ORDERING-FREE INFERENCE FROM LOCALLY DEPENDENT DATA

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ABSTRACT. This paper focuses on a situation where data exhibit local dependence but their dependence ordering is not known to the econometrician. Such a situation arises, for example, when many differentiated products are correlated locally and yet the correlation structure is not known, or when there are peer effects among students’ actions but precise information about their reference groups or friendship networks is absent. This paper introduces an ordering-free local dependence measure which is invariant to any permutation of the observations and can be used to express various notions of temporal and spatial weak dependence. The paper begins with the two-sided testing problem of a population mean, and introduces a randomized subsampling approach where one performs inference using U-statistic type test statistics that are constructed from randomized subsamples. The paper shows that one can obtain inference whose validity does not require knowledge of dependence ordering, as long as dependence is sufficiently “local” in terms of the ordering-free local dependence measure. The method is extended to models defined by moment restrictions. This paper provides results from Monte Carlo studies.

KEY WORDS. Randomized Subsampling Inference; Local Dependence; Cross-Sectional Dependence; Dependency Graphs; Large Networks; Ordering-Free Inference; Randomized Tests

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1. Introduction

A common practice in empirical analysis with a huge amount of data is to focus on a random subsample of much smaller size\(^1\). The practice is often for computational expediency. However, this paper draws attention to two other important consequences of using a random subsample from a huge data. First, using a random subsample, the inference can be robustified to unknown local dependence ordering. Second, using multiple random subsamples, one can enhance the accuracy of inference, without destroying the robustness property.

Inference that is robust to an unknown *strength* of dependence among observations has long received attention in the literature since seminal papers by White (1980), Newey and West (1987) and Andrews (1991). (See the literature review below for more recent developments.) This literature often assumes that the econometrician knows the dependence ordering correctly. This means that for each sample unit, he knows what other sample units are strongly dependent or weakly dependent with the given sample unit. Dependence ordering is naturally given in time series data or spatial data with geographic distances. However, when dependence arises from a latent process through which economic agents interact with each other, the econometrician hardly knows the dependence ordering with precision.

It is not this paper’s purpose to provide a general framework that includes various structural models of interactions among agents. Rather as an initial exploration of this issue, this paper focuses on a simple situation where the dependence pattern itself is not part of the interest in inference. While the situation excludes empirical studies such as a study of peer effects on a large network, it still accommodates many situations where the main interest is in the relation among variables within the same sample unit (e.g. the relation between \(Y_i\) and \(X_i\) for the same \(i\).)

This paper develops an inference approach which does not require specification of dependence ordering, as long as the dependence is “local” enough. In this paper, *local dependence* roughly refers to a dependence structure among a large number of observations where each observation has only a small number of other observations that it is strongly dependent with. Hence local dependence encompasses both temporal and spatial weak dependence.

To define the notion of “locality” of dependence precisely, this paper introduces a local dependence measure that is *ordering-free* in the sense that the set of observations has the same local dependence measure as any permutation of the observations. Roughly speaking, the local dependence measure is based on the likelihood that any randomly chosen small subset of observations turn out to be strongly dependent with one another. When the observations are dependent only locally, i.e., each observation has only a small number

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\(^1\)Varian (2014) notes, for example, “At Google, for example, I have found that random samples on the order of 0.1 percent work fine for analysis of business data.”
of other observations with which it is strongly correlated, this likelihood should be small, yielding a low measure of local dependence. As shown in this paper, the ordering-free measure of local dependence can be used to express various notions of temporal and spatial weak dependence such as strong mixing time series or random fields, $m$-dependent series, observations with a dependency graph, and observations with a group-dependence structure such that within-group dependence is permitted but between-group dependence is not.

This paper begins with a simple problem of testing on the population mean from locally dependent observations whose local dependence conditions are expressed solely in terms of the ordering-free local dependence measure. Then it focuses on a class of test statistics constructed by using a combination of random permutations and subsampling. The main intuition behind this approach is that when we randomly permute the observations and select subsamples where the subsample size is relatively small, we will have a set of observations that are close to be mutually independent. This paper calls this approach of combining random permutations with subsampling to construct a test statistic the randomized subsampling approach.

When applying this approach, it turns out that the choice of test statistics matters. For example, a naive choice of a sample-mean type estimator does not produce valid inference unless one makes a strong assumption on the number of random permutations and subsamples. This paper proposes a U-type test statistic (after “U-statistic”), and shows that under weaker conditions on the number of the permutations and the size of subsamples, the test becomes asymptotically valid and its limiting normal distribution does not involve a (temporal or spatial) long-run variance. These conditions can be checked for various triangular arrays with temporal and spatial weak dependence. For example, when the observations are locally dependent with a dependency graph with a bounded degree and the number of random permutations is set to be the full sample size $n$ along with the subsample size set to be $n^{1/3}$, the U-type test statistic converges in distribution to a distribution that involves only a short run variance. Thus using the test, one does not need to estimate the long run variance, despite the presence of local dependence.

The use of normal approximation may not deliver tests with stable finite sample properties in various contexts of application. This paper also considers permutation critical values, and shows its asymptotic validity under the same condition as that for asymptotic validity of the original test statistic. Our Monte Carlo simulation study shows that permutation critical values perform much more stably than asymptotic critical values.

The crucial question in this approach is its local power property. Certainly using only subsamples will reduce the power of the test as compared to using a full sample. However, we use many randomized subsamples in constructing a test statistic to enhance the power properties. It turns out that under the stated conditions in the randomized subsampling
approach, the tests proposed in the paper exhibit nontrivial power at the rate that is slightly slower than $\sqrt{n}$. Therefore, there is a price (in terms of the convergence rate of the test) to pay for obtaining robust inference without knowledge of dependence ordering.

This paper reports results from a Monte Carlo simulation study that investigates the finite sample performance of the randomized subsampling approach. The simulation study considered independent observations, dependency graphs and network dependent observations. Dependency graphs refer to graphs such that any two sets of observations that are not linked with an edge are independent. Network dependent observations refer to a set of observations where dependence between two observations decreases in the length of the shortest path between the two observations. Hence unlike dependency graphs, network dependent observations admit dependence between two indirectly connected observations.

The simulation results show that the randomized subsampling approach performs well for the case of dependency graphs. However, the performance in terms of the finite coverage probabilities becomes unstable for the network dependent observations unless the correlation is substantially weak. Certainly, when the sample size is large, the finite sample properties in the case of network dependent observations can be improved by choosing the subsample size and the number of the random permutations smaller. However, this stable size property carries a cost: it will lead to an increase in the false coverage probability in general. While this latter increase in the false coverage probability will not matter much if the sample size is already huge, it will if not.

**Literature Review:** The early literature of robust inference under weak dependence and heteroskedasticity has focused on consistent estimation of asymptotic covariance matrix (White (1980), Newey and West (1987) and Andrews (1991)). The consistent estimation involves a choice of a tuning parameter which needs to increase not too fast with the growing sample size to achieve consistency. A more recent strand of literature uses inconsistent scale normalizer that uses a fixed smoothing parameter (Kiefer, Vogelsang and Bunzel (2000) and Kiefer and Vogelsang (2002)). For example, Phillips, Sun and Jin (2007) and Sun, Phillips and Jin (2011) considered power kernels. Jansson (2004) and Sun, Phillips and Jin (2008) explored higher order accuracy of fixed smoothing asymptotics over increasing smoothing asymptotics. Sun (2014) established related results in a more general context of two-step GMM estimation. In the meanwhile, Müller (2007) showed non-robustness of HAC estimators to local perturbations of the DGP and proposed a class of quadratic long-run variance estimators. Sun and Kim (2015) proposed asymptotic F tests based on fixed smoothing asymptotics in the GMM framework with weakly dependent random fields. Also apart from this econometrics literature, it is worth noting that Shao and Politis (2013) applied the fixed smoothing approach to subsampling inference that uses a calibration method.
Robust estimation of asymptotic covariance matrix has received a great deal of attention in the literature of linear panel models and spatial models as well. Arellano (1987) proposed HAC estimation in linear panel models with fixed effects. Driscoll and Kraay (1998) suggested a simple approach of HAC estimation in linear panel models based on cross-sectional averages of moment functions. Vogelsang (2012) compares the two approaches using fixed smoothing asymptotics. See also Kelejian and Prucha (2007) and Kim and Sun (2011) for HAC estimation in spatial models.

A closely related approach in dealing with cross-sectional dependence is a clustered error approach. For example, Cameron, Gelbach and Miller (2008) proposed and studied bootstrap approaches to deal with clustered errors. This literature typically assumes many independent clusters in a linear set-up. Ibragimov and Müller (2010) proposed a novel approach based on t-statistics where the observations are divided into multiple clusters that are (approximately) independent from each other and the inference is based on within-cluster estimators that are asymptotically normal. Bester, Conley and Hansen (2011) elaborated this multiple-cluster approach in linear panel models and provided conditions that are more primitive than those of Ibragimov and Müller (2010). In contrast to the cluster error approach and various HAC or HAR inference procedures, the t-statistic approach of Ibragimov and Müller (2010) allows for the clusters to be few and to be heterogeneous on various dimensions such as size or within-cluster dependence strength. However, the approach requires knowledge of this group structure of (at least approximately) independent clusters.

Also, it is worth noting that a strand of literature has focused on developing tests for cross-sectional dependence using mainly cross-sectional variations. Pesaran (2004) developed a general test for cross-sectional dependence in linear panel models with a short time series dimension. See also Hsiao, Pesaran and Pick (2012) for an extension to limited dependent models. Robinson (2008) proposed a correlation test that can be applied for testing cross-sectional dependence in a spatial model.

Closer to the spirit of ordering-free inference under local dependence is a recent contribution by Kuersteiner and Prucha (2013). (See also Kuersteiner and Prucha (2015) for an extension to linear quadratic moment restrictions in dynamic panel models which accommodate social interactions and networks models.) They considered a linear panel model with a large cross-sectional dimension which does not require a long time series. Adopting a sequential exogeneity condition, and assuming conditional moment type restrictions, they obtained a limit theory for GMM estimators that accommodate unknown common shocks and various latent cross-sectional dependence structures. In their linear panel set-up, the conditional moment type restrictions render the errors to be conditionally uncorrelated given common shocks and other covariates, and thus permit the martingale approach which was used by Andrews (2005) for addressing the presence of common shocks in linear models with a short
time series. On the other hand, the current paper considers a more primitive set-up that
does not necessarily involve errors that are conditionally uncorrelated. Thus this paper’s
proposal is entirely different from their approach.

Lastly, it is worth noting that the use of permutations and subsamples in this paper
is different from permutation tests and subsampling-based inference in the literature. Most
importantly, the main use of permutations or subsamples in this paper is for constructing test
statistics rather than for finding critical values. This paper does propose permutation-based
critical values, but in this case, the original test statistic involves randomized subsamples,
and hence the permutation-based critical values are more like those from a Monte Carlo test
than from a standard permutation test.

Organization of the Paper: Section 2 introduces a local-dependence measure that is
ordering-free, and illustrates its meaning through examples. Section 3 focuses on the prob-
lem of inference on the population mean. The section introduces a randomized subsampling
approach and shows its validity using U-type test statistics. The section also introduces per-
mutation critical values. Section 4 extends the framework to a situation where the parameter
of interest is defined through moment restrictions. Then we move onto a Monte Carlo sim-
ulation study in Section 5, where small sample properties are investigated through various
spatial dependence configurations. Section 6 concludes. Technical proofs of the results in
this paper are found in Appendix that is Section 7.

2. Locally Dependent Triangular Arrays and Examples

2.1. Ordering-Free Local Dependence Measure

This section introduces a measure of local dependence for a triangular array of random
vectors which is invariant to any permutation of the observations. Given an integer \( n \geq 1 \),
we define

\[ [n] \equiv \{1, 2, ..., n\}, \]

and for each integer \( k \geq 1 \), we also write generically \([k] = \{1, 2, ..., k\}\). Because the set of
integers has a natural ordering, we do not distinguish between set \([k]\) and vector \((1, 2, ..., k)\).

For each \( A \subset [n] \), let

\[ \mathcal{P}(A) \equiv \{ \{A_1, A_2\} : A_1 \cup A_2 = A, A_1 \cap A_2 = \emptyset, \text{ and } A_1, A_2 \neq \emptyset \}, \]

i.e., the collection of the sets \( \{A_1, A_2\} \) of two nonempty subsets \( A_1, A_2 \) of \([n]\) which constitute
a partition of \( A \).
Let \( \{X_{i,n}\}_{i \in [n]} \) be a triangular array of random vectors in \( \mathbf{R}^m \). Let \( \mathcal{C}_n \) be a given \( \sigma \)-field that serves as a latent common shock to the triangular array \( \{X_{i,n}\}_{i \in [n]} \) and thus as a source of strong dependence (Andrews (2005)). Let \( \mathcal{F} \) be the union of \( \mathcal{F}_l \)'s over \( l \in [n] \), where \( \mathcal{F}_l \) is a given class of real measurable functions on \( \mathbf{R}^m \). (For this paper’s proposal, it suffices to take the class of functions \( \mathcal{F}_l \) as a finite set of polynomials of a certain degree. See a remark prior to Assumption 1 below.) For each \( A, A_2 \in \mathcal{P}(A) \) such that \( A = \{i_1, \ldots, i_a\} \) and \( A_2 = \{j_1, \ldots, j_b\} \), we define

\[
c_F(A, A_2; \mathcal{C}_n) = \sup_{(f_1, f_2) \in \mathcal{F}_A \times \mathcal{F}_b} |Cov(f_1(X_{i_1,n}, \ldots, X_{i_a,n}), f_2(X_{j_1,n}, \ldots, X_{j_b,n})| \mathcal{C}_n),
\]

where \( Cov(X,Y|\mathcal{C}_n) \) is conditional covariance between two random variables \( X \) and \( Y \) given \( \mathcal{C}_n \). Thus \( c_F(A, A_2; \mathcal{C}_n) \) measures the strength of the conditional dependence between \( \{X_{i,n} : i \in A_1\} \) and \( \{X_{i,n} : i \in A_2\} \) given \( \mathcal{C}_n \). For any \( A \subset [n] \), we let

\[
c_F(A; \mathcal{C}_n) = \min_{\{A_1, A_2\} \in \mathcal{P}(A)} c_F(A_1, A_2; \mathcal{C}_n),
\]

if \( |A| \geq 2 \), and let \( c_F(A; \mathcal{C}_n) = \sup_{f \in \mathcal{F}_A} Var(f(X_{i,n})|\mathcal{C}_n) \) if \( A = \{i\} \). (The minimum over an empty set in the above expression is taken to be zero.) For example, when there exists \( i \in A \) such that \( X_{i,n} \) is conditionally independent of \( \{X_{j,n} : j \in A \setminus \{i\}\} \) given \( \mathcal{C}_n \), we have \( c_F(A; \mathcal{C}_n) = 0 \).

Let \( \Pi \) be the collection of permutations on \( [n] \) with its generic element denoted by \( \pi \). We measure the strength of local dependence of the triangular array \( \{X_{i,n}\}_{i \in [n]} \) through function \( \lambda_F : [n] \to [0, \infty) \) defined as follows: for each \( k \in [n] \),

\[
\lambda_F(k; \mathcal{C}_n) = \frac{1}{|\Pi|} \sum_{\pi \in \Pi} c_F(\pi[k]; \mathcal{C}_n),
\]

where \( \pi[k] = (\pi(1), \ldots, \pi(k)) \). The local dependence of a large set of observations is measured by the convergence rate of \( \lambda_F(k; \mathcal{C}_n) \) (with each fixed \( k \)) to zero as the sample size \( n \) goes to infinity. We call \( \lambda_F \) the \( \lambda \)-coefficient of \( \{X_{i,n}\}_{i \in [n]} \) (with respect to a given class \( \mathcal{F} \) and a given common shock \( \mathcal{C}_n \)). We say that the triangular array \( \{X_{i,n}\}_{i \in [n]} \) is locally dependent if there exists \( q > 1 \) such that for all \( 1 \leq k \leq q \), \( \lambda_F(k; \mathcal{C}_n) \to 0 \) as \( n \to \infty \).

The local dependence measure \( \lambda_F \) is not scale-free, but this does not cause a problem for our purpose, because all we require of the measure \( \lambda_F(k) \) eventually is its (not necessarily sharp) convergence rate to zero as \( n \to \infty \) for each fixed \( k \).

For example, suppose that the random variables are such that each random variable is conditionally independent of all but a small number of other random variables given \( \mathcal{C}_n \). Then for most permutations, we will have at least one random variable, say, \( X_{\pi(i),n} \), which is conditionally independent of \( \{X_{\pi(j),n} : j \in [k] \setminus \{i\}\} \) given \( \mathcal{C}_n \). Thus for each fixed \( k \geq 2 \), \( \lambda_F(k; \mathcal{C}_n) \) goes to zero as \( n \to \infty \). Certainly the latter convergence will still follow if we weaken
the conditional independence assumption to some conditional weak dependence assumption. This observation is made precise when we go over examples in the next subsection.

The local dependence condition is weaker than the assumption that \( X_{i,n} \)’s are conditionally independent given \( C_n \). When \( X_{i,n} \)’s are conditionally independent given \( C_n \), we have

\[
\lambda_F(k; C_n) = 0,
\]

for any choice \( F \) and \( k \geq 1 \), because \( c_F(A; C_n) = 0 \) for any \( A \subset [n] \), and this excludes many models of weakly dependent time series or cross-sectional observations. On the other hand, the local dependence condition includes them as it requires only that \( \lambda_F(k; C_n) \) converges to zero at a certain rate when \( n \) goes to infinity.

The most distinctive feature of the \( \lambda \)-coefficient is that it is ordering-free. In other words, two triangular arrays \( \{X_{i,n}\}_{i \in [n]} \) and \( \{Y_{i,n}\}_{i \in [n]} \) have the same \( \lambda \)-coefficient if \( X_{i,n} = Y_{\pi(i),n} \) for all \( i \in [n] \), for some \( \pi \in \Pi \).

When there is no common shock (so that we write \( c_F(A_1, A_2; C_n) \) and \( c_F(A; C_n) \) simply as \( c_F(A_1, A_2) \) and \( c_F(A) \)) and the strong mixing conditions are used to characterize dependence among observations, one can translate the conditions on the \( \lambda \)-coefficient into those on the strong mixing coefficients using a covariance inequality. To see this, define

(2.1) \[
\alpha(A_1, A_2) = \sup_{i \in A_1} \sup_{j \in A_2} \alpha(\sigma(X_{i,n}), \sigma(X_{j,n})),
\]

where \( \alpha(\sigma(X_{i,n}), \sigma(X_{j,n})) \) denotes the strong mixing coefficient between the two \( \sigma \)-fields \( \sigma(X_{i,n}), \sigma(X_{j,n}) \) generated by \( X_{i,n} \) and \( X_{j,n} \), i.e.,

\[
\alpha(\sigma(X_{i,n}), \sigma(X_{j,n})) = \sup_{A, B} \left| P\{X_{i,n} \in A, X_{j,n} \in B\} - P\{X_{i,n} \in A\} P\{X_{j,n} \in B\} \right|,
\]

with the supremum being that over all the Borel sets \( A \) and \( B \). Then by covariance inequality (e.g. Corollary A.2 of Hall and Heyde (1980), p.278), for any disjoint subsets \( A_1 \) and \( A_2 \) of \( [n] \) such that \( k = |A_1| + |A_2| \), we have for some constant \( C > 0 \), and \( p > 2 \),

\[
c_F(A_1, A_2) \leq CH^2_{p,F}(k) \alpha(A_1, A_2)^{1-(2/p)},
\]

where

(2.2) \[
H_{p,F}(k) = \max_{\pi \in \Pi} \max_{1 \leq a \leq k} \sup_{f \in F_a} \left( \mathbb{E}[|f(X_{\pi(1)}, \ldots, X_{\pi(a),n})|^p] \right)^{1/p}.
\]

Using this relation, one can characterize sufficient conditions for local dependence in terms of the strong mixing coefficients. See Section 2.2.2 for an example in the case of weakly dependent random fields.

The notion of local dependence is based on covariance between functions of sets of random variables, and is similar to the weak dependence notion proposed by Doukhan and
except that the dependence measure in this paper is made invariant to the permutations of the observations, and hence does not require any reference to the underlying dependence ordering.

2.2. Examples of Locally Dependent Triangular Arrays

In this section, we relate the $\lambda$-coefficient to existing dependence concepts in the literature.

2.2.1. Triangular Arrays with a Dependency Graph. Let a graph $G_n = ([n], E_n)$ over $[n]$ be given, where $E_n$ denotes the collection of pairs $ij$, with $i, j \in [n]$ representing an edge (or a link) between vertices (or nodes) $i$ and $j$. (We exclude loops, i.e., for all $ij \in E_n$, $i \neq j$. Also we assume that the graph is undirected so that whenever $ij \in E_n$, $ji \in E_n$.) Let $(X_{i,n})_{i \in [n]}$ be a given triangular array of random variables. Let us say that $G_n$ is a conditional dependency graph for $(X_{i,n})_{i \in [n]}$ given a $\sigma$-field $C_n$, if for any two subsets $A_1$ and $A_2$ of $[n]$ such that $\{ij \in E_n : i \in A_1, j \in A_2\} = \emptyset$, $(X_{i,n})_{i \in A_1}$ and $(X_{i,n})_{i \in A_2}$ are conditionally independent given $C_n$.

A conditional dependency graph specifies which random variables are conditionally independent given $C_n$. Having a conditional dependency graph does not put restrictions on stochastic dependence between any pair of random variables joined by an edge in the graph, but imposes conditional independence on all other pairs. When $C_n$ is a trivial $\sigma$-field, the notion of conditional dependency graphs is reduced to the well-known notion of dependency graphs. (See e.g. [Penrose (2003), p.22].)

The case of a triangular array with a dependency graph having a bounded maximum degree includes $m$-dependent time series as a special case. Also dependence with many clusters, where there is a partition of the observations into clusters and dependence is restricted to within-cluster observations, not between clusters, is a special case of local dependence with a dependency graph.

Let us find a bound for the $\lambda$-coefficient of the triangular array $(X_{i,n})_{i \in [n]}$. A simple combinatoric argument gives the following lemma.

**Lemma 2.1.** Suppose that $(X_{i,n})_{i \in [n]}$ is a triangular array of random variables having $G_n = ([n], E_n)$ as a conditional dependency graph given $C_n$. Furthermore, let us assume that for some $q \geq 2$, there exists $C_q > 0$ such that for all $n \geq 1$,

\[
\max_{\pi \in \Pi} \sup_{1 \leq a \leq q} \max_{f \in F_n} E[|f(X_{\pi(1),n}, \ldots, X_{\pi(a),n})|^2 | C_n] < C_q.
\]

\[\text{(2.3)}\]

A degree of a node $i$, denoted by $d_n(i)$, is the size of its neighborhood, i.e., $d_n(i) = |\{j \in [n] : ij \in E_n\}|$. The maximum degree is $\max_{i \in [n]} d_n(i)$, i.e., the maximum of $d_n(i)$ over $i \in [n]$. 

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Then for each integer \( 1 \leq k \leq q \), there exists \( C_k > 0 \) such that for all \( n \geq 1 \),
\[
\lambda_F(k; C_n) \leq C_k n^{-k} d_n^{k+1/2},
\]
where \([a]\) for a real number \( a \) is the greatest integer that is not larger than \( a \).

Suppose that \( d_n \) is bounded. Then at any \( 1 \leq k \leq q \), the \( \lambda \)-coefficient converges to zero as \( n \to \infty \), at the rate of \( n^{k/2} \).

2.2.2. Weakly Dependent Random Fields. Suppose that a random field \((Y_{j,n})_{j \in \mathbb{Z}_d^q}\) is
given, where \( \mathbb{Z}_d^q \) is a subset of \( \mathbb{Z}_d \) such that \( n = |\mathbb{Z}_d^q| \) and \( \mathbb{Z}_d \) is a lattice as a subset of \( \mathbb{R}^d \).

For each pair \((j_1, j_2) \in \mathbb{Z}_d \times \mathbb{Z}_d\), we define distance
\[
d(j_1, j_2) = \max_{1 \leq k \leq d} |j_{1,k} - j_{2,k}|
\]
where \( j_{1,k} \) and \( j_{2,k} \) are the \( k \)-th entry of \( j_1 \) and \( j_2 \) respectively. We assume that the lattice \( \mathbb{Z}_d \) is
infinite countable having \( d_0 > 0 \) such that for all \( j_1, j_2 \in \mathbb{Z}_d \), we have \( d(j_1, j_2) \geq d_0 \). Hence as in Conley (1999) and Jenish and Prucha (2009a), we exclude the infill asymptotics where the sampling points become dense in a given domain as the sample size increases.

To map the random field to a triangular array, let \( \mu : [n] \to \mathbb{Z}_d^q \) be a one-to-one map, so
that we let \( X_{i,n} = Y_{\mu(i),n}, i = 1, \ldots, n \), and define \( \mu(i,j) = d(\mu(i), \mu(j)) \) for simplicity. Also, for given subsets \( A, A' \) of \([n]\), let
\[
d_{\mu}(A, A') = \min_{i \in A, j \in A'} d(\mu(i), \mu(j)).
\]

For simplicity, let us assume that there is no common shock. Define for \( m \geq 0 \),
\[
\bar{c}_{m,F}(A) = \min_{\{A_1, A_2\} \in \mathcal{P}(A) : m \leq d_{\mu}(A_1, A_2) < m+1} c_F(A_1, A_2).
\]
Hence \( \bar{c}_{m,F}(A) \) measures stochastic dependence among \( X_{i,n} \)'s with \( i \in A \), when there exists a
partition \( (A_1, A_2) \) of \( A \) such that \( m \leq d_{\mu}(A_1, A_2) \leq m+1 \). The following lemma characterizes
a bound for the \( \lambda \)-coefficient of \( \{X_{i,n}\}_{i \in [n]} \) in terms of \( \bar{c}_{m,F}(A) \).

**Lemma 2.2.** Suppose that \((Y_{j,n})_{j \in \mathbb{Z}_d^q}\) is a random field and let \( X_{i,n} = Y_{\mu(i),n}, i = 1, \ldots, n \).
Suppose further that the moment condition in (2.3) holds for \( \{X_{i,n}\} \) with some \( q \geq 2 \), and
for each \( 2 \leq k \leq q \), there exists \( C_k > 0 \) such that
\[
\sum_{m=1}^{\infty} m^{(k-1)(d-1)} \max_{\pi \in \Pi} \bar{c}_{m,F}(\pi[k]) \leq C_k, \text{ for all } n \geq 1.
\]
(2.4)

Then, for any fixed \( 1 \leq k \leq q \), there exists \( C_k > 0 \) such that
\[
\lambda_F(k) \leq C_k n^{-k+1}, \text{ for all } n \geq 1.
\]

When the random field is a strong-mixing random field, the condition (2.4) can be verified
in terms of the strong-mixing coefficient \( \alpha \) defined in (2.1). More specifically, suppose that
for some \( p > 2 \), \( H_{p,F}(k) < \infty \) for all \( k \in [2q] \), where \( H_{p,F}(k) \) is as defined in (2.2). Then
through covariance inequality, the condition (2.4) is expressed as the existence of a constant $C > 0$ such that for all $n \geq 1$,

$$
\sum_{m=1}^{\infty} m^{(k-1)(d-1)} \bar{\alpha}_m(k)^{1-(2/p)} < C,
$$

where

$$
\bar{\alpha}_m(k) = \max_{\pi \in \Pi} \min_{\{A_1,A_2\} \in \mathcal{P}(\pi[k]):m \leq d_{\mu}(A_1,A_2)<m+1} \alpha(A_1,A_2).
$$

Thus we obtain a sufficient condition for (2.4) in terms of the strong mixing coefficients on the random field.

### 3. Ordering-Free Inference on the Population Mean

#### 3.1. A Randomized Subsampling Approach

This section focuses on the simple set-up of estimating a mean from locally dependent data. Suppose that we are given observed random vectors $X_{1,n}, \ldots, X_{n,n} \in \mathbb{R}^m$ and a (potentially latent) common shock $C_n$ such that

$$
E[X_{i,n}] = E[X_{i,n}|C_n],
$$

for all $i \in [n]$. Thus we assume that $X_{i,n}$ is conditionally mean independent of the common shock $C_n$. (In a more general model with moment restrictions, this condition is translated into a conditional moment restriction given a common shock. See Section 4 below for more details.)

Let us assume that $X_{i,n}$’s have the same mean, so that we define

$$
\mu_0 \equiv E[X_{i,n}].
$$

The main goal here is to develop a procedure to yield an asymptotically valid confidence set for $\mu_0$. For this, consider the following testing problem of null and alternative hypotheses.

$$
H_0 : \mu_0 = \mu, \text{ against } \quad H_1 : \mu_0 \neq \mu,
$$

for some given number $\nu$.

Suppose that local dependence among observations across cross-sectional units is a nuisance to inference, rather than a feature of interest. In order to build up an inference procedure that is robust to any dependence ordering of observations, we construct a (randomized) test statistic as follows.
First, let $\Pi$ be the space of permutations on $[n]$ and $\pi_1, \ldots, \pi_R$ be i.i.d. draws from the uniform distribution on $\Pi$. Given $\tilde{\pi} = (\pi_1, \ldots, \pi_R)$ and $\mu \in \mathbb{R}^m$, we define

$$U_{n,r}(\mu; \tilde{\pi}) = \frac{1}{mb_n} \sum_{i,j \in [b_n]: i \neq j} (X_{\pi_r(i),n} - \mu)' \hat{\Sigma}^{-1} (X_{\pi_r(j),n} - \mu),$$

where

$$\hat{\Sigma} = \frac{1}{n} \sum_{i=1}^{n} (X_{i,n} - \bar{X})(X_{i,n} - \bar{X})',$$

and $b_n \leq n$ and $b_n \to \infty$. Then we define

$$S_n(\mu; \tilde{\pi}) = \frac{1}{\sqrt{R}} \sum_{r=1}^{R} U_{n,r}(\mu; \tilde{\pi}).$$

We introduce a test statistic as follows:

$$T_n(\mu; \tilde{\pi}) = S_n(\mu; \tilde{\pi}) - \frac{\sqrt{R}b_n}{n}.$$

This paper calls the test statistic $T_n(\mu; \tilde{\pi})$ a $U$-type test statistic (after “U-statistic”). The term $\sqrt{R}b_n/n$ is a bias adjustment term which converges to zero as $n \to \infty$ under Assumption 3.1 below and yet we include to enhance the small sample stability of the test statistics. (See Section 4 below for details.) As we shall see in the local power analysis, the test statistic $T_n(\mu; \tilde{\pi})$ is for a two-sided test, even though it does not involve absolute value.

This paper’s first main result shows that the test statistics converge in distribution to a standard normal distribution under certain conditions for local dependence. From here on, it suffices to take the class of functions $\mathcal{F}$ to be the union of $\mathcal{F}_l$ over $l \in [n]$, where $\mathcal{F}_l$ is the collection of real maps $\phi$ on $\mathbb{R}^{lm}$ of the form: $\phi(x_1, \ldots, x_l) = x_1^{k_1} x_2^{k_2} \cdots x_l^{k_l}$, where $|k_1| + \cdots + |k_l| \leq 8$ with $k_v = (k_{v,1}, \ldots, k_{v,m})$, $x_v^{k_v} = x_{v,1}^{k_{v,1}} \cdots x_{v,m}^{k_{v,m}}$, $|k_v| = \sum_{k=1}^{m} k_{v,k}$, and $k_{v,1}, \ldots, k_{v,m} \in [0,8]^m$. And with this definition, we simply write $\lambda(k; \mathcal{C}_n)$ instead of $\lambda_{\mathcal{F}}(k; \mathcal{C}_n)$, and define

$$\lambda_n(k) = E[\lambda(k; \mathcal{C}_n)].$$

The following assumption specifies local dependence in terms of the $\lambda$-coefficient.

**Assumption 3.1.** (i) $Rb_n^2 \{\lambda_n(4) + n^{-1}(\lambda_n(3) + \lambda_n(2)) + n^{-2}\} = o(1)$, as $n \to \infty$.

(ii) There exists $C > 0$ such that $b_n^{k-1} \lambda(k; \mathcal{C}_n) \leq C$ for all $n \geq 1$, for each $1 \leq k \leq 8$.

Assumption 3.1 specifies the requirement on local dependence of the triangular array $(X_{i,n})_{i \in [n]}$. For example in the case of a dependency graph with a bounded maximum degree, the condition requires that

$$n^{-2}b_n^2 R \to 0, \text{ and } b_n^2 n^{-1} = O(1).$$
Hence if we take $R = n$ and $b_n$ such that $b_n^2/n \to 0$ as $n \to \infty$, Assumption 3.1 is satisfied.

**Assumption 3.2.**

(i) $\sup_{n \geq 1} \max_{1 \leq i \leq n} \mathbb{E}[|X_{i,n}|^8|\mathcal{C}_n] < \infty$.

(ii) There exists $c > 0$ such that for all $n \geq 1$, the minimum eigenvalue of $\Sigma(\mathcal{C}_n)$ is greater than $c$, where

$$
\Sigma(\mathcal{C}_n) \equiv \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[(X_{i,n} - \mathbb{E}[X_{i,n}|\mathcal{C}_n])(X_{i,n} - \mathbb{E}[X_{i,n}|\mathcal{C}_n])'|\mathcal{C}_n].
$$

In Assumption 3.2, we require moment conditions and nondegenerate variances.

**Theorem 3.1.** Suppose that Assumptions 3.1-3.2 hold. Then,

$$
T_n(\mu_0; \tilde{\pi}) \overset{d}{\rightarrow} N(0,1).
$$

The test is asymptotically pivotal. The construction of the test statistic and its conditions do not require reference to the underlying dependence ordering at all. The sufficient condition is fully given in terms of the local dependence measure that is ordering-free.

### 3.2. Permutation Critical Values

The asymptotic normality in Theorem 3.1 suggests use of an asymptotic critical value from the standard normal table. In this section, we investigate an alternative approach in which one computes critical values based on random permutations.

For given $L$, we draw $\tilde{\pi}_l$, $l = 1, ..., L$, i.i.d., similarly as before, where $\tilde{\pi}_l = (\pi_{1,l}, ..., \pi_{R,l})$ and $\pi_{r,l}$’s are i.i.d. draws from the uniform distribution on $\Pi$. Define

$$
\tilde{c} = \inf \left\{ c > 0 : \frac{1}{L} \sum_{l=1}^{L} 1\{|S_n(\tilde{X}; \tilde{\pi}_l) > c\} \leq \alpha : l = 1, ..., L \right\},
$$

where $S_n(\tilde{X}; \tilde{\pi}_l)$ replaces $\mu$ in $S_n(\mu; \tilde{\pi}_l)$ by $\tilde{X}$. Note that we replace $\mu$ by the sample mean $\tilde{X}$ to ensure that the test has power when $\mu \neq \mu_0$. The following result establishes that the permutation critical values are asymptotically valid.

**Theorem 3.2.** Suppose that Assumptions 3.1-3.2 hold. Then as $n, L \to \infty$,

$$
P\{T_n(\mu_0; \tilde{\pi}) > \tilde{c}\} \to \alpha.
$$

The intuition behind the asymptotic validity of the permutation critical values is that the difference between $S_n(\tilde{X}; \tilde{\pi}_l)$ and $S_n(\mu_0; \tilde{\pi}_l)$ is asymptotically negligible under the stated conditions. The asymptotic normality of the test statistic comes from the weak convergence of the conditional distribution of $S_n(\tilde{X}; \tilde{\pi}_l)$ given observations to the standard normal distribution. However, the permutation critical values are directly based on the conditional
distribution of \( S_n(\bar{X}; \bar{\pi}) \) given observations. Thus they are expected to exhibit better finite sample properties than the asymptotic critical values.

3.3. Heuristics and Discussions

Let us see how the asymptotic validity of the randomized subsampling approach follows. For simplicity, we assume that there is no common shock \( C_n \) so that we write the short-run variance \( \Sigma(C_n) \) defined in Assumption 3.2(ii) as \( \Sigma \), and assume that we know \( \Sigma \).

First, we let \( Z_{i,n} = \Sigma^{-1/2}(X_{i,n} - \mu_0) \) and write

\[
S_n(\bar{X}; \bar{\pi}) = S_n(\mu_0; \bar{\pi}) + B_n = \frac{1}{m} \sqrt{R} \sum_{r=1}^{R} \frac{1}{b_n} \sum_{i=1}^{b_n} \sum_{j=1, j \neq i}^{b_n} Z_{\pi,r}(i,n)Z_{\pi,r}(j,n) + B_n,
\]

where \( \bar{Z} = n^{-1} \sum_{i=1}^{n} Z_{i,n} \) and

\[
B_n = \frac{\sqrt{R}(b_n - 1)}{m} \bar{Z}' \bar{Z} - \frac{\bar{Z}'}{m} \sqrt{R} \sum_{r=1}^{R} \frac{1}{b_n} \sum_{i=1}^{b_n} \sum_{j=1, j \neq i}^{b_n} (Z_{\pi,r}(i,n) + Z_{\pi,r}(i,n)).
\]

As for the last term in the definition of \( B_n \), note that

\[
\frac{\bar{Z}'}{\sqrt{R}} \sum_{r=1}^{R} \frac{1}{b_n} \sum_{i=1}^{b_n} \sum_{j=1, j \neq i}^{b_n} Z_{\pi,r}(i) = \frac{\bar{Z}'}{\sqrt{R}} \frac{b_n - 1}{b_n} \frac{1}{\sqrt{R}b_n} \sum_{r=1}^{R} \sum_{i=1}^{b_n} Z_{\pi,r}(i)
\]

\[
= \bar{Z}' \bar{Z} \sqrt{R}(b_n - 1) + A_n,
\]

where

\[
A_n = \frac{\bar{Z}'}{b_n} \frac{\bar{Z}'}{\sqrt{R}b_n} \sum_{r=1}^{R} \sum_{i=1}^{b_n} (Z_{\pi,r}(i) - \bar{Z}).
\]

One can show that the conditional expectation of \( A_n^2 \) given \((Z_{i,n})_{i=1}^{n}\) is \( O_P(b_n/n) \). Thus,

\[
B_n = -\frac{\sqrt{R}(b_n - 1)}{m} \bar{Z}' \bar{Z} + O_P(\sqrt{b_n/n}).
\]

Therefore, we conclude that

\[
S_n(\bar{X}; \bar{\pi}) = S_n(\mu_0; \bar{\pi}) - \frac{\sqrt{R}(b_n - 1)}{m} \bar{Z}' \bar{Z} + O_P(\sqrt{b_n/n})
\]

\[
= T_n^*(\mu_0; \bar{\pi}) - \frac{\sqrt{R}(b_n - 1)}{n} (n\bar{Z}' \bar{Z} - n - n\lambda_n(2)) + O_P(\sqrt{b_n/n}),
\]

where

\[
T_n^*(\mu_0; \bar{\pi}) = S_n(\mu_0; \bar{\pi}) - \frac{\sqrt{R}(b_n - 1)(1 + n\lambda_n(2))}{n},
\]

(3.2)
Note that
\[ n\mathbb{E}[\bar{Z}'\bar{Z}] - m = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[Z'_{i,n}Z_{i,n}] - m + \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1; j \neq i}^{n} \mathbb{E}[Z'_{i,n}Z_{j,n}] \]
and the absolute value of the last term is bounded by \( mn\lambda_n(2) = o(1) \). Since \( \sqrt{R}b_n/n = o(1) \) by Assumption 3.1(i), it follows that
\[ S_n(\bar{X}; \tilde{\pi}) = T_n^*(\mu_0; \tilde{\pi}) + o_P(1). \]

Since \( S_n(\bar{X}; \tilde{\pi}) \) is a sum of i.i.d. random variables conditional on \( (Z_{i,n})_{i=1}^{n} \) (due to the i.i.d. property of random permutations), we can apply the central limit theorem to establish its asymptotic normality.

As for the test statistic, \( T_n^*(\mu_0; \tilde{\pi}) \) is not feasible, because the last term \( \sqrt{R}(b_n - 1)(1 + n\lambda_n(2))/n \) in (3.2) is not consistently estimable without knowledge of dependence ordering. Thus we take the case of \( X_{i,n}'s \) being independent as a benchmark in which case \( \lambda_n(2) = 0 \), and replace \( \lambda_n(2) \) by zero. We obtain the following form of a test statistic:
\[ (3.4) \quad T_n(\mu_0; \tilde{\pi}) = S_n(\mu_0; \tilde{\pi}) - \frac{\sqrt{R}b_n}{n}, \]
(Replacing \( b_n - 1 \) by \( b_n \) for simplicity.) Under Assumption 3.1 the bias adjustment term \( \sqrt{R}b_n/n \) is asymptotically negligible, as \( n \to \infty \). However, including it stablizes the finite sample size properties of the test.

The approximation in (3.3) suggests that instead of using a normal distribution, one can directly use the distribution of \( S_n(\bar{X}; \tilde{\pi}) \) by drawing \( \tilde{\pi} = (\pi_1, ..., \pi_R) \) from the uniform distribution on \( \Pi^R \). The critical value obtained this way is nothing but the permutation critical value that we saw before. Since the permutation critical value is directly based on the term \( S_n(\bar{X}; \tilde{\pi}) \) that we apply the central limit theorem, one can expect that the finite sample property of the permutation critical values tends to be better than asymptotic critical values. This is confirmed in the simulation study which is reported later.

One might ask what if we simply use instead the following sample mean type test statistic: (for example in the case of \( m = 1 \))
\[ T_{n,M}(\mu; \tilde{\pi}) = \left| \frac{1}{\sqrt{R}} \sum_{r=1}^{R} \frac{1}{\sqrt{b_n}} \sum_{i=1}^{b_n} X_{\pi_r(i),n} - \mu \right|, \]
where \( \sigma^2 \) is the univariate counterpart of \( \Sigma \). It turns out that the randomized subsampling approach does not work with this test statistic unless one makes a much stronger restriction.
on $R$ and $b_n$. To see this, we write

$$\frac{1}{\sqrt{R}} \sum_{r=1}^{R} \frac{1}{\sqrt{b_n}} \sum_{i=1}^{b_n} X_{\pi_r(i),n} - \mu_0 \sim \frac{\sqrt{R}b_n}{n} \sum_{i=1}^{n} Z_{i,n} \to_d N(0,1).$$

The term $(1/\sqrt{n}) \sum_{i=1}^{n} Z_{i,n}$ converges to a normal distribution with the variance equal to the long-run variance, say, $\sigma_{LR}^2$, of $Z_{i,n}$’s. Thus we write

$$\frac{1}{\sqrt{R}} \sum_{r=1}^{R} \frac{1}{\sqrt{b_n}} \sum_{i=1}^{b_n} X_{\pi_r(i),n} - \mu \sim \frac{\sqrt{R}b_n}{n} \cdot A_{n,1} + A_{n,2},$$

where $A_{n,1} \to_d N(0, \sigma_{LR}^2)$ and $A_{n,2} \to_d N(0,1)$. Hence the test statistic is not asymptotically pivotal unless $Rb_n/n \to 0$ as $n \to \infty$. The latter condition is much stronger than the rate condition $\sqrt{Rb_n/n} \to 0$. (Simulation studies unreported here confirm the superiority of the U-type test statistic to the sample mean type test statistic in terms of the finite sample coverage probabilities.) Hence the randomized subsampling approach needs to be combined with a carefully chosen class of test statistics.

3.4. Local Power Analysis

One might wonder whether the use of the U-type test statistic leads to a test that achieves the $\sqrt{n}$ convergence rate, i.e., the same convergence rate that can be achieved with knowledge of local dependence ordering. In this section, it is shown that the answer is negative. To make the answer precise, consider the following Pitman local alternative:

$$\mu_0 = E[X_{i,n}] + \frac{\delta}{R^{1/4}b_n^{1/2}},$$

where $\delta \in \mathbb{R}^m \setminus \{0\}$ is a constant vector.

**Theorem 3.3.** Suppose that Assumptions 3.1-3.2 hold. Then for any constant $c > 0$,

$$P\{T_n(\mu_0; \hat{\pi}) > c\} \to 1 - E \left[ \Phi \left( c - \frac{1}{m} \delta^\prime \Sigma^{-1}(C_n) \delta \right) \right].$$

The local power function depends on the quadratic form of the drift term $\delta$. This shows that the test is for two-sided testing. The rate of convergence of the test is $R^{-1/4}b_n^{-1/2}$ which is slower than $\sqrt{n}$ by Assumption 3.1(i). The slower convergence rate of the test is essentially the cost of obtaining robust inference without requiring knowledge of local dependence ordering.
3.5. Non-randomized Inference

The tests based on $T_n$ are randomized tests, where there is randomness of the test statistic apart from that of the samples. Hence different researchers may have different results using the same data and the same model. To address this issue, this paper proposes the following approach of constructing confidence intervals. For a given large positive integer $S$, for each $s \in [S]$, we let $\tilde{\pi}_s = (\pi_{1,s}, \ldots, \pi_{R,s})$, where $\pi_{r,s}$’s, $r \in [R]$ and $s \in [S]$, are i.i.d. draws from the uniform distribution on $\Pi$. Then the randomized confidence interval at level $1 - \alpha$ is given as follows:

$$q(\mu; \alpha) = \frac{1}{S} \sum_{s=1}^{S} 1\{T_n(\mu; \tilde{\pi}_s) \leq c\},$$

where $c = z_{1-\alpha}$, i.e., the $1 - \alpha$ percentile of $N(0,1)$, when we use an asymptotic critical value and $c = \tilde{c}$ when we use a permutation critical value. We call $q(\cdot; \alpha)$ the randomized confidence function.

**Corollary 3.1.** Suppose that Assumptions 3.1-3.2 hold. Then for each $\mu \in \mathbb{R}^m$,

$$q(\mu; \alpha) \rightarrow_p \begin{cases} 1 - \alpha, & \text{if } \mu = \mu_0 \\ 0, & \text{if } \mu \neq \mu_0, \end{cases}$$

as $n, S \to \infty$ jointly.

The corollary says that the randomized confidence function converges in probability to $1 - \alpha$ at $\mu = \mu_0$ and 0 otherwise. The convergence of $q(\mu; \alpha)$ to 0 at $\mu \neq \mu_0$ reflects the consistency property of the randomized test. The result of Corollary 3.1 can be shown by slightly modifying the proof of Theorem 3.3.

Note that the randomness of the randomized confidence function disappears as the sample size goes to infinity. This makes contrast to the randomized test statistic $T_n(\mu; \tilde{\pi})$ whose randomness due to the random permutations does not disappear even with large samples. The main intuition behind this contrast comes from the fact that while we cannot increase $R$ to infinity arbitrarily fast, we can increase $S$ so as long as the computational limit permits. This is because the condition required of $R$ is made to ensure the asymptotic pivotalness of the limiting distribution, whereas the condition for $S$ is made to ensure that the simulated average of $1\{T_n(\mu; \tilde{\pi}_s) \leq c\}$ over $s = 1, \ldots, S$, converges to its population version.

While one may report the randomized confidence function as a measure of uncertainty regarding the parameter of interest, this may still pose difficulty interpreting, as these concepts are not familiar to many researchers in econometrics.\(^3\)

\(^3\)The notion of the randomized confidence function in this paper coincides with what Geyer and Meeden (2005) referred to as the membership function of fuzzy confidence intervals. The way randomized tests are
Thus this paper proposes considering a nonrandomized confidence set of the following form. Take $\beta \in (0, \alpha)$ and define

$$C_\alpha = \{ \mu : q(\mu; \alpha - \beta) \geq 1 - \alpha \}. \quad (3.5)$$

Then it is not hard to see from Corollary 3.1 that

$$\liminf_{n \to \infty} P\{\mu_0 \in C_\alpha\} \geq 1 - \alpha.$$ 

as $n \to \infty$. For example, we may take $\beta = 0.005$, which is used in simulation studies in this paper. Using a fixed value of $\beta$ yields a conservative inference result, and this can be remedied by choosing $\beta$ to decrease to zero satisfying appropriate rate conditions. However, in a spirit similar to Romano, Shaikh and Wolf (2014) (though in a different context), this paper proposes using asymptotics with a fixed value of $\beta$ for stable finite sample properties. (In simulation studies unreported in this paper, the choice of $\beta = 0$ was also used, and the results were not very different.)

motivated in this paper is different. Randomized tests here arise because the asymptotic pivotalness of the test (thus permitting ordering-free inference) prevents us from drawing an arbitrarily large number of random permutations.
4. Ordering-Free Inference on Models with Moment Restrictions

Let us extend the result to the case of moment restrictions. Suppose that we have a locally dependent triangular array of random vectors \( \{X_{i,n}\}_{i \in [n]} \), with \( X_{i,n} \in \mathbb{R}^{d_x} \), and that there is a true parameter \( \theta_0 \in \Theta \subset \mathbb{R}^{d_\theta} \) such that
\[
E[g(X_{i,n}; \theta_0) | C_n] = 0,
\]
for all \( i = 1, \ldots, n \), where \( g(\cdot; \theta) : \mathbb{R}^{d_x} \rightarrow \mathbb{R}^m \) is a given moment function. Here we do not assume that the moment restriction point-identifies \( \theta_0 \), nor do we require that \( g(\cdot; \theta) \) is continuous in \( \theta \).

The randomized subsampling approach developed for inference on the population mean applies in this set-up straightforwardly, by inverting the U-type test statistic. First, define
\[
\hat{\Sigma}(\theta) = \frac{1}{n} \sum_{i=1}^{n} (g_i(\theta) - \bar{g}(\theta)) (g_i(\theta) - \bar{g}(\theta))^\prime,
\]
where \( g_i(\theta) = g(X_{i,n}; \theta) \) and
\[
\bar{g}(\theta) = \frac{1}{n} \sum_{i=1}^{n} g_i(\theta).
\]
Let \( \tilde{\pi} = (\pi_1, \ldots, \pi_R) \) be given as before. Define
\[
S_n(\theta; \tilde{\pi}) = \frac{1}{\sqrt{R}} \sum_{r=1}^{R} \frac{1}{mb_n} \sum_{i,j \in [bn]; i \neq j} g_{\pi_r(i)}(\theta)^\prime \hat{\Sigma}^{-1}(\theta) g_{\pi_r(j)}(\theta),
\]
where \( m \) refers to the number of the moment restrictions. Then, we define our test statistic as follows:
\[
T_n(\theta_0; \tilde{\pi}) = S_n(\theta_0; \tilde{\pi}) - \frac{\sqrt{Rb_n}}{n}.
\]

Let us introduce the following assumption.

**Assumption 4.1.** Assumptions 3.1-3.2 hold when we replace \( (X_{i,n})_{i \in [n]} \) by \( (g_i(\theta_0))_{i \in [n]} \).

Assumption 4.1 can be readily checked. In particular, the local dependence conditions in Assumption 3.1 follows from those for the triangular arrays \( \{X_{i,n}\}_{i \in [n]} \). The following result is obtained similarly as in Theorem 3.1.

**Theorem 4.1.** Suppose that Assumption 4.1 holds. Then
\[
T_n(\theta_0; \tilde{\pi}) \rightarrow^d N(0,1).
\]

The normalization by \( \hat{\Sigma}^{-1}(\theta) \) eliminates asymptotically the dependence among the moment restrictions in the limiting distribution. Again the construction of the test statistic
and the critical values do not require knowledge of the dependence ordering of the triangular array \( \{X_{i,n}\}_{i \in [n]} \).

As for the permutation critical values, we draw \( \tilde{\pi}_l = (\pi_{1,l}, \ldots, \pi_{R,l}), \ l = 1, \ldots, L, \) i.i.d., similarly as before, and define

\[
\tilde{S}_n(\theta; \tilde{\pi}_l) = \frac{1}{\sqrt{R}} \sum_{r=1}^{R} \frac{1}{mb_n} \sum_{i,j \in [b_n]; i \neq j} (g_{\pi_{r,l}(i)}(\theta) - \bar{g}(\theta))^t \tilde{\Sigma}^{-1}(\theta)(g_{\pi_{r,l}(j)}(\theta) - \bar{g}(\theta)).
\]

Then we construct

\[
\tilde{c}(\theta) = \inf \left\{ c > 0 : \frac{1}{L} \sum_{l=1}^{L} 1\{ \tilde{S}_n(\theta; \tilde{\pi}_l) > c \} \leq \alpha \right\}.
\]

Let us turn to the randomized confidence function. For a given large positive integer \( S \), for each \( s \in [S] \), we let \( \tilde{\pi}_s = (\pi_{1,s}, \ldots, \pi_{R,s}) \) as before. Define

\[
q(\theta; \alpha) = \frac{1}{S} \sum_{s=1}^{S} 1\{ T_n(\theta; \tilde{\pi}_s) \leq c \},
\]

where \( c = z_{1-\alpha} \), i.e., the \( 1 - \alpha \) percentile of \( N(0, 1) \), or \( c = \tilde{c}(\theta) \) for the case of a permutation critical value.

Let us make the following assumption.

**Assumption 4.2.** (i) For each \( \theta \in \Theta \), \( E[g(X_{i,n}; \theta)|\mathcal{C}_n] \) is identical across all \( i \in [n] \).

(ii) For each \( \theta \in \Theta \), the smallest eigenvalue of \( \Sigma(\theta; \mathcal{C}_n) \) is bounded away from zero with probability one, where

\[
\Sigma(\theta; \mathcal{C}_n) = \frac{1}{n} \sum_{i=1}^{n} E \left[ (g_i(\theta) - E[g_i(\theta)|\mathcal{C}_n]) (g_i(\theta) - E[g_i(\theta)|\mathcal{C}_n])' |\mathcal{C}_n \right].
\]

Assumption 4.2 is made to simplify the results of the randomized confidence function in Corollary 4.1 below. One can relax the assumption at the expense of a more complex formulation of the conditions and the proofs.

Let us define for each \( \gamma \geq 0 \), \( \Theta_0(\mathcal{C}_n) = \{ \theta \in \Theta : \Sigma^{-1/2}(\theta; \mathcal{C}_n)E[g(X_{i,n}; \theta)|\mathcal{C}_n] = 0 \} \) and 

\[
\Theta(\gamma; \mathcal{C}_n) = \{ \theta \in \Theta : ||\Sigma^{-1/2}(\theta; \mathcal{C}_n)E[g(X_{i,n}; \theta)|\mathcal{C}_n|| \geq \gamma \}.
\]

When \( X_{i,n}'s \) are i.i.d. (conditional on \( \mathcal{C}_n \)), the set \( \Theta_0(\mathcal{C}_n) \) is an identified set for \( \theta_0 \) with respect to the conditional distribution of \( X_{i,n} \) given \( \mathcal{C}_n \). The set \( \Theta(\gamma; \mathcal{C}_n) \) is the collection of \( \theta \)'s that are away from the identified set. We are prepared to give the asymptotic result for the randomized confidence function for \( \theta_0 \).
**Corollary 4.1.** Suppose that Assumptions 4.1-4.2 hold. Then for each \( \theta \in \Theta \),

\[
q(\theta; \alpha) \xrightarrow{p} \begin{cases} 
1 - \alpha, & \text{if } P\{\theta \in \Theta_0(C_n)\} \to 1, \\
0, & \text{if } P\{\theta \in \Theta(\gamma; C_n)\} \to 1, \text{ for some } \gamma > 0.
\end{cases}
\]

Using the function \( q \), we can construct confidence sets as in (4.1). More specifically, take \( \beta \in (0, \alpha) \) and define

(4.1) \[ C_\alpha = \{ \theta \in \Theta : q(\theta; \alpha - \beta) \geq 1 - \alpha \}. \]

When the dimension of \( \theta \) is high, computing the function can be cumbersome for permutation critical values \( \tilde{c}(\theta) \), because one needs to compute the critical value for each \( \theta \). Then we may consider the following profiling method. Suppose that we are interested in \( \theta_1 \) which is a subvector of \( \theta \). Let \( \theta_2 \) be such that \( \theta = (\theta_1, \theta_2) \). Define

\[
q(\theta_1; \alpha) = \inf_{\theta_2} \frac{1}{S} \sum_{s=1}^{S} 1 \{ T_n(\theta; \tilde{\pi}_s) \leq \tilde{c}(\theta_1) \},
\]

where \( \tilde{c}(\theta_1) \) is defined as

\[
\tilde{c}(\theta_1) = \inf \left\{ c > 0 : \inf_{\theta_2} \frac{1}{L} \sum_{l=1}^{L} 1 \{ \tilde{S}_n(\theta; \tilde{\pi}_l) > c \} \leq \alpha \right\}.
\]

The use of this profiling method in general gives a conservative inference. The actual extent of conservativeness will depend on the specific application.

### 5. Simulation Studies

#### 5.1. Data Generating Process

This section presents and discusses a Monte Carlo simulation study which investigates the finite sample properties of the randomized subsampling approach in various situations with local dependence. As for local dependence, this study considered three kinds of data generating processes: (i) i.i.d. variables, (ii) variables having a dependency graph, (iii) network dependent variables. Both asymptotic and permutation critical values are considered.

As for the dependency graph case, there are two kinds of graphs considered. One is based on Erdös-Renyi graphs, and the other is based on Barabasi-Albert graphs of preferential attachment. In an Erdös-Renyi random graph, each pair of the vertices form an edge with equal probability \( p = \lambda/(n - 1) \). The simulation study here chose \( \lambda \) from \{1, 3, 5\}. Thus each vertex from this random graph has degree \( \lambda \) on average, and the degree distribution is approximately a Poisson distribution with parameter \( \lambda \) when the graph is large. For a
Barabasi-Albert random graph of preferential attachment, we first began with an Erdős-Rényi random graph of size 20 with $\lambda = 1$. Then we let the graph grow by adding each vertex sequentially and let the vertex form edges with $m$ other existing vertices. (We chose $m$ from $\{1, 2, 3\}$ for this study.) The probability of a new vertex forming an edge with an existing vertex is proportional to the number of the neighbors of the existing vertex. We keep adding new vertices until the size of the graph becomes $n$.

As for the generation of the random variables, we follow the design in Song (2015). We first generate $\{Y^*_i\}_{i=1}^n$ i.i.d. from $N(0, 1)$. Let $E = \{e_1, ..., e_S\}$ be the set of edges in the graph with redundant edges removed from $E$ (i.e., remove $ji$ with $j < i$) and let $M$ be two-column matrix whose entries are of the form $[i, j]$ for $e_s = i_j$. Let $M$ be sorted on the first column so that $i_s \leq i_{s+1}$.

**Step 1:** For $s = 1$, such that $e_1 = i_1j_1$, we draw $Z_1 \sim N(0, 1)$ and set

$$(Y_{i_1}, Y_{j_1}) = \sqrt{1 - c^2} \times (Y^*_{i_1}, Y^*_{j_1}) + c \times Z_1,$$

where $c$ is a parameter that determines the strength of graph dependence. We replace $(Y^*_{i_1}, Y^*_{j_1})$ by $(Y_{i_1}, Y_{j_1})$, and redefine the series $\{Y^*_i\}_{i=1}^n$.

**Step s:** For $s > 1$ such that $e_s = (i_s, j_s)$, we draw $Z_s \sim N(0, 1)$ and set

$$(Y_{i_s}, Y_{j_s}) = \sqrt{1 - c^2} \times (Y^*_{i_s}, Y^*_{j_s}) + c \times Z_s.$$ 

We replace $(Y^*_{i_s}, Y^*_{j_s})$ by $(Y_{i_s}, Y_{j_s})$, and redefine the series $\{Y^*_i\}_{i=1}^n$.

Let us say that random variables on a network are *weakly network dependent*, if dependence between any two random variables decreases with the distance between the two indices of the random variables. Here the distance between the two indices is defined to be the length of the shortest path between them in a given network. Therefore, unlike the case with a dependency graph, weakly network dependent random variables can be correlated with each other even if they are not joined by an edge, as long as they are connected in the network.

In this study, the observations are drawn from a jointly normal random vector with mean zero and a covariance matrix such that the correlation between two random variables at the distance $D$ is set to be $\exp(-\rho D)$, where $\rho$ is a parameter that determines the strength of the correlation. The parameter $\rho$ was taken from $\{1.0, 1.5, 2.0\}$. (Note that $\exp(-1)$ is roughly 0.3674 and $\exp(-2)$ is roughly 0.1353.) For the graph underlying the network-dependent observations, the simulation study considered a realization of a Erdős-Renyi random graph with $\lambda \in \{1, 2\}$.

The size $n$ of the networks was taken from $\{500, 1000, 3000\}$. As for $R$ and $b_n$, we chose $R = n$ and $b_n = n^{1/3}$. The Monte Carlo simulation number and the permutation number
Table 1. The Empirical Coverage Probability of Confidence Interval: Independent Observations

|       | Normal. |       |       |       |       |       |       |
|-------|---------|-------|-------|-------|-------|-------|-------|
|       | 99%     | 95%   | 90%   | 99%   | 95%   | 90%   |       |
| n = 500 | 0.9702  | 0.9237| 0.8813| 0.9913| 0.9487| 0.9002|
| n = 1000 | 0.9679  | 0.9178| 0.8730| 0.9916| 0.9488| 0.8998|
| n = 3000 | 0.9635  | 0.9089| 0.8613| 0.9916| 0.9475| 0.8971|

(denoted by $L$) in the computation of the critical values were set to be 1000. The tuning parameter $\beta$ was set to be 0.005 for the study. (As mentioned before, the results unreported here showed very similar performance when we took $\beta = 0$.)

5.2. Results

5.2.1. Finite Sample Size Properties. First, let us report results on finite sample size properties. The results are shown in Tables 1 - 2. Table 1 presents the results from i.i.d. observations as a benchmark case. Table 2 uses simulated observations with dependency graphs where graphs are chosen to be from a single realization from two random graphs: Erdős-Renyi random graph and Barabasi-Albert random graph. Finally, Table 3 shows the results from using network dependent observations.

First, for all the cases considered, the permutation critical values perform conspicuously better than asymptotic critical values. This is expected as we saw in the previous section devoted to heuristics. The normal approximation from the randomized subsampling approach does not perform very well in the case of independent observations. (See Table 1.)

Second, permutation critical values show fine performance in the case of i.i.d. observations and in the case of observations with dependency graphs. This shows the advantage of the randomized subsampling approach. Interestingly, the performance does not seem to worsen much as we make the graph denser and the correlation stronger. This is perhaps partly because as one observation has more neighbors, the correlation between the observation and each neighbor tends to be weaker by the design of the data generating process.

Third, the randomized subsampling approach tends to over-reject the null hypothesis in the case of network dependence case (Table 3), where the overjection becomes severe as the correlation between linked observations gets stronger.

Throughout the simulation study, the increase in the sample size does not necessarily show better size properties. This may be because the graph tends to have more nodes with more
Table 2. Empirical Coverage Probability: 95% for Dependency Graphs

|               | E-R           | B-A           |       |       |       |
|---------------|---------------|---------------|-------|-------|-------|
|               | $\lambda = 1$ | $\lambda = 3$ | $\lambda = 5$ | $m = 1$ | $m = 2$ | $m = 3$ |
| $n = 500$     | 0.9210        | 0.9221        | 0.9219 | 0.9212 | 0.9184 | 0.9210 |
| $n = 1000$    | 0.9166        | 0.9191        | 0.9179 | 0.9130 | 0.9174 | 0.9190 |
| $n = 3000$    | 0.9124        | 0.9114        | 0.9141 | 0.9114 | 0.9081 | 0.9111 |
| $n = 500$     | 0.9223        | 0.9246        | 0.9218 | 0.9242 | 0.9184 | 0.9201 |
| $n = 1000$    | 0.9179        | 0.9166        | 0.9210 | 0.9195 | 0.9179 | 0.9146 |
| $n = 3000$    | 0.9129        | 0.9109        | 0.9093 | 0.9111 | 0.9114 | 0.9097 |
| $n = 500$     | 0.9467        | 0.9480        | 0.9475 | 0.9472 | 0.9447 | 0.9467 |
| $n = 1000$    | 0.9484        | 0.9498        | 0.9493 | 0.9460 | 0.9486 | 0.9500 |
| $n = 3000$    | 0.9500        | 0.9503        | 0.9510 | 0.9490 | 0.9471 | 0.9494 |
| $n = 500$     | 0.9482        | 0.9496        | 0.9472 | 0.9496 | 0.9478 | 0.9461 |
| $n = 1000$    | 0.9493        | 0.9485        | 0.9516 | 0.9502 | 0.9494 | 0.9463 |
| $n = 3000$    | 0.9506        | 0.9492        | 0.9471 | 0.9494 | 0.9492 | 0.9485 |

Notes: The E-R represents Erdös-Renyi Random Graph with probability equal to $p = \lambda/(n-1)$, and $\lambda$ chosen from 1, 3, 5, and the B-A represents Barabasi-Albert random graph of preferential attachment, with $m$ referring to the number of links each new node forms with other existing nodes. A larger parameter $c$ represents a stronger correlation between two linked observations.

neighbors as the sample size becomes larger, and this may offset the improvement in size properties partially.

5.2.2. Power Properties. Let us turn to the power properties of the randomized tests. The results are shown in Figure 2. The figure shows selected results from using independent observations, dependency graphs and network dependent observations. In the case of dependency graphs, the result is based on a realization of Erdös-Reny random graph with $\lambda = 3$ and in the case of Barabasi-Albert random graph, it is with $m = 2$. For both cases, the parameter $c$ was chosen to be 0.6. Recall that the true value is $\mu_0 = 0$.

The false coverage probabilities using asymptotic critical values are lower than those using permutation critical values. This is not surprising given that the asymptotic critical values exhibit lower coverage probabilities at the true value of $\mu_0 = 0$ than permutation critical values.

Interestingly, the case of independent observations and the case of dependency graphs show similar results. This demonstrates the robustness properties of the randomized subsampling approach. However, in the case of network dependent observations, the false coverage probabilities are higher. This shows once again that the performance of randomized subsampling...
Table 3. Empirical Coverage Probability at 95% for Network Dependent Observations

|       | Normal. |       | Normal. |       |       |       |       |       |
|-------|---------|-------|---------|-------|-------|-------|-------|-------|
|       | $\rho = 2.0$ | $\rho = 1.5$ | $\rho = 1.0$ |       |       |       |       |       |
| $\lambda = 1$ | $n = 500$ | 0.9176 | 0.9083 | 0.8988 |       | 0.9439 | 0.9362 | 0.9285 |
|       | $n = 1000$ | 0.9092 | 0.9051 | 0.8880 |       | 0.9418 | 0.9051 | 0.8880 |
|       | $n = 3000$ | 0.9049 | 0.8993 | 0.8919 |       | 0.9446 | 0.9404 | 0.9351 |
| $\lambda = 2$ | $n = 500$ | 0.9002 | 0.8843 | 0.8059 | 0.9292 | 0.9152 | 0.8447 |
|       | $n = 1000$ | 0.9018 | 0.8763 | 0.7956 | 0.9366 | 0.9151 | 0.8425 |
|       | $n = 3000$ | 0.8928 | 0.8733 | 0.8032 | 0.9351 | 0.9198 | 0.8580 |

Notes: The Erdős-Renyi Random Graph with probability equal to $p = \lambda/(n - 1)$ was used. The correlation between linked observations is set to be $\exp(-\rho D)$ where $D$ represents the length of the shortest path between the two indices of the observations on the graph.

approach within the boundary of the simulation designs and the choice of $R$, $b_n$, and $n$ does not seem stable for network dependent observations.

6. Conclusion

This paper proposes a randomized subsampling approach to perform inference with locally dependent data when the dependence ordering is not known. This paper first introduces the notion of local dependence that does not invoke any reference to the underlying dependence ordering, and pursues ordering-free inference which does not require knowledge of the dependence ordering for implementation. For this, it is shown that the sample mean type test statistics require overly strong conditions for the number of random permutations and the subsample sizes. Hence this paper proposes U-type test statistics which require weaker conditions. The test statistics have a limiting distribution that is free of the long run variance, and one can perform inference without any reference to the underlying ordering. However, the tests are randomized tests. This paper introduces a randomized confidence function that is an average of simulated confidence intervals, and proposes a non-randomized confidence set as well. Finally, this paper shows how this method can be applied to models with moment restrictions.

In general, there is a tradeoff between size and power in the randomized subsampling approach. Of course, when we have a huge number of observations, choosing $R$ and $b_n$ much smaller than $n$ can improve the small sample size property without hurting much its power. Some theoretical results behind a good combination of $R$ and $b_n$ in general would be desirable.
Figure 2. Empirical Coverage Probability at 95% for Different $\mu$'s: For dependency graphs with E-R and B-A graphs (the second and the third rows of panels), we used $\lambda = 3$ and $m = 2$ respectively, with $c = 0.6$. For the network dependent case, we used $\lambda = 1$ and $\rho = 1.5$. 
7. Appendix: Mathematical Proofs

We begin with the proof of Lemmas 2.1 and 2.2.

Proof of Lemma 2.1. We fix $1 \leq k \leq q$ and compute a bound for the number of permutations $\pi \in \Pi$ such that $c_F(\pi[k]; \mathcal{C}_n) > 0$. Whenever there is $i \in [k]$ such that $\pi(i)$ is not in the neighborhood of any point in $\pi[k] \setminus \{\pi(i)\}$, we have

$$c_F(\pi[k]; \mathcal{C}_n) = c_F(\{\pi(i)\}, \pi[k] \setminus \{\pi(i)\}; \mathcal{C}_n) = 0.$$ 

Hence for a permutation $\pi$ such that $c_F(\pi[k]; \mathcal{C}_n) > 0$, we consider the following example. First we place $\pi(1)$ in one of the $n$ places of 1, 2, ..., $n$. Then we place $\pi(2)$ in its neighborhood, and place $\pi(3)$ in one of the $n - 2$ places, and then we place $\pi(4)$ in its neighborhood, etc, when $k$ is even. However, when $k$ is odd, we have $\pi(k)$ left which is not paired with its neighbor. We place $\pi(k)$ in one of the neighborhoods of $\pi(1), ..., \pi(k-1)$. After placing $\pi(1), ..., \pi(k)$ this way, we place $\pi(k+1), ..., \pi(n)$ in the remaining $n - k$ places. In fact, any permutation $\pi \in \Pi$ such that $c_F(\pi[k]; \mathcal{C}_n) > 0$ can be obtained in this way, except with a different ordering of $\pi(1), ..., \pi(k)$ in this process.

Thus, we have

$$|\{\pi \in \Pi : c_F(\pi[k]; \mathcal{C}_n) > 0\}| \leq C_k n^{[k/2]} d_n^{[(k+1)/2]} (n - k)!,$$

where $C_k > 0$ is a constant depending only on $k$ and $d_n$ is the maximum degree of $G_n$. Therefore, we obtain for each $1 \leq k \leq q$,

$$\lambda_F(k; \mathcal{C}_n) \leq C'_k n^{[k/2]} d_n^{[(k+1)/2]} (n - k)! / n!,$$

for some constant $C'_k > 0$. ■

Proof of Lemma 2.2 is a bit more involved. Let us introduce notation and an auxiliary lemma. For each nonnegative integer $m$ and $j_1, j_2 \in N$, let us define

$$H'_m(j_1, j_2) = \{i \in N : \min\{d_\mu(i, j_1), d_\mu(i, j_2)\} < m + 1\},$$

and let for $m \geq 1$,

$$H_m(j_1, j_2) = H'_m(j_1, j_2) \setminus \bigcup_{v=1}^{m} H'_v(j_1, j_2).$$

Also define $H_0(j_1, j_2) = H'_0(j_1, j_2)$, and $H_{-1}(j_1, j_2) = \{j_1, j_2\}$. The sets $H_m(j_1, j_2)$, $m = -1, 0, 1, 2, ...$ constitute a partition of $N$, where each $i \in H_m(j_1, j_2)$ is such that either the distance between $\mu(i)$ and $\mu(j_1)$ or the distance between $\mu(i)$ and $\mu(j_2)$ is precisely between $m$ and $m + 1$ (possibly including $m$).
Fix any integer $k \geq 1$ and let $\Pi[k] = \{\pi[k] : \pi \in \Pi\}$, i.e., the collection of all the permutations of $(1, \ldots, k)$. Define

$$\Pi'_m[k] = \{\pi \in \Pi[k] : \exists A_1, A_2 \in \mathcal{P}([k]), \text{ s.t. } m \leq d_\mu(\pi(A_1), \pi(A_2)) < m + 1\},$$

where $\pi(A_1) = (\pi(j_1), \ldots, \pi(j_r))$ when $A_1 = (j_1, \ldots, j_r)$, and let

$$\Pi_m[k] = \Pi'_m[k] \setminus \bigcup_{j=m+1}^{\infty} \Pi'_j([k]).$$

**Lemma 7.1.** For each $m \geq 1$ and $\pi \in \Pi_m[k]$, there exist $j_1, j_2 \in [k]$ and $s_k > 0$ such that $m \leq d_\mu(\pi(j_1), \pi(j_2)) \leq m + 1$ and for each integer $0 \leq s \leq s_k$, $H_{sm}(\pi(j_1), \pi(j_2)) \cap \pi[k] \neq \emptyset$ and for each integer $s > s_k$, $H_{sm}(\pi(j_1), \pi(j_2)) \cap \pi[k] = \emptyset$.

**Proof:** For each $\pi \in \Pi_m[k]$, there exists a partition $(A_1, A_2)$ of $[k]$ such that

$$m \leq d_\mu(\pi(A_1), \pi(A_2)) < m + 1,$$

but there exists no partition $(A'_1, A'_2)$ of $[k]$ such that $d_\mu(\pi(A'_1), \pi(A'_2)) \geq m + 1$. First note that by the definition of $\Pi_m[k]$, for each $\pi \in \Pi[k]$, there exist $j_1, j_2 \in [k]$ such that $m \leq d_\mu(\pi(j_1), \pi(j_2)) < m + 1$. Certainly, we cannot have $H_{sm}(\pi(j_1), \pi(j_2)) \cap \pi[k] \neq \emptyset$ for infinite $s$’s, because $|\pi[k]| = k$. Let $\bar{s}$ be the smallest integer such that for all $s > \bar{s}$, $H_{sm}(\pi(j_1), \pi(j_2)) \cap \pi[k] = \emptyset$. Let $s'$ be the largest integer such that for all $0 \leq s \leq s'$, $H_{sm}(\pi(j_1), \pi(j_2)) \cap \pi[k] \neq \emptyset$. Then it must be that $0 \leq s' < \bar{s}$. For the lemma, it suffices to show that $s' = \bar{s}$.

To the contrary, assume that $0 \leq s' < \bar{s}$. We show that it contradicts that $\pi \in \Pi_m[k]$. First, by the definition of $s'$ and $\bar{s}$, there must exist $s_1, s_2$ such that $s' < s_1 < s_2 < \bar{s}$ and $H_{s_1m}(\pi(j_1), \pi(j_2)) \cap \pi[k] = \emptyset$ and $H_{s_2m}(\pi(j_1), \pi(j_2)) \cap \pi[k] \neq \emptyset$. We take a partition $(A'_1, A'_2)$ of $[k]$ such that

$$\pi(A'_1) = \bigcup_{s=0}^{s_1} H_{sm}(\pi(j_1), \pi(j_2)) \cap \pi[k]$$

(by the choice of $s_1$) and

$$\pi(A'_2) = \bigcup_{j=s_1+1}^{\bar{s}} H_{sm}(\pi(j_1), \pi(j_2)) \cap \pi[k].$$

Since $s' < s_1 < s_2$, both $A'_1$ and $A'_2$ are not empty. Furthermore, by the definition of $H_{s_1m}(\pi(j_1), \pi(j_2))$, for all $i', j' \in [k]$ such that $\pi(i') \in H_{sm}(\pi(j_1), \pi(j_2))$ for some $s =
0, 1, ..., $s_1 - 1$ and $\pi(j') \in H_{sm}(\pi(j_1), \pi(j_2))$ for some $s = s_1 + 1, ..., s$, we have $d_{\mu}(\pi(i'), \pi(j')) \geq m + 1$. Hence $d_{\mu}(\pi(A'_1), \pi(A'_2)) \geq m + 1$. This contradicts that $\pi \in \Pi_m[k]$. ■

Proof of Lemma 2.2: Fix $k \geq 2$. For each $m \geq 1$, let us first compute a bound for $|\Pi_m[k]|$. Let $\pi \in \Pi_m[k]$ and $j_1$ and $j_2$ be such that $m \leq d_{\mu}(\pi(j_1), \pi(j_2)) < m + 1$. Let $\bar{s}$ be the smallest integer such that for all $s > \bar{s}$, $H_{sm}(\pi(j_1), \pi(j_2)) \cap \pi[k] = \emptyset$. Then by Lemma 7.1 for each integer $0 \leq s \leq \bar{s}$, $|H_{sm}(\pi(j_1), \pi(j_2)) \cap \pi[k]| \geq 1$. We divide the proof into two cases.

First, suppose that for each integer $0 \leq s \leq \bar{s}$, $|H_{sm}(\pi(j_1), \pi(j_2)) \cap \pi[k]| = 1$. Then we have $\pi(i_1) \in H_m(\pi(j_1), \pi(j_2))$, $\pi(i_2) \in H_{2m}(\pi(j_1), \pi(j_2))$, ..., $\pi(i_{k-2}) \in H_{(k-2)m}(\pi(j_1), \pi(j_2))$, where $\{i_1, ..., i_{k-2}\} = [k] \setminus \{j_1, j_2\}$. A bound for the number of such permutations is computed as follows. First we place $\pi(j_1)$ in one of $n$ places, and then place $\pi(j_2)$ so that $m \leq d_{\mu}(\pi(j_1), \pi(j_2)) < m + 1$. The total number of fixing $j_1$ and $j_2$ this way is bounded by $Cnm^{d-1}$ for some constant $C > 0$. (See Lemma A.1 of Jenish and Prucha (2009b).) Now, we choose $\pi(i_1)$ from $H_m(\pi(j_1), \pi(j_2))$ and the number of choosing $\pi(i_1)$ this way is bounded by $Cm^{d-1}$, and choose $\pi(i_2)$ from $H_{2m}(\pi(j_1), \pi(j_2))$ and the number of choosing $\pi(i_1)$ this way is bounded by $C(2m)^{d-1}$. We keep choosing $\pi(i_2), ..., \pi(i_{k-2})$ this way. Then we choose the remaining $\pi(k + 1), ..., \pi(n)$. Hence the total number of choosing $\pi$ is bounded by

$$Cnm^{d-1}m^{d-1}(2m)^{d-1}((k - 2)m)^{d-1}(n - k)! \leq Cnm^{(k-1)(d-1)}((k - 2)!)^{d-1}(n - k)!,$$

for some $C > 0$ that does not depend on $n$.

Second, suppose that for some $0 \leq s \leq \bar{s}$, $|H_{sm}(\pi(j_1), \pi(j_2)) \cap \pi[k]| > 1$. Then it is not hard to show that the number of $\pi$’s with this property is bounded by the previous bound, because we can choose a bound for the number of choosing both $\pi(i)$ and $\pi(i')$ from the same $H_{sm}(\pi(j_1), \pi(j_2))$ in a way that the bound is smaller than the previous bound for the number of choosing $\pi(i)$ from $H_{sm}(\pi(j_1), \pi(j_2))$ and $\pi(i')$ from $H_{(s+1)m}(\pi(j_1), \pi(j_2))$. Thus we obtain the desired bound.

Therefore,

$$\frac{1}{|\Pi|} \sum_{\pi \in \Pi} c_{\mathcal{F}}(\pi[k]) \leq \frac{1}{|\Pi|} \sum_{m=1}^{\infty} \sum_{\pi \in \Pi_m[k]} c_{\mathcal{F}}(\pi[k]) \leq \frac{1}{|\Pi|} \sum_{m=1}^{\infty} \sum_{\pi \in \Pi_m[k]} \bar{c}_{m,\mathcal{F}}(\pi[k]) \leq C_kn(n - k)! \sum_{m=1}^{\infty} m^{(k-1)(d-1)} \max_{\pi \in \Pi} \bar{c}_{m,\mathcal{F}}(\pi[k]).$$

Then it follows from (2.4) that for any fixed $k \geq 1$,

$$\lambda_{\mathcal{F}}(k) \leq C_kn^{-k+1},$$
for some constant $C_k > 0$. ■

Let us focus on the proof of Theorem 3.1. The proof proceeds in multiple steps. First, we introduce a basic inequality that involves permutations.

**Lemma 7.2.** For some positive integers $k_1, k_2 \in [n]$, let $f : [n]^{k_1} \to [0, \infty)$ and $g : [n]^{k_2} \to [0, \infty)$ be given nonnegative maps. Then for any disjoint subsets $A_1, A_2 \subset [n]$ such that $|A_1| = k_1$ and $|A_2| = k_2$,

\[
\frac{1}{|\Pi|} \sum_{\pi \in \Pi} f(\pi(A_1))g(\pi(A_2)) \leq \frac{C_{k_1,k_2}}{|\Pi|^2} \sum_{\pi \in \Pi} f(\pi(A_1)) \sum_{\pi \in \Pi} g(\pi(A_2)),
\]

where $\pi(A) = (\pi(i))_{i \in A}$, $A \subset [n]$, and $C_{k_1,k_2} > 0$ is a constant that depends only on $k_1, k_2$.

**Proof:** Let $N(k)$ be the set of $(i_1, \ldots, i_k)$ such that $i_1, \ldots, i_k$ are from 1, ..., $n$ and all $i_1, \ldots, i_k$ are different. We write the left hand side of (7.1) by (letting $k = k_1 + k_2$)

\[
\frac{(n-k)!}{n!} \sum_{(i_1, \ldots, i_{k_1+k_2}) \in N(k_1+k_2)} f(i_1, \ldots, i_{k_1}) g(i_{k_1+1}, \ldots, i_{k_1+k_2})
\]

\[
= \frac{(n-k)!}{n!} \sum_{(i_1, \ldots, i_{k_1}) \in N(k_1)} f(i_1, \ldots, i_{k_1}) \sum_{(j_1, \ldots, j_{k_2}) \in N(k_2): (j_1, \ldots, j_{k_2}) \cap (i_1, \ldots, i_{k_1}) \neq \emptyset} g(j_1, \ldots, j_{k_2}).
\]

Since $f$ and $g$ are nonnegative, we bound the last term by

\[
\frac{(n-k)!}{n!} \frac{n!}{(n-k_1)!(n-k_2)!} \times \frac{(n-k_1)!}{n!} \sum_{(i_1, \ldots, i_{k_1}) \in N(k_1)} f(i_1, \ldots, i_{k_1}) \frac{(n-k_2)!}{n!} \sum_{(j_1, \ldots, j_{k_2}) \in N(k_2)} g(j_1, \ldots, j_{k_2})
\]

\[
= \frac{(n-k)!}{n!} \frac{n!}{(n-k_1)!(n-k_2)!} \frac{1}{|\Pi|} \sum_{\pi \in \Pi} f(\pi(A_1)) \frac{1}{|\Pi|} \sum_{\pi \in \Pi} g(\pi(A_2)).
\]

Note that

\[
\frac{(n-k)!}{n!} \frac{n!}{(n-k_1)!(n-k_2)!} = \frac{(n-k)!}{(n-k_1)!(n-k_2)!} \leq C_{k_1,k_2},
\]

for some constant $C_{k_1,k_2} > 0$ that depends only on $k_1, k_2$. ■

Let us introduce some notation. Let $W = \{W_{i,n}\}_{i \in [n]}$ be a given triangular array of random variables. Suppose that $i_1, \ldots, i_q, j_1, \ldots, j_q$ have precisely $k \in [2q]$ distinct elements with precisely $r \in [k]$ uniquely distinctive elements (for example, if $(i_1, \ldots, i_q, j_1, \ldots, j_q) = (1, 1, 3, 4, 4, 5, 6)$, we have $k = 5$ because we have 5 different numbers and $r = 3$ because we have three numbers, 3, 5, and 6 which appear only once) so that there exist nonnegative
integers $r_1, \ldots, r_k$ such that $r_1 + \ldots + r_k = 2q$, and
\[ |\{ j \in [k] : r_j = 1 \}| = r, \]
i.e., exactly $r$ number of $r_1, \ldots, r_k$ are 1, and a vector of indices $(s_1, \ldots, s_k) \in [b_n]^k$, where all $s_1, \ldots, s_k$ are distinct and for each $v \in [k]$, exactly $r_v$ number of $s_v$’s are in $(i_1, \ldots, i_q, j_1, \ldots, j_q)$. In this case, we define
\[
\Psi(\{s_1, \ldots, s_k\}; [k]) = \mathbb{E} \left[ W_{i_1,n} W_{j_1,n} \cdots W_{i_q,n} W_{j_q,n} | \mathcal{C}_n \right] = \mathbb{E} \left[ W_{r_1}^{r_1} W_{r_2}^{r_2} \cdots W_{r_k}^{r_k} | \mathcal{C}_n \right],
\]
where the index vector $[k]$ in the notation $\Psi(\cdot; [k])$ represents the indices of $r_1, \ldots, r_k$. Also, for \{m_1, \ldots, m_t\} $\subset [k]$ with $t < k$, we write
\[
\Psi(\{s_{m_1}, \ldots, s_{m_t}\}; \{m_1, \ldots, m_t\}) = \mathbb{E} \left[ W_{s_{m_1},n} W_{s_{m_2},n} \cdots W_{s_{m_t},n} | \mathcal{C}_n \right].
\]
And for $i_1, \ldots, i_q, j_1, \ldots, j_q$, we define
\[
B_n(k; [k]) = \frac{1}{|\Pi|} \sum_{\pi \in \Pi} |\Psi(\pi [k]; [k])| = \frac{1}{|\Pi|} \sum_{\pi \in \Pi} \left| \mathbb{E} [W_{\pi(1),n} W_{\pi(2),n} \cdots W_{\pi(k),n} | \mathcal{C}_n] \right|.
\]
Thus $B_n(k; [k])$ depends on $(i_1, \ldots, i_q, j_1, \ldots, j_q)$ only through $(r_1, \ldots, r_k)$. Also, similarly, for any \{m_1, \ldots, m_t\} $\subset [k]$ with $t < k$, we let
\[
B_n(t; \{m_1, \ldots, m_t\}) = \frac{1}{|\Pi|} \sum_{\pi \in \Pi} |\Psi(\pi \{m_1, \ldots, m_t\}; \{m_1, \ldots, m_t\})| = \frac{1}{|\Pi|} \sum_{\pi \in \Pi} \left| \mathbb{E} [W_{\pi(s_{m_1},n)} W_{\pi(s_{m_2},n)} \cdots W_{\pi(s_{m_t},n)} | \mathcal{C}_n] \right|.
\]
Therefore, for example,
\[
B_n(1; \{j\}) = \frac{1}{|\Pi|} \sum_{\pi \in \Pi} \left| \mathbb{E} [W_{\pi(j),n} | \mathcal{C}_n] \right|.
\]
And hence if $\mathbb{E} [W_{i,n} | \mathcal{C}_n] = 0$ for all $i \in [n]$, we have $B_n(1; \{j\}) = 0$ whenever $r_j = 1$. The following lemma gives a moment bound that plays a crucial role for the remainder of the proofs.

**Lemma 7.3.** Suppose that Assumption 3.2 holds, and that \{W_{i,n}\}_{i \in [n]} is a given triangular array of random variables having $\lambda(\cdot; \mathcal{C}_n)$ as its $\lambda$-coefficient for some $\sigma$-field $\mathcal{C}_n$.

Then for any positive integer $1 \leq q \leq 4$, there exists a constant $C_q > 0$ that depends only on $q$ such that for any $(i_1, \ldots, i_q, j_1, \ldots, j_q) \in [b_n]^{2q}$ having precisely $k \in [2q]$ distinct elements,
we have
\[
\frac{1}{|\Pi|} \sum_{\pi \in \Pi} \mathbb{E} \left[ W_{\pi(i_1),n} W_{\pi(j_1),n} \ldots W_{\pi(i_q),n} W_{\pi(j_q),n} | C_n \right] \]
(7.3)
\[
\leq C_q \left\{ \prod_{j=1}^{k} B_n(1; \{j\}) + \max_{(v_1, \ldots, v_k) \in J(k)} \prod_{j=1}^{k} \lambda(v_j; C_n) \right\},
\]
(7.4)
where \( J(k) = \{(v_1, \ldots, v_k) \in (\{0\} \cup [k])^k : v_1 + \ldots + v_k = k \} \).

**Proof:** The proof uses arguments similar to those in the proof of Lemma 3 of Andrews and Pollard (1994). Write
\[
\frac{1}{|\Pi|} \sum_{\pi \in \Pi} \mathbb{E} \left[ W_{\pi(i_1),n} W_{\pi(j_1),n} \ldots W_{\pi(i_q),n} W_{\pi(j_q),n} | C_n \right] = \frac{1}{|\Pi|} \sum_{\pi \in \Pi} |\Psi(\pi[k]; [k])| = B_n(k; [k]).
\]
For each \( \{A_1, A_2\} \in \mathcal{P}([k]) \), we let \( \Pi(A_1, A_2) \subset \Pi \) be the collection of \( \pi \)'s such that
\[
c_F(\pi[k]) = c_F(\pi(A_1), \pi(A_2)),
\]
i.e., the collection of \( \pi \)'s such that \( (\pi(A_1), \pi(A_2)) \in \mathcal{P}(\pi[k]) \) is a minimizer of \( c_F(A'_1, A'_2) \) over all \( (A'_1, A'_2) \in \mathcal{P}(\pi[k]) \). Using the fact \( \text{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \), we bound
\[
B_n(k; [k]) \leq \sum_{\{A_1, A_2\} \in \mathcal{P}([k])} \frac{1}{|\Pi|} \sum_{\pi \in \Pi(A_1, A_2)} |\Psi(\pi(A_1); A_1)||\Psi(\pi(A_2); A_2)| + C_k \lambda(k; C_n),
\]
for some constant \( C_k > 0 \) that depends only on \( k \). By Lemma 7.2, the leading term on the right hand side is bounded by
\[
\sum_{\{A_1, A_2\} \in \mathcal{P}([k])} \frac{C'_k}{|\Pi|} \sum_{\pi \in \Pi(A_1, A_2)} |\Psi(\pi(A_1); A_1)| \frac{1}{|\Pi|} \sum_{\pi \in \Pi} |\Psi(\pi(A_2); A_2)|
\]
\[
\leq C'_k \sum_{\{A_1, A_2\} \in \mathcal{P}([k])} B_n(a_1; A_1)B_n(a_2; A_2),
\]
where \( a_1 = |A_1| \) and \( a_2 = |A_2| \) and \( C'_k > 0 \) is another constant that depends only on \( k \). Thus we conclude that
\[
B_n(k; [k]) \leq C'_k \sum_{\{A_1, A_2\} \in \mathcal{P}([k])} B_n(a_1; A_1)B_n(a_2; A_2) + C_k \lambda(k; C_n).
\]
Similarly, we have for \( j = 1, 2 \),
\[
B_n(a_j; A_j) \leq C''_{a_j} \sum_{(A_{j,1}, A_{j,2}) \in \mathcal{P}(A_j)} B_n(a_{j,1}; A_{j,1})B_n(a_{j,2}; A_{j,2}) + C''_{a_j} \lambda(a_j; C_n),
\]
where \( a_{j,1} = |A_{j,1}| \) and \( a_{j,2} = |A_{j,2}| \), and \( C''_{a_j}, C''_{a_j} > 0 \) are constants. Thus, if we apply this recursive split of \( B_n \), we have a sum of terms where each term is a product of \( B_n \)'s.
We continue this splitting until we have all the terms are products of the form $B_n(1, \{j\})$. The minimum number of such splits cannot exceed $k$ because that is the number of distinct integers in $(i_1, \ldots, i_q, j_1, \ldots, j_q)$. Each split results in a weighted sum of terms of the form $\lambda(k_1; C_n)\lambda(k_2; C_n)\ldots\lambda(k_l; C_n)$, where $k_1, \ldots, k_l$ are positive integers such that $k_1 + \ldots + k_l = k$. Let $D_n$ be the weighted sum of such terms that we are left with after completing the previous splitting process. Hence we find that

$$B_n(k; [k]) \leq C_{k,q} \left\{ \prod_{j=1}^{k} B_n(1; \{j\}) + D_n \right\} ,$$

where $C_{k,q} > 0$ is a constant that depends only on $k$ and $q$. Since the total number of splits cannot exceed the number $k$, by bounding $D_n$ by $C_k$ times the maximum on the right hand side of (7.3), for some $C_k > 0$ that depends only on $k$, and by taking $C_q$ to be the maximum over $C_k$’s and $C_{k,q}$’s over $k$, we obtain the desired bound. \[\blacksquare\]

Recall the definition of $Z_{i,n} = \Sigma^{-1/2}(C_n)(X_{i,n} - \mu_0)$, where

$$\Sigma(C_n) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}[(X_{i,n} - \mu_0)(X_{i,n} - \mu_0)'|C_n].$$

**Lemma 7.4.** Suppose that Assumption 3.2 holds, and that for some positive integer $1 \leq q \leq 4$, there exists $C > 0$ such that $b_n^{k-q}\lambda(k; C_n) \leq C$ for all $n \geq 1$, for each $1 \leq k \leq 2q$.

Then there exists a constant $C_q > 0$ that depends only on $q$ such that for all $n \geq 1$,

$$\frac{1}{|\Pi|} \sum_{\pi \in \Pi} \mathbb{E} \left( \left\| \frac{1}{\sqrt{b_n}} \sum_{i=1}^{b_n} Z_{\pi(i),n} \right\|^{2q} | C_n \right) \leq C_q.$$

**Proof:** Define for positive integers $q, k \in [2q]$,

$$A_n(k) = \{ i \in [b_n]^{2q} : i \ has \ exactly \ k \ different \ numbers. \}.$$ 

Let us write

$$\frac{1}{|\Pi|} \sum_{\pi \in \Pi} \mathbb{E} \left( \left\| \frac{1}{\sqrt{b_n}} \sum_{i=1}^{b_n} Z_{\pi(i),n} \right\|^{2q} | C_n \right)$$

$$\leq \frac{1}{b_n^q} \sum_{k=1}^{2q} \sum_{(i_1, \ldots, i_q, j_1, \ldots, j_q) \in A_n(k)} \left\| \frac{1}{|\Pi|} \sum_{\pi \in \Pi} \mathbb{E} \left[ Z_{\pi(i_1),n} Z_{\pi(j_1),n} \ldots Z_{\pi(i_q),n} Z_{\pi(j_q),n} | C_n \right] \right\|.$$ 

Using Lemma 7.3 we bound the term on the right hand side by

$$\frac{C_q}{b_n^q} \sum_{k=1}^{2q} \sum_{(i_1, \ldots, i_q, j_1, \ldots, j_q) \in A_n(k)} \left\{ \prod_{j=1}^{k} B_n(1; \{j\}) + \max_{(v_1, \ldots, v_k) \in J(k)} \prod_{j=1}^{k} \lambda(v_j; C_n) \right\} ,$$
for some constant $C_q > 0$ that depend on only $q$, where $B_n(1; \{j\})$ is as defined in (7.2) except that we use $\{Z_{i,n}\}$ in place of $\{W_{i,n}\}$. However, for each $(i_1, \ldots, i_q, j_1, \ldots, j_q) \in A_n(k)$ with $k > q$, we have $\prod_{j=1}^{k} B_n(1; \{j\}) = 0$ because there exists $j \in [k]$ such that $r_j = 1$, i.e., there exists an integer in $(i_1, \ldots, i_q, j_1, \ldots, j_q)$ which appears uniquely in the vector, and for such $j$, $B_n(1; \{j\}) = 0$ because $E[Z_{i,n}C_n] = 0$ for all $i \in [n]$. Therefore, the bound in (7.8) is written as

\[
(7.9) \quad \frac{C_q}{b_n^q} \sum_{k=1}^{q} |A_n(k)| + \frac{C_q}{b_n^q} \sum_{k=1}^{2q} |A_n(k)| \left( \max_{(v_1, \ldots, v_k) \in J(k)} \prod_{j=1}^{k} \lambda(v_j; C_n) \right)
\]

\[
\leq C_q' \sum_{k=1}^{2q} b_n^{k-q} + C_q'' \sum_{k=1}^{2q} b_n^{k-q} \left( \max_{(v_1, \ldots, v_k) \in J(k)} \prod_{j=1}^{k} \lambda(v_j; C_n) \right)
\]

\[
\leq C_q'' + C_q' \sum_{k=1}^{2q} \max_{(v_1, \ldots, v_k) \in J(k)} \prod_{j=1}^{k} b_n^{-q} \lambda(v_j; C_n)
\]

for some $C_q, C_q', C_q'' > 0$ that depend only on $q$, because $|A_n(k)| \leq C_k b_n^k$ for some $C_k > 0$. Applying the condition of the lemma, we obtain the desired result. ■

Let us define

\[
\xi_{r,n} \equiv \frac{1}{b_n} \sum_{i=1}^{b_n} \sum_{j=1, i \neq j}^{b_n} Z_{\pi_r(i),n} Z_{\pi_r(j),n}, \text{ and } S_{n}^*(\tilde{\pi}) \equiv \frac{1}{\sqrt{R}} \sum_{r=1}^{R} \xi_{r,n}.
\]

We focus on the asymptotic properties of $S_{n}^*(\tilde{\pi})$. For this, we define $\mathcal{Z} \equiv (Z_{i,n})_{i=1}^{n}$ and write

\[
(7.10) \quad S_{n}^*(\tilde{\pi}) = S_{n,A}^*(\tilde{\pi}) + S_{n,B}^*(\tilde{\pi}),
\]

where

\[
S_{n,A}^*(\tilde{\pi}) = \frac{1}{\sqrt{R}} \sum_{r=1}^{R} (\xi_{r,n} - E[\xi_{r,n}|\mathcal{Z}]) \text{ and } S_{n,B}^*(\tilde{\pi}) = \frac{1}{\sqrt{R}} \sum_{r=1}^{R} E[\xi_{r,n}|\mathcal{Z}].
\]

The following lemma gives the convergence rate of $S_{n,B}^*(\tilde{\pi})$.

**Lemma 7.5.** Suppose that Assumption 3.2 holds. Then

\[
E[(S_{n,B}^*(\tilde{\pi}))^2] = O(Rb_n^2\{\lambda_n(4) + n^{-1}(\lambda_n(2) + \lambda_n(3)) + n^{-2}\}).
\]

**Proof:** For notational brevity, denote

\[
(7.11) \quad \zeta_{i,j,n} \equiv Z_{i,n}' Z_{j,n}, \text{ and } \tilde{\zeta}_{i,j,n} \equiv Z_{i,n} Z_{j,n}'.
\]

Since we draw $\pi_r$’s i.i.d. from the uniform distribution on $\Pi$, we can rewrite

\[
S_{n,B}^*(\tilde{\pi}) = \frac{\sqrt{R}(b_n - 1)}{n(n - 1)} \sum_{i=1}^{n} \sum_{j=1: j \neq i}^{n} \zeta_{i,j,n}.
\]
Write
\[ \mathbf{E}[\{S_n^* B(\tilde{\pi})\}] = R(b_n - 1)^2 \mathbf{E} \left[ \frac{1}{n^2(n - 1)^2} \sum_{i=1}^{n} \sum_{j=1; j \neq i}^{n} \zeta_{i,j}^2 \right] + R(b_n - 1)^2 A_n, \]
where
\[ A_n = \frac{1}{n^2(n - 1)^2} \sum_{(i_1, j_1, i_2, j_2)} \mathbf{E}[\zeta_{i_1,j_1,n}\zeta_{i_2,j_2,n}], \]
and the sum over \((i_1, j_1, i_2, j_2)\) includes 4-tuples of positive integers from 1 to \(n\) such that \(i_1 \neq j_1, i_2 \neq j_2,\) and \((i_1, j_1) \neq (i_2, j_2)\). The leading term in the decomposition above is \(O(n^{-2}R\beta_n^2)\) by the moment conditions in Assumption 3.2.

Let us analyze \(A_n\) which we write
\[ A_n = 4B_{1,n} + B_{2,n}, \]
where
\[ B_{1,n} = \frac{1}{n^2(n - 1)^2} \sum_{(i_1, j_1, j_2)} \mathbf{E}[\zeta_{i_1,j_1,n}\zeta_{i_1,j_2,n}], \quad \text{and} \]
\[ B_{2,n} = \frac{1}{n^2(n - 1)^2} \sum_{(i_1, j_1, j_2)^*} \mathbf{E}[\zeta_{i_1,j_1,n}\zeta_{i_2,j_2,n}], \]
where the sum over \((i_1, j_1, j_2)^*\) is over all 3-tuples of distinct integers in \([n]\) and the sum over \((i_1, j_1, j_2)^*\) is over all 4-tuples of distinct integers in \([n]\). The factor 4 in front of \(B_{1,n}\) appears because for each 4-tuple, say \((i_1', j_1', i_2', j_2')\), there are four ways to form a pair with one from \((i_1', j_1')\) and the other from \((i_2', j_2')\). Then we set the pair, say, \((i_1', j_2')\), to be such that \(i_1' = j_2'\) so that we form \((i_1, j_1')\). We write
\[ (7.13) \quad B_{1,n} = \frac{1}{n^2(n - 1)^2} \sum_{(i_1, j_1, j_2)^*} \text{tr} \left( \mathbf{E}[\tilde{\zeta}_{i_1,j_1,n}\tilde{\zeta}_{j_2,j_1,n}] - \mathbf{E}[\mathbf{E}[\tilde{\zeta}_{i_1,j_1,n}|C_n]\mathbf{E}[\tilde{\zeta}_{j_2,j_1,n}|C_n]] \right) \]
\[ + \frac{1}{n^2(n - 1)^2} \sum_{(i_1, j_1, j_2)^*} \mathbf{E} \left[ \text{tr} \left( \mathbf{E}[\tilde{\zeta}_{i_1,j_1,n}|C_n]\mathbf{E}[\tilde{\zeta}_{j_2,j_1,n}|C_n] \right) \right]. \]

By the definition of \(\lambda\) and Assumption 3.2(i), the last term is bounded by \(Cn^{-1}\mathbf{E}[\lambda(2; C_n)] = Cn^{-1}\lambda_n(2)\), for some constant \(C > 0\) that does not depend on \(n\).

We turn to the leading term on the right hand side of (7.13) and find a bound in terms of the \(\lambda\)-coefficient. Note that
\[ P(\{i_1, j_1, j_2\}) = \{\{i_1\}, \{j_1, j_2\}\}, \{\{i_1, j_1\}, \{j_2\}\}, \{\{i_1, j_2\}, \{j_1\}\}. \]
The trace in the leading sum in (7.13) is bounded by $m \mathbb{E}[c_{\mathcal{F}}(\{i_1, j_1\}, \{j_1, j_2\})]$, where $\mathcal{F}$ is chosen as prior to Assumption 3.1. As we can also write (using the fact that $\mathbb{E}[Z_{j_1, n}|C_n] = 0$)

\[
\mathbb{E}[\tilde{\zeta}_{i_1, j_1, n, \tilde{\zeta}_{j_2, j_1, n}|C_n] = \mathbb{E}[\tilde{\zeta}_{i_1, j_1, n, \tilde{\zeta}_{j_2, j_1, n}|C_n] - \mathbb{E}[Z_{i_1, n}Z'_{i_1, n}Z_{j_2, n}|C_n]\mathbb{E}[Z'_{j_1, n}|C_n],
\]

the trace in the leading sum of (7.13) is bounded by

\[
m \mathbb{E}[c_{\mathcal{F}}(\{i_1, j_2\}, \{j_1\})] + mC \mathbb{E}[c_{\mathcal{F}}(\{j_2\}, \{j_1\})],
\]

for some constant $C > 0$, where $c_{\mathcal{F}}(\{j_2\}, \{j_1\})$ is due to $\mathbb{E}[\tilde{\zeta}_{j_2, j_1, n}|C_n]$. Similarly, the same trace is also bounded by $m \mathbb{E}[c_{\mathcal{F}}(\{i_1, j_1\}, \{j_2\})] + mC \mathbb{E}[c_{\mathcal{F}}(\{j_2\}, \{j_1\})]$. Therefore, the leading sum on the right hand side of (7.13) is bounded by $Cn^{-1}(\mathbb{E}[\lambda(3; C_n)] + \mathbb{E}[\lambda(2; C_n)]) = Cn^{-1}(\lambda_n(2) + \lambda_n(3))$. Combined with the bound $Cn^{-1}\lambda_n(2)$ for the last term in (7.13), we conclude that

\[
B_{1, n} \leq Cn^{-1}(\lambda_n(2) + \lambda_n(3)).
\]

Let us turn to $B_{2, n}$. By the definition of $\lambda(\cdot; C_n)$, we have

\[
B_{2, n} \leq C \mathbb{E}[\lambda(4; C_n)] = C\lambda_n(4),
\]

for some $C > 0$. We conclude that

\[
R(b_n - 1)^2 A_n = O(Rb_n^2 \{\lambda_n(4) + n^{-1}(\lambda_n(3) + \lambda_n(2))\}),
\]

completing the proof. }

**Lemma 7.6.** Suppose that Assumptions 3.1(i) - 3.2 hold. Then

\[
\text{Var}(S_{n,A}^*|Z) = m + o_P(1).
\]

**Proof:** Let us first consider

\[
\text{Var}(S_{n,A}^*|Z) = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1: j \neq i}^{n} \left( \zeta_{i,j,n} - \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1: j \neq i}^{n} \zeta_{i,j,n} \right)^2
\]

\[
= \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1: j \neq i}^{n} \zeta_{i,j,n}^2 - \left( \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1: j \neq i}^{n} \zeta_{i,j,n} \right)^2
\]

\[
= \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1: j \neq i}^{n} \zeta_{i,j,n}^2 + o_P(1),
\]

because, as we saw in the proof of Lemma 7.5,

\[
\frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1: j \neq i}^{n} \zeta_{i,j,n} = o_P(1).
\]
Now we write
\[
\frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1:j \neq i}^{n} \zeta_{i,j,n}^2 = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1:j \neq i}^{n} E[\zeta_{i,j,n}^2 | C_n] + \Delta_n,
\]
where
\[
\Delta_n = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1:j \neq i}^{n} \left( \zeta_{i,j,n}^2 - E[\zeta_{i,j,n}^2 | C_n] \right).
\]
Rewrite the leading term on the right hand side as
\[
m + \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1:j \neq i}^{n} \text{tr} \left( E[(Z_{i,n}Z'_{i,n} - I_m)(Z_{j,n}Z'_{j,n} - I_m) | C_n] \right),
\]
where \(I_m\) is the identity matrix of dimension \(m\). (Recall that \(\frac{1}{n} \sum_{i=1}^{n} E[Z_{i,n}Z'_{i,n} | C_n] = I_m\). The last term is bounded by \(C\lambda(2; C_n) = o_P(1)\).

We focus on \(\Delta_n\). We write
\[
E[\Delta_n^2] = \frac{1}{n^2(n-1)^2} \sum_{(i_1,j_1,i_2,j_2)} \text{E} [\zeta_{i_1,j_1,n}^2 \zeta_{i_2,j_2,n}^2] - \frac{1}{n^2(n-1)^2} \sum_{(i_1,j_1,i_2,j_2)} \text{E} [E[\zeta_{i_1,j_1,n}^2 | C_n]E[\zeta_{i_2,j_2,n}^2 | C_n]],
\]
where the sum over \((i_1,j_1,i_2,j_2)\) is over all the 4-tuples of positive integers from \([n]\) such that 4-tuples of positive integers from 1 to \(n\) such that \(i_1 \neq j_1, i_2 \neq j_2, \) and \((i_1,j_1) \neq (i_2,j_2)\). Again, we write
\[
E[\Delta_n^2] = \Delta_{1,n} + \Delta_{2,n},
\]
where
\[
\Delta_{1,n} = \frac{1}{n^2(n-1)^2} \sum_{(i_1,j_1,i_2,j_2)} \text{E} [\zeta_{i_1,j_1,n}^2 \zeta_{i_2,j_2,n}^2] - \frac{1}{n^2(n-1)^2} \sum_{(i_1,j_1,i_2,j_2)} \text{E} [E[\zeta_{i_1,j_1,n}^2 | C_n]E[\zeta_{i_2,j_2,n}^2 | C_n]],
\]
and
\[
\Delta_{2,n} = \frac{1}{n^2(n-1)^2} \sum_{(i_1,j_1,i_2,j_2)} \text{E} [\zeta_{i_1,j_1,n}^2 \zeta_{i_2,j_2,n}^2] - \frac{1}{n^2(n-1)^2} \sum_{(i_1,j_1,i_2,j_2)} \text{E} [E[\zeta_{i_1,j_1,n}^2 | C_n]E[\zeta_{i_2,j_2,n}^2 | C_n]].
\]
As in the proof of Lemma 7.5, the sum over \((i_1,j_1,j_2)^*\) is over all 3-tuples of distinct integers in \([n]\) and the sum over \((i_1,j_1,i_2,j_2)^*\) is over all 4-tuples of distinct integers in \([n]\). Note that
by Assumption \[3.2\] and by counting the number of all 3-tuples of distinct integers in \([n]\), we have

\[
\Delta_{1,n} = O(n^{-1}).
\]

As for \(\Delta_{2,n}\), we write

\[
\Delta_{2,n} = \frac{1}{n^2(n-1)^2} \sum_{(i_1,j_1,i_2,j_2) \neq \ast} E \left[ \text{Cov} \left( \xi_{i_1,j_1,n}^2, \xi_{i_2,j_2,n}^2 | C_n \right) \right] \leq E \left[ \lambda(2; C_n) \right] = o(1).
\]

We conclude that \(\Delta_n = o_P(1)\).

\[\square\]

**Lemma 7.7.** Suppose that the conditions of Lemma \[7.4\] hold. Then

\[
E \left[ |\xi_{r,n} - E[\xi_{r,n} | Z]|^3 \right] = O(1).
\]

**Proof:** We bound for some \(C > 0\),

\[
E \left[ |\xi_{r,n} - E[\xi_{r,n} | Z]|^3 \right] \leq C \left( E[|\xi_{r,n} | Z]|^3 \right] \leq C \left( E[\xi_{r,n}^4] \right)^{3/4}.
\]

Note that

\[
E[\xi_{r,n}^4] \leq \frac{1}{|\Pi|} \sum_{\pi \in \Pi} \left[ \frac{1}{b_n} \sum_{i=1}^{b_n} \sum_{j=1, j \neq i}^{b_n} \xi_{\pi(i), \pi(j), n} \right]^4.
\]

We bound the last term by

\[
\frac{2^3}{|\Pi|} \sum_{\pi \in \Pi} E \left[ \left( \frac{1}{\sqrt{b_n}} \sum_{i=1}^{b_n} Z_{\pi(i), n} \right) \right]^4 \leq C,
\]

for some \(C > 0\), completing the proof. \[\square\]
Lemma 7.8. Suppose that Assumption 3.2 holds, and that \( \lambda_n(2) = o(1) \) as \( n \to \infty \). Then

\[
\frac{1}{n} \sum_{i=1}^{n} (X_{i,n} - \bar{X})(X_{i,n} - \bar{X})' = \Sigma(C_n) + O_P(n^{-1/2} + \sqrt{\lambda_n(2)}). \tag{7.14}
\]

**Proof:** First, let us write that

\[
\frac{1}{n} \sum_{i=1}^{n} (X_{i,n} - \bar{X})(X_{i,n} - \bar{X})' = \Sigma(C_n) + \sum_{i=1}^{n} D_n Z_{i,n} - Z_n Z_n',
\]

where

\[
D_n = \frac{1}{n} \sum_{i=1}^{n} (Z_{i,n} Z_{i,n}' - \E[Z_{i,n} Z_{i,n}' | C_n]) - \bar{Z}_n \bar{Z}_n'.
\]

Note that for the first term of \( D_n \),

\[
\E \left( \left\| \frac{1}{n} \sum_{i=1}^{n} (Z_{i,n} Z_{i,n}' - \E[Z_{i,n} Z_{i,n}' | C_n]) \right\|^2 \right) = \frac{1}{n^2} \sum_{i=1}^{n} \text{tr} \left( \E \left[ (Z_{i,n} Z_{i,n}' - \E[Z_{i,n} Z_{i,n}' | C_n])^2 \right] \right)
\]

\[
+ \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} \text{tr} \left( \E \left[ (Z_{i,n} Z_{i,n}' - \E[Z_{i,n} Z_{i,n}' | C_n]) (Z_{j,n} Z_{j,n}' - \E[Z_{j,n} Z_{j,n}' | C_n])' \right] \right).
\]

The left hand side of the above equality is equal to \( O(n^{-1} + \lambda_n(2)) \). In the same way, we obtain

\[
\bar{Z} = \frac{1}{n} \sum_{i=1}^{n} Z_{i,n} = O_P(n^{-1/2} + \sqrt{\lambda_n(2)}),
\]

so that we have \( D_n = O_P(n^{-1/2} + \sqrt{\lambda_n(2)}) \). This completes the proof. ■

**Proof of Theorem 3.1:** Recall the definition of \( S_{n,A}^*(\tilde{\pi}) \) prior to Lemma 7.5. By Lemma 7.5-7.6, we find that

\[
\E \left[ \left| S_{n,A}^*(\tilde{\pi}) \right| \right] = O_P(1).
\]

Combining this with (7.14) and using (7.10) and Lemma 7.5, we obtain that

\[
S_n(\mu_0; \tilde{\pi}) = S_{n,A}^*(\tilde{\pi}) + o_P(1).
\]

We focus on \( S_{n,A}^*(\tilde{\pi}) \). Since \( \xi_{r,n} \) is a triangular array that is rowwise i.i.d. conditional on \( Z \), we deduce that by Berry-Esseen lemma (e.g. Theorem 3 of Chow and Teicher (1988),
p.304), for some $C > 0$,
\[
\left| P \left\{ \frac{S_{n,A}(\tilde{\pi})}{\sigma_n(Z)} \leq t | Z \right\} - \Phi(t) \right| \leq \frac{C}{R^{3/2}} \sum_{r=1}^{R} \mathbb{E}[|\xi_{r,n} - \mathbb{E}[\xi_{r,n}|Z]|^3|Z],
\]
where $\sigma^2_n(Z) = \text{Var}(S_{n,A}(\tilde{\pi})|Z)$. Now, observe that
\[
\left| P \left\{ \frac{S_{n,A}(\tilde{\pi})}{\sigma_n(Z)} \leq t \right\} - \Phi(t) \right| \leq \mathbb{E} \left[ \left| P \left\{ \frac{S_{n,A}(\tilde{\pi})}{\sigma_n(Z)} \leq t | Z \right\} - \Phi(t) \right| \right] \leq \frac{1}{R^{3/2}} \sum_{r=1}^{R} \mathbb{E}[|\xi_{r,n} - \mathbb{E}[\xi_{r,n}|Z]|^3].
\]
By Lemma 7.7, the last bound has a rate $O(R^{-1/2})$, so that
\[
\frac{S_{n,A}(\tilde{\pi})}{\sigma_n(Z)} \xrightarrow{d} N(0, 1).
\]
In view of Lemma 7.8, this completes the proof. ■

Let us now prove the asymptotic validity of the permutation critical values. We first begin with a lemma that deals with a sample mean type test statistic.

**Lemma 7.9.** Suppose that Assumption 3.2 holds. Then
\[
\text{Var} \left( \frac{1}{\sqrt{R}} \sum_{r=1}^{R} \frac{1}{\sqrt{b_n}} \sum_{i=1}^{b_n} Z_{\pi_r(i),n} | Z \right) = I_m + o_P(1).
\]

**Proof:** We write
\[
\text{Var} \left( \frac{1}{\sqrt{R}} \sum_{r=1}^{R} \frac{1}{\sqrt{b_n}} \sum_{i=1}^{b_n} Z_{\pi_r(i),n} | Z \right) = \frac{1}{R} \sum_{r=1}^{R} \mathbb{E} \left[ \left( \frac{1}{\sqrt{b_n}} \sum_{i=1}^{b_n} (Z_{\pi_r(i),n} - \bar{Z}) \right) \left( \frac{1}{\sqrt{b_n}} \sum_{i=1}^{b_n} (Z_{\pi_r(i),n} - \bar{Z}) \right)' | Z \right].
\]
We write the last term as $E_{1,n} + E_{2,n}$, where
\[
E_{1,n} = \frac{1}{R} \sum_{r=1}^{R} \mathbb{E} \left[ \frac{1}{b_n} \sum_{i=1}^{b_n} (Z_{\pi_r(i),n} - \bar{Z})(Z_{\pi_r(i),n} - \bar{Z})' | Z \right] \quad \text{and}
\]
\[
E_{2,n} = \frac{1}{R} \sum_{r=1}^{R} \mathbb{E} \left[ \frac{1}{b_n} \sum_{i,j=1,i\neq j} (Z_{\pi_r(i),n} - \bar{Z})(Z_{\pi_r(j),n} - \bar{Z})' | Z \right].
\]
As for the first term, we write

\[ E_{1,n} = \frac{1}{n} \sum_{i=1}^{n} Z_{i,n}Z'_{i,n} - \bar{Z}\bar{Z}' \]

\[ = \frac{1}{n} \sum_{i=1}^{n} (Z_{i,n}Z'_{i,n} - \mathbf{E}[Z_{i,n}Z'_{i,n}]) - \bar{Z}\bar{Z}' + \frac{1}{n} \sum_{i=1}^{n} \mathbf{E}[Z_{i,n}Z'_{i,n}]. \]

The last term is \( I_m \) and by (7.15),\( \bar{Z}\bar{Z}' = \mathbf{O}(n^{-1} + \lambda_n(2)). \)

(7.16)

As for the leading term,

\[
\mathbf{E} \left\| \frac{1}{n} \sum_{i=1}^{n} (Z_{i,n}Z'_{i,n} - I_m) \right\|^2 = \frac{1}{n^2} \sum_{i,j=1,i\neq j}^{n} \text{tr} \left( \mathbf{E} \left[ (Z_{i,n}Z'_{i,n} - I_m)(Z_{j,n}Z'_{j,n} - I_m)^t \right] \right) + \mathbf{O}(n^{-1}).
\]

The leading term is bounded by \( C\mathbf{E}[\lambda(2; C_n)] = C\lambda_n(2). \) Hence

\[ E_{1,n} = I_m + \mathbf{O}(n^{-1/2} + \sqrt{\lambda_n(2)}). \]

Let us turn to \( E_{2,n} \) which we write as

(7.17)

\[ \frac{b_n - 1}{n(n-1)} \sum_{i,j=1,i\neq j}^{n} (Z_{i,n}Z'_{j,n} - \mathbf{E}[Z_{i,n}Z'_{j,n}]) + \frac{b_n - 1}{n(n-1)} \sum_{i,j=1,i\neq j}^{n} \mathbf{E}[Z_{i,n}Z'_{j,n}] + (b_n - 1)B_n, \]

where

\[ B_n = \bar{Z}\bar{Z}' - \frac{1}{n(n-1)} \sum_{i,j=1,i\neq j}^{n} \bar{Z}Z'_{j,n} - \frac{1}{n(n-1)} \sum_{i,j=1,i\neq j}^{n} Z_{i,n}Z'_{j,n} \]

\[ = \mathbf{O}(n^{-1} + \lambda_n(2)), \]

using the same argument as in (7.16).

The second term in (7.17) is bounded by \( b_n\lambda_n(2). \) As for the leading term in (7.17), we can follow the same arguments in the proof of Lemma 7.5 and show that it is equal to \( \mathbf{o}(1). \)

Therefore, we conclude that

\[ E_{2,n} = \mathbf{O}(b_n\lambda_n(2) + n^{-1/2} + \sqrt{\lambda_n(2)}) + \mathbf{o}(1). \]

Thus we obtain the desired result by Assumption 3.1(ii).
Proof of Theorem 3.2: First, note that
\[ Z'_{\pi_r(i), n} Z_{\pi_r(j), n} - (Z_{\pi_r(i), n} - \bar{Z})'(Z_{\pi_r(j), n} - \bar{Z}) = \bar{Z}' Z_{\pi_r(j), n} + \bar{Z}' Z_{\pi_r(i), n} - \bar{Z}' \bar{Z}, \]
where \( \bar{Z} = n^{-1} \sum_{i=1}^n Z_{i,n} \). Thus, we have
\[ S_n(\bar{X}; \tilde{\pi}) = \frac{1}{m \sqrt{R}} \sum_{r=1}^R \frac{1}{b_n} \sum_{i=1}^{b_n} \sum_{j=1, j \neq i}^{b_n} Z'_{\pi_r(i), n} Z_{\pi_r(j), n} + B_n + o_P(1), \]
where \( o_P(1) \) is due to the estimation error in \( \hat{\Sigma} \), and \( B_n \) is as defined in (3.1). In view of the heuristics in Section 3.3, \( B_n = o_P(1) \), if
\[ \frac{1}{\sqrt{R b_n}} \sum_{r=1}^R \sum_{i=1}^{b_n} (Z_{\pi_r(i)} - \bar{Z}) = O_P(1). \] (7.18)

Since convergence of the first moment of a nonnegative bounded random variable to zero implies convergence in probability to zero of the random variable, it follows that for each \( \varepsilon > 0 \), we have \( P\{|B_n| > \varepsilon|\bar{Z}\} \to_P 0 \) as \( n \to \infty \).

But this statement in (7.18) immediately follows from Lemma 7.9. Therefore, we find that
\[ S_n(\bar{X}; \tilde{\pi}) = S_n(\mu_0; \tilde{\pi}) + o_P(1). \]

Hence the conditional distribution of \( S_n(\bar{X}; \tilde{\pi}) \) given \( \bar{Z} \) is the same as that of \( S_n(\mu_0; \tilde{\pi}) \) up to an \( o_P(1) \) term. From the proof of Theorem 3.1, we find that this cumulative distribution function of \( S_n(\mu_0; \tilde{\pi}) \) converges in probability to that of \( N(0, 1) \), which completes the proof.

We turn to the local power analysis.

Proof of Theorem 3.3: First, we write
\[ S_n(\mu; \tilde{\pi}) = S_n(\mu_0; \tilde{\pi}) + D_n, \]
where
\[ D_n = \frac{1}{\sqrt{R}} \sum_{r=1}^R \frac{1}{mb_n} \sum_{i=1}^{b_n} \sum_{j=1, j \neq i}^{b_n} (X_{\pi_r(i)} - \mu)'(X_{\pi_r(j)} - \mu) - \frac{1}{\sqrt{R}} \sum_{r=1}^R \frac{1}{mb_n} \sum_{i=1}^{b_n} \sum_{j=1, j \neq i}^{b_n} (X_{\pi_r(i)} - \mu_0)'(X_{\pi_r(j)} - \mu_0). \]
We write $D_n = E_{1,n} + E_{2,n}$, where

$$E_{1,n} = (\mu_0 - \mu)' \hat{\Sigma}^{-1} \frac{1}{\sqrt{R}} \sum_{r=1}^{R} \sum_{i=1}^{b_n} \sum_{j=1:j \neq i}^{b_n} \left( X_{\pi_r(i)} + X_{\pi_r(j)} - 2\mu_0 \right),$$

and

$$E_{2,n} = \frac{\sqrt{R}b_n (\mu_0 - \mu)' \hat{\Sigma}^{-1} (\mu_0 - \mu)}{m}.$$ 

We rewrite

$$E_{1,n} = (\mu_0 - \mu)' \hat{\Sigma}^{-1} \frac{1}{\sqrt{R}} \sum_{r=1}^{R} \sum_{i=1}^{b_n} \sum_{j=1:j \neq i}^{b_n} \left( X_{\pi_r(i)} + X_{\pi_r(j)} - 2\bar{X} \right)$$

$$+ \frac{2\sqrt{R}(b_n - 1) (\mu_0 - \mu)' \hat{\Sigma}^{-1} (\bar{X} - \mu_0)}{m}.$$ 

By Lemma 7.8 and as we are under the local alternatives, the leading term is equal to $O_P \left( R^{-1/4} \right) = o_P(1)$ and the last term is

$$O_P \left( \left( \frac{\sqrt{R}b_n}{R^{1/4}b_n^{1/2}} \right) \left( \frac{1}{n^{1/2}} \right) \right) = O_P \left( \frac{R^{1/4}b_n^{1/2}}{n^{1/2}} \right) = o_P(1)$$

by Assumption [3.1](i). As for $E_{2,n}$, we write it as

$$\frac{\delta' \hat{\Sigma}^{-1} \delta}{m} = \frac{\delta' \Sigma^{-1}(C_n)\delta}{m} + o_P(1),$$

by (7.14). Hence we conclude that

$$S_n(\mu; \tilde{\pi}) = S_n(\mu_0; \tilde{\pi}) + \frac{\delta' \Sigma^{-1}(C_n)\delta}{m} + o_P(1).$$

By using the same arguments in the proof of Theorem 1, the conditional distribution of $S_n(\mu_0; \tilde{\pi})$ given $C_n$ is asymptotically $N(0, 1)$. Therefore,

$$P\{S_n(\mu; \tilde{\pi}) > c | C_n \} = 1 - \Phi \left( c - \frac{\delta' \Sigma^{-1}(C_n)\delta}{m} \right) + o_P(1).$$

A close inspection of the proof reveals that in fact the $L_1$-norm of the term $o_P(1)$ on the right hand side converges to zero. Taking the expectation on both sides of the equality and sending $n \to \infty$, we obtain the desired result. ■

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