Irreducibility of the moduli space of vector bundles on surfaces and Brill-Noether theory on singular curves

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Abstract

We prove the irreducibility of the moduli space of rank 2 semistable torsion free sheaves (with a generic polarization and any value of $c_2$) on a $K3$ or a del Pezzo surface. Under these conditions the moduli space is smooth of the expected dimension.

If the surface $S$ is a $K3$, then by a recent theorem of O’Grady the moduli space is known to be irreducible. We present a new proof of this result using a different technique.

First we consider the case in which $\text{Pic}(S) = \mathbb{Z}$. We prove this case by constructing a connected family of rank 2 semistable torsion free sheaves that maps onto a dense set of the moduli space. To prove that this family is connected we need a result from Brill-Noether theory. For a smooth curve it is known that the Brill-Noether locus is connected if the expected dimension is positive. We need to generalize this for irreducible singular curves that lie on a $K3$ surface (we prove it for any surface whose anticanonical line bundle is generated by global sections). Finally we remove the condition $\text{Pic}(S) = \mathbb{Z}$ by considering families of surfaces.

If $S$ is a del Pezzo surface we reduce the problem to the case of $\mathbb{P}^2$ by studying the relationship of moduli spaces corresponding to different polarizations and then comparing the moduli space for a surface with the moduli space for the blown up surface at a point.
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Chapter 1

Introduction

In this chapter we will explain the main results of the thesis using as little mathematical background as possible. We will always work over the complex numbers, i.e all manifolds will be complex manifolds. Also we assume that manifolds are projective, i.e. there is an embedding in \( \mathbb{P}_\mathbb{C}^n \) (in particular they are Kähler).

\( X \) will be a variety (or a manifold with singularities). We will consider holomorphic vector bundles over \( X \) (the transition functions are assumed to be holomorphic). To distinguish non-isomorphic vector bundles of fixed rank we can define certain invariants called Chern classes. These are cohomology classes \( c_i(V) \in H^{2i}(X, \mathbb{Z}) \), \( 1 \leq i \leq \dim \mathbb{C} X \). We have \( c_i(V) = 0 \) if \( i > \text{rank}(V) \). Even after fixing these discrete invariants, we can have continuous families of non-isomorphic vector bundles. I.e., to specify the isomorphism class of a vector bundle it is not enough to fix some discrete invariants, but we have to fix also some continuous parameters.

For fixed rank and Chern classes we would like to define a variety called the moduli space of vector bundles. Each point of this variety will correspond to a different vector bundle. In this thesis we will study the irreducibility of this space for rank 2 vector bundles over certain surfaces.

Unfortunately, in order to construct this moduli space we have to restrict our attention to stable vector bundles (this is not a very strong restriction, since it can be proved that in some sense all vector bundles can be constructed starting from stable ones). There are different notions of stability (see chapter 2). Here we will only discuss Mumford stability (also called slope stability). Let \( X \) be a projective variety and \( H \) an ample divisor. For
a vector bundle $V$ we define the slope of $V$ with respect to $H$ as:

$$\mu_H(V) = \frac{c_1(V)H^{n-1}}{\text{rank}(V)}$$

(the product is the intersection product or cup product in cohomology) $V$ is called $H$-stable if for every subbundle $W$ of $V$ we have

$$\mu_H(W) < \mu_H(V).$$

It can be proved that there is a space $\mathcal{M}_H^0(r,c_i)$, called the moduli space of $H$-stable vector bundles of rank $r$ and Chern classes $c_i$ (if the rank is clear from the context we will drop it from the notation). In many situations it will be a variety (maybe with singularities) but in general it can have a very singular behavior.

In general $\mathcal{M}_H^0(r,c_i)$ is not compact. To get a compact moduli space we need to consider a larger class of objects. Instead of vector bundles we consider torsion free sheaves (they can be thought as “singular” vector bundles that fail to be locally a product on a subvariety of $X$). Also we have to relax the stability condition, and we will consider Gieseker semistable sheaves (see chapter 2 for the definition). The moduli space of Gieseker semistable torsion free sheaves is compact, and we denote it by $\mathcal{M}_H(r,c_i)$.

There is an important relationship between the theory of holomorphic vector bundles and gauge theory: there is a bijection between the set of $H$-stable vector bundles and differentiable bundles with a Hermite-Einstein connection. If $c_1 = 0$ and $\dim \mathbb{C}X = 2$ then a Hermite-Einstein connection is the same thing as an anti-selfdual (ASD) connection. Note that the metric of the manifold appears in the ASD equation. This is reflected in the fact that the stability condition depends on the polarization. This relationship has been used, for instance, to calculate Donaldson polynomials for the study of the topology of four-manifolds.

Now we are going to consider some particular cases. Let $C$ be a curve (i.e., a Riemann surface) of genus $g$. Then the moduli space of rank $r$ vector bundles on $C$ is a smooth variety of dimension $(g-1)r^2 + 1$.

For line bundles (rank=1) over a variety $X$ ($\dim \mathbb{C}X = n$) the moduli space is called the Jacobian and we have a very explicit description of it. First we note that all line bundles are stable (because they don’t have subbundle). Recall that $q = h^1(O_X) = b_1/2$, where $b_1$ is the first Betti number (if $X$ is simply connected then $b_1 = 0$. If $X$ is a curve then $b_1 = 2g$). The Jacobian $J$ is the moduli space of line bundles with $c_1 = 0$ and it is of the form $\mathbb{C}^q/\mathbb{Z}^{2q}$, where $\mathbb{Z}^{2q}$ is a lattice in $\mathbb{C}^q$. If $c_1 \neq 0$ then the moduli space $J^{c_1}$ is isomorphic to the Jacobian, but the isomorphism is not canonical.
If $X$ is a curve $c_1$ is called the degree $d$. We define some subsets of $J^d$ as follows

$$W_d^a = \{ L \in J^d : \dim(H^0(L)) = a + 1 \}$$

The study of the properties of these subsets is called Brill-Noether theory. If the curve is generic, then $W_d^a$ is a subvariety of the expected dimension $\rho(a,d) = g - (a + 1)(g - d + a)$. If $\rho(a,d) > 0$ then (for any curve) $W_d^a$ is connected. If the curve is singular, one has to consider also torsion free sheaves in order to get a compact moduli space (in this case the moduli space will be singular). This moduli space has been constructed, but little is known about its Brill-Noether theory.

In this thesis, to prove the irreducibility of the moduli space of rank 2 vector bundles on a $K3$ surface, we will need the connectivity of $W_d^a$ for certain singular curves that lie in the surface. In chapter 3, theorem I, we will prove that if $\rho(a,d) > 0$, $W_d^a$ is still connected for singular curves satisfying certain conditions.

If $X$ is a complex surface ($\dim \mathbb{C} X = 2$), then in general the moduli space of vector bundles of rank $r$ is very singular, but in many situations it will be a variety (maybe with singularities) of the expected dimension

$$\dim \mathcal{M}_H^0(r,c_1,c_2) = 2rc_2 - (r - 1)c_1^2 - (r^2 - 1)\chi(\mathcal{O}_X) + h^1(\mathcal{O}_X).$$

By a slight abuse of language we have denoted by $c_2$ the integral of the second Chern class on the variety $\int_X c_2(V)$, and by $c_1^2$ the integral $\int_X c_1(V) \wedge c_1(V)$. Recall that $\chi(\mathcal{O}_X) = 1 - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X)$.

In this thesis we will consider the case rank=2. For rank 2, $\mathcal{M}_H^0(c_1,c_2)$ is known to be irreducible and of the expected dimension if $c_2 > N$, where $N$ is a constant that depends on $X$, $c_1$ and $H$. In chapter 3 we present a new proof of the theorem of O’Grady.

**Theorem II.** Let $X$ be a projective $K3$ surface (a $K3$ surface is a simply connected complex surface with $c_1(T_X) = 0$, where $T_X$ is the tangent bundle). Let $H$ be a generic polarization (see chapter 2 for a definition). Assume that $c_1$ is a nonzero primitive element of $H^2(X,\mathbb{Z})$ (under this condition $\mathcal{M}_H^0(c_1,c_2)$ is smooth of the expected dimension).

Then $\mathcal{M}_H^0(c_1,c_2)$ is irreducible.

We should note that in chapter 3 we work with the moduli space of Gieseker semistable torsion free sheaves $\mathcal{M}_H(c_1,c_2)$. But it can be shown that under the conditions of the theorem the points of $\mathcal{M}_H(c_1,c_2)$ that are not in $\mathcal{M}_H^0(c_1,c_2)$ are in a subvariety of positive codimension, then the irreducibility of one of them is equivalent to the irreducibility of the other.
Using the relationship of holomorphic vector bundles with gauge theory this theorem can be stated as follows:

**Theorem 1.0.1** Let $X$ be a projective $K3$ surface with a generic Kähler metric $g$. Then the moduli space of $SO(3)$ anti-selfdual connections (with fixed instanton number $k$ and second Stiefel-Whitney class $w_2$) is smooth and connected.

In chapter 5 we study moduli spaces of vector bundles on del Pezzo surfaces. A del Pezzo surface is a surface whose anticanonical bundle is ample. It can be shown that these are all the del Pezzo surfaces: $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or the projective plane $\mathbb{P}^2$ blown up at most at 8 generic points. For rank 2 and fixed Chern classes, the moduli space of stable vector bundles is known to be empty or irreducible for $X = \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. In chapter 5 we prove this result for a projective plane $\mathbb{P}^2$ blown up at most at 8 generic points, and then we get the general result:

**Theorem III.** Let $X$ be a del Pezzo surface with a generic polarization $H$. Fix some Chern classes $c_1$ and $c_2$. Then $M_H^{\mathbb{O}}(c_1,c_2)$ is irreducible (or empty).

As in the $K3$ case, in this case irreducibility is equivalent to connectedness, and we have the same result for $M_{H}(c_1,c_2)$. We can also translate this theorem into gauge theory language:

**Theorem 1.0.2** Let $X$ be a del Pezzo surface with a generic Kähler metric $g$. Then the moduli space of $SO(3)$ anti-selfdual connections (with fixed Stiefel-Whitney class $w_2$ and instanton number $k$) is connected (or empty). The same is true for $SU(2)$ anti-selfdual connections with fixed instanton number $k$. 

2.1 Brill-Noether theory

Let $C$ be a smooth curve of genus $g$ (we will always assume that the base field is $\mathbb{C}$), $J(C)$ its Jacobian, and $W^r_d(C)$ the Brill-Noether locus corresponding to line bundles $L$ of degree $d$ and $h^0(L) \geq r+1$ (see [ACGH]). The expected dimension of this subvariety is $\rho(r,d) = g - (r + 1)(g - d + r)$. Fulton and Lazarsfeld [F-L] proved that $W^r_d(C)$ is connected when $\rho > 0$. We are going to generalize this result for certain singular curves, but before stating our result (theorem I), we need to recall some concepts.

Let $C$ be an integral curve (not necessarily smooth). We still have a generalized Jacobian $J(C)$, defined as the variety parametrizing line bundles, but it will not be complete in general. Define the degree of a rank one torsion-free sheaf on $C$ to be

$$\deg(A) = \chi(A) + p_a - 1,$$

where $p_a$ is the arithmetic genus of $C$. One can define a scheme $\mathcal{J}^d_j(C)$ parametrizing rank one torsion-free sheaves on $C$ of degree $d$ (see [AIK], [D], [R]). If $C$ lies on a surface, then $\mathcal{J}^d_j(C)$ is integral, and furthermore the generalized Jacobian $J(C)$ is an open set in $\mathcal{J}^d_j(C)$, and then $\mathcal{J}^d_j(C)$ is a natural compactification of $J(C)$.

We will need to consider families of sheaves parametrized by a scheme $T$, and furthermore the curve will vary as we vary the parameter $t \in T$.

All this can be done using a relative version of $\mathcal{J}^d_j(C)$, but we will proceed in a different way. We will use the fact that all these curves are going to lie on a fixed surface $S$. Then we will think of the coherent sheaves on $C$ as
torsion sheaves on $S$ (all sheaves in this paper will be coherent). To define precisely which sheaves we will consider we need some notation. For any sheaf $F$ on $S$, let $d(F)$ be the dimension of its support. We say that $F$ has pure dimension $n$ if for any subsheaf $E$ of $F$ we have $d(E) = d(F) = n$. Note that if the support is irreducible, then having pure dimension $n$ is equivalent to being torsion-free when considered as a sheaf on its support.

The following theorem follows from [S, theorem 1.21].

**Theorem 2.1.1 (Simpson)** Let $C$ be an integral curve on a surface $S$. Let $\mathcal{J}^d_{|C|}$ be the functor which associates to any scheme $T$ the set of equivalence classes of sheaves $A$ on $S \times T$ with

(a) $A$ is flat over $T$.

(b) The induced sheaf $A_t$ on each fiber $S \times \{t\}$ has pure dimension 1, and its support is an integral curve in the linear system $|C|$.

(c) If we consider $A_t$ as a sheaf on its support, it is torsion-free and has rank one and degree $d$.

Sheaves $A$ and $B$ are equivalent if there exists a line bundle $L$ on $T$ such that $A \cong B \otimes p_T^*L$, where $p_T : S \times T \to T$ is the projection on the second factor.

Then there is a coarse moduli space that we also denote by $\mathcal{J}^d_{|C|}$. I.e., the points of $\mathcal{J}^d_{|C|}$ correspond to isomorphism classes of sheaves, and for any family $A$ of such sheaves parametrized by $T$, there is a morphism

$$\phi : T \to \mathcal{J}^d_{|C|}$$

such that $\phi(t)$ corresponds to the isomorphism class of $A_t$.

Note that $\mathcal{J}^d_{|C|}$ parametrizes pairs $(C', A)$ with $C'$ an integral curve linearly equivalent to $C$ and $A$ a torsion-free rank one sheaf on $C$.

We denote by $\pi : \mathcal{J}^d_{|C|} \to U \subset |C|$ the obvious projection giving the support of each sheaf, where $U$ is the open subset of $|C|$ corresponding to integral curves.

A family of curves on a surface $S$ parametrized by a curve $T$ is a subvariety $C \subset S \times T$, flat over $T$, such that the fiber $C|_t = C_t$ over each $t \in T$ is a curve on $S$. Analogously, a family of sheaves on a surface $S$ parametrized by a curve $T$ is a sheaf $\mathcal{A}$ on $S \times T$, flat over $T$. For each $t \in T$ we will denote the corresponding member of the family by $\mathcal{A}_t = \mathcal{A}|_t$.

Altman, Iarrobino and Kleiman [AIK] proved the following theorem

**Theorem 2.1.2 (Altman–Iarrobino–Kleiman)** With the same notation as before, $\mathcal{J}^d_{|C|}$ is flat over $U$ and its geometric fibers are integral. The subset
of $\mathcal{J}_{|C|}$ corresponding to line bundles (i.e., the relative generalized Jacobian) is open and dense in $\mathcal{J}_{|C|}$.

We also consider the family of generalized Brill-Noether loci $\mathcal{W}_{d,|C|} \subset \mathcal{J}_{|C|}$, and the projection $q : \mathcal{W}_{d,|C|} \to U$.

We can define the generalized Brill-Noether locus $\mathcal{W}_d(C)$ as the set of points in $\mathcal{J}_d(C)$ corresponding to sheaves $A$ with $h^0(A) \geq r + 1$ (note that it is complete because of the upper semicontinuity of $h^0(\cdot)$). There is also a determinantal description that gives a scheme structure. This description is a straightforward generalization of the description for smooth curves (see [ACGH]), but we are only interested in the connectivity of $\mathcal{W}_d(C)$, so we can give it the reduced scheme structure.

We will consider curves that lie on a surface $S$ with the following property:

\[ h^1(\mathcal{O}_S) = 0, \quad \text{and} \quad -K_S \quad \text{is generated by global sections}. \quad (*) \]

We will need this condition to prove proposition 3.2.5. For instance, $S$ can be a K3 surface or a del Pezzo surface with $K_S^2 \neq 1$. Now we can state the theorem that we are going to prove in chapter 3.

**Theorem I.** Let $C$ be a reduced irreducible curve of arithmetic genus $\rho_a$ that lies in a surface $S$ satisfying $(*)$. Let $\mathcal{J}_d(C)$, $d > 0$, be the compactification of the generalized Jacobian. Then for any $r \geq 0$ such that $\rho(r, d) = \rho_a - (r + 1)(\rho_a - d + r) > 0$, the generalized Brill-Noether subvariety $\mathcal{W}_d(C)$ is nonempty and connected.

### 2.2 Moduli space of torsion free sheaves

Let $X$ be a smooth projective variety over $\mathbb{C}$ of dimension $n$. We are interested in constructing a moduli space that will parametrize torsion free sheaves on $X$, with some fixed rank $r$ and Chern classes $c_i$. To do so, we have to introduce some notion of stability.

**Definition 2.2.1 (Mumford-Takemoto stability)** Fix a polarization $H$. For any nonzero torsion free sheaf $F$ we define the slope of $F$ with respect to $H$ as

\[
\mu_H(F) = \frac{c_1(F) \cdot H^{n-1}}{\text{rk}(F)}
\]
We will say that a torsion free sheaf $V$ on $X$ is Mumford $H$-stable (resp. semistable) if for every nonzero subsheaf $W$ of $V$ we have

$$\mu_H(W) < \mu_H(V) \quad (\text{resp. } \leq).$$

Using this definition one can construct the coarse moduli space of Mumford stable vector bundles. This notion of stability turns out to have an analog in differential geometry: Mumford stable holomorphic vector bundles are in one to one correspondence with differentiable vector bundles with a Hermite-Einstein connection (the choice of a polarization is replaced by the choice of a Riemannian metric). This correspondence was proved by Narasimhan and Seshadri [N-S] if $X$ is a curve, by Donaldson [Do1, D-K] for a surface, and it was then generalized for any dimension by [U-Y]. This correspondence has been useful to calculate differentiable invariants of 4-manifolds (the so called Donaldson invariants).

The moduli space of Mumford stable sheaves is in general not compact. To define a compactification we have to introduce a refined notion of stability.

**Definition 2.2.2 (Gieseker stability).** Fix a polarization $H$. For any nonzero torsion free sheaf $F$ define the Hilbert polynomial of $F$ with respect to $H$ as

$$p_H(F)(n) = \frac{\chi(F \otimes \mathcal{O}_X(nH))}{\text{rank}(F)}.$$ 

Given two polynomials $f$ and $g$, we will write $f < g$ (resp. $\leq$) if $f(n) < g(n)$ (resp. $\leq$) for $n \gg 0$.

We will say that $V$ is Gieseker stable (resp. semistable) if for every nonzero torsion free subsheaf $W$ we have

$$p_H(W) \leq p_H(V) \quad (\text{resp. } \leq).$$

Using Hirzebruch-Riemann-Roch theorem we can see that Gieseker semistability implies Mumford semistability, and Mumford stability implies Gieseker stability.

In order to have a separated moduli space it is not enough to consider isomorphism classes of sheaves. We will introduce the notion of S-equivalence. For any Gieseker semistable sheaf $V$ there is a filtration $(\mathfrak{G}, \mathfrak{M})$

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_t = V$$

such that $V_i/V_{i-1}$ is stable and $p_H(V_i) = p_H(V)$. We define $\text{gr}(V) = \oplus(V_i/V_{i-1})$. It can be proved that $\text{gr}(V)$ doesn’t depend on the filtration chosen. We will say that $V$ is S-equivalent to $V'$ if $\text{gr}(V) = \text{gr}(V')$. 

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There is another characterization of S-equivalence that is more illuminating from the point of view of moduli problems. Assume that we have a family of Gieseker semistable sheaves parametrized by a curve $T$. I.e., we have a sheaf $V$ on $X \times T$, flat over $T$ inducing torsion free Gieseker stable sheaves $V|_t$ on the slices $X \times \{t\}$. Assume that for one point $0 \in T$ we have $V|_0 \cong V$ and for the rest of the points $V|_t$ is isomorphic to some other fixed $V'$. We will say that $V$ and $V'$ are equivalent. The equivalence relation generated by this definition is S-equivalence.

It can be proved that there is a coarse moduli space for S-equivalence classes of Gieseker semistable torsion free sheaves (with fixed rank and Chern classes). This moduli space is projective.

In general the moduli space can be very singular, but if $X$ is a surface and for fixed rank $r$, $c_1$ and polarization, it is known that for $c_2$ large enough the singular locus is a proper subset of positive codimension of the moduli space Do2, F, G-L2. The moduli space has the expected dimension

$$2rc_2 - (r - 1)c_1^2 - (r^2 - 1)\chi(O_X),$$

and is irreducible G-L1, G-L2, O1, O2. In the rank two case it is also known, again for $c_2$ large enough, that the moduli space is normal and has local complete intersection singularities at points corresponding to stable sheaves, and if the surface $X$ is of general type (with some technical conditions), then also the moduli space is of general type L2.

It is natural to ask what is the effect of changing the choice of polarization. From now on we will assume that $X$ is a surface $S$ and that the rank is 2 unless otherwise stated. We will denote the moduli space of rank 2 torsion free sheaves that are Gieseker semistable with respect to the polarization $H$ by $\mathfrak{M}_H(c_1, c_2)$.

**Definition 2.2.3** Fix $S$, the first and second Chern classes $c_1$, $c_2$. Let $\zeta$ be some class in $H^2(S, \mathbb{Z})$ with

$$\zeta \equiv c_1 \text{ (mod 2)}, \quad c_1^2 - 4c_2 \leq \zeta^2 < 0.$$  

The wall of type $(c_1, c_2)$ associated to $\zeta$ is a hyperplane of $H^2(S, \mathbb{Q})$ with nonempty intersection with the ample cone $\Omega_S$

$$W^{\zeta} = \{x \in \Omega_S \mid x \cdot \zeta = 0\}.$$  

The connected components of the complement of the walls in the ample cone are called chambers. A polarization $H$ is called $(c_1, c_2)$-generic if it doesn’t lie on a wall (i.e., it lies in a chamber).
The walls of type \((c_1, c_2)\) are known to be locally finite on the ample cone \([F-M]\).

If a polarization is \((c_1, c_2)\)-generic it is easy to see that Mumford and Gieseker stability coincide, and furthermore there are no strictly semistable sheaves. Stable sheaves are \textit{simple} (\(\text{Hom}(V, V) = \mathbb{C}\)). If \(-K_S\) is effective then this fact and the Kuranishi local model for the moduli space proves that the moduli space is smooth of the expected dimension (if not empty) \([F]\).

If \(H_1\) and \(H_2\) are two generic polarizations in the same chamber, then every sheaf that is \(H_1\)-stable is also \(H_2\)-stable, and we can identify the corresponding moduli spaces \([F, Q1, Q2]\).

If we restrict our attention to some particular class of algebraic surfaces we can obtain more properties of the moduli space (without the condition on \(c_2\)).

The moduli space of sheaves on a \(K3\) surface with a generic polarization has been studied by Mukai \([M]\) when the expected dimension is 0 or 2. In particular he proved that the moduli space is irreducible. O’Grady \([O3]\) has proved irreducibility for any \(c_2\) (and any rank), as well as having obtained results about the Hodge structure. In chapter \(\text{III}\) (theorem II), we give a new proof of the irreducibility for any \(c_2\) (and rank 2) based on our results about Brill-Noether theory on singular curves.

Now let’s consider the case \(S = \mathbb{P}^2\) and rank 2. Tensoring with a line bundle we can assume that \(c_1\) is either 0 or 1. If \(c_1 = 0\), then the moduli space is empty for \(c_2 < 2\) and is irreducible of the expected dimension for \(c_2 \geq 2\). If \(c_1 = 1\) then the moduli space is empty for \(c_2 < 1\) and irreducible of the expected dimension for \(c_2 \geq 1\).

In the case \(S = \mathbb{P}^1 \times \mathbb{P}^1\) (and also rank 2), if we take a \((c_1, c_2)\)-generic polarization, the moduli space is also known to be either empty or irreducible. We will generalized this result for any del Pezzo surface (i.e., a surface with \(-K_S\) ample) in chapter \(\text{IV}\) (theorem III).
Chapter 3

Connectivity of Brill-Noether loci for singular curves

Recall (see chapter 2) that we are going to study the Brill-Noether locus of singular irreducible curves that lie on a smooth surface $S$ satisfying

$$h^1(O_S) = 0 \text{, and } -K_S \text{ is generated by global sections.} \hspace{1cm} (*)$$

Now we state the theorem that we are going to prove:

**Theorem I.** Let $C$ be a reduced irreducible curve of arithmetic genus $p_a$ that lies in a surface $S$ satisfying $(*)$. Let $\overline{J}^d(C)$, $d > 0$, be the compactification of the generalized Jacobian. Then for any $r \geq 0$ such that $\rho(r, d) = p_a - (r + 1)(p_a - d + r) > 0$, the generalized Brill-Noether subvariety $\overline{W}^r_d(C)$ is nonempty and connected.

**Remark 3.0.1** If $r \leq d - p_a$, by Riemann-Roch inequality we have $\overline{W}^r_d(C) = J^d(C)$, and this is connected. Then, in order to prove theorem I we can assume $r > d - p_a$. Note that if $A$ corresponds to a point in $\overline{W}^r_d(C)$ with $r > d - p_a$, then by Riemann-Roch theorem $h^1(A) > 0$.

**Outline of the proof of theorem I**

Note that $\overline{W}^r_d(C)$ is the fiber of $q$ over the point $u_0 \in |C|$ corresponding to the curve $C$. Let $U$ be the open subset of $|C|$ corresponding to integral curves, and $V$ the subset of smooth curves. Define $\overline{W}^r_d(V)$ to be the Brill-Noether locus of sheaves with smooth support, i.e. $\overline{W}^r_d(V) = q^{-1}(V)$. By [F-L], the restriction $q_V : (\overline{W}^r_d)_V \to V$ has connected fibers. We want to use
this fact to show that $W_d^r(C)$ is connected. Let $A$ be a rank one torsion-free sheaf on $C$ corresponding to a point in $W_d^r(C)$, and assume that it is generated by global sections. We think of $A$ as a torsion sheaf on $S$. Then we have a short exact sequence on $S$

$$0 \to E \xrightarrow{f_0} H^0(A) \otimes O_S \to A \to 0,$$

where the map on the right is evaluation. This sequence has already appeared in the literature (see [La], [Ye]). Our idea is to deform $f_0$ to a family $f_t$. The cokernel of $f_t$ will define a family of sheaves $A_t$ with $h^0(A_t) \geq h^0(A)$ (because $h^0(E) = 0$), and then for each $t$ the point in $\mathcal{J}_{|C|}$ corresponding to $A_t$ lies in $\overline{W}_{d,|C|}$. Assume that there are 'enough' homomorphisms from $E$ to $H^0 \otimes O_S$ and the family $f_t$ can be chosen general enough, so that for a general $t$, the support of $A_t$ is smooth (the details of this construction are in section 3.2). The family $A_t$ shows that the point in $\overline{W}_d^r(C)$ corresponding to $A$ is in the closure of $(\overline{W}_d^r)_{V'}$ in $\mathcal{J}_{|C|}'$. It can be shown that this closure has connected fibers. Let $X$ be the fiber over $u_0$ of this closure. Then all sheaves for which this construction works are in the connected component $X$ of $\overline{W}_d^r(C)$. If this could be done for all sheaves in $\overline{W}_d^r(C)$ this would finish the proof, but there are sheaves for which this construction doesn’t work. For these sheaves we show in section 3.3 that they can be deformed (keeping the support $C$ unchanged) to a sheaf for which a refinement of this construction works. This shows that all points in $\overline{W}_d^r(C)$ are in the connected component $X$.

### 3.1 The main lemma

The precise statement that we will use to prove theorem I is the following lemma.

**Lemma 3.1.1** Let $C$ be an integral complete curve in a surface $S$. Assume that for each rank one torsion-free sheaf $A$ on $C$ with $h^0(A) = r + 1 > 0$ and $\deg(A) = d > 0$ such that $\rho(r, d) > 0$ we have the following data:

(a) A family of curves $\mathcal{C}$ in $S$ parametrized by an irreducible curve $T$ (not necessarily complete).

(b) A connected curve $T'$ (not necessarily irreducible nor complete) with a map $\psi : T' \to T$.

(c) A rank one torsion-free sheaf $\mathcal{A}$ on $\mathcal{C}' = C \times_T T'$, flat over $T'$, inducing rank one torsion-free sheaves on the fibers of $\mathcal{C}' \to T'$. 

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Assume that the following is satisfied:

(i) $C|_{t_0} \cong C$ for some $t_0 \in T$, $C|_t$ is linearly equivalent to $C$ for all $t \in T$, and $C|_t$ is smooth for $t \neq t_0$.

(ii) One irreducible component of $T'$ is a finite cover of $T$, and the rest of the components of $T'$ are mapped to $t_0 \in T$.

(iii) $A|_{t'_0} \cong A$ for some $t'_0 \in T'$ mapping to $t_0 \in T$.

(iv) $h^0(A|_{t'}) \geq r + 1$ for all $t' \in T'$.

Then the generalized Brill-Noether subvariety $W^r_d(C)$ of the compactified generalized Jacobian $J^d[C]$ is connected.

Proof. We will use the notation introduced in the previous section. The map $q : W^r_d|C| \to U$ is a projective morphism. Recall that $W^r_d(C)$ is the fiber of $q$ over $u_0$, where $u_0$ is the point corresponding to $C$. By [F-L] the morphism $q$ has connected fibers over $V$, thus a general fiber of $q$ is connected, and we want to prove that the fiber over $u_0 \in U$ is also connected.

Let $W^r_d|C| \xrightarrow{q'} U' \to U$ be the Stein factorization of $q$ (see [I] III Corollary 11.5), i.e. $q'$ has connected fibers and $g$ is a finite morphism. A general fiber of $q$ is connected, and then $U'$ has one irreducible component $Z$ that maps to $U$ birationally. The subset $U$ is open in $|C|$ and hence normal, the restriction $g|Z : Z \to U$ is finite and birational, $Z$ and $U$ are integral, thus by Zariski’s main theorem (see [I] III Corollary 11.4) each fiber of $g|Z$ consists of just one point. Let $z_0$ be the point of $Z$ in the fiber $g^{-1}(u_0)$.

Claim. Let $y_0$ be a point in the fiber $q^{-1}(u_0) = W^r_d(C)$. Then $y_0$ is mapped by $q'$ to $z_0$.

This claim implies that that $W^r_d(C)$ is connected. Now we will prove the claim.

Let $A$ be the sheaf on $S$ corresponding to the point $y_0$. Let $T'$, $T$, $t'_0 \in T'$, $t_0 \in T$, $\psi : T' \to T$ be the curves points and morphism given by the hypothesis of the lemma. Let $\phi : T' \to J^d|C|$ be the morphism given by the universal property of the moduli space $J^d|C|$. Item (iv) imply that the image of $\phi$ is in $W^r_d|C|$.
The restriction of $q' \circ \phi$ to $T' \setminus \psi^{-1}(t_0)$ maps to $Z$, because for $t' \in T' \setminus \psi^{-1}(t_0)$ the sheaf $\mathcal{A}|_{t'}$ has smooth support by item (i). Items (c) and (i) imply that $g \circ q' \circ \phi(\psi^{-1}(t_0)) = u_0$. Thus $q' \circ \phi(\psi^{-1}(t_0))$ is a finite number of points (because it is in the fiber of $g$ over $u_0$).

The facts that $q' \circ \phi(T' \setminus \psi^{-1}(t_0))$ is in $Z$ and that $q' \circ \phi(\psi^{-1}(t_0))$ is a finite number of points imply that $q' \circ \phi(\psi^{-1}(t_0))$ is also in $Z$ (because by item (b) the curve $T'$ is connected and thus also its image under $q' \circ \phi$), and in fact $q' \circ \phi(\psi^{-1}(t_0)) = z_0$ because $q' \circ \phi(\psi^{-1}(t_0))$ is in the fiber of $g$ over $u_0$.

By item (ii), $t'_0 \in \psi^{-1}(t_0)$. Then $q' \circ \phi(t'_0) = z_0$, and by item (iii) we have $y_0 = \phi(t'_0)$, then $q'(y_0) = q'(\phi(t'_0)) = z_0$ and the claim is proved.

In section 3.2 we will construct this family under some assumptions on $A$ (proposition 3.2.5), and in section 3.3 we will show how to use that to construct a family for any $A$. Note that because of remark 3.0.1 we can...
3.2 A particular case

Given a rank one torsion-free sheaf $A$ on an integral curve lying on a surface $S$, we define another sheaf $A^\ast$ that is going to be some sort of dual. Let $j$ be the inclusion of the curve $C$ in the surface $S$. We define $A^\ast$ as follows:

$$A^\ast = \text{Ext}^1(j_*A, \omega_S).$$

The operation $A \to A^\ast$ is a contravariant functor. Note that the support of $A^\ast$ is $C$. It will be clear from the context when we are referring to $A^\ast$ as a torsion sheaf on $S$ or as a sheaf on $C$. In the case in which $A$ is a line bundle, then $A^\ast = A^\vee \otimes \omega_C$. Now we prove some properties of this “dual”.

Lemma 3.2.1 Let $A$ be a rank one torsion-free sheaf on an integral curve lying on a surface. Then $A^{**} = A$

Proof. First observe that if $L$ is a line bundle on $C$, then $(A \otimes L)^\ast \cong A^\ast \otimes L^\vee$. To see this, take an injective resolution of $\omega_S$

$$0 \to \omega_S \to \mathcal{I}_0 \to \mathcal{I}_1 \to \cdots$$

Now we use this resolution to calculate the $\text{Ext}$ sheaf.

$$\text{Ext}^1(A \otimes L, \omega_S) = h^1(\text{Hom}(A \otimes L, \mathcal{I}_\bullet)) = h^1(L^\vee \otimes \text{Hom}(A, \mathcal{I}_\bullet)) =$$

$$= L^\vee \otimes h^1(\text{Hom}(A, \mathcal{I}_\bullet)) = L^\vee \otimes \text{Ext}^1(A, \omega_S)$$

The third equality follows from the fact that $\text{Hom}(A, \mathcal{I}_\bullet)$ is supported on the curve and $L^\vee$ is locally free.

It follows that $(L \otimes A)^{**} \cong L \otimes A^{**}$, and then proving the lemma for $A$ is equivalent to proving it for $L \otimes A$. Multiplying with an appropriate very ample line bundle, we can assume that $A$ is generated by global sections. Then we have an exact sequence

$$0 \to E \to V \otimes \mathcal{O}_S \to A \to 0,$$  \hspace{1cm} (3.1)

where $V = H^0(A)$. The following lemma proves that $E$ is locally free.

Lemma 3.2.2 Let $M$ be a torsion-free sheaf on an integral curve $C$ that lies on a smooth surface $S$. Let $j : C \to S$ be the inclusion. Let $F$ be a
locally free sheaf on the surface. Let $f : F \to j_*M$ be a surjection. Then the elementary transformation $F'$ of $F$, defined as the kernel of $f$

$$0 \to F' \to F \xrightarrow{f} j_*M \to 0,$$

(3.2)
is a locally free sheaf.

Proof. $M$ is torsion-free sheaf on $C$, and then $j_*M$ has depth at least one, and because $S$ is smooth of dimension 2, this implies that the projective dimension of $j_*M$ is at most one ($Ext^i(j_*M, \mathcal{O}_S) = 0$ for $i \geq 2$). Now $Ext^i(F, \mathcal{O}_S) = 0$ for $i \geq 1$ because $F$ is locally free, and then from the exact sequence (3.2), we get

$$0 \to Ext^i(F', \mathcal{O}_S) \to Ext^{i+1}(j_*M, \mathcal{O}_S) \to 0, \quad i \geq 1,$$

and then $Ext^i(F', \mathcal{O}_S) = 0$ for $i \geq 1$, and this implies that $F'$ is locally free.

In particular, $E^\vee\vee = E$. Applying the functor $Hom(\cdot, \omega_S)$ twice to the sequence (3.1), we get

$$0 \to E \to V \otimes \mathcal{O}_S \to A^{**} \to 0.$$

Comparing with (3.1) we get the result (because the map on the left is the same for both sequences).

Lemma 3.2.3 $Ext^1(A, \omega_S) \cong H^0(A^*), \text{ and this is dual to } H^1(A)$.

Proof. The local to global spectral sequence for Ext gives the following exact sequence

$$0 \to H^1(Hom(A, \omega_S)) \to Ext^1(A, \omega_S) \to H^0(A^*) \to H^2(Hom(A, \omega_S))$$

But $Hom(A, \omega_S) = 0$ because $A$ is supported in $C$ and then the first and last terms in the sequence are zero and we have the desired isomorphism.

Now we will prove a lemma that we will need. The proof can also be found in [O], but for convenience we reproduce it here.

Lemma 3.2.4 Let $E$ and $F$ be two vector bundles of rank $e$ and $f$ over a smooth variety $X$. Assume that $E^\vee \otimes F$ is generated by global sections. If $\phi : E \to F$ is a sheaf morphism, we define $D_k(\phi)$ to be the subset of $X$
where \( \text{rk}(\phi_x) \leq k \) (there is an obvious determinantal description of \( D_k(\phi) \) that gives a scheme structure). Let \( d_k \) be the expected dimension of \( D_k(\phi) \)

\[
d_k = \dim(X) - (e - k)(f - k).
\]

Then there is a Zariski dense set \( U \) of \( \text{Hom}(E,F) \) such that if \( \phi \in U \), then we have that \( D_k(\phi) \setminus D_{k-1}(\phi) \) is smooth of the expected dimension (if \( d_k < 0 \) then it will be empty).

**Proof.** Let \( M_k \) be the set of matrices of dimension \( e \times f \) and of rank at most \( k \) (there is an obvious determinantal description that gives a scheme structure to this subvariety). It is well known that the codimension of \( M_k \) in the space of all matrices is \( (e - k)(f - k) \), and that the singular locus of \( M_k \) is \( M_k - 1 \).

Now, because \( E^\vee \otimes F \) is generated by global sections, we have a surjective morphism

\[
H^0(E^\vee \otimes F) \otimes \mathcal{O}_X \to E^\vee \otimes F
\]

that gives a morphism of maximal rank between the varieties defined as the total space of the previous vector bundles

\[
p : X \times H^0(E^\vee \otimes F) \to \mathbb{V}(E^\vee \otimes F).
\]

Define \( \Sigma_k \subset \mathbb{V}(E^\vee \otimes F) \) as the set such that \( \text{rk}(\phi_x) \leq k \). The fiber of \( \Sigma_k \) over any point in \( X \) is obviously \( M_k \). Define \( Z_k \) to be \( p^{-1}(\Sigma_k) \). The fact that \( p \) has maximal rank implies that \( Z_k \) has codimension \( (e - k)(f - k) \) in \( X \times H^0(E^\vee \otimes F) \) and that the singular locus of \( Z_k \) is \( Z_{k-1} \).

Now observe that the restriction of the projection

\[
q|_{Z_k \setminus Z_{k-1}} : Z_k \setminus Z_{k-1} \to H^0(E^\vee \otimes F)
\]

has fiber \( q|_{Z_k \setminus Z_{k-1}}^{-1}(\phi) \cong D_k(\phi) \setminus D_{k-1}(\phi) \). Finally, by generic smoothness, for a general \( \phi \in H^0(E^\vee \otimes F) \) this is smooth of the expected dimension (or empty).

\[\square\]

Now we will construct the deformation of \( A \) that we described in the section 3.1 in the particular case in which both \( A \) and \( A^* \) are generated by global sections.

**Proposition 3.2.5** Let \( A \) be a rank one torsion-free sheaf on an integral curve \( C \) lying on a surface \( S \) with \( h^1(\mathcal{O}_S) = 0 \) and \( -K_S \) generated by global sections. Denote \( j : C \hookrightarrow S \). If \( A \) and \( A^* \) are both generated by global
sections, then there exists a (not necessarily complete) smooth irreducible curve $T$ and a sheaf $A$ on $S \times T$ flat over $T$, such that

(a) the sheaf induced on the fiber of $S \times T \to T$ over some $t_0 \in T$ is $j_* A$
(b) the sheaf $A_t$ induced on the fiber over any $t \in T$ with $t \neq t_0$ is supported on a smooth curve $C_t$ and it is a rank one torsion-free sheaf when considered as a sheaf on $C_t$
(c) $h^0(A_t) \geq h^0(A)$ for every $t \in T$.

Note that these are the hypothesis of lemma 3.1.1 for the particular case in which both $A$ and $A^*$ are generated by global sections. We will lift this condition in the next section.

Proof. The fact that $A$ is generated by global sections implies that there is an exact sequence

$$0 \to E \xrightarrow{f_0} V \otimes \mathcal{O}_S \to A \to 0 \quad V = H^0(A),$$

with $E$ locally free (by proposition 3.2.2). Taking global sections in this sequence we see that $H^0(E) = 0$, because

$$0 \to H^0(E) \to V \cong H^0(A).$$

Consider a curve $T$ mapping to $\text{Hom}(E, V \otimes \mathcal{O}_S)$ with $t_0 \in T$ mapping to $f_0$ (so that item (a) is satisfied). Denote by $f_t$ the morphism given for $t \in T$ by this map. After shrinking $T$ we can assume that $f_t$ is still injective. Let $\pi_1$ be the projection of $S \times T$ onto the first factor and let $\mathcal{E} = \pi_1^* E$. Using the universal sheaf and morphism on $\text{Hom}(E, V \otimes \mathcal{O}_S)$ we can construct (by pulling back to $S \times T$) an exact sequence on $S \times T$

$$0 \to \mathcal{E} \xrightarrow{f_t} V \otimes \mathcal{O}_{S \times T} \to A_t \to 0$$

that restricts for each $t$ to an exact sequence

$$0 \to E \xrightarrow{f_t} V \otimes \mathcal{O}_S \to A_t \to 0,$$

where $A_t$ is a sheaf supported in the degeneracy locus of $f_t$. It is clear that $\text{deg}(A) = \text{deg}(A_t)$.

Now we are going to prove that if the curve $T$ and the mapping to $\text{Hom}(E, V \otimes \mathcal{O}_S)$ are chosen generically, the quotient of the map gives the desired deformation.

The flatness of $A$ over $T$ follows from the fact that it has a short resolution and from the local criterion of flatness (we can apply [I, III Lemma 10.3.A]).
The condition on $h^0(A_t)$ follows because $H^0(E) = 0$ and we have a sequence

$$0 \to H^0(E) = 0 \to V \to H^0(A_t),$$

and then $h^0(A) \leq h^0(A_t)$. This proves item (c).

Using the long exact sequence obtained by applying $Hom(\cdot, O_S)$ to 3.3, and the fact that $E$ is locally free, we obtain that $Ext_i(A_t, O_S)$ vanishes for $i \geq 2$, and so the projective dimension of $A_t$ is 1, and this implies that $A_t$, when considered as a sheaf on its support $C_t$, is torsion-free.

We have to prove that we can choose the curve $T$ and the map to $Hom(E, V \otimes O_S)$ such that $C_t$ is smooth for $t \neq t_0$ (here we will use that $A^*$ is generated by global sections).

First note that $Ext^1(A_t, O_S)$ is generated by global sections, because $Ext^1(A_t, O_S) = A^* \otimes \omega_{-1}^S$, and both $A^*$ and $\omega_{-1}^S$ are generated by global sections. Now we see that $E^\vee$ is generated by global sections, because we have

$$0 \to V^\vee \otimes O_S \to E^\vee \to Ext^1(A_t, O_S) \to 0,$$

$Ext^1(A_t, O_S)$ is generated by global sections and $H^1(V^\vee \otimes O_S) = 0$. Then $E^\vee \otimes (V \otimes O_S)$ is generated by global sections.

Now apply lemma 3.2.4 with $F = V \otimes O_S$. Then $n = m = r + 1$, $k = r$ and the expected dimension is 1. And the lemma gives that for $\phi$ in a Zariski open subset of $Hom(E, V \otimes O_S)$, the degeneracy locus $D_r(\phi)$ of $\phi$ is smooth away from the locus $D_{r-1}(\phi)$ where $\phi$ has rank $r - 1$, but again by lemma 3.2.4 the locus $D_{r-1}(\phi)$ is empty. This proves item (b).

\[\square\]

### 3.3 General case

Now we don’t assume that $A$ satisfies the properties of the particular case (i.e., $A$ and $A^*$ now might not be generated by global sections). We will find a new sheaf that satisfies those conditions. We know how to deform this new sheaf, and we will show how we can use this deformation to construct a deformation of the original $A$.

We start with a rank one torsion-free sheaf $A$ with $h^0(A), h^1(A) > 0$ on an integral curve $C$ lying on a surface. First we define $A'$ as the base point free part of $A$, i.e. $A'$ is the image of the evaluation map

$$H^0(A) \otimes O_C \to A.$$
We have assumed that $h^0(A) > 0$, and then $A'$ is a (nonzero) rank one torsion-free sheaf. Obviously, $H^0(A) = H^0(A')$. We have a short exact sequence
\[ 0 \to A' \to A \to Q \to 0, \]
where $Q$ has support of dimension 0. Now consider $A'^*$, and define $B$ to be its base point free part. We have $h^0(A'^*) = h^1(A') = h^1(A) + h^0(Q) \geq h^1(A) > 0$. The first equality by lemma 3.2.3, and the last inequality by assumption. Then $B$ is a (nonzero) rank one torsion-free sheaf. Finally define $A''$ to be equal to $B^*$.

**Lemma 3.3.1** Both $A''$ and $A'^*$ are generated by global sections.

**Proof.** Since $B$ is the base point free part of $A'^*$, we have a sequence
\[ 0 \to B \to A'^* \to R \to 0 \]
where $R$ has support of dimension zero. Applying $\text{Hom}(\cdot, \omega_S)$ we get
\[ 0 \to A' \to B^* = A'' \to \tilde{R} \to 0 \]
whose associated cohomology long exact sequence gives
\[ 0 \to H^0(A') \to H^0(B^*) \to H^0(\tilde{R}) \to H^1(A') \to H^1(B^*) \to 0. \]
To see that $A''$ is generated by global sections, it is enough to prove that the last map is an isomorphism, because then the first three terms make a short exact sequence, and the fact that $A'$ and $\tilde{R}$ are generated by global sections (the first by definition, the second because its support has dimension zero) will imply that $B^*$ (that is equal to $A''$ by definition) is generated by global sections.

To prove that the last map is an isomorphism, we only need to show that $h^1(A') = h^1(B^*)$, and this is true because
\[ h^1(A') = h^0(A'^*) = h^0(B) = h^0(B^{**}) = h^1(B^*). \]
The first equality is by lemma 3.2.3, the second because $B$ is the base point free part of $A'^*$, the third by lemma 3.2.1, and the last again by lemma 3.2.3.

To see that $A'^*$ is generated by global sections, note that by definition $A'^* = B^{**} = B$, and this is generated by global sections.

\[ \square \]
We started with a rank one torsion-free sheaf $A$ with $h^0(A)$ and $h^1(A) > 0$, and we have constructed new sheaves $A'$ and $A''$ with (nontrivial) maps $A' \to A$ and $A' \to A''$. They give rise to exact sequences

\[ 0 \to A' \to A \to Q \to 0 \]
\[ 0 \to A' \to A'' \to \tilde{Q} \to 0 \]

(3.4)

Lemma 3.3.2 With the previous definitions we have $h^0(A') = h^0(A)$ and $h^1(A'') = h^1(A')$.

Proof. By construction $h^0(A') = h^0(A)$ and $h^0(A'') = h^0(A'')$. By lemma 3.2.3 this last equality is equivalent to $h^1(A'') = h^1(A')$. \qed

As $A''$ and $A''$ are generated by global sections, then by proposition 3.2.3 the sheaf $A''$ can be deformed in a family $A''_t$ in such a way that the support of a general member of the deformation is smooth. The idea now is to find (flat) deformations of $A'$ and $A$, so that for every $t$ we still have maps like 3.4. From the existence of these maps we will be able to obtain the condition that $h^0(A_t) \geq h^0(A)$, then we will be able to apply lemma 3.1.1 and then theorem I will be proved. The details are in section 3.4. We will start by showing how the condition on $h^0(A_t)$ is obtained, and then how we can find the deformations of $A'$ and $A$.

Proposition 3.3.3 Let $A$, $A'$, $A''$ be rank one torsion-free sheaves on an integral curve $C$. Assume that they fit into exact sequences like 3.4 and that $h^0(A') = h^0(A)$ and $h^1(A'') = h^1(A')$. Let $P$ be a curve (not necessarily complete), and let $A$, $A'$, and $A''$ be sheaves on $S \times P$, flat over $P$, inducing for each $p \in P$ rank one torsion-free sheaves $A_p$, $A'_p$, $A''_p$, supported on a curve $C_p$ of $S$, where $A_{p_0} = A$, $A'_{p_0} = A'$, and $A''_{p_0} = A''$ for some $p_0 \in P$. Assume that $h^0(A''_p) \geq h^0(A''_{p_0})$ for all $p \in P$ and that we have short exact sequences

\[ 0 \to A' \to A \to Q \to 0 \]
\[ 0 \to A' \to A'' \to \tilde{Q} \to 0 \]

with $Q$ and $\tilde{Q}$ flat over $P$ (i.e., the induced sheaves $Q_p$, $\tilde{Q}_p$ have constant length, equal to $l(Q)$ and $l(\tilde{Q})$ respectively).

Then we have $h^0(A_p) \geq h^0(A_{p_0})$ for all $p \in P$. 24
Proof. For each \( p \in P \) we have sequences

\[
0 \to A'_p \to A_p \to Q_p \to 0
\]

\[
0 \to A'_p \to A''_p \to \tilde{Q}_p \to 0.
\]

The maps on the left are injective because they are nonzero and the sheaves have rank one and are torsion-free. Using the associated long exact sequences and the hypothesis we have

\[
h^0(A_p) \geq h^0(A'_p) \geq h^0(A''_p) - l(\tilde{Q}_p) \geq h^0(A'') - l(\tilde{Q}) = h^0(A) = h^0(A).
\]

\( \square \)

It only remains to prove that those sheaves can be “deformed along”, and that those deformations are flat, i.e. that given \( A, A' \) and \( A'' \) we can construct \( A' \) and \( A'' \). This is proved in the following propositions.

**Proposition 3.3.4** Let \( L \) and \( M \) be rank one torsion-free sheaves on an integral curve \( C \) that lies on a surface \( S \). Assume we have a short exact sequence

\[
0 \to L \to M \to Q \to 0.
\] (3.5)

Assume furthermore that we are given a sheaf \( \mathcal{M} \) on \( S \times P \) (where \( P \) is a connected but not necessarily irreducible curve) that is a deformation of \( M \), flat over \( P \). I.e., \( \mathcal{M}|_{p_0} \cong M \) for some \( p_0 \in P \), and for all \( p \in P \) we have that \( M_p = \mathcal{M}|_p \) are torsion-free sheaves on \( C_p \), where \( C_p \) is a curve on \( S \).

Then, there is a connected curve \( P' \) with a map \( f : P' \to P \) and a sheaf \( \mathcal{L}' \) over \( S \times P' \) with the following properties:

One irreducible component of \( P' \) is a finite cover of \( P \) and the rest of the components map to \( p_0 \in P \). The sheaf \( \mathcal{L}' \) is a deformation of \( L \), in the sense that \( \mathcal{L}'|_{p'_0} \cong L \) for some \( p'_0 \in P' \) mapping to \( p_0 \in P \), the sheaf \( \mathcal{L}' \) is flat over \( P' \) and induces rank one torsion-free sheaves on the fibers over \( P' \). And if we define \( \mathcal{M}' \) to be the pullback of \( \mathcal{M} \) to \( S \times P' \), there exists an exact sequence

\[
0 \to \mathcal{L}' \to \mathcal{M}' \to Q' \to 0,
\]

inducing short exact sequences

\[
0 \to L'_{p'} \to M'_{p'} \to Q'_{p'} \to 0
\]

for every \( p' \in P' \).
Proof. If the support of $Q$ were in the smooth part of the curve, we would have $M = L \otimes \mathcal{O}_C(D)$, with $D$ an effective divisor of degree $l(Q)$. Then, if we are given a deformation $M_p$ of $M$, we only need to find a deformation $D_p$ of the effective divisor $D$, with the only condition that $D_p$ is an effective divisor on $C_p$, with degree $l(Q)$. This can easily be done if we are in the analytic category. In general we might need to do a base change of the parametrizing curve $P$ and we will obtain a finite cover $P'$ of $P$ (What we are doing is moving a dimension zero and length $l(Q)$ subscheme of $S$, with the only restriction that for each $p$ the corresponding scheme is in $C_p$). Then we only need to define $L' = M \otimes \mathcal{O}_{C_p}(−D_p')$ and the proposition would be proved (with $P'$ a finite cover of $P$).

To be able to apply this, we will have to make first a deformation of $L$, keeping $M$ fixed, until we get $Q$ to be supported in the smooth part of $C$ (the curve $C$ also remains fixed in this deformation). This is the reason for the need of the curve $P'$ with some irreducible components mapping to $p_0$.

We will prove this by induction on the length of the intersection of the support of $Q$ and the singular part of $C$.

Lemma 3.3.5 Let $L$ and $M$ be rank one torsion-free sheaves on an integral curve $C$ that lies on a surface $S$. Assume we have a short exact sequence

$$0 \to L \to M \to Q \to 0.$$  

Assume that $Q = R \oplus Q'$ where $Q'$ has length $l(Q) - 1$ and it is supported in the smooth part of $C$, and $R$ has length one as it is supported in a singular point of $C$ (“the length of the intersection of the support of $Q$ and the singular part of $C$ is one”).

Then there is a flat deformation $L_y$ of $L$ parametrized by a connected curve $Y$ (it might not be irreducible) such that $L_{y_0} = L$ for some $y_0 \in Y$ and for every $y \in Y$ there is an exact sequence

$$0 \to L_y \to M \to Q_y \to 0$$

and there is some $y_1 \in Y$ such that the support of $Q_{y_1}$ is in the smooth part of $C$.

Proof. In this situation, the exact sequence (3.3) gives rise to another exact sequence

$$0 \to L \otimes I_Z' \to M \to R \to 0$$

where the map on the right is the composition of $M \to Q$ and the projection $Q \to R$, and we denote by $I_Z$ the ideal sheaf of the support $Z$ of $Q'$. Because
$Z$ is in the smooth part of $C$, $I_Z$ is an invertible sheaf. Note that $Q'$ is the quotient of $\mathcal{O}_C$ by this ideal sheaf. Define $\widehat{L}$ to be $L \otimes I_Z^\vee$. If we know how to make a flat deformation $\widehat{L}_y$ of $\widehat{L}$ so that the quotient $R_y$ is supported in the smooth part of $C$ for some $y_1 \in Y$, then we can construct a deformation $L_y$ of $L$ defined as

$$L_y = \widehat{L}_y \otimes I_Z.$$ 

Note that this deformation is also flat. The cokernel $Q_y$ of $L_y \rightarrow M$ is supported in the smooth part of $C$ for the points $y \in Y$ for which $R_y$ is supported in the smooth part of $C$.

This shows that to prove the lemma we can assume that $Q$ has length one and its support is a singular point of $C$, i.e. $Q = \mathcal{O}_x$, where $x$ is a singular point of $C$.

Consider the scheme $\text{Quot}^1(M)$ representing the functor of quotients of $M$ of length 1. If the support $x$ of the quotient $Q$ is in the smooth part of $C$, then there is only one surjective map (up to scalar) because $\dim \text{Hom}(M, Q) = 1$, whose kernel is $M \otimes \mathcal{O}_C(-x)$.

If $x$ is in the singular part, then in general $\dim \text{Hom}(M, Q) > 1$, and the quotients are parametrized by $\mathbb{P}\text{Hom}(M, Q)$ (the universal bundle is flat over $\mathbb{P}\text{Hom}(M, Q)$). We want to show that $\text{Quot}^1(M)$ is connected by constructing a flat family of quotients $M \rightarrow \widetilde{Q}_\tilde{c}$ (the family $\widetilde{Q}_\tilde{c}$ will be parametrized by an open set of the normalization $\tilde{C}$ of $C$) such that for a general $\tilde{c}$ the support of $\widetilde{Q}_\tilde{c}$ is in the smooth part of $C$, and for some point $\tilde{c}_0$ the support of $\widetilde{Q}_{\tilde{c}_0}$ is a singular point of $C$.

Consider the normalization $\overline{C}$ of $C$, and let $F$ be an open set of $\overline{C}$

$$F \xrightarrow{j} C \times F \xrightarrow{\pi_1} C$$

(3.6)

Where $\pi_1$ is the projection to the first factor and $j = (\nu, i)$, the morphism $\nu: F \hookrightarrow \overline{C} \rightarrow C$ being the restriction to $F$ of the normalization map and $i$ the identity map. Note that $j$ is a closed immersion, and its image is just $C \times_C F \cong F$.

Let $\tilde{c}_0$ be a point of $\overline{C}$ in $\nu^{-1}(x)$ (the family is going to be parametrized by an open neighborhood $F$ of $\tilde{c}_0$). We have to construct a surjection of $\overline{M} = \pi_1^* M$ onto $\overline{Q} = j_* \mathcal{O}_F$. Note that $\overline{Q}|_{\overline{C} \times \tilde{c}} = \overline{Q}_{\tilde{c}} \cong \mathcal{O}_\nu(\tilde{c})$ and that $\overline{Q}$ is flat over $F$.

Now, to define that quotient, it is enough to define it in the restriction to the image of $j$ (because this is exactly the support of $\overline{Q}$). So the map we
have to define is
\[ j^*\tilde{M} \to \mathcal{O}_F. \]
But \( j^*\tilde{M} = \nu^*M \) is a rank one sheaf on the smooth curve \( F \), so it is the direct sum of a line bundle and a torsion part \( T \). Shrinking \( F \) if necessary, the line bundle part is isomorphic to \( \mathcal{O}_F \), and we have
\[ j^*\tilde{M} \cong T \oplus \mathcal{O}_F, \]
and then to define the quotient we just take an isomorphism in the torsion-free part. This finishes the proof of the lemma.

Now we go to the general case: the intersection of the support of \( Q \) with the singular part of \( C \) has length \( n \). We are going to see how this can be reduced to the case \( n = 1 \).

Take a surjection from \( Q \) to a sheaf \( Q' \) of length \( n - 1 \), such that \( Q \) is isomorphic to \( Q' \) at the smooth points. The kernel \( R \) of this surjection will have length 1, and will be supported in a singular point of \( C \). It is isomorphic to \( \mathcal{O}_x \), for some singular point \( x \). We have a diagram

\[
\begin{array}{cccccc}
0 & \to & L & \to & L' & \to & R & \to & 0 \\
| & | & | & | & | & | & | \\
0 & \to & L & \to & M & \to & Q & \to & 0 \\
| & | & | & | & | & | & | \\
Q' & \to & \quad & Q' & \\
| & | & | & | & | & | & | \\
0 & \to & 0 & \to & 0
\end{array}
\]

Observe that \( L, L' \) and \( R \) satisfy the hypothesis of lemma 3.3.5, so we can find deformations \( L_y, R_y \) (parametrized by some curve \( Y \) and with \( L_{y_0} = L \) and \( R_{y_0} = R \) for some \( y_0 \in Y \)) such that for some \( y_1 \in Y \) we have that the support of the corresponding sheaf \( R_{y_1} \) is a smooth point of \( C \). All the maps of the previous diagram can be deformed along. To do this, we change \( L \) by \( L_y \), \( R \) will be deformed to \( R_y \) and \( L' \) is kept constant. Then \( Q \) is deformed to a family \( Q_y \) defined as \( M/L_y \). The cokernel of \( R_y \to Q_y \) will
be $Q_y/R_y = M/L' = Q'$, and hence we keep it constant. Then for each $y$ we still have a commutative diagram, and furthermore it is easy to see that all deformations are flat (note that $R_y$ is a flat deformation and $Q'$ is kept constant, and then $Q_y$ is a flat deformation). An important point is that $M$ remains fixed, and the injection $L \to M$ is deformed to $L_y \to M$.

\[
\begin{array}{ccc}
0 & 0 \\
\downarrow & \downarrow \\
0 & \to & L_y \to L' \to R_y \to 0 \\
\downarrow & \downarrow \\
0 & \to & L_y \to M \to Q_y \to 0 \\
\downarrow & \downarrow \\
Q' & \to & Q' \\
\downarrow & \downarrow \\
0 & 0
\end{array}
\]

For $y_1$ we have that the length of the intersection of the support of $Q_{y_1}$ with the singular part of $C$ is $n - 1$. We repeat the process (starting now with $L_{y_1}$, $M$ and $Q_{y_1}$), until all the points of the support of $Q$ are moved to the smooth part of $C$. This finishes the proof of the proposition.

The following proposition is similar to proposition 3.3.4, but now the roles of $L$ and $M$ are changed: we are given a deformation of $L$ and we have to deform $M$ along.

**Proposition 3.3.6** Let $L$ and $M$ be rank one torsion-free sheaves on an integral curve $C$ that lies on a surface $S$. Assume we have a short exact sequence

\[0 \to L \to M \to Q \to 0.\]  \hspace{1cm} (3.7)

Assume furthermore that we are given a sheaf $\mathcal{L}$ on $S \times P$ (where $P$ is a connected but not necessarily irreducible curve) that is a deformation of $L$, flat over $P$, i.e., $\mathcal{L}|_{p_0} \cong L$ for some $p_0 \in P$, and for all $p \in P$, we have that $L_p = \mathcal{L}|_p$ are torsion-free sheaves on $C_p$, where $C_p$ is a curve on $S$.

Then, there is a connected curve $P'$ with a map $f : P' \to P$ and a sheaf $\mathcal{M}'$ over $S \times P'$ with the following properties:
One irreducible component of \( P' \) is a finite cover of \( P \) and the rest of the components map to \( p_0 \in P \). The sheaf \( M' \) is a deformation of \( M \), in the sense that \( M'_{p_0'} \cong M \) for some \( p_0' \in P' \) mapping to \( p_0 \in P \), the sheaf \( M' \) is flat over \( P' \) and induces rank one torsion-free sheaves on the fibers over \( P' \). And if we define \( L' \) to be the pullback of \( L \) to \( S \times P' \), there exists an exact sequence

\[
0 \to L' \to M' \to Q' \to 0,
\]

inducing short exact sequences

\[
0 \to L'_{p'} \to M'_{p'} \to Q'_{p'} \to 0
\]

for every \( p' \in P' \).

**Proof.** The proof is very similar to the proof of proposition 3.3.4. Again we start by observing that if the support of \( Q \) were in the smooth part of the curve, we would have \( M \cong L \otimes \mathcal{O}_C(D) \), with \( D \) an effective divisor. Then if we are given a flat deformation \( L_p \) of \( L \), we find a deformation \( D_p \) of \( D \) as in the first part, and the proposition would be proved. So again we need a lemma that deforms \( Q \) so that its support is in the smooth part of \( C \).

**Lemma 3.3.7** Let \( L \) and \( M \) be rank one torsion-free sheaves on an integral curve \( C \) that lies on a surface \( S \). Assume we have a short exact sequence

\[
0 \to L \to M \to Q \to 0.
\]

Assume that the part of \( Q \) with support in the smooth part of \( C \) has length \( l(Q) - 1 \), i.e. \( Q = R \oplus Q' \), where \( R \) has length one and is supported in a singular point of \( C \) and \( Q' \) has length \( l(Q) - 1 \) and is supported in the smooth part of \( C \). Then there is a flat deformation \( M_y \) of \( M \) parametrized by a curve \( Y \), such that for every \( y \in Y \) there is an exact sequence

\[
0 \to L \to M_y \to Q_y \to 0
\]

with \( M_y \) a torsion-free sheaf, and there is some \( y_1 \in Y \) such that the support of \( Q_{y_1} \) is in the smooth part of \( C \).

**Proof.** Arguing as in the proof of lemma 3.3.5, we see that it is enough to prove the case \( l(Q) = 1 \), and \( Q = \mathcal{O}_x \) for \( x \) a singular point of \( C \), then we can assume that the extension of the hypothesis of the lemma is

\[
0 \to L \to M \to \mathcal{O}_x \to 0. \tag{3.8}
\]
Now we will consider all extensions of $\mathcal{O}_x$ (for $x$ any point in $C$) by $L$. If $x$ is a smooth point, then there is only one extension that is not trivial (up to equivalence)

$$0 \rightarrow L \rightarrow M \cong L \otimes \mathcal{O}_C(x) \rightarrow \mathcal{O}_x \rightarrow 0.$$ 

All these extensions are then parametrized by the smooth part of $C$.

But if $x$ is a singular point, we could have more extensions, because in general $s = \dim \text{Ext}^1(\mathcal{O}_x, L) > 1$. They will be parametrized by a projective space $\mathbb{P}^{s-1}$. We call this space $E_x$. Note that there is a universal extension on $C \times E_x$ that is flat over $E_x$. We denote by $e_1$ the point in $E_x$ corresponding to the extension $3.8$.

Assume that $\tilde{Q}_c$ is a family of torsion sheaves on $C$ with length 1, parametrized by a curve $F$ such that for a general point $\tilde{c} \in F$ of the parametrizing curve the support of $\tilde{Q}_{\tilde{c}}$ in $C$ is a smooth point, and for a special point $\tilde{c}_0 \in F$ the support of $\tilde{Q}_{\tilde{c}_0}$ is a singular point. Now assume that we can construct a flat family (parametrized by $F$) of nontrivial extensions of $\tilde{Q}_{\tilde{c}}$ by $L$. The extension corresponding to $\tilde{c}_0$ gives a point $e_2$ in $E_x$. The space $E_x$ is a projective space, thus connected, and then there is a curve containing $e_1$ and $e_2$. Using this curve (together with the universal extension for $E_x$) and the curve $F$ (together with the family of extensions that it parametrizes) we construct the curve $Y$ that proves the lemma.

Now we need to construct $F$. As in the proof of lemma 3.3.5, the parametrizing curve $F$ will be an affine neighborhood of $\tilde{c}_0$ in the normalization $\tilde{C}$ of $C$, where $\tilde{c}_0$ is a point that maps to the singular point $x$ of $C$. Consider again the diagram 3.6 of the proof of lemma 3.3.5. The family will be given by an extension of $\tilde{Q} = j_*\mathcal{O}_F$ by $\tilde{L} = \pi^*L$ on $C \times F$. These extensions are parametrized by the group $\text{Ext}^1(\tilde{Q}, \tilde{L})$. The following lemma gives information about this group and relates this extension with the extensions that we get after restriction for each slice $C \times \tilde{c}$. We will call $\tilde{Q}_c$ and $\tilde{L}_c$ the restrictions of $\tilde{Q}$ and $\tilde{L}$ to the slice $C \times \tilde{c}$. Note that the restriction $\tilde{L}_c$ is isomorphic to $L$.

**Lemma 3.3.8** With the previous notation, we have

1) $\text{Ext}^1(\tilde{Q}, \tilde{L}) \cong H^0(\text{Ext}^1(\tilde{Q}, \tilde{L}))$

2) $\text{Ext}^1(\tilde{Q}, \tilde{L})$ has rank zero outside of the support of $\tilde{Q}$, and rank 1 on the smooth points of the support of $\tilde{Q}$

3) Let $I$ be the ideal sheaf corresponding to a slice $C \times \tilde{c}$. Then the natural map

$$\text{Ext}^1_{\mathcal{O}_{C \times F}}(\tilde{Q}, \tilde{L}) \otimes \mathcal{O}_{C \times F}/I \rightarrow \text{Ext}^1_{\mathcal{O}_{C \times \tilde{c}}}(\tilde{Q}_{\tilde{c}}, \tilde{L}_{\tilde{c}})$$

is injective.
Proof. Item 1 follows from the fact that $\text{Hom}(\tilde{Q}, \tilde{L}) = 0$ and the exact sequence

$$0 \to H^1(\text{Hom}(\tilde{Q}, \tilde{L})) \to \text{Ext}^1(\tilde{Q}, \tilde{L}) \to H^0(\text{Ext}^1(\tilde{Q}, \tilde{L})) \to H^2(\text{Hom}(\tilde{Q}, \tilde{L})).$$

To prove item 2 note that the stalk of $\text{Ext}^1(\tilde{Q}, \tilde{L})$ at a point $p$ is isomorphic to $\text{Ext}^1(R/I, R)$, where $R$ is the local ring at the point $p$, and $I$ is the ideal defining the support or $\tilde{Q}$. The ideal $I$ is principal if the point $p$ is smooth, then $R/I$ has a free resolution

$$0 \to I \to R \to R/I \to 0$$

and it follows that $\text{Ext}^1(R/I, R) \cong R/I$.

For item 3, consider the exact sequence

$$0 \to \tilde{Q} \xrightarrow{f} \tilde{Q} \to \tilde{Q}_c \to 0$$

where the first map is multiplication by the local equation $f$ of the slice $C \times \tilde{c}$. Applying $\text{Hom}(\cdot, \tilde{L}_c)$ we get

$$\text{Hom}(\tilde{Q}, \tilde{L}_c) = 0 \to \text{Ext}^1(\tilde{Q}_c, \tilde{L}_c) \to \text{Ext}^1(\tilde{Q}, \tilde{L}_c) \to \text{Ext}^1(\tilde{Q}, \tilde{L}_c),$$

but the last map is zero. To see this, take a locally free resolution of $\tilde{Q}$. The map induced on the resolution by the multiplication with the equation $f$ is just multiplication by the same $f$ on each term

$$\begin{array}{ccc}
\mathcal{F}^\bullet & \longrightarrow & \tilde{Q} \\
\cdot f & \downarrow & \cdot f \\
\mathcal{F}^\bullet & \longrightarrow & \tilde{Q} \\
\end{array} \longrightarrow 0$$

A local section of the sheaf $\text{Ext}^1(\tilde{Q}, \tilde{L}_c)$ is represented by some local section $\varphi(\cdot)$ of $\text{Hom}(\mathcal{F}^\bullet, \tilde{L}_c)$, and the endomorphism induced by multiplication by $f$ on $\text{Ext}^1(\tilde{Q}, \tilde{L}_c)$ is given by precomposition with multiplication $\varphi(f\cdot)$, but $\varphi$ is a morphism of sheaves of modules and then this is equal to $f\varphi(\cdot)$, and this is equal to zero because $f\tilde{L}_c = 0$. Then we have that

$$\text{Ext}^1(\tilde{Q}_c, \tilde{L}_c) \cong \text{Ext}^1(\tilde{Q}, \tilde{L}_c). \quad (3.9)$$

Taking the exact sequence

$$0 \to \tilde{L} \xrightarrow{f} \tilde{L} \to \tilde{L}_c \to 0$$

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and applying $Hom(\tilde{Q}, \cdot)$ we get

$$Ext^1(\tilde{Q}, \tilde{L}) \to Ext^1(\tilde{Q}, \tilde{\mathcal{L}}) \to Ext^1(\tilde{Q}, \tilde{L}_c)$$

and using this and the isomorphism $3.3$ we have an injection

$$Ext^1(\tilde{Q}, \tilde{L}) \otimes O_{C \times F}/I \cong Ext^1(\tilde{Q}, \tilde{L})/(f \cdot Ext^1(\tilde{Q}, \tilde{L})) \hookrightarrow Ext^1(\tilde{Q}_c, \tilde{L}_c).$$

Now we are going to construct the family of extensions. By item 2 of the lemma the sheaf $\mathcal{E} = Ext^1(\tilde{Q}, \tilde{L})$ is isomorphic to $O_X \oplus T(E)$ (shrinking $F$ if necessary) where $X$ is the support of $\tilde{Q}$ and $T(E)$ is the torsion part. Take a nonvanishing section of the torsion-free part, and by item 1 this gives a nonzero element $\psi$ of $Ext^1(\tilde{Q}, \tilde{L})$. This element gives a nontrivial extension

$$0 \to \tilde{L} \to \tilde{M} \to \tilde{Q} \to 0.$$

Observe that $\tilde{M}$ is flat over the base, because both $\tilde{L}$ and $\tilde{Q}$ are flat.

By items 3 and 1 we have that the image of $\psi$ under the restriction map

$$Ext^1(\tilde{Q}, \tilde{L}) \to Ext^1(\tilde{Q}_c, L)$$

is nonzero for any $\tilde{c}$ (recall that $\tilde{L}_c = L$ for all $\tilde{c}$), and this means that the extensions that we obtain after restriction to the corresponding slices

$$0 \to L \to \tilde{M}_c \to \tilde{Q}_c \to 0 \quad (3.10)$$

are non trivial. Furthermore $\tilde{M}_c$ is torsion-free. To prove this claim, let $T(M_c)$ be the torsion part of $M_c$. The map $L \to T(M_c)$ coming from $3.10$ is zero, because $L$ is torsion-free, i.e. $T(M_c)$ injects in $\tilde{Q}_c$. Then we have

$$\tilde{Q}_c \cong \frac{\tilde{M}_c}{L} \cong \frac{\tilde{M}_c/T(M_c) \oplus T(M_c)}{L} \cong \frac{\tilde{M}_c/T(M_c)}{L} \oplus T(M_c).$$

$\tilde{Q}_c$ doesn’t decompose as the direct sum of two sheaves, and then one of these summands must be zero. The first summand cannot be zero, because this would imply that $L \cong \tilde{M}_c/T(M_c)$ and then $\tilde{M}_c \cong L \oplus \tilde{Q}_c$, contradicting the hypothesis that the extension is not trivial. Then we must have $T(M_c) = 0$, and the claim is proved.

Now we are going to consider the general case, in which the part of $Q$ supported in singular points has length $n$. We are going to see that this can be reduced to the case $n = 1$, in a similar way to proposition $3.3.4$. 

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Let $R = \mathcal{O}_x$, where $x$ is a singular point in the support of $Q$, and take a surjection from $Q$ to $R$. We have a diagram

\[
\begin{array}{ccccccc}
0 & \rightarrow & L & \rightarrow & M & \rightarrow & Q & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & L' & \rightarrow & M & \rightarrow & R & \rightarrow & 0 \\
\end{array}
\]

Note that $L'$, $M$ and $R$ satisfy the hypothesis of lemma 3.3.7, then we can find (flat) deformations $M_y$ and $R_y$ parametrized by a curve $Y$ such that for some $y_1 \in Y$ we have that the support of the corresponding sheaf $R_{y_1}$ is a smooth point of $C$. All sheaves and maps can be deformed along. To do this we define $Q_y = M_y/L$ (we have $L \mapsto L' \mapsto M_y$, thus this quotient is well defined). The kernel of $Q_y \rightarrow R_y$ is $L'/L$. Then $Q_y$ is a flat deformation (being the extension of a flat deformation $R_y$ by a constant and hence flat deformation $L'/L$). Then for each $y$ we have a commutative diagram

\[
\begin{array}{ccccccc}
0 & \rightarrow & L & \rightarrow & M_y & \rightarrow & Q_y & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & L' & \rightarrow & M_y & \rightarrow & R_y & \rightarrow & 0 \\
\end{array}
\]

Observe that the length of the part of $Q_{y_1}$ supported in singular points is
$n - 1$, so repeating this process we can deform $Q$ until its support lies in the smooth part of $C$. This finishes the proof of the proposition.

\[\Box\]

### 3.4 Proof of theorem I

In this section we will prove theorem I:

**Proof.** Nonemptyness follows from the fact that the Brill-Noether loci for smooth curves is nonempty, and by upper semicontinuity of $h^0(\cdot)$. By remark 3.0.1 we can assume $r > d - p_a$. We will prove theorem I by applying lemma 3.1.1.

We start with a rank one torsion-free sheaf $A$ corresponding to a point in $W_d$, $d > 0$, $r \geq 0$, with $\rho(r, d) > 0$ (recall that we are assuming $r > d - p_a$). We have $h^0(A)$, $h^1(A) > 0$. As we explained at the beginning of the section 3.3 we call $A'$ its base point free part. Then we take $B$ to be the base point free part of $A''$, and finally define $A''$ to be $B^*$. By lemma 3.3.1, $A''$ and $A'^*$ are rank one locally free sheaves on $C$ generated by global sections. Then by proposition 3.2.5 we find a deformation $A''$ of $A'$ parametrized by some smooth irreducible curve $T$.

The support of $A''$ defines a family of curves $C$ parametrized by the irreducible curve $T$. Note that $C|_t$ is smooth for $t \neq 0$.

By the definition of $A'$ and $A''$ we have exact sequences

\[0 \to A' \to A \to Q \to 0 \tag{3.11}\]

\[0 \to A' \to A'' \to \bar{Q} \to 0 \tag{3.12}\]

with $h^0(A') = h^0(A)$ and $h^1(A'') = h^1(A')$ (lemma 3.3.2). If we look at 3.12 we see that we are in the situation of proposition 3.3.4, with $L = A'$, $M = A''$, $M = A''$, $P = T$. Then we get a family $A'$ (parametrized by some connected but in general not irreducible curve). Now we use this family $A'$ and the sequence 3.11 to apply 3.3.6 with $L = A'$, $M = A'$ and $L = A'$. We get a new family $A$. We denote by $T'$ the curve parametrizing the family $A$.

This family satisfies all the hypothesis of lemma 3.1.1 (item (iv) is given by proposition 3.3.3), and then theorem I is proved.

\[\Box\]
Chapter 4

Irreducibility of the moduli space for $K3$ surfaces

In this chapter we will prove the following theorem:

**Theorem II.** With the notation of chapter 3, if $L$ is a primitive nonzero element of Pic($S$), and $H$ is an ($L, c_2$)-generic polarization, then $\mathfrak{M}_H(L, c_2)$ is irreducible.

Due to the fact that the moduli space is smooth, irreducibility is equivalent to connectedness.

**Outline of the proof of theorem II**

First we will prove the theorem for the case in which Pic($S$) = $\mathbb{Z}$. For $H$ to be ($L, c_2$)-generic we need $L$ to be an odd multiple of a generator of Pic($S$), and tensoring the vector bundles with a line bundle we can assume that $H = L$ is a generator of Pic($S$). After proving the theorem for this case, in section 4.1 we show, by considering families of surfaces, that if the result is true for Pic($S$) = $\mathbb{Z}$, then it is also true under the conditions of the theorem (this part is very similar to an argument in [G-H]). From now on we will assume that Pic($S$) = $\mathbb{Z}$ and that $H = L$ is the ample generator.

The proof is divided into two parts. In section 4.2 we handle the case in which $c_2 \leq \frac{1}{2}L^2 + 3$. First we see (proposition 4.2.1) that the sheaves satisfying this inequality are exactly those which are nonsplit extensions of the form

$$0 \to \mathcal{O}_S \to V \to L \otimes I_Z \to 0,$$
with \( l(Z) = c_2 \). Then we study the set \( X \subseteq \text{Hilb}^{c_2}(S) \) for which there exist non-split extensions like these above, and we see, using theorem I, that it is connected (proposition 4.2.3). Finally we use this to prove (proposition 1.2.4) the connectedness of \( \mathcal{M}(L, c_2) \) for \( \dim \mathcal{M}(L, c_2) > 0 \) (if the dimension is zero the result is known \([M]\)).

Note that for \( c_2 = \frac{1}{2}L^2 + 3 \) we have \( \dim \mathcal{M}(L, c_2) = L^2 + 6 > 0 \), and then we can continue the proof by induction on \( c_2 \).

Let \( C(n) \) be the set of irreducible components of \( \mathcal{M}(L, n) \). We construct a map

\[
\Phi_n : C(n) \to C(n + 1).
\]

To define this map, take a sheaf \( E \) in a component \( A \) of \( \mathcal{M}(L, n) \). Take a point \( p \in S \) and a surjection \( E \to \mathcal{O}_p \). Let \( F \) be the kernel

\[
0 \to F \to E \to \mathcal{O}_p \to 0.
\]

\( F \) is clearly stable, and \( c_2(F) = c_2(E) + 1 \). Now we define \( \Phi_n(A) \) to be the component in which \( F \) lies. It is easy to see that this is independent of all the choices made, so that \( \Phi_n \) is well defined.

Now we assume that \( \mathcal{M}(L, c_2) \) is irreducible for \( c_2 < n \). We are going to see that if every connected component of \( \mathcal{M}(L, n) \) has a non-locally free sheaf \( F \), \( \Phi_{n-1} \) is surjective, and then by induction \( \mathcal{M}(L, n) \) will be irreducible.

Let \( B \) be a component of \( \mathcal{M}(L, n) \) with non-locally free sheaves. By lemma 1.3.3, it has a non-locally free sheaf \( F \) such that \( F^{\vee \vee} \in \mathcal{M}(L, n - 1) \). By smoothness of the moduli space, \( F^{\vee \vee} \) is in only one irreducible component. Call this component \( A \). By construction \( \Phi_{n-1}(A) = B \).

In other words, we have seen that if \( \mathcal{M}(L, n - 1) \) is irreducible, then there is only one component \( \mathcal{M}_0 \) of \( \mathcal{M}(L, n) \) that has sheaves that are not locally free, and then to prove that the later has only one component, it will be enough to check that every component has a non-locally free sheaf.

We divide the possible values of \( c_2 \) in regions labeled by \( n \geq 1 \), with \( c_2 \) satisfying

\[
((n - 1)^2 + (n - 1) + \frac{1}{2})L^2 + 3 < c_2 \leq (n^2 + n + \frac{1}{2})L^2 + 3.
\]

If \( V \) is locally free, we prove that then \( V \) fits in a short exact sequence

\[
0 \to L^{\otimes -m} \to V \to L^{\otimes m + 1} \otimes I_Z \to 0
\]

with \( 0 \leq m \leq n \) (proposition 1.3.3). We call it an extension of type \( m \). We will also say that \( V \) is of type \( m \).
Next (proposition 4.3.3) we show that the set of sheaves that are not locally free has positive codimension, and then we prove (proposition 4.3.4) that the generic sheaf is a vector bundle of type $n$.

But this is not enough, and we need more information about the generic vector bundle. Let $C$ be the set of vector bundles $V$ such that for any exact sequence

$$0 \to L^{-n} \to V \to L^{\otimes n+1} \otimes I_{Z_n} \to 0,$$

$L^{\otimes n+1} \otimes I_{Z_n}$ has no sections whose zero locus is an irreducible reduced curve. In proposition 4.3.7 we prove that this set has positive codimension. The reason to look at this set is because it is precisely because of these sheaves that we cannot apply the generalization of Fulton-Lazarsfeld’s theorem to proof that the set of type $n$ vector bundles is connected. But now we know that we can ignore $C$, because it has positive codimension, and then conclude that the generic vector bundle $V$ sits in an extension like (4.1) such that $L^{\otimes 2n+1} \otimes I_{Z_n}$ has a section whose zero locus is an irreducible reduced curve. In proposition 4.3.6 we prove that those vector bundles make a connected set. We will need the induction hypothesis to prove this proposition.

### 4.1 Preliminaries

In this section we will prove some propositions that will be useful later.

**Lemma 4.1.1** Let $S$ be a smooth surface and $C$ a smooth (not necessarily complete) curve. Let $p$ be a point in the curve and $j : S \hookrightarrow S \times C$ the corresponding injection. Let $L$ be a line bundle on $S$ and $I_W$ an ideal sheaf on $S$ corresponding to a subscheme of dimension zero. Let $\mathcal{V}$ be a family of rank two sheaves on $S$, i.e. a sheaf on $S \times C$ flat over $C$. If we have the following elementary transformation:

$$0 \to \mathcal{W} \to \mathcal{V} \to j_*(L \otimes I_W) \to 0$$

then $\mathcal{W}$ is a flat family of rank two sheaves on $S$, and furthermore

$$c_i(\mathcal{W}_{p'}) = c_i(\mathcal{V}_{p'})$$

for $i = 1, 2$ and $p'$ any point of $C$. 


Proof. We calculate the Chern classes of $j_*(L \otimes I_W)$ by the Grothendieck-Riemann-Roch theorem, and then the classes of $W$ by Whitney’s formula. The fact that $W$ is flat is proved in [4].

Now we will apply this lemma to take limits of stable extensions. Let $S$ be a smooth surface with $\text{Pic}(S) = \mathbb{Z}$. Consider a family of extensions parametrized by a curve $T$

$$0 \to L^{\otimes -n} \to V_t \to L^{\otimes n+1} \otimes I_Z \to 0,$$

where $L$ is a generator of $\text{Pic}(S)$, $t \in T$, and $Z$ is a subscheme of dimension zero. Assume that $V_t$ is stable for $t \neq 0$, where 0 is some fixed point of T, and unstable for $t = 0$. This defines a map $\varphi : T - \{0\} \to \mathfrak{M}$ to the moduli space of stable sheaves. By properness of $\mathfrak{M}$, this can be extended to a map $\varphi : T \to \mathfrak{M}$, i.e. we can take the limit of the family as $t$ goes to 0 and we obtain a stable sheaf corresponding to $\varphi(0)$.

**Proposition 4.1.2** The stable sheaf $V'$ corresponding to $\varphi(0)$ is not locally free or can be written as an extension

$$0 \to L^{\otimes -m} \to V' \to L^{\otimes m+1} \otimes I_{Z'} \to 0 \quad (4.2)$$

with $m < n$.

Proof. $V_0$ is unstable, so we have

$$0 \to L^{\otimes a} \otimes I_W \to V_0 \to L^{\otimes 1-a} \otimes I_W' \to 0$$

with $0 < a \leq n$. Consider the elementary transformation

$$0 \to W \to V \to j_*[L^{\otimes 1-a} \otimes I_W'] \to 0.$$

By lemma 4.1.1 we have a new family $W$. By standard arguments the member $W_0$ of the new family corresponding to $t = 0$ can be written as an extension

$$0 \to L^{\otimes 1-a} \otimes I_W \to W_0 \to L^{\otimes a} \otimes I_W \to 0. \quad (4.3)$$

Note that $0 \geq 1 - a > -n$. If $W_0$ is not stable, repeat the process: unstability gives an injective map $L^{\otimes a'} \otimes I_{W''} \to W_0$, and by 4.1.3 we have $a' < a$. Then in each step $1 - a$ grows. We are going to see that eventually we are going to get a stable sheaf. Assume we reach $1 - a = 0$ and $W_0$ is still unstable.
The destabilizing sheaf has to be $L \otimes I_{Z_d}$ with $l(Z_d) > l(W)$ and gives a short exact sequence

$$0 \to L \otimes I_{Z_d} \to W_0 \to I'_{Z_d} \to 0.$$  

Note that $l(Z'_d) < l(W')$. Performing the corresponding elementary transformation we get a new family $\overline{W}$ and the sheaf corresponding to 0 sits in an exact sequence

$$0 \to I_{Z'_d} \to W_0 \to L \otimes I_{Z_d} \to 0. \quad (4.4)$$

This sequence is like (4.3) but with $0 \leq l(Z'_d) < l(W')$. If we still don’t get a stable sheaf, repeat this. In each step $l(Z'_d)$ decreases, but this must stop because if $l(Z'_d) = 0$, the sheaf given by (4.4) is stable, as the following lemma shows.

Now, once we have obtained a stable sheaf, if it is not locally free, we are done. If it is locally free, then necessarily the subscheme $W'$ is empty, and we get an extension like (4.2) as desired. 

\[\Box\]

**Lemma 4.1.3** Let $V$ be a torsion free sheaf on a surface $S$ with $\text{Pic}(S) = \mathbb{Z}$, given by an extension

$$0 \to \mathcal{O}_S \to V \to L \otimes I_Z \to 0,$$

where $L$ is the effective generator of $\text{Pic}(S)$. Then $V$ is stable.

**Proof.** A destabilizing subsheaf should be of the form $L^m \otimes I_W$, with $m > 0$. By standard arguments, it is enough to check stability with subsheaves whose quotients are torsion free, so we can assume this.

The composition $L^m \otimes I_W \to V \to L \otimes I_Z$ is nonzero, because otherwise it would factor through $\mathcal{O}_S$, but this is impossible because $m > 0$. Then $m = 1$ and we have $I_W \hookrightarrow I_Z$. Furthermore, $l(W) > l(Z)$ because if $W = Z$, the sequence would split.

Then we have a sequence

$$0 \to L \otimes I_W \to V \to I'_{W'} \to 0,$$

but we reach a contradiction because $c_2 = l(W') + l(W') > l(Z) + l(W') = c_2 + l(W')$. Then there is no destabilizing subsheaf, and $V$ is stable. \[\Box\]
Proposition 4.1.4 Let $S$ be a smooth K3 surface with Picard group $\text{Pic}(S) = \mathbb{Z}$. If $\dim \text{Ext}^1(L' \otimes I_Z, L) \geq 2$, then there is a nonsplit extension

$$0 \to L \to V \to L' \otimes I_Z \to 0$$

such that $V$ is not locally free.

Proof. We have an exact sequence

$$0 \to H^1(L \otimes (L')^{-1}) \to \text{Ext}^1(L' \otimes I_Z, L) \to H^0(\mathcal{O}_Z).$$

If $L = L'$, then $H^1(L \otimes (L')^{-1}) = 0$ because $S$ is a K3 surface. If $L \neq L'$, then due to the fact that $\text{Pic}(S) = \mathbb{Z}$, applying Kodaira’s vanishing theorem we also have $H^1(L \otimes (L')^{-1}) = 0$. We have then an injection

$$0 \to \text{Ext}^1(L' \otimes I_Z, L) \to H^0(\mathcal{O}_Z).$$

An extension corresponding to $\xi$ is locally free iff the section $f(\xi)$ generates the sheaf $\mathcal{O}_Z$, i.e., iff

$$f(\xi) \notin W = \{s \in H^0(\mathcal{O}_Z) : 0 = s \otimes k(p) \in H^0(\mathcal{O}_p) \text{ for some } p \in \text{Supp}(Z)\}.$$

$W$ is a union of codimension 1 linear subspaces, hence if $\dim \text{Ext}^1(L' \otimes I_Z, L) \geq 2$, then $\dim \text{im}(f) \cap W > 0$, and we have a nonzero $\xi$ corresponding to an extension 4.5 with $V$ not locally free. $\square$

Usually we will apply the following corollary

Corollary 4.1.5 Let $S$ be a smooth K3 surface with $\text{Pic}(S) = \mathbb{Z}$. If

$$\dim \text{Ext}^1(L^\otimes n+1 \otimes I_Z, L^\otimes -n) \geq 2,$$

and there is a stable extension

$$0 \to L^\otimes -n \to V \to L^\otimes n+1 \otimes I_Z \to 0,$$

then there is a sheaf $V'$, in the same irreducible component of $\mathfrak{M}(L, c_2)$ as $V$, that is not locally free or sits in an extension

$$0 \to L^\otimes -m \to V \to L^\otimes m+1 \otimes I_Z \to 0$$

for some $m < n$. 

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Proof. There is an open set in $\mathbb{P}(\text{Ext}^1(L \otimes I_Z, O_S))$ whose points correspond to stable extensions, due to the openness of the stability condition. All these points get mapped to the same irreducible component of $\mathcal{M}(L, c_2)$. By proposition 4.1.4, there is an extension $V$ that is not locally free. If it is not stable, we can take a curve as in proposition 4.1.2, and applying the proposition we get a family of stable sheaves. All get mapped to the same component of $\mathcal{M}(L, c_2)$, and the sheaf corresponding to $t = 0$ has the required properties. 

4.2 Small second Chern class

In this section we will consider the case in which $c_2 \leq \frac{1}{2}L^2 + 3$. Recall that we are assuming that $S$ is a K3 surface with $\text{Pic}(S) = \mathbb{Z}$. In this case we have the following characterization of the stable torsion free sheaves.

**Proposition 4.2.1** Let $V$ be a torsion free stable rank two sheaf with $c_1 = L$, $c_2 \leq \frac{1}{2}L^2 + 3$. Then $V$ fits in an exact sequence

$$0 \rightarrow O_S \rightarrow V \rightarrow L \otimes I_Z \rightarrow 0.$$ 

(4.6)

Conversely, every nonsplit extension of $L \otimes I_Z$ by $O_S$ is a torsion free stable sheaf.

**Proof.** Take $V$ stable. Using the Riemann-Roch theorem,

$$h^0(V) + h^2(V) \geq \frac{L^2}{2} - c_2 + 4 \geq 1.$$ 

If $h^2(V)$ were different from zero, by Serre duality we would have $\text{Hom}(V, O) \neq 0$, contradicting stability because this would give a map $V \rightarrow O_S$ with image $L^{\otimes -n} \otimes I_Z$ ($n \geq 0$) and kernel $L^{\otimes n+1} \otimes I_Z$. Then $h^0(V) \neq 0$. Take a section of $V$. By stability, the quotient of the section is torsion free, and we have an extension like (4.6). The extension is not split because $V$ is stable.

The converse is lemma 4.1.3. \square

Now that we know that all sheaves can be written as extensions of $L \otimes I_Z$ by $O_S$, the obvious strategy is to construct families of extensions $\mathbb{P}(\text{Ext}^1(L \otimes I_Z, O_S))$ for each $Z$ such that $\dim \text{Ext}^1(L \otimes I_Z, O_S) \geq 1$. Ideally we would like to put all these families together in a bigger family parametrized by a variety $M$. This $M$ would map to $\mathcal{M}(L, c_2)$ surjectively, so it would be
enough to prove the connectedness of $M$, and because $M$ maps to \Hilb^c(S) with connected fibers, it would be enough to prove that the set $X = \{ Z \in \Hilb^c(S) : \dim \Ext^1(L \otimes I_Z, \mathcal{O}_S) \geq 1 \}$ (i.e., the image of the map $M \to \Hilb^c(S)$) is connected.

Unfortunately we cannot construct $M$ because $\dim \Ext^1(L \otimes I_Z, \mathcal{O}_S)$ is not constant. We will use a somewhat more elaborate argument to bypass this difficulty, but we will still use the connectivity of $X$, that we prove in the following proposition.

**Proposition 4.2.2** The set $X = \{ Z \in \Hilb^c(S) : \dim \Ext^1(L \otimes I_Z, \mathcal{O}_S) \geq 1 \}$ is connected.

**Proof.** By Serre duality and looking at the sequence

$$0 \to H^0(L \otimes I_Z) \to H^0(L) \to H^0(\mathcal{O}_Z) \to H^1(L \otimes I_Z) \to 0,$$

we have $\dim \Ext^1(L \otimes I_Z, \mathcal{O}_S) \geq 1 \iff h^0(L \otimes I_Z) \geq \frac{1}{2} L^2 + 3 - c_2$. Now consider the following commutative diagram

\[
\begin{array}{ccccccccc}
0 & & 0 & & \\
\downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_S & \longrightarrow & L \otimes I_Z & \longrightarrow & j_*(\omega_C \otimes I_Z) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_S & \longrightarrow & L & \longrightarrow & L|_C = j_*\omega_C & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{O}_Z & \longrightarrow & \mathcal{O}_Z & \longrightarrow & 0 & \longrightarrow & 0 \\
\end{array}
\]

where $C \in \mathbb{P}(H^0(L \otimes I_Z))$ (maybe $C$ is singular, but we know it is irreducible and reduced because $\text{Pic}(S) = \mathbb{Z}$ and $L$ is a generator of the group), $j : C \hookrightarrow S$ is the inclusion, and $\omega_C = L|_C$ is the dualizing sheaf on $C$.

Using the top row we get $h^0(L \otimes I_Z) \geq \frac{1}{2} L^2 + 3 - c_2 \iff h^0(\omega_C \otimes I_Z) \geq \frac{1}{2} L^2 + 2 - c_2$. This condition can be restated in terms of Brill-Noether sets $W_d^r$:

$$\omega_C \otimes I_Z \in W_d^r$$

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where \( r = \frac{1}{2}L^2 + 1 - c_2 \), and \( d = L^2 - c_2 \).

By a theorem of Fulton and Lazarsfeld \([F-L]\), the Brill-Noether set \( W^r_d \) of a smooth curve is nonempty and connected if the expected dimension \( \rho(r, d) = g - (r + 1)(g - d + r) \) is greater than zero. In the case of an irreducible reduced curve lying on a K3 the generalized Jacobian can be compactified, and the connectedness result is still true (theorem I). In our case we have

\[
\rho(r, d) = 2c_2 - \frac{L^2}{2} - 3 = \dim \mathcal{M}(L, c_2) > 0,
\]

(recall that for \( \dim \mathcal{M}(L, c_2) = 0 \) the irreducibility of the moduli space is known by the work of Mukai \([M]\) and we can apply the theorem. Now consider the variety

\[
N = \{(Z, C) : Z \subset C, \dim \text{Ext}^1(L \otimes I_Z, \mathcal{O}_S) \geq 1\} \subset \text{Hilb}^{c_2}(S) \times \mathbb{P}(H^0(L))
\]

and the projections

\[
\begin{array}{ccc}
N & \xrightarrow{p_2} & \mathbb{P}(H^0(L)) \\
\downarrow{p_1} & & \downarrow{p_1} \\
\text{Hilb}^{c_2}(S) & & \text{Hilb}^{c_2}(S)
\end{array}
\]

By theorem I, \( p_2 \) is surjective with connected fibers. Then \( N \) is connected, and also the image of \( p_1 \), that is equal to \( X \).

For the following proposition we will need this lemma:

**Lemma 4.2.3** Let \( T \) be a smooth curve, \( p \) a point in \( T \) and \( S \) a variety. Consider the diagram

\[
\begin{array}{ccc}
S & \xrightarrow{v} & S \times T \\
\downarrow{g} & & \downarrow{f} \\
p & \xrightarrow{u} & T
\end{array}
\]

Then for every coherent torsion free sheaf \( \mathcal{F} \) on \( S \times T \), there exist a natural map

\[
(f_* \mathcal{F})(p) \rightarrow H^0(\mathcal{F}_{S \times \{p\}}),
\]

where \( \mathcal{F}(p) = v^* \mathcal{F} \). Furthermore, this map is injective.
Proof. The question is local in $T$, so we can assume that $T$ is affine, $T = \text{Spec } A$, and there is an element $x \in A$ such that $p$ is the zero locus of $x$. We have

$$0 \to \mathcal{F} \xrightarrow{x} \mathcal{F} \to \mathcal{F}/x \cdot \mathcal{F} \to 0,$$

where the map on the left is multiplication by $f^*x$. On the other hand we have the sequence

$$0 \to \mathcal{O}_{S \times T}(-f^*p) \to \mathcal{O}_{S \times T} \to \mathcal{O}_{S \times \{p\}} \to 0. $$

Tensoring with $\mathcal{F}$ is right exact, so we get an exact sequence

$$\mathcal{F} \otimes \mathcal{O}_{S \times T}(-f^*p) \to \mathcal{F} \to \mathcal{F}_p \to 0.$$

Note that the image of the left map is $x \cdot \mathcal{F}$, and then we conclude that $\mathcal{F}/x \cdot \mathcal{F}$ is isomorphic to $\mathcal{F}_p$. Taking cohomology in the sequence (4.7) we get

$$H^0(\mathcal{F})/(x \cdot H^0(\mathcal{F})) \hookrightarrow H^0(\mathcal{F}_p),$$

but the first group is exactly $(f_*\mathcal{F})|_p$. □

Finally we can prove:

**Proposition 4.2.4** The moduli space $M(L, c_2)$ of torsion free, rank two sheaves with $c_2 \leq \frac{1}{2}L^2 + 3$ over a K3 surface with $\text{Pic}(S) = \mathbb{Z}$ is connected (hence irreducible, because we know it is smooth).

Proof. We have a stratification of $X$

$$X = \bigcup_{r \geq 1} H_r, H_r = \{Z \in \text{Hilb}^{c_2}(S) : \dim \text{Ext}^1(L \otimes I_Z, \mathcal{O}_S) = r\}.$$

On each stratum $H_r$ we can construct a projective bundle $M_r \to H_r$ with fiber $\mathbb{P}((\text{Ext}^1(L \otimes I_Z, \mathcal{O}_S)))$, because the dimension of the group is constant. Each point of $M_r$ corresponds to an extension (up to weak isomorphism of extensions). We have then morphisms $M_r \to \mathcal{M}(L, c_2)$ with fiber $\mathbb{P}(H^0(V))$ over $V$ (see proposition 4.3.8). We have $h^0(V) = \frac{1}{2}L^2 + 3 - c_2 + h^1(L \otimes I_Z)$, and a corresponding stratification of $\mathcal{M}(L, c_2)$

$$\mathcal{M}(L, c_2) = \bigcup_{r \geq 1} \mathcal{M}_r, \quad \mathcal{M}_r = \{V \in \mathcal{M}(L, c_2) : h^0(V) = \frac{1}{2}L^2 + 3 - c_2 + r\}$$

(the reason for this dependence on $r$ in the definition is that $h^0(V) = \frac{1}{2}L^2 + 3 - c_2 + h^1(L \otimes I_Z)$). The condition is equivalent to requiring that if $V \in$
\( \mathcal{M}(L, c_2) \) and \( V \) is an extension of \( L \otimes I_Z \) by \( \mathcal{O}_S \), then \( h^1(L \otimes I_Z) = r \). The previous formula proves that \( r \) only depends on \( V \).

Note that \( M_r \) can be thought also as a projective bundle over \( \mathcal{M}_r \) with fiber \( \mathbb{P}(H^0(V)) \).

To prove that \( \mathcal{M}(L, c_2) \) is connected, we will show that for any two sheaves \( V_a \in \mathcal{M}_a, V_b \in \mathcal{M}_b \), we can construct a family \( \mathcal{V} \) of stable sheaves on a connected parameter space, with \( \mathcal{V}|_0 = V_a, \mathcal{V}|_1 = V_b \).

Due to the fact that \( X \) is connected, it is enough to prove this for \( V_a, V_b \) given by extensions

\[
0 \to \mathcal{O}_S \xrightarrow{s_a} V_a \to L \otimes I_{Z_a} \to 0
\]

\[
0 \to \mathcal{O}_S \xrightarrow{s_b} V_b \to L \otimes I_{Z_b} \to 0
\]

such that \( Z_a \in H_a, Z_b \in H_b \) and \( Z_a \in \overline{\mathcal{M}}_b \), the closure of \( H_b \).

Take a curve \( f : T \to \text{Hilb}^2(S) \) with \( f(0) = Z_a, f(1) = Z_b \) and \( \text{im}(T - \{0\}) \in H_b. T \) doesn’t need to be complete. By shrinking \( T \) to a smaller open set, we can assume that there is a lift \( f \) to a map \( f : T \setminus \{0\} \to M_b \). This gives a family of sheaves \( \mathcal{V} \) and sections \( s_t \in H^0(\mathcal{V}|_t) \) parametrized by \( T \setminus \{0\} \).

We want to extend this to a family \( \mathcal{V} \) of sheaves and sections parametrized by \( T \), in such a way that the cokernel of \( s_0 : \mathcal{O}_S \to \mathcal{V}|_0 \) is \( L \otimes I_{Z_a} \). Maybe \( \mathcal{V}|_0 \) won’t be isomorphic to \( V_0 \), but at least both are extensions of the same sheaf \( L \otimes I_{Z_a} \) by \( \mathcal{O}_S \), and then they are in the same connected component, and this is enough.

This family gives a morphism \( T - \{0\} \to \mathcal{M}(L, c_2) \) that extends to a unique \( T \to \mathcal{M}(L, c_2) \) by properness (see section [4.1] just before proposition [4.1.2]). With this we have already extended the family \( \tilde{\mathcal{V}} \) to a family \( \mathcal{V} \) parametrized by \( T \), and we only need to extend the sections \( s_t \). Shrinking \( T \) to a smaller neighborhood of \( \{0\} \) if necessary, we can assume that \( \pi_{2a} \mathcal{V} \) is trivial \((\pi_1 \text{ and } \pi_2 \text{ are the first and second projections of } S \times T) \). Then the sections \( s_t \) fit together to give \( \mathcal{O}_{S \times (T - \{0\})} \to \tilde{\mathcal{V}}, \text{i.e. an element } s \in (\pi_{2a} \mathcal{V})(T - \{0\}) \). This can be extended to some \( s \in (\pi_{2a} \mathcal{V})(T) \) that is nonzero on the fiber of \( t = 0 \).

Using the previous lemma, we have an injection \( (\pi_{2a} \mathcal{V})|_{t=0} \to H^0(\mathcal{V}|_0) \) that gives a nonzero section \( s_0 \) of \( \mathcal{V}|_0 \). Now we only have to check that the cokernel of this section is \( L \otimes I_{Z_a} \). We have a short exact sequence over \( S \times T \)

\[
0 \to \mathcal{O}_{S \times T} \xrightarrow{s} \mathcal{V} \to \mathcal{Q} \to 0.
\]

Then \( \mathcal{Q} \) is torsion free, flat over \( T \), and then it is of the form

\[
\mathcal{Q} = \pi_1^*(L) \otimes I_Z \otimes \pi_2^*(L')
\]
for some line bundle $L'$ over $T$ and some subscheme $Z$ of $S \times T$ flat over $T$. This subscheme gives a morphism $g : T \to \text{Hilb}^{c_2}(S)$. By construction $f(t) = g(t)$ for $t \neq 0$, and by properness the equality also holds for $t = 0$, then $Z_0 = Z$ as desired.

\section*{4.3 Large second Chern class}

In this section we will handle the case in which $c_2$ is large.

\begin{proposition}
Assume that $c_2$ satisfies
\[(n-1)^2 + (n-1) + \frac{1}{2}L^2 + 3 < c_2 \leq (n^2 + n + 1)L^2 + 3.\]
If $V$ is locally free, then $V$ fits in a short exact sequence
\[0 \to L^{\otimes -m} \to V \to L^{\otimes m+1} \otimes I_{Z_m} \to 0, l(Z_m) = c_2 + (m+1)mL^2,\]
with $0 \leq m \leq n$. We will call such an exact sequence an extension of type $m$.
\end{proposition}

\begin{proof}
For any sheaf $V$, $h^2(V \otimes L^{\otimes n}) = \dim \text{Hom}(V, L^{\otimes -n}) = 0$ by stability, and then using the Riemann-Roch theorem we have
\[h^0(V \otimes L^{\otimes n}) \geq \frac{L^2}{2} - c_2 + 4 + (n + n^2)L^2 \geq 1,\]
so that we have an inclusion $L^{\otimes -n} \hookrightarrow V$. If $V$ is locally free, this will give an exact sequence on type $m$, with $m \leq n$.
\end{proof}

\begin{proposition}
If a component $\mathfrak{M}'$ has sheaves of type $m$, with $0 \leq m \leq n - 1$, then it has sheaves that are not locally free.
\end{proposition}

\begin{proof}
Choose $m$ such that there is no $V$ of type $m'$ for $m' < m$. Let $V$ be of type $m$. By Serre duality
\[\dim \text{Ext}^1(L^{\otimes m+1} \otimes I_{Z_m}, L^{\otimes -m}) = h^1(L^{\otimes 2m+1} \otimes I_{Z_m}),\]
and this is greater than 2. By proposition 4.1.5, $\mathfrak{M}'$ has a sheaf that is not locally free or is of type $m' < m$, but the later cannot happen because of the choice of $m$.
\end{proof}
Lemma 4.3.3 Let $X$ be an irreducible component of $\mathfrak{M}(L,c_2)$. Let $X_0$ be the subset corresponding to non-locally free sheaves. If $X_0$ is not empty, then it has codimension one. Furthermore, there is a dense subset $Y$ of $X_0$ such that for any $F \in Y$, we have

$$c_2(F^\vee) = c_2(F) + 1.$$ 

Proof. By [O1], prop. 7.1.3, we know that

$$\text{codim}(X_0, X) \leq 1. \tag{4.9}$$

On the other hand, let $F \in X_0$. It fits in an exact sequence

$$0 \to F \to F^\vee \to Q \to 0$$

where $Q$ is an Artinian sheaf with length $l = H^0(Q) = c_2(F^\vee) - c_2(F)$. We use this to bound the dimension of $X_0$ by a parameter count. First we choose a locally free sheaf $E \in \mathfrak{M}(L, c_2 - l)$. These requires $4(c_2 - l) - L^2 - 6$ parameters. Now we have to choose a quotient to a sheaf of length $l$ concentrated on a subset of dimension zero. These quotients are parametrized by the Grothendieck Quot scheme $\text{Quot}(E, l)$, whose dimension is $3l$ (this follows from [L1] Appendix, where it is proved that $\text{Quot}^0(E, l)$, the Quot scheme corresponding to quotients supported in $l$ distinct points, is dense in $\text{Quot}(E, l)$).

Define the following stratification on $X_0$:

$$X_0 = \bigcup_{l \geq 1} X_0^l, \quad X_0^l = \{ F : c_2(F^\vee) - c_2(F) = l \}$$

Then we have

$$\dim(X_0^l) \leq 4(c_2 - l) - L^2 - 6 + 3l$$

and together with the previous bound 4.9 we obtain that $Y = X_0^1$ is dense and $\text{codim}(X_0, X) = 1$. □

Proposition 4.3.4 If $c_2$ satisfies the inequalities of the hypothesis of proposition 4.3.1, then there is an open dense set on $\mathfrak{M}(L, c_2)$ that corresponds to sheaves of type $n$.

Proof. We will prove this by showing that the codimension of sheaves of type $m \leq n - 1$ is greater than zero.

We will divide the proof into two cases:

Case 1. Extensions of type $m$ with $Z_m$ such that $h^0(L^\otimes m + 1 \otimes I_{Z_m}) = 0$. 

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We have then $h^1(L^\otimes 2m + 1 \otimes I_Z) = c_2 - (m^2 + m + \frac{1}{2})L^2 - 2$. The dimension of the family $M$ of extensions of this kind is bounded (via Serre duality) by

$$\dim M \leq 2l(Z_m) + h^1(L^\otimes 2m + 1 \otimes I_Z) - 1 = 3c_2 + (m^2 + m - \frac{1}{2})L^2 - 3.$$ 

There is a map $\pi : M \to \mathfrak{M}(L,c_2)$ with fiber over each $V$ equal to $P(H^0(V \otimes L \otimes n))$, and we can give a bound to its dimension (see proof of proposition 4.3.1).

$$\dim P(H^0(V \otimes L \otimes n)) \geq (n^2 + n + \frac{1}{2})L^2 - c_2 + 3$$

Then the dimension of the image of $\pi$ is bounded by

$$\dim(\text{im}\, \pi) \leq \dim(M) - \min \dim P(H^0(V \otimes L \otimes n)) \leq 4c_2 - L^2 - 6 - 2nL^2$$

And then

$$\text{codim}(\text{im}\, \pi) \geq 2nL^2 > 0$$

**Case 2.** Extensions of type $m$ with $Z_m$ such that $h^0(L^\otimes 2m + 1 \otimes I_Z) \geq 1$.

Let $M$ be the family of stable extensions with $Z_m$ satisfying this inequality. Let $l = l(Z_m) = c_2 + (m + 1)\sqrt{m}L$. The subscheme $Z_m$ is in a curve $C$ defined as the zeroes of a section of $L^\otimes 2m + 1$, i.e. $Z_m \in \text{Hilb}^1(C)$. Although this curve will be reducible and not reduced in general, the fact that $C$ is in a smooth surface allows us to prove:

**Lemma 4.3.5** $\dim \text{Hilb}^d(C) = d$

**Proof.** We have a natural stratification of $\text{Hilb}^d(C)$ given by the number of points in the support of a subscheme

$$\text{Hilb}^d(C) = \bigcup_{1 \leq r \leq d} \text{Hilb}^d_r(C),$$

where $\text{Hilb}^d_r(C) = \{Z : \# \text{Supp } Z = r\}$. We have natural maps giving the support of a subscheme:

$$z^C_r : \text{Hilb}^d_r(C) \to \text{Sym}^r(C).$$

In the same way we define maps $z^S_r$, when we consider subschemes of the surface $S$. Clearly, if $x \in \text{Sym}^r(C) \subset \text{Sym}^r(S)$, then $(z^S_r)^{-1}(x) \subset$
Taking this into account, and using a result of Iarrobino about zero dimensional subschemes of a smooth surface:

\[
\dim \text{Hilb}^d(C) \leq \dim \text{Sym}^r(C) + \max \dim (z^S_r)^{-1}(x)
\]

\[
\leq \dim \text{Sym}^r(C) + \max \dim (z^S_r)^{-1}(x) = r + d - r = d
\]

This gives \(\dim \text{Hilb}^d(C) \leq d\), and the opposite direction is trivial. This finishes the proof the lemma.

Now we are going to bound the dimension of the set

\[
H' = \{ Z \in \text{Hilb}^l(S) : h^0(L^{\otimes 2m+1} \otimes I_Z) \geq 1 \}
\]

Consider the diagram

\[
Y = \{(Z, C) \in \text{Hilb}^l \times \mathbb{P}(H^0(L^{\otimes 2m+1}) : Z \in C) \xrightarrow{p_2} \mathbb{P}(H^0(L^{\otimes 2m+1}))
\]

\[
p_1 \downarrow \\
\text{Hilb}^l(S)
\]

We have \(H' = \text{im}(p_1)\) and the fiber of \(p_2\) is \(\text{Hilb}^l(C)\). \(p_2\) is clearly surjective and the fiber of \(p_1\) is \(\mathbb{P}(H^0(L^{\otimes 2m+1} \otimes I_Z))\). Then

\[
\dim H' = \dim \mathbb{P}(H^0(L^{\otimes 2m+1}))+\dim \text{Hilb}^l(C) - \dim \mathbb{P}(H^0(L^{\otimes 2m+1} \otimes I_Z)) = 2l - 1.
\]

Again we have a map \(\pi : M \rightarrow \mathcal{M}(L, c_2)\) with fiber \(\mathbb{P}(H^0(V \otimes L^{\otimes m}))\), and then

\[
\text{codim}(\text{im} \pi) \geq (2m + 1)L^2 + 1 > 0.
\]

As a corollary to this proposition we learn that to prove connectedness of \(\mathcal{M}(L, c_2)\) it is enough to prove that all type \(n\) sheaves are in one component.

**Proposition 4.3.6** All stable extensions of \(L^{\otimes n+1} \otimes I_Z\) by \(L^{\otimes -n}\) such that \(L^{\otimes 2n+1} \otimes I_Z\) has a section corresponding to an integral curve, are in one component.
Proof. Define the sets

\[
\tilde{X}_r = \{ Z \in \text{Hilb}^{c_2+n(n+1)L^2}(S) : \dim \text{Ext}^1(L^{\otimes n+1} \otimes I_Z, L^{\otimes -n}) = r \text{ and } L^{\otimes 2n+1} \otimes I_Z \text{ has a section corresponding to an integral curve} \}
\]

\[
X_r = \{ Z \in \tilde{X}_r : \text{there is a stable extension of } L^{\otimes n+1} \otimes I_Z \text{ by } L^{\otimes -n} \}
\]

\[
\tilde{M}_r = \{ \text{extensions of } L^{\otimes n+1} \otimes I_Z \text{ by } L^{\otimes -n} \text{ with } Z \in X_r \}
\]

\[
M_r = \{ m \in M_r : m \text{ corresponds to a stable extension} \}
\]

\[
N_r = \{(Z, C) \in \text{Hilb}^{c_2+n(n+1)L^2}(S) \times \mathbb{P}(H^0(L^{\otimes 2n+1})) : Z \subset C \text{ and } \dim \text{Ext}^1(L^{\otimes n+1} \otimes I_Z, L^{\otimes -n}) = r \}
\]

\[
U = \{ C \in \mathbb{P}(H^0(L^{\otimes 2m+1})) : C \text{ is irreducible and reduced} \}
\]

We construct \( \tilde{M}_r \) as parameter spaces of universal families of extensions by standard techniques. These techniques require that the dimension of the \( \text{Ext}^1 \) group is constant on the whole family. This is why we have to introduce the subscript \( r \) and break everything into pieces according to the dimension of the group \( \text{Ext}^1 \). We also consider the unions

\[
\tilde{M} = \bigcup_{r \geq 1} \tilde{M}_r, \quad M = \bigcup_{r \geq 1} M_r, \quad X = \bigcup_{r \geq 1} X_r, \quad \ldots
\]

Note that \( X \), being a subset of \( \text{Hilb}^{c_2+n(n+1)L^2}(S) \), has a natural scheme structure. This is also true for \( N \subset C \times \mathbb{P}(H^0(L^{\otimes 2n+1})) \). On the other hand, for \( M_r \), there is no natural way of “putting them together”, so we take just the disjoint union. We have the following maps

\[
\begin{array}{ccc}
\tilde{M} & \leftarrow & M \\
\downarrow & & \downarrow \\
\tilde{X} & \leftarrow & X \\
\end{array}
\]

\[
\begin{array}{ccc}
N & \overset{p_2}{\longrightarrow} & \mathbb{P}(H^0(L^{\otimes 2n+1})) \\
\overset{p_1}{\nearrow} & & \\
p_1^{-1}(U) & \rightarrow & U
\end{array}
\]

By construction \( \tilde{X} = p_1p_2^{-1}(U) \). Now we are going to prove that the fibers of \( p_2 \) over \( U \) are nonempty and connected. For each point in \( N \) we
have a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & \mathcal{O}_S & \to & L^\otimes 2n+1 \otimes I_Z & \to & j_*(\omega_C \otimes I_Z) & \to & 0 \\
0 & \to & \mathcal{O}_S & \to & L^\otimes 2n+1 & \to & L^\otimes 2n+1|_C = j_*\omega_C & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{O}_Z & & \mathcal{O}_Z & & \mathcal{O}_Z & & \mathcal{O}_Z & & 0
\end{array}
\]

We argue in the same way as in proposition 4.2.2. Here we have

\[r = (n^2 + n + \frac{1}{2})L^2 + 1 - c_2, \quad d = (3n^2 + 3n + 1)L^2 - c_2\]

\[\rho(r, d) = 2c_2 - \frac{L^2}{2} - 3 = \frac{\dim \mathcal{M}(L, c_2)}{2} > 0.\]

But now we don’t know if \(p_2\) is surjective with connected fibers, because theorem I only applies for irreducible reduced curves. This is the reason why we introduce the open set \(U\). For the fibers on \(U\) we can apply the theorem, and we conclude that \(p_2\) is surjective over \(U\) with connected fibers. Then \(\bar{X} = p_1p_2^{-1}(U)\) is connected.

**Case 1.** \(p_1p_2^{-1}(U) = X\).

If \(X_1 = X\), then we can construct a (connected, because \(X\) is connected) family \(M_1\) parametrizing all sheaves with the required properties, and we are done.

Now, if \(X_1 \neq X\), then there are extensions with \(r \geq 2\). By corollary 1.1.5, \(M_r\) with \(r \geq 2\) is mapped to \(\mathcal{M}_0\), the irreducible component that has sheaves that are not locally free. There is only one irreducible component with this property, because by induction hypothesis the moduli space when the second Chern class is smaller than \(c_2\) is irreducible (see the outline of the proof). Now we have to show that all the connected components of \(M_1\) also go to this component \(\mathcal{M}_0\).

The connectivity of \(X = \bar{X}\) and the fact that \(\dim \text{Ext}^1(L^\otimes n+1 \otimes I_Z, L^\otimes -n)\) is upper semicontinuous allows us to take a curve \(f : T \to X\) with \(f(T - \{0\})\) in any given connected component of \(X_1\) and \(f(0) \in X_r\), with \(r \geq 2\).
Lift $f$ to a map $f : T - \{0\} \to M_1$. Note that $M_1$ won’t be in general a projective bundle because we have removed the points corresponding to unstable extensions, but these make a closed set, and (maybe after restricting $T$ to a smaller open set) we can construct the lift without hitting this set.

$M_1$ maps to $\mathcal{M}(L, c_2)$, and then we have a map $\phi : T - \{0\} \to \mathcal{M}(L, c_2)$. As in the proof of proposition 4.2.4, this gives us a family of stable sheaves and sections parametrized by $T - \{0\}$ that we can extend to a family parametrized by $T$. Now there are two possibilities:

If $\phi(0)$ is of type $n$, then we have a family of extensions

$$0 \to L^\otimes -n \to V_l \to \mathcal{O}_S(L^\otimes n + 1 \otimes I_Z) \to 0$$

and a corresponding map $\psi : T \to \text{Hilb}^{c_2 + n(n+1)L^2}(S), t \mapsto Z_t$. By construction $\psi(t) = f(t)$ for $t \neq 0$, and by properness also for $t = 0$. The extension corresponding to $t = 0$ has to be in $M_r$ with $r \geq 2$, and then $M_1$ is also mapped to $\mathcal{M}_0$.

On the other hand, if $\phi(0)$ is not a vector bundle of type $n$, then it is either of type $m$ for $m < n$ or it is not locally free. In either case, we conclude that $M_1$ is also mapped to $\mathcal{M}_0$.

**Case 2.** $p_1p_2^{-1}(U) \neq X$.

Again, $M_r, r \geq 2$, gets mapped to $\mathcal{M}_0$.

No connected component of $X_1$ can be closed, because by connectedness of $\tilde{X}$ and upper semicontinuity of $\dim \text{Ext}^1(L^\otimes n + 1 \otimes I_Z, L^\otimes -n)$, we would have $\tilde{X} = X_1$, and then $X = \tilde{X}$, contrary to the hypothesis.

Now we can prove that every connected component of $M_1$ gets mapped to $\mathcal{M}_0$. Take the corresponding connected component of $X_1$. Take a curve $f : T \to \tilde{X}$, with $f(T - \{0\})$ in the given connected component of $X_1$, and $f(0) \notin X_1$. As in the previous case, lift $f$ to a map $f : T - \{0\} \to M_1$, and now the proof finishes like the end of case 1.

**Proposition 4.3.7** The set of sheaves $V$ of type $n$ such that for any extension

$$0 \to L^\otimes -n \to V \to L^\otimes n + 1 \otimes I_Z \to 0,$$

$L^\otimes 2n + 1 \otimes I_Z$ has no section whose zero locus is an integral curve, has positive codimension.

**Proof.** Define $\tilde{P} = \{ Z \in \text{Hilb}^{c_2 + n(n+1)L^2}(S) : L^\otimes 2n + 1 \otimes I_Z \text{ has no sections whose zero locus is an irreducible reduced curve } \}$. For each point of $\tilde{P}$ we have a family of extension of type $n$ given by the projectivization of the corresponding $\text{Ext}^1$ group. Writing $\tilde{P} = \cup \tilde{P}_r$, with $r$ equal to the dimension
of the group, we can construct a family of extensions \( \tilde{M}_r \) for each \( r \). As is the previous proposition, let \( P_r \subset \tilde{P}_r \) be the subset that has stable extensions. We have a natural map \( \pi_1 : M_r \rightarrow \mathfrak{M}(L, c_2) \), where \( M_r \) is the subset of \( \tilde{M}_r \) corresponding to stable sheaves.

**Lemma 4.3.8** The fiber of \( \pi_1 \) over \( V \in \mathfrak{M}(L, c_2) \) is \( \mathbb{P}(H^0(V \otimes L^\otimes n)) \).

**Proof.** The fiber consists of all extensions giving the same \( V \). Now, given a point in \( \mathbb{P}(H^0(V \otimes L^\otimes n)) \), we have an injection \( f : L^\otimes n \rightarrow V \) (up to scalar). \( V \) is locally free and of type \( n \), then the quotient is torsion free and we get an element \( Z_f \) of \( \text{Hilb}^{c_2+n(n+1)}L^2(S) \):

\[
0 \rightarrow L^\otimes n \rightarrow V \rightarrow L^\otimes n + 1 \otimes I_{Z_f} \rightarrow 0.
\]

This defines a map from \( \mathbb{P}(H^0(V \otimes L^\otimes n)) \) to the fiber of \( \pi \). It is clearly surjective. Now we will check that it is also injective.

If \( Z_f = Z_{f'} \), then \( f \) and \( f' \) have to differ at most by scalar multiplication, because all nonsplit extensions of \( L^\otimes n + 1 \otimes I_{Z_f} = L^\otimes n + 1 \otimes I_{Z_{f'}} \) by \( L^\otimes n \) that give the same \( V \) are weak isomorphic, so we get a diagram:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & L^\otimes n & \xrightarrow{f} & V & \rightarrow & L^{\otimes n+1} \otimes I_{Z_f} & \rightarrow & 0 \\
\downarrow{\alpha} & & \downarrow{\cong} & & \downarrow{\beta} & & \\
0 & \rightarrow & L^\otimes n & \xrightarrow{f'} & V & \rightarrow & L^{\otimes n+1} \otimes I_{Z_{f'}} & \rightarrow & 0
\end{array}
\]

where \( \alpha \) is multiplication by scalar. \( \square \)

In \( \mathbb{P}(H^0(L^{\otimes 2n+1})) \) we have a subvariety \( Y \) corresponding to reducible curves. This subvariety is the image of the natural map

\[
\bigcup_{0 < a < 2n+1} \mathbb{P}(H^0(L^{\otimes a})) \times \mathbb{P}(H^0(L^{\otimes 2n+1-a})) \rightarrow \mathbb{P}(H^0(L^{\otimes 2n+1}))
\]

We define the set

\[
\tilde{N}_r = \left\{(Z, C) \in X_r \times \mathbb{P}(H^0(L^{\otimes 2n+1})) : Z \subset C, \dim(p_1)^{-1}(Z) = (n^2 + n + \frac{1}{2})L^2 - c_2 + 1 + r\right\}
\]
and the maps

\[
\begin{array}{ccc}
\tilde{N}_t^P & \xrightarrow{p_2} & \mathbb{P}(H^0(L^\otimes 2^{n+1})) \\
\downarrow p_1 & & \downarrow \\
\text{Hilb}^{2^n + n(n+1)L^2}(S) & \xrightarrow{\pi_2} & \mathcal{M}(L,c^2)
\end{array}
\]

By construction we have \( P_r \subset \tilde{P}_r \subset p_2^{-1}(Y) \). Then

\[\dim P_r \leq \dim p_2((p_1)^{-1}(Y)) = \dim Y + \dim \text{fiber}(p_2) - \dim \text{fiber}(p_1),\]

where \( \dim Y \) is the maximum of the dimensions of its irreducible components. Finally

\[\text{codim}(\text{im} \, \pi_1) = \dim \mathcal{M}(L,c^2) - \dim P_r - \dim \text{fiber} \pi_2 + \dim \text{fiber} \pi_1,\]

and putting everything together we have \( \text{codim}(\text{im} \, \pi_1) > (2n - a)(a - 1)L^2 \) for every \( 0 < a < n + 1 \), and then \( \text{codim}(\text{im} \, \pi_1) > 0 \).

\[\square\]

### 4.4 General K3 surface (proof of theorem II)

In this section we finally prove theorem II by showing that if the result is true for a surface \( S \) with \( \text{Pic}(S) = \mathbb{Z} \), then it also holds under the hypothesis of theorem II. The idea is to deform the given surface to a generic surface with \( \text{Pic}(S) = \mathbb{Z} \). We also deform the moduli space, and then the irreducibility of the moduli space for the deformed surface will imply the irreducibility for the surface we started with. This is very similar to an argument in [G-H].

Because we are going to vary the surface, in this section we will denote the moduli space of semistable sheaves with \( \mathcal{M}_H(S,L,c^2) \), where \( S \) is the surface on which the sheaves are defined.

**Proof of theorem II** Recall that we have a surface \( S \) with a \((L,c^2)\)-generic polarization \( H \). By 2.1.1 in [G-H], there is a connected family of surfaces \( S \) parametrized by a curve \( T \) and a line bundle \( L \) on \( S \) such that \((S_0,L_0) = (S,L) \) and \( \text{Pic}(S_t) = L_t \cdot \mathbb{Z} \) for \( t \neq 0 \). By proposition 2.3 in [G-H], there is a connected smooth proper family \( Z \to T \) such that \( Z_0 \cong \mathcal{M}_H(S,L_0,c^2) \) (note that the polarization is \( H \) and not \( L_0 \)) and \( Z_t \cong \mathcal{M}_{L_t}(S_t,L_t,c^2) \) for \( t \neq 0 \). 

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By propositions \[4.2.4\] and \[4.3.4\] we know that \(Z_t\) is irreducible for \(t \neq 0\), and then by an argument parallel to lemma \[3.1.1\] we obtain that \(Z_0\) is connected, but \(Z_0\) is smooth (because \(H\) is generic), and then this implies that \(Z_0\) is irreducible. \(\square\)
Chapter 5

Irreducibility of the moduli space for del Pezzo surfaces

In this chapter we will consider the case in which $S$ is a del Pezzo surface. We will prove the following theorem (see chapter 2 for the notation).

**Theorem III.** Let $S$ be a del Pezzo surface. Let $\mathcal{M}_L(S,c_1,c_2)$ be the moduli space of rank 2, Gieseker $L$-semistable torsion free sheaves with Chern classes $c_1, c_2$, with $L$ a $(c_1,c_2)$-generic polarization. Then $\mathcal{M}_L(S,c_1,c_2)$ is either empty or irreducible.

As we explained in chapter 2, this is already known for $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, so we will assume from now on that $S$ is a surface isomorphic to $\mathbb{P}^2$ blown up at most at 8 points in general position. We denote the blow up map $\pi: S \rightarrow \mathbb{P}^2$. We will denote by $H$ the effective generator of $\text{Pic}(\mathbb{P}^2)$. We also denote by $H$ the pullback $\pi^*(H)$. $\mathcal{M}_L(X,c_1,c_2)$ will denote the moduli space of $L$-semistable torsion free rank two sheaves on $X$ with Chern classes $c_1, c_2$. In the case $X = \mathbb{P}^2$ we will drop $L$ from the notation, because there is only one possible polarization.

This is a particular case of the conjecture of Friedman and Qin [F-Q] that states that the moduli space of stable vector bundles on a rational surface with effective anticanonical line bundle is either empty or irreducible (for any choice of polarization and Chern classes).

**Proposition 5.0.1** $\mathcal{M}_{L_0}(S,c_1,c_2)$ is either irreducible or empty, where $L_0$ is a polarization of $S$ that lies in a chamber whose closure contains $H$.

**Proof.** By [3], $\mathcal{M}_{L_0}(S,bH + a_1E_1 + \cdots + a_nE_n,c_2)$ is birational to a $(\mathbb{P}^1)^m$ bundle over $\mathcal{M}(\mathbb{P}^2,bH,c_2)$ (where $m$ is the number of $a_i$'s that are odd), but it is well known that this moduli space is either irreducible or empty. \qed
To prove this statement for any generic polarization, we will need some lemmas about the following system of equations on integer numbers:

\[
\begin{align*}
  a_1^2 + \cdots + a_8^2 &= x + b^2 \\
  -a_1 - \cdots - a_8 &= x - 2 + 3b
\end{align*}
\] (†)

where \(x\) is some given (integer) number, and \(b, a_1, \ldots, a_8\) are the unknowns.

**Lemma 5.0.2** If \(x \geq 3\) then any integer solution of (†) with \(b > 0\) has \(b \leq 2\).

**Proof.** This is obtained by an elementary argument. We can interpret geometrically these equations as the intersection of a one-sheeted hyperboloid and a plane. We will look first at real solutions. Rewrite (†) as

\[
\begin{align*}
  b &= \frac{1}{3}(-a_1 - \cdots - a_8 + 2 - x) \\
  a_1^2 + \cdots + a_8^2 &= x + \frac{1}{9}(-a_1 - \cdots - a_8 + 2 - x)^2
\end{align*}
\]

The first equation defines a function, and the second equation is a constrain. Using the method of Lagrange multipliers we obtain that the maximum and minimum values of \(b\) are at points of the form \(a_i = t\) for some \(t\). Then (†) become

\[
\begin{align*}
  8t^2 &= b^2 + x \\
  -8t - 3b &= -2 + x
\end{align*}
\]

Looking at the real solutions of these equations, we find

\[
b^\pm = \frac{-6(x - 2) \pm \sqrt{36(x - 2)^2 - 4((x - 2)^2 - 8x)}}{2}. \quad (5.1)
\]

For \(x \geq 3\), \(b^-\) is always negative. If we want to have solutions with \(b > 0\) we need \(b^+ > 0\). Using \([5.1]\), this is equivalent to \((x - 2)^2 < 8x\), and this implies \(x < (12 + \sqrt{128})/2 < 12\). This bound, together with the hypothesis \(3 \leq x\) and \([5.1]\) implies \(b^+ < 3\), but if we are only interested in integer solutions this gives \(b \leq 2\). \(\Box\)

**Lemma 5.0.3** If \(1 \leq b \leq 2\), then the integer solutions of (†) (up to permutation of \(a_i\)) are:

\[
\begin{align*}
  b = 1, \ a_1 = \cdots = a_{x+1} &= -1, \ a_{x+2} = \cdots = a_8 = 0 \\
  b = 2, \ a_1 = \cdots = a_{x+4} &= -1, \ a_{x+5} = \cdots = a_8 = 0
\end{align*}
\]

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Proof. In both cases \((b = 1\) or \(b = 2\)), the right hand sides of the two equations are equal, and then, subtracting the equations we have

\[ \sum a_i^2 + a_i = 0. \]

But for \(a_i\) integer we have \(a_i^2 + a_i \geq 0\), and then \(a_i\) must be equal to \(-1\) or \(0\).

Now, looking at the equations we see that the number of nonzero \(a_i\)'s is given by \(x + b^2\), and we obtain the result. □

Theorem 5.0.4 \(\mathcal{M}_L(S, c_1, c_2)\), for any generic polarization \(L\), is either empty or irreducible.

Proof. We will denote by \(L_0\) a polarization lying in a chamber whose closure contains \(H\).

If \(\mathcal{M}_L(S, c_1, c_2)\) has more than one irreducible component, then there must be a wall between \(L\) and \(L_0\) that created the extra component. Recall that a wall \(W_\zeta\) is a hyperplane in the ample cone perpendicular to a class \(\zeta \equiv c_1 \mod 2\), and \(c_1^2 - 4c_2 \leq \zeta^2 < 0\). By [P-Q], if \(L_1 \cdot \zeta > 0 > L_2 \cdot \zeta\), then the sheaves that are \(L_1\)-unstable and \(L_2\)-stable make an irreducible family of dimension \(N_\zeta + 2l_\zeta\), where \(N_\zeta = h^1(\zeta) + l_\zeta - 1\) and \(l_\zeta = (4c_2 - c_1^2 + \zeta^2)/4\). In the case of a rational surface we have

\[ h^1(\zeta) = \frac{\zeta \cdot K_S}{2} - \frac{\zeta^2}{2} - 1. \]

The wall creates a new component if \(N_\zeta + 2l_\zeta\) is equal to the dimension of the moduli space, in our case \(4c_2 - c_1^2 - 3\). For a rational surface we have \(N_\zeta + N_{-\zeta} + 2l_\zeta = 4c_2 - c_1^2 - 4\), and then this condition is equivalent to \(N_{-\zeta} = -1\). Denoting \(\zeta = bH + a_1E_1 + \cdots + a_nE_n\) and \(x = 4c_2 - c_1^2\) we get the system of equations (†) \((n \leq 8\) so without loss of generality we can study the equations with \(n = 8\)). Furthermore, this wall will create a new component if \(L_0 \cdot \zeta < 0 < L \cdot \zeta\). By the definition of chamber, the last equality is equivalent to \(0 < H \cdot \zeta\), and this translates to \(b > 0\).

We will prove the proposition by showing that for given \((S, c_1, c_2)\) there is at most one such wall in the ample cone and that in this case \(\mathcal{M}_{L_0}(S, c_1, c_2)\) is empty, so that \(\mathcal{M}_L(S, c_1, c_2)\) is always either empty or only has one irreducible component.

If the moduli space is not empty, its dimension should be nonnegative, and this translates to \(x \geq 3\). Then by lemmas 5.0.2 and 5.0.3 we know all the solutions of (†), i.e. all the walls creating components.
By tensoring with a line bundle (and relabeling the exceptional curves), we can assume that $c_1$ is either $H + E_1 + \cdots + E_m$ or $E_1 + \cdots + E_m$ for some $m \leq 8$. Now we will use the condition $c_1 \equiv \zeta \pmod{2}$.

In the first case this implies that the only possible solution for ($\dagger$) is $\zeta = H - E_1 - \cdots - E_m$, and $m = x + 1$. Then $4c_2 - c_1^2 = x$ implies $c_2 = 0$. By [13], $M_{L_0}(S, H + E_1 + \cdots + E_m, 0)$ is birational to a $(\mathbb{P}^1)^m$ bundle over $M(\mathbb{P}^2, H, 0)$, but it is well known that this moduli space is empty, then the same is true for $M_{L_0}(S, H + E_1 + \cdots + E_m, 0)$.

In the second case ($c_1 = E_1 + \cdots + E_m$), the condition $c_1 \equiv \zeta \pmod{2}$ implies that the only possible solution is $\zeta = 2H - E_1 - \cdots - E_m$ with $m = x + 4$. Then $c_2 = -1$ and then $M_{L_0}(S, c_1, c_2)$ is empty (because its expected dimension is negative).

This techniques can also be used to study the irreducibility of the moduli space of stable torsion free sheaves on $\mathbb{P}^2$ with more than 8 blown up points. We obtain equations similar to ($\dagger$), but with more variables. Unfortunately now we don’t have a bound on $b$ like the one given by lemma [5.0.2], and then it becomes more difficult to classify the solutions. This is still work in progress and it will appear elsewhere.
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