\[ N = 1 \text{ supersymmetry and the three loop gauge } \beta\text{-function} \]

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We calculate the three loop gauge \( \beta \)-function for an abelian \( N = 1 \) supersymmetric gauge theory, using DRED. We construct a coupling constant redefinition that relates the result to the corresponding term in the NSVZ \( \beta \)-function, and by generalising this redefinition to the non-abelian case we derive the DRED three loop gauge \( \beta \)-function for the non-abelian case.
In a previous paper [1] we calculated the three-loop contribution $\gamma^{(3)}$ to the anomalous dimension of the chiral supermultiplet in a general $N = 1$ supersymmetric gauge theory, using DRED. † Here we present the analogous result for the gauge $\beta$-function. In fact we perform the explicit calculation only for the abelian case, and infer the non-abelian result by comparing our result to the all-orders NSVZ $\beta$-function[3].

The Lagrangian $L_{\text{SUSY}}(W)$ for a $N = 1$ supersymmetric theory is defined by the superpotential

$$W = \frac{1}{6} Y^{ijk} \Phi_i \Phi_j \Phi_k + \frac{1}{2} \mu^{ij} \Phi_i \Phi_j. \quad (1)$$

$L_{\text{SUSY}}$ is the Lagrangian for the $N = 1$ supersymmetric gauge theory, containing the gauge multiplet and a chiral superfield $\Phi_i$ with component fields $\{\phi_i, \psi_i\}$ transforming as a representation $R$ of the gauge group $\mathcal{G}$. We assume that there are no gauge-singlet fields and that $\mathcal{G}$ is simple.

The $\beta$-functions for the Yukawa couplings $\beta_Y^{ijk}$ are given by

$$\beta_Y^{ijk} = Y^{p(ij} \gamma^k)_{p} = Y^{ijp} \gamma^k_{p} + (k \leftrightarrow i) + (k \leftrightarrow j), \quad (2)$$

where $\gamma$ is the anomalous dimension for $\Phi$. The one-loop results for the gauge coupling $\beta_g$ and for $\gamma$ are given by

$$16\pi^2 \beta_{g}^{(1)} = g^3 Q, \quad \text{and} \quad 16\pi^2 \gamma^{(1)i}_{j} = P^{i}_{j}, \quad (3)$$

where

$$Q = T(R) - 3C(G), \quad \text{and} \quad P^{i}_{j} = \frac{1}{2} Y^{ikl} Y_{jkl} - 2g^2 C(R)^{i}_{j} \quad (4a)$$

Here

$$T(R)\delta_{AB} = \text{Tr}(R_AR_B), \quad C(G)\delta_{AB} = f_{ACD} f_{BCD} \quad \text{and} \quad C(R)^{i}_{j} = (R_AR_A)^{i}_{j}. \quad (5)$$

The two-loop $\beta$-functions for the dimensionless couplings were calculated in Refs. [4]–[8]:

$$\begin{align}
(16\pi^2)^2 \beta_{g}^{(2)} &= 2g^5 C(G)Q - 2g^3 r^{-1} C(R)^{i}_{j} P^{i}_{j}, \quad (6a) \\
(16\pi^2)^2 \gamma^{(2)i}_{j} &= [-Y_{jmn} Y^{mpi} - 2g^2 C(R)^{p}_{j} \delta^{i}_{n}] P^{n}_{p} + 2g^4 C(R)^{i}_{j} Q, \quad (6b)
\end{align}$$

† By DRED we mean dimensional reduction [2] with minimal (or modified minimal) subtraction.
where \( Q \) and \( P^i_j \) are given by Eq. (3), and \( r = \delta_{AA} \).

In our notation the NSVZ formula for \( \beta_g \) is

\[
\beta_g^{NSVZ} = \frac{g^3}{16\pi^2} \left[ \frac{Q - 2r^{-1}\text{Tr}[\gamma C(R)]}{1 - 2C(G)g^2(16\pi^2)^{-1}} \right],
\]

which leads to

\[
(16\pi^2)^3 \beta_g^{(3)NSVZ} = 4g^7Q C(G)^2 - 4g^5C(G)r^{-1}16\pi^2\text{Tr}[\gamma^{(1)}C(R)]
- 2g^3r^{-1}(16\pi^2)^2\text{Tr}[\gamma^{(2)}C(R)].
\]

Note that \( \beta_g^{(3)NSVZ} \) vanishes for a one-loop finite theory. This holds also for \( \beta_g^{(3)DRED} \), as explicitly verified by Parkes and West\,[9]; see also Ref. [1]. We also have for a theory with \( N = 2 \) supersymmetry that \( \beta_g^{(3)NSVZ} = 0 \), as is easy to verify from Eq. \( \text{(8)} \), and \( \beta_g^{(3)DRED} = 0 \), because \( N = 2 \) theories have one loop divergences only\,[10]. Nevertheless we shall see that DRED does not give the NSVZ result at three loops.

Let us turn to the explicit calculation. In the abelian case, this amounts to a straightforward determination using standard Feynman rules\,[11] of the vector supermultiplet self–energy. Now in the special case of one-loop finite theories, Parkes and West\,[8] were able to derive the result \( \beta_g^{(3)DRED} = 0 \) in the non-abelian case via an essentially abelian calculation, by using the fact that the same result holds for \( N = 2 \) theories (whether one-loop finite or not). As we shall see, this property of \( N = 2 \) theories will be useful for us as well.

Our result is

\[
(16\pi^2)^3 \beta_g^{(3)DRED} = 3r^{-1}g^3Y^{ikm}Y_{jkn}P^n_mC(R)^{ji} + 6r^{-1}g^5\text{tr}[PC(R)^2]
+ r^{-1}g^3\text{tr}[P^2C(R)] - 6r^{-1}Qg^7\text{tr}[C(R)^2]
\]

where here \( Q = T(R) \). Now we compare this with the corresponding result for \( \beta_g^{(3)NSVZ} \), obtained by setting \( C(G) = 0 \) in Eq. \( \text{(8)} \). They are not the same, but if we redefine the DRED coupling \( g \) as follows:

\[
g \rightarrow g + \delta g, \quad \text{where} \quad \delta g = -(16\pi^2)^{-2} \frac{1}{2\pi^2}g^3\text{tr}[PC(R)]
\]

then the resulting change in \( \beta_g, \delta \beta_g \) satisfies

\[
\delta \beta_g = \left[ \beta_Y \cdot \frac{\partial}{\partial Y} + \beta_Y^* \cdot \frac{\partial}{\partial Y^*} + \beta_g \cdot \frac{\partial}{\partial g} \right] \delta g - \delta g \cdot \frac{\partial}{\partial g} \beta_g
= -r^{-1}g^3Y^{ikm}Y_{jkn}P^n_mC(R)^{ji} - 2r^{-1}g^5\text{tr}[PC(R)^2]
- r^{-1}g^3\text{tr}[P^2C(R)] + 2r^{-1}Qg^7\text{tr}[C(R)^2],
\]
and it is easy to show that

\[
\beta^{(3)\NSVZ}_g = \beta^{(3)\DRED}_g + \delta\beta_g. \tag{12}
\]

Notice that it is quite non-trivial that \(\beta^{(3)\NSVZ}_g\) and \(\beta^{(3)\DRED}_g\) can be related in this way; \(\delta g\) as defined in Eq. (10) leads to four distinct tensor structures in Eq. (11).

We turn now to the non-abelian case. The crucial observation is that \(\delta g\) as defined in Eq. (10) does not vanish for a \(N = 2\) theory in general (though it does in the abelian case, as may be easily verified). There is, however, an obvious generalisation of it to the non-abelian case, to wit

\[
\delta g = (16\pi^2)^{-2}g^3 \left[ r^{-1}\text{tr} \left[ PC(R) \right] - g^2QC(G) \right] \tag{13}
\]

where we have reversed the overall sign (compared to Eq. (10)) because we plan to use this \(\delta g\) to go back from \(\beta^{(3)\NSVZ}_g\) to \(\beta^{(3)\DRED}_g\). It is easy to verify that Eq. (13) leads to \(\delta g = 0\) in the \(N = 2\) case. Are there any other candidate terms for inclusion in \(\delta g\)? We are constrained by the following requirements:

(1) \(\delta g = 0\) for a one-loop finite theory.
(2) \(\delta g = 0\) for a \(N = 2\) theory.
(3) Eq. (10) must hold in the abelian case.
(4) The resulting terms in \(\delta \beta_g\) must correspond to possible 1PI Feynman graphs.

It is easy to convince oneself that Eq. (13) represents the only possible transformation (up to an overall constant, which we have fixed by the abelian calculation). With hindsight, in fact, simply calculating the coefficient of the \(\text{tr} \left[ P^2C(R) \right]\) term in \(\beta^{(3)\DRED}_g\) would have sufficed: much easier than the full abelian calculation. By performing this we have, however, verified that the NSVZ \(\beta\)-function corresponds to a scheme equivalent to DRED.

Our result for \(\beta^{(3)\DRED}_g\) in the non-abelian case is therefore:

\[
(16\pi^2)^3\beta^{(3)\DRED}_g = 3r^{-1}g^3Y^{ikm}Y_{jkn}P_{\;\;m}^nC(R)^{\;j}_i + 6r^{-1}g^5\text{tr} \left[ PC(R)^2 \right] \\
+ r^{-1}g^3\text{tr} \left[ P^2C(R) \right] - 6r^{-1}Qg^7\text{tr} \left[ C(R)^2 \right] - 4r^{-1}g^5C(G)\text{tr} \left[ PC(R) \right] \\
+ g^7QC(G) \left[ 4C(G) - Q \right]. \tag{14}
\]

Of course our method has been somewhat indirect so it would be interesting to confirm Eq. (14) by an explicit calculation. Remarkably enough, in the special case of no
chiral superfields, there does exist one in the literature, by Avdeev and Tarasov\[12\]. They obtained the result

\[
\beta_g^{\text{DRED}} = -3g^3C(G)(16\pi^2)^{-1} - 6g^5C(G)^2(16\pi^2)^{-2} - 21g^7C(G)^3(16\pi^2)^{-3} + \cdots \quad (15)
\]

which precisely agrees with Eq. (14). Now Ref. \[12\] employed DRED with component fields rather than superfields, and hence a very different (and not manifestly supersymmetric) gauge; as we should perhaps have expected, however, within DRED $\beta_g$ is gauge invariant. (For discussion of the gauge invariance of $\beta_g$ in the context of ordinary dimensional regularisation, see Ref. \[13\]). Thus our conjecture in Ref. \[1\] that DRED would reproduce $\beta_g^{(3)NSVZ}$ was perhaps misguided.

In conclusion: our main result is the DRED result for the three loop gauge $\beta$-function, Eq. (15). In Ref. \[14\] the results of Ref. \[1\] and this paper are used to derive the three-loop supersymmetric standard model $\beta$-functions, and investigate their effect on the standard running coupling analysis. (For an interesting alternative approach to this running analysis, see Ref. \[15\]).

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