Arithmetics in number systems with a negative base

Z. Masáková, E. Pelantová, T. Vávra
Department of Mathematics FNSPE, Czech Technical University in Prague
Trojanova 13, 120 00 Praha 2, Czech Republic

Abstract

We study the numeration system with a negative base, introduced by Ito and Sadahiro. We focus on arithmetic operations in the set \( \text{Fin}(\beta) \) and \( \mathbb{Z}_{-\beta} \) of numbers having finite resp. integer \((-\beta)\)-expansions. We show that \( \text{Fin}(\beta) \) is trivial if \( \beta \) is smaller than the golden ratio \( \frac{1}{2}(1 + \sqrt{5}) \). For \( \beta \geq \frac{1}{2}(1 + \sqrt{5}) \) we prove that \( \text{Fin}(\beta) \) is a ring, only if \( \beta \) is a Pisot or Salem number with no negative conjugates. We prove the conjecture of Ito and Sadahiro that \( \text{Fin}(\beta) \) is a ring if \( \beta \) is a quadratic Pisot number with positive conjugate. For quadratic Pisot units we determine the number of fractional digits that may appear when adding or multiplying two \((-\beta)\)-integers.

1 Introduction

There exist many ways to represent real numbers. Besides commonly used decimal and binary number system, there is for example the well known expansion of numbers in the form of a continued fraction. The choice of number representation depends on the purpose for which it will be used. The conventional representation of real numbers in base \( b \in \mathbb{N} \), \( b \geq 2 \), has been generalized in 1957 by A. Rényi [16] when he considered writing non-negative real \( x \) in the form

\[
x = \sum_{i=-\infty}^{k} x_i \beta^i, \quad x_i \in \{0, 1, \ldots, [\beta] - 1\},
\]

where for the base \( \beta \) one chooses arbitrary real \( \beta > 1 \). The choice of an irrational base brings into play new phenomena which are not found in the numeration systems with integer base. For example, the set \( \text{Fin}(\beta) \) of such real \( x \) that have only a finite number of non-zero digits \( x_i \) in the expression (1) need not to be closed under addition. An important motivation for the study of Rényi \( \beta \)-expansions was given by mathematical models of non-crystalline solids with long-range order, the so-called quasicrystals. It turned out that a suitable discrete set for labelling coordinates of atoms in quasicrystals is formed by the \( \beta \)-integers. These are real numbers \( x \) which can be written as \( \sum_{i=-\infty}^{k} x_i \beta^i \), where the choice of base \( \beta \) is related to the rotational symmetry displayed by the material. Many papers since then were devoted to the study of properties of the set \( \text{Fin}(\beta) \) and \( \beta \)-integers \( \mathbb{Z}_\beta \), to algorithms for addition and multiplication, their relation to aperiodic tilings of the space, etc.

Recently, Ito and Sadahiro [13] have suggested to study representations of real numbers in base \(-\beta \), where \( \beta > 1 \), i.e. in the form

\[
x = \sum_{i=-\infty}^{k} x_i (-\beta)^i, \quad x_i \in \{0, 1, \ldots, [\beta]\}.
\]

The advantage of this numeration system with negative base stems from the fact that both positive and negative real numbers can be represented with non-negative digits, without the necessity to mark the sign of the number. In their paper, Ito and Sadahiro derive basic properties of the digit strings \( x_k x_{k-1} x_{k-2} \cdots \) corresponding to \((-\beta)\)-expansions of real numbers \( x \).

The aim of this article is to study arithmetical properties of the numeration system with negative base. The paper is organized as follows: In Section 2 we recall the definition of Rényi \( \beta \)-expansions, and several number-theoretical notions which allow us to state known results about sets \( \text{Fin}(\beta) \) and
\( \mathbb{Z}_\beta \). We give the definition of \((-\beta)\)-expansions and cite relevant facts from the paper \cite{13}. In Section 4 we study the behaviour of the set Fin\((-\beta)\) and \(\mathbb{Z}_{-\beta}\) of finite resp. integer \((-\beta)\)-expansions with respect to arithmetic operations. Section 5 is devoted to the properties of Fin\((-\beta)\) when \(\beta\) is a quadratic number. We prove the conjecture of Ito and Sadahiro \cite{13} that Fin\((-\beta)\) is a ring if \(\beta\) is a quadratic Pisot number with a positive conjugate. For quadratic Pisot units we determine the number of fractional digits that may appear when adding or multiplying two \((-\beta)\)-integers.

## 2 Rényi expansions

Let us recall representations of numbers in the numeration system with a positive real base, as introduced by Rényi \cite{16}. Let \(\beta > 1\). For a non-negative real number \(x\), the well known greedy algorithm yields a representation of \(x\) in the form

\[
x = \sum_{i=-\infty}^{k} x_i \beta^i,
\]

where the digits \(x_i\) are integers \(\{0, 1, \ldots, \lfloor \beta \rfloor - 1\}\). Such representation is called the \(\beta\)-expansion of \(x\). If \(\beta \notin \mathbb{N}\), then not all digit strings can arise as \(\beta\)-expansion of some real number \(x\). In order to describe the digit strings that are admissible as \(\beta\)-expansions, one defines the so-called Rényi expansion of 1, denoted by \(d_\beta(1)\). It is a sequence \(d_\beta(1) = t_1 t_2 t_3 \cdots\) of digits in \(\{0, 1, \ldots, \lfloor \beta \rfloor - 1\}\), such that \(t_1 = \lfloor \beta \rfloor\) and \(\sum_{i=1}^\infty t_i \beta^i\) is a \(\beta\)-expansion of \(\beta - \lfloor \beta \rfloor\). A real number \(\beta > 1\) is called Parry, if the Rényi expansion of 1 is an eventually periodic sequence, \(d_\beta(1) = t_1 \cdots t_m (t_{m+1} \cdots t_{m+p})^\omega\), \(t_m \neq t_{m+p}\). Note that \(d_\beta(1)\) is never purely periodic. We distinguish simple Parry numbers as those with period formed only by 0’s. We define

\[
d_\beta^\omega(1) = \begin{cases} (t_1 \cdots t_{m-1}(t_m-1))^\omega & \text{if } d_\beta(1) = t_1 \cdots t_m 0^\omega, \ t_m \neq 0, \\ d_\beta(1) & \text{otherwise.} \end{cases}
\]

Parry \cite{15} has shown that a digit string \(x_k x_{k-1} \cdots\) is admissible if and only if each of its suffixes satisfies

\[
x_i x_{i-1} \cdots < d_\beta^\omega(1),
\]

where \(<\) is the lexicographic order on strings.

The lexicographic order on the digit strings obtained by the greedy algorithm corresponds to the natural order on the real line. More precisely, if \(x = \sum_{i=-\infty}^{k} x_i \beta^i\), \(y = \sum_{i=-\infty}^{k} y_i \beta^i\) are the \(\beta\)-expansions of \(x\) and \(y\), respectively, then

\[
x < y \iff x_k x_{k-1} x_{k-2} \cdots < y_k y_{k-1} y_{k-2} \cdots.
\]

Several classes of numbers display exceptional properties when taken as bases \(\beta\) of Rényi numeration systems. Except Parry numbers, these are namely Pisot numbers and Salem numbers. A real number \(\beta > 1\) is called Pisot, if \(\beta\) is an algebraic integer with all conjugates in the interior of the unit disc. \(\beta\) is a Salem number, if all conjugates lie inside the unit disc and at least one on the unit circle. It can be shown that all Pisot numbers are Parry, the same statement for Salem numbers is an unproved conjecture.

When studying arithmetical properties of the \(\beta\)-numeration system, one is interested in the set of finite \(\beta\)-expansions, denoted by Fin\((\beta)\). Question arises, for which bases \(\beta\) this set is closed under addition, subtraction and multiplication, i.e. has a ring structure. Such bases are said to have the finiteness property. Frougny and Solomyak \cite{10} have shown that a necessary condition for \(\beta\) to have the finiteness property is to be a Pisot number. The converse is not true. Akiyama \cite{1} has described all cubic Pisot units for which Fin\((\beta)\) is a ring, other results about this problem are found in \cite{10} \cite{12}, and its connection to tilings is explained in \cite{2}. Another question to ask is about the time and space complexity of the arithmetical operations over finite \(\beta\)-expansions. One measure for this are the values \(L_\beta(\beta)\), \(L_\beta(\beta)\) denoting the maximal length of the \(\beta\)-fractional part arising in addition, resp. multiplication of numbers. More precisely, denoting by \(\mathbb{Z}_\beta\) the set of numbers whose \(\beta\)-expansion has non-zero digits only for non-negative indices,

\[
\mathbb{Z}_\beta = \left\{ \pm x \mid x = \sum_{i=0}^{k} x_i \beta^i \text{ is the } \beta\text{-expansion of } x \right\},
\]
Sadahiro [13]. Let \( \beta > 1 \). In this paper we study arithmetical properties of the \((-\beta)\)-numeration system, introduced by Ito and Sadahiro [13]. Let \( \beta > 1 \). Any real number \( x \) can be expressed in the form
\[
x = \sum_{i=-\infty}^{k} x_i (-\beta)^i, \quad x_i \in \mathbb{Z}.
\]
Symbolically, the number \( x \) can be written as a sequence of digits \( x_i \), where the delimiter \( \bullet \) separates between coefficients at non-negative and negative powers of \((-\beta)\), i.e.
\[
x_kx_{k-1} \cdots x_1x_0\bullet x_{-1}x_{-2} \cdots \text{ if } k \geq 0, \quad \text{or} \quad 0\bullet 0^{-k-1}x_kx_{k-1} \cdots \text{ if } k < 0.
\]
Ito and Sadahiro give a prescription to obtain a \((-\beta)\)-representation of numbers \( x \in I_\beta = [-\beta, \frac{1}{\beta+1}) \) using the transformation \( T_{-\beta} : I_\beta \mapsto I_\beta \),
\[
T_{-\beta}(x) = -\beta x - \left\lfloor -\beta x + \frac{\beta}{\beta + 1} \right\rfloor,
\]
and set
\[
x_i = \left\lfloor -\beta T_{-\beta}^{i-1}(x) + \frac{\beta}{\beta + 1} \right\rfloor \quad \text{and} \quad d_{-\beta}(x) = x_1x_2x_3 \cdots.
\]
As shown in [13], a very important role is played by the digit string
\[
d_{-\beta}(\ell_\beta) = d_1d_2d_3 \cdots, \quad \text{where} \quad \ell_\beta = \frac{-\beta}{\beta + 1}.
\]
Obviously, \( 0\bullet x_1x_2x_3 \cdots \) is a \((-\beta)\)-representation of \( x \) for any \( x \in I_\beta \). The transformation \( T_{-\beta} \) can be used to find a \((-\beta)\)-representation of any real number \( x \), by using \( d_{-\beta}(x(-\beta)^{-j}) \) for a suitable \( j \in \mathbb{N} \) and by suitable placement of the delimiter \( \bullet \). However, the \((-\beta)\)-representation obtained by such a procedure depends on the choice of \( j \), since for the left end-point \( \ell_\beta = \frac{-\beta}{\beta + 1} \) of the interval \( I_\beta \),
\[
d_{-\beta}(\ell_\beta) = d_1d_2d_3 \cdots \quad \text{and} \quad d_{-\beta}((-\beta)^{-2}\ell_\beta) = 1d_1d_2d_3 \cdots.
\]
In order to define a unique \((-\beta)\)-expansion for every real number \( x \), we choose the representation which satisfies the natural property of usual number representations that multiplication by the base results in shifting the digit sequence. More formally, we require that for all \( x \in \mathbb{R} \) and for all \( j \in \mathbb{Z} \),
\[
x = \sum_{i=-\infty}^{k} x_i(-\beta)^i \quad \text{is the \((-\beta)\)-expansion of} \quad x
\]
\[
\mathcal{D}
\]
\[
(-\beta)^{j}x = \sum_{i=-\infty}^{k+j} x_{i-j}(-\beta)^i \quad \text{is the \((-\beta)\)-expansion of} \quad (-\beta)^{j}x.
\]
Among all \((-\beta)\)-representations of \( x \), we define the \((-\beta)\)-expansion of \( x \) as the one which is obtained by the following algorithm.
Require: $x \in \mathbb{R}$
Find the minimal non-negative integer $j$ such that $y = x(\beta)^{-j} \in \left(\frac{-\beta}{\beta+1}, \frac{1}{\beta+1}\right)$. 
Find $d_{-\beta}(y) = y_1y_2y_3 \cdots$
if $j = 0$ then
\[ \text{put } (x)_{-\beta} = 0 \bullet y_1y_2y_3 \cdots. \]
else
\[ \text{put } (x)_{-\beta} = y_1 \cdots y_j \bullet y_{j+1}y_{j+2} \cdots. \]
end if

Applying the algorithm to the left end-point $\ell_{\beta}$ of the interval $I_{\beta}$, we obtain
\[ (\ell_{\beta})_{-\beta} = 1d_1d_2d_3 \cdots, \]
and $0 \bullet d_1d_2d_3 \cdots$ is a different $(-\beta)$-representation of $\ell_{\beta}$. Nevertheless, the sequence $d_{-\beta}(\ell_{\beta}) = d_1d_2d_3 \cdots$ is of great importance in describing the admissible digit strings. Before stating Theorem 2, taken from [13], recall the alternate order on sequences of digits. We define
\[ v_1v_2v_3 \cdots \prec_{alt} w_1w_2w_3 \cdots \text{ if } \left\{ \begin{array}{ll} v_j < w_j & \text{when } j \text{ is even}, \\ v_j > w_j & \text{when } j \text{ is odd}, \end{array} \right. \text{ where } j := \min\{i \mid v_i \neq w_i\}. \]

Recall that such ordering is well known from continued fractions. The following theorem is a slight modification of the result of [13], taking into account the requirement (3).

**Theorem 1.** Let $\beta > 1$. Let $d_{-\beta}(\ell_{\beta}) = d_1d_2d_3 \cdots$ Define
\[ d_{-\beta}(r_{\beta}) = \left\{ \begin{array}{ll} (0d_1 \cdots d_{m-1}(d_m - 1))^\omega & \text{if } d_{-\beta}(\ell_{\beta}) = (d_1 \cdots d_m)^\omega, \text{ and } m \text{ is odd}, \\ 0d_1d_2d_3 \cdots & \text{otherwise}. \end{array} \right. \]
Then the digit sequence $x_kx_{k-1}x_{k-2} \cdots$ is admissible as the $(-\beta)$-expansion of a real number $x$ if and only if each of the suffixes $u$ of the sequence $0x_kx_{k-1}x_{k-2}$ satisfies
\[ d_{-\beta}(\ell_{\beta}) \preceq_{alt} u \prec_{alt} d_{-\beta}(r_{\beta}). \]

In this paper we focus on the set $\text{Fin}(\beta)$ of real numbers with finite number of non-zero digits in their $(-\beta)$-expansion. An important subset of $\text{Fin}(\beta)$ is given by the set of so-called $(-\beta)$-integers, denoted by $\mathbb{Z}_{-\beta}$,
\[ \mathbb{Z}_{-\beta} = \{ x \in \mathbb{R} \mid (x)_{-\beta} = x_kx_{k-1} \cdots x_1x_0 \bullet \}. \]
In this notation,
\[ \text{Fin}(\beta) = \bigcup_{n \in \mathbb{N}} (\beta)^{-n} \mathbb{Z}_{-\beta}. \]

In analogy with the Rényi expansions, we study the maximal length of fractional part arising in arithmetical operations, i.e. we investigate
\[ L_{\bullet}(\beta) = \min\{ l \in \mathbb{N} \mid \forall x, y \in \mathbb{Z}_{-\beta}, x + y \in \text{Fin}(\beta) \Rightarrow x + y \in (-\beta)^{-l}\mathbb{Z}_{-\beta} \}, \]
\[ L_{\circ}(\beta) = \min\{ l \in \mathbb{N} \mid \forall x, y \in \mathbb{Z}_{-\beta}, x \cdot y \in \text{Fin}(\beta) \Rightarrow x \cdot y \in (-\beta)^{-l}\mathbb{Z}_{-\beta} \}. \]

## 4 Arithmetics on $(-\beta)$-expansions

The biggest difference between properties of Rényi $\beta$-expansions and Ito-Sadahiro $(-\beta)$-expansions is observed on the set of real numbers which have in their expansion only finitely many non-zero digits. While $\text{Fin}(\beta)$ is dense in $\mathbb{R}$ for every base $\beta > 1$, the set $\text{Fin}(\beta)$ may sometimes contain only the point 0. (Note that $T_{-\beta}(0) = 0$ and therefore the string $0^\omega$ is always admissible.)

**Theorem 2.** Let $\beta > 1$. Then $\text{Fin}(\beta) = \{0\}$ if and only if $\beta < \frac{1}{2}(1 + \sqrt{5})$.

**Proof.** If $x = -\frac{1}{\beta} \in \left[ -\frac{\beta}{\beta+1}, \frac{1}{\beta+1} \right)$, then $\lfloor -\beta x + \frac{\beta}{\beta+1} \rfloor = 1$ and $T_{-\beta}(x) = 0$. Therefore $d_{-\beta}(x) = 10^\omega$, and thus $-\frac{1}{\beta} \in \text{Fin}(\beta)$. On the other hand, if $a_{k-1} \cdots a_1a_00^\omega$ is admissible, and at least one digit is non-zero, then Theorem 1 implies that also $10^\omega$ is admissible, and thus $-\frac{1}{\beta} \in \left[ -\frac{\beta}{\beta+1}, \frac{1}{\beta+1} \right)$.
By that, we have shown that \( \text{Fin}(\beta) \neq \{0\} \) if and only if \( x = -\frac{1}{\beta} \in [-\frac{\beta}{\beta+1}, \frac{1}{\beta+1}] \), which, in turn, is equivalent to the fact that

\[
-\frac{\beta}{\beta+1} \leq -\frac{1}{\beta}.
\]

This is satisfied if and only if \( \beta \geq \frac{1}{2}(1 + \sqrt{5}) \). \( \square \)

The proof of the previous theorem implies that if \( \beta \geq \tau \), then \( 1 \in \text{Fin}(\beta) \). We have \((1)_{-\beta} = 1\bullet \) for \( \beta > \tau \) and \((1)_{-\beta} = 100\bullet \) for \( \beta = \tau \) (note that \( 1 = -\tau \ell \)). Our aim is to study for which \( \beta \geq \frac{1}{2}(1 + \sqrt{5}) \) the set \( \text{Fin}(\beta) \) is a ring. A necessary condition is that \(-1 \in \text{Fin}(\beta) \), i.e. \(-1 = \sum_{i=k}^{n} a_i(-\beta)^i \), \( k, n \in \mathbb{Z} \), \( k \leq n \). From this we can derive the following statement.

**Lemma 3.** Let \( \beta > 1 \) be such that \(-1 \in \text{Fin}(\beta) \). Then \( \beta \) is an algebraic number without negative conjugates.

Assuming that not only \(-1 \), but all negative integers have finite \((-\beta)\)-expansions, we obtain a stronger necessary condition for \( \text{Fin}(\beta) \) to be a ring. In order to show this, we recall a lemma from [3].

**Lemma 4.** Let \( a_1a_2a_3 \cdots \) be a \((-\beta)\)-admissible digit string with \( a_1 \neq 0 \). For fixed \( k \in \mathbb{Z} \), denote

\[
z = \sum_{i=1}^{\infty} a_i(-\beta)^{k-i}.
\]

Then

\[
z \in \left[ \frac{\beta^{k-1}}{\beta + 1}, \frac{\beta^{k+1}}{\beta + 1} \right], \text{ for } k \text{ odd, } \quad \text{and} \quad z \in \left[ -\frac{\beta^{k+1}}{\beta + 1}, -\frac{\beta^{k-1}}{\beta + 1} \right], \text{ for } k \text{ even.}
\]

**Proposition 5.** Let \( \mathbb{Z}^- \subset \text{Fin}(\beta) \). Then \( \beta \) is a Pisot or a Salem number.

**Proof.** If \( \beta \) is an integer, the statement is obvious. Let \( \beta \notin \mathbb{N} \). Theorem [2] implies that \( \beta \geq \tau \). If \( \beta = \tau \), the proof is finished. Consider \( \beta > \tau \), i.e. \( \beta^2 > \beta + 1 \). First we show that \( \beta \) is an algebraic integer. Consider the negative integer \(-|\beta^{2k+1}| \) for a fixed \( k \in \mathbb{N} \). Using \( \beta^2 > \beta + 1 \) we easily verify that

\[
\frac{\beta^{2k+1}}{\beta + 1} < |\beta^{2k+1}| - 1 < \frac{\beta^{2k+1}}{\beta + 1},
\]

and thus \(-|\beta^{2k+1}| \in \left( \frac{-\beta^{2k+1}}{\beta + 1}, \frac{-\beta^{2k+1}}{\beta + 1} \right) \). As \(-|\beta^{2k+1}| = (\beta)^{2k+1} + \varepsilon_{\beta} \), with \( \varepsilon_{\beta} \in [0, 1) \), we can use Lemma 4 to obtain the \((-\beta)\)-expansion of \(-|\beta^{2k+1}| \) in the form

\[
|\beta^{2k+1}| = (\beta)^{2k+1} + a_0 + \frac{a_1}{(-\beta)} + \cdots + \frac{a_l}{(-\beta)^l} \varepsilon_{\beta},
\]

This means that \( \beta \) is a root of a monic polynomial with integer coefficients, i.e. \( \beta \) is an algebraic integer. In order to show by contradiction that \( \beta \) is a Pisot or Salem number, suppose that \( \beta \) has a conjugate \( \gamma \) such that \( |\gamma| > 1 \). For \( \gamma \), we have

\[
|\beta^{2k+1}| = (\gamma)^{2k+1} + a_0 + \frac{a_1}{(-\gamma)} + \cdots + \frac{a_l}{(-\gamma)^l} \varepsilon_{\gamma}.
\]

Set \( M = |\beta| |\gamma|^{-1} \). Certainly, there exists a \( k \in \mathbb{N} \) such that

\[
|(-\beta)^{2k+1} - (\gamma)^{2k+1}| > 1 + M.
\]

Since \( a_i \in \{0, 1, \ldots, |\beta|\} \), we have

\[
|\varepsilon_{\beta}| \leq 1 \quad \text{and} \quad |\varepsilon_{\gamma}| \leq M \quad \text{and} \quad a_0 + \frac{a_1}{(-\gamma)} + \cdots + a_l |\gamma|^{-l}.
\]

Subtracting (4) from (5) and comparing with (6) we obtain

\[
1 + M < |(-\beta)^{2k+1} - (\gamma)^{2k+1}| = |\varepsilon_{\beta} - \varepsilon_{\gamma}| \leq |\varepsilon_{\beta}| + |\varepsilon_{\gamma}| \leq 1 + M,
\]

which is a contradiction. \( \square \)
In the following section we study \((-\beta)\)-expansions for \(\beta\) a quadratic Pisot number and show that there exist bases \(\beta\) for which \(\text{Fin}(\beta)\) is a ring. Simpler question is, whether the set \(\mathbb{Z}_-\beta\) may be closed under addition for some \(\beta\). It is not difficult to show that if \(\beta \in \mathbb{N}\), then \(\mathbb{Z}_-\beta = \mathbb{Z}\) and the answer about arithmetical operations is obvious.

**Proposition 6.** Let \(\beta \geq \frac{1}{2}(1 + \sqrt{5})\). Then \(\mathbb{Z}_-\beta\) is a ring if and only if \(\beta \in \mathbb{N}\).

**Proof.** Let \(\beta \notin \mathbb{N}\) and assume that \(\mathbb{Z}_-\beta\) is a ring. We have \(|\beta^2| \in \mathbb{Z}_-\beta\). Since

\[
\frac{\beta^2}{\beta + 1} < |\beta^2| < \frac{\beta^4}{\beta + 1},
\]

according to Lemma 4 the \((-\beta)\)-expansion of \(|\beta^2|\) is of the form

\[
|\beta^2| = x_2(-\beta)^2 + x_1(-\beta) + x_0, \quad \text{where } x_0, x_1, x_2 \in \{0, 1, \ldots, |\beta|\}, \ x_2 \geq 1.
\]

(7)

This, in turn, means that \(\beta\) is a quadratic number.

The assumption \(\mathbb{Z}_-\beta\) is a ring implies that also \(\text{Fin}(\beta)\) is a ring and by Proposition 5 \(\beta\) is either a Pisot or a Salem number. Since there are no quadratic Salem numbers, \(\beta\) is a quadratic Pisot number, i.e. \(\beta\) satisfies

\[
x^2 = mx + n, \quad \text{where } m, n \in \mathbb{N}, m \geq n, \ \text{or}
\]

\[
x^2 = mx - n, \quad \text{where } m, n \in \mathbb{N}, m \geq n + 2 \geq 3.
\]

Since the conjugate \(\beta'\) of \(\beta\) satisfying the equation \(x^2 = mx + n\) is negative, by Lemma 8 it suffices to consider the case \(\beta^2 = m\beta - n\). As \(|\beta| = m - 1\), the digits \(x_0, x_1, x_2\) belong to \(\{0, 1, \ldots, m - 1\}\). From (7), we obtain

\[
|\beta^2| = x_2(m\beta - n) - x_1\beta + x_0.
\]

Since numbers \(\beta, 1\) are linearly independent over \(\mathbb{Q}\), we must have \(x_2m - x_1 = 0\), which together with the condition \(x_2 \geq 1\) and \(x_1 \leq m - 1\) gives a contradiction. \(\square\)

Determination of values \(L_\oplus(\beta), L_\ominus(\beta)\) is in general complicated. For some algebraic numbers \(\beta\) with at least one conjugate \(\beta'\) in modulus smaller than 1, we have an easy estimate which uses the fact that the set \(\{|z'| \mid \mathbb{Z}_-\beta\}\) is bounded. Here the notation \(z'\) for a number \(z \in \mathbb{Q}(\beta)\) stands for the image of \(z\) under the field isomorphism \(\mathbb{Q}(\beta) \mapsto \mathbb{Q}(\beta')\).

**Theorem 7.** Let \(\beta\) be an algebraic number, and let \(\beta'\) be one of its conjugates satisfying \(|\beta'| < 1\). Denote

\[
\begin{align*}
H & := \sup \{|z'| \mid z \in \mathbb{Z}_-\beta\}, \\
K & := \inf \{|z'| \mid z \in \mathbb{Z}_-\beta \setminus (-\beta)\mathbb{Z}_-\beta\}. 
\end{align*}
\]

(8)

If \(K > 0\), then

\[
\frac{1}{|\beta'|^2} \leq \frac{2H}{K} \quad \text{and} \quad \frac{1}{|\beta'|^2} \leq \frac{H^2}{K}.
\]

(9)

Moreover, if the supremum or infimum in (8) is not reached, then strict inequality holds in both of (9).

**Proof.** Let \((x \pm y)_-\beta = z_k \cdots z_1 z_0 \cdot z_{-1} \cdots z_{-l}\), for some \(x, y \in \mathbb{Z}_-\beta\), where \(z_{-l} \neq 0\), i.e.

\[
(x \pm y)(-\beta)^l \in \mathbb{Z}_-\beta \setminus (-\beta)\mathbb{Z}_-\beta.
\]

Thus

\[
K \leq \left|(-\beta)^l (x' \pm y')\right| \leq |\beta|^l (|x'| + |y'|) \leq |\beta|^l \cdot 2H,
\]

which implies

\[
\frac{1}{|\beta'|^2} \leq \frac{2H}{K}.
\]

The statement for \(L_\oplus\) is now simple to see. Similarly, we derive the upper bound for \(L_\ominus\). \(\square\)
5 Quadratic bases in \((-\beta)\)-expansions

The aim of this section is mainly to prove the conjecture of Ito and Sadahiro [13], which says that for a Pisot number \(\beta\), root of \(x^2 - mx + n, m \geq n + 2 \geq 3\), the set \(\text{Fin}(-\beta)\) is a ring. For the other class of quadratic Pisot numbers, roots of \(x^2 - mx - n, m \geq n \geq 1\), this is not valid. For quadratic Pisot units \(\beta\) we obtain values of \(L_\beta(-\beta), L_\beta(-\beta)\).

Let us first show that the only quadratic numbers for which \(d_{-\beta}(l_\beta)\) is eventually periodic are quadratic Pisot numbers. In this, we have an analogue to the case of Rényi \(\beta\)-expansions, as proved by Bassino [5].

**Proposition 8.** Let \(\beta > 1\) be a quadratic number with eventually periodic \(d_{-\beta}(l_\beta)\). Then \(\beta\) is a quadratic Pisot number.

**Proof.** Let first \(d_{-\beta}(l_\beta) = d_1 d_2 \cdots d_k\) with \(d_k \neq 0\). Then
\[-\frac{\beta}{\beta + 1} = \frac{d_1}{(-\beta)} + \frac{d_2}{(-\beta)^2} + \cdots + \frac{d_k}{(-\beta)^k},\]
which implies
\[-(\beta)^{k+1} = d_1 (-\beta)^{k-1} (\beta + 1) + d_2 (-\beta)^{k-2} (\beta + 1) + \cdots + d_k (\beta + 1).\]
Therefore \(\beta\) is a root of a monic polynomial, say \(P(x)\), of degree \(k + 1\) with integer coefficients. Thus \(\beta\) is a quadratic integer. Its minimal polynomial is of the form \(Q(x) = x^2 - mx - n, m, n \in \mathbb{Z}\). As \(Q\) divides \(P\), necessarily \(n\) divides the constant term \(P(0)\), which satisfies \(|P(0)| = d_k\). Since \(d_k\) is a digit of a \((-\beta)\)-expansion, we have \(d_k \leq |\beta|\), and hence \(|n| \leq |\beta|\).

The absolute coefficient of a quadratic polynomial is the product of \(\beta\) and its conjugate \(\beta'\), i.e. \(|\beta\beta'| = |n| \leq |\beta| < \beta\). This implies that \(|\beta'| < 1\) and thus \(\beta\) is a Pisot number.

Consider now the situation \(d_{-\beta}(l_\beta) = d_1 d_2 \cdots d_k (d_{k+1} \cdots d_{k+p})\). Then \(k, p\) are as small as possible, we have \(d_k \neq d_{k+p}\). Equality
\[-\frac{\beta}{\beta + 1} = \frac{d_1}{(-\beta)} + \frac{d_2}{(-\beta)^2} + \cdots + \frac{d_k}{(-\beta)^k} + \frac{d_{k+1}}{(-\beta)^{k+1}} + \frac{d_{k+p}}{(-\beta)^{k+p}} (1 + \frac{1}{(-\beta)^p} + \frac{1}{(-\beta)^{2p}} + \cdots),\]
implies that \(\beta\) is a root of a monic polynomial of degree \(m + p + 1\) with integer coefficients. Moreover, the absolute coefficient of this polynomial is in modulus equal to \(|d_{k+p} - d_k|\). Since \(|d_{k+p} - d_k| \leq |\beta|\), we can derive as before that \(|\beta'| < 1\).

As a consequence of the above proposition, quadratic numbers \(\beta\) with eventually periodic expansion \(d_{-\beta}(l_\beta)\) are precisely the quadratic Pisot numbers, where we have
\[d_{-\beta}(l_\beta) = ((m-1)n)^\omega \quad \text{for} \quad \beta^2 = m\beta - n, \quad m - 2 \geq n \geq 1, \quad (10)\]
\[d_{-\beta}(l_\beta) = m(n-1)^\omega \quad \text{for} \quad \beta^2 = m\beta + n, \quad m \geq n \geq 1, \quad (11)\]
see [13]. In the following we study arithmetics for the two classes of quadratic Pisot numbers separately.

5.1 Case \(\beta^2 = m\beta - n, \quad m - 2 \geq n \geq 1\)

Let us now focus on the set \(\text{Fin}(-\beta)\) for quadratic Pisot numbers \(\beta\), solutions to \(x^2 = mx - n, \quad m, n \in \mathbb{N}, \quad m - 2 \geq n \geq 1\). For such \(\beta\), we have \(d_{-\beta}(l_\beta) = ((m-1)n)^\omega\), and the conjugate of \(\beta\) satisfies \(\beta' \in (0, 1)\). Since \(\beta \tau\), the set \(\text{Fin}(-\beta)\) is non-trivial, it contains 1. A necessary condition so that \(\text{Fin}(-\beta)\) is closed under addition is that the following implication holds for all \(x,\)
\[x \in \text{Fin}(-\beta) \quad \Rightarrow \quad x + 1 \in \text{Fin}(-\beta) .\]
Since we have \((-\beta)\text{Fin}(-\beta) = \text{Fin}(-\beta)\), this condition is also sufficient. Of course, closedness under addition implies closedness under multiplication. Therefore, if we verify that \(-1 \in \text{Fin}(-\beta)\), then closedness under addition already implies that \(\text{Fin}(-\beta)\) is a ring. For the considered bases \(\beta\) we have
\[-(-1)_{-\beta} = 1(m - 1) \cdot n .\]
As a consequence, in order to show that \(\text{Fin}(-\beta)\) is a ring, it suffices to verify the validity of the following statement.
Lemma 9. Let $\beta > 1$ satisfy $\beta^2 = m\beta - n$, for $m, n \in \mathbb{N}$, $m - 2 \geq n \geq 1$. Then $x + 1 \in \text{Fin}(-\beta)$ for every $x \in \text{Fin}(-\beta)$.

Before proving the lemma, let us first describe the digit strings that are admissible as $(-\beta)$-expansions. The following is a simple consequence of Theorem (1), taking into account expression (11).

Lemma 10. Let $\beta$ be the larger root of $x^2 - mx + n$, $m - 2 \geq n \geq 1$. A digit string $x_kx_{k-1}x_{k-2} \cdots$ with finitely many non-zero digits is admissible as $(-\beta)$-expansion if and only if $x_i \in \{0, 1, \ldots, m - 1\}$ and $x_i = m - 1 \Rightarrow x_{i-1} \geq n$.

Proof of Lemma 10. Let us first realize that since $\beta$ is the root of $x^2 - mx + n$, we have the following representations of 0,

$$1 \text{ } m \text{ } n \cdot \mathbb{T} \mathbb{m} \mathbb{n} \cdot = 0,$$

where $\mathbb{A}$ is a compact form of $-A$. By repeated application of this relation, we obtain for every $k \in \mathbb{N}$,

$$1 \text{ } (m-1) \frac{(m-n-1)}{(m-1)} \frac{(m-n-1)^k}{(m-n-1)^k} \frac{(m-n)}{(m-n)} n \cdot = 0 \quad (12)$$

Adding 1 to a number $x$ written as an admissible digit string may result in a non-admissible digit string, which, nevertheless, represents the number $x + 1$. We show that $x + 1$ belongs to $\text{Fin}(-\beta)$ by providing its finite $(-\beta)$-expansion. In order to see that the two strings represent the same number, one can verify that the second one is obtained from the first one by adding digit-wise a zero which is in the form (12). We give a list of cases. One verifies by inspection that the list contains all cases of non-admissible strings that arise from admissible ones by adding 1.

According to Lemma 10, a digit string may be non-admissible by breaking one of the two conditions, namely, either it is not over the alphabet $\{0, 1, \ldots, m - 1\}$, or it contains the subsequence $(m - 1)A$, where $A \leq n - 1$.

Case 1. Consider an $x \in \text{Fin}(\beta)$ such that its $(-\beta)$-expansion has digit $m - 1$ at $(-\beta)^0$. Then necessarily the digit at position $(-\beta)^{-1}$, denote it by $C$, is at least $n$. Find $k \in \{0, 1, 2, \ldots\}$ such that we have a representation of $x + 1$ in the form

$$x + 1 = \cdots A \text{ } B \text{ } [(m-1) \text{ } n]^k \text{ } m \cdot C \cdots$$

where the string $A \text{ } B \neq (m-1) \text{ } n$.

Case 1.1. First take $B = 0$. To the representation (13) of the number $x + 1$ we add digit-wise a representation of 0,

$$x + 1 = \cdots 0 \text{ } A \text{ } 0 \text{ } [(m-1) \text{ } n]^k \text{ } m \cdot C \cdots$$

Before proving the lemma, let us first describe the digit strings that are admissible as $(-\beta)$-expansions. The following is a simple consequence of Theorem (1), taking into account expression (11).

$$x + 1 = \cdots A \text{ } B \text{ } [(m-1) \text{ } n]^k \text{ } m \cdot C \cdots$$

Since $B = 0$ in the $(-\beta)$-expansion of $x$, we necessarily have $A \leq m - 2$. Therefore also the resulting representation of $x + 1$ is admissible as $(-\beta)$-expansion of $x + 1$.

The case $B \geq 1$ is divided into two.

Case 1.2. Let $B \geq 1$ and $k = 0$. Again, we add to the non-admissible representation of $x + 1$ in the form (13) a suitable representation of 0,

$$x + 1 = \cdots A \text{ } B \text{ } [(m-1) \text{ } n]^k \text{ } m \cdot C \cdots$$

Since $A \text{ } B \neq (m-1) \text{ } n$, the resulting representation of $x + 1$ is the $(-\beta)$-expansion of $x + 1$.

Case 1.3. Let $B \geq 1$ and $k \geq 1$. In this case we rewrite

$$x + 1 = \cdots A \text{ } (B-1) \text{ } 0 \text{ } [(m-1) \text{ } n]^k \text{ } m \cdot C \cdots$$

Since $A \text{ } B \neq (m-1) \text{ } n$, the resulting representation of $x + 1$ is the $(-\beta)$-expansion of $x + 1$.
**Case 2.** Consider an \( x \in \text{Fin}(\beta) \) such that its \((-\beta)\)-expansion has digit \( m - 2 \) at the position \((-\beta)^0\). In order that after adding 1 one obtains a non-admissible string, necessarily the digit \( C \) at position \((-\beta)^{-1}\) satisfies \( C \leq n - 1 \). Denote by \( M \) the set of pairs of digits

\[
M := \{ X \ Y \mid X \in \{m-n-1, \ldots, m-1\}, \ Y \in \{0, 1, \ldots, n-1\} \}.
\]

Then we can find \( k, l \in \{0, 1, 2, \ldots\} \) such that

\[
x + 1 = \cdots A \ B \ [(m-1) \ n]^k \ (m-1) \ \bullet \ C \ X_1 \ Y_1 \cdots X_l \ Y_l \ D \ E \ \cdots
\]

where the string \( A \ B \neq (m-1) \ n \) and the string \( D \ E \) does not belong to \( M \). Denote by \( p_1, p_2 \) the \((-\beta)\)-integers

\[
p_1 = 1 \ (m-1) \ (m-n-1) \ [(m-n-1)]^k \ \bullet
\]

\[
p_2 = 1 \ (m-1) \ (m-n-1) \ [(m-n-1)]^k \ \bullet
\]

and by \( z_1, z_2 \) the following numbers with only \((-\beta)\)-fractional part,

\[
z_1 = \bullet \ [(m-n-1)]^l \ (m-n) \ n
\]

\[
z_2 = \bullet \ [(m-n-1)]^l \ (m-n-1) \ (m-n) \ \pi
\]

Using \[12\] one can easily see that \( p_i + z_j = 0 \) for \( i, j \in \{1, 2\} \). We shall work separately with the \((-\beta)\)-integer and \((-\beta)\)-fractional part of \( x + 1 \).

First consider the \((-\beta)\)-fractional part of \( x + 1 \). Recall that \( C \leq n - 1 \) and the string \( D \ E \not\in M \). If \( D \leq m-n-2 \) or \( D = m-n-1 \) (the latter implies \( E \geq n \)), then we add \( z_1 \) to the \((-\beta)\)-fractional part of \( x + 1 \). We obtain

\[
\begin{array}{cccccccc}
\bullet & C & (m-n-1) & X_1 & Y_1 & \cdots & X_l & Y_l & D & E & \cdots \\
\bullet & C+m-n-1 & X_1-(m-n-1) & Y_1+m-n-1 & \cdots & X_l-(m-n-1) & Y_l+m-n & D+n & E & \cdots \\
\end{array}
\]

The resulting fractional part is an admissible digit string. If \( D \geq m-n \) (which implies \( E \geq n \)), then we add \( z_2 \) to the \((-\beta)\)-fractional part of \( x + 1 \). We obtain

\[
\begin{array}{cccccccc}
\bullet & C & (m-n-1) & X_1 & Y_1 & \cdots & X_l & Y_l & D & E & \cdots \\
\bullet & C+m-n-1 & X_1-(m-n-1) & Y_1+m-n-1 & \cdots & X_l-(m-n-1) & Y_l+m-n-1 & D+m+n & E-n & \cdots \\
\end{array}
\]

Again, the resulting string is admissible.

Let us now take the \((-\beta)\)-integer part of \( x + 1 \). Recall that \( A \ B \neq (m-1) \ n \). If \( B = 0 \), then to the \((-\beta)\)-integer part of \( x + 1 \) we add \( p_2 \), if \( B \geq 1 \), we add \( p_1 \). We obtain

\[
\begin{array}{cccccccc}
\cdots & A & 0 & [(m-1) \ n]^k & (m-1) & \bullet \\
\cdots & (A+1) & (m-1) & n & (m-1)^k & n & \bullet \\
\end{array}
\]

and

\[
\begin{array}{cccccccc}
\cdots & A & 0 & [(m-1) \ n]^k & (m-1) & \bullet \\
\cdots & A & (B-1) & 0 & [(m-1) \ n]^k & \bullet \\
\end{array}
\]

In both cases, the result is an admissible string with last digit equal to \( n \). Concatenating such a string with an admissible digit string resulting from the \((-\beta)\)-fractional part, we obtain an admissible digit string. Therefore, we have provided a prescription to rewrite the original non-admissible representation of \( x + 1 \) of the form \[13\] by adding 0 in the form \( p_i + z_j \), into the \((-\beta)\)-expansion of \( x + 1 \). This completes the proof. \( \square \)
Corollary 11. Let $\beta > 1$ satisfy $\beta^2 = m\beta - n$, for $m, n \in \mathbb{N}$, $m - 2 \geq n \geq 1$. Then $\text{Fin}(\beta)$ is a ring.

Addition of two $(-\beta)$-integers thus always yields a number whose $(-\beta)$-expansion has a finite number of fractional digits. We can give the upper bound to the length of such a fractional part in case that $\beta$ is a unit.

Example 12. Let $\beta > 1$ satisfy $\beta^2 = m\beta - 1$, for $m \in \mathbb{N}$, $m \geq 3$. The digits of $(-\beta)$-expansions thus take values in the set $\{0, 1, \ldots, m - 1\}$, and $(m - 1)0$ is a forbidden string. Put $x = m - 2$, $y = 1$. Obviously $x, y \in \mathbb{Z}_-\beta$. In order to find the $(-\beta)$-expansion of $z = x + y = m - 1$, we rewrite

$$m - 1 = (-\beta)^2 + (m - 1)(-\beta) + 1 + \frac{m - 1}{(-\beta)} + \frac{1}{(-\beta)^2},$$

and hence

$$(z)_{-\beta} = (m - 1)1 \bullet (m - 1)1 \in \left(\frac{1}{(-\beta)^2}\right)\mathbb{Z}_-\beta \setminus \left(\frac{1}{(-\beta)}\right)\mathbb{Z}_-\beta.$$

From this, we can conclude that $L_\oplus(\beta) \geq 2$.

Now put $x = (m - 2)(1 + \beta^2)$, $y = 1 + \beta^2$. The corresponding $(-\beta)$-expansions are $(x)_{-\beta} = (m - 2)0(m - 2)\bullet$ and $(y)_{-\beta} = 101\bullet$, i.e. $x, y \in \mathbb{Z}_-\beta$. One easily verifies that

$$z = xy = (m - 2)^2\beta^4 + 2(m - 2)\beta^2 + (m - 1) =$$

$$= (m - 1)(-\beta)^4 + (m - 1)(-\beta)^3 + (m - 2)(-\beta)^2 + (m - 2)(-\beta) + \frac{m - 1}{(-\beta)} + \frac{1}{(-\beta)^2},$$

and hence

$$(z)_{-\beta} = (m - 1)(m - 1)(m - 2)(m - 2)0 \bullet (m - 1)1 \in \left(\frac{1}{(-\beta)^2}\right)\mathbb{Z}_-\beta \setminus \left(\frac{1}{(-\beta)}\right)\mathbb{Z}_-\beta,$$

and we can conclude that $L_\oplus(\beta) \geq 2$.

Theorem 13. Let $\beta > 1$ satisfy $\beta^2 = m\beta - 1$, for $m \in \mathbb{N}$, $m \geq 3$. Then

$$L_\oplus(\beta) = 2 = L_\oplus(\beta).$$

Proof. We use Theorem 7. We first show that

$$H = \frac{1 - \beta'}{\beta'(1 + \beta')} \quad \text{and} \quad K = \frac{\beta'(1 - \beta')}{1 + \beta'}.$$

Consider $z \in \mathbb{Z}_-\beta$ with the expansion $(z)_{-\beta} = z_kz_{k-1}\cdots z_1z_0\bullet$, where $z_i = m - 1$ for at least one index $i$. Since the string $(m - 1)0$ is forbidden, we have $i \geq 1$ and, moreover, $z_i - 1 \geq 1$. Since the string $z_kz_{k+1}(m - 1)z_{i-1}\cdots z_0\bullet$ is admissible, also the strings $z_kz_{k+1}(m - 2)z_{i-1}\cdots z_0\bullet$ and $z_kz_{k+1}(m - 2)(z_{i-1} - 1)\cdots z_0\bullet$ are admissible, denote by $x$ and $y$ the corresponding $(-\beta)$-integers. For the field conjugates $x', y'$, $z'$ of $x$, $y$, $z$ we have

$$x' + (-\beta)^i = z' = y' + (-\beta)^i + (-\beta)^{i-1} = y' + (-\beta)^{i-1}(1 - \beta').$$

Since $\beta' \in (0, 1)$, the number $z'$ lies between numbers $x'$ and $y'$. From this, we can derive that

$$H = \sup \{||z'|| \mid z \in \mathbb{Z}_-\beta, (z)_{-\beta} \text{ does not contain the digit } (m - 1)\}.$$

Note that arbitrary string of digits $\{0, 1, \ldots, m - 2\}$ is admissible. For $z \in \mathbb{Z}_-\beta$ we can thus write

$$z' = \sum_{i=0}^{k} z_i(-\beta)^i \leq \sum_{0 \leq 2i \leq k} (m - 2)(-\beta)^{2i} \leq \frac{m - 2}{1 - (\beta')^2}.$$

Similarly, we have

$$z' = \sum_{i=0}^{k} z_i(-\beta)^i \geq \sum_{0 \leq 2i+1 \leq k} (m - 2)(-\beta)^{2i+1} \geq -\beta' \frac{m - 2}{1 - (\beta')^2}.$$
Equality \((\beta')^2 = m\beta' - 1\) implies \((m - 2)\beta' = (\beta' - 1)^2\) and therefore

\[
H = \frac{m - 2}{1 - (\beta')^2} = \frac{1}{\beta'} \frac{(1 - \beta')^2}{1 - (\beta')^2} = \frac{1}{\beta'} \frac{1 - \beta'}{1 + \beta'},
\]

as we wanted to show.

In order to determine \(K\), we study the field conjugate \(z'\) for \(z \in \mathbb{Z}_{-\beta} \setminus (-\beta)\mathbb{Z}_{-\beta}\). Again, for \(z\) whose \((-\beta)\)-expansion contains the digit \((m - 1)\) we can find \(x, y \in \mathbb{Z}_{-\beta} \setminus (-\beta)\mathbb{Z}_{-\beta}\) with digits in \(\{0, 1, \ldots, m - 2\}\) such that \(x' < z' < y'\). The only exception is the case when the last two digits of \(z\) equal to \(z_1 z_0 = (m - 1)1\). Here we have

\[
z' = \sum_{i=0}^{k} z_i (-\beta')^i > 1 + (m - 1)(-\beta') + \sum_{i=0}^{\infty} (m - 2)(-\beta')^{2i+1} = 1 - \beta' - \beta'H = \frac{\beta'(1 - \beta')}{1 + \beta'} = K.
\]

Let us put the computed values of \(H, K\) into Theorem 7. With the use of \(\beta' = \beta^{-1}\) we obtain that

\[
\beta^L = \frac{2H}{K} = \frac{2}{(\beta')^2} = 2\beta^2 < \beta^3,
\]

\[
\beta^L = \frac{2H}{K} = \frac{1 - \beta'}{(\beta')^3 1 + \beta'} = \beta^3 1 - \beta' < \beta^3.
\]

Together with Example 12, we have the statement of the theorem.

5.2 Case \(\beta^2 = m\beta + n, m \geq n \geq 1\)

Consider for \(\beta\) the larger root of \(x^2 = mx + n, m \geq n \geq 1\). Its conjugate \(\beta'\) is negative, and therefore by Lemma 3 the set \(\text{Fin}(-\beta)\) cannot be a ring. In particular, we have an infinite expansion for the number \(-1\),

\[
\langle -1 \rangle_{-\beta} = 1m \cdot (m - n + 1)^\omega.
\]

Nevertheless, this fact does not prevent \(\text{Fin}(\beta)\) to be closed under addition, as it is shown in 13. Even if by summing and multiplying \((-\beta)\)-integers leads sometimes to infinite \((-\beta)\)-expansions, we can estimate the number of arising fractional digits in case that the result of addition and multiplication belongs to \(\text{Fin}(\beta)\). Theorem 7 leads a simple result for bases \(\beta\) which are units.

**Theorem 14.** Let \(\beta > 1\) satisfy \(\beta^2 = m\beta + 1\), for \(m \in \mathbb{N}, m \geq 2\). Then

\[
L_{\oplus}(-\beta) = 1 = L_{\oplus}(-\beta).
\]

**Proof.** First we show that the values \(L_{\oplus}(-\beta) = 1, L_{\oplus}(-\beta) = 1\) can be reached. The digits of \((-\beta)\)-expansions take values in the set \(\{0, 1, \ldots, m\}\). Using 11 and 14 we derive that forbidden are the strings \(m(m - 1)^{2k+1}m\) and \(m(m - 1)^{2k}A\), where \(k \geq 0\) and \(A \leq m - 2\). Put \(x = -m\beta + m - 1\). Obviously \(x \in \mathbb{Z}_{-\beta}\). We rewrite

\[
z = x + x = 2m(-\beta) + (m - 1) = (-\beta)^3 + (m - 1)(-\beta)^2 + (m - 2)(-\beta) + m - 1 + \frac{1}{(-\beta)},
\]

and hence

\[
\langle z \rangle_{-\beta} = 1(m - 1)(m - 2)(m - 1) \cdot 1 \in \frac{1}{(-\beta)} \mathbb{Z}_{-\beta} \setminus \mathbb{Z}_{-\beta}.
\]

From this, we can conclude that \(L_{\oplus}(-\beta) \geq 1\). In order to find the lower bound to \(L_{\oplus}(-\beta)\), it suffices to realize that \(y = 2 \in \mathbb{Z}_{-\beta}\). For, we have \(\langle y \rangle_{-\beta} = 2\bullet\) if \(m \geq 3\), and \(\langle y \rangle_{-\beta} = 121\bullet\) if \(m = 2\). We therefore have \(xy = z\) as above and \(L_{\oplus}(-\beta) \geq 1\).

In order to find upper bounds on \(L_{\oplus}(-\beta), L_{\oplus}(-\beta)\), let us compute the values \(H, K\) for use in Theorem 7. Consider \(z \in \mathbb{Z}_{-\beta}\) with the expansion \(\langle z \rangle_{-\beta} = z_k z_{k-1} \cdots z_1 z_0\). Since the conjugate \(\beta' = -\beta^{-1}\) of \(\beta\) belongs to \((-1, 0)\), we have

\[
0 \leq z' = \sum_{i=0}^{k} z_i (-\beta')^i \leq \sum_{i=0}^{m} m |\beta'|^i = \frac{m \beta}{\beta - 1} - 1 = \beta,
\]
where we have taken account of the fact that a \((-\beta)\)-integer cannot have the digit \(m\) at the position \((-\beta)^0\). Therefore \(H = \sup\{|z'| \mid z \in \mathbb{Z}_{-\beta}\} = \beta\) and the supremum is not reached. Similarly, the fact that \(\beta' \in (-1, 0)\) implies that \(K = 1\). We therefore have

\[
\begin{align*}
\beta^{L_o} &< \frac{2H}{K} = 2\beta < \beta^2, \\
\beta^{L_o} &< \frac{H^2}{K} = \beta^2,
\end{align*}
\]

where we have used that \(\beta > 2\), which is valid, as we consider \(m \geq 2\). This implies the statement of the theorem.

Note that we derive the exact values of \(L_\oplus(-\beta)\), \(L_\otimes(-\beta)\) only for quadratic Pisot units not equal to the golden ratio. In fact, Theorem \[1\] leads in the case \(\beta = \frac{1}{2}(1+\sqrt{5})\) only to the values \(L_\oplus(-\beta) \leq 3\) and \(L_\otimes(-\beta) \leq 3\), but the actual values are \(L_\oplus(-\beta) = 2 = L_\otimes(-\beta)\). The proof of this fact is found in \[13\].

**Acknowledgements**

We acknowledge financial support by the Czech Science Foundation grant 201/09/0584 and by the grants MSM6840770039 and LC06002 of the Ministry of Education, Youth, and Sports of the Czech Republic. The work was also partially supported by the CTU student grant SGS10/085/OHK4/1T/14.

**References**

[1] S. Akiyama, *Cubic Pisot Units with finite beta expansions*, in ‘Algebraic Number Theory and Diophantine Analysis’, ed. by F.Halter-Koch and R.F. Tichy, de Gruyter (2000), 11–26.

[2] S. Akiyama, *Self affine tiling and Pisot numeration system*, Number Theory and its Applications, ed. by K. Gyory and S. Kanemitsu, 7-17 Kluwer 1999.

[3] P. Ambrož, D. Dombek, Z. Masáková, E. Pelantová, *Numbers with integer expansion in the numeration system with negative base*, preprint 2009, 13pp. http://arxiv.org/abs/0912.4597

[4] P. Ambrož, C. Frougny, Z. Masáková, E. Pelantová, *Arithmetics on number systems with irrational bases*, Bull. Belgian Math. Soc. Simon Stevin 10 (2003), 641-659.

[5] F. Bassino, *\(\beta\)-expansions for cubic Pisot numbers*, 5th Latin American Theoretical INformatics Symposium (LATIN’02), 2286 LNCS. Cancun, Mexico. April, 2002. pp. 141-152. Springer-Verlag.

[6] J. Bernat, *Arithmetics in \(\beta\)-numeration*, Discr. Math. Theor. Comp. Sci. 9 (2007), 85–106.

[7] J. Bernat, *Computation of \(L_\oplus\) for several cubic Pisot numbers*, Discr. Math. Theor. Comput. Sci. 9 (2007), 175–193.

[8] Č. Burdík, Ch. Frougny, J. P. Gazeau, R. Krejcar, *Beta-Integers as Natural Counting Systems for Quasicrystals*, J. Phys. A: Math. Gen. 31 (1998) 6449–6472.

[9] Ch. Frougny, A. C. Lai, *On negative bases*, Proceedings of DLT 09, Lectures Notes in Computer Science, 5583 (2009).

[10] Ch. Frougny and B. Solomyak, *Finite \(\beta\)-expansions*, Ergodic Theory Dynamical Systems 12 (1994), 713–723.

[11] L. S. Guimond, Z. Masáková, E. Pelantová, *Arithmetics of beta-expansions*, Acta Arith. 112 (2004), 23–40.

[12] M. Hollander, *Linear Numeration systems, Finite Beta Expansions, and Discrete Spectrum of Substitution Dynamical Systems*, Ph.D. thesis, University of Washington, 1996.

[13] S. Ito, T. Sadahiro, *\((-\beta)\)-expansions of real numbers*, Integers 9 (2009), 239–259.
[14] Z. Masáková, T. Vávra, *Arithmetics in numeration systems with negative quadratic base*, Kybernetika 47 (2011), (in press).

[15] W. Parry, *On the β-expansions of real numbers*, Acta Math. Acad. Sci. Hung. 11 (1960), 401–416.

[16] A. Rényi, *Representations for real numbers and their ergodic properties*, Acta Math. Acad. Sci. Hung. 8 (1957), 477–493.