Chiral corrections to $\pi^-\gamma \to 3\pi$ processes at low energies

N. Kaiser

Physik-Department T39, Technische Universität München, D-85747 Garching, Germany

Abstract

We calculate in chiral perturbation theory the double-pion photoproduction processes $\pi^-\gamma \to \pi^-\pi^0\pi^0$ and $\pi^-\gamma \to \pi^+\pi^-\pi^-$ at low energies. At leading order these reactions are governed by the chiral pion-pion interaction. The next-to-leading order corrections arise from pion-loop diagrams and chiral-invariant counterterms involving the low-energy constants $\ell_1$, $\ell_2$, $\ell_3$ and $\ell_4$. The pertinent production amplitudes $A_1$ and $A_2$ depending on five kinematical variables are given in closed analytical form. We find that the total cross section for neutral pion-pair production $\pi^-\gamma \to \pi^-\pi^0\pi^0$ gets enhanced in the region $\sqrt{s} < 7m_\pi$ by a factor $1.5 - 1.8$ by the next-to-leading order corrections. In contrast to this behavior the total cross section for charged pion-pair production $\pi^-\gamma \to \pi^+\pi^-\pi^-$ remains almost unchanged in the region $\sqrt{s} < 6m_\pi$ in comparison to its tree-level result. Although the dynamics of the pion-pair production reactions is much richer, this observed pattern can be understood from the different influence of the chiral corrections on the pion-pion final state interaction ($\pi^+\pi^- \to \pi^0\pi^0$ versus $\pi^-\pi^- \to \pi^-\pi^-$). We present also results for the complete set of two-pion invariant mass spectra. The predictions of chiral perturbation theory for the $\pi^-\gamma \to 3\pi$ processes can be tested by the COMPASS experiment which uses Primakoff scattering of high-energy pions in the Coulomb field of a heavy nucleus to extract cross sections for $\pi^-\gamma$ reactions with various final states.

PACS: 12.20.Ds, 12.39.Fe, 13.60.Fz, 13.75.Lb

1 Introduction and summary

The pions ($\pi^+, \pi^0, \pi^-$) are the Goldstone bosons of spontaneous chiral symmetry breaking in QCD: $SU(2)_L \times SU(2)_R \to SU(2)_V$. Their low-energy dynamics can therefore be calculated systematically (and accurately) with chiral perturbation theory in form of a loop-expansion based on an effective chiral Lagrangian. The accurate two-loop prediction [1] for the isospin-zero S-wave $\pi\pi$-scattering length $a_0^0 = (0.220 \pm 0.005)m_\pi^{-1}$ has been confirmed in the E865 [2] and NA48/2 [3] experiments by analyzing the $\pi^+\pi^-$ invariant mass distribution of the rare kaon decay mode $K^+ \to \pi^+\pi^-\nu_\mu$. One particular implication of that good agreement between theory and experiment is that the quark condensate $\langle 0|\bar{q}q|0\rangle$ constitutes the dominant order parameter [4] of spontaneous chiral symmetry breaking (considering the two-flavor sector of QCD). Likewise, the DIRAC experiment [5] has been proposed to determine the difference of the isospin-zero and isospin-two S-wave $\pi\pi$-scattering lengths $a_0^0 - a_0^2$ by measuring the life time ($\tau \simeq 3$fs) of pionium (i.e. $\pi^+\pi^-$ bound electromagnetically and decaying into $\pi^0\pi^0$). In the meantime the NA48/2 experiment [6] has accumulated very high statistics for the charged kaon decay modes $K^{\pm} \to \pi^{\pm}\pi^{0}\pi^{0}$, which allowed to extract the value $a_0^0 - a_0^2 = (0.268 \pm 0.010)m_\pi^{-1}$ for the $\pi\pi$-scattering length difference from the cusp effect in the $\pi^0\pi^0$ mass spectrum at the $\pi^+\pi^-$ threshold. This experimental result is again in very good agreement with the two-loop prediction $a_0^0 - a_0^2 = (0.265 \pm 0.004)m_\pi^{-1}$ of chiral perturbation theory [1]. For a discussion of isospin breaking corrections which have to be included in a meaningful comparison between
theory and experiment, see ref.[7]. Clearly, these remarkable confirmations give confidence that chiral perturbation theory is the correct framework to calculate reliably and accurately the strong interaction dynamics of the pions at low energies.

Electromagnetic processes offer further possibilities to probe the internal structure of the pion. For example, pion Compton scattering $\pi^-\gamma \to \pi^-\gamma$ at low energies allows one to extract the electric and magnetic polarizabilities ($\alpha_\pi$ and $\beta_\pi$) of the charged pion. Chiral perturbation theory at two-loop order gives for the dominant pion polarizability difference the firm prediction $\alpha_\pi - \beta_\pi = (5.7 \pm 1.0) \cdot 10^{-4} \text{fm}^3$ [8]. It is however in conflict with the existing experimental results from Serpukhov $\alpha_\pi - \beta_\pi = (15.6 \pm 7.8) \cdot 10^{-4} \text{fm}^3$ [9] and MAMI $\alpha_\pi - \beta_\pi = (11.6 \pm 3.4) \cdot 10^{-4} \text{fm}^3$ [10] which amount to values more than twice as large. Certainly, these existing experimental determinations of $\alpha_\pi - \beta_\pi$ raise doubts about their correctness since they violate the chiral low-energy theorem notably by a factor 2. The chiral low-energy theorem [11] relates $\alpha_\pi - \beta_\pi = 0$. The two-loop corrections of refs.[8, 13, 14] assure that the $\mathcal{O}(m_\pi)$ corrections to it are in fact small.

In that contradictory situation, it is promising that the ongoing COMPASS experiment [15] at CERN aims at re-measuring the pion polarizabilities, $\alpha_\pi$ and $\beta_\pi$, with high statistics using the Primakoff effect. The scattering of high-energy negative pions in the Coulomb field of a heavy nucleus (of charge $Z$) gives access to cross sections for $\pi^-\gamma$ reactions through the equivalent photon method:

$$\frac{d\sigma}{ds dQ^2} = \frac{Z^2 \alpha}{\pi(s - m_\pi^2)} \frac{Q^2 - Q_{\text{min}}^2}{Q^4} \sigma_{\pi^-\gamma}(s), \quad Q_{\text{min}} = \frac{s - m_\pi^2}{2E_{\text{beam}}}. \quad (1)$$

Here, $Q$ denotes the momentum transferred by the virtual photon to the heavy nucleus of charge $Z$, and one aims at isolating the Coulomb peak $Q \to 0$ from the strong interaction background. The last factor $\sigma_{\pi^-\gamma}(s)$ is the total cross section for a $\pi^-\gamma$ reaction induced by real photons with $\sqrt{s}$ the corresponding $\pi^-\gamma$ center-of-mass energy. Note that eq.(1) applies in the same form to differential cross sections on both sides. The COMPASS experiment is set up to detect simultaneously various (multi-particle) hadronic final states which are produced in the Primakoff scattering process of high-energy pions. In addition to pion Compton scattering $\pi^-\gamma \to \pi^-\gamma$ (which is of primary interest for determining the pion polarizabilities $\alpha_\pi$ and $\beta_\pi$) the reaction $\pi^-\gamma \to \pi^-\pi^0$ serves as a test of the QCD chiral anomaly (i.e. the anomalous $V_{AAA}$ rectangle quark diagram) by measuring the $\gamma 3\pi$ coupling constant $F_{\gamma 3\pi} = e/(4\pi f_\pi^2) = 9.72 \text{GeV}^{-3}$. For the two-body process $\pi^-\gamma \to \pi^-\pi^0$ the one-loop [16, 17] and two-loop corrections [18] of chiral perturbation theory as well as QED radiative corrections [19] have been worked out. Thus an accurate theoretical framework is available to analyze the upcoming data. The $\pi^-\gamma$ reaction with three charged pions in the final state is used by the COMPASS collaboration in the energy range 1 GeV $\sqrt{s}$ $< 2.5$ GeV to study the spectroscopy of non-strange meson resonances [20] ($a_1(1260)$, $a_2(1320)$, $\rho_0(1670)$, $\pi(1800)$, $a_4(2040)$ etc.) and to search for so-called exotic meson resonances [21] (e.g. $\pi_1(1600)$) with quantum numbers different from simple (constituent) quark-antiquark bound states. The statistics of the COMPASS experiment is actually so high that the event rates with three pions in the final state can even be continued downward to the threshold. The cross sections (and other more exclusive observables) of the $\pi^-\gamma \to 3\pi$ reactions in the low-energy region $\sqrt{s} < 1$ GeV offer new possibilities to test the strong interaction dynamics of the pions as predicted by chiral perturbation theory. The total cross sections for the processes $\pi^-\gamma \to \pi^-\pi^0\pi^0$ and $\pi^-\gamma \to \pi^+\pi^-\pi^-$ at tree-level have been calculated recently in ref.[17] and it was found that the cross section for the charged channel ($\pi^+\pi^-\pi^-$) comes
out about a factor of 2.5 larger than the one for the neutral channel ($\pi^-\pi^0\pi^0$). In both cases it is the chiral pion-pion interaction (together with the electromagnetic pion-photon coupling) which governs these reactions at leading order. A preliminary analysis [22] of the COMPASS data for $\pi^-\gamma \rightarrow \pi^+\pi^-\pi^-$ in the low-energy range $0.5\text{ GeV} < \sqrt{s} < 0.8\text{ GeV}$ leads to a total cross section which seems to be in agreement with the tree-level result [17] of chiral perturbation theory. An arguable aspect of that analysis is the absolute normalization which has to known well in order to convert count rates of events into a cross section. However, before any conclusions about an agreement between theory and experiment can be drawn the predictions of chiral perturbation theory need to be sharpened further by including higher orders in the small momentum expansion.

This is precisely the purpose of the present paper. We evaluate for the double-pion photoproduction processes $\pi^-\gamma \rightarrow \pi^-\pi^0\pi^0$ and $\pi^-\gamma \rightarrow \pi^+\pi^-\pi^-$ the next-to-leading order corrections as they arise from pion-loop diagrams and chiral-invariant counterterms (proportional to the low-energy constants $\ell_j$). Our paper is organized as follows. In section 2, we treat first the somewhat simpler case of neutral pion-pair production. We give for individual diagrams the analytical expressions for the pertinent production amplitudes, $A_1$ and $A_2$, which depend on five independent Lorentz-invariant kinematical variables. In this detailed exposition one can check the exact cancellation of ultraviolet divergences between pion-loops and counterterms. Combined with the tree-level amplitudes, $A_1$ and $A_2$ are employed to calculate the total cross section and the two-pion mass spectra for $\pi^-\gamma \rightarrow \pi^-\pi^0\pi^0$. We find that the total cross sections (as well as the two-pion mass spectra) get enhanced by a factor $1.5 - 1.8$ after including the next-to-leading order chiral corrections. Although the dynamics of the whole process is much richer this enhancement can be understood (in an approximate way) from the $\pi^+\pi^- \rightarrow \pi^0\pi^0$ final state interaction. In section 3, the same analytical calculation is performed for charged pion-pair production $\pi^-\gamma \rightarrow \pi^+\pi^-\pi^-$. In that case many more diagrams do contribute since the photon can now couple to all three outgoing (charged) pions. For that reason we restrict ourselves to specifying the finite parts of the pion-loop diagrams and the total counterterm contribution (rewritten in terms of the low-energy constants $\ell_j$ which subsume the chiral logarithm $\ln(m_\pi/\lambda)$ generated by pion-loops). Interestingly, we find that the total cross section for $\pi^-\gamma \rightarrow \pi^+\pi^-\pi^-$ in the low-energy region $3m_\pi < \sqrt{s} < 6m_\pi$ remains almost unchanged by the inclusion of the next-to-leading order corrections. Although the dynamics of the process is much richer the (known) weak influence of chiral corrections on the isospin-two $\pi^-\pi^- \rightarrow \pi^-\pi^-$ scattering length $a_0^2$ [1] can provide an explanation for this feature. In that case the $\pi^+\pi^-$ mass spectrum reveals better certain dynamical details which get averaged out in the total cross section. We estimate also the uncertainties of the observables for the $\pi^-\gamma \rightarrow 3\pi$ reactions which are induced by the present errorbars of the low-energy constants $\ell_j$, and find about $\pm 5\%$.

In summary, we have calculated the processes $\pi^-\gamma \rightarrow \pi^-\pi^0\pi^0$ and $\pi^-\gamma \rightarrow \pi^+\pi^-\pi^-$ at next-to-leading order in chiral perturbation theory. The predictions for the total cross sections and other more exclusive observables can be tested soon by the COMPASS experiment at CERN.

### 2 Neutral pion-pair production

In this section we treat the neutral pion-pair production process: $\pi^-(p_1) + \gamma(k, \epsilon) \rightarrow \pi^-(p_2) + \pi^0(q_1) + \pi^0(q_2)$. We choose for the (transversal) real photon the Coulomb-gauge in the center-of-mass frame, which entails the conditions $\epsilon \cdot p_1 = \epsilon \cdot k = 0$. These conditions imply that all diagrams for which the photon couples to the in-coming pion $\pi^-(p_1)$ vanish identically. Furthermore, it is advantageous to parametrize the special-unitary matrix-field $U$ in the chiral
Figure 1: Tree-level diagrams for $\pi^-\gamma \rightarrow \pi^-\pi^0\pi^0$ and $\pi^+\pi^-\pi^-$. Arrows indicate out-going pions. Only the left diagram contributes to $2\pi^0$-production.

Lagrangian $\mathcal{L}_{\pi\pi}$ through an interpolating pion-field $\bar{\pi}$ in the form $U = \sqrt{1 - \bar{\pi}^2/f_{\pi}^2} + i\bar{\pi} \cdot \pi/f_{\pi}$.

It has the consequence that no $\gamma 4\pi$ and $\gamma 6\pi$ contact vertices exist at leading order. Under these assumptions one is left with one single non-vanishing tree diagram shown in Fig. 1. Let us recall the expression for the total cross section for $\pi^-\gamma \rightarrow \pi^-\pi^0\pi^0$ at tree-level:

$$\sigma_{\text{tot}}(s) = \frac{\alpha}{32\pi^2f_{\pi}^4(s - m_{\pi}^2)^3} \int_{2m_{\pi}\sqrt{s}} \frac{dw}{s - w + m_{\pi}^2 - 4m_{\pi}^2} \left\{ w \ln \left[ \frac{w + \sqrt{w^2 - 4m_{\pi}^2}s}{2m_{\pi}\sqrt{s}} \right] - \sqrt{w^2 - 4m_{\pi}^2}s \right\}. \tag{2}$$

Here, $s = (p_1 + k)^2$ denotes the squared center-of-mass energy, $\alpha = e^2/4\pi = 1/137.036$ is the fine structure constant, and $f_{\pi} = 92.4$ MeV stands for the pion decay constant. In comparison to eq.(16) in ref.[17] we have included (some) isospin breaking effects by distinguishing the mass of the charged pion $m_{\pi} = 139.570$ MeV from the mass of the neutral pion $m_{\pi^0} = 134.977$ MeV. Since it turns out that these effects are very small (see Fig. 7) we will perform the whole calculation in the limit of isospin symmetry. In order to present the next-to-leading order corrections from chiral loops and counterterms one has to start from the general form of the T-matrix, which reads (in Coulomb-gauge):

$$T = \frac{2e}{f_{\pi}^2} \left[ \bar{\epsilon} \cdot \vec{q}_1 A_1 + \bar{\epsilon} \cdot \vec{q}_2 A_2 \right]. \tag{3}$$

In this decomposition $A_1$ and $A_2$ are two dimensionless production amplitudes which depend on $s = (p_1 + k)^2$ and on four other independent (Lorentz-invariant) Mandelstam variables:

$$s_1 = (p_2 + q_1)^2, \quad s_2 = (p_2 + q_2)^2, \quad t_1 = (q_1 - k)^2, \quad t_2 = (q_2 - k)^2. \tag{4}$$

This set is very convenient for describing the permutation of the two identical neutral pions in the final state via ($s_1 \leftrightarrow s_2$, $t_1 \leftrightarrow t_2$). For most diagrammatic contributions $A_1 = A_2$, but there are a few exceptions which require their separate listing and specification.

With the form of the T-matrix defined in eq.(3) and the kinematical variables introduced in eq.(4) the tree-level amplitudes read:

$$A_1^{\text{(tree)}} = A_2^{\text{(tree)}} = \frac{2m_{\pi}^2 + s - s_1 - s_2}{3m_{\pi}^2 - s - t_1 - t_2}. \tag{5}$$

Note that $T^{\text{(tree)}}$ is written with physical parameters ($f_{\pi}^2$, $m_{\pi}^2$) and not with leading order parameters as given by the chiral Lagrangian. It requires an extra renormalization contribution (see eq.(26)) to properly account for this difference.
2.1 Diagrammatic calculation

In this subsection we present analytical expressions for the amplitudes $A_1$ and $A_2$ as they arise from the next-to-leading order chiral loops and counterterms. We go step by step through the whole set of contributing diagrams. The one-loop diagrams (I), (II), (III) shown in Fig. 2 include an additional $\pi\pi$-interaction but leave the pion-photon coupling (of the tree diagram) unchanged. One finds from diagram (I):

\[
A_1^{(I)} = A_2^{(I)} = \frac{1}{(4\pi f_\pi)^2} \left( \frac{2m_\pi^2}{3m_\pi^2 - s - t_1 - t_2} \left\{ (\xi + \ln \frac{m_\pi}{\lambda})(s_1 + s_2 + t_1 + t_2 - 11m_\pi^2) 
+ (s_1 + s_2 + t_1 + t_2 - 7m_\pi^2) \left[ J(3m_\pi^2 + s - s_1 - s_2) - \frac{1}{2} \right] \right\} \right),
\]

with the abbreviation

\[
\xi = \lambda^{d-4} \left\{ \frac{1}{d-4} + \frac{1}{2}(\gamma_E - 1 - \ln 4\pi) \right\},
\]

for standard ultraviolet divergence in dimensional regularization. Note that $\xi$ is always accompanied by the chiral logarithm $\ln(m_\pi/\lambda)$. The complex-valued loop function $J(s)$ has the form:

\[
J(s) = \sqrt{\frac{s - 4m_\pi^2}{s}} \left[ \ln \sqrt{\frac{|s - 4m_\pi^2| + \sqrt{|s|}}{2m_\pi}} - i\frac{\pi}{2}(s - 4m_\pi^2) \right], \quad s < 0 \quad \text{or} \quad s > 4m_\pi^2,
\]

\[
J(s) = \sqrt{\frac{4m_\pi^2 - s}{s}} \arcsin \sqrt{\frac{s}{2m_\pi}}, \quad 0 < s < 4m_\pi^2.
\]

Similarly, one gets from diagram (II):

\[
A_1^{(II)} = A_2^{(II)} = \frac{1}{3(4\pi f_\pi)^2} \left( \frac{2m_\pi^2}{3m_\pi^2 - s - t_1 - t_2} \left\{ (\xi + \ln \frac{m_\pi}{\lambda})(s_1 + s_2 + t_1 + t_2) 
+ (5s_2 + 5t_2 - 4s_1 - 4t_1)m_\pi^2 - 13m_\pi^4 \right] + (s_1 + s_2 + t_1)(s_1 + s_2 + 2t_2) 
+ (s_1 + s_2 + t_1)(s_1 + s_2 + 2t_2) \right\},
\]

\[
	imes (5s_1 - 5s_2 - 2s_1 - 7t_2) + \frac{m_\pi^2}{6}(14s_1 + 11t_1 - 3s - 4s_2 - 7t_2 - 7m_\pi^2) \right\},
\]

and the contribution from diagram (III) follows by interchanging the two neutral pions:

\[
A_1^{(III)} = A_2^{(III)} = A_1^{(II)}|_{s_1 \leftrightarrow s_2, t_1 \leftrightarrow t_2}.
\]

The one-loop diagrams (IV), (V), (VI) shown in Fig. 3 include also an additional $\pi\pi$-interaction but the photon couples now to a charged pion inside the loop. One finds from diagram (IV):

\[
A_1^{(IV)} = A_2^{(IV)} = \frac{2m_\pi^2 + s - s_1 - s_2}{(4\pi f_\pi)^2} \left\{ (\xi + \ln \frac{m_\pi}{\lambda} - \frac{1}{2} + J(3m_\pi^2 + s - s_1 - s_2) 
+ \frac{m_\pi^2 - s}{2m_\pi^2 - t_1 - t_2} \left( s_1 + s_2 + s - m_\pi^2 - t_1 - t_2 \right) 
\times \left[ J(m_\pi^2 + s - s_2 + t_1 + t_2) - J(3m_\pi^2 + s_1 - s_2) \right] 
+ 2m_\pi^2 \left[ G(m_\pi^2 + s - s_1 - s_2 + t_1 + t_2) - G(3m_\pi^2 + s_1 - s_2) \right] \right\},
\]
Figure 2: One-pion loop diagrams for $\pi^-\gamma \rightarrow \pi^-\pi^0\pi^0$. Their combinatoric factor is 1/2.

Figure 3: One-pion loop diagrams for $\pi^-\gamma \rightarrow \pi^-\pi^0\pi^0$. Their combinatoric factor is 1.

where the complex-valued loop function $G(s)$ has the form:

$$G(s) = \left[ \ln \frac{\sqrt{|s - 4m^2_\pi|} + \sqrt{|s|}}{2m_\pi} - \frac{i\pi}{2} \theta(s - 4m^2_\pi) \right]^2, \quad s < 0 \text{ or } s > 4m^2_\pi,$$

$$\text{or}
\begin{align*}
G(s) &= -\arcsin^{2} \frac{s}{2m_\pi}, \quad 0 < s < 4m^2_\pi.
\end{align*}$$

(13) (14)

Considerably more lengthy are the expressions which one obtains from evaluating diagram (V):

$$A_1^{(V)} = \frac{1}{3(4\pi f^2)} \left\{ \left( \xi + \ln \frac{m_\pi}{\lambda} \right) (6m^2_\pi + 2s - 2s_1 + t_2) + \frac{1}{6} (13s_1 - 13s_1 - 5t_2) ight.$$

$$\begin{align*}
+ &\frac{(s_1 + 8m^2_\pi)(m^2_\pi - t_2)}{2(s + t_2 - 2m^2_\pi)} + 6m^2_\pi \left[ G(2m^2_\pi + s_1 - s - t_2) - G(s_1) \right] \\
\times &\frac{s(s + t_2 - s_1) + (s_1 - 2s)m^2_\pi}{(s + t_2 - 2m^2_\pi)^2} + \left\{ 2s_1 + 4m^2_\pi + \frac{s_1(s_1 + 8m^2_\pi)(t_2 - m^2_\pi)}{(s + t_2 - 2m^2_\pi)^2} \right. \\
&\left. + \frac{s_1(m^2_\pi - 2t_2 - 2s_1) + 4m^2_\pi(m^2_\pi - t_2)}{s + t_2 - 2m^2_\pi} \right\} [J(s_1) - J(2m^2_\pi + s_1 - s - t_2)] \\
&\left. + \left[ 2s - 2s_1 + t_2 + \frac{4m^2_\pi(t_2 - m^2_\pi)}{s + t_2 - 2m^2_\pi} \right] J(2m^2_\pi + s_1 - s - t_2) \right\},
\end{align*}$$

(15)

$$A_2^{(V)} = \frac{1}{3(4\pi f^2)} \left\{ \left( \xi + \ln \frac{m_\pi}{\lambda} \right) (s + s_1 - s_2 + t_1 - 3m^2_\pi) + \frac{1}{6} (5s_2 + t_1 - 5s_1) ight.$$

$$\begin{align*}
+ &\frac{s_1(2t_2 - 3t_1) - t_1t_2 + m^2_\pi(s_1 + 2t_2 - 8t_1 + 7m^2_\pi)}{2(s + t_2 - 2m^2_\pi)} + \frac{(s_1 + 8m^2_\pi)(t_1 - m^2_\pi)(t_2 - m^2_\pi)}{(s + t_2 - 2m^2_\pi)^2} \\
+ &\frac{s_1(m^2_\pi - 2t_2 - 2s_1) + 4m^2_\pi(m^2_\pi - t_2)}{s + t_2 - 2m^2_\pi} \right\},
\end{align*}$$

(16)
Figure 4: One-pion loop diagrams for $\pi^-\gamma \to \pi^-\pi^0\pi^0$. Their combinatoric factor is $1/2$.

\[
+6m_\pi^2\left[G(2m_\pi^2 + s_1 - s - t_2) - G(s_1)\right] \left\{ \frac{s_1 - t_1}{s + t_2 - 2m_\pi^2} + \frac{2s_1(t_1 - m_\pi^2)(m_\pi^2 - t_2)}{(s + t_2 - 2m_\pi^2)^3} \right. \\
\left. + \frac{2(3t_1 - t_2 - 2m_\pi^2)}{s + t_2 - 2m_\pi^2} + \frac{8(t_1 - m_\pi^2)(m_\pi^2 - t_2)}{(s + t_2 - 2m_\pi^2)^2} \right\} J(2m_\pi^2 + s_1 - s - t_2) \\
+ \left\{ m_\pi^2 - s_1 + \frac{s_1(3s_1 - s_2 - 2t_1 + t_2) + m_\pi^2(3m_\pi^2 + 4s_2 - 4t_1 - t_2)}{s + t_2 - 2m_\pi^2} \\
+ \frac{s_1(3t_1 - 2t_2 - m_\pi^2)}{(s + t_2 - 2m_\pi^2)^2} + 4(t_1 - m_\pi^2)(t_2 - m_\pi^2) \right\} \left\{ J(s_1) - J(2m_\pi^2 + s_1 - s - t_2) \right\} .
\]  

(16)

In the process of evaluation one encounters loop integrals over three pion-propagators with tensors up to third rank $(l_\mu, l_\mu l_\nu, l_\mu l_\nu l_\rho)$ in the numerator. After the pertinent tensor reduction all the occurring scalar loop functions can be expressed as linear combinations of $J(\ldots)$ and $G(\ldots)$ with rational coefficient functions. Note that the quality $A_1 = A_2$ does no more hold for diagram (V) since the four-momenta $q_1$ and $q_2$ of the two neutral pions appear in the loop in a non-symmetrical way. The permutational symmetry between both $\pi^0$ is restored by the contribution from diagram (VI):

\[
A_1^{(VI)} = A_2^{(V)}(s_1 \leftrightarrow s_2, t_1 \leftrightarrow t_2), \quad A_2^{(VI)} = A_1^{(V)}(s_1 \leftrightarrow s_2, t_1 \leftrightarrow t_2).
\]  

(17)

Diagram (VII) in Fig. 4 generates a (constant) vertex correction to the pion-photon coupling and therefore leads to the amplitudes:

\[
A_1^{(VII)} = A_2^{(VII)} = \frac{2m_\pi^2}{(4\pi f_\pi)^2} \frac{2m_\pi^2 + s - s_1 - s_2}{3m_\pi^2 - s - t_1 - t_2} \left( \xi + \ln \frac{m_\pi}{\lambda} \right),
\]  

(18)

which are proportional to the tree amplitudes written in eq.(5). The chiral six-pion vertex appearing in diagrams (VIII) and (IX) represents some challenge with respect to tackling the combinatorics involved. Altogether, one finds from diagram (VIII) the momentum-dependent amplitudes:

\[
A_1^{(VIII)} = A_2^{(VIII)} = \frac{m_\pi^2}{(4\pi f_\pi)^2} \frac{31m_\pi^2 + 2(4s - 5s_1 - 5s_2 - t_1 - t_2)}{3m_\pi^2 - s - t_1 - t_2} \left( \xi + \ln \frac{m_\pi}{\lambda} \right),
\]  

(19)
Figure 5: Diagrams for $\pi^-\gamma \to \pi^-\pi^0\pi^0$ involving the pion wave function renormalization factor. The black square symbolizes the counterterm proportional to $\ell_4$. Not all possible diagrams of this kind are shown.

Figure 6: Next-to-leading order tree diagrams for $\pi^-\gamma \to \pi^0\pi^0$. The black square symbolizes the chiral-invariant counterterm proportional to $\ell_4$, while diagram (IX) leads to constant amplitudes:

$$A_{1(IX)} = A_{2(IX)} = -\frac{2m_\pi^2}{(4\pi f_\pi)^2} \left( \xi + \ln \frac{m_\pi}{\lambda} \right). \quad (20)$$

Fig. 5 shows diagrams with self-energy insertions on external or internal pion-lines. Their overall effect is to multiply the tree amplitudes $A_{1,2}^{(tree)}$ with three times the pion wave function renormalization factor $Z_{\pi^-} - 1$ (see eq.(31) in ref.[23]):

$$A_{1(X)} = A_{2(X)} = \frac{6m_\pi^2}{(4\pi f_\pi)^2} \left( \frac{s_1 + s_2 - s - 2m_\pi^2}{3m_\pi^2 - s - t_1 - t_2} \right) \left( 3\xi + 16\pi^2\ell_4^r + \ln \frac{m_\pi}{\lambda} \right). \quad (21)$$

Note that we have combined the contribution from the pion-loop proportional to $\xi + \ln(m_\pi/\lambda)$ with that from the counterterm proportional to $\ell_4 = \ell_4^r + \xi/8\pi^2$.

Fig. 6 shows the remaining tree diagrams for $\pi^-\gamma \to \pi^-\pi^0\pi^0$ generated by the chiral counterterm Lagrangian $L_{\pi\pi}^{(4)}$. We use the form of $L_{\pi\pi}^{(4)}$ as written in eq.(20) of ref.[23]. Diagram (XI) with a vertex correction at the pion-photon coupling gives rise to the amplitudes:

$$A_{1(XI)} = A_{2(XI)} = \frac{4m_\pi^2}{(4\pi f_\pi)^2} \left( \frac{2m_\pi^2 + s - s_1 - s_2}{3m_\pi^2 - s - t_1 - t_2} \right) \left( \xi + 8\pi^2\ell_4^r \right), \quad (22)$$

which are obviously proportional to the tree amplitudes. Diagram (XII) involves the four-pion vertex from $L_{\pi\pi}^{(4)}$. For our choice of the interpolating pion field it has no term proportional to the low-energy constant $\ell_3$. With that simplification one finds from diagram (XII):

$$A_{1(XII)} = A_{2(XII)} = \frac{1}{(4\pi f_\pi)^2} \left( \frac{1}{3m_\pi^2 - s - t_1 - t_2} \left( \frac{\xi}{3} + 16\pi^2\ell_1^r \right) (s + m_\pi^2 - s_1 - s_2) \right)$$

while diagram (IX) leads to constant amplitudes:

$$A_{1(IX)} = A_{2(IX)} = -\frac{2m_\pi^2}{(4\pi f_\pi)^2} \left( \xi + \ln \frac{m_\pi}{\lambda} \right). \quad (20)$$
\[ \times (2s - 2m_{\pi}^2 - s_1 - s_2 + t_1 + t_2) + \left( \frac{\xi}{3} + 8\pi^2 \ell_2^r \right) \{ s_1 (s_1 + t_1 - t_2) \\
+ s_2 (s_2 + t_2 - t_1) - s (s_1 + s_2 + t_1 + t_2) + 3m_{\pi}^2 (2s - s_1 - s_2 + t_1 + t_2) \\
- 2t_1 t_2 \} + \left( \xi + 8\pi^2 \ell_1^r \right) m_{\pi}^2 \{ 5s + 5m_{\pi}^2 - 4s_1 - 4s_2 + t_1 + t_2 \} \}. \] (23)

Finally, there is diagram (XIII) involving the (next-to-leading order) \( \gamma 4\pi \) contact vertex from \( \mathcal{L}_{\pi \pi}^{(4)} \). Its polynomial contribution to the amplitudes:

\[ A_1^{(XIII)} = \frac{2}{(4\pi f_{\pi})^2} \left\{ \left( \frac{\xi}{3} + 16\pi^2 \ell_1^r \right) (s + m_{\pi}^2 - s_1 - s_2) \\
+ \left( \frac{\xi}{3} + 8\pi^2 \ell_2^r \right) (2m_{\pi}^2 - s + s_1 - s_2 - t_2) + \left( \xi + 8\pi^2 \ell_4^r \right) m_{\pi}^2 \right\}, \] (24)

\[ A_2^{(XIII)} = \frac{2}{(4\pi f_{\pi})^2} \left\{ \left( \frac{\xi}{3} + 16\pi^2 \ell_1^r \right) (s + m_{\pi}^2 - s_1 - s_2) \\
+ \left( \frac{\xi}{3} + 8\pi^2 \ell_2^r \right) (2m_{\pi}^2 - s - s_1 + s_2 - t_1) + \left( \xi + 8\pi^2 \ell_4^r \right) m_{\pi}^2 \right\}, \] (25)

is exceptional in the sense that \( A_1 \) and \( A_2 \) are not equal. Up to that point the compilation of the production amplitudes \( A_{1,2} \) for the process \( \pi^- \gamma \rightarrow \pi^- \pi^0 \pi^0 \) as they arise from chiral loops and counterterms is completed. There is still one issue which has to be addressed, namely the renormalization of the squared pion decay constant \( f_{\pi}^2 \) and squared pion mass \( m_{\pi}^2 \) by chiral loops and counterterms (see herefore e.g. eqs.(29,30) in ref.[23]). The chiral Lagrangian \( \mathcal{L}_{\pi \pi} \) operates with their leading-order values whereas the tree amplitudes in eqs.(3,5) have been written already in terms their physical values. In order to correct for this difference of order \( (m_{\pi}/2\pi f_{\pi})^2 \) one has to add an extra renormalization contribution of the form:

\[ A_1^{(\text{ren})} = A_2^{(\text{ren})} = \frac{m_{\pi}^2}{(4\pi f_{\pi})^2} \frac{1}{3m_{\pi}^2 - s - t_1 - t_2} \{ 4(2m_{\pi}^2 + s - s_1 - s_2) \\
\times \left( 8\pi^2 \ell_4^r - \ln \frac{m_{\pi}}{\lambda} \right) + m_{\pi}^2 \left( 32\pi^2 \ell_3^r + \ln \frac{m_{\pi}}{\lambda} \right) \}. \] (26)

A first crucial check of our calculation is provided by the fact that the ultraviolet divergence \( \xi \) drops out in the total sums for \( A_1 \) and \( A_2 \). Next, one can further simplify the expressions for the amplitudes by introducing via the relation:

\[ \ell_j^r = \frac{\gamma_j}{32\pi^2} \{ \tilde{\ell}_j^r + 2 \ln \frac{m_{\pi}}{\lambda} \}, \quad \gamma_1 = \frac{1}{3}, \quad \gamma_2 = \frac{2}{3}, \quad \gamma_3 = -\frac{1}{2}, \quad \gamma_4 = 2, \] (27)

the "barred" low-energy constants \( \tilde{\ell}_j^r \) which subsume the chiral logarithm \( \ln \frac{m_{\pi}}{\lambda} \). After summing up the expressions in eqs.(18-26) and adding the terms proportional to the chiral logarithm \( \ln \frac{m_{\pi}}{\lambda} \) in the loop amplitudes eqs.(6,10,11,12,15,16,17) one gets the following modified and complete counterterm contributions:

\[ A_1^{(\text{ct})} = \frac{1}{(4\pi f_{\pi})^2} \frac{1}{3m_{\pi}^2 - s - t_1 - t_2} \left\{ \frac{\tilde{\ell}_1^r}{3} (s_1 + s_2 - s - m_{\pi}^2)^2 + \frac{\tilde{\ell}_2^r}{3} \left[ s^2 + s_1^2 + s_2^2 + t_2^2 - 2st_1 + (s_2 - s_1 + 2s_2 - t_1) t_2 + m_{\pi}^2 (s - 6s_2 + t_1 - 2t_2 + 6m_{\pi}^2) \right] \\
- \frac{\tilde{\ell}_3^r}{2} m_{\pi}^4 + 2\tilde{\ell}_4^r m_{\pi}^2 (s + 2m_{\pi}^2 - s_1 - s_2) \right\}, \] (28)
\[ A^{(ct)}_2 = \frac{1}{(4\pi f_\pi^2)} \frac{1}{3m_\pi^2} \left\{ \frac{\ell_1}{2} (s_1 + s_2 - s - m_\pi^2) + \frac{\ell_2}{3} \left[ s^2 + s_1^2 + s_2^2 + t_1^2 - 2ss_2 + (s + 2s_1 - 2s_2 - t_2) t_1 + m_\pi^2 (s - 6s_1 + t_2 + 6m_\pi^2) \right] - \frac{\ell_3}{2} m_\pi^4 + 2\ell_4 m_\pi^2 (s + 2m_\pi^2 - s_1 - s_2) \right\}. \] (29)

The remaining finite parts of the loop amplitudes are then given by eqs. (6,10,11,12,15,16,17) with the \( \xi + \ln(m_\pi/\lambda) \) terms deleted altogether.

Let us also mention that the other three terms in the chiral Lagrangian \( \mathcal{L}_{4\pi}^{(4)} \) proportional to the low-energy constants \( \ell_5,6,7 \) [13] do not contribute to the \( \pi^- \gamma \to 3\pi \) processes considered in this work. The \( \ell_7 \) term breaks isospin symmetry and is ignored for this reason. The \( \ell_5 \) term requires (for electromagnetic process) at least two external photons. The \( \ell_6 \) term gives rise to a correction to the pion-photon coupling, which however vanishes for real photons (\( k^2 = 0 = \epsilon \cdot k \)). Let us remind that \( \ell_6 \) is related to the mean square charge radius of the pion via: \( \langle r^2_\pi \rangle = (\ell_6 - 1)/(4\pi f_\pi^2) + O(m_\pi^2) \) [24]. The associated \( \gamma 4\pi \) vertex vanishes also since the commutator in tr\( (\tau_3 [\partial_\mu U, \partial_\nu U^\dagger]) \) terminates at the quadratic order in the pion-field (for our choice of interpolating pion-field).

### 2.2 Results for \( \pi^- \gamma \to \pi^- \pi^0 \pi^0 \)

In this section we present and discuss the results for the neutral pion-pair production process \( \pi^- \gamma \to \pi^- \pi^0 \pi^0 \) at next-to-leading order in chiral perturbation theory. We use for the low-energy constants \( \ell_j \) the values: \( \ell_1 = -0.4 \pm 0.6, \ell_2 = 4.3 \pm 0.1, \ell_3 = 2.9 \pm 2.4, \ell_4 = 4.4 \pm 0.2 \), as determined (with improved empirical input) in ref.[1]. Applying the flux and symmetry factors the total cross section \( \sigma_{\text{tot}}(s) \) is obtained by integrating the squared (transversal) T-matrix over the three-pion phase space:

\[ \sigma_{\text{tot}}(s) = \frac{\alpha}{32\pi^3 f_\pi^2(s - m_\pi^2)} \int \frac{d\omega_1 d\omega_2}{z \epsilon^2} \int_{-1}^{1} dx \int_{0}^{\pi} d\phi \left| \hat{k} \times (\vec{q}_1 A_1 + \vec{q}_2 A_2) \right|^2. \] (30)

Here, \( \omega_{1,2} \) are the center-of-mass energies of the out-going neutral pions and \( q_{1,2} = \sqrt{\omega_{1,2}^2 - m_\pi^2} \).

In terms of the directional cosines \( x, y, z \) the squared cross products in eq.(30) take the form:

\[ (\hat{k} \times \vec{q}_1)^2 = q_1^2 (1 - x^2), \quad (\hat{k} \times \vec{q}_2)^2 = q_2^2 (1 - y^2), \quad (\hat{k} \times \vec{q}_1) \cdot (\hat{k} \times \vec{q}_2) = q_1 q_2 (z - xy), \] (31)

together with the relations:

\[ q_1 q_2 z = \omega_1 \omega_2 - \sqrt{s} (\omega_1 + \omega_2) + \frac{s + m_\pi^2}{2}, \quad y = xz + \sqrt{1 - x^2}(1 - z^2) \cos \phi. \] (32)

The Mandelstam variables \( s_1, s_2, t_1, t_2 \) follow as:

\[ s_1 = s + m_\pi^2 - 2\sqrt{s} \omega_1, \quad s_2 = s + m_\pi^2 - 2\sqrt{s} \omega_1, \]

\[ t_1 = m_\pi^2 + \frac{m_\pi^2 - s}{\sqrt{s}} (\omega_1 - q_1 x), \quad t_2 = m_\pi^2 + \frac{m_\pi^2 - s}{\sqrt{s}} (\omega_2 - q_2 y). \] (33)

The more definite expression of the tree-level cross section given in eq.(2) is very helpful for checking the numerical accuracy of the four-dimensional integration involved in eq.(30).

Fig.7 show the total cross section \( \sigma_{\text{tot}}(s) \) for the reaction \( \pi^- \gamma \to \pi^- \pi^0 \pi^0 \) in the low-energy region from threshold \( \sqrt{s} = 3m_\pi \) up to \( \sqrt{s} = 7m_\pi \). The dashed line corresponds to the tree
approximation and the full line includes in addition the next-to-leading order corrections from chiral loops and counterterms. The dotted line follows if the charged and neutral pion mass are distinguished (134.977 MeV = $m_{\pi^0} < m_\pi = 139.570$ MeV) in the tree-level amplitude and the three-pion phase space integral (see eq.(2)). As it could be expected such isospin-breaking effects are very small. By inspection of Fig. 7 one observes that the total cross section for $\pi^+\pi^- \rightarrow \pi^0\pi^0$ gets enhanced sizeably (by a factor of 1.5 - 1.8) after inclusion of the next-to-leading order chiral corrections. Although the dynamics of the whole process is much richer this feature can be understood (in an approximate way) from the $\pi^+\pi^- \rightarrow \pi^0\pi^0$ final state interaction. The $\pi^+\pi^- \rightarrow \pi^0\pi^0$ interaction strength at threshold is determined by the difference of the isospin-zero and isospin-two S-wave $\pi\pi$-scattering lengths. When considering the corresponding one-loop expression \cite{24}:

$$\frac{1}{3}(a_0^1 - a_0^2) = \frac{3m_\pi}{32\pi f_\pi^2} \left[ 1 + \frac{m_\pi^2}{36\pi^2 f_\pi^2} \left( \bar{\ell}_1 + 2\bar{\ell}_2 - \frac{3\bar{\ell}_3}{8} + \frac{9\bar{\ell}_4}{2} + \frac{33}{8} \right) \right], \quad (34)$$

one sees that the correction to 1 inside the square bracket amounts to about 0.20 (inserting the central values of $\bar{\ell}_j$). Indeed, the square $1.2^2 = 1.44$ is close to the enhancement factor 1.5 of the total cross section at $\sqrt{s} = 4m_\pi$.

It is also important to investigate the uncertainties which are induced by the present errorbars $\delta\bar{\ell}_j$ of the low-energy constants $\bar{\ell}_j$. Taking the total cross section at $\sqrt{s} = 6m_\pi$ as a measure one finds relative uncertainties of: ±5.1% from $\delta\bar{\ell}_1$, ±1.4% from $\delta\bar{\ell}_2$, ±0.2% from $\delta\bar{\ell}_3$, and ±0.9% from $\delta\bar{\ell}_4$. It is comforting that the badly known low-energy constant $\bar{\ell}_3 = 2.9 \pm 2.4$ has very little influence on the observables considered here. The largest uncertainty is actually connected with $\bar{\ell}_1$ and adding them all in quadrature one gets a total relative uncertainty of ±5.4%. This amounts to a fairly accurate prediction.

In addition to the total cross section there are more exclusive observables such as the two-pion mass spectra. The $\pi^0\pi^0$ invariant mass $\mu$ is defined by $\mu^2 = (q_1 + q_2)^2 = s - s_1 - s_2 + 3m_\pi^2$ and it varies over the kinematically allowed range $2m_\pi < \mu < \sqrt{s} - m_\pi$. In tree approximation

Figure 7: Total cross section for the reaction $\pi^- \gamma \rightarrow \pi^0\pi^0$ as a function of the center-of-mass energy $\sqrt{s}$. 
Figure 8: \( \pi^0\pi^0 \) mass spectrum for the reaction \( \pi^-\gamma \to \pi^-\pi^0\pi^0 \) as a function of the \( \pi^0\pi^0 \) invariant mass \( \mu \). The curves correspond to center-of-mass energies \( \sqrt{s} = (4, 5, 6, 7)m_\pi \) in ascending order.

The \( \pi^0\pi^0 \) mass spectrum can be given in closed analytical form:

\[
\frac{d\sigma}{d\mu} = \frac{\alpha\sqrt{\mu^2 - 4m^2_\pi}}{16\pi^2 f^4(\pi)(s - m^2_\pi)^3} \left( \frac{\ln}{2m_\pi\sqrt{s}} \right) \left( s + m^2_\pi - \mu^2 + \sqrt{W} - \sqrt{W} \right),
\]

(35)

with the abbreviation \( W = [s - (\mu + m_\pi)^2][s - (\mu - m_\pi)^2] \). In order to obtain \( d\sigma/d\mu \) in general one introduces new energy variables \( \omega_+ = \omega_1 + \omega_2 \) and \( \omega_- = (\omega_1 - \omega_2)/2 \) such that \( \mu^2 = 2\sqrt{s}\omega_+ + m^2_\pi - s \). The \( d\omega_- \) integration in eq.(30) is omitted and a normalization factor \( \mu/\sqrt{s} \) is applied. The lower and upper limit for the \( d\omega_+ \) integration are \( \pm(\mu^2 - 4m^2_\pi)W/(4\mu\sqrt{s}) \).

Fig.8 shows the calculated \( \pi^0\pi^0 \) mass spectrum \( m_\pi d\sigma/d\mu \). We have multiplied it by the constant factor \( m_\pi \) in order to keep the units (\( \mu \text{b} \)) of a cross section. The four pairs of (full and dashed) curves correspond to sections at center-of-mass-energies \( \sqrt{s} = (4, 5, 6, 7)m_\pi \) in ascending order. In essence Fig.8 reproduces the enhancement of the total cross section by the next-to-leading order chiral corrections. No further specific dynamical details to which the \( \pi^0\pi^0 \) mass spectrum could be selectively sensitive are visible.

The alternative combination is to couple one of neutral pions with the outgoing \( \pi^- \). The \( \pi^0\pi^- \) invariant mass is \( \sqrt{s_1} \) (or equivalently \( \sqrt{s_2} \)). Making use of the relation \( s_1 = s + m^2_\pi - 2\sqrt{s}\omega_2 \) the \( \pi^0\pi^- \) mass spectrum \( d\sigma/d\sqrt{s_1} \) is obtained by omitting the \( d\omega_2 \) integration in eq.(30) and applying an additional factor \( \sqrt{s_1/s} \). Fig.9 shows the calculated \( \pi^0\pi^- \) mass spectrum \( m_\pi d\sigma/d\sqrt{s_1} \). Again, it is only the enhancement of the total cross section which can be inferred from the comparison of the full and dashed curves in Fig.9. The shape of the \( \pi^0\pi^- \) mass spectrum (i.e. its dependence on \( \sqrt{s_1} \)) does not distinguish the tree-approximation from the full calculation in a noticeable way.
3 Charged pion-pair production

In this section we perform the same calculation and analysis for the charged pion-pair production process: \( \pi^- (p_1) + \gamma (k, \epsilon) \to \pi^+ (p_2) + \pi^- (q_1) + \pi^- (q_2) \). By assigning the four-momentum \( p_2 \) to the out-going positively charged pion \( \pi^+ (p_2) \) we can exploit the complete equivalence to the \( \pi^- \gamma \to \pi^- \pi^0 \pi^0 \) reaction concerning its kinematical description. In Coulomb-gauge \((\epsilon \cdot p_1 = \epsilon \cdot k = 0)\), eq.(3) constitutes the general form of the T-matrix for \( \pi^- \gamma \to \pi^+ \pi^- \pi^- \) and the corresponding Mandelstam variables are defined as in eq.(4). The interchange of the two identical \( \pi^- \) in the final state is now described by \((s_1 \leftrightarrow s_2, t_1 \leftrightarrow t_2)\). The three non-vanishing tree diagrams for \( \pi^- \gamma \to \pi^+ \pi^- \pi^- \) are shown in Fig. 1 and their evaluation leads to the following tree amplitudes:

\[
A_1^{(\text{tree})} = \frac{s + m_{\pi}^2 - s_1 - s_2}{3m_{\pi}^2 - s - t_1 - t_2} + \frac{s - s_1 - s_2 + t_2}{t_1 - m_{\pi}^2} - 1, \tag{36}
\]

\[
A_2^{(\text{tree})} = \frac{s + m_{\pi}^2 - s_1 - s_2}{3m_{\pi}^2 - s - t_1 - t_2} + \frac{s - s_1 - s_2 + t_1}{t_2 - m_{\pi}^2} - 1. \tag{37}
\]

One can see from the denominators of \( A_1^{(\text{tree})} \) and \( A_2^{(\text{tree})} \) that two diagrams contribute to each amplitude.

3.1 Amplitudes from chiral loops and counterterms

Beyond leading order the dynamical content of charged pion-pair production \( \pi^- \gamma \to \pi^+ \pi^- \pi^- \) is considerably more extensive than that of neutral pion-pair production \( \pi^- \gamma \to \pi^- \pi^0 \pi^0 \) because the photon can now couple to all three out-going (charged) pions. Many more diagrams with chiral pion-loops and counterterms do contribute. We have evaluated them individually.
Figure 10: One-pion loop diagrams for $\pi^-\gamma \to \pi^+\pi^-\pi^-$ with three possible couplings of the external photon.

Figure 11: One-pion loop diagrams for $\pi^-\gamma \to \pi^+\pi^-\pi^-$ with three possible couplings of the external photon.

and checked the exact cancellation of ultraviolet divergences $\xi$ in the total sums for the amplitudes $A_1$ and $A_2$. Without loss of information we can restrict the presentation of the analytical results to the finite parts of the pion-loop diagrams and to the complete counterterm contribution (reexpressed in terms of the low-energy constants $\bar{\ell}_j$ which subsume the chiral logarithm $\ln(m_\pi/\lambda)$ generated by the pion-loops).

The three possible couplings of the external photon for loop diagram (I) (see Fig. 2) are distinguished by labelling them (Ia), (Ib), (Ic) and are shown in Fig. 10. Omitting the terms proportional to $\xi + \ln(m_\pi/\lambda)$ the corresponding finite parts read:

\[
A_1^{(Ia)} = A_2^{(Ia)} = \frac{1}{(4\pi f_\pi)^2} \left( \frac{s + m_\pi^2 - s_1 - s_2}{3m_\pi^2 - s - t_1 - t_2} \right) (2s - 2m_\pi^2 - s_1 - s_2 + t_1 + t_2) \\
\times J(3m_\pi^2 + s - s_1 - s_2) - \frac{1}{2},
\]

\[
A_1^{(Ib)} = \frac{1}{(4\pi f_\pi)^2} \left( \frac{s - s_1 - s_2 + t_2}{t_1 - m_\pi^2} \right) (s - m_\pi^2 - s_1 - s_2 + t_1 + t_2) \\
\times J(m_\pi^2 + s - s_1 - s_2 + t_1 + t_2) - \frac{1}{2},
\]

\[
A_2^{(Ic)} = A_1^{(Ib)} \big|_{s_1 \leftrightarrow s_2, t_1 \leftrightarrow t_2}.
\]

Note that we do not list vanishing contributions to $A_1$ or $A_2$ from a diagram under consideration. Fig. 11 shows the one-pion loop diagrams (IIa), (IIb), (IIc) obtained from the
rescattering diagram (II) (see Fig. 2) by attaching the external photon in the three possible ways. One finds the following finite parts:

\[ A^{(IIa)}_1 = A^{(IIa)}_2 = \frac{1}{3(4\pi f_\pi)^2} \frac{1}{3m_\pi^2 - s - t_1 - t_2} \left\{ \frac{m_\pi^2}{3} (21s + 7s_1 - 20s_2 + 28t_1 + t_2) \\
+ \frac{1}{6} (s - s_2 + t_1)(5s_1 + 16s_2 - 23s - 18t_1 - 7t_2) - \frac{61m_\pi^4}{6} \\
+ \left[ m_\pi^2 (19m_\pi^2 - 17s - 2s_1 + 15s_2 - 19t_1 - 2t_2) + (s - s_2 + t_1) \right] \times (7s - s_1 - 5s_2 + 6t_1 + 2t_2) \right\}, \]

\[ (41) \]

\[ A^{(IIb)}_1 = \frac{1}{3(4\pi f_\pi)^2} \frac{1}{t_1 - m_\pi^2} \left\{ \frac{m_\pi^4}{2} + \frac{2m_\pi^2}{3} (6s_1 + 8s_2 - 6s - 3t_1 - 6t_2) \\
+ \frac{s_2}{6} (5s - 5s_1 - 16s_2 - 2t_1 + 5t_2) + \left[ s_2 (5s_2 + s_1 + t_1 - s - t_2) \right. \right. \\
+ \left. \left. m_\pi^2 (4s + m_\pi^2 - 4s_1 - 9s_2 + 2t_1 + 4t_2) \right] J(s_2) \right\}, \]

\[ (42) \]

\[ A^{(IIc)}_2 = \frac{1}{3(4\pi f_\pi)^2} \frac{1}{t_2 - m_\pi^2} \left\{ \frac{m_\pi^2}{3} (11s + 7s_1 - 11s_2 + 11t_1 - 8t_2) \\
- \frac{25m_\pi^4}{6} + \frac{1}{6} (s - s_2 + t_1)(5s_1 + 16s_2 + 2t_2 - 16s - 16t_1) \\
+ \left[ m_\pi^2 (9s_2 + 4t_2 - 9s - 2s_1 - 9t_1 + 7m_\pi^2) + (s - s_2 + t_1) \right] \times (5s - s_1 - 5s_2 + 5t_1 - 2t_2) \right\}. \]

\[ (43) \]

The additional contributions from the loop diagrams (IIIa), (IIIb), (IIIc) with crossed out-going \( \pi^- \) lines (see Fig. 2) follow immediately via the substitution \( q_1 \leftrightarrow q_2 \) as:

\[ A^{(IIIa)}_1 = A^{(IIIa)}_2 = A^{(IIa)}_1 \bigg| (s_1 \leftrightarrow s_2, t_1 \leftrightarrow t_2), \]

\[ (44) \]

\[ A^{(IIIb)}_1 = A^{(IIc)}_2 \bigg| (s_1 \leftrightarrow s_2, t_1 \leftrightarrow t_2), \quad A^{(IIIc)}_2 = A^{(IIIb)}_1 \bigg| (s_1 \leftrightarrow s_2, t_1 \leftrightarrow t_2). \]

\[ (45) \]

Next, we come to the irreducible one-pion loop diagrams (VI), (V), (VI) with internal photon coupling shown in Fig. 3 and interpreted now as diagrams for \( \pi^- \gamma \rightarrow \pi^+ \pi^- \pi^- \). One finds the following finite parts:

\[ A^{(IV)}_1 = A^{(IV)}_2 = \frac{2(s - s_1 - s_2) + t_1 + t_2 \left\{ \frac{1}{2} + \frac{1}{2m_\pi^2 - t_1 - t_2} \right\} \times \left[ (s - m_\pi^2 - s_1 - s_2 + t_1 + t_2) J(m_\pi^2 + s - s_1 - s_2 + t_1 + t_2) \right. \right. \\
\left. \left. + (s_1 + s_2 - s - m_\pi^2) J(3m_\pi^2 + s - s_1 - s_2) \right\} \right\}, \]

\[ (46) \]

\[ A^{(V)}_1 = \frac{1}{3(4\pi f_\pi)^2} \left\{ 10m_\pi^2 - 3s - 3t_2 + \frac{13s_1}{6} + 6m_\pi^2 \left( 1 - \frac{s_1}{s + t_2 - 2m_\pi^2} \right) \times \left[ G(2m_\pi^2 - s + s_1 - t_2) - G(s_1) \right] + (3s - 2s_1 + 3t_2 - 10m_\pi^2) J(s_1) \right. \right. \\
\left. \left. + \frac{3(s - s_1 + t_2 - 2m_\pi^2)}{s + t_2 - 2m_\pi^2} \left[ J(2m_\pi^2 - s + s_1 - t_2) - J(s_1) \right] \right\}, \]

\[ (47) \]
\[ A_2^{(V)} = \frac{1}{3(4\pi f_\pi f)^2} \left\{ \frac{1}{6} (19m_\pi^2 + 13s_1 + 9t_1 - 2t_2 - 2s) + \frac{3s_1(2m_\pi^2 - t_1 - t_2)}{2(s + t_2 - 2m_\pi^2)} \right. \\
+ 6m_\pi^2 \left[ \frac{m_\pi^2 - t_1}{s + t_2 - 2m_\pi^2} + \frac{s_1(t_1 - s)}{(s + t_2 - 2m_\pi^2)^2} \right] G(2m_\pi^2 - s + s_1 - t_2) - G(s_1) \\
+ \left[ s_1 + 2m_\pi^2 + \frac{3s_1^2(s - t_1)}{(s + t_2 - 2m_\pi^2)^2} + \frac{3s_1(t_1 - s)}{s + t_2 - 2m_\pi^2} \right] J(2m_\pi^2 - s + s_1 - t_2) \\
- J(s_1) + (s - 2s_1 + t_2 - 6m_\pi^2) J(2m_\pi^2 - s + s_1 - t_2) \} , \\
(48) \\

A_1^{(VI)} = A_2^{(V)} \big| (s_1 \leftrightarrow s_2 , t_1 \leftrightarrow t_2) , \quad A_2^{(VI)} = A_1^{(V)} \big| (s_1 \leftrightarrow s_2 , t_1 \leftrightarrow t_2) , \\
(49) \\

where the contributions from diagram (VI) are obtained from those of diagram (V) by applying the crossing transformation \( q_1 \leftrightarrow q_2 \). The set of next-to-leading order corrections to \( \pi^- \gamma \rightarrow \pi^+ \pi^+ \pi^- \) is completed by the total counterterms contribution which reads:

\[ A_1^{(ct)} = \frac{1}{(4\pi f_\pi f)^2} \left\{ \frac{\ell_1}{3} \left[ s + 3m_\pi^2 - 2s_2 + t_1 - t_2 - (s - s_1 + t_2)^2 + (s_2 - 2m_\pi^2)^2 \right] \right. \\
- \frac{(s - s_1 + t_2)^2 + (s_2 + t_2 - 3m_\pi^2)^2}{3m_\pi^2 - s - t_1 - t_2} \left. + \frac{\ell_2}{3} \left[ 5s + 5m_\pi^2 - 4s_1 - 6s_2 + 3t_1 - t_2 \right] \\
+ \frac{4s_2(s + m_\pi^2 - s_1 + t_2) - 3s_1^2 - 4m_\pi^2 - 3(s - s_1 + t_2)^2}{t_1 - m_\pi^2} + \frac{1}{3m_\pi^2 - s - t_1 - t_2} \right. \\
\times \left[ 2(s - s_1)(2s_2 + t_2 - 5m_\pi^2) - 3(s - s_1 + t_2 - m_\pi^2)^2 + t_2 - 8m_\pi^4 \right] \\
+ \frac{s_2(10m_\pi^2 - 3s_2 - 2t_2)}{m_\pi^4} \left. + \frac{m_\pi^4}{(t_1 - m_\pi^2) \left[ 3m_\pi^2 - s - t_1 - t_2 \right]} \right. \\
\left. + \frac{m_\pi^4}{3m_\pi^2 - s - t_1 - t_2} \right] \left. \right\} , \\
(50) \\

A_2^{(ct)} = A_1^{(ct)} \big| (s_1 \leftrightarrow s_2 , t_1 \leftrightarrow t_2) . \\
(51) \\

The relation \( A_2 = A_1 \big| (s_1 \leftrightarrow s_2 , t_1 \leftrightarrow t_2) \) holds also for the total sum of the loop amplitudes and it applies to both reactions \( \pi^- \gamma \rightarrow \pi^+ \pi^- \pi^- \) and \( \pi^- \gamma \rightarrow \pi^- \pi^0 \pi^0 \) in the same way.

### 3.2 Results for \( \pi^- \gamma \rightarrow \pi^+ \pi^- \pi^- \)

We are now in the position to present numerical results for the charged pion-pair production process \( \pi^- \gamma \rightarrow \pi^+ \pi^- \pi^- \) at next-to-leading order in chiral perturbation theory. The formula for calculating the total cross section \( \sigma_{tot}(s) \) is given in unchanged form by eq.(30). We use consistently the same values: \( \ell_1 = -0.4 \pm 0.6 , \ell_2 = 4.3 \pm 0.1 , \ell_3 = 2.9 \pm 2.4 , \ell_4 = 4.4 \pm 0.2 \), for the low-energy constants \( \ell_j \) as in subsection 2.2.

Fig. 12 shows the total cross section for \( \sigma_{tot}(s) \) for the reaction \( \pi^- \gamma \rightarrow \pi^+ \pi^- \pi^- \) in the low-energy region from threshold \( \sqrt{s} = 3m_\pi \) up to \( \sqrt{s} = 7m_\pi \). The dashed line corresponds to the tree approximation and the full line includes in addition the next-to-leading order corrections from chiral loops and counterterms. By inspection of Fig. 12 one observes that the total cross section for \( \pi^- \gamma \rightarrow \pi^+ \pi^- \pi^- \) remains almost unchanged in the region \( \sqrt{s} < 6m_\pi \) after inclusion of the next-to-leading order chiral corrections. This striking result is in marked contrast to the behavior of the total cross section for neutral pion-pair production \( \pi^- \gamma \rightarrow \pi^- \pi^0 \pi^0 \) (see Fig. 7). Although the dynamics of the whole process is much richer this feature can be understood (in
Figure 12: Total cross section for the reaction $\pi^-\gamma \rightarrow \pi^+\pi^-\pi^-$ as a function of the center-of-mass energy $\sqrt{s}$.

A suggestive way) from the $\pi^-\pi^- \rightarrow \pi^-\pi^-$ final state interaction. By considering the one-loop expression for the isospin-two S-wave $\pi\pi$ scattering length [24]:

$$a_0^2 = -\frac{m_\pi}{16\pi^2 f_\pi^2} \left[ 1 - \frac{m_\pi^2}{12\pi^2 f_\pi^2} \left( \ell_1 + 2\ell_2 - \frac{3\ell_3}{8} - \frac{3\ell_4}{2} + \frac{3}{8} \right) \right], \quad (52)$$

one sees that the correction to 1 inside the square bracket amounts to the very small number $-0.017$ (inserting the central values of $\bar{\ell}_j$). Chiral corrections (even at two-loop order [1]) affect the isospin-two $\pi\pi$-interaction only very weakly and this feature seems to be reflected by $\sigma_{tot}(s)$ in Fig.12. Note however, that the argument made here is only suggestive and not rigorous, because the on-shell $\pi^-\pi^- \rightarrow \pi^-\pi^-$ final state interaction does not factor out of the production amplitudes $A_1$ and $A_2$ in an obvious way. The same caveat applies to the $\pi^+\pi^- \rightarrow \pi^0\pi^0$ final state interaction which has been used as an argument for the observed enhancement in subsection 2.2.

Let us also comment on the theoretical uncertainties which are induced by the present errorbars $\delta\bar{\ell}_j$ of the low-energy constants $\bar{\ell}_j$. Taking again the total cross section at $\sqrt{s} = 6m_\pi$ as a measure one finds relative uncertainties of: $\pm 4.8\%$ from $\delta\ell_1$, $\pm 1.6\%$ from $\delta\ell_2$, $\pm 0.3\%$ from $\delta\ell_3$, and $\pm 1.0\%$ from $\delta\ell_4$. As for the reaction $\pi^-\gamma \rightarrow \pi^-\pi^0\pi^0$ the largest uncertainty goes along with $\ell_1$ and adding them in quadrature one gets a total relative uncertainty of $\pm 5.2\%$. This amounts again to a fairly accurate prediction.

The more exclusive observables than the total cross section are the two-pion mass spectra. Fig.13 shows the calculated $\pi^-\pi^-$ mass spectrum $m_\pi \frac{d\sigma}{d\mu}$ as a function of the $\pi^-\pi^-$ invariant mass $\mu$. The four pairs of (full and dashed) curves correspond to sections at center-of-mass-energies $\sqrt{s} = (4,5,6,7)m_\pi$ in ascending order. In essence Fig.13 reproduces the features of the total cross section, namely a slight enhancement above $\sqrt{s} = 6m_\pi$ by the next-to-leading order chiral corrections. The $\pi^+\pi^-$ mass spectrum $m_\pi \frac{d\sigma}{d\sqrt{s_1}}$ shown in Fig.14 indicates some more interesting structures. The dip of the $\pi^+\pi^-$ mass spectrum at intermediate $\pi^+\pi^-$ invariant masses $\sqrt{s_1}$ becomes much more pronounced when including the next-to-leading order
Figure 13: $\pi^-\pi^-$ mass spectrum for the reaction $\pi^-\gamma \to \pi^+\pi^-\pi^-$ as a function of the $\pi^-\pi^-$ invariant mass $\mu$. The curves correspond to center-of-mass energies $\sqrt{s} = (4, 5, 6, 7) m_\pi$ in ascending order.

Chiral corrections. This distinctive feature holds e.g. at the center-of-mass energy $\sqrt{s} = 5 m_\pi$ where the total cross sections in tree and one-loop approximation are equal. The $\pi^+\pi^-$ mass spectrum of the reaction $\pi^-\gamma \to \pi^+\pi^-\pi^-$ therefore seems to provide an interesting indicator for the role of chiral (pion-loop) dynamics beyond leading order. It is expected that the upcoming high-statistics data of the COMPASS experiment at CERN can reveal such dynamical details. Of course, the squared (transversal) T-matrix $|\hat{k} \times (\vec{q}_1 A_1 + \vec{q}_2 A_2)|^2$ with its full dependence on pion energies and angles incorporates still much more dynamical information.

In passing we note that $\sqrt{s} = (6 - 7) m_\pi$ is presumably the maximal center-of-mass energy up to which a one-loop calculation of the processes $\pi^-\gamma \to 3\pi$ in chiral perturbation theory can be trusted. At still higher energies the contributions from meson resonances (such as $a_1(1260)$, $a_2(1320)$ etc.) will start to play a prominent role. In the context of such considerations it should be kept in mind that the effects of the resonance tails at low-energies are encoded in the empirical values of low-energy constants $\tilde{\ell}_j$. The role of the low-lying $\rho(770)$ resonance occurring in the isospin-one 2$\pi$-subsystem needs to be investigated by studying an appropriate resonance model for $\pi^-\gamma \to 3\pi$. Respecting fully gauge-invariance in the construction of such a resonance model (with inclusion of finite resonance widths) represents some challenge.

Acknowledgments
I thank Jan Friedrich for many informative discussions.

References
[1] G. Colangelo, J. Gasser, and H. Leutwyler, Nucl. Phys. B603, 125 (2001).
[2] S. Pislak et al., Phys. Rev. D67, 072004 (2003).
[3] J.R. Batley et al., Eur. Phys. J. C54, 411 (2008).
[4] G. Colangelo, J. Gasser, and H. Leutwyler, Phys. Rev. Lett. 86, 5008 (2001).
Figure 14: $\pi^+\pi^-$ mass spectrum for the reaction $\pi^-\gamma \rightarrow \pi^+\pi^-\pi^-$ as a function of the $\pi^+\pi^-$ invariant mass $\sqrt{s_1}$. The curves correspond to center-of-mass energies $\sqrt{s} = (4, 5, 6, 7)m_\pi$ in ascending order.

[5] B. Adeva et al., J. Phys. G30, 1929 (2004).
[6] J.R. Batley et al., Phys. Lett. B633, 173 (2006).
[7] J. Gasser, hep-ph/0710.3048; Proceedings: PoSKAON, 033 (2008).
[8] J. Gasser, M.A. Ivanov, and M.E. Sainio, Nucl. Phys. B745, 84 (2006); and refs. therein.
[9] Y.M. Antipov et al., Phys. Lett. B121, 445 (1983); Z. Phys. C26, 495 (1985).
[10] J. Ahrens et al., Eur. Phys. J. A23, 113 (2005).
[11] M.V. Terentev, Sov. J. Nucl. Phys. 16, 87 (1973).
[12] E. Frlez et al., Phys. Rev. Lett. 93, 181804 (2004); E. Frlez, Nucl. Phys. Proc. Suppl. 162, 148 (2006).
[13] U. Bürgi, Phys. Lett. B377, 147 (1996); Nucl. Phys. B479, 392 (1996).
[14] J. Bijnens and P. Talavera, Nucl. Phys. B489, 387 (1997); C.Q. Geng, I.L. Ho, and T.H. Wu, Nucl. Phys. B684, 2815 (2004).
[15] COMPASS: P. Abbon et al., Nucl. Instrum. Meth. A577, 455 (2007); hep-ex/0703049.
[16] J. Bijnens, A. Bramon, and F. Cornet, Phys. Lett. B237, 488 (1990); J. Bijnens, Int. J. Mod. Phys. A8, 3045 (1993).
[17] N. Kaiser and J.M. Friedrich, Eur. Phys. J. A36, 181 (2008).
[18] T. Hannah, Nucl. Phys. B593, 577 (2001).
[19] L. Ametller, M. Knecht, and P. Talavera, Phys. Rev. D64, 094009 (2001).
[20] B. Grube, hep-ex/1020.1272.
[21] COMPASS collaboration: M.G. Alekseev et al., Phys. Rev. Lett. 104, 241803 (2010).
[22] Dimitri Ryabchikov and Jan Friedrich, TU München E18, private communications.
[23] V. Bernard, N. Kaiser and Ulf-G. Meißner, Nucl. Phys. A457, 147 (1995).
[24] J. Gasser and H. Leutwyler, Ann. Phys. (NY) 158, 142 (1984).