Singularity analysis of a spherical
Kadomtsev–Petviashvili equation

AYŞE KARASU-KALKANLI¹, SERGEI YU. SAKOVICH²

¹)Department of Physics, Middle East Technical University,
06531 Ankara, Turkey. E-mail: akarasu@metu.edu.tr

²)Institute of Physics, National Academy of Sciences,
220072 Minsk, Belarus. E-mail: saks@tut.by

Abstract
The (2+1)-dimensional spherical Kadomtsev–Petviashvili (SKP) equation of J.-K. Xue [Phys. Lett. A 314:479–483 (2003)] fails the Painlevé test for integrability at the highest resonance, where a non-trivial compatibility condition for recursion relations appears. This compatibility condition, however, is sufficiently weak and thus allows the SKP equation to possess an integrable (1+1)-dimensional reduction, which is detected by the Weiss method of truncated singular expansions.

1 Introduction
Recently, Xue [1] deduced a spherical Kadomtsev–Petviashvili (SKP) equation for nonlinear dust acoustic waves in unmagnetized dusty plasmas with effects of a nonplanar spherical geometry and a transverse perturbation. This SKP equation can be brought into the form

\[ u_{xxxx} - 12u_{xx} - 12u_x^2 + u_{xt} + \frac{1}{t^2}u_{yy} + \frac{1}{t}u_x + \frac{1}{yt^2}u_y = 0 \]  \hspace{1cm} (1)

by rescaling its dependent and independent variables.
Xue [1] noticed that the SKP equation (1) admits two exact reductions to (1+1)-dimensional equations: first, the reduction \( u = u(x,t) \) to the spherical Korteweg–de Vries (SKdV) equation \( u_t + uu_x + u_{xxx} - 6u_{tx} = 0 \), which is believed to be a nonintegrable equation [2]; and second, the reduction \( u = u(z,t) \) with

\[
    z = x - \frac{1}{2}y^2t 
\]

(2)
to the Korteweg–de Vries (KdV) equation \( u_t + uu_z - 12u_{zz} = 0 \). Having used this reduction to the integrable KdV equation, Xue [1] obtained an exact solitary wave solution of the SKP equation.

Owing to the reduction to the KdV equation, the SKP equation (1) automatically possesses \( N \)-soliton solutions, with any \( N \), derivable from those of the KdV equation through the variable \( z \) (2). Of course, the existence of such (1+1)-dimensional solitons does not imply that the (2+1)-dimensional SKP equation is integrable. Moreover, the existence of the reduction to the SKdV equation suggests that the SKP equation is nonintegrable.

In the present paper, we study the integrability of the SKP equation (1) directly, not using its reductions. In Section 2 we show that the SKP equation does not pass the Painlevé test for integrability due to some nondominant logarithmic branching of its general solution. The singularity analysis indicates, however, that many special solutions of the SKP equation are free from this logarithmic branching. In order to select those single-valued special solutions, we use the Weiss method of truncated singular expansions in Section 3 and in this way surprisingly obtain the reduction of the SKP equation to the KdV equation. Section 4 contains concluding remarks.

2 Singularity analysis

Let us show that the SKP equation (1) does not pass the Painlevé test for integrability. We follow the so-called Weiss–Kruskal algorithm of singularity analysis [3, 4].

Starting the singularity analysis, we substitute \( u = u_0(y,t)\phi^\alpha + \cdots + u_r(y,t)\phi^{r+\alpha} + \cdots \) with \( \partial_x\phi(x,y,t) = 1 \) into the SKP equation (1), and find that the singular behavior of a solution \( u \) corresponds to \( \alpha = -2 \) with \( u_0 = 1 \), the positions \( r \) of resonances being \( r = -1, 4, 5, 6 \). This is the generic branch representing the general solution.
Then, assuming that the singular behavior of \( u \) near a hypersurface \( \phi(x, y, t) = 0 \) with \( \phi_x = 1 \) is determined by the expansion

\[
u = \sum_{n=0}^{\infty} u_n(y, t) \phi^{n-2}, \tag{3}\]

we obtain from (1) the following recursion relations for the coefficients \( u_n \):

\[
(n - 2)(n - 3)(n - 4)(n - 5)u_n \\
+ (n - 4)(n - 5) \left( -6 \sum_{i=0}^{n} u_i u_{n-i} + \left( \phi_t + \frac{1}{t^2} \phi_y^2 \right) u_{n-2} \right) \\
+ (n - 5) \left( \left( \frac{1}{t} + \frac{1}{t^2} \phi_{yy} + \frac{1}{yt^2} \phi_y \right) u_{n-3} + \left( 1 + \frac{2}{t^2} \phi_y \right) \partial_t u_{n-3} \right) \\
+ \frac{1}{t^2} \partial_y^2 u_{n-4} + \frac{1}{yt^2} \partial_y u_{n-4} = 0, \quad n = 0, 1, 2, \ldots \tag{4}
\]

where \( u_{-4} = u_{-3} = u_{-2} = u_{-1} = 0 \) formally.

At \( n = 0, 1, 2, 3 \), the recursion relations (4) give us, respectively,

\[
u_0 = 1, \tag{5}\]
\[
u_1 = 0, \tag{6}\]
\[
u_2 = \frac{1}{12} \left( \frac{1}{t} + \frac{1}{t^2} \phi_y^2 \right), \tag{7}\]
\[
u_3 = -\frac{1}{12} \left( \frac{1}{t} + \frac{1}{t^2} \phi_{yy} + \frac{1}{yt^2} \phi_y \right). \tag{8}\]

At the resonances \( n = 4 \) and \( n = 5 \), where the coefficients \( u_4(y, t) \) and \( u_5(y, t) \) are not determined, the recursion relations (4) turn out to be compatible. However at the highest resonance, \( n = 6 \), where the coefficient \( u_6(y, t) \) is not determined, we obtain from (4) the following nontrivial compatibility condition:

\[
\left( \phi_y + \frac{1}{2} yt \right) \left( \phi_{yy} + \frac{1}{2} t \right) = 0, \tag{9}\]

which means that we should modify the expansion (3) by introducing additional logarithmic terms, starting from the one proportional to \( \phi^4 \log \phi \).
Consequently, the SKP equation (1) does not pass the Painlevé test for
integrability due to the nondominant logarithmic branching of its solutions.
The observed analytic properties of the SKP equation suggest that it cannot
possess any good Lax pair.

There is an interesting conjecture, formulated by Weiss [5], that the dif-
ferential constraints, which arise in the singularity analysis of nonintegrable
equations, are always integrable themselves (see [6] for further discussion on
this conjecture). In the present case of the SKP equation (1), we find that, in
accordance with the Weiss conjecture, the compatibility condition (9) with
\( \phi_x = 1 \) can be solved exactly, the result being

\[
\phi = x - \frac{1}{4} y^2 t + y f(t) + g(t)
\]  

(10)

with any \( f(t) \) and \( g(t) \).

We see from (10) that the compatibility condition (9) is not very restric-
tive. The class of single-valued solutions of the SKP equation (1), which are
free from the nondominant logarithmic branching, is wide: it is determined
by the Laurent type expansion (3) containing three arbitrary functions of
two variables and two arbitrary functions of one variable, namely, \( u_4(y,t) \),
\( u_5(y,t) \), \( u_6(y,t) \), \( f(t) \) and \( g(t) \). For this reason, one can hope to find many
special single-valued solutions of the SKP equation (1) in some closed form,
whereas the existing techniques provide no closed expressions for solutions
possessing nondominant logarithmic singularities.

3 Truncated expansion

Let us apply the method of truncated singular expansions of Weiss [7] to
the SKP equation (1). This method, which is able to produce Bäcklund
transformations and Lax pairs for integrable nonlinear systems, is also useful
in nonintegrable cases for finding explicit special solutions [8].

Substituting the truncated singular expansion

\[
u = \frac{u_0(x, y, t)}{\phi(x, y, t)^2} + \frac{u_1(x, y, t)}{\phi(x, y, t)} + u_2(x, y, t)
\]

(11)

into the SKP equation (1) and collecting terms with equal degrees of \( \phi \), we
obtain the following:

\[ u_0 = \phi_x^2, \quad (12) \]
\[ u_1 = -\phi_{xx}, \quad (13) \]
\[ u_2 = \frac{\phi_{xxx}}{3\phi_x} - \frac{\phi_{xx}^2}{4\phi_x^2} + \frac{\phi_{yy}^2}{12t^2\phi_x^2} + \frac{\phi_t}{12\phi_x}, \quad (14) \]

\[ \phi_{xxxx} - \frac{4\phi_{xx}\phi_{xxx}}{\phi_x} + \frac{3\phi_{xx}^3}{\phi_x^2} - \frac{\phi_{yy}\phi_{xx}}{t^2\phi_x^2} - \frac{\phi_t\phi_{xx}}{\phi_x} + \phi_{xt} + \frac{\phi_{yy}}{t^2} + \frac{\phi_x}{t} + \frac{\phi_y}{yt^2} = 0, \quad (15) \]

\[ (yt\phi_x + 2\phi_y) \left( \phi_{yy}^2 - 2\phi_x\phi_y\phi_{xy} + \phi_x^2\phi_{yy} + \frac{1}{2}t\phi_x^3 \right) = 0. \quad (16) \]

We see that, for any solution \( \phi(x, y, t) \) of the overdetermined nonlinear system \((15)–(16)\), the truncated singular expansion \((11)\) with the coefficients given by \((12)–(14)\) generates a solution \( u(x, y, t) \) of the SKP equation \((1)\). However, in order to use this fact, we need to solve the system \((15)–(16)\).

First, we solve the equation \((16)\), which is equivalent to

\[ \phi_{yy}^2 - 2\phi_x\phi_y\phi_{xy} + \phi_x^2\phi_{yy} + \frac{1}{2}t\phi_x^3 = 0 \quad (17) \]

because any solution of \(yt\phi_x + 2\phi_y = 0\) satisfies \((17)\). Setting \(x\) to be the new dependent variable,

\[ x = \psi(\phi, y, t), \quad (18) \]

we rewrite \((17)\) in the linear form

\[ \psi_{yy} = \frac{1}{2}t. \quad (19) \]

The general solution of \((19)\) and the expression \((18)\) give us the following implicit general solution of the equation \((16)\):

\[ x = \frac{1}{2}y^2t + a(\phi, t)y + b(\phi, t), \quad (20) \]

where \(a\) and \(b\) are arbitrary functions.
Now, having solved the equation (16), we use (20) and find that the equation (15) is equivalent to
\[ a = 0, \tag{21} \]
\[ b_{\phi t} + \frac{b_{\phi \phi \phi \phi}}{b_{\phi}^3} - \frac{6b_{\phi \phi}b_{\phi \phi \phi}}{b_{\phi}^4} + \frac{6b_{\phi \phi}^3}{b_{\phi}^5} = 0. \tag{22} \]
Owing to the condition (21), we get from (20) that \( \phi \) is in fact a function of two variables,
\[ \phi(x, y, t) = \omega(z, t), \tag{23} \]
where \( z \) is given by (2). Then the condition (22) is equivalent to
\[ \left( \frac{\omega_t}{\omega_z} + \frac{\omega_{zzz}}{\omega_z} - \frac{3\omega_z^2}{2\omega_z^2} \right)_z = 0. \tag{24} \]

We have solved the overdetermined nonlinear system (15)–(16): its general solution is (23), where \( z \) is given by (2) and \( \omega \) is any solution of (24). Using this, we find from (11) with (12)–(14) that the most general solution \( u \), obtainable for the SKP equation (1) by the method of truncated singular expansions, is
\[ u(x, y, t) = v(z, t) \tag{25} \]
with \( z \) given by (2) and any function \( v \) satisfying the equation
\[ (v_t + v_{zzz} - 12vv_z)_z = 0. \tag{26} \]
Consequently, the Weiss method [7], being applied to the SKP equation, rediscovers the reduction of this (2+1)-dimensional nonintegrable equation to the (1+1)-dimensional integrable KdV equation.

## 4 Conclusion

Let us summarize the obtained results. The discovered analytic properties of the SKP equation (1) suggest that this equation cannot admit any good Lax pair but can possess many single-valued solutions. The attempt of selecting those single-valued solutions by using the truncated singular expansion leads
not to a class of explicit special solutions but to an exact reduction of the studied nonintegrable equation to a lower dimensional integrable equation (as far as we know, this phenomenon is observed for the first time). It is very likely, however, that the truncated singular expansion represents not all single-valued solutions of the SKP equation \(^{[11]}\). Indeed, positions \(\phi = 0\) of singularities of a generic single-valued solution are determined by \(^{[10]}\) with arbitrary \(f(t)\) and \(g(t)\), whereas positions of singularities of solutions \(^{[25]}-^{[26]}\) are restricted by \(f = 0\). Therefore it would be interesting to find any closed form solution of the SKP equation \(^{[1]}\) without this restriction \(f = 0\) imposed on positions of its singularities.

References

[1] J.-K. Xue. A spherical KP equation for dust acoustic waves. Phys. Lett. A 314:479–483 (2003).

[2] F. Calogero, A. Degasperis. Spectral Transform and Solitons: Tools to Solve and Investigate Nonlinear Evolution Equations. North-Holland, New York, 1982. P. 51.

[3] J. Weiss, M. Tabor, G. Carnevale. The Painlevé property for partial differential equations. J. Math. Phys. 24:522–526 (1983).

[4] M. Jimbo, M. D. Kruskal, T. Miwa. Painlevé test for the self-dual Yang–Mills equations. Phys. Lett. A 92:59–60 (1982).

[5] J. Weiss. Bäcklund transformations and the Painlevé property. In: P. J. Olver, D. H. Sattinger (ed.). Solitons in Physics, Mathematics and Nonlinear Optics. Springer, New York, 1990. P. 175–202.

[6] S. Yu. Sakovich. On integrability of differential constraints arising from the singularity analysis. J. Nonlinear Math. Phys. 9:21–25 (2002); arXiv: nlin.SI/0004037

[7] J. Weiss. The Painlevé property for partial differential equations. II. Bäcklund transformation, Lax pairs, and the Schwarzian derivative. J. Math. Phys. 24:1405–1413 (1983).

[8] F. Cariello, M. Tabor. Painlevé expansions for nonintegrable evolution equations. Physica D 39:77–94 (1989).