Mean-field theory of a quasi-one-dimensional superconductor in a high magnetic field

N. Dupuis
Laboratoire de Physique des Solides, Université Paris-Sud, 91405 Orsay, France

Abstract

At high magnetic field, the semiclassical approximation which underlies the Ginzburg-Landau (GL) theory of the mixed state of type II superconductors breaks down. In a quasi-1D superconductor (weakly coupled chains system) with an open Fermi surface, a high magnetic field stabilizes a cascade of superconducting phases which ends in a strong reentrance of the superconducting phase. The superconducting state evolves from a triangular Abrikosov vortex lattice in the semiclassical regime towards a Josephson vortex lattice in the reentrant phase. We study the properties of these superconducting phases from a microscopic model in the mean-field approximation. The critical temperature is calculated in the quantum limit approximation (QLA) where only Cooper logarithmic singularities are retained while less divergent terms are ignored. The effects of Pauli pair breaking (PPB) and impurity scattering are taken into account. The Gor’kov equations are solved in the same approximation but ignoring the PPB effect. We derive the GL expansion of the free energy and obtain the specific heat jump at the transition. We find that each phase is first paramagnetic and then diamagnetic for increasing field, except the reentrant phase which is always paramagnetic. We also show that a gap opens at the Fermi level in the quasi-particle excitation spectrum. The QLA clearly shows how the system evolves from a quasi-2D and BCS-like behavior in the reentrant phase towards a gapless behavior at weaker field. The calculation is extended beyond the QLA taking into account all the pairing channels and the validity of the QLA is discussed in detail. We show that the complete excitation spectrum exhibits gaps at, below, and above the Fermi level. We also calculate the current distribution.

PACS numbers: 74.20-z, 74.70-Kn, 74.90+n, 74.60-w
I. INTRODUCTION

The equilibrium state of type II superconductors was first described by Abrikosov using a phenomenological Ginzburg-Landau (GL) theory, which was later justified by Gor’kov in a microscopic model. The Ginzburg-Landau-Abrikosov-Gor’kov (GLAG) theory treats the magnetic field semiclassically and therefore can be justified in clean materials only at high temperature or low magnetic field. In the last few years, there has been a lot of work devoted to the theoretical understanding of the effect of magnetic fields on the mean-field theory of the superconducting instability from a completely quantum point of view.

Most of these works have been concerned with the effects of Landau level quantization in superconductors with an isotropic dispersion law. On the one hand, the quantum effects of the field have been studied in the vicinity of the semiclassical critical field $H_{c2}(T = 0)$, with emphasis on the precise vortex lattice structure, the quasi-particle excitation spectrum, and the de Haas-van Alphen oscillations arising in the mixed state as a consequence of Landau level quantization. The first observation of these quantum magnetic oscillations in the mixed state occurred nearly twenty years ago in the layer compound $2H$-NbSe$_2$ and interest has been renewed recently with their observation in several other materials. On the other hand, it has been proposed that Landau level quantization can lead to reentrant behavior at very high magnetic field when the cyclotron energy becomes larger than the Fermi energy ($\omega_c \gg E_F$). This effect is absent from the GLAG theory which predicts a complete disappearance of the superconducting phase due to the orbital frustration of the order parameter in the magnetic field. This reentrant behavior originates in the suppression of the orbital frustration when the electrons reside in only one or the few lowest Landau levels. Indeed, when only one Landau level is occupied, the supercurrents can be made to coincide with the orbital motion of the electrons in this Landau level if the periodicity of the vortex lattice is approximately equal to the orbit radius of the lowest Landau level. Moreover, in this very high field limit, it has been argued that the destruction of superconductivity by the Pauli pair breaking (PPB) effect can be avoided because the effective 1D dispersion law allows one to construct a Larkin-Ovchinnikov-Fulde-Ferrell (LOFF) state which can exist far above the Pauli limited field. The reality of this reentrant superconductivity remains however controversial and there has been no experimental result up to now.

The quantum effects of the magnetic field were also studied in the case of quasi-one-dimensional superconductors (weakly coupled chains systems) with an open Fermi surface. These effects are especially pronounced when the zero-field critical temperature $T_{c0}$ is smaller (but not much smaller) than the interchain coupling $t_z$ in the direction perpendicular to the field (we will only consider this limit in this paper). (The chains are parallel to the $x$ axis. The external field is along the $y$ direction and the interchain hopping $t_y$ in this direction is assumed to be much larger than $t_z$). In this case, the superconductivity is well described semiclassically by the anisotropic GL theory. In particular, there is no Josephson coupling between chains even at $T = 0$. Because of the quasi-1D structure of the Fermi surface, the semiclassical orbits in the presence of the field are open. Consequently, there is no Landau level quantization but the field induces a 3D/2D crossover to a Larkin-Ovchinnikov-Fulde-Ferrell (LOFF) state which can exist far above the Pauli limited field. The electronic motion remains extended along the chains and along the direction parallel to the field, but becomes confined in the $z$ direction with an extension $\sim c t_z / \omega_c$ ($c$ is the interchain spacing in the $z$ direction and $\omega_c$ is the frequency of the semiclassical orbits). This dimen-
sional crossover is at the origin of a very unusual phase diagram. In particular, it leads to a restoration of time-reversal symmetry (as far as the Zeeman splitting is ignored) in very high magnetic field ($\omega_c \gg t_z$) which results in a reentrant behavior of the superconducting phase with a Josephson coupling between chains. This high-field-superconductivity can survive even in the presence of PPB because the quasi-1D Fermi surface allows one to construct a LOFF state (for any value of the magnetic field) which can exist far above the Pauli limited field. Although the origin of the reentrant behavior is very different in the quasi-1D case and in the isotropic case, in both cases it appears as a consequence of a reduction of dimensionality, from 3D to 2D in the quasi-1D case, and from 3D to 1D in the isotropic case. The suppression of the orbital frustration originates in this reduction of dimensionality.

Besides the qualitative differences between isotropic and quasi-1D superconductors, there is also an important quantitative difference. In the isotropic case, the temperature and magnetic ranges where quantum effects are expected to be important are determined by the Fermi energy $E_F$. For this reason, superconductivity is destroyed for intermediate fields, i.e. for fields much larger than in the semiclassical regime but much smaller than in the reentrant regime. Moreover, the reentrant behavior can be observed only at very low temperatures and very high fields. This restricts considerably the possible candidates to the experimental observation of very high-field superconductivity and is one of the reasons which explain the absence of experimental results. In the quasi-1D case, it is the coupling $t_z$ between chains which plays the crucial role. Since $t_z$ can be smaller than 10 K in organic conductors, the temperature and magnetic field ranges where very high-field superconductivity is expected can be experimentally accessible if the appropriate (i.e., sufficiently anisotropic) materials are chosen. This also means that superconductivity can survive even for intermediate fields between the GL and the very high field regimes. The interest in quasi-1D superconductors has been recently raised by experimental results on the organic compound (TMTSF)$_2$ClO$_4$. Resistive measurements have shown an anomalous behavior of the critical field $H_{c2}$. Although they do not give a definite answer for the existence of high-field superconductivity in (TMTSF)$_2$ClO$_4$, these results might be interpreted as the signature of a high-field superconducting phase.

The main features of the phase diagram of a quasi-1D superconductor are now well understood. Between the GL regime (where the superconducting state is a triangular Abrikosov vortex lattice) and the reentrant phase (where the superconducting state is a triangular Josephson vortex lattice), the magnetic field stabilizes a cascade of superconducting phases separated by first order transitions. In these quantum phases, the behavior of the system (and in particular the periodicity of the order parameter) is not determined any more by the (semiclassical) GL coherence length $\xi_z(T) \sim 1/\sqrt{T}$ but by the transverse (i.e., perpendicular to the chains) magnetic length $ct_z/\omega_c \sim 1/H$ ($H$ being the external magnetic field). When entering the quantum regime, $\omega_c \sim T$, the transverse magnetic length is much larger than the GL coherence length $\xi_z(T)$. This results in an increase of the transverse periodicity of the order parameter and in a strong modification of the vortex lattice. In the quantum regime, the amplitude of the order parameter and the current distribution show a symmetry of a laminar type while the vortices still describe a triangular lattice. The existence of this somehow new superconducting state is due to the symmetry of the one-particle wave-functions which is incompatible with the symmetry of the Abrikosov vortex lattice. The cascade of first order phase transitions originates in commensurability effects.
between the periodicity of the order parameter (i.e., the transverse magnetic length) and
the crystalline lattice spacing.

In this paper, we study the transition line and the properties of the superconducting
phases in the mean-field approximation starting from a microscopic model. Some of the
results presented here were published elsewhere. We assume that the superconductivity
is due to an effective attractive electron-electron interaction of the BCS type. We also
assume that the quasi-1D conductor is well described above the transition line by the Fermi
liquid theory, which justifies the use of a mean-field theory. This situation will be realized
if the system undergoes a single particle dimensionality crossover at a temperature $T_{x1} > T_c$. Below $T_{x1}$, the particle-particle (Cooper) and particle-hole (Peierls) channels decouple
so that the usual mean-field (or ladder) approximation is justified provided that the bare
parameters of the Hamiltonian are replaced by renormalized ones in order to take into
account the effects of 1D fluctuations (see also Refs.[19, 21] for a discussion of the validity
of the mean-field approximation, in particular for the organic conductors of the Bechgaard
salts family). As pointed out by Yakovenko, it is very important that the electrons in
the $(x, y)$ planes have a 2D behavior below $T_{x1}$. Even if the magnetic field suppresses
the electron hopping in the $z$ direction, it has no effect on the electron motion in the $(x, y)$
planes and the Cooper and Peierls channels remain decoupled. This does not exclude
the existence of thermodynamical fluctuations, in particular in the reentrant phase ($\omega_c \gg t_z$)
where the system becomes effectively quasi-2D. In this very high field limit, the transition
from the metallic phase towards a phase with real superconducting long-range order might
be replaced by a Kosterlitz-Thouless transition. This aspect will however not be considered
any further and we will restrict ourselves to the mean-field analysis.

In the next section, we calculate the eigenstates and the Green’s functions of the normal
phase in the presence of a uniform magnetic field $\mathbf{H}(0, H, 0)$. We use the gauge $\mathbf{A}(Hz, 0, 0)$
which presents the advantage to yield a very clear physical picture of the dimensional
crossover induced by the magnetic field. In Sec. we derive the transition line in the quan-
tum limit approximation (QLA) where only Cooper logarithmic singularities are retained
while less divergent terms are ignored. Although this approximation strongly underestimates
the critical temperature, it provides a clear physical picture of the pairing mechanism re-
 sponsible for the superconducting instability. Moreover, the effects of disorder and PPB can
easily be incorporated in this approach. In Sec. we study the superconducting phases
in the QLA ignoring the PPB effect. We first construct a variational order parameter using
the results of Sec. and then solve the Gor’kov equations. We derive the GL expansion
of the free energy and obtain the specific heat jump at the transition. The discontinuity
of the specific heat jump at the first order transition is related to the slope of the first order
transition lines. We find that each phase is first paramagnetic and then diamagnetic for
increasing field, except the reentrant phase which is always paramagnetic. We also show that
a gap opens at the Fermi level in the quasi-particle excitation spectrum. The QLA clearly
shows how the system evolves from a quasi-2D and BCS-like behavior in the reentrant phase
towards a gapless behavior at weaker field. In Sec. we go beyond the QLA. We first
obtain the transition line and construct a variational order parameter, thus recovering the
results obtained in Ref. in the gauge $\mathbf{A}'(0, 0, -Hx)$. We then derive the GL expansion
of the free energy. We discuss the importance of the screening of the external field by the
supercurrents and also compare the results with those obtained in the QLA. The quasi-
particle excitation spectrum is obtained from the Bogoliubov-de Gennes equations. Besides

gaps which open at the Fermi level as obtained in the QLA, gaps open below and above the
Fermi level. This excitation spectrum is very reminiscent of the one of the field-induced-spin-
density-wave (FISDW) phases which appear when the effective electron-electron interaction

is repulsive. Finally, the current distribution is calculated.

II. GREEN’S FUNCTIONS OF THE NORMAL PHASE

In this section, we derive the Green’s functions in the normal metallic phase in the

presence of a uniform magnetic field \( H(0,H,0) \). Contrary to what can be found in general

in the literature concerning quasi-1D conductors in a magnetic field, we work in the gauge

\( A(Hz,0,0) \). The one-particle Hamiltonian is obtained from the Peierls substitution

\( \mathcal{H}_0 = E(k \rightarrow -i\nabla - eA) \). The dispersion law is given by \( (\hbar^2 = k_B = 1 \text{ throughout the paper and}

the Fermi energy } E_F \text{ is chosen as the origin of the energies) }

\[
E(k) = v(|k_x| - k_F) + t_y \cos(k_yb) + t_z \cos(k_zc),
\]

where \( v \) is the Fermi velocity for the motion along the chains \( (x \text{ axis}) \) and \( t_y, t_z \) are the

couplings between chains separated by the distance \( b \) and \( c \). The condition \( t_y, t_z \ll E_F \)

ensures that the Fermi surface is open. Except in a few cases which will be pointed out

when necessary, we will not explicitly consider the \( y \) direction parallel to the magnetic field

which does not play any role for a linearized dispersion law (as long as Cooper pairs are

formed with states of opposite momenta in this direction). In order to take into account the

\( y \) direction, we just have to replace the 2D density of states per spin \( N(0) = 1/\pi vc \)

by its 3D value \( 1/\pi vbc \). It should be noted here that no generality is lost at the mean-field level

when studying a 2D system instead of a 3D system. This is due to the fact that the kinetic

energy mainly comes from the motion along the chains, which is not affected by the field (see

below), so that the electron-electron interaction can still be treated in perturbation for a 2D

system. This should be contrasted with 2D isotropic systems in high magnetic field where

the perturbative treatment (i.e. the mean-field analysis) of the superconducting instability

is highly questionable.

Since the magnetic field does not couple the two sheets of the Fermi surface, we can

write an Hamiltonian for each sheet of the Fermi surface:

\[
\mathcal{H}_{0,\sigma}^\alpha = v(-i\alpha \partial_x - k_F) + \alpha \hat{m} \omega_c + t_z \cos(-ic\partial_z) + \sigma h ,
\]

where \( \alpha = +(-) \) labels the right (left) sheet of the Fermi surface. \( \hat{m} \) is the (discrete)

position operator in the \( z \) direction and \( \sigma = +(-) \) for \( \uparrow (\downarrow) \) spin. \( \sigma h = \sigma \mu_B H \)

is the Zeeman energy for a \( g \) factor equal to 2. We have introduced the energy \( \omega_c = Gv \) where

\( G = -eHc \) is a magnetic wave vector. The operator \( -i\partial_x \) commutes with the Hamiltonian

so that the momentum \( k_x \) along the \( x \) axis is a good quantum number. We therefore look

for a solution \( \phi_{k_z}^\alpha(x,m) = e^{ik_x x} \phi_{k_z}^\alpha(m) \). The Fourier transform \( \phi_{k_z}^\alpha(k_z) \) of \( \phi_{k_z}^\alpha(m) \) is solution

of the Schrödinger equation (setting \( c = 1 \) for simplicity)

\[
[i\alpha \omega_c \partial_{k_z} + t_z \cos(k_z)]\phi_{k_z}^\alpha(k_z) = \tilde{c}\phi_{k_z}^\alpha(k_z) ,
\]
where \( \tilde{\epsilon} = \epsilon - v(\alpha k_x - k_F) - \sigma h \) and \( \epsilon \) is the eigenenergy of the eigenstate \( \phi_{k_x}^\alpha(x, m) \). The solution of (2.3) is

\[
\phi_{k_x}^\alpha(k_x) \sim e^{-\frac{i}{\omega_c}[(k_x - L_x) + t_z \sin(k_z)]]}
\]

up to a normalization factor. Going back to real space, we obtain

\[
\phi_{k_x}^\alpha(x, m) \sim \int_{-\pi}^{\pi} \frac{dk_z}{2\pi} e^{i(k_x(m - \frac{L_x}{c}) + t_z \sin(k_z))} \]

(2.5)

\( \phi_{k_x}^\alpha(x, m) \) is non zero only if \( \tilde{\epsilon} = \alpha \omega_c \) where \( l \) is an integer. Therefore, the normalized eigenstates and the eigenenergies of the Hamiltonian \( \mathcal{H}_{x,0,\sigma}^\alpha \) are given by (writing explicitly the interchain spacing \( c \))

\[
\phi_{k_x,l}^\alpha(x, m) = \frac{1}{\sqrt{c L_x}} e^{ik_x x} J_{l-m}(\alpha \tilde{\epsilon}),
\]

(2.6)

\[
\epsilon_{k_x,l,\sigma} = v(\alpha k_x - k_F) + \alpha \omega_c + \sigma h,
\]

(2.7)

where \( r = (x, m) \) and \( J_l \) is the \( l \)th order Bessel function. \( L_x \) is the length of the system in the \( x \) direction and \( \tilde{\epsilon} = t_z / \omega_c \) is a reduced interchain coupling. The state \( \phi_{k_x,l}^\alpha \) is localized around the \( l \)th chain with a spatial extension in the \( z \) direction of the order of \( \tilde{\epsilon} c \) which corresponds to the amplitude of the semiclassical orbits. The advantage of working with the vector potential \( \mathbf{A}(Hz, 0, 0) \) is now clear: in this gauge, the eigenstates are localized, which corresponds to the real physical behavior of the particles as can be seen by examining the one-particle Green’s function in real space. Moreover, since the momentum \( k_z \) remains a good quantum number, the motion along the chains is in some sense trivial. Note that the states \( \phi_{k_x,l}^\alpha \) can be deduced by the appropriate gauge transformation from the localized states introduced by Yakovenko in the gauge \( \mathbf{A}'(0, 0, -Hx) \).

In the Hamiltonian (2.2), the magnetic field appears only through the additional term \( \alpha \hat{m} \omega_c \). The effect of the magnetic field is therefore similar to an “electric field” \(-\alpha \hat{v} H\) whose sign would depend on the sheet of the Fermi surface. This fictitious electric field introduces an additional “potential energy” \( \alpha m \omega_c \) on each chain \( m \) which competes with the hopping term \( t_z \cos(-i c \partial_z) \) and tends to prevent the electronic motion in the \( z \) direction. The localization in the \( z \) direction is almost complete when the difference of “potential energy” \( \omega_c \) between two chains is much larger than the transfer integral \( t_z \) between chains \( (\omega_c \gg t_z) \). The fact that an electric field can localize the wave functions (as obtained here in a tight-binding model) has been known for a long time. This effect has recently attracted a lot of attention in connection with the studies of semiconductor superlattices submitted to an electric field. The quantized spectrum which results from this localization is known as a Wannier-Stark ladder. The semiclassical picture of this effect yields the well-known Bloch oscillations of a band electron in an electric field. Continuing this analogy between electric field and magnetic field in a quasi-1D conductor, we can interpret the semiclassical trajectories \( z = z_0 + c(t_z / \omega_c) \cos(Gx) \) obtained from \( \mathcal{H}_0 \) as Bloch oscillations of the electrons in the magnetic field. Most of the effects which are induced by the magnetic field in a quasi-1D conductor can be understood as the consequence of these Bloch oscillations.

In the next section, it will be useful to use Green’s functions in the representation of the states \( \phi_{k_x,l}^\alpha \). Introducing the creation and annihilation operators \( b_{k_x,l,\sigma}^\dagger \), \( b_{k_x,l,\sigma}^\alpha \) of a particle of spin \( \sigma \) in the state \( \phi_{k_x,l}^\alpha \), we define the Matsubara Green’s function \( G^\alpha_{\sigma}(k_x, l, \omega) \) by
\[
G^\sigma_{\alpha}(k_x, l, \omega) = -\int_0^{1/T} d\tau e^{i\omega\tau} \langle T_\tau b^{\alpha}_{k_x, l, \sigma}(\tau) b^{\alpha \dagger}_{k_x, l, \sigma}(0) \rangle \\
= (i\omega - \epsilon^\alpha_{k_x, l, \sigma})^{-1},
\]
where \(\omega \equiv \omega_n = \pi T(2n + 1)\) is a Matsubara frequency.

### III. TRANSITION LINE IN THE QUANTUM LIMIT APPROXIMATION

The exact mean-field critical temperature \(T_c\) has been calculated numerically and discussed in detail elsewhere. In this section, we calculate \(T_c\) in the QLA, where only Cooper logarithmic singularities are retained while less divergent terms are ignored. Although this approximation yields a critical temperature several orders of magnitude smaller than the exact mean-field result, it provides a clear physical picture of the pairing mechanism responsible for the superconducting instability. It will also allow us to give a simple description of the properties of the ordered phase below \(T_c\) (see Sec. IV). Moreover, the effect of disorder can be easily incorporated at this level of approximation.

The total Hamiltonian is now \(\mathcal{H}_0 + \mathcal{H}_{\text{int}}\) where the effective attractive electron-electron Hamiltonian is described by the BCS model with coupling constant \(\lambda > 0\):

\[
\mathcal{H}_{\text{int}} = -\frac{\lambda}{2} \sum_{\alpha, \alpha'} \int d^2 r \, \psi_{\alpha}^{\sigma \dagger}(r) \psi_{\alpha'}^{\overline{\sigma} \dagger}(r) \psi_{\overline{\sigma}}^\sigma(r) \psi_\alpha^\sigma(r).
\] (3.1)

We use the notation \(\int d^2 r = c \sum_m \int dx\) and \(\overline{\sigma} = -\alpha, \sigma = -\sigma\). The \(\psi_\alpha^\sigma(r)\)'s are fermionic operators for particles moving on the sheet \(\alpha\) of the Fermi surface. The interaction is effective only between particles whose energies are within \(\Omega\) of the Fermi level.

We first note that the superconducting instability can be qualitatively understood from the Wannier-Stark ladder (2.7). In zero-field, time-reversal symmetry ensures that \(E_\uparrow(k) = E_\downarrow(-k)\) so that the pairing at zero total momentum presents the usual (Cooper) logarithmic singularity \(\sim \ln(2\gamma \Omega / \pi T)\) \((\gamma \sim 1.781\) is the exponential of the Euler constant\) which results in an instability of the metallic state at a finite temperature \(T_{c0}\). A finite magnetic field breaks down time-reversal symmetry. Nonetheless, we still have \(\epsilon^\alpha_{k_x, l_1, \uparrow} = \epsilon^\overline{\alpha}_{q_x - k_x, l_2, \downarrow}\) for \(q_x = -(l_1 + l_2)G\) (if we ignore the Zeeman splitting). Thus, whatever the value of the field, some pairing channels present the Cooper singularity if the total momentum along the chain \(q_x\) is a multiple of \(G\). This results in logarithmic divergences at low temperature in the linearized gap equation, which destabilize the metallic state at a temperature \(T_c\) \((0 < T_c < T_{c0})\). This reasoning holds also in presence of the Zeeman splitting if we consider pairing at total momentum \(q_x = -(l_1 + l_2)G \pm 2h/v\) for two bars \(l_1, l_2\) of the Wannier-Stark ladder. The shift \(\pm 2h/v\) of the total pairing momentum, which displaces the Fermi surfaces of spin \(\uparrow\) and spin \(\downarrow\) relative to each other, partially compensates the PPB effect and yields to the formation of a LOFF state.

Besides the most singular channels which present the Cooper singularity, there exist less singular channels with singularities \(\sim \ln |2\Omega / n\omega_c|\) \((n \neq 0)\) for \(T \ll \omega_c\) as will be shown below. In this quantum limit \((\omega_c \gg T)\), a natural approximation consists in retaining only the most singular channels. This QLA has been used previously in the mean-field theory of isotropic superconductors in a high magnetic field.
It is worth pointing out that the same kind of reasoning can explain the appearance of the FISDW phases in the presence of a repulsive electron-electron interaction. Even if we add to the Hamiltonian a second neighbor hopping term $t''_x \cos(2k_x c)$ (assumed to be large enough to suppress any SDW instability in zero magnetic field), the spectrum remains a Wannier-Stark ladder (Eq. (2.7)) although the expression of the eigenstates differ from (2.3). Since $\epsilon^\sigma_{k_x,l_1} = -\epsilon^\sigma_{k_x+l_1,2} + \pi$ for $Q_x = 2k_F + (l_1 - l_2)G$, some electron-hole pairing channels present logarithmic singularities if $Q_x - 2k_F$ is a multiple of $G$. At low temperature and high enough field, this leads to an instability of the metallic phase with respect to a SDW phase at wave vector $Q_x = 2k_F + NG$ ($N$ integer). Thus, the gauge $A(H_z, 0, 0)$ provides a very natural picture of the quantized nesting mechanism, which is at the origin of the FISDW phases in quasi-1D conductors. The QLA approximation has also been used in this context where it is known as the single gap approximation (SGA).

A. Without disorder and without PPB

We first consider the simplest case where both PPB and disorder are neglected. In order to obtain the critical temperature, we consider the two-particle vertex function $\Gamma^{\alpha\alpha'}(r_1, r_2; r'_1, r'_2)$ for a pair of particles on opposite sides of the Fermi surface and with opposite spins and Matsubara frequencies. $\Gamma^{\alpha\alpha'}$ is evaluated in the ladder approximation shown diagrammatically in Fig. 1. We first write the two-particle vertex function in the representation of the eigenstates $\phi^\alpha_{k_x,l}$:

$$\Gamma^{\alpha\alpha'}(r_1, r_2; r'_1, r'_2) = \frac{1}{cL_x} \sum_{q_x, k_x,l_1,l_2,k'_x,l'_1,l'_2} \sum_{\alpha, \alpha'} \Gamma^{\alpha\alpha'}_{q_x}(l_1, l_2; l'_1, l'_2) \times \phi^\alpha_{k_x,l_1}(r_1) \phi^{\alpha'}_{q_x-k_x,l_2}(r_2) \phi^{\alpha'}_{k'_x,l'_1}(r'_1) \phi^\alpha_{q_x-k'_x,l'_2}(r'_2)^*. \tag{3.2}$$

Here we have used the fact that the total momentum $q_x$ of the Cooper pair along the chains is conserved. In the ladder approximation, the vertex $\Gamma^{\alpha\alpha'}_{q_x}(l_1, l_2; l'_1, l'_2)$ is solution of the equation

$$\Gamma^{\alpha\alpha'}_{q_x}(l_1, l_2; l'_1, l'_2) = \langle l_1, \alpha; l_2, \alpha' | \mathcal{H}_{\text{int}} | l'_1, \alpha'; l'_2, \alpha'' \rangle - \sum_{\alpha''} \langle l_1, \alpha; l_2, \alpha' | \mathcal{H}_{\text{int}} | l''_1, \alpha''; l''_2, \alpha'' \rangle \times \chi^{\alpha''}(q_x + (l''_1 + l''_2)G) \Gamma^{\alpha''\alpha'}_{q_x}(l''_1, l''_2; l'_1, l'_2), \tag{3.3}$$

where

$$\chi^{\alpha''}(q_x + (l''_1 + l''_2)G) = \frac{T}{L_x e} \sum_{\omega, k''_x} G^{\alpha''}_{\nu}(k''_x, l''_1, \omega) G^{\alpha''}_{\nu}(q_x - k''_x, l''_2, -\omega)$$

is the two-particle propagator in the representation of the states $\phi^\alpha_{k_x,l}$. Using the expression (2.8) for the Green’s function $G^\alpha_{\nu}(k_x, l, \omega)$, we have

$$\chi^{\alpha}(q_x) = \frac{N(0)}{2} \left[ \ln \left( \frac{2\gamma \Omega}{\pi T} \right) + \Psi \left( \frac{1}{2} \right) - \Re \Psi \left( \frac{1}{2} + \frac{\alpha v q_x}{4i\pi T} \right) \right], \tag{3.5}$$
where $\Psi$ is the digamma function. $N(0) = 1/\pi vc$ is the density of states per spin at the Fermi level. Note that $\chi(q_x) = \sum_\alpha \chi^\alpha(q_x)$ is the pair susceptibility at zero magnetic field evaluated at the total momentum $q_x$. The first term on the right-hand side of (3.3) is the two-particle matrix element of the electron-electron interaction $H_{\text{int}}$:

$$
\langle l_1, \alpha; l_2, \beta | H_{\text{int}} | l_1', \alpha'; l_2', \beta' \rangle \delta_{k_1 x + k_2 x, k_{1'} x + k_{2'} x} = -\lambda \int d^2 r \phi_{k_1 x, l_1}^*(r) \phi_{k_2 x, l_2}^*(r)^* \phi_{k_{1'} x, \mu}^*(r) \phi_{k_{2'} x, \nu}^*(r) = -\lambda \delta_{k_1 x + k_2 x, k_{1'} x + k_{2'} x} \alpha_{l_1 - l_2} \alpha_{l_1' - l_2'} \int_0^{2\pi} dx \int_0^{2\pi} \alpha_{l_1 - l_2} \alpha_{l_1' - l_2'} \bar{\chi}(l_{1} - l_{2}) \bar{\chi}(l_{1}' - l_{2}') x J_{l_1 - l_2}(2\bar{t} \cos x) J_{l_1' - l_2'}(2\bar{t} \cos x) .
$$

(3.6)

It is clear from (3.3) that Cooper logarithmic singularities $\chi^\alpha(0) \sim \ln(2\gamma \Omega / \pi T)$ appear when the intermediate two-particle state corresponds to $q_x + 2L''G = 0$ (which imposes $q_x$ to be a multiple of $G$) where $L''$ is the center of gravity of the Cooper pair with respect to the $z$ direction. Intermediate states with $q_x + 2L''G = nG$ ($n \neq 0$) lead to weaker logarithmic singularities $\chi^\alpha(nG) \sim \ln(n \omega_c / \Omega)$ for $\omega_c \gg T$. The QLA, which is expected to be valid for $\omega_c \gg T$, restricts the Hilbert space to the intermediate states such that $q_x = 2L''G = 0$. Since $q_x$ is a constant of motion, the center of gravity $L = (l_1 + l_2)/2 = -q_x/2G$ also becomes a constant of motion of the Cooper pair in the QLA. Thus, (3.3) reduces to

$$
\Gamma_{q_x(L), l}(l', l') = -\lambda V_{l, l'}^\alpha + \frac{\lambda}{2} \chi(0) \sum_{\alpha''} V_{l, l'}^{\alpha''} \Gamma_{q_x(L), l}(l', l') ,
$$

(3.7)

where $l = l_1 - l_2$ and $l' = l_{1}' - l_{2}'$ describe the relative motion of the pair in the $z$ direction. $q_x(L) = -2LG$ and $V_{l, l'}^{\alpha''} = \alpha^l \alpha'^{l'} V_{l, l'}$ with

$$
V_{l, l'} = \int_0^{2\pi} \frac{dx}{2\pi} J_{l}(2\bar{t} \cos x) J_{l'}(2\bar{t} \cos x) .
$$

(3.8)

Note that $V_{l, l'}^{\alpha''}$ is independent of the center of gravity $L$ of the Cooper pairs. To eliminate the dependence on $\alpha$ and $\alpha'$, we write $\Gamma_{q_x(L), l}(l, l') = -\lambda \alpha^l \alpha'^{l'} \Gamma_{q_x(L), l}(l, l')$. $\Gamma_{q_x(L), l}$ is solution of the equation

$$
\Gamma_{q_x(L), l}(l, l') = V_{l, l'} + \lambda \chi(0) \sum_{l''} V_{l, l''} \Gamma_{q_x(L), l}(l'' , l') .
$$

(3.9)

The preceding matrix equation is solved by introducing the orthogonal transformation $U_{l, l'}$ which diagonalizes the matrix $V_{l, l'}$: $(U^{-1} V U)_{l, l'} = \delta_{l, l'} V_{l, l'}$. The matrix $\bar{\Gamma}_{q_x(L), l} = U^{-1} \Gamma_{q_x(L), l} U$ is diagonal:

$$
\bar{\Gamma}_{q_x(L), l}(l, l') = \delta_{l, l'} \frac{V_{l, l}}{1 - \lambda \chi(0) V_{l, l}} ,
$$

(3.10)

and we obtain

$$
\Gamma_{q_x(L), l}(l, l') = -\lambda \alpha^l \alpha'^{l''} \sum_{l''} \frac{U_{l, l''} V_{l''} (U^{-1})_{l'' l'} }{1 - \lambda \chi(0) V_{l'', l''}} .
$$

(3.11)

The metallic state becomes instable when a pole appears in the two-particle vertex function. Using $\chi(0) = N(0) \ln(2\gamma \Omega / \pi T)$, we obtain the critical temperature.
where $\bar{V}_{l_0,l_0}$ is the highest eigenvalue of the matrix $V$. The critical temperatures are shown in Fig. 2 for the two highest eigenvalues of $V$. The parameters used in the numerical calculations ($T_{c0} = 1.5$ K and $t_z = 20$ K) are the same as those of Fig. 1 and Figs. 4-10 of Ref. The existence of two lines of instability results from the fact that $V_{l,l'} = 0$ if $l$ and $l'$ do not have the same parity. Diagonalizing the matrix $V_{l,l'}$ is then equivalent to separately diagonalizing the matrices $V_{2l,2l'}$ and $V_{2l+1,2l'+1}$. In the following, we label these two lines by $l_0 = 0, 1$ so that $\bar{V}_{l_0,l_0} = \max_l \bar{V}_{2l+1,2l+0}$. Since $2L = l_1 + l_2$ and $l_1 - l_2$ have the same parity, $L$ is integer (half-integer) for $l_0 = 0$ ($l_0 = 1$) and can be written as $L = -l_0/2 + p$ with $p$ integer. Correspondingly, we have $q_x(L) = (l_0 - 2p)G$. It is clear that the instability line $l_0$ corresponds to the instability line $Q = l_0G$ which was previously obtained in another approach where the magnetic Bloch wave vector $Q$ plays the role of a pseudo-momentum for the Cooper pairs in the magnetic field.

As can be seen from Fig. 2, $T_c$ calculated in the QLA is several orders of magnitude below the exact critical temperature except in the reentrant phase: it has been pointed out previously that in general the QLA strongly underestimates the critical temperature. In the QLA, we neglect intermediate pair states with a center of gravity $L'' \neq L = -q_x(L)/2G$. Since the one-particle states $\hat{\phi}^{\alpha}_{L,l}$ are localized, $|L'' - L|$ is bounded by $\sim t_z/\omega_c$. This means that the QLA neglects the logarithmic divergences $\ln(2\Omega/\omega_c), \ln(\Omega/\omega_c), \ldots, \ln(2\Omega/t_z)$. There are therefore two cases where the QLA becomes quantitatively correct. Either $\omega_c > t_z$ (which corresponds to the reentrant phase) so that the only logarithmic divergence to be considered is the Cooper singularity $\ln(2\gamma\Omega/\pi T)$. Or the cutoff energy $\Omega$ is sufficiently low for the condition $\omega_c \sim \Omega$ to hold. In a conventional (isotropic) superconductor where the attractive electron-electron interaction is due to the electron-phonon coupling, $\Omega$ is the Debye frequency and the condition $\omega_c \sim \Omega$ can never be satisfied for reasonable values of the magnetic field. In a quasi-1D superconductor, it has been argued that $\Omega \sim T_{x,1}$, where $T_{x,1}$ is the single particle dimensionality crossover temperature below which the system becomes 3D. The reason is that the superconducting instability cannot develop at energies $\epsilon > T_{x,1}$ (or equivalently at length scales $< v/T_{x,1}$) where the behavior of the system is 1D. In organic superconductors like the Bechgaard salts, $T_{x,1}$ can be of the order of 10 – 30 K, so that the condition $\omega_c \sim \Omega$ could be realized in particular cases although this remains quite unlikely. Moreover, in the weak coupling limit $T_c \ll \Omega$, the QLA can never be quantitatively correct when entering the quantum regime (\omega_c \sim T).

From (3.11), one can see that the superconducting condensation in the channel $q_x(L), L, l_0$ corresponds to the following spatial dependence for the order parameter

$$\Delta_{q_x(L),L,l_0}(r) \sim \sum_{\alpha} \alpha^L U_{l,l_0} \phi_{k_z,L_l + \frac{L}{2}}^\alpha(r) \phi_{q_x(L),L_l}^\alpha(r).$$

(3.13)

Noting that the matrices $V$ and $U$ have a range of the order of $\tilde{t}$ (i.e., $V_{l,l'}, U_{l,l'}$ are important for $|l|, |l'| < \tilde{t}$), one can see that $\Delta_{q_x(L),L,l_0}$ has the form of a strip extended in the direction of the chains and localized in the perpendicular direction on a length of the order of $c\tilde{t}$.  

$$T_c = \frac{2\gamma\Omega}{\pi} \exp\left(\frac{-1}{\lambda N(0)\bar{V}_{l_0,l_0}}\right),$$

(3.12)
This is not surprising since $\Delta_{q_{\nu}(L),l_{\nu}}$ results from pairing between the localized states $\phi_{k_{\nu},l}^{\nu}$. The expression (3.13) of the order parameter at $T_{c}$ will be used in Sec. IV to construct a variational order parameter describing the ordered phase below $T_{c}$.

**B. Effect of disorder**

We evaluate the effect of disorder on the critical temperature calculated in the QLA. In presence of impurity scattering, the pair propagator appearing in the integral equation for the vertex function $\Gamma_{\alpha\alpha'}$ has to be modified by self-energy and vertex corrections as shown diagrammatically in Fig. 3. In the Born approximation, the self-energy is given by

$$\Sigma^{\text{dis}}(\omega) = -\frac{i}{2\tau}\text{sgn}(\omega),$$

(3.14)

where $1/\tau = 2\pi N(0)n_{i}V_{i}^{2}$. $n_{i}$ is the impurity density and $V_{i}$ is the strength of the electron-impurity interaction which is assumed to be local in real space. The elastic scattering time $\tau$ is not affected by the magnetic field because the density of states per spin $N(\epsilon, H) = N(0)$ is magnetic field independent. Thus, self-energy corrections can be taken into account by the usual replacement $\omega \rightarrow \tilde{\omega} = \omega + \text{sgn}(\omega)/2\tau$ in the expression of the one-particle Green’s function. Because of the vertex corrections shown in Fig. 3, the pair propagator $\Pi_{\omega}^{\alpha\alpha}(l_{1},l_{2})$ in the pairing channel $q_{x}(L)$, $L$ is determined by the matrix equation

$$\Pi_{\omega}^{\alpha\alpha}(l_{1},l_{2}) = \delta_{\alpha_{1},\alpha_{2}}\chi_{\omega}^{\alpha_{1}}(0) + u_{0}\chi_{\omega}^{\alpha_{1}}(0) \sum_{\alpha_{3},l_{3}}V_{l_{1},l_{3}}^{\alpha_{1}\alpha_{3}}\Pi_{\omega}^{\alpha_{3}\alpha_{2}}(l_{3},l_{2}),$$

(3.15)

where $u_{0} = 1/2\pi N(0)\tau$. The variables $l_{i}$ refer to the relative motion of the Cooper pair in the $z$ direction. In writing (3.15), we have used the fact that in the QLA, the only states which are allowed satisfy $q_{x}(L) = -2LG$ where $L$ is the center of gravity of the pair with respect to the $z$ direction. $\chi_{\omega}^{\alpha}(0)$ is defined by

$$\chi_{\omega}^{\alpha}(0) = \frac{1}{cL_{x}} \sum_{k_{x}} G_{1}^{\alpha} (k_{x}, L + \frac{l_{1}}{2}, \tilde{\omega}) G_{1}^{\alpha} (q_{x}(L) - k_{x}, L - \frac{l_{1}}{2}, -\tilde{\omega}).$$

(3.16)

Introducing the matrix

$$\Pi_{\tilde{\omega}}(l_{1},l_{2}) = \sum_{\alpha_{1},\alpha_{2}} (\alpha_{1}\alpha_{2})^{l_{1}}\Pi_{\omega}^{\alpha_{1}\alpha_{2}}(l_{1},l_{2}),$$

(3.17)

the matrix equation (3.15) becomes

$$\Pi_{\tilde{\omega}}(l_{1},l_{2}) = \chi_{\tilde{\omega}}(0) + u_{0}\chi_{\tilde{\omega}}(0) \sum_{l_{3}} V_{l_{1},l_{3}}^{l_{1}}\Pi_{\tilde{\omega}}(l_{3},l_{2}),$$

(3.18)

where $\chi_{\tilde{\omega}}(0) = \sum_{\alpha} \chi_{\tilde{\omega}}^{\alpha}(0)$. In order to obtain the preceding equation, we used the property that $V_{l,l'}$ is non zero only if $l$ and $l'$ have the same parity. The matrix $\Pi_{\tilde{\omega}}$ is diagonalized by the transformation $U$:

$$\Pi_{\tilde{\omega}}(l_{1},l_{2}) = (U^{-1}\Pi_{\tilde{\omega}}U)_{l_{1},l_{2}}$$

$$= \delta_{l_{1},l_{2}} \frac{\chi_{\tilde{\omega}}(0)}{1 - u_{0}\chi_{\tilde{\omega}}(0)V_{l_{1},l_{2}}^{l_{1}}},$$

(3.19)
In presence of impurity scattering, the two-particle vertex function is determined in the QLA by the equation
\[
\Gamma_{\alpha_1,\alpha_2}(q_x(L),L(l_1,l_2)) = -\lambda V_{\alpha_1,\alpha_2}(l_1,l_2) + \lambda T \sum_{\alpha_3,l_3} \sum_{\alpha_4,l_4} V_{\alpha_1,\alpha_2}(l_1,l_2) \Pi_{\alpha_3,\alpha_4}(l_3,l_4) \Gamma_{\alpha_3,\alpha_4}(L(L(l_1,l_2))).
\] (3.20)

The dependence on the indices \(\alpha_i\) is suppressed by writing \(\Gamma_{\alpha_1,\alpha_2}(l_1,l_2) = -\lambda \delta_{l_1,l_2} \Gamma(l_1,l_2)\).

Using the property that \(\Pi_{\alpha,\alpha'}(l,l')\) is non-zero only if \(l\) and \(l'\) have the same parity, we obtain
\[
\Gamma_{q_x(L),L}(l_1,l_2) = V_{l_1,l_2} + \lambda T \sum_{\omega} \sum_{l_3,l_4} V_{l_1,l_3} \Pi_{\omega}(l_3,l_4) \Gamma_{q_x(L),L}(l_4,l_2).
\] (3.21)

The preceding matrix equation is diagonalized by the transformation \(U\):
\[
\bar{\Gamma}_{q_x(L),L}(l,l') = (U^{-1} \Gamma_{q_x(L),L} U)_{l,l'} = \delta_{l,l'} \bar{V}_{l,l} - \lambda \bar{V}_{l,l} T \sum_{\omega} \bar{\Pi}_{\omega}(l,l).
\] (3.22)

From (3.19) and (3.22), we see that the appearance of a pole in the two-particle vertex function corresponds to
\[
1 - \lambda \bar{V}_{l_0,l_0} T \sum_{\omega} \frac{\chi_{\omega}(0)}{1 - u_0 \bar{V}_{l_0,l_0} \chi_{\omega}(0)} = 0.
\] (3.23)

This equation determines the critical temperature in the pairing channel \(q_x(L),L,l\). As in the pure system, the highest \(T_c\) is obtained for \(l = l_0\) defined by \(\bar{V}_{l_0,l_0} = \max \bar{V}_{l,l}\). The result (3.23) can be expressed as
\[
N(0) \ln \left( \frac{T_c^{\text{dis}}}{T_c} \right) = T_c^{\text{dis}} \sum_{\omega} \frac{\chi_{\omega}(0)}{1 - u_0 \bar{V}_{l_0,l_0} \chi_{\omega}(0)} - T_c^{\text{dis}} \sum_{\omega} \chi_{\omega}(0),
\] (3.24)

where \(T_c^{\text{dis}}\) is the critical temperature in presence of disorder and \(T_c\) the critical temperature of the pure system given by (3.12). Using
\[
\chi_{\omega}(0) = \frac{\pi N(0)}{|\omega|},
\] (3.25)
we obtain
\[
\ln \left( \frac{T_c^{\text{dis}}}{T_c} \right) = \Psi \left( \frac{1}{2} \right) - \Psi \left( \frac{1}{2} + \frac{1 - \bar{V}_{l_0,l_0}}{4\pi T_c^{\text{dis}}} \right).
\] (3.26)

For \(T_c - T_c^{\text{dis}} \ll T_c\), the preceding equation simplifies in
\[
\frac{T_c - T_c^{\text{dis}}}{T_c} \approx \frac{\pi}{8\tau T_c} (1 - \bar{V}_{l_0,l_0}).
\] (3.27)

In the preceding equation, \(T_c\) is the critical temperature calculated in the QLA without impurity scattering. Since \(T_c\) is several orders of magnitude below the exact mean-field
value, $T_{c}^{\text{dis}}$ will also be much smaller than the exact value. However, a reasonable estimation of $T_{c}^{\text{dis}}$ can be obtained using for $T_{c}$ the exact value instead of the QLA value. From Eq. (3.27), one can see that $V_{l_{0},l_{0}}$, which comes from vertex corrections in the pair propagator, tends to reduce the difference $T_{c} - T_{c}^{\text{dis}}$. According to (3.27), impurity scattering does not affect the critical temperature in the reentrant phase since $V_{l_{0},l_{0}} \to 1$ when $T_{c} \to T_{c0}$. This is a direct consequence of Anderson’s theorem which states that the critical temperature is independent of a (weak) disorder for a system with time-reversal symmetry. Obviously, (3.27) restricts the observation of high-field superconductivity to clean superconductors with a critical temperature not too small. As pointed out in Ref. [19], this latter condition advantages materials with a large anisotropy. The consequences of (3.27) were discussed in detail in Ref. [11] in the case of the Bechgaard salts.

C. Effect of PPB

We evaluate the effect of PPB on the critical temperature calculated in Sec. II A (but ignoring the effect of disorder). The equation (3.3) for the two-particle vertex function $\Gamma_{q_{0}}^{\alpha \alpha'}(l_{1},l_{2};l_{1}'l_{2}')$ involves the quantity $\chi_{q_{0}}^{\alpha''}(q_{x} + (l_{1}'' + l_{2}'')G)$ which is given by (3.5) with the replacement $\alpha v_{q_{x}} \to \alpha v_{q_{x}} + 2h$. Therefore, logarithmic singularities arise through $\chi_{q_{x}}^{\alpha''}(q_{x} + (l_{1}'' + l_{2}'')G)$ each time we have $q_{x} = -2L''G + q_{0}$ where $L''$ is the center of gravity of the pair in an intermediate state and $q_{0} = \pm 2h/v$. In the QLA, we retain only the intermediate states corresponding to these logarithmic singularities. If $h/\omega_{c}$ is not (and not too close to) an integer, the center of gravity $L$ of the pair becomes a constant of motion and is related to the total momentum by $q_{x}(L) = -2LG + q_{0}$. The equation which determines the two-particle vertex function then reduces to

$$
\Gamma_{q_{x}(L),L}^{\alpha \alpha'}(l,l') = -\lambda V_{l,l'}^{\alpha \alpha'} + \lambda \sum_{\alpha''} V_{l,l''}^{\alpha ' \alpha''} \chi_{q_{0}}^{\alpha''}(q_{0}) \Gamma_{q_{x}(L),L}^{\alpha ' \alpha''}(l'',l').
$$

(3.28)

Following the analysis developed in Sec. II A, we find that a pole appears in the two-particle vertex function when

$$
1 - \lambda V_{l,l} \chi(q_{0}) = 0.
$$

(3.29)

Using

$$
\chi(q_{0}) \simeq \frac{N(0)}{2} \ln \left( \frac{\gamma \Omega^{2}}{\pi Th} \right)
$$

(3.30)

for $h \gg T$, we obtain the critical temperature

$$
T_{c}^{P} \simeq \frac{\gamma \Omega^{2}}{\pi h} \exp \left( -\frac{2}{\lambda N(0) V_{l_{0},l_{0}}} \right)
$$

$$
\simeq \frac{\pi T_{c}^{2}}{4\gamma h},
$$

(3.31)

where $V_{l_{0},l_{0}} = \max_{l} V_{l,l}$. In the reentrant phase ($\omega_{c} \gg t_{z}$), $T_{c} \to T_{c0}$ so that $T_{c}^{P} \to \pi T_{c0}^{2}/4\gamma h$, a result which was obtained in Ref. [11]. The superconducting phase is always stable at $T = 0,$
whatever the value of the magnetic field. This divergence of the orbital critical field $H_{c2}(T)$ at low temperature is a characteristic of the LOFF state in a quasi-1D superconductor.

If $h/\omega_c$ is equal (or close) to an integer, the preceding analysis does not hold any more. In this case, the center of gravity of the Cooper pair can take the two values $-q_x/2G \pm h/\omega_c$. This enlarges the number of accessible intermediate states and should lead to an increase of the critical temperature with respect to the result (3.31). It has been proposed by Lebed that this situation could be reached by tilting the field in the $(x,y)$ plane. This lets the Zeeman energy $h$ unchanged but modifies the orbital frequency which becomes $\omega_c \cos \theta$ where $\theta$ is the angle between the field and the $y$ axis. For certain values of $\theta$, $h/(\omega_c \cos \theta)$ is an integer. The effect of the component $H_x = H \sin \theta$ is expected to be small (if $\theta$ is not too large), because the critical field $H_{c2}(0)$ parallel to the $x$ axis is very large compared to the critical fields in the other directions.

D. Effect of PPB and disorder

We evaluate the effects of both PPB and disorder on the critical temperature calculated in Sec. III A. Following the analysis developed in the two preceding sections, we find that the critical temperature $T_{c,\text{dis}}$ is determined by the equation

$$N(0) \ln \left( \frac{T}{T_{c, \text{P}}} \right) = T \sum_\omega \frac{\chi_\omega(q_0)}{1 - u_0 V_{l_0,l_0} \chi_\omega(q_0)} - T \sum_\omega \chi_\omega(q_0),$$

where $T_{c, \text{P}}$ is the critical temperature calculated in the preceding section. For $T_{c, \text{P}} - T_{c, \text{dis}} \ll T_{c, \text{P}}$, we can expand (3.32) to first order in $u_0$ (or $1/\tau$), which leads to

$$N(0) \ln \left( \frac{T}{T_{c, \text{P}}} \right) \simeq T \sum_\omega [\chi_\omega(q_0) - \chi_\omega(q_0)] + u_0 \tilde{V}_{l_0,l_0} T \sum_\omega \chi_\omega(q_0),$$

where

$$\chi_\omega(q_0) = \frac{i}{v_c} \text{sgn}(\omega) \left[ \frac{1}{2i\omega} + \frac{1}{2i\omega - 4h} \right],$$

$$\chi_\omega(q_0) - \chi_\omega(q_0) \simeq -\frac{\pi N(0)}{\tau} \left[ \frac{1}{4\omega^2} - \frac{1}{(2i\omega - 4h)^2} \right].$$

Performing the sum over the Matsubara frequencies, we obtain

$$\ln \left( \frac{T_{c, \text{P}}}{T_{c, \text{P}}} \right) \simeq -\frac{\pi}{16\tau T} \left( 1 - \frac{\tilde{V}_{l_0,l_0}}{2} \right) \left[ 1 + \frac{2}{\pi^2} \text{Re } \Psi' \left( \frac{1}{2} + i \frac{h}{\pi T} \right) \right] + \frac{\tilde{V}_{l_0,l_0}}{8\tau h} \text{Im } \Psi \left( \frac{1}{2} + i \frac{h}{\pi T} \right),$$

where $\Psi'$ is the first derivative of the digamma function. For $h \gg \pi T$, we can use $\Psi(z) \simeq \ln z - 1/2z$ for $|z| \gg 1$ to get the following expression to lowest order in $1/\tau$:

$$\frac{T_{c, \text{dis}} - T_{c, \text{P}}}{T_{c, \text{P}}} \simeq -\frac{\pi}{16\tau T_{c, \text{P}}} \left( 1 - \frac{\tilde{V}_{l_0,l_0}}{2} \right) + \frac{\pi}{16\tau T_{c, \text{P}}} \left( 1 + \frac{\tilde{V}_{l_0,l_0}}{2} \right) \frac{T_{c, \text{P}}^2}{h^2}. \quad (3.37)$$
As in Sec. III B, a reasonable estimation of the effect of disorder can be obtained from the preceding equation using for $T^c$ the exact mean-field value. The effect of vertex corrections is weaker than in the absence of PPB (Eq. (3.27)). Even in the reentrant phase where $\bar{V}_{l_0,l_0} \to 1$, the effect of impurities remains important since time-reversal symmetry is broken by the Zeeman splitting and Anderson’s theorem does not hold. The sensitivity of the LOFF state to elastic impurity scattering has been known for a long time. The consequences of (3.37) were discussed in detail in Ref. 19 in the case of the Bechgaard salts.

IV. ORDERED PHASE IN THE QUANTUM LIMIT APPROXIMATION

In this section, we study the ordered phase ($T < T_c$) in the QLA where only the pairing channels leading to Cooper logarithmic singularities are retained. We only consider the orbital effects of the magnetic field, i.e. we put the $g$ factor equal to zero. Using the results obtained in Sec. III A, we first construct a variational order parameter with two unknown parameters, the amplitude and the periodicity in the $z$ direction. We solve the Gor’kov equations in the QLA and obtain the normal and anomalous Green’s functions. We then deduce the GL expansion of the free energy, the specific heat jump at the transition and the quasi-particle excitation spectrum.

Following the original approach proposed by Abrikosov, we construct the order parameter for $T < T_c$ as a linear combination of the functions $\Delta_{q_x(L),L,l_0}(r)$ (Eq. (3.13)) describing the superconducting condensation in the channel $q_x(L), L, l_0$. Since $T_c$ does not depend on $L$, the most general linear combination can be written as

$$
\Delta(r) = \sum_L \gamma(L) \Delta_{q_x(L),L,l_0}(r).
$$

As shown in Sec. III A, $2L$ must have the parity of $l_0$. Since $\Delta_{q_x(L),L,l_0}(r)$ is localized in the $z$ direction with an extension of $\bar{\omega}_t$, a natural choice for the coefficients $\gamma(L)$ is to take $\gamma(L) \neq 0$ if $L = -l_0/2 + pN'$ ($p$ integer) where the unknown integer $N'$ is expected to be of order $\bar{\omega}_t$. In order to correctly describe the triangular Josephson vortex lattice in the last phase (which has periodicity 2c in the $z$ direction), we choose $\gamma(L) \equiv \gamma_p = 1 (i)$ for $p$ even (odd). This leads to (noting $N = 2N'$)

$$
\Delta_{l_0,N}(r) = \Delta \sum_{l,p} U_{2l+l_0,l_0} \gamma_p e^{i(l_0-pN)G_x} J_p \bar{\omega}_t^{l+l_0-m} J_p \bar{\omega}_t^{-l-l_0-m}(-\bar{\omega}_t),
$$

where the amplitude $\Delta$ is chosen real. Eq. (4.2) defines a variational order parameter where the two unknown parameters $\Delta$ and $N$ have to be determined by minimizing the free energy. It can be seen that $|\Delta(r)|$ has periodicity $a_x = 2\pi/NG$ and $a_z = Nc$ so that the unit cell contains two flux quanta $\phi_0$ for a particle of charge $2e$: $Ha_xa_z = 2\phi_0$ (when a triangular lattice is described with a square unit cell, the unit cell contains two flux quanta). In Ref. 19, the order parameter was constructed by imposing that it describe both the triangular Abrikosov vortex lattice in weak field ($\omega_c \ll T$) and the triangular Josephson vortex lattice in very strong field ($\omega_c \gg t_z$). Both approaches lead to the same order parameter when only the Cooper singularities are retained.
The order parameter is shown in Figs. 4-8, for the phases \(N = 2, 4, 6, 8\) and 10. The exact (at the mean-field level) order parameter corresponding to Figs. 4, 6, and 7, is shown in figures 10, 9 and 8 of Ref. 19. The order parameter calculated in the QLA is a good approximation of the exact order parameter for the last phases \(N = 2, 4\) and 6. For larger values of \(N\), there appears significant differences between the approximate and exact results. For example, in the phase \(N = 8\) at \(H = 1.7\) T, the order parameter obtained in the QLA shows zeros which are not present in the exact calculation.

**A. Gor’kov equations**

The starting point of our analysis is the mean-field (or generalized Hartree-Fock) Hamiltonian \(H_0 + H_{\text{int}}^{\text{MF}}\) with

\[
H_{\text{int}}^{\text{MF}} = \frac{1}{2} \int d^2 \mathbf{r} \sum_{\alpha, \sigma} \left[ \Delta_\sigma(r)^* \overline{\psi}_\sigma(r) \psi_\sigma(r) + \text{H.c.} \right],
\]

where

\[
\Delta_\uparrow(r) = \lambda \sum_\alpha \langle \overline{\psi}_\uparrow(r) \psi_\uparrow(r) \rangle.
\]

\(\Delta_\uparrow(r) = -\Delta_\downarrow(r)\) is the variational order parameter defined by (4.2). In order to derive the thermodynamics and the excitation spectrum in the superconducting phases, it is necessary to determine the normal and anomalous Green’s functions

\[
G_\sigma(r, r', \omega) = -\langle \overline{\psi}_\sigma(r, \omega) \psi_\sigma(r', 0) \rangle,
\]

\[
F_\sigma(r, r', \omega) = -\langle \overline{\psi}_\sigma(r, \omega) \psi_\sigma(r', 0) \rangle,
\]

whose Fourier transforms with respect to the imaginary time \(\tau\) are solutions of the Gor’kov equations

\[
(i\omega - H_0^a,\sigma)G_\sigma^a(r, r', \omega) - \Delta_\sigma(r) F_\sigma \overline{\psi}_\sigma(r, r', \omega) = \delta(r - r'),
\]

\[
(-i\omega - H_0^{\overline{\psi}_\sigma}) F_\sigma^\overline{\psi}_\sigma(r, r', \omega) + \Delta_\sigma(r) G_\sigma^a(r, r', \omega) = 0,
\]

with the self-consistency equation

\[
\Delta_\sigma^*(r) = \lambda T \sum_{\alpha, \omega} F_\sigma^{\overline{\psi}_\sigma}(r, r, \omega).
\]

In the QLA, the supercurrents vanish as will be shown in Sec. [5]. The magnetization \(M\) (which is parallel to the magnetic field by symmetry arguments) and the flux density \(B = H + 4\pi M\) are uniform. In the Gor’kov equations, we should therefore in principle replace the external magnetic field \(H\) by the uniform flux density \(B\). However, when minimizing the free energy with respect to \(B\), it turns out that the approximation \(B = H\) is very accurate (see Sec. [4B]). Moreover, the magnetization vanishing in the middle of each phase (see below), this approximation becomes exact at this point. In this section, we consider that the flux density is equal to the external magnetic field.
To exploit the translational symmetry of the order parameter, we introduce the magnetic Bloch states $\phi_{q,l}^\alpha$, obtained by diagonalizing simultaneously the three operators $H_0$, $e^{a_x \partial_x}$ and $e^{a_z \partial_z + iNGx}$ which define the magnetic translation group:

$$\phi_{q,l}^\alpha = \sqrt{\frac{N}{N_z}} \sum_p e^{-ipq\alpha} \phi_{q+l-pN,l-pN}^\alpha.$$  \hspace{1cm} (4.8)

These states are eigenstates of the Hamiltonian $H_{0,\sigma}$ with the eigenenergies $\epsilon_{q,l,\sigma}^\alpha = \epsilon_{q,l,\sigma}^\alpha$. $N_z$ is the number of chains. $q$ is restricted to the magnetic Brillouin zone (MBZ) which is chosen to be equal to

$$\left[ -k_F - l_0G - \frac{\pi}{a_x}, -k_F - l_0G + \frac{\pi}{a_x} \right] \times \left[ -\frac{\pi}{a_z}, \frac{\pi}{a_z} \right]$$  \hspace{1cm} (4.9)

for the left sheet of the Fermi surface, and to

$$\left[ k_F - \frac{\pi}{a_x}, k_F + \frac{\pi}{a_x} \right] \times \left[ -\frac{\pi}{a_z}, \frac{\pi}{a_z} \right]$$  \hspace{1cm} (4.10)

for the right sheet of the Fermi surface. This choice is made for later convenience. There are $N$ branches crossing the Fermi level on each sheet of the Fermi surface. The excitation spectrum of the normal phase in the representation of the magnetic Bloch states is shown in Fig. 9. We introduce Green’s functions in the representation of these magnetic Bloch states writing

$$G_{\sigma}^\alpha(r, r', \omega) = \sum_{1,2} \phi_1^\alpha(r) \phi_2^\alpha(r')^* G_{\sigma}^\alpha(1, 2, \omega),$$

$$F_{\sigma}^{\alpha\dagger}(r, r', \omega) = \sum_{1,2} \phi_1^\alpha(r) \phi_2^\alpha(r')^* F_{\sigma}^{\alpha\dagger}(1, 2, \omega),$$  \hspace{1cm} (4.11)

where we use the notation $i \equiv (q_i, l_i)$. In this representation, the Gor’kov equations become

$$\left( i\omega - \epsilon_1^\sigma \right) G_{\sigma}^\alpha(1, 2, \omega) - \sum_3 \Delta_{\sigma}^\alpha(1, 3) F_{\sigma}^{\beta\dagger}(3, 2, \omega) = \delta_{1,2},$$

$$\left( -i\omega - \epsilon_1^\sigma \right) F_{\sigma}^{\alpha\dagger}(1, 2, \omega) + \sum_3 \Delta_{\sigma}^\alpha(1, 3)^* G_{\sigma}^\alpha(3, 2, \omega) = 0,$$  \hspace{1cm} (4.12)

where the pairing amplitude $\Delta_{\sigma}^\alpha(1, 2)$ is defined by

$$\Delta_{\sigma}^\alpha(1, 2) = \int d^2r \, \phi_1^\alpha(r) \phi_2^\alpha(r)^* \Delta_{\sigma}(r).$$  \hspace{1cm} (4.13)

Multiplying (4.7) by $\Delta_{\sigma}(r)$ and summing over $r$, we can rewrite the self-consistency equation as

$$\int d^2r \, |\Delta_{\sigma}(r)|^2 = \lambda T \sum_{\omega, \alpha} \sum_{1,2} \Delta_{\sigma}^\alpha(1, 2) F_{\sigma}^{\alpha\dagger}(1, 2, \omega).$$  \hspace{1cm} (4.14)

A careful examination shows that the pairing between $\phi_{q,l}^\alpha$ and $\phi_{l_0G-q,-l_0-l}^\alpha$ (with $G = (G, 0)$) is the only pairing compatible with the spatial dependence of the order parameter $\Delta_1$ (4.2). Thus, in the QLA, we have
\[ \Delta_\sigma^\alpha(\mathbf{q}_1, l_1; \mathbf{q}_2, l_2) = \delta_{\mathbf{q}_1, \mathbf{q}_2, l_0} \delta_{l_1, l_2, -l_0} \Delta_\sigma^\alpha(\mathbf{q}_1, l_1). \]  

(4.15)

The preceding result can also be obtained noting that in the QLA the electron-electron interaction conserves the center of gravity of the particles with respect to the \( z \) direction (Sec. IV). Eq. (1.13) follows from this property of the interaction. The definition of the MBZ (1.9, 1.10) ensures that if \( \mathbf{q} \) belongs to the MBZ, then \( l_0 \mathbf{G} - \mathbf{q} \) also belongs to the MBZ. The pairing amplitude \( \Delta_\sigma^\alpha(\mathbf{q}, l) \) can be calculated using (2.3, 4.2, 1.8):

\[ \Delta_\sigma^\alpha(\mathbf{q}, l) \equiv \Delta_\sigma^\alpha(q_z, l) = \Delta_\sigma \alpha^0 \sum_p \gamma_p e^{-ipq_{\alpha z}} U_{2l + q_0 + pN, l_0}, \]  

(4.16)

and \( \Delta_\tau = -\Delta_\delta = \Delta \). The equality (1.13) and the Gor’kov equations (1.12) imply that the Green’s functions in the representation of the magnetic Bloch states are diagonal:

\[ G_\sigma^\alpha(\mathbf{q}_1, l_1; \mathbf{q}_2, l_2, \omega) = \delta_{\mathbf{q}_1, \mathbf{q}_2, l_1, l_2} G_\sigma^\alpha(\mathbf{q}_1, l_1, \omega), \]

\[ F_\sigma^\alpha(\mathbf{q}_1, l_1; \mathbf{q}_2, l_2, \omega) = \delta_{\mathbf{q}_1, \mathbf{q}_2, l_1, l_2} F_\sigma^\alpha(\mathbf{q}_1, l_1, \omega), \]

(4.17)

where \( G_\sigma^\alpha(\mathbf{q}, l, \omega) \) and \( F_\sigma^\alpha(\mathbf{q}, l, \omega) \) are given by:

\[ G_\sigma^\alpha(\mathbf{q}, l, \omega) = \frac{-i\omega - \epsilon_{\mathbf{q}, l, \sigma}}{\omega^2 + \epsilon_{\mathbf{q}, l, \sigma}^2 + |\Delta_\sigma^\alpha(\mathbf{q}, l)|^2}, \]

\[ F_\sigma^\alpha(\mathbf{q}, l, \omega) = \frac{\Delta_\sigma^\alpha(\mathbf{q}, l)^*}{\omega^2 + \epsilon_{\mathbf{q}, l, \sigma}^2 + |\Delta_\sigma^\alpha(\mathbf{q}, l)|^2}. \]  

(4.18)

In the next sections, we will use the preceding expressions of the Green’s functions to obtain the GL expansion of the free energy and the excitation spectrum.

**B. Ginzburg-Landau expansion and thermodynamics**

We first note that the sign of the magnetization in the ordered phase can be obtained from general thermodynamics arguments. The magnetization is obtained from \( M = -\partial F_e/\partial B \), where \( F_e(T, B) \) is the electronic contribution to the difference of the free energies of the normal and superconducting phases. Since \( T_c \) is determined by \( F_e(T_c, B) = 0 \), the magnetization \( M \) close to the transition line has the sign of \( dF_e/dH \). Each phase will therefore first be paramagnetic and then diamagnetic for increasing field (which implies that there is a value of the field in each phase for which the magnetization vanishes), except the reentrant phase which is always paramagnetic.

Since the flux density \( B \) is uniform in the QLA, the free energy density is equal to \( F(T, B) = F_e(T, B) + B^2/8\pi \) where the electronic contribution can be written:

\[ F_e(T, B) = \int_{0}^{\Delta} d\Delta' \frac{d\Delta}{d\Delta'} \Delta'^2 \int \frac{d^2r}{S} \left| \frac{\Delta_\sigma(r)}{\Delta} \right|^2 \]

\[ = 2 \int_{0}^{\Delta} \Delta' d\Delta' (\lambda^{-1} - g(\Delta')) \int \frac{d^2r}{S} \left| \frac{\Delta_\sigma(r)}{\Delta} \right|^2, \]  

(4.19)

18
where the function \( g(\Delta) = 1/\lambda \) is defined by (4.14). In (4.19), all quantities are calculated in presence of a constant flux density \( B \). In order to get the free energy per volume unit (and not per surface unit), the 2D density of states \( N(0) = 1/\pi vc \) should be replaced by its 3D expression \( 1/\pi vbc \). Expanding \( F_{\sigma}^{\alpha} \) in (4.14) in power of \( \Delta \) leads to

\[
F_e(T, B) = \alpha \Delta^2 + \frac{\beta}{2} \Delta^4 ,
\]

(4.20)

with

\[
\alpha = \frac{1}{\lambda} \int \frac{d^2 r}{S} \left| \frac{\Delta_\sigma(r)}{\Delta} \right|^2 - \frac{T}{S} \sum_{\alpha,\omega,\mathbf{q},l} \frac{\left| \Delta_\sigma^\alpha(\mathbf{q}, l)/\Delta \right|^2}{\omega^2 + \epsilon_{\mathbf{q},l,\sigma}^\alpha} ,
\]

\[
\beta = \frac{T}{S} \sum_{\alpha,\omega,\mathbf{q},l} \frac{\left| \Delta_\sigma^\alpha(\mathbf{q}, l)/\Delta \right|^4}{(\omega^2 + \epsilon_{\mathbf{q},l,\sigma}^\alpha)^2} .
\]

(4.21)

In the two preceding equations (and in the following), the sum over \( \mathbf{q} \) is restricted to the MBZ. Using (4.2,4.16), and the property

\[
\Delta_\sigma^\alpha(q, l + \mathbf{p}N) = e^{2ipq_\alpha z} \Delta_\sigma^\alpha(q, l) ,
\]

(4.22)

we obtain

\[
\alpha = \frac{2V_{l_0,l_0}}{N} (\lambda^{-1} - \bar{V}_{l_0,l_0} \chi(0)) \]

\[
\simeq 2N(0) \frac{V_{l_0,l_0}}{N} \frac{T - T_c}{T_c} ,
\]

\[
\beta = \beta_{BCS} \frac{V_{l_0,l_0}^4}{N} \sum_{l=1}^N \sum_{p_1,p_2,p_3} \gamma_{p_1} \gamma_{p_2} \gamma_{p_3}^* \gamma_{p_1+p_2-p_3}^* U_{2l+l_0+p_1N,l_0} U_{2l+l_0+p_2N,l_0} U_{2l+l_0+p_3N,l_0} U_{2l+l_0+(p_1+p_2-p_3)N,l_0} ,
\]

(4.23)

where \( \beta \) is calculated at \( T_c \) and \( \beta_{BCS} = 7\zeta(3)N(0)/8\pi^2T_c^2 \). The equilibrium state is determined by the minimum of the Gibbs free energy

\[
G(T, H) = F(T, B) - \frac{B H}{4\pi}
\]

\[
= F_e(T, B) + \frac{(B - H)^2}{8\pi} - \frac{H^2}{8\pi} .
\]

(4.24)

Minimizing \( G(T, H) \) (or equivalently \( F_e(T, B) \)) with respect to \( \Delta \), we obtain

\[
F_e(T, B) = -\frac{\alpha^2}{2\beta} .
\]

(4.25)

Since \( \alpha \) and \( \beta \) are known, we can minimize numerically \( G(T, H) \) with respect to \( B \) and then look for the integer \( N \) which minimizes the Gibbs free energy. It turns out that the minimum is reached for \( B = H \) with a very high accuracy. As pointed out above, this approximation becomes exact in the middle of each phase where the magnetization
vanishes. In the following, we therefore consider that the flux density is equal to the external magnetic field. Thus, we can consider the free energy \( F_e(T, H) \) instead of the Gibbs free energy \( G(T, H) \). In the reentrant phase, we find that the minimum of \( F_e(T, H) \) is obtained for \( N = 2 \). When the field is decreased from its value in the reentrant phase, the system undergoes a first order transition and the minimum of \( F_e \) is then obtained for \( N = 4 \). This is in agreement with Refs.\cite{18,19} where it is argued that the first order phase transitions are due to commensurability effects between the crystalline lattice spacing and the periodicity of the order parameter. Unlike what is expected, the best value of \( N \) switches to 6 before reaching the next first order transition. This indicates that the QLA gives a bad estimate of the free energy in the phases \( N \geq 4 \). A correct calculation of the free energy in these phases requires the inclusion of all the pairing channels and maybe also of the screening of the magnetic field. This point will be further discussed in Sec. V B.

The calculation of the free energy can easily be extended to the case of a more general order parameter defined by (4.2) but with \( \gamma_p = 1 \ (\gamma) \) for \( p \) even (odd) where \( \gamma \) is an unknown parameter. For a fixed periodicity \( N \), we have found that the free energy is stationary with respect to \( \gamma \) for \( \gamma = 0, 1 \) or \( i \) which corresponds to square lattices of periodicities \( a_z = Nc \) and \( a_z = Nc/2 \), and to a triangular lattice of periodicity \( a_z = Nc \). Numerically, we have verified that in the reentrant phase, the triangular lattice with periodicity \( N = 2 \) has a lower free energy than the square lattice with periodicity \( N = 1 \) or \( N = 2 \). This result (stationarity of the free energy for the square and triangular lattices, the latter corresponding to the minimum of the free energy) extends to the quantum regime \( \omega_c \gg T \) a property which is well-known in the GL regime.\cite{3,45}

The specific heat jump at the transition is obtained from \( \Delta C = -T \partial^2 F_e / \partial T^2 \). From (4.23,4.25), we obtain

\[
r = \left[ \frac{\Delta C/C_N}{(\Delta C/C_N)_{BCS}} \right]^{-1},
\]

where \( C_N \) is the specific heat of the normal state and \( (\Delta C/C_N)_{BCS} = 12/7\zeta(3) \) is the zero-field value. The ratio \( r \) is always smaller than 1 (but close to 1 in the reentrant phase) and discontinuous at each first order phase transition (Fig. 11). As in the case of the FISDW phases, these discontinuities can be related to the slopes \( \Delta T/\Delta H \) of the first order transition lines. Consider two consecutive superconducting phases denoted by 1 and 2. For each phase \( i = 1, 2 \), the free energy can be written as \( (T \leq T_c^{(i)}) \)

\[
F_i(T, H) = -a_i(T_c^{(i)}, H)(T - T_c^{(i)})^2,
\]

and the specific heat jump at the transition is \( \Delta C_i = 2a_i(T_c^{(i)}, H)T_c^{(i)} \). \( T_c^{(i)} \) denotes the transition temperature between the normal phase and the superconducting phase \( i \). The first order transition line is obtained by writing the equality of the free energy between the two phases:
Introducing the quantities $\Delta T = T - T^*$ and $\Delta H = H - H^*$ where the point $(H^*, T^*)$ is defined by the intersection of the two transition lines, i.e. $T^* = T_c^{(1)}(H^*) = T_c^{(2)}(H^*)$, the preceding equality is rewritten as

$$\sqrt{a_1(T_c^{(1)}, H)}\left(\frac{\Delta T}{\Delta H} - \frac{\Delta T_c^{(1)}}{\Delta H}\right) = \sqrt{a_2(T_c^{(2)}, H)}\left(\frac{\Delta T}{\Delta H} - \frac{\Delta T_c^{(2)}}{\Delta H}\right).$$  \hspace{1cm} (4.29)

Taking the limit $T \to T^*$ and $H \to H^*$, we obtain

$$\frac{\Delta T}{\Delta H} = \frac{\sqrt{\Delta C_1} \frac{d\Delta T_c^{(1)}}{dH} - \sqrt{\Delta C_2} \frac{d\Delta T_c^{(2)}}{dH}}{\sqrt{\Delta C_1} - \sqrt{\Delta C_2}},$$  \hspace{1cm} (4.30)

where $\Delta C_1 \equiv \Delta C_1(H^*)$ and $\Delta C_2 \equiv \Delta C_2(H^*)$. Since $d\Delta T_c^{(1)}/dH < 0$ and $d\Delta T_c^{(2)}/dH > 0$ (Fig. 2), it is clear that the slope $\Delta T/\Delta H$ of the first order transition line has the sign of $\Delta C_2 - \Delta C_1$: the slope is positive for the transition between the phases $N = 4$ and $N = 2$ and negative for the other transitions.

The calculation of the specific heat jump at the transition clearly shows an important difference between the study of the superconducting phases in the QLA and the study of the FISDW phases in the SGA. In the SGA, the electron-hole pairing amplitudes are given by a single number (corresponding to the gap which opens at the Fermi level) so that the thermodynamics of the FISDW reduces to that of the (zero-field) BCS model. For example, one obtains $r = 1$ for any value of the magnetic field. In order to get corrections to the BCS description, one has to go beyond the SGA. In the QLA used here for the study of the superconducting phases, the (electron-electron) pairing amplitudes $\Delta_{q}(q_z, l)$ do not reduce to a single number so that the thermodynamics is different from that of the (zero-field) BCS model.

C. Excitation spectrum

From the Green’s functions (1.18), we deduce the quasi-particle spectrum in the QLA:

$$E_{q,l,\sigma}^\alpha = \pm \sqrt{\epsilon_{q,l,\sigma}^\alpha + |\Delta_{q}^\alpha(q_z, l)|^2}.$$

For each branch $l$ which crosses the Fermi level, a gap $2|\Delta_{q}^\alpha(q_z, l)|$ opens at the Fermi level for $q_z = \alpha k_F - lG$ where $\epsilon_{q_z,l,\sigma}^\alpha = 0$. The pairing amplitudes $\Delta_{q}^\alpha(q_z, l)$ are shown in Fig. [11] for the last five phases $N = 2$, $N = 4$, $N = 6$, $N = 8$ and $N = 10$. Note that $|\Delta_{q}^\alpha(-q_z, l \pm N/2)| = |\Delta_{q}^\alpha(q_z, l)|$. In the very high field limit $\tilde{t} \ll 1$, the dispersion of $\Delta_{q}^\alpha(q_z, l)$ with respect to $q_z$ is very weak (of the order of $\tilde{t}^2$). The corresponding excitation spectrum for the right sheet of the Fermi surface is shown in Fig. [12] for the two branches $l = 0$ and $l = 1$. Comparing Figs. [9] and [12], we clearly see how the excitation spectrum is modified by the superconducting pairing. The dispersion of the pairing amplitude $\Delta_{q}^\alpha(q_z, l)$ with respect to $q_z$ increases with decreasing field. For large $N$ and some particular values of
the field (for example in the phase $N = 8$ at $H = 1.7$ T), $\Delta_\sigma^\alpha(q_z, l)$ vanishes at $q_z = \pm \pi/2a_z$ which results in a gapless excitation spectrum. It is rather surprising that the phase $N = 8$ is gapless at $H = 1.7$ T while the phase $N = 10$ is never gapless. This point, which seems to indicate again that the validity of the QLA is restricted to the last few phases, will be further discussed in Sec. V C. Nevertheless, the QLA clearly shows the general tendency: the system evolves from a quasi-2D (BCS-like) behavior in very high magnetic field ($\tilde{t} \ll 1$) towards a gapless (or at least with a weak minimum excitation energy) behavior at weaker field.

In the QLA, the zeros of the excitation spectrum are related to the zeros of the order parameter in real space. This results from the relation

$$ U_{2m+l_0, l_0} = e^{i\phi} \frac{(-1)^m}{\sqrt{V_{l_0, l_0}}} \sum_l U_{2l+l_0, l_0} J_{l-m}(-\tilde{t}) J_{l_0-l-m}(-\tilde{t}) , \quad (4.32) $$

where $\phi = 0$ or $\pi$ depending on the value of the magnetic field. Eq. (4.32) will be derived in Sec. V. Using (4.32), the order parameter (4.2) can be rewritten as

$$ \Delta_{l_0, N}(r) = \Delta e^{-i\phi} \sqrt{V_{l_0, l_0}}(-1)^m e^{i\delta G_{x}} \sum_p \gamma_p e^{-ipN_{G_{x}}} U_{2m-pN_{l_0}, l_0} . \quad (4.33) $$

Comparing the preceding equation with (4.16), we see that

$$ |\Delta_\sigma^\alpha(q_z = \pm \pi/2a_z, l)| \sim |\Delta_{l_0, N}(x = \mp a_x/4, m = l)| . \quad (4.34) $$

This shows that a gapless spectrum at $q_z = \pm \pi/2a_z$ in the $l$th branch implies that the order parameter in real space vanishes at the points $(x = \mp a_x/4, m = l)$. Such a relation between the zeros of the excitation spectrum and the nodes of the order parameter has also been obtained by Dukan and Tešanović in their study of the vortex lattice of isotropic superconductors in the QLA.

Unlike for thermodynamics quantities, the third direction has a non trivial effect on the excitation spectrum. For a 3D system, instead of (4.31), we have

$$ E_{\alpha, q_z, k_y, l, \sigma}^\alpha = \pm \sqrt{(\epsilon_{q_z, l, \sigma}^\alpha + t_y \cos(k_y b))^2 + |\Delta_\sigma^\alpha(q_z, l)|^2} , \quad (4.35) $$

where $k_y$ is the momentum along the $y$ direction ($-\pi/b < k_y \leq \pi/b$). $\Delta_\sigma^\alpha(q_z, l)$ is the pairing amplitude between the magnetic Bloch states $\phi_{q_z, k_y, l, \sigma}$ and $\phi_{l_0 G_{x} - q_z, k_y, l_0 - l, \sigma}$. It remains unchanged since the Cooper pairs are formed with states of opposite momenta in the $y$ direction. The effect of the third direction can be viewed as a shift of the MBZ which is now centered around $\pm k_F(k_y)$ defined by

$$ v(k_F(k_y) - k_F) + t_y \cos(k_y b) = 0 . \quad (4.36) $$

Each time $\Delta_\sigma^\alpha(q_z, l)$ vanishes (for a given Bloch momentum $q_z$ and a given branch $l$), there is a line of zeros in the excitation spectrum determined by

$$ \epsilon_{q_z, l, \sigma}^\alpha + t_y \cos(k_y b) = 0 , \quad (4.37) $$

22
where $q_x$ belongs to the MBZ and $-\pi/b < k_y \leq \pi/b$. When calculating thermodynamics quantities, one has to sum over the quantum numbers associated with the excitation spectrum. For a linearized dispersion law (Eq. (2.1)), the $k_y$ dependence of the spectrum disappears when summing over the momentum $q_x$ and only yields a factor $1/b$. This property results from the fact that $q$ is coupled only with $l_0 G - q$. Thus, the direction along the field modifies the quasi-particle excitation spectrum but not the thermodynamics quantities.

V. BEYOND THE QUANTUM LIMIT APPROXIMATION

In the two preceding sections (Secs. III and IV), we have studied the quantum superconducting phases in the QLA. This approximation allowed us to obtain a clear and simple physical picture. Most of the quantities of interest (critical temperature, Green’s functions, thermodynamics quantities...) could be expressed in a rather simple form. Nevertheless, the QLA suffers from a lot of weak points. It is restricted to the quantum regime $\omega_c \gg T$ and strongly underestimates the critical temperature. It fails to calculate the free energy in the phases $N \geq 4$. Moreover, as will be shown in this section, it does not correctly describe the entire excitation spectrum and the supercurrents vanish in this approximation. It is therefore necessary to extend the analysis of the two preceding sections to take into account all the pairing channels. In the QLA, we only considered the pairing amplitudes $\Delta^\sigma_\alpha(q, l; l_0 G - q, -l_0 - l)$. In this section, we will call these pairing amplitudes primary gaps. The other pairing amplitudes will be called secondary gaps. The reason for this designation is that the primary gaps open at the Fermi level while the secondary gaps open above and below the Fermi level (Sec. V C). Thus, in this section, we will study how the secondary gaps modify the results obtained in the QLA. In the following, we first obtain the exact (at the mean-field level) transition line and construct a variational order parameter, thus recovering the results obtained in Ref. 19. Then, from the Gor’kov equations, we derive the GL expansion of the free energy, and the excitation spectrum is obtained from the Bogoliubov-de Gennes (BdG) equations. Finally, we calculate the current distribution in the ordered phase. As in Sec. IV, the PPB effect is ignored.

A. Transition line and variational order parameter

In order to obtain the exact (at the mean-field level) transition line, we consider the linearized gap equation in a mixed representation $(q_x, m)$ by taking the Fourier transform with respect to the $x$ direction:

$$\lambda^{-1} \Delta(q_x, m) = e \sum_{m'} K(q_x, m, m') \Delta(q_x, m'),$$

(5.1)

$$K(q_x, m, m') = T \sum_{\omega, \alpha} \int dx \: e^{-i q_x (x - x')} G^\alpha_+ (r, r', \omega) G^\alpha_-(r, r', -\omega).$$

(5.2)

Writing the real space Green’s function $G^\alpha_\sigma(r, r', \omega)$ as
\[ G^0_\sigma(r, r', \omega) = \sum_{k_x,y} \phi^0_{k_x,y}(r) \phi^0_{k_x,y}(r')^* G^0_\sigma(k_x, l, \omega), \]  

(5.3)

where \( \phi^0_{k_x,y} \) and \( G^0_\sigma(k_x, l, \omega) \) are given in (2.6) and (2.8), we obtain

\[ cK(q_x, m, m') = (-1)^{m-m'} \sum_l V_{l-2m, l-2m'} \chi(q_x + lG). \]  

(5.4)

The matrix \( V_{l,l'} \) is defined by (3.8) and \( \chi(q_x) = \sum_\alpha \chi^\alpha(q_x) \) is the pair susceptibility at zero magnetic field evaluated at total momentum \( q_x \) (Eq. (3.5)). It is clear from (5.4) that the highest \( T_c \) is obtained when \( q_x \) is a multiple of \( G \) due to the appearance in this case of Cooper logarithmic singularities \( \chi(0) \). Since the change \( q_x \rightarrow q_x \pm 2G \) is equivalent to the change \( m \rightarrow m \pm 1 \), it is sufficient to consider the two cases \( q_x = l_0G \) where \( l_0 = 0, 1 \). Comparing (5.4) with Eq. (18) of Ref. [13], we note that \( \Delta(q_x = l_0G, m) = \Delta^{Q=0G}_2 \). The coefficients \( \Delta^{Q=0G}_2 \) were introduced in Ref. [13] through the Fourier expansion of the order parameter. Thus, we have rederived the linearized gap equation which was obtained in the gauge \( A'(0, 0, -Hx) \).

In the QLA, where only logarithmic singularities are retained, the linearized gap equation reduces to

\[ \lambda^{-1} \Delta^{QLA}(q_x = l_0G, m) = \chi(0) \sum_{m'} (-1)^{m-m'} V_{2m+0, 2m'+0} \Delta^{QLA}(q_x = l_0G, m'). \]  

(5.5)

This matrix equation is solved by the orthogonal transformation \( U \) which was introduced in Sec. [11] and which diagonalizes the matrix \( V_{l,l'} \). Obviously, we recover the expression (3.12) of the critical temperature and

\[ \Delta^{QLA}(q_x = l_0G, m) = (-1)^m U_{2m+l_0, l_0}. \]  

(5.6)

For \( q_x = (l_0 - 2p)G \), the order parameter is expressed as

\[ \Delta^{QLA}(x, m) = (-1)^{m-p} U_{2m-2p+l_0, l_0} \lambda_i(l_0-2p)Gx. \]  

(5.7)

Comparing this expression with (3.13), we note that we must have

\[ U_{2m+l_0, l_0} = \text{const} (-1)^m \sum_l U_{2l+l_0, l_0} J_{l-m}(\hat{t}) J_{l_0-l-m}(\hat{t}) \]  

(5.8)

in order for Eqs. (3.13) and (5.7) to be equivalent. The preceding equation was verified numerically and the exact value of the constant is given in (1.32).

It is not possible to find an analytical expression for the coefficients \( \Delta(q_x = l_0G, m) \). Thus, one has to solve Eq. (5.4) numerically and then calculate every physical quantity as a function of the coefficients \( \Delta(q_x = l_0G, m) \).

As in Sec. [11], we construct the order parameter by taking a linear combination of the solutions of the linearized gap equation. \( \Delta(q_x = (l_0 - 2p)G, m) \) has the form of a strip extended in the \( x \) direction and localized in the perpendicular direction around \( z = (l_0/2 + p)c \). Thus, the triangular vortex lattice with periodicity \( a_z = Ne \) can be constructed as in Sec. [11] which leads to

\[ \Delta(x, m) = \Delta \sum_p \gamma_p \Delta^{l_0}_{m-p} e^{i(l_0-pN)Gx}, \]  

(5.9)
where $\Delta_{m}^{\alpha} \equiv \Delta(q_{x} = l_{0}G, m)$ is the normalized solution of the linearized gap equation (5.1): $\sum_{m} |\Delta_{m}^{\alpha}|^2 = 1$. The amplitude $\Delta$ is chosen real. The coefficients $\gamma_{p}$ were introduced in Sec. \textcopyright4. The order parameter \textcopyright5 can also be obtained from Eq. (41) of Ref.\textcopyright3 by the appropriate gauge transformation and using $\Delta_{m}^{\alpha} = \Delta_{m}^{Q_{x}=l_{0}G}$. Contrary to the QLA, the order parameter \textcopyright5 correctly describes the entire phase diagram, from the weak field regime $(\omega_{c} \ll T)$ where the superconducting state is an Abrikosov vortex lattice up to the very high field regime $(\omega_{c} \gg \tau_{z})$ where the superconducting state is a Josephson vortex lattice. The above form of the order parameter, obtained by taking a linear combination of the solutions of the linearized gap equation, is expected to be a very good approximation as long as the region of interest in the phase diagram is close to the transition line.

\section*{B. Ginzburg-Landau expansion}

In order to derive the thermodynamics close to the transition line, we have to perform a GL expansion of the free energy. The situation is more complicated than in the QLA, because the supercurrents do not vanish and should therefore be taken into account. The Gor’kov equations have to be solved in presence of a non uniform flux density the supercurrents do not vanish and should therefore be taken in to account. The Gor’kov GL expansion of the free energy. The situation is more complicated than in the QLA, because

\begin{equation}
\sum_{\sigma} \Delta_{0}^{\sigma}(r_{1}) = \frac{2}{\omega_{c}} \int d^{2}r \left| \Delta_{0}(r) \right|^2 - \frac{T}{S} \sum_{\alpha,\omega,1,2} \frac{|\Delta_{0}^{\alpha}(1,2)/\Delta|^2}{(i\omega - \epsilon_{1}^{\alpha})(-i\omega - \epsilon_{2}^{\alpha})},
\end{equation}

\begin{equation}
\beta = \frac{T}{S} \sum_{\alpha,\omega} \sum_{1,2,3,4} \frac{\Delta_{0}^{\alpha}(1,2)\Delta_{0}^{\alpha}(1,3)^{*}\Delta_{0}^{\alpha}(4,3)\Delta_{0}^{\alpha}(4,2)^{*}}{\Delta^{4}(i\omega - \epsilon_{1}^{\alpha})(-i\omega - \epsilon_{2}^{\alpha})(i\omega - \epsilon_{3}^{\alpha})(-i\omega - \epsilon_{4}^{\alpha})},
\end{equation}

where $\Delta_{1}(r) = -\Delta_{1}(r)$ is the variational order parameter \textcopyright3. Again, we use the notation
\( i \equiv (q_i, l_i) \). In (5.10), the sums over \( q_i \) are restricted to the MBZ defined in (4.9, 4.10). The pairing amplitudes \( \Delta^\alpha_{l}(i, j) \) are defined by (4.13). Using (5.9), we obtain

\[
\Delta^\alpha_{l}(1, 2) \equiv \delta_{q_1+q_2, l_0}G \Delta^\alpha_{l}(q_{12}, l_1, l_2) = \delta_{q_1+q_2, l_0}G \Delta \sum_p \gamma_p e^{-ipq_{1+2}} \sum_m \Delta^l_0 J_{l_1+p \frac{x}{2} - m} (\alpha l J_{l_2-p \frac{x}{2} - m} (\alpha l)).
\tag{5.11}
\]

As in the QLA, the total Bloch momentum of two paired states is equal to \( l_0 G \). But the pairing is not diagonal any more in the branch index \( l \), i.e. \( \phi^\alpha_{l_0, l} \) is not coupled only to \( \phi^\alpha_{l_0, l} \). Using the property (\( p \) integer)

\[
\Delta^\alpha_{l}(q_z, l_1 + pN, l_2 - pN) = e^{2ipq_{1+2}} \Delta^\alpha_{l}(q_z, l_1, l_2),
\tag{5.12}
\]

we obtain

\[
\alpha = \frac{c}{L_x} \sum_{q_z} \sum_{l_1=1}^N \sum_{l_2} \frac{\Delta^\alpha_{l}(q_z, l_1, l_2)}{\Delta} \left[ \chi((l_0 + l_1 + l_2)G, T_c) - \chi((l_0 + l_1 + l_2)G, T) \right],
\]

\[
\beta = \frac{1}{L_x} \sum_{q_z} \sum_{l_1=1}^N \sum_{l_2} \frac{1}{\Delta} \Delta^\alpha_{l}(q_z, l_1, l_2) \Delta^\alpha_{l}(q_z, l_1, l_3) \Delta^\alpha_{l}(q_z, l_1, l_3) \times \Delta^\alpha_{l}(q_z, l_4, l_3) \Delta^\alpha_{l}(q_z, l_4, l_2) \Delta^\alpha_{l}(q_z, l_4, l_2) \times K_4(l_1, l_2, l_3, l_4),
\tag{5.13}
\]

where

\[
K_4(l_1, l_2, l_3, l_4) = \frac{T}{2L_x \sum_{\omega, \alpha, k_z} (i \omega - \alpha v k_x - \alpha l_1 \omega_c) (-i \omega - \alpha v k_x + \alpha (l_0 + l_3) \omega_c)} \times \frac{1}{(i \omega - \alpha v k_x - \alpha l_4 \omega_c) (-i \omega - \alpha v k_x + \alpha (l_0 + l_2) \omega_c)}. \tag{5.14}
\]

Unless otherwise specified, the sum over the integers \( l_i \) runs from \(-\infty \) to \( \infty \). In (5.14), the sum over \( k_z \) runs from \(-\infty \) to \( \infty \). \( \chi(q_z, T) \) is the pair susceptibility in zero magnetic field at the temperature \( T \). Writing \( \alpha = \alpha' (T - T_c) \) with \( \alpha' = \partial \alpha / \partial T |_{T_c} \), we obtain, using (3.5) and \( \Psi(z) \approx \ln(z) - 1/2z \) for \( |z| \gg 1 \):

\[
\alpha' \approx \frac{N(0) c}{T_c} \frac{1}{L_x} \sum_{q_z} \sum_{l=1}^N \frac{\Delta^\alpha_{l}(q_z, l, -l - l_0)}{\Delta} \left| \beta \right|^2.
\tag{5.15}
\]

We also have

\[
\beta \approx \beta_{BCS} \frac{c}{L_x} \sum_{q_z} \sum_{l=1}^N \frac{\Delta^\alpha_{l}(q_z, l, -l - l_0)}{\Delta} \left| \beta \right|^4.
\tag{5.16}
\]

The corrections to the above expressions of \( \alpha' \) and \( \beta \) are of the order of \( T_c^2 / \omega_c^2 \) (up to logarithmic corrections for \( \beta \)). In the quantum regime \( \omega_c \gg T \), it is therefore sufficient to take into account only the primary gaps in order to calculate the coefficients \( \alpha' \) and \( \beta \). The secondary gaps, which open at \( n \omega_c / 2 \) (\( n \neq 0 \)) above and below the Fermi level (see Sec. \[\text{V.C}\]), do not contribute to \( \alpha' \) and \( \beta \) to leading order in \( T_c / \omega_c \). However, they have to be
taken into account in the calculation of the critical temperature $T_c$. Comparing the primary gaps in Figs. 13-16 obtained numerically from (5.11), with those obtained analytically in the QLA (Eq. (4.16) and Fig. 11), it turns out that the result obtained in the QLA is very close to the exact result. This means that the expression of the coefficients $\alpha'$ and $\beta$ given by (4.23) is very accurate. Thus, if we calculate the free energy and the specific heat jump at the transition, we will recover the results of Sec. IV, the only difference being that the value of $T_c$ is now exact. Again, we find that the minimum of the free energy in the reentrant phase is minimized for $N = 2$. When the field decreases from its value in the reentrant phase, the system undergoes a first order transition and the minimum of the free energy is then obtained for $N = 4$. As in the QLA, the best value of $N$ switches to 6 before reaching the next first order phase transition. This result, which seems to contradict the assumption that the first order phase transitions are due to commensurability effects between the period of the order parameter and the crystalline lattice spacing, indicates that the contribution of the supercurrents to the free energy has to be taken into account in the phases $N \geq 4$.

C. Excitation spectrum

The excitation spectrum can be obtained from the BdG equations. Performing the BdG transformation

$$\psi^\alpha_\sigma(r) = \sum_n \left( u^\alpha_n(r) \gamma^\alpha_{n,\sigma} - \sigma v^\alpha_n(r)^* \gamma^\alpha_{n,\pi} \right),$$

and expanding the coefficients $u$ and $v$ in the basis of the magnetic Bloch states:

$$u^\alpha_n(r) = \sum_i u^\alpha_{n,i} \phi^\alpha_i(r),$$

$$v^\alpha_n(r) = \sum_i v^\alpha_{n,i} \phi^\alpha_i(r)^*,$$

we obtain the following BdG equations:

$$(E^\alpha_{n,\sigma} - \epsilon^\alpha_{q,l,\sigma}) u^\alpha_{n,\mathbf{q} - \mathbf{l}} - \sum_{l'} v^\alpha_{n,l_0 \mathbf{G} - q_{l,l'}} \Delta^\alpha_{l,l'}(q_z, l, l') = 0,$$

$$(E^\alpha_{n,\sigma} + \epsilon^\alpha_{l_0 \mathbf{G} - q_{l,l'}}) v^\alpha_{n,l_0 \mathbf{G} - \mathbf{q} - \mathbf{l}} - \sum_{l'} u^\alpha_{n,q_{l,l'}} \Delta^\alpha_{l,l'}(q_z, l', l')^* = 0.$$  

In the quantum regime, $\omega_c \gg T_c$ so that $\omega_c \gg \Delta$ even at $T = 0$. The latter inequality allows us to treat the pairing amplitudes $\Delta^\alpha_{q_{l,l'}}(q_z, l, l')$ perturbatively. At low temperature, the order parameter (5.9) is modified by contributions of superconducting condensation channels with critical temperature $< T_c$. However, the main contribution to the order parameter still comes from the channel with the highest critical temperature so that (5.9) should remain a good approximation. To leading order in $\Delta^2/\omega_c^2$, the effect of $\Delta^\alpha_{q_{l,l'}}(q_z, l, l')$ is to lift the degeneracy between $\epsilon^\alpha_{q_{l,l'}}$ and $\epsilon^\alpha_{l_0 \mathbf{G} - q_{l,l'}}$. Thus, up to corrections of the order of $\Delta^2/\omega_c^2$, a gap $2|\Delta^\alpha_{q_{l,l'}}|$ opens in the spectrum at $q_z = \alpha k_F - (l - l_0 - l')G/2$ (with the restriction that $q_z$ has to be in the MBZ). This gap is located at $\alpha(l_0 + l + l')\omega_c/2$ away from the Fermi level. The primary gaps $\Delta^\alpha_{q_{l,l}}$, the only ones which were considered in the QLA)
open at the Fermi level, while the secondary gaps open above and below the Fermi level. The resulting spectrum, shown schematically in Fig. 17, is very reminiscent of the one of the FISDW phases.\cite{28,31} Since the one-particle states are localized in the $z$ direction on a length of the order of $\tilde{c}t$, the pairing amplitudes $\Delta_\alpha(q_z,l,l')$ are important only if $|l - l'| < \tilde{t}$. This means that there are $\sim N \sim \tilde{t}$ secondary gaps with a significant value opening above and below the Fermi level and extending on an energy width of the order of $t_z$. In the reentrant phase, the secondary gaps are of the order of $\tilde{t}^2$ with respect to the primary gaps.

When the field decreases (at $T = 0$) below the semiclassical critical field $H_{c2}(0)$, the amplitude of the order parameter will grow so that $\Delta$ will become larger than $\omega_c$. The coherence length $\nu/\Delta$ then becomes smaller than the (longitudinal) magnetic length $2\pi/G$ and the quantum effect of the field (i.e., the bending of the semiclassical orbits by the field) can be ignored. Thus, the condition $\Delta \sim \omega_c$ (at $T = 0$) signals the crossover to the (anisotropic) Abrikosov vortex lattice state.

As noted above, the QLA very accurately describes the primary gaps and therefore provides a very good approximation of the minimum excitation energy. We also note that the zeros of the primary gaps which were obtained in the QLA are not destroyed by the off-diagonal pairings $(l, l')$ ($l' \not= -l_0 - l$). A similar result was obtained in the case of isotropic superconductors where it has been shown that the gapless behavior obtained in the QLA is not destroyed by a weak off-diagonal Landau level pairing.\cite{7,11}

As in the QLA, we obtain a non-monotonous behavior of the minimum excitation energy: it decreases with the field and vanishes in the phase $N = 8$ at $H = 1.7$ T, but becomes finite again in the phase $N = 10$. This gapless behavior turns out to strongly depend on the dispersion law of the non-interacting system and is therefore accidental. For example, if we add a second neighbor hopping term $t' \cos(2k_zc)$ to the dispersion law (2.1), the zeros in the excitation spectrum appear for different values of the magnetic field. A proper treatment of the screening of the magnetic field would modify the BdG equations and is expected to suppress this accidental gapless behavior. Nevertheless, when the field decreases, the minimum excitation energy decreases and becomes very small in the large $N$ phases as can be seen in Fig. 18. This figure shows the gap which opens at the Fermi level in the phases $N = 26$ and $N = 28$ (the order parameter $\Delta_{0,N}(r)$ corresponding to the phase $N = 26$ is shown in Fig. 7 of Ref.\cite{38}). Even if we believe that the gapless behavior of the phase $N = 26$ is accidental, Fig. 18 clearly shows that the large $N$ superconducting phases effectively become gapless (note that one can distinguish 21 different phases (i.e. $N = 2, \cdots, 42$) in the quantum regime: see Fig. 1b of Ref.\cite{19}). It is however difficult to conclude if the system evolves towards a real gapless behavior. Let us finally mention that the wrong estimation of the minimum excitation energy in the phases $N \geq 4$ is related to the poor estimation of the free energy in these phases.

**D. Current distribution**

In this section, we give the expression of the current distribution in the superconducting phases. We will show that the primary gaps do not contribute to the supercurrents. Since the calculation is very fastidious, we only give the final expressions. The current distribution is obtained from
where the function $j_x(r) = evT \sum_{\omega,\alpha,\sigma} \alpha \delta G^\alpha_\sigma(r, r, \omega)$,

\[
\delta G^\alpha_\sigma(r, r', \omega) = -\sum_{1,2,3} \phi^\alpha_1(r) \phi^\alpha_2(r')^* \Delta^\alpha_\sigma(1,3) \Delta^\sigma_T(3,2)^* G^\alpha_\sigma(1, \omega) G^\sigma_T(3, -\omega) G^\alpha_\sigma(2, \omega),
\]  

(5.21)

where $\phi^\alpha_i$ is the magnetic Bloch state defined by (4.8) and we use the notation $i \equiv (q_i, l_i)$. Note that to lowest order ($\Delta^2$), the screening of the external field can be ignored and the correction $\delta G^\alpha_\sigma$ can be calculated with the Green’s functions $G^\alpha_\sigma(i, \omega)$ of the normal phase in the presence of a uniform magnetic field $H$. From (5.20,5.21), we obtain the following expressions for the Fourier transform of $j(r)$ at wave vector $k = (p_1 N G, p_2 2\pi/Nc)$:

\[
j_x(p_1, p_2) = -2e \sum_{l_1=1}^{N} \sum_{l_2,l_3} F_{l_0 + l_1 + l_2 + l_3} e^{-i\pi N (l_1 - l_2 - p_1 N)} e^{-i\pi N (l_1 + l_2 - p_1 N)}
\]

\[
\times J_{p_1 N - l_1 + l_2} \left( 2\ell \sin \frac{p_2 \pi}{N} \right) \frac{1}{L_z} \sum_{q_x} e^{-ip_1 q_x} \Delta^\uparrow_\sigma(q_x, l_1, l_3) \Delta^\downarrow_\uparrow(q_x, l_2, l_3)^*,
\]

(5.22)

Here, $j_x(x, m, m + 1)$ is the current at point $x$ between the chains $m$ and $m + 1$. $\delta G^\alpha_\sigma$ is the correction to the Green's function which results from a non zero order parameter. To lowest order,

\[
\delta G^\alpha_\sigma(r, r', \omega) = -\sum_{1,2,3} \phi^\alpha_1(r) \phi^\alpha_2(r')^* \Delta^\alpha_\sigma(1,3) \Delta^\sigma_T(3,2)^* G^\alpha_\sigma(1, \omega) G^\sigma_T(3, -\omega) G^\alpha_\sigma(2, \omega),
\]  

(5.20)

where $\phi^\alpha_i$ is the magnetic Bloch state defined by (4.8) and we use the notation $i \equiv (q_i, l_i)$. Note that to lowest order $(\Delta^2)$, the screening of the external field can be ignored and the correction $\delta G^\alpha_\sigma$ can be calculated with the Green’s functions $G^\alpha_\sigma(i, \omega)$ of the normal phase in the presence of a uniform magnetic field $H$. From (5.20,5.21), we obtain the following expressions for the Fourier transform of $j(r)$ at wave vector $k = (p_1 N G, p_2 2\pi/Nc)$:

\[
j_x(p_1, p_2) = -2e \sum_{l_1=1}^{N} \sum_{l_2,l_3} F_{l_0 + l_1 + l_2 + l_3} e^{-i\pi N (l_1 - l_2 - p_1 N)} e^{-i\pi N (l_1 + l_2 - p_1 N)}
\]

\[
\times J_{p_1 N - l_1 + l_2} \left( 2\ell \sin \frac{p_2 \pi}{N} \right) \frac{1}{L_z} \sum_{q_x} e^{-ip_1 q_x} \Delta^\uparrow_\sigma(q_x, l_1, l_3) \Delta^\downarrow_\uparrow(q_x, l_2, l_3)^*,
\]

(5.22)

where the function $F_{N,M}$ is defined by

\[
F_{N,M} = \begin{cases}
\frac{1}{4\pi T} \text{Im} \left[ \frac{1}{2} - \frac{N\omega_c}{4\pi T} \right] - \text{Re} \left[ \frac{1}{2} - \frac{N\omega_c}{4\pi T} \right] & \text{if } N = M, \\
\frac{1}{\pi(N-M)\omega_c} & \text{if } N \neq M.
\end{cases}
\]  

(5.23)

The function $F_{N,M}$ has been introduced in Ref.14 where its expression is given by Eq. (50) for $\omega_c \gg T$. If we retain only the primary gaps $\Delta^\alpha_\sigma(q_z, l, -l - l_0)$, then $F = F_{0,0} = 0$ so that the supercurrents vanish. Thus, in the QLA where the secondary gaps are ignored, the supercurrents vanish. The current distribution can also be obtained as a function of the coefficients $\Delta^\alpha_\sigma$ using the relation

\[
\Delta^\uparrow_\sigma(q_z, l_1, l_2) = \Delta \sum_{m} \Delta^\alpha_\sigma \frac{1}{N} \sum_{p=1}^{N} [1 + i(-1)^p] e^{i(q_z c - p \frac{T}{N})(l_1 - l_2)}
\]

\[
\times e^{-i\frac{\pi}{2}(l_1 + l_2 - 2m)} J_{l_1 + l_2 - 2m} \left( 2\ell \sin \left[ q_z c - p \frac{T}{N} \right] \right).
\]  

(5.24)

Again, we only give the final expressions:
\[ j_x(p_1, p_2) = -\frac{4e\Delta^2}{Nc} \sum_{m_1,m_2,l_1,l_2} \Delta^l_{m_1} \Delta^l_{m_2} F_{l_0+l_1+2m_1,0+l_2+2m_2} e^{-ip_2 \frac{\pi}{N}(l_2+2m_2)} \]

\[ \times e^{i\frac{\pi}{4}(p_1N-2m_1+2m_2)} J_{p_1N-1-2m_1+l_2+2m_2} \left( 2\tilde{t} \sin \frac{p_2 \pi}{N} \right) \]

\[ \times \left( e^{-iu(p_1N-l_1-2m_1+l_2+2m_2)} J_{l_1} \left( -2\tilde{t} \sin \left[ u + p_2 \frac{\pi}{N} \right] \right) J_{l_2} \left( -2\tilde{t} \sin u \right) \right) \eta_{p_1,p_2}, \quad (5.25) \]

\[ j_z(p_1, p_2) = \frac{2ct \Delta^2}{ivN} \sum_{\beta=\pm} \beta \sum_{m_1,m_2,l_1,l_2} \Delta^l_{m_1} \Delta^l_{m_2} F_{l_0+l_1+2m_1,0+l_2+2m_2} e^{-ip_2 \frac{\pi}{N}(l_2+2m_2-1)} \]

\[ \times e^{i\frac{\pi}{4}(p_1N-2m_1+2m_2+\beta)} J_{p_1N-1-2m_1+l_2+2m_2+\beta} \left( 2\tilde{t} \sin \frac{p_2 \pi}{N} \right) \]

\[ \times \left( e^{-iu(p_1N-l_1-2m_1+l_2+2m_2)} J_{l_1} \left( -2\tilde{t} \sin \left[ u + p_2 \frac{\pi}{N} \right] \right) J_{l_2} \left( -2\tilde{t} \sin u \right) \right) \eta_{p_1,p_2}, \quad (5.26) \]

where we use the notation \( \langle \cdots \rangle = \int_0^{2\pi} \cdots \frac{du}{2\pi} \). The function \( \eta_{p_1,p_2} \) is defined by

\[ \eta_{p_1,p_2} = \begin{cases} 
1 & \text{if } p_1 \text{ and } p_2 \text{ are even}, \\
i & \text{if } p_1 \text{ and } p_2 \text{ are odd}, \\
0 & \text{otherwise}.
\end{cases} \quad (5.27) \]

Since \( \Delta^l_m = \Delta^{Q=0}_m \), Eqs. (5.25, 5.26) are analogous to Eqs. (48,49) of Ref.\[1] where the current distribution has been calculated in the gauge \( \mathbf{A}'(0,0,-Hx) \). The current distribution was calculated numerically in Ref.\[10] using (5.25,5.26). In the reentrant phase, (5.25,5.26) can be simplified by retaining only the terms of order \( \tilde{t}^2 \): the current distribution is characteristic of a triangular Josephson vortex lattice. In the phases \( N \geq 4 \), the current distribution shows a symmetry of a laminar type and is different from what is obtained in the Abrikosov or Josephson vortex lattice.\[4]

VI. CONCLUSION

We have presented a systematic study of the phase diagram of a quasi-1D superconductor in a high magnetic field. We have obtained the thermodynamics quantities and the quasi-particle excitation spectrum of the quantum superconducting phases which are stabilized at high magnetic field as a result of the magnetic-field-induced confinement of the electrons. The reentrant phase (very high field limit) is the natural limit for the study of superconductivity, since the orbital frustration of the order parameter is suppressed and the screening of the external magnetic field can be ignored, which considerably simplifies the analysis. Although we have not taken into account the screening of the field in the other phases, our analysis clearly shows how the properties of the system evolves when the field is decreased from its value in the reentrant phase.

The main results can be summarized as follows. i) In the reentrant phase \( (\omega_c \gg t_z) \), the superconducting state is a Josephson vortex lattice. The behavior of the system is very close to the zero-field BCS situation, up to corrections of the order of \( t_z^2/\omega_c^2 \). For instance, the
supercurrents are of the order of $t_z^2/\omega_c^2$ and the specific heat jump at the superconducting transition $\Delta C/C_N$ is very close to the zero-field BCS value $(\Delta C/C_N)_{BCS}$. The phase is paramagnetic due to the positive slope of transition temperature $T_c$ ($dT_c/dH > 0$). Gaps open at the Fermi level on the whole MBZ. The dispersion of these gaps is of the order of $t_z^2/\omega_c^2$ and the minimum excitation energy is finite. Gaps also open at $n\omega_c/2$ ($n$ integer) away from the Fermi level but are of the order of $t_z^2/\omega_c^2$ with respect to the gaps opening at the Fermi level. ii) In the other phases ($N \geq 4$), the behavior of the system deviates more substantially from the zero-field BCS situation. The phase is first paramagnetic and then diamagnetic, which is a consequence of the sign change of the slope of the transition line $T_c$. The specific heat jump at the transition becomes smaller than the zero-field BCS value $(\Delta C/C_N)_{BCS}$. There are $\sim N \sim \tilde{t}$ gaps with a significant value opening below and above the Fermi level. The dispersion of the gaps which open at the Fermi level increases. The minimum excitation energy decreases and the quasi-particle spectrum becomes gapless for large $N$.

Our result have been obtained within a simple model where the electrons interact through an effective local attractive interaction. Some of these results would be modified in a more complicated (or realistic) model. For example, in the case of a $d$-wave (with respect to the $x$ and $y$ axis) superconductor, the excitation spectrum would be gapless for any value of the field. The inclusion in our analysis of the PPB effect, which yields to the formation of a LOFF state, would also result in a gapless behavior. However, the general structure of the phase diagram does not depend on the details of the microscopic mechanism which is at the origin of the attractive electron-electron interaction: the existence of superconductivity at high magnetic field in a quasi-1D superconductor originates in the magnetic-field-induced dimensional crossover and does not rely on a particular model of superconducting pairing.

ACKNOWLEDGMENTS

The author would like to thank G. Montambaux for useful discussions on this work and M. Gabay for interesting discussions on related subjects. The Laboratoire de Physique des Solides is Unité Associée au CNRS.
REFERENCES

1 A.A. Abrikosov, Sov. Phys. JETP 5, 1174 (1957).
2 L.P. Gor’kov, Sov. Phys. JETP 9, 1364 (1959).
3 See, for example, Superconductivity, edited by R.D. Parks (Dekker, New-York, 1969).
4 A.K. Rajagopal and R. Vasudevan, Phys. Lett. 23, 539 (1966); L.W. Gruenberg and L. Gunther, Phys. Rev. 176, 606 (1968).
5 Z. Tešanović, M. Rasolt and L. Xing, Phys. Rev. Lett. 63, 2425 (1989) and Phys. Rev. B 43, 288 (1991).
6 M. Rasolt and Z. Tešanović, Rev. Mod. Phys. 64, 709 (1992).
7 S. Dukan, A.V. Andreev and Z. Tešanović, Physica C183, 355 (1991); S. Dukan and Z. Tešanović, Phys. Rev. B 49, 13017 (1994).
8 T. Maniv, A.I. Rom, I.D. Wagner and P. Wyder, Phys. Rev. B 46, 8360 (1992); M.J. Stephen, Phys. Rev. B 45, 5481 (1992).
9 J.C. Ryan and A.K. Rajagopal, Phys. Rev. B 47, 8843 (1993); A.K. Rajagopal, in Selected Topics in Superconductivity, edited by L.C. Gupta and M.S. Multani, Frontiers in Solid State Sciences Vol.1 (World Scientific, Singapore 1993).
10 H. Akera, A.H. MacDonald, and S.M. Girvin, Phys. Rev. Lett. 67, 2375 (1991); M.R. Norman, H. Akera, and A.H. MacDonald, Physica C196, 43 (1992).
11 M.R. Norman, A.H. MacDonald, and H. Akera, unpublished (1994).
12 C.T. Rieck, K. Sharnberg and R.A. Klemm, Physica C170, 195 (1990); K. Sharnberg and C.T. Rieck, Phys. Rev. Lett. 66, 841 (1991).
13 V.M. Yakovenko, Phys. Rev. B 47, 8851 (1993).
14 J.E. Graebner and M. Robbins Phys. Rev. Lett. 36, 422 (1976).
15 See, for example, Y. Onuki et al., J. Phys. Soc. Jpn 61, 692 (1992); R. Corcoran et al., Phys. Rev. Lett. 72, 701 (1994); F.M. Mueller et al., Phys. Rev. Lett. 68, 3928 (1992); N. Harrison et al., Phys. Rev. B 50, 4208 (1994); C.M. Fowler et al., Phys. Rev. Lett. 68, 534 (1992); G. Kido et al., J. Phys. Chem. Solids 53, 1555 (1992); E.G. Haanappel et al., J. Phys. Chem. Solids 54, 1261 (1993); R.G. Goodrich et al., J. Phys. Chem. Solids 54, 1251 (1993).
16 P. Fulde and R.A. Ferrell, Phys. Rev. 135, A550 (1964); A.I. Larkin and Yu.N. Ovchinnikov, Sov. Phys. JETP 20, 762 (1965).
17 A.G. Lebed’, JETP Lett. 44, 114 (1986); L.I. Burlachkov, L.P. Gor’kov and A.G. Lebed’, EuroPhys. Lett. 4, 941 (1987).
18 N. Dupuis, G. Montambaux and C.A.R. Sá de Melo, Phys. Rev. Lett. 70, 2613 (1993).
19 N. Dupuis and G. Montambaux, Phys. Rev. B. 49, 8993 (1994).
20 N. Dupuis, Phys. Rev. B 50, 9607 (1994).
21 N. Dupuis, submitted to Phys. Rev. B (1994).
22 K.B. Efetov, J. Phys. (Paris) Lett. 44, L369 (1983).
23 L.P. Gor’kov and A.G. Lebed’, J. Phys. (Paris) Lett. 45, L433 (1984).
24 I.J. Lee, A.P. Hope, M.J. Leone and M.J. Naughton, unpublished (1994).
25 C. Bourbonnais and L. Caron, Int. J. Mod. Phys. B 5, 1033 (1991).
26 The competition between superconductivity and spin-density-wave below $T_{c}$ is suppressed by assuming that the nesting of the dispersion law is sufficiently imperfect in the direction of the field. See Ref14 for a more detailed discussion.
This paramagnetic/diamagnetic crossover as the field is increased within a given phase has also been proposed by C.A.R. Sá de Melo (unpublished).

K. Yamaji, J. Phys. Soc. Jpn 54, 1034 (1985).

A. Virosztek, L. Chen and K. Maki, Phys. Rev. B 34, 3371 (1986).

G. Montambaux and D. Poilblanc, Phys. Rev. B 37, 1913 (1988).

For a review on the FISDW, see for example G. Montambaux, Physica Scripta T35, 188 (1991).

N. Dupuis and G. Montambaux, Phys. Rev. B 46, 9603 (1992).

V.M. Yakovenko, Sov. Phys. JETP 66, 355 (1987).

G.H. Wannier, Phys. Rev. 117, 432 (1960); G.H. Wannier, Rev. Mod. Phys. 34, 645 (1965); J. Callaway, Phys. Rev. 130, 549 (1963).

E.E. Mendez, F. Agulló-Rueda and J.M. Hong, Phys. Rev. Lett. 60, 2426 (1988); P. Voisin, J. Bleuse, C. Bouche, S. Gaillard, C. Alibert and A. Regreny, Phys. Lett. 61, 1639 (1988).

M. Héritier, G. Montambaux and P. Lederer, J. Phys. 45, L943 (1984).

A similar picture of the quantized nesting mechanism has been given by G. Montambaux, in Low Dimensional Conductors and Superconductors, NATO ASI, Vol. 155, Plenum, New-York (1987).

K. Maki, Phys. Rev. B 33, 4826 (1986).

L.P. Gor’kov, Sov. Phys. JETP 10, 998 (1960).

P.W. Anderson, J. Phys. Chem. Solids 11, 26 (1959).

A.G. Lebed’, private communication (unpublished).

L.G. Aslamazov, Sov. Phys. JETP 28, 773 (1969); S. Takada, Prog. Theor. Phys. 43, 27 (1970).

G. Montambaux et al., Phys. Rev. B 39, 885 (1989).

See, for example, A.L. Fetter and J.D. Walecka, Quantum Theory of Many Particle Systems, Chap. 13 (McGraw-Hill, 1971).

W.H. Kleiner, L.M. Roth, and S.H. Autler, Phys. Rev. 133, A1226 (1964).

D. Poilblanc, Ph.D thesis, Université Paris-Sud, unpublished (1988).

See, for example, P.G. de Gennes, Superconductivity of Metals and Alloys (Addison-Wesley, 1966).
FIGURES

FIG. 1. Diagrammatic representation of the ladder approximation for the two-particle vertex function $\Gamma^{\alpha\alpha'}(r_1, r_2; r_1', r_2')$. The zigzag line denotes the attractive electron-electron interaction. $\alpha$ and $\alpha'$ refer to the sheet of the Fermi surface.

FIG. 2. Solid lines: critical temperature vs magnetic field for $l_0 = 0$ and $l_0 = 1$ in the QLA. Dashed lines: exact mean-field critical temperature (see Fig. 1 of Ref. 19). $T_{c0} = 1.5$ K and $t_z = 20$ K.

FIG. 3. a) Self-energy correction in the Born approximation due to impurity scattering. b) Vertex correction due to impurity scattering for the pair propagator. The dashed lines with a cross denote impurity scattering.

FIG. 4. Amplitude and phase of the order parameter $\Delta_{l_0,N}(r)$ (Eq. (4.2)) obtained in the QLA in the phase $l_0 = 0, N = 2 (H = 5.8$ T). In order to compare with the exact order parameter obtained in Ref. 19, we have made a gauge transformation from $A(Hz, 0, 0)$ to $A'(0, 0, -Hz)$.

FIG. 5. As in Fig. 4, but for the phase $l_0 = 1, N = 4 (H = 4$ T).

FIG. 6. As in Fig. 4, but for the phase $l_0 = 0, N = 6 (H = 2.4$ T).

FIG. 7. As in Fig. 4, but for the phase $l_0 = 1, N = 8 (H = 1.7$ T).

FIG. 8. As in Fig. 4, but for the phase $l_0 = 0, N = 10 (H = 1.3$ T).

FIG. 9. Excitation spectrum $\epsilon_{q,l,\sigma}^\alpha = \epsilon_{qz,l,\sigma}^\alpha$ of the normal phase in the representation of the magnetic Bloch states $\phi_{q,l}^\alpha$. $q_x$ is restricted to the MBZ (4.9,4.10). The figure corresponds to $N = 2$.

FIG. 10. Ratio $r = (\Delta C/C_N)/(\Delta C/C_N)_{BCS}$ vs magnetic field.

FIG. 11. Pairing amplitudes $|\Delta_{q,z,l}^\alpha|$ in the QLA in the phases $N = 2, N = 4, N = 6, N = 8$ and $N = 10$. The units are chosen so that max $|\Delta_{q,z,l}^\alpha| = 1$.

FIG. 12. Quasi-particle excitation spectrum $E_{q,l,\sigma}^\pm$ in the QLA in the reentrant phase $l_0 = 0, N = 2$ for the branches $l = 0$ (a) and $l = 1$ (b). $q$ is restricted to the MBZ. The gap opens at the Fermi level.
FIG. 13. Pairing amplitudes $|\Delta_\sigma^\alpha(q_z, l, L - l - l_0)|$ in the exact mean-field analysis in the phase $N = 2$ ($H = 5.1$ T). A gap $2|\Delta_\sigma^\alpha(q_z, l, L - l - l_0)|$ opens in the branch $l$ at $\alpha L \omega_c/2$ away from the Fermi level. The units are chosen so that $\max |\Delta_\sigma^\alpha(q_z, l, L - l - l_0)| = 1$.

FIG. 14. As in Fig. 13 but for the phase $N = 4$ ($H = 2.8$ T).

FIG. 15. As in Fig. 13 but for the phase $N = 6$ ($H = 2$ T).

FIG. 16. As in Fig. 13 but for the phase $N = 8$ ($H = 1.5$ T).

FIG. 17. Schematic representation of the quasi-particle excitation spectrum for the right sheet of the Fermi surface. $q_z$ is restricted to the MBZ. Gaps open at $n\omega_c/2$ ($n$ integer) away from the Fermi level. The dispersion with respect to $q_z$ is not shown.

FIG. 18. Pairing amplitudes (primary gaps) $|\Delta_\sigma^\alpha(q_z, l, -l - l_0)|$ in the exact mean-field analysis in the phases $N = 26$ ($H = 0.48$ T) and $N = 28$ ($H = 0.45$ T).