KOSZUL BLOWUP ALGEBRAS ASSOCIATED TO THREE-DIMENSIONAL FERRERS DIAGRAMS

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Abstract. We investigate the Rees algebra and the toric ring of the squarefree monomial ideal associated to the three-dimensional Ferrers diagram. Under the projection property condition, we describe explicitly the presentation ideals of the Rees algebra and the toric ring. We show that the toric ring is a Koszul Cohen–Macaulay normal domain, while the Rees algebra is Koszul and the defining ideal is of fiber type.

1. Introduction

Given a graded ideal \( I \) in a standard graded ring \( R \) over a field \( \mathbb{K} \), one encounters the Rees algebra \( \mathcal{R}(I) = R[I] \) of \( I \), as well as the special fiber ring \( \mathcal{F}(I) = \mathcal{R}[I] \otimes_R \mathbb{K} \). These objects are important to commutative algebraists and geometers because the projective schemes of these rings define the blowup and the special fiber of the blowup of the scheme \( \text{Spec}(R) \) along \( V(I) \) respectively. The most challenging question of this topic is to describe those objects in term of generators and relationships, i.e., to find the presentation equations of these objects over some polynomial rings. When the ideal \( I \) is generated by forms of the same degree, these rings describe the image and the graph of the rational map between the projective spaces. The presentation equations of these algebras give implicit equations of the graph and of the variety parametrized by the map. Finding those presentation equations is known as the implicitization problem \([5]\). When \( I \) is a monomial ideal generated by the same degree in a polynomial over a field \( \mathbb{K} \), the special fiber ring \( \mathcal{F}(I) \) is the toric ring induced by \( I \). The presentation ideal of \( \mathcal{F}(I) \) is a prime binomial ideal, hence is a toric ideal by \([11]\) Proposition 1.1.11]. Toric ideals play an important role in polyhedral geometry, algebraic topology, algebraic geometry and statistics. As pointed out in \([29]\), even though it is known that the toric ideal is generated by binomials, “there are no simple formulas for a finite set of generators of a general toric ideal”. And it is an active research area to understand and find the toric ideals; see for example, \([2]\), \([9]\), \([22]\), \([24]\) and \([32]\).

Finding the presentation ideal of \( \mathcal{R}[I] \) when \( I \) is a monomial ideal is another active field; see for example, \([14]\) and \([31]\). Once we have the presentation ideal of \( \mathcal{R}[I] \), we can obtain the presentation ideal of \( \mathcal{F}(I) \) for free, simply because \( \mathcal{R}[I] \otimes_R \mathbb{K} = \mathcal{F}(I) \). Of course, the reverse process is generally complicated.

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Ideals of fiber type was introduced in [19] for investigating Rees algebras. If an ideal $I$ is of fiber type, then the presentation ideal of $R[I]$ can be obtained from the combination of linear relations from the first syzygy of $I$, with the presentation ideal of $F(I)$; see also Definition 2.1. There is no doubt that with respect to the Rees algebra $R(I)$, an ideal of fiber type provides the next best possibility if $I$ is not of linear type, i.e., $I$ is not defined by linear relations. Since an explicit description of the presentation ideal of $R(I)$ is in general much involved and difficult, if $I$ is of fiber type, then the focus of investigation of $R(I)$ can be shifted to that of $F(I)$. This is also the strategy employed in this paper. Finding ideals of fiber type is another active field; see for example, [19] and [20].

Recall that if $I$ is a graded ideal in a polynomial ring $S$ over the field $K$, the quotient algebra $R = S/I$ is called Koszul if the (in general infinite) minimal free resolution of $K$ over $R$ is linear. For instance, Avramov and Eisenbud [11] showed that if this $R$ is Koszul, then every graded finitely generated $R$-module has finite regularity over $R$. Koszul property is probably the best possibility when one encounters infinite free resolutions. This even makes the Koszul property a basic notion of the representation theory in the non-commutative case. As pointed out in [23, Problem 34.6 and Question 74.1], what classes of toric rings are Koszul is an open and attractive question; see also [3]. A related question is when projective toric varieties are defined by quadratics; see for example, [6], [7] and [27]. People are also interested in finding Cohen–Macaulay or normal toric rings. See [8] for more on the Koszul algebras and [31] for more on the toric variety and toric ideal.

The main purpose of this work is to answer above open questions with respect to the three-dimensional Ferrers diagram. As pointed out in the work of Corso and Nagel [9], “Ferrers graph/tableaux have a prominent place in the literature as they have been studied in relation to chromatic polynomials, Schubert varieties, hypergeometric series, permutation statistics, quantum mechanical operators, and inverse rook problems”; see [9] for detailed reference. It is known that the toric ring and the Rees algebra associated to a (two-dimensional) Ferrers diagram are Cohen–Macaulay normal domains; see [28] or [9]. More recently, the work of Corso, Nagel, Petrović, and Yuen [10] extends the results to specialized Ferrers diagram and shows that the toric ring is a Koszul Cohen-Macaulay normal domain.

Interestingly, the special fiber rings of Ferrers diagrams can also be deemed as the affine semigroup rings generated by some two-dimensional squarefree monomials. In particular, they are isomorphic to the toric rings of incidence matrices of graphs. This kind of rings were well-studied by Hibi and Ohsugi. From this point of view, the special fiber rings of Ferrers diagrams are isomorphic to the toric rings of bipartite graph whose cycles of length $\geq 6$ has a chord. Consequently, the associated toric ideals admits a squarefree initial ideal by [21, Theorem].

Since both papers [9] and [10] involve monomial ideals generated in degree two, it is natural to inquire about the degree three case. As a result, we consider the three-dimensional Ferrers diagram and the monomial ideal associated to it. Notice that the toric ideals of
toric rings generated by squarefree monomials of degree ≤ 3 are as complicated as any arbitrary toric ideals by [25, Theorem 3.2].

In some sense, the model we have here can be regarded as sub-configurations of the 3-fold Segre product. From this point of view, a common strategy is a quest for the existence of related algebra retracts. With that, properties like normality of domains, regularity, complete intersection, Koszul, Stanley-Reisner can descend along algebra retracts. However, this is not known for properties like Cohen–Macaulay, Gorenstein in general.

Indeed, no such an algebra retract exists in general. Unlike the two-dimensional case, not every three-dimensional Ferrers diagram induces a Koszul special fiber ring; see Example 2.4. This bad phenomenon happens due to the recurrence of high-dimensional entanglements. Roughly speaking, in the two-dimensional case, it is arguably easy to find extremal cells of the diagram. After deleting these extremal cells, one still gets a nice diagram of similar configuration. On the other hand, in the three-dimensional case, unless we have a cubic diagram, one can almost always expect an extremal cell of one side being hampered by other sides of the diagram. The special cubic case is essentially of matroidal type, hence has been investigated; see Example 2.3.

To circumvent the high-dimensional entanglements, we introduce the “projection property” condition. Three-dimensional Ferrers diagrams with this condition can be thought of as natural generalizations of two-dimensional Ferrers diagrams. See Remark 2.6 for a heuristic explanation of this introduction, as well as Remark 2.15 for its usage. Under the projection property condition, we demonstrate that the toric ideal is generated by quadratics. Indeed, it has a quadratic Gröbner basis and the toric ring is a Koszul Cohen–Macaulay normal domain. We find out that the Ferrers ideal satisfies the ℓ-exchange property in the sense of [19]. Hence, the presentation ideal of the Rees algebra also have a quadratic Gröbner basis and the Rees algebra is Koszul as well. Moreover, the ideal is of fiber type.

Here is the outline of this work. We start by setting the notations and definitions in Section 2. The main object is the generalized 2-minors, \( I_2(D) \), that we propose as the generators of the toric ideal associated to the three-dimensional Ferrers diagram \( D \) (see Definition 2.10). It is well-known that once the presentation ideal of an algebra has a quadratic Gröbner basis, the algebra is Koszul (see, for instance, [13]). We show the set \( I_2(D) \) has a quadratic Gröbner basis if it comes from a Ferrers diagram satisfying the projection property (see Definition 2.5 and Corollary 2.14). Not only that, we extend the quadratic Gröbner basis property to certain subdiagrams that we need for the later sections (see Proposition 2.17). Since the quadratic Gröbner basis of \( I_2(D) \) has a squarefree initial ideal, we can pass from the initial ideal to its Stanley-Reisner complex in Section 3. We demonstrate that the associated Stanley–Reisner complex is pure vertex-decomposable and hence is shellable. From this, we obtain the Cohen–Macaulayness of the ideal \( I_2(D) \) (see Theorem 4.1). In Section 5, we show that the ideal \( I_2(D) \) is prime, by using the Cohen–Macaulay property of this ideal (see Theorem 5.2). Its proof is inspired by the work of Corso, Nagel, Petrović, and Yuen [10]. Finally, we put all pieces together to show that \( I_2(D) \) gives rise to the presentation ideal of the toric ideal, and the toric ring, namely the
special fiber ring, is a Koszul Cohen–Macaulay normal domain (see Theorem 6.1). With the ℓ-exchange property of the ideal, we obtain the presentation ideals of the Rees algebra as well (see Definition 6.2 and Lemma 6.4).

2. Preliminaries

In this section, we fix basic definitions and standard notations used throughout the paper. More precisely, we define a set of binomials that sits inside the presentation ideal of the special fiber ring that we are interested in. Elementary properties of this set are provided. In particular, this set has a quadratic Gröbner basis.

Throughout this paper, $\mathbb{K}$ is a field of characteristic zero. Let $\mathcal{D}$ be a diagram of finite lattice points in $\mathbb{Z}^3_{+}$. Let

$$m := \max \{ i : (i, j, k) \in \mathcal{D} \}, \quad n := \max \{ j : (i, j, k) \in \mathcal{D} \} \quad \text{and} \quad p := \max \{ k : (i, j, k) \in \mathcal{D} \}.$$ 

Associated to $\mathcal{D}$ is the polynomial ring $R = \mathbb{K}[x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_p]$ and the monomial ideal

$$I_D := (x_i y_j z_k : (i, j, k) \in \mathcal{D}) \subset R.$$ 

This ideal will be called the defining ideal of $\mathcal{D}$.

If we write $m$ for the graded maximal ideal of $R$, the special fiber ring of $I_D$ is

$$F(I_D) := \bigoplus_{i \geq 0} I_D^i / m I_D^i \cong R[I_D t] \otimes_R R/m.$$ 

Sometimes, we also call it the toric ring of $I_D$ and denote it by $\mathbb{K}[I_D]$. Let

$$\mathbb{K}[T_D] := \mathbb{K}[T_{i,j,k} : (i, j, k) \in \mathcal{D}]$$

be the polynomial ring in the variables $T_{i,j,k}$ over the field $\mathbb{K}$. Consider the map $\varphi : \mathbb{K}[T_D] \to R$, given by $T_{i,j,k} \mapsto x_i y_j z_k$, and extend algebraically. Then $\mathcal{F}(I_D)$ is canonically isomorphic to $\mathbb{K}[T_D]/\ker(\varphi)$. We will denote the kernel ideal $\ker(\varphi)$ by $J_D$ and call it the special fiber ideal of $I_D$. Sometimes, we also call it the toric ideal of $I_D$ and the presentation ideal of $\mathcal{F}(I_D)$. It is well-known that $J_D$ is a graded binomial ideal; see, for instance, [30, Corollary 4.3] or [33]. We also observe that $\mathcal{F}(I_D)$, being isomorphic to a subring of $R$, is a domain. Hence $J_D$ is a prime ideal.

Related and more complicated is the Rees algebra of $I_D$:

$$\mathcal{R}(I_D) := R[I_D t] = R \oplus I_D t \oplus I_D^2 t^2 \oplus \cdots \subset R[t].$$ 

Let

$$R[T_D] := R[T_{i,j,k} : (i, j, k) \in \mathcal{D}]$$

be the polynomial ring in the variables $T_{i,j,k}$ over $R$. Consider the map $\psi : R[T_D] \to R[t]$, given by $T_{i,j,k} \mapsto x_i y_j z_k t$, and extend algebraically. Then $\mathcal{R}(I_D)$ is canonically isomorphic to $R[T_D]/\ker(\psi)$. The kernel ideal $\ker(\psi)$ will be referred to as the Rees ideal of $I_D$, or the presentation ideal of $\mathcal{R}(I_D)$. 
In fact, the epimorphism from \( R[T_D] \) to \( \mathcal{R}(I_D) \) factors through the symmetric algebra \( \text{Sym}(I_D) \) as
\[
R[T_D] \xrightarrow{\alpha} \text{Sym}(I_D) \xrightarrow{\beta} \mathcal{R}(I_D).
\]
When the epimorphism \( \beta \) is indeed an isomorphism, the ideal \( I_D \) will be called of \textit{linear type}. The next best possibility with respect to the presentation ideal of \( \mathcal{R}(I_D) \) is when \( I_D \) is of fiber type, a concept introduced by Herzog, Hibi and Vladoiu in [19]. Using notations above, it can be formulated as follows.

**Definition 2.1.** The defining ideal \( I_D \) is called of \textit{fiber type} if
\[
\ker(\psi) = \ker(\alpha) + \ker(\varphi)R[T_D].
\]

For the given diagram \( \mathcal{D} \), let
\[
a_D := \{ i : (i,j,k) \in \mathcal{D} \}, \quad b_D := \{ j : (i,j,k) \in \mathcal{D} \} \text{ and } c_D := \{ k : (i,j,k) \in \mathcal{D} \}
\]
be the \textit{essential length}, \textit{width} and \textit{height} of \( \mathcal{D} \) respectively. By abuse of notation, for \( u = (i_0,j_0,k_0) \in \mathcal{D} \), let
\[
a_D(u) := \max \{ i : (i,j_0,k_0) \in \mathcal{D} \}.
\]

In a similar vein, we can define \( b_D(u) \) and \( c_D(u) \).

**Definition 2.2.** Let \( \mathcal{D} \) be a diagram of finite lattice points in \( \mathbb{Z}^3 \). \( \mathcal{D} \) is called a \textit{three-dimensional Ferrers diagram} if for each \( (i_0,j_0,k_0) \in \mathcal{D} \), for every positive integers \( i \leq i_0, j \leq j_0 \) and \( k \leq k_0 \), one has \( (i,j,k) \in \mathcal{D} \).

When the diagram \( \mathcal{D} \) is a Ferrers diagram, the defining ideal \( I_D \) will also be called the \textit{Ferrers ideal} of \( \mathcal{D} \).

Obviously, if \( \mathcal{D} \) is a three-dimensional Ferrers diagram, then \( a_D = a_D(u_0), b_D = b_D(u_0) \) and \( c_D = c_D(u_0) \) for the point \( u_0 = (1,1,1) \). In addition, if one of the three numbers is one, then we get the classic two-dimensional Ferrers diagram.

Recall that a standard graded \( \mathbb{K} \)-algebra \( R \) is called \textit{Koszul} if the residue class field \( \mathbb{K} = R/R_+ \) has a linear \( R \)-resolution. If we can write \( R \cong S/J \) as the quotient of a polynomial ring \( S \), to show the Koszulness of \( R \), it suffices to show that the homogeneous ideal \( J \) has a quadratic Gröbner basis with respect to some monomial order by [13, Theorem 6.7]. The aim of this paper is to find classes of three-dimensional diagrams whose associated toric rings are Koszul. Therefore, we will focus on those with quadratic Gröbner bases in some monomial order.

For a positive integer \( n \), we denote by \([n]\) the finite set \( \{1,2,\ldots,n\} \).

**Example 2.3.** The easiest example is when \( \mathcal{D} \) is a full rectangular cylinder diagram, i.e., \( \mathcal{D} \) takes the form \([a_D] \times [b_D] \times [c_D]\). It follows from [17, Theorem 5.3(b)] that the toric ring is Koszul in this situation.

Let \( V : v_1, \ldots, v_m \) be a set of lattice points in \( \mathbb{Z}^2 \). The minimal three-dimensional Ferrers diagram containing \( V \) will be called \textit{the Ferrers diagram generated by} \( V \).

Unlike the two-dimensional case in [10, Theorem 4.2], not all toric rings associated to three-dimensional Ferrers diagrams are Koszul.
**Example 2.4.** Figure 1 provides a diagram \( \mathcal{D} \) generated by 
\[
(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1).
\]
Therefore, it consists of the following lattice points 
\[
(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (1, 2, 3), (1, 3, 1), (1, 3, 2),
(2, 1, 1), (2, 1, 2), (2, 1, 3), (2, 2, 1), (2, 3, 1), (3, 1, 1), (3, 1, 2), (3, 2, 1).
\]
This is a three-dimensional Ferrers diagram. If we use the base ring 
\[
S = \mathbb{K}[T_{1,1,1}, T_{1,1,2}, T_{1,1,3}, T_{1,2,1}, T_{1,2,2}, T_{1,2,3}, T_{1,3,1}, T_{1,3,2},
T_{2,1,1}, T_{2,1,2}, T_{2,1,3}, T_{2,2,1}, T_{2,3,1}, T_{3,1,1}, T_{3,1,2}, T_{3,2,1}],
\]
and apply the canonical epimorphism, the minimal generating set of the special fiber ideal \( J_\mathcal{D} \) contains the following degree three binomial 
\[
T_{1,2,3}T_{2,3,1}T_{3,1,2} - T_{1,3,2}T_{2,1,3}T_{3,2,1},
\]
as suggested by Macaulay2 [13]. Thus, by [13 Proposition 6.3], the toric ring associated to \( \mathcal{D} \) cannot be Koszul.

Meanwhile, one can easily find abundant three-dimensional Ferrers diagrams whose associated special fiber ideals are quadratic, while their Gröbner bases in the common monomial orders are not quadratic. To circumvent the recurrence of high-dimensional entanglement when treating the Gröbner basis, we introduce the following projection property for three-dimensional Ferrers diagrams. We will show that three-dimensional Ferrers diagram which satisfies the projection property will have a Koszul associated toric ring in Theorem 6.1.

**Definition 2.5.** Let \( \mathcal{D} \) be a three-dimensional Ferrers diagram. For each \( 2 \leq i \leq a_\mathcal{D} \), let 
\[
b_i = \max \{ j : (i, j, 1) \in \mathcal{D} \}
\]
and 
\[
c_i = \max \{ k : (i, 1, k) \in \mathcal{D} \}.
\]
Then the *projection* of the \( x = i \) layer is the set 
\[
\{ (i - 1, j, k) \in \mathbb{Z}_+^3 : j \leq b_i \text{ and } k \leq c_i \}.
\]
And $\mathcal{D}$ is said to satisfy the projection property if the $x = i - 1$ layer covers the projection of the $x = i$ layer for $2 \leq i \leq a_\mathcal{D}$, i.e., the following equivalent conditions hold:

(a) $(i - 1, b_i, c_i) \in \mathcal{D}$;
(b) if $(i, j_1, k_1) \in \mathcal{D}$ and $(i, j_2, k_2) \in \mathcal{D}$, then $(i - 1, j_1, k_2) \in \mathcal{D}$.

Trivially, the cubic diagram in Example 2.3 satisfies the projection property. On the other hand, for the diagram $\mathcal{D}$ in Example 2.4, one has $b_2 = c_2 = 3$. Since $(1, 3, 3) \notin \mathcal{D}$, the diagram $\mathcal{D}$ does not satisfy the projection property. One can attach $(2, 2, 2)$ and $(1, 3, 3)$ to get the “closure diagram” $\overline{\mathcal{D}}$ with respect to the projection property, which is illustrated in Figure 2.

![Figure 2. The closure diagram $\overline{\mathcal{D}}$](image)

Here, we explain the introduction of the projection property.

**Remark 2.6.** Let $\mathcal{D}$ be a three-dimensional Ferrers diagram. If $a_\mathcal{D} = 1$ or $b_\mathcal{D} = 1$ or $c_\mathcal{D} = 1$, then $\mathcal{D}$ is essentially a two-dimensional Ferrers diagram. In this case, $\mathcal{D}$ automatically satisfies the projection property. From this point of view, three-dimensional Ferrers diagrams which satisfy the projection property are more natural as generalizations of two-dimensional Ferrers diagrams to the three-dimensional case. As a matter of fact, the essential reason that we introduce the projection property lies in the desire to achieve Koszul property. As a necessary condition, the degree 3 generator $T_{1,2,3}T_{2,3,1}T_{3,1,2}T_{1,3,2}T_{2,1,3}T_{3,2,1}$ in Example 2.3 is not expected to exist in any minimal generating set, due to [13, Proposition 6.3]. To remove it with respect to a bigger three-dimensional diagram, one has to seek support from the ubiquitous quadratic generators in Definition 2.10 via Lemma 2.11. With this in mind, and if we stick to three-dimensional Ferrers diagrams, $(1, 3, 3)$ will be the optimal element to attach to the diagram $\mathcal{D}$ of Example 2.4. After this is done, the degree 3 generator of the toric ideal will soon be reduced by combinations of quadratic generators. Notice that in this maneuver, the position $(1, 3, 3)$ of $x = 1$ layer lies in the projection of the $x = 2$ layer, which only respects two adjacent layers. To make it more induction-friendly, we naturally come up with the projection property in Definition 2.5.
Observation 2.7. Suppose that $D$ is a three-dimensional Ferrers diagram satisfying the projection property. Then all the three truncated subdiagrams

$$\{(i,j,k) \in D : i \neq i_0 \}, \quad \{(i,j,k) \in D : j \neq j_0 \} \quad \text{and} \quad \{(i,j,k) \in D : k \neq k_0 \}$$

are essentially three-dimensional Ferrers diagrams which still satisfy the projection property.

Return to $S = \mathbb{K}[T_D] = \mathbb{K}[T_{i,j,k} : (i,j,k) \in D]$. Throughout this paper, the variables will be ordered such that $T_{i,j,k} > T_{i',j',k'}$ if and only if the tuple $(i,j,k)$ precedes $(i',j',k')$ lexicographically. With respect to this order of variables, we will consider the lexicographic monomial order $\prec_{lex}$ on the set of monomials in $S$. A binomial ideal is called lexicographically quadratic if all its minimal Gröbner basis elements with respect to the lexicographic order are quadratic. When the special fiber ideal $J_D$ corresponding to $D$ is lexicographically quadratic, we will simply say that $D$ is lexicographically quadratic.

For a monomial ideal $I$, we write $\text{gens}(I)$ for the set of its minimal monomial generators. Meanwhile, if $J$ is a binomial ideal, $\text{in}(J)$ is the initial ideal with respect to the lexicographic monomial order. This is also a monomial ideal.

In the meantime, given a three-dimensional diagram $D$ and a point $u \in D$, we always let $D_u$ be the diagram obtained from $D$ by removing those points preceding $u$ lexicographically. This notation benefits our induction argument later in this paper.

The following property shows that the generating set and its initial part can be inherited by suitable subdiagrams.

**Proposition 2.8.** Suppose that $D$ is a finite three-dimensional diagram and $u = (i_0, j_0, k_0) \in D$. Let $G$ be one of the following three truncated subdiagrams

$$\{(i,j,k) \in D : i \neq i_0 \}, \quad \{(i,j,k) \in D : j \neq j_0 \} \quad \text{and} \quad \{(i,j,k) \in D : k \neq k_0 \}$$

or $G = D_u$ as defined above. Then the following restriction formulas hold:

$$J_D \cap \mathbb{K}[T_G] = J_G \quad \text{and} \quad \text{gens}(\text{in}(J_D)) \cap \mathbb{K}[T_G] = \text{gens}(\text{in}(J_G)).$$

**Proof.** (a) Without loss of generality, we consider the subdiagram

$$G = \{(i,j,k) \in D : i \neq i_0 \}.$$ 

It is clear that $J_G \subseteq J_D$. On the other hand, take arbitrary binomial

$$f = T_{u_1}T_{u_2}\cdots T_{u_n} - T_{v_1}T_{v_2}\cdots T_{v_n}$$

in the binomial ideal $J_D$. It follows from the definition of $J_D$ that one of the lattice points $u_1, \ldots, u_n$ is on the $x = i_0$ layer if and only if one of the lattice points $v_1, \ldots, v_n$ is on the $x = i_0$ layer. This implies that

$$J_D \cap \mathbb{K}[T_G] = J_G.$$

When $f \in J_G$, it is also clear that $T_{u_1}T_{u_2}\cdots T_{u_n}$ is the leading term of $f$ in $\mathbb{K}[T_G]$ if and only if it is so in $\mathbb{K}[T_D]$. This implies that

$$\text{in}(J_D) \cap \mathbb{K}[T_G] \supseteq \text{in}(J_G).$$
On the other hand, take arbitrary monomial \( T_{u_1}T_{u_2} \cdots T_{u_n} \in \text{in}(J_D) \cap \mathbb{K}[T_g] \). By the definition of the initial ideal, \( T_{u_1}T_{u_2} \cdots T_{u_n} \) is the leading term of some
\[
f = T_{u_1}T_{u_2} \cdots T_{u_n} - T_{v_1}T_{v_2} \cdots T_{v_n} \in J_D.
\]
As argued above, since \( T_{u_1}T_{u_2} \cdots T_{u_n} \in \mathbb{K}[T_g] \), we also have \( f \in \mathbb{K}[T_g] \). This means that \( f \in J_D \cap \mathbb{K}[T_g] = J_g \). And the leading term \( T_{u_1}T_{u_2} \cdots T_{u_n} \in \text{in}(J_g) \). Therefore, we have shown that
\[
\text{in}(J_D) \cap \mathbb{K}[T_g] = \text{in}(J_g).
\]
Consequently,
\[
\text{gens}(\text{in}(J_D)) \cap \mathbb{K}[T_g] = \text{gens}(\text{in}(J_g)).
\]

(b) Now, consider the case when \( G = D_u \). One simply notice that, using the notation in (1), since \( T_{u_1}T_{u_2} \cdots T_{u_n} \) is the leading term of \( f \) with respect to the lexicographic order, \( T_{u_1}T_{u_2} \cdots T_{u_n} \in \mathbb{K}[T_g] \) if and only if \( f \in \mathbb{K}[T_g] \). The remaining proof is similar to the previous case. \( \square \)

**Corollary 2.9.** Suppose that \( D \) is a finite three-dimensional diagram. If \( D \) is lexicographically quadratic and \( u = (i_0, j_0, k_0) \in D \), then all the three truncated subdiagrams
\[
\{(i, j, k) \in D : i \neq i_0 \}, \quad \{(i, j, k) \in D : j \neq j_0 \} \quad \text{and} \quad \{(i, j, k) \in D : k \neq k_0 \}
\]
as well as \( D_u \) are again lexicographically quadratic.

We are now ready to introduce the main subject discussed in this paper.

**Definition 2.10.** Let \( D \) be a diagram of finite lattice points in \( \mathbb{Z}_+^3 \). For \( u = (i_1, j_1, k_1) \) and \( v = (i_2, j_2, k_2) \) in \( D \), define
\[
I_{2,x}(u, v) := \begin{cases} T_uT_v - T_{i_2,j_1,k_1}T_{i_1,j_2,k_2}, & \text{if } (i_2, j_1, k_1), (i_1, j_2, k_2) \in D, \\ 0, & \text{otherwise}. \end{cases}
\]
We will simply say *switching the x-coordinates* in the first case. We can similarly define \( I_{2,y} \) and \( I_{2,z} \). Now, let
\[
I_2(D) := \langle I_{2,x}(u, v), I_{2,y}(u, v), I_{2,z}(u, v) : u, v \in D \rangle \subset \mathbb{K}[T_D],
\]
and call it the 2-minors ideal of \( D \).

Obviously, when \( D \) is essentially a two-dimensional diagram, \( I_2(D) \) is the traditional 2-minors ideal of \( D \).

In the following, we will investigate \( I_2(D) \) and \( I_2(D_u) \). It is clear that \( I_2(D) \subseteq J_D \), the special fiber ideal corresponding to the diagram \( D \). Notice that the choice of \( I_{2,x}(u, v) \), \( I_{2,y}(u, v) \) and \( I_{2,z}(u, v) \) is not by accident. Those elements are actually the degree two binomials of \( J_D \).

**Lemma 2.11.** If the nonzero binomial \( f = T_uT_v - T_{u'}T_{v'} \) belongs to \( J_D \), then it is one of \( I_{2,x}(u, v) \), \( I_{2,y}(u, v) \) and \( I_{2,z}(u, v) \).
Proof. We may write \( u = (i_1, j_1, k_1), v = (i_2, j_2, k_2) \), \( u' = (i'_1, j'_1, k'_1) \) and \( v' = (i'_2, j'_2, k'_2) \). As multi-sets, we have

\[
\{i_1, i_2\} = \{i'_1, i'_2\}, \quad \{j_1, j_2\} = \{j'_1, j'_2\} \quad \text{and} \quad \{k_1, k_2\} = \{k'_1, k'_2\}.
\]

Without loss of generality, we may assume that \( i'_1 = i_1 \) and \( i'_2 = i_2 \). We have the following three cases.

(a) If \( j'_1 = j_1 \), then \( k'_1 = k_2 \) and \( f = I_{2,z}(u, v) \).
(b) If \( k'_1 = k_1 \), then \( j'_1 = j_2 \) and \( f = I_{2,y}(u, v) \).
(c) If \( j'_1 = j_2 \) and \( k'_1 = k_2 \), then \( f = I_{2,x}(u, v) \). \(\square\)

Remark 2.12. The amiable fact that the degree 2 generators only appear in the form of 2-minors cannot be directly generalized to four-dimensional case, as manifested from the above proof.

The main goal of this section is to show the quadratic binomials in \( I_2(D) \), defined in Definition 2.10, form a Gröbner basis of this ideal with respect to the lexicographic order. The next proposition observes that if the toric ring is defined by a three-dimensional Ferrers diagram which satisfies the projection property, then the toric ideal \( J_D \) cannot have any degree three element in the minimal Gröbner basis. Since the degree two part of \( J_D \) coincides with that of \( I_2(D) \), this leads to the lexicographically quadratic property of the latter ideal.

Proposition 2.13. Let \( D \) be a three-dimensional Ferrers diagram which satisfies the projection property. Then none of the minimal Gröbner basis element with respect to lexicographic order of the special fiber ideal has degree three.

Proof. Notice that all the points involved in such a potential binomial is contained in a \( 3 \times 3 \times 3 \) cube. But the Ferrers diagram property and the projection property are all preserved under layer truncations. Thus, by Proposition 2.8, it suffices to show that for any three-dimensional Ferrers diagram governed by the point \((3,3,3)\), if it satisfies the projection property, then it is lexicographically quadratic. For this, we can verify by running Macaulay2 \[15\] and exhaust all possible cases. One can check, for instance, by running the \texttt{All3()} in the script \texttt{Ferrers3D.m2}. The latter script is attached to the arXiv version (arXiv:1709.03251) of this work, and is also accessible at http://www.personal.psu.edu/kul20/Ferrers3D.m2. \(\square\)

Corollary 2.14. Let \( D \) be a three-dimensional Ferrers diagram which satisfies the projection property or is lexicographically quadratic. Then the 2-minors ideal \( I_2(D) \) is lexicographically quadratic.

Proof. Since \( I_2(D) \subseteq J_D \) and their degree two parts agree by Lemma 2.11, this result follows from Proposition 2.13 and the well-known Buchberger’s criterion \[16\] Theorem 7.3]. \(\square\)

Remark 2.15. When dealing with Gröbner basis by using Buchberger’s criterion, one needs to show those \( S \)-pairs can be reduced to 0. The projection property provides the
sufficient condition for this purpose when all the \(a_D\), \(b_D\) and \(c_D\) are at least three, in view of Proposition 2.13. Therefore, \(D\) in our mind is a relatively large and more general diagram. Of course, one can construct diagrams with very few elements, not satisfying the projection property condition, but still have quadratic Gröbner basis. See, for instance, the subsequent example. It implies that the projection property is not a necessary condition. It also demonstrates that the Gröbner basis is not necessarily quadratic if the monomial ordering is not lexicographic.

**Example 2.16.** Consider the three-dimensional Ferrers diagram, generated by the lattice points

\[(1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1).\]

Since the lattice point \((2, 2, 2)\) is missing, this diagram does not satisfy the projection property. However, using Macaulay2 [15], we can verify that this diagram is lexicographically quadratic. Furthermore, if we change the monomial ordering from the lexicographic order to the graded reverse lexicographic order, then the Gröbner basis of the presentation ideal is no longer quadratic. Indeed, non-squarefree binomials of degree three emerge.

With a minor restriction, the lexicographically quadratic property can be obtained in a more general setting.

**Proposition 2.17.** Let \(D\) be a three-dimensional diagram such that \(I_2(D)\) is lexicographically quadratic. Let \(G \subset D\) be a subdiagram satisfying one of the following conditions.

1. **(Detaching Condition)** For any \(v_1, v_2 \in G\) such that \(T_{v_1}T_{v_2} - T_{v'_1}T_{v'_2} \in I_2(D)\), we will always have \(v'_1, v'_2 \in G\).
2. **(Leading Monomial Condition)** \(G = D \setminus u\) and for each \(v \in G\) such that \(T_uT_v - T_{u'}T_{v'} \in I_2(D)\), the monomial \(T_uT_v\) is the leading monomial of this binomial.

Then the following restriction formulas hold:

\[I_2(D) \cap K[T_G] = I_2(G) \quad \text{and} \quad \text{gens}(\text{in}(I_2(D))) \cap K[T_G] = \text{gens}(\text{in}(I_2(G))).\]

In particular, \(I_2(G)\) is lexicographically quadratic.

**Proof.** For the first equality, it is clear that \(I_2(D) \cap K[T_G] \supseteq I_2(G)\). Thus, it suffices to show the reverse containment \(I_2(D) \cap K[T_G] \subseteq I_2(G)\). For this, we take arbitrary binomial \(f \in I_2(D) \cap K[T_G]\). Suppose for contradiction that \(f \notin I_2(G)\). We can replace \(f\) by its remainder with respect to \(\text{in}(I_2(G))\). Hence, none of the terms of \(f\) belongs to \(\text{in}(I_2(G))\).

Now we may assume that \(f = f_1 - f_2\) with \(f_1\) being the leading monomial and \(f_2 \notin \text{in}(I_2(G))\). Notice that \(f_1, f_2 \in K[T_G]\). Since \(f \in I_2(D)\) while \(I_2(D)\) is lexicographically quadratic, we can find some quadratic binomial \(g = g_1 - g_2 \in I_2(D)\) with \(g_1\) being the leading monomial and \(g_1\) being a factor of \(f_1\). Since \(f_1 \in K[T_G]\), we will have \(g_1 \in K[T_G]\) as well. Under the detaching condition, we directly get \(g_2 \in K[T_G]\). Under the leading monomial condition, since \(g_1\) is the leading monomial and \(T_u\) does not divide \(g_1, T_u\) does not divide \(g_2\) as well. This also means that \(g_2 \in K[T_G]\). In turn, we have \(g \in K[T_G]\) and consequently \(g \in I_2(G)\). This implies that \(f_1 \in \text{in}(I_2(G))\), a contradiction.
∆ is a simplex or equal to ∅. By the definition, we can find a binomial \( g = g_1 - g_2 \in I_2(\mathcal{D}) \) with \( g_1 \) being the leading monomial. Now, the arguments in the previous paragraph still shows that \( g \in K[\mathcal{T}_g] \) and \( g \in \text{in}(I_2(\mathcal{G})) \).

□

**Remark 2.18.** (a) The detaching condition is satisfied when \( u = (i_0, j_0, k_0) \in \mathcal{D} \) and \( \mathcal{G} \) is one of the following three truncated subdiagrams

\[
\{ (i, j, k) \in \mathcal{D} : i \neq i_0 \} \quad \{ (i, j, k) \in \mathcal{D} : j \neq j_0 \} \quad \{ (i, j, k) \in \mathcal{D} : k \neq k_0 \}.
\]

(b) The leading monomial condition is automatically satisfied when \( u \) is lexicographically the first point of \( \mathcal{D} \).

3. Simplicial complex of the initial ideal

Let \( \mathcal{D} \) be a three-dimensional Ferrers diagram which satisfies the projection property. Notice that the initial ideal of \( I_2(\mathcal{D}) \) is squarefree by Corollary 2.14. The Stanley–Reisner complex of this initial ideal will be denoted by \( \Delta(\mathcal{D}) \). To be more specific,

\[
\Delta(\mathcal{D}) := \left\{ F \subseteq \{ T_u : u \in \mathcal{G} \} : \prod_{T_v \in F} T_v \notin \text{in}(I_2(\mathcal{D})) \right\}.
\]

For a subdiagram \( \mathcal{G} \subseteq \mathcal{D} \), we use \( \Delta(\mathcal{D}, \mathcal{G}) \) to represent the restriction complex of \( \Delta(\mathcal{D}) \) to the set \( \{ T_u : u \in \mathcal{G} \} \). On the other hand, for a given simplicial complex \( \Delta \) over some set \( X \), let \( I_\Delta \) be the Stanley–Reisner ideal in the corresponding polynomial ring \( K[X] \).

The purpose of the next section is to show that when \( \mathcal{D} \) is a three-dimensional Ferrers diagram which satisfies the projection property, the ideal \( I_2(\mathcal{D}) \) is Cohen–Macaulay. By [13 Corollary 3.3.5], it suffices to show that \( \text{in}(I_2(\mathcal{D})) \) is Cohen–Macaulay. To achieve this, we prove in Theorem 4.1 that the Stanley–Reisner complex, \( \Delta(\mathcal{D}) \), is pure vertex-decomposable; see Definition 3.2. It is well-known that pure vertex-decomposable complexes are pure shellable, hence Cohen–Macaulay, or equivalently, their Stanley–Reisner ideals are Cohen–Macaulay.

In this section, we will recall and build additional tools for the proofs in the sequel. In particular, we need to determine the dimensions of the restriction complexes that are involved in the those proofs.

**Remark 3.1.** Throughout this paper, we use implicitly the following well-known fact. Let \( T_u \) be a vertex of a simplicial complex \( \Delta \) on \( V \). Then the cone over the link complex \( \text{link}_\Delta(T_u) \) with apex \( T_u \) considered as a complex on \( V \) has Stanley–Reisner ideal \( I_\Delta : T_u \), and the Stanley–Reisner ideal of the deletion complex \( \Delta \setminus T_u \) considered as a complex on \( V \) is \( (T_u, I'_\Delta \setminus T_u) \), where \( I'_\Delta \setminus T_u \) is the Stanley–Reisner ideal of \( \Delta \setminus T_u \) considered as a complex on \( V \setminus T_u \).

**Definition 3.2** ([25]). A pure simplicial complex \( \Delta \) is said to be vertex-decomposable if \( \Delta \) is a simplex or equal to \( \{ \emptyset \} \), or there exists a vertex \( v \) such that the link complex \( \text{link}_\Delta(v) \) and the deletion complex \( \Delta \setminus v \) are both pure and vertex-decomposable and \( \dim(\Delta) = \dim(\Delta \setminus v) = \dim(\text{link}_\Delta(v)) + 1 \). The vertex \( v \) here is called a shedding vertex.
Remark 3.3. Suppose that $\Delta$ is a pure simplicial complex and a cone with apex $v$. It follows from [26, Proposition 2.4] that $\Delta$ is vertex-decomposable if and only if $\Delta \setminus v$ is so.

For a general finite diagram $D$ in $\mathbb{Z}^3$, we use the superscript to denote the corresponding $x$ layers. For instance,

$$D^1 := \{(j, k) \in D \} \quad \text{and} \quad D^{2a} := \{(i, j, k) \in D : i \geq a \}.$$  

Definition 3.4. Let $<$ be a total order on $D$. We say that $<$ is a quasi-lexicographic order if it satisfies the following two conditions.

(QLO-1) The points in $D^1$ precede the points in $D^{2a}$ with respect to $<$.

(QLO-2) For distinct $u = (1, j_1, k_1)$ and $v = (1, j_2, k_2)$ in $D$, if $j_1 \leq j_2$ and $k_1 \leq k_2$, then $u$ precedes $v$ with respect to $<$.  

Obviously, the lexicographic order is a quasi-lexicographic order. Throughout this section, we always assume that $<$ is a quasi-lexicographic order.

Given a lattice point $u \in D^1$, let $A_u$ be the diagram obtained from $D$ by removing the points before $u$ with respect to $<$. We also write $A_u^+ := A_u \setminus u$. Notice that when our quasi-lexicographic order $<$ happens to be the lexicographic order, then $A_u = D_u$.

Remark 3.5. Let $D$ be a finite three-dimensional diagram such that $I_2(D)$ is lexicographically quadratic. Suppose that $u = (1, j_1, k_1) \in D^1$ is the first point with respect to $<$ and let $G = D \setminus u = A_u^+$. By (QLO-2), those $v = (1, j_2, k_2)$ preceding $u$ lexicographically must satisfy $j_2 < j_1$ and $k_2 > k_1$. In particular, $(1, j_2, k_1) \notin D$. This implies that the “leading monomial condition” in Proposition 2.17 holds. Thus, we have the following restriction formulas:

$$I_2(D) \cap \mathbb{K}[T_G] = I_2(G) \quad \text{and} \quad \text{gens}(\text{in}(I_2(D))) \cap \mathbb{K}[T_G] = \text{gens}(\text{in}(I_2(G))).$$

By induction, for any $w \in D^1$, we have similar formulas:

$$I_2(D) \cap \mathbb{K}[T_{A_w}] = I_2(A_w) \quad \text{and} \quad \text{gens}(\text{in}(I_2(D))) \cap \mathbb{K}[T_{A_w}] = \text{gens}(\text{in}(I_2(A_w))).$$

In particular, the restriction complexes $\Delta(D, A_w) = \Delta(A_w)$.

With respect to the diagram $D$ above, define

$$N(D) := \{ u \in D^1 : \text{in}(I_2(A_u)) \nsubseteq \text{in}(I_2(A_u^+)) \mathbb{K}[T_{A_u}] \}$$

and $\text{Phan}(D) := D^1 \setminus N(D)$ to be the set of normal points and phantom points with respect to the quasi-lexicographic order $<$ respectively.

Remark 3.6. Let $D$ be a finite three-dimensional diagram such that $I_2(D)$ is lexicographically quadratic. For each $u \in D^1$, as pointed out in Remark 3.5, the pair $A_u^+ \subset A_u$ satisfies the leading monomial condition. Thus, $u$ is a normal point if and only if $u$ can switch coordinates with some $v \in A_u^+$. Dually, $u$ is a phantom point if and only if $\Delta(A_u)$ is a cone over $\Delta(A_u) \setminus T_u = \Delta(A_u^+)$ with the apex $T_u$.  

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Indeed, when $\mathcal{D}$ is a three-dimensional Ferrers diagram satisfying the projection property, the sets of normal points and phantom points are independent of the specific quasi-lexicographic order that we choose.

**Discussion 3.7.** Let $\mathcal{D}$ be a three-dimensional Ferrers diagram satisfying the projection property and $a_D \geq 2$. It is clear that if $u = (1, j_1, k_1) \in \mathcal{D}$ is a phantom point, then it is a border point, i.e., $(1, j_1 + 1, k_1 + 1) \notin \mathcal{D}$. Indeed, if otherwise, then $u$ can make a $y$-coordinates switch with the point $(1, j_1 + 1, k_1 + 1)$.

Denote the set of such border points by $\mathcal{B}$. Now, we apply Remark 3.6 to exclude normal points in $\mathcal{B}$. The following border points are normal points.

1. $(B_y)$ The border points on the $y = 1, 2, \ldots, b_{D^2} - 1$ lines with minimal $z$-coordinates.
2. $(B_z)$ The border points on the $z = 1, 2, \ldots, c_{D^2} - 1$ lines with minimal $y$-coordinates.

For $u = (1, j_1, k_1) \in B_y$, we have $(1, j_1 + 1, k_1) \in \mathcal{D}$ and $u$ can switch $y$-coordinates with $(2, j_1 + 1, 1)$. On the other hand, if $u$ is a border point which can switch $y$-coordinates within $\mathcal{A}_u$, it can only do so with points in $\mathcal{D}^2$ by (QLO-2). Hence such $u \in B_y$. We can similarly talk about $B_z$. Thus, border points in these two sets are the border points such that $y$-coordinates switch or $z$-coordinates switch is available. Furthermore, these two sets are disjoint. To see this, suppose that $u = (1, j_1, k_1) \in B_y \cap B_z$. Since $u \in B_y$, $(2, j_1 + 1, 1) \in \mathcal{D}$. Similarly, $(2, 1, k_1 + 1) \in \mathcal{D}$. By the projection property, this leads to $(1, j_1 + 1, k_1 + 1) \in \mathcal{D}$. But $u$ is supposed to be a border point. This is a contradiction.

For $v = (1, j_2, k_2) \in B \setminus (B_y \cup B_z)$, it is a normal point if and only if it can switch $x$-coordinates with some $v$ later than $v$ with respect to $\prec$. A necessary condition for this to happen is $(2, j_2, k_2) \in \mathcal{D}$. Hence $b_{D^2} \geq j_2$ and $c_{D^2} \geq k_2$. By the projection property, $(1, b_{D^2}, c_{D^2}) \in \mathcal{D}$. Since $v$ is a border point, either $j_2 = b_{D^2}$ or $k_2 = c_{D^2}$. If $j_2 = b_{D^2}$ while $k_2 < c_{D^2}$, then $z$-coordinates switch is available and $v \in B_z$. We can similarly discuss the symmetric case. Since we assume $v \notin B_y \cup B_z$ in this case, we must have $v = (1, b_{D^2}, c_{D^2})$. By (QLO-2), this is the only point in $\mathcal{A}_v$ which can increase its $x$-coordinate. Thus, $x$-coordinates switch is not possible for $v$.

Therefore, the normal points and phantom points with respect to $\prec$ agree with those defined with respect to the lexicographic order. Furthermore, the above discussion shows that

$$|N(\mathcal{D}) \cap \mathcal{B}| = b_{D^2} + c_{D^2} - 2.$$  

Since $|\mathcal{B}| = b_D + c_D - 1$,

$$|\text{Phan}(\mathcal{D})| = |\mathcal{B}| - |N(\mathcal{D}) \cap \mathcal{B}| = (b_D - b_{D^2}) + (c_D - c_{D^2}) + 1. \tag{2}$$

This is the expected number in view of Lemma 3.9 and Theorem 4.1.

**Observation 3.8.** Let $\mathcal{D}$ be a finite three-dimensional Ferrers diagram such that $I_2(\mathcal{D})$ is lexicographically quadratic. Take arbitrary $u \in D^1$. Suppose that $\text{in}(I_2(A_u))$ and $\text{in}(I_2(A^*_u))$ are unmixed, or, equivalently, $\Delta(A_u)$ and $\Delta(A^*_u)$ are pure.

(i) If $u$ is a phantom point, then trivially $\text{codim} I_2(A_u) = \text{codim} I_2(A^*_u)$. 


Lemma 3.9. If dimension of $\Delta(\mathcal{D})$ is $a_{\mathcal{D}} + b_{\mathcal{D}} + c_{\mathcal{D}} - 3$.

Proof. One checks with ease that
\[
\{ T_{1,1,1}, T_{2,1,1}, \ldots, T_{a_{\mathcal{D}},1,1}, T_{1,2,1}, T_{1,3,1}, \ldots, T_{1,b_{\mathcal{D}},1}, T_{1,1,2}, T_{1,1,3}, \ldots, T_{1,1,c_{\mathcal{D}}} \}
\]
forms a facet of $\Delta(\mathcal{D})$. And its cardinality is exactly $a_{\mathcal{D}} + b_{\mathcal{D}} + c_{\mathcal{D}} - 2$. □

Lemma 3.10. Let $\mathcal{D}$ be a three-dimensional Ferrers diagram. Then $\dim(\mathcal{F}(\mathcal{D})) = a_{\mathcal{D}} + b_{\mathcal{D}} + c_{\mathcal{D}} - 2$.

Proof. One can check that the images of
\[
\{ T_{1,1,1}, T_{2,1,1}, \ldots, T_{a_{\mathcal{D}},1,1}, T_{1,2,1}, T_{1,3,1}, \ldots, T_{1,b_{\mathcal{D}},1}, T_{1,1,2}, T_{1,1,3}, \ldots, T_{1,1,c_{\mathcal{D}}} \}
\]
form a transcendental basis of the domain $\mathcal{F}(\mathcal{D})$ over $\mathbb{K}$. Now, we may apply [4, Theorem A.16]. □

Now, at the end of this section, we advertise the attacking tactics for Theorem [4,1] and Theorem [5,2]. Let $\mathcal{D}$ be a three-dimensional Ferrers diagram. In the proof of those results, we need to decompose the diagram $\mathcal{D}$ in a rather involved way. Because of this, take arbitrary $u = (1, j_0, k_0) \in \mathcal{D}$ with
\[\alpha = a_{\mathcal{D}}(u), \quad \beta = b_{\mathcal{D}}(u) \quad \text{and} \quad \gamma = c_{\mathcal{D}}(u)\]
as defined before Definition [2,2]. We divide $\mathcal{D}$ into the following six zones:
\[
\begin{align*}
Z_1(\mathcal{D}, u) & := \{ (i, j, k) \in \mathcal{D} : 1 \leq j \leq j_0 \text{ and } k > \gamma \}, \\
Z_2(\mathcal{D}, u) & := \{ (i, j, k) \in \mathcal{D} : 1 \leq j \leq j_0 \text{ and } k_0 < k \leq \gamma \}, \\
Z_3(\mathcal{D}, u) & := \{ (i, j, k) \in \mathcal{D} : 1 \leq j \leq j_0 \text{ and } 1 \leq k \leq k_0 \}, \\
Z_4(\mathcal{D}, u) & := \{ (i, j, k) \in \mathcal{D} : j_0 < j \leq \beta \text{ and } k_0 < k \leq \gamma \}, \\
Z_5(\mathcal{D}, u) & := \{ (i, j, k) \in \mathcal{D} : j_0 < j \leq \beta \text{ and } 1 \leq k \leq k_0 \}, \\
Z_6(\mathcal{D}, u) & := \{ (i, j, k) \in \mathcal{D} : j > \beta \text{ and } 1 \leq k < k_0 \}.
\end{align*}
\]
It is clear that $\mathcal{D}$ is the disjoint union of the above six zones. In the subsequent discussion, they will be called $Z$-zones with respect to $\mathcal{D}$ and $u$. We will omit some of the parameters, if they are clear from the context. Figure 3 gives the idea of the division of $\mathcal{D}$ with respect to these zones.

Suppose that $a_{\mathcal{D}} \geq 2$. We adopt the following induction process for considering both the vertex-decomposable property of $\Delta(\mathcal{D})$ and the primeness of $I_2(\mathcal{D})$. Consider the symmetry operation $S : \mathbb{Z}_+^3 \to \mathbb{Z}_+^3$ by sending $(i, j, k)$ to $(i, k, j)$. The common induction process is as follows:
First stage We remove lexicographically the initial points within
\[ C := \{ (1, j, k) \in \mathcal{D} : k \leq c_{D} \geq 2 \} . \]

Second stage After we remove above points in the first stage, we do a flip by \( S \). The current flipped diagram also comes from \( S(\mathcal{D}) \) by removing lexicographically initial points in the \( x = 1 \) layer:
\[ S(\mathcal{D} \setminus C) = (S(\mathcal{D}))_{u} \]
where \( u = (1, c_{D} \geq 2 + 1, 1) \). We remove lexicographically the initial points in the \( x = 1 \) layer of the remaining flipped diagram in this stage.

The above induction process leads to a total order on the points in the \( x = 1 \) layer of \( \mathcal{D} \), and we will refer to it as the induction order. Figure 4 gives an idea how this proceeds. The first point is \( \circ = (1, 1, 1) \). When \( c_{D} < c_{D} \), the last point is \( \bullet = (1, b_{D}((1, 1, c_{D})), c_{D}) \) of \( \mathcal{D} \). Otherwise, \( c_{D} \geq 2 = c_{D} \) with the second stage disappears and the last point is \( (1, b_{D}, c_{D}((1, b_{D}, 1))) \) of \( \mathcal{D} \). Meanwhile, in Figure 4 \( \star \) denotes the last point in the first stage while \( \ast \) denotes the first point in the second stage. We may also extend the induction order to the whole \( \mathcal{D} \) by ordering the points in \( \mathcal{D}^{2} \) in a suitable way and put them after the points in \( \mathcal{D}^{1} \). But this does not matter since we overall prove by induction on \( a_{D} \).

Remark 3.11. (1) The induction order is a quasi-lexicographic order.
(2) Suppose that \( u = (1, j_{0}, k_{0}) \) belongs to the first stage. Since now \( k_{0} \leq c_{D} \), we have \( (2, 1, k_{0}) \in \mathcal{D} \). For \( \beta = b_{D}(u) \), we have \( (1, \beta + 1, k_{0}) \notin \mathcal{D} \) by definition. Hence if \( \mathcal{D} \) satisfies the projection property, then \( (2, \beta + 1, 1) \notin \mathcal{D} \). This implies that the zone \( Z_{6} = Z_{6}^{1} \).
(3) Let \( \mathcal{G} \) be a subdiagram of \( \mathcal{D} \). Under the operation \( S \), the Ferrers property and the projection property of \( \mathcal{G} \) are preserved. We extend this operation to \( K[T_{\mathcal{G}}] \). Obviously, \( S(I_{2}(\mathcal{G})) = I_{2}(S(\mathcal{G})) \). If \( I_{2}(\mathcal{G}) \) is lexicographically quadratic, then \( S \) also preserves the initial ideal with respect to the lexicographic order:
\[ S(\text{in}(I_{2}(\mathcal{G}))) = \text{in}(I_{2}(S(\mathcal{G}))). \]
To see this, we notice that $T_uT_v$ is the leading term of $f := T_uT_v - T_wT_{w'} \in I_2(G)$ if and only if $T_{S(u)}T_{S(v)}$ is the leading term of $S(f)$. We also need the fact that $S$ preserves the graded Hilbert function, since $\text{in}(I_2(G))$ and $I_2(G)$ share the same graded Hilbert function as well as so for $\text{in}(I_2(S(G)))$ and $I_2(S(G))$.

4. Cohen–Macaulayness

The purpose of this section is to establish the Cohen–Macaulayness of the ideal $I_2(D)$. As advertised in the previous section, we will show that $\Delta(D)$ is pure vertex-decomposable. We give readers the road map of the proof of Theorem 4.1 here, because the proof is very involved. We apply induction with respect to the order introduced in the previous section and this is natural because the vertex-decomposable property is by definition an induction property. During the proof, we especially focus on the deletion and the link of the complex with respect to a particular vertex. We pay close attention to the generators of the link complex and its dimension because this is where we apply induction hypothesis. Moreover, we use a lot of restriction complexes, in the form of $\Delta(D, G)$, during the proof so that we can reduce to a smaller case.

**Theorem 4.1.** Assume that $D$ is a three-dimensional Ferrers diagram which satisfies the projection property. Then $\Delta(D)$ is pure vertex-decomposable of dimension $a_D + b_D + c_D - 3$.

**Proof.** We prove by the induction on $a_D$. When $a_D = 1$, this can be reduced to the two-dimensional case in [10, Theorem 3.3]. Now, we assume that $a_D \geq 2$. To finish the proof, we will proceed by removing the points in the $x = 1$ layer according to the induction order given in the previous section. Write $C := \{(1, j, k) \in D : k \leq c_D\}$.

**First stage**

We study the subdiagram $A_u$ for each $u \in C$, which is the subdiagram of $D$ by removing points preceeding $u$ with respect to the induction order. By induction, Remark 3.5 and Remark 3.6, $\Delta(D, A_u) = \Delta(A_u)$ is expected to have dimension

$$a_{D22} + b_{D22} + c_{D22} - 3 + |\text{Phan}(D) \cap A_u|.$$
We will show that $\Delta(A_u)$ is pure vertex-decomposable.

The minimal case is when we remove all the points in $C$. We will deal it in the second stage. Other than this minimal case, when $u$ is a phantom point, this case is clear by Remark 3.3 and Remark 3.6. Thus, we may assume that $u$ is a normal point. Say that $u = (1, j_0, k_0)$ with

$$\alpha = a_D(u), \quad \beta = b_D(u) \quad \text{and} \quad \gamma = c_D(u).$$

We only need to check that $T_u$ is the expected shedding vertex for the simplicial complex. Note that from our induction hypothesis, the deletion complex $\Delta(A_u) \setminus T_u = \Delta(A_u^+)$ is pure vertex-decomposable of the expected dimension given by (3). As for the link complex $\text{link}_{\Delta(A_u)}(T_u)$, its Stanley–Reisner ideal in $K[T_{A_u}]$ is the colon ideal $I_{\Delta(A_u)} \triangleright T_u$. We need to investigate this colon ideal in detail. Since the monomial generators of $I_{\Delta(A_u)}$ are all quadratic, the monomial generators of the colon ideal $I_{\Delta(A_u)} \triangleright T_u$ are either quadratic or linear.

As observed in Remark 3.5, $u$ satisfies the “leading monomial” condition in Proposition 2.1.7. Therefore, $T_u' \in I_{\Delta(A_u)} \triangleright T_u$ if and only if $T_u$ and $T_u'$ can switch coordinates. Now, the lattice points in $A_u$ that contribute the linear minimal generators of $I_{\Delta(A_u)} \triangleright T_u$ come from the following “linear” regions:

$$\{ (i, j, k) \in D : i \geq j_0 \text{ and } j > k \} \quad \text{by switching the } y\text{-coordinates},$$

$$\{ (i, j, k) \in D : j \geq 2 \text{ and } j_0 < j \leq \beta \} \quad \text{by switching the } y\text{-coordinates},$$

$$\{ (i, j, k) \in D : j \geq 2 \text{ and } k_0 < k \leq \gamma \} \quad \text{by switching the } z\text{-coordinates},$$

$$\{ (i, j, k) \in D : j > \beta \text{ and } 2 \leq i \leq \alpha \} \quad \text{by switching the } x\text{-coordinates}.$$

However, since $k_0 \leq c_{D\oplus 2}$, $Z_6 = Z_6^1$ by the projection property. In particular, the last region (†) is indeed empty.

Consequently, we see that

$$I_{\Delta(A_u)} \triangleright T_u = (T_u' : u' \in \mathbb{Z}_4 \cup \mathbb{Z}_2^{\oplus 2} \cup \mathbb{Z}_5^{\oplus 2}) + I_{\Delta(A_u)} \cap K[T_u': u' \in \mathcal{E}].$$

Here,

$$\mathcal{E} \coloneqq A_u^+ \setminus (\mathbb{Z}_4 \cup \mathbb{Z}_2^{\oplus 2} \cup \mathbb{Z}_5^{\oplus 2}).$$

So far, $\Delta(D, \mathcal{E})$ is the restriction complex of the link complex $\text{link}_{\Delta(D,A_u)}(T_u)$ by removing the “invisible vertices” in $(T_u' : u' \in \mathbb{Z}_4 \cup \mathbb{Z}_2^{\oplus 2} \cup \mathbb{Z}_5^{\oplus 2})$. Notice that $\Delta(D, A_u) = \Delta(A_u)$ and $u$ is a normal point. As we hope that $T_u$ is a shedding vertex at this step, we are reduced to show that $\Delta(D, \mathcal{E})$ is a pure vertex-decomposable complex of dimension

$$\dim \Delta(D, \mathcal{E}) = \dim \Delta(A_u) - 1. \quad (4)$$

Notice that for each $(1, j, k) \in \mathcal{E}$ with $1 \leq j \leq j_0$, we have

- $j = j_0$ and $k_0 < k \leq \gamma$, or
- $j < j_0$ and $k > c_{D\oplus 2}$.

Let

$$\mathcal{H} \coloneqq \mathcal{E} \setminus \{ (1, j_0, k) : k_0 < k \leq \min(\gamma, c_{D\oplus 2}) \}$$
This implies that for any $v \in \mathcal{H}$, we can similarly define $\overline{\mathcal{H}}$. Notice that $\mathcal{H} \subseteq \overline{\mathcal{H}}$. This implies that for any $v \in \mathcal{H}$, one has $T_{\overline{\mathcal{H}}} \notin I_{\Delta(A_u)}$. Therefore, $\Delta(D, \mathcal{E})$ is the join of $\Delta(D, \mathcal{H})$ with a simplex of dimension $\min(\gamma, c_D) - k_0 - 1$. Consequently
\begin{equation}
\dim(\Delta(D, \mathcal{H})) = \dim(\Delta(D, \mathcal{E})) - (\min(\gamma, c_D) - k_0),
\end{equation}
and by Remark 2.18
\begin{equation}
gens(I_{\Delta(D)} \cap \mathbb{K}[T_{u'} : u' \in \mathcal{E}]) = gens(I_{\Delta(A_u)} \cap \mathbb{K}[T_{u'} : u' \in \mathcal{E}]) = gens(I_{\Delta(D, \mathcal{H})}).
\end{equation}
To summarize, by combining (3), (4), and (5), we are expecting
\begin{equation}
\dim(\Delta(D, \mathcal{H})) = a_D + b_D + c_D - 3 + |\text{Phan}(D) \cap \mathcal{A}_u| - (\min(\gamma, c_D) - k_0 + 1).
\end{equation}
Now, it suffices to show that $\Delta(D, \mathcal{H})$ is pure vertex-decomposable of this expected dimension. We prove this in Lemma 4.3.

**Second stage**

After the first stage, we are dealing with the case where we removed all the points in $\mathcal{C}$. Now we flip $D$ to get $S(D)$, which will be written as $D'$ for simplicity. Notice that for the remaining points in $(D')^1$, the induced induction order is exactly the lexicographic order. Now, as before, we will write $D'_u$ for the restriction diagram from $D'$ by removing those points preceding $u$ lexicographically in $D'$. Using this notation, the current case is $S(D \setminus \mathcal{C}) = D'_u$, since by flipping the diagram, we switch the $y$ and $z$ coordinates. By Remark 3.11
\begin{equation}
\Delta(S(D), S(D \setminus \mathcal{C})) \cong \Delta(D'_{(1, e_D + 1, 1)}).
\end{equation}
In the following, we will remove the points in the $x = 1$ layer of $D'_{(1, e_D + 1, 1)}$ lexicographically, and prove the corresponding complex is pure vertex-decomposable of expected dimension.

The minimal case is when we remove all the $x = 1$ layer. Now, we have $(D')^2 = S(D^2)$. By induction, $\Delta((D')^2)$ is pure vertex-decomposable of dimension
\begin{equation}
a_D + b_D + c_D - 3.
\end{equation}
Now, consider a general $u = (1, j_0, k_0) \in D'_u$. By induction, Remark 3.5 and Remark 3.6 $\Delta(D'_u)$ has dimension
\begin{equation}
a_D + b_D + c_D - 3 + |\text{Phan}(D') \cap D'_u|.
\end{equation}
Notice that $\text{Phan}(D') \cap D'_u$ coincides with $\text{Phan}(D'_u)$ by Remark 2.18. We will show that $\Delta(D'_u)$ is pure vertex-decomposable.

For general $D'_u$, when $u$ is a phantom point, this case is clear by Remark 3.6. Thus, we may assume that $u$ is not a phantom point. Say that $\alpha := a_{D'}(u), \beta := b_{D'}(u)$ and $\gamma := c_{D'}(u)$.

We will check that $T_u$ is the expected shedding vertex for the simplicial complex. For this, we prove similarly as in the first stage.
Notice that, in the first stage, the “ceiling restriction” of choosing points \((1, j, k)\) with \(k \leq c_{D2}\) is mainly used to ensure that \(Z_6(D) = Z_6^1(D)\). In the current case we will automatically get \(Z_6(D') = Z_6^1(D')\). In the following, the \(\mathcal{Z}\)-zones are with respect to \(D'\).

Now, as in the first stage, it suffices to show that the link complex \(\text{link}_{\Delta(D_u')}(T_u)\) is pure vertex-decomposable of dimension one less. By a similar screening, we see the colon ideal

\[
I_{\Delta(D_u')} : T_u = (T_u' : u' \in \mathcal{Z}_4 \cup \mathcal{Z}_2 \cup \mathcal{Z}_6^2) + I_{\Delta(D_u')} \cap \mathbb{K}[T_u' : u' \in \mathcal{E}'].
\]

Here, for \((D_u')^+ := D_u' \setminus u\), we have

\[
\mathcal{E}' := (D_u')^+ \setminus (\mathcal{Z}_4 \cup \mathcal{Z}_2 \cup \mathcal{Z}_6^2).
\]

Notice that for each \((1, j, k) \in \mathcal{E}' \) with \(1 \leq j \leq j_0\), we have \(j = j_0\) and \(k_0 < k \leq \gamma\). Let \(\mathcal{H}' := \mathcal{E}' \setminus \mathcal{Z}_2^1\). Similarly, we see that \(\Delta(D', \mathcal{H}')\) is the join of \(\Delta(D', \mathcal{H}')\) with a simplex of dimension

\[
\gamma - k_0 - 1,
\]

and by Remark 2.18

\[
gens(I_{\Delta(D')} \cap \mathbb{K}[T_u' : u' \in \mathcal{E}']) = gens(I_{\Delta(D_u')} \cap \mathbb{K}[T_u' : u' \in \mathcal{E}'])
\]

By the previous discussion, we are similarly anticipating

\[
\dim(\Delta(D', \mathcal{H}')) = \dim(\Delta(D', \mathcal{E}')) - (\gamma - k_0) = \dim(\Delta(D_u')) - (\gamma - k_0 + 1)
\]

(8) \[= a_{D2} + b_{D2} + c_{D2} - 3 + |\text{Phan}(D') \cap D_u'| - (\gamma - k_0 + 1).
\]

Now, it suffices to show that \(\Delta(D', \mathcal{H}')\) is pure vertex-decomposable of this expected dimension. We prove this in Lemma 4.3. □

**Lemma 4.2.** Using the notation in the proof of Theorem 4.1, we have

\[
I_2(D) \cap \mathbb{K}[T_{\mathcal{H}}] = I_2(\mathcal{H}) \quad \text{and} \quad gens(\text{link}(I_2(D))) \cap \mathbb{K}[T_{\mathcal{H}}] = gens(\text{link}(I_2(\mathcal{H}))).
\]

In particular, \(\Delta(D, \mathcal{H}) = \Delta(\mathcal{H})\) and \(I_2(\mathcal{H})\) is lexicographically quadratic.

**Proof.** Notice that \(\mathcal{H} \subseteq \mathcal{A}_u^+\). Since we already have similar formulas for \(\mathcal{A}_u^+\) instead of \(\mathcal{H}\) in Remark 3.2, we may first replace \(D\) by \(\mathcal{A}_u^+\). Now, it suffices to verify directly that when \(\mathcal{H} \neq \mathcal{A}_u^+\), the pair \(\mathcal{H} \subset \mathcal{A}_u^+\) satisfies the detaching condition in Proposition 2.17. □

**Lemma 4.3.** Using the notation in the proof of Theorem 4.1, the complex \(\Delta(D, \mathcal{H}) = \Delta(\mathcal{H})\) is pure vertex-decomposable of dimension given by Equation (6).

**Proof.** Let

\[
\tilde{D} := Z_3 \cup Z_5^1 \cup Z_6 \cup \{ (i, j, k) \in \mathcal{D} : j \leq j_0 \text{ and } k > \min(\gamma, c_{D2}) \}.
\]

This diagram is essentially a three-dimensional Ferrers diagram which satisfies the projection property. Furthermore, notice that \(Z_6 = Z_6^1\) and \(Z_1 \subset \tilde{D}\). Like \(\mathcal{A}_u\) and \(\mathcal{A}_u^+\) for \(D\), we similarly define \(\tilde{\mathcal{A}}_u\) and \(\tilde{\mathcal{A}}_u^+\) for \(\tilde{D}\). As the first step, we notice that \(\mathcal{H} = \tilde{\mathcal{A}}_u^+\). Even when
\( \mathcal{D} = \tilde{\mathcal{D}} \), this is a strictly smaller case compared to \( \mathcal{A}_u \). Hence by induction, \( \Delta(\mathcal{H}) \) is pure vertex-decomposable of dimension

\begin{equation}
    a_{D_{22}} + b_{D_{22}} + c_{D_{22}} - 3 + |\text{Phan}(\tilde{\mathcal{D}}) \cap \mathcal{H}|.
\end{equation}

It remains to verify that this number agrees with (6).

Notice that by (2), we have

\[
|\text{Phan}(\mathcal{D}) \cap \mathcal{A}_u| = |\text{Phan}(\mathcal{D})| - |\text{Phan}(\mathcal{D}) \setminus \mathcal{A}_u| = (b_D + c_D) - (b_{D_{22}} + c_{D_{22}}) + 1 - |\text{Phan}(\mathcal{D}) \setminus \mathcal{A}_u|,
\]

and similarly

\[
|\text{Phan}(\tilde{\mathcal{D}}) \cap \mathcal{H}| = (b_{\tilde{D}} + c_{\tilde{D}}) - (b_{\tilde{D}_{22}} + c_{\tilde{D}_{22}}) + 1 - |\text{Phan}(\tilde{\mathcal{D}}) \setminus \mathcal{H}|.
\]

Since

\[
a_{D_{22}} = a_{\tilde{D}_{22}}, \quad b_D = b_{\tilde{D}} \quad \text{and} \quad c_D - (\min(\gamma, c_{D_{22}}) - k_0) = c_{\tilde{D}},
\]

we are reduced to show that

\begin{equation}
|\text{Phan}(\tilde{\mathcal{D}}) \setminus \mathcal{H}| = |\text{Phan}(\mathcal{D}) \setminus \mathcal{A}_u| + 1.
\end{equation}

Notice that if \( \mathbf{v} = (1, j, k) \in \text{Phan}(\mathcal{D}) \setminus \mathcal{A}_u \) or \( \text{Phan}(\tilde{\mathcal{D}}) \setminus \mathcal{H} \), then \( j \leq j_0 \).

(a) Consider the case when \( j < j_0 \).

(i) Suppose that \( c_{D_{22}} < \gamma \). It is clear that

- \( \mathbf{v} \) is in the border of \( \mathcal{D} \) if and only if it is in the border of \( \tilde{\mathcal{D}} \);
- \( \mathbf{v} \) is a phantom point with respect to \( \mathcal{D} \) if and only if it is so with respect to \( \tilde{\mathcal{D}} \) by Discussion 3.7 and
- when \( \mathbf{v} \) is a common phantom point, then \( \mathbf{v} \) belongs to \( \mathcal{A}_u \) if and only if it belongs to \( \mathcal{H} \).

(ii) Suppose that \( c_{D_{22}} \geq \gamma \). We may simply assume that \( k_0 = \gamma \) and argue as above.

(b) Consider the case for \( j = j_0 \). If \( \mathbf{v} \) is a border point of \( \mathcal{D}^1 \) not belonging to \( \mathcal{A}_u \), then \( k < k_0 \leq c_{D_{22}} \). Hence \( \mathbf{v} \) is a normal point, since a z-coordinates switch is always feasible. This means that we have no such phantom point in \( \text{Phan}(\mathcal{D}) \setminus \mathcal{A}_u \).

On the other hand, with respect to \( \tilde{\mathcal{D}} \), we first notice that

\[
c_{\tilde{D}}((1, j_0 + 1, 1)) = \min(c_{\tilde{D}}((1, j_0 + 1, 1)), k_0).
\]

Each border point \( \mathbf{v} = (1, j_0, k) \in \tilde{\mathcal{D}} \) satisfies \( c_{\tilde{D}}((1, j_0 + 1, 1)) \leq k < k_0 \) or \( \min(\gamma, c_{D_{22}}) < k \leq \gamma \).

- When \( c_{\tilde{D}}((1, j_0 + 1, 1)) \leq k < k_0 \), a z-coordinates switch is available. Hence the border point is a normal point.
- When \( \min(\gamma, c_{D_{22}}) < k \leq \gamma \), the border point belongs to \( \mathcal{H} \).

Hence, it suffices to show that the border point \( \mathbf{u} = (1, j_0, k_0) \) is a phantom point for \( \tilde{\mathcal{D}} \). Notice that for any \( (1, j_0, k_0) \in \tilde{\mathcal{A}}_u \), one has \( (2, j_0, k_0) \notin \tilde{\mathcal{D}} \). Thus, x-coordinates switch is forbidden for \( \mathbf{u} \) in \( \tilde{\mathcal{A}}_u \). Since \( (2, j_0 + 1, 1) \notin \tilde{\mathcal{D}} \), y-coordinates switch is also not possible for \( \mathbf{u} \) in \( \tilde{\mathcal{A}}_u \).

- When \( c_{D_{22}} \geq \gamma \), we cannot increase the z-coordinate of \( \mathbf{u} \) within \( \tilde{\mathcal{D}} \).
• When \( c_{D^2} < \gamma \), we have \( c_{\tilde{D}^2} = k_0 \).

In either case, \( z \)-coordinates switch is not possible for \( u \) in \( \tilde{A}_u \). Thus, \( u \) is a phantom point for \( \tilde{D} \). This only phantom point contributes to the number 1 in (10).

Thus, we have established (10). \( \square \)

**Lemma 4.4.** Using the notation in the proof of Theorem 4.1, we have

\[
\mathbf{I}_2(D') \cap \mathbb{K}[T_{H'}] = \mathbf{I}_2(H') \quad \text{and} \quad \text{gens}(\mathbf{I}_2(D')) \cap \mathbb{K}[T_{H'}] = \text{gens}(\mathbf{I}_2(H')).
\]

In particular, \( \Delta(D', H') = \Delta(H') \) and \( \mathbf{I}_2(H') \) is lexicographically quadratic.

**Proof.** The proof is similar to that of Lemma 4.2. \( \square \)

**Lemma 4.5.** Using the notation in the proof of Theorem 4.1, the complex \( \Delta(D', H') = \Delta(H') \) is pure vertex-decomposable of dimension given by Equation (8).

**Proof.** The \( Z \)-zones here are with respect to \( D' = S(D) \). Let

\[
\tilde{D}' := Z_1 \cup Z_3 \cup Z_5 \cup Z_6.
\]

This diagram is still essentially a three-dimensional Ferrers diagram which satisfies the projection property. Furthermore, notice that \( Z_5 = Z_5^1 \) and \( Z_6 = Z_6^1 \) since \( b_{(D')^2} = c_{D^2} \).

As in the first step, we notice that \( H' = \tilde{D}'(1,j_0+1,1) \), the subdiagram of \( \tilde{D}' \) by removing the points preceding \( (1, \gamma, 1) \) lexicographically. Even when \( D' = \tilde{D}' \), this is a strictly smaller case compared with \( S(D \setminus C) \). Hence by induction, \( \Delta(H') \) is pure vertex-decomposable of dimension

\[
(11) \quad a_{D^2} + b_{D^2} + c_{D^2} - 3 + |\text{Phan}(\tilde{D}') \cap H'|.
\]

It remains to verify that this number agrees with (8).

Notice that by (2), we have

\[
|\text{Phan}(D') \cap D'_u| = |\text{Phan}(D')| - |\text{Phan}(D') \setminus D'_u| = (b_{D'} + c_{D'}) - (b_{D'^2} + c_{D'^2}) + 1 - |\text{Phan}(D') \setminus D'_u|,
\]

and similarly

\[
|\text{Phan}(\tilde{D}') \cap H'| = (b_{\tilde{D}'} + c_{\tilde{D}'}) - (b_{\tilde{D}'^2} + c_{\tilde{D}'^2}) + 1 - |\text{Phan}(\tilde{D}') \setminus H'|.
\]

Since

\[
a_{D^2} = a_{\tilde{D}^2}, \quad b_{D'} = b_{\tilde{D}'}, \quad \text{and} \quad c_{D'} - (\gamma - k_0) = c_{\tilde{D}'},
\]

we are reduced to show that

\[
(12) \quad |\text{Phan}(\tilde{D}') \setminus H'| = |\text{Phan}(D') \setminus D'_u| + 1.
\]

The proof is similar to but simpler than that for Lemma 4.3. If \( v = (1, j, k) \in \text{Phan}(D') \setminus D'_u \) or \( \text{Phan}(\tilde{D}') \setminus H' \), then \( j = j_0 \).

Take arbitrary \( v \in \text{Phan}(D') \setminus D'_u \). Then,

- \( k < k_0 \) since \( v \notin D'_u \);
- \( k \geq \gamma' = c_{D'}((1, j_0 + 1, 1)) \) since \( v \) is a border point;
• $k \geq c_{(D')_{\geq 2}}$ by Discussion 3.7.

It is also clear that any $(1, j_0, k)$ satisfying the above requirements belongs to $\text{Phan}(D') \setminus D_u$.

The cardinality of this set is

\begin{equation}
\max \left( k_0 - \max (\gamma', c_{(D')_{\geq 2}}), 0 \right).
\end{equation}

Likewise, $v = (1, j_0, k) \in \text{Phan}(D') \setminus H'$ if and only if the following three requirements are satisfied.

(a) $k_0 \geq k$ since $c_{D'}((1, j_0, 1)) = k_0$.

(b) $k \geq \min(\gamma', k_0)$ since $c_{D'}((1, j_0 + 1, 1)) = \min(\gamma', k_0)$.

(c) When $c_{D'} < k_0$, $c_{D'_{\geq 2}} = c_{D_{\geq 2}}$ and the only additional requirement for $v$ is

\[ k \geq c_{(D')_{\geq 2}} \]

by Discussion 3.7. On the other hand, when $c_{D'_{\geq 2}} < k_0$, the corresponding requirement is

\[ k = k_0. \]

Taking account of (a) above, we can combine these two into the condition:

\[ k \geq \min(c_{(D')_{\geq 2}}, k_0). \]

Thus the cardinality of $\text{Phan}(D') \setminus H'$ is

\begin{equation}
k_0 + 1 - \max \left( \min(\gamma', k_0), \min(c_{(D')_{\geq 2}}, k_0) \right).
\end{equation}

One can verify directly that (14) minus (13) equals one, which corresponds to the 1 in (12). Thus, we have established (12). \hfill \square

It follows from the proof of Theorem 4.1 and the discussion in Section 3 that

Corollary 4.6. Let $D$ be a three-dimensional Ferrers diagram which satisfies the projection property. Take arbitrary $u \in D^1$. Then both $I_2(A_u)$ and $\text{in}(I_2(A_u))$ are Cohen–Macaulay of the same codimension given by (3). In particular, both $I_2(D)$ and $\text{in}(I_2(D))$ are Cohen–Macaulay.

5. Primeness

Let $D$ be a three-dimensional Ferrers diagram which satisfies the projection property. The theme of this section is to show $I_2(D)$ is a prime ideal. The primary strategy is to use the Cohen–Macaulayness shown in the previous section. In particular, the ideal $I_2(D)$ is unmixed. We also need to define suitable maps in view of the localization of variables, so that we can proceed by induction.

Lemma 5.1. Let $D$ be a finite three-dimensional diagram. Take arbitrary $u \in D$ and let $D' = D \setminus u$. Suppose that $I_2(D)$ is unmixed and $\text{codim}(I_2(D)) = \text{codim}(I_2(D')) + 1$. If the localization $(K[T_D]/I_2(D))[T_u^{-1}]$ is a domain and $I_2(D')$ is a prime ideal, then $I_2(D)$ is also a prime ideal.
Proof. Consider the ideal $I_2(D)$ in $S = \mathbb{K}[T_D]$, whose associated primes are $p_1, \ldots, p_m$. Since $T_u$ does not vanish at the point $(1,1,\ldots,1) \in \mathbb{K}[D]$ which is a zero of $I_2(D)$, $T_u \notin \sqrt{I_2(D)} = p_1 \cap \cdots \cap p_m$. Meanwhile, $I_2(D)S[T_u^{-1}]$ is a prime ideal by hypothesis. Hence, without loss of generality, we may assume that $T_u \notin p_1$ while $T_u \in p_i$ for $i \neq 1$.

When $m = 1$, $I_2(D)$ is $p_1$-primary. Hence

$$I_2(D) = I_2(D)p_1 \cap S \supseteq I_2(D)S[T_u^{-1}] \cap S \supseteq I_2(D),$$

which means $I_2(D)$ is a prime ideal.

When $m \geq 2$, by the unmixedness assumption,

$$\text{codim}(I_2(D)) = \text{codim}(I_2(D), T_u).$$

Since $I_2(D) \neq I_2(D')$ by our hypothesis, we can find some quadratic binomial $f = T_u T_v - T_{u'} T_{v'} \in I_2(D) \setminus I_2(D')$. Obviously, $u', v' \in D'$ and the ideal $(f, T_u) = (T_u T_{v'}, T_u)$. Since $I_2(D')$ is a prime ideal and $T_{u'} T_{v'} \notin J_{D'} \supseteq I_2(D')$, $\text{codim}(I_2(D'), T_u T_{v'}) = \text{codim}(I_2(D')) + 1 = \text{codim}(I_2(D)).$

However,

$$\text{codim}(I_2(D), T_u) \geq \text{codim}(I_2(D'), f, T_u) = \text{codim}(I_2(D'), T_u T_{v'}, T_u) = \text{codim}(I_2(D'), T_u T_{v'}) + 1 = \text{codim}(I_2(D)) + 1.$$

This is a contradiction to (15).

Therefore, $m = 1$ and the ideal $I_2(D)$ is a prime ideal. 

\[\square\]

Theorem 5.2. Let $D$ be a three-dimensional Ferrers diagram which satisfies the projection property. Then $I_2(D)$ is a prime ideal in $\mathbb{K}[T_D]$.

Proof. We prove by the induction on $a_D$. When $a_D = 1$, this can be reduced to the two-dimensional case in [10, Proposition 3.5]. Thus, in the following, we assume that $a_D \geq 2$. We proceed by removing the points in the $x = 1$ layer, using the induction order given in Section 3. Recall that $C := \{(1, j, k) \in D : k \leq c_D \geq 2\}$.

First stage

Suppose that $u = (i_0 = 1, j_0, k_0) \in C$. Recall that $A_u$ is obtained from $D$ by removing those points in $C$ that are lexicographically before $u$. And $A_u^* = A_u \setminus u$. We want to prove that $I_2(A_u)$ is a prime ideal. The minimal case is when we remove all the points in $C$. We will deal it in the second stage. Other than this minimal case, by induction, we may assume that $I_2(A_u^*)$ is a prime ideal. We may assume that $I_2(A_u^*) \neq I_2(A_u)$. Whence, $u$ is a normal point and $\text{codim} I_2(A_u) = \text{codim} I_2(A_u^*) + 1$. Using Observation 3.3, Theorem 4.1 and its proof, we are reduced to showing that $(\mathbb{K}[T_{A_u}]/I_2(A_u))[T_u^{-1}]$ is a domain, by Lemma 5.1.
Consider the $\mathbb{K}$-algebra homomorphism $\varphi : (\mathbb{K}[T_{A_u}])[T_u^{-1}] \to (\mathbb{K}[T_{A_u}])[T_u^{-1}]$ defined by

$$T_{ijk} \mapsto \begin{cases} T_{ijk} + T_{ijk}T_{i_0j_0k}T_u^{-1}, & \text{if } (i, j, k) \in Z_2^2(A_u), \\ T_{ijk} + T_{ijk}T_{i_0j_0k}T_u^{-2}, & \text{if } (i, j, k) \in Z_4(A_u), \\ T_{ijk} + T_{ijk}T_{i_0j_0k}T_u^{-1}, & \text{if } (i, j, k) \in Z_5^2(A_u), \\ T_{ijk}, & \text{otherwise.} \end{cases}$$

Here, by $Z_*(A_u)$, we mean $Z_*(D) \cap A_u$. And the six zones are partitioned with respect to $u$. The above map gives an isomorphism whose inverse map is

$$T_{ijk} \mapsto \begin{cases} T_{ijk} - T_{ijk}T_{i_0j_0k}T_u^{-1}, & \text{if } (i, j, k) \in Z_2^2(A_u), \\ T_{ijk} - T_{ijk}T_{i_0j_0k}T_u^{-2}, & \text{if } (i, j, k) \in Z_4(A_u), \\ T_{ijk} - T_{ijk}T_{i_0j_0k}T_u^{-1}, & \text{if } (i, j, k) \in Z_5^2(A_u), \\ T_{ijk}, & \text{otherwise.} \end{cases}$$

Take arbitrary $(i, j, k) \in Z_2^2(A_u)$, we have $g_1 := T_u(T_{ijk} - T_{ijk}T_{i_0j_0k}) \in I_2(A_u)$. Notice that $\varphi(g_1) = T_u(T_{ijk} + T_{ijk}T_{i_0j_0k}T_u^{-1}) - T_{ijk}T_{i_0j_0k} = T_uT_{ijk}$. So $T_{ijk} \in \varphi(I_2(A_u))$.

If we take the ideal

$$b' := (T_{ijk} : (i, j, k) \in Z_2^2(A_u) \cup Z_4(A_u) \cup Z_5^2(A_u))$$

in $\mathbb{K}[T_{A_u}][T_u^{-1}]$, then by similar arguments, we have $b' \subseteq \varphi(I_2(A_u))$.

Recall that in the proof of Theorem 4.1, we defined

$$\mathcal{E} \defeq A_u \setminus \left( Z_2^2(A_u) \cup Z_4(A_u) \cup Z_5^2(A_u) \right) $$

and

$$\mathcal{H} \defeq \mathcal{E} \setminus \{ (1, j_0, k) : k_0 < k \leq \min(\gamma, c_{D(z)}) \},$$

where $\gamma \defeq c_{D(u)}$. It is clear that $b' + I_2(\mathcal{H}) \subseteq \varphi(I_2(A_u))$. We claim that they are equal:

$$b' + I_2(\mathcal{H}) = \varphi(I_2(A_u)).$$

Notice that

- if $(i, j, k) \in Z_2^2(A_u)$, then $\varphi(T_{ijk}) \equiv T_{ijk}T_{i_0j_0k}T_u^{-1} \mod b'$;
- if $(i, j, k) \in Z_4^2(A_u)$, then $\varphi(T_{ijk}) \equiv T_{ijk}T_{i_0j_0k}T_u^{-1} \mod b'$;
- if $(i, j, k) \in Z_5^2(A_u)$, then $\varphi(T_{ijk}) \equiv \varphi(T_{ijk}T_{i_0j_0k})T_u^{-1} \equiv \varphi(T_{ijk}T_{i_0j_0k})T_u^{-1} \equiv \varphi(T_{ijk}T_{i_0j_0k})T_u^{-1} \equiv \varphi(T_{i_0j_0k}T_{i_0j_0k}T_u^{-2} \mod b'$;
- $I_2(\mathcal{E}) = I_2(\mathcal{H})$.

Roughly speaking, the reduction by $b'$ has the effect of projecting the points in the above three designated zones to the coordinate planes and axes centered at $u$, where the landing points lay in $\mathcal{H}$. Thus, when dealing with the 2-minors, in the following situation

- (†-1) the points of both columns of the underlying $2 \times 2$ matrix belong to $b'$, or
- (†-2) the points of one column belong to $b'$, while the points in the other belong to $\mathcal{H}$,
then, we should be fine after factoring common factors; we will explain this by an example later in (a)-(iii). Thus, to show the above claim of equality, it suffices to consider the irregular generators $I_2, (v_1, v_2)$. Say, $v_1 = (i_1, j_1, k_1)$ and $v_2 = (i_2, j_2, k_2)$. Since at this stage $c_{D_2} \geq k_0$, by the projection property, we have $Z_6^1(D) = Z_6(D)$.

(a) We consider the case when $* = x$.

(i) We first investigate 2-minors that involve $u = v_1$. Notice that if $u$ can exchange $x$-coordinates with $v_2$, then

$$v_2 \in Z_5^{22} \cup Z_5^{22} \cup \{ (1, j_0, k) \mid k_0 < k \leq \min(\gamma, c_{D_2}) \}.$$  

After reductions by $b'$, the only irregular 2-minors involving $T_u$ take the form $g_2 = T_uT_v - T_{i_2j_0k_0}T_{i_0j_2k_2}$ with $v_2 \in Z_5^{22}(A_u)$. We have

$$\varphi(g_2) = T_u(T_{i_2j_0k_2}T_{i_0j_2k_0}T_u^{-1}) - T_{i_2j_0k_0}T_{i_0j_2k_2} \mod b',$$

and $T_{i_2j_0k_2}T_{i_0j_2k_0} - T_{i_2j_0k_0}T_{i_0j_2k_0} \in I_2(H)$. (ii) The other irregular case is, by symmetry, when $v_1 \in Z_5^{22}(A), \ v_2 \in Z_5^1(A)$. Thus, $i_1 \geq 2$ and $i_2 = 1 = i_0$. By symmetry, we may also assume that $j_1 \leq j_2$. Now, $I_{2,x}(v_1, v_2) = T_{v_1}T_{v_2} - T_{i_2j_1k_1}T_{i_1j_2k_2}$. Therefore,

$$\varphi(I_{2,x}(v_1, v_2)) = T_{i_1j_0k_1}T_{i_0j_1k_0}T_{i_2j_2k_2} - T_{i_2j_1k_1}T_{i_1j_0k_0}T_{i_0j_2k_0} \mod b'.$$

However,

$$T_{i_1j_0k_1}T_{i_0j_1k_0}T_{i_2j_2k_2} - T_{i_2j_1k_1}T_{i_1j_0k_2}T_{i_0j_2k_0} = T_{i_0j_1k_0}(T_{i_1j_0k_1}T_{i_2j_2k_2} - T_{i_1j_0k_2}T_{i_2j_1k_1})$$

$$+ T_{i_1j_0k_2}(T_{i_0j_1k_0}T_{i_2j_2k_1} - T_{i_2j_1k_1}T_{i_0j_2k_0}) \in I_2(H).$$

Notice that $T_{i_2j_2k_1}$ exists with $(i_2, j_2, k_1) \in Z_5^1$. (iii) All other 2-minors are regular in the sense of $(\dag)$ and can be reduced by $b'$ to 2-minors in $I_2(H)$.

We first look at an example in $(\dag-1)$. Say $v_1 \in Z_5^{22}$ while $v_2 \in Z_5^{22}$. Suppose that $v_1$ can exchange $x$-coordinates with $v_2$. Now,

$$\varphi(I_{2,x}(v_1, v_2)) = \varphi(T_{i_1j_1k_1}T_{i_2j_2k_2} - T_{i_2j_1k_1}T_{i_1j_2k_2})$$

$$= T_{i_1j_1k_0}(T_{i_2j_0k_1}T_{i_2j_2k_2}T_{i_1j_0k_0}^{-1}) - T_{i_2j_1k_0}(T_{i_1j_0k_1}T_{i_2j_2k_2}T_{i_1j_0k_0}^{-1}) \mod b'$$

$$= T_{i_2}^{-2}T_{i_1j_1k_0}T_{i_0j_2k_0}(T_{i_1j_1k_0}T_{i_2j_2k_2} - T_{i_2j_1k_0}T_{i_1j_0k_0}) \in I_2(H).$$

Therefore, still, $\varphi(I_{2,x}(v_1, v_2)) \in b' + I_2(H)$. We may also look at one example in $(\dag-2)$. Say $v_1 \in Z_5^1$ while $v_2 \in Z_5^{22}$. If $v_1$ can exchange $x$-coordinates with $v_2$, then $(i_2, j_1, k_1), (i_1, j_2, k_2) \in A_u$. Now, since $i_1 = 1 = i_0$,

$$\varphi(I_{2,x}(v_1, v_2)) = \varphi(T_{i_1j_1k_1}T_{i_2j_2k_2} - T_{i_2j_1k_1}T_{i_1j_2k_2})$$

$$= T_{i_1j_1k_0}(T_{i_0j_0k_2}T_{i_2j_2k_0}T_{i_2j_0k_0}^{-1}).$$
choosing points settled by induction. Notice that in the proof of the first stage, the “ceiling restriction” of □ So the proof is similar and easier.

the presentation ideal of the special fiber ring

Theorem 4.1, the current case is C. Now we flip to get \( S(D) \), again written as \( D' \). Using the notation in the proof of Theorem 6.1, the current case is \( S(D \setminus C) = D'_{(1,c_{D'} + 1,1)} \). Similar to the first stage, we will prove by induction on removing lexicographically initial points in the \( x = 1 \) layer of \( D'_{(1,c_{D'} + 1,1)} \). The minimal case will be when we remove all the \( x = 1 \) points and this is settled by induction. Notice that in the proof of the first stage, the “ceiling restriction” of choosing points \((1,j,k)\) with \( k \leq c_{D'} \) is only used to ensure that \( Z_6 = Z_6' \). In the current case of \( D'_{(1,c_{D'} + 1,1)} \), for any point \( u \) in the \( x = 1 \) layer, we will automatically get \( Z_6 = Z_6' \). So the proof is similar and easier. □

6. Blowup algebras

It is time for the main theorems of this work. Indeed, we show that the ideal \( I_2(D) \) is the presentation ideal of the special fiber ring \( F(I_D) \). Since \( I_2(D) \) has nice properties, so does the special fiber ring \( F(I_D) \). Moreover, we can extend the result to the Rees algebra \( R(I_D) \) easily because the ideal \( I_D \) satisfies \( \ell \)-exchange property (Definition 6.2).

**Theorem 6.1.** Let \( D \) be a three-dimensional Ferrers diagram which satisfies the projection property. Then the special fiber ring \( F(I_D) \) is a Koszul Cohen–Macaulay normal domain.

**Proof.** Notice that \( I_2(D) \subseteq J_D \). Since these two homogeneous ideals are prime and have the same codimension by Theorem 5.2 Lemma 3.9 and Lemma 3.10 they actually coincide. The Cohen–Macaulay property follows from Theorem 4.1. Since \( I_2(D) \) has a squarefree initial ideal, the normal property follows from [13] Theorem 5.16. The Koszul property follows from [13] Theorem 6.7. □

Next, we consider the Rees algebra \( R(I_D) \) of \( I_D \). The strategy is similar to that in [12] Section 6.

**Definition 6.2 ([13] Definition 4.1).** Let \( I = (f_1, \ldots, f_m) \subseteq R = \mathbb{K}[x_1, \ldots, x_n] \) be a monomial ideal generated in one degree. Let \( \mathbb{K}[T] := \mathbb{K}[T_1, \ldots, T_m] \) and \( J \) be the toric ideal of
I, i.e., the kernel of the surjective homomorphism
\[ \psi : \mathbb{K}[T] \to \mathbb{K}[f_1, \ldots, f_m], \]
defined by \( \psi(T_i) = f_i \) for all \( i \). Let \( < \) be a monomial order on \( \mathbb{K}[T] \). A monomial \( T^a \) in \( \mathbb{K}[T] \) is called a standard monomial of \( J \) with respect to \( < \), if it does not belong to the initial ideal of \( J \).

The monomial ideal \( I \) satisfies the \( \ell \)-exchange property with respect to the monomial order \( < \) on \( \mathbb{K}[T] \), if the following condition is satisfied: let \( T^a \) and \( T^b \) be any two standard monomials of \( J \) with respect to \( < \) of the same degree, with \( u = \psi(T^a) \) and \( v = \psi(T^b) \) satisfying
\[
\begin{align*}
(\text{i}) \quad & \deg_{x_t}(u) = \deg_{x_t}(v) \quad \text{for} \quad t = 1, \ldots, q - 1 \quad \text{with} \quad q \leq n - 1, \\
(\text{ii}) \quad & \deg_{x_q}(u) < \deg_{x_q}(v). 
\end{align*}
\]
Then there exists an integer \( k \), and an integer \( q < j \leq n \) such that \( x_q f_k/x_j \in I \).

Similar to [10] Example 4.2, we have

**Lemma 6.3.** Let \( D \) be a three-dimensional Ferrers diagram. Then the Ferrers ideal \( I_D \subset R = \mathbb{K}[x_1, \ldots, x_m, y_1, \ldots, y_n, z_1, \ldots, z_p] \) satisfies the \( \ell \)-exchange property with respect to any monomial order \( < \) on \( \mathbb{K}[T] \).

**Proof.** We will use the notation in the previous definition. Without loss of generality, we may assume that \( \deg_{x_t}(u) = \deg_{x_t}(v) \) for \( t = 1, \ldots, q - 1 \) with \( q \leq m \) and \( \deg_{x_q}(u) < \deg_{x_q}(v) \). Since
\[
3 \sum_{i=1}^{m} \deg_{x_i}(u) = \deg(u) = \deg(v) = 3 \sum_{i=1}^{m} \deg_{x_i}(v),
\]
we can see indeed that \( q \leq m - 1 \). Thus, we can find some \( f_\delta \) and \( q < j \leq m \) with \( \deg_{x_j}(f_\delta) \geq 1 \). Notice that \( f_\delta = x_j y_s z_r \). Thus, \( x_q f_\delta/x_j \in I_D \), since \( D \) is a Ferrers diagram. \( \square \)

The crucial weapon for our final result is the following.

**Lemma 6.4 ([10] Theorem 5.1).** Let \( I = (f_1, \ldots, f_m) \subset R = \mathbb{K}[x_1, \ldots, x_n] \) be a monomial ideal generated in one degree, satisfying the \( \ell \)-exchange property. Let \( <_{lex} \) be the lexicographic order on \( R \) with respect to \( x_1 > \cdots > x_n \) and \( < \) an arbitrary monomial order on \( T \). Let \( <_{lex} \) be the product order of \( < \) and \( <_{lex} \). Then the reduced Gröbner basis of the Rees ideal of \( I \) with respect to \( <_{lex} \) consists of all binomials belonging to the reduced Gröbner basis of \( J \) with respect to \( < \) together with the binomials \( x_i T_k - x_j T_l \), where \( x_i > x_j \) with \( x_i f_k = x_j f_l \) and \( x_j \) is the smallest variable for which \( x_i f_k/x_j \) belongs to \( I \). In particular, \( I \) is of fiber type.

As an application, we have

**Theorem 6.5.** Let \( D \) be a three-dimensional Ferrers diagram which satisfies the projection property. Then the Rees algebra \( \mathcal{R}(I_D) \) is Koszul and the ideal \( I_D \) is of fiber type.

**Proof.** By Theorem 6.1, we know the toric ideal \( J_D \) of \( I_D \) has a quadratic Gröbner basis. Hence by Lemma 6.3 and Lemma 6.4, \( I_D \) is of fiber type and the Rees ideal of \( I_D \) has a quadratic Gröbner basis. It follows from [13] Theorem 6.7 that \( \mathcal{R}(I_D) \) is Koszul. \( \square \)
We close with some questions for future research.

**Question 6.6.** Let $D$ be a three-dimensional Ferrers diagram. Is the degree of minimal binomial generators of the special fiber ideal at most three? Is the special fiber ideal always Cohen–Macaulay or normal? What about the Rees algebra?

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