Approximation methods for the stability of complete synchronization on duplex networks

Wenchen Han and Junzhong Yang
School of Science, Beijing University of Posts and Telecommunications, Beijing, 100876, People’s Republic of China

Recently, the synchronization on multi-layer networks has drawn a lot of attention. In this work, we study the stability of the complete synchronization on duplex networks. We investigate effects of coupling function on the complete synchronization on duplex networks. We propose two approximation methods to deal with the stability of the complete synchronization on duplex networks. In the first method, we introduce a modified master stability function and, in the second method, we only take into consideration the contributions of a few most unstable transverse modes to the stability of the complete synchronization. We find that both methods work well for predicting the stability of the complete synchronization for small networks. For large networks, the second method still works pretty well.

I. INTRODUCTION

Network science has provided a fertile ground for understanding complex systems. The traditional network approach treats complex systems as monolayer networks by charting an elementary unit into a network node and representing each unit-unit interaction on an equivalent footing as a network link [1–7]. Recently, it has become clear that many complex systems in social, biological, technological systems should not be treated as monolayer ones but multi-layer ones [8–17]. Consider a group of members interacting with each other through different channels such as Twitter, blog, Facebook, and Wechat. The social network formed by these members is a typical example of multilayer network in which each interaction channel corresponds to a different layer.

Synchronization, one of the most interesting collective behaviors, has been investigated since the dawn of natural science [5, 18–21]. There are different types of synchronization such as complete synchronization (CS), phase synchronization, lag synchronization, general synchronization, cluster synchronization, partial synchronization, and remote synchronization. Among all types of synchronization, CS, where states of all oscillators are identical, is the simplest one [26]. CS on monolayer networks has been well studied by applying the master stability function (MSF) method [29, 30]. The MSF method shows that the stability of CS is affected by coupling functions (CFs) and network structures.

Recently, much attention has been paid to synchronization on multi-layer networks [15, 17, 31–35]. Aguirre et al. have shown that connecting the high-degree (or low-degree) nodes in different layers turns out to be the most (or the least) effective strategy to achieve synchronization in multi-layer networks [16]. Using the MSF method, Sorrentino et al. have studied CS on duplex networks, one special type of two-layer networks with the two layers sharing the same nodes [15], when the two layers are subject to constrains such as commuting Laplacians, unweighted and fully connected layers, and nondiffusive coupling. Genio et al. have provided a full mathematical framework to evaluate the stability of CS on multi-layer networks by generalizing the MSF method. However, N – 1 transverse modes are coupled in their framework and the stability of CS is hard to deal with for large N. Then, one question arises: can we develop approximation methods to reduce the stability of CS on multi-layer networks to a low dimensional problem?

In this work, we study the coupled identical chaotic oscillators on duplex networks. We develop two approximation methods to deal with the stability of CS on duplex networks. In the first method, we assume that all transverse modes to synchronous chaos have the same contribution to the stability of CS. Then we obtain a modified MSF similar to the MSF on monolayer networks. In the second approximation, we only consider the contributions from a few most unstable transverse modes of layers, which are responsible for the desynchronization on each isolated layer, to the stability of CS. The stability diagrams of CS on duplex networks produced by the two approximation methods are compared with the direct numerical results on coupled chaotic oscillators and we find that the second approximation method provides better prediction on the stability of CS when N is large.

II. THE MODEL

We consider N oscillators sitting on a duplex network, whose time evolution is governed by

\[
\dot{x}_i = F(x_i) + \sum_{j=1}^{N} \left[ \varepsilon^{(1)} L_{i,j}^{(1)} H^{(1)}(x_j) + \varepsilon^{(2)} L_{i,j}^{(2)} H^{(2)}(x_j) \right],
\]

(1)

where \(x_i\) is an \(m\)-dimensional state variable of node \(i\), \(F(x)\) is the dynamics of individual nodes, \(\varepsilon^{(1)}\) (or \(\varepsilon^{(2)}\)) is the coupling strength and \(H^{(1)}(x)\) (or \(H^{(2)}(x)\)) is a linear CF, determining the output signal from a node on
the layer 1 (or the layer 2). \(L^{(1)}\) (or \(L^{(2)}\)) is the Laplacian matrix of the layer 1 (or the layer 2), with elements \(L_{i,i}^{(1)} = -k_i^{(1)}\) (or \(L_{i,i}^{(2)} = -k_i^{(2)}\)), the degree of node \(i\) on the layer 1 (or the layer 2), \(L_{i,j}^{(1)} = 1\) (or \(L_{i,j}^{(2)} = 1\)) if node \(i\) and node \(j\) are connected with a link on the layer 1 (or the layer 2), and \(L_{i,j}^{(1)} = 0\) (or \(L_{i,j}^{(2)} = 0\)) otherwise.

We briefly review the MSF method on the stability of CS on monolayer networks

\[
\dot{x}_i = F(x_i) + \varepsilon \sum_{j=1}^{N} L_{i,j} H(x_j) \tag{2}
\]

The variational equations of Eq. (2) with respect to CS, \((x_1 = x_2 = \cdots = x_N = s)\), are diagonalized into \(N\) decoupled eigenmodes of the form

\[
\dot{\eta}_i = [DF(s) + \varepsilon\lambda_i DH(s)]\eta_i \tag{3}
\]

where \(\lambda_i\) (\(i = 1, 2, \ldots, N\)) are eigenvalues of the Laplacian matrix \(L\) and \(L_{ii} = \lambda_i \phi_i\). For an undirected network, where \(L\) is symmetric, \(\lambda_i\) are real and can be sorted in descending order, i.e., \(0 = \lambda_1 > \lambda_2 \geq \ldots \geq \lambda_N\). The eigenmode \(\phi_i\) with \(\lambda_1 = 0\) accounts for the synchronous mode and other \(\phi_i\), for \(i = 2, 3, \ldots, N\), are transverse modes. \(DF(s)\) and \(DH(s)\) are the \(m \times m\) Jacobian matrices of the corresponding vector functions evaluated at CS. Letting \(\sigma = \varepsilon\lambda\), the largest Lyapunov exponent (LLE) \(\Lambda(\sigma)\), determined from Eq. (3), is the MSF. Generally, \(\Lambda(\sigma)\) is negative when \(\sigma_1 < \sigma < \sigma_2\). Therefore, CS is stable when all transverse eigenmodes with \(i > 1\) have negative LLE, that is, \(\sigma_1 < \varepsilon\lambda_i < \sigma_2\) for any \(i > 1\). For a given node dynamics, CF can be categorized according to \(\sigma_1\) and \(\sigma_2\). There are three types of CF. For the type-i CF, \(\sigma_1 = \infty\) and CS is always unstable. For the type-ii CF, \(\sigma_2 = \infty\) and CS is stable provided that \(\varepsilon > \sigma/\lambda_2\). For the type-iii CF, both \(\sigma_1\) and \(\sigma_2\) are finite and CS is stable when \(\sigma_1/\sigma_2 < \varepsilon < \sigma_2/\lambda_2\).

In the following, we take chaotic Lorenz oscillator \((m = 3)\) as the node dynamics, which is described as \(F(x) = [10(y-x), 28x - y - zx, xy - z]^T\). We concern with CFs whose Jacobian matrices have only one nonzero element and we denote them with their nonzero element. Thereby, there are 9 different CFs. Fig. 1 shows the LLE, \(\Lambda(\sigma)\), for 9 different CFs. The type of the CF in each plot is marked by the index in the top-right corner.

### III. NUMERICAL SIMULATIONS

In this section, we numerically investigate the dependence of CS on CFs in coupled Lorenz oscillators on duplex networks with \(N = 6\). Both layers of duplex networks are modeled by random networks. We have tried different realizations of duplex networks and found qualitatively similar results. Without the loss of generality, we consider a specific duplex network with the

![FIG. 1: The largest Lyapunov exponent \(\Lambda\) (in red) against \(\sigma\) for coupled Lorenz oscillators on monolayer networks for different CFs. (a) \(DH_{1,1}\), (b) \(DH_{1,2}\), (c) \(DH_{1,3}\), (d) \(DH_{2,1}\), (e) \(DH_{2,2}\), (f) \(DH_{2,3}\), (g) \(DH_{3,1}\), (h) \(DH_{3,2}\), and (i) \(DH_{3,3}\). The index in the top-right corner in each plot denotes the type of the corresponding CF.](image)

Laplacians, \(L^{(1)} = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 1 \\ 1 & -3 & 1 & 0 & 0 \\ 0 & 1 & -4 & 1 & 1 \\ 0 & 1 & 1 & -3 & 0 \end{pmatrix}\) and \(L^{(2)} = \begin{pmatrix} -3 & 0 & 1 & 0 & 1 \\ 0 & -2 & 0 & 0 & 1 \end{pmatrix}\). We consider the synchronization error, which is defined as

\[
\Delta = \frac{2}{N(N-1)} \sum_{i,j;i \neq j} \langle ||x_j - \bar{x}_i||_2 \rangle_\omega, \tag{4}
\]

where \(\langle x \rangle_\omega\) means the time average of \(x\) and \(||x_j - \bar{x}_i||_2\) is the Euclidean norm \(\sqrt{\sum_{l=1}^{L} (x_{j,l} - \bar{x}_{i,l})^2}\). In each simulation, the synchronization error is averaged over 400 time units after a transient time around 2000 time units. When \(\Delta < 10^{-6}\), we say that the coupled Lorenz oscillators are completely synchronized.

As shown in Fig. 1 all three types of CF are presented for Lorenz oscillators on monolayer networks. Consider coupled Lorenz oscillators on the duplex network in which two layers take CFs with different types. There are 9 typical combinations of CFs for duplex networks. Then we numerically explore stable CS on the plane of \(\varepsilon^{(1)}\) and \(\varepsilon^{(2)}\) and investigate effects of combinations of different types of CFs on CS. The results are presented in Fig. 2 where the combinations of CFs are labelled in

![FIG. 2: The synchronization error \(\Delta\) (in red) against \(\varepsilon^{(1)}\) and \(\varepsilon^{(2)}\) for coupled Lorenz oscillators on duplex networks for different CF combinations. (a) \(DH_{1,1}\), (b) \(DH_{1,2}\), (c) \(DH_{1,3}\), (d) \(DH_{2,1}\), (e) \(DH_{2,2}\), (f) \(DH_{2,3}\), (g) \(DH_{3,1}\), (h) \(DH_{3,2}\), and (i) \(DH_{3,3}\). The index in the top-right corner in each plot denotes the type of the corresponding CF.](image)
each plot, for example (i-iii) indicating a type-i CF on the layer 1 and a type-ii CF on the layer 2.

The results can be summarized as follows. Firstly, the presence of type-i CFs always disfavors CS. When both layers take the type-i CF, it is impossible to realize CS [see Fig. 2 (a)]. As shown in Fig. 2 (b) [or (d)] with the combination of CFs taking (i-ii) [or (ii-i)], increasing the coupling strength of the type-i CF shrinks the range of the coupling strength of the type-ii CF supporting stable CS. The similar effects of the type-i CF on CS can be observed for the combination of CFs taking (i-iii) in Fig. 2 (c) and (iii-i) in Fig. 2 (g). Fig. 2 (c) suggests that, if a type-iii CF does not support stable CS, stable CS cannot be built on duplex networks with the combinations of CFs taking (i-iii) or (iii-i). Secondly, the presence of type-ii CFs always enhances synchronization. Increasing the coupling strength of the type-ii CF expands the range of the coupling strength of the CF on the other layer allowing for the stable CS. Especially for the combination (ii-i) in Fig. 2 (e), only a small region on the plane of $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ prohibits stable CS. Thirdly, the presence of type-iii CFs makes the dependence of stable CS on CFs more interesting. As shown in Fig. 2 (f) with the combination of CFs as (ii-iii), even if a type-iii CF does not support CS on monolayer networks, it might play a very positive role on the stable CS on duplex networks. When (iii-iii) is taken by duplex networks [see Fig. 2 (i)], the enhancement of synchronization by the interplay between type-iii CFs on two layers is extraordinary strong.

![FIG. 2: The stability diagram of CS on the $\varepsilon^{(1)} - \varepsilon^{(2)}$ plane in coupled Lorenz system on duplex networks for different combinations of CFs. (a) $\mathcal{D}H_{1,2}^{(1)} = 1$ and $\mathcal{D}H_{1,2}^{(3)} = 1$, (b) $\mathcal{D}H_{1,2}^{(1)} = 1$ and $\mathcal{D}H_{1,2}^{(2)} = 1$, (c) $\mathcal{D}H_{1,2}^{(1)} = 1$ and $\mathcal{D}H_{1,2}^{(3)} = 1$, (d) $\mathcal{D}H_{1,2}^{(1)} = 1$ and $\mathcal{D}H_{1,2}^{(2)} = 1$, (e) $\mathcal{D}H_{1,2}^{(1)} = 1$ and $\mathcal{D}H_{1,2}^{(3)} = 1$, (f) $\mathcal{D}H_{1,2}^{(1)} = 1$ and $\mathcal{D}H_{1,2}^{(2)} = 1$, (g) $\mathcal{D}H_{1,2}^{(1)} = 1$ and $\mathcal{D}H_{1,2}^{(3)} = 1$, (h) $\mathcal{D}H_{1,2}^{(1)} = 1$ and $\mathcal{D}H_{1,2}^{(2)} = 1$, (i) $\mathcal{D}H_{1,2}^{(1)} = 1$ and $\mathcal{D}H_{1,2}^{(3)} = 1$.]

![FIG. 3: The master stability function $\Lambda(\sigma^{(1)}, \sigma^{(2)})$ calculated according to Eq. 6. The cyan lines are for $\sigma^{(1)} = 0$ and $\sigma^{(2)} = 0$. The CFs combinations are identical to those in Fig. 2.]

**IV. ANALYSIS AND APPROXIMATION METHODS**

In this section, we theoretically investigate the stability of CS on duplex networks. Similar to monolayer networks, the stability of CS on duplex networks is determined by the variational equation of Eq. 1 with respect to CS

$$\frac{d}{dt} \eta_i^{(\alpha)} = [\mathcal{D}F(s) + \varepsilon^{(\alpha)} \lambda_i^{(\alpha)} \mathcal{D}H^{(\alpha)}(s)] \eta_i^{(\alpha)} + \varepsilon^{(\beta)} \sum_{j=1}^{N} \mathcal{L}_{i,j}^{(\beta,\alpha)} \mathcal{D}H^{(\beta)}(s) \eta_j^{(\alpha)},$$

with $\alpha, \beta = 1, 2$ and $\alpha \neq \beta$, where $i = 2, 3, ..., N$, $\lambda_i^{(\alpha)}$ are the eigenvalues of the Laplacian matrix $L^{(\alpha)} = (\alpha \phi_i^{(\alpha)} = \lambda_i^{(\alpha)} \phi_i^{(\alpha)}$, and $e^{(\beta,\alpha)} = (\phi^{(\alpha)})^T L^{(\beta)} \phi^{(\alpha)}$. Different from Eq. 2, where $N-1$ decoupled m-dimensional ordinary differential equations (ODEs) are obtained, Eq. 5 depicts $N-1$ coupled m-dimensional equations. Consequently, the MSF method, which reduces a high-dimensional [$N-1$ m-dimensional] stability problem to a low-dimensional (m-dimensional) one on monolayer networks, cannot be applied to the stability of CS on duplex networks. Only for some special cases where $\mathcal{L}^{(\beta,\alpha)}$ can be diagonalized, Eq. 5 is reduced to $N-1$ decoupled m-dimensional equations. For example, duplex networks in which the Laplacian matrices satisfy $L^{(1)} L^{(2)} = L^{(2)} L^{(1)}$ or duplex networks with one layer whose Laplacian matrix has $N-1$ same eigenvalues except for $\lambda_1 = 0$ [17]. For arbitrary duplex networks, the analysis of the stability of CS is not an easy job, especially for large $N$. In the following, we propose two approximation methods to reduce Eq. 5 to a low-dimensional
Secondly, we search for two sets of $(\sum\lambda)$ eigenvalues and eigenvectors, $\sigma$ MSF for CS on duplex networks? For this aim, we of CS on monolayer networks. Can we find a similar encoded MSF as the LLE determined from Eq. (6), which is duplex networks.

A. Approximation method I

The MSF method is powerful in stability analysis of CS on monolayer networks. Can we find a similar MSF on duplex networks? For this aim, we have to decouple the $N - 1$ $m$-dimensional equations in Eq. (5). A plausible way to do it is approximating the last term, $\sum_{i,j=2}^{N} L_{i,j}^{(\alpha)} D H^{(\beta)}(s) \eta^{(\alpha)}$, in Eq. (5) with $\sum_{i,j=2}^{N} L_{i,j}^{(\alpha)} D H^{(\beta)}(s) \eta^{(\alpha)}$. Following this way, Eq. (5) are reformulated as

$$\frac{d}{dt} \eta = [DF(s) + \sigma^{(\alpha)} DH^{(\alpha)}(s) + \sigma^{(\beta)} DH^{(\beta)}(s)] \eta, \quad (6)$$

with $\sigma^{(\alpha)} = \varepsilon^{(\alpha)} \lambda^{(\alpha)}_i$, $\sigma^{(\beta)} = \varepsilon^{(\beta)} \sum_{j=2}^{N} L_{i,j}^{(\beta,\alpha)}$, $\alpha, \beta = 1, 2$ and $\alpha \neq \beta$, where $i = 2, 3, ..., N$. To be stressed, $\sigma^{(\alpha)}$ and $\sigma^{(\beta)}$ encode the information on the structure of two layers of a duplex network. We define a modified MSF as the LLE determined from Eq. (6), which is the function of $\sigma^{(1)}$ and $\sigma^{(2)}$ at a given node dynamics and CFs. In Fig. 4 $\Lambda (\sigma^{(1)}, \sigma^{(2)}) = 0$ is plotted on the plane of $\sigma^{(1)}$ and $\sigma^{(2)}$ with the same CFs and node dynamics as those in Fig. 2. Using Fig. 3 we may determine the stability of CS for coupled Lorenz oscillators on the duplex network used in Fig. 2 at any given $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ as following. Firstly, we calculate the eigenvalues and eigenvectors, $\lambda^{(\alpha)}_i$ and $\phi^{(\alpha)}_i$ ($\alpha = 1, 2$ and $i = 2, 3, ..., N$) for both layers of the duplex network. Then we calculate $\sum_{j=2}^{N} L_{i,j}^{(2,1)}$ and $\sum_{j=2}^{N} L_{i,j}^{(1,2)}$. Secondly, we search for two sets of $\varepsilon^{(2)}$ which leads $(\sigma^{(1)}, \sigma^{(2)}) = (\varepsilon^{(1)} \lambda^{(2)}_i, \varepsilon^{(2)} \sum_{j=2}^{N} L_{i,j}^{(2,1)})$ and $(\sigma^{(1)}, \sigma^{(2)}) = (\varepsilon^{(1)} \sum_{j=2}^{N} L_{i,j}^{(1,2)}, \varepsilon^{(2)} \lambda^{(2)}_i)$, for $i = 2, 3, ..., N$, to satisfy $\Lambda (\sigma^{(1)}, \sigma^{(2)}) < 0$, respectively. In the intersection of these two sets of $\varepsilon^{(1)}$, $\varepsilon^{(2)}$, CS on the duplex network is stable. Using the modified MSF method on duplex networks, the stability diagrams of CS on the plane of $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ in Fig. 2 are reproduced in Fig. 4. It seems that the stable CS regions predicted by the modified MSF method are similar to those based on the synchronization error.

B. Approximation method II

The MSF method on monolayer networks has shown that the most dangerous transverse eigenmodes to the stability of CS are those with $\sigma$ close to $\sigma_1$ or $\sigma_2$. For example, the eigenmode $\phi_2$ with eigenvalue $\lambda_2$ for a type-ii CF or the eigenmodes $\phi_2$ and $\phi_N$ with eigenvalues $\lambda_2$ and $\lambda_N$ for a type-iii CF. Bearing these facts in mind, we assume that the stability of CS on duplex networks are mainly determined by four eigenmodes, $\phi_2^{(1)}$, $\phi_N^{(1)}$, $\phi_2^{(2)}$, and $\phi_N^{(2)}$. Thereby, the stability of CS on duplex networks are determined by

$$\begin{align*}
\frac{d}{dt} \eta^{(\alpha)}_2 &= [DF(s) + \varepsilon^{(\alpha)} \lambda^{(\alpha)}_2 \phi^{(\alpha)}_2] \eta^{(\alpha)}_2 + DH^{(\beta)}(s) \eta^{(\alpha)}_2 + L_{i,j}^{(\beta,\alpha)} \phi^{(\alpha)}_2 + L_{N,i}^{(\beta,\alpha)} \eta^{(\alpha)}_N, \\
\frac{d}{dt} \eta^{(\alpha)}_N &= [DF(s) + \varepsilon^{(\alpha)} \lambda^{(\alpha)}_N \phi^{(\alpha)}_N] \eta^{(\alpha)}_N + DH^{(\beta)}(s) \eta^{(\alpha)}_N + L_{i,N}^{(\beta,\alpha)} \phi^{(\alpha)}_N + L_{i,j}^{(\beta,\alpha)} \eta^{(\alpha)}_2, \quad (7)
\end{align*}$$

FIG. 4: The stable diagram of CS predicted by the approximation method I (in red) and obtained by the synchronization error (in black). The CFs are identical to those in Fig. 2

FIG. 5: The stable diagram of CS predicted by the approximation method II (in blue) and obtained by the synchronization error (in black). The CFs are identical to those in Fig. 2
with $\alpha, \beta = 1, 2$ and $\alpha \neq \beta$. When the LLE determined by Eq. (4) are negative, the CS on duplex networks is stable. Through the truncation approximation, Eq. (5), the $N - 1$ $m$-dimensional ODEs, are reduced to $4m$-dimensional ODEs. For large $N$, the approximation here greatly simplifies the problem of stability analysis on CS on duplex networks. Fig. 5 shows the region on the plane of $\varepsilon^{(1)}$ and $\varepsilon^{(2)}$ predicted by Eq. (4). Clearly, the boundary of the stable CS region is almost the same as those obtained by the synchronization error.

C. Approximation methods applied on systems with larger sizes

As shown in Figs. 4 and 5, the two approximation methods seem to work quite well on predicting the stability of CS on duplex networks with $N = 6$. Then we wonder the validity of the approximation methods for duplex networks with large $N$. We consider $N = 20$ and $N = 100$, respectively. In both examples, each layer of duplex networks is modelled by a regular random network with degree $k = 4$. The top panel and the bottom panel in Fig. 6 show stable CS regions calculated by the synchronization error (in black), the approximation method I (in red), and the approximation method II (in blue) for $N = 20$ and $N = 100$, respectively. Clearly, the approximation method II works much better than the approximation method I. Both approximation methods work pretty well on predicting the stable CS region at $N = 20$. At $N = 100$, the stable CS region predicted by the approximation method I deviates largely from those acquired by the synchronization error while the stable CS region predicted by the approximation method II only shows a little mismatch with those obtained by the synchronization error. Fig. 6 also shows that the approximation method I (or the approximation method II) predicts a smaller (or larger) stable region than those obtained by the synchronization error. In addition, we have checked the validity of two approximation methods on the stability of CS on duplex networks in which each layer is modelled by an Erdős-Rényi network or a scale free network and found the results similar to Fig. 6.

V. DISCUSSION AND CONCLUSIONS

To conclude, we studied the stability of the complete synchronization in coupled systems upon duplex networks in this work. We found that the combinations of CFs on different layers have strong effects on the stability of CS. We propose two approximation methods to predict the stable CS region on the $\varepsilon^{(1)} - \varepsilon^{(2)}$ plane. In the approximation I, we assumed that all transverse eigenmodes of each layer have the same contribution to the stability of CS and, as a result, we could introduce a modified MSF. In the approximation II, we only considered the contributions of a few most unstable eigenmodes in each layer to the stability of CS. We found that both approximation methods could work pretty well for small networks. For networks with large size, the approximation II could work much better than the approximation I. We expect our work should be instructive for the future works on synchronization on multiplex networks, such as optimal synchronization, which attracted much attention on monolayer networks [36–38], as well as the system of opinion dynamics, information spreading and so on.

VI. ACKNOWLEDGEMENTS

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