A NEW TRANSVERSALITY CONDITION ON ORBIFOLDS AND INTEGER-VALUED GROMOV–WITTEN TYPE INVARIANTS

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ABSTRACT. Following the proposal of Fukaya–Ono [FO01] and the exploration by B. Parker [Par13], we introduce a new transversality condition, the FOP transversality condition, for sections of orbifold vector bundles $E \to U$ when both $E$ and $U$ have “normal complex structures.” This notion allows one to define various integral virtual cycles on moduli spaces of pseudoholomorphic curves. Two immediate applications in symplectic topology are the definition of integer-valued Gromov–Witten type invariants in all genera for general compact symplectic manifolds using the global Kuranishi chart constructed by Abouzaid–McLean–Smith [AMS21][AMS23] and Hirschi–Swaminathan [HS22], and an alternative proof of the cohomological splitting theorem for Hamiltonian fibrations over $S^2$ with integer coefficients from [AMS21].

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1. INTRODUCTION

Gromov–Witten invariants are generally rational numbers but not integers. A conceptual reason for the non-integrality is that we need to consider curves with symmetries. In the algebraic geometric situation, this means the moduli spaces are stacks rather than schemes. In the technical construction of symplectic Gromov–Witten invariants (see [LT98],[FO99],[Rua99],[CM07],[Par16],[HWZ17],[Sie96], and [AMS21]), it is also the consequence of using multi-valued perturbations. For the same reason, Hamiltonian Floer homology of a general symplectic manifold is also defined over $\mathbb{Q}$ but not over $\mathbb{Z}$ (see [FO99],[LT99], and [Par16]). This defect limits...
our applications. For example, regarding the Arnold conjecture, the Hamiltonian Floer chain complex can only detect rational homology classes of the symplectic manifold\(^1\). One cannot either define quantum Steenrod operations for general symplectic manifolds as one needs \(\mathbb{Z}/p\) counts of genus zero curves (see [Wil20] and [SW22]).

The main motivation of this paper is to define integer-valued Gromov–Witten invariants and prove integral Arnold conjecture, following a proposal of Fukaya–Ono [FO01]. Roughly speaking, because moduli spaces of pseudoholomorphic curves carry a “normal complex structure,” one can use a specified type of single-valued perturbations to separate curves with different automorphism groups; as a result, one could count only curves with trivial automorphism group and obtain well-defined integral invariants.

The main contribution of this paper is to rigorously realize Fukaya–Ono’s proposal in an abstract, finite-dimensional setting which can be applied to symplectic geometric problems. In particular, we define a new notion of transversality for sections on orbifolds. The most important topological consequence is the existence of integer-valued Euler cycles of orbifold vector bundles. We include two geometric applications. The first one is the definition of integer-valued Gromov–Witten type invariants for general compact symplectic manifolds. As far as we know, this is the first general construction of integer-valued curve-counting invariants for all symplectic manifolds, which in particular include all smooth projective varieties over complex numbers. The second application is a proof of principle: we offer an alternative proof of Abouzaid–McLean–Smith’s cohomological splitting result for Hamiltonian fibrations [AMS21] not relying ideas inspired by Floer homotopy theory.

In the subsequent work [BX22], we prove the Arnold conjecture over \(\mathbb{Z}\) by further developing the ideas in this paper. We expect this paper to serve as a starting point of constructing refinements of curve-counting invariants for general symplectic manifolds. Moreover, the methods here should also be useful for tackling longstanding conjectures concerning quantitative lower bound for fixed/intersection points in various settings, beyond the integral version of homological Arnold conjecture.

1.1. Statement of the main technical result. In this paper we define a new notion of transversality for sections of orbifold vector bundles. This notion, called the FOP transversality condition, originates from the proposal of Fukaya–Ono [FO01] and which was further elaborated by B. Parker [Par13]. One of the upshots of this notion is to define integral virtual fundamental classes when the usual notion of transversality fails in the presence of symmetry.

Before we get into details we would like to discuss the general philosophy. It is not hard to realize that transversality and symmetry are two conflicting features: the former is generic while the latter is specific. However we need to reconcile them when discussing transversality for sections of orbifold vector bundles, the model for regularizing moduli spaces in symplectic topology. The usual transversality theorem can provide a regular zero locus only away from orbifold points. Indeed,

\(^1\)The recent advancement of the Arnold conjecture by Abouzaid–Blumberg [AB21] strengthens the lower bound to the total Betti number over finite fields using more involved stable homotopy theory; afterwards we use the method developed in this paper to obtain a stronger lower bound including contributions from all characteristics.
The usual transversality for sections can be achieved within each stratum, but may fail in the normal directions. If an isotropy type \( \gamma \) is represented by \((G, V, W)\), then one can decompose
\[
V = V_G \oplus \tilde{V}_G, \quad W = W_G \oplus \tilde{W}_G \quad (1.1)
\]
where \( V_G, \tilde{V}_G \) resp. \( W_G, \tilde{W}_G \) are the direct sums of irreducible trivial resp. non-trivial subrepresentations of \( V, W \). If we restrict the tangent bundle \( T\mathcal{U} \) and the bundle \( \mathcal{E} \) to \( \mathcal{U}_\gamma \), then one has corresponding decompositions
\[
T\mathcal{U}|_{\mathcal{U}_\gamma} = T\mathcal{U}_\gamma \oplus \mathcal{N}_\gamma, \quad \mathcal{E}|_{\mathcal{U}_\gamma} = \mathcal{E}_\gamma \oplus \tilde{\mathcal{E}}_\gamma.
\]
The “normal” components \( \mathcal{N}_\gamma \) and \( \tilde{\mathcal{E}}_\gamma \) are roughly bundles with fibers being \( \tilde{V}_G \) and \( \tilde{W}_G \) specified above. Notice that for any section \( S: \mathcal{U} \to \mathcal{E} \), when restricted to \( \mathcal{U}_\gamma \), the \( \tilde{\mathcal{E}}_\gamma \)-component must vanish. If we decompose
\[
S|_{\mathcal{U}_\gamma} = S_\gamma \oplus \tilde{S}_\gamma,
\]
then usually the best we can hope is the transversality of \( S_\gamma : \mathcal{U}_\gamma \to \mathcal{E}_\gamma \). The derivative of \( S \) in the direction \( \mathcal{N}_\gamma \) may not be enough to cover the missing direction \( \tilde{\mathcal{E}}_\gamma \), causing the failure of transversality.

A normal complex structure on \( \mathcal{U} \) resp. \( \mathcal{E} \) consists of \( G \)-invariant complex structures on fibers of \( \mathcal{N}_\gamma \) resp. \( \tilde{\mathcal{E}}_\gamma \) for all isotropy types \( \gamma \) which are “compatible”

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2 All orbifolds in this paper are effective unless otherwise declared.
between adjacent strata (see Section 2 for the rigorous definition). In particular, if \( \mathcal{U} \) and \( \mathcal{E} \) are normally complex, then the isotropy types appearing are those \((G, V, W)\) with \( \hat{V}_G, \hat{W}_G \) being complex representations; such isotropy types are called \textit{normally complex isotropy types}.

For the purpose of dealing with infinite-dimensional situations, we need to introduce the \textit{stabilization} of isotropy types. A normally complex isotropy type \((G, V, W)\) can be stabilized to \((G, V \oplus R, W \oplus R)\) where \( R \) is a normally complex representation (cf. Definition 2.13). Let \( \mathbb{C}^{\text{NC}} \) be the set of all complex isotropy types modulo stabilization, which has induced partial order induced from inclusions of groups and representations. Let \( \mathbb{C}_k^{\text{NC}} \subseteq \mathbb{C}^{\text{NC}} \) be the subset of elements whose virtual dimension \( \dim \mathcal{E} - \dim \mathcal{W} \) is equal to \( k \in \mathbb{Z} \). There is a special maximal element \( \gamma^*_k \in \mathbb{C}_k^{\text{NC}} \) represented by \((\{1\}, \mathbb{R}^m, \mathbb{R}^n)\) with \( m - n = k \).

Our main technical result is stated as the following blackbox-type theorem.

**Theorem 1.1.** Suppose \( \mathcal{U} \) is a normally complex orbifold without boundary and \( \mathcal{E} \to \mathcal{U} \) is a normally complex vector bundle. Let \( \Gamma(\mathcal{U}, \mathcal{E}) \) be the space of smooth sections. Then there is a \( C^0 \)-dense\(^3\) subset \( \Gamma^{\text{FOP}}(\mathcal{U}, \mathcal{E}) \subseteq \Gamma(\mathcal{U}, \mathcal{E}) \) whose elements are called FOP transverse sections satisfying the following properties.

1. **(Classical Transversality)** If \( \mathcal{U} \) is a manifold, FOP transversality is equivalent to classical transversality.
2. **(Locality)** The restrictions of FOP transverse sections to open subsets are still FOP transverse.
3. **(Extension Property I)** For any pair of closed subsets \( \mathcal{V} \subseteq \mathcal{V}' \) and open neighborhoods \( \mathcal{U} \subseteq \mathcal{U}' \) of \( \mathcal{V} \) and \( \mathcal{V}' \subseteq \mathcal{U}' \) of \( \mathcal{V}' \), if \( S \in \Gamma^{\text{FOP}}(\mathcal{V}, \mathcal{E}|_{\mathcal{V}}) \), then there exists \( S' \in \Gamma^{\text{FOP}}(\mathcal{V}', \mathcal{E}|_{\mathcal{V}'}) \) which agrees with \( S \) near \( \mathcal{V} \).
4. **(Extension Property II)** Let \( \mathcal{X} \subseteq \mathcal{U} \) be a suborbifold with ordinary normal bundle (Definition 2.4), which implies that \( \mathcal{X} \) and \( \mathcal{E}|_{\mathcal{X}} \) are normally complex. Let \( \mathcal{S}_X : \mathcal{X} \to \mathcal{E}|_{\mathcal{X}} \) be an FOP transverse section. Then there exists an FOP transverse extension of \( \mathcal{S}_X \) to \( \mathcal{U} \).
5. **(Product Property)** Let \( \mathcal{U}' \) be another normally complex orbifold and \( \mathcal{E}' \to \mathcal{U}' \) be a normally complex vector bundle. Then the product map

\[
\Gamma(\mathcal{U}, \mathcal{E}) \times \Gamma(\mathcal{U}', \mathcal{E}') \to \Gamma(\mathcal{U} \times \mathcal{U}', \mathcal{E} \boxplus \mathcal{E}')
\]

sends products of FOP transverse sections to FOP transverse sections.
6. **(Stabilization Property)** If \( \pi_F : F \to \mathcal{U} \) is a normally complex orbifold vector bundle,\(^4\) then the stabilization map

\[
\Gamma(\mathcal{U}, \mathcal{E}) \to \Gamma(F, \pi_F^* \mathcal{E} \boxplus \pi_F^* \mathcal{F}), \quad S \to \pi_F^* S \boxplus \tau_F,
\]

where \( \tau_F : F \to \pi_F^* \mathcal{F} \) is the tautological section, sends FOP transverse sections to FOP transverse sections.
7. **(Stratified Regularity)** For each \( S \in \Gamma^{\text{FOP}}(\mathcal{U}, \mathcal{E}) \) and each normally complex isotropy type \( \gamma \), if we write \( S|_{\mathcal{U}_\gamma} \) as the direct sum \( S_\gamma \oplus S_\gamma \), then \( S_\gamma : U_\gamma \to E_\gamma \) is transverse in the usual sense.

In addition, the structure of the zero loci of FOP transverse sections can be described as follows. For each integer \( k \), there exists a countable partially ordered set \( \mathcal{Z}_k^{\text{univ}} \) (called the set of universal strata), an order-preserving map \( \rho_k : \mathcal{Z}_k^{\text{univ}} \to \mathbb{C}_k^{\text{NC}} \), and a

\(^3\)In general it is not \( C^1 \)-dense. See Remark 2.43.

\(^4\)The total space \( F \) has a canonical concordance class of normal complex structures.
strictly order-preserving map \( n_k : \mathcal{Z}_{\text{univ}}^k \to 2\mathbb{Z} \) (cf. Definition 3.6) (all independent of \( U \) and \( E \)) satisfying the following conditions.

1. For any maximal element \( \theta \in \mathcal{Z}_{\text{univ}}^k \), if \( \rho_k(\theta) \in \Gamma_{\text{NC}}^k \) is represented by a triple \((G, V, W)\), then \( n_k(\theta) = -\dim_k(G) + \dim_k(W) \).
2. There is a special maximal \( \theta_k \in \mathcal{Z}_{\text{univ}}^k \) with \( \rho_k(\theta_k) = \gamma_k^* \in \Gamma_{\text{NC}}^k \).
3. Suppose \( \dim(U) - \text{rank}(E) = k \). For \( \mathcal{S} \in \Gamma_{\text{FOP}}(U, E) \), \( S^{-1}(0) \) admits a partition

\[
S^{-1}(0) = \bigcup_{\theta \in \mathcal{Z}_{\text{univ}}^k} S^{-1}(0)_\theta
\]

satisfying the following conditions.

(a) Each \( S^{-1}(0)_\theta \) is a \( k + n_k(\theta) \)-dimensional smooth submanifold of \( U_{\rho_k(\theta)} \).
(b) After removing empty pieces, the partition (1.2) makes \( S^{-1}(0) \) a Thom–Mather stratified space (Definition C.1) whose natural partial order among strata coincides with the partial order of \( \mathcal{Z}_{\text{univ}}^k \).

Remark 1.2. The FOP transversality condition arises from the original proposal of Fukaya–Ono [FO01] and the crucial contribution of B. Parker [Par13]. Many difficult points in proving the desired properties of FOP transverse perturbations have their roots in [Par13]. Besides these works, the notion of normally polynomial perturbations introduced in [FO01] was used in [FOOO07].

Remark 1.3. Theorem 1.1 (as well as a few other technical results proved in this paper) is stated for orbifolds without boundary. In applications, one needs to consider orbifolds with boundary. In different scenarios, the corresponding extensions are addressed specifically regarding the context. However the general principle is very simple: on an orbifold with boundary, it is easy to construct collar neighborhoods of the boundary. The structures we need to construct (such as an FOP transverse section), can be first chosen on the boundary, then extended to the collar neighborhood by a trivial product. Properties such as the (Extension Property I) of Theorem 1.1 allow us to further extend to whole interior of the orbifold.

We briefly explain the spirit of the above conditions. Conditions (1)—(5) show that the FOP transversality is almost as flexible as the classical transversality on manifolds. In particular, the condition (Extension Property II) is useful when considering incidence conditions for evaluation maps into manifolds; for example, moduli spaces of pseudoholomorphic curves in a symplectic manifold satisfying constraints at marked points are suborbifolds with ordinary normal bundles inside the moduli spaces of curves without such constraints. Regarding Condition (6), any finite-dimensional reduction of an infinite-dimensional problem is only well-defined up to stabilization; this condition is stating that the transversality condition is invariant under stabilization. Condition (7) says that the FOP transversality condition implies the stratified transversality.

1.1.2. FOP pseudocycles. To define integral cycles, it is convenient to introduce the notion of derived orbifolds. They are models for regularizing various kinds of moduli spaces of pseudoholomorphic curves used in symplectic geometry.

Definition 1.4. A derived orbifold is a triple \( \mathcal{D} = (U, E, S) \) where \( U \) is an orbifold, \( E \to U \) is a vector bundle and \( S : U \to E \) is a continuous section. \( \mathcal{D} \) is said to be

\footnote{Namely, \( \theta_1 < \theta_2 \) implies \( n_k(\theta_1) < n_k(\theta_2) \).}
compact if $S^{-1}(0)$ is compact; is normally complex if both $U$ and $E$ are normally complex; is oriented if the virtual vector bundle $TU - E$ is oriented. The virtual dimension of $D$ is defined as $\dim(U) - \operatorname{rank}(E)$.

Given a compact, oriented, and normally complex derived orbifold $D = (U, E, S_0)$ of virtual dimension $k$, consider an FOP transverse section $S \in \Gamma_{\text{FOP}}(U, E)$ which is $C^0$-close to $S_0$. Assuming compactness and orientation, by Theorem 1.1, $S^{-1}(0)$ is a compact Thom–Mather stratified space with oriented strata. Moreover, as the dimensions of any pair of adjacent strata differ by an even number, the closure $\overline{S^{-1}(0)}_\theta$ is an oriented pseudomanifold (Definition C.6), hence carries a fundamental class

$$[S^{-1}(0)_\theta] \in H_{k+n_k(\theta)}(\overline{S^{-1}(0)}_\theta; \mathbb{Z}) \rightarrow H_k(\overline{S^{-1}(0)}_\theta; \mathbb{Z}).$$

We call this the $\theta$-th FOP Euler cycle of the derived orbifold $D$. As a theorem we prove that it is independent of the choice of the FOP transverse perturbations.

**Theorem 1.5.** Given a compact, oriented, and normally complex derived orbifold $D = (U, E, S_0)$ of virtual dimension $k$, for any $C^0$-small FOP transverse perturbation $S$ of $S_0$, for any universal stratum $\theta \in \mathcal{Z}_k^{\text{univ}}$, the integral class $[S^{-1}(0)_\theta]$ is independent of the choice of $S$.

**Remark 1.6.** The reader would be curious about the relation between the rational-valued Euler cycles and the various FOP Euler cycles as it would imply a decomposition of the Gromov–Witten invariants into enumerative invariants in different isotropy types. It seems plausible to connect them by further perturbing an FOP transverse section via multivalued transverse sections. This will be discussed in forthcoming work.

![Figure 1. Smooth perturbations and FOP transverse perturbations.](image_url)

**1.2. Intuitions about the FOP transversality.** We would like to provide an intuitive understanding of the FOP transversality condition. Let $E \rightarrow U$ be an orbifold vector bundle. As we have mentioned, the best transversality condition one can hope for a generic smooth section $S : U \rightarrow E$ is the stratified transversality. As a result, the zero locus is the union of smooth manifolds contained in each stratum of the isotropy stratification. However, in the normal direction of each
stratum, the transversality is typically lost. As a result, we do not know how the frontier of the zero locus in a higher stratum behaves when approaching to a lower stratum, which could be a priori pathological (see the left picture of Figure 1).

One idea to regularize the frontier of a stratum of the zero locus is to consider perturbations in a category more restricted than the smooth category. Fukaya–Ono’s suggestion in [FO01] essentially says that one can require the section to be algebraic in normal directions of each stratum. Then the frontier of the zero locus of each stratum looks like a smooth family of algebraic varieties (see the right picture of Figure 1). Although the ordinary transversality still fails, this type of singular space can be characterized using languages of Whitney stratified space or Thom–Mather stratified space.

Using the normally polynomial type section, the zero locus then becomes a Thom–Mather stratified space. In order to support a fundamental class rather than a chain, one needs to guarantee that adjacent strata differ in dimensions at least by 2. The assumption of the normal complex structure makes this work, as the frontier of each stratum looks like a family of complex algebraic varieties.

1.3. Applications in symplectic topology.

1.3.1. Integral Gromov–Witten type invariants. Let \((X, \omega)\) be a compact symplectic manifold. Let \(A \in H_2(M; \mathbb{Z})\) be a homology class. Let \(g, n \geq 0\) be nonnegative integers. Upon choosing a compatible almost complex structure \(J\) on \(X\), one obtains the moduli space \(\overline{M}_{g,n}(X, J, A)\) of \(J\)-holomorphic stable maps of genus \(g\), degree \(A\), with \(n\) marked points. Let \(k\) be its virtual dimension. The following theorem is proved in Section 6.

**Theorem 1.7.** For each element \(\theta \in \mathcal{Z}_k^{\text{univ}}\), there is a symplectic deformation invariant

\[
\left[\overline{M}_{g,n}(X, J, A)\right]_\theta^{\text{vir}} \in H_{k+n_\theta}(\overline{M}_{g,n} \times X^n; \mathbb{Z}).
\]  

(1.3)

When \(\theta\) is the maximal element \(\theta_k\) provided in Theorem 1.1, this class “morally” counts \(J\)-holomorphic curves in class \(A\) whose automorphism group is trivial. More precisely, when all elements of \(\overline{M}_{g,n}(X, J, A)\) have trivial automorphism group (hence the ordinary virtual fundamental class is integral), the above virtual class for \(\theta_k\) coincides with the ordinary virtual class.

**Remark 1.8.** When \((X, \omega)\) is semi-positive, \(g = 0\), and \(n \geq 3\), the numerical invariants (correlation function) defined via the class (1.3) corresponding to the maximal element \(\theta_k\) should coincide with the usual genus zero Gromov–Witten invariant of \((X, \omega)\) defined by Ruan [Rua96] and Ruan–Tian [RT95, RT97] (see also [MS04]). A proof of this claim will be provided in forthcoming work.

**Remark 1.9.** One may expect the integral virtual class \(\left[\overline{M}_{g,n}(X, J, A)\right]_\theta^{\text{vir}}\) to define a cohomological field theory satisfying the Kontsevich–Manin axioms. However, there are a few notable exceptions. First, as the FOP perturbation scheme is sensitive to the symmetry group, the \(S_n\)-invariance axiom (with respect to the permutation of marked points) may not be true. Second, the axioms related to forgetting marked points (the fundamental class axiom and the divisor axiom) may not hold; this is because when forgetting a marked point, the isotropy types of curves may change and it is difficult to find FOP transverse perturbations which respect forgetting marked points. Third, the splitting axiom must be modified. Indeed, the usual splitting axiom of Gromov–Witten invariants is stated using the
classical Poincaré duality which is not quantum deformed. However, this is not a universal truth; for example, the splitting axiom of quantum K-theory (see [Lee04] or the theories recently constructed by Abouzaid–McLean–Smith [AMS23] both use quantum deformed Poincaré duality. For the FOP invariants, one also needs to deform the classical Poincaré duality in order to have a splitting property. The discussion of Kontsevich–Manin axioms for the FOP Gromov–Witten invariants will be given in forthcoming work.

1.3.2. **Homological splitting for Hamiltonian loops.** The second main result of this paper is an alternate proof of the main theorem of [AMS21] which is conceptually much simpler than the original one (see Section 7).

**Theorem 1.10.** (cf. [AMS21, Theorem 1.1]) Let \( P \to S^2 \) be a smooth Hamiltonian fibration over the 2-sphere with fiber being a closed symplectic manifold \((X, \omega)\). Then there is an isomorphism of graded abelian groups

\[
H^*(P; \mathbb{Z}) \cong H^*(X; \mathbb{Z}) \otimes_{\mathbb{Z}} H^*(S^2; \mathbb{Z}).
\]  

(1.4)

**Remark 1.11.** Although [AMS21] shows that the splitting (1.4) actually holds for any complex oriented generalized cohomology theory, for the case over \( \mathbb{Z} \), our short proof uses only the virtual cycle underlying the class (1.3) indexed by the trivial group without appealing to Morava K-theories or Atiyah duality for orbifolds.

1.4. **Relations with other work.**

1.4.1. **Z-valued enumerative invariants in symplectic topology.** There are other known \( \mathbb{Z} \)-valued Gromov–Witten type invariants in symplectic topology. Firstly, as mentioned above, for semi-positive symplectic manifolds, Ruan–Tian’s construction ([RT95]) shows that the genus-0 Gromov–Witten invariants with at least three insertions are indeed \( \mathbb{Z} \)-valued. Secondly, there are the celebrated \( \mathbb{Z} \)-valued Gopakumar–Vafa invariants of Calabi–Yau 3-folds [GV98a], [GV98b] which govern the \( \mathbb{Q} \)-valued Gromov–Witten invariants as demonstrated in full generality by Ionel–Parker [IP18]. It was speculated by some experts (e.g. [Joy07, Section 6.3]) that the invariants defined using Fukaya–Ono’s normally polynomial sections are related to the Gopakumar–Vafa invariants. However, recent developments [DW23], [BS24], which should be viewed as continuations of the groundbreaking work of Taubes in dimension 4 [Tau96], construct \( \mathbb{Z} \)-valued invariants of Calabi–Yau 3-folds by (virtually) counting *embedded* pseudo-holomorphic curves and these invariants seems to be better connected with the Gopakumar–Vafa invariants. It is an intriguing question to understand the relation between these two seemingly distinct proposals for geometrically constructing the Gopakumar–Vafa invariants. Lastly, the \( K \)-theoretic Gromov–Witten invariants, defined by Givental and Lee [Lee04] for algebraic objects (which are expected to exist for general symplectic manifolds in light of [AMS21, Section 6.12]), are also \( \mathbb{Z} \)-valued.

1.4.2. **Z-valued invariants in algebraic geometry.** There are several types of integral enumerative invariants constructed using algebraic geometry, most notably the Donaldson–Thomas invariants [Tho00] and the Pandharipande–Thomas invariants [PT09]. Although these invariants are closely tied with the Gromov–Witten invariants [MNOP06a, MNOP06b], their constructions depend on sheaf theory. Fukaya–Ono’s proposal is differential-topological in nature, as it is still a variant of the general position argument. It would be interesting to see if the virtual classes \([\mathcal{M}_{0,n}(X, J, A)]_{\theta}^{vir}\) admit a purely algebro-geometric interpretation.
1.4.3. **Wasserman’s theorem and stable homotopy theory.** A renowned theorem of Wasserman [Was69], recaptured in [Par23, Theorem 5.6], provides a sufficient condition for equivariant transversality to hold. Our result can be interpreted as a variant of Wasserman’s theorem given the presence of normal complex structures, whose existence allows us to extract more information. The homotopical cobordism perspective gives far-reaching corollaries of Wasserman’s theorem, see [Sch18, Theorem 6.2.33]. As mentioned by [Par23, Remark 5.7], it is an interesting question to understand Fukaya–Ono’s proposal on the homotopical cobordism side.

**Addendum 1.12.** After the first version of this paper was posted on arXiv, the results and methods herein have been used for several applications. As mentioned above, the integral counting scheme is a crucial tool for our proof of the integral Arnold conjecture [BX22]; we also obtained cohomological splitting result for Hamiltonian fibrations over general bases in [BPX24] using our integral Gromov–Witten type invariants. More recently, the FOP perturbation method is combined with resolution of singularities to construct complex-cobordism-valued Gromov–Witten invariants [AB24]. In forthcoming work [BX25], we will also derive applications to Hamiltonian dynamics based on constructing quantum power operations on general symplectic manifolds.

1.5. **Plan of the paper.** We start with basic notions related to effective orbifolds in Section 2, especially the notion of *normal complex structure* (Definition 2.17) and *normally complex sections* (Definition 2.49). In Section 3, we study the canonical Whitney stratification on the universal zero locus $Z^{d}(G, V, W)$ and understand its behavior under various operations. This is the technical core of this paper. The definition of the FOP transversality condition and the proof of Theorem 1.1 is given in Section 4. In Section 5, we provide the homological consequence of the FOP transversality condition. In Section 6 and Section 7 we provide our geometric applications and prove Theorem 1.7 and Theorem 1.10.

In Appendix A we provide the proof of technical results about straightenings on orbifolds. In Appendix B we provide technical details on Whitney stratifications supporting the proofs in Section 3. In Appendix C we prove that zero loci of FOP transverse sections are Thom–Mather stratified spaces, which is needed for constructing the integral Euler cycles.

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2. **Normally Complex Orbifolds and Bundles**

In this section we recall basic notions about effective orbifolds and the prerequisite structures for defining the FOP transversality condition. In Subsection 2.1 we recall notions and facts about effective orbifolds and orbifold vector bundles. In
Subsection 2.2 we specify the stratification of orbifolds (together with a vector bundle) induced by the isotropy types of points. In Subsection 2.3 we define the notion of normal complex structures (which comes from the original proposal of Fukaya–Ono [FO01]). In Subsection 2.4 we describe the auxiliary and technical structures called “straightening” which can be viewed as a system of normal tubular neighborhoods for the strata of the isotropy stratification. The technical construction of straightenings is deferred to Appendix A. In Section 2.5 we define the notion of normally complex sections which have more rigid behavior in normal directions to each stratum than general smooth sections.

2.1. Orbifolds and orbifold vector bundles. We recall the basic definition of effective orbifolds. We follow the definition of [ALR07, Section 1.1]. Working with effective orbifolds allows us to use orbifold charts exclusively without appealing to the language of groupoids.

Let $U$ be a Hausdorff and second countable topological space. An $n$-dimensional orbifold chart of $U$ is a triple $C = (G, U, \psi)$ where $U$ is a nonempty smooth manifold, $G$ is a finite group acting effectively and smoothly on $U$, and $\psi : U \to U$ is a $G$-invariant continuous map such that the induced map

$$\bar{\psi} : U/G \to U,$$

is a homeomorphism onto an open subset of $U$. If $p \in \psi(U)$ we also say that $p$ is contained in the chart $C$. If $\psi^{-1}(p) \subseteq U_G$, the fixed point locus of the $G$-action, then we say that the chart $C$ is centered at $p$.

A chart embedding from another chart $C' = (G', U', \psi')$ to $C$ is a smooth open embedding $\iota : U' \to U$ such that

$$\psi \circ \iota = \psi'.$$

It follows that (see [ALR07, Page 3]) given a chart embedding $\iota$ as above there exists a canonical group injection $G' \hookrightarrow G$ such that $\iota$ is equivariant. Therefore we often include the group injection as part of the data of a chart embedding.

We say two charts $C_i = (G_i, U_i, \psi_i), i = 1, 2$ are compatible if for each $p \in \psi_1(U_1) \cap \psi_2(U_2)$, there exists an orbifold chart $C_p = (G_p, U_p, \psi_p)$ containing $p$ and chart embeddings into both $C_1$ and $C_2$.

An orbifold atlas $\mathcal{A} = \{ C_i \mid i \in I \}$ on $X$ is a collection of mutually compatible charts $C_i$ which cover $U$. We say an atlas $\mathcal{A}' = \{ C'_j \mid j \in J \}$ is a refinement of $\mathcal{A}$ if for each $C'_j$ there exists a chart embedding $C'_j \hookrightarrow C_i$ for some $i \in I$. We say two orbifold atlases are equivalent if they have a common refinement. A Hausdorff and second countable topological space $U$ together with an equivalence class of orbifold atlases is called a smooth effective orbifold. Every smooth effective orbifold has a unique maximal atlas; two atlases are equivalent if they are contained in the common maximal atlas (see [ALR07, Page 3]). It is convenient to work with the maximal atlas. From now on, an orbifold chart of a smooth effective orbifold means a chart in the maximal atlas.

We often use $|U|$ to denote the underlying topological space (called the coarse space) of an effective orbifold $U$ while forgetting the orbifold structure.

We do not define the general form of orbifold morphisms. Below are a few special cases of morphisms between orbifolds. First, a continuous function on an effective orbifold (or more generally a continuous map into a smooth manifold) is
smooth if its pullback to each chart is a smooth function. On the other hand, an isomorphism of orbifolds from \( U \) to \( U' \) is a homeomorphism \( f : |U| \to |U'| \) such that the correspondence

\[(G, U, \psi) \mapsto (G, U, f \circ \psi)\]

is a one-to-one correspondence between their maximal atlases. An open embedding from \( U \) to \( U' \) is an isomorphism from \( U \) to an open subset of \( U' \).

Remark 2.1. One can see that orbifolds are all locally compact. As we also assume they are Hausdorff and second countable, they are paracompact spaces. Hence for any open cover by charts, there exists a subordinate continuous partition of unity; as one can approximate continuous functions by smooth functions on each chart, there always exist a subordinate smooth partition of unity.

Remark 2.2. We also need the notion of orbifolds with boundary. In that case, the domain of a chart \( C = (G, U, \psi) \) is allowed to be a smooth manifold with boundary and the group \( G \) acts by diffeomorphisms of \( (U, \partial U) \). As a consequence, the fixed point locus \( U_G \) is a manifold with boundary \( \partial U_G = U_G \cap \partial U \).

The definition of orbifold vector bundles is very similar to that of effective orbifolds. Let \( U \) be an effective orbifold, \( E \) be a topological space, and \( \pi_E : E \to U \) be a continuous map. A bundle chart of \( \pi_E : E \to U \) consists of an orbifold chart \( C = (G, U, \psi) \) of \( U \), a \( G \)-equivariant smooth vector bundle \( \pi_E : E \to U \), and a \( G \)-invariant continuous map \( \hat{\psi} : E \to E \) such that the induced map from \( E \) to \( \pi^{-1}_E(\psi(U)) \) and such that the following diagram commutes:

\[
\begin{array}{ccc}
E & \xrightarrow{\hat{\psi}} & E \\
\downarrow{\pi_E} & & \downarrow{\pi_E} \\
U & \xrightarrow{\psi} & U
\end{array}
\]

In notation we will use a quadruple \( \hat{C} = (G, U, E, \hat{\psi}) \) to denote the bundle chart where the map \( \hat{\psi} : U \to \hat{U} \) is determined by the map \( \hat{\psi} : E \to \hat{E} \). If \( \hat{C}' = (G', U', E', \hat{\psi}') \) is another bundle chart, a bundle chart embedding from \( \hat{C}' \) to \( \hat{C} \) consists of an orbifold chart embedding \( i : U' \hookrightarrow U \) (equivariant with respect to a group injection \( G' \hookrightarrow G \)) covered by a vector bundle embedding \( \hat{i} : E' \to E \) such that

\[\hat{\psi} \circ i = \hat{\psi}'.\]

We can similarly define the notions of compatibility between bundle charts and bundle atlases. Then an orbifold vector bundle structure over \( \pi_E : E \to U \) is defined to be an equivalence class of bundle atlases as before. Similarly, an orbifold vector bundle has a unique maximal atlas and two atlases are equivalent if and only if they are contained in a common maximal atlas. A bundle chart then means a chart in the maximal atlas.

Definition 2.3. An orbifold vector bundle \( E \to U \) is said to be ordinary if for each bundle chart \( \hat{C} = (G, U, E, \hat{\psi}) \) and any \( x \in U \), \( G_x \) acts trivially on the fiber \( E_x \).

We do not attempt to deal with general suborbifolds in this paper. Instead we consider the following very restricted type of suborbifolds.
**Definition 2.4.** Let $\mathcal{U}$ be an effective orbifold. A closed suborbifold with ordinary normal bundle of $\mathcal{U}$ is a closed subset $\mathcal{X} \subseteq |\mathcal{U}|$ such that for each chart $C = (G, U, \psi)$ of $\mathcal{U}$, $\psi^{-1}(\mathcal{X})$ is a closed submanifold of $U$ such that for each $x \in \psi^{-1}(\mathcal{X})$, $G_x$ acts trivially on the fiber of the normal bundle at $0$. In particular, $(G, \psi^{-1}(\mathcal{X}), \psi|_{\psi^{-1}(\mathcal{X})})$ is an orbifold chart of $\mathcal{X}$ and all such charts make $\mathcal{X}$ itself an effective orbifold.

We spell out the definition of sections of an orbifold vector bundle because of their importance in this paper.

**Definition 2.5** (Sections). Let $\mathcal{E} \to \mathcal{U}$ be an orbifold vector bundle.

1. Let $\hat{\mathcal{C}}_i = (G_i, U_i, E_i, \hat{\psi}_i)$, $i = 1, 2$ be two bundle charts. We say that a $G_1$-equivariant section $S_1 : U_1 \to E_1$ and a $G_2$-equivariant section $S_2 : U_2 \to E_2$ are compatible if for any bundle chart $\hat{\mathcal{C}}_0 = (G_0, U_0, E_0, \hat{\psi}_0)$ of $E$ and chart embeddings $i_1 : \hat{\mathcal{C}}_0 \hookrightarrow \hat{\mathcal{C}}_1$, $i_2 : \hat{\mathcal{C}}_0 \hookrightarrow \hat{\mathcal{C}}_2$ there holds
   \[
i^{-1}_1 \circ S_1 \circ i_1 = i^{-1}_2 \circ S_2 \circ i_2 \]
   as sections of $E_0 \to U_0$.

2. A section of $\mathcal{E}$, denoted by $\mathcal{S} : \mathcal{U} \to \mathcal{E}$, is a collection of mutually compatible $G_i$-equivariant sections $S_i : U_i \to E_i$ for all bundle charts $\hat{\mathcal{C}}_i$ belonging to the maximal atlas of $\mathcal{E}$.

On each single chart $\hat{\mathcal{C}} = (G, U, E, \hat{\psi})$, there are a lot of $G$-equivariant sections $S : U \to E$. The existence of partitions of unity implies that any orbifold vector bundle over an effective orbifold has a lot of smooth sections.

As a final topic of this subsection, we recall the definition of Riemannian metrics on orbifolds and connections on orbifold vector bundles.

**Definition 2.6.** Let $\mathcal{U}$ be an effective orbifold (possibly with boundary).

1. A Riemannian metric on $\mathcal{U}$, denoted by $g^{TU}$, is a collection $g^{TU}$ for all orbifold charts $\mathcal{C} = (G, U, \psi)$ where $g^{TU}$ is a $G$-invariant Riemannian metric on $U$, such that every chart embedding is isometric.

2. Let $\mathcal{F} \to \mathcal{U}$ be an orbifold vector bundle. A connection resp. inner product on $\mathcal{F}$, denoted by $\nabla^F$ resp. $h^F$, is a collection $\nabla^F$ resp. $h^F$ for all bundle charts $\hat{\mathcal{C}} = (G, U, F, \hat{\psi})$ where $\nabla^F$ resp. $h^F$ is a $G$-invariant connection resp. inner product on $F$ such that all bundle chart embeddings preserve the connections resp. inner products. We say that $\nabla^F$ preserves $h^F$ if $\nabla^F$ preserves $h^F$ on each chart.

One can use the standard way to construct Riemannian metrics, connections, and inner products using smooth partition of unity on orbifolds (see Remark 2.1).

### 2.2. The isotropy stratification.

The isomorphism class of the stabilizer of a point $x$ in an orbifold chart $\mathcal{C} = (G, U, \psi)$ only depends on the point $\psi(x) \in U$ (see [ALR07, Page 3]). One can use this isomorphism class to stratify the orbifold. Moreover, together with the data from a vector bundle, we can actually obtain a more refined decomposition.

We first spell out certain frequently used notions.

**Definition 2.7.** Let $G$ be a group acting on a set $A$. A subgroup $H \subseteq G$ is called $A$-essential, denoted by $H \subseteq_A G$, if there exists $a \in A$ with isotropy group $H$. 

Notice that $G \subseteq A$ if and only if the set of fixed points $A_G \neq \emptyset$; \{e\} $\subseteq A$ if and only if the action is effective. The set of $A$-essential subgroups is closed under conjugations. If $(G, U, \psi) \rightarrow (G', U', \psi')$ is a chart embedding of an orbifold, then the map $G \rightarrow G'$ sends $U$-essential subgroups of $G$ to $U'$-essential subgroups of $G'$.

**Definition 2.8.** (Isotropy triples and isotropy types)

1. An isotropy triple is a triple $(G, V, W)$ where $G$ is a finite group and $V, W$ are finite-dimensional real representations of $G$.
2. An isomorphism of isotropy triples from $(G, V, W)$ to $(G', V', W')$ consists of a group isomorphism $G \cong G'$ and equivariant linear isomorphisms $V \cong V'$ and $W \cong W'$. An isotropy type is an isomorphism class of isotropy triples, typically denoted by $\gamma, \delta$, etc. Let $\Gamma$ be the set of all isotropy types.
3. A partial order on $\Gamma$ is defined as follows. We write $\gamma \leq \delta$ if $\gamma$ can be represented by $(G, V, W)$, $\delta$ can be represented by $(H, V, W)$ such that $H$ is isomorphic to a $V$-essential subgroup of $G$ and such that $X \cong V, Y \cong W$ as representations of $H$.

**Definition 2.9.** Suppose $W$ is a finite-dimensional real representation of a finite group $G$. Then $W$ can be decomposed as the direct sum of irreducible representations. We call the (canonical) decomposition

$$W = W_G \oplus \tilde{W}_G$$

where $W_G$ is the direct sum of all trivial summands and $\tilde{W}_G$ is the direct sum of all nontrivial summands the basic decomposition of $W$ with respect to $G$. Notice that when $W$ is a complex representation, the basic decomposition is an isomorphism of complex vector spaces.

More generally, if $U$ is a $G$-manifold and $E \rightarrow U$ is a $G$-equivariant vector bundle, then over the $G$-fixed points $U_G \subseteq U$, the fiberwise basic decomposition of $E|_{U_G}$ induces a decomposition of vector bundles (possibly with varying ranks on different connected components)

$$E = E_G \oplus \tilde{E}_G$$

where $E_G \subseteq E|_{U_G}$ coincides with the set of $G$-fixed points of $E$.

Now consider an effective orbifold $U$ with an orbifold vector bundle $E \rightarrow U$. For each $p \in U$, consider a bundle chart $C = (U, G, E, \tilde{\psi})$ containing $p$ with $x \in \psi^{-1}(p) \subseteq U$. Define $\gamma_p$ to be the isotropy type represented by the triple $(G_x, T_x U, E_x)$. Then $\gamma_p$ is well-defined, independent of the choice of charts and the point $p \in \psi^{-1}(x)$.

**Lemma 2.10.** For each isotropy type $\gamma$, the subset

$$U_\gamma := \{ p \in U \mid \gamma_p = \gamma \}$$

is a topological manifold (possibly with boundary if $U$ has boundary) equipped with a natural smooth structure.\(^6\)

**Proof.** By definition, for each $p \in U_\gamma$, there exists a bundle chart $(G, U, E, \tilde{\psi})$ and $x \in \psi^{-1}(p) \subseteq U$ such that $G = G_x$ and $\gamma$ is represented by $(G, T_x U, E_x)$. As $U_G \subseteq U$ is a closed submanifold, it implies that $U_\gamma$ is a topological manifold and $\psi^{-1}: U_\gamma \cap \psi(U) \rightarrow U_G \subseteq U$ is a local chart.

\(^6\)It is more “correct” to view $U_\gamma$ as an non-effective orbifold. However we refrain from discussing non-effective cases in this paper.
To put a smooth structure on $U_\gamma$ one needs to choose an atlas of $C^\infty$-compatible charts. Let \( \hat{A}_\gamma \) be the set of bundle charts \( \hat{C} = (G, U, E, \hat{\psi}) \) of $E$ such that for each \( x \in U_G \subseteq U \), the triple \( (G, T_xU, E_x, \hat{\psi}_x) \) represents $\gamma$ (so that $\psi(U) \cap U_\gamma \neq \emptyset$). Define the atlas
\[
\{ \psi^{-1} : \mathcal{U} \cap \psi(U) \to U_G \mid \hat{C} = (G, U, E, \hat{\psi}) \in \hat{A}_\gamma \},
\]
(2.1)
We claim that this atlas is $C^\infty$-compatible, namely, if \( \hat{C}_i = (G_i, U_i, E_i, \hat{\psi}_i), i = 1, 2 \), are two such bundle charts of $E$, then the map
\[
\psi_2^{-1} \circ \psi_1 : \psi_1^{-1}(\mathcal{U} \cap \psi_1(U_1) \cap \psi_2(U_2)) \to U_{2,G_2}
\]
is smooth. Indeed, the smoothness at each point $q$ of the domain of this transition map can be checked by using a smaller bundle chart containing $q$ which embeds into both $\hat{C}_1$ and $\hat{C}_2$. Therefore, $\mathcal{U}_\gamma$ is equipped with a smooth structure. \( \square \)

Now we recall the definition of the basic concept of stratification, which is frequently used in this paper.

**Definition 2.11.** Let $X$ be a topological space.

1. A **stratification** of $X$ is a set $\mathcal{X}$ of subsets of $X$ satisfying
   (a) $\mathcal{X}$ is equal to the disjoint union of all elements $X_\alpha \in \mathcal{X}$.
   (b) Each $X_\alpha \in \mathcal{X}$ (called a **stratum**) is locally closed and nonempty.
   (c) $\mathcal{X}$ is locally finite.
   (d) (Axiom of frontier) If $X_\alpha \cap X_\beta \neq \emptyset$, then $X_\alpha \subseteq \overline{X_\beta}$.

2. Notice that the last condition induces a partial order relation among strata:
   \[ X_\alpha \leq X_\beta \iff X_\alpha \subseteq \overline{X_\beta}. \]
   A **top stratum** of a stratification $\mathcal{X}$ is a stratum $X_\alpha$ which is maximal with respect to the above partial order, or equivalently, an open stratum.

3. A homeomorphism $f : X \to X$ is said to **preserve** a stratification $\mathcal{X} = \{ X_\alpha \}$ on $X$ if $\mathcal{X} = \{ f^{-1}(X_\alpha) \}$ as sets.

4. Let $X$ resp. $Y$ be equipped with a stratification $\mathcal{X} = \{ X_\alpha \}$ resp. $\mathcal{Y} = \{ Y_\beta \}$.
   The **product stratification** on $X \times Y$ is the set
   \[
   \mathcal{X} \times \mathcal{Y} = \{ X_\alpha \times Y_\beta \mid X_\alpha \in \mathcal{X}, Y_\beta \in \mathcal{Y} \}.
   \]

The following lemma follows from inspecting the definition.

**Lemma 2.12.** The set $\{ \mathcal{U}_\gamma \neq \emptyset \mid \gamma \in \Gamma \}$ is a stratification of $\mathcal{U}$, called the isotropy stratification of $\mathcal{U}$. Moreover, the partial order induced from the stratification coincides with the restriction of the partial order in $\Gamma$. \( \square \)

2.3. **Normal complex structures.** Now we introduce the most important geometric condition which plays the central role in our construction. In applications, normal complex structures appear naturally as the Cauchy–Riemann operator has a complex linear principal symbol.

In this paper, we frequently abbreviate the phrase “normal(ly) complex” by NC.

2.3.1. **Linear algebra.** We first discuss the related linear algebra.

**Definition 2.13.** A **normally complex representation** of a group $G$ is a real representation $V$ together with a $G$-invariant complex structure $I^G$ on $V^G \subseteq V$.

The following lemma is obvious.
Lemma 2.14. Let $V$ be an NC representation of $G$.

1. If $H \subseteq G$ is a subgroup, then $V$ is naturally an NC representation of $H$.
2. Any $G$-invariant subspace $V' \subseteq V$ is naturally an NC representation of $G$.

Definition 2.15. (Normally complex isotropy types)

1. A normal complex isotropy triple (NC triple for short) consists of an isotropy triple $(G,V,W)$ where $V,W$ are NC representations of $G$.
2. An isomorphism of NC triples from $(G',V',W')$ to $(G,V,W)$ is an isomorphism of isotropy triples such that the induced isomorphisms $\bar{V}' \cong \bar{V}$ and $\bar{W}' \cong \bar{W}$ are complex linear. An NC isotropy type is an isomorphism class of NC triples. Let $\mathcal{N}^\text{NC}$ denote the set of NC isotropy types.
3. A partial order on $\mathcal{N}^\text{NC}$ is defined as follows. We write $\gamma \leq \delta$ if $\gamma$ can be represented by $(G,V,W)$, $\delta$ can be represented by $(H,V,W)$ such that $H$ is isomorphic to a $V$-essential subgroup of $G$ and such that $X \cong V,Y \cong W$ as NC representations of $H$.

2.3.2. NC structure on charts and orbifolds.

Definition 2.16. Let $G$ be a finite group and $U$ be a $G$-manifold. When $U$ has boundary, this implies that $\partial U$ is $G$-invariant.

1. Let $E \to U$ be a $G$-equivariant vector bundle. A normal complex structure (NC structure for short) on $E$, denoted by $I^E$, consists, for each $U$-essential subgroup $H \subseteq G$, an $H$-invariant complex structure $I^E_H$ on the bundle $\hat{E}_H \to U_H$ satisfying the following conditions.
   - For each pair of $U$-essential subgroups $H \subseteq K$, one has the $H$-equivariant and $I^E_K$-linear decomposition
     $$\hat{E}_K = (E_H \cap \hat{E}_K) \oplus \hat{E}_H|_{U_K}.$$
     We require that the restriction of $I^E_K$ on the second summand $\hat{E}_H|_{U_K}$ coincides with $I^E_H$ restricted to $\hat{U}_K$.
   - For each $g \in G$ and $H' = gHg^{-1}$, the bundle isomorphism $\hat{E}_H \to \hat{E}_{H'}$ induced by $g$ is complex linear.
2. An NC structure on $U$ is an NC structure on the tangent bundle $TU \to U$.

The extension to orbifolds (with boundary) is straightforward.

Definition 2.17. (1) An NC structure on an orbifold vector bundle $E \to U$, denoted by $I^E$, consists, for each chart $\hat{C} = (G,U,E,\bar{\psi})$ of $E$, an NC structure $I^E = (I^E_U)$ on $E$ satisfying the following conditions. For each chart embedding from $\hat{C}' = (G',U',E',\bar{\psi}')$ to $\hat{C} = (G,U,E,\bar{\psi})$ given by a group injection $G' \to G$, an equivariant open embedding $i : U' \to U$ covered by an equivariant bundle isomorphism $\hat{i} : E' \to E$, for any $U'$-essential subgroup $H' \subseteq G'$ mapped onto $H \subseteq G$, $i$ maps $E'|_{U'_{H'}}$ into $E|_{U_H}$, we require that the induced bundle isomorphism
   $$\hat{E}'_{H'}|_{U'_{H'}} \to \hat{E}_H|_{U_H}$$
   is complex linear with respect to the complex structures $I^E_{H'}$ and $I^E_H$.
(2) An NC structure on an orbifold $U$ is an NC structure on $TU$. 

(3) An NC vector bundle is an orbifold vector bundle $\mathcal{E}$ equipped with an NC structure $\mathcal{F}$. An NC orbifold is an orbifold $\mathcal{U}$ equipped with an NC structure $\mathcal{T}^{\mathcal{U}}$. An NC pair is a pair $(\mathcal{U}, \mathcal{E})$ where $\mathcal{U}$ is an NC orbifold and $\mathcal{E}$ is an NC vector bundle over $\mathcal{U}$.

For example, if a vector bundle $\mathcal{E}$ is equipped with a complex structure $I^\mathcal{E}$, then there is a naturally induced NC structure $\mathcal{T}^\mathcal{E}$ on $\mathcal{E}$.

Notice that when $(\mathcal{U}, \mathcal{E})$ is an NC pair, the associated isotropy stratification of $\mathcal{U}$ can be indexed by NC isotropy types.

2.3.3. Induced NC structures on suborbifolds with ordinary normal bundle. Let $\mathcal{U}$ be an effective orbifold equipped with an NC structure $\mathcal{T}^{\mathcal{U}}$. Let $X \subseteq \mathcal{U}$ be a suborbifold with ordinary normal bundle (Definition 2.4). We claim that there is a naturally induced NC structure $\mathcal{T}^X$. Indeed, given any local chart $C = (G, U, \psi)$, $X := \psi^{-1}(X) \subseteq U$ is a $G$-invariant submanifold. For each $U$-essential subgroup $H \subseteq U \ G$, the normal bundle of $X$ restricted to $X_H$ is the same as the normal bundle of $X_H$ inside $U_H$. Hence

$$NX_H = NU_H|_{X_H}.$$ 

Therefore, $I^{NX_H} := I^{NU_H}|_{X_H}$ defines an NC structure on $X$.

On the other hand, if $E \rightarrow U$ is a vector bundle equipped with an NC structure $\mathcal{T}^E$, then the restriction $E|_X$ carries an induced NC structure.

Remark 2.18. When $\mathcal{U}$ is an orbifold with boundary, the normal bundle to the boundary $\partial \mathcal{U}$ is an ordinary vector bundle. One can see that if $\mathcal{U}$ has an NC structure, then $\partial \mathcal{U}$ has an induced NC structure.

2.3.4. Nearby and concordant NC structures. In later constructions, one often needs to modify the given NC structures. To connect nearby NC structures, one needs to define the closeness properly. First recall some basic linear algebra. Let $V$ be an $2n$-dimensional real vector space. Then a complex structure $J$ on $V$ is equivalent to a $n$-dimensional complex subspace $R \subseteq V \otimes \mathbb{C}$ such that $R \oplus \overline{R} = V \otimes \mathbb{C}$. If $V$ has a linear $G$-action and $J$ is $G$-invariant, then the associated spaces $R$ and $\overline{R}$ are $G$-invariant complex subspaces of $V \otimes \mathbb{C}$. A nearby complex structure $J'$ on $V$ corresponds to another subspace $R' \subseteq V \otimes \mathbb{C}$ which is close to $R$. Then $R' = \text{graph}(\rho)$ for some linear map $\rho : R \rightarrow \overline{R}$ which is close to zero. If $R'$ is also $G$-invariant, then $\rho$ is $G$-equivariant.

We want to characterize the closeness without referring to any extra structure. Let $\mathcal{F}(V)$ be the space of complex structures on $V$.

Definition 2.19. Given $J \in \mathcal{F}(V)$ corresponding to a subspace $R \subseteq V \otimes \mathbb{C}$, we say that $J' \in \mathcal{F}(V)$ is close to $J$ if the corresponding space $R' \subseteq V \otimes \mathbb{C}$ coincides with $\text{graph}(\rho)$ for a linear map $\rho : R \rightarrow \overline{R}$ such that for all $t \in [0, 1]$ and $R_t := \text{graph}(t\rho)$, there holds $R_t \cap \overline{R_t} = \{0\}$; in other words, $R_t$ is associated to a complex structure $J_t \in \mathcal{F}(V)$. We call $J_t$ the $t$-interpolation from $J$ to $J'$.

Notice that in the above definition, if both $J$ and $J'$ are invariant under a linear $G$-action on $V$, then $J_t$ is $G$-invariant. In addition, if $H \subseteq G$ is a $V$-essential subgroup, then one has the basic decompositions

$$R = R_H \oplus \tilde{R}_H,$$

$$\overline{R} = \overline{R}_H \oplus \tilde{\overline{R}}_H.$$
The $H$-equivariant linear map $\rho : R \to \overline{R}$ is block-diagonal with respect to this decomposition with diagonal blocks $\rho_H$ and $\hat{\rho}_H$. Then the restriction of $J'$ to $\hat{V}_H$ is close to $J|_{\hat{V}_H}$ and the restriction $J|_{\hat{V}_H}$ is the $t$-interpolation from $J|_{\hat{V}_H}$ to $J'|_{\hat{V}_H}$.

**Definition 2.20.** Suppose $\mathcal{E} \to \mathcal{U}$ is equipped with an NC structure $\mathcal{I}$. We say that another NC structure $\mathcal{I}_0$ on $\mathcal{E}$ is close to $\mathcal{I}$ if the following holds. For each bundle chart $\tilde{C} = (G, U, E, \psi)$ of $\mathcal{E}$ and $x \in U$, let $I^E_{Gx}$, $I^E_{0Gx}$ be the $G_x$-invariant complex structures on $E_{Gx} \to U_{Gx}$. Then $I^E_{0Gx}$ is close to $I^E_{Gx}$.

**Lemma 2.21.** Let $\mathcal{E} \to \mathcal{U}$ be equipped with NC structures $\mathcal{I}^E$. Let $\mathcal{Y} \subseteq \mathcal{U}$ be a closed subset and let $\mathcal{I}_0^E$ be another NC structure on $\mathcal{E}$ defined near $\mathcal{Y}$ which is close to $\mathcal{I}^E$. Then there exists an NC structure $\mathcal{I}_1^E$ which agrees with $\mathcal{I}_0^E$ near $\mathcal{Y}$ and which is close to $\mathcal{I}^E$.

**Proof.** Choose a smooth cut-off function $\eta : \mathcal{U} \to [0, 1]$ supported in the domain of $\mathcal{I}_1^E$ which is identically zero near $\mathcal{Y}$. For each bundle chart $\tilde{C} = (G, U, E, \psi)$ and each point $x \in U$, define $I_1^{E_{Gx}}(x)$ to be the $(1 - \eta(x))$-interpolation from $I^{E_{Gx}}(x)$ to $I_0^{E_{Gx}}(x)$. Then the collection $I_1^{E_{Gx}}$ defines an NC structure $\mathcal{I}_1^E$ on $\mathcal{E}$ which is close to $\mathcal{I}^E$ and which agrees with $\mathcal{I}_0^E$ near $\mathcal{Y}$. \hfill $\Box$

**Definition 2.22.** Let $\mathcal{E} \to \mathcal{U}$ be an orbifold vector bundle and let $\mathcal{I}_0^E$, $\mathcal{I}_1^E$ be two NC structures on $\mathcal{E}$. A concordance from $\mathcal{I}_0^E$ to $\mathcal{I}_1^E$ is a family of NC structures $\mathcal{I}_t^E$, $t \in [0, 1]$ which depends smoothly on $t$ and domain variables.

Above discussions essentially says that if $\mathcal{I}_1^E$ is close to $\mathcal{I}_0^E$, then there exists a canonical concordance from $\mathcal{I}_0^E$ to $\mathcal{I}_1^E$.

2.3.5. *Stabilization of normal complex structures.* We first describe an important model of normally complex orbifolds.

**Example 2.23.** Let $G$ be a finite group. Let $X$ be a smooth manifold and $F \to X$ be a complex vector bundle equipped with a fiberwise linear $G$-action. Then a $G$-equivariant complex structure $\mathcal{I}^F$ on $F$ induces a canonical NC structure on the total space of $F$. Indeed, for each essential subgroup $H \subseteq F G$, the fixed point set of $H$ is the total space of $F_H \subseteq F$ and the normal bundle $N F_H$ is $\pi_{F_H}^* F_H \to F_H$. The restriction of $\mathcal{I}^F$ to $\hat{F}_H$ is pulled back to an $H$-invariant complex structure on $N F_H$, giving an NC structure on the $G$-manifold $F$.

Now consider the following situation. Let $\mathcal{U}$ be a $G$-manifold. Let $F \to \mathcal{U}$ be a $G$-equivariant real vector bundle. Consider the total space $F$ which is a $G$-manifold. A $G$-invariant connection $\nabla^F$ provides an $G$-equivariant splitting of the exact sequence

$$0 \longrightarrow \pi_{F}^* F \longrightarrow TF \longrightarrow \pi_{F}^* TU \longrightarrow 0.$$  

If $H \subseteq G$ is an $F$-essential subgroup, then along the $H$-fixed point $F_H \subseteq F$, there is an induced split exact sequence

$$0 \longrightarrow \pi_{F_H}^* \hat{F}_H \longrightarrow N F_H \longrightarrow \pi_{F_H}^* N U_H \longrightarrow 0.$$  

If $TU$ and $F$ are normally complex, the normal bundle $N F_H \to F_H$ carries an $H$-invariant complex structure $I^{N F_H} = \pi_{F_H}^* I^{\hat{F}_H} \oplus \pi_{F_H}^* I^{N U_H}$.
We do not need to require that $\nabla^F$ preserves the NC structure. Therefore, the total space $F$ carries an NC structure $I^TF$ determined by $I^{TU}$, $I^F$, and the connection $\nabla^F$. We call $I^{TF}$ the bundle NC structure. Notice that the situation described by Example 2.23 is the case when $NU_H = 0$, where a connection is not necessary.

The above notion can be easily extends to orbifolds. Let $(\mathcal{U}, I^{\mathcal{T}U})$ be an NC orbifold and $\mathcal{F} \to \mathcal{U}$ be an NC vector bundle. Let $\nabla^\mathcal{F}$ be a connection on $\mathcal{F}$. Then there is a bundle NC structure $I^\mathcal{F}$. Moreover, as connections form an affine space, different choices of connections result in concordant NC structures.

One considers the analogous situation of NC structures on bundles under pullback. A typical example is as follows. Let $U$ be a $G$-manifold, $F \to U$ be a $G$-equivariant vector bundle, and $E \to U$ be a $G$-equivariant vector bundle equipped with an NC structure $I^E$. Consider the pullback $\pi^\ast F E$. For each $F$-essential subgroup $H \subseteq G$, one has

$$\pi^\ast F E|_{F H} = \pi^\ast F_H E_H \oplus \pi^\ast F_H E_H$$

where the second summand carries the pullback complex structure $\pi^\ast F_H I^{E_H}$. This makes the pullback bundle $\pi^\ast F E \to F$ an NC vector bundle. This example extends to orbifolds. Suppose $\mathcal{F} \to \mathcal{U}$ is an orbifold vector bundle and $\mathcal{E} \to \mathcal{U}$ is an NC vector bundle. Then $\pi^\ast \mathcal{E} \to \mathcal{F}$ carries a pullback NC structure.

**Definition 2.24** (Stabilization of NC pairs). Let $(\mathcal{U}, \mathcal{E})$ be an NC pair. Let $F \to \mathcal{U}$ be an NC vector bundle equipped with a connection $\nabla^F$. The stabilization of $(\mathcal{U}, \mathcal{E})$ is the pair $(\mathcal{F}, \pi^\ast \mathcal{E} \oplus \pi^\ast \mathcal{F})$ where $\mathcal{F}$ is equipped with the bundle NC structure induced from the NC structure $I^{\mathcal{T}U}$ on $\mathcal{T}U$, the NC structure $I^\mathcal{F}$, and the connection $\nabla^\mathcal{F}$, and where $\pi^\ast \mathcal{E} \oplus \pi^\ast \mathcal{F}$ is equipped with the pullback NC structure.

2.4. **Straightened geometric structures.** The local models for FOP perturbations are equivariant complex polynomial maps between two complex representations. To consider this notion on an orbifold, one needs to locally linearize the orbifold and the bundle so that the notion of polynomial maps (or certain generalization) transits between different local models. This is the purpose of introducing straightened geometric structures. A convenient way to construct such structures is to use exponential maps associated to a suitable Riemannian metric on the orbifold and parallel transport associated to a suitable connection on the vector bundle.

Since the discussion here is rather technical, we would like to begin with the case of straightening domain structures, while deferring bundle structures to the second part of this subsection.

Note: in this subsection as well as Appendix A, since the discussion is rather technical, we will restrict ourselves to the case where the manifolds or orbifolds do not have boundary. When dealing with spaces with boundary in applications, we construct collar neighborhoods of the boundary and extend structures on the boundary to interior via trivial products (cf. Remark 1.3).

2.4.1. **Notions and notations.** The model case for our purpose is the total space of a vector bundle equipped with some canonical metric.

**Definition 2.25** (Bundle metric). Let $(X, g^{TX})$ be a smooth Riemannian manifold, $F \to X$ be a smooth real vector bundle equipped with an inner product $h^F$ and an $h^F$-preserving connection $\nabla^F$. The bundle metric $g^{TF}$ on $F$ is the Riemannian
metric on the total space of $F$ determined as the direct sum of
\[ g^{TF} = \pi^*_F g^{TX} \oplus \pi^*_F h^F \]
under the isomorphism of vector bundles over $F$
\[ TF \cong \pi^*_F TX \oplus \pi^*_F F \]
determined by the connection $\nabla^F$.

We fix notations related to exponential maps and tubular neighborhoods. Let $(X, g^{TX})$ be a Riemannian manifold and $S \subseteq X$ be a closed submanifold. Identify the normal bundle $NS$ with the orthogonal complement of $TS$ in $TM|_S$. For any smooth function $\epsilon : S \to \R$, denote
\[ N^\epsilon S := \{(x, v) \mid x \in S, v \in NS|_x, \|v\|^2 < \epsilon(x)\} \subseteq NS. \tag{2.2} \]
Then for $\epsilon$ small enough, the exponential map in the normal direction to $S$ defines an open embedding
\[ \exp^{NS} : N^\epsilon S \to X. \]
Let the image be $|N^\epsilon S|$.

On the other hand, on the normal bundle $NS$ there is an induced inner product $h^{NS}$. Let $p^{NS} : TX|_S \to NS$ be the orthogonal projection induced from $g^{TX}$. Then there is the connection
\[ \nabla^{NS} := p^{NS} \circ \nabla^{TX}|_NS \]
induced from the Levi–Civita connection $\nabla^{TX}$, which preserves $h^{NS}$. Together with the restricted Riemannian metric $g^{TS}$ on $S$ there is a canonically induced bundle metric $g^{TNS}$ on $NS$.

**Definition 2.26.**
(1) Let $U$ be a manifold and $S \subseteq U$ be a closed submanifold. A Riemannian metric $g^{TU}$ on $U$ is said to be **straightened along $NS$** if for $\epsilon : S \to \R_+$ sufficiently small, the exponential map $\exp^{NS} : N^\epsilon S \to |N^\epsilon S|$ is an isometry between the bundle metric on $N^\epsilon S$ and the restriction $g^{TU}|_{|N^\epsilon S}$.

(2) Let $U$ be a $G$-manifold. A $G$-invariant Riemannian metric $g^{TU}$ is said to be **straightened** if it is straightened along $NU_H$ for all $U$-essential subgroup $H \subseteq G$. $(U, g^{TU})$ is called a **straightened Riemannian $G$-manifold**.

In the presence of normal complex structures, we would like the NC structure to be "fiberwise constant" in normal directions. Let $U$ be a $G$-manifold equipped with an NC structure $I^{TU}$. For each $U$-essential subgroup $H \subseteq G$, the total space of $NU_H$ carries a bundle NC structure $I^{TNU_H}$ induced from the $H$-invariant complex structure $I^{NU_H}$ (see Example 2.23).

**Definition 2.27.** Let $(U, g^{TU})$ be a Riemannian $G$-manifold equipped with an NC structure $I^{TU}$. We say that $I^{TU}$ is **straightened with respect to $g^{TU}$** if for each $U$-essential subgroup $H \subseteq G$ the following conditions are satisfied.

(1) The inner product $h^{NU_H}$ on $NU_H \to U_H$ induced from $g^{TU}$ is Hermitian with respect to $I^{NU_H}$ and the induced connection $\nabla^{NU_H}$ is $I^{NU_H}$-linear.

(2) For $\epsilon : U_H \to \R_+$ sufficiently small, the exponential map $\exp^{NS} : N^\epsilon U_H \to |N^\epsilon U_H|$ sends the bundle NC structure $I^{TNU_H}$ to the NC structure $I^{TU}$.

Now we turn to orbifolds. The definition is straightforward.

**Definition 2.28.** Let $U$ be an effective orbifold.
(1) A Riemannian metric $g^{TU}$ is said to be straightened if its pullback to each chart $C = (G, U, \psi)$ is a straightened Riemannian metric on $U$.

(2) An NC structure $\mathcal{I}^{TU}$ is said to be straightened with respect to $g^{TU}$ if its pullback to each chart is a straightened NC structure with respect to the chartwise metric.

Note that the above definitions are well-defined. Indeed, once $U$ is equipped with a Riemannian metric, the chart embeddings become isometric embeddings respecting the group action. Therefore, the property of being straightened for a Riemannian metric is compatible for different charts as dictated by Definition 2.26. A similar discussion can be applied to NC structures as well.

We give the detailed proof of the following existence result for straightened metrics in Appendix A.

**Proposition 2.29.** Let $(U, \mathcal{I}^{TU})$ be an NC orbifold and $Y \subseteq U$ be a closed set. Let $g_0^{TU}$ be a Riemannian metric on $U$ and $\mathcal{I}_0^{TU}$ be an NC structure on $U$ satisfying the following conditions.

1. $\mathcal{I}_0^{TU}$ is close to $\mathcal{I}^{TU}$ (see Definition 2.20).
2. $g_0^{TU}$ is straightened near $Y$.
3. $\mathcal{I}_0^{TU}$ is straightened near $Y$ with respect to $g_0^{TU}$.

Then there exists a straightened Riemannian metric $g_1^{TU}$, a straightened NC structure $\mathcal{I}_1^{TU}$ with respect to $g_1^{TU}$ such that

1. $\mathcal{I}_1^{TU}$ is close to $\mathcal{I}^{TU}$.
2. $(g_1^{TU}, \mathcal{I}_1^{TU})$ coincides with $(g_0^{TU}, \mathcal{I}_0^{TU})$ near $Y$.

Proof. See Appendix A.1. \qed

### 2.4.2. Straightened bundle structures

We would also like the vector bundle to be “straightened” in the direction normal to fixed point loci in a way analogous to Riemannian metrics. Specifically, we consider connections and inner products on vector bundles and their behavior along normal geodesics.

We first specify a few notations regarding behaviors of vector bundles within tubular neighborhoods of submanifolds.

**Definition 2.30.** Let $(U, g^{TU})$ be a Riemannian manifold and $S \subseteq U$ be a closed submanifold. Let $\exp^{NS} : N^*S \to U$ be the normal exponential map. Let $E \to U$ be a real vector bundle equipped with a connection $\nabla^E$. Then the parallel transport along normal geodesics induces a bundle isomorphism

$$
\begin{align*}
\pi^{*}_{NS}(E|_{N^*S}) & \xrightarrow{\text{par}^{NS}} E|_{N^*S} \\
\pi_{NS} & \downarrow \quad \quad \downarrow \pi_E \\
N^*S & \xrightarrow{\exp^{NS}} |N^*S|
\end{align*}
$$

We say that $\nabla^E$ is straightened along $NS$ with respect to $g^{TU}$ if $\text{par}^{NS}$ sends the pullback connection $\pi^{*}_{NS}(\nabla^E|_{NS})$ to the connection $\nabla^E$.

On the other hand, the corresponding straightenedness for bundle inner products is a consequence of requiring that the connection preserves the inner product. Therefore we do not need a more specific definition.

Now we consider the case with group actions.
Definition 2.31. Let \((U, g^{TU})\) be a Riemannian \(G\)-manifold. Let \(E \to U\) be a \(G\)-equivariant vector bundle equipped with a \(G\)-invariant connection \(\nabla^E\), a \(G\)-invariant inner product \(h^E\), and an NC structure \(\mathcal{I}^E\).

1. We say that \(\nabla^E\) is straightened with respect to \(g^{TU}\) if for each \(U\)-essential subgroup \(H \subseteq G\), \(\nabla^E\) is straightened along \(NU_H\).
2. We say that \(\mathcal{I}^E\) is straightened (with respect to \(g^{TU}\) and \(\nabla^E\)) if for each \(U\)-essential subgroup \(H \subseteq G\), the bundle map \(\text{par}^{NS}\) sends the pullback NC structure on \(\pi_{NU_H}^*(E|_{U_H})\) to the NC structure on \(E|_{N\cdot U_H}\).
3. We say that the triple \((\nabla^E, h^E, \mathcal{I}^E)\) is normally Hermitian if for each \(H \subseteq U \subseteq G\), the restriction \(h^E|_{E^H}\) is Hermitian with respect to \(I^E_H\), and the induced connection \(\nabla^E_{H^*}\) is a Hermitian connection.

Notice that the bundle isomorphism \(\text{par}^{NS}\) is defined by parallel transport. Then if for all \(H \subseteq U \subseteq G\), \(\nabla^E_{H^*}\) is complex linear with respect to \(I^E_H\), then \(\mathcal{I}^E\) is straightened. In particular, one has the following situation.

Lemma 2.32. If \(E \to U\) is a \(G\)-equivariant complex vector space with complex structure \(I^E\). Let \(\mathcal{I}^E\) be the induced NC structure. Let \(h^E\) be a Hermitian inner product and \(\nabla^E\) is a Hermitian connection. Then \(\mathcal{I}^E\) is straightened with respect to \(g^{TU}\) and \(\nabla^E\) and the triple \((\nabla^E, h^E, \mathcal{I}^E)\) is normally Hermitian.

The extension of these notions to orbifold vector bundles is straightforward.

Definition 2.33. Let \((\mathcal{U}, g^{TU})\) be a Riemannian orbifold. Let \(\mathcal{E} \to \mathcal{U}\) be an orbifold vector bundle equipped with a connection \(\nabla^\mathcal{E}\), an inner product \(h^\mathcal{E}\), and an NC structure \(\mathcal{I}^\mathcal{E}\).

1. We say that \(\nabla^\mathcal{E}\) is straightened with respect to \(g^{TU}\) if its pullback to each chart \(C = (G, U, E, \hat{\psi})\) is a straightened connection \(\nabla^E\) with respect to the chartwise metric \(g^{TU}\).
2. We say that \(\mathcal{I}^\mathcal{E}\) is straightened with respect to \(g^{TU}\) and \(\nabla^\mathcal{E}\) if its pullback to each chart is straightened with respect to the chartwise pullbacks of the bundle connection.
3. We say that the triple \((\nabla^\mathcal{E}, h^\mathcal{E}, \mathcal{I}^\mathcal{E})\) is normally Hermitian if its pullback to each bundle chart \(C = (G, U, E, \hat{\psi})\) is normally Hermitian.

Then one has the existence result for straightened structures. Its proof is simpler than the case of straightened Riemannian metrics.

Proposition 2.34. Let \((\mathcal{U}, g^{TU})\) be a straightened Riemannian orbifold. Let \(\mathcal{Y} \subseteq \mathcal{U}\) be a closed subset. Let \(\mathcal{E} \to \mathcal{U}\) be a vector bundle equipped with an NC structure \(\mathcal{I}^\mathcal{E}\). Let \(\nabla^\mathcal{E}_0\) be a connection on \(\mathcal{E}\), \(h^\mathcal{E}_0\) be an inner product on \(\mathcal{E}\), and \(\mathcal{I}^\mathcal{E}_0\) be an NC structure on \(\mathcal{E}\), satisfying the following conditions.

1. \(\mathcal{I}^\mathcal{E}_0\) is close to \(\mathcal{I}^\mathcal{E}\).
2. \(\nabla^\mathcal{E}_0\) is straightened near \(\mathcal{Y}\) with respect to \(g^{TU}\).
3. \(h^\mathcal{E}_0\) is preserved by \(\nabla^\mathcal{E}_0\) near \(\mathcal{Y}\).
4. \(\mathcal{I}^\mathcal{E}_0\) is straightened with respect to \(g^{TU}\) and \(\nabla^\mathcal{E}\) near \(\mathcal{Y}\).
5. \((\nabla^\mathcal{E}_0, h^\mathcal{E}_0, \mathcal{I}^\mathcal{E}_0)\) is normally Hermitian near \(\mathcal{Y}\).

Then there exist a connection \(\nabla^\mathcal{E}_1\), an inner product \(h^\mathcal{E}_1\), and an NC structure \(\mathcal{I}^\mathcal{E}_1\) satisfying the following conditions.

1. \(\mathcal{I}^\mathcal{E}_1\) is close to \(\mathcal{I}^\mathcal{E}\).
(2) $\nabla^E$ is straightened with respect to $g^{TU}$.
(3) $h^E_1$ is preserved by $\nabla^E_1$.
(4) $\mathcal{I}^E_1$ is straightened with respect to $g^{TU}$ and $\nabla^E_1$.
(5) $(\nabla^E_1, h^E_1, \mathcal{I}^E_1)$ is normally Hermitian.
(6) $(\nabla^E_1, h^E_1, \mathcal{I}^E_1)$ coincides with $(\nabla^E_0, h^E_0, \mathcal{I}^E_0)$ near $\mathcal{Y}$.

In addition, if $\mathcal{E}$ is a complex vector bundle, $\mathcal{I}^E$ is the NC structure induced from the complex structure, $h^E_0$ is a Hermitian inner product, $\nabla^E_0$ is a Hermitian connection, and $\mathcal{I}^E_0 = \mathcal{I}^E$, then we can require that $\mathcal{I}^E_1 = \mathcal{I}^E_0 = \mathcal{I}^E$, $h^E_1 = h^E_0$, and $\nabla^E_1$ is Hermitian.

**Proof.** See Appendix A.2. \[\square\]

Choosing a system of the above “straightened” structures are the prerequisite for the FOP perturbation method. We introduce the following concept.

**Definition 2.35.** Let $(\mathcal{U}, \mathcal{I}^{TU})$ be an NC orbifold and $(\mathcal{E}, \mathcal{I}^E)$ be an NC vector bundle over $\mathcal{U}$. A straightening of $(\mathcal{U}, \mathcal{E})$ is a pair $(\mathcal{U}^\#, \mathcal{E}^\#)$ where

1. $\mathcal{U}^\# = (\mathcal{U}, g^{TU}_1, \mathcal{I}^{TU}_1)$ consists of a straightened Riemannian metric $g^{TU}_1$ on $\mathcal{U}$ and an NC structure $\mathcal{I}^{TU}_1$ which is straightened with respect to $g^{TU}_1$ and which is concordant to $\mathcal{I}^{TU}$, and
2. $\mathcal{E}^\# = (\mathcal{E}, \nabla^E_1, h^E_1, \mathcal{I}^E_1)$ consists of a connection $\nabla^E_1$ on $\mathcal{E}$ which is straightened with respect to $g^{TU}_1$, $h^E_1$ is an inner product on $\mathcal{E}$ which is preserved by $\nabla^E_1$, and $\mathcal{I}^E_1$ is an NC structure on $\mathcal{E}$ which is straightened with respect to $g^{TU}_1$ and $\nabla^E_1$ which is concordant to $\mathcal{I}^E$, such that $(\nabla^E_1, h^E_1, \mathcal{I}^E_1)$ is normally Hermitian.

**Remark 2.36.** Later we will see that the bundle inner product will not enter the definition of normally complex perturbation.

The following corollary is a direct consequence of Proposition 2.29 and Proposition 2.34.

**Corollary 2.37.** Let $(\mathcal{U}, \mathcal{E})$ be an NC pair and $\mathcal{Y} \subseteq \mathcal{U}$ be a closed subset. Suppose a straightening of $(\mathcal{U}, \mathcal{E})$ is given over a neighborhood of $\mathcal{Y}$. Then there exists a straightening of $(\mathcal{U}, \mathcal{E})$ which agrees with the given one near $\mathcal{Y}$.

Another direct corollary to Proposition 2.29 and Proposition 2.34 is that two straightenings can be connected in a one-parameter family.

**Corollary 2.38.** Let $(\mathcal{U}, \mathcal{I}^{TU})$ be an NC orbifold without boundary. Let $(\mathcal{E}, \mathcal{I}^E)$ be an NC vector bundle over $\mathcal{U}$. Let

$$(\mathcal{U}^\#_i = (\mathcal{U}, g^{TU}_i, \mathcal{I}^{TU}_i), \mathcal{E}^\#_i = (\mathcal{E}, \nabla^E_i, h^E_i, \mathcal{I}^E_i)), \; i = 0, 1$$

be two straightenings of $(\mathcal{U}, \mathcal{E})$. Let $\tilde{\mathcal{U}} = \mathcal{U} \times [0, 1]$ equipped with the product NC structure $\mathcal{I}^{TU}$ and $\tilde{\mathcal{E}} = \mathcal{E} \times [0, 1]$ equipped with the product NC structure $\mathcal{I}^E$. Then there exists a straightening $(\tilde{\mathcal{U}}^\#, \tilde{\mathcal{E}}^\#)$ of $(\tilde{\mathcal{U}}, \tilde{\mathcal{E}})$ whose restriction to $\mathcal{U} \times [0, \epsilon)$ resp. $\mathcal{U} \times (1-\epsilon, 1]$ coincides with the product of $(\mathcal{U}^\#_0, \mathcal{E}^\#_0)$ resp. $(\mathcal{U}^\#_1, \mathcal{E}^\#_1)$ with the interval $[0, \epsilon)$ resp. $(1-\epsilon, 1]$.

**Proof.** We first choose a concordance of the NC structures. As $\mathcal{I}^{TU}_i$ resp. $\mathcal{I}^E_i$ is concordant to $\mathcal{I}^{TU}$ resp. $\mathcal{I}^E$, there exists an NC structure $\mathcal{I}^{TU}_{01}$ on $\tilde{\mathcal{U}}$ resp. $\mathcal{I}^E_{01}$ on $\tilde{\mathcal{E}}$ which is concordant to $\mathcal{I}^{TU}$ resp. $\mathcal{I}^E$ and which agrees with the product of...
\[ T_{TU} \] resp. \( T^E \) near the corresponding boundaries. For \( \epsilon \) sufficiently small, denote 
\[ \tilde{U} = U \times ([0, \epsilon] \cup [1 - \epsilon, 1]). \] Then the desired straightening can be constructed using Proposition 2.29 and Proposition 2.34. 

2.4.3. Straightenings and stabilizations. Recall that for an NC pair \( (U, \mathcal{E}) \) and an NC vector bundle \( F \to U \), the stabilization is
\[ (F, \pi^*_{F} \mathcal{E} \oplus \pi^*_F F). \]

Upon choosing a connection \( \nabla^F \) on \( F \), the stabilization becomes an NC pair. We need to consider straightenings on the stabilization. First, similar to the manifold case, if \( U \) is equipped with a metric \( g^{TU} \) and if \( h^F \) is an inner product on \( F \) preserved by \( \nabla^F \), then on the total space \( F \) there is a canonically induced bundle metric \( g^{TF} \).

There are similar pullback connections.

**Proposition 2.39.** Let \( (U^\#, E^\#) \) be a straightened NC pair with domain metric \( g^{TU} \), the domain NC structure \( T^{TU} \), the bundle connection \( \nabla^E \), the bundle inner product \( h^E \), and the bundle NC structure \( T^E \).

Let \( \pi_F : F \to U \) be a vector bundle equipped with an inner product \( h^F \), an \( h^F \)-preserving connection \( \nabla^F \), and an NC structure \( T^F \) such that the triple \( (\nabla^F, h^F, T^F) \) is normally Hermitian (Definition 2.33). Suppose \( \nabla^F \) is straightened with respect to \( g^{TU} \). Consider the pair
\[ (F, \pi^*_{F} \mathcal{E}) \]

Then the domain metric \( g^{TF} \) as the bundle metric on \( F \), the domain NC structure as the bundle NC structure on \( F \), the bundle connection \( \pi^*_F \nabla^E \), the bundle inner product \( \pi^*_F h^E \), the bundle NC structure \( \pi^*_F T^E \) give a straightening of \( (F, \pi^*_{F} \mathcal{E}) \).

**Proof.** See Appendix A.3. 

2.4.4. Suborbifolds with ordinary normal bundle. Now we discuss extending straightenings on an suborbifold with ordinary normal bundle to the ambient orbifold.

We first reorganize the notations. Let \( U \) be an orbifold and \( \mathcal{X} \subseteq U \) be a suborbifold with ordinary normal bundle. Recall that for each chart \( C = (G, U, \psi) \), \( \mathcal{X} := \psi^{-1}(X) \subseteq U \) is a \( G \)-invariant closed submanifold. Let \( \pi^{NX} : N^* \mathcal{X} \to \mathcal{X} \) be the normal bundle and \( N^* \mathcal{X} \) is the disk bundle of radius \( \epsilon : \mathcal{X} \to \mathbb{R}_+ \). If \( g^{TU} \) is a Riemannian metric on \( U \), then there is a normal exponential map
\[ \exp^{NX} : N^* \mathcal{X} \to U \]
which is an open embedding of orbifolds. Notice that the total space \( N^* \mathcal{X} \) carries a bundle metric. When \( T^{TN} \) is an NC structure on \( \mathcal{X} \) induced from an NC structure \( T^{TU} \), there is a bundle NC structure on \( N^* \mathcal{X} \).

In addition, if \( \mathcal{E} \to U \) is a vector bundle and \( \nabla^E \) is a connection on \( \mathcal{E} \). Abbreviate \( \mathcal{E}_X := \mathcal{E}|_{\mathcal{X}} \). Then along normal geodesics, there is a bundle map covering the normal exponential map
\[ \text{par}^{NX} : \left( \pi^*_N \mathcal{E}_X \right)|_{N^* \mathcal{X}} \to \mathcal{E}|_{N^* \mathcal{X}}. \]

**Definition 2.40.** Let \( g^{TU} \) be a Riemannian metric on \( U \) and \( T^{TU} \) is an NC structure on \( U \). Let \( \nabla^E, h^E, T^E \) be respectively a connection, an inner product, an NC structure on \( \mathcal{E} \to U \).

(1) \( g^{TU} \) is said to be straightened along \( N^* \mathcal{X} \) if for \( \epsilon : \mathcal{X} \to \mathbb{R}_+ \) sufficiently small, the normal exponential map \( \exp^{NX} \) is an isometric open embedding.
(2) $T^U$ is said to be straightened along $N\mathcal{X}$ if for $\epsilon : \mathcal{X} \to \mathbb{R}_+$ sufficiently small, $\exp^{N\mathcal{X}}_\epsilon$ is an open embedding of NC orbifolds.

(3) $\nabla^E_\epsilon$ is said to be straightened along $N\mathcal{X}$ if $\epsilon$ sends the pullback connection $\pi^*_{N\mathcal{X}}\nabla^{E_X}$ to $\nabla^E_\epsilon$.

(4) $T^E$ is said to be straightened along $N\mathcal{X}$ with respect to $g^U$ if $\epsilon$ sends the pullback $\pi^*_{N\mathcal{X}}T^E_X$ to $T^E$.

**Proposition 2.41.** Let $(\mathcal{U}, \mathcal{E})$ be an NC pair. Let $\mathcal{X} \subseteq \mathcal{U}$ be a suborbifold with ordinary normal bundle and $E_X := E|_{\mathcal{X}}$. Then $(\mathcal{X}, E_X)$ carries an induced NC structure. Suppose $(\mathcal{X}^\#, E_X^\#)$ is a straightening of $(\mathcal{X}, E_X)$. Then there exists a straightening $(\mathcal{U}^\#, E^\#)$ of $(\mathcal{U}, E)$ whose restriction to $\mathcal{X}$ is $(g_0^{TX}\mathcal{X}, I_0^T\mathcal{X})$ such that

1. The metric $g_0^\mathcal{U}$ is straightened along $N\mathcal{X}$.
2. The NC structure $I_0^T\mathcal{U}$ is straightened along $N\mathcal{X}$ with respect to $g_0^\mathcal{U}$.
3. The connection $N^\mathcal{U}$ is straightened along $N\mathcal{X}$ with respect to $g_0^\mathcal{U}$.
4. The NC structure $I_0^T\mathcal{U}$ is straightened along $N\mathcal{X}$ with respect to $g_0^\mathcal{U}$ and $N^\mathcal{U}$.

**Proof.** We only give the proof of the extension of the straightened metric. The extensions of other structures are similar and omitted. Let $g_0^\mathcal{U}$ be the straightened metric on $\mathcal{X}$. Let $g^\mathcal{U}$ be an arbitrary Riemannian metric on $\mathcal{U}$ extending $g_0^\mathcal{X}$, which gives a normal exponential map $\exp^{N\mathcal{X}} : N^\mathcal{X} \to \mathcal{U}$. Moreover, choose a connection $\nabla^N_0$ on the normal bundle $N\mathcal{X} \to \mathcal{X}$ which is straightened with respect to $g_0^\mathcal{X}$ and choose an inner product $h_0^{N\mathcal{X}}$ which is preserved by $\nabla^N_0$ and which is straightened with respect to $g_0^\mathcal{X}$ and $\nabla^N_0$ (the existence of such a connection and an inner product is guaranteed by Proposition 2.34). The triple $(g_0^\mathcal{X}, h_0^{N\mathcal{X}}, \nabla^N_0)$ induces a bundle metric $g_0^{TX}$ on the total space $N\mathcal{X}$, which is a straightened metric. Then using the normal exponential map to pushforward the bundle metric to a tubular neighborhood $|N^\mathcal{X}|$. This gives a straightened metric in $|N^\mathcal{X}|$ which extends $g_0^\mathcal{X}$. Then using the relative version of Proposition 2.29 to find a straightened metric $g^\mathcal{U}$ on $\mathcal{U}$ without altering its value near $\mathcal{X}$. \qed

2.5. **Normally complex sections.** Now we discuss the notion of normally complex sections originally introduced by Parker [Par13] which generalizes Fukaya–Ono’s notion of normally polynomial sections [FO01]. Such sections will form the category of perturbations we use.

2.5.1. **Normally complex maps between representations.** We first discuss equivariant polynomial maps between representations. Let $G$ be a finite group and $V, W$ be complex representations of $G$. Let $\text{Poly}_G(V, W)$ be the space of $G$-equivariant polynomial maps. For $d \geq 0$, let

$$\text{Poly}^d_G(V, W) \subseteq \text{Poly}_G(V, W)$$

be the subspace of maps of degrees at most $d$. More generally, for an NC triple $(G, V, W)$, define

$$\widehat{\text{Poly}}_G(V, W) := \text{Poly}_G(V_G, W_G) \oplus W_G.$$ 

Its elements, typically denoted by $P$, can be viewed as $G$-equivariant maps from $V$ to $W$ which are independent of the variable $v_G \in V_G$ and whose $W_G$-components are constant maps. Hence there is an inclusion

$$\text{Poly}_G(V, W) \subseteq C^G_\infty(V, W)$$.
where the latter is the space of smooth $G$-equivariant maps. Let $\widehat{\text{Poly}}^d_G(V,W) \subseteq \text{Poly}_G(V,W)$ be the subset of maps of degree at most $d$. It is the model of maps considered in [FO01]. Parker [Par13] generalizes it to the following larger space of equivariant maps.

**Definition 2.42.** Let $(G,V,W)$ be an NC triple. A $G$-equivariant normally complex map from $V$ to $W$ is an element of the module over the ring of $G$-invariant smooth functions $C^G_\text{in}(V;\mathbb{R})$ generated by $\text{Poly}_G(V,W)$, denoted by $C^\text{NC}_G(V,W)$.

A lift of an NC map $S \in C^\text{NC}_G(V,W)$ is a $G$-invariant smooth map $p \in C^G_\text{in}(V,\text{Poly}^d_G(V,W))$ such that $S(v) = p(v)(v)$. We denote $S_p(v) = p(v)(v)$.

Notice that there is a natural diagram

$$
\text{colim}_d C^G_\text{in}(V,\text{Poly}^d_G(V,W)) \rightarrow C^\text{NC}_G(V,W).
$$

**Remark 2.43.** In general $C^\text{NC}_G(V,W)$ is only a $C^1$-dense subspace of $C^G_\text{in}(V,W)$. For example, when $G = \mathbb{Z}_2$ and $V \cong W \cong \mathbb{C}$ is the nontrivial complex representation, the map $z \mapsto \overline{z}$, which is $G$-equivariant, cannot be approximated by an NC map in $C^1$-topology.

### 2.5.2. Change of symmetry groups

Let $(G,V,W)$ be an NC triple. If $H \subseteq G$ is a $V$-essential subgroup, then any $G$-equivariant map $f : V \rightarrow W$ is automatically $H$-equivariant. Hence there is an inclusion

$$
C^G_\text{in}(V,W) \hookrightarrow C^H_\text{in}(V,W).
$$

**Lemma 2.44.** Under the above inclusion, the space $C^\text{NC}_G(V,W)$ is mapped into $C^\text{NC}_H(V,W)$. Moreover, there is a natural map $\mu$ (the dashed arrow below) making the following diagram commute.

$$
\text{colim}_d C^G_\text{in}(V,\text{Poly}^d_G(V,W)) \rightarrow C^\text{NC}_G(V,W) \quad \mu
$$

**Proof.** A map $p : V \rightarrow \text{Poly}^d_G(V,W)$ is determined by its graph

$$
\text{graph}(p) \subseteq M^d(G,V,W) := V \times \text{Poly}^d_G(V,W).
$$

We would like to define a natural map

$$
\mu : M^d(G,V,W) \rightarrow M^d(H,V,W)
$$

which will play an important role. Given $v \in V$ and $P \in \text{Poly}^d_G(V,W)$, with respect to the basic decomposition (for the $H$-action) $V = V_H \oplus \tilde{V}_H$, write $v = v_H + \tilde{v}_H$;
with respect to the basic decomposition $W = W_H \oplus W_{\hat{H}}$, write $P = P_H + \hat{P}_H$. Then define
\[
\mu(v, P) := (v, \hat{P}_H(v_H, \cdot), P_H(v)) \in M^d(H, V, W).
\]
(2.3)
It is an $H$-equivariant map. Then there is a unique $H$-invariant map $q : V \to Poly_H^d(V, W)$ whose graph is $\mu(\text{graph}(p))$. By abuse of notation, define
\[
\mu(p) := q.
\]
From the definition one can see that $S_q = P$ as maps from $V$ to $W$. □

2.5.3. Family case. We also consider the family case.

**Definition 2.45.** Let $X$ be a manifold.

1. A family NC triple over $X$ is a triple $(G, F, E)$ consisting of a pair of real vector bundles $\pi_F : F \to X$, $\pi_E : E \to X$ with fiberwise linear $G$-actions, a $G$-invariant complex structure $I^F_G$ on $F_G \subseteq F$ and a $G$-invariant complex structure $I^E_G$ on $E_G \subseteq E$.

2. Given a family NC triple $(G, F, E)$, let $C^d_G(F, E)$ be the space of $G$-equivariant smooth bundle maps $S : F \to E$. Let
\[
\text{Poly}_G(F, E) \subseteq C^d_G(F, E)
\]
be the subbundle whose fiber at $x \in X$ is the space $\text{Poly}_G(F_x, E_x)$, and $\text{Poly}^d_G(F, E) \subseteq \text{Poly}_G(F, E)$ be the subbundle of fiberwise polynomial maps of degree at most $d$.

3. An element $S \in C^d_G(F, E)$ is called a normally complex bundle map if there exists $d \geq 0$ such that it is an element of the module over $\text{Poly}_G^d(F; \mathbb{R})$ generated by $\Gamma(\text{Poly}_G^d(F, E))$, the global sections of $\text{Poly}_G^d(F, E)$. Denote
\[
C^{NC}_G(F, E) := \colim_d C^d_G(F; \mathbb{R}) \cdot \Gamma(\text{Poly}_G^d(F, E)) \subseteq C^d_G(F, E).
\]

When we discuss transversality, we need to choose a particular presentation of a NC bundle map. Notice that for each $d \geq 0$, there is a natural map
\[
C^d_G(F, \text{Poly}_G^d(F, E)) \to C^{NC}_G(F, E)
\]
\[
p \mapsto S_p := (F \ni v \mapsto p(v)(v)).
\]
(2.4)

**Definition 2.46.** (cf. [Par13, Definition 4.12]) A lift of an NC bundle map $S \in C^{NC}_G(F, E)$ is a $G$-invariant bundle map $p : F \to \text{Poly}_G^d(F, E)$ such that $S = S_p$.

It is important to consider how the NC bundle maps behave under restriction to a smaller symmetry group. Let $H \subseteq G$ be an $F$-essential subgroup. Then there is the natural inclusion
\[
C^d_G(F, E) \to C^d_H(F, E).
\]
(2.5)
Lemma 2.44 implies that (2.5) sends $C^{NC}_G(F, E)$ into $C^{NC}_H(F, E)$. Moreover, there is a natural map on the level of lifts.
Lemma 2.47. There is a natural arrow $\mu$ (the dotted one) making the following diagram commutes.

\[
\begin{array}{ccc}
\colim_d C^G_G \left( F, \text{Poly}^d_G(F, E) \right) & \rightarrow & C^NC_G(F, E) \\
\mu \downarrow & & \downarrow \\
\colim_d C^G_H \left( F, \text{Poly}^d_H(F, E) \right) & \rightarrow & C^NC_H(F, E)
\end{array}
\]

Proof. One simply applies the map $\mu$ provided by Lemma 2.44 along each fiber of $F$ and the proof follows from the same argument. $\square$

One can view the total space $F$ as the total space of the pullback bundle $\pi^*_F \hat{F}_H$. Then an $H$-equivariant bundle map $S : F \rightarrow E$ can also be viewed as an $H$-equivariant bundle map from $\pi^*_F \hat{F}_H$ to $\pi^*_F E$. Hence one has the natural identification

\[ C^G_H(F, E) \cong C^G_H(\pi^*_F \hat{F}_H, \pi^*_F E), \] 

which induces the identification

\[ \pi^*_F \text{Poly}^d_H(F, E) \cong \text{Poly}^d_H(\pi^*_F \hat{F}_H, \pi^*_F E). \] 

The following lemma is readily to check.

Lemma 2.48. The map (2.6) sends $C^NC_H^G(F, E)$ to $C^NC_H^G(\pi^*_F \hat{F}_H, \pi^*_F E)$ and the following diagram commutes.

\[
\begin{array}{ccc}
C^G_H \left( F, \text{Poly}^d_H(F, E) \right) & \rightarrow & C^NC_H(F, E) \\
\cong & & \cong \\
C^G_H \left( \pi^*_F \hat{F}_H, \text{Poly}^d_H(\pi^*_F \hat{F}_H, \pi^*_F E) \right) & \rightarrow & C^NC_H(\pi^*_F \hat{F}_H, \pi^*_F E)
\end{array}
\]

2.5.4. Normally complex sections. Now we define the notion of normally complex sections over orbifolds. Let $(U, \mathcal{E})$ be an NC pair equipped with a straightening $(\mathcal{U}^#, \mathcal{E}^#)$. Let $\hat{C} = (G, U, E, \psi)$ be a bundle chart. Let $H \subseteq G$ be a $U$-essential subgroup. Recall that the straightening induces exponential maps

\[ \exp^{NU_H} : N^U \rightarrow |N^U_H| \subseteq U \]

and the bundle isomorphism

\[ \text{par}^{NU_H} : \pi^{NU_H}_E|_{|N^U_H|} \cong E|_{|N^U_H|}. \]

Then any $G$-equivariant section $S : U \rightarrow E$ induces a $G$-equivariant bundle map

\[ S_H^# : N^U \rightarrow E|_{|U^H_H|}. \]

Definition 2.49. Under the above setting, let $\hat{C} = (G, U, E, \psi)$ be a bundle chart of $\mathcal{E}$ and $S : U \rightarrow E$ is a $G$-equivariant section.

(1) Let $H \subseteq G$ be a $U$-essential subgroup. We say that $S$ is normally complex along $NU_H$ if for $\epsilon : U_H \rightarrow \mathbb{R}_+$ sufficiently small, the bundle map $S_H^#$ is the restriction of an NC bundle map from $NU_H$ to $E|_{|U^H_H|}$ to $N^U_H$.

(2) $S$ is said to be normally complex if it is NC along $NU_H$ for all $U$-essential subgroups $H \subseteq G$. 

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(3) A smooth section \( S : \mathcal{U} \rightarrow \mathcal{E} \) is an NC section if its pullback to each bundle chart is an NC section.

The importance of straightening is stated in the following lemma, which guarantees that the compatibility of being normally complex along tubular neighborhoods of fixed point loci of different groups.

**Lemma 2.50.** Let \( \hat{\mathcal{C}} = (G, U, E, \hat{\psi}) \) be a bundle chart. Suppose \( S : U \rightarrow E \) is NC along \( NU_G \), then for any \( U \)-essential subgroup \( H \subseteq G \), \( S \) is NC along \( NU_H \) in a neighborhood of \( U_G \).

**Proof.** By the definition of straightening, near \( U_G \), we may regard \( U \) with a disk bundle of \( NU_G \) equipped with the bundle metric and bundle NC structure, and identify \( E \) with the pullback bundle \( \pi^*_NU_G (E_{|U_G}) \). By the definition of NC sections, \( S^g_{\Gamma} \) is the restriction of a section in \( C^{\infty}_{\Gamma_G}(NU_G, E_{|U_G}) \). Let \( H \subseteq U \) be an essential subgroup. In the neighborhood \( |N'U_G| \) for \( \epsilon \) sufficiently small, the \( H \)-fixed point set is the total space of \( (NU_G)_H \). By Lemma 2.47, Lemma 2.48, and the definition of straightenings, the restriction of \( S^g_{\Gamma} \) to a neighborhood of \( (NU_G)_H \) is also normally complex. \( \square \)

Next we prove some simple formal properties of the space of normally complex sections. Let \( \Gamma(\mathcal{U}, \mathcal{E}) \) be the space of smooth sections of \( \mathcal{E} \). Let \( (\mathcal{U}^#, \mathcal{E}^#) \) be a straightening of \( (\mathcal{U}, \mathcal{E}) \). Let

\[
\Gamma^{\text{NC}}(\mathcal{U}^#, \mathcal{E}^#) \subseteq \Gamma(\mathcal{U}, \mathcal{E})
\]

be the subset of normally complex sections (with respect to the given straightening).

**Lemma 2.51.** \( \Gamma^{\text{NC}}(\mathcal{U}^#, \mathcal{E}^#) \) is a \( C^\infty(\mathcal{U}) \)-submodule of \( \Gamma(\mathcal{U}, \mathcal{E}) \).

**Proof.** It follows from the fact that \( C^{\infty}_{\Gamma_G}(F, E) \) is a \( C^\infty(F; \mathbb{R}) \)-submodule (cf. Definition 2.45). \( \square \)

Lastly we prove the \( C^0 \)-density of the space of normally complex sections.

**Proposition 2.52.** \( \Gamma^{\text{NC}}(\mathcal{U}^#, \mathcal{E}^#) \) is dense in \( \Gamma(\mathcal{E}) \) with respect to the \( C^0 \)-norm induced by any inner product on \( \mathcal{E} \).

To prove this result, we first recall a fundamental lemma proved by Fukaya–Ono [FO01]. For the convenience of the reader we provide the proof here.

**Lemma 2.53 (Fukaya–Ono’s lemma),** [FO01, Lemma 5] Let \( (G, V, W) \) be an NC triple. Then for any \( d \geq |G| \) and any \( v_0 \in V \) with isotropy subgroup \( H \subseteq G \) and \( w_0 \in W_H \), there exists \( P \in \text{Poly}_d(G, V, W) \) such that \( P(v_0) = w_0 \).

**Proof.** It is easy to see that it suffices to consider the case when \( V_G = W_G = \{0\} \); in particular, \( V \) and \( W \) are complex representations. Then by decomposing \( W \) into irreducible components, we may assume that \( W \) is an irreducible (complex) representation of \( G \). Define the \( G \)-vector space

\[
\mathcal{R} := \bigoplus_{\gamma \in G} \mathbb{C} \{ \langle \gamma \rangle \}
\]

with \( G \)-action defined as

\[
g \left( \sum_{\gamma} c_\gamma \cdot \langle \gamma \rangle \right) = \sum_{\gamma} c_\gamma \cdot \langle \gamma g^{-1} \rangle = \sum_{\gamma} c_\gamma g \cdot \langle \gamma \rangle.
\]
Since $R$ is a regular representation, there is a $G$-equivariant homomorphism $\Psi : R \to W$ and an element $r \in R$ such that

$$\Psi(r) = \Psi \left( \sum_{\gamma} w_{\gamma} \cdot \langle \gamma \rangle \right) = w_0.$$ 

Since $w \in W_H$, for all $h \in H$, one has

$$\Psi(hr) = h \Psi(r) = w_0.$$ 

Hence by taking average over $H$, one may assume that

$$\gamma' \cdot H = \gamma'' \cdot H \implies w_{\gamma'} = w_{\gamma''}.$$ 

Now we claim that one can choose a complex polynomial $f : V \to \mathbb{C}$ (not necessarily $G$-invariant) of degree at most $|G|$ such that

$$\forall \gamma \in G, \quad f(\gamma v_0) = w_{\gamma}.$$ 

Indeed, there are $n := |G/H|$ distinct elements in the $G$-orbit of $v_0$. One can choose a linear decomposition $V = V_1 \oplus V_2$ such that $V_1$ is one-dimensional and that the projection of these $n$ distinct elements are still distinct in $V_1$. Then by Lagrange’s method of interpolation, one can find a complex polynomial $f : V_1 \to \mathbb{C}$ of degree at most $|G|$ taking the prescribed values $w_{\gamma}$ at the corresponding projection image of $\gamma v_0$ in $V_1$. Extend $f$ trivially to $V$ one obtains a polynomial $f : V \to \mathbb{C}$ satisfying the required conditions. Now define $P : V \to W$ by

$$P(v) = \Psi \left( \sum_{\gamma \in G} f(\gamma v) \cdot \langle \gamma \rangle \right).$$

Then this is a $G$-equivariant polynomial map sending $v_0$ to $w_0$. 

**Proof of Proposition 2.52.** Fix an inner product on $E$. Given $S_0 \in \Gamma(U, E)$ and $\delta > 0$, it suffices to find a section $S \in \Gamma^{NC}(U^\#, E^\#)$ such that $\|S - S_0\|_{C^0} \leq \delta$. We construct $S$ by local approximations and gluing using a partition of unity.

For each $p \in U$, one can find a bundle chart $C = (G, U, E, \psi)$ centered at $p$ satisfying the following conditions.

1. The normal bundle $NU_G \to U_G$ is trivial with fiber being a representation $V$ of $G$. The tubular neighborhood $|N^U G|$ is the whole chart $U$, identified with a subset of $U_G \times V$.
2. $E|_{U_G}$ is trivial with fiber being an NC representation $W$ of $G$. Then the straightening induces a trivialization $E \cong U \times W$ and the section $S_0 : U \to E$ is equivalent to a $G$-equivariant map $S_0 : U \to W$.

**Claim.** For $d \geq |G|$ and any $\epsilon > 0$ there is a $G$-invariant map

$$p : U_G \times V \to \text{Poly}^d_G(V, W)$$

providing a lift of an NC map $S_p$ such that

$$\sup_{(y, v) \in U} \|S_0(y, v) - S_p(y, v)\| \leq \delta.$$ 

Assuming this claim is true, we prove the proposition as follows. As $U$ is second-countable, it is Lindelöf, hence there is a countable collection of points $p_i \in U$ and bundle charts $\tilde{C}_i = (G_i, U_i, E_i, \psi_i)$, $i = 1, \ldots$, which satisfy the above conditions such that the collection $\psi_i(U_i)$ cover $U$. Let $S_{0,i} : U_i \to E_i$ be the pullback of $S_0$...
to the chart $\hat{C}_i$. The above claim implies that there exists a $G_i$-equivariant NC section $S_i : U_i \to E_i$ such that
\[ \|S_{0,i} - S_i\|_{C^0(U_i)} \leq \frac{\delta}{2^i}. \]

Then choose a smooth partition of unity $\rho_i$ subordinate to the open cover $\psi_i(U_i)$. Consider
\[ S_i := \sum_{i=1}^{\infty} \rho_i S_i \]
which belongs to $\Gamma^{\text{NC}}(U^\#, E^\#)$ by Lemma 2.51. Then one has
\[ \|S_0 - S\|_{C^0} \leq \sum_{i=1}^{\infty} \|\rho_i S_{0,i} - \rho_i S\|_{C^0} \leq \sum_{i=1}^{\infty} \|S_{0,i} - S_i\|_{C^0(U_i)} \leq \sum_{i=1}^{\infty} \frac{\delta}{2^i} = \delta. \]

Now we prove the above claim. For each $x = (y_0, v_0) \in U$, the equivariance condition on $S_0(x) \in W_{G_x} \subseteq W$. Then by Lemma 2.53, for $d \geq |G|$, there exists $P_x \in \text{Poly}^d_G(V, W)$ such that $P_x(v_0) = S_0(x)$. We view $P_x$ as a map
\[ P_x : U \to W, \quad P_x(y, v) = P_x(v), \]
which is a $G$-equivariant NC section of $E \to U$. Then for the given $\delta > 0$, one can find an open neighborhood $O_x \subseteq U$ of $x$ such that
\[ \sup_{x' \in O_x} |S_0(x) - S_0(x')| + \sup_{x' \in O_x} |P_x(x) - P_x(x')| \leq \delta. \]

As $U$ is Lindelöf and paracompact, one can find countably many points $x_j$ such that the collection $O_{x_j}$ is a locally finite cover of $U$. Choose a partition of unity $\rho_j$ on $U$ subordinate to $O_{x_j}$. Define
\[ q : U \to \text{Poly}^d_G(V, W), \quad q(x) = \sum_{j=1}^{\infty} \rho_j(x) P_{x_j}. \]

We first check that this (not necessarily $G$-invariant) map defines a section that is close to the original section $S_0$. Indeed, for each $x = (y, v) \in U$, then one has
\[ |S_0(x) - q(x)(v)| \leq \sum_{j=1}^{\infty} \rho_j(x) |S_0(x) - P_{x_j}(x)|. \]

For $j = 1, \ldots, l$, if $x \notin O_{x_j}$, then $\rho_j(x) = 0$ and the corresponding summand above vanishes; if $x \in O_{x_j}$, then
\[ |S_0(x) - P_{x_j}(x)| \leq |S_0(x) - S_0(x_j)| + |P_{x_j}(x) - P_{x_j}(x_j)| \leq \delta. \]

Hence one has
\[ \sup_{x \in U} |S_0(x) - q(x)(v)| \leq \delta. \]

Now we make $q$ invariant by setting
\[ p(x) = \frac{1}{|G|} \sum_{\tau \in G} q(\tau x) \in \text{Poly}^d_G(V, W). \]

Then $p$ satisfies the condition stated in the above claim. \hfill \Box
3. Whitney Stratifications on the Variety $\mathbb{Z}$

This section serves as another facet of the technical foundation of the FOP transversality condition, which depends on an intricate analysis of a special class of affine varieties, especially their Whitney stratifications. Many technical discussions we carry out here originated from Parker’s work [Par13].

Before the technical discussion, we explain why such study of Whitney stratifications is necessary. As mentioned in the introduction, by doing single-valued perturbation to a section of an orbifold vector bundle, the best hope is to have a stratified zero locus whose isotropy free part has codimension two or higher frontier. The local models for such a stratified zero locus are certain “universal” zero locus, i.e., the varieties labelled by $\mathbb{Z}$. The Whitney stratification on $\mathbb{Z}$ we specify in this section will give models for stratifying the perturbed zero locus. The challenge in the discussion is that one needs to identify a kind of canonical Whitney stratifications which have sufficient functorial properties so that the corresponding transversality condition is well-defined and is an open condition on an orbifold. One is then compelled to revisit arguments of Whitney and Mather etc. about the existence and properties of canonical Whitney stratifications on algebraic varieties and adopt them to the current situation.

In Subsection 3.1 we review basics about Whitney stratifications. Some technical results stated here are proved in Appendix B. In Subsection 3.2 we specify the consideration to a special class of algebraic varieties whose Whitney stratifications are essential for defining the FOP transversality condition. The proofs of several important propositions are given at the end of this section. In Subsection 3.3, we describe a universal set which will index the strata of the zero loci of FOP transverse sections. The rest of this section contains proofs of results stated in Subsection 3.2.

3.1. Basic notions and facts about Whitney stratifications. We first recall basic notions and facts about Whitney stratifications. Proofs of some results will be given in Appendix B.

**Definition 3.1.** Let $M$ be a smooth manifold.

(1) Let $X, Y \subseteq M$ be disjoint smooth submanifolds. We say that $(X, Y)$ satisfies Whitney’s condition (b) at $y \in Y$ if the following is true. Given a sequence $x_i \in X$ converging to $y$ and $y_i \in Y$ converging to $y$, suppose the secant line $P_{y_i}^{x_i}$ converges to a line $L_y \subseteq T_y M$ and $T_{x_i} X$ converges to a dim$X$-dimensional subspace $H_y \subseteq T_y M$, then $L_y \subseteq H_y$. We say that $(X, Y)$ is Whitney regular if $(X, Y)$ satisfies Whitney’s condition (b) at all points $y \in Y$.

(2) A stratification (Definition 2.11) of a subset $Z \subseteq M$, denoted by $\mathcal{Z} = \{Z_\lambda\}$, is called a Whitney stratification if all strata are smooth submanifolds and each distinct pair $(Z_\lambda, Z_\mu)$ of strata is Whitney regular.

(3) Let $N$ be a smooth manifold and $f : N \to M$ be a smooth map. For a Whitney stratified subset $Z \subseteq M$, $f$ is said to be transverse to $Z$ with respect to a Whitney stratification $\mathcal{Z}$ on $Z$, if $f$ is transverse to each stratum of $\mathcal{Z}$. The naive pullback Whitney stratification on $f^{-1}(Z)$ is the stratification $f^* \mathcal{Z} := \{f^{-1}(Z_\lambda) \neq \emptyset \mid Z_\lambda \in \mathcal{Z}\}$

\footnote{The secant line depends on a choice of local coordinate system near $y$, but the condition does not depend on such choices.}
(see Lemma B.2 proving that this partition satisfies the axiom of frontier). Notice that there is a canonical map between the sets of strata:

\[ f_\# : f^* \mathcal{Z} \to \mathcal{Z}. \]

In particular, for any open subset \( U \subseteq M \), the restriction of \( \mathcal{Z} \) to \( U \cap Z \) is the pullback of \( \mathcal{Z} \) by the natural inclusion \( \iota : U \hookrightarrow M \), denoted by \( \mathcal{Z}|_{U \cap Z} \).

(4) Suppose \( f : N \to M \) is transverse to \( \mathcal{Z} \) as above. The refined pullback of \( \mathcal{Z} \) is the Whitney stratification (cf. Lemma B.3) on \( f^{-1}(Z) \)

\[ f^! \mathcal{Z} := \left\{ \text{connected components of } f^{-1}(Z_\lambda) \mid Z_\lambda \in \mathcal{Z} \right\}. \]

In particular, denote by \( \mathcal{Z}^! \) be the refined pullback by the identity map; in other words, \( \mathcal{Z}^! \) is obtained from \( \mathcal{Z} \) by taking connected components. Notice that there is canonically induced order-preserving map

\[ f_! : f^! \mathcal{Z} \to \mathcal{Z}. \]

Whitney stratification may not exist on an arbitrary subset. It was originally proved by Whitney [Whi65] in the complex analytic category that analytic sets have Whitney stratifications. We make a few more definitions.

**Definition 3.2.** Let \( M \) be a complex manifold.

1. A subset \( U \subseteq M \) is called a strongly analytic subset if both \( \overline{U} \) and \( \overline{U \setminus U} \) are closed analytic subsets. \( U \) is called a strongly analytic submanifold if it is an analytic submanifold and a strongly analytic set.
2. A Whitney stratification \( \mathcal{Z} \) on a subset \( Z \subseteq M \) is called strongly analytic if its strata are all strongly analytic submanifolds.

Whitney’s construction actually provides an “optimal” Whitney stratification. Below we define how to compare different Whitney stratifications.

**Definition 3.3.** Let \( M \) be an \( m \)-dimensional manifold and \( Z \subseteq M \).

1. The dimension filtration of a Whitney stratification \( \mathcal{Z} = \{Z_\lambda\} \) on \( Z \) is

\[ Z = \mathcal{Z}_m \supseteq \mathcal{Z}_{m-1} \supseteq \cdots \supseteq \mathcal{Z}_0 \]

where

\[ \mathcal{Z}_k := \bigcup_{\dim(Z_\lambda) \leq k} Z_\lambda. \]

2. Two Whitney stratifications \( \mathcal{Z} \) and \( \mathcal{Z}' \) on \( Z \) are called equivalent, denoted by \( \mathcal{Z} \equiv \mathcal{Z}' \), if their dimension filtrations coincide. We write \( \mathcal{Z} \lesssim \mathcal{Z}' \) if there exists \( k \) such that

\[ \mathcal{Z}_l = \mathcal{Z}'_l \ \forall l > k, \quad \mathcal{Z}_k \subseteq \mathcal{Z}'_k. \]

We write \( \mathcal{Z} \lesssim \mathcal{Z}' \) if either \( \mathcal{Z} \equiv \mathcal{Z}' \) or \( \mathcal{Z} \lesssim \mathcal{Z}' \).

3. Let \( \mathcal{WS}^\times(Z) \) be the set of all smooth Whitney stratifications on \( Z \) and let \( \mathcal{N}(Z) \subseteq \mathcal{WS}^\times(Z) \) be a subset. We say \( \mathcal{Z} \in \mathcal{N}(Z) \) is minimal within \( \mathcal{N}(Z) \) if for any \( \mathcal{Z}' \in \mathcal{N}(Z) \), either \( \mathcal{Z} \equiv \mathcal{Z}' \) or \( \mathcal{Z} \lesssim \mathcal{Z}' \).

It is easy to see that if a minimal Whitney stratification exists, then it is unique up to equivalence.
Definition 3.4. Let $M$ be a smooth manifold equipped with a stratification $\mathcal{M} = \{ M_\alpha \}$ where each stratum $M_\alpha$ is a locally closed smooth submanifold ($\mathcal{M}$ itself may not be a Whitney stratification). We say a Whitney stratification $\mathfrak{S}$ on a subset $Z \subseteq M$ respects $\mathcal{M}$ if each $Z \cap M_\alpha$ is a union of strata in $\mathfrak{S}$. Denote by

$$WS^X(Z; \mathcal{M}) \subseteq WS^X(Z)$$

the subset of $\mathcal{M}$-respecting Whitney stratifications on $Z$.

The case we are interested in comes from group actions.

Definition 3.5. Let $G$ be a finite group acting on a smooth manifold $M$. The action stratification on $M$ is the stratification $M$ indexed by $M$-essential subgroups of $G$ whose strata are

$$M^G_H := \left\{ x \in M \mid G_x = H \right\}.$$

For each $G$-invariant subset $Z \subseteq M$, let

$$WS^X(Z; G) \subseteq WS^X(Z)$$

be the subset of Whitney stratifications which respects the action stratification.

In the appendix we prove the following theorem, which can be viewed as a refinement of Whitney’s original theorem.

Proposition 3.6. In the situation of Definition 3.4, suppose $M$ is a complex manifold, each $M_\alpha$ is a strongly analytic submanifold, and $Z \subseteq M$ is a closed analytic subset. Then there exists a unique minimal element $\mathfrak{S} \in WS^X(Z; \mathcal{M})$ that has connected strata. In addition, $\mathfrak{S}$ satisfies the following properties.

1. For each open subset $U \subseteq M$, $\mathfrak{S}|_{U \cap Z}$ (which may not have connected strata) is minimal in $WS^X(U \cap Z, \mathcal{M}|_U)$.
2. $\mathfrak{S}$ is a strongly analytic Whitney stratification (Definition 3.2).
3. If $f : M \to M$ is a diffeomorphism such that $f^*\mathcal{M} = \mathcal{M}$ and $f(Z) = Z$, then $f^*\mathfrak{S} = \mathfrak{S}$.

Proof. See Appendix B.4. □

Remark 3.7. The construction needed for the proof was provided in the original paper of Whitney (see [Whi65, Section 18–21]). However, as Whitney’s proof does not explicitly discuss the minimality (which is important for our purpose), we need to revisit his argument and reproduce the proof in Appendix B.

For certain applications (for example when proving the Arnold conjecture over $\mathbb{Z}$ in [BX22]), we need the property that the certain Whitney stratifications respect direct product. Notice that products of Whitney stratifications are still Whitney stratifications (see [GWdPL76, (1.2)]). In the appendix we also prove the following.

Proposition 3.8. Let $M$ resp. $N$ be complex manifolds equipped with stratifications $\mathcal{M}$ resp. $\mathcal{N}$ by strongly analytic submanifolds. Let $S \subseteq M$ resp. $T \subseteq N$ be closed analytic sets. Let $\mathcal{S}$ resp. $\mathcal{T}$ be the minimal $\mathcal{M}$-respecting resp. $\mathcal{N}$-respecting Whitney stratification on $S$ resp. $T$ provided by Proposition 3.6. Then $\mathcal{S} \times \mathcal{T}$ is a minimal $\mathcal{M} \times \mathcal{N}$-respecting Whitney stratification on $S \times T$.

Proof. See Appendix B.5. □
3.2. The canonical Whitney stratifications on the variety $Z$. Now we use the existence results (Theorem 3.6) to specify the canonical Whitney stratifications on the class of algebraic varieties $Z$. We then state the properties of such Whitney stratifications which will be proved later in this section.

3.2.1. The variety $Z$. Recall that for each NC triple $(G, V, W)$ (Definition 2.15) and $d \geq 0$, there is the vector space introduced in the proof of Lemma 2.44

$$M^d := M^d(G, V, W) = V \times \text{Poly}^d_G(V, W)$$

where $G$ acts on the factor $V$. There is the natural evaluation map

$$ev : M^d \to W, \ (v, P) \mapsto P(v) \in W$$

whose zero locus is a $G$-invariant subvariety

$$Z^d = Z^d(G, V, W) := \{(v, P) \in M^d(G, V, W) \mid ev(v, P) = 0\}.$$ 

In particular, the $W_G$-component of $P$ for a point in $Z^d$ is 0. The group $G$ acts on the factor $V$ and $Z^d$ is $G$-invariant. Hence there is an action stratification on $M^d$ (Definition 3.5). The strata are indexed by $V$-essential subgroups $H \subseteq V$ with

$$V^d_H = \{v \in V_H \mid G_v = H\}, \quad M^d(G, V, W)_H = V^d_H \times \text{Poly}^d_G(V, W).$$

For each $d \geq 0$, denote

$$Z^d(G, V, W)_H = Z^d(G, V, W) \cap M^d(G, V, W)_H.$$

We first look at this induced partition on $Z^d$.

**Proposition 3.9.** For all $d \geq |G|$ and $H \subseteq V$, $Z^d(G, V, W)_H$ is a smooth submanifold of dimension $\dim_R(\text{Poly}^d_G(V, W)) + \dim_R(V_H) - \dim_R(W_H)$.

**Proof.** This is a consequence of Fukaya–Ono’s lemma (Lemma 2.53). For each $H \subseteq V$, $G$ and $P \in \text{Poly}^d_G(V, W)$, with respect to the basic decomposition $W = W_H \oplus W_H$, we can write $P = P_H + P_{\bar{H}}$. Then the $H$-equivariance of $P$ implies that

$$P_H|_{V_H} \equiv 0.$$ 

Then

$$Z^d(G, V, W)_H = \{(v_H, P) \in V^d_H \times \text{Poly}^d_G(V, W) \mid P_H(v_H) = 0\}.$$ 

Then Lemma 2.53 implies that when $d \geq |G|$, $Z^d(G, V, W)_H$ is transversely cut out, having dimension being $\dim_R(\text{Poly}^d_G(V, W)) + \dim_R(V_H) - \dim_R(W_H)$. Indeed, the linearization of the defining equation of $Z^d(G, V, W)_H$ is surjective: given $(v_H, P) \in Z^d(G, V, W)_H$, Lemma 2.53 ensures that for any $w_0 \in W_H$, we can find a polynomial $P$ such that $P(v_H) = w_0$, and we can linearize the equation $P_H(v_H) = 0$ along the direction specified by $P$ along the polynomial component. So, the smoothness follows accordingly, and the dimension formula follows by a count of number of parameters.

**Remark 3.10.** Proposition 3.9 implies that $Z^d(G, V, W)$ has a natural partition into smooth manifolds. However, this partition is not a stratification, as the axiom of frontier fails (see Definition A.1).
3.2.2. The canonical Whitney stratification. Recall that for the NC pair \((G, V, W)\), one has the basic decompositions

\[ V = V_G \oplus \tilde{V}_G, \quad W = W_G \oplus \tilde{W}_G. \]

Moreover, recall the definition

\[ \text{Poly}^d_G(V, W) = \text{Poly}^d_G(\tilde{V}_G, \tilde{W}_G) \times W_G. \]

Then

\[ M^d = V_G \times M^d(G, \tilde{V}_G, \tilde{W}_G) \times W_G \]

where \( \tilde{M}^d := M^d(G, \tilde{V}_G, \tilde{W}_G) \) is a complex vector space and

\[ Z^d = V_G \times Z^d(G, \tilde{V}_G, \tilde{W}_G) \times \{0\} \]

where \( \tilde{Z}^d := Z^d(G, \tilde{V}_G, \tilde{W}_G) \subseteq \tilde{M}^d \) is a complex algebraic variety with a \( G \)-action. In this case, by Proposition 3.6, there exists a unique minimal element

\[ \tilde{3}^d \in WS^x(\tilde{Z}^d; G) \]

with connected strata. Using the identification \( Z^d = V_G \times \tilde{Z}^d \times \{0\} \), there is a canonically induced Whitney stratification \( 3^d \) on \( Z^d \) given by

\[ 3^d = \left\{ V_G \times \tilde{Z}^d_\alpha \times \{0\} \mid \tilde{Z}^d_\alpha \in 3^d \right\} \]

It is easy to check that \( 3^d \) is a minimal Whitney stratification in \( WS^x(Z^d; G) \). We call \( 3^d \) the canonical Whitney stratification on \( Z^d \).

Example 3.11. Consider the situation where \( G = \mathbb{Z}_3 \), \( V \cong \mathbb{C} \) is the weight 1 representation and \( W \cong \mathbb{C} \) is the weight 2 representation. Let \( d = 5 \). Then

\[ \text{Poly}^5_{\mathbb{Z}_3}(V, W) = \{ P(v) = av^2 + bv^5 \mid a, b \in \mathbb{C} \} \]

and

\[ Z^5 = \{(a, b, v) \in \mathbb{C}^3 \mid av^2 + bv^5 = 0\}. \]

The canonical Whitney stratification has three strata. Two top strata are

\[ 3^5_{\lambda_1} = \{(a, b, v) \mid v \neq 0, \ a + bv^3 = 0\} \]

and

\[ 3^5_{\lambda_2} = \{(a, b, v) \mid v = 0, \ a \neq 0\}. \]

The intersection of their closure is the lowest stratum

\[ 3^5_{\lambda_3} = \{(a, b, v) \mid v = 0, \ a = 0\}. \]

Notice that this Whitney stratification is the same as the minimal Whitney stratification of \( Z^5 \) without requiring to respect the action stratification.

Example 3.12. Consider \( G = \mathbb{Z}_k \) and \( R \cong \mathbb{C} \) being the weight 1 representation. Choose \( V = R \oplus R \), \( W = R \), and \( d = 1 \). Let the variable of \( V \) be \((x, y)\). Then

\[ \text{Poly}^1_{\mathbb{Z}_k}(V, W) = \{ P(x, y) = ax + by \mid a, b \in \mathbb{C} \} \]

and \( Z^1 \) is the cone \( ax + by = 0 \). It only singularity is the origin and its minimal Whitney stratification (without referring to the group action) has two strata: the smooth locus and the cone point. However, with respect to the group action, the canonical Whitney stratification is finer: its top stratum is

\[ 3^1_{\lambda_1} = \{(a, b, x, y) \mid (x, y) \neq (0, 0), \ ax + by = 0\} \]

which is properly contained in the smooth locus of \( Z^1 \).
The various properties of the FOP transversality condition listed in Theorem 1.1 relies on the following important properties of the canonical Whitney stratification.

3.2.3. Invariance.

**Proposition 3.13.** Given an isomorphism \((G, V, W) \cong (G', V', W')\) of NC triples which induces a linear isomorphism \(f : M^d(G, V, W) \to M^d(G', V', W')\) which sends \(Z^d(G, V, W)\) to \(Z^d(G', V', W')\), there holds

\[
f^* Z^d(G', V', W') = Z^d(G, V, W).
\]

In particular, \(Z^d(G, V, W)\) is \(\text{Aut}(G, V, W)\)-invariant, where \(\text{Aut}(G, V, W)\) denotes the group consisting of a pair of \(G\)-equivariant linear isomorphisms of \(V\) and \(W\).

**Proof.** It is equivalent to consider the case when \((G, V, W) = (G', V', W')\) and \(f\) is induced from a pair of \(G\)-equivariant automorphisms \(f_V : V \to V\) and \(f_W : W \to W\). Then \(f\) induces a complex linear automorphism of the subspace \(M^d(G, V_G, W_G)\) preserving the complex variety \(Z^d(G, V_G, W_G)\) as well as the action stratification and \(Z^d\). Then by Proposition 3.6, \(f^* Z^d = Z^d\). \qed

Proposition 3.13 allows us to extend the canonical Whitney stratification to the family case. Let \((G, F, E)\) be a family NC triple over a manifold \(X\) (Definition 2.45). The one can define the vector bundle

\[
M^d(G, F, E) \to X
\]

and the \(G\)-invariant subbundle

\[
Z^d(G, F, E) \subseteq M^d(G, F, E).
\]

As the structure group of \(M^d(G, F, E)\) is contained in \(\text{Aut}(G, V, W)\), by Proposition 3.13, there is a locally trivial Whitney stratification \(Z^d(G, F, E)\) on \(Z^d(G, F, E)\), which is still called the canonical Whitney stratification on \(Z^d(G, F, E)\).

3.2.4. Change of degrees. The following proposition is necessary for showing the FOP transversality condition is well-defined and independent of the cut-off degree \(d\). Consider \(d \geq 0\) and \(d' > d\). Then there is the natural inclusion map

\[
\sigma_{d, d'} : M^d(G, V, W) \to M^{d'}(G, V, W), \quad (v, P) \mapsto (v, P)
\]

which respects the action stratifications such that \(\sigma_{d, d'}^{-1}(Z^{d'}) = Z^d\).

**Proposition 3.14.** For any NC triple \((G, V, W)\), there exists \(d_1 = d_1(G, V, W) \geq |G|\) such that when \(d > d' \geq d_1\), there holds

\[
\sigma^*_{d, d'} Z^{d'} = Z^d.
\]

Moreover, the corresponding map between the sets of strata

\[
(\sigma_{d, d'})_* : Z^d \to Z^{d'}
\]

is bijective.

**Proof.** See Subsection 3.4. \qed
3.2.5. Change of groups. The following proposition is necessary for showing the FOP transversality condition is an open condition on orbifolds. It is also the hardest one to prove. We first recall and introduce some notations. Let \((G, V, W)\) be an NC triple and \(H \subseteq V \subseteq G\) be an essential subgroup. Using the basic decomposition \(V = V_H \oplus \hat{V}_H\), a point of \(V\) is denoted by \((v_H, \hat{v}_H)\). Given an element \(P \in \text{Poly}_G(V, W)\) viewed as a function from \(V\) to \(W\), using the basic decomposition \(W = \hat{W}_H \oplus W_H\), we also write

\[
P = \hat{P}_H + P_H.
\]

Then one has defined the map in the proof of Lemma 2.44

\[
\mu : M^d(G, V, W) \rightarrow M^d(H, V, W)
\]

\[
(v, P) \mapsto \left(v, \hat{P}_H(v_H, \cdot) + P_H(v)\right).
\]

Here the term \(P_H(v)\) also contains the constant map component with value in \(W_G\). Notice that \(\mu\) maps \(Z^d(G, V, W)\) into \(Z^d(H, V, W)\). We shall not expect that the pullback Whitney stratification by \(\mu\) coincides with \(Z^d(G, V, W)\), because the weaker symmetry required by the subgroup \(H\) may not see some deeper strata of \(Z^d(G, V, W)\). However, they do coincide when restricted to the open subset of points whose isotropy group under the \(G\)-action is a subgroup of \(H\).

**Proposition 3.15.** Given an NC triple \((G, V, W)\), there exists \(d_2 \geq d_1\) such that for all \(d \geq d_2\) the following is true. For each \(H \subseteq V \subseteq G\), consider the open subset \(V^+_H := \{v \in V \mid G_v \subseteq H\} \subseteq V\).

Denote

\[
M^d(G, V, W)^+_H := V^+_H \times \text{Poly}_G^d(V, W), \quad M^d(H, V, W)^+_H := V^+_H \times \text{Poly}_H^d(V, W)
\]

\[
Z^d(G, V, W)^+_H := Z^d(G, V, W) \cap M^d(G, V, W)^+_H.
\]

Then \(\mu|_{M^d(G, V, W)^+_H}\) is transverse to \(Z^d(H, V, W)\) and

\[
\left(\mu^* Z^d(H, V, W)\right)|_{Z^d(G, V, W)^+_H} = Z^d(G, V, W)|_{Z^d(G, V, W)^+_H}.
\]

Moreover, the induced maps between the sets of strata

\[
\mu_* : Z^d(G, V, W)|_{Z^d(G, V, W)^+_H} \rightarrow Z^d(H, V, W)
\]

is invertible and makes the following diagram commute (when \(d' \geq d\)).

\[
\begin{array}{ccc}
Z^d(H, V, W) & \xrightarrow{\mu_*^{-1}} & Z^d(G, V, W) \\
\downarrow & & \downarrow \\
Z^{d'}(H, V, W) & \xrightarrow{\mu_*^{-1}} & Z^{d'}(G, V, W)
\end{array}
\]

\[
(3.3)
\]

**Proof.** See Subsection 3.7. \(\square\)
3.2.6. Product. Another property of the canonical Whitney stratification is that it respects direct product. This property is responsible for the Product Property stated in Theorem 1.1 which is important in proving the integral Arnold conjecture in [BX22] and in other potential chain-level applications.

For $i = 1, 2$, let $(G_i, V_i, W_i)$ be NC triples and denote

$$(G, V, W) = (G_1 \times G_2, V_1 \oplus V_2, W_1 \oplus W_2).$$

Proposition 3.16. For $d$ sufficiently large, consider the map

$$
\xi : M^d(G_1, V_1, W_1) \times M^d(G_2, V_2, W_2) \to M^d(G, V, W)
$$

$$(v_1, P_1, v_2, P_2) \mapsto \left( \begin{bmatrix} v_1 \\ v_2 \\ P_1 \\ 0 \\ P_2 \end{bmatrix} \right).$$

Then $\xi^*Z^d(G, V, W) = Z^d(G_1, V_1, W_1) \times Z^d(G_2, V_2, W_2)$.

Proof. See Subsection 3.5. \hfill \Box

3.2.7. Stabilization property. The following proposition is responsible for the Stabilization Property of Theorem 1.1. We prove the following proposition.

Proposition 3.17. Let $(G, V, W)$ be an NC triple and $R$ be an NC representation. For any $d \geq 1$, consider the map

$$
\eta : M^d(G, V, W) \to M^d(G, V \oplus R, W \oplus R)
$$

$$(v, P) \mapsto \left( \begin{bmatrix} v \\ P \end{bmatrix}, \begin{bmatrix} 0 \\ \text{proj}_{\tilde{R}_G} \end{bmatrix} \right)$$

where $\text{proj}_{\tilde{R}_G} : R \to \tilde{R}_G$ is the canonical projections induced from the basic decomposition $R = R_G \oplus \tilde{R}_G$. Then for $d \geq d_1$ (given by Proposition 3.14), $\eta$ is transverse to $\tilde{Z}^d(G, V \oplus R, W \oplus R)$ and

$$
\eta^*\tilde{Z}^d(G, V \oplus R, W \oplus R) = \tilde{Z}^d(G, V, W)
$$

(where $\eta^*$ is the refined pullback, see Definition 3.1). Moreover, the induced map

$$
\eta : \tilde{Z}^d(G, V, W) \to \tilde{Z}^d(G, V \oplus R, W \oplus R)
$$

makes the following diagram commute.

$$
\begin{array}{ccc}
\tilde{Z}^d(G, V, W) & \xrightarrow{\eta} & \tilde{Z}^d(G, V \oplus R, W \oplus R) \\
(\sigma_{d, d})_* & \downarrow & (\sigma_{d, d})_* \\
3^d(G, V, W) & \xrightarrow{\eta} & 3^d(G, V \oplus R, W \oplus R)
\end{array}
$$

(3.5)

Proof. See Subsection 3.6. \hfill \Box

Remark 3.18. Proposition 3.14—3.17 are stated for general NC triples $(G, V, W)$. However, it is very easy to reduce the proofs to the case when $V_G = W_G = \{0\}$; in particular, $V$ and $W$ are complex representations. In the subsections proving these propositions, we always make this simplifying assumption.
3.3. The set of universal strata. Now we use the previous technical results about the Whitney stratifications on the various varieties $\mathbb{Z}^d(G,V,W)$ to define a set of universal strata $\mathfrak{Z}^\text{univ}_k$ whose elements can label the strata of the zero locus of an FOP transverse section (see Theorem 1.1). Roughly, we define $\mathfrak{Z}^\text{univ}_k$ by taking the disjoint union of $\mathfrak{Z}^\text{d}_p \mathbb{G}, V, W_{q}$ for all NC triples of virtual dimension $k$ modulo a certain equivalence relation. To make this idea more precise, we need to specify a certain category and define $\mathfrak{Z}^\text{univ}_k$ as a colimit.

3.3.1. Category of NC triples.

Definition 3.19. (1) A morphism of NC triples from $(H, V, W)$ to $(G, X, Y)$ is a finite composition of the following two types of morphisms

(a) A stabilization consists of a group isomorphism $H \cong G$, an NC representation $R$ of $G$, and a pair of isomorphisms $X \cong V \oplus R$ and $Y \cong W \oplus R$ of NC representations of $G$.

(b) An embedding consists of a group embedding $H \hookrightarrow G$ and a pair of isomorphisms $V \cong X$ and $W \cong Y$ of NC representations of $H$.

(2) Two NC triples $(H, V, W)$ and $(G, X, Y)$ are called stably equivalent if they admit stabilizations that are isomorphic (isomorphism of groups + induced isomorphisms of representations). A stable equivalence class is called a stable NC isotropy type. The set of stable NC isotropy types is denoted by $\Gamma^\text{NC}$.

(3) Define a partial order in $\Gamma^\text{NC}_k$ as follows. We denote $\gamma \leq \delta$ if there exist a representative $(H, V, W)$ of $\gamma$, a representative $(G, X, Y)$ of $\delta$, and an embedding from $(H, V, W)$ to $(G, X, Y)$.

Notice that there is a natural poset map $\Gamma^\text{NC}_k \to \Gamma^\text{NC}$. For each NC pair $(U, E)$ of orbifolds and vector bundles whose virtual dimension is $k$ and a stable NC isotropy triple $\gamma \in \Gamma^\text{NC}_k$, define

$$\mathcal{U}_\gamma := \bigcup_{\gamma \leq \delta} \mathcal{U}_\delta$$

where we take the (disjoint) union of all isotropy types $\gamma$ which are sent to $\gamma$.

3.3.2. A particular category. The above technical results on Whitney stratifications allow us to gather all strata for different isotropy types. For our purpose related to Theorem 1.1, we first consider a particular category, denoted by $\text{Str}^\text{NC}_k$.

Its objects are triples $(\mathfrak{Z}, \rho, n)$ where $\mathfrak{Z}$ is a partially ordered set, $\rho : \mathfrak{Z} \to \Gamma^\text{NC}_k$ is a monotone map, and $n : \mathfrak{Z} \to 2\mathbb{Z}$ is a strictly monotone map. A morphism from $(\mathfrak{Z}_1, \rho_1, n_1)$ to $(\mathfrak{Z}_2, \rho_2, n_2)$ is a strictly monotone poset map $\iota_{21} : \mathfrak{Z}_1 \to \mathfrak{Z}_2$ such that $\rho_1 = \rho_2 \circ \iota_{21}$ and $n_1 = n_2 \circ \iota_{21}$.

Lemma 3.20. Colimits exist in $\text{Str}^\text{NC}_k$.

Proof. Let $\text{Poset}$ be the category of posets with morphisms being poset maps and let $\text{Poset}^\ast$ be the category of posets with morphisms being strictly monotone poset maps. In both categories colimits exist. On the other hand, there are forgetful functors

$$\begin{align*}
\text{Str}^\text{NC}_k \xrightarrow{\pi} \text{Poset}^\ast \xrightarrow{\iota} \text{Poset}
\end{align*}$$
where $\pi$ takes a triple to its underlying poset. Let $F : A \to \text{Str}^{NC}$ be a diagram (functor) where $A$ is any category. Then one obtains a colimit
\[
\colim_a \pi(F(a)) \in \text{ObPoset}^*.
\]
As the maps $n : 3 \to 2\mathbb{Z}$ is a morphism of $\text{Poset}^*$, it follows that $n$ descends to a strictly monotone map
\[
n : \colim_a \pi(F(a)) \to 2\mathbb{Z}.
\]
Moreover, $\iota : \colim_a \pi(F(a))$ is a colimit of the induced diagram $A \to \text{Poset}$, hence there exists a natural map
\[
\rho : \colim_a \pi(F(a)) \to \Gamma^{NC}.
\]
It is straightforward to check that $(\colim_a \pi(F(a)), n, \rho)$ is a colimit of $F(a)$. □

3.3.3. The universal strata. The sets of strata of the canonical Whitney stratifications we considered give rise to objects of the category $\text{Str}^{NC}$. Let $p : G, V, W \to A$ be an $\text{NC}$ triple of virtual dimension $k$. Define a map $\rho_k : Z^{d}(G, V, W) \to \Gamma^{NC}$
\[
as follows: if a stratum $Z^{d}_d \in 3^{d}(G, V, W)$ is contained in $V^*_H \times \text{Poly}^{d}_G(V, W)$ for some $V$-essential subgroup $H$, then define $\rho(Z^{d}_d)$ to be the isotropy type represented by the triple $(H, V, W)$. On the other hand, define a map
\[
n_k : 3^{d}(G, V, W) \to 2\mathbb{Z}
\]
which assigns to a stratum $Z^{d}_d$ the number
\[
n_k(Z^{d}_d) := \dim_{\mathbb{R}}(Z^{d}_d) - \dim_{\mathbb{R}}(\text{Poly}^{d}_G(V, W)) - k \in 2\mathbb{Z}. \quad (3.6)
\]
Then $(3^{d}(G, V, W), \rho_k, n_k)$ is an object of $\text{Str}^{NC}$. Note that Proposition 3.14 allows us to relate this set for different $d$. Indeed, the bijective map (for $d \leq d'$) induces a morphism
\[
\sigma_{d,d'} \in \text{Hom}_{\text{Str}^{NC}} \left(3^{d}(G, V, W), 3^{d'}(G, V, W)\right).
\]
Then define the colimit
\[
\mathfrak{Z}(G, V, W) := \colim_d 3^{d}(G, V, W) \in \text{Str}^{NC}_k.
\]
We would like to make it a functor from the category of NC triples to $\text{Str}^{NC}_k$.

**Lemma 3.21.** Suppose $(H, V, W) \to (G, X, Y)$ is a stabilization morphism. Then there is a canonical map
\[
\eta : \mathfrak{Z}(H, V, W) \to \mathfrak{Z}(G, X, Y).
\]

**Proof.** We identify $H$ with $G$, $V$ with a subspace of $X$, and $W$ with a subspace of $Y$. By definition, there exist a representation $R$ of $G$ and an equivariant decomposition $X \cong V \oplus R, Y \cong W \oplus R$.

By Proposition 3.17, each such decomposition induces a map
\[
\eta_1 : 3^{d}(H, V, W) \to 3^{d}(G, X, Y)
\]
which is compatible with the maps $\sigma_{d,d}$ when $d' > d$. Moreover, as the spaces of above equivariant decompositions are connected, the induced map is independent of the choices of the decompositions. Hence there is a well-defined morphism

$$\mathfrak{Z}(H, V, W) \to \mathfrak{Z}(G, X, Y).$$

$\square$

Similarly, from Proposition 3.15 one can define a morphism for embeddings of NC triples.

**Lemma 3.22.** Suppose $(H, V, W) \to (G, X, Y)$ be an embedding of NC triples. Then there is a canonical morphism

$$(\mu_*)^{-1} : \mathfrak{Z}(H, V, W) \to \mathfrak{Z}(G, X, Y).$$

Then the association $(G, V, W) \mapsto \mathfrak{Z}(G, V, W)$ is a functor from the category $\mathfrak{C}^{\text{NC}}$ to the category $\text{Str}^{\text{NC}}$. Then we define

$$\mathfrak{Z}^{\text{univ}} := \colim_{(G, V, W)} \mathfrak{Z}(G, V, W)$$

which we call the set of universal strata (of virtual dimension $k$).

**3.4. Proof of Proposition 3.14.** First notice that Proposition 3.14 reduces to the case that $V_G = W_G = 0$. Hence within this subsection, $V$ are $W$ are complex representations of $G$ containing no trivial subrepresentations. As a consequence

$$\text{Poly}_G^d(V, W) = \text{Poly}_G^d(V, W).$$

Our proof closely resembles that of [Par13, Lemma 4.11].

First we prove an algebraic result also used in [Par13] without providing a proof or reference.

**Lemma 3.23.** Suppose $V$ and $W$ are finite-dimensional complex representations of $G$. Then $\text{Poly}_G(V, W)$ is a finitely generated module over the ring $\text{Poly}_G(V, \mathbb{C})$ of $G$-invariant polynomials on $V$.

**Proof.** The proof follows from [hk], we present it here for completeness. By Hilbert’s basis theorem, given any finite-dimensional complex $G$-representation $V''$, the ring $\text{Poly}_G(V'', \mathbb{C})$ is finitely generated. Now let $V' = V \oplus W'$ where $W'$ is the dual to $W$ endowed with the corresponding $G$-action. The ring $\text{Poly}_G(V', \mathbb{C})$ has a bigrading by keeping track of the degree of the $V$-coordinates and $W'$-coordinates respectively. Choose $h_1, \ldots, h_r, h_{r+1}, \ldots, h_{r+m}, \ldots, h_n$ which generate $\text{Poly}_G(V', \mathbb{C})$ such that $h_1, \ldots, h_r$ have $W'$-degree 0, $h_{r+1}, \ldots, h_{r+m}$ have $W'$-degree 1, and $h_{r+m+1}, \ldots, h_n$ have higher $W'$-degrees.

Note that $\text{Poly}_G(V, W)$ can be identified with the subset of $\text{Poly}_G(V', \mathbb{C})$ consisting of elements with $W'$-degree 1. Then any element in $\text{Poly}_G(V, W)$ can be written as a linear combination of products of $h_1, \ldots, h_r, h_{r+1}, \ldots, h_{r+m}$. Observe that $h_1, \ldots, h_r$ are indeed elements of $\text{Poly}_G(V, \mathbb{C})$, it shows $\text{Poly}_G(V, W)$ is generated by $h_{r+1}, \ldots, h_{r+m}$ as a module over $\text{Poly}_G(V, \mathbb{C})$. $\square$

Next we construct a left inverse of the map $\sigma_{d,d}$ of (3.1).

**Lemma 3.24.** (cf. [Par13, Lemma 4.11]) Given an NC triple $(G, V, W)$, there exists $d_1$ such that for all $d' > d \geq d_1$, there is a $G$-invariant holomorphic map

$$\rho_{d,d} : M^d(G, V, W) \to M^{d'}(G, V, W)$$

satisfying the following properties.
(1) $\rho_{d,d}$ is a submersion with connected fibers and preserves the action stratification.

(2) $\rho_{d,d} \circ \sigma_{d,d}$ is the identity map of $M^d(G, V, W)$.

(3) ev $\circ \rho_{d,d} = ev : M^d(G, V, W) \to W$.

**Proof of Proposition 3.14.** Lemma 3.24–(3) implies that $Z^d = \rho_{d,d}^{-1}(Z^d)$. By Lemma 3.24–(1) and Proposition B.20, $\rho_{d,d}^*3^d = 3^d$. On the other hand, because $\rho_{d,d} \circ \sigma_{d,d} = \Id$ is transverse $3^d$ and $\sigma_{d,d}$ is transverse to $\rho_{d,d}^*3^d$ (see Lemma B.4). Hence

$$\sigma_{d,d}^*3^d = \sigma_{d,d}^*\rho_{d,d}^*3^d = (\rho_{d,d} \circ \sigma_{d,d})^*3^d = 3^d.$$ 

Moreover, because $\rho_{d,d}$ is surjective and has connected fibers, for each stratum $Z^d$, $\rho_{d,d}^{-1}(Z^d)$ is a single stratum of $3^d$ and $\sigma_{d,d}$ maps $Z^d$ into this stratum. Hence $\sigma$ induces a bijection between the sets of strata. 

**Proof of Lemma 3.24.** As $\Poly_G(V, W)$ is finitely generated over $\Poly_G(V, C)$ (see Lemma 3.23), one can find a sufficiently large $d_0$ such that $\Poly_{G,0}(V, W)$ contains a set of generators $Q_1, \ldots, Q_m$. Then when $d \geq d_0$, let $\Homo_G^d(V, W) \subseteq \Poly_G^d(V, W)$ be the subset of $G$-equivariant homogeneous polynomial maps of degree $d$. Then

$$\Poly_G^d(V, W) = \Poly_G^d(V, W) \oplus \bigoplus_{l=d+1}^d \Homo_G^l(V, W).$$

Choose a basis $P_1, \ldots, P_k$ of $\bigoplus_{l=d+1}^d \Homo_G^l(V, W)$, viewed as a $C$-vector space. Then there exist polynomials $h_{ij} \in \Poly_G(V, C)$ for $1 \leq i \leq k, 1 \leq j \leq m$ such that

$$P_i = \sum_{j=1}^m h_{ij}Q_j.$$ 

Then for each $P \in \Poly_G^d(V, W)$, write $P = P' + P''$ with $P' \in \Poly_G^d(V, W)$ and $P'' \in \bigoplus_{l=d+1}^d \Homo_G^l(V, W)$. We can write

$$P'' = \sum_{i=1}^k a_iP_i.$$ 

Then define

$$\rho_{d,d}(v, P) = \left( v, P' + \sum a_ih_{ij}(v)Q_j \right).$$ (3.7)

Abbreviate $\rho_{d,d}$ by $\rho$ and $\sigma_{d,d}$ by $\sigma$. Then $\rho$ is a submersion, preserves the action stratification, and satisfies $\rho \circ \sigma = \Id_{M^d}$. Moreover, for all $(v, P) \in M^d$ one has

$$ev(\rho(v, P)) = P'(v) + \sum a_ih_{ij}(v)Q_j(v) = P'(v) + P''(v) = P(v).$$

Lastly, we prove that $\rho$ has connected fibers. For any $(v, P) \in M^d$ with $P = P' + P''$ decomposed as above, consider the path

$$(v, P_t) = (v, P' + tP'') = \left( v, P' + t \sum a_ih_{ij}(v)Q_j + (1 - t)P'' \right).$$

Then $\rho(v, P_t) = \rho(v, P)$ for all $t$. Notice that $(v, P_t) = \sigma(\rho(v, P))$. Therefore, every point in the fiber of $\rho(v, P)$ can be connected with $\sigma(\rho(v, P))$. Hence fibers of $\rho$ are all connected. 

□
3.5. **Proof of Proposition 3.16.** Again, it suffices to consider the case that $V_G = W_G = \{0\}$.

**Lemma 3.25.** There exists a surjective submersion with connected fibers

$$
\xi^\prime: M^d(G_1 \times G_2, V_1 \oplus V_2, W_1 \oplus W_2) \rightarrow M^d(G_1, V_1, W_1) \times M^d(G_2, V_2, W_2)
$$
satisfying the following conditions.

1. $\xi^\prime \circ \xi$ is the identity map.
2. For each $(v, P)$ in the domain of $\xi^\prime$, $\text{ev}(\xi^\prime(v, P)) = \text{ev}(v, P)$.

**Proof.** For each $P \in \text{Poly}^d_{\text{Id}_{G_1} \times G_2}(V_1 \oplus V_2, W_1 \oplus W_2)$, denote its $W_1$-component by $P_1$ and its $W_2$-component by $P_2$. Then we can regard $P_1$ as a $G_2$-invariant map

$$P_1 \in \text{Poly}^d_{G_2}(V_2, \text{Poly}^d_{G_1}(V_1, W_1))$$

and regard $P_2$ as a $G_1$-invariant polynomial map

$$P_2 \in \text{Poly}^d_{G_1}(V_1, \text{Poly}^d_{G_2}(V_2, W_2)).$$

Then for $v = (v_1, v_2) \in V_1 \oplus V_2$, define

$$\xi^\prime(v, P) = (P_1(\cdot, v_2), P_2(v_1, \cdot)).$$

Then it is easy to verify that $\xi^\prime \circ \xi$ is the identity map and that

$$\text{ev}(\xi^\prime(v, P)) = \text{ev}(v, P).$$

Further, it is obvious that $\xi^\prime$ is a surjective submersion. Moreover, if $\xi^\prime(v, P) = \xi^\prime(u, Q)$, then $v = u$ and $\xi^\prime(v, P) = \xi^\prime(v, tP + (1 - t)Q)$ for all $t \in [0, 1]$. Hence $\xi^\prime$ has connected fibers. □

**Proof of Proposition 3.16.** Notice that the analytic submersion $\xi^\prime$ preserves the action stratification. Then by Proposition B.20, $\xi^\prime|_{U^d}$ pulls back the canonical Whitney stratification on $Z^d_1 \times Z^d_2$ to the canonical one on $Z^d$. The former is the product $Z^d_1 \times Z^d_2$ by Proposition 3.8. Therefore

$$(\xi^\prime)^*(Z^d_1 \times Z^d_2) = Z^d.$$

Then as $\xi^\prime \circ \xi$ is the identity, $\xi$ is transverse to $(\xi^\prime)^*(Z^d_1 \times Z^d_2)$. Hence

$$\xi^\prime(\xi^\prime)^*(Z^d_1 \times Z^d_2) = (\xi^\prime \circ \xi)^*(Z^d_1 \times Z^d_2) = Z^d_1 \times Z^d_2.$$ □

3.6. **Proof of Proposition 3.17.** Again, it suffices to consider the case when $V_G = W_G = R_G = 0$; in particular, $\text{proj}_{R_G}$ is the identity map of $R$. Consider equivariant polynomial maps from $V \oplus R$ to $W \oplus R$. A polynomial map from $V \oplus R$ to $W \oplus R$ consists of a polynomial map from $V \oplus R$ to $W$ and a polynomial map from $V \oplus R$ to $R$. For convenience we write them as

$$P \in \text{Poly}^d_{G}(V \oplus R, W), \quad \text{Id}_R + Q \in \text{Poly}^d_{G}(V \oplus R, R).$$

Denote variables of $V \oplus R$ be $(x, y)$ and the image of the map $\eta$ in (3.4) by

$$\tilde{\text{Poly}}^d_{G}(V \oplus R, W \oplus R) \subseteq \text{Poly}^d_{G}(V \oplus R, W \oplus R),$$

i.e., polynomial maps $(P_0, \text{Id}_R)$ where $P_0: V \rightarrow W$ does not depends on $y \in R$. We introduce an intermediate space

$$\text{Poly}^d_{G}(V \oplus R, W \oplus R).$$
consisting of pairs of the form
\[(x, y) \mapsto (P_0(x), y + Q_0(x)), \text{ where } P_0 \in \text{Poly}_G(V, W), \ Q_0 \in \text{Poly}_G(V, R).\]

Then consider
\[
\tilde{Z}^d(G, V \oplus R, W \oplus R) = Z^d(G, V \oplus R, W \oplus R) \cap \left( (V \oplus R) \times \text{Poly}_G(V \oplus R, W \oplus R) \right) 
\]

By Proposition 3.6, there are a canonical Whitney stratification \(\eta\) factorizing the map.

\[ \eta \text{ factors the map on } r \text{ on } q \text{ where } \eta \text{ is the map } G \times V \rightarrow R \times W \text{ as } \eta = \eta_1 \circ \eta_2 \]

where
\[
\eta_1 : (V \oplus R) \times \text{Poly}_G^d(V \oplus R, W \oplus R) \rightarrow (V \oplus R) \times \text{Poly}_G^d(V \oplus R, W \oplus R)
\]

and
\[
\eta_2 : (V \oplus R) \times \text{Poly}_G^d(V \oplus R, W \oplus R) \rightarrow (V \oplus R) \times \text{Poly}_G^d(V \oplus R, W \oplus R)
\]

are both the natural inclusions.

**Lemma 3.26.** There exists \(d_3\) (depending on \(G, V, W, \text{ and } R\)) such that when \(d \geq d_2\), \(\eta_2\) is transverse to \(\tilde{Z}^d(G, V \oplus R, W \oplus R)\) and

\[ \eta_2^* \tilde{Z}^d(G, V \oplus R, W \oplus R) = \tilde{Z}^d(G, V \oplus R, W \oplus R). \]

**Lemma 3.27.** There exists \(d_4\) (depending on \(G, V, W, \text{ and } R\)) such that when \(d \geq d_3\), \(\eta_1\) is transverse to \(\tilde{Z}^d(G, V \oplus R, W \oplus R)\) and

\[ \eta_1^* \tilde{Z}^d(G, V \oplus R, W \oplus R) = \tilde{Z}^d(G, V \oplus R, W \oplus R). \]

Assuming these two lemmas, we can prove Proposition 3.17.

**Proof of Proposition 3.17.** Let \(d\) be no smaller than \(d_1\) of Proposition 3.14, \(d_3\) of Lemma 3.26, and \(d_4\) of Lemma 3.27. Lemma 3.26 and Lemma 3.27 imply that \(\eta\) is transverse to \(\tilde{Z}^d = \tilde{Z}^d(G, V \oplus R, W \oplus R)\) and

\[ \eta^* \tilde{Z}^d = (\eta_1 \circ \eta_2)^* \tilde{Z}^d = \eta_2^* \eta_1^* \tilde{Z}^d = \eta_2^* \tilde{Z}^d = \tilde{Z}^d. \]

If \(l \geq d_1\) and \(d \geq l\), then by Proposition 3.14, the composition \(\eta \circ \sigma_{dl}\) is also transverse to \(\tilde{Z}^d\) and

\[ (\eta \circ \sigma_{dl})^* \tilde{Z}^d = \sigma_{dl}^* \tilde{Z}^d = \tilde{Z}^d. \]

Therefore the transversality claim about \(\eta\) is true for \(d \geq d_1(G, V, W)\). As \(\tilde{Z}^d\) has connected strata, it is equal to the refined pullback \(\eta^* \tilde{Z}^d\). Moreover, when \(d' > d\), the arrows of the diagram (3.5) are all induced from refined pullbacks of Whitney stratifications. As refined pullback is functorial, the diagram commutes. \(\square\)

The proof strategy of Lemma 3.26 and Lemma 3.27 is similar to those of Proposition 3.14 and Proposition 3.16. We need to construct a submersion in the opposite direction.
Proof of Lemma 3.26. Define
\[
\zeta_2 : (V \oplus R) \times \overline{\text{Poly}}_G^d(V \oplus R, W \oplus R) \to (V \oplus R) \times \overline{\text{Poly}}_G^d(V \oplus R, W \oplus R)
\]
\[
(x, y, P_0, \text{Id}_R + Q_0) \mapsto (x, y + Q_0(x), P_0, \text{Id}_R).
\]
It is straightforward to check that \(\zeta_2\) is a surjective submersion, preserves the action stratification, and \(\zeta_2^{-1}(\tilde{Z}^d) = \tilde{Z}^d\). Hence by Proposition B.20, one has \(\zeta_2^* \tilde{Z}^d = \tilde{Z}^d\). Moreover, it is easy to see that \(\zeta_2\) has connected fibers. So \(\zeta_2^* \tilde{Z}^d = \tilde{Z}^d\). Moreover, since \(\zeta_2 \circ \eta_2\) is the identity, one has
\[
\eta_2^* \tilde{Z}^d = \eta_2^* (\zeta_2^* \tilde{Z}^d) = (\zeta_2 \circ \eta_2)^* \tilde{Z}^d = \tilde{Z}^d.
\]

Now we prove Lemma 3.27. The construction of a left inverse of \(\eta_1\) is more involved. First we need the following result.

Proposition 3.28. [Sch80, Proposition 6.8] Let \(V_1\) and \(V_2\) be finite-dimensional representations of \(G\) and let \(U_1 \subseteq V_1\) be a \(G\)-invariant open subset. Then the space of \(G\)-invariant holomorphic maps \(\text{Map}_G^d(U_1, V_2)\) from \(U_1\) to \(V_2\) is a module over the space of \(G\)-invariant holomorphic functions \(\text{Map}_G^d(U_1, \mathbb{C})\) generated by \(\text{Poly}_G^d(V_1, V_2)\).

Proof of Lemma 3.27. Define a kind of “left inverse” to \(\eta_1\). For any \(d \geq 0\) and any \(Q \in \text{Poly}_G^d(V \oplus R, R)\), consider the equation
\[
y + Q(x, y) = 0.
\]
By the implicit function theorem, there exists a \(G\)-invariant open neighborhood
\[
O^d \subseteq V \times \text{Poly}_G^d(V \oplus R, R)
\]
of \(V \times \text{Poly}_G^d(V, R)\), where the second factor is viewed as elements in \(\text{Poly}_G^d(V \oplus R, R)\) not depending on \(R\), such that when \((x, Q) \in O^d\), we can always solve \(y\) in terms of \(x\). More precisely, there is a \(G\)-equivariant complex analytic map
\[
\mathcal{F} : O^d \to R
\]
such that
\[
\mathcal{F}(x, Q) + Q(x, \mathcal{F}(x, Q)) = 0 \text{ and } \mathcal{F}(x, Q_0) = -Q_0(x), \forall Q_0 \in \text{Poly}_G^d(V, R).
\]
Notice that \(\mathcal{F} : O^d \to R\) is no longer a polynomial map. However, by Proposition 3.28, we can write it as a combination
\[
\mathcal{F}(x, Q) = \sum_{j=1}^n f_j(x, Q)P_j(x, Q)
\]
where \(f_j : O^d \to \mathbb{C}\) is a \(G\)-invariant holomorphic function and \(P_j\) is \(G\)-equivariant polynomial map from \(V \oplus \text{Poly}_G^d(V \oplus R, R)\) to \(R\).

Now set
\[
U^d := \{ (x, y, P, \text{Id}_R + Q) \in (V \oplus R) \times \text{Poly}_G^d(V \oplus R, W \oplus R) \mid \text{Id}_R + Q \in O^d \}
\]
which is a \(G\)-invariant open subset of \((V \oplus R) \times \text{Poly}_G^d(V \oplus R, W \oplus R)\). Define
\[
\tilde{\zeta}_1 : U^d \to (V \oplus R) \times \text{Poly}_G^d(V \oplus R, W \oplus R)
\]
\[
(x, y, P, \text{Id}_R + Q) \mapsto (x, y, \tilde{P}, \text{Id}_R + \tilde{Q}).
\]
where

\[
\tilde{P}(\cdot) = P \left( \cdot, \sum_{j=1}^n f_j(x, Q) P_j(\cdot, Q) \right), \quad \tilde{Q}(\cdot) = Q \left( \cdot, \sum_{j=1}^n f_j(x, Q) P_j(\cdot, Q) \right).
\]

Indeed, \( \tilde{P} \) and \( \tilde{Q} \) are polynomial maps in variable \( x \in V \) because \( P, Q \) are polynomials in \( x, y \) and \( P_j \) are polynomials in \( x \). As \( f_j \) is \( G \)-invariant and \( P_j \) is \( G \)-equivariant, one can also check \( \tilde{P} \) and \( \tilde{Q} \) are equivariant. Moreover, there exists \( d \geq d \) (depending on \( d, G, V, W \)) such that \( \tilde{P} \) and \( \tilde{Q} \) have degrees at most \( d \). Hence we obtain an analytic map

\[
\zeta_1 : U^d \to (V \oplus R) \times \overline{\text{Poly}}_G^d(V \oplus R, W \oplus R).
\]

Claim. Let

\[
\tilde{\phi} : (V \oplus R) \times \overline{\text{Poly}}_G^d(V \oplus R, W \oplus R) \to (V \oplus R) \times \overline{\text{Poly}}_G^d(V \oplus R, W \oplus R)
\]

be the natural inclusion. Then \( \tilde{\zeta}_1 \circ \eta_1 = \tilde{\phi} \).

Proof of the claim. Indeed, if we insert \( P = P_0 \) and \( Q = Q_0 \) into the above formula of \( \tilde{P} \) and \( \tilde{Q} \), as \( P_0 \) and \( Q_0 \) only depend on \( x \), it follows that \( \tilde{P} = P_0 \) and \( \tilde{Q} = Q_0 \).

End of the proof of the claim.

On the other hand, similar to the proof of Proposition 3.14, there exists a \( G \)-invariant left inverse obtained by expressing high degree monomials in terms of low degree ones

\[
\tilde{\psi} : (V \oplus R) \times \overline{\text{Poly}}_G^d(V \oplus R, W \oplus R) \to (V \oplus R) \times \overline{\text{Poly}}_G^d(V \oplus R, W \oplus R)
\]
to \( \tilde{\phi} \). Consider the composition

\[
\zeta_1 := \tilde{\psi} \circ \tilde{\zeta}_1 : U^d \to (V \oplus R) \times \overline{\text{Poly}}_G^d(V \oplus R, W \oplus R)
\]

Now we check the following items.

1. \( \zeta_1 \) preserves the action stratification.
2. By the above claim, \( \zeta_1 \) is a left inverse to \( \eta_1 \).
3. \( \zeta_1^{-1}(\tilde{Z}^d) = Z^d \cap U^d \).\(^8\) Indeed, if \( \tilde{P}(x) = 0 \) and \( y + \tilde{Q}(x) = 0 \), then

\[
P(x, F(x, Q)) = 0, \quad y + Q(x, F(x, Q)) = 0,
\]

then by the definition of \( F(x, Q) \), it follows that \( y = F(x, Q) \). Hence \( P(x, y) = 0 \) and \( y + Q(x, y) = 0 \). Hence \( \zeta_1^{-1}(\tilde{Z}^d) \subseteq Z^d \). On the other hand, if \( (x, y, P, \text{id}_R + Q) \in Z^d \cap U^d \), then \( P(x, y) = 0 \) and \( y + Q(x, y) = 0 \) and \( y = F(x, Q) \) is the only solution. Hence \( \zeta_1(Z^d \cap U^d) = \tilde{Z}^d \).

Then by Proposition B.20, one has

\[
\zeta_1^* \tilde{Z}^d \equiv \tilde{Z}^d |_{Z^d \cap U^d}.
\]

On the other hand, as \( \zeta_1 \circ \eta_1 \) is the identity map, \( \eta_1 \) is transverse to \( \tilde{Z}^d \). Hence

\[
\eta_1^* \tilde{Z}^d = \eta_1^* (\tilde{Z}^d |_{Z^d \cap U^d}) \equiv \eta_1^* (\zeta_1^* \tilde{Z}^d) = (\zeta_1 \circ \eta_1)^* \tilde{Z}^d = \tilde{Z}^d.
\]

\(^8\)Notice that \( \zeta_1 \) does not preserve the evaluation but only preserves the zero set.
3.7. Proof of Proposition 3.15. From the previous proofs of various properties of the canonical Whitney stratification, one can see the pattern is to consider the inclusion map of a space of special polynomial maps to a larger space of more general polynomial maps and then create a left inverse, which is typically an analytic submersion. In those situations, the comparisons of Whitney stratifications appeal to Proposition B.20. However, the construction for proving Proposition 3.15 here is more complicated. The map we constructed is no long an analytic submersion nor a left inverse. As a result, one needs to appeal to Lemma B.6 which is more technical than Proposition B.20.

Definition 3.29. An averaging map is an $H$-equivariant smooth map of the form
\[ \kappa: M^d(H, V, W)^+_H \to M^d(G, V, W)^+_H, \quad \kappa(v, Q) = (v, \kappa(v, Q)). \]

such that $ev(\kappa(v, Q)) = Q(v) \in W$.

Roughly speaking, an averaging map provides a way of producing a (more symmetric) $G$-equivariant map from a (less symmetric) $H$-equivariant map which preserves the evaluation map.

The proof of Proposition 3.15 is based on the following three lemmas.

Lemma 3.30. When $d$ is sufficiently large, there exists an averaging map.

Proof. See Subsubsection 3.7.1. □

The next two lemmas are to verify part of the assumptions of Lemma B.6.

Lemma 3.31. For any averaging map $\kappa$, the following are true.

1. $\kappa \circ \mu: M^d(G, V, W)^+_H \to M^d(G, V, W)^+_H$ is transverse to $Z^d(G, V, W)|_{Z^d(G, V, W)^+_H}$.

2. $\mu \circ \kappa: M^d(H, V, W)^+_H \to M^d(H, V, W)^+_H$ is transverse to $Z^d(H, V, W)|_{Z^d(H, V, W)^+_H}$.

Proof. See Section 3.7.3. □

Lemma 3.32. Suppose there exists an averaging map $\kappa$. Then $\mu|_{M^d(G, V, W)^+_H}$ is transverse to $Z^d(H, V, W)$ and $\kappa$ is transverse to $Z^d(G, V, W)|_{Z^d(G, V, W)^+_H}$.

Proof. By Lemma 3.31 (cf. Lemma B.4), $\mu$ is transverse to $Z^d(H, V, W)$ at points on $\text{Im}(\kappa)$ where $\kappa$ is any averaging map. To show that $\mu$ is transverse to $Z^d(H, V, W)$, it suffices to show that any $(v_0, P_0) \in Z^d(G, V, W)^+_H$ lies in the image of some averaging map. Let $\kappa$ be an averaging map. Consider the following vector field on $M^d(G, V, W)^+_H$
\[ w_{P_0}(v, P) = \left(0, P_0 - \kappa(v, P_0)\right). \]

Its flow is the family of diffeomorphisms
\[ F_{P_0, t}(v, P) = \left(v, P + t(P_0 - \kappa(v, P_0))\right). \]

It is easy to see that $F_{P_0, t}$ preserves the evaluation map and is $H$-equivariant. Hence the composition
\[ F_{P_0, t} \circ \kappa: M^d(H, V, W)^+_H \to M^d(G, V, W)^+_H \]
is an averaging map. Moreover,
\[ F_{P_0,1} \circ \kappa(\mu(v_0, P_0)) = (v_0, \kappa(\mu(v_0, P_0)) + (P_0 - \kappa(\mu(v_0, P_0)))) = (v, P_0). \]
Hence \((v_0, P_0)\) is on the image of some averaging map. Therefore, \(\mu\) is transverse to \(\mathcal{S}^d(H, V, W)\).

The case for \(\kappa\) is similar. Lemma 3.31 (cf. Lemma B.4) implies that \(\kappa\) is transverse to \(\mathcal{S}^d(G, V, W)\) at points on \(\text{Im}(\mu)\). We would like to show that \(\kappa\) is transverse to \(\mathcal{S}^d(G, V, W)\) at any given \((v_0, Q_0) \in Z^d(H, V, W)_H^+\). Define the diffeomorphism
\[ g_{Q_0}(v, Q) = (v, Q + Q_0 - \mu(\kappa(v, Q_0))). \]
Then \(g_{Q_0}\) preserves the evaluation map. Then the composition \(\kappa \circ g_{Q_0}\) is also an averaging map and hence \(\kappa \circ g_{Q_0}\) is transverse at points on \(\text{Im}(\mu)\). In particular, it is transverse at the point \(\mu(\kappa(v_0, Q_0))\). But this is equivalent to that \(\kappa\) is transverse to \(\mathcal{S}^d(G, V, W)\) at the point
\[ g_{Q_0}(\mu(\kappa(v_0, Q_0))) = (v_0, \mu(\kappa(v_0, Q_0)) + Q_0 - \mu(\kappa(v_0, Q_0))) = (v_0, Q_0). \]
Therefore \(\kappa\) is transverse to \(\mathcal{S}^d(G, V, W)\).

**Proof of Proposition 3.15.** Notice that on the open subset
\[ M^d(G, V, W)_H^+ \subseteq M^d(G, V, W) \]
the action stratification coming from the \(G\)-action coincides with the action stratification coming from the \(H\)-action, as all the strata on both sides are indexed by \(V\)-essential subgroups of \(H\). Denote by
\[ \mathcal{WS}^\infty(Z^d(G, V, W)^+_H; H) \subseteq \mathcal{WS}^\infty(Z^d(G, V, W)^+_H) \]
the set of smooth Whitney stratifications which respect the action stratification. Similarly, one has the corresponding set \(\mathcal{WS}^\infty(Z^d(H, V, W)^+_H; H)\) of Whitney stratifications. Then all the Whitney stratifications contained in the statement of Proposition 3.15 belong to the corresponding subsets. Then by Lemma 3.31, 3.32 and Lemma B.6, it follows that
\[ \mu^* \mathcal{S}^d(H, V, W)|_{Z^d(G, V, W)^+_H} \equiv \mathcal{S}^d(G, V, W)|_{Z^d(G, V, W)^+_H}. \]
Moreover, as both \(\mathcal{S}^d(H, V, W)\) and \(\mathcal{S}^d(G, V, W)|_{Z^d(G, V, W)^+_H}\) have connected strata, one has
\[ \mu^* \mathcal{S}^d(H, V, W)|_{Z^d(G, V, W)^+_H} = \mathcal{S}^d(G, V, W)|_{Z^d(G, V, W)^+_H}. \]
Then consider the induced map
\[ \mu_* : \mathcal{S}^d(G, V, W)|_{Z^d(G, V, W)^+_H} \to \mathcal{S}^d(H, V, W)|_{Z^d(G, V, W)^+_H}. \]
Using an average map \(\kappa\) provided by Lemma 3.30 and its property provided by Lemma 3.31, we see \(\kappa_*\) is an inverse of \(\mu_*\). Lastly, since all arrows of the diagram (3.3) are induced by pullbacks of Whitney stratifications and pullback is functorial, the diagram (3.3) commutes. \(\square\)
3.7.1. Proof of Lemma 3.30.

**Lemma 3.33.** Denote $d_0 = |G|$. There exists an $H$-invariant smooth map

$$V_H^+ \to \text{Poly}^{d_0}_H(V, \mathbb{C})$$

denoted by $v \mapsto L_v$, such that

$$L_v(\tau v) = \begin{cases} 1, & \text{if } \tau \in H, \\ 0, & \text{if } \tau \notin H. \end{cases}$$  \hspace{1cm} (3.8)

**Proof.** We first construct a map $v \mapsto L''_v \in \text{Poly}^{d_0}(V, \mathbb{C})$ satisfying (3.8) without the $H$-invariance condition. For each $v \in V_H^+$, because $G_v$ is contained in $H$, $\tau_1 v \neq \tau_2 v$ if $\tau_1 \in H$ and $\tau_2 \notin H$. Consider a 1-dimensional subspace $S_v \subseteq V$ and a complement $S^+_v$. Let $p_v : V \to S_v$ be the projection induced from the splitting $V \cong S_v \oplus S^+_v$. One can choose $S_v$ and $S^+_v$ such that for a small open neighborhood $O(p_v)$ of $v$,

$$p_v(\tau_1 v') \neq p_v(\tau_2 v') \text{ if } \tau_1 \in H, \tau_2 \notin H, v' \in O(v).$$

Then by the Lagrange interpolation method, there exists a complex polynomial $f_{v,v'} : S_v \to \mathbb{C}$ (given by the exact formula of Lagrange interpolation) of degree at most $d_0 = |G|$ such that

$$f_{v,v'}(p_v(\tau v')) = \begin{cases} 1, & \tau \in H, \\ 0, & \tau \notin H. \end{cases}$$

Then define the polynomial

$$L_{v,v'} := f_{v,v'} \circ p_v : V \to \mathbb{C}, \forall v' \in O(v).$$

Then $L_{v,v'}(\tau v') = 1$ if $\tau \in H$ and $L_{v,v'}(\tau v') = 0$ otherwise.

For each $v \in V_H^+$, we can find as above an open neighborhood $O(v_i)$ of $v_i$ and a function $v' \mapsto L_{v,v'} \in \text{Poly}^{d_0}(V, \mathbb{C})$. As $V_H^+$ is Lindelöf, one can cover it by a countable subcover $O(v_i)$, $i = 1, \ldots$. Find a subordinate partition of unity $\rho_i$ and define

$$v \mapsto L''_v := \sum_{i=1}^{\infty} \rho_i(v) L_{v_i,v}.$$ 

The local finiteness of the partition of unity implies that this is a smooth map from $V_H^+$ to $\text{Poly}^{d_0}(V, \mathbb{C})$. Moreover,

$$L''_v(\tau v')|_{v' = v} = \sum_{i=1}^{\infty} \rho_i(v) L_{v_i,v}(\tau v')|_{v' = v} = \begin{cases} 1, & \tau \in H, \\ 0, & \tau \notin H. \end{cases}$$

Then define

$$L'_v := \frac{1}{|H|} \sum_{\tau \in H} L''_v \circ \tau \in \text{Poly}^{d_0}_H(V, \mathbb{C})$$

which still satisfies (3.8), and define

$$L_v := \frac{1}{|H|} \sum_{\tau \in H} L'_{\tau v} \in \text{Poly}^{d_0}_H(V, \mathbb{C})$$

so the map $v \mapsto L_v$ is $H$-invariant satisfying (3.8). \qed
Now we define a map
\[ \kappa : M_{d+p}^d(V, H, W) \to \text{Poly}_{d+d_0}(V, W) \]
\[ (v, Q) \mapsto \frac{1}{|H|} \sum_{\tau \in G} \tau^{-1} \circ (L_o Q) \circ \tau. \]
Define
\[ \kappa'(v, Q) = (v, g'(v, Q)) \in M_{d+d_0}(G, V, W). \]
On the other hand, when \( d \geq d_1 \), Lemma 3.24 provides a \( G \)-invariant holomorphic map
\[ \rho : V \times \text{Poly}_{d+d_0}(V, W) \to V \times \text{Poly}_{d}^d(V, W) \]
satisfying the two conditions of Lemma 3.24. Then define
\[ \kappa := \rho \circ \kappa'. \]
As both \( \rho \) and \( \kappa' \) preserve the evaluation map, \( \kappa \) is an averaging map. This concludes the proof of Lemma 3.30.

3.7.2. More preparations for Lemma 3.31.

**Lemma 3.34.** Let \( (G, F, E) \) be a family NC triple over \( X \) (Definition 2.45). Let \( \phi : M^d(G, F, E) \to M^d(G, F, E) \) be a fiberwise diffeomorphism which preserves the action stratification and which preserves the subbundle \( Z^d(G, F, E) \) set-wise. Then \( \phi^*Z^d(G, F, E) = Z^d(G, F, E). \)

**Proof.** Notice that the fiberwise \( G \)-action gives a family version of the action stratification \( \mathcal{M} \) on \( M^d(G, F, E) \). We first observe that \( Z^d(G, F, E) \) is a minimal Whitney stratification which respects \( \mathcal{M} \) (Definition 3.4). Indeed, as the bundles are locally trivial, the restriction of \( Z^d(G, F, E) \) to the preimage to any small open subset \( U \subseteq X \) is minimal among all Whitney stratifications which respects \( \mathcal{M} \). As the minimality is defined via the comparison of the dimension filtration, which is a local condition, the minimality of \( Z^d(G, F, E) \) follows. Then by Proposition B.15, it follows that \( \phi^*Z^d(G, F, E) = Z^d(G, F, E) \). Moreover, when \( X \) is connected, \( Z^d(G, F, E) \) has connected strata; as \( \kappa \) is a diffeomorphism, the pullback \( \phi^*Z^d(G, F, E) \) also has connected strata over each connected component of \( X \). Hence one has the equality as claimed.

Recall that for any lift \( p : F \to \text{Poly}_{d}^d(F, E) \) of an NC bundle map \( S \in C_G^d(F, E) \) (see Definition 2.46), its graph is the subset
\[ \text{graph}(p) = \{(v, p(v)) \mid v \in F\} \subseteq M^d(G, F, E). \]
Parker [Par13] observed the following. Two maps \( p_1, p_2 : F \to \text{Poly}_{d}^d(F, E) \) are lifts of the same bundle map \( S \in C_G^d(F, E) \) if and only if
\[ \text{graph}(p_1 - p_2) \subseteq Z^d(G, F, E). \]
We reprove a result of Parker [Par13].

**Lemma 3.35.** (cf. [Par13, Lemma 4.10]) Let \( (G, F, E) \) be as above. Fix an integer \( d \geq 0 \). Let \( F' \subseteq F \) be an open subset of the total space \( F \) and let \( p_1, p_2 : F' \to \text{Poly}_{d}^d(F, E) \) be two smooth bundle maps such that
\[ \text{graph}(p_1 - p_2) \subseteq Z^d(G, F, E). \]
Then $\text{graph}(p_1)$ is transverse to $Z^d(G, F, E)$ if and only if $\text{graph}(p_2)$ is transverse to $Z^d(G, F, E)$. Moreover, in this case, the two Whitney stratifications on $S_{p_1}^{-1}(0) = S_{p_2}^{-1}(0)$ pulled back by $\text{graph}(p_1)$ and by $\text{graph}(p_2)$ coincide.

**Proof.** Suppose the graph of $p_1$ is transverse to $Z^d(G, F, E)$. For any $(v_0, p_2(v_0)) \in Z^d(G, F, E)$, we would like to show that the graph of $p_2$ is transverse to $Z^d(G, F, E)$ at this point $(v_0, p_2(v_0))$. Choose a compactly supported cut-off function $\rho_0 : F' \to [0, 1]$ which is identically 1 near $v_0$. Consider the smooth vertical vector field on the total space of the vector bundle $F \oplus \overline{\text{Poly}}^d_G(F, E)$ defined by

$$f(v, P) = (0, \rho_0(v)(p_1(v) - p_2(v))).$$

(3.9)

Accordingly, the flow of $f$ is the 1-parameter family of fiber-preserving diffeomorphisms

$$\phi_t(v, P) = (v, P + t\rho_0(v)(p_1(v) - p_2(v)))$$

of $M^d(G, F, E)$ which exists for all time $t$. It is also easy to see that $\phi_t$ preserves the action stratification on $M^d(G, F, E)$ and the set $Z^d(G, F, E)$. Hence by Lemma 3.34, $\phi_t$ pulls back the canonical Whitney stratification on $Z^d(G, F, E)$ to itself. Moreover, $\phi_t$ maps a neighborhood of $(v_0, p_2(v_0))$ in $\text{graph}(p_2)$ to a neighborhood of $(v_0, p_1(v_0))$ in $\text{graph}(p_1)$. Hence the graph of $p_2$ is also transverse to $Z^d$ at $v_0$.

To show that the two pullback Whitney stratifications agree, we just need to check locally around a point $v_0 \in \text{graph}(p_1)^{-1}(Z^d) = \text{graph}(p_2)^{-1}(Z^d) \subseteq F'$. Let $\phi_1$ be the time-1 map of the flow of the vector field (3.9). Then in a neighborhood of $v_0 \in F'$, as maps, one has

$$\text{graph}(p_1) = \phi_1 \circ \text{graph}(p_2).$$

On the other hand, as $\phi_1$ preserves the canonical Whitney stratification $Z^d$, it follows that near $v_0$, $\text{graph}(p_1)^*Z^d = \text{graph}(p_2)^*\phi_1^*Z^d = \text{graph}(p_2)^*Z^d$. □

3.7.3. **Proof of Lemma 3.31.** Consider a family NC triple $(G, F, E)$ over the base $X = \overline{\text{Poly}}^d_G(V, W)$ with

$$F := X \times V, \quad E := X \times W.$$ 

Consider the subbundle

$$F^+_H := X \times V^+_H \subseteq F.$$ 

Then $\kappa \circ \mu$ can be viewed as a bundle map

$$p : F^+_H \to \overline{\text{Poly}}^d_G(F, E) \cong X \times \text{Poly}^d_G(V, W)$$

$$(P, v) \mapsto (P, \mu(v, P))$$

On the other hand, there is another natural bundle map

$$p_0 : F^+_H \to \overline{\text{Poly}}^d_G(F, E),$$

$$(P, v) \mapsto (P, P).$$

Then the construction implies that

$$\text{graph}(p - p_0) \subseteq Z^d(G, F, E).$$

As $\text{graph}(p_0)$ is transverse to the canonical Whitney stratification on $Z^d(G, F, E)$, by Lemma 3.35 the graph of $p$ is also transverse to it. Equivalently, it means
Let $\kappa \circ \mu$ is transverse to $\mathfrak{Z}^d(G,V,W)$. As $\text{graph}(p_0)$ pulls back the canonical Whitney stratification on $Z^d(G,F,E)$ to itself, it is the same for $\text{graph}(p_1)$. Equivalently, this means $(\kappa \circ \mu)^* \mathfrak{Z}^d(G,V,W) = \mathfrak{Z}^d(G,V,W)|_{Z^d(G,V,W)_H^+}$.

For the case of $\mu \circ \kappa$, consider another family NC triple $(H, \tilde{F}, \tilde{E})$ over the base $Y = \text{Poly}^d_H(V,W)$ with

\[ \tilde{F} := Y \times V, \quad \tilde{E} := Y \times W. \]

Then there is a subbundle

\[ \tilde{F}^+ := Y \times V^+ \subseteq F. \]

Then $\mu \circ \kappa$ can be viewed as a bundle map

\[ \hat{\mu} : \tilde{F}^+ \to \text{Poly}^d_H(\tilde{F}, \tilde{E}) \]

\[ (Q,v) \mapsto (Q, \mu(\kappa(v,Q))). \]

There is another bundle map

\[ \hat{\mu}_0(Q,v) = (Q,Q). \]

Then $\text{graph}(\hat{\mu} - \hat{\mu}_0) \subseteq Z^d(H,\tilde{F},\tilde{E})$. Then by Lemma 3.35, $\text{graph}(\hat{\mu})$ is transverse to $\mathfrak{Z}^d(H,\tilde{F},\tilde{E})$ and pulls it back to the canonical Whitney stratification on $\tilde{F}^+$. It is equivalent to that $\mu \circ \kappa$ is transverse to $\mathfrak{Z}^d(H,V,W)$ and pulls it back to $\mathfrak{Z}^d(H,V,W)_H^+$.

4. FOP Transverse Sections

In this section we define the space of FOP transverse sections and prove Theorem 1.1. Roughly speaking, a normally complex section is FOP transverse if it induces a certain map that is transverse to the variety $Z$ equipped with the canonical Whitney stratification specified in the previous section. The properties of the canonical Whitney stratifications allow us to deduce properties of FOP transverse sections. Within this section, let $(U, \mathcal{E})$ be an NC pair (Definition 2.17), i.e., a normally complex vector bundle $\mathcal{E}$ over a normally complex orbifold $U$.

4.1. FOP transverse bundle maps. We first discuss the FOP transversality on the local model. Let $(G,F,E)$ be a family NC triple over a smooth manifold $X$. Recall that one has the space

\[ C^NC_G(F,E) \subseteq C^NC_G(F,E) \]

of normally complex bundle maps (Definition 2.45).

**Definition 4.1.** Consider an NC bundle map $S \in C^NC_G(F,E)$ and a lift $p : F \to \text{Poly}^d_G(F,E)$ (Definition 2.46).

1. The lift $p$ is said to be **FOP transverse** at $(x,v) \in F$ if $d \geq d_1(G,V,W)$ of Proposition 3.14 where $V$ and $W$ are fibers of $F$ and $E$ respectively, and if the graph $\text{graph}(p) \subseteq M^d(G,F,E)$ is transverse to the canonical Whitney stratification of $Z^d(G,F,E)$ at $(x,v,p(x,v))$.

2. $S$ is said to be **FOP transverse** at $(x,v) \in F$ if there exists a lift $p$ which is FOP transverse at $(x,v)$.

We show that the transversality condition is independent of the lift.
Lemma 4.2. Let \( p : F \to \text{Poly}^d_G(F, E) \) and \( p' : F \to \text{Poly}^d_G(F, E) \) be two lifts of \( S : F \to E \), then \( p \) is FOP transverse at \((x, v)\) if and only if \( p' \) is FOP transverse at \((x, v)\).

Proof. If \( d \leq d' \), we may identify \( p \) with a map \( q : F \to \text{Poly}^d_G(F, E) \). Then by Proposition 3.14, \( p \) is transverse to \( 3^d(G, F, E) \) at \((x, v)\) if and only if \( q \) is transverse to \( 3^{d'}(G, F, E) \) at \((x, v)\). Hence we may assume \( d' = d \). Moreover, as \( p \) and \( p' \) induce the same section \( S \), it follows that graph\( (p - p') \subseteq Z^d(G, F, E) \). Then by Lemma 3.35, the two transversality conditions are equivalent.

Lemma 4.3. The set of points \((x, v) \in F\) at which an NC bundle map \( S \in C^\text{NC}_G(F, E) \) is FOP transverse is open in \( F \).

Proof. Choose a lift \( p : F \to \text{Poly}^d_G(F, E) \) for a sufficiently large \( d \). Consider the intersection between graph\( (p) \) and \( Z^d(G, F, E) \). Trotman [Tro78] proved that the set of transverse points to a Whitney stratified subset is open. Therefore, the set of points on graph\( (p) \) where the intersection is transverse is open. As graph\( (p) \) is homeomorphic to \( F \), the lemma follows.

Another crucial feature is related to the change of group. Recall that if \( H \subseteq_F G \) is an essential subgroup, then there is the basic decomposition

\[
F = F_H \oplus \bar{F}_H
\]

and the total space \( F \) is canonically identified with the total space of \( \pi^*_F \bar{F}_H \to F_H \). Then consider the canonical map defined in the proof of Lemma 2.44

\[
\mu : M^d(G, F, E) \to M^d(H, F, E)
\]

which induces a natural map between spaces of NC bundle maps, which, by abuse of notation, was denoted by

\[
\mu : C^\text{NC}_G(F, E) \to C^\text{NC}_H(\pi^*_F \bar{F}_H, \pi^*_F E).
\]

Lemma 4.4. Suppose \( S \) is FOP transverse at a point \((x, v)\) whose stabilizer is contained in \( H \), then \( \mu(S) \) is FOP transverse at \((x, v) \in F \).

Proof. Choose a lift \( p : F \to \text{Poly}^d_G(F, E) \) of \( S \). Using the map (4.1) one obtains an \( H \)-invariant bundle map \( q : F \to \text{Poly}^d_H(F, E) \) so that

\[
\mu(\text{graph}(p)) = \text{graph}(q).
\]

Since the stabilizer of \((x, v)\) is contained in \( H \), using the family version (which is a trivial extension) of Proposition 3.15, one can see that graph\( (q) \) is transverse to \( 3^d(H, F, E) \) at \((x, v, q(x, v))\). Therefore, as an \( H \)-equivariant NC bundle map, \( S \) is FOP transverse at \((x, v)\).

On the other hand, under the natural identification

\[
M^d(H, F, E) \cong M^d(H, \pi^*_F \bar{F}_H, \pi^*_F E)
\]

\( q \) becomes a lift of the (same) \( H \)-equivariant bundle map \( S : \pi^*_F \bar{F}_H \to \pi^*_F E \) and the Whitney stratification \( 3^d(H, F, E) \) is identical to \( 3^d(H, \pi^*_F \bar{F}_H, \pi^*_F E) \). Therefore, \( \mu(S) \) is transverse to the latter at the corresponding point and hence \( \mu(S) \) is FOP transverse at \((x, v)\).
Next we examine more carefully the FOP transversality condition aiming at proving the existence and density of transverse elements. Let $G, F, E$ be as before. Let $\text{proj}_2 : M^d(G, F, E) \to \text{Poly}^d_G(F, E)$ be the vector bundle projection.

**Lemma 4.5.** Suppose $F_G = 0$ (so $F$ is a complex vector bundle). Then the graph of $p : F \to \text{Poly}^d_G(F, E)$ is transverse to a stratum $Z_\alpha \in Z^d(G, F, E)$ at a point $(x, 0) \in X \subseteq F$ if and only if $p|_X : X \to \text{Poly}^d_G(F, E)$ is transverse to the smooth map $\text{proj}_2 : Z_\alpha \to \text{Poly}^d_G(F, E)$ at $x$.

**Proof.** We may assume that $F$ and $E$ are trivial with fibers $V$ and $W$ respectively. Then $p$ is equivalent to a $G$-invariant map

$$p : X \times V \to \text{Poly}^d_G(V, W).$$

Given $(x, 0) \in X \subseteq F$, the tangent space of graph($p$) at the corresponding point is

$$\left\{ \left( \eta, \xi, D_x p(x, 0)(\eta) + D_z p(x, 0)(\xi) \right) : \eta \in V, \ \xi \in T_x X \right\} \subseteq T_x X \oplus V \oplus \text{Poly}^d_G(V, W).$$

As $V$ contains no trivial representations and $p$ is $G$-invariant, the partial derivative $D_z p$ vanishes along the zero section. Hence the above tangent space to graph($p$) is $T_x X \oplus V \oplus \text{Im}(D_z p)$. As $\text{proj}_2$ collapses the $V$-direction, the transversality is equivalent to the transversality of $\text{Im}(D_x p)$ to $\text{proj}_2 : Z_\alpha \to \text{Poly}^d_G(V, W)$.

**Lemma 4.6.** Suppose $F_G = 0$ and $p_0 : F \to \text{Poly}^d_G(F, E)$ is FOP transverse near a closed subset $Y \subseteq F$, then there exists $p : F \to \text{Poly}^d_G(F, E)$ which is FOP transverse near the base $X$ such that $p = p_0$ near $Y$. Moreover, if we equip $\text{Poly}^d_G(F, E)$ with a norm and are given $\delta > 0$, then one can require that $\| p - p_0 \| \leq \delta$.

**Proof.** By Lemma 4.5, $p_0|_X$ is transverse to $\text{proj}_2 : Z_\alpha \to \text{Poly}^d_G(F, E)$ near $Y \cap X$ for each stratum $Z_\alpha \in Z^d(G, F, E)$. Then the standard transversality result shows that there exists a map $p_1 : X \to \text{Poly}^d_G(F, E)$ which is transverse to $\text{proj}_2 : Z_\alpha \to \text{Poly}^d_G(F, E)$ for all $\alpha$, such that $p_1$ agrees with $p_0|_X$ near $Y \cap X$ and such that $p_1$ is sufficiently close to $p_0|_X$. Consider the pullback $\pi^*_F p_1 : F \to \text{Poly}^d_G(F, E)$; by abuse of notation, still denote it by $p_1$.

Now choose a pair of $G$-invariant neighborhoods $O' \subseteq O''$ of $Y$ such that the closure of $O'$ is contained in $O''$ and such that $p_0$ is FOP transverse in $O''$. Choose a $G$-invariant cut-off function $\rho : F \to [0, 1]$ supported in a sufficiently small neighborhood of $X \setminus O' \subseteq F \setminus Y$ which is also identically 1 near $X \setminus O''$. Define

$$p = (1 - \rho)p_0 + \rho p_1 : F \to \text{Poly}^d_G(F, E)$$

which agrees with $p_0$ near $Y \subseteq F$ and which is sufficiently close to $p_0$. Moreover, its restriction to the zero section is transverse to each stratum of $Z^d(G, F, E)$ under $\text{proj}_2$. Then by Lemma 4.5, $p$ is FOP transverse at each point of $X$. By Lemma 4.3, it is FOP transverse near $X$.

### 4.2. FOP transverse sections

Suppose $(\mathcal{U}, \mathcal{E})$ is equipped with a straightening $(\mathcal{U}^\#, \mathcal{E}^\#)$ (Definition 2.35). Let $\hat{\mathcal{C}} = (G, U, E, \hat{\psi})$ be a bundle chart for $E \to \mathcal{U}$. Then for each $U$-essential subgroup $H \subseteq G$, near $U_H$ a section $S : U \to E$ can be identified with a nonlinear bundle map

$$S^\#_H : N^c U_H \to E|_{U_H}.$$
Choose a lift $\hat{\psi}$. Then $S_1$ is FOP transverse at $x_1$ if and only if $S_2$ is FOP transverse at $x_2$.

Proof. It follows from the invariance property of the canonical Whitney stratification provided by Proposition 3.13. □

Lemma 4.9. Let $S : U \to E$ be a $G$-equivariant NC section. Then the set of points $x \in U$ where $S$ is FOP transverse is open.

Proof. We may assume $G_x = G$. Using the straightening, we may assume $S$ is an NC bundle map $S^\#_G \in C^\infty_G(F, E)$ with $F = NU_G$ and $x$ is a point in the base $X = U_G$. Let $p : F \to Poly^d_G(F, E)$ be a lift of $S^\#_G$. Then by definition, graph$(p) \subseteq M^d(G, F, E)$ is transverse to $Z^d(G, F, E)$ at $(x, 0, p(x, 0))$. By Trotman’s theorem, they are transverse at all nearby points $(y, v)$. If $H = G_v \subseteq G$, then the map $\mu$ in (4.1) changes $p$ to an $H$-invariant lift $\mu$ of $S^\#_G$ near $(y, v)$. By Lemma 4.4, $S^\#_G$ is FOP transverse at $(y, v)$. Therefore, the set of FOP transverse points is open. □

We can then prove the chartwise existence and $C^0$-density of FOP transverse sections.

Proposition 4.10. Let $\hat{\mathcal{C}} = (G, U, E, \hat{\psi})$ be a bundle chart of $\mathcal{E}$ and $S_0 : U \to E$ be an NC section (with respect to the fixed straightening), then for each $\delta > 0$, there exists an FOP transverse section $S : U \to E$ such that $\|S - S_0\|_{C^0} \leq \delta$. Moreover, if $Y \subseteq U$ is a $G$-invariant closed subset and $S$ is FOP transverse near $Y$, then $S$ can be chosen to agree with $S_0$ near $Y$.

Proof. Via an induction on essential subgroups, one can see it suffices to prove the following statement. Let $H \subseteq G$ be a maximal $U$-essential subgroup. Then there exists an NC section $S : U \to E$ which is FOP transverse near $Y \cup GU_H$ and which agrees with $S_0$ near $Y$. To prove this statement, first choose a $G$-invariant $\epsilon : GU_H \to \mathbb{R}_+$. As $H$ is a maximal $U$-essential subgroup, it is possible to choose $\epsilon$ such that for any $H'$ conjugate to $H$,

$$|N^\epsilon U_H| \cap |N^\epsilon U_{H'}| \neq \emptyset \implies H = H'.$$

Next, consider the induced NC bundle map $S^\#_{0, H} : N^\epsilon U_H \to E|_{U_H}$.

Choose a lift $p_0 : N^\epsilon U_H \to Poly^d_H(NU_H, E|_{U_H})$. 
Then by Lemma 4.6, there exists
\[ p_1 : N^* U_H \to \text{Poly}_H^d(NU_H, E|_{U_H}) \]
which is sufficiently close to \( p_0 \), agrees with \( p_0 \) near \( Y \cap |N^* U_H| \), and is FOP transverse near \( U_H \). Let \( S_{p_1} : N^* U_H \to E|_{N^* U_H} \) be the corresponding section defined in the tubular neighborhood \( |N^* U_H| \) of \( U_H \). Via \( G \)-action, the lift \( p_1 \) induces sections defined on \( |N^* U_H| \) for all \( H' \) conjugate to \( H \) such that the totality defines a \( G \)-equivariant NC section \[ S_1 : G|N^* U_H| \to E|_{G|N^* U_H|}. \]

Now choose a \( G \)-invariant cut-off function \( \rho : U \to [0, 1] \) supported near \( GU_H \).

Consider the section \[ S = (1 - \rho)S_0 + \rho S_1. \]

Then by Lemma 2.51, \( S \) is an NC section. Moreover, \( S \) is FOP transverse near \( GU_H \) and agrees with \( S_0 \) near \( Y \). \( \square \)

4.3. **Proof of Theorem 1.1.**

4.3.1. **FOP transverse sections on orbifolds, classical transversality, and locality.** We first formally define the space of FOP transverse sections.

**Definition 4.11.** Let \( \mathcal{U} \) be an NC orbifold (without boundary) and \( \mathcal{E} \to \mathcal{U} \) be an NC vector bundle. Recall the definition of straightenings (Definition 2.35).

1. Let \((\mathcal{U}^#, \mathcal{E}^#) \) be a straightening of \((\mathcal{U}, \mathcal{E})\), an NC section \( S \in \Gamma^{NC}(\mathcal{U}^#, \mathcal{E}^#) \) is called **FOP transverse** if for each bundle chart \( \mathcal{C} = (G, U, E, \hat{\psi}) \), the pullback section \( S : U \to E \) is FOP transverse (Definition 4.7).

2. Define \( \Gamma^{\text{FOP}}(\mathcal{U}^#, \mathcal{E}^#) \subseteq \Gamma^{NC}(\mathcal{U}^#, \mathcal{E}^#) \) to be the subset of sections which are FOP transverse.

3. By taking the union over all straightenings on \((\mathcal{U}, \mathcal{E})\), define \[ \Gamma^{\text{FOP}}(\mathcal{U}, \mathcal{E}) = \bigcup_{(\mathcal{U}^#, \mathcal{E}^#)} \Gamma^{\text{FOP}}(\mathcal{U}^#, \mathcal{E}^#) \]

and call its elements **FOP transverse sections** of \( \mathcal{E} \).

The (Classical Transversality) and (Locality) stated in Theorem 1.1 are straightforward. If \( \mathcal{U} \) is a manifold, then there is a canonical (trivial) NC structure and all smooth sections are NC sections. The equivalence between FOP transversality and classical transversality can be traced via the definition. On the other hand, if \( S \in \Gamma^{\text{FOP}}(\mathcal{U}, \mathcal{E}) \), then it is FOP transverse with respect to a certain straightening \((\mathcal{U}^#, \mathcal{E}^#)\). The straightening restricts to one over any open subset \( \mathcal{U}' \). As the FOP transversality condition is defined pointwise, it follows that \( S|_{\mathcal{U}'} \in \Gamma^{\text{FOP}}(\mathcal{U}', \mathcal{E}|_{\mathcal{U}'}) \).

4.3.2. **Density of FOP transverse sections and Extension Property I.**

**Proposition 4.12.** Given a smooth section \( S_0 \in \Gamma(\mathcal{U}, \mathcal{E}) \) which is FOP transverse near a closed subset \( \mathcal{Y} \subseteq \mathcal{U} \), for any \( \delta > 0 \), there exists an FOP transverse section \( S : \mathcal{U} \to \mathcal{E} \) which agrees with \( S_0 \) near \( \mathcal{Y} \) such that \( \| S_0 - S \|_{C^0} \leq \delta \).

**Proof.** By definition, there exists a straightening near \( \mathcal{Y} \) with respect to which \( S_0 \) is normally complex and FOP transverse. By Corollary 2.37, there exists a straightening \((\mathcal{U}^#, \mathcal{E}^#)\) of \((\mathcal{U}, \mathcal{E})\) which agrees with the existing one near \( \mathcal{Y} \). As normally complex sections form a submodule over \( C^\infty(\mathcal{U}) \), by using a cut-off function, we
may assume that $S_0$ is normally complex with respect to this straightening by the density property stated in Proposition 2.52. For the rest of the proof, we will use this straightening.

On the other hand, as $|U|$ is Lindelöf and paracompact, one can find a countable and locally finite cover of $|U|$ by bundle charts $\hat{C}_i = (G_i, U_i, E_i, \hat{\psi}_i)$, $i = 1, \ldots, \infty$. We may also find precompact $G_i$-invariant open subsets $U_i' \subseteq U_i$ such that $\{\psi_i(U_i')\}$ still covers $U$. We prove the following statement by induction on $k = 1, 2, \ldots$.

- For each $k$, there exists a section $S_k \in \Gamma^{NC}(U^\#_1, \mathcal{E}^\#_1)$ such that $S_k$ is FOP transverse near $\bigcup_{i \leq k} \psi_i(U_i')$ and which agrees with $S_0$ near $\mathcal{Y}_1$. Moreover

$$\|S_k - S_0\|_{C^0} \leq \left(1 - \frac{1}{2^k}\right) \delta.$$ 

For the $k = 1$ case, let $S_{0,1} : U_1 \to E_1$ be the pullback of $S_0$ to the chart $\hat{C}_1$. Then by Proposition 4.10, there exists $S_1' : U_1 \to E_1$ which is FOP transverse on $U_1$ and which agrees with $S_{0,1}$ near $Y_1 := \psi_1^{-1}(\mathcal{Y})$ such that $\|S_1' - S_{0,1}\|_{C^0} \leq \frac{\delta}{2}$. Choose a $G_1$-invariant cut-off function $\rho_1 : U_1 \to [0, 1]$ supported near $\overline{U_1'}$ such that $\rho_1|_{U_1'} \equiv 1$. Regarding $\rho_1$ as a function on $U$, then $\rho_1 S_1'$ defines a normally complex section of $\mathcal{E}$. Then define

$$S_1 = (1 - \rho_1)S_0 + \rho_1 S_1'$$

which is FOP transverse near $\psi_1(U_1')$ and which agrees with $S_0$ near $\mathcal{Y}$. Moreover,

$$\|S_1 - S_0\|_{C^0} \leq \frac{\delta}{2}.$$

Suppose we have proved the above statement for $k - 1$. Denote

$$\mathcal{Y}_k := \mathcal{Y} \cup \bigcup_{i < k} \psi_i(U_i') \subseteq \mathcal{Y}, \quad Y_k := \psi_k^{-1}(\mathcal{Y}_k) \subseteq U_k.$$

The latter is a $G_k$-invariant closed subset. Let $S_{k-1,k} : U_k \to E_k$ be the pullback of $S_{k-1}$ to $\hat{C}_k$. Then $S_{k-1,k}$ is FOP transverse near $Y_k$. Then by Proposition 4.10, there exists an FOP transverse section $S_k' : U_k \to E_k$ which agrees with $S_{k-1,k}$ near $Y_k$ such that

$$\|S_k' - S_{k-1,k}\|_{C^0} \leq \frac{\delta}{2^k}.$$ 

Choose a $G_k$-invariant cut-off function $\rho_k : U_k \to [0, 1]$ supported near $\overline{U_k'}$ such that $\rho_k = 1$ over $U_k'$. Also regard $\rho_k$ as a function on $U$. Then $\rho_k S_k'$ is a normally complex section of $\mathcal{E}$. Define

$$S_k = (1 - \rho_k)S_{k-1} + \rho_k S_k'.$$

By construction, the section agrees with $S_{k-1}$ near $\mathcal{Y}_k$. Moreover, by construction, it is FOP transverse near $\psi_k(U_k')$. Lastly,

$$\|S_k - S_0\|_{C^0} \leq \|S_k - S_{k-1}\|_{C^0} + \|S_{k-1} - S_0\|_{C^0} \leq \left(1 - \frac{1}{2^k}\right) \delta.$$ 

This proves the $k$-th case of the induction.

Finally, as the countable cover is locally finite, on any compact subset, the sequence $S_k$ stabilizes for $k$ sufficiently large. Hence in the limit one obtains an FOP transverse section $S$ satisfying the required conditions. $\square$
4.3.3. Extension property II. We consider the problem of extending FOP transverse sections from a suborbifold \( \mathcal{X} \subseteq \mathcal{U} \) with an ordinary normal bundle (Definition 2.4). Denote \( \mathcal{E}_\mathcal{Y} := \mathcal{E}|_\mathcal{X} \). By the discussion of Subsection 2.3, \((\mathcal{X}, \mathcal{E}_\mathcal{X})\) is an NC pair. Let \( \mathcal{S}_\mathcal{X} : \mathcal{X} \to \mathcal{E}_\mathcal{X} \) be an FOP transverse section. Then by definition, there exists a straightening \((\mathcal{X}^\#, \mathcal{E}_\mathcal{X}^\#)\) of \((\mathcal{X}, \mathcal{E}_\mathcal{X})\) such that \( \mathcal{S}_\mathcal{X} \) is an FOP transverse section with respect to this straightening. Then by Proposition 2.41, there exists a straightening \((\mathcal{U}^\#, \mathcal{E}^\#)\) which extends \((\mathcal{X}^\#, \mathcal{E}_\mathcal{X}^\#)\). Moreover, the involved structures are all straightened along \( \mathcal{N}\mathcal{X} \). Let \( \epsilon : \mathcal{X} \to \mathbb{R}_+ \) be a sufficiently small smooth function with an associated disk bundle \( \mathcal{N}\mathcal{X} \). Then the normal exponential map \( \exp^{\mathcal{N}\mathcal{X}} : \mathcal{N}\mathcal{X} \to \mathcal{U} \) is an open embedding, covered by a bundle map

\[
\text{par}^{\mathcal{N}\mathcal{X}} : \pi^{\mathcal{N}\mathcal{X}}_X \mathcal{E}|_X \to \mathcal{E}
\]

which defines a bundle isomorphism when restricting \( \mathcal{E} \) along the image of \( \exp^{\mathcal{N}\mathcal{X}} \). We claim that the pullback

\[
\text{par}^{\mathcal{N}\mathcal{X}} \circ (\pi^{\mathcal{N}\mathcal{X}}_X \mathcal{S}_X) \circ (\exp^{\mathcal{N}\mathcal{X}})^{-1} : |\mathcal{N}\mathcal{X}| \to \mathcal{E}
\]

is FOP transverse. It suffices to consider this claim locally. Let \( \tilde{\mathcal{C}} = (G, U, E, \hat{\psi}) \) be a bundle chart centered at a point \( x \in \mathcal{X} \). Then by definition, \( \psi^{-1}(\mathcal{X}) \) is a \( G \)-invariant submanifold \( \mathcal{X} \subseteq \mathcal{U} \) with a normal bundle \( \mathcal{N}\mathcal{X} \to \mathcal{X} \). The section \( \mathcal{S}_\mathcal{X} \) is pulled back to an NC section

\[
\mathcal{S}_\mathcal{X} : X \to E|_X
\]

which induces an NC bundle map (for some \( \delta : X_{G_x} \to \mathbb{R}_+ \))

\[
\mathcal{S}_{X,G_x}^\# : N^\# X_{G_x} \to E|_{X_{G_x}}.
\]

As \( G_x \) acts trivially on the normal bundle \( \mathcal{N}\mathcal{X} \to \mathcal{X} \), the trivial factor in the normal direction in \( \mathcal{N}\mathcal{X} \) does not affects the FOP transversality. Hence the pullback

\[
(\pi^{\mathcal{N}\mathcal{X}})^* \mathcal{S}_{X,G_x}^\#
\]

is still FOP transverse. Therefore, after extending the FOP transverse section \( \mathcal{S}_\mathcal{X} \) using a smooth cut-off function, we can apply Proposition 4.12 to get the desired extension.

4.3.4. Product property. For \( i = 1, 2 \), let \( (\mathcal{U}_i, \mathcal{E}_i) \) be an NC pair. Given \( \mathcal{S}_i \in \Gamma^{\text{FOP}}(\mathcal{U}_i, \mathcal{E}_i) \). Denote \( \mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2, \mathcal{E} = \mathcal{E}_1 \boxplus \mathcal{E}_2, \) and \( \mathcal{S} = \mathcal{S}_1 \boxplus \mathcal{S}_2 \). We prove that \( \mathcal{S} \) is FOP transverse. Indeed, by definition, there exists a straightening \((\mathcal{U}_i^\#, \mathcal{E}_i^\#)\) on \((\mathcal{U}_i, \mathcal{E}_i)\) such that

\[
\mathcal{S}_i \in \Gamma^{\text{FOP}}(\mathcal{U}_i^\#, \mathcal{E}_i^\#).
\]

The two straightenings induce a straightening on \((\mathcal{U}, \mathcal{E})\) by taking the product. We check that \( \mathcal{S} \) is FOP transverse at each point \( p = (p_1, p_2) \) with respect to the product straightening. If \( \tilde{\mathcal{C}}_i = (G_i, U_i, E_i, \psi_i) \) is a bundle chart of \( \mathcal{E}_i \) centered at \( p_i \), then

\[
\tilde{\mathcal{C}} = (G, U, E, \hat{\psi}) := \tilde{\mathcal{C}}_1 \times \tilde{\mathcal{C}}_2 := (G_1 \times G_2, U_1 \times U_2, E_1 \boxplus E_2, \hat{\psi}_1 \boxplus \hat{\psi}_2)
\]

is a bundle chart of \( \mathcal{E} \). Let

\[
\mathcal{S}_{G_i}^\# : N^\epsilon U_{i,G_i} \to E_i|_{U_{G_i}}
\]

be the NC bundle maps induced from the straightenings. Then for \( \epsilon < \epsilon_1, \epsilon_2 \), the bundle map induced from the product straightening

\[
\mathcal{S}_G^\# = (\mathcal{S}_{G_1}^\#, \mathcal{S}_{G_2}^\#) : N^\epsilon U_G \to E|_{U_G}
\]
is well-defined, and it is obviously an NC bundle map. Abbreviate $X_i = U_{i,G_i}$, $F_i = NU_{i,G_i}$, $E_i = E_i|_{U_{i,G_i}}$. Let $p_i : F_i \to \text{Poly}^d_{G_i}(F_i, E_i)$ be a lift of $S^\#_i$. Then

$$p := (p_1, p_2) : F_1 \boxplus F_2 \to \text{Poly}^d_G(F_1 \boxplus F_2, E_1 \boxplus E_2)$$

is a lift of $S^\#$. Then by Proposition 3.16, the FOP transversality of $p_1$ and $p_2$ implies the FOP transversality of $p$. Hence $\mathcal{S}$ is FOP transverse.

4.3.5. **Stratified transversality.** Suppose $\mathcal{S} \in \Gamma^{\text{FOP}}(\mathcal{U}^\#, \mathcal{E}^\#) \subseteq \Gamma^{\text{FOP}}(\mathcal{U}, \mathcal{E})$. Let $\gamma$ be represented by an NC triple $(G, V, W)$. Let $p \in \mathcal{U}_\gamma$. Then it suffices to consider on a chart containing $p$. Let $\hat{\mathcal{C}} = (G, U, E, \hat{\psi})$ be a bundle chart centered at $p$. Then $\psi : U_G \to \mathcal{U}_\gamma$ gives a manifold chart of $\mathcal{U}_\gamma$. Let $\gamma : U \to E$ be the section corresponding to $\mathcal{S}$. Using the straightening one obtains the bundle map

$$S^\#_G : N^\# U_G \to E|_{U_G}.$$

Using the basic decomposition $E|_{U_G} = E_G \oplus \hat{E}_G$, one obtains the two components $S_G$ and $\hat{S}_G$ of $S^\#_G$. Then by the definition of FOP transversality, the section

$$S_G : N^\# U_G \to E_G$$

is transverse at $\psi^{-1}(p) \in U_G$. As the derivative of $S_G$ in the normal direction to $U_G$ vanishes, this is equivalent to that $S_G|_{U_G}$ is transverse at $0$. This implies the stratified transversality.

4.3.6. **Stabilization property.** Start with $\mathcal{S} \in \Gamma^{\text{FOP}}(\mathcal{U}^\#, \mathcal{E}^\#) \subseteq \Gamma^{\text{FOP}}(\mathcal{U}, \mathcal{E})$. Let $\mathcal{F} \to \mathcal{U}$ be an NC vector bundle. Then by Proposition 2.34, there exist a connection $\nabla^\mathcal{F}$ straightened with respect to $g^{TU}$, an inner product $h^\mathcal{F}$ preserved by $\nabla^\mathcal{F}$, and a nearby NC structure $\mathcal{I}^\mathcal{F}$ which is straightened with respect to $g^{TU}$ and $\nabla^\mathcal{F}$ such that $(\nabla^\mathcal{F}, h^\mathcal{F}, \mathcal{I}^\mathcal{F})$ is normally Hermitian. By Proposition 2.39, the bundle metric $g^{\mathcal{I}^\mathcal{F}}$ and the bundle NC structure $\mathcal{I}^{\mathcal{I}^\mathcal{F}}$ induce a straightening $\mathcal{F}^\#$ of $\mathcal{F}$. There are also straightenings

$$(\pi^\#_F \mathcal{E})^\# \text{ and } (\pi^\#_F \mathcal{F})^\#.$$  

We claim

$$\pi^\#_F \mathcal{S} \oplus \tau^\#_F \in \Gamma^{\text{FOP}}\left(\mathcal{F}^\#, (\pi^\#_F \mathcal{E})^\# \oplus (\pi^\#_F \mathcal{F})^\#\right).$$

Still, it suffices to consider this problem locally on a single chart. Let $\hat{\mathcal{C}} = (G, U, E, \hat{\psi})$ be a bundle chart of $\mathcal{E}$. Assume $\mathcal{F}$ is pulled back to a $G$-equivariant bundle $F \to U$ on this chart, which can be achieved by shrinking $U$ if necessary. Then $\mathcal{S}$ is pulled back to an FOP transverse NC section $\tilde{S} : U \to E$. Consider the section

$$\tilde{S} := \pi^\#_F \mathcal{S} \oplus \tau^\#_F : F \to \pi^\#_F E \oplus \pi^\#_F F.$$

The bundle straightening construction implies that $\tilde{S}$ is an NC section. Moreover, if $p : N^\# U_G \to \text{Poly}^d_G(NU_G, E|_{U_G})$ is a lift of $S^\#_G$, then there is a natural lift

$$\tilde{p} : NF_G \to \text{Poly}^d_G(NF_G, \pi^\#_F E \oplus \pi^\#_F F)$$

$$= \text{Poly}^d_G(NF_G, \pi^\#_F E \oplus \pi^\#_F F_G \oplus \pi^\#_F \hat{F}_G)$$

$$= \pi^\#_F \text{Poly}^d_G(NU_G, E|_{U_G}) \oplus \pi^\#_F \text{Poly}^d_G(\hat{F}_G, \hat{F}_G) \oplus \pi^\#_F F_G$$
of $\tilde{S}_G^\#$ defined as follows. Use $u$ to denote fiber coordinates of $F$; with respect to the basic decomposition $F = F_G \oplus \tilde{F}_G$, write $u = (u_G, \tilde{u}_G)$. Then

$$\tilde{p}(x, v, u_G, \tilde{u}_G) = \left(p(x, v), \text{Id}_{\tilde{F}_G}, u_G\right).$$

In terms of graphs, it follows that

$$\eta^{-1}(\text{graph}(\tilde{p})) = \text{graph}(p)$$

where $\eta$ is the (family version of the) map (3.4). Then by Proposition 3.17, $p$ is FOP transverse at $x \in U_G$ if and only if $\tilde{p}$ is FOP transverse at $(x, 0) \in F$.

4.3. Stratification of zero locus. Suppose the virtual dimension of $(\mathcal{U}, \mathcal{E})$ is $k$. Let $\mathcal{S} : \mathcal{U} \to \mathcal{E}$ be an FOP transverse section. For each universal stratum $\theta \in \mathcal{Z}_k^{\text{univ}}$, we define the subset

$$\mathcal{S}^{-1}(0)_{\theta} \subseteq \mathcal{S}^{-1}(0)$$

by identifying its intersection with each chart. Recall that one has the map $\rho_k : \mathcal{Z}_k^{\text{univ}} \to \prod_{NC}$. For a bundle chart $\tilde{C} = (G, U, E, \tilde{\psi})$, if $\psi(U) \cap \mathcal{U}_{\rho_k(\theta)} = \emptyset$, then $\mathcal{S}^{-1}(0)_{\theta} \cap \psi(U) = \emptyset$. Otherwise, suppose $\rho_k(\theta)$ is represented by a triple $(H, V, W)$ where $H \subseteq U$, $G$, $V$ is isomorphic to fibers of $TU|_{U_{\theta}}$ and $W$ is isomorphic to fibers of $E|_{U_{\theta}}$. Choose a lift

$$p : N\epsilon U_H \to \text{Poly}^d(H, NU_H, E|_{U_{\theta}})$$

of the induced bundle map $S^\#_H : N\epsilon U_H \to E|_{U_{\theta}}$. Then one has

$$(S^\#_H)^{-1}(0) = \text{proj}_1 \left(\text{graph}(p) \cap Z^d(H, NU_H, E|_{U_{\theta}})\right)$$

where

$$\text{proj}_1 : M^d(H, NU_H, E|_{U_{\theta}}) = NU_H \oplus \text{Poly}^d(H, NU_H, E|_{U_{\theta}}) \to NU_H$$

is the projection onto the first factor. Notice that $\theta$ is represented by an element in $Z^d_H \in \mathcal{Z}_k^d(H, V, W)$. Then using the local triviality of the canonical Whitney stratification, we can define

$$(S^\#_H)^{-1}(0)_{\theta} := \text{proj}_1 \left(\text{graph}(p) \cap Z^d_H\right).$$

Then define $\mathcal{S}^{-1}(0)_{\theta}$ to be the union of images of all $(S^\#_H)^{-1}(0)_{\theta}$ under $\psi$. The FOP transversality condition implies that $\mathcal{S}^{-1}(0)_{\theta}$ is a smooth submanifold of $\mathcal{U}_{\rho_k(\theta)}$ of dimension $k + n_k(\theta)$. Therefore we have constructed a decomposition

$$\mathcal{S}^{-1}(0) = \bigsqcup_{\theta \in \mathcal{Z}_k^{\text{univ}}} \mathcal{S}^{-1}(0)_{\theta}.$$
stratum $Z_{\alpha'} \geq Z_\alpha$. Then if $y \in S^{-1}(0)_\theta$, choose a bundle chart centered at $y$ and denoted, by abuse of notation, by $\hat{C} = (G, U, E, \hat{\psi})$. Let $\hat{S} : U \to E$ be the pullback of $S$ to $\hat{C}$. Then choose a lift of $S^\#_G$

$$p : N^U_G \to \text{Poly}_G^d(NU_G, E|U_G).$$

Then

$$y \in \hat{\psi}\left(\text{proj}_1\left(\text{graph}(p) \cap Z^d_\alpha(G, NU_G, E|U_G)\right)\right).$$

As $Z_\alpha \subseteq Z_{\alpha'}$ (which means $Z_\alpha \subseteq \tilde{Z}_\alpha$), by Lemma B.2 and its proof, one sees

$$y \in \hat{\psi}\left(\text{proj}_1\left(\text{graph}(p) \cap Z^d_{\alpha'}(G, NU_G, E|U_G)\right)\right).$$

It follows that $y \in \tilde{S}^{-1}(0)_\theta$.

Lastly we need to show that the decomposition of $S^{-1}(0)$ by the smooth manifolds $S^{-1}(0)_\theta$ makes the zero locus a Thom–Mather stratified space. Indeed, locally the preimage of $S^{-1}(0)$ in each chart is a Whitney stratified space, hence it is a Whitney stratified set in $U$ in the sense of Definition C.16. Then by Theorem C.19, Proposition C.18, and Definition C.1, $S^{-1}(0)$ is a Thom–Mather stratified space.

5. Integral Euler Cycles

In this section, we introduce aspects of the bordism theory of derived orbifolds and derive homological consequences from zero loci of FOP transverse sections. We will not discuss such bordism theories in full generality; instead, we only introduce concepts that are needed to state our results. The interested reader can refer to [Par22] and [Par23] for a comprehensive treatment.

5.1. Derived orbifold bordism.

**Definition 5.1.** A derived orbifold (with or without boundary) is a triple $\mathcal{D} = (\mathcal{U}, \mathcal{E}, \mathcal{S})$ where $\mathcal{U}$ is an effective orbifold (with or without boundary), $\mathcal{E} \to \mathcal{U}$ is an orbifold vector bundle, and $\mathcal{S} : \mathcal{U} \to \mathcal{E}$ is a continuous section.

- (1) $\mathcal{D}$ is called compact if $\mathcal{S}^{-1}(0)$ is compact.
- (2) An orientation on $\mathcal{D}$ is an orientation on the orbifold real vector bundle $TU \otimes \mathcal{E}^*$. 
- (3) A normal complex structure on $\mathcal{D}$ consists of an NC structure on $\mathcal{U}$ and an NC structure on $\mathcal{E}$. Two NC structures are concordant if the corresponding pairs of NC structures on $\mathcal{U}$ and $\mathcal{E}$ are concordant.
- (4) An oriented and normally complex (ONC for short) derived orbifold chart is a derived orbifold chart $\mathcal{D}$ together with an orientation and an NC structure.
- (5) The virtual dimension of $\mathcal{D}$ is the integer

$$\dim_{\text{vir}}(\mathcal{D}) = \dim_{\mathbb{R}}(\mathcal{U}) - \text{rank}_{\mathbb{R}}(\mathcal{E}).$$

We also consider a more special structure called stable complex structure.

**Definition 5.2.** Let $\mathcal{D} = (\mathcal{U}, \mathcal{E}, \mathcal{S})$ be a derived orbifold.

- (1) A stable complex structure on $\mathcal{E}$ is an equivalence class of quadruples

$$(k, F_0, F_1, \phi)$$

where $k \geq 0$ is an integer, $F_0, F_1 \to \mathcal{U}$ are complex vector bundles, and

$$\phi : \mathbb{R}^{\otimes k} \otimes \mathcal{E} \oplus F_0 \cong F_1$$
is an isomorphism of orbifold vector bundles. Here \( \mathbb{R}^\oplus k \) denotes the trivial rank \( k \) bundle. The equivalence relation \( \sim \) is generated by the following two relations: 1) we require that \( (k, \mathcal{F}_0, \mathcal{F}_1, \phi) \sim (k + 2, \mathcal{F}_0, \mathcal{F}_1 \oplus \mathcal{F}_1, \phi_0 \oplus \phi) \) where \( \phi_0 : \mathbb{R}^\oplus 2 \to \mathbb{C} \) is the map \( (x, y) \mapsto x + iy \); 2) we require that \( (k, \mathcal{F}_0, \mathcal{F}_1, \phi) \sim (k, \mathcal{F}_0 \oplus \mathcal{F}, \mathcal{F}_1 \oplus \mathcal{F}, \phi \oplus \text{Id}_\mathcal{F}) \) where \( \mathcal{F} \to \mathcal{U} \) is an arbitrary complex vector bundle.

(2) A stable complex structure on \( \mathcal{D} \) consists of a stable complex structure on \( T\mathcal{U} \) and a stable complex structure on \( \mathcal{E} \).

There are a few standard and basic operations on derived orbifolds. For a derived orbifold with boundary \( \mathcal{D} = (\mathcal{U}, \mathcal{E}, \mathcal{S}) \), one can restrict to the boundary \( \partial \mathcal{D} = (\partial \mathcal{U}, \partial \mathcal{E} \mid \partial \mathcal{U}, \partial \mathcal{S} \mid \partial \mathcal{U}) \).

If \( \mathcal{D} \) is ONC, so is \( \partial \mathcal{D} \). For derived orbifolds without boundary, there are the operation \( \mathcal{D} \mapsto -\mathcal{D} \) by reversing the orientation and the operation of taking disjoint union.

Now we consider the derived bordism group of topological spaces. We refrain from using the language of generalized homology theories. The introduction of the bordism group is merely for the convenience of stating the results.

**Definition 5.3.** Let \( X \) be a topological space. The oriented and normally complex derived orbifold bordism group resp. stably complex derived orbifold group of \( X \), denoted by \( d\Omega^\text{ONC}_*(X) \) resp. \( d\Omega^\text{U}_*(X) \), is the set of equivalence classes of quadruples \((\mathcal{U}, \mathcal{E}, \mathcal{S}, \kappa)\) where \( \mathcal{D} = (\mathcal{U}, \mathcal{E}, \mathcal{S}) \) is a compact ONC resp. stably complex derived orbifold without boundary and \( \kappa : \mathcal{U} \to X \) is a continuous map, while the equivalence relation is generated by the following relations

1. **(Shrinking)** \((\mathcal{U}, \mathcal{E}, \mathcal{S}, \kappa) \sim (\mathcal{U}', \mathcal{E}', \mathcal{S}', \kappa')\) if \( \mathcal{U}' \subseteq \mathcal{U} \) is an open neighborhood of \( S^{-1}(0) \), \( \mathcal{E}' = \mathcal{E} \mid \mathcal{U}', \mathcal{S}' = \mathcal{S} \mid \mathcal{U}' \), and \( \kappa' = \kappa \mid \mathcal{U}' \).

2. **(Stabilization)** \((\mathcal{U}, \mathcal{E}, \mathcal{S}, \kappa) \sim (\mathcal{U}, \mathcal{E}, \hat{\mathcal{S}}, \kappa)\) if there is a NC resp. complex vector bundle \( \pi_\mathcal{F} : \mathcal{F} \to \mathcal{U} \) such that \( \hat{\mathcal{U}} = \mathcal{U}, \hat{\mathcal{E}} = \pi_\mathcal{F} \mathcal{E} \oplus \pi_\mathcal{F} \mathcal{S}, \hat{\mathcal{S}} = \pi_\mathcal{F} \mathcal{S} \oplus \pi_\mathcal{F} T\mathcal{F}, \) and \( \hat{\kappa} : \mathcal{F} \to X \) is an extension of the induced map along the zero section.

3. **(Cobordism)** \((\mathcal{U}_0, \mathcal{E}_0, S_0, \kappa_0) \sim (\mathcal{U}_1, \mathcal{E}_1, S_1, \kappa_1)\) if there is a compact ONC resp. stably complex derived orbifold \( \hat{\mathcal{D}} = (\hat{\mathcal{U}}, \hat{\mathcal{E}}, \hat{\mathcal{S}}) \) with \( \partial \hat{\mathcal{D}} \cong (-D_0) \sqcup D_1 \) and a continuous map \( \hat{\kappa} : \hat{\mathcal{U}} \to X \) such that \( \hat{\kappa} \mid \partial \hat{\mathcal{U}} \) coincides with \( \kappa_0 \sqcup \kappa_1 \).

The bordism sets \( d\Omega^\text{ONC}_*(X) \) and \( d\Omega^\text{U}_*(X) \) are graded abelian groups with addition induced by disjoint union and grading defined by the virtual dimension. It is easy to see that \( d\Omega^\text{ONC}_* \) is a functor from topological spaces to graded abelian groups. Recall that there are the more classical bordism theories, the oriented bordism \( \Omega^\text{SO}_* \) and stable complex bordism \( \Omega^\text{U}_* \).

**Proposition 5.4.** There is a natural transformation (the dotted arrow) making the following diagram commute (the other arrows are all naturally defined).
Proof. Since complex vector bundles are canonically oriented, a stable complex structure on a vector bundle naturally induces an orientation. To obtain an NC structure, one needs to make certain choices. Suppose \( \mathcal{D} = (\mathcal{U}, \mathcal{E}, \mathcal{S}) \) is a stably complex derived orbifold. By definition, there exist two quadruples \((k, \mathcal{F}_0, \mathcal{F}_1, \phi)\) and \((l, \mathcal{E}_0, \mathcal{E}_1, \eta)\) where

\[
\phi: \mathbb{R}^{\oplus k} \oplus \mathcal{T}\mathcal{U} \oplus \mathcal{F}_0 \cong \mathcal{F}_1, \quad \eta: \mathbb{R}^{\oplus l} \oplus \mathcal{E} \oplus \mathcal{E}_0 \cong \mathcal{E}_1
\]

are isomorphisms. By taking stabilizations, which induce equivalent quadruples, we can assume \( \mathcal{F}_0 = \mathcal{E}_0 \). The above two isomorphisms can be pulled back to the total space of \( \mathcal{F}_0 \) as

\[
\pi^*_\mathcal{F}_0 \phi: \mathbb{R}^{\oplus k} \oplus \pi^*_\mathcal{F}_0 \mathcal{T}\mathcal{U} \oplus \pi^*_\mathcal{F}_0 \mathcal{F}_0 \cong \pi^*_\mathcal{F}_0 \mathcal{F}_1
\]

and

\[
\pi^*_\mathcal{F}_0 \eta: \mathbb{R}^{\oplus l} \oplus \pi^*_\mathcal{F}_0 \mathcal{E} \oplus \pi^*_\mathcal{F}_0 \mathcal{E}_0 \cong \pi^*_\mathcal{F}_0 \mathcal{E}_1.
\]

Upon choosing a connection on \( \mathcal{F}_0 \) inducing a bundle isomorphism \( \Delta: \pi^*_\mathcal{F}_0 \mathcal{T}\mathcal{U} \oplus \pi^*_\mathcal{F}_0 \mathcal{F}_0 \cong \mathcal{T}\mathcal{F}_0 \), the pullback isomorphism \( \pi^*_\mathcal{F}_0 \phi \) becomes an isomorphism

\[
\mathbb{R}^{\oplus k} \oplus \mathcal{T}\mathcal{F}_0 \cong \pi^*_\mathcal{F}_0 \mathcal{F}_1.
\]

As \( \mathbb{R}^{\oplus k} \) and \( \mathbb{R}^{\oplus l} \) do not affect the normal direction, the above two isomorphism induce a normal complex structure on the stabilization of \( \mathcal{D} \) by \( \mathcal{F}_0 \). Note that the space of connections on \( \mathcal{F}_0 \) is contractible; moreover, for different presentations of the stable complex structures on \( \mathcal{D} \), Definition 5.2 implies that the induced normal complex structures agree by an extension of the connection. Therefore, although the specific construction requires choices, the resulting bordism class in \( d\Omega^{\text{ONC}} \) is canonically defined. \( \square \)

5.2. Definition of the Euler cycles. There is a natural transformation from the oriented bordism to the integral homology

\[
\Omega^\text{SO}_\ast (\cdot) \to H_\ast (\cdot; \mathbb{Z})
\]

defined by pushing forward the fundamental classes of compact oriented manifolds. A virtual fundamental class can be viewed as a lift of this natural transformation to the derived orbifold bordism, possibly after changing the coefficients.

Theorem 5.5. For \( k \in \mathbb{Z} \) and a universal stratum \( \theta \in 3^\text{univ}_k \), there is a natural transformation

\[
\text{FOP}_\theta : d\Omega^\text{ONC}_k (\cdot) \to H_{k+n_k(\theta)} (\cdot; \mathbb{Z})
\]

such that for the special maximal \( \theta_k \in 3^\text{univ}_k \) (corresponding to the isotropy free part of the zero locus), the composition

\[
\Omega^\text{SO}_k \to d\Omega^\text{ONC}_k \xrightarrow{\text{FOP}_k} H_k (\cdot; \mathbb{Z})
\]

agrees with the natural transformation \( \Omega^\text{SO}_k (\cdot) \to H_k (\cdot; \mathbb{Z}) \) which takes a closed oriented manifold to its fundamental class.

The proof is to construct the Euler cycles via FOP transverse perturbations.
Definition 5.6. Let $\mathcal{D} = (\mathcal{U}, \mathcal{E}, S_0)$ be a compact derived orbifold. A perturbation of $\mathcal{D}$ is a section $S : \mathcal{U} \to \mathcal{E}$ satisfying the following condition: there exists a precompact open neighborhood $\mathcal{U}'$ of $S_0^{-1}(0)$ and an inner product on $\mathcal{E}$ such that

$$|S_0(x) - S(x)| < |S_0(x)| \quad \forall x \notin \mathcal{U}'.$$ 

In particular, $S$ does not vanish outside $\mathcal{U}'$.

Lemma 5.7. Let $\mathcal{D} = (\mathcal{U}, \mathcal{E}, S_0)$ be a compact and NC derived orbifold. Then there exists a smooth FOP transverse perturbation $S : \mathcal{U} \to \mathcal{E}$. Moreover, if $\mathcal{D}$ has boundary and $S_\partial$ is an FOP transverse perturbation of $\partial \mathcal{D}$, then one can take $S$ such that $\mathcal{S}|_{\partial \mathcal{U}} = S_\partial$.

Proof. We can choose a precompact open neighborhood $\mathcal{U}'$ of $S_0^{-1}(0)$ and an inner product on $\mathcal{E}$ such that $|S_0(x)| \geq 2\delta$ for some $\delta > 0$ and all $x \notin \mathcal{U}'$. Then by the $C^0$-density of the space of FOP transverse sections, there exists an FOP transverse $\mathcal{S} : \mathcal{U} \to \mathcal{E}$ such that $\|\mathcal{S} - S_0\|_{C^0} \leq \delta$, which defines a perturbation of $\mathcal{D}$. If $S_\partial$ is given, notice that $\partial \mathcal{U} \subseteq \mathcal{U}$ can be viewed as a suborbifold with ordinary normal bundle (the collar). Then by (Extension Property II) of Theorem 1.1, there exists an FOP perturbation extending $S_\partial$ to the interior. \qed

Proposition 5.8. Let $\mathcal{D} = (\mathcal{U}, \mathcal{E}, S_0)$ be a compact NC derived orbifold.

1. For each FOP transverse perturbation $\mathcal{S}$ and each universal stratum $\theta$, the closure $S^{-1}(0)_{\theta}$ is a compact oriented Thom–Mather stratified pseudomanifold (Definition C.6).

2. If $\mathcal{S}$ is an FOP transverse perturbation on the product $\mathcal{D} = [1, 2] \times \mathcal{D}$ with boundary restrictions denoted by $S_1$ and $S_2$ respectively, then the closure $S^{-1}(0)_{\theta}$ is a cobordism of oriented Thom–Mather stratified pseudomanifolds (Definition C.8).

Proof. For (1), by Theorem 1.1, the closure $S^{-1}(0)_{\theta}$ is a compact oriented Thom–Mather stratified space; furthermore, based on the discussion in Section 4.3.7, the frontier of $S^{-1}(0)_{\theta}$ is the union of smooth manifolds of codimension two or higher as the same assertion holds for the universal zero locus. Hence $S^{-1}(0)_{\theta}$ satisfies conditions for pseudomanifolds. For the same reason, in (2) the closure $S^{-1}(0)_{\theta}$ provides a cobordism. \qed

Proof of Theorem 5.5. Given an element $a \in d\Omega_{k}^{\text{ONC}}(X)$ represented by $(\mathcal{U}, \mathcal{E}, S_0, \kappa)$. Consider an FOP transverse perturbation $\mathcal{S}$. By (2) of Proposition 5.8, the Thom–Mather stratified pseudomanifold $S^{-1}(0)_{\theta}$ carries a fundamental class

$$[S^{-1}(0)_{\theta}] \in H_{n_k(\theta)}(S^{-1}(0)_{\theta}; \mathbb{Z}),$$

see Appendix C. Via the inclusion into $\mathcal{U}$ and the map $\kappa : \mathcal{U} \to X$, we define

$$\chi^\text{FOP}_\theta(a) := \kappa_*[S^{-1}(0)_{\theta}] \in H_{n_k(\theta)}(X; \mathbb{Z}).$$

We only need to prove that this element only depends on the element in $d\Omega_{k}^{\text{ONC}}(X)$.

We first show that it is independent of the choice of FOP transverse perturbations. Indeed, if $S_1$ and $S_2$ are two such perturbations, then (3) of Proposition 5.8 and Lemma C.9 imply that the corresponding cycles are equal.

We then need to show that the class $\chi^\text{FOP}_\theta(a)$ is invariant under the equivalence relation defining $d\Omega_{k}^{\text{ONC}}(X)$. Indeed, the subsequent Lemma 5.9, Lemma 5.10, and
Lemma 5.9. Let $\mathcal{D} = (U, E, S_0)$ be a compact ONC derived orbifold without boundary and $\iota : U' \hookrightarrow U$ be an open embedding onto a neighborhood of $S_0^{-1}(0)$ and $D'$ be the corresponding shrinking of $\mathcal{D}$. Then for any $\theta \in \Theta$, 

$$\chi^{\text{FOP}}(D) = \iota_* (\chi^{\text{FOP}}(D')) .$$

**Proof.** One can first choose an FOP transverse perturbation $S'$ of $D'$. Then by using a cut-off function supported in $U'$, one can find an FOP transverse perturbation $S$ of $D'$ such that $\iota^* S$ coincides with $S'$ on a smaller neighborhood of $S_0^{-1}(0)$ such that $S^{-1}(0) = \iota((S'))^{-1}(0))$. Then the lemma follows. □

Lemma 5.10. Let $\mathcal{D} = (U, E, S_0)$ be a compact ONC derived orbifold without boundary. Let $\mathcal{D}$ be the stabilization of $\mathcal{D}$ by an NC vector bundle $F \to U$ and let $\iota : U \to F$ be the zero section. Then for each universal stratum $\theta \in \Theta$, 

$$\chi^{\text{FOP}}(\mathcal{D}) = \iota_* (\chi^{\text{FOP}}(D)) .$$

(5.1)

**Proof.** Choose an FOP transverse perturbation $S$ of $\mathcal{D}$. By the (Stabilization Property) of Theorem 1.1, the stabilization $\hat{S}$ is FOP transverse; it is easy to check it is also a perturbation of $\mathcal{D}$. Then 

$$\iota(S^{-1}(0)_\theta) = \hat{S}^{-1}(0)_\theta .$$

This is indeed an isomorphism of Thom–Mather stratified spaces and preserves orientation. Hence (5.1) holds. □

Lemma 5.11. Let $\mathcal{D}_i = (U_i, E_i, S_{0,i})$, $i = 1, 2$, be compact ONC derived orbifolds without boundary. Let $\mathcal{D}$ be a compact ONC cobordism from $\mathcal{D}_1$ to $\mathcal{D}_2$. Let $\iota : U_1 \sqcup U_2 \to \partial U$ be the diffeomorph contained in the cobordism. Then for each universal stratum $\theta \in \mathcal{Z}^{\text{mix}}$, there exists a class $a_\theta \in H_*(U, \partial U; \mathbb{Z})$ such that 

$$\iota_* (\chi^{\text{FOP}}(\mathcal{D}_2) - \chi^{\text{FOP}}(\mathcal{D}_1)) = \partial a_\theta .$$

**Proof.** Choose FOP transverse perturbations $S_i : U_i \to E_i$ of $\mathcal{D}_i$. Then by Lemma 5.7, one can find an extension $\hat{S} : \hat{U} \to \hat{E}$ as a perturbation of $\mathcal{D}$ whose boundary restriction is $S_{0,i}$ respectively. Consider the stratum $\hat{S}^{-1}(0)_\theta \subseteq \hat{U}$ which is an oriented manifold with boundary identified with $(-S_1^{-1}(0)_\theta) \sqcup (S_2^{-1}(0)_\theta)$. Its closure is a cobordism of Thom–Mather stratified pseudomanifolds. Then there is a relative fundamental class 

$$[\hat{S}^{-1}(0)_\theta] \in H_*(\hat{S}^{-1}(0)_\theta, \partial \hat{S}^{-1}(0)_\theta; \mathbb{Z}) \to H_*(\hat{U}, \partial \hat{U}; \mathbb{Z}) .$$

By Lemma C.9, its boundary is the difference of $[S_2^{-1}(0)_\theta] - [S_1^{-1}(0)_\theta]$. □

Hence the map $\chi^{\text{FOP}}$ is well-defined. It follows from the construction that this is a natural transformation from $d\Omega^{\text{ONC}}(\cdot)$ to $H_{k+m_\theta}(\cdot; \mathbb{Z})$. Because any cobordism class of compact oriented manifolds tautologically gives rise to a cobordism class of ONC derived orbifold, this trivial section is an FOP transverse section and the homology class indexed by $\theta_k$ is indeed the fundamental class. □

**Remark** 5.12. The product property of the FOP transversality condition allows us to derive certain multiplicative properties of the FOP natural transformations.
6. Integer-valued Gromov–Witten Type Invariants

In this section we construct the integer-valued curve counting invariants of a general compact symplectic manifold and prove Theorem 1.7. Let \((X, \omega)\) be a compact symplectic manifold. Let \(J\) be an \(\omega\)-compatible almost complex structure on \(X\). Let \(A \in H_2(M; \mathbb{Z})\) be a homology class. Choose nonnegative integers \(g\) and \(n\). Then there is the moduli space of genus \(g\), \(n\)-marked stable maps in class \(A\)

\[ \mathcal{M}_{g,n}(X, J, A). \]

Denote its virtual dimension by \(k\). Let \(\mathcal{M}_{g,n}\) be the Deligne–Mumford space when \(2g + n \geq 3\) and be a single point otherwise. We would like to define symplectic deformation invariants

\[ \mathfrak{r}_{\mathcal{M}_{g,n}(X, J, A)}(\theta) \in H^k_{k+n}(\theta)(X^n \times \mathcal{M}_{g,n}; \mathbb{Z}) \forall \theta \in \mathfrak{F}_{k}^{\text{univ}} \]

(6.1)

using the FOP perturbation method and the global Kuranishi chart construction of Abouzaid–McLean–Smith [AMS21, AMS23] and Hirschi–Swaminathan [HS22]. In view of Proposition 5.4 and Theorem 5.5, to define these invariants, it suffices to construct a canonical element

\[ \mathfrak{r}_{\mathcal{M}_{g,n}(X, J, A)}(\theta) \in \mathcal{D}_{\mathfrak{r}}(X^n \times \mathcal{M}_{g,n}). \]

6.1. Stable complex Kuranishi bordism. The Kuranishi approach of regularizing moduli spaces was originally introduced by Fukaya–Ono [FO99] (see [KFOOO20] for a comprehensive discussion). Typically moduli spaces of pseudoholomorphic curves always admit local Kuranishi charts. The global Kuranishi charts on the stable map moduli spaces were firstly constructed by Abouzaid–McLean–Smith [AMS21] in the genus zero case and then extended by Hirschi–Swaminathan [HS22] and Abouzaid–McLean–Smith [AMS23] to the higher genus case. We first recall the notion of stable complex structure in the equivariant setting.

**Definition 6.1.** Let \(G\) be a compact Lie group, \(U\) be a smooth \(G\)-manifold, and \(E \to U\) be a \(G\)-equivariant real vector bundle. A \(G\)-equivariant stable complex structure on \(E\) is an equivalence class of quadruples

\[ (k, F_0, F_1, \phi) \]

where \(k\) is a nonnegative integer, \(F_0, F_1 \to V\) are \(G\)-equivariant complex vector bundles, and

\[ \phi : F \cong E \oplus F_0 \cong F_1 \]

is an isomorphism of \(G\)-equivariant real vector bundles. The equivalence relation is generated by the following two relations: 1) we require

\[ (k, F_0, F_1, \phi) \sim (k + 2, F_0, C \oplus F_1, \phi_0 \oplus \phi) \]

where \(\phi_0 : C \to C\) is \((x, y) \mapsto x + iy\); 2) we require

\[ (k, F_0, F_1, \phi) \sim (k, F_0 \oplus F, F_1 \oplus F, \phi \oplus \text{Id}_F) \]

where \(F \to V\) is any \(G\)-equivariant complex vector bundle.

**Definition 6.2.** ([AMS21, Definition 4.1, 4.3])

(1) A smooth Kuranishi space (K-space for short) is a quadruple

\[ K = (G, U, E, S) \]

where \(G\) is a compact Lie group, \(U\) is a smooth \(G\)-manifold such that the \(G\)-action is effective and has only finite isotropy groups, \(E \to U\) is a smooth
G-equivariant vector bundle, and \( S : U \to E \) is a continuous G-equivariant section. The K-space is called compact if \( S^{-1}(0) \) is compact.

(2) The virtual dimension of a K-space \( K = (G, U, E, S) \) is
\[
\dim^\text{vir}(K) := \dim_G(U) - \dim_G(G) - \operatorname{rank}_G(E).
\]

(3) A stable complex structure on a smooth K-space \( K = (G, U, E, S) \) consists of a G-equivariant stable complex structure on \( TU/\mathbb{R} \) (where \( \mathbb{R} \subseteq TU \) is the trivial subbundle generated by infinitesimal actions, thanks to the condition that the G-action has at no continuous stabilizers) and a G-equivariant stable complex structure on E. A stably complex K-space consists of a smooth K-space together with a stable complex structure.

Similar to the case of derived orbifold bordism, we can define the Kuranishi version of the bordism groups.

**Definition 6.3.** Let \( X \) be a topological space. The stable complex Kuranishi bordism group of \( X \), denoted by \( \Omega^U_* (X) \), is the set of equivalence classes of quintuples \( (G, U, E, S, \kappa) \) where \( K = (G, U, E, S) \) is a compact and stably complex K-space and \( \kappa : U \to X \) is a G-invariant continuous map, and where the equivalence relation is generated by the following relations.

1. **(Shrinking)** \( (G, U, E, S, \kappa) \sim (G, U', E', S', \kappa') \) if \( U' \subseteq U \) is a G-invariant open neighborhood of \( S^{-1}(0) \), \( E' = E|_{U'} \), \( S' = S|_{U'} \), and \( \kappa' = \kappa|_{U'} \).
2. **(Stabilization)** \( (G, U, E, S, \kappa) \sim (G, U, \hat{E}, \hat{S}, \hat{\kappa}) \) if there is a G-equivariant complex vector bundle \( \pi_F : F \to U \) and \( \hat{U} = F, \hat{E} = \pi^*_F E \oplus \pi^*_F F, \hat{S} = \pi^*_F S \oplus \tau_F, \) and \( \hat{\kappa} = \pi^*_F \kappa \).
3. **(Group enlargement)** \( (G, U, E, S, \kappa) \sim (G \times G', U', E', S', \kappa') \) where \( G' \) is another compact Lie group, if there is a G-equivariant principal \( G' \)-bundle \( \pi_P : P \to U \) such that \( U' = P, E' = \pi^*_P E, S' = \pi^*_P S, \) and \( \kappa' = \pi^*_P \kappa \).
4. **(Cobordism)** \( (G, U_0, E_0, S_0, \kappa_0) \sim (G, U_1, E_1, S_1, \kappa_1) \) if there is a compact and stably complex K-space with boundary \( K = (G, U, E, S) \) whose boundary restriction is isomorphic to the disjoint union \( (-K_0) \sqcup K_1 \) such that \( \hat{\kappa} : \hat{U} \to X \) is an extension of \( \kappa_0 \sqcup \kappa_1 \).

Notice that there is a natural transformation
\[
\Omega^U_* \to \kappa \Omega^U_*
\]
where we view a compact stably complex manifold as a stably complex K-space with trivial symmetry group and trivial obstruction bundle.

**Lemma 6.4.** There is a natural transformation (the dotted arrow) making the following diagram commute.

\[ \begin{array}{ccc}
\Omega^U_* & \longrightarrow & \kappa \Omega^U_* \\
\downarrow & & \downarrow \\
d \Omega^U_* & \longrightarrow & d \kappa \Omega^U_*
\end{array} \]

**Proof.** Given a K-space \( K = (G, U, E, S) \), one obtains a derived orbifold \( D = (\mathcal{U}, \mathcal{E}, \mathcal{S}) \) by taking the G-quotient. More precisely, we set
\[
\mathcal{U} := U/G, \quad \mathcal{E} := E/G
\]
and $S$ is induced from $S$. Moreover, a $G$-invariant map $\kappa : U \to X$ descends to a continuous map from $\mathcal{U}$ to $X$.

Now we consider the correspondence of stable complex structures. If $K = (G, U, E, S)$ has a stable complex structure, then it induces a stable complex structure on $\mathcal{D}$ as follows. By Definition 6.1, there exist $(k, F_0, F_1, \phi)$ and $(l, E_0, E_1, \eta)$ such that

$$\phi : \mathbb{R}^{\mathbb{G}^k} \oplus (TU/\mathfrak{g}) \oplus F_0 \cong F_1, \quad \eta : \mathbb{R}^{\mathbb{G}^l} \oplus E \oplus E_0 \cong E_1.$$

Notice that $(TU/\mathfrak{g})/G$ is canonically identified with $TU$. Therefore the two $G$-equivariant isomorphisms induce isomorphisms of orbifold vector bundles

$$\phi/G : \mathbb{R}^{\mathbb{G}^k} \oplus TU \oplus (F_0/G) \cong F_1/G, \quad \eta/G : \mathbb{R}^{\mathbb{G}^l} \oplus E \oplus (E_0/G) \cong E_1/G.$$

It is straightforward to check that the induced stable complex structure on $\mathcal{D}$ does not depend on the choices of the quadruples defining the stable complex structure. Moreover, one is readily to check that shrinkings resp. stabilizations of $K$-charts induce shrinkings resp. stabilizations of $D$-charts. On the other hand, group enlargements of $K$-charts induce isomorphic $D$-charts. Finally, cobordisms between $K$-spaces descend to cobordisms of derived orbifolds, and the commutativity of the diagram follows from definition.

\section*{6.2. Proof of Theorem 1.7.} Global topological Kuranishi charts on the moduli spaces of stable maps are constructed in [AMS21][HS22][AMS23].

\begin{theorem}
Let $X, \omega, J, A, g, n$ be as in Theorem 1.7. There exists a distinguished (nonempty) collection of quintuples $(G, U, E, S, L)$ (called the AMS charts) where $K = (G, U, E, S)$ is a stably complex $K$-space and $L : S^{-1}(0)/G \to \overline{M}_{g,n}(X, J, A)$ is a homeomorphism. Each AMS chart is associated to a choice of an auxiliary datum. Moreover, the following is true.

(1) For each AMS chart $(G, U, E, S, L)$ there exists a $G$-invariant map $\kappa : U \to X^n \times \overline{M}_{g,n}$ which extends the evaluation-stabilization map

$$(ev \times st) \circ L^{-1} : S^{-1}(0)/G \to X^n \times \overline{M}_{g,n}.$$

(2) The element of $k\Omega^U_*(X^n \times \overline{M}_{g,n})$ represented by $(G, U, E, S, \kappa)$ is independent of choices and the extension $\kappa$. Denote this element by $[\overline{M}_{g,n}(X, J, A)]$.

(3) Suppose $\omega$ can be deformed via symplectic forms to another symplectic form $\omega'$ on $X$ and $J'$ is $\omega'$-compatible. Then

$$[\overline{M}_{g,n}(X, J, A)] = [\overline{M}_{g,n}(X, J', A)] \in k\Omega^U_*(X^n \times \overline{M}_{g,n}).$$

\end{theorem}

\begin{proof}
The construction of the AMS charts are given in [AMS23, Section 4] and the statement where part (1), (2), and the case of fixing $\omega$ and varying $J$ in (3) are summarized in [AMS23, Corollary 4.70]. The case of deforming both $\omega$ and $J$ can be carried out in the same way as in [AMS23, Section 4].

\end{proof}

\begin{proof}[Proof of Theorem 1.7]
By applying the composition of natural transformations

$$k\Omega^U_* \to d\Omega^U_* \to d\Omega^{\text{ONC}}_*$$

to the element $[\overline{M}_{g,n}(X, J, A)]$ given by Theorem 6.5, one obtains a canonical element in $d\Omega^{\text{ONC}}_k(X^n \times \overline{M}_{g,n})$. Then for each universal stratum $\theta \in \mathcal{F}^{\text{univ}}$ where
$k$ is the virtual dimension, applying the FOP natural transformation, one obtains the $\theta$-th FOP Euler class
\[ \theta \in H_{n_k(\theta)}(X^n \times \overline{\mathcal{M}}_{g,n}; \mathbb{Z}). \]
Lastly, when $\overline{\mathcal{M}}_{g,n}(X, J, A)$ contains no elements with nontrivial automorphism groups, for an AMS chart $(G, U, E, S, L)$ the $G$-action can be made free. Hence the only nontrivial FOP class is the one corresponding to the special element $\theta_k$ and it agrees with the ordinary virtual fundamental class. \hfill \square

7. A Simple Proof of Abouzaid–McLean–Smith’s Splitting Theorem

Let $(X, \omega)$ be a closed symplectic manifold. Denote by $\text{Ham}(X, \omega)$ the group of Hamiltonian diffeomorphisms of $X$. Given an element of $\pi_1(\text{Ham}(X, \omega))$ represented by a smooth loop $\phi : S^1 \to \text{Ham}(X, \omega)$ one can construct a fibration $P_\phi \to S^2$ with fiber symplectomorphic to $X$ and the transition function along the equator being the loop $\phi$. We use the construction of integer-valued Gromov–Witten type invariants to give an alternative proof of [AMS21, Theorem 1.1]:

**Theorem 7.1.** There is an additive isomorphism
\[ H^*(P_\phi; \mathbb{Z}) \cong H^*(X; \mathbb{Z}) \otimes_{\mathbb{Z}} H^*(S^2; \mathbb{Z}). \]

For a discussion on the context of such cohomological splitting result, the reader may consult [BPX24, Section 1]. The proof follows from constructing a splitting map in integral cohomology (cf. [AMS21, Section 3.4]), whose proof is provided in this section.

**Proposition 7.2.** The map $\text{res} : H^*(P_\phi; \mathbb{Z}) \to H^*(X; \mathbb{Z})$ induced by the inclusion of a fiber $X \hookrightarrow P_\phi$ has a right inverse.

7.1. The moduli spaces and the idea of the proof. A typical idea to prove Proposition 7.2 is to consider moduli spaces of holomorphic sections of $P_\phi \to S^2$ with two marked points, whose fundamental classes can be used to define Seidel representations, see [LMP99] and [McD00]. Here we follow the alternate approach provided in [AMS21, Section 3] where we embed the total space $P_\phi$ into a fibration over a 4-dimensional base to bypass the necessity of introducing Seidel elements in quantum cohomology. Our proof essentially follows the same strategy as [AMS21] where we replaces the Morava virtual fundamental class by the integral Euler class defined in this paper.

Consider the product $\mathbb{CP}^1 \times \mathbb{CP}^1$. For convenience we denote the first factor by $B$ and the second factor by $S^2$. Let $M$ be the blow-up of $B \times S^2 \cong \mathbb{CP}^1 \times \mathbb{CP}^1$ at $(0,0)$. Let
\[ \pi_M : M \to B \]
be the composition of the blow-down map and the projection to $B$. For each $t \in B$, write $M_t := \pi^{-1}_M(t)$. Then $M_t \cong S^2$ except at $t = 0 \in B$ where $M_0$ is a reducible rational curve with two components. We may choose a Kähler form $\omega_M$ on $M$ such that for an open neighborhood $W_\infty \subset B$ of $\infty \in B \cong \mathbb{CP}^1$,
\[ \pi^{-1}_M(W_\infty) \cong W_\infty \times S^2 \]
as Kähler manifolds.

Further, there exists a symplectic fibration $\pi_P : P \to M$ equipped with a symplectic form $\omega_P$ on the total space $P$ as well as a compatible almost complex structure $J_P$ satisfying the following conditions (see [McD00, Section 3]).
(1) $\pi_P : P \to M$ is pseudo-holomorphic.
(2) $\pi_P^{-1}(\pi_M^{-1}(W_\infty))$ is symplectomorphic to the product $W_\infty \times S^2 \times X$ and $J_P$ is the product of given (almost) complex structures on the three factors.
(3) $\pi_M^{-1}(M_0)$ is the singular space $P_0 \cup_X P_{0-1}$ such that the restriction of $\omega_P$ to both $P_0$ and $P_{0-1}$ are in the deformation equivalence class of the symplectic structures determined by the loops $\phi$ and $\phi^{-1}$ (cf. [MS04, Section 8.2]). Denote the two components of $M_0$ by $M_\phi$ and $M_{\phi^{-1}}$, which are the bases of $P_\phi$ and $P_{\phi^{-1}}$ respectively.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fibration.png}
\caption{The fibration $P \to M$.}
\end{figure}

Next, choose a holomorphic section $h : B \to M$ whose image $M_h := h(B) \cong B$ passes through $M_\phi$ but not $M_{\phi^{-1}}$. Over $M_h$, denote
\[ P_h := \pi_P^{-1}(M_h) \subseteq P. \]

One can fix a diffeomorphism
\[ P_h \cong B \times X \]
Denote $P_\infty = \pi_P^{-1}(M_\infty)$. See Figure 2 for illustration.

We now describe a moduli space of stable maps. We fix a curve class $A \in H_2(P;\mathbb{Z})$ represented by $\{\infty\} \times S^2 \times \{pt\} \subseteq W_\infty \times S^2 \times X$. We consider the moduli space $\overline{\mathcal{M}}_{0,2}(P,J_P,A)$ with the evaluation map
\[ ev = (ev_1, ev_2) : \overline{\mathcal{M}}_{0,2}(P,J_P,A) \to P \times P. \]

Define
\[ \overline{\mathcal{M}}_h := ev^{-1}(P \times P_h) \]
and its subspace
\[ \overline{\mathcal{M}}_\infty := ev^{-1}(P_\infty \times P_h). \]
Notice that there is a special family of curves near $P := \pi_1^{-1}(M_t)$. As near $\pi_1^{-1}(W_{\infty})$, $J$ is the product $J_B \times J_S \times J_X$, there is a subset

$$\mathcal{M}_{W_{\infty}} \subseteq \overline{\mathcal{M}}$$

of curves contained in $\pi_1^{-1}(M_t)$ for $t \in W_{\infty}$, which is constant in the $X$-direction, where the first marked point is mapped into $P_t \cap P_l$ and the second marked point is arbitrary in that line.

**Lemma 7.3.** $\mathcal{M}_{W_{\infty}}$ is an open neighborhood of $\mathcal{M}$ in $\overline{\mathcal{M}}$ and is transversely cut out.

**Proof.** First, it is easy to see that $\pi_P: P \to M$ pushes forward the class $A \in H_2(P)$ to the class $\pi_P(A) \in H_2(M)$ represented by $\{t\} \times S^2 \cong M$ for all $t \in W_{\infty}$. Then any point of $\overline{\mathcal{M}}$ close to the subset $\mathcal{M}_{W_{\infty}}$ is projected to a holomorphic curve representing $\pi_P(A)$, which must be a sphere contained in $\{t\} \times S^2$ for $t$ close to $\infty$. Such curves are exactly those ones contained in $\mathcal{M}_{W_{\infty}}$. The transversality of $\mathcal{M}_{W_{\infty}}$ is standard. □

### 7.2. Proof using FOP cycles

We first equip the moduli spaces with derived orbifold charts.

**Lemma 7.4.** There exists a compact and stably complex derived orbifold $D = (\mathcal{U}, \mathcal{E}, S)$ with an isomorphism $S^{-1}(0) \cong \mathcal{M}_{0,2}(P, J_P, A)$. Moreover, the evaluation map $ev$ extends to a smooth submersion

$$\kappa: \mathcal{U} \to P.$$ 

**Proof.** This is a reformulation of the results from [AMS21, Section 5.8]. The construction of stable complex structure is standard in Gromov–Witten theory. □

**Lemma 7.5.** Define $\mathcal{U}_h$ by

$$\mathcal{U}_h := \kappa^{-1}(P \times P_h) \subseteq \mathcal{U}.$$ 

Then $\mathcal{U}_h \subseteq \mathcal{U}$ is a suborbifold with ordinary and complex normal bundle. Similarly, define $\mathcal{U}_c$ by

$$\mathcal{U}_c = \kappa^{-1}(P_c \times P_h) \subseteq \mathcal{U}.$$ 

Then $\mathcal{U}_c \subseteq \mathcal{U}_h$ is a suborbifold with ordinary and complex normal bundle.

**Proof.** This follows from [AMS21, Corollary 5.36]. Note that the submersive property of the evaluation maps is used here. □

Now we introduce more notations. From now on all homology and cohomology are assumed to be in $\mathbb{Z}$-coefficients. If $Y$ is a closed oriented manifold, $\alpha \in H_k(Y)$, and $S \subseteq Y$ is a codimension $l$ closed oriented submanifold, then denote by

$$\alpha|_S \in H_{k-l}(S)$$

the class obtained by intersecting with $S$. Geometrically, if $\alpha$ is represented by a pseudocycle in $Y$ which intersects transversely with $S$ (as well as its frontier), then $\alpha|_S$ is represented by the pseudocycle in $S$ obtained by the intersection.

If $W, Y$ are closed oriented manifolds, $\alpha \in H_*(W \times Y)$, then there is an associated map $\Psi_{\alpha}: H^*(W) \to H^*(Y)$ defined as the composition

$$H^*(W) \xrightarrow{\alpha/} H_*(Y) \xrightarrow{\text{PD}} H^*(Y)$$
where $\alpha/\iota$ is the slant product. If $\iota : Y_0 \to Y$ is the inclusion of a closed oriented submanifold and $\alpha_0 = \alpha|_{W \times Y_0}$, then the following diagram commutes.

\[
\begin{array}{ccc}
H^*(Y) & \xrightarrow{\Psi_\alpha} & H^*(Y_0) \\
\downarrow \iota^* & & \downarrow \iota^*_0 \\
H^*(W) & \xrightarrow{\Psi_{\alpha_0}} & H^*(Y_0)
\end{array}
\]

Now consider the stably complex derived orbifold

\[\mathcal{D}_h = (U_h, E_h, S_h) := (U_h, E|_{U_h}, S|_{U_h})\]

where $U_h$ is constructed in Lemma 7.5; up to concordance we may regard $\mathcal{D}_h$ as oriented and normally complex. Let $k$ be the virtual dimension of $\mathcal{M}_h$. By choosing an FOP transverse perturbation on $\mathcal{D}_h$, for the stratum $\theta_k \in \mathcal{Z}(\mathcal{M}_h)$, there is an integral homology class

\[\chi_{\text{FOP}}^{\theta_k} \in H_k(U_h; \mathbb{Z}),\]

which is pushed forward by the evaluation map to a class

\[\alpha := \text{ev}_*(\chi_{\text{FOP}}^{\theta_k}(\mathcal{D}_h)) \in H_k(P_h \times P; \mathbb{Z}).\]

Define

\[\alpha_{X} := \alpha|_{P_h \times \mathcal{P}_\phi} \in H_{k-2}(P_h \times \mathcal{P}_\phi; \mathbb{Z})\]

and

\[\alpha_{\phi} := \alpha|_{P_h \times P} \in H_k(P_h \times P_{\phi}; \mathbb{Z}).\]

Then there are the associated maps $\Psi_\alpha$, $\Psi_{\alpha_{X}}$, and $\Psi_{\alpha_{\phi}}$ which fit into the following commutative diagram

\[
\begin{array}{ccc}
H^*(P_{\phi}) & \xrightarrow{\Psi_{\alpha_{\phi}}} & H^*(P_h) & \xrightarrow{\Psi_{\alpha_{X}}} & H^*(X) \\
\downarrow \iota_{\phi}^* & & \downarrow \iota_{X}^* & & \downarrow \iota_X^* \\
H^*(P_h) & \xrightarrow{\text{proj}_2^*} & H^*(P_h) & \xrightarrow{\Psi_{\alpha}} & H^*(P_h) & \xrightarrow{\iota_X^*} & H^*(X)
\end{array}
\]

where $\text{proj}_2 : P_h \to X$ is the projection induced from the identification (7.1) and the three $\iota_{X}^*$ are induced from the inclusion of a fiber.

**Lemma 7.6.**

1. Near $\mathcal{M}_X$, the original section $S_h : \mathcal{D}_h \to \mathcal{E}_h$ is transverse in both classical and the FOP sense.
2. $\alpha|_{P_h \times P_{\phi}} = \text{ev}_*[\mathcal{M}_X]$.
3. There holds $\iota_{X}^* \circ \Psi_{\alpha_{X}} \circ \text{proj}_2^* = \text{Id}_{H^*(X)}$.

**Proof.** By Lemma 7.3, the open neighborhood of $\mathcal{M}_X$ in $\mathcal{M}_{W_{\phi}} \subseteq \mathcal{M}_h$ is transversely cut out. Moreover, it contains no curves with nontrivial automorphism groups. Then by the condition **Classical Transversality** of Theorem 1.1, $S_h$ is transverse in the FOP sense near $\mathcal{M}_{W_{\phi}}$. 
Next, by the condition **Extension Property I** of Theorem 1.1, after appropriately shrinking the chart $D_h$, one can find an FOP transverse perturbation $S'_h : U_h \to E_h$ which agrees with $S_h$ near $\overline{M}_X$. Then we see
\[ \text{ev}((S'_h)^{-1}(0)) \cap (P_h \times P_{\lambda e}) = \text{ev}((S_h)^{-1}(0)) \cap (P_h \times P_{\lambda e}) = \text{ev}(\overline{M}_X) \]
and the intersection is transverse. Homologically this means $\alpha|_{P_h \times P_{\lambda e}} = \text{ev}_*[\overline{M}_X]$.
Lastly, as the moduli space $\overline{M}_X$ is identified with $(x) \times S^2 \times X$, the relation $\iota^* \circ \Psi_{\alpha_v} \circ \text{proj}^* = \text{Id}_{H^*(X)}$. □

By the commutativity of the diagram (7.2), Lemma 7.6 implies that
\[ \iota^* \circ \Psi_{\alpha_v} \circ \text{proj}^* = \text{Id}_{H^*(X)}. \]
So Proposition 7.2, and hence Theorem 7.1 are proved.

**APPENDIX A. TECHNICAL RESULTS ABOUT STRAIGHTENED STRUCTURES**

In this appendix we prove Proposition 2.29, Proposition 2.34, and Proposition 2.39.

**A.1. PROOF OF PROPOSITION 2.29.** We first briefly sketch the idea of the proof. We would like to construct the straightened metric and NC structure inductively, first near the deepest strata of the isotropy stratification of the underlying orbifold, then move up until the top stratum. The model for each construction is the bundle metric and bundle NC structure on a total space of a vector bundle with fiberwise group actions. Hence we first prove below that the metric and NC structure in the model case can be straightened. We then carry out the inductive construction using chartwise constructions and partition-of-unity arguments.

**A.1.1. The straightening model.**

**Definition A.1.**

1. A **straightening model** is a triple $(G, X, F)$ where $G$ is a finite group, $X = (X, g^{TX})$ is a Riemannian manifold, $F = (F, h^F, \nabla^F)$ consists of a vector bundle $F \to X$ with a fiberwise linear $G$-action, a $G$-invariant inner product $h^F$, and an $h^F$-preserving connection $\nabla^F$.
2. Let $F \to X$ be a real vector bundle. A **Hermitian triple** is a triple $(\nabla^F, h^F, I^F)$ where $I^F$ is a complex structure on $F$, $h^F$ is a Hermitian inner product on $F$, and $\nabla^F$ is a Hermitian connection on $F$.
3. A **complex straightening model** consists of a straightening model $(G, X, F)$ and a $G$-invariant Hermitian triple $(\nabla^F, h^F, I^F)$.

Associated to a straightening model there is a Riemannian $G$-manifold $(F, g^{TF})$ where $g^{TF}$ is the bundle metric coupled with the Riemannian metric on the base. Moreover, for a complex straightening model, $I^F$ induces a bundle NC structure $I^{TF}$. The inductive proof of Proposition 2.29 relies on the following technical lemma.

**Proposition A.2.** Let $(G, X, F)$ be a straightening model. Then the bundle metric $g^{TF}$ is straightened. In addition, if $(G, X, F)$ is complex, then the bundle NC structure $I^{TF}$ is straightened with respect to $g^{TF}$.

To prove that the bundle metrics are straightened, we need to verify certain elementary facts about such metrics.
Lemma A.3. Let \((X, g^TF)\) be a Riemannian manifold and \(\pi_F : F \rightarrow X\) be a smooth real vector bundle equipped with an inner product \(h^F\) and an \(h^F\)-preserving connection \(\nabla^F\). Let \(g^TF\) be the bundle metric on \(F\). Then

1. fibers of \(F\) are totally geodesic and fiberwise geodesics are straight lines.
2. Let \(p^*\pi^F_F : TF \rightarrow \pi^*\pi_F^F\) be the orthogonal projection onto the vertical subbundle \(\pi^*\pi_F^F \subseteq TF\). Then the connection on \(TF\) induced from the Levi–Civita connection \(\nabla^TF\) of \(g^TF\), which in precise terms reads

\[
p^*\pi^F_F \circ \nabla^TF \circ p^*\pi^F_F,
\]

coincides with the pullback connection \(\pi^*\pi_F^F \nabla^F\).

Proof. Choose local coordinates \((x^1, \ldots, x^m)\) on \(X\) and local orthonormal frame \((e_1, \ldots, e_k)\) of \(F\). Write \(\nabla^F\) in terms of the connection matrix \(\omega_{\alpha\beta}^i\) as

\[
\nabla^F_{\partial_i} e_{\alpha} = \sum_{\beta=1}^{k} \omega_{\alpha\beta}^i e_{\beta}, \quad 1 \leq i \leq m, \quad 1 \leq \alpha \leq k.
\]

Let \((y^1, \ldots, y^k)\) be the fiber coordinates dual to the basis \(e_1, \ldots, e_k\). Abbreviate

\[
\partial_i = \frac{\partial}{\partial x^i}, \quad \partial_{\alpha} = \frac{\partial}{\partial y^\alpha}.
\]

Then by the definition of the bundle metric, one has

\[
\langle \partial_i, \partial_{\alpha} \rangle_{g^TF} = \sum_{\gamma=1}^{k} \omega_{i\gamma}^{\alpha} y^\gamma, \quad \langle \partial_{\alpha}, \partial_{\beta} \rangle_{g^TF} = \delta_{\alpha\beta}.
\] (A.1)

The two assertions follows from the identities

\[
\nabla^TF_{\partial_{\alpha}} \partial_{\beta} = 0,
\]

(A.2)

\[
\langle \nabla^TF_{\partial_{\alpha}} \partial_{\beta}, \partial_{\gamma} \rangle_{g^TF} = \omega_{\alpha\gamma}^\beta.
\] (A.3)

For (A.2), it is easy to see that for \(\gamma = 1, \ldots, k\), \(\langle \nabla^TF_{\partial_{\alpha}} \partial_{\beta}, \partial_{\gamma} \rangle \equiv 0\). Moreover, by the formula of the Levi–Civita connection and (A.1),

\[
\langle \nabla^TF_{\partial_{\alpha}} \partial_{\beta}, \partial_{\gamma} \rangle = \frac{1}{2} \left( \partial_{\alpha} \langle \partial_{\gamma}, \partial_{\beta} \rangle_{g^TF} - \partial_{\beta} \langle \partial_{\alpha}, \partial_{\gamma} \rangle_{g^TF} + \partial_{\delta} \langle \partial_{\gamma}, \partial_{\delta} \rangle_{g^TF} \right)
\]

\[
= \frac{1}{2} \left( \partial_{\alpha} \omega_{\gamma\beta}^\gamma + \partial_{\beta} \omega_{\alpha\gamma}^\gamma \right)
\]

\[
= \frac{1}{2} \left( \omega_{\gamma\beta}^\gamma + \omega_{\alpha\gamma}^\gamma \right)
\]

which vanishes as \((e_1, \ldots, e_k)\) is an orthonormal frame and \(\nabla^F\) preserves the inner product. This proves (A.2). For (A.3), using the formula for the Levi–Civita connection again, one has

\[
\langle \nabla^TF_{\partial_{\alpha}} \partial_{\beta}, \partial_{\gamma} \rangle_{g^TF} = \frac{1}{2} \left( \partial_{\alpha} \langle \partial_{\beta}, \partial_{\gamma} \rangle_{g^TF} + \partial_{\beta} \langle \partial_{\alpha}, \partial_{\gamma} \rangle_{g^TF} - \partial_{\gamma} \langle \partial_{\alpha}, \partial_{\beta} \rangle_{g^TF} \right)
\]

\[
= \frac{1}{2} \left( \partial_{\alpha} \sum_{\gamma=1}^{k} \omega_{\beta\gamma}^\gamma y^\gamma - \partial_{\beta} \sum_{\gamma=1}^{k} \omega_{\alpha\gamma}^\gamma y^\gamma \right)
\]

\[
= \frac{1}{2} \left( \omega_{\beta\gamma}^\gamma - \omega_{\alpha\gamma}^\gamma \right)
\]

\[
= \omega_{\beta\gamma}^\gamma.
\]
This proves (A.3).

We also need to consider bundles over bundles and bundle metrics obtained in a two-step construction. Let $F$ resp. $\tilde{F}$ be equipped with an inner product $h^F$ resp. $h^{\tilde{F}}$ and a metric connection $\nabla^F$ resp. $\nabla^{\tilde{F}}$. Consider the total space $E := F \oplus \tilde{F}$, which has the bundle metric $g^{TE}$ determined by $(g^{TX}, h^F \oplus h^{\tilde{F}}, \nabla^F \oplus \nabla^{\tilde{F}})$. Notice that there is a natural identification between the total space of $E$ and the total space of the pullback bundle $\pi^* E \to F$. The second bundle structure gives another bundle metric. We would like to prove that the two bundle metrics agree.

**Lemma A.4.** (1) The bundle isomorphism

$$TE|_F \cong TF \oplus \pi^*_F \tilde{F}.$$ 

induced from the identification $E \cong \pi^*_F \tilde{F}$ and the connection $\pi^*_F \nabla^{\tilde{F}}$ is an isometry between the restriction of the bundle metric $g^{TE}$ to $TE|_F$ and the direct sum $g^{TF} \oplus \pi^*_F h^{\tilde{F}}$.

(2) Let $p^* \pi^*_F F : TE|_F \to \pi^*_F \tilde{F}$ be the orthogonal projection. Then

$$p^* \pi^*_F F \circ \nabla^F \circ \pi^*_F F = \pi^*_F \nabla^{\tilde{F}}.$$ 

(3) With respect to the natural identification $E \cong \pi^*_F \tilde{F}$ of total spaces, the bundle metric $g^{TE}$ coincides with the bundle metric $g^{TF} \pi^*_F F$ induced from $(g^{TF}, \pi^*_F h^F, \pi^*_F \nabla^{\tilde{F}})$.

**Proof.** We would like to write down the identification $E \cong \pi^*_F \tilde{F}$ of total spaces in local coordinates. Let $(x^i)_{i=1}^m$ be local coordinates on $X$, $(e_\alpha)_{\alpha=1}^m$ be local orthonormal basis of $F$ with dual fiber coordinates $y^\alpha$, and $(\tilde{e}_\gamma)_{\gamma=1}^m$ be local orthonormal basis of $\tilde{F}$ with dual fiber coordinates $\tilde{y}^\gamma$. Then the identification is

$$E_x \cong (x, y^\alpha e_\alpha(x) + \tilde{y}^\gamma \tilde{e}_\gamma(x)) \mapsto (x, y^\alpha e_\alpha(x), \tilde{y}^\gamma \pi^*_F F e_\alpha(x)) \in \pi^*_F \tilde{F}|_{(x, y^\alpha e_\alpha(x))}.$$ 

Hence if we regard $\pi^*_F F e_\alpha(x)$ as a local basis of $\pi^*_F \tilde{F}$, then using the fiber coordinates induced by the given base coordinates and the chosen fiber bases, the map $E \to \pi^*_F \tilde{F}$ is just the identity map in $(x^i, y^\alpha, \tilde{y}^\gamma)$.

For (1), one only needs to compare along a single fiber of $F$ at some point $x \in X$. One can choose $e_\alpha$ and $\tilde{e}_\gamma$ such that the connection matrices at $x$ vanish. Then in the bundle coordinates of $E$, along the fiber $F_x$, one has

$$g^{TE} = g_{ij} dx^i \otimes dx^j + \sum_{\alpha} dy^\alpha \otimes dy^\alpha + \sum_{\gamma} \tilde{y}^\gamma \otimes \tilde{y}^\gamma.$$ 

Then (1) follows. For (2), we know from (2) of Lemma A.3 that the restriction of $\nabla^E$ to $\pi^*_F E$ is the pullback $\pi^*_F \nabla^E$. As $\nabla^F$ is the direct sum, it follows that the restriction of $\nabla^E$ to $\pi^*_F \tilde{F}$ is the pullback $\pi^*_F \nabla^{\tilde{F}}$. Then (2) follows from by further restricting to the submanifold $F \subseteq E$. Then (3) follows from the explicit form of the bundle metric on $\pi^*_F \tilde{F} \to F$. 

**Proof of Proposition A.2.** First notice that the bundle metric $g^{TF}$ is $G$-invariant. For each $H \subseteq F G$, the $H$-fixed point set of $F$ is the total space of the subbundle $F_H \subseteq F$. Its normal bundle $NF_H$ is then naturally the pullback $\pi^*_F \tilde{F}_H$, which carries the inner product induced $h^{NF_H}$ from $g^{TF}$ and the connection $\nabla^{NF_H}$ induced
from $\nabla^TF$. Then by definition of straightenedness (see Definition 2.26), one needs to show that the normal exponential map
\[
\exp^{NF_H} : N^sF_H \to F
\]
is isometric from the bundle metric on $NF_H \cong \pi_{F_H}^* F_H$ determined the triple $(g^{TF_H}, h^{NF_H}, \nabla^{NF_H})$. Indeed, by Lemma A.3 (1), the normal exponential map is just the linear isomorphism $F_H \oplus \tilde{F}_H \cong F$ on each fiber of $F$. Hence $\exp^{NF_H}$ is just the natural identification $NF_H \cong \pi_{F_H}^* F_H \cong F$. Then by Lemma A.4 (3), this natural identification is an isometry.

Now assume that $(G, X, F)$ is complex. By Definition 2.27, one needs to show that, for each $H \subseteq G$, there hold (a) the triple $(\nabla^{NF_H}, h^{NF_H}, I^{NF_H})$ is Hermitian and (b) $\exp^{NF_H}$ preserves the NC structure. For (a), we know from the definition of $I^{F_H}$, Lemma A.3 and Lemma A.4 that
\[
(\nabla^{NF_H}, h^{NF_H}, I^{NF_H}) = (\pi_{F_H}^* \nabla^F_H, \pi_{F_H}^* h^F_H, \pi_{F_H}^* I^F_H).
\]
As the corresponding triple on $\tilde{F}_H \to X$ is Hermitian, so is the pullback triple. For (b), notice that all involved normal bundles are tangent to fibers of $F$. As the exponential map is the linear addition in fibers and the NC structures are all pullbacks on fibers, the exponential map preserves all involved complex structures.

A.1.2. Inductive construction. To prepare the proof of Proposition 2.29, we first work on the case of a single chart. Let $G$ be a finite group and $U$ be a $G$-manifold.

**Lemma A.5.** Let $g_0^{TU}$ be a $G$-invariant metric on $U$ and $I_0^{TU}$ be an NC structure close to $I^{TU}$. Let $H \subseteq U$ be a maximal essential subgroup. Let $h_0^{NU_H}$ be the inner product on $NU_H \to U$, $\nabla_0^{NU_H}$ be the connection on $NU_H$ induced from $g_0^{TU}$ and $\nabla^{NU_H}$ be the connection on $NU_H$ induced from the Levi–Civita connection of $g_0^{TU}$. Given a $G$-invariant closed subset $Y$, suppose the following conditions hold.

1. Near $Y$, $g_0^{TU}$ is straightened and $I_0^{TU}$ is straightened with respect to $g_0^{TU}$.
2. Along $NU_H \to U$, the triple $(\nabla_0^{NU_H}, h_0^{NU_H}, I_0^{NU_H})$ is Hermitian.

Then there exist a $G$-invariant metric $g_1^{TU}$ and an NC structure $I_1^{TU}$ close to $I^{TU}$ such that

1. Near $Y \cap GU_H$, $g_1^{TU}$ is straightened and $I_1^{TU}$ is straightened with respect to $g_1^{TU}$.
2. $(g_1^{TU}, I_1^{TU})$ coincides with $(g_0^{TU}, I_0^{TU})$ near $Y$.

**Proof.** We would like to modify the metric near $GU_H$. Indeed, there is a straightening model $(H, U, NU_H)$ where $U$ is equipped with the metric restricted from $g_0^{TU}$ and $NU_H$ is equipped with the Hermitian triple $(\nabla_0^{NU_H}, h_0^{NU_H}, I_0^{NU_H})$. Hence by Proposition A.2, the total space $NU_H$ is equipped with the straightened metric $g^{NU_H}$ and the straightened bundle NC structure $I^{NU_H}$. Consider the normal exponential map
\[
\exp^{NU_H} : N^sU_H \to U.
\]
As there is no points fixed by groups larger than $H$, we may choose a $G$-invariant function $\epsilon : GU_H \to \mathbb{R}_+$ small enough such that for any $H' \subseteq G$ conjugate to $H$,
\[
|N^sNU_H| \cap |N^sNU_{H'}| \neq \emptyset \implies H = H'.
\]
Then we equip the neighborhood \(|N'U_H|\) with the pushforward of the the bundle metric \(g^{TNU_H}\) and the bundle NC structure \(I^{TNU_H}\). Using \(G\)-action one obtains similar pushforward to other conjugate neighborhoods \([N'U_H']\). Denote the new metric by \(g_1^{TU}\) and the new NC structure by \(I_1^{TU}\).

By our assumption, \(g_0^{TU}\) is straightened near \(Y\) and \(I_0^{TU}\) is straightened with respect to \(g_0^{TU}\) near \(Y\); therefore, by construction, near each point \(x \in Y \cap GU_H\), \(g_0^{TU} = g_1^{TU}\) and \(I_0^{TU} = I_1^{TU}\). Hence there is a \(G\)-invariant open neighborhood of \(Y \cup GU_H\) such that \(g_0^{TU}\) and \(g_1^{TU}\) glue together to a straightened metric, still denoted by \(g_1^{TU}\); on the same neighborhood, \(I_0^{TU}\) and \(I_1^{TU}\) glue together to an NC structure straightened with respect to \(g_1^{TU}\), which is still denoted by \(I^{TU}\). Notice that since \(I_1^{TU} = I_0^{TU}\) along \(Y \cup GU_H\), we may take the neighborhood sufficiently small so that the new one is still close to \(I^{TU}\). Then by Lemma 2.21, one can extend \(I_1^{TU}\) to the whole \(U\) without altering its value near \(Y \cup GU_H\) and still denoted by \(I_1^{TU}\), such that it is still close to \(I^{TU}\). One then use a cut-off function to extend \(g_1^{TU}\) to the whole of \(U\) without altering its value near \(Y \cup GU_H\) and still denoted by \(g_1^{TU}\).  

To extend this argument to an orbifold, one can first get the following easy corollary. Continue the setting of Lemma A.5. For each \(d \geq -1\), denote

\[
U_d := \bigcup_{\dim(U_H) \leq d} U_H \subseteq U.
\]

**Corollary A.6.** Suppose \(U_{d-1} = \emptyset\) (which implies that \(U_d \subseteq U\) is a disjoint union of fixed point sets and a \(G\)-invariant submanifold). Let \(NU_d \rightarrow U_d\) be the normal bundle which carries an induced inner product \(I_0^{NU_d}\), an induced connection \(\nabla_0^{NU_d}\), and a complex structure \(I_0^{NU_d}\). Suppose

1. Near \(Y\), \(g_0^{TU}\) is straightened and \(I_0^{TU}\) is straightened with respect to \(g_0^{TU}\).
2. Along \(NU_d \rightarrow U_d\), the triple \((\nabla_0^{NU_d}, h_0^{NU_d}, i_0^{NU_d})\) is Hermitian.

Then there exist a \(G\)-invariant metric \(g_1^{TU}\) and an NC structure \(I_1^{TU}\) close to \(I^{TU}\) such that

1. Near \(Y \cup U_d\), \(g_1^{TU}\) is straightened and \(I_1^{TU}\) is straightened with respect to \(g_1^{TU}\).
2. \((g_1^{TU}, I_1^{TU})\) coincides with \((g_0^{TU}, I_0^{TU})\) near \(Y\).

**Proof.** Apply the argument of Lemma A.5 to all fixed point sets \(U_K\) with \(\dim(U_K) = d\) (which are disjoint).

**Proof of Proposition 2.29.** For \(d \geq -1\), let \(U_d \subseteq U\) be the union of strata \(U_d\) of the isotropy stratification whose dimensions are at most \(d\). Then \(U_d\) is closed. We would like to prove the following statement by induction on \(d\):

**Claim I.** For each \(d\), there exist a Riemannian metric \(g_1^{TU}\) and NC structure \(I_1^{TU}\) which is close to \(I^{TU}\) such that the pair \((g_1^{TU}, I_1^{TU})\) is a straightened NC structure near \(Y\) and such that \((g_1^{TU}, I_1^{TU}) = (g_0^{TU}, I_0^{TU})\) near \(Y\).

Notice that the proposition is equivalent to Claim I for \(d = \dim(U)\) case.

**Claim II.** Suppose Claim I holds for \(d - 1\) as well for \(d\) under the special restriction \(U_{d-1} = \emptyset\), then Claim I holds for \(d\) without this restriction.

**Proof of Claim II.** Let \(U_{d-1}^+ \subseteq U\) be a closed neighborhood of \(U_{d-1}\) such that \((g_0^{TU}, I_0^{TU})\) is a straightened NC structure near \(Y \cup U_{d-1}^+\). Then define \(\hat{U} := U \setminus U_{d-1}^\prime\).
and \( \tilde{\mathcal{Y}} := (\mathcal{Y} \cup \mathcal{U}_{d-1}) \setminus \mathcal{U}_{d-1} \). We have \( \mathcal{U}_{d-1} = \emptyset \). Then there exists a pair \((\tilde{g}^{\mathcal{T}_d}, \tilde{\mathcal{T}}^{\mathcal{T}_d})\) defined on \( \mathcal{U} \) which is a straightened NC structure near \( \tilde{\mathcal{Y}} \cup \mathcal{U}_d \) and agrees with \((g^{\mathcal{T}_d}_0, \mathcal{T}^{\mathcal{T}_d}_0)\) such that \( \tilde{\mathcal{T}}^{\mathcal{T}_d}_1 \) is close to \( \mathcal{T}^{\mathcal{T}_d} \). Together with \((g^{\mathcal{T}_d}_0, \mathcal{T}^{\mathcal{T}_d}_0)\) near \( \mathcal{U}_{d-1} \), one obtains a pair \((g^{\mathcal{T}_d}_1, \mathcal{T}^{\mathcal{T}_d}_1)\) defined on \( \mathcal{U} \) which satisfies the requirement. □

Now we head towards proving Claim I for \( d \) under the special restriction \( \mathcal{U}_{d-1} = \emptyset \). We first modify the normal inner products and connections along all \( d \)-dimensional strata. For each chart \( C = (G, U, \psi) \), denote \( U_d := \psi^{-1}(\mathcal{U}_d) \). As \( \mathcal{U}_{d-1} = \emptyset \), it follows that \( U_d \) is a closed \( G \)-invariant submanifold of \( U \). Moreover, if \( \varepsilon : (G', U', \psi') \to (G, U, \psi) \) is a chart embedding, then \( \varepsilon : U'_d \to U_d \) is an open embedding inducing an equivariant bundle map \( NU'_d \to NU_d \). Let \( NU_d \to U_d \) be the normal bundle, which carries a complex structure \( I_0^{NU_d} \), an inner product \( h_0^{NU_d} \), and an \( h_0^{NU_d} \)-preserving connection \( \nabla_0^{NU_d} \) such that near \( \psi^{-1}(\mathcal{Y}) \cap U_d \), \( h_0^{NU_d} \) and \( \nabla_0^{NU_d} \) are Hermitian.

Claim III. For all charts \( C = (G, U, \psi) \), there exist a \( G \)-invariant Hermitian inner product \( h_1^{NU_d} \) on \( NU_d \) such that near \( \psi^{-1}(\mathcal{Y}) \cap U_d \), \( h_1^{NU_d} = h_0^{NU_d} \), and such that the collection \( h_1^{NU_d} \) is invariant under chart embeddings.

Proof of Claim III. We first construct \( h_1^{NU_d} \) on each chart. Indeed, take an arbitrary Hermitian inner product \( h_0^{NU_d} \) on \( NU_d \) which coincides with \( h_0^{NU_d} \) near \( \psi^{-1}(\mathcal{Y}) \cap U_d \), then take average over \( G \). By abuse of notation, let \( h_1^{NU_d} \) denote \( h_1^{NU_d} \). Then \( h_1^{NU_d} \) is \( G \)-invariant. Choose a countable cover of \( U \) by charts \( C_i = (G_i, U_i, \psi_i) \) and a subordinate smooth partition of unity \( \rho_i : U \to [0, 1] \). For each chart \( C = (G, U, \psi) \), denote \( \rho^C_i := \rho_i \circ \psi \). If \( \text{supp} \rho^C_i \neq \emptyset \), then \( \rho^C_i \big|_{U_d} h_1^{NU_i,d} \) is a Hermitian 2-tensor on the bundle \( NU_d \). Then

\[
\sum_{i=1}^{\infty} \rho^C_i \big|_{U_d} h_1^{NU_i,d}
\]

is a Hermitian inner product on \( NU_d \) which respects chart embedding. □

Claim IV. For each chart \( C = (G, U, \psi) \), there exists a \( G \)-invariant connection \( \nabla^{NU_d} \) on \( NU_d \to U_d \) which is Hermitian with respect to \( I_0^{NU_d} \) and \( h_1^{NU_d} \).

Proof of Claim IV. It is similar to the previous case. We first construct the Hermitian connection on a chart \( C_i \) and then use a partition of unity to glue together. □

Next we construct the new metric and NC structure. Choose a sufficiently small smooth function \( \varepsilon : U_d \to \mathbb{R}_+ \). By abuse of notation, let \( \varepsilon : U_d \to \mathbb{R}_+ \) be the pullback to any chart \( C = (G, U, \psi) \). Then the exponential map \( \exp^{NU_d} : N'U_d \to U \) is an open embedding. Replace \( g^{\mathcal{T}_d}_0 \big|_{N'U_d} \) by the bundle metric on \( N'U_d \) determined by \( \langle g^{\mathcal{T}_d}_0, h_1^{NU_d}, \nabla^{NU_d}_1 \rangle \) and replace the NC structure \( \mathcal{T}^{\mathcal{T}_d}_0 \big|_{N'U_d} \) by the bundle NC structure. Denote \( |N'U_d| \subseteq U \) the union of \( \psi(|N'U_d|) \). Then we obtain a Riemannian metric \( g^{\mathcal{T}_d}_1 \) and an NC structure \( \mathcal{T}^{\mathcal{T}_d}_1 \) defined in \( |N'U_d| \). If \( \varepsilon \) is sufficiently small, one can guarantee that \( \mathcal{T}^{\mathcal{T}_d}_1 \) is close to \( \mathcal{T}^{\mathcal{T}_d} \). One can check that since \( (g^{\mathcal{T}_d}_0, \mathcal{T}^{\mathcal{T}_d}_0) \) is straightened near \( \mathcal{Y} \), the new pair \( (g^{\mathcal{T}_d}_1, \mathcal{T}^{\mathcal{T}_d}_1) \) coincides with \( (g^{\mathcal{T}_d}_0, \mathcal{T}^{\mathcal{T}_d}_0) \) near \( \mathcal{Y} \cap |N'U_d| \). Then the old and new pairs determines a straightened pair near the closed set \( \mathcal{Y} \cup \mathcal{U}_d \). By Lemma 2.21, one can extend \( \mathcal{T}^{\mathcal{T}_d}_1 \) to the whose \( U \) without altering its current value near \( \mathcal{Y} \cup \mathcal{U}_d \). Then using a cut-off function to extend \( g^{\mathcal{T}_d}_1 \) to the whole \( U \) without altering its current value near \( \mathcal{Y} \cup \mathcal{U}_d \). This proves Claim I for \( d \) under the assumption \( \mathcal{U}_{d-1} = \emptyset \). By Claim II, this finishes the proof of the proposition. □
A.2. **Proof of Proposition 2.34.** The strategy of proving Proposition 2.34 is similar to that of Proposition 2.29, i.e., inductive construction of straightened structures near strata of the isotropy stratification, from the deepest up.

A.2.1. **Bundle straightening model.**

**Definition A.7.**

1. A bundle straightening model is a quadruple \((G, X, F, E)\) where \((G, X, F)\) is a stratification model (Definition A.1) and \(E = (E, h^E, \nabla^E, I^E)\) consists of a \(G\)-equivariant vector bundle \(E \to X\) (where \(X\) has the trivial \(G\)-action), \(h^E\) is an invariant inner product, \(\nabla^E\) is an \(h^E\)-preserving \(G\)-invariant connection, and \(I^E\) is a \(G\)-invariant complex structure on \(E_G \subset E\) such that the triple \((\nabla^E, h^E, I^E)\) is Hermitian.

2. A complex bundle straightening model is a straightening model \((G, X, F, E)\) equipped with a \(G\)-invariant complex structure \(I^E\) on \(E\), such that the triple \((\nabla^E, h^E, I^E)\) is Hermitian and such that \(I^E\) is induced from \(I^E_{G} \subset E\).

Associated to a bundle straightening model \((G, X, F, E)\) there are the following objects. On the pullback bundle \(E^*_F \to F\) there is the pullback connection \(\pi^* E^*_F \to F\), the pullback inner product \(\pi^* h^E\), and the pullback NC structure \(I^E\).

**Proposition A.8.**

1. The pullback connection \(\pi^* E^*_F \to F\) is straightened with respect to the bundle metric \(g^{TF}\).

2. The pullback inner product \(\pi^* h^E\) on \(E^*_F \to F\) is straightened with respect to \(g^{TF}\) and \(\pi^* E^*_F \to F\).

3. The pullback NC structure \(I^E\) on \(E^*_F \to F\) is straightened with respect to \(g^{TF}\) and \(\pi^* E^*_F \to F\).

**Proof.** Choose \(H \subset F\), whose fixed point set in the total space of \(F\) is the total space of \(F_H\). Along each geodesic normal to \(F_H\) contained in the fiber \(F_x \subset F\), the parallel transport of \(\pi^* E^*_F \to F\) with respect to \(\pi^* E^*_F \to F\) is simply the identity map of \(E_x\). As all involved structures on \(\pi^* E^*_F \to F\) are pullbacks, they are naturally straightened. \(\square\)

There is a way to obtain a bundle straightening model by looking at restrictions at fixed point sets. Let \((U, g^{TU})\) be a straightened Riemannian \(G\)-manifold, \(E \to U\) be a \(G\)-equivariant vector bundle equipped with \(G\)-invariant inner product \(h^E\), connection \(\nabla^E\), and NC structure \(I^E\). For \(H \subset U\), if \((h^{E_H}, \nabla^{E_H}, I^{E_H})\) is a Hermitian triple, then the quadruple

\[(H, U_H, N U_H, E|_{U_H})\]

with the naturally induced structures defines a bundle straightening model.

A.2.2. **The inductive construction.** Analogous to Lemma A.5 and Corollary A.6, one needs the following lemma for the inductive construction. Recall that \(U_d \subset U\) is the union of fixed point loci with dimension at most \(d\). Assume \(U_{d-1} = \emptyset\). Then \(U_d\) is a \(G\)-invariant closed submanifold. One can decompose

\[E|_{U_d} = E_d \oplus \tilde{E}_d\]

although \(E_d\) and \(\tilde{E}_d\) may have varying dimensions on different components of \(U_d\). Then connections, inner products, and NC structures induce corresponding structures on \(\tilde{E}_d\).
Lemma A.9. Let $U$ be a $G$-manifold equipped with a straightened $G$-invariant Riemannian metric $g^TU$. Suppose $d \geq 0$ and $U_{d-1} = \emptyset$. Let $Y \subseteq U$ be a $G$-invariant closed subset.

Let $E \to U$ be a $G$-equivariant vector bundle equipped with an NC structure $I^E$. Let $\nabla_0^E$ be a $G$-invariant connection on $E$, $h_0^E$ be a $G$-invariant inner product on $E$, and $I_0^E$ is an NC structure on $E$ which is close to $I^E$. Suppose

1. Near $Y$, $\nabla_0^E$ is straightened with respect to $g^TU$, $h_0^E$ is preserved by $\nabla_0^E$, and $I_0^E$ is straightened with respect to $g^TU$ and $\nabla_0^E$.
2. Along $\hat{E}_d \to U_d$, the triple $(\nabla_0^{\hat{E}_d}, h_0^{\hat{E}_d}, I_0^{\hat{E}_d})$ is Hermitian.

Then there exists a $G$-invariant connection $\nabla_1^E$, a $G$-invariant inner product $h_1^E$, and an NC structure $I_1^E$ which is close to $I^E$ such that

1. Near $Y \cup U_d$, $\nabla_1^E$ is straightened with respect to $g^TU$, $h_1^E$ is preserved by $\nabla_1^E$, and $I_1^E$ is straightened with respect to $g^TU$ and $\nabla_1^E$.
2. $(\nabla_1^E, h_1^E, I_1^E)$ coincides with $(\nabla_0^E, h_0^E, I_0^E)$ near $Y$.

In addition, if $E$ is a complex vector bundle, $h_0^E$ and $\nabla_0^E$ are Hermitian, and $I_0^E$ is induced from the complex structure, and $I_0^E = I^E$, then we can make $I_1^E = I_0^E = I^E$, $h_1^E = h_0^E$, and $\nabla_1^E$ be Hermitian.

Proof. It suffices to consider the case that $U_d = GU_H$ for an essential subgroup $H$. Consider the quadruple $(H, U_H, NU_H, E|_{U_H})$ where $(H, U_H, NU_H)$ is the induced straightening model with a straightened metric $g^{TNU_H}$ on the total space $NU_H$ and $E|_{U_H}$ is equipped with the induced connection $\nabla_0^E$, the induced inner product $h_0^{E|_{U_H}}$, and a complex structure $I^{E_H}$ on $E_H \subseteq E|_{U_H}$. By our assumption, the quadruple $(H, U_H, NU_H, E|_{U_H})$ is a bundle straightening model (Definition A.7). Hence by Proposition A.8, the pullback bundle $\pi_N^{U_H}(E|_{U_H})$, equipped with the pullback triple $(\pi_N^{U_H} \nabla_0^{E|_{U_H}}, \pi_N^{U_H} h_0^{E|_{U_H}}, \pi_N^{U_H} I_0^{E|_{U_H}})$, is straightened with respect to the bundle metric $g^{TNU_H}$.

Now let $\epsilon : GU_H \to \mathbb{R}_+$ be a small enough $G$-invariant function. Using the bundle isomorphism $\pi_N^{U_H}(E|_{U_H})|_{N \cdot U_H} \to E|_{N \cdot U_H}$ determined by the connection, one obtains a straightened triple $(\nabla_0^E, h_0^E, I_0^E)$ on $E$ near $U_d = GU_H$. By the same argument as Lemma A.5, this triple agrees near $Y \cup U_d$ with the old one $(\nabla_0^E, h_0^E, I_0^E)$, together with which they define a triple which is straightened near $Y \cup U_d$ and which agrees with the old triple near $Y$.

Lastly, suppose $E$ is complex with induced NC structure $I^E$, $h_0^E$ and $\nabla_0^E$ are Hermitian, and $I_0^E = I^E$. Then the bundle isomorphism

$$\pi_N^{U_H}(E|_{U_H})|_{N \cdot U_H} \to E|_{N \cdot U_H}$$

is unitary, hence already preserves the NC structures and the inner product. Hence in the procedure of constructing the new triple, one does not need to modify $h_0^E$ and $I_0^E$. Moreover, the pullback connection $\nabla_1^E$ is Hermitian. \hfill $\square$

Proof of Proposition 2.34. The inductive argument is very similar to the proof of Proposition 2.29. Let $U_d \subseteq U$ be the union of strata of the isotropy stratification whose dimensions are at most $d$. Using the same argument as in the proof of Proposition 2.29, it suffices to prove the following claim.

Claim I. Suppose $U_{d-1} = \emptyset$. Then there exists a triple $(\nabla_1^E, h_1^E, I_1^E)$ which satisfies the requirement of Proposition 2.34 near $Y \cup U_d$ and which agrees with the existing triple near $Y$.
We first need to modify the inner product and connection in the normal direction of $U_d$ to make them Hermitian.

**Claim II.** There exists a $G$-invariant Hermitian inner product $h_1^E$ on $E$ which agrees with $h_0^E$ near $\mathcal{Y}$ such that for each chart $\hat{C} = (G, U, E, \hat{\psi})$, the restriction of $h_1^E$ to $\hat{E}_d \to U_d$ is Hermitian with respect to $I_0^E_{ad}$

**Proof of Claim II.** We can first modify on each chart. Indeed, let $h_0^E$ be the restriction of $h_0^E$ to $\hat{E}_d$. Then it is Hermitian near $\psi^{-1}(\mathcal{Y}) \cap U_d$. One can find $h_1^E$ which is Hermitian and which agrees with $h_0^E$ near $\psi^{-1}(\mathcal{Y}) \cap U_d$. Then one extend $h_1^E$ to an inner product $h_1^E$ on $E$ which agrees with $h_0^E$ near $\psi^{-1}(\mathcal{Y})$. Then using a partition of unity argument, one can find the desired ones.

**Claim III.** For each chart $\hat{C} = (G, U, E, \hat{\psi})$, there exists a $G$-invariant connection $\nabla^{E,\hat{\psi}}$ on $\hat{E}_d \to U_d$ which is Hermitian with respect to $I_0^E$ and $h_1^E$.

**Proof of Claim III.** It is similar to the proof of the above claim. Indeed, on each chart $\hat{C} = (G, U, E, \hat{\psi})$, one first modifies $\nabla^{E,\hat{\psi}}$ to obtain a Hermitian connection $\nabla^{E,\hat{\psi}}$ which agrees with $\nabla^{E,\hat{\psi}}$ near $\psi^{-1}(\mathcal{Y}) \cap U_d$, and then find an $h_1^E$-preserving connection $\nabla^{E,\hat{\psi}}$ extending $\nabla^{E,\hat{\psi}}$ which agrees with $\nabla^{E,\hat{\psi}}$ near $\psi^{-1}(\mathcal{Y})$. Then using a partition of unity argument to find the desired collection of connections.

Then we modify the triple $(\nabla^{E,\hat{\psi}}, h_0^E, I_0^E)$ in a neighborhood of $U_d$ using the bundle isomorphism

\[(\pi_{N\cdot U_d}^* (E|_{U_d}))|_{N\cdot U_d} \to E|_{(N\cdot U_d)}.\]

induced from the connection $\nabla^{E,\hat{\psi}}$. By abuse of notation, let the new triple be $(\nabla^{E,\hat{\psi}}, h_1^E, I_1^E)$, which together with the old triple $(\nabla^{E,\hat{\psi}}, h_0^E, I_0^E)$, defines a straightened triple near $\mathcal{Y} \cup U_d$.

Lastly, suppose $E$ is a complex vector bundle, $h_0^E$ and $\nabla_0^E$ are Hermitian, $I_0^E = I^E$ is induced from the complex structure, then the constructions covered by Claim II and Claim III are unnecessary. Moreover, the chartwise bundle isomorphism $\Phi^E$ near $U_d$ preserves the NC structure. Therefore one does not need to modify the NC structure and the inner product on $E$ during the construction.

**A.3. Proof of Proposition 2.39.** One considers the case on a single chart first.

**Lemma A.10.** Let $(U, g^{TU})$ be a straightened Riemannian $G$-manifold. Let $F \to U$ be a $G$-equivariant vector bundle equipped with an inner product $h^F$ and an $h^F$-preserving connection $\nabla^F$. Suppose $\nabla^F$ is straightened with respect to $g^{TU}$ and $h^F$ is straightened with respect to $g^{TU}$ and $\nabla^F$. Then the bundle metric $g^{TF}$ determined by $(g^{TU}, \nabla^F, h^F)$ on the total space of $F$ is straightened.

**Proof.** Let $H \subset G$ be an essential subgroup whose fixed point set in $F$ is the total space of $F_H \to U_H$. As $g^{TU}$ is straightened, we may identify a neighborhood of $U_H$ isometrically with a disk bundle $N^\mathcal{Y} U_H$. As $\nabla^F$ and $h^F$ are straightened, we may identify $F|_{(N^\mathcal{Y} U_H)}$ with the pullback of $F|_{U_H}$ with the pullback connection and inner products. Then near $U_H$, the total space $F$ can be identified naturally with other total spaces

\[F|_{(N^\mathcal{Y} U_H)} \cong NU_H \oplus F_H \oplus \tilde{F}_H \cong \pi^*_H (NU_H \oplus \tilde{F}_H) \cong NF_H\]

By Lemma A.4 (3), bundle metrics obtained from different perspectives are identical. Hence $g^{TF} = g^{TNF_H}$, meaning $g^{TF}$ is straightened along $NF_H$. □
Next we treat the straightenedness of bundle NC structures.

**Lemma A.11.** Let \((U, g^{TU})\) be a straightened Riemannian \(G\)-manifold and \(I^{TU}\) is a straightened NC structure. Let \(F \to U\) be a \(G\)-equivariant vector bundle equipped with an NC structure \(I^F\), a Hermitian metric \(h^F\), a Hermitian connection \(\nabla^F\) such that \(\nabla^F\) is straightened with respect to \(g^{TU}\) and \(I^F\) is straightened with respect to \(g^{TU}\) and \(\nabla^F\). Then the bundle NC structure \(I^{TF}\) induced from \(I^{TU}, I^F,\) and \(\nabla^F\) is straightened with respect to the bundle metric \(g^{TF}\).

**Proof.** Using notations from the previous proof, we know near \(U_H, F\) is naturally identified with the total space of \(NU_H \oplus F_H \oplus \tilde{F}_H \to U_H\) and the normal exponential map along \(NF_H\) is the same as linear additions in fibers of the triple direct sum. As the complex structure \(I^F_H\) is the same as the pullback of the direct sum \(I^{NU_H} \oplus I^{F_H}\) to \(F_H\), the normal exponential map preserves the NC structure. □

Now we turn to straightened structures on pullback vector bundles.

**Lemma A.12.** Let \((U, g^{TU})\) and \((F, \nabla^F, h^F)\) be as in Lemma A.10. Let \(E \to U\) be a \(G\)-equivariant vector bundle equipped with a connection \(\nabla^E\) straightened with respect to \(g^{TU}\), an inner product \(h^E\) preserved by \(\nabla^E\), and an NC structure \(I^E\) straightened with respect to \(g^{TU}\) and \(\nabla^E\) such that \((\nabla^E, h^E, I^E)\) is normally Hermitian. Then

1. \(\pi^*_F \nabla^E\) is straightened with respect to the bundle metric \(g^{TF}\).
2. \(\pi^*_F h^E\) is preserved by \(\pi^*_F \nabla^E\) and \(\pi^*_F I^E\) is straightened with respect to \(g^{TF}\) and \(\pi^*_F \nabla^E\).
3. \((\pi^*_F \nabla^E, \pi^*_F h^E, \pi^*_F I^E)\) is normally Hermitian.

**Proof.** For any essential subgroup \(H \subseteq F, G\), one has the identification of the normal bundle

\[NF_H \cong \pi^*_F NU_H \oplus \pi^*_F \tilde{F}_H\]

which has the equipped bundle metric \(g^{TNF_H}\) which agrees with \(g^{TF}\) in a tubular neighborhood. As the normal geodesics are straight line segments in fibers of \(NF_H\), it is easy to see that \(\pi^*_F \nabla^E\) is just the pullback from the restriction to \(F_H\). The straightenedness of other structures follows similarly. □

**Proof of Proposition 2.39.** The straightenedness of \(g^{TF}, I^{TF}, \pi^*_F \nabla^E, \pi^*_F h^E, \pi^*_F I^E\) can be verified chartwise. These verifications follow from Lemma A.10, A.11, and A.12. □

**Appendix B. Whitney Stratifications**

In this appendix we prove the existence of a canonical Whitney stratification of complex analytic sets relative to a given stratification (Proposition 3.6), properties of such Whitney stratifications (Proposition B.14 and Proposition B.20) as well as Proposition 3.8.

**B.1 Most basic results.** Recall the definition of Whitney stratification (Definition 3.1). A first basic fact is that the partial order relation among strata of a Whitney stratification is compatible with dimensions.

**Lemma B.1.** [GWdPL76, (1.1)] Let \(Z_\alpha, Z_\beta\) be two strata of a Whitney stratification on \(Z \subseteq M\). If \(Z_\alpha < Z_\beta\), then \(\dim(Z_\alpha) < \dim(Z_\beta)\).
Next we consider the pullback operation on Whitney stratifications.

**Lemma B.2.** Let \( Z \subseteq M \) be equipped with a Whitney stratification \( \mathfrak{Z} \) and \( f : N \to M \) be a smooth map transverse to \( \mathfrak{Z} \), then the partition

\[
f^* \mathfrak{Z} := \left\{ f^{-1}(Z_\alpha) \neq \emptyset \mid Z_\alpha \in \mathfrak{Z} \right\}
\]

is a Whitney stratification on \( f^{-1}(Z) \subseteq N \).

**Proof.** The only non-obvious condition for \( f^* \mathfrak{Z} \) being a Whitney stratification is the axiom of frontier. Namely, we need to prove

\[
f^{-1}(Z_\alpha) \cap f^{-1}(Z_\beta) \neq \emptyset \implies f^{-1}(Z_\alpha) \subseteq f^{-1}(Z_\beta).
\]

Suppose \( f^{-1}(Z_\alpha) \cap f^{-1}(Z_\beta) \neq \emptyset \), then there exist a point \( x \in f^{-1}(Z_\alpha) \) and a sequence of points \( x_i \in f^{-1}(Z_\beta) \) such that \( x_i \to x \). Then \( Z_\beta \ni f(x_i) \to f(x) \in Z_\alpha \). Therefore, \( f(x) \in Z_\alpha \cap Z_\beta \neq \emptyset \). Hence \( Z_\alpha < Z_\beta \). On the other hand, \( f \) is transverse to \( \mathfrak{Z} \) equivalent to that \( \text{graph}(f) \subseteq N \times M \) is transverse to each \( N \times Z_\alpha \). Moreover, \( \{N \times Z_\alpha \mid Z_\alpha \in \mathfrak{Z}\} \) is a Whitney stratification of \( N \times Z \subseteq N \times M \). Then by [Mat12, Corollary 10.4],

\[
\text{graph}(f) \cap (N \times Z_\alpha) \subseteq \text{graph}(f) \cap (N \times Z_\beta) \implies f^{-1}(Z_\alpha) \subseteq f^{-1}(Z_\beta).
\]

It is often helpful to refine a Whitney stratification by taking connected components of all strata. The following lemma guarantees that one still obtains a Whitney stratification.

**Lemma B.3.** [Mat73, Proposition 8.7][Mat12, Corollary 10.5] Let \( M \) be a smooth manifold and \( \mathfrak{Z} \) be a locally finite partition of a closed subset \( Z \subseteq M \) into locally closed and connected submanifolds such that each disjoint pair \( (Z_\alpha, Z_\beta) \) of \( \mathfrak{Z} \) is Whitney regular, then \( \mathfrak{Z} \) is a Whitney stratification of \( Z \).

The following lemma is also frequently used in this paper.

**Lemma B.4.** Let \( X, M, N \) be smooth manifolds and \( h : X \to M \) and \( f : M \to N \) be smooth maps. Let \( Z \subseteq N \) be a closed subset equipped with a Whitney stratification \( \mathfrak{Z} \). Then \( f \circ h \) is transverse to \( \mathfrak{Z} \) if and only if \( f \) is transverse to \( \mathfrak{Z} \) at points in \( \text{Im}(h) \). Moreover, suppose \( f \) is transverse to \( \mathfrak{Z} \) hence pulls back a Whitney stratification \( f^* \mathfrak{Z} \) on \( f^{-1}(Z) \). Then \( h \) is transverse to \( f^* \mathfrak{Z} \) if and only if \( f \circ h \) is transverse to \( \mathfrak{Z} \).

**Proof.** The argument is the same as if \( Z \) is a smooth closed submanifold and follows from basic linear algebra. \( \square \)

**B.1.1. Invariance properties of minimal Whitney stratifications.** The purpose of this part is to show certain functorial properties of minimal Whitney stratifications. We do not only consider absolute minimal Whitney stratifications, but also those which are minimal in a restricted class. This is because the Whitney stratification we need in our applications must respect, in a certain sense, the group action on the ambient space (see Definition 3.5).

**Remark B.5.** As the following lemma is rather technical, we would like to try to provide some motivations. Indeed, Lemma B.6 serves as a generalization of Proposition B.20 which requires more setup in the complex analytic category. Both results deal with the following situation. Let \( f : M \to N \) be an embedding of manifolds. Suppose \( Z \subseteq N \) is equipped with a minimal Whitney stratification \( \mathfrak{Z} \) and \( f \) is transverse to \( \mathfrak{Z} \), then \( f^* \mathfrak{Z} \) is a Whitney stratification on \( Y := f^{-1}(Z) \) (see...
Figure 3). However, we would prefer to have that $f^*\mathcal{Z}$ is (equivalent to) a minimal Whitney stratification $\mathcal{Y}$ (which exists for other reasons), which may not be true in general. This is the typical scenario we have in Section 3. In order to show the minimality of $f^*\mathcal{Z}$, we typically construct a submersive left inverse $g: N \to M$. In the complex analytic category, when the left inverse $g$ is a submersion, one can directly compare $g^*\mathcal{Y}$ with $\mathcal{Z}$ and then derive $f^*\mathcal{Z} \equiv \mathcal{Y}$. However, there is one case, i.e. in the proof of Proposition 3.15, where the construction only provides a smooth map $g$ which is not a left inverse either. In that scenario, we need to appeal to Lemma B.6 to compare the involved Whitney stratifications. Lastly, we would like to mention that the idea comes from Parker [Par13].

Let $M$ be a smooth manifold, $Z \subseteq M$ be a subset, and $\mathcal{W}^\infty(Z)$ be the set of all smooth Whitney stratifications on $Z$. Recall that one can compare different Whitney stratifications using the dimension filtration (Definition 3.3).

Let $M$, $N$ be smooth manifolds and $Y \subseteq M$, $Z \subseteq N$ be subsets. Let $\mathcal{N}(Y) \subseteq \mathcal{W}^\infty(Y)$ resp. $\mathcal{N}(Z) \subseteq \mathcal{W}^\infty(Z)$ be a subset of smooth Whitney stratifications which is closed under the equivalence relation $\equiv$. Suppose $\mathcal{Y} \in \mathcal{N}(Y)$ resp. $\mathcal{Z} \in \mathcal{N}(Z)$ is a minimal one.

Let $f: M \to N$ and $g: N \to M$ be smooth maps satisfying the following.

1. $f^{-1}(Z) = Y$ and $g^{-1}(Y) = Z$.
2. $f$ resp. $g$ is transverse to $\mathcal{Z}$ resp. $\mathcal{Y}$.
3. $f^*\mathcal{Z} \in \mathcal{N}(Y)$ and $g^*\mathcal{Y} \in \mathcal{N}(Z)$.
4. $g \circ f$ is transverse to $\mathcal{Y}$ and $(g \circ f)^*\mathcal{Y} \equiv \mathcal{Y}$.
5. $f \circ g$ is transverse to $\mathcal{Z}$ and $(f \circ g)^*\mathcal{Z} \equiv \mathcal{Z}$.

Then $f^*\mathcal{Z} \equiv \mathcal{Y}$ and $g^*\mathcal{Y} \equiv \mathcal{Z}$.

Proof. Denote $\mathcal{Y} := f^*\mathcal{Z}$ and $\mathcal{Z} := g^*\mathcal{Y}$. By the minimality of $\mathcal{Y}$ resp. $\mathcal{Z}$, one has $\mathcal{Y} \equiv \mathcal{Z} \equiv \mathcal{Z}$. Denote $m = \dim(M)$, $n = \dim(N)$. We prove inductively that for all $l \geq 0$

$$\mathcal{Y}_{m-l} = \mathcal{Z}_{m-l}, \quad \mathcal{Z}_{n-l} = \mathcal{Y}_{n-l}.$$ 

The $l = 0$ case is automatically true. Suppose the above is true for all $l < k$. We first show that $\mathcal{Y}_{m-k} = \mathcal{Z}_{m-k}$. Suppose on the contrary that $\mathcal{Y}_{m-k} \neq \mathcal{Z}_{m-k}$.
Then as \( \mathcal{Y} \subseteq \mathcal{Z} \), one must have \( \mathcal{Y}_{m-k} \subset \mathcal{Z}_{m-k} \). Hence there exists a point \( y \in \mathcal{Y}_{m-k+1} = \mathcal{Z}_{m-k+1} \) and \( y \in \mathcal{Y}_{m-k} \setminus \mathcal{Y}_{m-k} \). Let \( \mathcal{Y}_y \) and \( \mathcal{Z}_y \) be the strata through \( y \) associated to \( \mathcal{Y} \) and \( \mathcal{Z} \) respectively. Then

\[
\dim(\mathcal{Y}_y) = m - k + 1, \quad \dim(\mathcal{Z}_y) \leq m - k. \tag{B.1}
\]

Condition (4) implies that \( f \) is transverse to \( \mathfrak{S} \) and \( f^*\mathfrak{S} \equiv \mathcal{Y} \). Then (B.1) implies

\[
\dim(\mathfrak{S}_{f(y)}) = n - m + \dim(\mathcal{Y}_y) = n - k + 1.
\]

where \( \mathfrak{S}_{f(y)} \) is the stratum containing \( f(y) \) associated to the Whitney stratification \( \mathfrak{S} \). On the other hand, by (B.1) and the transversality of \( f \) to \( \mathfrak{S} \), one has

\[
\dim(\mathfrak{S}_{f(y)}) = n - m + \dim(\mathcal{Z}_y) \leq n - k.
\]

Therefore,

\[
f(y) \in \mathfrak{S}_{n-k} \setminus \mathfrak{S}_{n-k} \implies \mathfrak{S}_{n-k} \subset \mathfrak{S}_{n-k}.\]

As \( \mathfrak{S}_{n-l} = \mathfrak{S}_{n-l} \) for all \( l < k \), this contradicts the fact that \( \mathfrak{S} \leq \mathfrak{S} \). Therefore, \( \mathcal{Y}_{m-k} = \mathcal{Z}_{m-k} \). Similarly, one also obtains that \( \mathfrak{S}_{n-k} = \mathfrak{S}_{n-k} \). Therefore, the inductive step is complete, and we conclude that \( \mathcal{Y} = \mathcal{Z} \) and \( \mathfrak{S} \equiv \mathfrak{S} \). \( \square \)

It is useful to consider a special case.

**Corollary B.7.** Let \( M \) be a smooth manifold, \( Y \subseteq M \) be a subset and \( \mathcal{N}(Y) \subseteq \mathcal{W}\mathcal{S}^Z(Y) \) be a subset of smooth Whitney stratifications on \( Y \) which is closed under equivalences. Suppose \( \mathcal{Y} \in \mathcal{N}(Y) \) is a minimal element. Suppose \( f : M \to M \) is a diffeomorphism such that \( f(Y) = Y \), \( f^*\mathcal{Y} \in \mathcal{N}(Y) \), and \( (f^{-1})^*\mathcal{Y} \in \mathcal{N}(Y) \). Then \( f^*\mathcal{Y} \equiv \mathcal{Y} \).

**Proof.** This follows from Lemma B.6 where \( M = N, Y = Z \), and \( g = f^{-1} \). \( \square \)

**B.2. Existence results in the complex analytic case.** The theory of canonical Whitney stratifications can be discussed in many different categories such as semianalytic, semialgebraic, real analytic, complex analytic, or complex algebraic (see [Mat73, Section 4]). In this paper we consider either the complex algebraic category or the complex analytic category.

**Lemma B.8.** A subset \( U \subseteq M \) is strongly analytic (Definition 3.2) if and only if there exist closed subsets \( V \) and \( W \) such that \( U = V \setminus W \).

**Proof.** If \( U \) is strongly analytic, then one can take \( V = \overline{U} \) and \( W = \overline{U} \setminus U \). Conversely, if \( U = V \setminus W \), then by [Loj91, IV.10. Theorem 5]), \( \overline{U} \) is analytic. Moreover, as \( \overline{U} \subseteq V \), one has

\[
\overline{U} \setminus (\overline{U} \cap W) = \overline{U} \setminus W \subseteq V \setminus W = U \implies \overline{U} \setminus U \subseteq \overline{U} \cap W.
\]

As \( U \cap W = \emptyset \), it follows that \( \overline{U} \setminus U = \overline{U} \cap W \) which is a closed analytic set. Then by definition, \( U \) is strongly analytic. \( \square \)

**Definition B.9.** Let \( Z \subseteq M \) be a strongly analytic subset. A point \( p \in Z \) is called a smooth regular point resp. analytic regular point if there is an open neighborhood \( N \subseteq M \) of \( p \) such that \( N \cap Z \) is a smooth resp. complex analytic submanifold.
By the theorem of Bloom [Blo69], smooth regular points are also analytic regular points. Hence we do not distinguish these two names. A point \( p \in Z \) is **singular** if it is not a regular point. Using the above definition, a regular point \( p \) has a well-defined local dimension \( \dim_p(Z) = -\infty \).

For each dimension \( k \), denote

\[ Z_k, \text{reg} = \{ p \in Z \text{ regular} \mid \dim_p(Z) = k \}. \]

We also denote

\[ \dim(Z) \leq k \iff \dim_p(Z) \leq k, \quad \forall p \in Z. \]

**Lemma B.10.** Suppose \( Z \subseteq M \) is a closed analytic subset.

1. The set of singular points \( Z_{\text{sing}} \) is a closed analytic subset.
2. For each \( k \), \( Z_k, \text{reg} \) is a strongly analytic submanifold.

**Proof.** For \( Z_{\text{sing}} \), it follows from [Loj91, Theorem IV.2.1]. On the other hand, notice that \( Z_k, \text{reg} \) is open and closed in \( Z \setminus Z_{\text{sing}} \), hence is the union of connected components. Indeed, the openness follows from the definition; for closedness, for \( p \in Z_k, \text{reg} \), if \( \dim_p(Z) \neq k \), then any point sufficiently close to \( p \) must also have local dimension not equal to \( k \), so \( p \in Z_{\text{sing}} \). By [Loj91, Theorem IV.2.3], the family of connected components of \( Z \setminus Z_{\text{sing}} \) is locally finite and each component’s closure is analytic. Hence \( Z_k, \text{reg} \) is analytic. As \( Z_k, \text{reg} = \overline{Z_k, \text{reg}} \setminus Z_{\text{sing}} \), by Lemma B.8, \( Z_k, \text{reg} \) is a strongly analytic submanifold. \( \square \)

Now we give a proof of the existence results of minimal Whitney stratifications following Mather [Mat73]. We restrict our consideration to the complex analytic case. The construction relies on the following fundamental result, whose original version was proved by Whitney [Whi65] in the complex analytic setting.

**Theorem B.11.** (see [Mat73, Theorem 4.1, Addendum 4.3] [Loj65, Section 25]) Let \( X \) and \( Y \) be disjoint strongly analytic submanifolds of a complex manifold \( M \). Let \( S_b(X, Y) \) be the set of points \( y \in Y \) where Whitney’s condition \((b)\) fails for \((X, Y)\) at \( y \). Then \( S_b(X, Y) \) is an analytic set. Moreover, if \( Y \subseteq X \), then \( S_b(X, Y) \) is nowhere dense in \( Y \).

Now we can construct a canonical Whitney stratification on a closed analytic set. Indeed the following theorem is the “absolute” version of Theorem B.13 below. However we would like to give a separate proof to show the reader the basic idea of the construction.

**Theorem B.12.** [Mat73, Theorem 4.9, Addendum 4.11, Addendum 4.12] Let \( Z \) be a closed analytic subset of a smooth complex manifold \( M \). Then there exists a smooth Whitney stratification \( \mathcal{S} \) of \( Z \) such that for any open subset \( U \subseteq M \) (in Euclidean topology), the restriction of \( \mathcal{S} \) to \( Z \cap U \) is minimal. Moreover, \( \mathcal{S} \) is strongly complex analytic, i.e., its strata are all strongly complex analytic submanifolds.

**Proof.** Let \( n = \dim_R(M) \). We construct inductively a decreasing sequence of subsets

\[ Z = Z_n \supseteq Z_{n-1} \supseteq \cdots \supseteq Z_k \]

such that for each \( l > k \) the following conditions are satisfied.

1. \( Z_l \) is a closed analytic subset of \( M \) of real dimension at most \( l \).

\[ ^9 \text{In this appendix “dimension” always means “real dimension.”} \]
(2) $\mathfrak{S}^*_l := \mathfrak{S}_l \setminus \mathfrak{S}_{l-1}$ is a strongly analytic submanifold of real dimension $l$ (empty if $l$ is odd).

(3) For any $m > l$, $(\mathfrak{S}^*_m, \mathfrak{S}^*_l)$ satisfies Whitney’s condition (b) at all points $x \in \mathfrak{S}^*_l$.

We start with $\mathfrak{S}_n = Z$. Suppose we have constructed $\mathfrak{S}_l$ for all $l \geq k$ satisfying the induction hypothesis. Then define $\mathfrak{S}_{k-1}$ to be the closure of points $x \in \mathfrak{S}_k$ satisfying one of the following conditions.

1. $x$ is a singular point of $\mathfrak{S}_k$ or a regular point with $\text{dim}_x(\mathfrak{S}_k) < k$;
2. $x$ is a regular point of $\mathfrak{S}_k$ with local dimension equal to $k$, and there exists $l > k$ such that the pair $(\mathfrak{S}^*_l, \mathfrak{S}^*_k, \text{reg})$ does not satisfy Whitney’s condition (b) at $x$ (in particular $x \in \overline{\mathfrak{S}^*_l}$).

Then by construction $\mathfrak{S}_{k-1}$ has dimension at most $k - 1$. The set of points satisfying (1) above is a closed analytic subset. On the other hand, by Lemma B.10 and Theorem B.11 above, the closure of $S_b(\mathfrak{S}^*_l, \mathfrak{S}^*_k, \text{reg})$ is a closed analytic set and has dimension at most $k - 1$. Hence $\mathfrak{S}_{k-1}$ is a closed analytic subset of $M$ of dimension at most $k - 1$ and $\mathfrak{S}_k := \mathfrak{S}_k \setminus \mathfrak{S}_{k-1}$ is a strongly analytic submanifold of dimension $k$ (by Lemma B.8). Other properties required for the induction hypothesis for $\mathfrak{S}_{k-1}$ are automatically satisfied. This completes the induction step.

Let $\mathfrak{S}$ be the collection of connected components of $\mathfrak{S}^*_k$ for all $k$. Then the collection $\mathfrak{S}$ is locally finite. Then $\mathfrak{S}$ satisfies all conditions for a Whitney stratification except for the axiom of frontier. The fact that $\mathfrak{S}$ satisfies the axiom of frontier follows from [Mat73, Proposition 8.7] (cf. Lemma B.3). Therefore, $\mathfrak{S}$ is a Whitney stratification on $Z$. We show that $\mathfrak{S}$ is strongly complex analytic. Indeed, for each $k$, $\mathfrak{S}^*_k$ is the union of irreducible components, each of which is a strongly analytic submanifold. As $\mathfrak{S}^*_k$ is nonsingular, its irreducible components do not intersect. On the other hand, each irreducible component is connected as it is regular (see [Chi12, Section 5.4, Theorem]). Hence all connected components of $\mathfrak{S}^*_k$ for all $k$, namely all strata of $\mathfrak{S}$, are strongly complex analytic submanifolds.

One needs to show that for all open subset $U \subseteq M$, $\mathfrak{S}|_{U \cap Z}$ is minimal among all smooth Whitney stratifications of $Z \cap U$. Indeed, let $\tilde{\mathfrak{S}}$ be any smooth Whitney stratification on $Z \cap U$ with dimension filtration $\tilde{\mathfrak{S}}$. We need to show that either $\tilde{\mathfrak{S}} \cap U = \tilde{\mathfrak{S}}_l$ for all $l$ or it is true for $l > k$ and $\tilde{\mathfrak{S}}_k \cap U \subseteq \tilde{\mathfrak{S}}_k$. We prove this claim inductively. For $n = \dim_\mathbb{R}(M)$, one has $\tilde{\mathfrak{S}}_n \cap U = \tilde{\mathfrak{S}}_n = Z \cap U$. Suppose we have proved that $\tilde{\mathfrak{S}}_l \cap U = \tilde{\mathfrak{S}}_l$ for all $l > k$ and suppose by contradiction that $\tilde{\mathfrak{S}}_k \cap U \nsubseteq \tilde{\mathfrak{S}}_k$. Then there exists $x \in \tilde{\mathfrak{S}}_k \cap U$ but $x \notin \tilde{\mathfrak{S}}_k$. Hence $x \in \tilde{\mathfrak{S}}_{k+1} \setminus \tilde{\mathfrak{S}}_k$. As $\tilde{\mathfrak{S}}_k$ is closed, it follows $\tilde{\mathfrak{S}}_{k+1} = \tilde{\mathfrak{S}}_{k+1} \cap U$ is locally a smooth submanifold near $x$ of dimension $k + 1$. Then $x$ is a regular point and hence $x \in \tilde{\mathfrak{S}}_{k+1, \text{reg}}$. Then by the construction of $\tilde{\mathfrak{S}}_k$, there must be some $m > k$ and a sequence $x_i \in \tilde{\mathfrak{S}}_{k+1, \text{reg}}$ such that $x_i \rightarrow x$ and $(\tilde{\mathfrak{S}}^*_{m+1}, \tilde{\mathfrak{S}}^*_{k+1, \text{reg}})$ does not satisfy Whitney’s condition (b) at $x_i$. As $\tilde{\mathfrak{S}}^*_{m+1} \cap U = \tilde{\mathfrak{S}}^*_{m+1}$, this contradicts the assumption that $\tilde{\mathfrak{S}}$ is a smooth Whitney stratification of $Z \cap U$.

One can easily extend the above results from the case of closed analytic sets to strongly analytic subsets. Indeed, if $Z$ is strongly analytic, one can first construct the canonical Whitney stratification for the closure $\overline{Z}$ whose strata $\overline{Z}_n$ are strongly analytic submanifolds. Then its restriction to the open subset $Z \subseteq \overline{Z}$ is a minimal Whitney stratification on $Z$. Moreover, the induced strata $\overline{Z}_\alpha \setminus (\overline{Z}\setminus Z)$ are again strongly analytic submanifolds (see Lemma B.8).
B.3. Relative minimal Whitney stratifications. To prove Proposition 3.6, we need to consider Whitney stratifications which are minimal in some relative sense. Suppose \( M \) is a smooth complex manifold. Let \( Z \subseteq M \) be a closed analytic subset and \( Z' \subseteq Z \) be an open subset such that \( Z \setminus Z' \) is a closed analytic set. Suppose \( Z' \) is equipped with a strongly analytic Whitney stratification \( \mathcal{S}' \). Define

\[
\mathcal{WS}^\mathcal{S}(Z; \mathcal{S}') \subseteq \mathcal{WS}^\mathcal{S}(Z)
\]

to be the subset of smooth Whitney stratifications on \( Z \) whose restriction to \( Z' \) coincides with \( \mathcal{S}' \); we say that such Whitney stratifications extend \( \mathcal{S}' \). Then for each open subset \( U \subseteq M \), the restriction of Whitney stratifications induces a map

\[
\mathcal{WS}^\mathcal{S}(Z; \mathcal{S}') \rightarrow \mathcal{WS}^\mathcal{S}(Z \cap U; \mathcal{S}'|_{Z \cap U}).
\]

**Theorem B.13.** There is a minimal element of \( \mathcal{S} \in \mathcal{WS}^\mathcal{S}(Z; \mathcal{S}') \) unique up to equivalence such that for any open subset \( U \subseteq M \), the restriction \( \mathcal{S}|_{Z \cap U} \) is a minimal element of \( \mathcal{WS}^\mathcal{S}(Z \cap U; \mathcal{S}'|_{Z \cap U}) \). In addition, \( \mathcal{S} \) is strongly analytic.

**Proof.** We follow the same strategy of the proof of Theorem B.12. As \( \mathcal{S}' \) is strongly complex analytic, for each \( k \), \( \mathcal{S}'_k \mathcal{S}'_{k-1} \) is a strongly complex analytic submanifold. Suppose \( \dim_{\mathcal{S}}(M) = n \). For each \( k \leq n \), we would like to define a decreasing sequence of closed analytic subsets

\[
Z \setminus Z' = \mathcal{S}_n' \supseteq \mathcal{S}_{n-1}' \cdots \supseteq \mathcal{S}_k'
\]
such that for all \( l \geq k \) the following conditions are satisfied.

1. \( \mathcal{S}_l' \) is a closed analytic set of dimension at most \( l \).
2. \( \mathcal{S}_l' \setminus \mathcal{S}_{l-1}' \) is a strongly complex submanifold of real dimension \( l \).
3. \( \mathcal{S}_m' \setminus \mathcal{S}_{m-1}' \) is disjoint from the closure of \( \mathcal{S}_l' \).
4. For each \( x \in \mathcal{S}_m' \setminus \mathcal{S}_{m-1}' \) and \( m > l \), the pair \( (\mathcal{S}_m' \setminus \mathcal{S}_{m-1}', \mathcal{S}_l') \) and the pair \( (\mathcal{S}_m' \setminus \mathcal{S}_{m-1}', \mathcal{S}_l') \) satisfy Whitney’s condition (b) at \( x \).

Now we start with \( \mathcal{S}_n' := Z \setminus Z' \) which is indeed a closed analytic subset of \( M \). Suppose we have constructed \( \mathcal{S}_m', \ldots, \mathcal{S}_k' \) satisfying the above conditions. Now consider the closure of points \( x \in \mathcal{S}_k' \) satisfying one of the following conditions, denoted by \( \mathcal{S}_{k-1}' \)

1. \( x \) is contained in the closure of \( \mathcal{S}_k' \).
2. \( x \) is either a singular point of \( \mathcal{S}_k' \) or a regular point with \( \dim_{\mathcal{S}}(\mathcal{S}_k') < k \).
3. \( x \in \mathcal{S}_k', \text{reg} \) and there exists \( m > k \) such that the pair \( (\mathcal{S}_m' \setminus \mathcal{S}_{m-1}', \mathcal{S}_k', \text{reg}) \) does not satisfy Whitney’s condition (b) at \( x \).
4. \( x \in \mathcal{S}_k', \text{reg} \) and there exists \( m > k \) such that the pair \( (\mathcal{S}_m' \setminus \mathcal{S}_{m-1}', \mathcal{S}_k', \text{reg}) \) does not satisfy Whitney’s condition (b) at \( x \).

Then following the same argument as the proof of Theorem B.12, \( \mathcal{S}_{k-1}' \) is a closed analytic set of real dimension at most \( k - 1 \) and hence \( \mathcal{S}_{k}' \setminus \mathcal{S}_{k-1}' \) is a strongly analytic submanifold. This finishes the inductive step. Then similar to Theorem B.12, one can find a Whitney stratification (of \( Z \setminus Z' \)) by taking all connected components of \( \mathcal{S}_{k}' \setminus \mathcal{S}_{k-1}' \). Together with strata of \( \mathcal{S}' \), one obtains an extension of \( \mathcal{S}' \), denoted by \( \mathcal{S} \).

We need to prove that \( \mathcal{S} \) satisfies the minimality condition. Let \( U \subseteq M \) be an open subset. Let \( \mathcal{S} \) be another Whitney stratification on \( Z \cap U \) which extends \( \mathcal{S}'|_{Z \cap U} \). Then the dimension filtrations of \( \mathcal{S}|_{Z \cap U} \) and \( \mathcal{S} \) are

\[
\mathcal{S}_k \cap U = (\mathcal{S}_k' \cap U) \cup (\mathcal{S}_k' \cap U), \quad \mathcal{S}_k = (\mathcal{S}_k' \cap U) \cup \tilde{\mathcal{S}}_k
\]
where $\tilde{Z}_k \subseteq (Z \setminus Z') \cap U$ is the dimension filtration of the restriction of $\tilde{Z}$ to $(Z \setminus S') \cap U$. Then we need to show that either $\tilde{Z}_k \cap U = \tilde{Z}_k''$ for all $l$ or there exists $k$ such that it is true for all $l > k$ and $\tilde{Z}_k'' \cap U \subseteq \tilde{Z}_k''$. We argue inductively. For $n = \dim(M)$, one has $\tilde{Z}_k'' \cap U = \tilde{Z}_k''' = (Z \setminus Z') \cap U$ by definition. Suppose we have proved that $\tilde{Z}_k'' \cap U = \tilde{Z}_k''$ for all $l > k$. Assume in contradiction that $\tilde{Z}_k'' \cap U \not\subseteq \tilde{Z}_k''$. Then there exists a point $x \in \tilde{Z}_k'' \cap U$ which is not contained in $\tilde{Z}_k''$, namely, $x \in \tilde{Z}_k' \setminus \tilde{Z}_k''$. As $\tilde{Z}_k''$ is closed, $x$ is a smooth (equivalently analytic) regular point of $\tilde{Z}_k' = \tilde{Z}_k''' \cap U$ of local (real) dimension $k + 1$.

Now because $\tilde{Z}$ is a Whitney stratification on $Z \cap U$ and $x \in \tilde{Z}_{k+1}' \setminus \tilde{Z}_k''$, it follows from Lemma B.1, $x \notin \tilde{Z}_{k+1}' \cap U$. Then as $x \in \tilde{Z}_k'' \cap U \subseteq \tilde{Z}_k''$, by the construction of $\tilde{Z}$, it follows that there is a sequence $x_i \to x$ and $m > k$ such that either $(\tilde{Z}_{m+1}' \setminus \tilde{Z}_m'', \tilde{Z}_{m+1, \text{reg}}')$ or $(\tilde{Z}_{m+1}' \setminus \tilde{Z}_m'', \tilde{Z}_{m+1, \text{reg}}''')$ does not satisfy Whitney’s condition (b) at $x_i$. This contradicts the hypothesis that $\tilde{Z}$ is a Whitney stratification on $Z \cap U$. Therefore $\tilde{Z}_k'' \cap U \subseteq \tilde{Z}_k''$, hence inductively we conclude that $\tilde{Z}_k'' \cap U \subseteq \tilde{Z}_k''$. □

B.4. Proof of Proposition 3.6. We first construct the Whitney stratification. Choose an increasing order of $\{M_i\}$ as $M_{i_1}, \ldots, M_{i_m}$ such that $M_{i_1} \subseteq M_{i_2} \subseteq \cdots \subseteq M_{i_m}$. For each $k$, denote

$$M^{(k)} := M \setminus \bigcup_{i < k} M_{i_k}$$

which gives a filtration

$$M = M^{(1)} \supset M^{(2)} \supset \cdots \supset M^{(m)}.$$  

Denote

$$Z^{(k)} := Z \cap M^{(k)}$$

which is a strongly complex analytic subset of $M$. Then for each $k$, $Z^{(k)}$ is analytic.

As the first step of the induction, by Theorem B.12, there exists a canonical Whitney stratification of the closure $Z^{(m)}$ whose strata are strongly complex analytic submanifolds of $M$. Restrict to the open subset $Z^{(m)}$ to obtain a Whitney stratification $Z^{(m)}$ which is still strongly complex analytic.

We assume, as the induction hypothesis, that for a $k \leq m$, we have constructed strongly complex analytic Whitney stratifications $Z^{(k)}$ on $Z^{(k)}$ such that for each $i \geq k$, $Z_{i_k}$ is the union of strata. Now consider the closure $Z^{(k-1)}$ which contains the open subset $Z^{(k)}$ such that $Z^{(k-1)} \setminus Z^{(k)}$ is closed analytic. Then by Theorem B.13, there exists a minimal extension of $Z^{(k)}$ to $Z^{(k-1)}$. Restrict the extension to $Z^{(k-1)}$ (which contains $Z^{(k)}$), one obtains a strongly complex analytic Whitney stratification $Z^{(k-1)}$. Then the induction can continue until we find a nice strongly complex analytic Whitney stratification on $Z$.

Lastly, one can combine induction with the proof of the minimality part of Theorem B.13 to show that the constructed Whitney stratification is minimal among all smooth Whitney stratifications of $Z$ which respect $\mathcal{M}$. □

We have the following invariance property of the canonical Whitney stratification which respects the given stratification.

**Proposition B.14.** Let $M$ be a smooth manifold equipped with a partition $\mathcal{M}$. Let $Z \subseteq M$ be a closed subset and let $\mathcal{Z} \in \mathcal{W}^Z(S, \mathcal{M})$ be a minimal element. Let $f$ be a diffeomorphism of $M$ which preserves $\mathcal{M}$ such that $f(Z) = Z$. Then $f^* \mathcal{Z} \equiv \mathcal{Z}$. If $\mathcal{Z}$ has connected strata, then $f^* \mathcal{Z} = \mathcal{Z}$. 

Proof. The assumption implies that $f^*\mathcal{Z} \in WS^X(Z; \mathcal{M})$. Then by Corollary B.7, one has $f^*\mathcal{Z} \equiv \mathcal{Z}$. Suppose $\mathcal{Z}$ has connected strata. As $f$ is a diffeomorphism, for each stratum $Z_\alpha \in \mathcal{Z}$ which is connected, $f^{-1}(Z_\alpha)$, a stratum of $f^*\mathcal{Z}$, is connected. Hence $f^*\mathcal{Z} = \mathcal{Z}$. 

B.5. Proof of Proposition 3.8. In this subsection we prove Proposition 3.8. First we gather some basic facts about product Whitney stratifications. Let $M$ and $N$ be smooth manifolds, and $S \subseteq M$ and $T \subseteq N$ be subsets equipped with a Whitney stratification $\mathcal{S}$ resp. $\mathcal{T}$. Denote $R = S \times T$, $\mathcal{R} = \mathcal{S} \times \mathcal{T}$.

Then the dimension filtration of $\mathcal{R}$ is

$$\mathcal{R}_m = \bigcup_{k+l=m} \mathcal{S}_k \times \mathcal{T}_l$$

while

$$\mathcal{R}_m^* = \bigcup_{k+l=m} \mathcal{S}_k^* \times \mathcal{T}_l^*$$

which is the disjoint union.

We first consider the absolute case of Proposition 3.8.

Proposition B.15. Let $M$ resp. $N$ be a complex manifold and $S \subseteq M$ resp. $T \subseteq N$ be a closed analytic set equipped with the canonical Whitney stratification $\mathcal{S}$ resp. $\mathcal{T}$ (provided by Proposition 3.6). Then the canonical Whitney stratification of $R := S \times T \subseteq M \times N$ is equal to $\mathcal{S} \times \mathcal{T}$.

Proof. Let $\mathcal{R}$ be the canonical Whitney stratification on $R$. Then $\mathcal{R} \subseteq \mathcal{R}$. As both stratifications have connected strata, we only need to prove that they are equivalent, i.e., having the same dimension filtration. We prove by contradiction. Suppose $\mathcal{R}$ is not equivalent to $\mathcal{R}$. Then by definition, there exists an $m$ such that

$$\mathcal{R}_m = \mathcal{R}_m, \forall n \geq m \text{ and } \mathcal{R}_m \not\subseteq \mathcal{R}_m.$$ 

Therefore, there exists a point $r = (s, t) \in \mathcal{R}_m = \mathcal{R}_m = \bigcup_{k+l=m} \mathcal{S}_k \times \mathcal{T}_l$,

and $r \in \mathcal{R}_{m-1}$, $r \not\in \mathcal{R}_{m-1}$. Then $r = (s, t) \in \mathcal{R}_m^* \subseteq \mathcal{R}_{m, \text{reg}} = \mathcal{R}_{m, \text{reg}}$. Hence there exist $k, l$ with $k + l = m$ such that $s \in \mathcal{S}_k$ and $t \in \mathcal{T}_l$. On the other hand, the condition that $r = (s, t) \in \mathcal{R}_{m-1}$ implies that either $s \in \mathcal{S}_k$ or $t \in \mathcal{T}_l$. Indeed, if $s \in \mathcal{S}_k^*$ and $t \in \mathcal{T}_l^*$, then $r \in \mathcal{S}_k^* \times \mathcal{T}_l^* \subseteq \mathcal{R}_m^*$, contradicting that $r \in \mathcal{R}_{m-1}$.

Without loss of generality, we assume $s \in \mathcal{S}_k$. Then there exist a sequence $s'_\mu \in \mathcal{S}_k^*$, a sequence $s''_\mu \in \mathcal{S}_k$, both converging to $s$, such that $T_{r_\mu}^* \mathcal{S}_k$ converges to a subspace $H_s \subseteq T_sM$, the secant line $s'_\mu s''_\mu$ converges to a line $L_s \subseteq T_sM$, and $L_s \not\subseteq H_s$. Now depending on the position of $t$ we discuss in two cases.

1. $t \not\in \mathcal{T}_{l-1}$. Then consider the sequence of points $r_\mu = (s_\mu, t) \in \mathcal{S}_k \times \mathcal{T}_l \subseteq \mathcal{R}_{k+l}$ which converges to $r = (s, t) \in \mathcal{R}_{k+l, \text{reg}}$. For $\mu$ sufficiently large, $r_\mu \in \mathcal{R}_{k+l, \text{reg}}$. Consider the sequence of points $s''_\mu \in \mathcal{S}_k \times \mathcal{T}_l \subseteq \mathcal{R}_{k+l, \text{reg}}$ which also converges to $r$. Then the sequence of tangent planes $T_{r_\mu}^* \mathcal{S}_k \mathcal{S}_k^* = T_{r_\mu}^* \mathcal{S}_k^* \oplus T_t \mathcal{T}_l^*$ converges to $H_s \oplus T_t \mathcal{T}_l^*$; the sequence of secant
lines $\overline{r_{\mu}r_{\mu}'}$ converges to the line $L_s \oplus \{0\}$ which is not contained in $H_s \oplus T_l \mathcal{I}^s_{\beta}$. Hence condition (b) for $(\mathcal{I}^s_{\alpha+1}, \mathcal{R}_{k+1, reg})$ fails at $r$.

(2) $t \in \mathcal{I}_{l-1}$. Then it follows that for some $l' > l$, condition (b) for $(\mathcal{I}^s_{\alpha}, \mathcal{I}_{l, reg})$ fails at $t$. Hence there exist a sequence $l'_\mu \in \mathcal{I}^s_{\alpha}$ and a sequence $l_\mu \in \mathcal{I}_{l, reg}$ both converging to $t$ such that $T_{l'_\mu} \mathcal{I}^s_{\beta}$ converges to a subspace $H_t \subseteq T_l N$, the secant lines $\overline{l_\mu l'_\mu}$ converge to a line $L_t \subseteq T_l N$, and $L_t \not\subset H_t$. Then consider the sequence $l_\mu = (s_\mu, t_\mu)$ which converges to $r \in \mathcal{R}_{k+l, reg}$. Then for $l$ sufficiently large, $r_\mu \in \mathcal{R}_{k+l, reg}$. Consider the sequence $r'_\mu = (s'_\mu, t'_\mu) \in \mathcal{G}_k \times \mathcal{I}^s_{\alpha} \subseteq \mathcal{R}_{k+l, \alpha}$ which also converges to $r$. The tangent planes $T_{l'_\mu} \mathcal{R}_{k+l, \alpha}$ converge to $H_s \oplus H_t$ while the secant lines $\overline{r'_\mu r_{\mu}}$ converge to a line $L_r \subseteq L_s \oplus L_t$. Then $L_r \not\subset H_s \oplus H_t$. Hence condition (b) for $(\mathcal{R}_{k+l, \alpha}, \mathcal{R}_{k+l, \alpha})$ fails at $r$.

In both cases, by the induction hypothesis, it follows that $r \in \mathcal{R}_{k+l-1} = \mathcal{R}_{m-1}$, contradicting the assumption that $r \in \mathcal{R}_{m}^s$.

**Proof of Proposition 3.8.** Abbreviate $S_\alpha = S \cap M_\alpha, T_\beta = T \cap N_\beta$ with corresponding Whitney stratifications $\mathcal{S}_\alpha$ and $\mathcal{I}_\beta$. Abbreviate $R = S \times T$ and $R_{\alpha\beta} = S_\alpha \times T_\beta$. Let the canonical Whitney stratification on $R$ resp. $R_{\alpha\beta}$ be $\mathcal{R}$ resp. $\mathcal{R}_{\alpha\beta}$ and the product Whitney stratification be $\mathcal{R}_p$ resp. $\mathcal{R}_{\alpha\beta}$.

We prove this proposition by induction on $(\alpha, \beta)$. First, Proposition B.15 implies that for any top stratum $R_{\alpha\beta} \subseteq M \times N$, $\mathcal{R} = $ $\mathcal{R}_{\alpha\beta}$. Now fix a pair $(\alpha, \beta)$, suppose we have shown that for any $R_{\gamma\delta} > R_{\alpha\beta}$, $\mathcal{R}_{\gamma\delta} = \mathcal{R}. \delta$. We would like to show that $\mathcal{R}_{\alpha\beta} = \mathcal{R}_{\alpha\beta}$. Suppose our claim is false. Then there exists $m \geq 0$ such that

$$\mathcal{R}_{\alpha\beta,n} = \mathcal{R}_{\alpha\beta,n} (\forall n \geq m) \text{ and } \mathcal{R}_{\alpha\beta,m-1} \subseteq \mathcal{R}_{\alpha\beta,m-1}.$$ 

Then one can choose a point $r = (s, t)$ such that

$$r \in \mathcal{R}_{\alpha\beta,m}, \quad r \in \mathcal{R}_{\alpha\beta,m-1}.$$ 

Then

$$r = (s, t) \in \mathcal{R}_{\alpha\beta,m-1} \subseteq \mathcal{R}_{\alpha\beta,m} = \bigcup_{p + q = m} \mathcal{S}_{\alpha,p} \times \mathcal{I}_{\beta,q} = \mathcal{R}_{\alpha\beta,m}. \quad (B.2)$$

Then $r \in \mathcal{R}_{\alpha\beta,m} \subseteq \mathcal{R}_{\alpha\beta,m, reg} = \mathcal{R}_{\alpha\beta,m, reg}$. Therefore for some pair $(p, q)$ with $p + q = m$, $(s, t)$ is a regular point of $\mathcal{S}_{\alpha,p} \times \mathcal{I}_{\beta,q}$ of local dimension $p + q$. Hence

$$s \in \mathcal{S}_{\alpha,p, reg}, \quad t \in \mathcal{I}_{\beta,q, reg}.$$ 

Similar to the proof of the absolute case, we know that either $s \in \mathcal{S}_{\alpha,p-1}$ or $t \in \mathcal{I}_{\beta,q-1}$. Without loss of generality, assume that $s \in \mathcal{S}_{\alpha,p-1}$.

**Claim.** For all $r > \alpha, p' \leq p, \delta > \beta, q' \leq q$,

$$s \notin \partial \mathcal{S}_{\gamma, p'}, \quad t \notin \partial \mathcal{I}_{\beta,q'}^s.$$ 

**Proof of the claim.** Suppose for some $\gamma > \alpha$ and $p' \leq p$, $s \in \partial \mathcal{S}_{p, p'}$. Let $q' \leq q$ be such that $t \in \mathcal{I}_{\beta,q'}$. Then we see that

$$r = (s, t) \in \partial \mathcal{S}_{p, p'} \times \mathcal{I}_{\beta,q'} \subseteq \partial \left( \mathcal{S}_{p, p'} \times \mathcal{I}_{\beta,q'} \right) \subseteq \partial \mathcal{R}_{\gamma, \beta, p'+q'} = \partial \mathcal{R}_{\gamma, \beta, p'+q'}. \quad \gamma \neq \alpha.$$ 

Here the last equality follows from the induction hypothesis. Notice that $p' + q' \leq m$ and $R_{\gamma\beta} > R_{\alpha\beta}$. This contradicts the fact that $r \notin \mathcal{R}_{\alpha\beta,m-1}$ and the way to construct the canonical nice Whitney stratification $\mathcal{R}$. The claimed condition for $t$ can be proved in the same way.

**End of the proof of the claim.**
The rest of the argument is similar to the last part of the proof of Proposition B.15, except that we need to use the above claim. The assumption \( s \in \mathcal{G}_{\alpha,p-1} \) and the above claim imply that there exists \( k' > p \) such that condition (b) for \((\mathcal{G}_{\alpha,k'}, \mathcal{G}_{\alpha,k',\text{reg}})\) fails at \( s \). Then there exist a sequence \( s'_\mu \in \mathcal{G}_{\alpha,k'} \), a sequence \( s_\mu \in \mathcal{G}_{\alpha,k',\text{reg}} \), both converging to \( s \), such that \( T_{s'_\mu} \mathcal{G}_{\alpha,k'} \) converges to a subspace \( H_s \subseteq T_s M_\alpha \), the secant line \( s_\mu s'_\mu \) converges to a line \( L_s \subseteq T_s M_\alpha \), and \( L_s \not\subseteq H_s \). Now depending on the position of \( t \) we discuss in two cases.

1. Assume \( t \not\in \mathcal{I}_{\beta,l-1} \). Then consider the sequence of points \( r_\mu = (s_\mu,t) \in \mathcal{G}_{\alpha,k} \times \mathcal{I}_{\beta,l} \subseteq \mathcal{R}_{\alpha\beta,k+l} \) which converges to \( r = (s,t) \in \mathcal{R}_{\alpha\beta,k+l,\text{reg}} \). Then for \( \mu \) sufficiently large, \( r_\mu \in \mathcal{R}_{\alpha\beta,k+l,\text{reg}} \). Consider the sequence of points \( r'_\mu = (s'_\mu,t) \in \mathcal{G}_{\alpha,k'} \times \mathcal{I}_{\beta,l'} \subseteq \mathcal{R}_{\alpha\beta,k'+l} \) which also converges to \( r \). Then the sequence of tangent planes \( T_{r_\mu} \mathcal{G}_{\alpha,k'} \) converges to \( H_s \oplus T_r \mathcal{I}_{\beta,l} \); the sequence of secant lines \( \overline{r_\mu r'_\mu} \) converges to the line \( L_s \oplus \{0\} \) which is not contained in \( H_s \oplus T_r \mathcal{I}_{\beta,l} \). Hence condition (b) for \((\mathcal{R}_{\alpha\beta,k'+l}, \mathcal{R}_{\alpha\beta,k+l,\text{reg}})\) fails at \( r \).

2. Suppose \( t \in \mathcal{I}_{\beta,l-1} \). Then the above claim implies that for some \( l' > l \), condition (b) for \((\mathcal{I}_{\beta,l'}, \mathcal{I}_{\beta,l,\text{reg}})\) fails at \( t \). Hence there exist a sequence \( t'_\mu \in \mathcal{I}_{\beta,l'} \) and a sequence \( t_\mu \in \mathcal{I}_{\beta,l,\text{reg}} \) both converging to \( t \) such that \( T_{t'_\mu} \mathcal{I}_{\beta,l'} \) converges to a subspace \( H_t \subseteq T_t N_{\beta} \), the secant lines \( t_\mu t_\mu' \) converge to a line \( L_\mu \subseteq T_t N_{\beta} \), and \( L_\mu \not\subseteq H_t \). Then consider the sequence \( r_\mu = (s_\mu,t_\mu) \) which converges to \( r \in \mathcal{R}_{\alpha\beta,k+l,\text{reg}} \). Then for \( \mu \) sufficiently large, \( r_\mu \in \mathcal{R}_{\alpha\beta,k+l,\text{reg}} \). Consider the sequence \( r'_\mu = (s'_\mu,t'_\mu) \in \mathcal{G}_{\alpha,k'} \times \mathcal{I}_{\beta,l'} \subseteq \mathcal{R}_{\alpha\beta,k'+l'} \) which also converges to \( r \). The tangent planes \( T_{r'_\mu} \mathcal{R}_{\alpha\beta,k'+l'} \) converge to \( H_s \oplus H_t \) while the secant lines \( \overline{r_\mu r'_\mu} \) converge to a line \( L_\mu \subseteq L_s \oplus H_t \). Then \( L_\mu \not\subseteq H_s \oplus H_t \). Hence condition (b) for \((\mathcal{R}_{\alpha\beta,k'+l'}, \mathcal{R}_{\alpha\beta,k+l,\text{reg}})\) fails at \( r \).

In both cases, by the induction hypothesis it follows that \( r \in \mathcal{R}_{\alpha\beta,m-1} \) which contradicts the assumption that \( r \in \mathcal{R}_{\alpha\beta,m} \).

\[ \square \]

B.6. Pullback under submersions. This subsection serves as a technical preparation for proving several statements in Section 3.

First notice that Whitney’s condition (b) is invariant under diffeomorphisms. In fact it is preserved under pullbacks by submersions.

**Lemma B.16.** Let \( X, Y \) be disjoint smooth submanifolds of \( M \). Let \( \pi : \tilde{M} \to M \) be a submersion. Denote \( \tilde{X} = \pi^{-1}(X), \tilde{Y} = \pi^{-1}(Y) \). Then \( (X,Y) \) satisfies Whitney’s condition (b) at \( y \in \pi(Y) \subseteq Y \) if and only if \( (\tilde{X},\tilde{Y}) \) satisfies Whitney’s condition (b) at \( \tilde{y} \in \pi(\tilde{Y}) \).

**Proof.** As \( \pi \) is a submersion, it is transverse to both \( X \) and \( Y \). Then the Whitney condition is preserved under transversal pullbacks (see for example [GWdPL76, (1.3)]). Conversely, suppose \((\tilde{X},\tilde{Y})\) satisfies condition (b) at all points \( \tilde{y} \in \pi^{-1}(y) \). Suppose \( x_\mu \in X \) converges to \( y_\mu \in Y \) converges to \( y \), \( \pi_\mu \) converge to a line \( l \subseteq T_{y_\mu} M \), and \( T_{x_\mu} X \) converges to \( \tau \subseteq T_y M \). Choose a preimage \( \tilde{y}_\mu \in \pi^{-1}(y) \). Then there exist nearby points \( \tilde{x}_\mu \in \pi^{-1}(x_\mu) \) converging to \( \tilde{y} \) and \( \tilde{y}_\mu \in \pi^{-1}(y_\mu) \) converging to \( \tilde{y}_\mu \). By choosing a subsequence, we may assume \( \tilde{x}_\mu \tilde{y}_\mu \) converges to \( \tilde{l} \subseteq T_{\tilde{y}} M \) and \( T_{\tilde{x}_\mu} \tilde{X} \) converges to \( \tilde{\tau} \subseteq T_{\tilde{y}} M \). We may also move \( \tilde{y}_\mu \) in the same fiber
of $\pi$ to guarantee that $\tilde{l} \notin \ker(d\pi)$. Then $d\pi(\tilde{l}) = l$ and

$$d\pi(\tilde{r}) = d\pi \left( \lim_{i \to \infty} T_{\tilde{x}_i} U \right) = \lim_{i \to \infty} d\pi(T_{\tilde{x}_i} U) = \lim_{i \to \infty} T_{x_i} U = \tau.$$ 

Then $\tilde{l} \subseteq \tilde{r}$ which implies $l \subseteq \tau$. Hence $(X, Y)$ satisfies condition (b) at $y$. □

One has the following straightforward corollary.

**Corollary B.17.** When $\pi$ is a surjective submersion, $S_b(\tilde{X}, \tilde{Y}) = \pi^{-1}(S_b(X, Y))$.

There are some other easy results about surjective submersions in complex analytic category.

**Lemma B.18.** Let $M$ be a complex manifold with $\text{dim}_R(M) = m$ and $U \subseteq M$. Let $\pi : M \to M$ be a surjective analytic submersion and $\text{dim}_R(M) = \tilde{m}$. Then

1. $\pi^{-1}(U) = \pi^{-1}(U)$.
2. If $U$ is a closed analytic subset, then 
   $$\pi^{-1}(U) \text{sing} = \pi^{-1}(U) \text{sing}$$
   and for each $l$
   $$\pi^{-1}(U)_{m-l, \text{reg}} = \pi^{-1}(U_{m-l, \text{reg}}).$$
3. If $U \subseteq M$ is strongly analytic, then $\pi^{-1}(U) \subseteq \tilde{M}$ is also strongly analytic.

**Proof.** For item (1), the inclusion $\pi^{-1}(U) \subseteq \pi^{-1}(U)$ follows from definition; for the reversed inclusion, it can be proven using the surjective property of $\pi$ and choosing a product neighborhood of limiting points in $U$. For item (2), note that $\tilde{p} \in \pi^{-1}(U)$ is a smooth point if and only if $p = \pi(\tilde{p}) \in U$ is a smooth point, so it follows by taking the complement. For item (3), by assumption, both $\overline{U}$ and $\overline{U \setminus U}$ are closed and analytic; it follows from (1) that both $\overline{\pi^{-1}(U)} = \pi^{-1}(\overline{U})$ and $\overline{\pi^{-1}(U) \setminus \pi^{-1}(U)}$ are closed and analytic subsets, so (3) is proved. □

**B.6.1. Pulling back minimal Whitney stratifications.** Let $M$ be a complex manifold and $\pi : M \to M$ be a surjective analytic submersion. Suppose $Z \subseteq M$ is a strongly analytic subset and $\tilde{3}$ is a strongly analytic Whitney stratification of $Z$, then by Lemma B.18, the pullback Whitney stratification $\pi^*\tilde{3}$ is also strongly analytic.

**Corollary B.19.** Let $M$ be a complex manifold, $Z \subseteq M$ be a closed analytic subset, $Z' \subseteq Z$ be an open subset such that $Z \setminus Z'$ is a closed analytic subset. Suppose $\pi : M \to M$ is a complex analytic submersion with $Z \subseteq \pi(M)$. Let $\tilde{3}'$ be a strongly analytic Whitney stratification of $Z'$ and let $\tilde{3}$ be a minimal extension to $Z$. Then $\pi^*\tilde{3}$ is a minimal extension to $\pi^{-1}(Z)$ of the pullback $\pi^*\tilde{3}'$ on $\pi^{-1}(Z')$.

**Proof.** The proof is based on tracing the steps in the proof of Theorem B.13. Suppose $\text{dim}_R(M) = m$ and $\text{dim}_R(M) = \tilde{m}$. Following the recipe of the proof of Theorem B.13, one can construct a minimal extension $\tilde{3}$ of $\pi^*\tilde{3}'$ by constructing a decreasing sequence 

$$\pi^{-1}(Z \setminus Z') = 3''_m \supseteq 3''_{m-1} \cdots \supseteq 3''_1.$$ 

We claim that for $l = 0, 1, \ldots,$

$$3''_{m-l} = \pi^{-1}(3''_{m-l}).$$

This would imply that $\tilde{3} = \pi^*\tilde{3}$. Indeed, the set $3''_k$ is defined as the closure of points in $3''_{k-1}$ which satisfy one of the conditions listed in the proof of Theorem
Corollary B.17 and Lemma B.18 imply that these conditions are preserved under the pullback by the submersion $\pi$. Therefore, $\tilde{Z}_{m-l} \equiv \pi^{-1}(Z_{m-l})$. □

There is another technical situation we need to consider.

**Proposition B.20.** Let $M, \mathfrak{M}, Z$ be as in Proposition 3.6. Let $Z$ be a minimal Whitney stratification of $Z$ which respects $\mathfrak{M}$. Let $\tilde{M}$ be a complex manifold and $\pi: \tilde{M} \to M$ be a surjective analytic submersion. Notice that there is the pullback stratification $\tilde{M}^{\alpha}$ of $\tilde{M}$ whose strata are strongly analytic submanifolds. Then the pullback Whitney stratification $\pi^* Z$ is a minimal one on $\pi^{-1}(Z)$ which respects $\tilde{M}$.

**Proof.** We go through the construction in the proof of Proposition 3.6 for both $Z$ and $\tilde{Z} := \pi^{-1}(Z)$ in parallel. Order the strata $M_\alpha$ increasingly as

$$M_\alpha, \ldots, M_{\alpha_m}$$

such that $M_\alpha \leq M_j$ implies $i \leq j$. Define

$$M^{(k)} := M \setminus \bigcup_{i < k} M_\alpha, \quad \tilde{M}^{(k)} := \tilde{M} \setminus \bigcup_{i < k} \tilde{M}_\alpha = \pi^{-1}(M^{(k)}).$$

Set

$$Z^{(k)} := Z \cap M^{(k)}, \quad \tilde{Z}^{(k)} := \tilde{Z} \cap \tilde{M}^{(k)} = \pi^{-1}(Z^{(k)}).$$

Then following the proof of Proposition 3.6 for both $Z$ and $\tilde{Z}$, we have the strongly analytic Whitney stratifications $Z$ and $\tilde{Z}$ respecting $\mathfrak{M}$ and $\tilde{\mathfrak{M}}$ respectively. For each $k$, let $Z^{(k)}$, resp. $\tilde{Z}^{(k)}$ be the restriction of $Z$, resp. $\tilde{Z}$ on the open subset $Z^{(k)}$, resp. $\tilde{Z}^{(k)}$. We prove by a reversed induction on $k$ that for each $k$, one has

$$\tilde{Z}^{(k)} = \pi^* Z^{(k)}. \quad (B.3)$$

When $k = m$, recall that $Z^{(m)}$ is the restriction of the canonical Whitney stratification on $Z^{(m)}$ to $Z^{(m)}$. Then the absolute case (i.e. the $Z = \emptyset$ case) of Corollary B.19 shows that the canonical Whitney stratification on $\overline{Z^{(m)}} = \pi^{-1}(\overline{Z^{(m)}})$ is equivalent to the pullback of the canonical Whitney stratification on $Z^{(m)}$. Then the restriction to $\tilde{Z}^{(m)}$ is equivalent to $\pi^* Z^{(m)}$. The relative case of Corollary B.19 allows us to prove $(B.3)$ inductively. □

**Appendix C. Thom–Mather Stratified Spaces**

In this appendix we review the notion of Thom–Mather stratified spaces (or called abstract stratified spaces), especially their fundamental classes. It was proved by Mather [Mat12] that Whitney stratified sets in smooth manifolds are Thom–Mather stratified spaces. The zero locus of an FOP transverse section is a stratified space inside an effective orbifold which satisfies similar properties as a Whitney stratified set in a manifold. In order to construct a fundamental class, we extend Mather’s result to this kind of stratified subsets in orbifolds.
C.1. **Thom–Mather stratified spaces.** Thom–Mather stratified spaces are spaces stratified by smooth manifolds with certain structures regularizing the normal behavior near each stratum. Since the definitions in standard references ([Tho64, Tho69][Mat12]) require the control data as part of the structure, which is not canonically given in relevant constructions, we modify the definition in the following way.

**Definition C.1.** A *Thom–Mather stratified space* is a stratified space \( (X, \mathcal{X}) \) satisfying the following conditions.

1. \( X \) is a locally compact, Hausdorff, and second countable space.
2. \( X \) is a stratification such that each stratum \( X_\alpha \in \mathcal{X} \) is a smooth manifold.
3. There exists a set of control data, which is a collection \( J = \{ (N_\alpha, \pi_\alpha, \rho_\alpha) \mid X_\alpha \in \mathcal{X} \} \)

   where \( N_\alpha \subseteq X \) is an open neighborhood of \( X_\alpha \), \( \pi_\alpha : N_\alpha \to X_\alpha \) is a continuous retraction, and \( \rho_\alpha : N_\alpha \to [0, +\infty) \) is a continuous function. Moreover, these triples satisfy the following conditions.

   a. For each pair of strata \( X_\alpha, X_\beta \in \mathcal{X} \), define \( N_{\beta\alpha} = N_\alpha \cap X_\beta \) (which is an open subset of the manifold \( X_\beta \)), \( \pi_{\beta\alpha} = \pi_\alpha |_{N_{\beta\alpha}} \), and \( \rho_{\beta\alpha} = \rho_\alpha |_{N_{\beta\alpha}} \).

   b. We require that the map \( (\pi_{\beta\alpha}, \rho_{\beta\alpha}) : N_{\beta\alpha} \to X_\alpha \times (0, +\infty) \) is a smooth submersion.

   c. For any three strata \( X_\alpha, X_\beta, X_\gamma \) one has

      \[ \pi_{\beta\alpha} \circ \pi_{\gamma\beta} = \pi_{\gamma\alpha}, \quad \rho_{\beta\alpha} \circ \pi_{\gamma\beta} = \rho_{\gamma\alpha} \]

      whenever both sides of the equations are defined.

The *dimension* of a Thom–Mather stratified space \( X \) is the maximum of dimensions of its strata. The *dimension filtration* of \( X \) is the filtration

\[ X_n \supseteq X_{n-1} \supseteq \cdots \]

where \( X_k \) is the union of strata of dimension at most \( k \). Notice that the union of any subset of strata is naturally a Thom–Mather stratified space. On the other hand, any open subset of \( X \) is also a Thom–Mather stratified space.

**Definition C.2.** An *isomorphism* of Thom–Mather stratified spaces from \( X \) to \( Y \) is a homeomorphism \( f : X \to Y \) whose restriction to each stratum \( X_\alpha \subseteq X \) is a diffeomorphism onto a stratum of \( Y \). An *open embedding* of Thom–Mather stratified space is an isomorphism of Thom–Mather stratified space onto an open subset of the codomain.

Goresky [Gor78] proved that Thom–Mather stratified spaces admits triangulations.

**Theorem C.3.** [Gor78, 5. Proposition] Let \( (X, \mathcal{X}) \) be a Thom–Mather stratified space. Then there exists a polyhedral complex \( K \) and a homeomorphism \( f : |K| \to X \) such that for each stratum \( X_\alpha \), \( f^{-1}(X_\alpha) \) is a subcomplex and \( f : f^{-1}(X_\alpha) \to X_\alpha \) is a smooth triangulation.\(^{10}\)

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\(^{10}\)Goresky’s construction *a priori* depends on the control data.
We would like to show that certain Thom–Mather stratified spaces admit fundamental classes. Let $X$ be a compact $n$-dimensional Thom–Mather stratified space.

**Lemma C.4.** If $k > l$, then $H_k(X; Z) = 0$.

**Proof.** Each $X_l$ itself is a Thom–Mather stratified space. Then the lemma follows from Goresky’s theorem. □

**Lemma C.5.** [Gor76, 0.5.1 Proposition] Given a control data on $(X, X)$, for $\epsilon > 0$ sufficiently small, let $N_\alpha(\epsilon) := \rho_\alpha^{-1}([0, \epsilon)) \subseteq N_\alpha \subseteq X$. Define

$$O_{n-1}(\epsilon) := \bigcup_{\dim(X_n) \leq n-1} N_\alpha(\epsilon).$$

Then for $\epsilon$ sufficiently small, the inclusion map $X_{n-1} \to O_{n-1}(\epsilon)$ is a homotopy equivalence.

Now suppose each top stratum of an $n$-dimensional Thom–Mather stratified space $X$ is oriented. Abbreviate $O_{n-1}(\epsilon)$ by $O_{n-1}$. Then there exists a fundamental class

$$[X] \in H_n(X, O_{n-1})$$

(in integer coefficients). The homotopy equivalence $X_{n-1} \cong O_{n-1}$ implies

$$H_n(X, X_{n-1}) \cong H_n(X, O_{n-1}).$$

Hence we regard $[X]$ as a class in $H_n(X, X_{n-1})$. We call it the fundamental class of $X$ (with respect to the orientations on the top strata). If we are given a triangulation of $X$, then the fundamental class is represented by the singular cycle which is the sum of all the oriented top-dimensional cells of the triangulation.

**Definition C.6.** An $n$-dimensional Thom–Mather stratified pseudomanifold is an $n$-dimensional Thom–Mather stratified space $X$ such that $X_{n-1} = X_{n-2}$.

If $X$ is a compact oriented Thom–Mather stratified pseudomanifold, the exact sequence

$$H_n(X) \to H_n(X, X_{n-1}) \to H_{n-1}(X_{n-1}) = H_{n-1}(X_{n-2}) = \{0\},$$

where the last equality follows from Lemma C.4, implies that $[X]$ lives in $H_n(X; Z)$. We record this discussion as follows.

**Proposition C.7.** Any $n$-dimensional Thom–Mather stratified pseudomanifold $X$ with oriented top-dimensional strata has a well-defined fundamental class

$$[X] \in H_n(X; Z).$$

**Definition C.8.** A cobordism of compact oriented $n$-dimensional Thom–Mather stratified pseudomanifolds from $Y$ to $Z$ consists of a compact oriented $n+1$-dimensional Thom–Mather stratified space $X$, a closed union of strata $\partial X \subseteq X$, and an isomorphism $Y \sqcup Z \cong \partial X$ satisfying the following conditions.

1. $X \setminus \partial X$ is an $n+1$-dimensional Thom–Mather stratified pseudomanifold.
2. There exists an open embedding $\partial X \times [0, \epsilon) \to X$ whose restriction to $\partial X \times \{0\}$ is the identity map of $\partial X$. This implies that top strata of $\partial X$ have naturally induced orientations.
3. The isomorphism $Y \sqcup Z \cong \partial X$ is orientation preserving if $Y \sqcup Z$ is oriented as $(-Y) \sqcup Z$. 
In the situation of the above definition, \( X_n = X_{n-1} \cup \partial X \). Consider the fundamental class \([X] \in H_{n+1}(X, X_n)\) and the exact sequence

\[
H_{n+1}(X_n, \partial X) \to H_{n+1}(X, \partial X) \to H_n(X_n, \partial X) \to H_n(X, \partial X).
\]

By the collar structure, excision, and the fact that \( X_n \setminus \partial X = X_{n-1} \setminus \partial X \), it follows that \( H_{n+1}(X_n, \partial X) = H_n(X_n, \partial X) = 0 \). Hence \([X]\) lives in \( H_{n+1}(X, \partial X)\).

**Lemma C.9.** The image of \([X]\) under the boundary map \( H_{n+1}(X, \partial X) \to H_n(\partial X)\) is \([\partial X] = [Z] - [Y]\).

**Proof.** By Theorem C.3, one can choose a triangulation of \( X \). We see that the codimension one faces of the top-dimensional cells, if not cancelled in pairs, are exactly the top-dimensional faces of the boundary \( \partial X \). \(\Box\)

### C.2. Tubular neighborhoods in orbifolds.

**C.2.1. Tubular neighborhoods in manifolds.** We recall several useful notions and results about tubular neighborhoods in smooth manifolds towards constructing control data on Whitney stratified subsets of orbifolds. The main reference is [Mat12, Section 6]. Let \((M, g)\) be a smooth manifold and \(S \subseteq M\) be a closed submanifold. Recall the notion of normal bundle \(NS \to S\) and disk bundle \(N^s S \subseteq NS\) of radius \(\sqrt{\epsilon}\) (defined with respect to the inner product on \(NS\) induced from \(g\)) associated to a smooth function \(\epsilon : S \to \mathbb{R}_+\) (cf. (2.2)). We need to consider identifications of disk bundles in \(NS\) and tubular neighborhoods beyond those obtained by normal exponential maps.

**Definition C.10.** Let \((M, g)\) and \(S\) be as above.

1. A tubular neighborhood of \(S\) is a pair \(T_S = (\epsilon, \varphi)\) where \(\epsilon : S \to \mathbb{R}_+\) is a smooth function and \(\varphi : N^s S \to M\) is an open embedding satisfying the following condition. For any \(x \in S\), \(\varphi(x) = x\) and composed linear map

\[
N_x S \xrightarrow{d\varphi_x} T_x M \longrightarrow N_x S
\]

is the identity map of \(N_x S\). In particular, \(d\varphi|_S\) gives a splitting of the canonical exact sequence

\[
0 \longrightarrow TS \longrightarrow TM|_S \longrightarrow NS \longrightarrow 0.
\]

2. Associated to each tubular neighborhood \(T_S\), denote

\[
|T_S| = \varphi(N^s S),
\]

over which there is a tubular projection

\[
\pi_S := \pi_{NS} \circ \varphi^{-1} : |T_S| \to S
\]

and the tubular function

\[
\rho_S : |T_S| \to [0, +\infty), \quad \rho_S(y) = |\varphi^{-1}(y)|^2.
\]

3. The restriction of a tubular neighborhood \(T_S\) of \(S\) to an open subset \(U \subseteq S\) is the tubular neighborhood \(T_S|_U := (\epsilon|_U, \varphi|_U)\) of \(U\).

4. Two tubular neighborhoods \(T_S = (\epsilon, \varphi)\) and \(T'_S = (\epsilon', \varphi')\) of \(S\) are equivalent if

\[
(\pi_S, \rho_S) = (\pi'_S, \rho'_S)
\]
as germs of maps defined near $S$. Equivalently, $T_S$ is equivalent to $T'_S$ if there exist a smooth function $\epsilon'' : S \to \mathbb{R}_+$ satisfying $\epsilon'' \leq \min(\epsilon, \epsilon')$ and a (nonlinear) bundle map

$$\tau : N\epsilon'' S \to N\epsilon'' S$$

which is fiberwise diffeomorphism such that $\|\tau(y)\| = \|y\|$, such that the linearization along the zero section

$$d\tau|_S : NS \to NS$$

is the identity map of $NS$, and such that

$$\varphi|_{N\epsilon'' S} = \varphi' \circ \tau|_{N\epsilon'' S}.$$  

If $T_S$ and $T'_S$ are equivalent, we denote $T_S \sim T'_S$.

**Definition C.11.** Let $M, P$ be smooth manifolds and $f : M \to P$ be a smooth map. Let $S \subseteq M$ be a submanifold. We say a tubular neighborhood $T_S$ is compatible with $f$ if $f \circ \pi_S = f$ in $|T_S|$.

Below is a lemma useful in the induction argument. As its argument will be used in the orbifold setting, we provide an alternate self-contained proof.

**Lemma C.12.** [Mat12, Proposition 6.2] In the situation of Definition C.11, suppose restriction $f|_S : S \to P$ is a submersion. Let $O \subseteq S$ be an open subset and $D \subseteq S$ is a closed subset contained in $O$. Suppose $T_0 = (\epsilon_0, \varphi_0)$ is a tubular neighborhood of $O$ in $M$ which is compatible with $f$, then there exists a tubular neighborhood $T_S = (\epsilon, \varphi)$ of $S$ which is compatible to $f$ such that $T_S$ and $T_0$ are equivalent near $D$. In particular, the old and new tubular projections and tubular functions agree near $D$.

**Proof.** For each $p \in P$, denote $M_p = f^{-1}(p)$ and $S_p = f^{-1}(p) \cap S$. By the condition that $f|_S : S \to P$ is a submersion, $S_p$ is a smooth submanifold and $M_p$ is a smooth submanifold near $S_p$. Moreover, the normal bundle of $S_p$ inside $M_p$ is isomorphic to $NS|_{S_p}$.

We first extend the tubular neighborhood in the linear level. Consider the exact sequence of vector bundles with a splitting $\tau : NS \to TM|_S$

$$0 \longrightarrow TS \longrightarrow TM|_S \overset{\tau}{\longrightarrow} NS \longrightarrow 0.$$  

A splitting $\tau$ is said to be compatible with $f$ if for each $x \in S$, $\tau(N_x S) \subseteq T_x M_{f(x)}$. For the existing tubular neighborhood $T_0 = (\epsilon_0, \varphi_0)$, the linearization

$$\tau_0 := d\tau|_O : NS|_O \to TM|_O$$

gives a splitting of the exact sequence over $O$ which is compatible with $f$. Then one can find a splitting $\tau_1 : NS \to TM|_S$ compatible with $f$ such that $\tau_1$ agrees with $\tau_0$ over an open neighborhood $O' \subseteq O$ of $D$.

We use the extended splitting $\tau_1$ to define a tubular neighborhood. For each $p \in P$, let $\exp_{M_p}$ be the exponential map inside the submanifold $M_p$ with respect to the restriction of $g^{TM}$ to $M_p$. For $\epsilon : S \to \mathbb{R}_+$ sufficiently small, define

$$\varphi : N\epsilon S \to M, \varphi(x, v) = \exp_{M_{f(x)}}(\tau_1(x, v)) \in M_{f(x)} \subseteq M.$$  

It is easy to check that this is a tubular neighborhood which is compatible with $f$. However it does not necessarily match with the existing one.
We interpolate $\varphi_0$ and $\varphi_1$ over the region $O' \setminus D$. Choose a cut-off function $\nu_{O' \setminus D} : S \to [0, 1]$ supported in $O'$ such that $\nu_{O' \setminus D} \equiv 1$ near $D$. For each $x \in O' \setminus D$, compare the two fibers $\varphi_1(N_x^S)$ and $\varphi_0(N_x^S)$.

As $\tau_1$ and $\tau_0$ has the same linearization along $O'$, we see that for all $(x, v) \in NS|_{O'}$, one has

$$d_{M_f(x)}(\varphi_0(x, v), \varphi_1(x, v)) = O(\|v\|^2).$$

Then if $\epsilon$ is sufficiently small, for all $(x, v) \in N^*O'$, there exists a unique shortest geodesic $\gamma_{(x,v)} : [0, 1] \to M_f(x)$

such that

$$\gamma_{(x,v)}(0) = \varphi_0(x, v), \quad \gamma_{(x,v)}(1) = \varphi_1(x, v).$$

Then define $\varphi : N^*S \to M$ by

$$\varphi(x, v) = \begin{cases} 
\varphi_0(x, v), & x \in D, \\
\gamma_{(x,v)}(1 - \nu_{O' \setminus D}(x)), & x \in O' \setminus D, \\
\varphi_1(x, v), & x \notin O'.
\end{cases}$$

(See Figure 4 for an illustration of this interpolation.)

Figure 4. The fibers of $\varphi_1$ and $\varphi_0$ over the same point are contained in the same preimage $M_p = f^{-1}(p)$. Their tangents at the zero section are the same so their distance is small compared to the distance from the zero section. One interpolates between them along the shortest geodesics $\gamma \subseteq M_p$ in a region near $D$.

One can easily check that $\varphi$ is a tubular neighborhood, compatible with $f$, and agrees with $\varphi_0$ near $D$. Hence $T_S = (\epsilon, \varphi)$ is a tubular neighborhood of $S$ satisfying the desired conditions. \qed
C.2.2. Normal bundles of the isotropy stratification. We would like to consider tubular neighborhoods of smooth manifolds inside an orbifold (without boundary). We first discuss the normal bundle to strata of the isotropy stratification (Lemma 2.12). Here we do not consider the vector bundle $E \to U$ and the isotropy stratification of $U$ means that for the rank zero vector bundle.

Let $\gamma$ be an isotropy type represented by $(G_0, V_0)$ and $U_\gamma \subseteq U$ be the stratum of the isotropy stratification. In the normal direction, locally, a neighborhood of $U_\gamma$ is a fiberwise quotient of a vector bundle by the group $G_0$. It is tempting to define a normal bundle to $U_\gamma$ as a fiberwise quotient of a global vector bundle over $U_\gamma$ such that for each orbifold chart $C = (G, U, \psi)$ containing a point of $U_\gamma$, the vector bundle is the normal bundle over $UG \subseteq U$. However, there is no canonical transition functions. Indeed, the correct viewpoint is that $U_\gamma$ is a non-effective orbifold and the local pieces $NU_G$ define an orbifold vector bundle. However, we want to refrain from discussing non-effective orbifolds as we always do in this paper.

Let $A_\gamma$ be the set of charts of $U$ of the form $(G, U, \psi)$ such that $G \cong G_0$ and the fiber of $TU|U_G$ is isomorphic to $V_0$ as representations. Then each such chart provides a manifold chart of $U_\gamma$. Then define a space

$$N_\gamma := \left( \bigsqcup_{C = (G, U, \psi) \in A_\gamma} NU_G \right) / \sim$$

where the equivalence relation is defined as follows: a point $(x_1, v_1) \in NU_{1,G_1}$ (where $x_1 \in U_{1,G_1}$ and $v_1 \in NU_{1,G_1}$) coming from a chart $C_1 = (G_1, U_1, \psi_1)$ and a point $(x_2, v_2) \in NU_{2,G_2}$ coming from a chart $C_2 = (G_2, U_2, \psi_2)$ are equivalent if there exists a chart $C = (G, U, \psi) \in A_\gamma$, $(x, v) \in NU_G$, and chart embeddings

$$\iota_i : U \to U_i, \ i = 1, 2$$

such that $\iota_i(x) = x_i$ and $d\iota_i(x)(v) = v_i$. We claim that $N_\gamma$ is naturally an effective orbifold. Indeed, for each $C = (G, U, \psi) \in A_\gamma$, the natural map

$$d\psi : NU_G \to N_\gamma$$

is $G$-invariant and a homeomorphism onto an open subset. Hence $(G, NU_G, d\psi)$ is an orbifold chart of $N_\gamma$. Moreover, the compatibility of charts in $A_\gamma$ guarantees the compatibility of the collection of orbifold charts of $N_\gamma$. Therefore, $N_\gamma$ is an effective orbifold. Notice that there is a canonical projection map $N_\gamma \to U_\gamma$ and a canonical zero section $U_\gamma \to N_\gamma$. We call $N_\gamma$ the normal bundle to $U_\gamma$. Notice that $N_\gamma$ has its isotropy stratification whose strata are indexed by isotropy types $\delta \geq \gamma$.

Moreover, suppose $U$ is equipped with a straightened Riemannian metric $g^{TU}$, then there is a smooth function on $N_\gamma$ whose pullback to each chart $(G, NU_G, d\psi)$ is the function $(x, v) \mapsto \|v\|^2$ where $\|v\|$ is calculated by the Riemannian metric.

C.2.3. Tubular neighborhoods in orbifolds. Let $U$ be an effective orbifold equipped with a straightened Riemannian metric $g^{TU}$. Within this appendix, a submanifold of $U$ means a submanifold of a stratum of the isotropy stratification of $U$. On the other hand, let $P$ be a smooth manifold. Recall that a continuous map $f : U \to P$ is called smooth if its pullback to each orbifold chart is a smooth map to $P$; it is a submersion if its pullback to each orbifold chart is a submersion (see Section 2). As an example, the projection $N_\gamma \to U_\gamma$ is a submersion.
For each smooth function \( \epsilon : Z \to \mathbb{R}_+ \): \( A^p \pi \) with a natural projection map is an orbifold open embedding satisfying the following conditions.

For each smooth function \( \epsilon : Z \to \mathbb{R}_+ \), denote

\[
N^\epsilon Z = \left\{ (x, v, w) \in NZ \mid \rho_{NZ}(x, v, w) < \epsilon(x) \right\}.
\]

Definition C.13. Let \( Z \subseteq U \) be a submanifold contained in \( U \). Let \( \pi_{N, Z} : N, Z \to Z \) be the normal bundle of \( Z \) inside \( U \). The normal bundle of a submanifold \( Z \) in \( U \) is the orbifold

\[
NZ := \pi_{N, Z}^* N|_Z
\]

with a natural projection map \( \pi_{NZ} : NZ \to Z \). A point of \( NZ \) is denoted by \((x, v, w)\) where \( x \in Z, v \in N, Z|_x \), and \( w \in N|_x \). Define

\[
\rho_{NZ}(x, v, w) := \|v\|^2 + \|w\|^2.
\]

For each smooth function \( \epsilon : Z \to \mathbb{R}_+ \), denote

\[
N^\epsilon Z = \{ (x, v, w) \in NZ \mid \rho_{NZ}(x, v, w) < \epsilon(x) \}.
\]

Definition C.14. A tubular neighborhood of \( Z \) inside \( U \) is a pair \( T_Z = (\epsilon, \varphi) \) where \( \epsilon : Z \to \mathbb{R}_+ \) is a smooth function and \( \varphi : N^\epsilon Z \to U \) is an orbifold open embedding satisfying the following conditions.

1. \( \varphi|_Z \) is the inclusion of \( Z \) into \( U \).
2. For each \( x \in Z \), if we lift \( \varphi \) to a chart \( C = (G, U, \psi) \) containing \( x \), then \( \varphi \) is a \( G \)-equivariant tubular neighborhood of \( \psi^{-1}(Z) \) inside \( U \).
3. Associated to each tubular neighborhood \( T_Z \), define

\[
|T_Z| := \varphi(N^\epsilon Z)
\]

over which there is a tubular projection \( \pi_Z : |T_Z| \to Z \) and a tubular function \( \rho_Z : |T_Z| \to [0, +\infty) \). Both are smooth maps on orbifolds.
4. Two tubular neighborhoods \( T_Z = (\epsilon, \varphi) \) and \( T_Z' = (\epsilon', \varphi') \) are equivalent if there exist smooth functions \( \epsilon'' : Z \to \mathbb{R}_+ \) such that \( \epsilon'' \leq \min(\epsilon, \epsilon') \), an orbifold isomorphism

\[
\tau : N^\epsilon'' Z \to N^{\epsilon''} Z
\]

satisfying 1) \( \pi_{NZ} \circ \tau = \pi_{NZ} \), 2) \( \rho_{NZ} \circ \tau = \rho_{NZ} \), 3), for each chart \( C = (G, U, \psi) \in \mathcal{A}_U \), \( \tau \) is lifted to a \( G \)-invariant map \( \tau : NU_G \to NU_G \) such that the linearization along the zero section is the identity map of \( NU_G \), and 4) \( \varphi|_{N^{\epsilon''} Z} = \varphi' \circ \tau|_{N^{\epsilon''} Z} \).
5. A tubular neighborhood \( T_Z \) is compatible with a smooth map \( f : U \to P \) if \( f \circ \pi_Z = f \) on \(|T_Z|\).

Now we can generalize the tubular neighborhood theorem to the orbifold setting.

Proposition C.15. Let \( Z \subseteq U \) be a submanifold. Let \( f : U \to P \) be a smooth map such that \( f|_Z : Z \to P \) is a submersion. Let \( O \subseteq Z \) be an open subset and \( D \subseteq Z \) be a closed subset contained in \( O \). Suppose \( T_0 = (\epsilon_0, \varphi_0) \) is a tubular neighborhood of \( O \) which is compatible with \( f \). Then there exists a tubular neighborhood \( T_Z = (\rho, \varphi) \) of \( O \) compatible with \( f \) such that \( T_Z \) is equivalent to \( T_0 \) near \( D \).

Proof. The argument is analogous to the manifold case. We construct the tubular neighborhood on the chart level and prove that the chartwise constructions are compatible with chart embeddings.

We introduce similar notations first. For each \( x \in Z \), choose an orbifold chart \( C = (G, U, \psi) \) centered at \( x \). Denote \( \tilde{Z} = \psi^{-1}(Z) \subseteq U_G \), \( \tilde{O} = \psi^{-1}(O) \) and \( \tilde{D} = \psi^{-1}(D) \). Then by definition \( \tilde{f} := f \circ \psi : U \to P \) is smooth. For each \( p \), denote...
$U_p = \tilde{f}^{-1}(p)$ and $\tilde{Z}_p = U_p \cap \tilde{Z}$. Any smooth function $\epsilon : Z \to \mathbb{R}_+$ is lifted to a smooth function $\epsilon : \tilde{Z} \to \mathbb{R}_+$. Then the existing tubular neighborhood $T_0$ is lifted to

$$\tilde{\varphi}_0 : N' \tilde{O} \to U$$

after appropriately shrinking the function $\epsilon$.

Consider the $G$-equivariant exact sequence

$$0 \longrightarrow T\tilde{Z} \longrightarrow TU|_{\tilde{Z}} \longrightarrow N\tilde{Z} \longrightarrow 0.$$

A $G$-equivariant splitting $\tau : N\tilde{Z} \to TU|_{\tilde{Z}}$ is said to be compatible with $f$ if for each $x \in \tilde{Z}$, $\tau(N_x\tilde{Z}) \subseteq TU_{f(\phi(x))}$. Then the linearization of $\tilde{\varphi}_0$ defines a splitting over $\tilde{O}$ which is compatible with $f$, denoted by $\tau_0$. Then we can choose a $G$-equivariant splitting

$$\tau : N\tilde{Z} \to TU|_{\tilde{Z}}$$

which coincides with $\tau_0$ over an open neighborhood $\tilde{O}' \subseteq \tilde{O}$ of $\tilde{D}$. Then define the map

$$\tilde{\varphi}_1 : N'\tilde{Z} \to U, \tilde{\varphi}_1(x, v) = \exp_{U_f(x)}(\tau(x, v)) \in U_{f(x)}.$$

Next, similar to the proof of Lemma C.12, we interpolate $\tilde{\varphi}_1$ and $\tilde{\varphi}_0$ within $\tilde{O}' \setminus \tilde{D}$. Indeed, if $x \in \tilde{O}'$, then

$$d_{U_{f(x)}}(\tilde{\varphi}_1(x, v), \tilde{\varphi}_0(x, v)) = O(\|v\|^2).$$

Hence there is a shortest geodesic $\gamma(x, v) : [0, 1] \to U_{f(x)}$ connecting $\tilde{\varphi}_0(x, v)$ and $\tilde{\varphi}_1(x, v)$. Choose a smooth cut-off function $\nu_{\tilde{O}', \tilde{D}} : \tilde{Z} \to \mathbb{R}_+$ supported inside $\tilde{O}'$ such that $\nu_{\tilde{O}', \tilde{D}} \equiv 1$ near $\tilde{D}$. Then define

$$U_{\tilde{f}(x)} \ni \tilde{\varphi}(x, v) = \begin{cases} 
\tilde{\varphi}_0(x, v), & x \in \tilde{D}, \\
\gamma(x,v)(1 - \nu_{\tilde{O}', \tilde{D}}(x)), & x \in \tilde{O}' \setminus \tilde{D}, \\
\tilde{\varphi}_1(x, v), & x \notin \tilde{O}'.
\end{cases}$$

As the submanifolds $U_{\tilde{f}(x)}$ are $G$-invariant and the Riemannian metric is $G$-invariant, the chartwise tubular neighborhoods are $G$-equivariant. Also easy to check that these tubular neighborhoods are compatible with chart embedding. Hence this is a tubular neighborhood of $Z$. From the construction we can see it agrees with $T_0$ near $\tilde{D}$. \hfill \square

**C.3. Whitney stratified sets in orbifolds are Thom–Mather sets.**

**Definition C.16.** Let $\mathcal{U}$ be an effective orbifold (without boundary). A *Whitney stratified subset* of $\mathcal{U}$ is a subset $Z \subseteq \mathcal{U}$ equipped with a stratification $\mathfrak{S}$ satisfying the following conditions.

1. Each stratum $Z_\alpha \in \mathfrak{S}$ is a submanifold contained in a stratum $\mathcal{U}_{\gamma(\alpha)}$ of the isotropy stratification.
2. For each orbifold chart $C = (G, U, \psi)$, the above conditions imply that for each stratum $Z_\alpha \in \mathfrak{S}$, $\psi^{-1}(Z_\alpha) \subseteq U$ is a (possibly empty) smooth submanifold. Then for each pair $Z_\alpha < Z_\beta$, $(\psi^{-1}(Z_\beta), \psi^{-1}(Z_\alpha))$ is Whitney regular.
Definition C.17. Let $(Z, 3)$ be a Whitney stratified subset of $\mathcal{U}$. A set of ambient control data for $3$ is a collection $\mathcal{J} = \{T_{\alpha} \mid Z_{\alpha} \in \mathfrak{S}\}$ of tubular neighborhoods $T_{\alpha}$ of $Z_{\alpha}$ such that whenever $Z_{\alpha} < Z_{\beta}$, one has
\[
\pi_{\alpha} \circ \pi_{\beta} = \pi_{\alpha}, \quad \rho_{\alpha} \circ \pi_{\beta} = \rho_{\alpha}
\]
as germs of maps defined near $Z_{\alpha}$.

The following is an obvious consequence of Definition C.1 and Definition C.17.

Proposition C.18. Suppose one has a set of ambient control data $\mathcal{J} = \{(T_{\alpha}) \mid Z_{\alpha} \in 3\}$ for $(Z, 3)$. Let $\mathcal{J}$ be the data
\[
\mathcal{J} = \left\{(\pi_{\alpha}|_Z, \rho_{\alpha}|_Z) \mid Z_{\alpha} \in 3\right\}.
\]
Then the triple $(Z, 3, \mathcal{J})$ is a Thom–Mather stratified space.

Our theorem is

Theorem C.19. Let $\mathcal{U}$ be an orbifold (without boundary) and $(Z, 3)$ is a closed Whitney stratified subset. Then there exists a set of ambient control data for $3$.

The proof resembles that of [Mat12, Proposition 7.1]. Since the proof (as well as Mather’s proof) is complicated, we would like to demonstrate the main idea for the simple case when there are only three strata $Z_1 < Z_2 < Z_3$. We can first construct a tubular neighborhood $T_1 = (\epsilon_1, \varphi_1)$ of $Z_1$, which comes with the tubular projection
\[
\pi_1 : |T_1| \to Z_1
\]
and tubular function
\[
\rho_1 : |T_1| \to [0, +\infty).
\]

Lemma C.20. There exists an open neighborhood $O_1 \subseteq |T_1|$ of $Z_1$ such that
\[
(\pi_1, \rho_1) : Z_2 \cap O_1 \to Z_1 \times (0, +\infty)
\]
is a submersion.

Proof. It suffices to prove the assertion locally. For each $p \in Z_1$, choose an orbifold chart $C = (G, U, \psi)$ containing $p$ so that $\psi^{-1}(Z_1) \subseteq U_G$ is a submanifold. The two maps $\pi_1$ and $\rho_1$ are lifted to $G$-invariant maps near $\psi^{-1}(Z_1)$. Denote them by $\tilde{\pi}_1$ and $\tilde{\rho}_1$. On the other hand, $\psi^{-1}(Z_2) \subseteq U$ is a $G$-invariant smooth submanifold. By definition, $(\psi^{-1}(Z_2), \psi^{-1}(Z_1))$ satisfies Whitney’s condition (b) at each point of $\psi^{-1}(Z_1)$. This implies (see [Mat73, Lemma 7.3]) that there exists a $G$-invariant neighborhood $\tilde{O}_1$ of $\psi^{-1}(Z_1)$ such that the projection
\[
(\tilde{\pi}_1, \tilde{\rho}_1) : \psi^{-1}(Z_2) \cap \tilde{O}_1 \to \psi^{-1}(Z_1) \times (0, +\infty)
\]
is a submersion. This is equivalent to say that $(\pi_1, \rho_1)$ is a submersion in $Z_2 \cap \psi(\tilde{O}_1)$. Then $O_1$ can be constructed by taking union of these local pieces. \qed

Then by the relative case of Proposition C.15, if we shrink $|T_1|$ appropriately (by taking a function $\epsilon_1' \leq \epsilon_1$), then there exists a tubular neighborhood $T_2$ of $Z_2 \cap |T_1|$ which is compatible with $(\pi_1, \rho_1)$. Then use the absolute case of Proposition C.15 again, we can extend it to a tubular neighborhood $T_2$ of $Z_2$ which is compatible with $T_1$ near $Z_1$.

Now we consider the third stratum $Z_3$. Notice that we can freely shrink $|T_1|$ and $|T_2|$ without altering the compatibility between $T_1$ and $T_2$. First consider the open
subset \( Z_3 \cap |T_2| \subseteq Z_3 \). In the same way as before, after shrinking \(|T_2|\) appropriately, there exists a tubular neighborhood \( T_3 \) of \( Z_3 \cap |T_2| \) inside \(|T_2|\) which is compatible with \( T_2 \). We need to show that this is also compatible with \( T_1 \). Indeed,
\[
\pi_1 \circ \pi_3 = (\pi_1 \circ \pi_2) \circ \pi_3 = \pi_1 \circ (\pi_2 \circ \pi_3) = \pi_1 \circ \pi_2 = \pi_1,
\]
where the equality holds near \(|T_1| \cap Z_2\). Similarly \( \rho_1 \circ \pi_3 = \rho_1 \) over the same region. Therefore, the tubular neighborhood \( T_3 \) is compatible with \((\pi_1, \rho_1)\).

Then by Proposition C.15, after appropriately shrinking \(|T_1|\) and \(|T_2|\), one can find a tubular neighborhood \( T_3 \) of \( Z_3 \) within \(|T_1| \cup |T_2|\). Using this proposition again, we can construct a global tubular neighborhood \( T_3 \) which is compatible with both \( T_1 \) and \( T_2 \). This finishes the proof in this sample case.

**Proof of Theorem C.19.** We follow the proof of Mather [Mat12, Section 7]. Suppose \( Z \) has dimension \( n \) and let \( Z_k \subseteq Z \) be the union of strata of dimension at most \( k \) and \( \mathfrak{Z}_k \) be the restricted stratification. We construct inductively for \( k \). The induction hypothesis is that there exists a set of ambient control data \( \mathcal{I}_k \) for \((Z_k, \mathfrak{Z}_k)\) (which is itself a Whitney stratified set in \( U \)). Now we construct a set of ambient control data \( \mathcal{I}_{k+1} \) for \((Z_{k+1}, \mathfrak{Z}_{k+1})\) which extends \( \mathcal{I}_k \). Notice that it is always the freedom to shrink the existing tubular neighborhood without breaking the compatibility conditions.

We construct \( \mathcal{I}_{k+1} \). As strata of the same dimension do not intersect, one can construct a tubular neighborhood for each individual strata of dimension \( k + 1 \) separately. Without loss of generality, assume that there is only one stratum \( Z_\beta \) of dimension \( k + 1 \). The construction of the tubular neighborhood \( T_\beta \) uses a reversed induction on \( l = 0, 1, \ldots, k + 1 \). For each \( l \), denote
\[
O_l = \bigcup_{l \leq \dim(Z_\alpha) \leq k} |T_\alpha|
\]
which is an open subset of \( U \). We construct inductively a tubular neighborhood \( T_{\beta,l} \) of \( Z_\beta \cap O_l \) inside \( O_l \) which is compatible with all existing tubular neighborhoods. The \( l = k + 1 \) case has nothing to prove. Suppose we have constructed a tubular neighborhood for \( Z_\beta \cap T_{l+1} \) satisfying the requirement. Then for each \( Z_\alpha \) of dimension \( l \), we can check that
\[
\pi_\alpha \circ \pi_\beta = \pi_\alpha, \quad \rho_\alpha \circ \pi_\beta = \rho_\alpha
\]
near \( Z_\alpha \). Therefore, \( T_{\beta,l+1} \) is compatible with \((\pi_\alpha, \rho_\alpha)\). Then one can extend \( T_{\beta,l+1} \) to a tubular neighborhood \( T_{\beta,l} \) of \( Z_\beta \cap O_l \) inside \( O_l \) using Proposition C.15. Then finally the tubular neighborhood \( T_{\beta,0} \) is a tubular neighborhood of \( Z_\beta \) inside \( U \) which is compatible with all existing tubular neighborhoods. The induction on \( k \) can be carried on and stops after finitely many stages.

C.3.1. The case for orbifolds with boundary. We extend Definition C.16 and Theorem C.19 to the case when the orbifold has boundary. Let \( U \) be an orbifold with boundary and \( Z \subseteq U \) be a subset. Denote
\[
\operatorname{Int} Z := Z \cap \operatorname{Int} U, \quad \partial Z := Z \cap \partial U.
\]
If \( \mathfrak{Z} \) is a stratification on \( Z \) with strata \( Z_\alpha \), then denote
\[
\operatorname{Int} Z_\alpha := Z_\alpha \cap \operatorname{Int} U, \quad \partial Z_\alpha := Z_\alpha \cap \partial U.
\]

**Definition C.21.** A **collared Whitney stratified subset** of \( U \) is a subset \( Z \subseteq U \) equipped with a stratification \( \mathfrak{Z} \) satisfying the following conditions.
The interior $\text{Int}Z$ with the induced stratification is a Whitney stratified subset of $\text{Int}U$. Let $U_{\gamma}(\alpha)$ be the stratum of $U$ whose interior contains $\text{Int}Z_{\alpha}$.

(2) $\partial Z_{\alpha}$ is a submanifold of $\partial U$ contained in $\partial U_{\gamma}(\alpha)$.

(3) There is an open embedding $i : \partial U \times [0, \epsilon) \to U$ (a collar neighborhood of the boundary) sending $\partial U \times \{0\}$ identically onto $\partial U$ such that the restriction of $i$ onto $\partial Z_{\alpha} \times [0, \epsilon)$ is an open embedding into $Z_{\alpha}$.

If $Z \subseteq U$ is a collared Whitney stratified subset, then one can refine the stratification as

$$Z = \bigsqcup_{\alpha} \text{Int}Z_{\alpha} \cup \bigsqcup_{\alpha} \partial Z_{\alpha}.$$  

**Corollary C.22.** Let $Z \subseteq U$ be a collared Whitney stratified subset. Then $Z$ with the refined stratification specified above admits a set of ambient control data.

**Proof.** The proof can be carried out in the same way as that of Theorem C.19. The collared structure allows one to choose product type ambient control data near boundary strata. □

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