COEFFICIENT CONDITIONS FOR
HARMONIC CLOSE-TO-CONVEX FUNCTIONS

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Abstract. New sufficient conditions, concerned with the coefficients of harmonic functions $f(z) = h(z) + \overline{g(z)}$, in the open unit disk $U$ normalized by $f(0) = h(0) = h'(0) - 1 = 0$, for $f(z)$ to be harmonic close-to-convex functions are discussed. Furthermore, several illustrative examples and the image domains of harmonic close-to-convex functions satisfying the obtained conditions are enumerated.

1. Introduction

For a continuous complex-valued function $f(z) = u(x,y) + iv(x,y)$, we say that $f(z)$ is harmonic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ if both $u(x,y)$ and $v(x,y)$ are real harmonic in $U$, that is, $u(x,y)$ and $v(x,y)$ satisfy the Laplace’s equations

$$
\Delta u = u_{xx} + u_{yy} = 0 \quad \text{and} \quad \Delta v = v_{xx} + v_{yy} = 0.
$$

A complex-valued harmonic function $f(z)$ in $U$ is given by $f(z) = h(z) + \overline{g(z)}$ where $h(z)$ and $g(z)$ are analytic in $U$. We call $h(z)$ and $g(z)$ the analytic part and the co-analytic part of $f(z)$, respectively. A necessary and sufficient condition for $f(z)$ to be locally univalent and sense-preserving in $U$ is $|h'(z)| > |g'(z)|$ in $U$ (see, [2] or [3]). Let $\mathcal{H}$ denote the class of harmonic functions $f(z)$ in $U$ with $f(0) = h(0) = 0$ and $h'(0) = 1$. Thus, every normalized harmonic function $f(z)$ can be written by

$$
f(z) = h(z) + \overline{g(z)} = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n \in \mathcal{H}
$$

where $a_1 = 1$ and $b_0 = 0$, for convenience.

We next denote by $\mathcal{S}_\mathcal{H}$ the class of functions $f(z) \in \mathcal{H}$ which are univalent and sense-preserving in $U$. Since the sense-preserving property of $f(z)$, we see that $|b_1| = |g'(0)| < |h'(0)| = 1$. If $g(z) \equiv 0$, then $\mathcal{S}_\mathcal{H}$ reduces to the class $\mathcal{S}$ consisting of normalized analytic univalent functions. Furthermore, for every function $f(z) \in \mathcal{S}_\mathcal{H}$, the function

$$
F(z) = \frac{f(z) - b_1 f(z)}{1 - |b_1|^2} = z + \sum_{n=2}^{\infty} \frac{a_n - b_1 b_n}{1 - |b_1|^2} z^n + \sum_{n=2}^{\infty} \frac{b_n - b_1 a_n}{1 - |b_1|^2} z^n
$$

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is also a member of $\mathcal{S}_H$. Therefore, we consider the subclass $\mathcal{S}_H^0$ of $\mathcal{S}_H$ defined as

$$\mathcal{S}_H^0 = \{ f(z) \in \mathcal{S}_H : b_1 = g'(0) = 0 \}.$$ 

Conversely, if $F(z) \in \mathcal{S}_H^0$, then $f(z) = F(z) + b_1 F'(z) \in \mathcal{S}_H$ for any $b_1 (|b_1| < 1)$.

We say that a domain $\mathbb{D}$ is a close-to-convex domain if the complement of $\mathbb{D}$ can be written as a union of non-intersecting half-lines (except that the origin of one half-line may lie on one of the other half-lines). Let $\mathcal{C}$, $\mathcal{C}_H$ and $\mathcal{C}_H^0$ be the respective subclasses of $\mathcal{S}$, $\mathcal{S}_H$ and $\mathcal{S}_H^0$ consisting of all functions $f(z)$ which map $U$ onto a certain close-to-convex domain.

Bshouty and Lyzzaik [1] have stated the following result.

**Theorem 1.1.** If $f(z) = h(z) + g(z) \in \mathcal{H}$ satisfies

$$g'(z) = zh'(z) \quad \text{and} \quad \text{Re} \left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}$$

for all $z \in U$, then $f(z) \in \mathcal{C}_H^0 \subset \mathcal{S}_H^0$.

A simple and interesting example is below.

**Example 1.1.** The function

$$f(z) = \frac{1 - (1 - z)^2}{2(1 - z)^2} + \frac{z^2}{2(1 - z)^2} = z + \sum_{n=2}^{\infty} \frac{n + 1}{2} z^n + \sum_{n=2}^{\infty} \frac{n - 1}{2} z^n$$

satisfies the conditions of Theorem 1.1 and therefore $f(z)$ belongs to the class $\mathcal{C}_H^0$. We now show that $f(U)$ is actually close-to-convex domain. It follows that

$$f(z) = \left(\frac{z}{2(1 - z)^2} + \frac{z}{2(1 - z)^2}\right) + \left(\frac{z}{2(1 - z)^2} - \frac{z}{2(1 - z)^2}\right)$$

$$= \text{Re} \left(\frac{z}{1 - z}\right) + \text{Im} \left(\frac{z}{1 - z}\right).$$

Setting

$$f(re^{i\theta}) = \frac{-2r^2 + r(1 + r^2) \cos \theta}{(1 + r^2 - 2r \cos \theta)^2} + \frac{r \sin \theta}{1 + r^2 - 2r \cos \theta} i = u + iv$$

for any $z = re^{i\theta} \in U \ (0 \leq r < 1, \ 0 \leq \theta < 2\pi)$, we see that

$$-4(u + v^2) = \frac{4r(r - \cos \theta)(1 - r \cos \theta)}{(1 + r^2 - 2r \cos \theta)^2} = \frac{4r(r - t)(1 - rt)}{(1 + r^2 - 2rt)^2} \equiv \phi(t) \quad (-1 \leq t = \cos \theta \leq 1).$$

Since

$$\phi'(t) = \frac{-4r(1 - r^2)^2}{(1 + r^2 - 2rt)^3} \leq 0,$$

we obtain that

$$\phi(t) \leq \phi(-1) = \frac{4r}{(1 + r)^2} \equiv \psi(r).$$

Also, noting that

$$\psi'(r) = \frac{4(1 - r)}{(1 + r)^3} > 0,$$
we know that
\[ \psi(r) < \psi(1) = 1 \]
which implies that
\[ u > -\psi^2 - \frac{1}{4}. \]
Thus, \( f(z) \) maps \( U \) onto the following close-to-convex domain.

**Figure 1.** The image of \( f(z) = \frac{1 - (1 - z)^2}{2(1 - z)^2} + \frac{z^2}{2(1 - z)^2} \).

**Remark 1.1.** Let \( \mathcal{M} \) be the class of all functions satisfying the conditions of Theorem 1.1. Then, it was earlier conjectured by Mocanu \[9, 10\] that \( \mathcal{M} \subset \mathcal{S}_H^0 \). Furthermore, we can immediately see that the function \( f(z) \) in Example 1.1 is a member of the class \( \mathcal{M} \) and it shows that \( f(z) \in \mathcal{M} \) is not necessarily starlike with respect to the origin in \( U \) (\( f(z) \) is starlike with respect to the origin in \( U \) if and only if \( tw \in f(U) \) for all \( w \in f(U) \) and \( t \), \( 0 \leq t \leq 1 \)).

**Remark 1.2.** For the function \( f(z) = h(z) + \overline{g(z)} \in \mathcal{H} \) given by
\[ g'(z) = z^{n-1} h'(z) \quad (n = 2, 3, 4, \ldots), \]
letting \( w(t) = f(e^{it}) = h(e^{it}) + \overline{g(e^{it})} \) \( (-\pi \leq t < \pi) \), we know that
\[ \text{Im} \left( \frac{w''(t)}{w'(t)} \right) \leq 0 \quad (-\pi \leq t < \pi) \]
which means that \( f(z) \) maps the unit circle \( \partial \mathcal{U} = \{ z \in \mathbb{C} : |z| = 1 \} \) onto a union of several concave curves (see, \[8\] Theorem 2.1)).
Jahangiri and Silverman \[7\] have given the following coefficient inequality for \( f(z) \in \mathcal{H} \) to be in the class \( \mathcal{C}_H \).

**Theorem 1.2.** If \( f(z) \in \mathcal{H} \) satisfies
\[
\sum_{n=2}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n|b_n| \leq 1,
\]
then \( f(z) \in \mathcal{C}_H \).

**Example 1.2.** The function
\[
f(z) = z + \frac{1}{5}z^5
\]
belongs to the class \( \mathcal{C}_H^0 \subset \mathcal{C}_H \) and satisfies the condition of Theorem 1.2. Indeed, \( f(z) \) maps \( U \) onto the following hypocycloid of six cusps (cf. \[3\] or \[6\]).

\[ \text{Figure 2. The image of } f(z) = z + \frac{1}{5}z^5. \]

The object of this paper is to find some sufficient conditions for functions \( f(z) \in \mathcal{H} \) to be in the class \( \mathcal{C}_H \). In order to establish our results, we have to recall here the following lemmas due to Clunie and Sheil-small \[2\].

**Lemma 1.1.** If \( h(z) \) and \( g(z) \) are analytic in \( U \) with \( |h'(0)| > |g'(0)| \) and \( h(z) + \varepsilon g(z) \) is close-to-convex for each \( \varepsilon \) (\( |\varepsilon| = 1 \)), then \( f(z) = h(z) + g(z) \) is harmonic close-to-convex.

**Lemma 1.2.** If \( f(z) = h(z) + g(z) \) is locally univalent in \( U \) and \( h(z) + \varepsilon g(z) \) is convex for some \( \varepsilon \) (\( |\varepsilon| \leq 1 \)), then \( f(z) \) is univalent close-to-convex.
We also need the following result due to Hayami, Owa and Srivastava [5].

**Lemma 1.3.** If a function $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$ is analytic in $U$ and satisfies

$$
\sum_{n=2}^{\infty} \left| \sum_{k=1}^{n} \left\{ \sum_{j=1}^{k} (-1)^{k-j} j(j+1) \left( \begin{array}{c} \alpha \\ k-j \end{array} \right) A_j \right\} \left( \begin{array}{c} \beta \\ n-k \end{array} \right) \right| \leq 2
$$

for some real numbers $\alpha$ and $\beta$, then $F(z)$ is convex in $U$.

2. **Main results**

Our first result is contained in

**Theorem 2.1.** If $f(z) \in \mathcal{H}$ satisfies the following condition

$$
\sum_{n=2}^{\infty} |na_n - e^{i\varphi} (n-1)a_{n-1}| + \sum_{n=1}^{\infty} |nb_n - e^{i\varphi} (n-1)b_{n-1}| \leq 1
$$

for some real number $\varphi$ ($0 \leq \varphi < 2\pi$), then $f(z) \in \mathcal{C}_H$.

**Proof.** Let $F(z) = z + \sum_{n=2}^{\infty} A_n z^n$ be analytic in $U$. If $F(z)$ satisfies

$$
\sum_{n=2}^{\infty} |nA_n - e^{i\varphi} (n-1)A_{n-1}| \leq 1
$$

then it follows that

$$
| (1 - e^{i\varphi} z)F'(z) - 1 | = \left| \sum_{n=2}^{\infty} (nA_n - e^{i\varphi} (n-1)A_{n-1}) z^{n-1} \right|
$$

$$
\leq \sum_{n=2}^{\infty} |nA_n - e^{i\varphi} (n-1)A_{n-1}| \cdot |z|^{n-1}
$$

$$
< \sum_{n=2}^{\infty} |nA_n - e^{i\varphi} (n-1)A_{n-1}| \leq 1 \quad (z \in U).
$$

This gives us that

$$
\text{Re} \left( (1 - e^{i\varphi} z)F'(z) \right) > 0 \quad (z \in U),
$$

that is, that $F(z) \in \mathcal{C}$. Then, it is sufficient to prove that

$$
F(z) = \frac{h(z) + \varepsilon g(z)}{1 + \varepsilon b_1} = z + \sum_{n=2}^{\infty} a_n + \varepsilon b_n z^n \in \mathcal{C}
$$
for each $\varepsilon (|\varepsilon| = 1)$ by Lemma 1.1. From the assumption of the theorem, we obtain that

$$\sum_{n=2}^{\infty} \left| \frac{a_n + \varepsilon b_n}{1 + \varepsilon b_1} - e^{i\varphi} (n - 1) \frac{a_{n-1} + \varepsilon b_{n-1}}{1 + \varepsilon b_1} \right|$$

$$\leq \frac{1}{1 - |b_1|} \sum_{n=2}^{\infty} \left| na_n - e^{i\varphi} (n - 1)a_{n-1} \right| + \left| nb_n - e^{i\varphi} (n - 1)b_{n-1} \right| \leq \frac{1 - |b_1|}{1 - |b_1|} = 1.$$

This completes the proof of the theorem. \qed

**Example 2.1.** The function

$$f(z) = -\log(1 - z) + \left(-mz - \log(1 - z)\right) = z + \sum_{n=2}^{\infty} \frac{1}{n} z^n + (1 - m)z + \sum_{n=2}^{\infty} \frac{1}{n} z^n \quad (0 < m \leq 1)$$

satisfies the condition of Theorem 2.1 with $\varphi = 0$ and belongs to the class $\mathcal{C}_H$. In particular, putting $m = 1$, we obtain the following.

![Figure 3. The image of $f(z) = -\varphi - 2\log|1 - z|$.](image)

By making use of Lemma 1.2 with $\varepsilon = 0$ and applying Lemma 1.3, we readily obtain the next theorem.

**Theorem 2.2.** If $f(z) \in \mathcal{H}$ is locally univalent in $\mathbb{U}$ and satisfies

$$\sum_{n=2}^{\infty} \left[ \sum_{k=1}^{n} \left( -1 \right)^{k-j} j(j+1) \left( \begin{array}{c} \alpha \\ k - j \end{array} \right) a_j \left( \begin{array}{c} \beta \\ n - k \end{array} \right) \right]$$

$$+ \sum_{k=1}^{n} \left[ \sum_{j=1}^{k} \left( -1 \right)^{k-j} j(j-1) \left( \begin{array}{c} \alpha \\ k - j \end{array} \right) a_j \left( \begin{array}{c} \beta \\ n - k \end{array} \right) \right] \leq 2$$

for some real numbers $\alpha$ and $\beta$, then $f(z) \in \mathcal{C}_H$. 


Putting $\alpha = \beta = 0$ in the above theorem, we arrive at the following result due to Jaha ngiri and Silverman [7].

**Theorem 2.3.** If $f(z) \in \mathcal{H}$ is locally univalent in $\mathbb{U}$ with

$$\sum_{n=2}^{\infty} n^2|a_n| \leq 1,$$

then $f(z) \in \mathcal{C}_\mathcal{H}$.

Furthermore, taking $\alpha = 1$ and $\beta = 0$ in the theorem, we have

**Corollary 2.1.** If $f(z) \in \mathcal{H}$ is locally univalent in $\mathbb{U}$ and satisfies

$$\sum_{n=2}^{\infty} \{n|(n+1)a_n - (n-1)a_{n-1}| + (n-1)|na_n - (n-2)a_{n-1}|\} \leq 2,$$

then $f(z) \in \mathcal{C}_\mathcal{H}$.

**Example 2.2.** The function

$$f(z) = -\int_{0}^{z} \frac{\log(1-t)}{t}dt + \left(z + (1-z) \log(1-z)\right) = z + \sum_{n=2}^{\infty} \frac{1}{n^2}z^n + \sum_{n=2}^{\infty} \frac{1}{n(n-1)}z^n$$

satisfies the conditions of Corollary 2.1 and belongs to the class $\mathcal{C}_\mathcal{H}$.

![Figure 4](image-url)
A sequence \( \{c_n\}_{n=0}^{\infty} \) of non-negative real numbers is called a convex null sequence if \( c_n \to 0 \) as \( n \to \infty \) and

\[
c_n - c_{n+1} \geq c_{n+1} - c_{n+2} \geq 0
\]

for all \( n (n = 0, 1, 2, \cdots) \).

The next lemma was obtained by Fejér [4].

**Lemma 3.1.** Let \( \{c_n\}_{k=0}^{\infty} \) be a convex null sequence. Then, the function

\[
p(z) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n z^n
\]

is analytic and satisfies \( \Re(p(z)) > 0 \) in \( U \).

Applying the above lemma, we deduce

**Theorem 3.1.** For some \( b \ (|b| < 1) \) and some convex null sequence \( \{c_n\}_{n=0}^{\infty} \) with \( c_0 = 2 \), the function

\[
f(z) = h(z) + g(z) = z + \sum_{n=2}^{\infty} \frac{c_{n-1}}{n} z^n + b \left( z + \sum_{n=2}^{\infty} \frac{c_{n-1}}{n} z^n \right)
\]

belongs to the class \( C_H \).

**Proof.** Let us define \( F(z) \) by

\[
F(z) = \frac{h(z) + \varepsilon g(z)}{1 + \varepsilon b} = z + \sum_{n=2}^{\infty} \frac{c_{n-1}}{n} z^n
\]

for each \( \varepsilon \ (|\varepsilon| = 1) \). Then, we know that

\[
F'(z) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n z^n \quad (c_0 = 2).
\]

By virtue of Lemma 1.1 and Lemma 3.1 it follows that \( \Re(F'(z)) > 0 \ (z \in U) \), that is, \( F(z) \in C \). Thus, we conclude that \( f(z) = h(z) + g(z) \in C_H \). \( \square \)

In the same manner, we also have

**Theorem 3.2.** For some \( b \ (|b| < 1) \) and some convex null sequence \( \{c_n\}_{n=0}^{\infty} \) with \( c_0 = 2 \), the function

\[
f(z) = h(z) + g(z) = z + \sum_{n=2}^{\infty} \frac{1}{n} \left( 1 + \sum_{j=1}^{n-1} c_j \right) z^n + b \left( z + \sum_{n=2}^{\infty} \frac{1}{n} \left( 1 + \sum_{j=1}^{n-1} c_j \right) z^n \right)
\]

belongs to the class \( C_H \).
Proof. Let us define $F(z)$ by

$$F(z) = \frac{h(z) + \varepsilon g(z)}{1 + \varepsilon b} = z + \sum_{n=2}^{\infty} \frac{1}{n} \left(1 + \sum_{j=1}^{n-1} c_j\right) z^n$$

for each $\varepsilon (|\varepsilon| = 1)$. Then, we know that

$$(1 - z)F'(z) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n z^n \quad (c_0 = 2).$$

Therefore, by the help of Lemma 1.1 and Lemma 3.1, we obtain that $\text{Re} ((1 - z)F'(z)) > 0 \ (z \in U)$, that is, $F(z) \in C$ which implies that $f(z) = h(z) + g(z) \in \mathcal{C}_H$. □

Remark 3.1. The sequence 

$$\{c_n\}_{n=0}^{\infty} = \left\{2, \frac{2}{3}, \frac{2}{4}, \cdots, \frac{2}{n+1}, \cdots\right\}$$

is a convex null sequence because

$$\lim_{n \to \infty} c_n = \lim_{n \to \infty} \left(\frac{2}{n+1}\right) = 0, \quad c_n - c_{n+1} = \frac{2}{(n+1)(n+2)} \geq 0$$

and

$$\left(c_n - c_{n+1}\right) - \left(c_{n+1} - c_{n+2}\right) = \frac{4}{(n+1)(n+2)(n+3)} \geq 0 \quad (n = 0, 1, 2, \cdots).$$

Setting $b = \frac{1}{4}$ in Theorem 3.1 with the above sequence $\{c_n\}_{n=0}^{\infty}$, we derive

Example 3.1. The function

$$f(z) = -z - 2 \int_0^z \frac{\log(1-t)}{t} \, dt - \frac{1}{4} \left( z + 2 \int_0^z \frac{\log(1-t)}{t} \, dt \right) = z + \sum_{n=2}^{\infty} \frac{2}{n^2} z^n + \frac{1}{4} \left(z + \sum_{n=2}^{\infty} \frac{2}{n^2} z^n\right)$$

is in the class $\mathcal{C}_H$.

![Figure 5. The image of $f(z)$ in Example 3.1](image)
Moreover, we know

**Remark 3.2.** The sequence

\[
\{c_n\}_{n=0}^\infty = \left\{2, 1, \frac{1}{2}, \ldots, 2^{1-n}, \ldots\right\}
\]

is a convex null sequence because

\[
\lim_{n \to \infty} c_n = \lim_{n \to \infty} 2^{1-n} = 0, \quad c_n - c_{n+1} = 2^{-n} \geq 0
\]

and

\[
(c_n - c_{n+1}) - (c_{n+1} - c_{n+2}) = 2^{-(n+1)} \geq 0 \quad (n = 0, 1, 2, \ldots).
\]

Hence, letting \(b = \frac{1}{4}\) in Theorem 3.2 with the sequence \(\{c_n\}_{n=0}^\infty = \{2^{1-n}\}_{n=0}^\infty\), we have

**Example 3.2.** The function

\[
f(z) = -3 \log(1 - z) + 4 \log \left(1 - \frac{z}{2}\right) + \left(-\frac{3}{4} \log(1 - z) + \log \left(1 - \frac{z}{2}\right)\right)
\]

\[
= z + \sum_{n=2}^{\infty} \frac{1}{n} \left(1 + \sum_{j=1}^{n-1} 2^{1-j}\right) z^n + \frac{1}{4} \left(z + \sum_{n=2}^{\infty} \frac{1}{n} \left(1 + \sum_{j=1}^{n-1} 2^{1-j}\right) z^n\right)
\]

is in the class \(C_H\).

**Figure 6.** The image of \(f(z)\) in Example 3.2
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