Notes on Connes’ Construction of the Standard Model
(Based on a talk presented at the San Miniato meeting
“The Inevitable Rise of the Standard Model”, April 21-25, 1997)

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Abstract

The mathematical apparatus of non commutative geometry and operator algebras which Connes has brought to bear to construct a rational scheme for the internal symmetries of the standard model is presented from the physicist’s point of view. Gauge symmetry, anomaly freedom, conservation of electric charge, parity violation and charge conjugation all play a vital role. When put together with a relatively simple set of algebraic algorithms they deliver many of the features of the standard model which otherwise seem rather ad hoc.

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1 Introduction

In recent years the approach of Connes\(^1\),\(^2\),\(^3\),\(^4\),\(^5\),\(^6\), using techniques of operator algebra and non-commutative geometry, has given new insights into the symmetry structure of the standard model (SM). Many of the otherwise inexplicable features of SM have been fitted into a neat rational scheme, the success of which suggests that Connes’ methods may well constitute one of the ingredients of the physics of the future. This is particularly true since the geometric concepts that are used are generalizations of usual continuum geometry to discrete non-commutative spaces. And this seems to be where quantum gravity is pointing to.

As Connes’ works themselves are couched in mathematical terms which generally are not part of the baggage of the everyman physicist, several exposés have been written whose purpose is to bridge the information gap. The present paper is conceived in the same vein, but with still more emphasis on the physics, and with less use of notation and language that is unfamiliar to physicists. My aim is to draw the attention of a larger part of the physics community to this development than has been the case heretofore.

My plan is take up severally, certain peculiar features of SM, mostly concerned with internal symmetries, and to show how Connes’ methods deal with them. I emphasize strongly the gauge principle and its handmaiden, anomaly freedom. One of the more remarkable things that has come up as the uncanny consistency of Connes’ approach with the constraints of anomaly freedom.

From the above it is clear that what follows will be very far from a systematic presentation. For that the reader may consult the very fine exposés on the subject. Furthermore their existence makes it pointless to include calculational detail in the present survey. Rather I sketch the ideas behind each point I take up and then refer to this or that reference which I esteem detailed and pedagogical concerning the item under consideration. Obviously nothing in the foregoing is original, if not perhaps a fresh appreciation of certain points.
2 Why Fundamental Representation

In SM, leptons and quarks serve as representations of non-Abelian groups. These reps are the fundamentals. Why?

The basic tool of Connes is the action of algebraic elements, built on points of a manifold, on a Hilbert space, $\mathcal{H}$, the latter being spanned by the elementary fermions. Internal symmetries are developed through use of the algebras $M_n(C) = n \times n$ matrices whose entries are complex numbers. (For $n = 2$, there is an important subalgebra of $M_2$, the quaternions, $\mathbb{H}$, of the form $A + i \vec{B} \cdot \vec{\sigma}$ with $A, \vec{B}$ real and $\vec{\sigma}$ the Pauli matrices and for $n = 1$, the algebra is that of the complex numbers $\mathbb{C}$.) Unlike groups, algebras have the additional property of linearity: if $a, b$ are elements of the algebra then $c \in \mathcal{A}$ if $c = \lambda_1 a + \lambda_2 b$ with $\lambda_1, \lambda_2 \in \mathbb{C}$ and $a, b \in \mathcal{A}$ in addition to the group property $c = ab$. This linearity restricts representations to the fundamental - one checks that tensor products reps of $M_n(C)$ violate linearity.

Lest there be any misunderstanding the linearity property does admit tensor products of different algebras and indeed in Section 6 we shall consider algebras which are sums of simple algebras, flavor + color. Members of $\mathcal{H}$ can have both attributes i.e. be tensor products. But the action of each subalgebra on the appropriate index of a component of such a representation is necessarily in the fundamental in order that it be an algebra. This property, in itself, to my mind is quite sufficient to motivate the algebraic approach.

3 Fermionic Mass, Spontaneously Broken Chiral Symmetry and Parity Violation

It is universally granted that one of the marvels of SM is the simultaneous occurrence of gauge symmetry governing the dynamics of all interacting vector and axial currents, and the existence of massy fermions. The reconciliation is, of course, effected by the mechanism of spontaneous broken chiral symmetry (S B $\chi$ S). And, of course, it is the gauge symmetry that makes the theory renormalizable, as was anticipated shortly after the discovery of massy gauge mesons\(^7\) and subsequently proven in the important works of Veltman and ’t Hooft \(^8\).
SBχS, in the context of gauge theory, was developed simultaneously and independently by Englert and myself \(^9\) and by Higgs \(^10\). In the former work two possibilities were considered: 1) the scalar responsible for SBχS was expressed in terms of a dynamically generated composite field, as proposed in prior work of Nambu or 2) it was an elementary scalar field; Higgs only considered this latter possibility. The issue is still not settled. In Connes’ approach, SBχS is realized in terms of an elementary scalar field. But it is wise to bear in mind that the theory so obtained, it is wise to bear in mind that the theory so obtained is “effective” i.e. thought to be valid at a certain length scale. For an example of how this might be constructed see ref.\(^6\). It is not excluded at a smaller scale that new phenomena will occur which would lead to compositeness, such as encountered in efforts to make SBχS a dynamical theory. Thus I prefer to reserve judgement and consider Connes’ approach, at its present stage, to be phenomenological.

Now for one of Connes’ main ideas. Scalars are gauge bosons that serve as connections in a manifold comprised of 2 points: left (L) and right (R), which I now explain at length.

Fermions, in the absence of mass, are displaced in space-time, on two different surfaces L and R, through the action of the Dirac operator. The usual Yang-Mills fields (YM) supply connections on these surfaces so as to allow internal symmetries to be gauged. The vector and axial currents coupled to the YM are sums of bilinears in L or R fermions and do not mix L and R. i.e. the coupling keeps an L (R) fermion on the L (R) surface. The role of mixing L and R is taken on by the scalars \(\phi\) through a coupling \((\psi_L^+ \psi_R \phi + h.c.)\). The bilinear \(\psi_L^+ \psi_R\) is a sort of current between the 2 corresponding points L and R at the same space-time point \(x^\mu\), \(\phi\) being the “gauge scalar”. The thought is not only engaging, but it ties in most elegantly with Connes’ rewriting of geometry on discrete spaces in terms of axioms which permit a natural generalization of all continuum concepts. These discrete spaces can be composed of but 2 points -L and R in particular, and so are applicable to SM gauge geometry.

To implement this extended sense of the gauge principle Connes defines generalized differentiation from the commutator with the Dirac operator, D. In L, R representation one has for a single fermion of mass, M
\[ D = D_0 + D_M = \begin{pmatrix} i\partial & 0 \\ 0 & i\partial \end{pmatrix} + \begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix} \]

(1)

where each entry is a diagonal 2 \times 2 matrix. Usual differentiation of a function \( f(x) \), now regarded as the representative of an algebraic element on the point, \( x \), of the space-time manifold, is obtained from \([D_0, f] = i\partial f\). In this way one can proceed systematically to invent the whole system of exterior derivatives so as to recover Riemannian geometry on one hand, and usual Yang-Mills theory on the other, when \( f \) is replaced by matrices of functions. This is not the subject of this paper. Rather we have introduced this notion of differentiation to motivate the construction of the “derivative” in the space of 2 points L, R through

\[ \delta a = [D_M, a] . \]

(2)

\( D_M \) being L, R non diagonal, thereby gives the sense of displacement L \( \leftrightarrow \) R. If there is more than one fermion in \( \mathcal{H} \) than \( a \) is a representative of some matrix algebra, to be specified, that is erected on the points L and R. And M is then a matrix whose structure reflects internal symmetries. For example for quarks \( u, d \) of one generation M is a diagonal matrix with elements \( M_u, M_d \). In the presence of more than one generation, mixing is expressed through Kobabayahshi-Maskawa mixing in the matrix \( M_d \). What is of primary importance is charge conservation. There is no \( u, d \) mixing in the matrix \( M \). Axiomatically, there is a charge operator \( Q \) such that \([D, Q] = 0\). This will be used extensively in what follows.

As has been stated, the algebra representative \( \rho(a) \) is erected on the points L and R. \( \rho(a) \) comes in two blocks \( \rho_L(a) \) and \( \rho_R(a) \), corresponding to the 2 points of the manifold. Axiomatically, there is an operator \( \chi \) such that \( \chi^2 = 1 \). The action of \( \chi \) on \( \mathcal{H} \) gives +1 on components in the L sector and -1 on the R sector. \( \chi \) anticommutes with \( D_M \) and commutes with \( a \) (for \( a \in \mathcal{A} \)). This is enough to give the representations

\[ \rho(a) = \begin{pmatrix} \rho_L(a) & 0 \\ 0 & \rho_R(a) \end{pmatrix} ; \quad D_M = \begin{pmatrix} 0 & M \\ M^+ & 0 \end{pmatrix} ; \quad \chi = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \]

(3)
where, as previously stipulated \([D_M, Q] = 0\), so \(M\) has no non diagonal elements between different charge states.

This last point warrants a digression into the underlying physics. In SM it is supposed that one symmetry is left unbroken. This is a natural condition in that the induced mass term is of the form \(\psi_R \psi_L^\dagger \langle \Phi \rangle\) (where \(\langle \Phi \rangle\) is the vacuum expectation value of a scalar or pseudoscalar field). This has as consequence a symmetry wherein, for a given component, \(\psi_R\), and its counterpart, \(\psi_L\), one multiplies each by the same phase factor - a universal phase angle multiplied by the electric charge of the component in question. This phase is the electromagnetic phase and the unbroken symmetry results in one gauge field retaining zero mass. The Hilbert space can thus be classed by eigenfunctions of this charge, \(Q\), and this must be respected by the dynamics. Connes has incorporated this physical requirement into the formalism through \([D_M, Q] = 0\).

An interesting aside that excites the imagination: in the geometric formulation of this concept of differentiation, Connes has generalized the notion of distance between 2 points of a manifold. For a single fermion at fixed \(x^\mu\), the distance between L and R is \(M^{-1}\).

Now to internal symmetries and the gauge principle. Because a Hilbert space exists one can invent an action \(\psi^+ D \psi\) and one is lead to investigate its invariances under automorphisms of the algebra. The components of \(\psi\) are in \(\mathcal{H}\). Henceforth unless otherwise specified we deal only with the internal symmetries so that \(D\) will be the internal part, \(D_M\), only.

The bilinear \(\psi^+ \psi\) is invariant under the subensemble of \(\mathcal{A}\) which is its set of unitary matrices. For our example this set is represented by the group \(U_L \times U_R\) i.e.

\[
\rho(u) = \begin{pmatrix} u_L & 0 \\ 0 & u_R \end{pmatrix}
\]  

with \(u_L \in U_L, u_R \in U_R\) (For quaternions \(U_L\) is not \(U_2\), but \(SU_2\) and from now on when the internal space is 2 dimensional we will work with \(\mathbb{H}\) and not \(M_2(\mathbb{C})\). In that case, the group is i-spin. This choice is made because it tallies correctly with the observed hypercharge assignments. (See Sections 5 and 6.)
If $\rho_L \neq \rho_R$, then $[D_M, \rho] \neq 0$, whence $\psi^+ D_M \psi$ is not invariant under transformation of $\psi$ by the unitaries. Following time honored procedure, we therefore introduce “vector or gauge potentials” to construct an invariant action beginning with $\psi^+ D_M \psi$. Since $\psi \to U \psi$ induces the change

$$\delta \psi^+ D_M \psi = \psi^+ U^+[D_M, U] \psi,$$

Connes invents a gauge potential of the form $\sum \alpha_i [D, \alpha'_i]$ with $\alpha_i, \alpha'_i \in A$ and for internal symmetries these elements are decomposed into $\rho_L$ and $\rho_R$ of Eq. 3. Since $U^+[D_M, U]$ is a hermitian form the gauge potentials are taken to be hermitian as well their role being to compensate for the variation 5). Indeed under $\psi \to U \psi$ one requires the gauge potential $V$ to transform as

$$V \to UVU^+ - U^+[D, U]$$

thereby ensuring the invariance of the action $A$ under unitary transformation, where

$$A = \psi^+[D + V] \psi$$

$V$'s which transform like 6) are of the form $\sum_i a_i [D, b_i]$ which are hermitian; $a_i, b_i \in A$.

[I have explicitly not used $D_M$ but $D$ in Eq. (7) in order to suggest to the reader to carry out the steps to recover $YM$ potentials in the case where $D = D_0$ and $U$ is taken to depend on the space-time variables $x^\mu$. He may also wish to construct the covariant fields $F_{\mu\nu}$. For this consult refs 1,2,11,12].

The law of transformation (6) has two parts. The first term, covariant, is simply a unitary transformation of all the members of the algebra -a one to one mapping to which corresponds the unitary transformation of $\mathcal{H}$. The second term is not covariant. Connes says that $D + V$ corresponds to a deformation of the space -a modification of its metric properties. This is understood from the fact that the operator $D$ (which is after all a sort
of square root of a laplacian) encodes this metric. One may think of the deformation of $\phi$ by gravity in the vierbein formulation.

For the case $D = D_M$, the gauge potentials are thus sums of terms of the form

$$V = \left( \begin{array}{c} \rho'_L \\ \rho'_R \end{array} \right) \left( \begin{array}{cc} 0 & \rho_L M - M \rho_R \\ M^+ \rho_L - \rho_R M^+ & 0 \end{array} \right) + \text{h.c.}$$

As an example drawn from the Glashow Weinberg Salam scheme (GWS) for quarks one picks $(\rho_L \in \mathbb{H}, \rho_R = \text{diag}(\lambda, \lambda^*)$ with $\lambda \in \mathbb{C}$). Correspondingly $u_L \in SU_2, u_R = \text{diag}(e^{i\varphi}, e^{-i\varphi})$. The Hilbert space is $u_L, d_L$ (an i-spin doublet) and $u_R, d_R$ (singlets). A typical term in $V$ is proportional to $qM$ where $q \in \mathbb{H}$. That this is an appropriate representation of the scalar potential of the GWS scheme is well known. For details see ref. $^{12}$. 

The point I want to stress here is that non trivial dynamics arises only in virtue of $[D_M, \rho] \neq 0$. This excludes a pure vector theory with unbroken symmetry, like QED or QCD wherein $\rho_L = \rho_R$ and $M$ proportional to $I$. The physically realized theory, SBCHS with L and R represented by distinct algebras fits nicely into Connes’ algorithm. But this method does not exclude parity non-violating vector theories with $M$ having different diagonal entries. An example is where the unitaries are $(SU_2)_L \times (SU_2)_R)$. So Connes algorithm only goes part of the way towards picking out the correct theory. More detail on this point is in ref.$^{13}$. 

It is important to be conscious of the novel feature of Connes’algorithm wherein the dynamics is postulated in terms of the c-number entries of the matrix $M$. In usual gauge theory one introduces a scalar field, an operator with certain quantum numbers (or possibly a non-local construction as envisaged in theories of dynamically broken symmetry.) There is no question of postulating commutation properties for the expectation value of that field. They emerge, in principle from the field theory. Connes reverses the order. He handles masses as inputs -and to this extent what he does is low energy phenomenology, building on the form of an effective theory as it were. Thus where the field theorist would have no objection, à priori, to SBCHS in QCD or QED as conventionally formulated, Connes effectively discards this possibility at the outset. The response of some physicists is then to discard the theory as well. And others will play along once noticing that Connes’algorithms, come out systematically on top. This seems to be the
way nature works. Such physicists will then start to ask why. It is purpose here to bring this option to his attention.

To see how \( \chi S \) in Connes’ approach comes about one derives an action for \( q \) by following the YM analogy. “Exterior derivatives” which are the analogies of \( \partial_\mu A_\nu - \delta_\nu A_\mu \) are found from the rule \( \delta(a_i[D,b_i]) = \delta(a_i\delta b_i) = [D,a_i][D,b_i] = [\delta a_i,\delta b_i] \). This rule is dictated by the construction of the algebra of exterior forms wherein \( \delta^2 = 0 \). (For those unfamiliar with this concept an introduction is to be found in ref. 14) and more formally in ref. 15). The covariant field, often called the curvature \( C \), is constructed from \( \delta V + V \times V \). [It is a nice exercise to prove that this form is covariant in consequence of (6)].

\( C \) being covariant, \( trC^2 \) is invariant and from the hermitian character of \( V \) it is easy to prove that this form is non negative. The trace is over the Hilbert space indices. For our GWS example one finds \( trC^2 \) proportional to \( tr[(q+1)^2 - 1]^2 \), therefore having a minimum at \( q = 0 \). It is assumed that the action is \( trC^2 \) or a sum of positive powers thereof. This latter possibility is not generally considered in analogy to the YM action being \( trF^2 \). But whereas this latter is supported by a renormalization group (RG) analysis wherein “irrelevant terms” are shown, indeed, to be irrelevant, there is no such calculation carried out in the present case. The proposition of ref. 6) requires further investigation in this regard. It may lead to interesting insights into usual RG analyses and quantum gravity.

The reader may be perplexed that \( \chi S \) is encoded in the minimum at \( q = 0 \). But this is as it should be. The fermion action is \( \psi^+[M+V]\psi \), which in our case takes on forms like \( \psi_R^+[M(q+1)\psi_L \). Therefore the scalar field \( \Phi \) is proportional to \( q + 1 \), since the Yukawa couplings are contained in the components of \( M \). Precisely one has \( q + 1 = (\Phi/\langle \Phi \rangle) \). The phenomenon of \( \chi S \) is \( M \neq 0 \). Fluctuations around \( M \) are thus encoded in \( q \) and \( <q> = 0 \) in vacuum. Moreover, from 7, one has that \( D_M + V \) is covariant, (by construction), so that \( \Phi \) is covariant. Since \( q + 1 \) and \( \Phi \) are related by multiplication by \( M \), it is then no surprise that the effective potential (\( = trC^2 \)) is expressible in powers of \( (q+1) \times \) powers of masses. More detail on the relation of \( q \) to \( \Phi \) and \( M \) is to be found in refs 11,12).

Generalization to leptons is obtained by eliminating \( u_R \) and setting \( m_u = 0 \). Many generations are handled by extending \( \mathcal{H} \) including KM mixing the \( d's \).
The above internal algebraic constructions are then combined with space-time through use of the part of $D$ equal to $i\partial$. Components of $\mathcal{H}$ now depend on $x^\mu$ as well. One follows the YM procedure in its Connesian algebraic guise. At the (quadratic + quartic) level one finds for the bosonic action what one expects, a sum of the gauge invariant terms, the YM action ($= \text{tr} F^2$), the kinetic term of the scalar properly covariantized $= (D_\mu \Phi)^2$ (where $\Phi = q + 1$ for the GWS example) and a term proportional to $\text{tr} C^2$. An interesting complication arises with respect to this latter, to wit:

4 Why More Than One Generation

All of these algebraic manipulations involve homomorphisms of the algebras (internal and space-time) on to their representations (matrices and functions of $x^\mu$). One encounters kernels; more precisely $\delta(\text{Kernel})$ is non zero. These must be divided out. As an example, a term arises in the second order exterior derivative which is represented at the first order level by zero. Thus, physically, one would get a field from a zero potential. This term is non gauge invariant and must be eliminated.

The interesting point that arises is that, upon tensoring the internal and external algebras, the two kernels intersect. The external algebra gives rise to a kernel which has a piece of the second order exterior derivative which is the algebra itself. In consequence, upon quotientizing one loses the term in $\text{tr} C^2$ when there is only one generation. In short, following the algebraic rules results in the loss of $\text{SB}_\chi S$ when there is only one generation. Then, eureka, it comes back for more than one generation and once more yields a non-negative effective potential in $|\Phi|^2$ of minimum $\Phi \neq 0$. For ample detail on this calculation a good reference is ref. ¹¹).

As far as I know this is the only place in standard model physics where a rationale has been supplied to this otherwise inexplicable physical phenomenom of the existence of several copies of the same representation. Conne’s algorithms ties non trivial dynamics to their existence, but at present do not explain why there are just three.

Let us now return to the main line of interest of this survey -the characterization of internal symmetries in terms of Connes’ constructions.
In the above GWS example there has been no question of hypercharge. The $u_L, d_L$ (or $e_L, \nu_L$) doublet thus gives an electric charge splitting of the doublet into equal and opposite values. The principle of charge conservation ($[D, Q] = 0$) then forces $\rho_R$ to be of the form diag ($\lambda, \lambda^*$) as well) as we have postulated above -since the R fermions must be split in electric charge just as their L partners. It is the unitary part of $C (= U_1)$ which in the R algebra is represented by $U_R \in U_1 \times U_1$ diag ($e^{i\varphi}, e^{-i\varphi}$) which does the job.

This version of GWS is at odds with observation and the scheme must be modified. I shall discuss this elaboration, first from the point of view of anomaly freedom (Section 5) and then present Connes’s construction of bimodules whereupon it will be seen how nicely they fit together to lead to the structure of the observed SM.

5 Anomaly Freedom

Anomalies arise from the sum of the vertex corrections, of the axial currents \(^{16}\). If they do not vanish the current coupled to the $\gamma_\mu \gamma_5$ vertex is not conserved; gauge invariance is violated and the theory is not renormalizable. In brief, consistency requires vanishing of the anomaly. [Note that the above remarks apply to those axial currents which are coupled to gauge fields. Sometimes in physics one encounters anomalous currents which are not so coupled, such as in the famous $U(1)$ problem of P C A C.]

We shall begin with the lepton sector, assuming some of the elements of the G W S scheme. Since the latter is anomalous, fermions other than leptons must exist. We introduce quarks and show how they manage to cancel the anomaly. I shall not discuss hypothetical schemes based on GUTS. They seem to have no basis in observation. What is remarkable is that there is one simple scheme, the observed one, that does work. And that this scheme follows in great measure from Connes’ axioms concerning real algebraic structures.

Several important results follow from the anomaly free G W S scheme (an $e_L, \nu_L$ i-spin doublet and $e_R$ singlet) taken together with the simplest and most natural i-spins of quarks ($u_L, d_L$ doublet and $u_R, d_R$ singlets). Firstly one finds that the neutrino has electric charge zero ($Q_\nu = 0$). The number of colors, $C$, is not determined, but one does find that if $C$ is odd, a collection of $C$ quarks—which is what is required to make a hadron,
a totally antisymmetric color state, has integer charge. It is then always possible to build a "proton", a hadron of equal and opposite charge to the electron. Finally the sum of all fermion charges is zero. This is called the unimodular condition and it played an important role in the formulation of GUTS. This latter, as mentioned, seems not in accord with nature; the proton refuses to decay. Nevertheless unimodularity has an interesting theoretical status which will be discussed in Section 6. For completeness we now present a brief deviation of these results.

It suffices to work with one generation since the quantum numbers of the members of each generation are in one to one correspondence. The anomaly is proportional to

\[ tr' \lambda_a \{ \lambda_b, \lambda_c \} \]

and this is required to vanish. Here \( tr' = \sum_L - \sum_R \), the minus sign because \( \gamma_5 = \pm 1 \) for L, R resp.. The \( \lambda \)'s are group generators coupled to the gauge fields at each vertex where \( a \) is an axial and \( b, c \) are vectors.

Consider leptons. Were there only \( \nu_L, e_L \), an i-spin doublet, there would be no anomaly. Nor would \( e \) have a mass. For this one needs \( e_R \neq 0 \). Of course one could also have \( \nu_R \neq 0 \) and \( (e_R, \nu_R) \) another doublet. Nature does not work that way. Let us then follow nature, exclude \( \nu_R \), and study the consequences. The result is that the leptonic sector is anomalous, which is clear from (9) since L, R cannot balance out. We now introduce hypercharge, defined through the Gell-Mann Nishijima relation \( Q = T_3 + Y/2 \), and for reasons discussed in Section 4), impose the condition that, when a fermion exists in both L and R versions, one has \( Q_L = Q_R \).

From the properties of i-spin, one readily sees that all combinations of \( a, b, c \) of Eq. 9) reduce to two conditions: \( a = Y, b, c = \) i-spin (with \( \{ T_b, T_c \} = \frac{1}{2} \delta_{bc} \), and \( a, b, c \) all = \( Y \). In the first instance only L contributes and we find a leptonic anomaly given by

\[ \sum Y_L^{lept} = \sum Q_L^{lept} = Q_\nu + Q_e = -1 + 2Q_\nu \]

The cubic anomaly is
\[ \sum (Y_L^{\text{lept}})^3 - \sum (Y_R^{\text{lept}})^3 = 6 - 12Q_\nu + 8Q_\nu^3 \]  

(11)

[This follows from \( \sum (Q_L^{\text{lept}})^3 - (Q_R^{\text{lept}})^3 = Q_\nu^3 \) and use of the G M N relation in conjunction with (10)].

Since there is no \( Q_\nu \) that sets (10) and (11) simultaneously to zero, there must exist other fermions to cancel the leptonic anomalies. The G W S leptonic scheme of itself is not quantum mechanically viable. Let us assume, in accord with Nature, that the accommodation to the leptonic anomaly is as simple as possible, to wit: there is only one other kind of fermion, the quark. To distinguish it from leptons it must bear another quantum number, color. Here we postulate that there are \( C \) colors which form the basis for a gauged unitary group (Q C D).

The assumption that the (colored) quarks are the only additional type of fermions implies that the color gauge group is pure vector. For otherwise there would be an anomaly involving gluons. To cancel this would then require other kinds of colored fermions.

To cancel Eq. 10, the L quarks must be an i-spin doublet \((u_L, d_L - \text{each in } C \text{ versions})\). They must also come in an R version to avoid a color anomaly, as well as to accommodate (11) and we postulate that there is both \( u_R \) and \( d_R \). They cannot be an i-spin doublet since this would undo the cancellation mission for which \((u_L, d_L)\) were invented. Therefore we postulate each to be i-spin singlets. The rest is simple arithmetic.

\[ 0 = \sum Y_L = \sum Q_L \]  

(12)

and (11) reads

\[ 0 = \frac{1}{8} \sum (Y_L^3 - Y_R^3) = \sum (Q_L - T_3)^3 - \sum Q_R^3 \]
\[ = \sum (Q_L^3 - Q_R^3) + \frac{3}{4} \sum Q_L = Q_\nu^3 \]  

(13)
The first conclusion is that the neutrino must be electrically neutral. Quantum mechanics assures that if parity is broken, it does so most elegantly. Thus one recovers the G W S assignments $Y_{lept}^L = -1$, $Y_{lept}^R = -2$ for the leptons. It then follows from (12) that

$$Y_{quark}^L = \left(\frac{1}{C}\right)$$
$$Q^{up} = \frac{1}{2}(1 + 1/C)$$
$$Q^{down} = \frac{1}{2}(-1 + 1/C)$$

and one has established

$$Y_{up}^R = (1 + 1/C)$$
$$Y_{down}^R = (-1 + 1/C)$$

and one has established

$$\sum Q_L = \sum Q_R = \sum Y_L = \sum Y_R = 0$$

Eq. (16) implies the unimodularity condition (i.e. $\sum (Y_L + Y_R) = 0$) discussed further in Section 6. In addition from (14) one sees that a collection of $C$ quarks has integer (half integer) charge according to $C$ odd (even). For $C$ odd, it is then always possible to form a “proton”, color antisymmetric state composed $(C \pm 1)/2$ up and down quarks respectively. Hence one can make neutral atoms.

A further remarkable element of consistency occurs when one includes gravity in the game. A potential anomaly occurs when the indices $b$ and $c$ of Fig.1 are gravitons and $a$ is the hypercharge. Cancellation requires

$$\sum Y_L - \sum Y_R = 0$$

resulting once again in $Q_\nu = 0$.

Strange that the neutrality of the universe should depend on the vicissitudes of an accommodation to a quantum anomaly.
6 Charge Conjugation and the Construction of Bimodules

For Connes’ construction of bimodules it is assumed that the algebra has both the GWS structure of quaternions and complex numbers \((\mathbb{H} \oplus \mathbb{C})\), and takes into account color as well. So for \(C\) colors the algebra is \((\mathbb{H} + \mathbb{C} + M_c(\mathbb{C}))\). Once more in what follows it suffices to work with one generation and the Hilbert space \(\mathcal{H}\) is modelled after GWS: \((e_L, \nu_L)(u_L, d_L), e_R, u_R, d_R\) having \(3 + 4C\) components. The bimodular construction is based on the fact that the complete Hilbert space is \(\mathcal{H} + \overline{\mathcal{H}}\) where \(\overline{\mathcal{H}}\) contains \(3 + 4C\) antiparticles. This is obtained from \(\mathcal{H}\) by taking its CP conjugate, the latter being an antilinear involution. It is assumed, that \([D, CP] = 0\). (Violation of CP through KM mixing does not influence the subsequent arguments which are designed to reveal the symmetry structure of the representation of the algebra acting on \(\mathcal{H} + \overline{\mathcal{H}}\)).

To carry out this program Connes postulates 2 axioms concerning the involution. Let \(J = CP\) and \(A\) the algebra. If \(\rho(a)\) is a representation of \(a(a \in A)\) one forms the conjugate representation \(J\rho(b)J^{-1}(b \in A)\). Thus the action of the conjugate representation on \(\mathcal{H}\) is to first send \(\mathcal{H}\) to \(\overline{\mathcal{H}}\), then operate with a representation of \(A\) on antiparticles and finally send the result of this last operation back on to \(\mathcal{H}\). It is natural to postulate Axiom I

\[
[\rho(a), \rho^0(b)] \equiv [\rho(a), J\rho(b)J^{-1}] = 0, \quad a, b \in A
\]  

(18)

since \(\mathcal{H} \oplus \overline{\mathcal{H}}\) is a direct sum and one should be allowed to operate on each of its sectors independently. But the consequence of (18) is far-reaching in that one may then construct \(\mathcal{H}\) which is a direct product representation wherein \(\rho(a)\) acts on the first index of the tensor product and the representation \(\rho^0(b)\) of the conjugate algebra acts on the second index (I refrain from the use of the terms left and right indices to avoid confusion with L, R). Specifically \(\rho^0(b)\) acts on the first index of \(\overline{\mathcal{H}}\). Two equivalent expressions to write these algebraic actions are

\[
\rho(a)\xi\rho^0(b) = \rho(a)J\rho(b^*)J^{-1}\xi \quad ; \xi \in \mathcal{H}
\]  

(19)
In tensor representations of groups, one is accustomed to this construction - say to make an
adjoint representation out of the fundamentals. What is not so customary is the explicit
use of Axiom 1 to make this construction. It permits one to use different members of
the direct sum which defines \( A \) to construct the representations \( \rho \) and \( \rho^0 \). Their joint ac-
tion, (19) clearly realizes (18) provided one can effect the 2 operations in arbitrary order.
What Connes realized is that the involution \( \mathcal{H} \rightarrow \mathcal{H} \) permitted this generalization of the
usual adjoint representation. Thus the bimodular representations (i.e. having 2 indices)
is deeply rooted in the charge conjugation invariance of physics (i.e. CP). Majorana
neutrinos won’t do, in this particular construction.

It is to be noted that this type of representation of an algebra \( A \) requires that \( A \) be a
direct sum. If not one would construct a bimodule which is the adjoint representation of
a group and lose contact with the fundamental notion that the use of algebras restricts
the representations to the fundamentals (Section 2). In using (18) where \( a \) and \( b \) refer to
subalgebras which commute, one retains the linearity property of the subalgebras.

But to exploit this idea, a second axiom is required, which as we shall see is rooted in
the physical requirement of independent gauge fields associated with the unitaries of the
2 representations \( \rho \) and \( \rho^0 \). The axiom is suggested by the fact that \([D, ]\) is a derivative,
a first order operator obeying a Leibniz rule. It is formulated through Axiom 2

\[
[\rho(a), [D_M, \rho^0(b)]] = 0
\]

which from (15), also implies \([\rho^0(b), [D_M, \rho(a)]] = 0\).

Axiom 2 will be seen, in what follows, to deliver the powerful result that color
symmetry is unbroken i.e. quark masses do not depend on their color and QCD is
a pure vector theory. Such a powerful result therefore calls for a closer analysis of the
physics behind (20). It is gauge symmetry that lurks in the background. To see this
it suffices to present the problem in a more general but more familiar setting. Suppose
\( \psi \) is bimodular, a 2 index entity which is covariant under transformation by 2 groups:
\( \psi \rightarrow UV\psi \). (For us \( V = JU'J^{-1} \), but this fact is not cogent to the present discussion).
The gauge principle then must be generalized to cover this case, and the way one does this is to expand $U$ and $V$ to first order about unity, expressing thereby each in terms of group generators multiplying infinitessimal space-time dependent parameters. One then proceeds to ensure the gauge invariance of the action by inventing two vector potentials, say $A_U$, and $A_V$ each of which transforms in the usual way for infinitessimal transformation

$$\delta A_{\mu,U} = -[A_{\mu,U}, \lambda_{a,U}] \in_{a,U} + \partial_{\mu} \in_{a,U}$$
$$\delta A_{\mu,V} = -[A_{\mu,V}, \lambda_{b,V}] \in_{b,V} + \partial_{\mu} \in_{b,V}$$

(21)

Given that $\delta \psi = \in_{a,U} \lambda_{a,U} \psi + \in_{a,V} \lambda_{b,V} \psi$, equation (21) ensures that $\delta \langle \text{Action} \rangle = \delta [\psi^+(D + A)\psi] = 0$ wherein I have abbreviated $A_U + A_V$ and $D = i\partial$. In (21) the $\lambda$'s are the group generators and $\in$'s are the infinitessimal parameters of the transformations.

It is then tacitly assumed that this procedure carries over to finite transformations $\psi \rightarrow UV\psi$ and

$$A_{\mu,U} \rightarrow UA_{\mu,U}U^+ + U\partial_{\mu}U^+$$
$$A_{\mu,V} \rightarrow VA_{\mu,V}V^+ + V\partial_{\mu}V^+$$

(22)

But it will be seen that the invariance of the action requires not only the condition that $[U, V] = 0$ but also $[U, [D, V]] = 0$. Indeed one has, assuming $[U, V] = 0$

$$\delta \Psi^+ D\psi = \psi^+ V^+ [D, V] \psi + \psi^+ V^+ [D, U] V \psi$$

(23)

For this variation to be compensated by the gauge potentials transforming as in (22) then requires this additional rule of commutation. In short not only must i-spin and color commute, but the variation of one of them, must be invariant under transformations of the other.

Since Connes is taking over the gauge principle in algebraic form for the gauging of symmetries under displacements $L \leftrightarrow R$, he then has little choice but to require (20). What normally is tacitly taken for granted in $YM$ theory requires this explicitation.
As in Section 4 wherein the matrix $D_M$ led the way towards the dynamics of $SB\chi S$, we once again are confronted with the novelty of Connes’ approach. The entries in $D_M$ are $c$ numbers, masses, and from the extended gauge hypothesis, field properties -here those of a scalar field- are deduced. Not the inverse! The theory so derived is much more constraining. For example, as we shall see, color is unbroken, quark masses do not depend on their color. Yet in field theory, nothing in principle prevents such breaking. Scalars can depend on color as well as flavor and so can their expectation values. But nature doesn’t work that way. As far as internal symmetries are concerned, she conforms to Connes’axioms. Then one must ask: at bottom, is the Connesian phenomenology any more outrageous than postulating a certain set of scalar fields? Indeed given that fermions have masses, it is rather more inductive than the usual approach -in that it begins “after the fact”.

I now proceed to Connes’constructions. For $\rho(a)$, the action of $A$ on the first index, he uses the GWS scheme modified so that in combination with the action on the second index, one recovers the usual leptonic assignments. It is assumed that for one generation (and once again it suffices to work with one generation), $H$ is $(\nu_L, e_L), (u_L, d_L)u_R, d_R$ wherein bracketed fermions are i-spin doublets. That $u_R$ and $d_R$ are i-spin singlets is an assumption that finds its basis in the reasoning of the previous section. The action $\rho(a)$ on $H$ is taken to be color blind; one uses $C \oplus |H$ and choses the phases of operations in both $\rho$ and $\rho^0$ to satisfy $Q_R = Q_L$.

\[
\begin{align*}
\rho(a) & \quad \begin{array}{c|c|c}
\nu_L & q & \nu_L \\
e_L & \lambda^* & e_L \\
e_R & q & e_R \\
u_L & \lambda & u_L \\
u_R & \lambda^* & u_R \\
\end{array} \\
u_L & u_L \\
\end{align*}
\]

In Eq. 24 read in descending order: $q$ is a $2 \times 2$ matrix, $\lambda^*$ is $1 \times 1$; $q$ that acts on $(u_L, d_L)$ is the collection of $C$ matrices each of which is $2 \times 2$ and $\lambda, \lambda^*$ acting on $u_R, d_R$ resp. are multiplied by unit $C \times C$ matrices.

Color is legislated into the theory by taking $u_L, d_L, u_R, d_R$ to be $C$ dimensional vectors and one uses $\rho^0(b)$ to act on the color indices. Clearly the choice $m(m \in M_c(C))$ for $\rho^0(b)$
acting on the quarks permits one to satisfy $[\rho(a), \rho^0(b)] = 0$. Another option could be to take for $\rho^0(b)$, the matrix $\lambda \otimes I$ to operate on $(u_L, d_l)$ and $M_c$ to operate on $u_R, d_R$. This is however disallowed by Axiom 2 which takes on the form

$$[M\rho_R(a) - \rho_L(a)M]\rho^0_R(b) = \rho^0_L(b)[M\rho_R(a) - \rho_L(a)M]$$  \hspace{1cm} (25)

(for all $a, b$!) wherein we have decomposed $\rho$ into its chiral sectors and used the anti-diagonality of $D_M$ in chiral representation. Applying (25) to the quark sector and using the fact that $[M\rho_R(a) - \rho_L(a)M] \neq 0$ since $\rho_R \neq \rho_L$ due to flavor splitting it is seen that $\rho^0_R(b) = \rho^0_L(b)$ and that each commutes with $[M\rho_R - \rho_L M]$, Axiom 2 delivers the result that color is a pure vector theory. Moreover, since $[M\rho_R(a) - \rho_L(a)M]$ commute with $\rho^0_R$ (or $\rho^0_L$) in the quark sector, we have the additional strong result that this mass breaking term commutes with color.

As we have seen in Section 4, this matrix is essentially the scalar field, so that the result can be stated that the scalar field is color independent.

The same reasoning yields that $\rho^0_R = \rho^0_L$ on leptons as well. Furthermore Axiom 1 also implies that $\rho^0$ acting on $(\nu_L, e_L)$ is proportional to the unit matrix. Therefore $\rho^0(b)$ acting on leptons is either $\lambda \otimes I$ or $\lambda^* \otimes I$. The condition $Q_L = Q_R$ for $e$ delivers the second option so that, in fine, one gets using the same convention as previously, the block form

$$\rho^0(b) = \begin{pmatrix} \lambda^* \\ \lambda^* \\ \lambda^* \\ m \\ m \\ m \\ m \end{pmatrix}$$  \hspace{1cm} (26)

One last step is required to complete the construction. The set of unitaries of $M_c$ is $U_c = U_1 \times SU_c$ (to within irrelevant homotopy). It is therefore necessary to fix this one last phase. It is to be noted that unlike $M_2(C)$ which has $|H$ as subalgebra, hence a set of unitaries $SU_2$, the algebras $M_c$ do not enjoy this property. Thus for $C \neq 2$
this extra phase is necessarily present. And it is precisely this phase which permits an adjustment so as to satisfy anomaly freedom: to wit this phase is \((1/2C)\). Hypercharge is then obtained from the sum of phases carried by \(\rho(a)\) and \(\rho^0(b)\). Reading off (24) and (26), wherein each entry is in the unitary subset of the corresponding algebraic element, then gives back Eqs. (12) to (16).

Connes makes an important remark concerning this last point. One easily checks that with the above assignments the total phase of \(\rho(a)\rho^0(b)\) vanishes. [The count is obtained from 4 factors of \(m\) with 4 factors of \(\lambda^*\); the \(SU_2\) part of \(q\) carries no total phase]. This is what anomaly freedom has given as result. But Connes remarks that were there a non trivial total phase, it would be a factor of the algebra which would multiply the \(15\times15\) unit matrix. Its commutator with \(D_M\) would vanish. Therefore it would not be associated with a scalar gauge potential. Of course the vanishing of this phase is the unimodular condition. It is then seen that this total phase has nothing to do with the dynamics coupled to internal symmetries. This then is another one of these remarkable facts. The algebraic approach leads to zero coupling of total phase in the internal algebra and the anomaly structure requires that this phase not be present. There is an overlap which remains to be understood. The algebraic approach appears to express “quantum roots” which are hidden in the formalism.

In summary, once it is admitted that L-R displacements lead to gauge theory in the manner of Connes, then the construction of bimodules conforming to the two axioms on real structures leads to a result in complete conformity with anomaly freedom wherein color is a pure vector theory (given flavor breaking) and moreover quark masses independent of color. This latter has nothing to do with anomalies, but of course is the way nature works.

7 Further Comments

a) Connes has elaborated an elegant topological argument which gives a deeper sense to his bimolecular construction. It is based on the powerful techniques of \(K\) theory developed over the past few decades and deserves a paper in itself. I shall simply sketch here some of the ingredients, since at the moment the argument is more one
of consistency then of an independent construction.

One forms a matrix \( K_{ij} \) where \( i \) and \( j \) take on values 1, 2, 3 corresponding to the members of \( \mathcal{A} = (\mathbb{C} \oplus \mathcal{H} \oplus \mathcal{M}_c) \) (i.e. the subalgebras span a vector space). \( K_{ij} \) receives a contribution, which is calculated according to a rule of projection. It is non vanishing when there are one or more members of \( \mathcal{H} \) whose first index transforms according to \( \mathcal{A}_i \) in \( \rho(a) \) [where \( \mathcal{A}_1 = \mathbb{C}_1; \mathcal{A}_2 = [H] \) and whose second index transforms according in \( \mathcal{A}_j \) in \( \rho^0(b) \) [where \( \mathcal{A}_1 = \mathbb{C}, \mathcal{A}_3 = \mathcal{M}_c \)] and vice versa. Thus \( K_{ij} \) is a sort of measure of whether or not there are common attributes in the two indices that make up the bimodule. It is also important that the sign of the contribution change with chirality. The theorem is that if \( |K| \neq 0 \), the manifold on which is built the algebra is a topologically acceptable space. If it is singular, the whole bimodular algebraic construction makes no sense. For SM with Connes’ assignments it works. If one tries to add in \( \nu_R \) where the latter has zero charge and zero i-spin, one finds \( |K| = 0 \). Upon performing the calculation one sees that to have \( |K| \neq 0 \) requires L, R dissymmetry in \( \mathcal{H} \) of the GSW type. Otherwise the columns which make up \( K_{ij} \) become linearly dependent.

The above result is satisfying since such a \( \nu_R \) would decouple from both the YM fields and the scalar field of SM. Unfortunately there has been no systematic study on what would be other acceptable models based on this criterion. Nevertheless the above calculation does suggest that parity violation is essential. And this is in keeping with remarks I made in Section 3.

It is rather important that this question be studied more extensively. For if my conjecture is borne out, parity violation would be elevated to a dynamical principle, based on topological arguments.

b) Similar schemes have been developed by other authors \(^{17},^{18}\). I have not reviewed them here, not because of their lack of interest, but rather because only Connes’s scheme has been developed in sufficient detail, and appears sufficiently constrained, to make contact with phenomenology. In particular there is now a body of quantitative work which I briefly mention in point c) below.
Chamsedinne and Connes have calculated an effective action for bosonic fields as follows. One first constructs the full Dirac operator $D_{Tot}$ in the presence of gravity, YM and Higgs fields. It is possible to introduce a set of coupling constants which is consistent with all the algebraic constraints so that $D$ is of the form used in SM phenomenology. These coupling constants are obtained by introducing different weights to various independent sectors of the total trace that makes up scalar products. The weight factors are diagonal matrices which commute with the matrices that represent $A$.

The calculation then consists in the evaluation of the number of states whose eigenvalue of $D^2(=\lambda^2)$ is such that $\lambda^2 < \Lambda^2$ when $\Lambda$ is a cut-off parameter and one works in the euclidean domain. The evaluation is made in descending powers of $\Lambda$ and the first three terms are retained these being considered “relevant”. For flat space only the $\Lambda$ independent term contributes, whereas in curred space one has a $\Lambda^4$ cosmological constant $+\Lambda^2 \times$ (Einstein-Hilbert action). Note that the calculation is entropistic in character and not the same thing that one gets on integrating over fermi fields (as has sometimes been done in an effort to generate gravity dynamically). Indeed to the above bosonic action one adds the fermionic action.

It is interesting to remark that the Bekenstein-Hawking (BH) entropy which arises in the presence of event horizons is related to the Connes-Chamedinne (CC) effective action, calculated as it is, in the euclidean. Indeed when the BH entropy is calculated as a functional integral over fields in a periodic domain, the integrand is the exponential of the total action. And this according to CC is obtained from a count of states. Thus, in this vision the BH entropy is the average over one cycle of the CC count of states. Of course for this to be truly useful one will have to supplement CC by a viable theory of gravity.

What is interesting in the present context is that in recent calculations carried out in flat space, the renormalization group, which shows how all couplings run with $\Lambda$, yields the acceptable result, that given their present experimental values one reaches symmetry (electroweak and strong) at $\Lambda$ about $10^{15}$ GeV. Moreover if one uses the value of the top mass ($\sim 175$ GeV), one predicts a Higgs mass of $\sim 200$ GeV.
Previous calculations along the same lines \textsuperscript{19}, but based on the construction sketched in Section 3 gives similar results, but according to the authors, accord less well with certain experimental facts.

It also has been pointed out to me by T. Schucker that the flexibility induced by the choice of weights in the trace when color is present destroys the no-go theorem of Section 4. It remains to be seen how this statement should be interpreted when the renormalization group is applied so as to reach the scale of symmetry restoration since then the weights are once more all the same. Thus one must reserve judgement as to whether the scheme does or does not require more than one generation.

As previously mentioned, there is at present no explanation of why there are three generations. Nor is there any indication in the scheme to explain the widely disparate mass scales from generation to generation, nor the KM mixing and CP violation. One must also be prepared for the eventuality that the Higgs scalar is not a local field. The whole phenomenology may break down at some intermediate large mass scale. As I have emphasized Connes’ construction provides a rational framework in which to set SM phenomenology. Many facts fall into place which seem to have no rhyme or reason, and in this sense I consider it significant.

d) With regard to gravity, since the symmetry value of $\Lambda (\equiv \Lambda_s)$ is less than $m_{\text{planck}}$ by some orders of magnitude, it would seem that the model used to construct $D$, (i.e. the usual continuum approach of Riemann geometry) must be modified at larger mass scales than $\Lambda_s$. This is quantum gravity - the formulation of which is the primary goal of present-day physics. At present we are still far from this goal. Nevertheless it is suggested that the reader look into refs \textsuperscript{5,6} which contain considerations on automorphisms of the algebra. Those which are norm conserving are concerned with internal symmetries and the others with gravity. This novel approach may help point the way. Also, Connes has pointed the possible relevance of quantum groups \textsuperscript{5}. (The first extension of groups to quantum groups contains the algebra $\mathcal{C} \oplus M_2(\mathcal{C}) \oplus M_3(\mathcal{C})$ - Of all things!).

e) It has been shown by Lizzi\textsuperscript{20} and collaborators that unless one adds new fermions, it is not possible to construct GUTS schemes which conform to Connes’ axioms. It
appears that SM is the unique solution.

8 Acknowledgements

This is to express my profound gratitude to Alain Connes for his explanations. I hope that these Notes reflect in some measure his scientific acumen and imagination.

This paper also reflects conversations with many colleagues whom I here thank with pleasure: F. Bastionelli, R. Coquereaux, J.-M. Frère, B. Jochum, T. Krajewski, T. Schucker, D. Testard and above all D. Kastler who not only has offered me gracious hospitality but who has taken upon himself the difficult task of teaching me operator algebras. Section 5 on anomalies was instigated by a suggestion of J.-M. Frère and implemented by important remarks of F. Bastionelli who showed me how the condition $Q_L = Q_R$ led to a clear understanding of internal consistency. Bastionelli also alerted me to the consistency with the gravitational anomaly.

REFERENCES

1) A Connes in The interface of Mathematics and Physics D.G. Quillen, B.B. Segal and S.T. Tsou, eds Clarendon Press Oxford (1990).

2) A. Connes and J. Lett. Nucl. Phys. B (Proc Suppl.) 18 (1990), 29.

3) A. Connes Non Commutative Geometry, Academic Press, London (1994).

4) A. Connes, J. Math. Phys. 36 (1995) 6194.

5) A. Connes, [hep-th/9603053]

6) A.H. Chamsedinne and A. Connes, [hep-th/9606001].

7) F. Englert, R. Brout and M.F. Thiry, Nuov Cim 43, 244 (1966).

8) For a good review see M. Veltman in the forthcoming proceedings of the History of the Standard Model Stanford University, 1992.
9) F. Englert, R. Brout, Phys. Rev. Lett. 13, 32 (1964).

10) P. Higgs, Phys. Rev. Lett. 13, 508 (1964).

11) T. Schücker, J-M Zylinski, J. Geom. Phys. 16 (1995), 207.

12) A good review is C.P. Martin, J.M. Garcia-Bondia, J.C. Varilly, hep-th/9605001.

13) B. Iochum, T. Schücker, Lett. Math. Phys. 32 (1994), 153.

14) C. Misner, K. Thorne and J. Wheeler, Gravitation, W.H. Freeman, New York (1970).

15) C. Nash, S. Sen Topology and Geometry for Physicists, Academic Press, London (1983).

16) See for example W. Greiner and B. Müller, Gauge Theory of Weak Interactions, Springer, Berlin 1993.

17) M. Dubois-Violette, R. Kerner and J. Madore, J. Math. Phys. 31 (1990) 316.

18) R. Coquereaux, G. Esposito-Farese and G. Valiant, Nucl. Phys. B 353 (1991), 689.

19) B. Iochum, D. Kastler and T. Schücker, hep-th/9507150.

20) Reported by T. Schücker at the Marseille, Workshop on NCG, March 1997.

21) Reported by F. Lizzi at the Marseille Workshop on NCG March 1997.