Introduction

This article is the third part of the series of papers on quantization of Lie bialgebras which we started in 1995. However, its object of study is much less general than in the previous two parts. While in the first and second paper we deal with an arbitrary Lie bialgebra, here we study Lie bialgebras of \( g \)-valued functions on a punctured rational or elliptic curve, where \( g \) is a finite dimensional simple Lie algebra. Of course, the general result of the first paper, which says that any Lie bialgebra admits a quantization, applies to this particular case. However, this result is not sufficiently effective, as the construction of quantization utilizes a Lie associator, which is computationally unmanageable. The goal of this paper is to give a more effective quantization procedure for Lie bialgebras associated to punctured curves, i.e. a procedure which will not use an associator. We will describe a general quantization procedure which reduces the problem of quantization of the algebra of \( g \)-valued functions on a curve with many punctures to the case of one puncture, and apply this procedure in a few special cases to obtain an explicit quantization.

The main object of study in this paper are Lie bialgebras associated to rational and elliptic curves with punctures, which can be described as follows.

We work over an algebraically closed field \( k \) of characteristic zero. Let \( \Sigma \) be a 1-dimensional algebraic group over \( k \) (i.e. \( G_a \), \( G_m \), or an elliptic curve), and \( u \) be an additive formal parameter near the origin. Let \( r \in \mathfrak{g} \otimes \mathfrak{g}(\Sigma) \) be a rational \( \mathfrak{g} \otimes \mathfrak{g} \)-valued function on \( \Sigma \) with the Laurent expansion near 0 of the form

\[
\sum \alpha \sum m \geq 1 (X_\alpha \otimes X_\alpha^* t^{-m}) u^m - 1 \in \mathfrak{g} \otimes \mathfrak{a}[[u]].
\]

Define a Lie bialgebra structure on \( \mathfrak{a} \) by the formulas

\[
[r_{13}(u), r_{23}(v)] = [r_{12}(u - v), r_{13}(u) + r_{23}(v)],
\]

\[
\delta(r(u)) = [r_{12}(u), r_{13}(u)].
\]

(1)
It is convenient to understand the classical r-matrix as a bilinear form \( \beta \) on \( \mathfrak{a} \) with values in \( k((u)) \), defined by the rule \( \beta(x, y)(u) = -\text{Res}_{v, w=0}(x(v) \otimes y(w), r(v - w + u)) \), where \( \langle, \rangle \) is the invariant form on \( \mathfrak{g} \), and \( r(v - w + u) \) is understood as an element of \( \mathfrak{g} \otimes \mathfrak{g}((u))[[v, w]] \) (that is, the function \( \frac{1}{v-w+u} \) is expanded as \( \sum u^{-r-1}(w-v)r \)). We can regard \( \beta \) as an element of \( \mathfrak{a}^* \otimes \mathfrak{a}^*((u)) \), where the tensor product is understood in the completed sense. The form \( \beta \) satisfies a set of axioms which are dual to the axioms of a pseudotriangular structure on a Lie bialgebra \([\text{Dr}1]\). Thus we call \( \beta \) a copseudotriangular structure.

Now we describe Lie bialgebras corresponding to punctured curves. Let \( \Gamma \) be the set of poles of \( r \). For any collection of points \( z = (z_1, ..., z_n) \) on \( \Sigma \) such that \( z_i - z_j \not\in \Gamma \) when \( i \neq j \), define the Lie bialgebra \( \mathfrak{a}_z \) to be the direct sum \( \mathfrak{a}_z = \mathfrak{a}_{z_1} \oplus ... \oplus \mathfrak{a}_{z_n} \), where \( \mathfrak{a}_{z_i} \) are Lie subbialgebras in \( \mathfrak{a}_z \), identified with \( \mathfrak{a} \), and the commutation relations between \( \mathfrak{a}_{z_i} \) and \( \mathfrak{a}_{z_j} \) are given by the formula

\[
[r_{i}^{13}(u), r_{j}^{23}(v)] = [r^{12}(u - v + z_i - z_j), r_{i}^{13}(u) + r_{j}^{23}(v)],
\]

where \( r_i \) is the image of \( r \) under the identification \( \mathfrak{a} \to \mathfrak{a}_{z_i} \). Here \( r(u - v + z_i - z_j) \) is regarded as an element of \( \mathfrak{g} \otimes \mathfrak{g}[[u, v]] \). This is possible as \( r \) is regular at \( z_i - z_j \) by the choice of \( z \).

The simplest example of this situation occurs when \( \Sigma = \mathbb{G}_a \) and \( r(u) = \sum \mathbb{X}_a \otimes \mathbb{X}_a/u \) (the Yang’s r-matrix). In this case it is easy to check that \( \mathfrak{a} \) is the Lie algebra \( t^{-1}\mathfrak{g}[t^{-1}] \), with cobracket dual to the standard bracket in \( \mathfrak{g}[t] \) \([\text{Dr}1]\), and \( \mathfrak{a}_z \) is the Lie algebra of rational functions on the line with values in \( \mathfrak{g} \) which have no poles outside \( z_1, ..., z_n \) and vanish at infinity, with a Lie coalgebra structure dual to the standard bracket in \( \mathfrak{g}[t_1] \oplus ... \oplus \mathfrak{g}[t_n] \) \([\text{Dr}1]\).

The Lie bialgebra \( \mathfrak{a}_z \) has two essential properties:

1. **Local factorization in a product of equal components.**

   - **factorization:** As a Lie coalgebra, \( \mathfrak{a}_z \) admits a decomposition \( \mathfrak{a}_z = \mathfrak{a}_1 \oplus ... \oplus \mathfrak{a}_n \), where \( \mathfrak{a}_i \) are Lie subbialgebras of \( \mathfrak{a}_z \).
   - **locality:** \( [\mathfrak{a}_i, \mathfrak{a}_j] \subset \mathfrak{a}_i \oplus \mathfrak{a}_j \).
   - **equal components:** All the Lie bialgebras \( \mathfrak{a}_i \) are identified with the same Lie bialgebra \( \mathfrak{a} \).

2. **Expression of commutator via the classical r-matrix.**

   The commutator between the components \( \mathfrak{a}_i \) and \( \mathfrak{a}_j \) is given in terms of the classical r-matrix via formula (2).

   Thus, we see that the structure of \( \mathfrak{a}_z \) is completely determined by the pair \((\mathfrak{a}, \beta)\) of a Lie bialgebra and a copseudotriangular structure on it.

Our purpose in this paper is to describe an “explicit” quantization of \( \mathfrak{a}_z \). The construction of such quantization consists of two parts.

**Part 1.** We define the notion of a copseudotriangular structure \( B \) on a Hopf algebra \( A \), by dualizing the notion of a pseudotriangular structure, introduced by Drinfeld \([\text{Dr}1]\). We show that any nondegenerate copseudotriangular Lie bialgebra \((\mathfrak{a}, \beta)\) can be quantized, i.e. that there exists a copseudotriangular Hopf algebra \((A, B)\) whose quasiclassical limit is \((\mathfrak{a}, \beta)\). This is done using the methods of \([\text{EK}1, \text{EK}2]\).

**Part 2.** We define the notion of a factored Hopf algebra, which is a quantization of the above notion of a factored Lie bialgebra, and give a construction of a factored Hopf algebra \( A_z \) corresponding to points \((z_1, ..., z_n)\) starting from a copseudotriangular Hopf algebra \((A, B)\). We show that if \((A, B)\) is a quantization of \((\mathfrak{a}, \beta)\) then \( A_z \) is a quantization of \( \mathfrak{a}_z \).
In some cases, this quantization of $a_z$ can be described by explicit formulas. For example, in the case of the Yang’s r-matrix the quantization of $(a, \beta)$ is $(Y^*(g), R^{-1})$, where $Y^*(g)$ is the dual algebra to the Yangian $Y(g)$, with opposite product, and $R \in Y(g) \otimes Y(g)((u))$ is the pseudotriangular structure on $Y(g)$ defined by Drinfeld. In this case, $A_z$ is defined by (3.5), (3.6), where $R$ is the Yang’s quantum R-matrix.

Let us describe briefly the structure of the paper.

In Chapter I we introduce the notion of a locally factored Lie bialgebra, which is a Lie bialgebra with properties 1(a) and 1(b), and the corresponding quantum notion of a locally factored Hopf algebra. We also introduce the notion of a weak $(\partial)$-copseudotriangular Lie bialgebra, which is a Lie bialgebra $a$ with a derivation $\partial$ and a form $\beta : a \otimes a \to k((u))$ having analogous properties to the form $\beta$ above, and the corresponding quantum notion of a weakly $(\partial)$-copseudotriangular Hopf algebra. We explain how to introduce a factored Lie bialgebra structure on $a^\otimes n$ given a weak $\partial$-copseudotriangular structure $\beta$ on a Lie bialgebra $a$, and how to introduce a factored Hopf algebra structure on $A^\otimes n$ given a weak $\partial$-copseudotriangular structure $B$ on a Hopf algebra $A$. Then we explain how to quantize a given weak $\partial$-copseudotriangular structure on a Lie bialgebra. This allows us to describe the quantization (constructed in [EK1,EK2]) of the factored algebra $a^\otimes n$ purely in terms of the quantization of $(a, \beta)$.

In Chapter II we describe in detail the Lie bialgebras $a$ and $a_z$ discussed above, and consider various examples (rational, trigonometric, and elliptic). The results of this Chapter are not original.

In Chapter III we give the main results of the paper. In the first part we completely describe the quantization $U_{A_z}$ of $a_z$ in the case of the Yang’s r-matrix in terms of the Yangian $Y(g)$ and its universal R-matrix. In the second part we consider the case $g = sl_N$ and $gl_N$, and describe quantizations of the simplest rational, trigonometric, and elliptic algebras $a_z$ explicitly by generators and relations containing the quantum R-matrices $R(u) \in \text{End}(k^N \otimes k^N)[[h]]$. This construction is analogous to the Faddeev-Reshetikhin-Takhtajan construction of quantum $GL_N$ [FRT], and the similar construction of the Yangian of $gl_N$ [Dr1].

Remark. Algebras similar to $U_{A_z}$ were considered in [Ch1] and [AGS1,AGS2].

In a subsequent paper we will use the algebras $U_{A_z}$ to describe a deformed version of the holomorphic part of the Wess-Zumino-Witten conformal field theory (in genus zero). More specifically, we will consider representation theory of $U_{A_z}$, the notion of quantum conformal blocks, quantum vertex operators, and show how the quantum Knizhnik-Zamolodchikov equations of Frenkel-Reshetikhin appear naturally in this context.

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1. COSEQUOTRIANGULAR LIE BIALGEBRAS AND THEIR QUANTIZATION

Throughout the paper, $k$ will denote an algebraically closed field of characteristic zero, and “an algebra” means “an associative $k$-algebra with 1”.  

3
1.1. Factored algebras.

Let $A$ be an algebra over $k$.

**Definition.** A factorization of $A$ is a collection of algebras $A_1,...,A_n$, which are subalgebras in $A$, such that for any $\sigma \in S_n$ the multiplication map defines a bijection $A_{\sigma(1)} \otimes \cdots \otimes A_{\sigma(n)} \to A$. Such a factorization is called local if the image of $A_i \otimes A_j$ in $A$ under the multiplication map is a subalgebra of $A$ for all $i,j$.

It is clear that a factorization of $A$ is local if and only if $A_iA_j = A_jA_i$ for all $i,j$.

We call an algebra $A$ equipped with a (local) factorization a (locally) factored algebra.

Let $A_1,...,A_n$ be algebras. We want to describe locally factored algebras with factors $A_1,...,A_n$.

Suppose we are given a locally factored algebra $A$, with factors $A_1,...,A_n$, $n \geq 2$, and product $m : A \otimes A \to A$. Let $m_{ij} : A_i \otimes A_j \to A$ be the restriction of $m$ to $A_i \otimes A_j$. By the definition, the map $m_{ij}$ is injective for $i \neq j$, and the images of $m_{ij}$ and $m_{ji}$ are the same. Therefore, for any $i < j$, $i,j \in \{1,...,n\}$, we can define a linear isomorphism $X_{ij} : A_i \otimes A_j \to A_i \otimes A_j$ by the formula $X_{ij} = m_{ij}^{-1} \circ m_{ji} \circ \sigma$, where $\sigma$ is the permutation of components.

It is easy to see that the maps $X_{ij}$ satisfy the following conditions:

(i)

$$X_{ij}X_{ik}X_{jk} = X_{jk}X_{ik}X_{ij}, i < j < k; \quad (1.1)$$

(ii)

$$X_{ij}(a \otimes bc) = (1 \otimes m)(X_{ij}^{12}X_{ij}^{13}(a \otimes b \otimes c)), \quad X_{ij}(ab \otimes c) = (m \otimes 1)(X_{ij}^{23}X_{ij}^{13}(a \otimes b \otimes c)), \quad (1.2)$$

(iii)

$$X_{ij}(a \otimes 1) = a \otimes 1, a \in A_i, \quad X_{ij}(1 \otimes a) = 1 \otimes a, a \in A_j. \quad (1.3)$$

**Remark.** Condition (i) is vacuous if $n = 2$.

Conversely, let $X_{ij} : A_i \otimes A_j \to A_i \otimes A_j$ be an arbitrary collection of invertible operators satisfying (i)-(iii), and $I(\{X_{ij}\})$ be the ideal in the free product $A_1*\cdots*A_n$ generated by the relations $ba = m(X_{ij}(a \otimes b)), a \in A_i, b \in A_j, i < j$. Denote by $P_n(A_1,...,A_n,\{X_{ij}\})$ the quotient of $A_1*\cdots*A_n$ by this ideal.

Let $\phi : A_1 \otimes \cdots \otimes A_n \to P_n(A_1,...,A_n,\{X_{ij}\})$ be the linear map defined by the rule $a_1 \otimes \cdots \otimes a_n \to a_1...a_n$.

**Proposition 1.1.** The map $\phi$ is a bijection.

To prove this proposition, we define a product on the space $A_1 \otimes \cdots \otimes A_n$ by

$$(a_1 \otimes \cdots \otimes a_n) \cdot (b_1 \otimes \cdots \otimes b_n) = \quad (1.4)$$

$$(m \otimes \cdots \otimes m)(X_{n-1,n}^{2n-1} \cdots X_{23}^{n+2} \cdots X_{2n}^{n+2} \cdots X_{12}^{n+1,2} \cdots X_{1n}^{n+1,n}(a_1 \otimes \cdots \otimes a_n \otimes b_1 \otimes \cdots \otimes b_n)), \quad$$

where the superscripts denote the components where the corresponding $X$-operator acts.
Lemma 1.2. The product defined by (1.4) endows $A_1 \otimes \ldots \otimes A_n$ with the structure of an associative algebra, in which the unit is $1 \otimes \ldots \otimes 1$.

Proof. The associativity follows directly from properties (i),(ii) of $X_{ij}$. The unit axiom follows from property (iii) of $X_{ij}$. □

We denote the algebra of Lemma 1.2 by $E$. We have a natural homomorphism $\psi : P_n(A_1, \ldots, A_n, \{X_{ij}\}) \to E$, induced by the embeddings $A_i \to E$.

Lemma 1.3. $\psi$ is an isomorphism.

Proof. This homomorphism is surjective, as $E$ is generated by $A_i$. It is also injective, as $P_n(A_1, \ldots, A_n, \{X_{ij}\})$ is obviously spanned by the elements of the form $a_1 \ldots a_n$, $a_i \in A_i$. The Lemma is proved. □

Proof of Proposition 1.1. It is clear that $\psi \circ \phi = \text{Id}$, so $\phi$ is an isomorphism. □

Corollary 1.4. The assignment $A \to \{X_{ij}(A)\}$ is a 1-1 correspondence between locally factored algebra structures on $A_1 \otimes \ldots \otimes A_n$, with factors $A_1, \ldots, A_n$, and collections of invertible operators $\{X_{ij}\}$ satisfying (i)-(iii).

The same definitions and results apply to the case when $A$ is an algebra over $k[[h]]$ which is a deformation of an algebra over $k$. In this case, the sign $\otimes$ denotes the completed (in the h-adic topology) tensor product over $k[[h]]$.

1.2. Factored Lie bialgebras and Hopf algebras.

Let $a$ be a Lie bialgebra over $k$.

Definition. A factorization of $a$ is a decomposition $a = a_1 \oplus \ldots \oplus a_n$, where $a_i \subset a$ are Lie subbialgebras. Such a factorization is called local if $a_i \oplus a_j$ is a Lie subbialgebra of $a$ for $i \neq j$, i.e. if $[a_i, a_j] \subset a_i \oplus a_j$ for $i \neq j$.

Let $U$ be a Hopf algebra.

Definition. A factorization of $U$ is a collection of Hopf algebras $U_1, \ldots, U_n$, which are Hopf subalgebras in $U$, such that for any $\sigma \in S_n$ the multiplication map defines a bijection $U_{\sigma(1)} \otimes \ldots \otimes U_{\sigma(n)} \to U$. Such a factorization is called local if the image of $U_i \otimes U_j$ in $U$ under the multiplication map is a Hopf subalgebra of $U$ for $i \neq j$, i.e. if $U_i U_j = U_i U_j$ for $i \neq j$.

We will call a Lie bialgebra (Hopf algebra) with a (local) factorization a (locally) factored Lie bialgebra (Hopf algebra).

It is clear that if a Lie bialgebra $a$ is the quasiclassical limit of a quantized universal enveloping (QUE) algebra $U$ ([Dr1]), then any factorization of $U$ (as a QUE algebra) defines a factorization of $a$. Moreover, if the first factorization is local, so is the second one.

Fix a universal Lie associator $\Phi$. Let $a \to U_h(a)$ be the functor of quantization from the category of Lie bialgebras to the category of QUE algebras, defined in [EK2] using $\Phi$. Suppose we are given a factorization $a = a_1 \oplus a_n$. Using the functoriality of $U_h$, we obtain embeddings of Hopf algebras $U_h(a_1) \to U_h(a)$, so we can regard $U_h(a_i)$ as Hopf subalgebras of $U_h(a)$. These Hopf subalgebras define a factorization $U_h(a) = U_h(a_1) \otimes \ldots \otimes U_h(a_n)$. Moreover, it follows from the functoriality of $U_h$ that if the factorization of $a$ is local, so is the factorization of $U_h(a)$.

Thus, we have the following proposition.
Proposition 1.5. To any (local) factorization of a Lie bialgebra $a$ one can assign its quantization, i.e. a (local) factorization of $U_h(a)$, whose quasicalssical limit is the initial factorization of $a$.

Now we give an analogue of Corollary 1.4 for Hopf algebras.

Let $U_1, ..., U_n$ be Hopf algebras. Then the free product $U_1 \ast ... \ast U_n$ has a natural Hopf algebra structure induced by the Hopf algebra structures on $U_1, ..., U_n$.

Proposition 1.6. The assignment $U \to \{X_{ij}(U)\}$ is a 1-1 correspondence between locally factored Hopf algebra structures on $U_1 \otimes ... \otimes U_n$, with factors $U_1, ..., U_n$, and such collections of invertible operators $\{X_{ij}\}$ satisfying (i)-(iii), that the ideal $I(\{X_{ij}\})$ is a Hopf ideal in $U_1 \ast ... \ast U_n$.

Proof. Clear. □

1.3. Copseudotriangular Lie bialgebras.

Let $b$ be a Lie bialgebra over $k$, with commutator $\mu$ and cocommutator $\delta$.

Recall [Dr1] that a quasitriangular structure on $b$ is an element $r \in b \otimes b$ satisfying the classical Yang-Baxter equation

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0,$$

such that

$$\delta(x) = [x \otimes 1 + 1 \otimes x, r], x \in b.$$  

It is easy to see that relation (1.5) can be replaced by any one of the two relations

$$(\delta \otimes 1)(r) = [r^{13}, r^{23}], (1 \otimes \delta)(r) = [r^{13}, r^{12}].$$

Indeed, given (1.6), relation (1.5) is equivalent to either of (1.7).

Let $b$ be a finite dimensional quasitriangular Lie bialgebra, and $a = b^*$. In this case, the element $r$ defines a bilinear form $\beta : a \otimes a \to k$, by

$$\beta(x, y) = (r, x \otimes y), x, y \in a.$$  

This form has the following properties, which are equivalent to properties (1.6),(1.7) of $r$:

$$\beta([xy], z) = \beta(x \otimes y, \delta(z)), \beta(x, [zy]) = \beta(\delta(x), y \otimes z);$$

$$[yx] = \beta_{12}(x \otimes \delta(y)) + \beta_{13}(\delta(x) \otimes y).$$

Definition. A bilinear form $\beta$ on a Lie bialgebra $a$ satisfying (1.9)-(1.10) is called a coquasitriangular structure on $a$.

Thus, if $a$ is finite-dimensional, a coquasitriangular structure on $a$ is the same thing as a quasitriangular structure on $a^*$.

The following generalization of this definition is essentially due to Drinfeld, [Dr1].

Let $a$ be a Lie bialgebra over $k$. Let $\partial$ be a derivation of $a$ as a Lie bialgebra. Define the map $\alpha_u : a \to a[[u]]$ by

$$\alpha_u(x) = e^{u\partial} x.$$
Definition. A $\partial$-copseudotriangular structure on a Lie bialgebra $\mathfrak{a}$ is a bilinear form
\begin{equation}
\beta : \mathfrak{a} \otimes \mathfrak{a} \rightarrow k((u)),
\end{equation}
which satisfies the following conditions:
\begin{align}
\beta([xy], z) &= \beta(x \otimes y, \delta(z)), \beta(x, [zy]) = \beta(\delta(x), y \otimes z), \\
[y, \alpha_u(x)] &= \beta_1(x \otimes \delta(y))(u) + \alpha_u(\beta_{13}(\delta(x) \otimes y))(u). \\
\beta(\alpha_v(x), y)(u) &= \beta(x, \alpha_{-v}(y))(u) = \beta(x, y)(u + v).
\end{align}

If $\mathcal{O} \subset k((u))$ is a subalgebra, and in addition to (1.13)-(1.15) $\beta$ takes values in $\mathcal{O}$, we say that $\beta$ is a $\partial$-copseudotriangular $\mathcal{O}$-structure.

Consider the linear map $d = -\frac{1}{2} \mu \circ \delta : \mathfrak{a} \rightarrow \mathfrak{a}$. It is easy to check that this map is a derivation of $\mathfrak{a}$ as a Lie bialgebra [Dr1]. We call it the canonical derivation of $\mathfrak{a}$. If $\partial = d$, we will call $\beta$ a copseudotriangular structure (without specification of $\partial$).

Example. If $\mathcal{O} = k$ then a $\partial$-copseudotriangular $\mathcal{O}$-structure is the same thing as a coquasitriangular structure, which vanishes on the image of $\partial$.

Proposition 1.7. Let $\beta$ be a $\partial$-copseudotriangular structure on $\mathfrak{a}$ such that for a suitable $m \in \mathbb{Z}$ we have $\beta(x, y) \in u^m k[[u]]$ for any $x, y \in \mathfrak{a}$. Then $\beta$ is constant (does not depend on $u$), and therefore defines a coquasitriangular structure on $\mathfrak{a}$, which vanishes on the image of $\partial$.

Proof. Consider the form $\beta_0 : \mathfrak{a} \otimes \mathfrak{a} \rightarrow k((u))$ defined by $\beta_0(x, y)(u) := \beta(\alpha_{-u}(x), y)(u)$. Since $\beta$ takes values in $u^m k[[u]]$, this expression makes sense. From (1.15) we get that $\beta_0(u + v) = \beta_0(u)$, so $\beta_0$ is constant. Identities (1.13)-(1.14) for $\beta$ imply (1.9)-(1.10) for $\beta_0$. Thus, $\beta_0$ is a quasitriangular structure on $\mathfrak{a}$. It is obvious that $\beta_0$ vanishes on the image of $\partial$. □

Remark. Proposition 1.7 shows that interesting (i.e. not coquasitriangular) examples of $\partial$-copseudotriangular structures can only arise when $\beta(x, y)(u)$ can have a pole of arbitrary order at $u = 0$, for which $\mathfrak{a}$ has to be infinite dimensional.

1.4. Copseudotriangular Hopf algebras.

Let $A$ be a Hopf algebra over $k[[h]]$, such that $A_0 = A/hA$ is commutative, and $A$ is a deformation of $A_0$. Let $m, 1, \Delta, \varepsilon, S$ be the product, unit, coproduct, counit, and the antipode of $A$.

Let $\partial$ be a derivation of $A$. Define a Hopf algebra homomorphism
\begin{equation}
\alpha_u := e^{u\partial} : A \rightarrow A[[u]].
\end{equation}
Definition. A $k[[h]]$-bilinear form $B : A \otimes A \to k((u))[[h]]$ is called a $\partial$-copseudotriangular structure on $A$ if it satisfies the following conditions:

\[(1.17) \quad B(xy, z) = B(x \otimes y, \Delta(z)), B(x, zy) = B(\Delta(x), y \otimes z), x, y, z \in A;\]

\[(1.18) \quad m((\alpha_u \otimes 1)[B_{13}(\Delta(x) \otimes \Delta(y))(u)]) = m^{op}((\alpha_u \otimes 1)[B_{24}(\Delta(x) \otimes \Delta(y))(u)]);\]

\[(1.19) \quad B(\alpha_v(x), y)(u) = B(x, \alpha_{-v}(y))(u) = B(x, y)(u + v);\]

\[(1.20) \quad B(1, x) = B(x, 1) = \varepsilon(x), x \in A; B(x, y) = \varepsilon(x)\varepsilon(y) + O(h),\]

where $B_{ij}$ means that $B$ is evaluated on the $i$-th and $j$-th components of the product.

Let $O$ be a subalgebra of $k((u))$. We say that $B$ is a $\partial$-copseudotriangular $O$-structure on $A$ if $B$ takes values in $O[[h]]$.

Remark. Commutativity of $A_0$ is essential to enable the equality $B(x, y) = \varepsilon(x)\varepsilon(y) + O(h)$ in presence of condition (1.18).

Since $A_0$ is commutative, we have $S^2 = 1 + O(h)$. Let $D := \frac{1}{h}\ln S^2$. It is clear that $D$ is a derivation of $A$. We call $D$ the canonical derivation of $A$. If $\partial = D$, we will call $B$ a copseudotriangular structure (without specification of $\partial$).

1.5. Weak $\partial$-copseudotriangular structures.

We will need the following weaker version of the notion of a $\partial$-copseudotriangular structure.

Definition. (i) Let $\mathfrak{a}$ be a Lie bialgebra over $k$. A $k((u))$-valued bilinear form $\beta$ on $\mathfrak{a}$ is called a weak $\partial$-copseudotriangular structure on $\mathfrak{a}$ if it satisfies equations (1.13), (1.15), and the equation

\[(1.21) \quad \beta([y, \alpha_u(x)], z)(v) = \beta_{12}(u)\beta_{34}(v)(x \otimes \delta(y) \otimes z) - \beta_{14}(v)\beta_{23}(u)((\alpha_u \otimes 1)(\delta(x)) \otimes y \otimes z).\]

(ii) Let $A$ be a Hopf algebra, as in Section 1.4. A $k((u))[[h]]$-valued bilinear form $B$ on $A$ is called a weak $\partial$-copseudotriangular structure on $A$ if it satisfies equations (1.17), (1.19), (1.20) and the equation

\[(1.22) \quad B(m((\alpha_u \otimes 1)[B_{13}(\Delta(x) \otimes \Delta(y))(u)]), z)(v) = B(m^{op}((\alpha_u \otimes 1)[B_{24}(\Delta(x) \otimes \Delta(y))(u)]), z)(v).\]

If $\beta$ or $B$ takes values in $O \subset k((u))$, it is called a weak $\partial$-copseudotriangular $O$-structure.

If $\partial = d$ (respectively, $\partial = D$), we will call $\beta$ (respectively, $B$) a weak copseudotriangular structure (without specification of $\partial$).

Remark. Equations (1.21), (1.22) are obtained by applying the functionals $\beta(\ast, z) = B(\ast, z)$ to equations (1.14), (1.18). Thus, for a left-nondegenerate bilinear form (i.e. a form with trivial left kernel), the property to be a weak $\partial$-copseudotriangular structure is the same as to be a $\partial$-copseudotriangular structure.
1.6. $\partial$-copseudotriangular structure on $h$-formal groups.

Let $a_0$ be a Lie coalgebra, $A_0 = \prod_{j \geq 0} S^j a_0$ be the ring of functions on the corresponding formal group, and $A$ be a deformation of $A_0$ as a topological Hopf algebra. Let $m$ be the maximal ideal in $A$, i.e. the kernel of the projection $A \to k$.

Let $B$ be a $\partial$-copseudotriangular $O$-structure on $A$. We define the quasiclassical limit of $B$ as follows.

Let $U_A$ be the $h$-adic completion of the direct sum $\oplus_{j \geq 0} h^{-j} m^j$ (see [Dr1]). Then $U_A$ is a quantized universal enveloping algebra.

Let $a$ be the Lie bialgebra over $k$ which is the quasiclassical limit of $U_A$ [Dr1]. Let $\mu, \delta$ be the commutator and the cocommutator in $a$, and $\partial_0 : a \to a$ be the quasiclassical limit of $\partial$. In particular if $\partial = D$, then $\partial_0 = d$ (see [Dr1], Section 8).

Consider the pairing $B : m \otimes m \to O[[h]]$. It is clear that $B|_{m \otimes m} = O(h)$, so we define $\beta : m \otimes m \to O$ by $\beta(x, y) := \frac{B(x, y)}{h} \mod h$.

Let $m_0 = m/hm$. It is clear that $\beta$ descends to a bilinear form $\beta : m_0 \otimes m_0 \to O$, which vanishes on $m_0^2$ on the left and on the right. As $m_0/m_0^2$ is naturally identified with $a$, we get a bilinear form $\beta : a \otimes a \to O$.

**Proposition 1.8.** The form $\beta$ satisfies equations (1.13)-(1.15).

**Proof.** Relations (1.13)-(1.15) for $\beta$ are easily obtained from relations (1.17)-(1.19) for $B$. □

Thus, Proposition 1.8 states that the quasiclassical limit of $U_A$ is endowed with a natural $\partial$-copseudotriangular $O$-structure $\beta$. We call $\beta$ the quasiclassical limit of $B$, and $B$ a quantization of $\beta$.

Similar definitions and statements apply to weak $\partial$-copseudotriangular structures.

1.7. Factorizations associated to $\partial$-copseudotriangular structures.

Consider the field $F_n = k((u_1))( (u_2)) ... ( (u_n))$. For $i < j$, we have a subfield $k((u_i))( (u_j)) \subset F_n$. Consider the embedding $k((t)) \to k((u))((v))$ by the formula

$$f(t) \to f(u - v) := \sum_{m \geq 0} f^{(m)}(u)(-v)^m / m!.$$ 

Using this embedding, we define subfields $k((u_i - u_j)) \subset k((u_i))( (u_j)) \subset F_n$.

Let $a$ be a Lie bialgebra over $k$ with a weak $\partial$-copseudotriangular structure $\beta$. Let $a_{F_n} := a \otimes_k F_n$ be a Lie bialgebra over the field $F_n$ obtained by extension of scalars from $a$. In this section we will define a structure of a factored Lie bialgebra on the space $a_n := a_{F_n}$.

Let $a_i = a_{F_n}$, $i = 1, \ldots, n$. For any $i < j$, we define a linear map $a_i \otimes a_j \to a_i \oplus a_j$ by the formula

$$\mu_{ij}(x, y) = \beta_{13}(\delta(x) \otimes y)(u_i - u_j) \oplus \beta_{12}(x \otimes \delta(y))(u_i - u_j)$$

For $a, b \in a_n$, such that $a = \sum_{i=1}^n a_i$, $b = \sum_{i=1}^n b_i$, $a_i, b_i \in a_i$, set

$$[a_1 + \ldots + a_n, b_1 + \ldots + b_n] = \sum_{j=1}^n [a_j, b_j] - \sum_{i<j} (\mu_{ij}(a_i, b_j) - \mu_{ij}(b_i, a_j)).$$

The space $a_n$ has a natural Lie coalgebra structure $\delta$, coming form the Lie coalgebra structures on $a_1, \ldots, a_n$. 
**Proposition 1.9.** The bracket $[,]$ is a Lie bracket on $\mathfrak{a}^n_u$, and $\delta$ is a Lie bialgebra structure on $(\mathfrak{a}^n_u, [,])$.

**Proof.** Skew symmetry of $[,]$ is obvious. The Jacobi identity and the cocycle condition for $\delta$ is verified by a direct computation. □

Proposition 1.9 shows that any weak $\partial$-copseudotriangular structure on $\mathfrak{a}$ defines a natural structure of a factored Lie bialgebra on $\mathfrak{a}^n_u$.

Now consider the quantum analogue of this construction. Let $A$ be a Hopf algebra with a weak copseudotriangular structure $B$. Define linear operators $X_i, X_r, X : A \otimes A \rightarrow A \otimes A((u))$ by the formula

\[(1.25) \quad X_i(a \otimes b) = B_{13}(\Delta(a) \otimes \Delta(b)), \quad X_r(a \otimes b) = B_{24}(\Delta(a) \otimes \Delta(b)), \quad X = X_lX_r^{-1}.\]

Because of property (1.20) of $B$, we have $X_i, X_r = 1 + O(h)$.

**Proposition 1.10.** (i) For any $i,j,p,q \in \{1, \ldots, n\}$ $[X^{ij}_i(u), X^{pq}_r(v)] = 0$.

Also, $Y = X_r^{-1}, X_l$ satisfy the following equations:

(ii) The quantum Yang-Baxter equation

\[(1.26) \quad Y^{12}(u_1 - u_2)Y^{13}(u_1 - u_3)Y^{23}(u_2 - u_3) = Y^{23}(u_2 - u_3)Y^{13}(u_1 - u_3)Y^{12}(u_1 - u_2).\]

(iii) $Y(a \otimes bc) = (1 \otimes m)(Y^{12}Y^{13}(a \otimes b \otimes c)), \quad Y(ab \otimes c) = (m \otimes 1)(Y^{23}Y^{13}(a \otimes b \otimes c));$

(iv) $Y(1 \otimes a) = 1 \otimes a, \quad Y(a \otimes 1) = a \otimes 1, \quad a \in A$.

**Proof.** The identity $[X^{ij}_i(u), X^{pq}_r(v)] = 0$ is obvious from the definition. The Yang-Baxter equation follows from the properties (1.17)-(1.19) of $B$. Identities (iii),(iv) follow directly from properties (1.17),(1.20) of $B$. □

Let $A_{F_n}$ be the $h$-adic completion of $A \otimes_k F_n$. $A_i = A_{F_n}$, $i = 1, \ldots, n$. For $i < j$, let $X_{ij} : A_i \otimes A_j \rightarrow A_i \otimes A_j$ be the operators defined by the formula $X_{ij} := X(u_i - u_j)$. Proposition 1.10 implies that $X_{ij}$ satisfy properties (i)-(iii) in Section 1.1, so they define a factored algebra $P_n(A_1, \ldots, A_n, \{X_{ij}\})$.

**Proposition 1.11.** The ideal $I(\{X_{ij}\})$ is a Hopf ideal with respect to the coproduct on $A_1 \ast \ldots \ast A_n$ induced by the coproduct on $A_i$.

**Proof.** It is convenient to write the relations of the ideal $I(\{X_{ij}\})$ in the form $m(X_i(a \otimes b)) = m^{op}(X_r(a \otimes b)), \quad a \in A_i, b \in A_j, \quad i < j$. Then it is easy to verify directly that this relation is invariant under $\Delta$. □

Proposition 1.11 shows that $P_n(A_1, \ldots, A_n, \{X_{ij}\})$ has a natural Hopf algebra structure. We denote this Hopf algebra by $A^n_u$.

Now let $A$ be an $h$-formal group, and $U_A$ the corresponding QUE algebra. Let $B$ be a $\partial$-copseudotriangular structure on $A$. Although $B$ does not extend to $U_A$, we have the following proposition.

**Proposition 1.12.** The operator $X$ extends to an invertible, $h$-adically continuous operator $X : U_A \otimes U_A \rightarrow U_A \otimes U_A((u))$.

**Proof.** It is easy to see that for any $x,y \in \mathfrak{m}$, $X_l(u)(x \otimes y)$ and $X_r(u)(x \otimes y)$ belong to the coset $B(x,y)(u)1 \otimes 1 + \mathfrak{m} \otimes \mathfrak{m}$. On the other hand, as $X_l, X_r = 1 + O(h)$, we
have $X = X_lX_r^{-1} = 1 + X_l - X_r + O(h^2)$. Therefore, $X(m \otimes m) \subset m \otimes m$. Using Proposition 1.10, we see that $X(m^r \otimes m^s) \subset m^r \otimes m^s$ for any $r, s \geq 0$. Therefore, $X$ extends to $U_A \otimes U_A$. Similarly, we can extend $X^{-1} = X_rX_l^{-1}$ to $U_A \otimes U_A$. □

Proposition 1.12 enables us to construct the factored Hopf algebra $P_n(U_A, ..., U_{A_u}, \{X_{ij}\})$.

We denote this Hopf algebra by $U_{A_u}$.

Remark. It is easy to see that $A_u$ is an $h$-formal group over $F_n$, and $U_{A_u}$ is the corresponding QUE algebra, so this notation is consistent with the previous notation.

Now let us consider copseudotriangular structures and factorizations which are defined over the ring of functions on some algebraic variety.

Let $\Sigma$ be a connected 1-dimensional algebraic group over $k$, a finite subset of $\Sigma(k)$, and $O$ be the algebra of regular functions on $\Sigma \setminus \Gamma$. We can regard $O$ as a subalgebra in $k((u))$ using the canonical formal parameter $u$ near the origin (the parameter whose differential is $du$).

Let $\Sigma_n(\Gamma)$ be the variety of all $z = (z_1, ..., z_n) \in \Sigma^n$ such that $z_i - z_j \notin \Gamma$. Let $O_n$ be the ring of regular functions on $\Sigma_n(\Gamma)$. We have a natural embedding $O_n \rightarrow F_n$, which acts by taking the Laurent expansion of a function $f \in O_n$ near the origin, consecutively in the variables $z_n, ..., z_1$.

Let $\beta$ be a weak $\partial$-copseudotriangular $O$-structure on a Lie bialgebra $\mathfrak{a}$. Then the Lie bialgebra $\mathfrak{a}_u^O$ over $F_n$ has a natural $O_n$-structure. Indeed, the $O_n$-submodule $\mathfrak{a}_u^{O_n} := \oplus_{i=1}^n \mathfrak{a} \otimes_k O_n \subset \mathfrak{a}_u^O$ is a Lie bialgebra over $O_n$, and $\mathfrak{a}_u^n = \mathfrak{a}_u^{O_n} \otimes O_n F_n$. For any $z \in \Sigma_n(\Gamma)(k)$, define the Lie bialgebra $\mathfrak{a}_z := \mathfrak{a}_u^{O_n} / I(z)$, where $I(z) \subset O_n$ is the ideal of functions vanishing at $z$. Then $\mathfrak{a}_z$ is a factored Lie bialgebra over $k$, with $n$ factors isomorphic to $\mathfrak{a}$.

Similarly, one defines factored Hopf algebras $A_u^{O_n}, A_z, U_{A_u}^{O_n}, U_{A_z}$.

**Proposition 1.16.** If $U_A$ is a quantization of a Lie bialgebra $\mathfrak{a}$, and $B$ is a quantization of a weak $\partial$-copseudotriangular structure $\beta$ on $\mathfrak{a}$, then the QUE algebra $U_{A_u}$ is a quantization of the Lie bialgebra $\mathfrak{a}_u^n$. If in addition $\beta$, $B$ are $O$-structures, then $U_{A_u}^{O_n}, U_{A_z}$ are quantizations of $\mathfrak{a}_u^{O_n}, \mathfrak{a}_z$.

**Proof.** Easy. □

**1.8. Quantization of weak $\partial$-copseudotriangular structures.**

In this section we will show that any weak $\partial$-copseudotriangular structure on a Lie bialgebra $\mathfrak{a}$ admits a quantization.

Let $\mathfrak{a}$ be a Lie bialgebra over $k$ with a weak $\partial$-copseudotriangular structure $\beta$. Consider the Lie bialgebra $\mathfrak{a}_u^n$ over $F_n$ defined above. Define a linear map $\theta_n : \mathfrak{a}_u^n \rightarrow \mathfrak{a}^{*}_{F_n}$ by

\begin{equation}
\theta_n(a_1 + ... + a_n)(b) = \beta(a_1, b)(u_1) + ... + \beta(a_n, b)(u_n).
\end{equation}

**Proposition 1.17.**

(i) $\theta_n$ is a homomorphism of Lie bialgebras $\mathfrak{a}_u^n \rightarrow \mathfrak{a}^{*}_{F_n}$ (where $b^{*}$ is $b$ with opposite counit, for any Lie bialgebra $b$).

(ii) $\theta_1(\alpha_v(x)) = \alpha_v^*(\theta_1(x)(u)) = \theta_1(x)(u + v)$.

(iii) $\theta_n(a_1 + ... + a_n)(u_1, ..., u_n) = \theta_1(a_1)(u_1) + ... + \theta_1(a_n)(u_n), a_i \in \mathfrak{a}_i$.

(iv) $\theta_1$ is injective if and only if $\beta$ is left-nondegenerate.

**Proof.** Properties (i)-(iv) follow from (1.13),(1.15),(1.21). We check only the case of property (i), which is less obvious than others. For $n = 1$, (i) follows from (1.13).
Consider the case \( n \geq 2 \). It is clear that we can assume \( n = 2 \). In this case (i) reduces to the identity

\[
\theta_2(\mu(x, y)) = [\theta_1(y), \theta_1(x)] , x, y \in a.
\]

Using the definition (1.23) of \( \mu \), we rewrite (1.29) in the form

\[
\theta_1(u)(\beta_{13}(\delta(x) \otimes y)(u - v)) + \theta_1(v)(\beta_{12}(x \otimes \delta(y))(u - v)) = [\theta_1(v), \theta_1(u)](x).
\]

Evaluating both sides of (1.30) on an element \( z \in a \), we rewrite (1.30) in the form

\[
\beta_{13}(u-v)\beta_{24}(u)(\delta(x) \otimes y \otimes z) + \beta_{12}(u-v)\beta_{34}(v)(x \otimes \delta(y) \otimes z) + \beta_{13}(u)\beta_{24}(v)(x \otimes y \otimes \delta(z)) = 0.
\]

On the other hand, and using (1.13), (1.15), we can rewrite (1.21) in the form

\[
\beta_{13}(u)\beta_{24}(u+v)(\delta(x) \otimes y \otimes z) + \beta_{12}(u)\beta_{34}(v)(x \otimes \delta(y) \otimes z) + \beta_{13}(u+v)\beta_{24}(v)(x \otimes y \otimes \delta(z)) = 0.
\]

It is easy to see that (1.32) is transformed to (1.31) by the change of variable \( u = u' - v', v = v' \). The proposition is proved. \( \square \)

Let \( a \) be a Lie bialgebra over \( k \), \( \partial_0 \) a derivation of \( a \), \( \beta : a \otimes a \to k((u)) \) a weak \( \partial_0 \)-copseudotriangular structure on \( a \). Let \( U_h \) be the functor of quantization of Lie bialgebras (see section 1.2). Let \( U_A = U_h(a) \), \( \partial = U_h(\partial_0) \), and \( A \) be the \( h \)-formal group corresponding to \( U_A \). ([Dr1], Section 7).

**Theorem 1.18.** The Hopf algebra \( A \) admits a \( \partial \)-copseudotriangular structure \( B \), which is a quantization of \( \beta \), such that \( B \) is left-nondegenerate if and only if so is \( \beta \).

**Proof.** Since \( U_h \) is a functor, the Hopf algebra \( U_h(a^n) \) admits a factorization \( U_{A_1} \otimes ... \otimes U_{A_n} \), where \( U_{A_i} \) is the \( h \)-adic completion of \( U_A \otimes_k F_n \).

By Proposition 1.17, the linear map \( \theta_n \) constructed in the previous section, is a homomorphism of Lie bialgebras. Therefore, by functoriality of quantization [EK2], it defines a homomorphism of Hopf algebras \( \tilde{\Theta}_n : U_h(a^n) \to U_h(a^*_{F_n}) \), by \( \tilde{\Theta}_n = U_h(\theta_n) \).

Therefore, Proposition 1.17 implies the following properties of \( \tilde{\Theta}_n \).

(i) \( \tilde{\Theta}_n \) is a homomorphism of Hopf algebras \( U_h(a^n) \to U_h(a^*_{F_n}) \).

(ii) \( \tilde{\Theta}_1(\alpha(\nu(x)))(u) = \alpha(\nu(\tilde{\Theta}_1(x))(u)) = \Theta_1(x)(u + v) \).

(iii) \( \tilde{\Theta}_n(a_1 \otimes ... \otimes a_n)(u_1, ..., u_n) = \tilde{\Theta}_1(a_1)(u_1) ... \tilde{\Theta}_1(a_n)(u_n) \), \( a_i \in U_{A_i} \).

(iv) \( \tilde{\Theta}_1 \) is injective if and only if \( \beta \) is left-nondegenerate.

Let \( I \) be the kernel of \( \tilde{\Theta}_1 \). It is clear that \( I \) is a Hopf ideal in \( U_h(a) \).

For any Lie bialgebra \( b \), we have a natural Hopf algebra isomorphism \( \psi : U_h(b^*_{op}) \to U_h(b)^{op} \) defined as follows. Consider the natural Lie bialgebra maps \( \eta : b \to D(b) \), \( \eta^* : b^*_{op} \to D(b) \), where \( D(b) = b \oplus b^*_{op} \) is the double of \( b \). These maps are given by \( \eta(x) = (x, 0) \), \( \eta^*(y) = (0, y) \). Let \( U_h(\eta) : U_h(b) \to U_h(D(b)) \), \( U_h(\eta^*) : U_h(b^*_{op}) \to U_h(D(b)) \) be their quantizations. Let \( R \) be the universal R-matrix of \( U_h(D(b)) \). By [EK1,EK2], we have \( R \in ImU_h(\eta) \otimes ImU_h(\eta^*) \).

Thus, \( R \) can be regarded as an element of \( U_h(b) \otimes U_h(b^*_{op}) \), hence as a linear map \( \phi : U_h(b^*_{op}) \to U_h(b)^{op} \). It is clear that \( \phi \) is an isomorphism of Hopf algebras, so we can define \( \psi = \phi^{-1} \).
Now set $\Theta_n = \psi \circ \hat{\Theta}_n$. It is clear that $\Theta_n$ is a homomorphism of Hopf algebras $U_h(a_n^u) \to U_h(a_{F_n})$, which satisfies properties (ii)-(iv) above.

Define the bilinear form $B : A \otimes A \to k((u))[[h]]$ by $B(a, b) = \Theta_1(a)(b)$.

We claim that $B$ is a weak $\partial$-copseudotriangular structure.

Indeed, identity (1.17) follows from the fact that $\Theta_1$ is a homomorphism. Property (iv) of $\Theta_n$ implies (1.19). Property (1.20) is clear. It remains to establish property (1.22).

Let us prove (1.22). Let $a_n^2$ be as above, and $D(a) = a \oplus a^{*\text{op}}$ be the double of $a$. Define a linear map $\chi : a_n^2 \to D(a) \otimes F_2$ defined by $(a, b) \to (b, \theta_1(a)(u_1 - u_2))$.

**Lemma 1.19.** $\chi$ is a Lie bialgebra homomorphism.

*Proof.* Straightforward.

Lemma 1.19 allows us to define a homomorphism $\hat{\chi} = U_h(\chi) : U_h(a_n^2) \to U_h(D(a)) = U_h(a) \otimes U_h(a^{*\text{op}})$. This homomorphism satisfies the equation $\hat{\chi}(ab) = \Theta_1(a)b$, where $a$ is from $U_h(a)_1$ and $b$ from $U_h(a)_2$, and $U_h(a)_1,2$ denote the first and second components of $U_h(a)$ in $U_h(a_n^2)$. This shows that we have:

**Lemma 1.20.** The linear map $\xi : U_h(a_n^2) \to U_h(a) \otimes U_h(a)^{*\text{op}} = D(U_h(a))$ given by $\xi(ab) = \Theta_1(a)b$, where $a$ is from $U_h(a)_1$ and $b$ from $U_h(a)_2$, is a Hopf algebra homomorphism.

Lemma 1.20 allows us to get information about commutation relations between the two components of $U_h(a)$ inside of $U_h(a_n^2)$.

**Lemma 1.21.** Modulo $I \otimes U_h(a)$, the multiplication in $U_h(a_n^2)$ satisfies the equation $m^{*\text{op}}((X_r(u_1 - u_2))(a \otimes b)) = m(X_l(u_1 - u_2)(a \otimes b))$, where $X_{l,r}(u) : U_h(a)^{\otimes 2} \to U_h(a)^{\otimes 2}$ are the operators defined by $B$ according to formula (1.25).

**Remark.** Note that since $I$ is a Hopf ideal, the operators $X_{l,r}$ preserve $I \otimes U_h(a)$.

*Proof.* It is enough to check that the relation of Lemma 1.21 is satisfied after applying the map $\xi$. This easily follows from property (1.17) of $B$ and the commutation relation between $U_h(a)$ and its dual in the double.

Now we can prove property (1.22) of $B$. For this purpose we will apply property (iii) of $\Theta_n$ for $n = 2$. Namely, using Lemma 1.21, we see that the fact that the map $\Theta_2$ defined by $\Theta_2(ab)(u_1, u_2) = \Theta_1(a)(u_1)\Theta_1(b)(u_2)$ is a homomorphism of Hopf algebras gives exactly (1.22).

Thus, $B$ is a weak $\partial$-copseudotriangular structure on $U_A$. It is easy to show that $B$ is a quantization of $\beta$. It is also clear that $B$ is left-nondegenerate iff $\Theta_1$ is injective, so the nondegeneracy of $B$ is equivalent to the nondegeneracy of $\beta$. The theorem is proved.

**Remark.** Since the Lie algebra $a$ is allowed to be infinite-dimensional (cf. Proposition 1.7), it is necessary to clarify what is meant by $U_h(a^{*\text{op}})$ and $U_h(a)^{*\text{op}}$. As vector spaces, these algebras are equal to $(\oplus_{n \geq 0}(S^n a)^*)[[h]]$, with the operations defined in the same way as in the finite-dimensional case. $\square$

The form $B$ constructed in the proof of Theorem 1.18 will be denoted by $U_h(\beta)$.

**Proposition 1.22.** Suppose $\beta$ is a left-nondegenerate $\partial_0$-copseudotriangular structure on $a$. Then the factorization of $U_h(a_n^u)$ into $n$ copies of $U_h(a)$ is defined by the $\partial$-copseudotriangular structure $B = U_h(\beta)$.

*Proof.* This follows from Lemma 1.21 and the fact that $I = 0$ in the nondegenerate case.
Proposition 1.23. Any weak copseudotriangular structure on a Lie bialgebra admits a quantization.

Proof. The Proposition follows from Theorem 1.18 and the equality $U_k(d) = D$, which is proved in Appendix A (Proposition A3). □

2. Lie bialgebras of functions on a curve with punctures

In this chapter, we will give some important examples of $\partial$-copseudotriangular structures, which arise from solutions of the classical Yang-Baxter equations with a spectral parameter.

2.1. Classical r-matrices.

In this section, we remind the definition of certain Lie bialgebras associated with a rational or elliptic curve with punctures, which was introduced by Drinfeld, [Dr1]. As this material is known, and proofs are easy, we omit proofs of most statements.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $k$, $\langle , \rangle$ be the Killing form on $\mathfrak{g}$ divided by 2. Let $\Omega \in (S^2\mathfrak{g})^0$ be the inverse element to $\langle , \rangle$.

Let $\mathfrak{g}_+ := \mathfrak{g}((t))$ be the Lie algebra of formal Laurent series with values in $\mathfrak{g}$. The algebra $\mathfrak{g}_+$ has a natural nondegenerate invariant inner product $\langle , \rangle$ defined by $(a,b) = \text{Res}(\langle a(t), b(t) \rangle)$, where $\text{Res}(\sum a_m t^m) := a_{-1}$. Denote by $\mathfrak{g}_n^a$ the direct sum of $n$ copies of $\mathfrak{g}$. The inner product on $\mathfrak{g}_+$ induces one on $\mathfrak{g}_n^a$.

Let $\mathfrak{g}_- := \mathfrak{g}[[t]]$, and $\mathfrak{g}_n^-$ be the direct sum of $n$ copies of $\mathfrak{g}_-$. It is clear that for any $n \geq 1$, $\mathfrak{g}_n^-$ is a Lie subalgebra in $\mathfrak{g}_n^a$, isotropic under the inner product $\langle , \rangle$.

Let $r \in \mathfrak{g} \otimes \mathfrak{g}((u))$, $r(u) = \Omega u + O(1)$.

Definition. One says that $r(u)$ is a classical r-matrix if it satisfies the classical Yang-Baxter equation

\begin{equation}
[r^{12}(u_1-u_2), r^{13}(u_1-u_3)] + [r^{12}(u_1-u_2), r^{23}(u_2-u_3)] + [r^{13}(u_1-u_3), r^{23}(u_2-u_3)] = 0.
\end{equation}

Belavin and Drinfeld [BD] showed that any classical r-matrix also satisfies the unitarity condition

\begin{equation}
r(u) = -r^{21}(-u).
\end{equation}

2.2. Lie bialgebras associated to classical r-matrices.

Given a classical r-matrix $r(u)$ on $\mathfrak{g}$, one can construct an infinite dimensional Lie algebra $\mathfrak{g}(r)$ as follows. We regard $r^{21}(t-u)$ as an element of $\mathfrak{g} \otimes \mathfrak{g}((t))[u]$, expanding $\frac{1}{t-u}$ as $\sum_{m \geq 0} t^{-m-1}u^m$. Define

\begin{equation}
\mathfrak{g}(r) := \{ \text{Res}(\langle X(u) \otimes 1, r^{21}(t-u) \rangle) : X(u) \in u^{-1}\mathfrak{g}[u^{-1}] \} \subset \mathfrak{g}((t)).
\end{equation}

We have the following well known propositions.

Proposition 2.1. $\mathfrak{g}(r)$ is a Lie subalgebra of $\mathfrak{g}((t))$ isotropic under $\langle , \rangle$.

Proof. The first statement follows from (2.1), and the second one from (2.2). □

Proposition 2.2. $\mathfrak{g}_+ = \mathfrak{g}(r) \oplus \mathfrak{g}_-$, and $(\mathfrak{g}, \mathfrak{g}(r), \mathfrak{g}_-)$ is a Manin triple.

Proof. Clear. □

Thus, $\mathfrak{g}(r)$ has a natural Lie bialgebra structure.

The commutation and cocommutation relations of $\mathfrak{g}(r)$ have the following convenient explicit representation. Define the generating function $\tau(u) := r^{21}(t-u) \in \mathfrak{g} \otimes \mathfrak{g}(r)[[u]]$. 
Proposition 2.3. One has

\[
[\tau^{13}(u), \tau^{23}(v)] = [r^{12}(u - v), \tau^{13}(u) + \tau^{23}(v)],
\]
(2.4)
\[
\delta(\tau(u)) = [\tau^{12}(u), \tau^{13}(u)].
\]

Remark 1. Here and below we use the usual notation \(\tau^{12} := \tau \otimes 1, \tau^{23} := 1 \otimes \tau\), and so on.

Remark 2. It is obvious that \([\Omega^{12}, \tau^{13}(u) + \tau^{23}(u)] = 0\), so the right hand side of the first equation in (2.4) is regular at \(u, v = 0\), and thus (2.4) really defines a bracket on \(\mathfrak{g}(r)\).

Proof. Relations (2.4) follow from the classical Yang-Baxter equation for \(r(u)\).

2.3. The canonical derivation.

Let \(f \in \mathfrak{g} \otimes \mathfrak{g}\) be the free term in the Laurent expansion of \(r\), i.e. \(r(u) = \frac{\Omega}{u} + f + O(u)\). Let \(\mu\) be the commutator in \(\mathfrak{g}\), and \(\rho_r = \frac{1}{2} \mu(f)\).

Proposition 2.4. The canonical derivation \(d\) on \(\mathfrak{g}(r)\) (see Section 1.3) equals \(-\frac{d}{du} + \text{ad}(\rho_r)\).

Proof. Since \(\mathfrak{g}(r)\) is a subbialgebra of \(\mathfrak{g}(t)\), it is enough to show that \(d = -\frac{d}{dt} + \text{ad}(\rho_r)\) in \(\mathfrak{g}(t)\).

The cobracket in \(\mathfrak{g}(t)\) is given by

\[
\delta(xu^n) = [xu^n \otimes 1 + 1 \otimes xv^n, r(u - v)] = \delta_0(xu^n) + \phi(xu^n),
\]
(2.5)
\[
\delta_0(xu^n) := [xu^n \otimes 1 + 1 \otimes xv^n, \frac{\Omega}{u - v}], \quad \phi(xu^n) := [xu^n \otimes 1 + 1 \otimes xv^n, f + O(u - v)].
\]

Let \(d_0 = -\frac{1}{2} \mu \circ \delta_0\). Then from (2.5) we obtain

\[
d = d_0 + \text{ad}(\rho_r).
\]
(2.6)

It remains to show that \(d_0 = -\frac{d}{du}\). We have

\[
d_0(xu^n) = -\frac{1}{2} \mu([x \otimes 1, \Omega] \frac{u^n - v^n}{u - v}).
\]
(2.7)

The normalization of \(\Omega\) is such that \(\mu([x \otimes 1, \Omega]) = 2x\). Therefore, the right hand side of (2.7) equals \(-nxu^{n-1}\), as desired.

Proposition 2.5. Any classical \(r\)-matrix \(r(u)\) is invariant under the adjoint action of \(\rho_r\), and the function \(\tilde{r}(u) := (e^{-u\rho_r} \otimes 1)r(u)(e^{u\rho_r} \otimes 1)\) is also a classical \(r\)-matrix.

Proof. It is clear that \((d \otimes 1 + 1 \otimes d)(r(u - v)) = 0\), as \(d\) preserves the structure of a Manin triple in \(\mathfrak{g}(t)\). By Proposition 2.4, this implies that \([\rho_r \otimes 1 + 1 \otimes \rho_r, r(u)] = 0\), as desired.

Because of this, we can write

\[
\tilde{r}(u - v) = (\text{Ad} e^{-u\rho_r} \otimes \text{Ad} e^{-u\rho_r})(r(u - v)).
\]
(2.8)

It is clear from this formula that \(\tilde{r}\) is a classical \(r\)-matrix.

Proposition 2.5 shows that the operator \(\text{ad}(\rho_r)\) is a derivation of the Manin triple \((\mathfrak{g}, \mathfrak{g}(r), \mathfrak{g}_-\)). Define \(\partial = d - \text{ad}(\rho_r)\). This is also a derivation of this Manin triple. We have \(\partial = -\frac{d}{du}\), i.e. \(\partial(\tau(u)) = \frac{d\tau(u)}{du}\).
2.4. $\partial$-copseudotriangular structure on $g(r)$.

It turns out that the Lie bialgebra $g(r)$ has a natural $\partial$-copseudotriangular structure. Namely, set

$$\beta_3(\tau^{13}(v), \tau^{23}(w))(u) = -r^{12}(v - w + u).$$

**Proposition 2.6.** $\beta$ is a $\partial$-copseudotriangular structure on $g(r)$.

**Proof.** Formulas (1.13) follow from (2.4) and the Yang-Baxter equation for $r$. Property (1.15) follows from Proposition 2.4. Finally, formula (1.14) follows from the identity $\delta(x(u)) = [x(u) \otimes 1 + 1 \otimes x(v), r(u - v)]$. □

2.5. Factored Lie bialgebras associated with $g(r)$.

Let $\Sigma$ be a 1-dimensional connected algebraic group over $k$ (i.e. $G_a$, $G_m$, or an elliptic curve), and $du$ be an invariant differential on $\Sigma$. The differential $du$ defines a canonical formal parameter near any point of $\Sigma$. Let $u$ be the corresponding formal parameter at the origin.

The following deep theorem is due to Belavin and Drinfeld [BD].

**Theorem 2.7.** Let $r$ be a classical $r$-matrix. Then there exists a unique 1-dimensional algebraic group $\Sigma$ such that the formal series $r(u)$ is the Laurent expansion with respect to $u$ of a rational function on $\Sigma$ with values in $g \otimes g$, and the stabilizer of $r(u)$ in $\Sigma$ is trivial.

Let $\Gamma$ be the set of poles of $r(u)$ in $\Sigma$. Theorem 2.7 shows that the $\partial$-copseudotriangular structure $\beta$ on $g(r)$ is in fact an $O$-structure, where $O = O(\Sigma \setminus \Gamma) \subset k((u))$.

Let $\mathbf{z} = (z_1, \ldots, z_n) \in \Sigma(k)^n$ be such that $z_i - z_j \notin \Gamma$ for $i \neq j$. The form $\beta$ allows us to define the factored Lie bialgebras $g(r)^O_{\mathbf{z}}$ and $g(r)_{\mathbf{z}}$, as in Section 1.7.

It is easy to write down explicitly the commutator in $g(r)_{\mathbf{z}}$. Recall that $g(r)_{\mathbf{z}}$ is equal to $g(r)^{\otimes n}$ as a vector space. Let $\tau_i(u)$ be the series $\tau(u)$ for the $i$-th summand $g(r)$ in this direct sum.

**Proposition 2.8.** The bracket in $g(r)_{\mathbf{z}}$ has the form

$$[\tau_i^{13}(u), \tau_j^{23}(v)] = [r^{12}(u - v + z_i - z_j), \tau_i^{13}(u) + \tau_j^{23}(v)].$$

**Proof.** Formula (2.10) is obtained by substitution of (2.4) in (1.23). □

The formula for the cobracket in $g(r)_{\mathbf{z}}$ follows directly from the definition:

$$\delta(\tau_i(u)) = [\tau_i^{12}(u), \tau_i^{13}(u)].$$

2.6. The Manin triple associated to $g(r)_{\mathbf{z}}$.

Consider the map $f_{\mathbf{z}} : g(r)_{\mathbf{z}} \to \overline{g}_n$ defined by

$$f_{\mathbf{z}}(\tau_i(u)) = (r(t - u + z_1 - z_i), \ldots, r(t - u + z_n - z_i)).$$

It is easy to check that $f_{\mathbf{z}}$ is a Lie algebra embedding. Thus we can regard $g(r)_{\mathbf{z}}$ as a Lie subalgebra in $\overline{g}_n$. 
Proposition 2.9. (i) The algebra $g(r)_z$ is isotropic in $\mathfrak{g}^n$ with respect to the form $(,)$.

(ii) $\mathfrak{g}^n = g(r)_z \oplus \mathfrak{g}^n_-$.

(iii) The form $(,)$ defines a linear isomorphism $\mathfrak{g}^n_- \rightarrow g(r)_z^*$.

Proof. (i) follows from the unitarity property of $r(u)$. (ii) is clear. (iii) follows from (ii). $\Box$

Corollary 2.10. The triple of Lie algebras $(\mathfrak{g}^n, g_z, \mathfrak{g}^n)$ is a Manin triple.

Proposition 2.11. The Lie bialgebra structure on $g(r)_z$ coming from the Manin triple of Corollary 2.10 coincides with the one defined by formula (2.11).

2.7. Examples.

Example 1: Yang’s rational r-matrix.

Let $r(u)$ be the Yang’s r-matrix $r(u) = \frac{\Omega}{u}$. In this case, $\Sigma = \mathbb{G}_a$, and $u$ is an affine coordinate. The Lie algebra $g(r)$ coincides with the Lie algebra $a = t^{-1}g[t^{-1}]$. As a Lie bialgebra, $g(r)$ is the graded dual to the Yangian Lie bialgebra, defined by Drinfeld in [Dr1], with opposite commutator.

Let $z = (z_1, \ldots, z_n) \in k^n$, $z_i \neq z_j$ for $i \neq j$. Let $R_z$ be the algebra of all rational functions in one variable over $k$ with no poles outside of the points $z_1, \ldots, z_n$ which vanish at infinity. Consider the Lie algebra $a_z = g \otimes R_z$, and the Lie algebra embedding $\phi_z : a_z \rightarrow \mathfrak{g}^n$, defined by the formula $\phi_z(f) = (\phi_{z_1}(f), \ldots, \phi_{z_n}(f))$, where $\phi_{z_i}(f) \in \mathfrak{g}$ is the Laurent expansion of $f$ at $z_i$. Using this embedding, we can regard $a_z$ as a Lie subalgebra in $\mathfrak{g}^n$.

Proposition 2.12. $g(r)_z = a_z$.

Let $\tilde{r}(u)$ be any classical r-matrix. Then $\lim_{\varepsilon \to 0} \varepsilon \tilde{r}(\varepsilon u)$ equals the Yang’s r-matrix $r(u)$. Therefore, the bialgebras $g(r)$, $g(r)_z$ can be obtained as a degeneration of $g(\tilde{r})$, $g(\tilde{r})_z$.

Example 2: Trigonometric r-matrices.

Suppose that the Lie algebra $g$ is endowed with a standard polarization $g = n^- \oplus h \oplus n^+$, where $h$ is a Cartan subalgebra, $n^\pm$ the nilpotent subalgebras.

Let $L = L_+ + \frac{1}{2}L_0 \in g \otimes g$, where $L_+$ is the canonical element in $n^+ \otimes n^-$, and $L_0 \in h \otimes h$ is the element dual to $\langle , \rangle|_h$. It is easy to check that the element $L$ is a constant solution to the classical Yang-Baxter equation. Therefore, the element

\begin{equation}
(2.13) \quad r(u) = \frac{L^{21}e^u + L}{e^u - 1},
\end{equation}

is a classical r-matrix. In this case, $\Sigma = \mathbb{G}_m = A^1 \setminus 0$, $u = \ln x$, where $x$ is the affine coordinate on $A^1$.

Let $a$ be the Lie algebra of $g$-valued rational functions $X(x)$ on $\Sigma$, such that

(i) $X$ has no poles on $\Sigma$ outside of the identity;

(ii) $X$ is regular at the infinite points $0, \infty$ of $\Sigma$;

(iii) $X(0) \in h \oplus n_-$, $X(\infty) \in h \oplus n_+$, and $X(0) + X(\infty) \in n_+ \oplus n_-$.

Consider the embedding $\phi : a \rightarrow \mathfrak{g}$, which assigns to $X \in a$ its Laurent expansion at the identity with respect to $u$. This embedding allows us to regard $a$ as a subalgebra in $\mathfrak{g}$. 

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Proposition 2.13. \( g(r) = a \).

Analogously to the construction of \( g(r) \), for any \( z \in \Sigma^n, z_i - z_j \neq \emptyset, i \neq j \), define \( a_z \) to be the Lie algebra of \( g \)-valued rational functions \( X(x) \) on \( \Sigma \), such that
(i) \( X \) has no poles on \( \Sigma \) outside of \( z_1, ..., z_n \);
(ii) \( X \) is regular at the infinite points 0, \( \infty \) of \( \Sigma \);
(iii) \( X(0) \in h \oplus n_-, X(\infty) \in h \oplus n_+ \), and \( X(0) + X(\infty) \in n_+ \oplus n_- \).

Consider the embedding \( \phi_z : a_z \to \mathfrak{g}^n \), which assigns to \( X \in a_z \) the collection of its Laurent expansions at \( z_i \) with respect to the parameters \( u_i = \ln(x/x(z_i)) \). This embedding allows us to regard \( a_z \) as a subalgebra in \( \mathfrak{g}^n \).

Proposition 2.14. \( g(r)_z = a_z \).

Example 3: Elliptic r-matrices.

Let \( g = \mathfrak{sl}_N, \langle X, Y \rangle = N \text{tr}(XY) \). Let \( A : g \to g \) be the conjugation by the cyclic permutation \((12...N)\), and \( B : g \to g \) be the conjugation by the matrix \( \text{diag}(1, \varepsilon, ..., \varepsilon^{-N-1}) \), where \( \varepsilon \) is a primitive \( N \)-th root of 1. Let \( H \) be the subgroup in \( \text{Aut}(g) \) generated by \( A, B \). This group is isomorphic to \( \mathbb{Z}/NZ \times \mathbb{Z}/NZ \).

Let \( \Sigma \) be an elliptic curve over \( k \). Let \( \Gamma \) be the group of points of order \( N \) in \( \Sigma \). Fix an isomorphism \( \lambda : \Gamma \to H \). Let \( r \) be a \( g \)-valued rational function on \( \Sigma \) of the form \( r(u) = \frac{u}{u} + O(1) \), such that \( r(z + \gamma) = (1 \otimes \lambda(\gamma))(r(z)), \gamma \in \Gamma \), and \( r \) is regular outside of \( \Gamma \). It is easy to see that such a function is unique. It is known [BD] that it is a classical \( r \)-matrix.

Let \( \mathfrak{a} \) be the Lie algebra of all \( g \)-valued rational functions \( X \) on \( \Sigma \) such that \( X \) has no poles outside of \( \Gamma \), and \( X(z + \gamma) = \lambda(\gamma)(X(z)), \gamma \in \Gamma \). Define a Lie algebra embedding \( \phi : \mathfrak{a} \to \mathfrak{g} \) by assigning to a function \( X \) its Laurent expansion at the origin. We identify \( \mathfrak{a} \) with its image under this embedding. Then we have

Proposition 2.15. \( g(r) = a \).

Analogously, for any \( z \in \Sigma^n, z_i - z_j \notin \emptyset, i \neq j \), define \( a_z \) to be the Lie algebra of \( g \)-valued rational functions \( X \) on \( \Sigma \), such that \( X \) has no poles outside of \( \cup_i(z_i + \Gamma) \), and \( X(z + \gamma) = \lambda(\gamma)(X(z)), \gamma \in \Gamma \).

Consider the embedding \( \phi_z : a_z \to \mathfrak{g}^n \), which assigns to \( X \in a_z \) the collection of its Laurent expansions at \( z_i \). This embedding allows us to regard \( a_z \) as a subalgebra in \( \mathfrak{g}^n \).

Proposition 2.16. \( g(r)_z = a_z \).

2.8. Poisson groups.

In this section we consider Poisson groups corresponding to the constructed Lie bialgebras. They will be useful in the next Chapter.

Assume that \( g \) is the Lie algebra of an affine algebraic group \( G \) defined over \( k \). In this case, define the proalgebraic group \( \overline{G}_- := G[[t]] \). Any classical \( r \)-matrix \( r(u) \) on \( \Sigma \) defines a natural Poisson-Lie structure on this group, coming from the Manin triple of Proposition 2.2. We denote the obtained Poisson group by \( G(r) \).

Let \( \overline{G}^n := \prod_{i=1}^n \overline{G}_- \). The Lie algebra of this group is \( \overline{\mathfrak{g}}^n \). Therefore, for any element \( z = (z_1, ..., z_n) \in \Sigma(k)^n, z_i - z_j \notin \Gamma \), we can define a natural structure on this group, coming from the Manin triple of Corollary 2.10. We denote the obtained Poisson group by \( G(r)_z \). For example, if \( n = 1 \) then \( G(r)_z \) coincides with \( G(r) \).

At the level of formal groups, it is easy to describe the Poisson bracket on \( G(r)_z \) explicitly. Let \( F_{x,r}(G) \) be the algebra of regular functions on the Poisson
group $G(r)$. Let $F_{r,G}(G)$ be the completion of the algebra $F_{r,G}(G)$ at the identity. Using the exponential map, we can identify $F_{r,G}(G)$ with the algebra topologically generated by linear functions on the Lie algebra $\mathfrak{g}^\mathbb{C}$. So, it is enough to write down the Poisson bracket of such linear functions.

Using the dualities of Proposition 2.9 (iii), we can regard the expression $\langle X \otimes 1, \tau_i(u) \rangle$, $X \in \mathfrak{g}$, as a formal series whose coefficients are linear functions on $\mathfrak{g}^\mathbb{C}$:

$$\tau_i(u)(a_1, \ldots, a_n) = \langle r^{12}(t - u), a_i^2(t) \rangle,$$

Denote by $t_i(u)$ the formal series of linear functions corresponding to $\tau_i(u)$. Then the commutation relations for $\tau_i(u)$ can be rewritten as Poisson commutation relations for $t_i(u)$:

$$\{t_i^{13}(u), t_j^{23}(v)\} = [r^{12}(u - v + z_i - z_j), t_i^{13}(u) + t_j^{23}(v)].$$

These relations describe the Poisson bracket on the formal version of the group $G(r)$.  

3. Quantum groups associated to curves with punctures

3.1. Copseudotriangular structure of dual Yangians and associated factored Hopf algebras.

In this section we will explicitly quantize the Lie bialgebra $\mathfrak{g}(r)_z$, where $r$ is the Yang’s r-matrix.

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra, $\langle , \rangle$ be the Killing form on $\mathfrak{g}$ divided by 2. Let $Y(\mathfrak{g})$ be the Yangian of $\mathfrak{g}$, defined by Drinfeld, [Dr2]. It is known [Dr2] that $Y(\mathfrak{g})$ is a quantization of the Lie algebra $\mathfrak{g}[t]$, with the cobracket coming from the Manin triple $(\mathfrak{g}, \mathfrak{g}(r), \mathfrak{g})$, via the natural embedding $\mathfrak{g}[t] \to \mathfrak{g}$, where $r(u)$ is the Yang’s r-matrix $\Omega/u$ (See Chapter 2). Thus, as a $k[[h]]$-module, $Y(\mathfrak{g})$ is identified with $U(\mathfrak{g}[t])[h]$.  

Recall that $Y(\mathfrak{g})$ has a natural $\mathbb{Z}_+$-grading, with $\deg(Xz^j) = j$, $\deg(h) = 1$. Let $f : Y(\mathfrak{g}) \to k[[h]]$ be a continuous $k[[h]]$-linear functional. We say that $f$ is tempered if it can be interpreted in the form $\sum_{m \geq 0} f_m$, where $f_m$ is homogeneous and $\lim_{m \to \infty} \deg f_m = +\infty$. Let $Y(\mathfrak{g})'$ denote the space of all tempered linear functionals on $Y(\mathfrak{g})$. The $k[[h]]$-module $Y(\mathfrak{g})'$ has a structure of an associative algebra dual to the coalgebra structure in $Y(\mathfrak{g})$. Let $\mathfrak{m}$ be the ideal in $Y(\mathfrak{g})'$ consisting of all functionals $f$ such that $f(1) = O(h)$. Denote by $Y(\mathfrak{g})^\ast$ the $h$-adic completion of the direct sum $\oplus_{s \geq 0} h^{-s}\mathfrak{m}^\ast$. It is easy to check that $Y(\mathfrak{g})^\ast$ inherits the associative algebra structure from $Y(\mathfrak{g})'$. Besides, we have

**Lemma 3.1.**

(i) the map $m^\ast$ dual to the product $m : Y(\mathfrak{g}) \otimes Y(\mathfrak{g}) \to Y(\mathfrak{g})$ takes $Y(\mathfrak{g})^\ast$ into $Y(\mathfrak{g})^\ast \otimes Y(\mathfrak{g})^\ast$, and defines a coassociative coproduct on the topological algebra $Y(\mathfrak{g})^\ast$, which is cocommutative mod $h$;

(ii) the algebra $Y(\mathfrak{g})^\ast$, equipped with $m^\ast$, is a quantized universal enveloping algebra, whose quasiclassical limit is $\mathfrak{g}(r)$, with opposite cocommutator.

**Proof.** Part (i) is easy. Part (ii) follows from the fact that the quantized universal enveloping algebra $Y(\mathfrak{g})$ is a quantization of the Lie bialgebra $\mathfrak{g}[t]$, which is the graded dual to $\mathfrak{g}(r)$, with opposite coproduct. □
Denote by $Y^*(\mathfrak{g})$ the Hopf algebra $Y(\mathfrak{g})^*$, with opposite product. Then $Y^*(\mathfrak{g})$ is a quantization of $\mathfrak{g}(r)$, and the subalgebra $Y(\mathfrak{g})'_{op} \subset Y^*(\mathfrak{g})$ (i.e. $Y(\mathfrak{g})'$ with opposite product) is the corresponding h-formal group.

Let $\Sigma = \mathbb{G}_a$, $\Gamma = \{0\}$. Then $\mathcal{O} = k[u, u^{-1}]$. Now, using the pseudotriangular structure on $Y(\mathfrak{g})$ defined by Drinfeld [Dr2], we will define a copseudotriangular $\mathcal{O}$-structure on $Y(\mathfrak{g})'_{op}$.

**Proposition 3.2.** [Dr2] There exists a unique series

$$
R(u) = 1 + \sum_{m \geq 1} \mathbb{R}_m u^{-m}, \mathbb{R}_m \in Y(\mathfrak{g}) \otimes Y(\mathfrak{g})
$$

which satisfies the hexagon relations

$$
(\Delta \otimes 1)(R) = \mathbb{R}^{13}, (1 \otimes \Delta)(R) = \mathbb{R}^{13},
$$

and the property

$$
R(u)(\alpha_u \otimes 1)(\Delta(x)) = (\alpha_u \otimes 1)(\Delta^{op}(x))R(u),
$$

where $\alpha_u = e^{uD}$.

This series also satisfies the equations $(\varepsilon \otimes 1)(R) = (1 \otimes \varepsilon)(R) = 1$, $(\alpha_v \otimes 1)(R(u)) = (1 \otimes \alpha_{-v})(R(u)) = R(u + v) := \sum R^{(m)}(u)v^m/m!$.

The element $R$ is called the pseudotriangular structure on the Yangian $Y(\mathfrak{g})$.

Define the form $B : Y(\mathfrak{g})'_{op} \otimes Y(\mathfrak{g})'_{op} \to \mathcal{O}[[h]]$ by the formula

$$
B(x, y)(u) = (x \otimes y)(\mathbb{R}^{21}(-u)) = (x \otimes y)(\mathbb{R}(u)^{-1}).
$$

**Proposition 3.3.** The form $B$ is a copseudotriangular $\mathcal{O}$-structure on $Y(\mathfrak{g})'_{op}$.

**Proof.** Properties (1.17)-(1.20) of $B$ are dual to the identities of Proposition 3.2. □

**Remark.** This construction explains the terminology “copseudotriangular structure”.

Let $\mathfrak{g}[t]$ be the Yangian Lie bialgebra, considered by Drinfeld in [Dr1], Example 3.3. The coproduct in this Lie bialgebra is defined by

$$
\delta(x(u)) = [x(u) \otimes 1 + 1 \otimes x(v), \frac{\Omega}{u - v}].
$$

**Proposition 3.4.** $Y(\mathfrak{g}) = U_h(\mathfrak{g}[t])$.

**Proof.** Both $Y(\mathfrak{g})$ and $U_h(\mathfrak{g}[t])$ are graded quantizations of $\mathfrak{g}[t]$. Drinfeld showed that a graded quantization of $\mathfrak{g}[t]$ is unique up to an isomorphism (see [Dr2]). □

Let $r$ be the Yang’s r-matrix $\Omega/u$, $\mathfrak{g}(r)$ be the Lie bialgebra from Example 1 of Section 2.7, and $\beta$ be the copseudotriangular structure on $\mathfrak{g}(r)$ defined in Section 2.4.
Proposition 3.5. \(Y^*(g) = U_h(g(t)), \) and \(B = U_h(\beta).\)

Proof. By the definition, \(Y^*(g)\) is the graded dual algebra to \(Y(g),\) with opposite product, and \(g(t)\) is the graded dual bialgebra to \(g[t],\) with opposite commutator. Therefore, Proposition 3.4 implies that \(Y^*(g) = U_h(g(t)).\)

Let \(B' = U_h(\beta).\) Then \(B'\) defines an element \(\mathbb{R}' \in Y(g) \otimes Y(g)((u)),\) such that \(B'(x,y) = (x \otimes y, (\mathbb{R}'^{21})^{-1}).\) We know that \(B'\) is a weak cokpseudotriangular structure. Also, since \(\beta\) is left-nondegnerate, so is \(B'.\) Therefore, \(B'\) is a cokpseudotriangular structure. Thus, the element \(\mathbb{R}'\) satisfies properties (3.2),(3.3). Therefore, by Proposition 3.2, \(\mathbb{R}' = \mathbb{R},\) and thus \(B' = B.\) □

Let \((z_1, ..., z_n) \in k^n\) be a collection of distinct points, and \(Y(g)_{op}(z), Y^*(g)(z)\) be the factored Hopf algebras \(A_z, U_{A_z}\) obtained from \(A := Y(g)_{op}, U_A := Y^*(g)\) with the help of the cokpseudotriangular structure \(B,\) as described in Section 1.7. Let \(g(r)_z\) be the factored Lie bialgebra from Example 1 of Section 2.7.

Proposition 3.6. \(Y^*(g)(z) = U_h(g(r)_z).\)

Proof. Since \(g(r)_z\) is obtained from \(g(r)\) with the help of the form \(\beta,\) by Proposition 1.22 \(U_h(g(r)_z)\) is obtained from \(U_h(g(r))\) with the help of the form \(U_h(\beta).\) Therefore, the statement follows from Proposition 3.5. □

Remark. Proposition 3.6 gives an “explicit” description of the quantization \(U_h(g(r)_z),\) in the sense that it does not use a Drinfeld associator, which is used in the construction of \(U_h(a)\) for a general Lie bialgebra \(a,\) as in [EK1].

3.2. R-matrix Hopf algebras.

It this section we will consider quantum groups defined by FRT-type relations [FRT,RS].

Let \(R(u) \in \text{End}(k^N \otimes k^N)((u))[[h]]\) be an element such that \(R = 1 + O(h).\) Following [FRT,RS], one associates to \(R\) the h-adic completion \(F(R)\) of the algebra over \(k[[h]]\) whose generators are the entries of formal series \(T(u)^{\pm 1} \in \text{End}(k^N) \otimes F(R)[[u]], T(u) = T_0 + T_1 u + ...,\) and the defining relations are

\[
T(u)T(u)^{-1} = T(u)^{-1}T(u) = 1,
R^{12}(u - v)T^{13}(u)T^{23}(v) = T^{23}(v)T^{13}(u)R^{12}(u - v)
\]

(3.5)

Proposition 3.7. There exists a unique Hopf algebra structure on \(F(R)\) such that

\[
\Delta(T(u)) = T^{12}(u)T^{13}(u).
\]

(3.6)

Proof. Introduce the cokprodbduct on the free algebra generated by \(T(u)^{\pm 1}\) modulo the first relation of (3.5). It is easy to check that the ideal in this algebra generated by the second relation in (3.5) is a Hopf ideal. The proposition is proved. □

Proposition 3.8. The map \(\phi_0 : F(R)/hF(R) \to k[GL_N[[t]]\) defined by \(\phi_0(T(u))(g(t)) = g(u), g(\ast) \in GL_N(k)[[t]],\) is a Hopf algebra isomorphism.

Proof. Clear. □

It is interesting to determine when \(F(R)\) is a quantization of the proalgebraic group \(GL_N[[t]],\) i.e. when \(F(R)\) is a flat deformation of the function algebra
$k[GL_N[[t]]]$. Flatness is equivalent to the property that the operator of multiplication by $h$ on $F(R)$ is injective.

In general, the algebra $F(R)$ is not a flat deformation of the function algebra $k[GL_N[[t]]]$. The following proposition gives necessary and sufficient conditions of flatness for a large class of $R$-matrices.

**Proposition 3.9.** (a) If $F(R)$ is a flat deformation of $k[GL_N[[t]]]$ then $R(u)$ is a solution of the quantum Yang-Baxter equation

$$R^{12}(u_1-u_2)R^{13}(u_1-u_3)R^{23}(u_2-u_3) = R^{23}(u_2-u_3)R^{13}(u_1-u_3)R^{12}(u_1-u_2),$$

and $R^{21}(-u)R(u)$ is a scalar operator of the form $f(u)Id$, where $f(u) \in 1 + hk((u))[h]$, such that

$$f(u) = f(-u).$$

(b) Let $R(u, h) \in k[[h, u, h/u]]$, and suppose that $\lim_{u \to 0} R(su, sh) = R_{rat}(u)$, where $R_{rat}(u) = 1 - \frac{h(\sigma-1/N)}{N(u-h/N^2)}$ is the rational $R$-matrix. Then, if $R$ satisfies (3.7), $F(R)$ is a flat deformation of $k[GL_N[[t]]]$.

**Proof.** The proof is given in Section 3.4. □

**Remark.** Note that the converse to (a) does not hold. For example, if $R = 1 - h\Omega/u$, where $\Omega$ is the Casimir element of $O(N) \subset GL(N)$, then the conclusion of (a) holds but $F(R)$ is not flat. This is easily seen from considering the quasiclassical limit (one does not get a Poisson-Lie structure on $GL_N[[t]]$).

It is clear that rescaling of $R$, i.e. multiplication of $R$ by a scalar in $k((u))[h]$,

does not change the algebra $F(R)$. Therefore, Proposition 3.9 implies that if $F(R)$ is a flat deformation, one can arrange that $R$ satisfies the unitarity condition

$$R(u)R^{21}(-u) = 1.$$

We know that if $R(u)$ satisfies (3.7) and (3.9) then $R(u) = 1 - hr(u) + O(h^2)$, where $r(u) \in gl_N \otimes gl_N((u))$ is a unitary solution of the classical Yang-Baxter equation. In addition, if the assumption of Proposition 3.9(b) holds, we have $r(u) = (\sigma - 1/N)/u + O(1)$, where $\sigma$ is the permutation. It is explained in Section 2.8 that $r$ defines a Poisson group $G(r)$ ($G = GL_N$). Proposition 3.9(b) implies that the algebra $F(R)$ is always a quantization of the Poisson group $G(r)$.

Assume that $F(R)$ is a flat deformation.

Let $A(R)$ be the completion of $F(R)$ with respect to the kernel of the counit. Then $A(R)$ is an $h$-formal group, which is a flat deformation of the formal group corresponding to $GL_N[[t]]$. Denote the corresponding QUE algebra by $U(R)$.

Let $R(u) = 1 - hr(u) + O(h^2)$, where $r \in \text{End}(k^N \otimes k^N)((u))$. According to Proposition 3.9, $r(u)$ satisfies the classical Yang-Baxter equation (2.1), and $r(u) + r^{21}(-u)$ is a scalar. Define a Lie bialgebra $gl_N(r)$ by relations (2.4).

It is easy to describe the algebra $U(R)$ by generators and relations.

Namely, let $t(u) \in \text{End}(k^N) \otimes U(R)[[u]]$ be defined by the formula $T(u) = 1 + ht(u)$. Let $r_u(u) = h^{-1}(1 - R(u))$. It is easy to see that relations (3.5),(3.6) are equivalent to the following commutation and cocommutation relations:

$$[t^{13}(u), t^{23}(u)] = [r_u^{12}(u-v), t^{13}(u)] + h(r_u^{12}(u-v)t^{13}(u)t^{23}(v) - t^{23}(v)t^{13}(u)r_u^{12}(u-v)),

\Delta(t(u)) = t^{12}(u) + t^{13}(u) + ht^{12}(u)t^{13}(u).$$
It is easy to see that the QUE algebra $U(R)$ is a quantization of $\mathfrak{gl}_N(r)$, because relations (3.10) coincide with (2.4) modulo $h$.

**Remark.** R-matrix algebras were first considered by Cherednik [Ch1].

### 3.3. R-matrix realization of dual Yangians.

In this section we will describe dual Yangians for $\mathfrak{g} = \mathfrak{gl}_{N}$ and $\mathfrak{sl}_{N}$ in the R-matrix language.

Let $R(u)$ be the Yang’s quantum R-matrix $R_Y(u) := 1 - \frac{h\sigma}{Nu}$. It is well known (e.g. see [Ch1]) that $F(R)$ is a flat deformation of $k[GL_N[[t]]]$.  

**Proposition 3.11.** The algebra $U(R_Y)$ is a quantization of the Lie bialgebra $\mathfrak{gl}_N(r)$, where $r = \sigma/Nu$ is the Yang’s r-matrix for $\mathfrak{gl}_N$.

**Proof.** Clear. □

Now we will formulate the analogs of these results for the Lie algebra $\mathfrak{sl}_N$.

Let $A$ be an algebra over $k$. For any matrix $X(u) \in Mat_N(k) \otimes A[[u^{-1}]]$, define its quantum determinant by

\[
\text{qdet}(X) = \sum_{\sigma} (-1)^{\sigma} X(u - \frac{h(N - 1)}{2N}) 1_{\sigma(1)} X(u - \frac{h(N - 3)}{2N}) 2_{\sigma(2)} \ldots X(u + \frac{h(N - 1)}{2N}) N_{\sigma(N)}.
\]

Let $F_0(R_Y)$ denote the quotient of $F(R_Y)$ by the relation

\[
\text{qdet}(T(u)) = 1,
\]

It is easy to see that $F_0(R_Y)/hF_0(R_Y) = k[SL_N[[t]]]$. From the flatness of $F(R_Y)$ it can be deduced that $F_0(R_Y)$ is a flat deformation of the function algebra $k[SL_N[[t]]]$.  

Let $A_0(R_Y)$ be the h-formal group obtained by completion of $F_0(R_Y)$ with respect to the kernel of the counit, and $U_0(R_Y)$ be the QUE algebra corresponding to $A_0(R_Y)$. Analogously to Proposition 3.11 we have  

**Proposition 3.12.** The algebra $U_0(R_Y)$ is a quantization of the Lie bialgebra $\mathfrak{g}(r)$, where $\mathfrak{g} = \mathfrak{sl}_N$, and $r = \Omega/u$ is the Yang’s r-matrix for $\mathfrak{g}$.

**Proposition 3.13.** $U_0(R_Y)$ is isomorphic to $Y^*(\mathfrak{sl}_N)$.

**Proof.** It is easy to see that $U_0(R_Y)$ is a graded quantization of $\mathfrak{sl}_N(r)$, i.e. it admits a $\mathbb{Z}$-grading (with $\deg(h) = 1$) whose quasiclassical limit is the standard grading on $\mathfrak{g}(r)$. The Hopf algebra $Y^*(\mathfrak{sl}_N)$ has the same property, as it is dual to the graded Hopf algebra $Y(\mathfrak{sl}_N)$ with opposite coproduct. According to Drinfeld [Dr1], the graded quantization is unique. Therefore, Proposition 3.13 is proved. □

One can also consider the Yangian $Y(\mathfrak{g}_N)$. It is the quantized universal enveloping algebra with generators $t^*(u) = t_{u}^* u^{-1} t_{u}^* u^{-2} + \ldots$, $t_{u}^* \in Mat_N(k) \otimes Y(\mathfrak{g}_N)$, and defining relations (3.5),(3.6) (with $T^*$ instead of $T$), where $R = R_Y$, and $T^*(u) = 1 + ht^*(u) = 1 + T_{u}^* u^{-1} + T_{u}^* u^{-2} + \ldots$. Let $Y^*(\mathfrak{g}_N)$ be the dual Yangian, constructed as in Section 3.1, with opposite product.

**Proposition 3.14.** $U(R_Y)$ is isomorphic to $Y^*(\mathfrak{gl}_N)$.

**Proof.** Let $H$ be the universal enveloping algebra of the abelian Lie algebra $t^{-1}k[t^{-1}]$. It is easy to see that we have Hopf algebra isomorphisms $U(R_Y) = U_0(R_Y) \otimes H$, $Y^*(\mathfrak{g}_N) = Y^*(\mathfrak{sl}_N) \otimes H$. Thus, Proposition 3.14 follows from Proposition 3.13. □
3.4. The PBW theorem for R-matrix algebras.

In this section we prove Proposition 3.9. Note that part (b) of Proposition 3.9 plays the role of a Poincare-Birkhoff-Witt theorem for R-matrix algebras.

(a) Let \( Y = Y_r^{-1} Y_l \), where \( Y_r, Y_l \) are the right and left hand sides of (3.7). Then from (3.5) we get
\[
[Y \otimes 1, T^{14}(u_1)T^{24}(u_2)T^{34}(u_3)] = 0.
\]

Let \( Y = \sum_{m \geq 0} Y_m h^m \). We prove that \( Y_m \) is a scalar by induction. The base of induction follows from the fact that \( Y_0 = 1 \). Assume that we proved that \( Y_0, ..., Y_{m-1} \) are scalars. If \( F(R) \) is a flat deformation then, dividing the above equation by \( h^m \) and reducing it mod \( h \), we get \([Y_m, g(u_1) \otimes g(u_2)] = 0\) for any \( g \in GL_N[[t]](k) \). This implies that \( Y_m \) is a scalar.

Thus, we have shown that \( Y \) is a scalar, which equals 1 mod \( h \). Taking the determinants of both sides of (3.7), we get \( Y = 1 \).

Similarly one shows that \( R^{21}(-u)R(u) \) is a scalar. Indeed, let \( Z = R^{21}(v - u)R^{12}(u - v) \). Then from (3.5) we get
\[
[Z \otimes 1, T^{13}(u)T^{23}(v)] = 0.
\]

Let \( Z = \sum_{m \geq 0} Z_m h^m \). We prove that \( Z_m \) is a scalar by induction. The base of induction follows from the fact that \( Z_0 = 1 \). Assume that we proved that \( Z_0, ..., Z_{m-1} \) are scalars. If \( F(R) \) is a flat deformation then, dividing the above equation by \( h^m \) and reducing it mod \( h \), we get \([Z_m, g(u_1) \otimes g(u_2) \otimes g(u_3)] = 0\) for any \( g \in GL_N[[t]](k) \). This implies that \( Z_m \) is a scalar.

Thus, we have shown that \( Z \) is a scalar which equals 1 mod \( h \), as desired.

(b) First consider the case when \( R = R_{rat} = 1 - \frac{h^{(\sigma-1)/N}}{N^{u-h/N^2}} \). In this case the flatness of \( F(R) \) follows from the flatness of \( F(R_Y) \), since we have \( R_{rat} = \frac{u}{u-h/N^2} \).

Now let \( R \) be an arbitrary solution of (3.7), which satisfies the assumption of (b). Let \( R_h(u, h) = R(su, sh) \). Consider the algebra \( F(R_h) \).

Assume the opposite, i.e. that \( F(R) \) is not flat. Then \( F(R_h) \) is not flat for any \( s \neq 0 \) (since \( F(R_h) \) for \( s \neq 0 \) are all essentially the same). Denote by \( F_\tau(R) \) the algebra \( F(R_h) \) with \( s \) being a formal parameter \( \tau \). This is an algebra over \( k[[h, \tau]] \), which is not a flat deformation of \( k[GL_N[[t]]] \).

Let \( B \) be a basis of \( k[GL_N[[t]]] \) consisting of monomials in the generators \( t_{ij} \) (one has to choose an appropriate subset of the set of all monomials). It is clear that \( B \) can be naturally regarded as a spanning system for \( F_\tau(R) \), but it is linearly dependent. Pick a nontrivial linear relation \( \sum_{j=1}^m \alpha_j(h, t)b_j = 0 \), where \( b_j \in B \).

Now we will make use of the representation theory of the dual Yangian. Let \( V(a) \), \( a \in k^* \), be the representation of \( F(R_Y) \) on the vector space \( V = k^N \) defined by \( T(u) \rightarrow R_Y(u - a) \) (the shifted basic representation). We will consider all possible tensor products of these representations.

**Proposition 3.15.** The natural map \( F(R_Y) \rightarrow \oplus_n \oplus_{a_1,...,a_n} k^* \text{End}(V(a_1) \otimes ... \otimes V(a_2))[[h]] \) is injective.

**Proof.** It is enough to show that the map \( U(R_Y) \rightarrow \oplus_n \oplus_{a_1,...,a_n} k^* \text{End}(V(a_1) \otimes ... \otimes V(a_n))[[h]] \) is injective. This follows easily from the fact that \( U(R_Y) \) is a deformation of \( U(t^{-1}g[t^{-1}]) \). \( \square \)
Now let $R$ be as in (b). Then, since $R$ satisfies (3.7), one can define the representations $V(a)$ of $F_\tau(R)$ in the same way as for the Yangian. Choose a finite collection of representations $W_1, \ldots, W_l$ of the form $V(a_1) \otimes \cdots \otimes V(a_n)$ such that $b_i$ are linearly independent in $M = (\text{End}(W_1) \oplus \cdots \oplus \text{End}(W_l)) \otimes k[[h, \tau]]$. Let $P = \oplus_{j=1}^m k b_j \otimes_k k[[h, \tau]]$. We have a linear mapping $\theta : P \to M$, which assigns to $b_i$ the direct sum of the corresponding operators. We know that:

(i) $P, M$ are finitely generated, free $k[[h, \tau]]$ modules. (ii) The map $\theta$ is not injective. (iii) $\theta$ is injective modulo $\tau$ (by Proposition 3.15).

This is a contradiction, since by (ii) all the maximal minors of the matrix of $\theta$ in any basis are 0, which contradicts (iii).

3.5. $\partial$-copseudotriangular structure on $F(R)$.

In this section we describe a $\partial$-copseudotriangular structure on any $R$-matrix algebra.

Let $\partial$ be the derivation of $F(R)$ defined by the rule $\partial T(u) = \frac{dT(u)}{du}$.

**Proposition 3.16.** (i) Let $R$ satisfy (3.7). Then there exists a unique $\partial$-copseudotriangular structure $B$ on $A(R)$ such that

$$B(T^{13}(u), T^{23}(v))(y) = R^{12}(u - v + y).$$

(ii) The form $B$ is given by the formula

$$B(T^{1,p+q+1}(u_1) \ldots T^{p,p+q+1}(u_p), T^{p+1,p+q+1}(v_1) \ldots T^{p+q,p+q+1}(v_q))(y) =$$

$$\prod_{i=1}^p \prod_{j=q}^1 R^{i,p+q}(u_i - v_j + y)$$

**Proof.** Consider the form $B$ on the free algebra generated by $T(u)$ which is defined by (3.14). It follows from the Yang-Baxter equation for $R$ that the ideal generated by relations (3.5) is annihilated by $B$ on the right and on the left. This implies that $B$ descends to a form on $F(R)$ and defines a form on $A(R)$. The fact that $B$ is a $\partial$-copseudotriangular structure is easily obtained from the definition of $B$ and the properties of $R$. Uniqueness of $B$ is easily obtained from equations (1.17). $\square$

Let $f(u) \in 1 + hk((u))[[h]]$, and

$$R^f(u) = f(u)R(u).$$

Then $R^f_Y(u)$ satisfies (3.7). Thus we obtain the following proposition.

**Proposition 3.17.** For any $f \in 1 + hk((u))[[h]]$, there exists a unique $\partial$-copseudotriangular structure on $F(R)$ such that

$$B_f(T^{13}(u), T^{23}(v))(y) = (R^f)^{12}(u - v - y).$$

Let $f_0(u)$ be defined by the condition that $q\text{det}(R^f_0(u)) = 1$. By Proposition 3.17, $f_0$ defines a $\partial$-copseudotriangular structure $B_{f_0}$ on $Y(gl_N)'_{op}$. Since $q\text{det}(R^f_Y(u)) = 1$, this $\partial$-copseudotriangular structure descends to one on $Y(gl_N)'_{op}$. Thus, using Proposition 3.2, and the fact that $\partial = d$ in this case, we get the following proposition.
**Proposition 3.18.** There exists a unique copseudotriangular structure on $Y(\mathfrak{sl}_N)_\text{op}$ such that

$$B(T^{13}(u), T^{23}(v))(y) = (R^t_Y)^{12}(u - v + y).$$

This structure coincides with the one defined by (3.4).

### 3.6. Elliptic R-matrix algebras.

Here we consider R-matrix algebras associated to elliptic solutions of the quantum Yang-Baxter equation.

Let $\Sigma$ be an elliptic curve over $k$, and $k(\Sigma)$ be the field of rational functions on $\Sigma$. Let $du$ be an invariant differential on $\Sigma$, and $u$ the corresponding formal parameter near the origin. This parameter defines an embedding $k(\Sigma) \to k((u))$. Let $\Gamma$ be the group of points of order $N$ on $\Sigma$.

**Proposition 3.19.** There exists a unique element $R(u, h) \in \text{End}(k^N \otimes k^N) \otimes k(\Sigma)[[h]]$, such that $R = 1 + O(h)$, satisfying the following conditions.

(i) Let $R(u + h/N^2, h) = \sum_{m \geq 0} \rho_m(u)h^m$. Then $\rho_m(u)$ have at most simple poles at points of $\Gamma$, and no other singularities.

(ii) Quasiperiodicity: if $\gamma \in \Gamma$, then $R(u + \gamma, 1) = (1 \otimes \lambda(\gamma))(R(u)) = (\lambda(\gamma)^{-1} \otimes 1)(R(u))$, where $\lambda$ is as in Example 3 of Section 2.7.

(iii) $R(0) := \sum_{m \geq 0} \rho_m(-h/N^2)h^m = N\sigma$, where $\sigma$ is the permutation matrix.

**Proof.**

Existence. Let $r_{\text{ell}}(u)$ be the elliptic R-matrix on $\Sigma$ introduced in Example 3 of Section 2.7. For $\mathfrak{g} = \mathfrak{sl}_N$, the Casimir tensor $\Omega$ introduced in Chapter 2, is equal to $\frac{\sigma-1/N}{N}$, so $r_{\text{ell}}(u) = \frac{\sigma-1/N}{Nu} + O(1)$. Set

$$R(u, h) = N\sigma - h[r_{\text{ell}}(u - h/N^2) - r_{\text{ell}}(-h/N^2)].$$

It is easy to check that $R(u, h)$ satisfies conditions (i)-(iii).

Uniqueness. Let $R_1, R_2$ be two functions satisfying (i)-(iii). Let $R_1 - R_2 = h^mX + O(h^{m+1})$, $X \in \text{End}(k^N \otimes k^N) \otimes k(\Sigma)$.

From properties (i)-(iii) of $R_i$ we get the following properties of $X$, respectively:

(i)’ $X(u)$ has at most simple poles at $\Gamma$, and no other singularities.

(ii)’ $X(u + \gamma) = (1 \otimes \lambda(\gamma))(X(u)) = (\lambda(\gamma)^{-1} \otimes 1)(X(u))$, $\gamma \in \Gamma$.

(iii)’ $X(u)$ is regular at $u = 0$.

It is easy to see that any element $x \in \mathfrak{sl}_N$ which is invariant under the operators $\lambda(\gamma)$ is a scalar. Therefore, properties (i)’-(iii)’ imply that $X$ is a scalar constant. Let $R_1 - R_2 = 1 + h^mX + h^{m+1}Z(u) + O(h^{m+2})$. The function $Z(u)$ satisfies properties (i)’,(ii)’. Let $Z_0$ be the residue of $Z(u)$ at $u = 0$. From condition (iii) we get $X - N^2Z_0 = 0$. On the other hand, from conditions (i)’,(ii)’ for $Z(u)$ we get $\sum_{\gamma \in \Gamma}(\lambda(\gamma) \otimes 1)(Z_0) = 0$. This implies that $Z_0 = X = 0$, as desired. \qed

It is easy to check that the free term in the Laurent expansion of $r_{\text{ell}}(u)$ vanishes. This implies that $R(u, h) = 1 - hr_{\text{ell}}(u) + O(h^2)$. We will denote $R$ by $R_{\text{ell}}$.

**Proposition 3.22.** $R_{\text{ell}}(u, h)$ satisfies the quantum Yang-Baxter equation (3.7).

**Proof.** See [Be1,Ch2, Ta]. \qed
Corollary 3.23. The algebras $F(R_{\text{ell}})$, $A(R_{\text{ell}})$, $U(R_{\text{ell}})$ are flat deformations. The algebra $U(R_{\text{ell}})$ is a quantization of the Lie bialgebra $\mathfrak{gl}_{N}(r_{\text{ell}})$.

Proof. Follows from Proposition 3.9. □

3.7. Trigonometric R-matrix algebras.

In this section we consider the limit of the R-matrix of Section 3.6 when the elliptic curve degenerates into the multiplicative group $G$-algebra $U$.

Let $g = \mathfrak{sl}_N$. Let $\Sigma = \mathbb{G}_m$, $\Gamma$ be the group of points of order $N$, and $\lambda: \Gamma \to \text{Aut}(g)$ be the homomorphism defined by the rule $\lambda(\varepsilon) = \text{Ad}(\text{diag}(1, \varepsilon^r, ..., \varepsilon^{r(N-1)}))$, where $1 \leq r \leq N$ and $r$ is relatively prime to $N$.

Let $r_{\text{tr}}(u)$ be the rational function on $\Sigma$ with values in $g \otimes g$ such that $r_{\text{tr}}(u) = \frac{\Omega}{u} + O(1)$, $u \to 0$, $r_{\text{tr}}$ is regular in $\Sigma \setminus \Gamma$, where $\Sigma = \mathbb{P}^1$ is the projective closure of $\Sigma$, and $r_{\text{tr}}(u + \gamma) = (1 \otimes \lambda(\gamma))(r_{\text{tr}}(u))$ for $\gamma \in \Gamma$. It is clear that such a function is unique. It is obtained from $r_{\text{ell}}(u)$ (for a suitable choice of $\lambda$) by the limiting procedure in which the elliptic curve degenerates into a rational curve.

It is easy to write an explicit formula for $r_{\text{tr}}$ using the global coordinate on $\Sigma$. This formula can be found in [BD]. Namely, $r_{\text{tr}}$ can be obtained from the r-matrix (2.13) by the transformation described in Proposition 2.5.

Now we consider the quantum R-matrix. Define the function

$$R_{\text{tr}}(u, h) := N\sigma - h[r_{\text{tr}}(u - \frac{h}{N^2}) - r_{\text{tr}}(-\frac{h}{N^2})].$$

As this function is a limiting case of $R_{\text{ell}}(u, h)$, Proposition 3.22 holds for $R_{\text{tr}}$. Therefore, we have

Corollary 3.24. The algebras $F(R_{\text{tr}})$, $A(R_{\text{tr}})$, $U(R_{\text{tr}})$ are flat deformations. The algebra $U(R_{\text{tr}})$ is a quantization of the Lie bialgebra $\mathfrak{gl}_{N}(r_{\text{tr}})$.

3.8. The factored Hopf algebras $F(R)_{\mathbf{z}}$, $U(R)_{\mathbf{z}}$.

Here we quantize the Lie bialgebra $g(r)_{\mathbf{z}}$, where $g = \mathfrak{gl}_N$, and $r$ is a classical r-matrix which has a quantization $R$.

Let $\Sigma$ be a 1-dimensional algebraic group over $k$, and $u$ be a canonical formal parameter near the origin. Let $R \in \text{End}(k^{\Sigma} \otimes k^{\Sigma})([h])$ be a function of the form

$R = 1 - hr(u) + O(h^2)$

which satisfies the conditions of Proposition 3.9 (b). Let $A = A(R)$ be the h-formal group corresponding to $R$, then $U_A = U(R)$. Let $\mathbf{z} = (z_1, ..., z_n)$, $z_i \in \Sigma$, and $R(u)$ is regular at $z_i - z_j$ for $i \neq j$. In this case we can define the factored Hopf algebras $F(R)_{\mathbf{z}}$, $A(R)_{\mathbf{z}}$, $U(R)_{\mathbf{z}}$ using the $\partial$-copseudotriangular structure on $A(R)$ defined in Section 3.5.

Proposition 3.25. (i) The Hopf algebra $F(R)_{\mathbf{z}}$ is isomorphic to the $h$-adic completion of the algebra over $k[[h]]$ whose generators are the entries of formal series $T_i(u)^{\pm 1} \in \text{End}(k^{\Sigma}) \otimes F(R)_{\mathbf{z}}[[u]]$, $T_i(u) = T_0^i + T_1^i u + ...$, and the defining relations are

$$T_i(u)T_j(u)^{-1} = T_i(u)^{-1}T_j(u) = 1,$$

$$R^{12}(u - v + z_i - z_j)T_i^{13}(u)T_j^{23}(v) = T_j^{23}(v)T_j^{13}(u)R^{12}(u - v + z_i - z_j),$$

and the coproduct is defined by

$$\Delta(T_i(u)) = T_i^{12}(u)T_i^{13}(u).$$
(ii) The algebra $F(R)_z$ is a quantization of the Poisson group $G(r)_z$.

Proof. By the definition, we have Hopf subalgebras $F(R)_{z_i} \subset F(R)_z$ for all $i$, and multiplication induces an isomorphism $F(R)_{z_1} \otimes \ldots \otimes F(R)_{z_n} \to F(R)_z$. Let $T_i(u)$ be the image of the generating series $T(u)$ of $F(R)_{z_i}$ under the embedding $F(R)_{z_i} \to F(R)_z$. Using Proposition 3.16 and the definition of $F(R)_z$, it is easy to show that $T_i(u)$ satisfy relations (3.20). Thus, we have a surjective homomorphism from the algebra defined by (3.20) to $F(R)_z$. This homomorphism is an isomorphism modulo $h$, so it is an isomorphism.

Relation (3.21) is obvious. The fact that $F(R)_z$ is a quantization of $G(r)_z$ follows from the fact that $F(R)_z$ is a flat deformation. The Proposition is proved. □

Corollary 3.26. (i) The Hopf algebra $U(R)_z$ is isomorphic to the h-adic completion of the algebra over $k[[h]]$ whose generators are the entries of formal series $t_i(u)^{\pm 1} \in \text{End}(k^N) \otimes U(R)_z[u]$, $t_i(u) = t_i^0 + t_i^1 u + \ldots$, and the defining relations are

$$[t_i^{12}(u), t_j^{23}(v)] = [r_{ij}^{12}(u - v + z_i - z_j), t_i^{12}(u) + t_j^{23}(v)] +$$

$$h(r_{ij}^{12}(u - v + z_i - z_j)t_i^{12}(u)t_j^{23}(v) - t_j^{23}(v)t_i^{12}(u)r_{ij}^{12}(u - v + z_i - z_j)),$$

where $r_{ij} = h^{-1}(1 - R)$, and the coproduct is defined by

$$\Delta(t_i(u)) = t_i^{12}(u) + t_i^{13}(u) + h t_i^{12}(u)t_i^{13}(u).$$

(ii) The algebra $U(R)_z$ is a quantization of the Lie bialgebra $\mathfrak{g}(r)_z$.

Appendix A: Calculation of the Square of the Antipode

In this appendix we calculate the square of the antipode $S$ of a quantized universal enveloping algebra $U_h(\mathfrak{g}_+)$ obtained by quantization of a Lie bialgebra $\mathfrak{g}_+$ via the procedure of [EK1], Chapter 7-9. We will freely use the notation from [EK1].

Let $\mathfrak{g}_+$ be a Lie bialgebra, and $\mathfrak{g}$ the double of $\mathfrak{g}_+$. Let $M_-, M_+$ be the Verma modules for $\mathfrak{g}$, and $M^*_+$ the dual Verma module (see Chapter 7 of [EK1]).

Let $1_-, 1_+$ be the generating vectors in the Verma modules $M_+, M_-$ defined in Section 7.5 of [EK1], and $1^*_+ \in M^*_+$ be the $\mathfrak{g}$-invariant functional on $M_+$ normalized by $1^*_+(1_+) = 1$. Let $(M^*_+)\downarrow$ be the orthogonal complement to $1_+$ in $M^*_+$ (see the Appendix of [EK1]).

Let $\mathcal{M}^e$ be the Drinfeld category for $\mathfrak{g}$, defined in Chapter 7 of [EK1]. Let $F$ be the fiber functor on $\mathcal{M}^e$ defined in Chapter 8 of [EK1], given by $F(V) = \text{Hom}_{\mathcal{M}^e}(\mathcal{M}^e, M^*_+ \otimes V)$. Let $U_h(\mathfrak{g}) := \text{End}(F)$. For any $V \in \mathcal{M}^e$, we identify $F(V)$ and $V[[h]]$ using the map $\xi_V : F(V) \to V[[h]]$ defined by $\xi(a) = (1_+ \otimes 1)(a1_-).$

Set $M^h_+ = F(M_-)$, $M^h_+ = F(M^*_+)$, and denote by $1^h_+ \in F(M^*_+)$, $1^h_- \in M^*_+$, $(M^*_+)^h_1 \subset M^h_+$ the images of $1^*_+, 1_-, (M^*_+)\downarrow$ under $\xi$. Let $1^h_+ : M^h_+ \to k[[h]]$ be the linear function which vanishes on $(M^*_+)^h_1$ and $1^h_+(1^h_+) = 1$.

Recall that the quantization $U_h(\mathfrak{g}_+) \otimes \mathfrak{g}_+$ defined in Chapter 9 of [EK1] is equal, as a vector space, to $F(M_-) = \text{Hom}(M_-, M^*_+ \otimes M_-)$.

Proposition A1. For any $a \in U_h(\mathfrak{g}_+)$, $S^2(a) = \gamma^2 \circ a$, where $\gamma$ is the braiding in $\mathcal{M}^e$ defined in Section 7.7 of [EK1].

Proof. Let $R \in \text{End}(F \times F)$ be the universal $R$-matrix of $U_h(\mathfrak{g})$, i.e. $F(\gamma) = (\sigma R)^{-1}$, where $\sigma$ is a permutation. We can represent $R$ as an infinite sum $R = \sum a_\alpha \otimes b_\alpha$. 


\( a_\alpha \in U_h(g_+), b_\alpha \in U_h(g_-) \), where \( g_- \) is the dual algebra to \( g_+ \). This series is convergent in the appropriate topology. Set \( u := \sum S^{-1}(a_\alpha)b_\alpha \in U_h(g) \) (it is easy to see that this series is also convergent).

The following properties of \( u \) were found by Drinfeld [Dr3]:

(i) \( u^{-1}xu = S^2(x) \) in \( U_h(g) \), where \( x \in U_h(g_+) \).

(ii) \( \Delta(u) = (u \otimes u)(R^{21}R)^{-1} \).

Now we are ready to prove the proposition. Since the functor \( F \) is faithful, it is enough to check the identity

\[
(A1) \quad F(S^2(a)) = F(\gamma^2) \circ F(a) \text{ in } \text{Hom}_{U_h(g)}(M^h_+, M^h_+ \otimes M^h_-).
\]

Applying the transformation \( x \to (1_+^h \otimes 1)(x1_+^h) \) to both sides of (A1), and using the fact that \( F(\gamma^2) = (R^{21}R)^{-1} \), we obtain an equivalent identity

\[
(A2) \quad S^2(a) = (1_+^h \otimes 1)((R^{21}R)^{-1}F(a)1_+^h) \text{ in } M^h_+.
\]

Using property (ii) of \( u \), we can rewrite this identity in the form

\[
(A3) \quad S^2(a) = (1_+^h \otimes 1)((u^{-1} \otimes u^{-1})\Delta(u)F(a)1_+^h).
\]

Since \( (\varepsilon \otimes 1)(R) = (1 \otimes \varepsilon)(R) = 1 \), we see that \( u1_+^h = 1_+^h \), \( u(M^*_+) = (M^*_+) \). Using these properties of \( u \), we reduce (A3) to the form

\[
(A4) \quad S^2(a) = (1_+^h \otimes 1)((1 \otimes u^{-1})F(a)1_+^h).
\]

The right hand side of (A4) equals \( u^{-1} \ast a \), where \( u^{-1} \ast v \) denotes the action of \( u^{-1} \) on \( v \in F(V) \). Thus, we have reduced (A2) to

\[
(A5) \quad S^2(a) = u \ast a.
\]

On the other hand, from property (i) of \( u \), and the equation \( u1_+^h = 1_+^h \), we get \( S^2(a)1_+^h = uau^{-1}1_+^h = u1_+^h = u \ast a \), which coincides with (A5). The proposition is proved. \( \square \)

Let \( g_\alpha \) be a basis of \( g_+ \), \( g_\alpha^* \) the dual basis of \( g_- = g_+^* \), and \( C_+ = \sum_\alpha g_\alpha g_\alpha^* \) be the half-Casimir. \( C_+ \) is an endomorphism of the forgetful functor to \( k[[h]] \)-modules on the category \( \mathcal{M}^e \).

**Proposition A2.** We have

\[
(A6) \quad \xi_{M_-} \circ S^2 = e^{hC_+} \circ \xi_{M_-}.
\]

**Proof.** We start with a tautological identity

\[
(A7) \quad (1_+ \otimes 1)((1 \otimes e^{hC_+})a1_-) = e^{hC_+}(1_+ \otimes 1)(a1_-).
\]

Let \( \Omega \) be the Casimir operator for \( g \) (see Chapter 7 of [EK1]). Since \( \Omega = \Delta(C_+) - C_+ \otimes 1 - 1 \otimes C_+ \), and \( C_+1_- = 0 \), \( S(C_+)1_+ = 0 \), formula (A7) implies

\[
(A8) \quad (1_+ \otimes 1)(e^{-h\Omega}a1_-) = e^{hC_+}(1_+ \otimes 1)(a1_-).
\]

By Proposition A1, \( S^2(a) = \gamma^2 \circ a \). By the definition, in \( \mathcal{M}^e \) we have \( \gamma^2 = e^{-h\Omega} \); Thus, (A8) reduces to the form

\[
(A9) \quad (1_+ \otimes 1)(S^2(a)1_-) = e^{hC_+}(1_+ \otimes 1)(a1_-),
\]

which is equivalent to (A6). The proposition is proved. \( \square \)
**Proposition A3.** Let $d$ be the derivation of $\mathfrak{g}_+$ given by the formula $d = -\frac{1}{2} \mu \circ \delta$, where $\mu, \delta$ are the commutator and cocommutator in $\mathfrak{g}_+$. Let $D = \frac{1}{\hbar} \ln S^2$ be the canonical derivation of $U_h(\mathfrak{g}_+)$. Then the image of $d$ under the functor $U_h$ equals to $D$.

**Proof.** We first prove the following Lemma.

**Lemma.** For any $x \in \mathfrak{g}_+$, $dx = [C_+, x]$ in any $V \in \mathcal{M}$.\[\text{(A10)} [C_+, x] = \sum (g_\alpha [g_\alpha^*, x] + [g_\alpha, x] g_\alpha^*)\]

For any $y^* \in \mathfrak{g}_-$, we have $[y^*, x] = (1 \otimes y^*)(\delta(x)) - ad^* x(y^*)$. Using this identity and the invariance of the pairing between $\mathfrak{g}_+ \mathfrak{g}_-$ under $x$, we get\[\text{(A10)} [C_+, x] = \sum g_\alpha (1 \otimes g_\alpha^*)(\delta(x)) = m_{21}(\delta(x)),\]

where $m_{21}(a \otimes b) := ba$. Since $\delta(x)$ is skew symmetric, we have $m_{21}(\delta(x)) = -\frac{1}{2} \mu(\delta(x)) = dx$. The Lemma is proved.

Now we prove Proposition A3. It follows from proposition A2 that after $U_h(\mathfrak{g}_+)$ is identified with $M_-[[h]] = U(\mathfrak{g}_+)[[h]]$, as a vector space, via $\xi_{M_-}$, the derivation $D$ is given by $Dx = [C_+, x]$. Therefore, by the lemma, $D = d$, as desired. □

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