Stationary propagation of a wave segment along an inhomogeneous excitable stripe

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Abstract
We report a numerical and theoretical study of an excitation wave propagating along an inhomogeneous stripe of an excitable medium. The stripe inhomogeneity is due to a jump of the propagation velocity in the direction transverse to the wave motion. Stationary propagating wave segments of rather complicated curved shapes are observed. We demonstrate that the stationary segment shape strongly depends on the initial conditions which are used to initiate the excitation wave. In a certain parameter range, the wave propagation is blocked at the inhomogeneity boundary, although the wave propagation is supported everywhere within the stripe. A free-boundary approach is applied to describe these phenomena which are important for a wide variety of applications from cardiology to information processing.

Keywords: excitable medium, wave segment, inhomogeneous, propagation block

1. Introduction

Many active distributed systems in physics, biology or chemistry can be considered as excitable media, which are able to support an undamped propagation of excitation waves [1–4]. For instance, excitation waves have been observed experimentally in a wide variety of systems
including the oxidation reaction of CO on platinum single crystal [5], the chemical Belousov–Zhabotinsky reaction [6], slime mold aggregation [7], electrical activity of cardiac tissue [8], chicken retinas [9], and intracellular calcium waves [10]. Mathematical models of excitable media can usually be written in the form of essentially nonlinear reaction–diffusion systems, which are far away from thermodynamic equilibrium.

While a study of wave dynamics in homogeneous media is a necessary starting point for clarifying the corresponding theoretical backgrounds, wave processes in inhomogeneous media have always been considered as a fundamental problem for many applications and attract growing interest nowadays. Indeed, it is extremely important to elucidate the role of an inhomogeneity as a factor which affects wave structures existing in homogeneous media or even creates quite unusual ones [11–16]. Of course, this is very important for diverse applications because inhomogeneity is quite natural in all real systems. For instance, in cardiology the medium’s inhomogeneity is considered as a crucial reason for the wave breaking and the appearance of the arrhythmia [17]. On the other hand, the medium’s inhomogeneity can be efficiently used to suppress life treating arrhythmia [18, 19]. Moreover, there is a tendency to create a medium’s inhomogeneity artificially, e.g. to develop structured excitable media for information processing [20–22].

In this paper we are dealing with relatively simple spatial inhomogeneity within an infinitely long stripe oriented horizontally, as shown in figure 1. The inhomogeneity boundary is specified by a straight line between the top and the bottom parts of the stripe. It is assumed that the both parts of the stripe are able to support a propagating wave. However the top part of this stripe has different properties with respect to the bottom one. This results in a jump of the propagation velocity at the boundary between these two parts. Similar inhomogeneity has been considered already in the literature. In particular, it was shown that an initially planar wave propagating along the inhomogeneity boundary will be curved. In the course of time the wave approaches a stationary shape and a constant propagation velocity [23]. It was shown also that a similar inhomogeneity can be used to suppress wave front instabilities [24].

However, the simulation results obtained reveal unknown features of excitable wave dynamics in an excitable medium with such inhomogeneity. In particular, it is shown that the stationary wave structure strongly depends on the initial conditions used to initiate a wave. It is especially interesting that the wave initiated within one part of the stripe is not able to penetrate into the other one under certain conditions. It is also demonstrated that a free-boundary approach can be used to determine the velocity and the shape of a wave propagating through an excitable medium with such inhomogeneity.

We start with a description of the excitable medium model used below and represent the main results of the reaction–diffusion computations. After this, the free-boundary approach applied is described and then this approach is used in order to explain the data for the
reaction–diffusion simulation. In the final part of the paper we discuss a wave propagation block which has been observed unexpectedly in the simulations.

2. The excitable medium model

Many basic features of waves propagating in excitable media can be analyzed by means of a generic two-component reaction–diffusion model of the form [25]

\[
\frac{\partial u}{\partial t} = D V^2 u + F(u, v), \quad \frac{\partial v}{\partial t} = \varepsilon G(u, v), \tag{1}
\]

where the variables \(u\) and \(v\) represent the propagator and controller species, respectively. Typically the nullcline \(F(u, v) = 0\) is a non-monotone function creating the possibility for undamped wave propagation. The second nullcline \(G(u, v) = 0\) is monotone and intersects the first one at only one point \((u_0, v_0)\) as illustrated in Figure 2. For this illustration, the functions \(F(u, v)\) and \(G(u, v)\) are taken in the form used previously [26–28]:

\[
F(u, v) = A(3u - u^3 - v), \quad G(u, v) = u - \delta, \tag{2}
\]

where the constant \(\delta\) determines the single steady state as \((u_0, v_0) = (\delta, 3\delta - \delta^3)\). In the case \(\varepsilon = 0\) the system of equations (1) and (2) transforms into a bistable medium, since then \((u_0, v_0)\) represents another stable steady state. Here \(u_c\) is the largest root of the nonlinear equation \(F(u, v_0) = 0\). The bistable system is able to support stationary propagation of a trigger wave which corresponds to an abrupt transition from one steady state to another [2]. The velocity of a planar wave depends on the controller value \(v_0\) and vanishes at \(v_0 = v^*\). The propagation is supported if the deviation \(\Delta \equiv v^* - v_0 < \Delta_c\). If \(\Delta \ll \Delta_c\) the front velocity \(c_p\) is proportional to \(\Delta\) and can be written as

\[
c_p(v_0) = \alpha \sqrt{AD} \Delta. \tag{3}
\]
The constants $\alpha$, $v^*$ and $\Delta_c$ are determined by the function $F(u, v)$. In particular, for the function defined in equation (2), they can be determined analytically as $\alpha = 1/\sqrt{2}$, $v^* = 0$ and $\Delta_c = 2$.

If $0 < \varepsilon \ll 1$, there is a single steady state $(u_0, v_0)$ of the system (1) and (2). A suprathreshold perturbation induces propagation of a wave including an abrupt transition from $(u_0, v_0)$ to practically $(u_c, v_c)$ (wave front), slow motion along the right branch of the $u$-nullcline (wave plateau), and an abrupt transition to the left branch (wave back) as illustrated in figure 2. During the recovery process following the wave back the system returns to the steady state. In the simplest case of a homogeneous excitable stripe, such a wave exhibiting stationary propagation along the stripe has a planar front and a planar back, which have no common points.

3. Reaction–diffusion simulation results

The computations performed with an inhomogeneous stripe show that an initially planar wave evolves into a curved one exhibiting stationary propagation along the stripe as shown in figure 3. In these computations, the parameter $A$ was taken to be $A_1 = 1$ in the upper part of the stripe and $A_2 = 2$ in the bottom part. Hence the propagation velocity in the top part is lower than that in the bottom part. The existence of a curved wave exhibiting stationary propagation is in accordance with the data obtained earlier relating to inhomogeneous media [23].

However, a quite different wave pattern can be created in the same stripe, as shown in figure 4. In these computations, a part of the initially planar wave is cut off somewhere within the top part of the stripe. A phase change point, where the wave front coincides with the wave back, is created due to this procedure. We will refer to this wave pattern as a wave segment to
stress a close relation to similar localized wave patterns exhibiting stationary propagation, well studied for homogeneous media [26–29].

Thus, different initial conditions result in different wave patterns exhibiting stationary propagation, which is rather unusual in excitable medium dynamics.

The stationary shape of the propagating segment does not depend on the cutoff location. However, it strongly depends on the medium’s parameters and the width of the bottom part of the stripe, \( w_b \). For example, the wave segment width obtained when \( \epsilon = 0.005 \) is smaller, as can be seen in figure 5(a). The whole wave segment is practically located within the bottom part of the stripe. Under a further increase of \( \epsilon \), the induced wave segment becomes unstable and disappears in the course of time, as illustrated in figure 5(b).

4. The free-boundary approach

The free-boundary approach applied in this study is aimed at simplifying the underlying reaction–diffusion model, (1) and (2). Firstly, the front and the back of the propagating wave are assumed to be thin in comparison to the wave plateau and, hence, the shape of a wave segment is determined by the boundary of the excited region, e.g. the curve \( u(x, y, t) = (u_e + u_o)/2 \). It is appropriate to specify the shape and the normal velocity of the boundary via the arc length \( s \) measured from the phase change point \( q \), taking \( s > 0 \) at the front and \( s < 0 \) at the back [4].

Secondly, it is assumed that the normal velocity of the boundary obeys the linear eikonal equation

\[
c_n = c_p - Dk, \tag{4}
\]

where \( k \) is the local curvature and \( c_p = c_p(v) \) depends on the controller value \( v^+ (v^-) \) at the front (back) in accordance with equation (3) [4, 30]. At the phase change point, where the front
coincides with the back, the normal velocity vanishes. The velocity of this point in the tangent
direction, $c_t$, is equal to the translational velocity of the whole wave segment, as illustrated in
figure 4. Simple geometry shows that

$$c_p (v^\pm) = D k^\pm = c_i \cos (\Theta^\pm), \quad (5)$$

where $k^\pm$ and $\Theta^\pm$ specify the curvature and the normal angle with respect to the $x$ axis at the
front (+) and at the back (−).

Let us start the consideration from the top part of the stripe with lower velocity, assuming
that the translational velocity $c_i$ is given. The controller value at the wave front is constant,
$v^+ = v_0$. Hence the velocity $c_p (v^+)$ in equation (5) is also constant, $c_p (v^+) = c_p (v_0) \equiv c_0$. In
contrast to this, the controller value at the wave back is not constant. Indeed, due to equations
(1) and (2), the spatial gradient of the controller along the propagation direction is given by

$$\frac{\partial v}{\partial x} = -\varepsilon G \left( u_v (v), v \right) / c_i.$$ 

Under the assumption that $\Delta \ll \Delta$, this gradient remains practically constant along the wave plateau $G \left( u_v (v), v \right) \approx G \left( u_v (v^*), v^* \right) \equiv G^*$. Hence $v^-$ can be written as

$$v^- = v_0 + \frac{G^* \varepsilon}{c_i} \left[ x_i^+ - x_i^- \right], \quad (6)$$

where $x_i^+$ and $x_i^-$ determine the locations of the front and back at the same vertical coordinate $y_i$.

Then, since $k^\pm = -d\Theta^\pm/ds$, equation (5) transforms into the ordinary differential equation
for the angle $\Theta^+$:

$$D \frac{d\Theta^+}{ds} = c_i \cos (\Theta^+) - c_0. \quad (7)$$

Figure 5. Shape of a wave segment propagating through an inhomogeneous stripe. (a)
Stationary propagation obtained for $\varepsilon = 0.005$. (b) Dissipation of the wave segment
obtained for $\varepsilon = 0.006$. 
A similar transformation taking into account equation (6) yields the equation for the angle $\Theta^-$:

$$ D \frac{d\Theta^-}{ds} = \frac{G^s c_0 \alpha \sqrt{D}}{c_i} (x_i^+ - x_i^-) - c_0 + c_1 \cos (\Theta^-). $$

Equations (7) and (8) supplemented by the obvious relationships $dy \div ds = -\cos (\Theta^\pm)$ and $dx \div ds = \sin (\Theta^\pm)$ specify the shape of the traveling wave segment within the top part of the stripe with lower propagation velocity. The integration of the system has to be started with the following initial conditions: $\Theta_0^+ (0) = \Theta_0^- (0) = \pi/2$, $x_0^+ (0) = x_0^- (0) = x_0$ and $y_0^+ (0) = y_0^- (0) = y_0$. Here $x_0$ and $y_0$ can be chosen arbitrarily, e.g. $x_0 = y_0 = 0$.

To analyze possible solutions of equations (7) and (8), it is appropriate to use the value $c_0$ in order to rescale velocities, e.g. $C_i = c_i / c_0$, and space variables, e.g. $S = c_0 \alpha y / D$, $X = c_0 x / D$ and $Y = c_0 y / D$, which yields for the angle $\Theta_0^+$

$$ \frac{d\Theta_0^+}{dS} = C_i \cos (\Theta_0^+) - 1, $$

and for the angle $\Theta_0^-$

$$ \frac{d\Theta_0^-}{dS} = B (X_0^+ - X_0^-) \frac{C_i}{C_0} - 1 + C_i \cos (\Theta_0^-), $$

where

$$ B = \frac{G^s \alpha}{c_0 \Delta}. $$

After this rescaling, one can conclude that since the initial conditions for this system are well determined, the solution depends on the single dimensionless parameter $B$. Note that for the given model (1) and (2) with fixed $\Delta = 0.3$, the parameter $B$ is simply proportional to the parameter $\varepsilon$.

Three characteristic examples of the solutions of equations (7) and (8) are shown in figures 6(a)–(c). When $B = B_{rf}$, where

$$ B_{rf} = 0.535 + (C_i - 1) \times 0.63, $$

the solution of the system (7) and (8) corresponds to a retracting finger [27]. In this case, far away from the phase change point, the front and back are practically parallel to each other, as can be seen in figure 6(a). The normal angle to the wave front decreases from $\pi/2$ to $\arccos (1/C_i)$. When $B < B_{rf}$ the back approaches the front, creating a closed retracting finger, shown in figure 6(b). When $B > B_{rf}$ an open retracting finger is generated, as shown in figure 6(c).

In order to obtain the wave segment shape within the top part of the stripe, the system (7) and (8) should be integrated from $s = 0$ to $s = s_j$. The last value corresponds to the excitability jump between the top and bottom parts of the stripe. It is initially unknown and should be correctly chosen later to fulfill some boundary conditions. The Cartesian coordinates of the front on the jump line are given as $x_i^- (s_j)$ and $y_i^+ (s_j)$, while the coordinates of the back are $x_i^- (s_j)$ and $y_i^- (s_j)$, where $s_j$ can be found from the equality $y_i^- (s_j) = y_i^+ (s_j)$. 


The next aim is to continue this solution into the bottom part of the stripe, where the planar velocity $c_h$ is higher than $c_t$. Note that the motion of the front and the back of a wave segment in the bottom region obey a system of equations which are very similar to equations (7) and (8), namely

$$ D \frac{d\Theta^+}{ds} = c_i \cos(\Theta^+) - c_h. \quad (13) $$

and

$$ D \frac{d\Theta^-}{ds} = \frac{G^*\varepsilon\alpha \sqrt{DA_h} \left( x_h^+ - x_h^- \right)}{c_i} - c_h + c_i \cos(\Theta^-). \quad (14) $$

However, in contrast to the case for equations (7) and (8), the planar velocity $c_h$ is larger than $c_i$ in these equations. Therefore, the shape of a front determined by equation (13) looks quite

![Figure 6. Solutions of equations (7) and (8) obtained for three different values of the dimensionless parameter $B$. (a) $B = B_{rf} = 0.755611$, (b) $B = 0.755581 < B_{rf}$, (c) $B = 0.755631 > B_{rf}$. Other parameters are fixed as $c_0 = 0.212, c_i = 0.285$. (d) Solution of equation (13) obtained for $c_h = 0.3$ and the same $c_i$, with the initial condition $\Theta^+_h = \pi/2$. In all computations, $D = 1$.](image)
different in comparison to the solution of equation (7), as can be seen in figure 6(d). This shape is identical to one found earlier for stabilized wave segments [28]. Note that the normal angle in this case reduces from $\pi/2$ to zero. This range is broader than in the case of a retracting finger. Therefore, for any given $s_j$, it is possible to smoothly continue a solution previously found within the top part of the stripe as $\Theta_h^+(s) = \Theta_l^+(s)$. It is important to stress that the Cartesian coordinates should also satisfy smoothness conditions like $x_h(s) = x_l(s)$ and $y_h(s) = y_l(s)$.

This solution should be continued to a point $s = s_b$, where $\Theta_h^+(s_b) = 0$, as shown in figure 6(d). Such a point always exists, since it corresponds to a symmetry axis of a stabilized wave segment in a homogeneous medium. In the problem under consideration, this angle value $\Theta_h^+(s_b) = 0$ corresponds to the no-flux boundary conditions at the stripe bottom.

Note that the wave back should be orthogonal to the no-flux boundary at the bottom, as well. The solution of equation (14) is completely determined by the initial condition $\Theta_h^-(s_j) = \Theta_l^-(s_j)$ and the shape of the wave front obtained for an arbitrarily chosen $s_j$. The corresponding computations show that if $s_j$ is too small, $\Theta_h^-(s_j) < \pi$; and if $s_j$ is too large, $\Theta_h^-(s_j) > \pi$. By the trial and error method, a corrected value of $s_j$ should be determined.

Thus, there is an opportunity to obtain a solution of the above formulated free-boundary problem corresponding to boundary conditions for the underlying reaction–diffusion system (1) and (2) by varying the single parameter $s_j$. An example of such a solution is shown in figure 7. Here the planar velocity in the top part is fixed to $c_0 = 0.212$, which corresponds to $\Delta = 0.3$ in the system (1) and (2) due to equation (3). In the bottom part the planar velocity is $c_h = 0.3$, which corresponds to $A_h = 2$. The dimensionless parameter $B = 0.756\,086$, which corresponds to $\varepsilon = 0.002\,992$ due to equation (11). The translational velocity of the wave segments as a whole is $c_l = 0.285$. The solution of the free-boundary problem completely determines the wave
segment shape. The width of the segment in the top part is determined as \( w = y(0) - y(s_h) \). The width of the bottom part of the stripe is specified as \( w_b = y(s) - y(s_h) \).

5. The existence of a solution

The solution of the free-boundary problem presented in figure 7 is obtained for \( B > B_{tr} \), in the case of the open retracting finger shown in figure 6(c). The parameter \( B \) can be continuously decreased by decreasing of \( \epsilon \) without any significant effect on the propagation velocity. As a result, the width of the segment in the top part of the stripe is monotonically increasing. Moreover, the width diverges when \( B \to B_{tr} \). In this limit, the part of the wave segment within the top part of the stripe approaches a retracting finger, which by definition has an infinitely large width [27]. This means that a propagating wave segment can exist in this limiting case only within an infinitely broad stripe. If the stripe has a finite width and is restricted by a no-flux boundary, a solution in the form of a wave segment does not exist. Only a curved wave similar to the one shown in figure 3 can exist in this case.

An example of a relationship \( w = w(\epsilon) \) is shown in figure 8 by the thick solid line. The width diverges at some \( \epsilon \) corresponding to \( B_{tr} \).

The results of the corresponding reaction–diffusion computations are shown by the dashed line. It is indicated by the data obtained that the segment width also diverges at some \( B \). All attempts to initiate a propagating wave segment in the reaction–diffusion computations for \( B < B_{tr} \) lead to the creation of a curved wave.

In the framework of the free-boundary approach, it is also impossible to obtain a physically acceptable solution for \( B < B_{tr} \). In this case, the solution in the top part has the form of a closed retracting finger, shown in figure 6(b). While the front solution can be continued into the bottom part to a point where \( \Theta^+_h(s_h) = 0 \), it is impossible to reach the condition \( \Theta^-_h(s_h) = \pi \) for the back solution for any \( s_j \).

Note that the accuracy of the theoretical predictions can be improved. The data shown in figure 8 by the thick solid line are based on the analytically obtained value \( \alpha = 1/\sqrt{2} \) under the assumption that \( \epsilon \ll 1 \) and \( \Delta \ll 1 \). However, this coefficient can be obtained from direct reaction–diffusion computations, giving \( \alpha = 0.943/\sqrt{2} \). This small correction of the coefficient \( \alpha \) considerably increases the prediction accuracy, as is shown by the dotted line in figure 8.

There is another restriction for the existence of a propagating wave segment, which has to be taken into account. This restriction becomes visible in the framework of the free-boundary approach. Let us assume that all parameters are fixed (e.g., as in figure 7) except the translational velocity \( c_t \), which is continuously decreasing. Two examples of such computations are shown in figure 9. It is clear that the slower translation velocity corresponds to the larger front curvature near the bottom boundary of the stripe. Consequently the width of the bottom part of the stripe, \( w_b \), corresponding to this velocity becomes smaller. Simultaneously, the width of the wave segments within the top part of the stripe, \( w \), also becomes smaller. Figure 9(b) illustrates the limiting case where \( w \) vanishes at \( c_t = c_{tm} \). There is no solution of the free-boundary approach for \( c_t < c_{tm} \).

It is important to stress that the shape of the wave segment within the bottom part of the stripe obtained for \( c_t = c_{tm} \) should be identical to a stabilized wave segment shape observed
earlier in a homogeneous medium [29]. In the last case, the dimensionless wave segment width is completely determined by the dimensionless parameter $B$. Hence this parameter specified for the bottom part of the stripe as

$$B_b = \frac{G^*e}{A_b \alpha^2 \Delta^1}$$

(15)

can be used to predict the minimal width of the bottom part of the stripe. In accordance with equation (19) in [29], the translational velocity reads

$$c_i/c_h = \sqrt{B_b + (1 - 0.535)(\frac{B_b}{0.535})^n},$$

(16)

where $n = 2.502$. The corresponding value is depicted by the left dotted line in figure 10.

Figure 8. Width of the top part of the propagating wave segment, $w$, versus the parameter $\varepsilon$ obtained for $c_0 = 0.212, c_i = 0.285, c_h = 0.3$ and $\Delta = 0.3$ (thick solid line) and the results of reaction–diffusion computations (the dashed line). The dotted line corresponds to the free-boundary computations with $\alpha = 0.943/\sqrt{2}$.

Figure 9. Shape of a wave segment obtained as a solution of the free-boundary problem with $c_0 = 0.212, c_h = 0.3, B = 0.756\,086$. (a) $c_i = 0.275$, (b) $c_i = c_{im} = 0.2281$. There is no solution for $c_i < c_{im}$.
Note that a curved wave solution exists in an inhomogeneous stripe even if the part with high propagation velocity is very thin [23]. Thus, starting from a planar wave, a propagating wave solution can exist in the form of a curved wave, but a wave solution can disappear if a localized wave segment has been used as an initial condition.

Thus, if all parameters of the free-boundary system are fixed, a wave segment solution exists within a restricted interval of the translational velocity $C_t$ as shown in figure 10. At the left edge of this parameter range, the width of the wave segment within the top part of the stripe, $w$, vanishes and the solution corresponds to the one shown in figure 9(b). At the right edge the width of the wave segment within the top part of the stripe, $w$, diverges.

It is important that a well determined range of the width $W_b$ corresponds to this interval of $C_t$ in which a wave segment solution can exist, as illustrated by figure 10. Note that the translation velocity $C_t$ is used as a control parameter in the framework of the free-boundary approach. In contrast to this, in reaction–diffusion computations or real experiments the width of the bottom part of the stripe has to be considered as the control parameter which determines the translational velocity $C_t$. Thus, a wave segment solution exists in a restricted range of the bottom part width.

6. Two asymptotes for translational velocity

It follows from the above consideration that for given parameters $c_0$, $c_h$ and $c_t$, which satisfy the condition $c_0 < c_t < c_h$, there is a physically acceptable front solution. This solution corresponds to a well determined width, $w_b$, of the bottom part of the stripe. Thus, independently of $B$, there is a relationship between the three given velocities and the width $w_b$.

On the other hand, there is a curved wave solution shown in figure 3. As was mentioned in [23] in the case of an infinite width of the top part of the stripe, the normal angle near the
excitability jump approaches
\[
\Theta_i^+(s_i) = \arccos \left( \frac{c_i}{c_h} \right).
\] (17)

Moreover, a solution of equation (13) can be expressed analytically [23] as
\[
\frac{y^h_+}{D} = -\frac{\Theta^+_h}{c_i} + \frac{2c_h}{c_i \sqrt{c_h^2 - c_i^2}} \arctan \left( \frac{c_h + c_i \tan \frac{\alpha_t}{2}}{\sqrt{c_h^2 - c_i^2}} \right).
\] (18)

Since the width of the bottom part can be expressed as
\[
w_b = y^h_+ - \Theta^+_h (s_i) - y^h_+(0),
\]
substitution of equation (17) into equation (18) gives the width of the bottom part as
\[
\frac{w_b}{D} = -\frac{\arccos \left( \frac{c_0}{c_h} \right)}{c_i} + \frac{2c_h}{c_i \sqrt{c_h^2 - c_i^2}} \arctan \left( \frac{c_h + c_i \tan \frac{\arccos \left( \frac{c_0}{c_h} \right)}{2}}{\sqrt{c_h^2 - c_i^2}} \right). \] (19)

An example of this relationship is shown in figure 10 by the thin line. Here the dimensionless width \( W_b = w_b c_i / D \) is presented as a function of \( c_i / c_h \) for \( c_h c_0 = \sqrt{2} \). Note that the solution in the form of a wave segment approaches this relationship in the limiting case when the width \( w \) diverges. For a given value of the parameter \( B \), this happens when the translational velocity satisfies equation (12) (depicted by the right dotted line in figure 10).

It is necessary to stress that, at the left edge of the existence region, the free-boundary solution approaches a wave segment propagating within the bottom part of the stripe, as shown in figure 9(b). In order to get a relationship between the translational velocity and the width of such a segment, it is enough to repeat the above consideration keeping \( \Theta^+_i (s) = \pi/2 \), which gives
\[
\frac{w_b}{D} = -\frac{\pi}{2c_i} + \frac{2c_h}{c_i \sqrt{c_h^2 - c_i^2}} \arctan \sqrt{\frac{c_h + c_i}{c_h - c_i}}.
\] (20)

This relationship is shown by the dashed line in figure 10.

It can be seen that the relationship obtained in the framework of the free-boundary approach (the thick solid) is limited by these two asymptotes, shown by the thin solid and dashed lines.

### 7. The propagation block

In a previous part of the paper, it is demonstrated that the initiation of an excitation by application of a planar wave results in a curved wave exhibiting stationary propagation, as shown in figure 3. In contrast to this, in order to obtain wave segments exhibiting stationary propagation, the initial wave should be cut off (e.g. see figure 3). The exact location of the initial cutoff is not very important. The whole initial excitation wave can be located inside the bottom part of the stripe with a faster propagation velocity. This wave is propagating along the stripe and has a tendency to penetrate into the top part. If the value of \( B \) is too small, the wave segment remains within the bottom part, like in figure 5(a), or even disappears, as shown in figure 5(b).

Similar behavior has been expected in the case where the whole wave is initially located within the top part of the stripe. For example, the wave shown in figure 11(a) was initiated...
within the top part and, in the course of time, propagates along the stripe, and penetrates into its bottom part. Finally, this wave will evolve into a curved wave exhibiting stationary propagation along the stripe. This dynamics has been observed for $A_h < 1.4$.

However, if the parameter $A_h$, which determines the propagation velocity in the bottom part, is increased to $A_h = 1.6$, the wave stops penetrating into the bottom part, as shown in figure 11(b). This propagation block looks rather strange because the bottom part is able to support traveling waves. Moreover, the propagation velocity here is even larger than that within the top part. Hence the bottom part can be considered as more excitable than the top one.

The existence of such a propagation block on the boundary of the propagation velocity jump has also been observed during the numerical computations for 1D excitable media. The only small difference is in the critical value of the parameter $A_h$, i.e. the minimal value for which the block takes place.

In order to clarify the reason for this unusual propagation block, let us consider more carefully the wave pattern in figure 11(b). Note that the bottom boundary of the wave segment is rather flat. A similar segment shape can be expected if it is induced near a boundary of a passive medium. Such a passive medium can be described using the reaction–diffusion system (1) with the modified kinetic function $F(u, v)$. Let us take within the bottom part

$$ F(u, v) = A_h \left[ 3(u - u_b) - (v - v_b) \right], \quad G(u, v) = u - \delta. \quad (21) $$

The system of equations (1) and (21) has the same steady state as the system (1) and (2) describing the top part. Obviously two-dimensional computations performed with this modified kinetics within the bottom part will result in a wave pattern similar to the one shown in figure 11(b). At the boundary between the top and bottom parts, the activator value will reach some value $u_b$, which will, of course, depend on the parameter $A_h$. An increase of $A_h$ will reduce $u_b$ since a negative feedback stabilizing the steady state of the bottom part becomes stronger. It is clear that for sufficiently large $A_h$ it is possible to get $u_b < -1.2$. Note that this value corresponds to a practically linear part of the nullcline $F(u, v) = 0$ shown in figure 2.
This means that for such a value of $A_h$, the kinetic functions in equation (21) can be replaced by the old ones in equation (2), because the cubic term in equation (2) will be negligibly small.

This is the reason for the observed propagation block. Indeed, if the parameter $A_h$ is sufficiently large, the stabilizing negative feedback is so strong that a wave in the top part is not able to exceed the excitation threshold value near the velocity jump line and penetrate into the bottom part. There is an obvious similarity between the observed effect and the propagation blocks in inhomogeneous cardiac tissue, usually referred to as a source–sink mismatch [31]. Thus, the propagation block phenomenon discovered could potentially have an application in cardiology.

8. Summary

The numerical computations performed with a standard and widely used reaction–diffusion model of an excitable medium demonstrate interesting spatiotemporal patterns appearing in an inhomogeneous stripe. Unexpectedly, it is shown that the stationary wave pattern depends on the initial conditions creating the primary excitation. Application of the free-boundary approach allows us to explain the observed wave patterns. Moreover, the main parameters of the patterns observed in the reaction–diffusion computations can be predicted in the framework of this approach. The accuracy of these predictions is rather high, as can be seen in figure 8. The results obtained should be widely applicable to quite different excitable media because the free-boundary approach is based on very general properties of the excitation waves.

The existence of a propagation block on the boundary between two parts of a medium, both of which are supporting excitation waves, is very important for many applications. For instance, in cardiology such a phenomenon has to be taken into account as a possible mechanism of wave breaking leading to cardiac arrhythmia. It can also be used for the engineering of structured excitable media intended for information processing.

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