Asymptotic Analysis of Boltzmann Equation in Bounded Domains

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Abstract

Consider 3D Boltzmann equation in convex domains with diffusive boundary. We study the hydrodynamic limits as the Knudsen number and Strouhal number $\epsilon \to 0^+$. Using Hilbert expansion, we rigorously justify that the solution of stationary/evolutionary problem converges to that of the steady/unsteady Navier-Stokes-Fourier system. This is the first paper to justify the hydrodynamic limits of nonlinear Boltzmann equations with hard-sphere collision kernel in bounded domain in $L^\infty$ sense. The proof relies on a novel and detailed analysis on the boundary layer effect with geometric correction. The difficulty mainly comes from three sources: 3D domain, boundary layer regularity, and time dependence. To fully solve this problem, we introduce several techniques: (1) boundary layer with geometric correction; (2) remainder estimates with $L^2 - L^{2m} - L^\infty$ framework; (3) boundary layer regularity analysis.

Keywords: Boltzmann equation; boundary layer; Milne problem; geometric correction
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Chapter 1

Introduction

1.1 Stationary Boltzmann Equation

1.1.1 Problem Presentation

We consider the stationary Boltzmann equation in a three-dimensional smooth convex domain $\Omega \ni x = (x_1, x_2, x_3)$ with velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. The density function $\mathfrak{F}(x, v)$ satisfies

$$\begin{cases}
\epsilon v \cdot \nabla_x \mathfrak{F} = Q[\mathfrak{F}, \mathfrak{F}] \quad \text{in } \Omega \times \mathbb{R}^3, \\
\mathfrak{F}(x_0, v) = P^\epsilon[\mathfrak{F}](x_0, v) \quad \text{for } x_0 \in \partial \Omega \text{ and } v \cdot n(x_0) < 0,
\end{cases}$$

(1.1)

where $n(x_0)$ is the unit outward normal vector at $x_0$, the Knudsen number $\epsilon$ satisfies $0 < \epsilon << 1$, $Q$ is the hard-sphere collision operator (see [8, Chapter 1]), and the diffusive boundary

$$P^\epsilon[\mathfrak{F}](x_0, v) := \mu^\epsilon_b(x_0, v) \int_{u \cdot n(x_0) > 0} \mathfrak{F}(x_0, u) |u \cdot n(x_0)| \, du.$$  

(1.2)

Here the boundary Maxwellian

$$\mu^\epsilon_b(x_0, v) := \frac{\rho^\epsilon_b(x_0)}{2\pi \left( \theta^\epsilon_b(x_0) \right)^2} \exp \left( -\frac{|v - u^\epsilon_b(x_0)|^2}{2\theta^\epsilon_b(x_0)} \right),$$

(1.3)

is an $\epsilon$-perturbation of the standard Maxwellian

$$\mu(v) := \frac{1}{2\pi} \exp \left( -\frac{|v|^2}{2} \right).$$

(1.4)

We assume that both $\mu^\epsilon_b$ and $\mu$ satisfies the normalization condition

$$\int_{v \cdot n(x_0) > 0} \mu^\epsilon_b(x_0, v) |v \cdot n(x_0)| \, dv = \int_{v \cdot n(x_0) > 0} \mu(v) |v \cdot n(x_0)| \, dv = 1.$$  

(1.5)

In addition, we require that the particles are only reflected on $\partial \Omega$ without in-flow or out-flow, i.e.

$$\int_{\mathbb{R}^3} \mu^\epsilon_b(x_0, v) |v \cdot n(x_0)| \, dv = \int_{\mathbb{R}^3} \mu(v) |v \cdot n(x_0)| \, dv = 0.$$  

(1.6)

We also assume that $\rho^\epsilon_b$, $u^\epsilon_b$ and $\theta^\epsilon_b$ can be expanded into a power series with respect to $\epsilon$,

$$\rho^\epsilon_b(x_0) := 1 + \sum_{k=1}^{\infty} \epsilon^k \rho_{b,k}(x_0), \quad u^\epsilon_b(x_0) := 0 + \sum_{k=1}^{\infty} \epsilon^k u_{b,k}(x_0), \quad \theta^\epsilon_b(x_0) := 1 + \sum_{k=1}^{\infty} \epsilon^k \theta_{b,k}(x_0),$$

(1.7)
i.e. \((\rho_b, u_b, \theta_b)\) is an \(\epsilon\)-perturbation of \((1, 0, 1)\). Hence, we may also expand the boundary Maxwellian \(\mu_b\) into power series with respect to \(\epsilon\),

\[
\mu_b(x_0, v) = \mu(v) + \epsilon^b(v) \left( \sum_{k=1}^{\infty} \epsilon^k \mu_k(x_0, v) \right).
\]

(1.8)

In particular, we have

\[
\mu_1(x_0, v) := \mu^b(v) \left( \rho_{b,1}(x_0) + u_{b,1}(x_0) \cdot v + \theta_{b,1}(x_0) \right) \frac{|v|^2 - 3}{2}.
\]

(1.9)

We further assume that

\[
\left| (v)^{\vartheta} \epsilon^b \frac{\mu_b - \mu}{\mu^2} \right| \leq C_0(\vartheta, \vartheta) \epsilon,
\]

(1.10)

for any \(0 \leq \vartheta < \frac{1}{4}\) and \(\vartheta > 3\), and constant \(C_0 > 0\) is sufficiently small. Based on (1.5), (1.6) and (1.8), we know

\[
\int_{\mathbb{R}^3} \mu_k(x_0, v) \mu^b(v) |v \cdot n(x_0)| dv = 0 \quad \text{for} \quad k \geq 1,
\]

(1.11)

\[
\int_{v \cdot n(x_0) \leq 0} \mu_k(x_0, v) \mu^b(v) |v \cdot n(x_0)| dv = 0 \quad \text{for} \quad k \geq 1.
\]

(1.12)

Note that if \(\mathcal{F}^\epsilon\) is a solution to (1.1), then for any constant \(M \in \mathbb{R}\), \(\mathcal{F}^\epsilon + M \mu_b\) is also a solution. To guarantee uniqueness, we require the normalization condition

\[
\iint_{\Omega \times \mathbb{R}^3} \mathcal{F}^\epsilon(x, v) dv dx = \iint_{\Omega \times \mathbb{R}^3} \mu(v) dv dx = \sqrt{2\pi} |\Omega|.
\]

(1.13)

We intend to study the behavior of \(\mathcal{F}^\epsilon\) as \(\epsilon \to 0\).

### 1.1.2 Linearization

Considering (1.13), the solution \(\mathcal{F}^\epsilon\) to (1.1) can be expressed as a perturbation of the standard Maxwellian

\[
\mathcal{F}^\epsilon(x, v) = \mu(v) + \mu^b(v) f^\epsilon(x, v),
\]

(1.14)

with the normalization condition

\[
\iint_{\Omega \times \mathbb{R}^3} f^\epsilon(x, v) \mu^b(v) dv dx = 0.
\]

(1.15)

Here \(f^\epsilon(x, v)\) satisfies the equation

\[
\begin{cases}
\epsilon v \cdot \nabla_x f^\epsilon + \mathcal{L}[f^\epsilon] = \Gamma[f^\epsilon, f^\epsilon] \quad \text{in} \quad \Omega \times \mathbb{R}^3, \\
f^\epsilon(x_0, v) = \mathcal{P}^\epsilon[f^\epsilon](x_0, v) \quad \text{for} \quad x_0 \in \partial \Omega \quad \text{and} \quad v \cdot n(x_0) < 0,
\end{cases}
\]

(1.16)

where

\[
\mathcal{L}[f^\epsilon] := -2 \mu^b \frac{\mu^b}{\mu} Q \left[ \mu^b \right] f^\epsilon, \quad \Gamma[f^\epsilon, f^\epsilon] := \mu^b \frac{\mu^b}{\mu} \left[ \mu^b f^\epsilon, \mu^b f^\epsilon \right],
\]

(1.17)

and

\[
\mathcal{P}^\epsilon[f^\epsilon](x_0, v) := \mu_b(x_0, v) \mu^b(v) \int_{u \cdot n(x_0) > 0} \mu^b(u) f^\epsilon(x_0, u) |u \cdot n(x_0)| du + \mu^b(v) \left( \mu_b(x_0, v) - \mu(v) \right).
\]

(1.18)

Hence, in order to study \(\mathcal{F}^\epsilon\), it suffices to consider \(f^\epsilon\).
1.1.3 Linearized Boltzmann Operator

To clarify, we specify the hard-sphere collision operator $Q$ in (1.14) and (1.16)

$$Q[F, G] := \int_{\mathbb{R}^3} \int_{S^2} q(\omega, |u - v|) \left( F(u_*)G(v_*) - F(u)G(v) \right) d\omega du,$$

with

$$u_* := u + \omega \left( (v - u) \cdot \omega \right), \quad v_* := v - \omega \left( (v - u) \cdot \omega \right),$$

and the hard-sphere collision kernel

$$q(\omega, |u - v|) := q_0 \omega \cdot (v - u),$$

for a positive constant $q_0$.

[8] Chapter 3 describes the linearized Boltzmann operator $\mathcal{L}$ as

$$\mathcal{L}[f] = -2\mu^2 Q[\mu, \mu^2 f] := \nu(v)f - K[f],$$

where

$$\nu(v) = \int_{\mathbb{R}^3} \int_{S^2} q(\omega, |u - v|)\mu(u) d\omega du = \pi^2 q_0 \left( 2|v| + \frac{1}{|v|} \right) \int_0^{|v|} e^{-z^2} dz + e^{-|v|^2},$$

$$K[f](v) = K_2[f](v) - K_1[f](v) = \int_{\mathbb{R}^3} k(u, v)f(u) du,$$

$$K_1[f](v) = \mu^2(v) \int_{\mathbb{R}^3} \int_{S^2} q(\omega, |u - v|)\mu^2(u)f(u) d\omega du = \int_{\mathbb{R}^3} k_1(u, v)f(u) du,$$

$$K_2[f](v) = \int_{\mathbb{R}^3} \int_{S^2} q(\omega, |u - v|)\mu^2(u) \left( \mu^2(v_*)f(u_*) + \mu^2(u_*)f(v_*) \right) d\omega du = \int_{\mathbb{R}^3} k_2(u, v)f(u) du,$$

for some kernels

$$k(u, v) = k_2(u, v) - k_1(u, v),$$

$$k_1(u, v) = \pi q_0 |u - v| \exp \left( -\frac{1}{2} |u|^2 - \frac{1}{2} |v|^2 \right),$$

$$k_2(u, v) = \frac{2\pi q_0}{|u - v|} \exp \left( -\frac{1}{4} |u - v|^2 - \frac{1}{4} \frac{(|u|^2 - |v|^2)^2}{|u - v|^2} \right).$$

In particular, $\mathcal{L}$ is self-adjoint in $L^2(\mathbb{R}^3)$ and the null space $\mathcal{N}$ is five-dimensional spanned by

$$\mu^2 \left\{ 1, v, \frac{|v|^2 - 3}{2} \right\}.$$

We denote $\mathcal{N}^\perp$ the orthogonal complement in $L^2(\mathbb{R}^3)$.  

1.1.4 Previous Results

Hydrodynamic limits are central to connecting the kinetic theory and fluid mechanics. Since early 20th century, this type of problems have been extensively studied in many different settings: stationary or evolutionary, linear or nonlinear, strong solution or weak solution, etc.

The early result dates back to 1912 by Hilbert himself, using the so-called Hilbert’s expansion, i.e. an expansion of the distribution function $\mathfrak{F}^\epsilon$ as a power series of the Knudsen number $\epsilon$. Since then, a lot of works on Boltzmann equation in $\mathbb{R}^n$ or $\mathbb{T}^n$ have been presented, including [9], [19], [1], [2], [3], [4], for either smooth solutions or renormalized solutions.
The general theory of initial-boundary-value problems was first developed in \[10\], and then extended by \[20, 21, 22, 25\], for both the evolutionary and stationary equations. The classical books \[23, 24\] provides a comprehensive summary of previous results and gave a complete analysis of such approaches.

For stationary Boltzmann equation where the state of gas is close to a uniform state at rest, the expansion of the perturbation \(f'\) consists of two parts: the interior solution \(F\), which is based on a hierarchy of linearized Boltzmann equations and satisfies a steady Navier-Stokes-Fourier system, and the boundary layer \(\mathcal{F}\), which is based on a half-space kinetic equation and decays rapidly when it is away from the boundary.

The justification of hydrodynamic limits usually involves two steps:

1. Expanding \(F = \sum_{k=1}^{\infty} \epsilon^k F_k\) and \(\mathcal{F} = \sum_{k=1}^{\infty} \epsilon^k \mathcal{F}_k\) as power series of \(\epsilon\) and proving the coefficients \(F_k\) and \(\mathcal{F}_k\) are well-defined. This is doable by inserting above expansion ansatz into the Boltzmann equation to compare the order of \(\epsilon\) and get a hierarchy of equations for \(F_k\) and \(\mathcal{F}_k\). Traditionally, the estimates of interior solutions \(F_k\) are relatively straightforward. On the other hand, boundary layers \(\mathcal{F}_k\) satisfy one-dimensional half-space problems which lose some key structures of the original equations. The well-posedness of boundary layer equations are sometimes extremely difficult and it is possible that they are actually ill-posed (e.g. certain type of Prandtl layers).

2. Proving that \(R = f' - \epsilon F_1 - \epsilon \mathcal{F}_1 = o(\epsilon)\) as \(\epsilon \to 0\). Ideally, this should be done just by expanding to the leading-order level \(F_1\) and \(\mathcal{F}_1\). However, in singular perturbation problems, the estimates of the remainder \(R\) usually involve negative powers of \(\epsilon\), which require expansion to higher order terms \(F_N\) and \(\mathcal{F}_N\) for \(N \geq 2\) such that we have sufficient power of \(\epsilon\). In other words, we define \(R = f' - \sum_{k=1}^{N} \epsilon^k F_k - \sum_{k=1}^{N} \epsilon^k \mathcal{F}_k\) for \(N \geq 2\) instead of \(R = f' - \epsilon F_1 - \epsilon \mathcal{F}_1\) to get better estimate of \(R\).

Note that boundary layer plays a significant role in proving the asymptotic convergence in the \(L^\infty\) sense. If instead we consider \(L^p\) convergence for \(1 \leq p < \infty\), then the boundary layer \(\mathcal{F}_1\) is of order \(\epsilon^\frac{p}{2}\) due to rescaling, which is negligible compared with \(F_1\) as \(\epsilon \to 0\). \[6\] justifies the \(L^p\) convergence for 3D stationary Boltzmann equation with the \(L^2 - L^\infty\) framework. In a recent paper \[27\], we show the \(L^\infty\) convergence in 2D stationary settings. In this monograph, we follow the same formulation as \[6\]. The major upshot is to take the effect of boundary layers into consideration and justify the \(L^\infty\) convergence in 3D stationary and evolutionary settings. This requires a modified \(L^2 - L^6 - L^\infty\) framework and a thorough and delicate analysis of the well-posedness and regularity of the \(\epsilon\)-Milne problem with geometric correction. We list some recent development using \(L^2 - L^\infty\) path \[5, 7, 12, 14, 13, 17\]. Also, we record some recent papers on geometrically corrected boundary layer \[31, 29, 15, 16, 30, 28\].

### 1.1.5 Main Theorem

Let \(\langle \cdot, \cdot \rangle\) be the standard \(L^2\) inner product for \(v \in \mathbb{R}^3\). Define the \(L^p\) and \(L^\infty\) norms in \(\mathbb{R}^3\):

\[
|f(x)|_p := \left( \int_{\mathbb{R}^3} |f(x,v)|^p \, dv \right)^{\frac{1}{p}}, \quad |f(x)|_\infty := \text{ess sup}_{(v)\in\mathbb{R}^3} |f(x,v)|. \tag{1.31}
\]

Furthermore, we define the \(L^p\) and \(L^\infty\) norms in \(\Omega \times \mathbb{R}^3\):

\[
\|f\|_p := \left( \int_{\Omega \times \mathbb{R}^3} |f(x,v)|^p \, dv \, dx \right)^{\frac{1}{p}}, \quad \|f\|_\infty := \text{ess sup}_{(x,v)\in\Omega \times \mathbb{R}^3} |f(x,v)|. \tag{1.32}
\]

Define the weighted \(L^2\) norms:

\[
|f(x)|_\nu := \left| \nu^\frac{1}{2} f(x) \right|_2, \quad \|f\|_\nu := \left\| \nu^\frac{1}{2} f \right\|_2. \tag{1.33}
\]

Denote the Japanese bracket:

\[
\langle v \rangle = \left( 1 + |v|^2 \right)^{\frac{1}{2}}. \tag{1.34}
\]
Define the weighted $L^\infty$ norm for $\varrho, \vartheta \geq 0$:
\[
|f(x)|_{\infty, \varrho, \vartheta} = \sup_{v \in \mathbb{R}^3} \left( \langle v \rangle^\varrho \varrho|v|^2 |f(x,v)| \right), \quad \|f\|_{\infty, \varrho, \vartheta} = \sup_{(x,v) \in \Omega \times \mathbb{R}^3} \left( \langle v \rangle^\varrho \varrho|v|^2 |f(x,v)| \right).
\]  
In (1.1) and (1.10), based on the flow direction, we can divide the boundary $\gamma := \{(x_0,v): x_0 \in \partial \Omega, v \in \mathbb{R}^3\}$ into the in-flow boundary $\gamma_-$, the out-flow boundary $\gamma_+$, and the grazing set $\gamma_0$:
\[
\gamma_- := \{(x_0,v): x_0 \in \partial \Omega, \ v \cdot n(x_0) > 0\},
\gamma_+ := \{(x_0,v): x_0 \in \partial \Omega, \ v \cdot n(x_0) < 0\},
\gamma_0 := \{(x_0,v): x_0 \in \partial \Omega, \ v \cdot n(x_0) = 0\}.
\]  
It is easy to see $\gamma = \gamma_+ \cup \gamma_- \cup \gamma_0$. In particular, the boundary condition is only given on $\gamma_-$. Define $d\gamma = |v \cdot n| \, d\varpi dv$ on $\gamma$ for the surface measure $\varpi$ the surface measure. Define the $L^p$ and $L^\infty$ norms on the boundary:
\[
\|f\|_{\gamma, p} = \left( \int_\gamma |f(x,v)|^p \, d\gamma \right)^{\frac{1}{p}}, \quad \|f\|_{\gamma, \infty} = \sup_{(x,v) \in \gamma} |f(x,v)|.
\]  
Also, define the weighted $L^\infty$ norm for $\varrho, \vartheta \geq 0$:
\[
\|f\|_{\gamma, \infty, \varrho, \vartheta} = \sup_{(x,v) \in \gamma} \left( \langle v \rangle^\varrho \varrho|v|^2 |f(x,v)| \right).
\]  
The similar notation also applies to $\gamma_\pm$.

**Theorem 1.1.1.** For given $\mu_\varrho^0$ satisfying (1.8) and (1.10), there exists a unique positive solution $\mathcal{F}^\varrho = \mu_\varrho^0 + \vartheta^2 f^\varrho$ to the stationary Boltzmann equation (1.1) with (1.13). In particular, $f^\varrho$ satisfies the equation (1.16) with (1.15), and fulfils that for $0 \leq \varrho < \frac{1}{4}$ and $\vartheta > 3
\[
\|f^\varrho - \varepsilon F\|_{\infty, \varrho, \vartheta} \lesssim \varepsilon \delta^{\frac{1}{2} - \delta},
\]  
for any $0 < \delta << 1$. Here
\[
F = \mu_\varrho^{\frac{2}{3}} \left( \varrho + \vartheta \frac{|v|^2 - 3}{2} \right),
\]  
in which $(\varrho, \vartheta)$ satisfies the steady Navier-Stokes-Fourier system:
\[
\begin{align*}
\nabla \varrho &= \nabla \varrho + \gamma_1 \Delta u + \nabla p = 0, \\
\nabla \vartheta &= \gamma_2 \Delta \vartheta = 0,
\end{align*}
\]  
with boundary data:
\[
\varrho(x_0) = \rho_{\varrho,1}(x_0) + M(x_0), \quad u(x_0) = u_{\varrho,1}(x_0), \quad \vartheta(x_0) = \theta_{\varrho,1}(x_0),
\]  
where $\gamma_1 > 0$ and $\gamma_2 > 0$ are constants, $M(x_0)$ is a constant chosen such that the Boussinesq relation
\[
\rho + \vartheta = \text{constant},
\]  
and the normalization condition (1.15) hold.

**Remark 1.1.2.** The case $\rho_{\varrho,1}(x_0) = 0$, $u_{\varrho,1}(x_0) = 0$ and $\theta_{\varrho,1}(x_0) \neq 0$ is called the non-isothermal model, which represents a system that only has heat transfer through the boundary but has no particle exchange and no work done between the environment and the system. Based on the above theorem, the hydrodynamic limit is a steady Navier-Stokes-Fourier system with non-slip boundary condition. This provides a rigorous derivation of this important fluid model.
1.2 Evolutionary Boltzmann Equation

1.2.1 Problem Presentation

We consider the evolutionary Boltzmann equation in a three-dimensional smooth convex domain \( \Omega \ni x = (x_1, x_2, x_3) \) with velocity \( v = (v_1, v_2, v_3) \in \mathbb{R}^3 \). The density function \( \mathcal{F}(t, x, v) \) satisfies

\[
\begin{cases}
\epsilon^2 \partial_t \mathcal{F} + \epsilon v \cdot \nabla_x \mathcal{F} = Q[\mathcal{F}^e, \mathcal{F}^e] 	ext{ in } \mathbb{R}^+ \times \Omega \times \mathbb{R}^3, \\
\mathcal{F}(0, x, v) = \mathcal{F}_0(x, v) \text{ in } \Omega \times \mathbb{R}^3, \\
\mathcal{F}(t, x, v) = \mathcal{F}_0(x, v) \text{ for } x_0 \in \partial \Omega \text{ and } v \cdot n(x_0) < 0,
\end{cases}
\]

(1.46)

where \( n(x_0) \) is the unit outward normal vector at \( x_0 \), the Knudsen number \( \epsilon \) satisfies \( 0 < \epsilon << 1 \), the diffusive boundary

\[
P^e[\mathcal{F}^e](t, x_0, v) := \mu^e_b(t, x_0, v) \int_{u \cdot n(x_0) > 0} \mathcal{F}^e(t, x_0, u) |u \cdot n(x_0)| \, du.
\]

(1.47)

Boundary Assumption:

The boundary Maxwellian

\[
\mu_b^e(t, x_0, v) := \frac{\rho_b^e(t, x_0)}{2\pi (\theta_b^e(t, x_0))^2} \exp \left( -\frac{|v - u_b^e(t, x_0)|^2}{2\theta_b^e(t, x_0)} \right),
\]

(1.48)

is an \( \epsilon \)-perturbation of the standard Maxwellian

\[
\mu(v) = \frac{1}{2\pi} \exp \left( -\frac{|v|^2}{2} \right).
\]

(1.49)

We assume that both \( \mu_b^e \) and \( \mu \) satisfies the normalization condition

\[
\int_{v \cdot n(x_0) > 0} \mu_b^e(t, x_0, v) |v \cdot n(x_0)| \, dv = \int_{v \cdot n(x_0) > 0} \mu(v) |v \cdot n(x_0)| \, dv = 1.
\]

(1.50)

In addition, we require that the particles are only reflected on \( \partial \Omega \) without in-flow or out-flow, i.e.

\[
\int_{\mathbb{R}^3} \mu_b^e(t, x_0, v) |v \cdot n(x_0)| \, dv = \int_{\mathbb{R}^3} \mu(v) |v \cdot n(x_0)| \, dv = 0.
\]

(1.51)

We also assume that \( \rho_b^e, u_b^e \) and \( \theta_b^e \) can be expanded into a power series with respect to \( \epsilon \),

\[
\rho_b^e(t, x_0) := 1 + \sum_{k=1}^{\infty} \epsilon^k \rho_{b,k}(t, x_0), \quad u_b^e(t, x_0) := 0 + \sum_{k=1}^{\infty} \epsilon^k u_{b,k}(t, x_0), \quad \theta_b^e(t, x_0) := 1 + \sum_{k=1}^{\infty} \epsilon^k \theta_{b,k}(t, x_0),
\]

(1.52)

i.e. \( (\rho_b^e, u_b^e, \theta_b^e) \) is an \( \epsilon \)-perturbation of \((1, 0, 1)\). Hence, we may also expand the boundary Maxwellian \( \mu_b^e \) into power series with respect to \( \epsilon \),

\[
\mu_b^e(t, x_0, v) = \mu(v) + \mu^\frac{1}{2}(v) \left( \sum_{k=1}^{\infty} \epsilon^k \mu_k(t, x_0, v) \right).
\]

(1.53)

In particular, we have

\[
\mu_1(t, x_0, v) := \mu^\frac{1}{2}(v) \left( \rho_{b,1}(t, x_0) + u_{b,1}(t, x_0) \cdot v + \theta_{b,1}(t, x_0) \frac{|v|^2 - 3}{2} \right).
\]

(1.54)
We further assume that for some $K_0 > 0$,
\[
|e^{K_0 t} \langle v \rangle^q e^{\|v\|^2} \frac{\mu_k}{\mu} | + |e^{K_0 t} \langle v \rangle^q e^{\|v\|^2} \mu^{-\frac{1}{2}} \partial_j \mu_k| \leq C_0(\varrho, \vartheta)\epsilon, \quad (1.55)
\]
for any $0 \leq \varrho < \frac{1}{4}$ and $\vartheta > 3$, and constant $C_0 > 0$ is sufficiently small. Based on (1.50), (1.51) and (1.53), we know
\[
\int_{\mathbb{R}^3} \mu_k(t, x_0, v) \mu^\frac{1}{2}(v) |v \cdot n(x_0)| \, dv = 0 \quad \text{for} \quad k \geq 1, \quad (1.56)
\]
\[
\int_{v \cdot n(x_0) \leq 0} \mu_k(t, x_0, v) \mu^\frac{1}{2}(v) |v \cdot n(x_0)| \, dv = 0 \quad \text{for} \quad k \geq 1.
\]

**Initial Assumption:**
We assume that the initial data $\mathfrak{F}_0$ is a perturbation of the standard Maxwellian
\[
\mathfrak{F}_0(x, v) := \mu(v) + \mu^{\frac{1}{2}}(v)f_0(x, v) := \mu(v) + \mu^{\frac{1}{2}}(v) \sum_{k=1}^{\infty} \epsilon^k f_{0,k}(x, v), \quad (1.57)
\]
satisfying
\[
\iint_{\Omega \times \mathbb{R}^3} \mu^{\frac{1}{2}}(v)f_0(x, v) \, dv \, dx = 0, \quad (1.58)
\]
which means that
\[
\iint_{\Omega \times \mathbb{R}^3} \mu^{\frac{1}{2}}(v)f_{0,k}(x, v) \, dv \, dx = 0 \quad \text{for} \quad k \geq 1. \quad (1.59)
\]
In particular, we assume that the initial data $f_{0,1} \in \mathcal{N}$, i.e.
\[
f_{0,1}(x, v) = \mu^{\frac{1}{2}}(v) \left( \rho_{0,1}(x) + u_{0,1}(x) \cdot v + \theta_{0,1}(x) \frac{|v|^2 - 3}{2} \right). \quad (1.60)
\]
Also, we assume the smallness of initial perturbation
\[
|\langle v \rangle^q e^{\|v\|^2} f_0| \leq C_0(\varrho, \vartheta)\epsilon, \quad (1.61)
\]
for any $0 \leq \varrho < \frac{1}{4}$ and $\vartheta > 3$, and constant $C_0 > 0$ is sufficiently small.

**Compatibility Assumption:**
Also, the initial and boundary data satisfy the compatibility condition at $t = 0$ and $x_0 \in \partial \Omega$
\[
\mu_k(0, x_0, v) = 0, \quad \partial_t \mu_k(t, x_0, v) = 0 \quad \text{for} \quad k \geq 1, \quad (1.62)
\]
\[
f_{0,k}(x_0, v) = \rho_{0,k}(x_0) \mu^{\frac{1}{2}}, \quad \nabla_x f_{0,k}(x_0, v) = 0, \quad \nabla_x^2 f_{0,k}(x_0, v) = 0 \quad \text{for} \quad k \geq 1.
\]
We may directly check that the solution $\mathfrak{F}_\epsilon$ satisfies
\[
\iint_{\Omega \times \mathbb{R}^3} \mathfrak{F}_\epsilon(t, x, v) \, dv \, dx = \iint_{\Omega \times \mathbb{R}^3} \mathfrak{F}_0(x, v) \, dv \, dx = \iint_{\Omega \times \mathbb{R}^3} \mu(v) \, dv \, dx = \sqrt{2\pi} |\Omega|. \quad (1.63)
\]
We intend to study the behavior of $\mathfrak{F}_\epsilon$ as $\epsilon \to 0$. 
1.2.2 Linearization

Considering (1.63), the solution $\mathfrak{F}^\varepsilon$ can be expressed as a perturbation of the standard Maxwellian

$$\mathfrak{F}^\varepsilon(t, x, v) = \mu(v) + \mu^\frac{1}{2}(v)f^\varepsilon(t, x, v), \quad (1.64)$$

satisfying the normalization condition

$$\iint_{\Omega \times \mathbb{R}^3} f^\varepsilon(t, x, v) \mu^\frac{1}{2}(v) dv dx = 0. \quad (1.65)$$

Then $f^\varepsilon$ satisfies the equation

$$\begin{cases}
\varepsilon^2 \partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon + \mathcal{L}[f^\varepsilon] = \Gamma[f^\varepsilon, f^\varepsilon] \quad \text{in} \quad \mathbb{R}^+ \times \Omega \times \mathbb{R}^3, \\

f^\varepsilon(0, x, v) = f_0(x, v) \quad \text{in} \quad \Omega \times \mathbb{R}^3, \\
f^\varepsilon(t, x_0, v) = \mathcal{P}^\varepsilon[f^\varepsilon](t, x_0, v) \quad \text{for} \quad x_0 \in \partial \Omega \quad \text{and} \quad v \cdot n(x_0) < 0,
\end{cases} \quad (1.66)$$

where

$$\mathcal{L}[f^\varepsilon] := -2\mu^\frac{1}{2}Q\left[\mu, \mu^\frac{1}{2} f^\varepsilon\right], \quad \Gamma[f^\varepsilon, f^\varepsilon] := \mu^\frac{1}{2}Q\left[\mu^\frac{1}{2} f^\varepsilon, \mu^\frac{1}{2} f^\varepsilon\right], \quad (1.67)$$

and

$$\mathcal{P}^\varepsilon[f^\varepsilon](t, x_0, v) := \mu^\varepsilon(t, x_0, v)\mu^\frac{1}{2}(v) \int_{\mathbb{R}^3} \mu^\varepsilon(u)f^\varepsilon(t, x_0, u)|u \cdot n(x_0)| du + \mu^\frac{1}{2}(v)\left(\mu^\varepsilon(t, x_0, v) - \mu(v)\right). \quad (1.68)$$

Here we use the same notation as in Section 1.1.3 to define $Q$, $\mathcal{L}$ and $\mathcal{N}$. Hence, in order to study $\mathfrak{F}^\varepsilon$, it suffices to consider $f^\varepsilon$.

1.2.3 Main Theorem

Let $\langle \cdot, \cdot \rangle$ be the standard $L^2$ inner product for $v \in \mathbb{R}^3$. Define the $L^p$ and $L^\infty$ norms in $\mathbb{R}^3$:

$$|f(t, x)|_p := \left(\int_{\mathbb{R}^3} |f(t, x, v)|^p dv \right)^{\frac{1}{p}}, \quad |f(t, x)|_\infty := \text{ess sup}_{(v) \in \mathbb{R}^3} |f(t, x, v)|. \quad (1.69)$$

Furthermore, we define the $L^p$ and $L^\infty$ norms in $\Omega \times \mathbb{R}^3$:

$$\|f(t)\|_p := \left(\int_{\Omega \times \mathbb{R}^3} |f(t, x, v)|^p dv dx \right)^{\frac{1}{p}}, \quad \|f(t)\|_\infty := \text{ess sup}_{(x, v) \in \Omega \times \mathbb{R}^3} |f(t, x, v)|. \quad (1.70)$$

Moreover, we define the $L^p$ and $L^\infty$ norms in $[0, t] \times \Omega \times \mathbb{R}^3$:

$$\|f\|_p := \left(\int_{\mathbb{R}^+} \int_{\Omega \times \mathbb{R}^3} |f(x, v)|^p dv dx \right)^{\frac{1}{p}}, \quad \|f\|_\infty := \text{ess sup}_{(t, x, v) \in \mathbb{R}^+ \times \Omega \times \mathbb{R}^3} |f(t, x, v)|. \quad (1.71)$$

Define the weighted $L^2$ norms:

$$|f(t, x)|_\nu := \nu^\frac{1}{2}|f(t, x)|_2, \quad \|f(t)\|_\nu := \|\nu^\frac{1}{2} f(t)\|_2, \quad \|f\|_\nu := \|\nu^\frac{1}{2} f\|_2. \quad (1.72)$$

Denote the Japanese bracket:

$$\langle v \rangle = \left(1 + |v|^2\right)^{\frac{1}{2}} \quad (1.73)$$
Define the weighted $L^\infty$ norm for $\varrho, \vartheta \geq 0$:

$$
|f(t, x)|_{\infty, \varrho, \vartheta} = \text{ess sup}_{x \in \mathbb{R}^3} \left( \langle v \rangle^\varrho \vartheta e^{|v|^2} |f(t, x, v)| \right),
$$

(1.74)

$$
\|f\|_{\infty, \varrho, \vartheta} = \text{ess sup}_{(x, v) \in \Omega \times \mathbb{R}^3} \left( \langle v \rangle^\varrho \vartheta e^{|v|^2} |f(t, x, v)| \right),
$$

(1.75)

$$
\|f\|_{\infty, \varrho, \vartheta} = \text{ess sup}_{(t, x, v) \in \mathbb{R}^+ \times \Omega \times \mathbb{R}^3} \left( \langle v \rangle^\varrho \vartheta e^{|v|^2} |f(t, x, v)| \right).
$$

(1.76)

The similar notation also applies to $\gamma_{\pm}$. In all above notation, we can replace $\mathbb{R}^+$ by $[0, t)$ or even $[s, t]$, and it can be understood from the context without confusion.

**Theorem 1.2.1.** For given $\mu_0^*$ satisfying (1.53) and (1.55) and $f_0$ satisfying (1.61) and (1.62), there exists a unique positive solution $\tilde{\varrho}^\ast = \mu + \mu^2 f^\ast$ to the evolutionary Boltzmann equation (1.46). In particular, $f^\ast$ satisfies the equation (1.66) with (1.65), and fulfills that for $0 \leq \varrho < \frac{1}{4}$ and $\vartheta > 3$

$$
\left\| e^{K\varrho t} (f^\ast - e F) \right\|_{\infty, \varrho, \vartheta} \lesssim C(\delta) e^\frac{1}{2} - \delta,
$$

(1.77)

for any $0 < \delta << 1$, where

$$
F = \mu^\ast \left( \rho + u \cdot v + \vartheta \frac{|v|^2 - 3}{2} \right),
$$

(1.78)

satisfies the unsteady Navier-Stokes-Fourier system

$$
\begin{cases}
\nabla x (\rho + \vartheta) = 0, \\
\partial_t u + u \cdot \nabla x u - \gamma_1 \Delta x u + \nabla x p = 0, \\
\nabla x \cdot u = 0, \\
\partial_t \theta + u \cdot \nabla x \theta - \gamma_2 \Delta x \theta = 0,
\end{cases}
$$

(1.79)

with initial and boundary data

$$
\begin{align*}
\rho(0, x) &= \rho_{0,1}, \quad u(0, x) = u_{0,1}, \quad \theta(0, x) = \theta_{0,1}, \\
\rho(t, x_0) &= \rho_{b,1}(t, x_0) + M(t, x_0), \quad u(t, x_0) = u_{b,1}(t, x_0), \quad \theta(t, x_0) = \theta_{b,1}(t, x_0),
\end{align*}
$$

(1.80)

(1.81)

(1.82)

where $\gamma_1 > 0$ and $\gamma_2 > 0$ are some constants, $M(t, x_0)$ is a constant such that the Boussinesq relation

$$
\rho + \vartheta = \text{constant},
$$

(1.83)

and the normalization condition (1.65) hold for all time $t$. 


1.2.4 Notation and Convention

Throughout this paper, $C > 0$ denotes a constant that only depends on the domain $\Omega$, but does not depend on the data or $\epsilon$. It is referred as universal and can change from one inequality to another. When we write $C(z)$, it means a certain positive constant depending on the quantity $z$. We write $a \lesssim b$ to denote $a \leq Cb$. This paper is organized as follows: in Chapter 2, we study the stationary problem, and in Chapter 3, we study the evolutionary problem. Chapter 4 focuses on the analysis of boundary layer equation, i.e. the $\epsilon$-Milne problem with geometric correction.
Chapter 2

Stationary Boltzmann Equation

2.1 Asymptotic Expansion

2.1.1 Interior Expansion

We define the interior expansion

$$F(x,v) \sim \sum_{k=1}^{3} \epsilon^k F_k(x,v).$$  \hfill (2.1)

Plugging it into the equation (1.16) and comparing the order of $\epsilon$, we obtain

$$L[F_1] = 0,$$  \hfill (2.2)

$$L[F_2] = -v \cdot \nabla_x F_1 + \Gamma[F_1,F_1],$$  \hfill (2.3)

$$L[F_3] = -v \cdot \nabla_x F_2 + 2\Gamma[F_1,F_2].$$  \hfill (2.4)

The analysis of $F_k$ solvability is standard and well-known. Note that the null space $N$ of the operator $L$ is spanned by

$$\mu^\frac{1}{2}\left\{1, v_1, v_2, v_3, \frac{|v|^2 - 3}{2}\right\} = \{\phi_0, \phi_1, \phi_2, \phi_3, \phi_4\}.$$  \hfill (2.5)

Then $L[f] = S$ is solvable if and only if $S \in N^\perp$ the orthogonal complement of $N$ in $L^2(\mathbb{R}^3)$. As [23, Chapter 4] and [24, Chapter 3] reveal, each $F_k$ consists of three parts:

$$F_k(x,v) := A_k(x,v) + B_k(x,v) + C_k(x,v).$$  \hfill (2.6)

- Principal contribution $A_k := \sum_{i=0}^{4} A_{k,i} \phi_i \in N$, where the coefficients $A_{k,i}$ must be determined at each order $k$ independently.

- Connecting contribution $B_k := \sum_{i=0}^{4} B_{k,i} \phi_i \in N$, where the coefficients $B_{k,i}$ depends on $A_s$ for $1 \leq s \leq k-1$. In other words, $B_k$ is accumulative information from previous orders and thus is not independent.
This term is present due to the nonlinearity in $\Gamma$. In detail,

\begin{align*}
B_{k,0} &= 0, \quad B_{k,1} = \sum_{i=1}^{k-1} A_{i,0} A_{k-i,1}, \quad B_{k,2} = \sum_{i=1}^{k-1} A_{i,0} A_{k-i,2}, \quad B_{k,3} = \sum_{i=1}^{k-1} A_{i,0} A_{k-i,3}, \quad (2.7) \\
B_{k,4} &= \sum_{i=1}^{k-1} \left( A_{i,0} A_{k-i,4} + A_{i,1} A_{k-i,1} + A_{i,2} A_{k-i,2} + A_{i,3} A_{k-i,3} \\
&\quad + \sum_{j=1}^{k-i} A_{i,0} (A_{j,1} A_{k-i-j,1} + A_{j,2} A_{k-i-j,2} + A_{j,3} A_{k-i-j,3}) \right).
\end{align*}

- Orthogonal contribution $C_k \in N^\perp$ satisfying

\[ \mathcal{L}[C_k] = -v \cdot \nabla_x F_{k-1} + \sum_{i=1}^{k-1} \Gamma[F_i, F_{k-i}], \quad (2.8) \]

which can be uniquely determined. Similar to $B_k$, here $C_k$ is also accumulative information from previous orders and thus is not independent.

All in all, we will focus on how to determine $A_k$. Traditionally, we write

\[ A_k = \mu^2 \left( \rho_k + u_k \cdot v + \theta_k \left( \frac{|v|^2 - 3}{2} \right) \right), \quad (2.9) \]

where the coefficients $\rho_k$, $u_k$ and $\theta_k$ represent density, velocity and temperature in the macroscopic scale. \[23\,\text{Chapter 4}] and \[24\,\text{Chapter 3}] states that $A_k$ satisfies the equations as follows:

1\textsuperscript{st}-order expansion:

\begin{align*}
 p_1 - (\rho_1 + \theta_1) &= 0, \quad (2.10) \\
 \nabla_x p_1 &= 0, \quad (2.11) \\
 \nabla_x \cdot u_1 &= 0, \quad (2.12)
\end{align*}

2\textsuperscript{nd}-order expansion:

\begin{align*}
 p_2 - (\rho_2 + \theta_2 + \rho_1 \theta_1) &= 0, \quad (2.13) \\
 u_1 \cdot \nabla_x u_1 - \gamma_1 \Delta_x u_1 + \nabla_x p_2 &= 0, \quad (2.14) \\
 u_1 \cdot \nabla_x \theta_1 - \gamma_2 \Delta_x \theta_1 &= 0, \quad (2.15) \\
 \nabla_x \cdot u_2 + u_1 \cdot \nabla_x \rho_1 &= 0. \quad (2.16)
\end{align*}

Here $p_1$ and $p_2$ represent the pressure, $\gamma_1 > 0$ and $\gamma_2 > 0$ are constants. (In particular, for different collision kernel, these constants may be different.) The higher-order expansion produces more complicated fluid equations, which can be found in \[23\,\text{Chapter 4}]. If the interior solution $F_k$ cannot satisfy the boundary condition, then we have to introduce boundary layer $\mathcal{F}_k$ to handle the gap.

2.1.2 Quasi-Spherical Coordinate System

In this section, we focus on three-dimensional transport operator $v \cdot \nabla_x$ and try to rewrite it using the normal and tangential variables near the boundary. This is basically textbook-level differential geometry, so we omit the details.

Substitution 1: Spacial Substitution:
We choose the simplest coordinate system to parameterize the surface $\partial \Omega$. For smooth manifold $\partial \Omega$, there
exists an orthogonal curvilinear coordinates system \((l_1, l_2)\) such that the coordinate lines coincide with the principal directions at \(x_0\) (at least locally).

Assume \(\partial \Omega\) is parameterized by \(r = r(l_1, l_2)\). Let \(|\cdot|\) denote the length and \(\partial_i\) denote the derivative with respect to \(l_i\) for \(i = 1, 2\). Hence, \(\partial_1 r\) and \(\partial_2 r\) represent two orthogonal tangential vectors. Denote \(P_i = |\partial_i r|\) for \(i = 1, 2\). Then define the two orthogonal unit tangential vectors

\[
\varsigma_1 := \frac{\partial_1 r}{P_1}, \quad \varsigma_2 := \frac{\partial_2 r}{P_2}.
\]  

(2.17)

Also, the outward unit normal vector is

\[
n := \frac{\partial_1 r \times \partial_2 r}{|\partial_1 r \times \partial_2 r|} = \varsigma_1 \times \varsigma_2.
\]  

(2.18)

Obviously, \((\varsigma_1, \varsigma_2, n)\) forms a new orthogonal frame. Hence, consider the corresponding new coordinate system \((\mu, l_1, l_2)\), where \(\mu\) denotes the normal distance to boundary surface \(\partial \Omega\), i.e.

\[
x = r - \mu n.
\]  

(2.19)

Note that \(\mu = 0\) means \(x \in \partial \Omega\) and \(\mu > 0\) means \(x \in \Omega\) (before reaching the other side of \(\partial \Omega\)). Using this new coordinate system, the transport operator becomes

\[
v \cdot \nabla_x = -\left( \frac{(\partial_1 r - \mu \partial_1 n) \times (\partial_2 r - \mu \partial_2 n)}{(\partial_1 r - \mu \partial_1 n) \times (\partial_2 r - \mu \partial_2 n)} \cdot v \right) \frac{\partial f}{\partial \mu} - \left( \frac{(\partial_2 r - \mu \partial_2 n) \times n}{(\partial_1 r - \mu \partial_1 n) \times (\partial_2 r - \mu \partial_2 n)} \cdot v \right) \frac{\partial f}{\partial \mu} + \left( \frac{(\partial_2 r - \mu \partial_2 n) \times n}{(\partial_1 r - \mu \partial_1 n) \times (\partial_2 r - \mu \partial_2 n)} \cdot v \right) \frac{\partial f}{\partial l_2}.
\]  

(2.20)

We may further simplify this expression utilizing the orthogonality. Denote the first fundamental form

\[
(E, F, G) := \left( \partial_1 r \cdot \partial_1 r, \partial_1 r \cdot \partial_2 r, \partial_2 r \cdot \partial_2 r \right),
\]  

(2.21)

and second fundamental form

\[
(L, M, N) := \left( \partial_{l_1} r \cdot n, \partial_{l_2} r \cdot n, \partial_{l_2} r \cdot n \right).
\]  

(2.22)

Then we have \(F = M = 0\) due to the orthogonality. Two principal curvatures are given by

\[
\kappa_1 := \frac{L}{E}, \quad \kappa_2 := \frac{N}{G}.
\]  

(2.23)

Also, we know the relation

\[
\partial_1 n = \kappa_1 \partial_1 r, \quad \partial_2 n = \kappa_2 \partial_2 r.
\]  

(2.24)

Hence, direct computation using \(2.18\) and \(2.23\) reveals that

\[
\left( \frac{(\partial_1 r - \mu \partial_1 n) \times (\partial_2 r - \mu \partial_2 n)}{(\partial_1 r - \mu \partial_1 n) \times (\partial_2 r - \mu \partial_2 n)} \cdot n \right) = (1 - \kappa_1 \mu)(1 - \kappa_2 \mu)(\partial_1 r \times \partial_2 r) \cdot n
\]  

(2.25)

\[
= (1 - \kappa_1 \mu)(1 - \kappa_2 \mu)P_1 P_2,
\]

\[
\left( \frac{(\partial_1 r - \mu \partial_1 n) \times (\partial_2 r - \mu \partial_2 n)}{(\partial_1 r - \mu \partial_1 n) \times (\partial_2 r - \mu \partial_2 n)} \cdot v \right) = (1 - \kappa_1 \mu)(1 - \kappa_2 \mu)(\partial_1 r \times \partial_2 r) \cdot v
\]  

(2.26)

\[
= (1 - \kappa_1 \mu)(1 - \kappa_2 \mu)P_1 P_2(v \cdot n),
\]
and
\[
\left( (\partial_2 r - \mu \partial_2 n) \times n \right) \cdot v = (1 - \kappa_2 \mu)(\partial_2 r \times n) \cdot v = (1 - \kappa_2 \mu)P_2(v \cdot \varsigma_1),
\]
\[
\left( (\partial_1 r - \mu \partial_1 n) \times n \right) \cdot v = (1 - \kappa_1 \mu)(\partial_1 r \times n) \cdot v = -(1 - \kappa_1 \mu)P_1(v \cdot \varsigma_2).
\]

Hence, plugging (2.24), (2.26), (2.27) and (2.28) into (2.20), we have the transport operator
\[
v \cdot \nabla_x = -(v \cdot n) \frac{\partial}{\partial \mu} - \frac{v \cdot \varsigma_1}{P_1(\kappa_1 \mu - 1)} \frac{\partial}{\partial v_1} - \frac{v \cdot \varsigma_2}{P_2(\kappa_2 \mu - 1)} \frac{\partial}{\partial v_2}.
\]

Therefore, under substitution \((x_1, x_2, x_3) \to (\mu, \iota_1, \iota_2)\), the equation (1.10) is transformed into
\[
\left\{ \begin{array}{l}
\epsilon \left( -(v \cdot n) \frac{\partial f^*}{\partial \mu} - \frac{v \cdot \varsigma_1}{P_1(\kappa_1 \mu - 1)} \frac{\partial f^*}{\partial v_1} - \frac{v \cdot \varsigma_2}{P_2(\kappa_2 \mu - 1)} \frac{\partial f^*}{\partial v_2} \right) + f^* + \mathcal{L}[f^*] = \Gamma[f^*, f^*] \text{ in } \Omega \times \mathbb{R}^3,
\end{array} \right.
\]
\[
f^*(0, t_1, t_2, v) = \mathcal{P}[f^*](0, t_1, t_2, v) \text{ for } v \cdot n < 0.
\]

Substitution 2: Velocity Substitution.
Define the orthogonal velocity substitution for \(v := (v_\eta, v_\phi, v_\psi)\) as
\[
\left\{ \begin{array}{l}
-v \cdot n := v_\eta, \\
-v \cdot \varsigma_1 := v_\phi, \\
-v \cdot \varsigma_2 := v_\psi.
\end{array} \right.
\]

Then using chain rule, fundamental forms and (2.24), we have
\[
\frac{\partial}{\partial v_1} \to \frac{\partial}{\partial v_1} + \frac{\partial}{\partial v_\eta} \frac{\partial}{\partial \iota_1} + \frac{\partial}{\partial v_\phi} \frac{\partial}{\partial \iota_1} + \frac{\partial}{\partial v_\psi} \frac{\partial}{\partial \iota_1}
\]
\[
= \frac{\partial}{\partial \iota_1} - \kappa_1 P_1 v_\phi \frac{\partial}{\partial v_\eta} + \left( \left( \partial_{11} r \cdot n \right) \frac{1}{P_1} v_\eta + \left( \partial_{12} r \cdot \partial_2 r \right) \frac{1}{P_1 P_2} v_\psi \right) \frac{\partial}{\partial v_\phi}
\]
\[
+ \left( \left( \partial_{12} r \cdot n \right) \frac{1}{P_2} v_\eta + \left( \partial_{12} r \cdot \partial_1 r \right) \frac{1}{P_1 P_2} v_\phi \right) \frac{\partial}{\partial v_\psi},
\]
and
\[
\frac{\partial}{\partial v_2} \to \frac{\partial}{\partial v_2} + \frac{\partial}{\partial v_\eta} \frac{\partial}{\partial v_2} + \frac{\partial}{\partial v_\phi} \frac{\partial}{\partial v_2} + \frac{\partial}{\partial v_\psi} \frac{\partial}{\partial v_2}
\]
\[
= \frac{\partial}{\partial v_2} - \kappa_2 P_2 v_\psi \frac{\partial}{\partial v_\eta} + \left( \left( \partial_{22} r \cdot n \right) \frac{1}{P_1} v_\eta + \left( \partial_{22} r \cdot \partial_1 r \right) \frac{1}{P_1 P_2} v_\phi \right) \frac{\partial}{\partial v_\psi}
\]
\[
+ \left( \left( \partial_{22} r \cdot n \right) \frac{1}{P_2} v_\eta + \left( \partial_{22} r \cdot \partial_1 r \right) \frac{1}{P_1 P_2} v_\phi \right) \frac{\partial}{\partial v_\psi}.
\]
Here, we utilize $\partial_{12} \eta \cdot n = M = 0$ in the second fundamental form and $\partial_i r \cdot n = -\partial_r \cdot \partial_i n = -\kappa_i |\partial_i r|^2$ for $i = 1, 2$. Then the transport operator in (\ref{eq:asymptotic}) becomes

\begin{equation}
\begin{aligned}
v \cdot \nabla_x = & v_\eta \frac{\partial}{\partial \mu} - \frac{1}{R_1 - \mu} \left( v_\phi^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} \right) - \frac{1}{R_2 - \mu} \left( v_\phi^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} \right) \\
& - \frac{1}{P_1 P_2} \left( \frac{\partial_{11} r \cdot \partial_{2r}}{P_1 (\kappa_1 \mu - 1)} v_\phi v_\psi + \frac{\partial_{12} r \cdot \partial_{2r}}{P_2 (\kappa_2 \mu - 1)} v_\psi^2 \right) \frac{\partial}{\partial v_\eta} - \frac{1}{P_1 P_2} \left( \frac{\partial_{22} r \cdot \partial_{1r}}{P_2 (\kappa_2 \mu - 1)} v_\phi v_\psi + \frac{\partial_{12} r \cdot \partial_{1r}}{P_1 (\kappa_1 \mu - 1)} v_\psi^2 \right) \frac{\partial}{\partial v_\phi} \\
& - \frac{v_\phi}{P_1 (\kappa_1 - 1) \partial_1 v_\eta + \frac{v_\psi}{P_2 (\kappa_2 - 1) \partial_2 v_\eta}} \frac{\partial}{\partial v_\eta} + \frac{v_\psi}{P_2 (\kappa_2 - 1) \partial_2 v_\phi} \frac{\partial}{\partial v_\phi},
\end{aligned}
\end{equation}

where $R_1 = \frac{1}{\kappa_1}$ and $R_2 = \frac{1}{\kappa_2}$ represent the radius of principal curvature. Hence, under substitution $v \rightarrow v$, the equation (\ref{eq:asymptotic}) is transformed into

\begin{equation}
\begin{aligned}
\epsilon \nu_i \frac{\partial f^c}{\partial \mu} = & \frac{\epsilon}{R_1 - \mu} \left( v_\phi^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} \right) - \frac{\epsilon}{R_2 - \mu} \left( v_\phi^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} \right) \\
& - \frac{\epsilon}{P_1 P_2} \left( \frac{\partial_{11} r \cdot \partial_{2r}}{P_1 (\kappa_1 \mu - 1)} v_\phi v_\psi + \frac{\partial_{12} r \cdot \partial_{2r}}{P_2 (\kappa_2 \mu - 1)} v_\psi^2 \right) \frac{\partial}{\partial v_\eta} - \frac{\epsilon}{P_1 P_2} \left( \frac{\partial_{22} r \cdot \partial_{1r}}{P_2 (\kappa_2 \mu - 1)} v_\phi v_\psi + \frac{\partial_{12} r \cdot \partial_{1r}}{P_1 (\kappa_1 \mu - 1)} v_\psi^2 \right) \frac{\partial}{\partial v_\phi} \\
& - \frac{\epsilon}{P_1 (\kappa_1 - 1) \partial_1 v_\eta + \frac{\epsilon}{P_2 (\kappa_2 - 1) \partial_2 v_\eta}} \frac{\partial}{\partial v_\eta} + \frac{\epsilon v_\psi}{P_2 (\kappa_2 - 1) \partial_2 v_\phi} \frac{\partial}{\partial v_\phi} + L[f^c] = \Gamma[f^c, f^c] \text{ in } \Omega \times \mathbb{R}^3,
\end{aligned}
\end{equation}

\begin{equation}
f^c(0, t_1, t_2, v) = \mathcal{P}[f^c](0, t_1, t_2, v) \text{ for } v > 0.
\end{equation}

Substitution 3: Scaling Substitution.

Finally, we define the scaled variable $\eta = \frac{\mu}{\epsilon}$, which implies $\frac{\partial}{\partial \mu} = \frac{1}{\epsilon} \frac{\partial}{\partial \eta}$. Then, under the substitution $\mu \rightarrow \eta$, the equation (\ref{eq:asymptotic}) is transformed into

\begin{equation}
\begin{aligned}
v_\eta \frac{\partial f^c}{\partial \eta} = & \frac{\epsilon}{R_1 - \epsilon \eta} \left( v_\phi^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} \right) - \frac{\epsilon}{R_2 - \epsilon \eta} \left( v_\phi^2 \frac{\partial}{\partial v_\eta} - v_\eta v_\phi \frac{\partial}{\partial v_\phi} \right) \\
& - \frac{\epsilon}{P_1 P_2} \left( \frac{\partial_{11} r \cdot \partial_{2r}}{P_1 (\kappa_1 \eta - 1)} v_\phi v_\psi + \frac{\partial_{12} r \cdot \partial_{2r}}{P_2 (\kappa_2 \eta - 1)} v_\psi^2 \right) \frac{\partial}{\partial v_\eta} - \frac{\epsilon}{P_1 P_2} \left( \frac{\partial_{22} r \cdot \partial_{1r}}{P_2 (\kappa_2 \eta - 1)} v_\phi v_\psi + \frac{\partial_{12} r \cdot \partial_{1r}}{P_1 (\kappa_1 \eta - 1)} v_\psi^2 \right) \frac{\partial}{\partial v_\phi} \\
& - \frac{\epsilon}{P_1 (\kappa_1 \eta - 1) \partial_1 v_\eta + \frac{\epsilon}{P_2 (\kappa_2 \eta - 1) \partial_2 v_\eta}} \frac{\partial}{\partial v_\eta} + \frac{\epsilon v_\psi}{P_2 (\kappa_2 \eta - 1) \partial_2 v_\phi} \frac{\partial}{\partial v_\phi} + L[f^c] = \Gamma[f^c, f^c] \text{ in } \Omega \times \mathbb{R}^3,
\end{aligned}
\end{equation}

\begin{equation}
f^c(0, t_1, t_2, v) = \mathcal{P}^c[f^c](0, t_1, t_2, v) \text{ for } v > 0.
\end{equation}

2.1.3 Boundary Layer Expansion

We define the boundary layer expansion:

\begin{equation}
\mathcal{F}(\eta, t_1, t_2, v) \sim \sum_{k=1}^{2} \epsilon^k \mathcal{F}_k(\eta, t_1, t_2, v),
\end{equation}
where $\mathcal{F}_k$ can be defined by comparing the order of $\epsilon$ via plugging (2.37) into the equation (2.36). Thus, in a neighborhood of the boundary, we have

$$
\begin{align*}
\dot{v}_n &= -\frac{\epsilon}{R_1 - \epsilon_1} \left( v_\phi^2 \frac{\partial \mathcal{F}_1}{\partial v_\phi} - v_n v_\phi \frac{\partial \mathcal{F}_1}{\partial v_\phi} \right) - \frac{\epsilon}{R_2 - \epsilon_1} \left( v_\phi^2 \frac{\partial \mathcal{F}_1}{\partial v_\phi} - v_n v_\phi \frac{\partial \mathcal{F}_1}{\partial v_\phi} \right) + \mathcal{L}[\mathcal{F}_1] = 0, \\
\dot{v}_n &= -\frac{\epsilon}{R_1 - \epsilon_1} \left( v_\phi^2 \frac{\partial \mathcal{F}_2}{\partial v_\phi} - v_n v_\phi \frac{\partial \mathcal{F}_2}{\partial v_\phi} \right) - \frac{\epsilon}{R_2 - \epsilon_1} \left( v_\phi^2 \frac{\partial \mathcal{F}_2}{\partial v_\phi} - v_n v_\phi \frac{\partial \mathcal{F}_2}{\partial v_\phi} \right) + \mathcal{L}[\mathcal{F}_2] = Z,
\end{align*}
$$

where $Z = Z \left[ F_1, \mathcal{F}_1, \frac{\partial \mathcal{F}_1}{\partial v_\phi}, \frac{\partial \mathcal{F}_1}{\partial v_\phi}, \frac{\partial \mathcal{F}_1}{\partial \eta_1}, \frac{\partial \mathcal{F}_1}{\partial \eta_2} \right]$ as

$$
Z := 2\Gamma[F_1, \mathcal{F}_1] + \Gamma[\mathcal{F}_1, \mathcal{F}_1] + \frac{1}{P_1 P_2} \left( \frac{\partial_1 r \cdot \partial_2 r}{P_1 (\epsilon_1 \eta_1 - 1)} v_\phi v_\phi + \frac{\partial_1 r \cdot \partial_2 r}{P_2 (\epsilon_2 \eta_2 - 1)} \right) \frac{\partial \mathcal{F}_1}{\partial v_\phi} + \frac{v_\phi}{P_1 (\epsilon_1 \eta_1 - 1)} \frac{\partial \mathcal{F}_1}{\partial \eta_1} + \frac{v_\phi}{P_2 (\epsilon_2 \eta_2 - 1)} \frac{\partial \mathcal{F}_1}{\partial \eta_2}.
$$

### 2.1.4 Boundary Condition Expansion

The bridge between the interior solution and boundary layer is the boundary condition. Define

$$
P[f](x_0, v) := \mu^2(v) \int_{u \cdot n(x_0) > 0} \mu^2(u) f(x_0, u) |u \cdot n(x_0)| \, du.
$$

Plugging the combined expansion from (2.1) and (2.37)

$$
f^* \sim \sum_{k=1}^{3} \epsilon^k F_k + \sum_{k=1}^{2} \epsilon^k \mathcal{F}_k
$$

into the boundary condition (1.10) and (1.18), and comparing the order of $\epsilon$, we obtain

$$
F_1 + \mathcal{F}_1 = p[F_1 + \mathcal{F}_1] + \mu_1(x_0, v),
$$

$$
F_2 + \mathcal{F}_2 = p[F_2 + \mathcal{F}_2] + \mu_1(x_0, v) \int_{u \cdot n(x_0) > 0} \mu^2(u)[F_1 + \mathcal{F}_1] |u \cdot n(x_0)| \, du + \mu_2(x_0, v).
$$

In particular, we do not further expand the boundary layer, so we directly require

$$
F_3 = p[F_3] + \mu_2(x_0, v) \int_{u \cdot n(x_0) > 0} \mu^2(u)[F_1 + \mathcal{F}_1] |u \cdot n(x_0)| \, du
$$

$$
+ \mu_1(x_0, v) \int_{u \cdot n(x_0) > 0} \mu^2(u)[F_2 + \mathcal{F}_2] |u \cdot n(x_0)| \, du + \mu_3(x_0, v).
$$

These are the boundary conditions $F_k$ and $\mathcal{F}_k$ need to satisfy.

### 2.1.5 Matching Procedure

Define the length of boundary layer in rescaled variable $L := \epsilon^{-\frac{1}{2}}$. Also, denote $\mathcal{B}[v_n, v_\phi, v_\psi] = (-v_n, v_\phi, v_\psi)$.

Step 1: Construction of $F_1$ and $\mathcal{F}_1$.

Based on Section 2.1.1 we know $F_1 = A_1$ since there is no contribution of $B_1$ and $C_1$. Considering the boundary Maxwellian expansion (1.54) and reorganizing (2.10)–(2.19), we define

$$
F_1 := \mu^2 \left( \rho_1 + u_1 \cdot v + \theta_1 \frac{|v|^2}{2} - 3 \right),
$$
where \((\rho_1, u_1, \theta_1)\) satisfies the Navier-Stokes-Fourier system

\[
\begin{align*}
\nabla_x (\rho_1 + \theta_1) &= 0, \\
u_1 \cdot \nabla_x u_1 - \gamma_1 \Delta_x u_1 + \nabla_x p_1 &= 0, \\
\nabla_x \cdot u_1 &= 0, \\
u_1 \cdot \nabla_x \theta_1 - \gamma_2 \Delta_x \theta_1 &= 0,
\end{align*}
\] (2.47)

with the boundary condition

\[
\rho_1(x_0) = \rho_{b,1}(x_0) + M_1(x_0), \quad u_1(x_0) = u_{b,1}(x_0), \quad \theta_1(x_0) = \theta_{b,1}(x_0).
\] (2.48)

Here \(M_1(x_0)\) is chosen such that the Boussinesq relation

\[\rho_1 + \theta_1 = \text{constant}\] (2.49)

is satisfied (which is part of (2.10)–(2.16)). Note that the above constant is determined by the normalization condition

\[
\int_{\Omega \times \mathbb{R}^3} F_1(x, v) \mu^{\frac{1}{2}}(v) \, dv \, dx = 0.
\] (2.50)

which is a requirement from (1.15). On the other hand, based on (1.11), we naturally obtain

\[\mathcal{P}[F_1] = M_1 \mu^{\frac{1}{2}},\] (2.51)

which means

\[F_1 = \mathcal{P}[F_1] + \mu_1 \text{ on } \partial \Omega.\] (2.52)

Therefore, compared with (2.43), since \(F_1\) already satisfies the boundary condition, it is not necessary to introduce the boundary layer at this order and we simply take \(\mathcal{F}_1 = 0\).

Step 2: Construction of \(F_2\) and \(\mathcal{F}_2\).

Define \(F_2 = A_2 + B_2 + C_2\), where \(B_2\) and \(C_2\) can be uniquely determined following previous analysis, and

\[A_2 = \mu^{\frac{1}{2}} \left( \rho_2 + u_2 \cdot v + \theta_2 \frac{|v|^2 - 3}{2} \right),\] (2.53)

satisfying a fluid-type equation (see [23 Page 92])

\[
\begin{align*}
p_2 - (\rho_2 + \theta_2 + \rho_1 \theta_1) &= 0, \\
u_1 \cdot \nabla_x u_1 + (\rho_1 u_1 + u_2) \cdot \nabla_x u_1 - \gamma_1 \Delta_x u_2 + \nabla_x p_3 &= -\gamma_2 \nabla_x \cdot \Delta_x \theta_1 - \gamma_4 \nabla_x \cdot \left( \theta_1 \left( \nabla_x u_1 + (\nabla_x u)^T \right) \right), \\
\nabla_x \cdot u &= -u_1 \cdot \nabla_x \rho_1, \\
u_1 \cdot \nabla_x \theta_2 + (\rho_1 u_1 + u_2) \cdot \nabla_x \theta_1 - u_1 \cdot \nabla_x p_2 &= \gamma_1 \left( \nabla_x u_1 + (\nabla_x u)^T \right)^2 + \Delta_x \left( \gamma_2 \theta_2 + \gamma_3 \theta_1^2 \right),
\end{align*}
\] (2.54)

where \(\gamma_3, \gamma_4, \gamma_5\) are constants. Now \(F_2\) does not satisfy (2.44) alone, so we have to introduce boundary layer. Let \(\mathcal{F}_2\) satisfy the \(\epsilon\)-Milne problem with geometric correction

\[
\begin{align*}
\frac{\partial \mathcal{F}_2}{\partial \eta} - \frac{\epsilon}{R_1 - \epsilon \eta} \left( v_{\psi}^{\mu} \frac{\partial \mathcal{F}_2}{\partial v_\eta} - v_\eta v_\psi \frac{\partial \mathcal{F}_2}{\partial v_\psi} \right) - \frac{\epsilon}{R_2 - \epsilon \eta} \left( v_{\psi}^{\mu} \frac{\partial \mathcal{F}_2}{\partial v_\eta} - v_\eta v_\psi \frac{\partial \mathcal{F}_2}{\partial v_\psi} \right) + \mathcal{L}[\mathcal{F}_2] &= 0, \\
\mathcal{F}_2(0, \iota_1, \iota_2, v) &= h(\iota_1, \iota_2, v) - \tilde{h}(\iota_1, \iota_2, v) \text{ for } \nu_\eta > 0, \\
\mathcal{F}_2(L, \iota_1, \iota_2, v) &= \mathcal{F}_2(L, \iota_1, \iota_2, \mathcal{F}[v]),
\end{align*}
\] (2.55)
with the in-flow boundary data

\[ h(\tau_1,\tau_2, \mathbf{v}) = M_1 \mu_1(x_0, v) + \mu_2(x_0, v) - \left( (B_2 + C_2) - \mathcal{P}[B_2 + C_2] \right). \]  

(2.56)

Using (1.11), considering \( B_2 \) and \( C_2 \) given in Section 2.1.1, we may directly check that

\[ \int_{v_0 > 0} \mu \hat{\xi} (\mathbf{v}) h(\tau_1, \tau_2, \mathbf{v}) \mid v_0 \mid \, d\mathbf{v} = - \int_{\mathbb{R}^3} (B_2 + C_2)(x_0) \left( v \cdot n(x_0) \right) \, dv = 0. \]  

(2.57)

Based on Theorem 4.1.15 and 4.1.24, there exists a unique

\[ \hat{h}(\tau_1, \tau_2, \mathbf{v}) = \mu \frac{1}{2} \sum_{k=0}^{4} \hat{D}_k(\tau_1, \tau_2) \mathbf{e}_k, \]  

(2.58)

such that (2.55) is well-posed and the solution decays exponentially fast (here \( \mathbf{e}_k \) with \( k = 0, 1, 2, 3, 4 \) form a basis of null space \( \mathcal{N} \) of \( \mathcal{L} \)). In particular, \( \hat{D}_1 = 0 \). Then we further require that \( A_2 \) satisfies the boundary condition

\[ A_2(x_0, v) = \hat{h}(\tau_1, \tau_2, \mathbf{v}) + M_2(x_0) \mu \frac{1}{2} (\mathbf{v}). \]  

(2.59)

Here \( x_0 \) corresponds to \( (\tau_1, \tau_2) \) and \( v \) corresponds to \( \mathbf{v} \), based on substitution in Section 2.1.2. Here, the constant \( M_2(x_0) \) is chosen to enforce the Boussinesq relation

\[ p_2 - (\rho_2 + \theta_2 + \rho_1 \theta_1) = 0, \]  

(2.60)

where \( p_2 \) is the pressure solved from (2.47). Similar to the construction of \( F_1 \), due to (1.15), we can choose the constant to satisfy the normalization condition

\[ \int \int_{\Omega \times \mathbb{R}^3} (F_2 + \mathcal{F}_2)(x, v) \mu \frac{1}{2} (v) \, dv \, dx = 0. \]  

(2.61)

Also, based on (2.61), \( \mathcal{F}_1 = 0 \), (3.66) and (2.55), we have

\[ A_2 + \mathcal{F}_2 = M_2 \mu \frac{1}{2} + h \]  

(2.62)

\[ = M_2 \mu \frac{1}{2} + \mu_1 \int_{u \cdot n(x_0) > 0} \mu \hat{\xi}(u)(F_1 + \mathcal{F}_1) \mid u \cdot n(x_0) \mid \, du + \mu_2 - \left( (B_2 + C_2) - \mathcal{P}[B_2 + C_2] \right). \]

Comparing this with the desired boundary expansion (2.44), i.e.

\[ A_2 + B_2 + C_2 + \mathcal{F}_2 = \mathcal{P}[A_2 + B_2 + C_2 + \mathcal{F}_2] + \mu_1 \int_{u \cdot n(x_0) > 0} \mu \hat{\xi}(u)(F_1 + \mathcal{F}_1) \mid u \cdot n(x_0) \mid \, du + \mu_2; \]  

(2.63)

we only need to verify that

\[ \mathcal{P}[A_2 + \mathcal{F}_2] = M_2 \mu \frac{1}{2}. \]  

(2.64)

Based on Theorem 4.1.15 the equation (2.55) implies the zero mass-flux condition of \( \mathcal{F}_2 \) as

\[ \int_{\mathbb{R}^3} \mu \hat{\xi}(u) \mathcal{F}_2(x, u) (u \cdot n) \, du = 0. \]  

(2.65)

Since \( \mu_1 \) and \( \mu_2 \) satisfy (1.11), based on (3.66), we have

\[ \mathcal{P}[A_2 + \mathcal{F}_2] = \mu \hat{\xi} \int_{u \cdot n > 0} \mu \hat{\xi}(u) A_2(x, u) (u \cdot n) \, du + \mu \hat{\xi} \int_{u \cdot n > 0} \mu \hat{\xi}(u) \mathcal{F}_2(x, u) (u \cdot n) \, du \]  

(2.66)

\[ = M_2 \mu \frac{1}{2} + \mu \frac{1}{2} \int_{u \cdot n > 0} \mu \hat{\xi}(u) \hat{h}(x, u) (u \cdot n) \, du + \mu \frac{1}{2} \int_{u \cdot n > 0} \mu \hat{\xi}(u) \mathcal{F}_2(x, u) (u \cdot n) \, du. \]
Using (2.65) and (2.55), noting $v_\eta > 0$ represents in-flow boundary, we know

$$P[A_2 + F_2] = M_2 \mu^2 + \mu^3 \int_{u \cdot n > 0} \mu^2 (u) \tilde{h}(x, u)(u \cdot n) du - \mu^2 \int_{u \cdot n < 0} \mu^2 (u) F_2 (x, u)(u \cdot n) du$$

(2.67)

$$= M_2 \mu^2 + \mu^3 \int_{u \cdot n > 0} \mu^2 (u) \tilde{h}(x, u)(u \cdot n) du - \mu^2 \int_{u \cdot n < 0} \mu^2 (u) (h - \tilde{h})(x, u)(u \cdot n) du.$$

Then direct computation reveals that

$$P[A_2 + F_2] = M_2 \mu^2 + \mu^3 \int_{u \cdot n > 0} \mu^2 (u) \tilde{h}(x, u)(u \cdot n) du - \mu^2 \int_{u \cdot n < 0} \mu^2 (u) h(x, u)(u \cdot n) du$$

(2.68)

$$+ \mu^2 \int_{u \cdot n < 0} \mu^2 (u) h(x, u)(u \cdot n) du$$

$$= M_2 \mu^2 + \mu^3 \int_{u \cdot n > 0} \mu^2 (u) \tilde{h}(x, u)(u \cdot n) du - \mu^2 \int_{u \cdot n < 0} \mu^2 (u) h(x, u)(u \cdot n) du.$$

Finally, using (2.56), (1.11), (2.57) and $\tilde{D}_1 = 0$, we obtain

$$P[A_2 + F_2] = M_2 \mu^2 + \tilde{D}_1 - 0 = M_2 \mu^2.$$

(2.69)

$F_3$ can be defined in a similar fashion which satisfies an even more complicated fluid-type system (see [23 Page 92]).
2.2 Remainder Estimates

We consider the linearized stationary Boltzmann equation
\[
\begin{align*}
\left\{ \begin{array}{l}
ev \cdot \nabla_x f + \mathcal{L}[f] = S(x,v) \quad \text{in} \quad \Omega \times \mathbb{R}^3, \\
f(x_0,v) = \mathcal{P}[f](x_0,v) + h(x_0,v) \quad \text{for} \quad x_0 \in \partial \Omega \quad \text{and} \quad v \cdot n < 0,
\end{array} \right.
\end{align*}
\tag{2.70}
\]
where
\[
\mathcal{P}[f](x_0,v) = \mu_{\pm}^{\frac{1}{2}}(v) \int_{\gamma_{\pm}} f(x_0,v) \mu_{\pm}^{\frac{1}{2}}(v) d\gamma.
\tag{2.71}
\]

The data \( S \) and \( h \) satisfy the compatibility condition
\[
\int_{\Omega \times \mathbb{R}^3} S(x,v) \mu_{\pm}^{\frac{1}{2}}(v) dv dx + \int_{\gamma_{\pm}} h(x,v) \mu_{\pm}^{\frac{1}{2}}(v) d\gamma = 0.
\tag{2.72}
\]

It is easy to see if \( f \) is a solution to (2.70), then \( f + C \mu_{\pm}^{\frac{1}{2}} \) is also a solution for arbitrary \( C \in \mathbb{R} \). Hence, to guarantee uniqueness, the solution should satisfy the normalization condition
\[
\int_{\Omega \times \mathbb{R}^3} f(x,v) \mu_{\pm}^{\frac{1}{2}}(v) dv dx = 0.
\tag{2.73}
\]

Our analysis is based on the ideas in [5, 12, 29] and [27]. Since proof of the well-posedness of (2.70) is standard, we will focus on the a priori estimates here.

2.2.1 Preliminaries

We first introduce the well-known micro-macro decomposition. Define \( \mathbb{P} \) as the orthogonal projection onto the null space of \( \mathcal{L} \):
\[
\mathbb{P}[f] := \mu_{\pm}^{\frac{1}{2}}(v) \left( a_f(x) + v \cdot b_f(x) + \frac{|v|^2 - 3}{2} c_f(x) \right) \in \mathcal{N},
\tag{2.74}
\]
where \( a_f, b_f \) and \( c_f \) are coefficients. When there is no confusion, we will simply write \( a, b, c \). Definitely, \( \mathcal{L} [\mathbb{P}[f]] = 0 \). Then the operator \( I - \mathbb{P} \) is naturally
\[
(I - \mathbb{P})[f] := f - \mathbb{P}[f],
\tag{2.75}
\]
which satisfies \((I - \mathbb{P})[f] \in \mathcal{N}^\bot\), i.e. \( \mathcal{L}[f] = \mathcal{L} [(I - \mathbb{P})[f]] \).

**Lemma 2.2.1.** The linearized collision operator \( \mathcal{L} = v I - K \) defined in (1.22) is self-adjoint in \( L^2 \). It satisfies
\[
\langle v \rangle \lesssim \nu(v) \lesssim \langle v \rangle,
\tag{2.76}
\]
\[
\langle f, \mathcal{L}[f] \rangle (x) = \left\langle (I - \mathbb{P})[f], \mathcal{L} [(I - \mathbb{P})[f]] \right\rangle (x),
\tag{2.77}
\]
\[
|\langle (I - \mathbb{P})[f(x)] \rangle |^2 \lesssim \langle f, \mathcal{L}[f] \rangle (x) \lesssim |\langle (I - \mathbb{P})[f(x)] \rangle |^2.
\tag{2.78}
\]

**Proof.** These are standard properties of \( \mathcal{L} \). See [8, Chapter 3] and [12, Lemma 3].

**Lemma 2.2.2.** For \( 0 < \delta << 1 \), define the near-grazing set of \( \gamma_{\pm} \):
\[
\gamma_{\pm}^{\delta} := \left\{ (x,v) \in \gamma_{\pm} : |n(x) \cdot v| \leq \delta \text{ or } |v| \geq \frac{1}{\delta} \text{ or } |v| \leq \delta \right\}.
\tag{2.79}
\]

Then
\[
\left\| f 1_{\gamma_{\pm} \setminus \gamma_{\pm}^{\delta}} \right\|_{\gamma_{\pm}} \leq C(\delta) \left( \| f \|_1 + \| v \cdot \nabla_x f \|_1 \right).
\tag{2.80}
\]

Here \( 1 \) denotes the indicator function.
Step 1: Estimates of $c$

We choose the test function

$$\psi = \mu^\frac{1}{2}(v) \left( \frac{v^2}{2} - \beta_c \right) \left( v \cdot \nabla_x \phi_c(x) \right),$$

where

$$\begin{cases}
-\Delta_x \phi_c = c |c|^{2m-2} (x) & \text{in } \Omega, \\
\phi_c = 0 & \text{on } \partial \Omega,
\end{cases}$$

and $\beta_c \in \mathbb{R}$ will be determined later. Based on the standard elliptic estimates in [13], we have

$$\|\phi_c\|_{W^{2, \frac{2m}{2m-1}}(\Omega)} \lesssim \|c|^{2m-1}\|_{L^\frac{2m}{2m-1}(\Omega)} \lesssim \|c|^{2m-1}\|_{L^\infty(\Omega)}.$$
Hence, by Sobolev embedding theorem, we know
\[ \|\psi_c\|_2 \lesssim \|\phi_c\|_{H^1(\Omega)} \lesssim \|\phi_c\|_{W^{2, \frac{2m}{m+4}}(\Omega)} \lesssim \|c\|_{L^{2m}(\Omega)}^{2m-1}, \tag{2.89} \]
\[ \|\phi_c\|_{W^{1, \frac{2m}{m+4}}(\Omega)} \lesssim \|\phi_c\|_{W^{2, \frac{2m}{m+4}}(\Omega)} \lesssim \|c\|_{L^{2m}(\Omega)}^{2m-1}. \tag{2.90} \]

Also, for \(1 \leq m \leq 3\), using Sobolev embedding theorem and trace estimates, we have
\[ |\nabla_x \phi_c|_{L^{\frac{2m}{m+4}}(\partial \Omega)} \lesssim |\nabla_x \phi_c|_{W^{1, \frac{2m}{m+4}}(\partial \Omega)} \lesssim |\nabla_x \phi_c|_{W^{2, \frac{2m}{m+4}}(\partial \Omega)} \lesssim \|\phi_c\|_{W^{2, \frac{2m}{m+4}}(\Omega)} \lesssim \|c\|_{L^{2m}(\Omega)}^{2m-1}. \tag{2.91} \]

We first consider the right-hand side (RHS) of (2.85). With the choice of (2.86) and Hölder’s inequality, using (2.83) and Lemma 2.2.1, we have
\[ \left| \int_{\Omega \times \mathbb{R}^3} \psi_c \mathcal{L}\left((\mathbb{I} - \mathcal{P})[f]\right) \right| = \left| \int_{\Omega \times \mathbb{R}^3} \mathcal{L}[\psi_c](\mathbb{I} - \mathcal{P})[f] \right| \lesssim \|\mathcal{L}[\psi_c]\|_2 \|(\mathbb{I} - \mathcal{P})[f]\|_2 \tag{2.92} \]
\[ \lesssim \|\psi_c\|_2 \|(\mathbb{I} - \mathcal{P})[f]\|_2 \lesssim \|c\|_{L^{2m}(\Omega)}^{2m-1} \|\mathbb{I} - \mathcal{P}\|_2 \|\psi_c\|_2, \]
and
\[ \left| \int_{\Omega \times \mathbb{R}^3} S \psi_c \right| \lesssim \|\psi_c\|_2 \left\|\nu^{-\frac{1}{2}} S\right\|_2 \lesssim \|c\|_{L^{2m}(\Omega)}^{2m-1} \left\|\nu^{-\frac{1}{2}} S\right\|_2. \tag{2.93} \]

Therefore, we know
\[ \text{RHS} \lesssim \left( \|(\mathbb{I} - \mathcal{P})[f]\|_2 + \left\|\nu^{-\frac{1}{2}} S\right\|_2 \right) \|c\|_{L^{2m}(\Omega)}^{2m-1}. \tag{2.94} \]

Then we turn to the left-hand side (LHS) of (2.85). Based on (2.70), note the decomposition
\[ f|_{\gamma} = 1_{\gamma_+} f + 1_{\gamma_-} \mathcal{P}[f] + 1_{\gamma_-} h = 1_{\gamma_+} \mathcal{P}[f] + 1_{\gamma_+} (1 - \mathcal{P})[f] + 1_{\gamma_-} h. \tag{2.95} \]

Then, we will choose \(\beta_c\) such that
\[ \int_{\mathbb{R}^3} \mu^\frac{1}{2} (v_i)^2 (v_i - \beta_c) v_i^2 dv = 0 \quad \text{for} \quad i = 1, 2, 3, \tag{2.96} \]
which, combined with oddness, implies
\[ \int_{\mathbb{R}^3} \mu^\frac{1}{2} \psi_c d\gamma - \int \mu^\frac{1}{2} \psi_c d\gamma = 0. \tag{2.97} \]

Hence, the boundary term on the LHS of (2.85) can be simplified as
\[ \epsilon \int_{\gamma_+} f \psi_c d\gamma - \epsilon \int_{\gamma_-} f \psi_c d\gamma \tag{2.98} \]
\[ = \left( \int_{\gamma_+} \mathcal{P}[f] \psi_c d\gamma - \int_{\gamma_-} \mathcal{P}[f] \psi_c d\gamma \right) + \epsilon \int_{\gamma_+} (1 - \mathcal{P})[f] \psi_c d\gamma - \epsilon \int_{\gamma_-} h \psi_c d\gamma \]
\[ = \epsilon \int_{\gamma_+} (1 - \mathcal{P})[f] \psi_c d\gamma - \epsilon \int_{\gamma_-} h \psi_c d\gamma. \]

Applying Hölder’s inequality and (2.91) to (2.83), we have
\[ \left| \epsilon \int_{\gamma_+} f \psi_c d\gamma - \epsilon \int_{\gamma_-} f \psi_c d\gamma \right| \lesssim \epsilon \left( \|(1 - \mathcal{P})[f]\|_{\gamma_+, \frac{4m}{m+4}} + \|h\|_{\gamma_-, \frac{4m}{m+4}} \right) \|\psi_c\|_{\gamma_+, \frac{4m}{m+4}} \tag{2.99} \]
\[ \lesssim \epsilon \left( \|(1 - \mathcal{P})[f]\|_{\gamma_+, \frac{4m}{m+4}} + \|h\|_{\gamma_-, \frac{4m}{m+4}} \right) \|\nabla_x \phi_c\|_{L^{\frac{2m}{m+4}}(\partial \Omega)} \]
\[ \lesssim \epsilon \left( \|(1 - \mathcal{P})[f]\|_{\gamma_+, \frac{4m}{m+4}} + \|h\|_{\gamma_-, \frac{4m}{m+4}} \right) \|c\|_{L^{2m}(\Omega)}^{2m-1}. \]
For the bulk term on the LHS of (2.85), noting the decomposition
\[
f = P[f] + (I - P)[f] = \mu^\frac{1}{2} \left( a + v \cdot b + \frac{|v|^2 - 3}{2} c \right) + (I - P)[f],
\]
we have
\[
-\epsilon \int_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \psi_c) f = -\epsilon \int_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \psi_c) \mu^\frac{1}{2}(v) \left( a + v \cdot b + \frac{|v|^2 - 3}{2} c \right) - \epsilon \int_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \psi_c)(I - P)[f].
\]
Considering (2.86), we directly compute
\[
v \cdot \nabla_x \psi_c = \mu^\frac{1}{2}(v) \left( \sum_{i,j=1}^3 v_i v_j \partial_i \partial_j \phi_c \right).
\]
Due to oddness, the \( b \) contribution in (2.101) vanishes. (2.96) implies that the \( a \) contribution in (2.101) also vanishes. For the \( c \) contribution, using (2.102) and oddness, with
\[
\int_{\mathbb{R}^3} \mu(v) |v|^2 \left( |v|^2 - \beta_c \right) \frac{|v|^2 - 3}{2} dv \neq 0 \quad \text{for} \quad i = 1, 2, 3,
\]
we know
\[
-\epsilon \int_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \psi_c) \mu^\frac{1}{2}(v) \left( |v|^2 - \beta_c \right) \frac{|v|^2 - 3}{2} c = -\int_{\Omega} c \Delta_x \phi_c = ||c||_{L^{2m}(\Omega)}^{2m}.
\]
Also, Hölder’s inequality and (2.90) yield
\[
\left| \epsilon \int_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \psi_c) (I - P)[f] \right| \lesssim \epsilon \|v \cdot \nabla_x \psi_c\|_{L^{2m}(\Omega)} \|I - P\|_{L^{2m}(\Omega)}\|f\|_{L^{2m}(\Omega)}
\]
\[
\lesssim \epsilon \|\phi_c\|_{W^{2,\frac{2m}{2m-1}}(\Omega)} \|I - P\|_{L^{2m}(\Omega)}\|f\|_{L^{2m}(\Omega)} \lesssim \epsilon \|c\|_{L^{2m-1}(\Omega)} \|I - P\|_{L^{2m}(\Omega)}\|f\|_{L^{2m}(\Omega)}.
\]
Collecting (2.94), (2.99), (2.104), (2.105), and cancelling \( ||c||_{L^{2m}(\Omega)}^{2m-1} \), we have
\[
\epsilon ||c||_{L^{2m}(\Omega)} \lesssim \epsilon \|I - P\|_{\gamma, \frac{1}{4m} + \|I - P\|_{\gamma, \frac{1}{2}} + \epsilon \|I - P\|_{\gamma, \frac{1}{2m}} + \left\| \nu^{-\frac{1}{2}} S \right\|_{2} + \epsilon \|h\|_{\gamma, \frac{1}{4m}}.
\]
Step 2: Estimates of \( b \).
We further divide this step into several sub-steps:

Sub-Step 2.1: Estimates of \( \left( \partial_i \partial_j \Delta_x^{1} \left( b_j |b_j|^{2m-2} \right) \right) b_i \) for \( i, j = 1, 2, 3 \).
Let \( b = (b_1, b_2, b_3) \). We choose the test functions for \( i, j = 1, 2, 3 \),
\[
\psi = \psi_{b,i,j} = \mu^\frac{1}{2}(v) \left( v_i^2 - \beta_{b,i,j} \right) \partial_j \phi_{b,j},
\]
where
\[
\begin{cases} 
-\Delta_x \phi_{b,j} = b_j |b_j|^{2m-2} (x) \text{ in } \Omega, \\
\phi_{b,j} = 0 \text{ on } \partial \Omega,
\end{cases}
\]
and \( \beta_{b,i,j} \in \mathbb{R} \) will be determined later. This is very similar to Step 1. We can recover the elliptic estimates and trace estimates as in (2.89), (2.90) and (2.91). With the choice of (2.107), the right-hand side (RHS) of (2.85) is bounded by
\[
\text{RHS} \lesssim \left( \|I - P\|_{\gamma, \frac{1}{2}} + \left\| \nu^{-\frac{1}{2}} S \right\|_{2} \right) ||b||_{L^{2m}(\Omega)}^{2m-1}.
\]
We will choose $\beta_b$ such that
\[ \int_{\mathbb{R}^3} \mu(v) \left( |v_i|^2 - \beta_{b,i,j} \right) \, dv = 0, \tag{2.110} \]
which means for the boundary term on the left-hand side of (2.85), there is no $P[f]$ contribution. We may recover estimates as (2.99)
\[ \left| \epsilon \iint_{\gamma_+} f \psi_{b,i,j} \, d\gamma - \epsilon \iint_{\gamma_-} f \psi_{b,i,j} \, d\gamma \right| \lesssim \epsilon \left( \| (1 - P)[f] \|_{\gamma_+} \frac{c_1}{m} + \| h \|_{\gamma_-} \frac{c_1}{m} \right) \| b \|_{L^{2m}(\Omega)}^{2m-1}. \tag{2.111} \]
For the bulk term on the LHS of (2.85), the $a$ and $c$ contribution vanish due to oddness of (2.107). Then we focus on the $b$ contribution
\[ -\epsilon \iint_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \psi_{b,i,j}) \mu^b(v) (b \cdot v) = -\epsilon \iint_{\Omega \times \mathbb{R}^3} \sum_{k,s=1}^{3} \mu(v) \left( v_i^2 - \beta_{b,i,j} \right) v_k v_s b_k \partial_k \partial_j \phi_{b,j}. \tag{2.112} \]
Due to oddness, for $k \neq s$, the terms in (2.112) vanish. Hence, we have
\[ -\epsilon \iint_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \psi_{b,i,j}) \mu^b(v) (b \cdot v) = -\epsilon \iint_{\Omega \times \mathbb{R}^3} \sum_{k=1}^{3} \mu(v) \left( v_i^2 - \beta_{b,i,j} \right) v_i^2 b_k \partial_k \partial_j \phi_{b,j}. \tag{2.113} \]
Based on our choice of $\beta_{b,i,j}$ in (2.110), we directly compute
\[ \int_{\mathbb{R}^3} \mu(v) \left( |v_i|^2 - \beta_{b} \right) v_i^2 \, dv = 0 \quad \text{for} \quad k \neq i, \tag{2.114} \]
\[ \int_{\mathbb{R}^3} \mu(v) \left( |v_i|^2 - \beta_{b} \right) v_i^2 \, dv \neq 0. \tag{2.115} \]
Thus, for $k \neq i$, the terms in (2.113) vanish. Hence, we have
\[ -\epsilon \iint_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \psi_{b,i,j}) \mu^b(v) (b \cdot v) = -\epsilon \iint_{\Omega \times \mathbb{R}^3} \mu(v) \left( v_i^2 - \beta_{b,i,j} \right) v_i^2 b_i \partial_i \partial_j \phi_{b,j} \tag{2.116} \]
\[ = -\epsilon \int_{\Omega} b_i \partial_i \partial_j \phi_{b,j} = -\int_{\Omega} \left( \partial_i \partial_j \Delta_x^{-1} \left( b_j |b_j|^{2m-2} \right) \right) b_i. \tag{2.117} \]
Finally, the $(I - P)[f]$ contribution on the LHS of (2.85) can be estimated as
\[ \left| \epsilon \iint_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \psi_{b,i,j})(I - P)[f] \right| \lesssim \epsilon \| b \|_{L^{2m}(\Omega)}^{2m-1} \| (I - P)[f] \|_{2m}. \tag{2.118} \]
Collecting (2.109), (2.111), (2.116) and (2.118), we obtain
\[ \epsilon \left| \int_{\Omega} \left( \partial_i \partial_j \Delta_x^{-1} \left( b_j |b_j|^{2m-2} \right) \right) b_i \right| \lesssim \| b \|_{L^{2m}(\Omega)}^{2m-1} \left( \epsilon \| (1 - P)[f] \|_{\gamma_+} \frac{c_1}{m} + \| (I - P)[f] \|_2 + \epsilon \| (1 - P)[f] \|_{2m} + \| \nu^{-\frac{1}{2}} S \|_2 + \epsilon \| h \|_{\gamma_-} \frac{c_1}{m} \right). \tag{2.119} \]
Note that we cannot further simplify the LHS of (2.119) at this stage since we do not include all derivative terms in $\Delta_x b_j$. For example, $\partial_i \partial_j \Delta_x^{-1} \left( b_j |b_j|^{2m-2} \right) b_j$ is not controlled here.

Sub-Step 2.2: Estimates of $\left( \partial_i \partial_j \Delta_x^{-1} \left( b_j |b_j|^{2m-2} \right) \right) b_j$ for $i \neq j$. Notice that the $i = j$ case is included in Sub-Step 2.1. We choose the test function
\[ \psi = \psi_{b,i,j} = \mu^b(v) |v|^2 v_i v_j \partial_i \partial_j \phi_{b,j} \quad \text{for} \quad i \neq j. \tag{2.20} \]
Similar to Sub-Step 2.1, we focus on the $b$ contribution on the LHS of (2.85)

$$-\epsilon \iint_{\Omega \times \mathbb{R}^3} \left( v \cdot \nabla x \tilde{\psi}_{b,i,j} \right) \mu^b(v)(b \cdot v) = -\epsilon \iint_{\Omega \times \mathbb{R}^3} \sum_{k,s=1}^3 \mu(v) |v|^2 v_k v_s v_i v_j b_s \partial_k \partial_j \phi_{b,j}. \quad (2.121)$$

Due to oddness, the terms in (2.121) do not vanish only if $k = i, s = j$ or $k = j, s = i$. Hence, we are left with

$$-\epsilon \iint_{\Omega \times \mathbb{R}^3} \left( v \cdot \nabla x \tilde{\psi}_{b,i,j} \right) \mu^b(v)(b \cdot v) = -\epsilon \iint_{\Omega \times \mathbb{R}^3} \mu(v) |v|^2 v_i^2 v_j^2 (b_j \partial_i \partial_j \phi_{b,j} + b_i \partial_j \partial_i \phi_{b,j})$$

$$= -\epsilon \iint_{\Omega} \left( b_j \partial_i \partial_j \phi_{b,j} + b_i \partial_j \partial_i \phi_{b,j} \right). \quad (2.122)$$

Note that $\epsilon \left( \iint_{\Omega} b_i \partial_j \partial_i \phi_{b,j} \right)$ has been controlled by Sub-Step 2.2. Hence, we obtain

$$\epsilon \left| \int_{\Omega} \left( \partial_i \partial_j \Delta^{-1}_x \left( b_j \left| b_j \right|^{2m-2} \right) \right) b_j \right| \quad (2.123)$$

$$\leq \|b\|^{2m-1}_{L^{2m}(\Omega)} \left( \epsilon \| (1 - P) [f] \|_{\gamma_+, \frac{4m}{2m}} + \| (1 - P) [f] \|_2 + \epsilon \| (1 - P) [f] \|_{2m} + \left\| \nu^{-\frac{1}{2}} S \right\|_2 + \epsilon \| h \|_{\gamma_-, \frac{4m}{2m}} \right).$$

Sub-Step 2.3: Synthesis.
Summarizing (2.119) and (2.123), we may sum up over $j = 1, 2, 3$ to obtain, for any $i = 1, 2, 3,$

$$\epsilon \| b_i \|^{2m}_{L^{2m}(\Omega)} \lesssim \| b \|^{2m-1}_{L^{2m}(\Omega)} \left( \epsilon \| (1 - P) [f] \|_{\gamma_+, \frac{4m}{2m}} + \| (1 - P) [f] \|_2 + \epsilon \| (1 - P) [f] \|_{2m} + \left\| \nu^{-\frac{1}{2}} S \right\|_2 + \epsilon \| h \|_{\gamma_-, \frac{4m}{2m}} \right). \quad (2.124)$$

which further implies

$$\epsilon \| b \|_{L^{2m}(\Omega)} \lesssim \epsilon \| (1 - P) [f] \|_{\gamma_+, \frac{4m}{2m}} + \| (1 - P) [f] \|_2 + \epsilon \| (1 - P) [f] \|_{2m} + \left\| \nu^{-\frac{1}{2}} S \right\|_2 + \epsilon \| h \|_{\gamma_-, \frac{4m}{2m}}. \quad (2.125)$$

Step 3: Estimates of $a$.
We choose the test function

$$\psi = \psi_a = \mu^b(v) \left( |v|^2 - \beta_a \right) \left( v \cdot \nabla x \phi_a(x) \right), \quad (2.126)$$

where

$$\begin{cases} -\Delta_x \phi_a = a |a|^{2m-2} (x) - \frac{1}{|\Omega|} \int_\Omega a |a|^{2m-2} (x) dx \quad \text{in} \ \Omega, \\ \partial \phi_a \partial n = 0 \quad \text{on} \ \partial \Omega, \end{cases} \quad (2.127)$$

and $\beta_a \in \mathbb{R}$ will be determined later. Since

$$\int_{\Omega} \left( a^{2m-1}(x) - \frac{1}{|\Omega|} \int_{\Omega} a^{2m-1}(x) dx \right) dx = 0, \quad (2.128)$$

based on standard elliptic estimates, we may recover the estimates as (2.89), (2.90) and (2.91). Then similar to Step 1, we obtain that the right-hand side (RHS) of (2.85) is bounded as

$$\text{RHS} \lesssim \left( \| (1 - P) [f] \|_2 + \left\| \nu^{-\frac{1}{2}} S \right\|_2 \right) \| a \|^{2m-1}_{L^{2m}(\Omega)}. \quad (2.129)$$
For the left-hand side (LHS) of (2.85), the bulk term can be estimated as Step 1. There is no \( b \) contribution due to oddness. We will choose \( \beta_a \) such that

\[
\int_{\mathbb{R}^3} \mu^\perp(v) \left( |v|^2 - \beta_a \right) \frac{|v|^2 - 3}{2} v_i^2 dv = 0 \quad \text{for} \quad i = 1, 2, 3, \tag{2.130}
\]

which will eliminate \( c \) contribution. Hence, the remaining \( a \) contribution will be

\[
-\epsilon \int_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \psi_a) \mu^\perp(v) a = -\int_\Omega a \Delta_x \phi_a = \|a\|_{L^2_m(\Omega)}^{2m} . \tag{2.131}
\]

Here, we use the fact that \( \int_\Omega a = \int_{\Omega \times \mathbb{R}^3} f(x, v) = 0 \) due to (2.73). Similarly, \( (I - P)[f] \) contribution is

\[
\epsilon \int_{\Omega \times \mathbb{R}^3} (v \cdot \nabla_x \psi_a) (I - P)[f] \left| \right| \lesssim \epsilon \|a\|_{L^2_m(\Omega)}^{2m-1} \|(I - P)[f]\|_{2m} . \tag{2.132}
\]

Now the only difficulty is the boundary term in (2.85). In particular, as in (2.98), we are concerned with

\[
\int_{\gamma_+} \mathcal{P}[f] \psi_a d\gamma - \int_{\gamma_-} \mathcal{P}[f] \psi_a d\gamma = \int_{\partial \Omega \times \mathbb{R}^3} (v \cdot n) \mu^\perp(v) \left( |v|^2 - \beta_a \right) \left( v \cdot \nabla_x \phi_a(x_0) \right) \mathcal{P}[f](x_0, v) . \tag{2.133}
\]

This cannot be directly killed with the previous techniques (oddness and choice of \( \beta_a \) cannot do it). Notice that \( \mathcal{P}[f](x_0, v) = z(x_0) \mu^\perp(v) \) for \( z(x_0) = \int_{\gamma_+} f d\gamma \). We decompose the velocity into normal and tangential directions:

\[
v = n(v \cdot n) + n^\perp , \tag{2.134}
\]

where \( n^\perp \) is the tangential part. Then

\[
\int_{\partial \Omega \times \mathbb{R}^3} (v \cdot n) \mu^\perp(v) \left( |v|^2 - \beta_a \right) \left( v \cdot \nabla_x \phi_a(x_0) \right) \mathcal{P}[f](x_0, v) \tag{2.135}
\]

\[
= \int_{\partial \Omega \times \mathbb{R}^3} (v \cdot n)^2 \mu^\perp(v) \left( |v|^2 - \beta_a \right) \frac{\partial \phi_a(x_0)}{\partial n} \mathcal{P}[f](x_0, v) + \int_{\partial \Omega \times \mathbb{R}^3} (v \cdot n) \mu(v) \left( |v|^2 - \beta_a \right) \left( n^\perp \cdot \nabla_x \phi_a(x_0) \right) z(x_0) .
\]

Here, in the RHS, the first term vanishes due to the Neumann boundary condition in (2.127), and the second term vanishes due to oddness. Then in total, we have

\[
\int_{\gamma_+} \mathcal{P}[f] \psi_a d\gamma - \int_{\gamma_-} \mathcal{P}[f] \psi_a d\gamma = 0 . \tag{2.136}
\]

With this in hand, we can bound as (2.99) to get the boundary contribution

\[
\left| \epsilon \int_{\gamma_+} f \psi_a d\gamma - \epsilon \int_{\gamma_-} f \psi_a d\gamma \right| \lesssim \epsilon \left( \left\| (1 - \mathcal{P})[f] \right\|_{\gamma_+, \frac{4m}{3}} + \left\| h \right\|_{\gamma_-, \frac{4m}{3}} \right) \|a\|_{L^2_m(\Omega)}^{2m-1} . \tag{2.137}
\]

Collecting (2.103), (2.104), (2.105), (2.106), and cancelling \( \|a\|_{L^2_m(\Omega)}^{2m-1} \), we have

\[
\epsilon \|a\|_{L^2_m(\Omega)} \lesssim \epsilon \left( \left\| (1 - \mathcal{P})[f] \right\|_{\gamma_+, \frac{4m}{3}} + \left\| (I - \mathcal{P})[f] \right\|_2 + \epsilon \left\| (I - \mathcal{P})[f] \right\|_{2m} + \left\| \nu^\perp S \right\|_2 + \epsilon \left\| h \right\|_{\gamma_-, \frac{4m}{3}} . \tag{2.138}
\]

Step 4: Synthesis. Collecting (2.103), (2.105) and (2.138), we deduce

\[
\epsilon \|\mathcal{P}[f]\|_{2m} \lesssim \epsilon \left( \left\| (1 - \mathcal{P})[f] \right\|_{\gamma_+, \frac{4m}{3}} + \left\| (I - \mathcal{P})[f] \right\|_2 + \epsilon \left\| (I - \mathcal{P})[f] \right\|_{2m} + \left\| \nu^\perp S \right\|_2 + \epsilon \left\| h \right\|_{\gamma_-, \frac{4m}{3}} . \tag{2.139}
\]

This completes our proof.
Theorem 2.2.7. The solution $f(x, v)$ to the equation (2.70) satisfies the estimate

$$
\frac{1}{\epsilon} \| (1 - \mathcal{P}) [f] \|_{\gamma, 2} + \frac{1}{\epsilon} \| (\mathcal{I} - \mathcal{P}) [f] \|_{\nu} + \mathcal{P} [f] \|_{2m} \leq \frac{1}{\epsilon^2} \| \mathcal{P} [S] \|_{\infty} + \frac{1}{\epsilon^2} \| \nu^{-\frac{1}{2}} (\mathcal{I} - \mathcal{P}) [S] \|_{2} + \| h \|_{\gamma, \infty} + \frac{1}{\epsilon} \| h \|_{\gamma, 2}.
$$

Proof.

Step 1: Energy Estimate.

Multiplying $f$ on both sides of (2.70) and applying Green’s identity in Lemma 2.2.3 imply

$$
\frac{\epsilon}{2} \int_{\Omega \times \mathbb{R}^3} f \mathcal{L} [f] = \int_{\Omega \times \mathbb{R}^3} f.S.
$$

A direct computation shows that

$$
\| (1 - \mathcal{P}) [f] \|_{\gamma, 2}^2 = \int_{\gamma_+} \left( f - \mathcal{P} [f] \right)^2 d\gamma = \int_{\gamma_+} f^2 d\gamma + \int_{\gamma_+} \mathcal{P} [f]^2 d\gamma - 2 \int_{\gamma_+} f \mathcal{P} [f] d\gamma
$$

$$
= \| f \|_{\gamma, 2}^2 + \| \mathcal{P} [f] \|_{\gamma, 2}^2 - 2 \| \mathcal{P} [f] \|_{\gamma, 2}^2 = \| f \|_{\gamma, 2}^2 - \| \mathcal{P} [f] \|_{\gamma, 2}^2.
$$

Obviously, $\| \mathcal{P} [f] \|_{\gamma, 2}^2 = \| \mathcal{P} [f] \|_{\gamma, 2}^2$. Hence, we have

$$
\frac{\epsilon}{2} \| f \|_{\gamma, 2}^2 - \frac{\epsilon}{2} \| \mathcal{P} [f] \|_{\gamma, 2}^2 + \int_{\Omega \times \mathbb{R}^3} h \mathcal{L} [f] \geq \| (1 - \mathcal{P}) [f] \|_{\nu}^2 + \int_{\Omega \times \mathbb{R}^3} h \mathcal{L} [f] \geq \int_{\Omega \times \mathbb{R}^3} f.S.
$$

Inserting (2.143) and (2.144) into (2.141), we have

$$
\epsilon \| (1 - \mathcal{P}) [f] \|_{\gamma, 2}^2 + \| (\mathcal{I} - \mathcal{P}) [f] \|_{\nu}^2 \leq \frac{1}{\eta} \| \mathcal{P} [f] \|_{\gamma, 2}^2 + \frac{1}{\eta} \| h \|_{\gamma, 2}^2 + \int_{\Omega \times \mathbb{R}^3} f.S.
$$

Step 2: Estimate of $\| \mathcal{P} [f] \|_{\gamma, 2}$.

Multiplying $f$ on both sides of the equation (2.70), we have

$$
v \cdot \nabla_x (f^2) = \frac{2}{\epsilon} \left( - f \mathcal{L} [f] + f.S \right).
$$

Taking absolute value and integrating (2.146) over $\Omega \times \mathbb{R}^3$, using Lemma 2.2.1 we deduce

$$
\| v \cdot \nabla_x (f^2) \|_{1} \leq \frac{1}{\epsilon} \left( \| (1 - \mathcal{P}) [f] \|_{2}^2 + \int_{\Omega \times \mathbb{R}^3} f.S \right).
$$

On the other hand, applying Lemma 2.2.2 to $f^2$, for near grazing set $\gamma^5$, we have

$$
\| 1_{\gamma} f^2 \|_{\gamma, 2} \leq C(\delta) \left( \| f \|_{1}^2 + \| v \cdot \nabla_x (f^2) \|_{1} \right) = C(\delta) \left( \| f \|_{2}^2 + \| v \cdot \nabla_x (f^2) \|_{1} \right)
$$

$$
\leq C(\delta) \left( \| f \|_{2}^2 + \frac{1}{\epsilon} \| (1 - \mathcal{P}) [f] \|_{2}^2 + \frac{1}{\epsilon} \int_{\Omega \times \mathbb{R}^3} f.S \right).
$$
As in Step 3 of proof to Lemma 2.2.6 we can rewrite \( \mathcal{P}[f](x_0, v) = z(x)\mu^n(v) \). Then for \( \delta \) small, we deduce

\[
\|\mathcal{P}[1, \gamma^n f]\|_{\gamma^n, 2}^2 = \int_{\partial \Omega} |z(x)|^2 \left( \int_{\mathbb{R}^3} |v - n(x)| \geq \delta, |v| \leq \delta^{-1} \right. \mu(v) |v \cdot n(x)| \, dv \right) \, dx
\]

\[
\geq \frac{1}{2} \left( \int_{\partial \Omega} |z(x)|^2 \, dx \right) \left( \int_{\gamma^+} \mu(v) |v \cdot n(x)| \, dv \right) = \frac{1}{2} \|\mathcal{P}[f]\|_{\gamma^n, 2}^2,
\]

where we utilize the bounds that

\[
\int_{v \cdot n(x) \leq \delta} \mu(v) |v \cdot n(x)| \, dv \lesssim \delta,
\]

\[
\int_{|v| \leq \delta \text{ or } |v| \geq \delta^{-1}} \mu(v) |v \cdot n(x)| \, dv \lesssim \delta.
\]

Therefore, from (2.149) and the fact

\[
\|\mathcal{P}[1, \gamma^n f]\|_{\gamma^n, 2} \lesssim \|1, \gamma^n f\|_{\gamma^n, 2} \lesssim \|1, \gamma^n f\|_{\gamma, 2},
\]

we conclude

\[
\|\mathcal{P}[f]\|_{\gamma^n, 2}^2 \lesssim \|\mathcal{P}[1, \gamma^n f]\|_{\gamma^n, 2} \lesssim \|1, \gamma^n f\|_{\gamma, 2}.
\]

Considering (2.148), we have

\[
\|\mathcal{P}[f]\|_{\gamma^n, 2}^2 \leq C(\delta) \left( \|f\|_2 + \frac{1}{\epsilon} \|(1 - \mathcal{P})[f]\|_2 + \frac{1}{\epsilon} \int_{\Omega \times \mathbb{R}^3} f \mathcal{S} \right).
\]

For fixed \( 0 < \delta \ll 1 \) and using \( f = \mathcal{P}[f] + (1 - \mathcal{P})[f] \), we obtain

\[
\|\mathcal{P}[f]\|_{\gamma^n, 2}^2 \lesssim \|\mathcal{P}[f]\|_2^2 + \frac{1}{\epsilon} \|(1 - \mathcal{P})[f]\|_2^2 + \frac{1}{\epsilon} \int_{\Omega \times \mathbb{R}^3} f \mathcal{S} \bigg|.
\]

Step 3: Interpolation Estimates.

Plugging (2.155) into (2.149) with \( \epsilon \) sufficiently small to absorb \( \|(1 - \mathcal{P})[f]\|_\nu^2 \) into the left-hand side, we obtain

\[
\epsilon \|(1 - \mathcal{P})[f]\|_{\gamma^n, 2}^2 + \|(1 - \mathcal{P})[f]\|_\nu^2 \lesssim \eta \epsilon^2 \|\mathcal{P}[f]\|_2^2 + \frac{1}{\eta} \|h\|_{\gamma^n, 2}^2 + \int_{\Omega \times \mathbb{R}^3} f \mathcal{S} \bigg|.
\]

We square on both sides of (2.84) to obtain

\[
\epsilon^2 \|\mathcal{P}[f]\|_{2m}^2 \lesssim \epsilon^2 \|(1 - \mathcal{P})[f]\|_{\gamma^n, 2m}^2 + \|(1 - \mathcal{P})[f]\|_{\gamma^n, 2m}^2 + \epsilon^2 \|\mathcal{P}[f]\|_{2m}^2 + \|\nu^{\frac{1}{2}} S\|_2^2 + \epsilon^2 \|h\|_{\gamma^n, 2m}^2 + \int_{\Omega \times \mathbb{R}^3} f \mathcal{S} \bigg|.
\]

Hölder’s inequality implies

\[
\|\mathcal{P}[f]\|_2 \lesssim \|\mathcal{P}[f]\|_{2m}.
\]

Multiplying a small constant on both sides of (2.157) and adding to (2.156) with \( \eta > 0 \) sufficiently small to absorb \( \eta \epsilon^2 \|\mathcal{P}[f]\|_2^2 \) and \( \|(1 - \mathcal{P})[f]\|_2^2 \) into the left-hand side, we obtain

\[
\epsilon \|(1 - \mathcal{P})[f]\|_{\gamma^n, 2}^2 + \|(1 - \mathcal{P})[f]\|_\nu^2 + \epsilon^2 \|\mathcal{P}[f]\|_{2m}^2 \lesssim \epsilon^2 \|(1 - \mathcal{P})[f]\|_{\gamma^n, 2m}^2 + \epsilon^2 \|\mathcal{P}[f]\|_{2m}^2 + \|\nu^{\frac{1}{2}} S\|_2^2 + \epsilon^2 \|h\|_{\gamma^n, 2m}^2 + \int_{\Omega \times \mathbb{R}^3} f \mathcal{S} \bigg|.
\]
Now we need to handle the extra term $\epsilon^2 \|(1 - \mathcal{P})[f]\|_{\gamma_{+}, 4m}^2$ and $\epsilon^2 \|(\mathbb{I} - \mathcal{P})[f]\|_{2m}^2$ on the right-hand side of (2.159). By interpolation estimate and Young’s inequality, we have

\[
\|(1 - \mathcal{P})[f]\|_{\gamma_{+}, 4m} \leq \|(1 - \mathcal{P})[f]\|_{\gamma_{+}, 2}^{\frac{1}{2}} \|(1 - \mathcal{P})[f]\|_{\gamma_{+}, \infty}^{\frac{1}{2}} \leq \left( \frac{1}{\epsilon^{2m-1}} \right) \|(1 - \mathcal{P})[f]\|_{\gamma_{+}, 2}^{\frac{1}{2}} \left( \epsilon^{\frac{6m-3}{4m^2}} \|(1 - \mathcal{P})[f]\|_{\gamma_{+}, \infty}^{\frac{2m-3}{2m}} \right) \leq \frac{1}{\epsilon^{2m-1}} \|(1 - \mathcal{P})[f]\|_{\gamma_{+}, 2} + o(1) \epsilon^{\frac{1}{2m}} \|(1 - \mathcal{P})[f]\|_{\gamma_{+}, \infty}.
\]

Similarly, we have

\[
\|((\mathbb{I} - \mathcal{P})[f]\|_{2m} \leq \|((\mathbb{I} - \mathcal{P})[f]\|_{2}^{\frac{1}{2}} \|((\mathbb{I} - \mathcal{P})[f]\|_{\infty}^{\frac{1}{2}} \leq \left( \frac{1}{\epsilon^{2m-1}} \right) \|((\mathbb{I} - \mathcal{P})[f]\|_{2}^{\frac{1}{2}} \left( \epsilon^{\frac{6m-3}{4m^2}} \|((\mathbb{I} - \mathcal{P})[f]\|_{\infty}^{\frac{2m-3}{2m}} \right) \leq \frac{1}{\epsilon^{2m-1}} \|((\mathbb{I} - \mathcal{P})[f]\|_{2} + o(1) \epsilon^{\frac{1}{2m}} \|((\mathbb{I} - \mathcal{P})[f]\|_{\infty}.
\]

We need this extra $\epsilon^{\frac{1}{2m}}$ for the convenience of $L^\infty$ estimate. Then we know for sufficiently small $\epsilon$ and $\frac{3}{2} < m < 3$,

\[
\epsilon^2 \|(1 - \mathcal{P})[f]\|_{\gamma_{+}, 4m}^2 \leq \epsilon^2 \|((\mathbb{I} - \mathcal{P})[f]\|_{2}^2 + o(1) \epsilon^{2 + \frac{1}{2m}} \|(1 - \mathcal{P})[f]\|_{\gamma_{+}, \infty}^2 \leq o(1) \epsilon \|(1 - \mathcal{P})[f]\|_{\gamma_{+}, 2}^2 + o(1) \epsilon^{2 + \frac{1}{2m}} \|f\|_{2m}^2.
\]

Similarly, we have

\[
\epsilon^2 \|((\mathbb{I} - \mathcal{P})[f]\|_{2m}^2 \leq \epsilon^2 \|((\mathbb{I} - \mathcal{P})[f]\|_{2}^2 + o(1) \epsilon^{2 + \frac{1}{2m}} \|((\mathbb{I} - \mathcal{P})[f]\|_{\infty}^2 \leq o(1) \|(1 - \mathcal{P})[f]\|_{2}^2 + o(1) \epsilon^{2 + \frac{1}{2m}} \|f\|_{2m}^2.
\]

Inserting (2.162) and (2.163) into (2.159), we can absorb $o(1) \epsilon \|(1 - \mathcal{P})[f]\|_{\gamma_{+}, 2}^2$ and $o(1) \|(1 - \mathcal{P})[f]\|_{2m}^2$ into the left-hand side to obtain

\[
\epsilon \|(1 - \mathcal{P})[f]\|_{\gamma_{+}, 2}^2 + \|((\mathbb{I} - \mathcal{P})[f]\|_{2m}^2 + \epsilon^2 \|\mathcal{P}[f]\|_{2m}^2 \leq o(1) \epsilon^{2 + \frac{1}{2m}} \|f\|_{\gamma_{+}, \infty}^2 + \epsilon^2 \|h\|_{\gamma_{+}, 4m}^2 + \|h\|_{\gamma_{+}, 2m}^2 + \left| \int_{\Omega \times \mathbb{R}^3} fS \right|.
\]

Step 4: Synthesis.
We can decompose

\[
\int_{\Omega \times \mathbb{R}^3} fS = \int_{\Omega \times \mathbb{R}^3} \mathcal{P}[f] \mathcal{P}[S] + \int_{\Omega \times \mathbb{R}^3} (1 - \mathcal{P})[f] (\mathbb{I} - \mathcal{P})[S].
\]

Hölder’s inequality and Cauchy’s inequality imply

\[
\int_{\Omega \times \mathbb{R}^3} \mathcal{P}[f] \mathcal{P}[S] \leq \|\mathcal{P}[f]\|_{2m} \|\mathcal{P}[S]\|_{\frac{2m}{2m-1}} \leq o(1) \epsilon^2 \|\mathcal{P}[f]\|_{2m}^2 + \frac{1}{\epsilon^2} \|\mathcal{P}[S]\|_{\frac{2m}{2m-1}}^2.
\]

\[
\int_{\Omega \times \mathbb{R}^3} (1 - \mathcal{P})[f] (\mathbb{I} - \mathcal{P})[S].
\]
Remark 2.2.10. Roughly speaking, this definition describes one characteristic line with reflection (alternatively so-called stochastic cycle), starting from \((x_k, v_k) \in \gamma_+\), tracking back to \((x_{k+1}, v_k) \in \gamma_-, \) diffusively reflected to \((x_{k+1}, v_{k+1}) \in \gamma_+\), and beginning a new cycle. \(t_k\) the accumulative time the characteristic moves backward. Note that we are free to choose any \(v_k \cdot n(x_k) > 0\), so different sequence \(\{v_k\}_{k=1}^\infty\) represents different stochastic cycles.

\[\int_{\Omega \times \mathbb{R}^3} (1 - \mathbb{P})[f](1 - \mathbb{P})[S] \preceq o(1) \|(1 - \mathbb{P})[f]\|^2_v + \left\|\nu^{\frac{1}{2}}(1 - \mathbb{P})[S]\right\|^2_2. \tag{2.167}\]

Inserting (2.166) and (2.167) into (2.165) and further (2.164), absorbing \(o(1)\epsilon^2\|\mathbb{P}[f]\|^2_{2m}\) and \(o(1) \|(1 - \mathbb{P})[f]\|^2_v\) into the left-hand side, we get

\[
\epsilon \|(1 - \mathbb{P})[f]\|^2_{\gamma^+} + \|(1 - \mathbb{P})[f]\|^2_v + \epsilon^2 \|\mathbb{P}[f]\|^2_{2m} \preceq o(1) \epsilon^2 + \epsilon^2 \nu^{\frac{1}{2}}(\mathbb{P})[S]\|^2_2 + \epsilon^2 \|h\|^2_{\gamma^+} + \|h\|^2_{\gamma^-}.
\]

Therefore, we have

\[
\frac{1}{\epsilon^2} \|(1 - \mathbb{P})[f]\|^2_{\gamma^+} + \frac{1}{\epsilon} \|(1 - \mathbb{P})[f]\|^2_v + \frac{1}{\epsilon^2} \|\mathbb{P}[S]\|^2_{2m} + \frac{1}{\epsilon} \nu^{\frac{1}{2}}(\mathbb{P})[S]\|^2_2 + \|h\|^2_{\gamma^+} + \|h\|^2_{\gamma^-}.
\]

\[
\frac{1}{\epsilon^2} \|(1 - \mathbb{P})[f]\|^2_{\gamma^+} + \frac{1}{\epsilon} \|(1 - \mathbb{P})[f]\|^2_v + \frac{1}{\epsilon^2} \|\mathbb{P}[S]\|^2_{2m} + \frac{1}{\epsilon} \|h\|^2_{\gamma^+} + \|h\|^2_{\gamma^-}.
\]

\[
\frac{1}{\epsilon^2} \|(1 - \mathbb{P})[f]\|^2_{\gamma^+} + \frac{1}{\epsilon} \|(1 - \mathbb{P})[f]\|^2_v + \frac{1}{\epsilon^2} \|\mathbb{P}[S]\|^2_{2m} + \frac{1}{\epsilon} \|h\|^2_{\gamma^+} + \|h\|^2_{\gamma^-}.
\]

2.2.3 \(L^\infty\) Estimates

Now we begin to consider mild formulation. When tracking the solution backward along the characteristics, once it hits the in-flow boundary, due to diffusive reflection boundary, actually the information comes from the integral of characteristics hitting the out-flow boundary. Following this idea, we may define the backward stochastic cycles, with multiple hitting times and out-flow integrals.

**Definition 2.2.8** (Hitting Time and Position). For any \((x, v) \in \Omega \times \mathbb{R}^3\) with \((x, v) \notin \gamma_0\), define the backward hitting time

\[t_b(x, v) := \inf\{t > 0 : x - \epsilon t v \notin \Omega\}. \tag{2.170}\]

Also, define the hitting position

\[x_b := x - \epsilon t_b(x, v)v \notin \Omega. \tag{2.171}\]

**Definition 2.2.9** (Stochastic Cycle). For any \((x, v) \in \Omega \times \mathbb{R}^3\) with \((x, v) \notin \gamma_0\), let \((t_0, x_0, v_0) = (0, x, v)\). Define the first stochastic triple

\[(t_1, x_1, v_1) := \left(t_0(x_0, v_0), x_b(x_0, v_0), v_1\right), \tag{2.172}\]

for some \(v_1\) satisfying \(v_1 \cdot n(x_1) > 0\).

Inductively, assume we know the \(k\)th stochastic triple \((t_k, x_k, v_k)\). Define the \((k + 1)\)th stochastic triple

\[(t_{k+1}, x_{k+1}, v_{k+1}) := \left(t_k + t_b(x_k, v_k), x_k(x_k, v_k), v_{k+1}\right), \tag{2.173}\]

for some \(v_{k+1}\) satisfying \(v_{k+1} \cdot n(x_{k+1}) > 0\).

**Remark 2.2.10.** Roughly speaking, this definition describes one characteristic line with reflection (alternatively so-called stochastic cycle), starting from \((x_k, v_k) \in \gamma_+\), tracking back to \((x_{k+1}, v_k) \in \gamma_-, \) diffusively reflected to \((x_{k+1}, v_{k+1}) \in \gamma_+\), and beginning a new cycle. \(t_k\) the accumulative time the characteristic moves backward. Note that we are free to choose any \(v_k \cdot n(x_k) > 0\), so different sequence \(\{v_k\}_{k=1}^\infty\) represents different stochastic cycles.
**Definition 2.2.11** (Diffusive Reflection Integral). Define $\mathcal{V}_k = \{ v \in \mathbb{R}^3 : v \cdot n(x_k) > 0 \}$, so the stochastic cycle must satisfy $v_k \in \mathcal{V}_k$. Let the iterated integral for $k \geq 2$ be defined as

\[
\int_{\prod_{j=1}^{k-1} \mathcal{V}_j} \prod_{j=1}^{k-1} d\sigma_j := \int_{\mathcal{V}_1} \cdots \left( \int_{\mathcal{V}_{k-1}} d\sigma_{k-1} \right) \cdots d\sigma_1 \tag{2.174}
\]

where $d\sigma_j := \mu(v_j) |v_j \cdot n(x_j)| dv_j$ is a probability measure.

We define a weight function scaled with parameter $\xi$, for $0 \leq \vartheta < \frac{1}{4}$ and $\varphi \geq 0$,

\[
w(v) := (v)^\vartheta e^{\varphi |v|^2}, \tag{2.175}\]

and

\[
\tilde{w}(v) := \frac{1}{\mu^{\frac{1}{2}}(v) w(v)} = \frac{\sqrt{2\pi}}{(1 + |v|^2) \frac{\varphi}{2}} \tag{2.176}\]

**Lemma 2.2.12.** For $T_0 > 0$ sufficiently large, there exists constants $C_1, C_2 > 0$ independent of $T_0$, such that for $k = C_1 T_0^{\frac{1}{4}}$, and $(x, v) \in \times \Omega \times \mathbb{R}^3$,

\[
\int_{\prod_{j=1}^{k-1} \mathcal{V}_j} 1_{\{t_k(x,v,v_1,\ldots,v_{k-1}) < \frac{2m}{\epsilon} \}} \prod_{j=1}^{k-1} d\sigma_j \leq \left( \frac{1}{2} \right)^{C_2 T_0^{\frac{1}{4}}} . \tag{2.177}\]

**Proof.** This is a rescaled version of [5] Lemma 4.1. Since our hitting time in (2.170) is rescaled with $\epsilon$, we should rescale back in the statement of lemma.

**Remark 2.2.13.** Roughly speaking, Lemma 2.2.12 states that even though we have the freedom to choose $v_k$ in each stochastic cycle, in the long run, the accumulative time will not be too small. After enough reflections $\sim k$, most characteristics has the accumulative time that will exceed any set threshold $T_0$.

**Theorem 2.2.14.** Assume (2.72) and (2.79) hold. The solution $f(x, v)$ to the equation (2.70) satisfies for $\varphi \geq 0$ and $0 \leq \vartheta < \frac{1}{4}$,

\[
\|f\|_{\infty, \varphi, \vartheta} + \|f\|_{\gamma_+, \infty, \varphi, \vartheta} \tag{2.178}
\]

\[
\leq \frac{1}{\epsilon^{\frac{1}{2} + \frac{2m}{m}}\epsilon} \|\mathbb{P}[S]\|_{\frac{2m}{2m}} + \frac{1}{\epsilon^{\frac{1}{2} + \frac{2m}{m}}} \left\|\nu^{-\frac{1}{2}}(1 - \mathbb{P})[S]\right\|_2 + \|\nu^{-1} S\|_{\infty, \varphi, \vartheta}
\]

\[
+ \frac{1}{\epsilon^{\frac{1}{2} + \frac{2m}{m}}} \|h\|_{\gamma_-, \frac{4m}{m}} + \frac{1}{\epsilon^{\frac{1}{2} + \frac{2m}{m}}} \|h\|_{\gamma_-, 2} + \|h\|_{\gamma_-, \infty, \varphi, \vartheta}.
\]

**Proof.**

Step 1: Mild formulation.

Denote the weighted solution

\[
g(x, v) := w(v) f(x, v), \tag{2.179}\]

and the weighted non-local operator

\[
K_{w(v)}g(v) := w(v)K \left[ \frac{g}{w} \right](v) = \int_{\mathbb{R}^3} k_{w(v)}(v, u) g(u) du, \tag{2.180}\]

where

\[
k_{w(v)}(v, u) := k(v, u) \frac{w(v)}{w(u)}. \tag{2.181}\]
Multiplying \( w \) on both sides of (2.70), we have

\[
\begin{aligned}
&\{ \epsilon v \cdot \nabla_x g + \nu g = K_w(x, v) + w(v)S(x, v) \text{ in } \Omega \times \mathbb{R}^3, \\
g(x_0, v) = w(v)\mu^x(v) \int_{u \cdot n > 0} \tilde{w}(u)g(x_0, u)du + \nu h(x_0, v) \text{ for } x_0 \in \partial \Omega \text{ and } v \cdot n < 0,
\end{aligned}
\] (2.182)

We can rewrite the solution of the equation (2.182) along the characteristics by Duhamel’s principle as

\[
g(x, v) = w(v)h(x_1, v)e^{-\nu(v)t_1} + \int_0^{t_1} w(v)S\left(x - \epsilon(t_1 - s)v, v\right)e^{-\nu(v)(t_1-s)}ds
\] (2.183)
Similarly,

\[
\left| \frac{e^{-\nu(v)t_1}}{\bar{w}(v)} \sum_{\ell=1}^{k-1} \int_{\Pi_{j=1}^{\ell} v_j}^{t_{\ell+1}} \left( \int_{t_{\ell+1}}^{t_{\ell+1}} (w(v_t)S(x_t - \epsilon(t_{\ell+1} - s)v_t, v_t)e^{\nu(v_t)s})ds \right) \bar{w}(v_t) \left( \prod_{j=1}^{\ell} e^{-\nu(v_j)(t_{\ell+1} - t_j)}d\sigma_j \right) \right| \leq \nu^{-1}\|wS\|_{\infty} \sum_{\ell=1}^{k-1} \int_{\Pi_{j=1}^{\ell} v_j}^{t_{\ell+1}} \left( \int_{t_{\ell+1}}^{t_{\ell+1}} |\nu(v_t)e^{\nu(v_t)(s-(t_{\ell+1} - t_j))}ds| \bar{w}(v_t) \prod_{j=1}^{\ell} d\sigma_j \right) \leq CT_0^2 \nu^{-1}\|wS\|_{\infty}.
\]

Collecting all terms in (2.187), (2.188), (2.189) and (2.190), we have

Boundary Term Contribution \( \lesssim CT_0^{\frac{\nu}{2}} \|wh\|_{\gamma_{-\infty}} \lesssim \|wh\|_{\gamma_{-\infty}}, \)

(2.191)

and

Source Term Contribution \( \lesssim CT_0^{\frac{\nu}{2}} \|\nu^{-1}wS\|_{\infty} \lesssim \|\nu^{-1}wS\|_{\infty}. \)

(2.192)

Step 3: Estimates of Multiple Reflection.

We focus on the last term in (2.184), which can be decomposed based on accumulative time \( t_{k+1} \):

\[
\left| \frac{e^{-\nu(v)t_1}}{\bar{w}(v)} \int_{\Pi_{j=1}^{k} v_j} g(x_k, v_k) \bar{w}(v_k) \left( \prod_{j=1}^{k} e^{-\nu(v_j)(t_{j+1} - t_j)}d\sigma_j \right) \right|
\leq \frac{e^{-\nu(v)t_1}}{\bar{w}(v)} \int_{\Pi_{j=1}^{k} v_j} 1\{t_k \leq \frac{T}{2}\} g(x_k, v_k) \bar{w}(v_k) \left( \prod_{j=1}^{k} e^{-\nu(v_j)(t_{j+1} - t_j)}d\sigma_j \right) + \frac{e^{-\nu(v)t_1}}{\bar{w}(v)} \int_{\Pi_{j=1}^{k} v_j} 1\{t_k \geq \frac{T}{2}\} g(x_k, v_k) \bar{w}(v_k) \left( \prod_{j=1}^{k} e^{-\nu(v_j)(t_{j+1} - t_j)}d\sigma_j \right) \]
\[= J_1 + J_2. \]

Based on Lemma 2.2.12 we have

\[
J_1 \lesssim \|g\|_{\infty} \left| \int_{\Pi_{j=1}^{k} v_j} 1\{t_k \leq \frac{T}{2}\} \left( \int_{\Pi_{j=1}^{k} v_j} \bar{w}(v_k) d\sigma_k \right) \left( \prod_{j=1}^{k-1} d\sigma_j \right) \right| \leq \left( \frac{1}{2} \right)^{C_2 T_0^2} \|g\|_{\infty}.
\]

(2.194)

On the other hand, when \( t_k \) is large, the exponential terms become extremely small, so we obtain

\[
J_2 \lesssim \|g\|_{\infty} \left| \frac{e^{-\nu(v)t_1}}{\bar{w}(v)} \int_{\Pi_{j=1}^{k} v_j} 1\{t_k \geq \frac{T}{2}\} \left( \int_{\Pi_{j=1}^{k} v_j} \bar{w}(v_k) d\sigma_k \right) \left( \prod_{j=1}^{k-1} e^{-\nu(v_j)(t_{j+1} - t_j)}d\sigma_j \right) \right| \leq \frac{e^{-\frac{T}{2}}}{\bar{w}(v)} \|g\|_{\infty}.
\]

(2.195)

Summarizing (2.194) and (2.195), we get for \( \delta \) arbitrarily small

\[
\text{Multiple Reflection Term Contribution} \lesssim \delta \|g\|_{\infty}. \]

(2.196)
Step 4: Estimates of $K_w$ terms.
So far, the only remaining terms in (2.184) are related to $K_w$. We focus on
\[
\left| \int_0^{t_1} K_w(v)g(x - \epsilon(t_1 - s)v, v)e^{-\nu(t_1 - s)}ds \right| \lesssim \| K_w(v)g(x - \epsilon(t_1 - s)v, v) \|_{\infty}.
\] (2.197)
Denote $X(s; x, v) := x - \epsilon(t_1 - s)v$. Define the back-time stochastic cycle from $(s, X, v')$ as $(t'_1, x'_1, v'_1)$ with $(t'_0, x'_0, v'_0) = (s, X, v')$. Then we can rewrite $K_w$ along the stochastic cycle as (2.184)
\[
|K_w(v)| \left( x - \epsilon(t_1 - s)v, v \right) = |K_w(v)| \left( X, v \right) = \left| \int_{\mathbb{R}^3} k_w(v, v')g(X, v')dv' \right|
\leq \left| \int_{\mathbb{R}^3} \int_0^{t_1} k_w(v, v')K_w(v')g \left( X - \epsilon(t'_1 - r)v', v' \right)e^{-\nu(v')(t'_1 - r)}drdv' \right|
+ \int_{\mathbb{R}^3} \frac{e^{-\nu(v')}t'_1}{\bar{w}(v')} \sum_{e=1}^{k-1} \int_{\prod_{j=1}^{j_e} \nu_j} k_w(v, v')H_e(X, v')w(v') \left( \prod_{j=1}^{r} e^{-\nu(v')(t'_1 - t'_j)}d\nu_j \right)dv'
+ \int_{\mathbb{R}^3} k_w(v, v') \left( \text{boundary terms + source terms + multiple reflection terms} \right)dv'
:= I + II + III.
\] (2.198)
Using estimates (2.191), (2.192), (2.196) from Step 2 and Step 3, and Lemma 2.2.5, we can bound $III$ directly
\[
III \lesssim \| wh \| \gamma_-, \infty + \| \nu^{-1}wS \| \infty + \delta \| g \| \infty.
\] (2.199)
$I$ and $II$ are much more complicated. We may further rewrite $I$ as
\[
I = \left| \int_{\mathbb{R}^3} \int_0^{t_1} k_w(v, v')k_w(v''')(v', v''')g(X - \epsilon(t'_1 - r)v', v'')e^{-\nu(v')(t'_1 - r)}drdv' dvd'' \right|
\tag{2.200}
\]
which will estimated in four cases:
\[
I := I_1 + I_2 + I_3 + I_4.
\] (2.201)
Case I: $I_1 : \| v \| \geq N$.
Based on Lemma 2.2.5, we have
\[
\left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} k_w(v, v')k_w(v''')(v', v'')dv' dvd''' \right| \lesssim \frac{1}{1 + |v|} \lesssim \frac{1}{N}.
\] (2.202)
Hence, we get
\[
I_1 \lesssim \frac{1}{N} \| g \| \infty.
\] (2.203)
Case II: $I_2 : \| v \| \leq N$, $\| v' \| \geq 2N$, or $\| v'' \| \leq 2N$, $\| v''' \| \geq 3N$.
Notice this implies either $\| v' - v \| \geq N$ or $\| v'' - v''' \| \geq N$. Hence, either of the following is valid correspondingly:
\[
|k_w(v')(v, v')| \leq Ce^{-\delta N^2} |k_w(v)(v, v')| e^{\delta |v - v'|^2},
\] (2.204)
\[
|k_w(v')(v', v'')| \leq Ce^{-\delta N^2} |k_w(v')(v', v'')| e^{\delta |v' - v'''|^2}.
\] (2.205)
Based on Lemma 2.2.5, we know
\[\int_{\mathbb{R}^3} |k_{w(v)}(v, v')| e^{\delta |v-v'|^2} dv' < \infty,\] (2.206)
\[\int_{\mathbb{R}^3} |k_{w(v')}(v', v'')| e^{\delta |v-v''|^2} dv'' < \infty.\] (2.207)

Hence, we have
\[I_2 \lesssim e^{-\delta N^2} \|g\|_{\infty}.\] (2.208)

Case III: \(I_3 : t_1' - r \leq \delta\) and \(|v| \leq N, |v'| \leq 2N, |v''| \leq 3N\).
In this case, since the integral with respect to \(r\) is restricted in a very short interval, there is a small contribution as
\[I_3 \lesssim \int_{t_1' - \delta}^{t_1'} e^{-(t_1' - r)} dr \|g\|_{\infty} \lesssim \delta \|g\|_{\infty}.\] (2.209)

Case IV: \(I_4 : t_1' - r \geq \delta\) and \(|v| \leq N, |v'| \leq 2N, |v''| \leq 3N\).
This is the most complicated case. Since \(k_{w(v)}(v, v')\) has possible integrable singularity of \(\frac{1}{|v-v'|}\), we can introduce the truncated kernel \(k_N(v, v')\) which is smooth and has compactly supported range such that
\[\sup_{|v| \leq 3N} \int_{|v'| \leq 3N} |k_N(v, v') - k_{w(v)}(v, v')| dv' \leq \frac{1}{N}.\] (2.210)

Then we can split
\[k_{w(v)}(v, v')k_{w(v')}(v', v'') = k_N(v, v')k_N(v', v'') + \left(k_{w(v)}(v, v') - k_N(v, v')\right)k_{w(v')}(v', v'') + \left(k_{w(v')}(v', v'') - k_N(v', v'')\right)k_N(v, v').\] (2.211)

This means that we further split \(I_4\) into
\[I_4 := I_{4,1} + I_{4,2} + I_{4,3}.\] (2.212)

Based on (2.210), we have
\[I_{4,2} \lesssim \frac{1}{N} \|g\|_{\infty}, \quad I_{4,3} \lesssim \frac{1}{N} \|g\|_{\infty}.\] (2.213)

Therefore, the only remaining term is \(I_{4,1}\). Note that we always have \(X - \epsilon(t_1' - r)v' \in \Omega\). Hence, we define the change of variable \(v' \to y\) as \(y = (y_1, y_2, y_3) = X - \epsilon(t_1' - r)v'\). Then the Jacobian
\[\left|\frac{dy}{dv'}\right| = \left|\begin{array}{ccc}
\epsilon(t_1' - r) & 0 & 0 \\
0 & \epsilon(t_1' - r) & 0 \\
0 & 0 & \epsilon(t_1' - r)
\end{array}\right| = \epsilon^3(t_1' - r)^3 \geq \epsilon^3 \delta^3.\] (2.214)

Considering \(|v|, |v'|, |v''| \leq 3N\), we know \(|g| \lesssim |f|\). Also, since \(k_N\) is bounded, we estimate
\[I_{4,1} \lesssim \int_{|v| \leq 2N} \int_{|v'| \leq 3N} \int_0^{t_1'} 1\{X - \epsilon(t_1' - r)v' \in \Omega\} |f(X - \epsilon(t_1' - r)v', v'')| e^{-\nu(v')(t_1' - r)} dr dv' dv''.\] (2.215)
Using the decomposition \( f = \mathbb{P}[f] + (I - \mathbb{P})[f] \), (2.214) and Hölder’s inequality, we estimate them separately,

\[
\int_{|\nu'| \leq 2N} \int_{|\nu''| \leq 3N} \int_0^t \mathbf{1}_{\{X - \epsilon(t'_1 - r)\nu' \in \Omega\}} \left| \mathbb{P}[f](X - \epsilon(t'_1 - r)\nu', \nu'') \right| |e^{-\nu(\nu')(t'_1 - r)} dr d\nu' d\nu'' \quad (2.216)
\]

\[
\leq \left( \int_{|\nu'| \leq 2N} \int_{|\nu''| \leq 3N} \int_0^t \mathbf{1}_{\{X - \epsilon(t'_1 - r)\nu' \in \Omega\}} |e^{-\nu(\nu')(t'_1 - r)} dr d\nu'' \right)^{\frac{2m}{3}}
\]

\[
\times \left( \int_{|\nu'| \leq 2N} \int_{|\nu''| \leq 3N} \int_0^t \mathbf{1}_{\{X - \epsilon(t'_1 - r)\nu' \in \Omega\}} \left( (I - \mathbb{P})[f] \right)^2 (X - \epsilon(t'_1 - r)\nu', \nu'') |e^{-\nu(\nu')(t'_1 - r)} dr d\nu'' \right)^{\frac{1}{3}}
\]

\[
\lesssim \int_0^t \frac{1}{\epsilon^3 \delta^3} \int_{|\nu'| \leq 2N} \int_{|\nu''| \leq 3N} \mathbf{1}_{\{y \in \Omega\}} \left( \mathbb{P}[f] \right)^2 (y, \nu'') |e^{-(t'_1 - r)} d\nu'' dr \quad \lesssim \frac{1}{\epsilon^2 \delta^2} \|\mathbb{P}[f]\|_2.
\]

Inserting (2.216) and (2.217) into (2.215), we obtain

\[
I_{4,1} \lesssim \frac{1}{\epsilon^2 \delta^2} \|\mathbb{P}[f]\|_2 + \frac{1}{\epsilon^2 \delta^2} \|(I - \mathbb{P})[f]\|_2.
\]

Combined with (2.213), we know

\[
I_4 \lesssim \frac{1}{N} \|g\|_\infty + \frac{1}{\epsilon \delta^2 \frac{m}{2m} \|\mathbb{P}[f]\|_2 + \frac{1}{\epsilon^2 \delta^2} \|\mathbb{I} - \mathbb{P}[f]\|_2.
\]

Summarizing all four cases in (2.216), (2.217), (2.218) and (2.219), we obtain

\[
I \lesssim \left( \frac{1}{N} + \epsilon^{-\delta N^2 + \delta} \right) \|g\|_\infty + \frac{1}{\epsilon \delta^2 \frac{m}{2m} \|\mathbb{P}[f]\|_2 + \frac{1}{\epsilon^2 \delta^2} \|\mathbb{I} - \mathbb{P}[f]\|_2.
\]

Choosing \( \delta \) sufficiently small and then taking \( N \) sufficiently large, we have

\[
I \lesssim \delta \|g\|_\infty + \frac{1}{\epsilon \delta^2 \frac{m}{2m} \|\mathbb{P}[f]\|_2 + \frac{1}{\epsilon^2 \delta^2} \|\mathbb{I} - \mathbb{P}[f]\|_2.
\]

By a similar but tedious computation, we arrive at

\[
II \lesssim \delta \|g\|_\infty + \frac{1}{\epsilon \delta^2 \frac{m}{2m} \|\mathbb{P}[f]\|_2 + \frac{1}{\epsilon^2 \delta^2} \|\mathbb{I} - \mathbb{P}[f]\|_2.
\]

Combined with (2.199), we have

\[
\left| \int_0^t K_{w(\nu)}[g] (x - \epsilon(t_1 - s)\nu, v) e^{-\nu(\nu)(t_1 - s)} ds \right| \lesssim \delta \|g\|_\infty + \frac{1}{\epsilon \delta^2 \frac{m}{2m} \|\mathbb{P}[f]\|_2 + \frac{1}{\epsilon^2 \delta^2} \|\mathbb{I} - \mathbb{P}[f]\|_2
\]

\[
+ \|w\|_{\gamma_\infty} + \|\nu^{-1}w S\|_\infty.
\]
All the other terms in (2.183) related to $K_w$ can be estimated in a similar fashion. At the end of the day, we have

\[
K_w \text{ term contribution } \lesssim \delta \|g\|_\infty + \frac{1}{\epsilon^{2m} \delta^{2m}} \|P_f\|_{2m} + \frac{1}{\epsilon^2 \delta^2} \|(I - P)(S)\|_2 + \|w h\|_{\gamma - \infty} + \|\nu^{-1} w S\|_\infty.
\]  

(2.224)

Step 5: Synthesis.

Summarizing all above and inserting (2.191), (2.192), (2.196) and (2.224) into (2.184), we obtain for any $(x,v) \in \Omega \times \mathbb{R}^3$,

\[
|g(x,v)| \lesssim \delta \|g\|_\infty + \frac{1}{\epsilon^{2m} \delta^{2m}} \|P_f\|_{2m} + \frac{1}{\epsilon^2 \delta^2} \|(I - P)(S)\|_2 + \|w h\|_{\gamma - \infty} + \|\nu^{-1} w S\|_\infty.
\]  

(2.225)

Taking supremum over $(x,v) \in \gamma_+$ in (2.226), we have

\[
\|g\|_{\gamma_+, \infty} \lesssim \delta \|g\|_\infty + \frac{1}{\epsilon^{2m} \delta^{2m}} \|P_f\|_{2m} + \frac{1}{\epsilon^2 \delta^2} \|(I - P)(S)\|_2 + \|w h\|_{\gamma - \infty} + \|\nu^{-1} w S\|_\infty.
\]  

(2.226)

Based on Theorem (2.227) for $\frac{3}{2} < m < 3$, we obtain

\[
\|g\|_{\gamma_+, \infty} \lesssim \delta \|g\|_\infty + o(1) \left( \|f\|_{\gamma_+, \infty} + \|f\|_\infty \right) + E \lesssim \delta \|g\|_\infty + o(1) \left( \|g\|_{\gamma_+, \infty} + \|g\|_\infty \right) + E,
\]  

where

\[
E := \frac{1}{e^2 \gamma^2} \|P(S)\|_{2m} + \frac{1}{\epsilon^{2m} \delta^{2m}} \|P_f\|_{2m} + \frac{1}{\epsilon^2 \delta^2} \|(I - P)(S)\|_2 + \|w h\|_{\gamma - \infty} + \|\nu^{-1} w S\|_\infty.
\]  

(2.227)

Absorbing $o(1) \|g\|_{\gamma_+, \infty}$ into the left-hand side, we have

\[
\|g\|_{\gamma_+, \infty} \lesssim \delta \|g\|_\infty + o(1) \|g\|_\infty + E.
\]  

(2.228)

On the other hand, taking supremum over $(x,v) \in \Omega \times \mathbb{R}^3$ in (2.226), we have

\[
\|g\|_\infty \lesssim \delta \|g\|_\infty + \frac{1}{\epsilon^{2m} \delta^{2m}} \|P_f\|_{2m} + \frac{1}{\epsilon^2 \delta^2} \|(I - P)(S)\|_2 + \|w h\|_{\gamma - \infty} + \|\nu^{-1} w S\|_\infty.
\]  

(2.229)

Based on Theorem (2.227) we obtain

\[
\|g\|_\infty \lesssim \delta \|g\|_\infty + o(1) \left( \|g\|_{\gamma_+, \infty} + \|g\|_\infty \right) + E.
\]  

(2.230)

Absorbing $\delta \|g\|_\infty$ and $o(1) \|g\|_\infty$ into the left-hand side, we have

\[
\|g\|_\infty \lesssim o(1) \|g\|_{\gamma_+, \infty} + E.
\]  

(2.231)

Inserting (2.229) into (2.232), and absorbing $\delta \|g\|_\infty$ and $o(1) \|g\|_\infty$ into the left-hand side, we get

\[
\|g\|_\infty \lesssim E.
\]  

(2.233)

Then (2.229) implies

\[
\|g\|_{\gamma_+, \infty} \lesssim E.
\]  

(2.234)

In summary, we have

\[
\|g\|_\infty + \|g\|_{\gamma_+, \infty} \lesssim \frac{1}{e^2 \gamma^2} \|P(S)\|_{2m} + \frac{1}{\epsilon^{2m} \delta^{2m}} \|P_f\|_{2m} + \frac{1}{\epsilon^2 \delta^2} \|(I - P)(S)\|_2 + \|w h\|_{\gamma - \infty} + \|\nu^{-1} w S\|_\infty
\]  

\[
+ \frac{1}{\epsilon \gamma^2} \|h\|_{\gamma_-, \frac{3m}{4}} + \frac{1}{\epsilon^{1 + \frac{1}{2m}} \delta^{\frac{1}{m}}} \|h\|_{\gamma_-, \frac{2}{1}} + \|w h\|_{\gamma - \infty}.
\]

(2.235)

Then our result naturally follows. \qed
Remark 2.2.15. Inserting Theorem 2.2.14 into Theorem 2.2.7, we actually have

\[
\frac{1}{\epsilon^2} \left\| (1 - \mathcal{P})[f] \right\|_{\gamma, 2} + \frac{1}{\epsilon} \left\| (1 - \mathcal{P})[f] \right\|_{\nu} + \left\| \mathcal{P}[f] \right\|_{2m} \leq \frac{1}{\epsilon^2} \left\| \mathcal{P}[S] \right\|_{\frac{2m}{2m - 1}} + \frac{1}{\epsilon} \left\| \nu^{-\frac{1}{2}} (1 - \mathcal{P})[S] \right\|_{2} + \left\| \nu^{-1} S \right\|_{\infty, \vartheta, \varrho} + \left\| h_{\gamma, \frac{2m}{2m - 1}} \right\| + \frac{1}{\epsilon} \left\| h_{\gamma, 2} \right\| + \left\| h_{\gamma, \infty, \vartheta, \varrho} \right\|.
\]
2.3 Hydrodynamic Limit

2.3.1 Nonlinear Estimates

Lemma 2.3.1. The nonlinear term \( \Gamma \) defined in (1.17) satisfies \( \Gamma[f, g] \in N^2 \). Also, for \( 0 \leq \varrho < \frac{1}{4} \) and \( \varrho \geq 0 \),

\[
\| \Gamma[f, g] \|_2 \lesssim \left( \sup_{x \in \mathbb{R}^3} |v g(x)|_2 \right) \|v f\|_2, \tag{2.237}
\]

\[
\| \nu^{-1} \Gamma[f, g] \|_{\infty, \varrho, \varrho} \lesssim \| f \|_{\infty, \varrho, \varrho} \| g \|_{\infty, \varrho, \varrho}. \tag{2.238}
\]

Proof. The orthogonality is shown in [8, Section 3.8]. (2.237) can be shown following the idea in [11, Lemma 2.3]. From (1.17),

\[
\Gamma[f, g] := \mu^{-\frac{1}{2}} Q \left[ \mu^{\frac{1}{2}} f, \mu^{\frac{1}{2}} g \right] = \Gamma_{\text{gain}}[f, g] - \Gamma_{\text{loss}}[f, g], \tag{2.239}
\]

where using the energy conservation \( |u|^2 + |v|^2 = |u_*|^2 + |v_*|^2 \),

\[
\Gamma_{\text{gain}}[f, g] := q_0 \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} e^{-\frac{|u|^2}{2}} (\omega \cdot (v - u)) f(u_*) g(v_*) d\omega du,
\]

\[
\Gamma_{\text{loss}}[f, g] := q_0 \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} e^{-\frac{|u|^2}{2}} (\omega \cdot (v - u)) f(u) g(v) d\omega du,
\]

with

\[
\quad u_* := u + \omega \left( (v - u) \cdot \omega \right), \quad v_* := v - \omega \left( (v - u) \cdot \omega \right). \tag{2.242}
\]

For the loss term, we substitute \( u = v - u \), so we know

\[
\Gamma_{\text{loss}}[f, g] := q_0 g(v) \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} e^{-\frac{|v-u|^2}{2}} (\omega \cdot u) f(v-u) d\omega du. \tag{2.243}
\]

Hence, using Hölder’s inequality, we have

\[
\int_{\mathbb{R}^3} \left( \Gamma_{\text{loss}}[f, g](x) \right)^2 dv = q_0^2 \int_{\mathbb{R}^3} g^2(x, v) \left( \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} e^{-\frac{|v-u|^2}{2}} (\omega \cdot u) f(x, v-u) d\omega du \right)^2 dv \tag{2.244}
\]

\[
\lesssim \int_{\mathbb{R}^3} g^2(x, v) \left( \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} e^{-|v-u|^2} |u|^2 du \right) \left( \int_{\mathbb{R}^3} f^2(x, v-u) du \right) dv \lesssim \int f(x)_2^2 \nu g(x)_2^2, \tag{2.245}
\]

where we utilize the fact that

\[
\int_{\mathbb{R}^3} e^{-|v-u|^2} |u|^2 du \lesssim \nu^2(v). \tag{2.245}
\]

On the other hand, for the gain term, after substituting \( u = v - u \), we know

\[
\Gamma_{\text{gain}}[f, g] := q_0 \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} e^{-\frac{|v-u|^2}{2}} (\omega \cdot u) f(v-u) g(v) d\omega du, \tag{2.246}
\]

where

\[
u_\perp = u - \omega (u \cdot \omega), \quad \nu_\parallel = \omega (u \cdot \omega). \tag{2.247} \]
\[\begin{align*}
\int_{\mathbb{R}^3} \left( \Gamma_{\text{gain}}[f, g](x) \right)^2 \, dv &= \gamma^2_{0} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{-\frac{|\omega \cdot u|^2}{2}} (\omega \cdot u) f(x, v - u) g(x, v - u) \, dw \, du \, dv \\
&\lesssim \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} e^{-|v - u|^2} |u|^2 \, du \right) \left( \int_{\mathbb{R}^3} f^2(x, v - u) g^2(x, v - u) \, du \right) \, dv \\
&\lesssim \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \nu^2(v) f^2(x, v - u) g^2(x, v - u) \, dv \, du.
\end{align*}\] (2.248)

Denote \( u' = v - u \) and \( v' = v - u \). Consider substitution \((u, v) \to (u', v')\). It is well-known (see the proof of [11 Lemma 2.3]) that \( \text{d}udv = du'dv' \) and \(|v| \leq |u'| + |v'|\). Hence, we have

\[\begin{align*}
\int_{\mathbb{R}^3} \left( \Gamma_{\text{gain}}[f, g](x) \right)^2 \, dv &\lesssim \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (\nu^2(u') + \nu^2(v')) f^2(x, u') g^2(x, v') \, du' \, dv' \\
&\lesssim \left( \int_{\mathbb{R}^3} \nu^2(u') f^2(x, u') \, du' \right) \left( \int_{\mathbb{R}^3} \nu^2(v') g^2(x, v') \, dv' \right) \\
&\lesssim |\nu f(x)|^2 |\nu g(x)|^2.
\end{align*}\] (2.249)

Combining (2.244) and (2.249), we know

\[\int_{\mathbb{R}^3} \left( \Gamma_{\text{gain}}[f, g](x) \right)^2 \, dv \lesssim |\nu f(x)|^2 |\nu g(x)|^2,\] (2.250)

which further implies

\[\int_{\Omega} \int_{\mathbb{R}^3} \left( \Gamma_{[f, g]} \right)^2 \, dv \, dx \lesssim \left( \sup_{x \in \Omega} |\nu g(x)|^2 \right) ||\nu f||_2^2.\] (2.251)

Therefore, (2.237) naturally follows. Also, (2.238) is proved in [12 Lemma 5].

### 2.3.2 Perturbed Remainder Estimates

We consider the perturbed linearized stationary Boltzmann equation

\[\begin{align*}
\left\{ \begin{array}{l}
e v \cdot \nabla_x f + \mathcal{L}[f] = \Gamma_{[f, g]} + S(x, v) \quad \text{in} \quad \Omega \times \mathbb{R}^3, \\
f(x_0, v) = P[f](x_0, v) + (\mu'_0 - \mu) \mu^{-1} P[f] + h(x_0, v) \quad \text{for} \quad x_0 \in \partial \Omega \quad \text{and} \quad v \cdot n < 0.
\end{array} \right.
\end{align*}\] (2.252)

Assume that a priori

\[\int_{\Omega \times \mathbb{R}^3} f(x, v) \mu^\frac{1}{2}(v) \, dv \, dx = 0.\] (2.253)

and

\[\|g\|_{\infty, \partial \Omega} = o(1) \epsilon.\] (2.254)

The data \( S \) and \( h \) satisfy the compatibility condition

\[\int_{\Omega \times \mathbb{R}^3} S(x, v) \mu^\frac{1}{2}(v) \, dv \, dx + \int_{\gamma_-} h(x, v) \mu^\frac{1}{2}(v) \, d\gamma = 0.\] (2.255)

**Theorem 2.3.2.** Assume (2.272) and (2.273) hold. The solution \( f(x, v) \) to the equation (2.252) satisfies

\[\frac{1}{\epsilon^2} ||(1 - P)[f]||_{\gamma^+, 2} + \frac{1}{\epsilon} \||1 - P[f]||_{\nu} + \|P[f]\|_{2m} \]

\[\leq o(1) \epsilon^\frac{1}{2m} \left( ||f||_{\gamma^+, \infty} + ||f||_{\infty} \right) + \frac{1}{\epsilon^2} \|P[S]\|_{2m} + \frac{1}{\epsilon} \|\nu^{-\frac{1}{2}}(1 - P)[S]\|_2 + ||h||_{\gamma^-, 4m} + \frac{1}{\epsilon} ||h||_{\gamma^-, 2}.\] (2.256)
Proof. Since the perturbed term $\Gamma[f, g] \in N^{1, \epsilon}$, we apply Theorem 2.2.7 to (2.252) to obtain
\begin{align*}
\frac{1}{e^2} \| (1 - P) [f] \|_{\gamma_+^2} + \frac{1}{e} \| (1 - P) [f] \|_{\nu} + \| P [f] \|_{2m} \\
\leq o(1) \epsilon \frac{2}{e^2} \left( \| f \|_{\gamma_+^\infty} + \| f \|_{\infty} \right) + \frac{1}{e^2} \| P[S] \|_{2m} + \frac{1}{e} \| \nu^{-\frac{1}{2}} (1 - P) [S] \|_2 + \| h \|_{\gamma_+^\infty} + \frac{1}{e} \| h \|_{\gamma_-^2} \\
+ \frac{1}{e} \| \nu^{-\frac{1}{2}} \Gamma[f, g] \|_2 + \| (\mu_0^2 - \mu) \mu^{-1} P [f] \|_{\gamma_-^\infty} + \frac{1}{e} \| (\mu_0^2 - \mu) \mu^{-1} P [f] \|_{\gamma_-^2}.
\end{align*}
Using Lemma 2.3.1 and (2.254), we have
\begin{equation}
\frac{1}{e} \left\| \nu^{-\frac{1}{2}} \Gamma[f, g] \right\|_2 \lesssim o(1) \left\| \nu^{\frac{1}{2}} f \right\|_2 \lesssim o(1) \| P [f] \|_{\nu} + o(1) \| (1 - P) [f] \|_{\nu}.
\end{equation}
Note that direct computation reveals that
\begin{equation}
\| P [f] \|_{2m} \gtrsim \| P [f] \|_{\nu},
\end{equation}
so inserting (2.258) into (2.257), we can absorb $o(1) \| P [f] \|_{\nu}$ and $o(1) \| (1 - P) [f] \|_{\nu}$ into the left-hand side. On the other hand, due to (1.11), we know
\begin{equation}
\| (\mu_0^2 - \mu) \mu^{-1} P [f] \|_{\gamma_-^\infty} + \frac{1}{e} \| (\mu_0^2 - \mu) \mu^{-1} P'[f] \|_{\gamma_-^2} \lesssim o(1) \epsilon \| P [f] \|_{\gamma_-^\infty} + o(1) \| P [f] \|_{\gamma_-^2} \\
\lesssim o(1) \epsilon \| f \|_{\gamma_+^\infty} + o(1) \| P [f] \|_{\gamma_+^2}.
\end{equation}
Here, $o(1) \| f \|_{\gamma_+^\infty}$ can be combined with the corresponding term on the right-hand side of (2.257). Also, the bound of $\| P [f] \|_{\gamma_+^2}$ has been achieved in the proof of Theorem 2.2.7. Inserting (2.251) into (2.257), using (2.260), Hölder’s inequality and Theorem 2.2.7, we know
\begin{equation}
\| P [f] \|_{\gamma_+^2} \lesssim \| P [f] \|_2 + \frac{1}{e} \| (1 - P) [f] \|_2 + \frac{1}{e} \left( \left\| \int_{\Omega \times R^3} f S \right\| \right)^{\frac{1}{2}} \\
\lesssim o(1) \epsilon \frac{2}{e^2} \left( \| f \|_{\gamma_+^\infty} + \| f \|_{\infty} \right) \\
+ \frac{1}{e^2} \| P[S] \|_{2m} + \frac{1}{e} \| \nu^{-\frac{1}{2}} (1 - P) [S] \|_2 + \| h \|_{\gamma_+^\infty} + \frac{1}{e} \| h \|_{\gamma_-^2} + o(1) \| P [f] \|_{\gamma_+^2}.
\end{equation}
Then absorbing $o(1) \| P [f] \|_{\gamma_+^2}$ into the left-hand side, we get control of $\| P [f] \|_{\gamma_+^2}$. Then inserting it into (2.260) and further (2.257), we get the desired result. \hfill \square

Theorem 2.3.3. Assume (2.72) and (2.73) hold. The solution $f(x, v)$ to the equation (2.253) satisfies for $\phi \geq 0$ and $0 \leq \epsilon < \frac{1}{4}$,
\begin{equation}
\| f \|_{\infty, \phi, \epsilon} + \| f \|_{\gamma_+^\infty, \phi, \epsilon} \lesssim \frac{1}{e^2 + \frac{3}{e^2 \nu_0^2 (1 - P)[S]}} + \frac{1}{e^2 + \frac{3}{e^2 \nu_0^2 (1 - P)[S]}} + \nu^{-\frac{1}{2}} (1 - P) \| S \|_2 + \nu^{-\frac{1}{2}} (1 - P) \| S \|_2 \\
+ \frac{1}{e^2 + \frac{3}{e^2 \nu_0^2 (1 - P)[S]}} + \frac{1}{e^2 + \frac{3}{e^2 \nu_0^2 (1 - P)[S]}} + \| h \|_{\gamma_-^\infty} + \| h \|_{\gamma_-^2}.
\end{equation}
Proof. Since we already have bounds for $f$ in $L^{2m}$ as Theorem 3.3.1 following the proof of Theorem 2.2.1, we obtain
\begin{equation}
\| f \|_{\infty, \phi, \epsilon} + \| f \|_{\gamma_+^\infty, \phi, \epsilon} \lesssim \frac{1}{e^2 + \frac{3}{e^2 \nu_0^2 (1 - P)[S]}} + \frac{1}{e^2 + \frac{3}{e^2 \nu_0^2 (1 - P)[S]}} + \nu^{-\frac{1}{2}} (1 - P) \| S \|_2 + \nu^{-\frac{1}{2}} (1 - P) \| S \|_2 \\
+ \| h \|_{\gamma_-^\infty} + \| h \|_{\gamma_-^2} + \| h \|_{\gamma_-^\infty, \phi, \epsilon} + \| h \|_{\gamma_-^2} + \| h \|_{\gamma_-^\infty, \phi, \epsilon}.
\end{equation}
Using Lemma \[2.3.1\] and \[2.254\], we have
\[
\|v^{-1} \Gamma[f, g]\|_{\infty, \vartheta, \varrho} \lesssim \|f\|_{\infty, \vartheta, \varrho} \|g\|_{\infty, \vartheta, \varrho} \lesssim o(1) \|f\|_{\infty, \vartheta, \varrho}.
\]
Inserting \[2.264\] into \[2.263\], we can absorb \(o(1) \|f\|_{\infty, \vartheta, \varrho}\) into the left-hand side. Also, using \[1.10\], we have
\[
\|(\mu^0_b - \mu) v^{-1} \mathcal{P}[f]\|_{\gamma_0, \infty, \vartheta, \varrho} \lesssim o(1) \|f\|_{\gamma_0, \infty, \vartheta, \varrho}.
\]
Inserting \[2.265\] into \[2.263\] and absorbing \(o(1) \|f\|_{\gamma_+, \infty, \vartheta, \varrho}\) into the left-hand side, we obtain the desired result. \(\square\)

### 2.3.3 Analysis of Asymptotic Expansion

Based on the construction of interior solutions in Section \[2.1.5\] we know \(F_1, F_2\) and \(F_3\) satisfy certain fluid equations. For small data, the well-posedness and regularity of these equations are well-known, so we omit the proof and only present the main results.

**Theorem 2.3.4.** For \(K_0 > 0\) sufficiently small, the boundary layer satisfies
\[
\left\| \langle v \rangle^\vartheta e^{\vartheta|v|^2} F_1 \right\|_{H^1_{\vartheta} L^\infty_x} \leq 1, \quad \left\| \langle v \rangle^\vartheta e^{\vartheta|v|^2} F_2 \right\|_{H^1_{\vartheta} L^\infty_x} \leq 1, \quad \left\| \langle v \rangle^\vartheta e^{\vartheta|v|^2} F_3 \right\|_{H^1_{\vartheta} L^\infty_x} \leq 1.
\]

On the other hand, based on the construction of boundary layers in Section \[2.1.3\] we know \(\mathcal{F}_1 = 0\) and \(\mathcal{F}_2\) is well-defined. Using Theorem \[1.1.24\] Theorem \[1.2.14\] Theorem \[1.2.16\] and Theorem \[1.2.17\] we have for \(0 \leq \varrho < \frac{1}{4}\) and \(\vartheta > 3,

**Theorem 2.3.5.** For \(K_0 > 0\) sufficiently small, the boundary layer \(\mathcal{F}_2\) satisfies
\[
\left\| e^{K_0 \vartheta, \varrho} \mathcal{F}_2 \right\|_{\infty, \vartheta, \varrho} \lesssim 1,
\]
and
\[
\left\| e^{K_0 \vartheta, \varrho} \frac{\partial \mathcal{F}_2}{\partial \vartheta} \right\|_{\infty, \vartheta, \varrho} + \left\| e^{K_0 \vartheta, \varrho} \frac{\partial \mathcal{F}_2}{\partial v_1} \right\|_{\infty, \vartheta, \varrho} + \left\| e^{K_0 \vartheta, \varrho} \frac{\partial \mathcal{F}_2}{\partial v_2} \right\|_{\infty, \vartheta, \varrho} \lesssim |\ln(\epsilon)|^8,
\]
\[
\left\| e^{K_0 \vartheta, \varrho} \frac{\partial \mathcal{F}_2}{\partial \varrho} \right\|_{\infty, \vartheta, \varrho} + \left\| e^{K_0 \vartheta, \varrho} \frac{\partial \mathcal{F}_2}{\partial v_3} \right\|_{\infty, \vartheta, \varrho} + \left\| e^{K_0 \vartheta, \varrho} \frac{\partial \mathcal{F}_2}{\partial v_4} \right\|_{\infty, \vartheta, \varrho} \lesssim |\ln(\epsilon)|^8.
\]

**Remark 2.3.6.** Note that the norms defined in studying \(\epsilon\)-Milne problem with geometric correction can naturally be extended to include \((i_1, i_2)\) dependence and are consistent with the current format.

### 2.3.4 Proof of Main Theorem

Now we turn to the proof of the main result, Theorem \[1.1.1\]

Step 1: Remainder definitions.
Define the remainder as
\[
\epsilon^3 R := f' - \left( \epsilon F_1 + \epsilon^2 F_2 + \epsilon^3 F_3 \right) - \left( \epsilon \mathcal{F}_1 + \epsilon^2 \mathcal{F}_2 \right) = f' - Q - \mathcal{L},
\]
where
\[
Q := \epsilon F_1 + \epsilon^2 F_2 + \epsilon^3 F_3, \quad \mathcal{L} := \epsilon \mathcal{F}_1 + \epsilon^2 \mathcal{F}_2.
\]
We write \(\mathcal{L}\) to denote the linearized Boltzmann operator
\[
\mathcal{L}[f] = \epsilon v \cdot \nabla_x f + L[f].
\]
In studying boundary layers, we use substitutions to rewrite $\mathcal{L}$ into normal and tangential component as in (2.36):

$$
\mathcal{L}[f] = v_\eta \frac{\partial f}{\partial \eta} - \frac{\epsilon}{R_1 - \epsilon \eta} \left( v_\eta \frac{\partial f}{\partial v_\eta} - v_\phi \frac{\partial f}{\partial v_\phi} \right) - \frac{\epsilon}{R_2 - \epsilon \eta} \left( v_\phi \frac{\partial f}{\partial v_\phi} - v_\eta \frac{\partial f}{\partial v_\eta} \right) - \frac{\epsilon}{P_1 P_2} \left( \frac{\partial_1 r \cdot \partial_2 r}{P_1(\epsilon k_1 \eta - 1)} v_\phi v_\psi + \frac{\partial_2 r \cdot \partial_2 r}{P_2(\epsilon k_2 \eta - 1)} v_\phi^2 \right) \frac{\partial f}{\partial v_\phi}
$$

$$
- \frac{\epsilon}{P_1 P_2} \left( \frac{\partial_2 r \cdot \partial_1 r}{P_2(\epsilon k_2 \eta - 1)} v_\phi v_\psi + \frac{\partial_2 r \cdot \partial_1 r}{P_1(\epsilon k_1 \eta - 1)} v_\phi^2 \right) \frac{\partial f}{\partial v_\phi}
$$

$$
- \epsilon \left( \frac{v_\phi}{P_1(\epsilon k_1 \eta - 1)} \frac{\partial f}{\partial \tau_1} + \frac{v_\psi}{P_2(\epsilon k_2 \eta - 1)} \frac{\partial f}{\partial \tau_2} \right) + \mathcal{L}[f].
$$

(2.272)

Step 2: Representation of $\mathcal{L}[R]$.

The equation (1.16) is actually

$$
\mathcal{L}[f^*] = \Gamma[f^*, f^*],
$$

(2.273)

which means

$$
\mathcal{L}[Q + \mathcal{L} + \epsilon^3 R] = \Gamma[Q + \mathcal{L} + \epsilon^3 R, Q + \mathcal{L} + \epsilon^3 R].
$$

(2.274)

In (2.274), the nonlinear terms on the right-hand side can be decomposed as

$$
\Gamma[Q + \mathcal{L} + \epsilon^3 R, Q + \mathcal{L} + \epsilon^3 R] = \epsilon^6 \Gamma[R, R] + 2 \epsilon^3 \Gamma[R, Q + \mathcal{L}] + \Gamma[Q + \mathcal{L}, Q + \mathcal{L}].
$$

(2.275)

For the left-hand side of (2.274), based on the construction of interior solutions in Section 2.1.5, the interior contribution

$$
\mathcal{L}[Q] = \epsilon^4 v \cdot \nabla x \left( \epsilon F_1 + \epsilon^2 F_2 + \epsilon^3 F_3 \right) + \mathcal{L}[\epsilon F_1 + \epsilon^2 F_2 + \epsilon^3 F_3]
$$

(2.276)

$$
= \epsilon^4 v \cdot \nabla x F_3 + \epsilon^2 \Gamma[F_1, F_1] + 2 \epsilon^3 \Gamma[F_1, F_2].
$$

On the other hand, based on the construction of boundary layers in Section 2.1.5, we know the boundary layer contribution with $\mathcal{L} = \epsilon^2 \mathcal{F}_2$

$$
\mathcal{L}[\mathcal{L}] = - \epsilon^3 \left( \frac{1}{P_1 P_2} \left( \frac{\partial_1 r \cdot \partial_2 r}{P_1(\epsilon k_1 \eta - 1)} v_\phi v_\psi + \frac{\partial_2 r \cdot \partial_2 r}{P_2(\epsilon k_2 \eta - 1)} v_\phi^2 \right) \frac{\partial \mathcal{F}_2}{\partial v_\phi} \right)
$$

$$
- \epsilon^3 \left( \frac{1}{P_1 P_2} \left( \frac{\partial_2 r \cdot \partial_1 r}{P_2(\epsilon k_2 \eta - 1)} v_\phi v_\psi + \frac{\partial_2 r \cdot \partial_1 r}{P_1(\epsilon k_1 \eta - 1)} v_\phi^2 \right) \frac{\partial \mathcal{F}_2}{\partial v_\psi} \right)
$$

$$
- \epsilon^3 \left( \frac{v_\phi}{P_1(\epsilon k_1 \eta - 1)} \frac{\partial \mathcal{F}_2}{\partial \tau_1} + \epsilon^2 \frac{v_\psi}{P_2(\epsilon k_2 \eta - 1)} \frac{\partial \mathcal{F}_2}{\partial \tau_2} \right).
$$

(2.277)

Therefore, inserting (2.276), (2.277) and (2.274) into (2.274), we have

$$
\mathcal{L}[R] = \epsilon^3 \Gamma[R, R] + 2 \epsilon^3 \Gamma[R, Q + \mathcal{L}] + S_1 + S_2,
$$

(2.278)

where

$$
S_1 = - \epsilon v \cdot \nabla x F_3 + \frac{1}{P_1 P_2} \left( \frac{\partial_1 r \cdot \partial_2 r}{P_1(\epsilon k_1 \eta - 1)} v_\phi v_\psi + \frac{\partial_2 r \cdot \partial_2 r}{P_2(\epsilon k_2 \eta - 1)} v_\phi^2 \right) \frac{\partial \mathcal{F}_2}{\partial v_\phi}
$$

(2.279)

$$
+ \frac{1}{P_1 P_2} \left( \frac{\partial_2 r \cdot \partial_1 r}{P_2(\epsilon k_2 \eta - 1)} v_\phi v_\psi + \frac{\partial_2 r \cdot \partial_1 r}{P_1(\epsilon k_1 \eta - 1)} v_\phi^2 \right) \frac{\partial \mathcal{F}_2}{\partial v_\psi}
$$

$$
+ \left( \frac{v_\phi}{P_1(\epsilon k_1 \eta - 1)} \frac{\partial \mathcal{F}_2}{\partial \tau_1} + \epsilon^2 \frac{v_\psi}{P_2(\epsilon k_2 \eta - 1)} \frac{\partial \mathcal{F}_2}{\partial \tau_2} \right),
$$

$$
S_2 = 2 \epsilon \Gamma[F_1, \mathcal{F}_2] + 2 \epsilon^3 \Gamma[F_1, F_3] + \text{higher-order $\Gamma$ terms up to $\epsilon^3$}.
$$

(2.280)
Step 3: Representation of $R - \mathcal{P}[R]$.
The boundary condition of (2.16) is essentially

$$f' = \mu_0^\mu - 1 \mathcal{P}[f'] + \mu^{\frac{1}{2}} (\mu_0^\mu - \mu).$$  \hspace{1cm} (2.281)

which means

$$Q + \mathcal{L} + \epsilon^3 R = \mathcal{P}[Q + \mathcal{L} + \epsilon^3 R] + (\mu_0^\mu - \mu) \mu^{-1} \mathcal{P}[Q + \mathcal{L} + \epsilon^3 R] + \mu^{\frac{1}{2}} (\mu_0^\mu - \mu).$$  \hspace{1cm} (2.282)

Based on the boundary condition expansion in Section 2.1.4 we have

$$R - \mathcal{P}[R] = H[R] + h,$$  \hspace{1cm} (2.283)

where

$$H[R](x_0, v) = (\mu_0^\mu - \mu) \mu^{-1} \mathcal{P}[R],$$  \hspace{1cm} (2.284)

and

$$h = \epsilon^2 \left( \mu_0^\mu - \mu - \epsilon \mu^\mu \lambda_1 - \epsilon^2 \mu^2 \lambda_2 \right) \mathcal{P}[F_1 + \mathcal{F}_1] + \epsilon \left( \mu_0^\mu - \mu - \epsilon \mu^\mu \lambda_1 \right) \mathcal{P}[F_2 + \mathcal{F}_2]$$  \hspace{1cm} (2.285)

$$+ (\mu_0^\mu - \mu) \mu^{-1} \mathcal{P}[F_3] + \epsilon^3 \mu^{\frac{1}{2}} \left( \mu_0^\mu - \mu - \epsilon \mu^\mu \lambda_1 - \epsilon^2 \mu^2 \lambda_2 - \epsilon^3 \mu^3 \lambda_3 \right).$$

Step 4: Remainder Estimate.
The equation (2.278) and boundary condition (2.283) forms a system that fits into (2.252):

$$\left\{ \begin{array}{l}
\epsilon v \cdot \nabla x + \mathcal{L} = \mu \left[ R, 2(Q + \mathcal{L}) + \epsilon^3 R \right] + S_1(x, v) + S_2(x, v) \quad \text{in} \quad \Omega \times \mathbb{R}^3, \\
R(x_0, v) = \mathcal{P}[R](x_0, v) + H[R](x_0, v) + h(x_0, v) \quad \text{for} \quad x_0 \in \partial \Omega \quad \text{and} \quad v \cdot n < 0.
\end{array} \right.$$  \hspace{1cm} (2.286)

We assume that

$$\| \epsilon^3 R \|_{\infty, \partial \Omega} \lesssim o(1).$$

Then we can verify (2.286) satisfies the assumptions (2.254) (since $Q$ and $\mathcal{L}$ are small), (2.255). Also, the construction in Section 2.1.5 implies that the solution satisfies (2.254). Applying Theorem 2.3.2 to (2.254), we obtain

$$\left\| \frac{\partial I}{\partial x} \right\|_{\infty, \partial \Omega} \lesssim o(1) \epsilon.$$  \hspace{1cm} (2.287)

Step 5: Estimates of $S_1$ Terms.
Based on Theorem 2.3.3 we know

$$\| \epsilon v \cdot \nabla x F_3 \|_{\infty, \partial \Omega} \lesssim C_0.$$  \hspace{1cm} (2.288)

On the other hand, based on Theorem 2.3.5 using the rescaling $\eta = \frac{\mu}{\epsilon}$, we have

$$\left\| \frac{\partial F_2}{\partial x_1} \right\|_{\infty, \partial \Omega} \lesssim C_0,$$  \hspace{1cm} (2.289)

$$\left\| \frac{\partial F_2}{\partial x_2} \right\|_{\infty, \partial \Omega} \lesssim C_0,$$  \hspace{1cm} (2.290)

$$\left\| \frac{\partial F_2}{\partial x_3} \right\|_{\infty, \partial \Omega} \lesssim C_0.$$  \hspace{1cm} (2.291)

$$\left\| \frac{\partial F_2}{\partial \nu} \right\|_{\infty, \partial \Omega} \lesssim C_0.$$  \hspace{1cm} (2.292)
Collecting all terms, we have
\[
\|P[S_1]\|_{\text{asm}} \lesssim \epsilon^1 \frac{1}{m} |\ln(\epsilon)|^8, \quad \|\nu^{-\frac{1}{2}}(1 - P)[S_1]\|_2 \lesssim \epsilon^{\frac{1}{2}} |\ln(\epsilon)|^8, \quad \|\nu^{-1} S_1\|_{\infty, \theta, \vartheta} \lesssim |\ln(\epsilon)|^8. \tag{2.293}
\]

Step 6: Estimates of $S_2$ Terms.
Since $S_2$ are all nonlinear terms, Lemma 2.3.1 implies that $P[S_2] = 0$. Then the leading-order term is $\Gamma[F_1, \mathcal{F}_2]$. Hence, using Theorem 2.3.4, Theorem 2.3.5 and Lemma 2.3.1, we have
\[
\|\nu^{-\frac{1}{2}} \Gamma[F_1, \mathcal{F}_2]\|_2 \lesssim \sup_{x \in \Omega} \left( |\nu F_1(x)|_2 \right) \|\nu \mathcal{F}_2\|_2 \lesssim \epsilon^{\frac{1}{2}} |\ln(\epsilon)|^8, \tag{2.294}
\]
\[
\|\nu^{-1} \Gamma[F_1, \mathcal{F}_2]\|_{\infty, \theta, \vartheta} \lesssim \|F_1\|_{\infty, \theta, \vartheta} \|\mathcal{F}_2\|_{\infty, \theta, \vartheta} \lesssim |\ln(\epsilon)|^8. \tag{2.295}
\]
Hence, we have
\[
\|P[S_2]\|_{\text{asm}} = 0, \quad \|\nu^{-\frac{1}{2}}(1 - P)[S_2]\|_2 \lesssim \epsilon^{\frac{1}{2}} |\ln(\epsilon)|^8, \quad \|\nu^{-1} S_2\|_{\infty, \theta, \vartheta} \lesssim |\ln(\epsilon)|^8. \tag{2.296}
\]

Step 7: Estimates of $h$ Terms.
Note that all terms in $h$ are at least order of $\epsilon$. Hence, we directly bound
\[
\|h\|_{\gamma, \theta, \vartheta} \lesssim \epsilon, \quad \|h\|_{\gamma, \infty, \theta} \lesssim \epsilon, \quad \|h\|_{\gamma, \infty, \theta, \vartheta} \lesssim \epsilon. \tag{2.297}
\]

Step 8: Synthesis.
Inserting (2.293), (2.295) and (2.297) into (2.288), we have
\[
\|\mathcal{R}\|_{\infty, \theta, \vartheta} + \|\mathcal{R}\|_{\gamma, \infty, \theta, \vartheta} \lesssim \frac{1}{\epsilon^{\frac{1}{2} + \frac{1}{m}}} \left( \epsilon^{1 - \frac{1}{2m}} |\ln(\epsilon)|^8 \right) + \frac{1}{\epsilon^{1 + \frac{1}{m}}} \left( \epsilon^{\frac{1}{2}} |\ln(\epsilon)|^8 \right) + \left( |\ln(\epsilon)|^8 \right) \tag{2.298}
\]
\[
+ \frac{1}{\epsilon^{\frac{1}{2} + \frac{1}{m}}} (\epsilon) + \frac{1}{\epsilon^{1 + \frac{1}{m}}} (\epsilon) + (\epsilon) + \epsilon^{1 - \frac{1}{2m}} \|\mathcal{R}\|_{\infty, \theta, \vartheta}^2 \lesssim \epsilon^{-1 - \frac{1}{2m}} |\ln(\epsilon)|^8 + \epsilon^{1 - \frac{1}{2m}} |\mathcal{R}\|_{\infty, \theta, \vartheta}^2.
\]
In particular, we can verify the validity of assumption (2.287). Considering (2.269), this means that we have shown
\[
\frac{1}{\epsilon^{\frac{1}{2}}} \left| f^\epsilon - \left( \epsilon F_1 + \epsilon^2 F_2 + \epsilon^3 F_3 \right) - \left( \epsilon \mathcal{F}_1 + \epsilon^2 \mathcal{F}_2 \right) \right|_{\infty, \theta, \vartheta} \lesssim \epsilon^{-1 - \frac{1}{2m}} |\ln(\epsilon)|^8. \tag{2.299}
\]
Therefore, we know
\[
\|f^\epsilon - \epsilon F_2 - \epsilon F_3\|_{\infty, \theta, \vartheta} \lesssim \epsilon^{2 - \frac{1}{2m}} |\ln(\epsilon)|^8. \tag{2.300}
\]
Since $\mathcal{F}_1 = 0$, then we naturally have for $F = F_1$.
\[
\|f^\epsilon - \epsilon F\|_{\infty, \theta, \vartheta} \lesssim \epsilon^{2 - \frac{1}{2m}} |\ln(\epsilon)|. \tag{2.301}
\]
Here $\frac{3}{2} < m < 3$, so we may further bound
\[
\|f^\epsilon - \epsilon F\|_{\infty, \theta, \vartheta} \lesssim C(\delta) \epsilon^{\frac{1}{2} - \delta}, \tag{2.302}
\]
for any $0 < \delta < 1$. 

**ASYMPTOTIC ANALYSIS OF BOLTZMANN EQUATION**

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Chapter 3

Evolutionary Boltzmann Equation

3.1 Asymptotic Expansion

3.1.1 Interior Expansion

We define the interior expansion

\[ F(t, x, v) \sim \sum_{k=1}^{3} \epsilon^k F_k(t, x, v). \]  

(3.1)

Plugging it into the equation \((1.66)\) and comparing the order of \(\epsilon\), we obtain

\[ \mathcal{L}[F_1] = 0, \]  

(3.2)

\[ \mathcal{L}[F_2] = -v \cdot \nabla_x F_1 + \Gamma[F_1, F_1], \]  

(3.3)

\[ \mathcal{L}[F_3] = -\partial_t F_1 - v \cdot \nabla_x F_2 + 2\Gamma[F_1, F_2]. \]  

(3.4)

The analysis of \(F_k\) solvability is standard and well-known. Note that the null space \(\mathcal{N}\) of the operator \(\mathcal{L}\) is spanned by

\[ \mu^2 \{1, v_1, v_2, v_3, |v|^2 - 3 \} = \{\varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi_4\}. \]  

(3.5)

Then \(\mathcal{L}[f] = S\) is solvable if and only if \(S \in \mathcal{N}^\perp\) the orthogonal complement of \(\mathcal{N}\) in \(L^2(\mathbb{R}^3)\). As [23, Chapter 4] and [24, Chapter 3] reveal, similar to stationary problems in Section 2.1.1, each \(F_k\) consists of three parts:

\[ F_k(t, x, v) := A_k(t, x, v) + B_k(t, x, v) + C_k(t, x, v). \]  

(3.6)

• Principal contribution \(A_k := \sum_{i=0}^{4} A_{k,i} \varphi_i \in \mathcal{N}\), where the coefficients \(A_{k,i}\) must be determined at each order \(k\) independently.

• Connecting contribution \(B_k := \sum_{i=0}^{4} B_{k,i} \varphi_i \in \mathcal{N}\), where the coefficients \(B_{k,i}\) depends on \(A_s\) for \(1 \leq s \leq k-1\). In other words, \(B_k\) is accumulative information from previous orders and thus is not independent. This term is present due to the nonlinearity in \(\Gamma\).

• Orthogonal contribution \(C_k \in \mathcal{N}^\perp\) satisfying

\[ \mathcal{L}[C_k] = \partial_t F_{k-2} - v \cdot \nabla_x F_{k-1} + \sum_{i=1}^{k-1} \Gamma[F_i, F_{k-i}], \]  

(3.7)

which can be uniquely determined. Similar to \(B_k\), here \(C_k\) is also accumulative information from previous orders and thus is not independent.
We define the rescaled time variable $\tau = \frac{t}{\epsilon^2}$, which implies $\frac{\partial u^\epsilon}{\partial t} = \frac{1}{\epsilon^2} \frac{\partial u^\epsilon}{\partial \tau}$. Then, under the substitution $t \rightarrow \tau$, the equation (1.66) is transformed into

$$\left\{ \begin{array}{l}
\partial_\tau f^\epsilon + \epsilon v \cdot \nabla_x f^\epsilon + L[f^\epsilon] = \Gamma[f^\epsilon, f^\epsilon], \quad \text{in } \mathbb{R}^+ \times \Omega \times \mathbb{R}^3 \\
f^\epsilon(0, x, v) = f_0(x, v), \quad \text{in } \Omega \times \mathbb{R}^3 \\
f^\epsilon(\tau, x_0, v) = \mathcal{P}^\epsilon[f^\epsilon](\tau, x_0, v) \quad \text{for } x_0 \in \partial \Omega \quad \text{and} \quad v \cdot n(x_0) < 0,
\end{array} \right.$$

(3.16)

We define the initial layer expansion:

$$\mathcal{F}(\tau, x, v) \sim \sum_{k=1}^{4} \epsilon^k \mathcal{F}_k(\tau, x, v),$$

(3.17)

where $\mathcal{F}_k$ can be determined by comparing the order of $\epsilon$ via plugging (3.17) into the equation (3.16). Thus, we have

$$\partial_\tau \mathcal{F}_1 + L[\mathcal{F}_1] = 0,$$

(3.18)

$$\partial_\tau \mathcal{F}_2 + L[\mathcal{F}_2] = -v \cdot \nabla_x \mathcal{F}_1 + \Gamma[\mathcal{F}_1, \mathcal{F}_1] + 2\Gamma[\mathcal{F}_1, \mathcal{F}_1],$$

(3.19)

$$\partial_\tau \mathcal{F}_3 + L[\mathcal{F}_3] = -v \cdot \nabla_x \mathcal{F}_2 + 2\Gamma[\mathcal{F}_1, \mathcal{F}_2] + 2\Gamma[\mathcal{F}_1, \mathcal{F}_2] + 2\Gamma[\mathcal{F}_2, \mathcal{F}_1],$$

(3.20)

$$\partial_\tau \mathcal{F}_4 + L[\mathcal{F}_4] = -v \cdot \nabla_x \mathcal{F}_3 + 2\Gamma[\mathcal{F}_1, \mathcal{F}_3] + 2\Gamma[\mathcal{F}_2, \mathcal{F}_2] + 2\Gamma[\mathcal{F}_1, \mathcal{F}_3] + 2\Gamma[\mathcal{F}_1, \mathcal{F}_3] + 2\Gamma[\mathcal{F}_2, \mathcal{F}_2].$$

(3.21)

3.1.3 Boundary Layer Expansion

This is very similar to the stationary problem in Section 2.1.2. We need to introduce several geometric substitutions.
1. In a neighborhood of \( x_0 \in \partial \Omega \) define an orthogonal curvilinear coordinates system \((\iota_1, \iota_2)\) such that at \( x_0 \) the coordinate lines coincide with the principal directions. Let \( \mu \) be the normal distance to the boundary. Then \((\mu, \iota_1, \iota_2)\) forms a local orthogonal coordinate system. Let \( \kappa_1 \) and \( \kappa_2 \) denote two principal curvatures and \( R_1 \) and \( R_2 \) two radii of principal curvature.

2. We also decompose the velocity into normal and tangential directions

\[
\begin{align*}
-v \cdot n &= v_\eta, \\
-v \cdot \xi_1 &= v_\phi, \\
-v \cdot \xi_2 &= v_\psi.
\end{align*}
\]

Denote \( \mathbf{v} = (v_\eta, v_\phi, v_\psi) \).

3. Define the scaled variable \( \eta = \frac{\mu}{\epsilon} \), which implies \( \partial \partial_{\mu} = \frac{1}{\epsilon} \partial_{\eta} \).

Under these substitutions, the equation (1.60) is transformed into

\[
\begin{align*}
\left\{ \begin{array}{l}
\epsilon^2 \partial_{\eta} f^* + v_\eta \frac{\partial f^*}{\partial \eta} - \frac{\epsilon}{R_1 - \epsilon} \left( v_\phi \frac{\partial f^*}{\partial \phi} - v_\eta v_\phi \frac{\partial f^*}{\partial v_\eta} - \frac{1}{R_2 - \epsilon} \left( v_\phi \frac{\partial f^*}{\partial \phi} - v_\eta v_\phi \frac{\partial f^*}{\partial v_\phi} \right) \\
- \frac{\epsilon}{P_1 P_2} \left( \partial_{11} \mathbf{r} \cdot \partial_{11} \mathbf{r} \right) v_\phi v_\phi + \frac{\partial_{12} \mathbf{r} \cdot \partial_{12} \mathbf{r}}{P_2 (\epsilon \kappa_2 \eta - 1)} v_\psi^2 + \frac{\partial_{22} \mathbf{r} \cdot \partial_{22} \mathbf{r}}{P_1 (\epsilon \kappa_1 \eta - 1)} v_\psi^2 \\
- \frac{\epsilon}{P_1 P_2} \left( \partial_{11} \mathbf{r} \cdot \partial_{11} \mathbf{r} \right) v_\phi v_\phi + \frac{\partial_{12} \mathbf{r} \cdot \partial_{12} \mathbf{r}}{P_2 (\epsilon \kappa_2 \eta - 1)} v_\psi^2 + \frac{\partial_{22} \mathbf{r} \cdot \partial_{22} \mathbf{r}}{P_1 (\epsilon \kappa_1 \eta - 1)} v_\psi^2 \\
- \frac{\epsilon}{(P_1 (\epsilon \kappa_1 \eta - 1))} \frac{\partial f^*}{\partial \eta_1} + \frac{\epsilon}{v_\psi} \left( \frac{\partial f^*}{\partial \eta_1} + \frac{\partial f^*}{\partial \eta_2} \right) + L[f^*] = \Gamma[f^*, f^*] \text{ in } \mathbb{R}^+ \times \Omega \times \mathbb{R}^3,
\end{array} \right.
\end{align*}
\]

\( f^*(0, \eta_1, \iota_1, \iota_2, \mathbf{v}) = f_0(\eta_1, \iota_1, \iota_2, \mathbf{v}) \) in \( \Omega \times \mathbb{R}^3 \),

\( f^*(t, 0, \iota_1, \iota_2, \mathbf{v}) = \mathcal{P}[f^*](t, 0, \iota_1, \iota_2, \mathbf{v}) \) for \( v_\eta > 0 \).

We define the boundary layer expansion as follows:

\[
\mathcal{F}(t, \eta_1, \iota_1, \iota_2, \mathbf{v}) \sim \sum_{k=1}^{3} \epsilon^k \mathcal{F}_k(t, \eta_1, \iota_1, \iota_2, \mathbf{v}),
\]

where \( \mathcal{F}_k \) can be defined by comparing the order of \( \epsilon \) via plugging (3.24) into the equation (3.23). Thus, in a neighborhood of the boundary, we have

\[
\begin{align*}
0 &= \frac{v_\eta}{R_1 - \epsilon} \left( v_\phi \frac{\partial \mathcal{F}_1}{\partial \phi} - v_\eta v_\phi \frac{\partial \mathcal{F}_1}{\partial v_\eta} \right) - \frac{\epsilon}{R_2 - \epsilon} \left( v_\phi \frac{\partial \mathcal{F}_1}{\partial \phi} - v_\eta v_\phi \frac{\partial \mathcal{F}_1}{\partial v_\phi} \right) + L[\mathcal{F}_1] = 0, \\
0 &= \frac{v_\eta}{R_1 - \epsilon} \left( v_\phi \frac{\partial \mathcal{F}_2}{\partial \phi} - v_\eta v_\phi \frac{\partial \mathcal{F}_2}{\partial v_\eta} \right) - \frac{\epsilon}{R_2 - \epsilon} \left( v_\phi \frac{\partial \mathcal{F}_2}{\partial \phi} - v_\eta v_\phi \frac{\partial \mathcal{F}_2}{\partial v_\phi} \right) + L[\mathcal{F}_2] = Z_1,
\end{align*}
\]

where \( Z_1 = Z_1 \left[ F_1, \frac{\partial \mathcal{F}_1}{\partial \phi}, \frac{\partial \mathcal{F}_1}{\partial v_\phi}, \frac{\partial \mathcal{F}_1}{\partial \eta_1}, \frac{\partial \mathcal{F}_1}{\partial \eta_2} \right] \) as

\[
Z_1 := 2\Gamma[F_1, \mathcal{F}_1] + \Gamma[\mathcal{F}_1, \mathcal{F}_1] + \frac{1}{P_1 P_2} \left( \frac{\partial_{11} \mathbf{r} \cdot \partial_{11} \mathbf{r}}{P_1 (\epsilon \kappa_1 \eta - 1)} v_\phi v_\phi + \frac{\partial_{12} \mathbf{r} \cdot \partial_{12} \mathbf{r}}{P_2 (\epsilon \kappa_2 \eta - 1)} v_\psi^2 \right) \frac{\partial \mathcal{F}_1}{\partial \phi} + \frac{v_\phi}{P_1 (\epsilon \kappa_1 \eta - 1)} \frac{\partial \mathcal{F}_1}{\partial \eta_1} + \frac{v_\psi}{P_2 (\epsilon \kappa_2 \eta - 1)} \frac{\partial \mathcal{F}_1}{\partial \eta_2}.
\]
However, we define \( \mathcal{F}_3 \) in a completely different fashion. Let \( \mathcal{F}_3 \) satisfy
\[
\begin{align*}
\nu_\eta \frac{\partial \mathcal{F}_3}{\partial \eta} - & \frac{\epsilon}{\eta_1 - \eta_0} \left( \nu_\phi \frac{\partial \mathcal{F}_3}{\partial \nu_\phi} - \nu_\theta \nu_\phi \frac{\partial \mathcal{F}_3}{\partial \nu_\theta} \right) - \frac{\epsilon}{\eta_2 - \eta_0} \left( \nu_\phi \frac{\partial \mathcal{F}_3}{\partial \nu_\phi} - \nu_\eta \nu_\phi \frac{\partial \mathcal{F}_3}{\partial \nu_\eta} \right) \\
& - \frac{\epsilon}{P_1 P_2} \left( \frac{\partial_{11} r \cdot \partial_1 r}{P_1(\kappa_1 \eta - 1)} \nu_\phi v_\phi + \frac{\partial_{12} r \cdot \partial_1 r}{P_2(\kappa_1 \eta - 1)} v_\phi^2 \right) \frac{\partial \mathcal{F}_3}{\partial v_\phi} \\
& - \frac{\epsilon}{P_1 P_2} \left( \frac{\partial_{22} r \cdot \partial_1 r}{P_2(\kappa_2 \eta - 1)} \nu_\phi v_\phi + \frac{\partial_{22} r \cdot \partial_1 r}{P_1(\kappa_1 \eta - 1)} v_\phi^2 \right) \frac{\partial \mathcal{F}_3}{\partial v_\phi} \\
& - \epsilon \left( \frac{\nu_\phi}{P_1(\kappa_1 \eta - 1)} \frac{\partial \mathcal{F}_3}{\partial t_1} + \frac{\nu_\phi}{P_2(\kappa_2 \eta - 1)} \frac{\partial \mathcal{F}_3}{\partial t_2} \right) + \mathcal{L}[\mathcal{F}_3] = Z_2,
\end{align*}
\]
where
\[
Z_2 : = 2\Gamma[\mathcal{F}_1, \mathcal{F}_2] + 2\Gamma[F_1, \mathcal{F}_2] + 2\Gamma[F_2, \mathcal{F}_1] + \frac{1}{P_1 P_2} \left( \frac{\partial_{11} r \cdot \partial_1 r}{P_1(\kappa_1 \eta - 1)} \nu_\phi v_\phi + \frac{\partial_{12} r \cdot \partial_1 r}{P_2(\kappa_1 \eta - 1)} v_\phi^2 \right) \frac{\partial \mathcal{F}_2}{\partial v_\phi} \\
& + \frac{1}{P_1 P_2} \left( \frac{\partial_{22} r \cdot \partial_1 r}{P_2(\kappa_2 \eta - 1)} \nu_\phi v_\phi + \frac{\partial_{22} r \cdot \partial_1 r}{P_1(\kappa_1 \eta - 1)} v_\phi^2 \right) \frac{\partial \mathcal{F}_2}{\partial v_\phi} + \frac{\nu_\phi}{P_1(\kappa_1 \eta - 1)} \frac{\partial \mathcal{F}_2}{\partial t_1} + \frac{\nu_\phi}{P_2(\kappa_2 \eta - 1)} \frac{\partial \mathcal{F}_2}{\partial t_2}.
\]

Obviously, (3.28) actually contains all terms in (3.23) except the time derivative, so it is essentially
\[
\epsilon \nu_\phi v_\phi \cdot \nabla_x \mathcal{F}_3 + \mathcal{L}[\mathcal{F}_3] = Z_2.
\]

Hence, we will resort to the well-posedness and decay theory of linearized stationary problem instead of that of \( \epsilon \)-Milne problem with geometric correction.

### 3.1.4 Initial Condition Expansion

The bridge between the interior solution and initial layer is the initial condition. Plugging the combined expansion from (3.1) and (3.17)
\[
f' \sim \sum_{k=1}^{3} \epsilon^k F_k + \sum_{k=1}^{4} \epsilon^k \mathcal{F}_k
\]
into the initial condition (1.66), and comparing the order of \( \epsilon \), we obtain
\[
\begin{align*}
F_1 + \mathcal{F}_1 &= f_0, \\
F_2 + \mathcal{F}_2 &= 0, \\
F_3 + \mathcal{F}_3 &= 0.
\end{align*}
\]

Since we do not expand \( F \) to higher order, we simply require the initial condition such that \( \mathcal{F}_1 \) decays to zero as \( \tau \to \infty \).

### 3.1.5 Boundary Condition Expansion

Similar to the stationary problem in Section 2.1.4, we require the boundary condition matching
\[
\begin{align*}
F_1 + \mathcal{F}_1 &= \mathcal{P}[F_1 + \mathcal{F}_1] + \mu_1(x_0, v), \\
F_2 + \mathcal{F}_2 &= \mathcal{P}[F_2 + \mathcal{F}_2] + \mu_1(x_0, v) \int_{u \cdot n(x_0) > 0} \mu^h(u)(F_1 + \mathcal{F}_1) |u \cdot n(x_0)| \, du + \mu_2(x_0, v).
\end{align*}
\]

For \( \mathcal{F}_3 \) and \( \mathcal{F}_3 \), we can assign stronger version
\[
F_3 + \mathcal{F}_3 = \mathcal{P}[F_3 + \mathcal{F}_3] + \epsilon^2 \left( \mu_2' - \mu - \epsilon \mu^h \mu_1 \right) \mu^{-1} \mathcal{P}[F_1 + \mathcal{F}_1] + \epsilon^{-3} \mu^h \left( \mu_2' - \mu - \epsilon \mu^h \mu_1 - \epsilon^2 \mu^h \mu_2 \right).
\]
3.1.6 Matching Procedure

Define the length of boundary layer $L = \epsilon^{-\frac{1}{2}}$. Also, denote $\mathcal{F}[v_\eta,v_\phi,v_\psi] = (-v_\eta,v_\phi,v_\psi)$.

Step 1: Construction of $F_1$, $F_2$ and $F_3$.

A direct computation reveals that $F_1 = A_1 + B_1 + C_1$, where $B_1 = C_1 = 0$. Define

$$F_1 = \frac{\mu_1}{4} \left( \rho_1 + u_1 \cdot v + \theta_1 \frac{|v|^2 - 3}{2} \right), \quad (3.38)$$

where $(\rho_1, u_1, \theta_1)$ satisfies the Navier-Stokes-Fourier system

$$\begin{aligned}
\nabla_x (\rho_1 + \theta_1) &= 0, \\
\partial_t u_1 + u_1 \cdot \nabla_x u_1 - \gamma_1 \Delta_x u_1 + \nabla_x p_2 &= 0, \\
\nabla_x \cdot u_1 &= 0, \\
\partial_t \theta_1 + u_1 \cdot \nabla_x \theta_1 - \gamma_2 \Delta_x \theta_1 &= 0,
\end{aligned} \quad (3.39)$$

with the initial condition

$$\rho_1(0,x) = \rho_{0,1}(x), \quad u_1(0,x) = u_{0,1}(x), \quad \theta_1(0,x) = \theta_{0,1}(x), \quad (4.40)$$

and the boundary condition

$$\rho_1(t,x_0) = \rho_{b,1}(t,x_0) + M_1(t,x_0), \quad u_1(t,x_0) = u_{b,1}(t,x_0), \quad \theta_1(t,x_0) = \theta_{b,1}(t,x_0). \quad (4.41)$$

Here $M_1(t,x_0)$ is such that the Boussinesq relation

$$\rho_1 + \theta_1 = \text{constant} \quad (4.42)$$

is satisfied. Note that the above constant is determined by the normalization condition

$$\int_{\Omega \times \mathbb{R}^3} F_1(t,x,v) \mu_1^{\frac{1}{2}}(v) dv dx = 0. \quad (4.43)$$

Then based on the compatibility condition of $\mu_1$ which is

$$\int_{u \cdot n(x_0) > 0} \mu_1^{\frac{1}{2}}(u) \mu_1(t,x_0,u) |u \cdot n(x_0)| du = 0, \quad (4.44)$$

we naturally obtain $\mathcal{P}[F_1] = M_1 \mu_1^{\frac{1}{2}}$, which means

$$F_1 = \mathcal{P}[F_1] + \mu_1 \text{ on } \partial \Omega. \quad (4.45)$$

Therefore, compared with $(3.35)$, it is not necessary to introduce the boundary layer at this order and we simply take $F_1 = 0$. Also, the interior solution can already satisfy the initial data, so it is not necessary to introduce the initial layer at this order and we simply take $F_1 = 0$.

Step 2: Construction of $F_2$, $F_3$ and $F_2$.

Define $F_2 = A_2 + B_2 + C_2$, where $B_2$ and $C_2$ can be uniquely determined following previous analysis, and

$$A_2 = \frac{\mu_1}{4} \left( \rho_2 + u_2 \cdot v + \theta_2 \frac{|v|^2 - 3}{2} \right), \quad (3.46)$$

satisfying a linear fluid-type equation provided $F_1$ is known (we omit the detailed form of this equation here, which is an evolutionary version of $(2.54)$). Now $F_2$ does not satisfy $(3.44)$ alone, so we have to introduce
boundary layer. Let $\mathcal{F}_2$ satisfy the $\epsilon$-Milne problem with geometric correction
\[
\begin{cases}
 v_0 \frac{\partial \mathcal{F}_2}{\partial \eta} - \frac{\epsilon}{R_1 - \epsilon \eta} \left( v_0^2 \frac{\partial \mathcal{F}_2}{\partial v_0} - v_0 v_\psi \frac{\partial \mathcal{F}_2}{\partial v_\psi} \right) - \frac{\epsilon}{R_2 - \epsilon \eta} \left( v_0^2 \frac{\partial \mathcal{F}_2}{\partial v_0} - v_0 v_\psi \frac{\partial \mathcal{F}_2}{\partial v_\psi} \right) + \mathcal{L}[\mathcal{F}_2] = 0,
\end{cases}
\] (3.47)

with the in-flow boundary data
\[
h(t, \iota_1, \iota_2, \mathbf{v}) = M_1 \mu_1(t, x_0, v) + \mu_2(t, x_0, v) - \left( (B_2 + C_2) - \mathcal{P}[B_2 + C_2] \right).
\] (3.48)

Based on Theorem 4.1.15 and 4.1.24 there exists a unique
\[
\tilde{h}(t, \iota_1, \iota_2, \mathbf{v}) = \mu^2 \sum_{k=0}^{4} \hat{D}_k(t, \iota_1, \iota_2) e_k,
\] (3.49)

such that (3.47) is well-posed and the solution decays exponentially fast (here $e_k$ with $k = 0, 1, 2, 3, 4$ form a basis of null space $\mathcal{N}$ of $\mathcal{L}$). In particular, $\hat{D}_1 = 0$. Then we further require that $A_2$ satisfies the boundary condition
\[
A_2(t, x_0, v) = \tilde{h}(t, \iota_1, \iota_2, v) + M_2(t, x_0) \mu^2(v).
\] (3.50)

Here $x_0$ corresponds to $(\iota_1, \iota_2)$ and $v$ corresponds to $\mathbf{v}$, based on substitution in Section 3.1.3. Here, the constant $M_2(t, x_0)$ is chosen to enforce the Boussinesq relation
\[
p_2 - (\rho_2 + \theta_2 + \rho_1 \theta_1) = 0,
\] (3.51)

where $p_2$ is the pressure solved from (3.39). Similar to the construction of $F_1$, due to (1.65), we can choose the constant to satisfy the normalization condition
\[
\int_{\Omega} (F_2 + \mathcal{F}_2)(t, x, v) \mu^{1/2}(v) dv dx = 0.
\] (3.52)

Similar to stationary problem, we can verify that such construction satisfies the boundary condition (3.36).

Also, the initial layer is no longer zero at this order. It satisfies
\[
\begin{cases}
 \partial_\tau \mathcal{F}_2 + \mathcal{L}[\mathcal{F}_2] = 0,
 \mathcal{F}_2(0, x, v) = (B_2 + C_2)(0, x, v) - \mathcal{F}_{2, \infty},
\end{cases}
\] (3.53)

where $\mathcal{F}_{2, \infty}(x, v) \in \mathcal{N}$ is such that
\[
\lim_{\tau \to \infty} \mathcal{F}_2(\tau, x, v) = 0.
\] (3.54)

Then we further require that $A_2$ satisfies the initial condition
\[
A_2(0, x, v) = \mathcal{F}_{2, \infty}(x, v).
\] (3.55)

Step 3: Construction of $F_3$, $\mathcal{F}_3$ and $\mathcal{F}_3$.

This is almost the same as Step 2. Define $F_3 = A_3 + B_3 + C_3$, where $B_3$ and $C_3$ can be uniquely determined following previous analysis, and
\[
A_3 = \mu^2 \left( \rho_3 + u_3 \cdot v + \theta_3 \frac{|v|^2}{2} - 3 \right),
\] (3.56)
satisfying a linear fluid-type equation provided $F_1$ and $F_2$ are known. In particular, we define the boundary condition

$$A_3(t, x_0, v) = 0. \quad (3.57)$$

On the other hand, define the boundary layer $\mathcal{F}_3$

$$\left. \begin{aligned}
v_\eta \frac{\partial \mathcal{F}_3}{\partial \eta} - & \epsilon \left( \frac{v_\phi^2 \frac{\partial \mathcal{F}_3}{\partial v_\phi}}{R_1} - v_\eta \frac{\partial \mathcal{F}_3}{\partial v_\phi} \right) - \epsilon \left( \frac{v_\phi^2 \frac{\partial \mathcal{F}_3}{\partial v_\phi}}{R_2} - v_\eta \frac{\partial \mathcal{F}_3}{\partial v_\phi} \right) \right) = 0, \\
- & \left( \frac{\epsilon_{11} r \cdot \partial_{2r}}{P_1(\epsilon_{11} \eta - 1)} v_\phi v_\phi + \frac{\epsilon_{12} r \cdot \partial_{2r}}{P_2(\epsilon_{22} \eta - 1)} v_\phi v_\phi \right), \\
- \epsilon \left( \frac{v_\phi}{P_1(\epsilon_{11} \eta - 1)} \frac{\partial \mathcal{F}_3}{\partial v_\phi} + v_\phi \frac{\partial \mathcal{F}_3}{\partial v_\phi} \right) + \mathcal{L}[\mathcal{F}_3] = Z, \\
\end{aligned} \right. \quad (3.57)$$

where

$$Z := 2\Gamma[F_1, \mathcal{F}_2] + \frac{1}{P_1 P_2} \left( \frac{\epsilon_{11} r \cdot \partial_{2r}}{P_1(\epsilon_{11} \eta - 1)} v_\phi v_\phi + \frac{\epsilon_{12} r \cdot \partial_{2r}}{P_2(\epsilon_{22} \eta - 1)} v_\phi v_\phi \right) \frac{\partial \mathcal{F}_2}{\partial v_\phi} + \frac{1}{P_1 P_2} \left( \frac{\epsilon_{22} r \cdot \partial_{2r}}{P_2(\epsilon_{22} \eta - 1)} v_\phi v_\phi + \frac{\epsilon_{12} r \cdot \partial_{2r}}{P_1(\epsilon_{11} \eta - 1)} v_\phi v_\phi \right) \frac{\partial \mathcal{F}_2}{\partial v_\phi} + \frac{v_\phi}{P_1(\epsilon_{11} \eta - 1)} \frac{\partial \mathcal{F}_2}{\partial v_\phi} + \frac{v_\phi}{P_2(\epsilon_{22} \eta - 1)} \frac{\partial \mathcal{F}_2}{\partial v_\phi}.$$ 

The boundary condition is taken as

$$\mathcal{F}_3 = \mathcal{P}[\mathcal{F}_3] + \epsilon^{-2} \left( \mu_0 - \mu - \epsilon \mu_1 \right) \mathcal{P}[F_1 + \mathcal{F}_1] + \epsilon^{-3} \mu_1 \left( \mu_0 - \mu - \epsilon \mu_1 \mu_2 \right), \quad (3.63)$$

Also, the initial layer satisfies

$$\left\{ \begin{aligned}
& \partial_t \mathcal{F}_3 + \mathcal{L}[\mathcal{F}_3] = -v \cdot \nabla_x \mathcal{F}_2 + 2\Gamma[F_1, \mathcal{F}_2], \\
& \mathcal{F}_3(0, x, v) = (B_3 + C_3)(0, x, v) - \mathcal{F}_{3,\infty},
\end{aligned} \right. \quad (3.64)$$

where $\mathcal{F}_{3,\infty}(x, v) \in \mathcal{N}$ is such that

$$\lim_{\tau \to \infty} \mathcal{F}_3(\tau, x, v) = 0. \quad (3.65)$$

Then we further require that $A_3$ satisfies the initial condition

$$A_3(0, x, v) = \mathcal{F}_{3,\infty}(x, v). \quad (3.66)$$

In a similar fashion, we can define $\mathcal{F}_4$. 
3.2 Remainder Estimates

We consider the linearized evolutionary Boltzmann equation

\[
\begin{aligned}
\begin{cases}
\epsilon^2 \partial_t f + ev \cdot \nabla_x f + L[f] = S(t, x, v) & \text{in } \mathbb{R}^+ \times \Omega \times \mathbb{R}^3, \\
f(0, x, v) = z(x, v) & \text{in } \Omega \times \mathbb{R}^3, \\
f(t, x_0, v) = \mathcal{P}[f](t, x_0, v) + h(t, x_0, v) & \text{on } \mathbb{R}^+ \times \gamma_-, \\
\end{cases}
\end{aligned}
\]  \tag{3.67}

where

\[\mathcal{P}[f](t, x_0, v) = \mu^\frac{1}{2}(v) \int_{u \cdot n(x_0) > 0} \mu^\frac{1}{2}(u) f(t, x_0, u) |u \cdot n(x_0)| \, du.\]  \tag{3.68}

The data \(z, S\) and \(h\) satisfy the compatibility condition

\[\int_{\Omega \times \mathbb{R}^3} \mu^\frac{1}{2} z = 0, \quad \int_{\Omega \times \mathbb{R}^3} S(x, v) \mu^\frac{1}{2} (v) \, dv \, dx + \int_{\gamma_-} h(x, v) \mu^\frac{1}{2}(v) \, d\gamma = 0.\]  \tag{3.69}

Then we can easily derive

\[\int_{\Omega \times \mathbb{R}^3} \mu^\frac{1}{2} f(t) = 0.\]  \tag{3.70}

Our analysis is based on the ideas in [5], [12], [29] and [27]. In particular, we will invoke the results of [29, 30] standard, we will focus on the a priori estimates here.

3.2.1 Preliminaries

We first introduce the well-known micro-macro decomposition. Define \(\mathbb{P}\) as the orthogonal projection onto the null space of \(L\):

\[\mathbb{P}[f] := \mu^\frac{1}{2}(v) \left( a_f(t, x) + v \cdot b_f(t, x) + \frac{|v|^2 - 3}{2} c_f(t, x) \right) \in \mathcal{N},\]  \tag{3.71}

where \(a_f, b_f\) and \(c_f\) are coefficients. When there is no confusion, we will simply write \(a, b, c\). Definitely, \(L[\mathbb{P}[f]] = 0\). Then the operator \(I - \mathbb{P}\) is naturally

\[(I - \mathbb{P})[f] := f - \mathbb{P}[f],\]  \tag{3.72}

which satisfies \((I - \mathbb{P})[f] \in \mathcal{N}^\perp\), i.e. \(L[f] = L[(I - \mathbb{P})[f]]\).

**Lemma 3.2.1.** The linearized collision operator \(L = vI - K\) defined in \([12]\) is self-adjoint in \(L^2\). It satisfies

\[\langle v \rangle \lesssim \nu(v) \lesssim \langle v \rangle,\]  \tag{3.73}

\[\langle f, L[f] \rangle(t, x) = \langle (I - \mathbb{P})[f], L[(I - \mathbb{P})[f]] \rangle(t, x),\]  \tag{3.74}

\[\|(I - \mathbb{P})[f(t, x)]\|_p^2 \lesssim \langle f, L[f] \rangle(t, x) \lesssim \|(I - \mathbb{P})[f(t, x)]\|_p^2.\]  \tag{3.75}

**Proof.** These are standard properties of \(L\). See [8, Chapter 3] and [12, Lemma 3]. \(\square\)

**Lemma 3.2.2.** For \(0 < \delta << 1\), define the near-grazing set of \(\gamma_{\pm}\)

\[\gamma_{\pm} := \left\{ (x, v) \in \gamma_{\pm} : |n(x) \cdot v| \leq \delta \text{ or } |v| \geq \frac{1}{\delta} \text{ or } |v| \leq \delta \right\}.\]  \tag{3.76}

Then

\[\int_s^t \left\| f_{\gamma_{\pm} \setminus \gamma_{\pm}^\delta} \right\|_{\gamma_{\pm}, 1} \leq C(\delta) \left( \epsilon \|f(s)\|_1 + \int_s^t \left( \|f\|_1 + \|\epsilon \partial_v f + v \cdot \nabla_x f\|_1 \right) \right).\]  \tag{3.77}
Proof. See [5] Lemma 2.1 with a standard time rescaling argument. \hfill \Box

Lemma 3.2.3 (Time-Dependent Green’s Identity). Assume $f(t, x, v), g(t, x, v) \in L^2(\mathbb{R}^+ \times \Omega \times \mathbb{R}^3)$ and $\partial_t f + v \cdot \nabla_x f, \partial_t g + v \cdot \nabla_x g \in L^2(\mathbb{R}^+ \times \Omega \times \mathbb{R}^3)$ with $f, g \in L^2(\mathbb{R}^+ \times \gamma)$. Then for almost all $s, t \in \mathbb{R}^+$,

$$\int_s^t \int_{\Omega \times \mathbb{R}^3} \left( \partial_t f + v \cdot \nabla_x f \right) \nabla_x g + \left( \partial_t g + v \cdot \nabla_x g \right) f \right)$$

(3.78)

Proof. Assume \( f, g \in L^2(\mathbb{R}^+ \times \Omega \times \mathbb{R}^3) \) satisfying \( \epsilon \partial_t f + v \cdot \nabla_x f \in L^2(\mathbb{R}^+ \times \Omega \times \mathbb{R}^3) \) and \( g \in L^2(\mathbb{R}^+ \times \gamma) \), we have

$$\epsilon \int_0^t \int_{\Omega \times \mathbb{R}^3} f(0) \nabla_x f \right)$$

(3.80)

The proof follows the same idea as in stationary version of Lemma 2.2.6 with \( m = 1 \). Actually, we use almost the same test function $\psi \sim \mu^2 v \cdot \nabla_x \phi$ to estimate $a, b$ and $c$, where $\phi$ satisfies proper elliptic equations. Hence, we will omit the details and only present the main result. Compared with stationary estimate, the new terms only show up on the right-hand side of (3.80). Using Hölder’s inequality, we know

$$\left| \epsilon \int_0^t \int_{\Omega \times \mathbb{R}^3} \partial_t \psi \right| \lesssim \epsilon^2 \left\| \partial_t \psi \right\|_2 \lesssim \epsilon^2 \left\| \partial_t f \right\|_2 \left\| \partial_t \nabla_x \phi \right\|_2.$$  

(3.81)

In a similar fashion, we have

$$\left| \epsilon \int_{\Omega \times \mathbb{R}^3} f(t) \psi(t) \right| \lesssim \epsilon^2 \left\| f(t) \right\|_2 \left\| \psi(t) \right\|_2 \lesssim \epsilon^2 \left\| f(t) \right\|_2 \left\| \nabla_x \phi(t) \right\|_2.$$  

(3.82)

$$\left| \epsilon \int_{\Omega \times \mathbb{R}^3} f(0) \psi(0) \right| \lesssim \epsilon^2 \left\| f(0) \right\|_2 \left\| \psi(0) \right\|_2 \lesssim \epsilon^2 \left\| \nabla_x \phi(0) \right\|_2.$$  

(3.83)

Step 1: Estimates of $c$.

We choose the test function

$$\psi = \psi_c = \mu^2(v) \left( |v|^2 - \beta_0 \right) \left( v \cdot \nabla_x \psi_c(t, x) \right),$$  

(3.84)

where for fixed $t$,

$$\left\{ \begin{array}{ll}
-\Delta_x \phi_c = c(t, x) & \text{in } \Omega, \\
\phi_c = 0 & \text{on } \partial \Omega,
\end{array} \right.$$  

(3.85)
and $\beta_c \in \mathbb{R}$ will be determined as in stationary problem. Based on the standard elliptic estimates (see [18]), we have

$$\|\phi_c(t)\|_{H^2(\Omega)} \lesssim \|c(t)\|_{L^2(\Omega)}.$$  \hfill (3.86)

Eventually, we have

$$\epsilon \|c\|_{L^2_{[0,t] \times \Omega}}^2 \lesssim \left(\|\mathcal{I} - \mathcal{P}\|_{2}\|f\|_{2} + \left\|\nu^{-\frac{2}{\gamma}} S\right\|_{2} + \epsilon \|\mathcal{P}\|_{\gamma,2} + \epsilon \|h\|_{\gamma,2}\right) \|c\|_{L^2_{[0,t] \times \Omega}} \hfill (3.87)$$

$$+ \epsilon^2 \left(\|f\|_{2}\left\|\partial_t \nabla_x \phi_c\|_{2} + \|c(t)\|_{L^2(\Omega)} + \|z\|_{2}\|c(0)\|_{L^2(\Omega)}\right).$$

Step 2: Estimates of $b$.

We further divide this step into several sub-steps:

Sub-Step 2.1: Estimates of $\left(\partial_t \partial_j \Delta_x^{-1} b_j\right)$ for $i, j = 1, 2, 3$.

Let $b = (b_1, b_2, b_3)$. We choose the test functions for $i, j = 1, 2, 3,$

$$\psi = \psi_{b,i,j} = \mu^\ast(v) \left(\\nu^2 - \beta_{b,i,j}\right) \partial_j \phi_{b,j},$$  \hfill (3.88)

where

$$\left\{\begin{array}{l}
-\Delta_x \phi_{b,j} = b_j(t,x) \ \text{in} \ \Omega, \\
\phi_{b,j} = 0 \ \text{on} \ \partial\Omega,
\end{array}\right.$$  \hfill (3.89)

and $\beta_{b,i,j} \in \mathbb{R}$ will be determined as in stationary problem. Eventually, we obtain

$$\epsilon \int_0^t \int_\Omega \left(\partial_t \partial_j \Delta_x^{-1} b_j\right) b_i \lesssim \left(\|\mathcal{I} - \mathcal{P}\|_{2}\|f\|_{2} + \left\|\nu^{-\frac{2}{\gamma}} S\right\|_{2} + \epsilon \|\mathcal{P}\|_{\gamma,2} + \epsilon \|h\|_{\gamma,2}\right) \|b\|_{L^2_{[0,t] \times \Omega}} \hfill (3.90)$$

$$+ \epsilon^2 \left(\|f\|_{2}\left\|\partial_t \nabla_x \phi_{b,j}\|_{2} + \|c(t)\|_{L^2(\Omega)} + \|z\|_{2}\|b(0)\|_{L^2(\Omega)}\right).$$

Sub-Step 2.2: Estimates of $\left(\partial_t \partial_i \Delta_x^{-1} b_i\right)$ for $i \neq j$.

Notice that the $i = j$ case is included in Sub-Step 2.1. We choose the test function

$$\psi = \psi_{b,i,j} = \mu^\ast(v) \left(\\nu^2 - \beta_{b,i,j}\right) \partial_i \phi_{b,j} \ \text{for} \ i \neq j.$$  \hfill (3.91)

Eventually, we obtain

$$\epsilon \int_0^t \int_\Omega \left(\partial_t \partial_i \Delta_x^{-1} b_i\right) b_j \lesssim \left(\|\mathcal{I} - \mathcal{P}\|_{2}\|f\|_{2} + \left\|\nu^{-\frac{2}{\gamma}} S\right\|_{2} + \epsilon \|\mathcal{P}\|_{\gamma,2} + \epsilon \|h\|_{\gamma,2}\right) \|b\|_{L^2_{[0,t] \times \Omega}} \hfill (3.92)$$

$$+ \epsilon^2 \left(\|f\|_{2}\left\|\partial_t \nabla_x \phi_{b,j}\|_{2} + \|c(t)\|_{L^2(\Omega)} + \|z\|_{2}\|b(0)\|_{L^2(\Omega)}\right).$$

Sub-Step 2.3: Synthesis.

Summarizing (3.90) and (3.92), we may sum up over $j = 1, 2, 3$ to obtain, for any $i = 1, 2, 3,$

$$\epsilon \|b_i\|^2_{L^2_{[0,t] \times \Omega}} \lesssim \left(\|\mathcal{I} - \mathcal{P}\|_{2}\|f\|_{2} + \left\|\nu^{-\frac{2}{\gamma}} S\right\|_{2} + \epsilon \|\mathcal{P}\|_{\gamma,2} + \epsilon \|h\|_{\gamma,2}\right) \|b\|_{L^2_{[0,t] \times \Omega}} \hfill (3.93)$$

$$+ \epsilon^2 \left(\|f\|_{2}\sum_{j=1}^3 \left\|\partial_t \nabla_x \phi_{b,j}\|_{2} + \|c(t)\|_{L^2(\Omega)} + \|z\|_{2}\|b(0)\|_{L^2(\Omega)}\right).$$
which further implies

$$
\epsilon \|b\|_{L^2(0,t \times \Omega)}^2 \lesssim \left( \|\left( I - \mathcal{P}\right) f \|_2 + \left\| \nu^{-\frac{1}{2}} S \right\|_2 + \epsilon \|\left( I - \mathcal{P}\right) f \|_{\gamma_+,2} + \epsilon \|h\|_{\gamma_-,2} \right) \|b\|_{L^2(0,t \times \Omega)} + 2 \epsilon \left( \left\| f \right\|_2 \sum_{j=1}^3 \|\partial_t \nabla x \phi_{b,j}\|_2 + \|f(t)\|_2 \|b(t)\|_{L^2(\Omega)} + \|z\|_2 \|b(0)\|_{L^2(\Omega)} \right). 
$$

(3.94)

Step 3: Estimates of $a$.
We choose the test function

$$
\psi = \psi_a = \mu^\frac{1}{2}(v) \left( |v|^2 - \beta_a \right) \left( v \cdot \nabla x \phi_a(t, x) \right),
$$

(3.95)

where

$$
\left\{ \begin{array}{l}
-\Delta_x \phi_a = a(t, x) \text{ in } \Omega, \\
\frac{\partial \phi_a}{\partial n} = 0 \text{ on } \partial \Omega,
\end{array} \right.
$$

(3.96)

and $\beta_a$ is a real number to be determined as in stationary problem. Eventually, we get

$$
\epsilon \|a\|_{L^2((0,t] \times \Omega)}^2 \lesssim \left( \|\left( I - \mathcal{P}\right) f \|_2 + \left\| \nu^{-\frac{1}{2}} S \right\|_2 + \epsilon \|\left( I - \mathcal{P}\right) f \|_{\gamma_+,2} + \epsilon \|h\|_{\gamma_-,2} \right) \|a\|_{L^2((0,t] \times \Omega)} + 2 \epsilon \left( \left\| f \right\|_2 \sum_{j=1}^3 \|\partial_t \nabla x \phi_{a,j}\|_2 + \|f(t)\|_2 \|a(t)\|_{L^2(\Omega)} + \|z\|_2 \|a(0)\|_{L^2(\Omega)} \right).
$$

(3.97)

Step 4: First Synthesis.
Collecting (3.94), (3.95) and (3.96), we deduce

$$
\epsilon \|P\|_2^2 \lesssim \left( \|\left( I - \mathcal{P}\right) f \|_2 + \left\| \nu^{-\frac{1}{2}} S \right\|_2 + \epsilon \|\left( I - \mathcal{P}\right) f \|_{\gamma_+,2} + \epsilon \|h\|_{\gamma_-,2} \right) \|P\|_2 \nonumber + 2 \epsilon \left( \left\| f \right\|_2 \sum_{j=1}^3 \|\partial_t \nabla x \phi_{a,j}\|_2 + \|f(t)\|_2 \left\| a(t) \right\|_{L^2(\Omega)} + \|z\|_2 \left\| a(0) \right\|_{L^2(\Omega)} \right).
$$

(3.98)

In order to close the proof, we must bound $\|\partial_t \nabla x \phi_a\|_2, \|\partial_t \nabla x \phi_{b,j}\|_2$ and $\|\partial_t \nabla x \phi_c\|_2$.
Apply Green’s identity in Lemma 2.2.3 to the equation (3.67). Then for any $\psi \in L^2(\Omega \times \mathbb{R}^3)$ independent of time $t$ satisfying $v \cdot \nabla x \psi \in L^2(\Omega \times \mathbb{R}^3)$ and $\psi \in L^2(\gamma)$, we have

$$
\epsilon^2 \int_{\Omega \times \mathbb{R}^3} \partial_t f(t) \psi = -\epsilon \int_{\gamma_+} f(t) \psi d\gamma + \epsilon \int_{\gamma_-} f(t) \psi d\gamma + \epsilon \int_{\Omega \times \mathbb{R}^3} (v \cdot \nabla x \psi) f(t) \nonumber \\
- \int_{\Omega \times \mathbb{R}^3} \psi L\left( I - \mathcal{P}\right) f(t) \right) + \int_{\Omega \times \mathbb{R}^3} S(t) \psi.
$$

(3.99)

Step 5: Estimate of $\partial_t \nabla x \phi_c$.
For fixed $t$, taking $\psi = -\mu^\frac{1}{2} |v|^2 - \beta \partial_t \phi_c(t)$, using integration by parts, we have

$$
\epsilon^2 \int_{\Omega \times \mathbb{R}^3} \partial_t f(t) \psi = -\epsilon^2 \int_{\Omega \times \mathbb{R}^3} \partial_t f(t) \mu^\frac{1}{2} |v|^2 - \beta \partial_t \phi_c(t) = -\epsilon^2 \int_{\Omega} \partial_t \epsilon(t) \partial_t \phi_c(t) 
\nonumber \\
- \epsilon^2 \int_{\Omega} \Delta_x \partial_t \phi_c(t) \partial_t \phi_c(t) + \epsilon^2 \int_{\Omega} |\partial_t \nabla x \phi_c(t)|^2 = \|\partial_t \nabla x \phi_c(t)\|_{L^2(\Omega)}^2.
$$

(3.100)

Following a similar argument as in Step 1 - Step 3, we have

$$
\epsilon^2 \|\partial_t \nabla x \phi_c\|_2 \lesssim \epsilon \|b\|_{L^2((0,t] \times \Omega)} + \epsilon \|\left( I - \mathcal{P}\right) f \|_2 + \left\| \nu^{-\frac{1}{2}} S \right\|_2.
$$

(3.101)
Step 6: Estimate of $\partial_t \nabla_x \phi_{b,j}$.  
For fixed $t$, taking $\psi = -\mu^2 v_j \partial_t \phi_{b,j}(t)$, using integration by parts, we have

$$
\epsilon^2 \int_{\Omega \times \mathbb{R}^3} \partial_t f(t) \psi = \epsilon^2 \int_{\Omega \times \mathbb{R}^3} \partial_t f(t) \mu^2 v_j \partial_t \phi_{b,j}(t) = -\epsilon^2 \int_{\Omega} \partial_t \phi_{b,j}(t) \partial_t \phi_{b,j}(t) 
$$

(3.102)

Following a similar argument as in Step 1 - Step 3, we have

$$
\epsilon^2 \|\partial_t \nabla_x \phi_{b,j}\|_2 \lesssim \epsilon \|a\|_{L^2((0,t) \times \Omega)} + \epsilon \|c\|_{L^2((0,t) \times \Omega)} + \epsilon \|\phi\|_{L^2((0,t) \times \Omega)} + \epsilon \|\nu^{-\frac{1}{2}} S\|_2. 
$$

(3.103)

Step 7: Estimate of $\partial_t \nabla_x \phi_{a}$.  
For fixed $t$, taking $\psi = -\mu^2 \partial_t \phi_{a}(t)$, using integration by parts, we have

$$
\epsilon^2 \int_{\Omega \times \mathbb{R}^3} \partial_t f(t) \psi = \epsilon^2 \int_{\Omega \times \mathbb{R}^3} \partial_t f(t) \mu^2 \partial_t \phi_{a}(t) = -\epsilon^2 \int_{\Omega} \partial_t \phi_{a}(t) \partial_t \phi_{a}(t) 
$$

(3.104)

Following a similar argument as in Step 1 - Step 3, we have

$$
\epsilon^2 \|\partial_t \nabla_x \phi_{a}\|_2 \lesssim \epsilon \|b\|_{L^2((0,t) \times \Omega)} + \epsilon \|\nu^{-\frac{1}{2}} \partial^\gamma S\|_2. 
$$

(3.105)

Step 8: Second Synthesis.  
Inserting (3.101), (3.103) and (3.105) into (3.98), we have

$$
\epsilon \|P[f]\|_2^2 \lesssim \left( \epsilon \|\phi\|_{L^2((0,t) \times \Omega)} + \epsilon \|\nu^{-\frac{1}{2}} S\|_2 + \epsilon \|\nu^{-\frac{1}{2}} \partial^\gamma S\|_2 + \epsilon \|\nu^{-\frac{1}{2}} \partial^\gamma S\|_2 \right) \|P[f]\|_2 
$$

(3.106)

Applying Cauchy’s inequality, we have

$$
\epsilon \|P[f]\|_2^2 \lesssim o(1) \epsilon \|P[f]\|_2^2 + \epsilon \|P[f]\|_2^2 + \epsilon \|\nu^{-\frac{1}{2}} S\|_2^2 + \epsilon \|\nu^{-\frac{1}{2}} \partial^\gamma S\|_2^2 + \epsilon \|\nu^{-\frac{1}{2}} \partial^\gamma S\|_2^2. 
$$

(3.107)

Hence, absorbing $o(1) \epsilon \|P[f]\|_2^2$ into the left-hand side, we have

$$
\epsilon \|P[f]\|_2^2 \lesssim \epsilon \|\phi\|_{L^2((0,t) \times \Omega)} + \epsilon \|\nu^{-\frac{1}{2}} S\|_2^2 + \epsilon \|\nu^{-\frac{1}{2}} \partial^\gamma S\|_2 + \epsilon \|\nu^{-\frac{1}{2}} \partial^\gamma S\|_2. 
$$

(3.108)

This completes our proof. 

\[ \square \]

**Theorem 3.2.5.** Assume (3.69) and (3.70) hold. The solution $f(t, x, v)$ to the equation (3.67) satisfies

$$
\|f(t)\|_2 + \frac{1}{\epsilon^2} \|\nu^{-\frac{1}{2}} (\mathbb{I} - \mathbb{P})[f]\|_{\gamma,+,2} + \frac{1}{\epsilon} \|\nu^{-\frac{1}{2}} \partial^\gamma h\|_{\gamma,+,2} + \frac{1}{\epsilon} \|\nu^{-\frac{1}{2}} \partial^\gamma h\|_{\gamma,+,2} \lesssim \frac{1}{\epsilon^2} \|P[S]\|_2 + \frac{1}{\epsilon} \|\nu^{-\frac{1}{2}} (\mathbb{I} - \mathbb{P})[S]\|_2 + \frac{1}{\epsilon} \|\nu^{-\frac{1}{2}} \partial^\gamma h\|_{\gamma,+,2} + \|\nu^{-\frac{1}{2}} \partial^\gamma h\|_{\gamma,+,2}. 
$$

(3.109)
Inserting (3.111) and (3.112) into (3.110), we have

\[ \frac{\epsilon^2}{2} \left\| f(t) \right\|_2^2 + \frac{\epsilon}{2} \left\| \nabla f \right\|_{L^2}^2 + \frac{\epsilon}{2} \left\| (1 - P) f \right\|_{L^2}^2 = \frac{\epsilon^2}{2} \left\| \nabla \right\|_2^2 + \int_0^t \int_{\Omega \times \mathbb{R}^3} f \mathcal{L}[f] \, d\gamma + \int_0^t \int_{\Omega \times \mathbb{R}^3} f S. \]  

(3.110)

Direct computation reveals that

\[ \frac{\epsilon^2}{2} \left\| f \right\|_{L^2}^2 - \frac{\epsilon}{2} \left\| (1 - P) f \right\|_{L^2}^2 = \frac{\epsilon^2}{2} \left\| f \right\|_{L^2}^2 + \left( \frac{\epsilon}{2} \left\| (1 - P) f \right\|_{L^2}^2 - \frac{\epsilon}{2} \left\| h \right\|_{L^2}^2 + \epsilon \int_0^t \int_{\gamma} h (1 - P) f \, d\gamma \right) \geq \frac{\epsilon}{2} \left( \left\| (1 - P) f \right\|_{L^2}^2 - \frac{1}{\eta} \left\| h \right\|_{L^2}^2 \right), \]

(3.111)

where \( 0 < \eta \ll 1 \) will be determined later. On the other hand, based on Lemma 3.2.1, we know

\[ \int_0^t \int_{\Omega \times \mathbb{R}^3} f \mathcal{L}[f] \geq \left\| (1 - P) f \right\|_{L^2}^2. \]

(3.112)

Inserting (3.111) and (3.112) into (3.110), we have

\[ \epsilon^2 \left\| f(t) \right\|_2^2 + \epsilon \left\| (1 - P) f \right\|_{L^2}^2 + \left\| (1 - P) f \right\|_{L^2}^2 \leq \frac{1}{\eta} \left\| f \right\|_{L^2}^2 + \epsilon \left\| f \right\|_{L^2}^2 + \epsilon \left( \int_0^t \int_{\Omega \times \mathbb{R}^3} f S. \right). \]

(3.113)

Step 2: \( \left\| (1 - P) f \right\|_{L^2}^2 \).

Multiplying \( f \) on both sides of the equation (3.67), we have

\[ \epsilon \partial_t (f^2) + v \cdot \nabla_x (f^2) = \frac{2}{\epsilon} \left( (1 - P) f + f S \right). \]

(3.114)

Taking absolute value and integrating (3.114) over \([0, t] \times \Omega \times \mathbb{R}^3\), using Lemma 3.2.1 we deduce

\[ \left\| \epsilon \partial_t (f^2) + v \cdot \nabla_x (f^2) \right\|_1 \leq \frac{1}{\epsilon} \left( \left\| (1 - P) f \right\|_{L^2}^2 + \left\| f S \right\|_1 \right). \]

(3.115)

On the other hand, applying Lemma 3.2.2 to \( f^2 \), for near grazing set \( \gamma^5 \), we have

\[ \left\| (1 - P) f \right\|_{L^2}^2 \leq C(\delta) \left( \epsilon \left\| z \right\|_2^2 + \left\| f \right\|_2^2 + \left\| \epsilon \partial_t (f^2) + v \cdot \nabla_x (f^2) \right\|_1 \right) \]

(3.116)

We can rewrite \( P[f](t, x, v) = v(t, x) \mu \nabla \theta(v) \). Then for \( \delta \) small, we deduce

\[ \left\| P[1_{\gamma^5} f] \right\|_{L^2}^2 = \int_0^t \int_{\partial \Omega} \left| y(t, x) \right|^2 \left( \int_{\left| v - n(x) \right| \geq \delta, \delta \leq |v| \leq \delta^{-1}} \mu(v) |v \cdot n(x)| \, dv \right) dx \]

\[ \geq \frac{1}{2} \left( \int_0^t \int_{\partial \Omega} \left| y(t, x) \right|^2 \left( \int_{\gamma^5} \mu(v) |v \cdot n(x)| \, dv \right) dx \right) = \frac{1}{2} \left\| P[f] \right\|_{L^2}^2, \]

(3.117)
where we utilize the bounds that
\[ \int_{v \cdot n(x) \leq \delta} \mu(v) |v \cdot n(x)| \, dv \lesssim \delta, \tag{3.118} \]
\[ \int_{|v| \leq \delta \text{ or } |v| \geq \delta^{-1}} \mu(v) |v \cdot n(x)| \, dv \lesssim \delta. \tag{3.119} \]
Therefore, from (3.117) and the fact
\[ \|P[1_{\gamma \gamma} f]\|_{\gamma_2} \lesssim \|1_{\gamma \gamma} f\|_{\gamma_2} \lesssim \|1_{\gamma \gamma} f\|_{\gamma}, \tag{3.120} \]
we conclude
\[ \|P[f]\|_{\gamma_2} \lesssim \|P[1_{\gamma \gamma} f]\|_{\gamma_2} \lesssim \|1_{\gamma \gamma} f\|_{\gamma}. \tag{3.121} \]
Considering (3.119), we have
\[ \|P[f]\|_{\gamma_2} \lesssim C(\delta) \left( \epsilon \|z\|_2^2 + \|f\|_2^2 + \frac{1}{\epsilon} \|\|\| f \|\|_2^2 + \frac{1}{\epsilon} \int_0^t \int_{\Omega \times \mathbb{R}^3} f S \right). \tag{3.122} \]
For fixed \( 0 < \delta << 1 \) and using \( f = P[f] + (I - P)[f] \), we obtain
\[ \|P[f]\|_{\gamma_2} \lesssim \epsilon \|z\|_2^2 + \|P[f]\|_2^2 + \frac{1}{\epsilon} \|\|\| f \|\|_2^2 + \frac{1}{\epsilon} \int_0^t \int_{\Omega \times \mathbb{R}^3} f S. \tag{3.123} \]

Step 3: Synthesis.
Plugging (3.123) into (3.120) with \( \epsilon \) sufficiently small to absorb \( \|\|\| f \|\|_2^2 \) into the left-hand side, we obtain
\[ \epsilon^2 \|f(t)\|_2^2 + \frac{1}{\epsilon} \|\|\| f \|\|_2^2 \lesssim \eta \epsilon^2 \|P[f]\|_2^2 + \epsilon^2 \|z\|_2^2 + \frac{1}{\eta} \|h\|_{\gamma_2}^2 + \frac{1}{\epsilon} \int_0^t \int_{\Omega \times \mathbb{R}^3} f S. \tag{3.124} \]
We square on both sides of (3.79) to obtain
\[ \epsilon^2 \|P[f]\|_2^2 \lesssim \epsilon^2 \|f(t)\|_2^2 + \epsilon^2 \|\|\| f \|\|_2^2 + \frac{1}{\epsilon} \|\|\| f \|\|_2^2 + \epsilon^2 \|z\|_2^2 + \epsilon^2 \|h\|_{\gamma_2}^2. \tag{3.125} \]

Multiplying a small constant on both sides of (3.125) and adding to (3.124) with \( \eta \) sufficiently small to absorb \( \epsilon^2 \|\|\| f \|\|_2^2 \), \( \|\|\| f \|\|_2^2 \), \( \|f(t)\|_2^2 \), \( \|\|\| f \|\|_2^2 \), \( \eta \epsilon^2 \|P[f]\|_2^2 \) into the left-hand side, we obtain
\[ \epsilon^2 \|f(t)\|_2^2 + \epsilon \|\|\| f \|\|_2^2 + \|\|\| f \|\|_2^2 + \epsilon^2 \|P[f]\|_2^2 \lesssim \epsilon^2 \|f(t)\|_2^2 + \epsilon \|\|\| f \|\|_2^2 + \|\|\| f \|\|_2^2 + \epsilon^2 \|P[f]\|_2^2 \tag{3.126} \]
\[ \lesssim \|h\|_{\gamma_2}^2 + \epsilon^2 \|z\|_2^2 + \epsilon^2 \|\|\| f \|\|_2^2 + \epsilon \|\|\| f \|\|_2^2 + \int_0^t \int_{\Omega \times \mathbb{R}^3} f S. \]

Applying Cauchy's inequality, we have
\[ \int_0^t \int_{\Omega \times \mathbb{R}^3} f S \lesssim \int_0^t \int_{\Omega \times \mathbb{R}^3} (I - P)[f][f][S] + \int_0^t \int_{\Omega \times \mathbb{R}^3} P[f][P][S] \tag{3.127} \]
\[ \lesssim o(1) \|\|\| f \|\|_2^2 + \frac{1}{\epsilon} \|\|\| f \|\|_2^2 + o(1) \epsilon^2 \|P[f]\|_2^2 + \frac{1}{\epsilon^2} \|\|\| P[S] \|\|_2^2 \]

Inserting (3.127) into (3.128) to absorb \( o(1) \|\|\| f \|\|_2^2 \) and \( o(1) \epsilon^2 \|P[f]\|_2^2 \) into the left-hand side, we obtain
\[ \epsilon^2 \|f(t)\|_2^2 + \epsilon \|\|\| f \|\|_2^2 + \|\|\| f \|\|_2^2 + \epsilon^2 \|P[f]\|_2^2 \lesssim \epsilon^2 \|f(t)\|_2^2 + \epsilon \|\|\| f \|\|_2^2 + \|\|\| f \|\|_2^2 + \epsilon^2 \|P[f]\|_2^2 \tag{3.128} \]
\[ \lesssim \frac{1}{\epsilon^2} \|\|\| P[S] \|\|_2^2 + \frac{1}{\epsilon} \|\|\| f \|\|_2^2 + \epsilon^2 \|z\|_2^2 + \|h\|_{\gamma_2}^2. \]

Hence, our desired result naturally follows.
Corollary 3.2.6. Since \((3.67)\) is a linear equation, taking time derivative on both sides, we know \(\partial_t f\) satisfies
\[
\begin{aligned}
\epsilon^2 \partial_t (\partial_t f) + \epsilon v \cdot \nabla_x (\partial_t f) + \mathcal{L}(\partial_t f) &= \partial_t S(t, x, v) \quad \text{in} \quad \mathbb{R}^+ \times \Omega \times \mathbb{R}^3, \\
\partial_t f(0, x, v) &= -\frac{1}{\epsilon^2} \mathcal{L}[z(x, v)] - \frac{1}{\epsilon} v \cdot \nabla_x z(x, v) + \frac{1}{\epsilon^2} S(0, x, v) \quad \text{in} \quad \Omega \times \mathbb{R}^3, \\
\partial_t f(t, x_0, v) &= P[\partial_t f](t, x_0, v) + \partial_t h(t, x_0, v) \quad \text{on} \quad \mathbb{R}^+ \times \gamma_-,
\end{aligned}
\]
(3.129)
where we solve the initial data \(\partial_t f(0, x, v)\) from \((3.67)\). Then applying Lemma 3.2.5 to \((3.129)\), we obtain
\[
\|\partial_t f(t)\|_2 + \frac{1}{\epsilon^2} \|\mathcal{L}(\partial_t f)\|_{\gamma_-} + \frac{1}{\epsilon} \|\mathcal{L}_v(\partial_t f)\|_\nu + \|P[\partial_t f]\|_2 \leq \frac{1}{\epsilon^2} \|\partial_t S\|_2 + \frac{1}{\epsilon} \|\mathcal{L}(\partial_t f)\|_{\gamma_-} + \frac{1}{\epsilon^2} \|\mathcal{L}_v(\partial_t f)\|_\nu + \frac{1}{\epsilon} \|v \cdot \nabla_x z\|_2 + \frac{1}{\epsilon^2} \|S(0)\|_2.
\]
(3.130)

3.2.3 \(\ell^{2m}\) Estimates

Throughout this section, we need \(\frac{3}{2} < m < 3\). Let \(o(1)\) denote a sufficiently small constant.

Lemma 3.2.7. Assume \((3.69)\) and \((3.70)\) hold. The solution \(f(t, x, v)\) to the equation \((3.67)\) satisfies
\[
\epsilon \|P[f(t)]\|_{2m} \lesssim \epsilon\|\mathcal{L}(f(t))\|_{\gamma_-} + \epsilon\|\mathcal{L}_v(f(t))\|_\nu + \epsilon\|\mathcal{L}_v S(t)\|_2 + \epsilon\|h(t)\|_{\gamma_-} + \epsilon^2 \|\partial_t f(t)\|_2.
\]
(3.131)

Proof. This is very similar to the proof of Lemma 3.2.4 and the stationary version in Lemma 2.2.6. We apply Green’s identity to the equation \((3.67)\) and choose particular test functions to control \(a, b\) and \(c\). However, there is no simple way to get around the \(\partial_t \nabla \varphi\) terms as in Step 5 - Step 7 of the proof of Lemma 3.2.4. Here, we resort to stationary techniques, i.e. to use time-independent Green’s identity instead of time-dependent one.

Apply Green’s identity in Lemma 2.2.3 to the equation \((3.67)\). Then for any \(\psi(t) \in L^2(\Omega \times \mathbb{R}^3)\) satisfying \(v \cdot \nabla_x \psi(t) \in L^2(\Omega \times \mathbb{R}^3)\) and \(\psi(t) \in L^2(\gamma)\), we have
\[
\epsilon \int_{\gamma_-} f(t) \psi(t) d\gamma - \epsilon \int_{\gamma_-} f(t) \psi(t) d\gamma - \epsilon \int_{\Omega \times \mathbb{R}^3} \left(v \cdot \nabla_x \psi(t)\right) f(t) = -\epsilon \int_{\Omega \times \mathbb{R}^3} \psi(t) \mathcal{L}[f(t)] + \int_{\Omega \times \mathbb{R}^3} S(t) \psi(t) - \epsilon^2 \int_{\Omega \times \mathbb{R}^3} \partial_t f(t) \psi(t).
\]
(3.132)

Then except from \(-\epsilon^2 \int_{\Omega \times \mathbb{R}^3} \partial_t f(t) \psi(t)\), this is exactly the same as the stationary estimates in Lemma 2.2.6 so we just mimick the proof there and that of Lemma 3.2.4 and point out the major differences. In particular, we always use the bound
\[
\epsilon^2 \int_{\Omega \times \mathbb{R}^3} \partial_t f(t) \psi(t) \lesssim \epsilon^2 \|\partial_t f(t)\|_2 \|\psi(t)\|_2.
\]
(3.133)

Step 1: Estimates of \(c\).
We choose the test function
\[
\psi(t) = \psi_c(t) = \mu^\frac{1}{2}(v) \left( |v|^2 - \beta_c \right) \left( v \cdot \nabla_x \phi_c(t, x) \right),
\]
(3.134)
where
\[
\begin{cases}
-\Delta_x \phi_c = c \left| \epsilon \right|^{2m-2} (t, x) \quad \text{in} \quad \Omega, \\
\phi_c(t) = 0 \quad \text{on} \quad \partial \Omega,
\end{cases}
\]
(3.135)
and $\beta_c \in \mathbb{R}$ will be determined as in stationary problem. Based on the standard elliptic estimates in [18], we have

$$
\|\phi_c(t)\|_{W^{2, \frac{2m}{m-4}}(\Omega)} \lesssim \|c(t)\|_{L^2}^{2m-1} \lesssim \|c(t)\|_{L^{2m}(\Omega)}^{2m-1}. \tag{3.136}
$$

Hence, by Sobolev embedding theorem, we know

$$
\|\psi_c(t)\|_2 \lesssim \|\phi_c\|_{H^2(\Omega)} \lesssim \|\phi_c(t)\|_{W^{2, \frac{2m}{m-4}}(\Omega)} \lesssim \|c(t)\|_{L^{2m}(\Omega)}^{2m-1}, \tag{3.137}
$$

$$
\|\phi_c(t)\|_{W^{1, \frac{2m}{m-4}}(\Omega)} \lesssim \|\phi_c(t)\|_{W^{2, \frac{2m}{m-4}}(\Omega)} \lesssim \|c(t)\|_{L^{2m}(\Omega)}^{2m-1}. \tag{3.138}
$$

Also, for $1 \leq m \leq 3$, using Sobolev embedding theorem and trace estimates, we have

$$
|\nabla_x \phi_c(t)|_{L^{\frac{2m}{m-4}}(\partial\Omega)} \lesssim |\nabla_x \phi_c(t)|_{W^{1, \frac{2m}{m-4}}(\partial\Omega)} \lesssim \|\nabla_x \phi_c(t)\|_{W^{1, \frac{2m}{m-4}}(\Omega)} \lesssim \|c(t)\|_{L^{2m}(\Omega)}^{2m-1}. \tag{3.139}
$$

Eventually, we have

$$
\epsilon \|c(t)\|_{L^2(\Omega)} \lesssim \epsilon \|(1 - P)[f(t)]\|_{\gamma_+, \frac{4m}{m-4}} + \|[(1 - P)[f(t)]\|_2 + \epsilon \|(1 - P)[f(t)]\|_{2m} \tag{3.140}
$$

$$
+ \left\| \epsilon^{-\frac{4}{m-4}} S(t) \right\|_2 + \epsilon \|h(t)\|_{\gamma_-, \frac{4m}{m-4}} + \epsilon^2 \|\partial_t f(t)\|_2.
$$

Step 2: Estimates of $b$.

We further divide this step into several sub-steps:

Sub-Step 2.1: Estimates of $\left(\partial_i \partial_j \Delta_x^{-1} \left(b_j |b_j|^{2m-2}\right)\right)b_i$ for $i, j = 1, 2, 3$.

Let $b = (b_1, b_2, b_3)$. We choose the test functions for $i, j = 1, 2, 3$,

$$
\psi(t) = \psi_{b,i,j}(t) = \mu^{\frac{1}{2}}(v) \left(v_i^2 - \beta_{b,i,j}\right) \partial_j \phi_{b,i,j}, \tag{3.141}
$$

where

$$
\begin{cases}
-\Delta_x \phi_{b,j}(t) = b_j |b_j|^{2m-2}(t,x) \text{ in } \Omega, \\
\phi_{b,j}(t) = 0 \text{ on } \partial\Omega, \tag{3.142}
\end{cases}
$$

and $\beta_{b,i,j} \in \mathbb{R}$ will be determined as in stationary problem. We can recover the elliptic estimates and trace estimates. Eventually, we have,

$$
\epsilon \left| \int_{\Omega} \left(\partial_i \partial_j \Delta_x^{-1} \left(b_j |b_j|^{2m-2}\right)\right)b_i \right| \tag{3.143}
$$

$$
\lesssim \|b(t)\|_{L^{2m}(\Omega)} \left( \epsilon \|(1 - P)[f(t)]\|_{\gamma_+, \frac{4m}{m-4}} + \|[(1 - P)[f(t)]\|_2 + \epsilon \|(1 - P)[f(t)]\|_{2m} \right.
$$

$$
\left. + \left\| \epsilon^{-\frac{4}{m-4}} S(t) \right\|_2 + \epsilon \|h(t)\|_{\gamma_-, \frac{4m}{m-4}} + \epsilon^2 \|\partial_t f(t)\|_2 \right).
$$

Sub-Step 2.2: Estimates of $\left(\partial_i \partial_j \Delta_x^{-1} \left(b_j |b_j|^{2m-2}\right)\right)b_j$ for $i \neq j$.

Notice that the $i = j$ case is included in Sub-Step 2.1. We choose the test function

$$
\psi(t) = \tilde{\psi}_{b,i,j}(t) = \mu^{\frac{1}{2}}(v) v_i v_j \partial_i \phi_{b,j} \text{ for } i \neq j. \tag{3.144}
$$
Eventually, we have
\[ \epsilon \left| \int_\Omega \left( \partial_i \partial_i \Delta_x^{-1} (b_j |b_j|^{m-2}) \right) b_j \right| \]
\[ \leq \|b(t)\|_{L^{2m}(\Omega)}^{m-1} \left( \epsilon \| (1 - \mathcal{P}) f(t) \|_{\gamma_+, \Lambda} + \| (\mathbb{I} - \mathbb{P}) f(t) \|_2 + \epsilon \| (\mathbb{I} - \mathbb{P}) f(t) \|_{2m} \right) \]
\[ + \left\| \nu^{-\frac{1}{2}} S(t) \right\|_2 + \epsilon \| h(t) \|_{\gamma_-, \Lambda} + \epsilon^2 \| \partial_t f(t) \|_2. \]

Sub-Step 2.3: Synthesis.
Summarizing (3.43) and (3.145), we may sum up over \( j = 1, 2, 3 \) to obtain, for any \( i = 1, 2, 3 \),
\[ \epsilon \| b_i(t) \|_{L^{2m}(\Omega)}^{2m-1} \leq \|b(t)\|_{L^{2m}(\Omega)}^{m-1} \left( \epsilon \| (1 - \mathcal{P}) f(t) \|_{\gamma_+, \Lambda} + \| (\mathbb{I} - \mathbb{P}) f(t) \|_2 + \epsilon \| (\mathbb{I} - \mathbb{P}) f(t) \|_{2m} \right) \]
\[ + \left\| \nu^{-\frac{1}{2}} S(t) \right\|_2 + \epsilon \| h(t) \|_{\gamma_-, \Lambda} + \epsilon^2 \| \partial_t f(t) \|_2. \]

which further implies
\[ \epsilon \| b(t) \|_{L^{2m}(\Omega)} \leq \epsilon \| (1 - \mathcal{P}) f(t) \|_{\gamma_+, \Lambda} + \| (\mathbb{I} - \mathbb{P}) f(t) \|_2 + \epsilon \| (\mathbb{I} - \mathbb{P}) f(t) \|_{2m} \]
\[ + \left\| \nu^{-\frac{1}{2}} S(t) \right\|_2 + \epsilon \| h(t) \|_{\gamma_-, \Lambda} + \epsilon^2 \| \partial_t f(t) \|_2. \]

Step 3: Estimates of \( a \).
We choose the test function
\[ \psi(t) = \psi_a(t) = \mu^{\frac{\Lambda}{2}} (v^2 - \beta_a) \left( v \cdot \nabla_x \phi_a(t, x) \right), \]
where
\[ \begin{cases} -\Delta_x \phi_a(t) = a |a|^{2m-2} (t, x) - \frac{1}{|I|} \int_I a |a|^{2m-2} (t, x) \, dx \text{ in } \Omega, \\ \frac{\partial \phi_a(t)}{\partial n} = 0 \text{ on } \partial \Omega, \end{cases} \]
and \( \beta_a \in \mathbb{R} \) will be determined as in stationary problem. We can recover the elliptic estimates and trace estimates. Eventually, we have
\[ \epsilon \| a(t) \|_{L^{2m}(\Omega)} \leq \epsilon \| (1 - \mathcal{P}) f(t) \|_{\gamma_+, \Lambda} + \| (\mathbb{I} - \mathbb{P}) f(t) \|_2 + \epsilon \| (\mathbb{I} - \mathbb{P}) f(t) \|_{2m} \]
\[ + \left\| \nu^{-\frac{1}{2}} S(t) \right\|_2 + \epsilon \| h(t) \|_{\gamma_-, \Lambda} + \epsilon^2 \| \partial_t f(t) \|_2. \]

Step 4: Synthesis.
Collecting (3.140), (3.147) and (3.150), we deduce
\[ \epsilon \| \mathbb{P} f(t) \|_{2m} \leq \epsilon \| (1 - \mathcal{P}) f(t) \|_{\gamma_+, \Lambda} + \| (\mathbb{I} - \mathbb{P}) f(t) \|_2 + \epsilon \| (\mathbb{I} - \mathbb{P}) f(t) \|_{2m} \]
\[ + \left\| \nu^{-\frac{1}{2}} S(t) \right\|_2 + \epsilon \| h(t) \|_{\gamma_-, \Lambda} + \epsilon^2 \| \partial_t f(t) \|_2. \]
Theorem 3.2.8. Assume \([3.69]\) and \([3.70]\) hold. The solution \(f(t,x,v)\) to the equation \([3.67]\) satisfies
\[
\frac{1}{\epsilon^2} \epsilon \| (1 - \mathcal{P})[f(t)] \|_{\gamma_{\epsilon,2}} + \frac{1}{\epsilon} \| (1 - \mathcal{P})[f(t)] \|_2 + \mathcal{P}[f(t)]_{2m} \\
+ \frac{1}{\epsilon^2} \| (1 - \mathcal{P})[\partial_t f] \|_{\gamma_{\epsilon,2}} + \frac{1}{\epsilon} \| (1 - \mathcal{P})[\partial_t f] \|_2 + \mathcal{P}[\partial_t f]_{2m} \\
\lesssim o(1) \epsilon \frac{2m}{m+1} \left( \| f(t) \|_{\gamma_{\epsilon,\infty}} + \| f(t) \|_{\infty} \right) \\
+ \frac{1}{\epsilon^2} \| \mathcal{P}[S(t)] \|_{2m} + \left( \epsilon^2 \| \mathcal{P}[\partial_t S] \|_2 + \frac{1}{\epsilon} \| \mathcal{P}[\partial_t S] \|_2 \right) \\
+ \| h(t) \|_{\gamma_{\epsilon,4m}} + \frac{1}{\epsilon} \| h(t) \|_{\gamma_{\epsilon,2}} + \frac{1}{\epsilon} \| \partial_v h(t) \|_{\gamma_{\epsilon,2}} + \frac{1}{\epsilon^2} \| \partial_v h(t) \|_2 + \frac{1}{\epsilon^2} \| \partial_v h(t) \|_{\gamma_{\epsilon,\infty}}.
\]  

Proof.
Step 1: Energy Estimate.
Multiplying \(f\) on both sides of \([3.67]\) and use the similar estimates as in the proof of Lemma \([3.25]\) the stationary energy structure implies
\[
\epsilon \| (1 - \mathcal{P})[f(t)] \|_{\gamma_{\epsilon,2}}^2 + \| (1 - \mathcal{P})[f(t)] \|_{\infty}^2 \\
\lesssim \epsilon \epsilon \| \mathcal{P}[f(t)] \|_{\gamma_{\epsilon,2}}^2 + \frac{1}{\epsilon^2} \int_{\Omega \times \mathbb{R}^3} f(t)S(t) + \epsilon^2 \left( \int_{\Omega \times \mathbb{R}^3} f(t)\partial_t f(t) \right) \]

We square on both sides of \([3.131]\) to obtain
\[
\epsilon^2 \| \mathcal{P}[f(t)] \|_{2m}^2 \lesssim \frac{1}{\epsilon} \left( (1 - \mathcal{P})[f(t)] \right)_{\gamma_{\epsilon,\infty}} + \left( (1 - \mathcal{P})[f(t)] \right)_{\infty}^2 + \epsilon^2 \| (1 - \mathcal{P})[f(t)] \|_{2m}^2 \\
+ \epsilon^2 \| \nu^{-\frac{1}{2}} S(t) \|_{2}^2 + \epsilon^2 \| h(t) \|_{\gamma_{\epsilon,4m}}^2 + \epsilon^2 \| \partial_v h(t) \|_{\gamma_{\epsilon,2}}^2.
\]

Hölder’s inequality implies
\[
\| \mathcal{P}[f(t)] \|_{2m} \lesssim \| \mathcal{P}[f(t)] \|_{2m}^2.
\]

Multiplying a small constant on both sides of \([3.157]\) and adding to \([3.153]\) with \(\eta\) sufficiently small to absorb \(\eta^2 \| \mathcal{P}[f(t)] \|^2_{\gamma_{\epsilon,\infty}}\) and \(\| (1 - \mathcal{P})[f(t)] \|^2_{\infty}\) into the left-hand side, we obtain
\[
\epsilon \| (1 - \mathcal{P})[f(t)] \|_{\gamma_{\epsilon,2}}^2 + \| (1 - \mathcal{P})[f(t)] \|_{\infty}^2 + \epsilon^2 \| \mathcal{P}[f(t)] \|_{2m}^2 \\
\lesssim \epsilon^2 \left( (1 - \mathcal{P})[f(t)] \right)_{\gamma_{\epsilon,\infty}} + \left( (1 - \mathcal{P})[f(t)] \right)_{\infty}^2 + \epsilon^2 \left( (1 - \mathcal{P})[f(t)] \right)_{2m}^2 + \epsilon^2 \left( (1 - \mathcal{P})[f(t)] \right)_{2m}^2 \\
+ \epsilon^2 \| \nu^{-\frac{1}{2}} S(t) \|_{2}^2 + \epsilon^2 \| h(t) \|_{\gamma_{\epsilon,4m}}^2 + \epsilon^2 \| \partial_v h(t) \|_{\gamma_{\epsilon,2}}^2 + \epsilon^2 \left( \int_{\Omega \times \mathbb{R}^3} f(t)S(t) \right) + \epsilon^2 \left( \int_{\Omega \times \mathbb{R}^3} f(t)\partial_t f(t) \right).
\]

Now we have to handle \(\epsilon^2 \left( (1 - \mathcal{P})[f(t)] \right)_{\gamma_{\epsilon,4m}}^2\) and \(\epsilon^2 \left( (1 - \mathcal{P})[f(t)] \right)_{\gamma_{\epsilon,2}}^2\) on the right-hand side.

Step 2: Interpolation Argument.
By interpolation estimate and Young’s inequality, we have
\[
(1 - \mathcal{P})[f(t)] \lesssim (1 - \mathcal{P})[f(t)]^{\frac{2m-1}{\gamma_{\epsilon,\infty}}} (1 - \mathcal{P})[f(t)]^{\frac{2m-1}{2m-1}} \\
= \left( \frac{1}{\epsilon^2} \right) \left( (1 - \mathcal{P})[f(t)] \right)_{\gamma_{\epsilon,\infty}}^{\frac{2m-1}{2m-1}} + o(1) \left( \frac{6m-6}{4m^2} \right) \left( (1 - \mathcal{P})[f(t)] \right)_{\gamma_{\epsilon,\infty}}^{\frac{2m-1}{2m-1}} \\
\lesssim \left( \frac{1}{\epsilon^2} \right) \left( (1 - \mathcal{P})[f(t)] \right)_{\gamma_{\epsilon,\infty}}^{\frac{2m-1}{2m-1}} + o(1) \left( \frac{6m-6}{4m^2} \right) \left( (1 - \mathcal{P})[f(t)] \right)_{\gamma_{\epsilon,\infty}}^{\frac{2m-1}{2m-1}} \\
\lesssim \left( \frac{1}{\epsilon^2} \right) \left( (1 - \mathcal{P})[f(t)] \right)_{\gamma_{\epsilon,\infty}}^{\frac{2m-1}{2m-1}} + o(1) \left( \frac{6m-6}{4m^2} \right) \left( (1 - \mathcal{P})[f(t)] \right)_{\gamma_{\epsilon,\infty}}^{\frac{2m-1}{2m-1}}.
\]
Similarly, we have
\[
\|(\mathbb{I} - \mathcal{P})[f(t)]\|_{2m} \leq \|(\mathbb{I} - \mathcal{P})[f(t)]\|_{\gamma^+}^{\frac{3}{2}} \|(\mathbb{I} - \mathcal{P})[f(t)]\|_{\gamma^-}^{\frac{m-1}{2}} \tag{3.158}
\]
\[
= \left( \frac{1}{\epsilon^{2m-3}} \|(\mathbb{I} - \mathcal{P})[f(t)]\|_{\gamma^+}^{\frac{3}{2}} \right) \left( \epsilon^{3m-3} \|(\mathbb{I} - \mathcal{P})[f(t)]\|_{\gamma^-}^{\frac{m-1}{2}} \right)
\leq \left( \frac{1}{\epsilon^{2m-3}} \|(\mathbb{I} - \mathcal{P})[f(t)]\|_{\gamma^+}^{\frac{3}{2}} \right) \epsilon \left( \frac{3m-3}{2m-3} \|(\mathbb{I} - \mathcal{P})[f(t)]\|_{\gamma^-}^{\frac{m-1}{2}} \right) \epsilon^{\frac{3m-3}{2m-3}}
\leq \frac{1}{\epsilon^{2m-3}} \|(\mathbb{I} - \mathcal{P})[f(t)]\|_{\gamma^+}^{\frac{3}{2}} + o(1) \epsilon \frac{3m-3}{2m-3} \|(\mathbb{I} - \mathcal{P})[f(t)]\|_{\gamma^-}^{\frac{m-1}{2}}.
\]

We need this extra \( \epsilon^{3m-3} \) for the convenience of \( L^\infty \) estimate. Then we know for sufficiently small \( \epsilon \) and \( \frac{3}{2} < m < 3 \),
\[
\epsilon^2 \|((1 - \mathcal{P})[f(t)])\|_{\gamma^+, \frac{3m}{2}} \leq \epsilon \frac{3m-3}{2m-3} \|(1 - \mathcal{P})[f(t)]\|_{\gamma^+, \frac{3}{2}} + o(1) \epsilon^{2 + \frac{3}{m}} \|(1 - \mathcal{P})[f(t)]\|_{\gamma^+, \infty} \tag{3.159}
\]
\[
\approx o(1) \epsilon \|((1 - \mathcal{P})[f(t)])\|_{\gamma^+, \frac{3}{2}} + o(1) \epsilon^{2 + \frac{3}{m}} \|f(t)\|_{\gamma^+, \infty}.
\]

Similarly, we have
\[
\epsilon^2 \|\mathbb{I} - \mathcal{P})[f(t)]\|_{2m} \leq \epsilon \frac{3m-3}{2m-3} \|(\mathbb{I} - \mathcal{P})[f(t)]\|_{\gamma^+, \infty} + o(1) \epsilon^{2 + \frac{3}{m}} \|f(t)\|_{2m} \tag{3.160}
\]
\[
\approx o(1) \epsilon \|(\mathbb{I} - \mathcal{P})[f(t)]\|_{2m} + o(1) \epsilon^{2 + \frac{3}{m}} \|f(t)\|_{2m}.
\]

Inserting (3.159) and (3.160) into (3.156), and absorbing \( o(1) \epsilon \|((1 - \mathcal{P})[f(t)])\|_{\gamma^+, \frac{3}{2}} \) and \( o(1) \|(\mathbb{I} - \mathcal{P})[f(t)]\|_{2m} \) into the left-hand side, we obtain
\[
\epsilon \|(1 - \mathcal{P})[f(t)]\|_{\gamma^+, \frac{3}{2}} + \|(\mathbb{I} - \mathcal{P})[f(t)]\|_{\gamma^+, \infty} + \|f(t)\|_{\gamma^+, \infty} \] + \epsilon^2 \|\partial_t f(t)\|_{\gamma^+, \infty}^2
\]
\[
+ \nu^{-\frac{1}{2}} \mathcal{S}(t) \|_{\gamma^-, \frac{3m}{2}} + \|h(t)\|_{\gamma^-, \infty}^2 + \left| \int_{\Omega \times \mathbb{R}^3} f(t) \mathcal{S}(t) \right| + \epsilon^2 \left| \int_{\Omega \times \mathbb{R}^3} f(t) \partial_t f(t) \right|.
\]

**Step 3: Synthesis.**

We can decompose
\[
\int_{\Omega \times \mathbb{R}^3} f(t) \mathcal{S}(t) = \int_{\Omega \times \mathbb{R}^3} \mathbb{P}[f(t)] \mathbb{P}[\mathcal{S}(t)] + \int_{\Omega \times \mathbb{R}^3} (\mathbb{I} - \mathcal{P})[f(t)] (\mathbb{I} - \mathcal{P})[\mathcal{S}(t)]. \tag{3.162}
\]

Hölder’s inequality and Cauchy’s inequality imply
\[
\int_{\Omega \times \mathbb{R}^3} \mathbb{P}[f(t)] \mathbb{P}[\mathcal{S}(t)] \leq \mathbb{P}[f(t)]_{2m} \mathbb{P}[\mathcal{S}(t)]_{2m} \leq o(1) \epsilon^2 \mathbb{P}[f(t)]_{2m} + \frac{1}{\epsilon^2} \mathbb{P}[\mathcal{S}(t)]_{2m}, \tag{3.163}
\]
and
\[
\int_{\Omega \times \mathbb{R}^3} (\mathbb{I} - \mathcal{P})[f(t)] (\mathbb{I} - \mathcal{P})[\mathcal{S}(t)] \leq o(1) \|(\mathbb{I} - \mathcal{P})[f(t)]\|_{\gamma^+, \infty}^2 + \|\nu^{-\frac{1}{2}} (\mathbb{I} - \mathcal{P})[\mathcal{S}(t)]\|_{\gamma^-, \infty}^2, \tag{3.164}
\]

Inserting (3.163) and (3.164) into (3.162) and further (3.156), absorbing \( o(1) \epsilon^2 \mathbb{P}[f(t)]_{2m} \) and \( o(1) \|(\mathbb{I} - \mathcal{P})[f(t)]\|_{\nu}^2 \) into the left-hand side, we get
\[
\epsilon \|(1 - \mathcal{P})[f(t)]\|_{\gamma^+, \frac{3}{2}} + \|(\mathbb{I} - \mathcal{P})[f(t)]\|_{\gamma^+, \infty} + \|f(t)\|_{\gamma^+, \infty} \]
\[
+ \frac{1}{\epsilon^2} \mathbb{P}[\mathcal{S}(t)]_{2m} \leq o(1) \epsilon \|(1 - \mathcal{P})[f(t)]\|_{\gamma^+, \infty} + \|\nu^{-\frac{1}{2}} (\mathbb{I} - \mathcal{P})[\mathcal{S}(t)]\|_{\gamma^-, \infty}^2 + \|h(t)\|_{\gamma^-, \infty}^2 + \epsilon^2 \left| \int_{\Omega \times \mathbb{R}^3} f(t) \partial_t f(t) \right|. \tag{3.165}
\]
Now we handle the most difficult term:
\[
e^2 \int_{\Omega \times \mathbb{R}^3} f(t) \partial_t f(t) \leq e^2 \| f(t) \|_2 \| \partial_t f(t) \|_2 \lesssim o(1) e^2 \| f(t) \|_2 + e^2 \| \partial_t f(t) \|_2^2. \tag{3.166}
\]
Here \( o(1) e^2 \| f(t) \|_2^2 \) can be absorbed into the left-hand side of (3.165). Then we resort to (3.130) to tackle \( e^2 \| \partial_t f(t) \|_2^2 \):
\[
e^2 \| \partial_t f(t) \|_2^2 + \epsilon \| (1 - \mathcal{P})[\partial_t f] \|_{\gamma_{+,2}}^2 + 2 \| (I - \mathcal{P})[\partial_t f] \|_2^2 + e^2 \| \mathcal{P}[\partial_t f] \|_2^2 \tag{3.167}
\]
\[
\lesssim \frac{1}{e^2} \| \mathcal{P}[\partial_t S] \|_2^2 + \left( \| \nu^{-\frac{1}{2}}(I - \mathcal{P})[\partial_t \mathcal{S}] \|_2^2 + \| \partial_t h \|_{\gamma_{-,2}}^2 + \frac{1}{e^2} \| \nu \|_2^2 + \| v \cdot \nabla x \|_2^2 + \frac{1}{e^2} \| S(0) \|_2^2. \right)
\]
Multiplying a small constant on (3.165) and adding it to (3.167) to absorb \( e^2 \| \partial_t f(t) \|_2^2 \), we have
\[
e^2 \| \partial_t f(t) \|_{\gamma_{+,\infty}}^2 + \| f(t) \|_{\infty}^2 \tag{3.168}
\]
\[
\lesssim o(1) e^{2+\frac{2}{m}} \left( \| f(t) \|_{\gamma_{+,\infty}}^2 + \| f(t) \|_{\infty}^2 \right) + \frac{1}{e^2} \| \mathcal{P}[S(t)] \|_{2m+1}^2 + \left( \| \nu^{-\frac{1}{2}}(I - \mathcal{P})[S(t)] \|_2^2 + \frac{1}{e^2} \| \mathcal{P}[\partial_t \mathcal{S}] \|_2^2 + \| \nu^{-\frac{1}{2}}(I - \mathcal{P})[\partial_t \mathcal{S}] \|_2^2 \right) + \epsilon^2 \| h(t) \|_{\gamma_{-,4m}}^2 + \| \partial_z h \|_{\gamma_{-,2}}^2 + \frac{1}{e^2} \| \nu \|_2^2 + \| v \cdot \nabla x \|_2^2 + \frac{1}{e^2} \| S(0) \|_2^2.
\]
Then our desired result follows.

\[\square\]

**Remark 3.2.9.** Roughly speaking, Theorem 3.2.8 justifies that in order to bound instantaneous \( f \) in \( L^{2m} \), we need the accumulative bound for \( f \) and \( \partial_t f \) in \( L^2 \).

### 3.2.4 \( L^\infty \) Estimates

Now we begin to consider the mild formulation. When tracking the solution backward along the characteristics, once it hits the in-flow boundary or initial time, it either terminates (when hitting the initial time) or is diffusively reflected (when hitting the boundary). Following this idea, we may define the backward stochastic cycles, with multiple hitting times and out-flow integrals.

**Definition 3.2.10 (Hitting Time and Position).** For any \((t, x, v) \in \mathbb{R}^+ \times \Omega \times \mathbb{R}^3 \) with \((x, v) \notin \gamma_0\), define the backward hitting time
\[
t_b(t, x, v) := \inf \{ s > 0 : x - \epsilon v(s) \notin \Omega \text{ or } t = \epsilon^2 s \}. \tag{3.169}
\]
Also, define the hitting position
\[
x_b := x - \epsilon t_b(x, v)v. \tag{3.170}
\]
Note that \( x_b \in \Omega \) means the characteristic already hit the initial time, and \( x_b \in \partial \Omega \) means the characteristic hits the boundary, so it can be reflected and continue moving.

**Definition 3.2.11 (Stochastic Cycle).** For any \((t, x, v) \in \mathbb{R}^+ \times \Omega \times \mathbb{R}^3 \) with \((x, v) \notin \gamma_0\), let \((t_0, x_0, v_0) = (t, x, v)\). Define the first stochastic triple
\[
(t_1, x_1, v_1) := \left( t - \epsilon^2 t_b(x_0, v_0), x_b(x_0, v_0), v_1 \right), \tag{3.171}
\]
for some \( v_1 \) satisfying \( v_1 \cdot n(x_1) > 0 \).

Inductively, assume we know the \( k\)th stochastic triple \((t_k, x_k, v_k)\) with \( t_k > 0 \) (i.e. \( x_k \in \partial \Omega \)). Define the \((k + 1)\)th stochastic triple
\[
(t_{k+1}, x_{k+1}, v_{k+1}) := \left( t_k - \epsilon^2 t_b(x_k, v_k), x_b(x_k, v_k), v_{k+1} \right), \tag{3.172}
\]
for some \( v_{k+1} \) satisfying \( v_{k+1} \cdot n(x_{k+1}) > 0 \).
Remark 3.2.12. Roughly speaking, this definition describes one characteristic line with reflection (alternatively so-called stochastic cycle), starting from \((t_k, x_k, v_k) \in \gamma_+\), tracking back to \((t_{k+1}, x_{k+1}, v_k) \in \{0\} \times \Omega \times \mathbb{R}^3\) which will terminate, or \((t_{k+1}, x_{k+1}, v_k) \in (0, \infty) \times \gamma_\ast\), diffusively reflected to \((t_{k+1}, x_{k+1}, v_{k+1}) \in \gamma_\ast\), and beginning a new cycle. \(t_k\) the actual time the characteristic moves backward. Note that we are free to choose any \(v_k \cdot n(x_k) > 0\), so different sequence \(\{v_k\}_{k=1}^\infty\) represents different stochastic cycles.

Definition 3.2.13 (Diffusive Reflection Integral). Define \(V_k = \{v \in \mathbb{R}^3 : v \cdot n(x_k) > 0\}\), so the stochastic cycle must satisfy \(v_k \in V_k\). Let the iterated integral for \(k \geq 2\) be defined as

\[
\int_{\Pi_{j=1}^{k-1} v_j} \prod_{j=1}^{k-1} d\sigma_j := \int_{\nu_1} \ldots \left( \int_{\nu_{k-1}} d\sigma_{k-1} \right) \ldots d\sigma_1
\]

(3.173)

where \(d\sigma_j := \mu(v_j) |v_j \cdot n(x_j)| \, dv_j\) is a probability measure.

We define a weight function scaled with parameter \(\xi\), for \(0 \leq \varrho < \frac{1}{4}\) and \(\vartheta \geq 0\),

\[
w(v) := (v)^\varrho \exp(|v|^2)\]

(3.174)

and

\[
\bar{w}(v) := \frac{1}{\mu^2(v) w(v)} = \sqrt{2\pi} \frac{e^{(\frac{1}{2} - \varrho)|v|^2}}{\left(1 + |v|^2\right)^{\vartheta}}.
\]

(3.175)

Lemma 3.2.14. For \(T_0 > 0\) sufficiently large, there exists constants \(C_1, C_2 > 0\) independent of \(T_0\), such that for \(k = C_1 T_0^\frac{5}{8}\), and \((x, v) \in \times \Omega \times \mathbb{R}^3\),

\[
\int_{\Pi_{j=1}^{k-1} v_j} \prod_{j=1}^{k-1} d\sigma_j \leq \left(\frac{1}{2}\right)^{C_2 T_n^\frac{5}{8}}
\]

(3.176)

Proof. This is a rescaled version of [5] Lemma 4.1. Since our hitting time in (3.169) is rescaled with \(\epsilon\), we should rescale back in the statement of lemma. \(\square\)

Remark 3.2.15. Roughly speaking, Lemma [3.2.14] states that even though we have the freedom to choose \(v_k\) in each stochastic cycle, in the long run, the accumulative time will not be too small. After enough reflections \(\sim k\), most characteristics has the accumulative time that will exceed any set threshold \(T_0\).

Theorem 3.2.16. Assume (3.69) and (3.70) hold. The solution \(f(t, x, v)\) to the equation (3.67) satisfies for \(\vartheta \geq 0\) and \(0 \leq \varrho < \frac{1}{4}\),

\[
\begin{align*}
\|f\|_{\infty, \varrho, \vartheta} + \|f\|_{\gamma_+, \infty, \varrho, \vartheta} &\leq \frac{1}{\epsilon^{2^m + \frac{2}{m}}} \|P[S(t)]\|_{2m-1} + \frac{1}{\epsilon^{1 + \frac{2}{m}}} \|\nu^{-\frac{1}{2}} (\bar{P} - P)[S(t)]\|_{2} + \frac{1}{\epsilon^{2 + \frac{2}{m}}} \|P[\partial_t S]\|_{2} + \frac{1}{\epsilon^{1 + \frac{2}{m}}} \|\nu^{-\frac{1}{2}} (\bar{P} - P)[\partial_t S]\|_{2} \\
&+ \|\nu^{-1} S\|_{\infty, \varrho, \vartheta} + \frac{1}{\epsilon^{2 + \frac{2}{m}}} \|\bar{h}(t)\|_{\gamma_-, 2m} + \frac{1}{\epsilon^{1 + \frac{2}{m}}} \|\bar{h}(t)\|_{\gamma_-, 2} + \frac{1}{\epsilon^{1 + \frac{2}{m}}} \|\partial_t h\|_{\gamma_-, 2} + \|h\|_{\gamma_-, \infty, \varrho, \vartheta} \\
&+ \frac{1}{\epsilon^{2 + \frac{2}{m}}} \|\nu z\|_{2} + \frac{1}{\epsilon^{1 + \frac{2}{m}}} \|v \cdot \nabla x z\|_{2} + \|z\|_{\infty, \varrho, \vartheta} + \frac{1}{\epsilon^{2 + \frac{2}{m}}} \|S(0)\|_{2}
\end{align*}
\]

(3.177)

Proof.
Step 1: Mild formulation.
Denote the weighted solution

\[
g(t, x, v) := w(v) f(t, x, v),
\]

(3.178)
and the weighted non-local operator

\[ K_w(v)[g](v) := w(v)K \left[ \frac{g}{w} \right] (v) = \int_{\mathbb{R}^3} k_w(v, u) g(u) du, \]  

where

\[ k_w(v, u) := k(v, u) \frac{w(v)}{w(u)}. \]  

Multiplying \( w \) on both sides of (3.67), we have

\[
\begin{cases}
  \epsilon^2 \partial_t g + \epsilon v \cdot \nabla_x g + \nu g = K_w(t, x, v) + w(v)S(t, x, v) \quad \text{in } \mathbb{R}^+ \times \Omega \times \mathbb{R}^3, \\
  g(0, x, v) = w(v)z(x, v) \quad \text{in } \Omega \times \mathbb{R}^3, \\
  g(t, x, v) = w(v)\mu_{\frac{k}{\nu}}(v) \int_{\mathbb{R}^3} \tilde{w}(u) \delta(t, x, v) du + w(t, x, v) \quad \text{for } x_0 \in \partial \Omega \quad \text{and } v \cdot n < 0,
\end{cases}
\]

We introduce indicator function \( 1_{\{t_0 = 0\}} \) which implies the characteristic hits the initial time and \( 1_{\{t_k > 0\}} \) which implies the characteristic hits the boundary. We can rewrite the solution of the equation (3.67) along the characteristics by Duhamel’s principle as

\[
g(t, x, v) = \left(1_{\{t_0 = 0\}} w(v)z(x_1, v) e^{-\nu(v)\frac{t}{\epsilon^2}} + 1_{\{t_1 > 0\}} w(v)h(t_1, x_1, v) e^{-\nu(v)\frac{t}{\epsilon^2}} \right) \\
+ \int_0^{\frac{t}{\epsilon^2}} w(v)S\left(t - \epsilon^2 s, x - \epsilon sv, v\right) e^{-\nu(v)s} ds + \int_0^{\frac{t}{\epsilon^2}} K_w(v)[g](t - \epsilon^2 s, x - \epsilon sv, v) e^{-\nu(v)s} ds \\
+ e^{-\nu(v)\frac{t}{\epsilon^2}} \int_{\Omega} g(t_1, x_1, v_1) \tilde{w}(v_1) d\sigma_1,
\]

where the last term refers to \( P[f] \). We may further rewrite the last term using (3.182) along the stochastic cycle by applying Duhamel’s principle \( k \) times as

\[
g(t, x, v) = \left(1_{\{t_0 = 0\}} w(v)z(x_1, v) e^{-\nu(v)\frac{t}{\epsilon^2}} + 1_{\{t_k > 0\}} w(v)h(t_1, x_1, v) e^{-\nu(v)\frac{t}{\epsilon^2}} \right) \\
+ \int_0^{\frac{t}{\epsilon^2}} w(v)S\left(t - \epsilon^2 s, x - \epsilon sv, v\right) e^{-\nu(v)s} ds + \int_0^{\frac{t}{\epsilon^2}} K_w(v)[g](t - \epsilon^2 s, x - \epsilon sv, v) e^{-\nu(v)s} ds \\
+ e^{-\nu(v)\frac{t}{\epsilon^2}} \sum_{k=1}^{k-1} \int_{\Omega} \Pi_{j=1}^k \left( G_l[t, x, v] + H_l[t, x, v] \right) \tilde{w}(v_j) \left( \prod_{j=1}^{k} e^{-\nu(v)\frac{t_{j+1}}{\epsilon^2}} d\sigma_j \right) \\
+ e^{-\nu(v)\frac{t}{\epsilon^2}} \int_{\Omega} g(t_k, x_k, v_k) \tilde{w}(v_k) \left( \prod_{j=1}^{k} e^{-\nu(v)\frac{t_{j+1}}{\epsilon^2}} d\sigma_j \right),
\]

where

\[
G_l[t, x, v] := 1_{\{t_{\ell-1} = 0\}} w(v)z(x_{\ell-1} + v_t, v_t) + 1_{\{t_{\ell+1} > 0\}} w(v)h(t_{\ell+1}, x_{\ell+1}, v_t) \\
+ \int_0^{\frac{t_{\ell+1} - t_{\ell}}{\epsilon^2}} \left( S(t - \epsilon^2 s, x_{\ell} - \epsilon sv_t, v_t) e^{v_s} ds \right) \\
H_l[t, x, v] := \int_0^{\frac{t_{\ell+1} - t_{\ell}}{\epsilon^2}} \left( K_w(v)[g](t - \epsilon^2 s, x_{\ell} - \epsilon sv_t, v_t) e^{v_s} ds \right) ds.
\]

Step 2: Estimates of source terms initial terms and boundary terms.
We set \( k = CT_0^+ \) for \( T_0 \) defined in Lemma (3.2.14). Consider all terms in (3.183) related to \( h \) and \( S \).
Since \( t_1 \leq t \), we have

\[
\left| \mathbf{1}_{\{t_1=0\}} w(v)z(x_1, v)e^{-\nu(v)\frac{t-t_1}{\epsilon^2}} + \mathbf{1}_{\{t_1>0\}} w(v)h(t_1, x_1, v)e^{-\nu(v)\frac{t-t_1}{\epsilon^2}} \right| \leq \| wz \|_\infty + \| wh \|_{\gamma_\infty, \infty}. \tag{3.186}
\]

Also,

\[
\left| \int_0^{t_1} w(v)S(t - \epsilon^2 s, x - \epsilon sv, v)e^{-\nu(v)\epsilon^2 s}ds \right| \leq \| \nu^{-1} wS \|_\infty \int_0^{t_1} \nu(v)e^{-\nu(v)\epsilon^2 s}ds \leq \| \nu^{-1} wS \|_\infty. \tag{3.187}
\]

Then we turn to terms defined in \( G_\ell \) of (3.184). Noting that \( \frac{1}{\omega} \lesssim 1 \), we know

\[
\left| \mathbf{e}^{-\nu(v)\frac{t-t_1}{\epsilon^2}} \sum_{\ell=1}^{k-1} \int_{T_{\ell-1+1} \cap \sigma_j} \mathbf{1}_{\{t_{\ell+1}=0\}} w(v_{\ell})z(x_{\ell+1}, v_{\ell})\tilde{w}(v_{\ell}) \left( \prod_{j=1}^\ell e^{-\nu(v_{\ell})\frac{t_{\ell+1}-t_{\ell}}{\epsilon^2}} \right) ds_j \right| \lesssim \| wz \|_\infty \sum_{\ell=1}^{k-1} \int_{\sigma_j} \tilde{w}(v_{\ell}) \left( \prod_{j=1}^\ell \right) ds_j \lesssim CT_0^{\frac{7}{3}} \| wz \|_\infty, \tag{3.188}
\]

denote

\[
\left| \mathbf{e}^{-\nu(v)\frac{t-t_1}{\epsilon^2}} \sum_{\ell=1}^{k-1} \int_{T_{\ell-1+1} \cap \sigma_j} \mathbf{1}_{\{t_{\ell+1}>0\}} w(v_{\ell})h(t_{\ell+1}, x_{\ell+1}, v_{\ell})\tilde{w}(v_{\ell}) \left( \prod_{j=1}^\ell e^{-\nu(v_{\ell})\frac{t_{\ell+1}-t_{\ell}}{\epsilon^2}} \right) ds_j \right| \lesssim \| wh \|_{\gamma\infty, \infty} \sum_{\ell=1}^{k-1} \int_{\sigma_j} \tilde{w}(v_{\ell}) \left( \prod_{j=1}^\ell \right) ds_j \lesssim CT_0^{\frac{7}{3}} \| wh \|_{\gamma\infty, \infty}. \tag{3.189}
\]

Similarly,

\[
\left| \mathbf{e}^{-\nu(v)\frac{t-t_1}{\epsilon^2}} \sum_{\ell=1}^{k-1} \int_{T_{\ell-1} \cap \sigma_j} \int_0^{t_{\ell-1+1}/\epsilon^2} \left( w(v_{\ell})S(t_{\ell} - \epsilon^2 s, x_{\ell} - \epsilon sv_{\ell}, v_{\ell})e^{\nu(v_{\ell})\epsilon^2 s} \right) ds_w \tilde{w}(v_{\ell}) \left( \prod_{j=1}^\ell e^{-\nu(v_{\ell})\frac{t_{\ell+1}-t_{\ell}}{\epsilon^2}} \right) ds_j \right| \lesssim \| \nu^{-1} wS \|_\infty \sum_{\ell=1}^{k-1} \int_{\sigma_j} \nu(v_{\ell})e^{\nu(v_{\ell})\frac{t_{\ell+1}-t_{\ell}}{\epsilon^2}} ds_w \tilde{w}(v_{\ell}) \left( \prod_{j=1}^\ell \right) ds_j \lesssim CT_0^{\frac{7}{3}} \| \nu^{-1} wS \|_\infty. \tag{3.190}
\]

Collecting all terms in (3.186), (3.187), (3.188), (3.189) and (3.190), we have

\[
\text{Initial Term and Boundary Term Contribution} \lesssim CT_0^{\frac{7}{3}} \left( \| wz \|_\infty + \| wh \|_{\gamma\infty, \infty} \right) \tag{3.191}
\]

\[
\lesssim \| wz \|_\infty + \| wh \|_{\gamma\infty, \infty},
\]

and

\[
\text{Source Term Contribution} \lesssim CT_0^{\frac{7}{3}} \| \nu^{-1} wS \|_\infty \lesssim \| \nu^{-1} wS \|_\infty. \tag{3.192}
\]

Step 3: Estimates of Multiple Reflection.
We focus on the last term in (3.183), which can be decomposed based on accumulative time \( t_{k+1} \):

\[
\left| \frac{e^{-\nu(v) \frac{t-t_{k}}{2^k}}}{\tilde{w}(v)} \int_{\prod_{j=1}^{k} V_j} g(t, x, v) \tilde{w}(v) \left( \prod_{j=1}^{k} e^{-\nu(v_j) \frac{t-j+t_{k+1}}{2^j}} d\sigma_j \right) \right| \\
\leq \left| \frac{e^{-\nu(v) \frac{t-t_{k}}{2^k}}}{\tilde{w}(v)} \int_{\prod_{j=1}^{k} V_j} 1_{\left\{ \frac{t-t_{k}}{2^k} \leq \frac{t_0}{2^k} \right\}} g(t, x, v) \tilde{w}(v) \left( \prod_{j=1}^{k} e^{-\nu(v_j) \frac{t-j+t_{k+1}}{2^j}} d\sigma_j \right) \right|
\]

\[
+ \left| \frac{e^{-\nu(v) \frac{t-t_{k}}{2^k}}}{\tilde{w}(v)} \int_{\prod_{j=1}^{k} V_j} 1_{\left\{ \frac{t-t_{k}}{2^k} \geq \frac{t_0}{2^k} \right\}} g(t, x, v) \tilde{w}(v) \left( \prod_{j=1}^{k} e^{-\nu(v_j) \frac{t-j+t_{k+1}}{2^j}} d\sigma_j \right) \right| := J_1 + J_2.
\]

Based on Lemma 3.2.14, we have

\[
J_1 \lesssim \|g\|_{\infty} \left| \int_{\prod_{j=1}^{k-1} V_j} 1_{\left\{ \frac{t-t_{k}}{2^k} \leq \frac{t_0}{2^k} \right\}} \left( \int_{V_k} \tilde{w}(v_k) d\sigma_k \right) \left( \prod_{j=1}^{k-1} \int_{V_j} \tilde{w}(v_j) d\sigma_j \right) \right| \lesssim \left( \frac{1}{2} \right)^{C_2 T_0^*} \|g\|_{\infty}.
\]

On the other hand, when \( t_k \) is large, the exponential terms become extremely small, so we obtain

\[
J_2 \lesssim \|g\|_{\infty} \left| \int_{\prod_{j=1}^{k-1} V_j} 1_{\left\{ \frac{t-t_{k}}{2^k} \geq \frac{t_0}{2^k} \right\}} \left( \int_{V_k} \tilde{w}(v_k) d\sigma_k \right) \left( \prod_{j=1}^{k-1} \int_{V_j} \tilde{w}(v_j) d\sigma_j \right) \right| \lesssim \|g\|_{\infty} e^{-\frac{t_0}{2^k}}
\]

Summarizing (3.194) and (3.195), we get for \( \delta \) arbitrarily small

\[
\text{Multiple Reflection Term Contribution} \lesssim \delta \|g\|_{\infty}.
\]

Step 4: Estimates of \( K_w \) terms.

So far, the only remaining terms in (3.183) are related to \( K_w \). We focus on

\[
\left| \int_0^{t_{k+1}} \int_{R^3} K_{w(v)} g(t - \epsilon^2 s, x, -\epsilon sv, v) e^{-\nu(v) s} ds \right| \lesssim \left| \int_{R^3} K_{w(v)} g(t - \epsilon^2 s, x, -\epsilon sv, v) \right|_{\infty}.
\]

Denote \( T(s; t, x, v) := t - \epsilon^2 s \) and \( X(s; t, x, v) := x - \epsilon(t_1 - s)v \). Define the back-time stochastic cycle from \( (T, X, v') \) as \((t', x', v')\) with \((t_0', x_0', v_0') = (T, X, v') \). Then we can rewrite \( K_w \) along the stochastic cycle as (3.183)

\[
\left| K_{w(v)} g(t - \epsilon^2 s, x, -\epsilon(t_1 - s)v, v) \right| = \left| K_{w(v)} g(T, X, v) \right| = \left| \int_{R^3} k_{w(v)}(v, v') g(T, X, v') dv' \right|
\]

\[
\leq \left| \int_{R^3} \int_{R^3} k_{w(v)}(v, v') K_{w(v')} g(T - \epsilon^2 r, X - \epsilon rv', v') e^{-\nu(v') r} dr dv' \right|
\]

\[
+ \left| \int_{R^3} \frac{e^{-\nu(v') \frac{r^2}{2^k}}}{\tilde{w}(v')} \sum_{\ell=1}^{k-1} \int_{\prod_{j=1}^{\ell} V_j} k_{w(v)}(v, v') H_\ell[T, X, v'] \tilde{w}(v') \left( \prod_{j=1}^{\ell} e^{-\nu(v_j) \frac{r-j+t_{k+1}}{2^j}} d\sigma_j \right) dv' \right|
\]

\[
+ \left| \int_{R^3} k_{w(v)}(v, v') \left( \text{initial terms + boundary terms + source terms + multiple reflection terms} \right) dv' \right| := I + II + III.
\]
Using estimates (3.191), (3.192), (3.196) from Step 2 and Step 3, and Lemma 2.2.5, we can bound $III$ directly

$$III \lesssim \|wz\|_{\infty} + \||wh||_{\gamma, \infty} + \|\nu^{-1} wS\|_{\infty} + \|g\|_{\infty}. \quad (3.199)$$

$I$ and $II$ are much more complicated. We may further rewrite $I$ as

$$I = \left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{0}^{\frac{T-r}{r}} k_{w(v)}(v, v') k_{w(v')}(v', v'') g \left( T - e^2 r, X - e r v', v'' \right) e^{-\nu(v') r} dr dv' dv'' \right|, \quad (3.200)$$

which will estimated in four cases:

$$I := I_1 + I_2 + I_3 + I_4. \quad (3.201)$$

Case I: $I_1 : |v| \geq N$.
Based on Lemma 2.2.5 we have

$$\left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} k_{w(v)}(v, v') k_{w(v')}(v', v'') dv' dv'' \right| \lesssim \frac{1}{1 + |v|} \lesssim \frac{1}{N}. \quad (3.202)$$

Hence, we get

$$I_1 \lesssim \frac{1}{N} \|g\|_{\infty}. \quad (3.203)$$

Case II: $I_2 : |v| \leq N, \ |v'| \geq 2N$, or $|v'| \leq 2N, \ |v''| \geq 3N$.
Notice this implies either $|v' - v| \geq N$ or $|v' - v''| \geq N$. Hence, either of the following is valid correspondingly:

$$|k_{w(v)}(v, v')| \leq C e^{-\delta N^2} |k_{w(v)}(v, v')| e^{d|v - v'|^2}, \quad (3.204)$$

$$|k_{w(v')}(v', v'')| \leq C e^{-\delta N^2} |k_{w(v')}(v', v'')| e^{d|v' - v''|^2}. \quad (3.205)$$

Based on Lemma 2.2.5 we know

$$\int_{\mathbb{R}^3} |k_{w(v)}(v, v')| e^{d|v - v'|^2} dv' < \infty, \quad (3.206)$$

$$\int_{\mathbb{R}^3} |k_{w(v')}(v', v'')| e^{d|v' - v''|^2} dv'' < \infty. \quad (3.207)$$

Hence, we have

$$I_2 \lesssim e^{-\delta N^2} \|g\|_{\infty}. \quad (3.208)$$

Case III: $I_3 : 0 \leq r \leq \delta$ and $|v| \leq N, \ |v'| \leq 2N, \ |v''| \leq 3N$.
In this case, since the integral with respect to $r$ is restricted in a very short interval, there is a small contribution as

$$I_3 \lesssim \int_0^\delta e^{-r} dr \|g\|_{\infty} \lesssim \delta \|g\|_{\infty}. \quad (3.209)$$

Case IV: $I_4 : r \geq \delta$ and $|v| \leq N, \ |v'| \leq 2N, \ |v''| \leq 3N$. 


This is the most complicated case. Since \( k_{w(v)}(v, v') \) has possible integrable singularity of \( \frac{1}{|v - v'|} \), we can introduce the truncated kernel \( k_N(v, v') \) which is smooth and has compactly supported range such that

\[
\sup_{|v| \leq 3N} \int_{|v'| \leq 3N} \left| k_N(v, v') - k_{w(v)}(v, v') \right| \, dv' \leq \frac{1}{N}.
\] (3.210)

Then we can split

\[
k_{w(v)}(v, v')k_{w(v')}(v', v'') = k_N(v, v')k_N(v', v'') + \left( k_{w(v)}(v, v') - k_N(v, v') \right) k_{w(v')}(v', v'')
\] (3.211)

\[+ \left( k_{w(v')}(v', v'') - k_N(v', v'') \right) k_N(v, v').
\]

This means that we further split \( I_4 \) into

\[
I_4 := I_{4,1} + I_{4,2} + I_{4,3}.
\] (3.212)

Based on (3.210), we have

\[
I_{4,2} \lesssim \frac{1}{N} \| g \|_{L^\infty}, \quad I_{4,3} \lesssim \frac{1}{N} \| g \|_{L^\infty}.
\] (3.213)

Therefore, the only remaining term is \( I_{4,1} \). Note that we always have \( X - \epsilon rv' \in \Omega \). Hence, we define the change of variable \( v' \to y \) as \( y = (y_1, y_2, y_3) = X - \epsilon rv' \). Then the Jacobian

\[
\left| \frac{dy}{dv'} \right| = \begin{vmatrix}
-\epsilon r & 0 & 0 \\
0 & -\epsilon r & 0 \\
0 & 0 & -\epsilon r
\end{vmatrix} = \epsilon^3 r^3 \geq \epsilon^3 \delta^3.
\] (3.214)

Considering \( |v|, |v'|, |v''| \leq 3N \), we know \( |g| \lesssim |f| \). Also, since \( k_N \) is bounded, we estimate

\[
I_{4,1} \lesssim \int_0^{T-t} \int_{|v'| \leq 2N} \int_{|v''| \leq 3N} 1_{\{X-\epsilon rv' \in \Omega\}} |\mathbb{P}[f](T - \epsilon^2 r, X - \epsilon rv', v'')| e^{-\epsilon(r')^2} r \, dr \, dv' \, dv''
\] (3.215)

Using the decomposition \( f = \mathbb{P}[f] + (\mathbb{I} - \mathbb{P})[f] \), and Hölder’s inequality, we estimate them separately,

\[
\int_0^{T-t} \int_{|v'| \leq 2N} \int_{|v''| \leq 3N} 1_{\{X-\epsilon rv' \in \Omega\}} \left| \mathbb{P}[f](T - \epsilon^2 r, X - \epsilon rv', v'') \right| e^{-\epsilon(r')^2} r \, dr \, dv' \, dv''
\] (3.216)

\[
\leq \int_0^{T-t} \left( \int_{|v'| \leq 2N} \int_{|v''| \leq 3N} 1_{\{X-\epsilon rv' \in \Omega\}} \left| \mathbb{P}[f] \right| (T - \epsilon^2 r, X - \epsilon rv', v'') e^{-\epsilon(r')^2} r \, dr \, dv' \, dv'' \right)^{\frac{2m-1}{2m}}
\]

\[
\times \left( \int_{|v'| \leq 2N} \int_{|v''| \leq 3N} 1_{\{X-\epsilon rv' \in \Omega\}} \left| \mathbb{P}[f] \right|^2 (T - \epsilon^2 r, X - \epsilon rv', v'') e^{-\epsilon(r')^2} r \, dr \, dv' \, dv'' \right)^{\frac{1}{2m}} e^{-\epsilon r} dr
\]

\[
\lesssim \int_0^{T-t} \left( \frac{1}{\epsilon^3 \delta^3} \int_{|v'| \leq 2N} \int_{|v''| \leq 3N} 1_{\{y \in \Omega\}} \left| (\mathbb{I} - \mathbb{P})[f] \right|^2 (T - \epsilon^2 r, y, v'') dy \, dv'' \right)^{\frac{1}{2m}} e^{-\epsilon r} dr
\]

\[
\leq \frac{1}{\epsilon^3 \delta^3} \sup_{[0,T]} \left\| (\mathbb{I} - \mathbb{P})[f(t)] \right\|_{L^2},
\]

and

\[
\int_0^{T-t} \int_{|v'| \leq 2N} \int_{|v''| \leq 3N} 1_{\{X-\epsilon rv' \in \Omega\}} |(\mathbb{I} - \mathbb{P})[f](T - \epsilon^2 r, X - \epsilon rv', v'')| e^{-\epsilon(r')^2} r \, dr \, dv' \, dv''
\] (3.217)

\[
\leq \int_0^{T-t} \left( \int_{|v'| \leq 2N} \int_{|v''| \leq 3N} 1_{\{X-\epsilon rv' \in \Omega\}} \left| (\mathbb{I} - \mathbb{P})[f] \right| (T - \epsilon^2 r, X - \epsilon rv', v'') \, dv' \, dv'' \right)^{\frac{1}{2}} e^{-\epsilon r} dr
\]

\[
\times \left( \int_{|v'| \leq 2N} \int_{|v''| \leq 3N} 1_{\{X-\epsilon rv' \in \Omega\}} \left| (\mathbb{I} - \mathbb{P})[f] \right|^2 (T - \epsilon^2 r, X - \epsilon rv', v'') \, dv' \, dv'' \right)^{\frac{1}{2}} e^{-\epsilon r} dr
\]

\[
\lesssim \int_0^{T-t} \left( \frac{1}{\epsilon^3 \delta^3} \int_{|v'| \leq 2N} \int_{|v''| \leq 3N} 1_{\{y \in \Omega\}} \left| (\mathbb{I} - \mathbb{P})[f] \right|^2 (T - \epsilon^2 r, y, v'') dy \, dv'' \right)^{\frac{1}{2}} e^{-\epsilon r} dr
\]

\[
\leq \frac{1}{\epsilon^3 \delta^3} \sup_{[0,T]} \left\| (\mathbb{I} - \mathbb{P})[f(t)] \right\|_{L^2}.
\]
Inserting (3.216) and (3.217) into (3.215), we obtain

\[ I_{4,1} \leq \frac{1}{\epsilon^{2m} \delta^{2m}} \sup_{[0, T]} \| \mathbb{P}[f(t)] \|_{2m} + \frac{1}{\epsilon^{2m} \delta^{2m}} \sup_{[0, T]} \| (I - \mathbb{P})[f(t)] \|_2. \] (3.218)

Combined with (3.213), we know

\[ I_4 \leq \frac{1}{N} \| g \|_\infty + \frac{1}{\epsilon^{2m} \delta^{2m}} \sup_{[0, t]} \| \mathbb{P}[f(t)] \|_{2m} + \frac{1}{\epsilon^{2m} \delta^{2m}} \sup_{[0, t]} \| (I - \mathbb{P})[f(t)] \|_2. \] (3.219)

Summarizing all four cases in (3.203), (3.208), (3.209) and (3.219), we obtain

\[ I \lesssim \left( \frac{1}{N} + e^{-\delta N^2} + \delta \right) \| g \|_\infty + \frac{1}{\epsilon^{2m} \delta^{2m}} \sup_{[0, t]} \| \mathbb{P}[f(t)] \|_{2m} + \frac{1}{\epsilon^{2m} \delta^{2m}} \sup_{[0, t]} \| (I - \mathbb{P})[f(t)] \|_2. \] (3.220)

Choosing \( \delta \) sufficiently small and then taking \( N \) sufficiently large, we have

\[ I \lesssim \delta \| g \|_\infty + \frac{1}{\epsilon^{2m} \delta^{2m}} \sup_{[0, t]} \| \mathbb{P}[f(t)] \|_{2m} + \frac{1}{\epsilon^{2m} \delta^{2m}} \sup_{[0, t]} \| (I - \mathbb{P})[f(t)] \|_2. \] (3.221)

By a similar but tedious computation, we arrive at

\[ II \lesssim \delta \| g \|_\infty + \frac{1}{\epsilon^{2m} \delta^{2m}} \sup_{[0, t]} \| \mathbb{P}[f(t)] \|_{2m} + \frac{1}{\epsilon^{2m} \delta^{2m}} \sup_{[0, t]} \| (I - \mathbb{P})[f(t)] \|_2. \] (3.222)

Combined with (3.199), we have

\[ \left| \int_0^t K_{w(v)}[g] \left( t - \epsilon^2 s, x - \epsilon s v, v \right) e^{-\nu(v)s} \, ds \right| \lesssim \delta \| g \|_\infty + \frac{1}{\epsilon^{2m} \delta^{2m}} \sup_{[0, t]} \| \mathbb{P}[f(t)] \|_{2m} + \frac{1}{\epsilon^{2m} \delta^{2m}} \sup_{[0, t]} \| (I - \mathbb{P})[f(t)] \|_2 + \| wz \|_\infty + \| \nu^{-1} w S \|_\infty. \] (3.223)

All the other terms in (3.183) related to \( K_w \) can be estimated in a similar fashion. At the end of the day, we have

\[ K_w \text{ term contribution} \] (3.224)

\[ \lesssim \delta \| g \|_\infty + \frac{1}{\epsilon^{2m} \delta^{2m}} \sup_{[0, t]} \| \mathbb{P}[f(t)] \|_{2m} + \frac{1}{\epsilon^{2m} \delta^{2m}} \sup_{[0, t]} \| (I - \mathbb{P})[f(t)] \|_2 + \| wz \|_\infty + \| \nu^{-1} w S \|_\infty. \]

Step 5: Synthesis.

Summarizing all above and inserting (3.191), (3.192), (3.196) and (3.224) into (3.183), we obtain for any \((t, x, v) \in \mathbb{R}^+ \times \hat{\Omega} \times \mathbb{R}^3\),

\[ \| g(t, x, v) \|_{\gamma_+} \lesssim \delta \| g \|_\infty + \frac{1}{\epsilon^{2m} \delta^{2m}} \sup_{[0, t]} \| \mathbb{P}[f(t)] \|_{2m} + \frac{1}{\epsilon^{2m} \delta^{2m}} \sup_{[0, t]} \| (I - \mathbb{P})[f(t)] \|_2 + \| wz \|_\infty + \| \nu^{-1} w S \|_\infty. \] (3.225)

Taking supremum over \([0, t] \times \gamma_+\) in (3.225), we have

\[ \sup_{[0, t]} \| g(t) \|_{\gamma_+, \infty} \lesssim \delta \| g \|_\infty + \frac{1}{\epsilon^{2m} \delta^{2m}} \sup_{[0, t]} \| \mathbb{P}[f(t)] \|_{2m} + \frac{1}{\epsilon^{2m} \delta^{2m}} \sup_{[0, t]} \| (I - \mathbb{P})[f(t)] \|_2 + \| wz \|_\infty + \| \nu^{-1} w S \|_\infty. \] (3.226)
Based on Theorem 3.2.8 for $\frac{3}{2} < m < 3$, we obtain

\[
\sup_{[0,t]} \|g(t)\|_{\gamma_+,\infty} \lesssim \delta \|g\|_{\infty} + o(1) \left( \sup_{[0,t]} \|f(t)\|_{\gamma_+,\infty} + \sup_{[0,t]} \|f(t)\|_{\infty} \right) + E \tag{3.227}
\]

\[
\lesssim \delta \|g\|_{\infty} + o(1) \left( \sup_{[0,t]} \|g(t)\|_{\gamma_+,\infty} + \sup_{[0,t]} \|g(t)\|_{\infty} \right) + E,
\]

where

\[
E := \frac{1}{\epsilon^{2+\frac{m}{2}}} \|P[S(t)]\|_{\infty} + \frac{1}{\epsilon^{1+\frac{m}{2}}} \left\| \nu^{\frac{1}{2}} (I - P)[S(t)] \right\|_2 + \frac{1}{\epsilon^{2+\frac{m}{2}}} \|P[\partial_t S]\|_2 + \frac{1}{\epsilon^{1+\frac{m}{2}}} \left\| \nu^{\frac{1}{2}} (I - P)[\partial_t S] \right\|_2
\]

\[
+ \left\| \nu^{-1} w S \right\|_{\infty} + \frac{1}{\epsilon^{1+\frac{m}{2}}} \|h(t)\|_{\gamma_-,\infty} + \frac{1}{\epsilon^{1+\frac{m}{2}}} \|h(t)\|_{\gamma_-} + \frac{1}{\epsilon^{1+\frac{m}{2}}} \|\partial_t h\|_{\gamma_-} + \|wh\|_{\gamma_-}\infty
\]

\[
+ \frac{1}{\epsilon^{2+\frac{m}{2}}} \|\nu z\|_2 + \frac{1}{\epsilon^{1+\frac{m}{2}}} \|v \cdot \nabla x z\|_2 + \|wz\|_\infty + \frac{1}{\epsilon^{2+\frac{m}{2}}} \|S(0)\|_2.
\]

Absorbing $o(1) \sup_{[0,t]} \|g(t)\|_{\gamma_+,\infty}$ into the left-hand side, we have

\[
\sup_{[0,t]} \|g(t)\|_{\gamma_+,\infty} \lesssim \delta \|g\|_{\infty} + o(1) \sup_{[0,t]} \|g(t)\|_{\infty} + E. \tag{3.229}
\]

On the other hand, taking supremum over $[0, t] \times \Omega \times \mathbb{R}^3$ in (3.229), we have

\[
\sup_{[0,t]} \|g(t)\|_{\infty} \lesssim \delta \|g\|_{\infty} + \frac{1}{\epsilon^{2+\frac{m}{2}}} \|P[f(t)]\|_2 + \frac{1}{\epsilon^{1+\frac{m}{2}}} \sup_{[0,t]} \|P[f(t)]\|_2 + \frac{1}{\epsilon^{2+\frac{m}{2}}} \|P[f(t)]\|_2 + \|wz\|_\infty + \|wh\|_{\gamma_-\infty} + \|\nu^{-1} w S\|_{\infty}. \tag{3.230}
\]

Based on Theorem 3.2.8 we obtain

\[
\sup_{[0,t]} \|g(t)\|_{\infty} \lesssim \delta \|g\|_{\infty} + o(1) \left( \sup_{[0,t]} \|g(t)\|_{\gamma_+,\infty} + \sup_{[0,t]} \|g(t)\|_{\infty} \right) + E. \tag{3.231}
\]

Absorbing $\delta \|g\|_{\infty}$ and $o(1) \sup_{[0,t]} \|g(t)\|_{\infty}$ into the left-hand side, we have

\[
\sup_{[0,t]} \|g(t)\|_{\infty} \lesssim o(1) \sup_{[0,t]} \|g(t)\|_{\gamma_+,\infty} + E. \tag{3.232}
\]

Inserting (3.229) into (3.231), and absorbing $\delta \|g\|_{\infty}$ and $o(1) \|g\|_{\infty}$ into the left-hand side, we get

\[
\sup_{[0,t]} \|g(t)\|_{\infty} \lesssim E. \tag{3.233}
\]

Then (3.229) implies

\[
\sup_{[0,t]} \|g(t)\|_{\gamma_+,\infty} \lesssim E. \tag{3.234}
\]

In summary, we have

\[
\|g\|_{\infty} + \|g\|_{\gamma_+,\infty} \lesssim \frac{1}{\epsilon^{2+\frac{m}{2}}} \|P[S(t)]\|_{\infty} + \frac{1}{\epsilon^{1+\frac{m}{2}}} \left\| \nu^{\frac{1}{2}} (I - P)[S(t)] \right\|_2 + \frac{1}{\epsilon^{2+\frac{m}{2}}} \|P[\partial_t S]\|_2 + \frac{1}{\epsilon^{1+\frac{m}{2}}} \left\| \nu^{\frac{1}{2}} (I - P)[\partial_t S] \right\|_2
\]

\[
+ \left\| \nu^{-1} w S \right\|_{\infty} + \frac{1}{\epsilon^{1+\frac{m}{2}}} \|h(t)\|_{\gamma_-\infty} + \frac{1}{\epsilon^{1+\frac{m}{2}}} \|h(t)\|_{\gamma_-} + \frac{1}{\epsilon^{1+\frac{m}{2}}} \|\partial_t h\|_{\gamma_-} + \|wh\|_{\gamma_-\infty}
\]

\[
+ \frac{1}{\epsilon^{2+\frac{m}{2}}} \|\nu z\|_2 + \frac{1}{\epsilon^{1+\frac{m}{2}}} \|v \cdot \nabla x z\|_2 + \|wz\|_\infty + \frac{1}{\epsilon^{2+\frac{m}{2}}} \|S(0)\|_2.
\]

Then our result naturally follows.
Remark 3.2.17. In the above proof, we use the trace $\|g(t)\|_{\infty}, \|g(t)\|_{\gamma, \infty}$ and $\|g\|_{\gamma, \infty}$ interchangeably with $\|g\|_{\infty}$ to perform absorbing argument. Roughly speaking, we track the solution using mild formulation, so it is always continuous along the characteristics, which covers the whole domain $\mathbb{R}^+ \times \Omega \times \mathbb{R}^3$, so $\|g\|_{\gamma, \infty}$ will control all the rest. To be more precise, it actually relies on Ukai’s trace theorem in [20], which says that for transport operator $\partial_t + v \cdot \nabla_x$, such traces are always well-defined and controllable.

Remark 3.2.18 (Exponential Decay). Define $\tilde{f} = e^{K_0 t} f$. Then $\tilde{f}$ satisfies

$$
\begin{align*}
&\epsilon^2 \partial_t \tilde{f} + \epsilon v \cdot \nabla_x \tilde{f} + \mathcal{L} \tilde{f} = \epsilon^2 K_0 \tilde{f} + e^{K_0 t} S(t,x,v) \quad \text{in} \quad \mathbb{R}^+ \times \Omega \times \mathbb{R}^3, \\
&\tilde{f}(0,x,v) = z(x,v) \quad \text{in} \quad \Omega \times \mathbb{R}^3, \\
&\tilde{f}(t,x_0,v) = \mathcal{P} \tilde{f}(t,x_0,v) + e^{K_0 t} h(t,x_0,v) \quad \text{on} \quad \mathbb{R}^+ \times \gamma_-, 
\end{align*}
$$

(3.236)

where

$$
\mathcal{P} \tilde{f}(t,x_0,v) = \mu^+(v) \int_{u \cdot n(x_0) > 0} \mu^+(u) \tilde{f}(t,x_0,u) |u \cdot n(x_0)| \, du.
$$

(3.237)

The extra term is $\epsilon^2 K_0 \tilde{f}$. Thanks to $\epsilon^2$, based on $L^2$ and $L^{2m}$ energy estimates in Lemma 3.2.5 and Theorem 3.2.8 for $K_0$ small, we can absorb this term into the left-hand side. Therefore, we can recover all estimates as in Theorem 3.2.16.
3.3 Hydrodynamic Limit

3.3.1 Perturbed Remainder Estimates

We consider the perturbed evolutionary Boltzmann equation

\[
\left\{
\begin{array}{l}
\epsilon^2 \partial_t f + \epsilon v \cdot \nabla_x f + \mathcal{L}[f] = \Gamma[f, g] + \epsilon^3 \Gamma[f, f] + S(t, x, v) \quad \text{in} \quad \mathbb{R}^+ \times \Omega \times \mathbb{R}^3, \\
f(0, x, v) = z(x, v) \quad \text{in} \quad \Omega \times \mathbb{R}^3, \\
f(t, x_0, v) = \mathcal{P}[f](t, x_0, v) + (\nu^\epsilon - \mu) \mu^{-1} \mathcal{P}[f] + h(t, x_0, v) \quad \text{for} \quad x_0 \in \partial \Omega \quad \text{and} \quad v \cdot n < 0.
\end{array}
\right.
\]  

(3.238)

Assume that a priori

\[
\|g\|_{\infty, \partial \theta} + \|\partial_t g\|_{\infty, \partial \theta} + \|\epsilon^3 f\|_{\infty, \partial \theta} = o(1). 
\]

(3.239)

**Theorem 3.3.1.** Assume \((3.60)\) and \((3.70)\) hold. The solution \(f(t, x, v)\) to the equation \((3.238)\) satisfies

\[
\frac{1}{\epsilon^2} \|(1 - \mathcal{P})[f(t)]\|_{\infty, \gamma_+} + \frac{1}{\epsilon} \|\mathcal{P}[f(t)]\|_{\infty, \gamma_+} + \frac{1}{\epsilon} \|\mathcal{P}[f(t)]\|_2 
\]

(3.240)

\[
\|f(t)\|_2 + \frac{1}{\epsilon^2} \|(1 - \mathcal{P})[f]\|_{\infty, \gamma_+} + \frac{1}{\epsilon} \|\mathcal{P}[f]\|_{\infty, \gamma_+} + \frac{1}{\epsilon} \|\mathcal{P}[f]\|_2 
\]

\[
+ \frac{1}{\epsilon^2} \|(1 - \mathcal{P})[\partial_t f]\|_{\infty, \gamma_+} + \frac{1}{\epsilon} \|\mathcal{P}[\partial_t f]\|_{\infty, \gamma_+} + \frac{1}{\epsilon} \|\mathcal{P}[\partial_t f]\|_2 
\]

\[
\lesssim o(1) \epsilon^{\frac{2m}{2m+4}} \left( \|f(t)\|_{\infty, \gamma_+} + \|f(t)\|_\infty \right) + \frac{1}{\epsilon^2} \|\mathcal{P}[S(t)]\|_{\infty, \gamma_+} + \frac{1}{\epsilon} \nu^{-\frac{1}{2}} \|\mathcal{P}[S(t)]\|_2  
\]

\[
+ \frac{1}{\epsilon^2} \|\mathcal{P}[S(t)]\|_2 + \frac{1}{\epsilon} \nu^{-\frac{1}{2}} \|\mathcal{P}[S(t)]\|_2 + \frac{1}{\epsilon^2} \|\mathcal{P}[\partial_t S]\|_2 + \frac{1}{\epsilon} \nu^{-\frac{1}{2}} \|\mathcal{P}[\partial_t S]\|_2 
\]

\[
+ \|h(t)\|_{\infty, \gamma_+} + \frac{1}{\epsilon} \|h(t)\|_{\infty, \gamma_+} + \frac{1}{\epsilon} \|\partial_t h\|_{\infty, \gamma_+} + \frac{1}{\epsilon} \|\nu z\|_2 + \frac{1}{\epsilon} \|v \cdot \nabla x z\|_2 + \frac{1}{\epsilon^2} \|\mathcal{P}[S(0)]\|_2. 
\]

**Proof.** Since the perturbed term \(\Gamma[f, g] = \Gamma[f, f] \in \mathcal{N}^1\), we apply Theorem \((3.22)\) to \((3.23)\) to obtain

\[
\frac{1}{\epsilon^2} \|(1 - \mathcal{P})[f(t)]\|_{\infty, \gamma_+} + \frac{1}{\epsilon} \|\mathcal{P}[f(t)]\|_{\infty, \gamma_+} + \frac{1}{\epsilon} \|\mathcal{P}[f(t)]\|_2 
\]

(3.241)

\[
+ \frac{1}{\epsilon^2} \|(1 - \mathcal{P})[\partial_t f]\|_{\infty, \gamma_+} + \frac{1}{\epsilon} \|\mathcal{P}[\partial_t f]\|_{\infty, \gamma_+} + \frac{1}{\epsilon} \|\mathcal{P}[\partial_t f]\|_2 
\]

\[
\lesssim o(1) \epsilon^{\frac{2m}{2m+4}} \left( \|f(t)\|_{\infty, \gamma_+} + \|f(t)\|_\infty \right) 
\]

\[
+ \frac{1}{\epsilon^2} \|\mathcal{P}[S(t)]\|_{\infty, \gamma_+} + \frac{1}{\epsilon} \nu^{-\frac{1}{2}} \|\mathcal{P}[S(t)]\|_2 + \frac{1}{\epsilon^2} \|\mathcal{P}[\partial_t S]\|_2 + \frac{1}{\epsilon} \nu^{-\frac{1}{2}} \|\mathcal{P}[\partial_t S]\|_2 
\]

\[
+ \|h(t)\|_{\infty, \gamma_+} + \frac{1}{\epsilon} \|h(t)\|_{\infty, \gamma_+} + \frac{1}{\epsilon} \|\partial_t h\|_{\infty, \gamma_+} + \frac{1}{\epsilon} \|\nu z\|_2 + \frac{1}{\epsilon} \|v \cdot \nabla x z\|_2 + \frac{1}{\epsilon^2} \|\mathcal{P}[S(0)]\|_2. 
\]

Also, based on Lemma \((3.20)\), we have \(L^2\) estimate

\[
\|f(t)\|_2 + \frac{1}{\epsilon^2} \|(1 - \mathcal{P})[f]\|_2 + \frac{1}{\epsilon} \|\mathcal{P}[f]\|_2 
\]

(3.242)

\[
\lesssim \frac{1}{\epsilon^2} \|\mathcal{P}[S]\|_2 + \frac{1}{\epsilon} \nu^{-\frac{1}{2}} \|\mathcal{P}[S]\|_2 + \frac{1}{\epsilon} \|h\|_{\infty, \gamma_+} + \|z\|_2 
\]

\[
+ \frac{1}{\epsilon} \nu^{-\frac{1}{2}} \|\Gamma[f, g]\|_2 + \frac{1}{\epsilon} \|\nu^{-\frac{1}{2}} \Gamma[f, f]\|_2 + \frac{1}{\epsilon} \|\nu^{-\frac{1}{2}} \partial_t \Gamma[f, f]\|_2 + \frac{1}{\epsilon} \|\nu^{-\frac{1}{2}} \partial_t \Gamma[f, f]\|_2. 
\]
Step 1: Bulk Perturbation Terms.
Using Lemma 2.3.4 and (3.243), we have
\[
\frac{1}{\epsilon} \left\| \nu^{-\frac{1}{2}} \Gamma[f, g](t) \right\|_2 \lesssim o(1) \left\| \nu^{\frac{1}{2}} f(t) \right\|_2 + o(1) \left\| P[f(t)] \right\|_\nu + o(1) \left\| (1 - P)[f(t)] \right\|_\nu .
\] (3.243)

Note that direct computation reveals that
\[
\left\| P[f(t)] \right\|_{2m} \gtrsim \left\| P[f(t)] \right\|_\nu ,
\] (3.244)
so inserting (3.243) into (3.241), we can absorb \( o(1) \left\| P[f(t)] \right\|_\nu \) and \( o(1) \left\| (1 - P)[f(t)] \right\|_\nu \) into the left-hand side. On the other hand, Using Lemma 2.3.4 and (3.239), we have
\[
\frac{1}{\epsilon} \left\| \nu^{-\frac{1}{2}} \partial_t \Gamma[f, g] \right\|_2 \lesssim o(1) \left\| \nu^{\frac{1}{2}} f \right\|_2 + o(1) \left\| \nu^{\frac{1}{2}} \partial_t f \right\|_2.
\] (3.245)

Then \( o(1) \left\| \nu^{\frac{1}{2}} f \right\|_2 \) can be handled by \( L^2 \) estimates and \( o(1) \left\| \nu^{\frac{1}{2}} \partial_t f \right\|_2 \) can be absorbed into LHS. Similarly,
\[
\frac{1}{\epsilon} \left\| \nu^{\frac{1}{2}} \partial_t \Gamma[f, g] \right\|_2 \lesssim \frac{1}{\epsilon} \left\| \nu^{\frac{1}{2}} f \right\|_2 + o(1) \left\| \nu^{\frac{1}{2}} \partial_t f \right\|_2,
\] (3.246)
\[
\frac{1}{\epsilon} \left\| \nu^{\frac{1}{2}} \partial_t \Gamma[f, g] \right\|_2 \lesssim \frac{1}{\epsilon} \left\| \nu^{\frac{1}{2}} f \right\|_2 + o(1) \left\| \nu^{\frac{1}{2}} \partial_t f \right\|_2.
\] (3.247)
Both of them can be absorbed into LHS of (3.241). A similar argument justifies the absorbing in (3.242).

Step 2: Boundary Perturbation Terms.
On the other hand, due to (1.59), we know
\[
\left\| (\mu^b_\epsilon - \mu) \mu^{-1} P[f(t)] \right\|_{\gamma_-, 2m} \lesssim o(1) \epsilon \left\| f(t) \right\|_{\gamma_+, \infty},
\] (3.248)
which can be combined with the corresponding term on the right-hand side of (3.241). Also,
\[
\frac{1}{\epsilon} \left\| (\mu^b_\epsilon - \mu) \mu^{-1} P[f(t)] \right\|_{\gamma_-, 2} \lesssim o(1) \left\| P[f(t)] \right\|_{\gamma_+},
\] (3.249)
\[
\frac{1}{\epsilon} \left\| (\mu^b_\epsilon - \mu) \mu^{-1} P[\partial_t f] \right\|_{\gamma_-, 2} \lesssim o(1) \left\| P[\partial_t f] \right\|_{\gamma_+}.
\] (3.250)
Note that both of then involve \( P[f] \), which has been controlled by the proof of Theorem 3.2.5 (Step 2). Hence, adding (3.242) to (3.241) and absorbing all new terms into the LHS, we can close the proof. □

**Theorem 3.3.2.** Assume (3.69) and (3.70) hold. The solution \( f(t, x, v) \) to the equation (3.238) satisfies for \( \delta \geq 0 \) and \( 0 \leq \varrho < \frac{1}{4} \),
\[
\left\| f \right\|_{\infty, \theta, \varrho} + \left\| f \right\|_{\gamma_+, \infty, \delta, \theta}
\lesssim \frac{1}{\epsilon^{2 + \frac{m}{2}}} \left\| P[S(t)] \right\|_{2m} + \frac{1}{\epsilon^{2 + \frac{m}{2}}} \left\| \nu^{-\frac{1}{2}} (1 - P)[S(t)] \right\|_2 + \frac{1}{\epsilon^{1 + \frac{m}{2}}} \left\| P[S] \right\|_2 + \frac{1}{\epsilon^{1 + \frac{m}{2}}} \left\| \nu^{-\frac{1}{2}} (1 - P)[S] \right\|_2
+ \frac{1}{\epsilon^{1 + \frac{m}{2}}} \left\| \nu^{-\frac{1}{2}} (1 - P)[\partial_t S] \right\|_2 + \frac{1}{\epsilon^{1 + \frac{m}{2}}} \left\| \nu^{-\frac{1}{2}} (1 - P)[\partial_t S] \right\|_2 + \left\| \nu^{-1} S \right\|_{\infty, \theta, \varrho}
+ \frac{1}{\epsilon^{2 + \frac{m}{2}}} \left\| h(t) \right\|_{\gamma_-, 2m} + \frac{1}{\epsilon^{2 + \frac{m}{2}}} \left\| h(t) \right\|_{\gamma_-, 2} + \frac{1}{\epsilon^{1 + \frac{m}{2}}} \left\| h(t) \right\|_{\gamma_-, 2} + \frac{1}{\epsilon^{1 + \frac{m}{2}}} \left\| \partial_t h(t) \right\|_{\gamma_-, 2} + \left\| h(t) \right\|_{\gamma_-, \infty, \theta}
+ \frac{1}{\epsilon^{2 + \frac{m}{2}}} \left\| v \cdot \nabla_x z \right\|_2 + \left\| z \right\|_{\infty, \theta, \varrho} + \frac{1}{\epsilon^{2 + \frac{m}{2}}} \left\| S(0) \right\|_2.
\] (3.251)
Proof. Since we already have bounds for $f$ in $L^{2m}$ as in Theorem 3.3.1 following the proof of Theorem 3.2.16 we obtain

$$
\|\|f\|\|_{\infty, \partial, \varrho} + \|\|f\|\|_{\gamma_+, \infty, \varrho, \vartheta} \leq \frac{1}{\epsilon^{2 + \frac{2m}{3}}} \|\|P[S(t)]\|\|_{\frac{2m}{3} - 1} + \frac{1}{\epsilon^{1 + \frac{2m}{3}}} \|\|w^{-\frac{1}{2}}(I - P)[S(t)]\|\|_{2} + \frac{1}{\epsilon^{1 + \frac{2m}{3}}} \|\|w^{-\frac{1}{2}}(I - P)[S]\|\|_{2}
$$

(3.252)

+ \frac{1}{\epsilon^{2 + \frac{2m}{3}}} \|\|P[\partial_t S]\|\|_{2} + \frac{1}{\epsilon^{1 + \frac{2m}{3}}} \|\|w^{-\frac{1}{2}}(I - P)[\partial_t S]\|\|_{2} + \|\|w^{-1} S\|\|_{\infty, \vartheta, \varrho}

+ \frac{1}{\epsilon^{2 + \frac{2m}{3}}} \|\|h(t)\|\|_{\gamma_-, \infty, \vartheta} + \frac{1}{\epsilon^{1 + \frac{2m}{3}}} \|\|h(t)\|\|_{\gamma_-, \infty, \vartheta} + \frac{1}{\epsilon^{1 + \frac{2m}{3}}} \|\|\partial_t h\|\|_{\gamma_-, \infty, \vartheta} + \|\|h\|\|_{\gamma_-, \infty, \varrho, \vartheta}

+ \|\|v_0 \|\|_{2} + \frac{1}{\epsilon^{2 + \frac{2m}{3}}} \|\|P(0)\|\|_{2}

+ \|\|w^{-1} \Gamma[f, g]\|\|_{\infty, \partial, \varrho} + \|\|w^3 \|\|\gamma_+, \partial, \varrho\|\|_{\infty, \partial, \varrho} + \|\|(\mu_0^* - \mu)^{-1} P[f]\|\|_{\gamma_+, \infty, \varrho, \vartheta}.

Using Lemma 3.3.1 and (3.239), we have

$$
\|\|w^{-1} \Gamma[f, g]\|\|_{\infty, \partial, \varrho} \leq \|\|f\|\|_{\infty, \partial, \varrho} \|\|g\|\|_{\infty, \partial, \varrho} \leq o(1) \|\|f\|\|_{\infty, \partial, \varrho},

(3.253)

\|\|w^3 \|\|\gamma_+, \partial, \varrho\|\|_{\infty, \partial, \varrho} \leq \|\|f\|\|_{\infty, \partial, \varrho} \|\|e^3 f\|\|_{\infty, \partial, \varrho} \leq o(1) \|\|f\|\|_{\infty, \partial, \varrho}.

(3.254)

Inserting (3.253) into (3.252), we can absorb $o(1) \|\|f\|\|_{\infty, \partial, \varrho}$ into the left-hand side. Also, using (1.55), we have

$$
\|\|(\mu_0^* - \mu)^{-1} P[f]\|\|_{\gamma_+, \infty, \varrho, \vartheta} \leq o(1) \|\|f\|\|_{\gamma_+, \infty, \varrho, \vartheta}.

(3.255)

Inserting (3.255) into (3.252) and absorbing $o(1) \|\|f\|\|_{\gamma_+, \infty, \varrho, \vartheta}$ into the left-hand side, we obtain the desired result.

3.3.2 Analysis of Asymptotic Expansion

Analysis of Initial Layer

We first prove a theorem about well-posedness and decay of initial layer equation.

Theorem 3.3.3. For equation

$$
\begin{cases}
\partial_t g + L[g] = S(\tau, v) \text{ in } \mathbb{R}^+ \times \mathbb{R}^3, \\
g(0, v) = z(v),
\end{cases}
$$

(3.256)

satisfying

$$
|z|_{\infty, \partial, \varrho} \leq 1, \quad \|e^{K_0 t} g\|_{\infty, \partial, \varrho} \leq 1,
$$

(3.257)

there exists a unique solution $g(\tau, v)$ and a $g_\infty \in \mathcal{N}$ satisfying

$$
|g_\infty| \leq 1, \quad \|e^{K_0 \tau} (g - g_\infty)\|_{\infty, \partial, \varrho} \leq 1.
$$

(3.258)

Proof. This is very similar to the analysis of $\epsilon$-Milne problem with geometric correction, but just simpler. We decompose $g = r + q$, where $r \in \mathcal{N}^\perp$ and $q = \sum_{k=0}^4 q_k(\tau) \varphi_k(v) \in \mathcal{N}$. Then using the same $L^2 - L^\infty$ estimates, we can get the desired result.

With this theorem in hand, based on the analysis in Section 3.1.6 we know $\mathcal{F}_1 = 0$ and $\mathcal{F}_2, \mathcal{F}_3$ are well-defined.

Theorem 3.3.4. For $K_0 > 0$ sufficiently small, the initial layer satisfies

$$
\|e^{K_0 \tau} \mathcal{F}_3(x)\|_{\infty, \partial, \varrho} \leq 1, \quad \|e^{K_0 \tau} \mathcal{F}_4(x)\|_{\infty, \partial, \varrho} \leq 1.
$$

(3.259)
In particular, since $\partial_t = e^{-2}\partial_T$, we have the time derivative estimate

**Theorem 3.3.5.** For $K_0 > 0$ sufficiently small, the initial layer satisfies

$$\left\| e^{K_0\sigma} \frac{\partial \mathcal{F}_3(x)}{\partial t} \right\|_{\infty, \theta, \varphi} \lesssim \epsilon^{-2}, \quad \left\| e^{K_0\sigma} \frac{\partial \mathcal{F}_4(x)}{\partial t} \right\|_{\infty, \theta, \varphi} \lesssim \epsilon^{-2}. \quad (3.260)$$

The space derivative version follows the same fashion. Note that due to rescaling $\tau = \frac{t}{\epsilon^2}$, the bound for $\partial_t \mathcal{F}_k$ is much worse than $\mathcal{F}_k$. This is the main reason that we have to expand the initial layer to more orders than interior solution and boundary layer. Also, this is why we have to enforce the compatibility condition \((1.02)\) and let $\mathcal{F}_1$ vanish.

**Analysis of Boundary Layer**

Based on the analysis in Section \([3.1.6]\) we know $\mathcal{F}_1 = 0$ and $\mathcal{F}_2, \mathcal{F}_3$ are well-defined.

**Theorem 3.3.6.** For $K_0 > 0$ sufficiently small, the boundary layer $\mathcal{F}_2$ satisfies

$$\left\| e^{K_0\eta} \mathcal{F}_2(t) \right\|_{\infty, \theta, \varphi} \lesssim 1, \quad (3.261)$$

and

$$\left\| e^{K_0\eta} \frac{\partial \mathcal{F}_2(t)}{\partial \eta} \right\|_{\infty, \theta, \varphi} + \left\| e^{K_0\eta} \frac{\partial \mathcal{F}_2(t)}{\partial \nu} \right\|_{\infty, \theta, \varphi} + \left\| e^{K_0\eta} \frac{\partial \mathcal{F}_2(t)}{\partial \varphi} \right\|_{\infty, \theta, \varphi} \lesssim \left\| \ln(\epsilon) \right\|^3, \quad (3.262)$$

However, the tricky part is the estimate of $\mathcal{F}_3$, which essentially satisfies a stationary linearized Boltzmann equation

$$\begin{cases}
\epsilon v \cdot \nabla_x \mathcal{F}_3(t) + \mathcal{L}[\mathcal{F}_3(t)] = Z[\mathcal{F}_3(t)] & \text{in } \hat{\Omega} \times \mathbb{R}^3,
\mathcal{F}_3(t)|_{x_0} = \mathcal{P}[\mathcal{F}_3(t)](x_0, v) & \text{for } x_0 \in \partial\Omega \text{ and } v \cdot n < 0.
\end{cases} \quad (3.263)$$

Based on stationary $L^2$ estimates in Remark \([2.2.15]\) we obtain

$$\frac{1}{\epsilon^2} \left\| (1 - \mathcal{P})[\mathcal{F}_3(t)] \right\|_{\gamma + 2, \varphi} + \frac{1}{\epsilon} \left\| (1 - \mathcal{P})[\mathcal{F}_3(t)] \right\|_{\nu} + \left\| \mathcal{P}[\mathcal{F}_3(t)] \right\|_{2m} \lesssim \frac{1}{\epsilon^2} \left\| \ln(\epsilon) \right\|^3 \lesssim \frac{1}{\epsilon^{1 + \frac{2m}{3}}} \left\| \ln(\epsilon) \right\|^3, \quad (3.264)$$

where we strongly rely on the rescaling $\eta = \frac{\mu}{\epsilon}$ and the exponential decay of $Z$ in $\eta$. Then using the stationary $L^\infty$ estimates in Theorem \([2.2.14]\) we have

$$\left\| \mathcal{F}_3(t) \right\|_{\infty, \theta, \varphi} + \left\| \mathcal{F}_3(t) \right\|_{\gamma + \infty, \varphi, \varphi} \lesssim \frac{1}{\epsilon^{2 + \frac{2m}{3}}} \left\| \mathcal{P}[\mathcal{F}_3(t)] \right\|_{2m} \left\| \nu^{-\frac{1}{2}} (1 - \mathcal{P})[Z(t)] \right\|_2 + \left\| \nu^{-\frac{1}{2}} Z(t) \right\|_{\infty, \theta, \varphi} \lesssim \frac{1}{\epsilon^{1 + \frac{2m}{3}}} \left\| \ln(\epsilon) \right\|^3 \lesssim \frac{1}{\epsilon^{1 + \frac{2m}{3}}} \left\| \ln(\epsilon) \right\|^3. \quad (3.265)$$

The above is only instantaneous version. The corresponding accumulative version for both $\mathcal{F}_3$ and $\partial_t \mathcal{F}_3$ also hold. However, we lose the decay of $\mathcal{F}_3$ in $\eta$.

**Analysis of Interior Solution**

Based on the analysis in matching procedure, we know $F_k = 0$ are well-defined satisfy corresponding fluid equations.

**Theorem 3.3.7.** For $K_0 > 0$ sufficiently small, the boundary layer satisfies

$$\left\| \langle u \rangle^\theta e^{c|v|^2} F_1 \right\|_{L^\infty_{t} H^2_x L^\infty_v} \lesssim 1, \quad \left\| \langle u \rangle^\theta e^{c|v|^2} F_2 \right\|_{L^\infty_{t} H^2_x L^\infty_v} \lesssim 1, \quad \left\| \langle u \rangle^\theta e^{c|v|^2} F_3 \right\|_{L^\infty_{t} H^2_x L^\infty_v} \lesssim 1. \quad (3.266)$$
Analysis of Initial-Boundary Layer

The compatibility condition (3.162) implies that at the corner points \((0, x_0, v)\), the equation (1.66) is naturally satisfied. Also, we have the simplified expansion at these points:

- By our construction in Section 3.1.6, \(F_1 = 0\) and \(\mathcal{F} = 0\). Also,
\[
F_1(0, x_0, v) = A_1(t, x_0, v) + B_1(t, x_0, v) + C_1(t, x_0, v) = \rho_{0,1}(x_0),
\]
with
\[
A_1(t, x_0, v) = \rho_{0,1}(x_0)\mu^\frac{1}{2}(v), \quad B_1(t, x_0, v) = 0, \quad C_1(t, x_0, v) = 0.
\]

- By our construction in Section 3.1.6 at \((t, x_0, v)\), \(\mathcal{F}_2\) and \(\mathcal{F}_2\) satisfy trivial equations with zero source term and zero data, so \(\mathcal{F}_2(0, x, v) = 0\) and \(\mathcal{F}_2(t, x_0, v) = 0\). Also,
\[
F_2(0, x_0, v) = A_2(t, x_0, v) + B_2(t, x_0, v) + C_2(t, x_0, v) = \rho_{0,2}(x_0),
\]
with
\[
A_2(t, x_0, v) = \rho_{0,2}(x_0)\mu^\frac{1}{2}(v), \quad B_2(t, x_0, v) = 0, \quad C_2(t, x_0, v) = 0.
\]

Here the space derivative \(\nabla_x f_{0,1}(x_0, v) = 0\) plays a role.

- Based on our construction in Section 3.1.6 we know
\[
F_3(0, x_0, v) = A_3(t, x_0, v) + B_3(t, x_0, v) + C_3(t, x_0, v).
\]
In particular, have
\[
B_3(t, x_0, v) = 0, \quad C_3(t, x_0, v) = 0.
\]

Here the space derivative \(\nabla_x f_{0,1}(x_0, v) = \nabla_x f_{0,2}(x_0, v) = 0\) and \(\nabla_x^2 f_{0,1}(x_0, v) = 0\) play a role. Also, these space derivatives accompanied with \(\partial_t \mu_1(t, x_0, v) = 0\) yield \(v \cdot \nabla_x \mathcal{F}_2 = 0\). Hence, we know \(\mathcal{F}_3\) and \(\mathcal{F}_3\) satisfy trivial equation with zero source term and zero data, so \(\mathcal{F}_3(0, x, v) = 0\) and \(\mathcal{F}_3(t, x_0, v) = 0\). In the end, we know
\[
F_3(0, x_0, v) = A_3(t, x_0, v) = \rho_{0,3}(x_0)\mu^\frac{1}{2}(v).
\]

- In summary, we have shown that at the corner point \((0, x_0, v)\), both the initial layer and boundary layer are zero up to third order.

### 3.3.3 Proof of Main Theorem

Now we turn to the proof of the main result, Theorem 1.2.1. The asymptotic analysis already reveals that the construction of the interior solution, initial layer and boundary layer is valid. Here, we focus on the remainder estimates. We divide the proof into several steps:

Step 1: Remainder definitions.
Define the remainder as
\[
c^3 R = f^e - Q - \mathcal{Q} - \mathcal{O},
\]
where
\[
Q := \sum_{k=1}^3 c^k F_k, \quad \mathcal{Q} := \sum_{k=1}^3 c^k \mathcal{F}_k, \quad Q := \sum_{k=1}^4 c^k \mathcal{F}_k.
\]
In other words, we have

\[ f' = Q + \mathcal{L} + Q + \epsilon^3 R. \]  

(3.276)

We write \( \mathcal{L} \) to denote the linearized Boltzmann operator:

\[ \mathcal{L}[f] = \epsilon^2 \partial_t f + \epsilon v \cdot \nabla_x f + \mathcal{L}[f]. \]

(3.277)

In studying initial layer in Section 3.1.2 we utilize the equivalent form:

\[ \mathcal{L}[f] = \partial_t f + \epsilon v \cdot \nabla_x u + \mathcal{L}[f]. \]

(3.278)

In studying boundary layer in Section 3.1.3 we use another equivalent form:

\[ \mathcal{L}[f] = \epsilon^2 \partial_t f + \epsilon v \cdot \nabla_x f + \mathcal{L}[f]. \]

\[ \text{Step 2: Representation of } \mathcal{L}[R]. \]

The equation \( \text{(1.66)} \) is actually

\[ \mathcal{L}[f'] = \Gamma[f', f'], \]

(3.279)

which means

\[ \mathcal{L}[Q + \mathcal{L} + Q + \epsilon^3 R] = \Gamma[Q + \mathcal{L} + Q + \epsilon^3 R, Q + \mathcal{L} + Q + \epsilon^3 R]. \]

(3.280)

Note that the right-hand side of (3.280), i.e. the nonlinear term can be decomposed as

\[ \Gamma[Q + \mathcal{L} + Q + \epsilon^3 R, Q + \mathcal{L} + Q + \epsilon^3 R] = \epsilon^3 \Gamma[R, R] + 2 \epsilon^3 \Gamma[R, Q + \mathcal{L} + Q] + \Gamma[Q + \mathcal{L} + Q, Q + \mathcal{L} + Q]. \]

(3.281)

Then we turn to the left-hand side of (3.280). The interior contribution is

\[ \mathcal{L}[Q] = \epsilon^2 \partial_t \left( \epsilon F_1 + \epsilon^2 F_2 + \epsilon^3 F_3 \right) + \epsilon v \cdot \nabla_x \left( \epsilon F_1 + \epsilon^2 F_2 + \epsilon^3 F_3 \right) + \mathcal{L}[\epsilon F_1 + \epsilon^2 F_2 + \epsilon^3 F_3] \]

(3.282)

\[ = \epsilon^2 v \cdot \nabla_x F_3 + \epsilon^4 \partial_t F_2 + \epsilon^4 \partial_t F_3 + \epsilon^2 \Gamma[F_1, F_1] + 2 \epsilon^3 \Gamma[F_1, F_2]. \]

On the other hand, we consider the boundary layer contribution. Since \( \mathcal{F}_1 = 0, \mathcal{F}_2 \) and \( \mathcal{F}_3 \) terms are all included in boundary layer construction except the time derivatives, we compute

\[ \mathcal{L}[\mathcal{L}] = \epsilon^4 \partial_t \mathcal{F}_2 + \epsilon^4 \partial_t \mathcal{F}_3 + 2 \epsilon^3 \Gamma[F_1, F_2]. \]

(3.283)

Also, since \( \mathcal{F}_1 = 0 \), the initial layer contribution

\[ \mathcal{L}[Q] = \epsilon^3 v \cdot \nabla_x F_4 + \epsilon^3 \Gamma[F_1, F_4] + 2 \epsilon^4 \Gamma[F_2, F_4] + 2 \epsilon^4 \Gamma[F_3, F_4] + 2 \epsilon^4 \Gamma[F_1, F_3]. \]

(3.284)

Therefore, inserting (3.281), (3.282), (3.283) and (3.284) into (3.280), we have

\[ \mathcal{L}[R] = \epsilon^3 \Gamma[R, R] + 2 \Gamma[R, Q + \mathcal{L} + Q] + S_1 + S_2, \]

(3.285)
where
\[ S_1 = -\epsilon v \cdot \nabla_x F_3 - \epsilon \partial_t F_2 - \epsilon^2 \partial_t F_3 - \epsilon \partial_t \mathcal{F}_2 - \epsilon^2 \partial_t \mathcal{F}_3 - \epsilon^2 v \cdot \nabla_x \mathcal{F}_4, \]
\[ S_2 = \epsilon \left( 2\Gamma[\mathcal{F}_2, \mathcal{F}_2] + 2\Gamma[F_1, F_3] + 2\Gamma[F_2, F_2] + 2\Gamma[F_1, \mathcal{F}_3] \right) + \text{higher-order } \Gamma \text{ terms up to } \epsilon^4. \] (3.286)

Step 3: Representation of \( R - \mathcal{P}[R] \) and \( R(0) \).
The boundary condition of (1.66) is essentially
\[ f^c = \mu_b \mu^{-1} \mathcal{P}[f'] + \mu^{-\frac{1}{2}}(\mu_b - \mu). \] (3.288)

which means
\[ Q + \mathcal{I} + \epsilon^3 R = \mathcal{P}[Q + \mathcal{I} + \epsilon^3 R] + (\mu_b - \mu) \mu^{-1} \mathcal{P}[Q + \mathcal{I} + \epsilon^3 R] + \mu^{-\frac{1}{2}}(\mu_b - \mu). \] (3.289)

Based on the boundary condition expansion in Section [3.1.6] we have
\[ R - \mathcal{P}[R] = H[R] + h, \] (3.290)

where
\[ H[R](t, x_0, v) = (\mu_b - \mu) \mu^{-1} \mathcal{P}[R], \] (3.291)

and
\[ h = -\epsilon \mathcal{F}_4. \] (3.292)

In other words, the only contribution is from the initial layer \( \mathcal{F}_4 \) at the corner point. On the other hand, for initial data
\[ R(0) = z = \epsilon \mathcal{F}_4(0). \] (3.293)

In other words, the only contribution is from the initial data of initial layer \( \mathcal{F}_4 \).

Step 4: Remainder Estimate.
The equation (3.259), initial condition (3.293) and boundary condition (3.290) forms a system that fits into (3.283):
\[ \begin{align*}
\epsilon^2 \partial_t R + \epsilon v \cdot \nabla_x R + \mathcal{L}[R] &= \Gamma[R, 2(Q + \mathcal{I} + \mathcal{Q}) + \epsilon^3 R] + S_1(t, x, v) + S_2(t, x, v) \text{ in } \mathbb{R}^+ \times \Omega \times \mathbb{R}^3, \\
R(0, x, v) &= z(x, v) \text{ in } \Omega \times \mathbb{R}^3, \\
R(t, x_0, v) &= \mathcal{P}[R](t, x_0, v) + H[R](t, x_0, v) + h(t, x_0, v) \text{ for } x_0 \in \partial \Omega \text{ and } v \cdot n < 0.
\end{align*} \] (3.294)

Hence, by Theorem 3.3.2 we have
\[ \begin{align*}
\| R \|_{\infty, \partial \Omega} + \| R \|_{\gamma_+, \infty, \partial \Omega} & \lesssim \frac{1}{\epsilon^{2+\frac{1}{2m}}} \| \mathcal{P}[S_1(t)] \|_{\frac{2m}{2m+1}} + \frac{1}{\epsilon^{1+\frac{1}{2m}}} \| \nu^{-\frac{1}{2}}(1 - \mathcal{P})[S_1(t)] \|_2 + \frac{1}{\epsilon^{1+\frac{1}{2m}}} \| \mathcal{P}[S_1] \|_2 + \frac{1}{\epsilon^{1+\frac{1}{2m}}} \| \nu^{-\frac{1}{2}}(1 - \mathcal{P})[S_1] \|_2 \\
& + \frac{1}{\epsilon^{2+\frac{1}{2m}}} \| \mathcal{P}[\partial_t S_1] \|_2 + \frac{1}{\epsilon^{1+\frac{1}{2m}}} \| \nu^{-\frac{1}{2}}(1 - \mathcal{P})[\partial_t S_1] \|_2 + \| \nu^{-1} S \|_{\infty, \partial \Omega} + \frac{1}{\epsilon^{1+\frac{1}{2m}}} \| S_1(0) \|_2 \\
& + \frac{1}{\epsilon^{2+\frac{1}{2m}}} \| \mathcal{P}[S_2(t)] \|_{\frac{2m}{2m+1}} + \frac{1}{\epsilon^{1+\frac{1}{2m}}} \| \nu^{-\frac{1}{2}}(1 - \mathcal{P})[S_2(t)] \|_2 + \frac{1}{\epsilon^{2+\frac{1}{2m}}} \| \mathcal{P}[S_2] \|_2 + \frac{1}{\epsilon^{1+\frac{1}{2m}}} \| \nu^{-\frac{1}{2}}(1 - \mathcal{P})[S_2] \|_2 \\
& + \frac{1}{\epsilon^{2+\frac{1}{2m}}} \| \mathcal{P}[\partial_t S_2] \|_2 + \frac{1}{\epsilon^{1+\frac{1}{2m}}} \| \nu^{-\frac{1}{2}}(1 - \mathcal{P})[\partial_t S_2] \|_2 + \| \nu^{-1} S \|_{\infty, \partial \Omega} + \frac{1}{\epsilon^{2+\frac{1}{2m}}} \| S_2(0) \|_2 \\
& + \frac{1}{\epsilon^{1+\frac{1}{2m}}} \| h(t) \|_{\gamma_-, \infty} + \frac{1}{\epsilon^{1+\frac{1}{2m}}} \| h(t) \|_{\gamma_-, 2} + \frac{1}{\epsilon^{1+\frac{1}{2m}}} \| h(t) \|_{\gamma_-, 2} + \frac{1}{\epsilon^{1+\frac{1}{2m}}} \| \partial_t h(t) \|_{\gamma_-, 2} + \| h(t) \|_{\gamma_-, \infty, \partial \Omega} \\
& + \frac{1}{\epsilon^{1+\frac{1}{2m}}} \| v \cdot \nabla_x z \|_2 + \| z \|_{\infty, \partial \Omega}.
\end{align*} \] (3.295)
Step 5: Estimate of $S_1$. Using results in Section 3.3.2 for the interior contribution $S_{IS} := -\nu \cdot \nabla_s F_3 - \epsilon \partial_t F_2 - \epsilon^2 \partial_t F_3$:

$$
\| S_{IS}(t) \|_{L^2} + \| \nu^{-\frac{1}{2}} S_{IS}(t) \|_{L^2} + \| \nu^{-\frac{1}{2}} S_{IS} \|_{L^2} + \| \partial_t S_{IS} \|_{L^2} + \| \nu^{-\frac{1}{2}} \partial_t S_{IS} \|_{L^2} + \| \nu^{-1} S_{IS} \|_{L^\infty, \partial, \theta} + \| S_{IS}(0) \|_{L^2} \lesssim \epsilon. \tag{3.296}
$$

Using results in Section 3.3.2 for the boundary layer contribution $S_{BL} := -\epsilon^2 \partial_t \mathcal{F}_3$, note that $\| g(t) \|_{L^p} \lesssim \| g(t) \|_{L^{2m}}$ for $1 \leq p \leq 2m$:

$$
\| P[S_{BL}](t) \|_{L^2} \lesssim \epsilon^1 \frac{1}{\ln(\epsilon)} |\ln(\epsilon)|^8, \quad \| \nu^{-\frac{1}{2}} (I - P)[S_{BL}](t) \|_{L^2} \lesssim \epsilon^\frac{1}{2} \frac{1}{\ln(\epsilon)} |\ln(\epsilon)|^8, \tag{3.297}
$$

Using results in Section 3.3.2 for the initial layer contribution $S_{IL} := -\epsilon^2 \nu \cdot \nabla_x \mathcal{F}_4$, note the rescaling $\tau = \frac{t}{\epsilon^2}$:

$$
\| P[S_{IL}](t) \|_{L^2} \lesssim \epsilon^2, \quad \| \nu^{-\frac{1}{2}} (I - P)[S_{IL}](t) \|_{L^2} \lesssim \epsilon^2, \tag{3.299}
$$

Hence, we have

$$
\| P[S_1](t) \|_{L^2} \lesssim \epsilon^1 \frac{1}{\ln(\epsilon)} |\ln(\epsilon)|^8, \quad \| \nu^{-\frac{1}{2}} (I - P)[S_1](t) \|_{L^2} \lesssim \epsilon^\frac{1}{2} \frac{1}{\ln(\epsilon)} |\ln(\epsilon)|^8, \tag{3.301}
$$

Step 6: Estimate of $S_2$. It suffices to consider the leading-order term $2\epsilon \Gamma[\mathcal{F}_2, \mathcal{F}_2]$ which contains the most dangerous initial layer $\mathcal{F}_2$. Note that the time derivative estimate is the worst one. Using nonlinear estimates in Lemma 2.3.1 and rescaling $\eta = \frac{\mu}{\epsilon}$ and $\tau = \frac{t}{\epsilon^2}$, we have

$$
\| P[S_2](t) \|_{L^2} = 0, \quad \| \nu^{-\frac{1}{2}} (I - P)[S_2](t) \|_{L^2} \lesssim \epsilon^\frac{1}{2}, \tag{3.303}
$$

Step 7: Estimate of $h$ and $z$. 
For boundary data \( h = -\epsilon F_4 \), we have
\[
\|h(t)\|_{\gamma_{-\frac{1}{2^m}}} \lesssim \epsilon, \quad \|h(t)\|_{\gamma_{-2}} \lesssim \epsilon, \quad \|h\|_{\gamma_{-2}} \lesssim \epsilon^2,
\]
\[
\|\partial_t h\|_{\gamma_{-2}} \lesssim 1, \quad \|h\|_{\gamma_{-\infty,0,0}} \lesssim \epsilon.
\]

For initial data \( z = -\epsilon F_4(0) \), we have
\[
\|\nu z\|_2 \lesssim \epsilon, \quad \|\nu \cdot \nabla_x z\|_2 \lesssim \epsilon, \quad \|z\|_{\infty,0,0} \lesssim \epsilon.
\]

Step 8: Synthesis.
Summarizing all above, we have
\[
\left\| \|R\|_{\infty,0,0} + \|R\|_{\gamma_{+1,0,0}} \right\| \lesssim \frac{1}{\epsilon^{2+\frac{1}{2m}}} \left( \epsilon^{1-\frac{2}{m}} |\ln(\epsilon)|^8 \right) + \frac{1}{\epsilon^{1+\frac{2}{2m}}} \left( \epsilon^{2-\frac{2}{m}} |\ln(\epsilon)|^8 \right) + \frac{1}{\epsilon^{1+\frac{2}{2m}}} \left( \epsilon^{2-\frac{2}{m}} |\ln(\epsilon)|^8 \right) + \frac{1}{\epsilon^{1+\frac{2}{2m}}} \left( \epsilon^{2-\frac{2}{m}} |\ln(\epsilon)|^8 \right)
+ \frac{1}{\epsilon^{1+\frac{2}{2m}}} \left( \epsilon^{1-\frac{2}{m}} |\ln(\epsilon)|^8 \right) + \frac{1}{\epsilon^{1+\frac{2}{2m}}} \left( \epsilon^{2-\frac{2}{m}} |\ln(\epsilon)|^8 \right) + \frac{1}{\epsilon^{1+\frac{2}{2m}}} \left( \epsilon^{2-\frac{2}{m}} |\ln(\epsilon)|^8 \right)
+ \frac{1}{\epsilon^{2+\frac{1}{2m}}} \left( \epsilon \right) + \frac{1}{\epsilon^{2+\frac{1}{2m}}} \left( \epsilon \right) + \frac{1}{\epsilon^{2+\frac{1}{2m}}} \left( \epsilon \right) + \frac{1}{\epsilon^{2+\frac{1}{2m}}} \left( \epsilon \right) + \frac{1}{\epsilon^{2+\frac{1}{2m}}} \left( \epsilon \right) + \frac{1}{\epsilon^{2+\frac{1}{2m}}} \left( \epsilon \right) + \frac{1}{\epsilon^{2+\frac{1}{2m}}} \left( \epsilon \right)
\lesssim \frac{1}{\epsilon^{2+\frac{1}{2m}}} |\ln(\epsilon)|^8.
\]

We have shown
\[
\frac{1}{\epsilon^2} \left\| f^\epsilon - \sum_{k=1}^3 \epsilon^k F_k - \sum_{k=1}^4 \epsilon^k \mathcal{F}_k - \sum_{k=1}^4 \epsilon^k \mathcal{F}_k \right\|_{\infty,0,0} \lesssim \epsilon^{1-\frac{2}{m}} |\ln(\epsilon)|^8.
\]

Therefore, we know
\[
\| f^\epsilon - \epsilon F_1 - \epsilon \mathcal{F}_1 - \mathcal{F} \|_{\infty,0,0} \lesssim \epsilon^{2-\frac{2}{m}} |\ln(\epsilon)|^8.
\]

Since \( \mathcal{F}_1 = \mathcal{F} = 0 \), then we naturally have for \( F = F_1 \).
\[
\| f^\epsilon - \epsilon F\|_{\infty,0,0} \lesssim \epsilon^{2-\frac{2}{m}} |\ln(\epsilon)|^8.
\]

Here \( \frac{3}{2} < m < 3 \), so we may further bound
\[
\| f^\epsilon - \epsilon F\|_{\infty,0,0} \lesssim C(\delta) \epsilon^{\frac{3}{2}-\delta},
\]
for any \( 0 < \delta << 1 \). The exponential decay in time can be justified in a similar fashion using Remark 3.2.15.
Chapter 4

\( \varepsilon \)-Milne Problem with Geometric Correction

4.1 Well-Posedness and Decay

We consider the \( \varepsilon \)-Milne problem with geometric correction for \( g(\eta, v) \) in the domain \((\eta, v) \in [0, L] \times \mathbb{R}^3 \) as

\[
\begin{cases}
\frac{\partial g}{\partial \eta} - \frac{\varepsilon}{R_1 - \varepsilon \eta} \left( v_\phi^2 \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi} \right) - \frac{\varepsilon}{R_2 - \varepsilon \eta} \left( v_\psi^2 \frac{\partial g}{\partial v_\eta} - v_\eta v_\psi \frac{\partial g}{\partial v_\psi} \right) \quad & + \mathcal{L}[g] = S, \\
g(0, v) = h(v) \text{ for } v_\eta > 0, \\
g(L, v) = g(L, R[v]).
\end{cases}
\]

(4.1)

where \( R[v] = (-v_\eta, v_\phi, v_\psi) \) and \( L = \varepsilon^{-\frac{1}{2}} \). For simplicity, we temporarily ignore the dependence of \( \iota_1, \iota_2 \), but our estimates are uniform in these variables.

Since the null space \( \mathcal{N} \) of the operator \( \mathcal{L} \) is spanned by \( \mu^\perp \{1, v_\eta, v_\phi, v_\psi, \frac{|v|^2 - 3}{2} \} = \{e_0, e_1, e_2, e_3, e_4\} \), we can decompose

\[ g := w_g + q_g, \]

(4.2)

where

\[
q_g = \mu^\perp \left( g_{g,0} + g_{g,1} v_\eta + g_{g,2} v_\phi + g_{g,3} v_\psi + g_{g,4} \frac{|v|^2 - 3}{2} \right) = q_{g,0} e_0 + q_{g,1} e_1 + q_{g,2} e_2 + q_{g,3} e_3 + q_{g,4} e_4 \in \mathcal{N},
\]

(4.3)

and

\[ w_g \in \mathcal{N}^\perp, \]

(4.4)

where \( \mathcal{N}^\perp \) is the orthogonal complement of \( \mathcal{N} \) in \( L^2 \). When there is no confusion, we will simply write \( g = w + q \).

Our main goal is to study the well-posedness of \( g \). In addition, we plan to find

\[ \tilde{h}(v) := \sum_{i=0}^4 \tilde{D}_i e_i \in \mathcal{N}, \]

(4.5)

with \( \tilde{D}_1 = 0 \) such that the modified \( \varepsilon \)-Milne problem with geometric correction for \( G(\eta, v) \) in the domain
\((\eta, v) \in [0, L] \times \mathbb{R}^3\) as
\[
\begin{aligned}
&\left\{ v_\eta \frac{\partial G}{\partial \eta} - \frac{\epsilon}{R_1 - \epsilon \eta} \left( v_\eta^2 \frac{\partial G}{\partial v_\eta} - v_\eta v_v \frac{\partial G}{\partial v_v} \right) - \frac{\epsilon}{R_2 - \epsilon \eta} \left( v_\eta^2 \frac{\partial G}{\partial v_\eta} - v_\eta v_v \frac{\partial G}{\partial v_v} \right) \right\} + L[G] = S,
\end{aligned}
\]
\[
G(0, v) = h(v) - \tilde{h}(v) \quad \text{for} \quad v_\eta > 0,
\]
\[
G(L, v) = G(L, \mathbb{R}[b]),
\]
is well-posed, and \(G\) decays exponentially fast as \(\eta\) becomes larger and larger. The estimates and decaying rate should be uniform in \(\epsilon\).

In this section, we introduce some special notation to describe the norms for \((\eta, v) \in [0, L] \times \mathbb{R}^3\). Define the \(L^2\) norms as follows:
\[
|f(\eta)|_2 := \left( \int_{\mathbb{R}^3} |f(\eta, v)|^2 \, dv \right)^{\frac{1}{2}},
\]
\[
\|f\|_2 := \left( \int_0^L \int_{\mathbb{R}^3} |f(\eta, v)|^2 \, dv \, d\eta \right)^{\frac{1}{2}}.
\]
Define the inner product in \(v\)
\[
\langle f, g \rangle (\eta) := \int_{\mathbb{R}^3} f(\eta, v) g(\eta, v) \, dv.
\]
Define the weighted \(L^\infty\) norms as follows:
\[
|f(\eta)|_{\infty, \vartheta, \varphi} := \text{ess sup}_{v \in \mathbb{R}^3} \left( \langle v \rangle^\varphi e^{\vartheta |v|^2} |f(\eta, v)| \right),
\]
\[
\|f\|_{\infty, \vartheta, \varphi} := \text{ess sup} \left( \langle v \rangle^\varphi e^{\vartheta |v|^2} |f(\eta, v)| \right).
\]
Define the mixed \(L^2\) and weighted \(L^\infty\) norm as follows:
\[
\|f\|_{m, \vartheta} := \text{ess sup}_{\eta \in [0, L]} \left( \int_{\mathbb{R}^3} |e^{\vartheta |v|^2} f(\eta, v)|^2 \, dv \right)^{\frac{1}{2}}.
\]
Here, we require \(0 \leq \vartheta < \frac{1}{4}\) and \(\varphi > 3\).

Since the boundary data \(h(v)\) is only defined on \(v_\eta > 0\), we naturally extend above definitions to this half-domain as follows:
\[
|h|_2 := \left( \int_{v_\eta > 0} |h(v)|^2 \, dv \right)^{\frac{1}{2}},
\]
\[
|h|_{\infty, \vartheta, \varphi} := \sup_{v_\eta > 0} \left( \langle v \rangle^\varphi e^{\vartheta |v|^2} |h(v)| \right).
\]
Throughout this section, we assume
\[
|h|_{\infty, \vartheta, \varphi} \lesssim 1, \quad \|e^{K\eta} S\|_{\infty, \vartheta, \varphi} \lesssim 1,
\]
for some constant \(K > 0\) uniform in \(\epsilon\).

**Lemma 4.1.1.** For \(f \in \mathcal{K}^\perp\), we have
\[
|\mathcal{L}[f]|_2 \lesssim |\nu f|_2.
\]

**Proof.** Based on [28 Section 3], we know \(\mathcal{L} = \nu I - K\), where
\[
|K[f]|_2 \lesssim |f|_2,
\]
so \(\mathcal{L}\) estimate naturally follows. 

The existence of uniqueness of \(g\) and \(G\) follow from a standard iteration argument as in [29] and [31], so we will omit the proof here and focus on the a priori estimates.
4.1.1 $L^2$ Estimates

$S \in \mathcal{N}^\perp$ Case

Denote

\[ G_1(\eta) := -\frac{\epsilon}{R_1 - \epsilon\eta}, \quad G_2(\eta) := -\frac{\epsilon}{R_2 - \epsilon\eta}. \]  

(4.18)

and

\[ G(\eta) := G_1(\eta) + G_2(\eta). \]  

(4.19)

Let $W_i(\eta)$ satisfy

\[ G_i = -\frac{dW_i}{d\eta}, \quad W_i(0) = 0 \quad \text{for} \quad i = 1, 2. \]  

(4.20)

Hence, it is easy to check that

\[ W_i(\eta) = \ln \left( \frac{R_i}{R_i - \epsilon\eta} \right). \]  

(4.21)

Denote

\[ W(\eta) := W_1(\eta) + W_2(\eta). \]  

(4.22)

**Remark 4.1.2.** We know for $\epsilon << 1$, $\epsilon\eta \leq \epsilon L = \epsilon \frac{\epsilon}{2} << 1$, which implies $W(\eta) \sim 0$ and further $e^{W(\eta)} \sim 1$.

We will estimate $g = w + q$ separately and divide it into several steps.

**Lemma 4.1.3** (orthogonality estimate). Assume $S \in \mathcal{N}^\perp$. We have

\[ \langle v_{\eta} e_j, g \rangle (\eta) = 0 \quad \text{for} \quad j = 0, 2, 3, 4 \quad \text{and} \quad \eta \in [0, L]. \]  

(4.23)

**Proof.** Multiplying $e_j$ for $j = 0, 2, 3, 4$ on both sides of (4.1) and integrating over $v \in \mathbb{R}^3$, we have

\[ \frac{d}{d\eta} \langle v_{\eta} e_j, g \rangle + G_1 \left( v_{\eta}^2 \frac{\partial g}{\partial v_{\eta}} - v_{\eta} v_{\phi} \frac{\partial g}{\partial v_{\phi}}, e_j \right) + G_2 \left( v_{\eta}^2 \frac{\partial g}{\partial v_{\eta}} - v_{\eta} v_{\psi} \frac{\partial g}{\partial v_{\psi}}, e_j \right) = -\langle L[g], e_j \rangle + \langle S, e_j \rangle. \]  

(4.24)

Since $L$ is self-adjoint and $e_j \in \mathcal{N}$ as well as $S \in \mathcal{N}^\perp$, we have

\[ \langle L[g], e_j \rangle = \langle L[e_j], g \rangle = 0, \quad \langle S, e_j \rangle = 0. \]  

(4.25)

An integration by parts implies

\[ G_1 \left( v_{\eta}^2 \frac{\partial g}{\partial v_{\eta}} - v_{\eta} v_{\phi} \frac{\partial g}{\partial v_{\phi}}, e_j \right) = -G_1 \left( \frac{\partial}{\partial v_{\eta}} (e_j v_{\eta}^2) - \frac{\partial}{\partial v_{\phi}} (e_j v_{\eta} v_{\phi}), g \right) = C_1 G_1 (e_j v_{\eta}, g), \]  

(4.26)

\[ G_2 \left( v_{\eta}^2 \frac{\partial g}{\partial v_{\eta}} - v_{\eta} v_{\psi} \frac{\partial g}{\partial v_{\psi}}, e_j \right) = -G_2 \left( \frac{\partial}{\partial v_{\eta}} (e_j v_{\eta}^2) - \frac{\partial}{\partial v_{\psi}} (e_j v_{\eta} v_{\psi}), g \right) = C_1 G_2 (e_j v_{\eta}, g). \]  

(4.27)

where $C_1$ and $C_2$ are constants. Summarizing all above, we know that (4.24) is

\[ \frac{d}{d\eta} \langle e_j v_{\eta}, g \rangle = (C_1 G_1 + C_2 G_2) \langle e_j v_{\eta}, g \rangle. \]  

(4.28)

Considering the reflexive boundary which implies $\langle e_j v_{\eta}, g \rangle (L) = 0$, we have for any $\eta \in [0, L]$,

\[ \langle e_j v_{\eta}, g \rangle (\eta) = 0. \]  

(4.29)
Remark 4.1.4. Note that $\langle v_\eta, e_1, g \rangle (\eta)$ is not necessarily zero.

Lemma 4.1.5 ($L^2$ estimates of $L^2$). Assume (4.15) holds and $S \in \mathcal{N}^\perp$. We have

\[
\left\| v^{\frac{1}{2}} w \right\|_2 \lesssim 1.
\] (4.30)

Proof. Multiplying $g$ on both sides of (4.1) and integrating over $v \in \mathbb{R}^3$, we have

\[
\frac{1}{2} \frac{d}{d\eta} \langle v_\eta g, g \rangle + G_1 \left\langle v_\eta^2 \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi}, g \right\rangle + G_2 \left\langle v_\phi^2 \frac{\partial g}{\partial v_\phi} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi}, g \right\rangle = -\langle g, \mathcal{L}[g] \rangle + \langle g, S \rangle. \tag{4.31}
\]

An integration by parts implies

\[
\left\langle v_\eta^2 \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi}, g \right\rangle = \frac{1}{2} \left\langle v_\eta^2, \frac{\partial (g^2)}{\partial v_\eta} \right\rangle - \frac{1}{2} \left\langle v_\eta v_\phi, \frac{\partial (g^2)}{\partial v_\phi} \right\rangle = \frac{1}{2} \langle v_\eta g, g \rangle, \tag{4.32}
\]

\[
\left\langle v_\phi^2 \frac{\partial g}{\partial v_\phi} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi}, g \right\rangle = \frac{1}{2} \left\langle v_\phi^2, \frac{\partial (g^2)}{\partial v_\phi} \right\rangle - \frac{1}{2} \left\langle v_\eta v_\phi, \frac{\partial (g^2)}{\partial v_\phi} \right\rangle = \frac{1}{2} \langle v_\eta g, g \rangle. \tag{4.33}
\]

Also, since $\mathcal{L}$ is a self-adjoint operator with null space $\mathcal{N}$, we get

\[
\langle g, \mathcal{L}[g] \rangle = \langle g, \mathcal{L}[g] \rangle + \langle w, \mathcal{L}[g] \rangle + \langle g, \mathcal{L}[w] \rangle + \langle w, \mathcal{L}[w] \rangle = \langle w, \mathcal{L}[w] \rangle. \tag{4.34}
\]

Therefore, we simplify (4.31) to obtain

\[
\frac{1}{2} \frac{d}{d\eta} \langle v_\eta g, g \rangle + \frac{1}{2} G \langle v_\eta g, g \rangle = -\langle w, \mathcal{L}[w] \rangle + \langle w, S \rangle. \tag{4.35}
\]

Define

\[
\alpha(\eta) = \frac{1}{2} \langle v_\eta g, g \rangle (\eta). \tag{4.36}
\]

Then (4.35) may be rewritten as

\[
\frac{d\alpha}{d\eta} + G\alpha = -\langle w, \mathcal{L}[w] \rangle + \langle w, S \rangle. \tag{4.37}
\]

Then regarding the above as an ODE and solve it in $[\eta, L]$, we have

\[
\alpha(\eta) = \alpha(L) \exp \left( \int_{\eta}^{L} G(y) dy \right) + \int_{\eta}^{L} \exp \left( - \int_{y}^{L} G(z) dz \right) \langle w, \mathcal{L}[w] \rangle (y) + \langle w, S \rangle (y) dy. \tag{4.38}
\]

Note the fact that $\alpha(L) = 0$ due to the reflexive boundary condition. Also, $\langle w, \mathcal{L}[w] \rangle (\eta) \geq \left\| v^{\frac{1}{2}} w(y) \right\|_2^2$ due to coercivity. Hence, (4.38) implies that

\[
\alpha(\eta) \geq \int_{\eta}^{L} \exp \left( - \int_{y}^{L} G(z) dz \right) \left( \left\| v^{\frac{1}{2}} w(y) \right\|_2^2 + \langle w, S \rangle (y) \right) dy. \tag{4.39}
\]

In particular, taking $\eta = 0$, we have

\[
\alpha(0) \geq \int_{0}^{L} \exp \left( - \int_{0}^{y} G(z) dz \right) \left( \left\| v^{\frac{1}{2}} w(y) \right\|_2^2 + \langle w, S \rangle (y) \right) dy = \int_{0}^{L} e^{W(y)} \left( \left\| v^{\frac{1}{2}} w(y) \right\|_2^2 + \langle w, S \rangle (y) \right) dy, \tag{4.40}
\]

which yields

\[
\int_{0}^{L} e^{W(y)} \left\| v^{\frac{1}{2}} w(y) \right\|_2^2 dy \leq \alpha(0) - \int_{0}^{L} e^{W(y)} \langle w, S \rangle (y) dy. \tag{4.41}
\]
On the other hand, (4.15) implies
\[ \alpha(0) = \frac{1}{2} \langle v_\eta g, v \rangle (0) = \frac{1}{2} \int_{v_\eta > 0} v_\eta g^2(0, v) dv + \frac{1}{2} \int_{v_\eta < 0} v_\eta g^2(0, v) dv \leq \frac{1}{2} \int_{v_\eta > 0} v_\eta g^2(0, v) dv \]
\[ = \frac{1}{2} \int_{v_\eta > 0} v_\eta h^2(v) dv \lesssim 1. \]

Combined (4.41) and (4.42), we obtain
\[ \int_0^L e^{W(y)} \left| \nu^2 w_0(y) \right|^2 dy \lesssim 1 + \int_0^L e^{W(y)} \langle w, S \rangle (y) dy. \]

Using (4.20) and Remark 4.1.2 as well as Hölder’s inequality and Cauchy’s inequality, we get
\[ \int_0^L \left| \nu^2 w_0(\eta) \right|^2 d\eta \lesssim 1 + \int_0^L \left| \langle w, S \rangle (\eta) \right| d\eta \lesssim 1 + \int_0^L \left| \nu^2 w_0(\eta) \right| \left| \nu^{\frac{n}{2}} S(\eta) \right|^2 d\eta \]
\[ \lesssim 1 + \delta \int_0^L \left| \nu^2 w_0(\eta) \right|^2 d\eta + \delta^{-1} \int_0^L \left| \nu^{\frac{n}{2}} S(\eta) \right|^2 d\eta. \]

Therefore, for sufficiently small \( \delta \), we absorb \( \delta \int_0^L \left| \nu^2 w_0(\eta) \right|^2 d\eta \) into LHS and use (4.15) to obtain
\[ \int_0^L \left| \nu^2 w_0(\eta) \right|^2 d\eta \lesssim 1 + \int_0^L \left| \nu^{\frac{n}{2}} S(\eta) \right|^2 d\eta \lesssim 1. \]

\[ \square \]

**Remark 4.1.6.** Based on the proof of Lemma 4.1.5, (4.40) actually implies
\[ \alpha(0) \gtrsim \int_0^L e^{W(y)} \langle w, S \rangle (y) dy \gtrsim \int_0^L \left| \langle w, S \rangle (y) \right| dy. \]

Hence, with (4.42) holds, \( \alpha(0) \) actually has both upper and lower bounds, i.e.
\[ \int_0^L \left| \langle w, S \rangle (y) \right| dy \lesssim \alpha(0) \lesssim 1. \]

**Lemma 4.1.7** (point-wise estimate of \( q \)). Assume (4.15) holds and \( S \in \mathcal{N}^\perp \). We have \( q_j(\eta) = 0 \) and
\[ |q_j(\eta)| \lesssim 1 + \eta + \left| \nu^2 w_0(\eta) \right|_2 \quad \text{for } j = 0, 2, 3, 4. \]

**Proof.** Multiplying \( e_0 \) on both sides of (4.1) and integrating over \( v \in \mathbb{R}^3 \), we have
\[ \frac{d}{d\eta} \langle v_\eta e_0, g \rangle + G_1 \left( v_\phi v_\eta \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi}, e_0 \right) + G_2 \left( v_\phi \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi}, e_0 \right) = - \langle \mathcal{L}[g], e_0 \rangle + \langle S, e_0 \rangle. \]

Since \( \mathcal{L} \) is self-adjoint and \( e_0 \in \mathcal{N} \) as well as \( S \in \mathcal{N}^\perp \), we have
\[ \langle \mathcal{L}[g], e_0 \rangle = \langle \mathcal{L}[e_0], g \rangle = 0, \quad \langle S, e_0 \rangle = 0. \]

An integration by parts implies
\[ G_1 \left( v_\phi \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi}, e_0 \right) = -G_1 \left( \frac{\partial}{\partial v_\eta} (e_0 v_\phi^2) - \frac{\partial}{\partial v_\phi} (e_0 v_\eta v_\phi), g \right) = G_1 \langle e_0 v_\eta, g \rangle, \]
\[ G_2 \left( v_\phi \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi}, e_0 \right) = -G_2 \left( \frac{\partial}{\partial v_\eta} (e_0 v_\phi^2) - \frac{\partial}{\partial v_\phi} (e_0 v_\eta v_\phi), g \right) = G_2 \langle e_0 v_\phi, g \rangle. \]
Summarizing all above, we know that (4.49) is
\[
\frac{d}{d\eta} \langle e_0 v_\eta, g \rangle = -G \langle e_0 v_\eta, g \rangle.
\] (4.53)
Since \( \langle e_0 v_\eta, g \rangle = \langle e_1, g \rangle = q_1 \), (4.53) is actually
\[
\frac{d q_1}{d\eta} = -G q_1.
\] (4.54)
Considering the reflexive boundary which implies \( q_1(L) = 0 \), we have for any \( \eta \in [0, L] \),
\[
q_1(\eta) = 0.
\] (4.55)
Multiplying \( v_\eta e_j \) with \( j = 0, 2, 3, 4 \) on both sides of (4.1) and integrating over \( v \in \mathbb{R}^3 \), we obtain
\[
\frac{d}{d\eta} \langle v_\eta^2 e_j, g \rangle + G_1 \left\langle v_\eta e_j, v_\phi^2 \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi} \right\rangle + G_2 \left\langle v_\eta e_j, v_\phi^2 \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi} \right\rangle = -\langle v_\eta e_j, L[g] \rangle + \langle v_\eta e_j, S \rangle.
\] (4.56)
Define
\[
\beta_j(\eta) = \langle v_\eta^2 e_j, q \rangle(\eta),
\] (4.57)
\[
\beta(\eta) = \begin{pmatrix} \beta_0(\eta), \beta_2(\eta), \beta_3(\eta), \beta_4(\eta) \end{pmatrix}^T.
\] (4.58)
For \( j = 0, 2, 3, 4 \),
\[
\langle v_\eta^2 e_j, g \rangle = \langle v_\eta^2 e_j, q \rangle + \langle v_\eta^2 e_j, w \rangle = \beta_j + \langle v_\eta^2 e_j, w \rangle.
\] (4.59)
Using integration by parts, we have
\[
G_1 \left\langle v_\eta e_j, v_\phi^2 \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi} \right\rangle = -G_1 \left\langle \frac{\partial}{\partial v_\eta} (v_\eta v_\phi^2 e_j) - \frac{\partial}{\partial v_\phi} (v_\eta^2 v_\phi e_j), g \right\rangle,
\] (4.60)
\[
G_2 \left\langle v_\eta e_j, v_\phi^2 \frac{\partial g}{\partial v_\eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi} \right\rangle = -G_2 \left\langle \frac{\partial}{\partial v_\eta} (v_\eta v_\phi^2 e_j) - \frac{\partial}{\partial v_\phi} (v_\eta^2 v_\phi e_j), g \right\rangle.
\] (4.61)
Considering \( q = w + q \) and summarizing the above, we can simplify (4.56) into
\[
\frac{d}{d\eta} \langle v_\eta^2 e_j, g \rangle = G_1 \left\langle \frac{\partial}{\partial v_\eta} (v_\eta v_\phi^2 e_j) - \frac{\partial}{\partial v_\phi} (v_\eta^2 v_\phi e_j), g \right\rangle + G_2 \left\langle \frac{\partial}{\partial v_\eta} (v_\eta v_\phi^2 e_j) - \frac{\partial}{\partial v_\phi} (v_\eta^2 v_\phi e_j), g \right\rangle
\] (4.62)
which further implies
\[
\beta_j = G_1 \left\langle \frac{\partial}{\partial v_\eta} (v_\eta v_\phi^2 e_j) - \frac{\partial}{\partial v_\phi} (v_\eta^2 v_\phi e_j), q + w \right\rangle + G_2 \left\langle \frac{\partial}{\partial v_\eta} (v_\eta v_\phi^2 e_j) - \frac{\partial}{\partial v_\phi} (v_\eta^2 v_\phi e_j), q + w \right\rangle
\] (4.63)
Then we can write
\[
\left\langle \frac{\partial}{\partial v_\eta} (v_\eta v_\phi^2 e_j) - \frac{\partial}{\partial v_\phi} (v_\eta^2 v_\phi e_j), q \right\rangle(\eta) = \sum_i B^{(1)}_{j_i} q_i(\eta),
\] (4.64)
\[
\left\langle \frac{\partial}{\partial v_\eta} (v_\eta v_\phi^2 e_j) - \frac{\partial}{\partial v_\phi} (v_\eta^2 v_\phi e_j), q \right\rangle(\eta) = \sum_i B^{(2)}_{j_i} q_i(\eta).
\] (4.65)
for \(i, j = 0, 2, 3, 4\), where there is no \(q_1\) contribution since \(q_1 = 0\). Here \(B^{(1)}\) and \(B^{(2)}\) are \(4 \times 4\) constant matrices defined by

\[
B^{(1)}_{ji} = \left\langle \frac{\partial}{\partial v_\eta} (v_\eta v_\phi^2 e_j) - \frac{\partial}{\partial v_\phi} (v_\eta^2 v_\phi e_j), e_i \right\rangle,
\]
\[
B^{(2)}_{ji} = \left\langle \frac{\partial}{\partial v_\eta} (v_\eta v_\phi^2 e_j) - \frac{\partial}{\partial v_\phi} (v_\eta^2 v_\phi e_j), e_i \right\rangle.
\]  

(4.66)  

(4.67)

Moreover, we may rewrite

\[
\beta_j(\eta) = \sum_k A_{jk} q_k(\eta),
\]

for \(k, j = 0, 2, 3, 4\), where \(A\) is an invertible \(4 \times 4\) constant matrix defined by

\[
A_{jk} = \left\langle v_\phi^2 e_j, e_k \right\rangle.
\]

(4.69)

Thus, we can express back

\[
(q_0, q_2, q_3, q_4)^T = A^{-1}(\beta_0, \beta_2, \beta_3, \beta_4)^T.
\]

(4.70)

Hence, (4.63) can be rewritten in vector form as

\[
\frac{d\beta}{d\eta} = \left((G_1 B^{(1)} + G_2 B^{(2)}) A^{-1}\right) \beta + D + E - \frac{dF}{dy} \eta
\]

(4.71)

where the four-vector \(D\), \(E\) and \(F\) are defined for \(j = 0, 2, 3, 4\)

\[
D_j = G_1 \left\langle \frac{\partial}{\partial v_\eta} (v_\eta v_\phi^2 e_j) - \frac{\partial}{\partial v_\phi} (v_\eta^2 v_\phi e_j), w \right\rangle + G_2 \left\langle \frac{\partial}{\partial v_\eta} (v_\eta v_\phi^2 e_j) - \frac{\partial}{\partial v_\phi} (v_\eta^2 v_\phi e_j), w \right\rangle,
\]

\[
E_j = - \left\langle v_\phi e_j, [w] \right\rangle + \left\langle v_\phi e_j, S \right\rangle,
\]

\[
F_j = \left\langle v_\phi^2 e_j, w \right\rangle.
\]

(4.72)

(4.73)

(4.74)

(4.75)

(4.76)
and \( Z \) is a four-vector satisfying

\[
Z = D + E - (G_1 B^{(1)} + G_2 B^{(2)}) A^{-1} F.
\] (4.77)

Hence, considering \( A, B^{(1)}, B^{(2)} \) are all constant matrices and Remark 4.1.2, we can directly estimate to get

\[
|\beta_j(\eta)| \lesssim |\theta_j| + |F_j(\eta)| + \int_0^\infty |Z_j(y)| \, dy \quad \text{for} \quad j = 0, 2, 3, 4.
\] (4.78)

Using Hölder’s inequality and Lemma 4.1.1, we have

\[
|D_j(\eta)| \lesssim |G_1| \left| \left\langle \frac{\partial}{\partial v_\eta} (v_\eta v_\eta^2 e_j) - \frac{\partial}{\partial v_\psi} (v_\psi v_\psi^2 e_j), w \right\rangle (\eta) \right| + |G_2| \left| \left\langle \frac{\partial}{\partial v_\eta} (v_\eta v_\eta^2 e_j) - \frac{\partial}{\partial v_\psi} (v_\psi v_\psi^2 e_j), w \right\rangle (\eta) \right|
\] (4.79)

\[
\lesssim \epsilon \left| \left\langle \nu^\frac{3}{2} \left( \frac{\partial}{\partial v_\eta} (v_\eta v_\eta^2 e_j) - \frac{\partial}{\partial v_\psi} (v_\psi v_\psi^2 e_j) \right), \nu^\frac{3}{2} w(\eta) \right\rangle \right|_2 + \epsilon \left| \left\langle \nu^\frac{3}{2} \left( \frac{\partial}{\partial v_\eta} (v_\eta v_\eta^2 e_j) - \frac{\partial}{\partial v_\psi} (v_\psi v_\psi^2 e_j) \right), \nu^\frac{3}{2} w(\eta) \right\rangle \right|_2
\]

\[
\lesssim \epsilon \left| \nu^\frac{3}{2} w(\eta) \right|_2,
\]

and

\[
|E_j(\eta)| \lesssim |\langle v_\eta e_j, \mathcal{L}[w] \rangle (\eta)| + |\langle v_\eta e_j, S \rangle (\eta)| = |\langle \mathcal{L}[v_\eta e_j], w \rangle (\eta)| + |\langle v_\eta e_j, S \rangle (\eta)|
\] (4.80)

\[
\lesssim \left| \nu^\frac{3}{2} \mathcal{L}[v_\eta e_j] \right|_2 \left| \nu^\frac{3}{2} w(\eta) \right|_2 + \left| \nu^\frac{3}{2} v_\eta e_j \right|_2 \left| \nu^\frac{3}{2} S(\eta) \right|_2 \lesssim \left| \nu^\frac{3}{2} w(\eta) \right|_2 + \left| \nu^\frac{3}{2} S(\eta) \right|_2,
\]

as well as

\[
|F_j(\eta)| \lesssim |\langle v_\eta^2 e_j, w \rangle (\eta)| \lesssim \left| \nu^\frac{3}{2} v_\eta^2 e_j \right|_2 \left| \nu^\frac{3}{2} w \right|_2 \lesssim \left| \nu^\frac{3}{2} w(\eta) \right|_2.
\] (4.81)

Inserting (4.79), (4.80) and (4.81) into (4.77), we obtain

\[
|Z_j| \lesssim \left| \nu^\frac{3}{2} w(\eta) \right|_2 + \left| \nu^\frac{3}{2} S(\eta) \right|_2 \lesssim 1 + \left| \nu^\frac{3}{2} w(\eta) \right|_2.
\] (4.82)

On the other hand, for \( \theta_j \), using Hölder’s inequality, we have

\[
|\theta_j| = \left| \langle v_\eta^\frac{3}{2} e_j, g \rangle (0) \right| \lesssim \left| \left| v_\eta^\frac{3}{2} e_j \right|_2 \right| \left| \left| \nu^\frac{3}{2} g(0) \right|_2 \right| \lesssim \left| \left| v_\eta^\frac{3}{2} g(0) \right|_2 \right|,
\] (4.83)

where

\[
\left| \left| v_\eta^\frac{3}{2} g(0) \right|_2 \right| = \int_{v_\eta > 0} v_\eta h^2(v^2)dv - \int_{v_\eta < 0} v_\eta g^2(0, v)dv.
\] (4.84)

Using (4.30) and (4.47), we have

\[
\int_{v_\eta > 0} v_\eta h^2(v)dv + \int_{v_\eta < 0} v_\eta g^2(0, v)dv = 2a(0) \gtrsim \int_0^L |\langle w, S \rangle (y)| \, dy,
\] (4.85)

which implies

\[
- \int_{v_\eta < 0} v_\eta g^2(0, v)dv \lesssim \int_{v_\eta > 0} v_\eta h^2(v)dv - \int_0^L |\langle w, S \rangle (y)| \, dy.
\] (4.86)
Hence, inserting (4.86) into (4.84) and further (4.83), applying Hölder’s inequality and Cauchy’s inequality, we have
\[ |\theta_j| \lesssim \int_0^L |(w, S)(y)| \, dy + \int_{v_0 > 0} v_0 h^2(v) \, dv \lesssim 1 + \|v^+ w\|_2 \|v^- S\|_2 \lesssim 1 + \|v^+ w\|_2. \] (4.87)

In conclusion, inserting (4.82), (4.79) and (4.87) into (4.78), we have
\[ |\beta_j(\eta)| \lesssim 1 + \|v^+ w(\eta)\|_2 + \int_0^\eta \left( 1 + \|v^+ w(y)\|_2 \right) \, dy \quad \text{for } j = 0, 2, 3, 4, \] (4.88)
which, using (4.70) and Lemma 4.1.5 further implies
\[ |q_j(\eta)| \lesssim 1 + \|v^+ w(\eta)\|_2 + \int_0^\eta \|v^+ w(y)\|_2 \, dy \quad \text{for } j = 0, 2, 3, 4. \] (4.89)

An application of Hölder’s inequality, Cauchy’s inequality and Lemma 4.1.5 lead to
\[ |q_j(\eta)| \lesssim 1 + \|v^+ w(\eta)\|_2 + \eta \|v^+ w\|_2 \lesssim 1 + \|v^+ w(\eta)\|_2 \quad \text{for } j = 0, 2, 3, 4. \] (4.90)

**Remark 4.1.8.** Using a standard iteration argument, Lemma 4.1.3 and Lemma 4.1.7 justify the well-posedness of solution \( g = w + q \). However, the estimates in Lemma 4.1.7 are not uniform in \( \eta \), so we need a stronger version.

**Lemma 4.1.9** \((L^2 \text{ decay of } w)\). Assume (4.16) holds and \( S \in \mathcal{N}^1 \). There exists \( 0 < K_0 < K \) such that
\[ \left\| e^{K_0 \eta} v^+ w \right\|_2 \lesssim 1. \] (4.91)

**Proof.** Multiplying \( e^{2K_0 \eta} g \) on both sides of (4.1) and integrating over \( v \in \mathbb{R}^3 \), we obtain
\[
\frac{1}{2} \frac{d}{d\eta} \langle v_0 g, e^{2K_0 \eta} g \rangle + G_1 \left\langle v^2 \frac{\partial g}{\partial v_0} - v_0 v_\phi \frac{\partial g}{\partial v_\phi} e^{2K_0 \eta} g \right\rangle + G_2 \left\langle v^2 \frac{\partial g}{\partial \eta} - v_0 v_\phi \frac{\partial g}{\partial v_\phi} e^{2K_0 \eta} g \right\rangle = K_0 e^{2K_0 \eta} \langle v_0 g, g \rangle - e^{2K_0 \eta} \langle g, L[g] \rangle + e^{2K_0 \eta} \langle S, g \rangle.
\] (4.92)

We simplify each term here. The orthogonal properties in Lemma 4.1.3 implies
\[ \langle v_0 e_j, g \rangle (\eta) = \langle v_0 e_j, w \rangle (\eta) + \langle v_0 e_j, q \rangle (\eta) = 0 \quad \text{for } j = 0, 2, 3, 4. \] (4.93)

Based on Lemma 4.1.7 \( q_1 = 0 \). Combined with oddness, we know
\[ \langle v_0 e_j, q \rangle (\eta) = \sum_{k=0}^{4} q_k \langle v_0 e_j, e_k \rangle (\eta) = 0. \] (4.94)

Inserting (4.94) into (4.93), we obtain
\[ \langle v_0 e_j, w \rangle (\eta) = 0. \] (4.95)

Still by \( q_1 = 0 \), we have
\[ \langle v_0 q, w \rangle (\eta) = \sum_{j=0}^{4} q_j \langle v_0 e_j, w \rangle = 0, \] (4.96)
and also by oddness
\[ \langle v_0 q, q \rangle (\eta) = 0. \] (4.97)
Therefore, we deduce that
\[ \langle v_\eta g, g \rangle (\eta) = \langle v_\eta w, w \rangle (\eta) + 2 \langle v_\eta q, q \rangle (\eta) = \langle v_\eta w, w \rangle (\eta). \] (4.98)

On the other hand, (4.34) yields
\[ \langle v_\eta g, g \rangle (\eta) = \langle v_\eta w, w \rangle (\eta). \] (4.99)

Similar to the proof of Lemma 4.1.5, an integration by parts and (4.98) imply
\[ \langle v_\eta^2 \phi \partial g / \partial v_\eta - v_\eta v_\phi \partial g / \partial v_\phi, e^{2K_0\eta} g \rangle = \frac{1}{2} \langle v_\eta^2 \phi, \partial (e^{2K_0\eta} g^2) / \partial v_\eta \rangle - \frac{1}{2} \langle v_\eta v_\phi, \partial (e^{2K_0\eta} g^2) / \partial v_\phi \rangle \] (4.100)
\[ = \frac{1}{2} \langle v_\eta g, e^{2K_0\eta} g \rangle = \frac{1}{2} \langle v_\eta w, e^{2K_0\eta} w \rangle, \]
\[ \langle v_\eta^2 \phi \partial g / \partial v_\eta - v_\eta v_\phi \partial g / \partial v_\phi, e^{2K_0\eta} G \rangle = \frac{1}{2} \langle v_\eta^2 \phi, \partial (e^{2K_0\eta} g^2) / \partial v_\eta \rangle - \frac{1}{2} \langle v_\eta v_\phi, \partial (e^{2K_0\eta} g^2) / \partial v_\phi \rangle \] (4.101)
\[ = \frac{1}{2} \langle v_\eta g, e^{2K_0\eta} g \rangle = \frac{1}{2} \langle v_\eta w, e^{2K_0\eta} w \rangle. \]

Also,
\[ \langle S, g \rangle = \langle S, w \rangle. \] (4.102)

Summarizing all above, (4.92) is actually
\[ \frac{1}{2} \frac{d}{d\eta} \langle v_\eta^2 w, e^{2K_0\eta} w \rangle + \frac{1}{2} G(\eta) \langle v_\eta^2 w, e^{2K_0\eta} w \rangle = K_0 e^{2K_0\eta} \langle v_\eta w, w \rangle - e^{2K_0\eta} \langle \mathcal{L}[w], w \rangle + e^{2K_0\eta} \langle S, w \rangle. \] (4.103)

Since
\[ \langle \mathcal{L}[w], w \rangle \gtrsim \left| \nu^{1/2} w \right|^2, \] (4.104)
for \( K_0 \) sufficiently small, we have
\[ \langle \mathcal{L}[w], w \rangle - K_0 \langle v_\eta w, w \rangle \gtrsim \left| \nu^{1/2} w \right|^2. \] (4.105)

Then by a similar argument as in the proof of Lemma 4.1.5, we can show that
\[ \int_0^L e^{2K_0\eta} \left| \nu^{1/2} w(\eta) \right|^2 d\eta \lesssim 1. \] (4.106)

**Lemma 4.1.10** \((q - q_L)\) estimate. Assume (4.10) holds and \( S \in \mathcal{N}^\perp \). There exists
\[ q_L = \sum_{k=0}^4 q_{k,L} e_k \in \mathcal{N}, \] (4.107)
satisfying
\[ |q_{k,L}| \lesssim 1 \quad \text{for} \quad k = 0, 1, 2, 3, 4, \] (4.108)
and
\[ \|q - q_L\|_2 \lesssim 1. \] (4.109)
Proof. Recall (4.75)

\[ \beta(\eta) = \exp \left( - \left( W_1(\eta)B^{(1)} + W_2(\eta)B^{(2)} \right) A^{-1} \right) \theta - F(\eta) \quad (4.110) \]

\[ + \int_0^\eta \exp \left( \left( W_1(y) - W_1(\eta) \right) B^{(1)} A^{-1} + \left( W_2(y) - W_2(\eta) \right) B^{(2)} A^{-1} \right) Z(y) dy. \]

Define

\[ \beta_L := \exp \left( - \left( W_1(L)B^{(1)} + W_2(L)B^{(2)} \right) A^{-1} \right) \theta \quad (4.111) \]

\[ + \int_0^L \exp \left( \left( W_1(y) - W_1(L) \right) B^{(1)} A^{-1} + \left( W_2(y) - W_2(L) \right) B^{(2)} A^{-1} \right) Z(y) dy. \]

Here

\[ \beta_L = \left( \beta_{0,L}, \beta_{2,L}, \beta_{3,L}, \beta_{4,L} \right)^T, \quad (4.112) \]

is a four-vector. Based on (4.78),

\[ |\beta_j| \lesssim |\theta_j| + \int_0^L |Z_j(y)| dy \quad \text{for} \quad j = 0, 2, 3, 4. \quad (4.113) \]

Inserting (4.82), (4.79) and (4.87) into (4.113), we have

\[ |\beta_j| \lesssim 1 + \|\nu^{\frac{1}{2}} w\|_2 + \int_0^L \left( \|\nu^{\frac{1}{2}} S(y)\|_2 + \|\nu^{\frac{1}{2}} w(y)\|_2 \right) dy \quad \text{for} \quad j = 0, 2, 3, 4. \quad (4.114) \]

Using (4.15), we know

\[ \int_0^L \|\nu^{-\frac{1}{2}} S(y)\|_2 \, dy \lesssim 1. \quad (4.115) \]

Applying Hölder’s inequality and Lemma 4.1.9 we obtain

\[ \int_0^L \|\nu^{\frac{1}{2}} w(y)\|_2 \, dy \lesssim \|e^{-K_0\eta}\|_2 \|e^{K_0\eta}\nu^{\frac{1}{2}} w\|_2 \lesssim 1. \quad (4.116) \]

Inserting (4.115) and (4.116) into (4.114) and using Lemma 4.1.5 we have

\[ |\beta_j| \lesssim 1 \quad \text{for} \quad j = 0, 2, 3, 4. \quad (4.117) \]

Then by (4.70), define

\[ \left( q_{0,L}, q_{2,L}, q_{3,L}, q_{4,L} \right)^T := A^{-1} \left( \beta_{0,L}, \beta_{2,L}, \beta_{3,L}, \beta_{4,L} \right)^T. \quad (4.118) \]

We have

\[ |q_j| \lesssim 1. \quad (4.119) \]

Let \( q_{1,L} = 0 \). Then we know \( q_L \) is always well-defined and

\[ |q_k| \lesssim 1 \quad \text{for} \quad k = 0, 1, 2, 3, 4. \quad (4.120) \]

Then we investigate the estimate \( q - q_L \). Denote

\[ \Xi(\eta) = \left( W_1(\eta)B^{(1)} + W_2(\eta)B^{(2)} \right) A^{-1}. \quad (4.121) \]
Similarly, using Hölder’s inequality and Lemma 4.1.9, we have
\[
\beta(\eta) - \beta_L = e^{-\Xi(\eta)}\left( e^{\Xi(\eta)} \beta(\eta) \right) - e^{-\Xi(L)} \left( e^{\Xi(L)} \beta_L \right) \\
= e^{-\Xi(\eta)}\left( e^{\Xi(\eta)} \beta(\eta) - e^{\Xi(L)} \beta_L \right) + (e^{-\Xi(\eta)} - e^{-\Xi(L)}) \left( e^{\Xi(L)} \beta_L \right) := \Delta_1(\eta) + \Delta_2(\eta).
\] (4.122)

Here, using Remark 4.1.2, we have
\[
\int_0^L \Delta_1^2(\eta)d\eta \lesssim \int_0^L \left( e^{\Xi(\eta)} \beta(\eta) - e^{\Xi(L)} \beta_L \right)^2 d\eta, 
\] (4.123)
where
\[
e^{\Xi(\eta)}\Omega(\eta) - e^{\Xi(L)}\Omega_L = -e^{\Xi(\eta)}F(\eta) + \int_{\eta}^L e^{\Xi(y)Z(y)}dy.
\] (4.124)

Note that
\[
Z = D + E - (G_1B^{(1)} + G_2B^{(2)})A^{-1}F,
\] (4.125)
where \(D, E, F\) are defined in (4.72), and due to Remark 4.1.2,
\[
\int_0^L \Delta_2^2(\eta)d\eta \lesssim \int_0^L F^2(\eta)d\eta + \int_0^L \left( \int_{\eta}^L D(y) \right)^2 d\eta + \int_0^L \left( \int_{\eta}^L E(y) \right)^2 d\eta + \epsilon \int_0^L \left( \int_{\eta}^L F(y) \right)^2 d\eta.
\] (4.126)

We need to estimate each term. In the following, let \(j = 0, 2, 3, 4\), using Hölder’s inequality and Lemma 4.1.9 we have
\[
\int_0^L F_j^2(\eta)d\eta \lesssim \int_0^L \left| \nu^2 e_j \right|_2 \left| \nu^\frac{1}{2} w(\eta) \right|_2^2 d\eta \lesssim \| \nu^\frac{1}{2} w \|_2^2 \lesssim 1.
\] (4.127)
Similarly, using Hölder’s inequality and Lemma 4.1.9 we have
\[
\int_0^L \left( \int_{\eta}^L D(y) \right)^2 d\eta \lesssim \epsilon^2 \int_0^L \left( \int_{\eta}^L \left| \nu^\frac{1}{2} w(y) \right|_2 \right)^2 d\eta \\
\lesssim \epsilon^2 \int_0^L \left( \int_{\eta}^L e^{2K_0 y} \left| \nu^\frac{1}{2} w(y) \right|_2 \right) \left( \int_{\eta}^L e^{-2K_0 y} \right) d\eta \\
\lesssim \epsilon^2 \left\| e^{K_0 \eta} \nu^{\frac{1}{2}} w \right\|_2^2 \int_0^L e^{-2K_0 \eta} d\eta \lesssim \epsilon^2.
\] (4.128)

Similarly, using Hölder’s inequality, Lemma 4.1.9 and Lemma 4.1.1 we have
\[
\int_0^L \left( \int_{\eta}^L E(y) \right)^2 d\eta \lesssim \left\| e^{K_0 \eta} \nu^{\frac{1}{2}} w \right\|_2^2 + \left\| e^{K_0 \eta} \nu^{-\frac{1}{2}} S \right\|_2^2 \lesssim 1,
\] (4.129)
\[
\epsilon \int_0^L \left( \int_{\eta}^L F(y) \right)^2 d\eta \lesssim \left\| e^{K_0 \eta} \nu^{\frac{1}{2}} w \right\|_2^2.
\] (4.130)

Inserting (4.127), (4.128), (4.129) and (4.130) into (4.126), we have
\[
\int_0^L \Delta_2^2(\eta)d\eta \lesssim 1.
\] (4.131)
On the other hand, using Remark 4.1.2, since $|e^s - 1| \lesssim |s|$ and $|\ln(1 + s)| \lesssim |s|$ for $|s| << 1$, we have

\[
\int_0^L \Delta_x^2(\eta)d\eta \lesssim \int_0^L \left( e^{-\Xi(\eta)} - e^{-\Xi(L)} \right)^2 d\eta \lesssim \int_0^L \left( e^{\Xi(L) - \Xi(\eta)} - 1 \right)^2 d\eta \tag{4.132}
\]

\[
\lesssim \int_0^L \left( e^{W_1(\eta) - W_1(L) + W_2(\eta) - W_2(L)} - 1 \right)^2 d\eta
\]

\[
\lesssim \int_0^L \left( W_1(\eta) - W_1(L) \right)^2 d\eta + \int_0^L \left( W_2(\eta) - W_2(L) \right)^2 d\eta
\]

\[
\lesssim \int_0^L \ln^2 \left( \frac{R_1 - \epsilon L}{R_1 - \epsilon \eta} \right) d\eta + \int_0^L \ln^2 \left( \frac{R_2 - \epsilon L}{R_2 - \epsilon \eta} \right) d\eta
\]

\[
\lesssim \int_0^L (\epsilon(\eta - L))^2 d\eta \lesssim c^2 L^3 \lesssim \epsilon^4.
\]

Inserting (4.131) and (4.132) into (4.122), we obtain

\[
\int_0^L \left( \beta(\eta) - \beta_L \right)^2 d\eta \lesssim 1. \tag{4.133}
\]

Since $A$ is invertible, by (4.118), we know

\[
\int_0^L \left( q_j - q_{j,L} \right)^2 d\eta \lesssim 1 \quad \text{for } j = 0, 2, 3, 4. \tag{4.134}
\]

It is easy to see that $q_1(\eta) = q_{1,L} = 0$. Therefore, we prove that

\[
\|q - q_L\|_2 \leq C. \tag{4.135}
\]

**Remark 4.1.11.** This proof highly depends on the fact that $G \sim \epsilon$ and $L \sim \epsilon^{-\frac{1}{2}}$. Also, the $L^2$ decay of $w$ in Lemma 4.1.9 is indispensable.

**Lemma 4.1.12** (L^2 estimate of $g - g_L$). Assume (4.15) holds and $S \in \mathcal{N}^\perp$. There exists a unique solution $g(\eta, \upsilon)$ to the $\epsilon$-Milne problem with geometric correction (4.1) satisfying

\[
\|g - g_L\|_2 \lesssim 1, \tag{4.136}
\]

for some $g_L = \sum_{k=0}^4 g_{k,L} e_k \in \mathcal{N}$ satisfying $|g_{k,L}| \lesssim 1$.

**Proof.** Taking $g_L = q_L$ in Lemma 4.1.10 combined with Lemma 4.1.5 we can naturally obtain the desired result. 

**S \notin \mathcal{N}^\perp** Case

**Lemma 4.1.13** (L^2 well-posedness of $g$). Assume (4.15) holds. There exists a unique solution $g(\eta, \upsilon)$ to the $\epsilon$-Milne problem with geometric correction (4.1) satisfying

\[
\|g - g_L\|_2 \lesssim 1, \tag{4.137}
\]

for some $g_L = \sum_{k=0}^4 g_{k,L} e_k \in \mathcal{N}$ satisfying $|g_{k,L}| \lesssim 1$. 
There is no way to apply Lemma 4.1.12 to $Sg$ where $Sg \in \mathcal{N}$. In the following, we will construct a few auxiliary functions $g_i$ to handle $Sg$ and $S_W$ separately.

Step 1: Construction of $g_1$.
We first solve the problem for auxiliary function $g_1$ with source term $S_W$ as

$$
\begin{align*}
\left\{ \begin{array}{l}
\nu_\eta \frac{\partial g_1}{\partial \eta} + G_1 \left( \nu_\phi \frac{\partial g_1}{\partial \nu_\phi} - \nu_\eta \nu_\phi \frac{\partial g_1}{\partial \nu_\phi} \right) + G_2 \left( \nu_\psi \frac{\partial g_1}{\partial \nu_\psi} - \nu_\eta \nu_\psi \frac{\partial g_1}{\partial \nu_\psi} \right) + \mathcal{L}[g_1] = S_W, \\
g_1(0, \nu) = h(\nu) \quad \text{for} \quad \nu_\eta > 0, \\
g_1(L, \nu) = g_1(L, \mathcal{X}[\nu]).
\end{array} \right.
\end{align*}
$$

(4.139)

Applying Lemma [4.1.12] we know $g_1$ is well-posed.

Step 2: Construction of $g_2$.
There is no way to apply Lemma [4.1.12] to $S_Q$ part, so we resort to explicit formula and analyze it in the following two steps. First, we try to find an auxiliary function $g_2$ such that

$$
\begin{align*}
\nu_\eta \frac{\partial g_2}{\partial \eta} + G_1 \left( \nu_\phi \frac{\partial g_2}{\partial \nu_\phi} - \nu_\eta \nu_\phi \frac{\partial g_2}{\partial \nu_\phi} \right) + G_2 \left( \nu_\psi \frac{\partial g_2}{\partial \nu_\psi} - \nu_\eta \nu_\psi \frac{\partial g_2}{\partial \nu_\psi} \right) + S_Q \in \mathcal{N}^\perp,
\end{align*}
$$

(4.140)

which further means

$$
\int_{\mathbb{R}^3} \nu_\eta \frac{\partial g_2}{\partial \eta} + G_1 \left( \nu_\phi \frac{\partial g_2}{\partial \nu_\phi} - \nu_\eta \nu_\phi \frac{\partial g_2}{\partial \nu_\phi} \right) + G_2 \left( \nu_\psi \frac{\partial g_2}{\partial \nu_\psi} - \nu_\eta \nu_\psi \frac{\partial g_2}{\partial \nu_\psi} \right) + S_Q \right) d\nu = 0.
$$

(4.141)

for $j = 0, 1, 2, 3, 4$. Denote

$$
S_Q = \sum_{k=0}^{4} S_{Q,k}\mathbf{e}_k.
$$

(4.142)

We make an ansatz that

$$
g_2 := \mu^\frac{\perp}{\nu} \left( A(\eta) \nu_\eta + B_1(\eta) + B_2(\eta) \nu_\eta \nu_\phi + B_3(\eta) \nu_\eta \nu_\psi + C(\eta) \nu_\eta |\nu|^2 \right). 
$$

(4.143)

Hence, we can directly compute

$$
\begin{align*}
\frac{\partial g_2}{\partial \nu_\eta} = & -\nu_\eta g_2 + \mu^\frac{\perp}{\nu} \left( A + B_2 \nu_\psi + B_3 \nu_\psi + C |\nu|^2 + 2C \nu_\eta \right), \\
\frac{\partial g_2}{\partial \nu_\phi} = & -\nu_\phi g_2 + \mu^\frac{\perp}{\nu} \left( B_2 \nu_\eta + 2C \nu_\eta \nu_\phi \right), \\
\frac{\partial g_2}{\partial \nu_\psi} = & -\nu_\psi g_2 + \mu^\frac{\perp}{\nu} \left( B_3 \nu_\eta + 2C \nu_\eta \nu_\psi \right),
\end{align*}
$$

(4.144)-(4.146)

and further

$$
\begin{align*}
\nu_\phi^2 \frac{\partial g_2}{\partial \nu_\phi} - \nu_\eta \nu_\phi \frac{\partial g_2}{\partial \nu_\phi} = & \mu^\frac{\perp}{\nu} \left( A \nu_\phi^2 + B_2 \nu_\psi \nu_\phi^2 + B_3 \nu_\psi \nu_\phi + C \nu_\phi^2 |\nu|^2 \right), \\
\nu_\psi^2 \frac{\partial g_2}{\partial \nu_\psi} - \nu_\eta \nu_\psi \frac{\partial g_2}{\partial \nu_\psi} = & \mu^\frac{\perp}{\nu} \left( A \nu_\psi^2 + B_2 \nu_\phi \nu_\psi^2 + B_3 \nu_\phi \nu_\psi + C \nu_\psi^2 |\nu|^2 \right).
\end{align*}
$$

(4.147)-(4.148)
Also, note the Gaussian integral
\[
\int_{\mathbb{R}^3} |v| \mu dv = 1, \quad \int_{\mathbb{R}^3} |v|^2 \mu dv = 3, \quad \int_{\mathbb{R}^3} |v|^4 \mu dv = 15, \quad \int_{\mathbb{R}^3} |v|^6 \mu dv = 105. \quad (4.149)
\]
Plugging this ansatz into the equation (4.131), we obtain a system of linear ordinary differential equations
\[
\begin{aligned}
\frac{d}{d\eta} \begin{pmatrix}
A + 5C \\
B_1 \\
B_2 \\
B_3 \\
A + 10C 
\end{pmatrix}
+ \begin{pmatrix}
G_1 + G_2 & 0 & 0 & 0 & 5G_1 + 5G_2 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2G_1 + G_2 & 0 & 0 \\
0 & 0 & 0 & G_1 + 2G_2 & 0 \\
G_1 + G_2 & 0 & 0 & 0 & 10G_1 + 10G_2 
\end{pmatrix}
\begin{pmatrix}
A \\
B_1 \\
B_2 \\
B_3 \\
C 
\end{pmatrix}
&= -\begin{pmatrix}
S_{Q,0} \\
S_{Q,1} \\
S_{Q,2} \\
S_{Q,3} \\
S_{Q,4} 
\end{pmatrix}.
\end{aligned}
\quad (4.150)
\]
It is easy to check that all five variables \(A, B_1, B_2, B_3, C\) are well-defined as long as \(S_{Q,k}\) decays exponentially (by solving them explicitly). Furthermore, \(g_2\) decays exponentially to \(g_2(L) = 0\) as long as the boundary data are taken properly.

Step 3: Construction of \(g_3\).
Let
\[
\tilde{S} := v_\eta \frac{\partial g_2}{\partial \eta} + G_1 \left( v_\phi^2 \frac{\partial g_2}{\partial v_\phi} - v_\eta v_\phi \frac{\partial g_2}{\partial v_\phi} \right) + G_2 \left( v_\psi^2 \frac{\partial g_2}{\partial v_\psi} - v_\eta v_\psi \frac{\partial g_2}{\partial v_\psi} \right) + \mathcal{L}[g_2] + S_Q. \quad (4.151)
\]
We know \(\tilde{S} \in \mathcal{N}^-\) due to analysis in Step 2 and \(\mathcal{L}[g_2] \in \mathcal{N}^-\). Then we may define an auxiliary function \(g_3\) as the solution of the equation
\[
\left\{ \begin{array}{l}
v_\eta \frac{\partial g_3}{\partial \eta} + G_1 \left( v_\phi^2 \frac{\partial g_3}{\partial v_\phi} - v_\eta v_\phi \frac{\partial g_3}{\partial v_\phi} \right) + G_2 \left( v_\psi^2 \frac{\partial g_3}{\partial v_\psi} - v_\eta v_\psi \frac{\partial g_3}{\partial v_\psi} \right) + \mathcal{L}[g_3] = \tilde{S}, \\
g_3(0, v) = -g_2(0, v) \quad \text{for} \quad v_\eta > 0, \\
g_3(L, v) = g_3(L, \mathcal{R}[v]).
\end{array} \right. \quad (4.152)
\]
Applying Lemma 4.1.12, we know \(g_3\) is well-posed.

Step 4: Construction of \(g_4\).
We may directly verify that \(g_4 = g_2 + g_3\) satisfies the equation
\[
\left\{ \begin{array}{l}
v_\eta \frac{\partial g_4}{\partial \eta} + G_1 \left( v_\phi^2 \frac{\partial g_4}{\partial v_\phi} - v_\eta v_\phi \frac{\partial g_4}{\partial v_\phi} \right) + G_2 \left( v_\psi^2 \frac{\partial g_4}{\partial v_\psi} - v_\eta v_\psi \frac{\partial g_4}{\partial v_\psi} \right) + \mathcal{L}[g_4] = S_Q, \\
g_4(0, v) = 0 \quad \text{for} \quad v_\eta > 0, \\
g_4(L, v) = g_4(L, \mathcal{R}[v]).
\end{array} \right. \quad (4.153)
\]
In summary, by superposition, we know \(g = g_1 + g_4\) satisfies the equation (4.1) and is well-posed.

\(L^2\) Boundedness
Then we turn to the construction of \(\tilde{h}\) and the well-posedness of the equation (4.10).

**Theorem 4.1.14 (\(L^2\) well-posedness of \(G\)).** Assume (4.15) holds. Then there exists \(\tilde{h} \in \mathcal{N}\) such that there exists a unique solution \(G(\eta, v)\) to the \(\epsilon\)-Milne problem with geometric correction (4.10) satisfying
\[
\|G\|_2 \lesssim 1. \quad (4.154)
\]
**Proof.** Given \( h \) and \( S \), Lemma [4.1.12] tells us that the equation \((4.1)\) for \( g \) is well-posed and \( g_L \) is well-defined. By a similar argument, we know for any \( \tilde{h} \), \( G \) must also be well-posed. Hence, our main concern here is to delicately choose \( \tilde{h} \) such that \( \mathcal{G}_L = 0 \), and then Lemma [4.1.12] implies that \((4.15)\) holds.

**Step 1:**
Let \( \tilde{g} = g - \mathcal{G} \), which satisfies the equation
\[
\begin{align*}
v_\eta \frac{\partial \tilde{g}}{\partial \eta} + G_1 \left( v_\phi^2 \frac{\partial \tilde{g}}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \tilde{g}}{\partial v_\phi} \right) + & G_2 \left( v_\phi^2 \frac{\partial \tilde{g}}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \tilde{g}}{\partial v_\phi} \right) + \mathcal{L}[\tilde{g}] = 0, \\
g(0, v) = \tilde{h}(v) \quad \text{for} \quad v_\eta > 0, \\
g(L, v) = \tilde{g}(L, \mathcal{A}[v]).
\end{align*}
\]
(4.155)

In order for \( \mathcal{G}_L = 0 \), we must choose proper \( \tilde{h} \) such that
\[
\tilde{g}_L(v) = g_L(v) = q_0, L e_0 + q_1, L e_1 + q_2, L e_2 + q_3, L e_3 + q_4, L e_4.
\]
(4.156)

where \( \tilde{g}_L \) is defined as in Lemma [4.1.12].

In other words, \( g \) and \( \tilde{g} \) may have different in-flow boundary (\( h \) or \( \tilde{h} \)) and source terms (\( S \) or \( 0 \)), but they share the same \( \mathcal{G}_L \). Hence, \( \tilde{g}_L(v) = g_L(v) \in \mathcal{N} \).

**Step 2:**
Note that
\[
\tilde{h}(v) := D_0 e_0 + D_1 e_1 + D_2 e_2 + D_3 e_3 + D_4 e_4.
\]
(4.157)

Hence, in (4.155), we actually need to build a mapping between \( \tilde{h} \in \mathcal{N} \) and \( \tilde{g}_L \in \mathcal{N} \). We can take \( D_1 = q_1, L = 0 \). Then we consider the endomorphism \( \mathcal{M} \) in a four-dimensional space \( \mathcal{N} = \text{span}\{e_0, e_2, e_3, e_4\} \) defined as \( \mathcal{M} : \tilde{h} \rightarrow \mathcal{M}[\tilde{h}] = \tilde{g}_L \). Therefore, we only need to study the matrix of \( \mathcal{M} \) at the basis \( \{e_0, e_2, e_3, e_4\} \). It suffices to show that \( \mathcal{M} \) is invertible.

**Step 3:**
It is easy to check when \( \tilde{h} = e_0 \) and \( \tilde{h} = e_4 \), \( \mathcal{M} \) is an identity mapping, i.e.
\[
\mathcal{M}[e_0] = e_0, \quad \mathcal{M}[e_4] = e_4.
\]
(4.158)

The main obstacle is when \( \tilde{h} = e_2, e_3 \). Actually, \( \mathcal{M}[e_2] \) is almost \( e_2 \), so we only need to estimate the difference. For \( \tilde{h} = e_2 \) in (4.155), define \( \tilde{g}' = \tilde{g} - e_2 \). Then \( \tilde{g}' \) satisfies the equation
\[
\begin{align*}
v_\eta \frac{\partial \tilde{g}'}{\partial \eta} + G_1(\eta) \left( v_\phi^2 \frac{\partial \tilde{g}'}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \tilde{g}'}{\partial v_\phi} \right) + & G_2(\eta) \left( v_\phi^2 \frac{\partial \tilde{g}'}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \tilde{g}'}{\partial v_\phi} \right) + \mathcal{L}[\tilde{g}'] = G(\eta) \mu^2 v_\eta v_\phi, \\
\tilde{g}'(0, v) = 0 \quad \text{for} \quad v_\eta > 0, \\
\tilde{g}'(L, v) = \tilde{g}'(L, \mathcal{A}[v]).
\end{align*}
\]
(4.159)

Here we cannot directly apply Lemma [4.1.5] to Lemma [4.1.12] with \( S = G(\eta) \mu^2 v_\eta v_\phi \in \mathcal{N}^\perp \) and \( h = 0 \), since \( G(\eta) \mu^2 v_\eta v_\phi \) does not decay exponentially. At best, we only have \( \|S\|_{\infty, \theta} \lesssim \epsilon \) and have to modify the proof accordingly. In Lemma [4.1.5] we can show that
\[
\| \nu^\perp w \|_2 \lesssim \epsilon^2.
\]
(4.160)

Lemma [4.1.3] remains the same. Lemma [4.1.4] does not hold any more, so we need to use the smallness of \( S \) and \( \mu^2 \) in proving Lemma [4.1.10] instead of exponential decay. We focus on the derivation of \( q_L \). Here the estimates of \( D, E, F, \theta \) remains the same. Then we have
\[
|q_{j, L}| \lesssim \| \nu^\perp w \|_2 + \int_0^L \| \nu^\perp w(\eta) \|_2 d\eta + \int_0^L |S(\eta)|_2 d\eta \lesssim \epsilon^2 L^2 \lesssim \epsilon^2.
\]
(4.161)
In other other, the limit $\tilde{q}_L^\alpha$ to [4.159] is at the order $\epsilon^2$ and is very small, i.e.

$$\mathcal{M}[e_2] = e_2 + \tilde{q}_L^\alpha e_2 + \epsilon^\frac{1}{2}e_j.$$  (4.162)

A similar argument can justify $\tilde{g}'' = \tilde{g} - e_3$, case, i.e.

$$\mathcal{M}[e_3] = e_3 + \tilde{q}_L'' e_3 + \epsilon^\frac{1}{2}e_j.$$  (4.163)

Step 4:
In summary, we know the matrix of $M$ is just a small perturbation of identity matrix

$$\mathcal{M} \begin{bmatrix} e_0 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{pmatrix} 1 & \tilde{q}_{0,L}^\alpha & \tilde{q}_{0,L}'' & 0 \\ 0 & 1 + \tilde{q}_{2,L} & \tilde{q}_{2,L}'' & 0 \\ 0 & \tilde{q}_{3,L} & 1 + \tilde{q}_{3,L}'' & 0 \\ 0 & \tilde{q}_{4,L} & \tilde{q}_{4,L}'' & 1 \end{pmatrix} \begin{bmatrix} e_0 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}.$$  (4.164)

Here $\tilde{q}_{k,L}^\alpha$ and $\tilde{q}_{k,L}''$ are defined as in Step 3 and are of order $\epsilon^\frac{1}{2}$. For $\epsilon$ sufficiently small, this matrix is invertible, which means $\mathcal{M}$ is bijective. Therefore, we can always find $\tilde{h}$ such that $\tilde{g}_L = g_L$, which is desired. 

$L^2$ Decay

**Theorem 4.1.15** ($L^2$ decay). Assume (4.15) holds. Then there exists $0 < K_0 < K$ such that the solution $g(\eta, v)$ to (4.0) satisfying

$$\|e^{K_0\eta}g\|_2 \lesssim 1.$$  (4.165)

**Proof.** We decompose $G = w + q$ with $G_L = q_L = 0$. Lemma 4.1.9 already justifies the decay of $w$

$$\|e^{K_0\eta}w\|_2 \lesssim 1.$$  (4.166)

Hence, we focus on $q$ decay. Here, we use the same notation as in the proof of Lemma 4.1.7 and Lemma 4.1.10. Recall (4.11). $\beta_L = q_L = 0$ implies

$$\theta = - \int_0^L \exp\left(W_1(y)B^{(1)}A^{-1} + W_2(y)B^{(2)}A^{-1}\right)Z(y)dy.$$  (4.167)

Inserting (4.167) into (4.11), we obtain

$$\beta(\eta) = - F(\eta) - \int_0^L \exp\left(\left(W_1(y) - W_1(\eta)\right)B^{(1)}A^{-1} + \left(W_2(y) - W_2(\eta)\right)B^{(2)}A^{-1}\right)Z(y)dy.$$  (4.168)

Note that

$$Z = D + E - (G_1B^{(1)} + G_2B^{(2)})A^{-1}F,$$  (4.169)

where $D, E, F$ are defined in (4.72), and due to Remark 4.1.2

$$\|e^{K_0\eta}q\|_2^2 \lesssim \|e^{K_0\eta}\|_2^2 \lesssim \int_0^L e^{2K_0\eta}F^2(\eta)d\eta + \int_0^L e^{2K_0\eta}\left(\int_0^L Z(y)dy\right)^2 d\eta.$$  (4.170)

Then the proof is similar to that of Lemma 4.1.10 so we omit it here. Here, we take $K_0' \leq \frac{K_0}{2}$. 

**Remark 4.1.16.** In (4.1), $g - g_L$ does not necessarily decay exponentially. This is the main reason we have to introduce Theorem 4.1.14 to design the boundary data such that $G_L = 0$. 

4.1.2 $L^\infty$ Estimates

Characteristic Formulation

We rewrite (4.171) as the following $\epsilon$-transport problem for $g(\eta, v)$

\[
\begin{align*}
\frac{\partial g}{\partial \eta} + G_1(\eta) \left( v^2 \frac{\partial g}{\partial \eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi} \right) + G_2(\eta) \left( v_\phi^2 \frac{\partial g}{\partial \eta} - v_\eta v_\phi \frac{\partial g}{\partial v_\phi} \right) + \nu g &= Q(\eta, v), \\
g(0, v) &= h(v) \quad \text{for} \quad v_\eta > 0, \\
g(L, v) &= g(L, R[v]),
\end{align*}
\]

(4.171)

Here $Q := K[g] + S$.

Define the characteristics starting from $\left( \eta(0), v_\eta(0), v_\phi(0), v_\psi(0) \right)$ as $\left( \eta(s), v_\eta(s), v_\phi(s), v_\psi(s) \right)$ for some $s \in \mathbb{R}$ satisfying

\[
\frac{d\eta}{ds} = v_\eta, \quad \frac{dv_\eta}{ds} = G_1(\eta)v_\phi^2 + G_2(\eta)v_\psi^2, \quad \frac{dv_\phi}{ds} = -G_1(\eta)v_\eta v_\phi, \quad \frac{dv_\psi}{ds} = -G_2(\eta)v_\eta v_\psi,
\]

(4.172)

which leads to

\[
v_\eta^2(s) + v_\phi^2(s) + v_\psi^2(s) := E_1, \quad v_\phi(s)e^{-W_1(\eta(s))} := E_2, \quad v_\psi(s)e^{-W_2(\eta(s))} := E_3,
\]

(4.173)

where the energy $E_i$ are constants depending on the starting point.

Therefore, along the characteristics, $v_\eta^2 + v_\phi^2 + v_\psi^2$, $v_\phi e^{-W_1(\eta)}$ and $v_\psi e^{-W_2(\eta)}$ are conserved quantities and the equation (4.171) can be rewritten as

\[
\frac{dg}{ds} + \nu g = Q,
\]

(4.174)

or equivalently,

\[
v_\eta \frac{dg}{d\eta} + \nu g = Q.
\]

(4.175)

Let

\[
v'_\eta(\eta, v; \eta') := v_\phi e^{W_1(\eta') - W_1(\eta)}, \quad v'_\phi(\eta, v; \eta') := v_\psi e^{W_2(\eta') - W_2(\eta)}.
\]

(4.176)

On the characteristics, we should always have $E_1 \geq v_\eta^2 + v_\phi^2$. Define

\[
v'_\eta(\eta, v; \eta') := \sqrt{E_1 - v_\phi^2(\eta, v; \eta') - v_\psi^2(\eta, v; \eta')},
\]

(4.177)

\[
v'(\eta, v; \eta') := \left( v'_\eta(\eta, v; \eta'), v'_\phi(\eta, v; \eta'), v'_\psi(\eta, v; \eta') \right),
\]

(4.178)

\[
\mathcal{D}[v'(\eta, v; \eta')] := \left( -v'_\eta(\eta, v; \eta'), v'_\phi(\eta, v; \eta'), v'_\psi(\eta, v; \eta') \right).
\]

(4.179)

Basically, this means $(\eta, v_\eta, v_\phi, v_\psi)$ and $(\eta', v'_\eta, v'_\phi, v'_\psi)$ are on the same characteristics. In particular, this implies $v'_\eta \geq 0$.

We can rewrite the solution to the equation (4.171) along the characteristics using (4.173) as

\[
g(\eta, v) = \mathcal{K}[h](\eta, v) + \mathcal{T}[Q](\eta, v),
\]

(4.180)

where the operators $\mathcal{K}$ and $\mathcal{T}$ are defined as follows:

Case I: $v_\eta > 0$:

The characteristics directly tracks back to the in-flow boundary $\eta = 0$ and $v_\eta > 0$, i.e.

\[
\mathcal{K}[h](\eta, v) := h \left( \left. v'(\eta, v; 0) \right| \exp(-H_{\eta,0}) \right),
\]

(4.181)

\[
\mathcal{T}[Q](\eta, v) := \int_0^\eta \frac{Q(\eta', v'(\eta, v; \eta'))}{v'_\eta(\eta, v; \eta')} \exp(-H_{\eta,\eta'}) d\eta'.
\]

(4.182)
Here

\[ H_{\eta, \eta'} := \int_{\eta'}^{\eta} \frac{\nu \left( \nu'(\eta, \nu; y) \right)}{v'_\eta(\eta, \nu; y)} dy. \]  

(4.183)

Case II: \( v_\eta < 0 \) and \( v^2_\eta + v^2_\xi + v^2_\psi \geq v^2_\phi(\eta, \nu; L) + v^2_\phi(\eta, \nu; L) \):

The characteristics first goes a bit farther to the boundary \( \eta = L \), then gets reflected and tracks back to the in-flow boundary, i.e.

\[ \mathcal{K}[\eta](\eta, \nu) := h \left( \nu'(\eta, \nu; 0) \right) \exp(-H_{\eta+, 0} - \mathcal{R}[H_{\eta+, \eta}]), \]  

(4.184)

\[ \mathcal{T}[Q](\eta, \nu) := \left( \int_{0}^{L} Q \left( \eta', \nu'(\eta, \nu; \eta') \right) \frac{v'_\eta(\eta, \nu; \eta')}{v'_\eta(\eta, \nu; \eta')} \exp(-H_{\eta+, \eta'} - \mathcal{R}[H_{\eta+, \eta}]) d\eta' \right. \]

\[ + \left. \int_{\eta}^{L} Q \left( \eta', \mathcal{R}[\nu'(\eta, \nu; \eta')] \right) \frac{v'_\eta(\eta, \nu; \eta')}{v'_\eta(\eta, \nu; \eta')} \exp(\mathcal{R}[H_{\eta, \eta'}]) d\eta' \right). \]  

(4.185)

Here

\[ \mathcal{R}[H_{\eta, \eta'}] := \int_{\eta'}^{\eta} \frac{\nu \left( \mathcal{R}[\nu'(\eta, \nu; y)] \right)}{v'_\eta(\eta, \nu; y)} dy. \]  

(4.186)

Actually, since \( \nu \) only depends on \(|\nu|\), we must have \( H_{\eta, \eta'} = \mathcal{R}[H_{\eta, \eta'}] \). This distinction is purely for clarification and does not play a role in the estimates.

Case III: \( v_\eta < 0 \) and \( v^2_\eta + v^2_\xi + v^2_\psi \leq v^2_\phi(\eta, \nu; L) + v^2_\phi(\eta, \nu; L) \)

The characteristics reaches the line \( v_\eta = 0 \) before reaching the boundary \( \eta = L \), and then directly tracks back to the in-flow boundary, i.e.

\[ \mathcal{K}[\eta](\eta, \nu) := h \left( \nu'(\eta, \nu; 0) \right) \exp(-H_{\eta+, 0} - \mathcal{R}[H_{\eta+, \eta}]), \]  

(4.187)

\[ \mathcal{T}[Q](\eta, \nu) := \left( \int_{0}^{\eta^+} Q \left( \eta', \nu'(\eta, \nu; \eta') \right) \frac{v'_\eta(\eta, \nu; \eta')}{v'_\eta(\eta, \nu; \eta')} \exp(-H_{\eta^+, \eta'} - \mathcal{R}[H_{\eta^+, \eta}]) d\eta' \right. \]

\[ + \left. \int_{\eta}^{\eta^+} Q \left( \eta', \mathcal{R}[\nu'(\eta, \nu; \eta')] \right) \frac{v'_\eta(\eta, \nu; \eta')}{v'_\eta(\eta, \nu; \eta')} \exp(\mathcal{R}[H_{\eta, \eta'}]) d\eta' \right). \]  

(4.188)

Here \( \eta^+(\eta, \nu) \) is defined by

\[ E_1(\eta, \nu) = v^2_\phi(\eta, \nu; \eta^+) + v^2_\phi(\eta, \nu; \eta^+). \]  

(4.189)

locates the position that the characteristics touch \( v_\eta = 0 \) line, i.e. \((\eta^+, 0, v'_\xi', v'_\psi')\) is on the same characteristics as \((\eta, v_\eta, v_\xi, v_\psi)\).

In order to achieve the estimate of \( q \), we need to control \( \mathcal{K}[\eta] \) and \( \mathcal{T}[Q] \). Since we always assume that \((\eta, \nu)\) and \((\eta', \nu')\) are on the same characteristics, in the following, we will simply write \( \nu'(\eta') \) or even \( \nu' \) instead of \( \nu'(\eta, \nu; \eta') \) when there is no confusion.

\( L^\infty \ Boundedness \)

We first prove some important lemmas characterizing the operators \( \mathcal{K} \) and \( \mathcal{T} \).

**Lemma 4.1.17** (estimate of boundary term). There is a positive \( 0 < \beta < \nu_0 \) such that for any \( \vartheta \geq 0 \) and \( \varrho \geq 0 \),

\[ \| e^{\beta \eta} \mathcal{K}[\eta] \|_{\infty, \vartheta, \varrho} \lesssim |\eta|_{\infty, \vartheta, \varrho}. \]  

(4.190)
Proof. Consider (4.183), we know
\[ \frac{\nu(v')}{v'_\eta} \geq \nu_0, \quad \frac{\nu(\mathcal{R}[v'])}{v'_\eta} \geq \nu_0. \]  
(4.191)

It follows that
\[ \exp(-H_{0,0}) \leq e^{-\beta n}, \quad \exp(-H_{0,\eta} - \mathcal{R}[H_{L,0}]) \leq e^{-\beta n}, \quad \exp(-H_{0,0} + \mathcal{R}[H_{L,\eta}]) \leq e^{-\beta n}. \]  
(4.192) (4.193) (4.194)

Then our results are obvious. \[ \square \]

**Lemma 4.1.18** (estimate of bulk term). For any \( \vartheta \geq 0, \varrho \geq 0 \) and \( 0 \leq \beta \leq \frac{\nu_0}{2} \), there is a constant \( C \) such that
\[ \| e^{\beta n} T Q \|_{\infty, \vartheta, \varrho} \lesssim \| \nu^{-1} e^{\beta n} Q \|_{\infty, \vartheta, \varrho}. \]  
(4.195)

**Proof.** For \( v_\eta > 0 \) case, we have
\[ \beta(n - n') - H_{n,n'} \leq \beta(n - n') - \frac{\nu_0(n - n')}{2} - \frac{H_{n,n'}}{2} \leq - \frac{H_{n,n'}}{2}. \]  
(4.196)

It is natural that
\[ \int_0^n \frac{\nu(v'(\eta'))}{v'_\eta(\eta')} \exp \left( \beta(n - n') - H_{n,n'} \right) d\eta' \leq \int_0^\infty \exp \left( - \frac{z}{2} \right) dz = 2, \]  
(4.197)

for \( z = H_{n,n'} \). Notice that \( |v| = |v'| \). Then we estimate
\[ \| \| e^{\beta n} \| V \| e^{\beta n} T Q \| \leq e^{\beta n} \int_0^n \langle v \rangle e^{\beta n} \frac{|Q(\eta', v'(\eta'))|}{v'_\eta(\eta')} \exp(-H_{n,n'}) d\eta' \]  
(4.198)
\[ \leq \| \nu^{-1} e^{\beta n} Q \|_{\infty, \vartheta, \varrho} \int_0^n \frac{\nu(v'(\eta'))}{v'_\eta(\eta')} \exp \left( \beta(n - n') - H_{n,n'} \right) d\eta' \]  
\[ \lesssim \| \nu^{-1} e^{\beta n} Q \|_{\infty, \vartheta, \varrho}. \]

The \( v_\eta < 0 \) case can be proved in a similar fashion, so we omit it here. \[ \square \]

**Lemma 4.1.19** (further estimate of bulk term). For any \( \delta > 0 \), an integer \( \vartheta > 3 \) and \( \varrho \geq 0 \), there is a constant \( C(\delta) \) such that
\[ \| T Q \|_{m, \varrho} \leq C(\delta) \| \nu^{-\frac{1}{2}} \nu Q \|_2 + \delta \| Q \|_{\infty, \vartheta, \varrho}. \]  
(4.199)

**Proof.** In the following, we will repeatedly use the fact that \( |v| = |v'| \).

Case I: For \( v_\eta > 0 \), \( T Q \) is defined in (4.181). We need to estimate
\[ \int_{\mathbb{R}^3} e^{2|v|^2} \left( \int_0^n Q(\eta', v(\eta')) \frac{v'_\eta(\eta')}{v'_\eta(\eta')} \exp(-H_{n,n'}) d\eta' \right)^2 dv. \]  
(4.200)

Assume \( m > 0 \) is sufficiently small, \( M > 0 \) is sufficiently large and \( \sigma > 0 \) is sufficiently small, which will be determined in the following. We can split the above integral into four parts
\[ I := I_1 + I_2 + I_3 + I_4. \]  
(4.201)
In the following, we use $\chi_i$ for $i = 1, 2, 3, 4$ to represent the indicator function of each type.

**Case I - Type I:** $\chi_1: M \leq v^r_\eta(\eta')$ or $M \leq v^r_\phi(\eta')$ or $M \leq v_\psi(\eta')$.

We have

$$|u(\eta')| + 1 \lesssim \nu \left( v(\eta') \right). \quad (4.202)$$

Then for $\nu > 3$, since $|u|$ is conserved along the characteristics, we have

$$I_1 \lesssim \|Q\|_{\infty, \sigma, e}^2 \int_{\mathbb{R}^3} \chi_1 \left( \int_0^n \frac{1}{\nu} \frac{\exp(-H_{\eta,\eta'})}{v^r_\eta(\eta')} d\eta' \right)^2 d\nu \quad (4.203)$$

$$\lesssim \frac{1}{M^{\sigma}} \|Q\|_{\infty, \sigma, e}^2 \int_{\mathbb{R}^3} \frac{1}{\nu} \left( \int_0^n \frac{\exp(-H_{\eta,\eta'})}{v^r_\eta(\eta')} d\eta' \right)^2 d\nu \quad (4.204)$$

Case I - Type II: $\chi_2: v_\eta \geq \sigma, m \leq v_\phi(\eta') \leq M$ and $v_\psi(\eta') \leq M$.

Since along the characteristics, $|b|^2$ can be bounded by $3M^2$ and the integral domain for $b$ is finite, by Cauchy’s inequality, we have

$$I_2 \lesssim e^{\delta_0 M^2} \int_{\mathbb{R}^3} \left( \int_0^n \frac{Q^2}{\nu} (\eta', \nu'(\eta')) d\eta' \right) \left( \int_0^n \frac{\nu(\nu'(\eta'))}{v^r_\eta(\eta')} d\eta' \right) d\nu \quad (4.205)$$

$$\lesssim \frac{e^{\delta_0 M^2}}{m} \int_{\mathbb{R}^3} \left( \int_0^n \frac{Q^2}{\nu} (\eta', \nu'(\eta')) d\eta' \right) \left( \int_0^n \frac{\nu(\nu'(\eta'))}{v^r_\eta(\eta')} d\eta' \right) d\nu \quad (4.206)$$

where for $y = H_{\eta,\eta'}$,

$$\int_0^n \frac{\nu(\nu'(\eta'))}{v^r_\eta(\eta')} d\eta' \exp(-2H_{\eta,\eta'}) d\nu' \lesssim \int_0^\infty e^{-2y} dy = \frac{1}{2}. \quad (4.207)$$

and the Jacobian

$$\left| \frac{dv}{dv'} \right| = \left| \frac{R_1 - c\eta R_2 - c\eta v'_\eta}{R_1 - c\eta R_2 - c\eta v_\eta} \right| \lesssim \frac{v'_\eta}{v_\eta} \lesssim \frac{M}{\sigma}. \quad (4.208)$$

Case I - Type III: $\chi_3: v_\eta \geq \sigma, 0 \leq v_\phi(\eta') \leq M$ and $v_\psi(\eta') \leq M$.

We can directly verify the fact that

$$0 \leq v_\eta \leq v_\phi(\eta'). \quad (4.209)$$
for $\eta' \leq \eta$. Then we know the integral of $v_{\eta}$ is always in a small domain. We have for $\eta = H_{\eta,\eta'}$,

\[
I_3 \lesssim e^{6eM^2} \|Q\|^2_{\infty, \sigma, e} \int_{\mathbb{R}^3} \chi_3(\eta') \left( \int_{0}^{\eta} \exp(-H_{\eta,\eta'}) v_{\eta}(\eta') d\eta' \right)^2 d\eta \tag{4.209}
\]

\[
\lesssim e^{6eM^2} \|Q\|^2_{\infty, \sigma, e} \int_{\mathbb{R}^3} \chi_3(\eta') \left( \int_{0}^{\infty} e^{-\eta} d\eta \right)^2 d\eta
\]

\[
\lesssim e^{6eM^2} \|Q\|^2_{\infty, \sigma, e} \int_{\mathbb{R}^3} \chi_3(\eta') d\eta
\]

\[
\lesssim e^{6eM^2} \|Q\|^2_{\infty, \sigma, e} \int_{\mathbb{R}^3} \chi_3(\eta') d\eta
\]

Case I - Type IV: $v_{\eta} \leq \sigma, v_{\eta}(\eta') \leq M$ and $v_{\eta}(\eta') \leq M$ and $v_{\psi}(\eta') \leq M$.

Similar to Case I - Type III, we know the integral of $\eta$ is always in a small domain. We have for $\eta = H_{\eta,\eta'}$,

\[
I_4 \lesssim e^{6eM^2} \|Q\|^2_{\infty, \sigma, e} \int_{\mathbb{R}^3} \chi_4(\eta') \left( \int_{0}^{\eta} \exp(-H_{\eta,\eta'}) v_{\eta}(\eta') d\eta' \right)^2 d\eta \tag{4.210}
\]

\[
\lesssim e^{6eM^2} \|Q\|^2_{\infty, \sigma, e} \int_{\mathbb{R}^3} \chi_4(\eta') \left( \int_{0}^{\infty} e^{-\eta} d\eta \right)^2 d\eta
\]

\[
\lesssim e^{6eM^2} \|Q\|^2_{\infty, \sigma, e} \int_{\mathbb{R}^3} \chi_4(\eta') d\eta
\]

\[
\lesssim e^{6eM^2} \|Q\|^2_{\infty, \sigma, e} \int_{\mathbb{R}^3} \chi_4(\eta') d\eta
\]

Collecting all four types, we have

\[
I \lesssim \frac{Me^{6eM^2}}{m\sigma} \left| \nu^2 Q \right|_{2}^{2} + \left( \frac{1}{M\sigma} + e^{6eM^2} (m + \sigma) \right) \|Q\|^2_{\infty, \sigma, e} \tag{4.211}
\]

Taking $M$ sufficiently large, $m << e^{-6eM^2}$ and $\sigma << e^{-6eM^2}$ sufficiently small, we obtain the desired result.

Case II:

For $v_{\eta} < 0$ and $v_{\eta}^{2} + v_{\phi}^{2} + v_{\psi}^{2} \geq v_{\phi}^{2}(L) + v_{\psi}^{2}(L)$, $T[Q]$ is defined in [1831]. We first estimate

\[
\int_{\mathbb{R}^3} e^{2|\eta|^2} \left( \int_{\eta}^{L} Q(\eta', v_{\eta}(\eta')) \exp(\overline{R}[H_{\eta,\eta'}]) d\eta' \right)^2 d\eta.
\tag{4.212}
\]

We can split the above integral into four parts:

\[
II := II_1 + II_2 + II_3 + II_4.
\tag{4.213}
\]

Case II - Type I: $\chi_1$: $\eta'$

Similar to Case I - Type I, we have

\[
II_1 \lesssim \|Q\|^2_{\infty, \sigma, e} \int_{\mathbb{R}^3} \chi_1 \left( \int_{\eta}^{L} \frac{1}{v_{\eta}(\eta')} \exp(\overline{R}[H_{\eta,\eta'}]) d\eta' \right)^2 d\eta
\tag{4.214}
\]

\[
\lesssim \frac{1}{M\sigma} \|Q\|^2_{\infty, \sigma, e} \int_{\mathbb{R}^3} \frac{1}{|v_{\eta}(\eta')|} \left( \int_{\eta}^{L} \exp(\overline{R}[H_{\eta,\eta'}]) d\eta' \right)^2 d\eta
\]

\[
\lesssim \frac{1}{M\sigma} \|Q\|^2_{\infty, \sigma, e} \int_{\mathbb{R}^3} \frac{1}{|v_{\eta}(\eta')|} d\eta
\]

\[
\lesssim \frac{1}{M\sigma} \|Q\|^2_{\infty, \sigma, e}.
\]
since for $y = H_{\eta,\eta'}$, 

$$
\left| \int_{\eta}^{L} \frac{\exp(\mathcal{A}[H_{\eta,\eta'}])}{v_{\eta,\eta'}(\eta')} d\eta' \right| \leq \left| \int_{\eta}^{L} \nu \left( v'(\eta') \right) \exp(\mathcal{A}[H_{\eta,\eta'}]) v_{\eta,\eta'}(\eta') d\eta' \right| \lesssim \int_{-\infty}^{0} e^{\eta} dy = 1. \tag{4.215}
$$

Case II - Type II: $\chi_2$: $m \leq v_{\eta}'(\eta') \leq M$ and $v_{\eta}'(\eta') \leq M$ and $v_{\psi}'(\eta') \leq M$.

We can directly verify the fact that

$$0 \leq v_{\eta}'(\eta') \leq |v_{\eta}|, \tag{4.216}$$

for $\eta' \geq \eta$. Similar to Case I - Type II, by Cauchy’s inequality, we have

$$II_2 \lesssim e^{6M^2} \int_{\mathbb{R}^3} \left( \int_{0}^{\eta} \frac{Q^2}{v} \left( \eta', v(\eta') \right) d\eta' \right) \left( \int_{\eta}^{L} \nu \left( v'(\eta') \right) \exp(2\mathcal{A}[H_{\eta,\eta'}]) v_{\eta,\eta'}(\eta') d\eta' \right) d\nu \tag{4.217}
$$

$$\lesssim e^{6M^2} \frac{1}{m} \int_{\mathbb{R}^3} \left( \int_{\eta}^{L} \frac{Q^2}{v} \left( \eta', v(\eta') \right) d\eta' \right) \left( \int_{\eta}^{L} \nu \left( v'(\eta') \right) \exp(2\mathcal{A}[H_{\eta,\eta'}]) v_{\eta,\eta'}(\eta') d\eta' \right) d\nu \tag{4.217}
$$

$$\lesssim e^{6M^2} \frac{1}{m} \left( \int_{\mathbb{R}^3} \int_{\eta}^{L} \frac{Q^2}{v} \left( \eta', v(\eta') \right) d\eta' d\nu \right) \tag{4.217}
$$

$$\lesssim e^{6M^2} \frac{1}{m} \left( \int_{\mathbb{R}^3} \frac{Q^2}{v} \left( \eta', v(\eta') \right) d\eta' d\nu \right) \tag{4.217}
$$

where for $y = H_{\eta,\eta'}$,

$$\int_{0}^{\eta} \frac{\nu \left( v'(\eta') \right)}{v_{\eta,\eta'}(\eta')} \exp(-2H_{\eta,\eta'}) d\eta' d\nu \lesssim \int_{0}^{\infty} e^{-2y} dy = \frac{1}{2}. \tag{4.218}$$

and the Jacobian

$$\left| \frac{d\nu}{d\nu'} \right| = \frac{R_1 - c_{\eta} R_2 - c_{\eta} v_{\eta}^{'}}{R_1 - c_{\eta} R_2 - c_{\eta} v_{\eta}^{'}} \lesssim \frac{v_{\eta}^{'}}{v_{\eta}} \lesssim 1. \tag{4.219}$$

Case II - Type III: $\chi_3$: $0 \leq v_{\eta}'(\eta') \leq m$, $v_{\psi}'(\eta') \leq M$, $v_{\psi}'(\eta') \leq M$ and $\eta' - \eta \geq \sigma$.

We know

$$H_{\eta,\eta'} \leq - \frac{\sigma}{m}. \tag{4.220}$$

Then after substitution $y = H_{\eta,\eta'}$, the integral is not from zero, but from $-\frac{\sigma}{m}$. In detail, we have

$$II_3 \lesssim e^{6M^2} \left\| Q \right\|_{\infty, \nu, \psi}^2 \int_{\mathbb{R}^3} \frac{X_3}{(\nu)^3} \left( \int_{\eta}^{L} \frac{\exp(\mathcal{A}[H_{\eta,\eta'}])}{v_{\eta,\eta'}(\eta')} d\eta' \right)^2 d\nu \tag{4.221}
$$

$$\lesssim e^{6M^2} \left\| Q \right\|_{\infty, \nu, \psi}^2 \int_{\mathbb{R}^3} \frac{X_3}{(\nu)^3} \left( \int_{\eta}^{L} \nu \left( v'(\eta') \right) \exp(\mathcal{A}[H_{\eta,\eta'}]) v_{\eta,\eta'}(\eta') d\eta' \right)^2 d\nu \tag{4.221}
$$

$$\lesssim e^{6M^2} \left\| Q \right\|_{\infty, \nu, \psi}^2 \int_{\mathbb{R}^3} \frac{X_3}{(\nu)^3} \left( \int_{-\infty}^{0} e^{\eta} d\eta \right)^2 d\nu \tag{4.221}
$$

$$\lesssim e^{6M^2} e^{-2\sigma} \left\| Q \right\|_{\infty, \nu, \psi}^2 \int_{\mathbb{R}^3} \frac{X_3}{(\nu)^3} d\nu \tag{4.221}
$$

$$\lesssim e^{6M^2} e^{-2\sigma} \left\| Q \right\|_{\infty, \nu, \psi}^2. \tag{4.221}$$
Therefore, the integral domain for $\eta$ is very small. We have the estimate for $y = H_{\eta, \eta'}$

$$IH_4 \lesssim \|Q\|_{\infty, \eta, s}^2 \int_{\mathbb{R}^3} \frac{X_4}{(|y|)^3} \left( \int_\eta^L \frac{\exp(H[H_{\eta, \eta'}])}{v^2(\eta')} \, d\eta' \right)^2 \, dv \tag{4.223}$$

$$\lesssim \|Q\|_{\infty, \eta, s}^2 \int_{\mathbb{R}^3} \frac{X_4}{(|y|)^3} \left( \int_{-\infty}^0 e^y \, dy \right)^2 \, dv$$

$$\lesssim \|Q\|_{\infty, \eta, s}^2 \int_{\mathbb{R}^3} \frac{X_4}{(|y|)^3} \, dv$$

$$\lesssim \|Q\|_{\infty, \eta, s}^2 (m + \sqrt{\sigma}) \|Q\|^2_{\infty, \eta, \theta}.$$ 

Collecting all four types, we have

$$II \leq C \frac{e^{6\sigma M^2}}{m} \left\| \nu^{-1} \frac{Q}{2} \right\|_2 + C \left( \frac{1}{M^2} + e^{6\sigma M^2} \left( e^{-\tilde{\sigma}} + m + \sqrt{\sigma} \right) \right) \|Q\|^2_{\infty, \eta, \theta}. \tag{4.224}$$

Taking $M$ sufficiently large, $\sigma < e^{-6\sigma M^2}$ sufficiently small and $m < \min\{\sigma, e^{-4\sigma M^2}\}$ sufficiently small, we obtain the desired result.

Note that we have the decomposition

$$\int_0^L Q \left( \eta', v(\eta') \right) \frac{\exp(-H_{L, \eta'} - \mathcal{H}[H_{L, \eta}]) \, d\eta'}{v^2(\eta')} \tag{4.225}$$

$$= \int_0^\eta Q \left( \eta', v(\eta') \right) \frac{\exp(-H_{L, \eta'} - \mathcal{H}[H_{L, \eta}]) \, d\eta'}{v^2(\eta')} + \int_\eta^L Q \left( \eta', v(\eta') \right) \frac{\exp(-H_{L, \eta'} - \mathcal{H}[H_{L, \eta}]) \, d\eta'}{v^2(\eta')}.$$

Then this term can actually be bounded using the techniques in Case I and Case II.

Case III:

For $\eta < 0$ and $v^2 + v_0^2 + v_\gamma^2 \leq v_\phi^2(L) + v_\psi^2(L)$, $\mathcal{T}[Q]$ is defined in (4.157). This is a combination of Case I and Case II, so we omit the proof here.

**Lemma 4.1.20** ($L^\infty$ estimate of $g - g_L$). Assume (4.15) holds. Then the solution $g(\eta, \theta)$ to the $\epsilon$-Milne problem with geometric correction (4.1) satisfies for $\eta \geq 0$ and $\theta > 3$,

$$\|g - g_L\|_{\infty, \eta, \theta} \lesssim 1 + \|g - g_L\|_2. \tag{4.226}$$
Proof. Define \( u = g - g_L \). Then \( u \) satisfies the equation
\[
\begin{align*}
&\begin{cases}
    v_\eta \frac{\partial u}{\partial \eta} + G_1 \left( v_\eta \frac{\partial u}{\partial v_\eta} - v_\eta v_\phi \frac{\partial u}{\partial v_\phi} \right) + G_2 \left( v_\eta \frac{\partial u}{\partial v_\eta} - v_\eta v_\psi \frac{\partial u}{\partial v_\psi} \right) + \nu u - K[u] = \tilde{S}, \\
    u(0, \nu) = p(\nu) \quad \text{for} \quad \nu_\eta > 0, \\
    u(L, \nu) = u(L, \mathcal{\Phi}[\nu]),
\end{cases}
\end{align*}
\]  
(4.227)

where
\[
\begin{align*}
\tilde{S} &= S + g_{2,L} G_1 \frac{1}{2} v_\eta v_\phi + g_{1,L} G_2 \frac{1}{2} v_\eta v_\psi,
\end{align*}
\]  
(4.228)
\[
p = h(\nu) - g_L(\nu).
\]  
(4.229)

The using the operators \( \mathcal{K} \) and \( \mathcal{T} \) defined in (4.181), (4.184) and (4.187), we can write \( u = \mathcal{K}[p] + \mathcal{T} \left[ K[u] \right] + \mathcal{T}[\tilde{S}] \). Based on Lemma 4.1.17, Lemma 4.1.18 and Lemma 4.1.19, we have
\[
\|u\|_{m,\nu} \lesssim \|\mathcal{K}[p]\|_{m,\nu} + \left\| \mathcal{T} \left[ K[u] \right] \right\|_{m,\nu} + \left\| \mathcal{T} \left[ \tilde{S} \right] \right\|_{m,\nu}
\]  
(4.230)

\[
\lesssim \|\mathcal{K}[p]\|_{\infty,\nu} + \left\| \mathcal{T} \left[ K[u] \right] \right\|_{m,\nu} + \left\| \mathcal{T} \left[ \tilde{S} \right] \right\|_{\infty,\nu}
\]  
(4.231)

\[
\lesssim |p|_{\nu,\delta} + C(\delta) \left\| \nu^{-\frac{1}{2}} K[u] \right\|_2 + \delta \|K[u]\|_{\infty,\nu} + \left\| \nu^{-\frac{1}{2}} \tilde{S} \right\|_{\infty,\nu}
\]  
(4.232)

where [8] Section 3.5 verifies
\[
\left\| \nu^{-\frac{1}{2}} K[u] \right\|_2 \lesssim \|K[u]\|_2 \lesssim \|u\|_2.
\]  
(4.233)

In [8] Lemma 3.3.1], it is shown that
\[
\|K[u]\|_{\infty,\nu} \lesssim \|u\|_{\infty,\nu-1,\nu},
\]  
(4.234)
\[
\|K[u]\|_{\infty,0,\nu} \lesssim \|u\|_{m,\nu}.
\]  
(4.235)

Since \( u = \mathcal{K}[p] + \mathcal{T} \left[ K[u] \right] + \mathcal{T}[\tilde{S}] \), by Lemma 4.1.18 and Lemma 4.1.19 using (4.232), we can estimate
\[
\|u\|_{\infty,\nu} \lesssim \|\mathcal{K}[p]\|_{\infty,\nu} + \left\| \mathcal{T} \left[ K[u] \right] \right\|_{\infty,\nu} + \left\| \mathcal{T} \left[ \tilde{S} \right] \right\|_{\infty,\nu}
\]  
(4.236)

\[
\lesssim |p|_{\nu,\delta} + \|K[u]\|_{\infty,\nu} + \left\| \nu^{-1} \tilde{S} \right\|_{\infty,\nu}
\]  
(4.237)

\[
\lesssim |p|_{\nu,\delta} + \|u\|_{m,\nu} + \left\| \nu^{-1} \tilde{S} \right\|_{\infty,\nu}.
\]  
(4.238)

Note that now we have \( \|u\|_{\infty,\nu-1,\nu} \). Hence, it is available to redo the above estimate (4.234) for \( \nu - 1 \). This procedure can keep going until the zeroth order. Then using (4.233) and (4.237), we obtain
\[
\|u\|_{\infty,\nu} \lesssim |p|_{\nu,\delta} + \|K[u]\|_{\infty,\nu} + \left\| \nu^{-1} \tilde{S} \right\|_{\infty,\nu}
\]  
(4.239)
\[
\lesssim |p|_{\nu,\delta} + \|K[u]\|_{\infty,0,\nu} + \left\| \nu^{-1} \tilde{S} \right\|_{\infty,\nu}
\]  
(4.240)
\[
\lesssim |p|_{\nu,\delta} + \|u\|_{m,\nu} + \left\| \nu^{-1} \tilde{S} \right\|_{\infty,\nu}
\]  
(4.241)
\[
\lesssim |p|_{\nu,\delta} + C(\delta) \|u\|_2 + \delta \|K[u]\|_{\infty,\nu} + \left\| \nu^{-1} \tilde{S} \right\|_{\infty,\nu}.
\]  
(4.242)
Therefore, for \( \delta \) sufficiently small, absorbing \( \delta \| K[u] \|_{\infty, \vartheta, \varphi} \) into the right-hand side of the first inequality of (4.235), we get

\[
\| K[u] \|_{\infty, \vartheta, \varphi} \lesssim |p|_{\infty, \vartheta, \varphi} + \| u \|_2 + \| \nu^{-1} \tilde{S} \|_{\infty, \vartheta, \varphi} .
\]  
(4.236)

Therefore, inserting (4.236) into the first inequality of (4.235), we have

\[
\| u \|_{\infty, \vartheta, \varphi} \lesssim |p|_{\infty, \vartheta, \varphi} + \| u \|_2 + \| \nu^{-1} \tilde{S} \|_{\infty, \vartheta, \varphi} .
\]  
(4.237)

In particular, using Lemma 4.1.12 and (4.15), we know

\[
\text{the equation (4.15) leads to the desired result.}
\]

Lemma 4.1.21 \((L^\infty \text{ well-posedness of } g)\). Assume (4.15) holds. Then there exists a unique solution \( g(\eta, v) \) to the \( \epsilon \)-Milne problem with geometric correction (4.1) satisfying for \( \rho \geq 0 \) and \( \vartheta \geq 3 \),

\[
\| g - g_{L} \|_{\infty, \vartheta, \varphi} \lesssim 1.
\]  
(4.240)

**Proof.** Based on Lemma 4.1.12 and Lemma 4.1.20 this is obvious. \( \square \)

Theorem 4.1.22 \((L^\infty \text{ well-posedness of } G)\). Assume (4.15) holds. Then there exists a unique solution \( G(\eta, v) \) to the \( \epsilon \)-Milne problem with geometric correction (4.1) satisfying for \( \rho \geq 0 \) and integer \( \vartheta \geq 3 \),

\[
\| G \|_{\infty, \vartheta, \varphi} \lesssim 1.
\]  
(4.241)

**Proof.** Based on Theorem 4.1.14 and Lemma 4.1.21 this is obvious. \( \square \)

\( L^\infty \) Decay

Now we intend to show the \( L^\infty \) decay of solution to the equation (4.1). Define \( U = e^{K_\eta}G \). Then \( U \) satisfies the equation

\[
\begin{cases}
\nu \frac{\partial U}{\partial \eta} + G_1 \left( v_\nu^2 \frac{\partial U}{\partial v_\nu} - v_\eta v_\nu \frac{\partial U}{\partial v_\nu} \right) + G_2 \left( \nu \frac{\partial U}{\partial \varphi} - v_\eta v_\nu \frac{\partial U}{\partial v_\nu} \right) + \mathcal{L}[U] = K_0 v_\eta U + e^{K_\eta}S , \\
U(0, v) = h(v) - \hat{h}(v) \text{ for } v_\eta > 0 , \\
U(L, v) = U(L, \mathcal{S}[v]) .
\end{cases}
\]  
(4.242)

Lemma 4.1.23 \((\text{decay estimate})\). Assume (4.15) holds. Then there exists \( 0 < K_0 < K \) such that for \( \rho \geq 0 \) and \( \vartheta \geq 3 \)

\[
\| U \|_{\infty, \vartheta, \varphi} \lesssim 1 + \| U \|_2 .
\]  
(4.243)

**Proof.** Since \( U = K[p] + \mathcal{T} \left[ K[U] \right] + \mathcal{T} \left[ K_0 v_\eta U \right] + \mathcal{T} \left[ e^{K_\eta}S \right] \), similar to the proof of Lemma 4.1.20 we have

\[
\begin{align*}
\| U \|_{\infty, \vartheta, \varphi} & \lesssim |p|_{\infty, \vartheta, \varphi} + \| U \|_2 + \| \nu^{-1} K_0 v_\eta U \|_{\infty, \vartheta, \varphi} + \| \nu^{-1} e^{K_\eta}S \|_{\infty, \vartheta, \varphi} \\
& \lesssim |p|_{\infty, \vartheta, \varphi} + \| U \|_2 + K_0 \| U \|_{\infty, \vartheta, \varphi} + \| \nu^{-1} e^{K_\eta}S \|_{\infty, \vartheta, \varphi} .
\end{align*}
\]  
(4.244)

When \( K_0 > 0 \) is sufficiently small, we may absorb \( K_0 \| U \|_{\infty, \vartheta, \varphi} \) into the left-hand side to obtain

\[
\| U \|_{\infty, \vartheta, \varphi} \lesssim |p|_{\infty, \vartheta, \varphi} + \| U \|_2 + \| \nu^{-1} e^{K_\eta}S \|_{\infty, \vartheta, \varphi} .
\]  
(4.245)

Then (4.15) leads to the desired result. \( \square \)
Theorem 4.1.24 ($L^\infty$ decay). Assume (4.15) holds. Then there exists $0 < K_0 < K$ such that the solution $g(\eta, v)$ to (4.6) satisfying for $\varrho \geq 0$ and $\vartheta > 3$,

$$\|e^{K_0 \eta} g\|_{\infty, \varrho, \vartheta} \lesssim 1.$$  \hfill (4.246)

Proof. Based on Theorem 4.1.15 and Lemma 4.1.23 this is obvious. \qed
4.2 Regularity

Now we begin to study the regularity of the solution \( \mathcal{G} \) to (4.16). In this section, denote the boundary data \( p = h - \hat{h} \). Besides (4.15), throughout this section, we further require the regularity bound that for \( \varrho \geq 0 \) and \( \vartheta > 3 \)

\[
|\nabla_{\varrho} p|_{\infty, \varrho, \vartheta} \lesssim 1, \quad \|e^{K \varrho \partial_{\varrho}} S\|_{\infty, \varrho, \vartheta} + \|e^{K \varrho \nabla_{\varrho}} S\|_{\infty, \varrho, \vartheta} \lesssim 1. \tag{4.247}
\]

4.2.1 Preliminaries

Weight Function

Define a weight function

\[
\zeta(\eta, \varphi) = \left( v_\eta^2 + v_\varphi^2 + v_\psi^2 \right) - \left( \frac{R_1 - \epsilon \eta}{R_1} \right)^2 v_\phi^2 - \left( \frac{R_2 - \epsilon \eta}{R_2} \right)^2 v_\psi^2. \tag{4.248}
\]

It is easy to see that the closer a point \((\eta; \varphi, \psi, \psi)\) is to the grazing set \((\eta; \varphi, \psi, \psi) = (0; 0, v_\varphi, v_\psi)\), the smaller \(\zeta\) is. In particular, at the grazing set, \(\zeta(0; 0, v_\varphi, v_\psi) = 0\).

**Lemma 4.2.1** (weight function in \(\epsilon\) Milne problem). Let \(\zeta\) be defined as in (4.248). We have

\[
v_\eta \frac{\partial \zeta}{\partial \eta} - \frac{\epsilon}{R_1 - \epsilon \eta} \left( v_\phi^2 \frac{\partial \zeta}{\partial v_\phi} - v_\eta v_\psi \frac{\partial \zeta}{\partial v_\psi} \right) - \frac{\epsilon}{R_2 - \epsilon \eta} \left( v_\psi^2 \frac{\partial \zeta}{\partial v_\psi} - v_\eta v_\phi \frac{\partial \zeta}{\partial v_\phi} \right) = 0. \tag{4.249}
\]

**Proof.** We may directly compute

\[
\frac{\partial \zeta}{\partial \eta} = \frac{1}{\zeta} \left( \frac{R_1 - \epsilon \eta}{R_1} \epsilon v_\varphi^2 + \frac{R_2 - \epsilon \eta}{R_2} \epsilon v_\psi^2 \right),
\]

\[
\frac{\partial \zeta}{\partial v_\eta} = \frac{1}{\zeta} v_\eta, \quad \frac{\partial \zeta}{\partial v_\varphi} = \frac{1}{\zeta} \left( v_\varphi - \left( \frac{R_1 - \epsilon \eta}{R_1} \right)^2 v_\varphi \right), \quad \frac{\partial \zeta}{\partial v_\psi} = \frac{1}{\zeta} \left( v_\psi - \left( \frac{R_2 - \epsilon \eta}{R_2} \right)^2 v_\psi \right). \tag{4.251}
\]

Then we know

\[
v_\eta \frac{\partial \zeta}{\partial \eta} - \frac{\epsilon}{R_1 - \epsilon \eta} \left( v_\phi^2 \frac{\partial \zeta}{\partial v_\phi} - v_\eta v_\psi \frac{\partial \zeta}{\partial v_\psi} \right) - \frac{\epsilon}{R_2 - \epsilon \eta} \left( v_\psi^2 \frac{\partial \zeta}{\partial v_\psi} - v_\eta v_\phi \frac{\partial \zeta}{\partial v_\phi} \right) = 0. \tag{4.252}
\]

**Remark 4.2.2.** With this lemma in hand, we know for any function \(f\), we can put the weight \(\zeta\) inside the \(\epsilon\)-Milne operator, i.e.

\[
v_\eta \frac{\partial (\zeta f)}{\partial \eta} - \frac{\epsilon}{R_1 - \epsilon \eta} \left( v_\phi^2 \frac{\partial (\zeta f)}{\partial v_\phi} - v_\eta v_\psi \frac{\partial (\zeta f)}{\partial v_\psi} \right) - \frac{\epsilon}{R_2 - \epsilon \eta} \left( v_\psi^2 \frac{\partial (\zeta f)}{\partial v_\psi} - v_\eta v_\phi \frac{\partial (\zeta f)}{\partial v_\phi} \right) = 0. \tag{4.253}
\]
Important Lemmas

Lemma 4.2.3. For Boltzmann collision frequency \( \nu = \nu(\|v\|) \), we have

\[
\left| \frac{d\nu}{d\|v\|} \right| \lesssim 1. \tag{4.254}
\]

Proof. Based on [8] Chapter 3, we know

\[
\nu(\|v\|) \sim \left( 2 |v| + \frac{1}{|v|} \right) \int_0^{\|v\|} e^{-z^2}dz + e^{-|v|^2}. \tag{4.255}
\]

Then for \( |v| \geq 1 \), we have

\[
\left| \frac{d\nu}{d|v|} \right| \lesssim \left( 1 + \frac{1}{|v|^2} \right) \int_0^{|v|} e^{-z^2}dz + \left( |v| + \frac{1}{|v|} \right) e^{-|v|^2} \lesssim 1. \tag{4.256}
\]

For \( |v| \leq 1 \), the key difficulty is the fractional term. Taylor expansion implies

\[
\frac{1}{|v|} \int_0^{|v|} e^{-z^2}dz \sim \frac{1}{|v|} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} |v|^{2k+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} |v|^{2k} \lesssim 1. \tag{4.257}
\]

Hence, the desired result naturally follows. \( \square \)

Lemma 4.2.4. Let \( 0 \leq \varphi < \frac{1}{4} \) and \( \delta \geq 0 \). Then for \( \delta > 0 \) sufficiently small and any \( v \in \mathbb{R}^3 \),

\[
\int_{\mathbb{R}^3} e^{\delta |u-v|^2} \frac{1}{|u|} \left| k(u,v) \right| \frac{\langle v \rangle^{\varphi} e^{\varphi |v|^2}}{\langle u \rangle^{\varphi} e^{\varphi |u|^2}} du \lesssim 1. \tag{4.258}
\]

Proof. This proof is mainly motivated by [12] Lemma 3. Notice that

\[
\left| \frac{\langle v \rangle^{\varphi} e^{\varphi |v|^2}}{\langle u \rangle^{\varphi} e^{\varphi |u|^2}} \right| \lesssim \left( 1 + |u - v|^2 \right)^{\frac{\varphi}{2}} e^{-\varphi(|u|^2 - |v|^2)} \tag{4.259}
\]

Combining Lemma 4.2.3 and 4.259, we have

\[
|k(u,v)| \frac{\langle v \rangle^{\varphi} e^{\varphi |v|^2}}{\langle u \rangle^{\varphi} e^{\varphi |u|^2}} \lesssim \left( 1 + |u - v|^2 \right)^{\varphi} \left( |u - v| + \frac{1}{|u - v|} \right) e^{-\frac{1}{8}|u-v|^2 - \frac{1}{8} \frac{|u-v|^2}{|u-v|^2} - \varphi(|u|^2 - |v|^2)}. \tag{4.260}
\]

We first handle the exponential term in 4.260. Let \( \sigma = u - v \), so \( u = \sigma + v \). Then we have

\[
-\frac{1}{8} |u - v|^2 - \frac{1}{8} \frac{|u|^2 - |v|^2|^2}{|u - v|^2} - \varphi \left( |u|^2 - |v|^2 \right) \tag{4.261}
\]

\[
= -\frac{1}{8} |\sigma|^2 - \frac{1}{8} \frac{|\sigma + v|^2 - |v|^2|^2}{|\sigma|^2} - \varphi \left( |\sigma + v|^2 - |v|^2 \right) \tag{4.262}
\]

\[
= -\frac{1}{8} |\sigma|^2 - \frac{1}{8} \frac{|\sigma|^2 - 2\sigma \cdot v|^2}{|\sigma|^2} - \varphi \left( |\sigma|^2 - 2\sigma \cdot v \right) \tag{4.263}
\]

\[
= -\frac{1}{4} |\sigma|^2 + \frac{1}{2} \sigma \cdot v - \frac{1}{2} \frac{|\sigma|^2}{|\sigma|^2} - \varphi \left( |\sigma|^2 - 2\sigma \cdot v \right) \tag{4.264}
\]

\[
= \left( -\frac{1}{4} - \varphi \right) |\sigma|^2 + \left( \frac{1}{2} + 2\varphi \right) \sigma \cdot v - \frac{1}{2} \frac{|\sigma|^2}{|\sigma|^2}. \tag{4.265}
\]
Similarly, using spherical coordinates and substitution

\[ \text{Proof.} \]

Then we need to bound

\[ \delta \]

so the above quadratic form for \(|\sigma|\) and \(\frac{\sigma \cdot v}{|\sigma|}\) is negative definite. This implies

\[ -\frac{1}{8}|u - v|^2 - \frac{1}{8} \left| \frac{|u|^2 - |v|^2}{|u - v|^2} \right|^2 - \rho \left( |u|^2 - |v|^2 \right)^2 \lesssim - \left( |\sigma|^2 + \frac{|\sigma \cdot v|^2}{|\sigma|^2} \right) \lesssim - |u - v|^2. \tag{4.263} \]

In particular, for \(\delta\) small, the perturbed form is still negative definite, i.e.

\[ - \left( \frac{1}{8} - \delta \right) |u - v|^2 - \frac{1}{8} \left| \frac{|u|^2 - |v|^2}{|u - v|^2} \right|^2 - \rho \left( |u|^2 - |v|^2 \right)^2 \lesssim - \left( |\sigma|^2 + \frac{|\sigma \cdot v|^2}{|\sigma|^2} \right) \lesssim - |u - v|^2. \tag{4.264} \]

Hence, using Hölder’s inequality, (4.260) and (4.264), we may bound

\[ \int_{\mathbb{R}^3} e^{\delta|u|} \frac{1}{|u|} |k(u, v)| \frac{v}{u} \frac{e^{\theta|v|^2}}{e^{\theta|u|^2}} du \tag{4.265} \]

\[ \lesssim \int_{\mathbb{R}^3} \frac{1}{|u|} \left( |u - v| + \frac{1}{|u - v|} \right) e^{-|u - v|^2} du \]

\[ \lesssim \left( \int_{|u| \leq 1} \frac{1}{|u|^2} e^{-|u-v|^2} du \right)^{\frac{1}{2}} + \left( \int_{|u| \geq 1} \frac{1}{|u|} e^{-|u-v|^2} du \right)^{\frac{1}{2}} \]

\[ \lesssim \left( \int_{|u| \leq 1} \frac{1}{|u|^2} du \right)^{\frac{1}{2}} + \left( \int_{|u| \geq 1} e^{-|u-v|^2} du \right)^{\frac{1}{2}} \lesssim 1 + \left( \int_{\mathbb{R}^3} e^{-|\sigma|^2} d\sigma \right)^{\frac{1}{2}} \lesssim 1. \tag{4.266} \]

Similarly, using spherical coordinates and substitution \(v = u - v\), we have

\[ II \lesssim \left( \int_{\mathbb{R}^3} \left( |\sigma|^2 + \frac{1}{|\sigma|^2} \right) e^{-|\sigma|^2} d\sigma \right)^{\frac{1}{2}} \lesssim 1. \tag{4.267} \]

In summary, inserting (4.260) and (4.267) into (4.259), we obtain the desired result. \(\square\)

**Lemma 4.2.5.** Let \(0 \leq \varrho < \frac{1}{4}\) and \(\vartheta \geq 0\). We have

\[ \int_{\mathbb{R}^3} |\nabla k(u, v)| \frac{v}{u} \frac{e^{\theta|v|^2}}{e^{\theta|u|^2}} du \lesssim 1. \tag{4.268} \]

**Proof.** Based on [8] Chapter 3, for hard-sphere gas, \(k = k_1 + k_2\), where

\[ k_1(u, v) \sim |u - v| e^{-\frac{1}{2}|u|^2 - \frac{1}{2}|v|^2}, \tag{4.269} \]

\[ k_2(u, v) \sim \frac{1}{|u - v|} e^{-\frac{1}{2}|u|^2 - \frac{1}{2}|v|^2} \frac{|u|^2 - |v|^2}{|u - v|^2}. \tag{4.270} \]
Following the similar argument as in Lemma 2.2.5 and (4.260) in Lemma 2.2.4 we have

\[ |\nabla \varphi k(u, v)| \frac{\langle v \rangle^\varphi e^{\varphi|v|^2}}{\langle u \rangle^\varphi e^{\varphi|u|^2}} \lesssim \left(1 + |u - v|^2\right)^{\frac{\varphi}{2}} |\nabla \varphi k(u, v)| e^{-\varphi(|u|^2 - |v|^2)} \]  

(4.271)

Here, the key is to bound |\nabla \varphi k(u, v)|. Substituting \( u \to \sigma = u - v \), we get

\[ k_1(\sigma, v) = |\sigma| e^{-|\sigma|^2 - |\sigma - v - \frac{1}{2}v|^2}, \]  

(4.272)

\[ k_2(\sigma, v) = \frac{1}{|\sigma|} e^{-\frac{1}{2}|\sigma|^2 - \frac{1}{4} \frac{|\sigma|^2 - 2\sigma \cdot v}{|\sigma|^2}} \]  

(4.273)

Then we compute

\[ \nabla \varphi k_1(\sigma, v) = |\sigma| \left(-2 \sigma - \sigma \right) e^{-|\sigma|^2 - |\sigma - v - \frac{1}{2}v|^2}, \]  

(4.274)

which implies

\[ |\nabla \varphi k_1(\sigma, v)| \lesssim |\sigma|^2 e^{-|\sigma|^2 - |\sigma - v - \frac{1}{2}v|^2} + |\sigma| |v| e^{-|\sigma|^2 - |\sigma - v - \frac{1}{2}v|^2} := I_1 + I_2. \]  

(4.275)

Here, \( I_1 \) is covered by similar techniques as in the proof of Lemma 4.2.4, \( I_2 \) is covered in Lemma 2.2.5. We obtain

\[ I_1 \lesssim 1, \quad I_2 \lesssim \frac{|v|}{1 + |v|} \lesssim 1, \]  

(4.276)

which implies

\[ \int_{\mathbb{R}^3} \nabla \varphi k_1(u, v) \frac{\langle v \rangle^\varphi e^{\varphi|v|^2}}{\langle u \rangle^\varphi e^{\varphi|u|^2}} du \lesssim 1. \]  

(4.277)

On the other hand, we compute

\[ |\nabla \varphi k_2(\sigma, v)| = \frac{1}{|\sigma|} \left(\sigma - \frac{2\sigma \cdot v}{|\sigma|^2} \sigma\right) e^{-\frac{1}{2}|\sigma|^2 - \frac{1}{4} \frac{|\sigma|^2 - 2\sigma \cdot v}{|\sigma|^2}}, \]  

(4.278)

which implies

\[ |\nabla \varphi k_2(\sigma, v)| \lesssim e^{-\frac{1}{2}|\sigma|^2 - \frac{1}{4} \frac{|\sigma|^2 - 2\sigma \cdot v}{|\sigma|^2}} + \frac{|v|}{|\sigma|} e^{-\frac{1}{2}|\sigma|^2 - \frac{1}{4} \frac{|\sigma|^2 - 2\sigma \cdot v}{|\sigma|^2}} := II_1 + II_2. \]  

(4.279)

Still, \( II_1 \) is covered by similar techniques as in the proof of Lemma 4.2.4, \( II_2 \) is covered in Lemma 2.2.5. We obtain

\[ II_1 \lesssim 1, \quad II_2 \lesssim \frac{|v|}{1 + |v|} \lesssim 1, \]  

(4.280)

which implies

\[ \int_{\mathbb{R}^3} \nabla \varphi k_2(u, v) \frac{\langle v \rangle^\varphi e^{\varphi|v|^2}}{\langle u \rangle^\varphi e^{\varphi|u|^2}} du \lesssim 1. \]  

(4.281)

Then the desired results follow from (4.277) and (4.281).

**Lemma 4.2.6.** Let \( 0 \leq \varphi < \frac{1}{4} \) and \( \vartheta \geq 0 \). We have

\[ \int_{\mathbb{R}^3} |\nabla u k(u, v)| \frac{\langle v \rangle^\varphi e^{\varphi|v|^2}}{\langle u \rangle^\varphi e^{\varphi|u|^2}} du \lesssim \langle v \rangle^2. \]  

(4.282)
Proof. This is very similar to the proof of Lemma 4.2.5. Following a similar argument, we have

$$|\nabla \sigma k_1(u, v)| \leq \left(1 + |v - \sigma|^2\right) \frac{2}{|\sigma|} |\nabla \sigma k(u, v)| e^{-\varepsilon (|u|^2 - |\sigma|^2)}.$$

(4.283)

Here, the key is to bound $|\nabla \sigma k(u, v)|$. Substituting $u \rightarrow \sigma = u - v$, we get (4.272) and (4.273). Also, note that $\nabla \sigma = \nabla \sigma$. Then we compute

$$\nabla \sigma k_1(\sigma, v) = |\sigma| \left( -v - \sigma \right) e^{-|v|^2 - \frac{1}{2} |\sigma|^2} + \frac{\sigma}{|\sigma|} e^{-|v|^2 - \frac{1}{2} |\sigma|^2},$$

which implies

$$|\nabla \sigma k_1(\sigma, v)| \leq \left( |\sigma|^2 + 1 \right) e^{-|v|^2 - \frac{1}{2} |\sigma|^2} + |\sigma| |v| e^{-|v|^2 - \frac{1}{2} |\sigma|^2} := I_1 + I_2.$$

(4.284)

Here, using similar techniques as in the proof of Lemma 4.2.3, we obtain

$$I_1 \lesssim 1, \quad I_2 \lesssim \langle v \rangle,$$

(4.285)

which implies

$$\int_{\mathbb{R}^3} \nabla \sigma k_1(u, v) \frac{\langle v \rangle^2 e^{|v|^2}}{\langle u \rangle^2 e^{|u|^2}} \, du \lesssim |v|.$$

(4.286)

On the other hand, we compute

$$|\nabla \sigma k_2(\sigma, v)| = \frac{1}{|\sigma|} \left( -\sigma + v - \frac{2\sigma \cdot v}{|\sigma|^2} (v \cdot \mathcal{T}) \right) e^{-\frac{1}{4} |\sigma|^2 - \frac{1}{4} |v|^{2 - 2\sigma \cdot v} |\sigma|^2} - \frac{\sigma}{|\sigma|^2} e^{-\frac{1}{4} |\sigma|^2 - \frac{1}{4} |v|^{2 - 2\sigma \cdot v} |\sigma|^2},$$

(4.287)

for tensor

$$\mathcal{T} := \frac{1}{|\sigma|^2} \begin{pmatrix} \sigma_\sigma^2 + \sigma_\psi^2 & -\sigma_\eta \sigma_\phi & -\sigma_\phi \sigma_\psi \\ -\sigma_\eta \sigma_\phi & \sigma_\eta^2 + \sigma_\psi^2 & -\sigma_\phi \sigma_\psi \\ -\sigma_\phi \sigma_\psi & -\sigma_\phi \sigma_\psi & \sigma_\eta^2 + \sigma_\phi^2 \end{pmatrix},$$

(4.288)

which implies

$$|\nabla \sigma k_2(\sigma, v)| \lesssim \left(1 + \frac{1}{|\sigma|^2}\right) e^{-\frac{1}{4} |\sigma|^2 - \frac{1}{4} |v|^{2 - 2\sigma \cdot v} |\sigma|^2} + \frac{|v|}{|\sigma|} \frac{|v|^2}{|\sigma|^2} e^{-\frac{1}{4} |\sigma|^2 - \frac{1}{4} |v|^{2 - 2\sigma \cdot v} |\sigma|^2} := II_1 + II_2.$$n

(4.289)

Still, using similar techniques as in the proof of Lemma 4.2.4, we obtain

$$II_1 \lesssim 1, \quad II_2 \lesssim \langle v \rangle^2,$$

(4.290)

which implies

$$\int_{\mathbb{R}^3} \nabla \sigma k_2(u, v) \frac{\langle v \rangle^2 e^{|v|^2}}{\langle u \rangle^2 e^{|u|^2}} \, du \lesssim \langle v \rangle^2.$$

(4.291)

Then the desired results follow from (4.287) and (4.292). \qed

Lemma 4.2.7. For any $v \in \mathbb{R}^3$, we have

$$\int_{\mathbb{R}^3} \frac{1}{\zeta(\eta; u)} |k(u, v)| \frac{\langle v \rangle^2 e^{|v|^2}}{\langle u \rangle^2 e^{|u|^2}} \, du \lesssim 1 + |\ln(\epsilon_\eta)|.$$

(4.293)
Proof. Based on (4.260) and (4.264), we know

\[ |k(u,v)| \left( |v|^\theta e^{\phi|v|^2} \right) \lesssim \left( |u-v| + \frac{1}{|u-v|} \right) e^{-\delta|u-v|^2}. \]  

(4.294)

Based on (4.248), letting \( u = (u_\eta, u_\phi, u_\psi) \), we directly obtain

\[ \eta > 0 \quad \| u \| \lesssim \sqrt{u_\eta^2 + (\phi)u_\phi^2 + (\psi)u_\psi^2}. \]  

(4.295)

Hence, using (4.294) and (4.295), we bound

\[
\int_{\mathbb{R}^3} \frac{1}{\zeta(\eta; u)} |k(u,v)| \, du \lesssim \int_{\mathbb{R}^3} \frac{1}{\sqrt{u_\eta^2 + (\phi)u_\phi^2 + (\psi)u_\psi^2}} |u-v| e^{-\delta|u-v|^2} \, du \\
+ \int_{\mathbb{R}^3} \frac{1}{\sqrt{u_\eta^2 + (\phi)u_\phi^2 + (\psi)u_\psi^2}} |u-v| e^{-\delta|u-v|^2} \, du := I + II.
\]  

(4.296)

We need to estimate \( I \) and \( II \) separately. Since exponential term decays much faster than polynomial term, we have

\[
I \lesssim \int_{|u| \leq 1} \frac{1}{\sqrt{u_\eta^2 + (\phi)u_\phi^2 + (\psi)u_\psi^2}} e^{-|u-v|^2} \, du \\
\lesssim \int_{|u_\phi| \leq 1, |u_\psi| \leq 1} \left( \int_{|u_\eta| \leq 1} \frac{1}{\sqrt{u_\eta^2 + (\phi)u_\phi^2 + (\psi)u_\psi^2}} \, du_\eta \right) du_\phi du_\psi \\
\lesssim 1 + \int_{|u_\phi| \leq 1, |u_\psi| \leq 1} \left( \int_{|u_\eta| \leq 1} \frac{1}{\sqrt{u_\eta^2 + (\phi)u_\phi^2 + (\psi)u_\psi^2}} \, du_\eta \right) du_\phi du_\psi.
\]  

(4.297)

The key is to bound the inner integral for \( |u_\phi| \leq 1, |u_\psi| \leq 1, 0 < \eta \leq L = e^{-\frac{1}{2}}, \)

\[
J := \int_{|u_\phi| \leq 1} \frac{1}{\sqrt{u_\eta^2 + (\phi)u_\phi^2 + (\psi)u_\psi^2}} \, du_\eta \\
= 2 \ln \left( 1 + \sqrt{1 + (\phi)u_\phi^2 + (\psi)u_\psi^2} \right) - 2 \ln \left( \sqrt{(\phi)u_\phi^2 + (\psi)u_\psi^2} \right) \\
\lesssim 1 + (\phi)u_\phi^2 + (\psi)u_\psi^2 + \ln \left( (\phi)u_\phi^2 + (\psi)u_\psi^2 \right) \\
\lesssim 1 + \ln \left( (\phi)u_\phi^2 \right) + \ln \left( (\phi)u_\psi^2 \right) \lesssim 1 + \ln(\phi) + \ln |u_\phi| + \ln |u_\psi|.
\]  

(4.298)

Inserting (4.298) into (4.297), we obtain

\[
I \lesssim 1 + \int_{|u_\phi| \leq 1, |u_\psi| \leq 1} \left( 1 + \ln(\phi) + \ln |u_\phi| + \ln |u_\psi| \right) du_\phi du_\psi \\
\lesssim 1 + \ln(\phi) + \int_{|u_\phi| \leq 1} \ln |u_\phi| \, du_\phi + \int_{|u_\psi| \leq 1} \ln |u_\psi| \, du_\psi \lesssim 1 + \ln(\phi).
\]  

(4.299)
On the other hand, similar to (4.297), we have

\[
II \lesssim \int_{|u| \leq 1} \frac{1}{u_0 + (\epsilon \eta |u_\phi|^2 + (\epsilon \eta |u_\psi|^2)} \frac{1}{|u - v|^2} e^{-|u-v|^2} \, du
\]

\[
+ \int_{|u| \geq 1} \frac{1}{u_0 + (\epsilon \eta |u_\phi|^2 + (\epsilon \eta |u_\psi|^2)} \frac{1}{|u - v|^2} e^{-|u-v|^2} \, du
\]

\[
\lesssim \int_{|u_\phi| \leq 1, |u_\psi| \leq 1} \left( \int_{|u_\phi| \leq 1} \frac{1}{u_0 + (\epsilon \eta |u_\phi|^2 + (\epsilon \eta |u_\psi|^2)} \, du_\phi \right) \frac{1}{(u_\phi - v_\phi)^2 + (u_\phi - v_\psi)^2} \, du_\phi \, du_\psi.
\]

Inserting (4.298) into (4.300), and applying Hölder’s inequality, we obtain

\[
II \lesssim 1 + \int_{|u_\phi| \leq 1, |u_\psi| \leq 1} \frac{1 + |\ln(\eta)| + |\ln |u_\phi|| + |\ln |u_\psi||}{(u_\phi - v_\phi)^2 + (u_\phi - v_\psi)^2} \, du_\phi \, du_\psi \quad (4.301)
\]

\[
\lesssim 1 + |\ln(\eta)| + \int_{|u_\phi| \leq 1, |u_\psi| \leq 1} \frac{|\ln |u_\phi|| + |\ln |u_\psi||}{(u_\phi - v_\phi)^2 + (u_\phi - v_\psi)^2} \, du_\phi \, du_\psi
\]

\[
\lesssim 1 + |\ln(\eta)| + \left( \int_{|u_\phi| \leq 1, |u_\psi| \leq 1} \frac{1}{(u_\phi - v_\phi)^2 + (u_\phi - v_\psi)^2} \, du_\phi \, du_\psi \right) \frac{2}{3}
\]

\[
\times \left( \int_{|u_\phi| \leq 1, |u_\psi| \leq 1} \left( |\ln |u_\phi|| + |\ln |u_\psi|| \right)^3 \, du_\phi \, du_\psi \right) \frac{2}{3}.
\]

Note that using polar coordinates, we have

\[
\int_{|u_\phi| \leq 1, |u_\psi| \leq 1} \frac{1}{(u_\phi - v_\phi)^2 + (u_\phi - v_\psi)^2} \, du_\phi \, du_\psi \lesssim 1, \quad (4.302)
\]

\[
\int_{|u_\phi| \leq 1, |u_\psi| \leq 1} \left( |\ln |u_\phi|| + |\ln |u_\psi|| \right)^3 \, du_\phi \, du_\psi \lesssim 1. \quad (4.303)
\]

Hence, inserting (4.302) and (4.303) into (4.301), we get

\[
II \lesssim 1 + |\ln(\eta)|. \quad (4.304)
\]

Inserting (4.299) and (4.304) into (4.296), we obtain the desired result.

Remark 4.2.8. Lemma 4.2.4 and Lemma 4.2.7 are valid uniformly in \( v \in \mathbb{R}^3 \).

4.2.2 Mild Formulation

Taking \( \eta \) derivative in (4.6) and multiplying \( \zeta \) defined in (4.248) on both sides, we obtain the \( \epsilon \)-transport problem for \( \mathcal{A} : = \frac{\partial \mathcal{G}}{\partial \eta} \) as

\[
\left\{
\begin{array}{l}
\frac{\partial \mathcal{A}}{\partial \eta} + G_1(\eta) \left( v_\phi \frac{\partial \mathcal{A}}{\partial \eta} - v_\phi \frac{\partial \mathcal{A}}{\partial \phi} \right) + G_2(\eta) \left( v_\psi \frac{\partial \mathcal{A}}{\partial \eta} - v_\psi \frac{\partial \mathcal{A}}{\partial \psi} \right) + \nu \mathcal{A} = \mathcal{A} + S_{\mathcal{A}}, \\
\mathcal{A}(0, v) = p_{\mathcal{A}}(v) \quad \text{for} \quad v_\eta > 0, \\
\mathcal{A}(L, v) = \mathcal{A}(L, \mathcal{G}[v]),
\end{array}
\right.
\]

\[
(4.305)
\]
where the crucial non-local term

$$\mathcal{A}^\prime(\eta, \nu) = \int_{\mathbb{R}^3} \frac{\zeta(\eta, \nu)}{\zeta(\eta, u)} k(u, v) \mathcal{A}(\eta, u) du. \quad (4.306)$$

Here we utilize Lemma 4.2.1 to move $\zeta$ inside the derivative. $p_{\mathcal{A}}$ and $S_{\mathcal{A}}$ will be specified later. We need to derive the a priori estimate of $\mathcal{A}$. Note that $\mathcal{A}$ is different from $K[\mathcal{A}]$ since the denominator $\zeta(\eta, u)$ is possibly zero. Thus, this creates a strong singularity and becomes the major difficulty in this section. Here we use the notation as in $L^\infty$ estimates of Section 4.1.2 We can easily check that the weight function satisfies $\zeta = \sqrt{E_1 - E_2^2 - E_3^2}$. Along the characteristics, where $E_1$, $E_2$, $E_3$ and $\zeta$ are constants, the equation (4.305) can be rewritten as:

$$\nu_i \frac{d\mathcal{A}}{d\eta} + \mathcal{A} = \mathcal{A}^\prime + S_{\mathcal{A}}. \quad (4.307)$$

We can define the solution to (4.305) along the characteristics as follows:

$$\mathcal{A}(\eta, \nu) = K[p_{\mathcal{A}}] + T[\mathcal{A}^\prime + S_{\mathcal{A}}], \quad (4.308)$$

where the operators $K$ and $T$ are defined in (4.181) to (4.187). Based on Lemma 4.1.17 and Lemma 4.1.18, we can directly obtain

$$||K[p_{\mathcal{A}}]||_{\infty, \vartheta, \epsilon} \lesssim \|p_{\mathcal{A}}\|_{\infty, \vartheta, \epsilon}, \quad (4.309)$$

$$||T[S_{\mathcal{A}}]||_{\infty, \vartheta, \epsilon} \lesssim \|v^{-1} S_{\mathcal{A}}\|_{\infty, \vartheta, \epsilon}. \quad (4.310)$$

The next sections will be devoted to the estimate of $T[\mathcal{A}^\prime]$. Similar to Section 4.1.2 since we always assume that $(\eta, \nu)$ and $(\eta', \nu')$ are on the same characteristics, in the following, we will simply write $\nu'(\eta')$ or even $\nu'$ instead of $\nu'(\eta, \nu; \eta')$ when there is no confusion. In addition, we will use $\delta$ or $\delta_\epsilon$ to represent small quantities. They are not necessarily constants, but may depend on $\epsilon$ and need to be chosen later.

In the analysis below, we will repeatedly use the following packages of simple facts (PSF):

- Based on Theorem 4.1.22 and Theorem 4.1.23, we know $\|e^{K_0 G}\|_{\infty, \vartheta, \epsilon} \lesssim 1$.

- Based on Lemma 2.2.5, we have for $0 < \vartheta < \frac{1}{4}$ and $\vartheta > 3$

$$\|e^{K_0 G}\|_{\infty, \vartheta, \epsilon} \lesssim \|e^{K_0 G^{-1}}\|_{\infty, \vartheta, \epsilon} \lesssim 1. \quad (4.311)$$

- Based on Lemma 4.2.5, we know

$$\|e^{K_0 \nabla_x K[G]}\|_{\infty, \vartheta, \epsilon} \lesssim \|e^{K_0 G}\|_{\infty, \vartheta, \epsilon} \lesssim 1. \quad (4.312)$$

- Since $E_1$ is conserved along the characteristics, we must have $|\nu| = |\nu'|$ and further $(\nu')^{\vartheta} e^{(\nu')^2} = (\nu')^{\vartheta} e^{(\nu)^2}$.

**Region I:** $v_\eta > 0$

Based on (4.181), we need to bound

$$I = T[\mathcal{A}^\prime] = \int_0^\eta \mathcal{A}^\prime(\eta', \nu'(\eta, \nu; \eta')) \frac{v_\nu'(\eta, \nu; \eta')}{v_\eta'(\eta, \nu; \eta')} \exp(-H_{\eta, \nu'}) d\eta'. \quad (4.313)$$

Step 0: Preliminaries.

Based on (4.173) and (4.20), we have

$$E_2(\eta', \nu') = \frac{R_1 - \epsilon \nu'}{R_1} v_\phi', \quad E_3(\eta', \nu') = \frac{R_2 - \epsilon \nu'}{R_2} v_\phi'. \quad (4.314)$$
Then we can directly obtain for $0 < \eta' < L = e^{-\frac{1}{\epsilon}}$,

$$\zeta(\eta', \psi') = \sqrt{v_{\eta}^2 + v_{\phi}^2 + v_{\psi}^2} - \left( \frac{R_1 - \epsilon\eta'}{R_1} \right)^2 v_{\phi}^2 - \left( \frac{R_2 - \epsilon\eta'}{R_2} \right)^2 v_{\psi}^2$$

$$= \sqrt{v_{\eta}^2 + \frac{R_1^2 - (R_1 - \epsilon\eta')^2}{R_1^2} v_{\phi}^2 + \frac{R_2^2 - (R_2 - \epsilon\eta')^2}{R_2^2} v_{\psi}^2}$$

$$\leq \sqrt{v_{\eta}^2 + \frac{1}{R_1} \left( \sqrt{R_1^2 - (R_1 - \epsilon\eta')^2} \right) v_{\phi}^2 + \frac{1}{R_2} \left( \sqrt{R_2^2 - (R_2 - \epsilon\eta')^2} \right) v_{\psi}^2}$$

$$\leq |v_{\eta}'| + \sqrt{\epsilon\eta'} |v_{\phi}'| + \sqrt{\epsilon\eta'} |v_{\psi}'| \leq |\psi'|.$$  

and

$$\zeta(\eta', \psi') \geq \frac{1}{2} \left( \sqrt{v_{\eta}^2 + \frac{1}{R_1} \left( \sqrt{R_1^2 - (R_1 - \epsilon\eta')^2} \right) v_{\phi}^2 + \frac{1}{R_2} \left( \sqrt{R_2^2 - (R_2 - \epsilon\eta')^2} \right) v_{\psi}^2} \right)$$

$$\geq \sqrt{|v_{\eta}'| + \sqrt{\epsilon\eta'} |v_{\phi}'| + \sqrt{\epsilon\eta'} |v_{\psi}'|} \geq \sqrt{\epsilon\eta'} |\psi'|.$$  

Also, considering (4.173) and (4.20), we know for $0 \leq \eta' \leq \eta$,

$$v_{\eta}' = \sqrt{v_{\eta}^2 + v_{\phi}^2 + v_{\psi}^2 - v_{\phi}^2 - v_{\psi}^2} = \sqrt{v_{\eta}^2 + v_{\phi}^2 + v_{\psi}^2 - v_{\phi}^2 \left( \frac{R_1 - \epsilon\eta}{R_1 - \epsilon\eta'} \right)^2 - v_{\psi}^2 \left( \frac{R_2 - \epsilon\eta}{R_2 - \epsilon\eta'} \right)^2}$$

$$= \sqrt{v_{\eta}^2 + \left( 2R_1 - \epsilon\eta - \epsilon\eta' \right) \left( \epsilon\eta - \epsilon\eta' \right) v_{\phi}^2 + \left( 2R_2 - \epsilon\eta - \epsilon\eta' \right) \left( \epsilon\eta - \epsilon\eta' \right) v_{\psi}^2}.$$  

Since for $i = 1, 2$

$$0 \leq \left( 2R_i - \epsilon\eta - \epsilon\eta' \right) \left( \epsilon\eta - \epsilon\eta' \right) \leq \epsilon(\eta - \eta'),$$

$$1 \leq R_i - \epsilon\eta',$$  

we have

$$v_{\eta} \leq v_{\eta}' \leq \sqrt{v_{\eta}^2 + \epsilon(\eta - \eta') v_{\phi}^2 + \epsilon(\eta - \eta') v_{\psi}^2},$$

which means

$$\frac{1}{2} \frac{1}{\sqrt{v_{\eta}^2 + \epsilon(\eta - \eta') v_{\phi}^2 + \epsilon(\eta - \eta') v_{\psi}^2}} \leq \frac{1}{v_{\eta}'} \leq \frac{1}{v_{\eta}}.$$  

Therefore,

$$- \int_{\eta'}^{\eta} \frac{1}{v_{\eta}'(y)} dy \leq - \int_{\eta'}^{\eta} \frac{1}{2 \sqrt{v_{\eta}^2 + \epsilon(\eta - y) v_{\phi}^2 + \epsilon(\eta - y) v_{\psi}^2}} dy$$

$$= \frac{1}{\epsilon(v_{\phi}^2 + v_{\psi}^2)} \left( v_{\eta} - \sqrt{v_{\eta}^2 + \epsilon(\eta - \eta') v_{\phi}^2 + \epsilon(\eta - \eta') v_{\psi}^2} \right)$$

$$= - \frac{\eta - \eta'}{v_{\eta} + \sqrt{v_{\eta}^2 + \epsilon(\eta - \eta') v_{\phi}^2 + \epsilon(\eta - \eta') v_{\psi}^2}} \leq \frac{\eta - \eta'}{\sqrt{v_{\eta}^2 + \epsilon(\eta - \eta') v_{\phi}^2 + \epsilon(\eta - \eta') v_{\psi}^2}}.$$  

Define a $C^\infty$ cut-off function $\chi \in C^\infty[0, \infty)$ satisfying

$$\chi(v_{\eta}) = \begin{cases} 1 & \text{for } |v_{\eta}| \leq \delta, \\ 0 & \text{for } |v_{\eta}| \geq 2\delta. \end{cases}$$
We use $\chi$ to avoid discontinuous cut-off for the convenience of integration by parts. In the following, we will divide the estimate of $I$ in (4.305) into several cases based on the value of $v_\eta$, $v'_\eta$, $\eta'$ and $\epsilon(\eta - \eta')$. Assume the dummy variable $u = (u_\eta, u_\phi, u_\psi) = (u_\eta, \tilde{u})$. The similar notation also applies to $v = (v_\eta, v_\phi, v_\psi) = (v_\eta, \tilde{v})$.

Step 1: Estimate of $I_1: v_\eta \geq \delta_0$.
In this step, we will not resort to $\mathcal{A}$ equation (4.305), but rather directly bound

$$\left| \langle y \rangle^v e^{\delta|v|^2} I_1 \right| \leq \left| \langle y \rangle^v e^{\delta|v|^2} \frac{\partial G}{\partial \eta} \right| \leq \left| \langle y \rangle^{v+1} e^{\delta|v|^2} \frac{\partial G}{\partial \eta} \right|. \tag{4.324}$$

Hence, the key is to estimate $\frac{\partial G}{\partial \eta}$. As in (4.181), we rewrite the equation (4.6) along the characteristics as

$$G(\eta, v) = \exp \left( -H_{\eta,0} \right) p \left( \nu'(0) \right) + \int_0^\eta \frac{K[G] + S(\eta', v'(\eta'))}{v'_\eta(\eta')} \exp \left( H_{\eta',0} \right) d\eta'. \tag{4.325}$$

Taking $\eta$ derivative on both sides of (4.325), we have

$$\frac{\partial G}{\partial \eta} := X_1 + X_2 + X_3 + X_4 + X_5 + X_6, \tag{4.326}$$

where

$$X_1 = - \exp \left( -H_{\eta,0} \right) \frac{\partial H_{\eta,0}}{\partial \eta} \left( p \left( \nu'(0) \right) + \int_0^\eta \frac{K[G](\eta', v'(\eta'))}{v'_\eta(\eta')} \exp \left( H_{\eta',0} \right) d\eta' \right), \tag{4.327}$$

$$X_2 = \exp \left( -H_{\eta,0} \right) \frac{\partial p \left( \nu'(0) \right)}{\partial \eta}, \tag{4.328}$$

$$X_3 = \frac{(K[G] + S)(\eta, v)}{v_\eta}, \tag{4.329}$$

$$X_4 = - \exp \left( -H_{\eta,0} \right) \int_0^\eta \frac{\left( K[G] + S(\eta', v'(\eta')) \exp \left( H_{\eta',0} \right) \frac{1}{v'_\eta(\eta')} \frac{\partial v'_\eta(\eta')}{\partial \eta} \right) d\eta'}{v'_\eta(\eta')}, \tag{4.330}$$

$$X_5 = \exp \left( -H_{\eta,0} \right) \int_0^\eta \frac{\left( K[G] + S(\eta', v'(\eta')) \exp \left( H_{\eta',0} \right) \frac{\partial H_{\eta',0}}{\partial \eta} \right) d\eta'}{v'_\eta(\eta')}, \tag{4.331}$$

$$X_6 = \exp \left( -H_{\eta,0} \right) \int_0^\eta \frac{1}{v'_\eta(\eta')} \left( \nabla v' \left( K[G] + S(\eta', v'(\eta')) \frac{\partial v'(\eta')}{\partial \eta} \right) \exp \left( H_{\eta',0} \right) d\eta'. \tag{4.332}$$

We need to estimate each term. Below are some preliminary results:

- For $\eta' \leq \eta$, we must have $v'_\eta \geq v_\eta \geq \delta_0$, which means $1/v'_\eta \leq 1/v_\eta \leq 1/\delta_0$.

- Using substitution $y = H_{\eta,\nu}$, we know

$$\left| \int_0^\eta \frac{\nu'(\eta')}{v'_\eta(\eta')} \exp(-H_{\eta,\nu}) d\eta' \right| \leq \left| \int_0^\infty e^{-y} dy \right| = 1. \tag{4.333}$$

- For $t, s \in [0, \eta]$, based on (PSF), we have

$$|H_{t,s}| \lesssim \left| \int_s^t \frac{\nu'(y)}{v'_\eta(y)} dy \right| \lesssim |\frac{\nu|}{\delta_0}|t - s|. \tag{4.334}$$
• Considering

\[ v_\phi'(\eta') = v_\phi e^{W_1(\eta') - W_1(\eta)} = v_\phi \frac{R_1 - \epsilon \eta}{R_1 - \epsilon \eta'}, \]
\[ v'_\psi(\eta') = v_\psi e^{W_2(\eta') - W_2(\eta)} = v_\psi \frac{R_2 - \epsilon \eta}{R_2 - \epsilon \eta'}, \]
\[ v'_\eta(\eta') = \sqrt{v_\eta^2 + v_\phi^2 + v_\psi^2 - v'_\eta^2} = \sqrt{v_\eta^2 + v_\phi^2 + v_\psi^2 - \left( \frac{R_1 - \epsilon \eta}{R_1 - \epsilon \eta'} \right)^2 - \frac{R_2 - \epsilon \eta}{R_2 - \epsilon \eta'}^2}, \]

we know

\[ \frac{\partial v'_\phi(\eta')}{\partial \eta} = -\frac{\epsilon v_\phi}{R_1 - \epsilon \eta}, \quad \frac{\partial v'_\psi(\eta')}{\partial \eta} = -\frac{\epsilon v_\psi}{R_2 - \epsilon \eta}, \]
\[ \frac{\partial v'_\eta(\eta')}{\partial \eta} = \frac{2\epsilon}{v_\eta(\eta)} \left( \frac{R_1 - \epsilon \eta}{R_1 - \epsilon \eta'} + \frac{R_2 - \epsilon \eta}{R_2 - \epsilon \eta'} \right) . \]

This implies

\[ \left| \frac{\partial v'_\phi(\eta')}{\partial \eta} \right| \lesssim \epsilon |v|, \quad \left| \frac{\partial v'_\psi(\eta')}{\partial \eta} \right| \lesssim \epsilon |v|, \quad \left| \frac{\partial v'_\eta(\eta')}{\partial \eta} \right| \lesssim \frac{\epsilon |v|^2}{v'_\eta(\eta')} \lesssim \frac{\epsilon |v|^2}{\delta_0} . \]  

• For \( t, s \in [0, \eta] \), note that

\[ \frac{\partial H_{t,s}}{\partial \eta} = \int_s^t \frac{\partial}{\partial \eta} \left( \frac{\nu(\nu'(y))}{v'_\eta(y)} \right) dy \]
\[ = \int_s^t \frac{1}{v'_\eta(y)} \frac{\partial \nu(\nu'(y))}{\partial |v'|} (y) \frac{1}{|v'(y)|} \left( v'_\eta(y) \frac{\partial v'(y)}{\partial \eta} + v'_\psi(y) \frac{\partial v'(y)}{\partial \eta} + v'_\phi(y) \frac{\partial v'(y)}{\partial \eta} \right) dy - \int_s^t \frac{\nu(\nu'(y))}{v'_\eta(y)} \frac{\partial v'(y)}{\partial \eta} dy. \]

Based on (4.334), Lemma 12.2.3 and (PSF), we obtain

\[ \left| \frac{\partial H_{t,s}}{\partial \eta} \right| \lesssim \int_s^t \left( \frac{\nu(\nu'(y))}{v'_\eta(y)} \right) dy + \int_s^t \frac{\nu(\nu'(y)) |v|^2}{v'_\eta(y)} dy \]
\[ \lesssim \frac{\epsilon (v)^2}{\delta_0} |H_{t,s}| \lesssim \frac{\epsilon (v)^3}{\delta_0} |t - s| \lesssim \frac{\epsilon \eta (v)^3}{\delta_0} \lesssim \frac{(v)^3}{\delta_0}. \]

The estimate of \( X_1 \) is standard based on (PSF) and the above preliminaries. Using (4.326) and (4.411), we have

\[ \left| \langle v \rangle^\theta e^{(v)^2} X_1 \right| \lesssim \left| \frac{\partial H_{n,0}}{\partial \eta} \right| \left| \langle v \rangle^\theta e^{(v)^2} G \right| \lesssim \left( \epsilon \frac{|v|}{v_\eta} + \int_0^\eta \frac{\partial}{\partial \eta} \left( \frac{\nu(\nu'(y))}{v'_\eta(y)} \right) dy \right) \left| \langle v \rangle^\theta e^{(v)^2} G \right| \]
\[ \lesssim \left( \epsilon + \frac{(v)^3}{\delta_0} \right) \left| \langle v \rangle^\theta e^{(v)^2} G \right| \lesssim \frac{1}{\delta_0} \| G \|_{\infty, \theta + 3, \sigma} \lesssim \frac{1}{\delta_0}. \]

Based on (4.339) and (4.241), we know

\[ \left| \langle v \rangle^\theta e^{(v)^2} X_2 \right| \lesssim \left| \exp (-H_{n,0}) \right| \left| \langle v \rangle^\theta e^{(v)^2} \nabla_v p \right| \left| \frac{\partial v'(0)}{\partial \eta} \right| \lesssim \left( \epsilon |v| + \frac{|v|^2}{\delta_0} \right) \left| \langle v \rangle^\theta e^{(v)^2} \nabla_v p \right| \]
\[ \lesssim \frac{\epsilon}{\delta_0} \left| \nabla_v p \right|_{\infty, \theta + 2, \sigma} \lesssim \frac{\epsilon}{\delta_0}. \]

Also, using (4.18) and Lemma 2.2.5, we have

\[ \left| \langle v \rangle^\theta e^{(v)^2} X_3 \right| \lesssim \frac{1}{v_\eta} \left( \left| \langle v \rangle^\theta e^{(v)^2} K[G] \right| + \left| \langle v \rangle^\theta e^{(v)^2} S \right| \right) \lesssim \frac{1}{\delta_0} \left( 1 + \left| \langle v \rangle^\theta e^{(v)^2} \nu^{-1} G \right| \right) \lesssim \frac{1}{\delta_0}. \]
On the other hand, using (4.339), (4.333) and (1.13), we obtain

$$\left| (v)^\eta \exp(|v|^2) X_4 \right| \lesssim \int_0^\eta \left( \left| (v')^\eta \exp(|v'|^2) K[G] \right| + \left| (v')^\eta \exp(|v'|^2) S \right| \right) \exp(-H_{\eta,\eta'}) \frac{\epsilon |v|^2}{\delta_0} \, dy'$$

(4.45)

Using (4.331), (4.333) and (1.14), we know

$$\left| (v)^\eta \exp(|v|^2) X_5 \right| \lesssim \int_0^\eta \frac{\epsilon}{\delta_0} \left( \left| (v')^\eta \exp(|v'|^2) K[G] \right| + \left| (v')^\eta \exp(|v'|^2) S \right| \right) \exp(-H_{\eta,\eta'}) \frac{(v)^3}{\delta_0^3} \, dy'$$

(4.46)

Finally, using (4.339), (4.333) and (1.247), we have

$$\left| (v)^\eta \exp(|v|^2) X_6 \right| \lesssim \int_0^\eta \frac{\epsilon}{\delta_0} \left( \left| (v')^\eta \exp(|v'|^2) \nabla_v K[G] \right| + \left| (v')^\eta \exp(|v'|^2) \nabla_v S \right| \right) \exp(-H_{\eta,\eta'}) \left( \epsilon |v| + \frac{\epsilon |v|^2}{\delta_0} \right) \, dy'$$

(4.47)

Collecting all $X_i$ estimates, we have

$$\left| (v)^\eta \exp(|v|^2) I_1 \right| \lesssim \frac{\epsilon}{\delta_0} + \frac{1}{\delta_0}.$$  

(4.48)

Step 2: Estimate of $I_2$: $0 \leq \nu_\eta \leq \delta_0$ with $1 - \chi(u_\eta)$ term.

We naturally decompose $1 = \left( 1 - \chi(u_\eta) \right) + \chi(u_\eta)$. In this step, we focus on $1 - \chi(u_\eta)$ part, while $\chi(u_\eta)$ part will handled in following steps involving $I_3, I_4, I_5$. Based on (4.323), the cut-off $1 - \chi(u_\eta)$ is nonzero only when $|u_\eta| \geq \delta$. We have

$$I_2 := \int_0^\eta \left( \int_{\mathbb{R}^3} \zeta(\eta', \nu') \left( 1 - \chi(u_\eta) \right) k(u, v') \nu' \phi(\eta', u) |u| \, du \right) \frac{\exp(-H_{\eta,\eta'})}{\nu_\eta} \, dy'$$

(4.49)

$$= \int_0^\eta \left( \int_{\mathbb{R}^3} \left( 1 - \chi(u_\eta) \right) k(u, v') G(\eta', u) \frac{\partial G(\eta', u)}{\partial \eta'} \frac{\partial G(\eta', u)}{\partial \phi} \right) \, dy'.$$

We first handle the inner integral. Based on (1.6), $G(\eta', u)$ satisfies

$$u_\eta \frac{\partial G(\eta', u)}{\partial \eta'} + G_1(\eta') \left( u_\eta^2 \frac{\partial G(\eta', u)}{\partial u_\eta} - u_\eta u_{\nu'} \frac{\partial G(\eta', u)}{\partial u_{\nu'}} \right)$$

(4.50)

$$+ G_2(\eta') \left( u_\eta^2 \frac{\partial G(\eta', u)}{\partial u_\eta} - u_\eta u_{\nu'} \frac{\partial G(\eta', u)}{\partial u_{\nu'}} \right) + \nu G(\eta', u) - K[G](\eta', u) = S(\eta', u),$$

which implies

$$\frac{\partial G(\eta', u)}{\partial \eta'} = - \frac{1}{u_\eta} \left( G_1(\eta') \left( u_\eta^2 \frac{\partial G(\eta', u)}{\partial u_\eta} - u_\eta u_{\nu'} \frac{\partial G(\eta', u)}{\partial u_{\nu'}} \right) + G_2(\eta') \left( u_\eta^2 \frac{\partial G(\eta', u)}{\partial u_\eta} - u_\eta u_{\nu'} \frac{\partial G(\eta', u)}{\partial u_{\nu'}} \right) + \nu G(\eta', u) - K[G](\eta', u) - S(\eta', u) \right).$$

(4.51)
Hence, inserting (4.351) into the inner integral in (4.349), we have

\[
J := \int_{\mathbb{R}^3} \left(1 - \chi(u_\eta)\right) k(u, v') G(\eta', u) \frac{\partial G(\eta', u)}{\partial \eta'} \, du \tag{4.352}
\]

\[
= - \int_{\mathbb{R}^3} \left(1 - \chi(u_\eta)\right) \frac{1}{u_\eta} \left(\nu G(\eta', u) - K[G](\eta', u) - S(\eta', u)\right) \, du
\]

\[
- \int_{\mathbb{R}^3} \left(1 - \chi(u_\eta)\right) \frac{1}{u_\eta} G_1(\eta') \left(\frac{\partial G(\eta', u)}{\partial u_\eta} - u_\eta \frac{\partial G(\eta', u)}{\partial u_\phi}\right) \, du
\]

\[
- \int_{\mathbb{R}^3} \left(1 - \chi(u_\eta)\right) \frac{1}{u_\eta} G_2(\eta') \left(\frac{\partial G(\eta', u)}{\partial u_\psi} - u_\eta \frac{\partial G(\eta', u)}{\partial u_\psi}\right) \, du
\]

\[
:= J_1 + J_2 + J_3.
\]

Since \(|u_\eta| \geq \delta\), using Lemma 2.2.3 (4.15) and (PSF), we obtain

\[
\left| \left< (v')^\theta e^{|v'|^2} \right>_1 \right| \leq \frac{1}{\delta} \left( \left< |G|_{\infty, \theta+1, \phi} + \||S||_{\infty, \theta, \phi} \right> \right) \int_{\mathbb{R}^3} k(u, v') \left< (v')^\theta e^{|v'|^2} \right>_1 \, du \leq \frac{1}{\delta}. \tag{4.353}
\]

On the other hand, an integration by parts yields

\[
J_2 = \int_{\mathbb{R}^3} \frac{\partial}{\partial u_\eta} \left( \frac{1}{u_\eta} G_1(\eta') \left(1 - \chi(u_\eta)\right) k(u, v') \right) G(\eta', u) \, du \tag{4.354}
\]

\[
- \int_{\mathbb{R}^3} \frac{\partial}{\partial u_\phi} \left( \frac{1}{u_\eta} G_1(\eta') \left(1 - \chi(u_\eta)\right) k(u, v') \right) G(\eta', u) \, du,
\]

\[
= \int_{\mathbb{R}^3} \left( - \frac{u_\phi^2}{u_\eta} (1 - \chi(u_\eta)) - \frac{u_\phi^2}{u_\eta} \chi'(u_\eta) - \left(1 - \chi(u_\eta)\right) \right) G_1(\eta') k(u, v') G(\eta', u) \, du
\]

\[
+ \int_{\mathbb{R}^3} G_1(\eta') \left(1 - \chi(u_\eta)\right) \left( - \frac{u_\phi^2}{u_\eta} \frac{\partial k(u, v')}{\partial u_\eta} + u_\phi \frac{\partial k(u, v')}{\partial u_\phi} \right) G(\eta', u) \, du := J_{2,1} + J_{2,2}.
\]

Since \(|u_\eta| \geq \delta\), using Lemma 4.2.4 and Lemma 2.2.3 we have

\[
\left| \left< (v')^\theta e^{|v'|^2} \right>_2, J_{2,1} \right| \leq \frac{\epsilon}{\delta^2} \left| G \right|_{\infty, \theta+2, \phi} \sim \frac{\epsilon}{\delta^2}. \tag{4.355}
\]

Also, using Lemma 4.2.4 and Lemma 4.2.6 we have

\[
\left| \left< (v')^\theta e^{|v'|^2} \right>_2, J_{2,2} \right| \leq \frac{\epsilon}{\delta^2} \left| G \right|_{\infty, \theta+4, \phi} \sim \frac{\epsilon}{\delta}. \tag{4.356}
\]

(4.355) and (4.356) yield

\[
\left| \left< (v')^\theta e^{|v'|^2} \right>_2, J_2 \right| \leq \frac{\epsilon}{\delta^2}. \tag{4.357}
\]

Similarly,

\[
\left| \left< (v')^\theta e^{|v'|^2} \right>_2, J_3 \right| \leq \frac{\epsilon}{\delta^2}. \tag{4.358}
\]

In summary, the inner integral in (4.349)

\[
\left| \left< (v')^\theta e^{|v'|^2} \right>_1 \right| \leq \left| \left< (v')^\theta e^{|v'|^2} \right>_J \right| \leq \left| \left< (v')^\theta e^{|v'|^2} \right>_1 \right| + \left| \left< (v')^\theta e^{|v'|^2} \right>_2 \right| + \left| \left< (v')^\theta e^{|v'|^2} \right>_3 \right| \leq \frac{1}{\delta} + \frac{\epsilon}{\delta^2}. \tag{4.359}
\]
Then for the outer integral in (4.349), we can use (4.315) and (4.333) to show that
\[
\left| \int_0^\eta \zeta(\eta', \nu') \exp(-H_{\eta, \eta'}) d\eta' \right| \leq \left| \int_0^\eta \frac{\nu(\nu')}{v_{\eta}'(\nu')} \exp(-H_{\eta, \eta'}) d\eta' \right| \lesssim 1. \tag{4.360}
\]
Then we have
\[
|\langle \psi \rangle^\theta e^{\frac{1}{2} z}|^2 I_2 \lesssim |\langle \psi' \rangle^\theta e^{\frac{1}{2} z} J| \lesssim \frac{1}{\delta^2} + \epsilon \tag{4.361}
\]
Step 3: Estimate of $I_3$: $0 \leq v_{\eta} \leq \delta_0$, with $\chi(u_{\eta})$ term, and $\sqrt{c_{\eta}} \cdot |\tilde{v}| \geq v_{\eta}$.
Based on (4.323) and (4.349), we are left with $\chi(u_{\eta})$ part, which is nonzero only when $|u_{\eta}| \leq 2\delta$, i.e.
\[
\int_{0}^{\eta} \left( \int_{\mathbb{R}^3} \frac{\zeta(\eta', \nu')}{\zeta(\eta', u)} \chi(u_{\eta}) k(u, \nu') P(\eta', u) du \right) \frac{1}{v_{\eta}'} \exp(-H_{\eta, \eta'}) d\eta'. \tag{4.362}
\]
We will further decompose this integral into $I_3, I_4, I_5$. In this step, based on (4.315), $\sqrt{c_{\eta}} \cdot |\tilde{v}'| \geq v_{\eta}'$ implies
\[
\zeta(\eta', \nu') \lesssim |v_{\eta}'| + \sqrt{c_{\eta}} \cdot |\tilde{v}'| \lesssim \sqrt{c_{\eta}} \cdot |\tilde{v}'|. \tag{4.363}
\]
On the other hand, (4.316) implies
\[
\zeta(\eta', u) \gtrsim \sqrt{c_{\eta}} |u|. \tag{4.364}
\]
Then considering (4.363) and (4.364), the inner integral in (4.362)
\[
M : = \left| \int_{\mathbb{R}^3} \frac{\zeta(\eta', \nu')}{\zeta(\eta', u)} \chi(u_{\eta}) k(u, \nu') P(\eta', u) du \right| \lesssim |\tilde{v}'| \left| \int_{\mathbb{R}^3} \frac{1}{|u|} \chi(u_{\eta}) k(u, \nu') P(\eta', u) du \right|. \tag{4.365}
\]
Using Lemma 4.2.1, we know
\[
\left| \langle \psi \rangle^\theta e^{\frac{1}{2} z} M \right| \lesssim |\tilde{v}'| \|P\|_{\infty, \theta, \nu'} . \tag{4.366}
\]
This bound is too weak since we have not used the smallness $|u_{\eta}| \leq 2\delta$, which means the integral is actually over a very small domain. We naturally modify the proof of Lemma 4.2.1. The key step is (4.366). Here for either $|u| \leq 1$ or $|u| \geq 1$, the small domain of $u_{\eta}$ produces an extra smallness in integral. In precise,
\[
\left| \langle \psi \rangle^\theta e^{\frac{1}{2} z} M \right| \lesssim \delta |\tilde{v}'| \|P\|_{\infty, \theta, \nu'}. \tag{4.367}
\]
Here, this $|\tilde{v}'|$ will be handled in outer integral of (4.362) as in (4.360),
\[
\int_{0}^{\eta} \frac{|\tilde{v}'|}{v_{\eta}'} \exp(-H_{\eta, \eta'}) d\eta' \lesssim \int_{0}^{\eta} \frac{\nu(\nu')}{v_{\eta}'} \exp(-H_{\eta, \eta'}) d\eta' \lesssim 1. \tag{4.368}
\]
In total, we have
\[
\left| \langle \psi \rangle^\theta e^{\frac{1}{2} z} I_3 \right| \lesssim \delta \|P\|_{\infty, \theta, \nu'}. \tag{4.369}
\]
Step 4: Estimate of $I_4$: $0 \leq v_{\eta} \leq \delta_0$, with $\chi(u_{\eta})$ term, $\sqrt{c_{\eta}} \cdot |\tilde{v}'| \leq v_{\eta}'$ and $v_{\eta}' \leq \epsilon(\eta - \eta') |\tilde{v}|^2$.
$I_4$ is defined similar as (4.362). Based on (4.315), $\sqrt{c_{\eta}} \cdot |\tilde{v}'| \leq v_{\eta}'$ implies
\[
\zeta(\eta', \nu') \lesssim |v_{\eta}'| + \sqrt{c_{\eta}} \cdot |\tilde{v}'| \lesssim v_{\eta}'. \tag{4.370}
\]
Hence, similar to the derivation for $I_3$ in (4.365) and (4.367), using (4.364) and (4.370), we have
\[
M \lesssim \frac{v_{\eta}'}{\sqrt{c_{\eta}}} \left| \int_{\mathbb{R}^3} \frac{1}{|u|} \chi(u_{\eta}) k(u, \nu') P(\eta', u) du \right| . \tag{4.371}
\]
and
\[ |(\mathcal{V})^{\langle \eta \rangle} e^{\varphi|\mathcal{V}|^2} M| \lesssim \delta \frac{v_{\eta}^4}{\sqrt{\epsilon \eta}} \|\mathcal{A}\|_{\infty, \delta, \varrho}. \] (4.372)

Hence, we must handle \( \frac{v_{\eta}^4}{\sqrt{\epsilon \eta}} \) with the outer integral in (4.362). Based on (4.322), \( v_{\eta}^2 \leq \epsilon (\eta - \eta') |\bar{v}|^2 \) leads to
\[-H_{\eta, \eta'} = - \int_{\eta'}^{\eta} \frac{\nu(v)}{v_{\eta}(y)} dy \lesssim - \frac{\nu(v)(\eta - \eta')}{|\bar{v}| \sqrt{\epsilon (\eta - \eta')}} \lesssim - \frac{\nu(v)}{|\bar{v}|} \sqrt{\frac{\eta - \eta'}{\epsilon}}. \] (4.373)

Therefore, we know
\[ \int_{0}^{\eta} \frac{v_{\eta}^4}{\sqrt{\epsilon \eta} v_{\eta}} \exp(-H_{\eta, \eta'}) d\eta' = \int_{0}^{\eta} \frac{1}{\sqrt{\epsilon \eta}} \exp(-H_{\eta, \eta'}) d\eta' \]
\[ \lesssim \int_{0}^{\eta} \frac{1}{\sqrt{\epsilon \eta}} \exp\left(-\frac{\nu(v)}{|\bar{v}|} \sqrt{\frac{\eta - \eta'}{\epsilon}}\right) d\eta' = \int_{0}^{z} \frac{1}{\sqrt{z}} \exp\left(-\frac{\nu(v)}{|\bar{v}|} \sqrt{\frac{\eta - \eta'}{\epsilon}}\right) dz \]
\[ = \int_{1}^{\frac{z}{\sqrt{|\bar{v}|}}} \frac{1}{\sqrt{z}} \exp\left(-\frac{\nu(v)}{|\bar{v}|} \sqrt{\frac{\eta - \eta'}{\epsilon}}\right) dz + \int_{\frac{z}{\sqrt{|\bar{v}|}}}^{\frac{\eta}{\sqrt{|\bar{v}|}}} \frac{1}{\sqrt{z}} \exp\left(-\frac{\nu(v)}{|\bar{v}|} \sqrt{\frac{\eta - \eta'}{\epsilon}}\right) dz, \] (4.375)

where we define substitution \( \eta' \to z = \frac{\eta'}{\sqrt{\epsilon}}, \) which implies \( d\eta' = \epsilon dz. \) We can estimate these two terms separately.
\[ \int_{0}^{1} \frac{1}{\sqrt{z}} \exp\left(-\frac{\nu(v)}{|\bar{v}|} \sqrt{\frac{\eta - \eta'}{\epsilon}}\right) dz \leq \int_{0}^{1} \frac{1}{\sqrt{z}} dz = 2. \] (4.376)
\[ \int_{1}^{z} \frac{1}{\sqrt{z}} \exp\left(-\frac{\nu(v)}{|\bar{v}|} \sqrt{\frac{\eta - \eta'}{\epsilon}}\right) dz \leq \int_{1}^{z} \frac{1}{\sqrt{z}} \exp\left(-\frac{\nu(v)}{|\bar{v}|} \sqrt{\frac{\eta - \eta'}{\epsilon}}\right) dz \]
\[ \leq \int_{1}^{z} \frac{1}{\sqrt{z}} \exp\left(-\frac{\nu(v)}{|\bar{v}|} \sqrt{\frac{\eta - \eta'}{\epsilon}}\right) dz \]
\[ \lesssim \int_{0}^{\infty} te^{-\frac{\nu(v)}{|\bar{v}|} \sqrt{\frac{\eta - \eta'}{\epsilon}}} dt \lesssim \left(\frac{|\bar{v}|}{\nu(v)}\right)^2 \lesssim 1. \] (4.377)

Inserting (4.366) and (4.377) into (4.374), we know the outer integral in (4.362) is bounded. Therefore, we have
\[ |(\mathcal{V})^{\langle \eta \rangle} e^{\varphi|\mathcal{V}|^2} I_5| \lesssim \delta \|\mathcal{A}\|_{\infty, \delta, \varrho}. \] (4.378)

Step 5: Estimate of \( I_5: \) \( 0 \leq v_{\eta} \leq \delta_0, \) with \( \chi(u_{\eta}) \) term, \( \sqrt{\epsilon \eta'} |\bar{v}'| \leq v_{\eta}', v_{\eta}^2 \geq \epsilon (\eta - \eta') |\bar{v}|^2. \)
\( I_5 \) is defined similar as (4.362). Using (4.370), we have
\[ M \lesssim \left| \int_{\mathbb{R}^3} \frac{v_{\eta}'}{\zeta(\eta', u)} \chi(u_{\eta}) k(u, v') \mathcal{A}(\eta', u) du \right|. \] (4.379)

Using Lemma (4.2.7) we may bound
\[ |(\mathcal{V})^{\langle \eta \rangle} e^{\varphi|\mathcal{V}|^2} M| \lesssim v_{\eta}' \left(1 + |\ln(\epsilon \eta')|\right) \|\mathcal{A}\|_{\infty, \delta, \varrho}. \] (4.380)

Hence, we must handle \( v_{\eta}' \left(1 + |\ln(\epsilon \eta')|\right) \) with the outer integral in (4.362). Based on (4.322), \( v_{\eta}^2 \geq \epsilon (\eta - \eta') |\bar{v}|^2 \) implies
\[-H_{\eta, \eta'} = - \int_{\eta'}^{\eta} \frac{\nu(v')}{v_{\eta}'(y)} dy \lesssim - \frac{\nu(v')(\eta - \eta')}{v_{\eta}}. \] (4.381)
Therefore, we know
\[ \int_0^{\eta} v_{\eta}' \left( 1 + |\ln(\epsilon')| \right) \frac{1}{v_{\eta}'} \exp(-H_{\eta, \epsilon'}) d\eta' = \int_0^{\eta} \left( 1 + |\ln(\epsilon')| \right) \exp(-H_{\eta, \epsilon'}) d\eta' \]
\[ \lesssim \int_0^{\eta} \left( 1 + |\ln(\epsilon')| \right) \exp \left( -\frac{\nu(\epsilon')(\eta - \eta')}{v_{\eta}} \right) d\eta'. \]

Naturally,
\[ 1 + |\ln(\epsilon')| \lesssim \left( 1 + |\ln(\epsilon)| \right) + |\ln(\eta')|. \]

Since \( 0 \leq v_{\eta} \leq \delta_0 \), direct computation reveals that
\[ \int_0^{\eta} \left( 1 + |\ln(\epsilon)| \right) \exp \left( -\frac{\nu(\epsilon')(\eta - \eta')}{v_{\eta}} \right) d\eta' \lesssim \left( 1 + |\ln(\epsilon)| \right) \frac{v_{\eta}}{\nu(\epsilon')} \lesssim \delta_0 \left( 1 + |\ln(\epsilon)| \right). \]

Hence, it suffices to consider
\[ Q = \int_0^{\eta} |\ln(\eta')| \exp \left( -\frac{\nu(\epsilon')(\eta - \eta')}{v_{\eta}} \right) d\eta'. \]

If \( 0 \leq \eta \leq 2 \), applying Hölder's inequality, we have
\[ Q \lesssim \left( \int_0^{2} |\ln(\eta')|^2 d\eta' \right)^{\frac{1}{2}} \left( \int_0^{\eta} \exp \left( -\frac{2\nu(\epsilon')(\eta - \eta')}{v_{\eta}} \right) d\eta' \right)^{\frac{1}{2}} \lesssim \sqrt{\frac{v_{\eta}}{\nu(\epsilon')}} \lesssim \sqrt{\delta_0}. \]

If \( 2 \leq \eta \leq L = e^{-\frac{1}{2}} \), we decompose and apply Hölder’s inequality to obtain
\[ Q \lesssim \int_0^{2} |\ln(\eta')|^2 d\eta' + \int_2^{\eta} |\ln(\eta')| \exp \left( -\frac{\nu(\epsilon')(\eta - \eta')}{v_{\eta}} \right) d\eta'
\]
\[ \lesssim \left( \int_0^{2} |\ln(\eta')|^2 d\eta' \right)^{\frac{1}{2}} \left( \int_0^{\eta} \exp \left( -\frac{2\nu(\epsilon')(\eta - \eta')}{v_{\eta}} \right) d\eta' \right)^{\frac{1}{2}} + \ln(L) \int_2^{\eta} \exp \left( -\frac{\nu(\epsilon')(\eta - \eta')}{v_{\eta}} \right) d\eta'
\]
\[ \lesssim \sqrt{\delta_0} \left( 1 + |\ln(\epsilon)| \right). \]

In summary, we have
\[ \int_0^{\eta} |\ln(\eta')| \exp \left( -\frac{\nu(\epsilon')(\eta - \eta')}{v_{\eta}} \right) d\eta' \lesssim \sqrt{\delta_0} \left( 1 + |\ln(\epsilon)| \right). \]

This completes the bound of outer integral of (4.382). Hence, we know
\[ \left| \langle \nu \rangle^\theta e^{\theta |\nabla|^2} I_5 \right| \lesssim \sqrt{\delta_0} \left( 1 + |\ln(\epsilon)| \right) \| \mathcal{A} \|_{\infty, \delta, \epsilon}. \]

Step 6: Synthesis.
Collecting all estimates related to \( I_i \) in (4.345), (4.361), (4.369), (4.378) and (4.389), we have proved
\[ \left| \langle \nu \rangle^\theta e^{\theta |\nabla|^2} I \right| \lesssim \left( \delta + \sqrt{\delta_0} \left( 1 + |\ln(\epsilon)| \right) \| \mathcal{A} \|_{\infty, \delta, \epsilon} + \left( \frac{\epsilon}{\delta_0^3} + \frac{1}{\delta_0^4} + \frac{\epsilon}{\delta_0^2} + \frac{1}{\delta} \right) \right). \]

**Region II:** \( v_{\eta} < 0 \) and \( v^2_{\eta} + v^2_{\phi} \geq v^2_{\phi}(L) + v^2_{\phi}(L) \)

Based on (4.184), we only need to estimate
\[ T[\mathcal{A}] = \int_0^L \frac{\mathcal{A} \left( \eta, v'(\eta, v; \eta') \right)}{v_{\eta}'(\eta, v; \eta')} \exp(-H_{L, \epsilon'} - \mathcal{R}[H_{L, \eta}]) d\eta' \]
\[ + \int_\eta^L \frac{\mathcal{A} \left( \eta', \mathcal{R}[v'(\eta, v; \eta')] \right)}{v_{\eta}'(\eta, v; \eta')} \exp(\mathcal{R}[H_{L, \eta}]) d\eta'. \]
Here $\mathcal{R}[H] = H$ just for clarification. Notice that
\[ \exp(-H_{L,n'} - \mathcal{R}[H_{L,n}]) \lesssim \exp(-\mathcal{R}[H_{n',n}]). \] (4.392)

Also, we can decompose
\[ T[\mathcal{A}] = \int_0^\eta \mathcal{A}'(\eta', \nu(\eta')) \frac{\exp(-H_{L,n'} - \mathcal{R}[H_{L,n}])}{v_n'(\eta')} d\eta' \]
\[ + \int_\eta^L \mathcal{A}'(\eta', \nu(\eta')) \exp(-H_{L,n'} - \mathcal{R}[H_{L,n}]) d\eta' + \int_\eta^L \mathcal{A}'(\eta', \mathcal{R}[\nu(\eta')]) \frac{\exp(\mathcal{R}[H_{n',n}])}{v_n'(\eta')} d\eta'. \] (4.393)

The integral $\int_0^\eta$ part can be estimated as in Region I due to (4.392), so we focus on the integral $\int_\eta^L$ part. Also, due to (4.392), it suffices to estimate
\[ II = \int_\eta^L \mathcal{A}'(\eta', \nu(\eta')) \exp(-H_{n',n}) d\eta'. \] (4.394)

Here the proof is almost identical to that in Region I, so we only point out the key differences.

Step 0: Preliminaries.
(4.315) and (4.316) still holds, but the key result (4.322) needs to be updated. For $0 \leq \eta' \leq \eta'$,
\[ \nu' = \sqrt{E_1 - \nu'^2} = \sqrt{v_n^2 + v_\phi^2 + v_\psi^2} = \left( 1 - \frac{\nu'^2}{\nu} \right) \left( \frac{R_1 - \nu \eta'}{R_1 - \nu \eta'} \right) \left( \frac{R_2 - \nu \eta'}{R_2 - \nu \eta'} \right)^2 \leq |\nu| \] (4.395)

Then we have
\[ -\int_\eta^\eta' \frac{1}{\nu'(y)} dy \leq -\int_\eta^\eta' \frac{1}{|\nu|} dy = -\frac{\eta' - \eta}{|\nu|}. \] (4.396)

Here, note that $\nu' < 0$ but $\nu' \geq 0$ defined in (4.177).

Step 1: Estimate of $II_1$: $\nu' \leq -\delta_0$ and $\nu' \geq \frac{\delta_0}{2}$ for all $\eta' \in [0, L]$.

Since $\eta' \geq \eta$, we must have $\nu' \leq |\nu|$, so it is unclear whether $|\nu'| \geq \frac{\delta_0}{2}$ directly from $\nu \leq \delta_0$. Hence, we must put this as an additional requirement. If there exists some $\nu' \leq \frac{\delta_0}{2}$, it will be handled in $II_5$ estimate later. As for the estimate, this is very similar to the estimate of $I_1$. We will use the mild formulation of $G$ in (4.184) instead of $\mathcal{A}$ in (4.305).
\[ |(v)^\theta e^{\phi |v|^2} II_1| \lesssim |\zeta| |(v)^{\theta+1} e^{\phi |v|^2} \frac{\partial G}{\partial \eta}| \lesssim |(v)^{\theta+1} e^{\phi |v|^2} \frac{\partial G}{\partial \eta}|. \] (4.397)

Hence, the key is to estimate $\frac{\partial G}{\partial \eta}$. As in (4.184), we rewrite the equation (4.6) along the characteristics as
It suffices to consider
\[ G(\eta, \nu) \sim \exp(H_{n',0}) \int_\eta^L (K[G] + S)(\eta', \nu'(\eta')) \nu'(\eta') \exp(-H_{n',0}) d\eta'. \] (4.398)

where $\sim$ denotes that we only focus on $\int_\eta^L$ part due to the decomposition as in (4.393). The justification of $\int_0^\eta$ and boundary data $p$ part is covered by the estimate of $I_1$ and (4.392).
Taking $\eta$ derivative on both sides of (4.398), we have

$$\frac{\partial G}{\partial \eta} := Y_1 + Y_2 + Y_3 + Y_4 + Y_5,$$

where

$$Y_1 = \exp(H_{n,0}) \frac{\partial H_{n,0}}{\partial \eta} \int_{\eta}^{L} \frac{K[\mathcal{G}'](\eta', v'\eta')}{v_n'(\eta')} \exp(-H_{n',0}) \, d\eta',$$

$$Y_2 = \frac{(K[\mathcal{G}] + S)(\eta, v)}{v_n},$$

$$Y_3 = -\exp(H_{n,0}) \int_{\eta}^{L} \left(K[\mathcal{G}] + S\right)(\eta', v'(\eta')) \exp(-H_{n',0}) \frac{1}{v_n'^2(\eta')} \frac{\partial v_n'(\eta')}{\partial \eta} \, d\eta',$$

$$Y_4 = -\exp(H_{n,0}) \int_{\eta}^{L} \left(K[\mathcal{G}] + S\right)(\eta', v'(\eta')) \exp(-H_{n',0}) \frac{\partial H_{n',0}}{\partial \eta} \, d\eta',$$

$$Y_5 = \exp(H_{n,0}) \int_{\eta}^{L} \frac{1}{v_n'(\eta')} \left(\nabla v'(K[\mathcal{G}] + S)(\eta', v'(\eta')) \frac{\partial v_n'(\eta')}{\partial \eta}\right) \exp(-H_{n',0}) \, d\eta'. $$

We need to estimate each term. Below are some preliminary results, which are direct extension of (4.333), (4.334), (4.338) and (4.341):

- For $\eta' \geq \eta$, we must have $\frac{1}{v_n'} \lesssim \frac{1}{\delta_0}$.

- Using substitution $y = H_{n,n'}$, we know

$$\left|\int_{\eta}^{L} \frac{\nu \left(\nu'(\eta')\right)}{v_n'(\eta')} \exp(H_{n,n'}) \, dy\right| \leq \left|\int_{-\infty}^{0} e^y \, dy\right| = 1. \quad (4.405)$$

- For $t, s \in [\eta, L]$, based on (PSF), we have

$$|H_{t,s}| \lesssim \frac{|v|}{\delta_0} |t - s|. \quad (4.406)$$

- We have

$$\left|\frac{\partial v_n'(\eta')}{\partial \eta}\right| \lesssim \epsilon |v|, \quad \left|\frac{\partial v_n'(\eta')}{\partial \eta}\right| \lesssim \epsilon |v|, \quad \left|\frac{\partial v_n'(\eta')}{\partial \eta}\right| \lesssim \epsilon |v|^2 \lesssim \epsilon |v|^2. \quad (4.407)$$

- For $t, s \in [\eta, L]$, we obtain

$$\left|\frac{\partial H_{t,s}}{\partial \eta}\right| \lesssim \epsilon (v)^3 \frac{\delta_0^3}{\delta_0^3} |t - s| \lesssim \epsilon L (v)^3 \frac{\delta_0^3}{\delta_0^3} \lesssim (v)^3. \quad (4.408)$$

The estimate of $Y_i$ is standard based on (PSF) and the above preliminaries. Using Lemma (4.402) and (4.408), we have

$$\left|\langle v \rangle^\theta e^{\epsilon |v|^2} Y_1\right| \lesssim \left|\frac{\partial H_{n,0}}{\partial \eta}\right| \left|\int_{\eta}^{L} \frac{\nu \left(\nu'(\eta')\right)}{v_n'(\eta')} \exp(H_{n,n'}) \, d\eta'\right| \left|\langle v \rangle^\theta e^{\epsilon |v|^2} \nu^{-1} \mathcal{G}\right|$$

$$\lesssim \frac{(v)^3}{\delta_0^3} \left|\langle v \rangle^\theta e^{\epsilon |v|^2} \nu^{-1} \mathcal{G}\right| \leq \frac{1}{\delta_0^3} \left|\mathcal{G}\right|_{\infty, \theta + 2, \epsilon} \lesssim \frac{1}{\delta_0^3}. \quad (4.409)$$
Also, using (4.13) and Lemma 2.2.5, we have
\[
\left| \langle \psi \rangle^\theta e^{\psi \beta} Y_2 \right| \lesssim \frac{1}{v_\eta} \left( \left| \langle \psi \rangle^\theta e^{\psi \beta} K[G] \right| + \left| \langle \psi \rangle^\theta e^{\psi \beta} S \right| \right) \lesssim \frac{1}{\delta_0} \left( 1 + \left| \langle \psi \rangle^\theta e^{\psi \beta} v^{-1} \right| \right) \lesssim \frac{1}{\delta_0}. \tag{4.410}
\]

On the other hand, using (4.407), (4.405) and (4.13), we obtain
\[
\left| \langle \psi \rangle^\theta e^{\psi \beta} Y_3 \right| \lesssim \int_\eta^L \left( \left| \langle \psi \rangle^\theta e^{\psi \beta} v^2 \right| K[G] \right| + \left| \langle \psi \rangle^\theta e^{\psi \beta} v^2 S \right| \right) \exp \left( H_{\eta, \eta'} \right) \frac{1}{\delta_0} \frac{1}{\delta_0} d\eta' \tag{4.411}
\]
\[
\lesssim \frac{\epsilon}{\delta_0} \left( \| v^{-1} \|_{\infty, \eta + 2, e} + \| S \|_{\infty, \eta + 2, e} \right) \left( \int_\eta^L \exp \left( H_{\eta, \eta'} \right) d\eta' \right) \lesssim \frac{\epsilon}{\delta_0}.
\]

Using (4.408), (4.405) and (4.13), we know
\[
\left| \langle \psi \rangle^\theta e^{\psi \beta} Y_4 \right| \lesssim \int_\eta^L \frac{1}{\delta_0} \left( \left| \langle \psi \rangle^\theta e^{\psi \beta} v^2 \right| \nabla \psi K[G] \right| + \left| \langle \psi \rangle^\theta e^{\psi \beta} v^2 S \right| \right) \exp \left( H_{\eta, \eta'} \right) \frac{\langle \psi \rangle^3}{\delta_0^3} d\eta' \tag{4.412}
\]
\[
\lesssim \frac{\epsilon}{\delta_0} \left( \| v^{-1} \|_{\infty, \eta + 3, e} + \| S \|_{\infty, \eta + 3, e} \right) \left( \int_\eta^L \exp \left( H_{\eta, \eta'} \right) d\eta' \right) \lesssim \frac{1}{\delta_0}.
\]

Finally, using (4.407), (4.405) and (4.237), we have
\[
\left| \langle \psi \rangle^\theta e^{\psi \beta} Y_5 \right| \lesssim \int_\eta^L \frac{1}{\delta_0} \left( \left| \langle \psi \rangle^\theta e^{\psi \beta} v^2 \nabla \psi K[G] \right| + \left| \langle \psi \rangle^\theta e^{\psi \beta} v^2 S \right| \right) \exp \left( H_{\eta, \eta'} \right) \left( \epsilon v + \frac{1}{\delta_0} \right) d\eta' \tag{4.413}
\]
\[
\lesssim \frac{\epsilon}{\delta_0} \left( \| G \|_{\infty, \eta + 2, e} + \| S \|_{\infty, \eta + 2, e} \right) \left( \int_\eta^L \exp \left( H_{\eta, \eta'} \right) d\eta' \right) \lesssim \frac{\epsilon}{\delta_0}.
\]

Collecting all Y_i estimates, we have
\[
\left| \langle \psi \rangle^\theta e^{\psi \beta} I_1 \right| \lesssim \frac{\epsilon}{\delta_0} + \frac{1}{\delta_0^2}. \tag{4.414}
\]

Step 2: Estimate of I_2: \(-\delta_0 \leq \nu_\eta \leq 0\) with \(1 - \chi(u_\eta)\) term.
We decompose \(1 = \left( 1 - \chi(u_\eta) \right) + \chi(u_\eta)\).

\[
I_2 := \int_\eta^L \left( \int_{\mathbb{R}^3} \frac{\zeta'(\eta', \nu)}{\zeta'(\eta', \nu)} \left( 1 - \chi(u_\eta) \right) k(u, \nu) \mathcal{A}^\nu(\eta', u) du \right) \frac{1}{v_\eta} \exp(H_{\eta, \eta'}) d\eta'. \tag{4.415}
\]

Then by a similar argument as estimating I_2, we have
\[
\left| \langle \psi \rangle^\theta e^{\psi \beta} I_2 \right| \lesssim \frac{1}{\delta_2} + \frac{\epsilon}{\delta_2}. \tag{4.416}
\]

Step 3: Estimate of I_3: \(-\delta_0 \leq \nu_\eta \leq 0\), with \(\chi(u_\eta)\) term and \(\sqrt{\nu \eta' v_\phi} \geq \nu_{\phi}^*\).
This is similar to the estimate of I_3, we have
\[
\left| \langle \psi \rangle^\theta e^{\psi \beta} I_3 \right| \lesssim \delta \mathcal{A}^\nu_{\infty, \eta, \phi}. \tag{4.417}
\]

Step 4: Estimate of I_4: \(-\delta_0 \leq \nu_\eta \leq 0\), with \(\chi(u_\eta)\) term, and \(\sqrt{\nu \eta' v_\phi} \leq \nu_{\phi}^*\).
This step is different. We do not need to further decompose the cases like I_4 and I_5. Based on (4.396), we have,
\[
-H_{\eta, \eta'} \lesssim - \frac{\nu^2(\nu_{\phi}^* - \eta)}{v_\eta}. \tag{4.418}
\]
Then following the same argument in estimating $I_5$, we know
\[
\left| \langle \psi \rangle^0 e^{\|v\|^2} II_4 \right| \lesssim \sqrt{\delta_0 \left( 1 + |\ln(\epsilon)| \right)} \| \phi' \|_{\infty, \phi, \psi}.
\] (4.419)

Step 5: Estimate of $II_5$: $v_\eta \leq -\delta_0$ and $v'_\eta \leq \frac{\delta_0}{2}$ for some $\eta' \in [0, L]$.

Now we come back to study the leftover in Step 1, i.e. though the characteristic starts from a point with $|v_\eta| \geq \delta_0$, as it goes, we finally arrive at the region that $v'_\eta \leq \frac{\delta_0}{2}$.

Let $\left( \eta^*, -\frac{\delta_0}{2}, v^*_\phi, v^*_\psi \right)$ be on the same characteristic as $(\eta, v)$, i.e. this is the first time that the characteristic enters the region $v'_\eta \leq \frac{\delta_0}{2}$. In detail, we have
\[
v^*_\phi = \frac{R_1 - \epsilon \eta}{R_1 - \epsilon \eta^*} v^*_\phi, \quad v^*_\psi = \frac{R_2 - \epsilon \eta}{R_2 - \epsilon \eta^*} v^*_\psi, \quad (4.420)
\]
\[
v^2_\eta + v^2_\phi + v^2_\psi = \frac{\delta_0^2}{4} + \left( \frac{R_1 - \epsilon \eta}{R_1 - \epsilon \eta^*} \right)^2 v^2_\phi + \left( \frac{R_2 - \epsilon \eta}{R_2 - \epsilon \eta^*} \right)^2 v^2_\psi. \quad (4.421)
\]

Taking $\eta$ derivative in (4.211), we obtain
\[
\frac{\partial \eta^*}{\partial \eta} = \frac{\left( \frac{R_1 - \epsilon \eta}{R_1 - \epsilon \eta^*} \right)^2 v^2_\phi + \left( \frac{R_2 - \epsilon \eta}{R_2 - \epsilon \eta^*} \right)^2 v^2_\psi}{\left( \frac{R_1 - \epsilon \eta}{R_1 - \epsilon \eta^*} \right)^3 v^2_\phi + \left( \frac{R_2 - \epsilon \eta}{R_2 - \epsilon \eta^*} \right)^3 v^2_\psi}. \quad (4.422)
\]

Here we do not need to compute $\eta^*$ explicitly. Since $\eta < \eta^* \leq L$, we know $0 \leq \epsilon \eta < \epsilon \eta^* \leq \epsilon L = \epsilon^2$, which implies
\[
\frac{R_1}{2} \leq R_1 - \epsilon \eta^* < R_1 - \epsilon \eta \leq R_1, \quad \frac{R_2}{2} \leq R_2 - \epsilon \eta^* < R_2 - \epsilon \eta \leq R_2. \quad (4.423)
\]

Inserting (4.423) into (4.422), we have
\[
\left| \frac{\partial \eta^*}{\partial \eta} \right| \lesssim 1. \quad (4.424)
\]

Taking $\eta$ derivative in (4.420) and using (4.424) and (4.423), we obtain
\[
\frac{\partial v^*_\phi}{\partial \eta} = \epsilon |v_\phi| \left| \frac{\left( \frac{R_1 - \epsilon \eta}{R_1 - \epsilon \eta^*} \right)^2 v^*_\phi}{(R_1 - \epsilon \eta^*)^2} - \frac{1}{R_1 - \epsilon \eta^*} \right| \lesssim \epsilon \nu(v), \quad (4.425)
\]
\[
\frac{\partial v^*_\psi}{\partial \eta} = \epsilon |v_\psi| \left| \frac{\left( \frac{R_2 - \epsilon \eta}{R_2 - \epsilon \eta^*} \right)^2 v^*_\phi}{(R_2 - \epsilon \eta^*)^2} - \frac{1}{R_2 - \epsilon \eta^*} \right| \lesssim \epsilon \nu(v). \quad (4.426)
\]

Then we have the mild formulation between $\eta$ and $\eta^*$ as
\[
G(\eta, v) = G(\eta^*, -\frac{\delta_0}{2}, v^*_\phi, v^*_\psi) \exp(-H_{\eta^*, \eta}) + \int_{\eta^*}^{\eta} \frac{K[G] + S(\eta', v'(\eta, \eta'; v)'\psi)}{v^*_\psi(\eta, v; \eta')} \exp(H_{\eta', \eta})d\eta'. \quad (4.427)
\]

Similar to the estimate of $II_1$, taking $\eta$ derivative in (4.22) and multiplying $\zeta$ on both sides, we obtain
\[
\left| \langle \psi \rangle^0 e^{\|v\|^2} II_5 \right| = \left| \langle \psi \rangle^0 e^{\|v\|^2} \zeta(\eta, v) \frac{\partial G}{\partial \eta} \right| \lesssim \left| \langle \psi \rangle^0 e^{\|v\|^2} \zeta(P_1 + P_2) \right|, \quad (4.428)
\]
where
\[
P_1 = \frac{\partial G \left( \eta^*, -\frac{\delta_0}{2}, v_\phi^*, v_\psi^* \right)}{\partial \eta} \exp(-H_{\eta^*, \eta}),
\tag{4.429}
\]
\[
P_2 = -G \left( \eta^*, -\frac{\delta_0}{2}, v_\phi^*, v_\psi^* \right) \exp(-H_{\eta^*, \eta}) \frac{\partial H_{\eta^*, \eta}}{\partial \eta}
+ \frac{\partial}{\partial \eta} \left( \int_{\eta}^{\eta^*} \left( K[G] + S \left( \eta', v'(\eta, v; \eta') \right) \right) \exp(H_{\eta', \eta}) d\eta' \right).
\tag{4.430}
\]

Since for \( \eta' \in [\eta, \eta^*] \), we always have \( v' \geq \frac{\delta_0}{2} \), mimicking Step 1 to estimate \( H_1 \) and using (4.424), we may bound
\[
\left| \langle v \rangle^\theta e^{\varepsilon |v|^2} \zeta P_2 \right| \lesssim \frac{\varepsilon}{\delta_0^2} + \frac{1}{\delta_0}.
\tag{4.431}
\]

The key is the estimate of \( P_1 \): considering \( |\exp(-H_{\eta^*, \eta})| \lesssim 1 \) and using (4.424), (4.425) and (4.426), we have
\[
\left| \langle v \rangle^\theta e^{\varepsilon |v|^2} \zeta P_1 \right| \lesssim \left| \langle v \rangle^\theta e^{\varepsilon |v|^2} \right|
\times \frac{\partial G \left( \eta^*, -\frac{\delta_0}{2}, v_\phi^*, v_\psi^* \right)}{\partial \eta} \eta^* + \frac{\partial G \left( \eta^*, -\frac{\delta_0}{2}, v_\phi^*, v_\psi^* \right)}{\partial v_\phi^*} v_\phi^*
\tag{4.432}
\]
\[
+ \frac{\partial G \left( \eta^*, -\frac{\delta_0}{2}, v_\phi^*, v_\psi^* \right)}{\partial v_\psi^*} v_\psi^*
\]
\[
\lesssim \langle v \rangle^\theta e^{\varepsilon |v|^2} \eta^* + \varepsilon \left| \nu \left( \zeta \frac{\partial G}{\partial v_\phi} \right) \right|_{\infty, \theta, \varepsilon} + \varepsilon \left| \nu \left( \zeta \frac{\partial G}{\partial v_\psi} \right) \right|_{\infty, \theta, \varepsilon}.
\]

The estimate of \( \left| \langle v \rangle^\theta e^{\varepsilon |v|^2} \right| \) is achieved as in \( H_2, H_3, H_4 \) since now \( |v_\eta^*| \leq \frac{\delta_0}{2} \). However, we have to preserve the latter two terms related to \( \frac{\partial G}{\partial v_\phi} \) and \( \frac{\partial G}{\partial v_\psi} \). Hence, we have
\[
\left| \langle v \rangle^\theta e^{\varepsilon |v|^2} \zeta P_2 \right| \lesssim \left( \delta + \sqrt{\delta_0} \left( 1 + |\ln(\varepsilon)| \right) \right) \left| \mathcal{A} \right|_{\|v\|_{\infty, \theta, \varepsilon}} + \left( \frac{\varepsilon}{\delta_0^2} + \frac{1}{\delta_0} \right).
\tag{4.433}
\]

Inserting (4.31) and (4.33) into (4.28), we obtain
\[
\left| \langle v \rangle^\theta e^{\varepsilon |v|^2} \right| \lesssim \left( \delta + \sqrt{\delta_0} \left( 1 + |\ln(\varepsilon)| \right) \right) \left| \mathcal{A} \right|_{\|v\|_{\infty, \theta, \varepsilon}} + \left( \frac{\varepsilon}{\delta_0^2} + \frac{1}{\delta_0} \right)
\tag{4.434}
\]
\[
+ \varepsilon \left| \nu \left( \zeta \frac{\partial G}{\partial v_\phi} \right) \right|_{\infty, \theta, \varepsilon} + \varepsilon \left| \nu \left( \zeta \frac{\partial G}{\partial v_\psi} \right) \right|_{\infty, \theta, \varepsilon}.
\]

Step 6: Synthesis.
Collecting all estimates related to \( H_i \) in (4.31), (4.32), (4.33), (4.36) and (4.34), we have proved
\[
\left| \langle v \rangle^\theta e^{\varepsilon |v|^2} \right| \lesssim \left( \delta + \sqrt{\delta_0} \left( 1 + |\ln(\varepsilon)| \right) \right) \left| \mathcal{A} \right|_{\|v\|_{\infty, \theta, \varepsilon}} + \left( \frac{\varepsilon}{\delta_0^2} + \frac{1}{\delta_0} \right)
\tag{4.435}
\]
\[
+ \varepsilon \left| \nu \left( \zeta \frac{\partial G}{\partial v_\phi} \right) \right|_{\infty, \theta, \varepsilon} + \varepsilon \left| \nu \left( \zeta \frac{\partial G}{\partial v_\psi} \right) \right|_{\infty, \theta, \varepsilon}.
\]
**Region III:** \( v_\eta < 0 \) and \( v_\eta^2 + v_\phi^2 + v_\psi^2 \leq v_\eta^2(L) + v_\phi^2(L) \)

Based on (4.187), we only need to estimate

\[
III = T[\mathcal{A}] = \int_0^{\eta^+} \frac{\mathcal{A}(\eta', v'(\eta; \eta'))}{v_\eta(\eta')} \exp(-H_{\eta^+, \eta'} - \mathcal{B}[H_{\eta^+, \eta}]) d\eta'
\]

(4.436)

\[
+ \int_{\eta}^{\eta^+} \frac{\mathcal{A}(\eta', \mathcal{B}[v'(\eta; \eta')])}{v_\eta(\eta')} \exp(\mathcal{B}[H_{\eta, \eta'}]) d\eta'.
\]

Here \( \eta^+ \) is defined in (4.188) and \( \mathcal{B}[H] = H \). Notice that

\[
\exp(-H_{\eta^+, \eta'} - \mathcal{B}[H_{\eta^+, \eta}]) \lesssim \exp(-\mathcal{B}[H_{\eta', \eta}]).
\]

(4.437)

Also, we can decompose

\[
T[\mathcal{A}] = \int_0^{\eta} \frac{\mathcal{A}(\eta', v'(\eta'))}{v_\eta(\eta')} \exp(-H_{\eta^+, \eta'} - \mathcal{B}[H_{\eta^+, \eta}]) d\eta'
\]

(4.438)

\[
+ \int_{\eta}^{\eta^+} \frac{\mathcal{A}(\eta', v'(\eta'))}{v_\eta(\eta')} \exp(-H_{\eta^+, \eta'} - \mathcal{B}[H_{\eta^+, \eta}]) d\eta'.
\]

Due to (4.437), the integral \( \int_0^{\eta} \) part can be estimated as in Region I and the integral \( \int_{\eta}^{\eta^+} \) part can be estimated as in Region II, so we omit the details here. At the end of the day, we have

\[
|v|^3 e^{\theta|v|^2} III \lesssim \left( \delta + \sqrt{\delta_0} \left( 1 + |\ln(\epsilon)| \right) \right) \|\mathcal{A}\|_{\infty, \theta, \epsilon} + \left( \frac{\epsilon}{\delta_0} + \frac{1}{\delta_0} + \frac{\epsilon}{\delta^2} + \frac{1}{\delta} \right)
\]

(4.439)

\[
+ \epsilon \left\| \nu \left( \zeta \frac{\partial \mathcal{G}}{\partial v_\theta} \right) \right\|_{\infty, \theta, \epsilon} + \epsilon \left\| \nu \left( \zeta \frac{\partial \mathcal{G}}{\partial v_\psi} \right) \right\|_{\infty, \theta, \epsilon}.
\]

**4.2.3 Regularity Estimates**

**Estimates of Normal Derivative**

Collecting estimates (4.390), (4.435), (4.439) in these three regions, and inserting (4.309) and (4.310) into (4.308), we have

\[
\|\mathcal{A}\|_{\infty, \theta, \epsilon} \lesssim \left( \delta + \sqrt{\delta_0} \left( 1 + |\ln(\epsilon)| \right) \right) \|\mathcal{A}\|_{\infty, \theta, \epsilon} + \left( \frac{\epsilon}{\delta_0} + \frac{1}{\delta_0} + \frac{\epsilon}{\delta^2} + \frac{1}{\delta} \right)
\]

(4.440)

\[
+ \epsilon \left\| \nu \left( \zeta \frac{\partial \mathcal{G}}{\partial v_\theta} \right) \right\|_{\infty, \theta, \epsilon} + \epsilon \left\| \nu \left( \zeta \frac{\partial \mathcal{G}}{\partial v_\psi} \right) \right\|_{\infty, \theta, \epsilon} + |p, \mathcal{A}|_{\infty, \theta, \epsilon} + \|\nu^{-1} S_\mathcal{A}\|_{\infty, \theta, \epsilon}.
\]

Then we choose these constants to perform absorbing argument. First we choose \( 0 < \delta << 1 \) sufficiently small such that

\[
C \delta \leq \frac{1}{4},
\]

(4.441)

Then we take \( \delta_0 = \delta^2 (1 + |\ln(\epsilon)|)^{-2} \) such that

\[
C(1 + |\ln(\epsilon)|) \sqrt{\delta_0} \leq C \delta \leq \frac{1}{4},
\]

(4.442)

for \( \epsilon \) sufficiently small. Hence, we can absorb all the term related to \( \|\mathcal{A}\|_{\infty, \theta, \epsilon} \) on the right-hand side of (4.440) to the left-hand side to obtain the desired result.
Lemma 4.2.9. Assume (4.15) and (4.247) holds. We have
\[ \|A\|_{\infty, \varphi, e} \lesssim \|\ln(\epsilon)\|^8 + |p_A|_{\infty, \varphi, e} + \|\nu^{-1}S_A\|_{\infty, \varphi, e} + \epsilon \left\| \nu \left( \frac{\partial G}{\partial \varphi} \right) \right\|_{\infty, \varphi, e} + \epsilon \left\| \nu \left( \frac{\partial G}{\partial \psi} \right) \right\|_{\infty, \varphi, e}. \tag{4.443} \]

Estimates of Velocity Derivatives

Taking \(v_\eta\) derivative in (4.16) and multiplying \(\zeta\) defined in (4.248) on both sides, we obtain the \(\epsilon\)-transport problem for \(B := \zeta \frac{\partial G}{\partial \varphi}\) as
\[ \begin{cases} v_\eta \frac{\partial B}{\partial \eta} + G_1(\eta) \left( v_\varphi \frac{\partial B}{\partial \varphi} - v_\eta \frac{\partial B}{\partial \psi} \right) + G_2(\eta) \left( v_\psi \frac{\partial B}{\partial \psi} - v_\eta \frac{\partial B}{\partial \psi} \right) + \nu B = \hat{B} + S_B, \\ B(0, v) = p_B(v) \text{ for } v_\eta > 0, \\ B(L, v) = -B(L, B[v]), \end{cases} \tag{4.444} \]
where the crucial non-local term
\[ \hat{B}(\eta, v) = \int_{\mathbb{R}^3} \zeta(v) \partial_{v\eta} k(u, v) G(\eta, u) du. \tag{4.445} \]
Here we utilize Lemma 4.2.1 to move \(\zeta\) inside the derivative. \(p_B\) and \(S_B\) will be specified later. We need to derive the a priori estimate of \(B\). Compared with \(A\) defined in (4.306), the key difference is that \(B\) does not contain \(B\) directly but rather \(\hat{G}\). Hence, we no longer need the analysis in previous sections to tackle the strong singularities. Then directly tracking along the characteristics, by a similar but much simpler argument using Theorem 4.1.24, Lemma 4.2.5 and (4.15), (4.247), we obtain the desired result.

Lemma 4.2.10. Assume (4.15) and (4.247) holds. We have
\[ \|B\|_{\infty, \varphi, e} \lesssim 1 + |p_B|_{\infty, \varphi, e} + \|\nu^{-1}S_B\|_{\infty, \varphi, e}. \tag{4.446} \]
In a similar fashion, \(C := \zeta \frac{\partial G}{\partial \psi}\) and \(D := \zeta \frac{\partial G}{\partial \psi}\) can be estimated.

Lemma 4.2.11. Assume (4.15) and (4.247) holds. We have
\[ \|C\|_{\infty, \varphi, e} \lesssim 1 + |p_C|_{\infty, \varphi, e} + \|\nu^{-1}S_C\|_{\infty, \varphi, e}, \tag{4.447} \]
\[ \|D\|_{\infty, \varphi, e} \lesssim 1 + |p_D|_{\infty, \varphi, e} + \|\nu^{-1}S_D\|_{\infty, \varphi, e}. \tag{4.448} \]

A Priori Estimates

In this subsection, we combine above a priori estimates of normal and velocity derivatives.

Theorem 4.2.12. Assume (4.15) and (4.247) holds. We have
\[ \left\| \frac{\partial G}{\partial \eta} \right\|_{\infty, \varphi, e} + \left\| \nu \frac{\partial G}{\partial \psi} \right\|_{\infty, \varphi, e} \lesssim |\ln(\epsilon)|^8, \tag{4.449} \]
\[ \left\| \nu \frac{\partial G}{\partial \phi} \right\|_{\infty, \varphi, e} + \left\| \nu \frac{\partial G}{\partial \psi} \right\|_{\infty, \varphi, e} \lesssim 1. \tag{4.450} \]

Proof. Collecting the estimates for \(A, B, C\) and \(D\) in Lemma 4.2.9, Lemma 4.2.10 and Lemma 4.2.11, we have
\[ \|A\|_{\infty, \varphi, e} \lesssim |\ln(\epsilon)|^8 + |p_A|_{\infty, \varphi, e} + \|\nu^{-1}S_A\|_{\infty, \varphi, e} + \epsilon \left( \|\nu C\|_{\infty, \varphi, e} + \|\nu D\|_{\infty, \varphi, e} \right), \tag{4.451} \]
\[ \|B\|_{\infty, \varphi, e} \lesssim 1 + |p_B|_{\infty, \varphi, e} + \|\nu^{-1}S_B\|_{\infty, \varphi, e}, \tag{4.452} \]
\[ \|C\|_{\infty, \varphi, e} \lesssim 1 + |p_C|_{\infty, \varphi, e} + \|\nu^{-1}S_C\|_{\infty, \varphi, e}, \tag{4.453} \]
\[ \|D\|_{\infty, \varphi, e} \lesssim 1 + |p_D|_{\infty, \varphi, e} + \|\nu^{-1}S_D\|_{\infty, \varphi, e}. \tag{4.454} \]
Now we clear up these boundary terms and source terms. At \( \eta = 0 \), we know \( \zeta = v_\eta \). Hence, we may solve from (4.6) to get
\[
p_{\eta} = v_\eta \frac{\partial G}{\partial \eta}(0, v) = -\epsilon \left( v_\eta^2 \frac{\partial p}{\partial v_\eta} - v_\eta v_v \frac{\partial p}{\partial v_v} \right) - \epsilon \left( v_\eta^2 \frac{\partial p}{\partial v_\eta} - v_\eta v_v \frac{\partial p}{\partial v_v} \right) + v_\eta \nu = K \mathcal{G}(0, v)
\]
(4.455)

Therefore, using Theorem 4.12, Lemma 2.2.3 (4.15) and (4.247), we have
\[
|p_\eta|_{\infty, \delta, \varrho} \lesssim \epsilon |\nabla p|_{\infty, \delta, \varrho} + |p|_{\infty, \delta, \varrho} + ||p|^{-1}\mathcal{G}||_{\infty, \delta, \varrho} \lesssim 1.
\]
(4.466)

On the other hand, we can directly take derivative in the boundary data \( p \) to get
\[
p_{\varrho} = v_\eta \frac{\partial p}{\partial \eta}, \quad p_\varrho = v_\eta \frac{\partial p}{\partial \varrho}, \quad p_\varrho = v_\eta \frac{\partial p}{\partial \varrho},
\]
(4.457)

which, using (4.247), yield
\[
|p_{\varrho}|_{\infty, \delta, \varrho} \lesssim \epsilon |\nabla p|_{\infty, \delta, \varrho} + |p_\varrho|_{\infty, \delta, \varrho} \lesssim |\nabla p|_{\infty, \delta, \varrho} \lesssim 1.
\]
(4.458)

Directly Taking \( \eta \) and \( v \) derivatives on both sides of (4.50) and multiplying \( \zeta \), we obtain
\[
S_\eta = \frac{dG_1}{\eta} \left( v_\eta^2 \mathcal{B} - v_\eta v_v \mathcal{E} \right) + \frac{dG_2}{\eta} \left( v_\eta^2 \mathcal{B} - v_\eta v_v \mathcal{D} \right),
\]
\[
S_\varrho = \mathcal{A} - G_1 v_\eta \mathcal{E} - G_2 v_\eta \mathcal{D}, \quad S_\varrho = G_1 \left( 2v_\varrho \mathcal{B} - v_\eta \mathcal{E} \right), \quad S_\varrho = G_2 \left( 2v_\varrho \mathcal{B} - v_\eta \mathcal{D} \right).
\]
(4.459)

Note that fact that \( |G_1| + |G_2| \lesssim \epsilon \) and \( \left| \frac{dG_1}{\eta} \right| + \left| \frac{dG_2}{\eta} \right| \lesssim \epsilon^2 \). We have
\[
\left\| \nu^{-1} S_\eta \right\|_{\infty, \delta, \varrho} \lesssim \epsilon^2 \left( \left\| \nu \mathcal{B} \right\|_{\infty, \delta, \varrho} + \left\| \nu \mathcal{E} \right\|_{\infty, \delta, \varrho} + \left\| \nu \mathcal{D} \right\|_{\infty, \delta, \varrho} \right),
\]
(4.461)

\[
\left\| \nu^{-1} S_\varrho \right\|_{\infty, \delta, \varrho} \lesssim \nu^{-1} \mathcal{A} + \epsilon \left( \left\| \mathcal{B} \right\|_{\infty, \delta, \varrho} + \left\| \mathcal{D} \right\|_{\infty, \delta, \varrho} \right),
\]
(4.462)

\[
\left\| \nu^{-1} S_\varrho \right\|_{\infty, \delta, \varrho} \lesssim \epsilon \left( \left\| \mathcal{B} \right\|_{\infty, \delta, \varrho} + \left\| \mathcal{D} \right\|_{\infty, \delta, \varrho} \right),
\]
(4.463)

\[
\left\| \nu^{-1} S_\varrho \right\|_{\infty, \delta, \varrho} \lesssim \epsilon \left( \left\| \mathcal{B} \right\|_{\infty, \delta, \varrho} + \left\| \mathcal{D} \right\|_{\infty, \delta, \varrho} \right).
\]
(4.464)

Inserting (4.458) and (4.463) into (4.459), and absorbing \( \epsilon \left\| \mathcal{E} \right\|_{\infty, \delta, \varrho} \) into the left-hand side, we get
\[
\left\| \mathcal{B} \right\|_{\infty, \delta, \varrho} \lesssim 1 + \epsilon \left\| \mathcal{B} \right\|_{\infty, \delta, \varrho}.
\]
(4.465)

Similarly, inserting (4.458) and (4.464) into (4.454), and absorbing \( \epsilon \left\| \mathcal{D} \right\|_{\infty, \delta, \varrho} \) into the left-hand side, we get
\[
\left\| \mathcal{D} \right\|_{\infty, \delta, \varrho} \lesssim 1 + \epsilon \left\| \mathcal{D} \right\|_{\infty, \delta, \varrho}.
\]
(4.466)

Inserting (4.455) and (4.469) into (4.402), and further with (4.468) into (4.402), after absorbing \( \epsilon^2 \left\| \mathcal{B} \right\|_{\infty, \delta, \varrho} \) into the left-hand side, we have
\[
\left\| \mathcal{B} \right\|_{\infty, \delta, \varrho} \lesssim 1 + \epsilon \left\| \nu^{-1} \mathcal{A} \right\|_{\infty, \delta, \varrho}.
\]
(4.467)

Then inserting (4.467) into (4.455) and (4.466), we obtain
\[
\left\| \mathcal{E} \right\|_{\infty, \delta, \varrho} \lesssim 1 + \epsilon \left\| \nu^{-1} \mathcal{A} \right\|_{\infty, \delta, \varrho}, \quad \left\| \mathcal{D} \right\|_{\infty, \delta, \varrho} \lesssim 1 + \epsilon \left\| \nu^{-1} \mathcal{A} \right\|_{\infty, \delta, \varrho}.
\]
(4.468)

Finally, inserting (4.467) and (4.468) into (4.461), and further with (4.460) into (4.461), after absorbing \( \epsilon^2 \left\| \mathcal{A} \right\|_{\infty, \delta, \varrho} \) into the left-hand side, we obtain
\[
\left\| \nu^{-1} \mathcal{A} \right\|_{\infty, \delta, \varrho} \lesssim |\ln(\epsilon)|^8.
\]
(4.469)

Hence, inserting (4.469) into (4.467) and (4.468), we get the desired result.
Remark 4.2.13. The estimates of weighted velocity derivatives $\zeta \frac{\partial G}{\partial v_\eta}$, $\zeta \frac{\partial G}{\partial v_\phi}$ and $\frac{\partial G}{\partial v_\psi}$ have an extra $\nu$ in the estimates. This is crucial for the tangential derivative estimates.

Theorem 4.2.14. Assume (4.15) and (4.247) holds. For $K_0 > 0$ sufficiently small, we have

$$\left\| e^{K_0 \eta} \frac{\partial G}{\partial \eta} \right\|_{\infty, \theta, \varphi} + \left\| e^{K_0 \nu \zeta} \frac{\partial G}{\partial v_\eta} \right\|_{\infty, \theta, \varphi} \lesssim |\ln(\epsilon)|^8, \quad (4.470)$$

$$\left\| e^{K_0 \eta} \zeta \frac{\partial G}{\partial v_\psi} \right\|_{\infty, \theta, \varphi} + \left\| e^{K_0 \nu \zeta} \frac{\partial G}{\partial v_\psi} \right\|_{\infty, \theta, \varphi} \lesssim 1. \quad (4.471)$$

Proof. This proof is almost identical to that of Theorem 4.2.12. In each step, we need to multiple $e^{K_0 \eta}$ on both sides (sometimes inside the integral). When $K_0$ is sufficiently small, we can close the proof. □

Corollary 4.2.15. Assume (4.15) and (4.247) holds. We have

$$\epsilon \left\| e^{K_0 \eta} v_\phi^2 \frac{\partial G}{\partial v_\eta} \right\|_{\infty, \theta, \varphi} + \epsilon \left\| e^{K_0 \eta} v_\psi^2 \frac{\partial G}{\partial v_\eta} \right\|_{\infty, \theta, \varphi} \lesssim |\ln(\epsilon)|^8. \quad (4.472)$$

Proof. We rearrange the terms in (4.16) to obtain

$$(G_1 v_\phi^2 + G_2 v_\psi^2) \frac{\partial G}{\partial v_\eta} = (S - \nu G + K[G]) - v_\eta \frac{\partial G}{\partial \eta} + G_1 v_\eta v_\phi \frac{\partial G}{\partial v_\phi} + G_2 v_\eta v_\psi \frac{\partial G}{\partial v_\psi}. \quad (4.473)$$

Recall $\zeta$ definition in (4.248), we know $|v_\eta| \lesssim \zeta$. Therefore, using (4.15) and Theorem 4.2.12 we know

$$\left\| \left( G_1 v_\phi^2 + G_2 v_\psi^2 \right) \frac{\partial G}{\partial v_\eta} \right\|_{\infty, \theta, \varphi} \lesssim \|S\|_{\infty, \theta, \varphi} + \left\| \frac{\partial G}{\partial \eta} \right\|_{\infty, \theta, \varphi} + \left\| \frac{\partial G}{\partial v_\phi} \right\|_{\infty, \theta, \varphi} + \left\| \frac{\partial G}{\partial v_\psi} \right\|_{\infty, \theta, \varphi} \lesssim |\ln(\epsilon)|^8. \quad (4.474)$$

Since $G_1$ and $G_2$ have the same sign and $\epsilon \lesssim |G_1| \lesssim \epsilon$, $\epsilon \lesssim |G_2| \lesssim \epsilon$, we can separate the two terms in the left-hand side of (4.471) to obtain

$$\epsilon \left\| v_\phi^2 \frac{\partial G}{\partial v_\eta} \right\|_{\infty, \theta, \varphi} + \epsilon \left\| v_\psi^2 \frac{\partial G}{\partial v_\eta} \right\|_{\infty, \theta, \varphi} \lesssim |\ln(\epsilon)|^8. \quad (4.475)$$

We can perform the same analysis with an extra $e^{K_0 \eta}$ term. Hence, our result naturally follows. □

Estimates of Tangential Derivative

Now we pull the tangential variables $\iota_1$ and $\iota_2$ dependence back and study the tangential derivatives.

Theorem 4.2.16. Assume (4.15) and (4.247) holds. We have

$$\left\| e^{K_0 \eta} \frac{\partial G}{\partial \iota_1} \right\|_{\infty, \theta, \varphi} \lesssim |\ln(\epsilon)|^8, \quad \left\| e^{K_0 \eta} \frac{\partial G}{\partial \iota_2} \right\|_{\infty, \theta, \varphi} \lesssim |\ln(\epsilon)|^8. \quad (4.476)$$

Proof. Let $\mathcal{W} := \frac{\partial G}{\partial \iota_i}$ for $i = 1, 2$. Taking $\iota_i$ derivative on both sides of (4.16), we know that $\mathcal{W}$ satisfies the equation

$$\left\{ \begin{array}{l}
\nu_\eta \frac{\partial \mathcal{W}}{\partial \eta} + G_1(\eta) \left( v_\phi^2 \frac{\partial \mathcal{W}}{\partial v_\phi} - v_\eta \frac{\partial \mathcal{W}}{\partial v_\eta} \right) & \quad + G_2(\eta) \left( v_\psi^2 \frac{\partial \mathcal{W}}{\partial v_\psi} - v_\eta v_\phi \frac{\partial \mathcal{W}}{\partial v_\phi} \right) + \nu \mathcal{W} - K[\mathcal{W}] = S[\mathcal{W}], \\
\mathcal{W}(0, \iota_1, \iota_2, \nu) = \frac{\partial p}{\partial \iota_i}(\iota_1, \iota_2, \nu) \text{ for } \sin \phi > 0, \\
\mathcal{W}(L, \iota_1, \iota_2, \nu) = \mathcal{W}(L, \iota_1, \iota_2, \mathcal{A}[\nu]),
\end{array} \right. \quad (4.477)$$
where
\[
S_\omega = \frac{\partial S}{\partial t_i} + \frac{\partial_i R_1}{R_1 - \epsilon \eta} G_1(\eta) \left( v_\omega^2 \frac{\partial G}{\partial v_\omega} - v_\eta v_\phi \frac{\partial G}{\partial v_\phi} \right) + \frac{\partial_i R_2}{R_2 - \epsilon \eta} G_2(\eta) \left( v_\omega^2 \frac{\partial G}{\partial v_\omega} - v_\eta v_\phi \frac{\partial G}{\partial v_\phi} \right).
\] (4.478)

For \( \eta \in [0, L] \), we have
\[
\frac{\partial_i R_j}{R_j - \epsilon \eta} \leq \max_{i,j=1,2} \frac{\partial_i R_j}{1} \leq 1.
\] (4.479)

Therefore, noting that \( |v_\eta| \leq \zeta \), using (4.471), Theorem 4.2.13 and Corollary 4.2.15 we have
\[
\|S_\omega\|_{\infty, \theta, \varphi} \lesssim \left\| \frac{\partial S}{\partial t_i} \right\|_{\infty, \theta, \varphi} + \left\| G_1(\eta) \left( v_\omega^2 \frac{\partial G}{\partial v_\omega} - v_\eta v_\phi \frac{\partial G}{\partial v_\phi} \right) \right\|_{\infty, \theta, \varphi} + \left\| G_2(\eta) \left( v_\omega^2 \frac{\partial G}{\partial v_\omega} - v_\eta v_\phi \frac{\partial G}{\partial v_\phi} \right) \right\|_{\infty, \theta, \varphi}
\lesssim 1 + \epsilon \left\| \frac{\partial G}{\partial v_\omega} \right\|_{\infty, \theta, \varphi} + \epsilon \left\| \frac{\partial G}{\partial v_\eta} \right\|_{\infty, \theta, \varphi} + \epsilon \left\| \frac{\partial G}{\partial v_\phi} \right\|_{\infty, \theta, \varphi} \lesssim |\ln(\epsilon)|^8.
\] (4.480)

By a similar argument, we can add \( e^{K_0 \eta} \) contribution to obtain
\[
\left\| e^{K_0 \eta} S_\omega \right\|_{\infty, \theta, \varphi} \lesssim |\ln(\epsilon)|^8.
\] (4.481)

Therefore, applying Theorem 4.1.24 to (4.481), we have that
\[
\left\| e^{K_0 \eta} \psi(\eta, t_1, t_2, \nu) \right\|_{\infty, \theta, \varphi} \lesssim |\ln(\epsilon)|^8.
\] (4.482)

\textbf{Theorem 4.2.17.} Assume (4.15) and (4.247) holds. We have
\[
\left\| e^{K_0 \eta} \frac{\partial G}{\partial v_\phi} \right\|_{\infty, \theta, \varphi} \lesssim |\ln(\epsilon)|^8, \quad \left\| e^{K_0 \eta} \frac{\partial G}{\partial v_\psi} \right\|_{\infty, \theta, \varphi} \lesssim |\ln(\epsilon)|^8.
\] (4.483)

\textbf{Proof.} Let \( \psi := v_\phi \frac{\partial G}{\partial v_\phi} \). Taking \( v_\phi \) derivative on both sides of (4.40) and multiplying \( v_\phi \), we know that \( \psi \) satisfies the equation
\[
\begin{aligned}
&v_\phi \frac{\partial \psi}{\partial \nu} + G_1(\eta) \left( v_\phi^2 \frac{\partial \psi}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \psi}{\partial v_\phi} \right) + G_2(\eta) \left( v_\phi^2 \frac{\partial \psi}{\partial v_\eta} - v_\eta v_\phi \frac{\partial \psi}{\partial v_\phi} \right) + \nu \psi = S_\psi, \\
&\psi(0, t_1, t_2, \nu) = v_\phi \frac{\partial \psi}{\partial v_\phi}(t_1, t_2, \nu) \quad \text{for} \quad \sin \phi > 0, \\
&\psi(L, t_1, t_2, \nu) \equiv \psi(L, t_1, t_2, \mathcal{D}[\nu]),
\end{aligned}
\] (4.484)

where
\[
S_\psi = \int_{\mathbb{R}^3} v_\phi \partial_{v_\phi} k(u, \nu) du + v_\phi \frac{\partial S}{\partial v_\phi} + 2G_1 v_\phi^2 \frac{\partial G}{\partial v_\eta} - 2G_1 v_\eta v_\phi \frac{\partial G}{\partial v_\phi}.
\] (4.485)

Based on (4.247), Lemma 4.2.5 and Theorem 4.1.24 we have
\[
\left\| \int_{\mathbb{R}^3} v_\phi \partial_{v_\phi} k(u, \nu) du \right\|_{\infty, \theta, \varphi} + \left\| v_\phi \frac{\partial S}{\partial v_\phi} \right\|_{\infty, \theta, \varphi} \lesssim 1.
\] (4.486)

Using Corollary 4.2.15 we get
\[
\left\| 2G_1 v_\phi^2 \frac{\partial G}{\partial v_\eta} \right\|_{\infty, \theta, \varphi} \lesssim \epsilon \left\| v_\phi^2 \frac{\partial G}{\partial v_\eta} \right\|_{\infty, \theta, \varphi} \lesssim |\ln(\epsilon)|^8.
\] (4.487)
Using Theorem 4.2.14 we obtain
\[
\left\| 2G_1 v_\eta v_\phi \frac{\partial G}{\partial v_\phi} \right\|_{\infty, \theta, \varrho} \lesssim \nu \left\| \frac{\partial G}{\partial v_\phi} \right\|_{\infty, \theta, \varrho} \lesssim 1.
\] (4.488)

Hence, collecting all above, we have proved that
\[
\| S V \|_{\infty, \theta, \varrho} \lesssim |\ln(\nu)|^8.
\] (4.489)

Based on the analysis in Section 4.1.2 we have
\[
\| V \|_{\infty, \theta, \varrho} \lesssim \left| v_\eta \frac{\partial p}{\partial v_\phi} \right|_{\infty, \theta, \varrho} + \nu^{-1} S V \|_{\infty, \theta, \varrho} \lesssim |\ln(\nu)|^8.
\] (4.490)

By a similar argument, we can add \( e^{K_0 \eta} \) contribution to obtain
\[
\left\| e^{K_0 \eta} v_\eta \frac{\partial G}{\partial v_\phi} \right\|_{\infty, \theta, \varrho} \lesssim |\ln(\nu)|^8.
\] (4.491)

Similarly, we can show
\[
\left\| e^{K_0 \eta} v_\eta \frac{\partial G}{\partial v_\psi} \right\|_{\infty, \theta, \varrho} \lesssim |\ln(\nu)|^8.
\] (4.492)

Since \( |v_\eta| \lesssim \zeta \), Theorem 4.2.14 implies
\[
\left\| e^{K_0 \eta} v_\eta \frac{\partial G}{\partial v_\psi} \right\|_{\infty, \theta, \varrho} \lesssim |\ln(\nu)|^8.
\] (4.493)

Then our result naturally follows. The bounds can be shown in a similar fashion.

\[\square\]

Remark 4.2.18. Theorem 4.2.14, Corollary 4.2.15, Theorem 4.2.16 and Theorem 4.2.17 provide bounds of all kinds of normal and velocity derivatives. However, note that \( \frac{\partial G}{\partial \eta} \) estimate must be accompanied by the weight \( \zeta \) since it may have singularity near the grazing set. Similarly, \( \frac{\partial G}{\partial v_\eta} \) estimate should be with either \( \zeta \) or \( \epsilon \). On the other hand, \( \frac{\partial G}{\partial v_i} \), \( \frac{\partial G}{\partial v_\phi} \) and \( \frac{\partial G}{\partial v_\psi} \) can avoid the introduction of \( \zeta \) or \( \epsilon \), since they do not directly interact with grazing set.
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