Rogue waves statistics in the framework of one-dimensional Generalized Nonlinear Schrodinger Equation

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We measure evolution of spectra, spatial correlation functions and probability density functions (PDF) of waves appearance for one-dimensional generalized Nonlinear Schrödinger equation: (1) accounting for six- and eight-wave interactions terms and (2) accounting for six-wave interactions, dumping (linear dissipation and three-photon absorption) and pumping terms. All additional terms beyond the classical NLS equation are small. We observe strongly non-Gaussian PDFs with “fat tails” in the region of large amplitudes when higher waves appear more frequently. For generalized NLS equation with six-wave interactions, dumping and pumping terms we demonstrate absence of non-Gaussian addition to PDF for zeroth six-wave interactions coefficient and increase of non-Gaussian addition with six-wave interactions term.

1. Since the first observation by Solli et al. in 2007 \[1\], optical rogue waves - large wave events with statistics drastically different from Gaussian that appear randomly from initially smooth pulses - has drawn much scientific attention from both optical and hydrodynamic society. In case of optics rogue waves are huge waves that can damage optical systems and therefore their appearance must be controlled. For hydrodynamics optical rogue waves are interesting phenomenon that can be conveniently studied in laboratory conditions and that occurs in systems described by the similar equations and has similar to hydrodynamic rogue waves statistics \[2\]. The current study of optical rogue waves went in two main directions: 1) harnessing and control of rogue waves emergence by seeding the initial pulse \[3, 4\], and 2) understanding physical mechanisms underlying the phenomenon by establishing connection between different linear and nonlinear terms in the equations of motion and appearance of non-Gaussian tails in the probability of large waves occurrence \[5-8\]. The aim of the current publication is to study the influence of higher nonlinearity, namely six-wave interactions term, on the frequency of rogue waves appearance for two different nonlinear systems one of which is conservative and the other is non-conservative.

Our research can also be considered in the broader context. Nonlinear Schrodinger equation

\[i \Psi_t + \Psi_{xx} + |\Psi|^2 \Psi = 0,\]  \hspace{1cm} (1)

generalizations of which are studied in the framework of rogue waves phenomenon, describes many physical systems from Bose-Einstein condensate and plasma oscillations to propagation of waves in optics and hydrodynamics. We consider the first nonlinear term beyond Eq. \(1\) that naturally appears as next-order term in perturbation theory and investigate its influence on the map of extreme events of the system in the regime when dynamics of the system is close to that described by the classical NLS equation. In this sense our choice of parameters is more natural than in the recent publication \[9\] where the similar research was made for the quintic NLS equation.

Let us suppose that the current state of a system consists of multitude of uncorrelated linear waves,

\[\Psi = \sum_k a_k \exp(kx - \omega_k t + \phi_k).\]

If \(a_k\) and \(\phi_k\) are random uncorrelated values and number of linear waves is large enough, then probability density function (PDF) for amplitude \(|\Psi|\) is Gaussian-distributed,

\[PDF(|\Psi|) \sim \exp(-|\Psi|^2/2\sigma^2).\]
In the current publication we search for deviations from Gaussian distribution at large amplitudes, namely for signs of non-exponential tails when higher waves occur more frequently. Presence of such a tail would mean the probability of occurrence of a large wave event is by several orders of magnitude higher than that predicted by linear theory and therefore the map of extreme events is qualitatively different from the ordinary Gaussian.

In this work we perform numerical simulations of wave field $\Psi$ evolution in the framework of two different nonlinear equations for ensemble of 10000 initial distributions $\Psi(t=0)$ for each nonlinear system. Inside each ensemble initial distributions differ only by realization of stochastic noise with a fixed statistical properties. Based on the ensembles, we measure spectra $I_k = \langle |\Psi_k|^2 \rangle$ where $\langle \ldots \rangle$ stands for averaging over ensemble, spatial correlation functions $g(x) = \langle \Psi(y,t)\Psi^*(y+x,t) \rangle$ and PDFs for amplitudes of waves $|\Psi|$ and examine how these functions depend on time and different nonlinear terms included in the equations of motion.

The paper is organized as follows. Next section gives brief overview of nonlinear systems we examine and also numerical methods we use. Results of our numerical simulations and their comparisons are listed in Section 3. Section 4 contains conclusions and acknowledgements.

2. In this work we consider two different one-dimensional generalized Nonlinear Schrodinger equations: 1) accounting for six- and eight-wave interactions terms,

$$i\Psi_t + \Psi_{xx} + |\Psi|^2 \Psi + \alpha |\Psi|^4 \Psi - \beta |\Psi|^6 \Psi = 0, \quad \alpha, \beta > 0, \quad \alpha, \beta/\alpha \ll 1,$$

and 2) accounting for the collapsing six-wave interactions term, as well as dumping terms (linear dissipation and three-photon absorption) and a pumping term,

$$i\Psi_t + (1 - id_l)\Psi_{xx} + |\Psi|^2 \Psi + (\alpha + id_n)|\Psi|^4 \Psi = ip\Psi, \quad \alpha, d_l, d_n, p > 0, \quad \alpha, d_l, d_n, p \ll 1.$$  

Here $t$ is time, $x$ is spatial coordinate, $\alpha$ and $\beta$ correspond to six- and eight-wave interactions terms respectively, coefficients $d_l$ and $d_n$ determine linear and nonlinear dumping respectively and coefficient $p$ corresponds to deterministic forcing of the system. Eq. (2)-(3) correspond to anomalous group velocity dispersion (GVD) regime because optical rogue waves are huge pulses lying entirely in the anomalous GVD [1–8]. Eq. (2) is the Hamiltonian one,

$$i\Psi_t = \delta H / \delta \Psi^*,$$

with Hamiltonian

$$H = H_d + H_4 + H_6 + H_8, \quad H_d = \int |\Psi_x|^2 \ dx,$$

$$H_4 = -\int \frac{|\Psi|^4}{2} \ dx, \quad H_6 = -\int \frac{\alpha |\Psi|^6}{3} \ dx, \quad H_8 = \int \frac{\beta |\Psi|^8}{4} \ dx.$$  

In addition to Hamiltonian, Eq. (2) also conserves wave action $N = \int |\Psi|^2 \ dx$ and momentum $P = (i/2) \int (\Psi_x^* \Psi - \Psi \Psi_x^*) \ dx$. For Eq. (3) energy $H_d + H_4 + H_6$, wave action and momentum become functions of time.

Addition of six-wave interactions term only to the classical NLS equation (1) results in generation of blow-up collapses in a finite time. There are two ways to regularize these collapses: by adding next-order nonlinear term in the form of defocusing eight-wave interactions (2) or with the help of dumping terms (3). Thus, nonlinear dissipation in the form of three-photon absorption prevents formation of waves with too high amplitude, while linear dissipation may originate from optical filtering [10] and prevents appearance of too high gradients. Energy loss due to the dumping terms that manifests itself mainly as a result of regularization of collapses (compare to [9]) is compensated by inclusion of forcing term. In this work we limit ourselves with consideration of only deterministic forcing in Eq. (3) because according to [9] we do not expect
qualitative difference for other types of forcing. Specific values of coefficients for Eq. (2)-(3) \(\alpha, \beta, d_1, d_n, p\) were chosen to be small enough in order to make the dynamics of the systems close to that described by the classical NLS equation (1).

We solve Eq. (2)-(3) numerically in the box \(-16\pi \leq x < 16\pi\) with periodical boundary conditions starting from the initial data \(\Psi|_{t=0} = 1 + \epsilon(x)\), where \(\epsilon(x)\) is stochastic Gaussian-distributed noise. We did not find difference in our results using other statistical distributions of noise. We observe modulation instability with the fastest growing mode corresponding to wavelength \(2\pi\) that develops to time-shifts \(t \sim 10\) generating 16 peaks in such a box and leading to formation of one-dimensional wave turbulence.

For our initial conditions kinetic energy \(H_d\) is comparable to the potential one \(H_4\), that means we are working in the regime of solitonic turbulence when solitons or quasi-solitons play significant role in the turbulent re-distribution of energy inside the system (see also [11,12]). In the integrable case (1) the turbulence is called integrable and relaxes to one of infinite possible stationary states. Eq. (2) describes statistically irreversible movement to it's statistical attractor - one big soliton containing all the potential energy \(H\) and immersed in the field of small fluctuations. Up to time shifts \(t \sim 60\) wave field \(\Psi\) quickly (tens of nonlinear lengths) approaches to statistically steady state when high waves appear randomly in space and time.

In our numerical simulations we used the 2nd-order Split-Step method in which linear and nonlinear parts of the equations were calculated separately. In order to improve simulations and save computational resources we employed adaptive change of spacial grid size \(\Delta x\) reducing it when Fourier components of solution \(\Psi_k\) at high wave numbers \(k\) exceeded \(10^{-13}\max|\Psi_k|\) and increasing \(\Delta x\) when this criterion allowed. In order to prevent appearance of numerical instabilities, time step \(\Delta t\) also changed with \(\Delta x\) as \(\Delta t = h\Delta x^2\) with \(h \leq 0.1\) (see [14]).

Implementation of such numerical schema allowed us to safely shift up to \(t \sim 100\) nonlinear lengths, i.e. when

\[
||\Psi_h(t,x) - \Psi_{0.5h}(t,x)||/||\Psi_{0.5h}(t,x)|| \ll 1, \tag{5}
\]

where \(\Psi_h(t,x)\) is the numerical solution calculated with a fixed coefficient \(h = \Delta t/\Delta x^2\). We also did comparisons with 4th-order Split-Step method [15] as well as with 4th- and 5th-order Runge-Kutta methods that confirmed validity of our results for single simulations up to \(t \sim 100\) nonlinear lengths.

Thus, all methods we used, including comparisons with the results obtained with higher number of Fourier modes (lower \(\Delta x\)) with the same initial noise, gave virtually the same results for the classical NLS equation up to time shifts \(t \sim 60\). Beyond 60 nonlinear lengths all methods and all schema parameters \(\Delta t\) and \(\Delta x\) we used gave different results in the sense of criterion (5). This behavior is connected with quasi-periodical dynamics of the classical integrable NLS equation: near \(t \sim 60\) wave field again can be represented as \(\Psi = 1 + \xi(x)\) with \(|\xi(x)| \ll 1\), but this time \(\xi(x)\) contains numerical errors that are unique for numerical method and it's parameters \(\Delta t\) and \(\Delta x\). Therefore, beyond 60 nonlinear lengths modulation instability develops differently for different numerical methods and their parameters.

Nevertheless, comparison of our statistical results calculated with \(h_0 \sim 1/12\), in particular PDFs, revealed no difference with the results obtained with \(h = 0.25h_0\) or with the help of 4th-order Split-Step or 4th- and 5th-order Runge-Kutta methods far beyond 100 nonlinear lengths.

3. We start this section with our results concerning evolution of kinetic and potential energy for Eq. (2)-(3) as well as for evolution of total energy and wave action for Eq. (3). As shown on FIG. 1a, generalized NLS equation accounting for six- and eight-wave interactions (3) demonstrates asymptotic movement to its statistical attractor - one big soliton containing all the potential energy \(H_1 + H_6 + H_8\) and immersed in the field of small fluctuations. Up to time shifts \(t \sim 300\) for \(\alpha = 0.04, \beta = 0.001\) and \(t \sim 150\) for \(\alpha = 0.064, \beta = 0.002\) six- \(H_6\) and eight-wave interactions \(H_8\) are small compared to kinetic \(H_d\) and four-wave interactions energy \(H_4\). Beyond these
time shifts are seen the first signs of relaxation when $H_d$, $H_4$, $H_6$ and $H_8$ become comparable first and then approach to their asymptotic values after $t \sim 2000$ for $\alpha = 0.04$, $\beta = 0.001$ and $t \sim 1500$ for $\alpha = 0.064$, $\beta = 0.002$.

![FIG. 1](image)

**FIG. 1**: (Color on-line) Dependence on time shift $t$ of averaged over ensemble (a) kinetic energy $\langle H_d \rangle$ (black), four-$\langle H_4 \rangle$ (blue), six-$\langle H_6 \rangle$ (green) and eight-wave interactions energy $\langle H_8 \rangle$ (red) for generalized NLS equation accounting for six- and eight-wave interactions $[2]$, $\alpha = 0.040$, $\beta = 0.001$; (b) total energy $\langle H_4 + H_6 \rangle$, (c) wave action $N$ and (d) kinetic energy $\langle H_d \rangle$ (black), four-$\langle H_4 \rangle$ (blue) and six-wave interactions energy $\langle H_6 \rangle$ (green) for generalized NLS equation accounting for six-wave interactions, dumping and pumping terms $[3]$, $\alpha = 0.128$, $d_l = 0.04$, $d_n = 0.0004$, $p = 0.05$.

Another way of regularization of collapses due to six-wave interactions with the help of dumping and pumping terms in Eq. $[3]$ allows one to constantly stay in the regime where $|H_6| \ll |H_d|, |H_4|$, i.e. when dynamics of the system is close to that described by the classical NLS equation (see FIG. 1c). After the first 20-50 nonlinear lengths the system reaches statistically steady state when energy drain due to dissipation is compensated by energy income due to pumping and wave action, momentum and total energy as well as kinetic $H_d$, four- $H_4$ and six-wave interactions energy $H_4$ fluctuate near their mean values (see FIG. 1b,c). The statistically steady state is determined by coefficients $\alpha$, $d_l$, $d_n$ and $p$ only and does not depend on initial distribution $\Psi|_{t=0}$: thus, we observe the same results for $\Psi|_{t=0} = \epsilon(x)$ where $|\epsilon(x)| \ll 1$ is stochastic noise. In the steady state spectra $I_k$, spatial correlation functions $g(x)$, correlation length $x_{corr}$ defined as full width at half maximum for $g(x)$ and PDFs for wave field amplitude $|\Psi|$ measured at some specific time $t$ no longer depend on time that allows us to perform additional averaging over time.

**FIG. 2** shows evolution of spectra, spatial correlation function and PDF for generalized NLS equation accounting for six- and eight-wave interactions $[2]$. During relaxation process spectra gradually accepts some universal bell-shaped form with two exponential parts corresponding to small and large wave-numbers $k$ and can be divided by two parts - the upper one corresponding to growing soliton and the lower one corresponding to small fluctuations field. At small time-shifts $t \sim 50$ and for different sets of $\alpha$ and $\beta$ normalized spatial correlation function $g(x)/g(0)$ is close to universal form which approaches to Gaussian for small $|x| < x_{corr}$. At larger time-shifts universal form of correlation functions persists and becomes sharp, almost triangular in its maximum at $x = 0$. Evolution of correlation length $x_{corr}$ for Eq. $[2]-[4]$ is shown on FIG. 3.

Optical rogue waves are typically observed in the initial stage of modulation instability $[1]$ that occurs at time shifts $t \sim 10$ for our initial pulse and noise amplitude. At these time shifts even despite significantly nonlinear regime $H_d \sim H_4$ when weak wave turbulence approach does not work PDFs for Eq. $[2]$ and for the classical NLS equation fluctuate near Gaussian distribution. These fluctuations occur because different realizations of noise lead to different times of modulation instability development and diminish at larger times (see FIG. 4a for classical NLS equation for example).

Thus, and at $t = 40$ PDFs for Eq. $[2]$ turn out to be almost Gaussian with the exception of
small amplitudes regions. We observe such behavior until six- $H_6$ and eight-wave interactions $H_8$ are small compared with dispersion $H_d$ and four-wave interactions $H_4$, i.e. until dynamics of the system remains close to that described by the classical NLS equation. When $H_d$, $H_4$, $H_6$ and $H_8$ become comparable there appears power-law region at middle amplitudes on the PDFs that divides initial exponential decay at low amplitudes and the peak at high amplitudes corresponding to relaxation to one big soliton. Beyond this peak PDFs again decay exponentially that in our opinion must be connected with deviations in amplitude of this big soliton due to deviations in initial data $\Psi_{t=0}$. Spectrograms enlarged at pulse maximums shown on FIG. 5a, 5b demonstrate that typical large wave events for Eq. (2) are collisions of quasi-solitons at moderate time shifts $t \sim 100$ and single quasi-solitons at large time shifts $t \sim 2000$ where relaxation to statistical attractor occurs (see [8, 17]).

In contrast to Eq. (1)-(2), Eq. (3) demonstrates significantly different evolution of PDFs with time when long non-Gaussian tails appear already during modulation instability development $t \sim 10$. Beyond these time shits PDFs fluctuate preserving their shape as shown on FIG. 4b where PDFs are represented versus $|\Psi|^2/(|\Psi|^2)$ and $\langle |\Psi|^2 \rangle$ is the averaged over ensemble squared amplitude at the corresponding time shift. At $t \sim 50$ fluctuations of wave action and $\langle |\Psi|^2 \rangle$ diminish (compare with FIG. 1c) as the system reaches statistically steady state where energy drain due to dumping terms is compensated by the energy input from the pumping term. The corresponding “fat tail” in the region of large amplitudes decays faster than any power of amplitude $|\Psi|$ and turns out to be similar to const $\exp(-|\Psi|)$ as shown on FIG. 4c. Spectrogram on FIG. 5c demonstrates that a typical collapse in case of Eq. (3) originates as a collision of several quasi-solitons.

We now compare results obtained for Eq. (3) with different six-wave interactions coefficients $\alpha$ and dumping and pumping parameters $d_l$, $d_n$ and $p$. First, we did several numerical simulations for a set of different $d_l$, $d_n$ and $p$ with fixed $\alpha$ and found our results qualitatively do not depend on pumping and dumping parameters. Then we fixed $d_l = 0.04$, $d_n = 0.0004$ and $p = 0.05$ and did five simulations for different six-wave interactions coefficient from $\alpha = 0$ to $\alpha = 0.256$. Corresponding spectra, spacial correlation functions and PDFs are shown on FIG. 6. Thus, in the statistically steady state spectra consists of two exponential parts $\sim \exp(-q|k|)$ with increasing coefficient $q$ from small to large wave-numbers $k$ and a transition region between them corresponding to medium wave-numbers. Normalized spacial correlation functions $g(x)/g(0)$ approach to some universal function of $x/x_{corr}$ that for small $x$ is close to Gaussian.

The mean wave action at large time shifts changes with $\alpha$ from $N \sim 170$ at $\alpha = 0$ to $N \sim 70$ at $\alpha = 0.256$ because the main power drain from the system occurs during regularization of collapses [11], while the number of collapses per time unit depends on $\alpha$: thus, six-wave interactions dominate for $\alpha = 0.032$ starting from $|\Psi| > 6$ and for $\alpha = 0.256$ starting from $|\Psi| > 2$. In the framework of Eq. (3) there is no independent on $\alpha$ universal spatio-temporal dynamics for the collapses saturation. Since mean wave action at statistically steady state depends on six-wave interactions coefficient, mean square amplitude also depend on it: $\langle |\Psi|^2 \rangle$ decreases with increasing $\alpha$ from 1.7 at $\alpha = 0$ to 0.7 at $\alpha = 0.256$.

Therefore on FIG.6c we plot normalized PDFs depending on $|\Psi|^2/(|\Psi|^2)$ that allows us to examine PDFs for different $\alpha$, $d_l$, $d_n$ and $p$ all on one graph. At small amplitudes $|\Psi|$ PDFs demonstrate almost Gaussian decay that changes at moderate and high amplitudes to strongly non-Gaussian $\sim \exp(-|\Psi|)$ for $\alpha > 0$, while for $\alpha = 0$ PDF remains Gaussian even for high waves. The latter in the sense of Gaussian PDF for the classical NLS equation (1) shows that dumping and pumping terms in Eq. (3) do not affect the form of PDF. For non-zeroth $\alpha$ the corresponding non-Gaussian addition to PDF increase with $\alpha$, i.e. frequency of large wave occurrence increase with six-wave interactions coefficient.

We would like to underline two our main results. First, presence of nonlinearity and significantly nonlinear regime of a system do not necessarily mean non-Gaussian PDFs as demonstrated by our results for the classical NLS equation (1) shown on FIG. 4a and for generalized NLS equa-
tion accounting for pumping and dumping terms \(\alpha = 0\) as shown on FIG. 6c. Generalized NLS equation accounting for six- and eight-wave interactions also demonstrates Gaussian tails on the PDFs in the region of large amplitudes if its dynamics is close to that described by the classical NLS equation.

Second, for Eq. (2) we discovered emergence of power-law tails starting from moderate time-shifts when first signs of relaxation to statistical attractor appear. We also demonstrated presence of non-Gaussian tails \(\sim \exp(-|\Psi|)\) when higher waves appear more frequently for generalized NLS equation accounting for six-wave interactions, pumping and dumping terms even when six-wave interactions are small compared to four-wave interactions. The corresponding non-Gaussian addition does not qualitatively depend on dumping and pumping parameters, disappear for zeroth six-wave interactions coefficient \(\alpha = 0\) and increase with \(\alpha\).

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FIG. 2: (Color on-line) Solid lines are spectra $I_k$ (a), normalized spatial correlation functions $g(x)/g(0)$ (b) and PDFs in semi-log (c) and log-log scale (d) for generalized NLS equation accounting for six- and eight-wave interactions \[ \left[ \begin{array}{c}
\end{array} \right] \] for $\alpha = 0.040$, $\beta = 0.001$ (red) and $\alpha = 0.064$, $\beta = 0.002$ (blue) at time shifts $t = 40$, $t = 400$ and $t = 2000$. Dashed lines for graphs (b) and (c) are Gaussian distributions, for graphs (d) - power-law tails. Linear parts on graphs (d) demonstrate power-law decay and continue from $|\Psi|^2 = 3$ to $|\Psi|^2 = 20$. 
FIG. 3: Correlation length $x_{\text{corr}}$ defined as full width at half maximum for spatial correlation function $g(x)$ depending on time $t$ for (a) generalized NLS equation accounting for six- and eight-wave interactions [2], $\alpha = 0.040$, $\beta = 0.001$ and (b) generalized NLS equation accounting for six-wave interactions, dumping and pumping terms [3], $\alpha = 0.256$, $d_l = 0.04$, $d_n = 0.0004$, $p = 0.05$.

FIG. 4: (Color on-line) PDFs for (a) classical NLS equation [1] at time shifts $t = 10$ (red), $t = 12$ (blue), $t = 30$ (black) versus $|\Psi|^2$, (b) generalized NLS equation accounting for six-wave interactions, dumping and pumping terms [3] at time shifts $t = 6$ (purple), $t = 8$ (red), $t = 10$ (blue), $t = 12$ (green) and in the statistically steady state averaged over time $[50, 250]$ (black) versus $|\Psi|^2/\langle|\Psi|^2\rangle$, (c) Eq. [3] in the statistically steady state averaged over time $[50, 250]$ versus $|\Psi|$. Solid lines - PDFs, dashed lines - Gaussian distributions for graphs (a) and (b) and $\exp(-|\Psi|)$ tail for graph (c). Graphs (b) and (c) are calculated with parameters $d_l = 0.04$, $d_n = 0.0004$, $p = 0.05$, $\alpha = 0.128$.

FIG. 5: (Color on-line) Spectrograms of typical large wave events for generalized NLS equation accounting for six- and eight-wave interactions [2], $\alpha = 0.040$, $\beta = 0.001$, at moderate (a) $|\Psi|_{\text{max}} = 6.1$, $t = 72.7$ and large (b) time shifts $|\Psi|_{\text{max}} = 7.7$, $t = 1789$ and (c) generalized NLS equation accounting for six-wave interactions, dumping and pumping terms [3], $|\Psi|_{\text{max}} = 7.9$ at $t = 219.7$. 
FIG. 6: (Color on-line) Solid lines are (a) spectra $I_k$, (b) normalized spatial correlation functions $g(x)/g(0)$ and (c) normalized PDFs for generalized NLS equation accounting for six-wave interactions, dumping and pumping terms \( \left( [\text{3}] \right) \) with fixed $d_i = 0.04$, $d_\alpha = 0.0004$, $p = 0.05$ and $\alpha$ = 0 (purple), $\alpha$ = 0.032 (red), $\alpha$ = 0.064 (blue), $\alpha$ = 0.128 (green) and $\alpha$ = 0.256 (black) in the statistically steady states. Dashed lines for graphs (b) and (c) - Gaussian distributions.