Discrete Heisenberg group and its automorphism group

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In this note we give more easy and short proof of a statement previously proved by P. Kahn in [1] that the automorphism group of the discrete Heisenberg group Heis(3, \mathbb{Z}) is isomorphic to the group \((\mathbb{Z} \oplus \mathbb{Z}) \rtimes GL(2, \mathbb{Z})\). The method which we suggest to construct this isomorphism gives far more transparent picture of the structure of the automorphism group of the group Heis(3, \mathbb{Z}).

We consider the discrete Heisenberg group \(G = \text{Heis}(3, \mathbb{Z})\) which is the group of matrices of the form:

\[
\begin{pmatrix}
1 & a & c \\
0 & 1 & b \\
0 & 0 & 1
\end{pmatrix},
\]

where \(a, b, c\) are from \(\mathbb{Z}\). We can consider also the group \(G\) as a set of all integer triples endowed with the group law:

\[
(a_1, b_1, c_1)(a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2 + a_1 b_2).
\]

It is clear that \([(1, 0, 0), (0, 1, 0)] = (0, 0, 1)\) and the group \(G\) is generated by the elements \((1, 0, 0)\) and \((0, 0, 1)\).

Let \(\text{Aut}(G)\) be the group of automorphisms of the group \(G\). The starting point for this note was the group homomorphism \(\mathbb{Z} \rightarrow \text{Aut}(G)\) constructed in [3, \S 5] and given by the formula

\[
R_d((a, b, c)) = \left(a + db, b, c + \frac{b(b - 1)d}{2}\right),
\]

where \(R_d\) is the automorphism of the group \(G\) induced by an element \(d \in \mathbb{Z}\). Formula (2) was obtained in [3] after some calculation of automorphisms which are analogs of ”loop rotations” in loop groups. But the fact that formula (2) defines a homomorphism from the group \(\mathbb{Z}\) to the group \(\text{Aut}(G)\) is also an easy direct consequence of formulas (1) and (2).

For the group \(G\) we have the following exact sequence of groups

\[
1 \rightarrow C \rightarrow G \xrightarrow{\lambda} H \oplus P \rightarrow 1,
\]

where 1 is the trivial group, the group \(C = \{(0, 0, c) \mid c \in \mathbb{Z}\} \simeq \mathbb{Z}\) is the center of \(G\), and \(H = \{(a, 0, 0) \mid a \in \mathbb{Z}\} \simeq \mathbb{Z}\), \(P = \{(0, b, 0) \mid b \in \mathbb{Z}\} \simeq \mathbb{Z}\).
From exact sequence (3) we obtain the following sequence

\[ 1 \longrightarrow \text{Inn}(G) \xrightarrow{\theta} \text{Aut}(G) \xrightarrow{\vartheta} \text{GL}(2, \mathbb{Z}), \quad (4) \]

where the group of inner automorphisms \( \text{Inn}(G) \cong G/C \cong \mathbb{Z} \oplus \mathbb{Z} \), and the homomorphism \( \vartheta \) is the homomorphism \( \text{Aut}(G) \to \text{Aut}(\mathbb{Z} \oplus \mathbb{Z}) \), which is induced by the homomorphism \( \lambda \) from exact sequence (3). It is clear that \( \text{Im}(\theta) \subset \text{Ker}(\vartheta) \).

We claim that sequence (4) is exact in the term \( \text{Aut}(G) \). Indeed, it is enough to prove that \( \text{Ker}(\vartheta) \subset \text{Im}(\theta) \). Consider any \( \omega \in \text{Ker}(\vartheta) \). We have \( \omega((1, 0, 0)) = (1, 0, c_1) \) and \( \omega((0, 1, 0)) = (0, 1, c_2) \) for some integer \( c_1 \) and \( c_2 \). By direct calculations, we obtain \( (c_2, -c_1, 0)(1, 0, 0) = (1, 0, c_1)(c_2, -c_1, 0) \) and \( (c_2, -c_1, 0)(0, 1, 0) = (0, 1, c_2)(c_2, -c_1, 0) \). Since elements \((1, 0, 0)\) and \((0, 1, 0)\) generate the group \( G \), we obtain that the inner automorphism defined by the element \((c_2, -c_1, 0)\) coincides with the automorphism \( \omega \).

We claim also that the homomorphism \( \vartheta \) from sequence (4) is surjective. Indeed, by formula (2) we have a homomorphism from the group \( \mathbb{Z} \) to the group \( \text{Aut}(G) \). It is easy to see that this homomorphism is a section of the homomorphism \( \vartheta \) over the subgroup \( \left\{ \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \right\}_{d \in \mathbb{Z}} \) of the group \( \text{GL}(2, \mathbb{Z}) \). By formula (2), the action of the matrices from this subgroup on elements of the group \( G \) is given as:

\[ \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}(a, b, c) = \left( a + db, b, c + \frac{b(b - 1)d}{2} \right). \quad (5) \]

Symmetrically to the formula (5) we can write the following formula:

\[ \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix}(a, b, c) = \left( a, da + b, c + \frac{a(a - 1)d}{2} \right). \quad (6) \]

By an easy direct calculation we have that formula (6) defines a correct automorphism of the group \( G \). This automorphism depends on \( d \in \mathbb{Z} \) and defines the homomorphism from the subgroup \( \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}_{d \in \mathbb{Z}} \) of the group \( \text{GL}(2, \mathbb{Z}) \) to the group \( \text{Aut}(G) \). This homomorphism defines a section of the homomorphism \( \vartheta \) over this subgroup.

Besides, it is easy to see that the homomorphism from the subgroup \( \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \) of the group \( \text{GL}(2, \mathbb{Z}) \) to the group \( \text{Aut}(G) \) given by the formula

\[ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}(a, b, c) = (-a, b, -c - b) \quad (7) \]

defines a section of the homomorphism \( \vartheta \) over this subgroup.

By the classical result (see its proof, for example, in [2, Appendix A]), the group \( \text{SL}(2, \mathbb{Z}) \) has a presentation:

\[ \langle \rho, \tau \mid \rho \tau \rho = \tau \rho \tau, (\rho \tau \rho)^4 = 1 \rangle, \]
where the element $\rho$ corresponds to the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and the element $\tau$ corresponds to the matrix $B = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. The group $GL(2, \mathbb{Z})$ is generated by the elements of the group $SL(2, \mathbb{Z})$ and the matrix $D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Therefore the group $GL(2, \mathbb{Z})$ has a presentation:

$$< \rho, \tau, \kappa \mid \rho \tau \rho = \tau \rho \tau, (\rho \tau \rho)^4 = 1, \kappa \tau \kappa^{-1} = \tau^{-1}, \kappa \rho \kappa^{-1} = \rho^{-1}, \kappa^2 = 1 >,$$

where the element $\kappa$ corresponds to the matrix $D$. Thus we obtained that the homomorphism $\vartheta$ is surjective.

We note that the group $Aut(G)$ contains a distinguished subgroup $Aut^+(G) = \vartheta^{-1}(SL(2, \mathbb{Z}))$ of index 2. Since for any $\omega \in Aut(G)$ we have

$$\omega((0, 0, 1)) = \omega([(1, 0, 0), (0, 1, 0)]) = [\omega(1, 0, 0), \omega(0, 1, 0)] = (0, 0, \det(\vartheta(\omega))),$$

we obtain that the group $Aut^+(G)$ consists of elements $\omega$ of the group $Aut(G)$ such that $\omega((0, 0, 1)) = (0, 0, 1)$. In other words, the group $Aut^+(G)$ consists of automorphisms of the group $G$ which act identically on the center of the group $G$.

**Theorem 1** Partial sections of the homomorphism $\vartheta$ given by formulas (5), (6) and (7) are glued together and define a section of the homomorphism $\vartheta$ over the whole group $GL(2, \mathbb{Z})$. Hence, the group $Aut(G) \simeq (\mathbb{Z} \oplus \mathbb{Z}) \rtimes GL(2, \mathbb{Z})$, where an action of $GL(2, \mathbb{Z})$ on $\mathbb{Z} \oplus \mathbb{Z}$ (given by inner automorphisms in the group $Aut(G)$) is the natural matrix action. Besides, $Aut^+(G) \simeq (\mathbb{Z} \oplus \mathbb{Z}) \rtimes SL(2, \mathbb{Z})$.

**Proof.** From the above discussion we see that it is enough to check that automorphisms of the group $G$ which are given by the matrices $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (with the help of formulas (5), (6) and (7)) satisfy relations from formula (8). Since the group $G$ is generated by elements $(1, 0, 0)$ and $(0, 1, 0)$, it is enough to check these relations between automorphisms after application to these elements. We have

$$ABA(1, 0, 0) = AB(1, 0, 0) = A(1, -1, 0) = (0, -1, 1), \text{ and}$$

$$BAB(1, 0, 0) = BA(1, -1, 0) = B(0, -1, 1) = (0, -1, 1).$$

Besides, we have

$$ABA(0, 1, 0) = AB(1, 1, 0) = A(1, 0, 0) = (1, 0, 0), \text{ and}$$

$$BAB(0, 1, 0) = BA(0, 1, 0) = B(1, 1, 0) = (1, 0, 0).$$

Thus we see that the automorphism of the group $G$ given by the composition $ABA$ coincides with the automorphism of $G$ given by the composition $BAB$. We will check
now that the composition \((ABA)^4\) defines the identical automorphism of the group \(G\). Indeed, we have checked that \(ABA(1,0,0) = (0,-1,1) = (0,1,0)^{-1}\). Therefore, using again above calculations, we have
\[
(ABA)^2(1,0,0) = ABA((0,1,0)^{-1}) = (ABA(0,1,0))^{-1} = (1,0,0)^{-1}.
\]
Hence we obtain
\[
(ABA)^4(1,0,0) = (ABA)^2((1,0,0)^{-1}) = (1,0,0).
\]
Analogously we have
\[
(ABA)^2(0,1,0) = ABA(1,0,0) = (0,1,0)^{-1}, \text{ and hence } (ABA)^4(0,1,0) = (ABA)^2((0,1,0)^{-1}) = (0,1,0).
\]
Besides, we calculate
\[
DAD^{-1}(0,1,0) = DA(0,1,-1) = D(1,1,-1) = (−1,1,0) = A^{-1}(0,1,0), \text{ and }
DAD^{-1}(1,0,0) = DA(−1,0,0) = D(−1,0,0) = (1,0,0) = A^{-1}(1,0,0), \text{ and }
DBD^{-1}(0,1,0) = DB(0,1,-1) = D(0,1,-1) = (0,1,0) = B^{-1}(0,1,0), \text{ and }
DBD^{-1}(1,0,0) = DB(−1,0,0) = D(−1,1,−1) = (1,1,0) = B^{-1}(1,0,0).
\]
We have also directly from formula (7) that the automorphism of the group \(G\) given by the composition \(D^2\) is the identity automorphism. The theorem is proved.

**Remark 1** P. Kahn constructed in [1] a section of the homomorphism \(\vartheta\) over the group \(GL(2,\mathbb{Z})\) by direct calculations inside the group \(\text{Aut}(G)\) (in contrast to our approach, which we started from the explicit homomorphism given by formula (2) and this homomorphism was obtained in [3 § 5] from an analog of "loop rotations"). We note, that the section \(GL(2,\mathbb{Z}) \to \text{Aut}(G)\) constructed by P. Kahn does not coincide with the section from Theorem [1].

Let \(\alpha_1\) and \(\alpha_2\) be two sections of the homomorphism \(\vartheta\). We define \(\varphi(g) = \alpha_2(g)\alpha_1(g)^{-1} \in \mathbb{Z} \oplus \mathbb{Z}\) for any \(g \in GL(2,\mathbb{Z})\). Then \(\varphi(g_1g_2) = \varphi(g_1) + g_1 \cdot \varphi(g_2)\) for any \(g_1, g_2 \in GL(2,\mathbb{Z})\), i.e. the map \(\varphi : GL(2,\mathbb{Z}) \to \mathbb{Z} \oplus \mathbb{Z}\) is a 1-cocycle. Conversely, for any section \(\alpha\) of the homomorphism \(\vartheta\) and for any 1-cocycle \(\varphi : GL(2,\mathbb{Z}) \to \mathbb{Z} \oplus \mathbb{Z}\) we have that the map \((\varphi \cdot \alpha) : GL(2,\mathbb{Z}) \to \text{Aut}(G)\) is a homomorphism which is a section of the homomorphism \(\vartheta\). Using the Mayer-Vietoris sequence for the presentation \(SL(2,\mathbb{Z}) = \mathbb{Z}_4 \ast_{\mathbb{Z}_2} \mathbb{Z}_6\) as amalgamated free product P. Kahn computed in [1] that \(H^1(SL(2,\mathbb{Z}), \mathbb{Z} \oplus \mathbb{Z}) = 0\) for the natural matrix action of \(SL(2,\mathbb{Z})\) on \(\mathbb{Z} \oplus \mathbb{Z}\). Hence, from the Lyndon-Hochschild-Serre spectral sequence applied to \(SL(2,\mathbb{Z}) \hookrightarrow GL(2,\mathbb{Z})\) one immediately obtains that \(H^1(GL(2,\mathbb{Z}), \mathbb{Z} \oplus \mathbb{Z}) = 0\). Therefore \(\varphi\) is a 1-coboundary, i.e. there is an element \(a \in \mathbb{Z} \oplus \mathbb{Z}\) such that \(\varphi(g) = ga - a\) for any \(g \in GL(2,\mathbb{Z})\). Hence we obtain that the group of 1-cocycles of \(GL(2,\mathbb{Z})\) with values in \(\mathbb{Z} \oplus \mathbb{Z}\) is isomorphic to the group \(\mathbb{Z} \oplus \mathbb{Z}\). Thus, the set of all sections of the homomorphism \(\vartheta\) is a principal homogeneous space for the group \(\mathbb{Z} \oplus \mathbb{Z}\).
References

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