Asymmetric noncommutative torus

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Abstract

We compute the scalar curvature and prove the Gauss-Bonnet formula for a family of Dirac operators on a noncommutative torus, which are not (a priori) conformally related to "flat" Dirac operators.

1 Introduction

In the seminal works \cite{2}, \cite{5} Connes and Tretkoff initiated the investigation of curvature aspects on the noncommutative two torus and have shown the analogue of Gauss-Bonnet theorem for the conformally rescaled Dirac $D$ and the related spin Laplacian corresponding to the standard conformal structure. In \cite{8} these studies were extended to arbitrary conformal structure. The scalar curvature itself was defined and computed in \cite{4} and independently in \cite{9}.

The methods used in these papers build on Connes’ pseudodifferential calculus\cite{5} and heat kernel small time asymptotic expansion. The novelty therein is the employment of twisted spectral triples, non-tracial weight and the modular operator. For some related papers see \cite{1}, \cite{10}, \cite{11}.

In \cite{6} using the conformal (Weyl) factor from the commutant algebra and thus remaining on the level of the usual spectral triples, the Gauss-Bonnet has been established and the scalar curvature computed perturbatively up to the second order, for a wider class of Dirac operators.

In the present paper we establish the Gauss-Bonnet and compute the scalar curvature non-perturbatively for the class of Dirac operators obtained by asymmetric rescaling of only a part of the standard Dirac operator. For that we employ a positive element which can belong to

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the counterpart of coordinate algebra in the commutant, thus remaining on the level of usual spectral triples, or it can belong to the coordinate algebra itself, in which case however we are outside even of the class of twisted or modular spectral triples.

These results are possible due to a very recent neat generalization by Lesch [12] of the "rearrangement lemma", which is an important technical tool in [5].

2 Dirac operator

Let \( \mathbb{T}^2 \) be the classical torus with coordinates \( 0 \leq x, y \leq 2\pi \), equipped with the metric

\[
dx^2 + k^{-2}(x, y) dy^2,
\]

(2.1)

where \( k \) is a strictly positive function. The motivation for such a choice comes from the usual realization of \( \mathbb{T}^2 \) as an embedded surface in \( \mathbb{R}^3 \): The "usual" symmetric torus has the following parametrization:

\[
X = (c + \cos y) \cos x, \quad Y = (c + \cos y) \sin x, \quad Z = \sin y.
\]

With the induced metric this is a particular case of the „asymmetric torus” corresponding to \( k^{-1} = c + \cos y \) in (2.1).

The scalar curvature of the torus with the metric (2.1) reads

\[
R = 2k^{-1} \partial_x^2 (k) - 4k^{-2} (\partial_x(k))^2.
\]

(2.2)

In the commutative case such metric is, of course, conformally equivalent to some flat metric on the torus even though the explicit formula for the curvature depends on the chosen coordinate system. However, when passing to the noncommutative torus we are entering a new unexplored land, where one does not know what is metric and what exactly means conformally equivalent. As in the approach of Connes the natural object is the Dirac operator rather than the metric itself, for this reason we propose a new Dirac operator, which generalizes to the noncommutative situation the classical case of asymmetric torus.

We start with the commutative case of Dirac operator on \( L^2(\mathbb{T}^2, k^{-1} dx \, dy) \otimes \mathbb{C}^2 \) for the metric (2.1):

\[
\hat{D} = -i\sigma^1 (\partial_x - \frac{1}{2}k^{-1}\partial_x (k)) - i\sigma^2 k \partial_y,
\]
where
\[
\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.
\] (2.3)

Using the multiplication by \(\sqrt{k}\) we obtain the unitarily equivalent Dirac operator
\[
\tilde{D} = -i\sigma^1 \partial_x - i\sigma^2 \left( k \partial_y + \frac{1}{2} \partial_y(k) \right)
\] (2.4)
on \(L^2(T, dx \, dy) \otimes \mathbb{C}^2\). It is selfadjoint on the dense domain \(H^{1,2}(T)\).

Next we pass to the noncommutative torus \(T^2\), for which we refer to [5] for the needed information. As the Dirac operator we take \(D\) of the form (2.4), however, with \(-i\) times the partial derivatives replaced by the usual derivations on \(T^2\):
\[
D = \sigma^1 \delta_1 + \sigma^2 \left( k \delta_2 + \frac{1}{2} \delta_2(k) \right)\] (2.5)
It acts on \(\mathcal{H} = L^2(T^2, t) \otimes \mathbb{C}^2\), where \(t\) is the usual trace on \(T^2\). Such an operator is a differential operator in the sense of [5] and we can extract the associated scalar curvature following [5], [4].

### 3 The curvature

The square of \(D\) reads
\[
D^2 = \left( (\delta_1)^2 + k^2 (\delta_1)^2 \right) + \left( \frac{1}{2} k \delta_2(k) + \frac{1}{2} \delta_2(k) + i \sigma^3 \delta_1(k) \right) \delta_2 + \left( \frac{1}{4} (\delta_2(k))^2 + \frac{1}{2} i \sigma^3 \delta_{12}(k) + \frac{1}{2} k \delta_{22}(k) \right).
\]
and its symbol is
\[
\sigma(D^2) = a_0 + a_1 + a_2,
\]
where
\[
a_0 = (\xi_1^2 + k^2 \xi_2^2) \]
\[
a_1 = \left( \frac{3}{2} k \delta_2(k) + \frac{1}{2} \delta_2(k) + i \sigma^3 \delta_1(k) \right) \xi_2 \]
\[
a_2 = \left( \frac{1}{4} (\delta_2(k))^2 + \frac{1}{2} i \sigma^3 \delta_{12}(k) + \frac{1}{2} k \delta_{22}(k) \right).
\]

As was demonstrated first in [5] the value \(\zeta(0)\) at the origin of the zeta function of the operator \(D^2\) is given by
\[
\zeta(0) = -\int t(b_2(\xi)) \, d\xi,
\]
where \(b_2(\xi)\) is a symbol of order \(-4\) of the pseudodifferential operator \((D^2 + 1)^{-1}\). It can be computed by pseudodifferential calculus of symbols from the symbol \(a_2(\xi) + a_1(\xi) + a_0(\xi)\) of \(D^2\) as follows:
\[
b_2 = -(b_0 a_0 b_0 + b_1 a_1 b_0 + \partial_1(b_0) \delta_1(a_1) b_0 + \partial_2(b_0) \delta_2(a_1) b_0 + \partial_1(b_1) \delta_1(a_2) b_0 + \partial_2(b_1) \delta_2(a_2) b_0 + \frac{1}{2} \partial_{11}(b_0) \delta_1^2(a_2) b_0 + \frac{1}{2} \partial_{22}(b_0) \delta_2^2(a_2) b_0 + \partial_{12}(b_0) \delta_{12}(a_2) b_0).
\] (3.1)
where

\[
\begin{align*}
    b_1 &= -(b_0 a_1 b_0 + \partial_1 (b_0) \delta_1 (a_2) b_0 + \partial_2 (b_0) \delta_2 (a_2) b_0), \\
    b_0 &= (a_2 + 1)^{-1},
\end{align*}
\]

(3.2)

Since to obtain the curvature (or the zero of the \(\varepsilon_2\) function), we need to integrate with respect to \(\xi_1, \xi_2\), we notice that terms which contain odd powers of these variables shall vanish. Therefore, we can neglect them and keep only the relevant parts for the computations with even powers.

We have:

\[
b_2^e = A + B + C,
\]

where

\[
A = -2k b_0^2 \delta_1 (k) b_0 \delta_2 (k) b_0 \xi_2^4 + 4k b_0^2 \delta_1 (k) k b_0^2 \delta_1 (k) b_0 \xi_1^2 \xi_2^4 - 2k b_0^2 \delta_1 (k) b_0 \delta_1 (k) k b_0 \xi_2^4 \\
+ 4k b_0^2 \delta_1 (k) b_0^2 \delta_1 (k) k b_0 \xi_1^2 \xi_2^4 + 8k b_0^2 \delta_1 (k) k b_0 \delta_1 (k) b_0 \xi_1^2 \xi_2^4 + 8k b_0^2 \delta_1 (k) b_0 \delta_1 (k) k b_0 \xi_2^4 \\
- b_0 \delta_1 (k) b_0 \delta_1 (k) b_0 \xi_2^2 + 2b_0^2 \delta_1 (k) b_0 \delta_2 (k) b_0 \xi_2^2 - 2b_0^2 \delta_1 (k) k b_0 \delta_1 (k) k b_0 \xi_2^4 \\
+ 4b_0^2 \delta_1 (k) k b_0 \delta_1 (k) b_0 \xi_1^2 \xi_2^4 - 2b_0^2 \delta_1 (k) k b_0 \delta_1 (k) b_0 \xi_1^2 \xi_2^4 + 4b_0^2 \delta_1 (k) k b_0 \delta_1 (k) b_0 \xi_2^4 \\
- 8b_0^2 \delta_1 (k) b_0 \xi_1^2 \xi_2^4 + 8b_0^2 \delta_1 (k) k b_0 \delta_1 (k) k b_0 \xi_1^2 \xi_2^4 + 8b_0^2 \delta_1 (k) k b_0 \delta_1 (k) b_0 \xi_2^4 \xi_2^4,
\]

\[
B = \frac{15}{4} k b_0^2 \delta_2 (k) k b_0 \delta_2 (k) b_0 \xi_2^4 - 3k b_0 \delta_2 (k) k^2 b_0 \delta_2 (k) k b_0 \xi_2^4 - 3k b_0 \delta_2 (k) k^3 b_0 \delta_2 (k) b_0 \xi_2^4 \\
+ \frac{3}{4} k b_0 \delta_2 (k) k b_0 \delta_2 (k) b_0 \xi_2^4 + 6k^2 b_0^2 \delta_2 (k) \delta_2 (k) b_0 \xi_2^2 - 8k^2 b_0^2 \delta_2 (k) k b_0 \delta_2 (k) b_0 \xi_2^4 \\
- 10k^2 b_0^2 \delta_2 (k) k^2 b_0 \delta_2 (k) b_0 \xi_2^4 + 4k^2 b_0^2 \delta_2 (k) k^3 b_0^2 \delta_2 (k) k b_0 \xi_2^6 + 4k^2 b_0^2 \delta_2 (k) k^4 b_0^2 \delta_2 (k) b_0 \xi_2^6 \\
- 12k^3 b_0^2 \delta_2 (k) k b_0 \delta_2 (k) b_0 \xi_2^4 + 4k^3 b_0^2 \delta_2 (k) k^2 b_0 \delta_2 (k) b_0 \xi_2^4 + 4k^3 b_0^2 \delta_2 (k) k^3 b_0 \delta_2 (k) b_0 \xi_2^4 \\
- 10k^3 b_0^2 \delta_2 (k) b_0 \delta_2 (k) b_0 \xi_2^4 - 8k^4 b_0^3 \delta_2 (k) \delta_2 (k) b_0 \xi_2^4 + 8k^4 b_0^3 \delta_2 (k) k b_0 \delta_2 (k) b_0 \xi_2^4 \\
+ 8k^4 b_0^3 \delta_2 (k) k^2 b_0 \delta_2 (k) b_0 \xi_2^4 + 8k^5 b_0^3 \delta_2 (k) k b_0 \delta_2 (k) b_0 \xi_2^6 + 8k^5 b_0^3 \delta_2 (k) b_0 \delta_2 (k) b_0 \xi_2^6 \\
- \frac{1}{2} b_0 \delta_2 (k) \delta_2 (k) b_0 + \frac{3}{4} b_0 \delta_2 (k) k b_0 \delta_2 (k) k b_0 \xi_2^2 + \frac{5}{2} b_0 \delta_2 (k) k^2 b_0 \xi_2^4 \\
- b_0 \delta_2 (k) k^3 b_0^2 \delta_2 (k) k b_0 \xi_2^4 - b_0 \delta_2 (k) k^4 b_0^2 \delta_2 (k) b_0 \xi_2^4,
\]

and

\[
C = + k b_0^2 \delta_{11} (k) b_0 \xi_2^4 - 4k b_0^3 \delta_{11} (k) b_0 \xi_2^2 + b_0^2 \delta_{11} (k) k b_0 \xi_2^2 - 4b_0^2 \delta_{11} (k) k b_0 \xi_2^4 \\
- \frac{1}{2} k b_0 \delta_{22} (k) b_0 + 2k^2 b_0^2 \delta_{22} (k) k b_0 \xi_2^2 + 4k^3 b_0^2 \delta_{22} (k) k b_0 \xi_2^4 \\
- 4k^4 b_0^3 \delta_{22} (k) k b_0 \xi_2^4 - 4k^5 b_0^3 \delta_{22} (k) b_0 \xi_2^4,
\]

Similarly for the chiral part of the curvature:

\[
b_2^e = A^\gamma + B^\gamma + C^\gamma,
\]

where

\[
A^\gamma = -2k^2 b_0^2 \delta_1 (k) k b_0 \delta_2 (k) b_0 i \xi_2^4 - 2k^2 b_0^2 \delta_1 (k) b_0 \delta_2 (k) k b_0 i \xi_2^4 \\
+ \frac{3}{2} b_0 \delta_1 (k) k b_0 \delta_2 (k) k b_0 i \xi_2^2 - 2b_0 \delta_1 (k) k^2 b_0 \delta_2 (k) k b_0 i \xi_2^4 \\
- 2b_0 \delta_1 (k) k^3 b_0 \delta_2 (k) b_0 i \xi_2^4 + \frac{3}{2} b_0 \delta_1 (k) k b_0 \delta_2 (k) k b_0 i \xi_2^2 \\
B^\gamma = \frac{3}{2} k b_0 \delta_2 (k) k b_0 \delta_1 (k) b_0 i \xi_2^2 - 2k^2 b_0^2 \delta_2 (k) k b_0 \delta_1 (k) b_0 i \xi_2^4 \\
- 2k^3 b_0^2 \delta_2 (k) k b_0 \delta_1 (k) b_0 i \xi_2^4 + \frac{3}{2} b_0 \delta_2 (k) k b_0 \delta_1 (k) b_0 i \xi_2^2,
\]

\[
C^\gamma = + k b_0^2 \delta_{11} (k) k b_0 \xi_2^4 - 4k b_0^3 \delta_{11} (k) k b_0 \xi_2^2 + b_0^2 \delta_{11} (k) k b_0 \xi_2^2 - 4b_0^2 \delta_{11} (k) k b_0 \xi_2^4 \\
- \frac{1}{2} k b_0 \delta_{22} (k) b_0 + 2k^2 b_0^2 \delta_{22} (k) k b_0 \xi_2^2 + 4k^3 b_0^2 \delta_{22} (k) k b_0 \xi_2^4 \\
- 4k^4 b_0^3 \delta_{22} (k) k b_0 \xi_2^4 - 4k^5 b_0^3 \delta_{22} (k) b_0 \xi_2^4.
\]
and
\[ C' = 2k^2b_0^2\delta_{12}(k)b_0 i\xi_2^2 - \frac{1}{2}b_0\delta_{12}(k)b_0 i. \]

### 3.1 The classical limit

At this point we can check the classical (commutative) value of our expressions for \( \theta = 0 \). They become respectively:

\[
\begin{align*}
\gamma_2 = &\ 48b_0^5k^2\delta_2(k)^2\xi_2^4 - 8b_0^4k^5\delta_{22}(k)\xi_2^4 \\
& - 56b_0^4k^4\delta_2(k)^2\xi_2^4 - 8b_0^3k^2\delta_1(k)^2\xi_2^4 - 8b_0^2k^2\delta_{11}(k)\xi_2^2\xi_1^2 \\
& - 8b_0^2\delta_1(k)^2\xi_2^2\xi_1^2 + 6b_0^3k^3\delta_{22}(k)\xi_2^4 + 14b_0^3k^2\delta_2(k)^2\xi_2^2 \\
& + 2b_0^3k\delta_{11}(k)\xi_2^2 + b_0^3\delta_1(k)^2\xi_2^2 - 1/2b_0^2k\delta_{22}(k) - 1/4b_0^2\delta_2(k)^2,
\end{align*}
\]

and

\[
\gamma_2 = -12b_0^3k^3\delta_1(k)\delta_2(k)i\xi_2^4 + 2b_0^2k^2\delta_{12}(k)i\xi_2^2 + 6b_0^3k\delta_1(k)\delta_2(k)i\xi_2^2 - 1/2b_0^2\delta_{12}(k)i,
\]

which after integration gives:

\[
\int d\xi_1 d\xi_2 b_2 = -\frac{\pi}{3} \frac{(\delta_1(k))^2}{k^3} + \frac{\pi}{6} \frac{\delta_{11}(k)}{k^2},
\]

and

\[
\int d\xi_1 d\xi_2 b_{2\gamma} = 0.
\]

Taking into account that we compute the Gilkey-Seeley-deWitt coefficients for the asymptotic heat kernel expansion of the square of the Dirac operator and not the Laplace operator itself, and assuming that \( D \) has no zero eigenvalue, we have:

\[
\zeta(0) = \frac{1}{48\pi} \int \sqrt{g}R.
\]

Moreover, since \( t = \frac{1}{4\pi^2} \int dx dy \) for \( \theta = 0 \), taking into account the appropriate rescaling of the volume form and putting it all together we obtain:

\[
\sqrt{g}R = 48\pi \frac{1}{4\pi^2} \left( -\frac{\pi}{3} \frac{(\partial_1(k))^2}{k^3} + \frac{\pi}{6} \frac{\partial_{11}(k)}{k^2} \right) = (2k^{-2}\partial_{11}(k) - 4k^{-3}(\partial_1(k))^2),
\]

which agrees with the classical formula (2.2). Similarly,

\[
\sqrt{g}R_{\gamma} = 0.
\]

Before we can proceed with the noncommutative computation let us recall the general framework of computations as shown recently by Lesch [12].
3.2 Rearrangement Lemma

In [12] Lesch proved the following formula:

\[
\int_0^\infty f_0(uk^2) \cdot b_1 \cdot f_1(uk^2) \cdot b_2 \cdots b_p \cdot f_p(uk^2) du = \\
= k^{-2} F(\Delta_2^{(1)}, \Delta_2^{(1)} \Delta_2^{(2)}, \ldots, \Delta_2^{(1)} \cdots \Delta_2^{(p)})(b_1 \cdot b_2 \cdots b_p),
\]

where the function \( F(s_1, \ldots, s_p) \) is

\[
F(s) = \int_0^\infty f_0(u) f_1(us_1) \cdots f_p(us_p) du
\]

and \( \Delta_2^{(j)} \) signifies the square of the modular operator \( \Delta_2 = \Delta^2 \), acting on the \( j \)-th component of the product. Here we shall rather use \( \Delta = k^{-1} \cdot k \) instead of its square.

In our case we need to adapt the formula to a slightly different setting, when we integrate over two variables \( \xi_1 \) and \( \xi_2 \). A generic integral we have is of the form:

\[
J = \int_0^\infty d\xi_1 \int_0^\infty d\xi_2 \ k^{n_1} b_0^{m_1} (\xi_1, \xi_2) \ X \ k^{n_2} b_0^{m_2} (\xi_1, \xi_2) \ Y \ k^{n_3} b_0^{m_3} (\xi_1, \xi_2) \ \xi_1^{2k_1} \xi_2^{2k_2},
\]

where \( X, Y \) are derivations of \( k \) and

\[
b_0(\xi_1, \xi_2) = \frac{1}{1 + \xi_1^2 + k^2 \xi_2^2}.
\]

Extending the result of Lesch we see that

\[
J = F(\Delta^{(1)}, \Delta^{(1)} \Delta^{(2)})(X \cdot Y),
\]

where, after change of variables we obtain:

\[
F(s, t) = 2 \int_0^\infty dv \int_0^\infty du \ k^{n_1+n_2+n_3-1-2k_2} \frac{u^{k_2-\frac{1}{2}} v^{2k_1}}{(1 + v^2 + u)^{m_1}} \frac{s^{n_2}}{(1 + v^2 + us^2)^{m_2}} \frac{t^{n_3}}{(1 + v^2 + ut^2)^{m_3}}.
\]

In case \( Y = 1 \) the resulting function depends only on \( s \).

3.3 The curvature and its trace

In order to compute explicitly the expressions for the curvature we shall use the following lemma.

Lemma 3.1. Under the trace an entire function \( F \) of two variables satisfies

\[
t \left( F(\Delta^{(1)}, \Delta^{(1)} \Delta^{(2)})(X \cdot Y) \right) = t \left( F(\Delta^{(1)}, id)(XY) \right) = t(F(\Delta^{(1)}, 1)(X)Y).
\]

and in case of one variable:

\[
t \left( F(\Delta^{(1)})(X) \right) = t \left( F(1)X \right).
\]
Proof. We have:

\[ F(s, t) = \sum_{n, m \geq 0} f_{nm} s^n t^m, \]

so:

\[ t \left( F(\Delta^{(1)}, \Delta^{(2)})(X \cdot Y) \right) = \sum_{n, m \geq 0} f_{nm} t (\Delta^{n+m}(X) \Delta^m(Y)) \]
\[ = \sum_{n, m \geq 0} f_{nm} t (\Delta^n(X)Y) \]
\[ = \sum_{n, m \geq 0} f_{nm} t (\Delta^n(X)) \]
\[ = t \left( F(\Delta^{(1)}, 1)(X \cdot Y) \right). \]

The other identity is a simple consequence of the above one. \( \square \)

### 3.4 Curvature and chiral curvature

We first compute the chiral curvature.

**Lemma 3.2.** The chiral curvature for the asymmetric torus is:

\[ R_\gamma = G_{12}(\delta_1(k), \delta_2(k)) + G_{21}(\delta_2(k), \delta_1(k)) + G(\delta_{12}(k)), \]

where

\[ G_{12}(s, t) = \frac{\pi}{k^2} \frac{(t - 1)}{(t + 1)^2(s + 1)}, \]
\[ G_{21}(s, t) = \frac{\pi}{k^2} \frac{(t - 1)}{(t + 1)^2(s + t)}, \]

and

\[ G(s) = -\frac{\pi}{k} \frac{(s - 1)}{(s + 1)^2}. \]

and its trace vanishes.

**Proof.** By computation. Then the last statement follows from:

\[ G_{12}(s, 1) = G_{21}(s, 1) = G(1) = 0. \]

Next we compute the scalar curvature.

**Lemma 3.3.** The scalar curvature for the asymmetric torus is:

\[ R = F_{11}(\delta_1(k), \delta_1(k)) + F'_{11}(\delta_1(k)^2) + F_{22}(\delta_2(k), \delta_2(k)) + F'_{22}(\delta_2(k)^2) + F_1(\delta_{11}(k)) + F_2(\delta_{22}(k)), \]

where

\[ F_{11}(s, t) = -\frac{2\pi}{3k^3} \frac{(2s^2 + 4st + 4s + 3 + 8t + 3t^2)}{(t + 1)^3(s + 1)(s + t)}. \]
\[ F'_{11}(s) = \frac{4\pi}{3k^3} \frac{1}{(s+1)^3}, \]
\[ F_{22}(s, t) = \frac{\pi}{2k} \frac{(t^2 - 6t + 1)}{(t+1)^3}, \]
\[ F'_{2}(s) = -\frac{\pi}{2k} \frac{(s^2 - 6s + 1)}{(s+1)^3}, \]

and
\[ F_1(s) = \frac{2\pi}{3k^2} \frac{1}{(s+1)^2}. \]
\[ F_2(s) = 0. \]

and its trace vanishes.

Proof. First of all, observe that
\[ F_{22}(s, 1) + F'_{22}(1) = 0, \quad F_2(1) = 0, \]
so all terms containing \( \delta_2(k) \) and \( \delta_{22}(k) \) vanish.

For the terms containing \( \delta_1(k) \) we have:
\[ F_{11}(s, 1) + F'_{11}(1) = -\frac{\pi}{3k^3} \frac{s + 3}{(s+1)^2}, \]
then using the identity:
\[ t \left( (\delta^{-2}_{11}(k)) \right) = 2t \left( (\delta^{-2}_{11}(k))k^{-1}(\delta_{1}(k)) \right) = 2t \left( (\delta^{-3}_{11}(k)) \right), \]
which follows directly from the Leibniz rule and the fact that the trace is closed, we can rewrite all the terms:
\[ t \left( (F_{11}(\delta_{1}(k), \delta_{1}(k)) + F'_{11}(\delta_{1}(k)^2) + F_{1}(\delta_{11}(k))) \right) = t \left( (\delta^{-3}_{11}(k)) \right), \]
where
\[ H(s) = \frac{\pi}{3k^3} \frac{1 - s}{s(s+1)^2}. \]

Next, we observe that for any \( A \) and \( B \) and an entire function \( H \):
\[ t \left( (\delta^{-3}_{11}(k)) \right) = t \left( (H(\Delta)(\Delta^3(A)))k^{-3}B \right) = t \left( (\delta^{-3}_{11}(k)) \right), \]
and
\[ t \left( (\delta^{-3}_{11}(k)) \right) = t \left( (\delta^{-3}_{11}(k)) \right). \]

Now if \( A = B \) then both expressions on the right-hand side are identical. In our case, however:
\[ H(s)s^3 = \frac{\pi}{3k^3} \frac{s^2(1 - s)}{(s+1)^2}. \]
and
\[ H(s^{-1}) = \frac{\pi}{3k^3} \frac{s^2(s-1)}{(s+1)^2}, \]

and therefore since
\[ H(s)s^3 = -H(s^{-1}), \]

the trace of the above expression must vanish, hence, the Gauss-Bonnet theorem holds.

4 Conclusions

We have introduced a new class of Dirac operators on the noncommutative tori, computed the scalar curvature and shown that the Gauss-Bonnet theorem holds. Even though in the classical limit they arise from the metric, which is conformally equivalent to the flat one, this might not be the case in the noncommutative situation. This raises an interesting question how the metrics as defined above and the flat one are related to each other in the noncommutative case?

Moreover, it becomes now more evident that the class of admissible Dirac operators on the noncommutative torus is certainly bigger than the one-parameter family of "flat metric" (equivariant) Dirac operators. It is therefore necessary to study the conditions and the general setup of such construction.

Although in this paper we have concentrated on the 2-dimensional case it a natural task to generalise the results to 3 and more dimensions, in particular to study the curvature and minimality of Dirac operators introduced in [7].

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