonlinear local field redefinition which defines “graviton”
and “electron” fields as nonlinear functions of \( g_{\mu\nu} \) and \( \Psi_e \)
in such a way that the action becomes the conventional
standard model action when expressed in terms of these
new fields?” This is a valid concern (c.f. the discussion
near the end of Sec. IIA above) which we now discuss.
Consider the specific case of the scattering of two non-
relativistic electrons. The key point here is the equiv-
ance theorem discussed in Sec. IIA above \(^{10}\). This
theorem guarantees that the electron-electron scattering
cross section is invariant under nonlinear local field re-
definitions. Thus, we are free to use the form \(^{8,8}\) of the
action to compute that cross section.

\[ \Psi_e = e^{-im_e c^2 t} \left( \begin{array}{c} \Phi \\ \chi \end{array} \right), \]  

(3.4)
solving the equation of motion for \( \chi \), substituting that
solution back into the action, and discarding relativistic
corrections. The result at leading order is the following
action for the electron field \( \Phi \)

\[ S = \int d^3 x dt \left[ i \Phi^\dagger \dot{\Phi} - \frac{1}{2m_e} (\nabla \Phi)^\dagger (\nabla \Phi) + g (\Phi^\dagger \Phi)^2 \right], \]  

(3.5)

where the coupling constant is \( g = -\sqrt{3} m_e^2 / (48c m_e^2) \).

The action \(^{8,8}\) yields at leading order the follow-
ing Hamiltonian describing the interaction of two non-
relativistic electrons:

\[ H = \frac{\hat{p}_1^2}{2m_e} + \frac{\hat{p}_2^2}{2m_e} - \frac{e^2}{|\mathbf{r}_1 - \mathbf{r}_2|} - \frac{\beta m_e^2}{c m_e^4} \delta^3(\mathbf{r}_1 - \mathbf{r}_2). \]  

(3.6)

Here \( \beta \) is a dimensionless constant of order unity, \( \mathbf{r}_1 \), \( \mathbf{r}_2 \),
\( \mathbf{p}_1 \) and \( \mathbf{p}_2 \) are the positions and momenta of the two
electrons, we have added the Coulomb interaction term
and we have neglected spin effects. Thus, the leading
order correction is a contact interaction. The Hamil-
tonian \(^{8,8}\) yields for the total cross section for back-
ward\(^\text{13}\) scattering the conventional result \( \sigma \sim e^4 / \varepsilon^2 \) for
\( E \ll e^2 m_e^2 / \varepsilon^3 \sim 10^{-32} \text{ eV} \), where \( \varepsilon \) is the center of
mass energy, whereas for \( E \gg e^2 m_e^2 / \varepsilon^3 \) we obtain the
modified prediction

\[ \sigma \sim \frac{\varepsilon^6}{e^2 m_e^8}. \]  

(3.7)

Based on this prediction, we claimed in Ref. \(^{9}\) that the
action \(^{8,8}\) disagrees with experiments.

However, this claim is not quite correct, because the
prediction \(^{8,8}\) is only valid in an energy regime which
is inaccessible to experiments. This is so for two rea-
sons. First, the higher order corrections indicated by
the dots in Eq. (3.3) give rise to corrections to the ac-
tion for the total cross section would require an infi-
nite number of terms. Second, although the theory \(^{8,8}\)
is a weakly coupled effective quantum field theory at low
energies, there are indications that above a certain energy
scale it becomes strongly coupled, i.e. loop corrections
become large. This follows from the fact that the tree
level s-wave electron-electron scattering cross section
obtained from the action \(^{8,8}\) is of order \( m_e^2 g^2 \), which
exceeds the unitarity limit \( \sim 1 / (m_e \varepsilon) \) \(^{41}\) for energies
above

\[ E_{sc} \sim \frac{1}{m_e^2 g^2} \sim m_e^2 \left( \frac{m_e}{m_e} \right)^7 \sim 10^{-62} \text{ eV}. \]  

(3.8)

The same conclusion can also be reached by applying di-
imensional analysis \(^{42}\) to the action \(^{8,8}\). The coupling
\( g \) has dimensions \( [M]^{-3/2} [\varepsilon]^{-1/2} \), and so for a process of
energy \( \varepsilon \) the interaction term in the action \(^{8,8}\) will give a
contribution to \( S \) of order \( g m_e^{3/2} \varepsilon^{1/2} \sim (\varepsilon / E_{sc})^{1/2} \).

Therefore the effective description will break down at
\( \varepsilon \sim E_{sc} \). Of course, this conclusion might be modified
once one includes in the analysis all of the other fields
in the standard model besides the electron. Nevertheless
the indication is that the resulting theory probably be-
comes strongly coupled at energy scales that are too low
to be accessible to experiments.

Therefore, while one expects large corrections from the
extra terms in the action \(^{8,8}\), it is not possible to com-
pute these corrections reliably at accessible energy scales.
Thus, it is not possible to definitively rule out the theory.
However, one can make the following general arguments
against the theory \(^{8,8}\) as model of the Universe’s ac-
celeration. First, in this theory one looses the ability to

\(^{13}\) We choose to focus on this observable since the total forward
plus backward Coulomb scattering cross section diverges.
make predictions in atomic physics and low energy particle physics. That is, the theory replaces the standard model of particle physics with a matter field theory which is strongly coupled at low energies. Normally, in positing a modification to gravity one would like to retain the successful theoretical description of atomic physics provided by the standard model. Second, in order to make a quantitative model of the acceleration of the Universe, one needs to invoke the approximation that the Universe is homogeneous. This approximation is invalid for the theory (3.3), and so one cannot justify any Friedmann-Robertson-Walker models (3.4). For these reasons we feel that this theory does not provide a successful model of the Universe’s acceleration.

Acknowledgments

We thank Dan Vollick for useful criticisms, and Marc Favata, Daniel Grumiller, Scott Hughes, Sergei Odintsov, Saul Teukolsky, and Peng Wang for useful comments on the manuscript. This research was supported in part by NSF grant PHY-0140209.

[1] A. G. Riess et al., Observational evidence from supernova for an accelerating Universe and a cosmological constant, Astron. J. 116, 1009 (1998); S. Perlmutter et al., Measurements of Omega and Lambda from 42 high redshift supernovae, Astrophys. J. 517, 565 (1999).
[2] C. L. Bennett et al., First Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Preliminary Maps and Basic Results, Astrophys. J. Suppl. 148, 1 (2003) [also astro-ph/0302207].
[3] S. Capozziello et al., Quintessence without scalar fields, astro-ph/0303041; A. Lue, Differentiating between modified gravity and dark energy, Phys. Rev. D 69, 044005 (2004) [also astro-ph/0307034].
[4] S. M. Carroll, V. Duvvuri, M. Trodden, M.S. Turner, Is cosmic speed-up due to new gravitational physics? astro-ph/0306438.
[5] T. Chiba, 1/R gravity and scalar tensor gravity, Phys. Lett B 575, 1 (2003) [also astro-ph/0307338].
[6] S. Nojiri and S. D. Odintsov, Modified gravity with negative and positive powers of the curvature: unification of the inflation and of the cosmic acceleration, Phys. Rev. D 68, 123512 (2003) [also hep-th/0307288].
[7] X. Meng, P. Wang, Modified Friedman equations in 1/R modified gravity, Class. Quant. Grav. 20, 4949 (2003) [also astro-ph/0307354]; ibid, Cosmological evolution in 1/R gravity theory, 21, 951 (2004) [also astro-ph/0308031].
[8] N. Vollick, 1/R curvature corrections as the source of the cosmological acceleration, Phys. Rev. D 68, 063510, (2003) [also astro-ph/0306630].
[9] E. É. Flanagan, Palatini form of 1/R gravity, Phys. Rev. Lett. 92, 071101 (2004) [also astro-ph/0308111].
[10] N. Vollick, On the viability of the Palatini form of 1/R gravity, gr-qc/0312041.
[11] E. É. Flanagan, Higher order gravity theories and scalar tensor theories, Class. Quant. Grav. 21, 417 (2003) [also gr-qc/0309015].
[12] See, e.g., J.D. Jackson, Classical Electrodynamics, Wiley, New York (1975).
[13] T. Damour and G. Esposito-Farese, Tensor multi-scalar theories of gravitation, Class. Quant. Grav. 9, 2093 (1992).
[14] C. W. Misner et al., Gravitation (Freeman, 1973).
[15] See for example chapter 10 of R. M. Wald, General Relativity, University of Chicago Press, Chicago (1984).
[16] C.H. Brans, Nonlinear Lagrangians and the significance of the metric, Class. Quant. Grav. 5, L197 (1988).
[17] G. Magnano, Are there metric theories of gravity other than general relativity?, gr-qc/9511027.
[18] L.M. Sokolowski, Uniqueness of the metric line element in dimensionally reduced theories, Class. Quant. Grav. 6, 59 (1989).
[19] L.M. Sokolowski, Physical versions of nonlinear gravity theories and positivity of energy, Class. Quant. Grav. 6, 2045 (1989).
[20] M. Ferraris, M. Francaviglia and G. Magnano, Remarks on the physical metric in nonlinear theories of gravitation, Class. Quant. Grav 7, 261 (1990).
[21] G. Magnano, L.M. Sokolowski, On physical equivalence between nonlinear gravity theories and a general-relativistic self-gravitating scalar field, Phys. Rev. D 50, 5039 (1994) [also gr-qc/9312008].
[22] L.M. Sokolowski, Universality of Einstein’s general relativity, plenary talk at the 14th Conference on General Relativity and Gravitation, Florence, 1995 (gr-qc/9511073).
[23] V. Faraoni, E. Gunzig and P. Nardone, Conformal transformations in classical gravitation theories and in cosmology, Fundam. Cosm. Phys. 20, 121 (1999) [also gr-qc/9811047].
[24] S. Bellucci, V. Faraoni and D. Babusci, Scalar gravitational waves and Einstein frame, hep-th/0103180.
[25] V. Faraoni and E. Gunzig, Einstein frame or Jordan frame?, Int. J. Theor. Phys. 38, 217 (1999) [also astro-ph/9910176].
[26] S. Capozziello, R. de Ritis, A. A. Marino, Some aspects of the cosmological conformal equivalence between “Jordan frame” and “Einstein frame”, Class. Quant. Grav. 14, 3243 (1997) [also gr-qc/9612055].
[27] E.W. Kolb, D.S. Salopek, M.S. Turner, Origin of density fluctuations in extended inflation, Phys. Rev D 42, 3925 (1990).
[28] D. Kaiser, Frame-independent calculation of spectral indices from inflation, astro-ph/9507048.
[29] C. Armendariz-Picón, Predictions and Observations in theories with varying couplings, Phys. Rev. D 66, 064008 (2002).
[30] R. H. Dicke, Mach’s principle and invariance under transformation of units, Phys. Rev. 126, 2163 (1961).
[31] M. J. Duff, Comment on time variation of fundamental constants, hep-th/0208093.
[32] Y. Fujii and T. Nishioka, Model of a decaying cosmolog-
ical constant, Phys. Rev. D 42, 361 (1990).

[33] A. Ashtekar and A. Corichi, Non-minimal couplings, quantum geometry and black hole entropy, Class. Quant. Grav. 20, 4473 (2003) [also gr-qc/0305082].

[34] E. Elizalde, S. Naftulin and S.D. Odintsov, The renormalization structure and quantum equivalence of 2D dilaton gravities, Int. J. Mod. Phys. A 9, 933 (1994) [also hep-th/9304091].

[35] D. Grumiller, W. Kummer, and D. V. Vassilevich, Positive specific heat of the quantum corrected dilaton black hole, JHEP 07, 009 (2003) [also hep-th/0305036].

[36] D. Grumiller, W. Kummer, and D. V. Vassilevich, Dilaton gravity in two dimensions, Phys. Rep. 369, 327 (2002) [also hep-th/0204253].

[37] N. C. Tsamis and R. P. Woodard, No New Physics In Conformal Scalar - Metric Theory, Annals Phys. 168, 457 (1986).

[38] C.P. Burgess, Quantum Gravity in Everyday Life: General Relativity as an Effective Field Theory, gr-qc/0311082.

[39] J.F. Donoghue, Introduction to the Effective Field Theory Description of Gravity, in Advanced School on Effective Theories, ed. by F. Cornet and M.J. Herrero, (World Scientific) pp 217, (gr-qc/9512024).

[40] J. S. R. Chisholm, Nucl. Phys. 26, 469 (1961); S. Kamefuchi, L. O’ Raifeartaigh and A. Salam, Nucl. Phys. 28, 529 (1961); S. Coleman, J. Wess and B. Zumino, Structure of phenomenological Lagrangians. I., Phys. Rev. 177, 2239 (1969); A. Salam and J. Strathdee, Equivalent formulation of massive vector field theories, Phys Rev D 2, 2869 (1970); Y.-M.P. Lam, Equivalence theorem on Bogoliubov-Parasiuk-Hepp-Zimmermann-Renormalized Lagrangian Field Theories, Phys. Rev. D 7, 2943 (1973); M. C. Bergere and Y.-M.P. Lam, Equivalence theorem and Faddeev-Popov ghosts, Phys. Rev. D 13, 3247 (1976); A. Blasi, N. Maggiore, S. P. Sorella and L. C. Q. Vilar, Renormalizability of nonrenormalizable field theories, Phys. Rev D. 59, 121701 (1999).

[41] S. Weinberg, The Quantum Theory of Fields, Volume I, University of Cambridge press, Cambridge, UK (1995), section 3.7.

[42] J. Polchinski, Effective field theory and the Fermi surface, hep-th/9210046. The scaling method discussed by Polchinski for classifying operators in non-relativistic field theories is equivalent to the dimensional analysis method used here, where $\hbar = 1$ and the two independent units are taken to be mass and energy.