Ray–Singer Torsion and the Rumin Laplacian on Lens Spaces

Akira KITAOKA

Graduate School of Mathematical Sciences, The University of Tokyo, Japan
E-mail: akira5kitaoka@gmail.com
URL: https://akira5kitaoka.github.io/Akira5Kitaoka-en.github.io/

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Abstract. We express explicitly the analytic torsion functions associated with the Rumin complex on lens spaces in terms of the Hurwitz zeta function. In particular, we find that the functions vanish at the origin and determine the analytic torsions. Moreover, we have a formula between this torsion and the Ray–Singer torsion.

Key words: analytic torsion; Rumin complex; CR geometry; contact geometry

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1 Introduction

Let \((M, H)\) be a compact contact manifold of dimension \(2n + 1\) and \(E\) be the flat vector bundle with unitary holonomy on \(M\). Rumin [12] introduced a complex \((E^\bullet(M, E), d_R^\bullet)\), which is a subquotient of the de Rham complex of \(E\). A specific feature of the complex is that the operator \(D = d_n^R: E^n(M, E) \to E^{n+1}(M, E)\) in ‘middle degree’ is a second-order, while \(d_k^R: E^k(M, E) \to E^{k+1}(M, E)\) for \(k \neq n\) are first order which are induced by the exterior derivatives. Let \(a_k = 1/\sqrt{|n-k|}\) for \(k \neq n\) and \(a_n = 1\). Then, \((E^\bullet(M, E), d_R^\bullet)\), where \(d_k^E = a_k d_R^k\), is also a complex. We call \((E^\bullet(M, E), d_R^\bullet)\) the Rumin complex. In virtue of the rescaling, \(d_R^\bullet\) satisfies Kähler-type identities on Sasakian manifolds [13, equation (34)], which include the case of lens spaces.

Let \(\theta\) be a contact form of \(H\) and \(J\) be an almost complex structure on \(H\). Then we may define a Riemann metric \(g_{\theta,J}\) on \(T M\) by extending the Levi metric \(d\theta(\cdot, J\cdot)\) on \(H\) (see Section 2.1). Following [12], we define the Rumin Laplacians \(\Delta_k^E\) associated with \((E^\bullet(M, E), d_R^\bullet)\) and the metric \(g_{\theta,J}\) by

\[
\Delta_k^E := \begin{cases}
(d_E d_E^\dagger)^2 + (d_E^\dagger d_E)^2, & k \neq n, n+1, \\
(d_E d_E^\dagger)^2 + d^\dagger D, & k = n, \\
DD^\dagger + (d_E^\dagger d_E)^2, & k = n + 1.
\end{cases}
\]

Rumin showed that \(\Delta_k^E\) has discrete eigenvalues with finite multiplicities.

We next introduce the analytic torsion and metric of the Rumin complex \((E^\bullet(M, E), d_R^\bullet)\) by following [2, 8, 14]. We define the contact analytic torsion function associated with \((E^\bullet(M, E), d_R^\bullet)\) by

\[
\kappa_E(M, E, g_{\theta,J})(s) := \sum_{k=0}^n (-1)^{k+1}(n + 1 - k) \zeta(\Delta_k^E)(s),
\]

where \(\zeta(\Delta_k^E)(s)\) is the spectral zeta function of \(\Delta_k^E\). Since the Rumin Laplacians \(\Delta_k^E\) satisfies the Rockland condition by [12, p. 300], the spectral zeta function \(\zeta(\Delta_k^E)(s)\) extends to a meromorphic
Our main result is

\[ \kappa_{\mathcal{E}}(K, \mathbb{C}, g_{\theta,J})/s = -(n+1)(1 + 2^{2s+1} \mu^{-2s} \zeta(2s)) \]

\[(1.2)\]
where $\zeta$ is the Riemann zeta function. In particular, we have
\begin{align}
\kappa_{\mathcal{E}}(K, \mathbb{C}, g_{\theta, J})(0) &= 0, \\
T_{\mathcal{E}}(K, \mathbb{C}, g_{\theta, J}) &= \left(\frac{4\pi}{\mu}\right)^{n+1}.
\end{align}

(2) The contact analytic torsion function of $(K, E_u)$ for $u \in \{1, \ldots, \mu - 1\}$ is given by
\begin{align}
\kappa_{\mathcal{E}}(K, E_u, g_{\theta, J})(s) &= -2^{2s} \mu^{-2s} \sum_{j=1}^{n+1} \left(\zeta(2s, A_{\mu}(u \tau_j)/\mu) + \zeta(2s, A_{\mu}(-u \tau_j)/\mu)\right), \\
T_{\mathcal{E}}(K, E_u, g_{\theta, J}) &= \prod_{j=1}^{n+1} \left|e^{2\pi \sqrt{-1} \tau_j/\mu} - 1\right|.
\end{align}

The equations (1.2) and (1.5) extend the following results of $\kappa_{\mathcal{E}}$ the spheres to on lens spaces. Rumin and Seshadri [14, Theorem 5.4] showed (1.2) in the case of 3-dimensional lens spaces. The author [8] showed (1.2) in the case of $(S^{2n+1}, \mathbb{C})$ for arbitrary $n$.

From (1.3) and (1.6), we see that the metric $\|\varepsilon\|$ on $(K, E_u, g_{\theta, J})$ is invariant under the constant rescaling $\theta \mapsto C\theta$. The argument is exactly the same as the one in [14].

In the same way as [8], the fact that the representations determine the eigenvalues of $\Delta_{\mathcal{E}}$ cause several cancellations in the linear combination (1.1), which significantly simplifies the computation of $\kappa_{\mathcal{E}}(s)$. We cannot get such a simple formula for the contact analytic torsion function $\kappa_{\mathcal{R}}$ of $(\mathbb{E}^\bullet(M, E), d^\bullet_{\mathcal{R}})$ for dimensions higher than 3.

Let us compare the contact analytic torsion with the Ray–Singer torsion on lens spaces. Ray [10] showed that for $u(= 1, \ldots, \mu - 1)\$
\begin{align}
T_{d\mathcal{R}}(K, E_u, 4g_{\text{std}}) &= \prod_{j=1}^{n+1} \left|e^{2\pi \sqrt{-1} \tau_j/\mu} - 1\right|.
\end{align}

Weng and You [15] calculate the Ray–Singer torsion on spheres. We extend their results for the trivial bundle on lens spaces:

**Proposition 1.2.** In the setting of Theorem 1.1, we have
\begin{align}
T_{d\mathcal{R}}(K, \mathbb{C}, 4g_{\text{std}}) &= \left(\frac{4\pi}{n!\mu^{n+1}}\right)^{n+1}.
\end{align}

The metric $4g_{\text{std}}$ agrees with the metric $g_{\theta, J}$ defined from the contact form $\theta = \sqrt{-1} (\bar{\partial} - \partial)|z|^2$. Since the cohomology of $(\mathbb{E}^\bullet(M, E), d^\bullet_{\mathcal{R}})$ coincides with that of $(\Omega^\bullet(M, E), d)$ (e.g., [12, p. 286]), there is a natural isomorphism $\det H^\bullet(\mathbb{E}^\bullet(M, E), d^\bullet_{\mathcal{R}}) \cong \det H^\bullet(\Omega^\bullet(M, E), d)$, which turns out to be isometric for the $L^2$ metrics. Therefore (1.4) and (1.7) give

**Corollary 1.3.** In the setting of Theorem 1.1, for all unitary holonomy $\alpha$: $\pi_1(K) \rightarrow U(r)$, we have
\begin{align}
T_{\mathcal{E}}(K, E_\alpha, g_{\theta, J}) &= n!^{\dim H^0(K, E_\alpha)} T_{d\mathcal{R}}(K, E_\alpha, g_{\theta, J}), \\
\|\varepsilon(K, E_\alpha, g_{\theta, J}) &= n!^{\dim H^0(K, E_\alpha)} \|\|_{d\mathcal{R}}(K, E_\alpha, g_{\theta, J}),
\end{align}

via the isomorphism $\det H^\bullet(\mathbb{E}^\bullet(M, E_\alpha), d^\bullet_{\mathcal{R}}) \cong \det H^\bullet(\Omega^\bullet(M, E_\alpha), d)$. 

The paper is organized as follows. In Section 2, we recall the definition and properties of the Rumin complex on $S^{2n+1}$. In Section 3, we calculate the contact analytic torsion function $\kappa_\xi$ of flat vector bundles on lens spaces. In Section 4, we compute the Ray–Singer torsion $T_{dR}$ of the trivial vector bundle. In Section 5, we compare the Ray–Singer torsion and the contact analytic torsion.

2 The Rumin complex

2.1 The Rumin complex on contact manifolds

We call $(M, H)$ an orientable contact manifold of dimension $2n+1$ if $H$ is a subbundle of $TM$ of codimension 1 and there exists a 1-form $\theta$, called a contact form, such that $\text{Ker}(\theta: TM \to \mathbb{R}) = H$ and $\theta \wedge (d\theta)^n \neq 0$. The Reeb vector field of $\theta$ is the unique vector field $T$ satisfying $\theta(T) = 1$ and $\text{Int}_T d\theta = 0$, where $\text{Int}_T$ is the interior product with respect to $T$.

For $H$ and $\theta$, we call $J \in \text{End}(TM)$ an almost complex structure associated with $\theta$ if $J^2 = -\text{Id}$ on $H$, $JT = 0$, and the Levi form $d\theta(-, J-)$ is positive definite on $H$. Given $\theta$ and $J$, we define a Riemannian metric $g_{\theta, J}$ on $TM$ by

$$g_{\theta, J}(X, Y) := d\theta(X, JY) + \theta(X)d\theta(Y) \quad \text{for } X, Y \in TM.$$ 

Let $*$ be the Hodge star operator on $\wedge^* T^* M$ with respect to $g_{\theta, J}$.

Let $M$ be a manifold, $\tilde{M}$ be the universal covering of $M$, $\pi_1(M)$ be the fundamental group of $M$. For each unitary representation $\alpha: \pi_1(M) \to U(n)$, we denote the flat vector bundle associated with $\alpha$ by

$$E_\alpha := \tilde{M} \times_\alpha \mathbb{C}^n \to M.$$ 

Let $\nabla_\alpha$ be the flat connection on $E_\alpha$ induced from the trivial connection on $\tilde{M} \times \mathbb{C}^n$, and $d\nabla_\alpha$ be the exterior covariant derivative of $\nabla_\alpha$.

The Rumin complex [12] is defined on contact manifolds as follows. We set $L := d\theta \wedge$ and $\Lambda := *^{-1}L*$, which is the adjoint operator of $L$ with respect to the metric $g_{\theta, J}$ at each point. We set

\[
\Lambda_{\text{prim}}^k H^* := \{ v \in \Lambda^k H^* \mid \Lambda v = 0 \}, \\
\Lambda^k H^* := \{ v \in \Lambda^k H^* \mid L v = 0 \}, \\
\mathcal{E}^k(M, E_\alpha) := \begin{cases} C^\infty(M, \Lambda^k_{\text{prim}} H^* \otimes E_\alpha), & k \leq n, \\
C^\infty(M, \theta \wedge \Lambda^{k-1} H^* \otimes E_\alpha), & k \geq n+1. \end{cases}
\]

We embed $H^*$ into $T^* M$ as the subbundle $\{ \phi \in T^* M \mid \phi(T) = 0 \}$ so that we can regard

$$\Omega^k_H(M, E_\alpha) := C^\infty(M, \Lambda^k H^* \otimes E_\alpha)$$

as a subspace of $\Omega^k(M)$, the space of $k$-forms. We define $d_b: \Omega^k_H(M, E_\alpha) \to \Omega^{k+1}_H(M, E_\alpha)$ by

$$d_b \phi := d\nabla_\alpha \phi - \theta \wedge (\text{Int}_T d\nabla_\alpha \phi),$$

and then $D: \mathcal{E}^n(M, E_\alpha) \to \mathcal{E}^{n+1}(M, E_\alpha)$ by

$$D = \theta \wedge (L_T + d_b L^{-1} d_b),$$

where $L_T$ is the Lie derivative with respect to $T$, and we use the fact that $L: \Lambda^{n-1} H^* \to \Lambda^{n+1} H^*$ is an isomorphism.
Let \( P : \bigwedge^k H^* \to \bigwedge^k_{\text{prim}} H^* \) be the fiberwise orthogonal projection with respect to \( g_{\theta,J} \), which also defines a projection \( P : \Omega^k(M, E_\alpha) \to \mathcal{E}^k(M, E_\alpha) \). We set

\[
d_R^k := \begin{cases} 
P \circ d^{\nabla} & \text{on } \mathcal{E}^k(M, E_\alpha), \ k \leq n - 1, \\
D & \text{on } \mathcal{E}^n(M, E_\alpha), \\
d^{\nabla} & \text{on } \mathcal{E}^k(M, E_\alpha), \ k \geq n + 1.
\end{cases}
\]

Then \( (\mathcal{E}^\bullet(M, E_\alpha), d^\bullet_R) \) is a complex. Let \( d^k_R = a_k d^k_R \), where \( a_k = 1/\sqrt{|n-k|} \) for \( k \neq n \) and \( a_n = 1 \). We call \( (\mathcal{E}^\bullet(M, E_\alpha), d^\bullet_R) \) the Rumin complex.

We define the \( L^2 \)-inner product on \( \Omega^k(M, E_\alpha) \) by

\[
(\phi, \psi) := \int_M g_{\theta,J}(\phi, \psi) \, d \text{vol}_{g_{\theta,J}}
\]

and the \( L^2 \)-norm on \( \Omega^k(M, E_\alpha) \) by \( \|\phi\| := \sqrt{(\phi, \phi)} \). Since the Hodge star operator \( * \) induces a bundle isomorphism from \( \bigwedge^k_{\text{prim}} H^* \) to \( \theta \wedge \bigwedge^2_{n-k} H^* \), it also induces a map \( \mathcal{E}^k(M, E_\alpha) \to \mathcal{E}^{2n+1-k}(M, E_\alpha) \). We note that

\[
\mathcal{E}^k(M, E_\alpha) = \{ \phi \in \mathcal{E}^k(\tilde{M}, C^\bullet) \mid t_\ast \phi = \alpha(t)^{-1} \phi \text{ for } t \in \pi_1(M) \}.
\]

Let \( d^\mathcal{E}_k \) and \( D^\dagger \) denote the formal adjoint of \( d^\mathcal{E} \) and \( D \), respectively for the \( L^2 \)-inner product. We define the fourth-order Laplacians \( \Delta_{\mathcal{E}} \) on \( \mathcal{E}^k(M, E_\alpha) \) by

\[
\Delta_{\mathcal{E}}^k := \begin{cases} 
(d_{\mathcal{E}}^{k-1} d_{\mathcal{E}}^{k-1})^2 + (d_{\mathcal{E}}^k d_{\mathcal{E}}^k)^2, & k \neq n, n + 1, \\
(d_{\mathcal{E}}^{n-1} d_{\mathcal{E}}^{n-1})^2 + D^\dagger D, & k = n, \\
D D^\dagger + (d_{\mathcal{E}}^{n+1} d_{\mathcal{E}}^{n+1})^2, & k = n + 1.
\end{cases}
\]

We call it the Rumin Laplacian [12]. Since \( * \) and \( \Delta_{\mathcal{E}} \) commute, to determine the eigenvalue on \( \mathcal{E}^\bullet(M, E_\alpha) \), it is enough to compute them on \( \mathcal{E}^k(M, E_\alpha) \) for \( k \leq n \).

### 2.2 The Rumin complex on the CR spheres

Let \( S := \{ z \in \mathbb{C}^{n+1} \mid |z|^2 = 1 \} \) and \( \theta := -1(\bar{\partial} - \partial)|z|^2 \). (We will omit the dimension from \( S^{2n+1} \) for the simplicity of the notation.) Let \( g_{\text{std}} \) be the standard metric on \( S \). Then, \( g_{\theta,J} \) coincides with \( 4g_{\text{std}} \). With respect to the standard almost complex structure \( J \), we decompose the bundles defined in the previous subsection as follows:

\[
H^{*,1.0} := \{ v \in CH^* \mid Jv = -\sqrt{-1}v \},
\]

\[
H^{*,0.1} := \{ v \in CH^* \mid Jv = -\sqrt{-1}v \},
\]

\[
\bigwedge^{i,j} H^* := \bigwedge^i H^{*,1.0} \otimes \bigwedge^j H^{*,0.1},
\]

\[
\bigwedge_{\text{prim}}^{i,j} H^* := \{ \phi \in \bigwedge^{i,j} H^* \mid \Delta \phi = 0 \},
\]

\[
\mathcal{E}^{i,j} := C^\infty(S, \bigwedge^{i,j}_{\text{prim}} H^*).
\]

We decompose \( \mathcal{E}^{i,j} \) into a direct sum of irreducible representations of the unitary group \( U(n+1) \). Recall that irreducible representations of \( U(m) \) are parametrized by the highest weight \( \lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{Z}^m \) with \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \); the representation corresponding to \( \lambda \) will be denoted by \( V(\lambda) \). To simplify the notation, we introduce the following notation: for \( a_1, \ldots, a_k \in \mathbb{Z} \) and \( k_1, \ldots, k_l \in \mathbb{Z} \), \((a_{1k_1}, \ldots, a_{lk_l})\) denotes the \( k_1 + \cdots + k_l \)-tuple whose first \( k_1 \) entries are \( a_1 \), whose next \( k_2 \) entries are \( a_2 \), etc. For example,

\[
(1, 3, 0_2, -1_2) = (1, 1, 1, 0, 0, -1, -1).
\]

We note that \( a_1 \) is \( a \), and \( a_0 \) is the zero tuple.
In [7], it is shown that the multiplicity of \( V(q, 1_j, 0_{n-1-i-j}, \overline{1}_i, -p) \) in \( \mathcal{E}^{s,t} \) is at most one. Thus we may set

\[ \Psi^{(s,t)}_{(q,j,i,p)} := \mathcal{E}^{s,t} \cap V(q, 1_j, 0_{n-1-i-j}, \overline{1}_i, -p). \]

Proposition 2.1 ([7, Section 4(b)]). Given \( q, j, i, p \), we list up all \( s, t \) such that \( s + t \leq n \) and \( \Psi^{(s,t)}_{(q,j,i,p)} \neq \{0\} \) as the following:

Case I: For \( i = j = 0 \) and \( p = q = 0 \), the space is

\[ \Psi^{(0,0)}_{(0,0,0,0)}. \]

Case II: For \( i + j \leq n - 2, p \geq 1 \) and \( q \geq 1 \), the spaces are

\[ \Psi^{(i,j)}_{(q,j,i,p)}, \Psi^{(i+1,j)}_{(q,j,i,p)}, \Psi^{(i,j+1)}_{(q,j,i,p)}, \Psi^{(i+1,j+1)}_{(q,j,i,p)}. \]

Case III: For \( 0 \leq i \leq n - 1, j = 0, p \geq 1 \) and \( q = 0 \), the spaces are

\[ \Psi^{(i,0)}_{(0,0,i,p)}, \Psi^{(i+1,0)}_{(0,0,i,p)}. \]

Case IV: For \( i = 0, 0 \leq j \leq n - 1, p = 0 \) and \( q \geq 1 \), the spaces are

\[ \Psi^{(0,j)}_{(q,j,0,0)}, \Psi^{(0,j+1)}_{(q,j,0,0)}. \]

Case V: For \( i + j = n - 1, p \geq 1 \) and \( q \geq 1 \), the spaces are

\[ \Psi^{(i,j)}_{(q,j,i,p)}, \Psi^{(i+1,j)}_{(q,j,i,p)}, \Psi^{(i,j+1)}_{(q,j,i,p)}. \]

Case VI: \( i = n - 1, j = 0, p \geq 1 \) and \( q = -1 \), the space is

\[ \Psi^{(n,0)}_{(-1,0,n-1,p)}. \]

Case VII: \( i = 0, j = n - 1, p = -1 \) and \( q \geq 1 \), the space is

\[ \Psi^{(0,n)}_{(q,n-1,0,-1)}. \]

Remark 2.2. About Case VI, we substitute \( s = n, t = 0, q = -1, j = 0, i = n - 1 \), for \( \Psi^{(s,t)}_{(q,j,i,p)} \),

\[ \Psi^{(n,0)}_{(-1,0,n-1,p)} = \mathcal{E}^{n,0} \cap V(-1, \overline{1}_0, 0_0, -1_{n-1}, -p) = \mathcal{E}^{n,0} \cap V(-1_n, -p). \]

As the same way, about VII we obtain

\[ \Psi^{(0,n)}_{(q,n-1,0,-1)} = \mathcal{E}^{0,n} \cap V(q, \overline{1}_{n-1}, 0_0, -1_0, -(1)) = \mathcal{E}^{0,n} \cap V(q, 1_n). \]

In [8], \( \Delta_E \) acts as a scalar on these spaces, and eigenvalues of \( \Delta_E \) on the standard CR spheres are explicitly determined.

Proposition 2.3 ([8, Theorem 0.1]). Let \( S \) be the standard CR sphere with the contact form \( \theta = \sqrt{-1}(\partial - \overline{\partial})|z|^2 \). Then, on the subspaces of the complexification of \( \mathcal{E}^{\bullet}(S) \) corresponding to the representations \( \Psi^{(\bullet,\bullet)}_{(q,j,i,p)} \), the eigenvalue of \( \Delta_E \) is

\[ \frac{(p+i)(q+n-i) + (q+j)(p+n-j)}{4(n-i-j)^2}. \]
3 Contact analytic torsion of flat vector bundles

Let $\mu, \nu_1, \ldots, \nu_{n+1}$ be integers such that the $\nu_j$ are coprime to $\mu$. Let $\Gamma$ be the subgroup of $(S^1)^{n+1}$ generated by

$$\gamma = (\gamma_1, \ldots, \gamma_{n+1}) := \left( \exp(2\pi\sqrt{-1}\nu_1/\mu), \ldots, \exp(2\pi\sqrt{-1}\nu_{n+1}/\mu) \right).$$

We denote the lens space by

$$K := S^{2n+1}/\Gamma.$$

Fix $u \in \{1, \ldots, \mu\}$ and consider the unitary representation $\alpha_u: \pi_1(K) = \Gamma \to U(1)$, defined by

$$\alpha_u(\gamma^\ell) := \exp(2\pi\sqrt{-1}u\ell/\mu) \quad \text{for} \ \ell \in \mathbb{Z},$$

where

$$\gamma^\ell := (\gamma_1^\ell, \ldots, \gamma_{n+1}^\ell) = \left( \exp(2\pi\sqrt{-1}\nu_1\ell/\mu), \ldots, \exp(2\pi\sqrt{-1}\nu_{n+1}\ell/\mu) \right).$$

Let $E_\alpha$ be the flat vector bundle associated with the unitary representation $\alpha: \pi_1(K) = \Gamma \to U(r)$, and $E_u := E_{\alpha_u}$, which can be considered as $\alpha_u$-equivariant functions on $S^{2n+1}$.

For each unitary representation $(V, \rho)$ of $U(n+1)$, we define the vector subspace $V^{\alpha_u}$ of $V$ by

$$V^{\alpha_u} := \{ \phi \in V \mid \rho(\gamma)\phi = \alpha(\gamma)^{-1}\phi \}.$$  

Proposition 3.1. We have

$$\kappa_\varepsilon(K, E_u, g_{\theta, j})(s) = \kappa_1(K, E_u, g_{\theta, j})(s) + \kappa_2(K, E_u, g_{\theta, j})(s) + \kappa_3(K, E_u, g_{\theta, j})(s),$$

where

$$\kappa_1(K, E_u, g_{\theta, j})(s) := -(n+1) \dim V^{\alpha_u}(0_{n+1}) = \begin{cases} -(n+1), & u = 0, \\ 0, & u \neq 0, \end{cases}$$

$$\kappa_2(K, E_u, g_{\theta, j})(s) := (-1)^j(n+1) \sum_{q \geq 1} \frac{\dim V^{\alpha_u}(q, 0_n)}{(q/2)^2s} \left( \sum_{j=1}^{n} (-1)^{j+1}(n+1-j) \sum_{q \geq 1} \frac{\dim V^{\alpha_u}(q, 1_j, 0_{n-j})}{((q+j)/2)^2s} + \frac{\dim V^{\alpha_u}(q, 1_{j-1}, 0_{n-j+1})}{((q+j-1)/2)^2s} \right),$$

$$\kappa_3(K, E_u, g_{\theta, j})(s) := \kappa_2(K, E_{-u}, g_{\theta, j})(s).$$

Proof. From Proposition 2.3, we check that the terms of $\kappa_\varepsilon(K, E_u, g)(s)$ in Cases II and V in Proposition 2.1 cancel each other. “The sum of the terms of $\kappa_\varepsilon(K, E_u, g_{\theta, j})(s)$ in Case II” is

$$\sum_{a=0}^{n-2} \sum_{i+j=a} \sum_{p \geq 1} \sum_{q \geq 1} \frac{((-1)^{i+j+1}(n+1-i-j)}{((-1)^{i+j+2}(n-i-j) + (-1)^{i+j+3}(n-1-i-j)) \times \frac{\dim V^{\alpha_u}(q, 1_j, 0_{n-1-i-j}, -1_s, -p)}{((p+i)(q+n-i) + (q+j)(p+n-j))/(2(n-i-j))^2s}}.$$
\[ \sum_{a=0}^{n-2} \sum_{i+j=a, p \geq 1, q \geq 1} (0) \frac{\dim V^{\alpha_u}(q, \frac{1}{2}, 0_{n-1-i-j}, \frac{-1}{2}, -p)}{((p+i)(q+n-i) + (q+j)(p+n-j))/2(n-i-j))^{2s}} = 0. \]

Similarly, “the sum of the terms of \( \kappa_{\varepsilon}(K, E_u, g_{\theta,j})(s) \) in Case V” is

\[ \sum_{i+j=n-1, p \geq 1, q \geq 1} (-1)^n(n+1-(n-1)) + 2(-1)^{n+1}(n+1-n) \times \frac{\dim V^{\alpha_u}(q, \frac{1}{2}, -\frac{1}{2}, -p)}{((p+i)(q+n-i) + (q+j)(p+n-j))/2(n-i-j))^{2s}} = 0. \]

The function \( \kappa_{\varepsilon}(K, E_u, g_{\theta,j})(s) \) is the sum of the terms of \( \kappa_{\varepsilon}(K, E_u, g_{\theta,j})(s) \) in Case I.

Next we consider the sum of the terms of \( \kappa_{\varepsilon} \) in Cases III and VI. For \( j \), “the sum of the terms of \( \kappa_{\varepsilon}(K, E_u, g_{\theta,j}) \) in \( E^{0,j} \), Cases III and VI” is

\[ (-1)^{1}(n+1) \sum_{q \geq 1} \frac{\dim V^{\alpha_u}(q, \frac{1}{2}, 0_{n})}{(q/2)^{2s}}, \quad j = 0, \]

\[ (-1)^{j+1}(n+1-j) \sum_{q \geq 1} \frac{\dim V^{\alpha_u}(q, \frac{1}{2}, 0_{n-j}, 0_{(n-j)+1})}{((q+j)/2)^{2s}} + \frac{\dim V^{\alpha_u}(q, \frac{1}{2}, 0_{n-j+1})}{((q+j-1)/2)^{2s}} , \quad 1 \leq j \leq n. \]

“The sum of the terms of \( \kappa_{\varepsilon} \) in Cases III and VI” is

\[ (-1)^{1}(n+1) \sum_{q \geq 1} \frac{\dim V^{\alpha_u}(q, \frac{1}{2}, 0_{n})}{(q/2)^{2s}} + \sum_{j=1}^{n} (-1)^{j+1}(n+1-j) \sum_{q \geq 1} \left( \frac{\dim V^{\alpha_u}(q, \frac{1}{2}, 0_{n-j}, 0_{(n-j)+1})}{((q+j)/2)^{2s}} + \frac{\dim V^{\alpha_u}(q, \frac{1}{2}, 0_{n-j+1})}{((q+j-1)/2)^{2s}} \right) \]

\[ = \kappa_{2}(K, E_u, g_{\theta,j})(s). \quad (3.4) \]

Finally, we consider the sum of the terms of \( \kappa_{\varepsilon} \) in Cases IV and VII. As the same way (3.4) in Cases III and VI, “the sum of the terms of \( \kappa_{\varepsilon} \) in Cases IV and VII” is given by

\[ (-1)^{1}(n+1) \sum_{q \geq 1} \frac{\dim V^{\alpha_u}(0_{n}, -q)}{(q/2)^{2s}} + \sum_{j=1}^{n} (-1)^{j+1}(n+1-j) \sum_{q \geq 1} \left( \frac{\dim V^{\alpha_u}(0_{n-j}, -\frac{1}{2}, -q)}{((q+j)/2)^{2s}} + \frac{\dim V^{\alpha_u}(0_{n-j+1}, -\frac{1}{2}, -q)}{((q+j-1)/2)^{2s}} \right). \quad (3.5) \]

Let \( (V, \rho) \) be the unitary representation of \( U(n+1) \). We define the representation \( (\overline{V}, \overline{\rho}) \) by

\[ \bar{V} := V, \quad \overline{\rho}(U) := \rho(U) \quad \text{for } U \in U(n+1). \]

Since \( (V(q, \frac{1}{2}, 0_{n-j}), \rho) \) is the unitary representation, its conjugate representation is isomorphic to its dual representation as \( U(n+1) \)-module. From [6, Theorem 3.2.13], the conjugate representation of \( (V(q, \frac{1}{2}, 0_{n-j}), \rho) \) is isomorphic to \( (V(0_{n-j}, -\frac{1}{2}, -q), \overline{\rho}) \) as \( U(n+1) \)-module. Therefore, we have

\[ V^{\alpha-u}(q, \frac{1}{2}, 0_{n-j}) = \{ \phi \in V(q, \frac{1}{2}, 0_{n-j}) \mid \alpha_{u}(-\gamma)\phi = \rho(\gamma)\phi \} \]
\(\kappa = \{ \phi \in V(q, 1_j, 0_{n-j}) \mid \alpha_u(-\gamma)^{-1}\phi = \rho(\gamma)\phi \}\)

\(= \{ \phi \in V(q, 1_j, 0_{n-j}) \mid \alpha_u(\gamma)\phi = \rho(\gamma)\phi \}\)

\(\cong \{ \phi \in \overline{V(0_{n-j}, -1_j, -q)} \mid \alpha_u(\gamma)\phi = \overline{\rho(\gamma)\phi} \}\)

\(= \{ \phi \in V(0_{n-j}, -1_j, -q) \mid \alpha_u(\gamma)^{-1}\phi = \overline{\rho(\gamma)\phi} \}\)

\(= V(0_{n-j}, -1_j, -q)^{\alpha_u}\),

where \(\cong\) means isomorphic as real vector spaces via the complex conjugate. Then, from (3.4) and (3.5), “the sum of the terms of \(\kappa_{E}\) in Cases IV and VII” is given by

\[
(-1)^{1}(n + 1) \sum_{q \geq 1} \frac{\dim V^{\alpha_u}(0_{n-j}, -q)}{(q/2)^{2s}}
\]

\[
+ \sum_{j=1}^{n} (-1)^{j+1}(n + 1 - j) \sum_{q \geq 1} \left( \frac{\dim V^{\alpha_u}(0_{n-j}, -1_j, -q)}{(q+j)^{2s}} + \frac{\dim V^{\alpha_u}(0_{n-j+1}, -1_j-1, -q)}{(q+j-1)^{2s}} \right)
\]

\[
= (-1)^{1}(n + 1) \sum_{q \geq 1} \frac{\dim V^{\alpha-u}(q, 0_n)}{(q/2)^{2s}}
\]

\[
+ \sum_{j=1}^{n} (-1)^{j+1}(n + 1 - j) \sum_{q \geq 1} \left( \frac{\dim V^{\alpha-u}(q, 1_j, 0_{n-j})}{(q+j)^{2s}} + \frac{\dim V^{\alpha-u}(q, 1_j-1, 0_{n-j+1})}{(q+j-1)^{2s}} \right)
\]

\(= \kappa_2(K, E_{-u}, g_{\theta, j})(s).\)

We set for \(q \geq 1\),

\[V(q, 1_{-1}, 0_{n+1}) := \{0\}.
\]

We have

\[
\kappa_2(K, E_{u}, g_{\theta, j})(s) = (-1)^{1}(n + 1) \sum_{q \geq 1} \frac{\dim V^{\alpha-u}(q, 0_n)}{(q/2)^{2s}}
\]

\[
+ \sum_{j=1}^{n} (-1)^{j+1}(n + 1 - j) \sum_{q \geq 1} \left( \frac{\dim V^{\alpha-u}(q, 1_j, 0_{n-j})}{(q+j)^{2s}} + \frac{\dim V^{\alpha-u}(q, 1_j-1, 0_{n-j+1})}{(q+j-1)^{2s}} \right)
\]

\[
= \sum_{j=0}^{n} (-1)^{j+1}(n + 1 - j) \sum_{q \geq 1} \left( \frac{\dim V^{\alpha-u}(q, 1_j, 0_{n-j})}{(q+j)^{2s}} + \frac{\dim V^{\alpha-u}(q, 1_j-1, 0_{n-j+1})}{(q+j-1)^{2s}} \right)
\]

\[
= \sum_{j=0}^{n} (-1)^{j+1}(n + 1 - j) \sum_{q \geq 1} \frac{\dim V^{\alpha-u}(q, 1_j, 0_{n-j})}{(q+j)^{2s}} + \frac{\dim V^{\alpha-u}(q, 1_j-1, 0_{n-j+1})}{(q+j-1)^{2s}}
\]

\[
+ \sum_{j=1}^{n} (-1)^{j+1}(n + 1 - j) \frac{\dim V^{\alpha-u}(1_j, 0_{n-j+1})}{(j/2)^{2s}}.
\]

Let \(\chi_V\) be the character of the representation \((V, \rho)\) of \(U(n+1)\). We note that for each representation \((V, \rho)\),

\[
\dim V^{\alpha_u} = \sum_{t \in \Gamma} \chi_V(t)\alpha_u(t)/\#\Gamma,
\]

(cf. [5, equation (2.9)]). By Littlewood–Richardson’s rule (cf. [4, Corollary 3]), we have

\[
\chi_V(1_j, 0_{n-j+1})\chi_V(q, 0_n) = \chi_V(q, 1_j, 0_{n-j}) + \chi_V(q+1, 1_j-1, 0_{n-j+1}).
\]

(3.6)
From (3.6),
\[
\kappa_2(K, E_u, g_{\theta,j})(s) = \frac{1}{\mu} \sum_{j=0}^{n} (-1)^{j+1}(n + 1 - j) \sum_{q \geq 1} \sum_{\ell=0}^{\mu-1} \frac{\chi_V(q, \partial_{\tau^{\alpha_{n-j+1}}}) \chi_V(q, \partial_{\tau^{\alpha_{n-j+1}}}) \alpha_u(\gamma^q)}{(q + j)/2}^{2s} \\
+ \frac{1}{\mu} \sum_{j=1}^{n} (-1)^{j+1}(n + 1 - j) \sum_{\ell=0}^{\mu-1} \frac{\chi_V(q, \partial_{\tau^{\alpha_{n-j+1}}}) \chi_V(q, \partial_{\tau^{\alpha_{n-j+1}}}) \alpha_u(\gamma^q)}{(j/2)^{2s}} \\
= \frac{2^{2s}}{\mu \Gamma(2s)} \sum_{j=0}^{n} (-1)^{j+1}(n + 1 - j) \\
\times \sum_{q \geq 1} \sum_{\ell=0}^{\mu-1} \int_{0}^{\infty} \chi_V(q, \partial_{\tau^{\alpha_{n-j+1}}}) \chi_V(q, \partial_{\tau^{\alpha_{n-j+1}}}) \alpha_u(\gamma^q) e^{-(j+q)x} x^{2s-1} \, dx \\
+ \frac{2^{2s}}{\mu \Gamma(2s)} \sum_{j=1}^{n} (-1)^{j+1}(n + 1 - j) \sum_{\ell=0}^{\mu-1} \int_{0}^{\infty} \chi_V(q, \partial_{\tau^{\alpha_{n-j+1}}}) \chi_V(q, \partial_{\tau^{\alpha_{n-j+1}}}) \alpha_u(\gamma^q) e^{-jx} x^{2s-1} \, dx \\
= \frac{2^{2s}}{\mu \Gamma(2s)} \sum_{\ell=0}^{\mu-1} \int_{0}^{\infty} \left( \sum_{j=0}^{n} (-1)^{j+1}(n + 1 - j) \chi_V(q, \partial_{\tau^{\alpha_{n-j+1}}}) \chi_V(q, \partial_{\tau^{\alpha_{n-j+1}}}) \alpha_u(\gamma^q) e^{-jx} \sum_{q \geq 1} \chi_V(q, \partial_{\tau^{\alpha_{n-j+1}}}) \alpha_u(\gamma^q) e^{-qx} \right) x^{2s-1} \, dx \quad \text{(3.7)}
\]

We consider the contents of the integral for the last equation of (3.7). It is known that for \( t = (t_1, \ldots, t_{n+1}) \in (S^1)^{n+1} \),
\[
\chi_V(q, \partial_{\tau^{\alpha_{n-j+1}}})(t) = \sum_{\beta_1, \ldots, \beta_{n+1} = j, 0 \leq \beta_1, \ldots, \beta_{n+1} \leq 1} t_1^{\beta_1} \cdots t_{n+1}^{\beta_{n+1}} , \quad \text{ (3.8)}
\]
\[
\chi_V(q, \partial_{\tau^{\alpha_{n-j+1}}})(t) = \sum_{\alpha_1, \ldots, \alpha_{n+1} = 0} t_1^{\alpha_1} \cdots t_{n+1}^{\alpha_{n+1}} , \quad \text{ (3.9)}
\]

(cf. [5, equations (6.1) and (6.2)]). We set \( X := e^{-x} \) and
\[
F_1(t, X) := \sum_{j=0}^{n+1} (-1)^{j} \chi_V(q, \partial_{\tau^{\alpha_{n-j+1}}})(t)X^j. 
\]

Then (3.8) gives
\[
F_1(t, X) = \prod_{j=1}^{n+1} (1 - t_j X), \quad \text{(3.10)}
\]
\[
X \frac{\partial F_1}{\partial X}(t, X) = \sum_{j=0}^{n+1} (-1)^j t_j \chi_V(q, \partial_{\tau^{\alpha_{n-j+1}}})(t)X^j = - \sum_{i=1}^{n+1} t_i X \prod_{j=1, j \neq i}^{n+1} (1 - t_j X). \quad \text{(3.11)}
\]

From the definition of \( F_1 \) and (3.11), we have
\[
\sum_{j=0}^{n} (-1)^{j+1}(n + 1 - j) \chi_V(q, \partial_{\tau^{\alpha_{n-j+1}}})(t)X^j = -(n + 1)F_1(t, X) + X \frac{\partial F_1}{\partial X}(t, X). \quad \text{(3.12)}
\]
We set
\[ F_2(t, X) := \sum_{q \geq 1} \chi_{V(q, \mathbb{Z}_n)}(t) X^q. \]

From (3.9) and (3.10), we can rewrite \( F_2 \) as
\[
F_2(t, X) = \prod_{j=1}^{n+1} \frac{1}{1 - t_j X} - 1 = \frac{1}{F_1(t, X)} - 1. \tag{3.13}
\]

From (3.10)–(3.13), we can deduce that
\[
\sum_{j=0}^{n} (-1)^{j+1}(n + 1 - j) \chi_{V(\mathbb{Z}_n, \mathbb{Z}_n^{-j+1})}(t) X^j \sum_{q \geq 1} \chi_{V(q, \mathbb{Z}_n)}(t) X^q
+ \sum_{j=1}^{n} (-1)^{j+1}(n + 1 - j) \chi_{V(\mathbb{Z}_n, \mathbb{Z}_n^{-j+1})}(t) X^j
= - \left( (n + 1) F_1(t, X) - X \frac{\partial F_1}{\partial X}(t, X) \right) \left( \frac{1}{F_1(t, X)} - 1 \right)
- \left( (n + 1)(F_1(t, X) - 1) - X \frac{\partial F_1}{\partial X}(t, X) \right)
= \frac{X \frac{\partial F_1}{\partial X}(t, X)}{F_1(t, X)} = - \sum_{j=1}^{n+1} t_j X \frac{1}{1 - t_j X}. \tag{3.14}
\]

From (3.7) and (3.14), we see
\[
\kappa_2(K, E_u, g_{\theta, j})(s) = -\frac{2^{2s}}{\mu \Gamma(2s)} \sum_{\ell=0}^{\mu-1} \sum_{q=1}^{n+1} e^{\mu \ell} \int_{\mathbb{R}}^{\infty} \frac{\gamma_j e^{-x}}{1 - \gamma_j e^{-x}} e^{2\pi \sqrt{-1}(q \nu_j + u \ell) / \mu} x^{2s-1} \, dx
= - \sum_{j=1}^{n+1} \frac{2^{2s}}{\mu \Gamma(2s)} \sum_{\ell=0}^{\mu-1} \int_{\mathbb{R}}^{\infty} \sum_{q=1}^{\infty} \exp \left( 2\pi \sqrt{-1}(q \nu_j + u \ell) / \mu \right) e^{-q x} x^{2s-1} \, dx
= - \sum_{j=1}^{n+1} \sum_{\ell=0}^{\mu-1} \sum_{q=1}^{\infty} \exp \left( 2\pi \sqrt{-1}(q \nu_j + u \ell) / \mu \right) \frac{1}{q^{2s}}. \tag{3.15}
\]

Let \( \tau_j \) be the integers in \( \{1, \ldots, \mu\} \) such that \( \tau_j \nu_j \equiv 1 \mod \mu \). Since the multiplication of \( \nu_j \in (\mathbb{Z}/\mu\mathbb{Z})^\times \) induced the bijective map from \( \mathbb{Z}/\mu\mathbb{Z} \) to \( \mathbb{Z}/\mu\mathbb{Z} \), We have
\[
\sum_{\ell=0}^{\mu-1} \exp \left( 2\pi \sqrt{-1}(q \nu_j + u \ell) / \mu \right) = \sum_{\ell=0}^{\mu-1} \exp \left( 2\pi \sqrt{-1}(q + u \tau_j) \nu_j \ell / \mu \right)
= \sum_{\ell=0}^{\mu-1} \exp \left( 2\pi \sqrt{-1}(q + u \tau_j) \ell / \mu \right)
= \begin{cases} 0, & q \equiv -u \tau_j \mod \mu, \\ \mu, & q \equiv -u \tau_j \mod \mu. \end{cases}
\]

For \( w \in \mathbb{Z} \) let \( A_\mu(w) \) be the integer between 1 and \( \mu \) which is congruent to \( w \) modulo \( \mu \), then from (3.15), we can rewrite \( \kappa_2 \) as
\[
\kappa_2(K, E_u, g_{\theta, j})(s) = -2^{2s} \sum_{j=1}^{n+1} \sum_{q \equiv -u \tau_j \mod \mu} q^{-2s} = -2^{2s} \sum_{j=1}^{n+1} \sum_{q=0}^{\infty} \left( q \mu + A_\mu(-u \tau_j) \right)^{-2s}
\]
\[ \zeta(s, a) := \sum_{q=0}^{\infty} (q + a)^{-s} \text{ is the Hurwitz zeta function.} \]

Next, we calculate \( \kappa_3 \). As the same way in calculating \( \kappa_2 \), from (3.3), we can rewrite \( \kappa_3 \) as

\[ \kappa_3(K, E_u, g_{\theta, J})(s) = -2^{2s} \mu^{2s} \sum_{j=1}^{n+1} \zeta(2s, A_\mu(\tau_j) / \mu). \]  

(3.17)

From (3.1), (3.2), (3.16) and (3.17), we have

\[ 2 \log T_{\text{dr}}(M, E; g_{\text{std}}) = 1 + \sum_{\ell=0}^{2n} \sum_{j=0}^{2n} (-1)^{j+1} \zeta'(0; j, \gamma^\ell), \]  

where \( \zeta(\Delta_{\text{dr}, g}^k)(s) \) is the spectral zeta function of the \( k \)-th Hodge–de Rham Laplacian \( \Delta_{\text{dr}, g}^k \) with respect to \( g \). We define the Ray–Singer torsion \( T_{\text{dr}} \) by

\[ 2 \log T_{\text{dr}}(M, E; g) := \kappa_{\text{dr}}(M, E; g)'(0). \]

Following the derivation of [10, equation (3)], we have

\[ 2 \log T_{\text{dr}}(K, \mathbb{C}; g_{\text{std}}) = \frac{1}{\mu} \sum_{\ell=0}^{\mu-1} \sum_{j=0}^{2n} (-1)^{j+1} \zeta'(0; j, \gamma^\ell), \]  

as claimed.

4 Ray–Singer torsion of the trivial bundle

We compute the analytic torsion of the trivial bundle on lens spaces. We define the Ray–Singer torsion function associated with \( (\Omega^*(M, E), d^\nabla) \) by

\[ \kappa_{dR}(M, E, g)(s) := \sum_{k=0}^{2n+1} (-1)^k k \zeta(\Delta_{dR, g}^k)(s), \]

where \( \zeta(\Delta_{dR, g}^k)(s) \) is the spectral zeta function of the \( k \)-th Hodge–de Rham Laplacian \( \Delta_{dR, g}^k \) with respect to \( g \). We define the Ray–Singer torsion \( T_{dR} \) by

\[ 2 \log T_{dR}(M, E, g) := \kappa_{dR}(M, E, g)'(0). \]
where $\lambda_m$ is the $m$-th eigenvalue of $d^\dagger d$ and
\[
\zeta(s; j, \gamma^\ell) = \sum_{m=0}^{\infty} \lambda_m^{-s} \text{Tr} (\gamma^\ell|_{X_{j,m}}),
\]
\[
X_{j,m} := \{ \phi \in \Omega^j(K) \mid d^\dagger d\phi = \lambda_m \phi \}.
\]

We recall from [10, p. 123]
\[
\sum_{j=0}^{2n} (-1)^{j+1} \zeta(s; j, \gamma^\ell) = (\Gamma(s))^{-2} \int_{0}^{\infty} t^{2s-1} \int_{0}^{1} (u(1-u))^{s-1} (f_0(t,u) + f_1(t,\ell\nu/\mu)) \, du \, dt + O(s^2),
\]
where $\nu = (\nu_1, \ldots, \nu_{n+1})$ and for $\sigma = (\sigma_1, \ldots, \sigma_{n+1}) \in \mathbb{R}^{n+1}$,
\[
f_0(t,u) = \sum_{j=0}^{n+1} (-1)^j \binom{n}{j} \left( \frac{2 \sinh tu \sinh (1-u)}{\sinh t} \right)^j - \frac{e^{-(2n+1)tu}}{2 \sinh tu} - \frac{e^{-(2n+1)t(1-u)}}{2 \sinh t(1-u)},
\]
\[
f_1(t,\sigma) = \sum_{k} \left( 1 - \frac{\sinh t}{\cosh t - \cos 2\pi \sigma_k} \right).
\]

We set
\[
h_0(s) := (\Gamma(s))^{-2} \int_{0}^{\infty} t^{2s-1} \int_{0}^{1} (u(1-u))^{s-1} f_0(t,u) \, du \, dt,
\]
\[
h_1(s, \gamma^\ell) := (\Gamma(s))^{-2} \int_{0}^{\infty} t^{2s-1} \int_{0}^{1} (u(1-u))^{s-1} f_1(t,\ell\nu/\mu) \, du \, dt.
\]

From [10, p. 125], it is seen that
\[
h_1(s, \gamma^\ell) = -\frac{1}{2} \sum_{\ell=0}^{\mu-1} \text{Tr} (\gamma^\ell) \left( \zeta(2s, j/\mu) + \zeta(2s, 1-j/\mu) \right) - 2(n+1)\mu^{-2s} \zeta(2s),
\]
where
\[
\text{Tr} (\gamma^\ell) = \sum_{j=1}^{n+1} (e^{2\pi \sqrt{-1} \ell \sigma_j/\mu} + e^{-2\pi \sqrt{-1} \ell \sigma_j/\mu}).
\]

Taking the average of $h_1(s, \gamma^\ell)$, we have
\[
\frac{1}{\mu} \sum_{\ell=0}^{\mu-1} h_1(s, \gamma^\ell) = -2(n+1)\mu^{-2s} \zeta(2s)
\]

Using $\zeta'(0) = -\log(2\pi)/2$, we get
\[
\frac{1}{\mu} \sum_{\ell=0}^{\mu-1} h_1'(0, \gamma^\ell) = 2(n+1) \log \left( \frac{2\pi}{\mu} \right). \tag{4.1}
\]

We recall the Ray–Singer torsion on spheres,
Proposition 4.1 ([15]).
\[
T_{dR}(S, \mathbb{C}, g_{\text{std}}) = \frac{2\pi^{n+1}}{n!}
\]

Remark 4.2. The Ray–Singer torsion of spheres can easily be determined using the Cheeger–Müller theorem, a result which predates [15]. Details can be found in [3].

To put \( \mu = 1, \nu = (1, \ldots, 1) \), from (4.1), it follows that
\[
h'_0(0) = 2 \log T_{dR}(S, \mathbb{C}, g_{\text{std}}) - \frac{1}{\mu} \sum_{\ell=0}^{\mu-1} h'_1(0, \gamma^\ell)
\]
\[
= 2 \log \left( \frac{2\pi^{n+1}}{n!} \right) - 2(n+1) \log(2\pi) = 2 \log \left( \frac{2^{-n}}{n!} \right).
\] (4.2)

By (4.1) and (4.2), we conclude
\[
2 \log T_{dR}(K, \mathbb{C}, g_{\text{std}}) = \frac{1}{\mu} \sum_{\ell=0}^{\mu-1} h'_0(0) + \frac{1}{\mu} \sum_{\ell=0}^{\mu-1} h'_1(0, \gamma^\ell)
\]
\[
= 2 \log \left( 2^{-n} \right) + 2(n+1) \log \left( \frac{2\pi}{\mu} \right) = 2 \log \left( \frac{2\pi^{n+1}}{n!\mu^{n+1}} \right).
\] (4.3)

For the metric \( g \) on \( E \) over \( M \), we set \( g_\rho := e^{2\rho}g \) for \( \rho \in \mathbb{R} \). Then, the Hodge–de Rham Laplacian \( \Delta_{dR,g_\rho} \) with respect to \( g_\rho \) is given by
\[
\Delta_{dR,g_\rho} = e^{-2\rho} \Delta_{dR,g}
\] (e.g., see [11, equation (5.4)]). From (4.4), we see
\[
\kappa_{dR}(M, E, g_\rho)(s) = \sum_{k=0}^{2n+1} (-1)^k k \dim H^k_{dR}(M, E)
\]
\[
+ e^{2\rho s} \sum_{k=0}^{2n+1} (-1)^k k (\zeta(\Delta^k_{dR,g})(s) - \dim H^k_{dR}(M, E)).
\]

To derive the above equation,
\[
\kappa_{dR}(M, E, g_\rho)'(s) = 2\rho e^{2\rho s} \sum_{k=0}^{2n+1} (-1)^k k (\zeta(\Delta^k_{dR,g})(s) - \dim H^k_{dR}(M, E))
\]
\[
+ e^{2\rho s} \sum_{k=0}^{2n+1} (-1)^k k (\zeta(\Delta^k_{dR,g})'(s)).
\]

To substitute \( s = 0 \), since
\[
\zeta(\Delta^k_{dR,g})(0) = 0
\]
on manifolds with dimension \( 2n + 1 \), the Ray–Singer torsion is given by
\[
\log T_{dR}(M, E, g_\rho) = \log T_{dR}(M, E, g) - \rho \sum_{k=0}^{2n+1} (-1)^k k \dim H^k_{dR}(M, E).
\] (4.5)

To substitute \( M = K, E \) is the trivial bundle, \( \rho = \log 2 \) and \( g = g_{\text{std}} \), we obtain
\[
T_{dR}(K, \mathbb{C}, 4g_{\text{std}}) = 2^{2n+1} T_{dR}(K, \mathbb{C}, g_{\text{std}}).
\] (4.6)

By (4.3) and (4.6), we conclude Proposition 1.2.
5 Proof of Corollary 1.3

Since $\alpha(\gamma) \in U(r)$ is diagonalizable by a unitary matrix, we have

$$E_\alpha = E_{u_1} \oplus \cdots \oplus E_{u_r}.$$ 

We recall the Ray–Singer torsion on lens spaces,

**Proposition 5.1 ([10]).** For $u (= 1, \ldots, \mu - 1)$,

$$T_{dR}(K, E_u, 4g_{std}) = \prod_{j=1}^{n+1} \left| e^{2\pi \sqrt{-1} \tau_j/\mu} - 1 \right|.$$ 

From Proposition 5.1, Theorem 1.1, and Proposition 1.2, we conclude

$$T_{dR}(K, E_\alpha, g_{\theta,J}) = \prod_{j=1}^{\mu} T_{dR}(K, E_{u_j}, g_{\theta,J}) = \prod_{j=1}^{\mu} n!^{-\dim H^0(K, E_{u_j})} T_\xi(K, E_{u_j}, g_{\theta,J})$$

$$= n!^{-\dim H^0(K, E_{\alpha})} T_\xi(K, E_{\alpha}, g_{\theta,J}).$$

A Alternative shorter derivation of Proposition 1.2

The decomposition

$$\Omega^\bullet(S^{2n+1}, \mathbb{C}) = \bigoplus_{u=0}^{\mu-1} \Omega^\bullet(K, E_u)$$

gives

$$\kappa_{dR}(S^{2n+1}, \mathbb{C}, 4g_{std}) = \sum_{u=0}^{\mu-1} \kappa_{dR}(K, E_u, 4g_{std}).$$

In particular,

$$T_{dR}(S^{2n+1}, \mathbb{C}, 4g_{std}) = T_{dR}(K, \mathbb{C}, 4g_{std}) \prod_{u=1}^{\mu-1} T_{dR}(K, E_u, 4g_{std}).$$

Combining this with (4.5), (4.6), Propositions 4.1 and 5.1 and [3], we obtain

$$\frac{(4\pi)^{n+1}}{n!} = T_{dR}(K, \mathbb{C}, 4g_{std}) \prod_{u=1}^{\mu-1} \prod_{j=1}^{n+1} \left| e^{2\pi \sqrt{-1} \tau_j/\mu} - 1 \right| = T_{dR}(K, \mathbb{C}, 4g_{std}) \mu^{n+1},$$

whence Proposition 1.2.

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