MACH'S PRINCIPLE: EXACT FRAME-Dragging
BY ENERGY CURRENTS IN THE UNIVERSE

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We show that the dragging of axis directions of local inertial frames by a weighted average of the energy currents in the universe (Mach’s postulate) is exact for all linear perturbations of all Friedmann-Robertson-Walker universes.

1 Mach’s Principle

1.1 The Observational Fact: ’Mach zero’

The time-evolution of local inertial axes, i.e. the local non-rotating frame is experimentally determined by the spin axes of gyroscopes, as in inertial guidance systems in airplanes and satellites. This is true both in Newtonian physics (Foucault 1852) and in General Relativity.

It is an observational fact within present-day accuracy that the spin axes of gyroscopes do not precess relative to quasars. This observational fact has been named ’Mach zero’, where ’zero’ designates that this fact is not yet Mach’s principle, it is just the observational starting point. — There is an extremely small dragging effect by the rotating Earth on the spin axes of gyroscopes, the Lense-Thirring effect, which makes the spin axes of gyroscopes precess relative to quasars by 43 milli-arc-sec per year. It is hoped that one will be able to detect this effect by further analysis of the data which have been taken by Gravity Probe B.

1.2 The Question

What physical cause explains the observational fact ’Mach zero’? Equivalently: What physical cause determines the time-evolution of gyroscope axes? In the words of John A. Wheeler: Who gives the marching orders to the spin axes of gyroscopes, i.e. to inertial axes?

1.3 Mach’s Postulate

An answer to this fundamental question was formulated by Ernst Mach in his postulate (1883) that inertial axes (i.e. the spin axes of gyroscopes) exactly follow an average of the motion of the masses in the universe: Mach postulated exact frame dragging of inertial axes by the motion of cosmological masses, not merely a little bit of frame-dragging as in Lense-Thirring effect.

Mach did not know, what mechanism, what new force could do the job, he merely stated: ’the laws of motion could be conceived ...’. Mach also asked: “What share has every mass in the determination of direction ... in the law of inertia? No definite answer can be given by our experiences.”
1.4 Our Results

We have shown that exact dragging of inertial axis directions, i.e. Mach’s Principle, follows from Cosmological General Relativity for general, linear perturbations of FRW backgrounds with $K = (±1, 0)$. This also holds for FRW backgrounds with arbitrarily small energy density and pressure compared to $\rho_{\text{crit}}$ (Milne limit of FRW universe).

These results have been demonstrated for the first time in our paper\(^1\) for $K = 0$, and in our paper\(^2\) for $K = (±1, 0)$.

2 Theoretical Results and Tools

2.1 Cosmological Vorticity Perturbations

The vector sector of cosmological perturbations is the sector of vorticity perturbations. Two important theorems for the vorticity sector are needed to understand the following summary:

1. The slicing of space-time in slices $\Sigma_t$ of fixed time is unique. The lapse function (elapsed measured time between slices) and $g_{00}$ are unperturbed.

2. The intrinsic geometry of each slice $\Sigma_t$, i.e. of 3-space, remains unperturbed.

The coordinate choice uniquely adapted to our 3-geometry is comoving Cartesian coordinates for FRW with $K = 0$, resp. comoving spherical coordinates for $K = (±1, 0)$. Hence the only quantity referring to vorticity perturbations is the shift 3-vector $\beta^i$ (resp $\beta^i = g_{0i}$):

$$ds^2 = -dt^2 + a^2 [d\chi^2 + R_{\text{com}}^2(d\theta^2 + \sin^2 \theta \, d\phi^2)] + 2\beta^i dx^i dt,$$

$$R_{\text{com}} = (\chi, \sin \chi, \sinh \chi).$$

2.2 Gravitomagnetism

The general operational definitions of the gravitomagnetic and gravitoelectric fields are given via measurements by FIDO (Fiducial Observers) with LONB (Local Ortho-Normal Bases), where LONB components are denoted by hats over indices.

**Gravitoelectric field** $\vec{E}_g \equiv \vec{g}$:

$$\frac{d}{dt} \hat{p}_i \equiv m \vec{E}_g^i \quad \text{free-falling quasistatic test particle.}$$

**Gravitomagnetic field** $\vec{B}_g$:

$$\Omega^\text{gyro}_i \equiv -\frac{1}{2} B^g_i \quad \text{precession of gyro comoving with FIDO.}$$

**Gravitomagnetic vector potential** $\vec{A}_g$ : Because all 3-scalars must be unperturbed in the vector sector, $\text{div} \, \vec{A}_g \equiv 0$, and $\vec{A}_g$ is uniquely determined by $\vec{B}_g$,

$$\vec{B}_g =: \text{curl} \, \vec{A}_g \, \Rightarrow \, \vec{A}_g = \vec{\beta} \equiv \text{shift vector.}$$

*Our choice of FIDOs*: Our FIDOs are at fixed values of the spatial coordinates $x^i$, and the spatial axes are fixed in the direction of our coordinate basis vectors.
2.3 Einstein’s $G_k^0$ Equation: The Momentum Constraint

New result: The momentum constraint is form-identical for all three FRW background geometries, $K = (0, \pm 1):$

$$(-\Delta + \mu^2) \vec{A}_g = -16\pi G_N \vec{J}_\varepsilon,$$

(6)

where $(\mu/2)^2 \equiv -(dH/dt) \equiv (H$-dot radius$)^{-2}$, and $\vec{J}_\varepsilon$ = energy current density = momentum density. Since the source in Eq. (6) is the momentum density, this equation is called the 'momentum constraint'.

The momentum constraint is an elliptic equation, i.e. there are no partial time-derivatives of perturbations, although the momentum constraint refers to time-dependent gravitomagnetism.

Our new approach: For the source we have used the LONB components $\vec{J}_\varepsilon^k = T_0^k$, which is a measurable input, and which needs no prior knowledge of $g_{0i}$, which is the output. Einstein had emphasized that the coordinate-basis components $T_0^k$ are not a directly measurable input: 'If you have $T_{\mu\nu}$ and not a metric, the statement that matter by itself determines the metric is meaningless.'

New result: The momentum constraint for time-dependent gravitomagnetism for all three FRW background geometries has the same form as Ampère’s law for stationary magnetism, except for the term $\mu^2 \vec{A}_g$, which causes a Yukawa suppression beyond the $H$-dot radius. There are no curvature terms in Eq. (6).

2.4 The Laplacian on Vector Fields in Riemannian 3-Spaces

The Laplacian $\Delta$ acting on vector fields in Eq. (6) is the de Rham - Hodge Laplacian, which mathematicians simply call 'the Laplacian', and which differs from $\nabla^2$, which mathematicians call the 'rough Laplacian'. Unfortunately all publications on cosmological vector perturbations up to ours have used the 'rough Laplacian' $\nabla^2$. The difference between the two operators is given by the Weitzenböck formula:

$$(\Delta - \nabla^2) \vec{A} = -(2K/a_c^2) \vec{A},$$

(7)

where $K = (\pm 1, 0)$ is the curvature index for the FRW background, and $a_c$ is its curvature radius. For vorticity fields (divergence zero) the de Rham - Hodge Laplacian is defined by

$$(\Delta \vec{a})_\mu = -(\text{curl curl} \vec{a})_\mu = -(\star d \star d \vec{a})_\mu,$$

(8)

where we have given both the notation of elementary vector calculus and the notation of calculus of differential forms with $d$ = exterior derivative and $\star$ = Hodge dual.

The de Rham - Hodge Laplacian on vector fields is singled out by the following properties:

1. If all sources (curl and div) are zero $\Rightarrow$ the de Rham - Hodge Laplacian gives zero.

2. The de Rham - Hodge Laplacian commutes with curl, div, grad.

3. The identities of vector calculus in Euclidean 3-space (familiar from Classical Electrodynamics) remain true in Riemannian 3-spaces for the Hodge - de Rham Laplacian.

4. The action principle for Ampère magnetism in Riemannian 3-spaces directly produces Ampère’s equation for $\vec{A}$ with the Hodge - de Rham Laplacian and without curvature terms.

5. For electromagnetism in curved space-time the equivalence principle forbids curvature terms in equations with the Hodge - de Rham Laplacian.

Every one of these properties does not hold for the 'rough' Laplacian $\nabla^2$. 

3 The Bottom Lines

3.1 The Solution of the Momentum Constraint

Cosmological gravitomagnetism on a background of open FRW universes gives identical expressions for \( \vec{K} = (0, -1) \):

\[
\begin{align*}
\vec{B}_g(P) &= -2 \vec{\Omega}_{\text{gyro}}(P) = \\
&= -4 G_N \int d(\text{vol}_Q) [\vec{n}_{PQ} \times \vec{J}_Q] Y_{\mu}(r_{PQ}) \\
Y_{\mu}(r) &= \frac{-d}{dr} \left[ \frac{1}{R} \exp(-\mu r) \right] = \text{Yukawa force},
\end{align*}
\]

(9) (10)

where \( r = \) radial distance, and \( 2\pi R = \) circumference of the great circle through \( Q \) and centered at \( P \). The solution Eq. (9) is analogous to Ampère’s solution for stationary magnetism, but Eq. (9) is valid for time-dependent gravito-magnetodynamics, and it has a Yukawa suppression.

There is a fundamental difference between our solution Eq. (9) for cosmological gravitomagnetism and the corresponding solutions in other theories, Ampère’s magnetism, electromagnetism in Minkowski space, and General Relativity in the solar system: Our solution for Cosmological General Relativity is manifestly form-invariant when going to globally rotating frames. In contrast, the other theories give solutions, which are not form-invariant when going to globally rotating frames.

If the background is a closed FRW universe, one simply makes the following replacement in Eqs. (9, 10):

\[
\exp(-\mu r) \Rightarrow \sinh^{-1}(\mu \pi) \sinh[\mu(\pi - r)].
\]

(11)

3.2 Exact Dragging of Inertial Axes

From symmetry under rotations and reflections one concludes: The precession of a gyroscope can only be acted on by the component of the matter velocity field with \( J^P = 1^+ \) relative to the gyroscope. This component of the velocity field is equivalent to a rigid rotation of matter with angular velocity \( \vec{\Omega}_{\text{matter}}(r) \).

From Eq. (9) one concludes that inertial axes, i.e. the spin axes of gyroscopes, exactly follow the weighted average of the energy currents of cosmic matter,

\[
\begin{align*}
\vec{\Omega}_{\text{gyro}} &= < \vec{\Omega}_{\text{matter}} > \equiv \int_0^\infty dr \ \vec{\Omega}_{\text{matter}}(r) \ W(r) \\
W(r) &= \frac{1}{3} 16\pi G_N (\rho + p) R^3 Y_{\mu}(r),
\end{align*}
\]

(12) (13)

for perturbations of open FRW universes. The weight function \( W(r) \) is normalized to unity,

\[
\int_0^\infty dr \ W(r) = 1,
\]

(14)

as it must be for a proper average weighting function in any problem. — For perturbations of a closed FRW universe one again makes the replacement of Eq. (11).

References

1. C. Schmid, Phys. Rev. D 74, 044031 (2006).
2. C. Schmid, arXiv:0801.2907v1 [astro-ph].