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Elementary constituents of the group $SL(4, \mathbb{R})$, and classification of the Mueller matrices

Abstract

The goal of this paper is to develop a systematic method of locating the Mueller matrices within the class of the matrices of the real group $SL(4, \mathbb{R})$. The main idea is to construct the general transformation of the group $SL(4, \mathbb{R})$ (whose real matrices have unit determinant) is straightforward, but to analyze the adequacy of such a transformation for describing Mueller matrices is highly nontrivial. However, using the technique of Dirac matrices, we can quite easy and explicitly describe all the 16 one-parametric subgroups, from which, using the all possible products, emerges the whole group $SL(4, \mathbb{R})$. As a matter of fact, for these separate 1-parametric subgroups the question of their adequacy of describing Mueller matrices becomes sufficiently simple and thus we obtain in each case a definite answer.

1 Elementary 1-parametric generators of the group $GL(4, \mathbb{R})$

It is well known the cornerstone role played in polarization optics by the Mueller matrices. Distinguished subsets of the set of Mueller matrices generate group structures, which are isomorphic to the group of rotations or to the Lorentz group.

Since the Mueller matrices are real, of order $4 \times 4$, and acting on the 4-dimensional real Stokes vector, to investigate the sets of all possible Mueller matrices, one can use the parametrization of 4-dimensional matrices which is obtained on the ground of using the basis of Dirac matrices, developed in the works [1, 3, 2]. The main goal of this paper is to develop a systematic method of locating the Mueller matrices within the class of the matrices of the real group $SL(4, \mathbb{R})$. The main idea is the following. To construct the general transformation of the group $SL(4, \mathbb{R})$ (whose real matrices have unit determinant) is straightforward, but to analyze the adequacy of such a transformation for describing Mueller matrices is highly nontrivial (practically impossible). However, using the technique of Dirac matrices, we can quite easy and explicitly describe all the 16 one-parametric subgroups, from which, using the all possible products, emerges the whole group $SL(4, \mathbb{R})$. As a matter of fact, for these separate 1-parametric subgroups the question of their adequacy of describing Mueller matrices becomes sufficiently simple and thus we have reasons to expect in each case a definite answer.

The explicit form of the Dirac 16-dimensional basis (using the Weyl spinor representation)
is:
\[
\begin{align*}
\gamma^5 &= \begin{vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \quad \gamma^0 = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}, \quad i\gamma^5\gamma^0 = \begin{vmatrix} 0 & 0 & -i & 0 \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{vmatrix}, \\
\gamma^1 &= \begin{vmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{vmatrix}, \quad i\gamma^1 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & i \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}, \quad i\gamma^2 = \begin{vmatrix} 0 & 0 & 1 \end{vmatrix}, \\
\gamma^5\gamma^2 &= \begin{vmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{vmatrix}, \quad i\gamma^3 = \begin{vmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \\ 0 & i & 0 \end{vmatrix}, \quad \gamma^5\gamma^3 = \begin{vmatrix} 0 & 0 & 1 \end{vmatrix}, \\
2\sigma^{01} &= \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \quad 2\sigma^{02} = \begin{vmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{vmatrix}, \quad 2\sigma^{03} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \\
2i\sigma^{12} &= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}, \quad 2i\sigma^{23} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}, \quad 2i\sigma^{31} = \begin{vmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{vmatrix}.
\end{align*}
\]

All these 15 matrices (let us note them as \(\Lambda_k\)) are of Gell-Mann type: this means they have null trace and they are Hermitian too; moreover, the square of each of them is equal to 1:

\[\text{Sp} \Lambda = 0, \quad (\Lambda)^2 = I, \quad (\Lambda)^\dagger = \Lambda, \quad \Lambda \in \{\Lambda_k : k = 1, ..., 15 \}.\]

Due to the identity \((\Lambda)^2 = I\), the exponential of each from these matrices \(\Lambda\) is

\[U = e^{ia\Lambda} = \cos a + i \sin a\Lambda, \quad \det e^{ia\Lambda} = +1, \quad a \in \mathbb{R}.\]

We will use the following notations [1,3] for some specially chosen six generators \(\Lambda_i\):

\[
\begin{align*}
\alpha_1 &= \gamma^0\gamma^2, \quad \alpha_2 &= i\gamma^0\gamma^5, \quad \alpha_3 = \gamma^5\gamma^5, \\
\alpha_1^\dagger &= I, \quad \alpha_1\alpha_2 = i\alpha_3, \quad \alpha_2\alpha_1 = -i\alpha_3; \\
\beta_1 &= i\gamma^3\gamma^1, \quad \beta_2 = i\gamma^3, \quad \beta_3 = i\gamma^1, \\
\beta_1^\dagger &= I, \quad \beta_1\beta_2 = i\beta_3, \quad \beta_2\beta_1 = -i\beta_3.
\end{align*}
\]

Note that we have the commutation relations \(\alpha_j\beta_k = \beta_k\alpha_j\). Due to this commutation one can construct nine Abelian 2-parametric subgroups. For instance, \(e^{ia_1\alpha_1}e^{ib_1\beta_1} = e^{ib_1\beta_1}e^{ia_1\alpha_1}\) and so on.

All the possible products of matrices (1.2) provide us nine Lie algebra generators:

\[
\begin{align*}
A_1 &= \alpha_1\beta_1 = -\gamma^5, \quad B_1 = \alpha_1\beta_2 = \gamma^5\gamma^1, \quad C_1 = \alpha_1\beta_3 = \gamma^3\gamma^5, \\
A_2 &= \alpha_2\beta_1 = -i\gamma^2, \quad B_2 = \alpha_2\beta_2 = -i\gamma^1\gamma^2, \quad C_2 = \alpha_2\beta_3 = -i\gamma^2\gamma^3, \\
A_3 &= \alpha_3\beta_1 = \gamma^0, \quad B_3 = \alpha_3\beta_2 = \gamma^0\gamma^1, \quad C_3 = \alpha_3\beta_3 = \gamma^0\gamma^3.
\end{align*}
\]
We further specify the explicit form of the 15 elementary 1-parametric unitary transformations:

\[
\alpha_1 = \begin{bmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{bmatrix}, \quad \alpha_2 = \begin{bmatrix} 0 & iL_2 \\ -iL_2 & 0 \end{bmatrix}, \quad \alpha_3 = \begin{bmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{bmatrix},
\]

\[
U_1^\alpha = \begin{bmatrix} \cos \phi + i \sin \phi \sigma_2 & 0 \\ 0 & \cos \phi - i \sin \phi \sigma_2 \end{bmatrix}, \quad U_2^\alpha = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}, \quad U_3^\alpha = \begin{bmatrix} \cos \phi & i \sin \phi \sigma_2 \\ i \sin \phi \sigma_2 & \cos \phi \end{bmatrix}; (1.4)
\]

\[
\beta_1 = \begin{bmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{bmatrix}, \quad \beta_2 = \begin{bmatrix} 0 & -i\sigma_3 \\ i\sigma_3 & 0 \end{bmatrix}, \quad \beta_3 = \begin{bmatrix} 0 & -i\sigma_1 \\ i\sigma_1 & 0 \end{bmatrix},
\]

\[
U_1^\beta = \begin{bmatrix} \cos \phi + i \sin \phi \sigma_2 & 0 \\ 0 & \cos \phi - i \sin \phi \sigma_2 \end{bmatrix}, \quad U_2^\beta = \begin{bmatrix} \cos \phi & \sin \phi \sigma_3 \\ -\sin \phi \sigma_3 & \cos \phi \end{bmatrix}, \quad U_3^\beta = \begin{bmatrix} \cos \phi & \sin \phi \sigma_1 \\ -\sin \phi \sigma_1 & \cos \phi \end{bmatrix}; (1.5)
\]

\[
A_1 = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix},
\]

\[
U_1^A = \begin{bmatrix} \cos \phi + i \sin \phi & 0 \\ 0 & \cos \phi - i \sin \phi \end{bmatrix}, \quad U_2^A = \begin{bmatrix} \cos \phi & -\sin \phi \sigma_2 \\ \sin \phi \sigma_2 & \cos \phi \end{bmatrix}, \quad U_3^A = \begin{bmatrix} \cos \phi & i \sin \phi \\ i \sin \phi & \cos \phi \end{bmatrix}; (1.6)
\]

\[
B_1 = \begin{bmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -\sigma_3 & 0 \\ 0 & -\sigma_3 \end{bmatrix}, \quad B_3 = \begin{bmatrix} -\sigma_1 & 0 \\ 0 & \sigma_1 \end{bmatrix},
\]

\[
U_1^B = \begin{bmatrix} \cos \phi & i \sin \phi \sigma_1 \\ i \sin \phi \sigma_1 & \cos \phi \end{bmatrix}, \quad U_2^B = \begin{bmatrix} \cos \phi - i \sin \phi \sigma_3 & 0 \\ 0 & \cos \phi - i \sin \phi \sigma_3 \end{bmatrix}, \quad U_3^B = \begin{bmatrix} \cos \phi - i \sin \phi \sigma_1 & 0 \\ 0 & \cos \phi + i \sin \phi \sigma_1 \end{bmatrix}; (1.7)
\]

\[
C_1 = \begin{bmatrix} 0 & -\sigma_3 \\ -\sigma_3 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} -\sigma_1 & 0 \\ 0 & -\sigma_1 \end{bmatrix}, \quad C_3 = \begin{bmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{bmatrix},
\]

\[
U_1^C = \begin{bmatrix} \cos \phi & -i \sin \phi \sigma_3 \\ -i \sin \phi \sigma_3 & \cos \phi \end{bmatrix}, \quad U_2^C = \begin{bmatrix} \cos \phi - i \sin \phi \sigma_1 & 0 \\ 0 & \cos \phi - i \sin \phi \sigma_1 \end{bmatrix}, \quad U_3^C = \begin{bmatrix} \cos \phi + i \sin \phi \sigma_3 & 0 \\ 0 & \cos \phi - i \sin \phi \sigma_3 \end{bmatrix}; (1.8)
\]
and hence we easily get the 15 real 1-parametric $4 \times 4$-transformations from the group $SL(4,\mathbb{R})$:

$$U_1^\alpha(\phi) = \begin{vmatrix} \cos \phi & \sin \phi & 0 & 0 \\ -\sin \phi & \cos \phi & 0 & 0 \\ 0 & 0 & \cos \phi & -\sin \phi \\ 0 & 0 & \sin \phi & \cos \phi \end{vmatrix},$$

$$U_2^\alpha(\phi) = \begin{vmatrix} \cos \phi & 0 & -\sin \phi & 0 \\ 0 & \cos \phi & 0 & -\sin \phi \\ \sin \phi & 0 & \cos \phi & 0 \\ 0 & \sin \phi & 0 & \cos \phi \end{vmatrix},$$

$$U_3^\alpha(\phi) = \begin{vmatrix} \cos \phi & 0 & 0 & \sin \phi \\ 0 & \cos \phi & -\sin \phi & 0 \\ 0 & \sin \phi & \cos \phi & 0 \\ -\sin \phi & 0 & 0 & \cos \phi \end{vmatrix}; \quad (1.9)$$

$$U_1^\beta(\phi) = \begin{vmatrix} \cos \phi & \sin \phi & 0 & 0 \\ -\sin \phi & \cos \phi & 0 & 0 \\ 0 & 0 & \cos \phi & \sin \phi \\ 0 & 0 & -\sin \phi & \cos \phi \end{vmatrix},$$

$$U_2^\beta(\phi) = \begin{vmatrix} \cos \phi & 0 & \sin \phi & 0 \\ 0 & \cos \phi & 0 & -\sin \phi \\ -\sin \phi & 0 & \cos \phi & 0 \\ 0 & \sin \phi & 0 & \cos \phi \end{vmatrix},$$

$$U_3^\beta(\phi) = \begin{vmatrix} \cos \phi & 0 & 0 & \sin \phi \\ 0 & \cos \phi & \sin \phi & 0 \\ 0 & -\sin \phi & \cos \phi & 0 \\ -\sin \phi & 0 & 0 & \cos \phi \end{vmatrix}; \quad (1.10)$$

$$U_1^\Lambda(i\lambda) = \begin{vmatrix} e^{-\lambda} & 0 & 0 & 0 \\ 0 & e^{-\lambda} & 0 & 0 \\ 0 & 0 & e^\lambda & 0 \\ 0 & 0 & 0 & e^\lambda \end{vmatrix},$$

$$U_2^\Lambda(i\beta) = \begin{vmatrix} \cosh \beta & 0 & 0 & -\sinh \beta \\ 0 & \cosh \beta & \sinh \beta & 0 \\ 0 & \sinh \beta & \cosh \beta & 0 \\ -\sinh \beta & 0 & 0 & \cosh \beta \end{vmatrix},$$

$$U_3^\Lambda(i\beta) = \begin{vmatrix} \cosh \beta & 0 & -\sinh \beta & 0 \\ 0 & \cosh \beta & 0 & -\sinh \beta \\ -\sinh \beta & 0 & \cosh \beta & 0 \\ 0 & -\sinh \beta & 0 & \cosh \beta \end{vmatrix}; \quad (1.11)$$
\[
U_1^B(i\beta) = \begin{vmatrix}
\cosh \beta & 0 & 0 & -\sinh \beta \\
0 & \cosh \beta & -\sinh \beta & 0 \\
0 & -\sinh \beta & \cosh \beta & 0 \\
-\sinh \beta & 0 & 0 & \cosh \beta
\end{vmatrix},
\]
\[
U_2^B(i\lambda) = \begin{vmatrix}
e^{\lambda} & 0 & 0 & 0 \\
0 & e^{-\lambda} & 0 & 0 \\
0 & 0 & e^{\lambda} & 0 \\
0 & 0 & 0 & e^{-\lambda}
\end{vmatrix},
\]
\[
U_3^B(i\beta) = \begin{vmatrix}
\cosh \beta & \sinh \beta & 0 & 0 \\
\sinh \beta & \cosh \beta & 0 & 0 \\
0 & 0 & \cosh \beta & -\sinh \beta \\
0 & 0 & -\sinh \beta & \cosh \beta
\end{vmatrix}; \quad (1.12)
\]
\[
U_1^C(i\beta) = \begin{vmatrix}
\cosh \beta & 0 & \sinh \beta & 0 \\
0 & \cosh \beta & 0 & -\sinh \beta \\
\sinh \beta & 0 & \cosh \beta & 0 \\
0 & -\sinh \beta & 0 & \cosh \beta
\end{vmatrix},
\]
\[
U_2^C(i\beta) = \begin{vmatrix}
\cosh \beta & \sinh \beta & 0 & 0 \\
\sinh \beta & \cosh \beta & 0 & 0 \\
0 & 0 & \cosh \beta & \sinh \beta \\
0 & 0 & \sinh \beta & \cosh \beta
\end{vmatrix},
\]
\[
U_3^C(i\lambda) = \begin{vmatrix}
e^{-\lambda} & 0 & 0 & 0 \\
0 & e^{\lambda} & 0 & 0 \\
0 & 0 & e^{\lambda} & 0 \\
0 & 0 & 0 & e^{-\lambda}
\end{vmatrix}. \quad (1.13)
\]

Note that to the generator \(\lambda_0 = I\) corresponds the finite element
\[
U_0(i\lambda) = e^{-\lambda} \begin{vmatrix}1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1\end{vmatrix}.
\]

2 On the (pseudo) Euclidean rotations of Stokes 4-vectors

A real \(4 \times 4\)-matrix \(M\) may be considered as being of Mueller type and acting on polarized light (completely or partially) if \(M_{ab}S_a = S'_a\):
\[
S'_0 = M_{00}S_0 + M_{01}S_1 + M_{02}S_2 + M_{03}S_3, \\
S'_1 = M_{10}S_0 + M_{11}S_1 + M_{12}S_2 + M_{13}S_3, \\
S'_2 = M_{20}S_0 + M_{21}S_1 + M_{22}S_2 + M_{23}S_3, \\
S'_3 = M_{30}S_0 + M_{31}S_1 + M_{32}S_2 + M_{33}S_3,
\]
where the following two inequalities hold:
\[
S_0 \geq 0, \quad S^2 \geq 0, \quad (S^2 \equiv S'_0 - S'_1 - S'_2 - S'_3), \quad (2.1) \\
S'_0 \geq 0, \quad S'^2 \geq 0, \quad (S'^2 \equiv S'_0 - S'_1 - S'_2 - S'_3), \quad (2.2)
\]
In more detailed form, these inequalities look as

\[
M_{00}S_0 + M_{01}S_1 + M_{02}S_2 + M_{03}S_3 \geq 0,
\]

\[
(M_{00}S_0 + M_{01}S_1 + M_{02}S_2 + M_{03}S_3)^2 - (M_{10}S_0 + M_{11}S_1 + M_{12}S_2 + M_{13}S_3)^2 - (2.3)
\]

\[
(M_{20}S_0 + M_{21}S_1 + M_{22}S_2 + M_{23}S_3)^2 - (M_{30}S_0 + M_{31}S_1 + M_{32}S_2 + M_{33}S_3)^2 \geq 0.
\]

We recall that \(S_0\) is the intensity \(I\) of the light beam, and \(S_i = S_0p_i = Ip_i\), where \(p_i\) is the polarization vector. Correspondingly, the above inequalities read as

\[
p_1^2 + p_2^2 + p_3^2 \leq 1, \quad M_{00} + M_{01}p_1 + M_{02}p_2 + M_{03}p_3 \geq 0,
\]

\[
(M_{00} + M_{01}p_1 + M_{02}p_2 + M_{03}p_3)^2 - (M_{10} + M_{11}p_1 + M_{12}p_2 + M_{13}p_3)^2 - (2.4)
\]

\[
(M_{20} + M_{21}p_1 + M_{22}p_2 + M_{23}p_3)^2 - (M_{30} + M_{31}p_1 + M_{32}p_2 + M_{33}p_3)^2 \geq 0.
\]

The problem we face here is very complicated due to the big number of the independent parameters – elements of the matrix \(M_{ab}\). So, we may expect various solutions. In the first place, we are interested in Mueller matrix sets which exhibit a group structure.

The most evident and known such sets are the 3-parametric group of the Euclidean 3-rotations, and the 6-parametric group of the pseudo-Euclidean rotations, which form the Lorentz group. We shall describe them below. Moreover, we shall consider all the 1-parametric elementary generators of the real linear group \(SL(4, \mathbb{R})\). Examination of other subgroups is a subject for further concern.

Let us consider, by using the notation introduced above, the Lorentzian rotations of the Stokes 4-vectors. To this end let us specify the following two subgroups:

\[
R_\alpha(k_0, k_i) = k_0 I + k_i \alpha^i = \begin{vmatrix}
    k_0 & k_1 & k_2 & k_3 \\
    -k_1 & k_0 & -k_3 & k_2 \\
    -k_2 & k_3 & k_0 & -k_1 \\
    -k_3 & -k_2 & k_1 & k_0
\end{vmatrix},
\]

\[
k''_0 = k'_0 k_0 - k'_i k_i,
\]

\[
k''_n = k'_0 k_n + k'_i k_0 + \epsilon_{ijn} k'_i k_j;
\]

\[
R_\beta(m_0, m_i) = m_0 I + m_i \beta^i = \begin{vmatrix}
    m_0 & m_1 & m_2 & m_3 \\
    -m_1 & m_0 & m_3 & m_2 \\
    m_2 & -m_3 & m_0 & m_1 \\
    m_3 & m_2 & -m_1 & m_0
\end{vmatrix},
\]

\[
m''_m = m'_0 m_m - m'_i m_i,
\]

\[
m''_n = m'_0 m_n + m'_i m_0 - \epsilon_{ijn} m'_i m_j. \tag{2.5}
\]

Because the matrices of these two subgroups commute one with each other, we can multiply them and, in such a way, we can obtain a new subgroup. Moreover, this new subgroup allows us to impose the following constraints to the parameters:

\[
m_0 = k_0^*, \quad m_1 = -k^*_i;
\]

These constraints lead to

\[
R(k, k^*) = R_\alpha(k) R_\beta(k^*) = (k_0 I + k_i \alpha^i) (m_0 I + k^*_i \beta^i) =
\]

\[
k_0 k^*_0 + k^*_0 k_i \alpha^i - k_0 k^*_i \beta^i - k_i k^*_j \alpha^i \beta^j; \tag{2.6}
\]

\[
\]
We further specify a particular case:

\[
-k_0 k^*_0 + k_0^* k_i \alpha^i - k_0 k^*_i \beta^i = \\
= \begin{vmatrix}
    k_0 k^*_0 & k_0^* k_1 - k_0 k^*_1 & k_0^* k_2 - k_0 k^*_2 & k_0^* k_3 - k_0 k^*_3 \\
    -k_0 k^*_1 + k_0 k^*_1 & k_0^* k_0 - k_0 k^*_0 & -k_0 k^*_2 - k_0 k^*_2 & +k_0^* k_2 + k_0 k^*_2 \\
    -k_0 k^*_2 + k_0 k^*_2 & k_0^* k_3 + k_0 k^*_3 & k_0 k^*_0 & -k_0^* k_1 - k_0 k^*_1 \\
    -k_0 k^*_3 + k_0 k^*_3 & -k_0 k^*_2 - k_0 k^*_2 & +k_0^* k_1 + k_0 k^*_1 & k_0 k^*_0
\end{vmatrix},
\]

\[-k_j k^*_j \alpha^j \beta^j = \\
= \begin{vmatrix}
    k_j k^*_j & k_2 k^*_3 - k_3 k^*_2 & k_3 k^*_1 - k_1 k^*_3 & k_1 k^*_2 - k_2 k^*_1 \\
    k_2 k^*_3 - k_3 k^*_2 & (k_1 k^*_1 - k_2 k^*_2 - k_3 k^*_3) & k_1 k^*_2 + k_2 k^*_1 & k_1 k^*_3 + k_3 k^*_1 \\
    k_3 k^*_1 - k_1 k^*_3 & k_1 k^*_2 + k_2 k^*_1 & (k_2 k^*_2 - k_1 k^*_1 - k_3 k^*_3) & k_2 k^*_3 + k_3 k^*_2 \\
    k_1 k^*_2 - k_2 k^*_1 & k_3 k^*_3 + k_1 k^*_1 & k_2 k^*_3 + k_3 k^*_2 & (k_3 k^*_3 - k_1 k^*_1 - k_2 k^*_2)
\end{vmatrix}.
\]

The matrices \( R(k, \bar{k}^*) \) determine transformations in the 4-dimensional space with one real and three imaginary coordinates. Transition to all four real coordinates is achieved via

\[
y'_a = R_{ab} y_b, \quad \begin{vmatrix}
    y_0 \\
    y_1 \\
    y_2 \\
    y_3
\end{vmatrix} = \begin{vmatrix}
    1 & 0 & 0 & 0 \\
    0 & i & 0 & 0 \\
    0 & 0 & i & 0 \\
    0 & 0 & 0 & i
\end{vmatrix} \begin{vmatrix}
    x_0 \\
    x_1 \\
    x_2 \\
    x_3
\end{vmatrix},
\]

\[
Y = \Pi X, \quad L(k, \bar{k}^*) = \Pi^{-1} R(k, \bar{k}^*) \Pi, \quad (2.7)
\]

\[
\Pi^{-1} (k_0 k^*_0 + k_0^* k_i \alpha^i - k_0 k^*_i \beta^i) \Pi = \\
= \begin{vmatrix}
    k_0 k^*_0 & i(k_0^* k_1 - k_0 k^*_1) & i(k_0^* k_2 - k_0 k^*_2) & i(k_0^* k_3 - k_0 k^*_3) \\
    i(k_0^* k_1 - k_0 k^*_1) & k_0^* k_0 - k_0 k^*_0 & -k_0 k^*_2 - k_0 k^*_2 & +k_0^* k_2 + k_0 k^*_2 \\
    i(k_0^* k_2 - k_0 k^*_2) & k_0^* k_3 + k_0 k^*_3 & k_0 k^*_0 & -k_0^* k_1 - k_0 k^*_1 \\
    i(k_0^* k_3 - k_0 k^*_3) & -k_0 k^*_2 - k_0 k^*_2 & +k_0^* k_1 + k_0 k^*_1 & k_0 k^*_0
\end{vmatrix},
\]

\[
\Pi^{-1} (-k_j k^*_j \alpha^j \beta^j) \Pi = \\
= \begin{vmatrix}
    k_j k^*_j & i(+k_2 k^*_3 - k_3 k^*_2) & i(-k_1 k^*_3 + k_3 k^*_1) & i(-k_1 k^*_2 - k_2 k^*_1) \\
    i(+k_2 k^*_3 - k_3 k^*_2) & k_1 k^*_1 - k_2 k^*_2 - k_3 k^*_3 & k_1 k^*_2 + k_2 k^*_1 & k_1 k^*_3 + k_3 k^*_1 \\
    i(-k_1 k^*_3 + k_3 k^*_1) & k_1 k^*_2 + k_2 k^*_1 & k_2 k^*_2 - k_1 k^*_1 - k_3 k^*_3 & k_2 k^*_3 + k_3 k^*_2 \\
    i(-k_1 k^*_2 - k_2 k^*_1) & +k_1 k^*_3 + k_3 k^*_1 & +k_2 k^*_3 + k_3 k^*_2 & k_3 k^*_3 - k_1 k^*_1 - k_2 k^*_2
\end{vmatrix}.
\]

We further specify a particular case:

\[
k_0 \neq 0, \quad k_3 \neq 0
\]

\[
L(k, \bar{k}^*) = \begin{vmatrix}
    k_0 k^*_0 + k_3 k^*_3 & 0 & 0 & i(k_0^* k_3 - k_0 k^*_3) \\
    0 & k_0^* k_0 - k_0 k^*_0 & -k_0^* k_3 - k_0 k^*_3 & 0 \\
    0 & k_0^* k_3 + k_0 k^*_3 & k_0 k^*_0 - k_0 k^*_3 & 0 \\
    i(k_0^* k_3 - k_0 k^*_3) & 0 & 0 & k_0 k^*_0 + k_3 k^*_3
\end{vmatrix}.
\]
For real parameters we get Euclidean rotations:

\[
\begin{align*}
  k_0^* &= k_0 = D \cos \frac{\phi}{2}, \quad k_3^* = k_3 = D \sin \frac{\phi}{2}, \\
  L(k, \bar{k}^*) &= D^2 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \phi & -\sin \phi & 0 \\ 0 & \sin \phi & \cos \phi & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}; \tag{2.8}
\end{align*}
\]

For complex parameters we get pseudo-Euclidean transformations:

\[
\begin{align*}
  k_0 &= k_0^* = D \cosh \frac{\beta}{2}, \quad k_3 = -k_3^* = iD \sinh \frac{\beta}{2}, \\
  L &= \Pi^{-1} R \Pi = D^2 \begin{vmatrix} \cosh \beta & 0 & 0 & -\sinh \beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh \beta & 0 & 0 & \cosh \beta \end{vmatrix}. \tag{2.9}
\end{align*}
\]

The quantity \( D \) determines the determinant of \( L \), via

\[
\det L = D^8.
\]

Let us consider now another particular example:

\[
\begin{align*}
  k_0 \neq 0, \quad k_1 \neq 0 \\
  L &= \Pi^{-1} R \Pi = \begin{vmatrix} k_0 k_0^* + k_1 k_1^* & i(k_0^* k_1 - k_0 k_1^*) & 0 & 0 \\ i(k_0^* k_1 - k_0 k_1^*) & k_0 k_0^* + k_1 k_1^* & 0 & 0 \\ 0 & 0 & k_0 k_0^* - k_1 k_1^* & -k_0^* k_1 - k_0 k_1^* \\ 0 & 0 & +k_0^* k_1 + k_0 k_1^* & k_0 k_0^* - k_1 k_1^* \end{vmatrix}; \\
\text{whence it follows} \\
  k_0^* &= k_0 = D \cos \frac{\phi}{2}, \quad k_1^* = k_3 = D \sin \frac{\phi}{2}, \\
  L &= D^2 \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \phi & -\sin \phi \\ 0 & 0 & \sin \phi & \cos \phi \end{vmatrix}; \tag{2.10}
\end{align*}
\]

\[
\begin{align*}
  k_0 &= k_0^* = D \cosh \frac{\beta}{2}, \quad k_1 = -k_1^* = iD \sinh \frac{\beta}{2}, \\
  L &= D^2 \begin{vmatrix} \cosh \beta & -\sinh \beta & 0 & 0 \\ -\sinh \beta & \cosh \beta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}. \tag{2.11}
\end{align*}
\]

\(^1\text{Note that for the need of polarization optics the transformations having non-unit determinant may also be of interest.}\)
Let us write below the explicit form of the factors \(R_\alpha(k_0, k_j)\) and \(R_\beta(k^*_0, -k^*_j)\), after performing the above similarity transformation:

\[
K = \Pi^{-1} R_\alpha(k_0, k_j)\Pi = \begin{bmatrix}
k_0 & ik_1 & ik_2 & ik_3 \\
ik_1 & k_0 & -k_3 & k_2 \\
ik_2 & k_3 & k_0 & -k_1 \\
ik_3 & -k_2 & k_1 & k_0 \\
\end{bmatrix},
\]

\[
K^* = \Pi^{-1} R_\beta(k^*_0, -k^*_j)\Pi = \begin{bmatrix}
k^*_0 & -ik^*_1 & -ik^*_2 & -ik^*_3 \\
-ik^*_1 & k^*_0 & -k^*_3 & k^*_2 \\
-ik^*_2 & k^*_3 & k^*_0 & -k^*_1 \\
-ik^*_3 & -k^*_2 & k^*_1 & k^*_0 \\
\end{bmatrix}, \tag{2.12}
\]

According to the afore mentioned considerations, any arbitrary transformation of the Lorentz group may be factorized into two mutually commuting and conjugate terms as

\[L = K K^* = K^* K.\]

In Physics literature, that fact was firstly noted by Einstein and Mayer (in 1932–1933), while constructing the theory of semi-spinors \([5, 6, 7]\); further, a systematic approach of the Lorentz group was developed by Fedorov \([2]\).

Let us write below the explicit form of the 2-parametric commuting factors:

\[
K_1 = \begin{bmatrix}
k_0 & ik_1 & 0 & 0 \\
0 & k_0 & 0 & 0 \\
0 & 0 & k_0 & -k_1 \\
0 & 0 & k_1 & k_0 \\
\end{bmatrix}, \quad K_1^* = \begin{bmatrix}
k^*_0 & -ik^*_1 & 0 & 0 \\
0 & k^*_0 & 0 & 0 \\
0 & 0 & k^*_0 & -k^*_1 \\
0 & 0 & k^*_1 & k^*_0 \\
\end{bmatrix},
\]

\[
K_2 = \begin{bmatrix}
k_0 & 0 & ik_2 & 0 \\
0 & k_0 & 0 & k_2 \\
0 & k_2 & k_0 & 0 \\
0 & -k_2 & 0 & k_0 \\
\end{bmatrix}, \quad K_2^* = \begin{bmatrix}
k^*_0 & 0 & -ik^*_2 & 0 \\
0 & k^*_0 & 0 & k^*_2 \\
0 & -k^*_2 & 0 & k^*_0 \\
0 & k^*_2 & 0 & k^*_0 \\
\end{bmatrix},
\]

\[
K_3 = \begin{bmatrix}
k_0 & 0 & 0 & ik_3 \\
0 & k_0 & -k_3 & 0 \\
0 & k_3 & k_0 & 0 \\
0 & 0 & 0 & k_0 \\
\end{bmatrix}, \quad K_3^* = \begin{bmatrix}
k^*_0 & 0 & 0 & -ik^*_3 \\
0 & k^*_0 & -k^*_3 & 0 \\
0 & k^*_3 & k^*_0 & 0 \\
0 & 0 & 0 & k^*_0 \\
\end{bmatrix}. \tag{2.13}
\]

### 3 The action of the pseudo-Euclidean rotations on the partially polarized light

Let us consider the action of pure Lorentz transformations on 4-vectors in the context of Stokes formalism, and firstly let us restrict to partially polarized light. The corresponding Stokes 4-vectors are analogous to the velocity 4-vectors of massive particles:

\[S^a = (I, Ip), \quad U^a = (U^0, U^0V);\]

Moreover, these 4-vectors behave in the same way with respect to Lorentzian transformations

\[L = \begin{bmatrix}
cosh \beta & -e \sinh \beta \\
e \sinh \beta & \delta_{ij} + (cosh \beta - 1)e_i e_j \\
\end{bmatrix}, \quad e^2 = 1.
\]
They act on the Stokes vectors as follows:

\[ I' = I (\cosh \beta - \sinh \beta e p), \]

\[ I' p' = I \left[ -\sinh \beta e + p + (\cosh \beta - 1) e (e p) \right], \]

or

\[ I' = I (\cosh \beta - \sinh \beta e p), \]

\[ p' = \frac{-\sinh \beta e + p + (\cosh \beta - 1) e (e p)}{\cosh \beta - \sinh \beta e p}. \] (3.1)

Let us specify them in the following several special cases:

1:

\[ e p = 0, \quad I' = I \cosh \beta, \quad p' = -\sinh \beta e + p, \]

where the intensity of the light beam increases and the polarization degree decreases.

2:

\[ p = + p e, \quad I' = I (\cosh \beta - \sinh \beta p), \]

where the polarization vector changes according to the law

\[ p' = p' e = -\sinh \beta + \cosh \beta p e. \]

Therefore the polarization degree varies as

\[ p' = p - p^{-1} \tanh \beta. \]

Note that when \( \beta > 0 \), this decreases, and when \( \beta < 0 \), then it increases. In particular, there exists the "rest reference frame" where the partially polarized light becomes natural light:

\[ L_0 = L_0(\beta_0, n), \quad \tanh \beta_0 = p, \]

\[ p' = 0, \quad I' = I (\cosh \beta_0 - \sinh \beta_0 \beta \tanh \beta_0) = \frac{I}{\cosh \beta_0}, \] (3.2)

and here the polarization degree equals to zero.

3:

\[ p = - p e, \quad I' = I (\cosh \beta + \sinh \beta p), \]

\[ p' = p' e = -\sinh \beta + \cosh \beta p e. \] (3.3)

The polarization degree varies according to the law

\[ p' = p + p^{-1} \tanh \beta, \]

where for \( \beta > 0 \) increases, while for \( \beta < 0 \) it decreases. Again, there exist an analogue of the "rest reference frame" \( L_0 \), in which the polarization degree equals to zero.
Let us further consider the role of the relativistic ellipsoid in optics. We start with the simplest case, \( e = (0, 0, 1) \), while the formulas (3.2) give

\[
I' = I \left( \cosh \beta - p_3 \sinh \beta \right), \quad p_3' = \frac{\cosh \beta p_3 - \sinh \beta}{\cosh \beta - p_3 \sinh \beta},
\]

\[(3.4)\]

\[
p_1' = \frac{p_1}{(\cosh \beta - p_3 \sinh \beta)}, \quad p_2' = \frac{p_2}{(\cosh \beta - p_3 \sinh \beta)}.
\]

\[(3.5)\]

Due to the basic property of the Lorentz matrices, the following identity holds:

\[
I_2' (1 - p_2'^2) = I_2 (1 - p_2^2) \quad \text{or} \quad 1 - p_2'^2 = \frac{1 - p_2^2}{(\cosh \beta - p_3 \sinh \beta)^2}.
\]

\[(3.6)\]

Therefore, the polarization degree transforms according to

\[
p_2'^2 = 1 - \frac{1 - p_2^2}{(\cosh \beta - p_3 \sinh \beta)^2}.
\]

\[(3.7)\]

By expressing \( p_3 \) through \( p_3' \), we infer

\[
p_3 = \frac{\cosh \beta p_3' + \sinh \beta}{\cosh \beta + \sinh \beta p_3'}, \quad \Rightarrow \quad \text{cosh} \beta - p_3 \sinh \beta = \frac{1}{\cosh \beta + \sinh \beta p_3'},
\]

and (3.6) reduces to the form

\[
p_2'^2 = 1 - (1 - p_2^2)(\cosh \beta + \sinh \beta p_3')^2.
\]

\[(3.8)\]

Let us show that this equation (3.7) describes an ellipsoid. Indeed, (3.7) can be re-written as

\[
p_1'^2 + p_2'^2 + p_3'^2 + (1 - p_2^2)2 \cosh \beta \sinh \beta p_3' + (1 - p_2^2) \sinh^2 \beta p_3'^2 = 1 - (1 - p_2^2) \cosh^2 \beta,
\]

or

\[
p_1'^2 + p_2'^2 + (\cosh^2 \beta - p_2^2 \sinh^2 \beta) \left[ p_3' + \frac{(1 - p_2^2) \sinh \beta \cosh \beta}{\cosh^2 \beta - p_2^2 \sinh^2 \beta} \right]^2
= p_2^2 \cosh^2 \beta - \sinh^2 \beta + \frac{(1 - p_2^2)^2 \sinh^2 \beta \cosh^2 \beta}{\cosh^2 \beta - p_2^2 \sinh^2 \beta},
\]

and finally we obtain the ellipsoid equation

\[
p_1'^2 + p_2'^2 + (\cosh^2 \beta - p_2^2 \sinh^2 \beta) (p_3' + \gamma)^2 = \frac{p_2^2}{\cosh^2 \beta - p_2^2 \sinh^2 \beta},
\]

\[(3.8)\]

where

\[
\gamma = \frac{(1 - p_2^2) \sinh \beta \cosh \beta}{\cosh^2 \beta - p_2^2 \sinh^2 \beta}, \quad \cosh^2 \beta - p_2^2 \sinh^2 \beta = \cosh^2 \beta (1 - p_2^2) + p_2 > 0.
\]

Thus, the surface which has the form of a sphere \( (p^2 = p_2^2) \) will transform under the action of a Mueller matrix of Lorentz type into ellipsoid (3.8).
This result can be extended for more general Mueller matrices of Lorentz type of arbitrary
orientation:

\[
I' = I (\cosh \beta - \sinh \beta (ep)), \quad p' = \frac{p - e \sinh \beta + (\cosh \beta - 1) e (ep)}{\cosh \beta - \sinh \beta e p}.
\]

Considering the identity

\[
I'^2 (1 - p'^2) = I^2 (1 - p^2) \quad \Rightarrow \quad 1 - p'^2 = \frac{1 - p^2}{[\cosh \beta - \sinh \beta (ep)]^2}, \tag{3.9}
\]

then, excluding the variable \(p\), we get:

\[
p = \frac{p' + e \sinh \beta + (\cosh \beta - 1) e (e p')}{\cosh \beta + \sinh \beta e p'},
\]

or

\[
\cosh \beta - \sinh \beta (ep) = \frac{1}{\cosh \beta + \sinh \beta ep'}.
\]

From (3.9) we get that

\[
1 - p'^2 = (1 - p^2) (\cosh \beta + \sinh \beta ep'^2).
\]

This describes an ellipsoid equation with the orientation governed by the vector \(e\).

4 The action of a Lorentzian transformation on completely polarized light

Now let us consider the action of Lorentzian transformations on completely polarized light
(the analogous of the isotropic 4-vectors in Special Relativity):

\[
S^a = (I, In), \quad n^2 = 1. \tag{4.1}
\]

With respect to transformations of kind

\[
L = \begin{vmatrix}
\cosh \beta & -e \sinh \beta \\
-e \sinh \beta & [\delta_{ij} + (\cosh \beta - 1)e_ie_j]
\end{vmatrix},
\]

the Stokes 4-vector transforms as follows:

\[
I' = I (\cosh \beta - \sinh \beta en), \quad n' = \frac{-\sinh \beta e + n + (\cosh \beta - 1) e(en)}{\cosh \beta - \sinh \beta en}.
\]

Let us specify now several special cases:

1:

\[
\begin{align*}
e & = 0, \quad I' = I \cosh \beta, \quad n' = \frac{n - \sinh \beta e}{\cosh \beta}. \\
\end{align*}
\]

2:

\[
\begin{align*}
e & = n, \quad I'^{-\beta}, \quad n' = \frac{-\sinh \beta n + n + (\cosh \beta - 1) n}{\cosh \beta - \sinh \beta} = +n. \tag{4.1}
\end{align*}
\]
\[ e = -n, \quad I^{+\beta}, \quad n' = \frac{\sinh \beta n + n + (\cosh \beta - 1) n}{\cosh \beta + \sinh \beta} = +n \] (4.2)

The known non-existence of the "rest reference frame" for massless particles means, in the context of polarization optics, that one cannot completely transform the polarized light into a natural one.

Note that under the action of Mueller transformations of Lorentz type, the degree of the polarization of light does not change, that is \( p' = p = 1 \).

5 On the deformations of the Stokes 4-vectors

Now, let us consider in detail the Mueller matrices of diagonal form: – these describe simple deformations of the Stokes 4-vectors. Firstly, let it be the variant \( U_0(i\lambda) \):

\[
M = U_0(i\lambda) = \begin{vmatrix}
    e^{-\lambda} & 0 & 0 & 0 \\
    0 & e^{-\lambda} & 0 & 0 \\
    0 & 0 & e^{-\lambda} & 0 \\
    0 & 0 & 0 & e^{-\lambda}
\end{vmatrix}, \quad S'_a = e^{-\lambda} S_a.
\]

The inequalities (2.1) hold good:

\[
S'_0 = e^{-\lambda} S_0 \geq 0, \quad S'^2 = e^{-2\lambda} S'^2_0 \geq 0.
\]

The action of such Mueller matrices on the light is given by the relations

\[
I'^{-\lambda} I, \quad p'_i = p_i.
\]

We further consider the variant \( U^B_2(i\lambda) \):

\[
M = U^B_2(i\lambda) = \begin{vmatrix}
    e^{\lambda} & 0 & 0 & 0 \\
    0 & e^{\lambda} & 0 & 0 \\
    0 & 0 & e^{\lambda} & 0 \\
    0 & 0 & 0 & e^{-\lambda}
\end{vmatrix},
\]

\[
S'_0 = e^{\lambda} S_0, \quad S'_1 = e^{-\lambda} S_1, \quad S'_2 = e^{\lambda} S_2, \quad S'_3 = e^{-\lambda} S_3. \] (5.1)

The restrictions (2.1) take the form (the first inequality always holds good):

\[
S'_0 = e^{\lambda} S_0 \geq 0, \quad S'^2 = e^{2\lambda} (S'^2_0 - S'^2_2) - e^{-2\lambda} (S'^2_1 + S'^2_3) \geq 0.
\]

The second one is equivalent to

\[
e^{4\lambda} \geq \frac{S'^2_1 + S'^2_3}{S'^2_0 - S'^2_2}.
\]

Since the initial light obeys the restriction

\[
S'^2_0 - S'^2_1 - S'^2_2 - S'^2_3 \geq 0 \quad \Rightarrow \quad 1 \geq \frac{S'^2_2 + S'^2_3}{S'^2_0 - S'^2_1},
\]
where the inequality (2.2b) will be true when \( \lambda \) is positive, that is

\[
\lambda \in [0, +\infty)
\]

The action of these Mueller matrices on the light is given by the relations

\[
I'^\lambda I, \quad p'_1 = e^{-2\lambda} p_1, \quad p'_2 = p_2, \quad p'_3 = e^{-2\lambda} p_3.
\]

Now, consider the variant \( U^A_1(i\lambda) \):

\[
M = U^A_1(i\lambda) = \begin{pmatrix}
  e^{-\lambda} & 0 & 0 & 0 \\
  0 & e^{-\lambda} & 0 & 0 \\
  0 & 0 & e^{\lambda} & 0 \\
  0 & 0 & 0 & e^{\lambda}
\end{pmatrix},
\]

\[
S'_0 = e^{-\lambda} S_0, \quad S'_1 = e^{-\lambda} S_1, \quad S'_2 = e^{\lambda} S_2, \quad S'_3 = e^{\lambda} S_3,
\]

\[
S'^2 = e^{-2\lambda}(S_0^2 - S_1^2) - e^{2\lambda}(S_2^2 + S_3^2) \geq 0.
\]

The last inequality is equivalent to

\[
\frac{S_0^2 - S_1^2}{S_2^2 + S_3^2} \geq e^{4\lambda} \implies \lambda \in (-\infty, 0].
\]

The intensity of the light and the degree of polarization transform as follows:

\[
I'^{-\lambda} = I, \quad p'_1 = p_1, \quad p'_2 = e^{2\lambda} p_2, \quad p'_3 = e^{2\lambda} p_3.
\]

Now, consider the variant \( U^C_3(i\lambda) \):

\[
M = U^C_3(i\lambda) = \begin{pmatrix}
  e^{-\lambda} & 0 & 0 \\
  0 & e^{\lambda} & 0 \\
  0 & 0 & e^{\lambda}
\end{pmatrix},
\]

\[
S'_0 = e^{-\lambda} S_0, \quad S'_1 = e^{\lambda} S_1, \quad S'_2 = e^{\lambda} S_2, \quad S'_3 = e^{-\lambda} S_3,
\]

\[
S'^2 = e^{-2\lambda}(S_0^2 - S_1^2) - e^{2\lambda}(S_2^2 + S_3^2) \geq 0.
\]

The last inequality gives

\[
\frac{S_0^2 - S_1^2}{S_2^2 + S_3^2} \geq e^{4\lambda} \implies \lambda \in (-\infty, 0].
\]

The intensity of the light and the degree of polarization transform as follows:

\[
I'^{-\lambda} = I, \quad p'_1 = e^{2\lambda} p_1, \quad p'_2 = e^{2\lambda} p_2, \quad p'_3 = I p_3.
\]

Instead of the four diagonal Mueller transformations described above, one may introduce the
following simpler four elementary deformations:

\[
E_0 = \begin{pmatrix}
  e^\lambda & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{pmatrix}, \quad I^\lambda = I, \ p_i' = p_i, \ p_i'' = p_i, \ p_i''' = p_i,
\]

\[
E_1 = \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & e^\lambda & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1
\end{pmatrix}, \quad I' = I, \ p_1' = e^\lambda p_1, \ p_1'' = p_2, \ p_1''' = p_3,
\]

\[
E_2 = \begin{pmatrix}
  e^{-\lambda} & 0 & 0 & 0 \\
  0 & e^\lambda & 0 & 0 \\
  0 & 0 & e^\lambda & 0 \\
  0 & 0 & 0 & e^{-\lambda}
\end{pmatrix}, \quad I' = I, \ p_2' = e^\lambda p_2, \ p_2'' = p_3,
\]

\[
E_3 = \begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & e^\lambda
\end{pmatrix}, \quad I' = I, \ p_3' = p_1, \ p_3'' = p_2, \ p_3''' = e^\lambda p_3.
\]  \tag{5.4}

It is obvious that the above four matrices determine a 4-parametric Abelian subgroup of the mutually commuting transformations.

6 On other subgroups of \(SL(4, \mathbb{R})\)

Among the generators (1.5) one can emphasize the following triplets:

\[
K = \{A_1 = \alpha_1 \beta_1, B_2 = \alpha_2 \beta_2, C_3 = \alpha_3 \beta_3\},
\]

\[
L = \{C_1 = \alpha_1 \beta_3, A_2 = \alpha_2 \beta_1, B_3 = \alpha_3 \beta_2\},
\]

\[
M = \{B_1 = \alpha_1 \beta_2, C_2 = \alpha_2 \beta_3, A_3 = \alpha_3 \beta_1\}, \quad \tag{6.1}
\]

and

\[
K' = \{-C_1 = -\alpha_1 \beta_3, -B_2 = -\alpha_2 \beta_2, -C_3 = -\alpha_3 \beta_3\},
\]

\[
L' = \{-B_1 = -\alpha_1 \beta_2, -A_2 = -\alpha_2 \beta_1, -B_3 = -\alpha_3 \beta_2\},
\]

\[
M' = \{-A_1 = -\alpha_1 \beta_1, -C_2 = -\alpha_2 \beta_3, -B_3 = -\alpha_3 \beta_2\}. \quad \tag{6.2}
\]

For all 3-vectors of the above generators, we can accordingly construct multiplication laws of the same type:

\[
\Gamma_1 \Gamma_2 = -\Gamma_3, \quad \Gamma_2 \Gamma_1 = -\Gamma_3, \quad \Gamma_1 \Gamma_2 - \Gamma_2 \Gamma_1 = 0,
\]

and similar ones provided by cyclic permutations. Within each triple, the generators commute one with each other. This means that on the base of each triplet of generators one can construct 1-, 2-, 3-parametric Abelian subgroups.
Let us mention that using the 15 generators $\alpha_i$, $\beta_i$, $A_i$, $B_i$, $C_i$, one can build 20 isomorphic to $SU(2)$ Lie algebras:

$$(\alpha_1, \alpha_2, \alpha_3), \quad (\beta_1, \beta_2, \beta_3), \quad (\alpha_1, A_2, A_3), \quad (A_1, \alpha_2, A_3), \quad (A_1, A_2, \alpha_3),$$

$$(\alpha_1, B_2, B_3), \quad (B_1, \alpha_2, B_3), \quad (B_1, B_2, \alpha_3),$$

$$(\alpha_1, C_2, C_3), \quad (C_1, \alpha_2, C_3), \quad (C_1, C_2, \alpha_3),$$

$$(\beta_1, B_1, \alpha_3), \quad (\beta_1, B_2, C_2), \quad (\beta_1, B_3, C_3),$$

$$(A_1, \beta_2, C_1), \quad (A_2, \beta_2, C_2), \quad (A_3, \beta_2, C_3),$$

$$(A_1, B_1, \beta_3), \quad (A_2, B_2, \beta_3), \quad (A_3, B_3, \beta_3).$$

All these say that the reserve of the possible subgroups of $SL(4, \mathbb{R})$ is rather large, and that we may expect to separate some of them as being appropriate to define Mueller-type subgroups.

7 Examination of 16 elementary one-parametric subgroups in $SL(4, \mathbb{R})$

Let us consider the remained 12 one-parametric subgroups in $SL(4, \mathbb{R})$:

(7.1)

Variant $U_1^\alpha(\phi)$:

$$M = U_1^\alpha(\phi) = \begin{vmatrix} \cos \phi & \sin \phi & 0 & 0 \\ -\sin \phi & \cos \phi & 0 & 0 \\ 0 & 0 & \cos \phi & -\sin \phi \\ 0 & 0 & \sin \phi & \cos \phi \end{vmatrix},$$

or with the use of the block form

$$M = \begin{vmatrix} k_0 + k_3 & k_1 + k_2 & n_0 + n_3 & n_1 + n_2 \\ k_1 - k_2 & k_0 - k_3 & n_1 - n_2 & n_0 - n_3 \\ l_0 + l_3 & l_1 + l_2 & m_0 + m_3 & m_1 + m_2 \\ l_1 - l_2 & l_0 - l_3 & m_1 - m_2 & m_0 - m_3 \end{vmatrix},$$

where

$$k_0 = \cos \phi, \quad k_1 = 0, \quad k_2 = +\sin \phi, \quad k_3 = 0,$$

$$m_0 = \cos \phi, \quad m_1 = 0, \quad m_2 = -\sin \phi, \quad m_3 = 0,$$

$$n_0 = 0, \quad n_1 = 0, \quad n_2 = 0, \quad n_3 = 0,$$

$$l_0 = 0, \quad l_1 = 0, \quad l_2 = 0, \quad l_3 = 0.$$ 

The restrictions (2.1) give

$$S_0 \geq 0, \quad S_0^2 - S_1^2 - S_2^2 - S_3^2 \geq 0,$$

$$S_0' = \cos \phi S_0 + \sin \phi S_1 \geq 0,$$

$$(\cos \phi S_0 + \sin \phi S_1)^2 \geq (-\sin \phi S_0 + \cos \phi S_1)^2 + S_2^2 + S_3^2 \implies \cos 2\phi (S_0^2 - S_1^2) + \sin 2\phi 2S_0 S_1 - S_2^2 - S_3^2 \geq 0.$$
With respect to the variables
\[ \tan \phi = x, \quad p_1 = a, \quad p_2 = b, \quad p_3 = c, \]
the above inequalities take the form
\[
\frac{a \sin \phi + \cos \phi}{1 - \frac{x^2}{1 + x^2}(1 - a^2)} + \frac{2x}{1 + x^2} 2a - b^2 - c^2 \geq 0. \tag{7.2}
\]
The sets of solutions of the first inequality can be illustrated by the following formulas and by Fig. 1 (the interval \( \phi \in [1, 2\pi] \) is divided into four parts):
\[
\begin{align*}
\text{Case 1:} & \quad a > 0, \quad \cos \phi > 0, \quad \phi \in \{I, IV\}, \quad x \geq -\frac{1}{a}; \\
\text{Case 2:} & \quad a > 0, \quad \cos \phi < 0, \quad \phi \in \{II, III\}, \quad x \leq -\frac{1}{a}; \\
\text{Case 3:} & \quad a < 0, \quad \cos \phi > 0, \quad \phi \in \{I, IV\}, \quad x \leq -\frac{1}{a}; \\
\text{Case 4:} & \quad a < 0, \quad \cos \phi < 0, \quad \phi \in \{II, III\}, \quad x \geq -\frac{1}{a}. \tag{7.3}
\end{align*}
\]
The quadratic inequality (7.2) is equivalent to
\[
(a^2 - 1 - b^2 - c^2)x^2 + 4ax + (1 - a^2 - b^2 - c^2) \geq 0.
\]
whose solution is \( x \in [x_1, x_2], \) where \( x_1, x_2 \) are the roots of the equation
\[
x^2 - \frac{2x}{b^2 + c^2 + 1 - a^2} - \frac{1 - a^2 - b^2 - c^2}{b^2 + c^2 + 1 - a^2} = 0.
\]
These are the real numbers
\[
\begin{align*}
x_1 &= \frac{2a - \sqrt{4a^2 + (1 - b^2 - c^2 - a^2)(b^2 + c^2 + 1 - a^2)}}{b^2 + c^2 + 1 - a^2}, \\
x_2 &= \frac{2a + \sqrt{4a^2 + (1 - b^2 - c^2 - a^2)(b^2 + c^2 + 1 - a^2)}}{b^2 + c^2 + 1 - a^2}. \tag{7.4}
\end{align*}
\]
In the case of completely polarized light, the formulas become much simpler:
\[ b^2 + c^2 + a^2 = 1, \quad x_{1,2} = \left\{ 0, \frac{2a}{1 - a^2} \right\}; \]
and two different possibilities should be distinguished
\[
\begin{align*}
a &> 0, \quad 0 \leq x \leq \frac{2a}{1 - a^2} \quad \{ I, III \}; \\
a &< 0, \quad \frac{2a}{1 - a^2} \leq x \leq 0 \quad \{ II, IV \}; \\
a &= 0, \quad x_1 = x_2 = 0 \quad \text{(degenerate case)}.
\end{align*}
\]
For $a > 0$, the joint solution of both inequalities in (7.2) are possible only in the quadrant I:

$$a > 0, \quad 0 \leq x \leq \frac{2a}{1 - a^2}.$$  

For $a < 0$, the joint solution of both inequalities in (7.2) is possible only in the quadrant IV:

$$a < 0, \quad \frac{2a}{1 - a^2} \leq x \leq 0.$$  

Let us investigate now the more complicated case of partially polarized light. Since for any value of $a$ the root $x_1$ is negative, and the root $x_2$ is positive, turning to Fig. 1, we see that the joint solutions of both inequalities in (7.2) are possible.

\begin{align*}
(7.2) \quad M = U_2^a(\phi) = & \begin{vmatrix}
\cos \phi & 0 & \sin \phi & 0 \\
0 & \cos \phi & 0 & \sin \phi \\
-\sin \phi & 0 & \cos \phi & 0 \\
0 & -\sin \phi & 0 & \cos \phi
\end{vmatrix}, \\
\text{or in block form} \\
M = & \begin{vmatrix}
k_0 + k_3 & k_1 + k_2 & n_0 + n_3 & n_1 + n_2 \\
k_1 - k_2 & k_0 - k_3 & n_1 - n_2 & n_0 - n_3 \\
l_0 + l_3 & l_1 + l_2 & m_0 + m_3 & m_1 + m_2 \\
l_1 - l_2 & l_0 - l_3 & m_1 - m_2 & m_0 - m_3
\end{vmatrix},
\end{align*}

(7.5)
The restrictions (2.1) lead to

\[ S_0 \geq 0, \quad S_0^2 - S_1^2 - S_2^2 - S_3^2 \geq 0, \]
\[ S_0' - S_1' + S_2' + S_3' \geq 0, \]
\[ S_0' = \cos \phi S_0 + \sin \phi S_2 \geq 0, \]
\[ (\cos \phi S_0 + \sin \phi S_2)^2 - (\cos \phi S_1 + \sin \phi S_3)^2 - \]
\[ (-\sin \phi S_0 + \cos \phi S_2)^2 - (-\sin \phi S_1 + \cos \phi S_3)^2 \geq 0 \implies \]
\[ \implies \cos 2\phi (S_0^2 - S_3^2) + \sin 2\phi 2S_0S_2 - S_1^2 - S_3^2 \geq 0. \]

In the variables \( a, b, c, x \), these inequalities take the form

\[
a^2 + b^2 + c^2 \leq 1, \quad \cos \phi + b \sin \phi \geq 0, \]
\[
\frac{1 - x^2}{1 + x^2}(1 - b^2) + \frac{2x}{1 + x^2} 2b - a^2 - c^2 \geq 0. \tag{7.6}
\]

They differ from the previous ones from (7.2) only in formal notation, and therefore the results will be much the same.

(7.3)

**Variant \( U_3^a(\phi) \)**

\[
U_3^a(\phi) = \begin{vmatrix}
\cos \phi & 0 & 0 & \sin \phi \\
0 & \cos \phi & -\sin \phi & 0 \\
0 & \sin \phi & \cos \phi & 0 \\
-\sin \phi & 0 & 0 & \cos \phi
\end{vmatrix},
\]

or in block form

\[
M = \begin{pmatrix}
k_0 + k_3 & k_1 + k_2 & n_0 + n_3 & n_1 + n_2 \\
k_1 - k_2 & k_0 - k_3 & n_1 - n_2 & n_0 - n_3 \\
l_0 + l_3 & l_1 + l_2 & m_0 + m_3 & m_1 + m_2 \\
l_1 - l_2 & l_0 - l_3 & m_1 - m_2 & m_0 - m_3
\end{pmatrix},
\]

\[
k_0 = \cos \phi, \quad k_1 = 0, \quad k_2 = 0, \quad k_3 = 0, \]
\[
m_0 = \cos \phi, \quad m_1 = 0, \quad m_2 = 0, \quad m_3 = 0, \]
\[
n_0 = 0, \quad n_1 = 0, \quad n_2 = \sin \phi, \quad n_3 = 0, \]
\[
l_0 = 0, \quad l_1 = 0, \quad l_2 = \sin \phi, \quad l_3 = 0. \tag{7.7}
\]

The restrictions (2.1) lead to

\[ S_0 \geq 0, \quad S_0^2 - S_1^2 - S_2^2 - S_3^2 \geq 0, \]
\[ S_0' = \cos \phi S_0 + \sin \phi S_3 \geq 0, \]
\[ (\cos \phi S_0 + \sin \phi S_3)^2 \geq (-\sin \phi S_0 + \cos \phi S_3)^2 + S_1^2 + S_2^2 \implies \]
\[ \implies \cos 2\phi (S_0^2 - S_3^2) + \sin 2\phi 2S_0S_3 \geq S_1^2 + S_2^2. \]

In the variables \( a, b, c, x \), the inequalities take the form

\[
\cos \phi + c \sin \phi \geq , \]
\[
\frac{1 - x^2}{1 + x^2}(1 - c^2) + \frac{2x}{1 + x^2} 2c - a^2 - b^2 \geq 0. \tag{7.8}
\]
These differ from the previous ones from (7.2) only in formal notation, and therefore the results will be much the same.

(7.4)

Variant \( U_1^\beta (\phi) \)

\[
M = U_1^\beta (\phi) = \begin{bmatrix}
\cos \phi & \sin \phi & 0 & 0 \\
-\sin \phi & \cos \phi & 0 & 0 \\
0 & 0 & \cos \phi & \sin \phi \\
0 & 0 & -\sin \phi & \cos \phi
\end{bmatrix},
\]

or in block form

\[
M = \begin{bmatrix}
k_0 + k_3 & k_1 + k_2 & n_0 + n_3 & n_1 + n_2 \\
k_1 - k_2 & k_0 - k_3 & n_1 - n_2 & n_0 - n_3 \\
l_0 + l_3 & l_1 + l_2 & m_0 + m_3 & m_1 + m_2 \\
l_1 - l_2 & l_0 - l_3 & m_1 - m_2 & m_0 - m_3
\end{bmatrix},
\]

\[
k_0 = \cos \phi, \quad k_1 = 0, \quad k_2 = +\sin \phi, \quad k_3 = 0, \\
m_0 = \cos \phi, \quad m_1 = 0, \quad m_2 = +\sin \phi, \quad m_3 = 0, \\
n_0 = 0, \quad n_1 = 0, \quad n_2 = 0, \quad n_3 = 0, \\
l_0 = 0, \quad l_1 = 0, \quad l_2 = 0, \quad l_3 = 0.
\]

The restrictions (2.1) give

\[
S_0 \geq 0, \quad S_0^2 - S_1^2 \geq S_2^2 + S_3^2, \\
S_0' = \cos \phi S_0 + \sin \phi S_1 \geq 0, \\
(\cos \phi S_0 + \sin \phi S_1)^2 \geq (-\sin \phi S_0 + \cos \phi S_1)^2 + S_2^2 + S_3^2 \implies \\
\cos 2\phi(S_0^2 - S_1^2) + \sin 2\phi 2S_0 S_1 \geq S_2^2 + S_3^2.
\]

With respect to the variables \(a, b, c, x\), the inequalities become

\[
\cos \phi + a \sin \phi \geq 0, \\
\frac{1 - x^2}{1 + x^2}(1 - a^2) + \frac{2x}{1 + x^2}2a - b^2 - c^2 \geq 0.
\]

These differs from the previous ones from (7.2) only in formal notation, and therefore the results will be much the same.

(7.5)

Variant \( U_2^\beta (\phi) \)

\[
M = U_2^\beta (\phi) = \begin{bmatrix}
\cos \phi & 0 & \sin \phi & 0 \\
0 & \cos \phi & 0 & -\sin \phi \\
-\sin \phi & 0 & \cos \phi & 0 \\
0 & \sin \phi & 0 & \cos \phi
\end{bmatrix},
\]
or in block form

\[
M = \begin{vmatrix}
  k_0 + k_3 & k_1 + k_2 & n_0 + n_3 & n_1 + n_2 \\
  k_1 - k_2 & k_0 - k_3 & n_1 - n_2 & n_0 - n_3 \\
  l_0 + l_3 & l_1 + l_2 & m_0 + m_3 & m_1 + m_2 \\
  l_1 - l_2 & l_0 - l_3 & m_1 - m_2 & m_0 - m_3 \\
\end{vmatrix},
\]

(7.11)

The restrictions (2.1) give

\[
\begin{align*}
S_0 &\geq 0, \\
S_0^2 - S_1^2 - S_2^2 - S_3^2 &\geq 0, \\
S_0^2 - S_1^2 &\geq S_2^2 + S_3^2, \\
S_0' &= \cos \phi S_0 + \sin \phi S_2 \geq 0, \\
-\sin \phi S_0 + \cos \phi S_2 \geq 0.
\end{align*}
\]

\[
(-\sin \phi S_0 + \cos \phi S_2)^2 - (\sin \phi S_1 + \cos \phi S_3)^2 \geq 0 \quad \implies \\
\implies \cos 2\phi(S_0^2 - S_2^2) + \sin 2\phi 2S_0S_2 - S_1^2 - S_3^2 \geq 0.
\]

In the variables \(a, b, c, x\) the above inequalities take the form

\[
\frac{1}{1 + x^2} (1 - b^2) + \frac{2x}{1 + x^2} 2b - a^2 - c^2 \geq 0.
\]

(7.12)

They differ from the previous ones from (7.2) only in formal notation, therefore the results will be much the same.

(7.6)

\[U_3^\beta(\phi)\]

\[
M = U_3^\beta(\phi) = \begin{vmatrix}
  \cos \phi & 0 & 0 & \sin \phi \\
  0 & \cos \phi & \sin \phi & 0 \\
  0 & -\sin \phi & \cos \phi & 0 \\
  -\sin \phi & 0 & 0 & \cos \phi \\
\end{vmatrix},
\]

or in block form

\[
M = \begin{vmatrix}
  k_0 + k_3 & k_1 + k_2 & n_0 + n_3 & n_1 + n_2 \\
  k_1 - k_2 & k_0 - k_3 & n_1 - n_2 & n_0 - n_3 \\
  l_0 + l_3 & l_1 + l_2 & m_0 + m_3 & m_1 + m_2 \\
  l_1 - l_2 & l_0 - l_3 & m_1 - m_2 & m_0 - m_3 \\
\end{vmatrix},
\]

(7.13)
The restrictions (2.11) give

\[ S_0 \geq 0, \quad S_0^2 - S_1^2 - S_2^2 - S_3^2 \geq 0, \]

\[ S_0'^2 - S_1'^2 \geq S_2'^2 + S_3'^2, \]

\[ S'_0 = \cos \phi S_0 + \sin \phi S_3 \geq 0, \]

\[ (\cos \phi S_0 + \sin \phi S_3)^2 - (\cos \phi S_1 + \sin \phi S_2)^2 -
\]

\[ (- \sin \phi S_1 + \cos \phi S_2)^2 - (- \sin \phi S_0 + \cos \phi S_3)^2 \geq 0 \implies \]

\[ \implies \cos 2\phi(S_0^2 - S_3^2) + \sin 2\phi 2S_0S_3 - S_1^2 - S_2^2 \geq 0. \]

In the variables \(a, b, c, x\) the inequalities take the form

\[ a^2 + b^2 + c^2 \leq 1, \quad \cos \phi + c \sin \phi \geq 0, \]

\[ \frac{1 - x^2}{1 + x^2}(1 - e^2) + \frac{2x}{1 + x^2}2c - a^2 - b^2 \geq 0. \quad (7.14) \]

These differ from the previous ones from (7.2) only in formal notation, therefore the results will be much the same.

(7.7)

**Variant \( U_2^A(-i\beta) \)**

\[ U_2^A(i\beta) = \begin{vmatrix} \cosh \beta & 0 & 0 & \sinh \beta \\ 0 & \cosh \beta & -\sinh \beta & 0 \\ 0 & -\sinh \beta & \cosh \beta & 0 \\ \sinh \beta & 0 & 0 & \cosh \beta \end{vmatrix}, \]

or in block form

\[ M = \begin{pmatrix} k_0 + k_3 & k_1 + k_2 & n_0 + n_3 & n_1 + n_2 \\ k_1 - k_2 & k_0 - k_3 & n_1 - n_2 & n_0 - n_3 \\ l_0 + l_3 & l_1 + l_2 & m_0 + m_3 & m_1 + m_2 \\ l_1 - l_2 & l_0 - l_3 & m_1 - m_2 & m_0 - m_3 \end{pmatrix}, \]

\[ k_0 = \cosh \beta, \quad k_1 = 0, \quad k_2 = 0, \quad k_3 = 0, \]

\[ m_0 = \cosh \beta, \quad m_1 = 0, \quad m_2 = 0, \quad m_3 = 0, \]

\[ n_0 = 0, \quad n_1 = 0, \quad n_2 = -\sinh \beta, \quad n_3 = 0, \]

\[ l_0 = 0, \quad l_1 = 0, \quad l_2 = -\sinh \beta, \quad l_3 = 0. \quad (7.15) \]

The restrictions (2.11) give

\[ \cosh \beta S_0 + \sinh \beta S_3 \geq 0 \implies e^\beta(S_0 + S_3) + e^{-\beta}(S_0 - S_3) \geq 0, \]

which holds good for any \(\beta\). At the same time we have

\[ (\cosh \beta S_0 + \sinh \beta S_3)^2 - (\cosh \beta S_1 - \sinh \beta S_2)^2 -
\]

\[ -(- \sinh \beta S_1 + \cosh \beta S_2)^2 - (\sinh \beta S_0 + \cosh \beta S_3)^2 \geq 0 \implies \]

\[ \implies S_0^2 - S_3^2 - \cosh 2\beta(S_1^2 + S_2^2) + 2\sinh 2\beta S_1S_2 \geq 0. \]
In the variables $a, b, c$ and $y = \tanh \beta$, $y \in (-1, +1)$, the second inequality takes the form
\[
1 - c^2 - \frac{1 + y^2}{1 - y^2}(a^2 + b^2) + \frac{2y}{1 - y^2}2ab \geq 0,
\]
or
\[
-y^2(a^2 + b^2 + 1 - c^2) + 4aby + (1 - a^2 - b^2 - c^2) \geq 0.
\]
The coefficient of $y^2$ is positive, so the inequality holds good in the interval between the roots of the corresponding quadratic equation, that is
\[
y \in [y_1, y_2], \quad y_1 \leq \tanh \beta \leq y_2,
\]
where
\[
y_1 = \frac{2ab - \sqrt{4a^2b^2 + (1 - a^2 - b^2 - c^2)(a^2 + b^2 + 1 - c^2)}}{a^2 + b^2 + 1 - c^2} < 0,
\]
and
\[
y_2 = \frac{2ab + \sqrt{4a^2b^2 + (1 - a^2 - b^2 - c^2)(a^2 + b^2 + 1 - c^2)}}{a^2 + b^2 + 1 - c^2} > 0.
\]
(7.16)

Let us check the necessary condition $y_2 \leq +1$; we have
\[
\sqrt{4a^2b^2 + (1 - a^2 - b^2 - c^2)(a^2 + b^2 + 1 - c^2)} \leq (a^2 + b^2 + 1 - c^2) - 2ab,
\]
which after being squared, leads to
\[
4a^2b^2 + (1 - a^2 - b^2 - c^2)(a^2 + b^2 + 1 - c^2) \leq (a^2 + b^2 + 1 - c^2)^2 + 4ab(a^2 + b^2 + 1 - c^2) - 4ab(a^2 + b^2 + 1 - c^2).
\]
This is equivalent to
\[
0 \leq 2(a^2 + b^2) - 4ab \iff 0 \leq (a - b)^2.
\]
Therefore, the relation $y_2 \leq +1$ is satisfied.

Now let us verify the second necessary condition $y_1 \geq -1$: we have
\[
-\sqrt{4a^2b^2 - (1 - a^2 - b^2 - c^2)(a^2 + b^2 + 1 - c^2)} \geq -(a^2 + b^2 + 1 - c^2) - 2ab,
\]
which yields
\[
4a^2b^2 + (1 - a^2 - b^2 - c^2)(a^2 + b^2 + 1 - c^2) \leq (a^2 + b^2 + 1 - c^2)^2 + 4ab(a^2 + b^2 + 1 - c^2) - 4ab(a^2 + b^2 + 1 - c^2).
\]
This is equivalent to
\[
0 \leq 2(a^2 + b^2) - 4ab \iff 0 \leq (a - b)^2,
\]
and hence the relation $y_1 \geq -1$ holds good.

For the case of completely polarized initial light, we have $a^2 + b^2 + c^2 = 1$ and the formulas become much simpler:
\[
y \in [y_1, y_2], \quad y_1 \leq \tanh \beta \leq y_2,
\]
where
\[
y_1 = \frac{ab - \sqrt{a^2b^2}}{a^2 + b^2}, \quad y_2 = \frac{ab + \sqrt{a^2b^2}}{a^2 + b^2}.
\]
so that

\[ ab > 0, \quad y_1 = 0, \quad y_2 = \frac{2ab}{a^2 + b^2} \leq +1; \]

\[ ab < 0, \quad y_1 = \frac{2ab}{a^2 + b^2} \geq -1, \quad y_2 = 0. \]

(7.17)

\[(7.8)\]

**Variant** \( U^A_3(i\beta) \)

\[ U^A_3(i\beta) = \begin{vmatrix}
\cosh \beta & 0 & -\sinh \beta & 0 \\
0 & \cosh \beta & 0 & -\sinh \beta \\
-\sinh \beta & 0 & \cosh \beta & 0 \\
0 & -\sinh \beta & 0 & \cosh \beta
\end{vmatrix}, \]

or in block form

\[ M = \begin{vmatrix}
k_0 + k_3 & k_1 + k_2 & n_0 + n_3 & n_1 + n_2 \\
k_1 - k_2 & k_0 - k_3 & n_1 - n_2 & n_0 - n_3 \\
l_0 + l_3 & l_1 + l_2 & m_0 + m_3 & m_1 + m_2 \\
l_1 - l_2 & l_0 - l_3 & m_1 - m_2 & m_0 - m_3
\end{vmatrix}, \]

\[ k_0 = \cosh \beta, \quad k_1 = 0, \quad k_2 = 0, \quad k_3 = 0, \]

\[ m_0 = \cosh \beta, \quad m_1 = 0, \quad m_2 = 0, \quad m_3 = 0, \]

\[ n_0 = -\sinh \beta, \quad n_1 = 0, \quad n_2 = 0, \quad n_3 = 0, \]

\[ l_0 = -\sinh \beta, \quad l_1 = 0, \quad l_2 = 0, \quad l_3 = 0. \]

(7.18)

The restrictions (2.1) lead to

\[
\cosh \beta S_0 - \sinh \beta S_2 \geq 0 \quad \text{holds good for all } \beta,
\]

\[
(cosh \beta S_0 - \sinh \beta S_2)^2 - (cosh \beta S_1 - \sinh \beta S_3)^2 -
\]

\[-(\sinh \beta S_0 + \cosh \beta S_2)^2 - (-\sinh \beta S_1 + \cosh \beta S_3)^2 \geq 0 \implies
\]

\[ \implies S_0^2 - S_2^2 - \cosh 2\beta(S_1^2 + S_3^2) + 2\sinh 2\beta S_1 S_3 \geq 0. \]

In the variables \( a, b, c \) the second inequality becomes

\[ 1 - b^2 - \cosh 2\beta(a^2 + c^2) + \sinh 2\beta 2ac \geq 0. \]

This differs from (7.2) only by notations, so the results will be much the same.

(7.9)

**Variant** \( U^B_1(i\beta) \)

\[ U^B_1(i\beta) = \begin{vmatrix}
\cosh \beta & 0 & 0 & -\sinh \beta \\
0 & \cosh \beta & -\sinh \beta & 0 \\
0 & -\sinh \beta & \cosh \beta & 0 \\
-\sinh \beta & 0 & 0 & \cosh \beta
\end{vmatrix}, \]
or in block form

\[ M = \begin{vmatrix}
  k_0 + k_3 & k_1 + k_2 & n_0 + n_3 & n_1 + n_2 \\
  k_1 - k_2 & k_0 - k_3 & n_1 - n_2 & n_0 - n_3 \\
  l_0 + l_3 & l_1 + l_2 & m_0 + m_3 & m_1 + m_2 \\
  l_1 - l_2 & l_0 - l_3 & m_1 - m_2 & m_0 - m_3 \\
\end{vmatrix}, \]

(7.19)

The restrictions (2.1) give

\[ \cosh \beta S_0 - \sinh \beta S_3 \geq 0, \]

which holds good for all \( \beta \). At the same time we have

\[
\begin{align*}
(\cosh \beta S_0 - \sinh \beta S_3)^2 - (\cosh \beta S_1 - \sinh \beta S_2)^2 - \\
-(-\sinh \beta S_1 + \cosh \beta S_2)^2 - (-\sinh \beta S_0 + \cosh \beta S_3)^2 \geq 0 \implies \\
\implies S_0^2 - S_3^2 - \cosh 2\beta(S_1^2 + S_2^2) + 2\sinh 2\beta S_1 S_2 \geq 0.
\end{align*}
\]

In the variables \( a, b, c \) the second inequality becomes

\[
1 - c^2 - \cosh 2\beta(a^2 + b^2) + \sinh 2\beta 2ab \geq 0.
\]

This differs from (7.2) only in notation, so the results will be much the same.

(7.10)

Variant \( U_3^B(\beta) \)

\[ U_3^B(\beta) = \begin{vmatrix}
  \cosh \beta & \sinh \beta & 0 & 0 \\
  \sinh \beta & \cosh \beta & 0 & 0 \\
  0 & 0 & \cosh \beta & -\sinh \beta \\
  0 & 0 & -\sinh \beta & \cosh \beta \\
\end{vmatrix}, \]

or in block form

\[ M = \begin{vmatrix}
  k_0 + k_3 & k_1 + k_2 & n_0 + n_3 & n_1 + n_2 \\
  k_1 - k_2 & k_0 - k_3 & n_1 - n_2 & n_0 - n_3 \\
  l_0 + l_3 & l_1 + l_2 & m_0 + m_3 & m_1 + m_2 \\
  l_1 - l_2 & l_0 - l_3 & m_1 - m_2 & m_0 - m_3 \\
\end{vmatrix}, \]

(7.20)

The restrictions (2.1) give

\[ \cosh \beta S_0 + \sinh \beta S_1 \geq 0, \]
which holds good for all $\beta$. At the same time we have

$$(\cosh \beta S_0 + \sinh \beta S_1)^2 - (\sinh \beta S_0 + \cosh \beta S_1)^2 -
-(\cosh \beta S_2 - \sinh \beta S_3)^2 - (-\sinh \beta S_1 + \cosh \beta S_3)^2 \geq 0 \implies
\implies S_0^2 - S_1^2 - \cosh 2\beta(S_2^2 + S_3^2) + 2\sinh 2\beta S_2 S_3 \geq 0.$$

In the variables $a, b, c$ the second inequality becomes

$$1 - a^2 - \cosh 2\beta(b^2 + c^2) + \sinh 2\beta 2bc \geq 0.$$

This differs from (7.2) only in notation, so the results will be much the same.

(7.21)

**Variant $U_C^i(i\beta)$**

$$U_C^i(i\beta) = \begin{vmatrix}
cosh \beta & 0 & \sinh \beta & 0 \\
0 & \cosh \beta & 0 & -\sinh \beta \\
\sinh \beta & 0 & \cosh \beta & 0 \\
0 & -\sinh \beta & 0 & \cosh \beta \\
\end{vmatrix},$$

or in block form

$$M = \begin{vmatrix}
k_0 + k_3 & k_1 + k_2 & n_0 + n_3 & n_1 + n_2 \\
k_1 - k_2 & k_0 - k_3 & n_1 - n_2 & n_0 - n_3 \\
l_0 + l_3 & l_1 + l_2 & m_0 + m_3 & m_1 + m_2 \\
l_1 - l_2 & l_0 - l_3 & m_1 - m_2 & m_0 - m_3 \\
\end{vmatrix},$$

$$\begin{align*}
k_0 &= \cosh \beta, & k_1 &= 0, & k_2 &= 0, & k_3 &= 0, \\
m_0 &= \cosh \beta, & m_1 &= 0, & m_2 &= 0, & m_3 &= 0, \\
n_0 &= 0, & n_1 &= 0, & n_2 &= 0, & n_3 &= \sinh \beta, \\
l_0 &= 0, & l_1 &= 0, & l_2 &= 0, & l_3 &= \sinh \beta. \\
\end{align*}$$

The restrictions (7.21) give

$$\cosh \beta S_0 + \sinh \beta S_2 \geq 0,$$

which holds good for all $\beta$. We also have

$$(\cosh \beta S_0 + \sinh \beta S_2)^2 - (\cosh \beta S_1 - \sinh \beta S_3)^2 -
-(\sinh \beta S_0 + \cosh \beta S_2)^2 - (-\sinh \beta S_1 + \cosh \beta S_3)^2 \geq 0 \implies
\implies S_0^2 - S_2^2 - \cosh 2\beta(S_1^2 + S_3^2) + 2\sinh 2\beta S_1 S_3 \geq 0.$$

In the variables $a, b, c$ the second inequality becomes

$$e^{\beta}(1 + b) + e^{-\beta}(1 - b) \geq 0 \iff
1 - b^2 - \cosh 2\beta(a^2 + c^2) + \sinh 2\beta 2ac \geq 0.$$

(7.22)

This differs from (7.2) only by notation, so the results will be much the same.
(7.12)

Variant $U_2^C(-i\beta)$

$$U_2^C(-i\beta) = \begin{vmatrix}
\cosh \beta & -\sinh \beta & 0 & 0 \\
-\sinh \beta & \cosh \beta & 0 & 0 \\
0 & 0 & \cosh \beta & -\sinh \beta \\
0 & 0 & -\sinh \beta & \cosh \beta \\
\end{vmatrix},$$

or in block form

$$M = \begin{vmatrix}
k_0 + k_3 & k_1 + k_2 & n_0 + n_3 & n_1 + n_2 \\
k_1 - k_2 & k_0 - k_3 & n_1 - n_2 & n_0 - n_3 \\
l_0 + l_3 & l_1 + l_2 & m_0 + m_3 & m_1 + m_2 \\
l_1 - l_2 & l_0 - l_3 & m_1 - m_2 & m_0 - m_3 \\
\end{vmatrix},$$

$$k_0 = \cosh \beta, \quad k_1 = -\sinh \beta, \quad k_2 = 0, \quad k_3 = 0,$n_0 = 0, \quad n_1 = 0, \quad n_2 = 0, \quad n_3 = 0,
l_0 = 0, \quad l_1 = 0, \quad l_2 = 0, \quad l_3 = 0.$$

(7.23)

The restrictions (2.1) give

$$\cosh \beta S_0 - \sinh \beta S_1 \geq 0,$$

which holds good for all $\beta$. We also have

$$(\cosh \beta S_0 - \sinh \beta S_1)^2 - (-\sinh \beta S_0 + \cosh \beta S_1)^2 -$$

$$-(\cosh \beta S_2 - \sinh \beta S_3)^2 - (-\sinh \beta S_2 + \cosh \beta S_3)^2 \geq 0 \implies$$

$$\implies S_0^2 - S_1^2 - \cosh 2\beta(S_2^2 + S_3^2) + 2\sinh 2\beta S_2 S_3 \geq 0.$$

In the variables $a, b, c$ the second inequality becomes

$$1 - a^2 - \cosh 2\beta(b^2 + c^2) + \sinh 2\beta 2bc \geq 0.$$

It differs from (7.2) only in notation, so the results will be much the same.

8 Varying the degree of polarization of the light

Let us examine how the degree of polarization of the light changes for the above 12 special one-parametric Mueller transformations. The corresponding characteristics may be given by the difference

$$(a'^2 + b'^2 + c'^2) - (a^2 + b^2 + c^2) = D.$$
The $D$-entities for the first six cases (7.1)–(7.6) are

\[(7.1) \quad D = \frac{(a - x)^2 + (b^2 + c^2)(1 + x^2)}{(1 + ax)^2} - a^2 - b^2 - c^2,\]

\[(7.2) \quad D = \frac{(b - x)^2 + (a^2 + c^2)(1 + x^2)}{(1 + bx)^2} - a^2 - b^2 - c^2,\]

\[(7.3) \quad D = \frac{(c - x)^2 + (a^2 + b^2)(1 + x^2)}{(1 + ac)^2} - a^2 - b^2 - c^2,\]

\[(7.4) \quad D = \frac{(a - x)^2 + (b^2 + c^2)(1 + x^2)}{(1 + ax)^2} - a^2 - b^2 - c^2,\]

\[(7.5) \quad D = \frac{(b - x)^2 + (a^2 + c^2)(1 + x^2)}{(1 + bx)^2} - a^2 - b^2 - c^2,\]

\[(7.6) \quad D = \frac{(c - x)^2 + (a^2 + b^2)(1 + x^2)}{(1 + ac)^2} - a^2 - b^2 - c^2.\]  

(8.1)

It will be sufficient to consider in detail only one case, e.g., (7.1). In this case, we get:

\[D = \frac{(a - x)^2}{(1 + ax)^2} + \frac{(b^2 + c^2)(x^2 + 1)}{(1 + ax)^2} - (a^2 + b^2 + c^2).\]

The degree of polarization decreases when

\[D < 0, \quad (a - x)^2 + (b^2 + c^2)(x^2 + 1) < p^2(1 + ax)^2,\]

or when

\[x^2(1 + b^2 + c^2) - 2ax < x^2a^2p^2 + 2xap^2,\]

and this yields

\[D < 0, \quad x^2(1 - a^2) - 2ax < 0.\]

Thus, the light is depolarized if

\[D < 0, \quad 0 < x < \frac{2a}{1 - a^2}, \quad a > 0;\]

\[D < 0, \quad \frac{2a}{1 - a^2} < x < 0, \quad a < 0.\]  

(8.2)

In contrast, the degree of polarization of the light increases if

\[x^2(1 - a^2) - 2ax > 0,\]  

that is, when

\[D > 0, \quad x > \frac{2a}{1 - a^2}, \quad a > 0;\]

\[D > 0, \quad x < \frac{2a}{1 - a^2}, \quad a < 0.\]  

(8.4)

The light does not change the degree of polarization if

\[D = 0 \quad \Rightarrow \quad x = \frac{2a}{1 - a^2}.\]
This fact is illustrated by Fig. 2.

Now let us examine the other six cases (7.7)–(7.12).

The case (7.7) (we use the notation \( \tanh \beta = y \)) gives

\[
D = \frac{(a \cosh \beta - b \sinh \beta)^2 + (-a \sinh \beta + b \cosh \beta)^2 + (\sinh \beta + c \cosh \beta)^2}{(\cosh \beta + c \sinh \beta)^2} - a^2 - b^2 - c^2 = \frac{(a - by)^2 + (b - ay)^2 + (c + y)^2}{(1 + cy)^2} - a^2 - b^2 - c^2. \tag{8.5}
\]

The degree of polarization decreases when

\[
D < 0, \quad (a - by)^2 + (b - ay)^2 + (c + y)^2 < (a^2 + b^2 + c^2)(1 + cy)^2,
\]

or when

\[
y^2(a^2 + b^2 + 1) + 2y(c - 2ab) < y^2c^2p^2 + 2ycp^2,
\]

and this leads us to

\[
D < 0, \quad y^2(1 - c^2)(1 + p^2) - 2y[2ab - c(1 - p^2)] < 0.
\]

The solutions have the form

\[
D < 0, \quad 2ab - c(1 - p^2) > 0, \quad 0 < y < \frac{2}{(1 - c^2)(1 + p^2)} \left[2ab - c(1 - p^2)\right],
\]

\[
D < 0, \quad 2ab - c(1 - p^2) < 0, \quad \frac{2}{(1 - c^2)(1 + p^2)} \left[2ab - c(1 - p^2)\right] < y < 0. \tag{8.6}
\]

The degree of polarization increases when

\[
D > 0, \quad (a - by)^2 + (b - ay)^2 + (c + y)^2 > (a^2 + b^2 + c^2)(1 + cy)^2.
\]

\[\footnote{We use the constraint \( a^2 + b^2 + c^2 = p^2 \).} \]
This gives
\[ y^2(1-c^2)(1+p^2) - 2y[2ab - c(1-p^2)] > 0. \]
The solutions are
\[ D > 0, \quad 2ab - c(1-p^2) > 0, \quad y > \frac{2}{(1-c^2)(1+p^2)} [2ab - c(1-p^2)], \]
\[ D > 0, \quad 2ab - c(1-p^2) < 0, \quad y < \frac{2}{(1-c^2)(1+p^2)} [2ab - c(1-p^2)]. \] (8.7)
The light does not change the degree of polarization if
\[ D = 0, \quad y = \frac{2}{(1-c^2)(1+p^2)} [2ab - c(1-p^2)]. \]
The formulas become considerably simpler when the initial light is completely polarized. All the six variants of this type are characterized by the following \( D \)-entities:
\[ (7.7) \quad D = \frac{(a-by)^2 + (b-ay)^2 + (y+c)^2}{(1+cy)^2} - a^2 - b^2 - c^2, \]
\[ (7.9) \quad D = \frac{(a-by)^2 + (b-ay)^2 + (y-c)^2}{(1-ay)^2} - a^2 - b^2 - c^2, \]
\[ (7.8) \quad D = \frac{(a-cy)^2 + (c-ay)^2 + (y-b)^2}{(1-by)^2} - a^2 - b^2 - c^2, \]
\[ (7.11) \quad D = \frac{(a-cy)^2 + (c-ay)^2 + (y+b)^2}{(1+by)^2} - a^2 - b^2 - c^2, \]
\[ (7.10) \quad D = \frac{(b-cy)^2 + (c-ay)^2 + (y+a)^2}{(1+ay)^2} - a^2 - b^2 - c^2, \]
\[ (7.12) \quad D = \frac{(b-cy)^2 + (c-by)^2 + (y-a)^2}{(1-ay)^2} - a^2 - b^2 - c^2. \] (8.8)
It is obvious that the differences between these relations consist of notations only, so the derived results will be similar.

Examination of other subgroups (see Section 7) is a subject for further concern.

9 Acknowledgements

The present work was developed under the auspices of Grant 1196/2012 - BRFFR - RA No. F12RA-002, within the cooperation framework between Romanian Academy and Belarusian Republican Foundation for Fundamental Research.

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