Effect of size on necking of dynamically loaded notched bars

A. Needleman

Department of Materials Science & Engineering, Texas A&M University, College Station, TX 77843 USA

Abstract
The influence of material inertia on neck development in a notched round bar is analyzed numerically. Dynamic axisymmetric calculations are carried out for isotropically hardening elastic-viscoplastic solids so that both material strain rate sensitivity and material inertia are accounted for. The focus is on the effect of bar size on whether the notch triggers necking or necking initiates away from the notch. The governing equations are presented in non-dimensional form and two key non-dimensional groups that involve both material and loading parameters are identified. For both non-dimensional groups, with all parameters fixed except for bar size, it is found that for sufficiently small bars, the notch triggers necking, whereas for sufficiently large bars necking ultimately occurs away from the notch. With material properties fixed, for one non-dimensional group the transition to necking away from the notch corresponds to increasing imposed velocity whereas for the other non-dimensional group this transition takes place for decreasing imposed strain rate. Both these transitions correspond to increasing bar size. The results indicate that this transition is governed by material inertia but the bar size at which it occurs depends on the material properties, particularly strain hardening and strain rate hardening.

Keywords: Necking; dynamic instability; notch sensitivity; plasticity; size effects

1. Introduction

There is an extensive, more than 100 year old, literature on the mechanics of necking in the uniaxial tensile test. The classical criterion of Consideré (1885) holds for necking of a tensile bar in the limiting case of a infinitely long, thin bar and states that necking initiates at the maximum load. For any finite aspect ratio, there is a delay between the maximum load point and the onset of necking that increases as the bar becomes more stubby; Needleman (1972); Hutchinson and Miles (1974); Hutchinson and Neale (1977). The literature on the analyses of necking in tensile bars includes one dimensional analyses as well as full three dimensional finite element solutions, involving both quasi-static and dynamic formulations, and analyses that account for effects of various mechanical properties, such as thermal softening, porosity induced softening, bar geometry, etc. Reviews of tensile bar necking analyses are provided by Hutchinson (1973); Molinari at al. (2014).

For rate independent plasticity and quasi-static deformations, the onset of necking in a uniform circular cylindrical tensile bar is associated with a bifurcation from a state of homogeneous uniaxial tension Cheng et al. (1971); Needleman (1972); Hutchinson and Miles.
The bifurcation mode is associated with a sinusoidal variation in the radial dimension of the bar with the longest possible wavelength consistent with the bar geometry and the boundary conditions at the bar ends. A geometrical imperfection leads to the fairly abrupt development of this mode at an overall strain somewhat less (depending on the imperfection) than the bifurcation strain. Once the neck develops, the classic approximate analysis of Bridgman (1952), and subsequent full numerical solutions, e.g., Chen (1971), Needleman (1972), Argon et al. (1975), Norris et al. (1978), Tvergaard and Needleman (1984), Needleman and Tvergaard (1985), show that the neck curvature induces stress triaxiality that plays a key role in the ductile failure process.

For a viscoplastic solid under quasi-static loading conditions, the onset of necking is no longer associated with a bifurcation. However, the onset of necking can be analyzed as the growth of an initial inhomogeneity, Hutchinson and Neale (1977). As for a rate independent plastic solid, a notch serves as an imperfection that triggers necking and sets the neck location. Material rate sensitivity leads to a delay in the onset of necking, Hutchinson and Neale (1977).

In addition, for both rate independent and rate dependent plastic solids characterized by a classic plastic constitutive relation, there is no material length scale in a quasi-static analysis. Hence, the evolution of the neck with strain (at the same imposed strain rate for viscoplastic solids) is independent of specimen size.

The necking behavior under dynamic loading conditions, e.g., Needleman (1991), Knoche and Needleman (1993), Fressenges and Molinari (1994), Guduru and Freund (2002), Mercier and Molinari (2003), Rusinek et al. (2005), Osovskyi et al. (2013), Vaz-Romero et al. (2015), Rotbaum et al. (2015), can be quite different than under quasi-static conditions. Material inertia tends to slow neck development, Needleman (1991), Xue et al. (2008); multiple necking can occur, e.g., Knoche and Needleman (1993), Fressenges and Molinari (1994), Guduru and Freund (2002); there are size effects, e.g., Rusinek et al. (2005), Knoche and Needleman (1993), and neck development can ignore the presence of notches, Rotbaum et al. (2015). Experiments and modeling carried out in Rotbaum et al. (2015) showed that under dynamic loading conditions, the onset of necking in notched tensile bars could occur away from the notch location.

Since material inertia implicitly introduces a length scale, different size specimens deformed at the same strain rate may respond differently. As a consequence, there can be a dependence of the failure strain on specimen size, Knoche and Needleman (1993). Knoche and Needleman (1993) carried out finite deformation dynamics analyses aimed at modeling the effect of specimen size at a fixed imposed strain rate, on ductile failure in geometrically self-similar tensile bars having various sizes. The material was modeled as a viscoplastic progressively cavitating solid. It was found that the variation of the necking strain with specimen size was not monotonic; the response of sufficiently small specimens was essentially quasi-static and size independent, the failure strain then increased with specimen size before eventually decreasing for sufficiently large specimens.

In this study, a combination of the issues addressed in Knoche and Needleman (1993) and Rotbaum et al. (2015) is considered. In particular, a main focus in this paper is to continue exploring the issue raised by Rotbaum et al. (2015) concerning the circumstances,
for dynamic loading conditions, under which necking ignores the presence of a notch.

Calculations are carried out for geometrically similar, dynamically loaded notched circular cylindrical tensile bars of various sizes. Attention is restricted to axisymmetric deformations. The bar material is characterized as an isotropically hardening viscoplastic Mises solid. A non-dimensional form of the governing equations is presented and two key non-dimensional ratios are identified: one relates the bar length to a characteristic length that depends on material properties and the imposed velocity, while the other relates the imposed strain rate (the imposed velocity divided by the bar length) to a material characteristic strain rate. Both of these non-dimensional ratios involve the bar length. The focus of the results is on the transition from necking at the notch cross section to necking away from the notch cross section as the specimen size is varied.

2. Problem Formulation

As in Knoche and Needleman (1993), the calculations are based on a convected coordinate Lagrangian formulation of the field equations. The independent variables are taken to be the particle positions in the initial stress free configuration of the axisymmetric tensile bar and time. In the current configuration the material point initially at \( \mathbf{X} \) is at \( \mathbf{x} \). The displacement vector \( \mathbf{u} \) and the deformation gradient \( \mathbf{F} \) are defined by

\[
\mathbf{u} = \mathbf{x} - \mathbf{X}, \quad \mathbf{F} = \frac{\partial \mathbf{u}}{\partial \mathbf{X}} \tag{1}
\]

The principle of virtual work accounting for material inertia is written as

\[
\int_V \mathbf{S} : \delta \mathbf{F} dV = \int_B (\mathbf{S} \cdot \mathbf{n}) \cdot \delta \mathbf{u} dB - \int_V \rho \frac{\partial^2 \mathbf{u}}{\partial T^2} \cdot \delta \mathbf{u} dV \tag{2}
\]

Here, \( T \) is time, \( \mathbf{S} \) is the (unsymmetric) nominal stress tensor, \( \mathbf{S} = (\det \mathbf{F}) \mathbf{F}^{-1} \cdot \mathbf{\Sigma} \) with \( \mathbf{\Sigma} \) the Cauchy stress, \( \rho \) is the mass density, and \( V \) and \( B \) are, respectively, the volume and the surface of the body in the undeformed reference configuration.

Attention is confined to axisymmetric deformations and, for notational simplicity, we use \( r \) and \( z \) to denote the convected Lagrangian coordinates. The initial length of the bar is \( 2L_0 \) and the initial radius, which varies along the bar is denoted by \( R_0(z) \). The bar occupies the region \( -L_0 \leq z \leq L_0, 0 \leq r \leq R_0(z) \).

An axial velocity \( V(t) \) is imposed on \( z = L_0 \) together with symmetry about \( z = 0 \). This means that the loading is actually applied at \( z = -L_0 \) as well. The reason for imposing symmetry about \( z = 0 \) is that without this symmetry and with shear free conditions on the loading ends, the preferred quasi-static necking mode would be the long wavelength mode with the neck forming at one of the ends. Thus, under quasi-static loading conditions the deformation and stress concentrations associated with the centrally placed notch would be competing with those associated with the preferred necking mode. On the other hand, if shear constraints were imposed at the ends, then the onset of necking would be affected by both the notch and the deformation gradient imposed by the constraints, complicating
the interpretation of the notch effect. With symmetry about $z = 0$ imposed, the preferred quasi-static necking mode is driven by the presence of the notch and necking occurs at the notch cross section. The occurrence of necking away from the notch cross section, when it occurs, is a dynamic effect.

The boundary conditions on the region analyzed are $u_z(r, L_0, T) = V(T)$ where

$$V(T) = \begin{cases} V_1 T/T_r, & \text{for } T \leq T_r \\ V_1, & \text{for } T > T_r \end{cases}$$

(3)

Here, $V_1$ is the magnitude of the imposed velocity and $T_r$ is the rise time.

The other displacement boundary conditions imposed are $u_z(r, 0, T) = 0$ and $u_r(0, z, T) = 0$. All other boundary conditions correspond to zero imposed tractions.

The material is characterized as an elastic-viscoplastic Mises solid. The total rate of deformation, $D$, is written as the sum of an elastic (actually hypoelastic) part, $D^e$, and a viscoplastic part, $D^p$, with

$$D^e = \frac{1 + \nu}{E} T - \frac{\nu}{E} \text{tr}(T) \mathbf{I}$$

(4)

where $E$ is Young’s modulus, $\nu$ is Poisson’s ratio, $T = (\det \mathbf{F}) \Sigma$, $\dot{}$ denotes the Jaumann rate based on $T$, $\text{tr}( )$ denotes the trace and $\mathbf{I}$ is the identity tensor.

The viscoplastic flow rule is

$$D^p = \frac{3 \dot{\Lambda}^p}{2 \Sigma_e} T'$$

(5)

where $\mathbf{I}$ is the identity tensor, $\dot{\Lambda}^p$ is the effective plastic strain rate, and the Kirchhoff stress deviator $T'$ and effective stress $\Sigma_e$ are given by

$$T' = T - \Sigma_h \mathbf{I}, \quad \Sigma_e = \sqrt{\frac{3}{2} T' : T'}, \quad \Sigma_h = \frac{1}{3} \text{tr}(T) \mathbf{I}$$

(6)

The material response, the specimen geometry and the boundary value problem are characterized by a collection of non-dimensional quantities. To put the equations in non-dimensional form, we normalize all stress quantities by a reference stress $\sigma_0$, all length quantities by a reference length $L_c$ and all time quantities by a reference time $t_c$.

The principle of virtual work, Eq. (2), can be written as

$$\int s : \delta \mathbf{F} \, dv = \int_b (s \cdot n) \cdot \delta w \, db - \int_V \ddot{w} \cdot \delta w \, dv$$

(7)

provided

$$t_c = L_c \sqrt{\frac{\rho}{\sigma_0}}$$

(8)

In Eq. (7), $s = \sigma_0 \mathbf{S}$, $u = L_c \mathbf{w}$, $t = t_c T$, $dV = L_c^3 \, dv$, $dB = L_c^2 \, db$ and $\dot{}$ denotes $\partial( )/\partial t$.

As in Knoche and Needleman [1993], we take

$$L_c = L_0 \frac{c_0}{V_1}, \quad c_0 = \sqrt{\frac{E}{\rho}}$$

(9)
Hence, from Eq. (8)

\[ t_c = \frac{L_0}{V_1} \sqrt{\frac{E}{\sigma_0}} \]  

(10)

In non-dimensional form, the rate constitutive relation, Eqs. (4) and (5) become

\[ \text{d}^\varepsilon = \epsilon_0 \left[ (1 + \nu) \dot{\tau} - \nu \text{tr} (\dot{\tau}) \mathbf{I} \right], \quad \epsilon_0 = \frac{\sigma_0}{E} \]  

(11)

where (\dot{\cdot}) denotes the Jaumann rate based on \( t \), and

\[ \text{d}^p = \frac{3 \dot{\epsilon}^p}{2 \sigma_e} \tau' \]  

(12)

In Eq. (12), \( \dot{\epsilon}^p = t_c \dot{\Lambda}^p \).

The plastic response of the material is characterized by a power law rate hardening of the form

\[ \dot{\epsilon}_p = t_c \dot{\epsilon}_0 \left( \frac{\sigma_e}{g} \right)^{1/m} \]  

(13)

Here, \( \sigma_e = \Sigma_e / \sigma_0 \), \( \dot{\epsilon}_0 \) is a reference strain rate, \( m \) is the rate sensitivity exponent.

The flow strength function \( g \) in Eq. (13) is taken to be a function of \( \epsilon_p \) and have the form

\[ g (\epsilon_p) = \left[ 1 + \frac{\epsilon_p}{\epsilon_0} \right]^N \]  

(14)

Here, \( \epsilon_p = \int \dot{\epsilon}_p dt \) and \( N \) is the strain hardening exponent. Since \( g(0) = 1 \) in Eq. (14), \( \sigma_0 \) is now identified with the flow strength at zero plastic strain. Note that up to the identification in Eq. (14), \( \sigma_0 \) could be any convenient quantity having the dimension of stress.

The constitutive response is characterized by the non-dimensional material parameters. \( \epsilon_0, \nu, m \) and \( N \). For a given function \( R_0(z) \), the specimen geometry is characterized by the non-dimensional ratio \( R_0(0)/L_0 \).

There are two key non-dimensional groups involving both loading and material parameters. One is

\[ \frac{L_0}{L_c} = \frac{V_1}{\epsilon_0} \]  

(15)

and the other is

\[ \kappa = \frac{V_1/L_0}{\epsilon_0} \]  

(16)

which is the ratio of the imposed strain rate to the material characteristic strain rate. The ratio \( L_0/L_c \) provides a measure of the effect of material inertia while \( \kappa \) provides a measure of the effect of loading rate, independent of material inertia. In the quasi-static limit \( L_0/L_c \rightarrow 0 \) so that, of course, the effect of inertia on the response vanishes. On the other hand, the role of \( \kappa \) persists in the quasi-static limit.

There are other non-dimensional groups relating material parameters and loading parameters, but they are not independent of the ones already defined. For example, \( t_c \dot{\epsilon}_0 \), which
can be regarded as a ratio of dynamic and constitutive time scales is, from Eqs. (10) and (16), given by

\[ t_c \dot{\epsilon}_0 = \frac{L_0 \dot{\epsilon}_0}{V_1} \sqrt{\frac{E}{\sigma_0}} = \frac{1}{\kappa \sqrt{\dot{\epsilon}_0}} \]  

Eq. (13) then can be rewritten as

\[ \dot{\epsilon}_p = \frac{1}{\kappa \sqrt{\dot{\epsilon}_0}} \left( \frac{\sigma_e}{g} \right)^{1/m} \]  

and used to express the plastic flow rule, Eq. (12) in non-dimensional form as

\[ \mathbf{d}_p = \frac{3}{2} \frac{1}{\kappa \sqrt{\dot{\epsilon}_0}} \left( \frac{\sigma_e}{g} \right)^{1/m} \left( \frac{\tau'}{\sigma_e} \right) \]  

Eq. (19) explicitly shows the constitutive dependence on the non-dimensional parameter \( \kappa \) which gives rise to a coupling of the constitutive response to parameters involving the bar geometry and the imposed loading. This coupling occurs because of material rate sensitivity and does not occur for the corresponding rate independent plastic flow rule.

Another non-dimensional ratio is the ratio of the stress carried by the loading wave to the reference stress. A one dimensional linear elastic wave propagation analysis of a tensile bar subject to a prescribed end velocity \( V_1 \) gives the stress carried by the loading wave as \( \rho c_0 V_1 \), see e.g. Lee (1967). However, this is not an independent ratio. The ratio of this loading wave stress to the reference stress is

\[ \frac{\rho c_0 V_1}{\sigma_0} = \frac{1}{\epsilon_0} \frac{V_1}{c_0} = \frac{1}{\epsilon_0} \frac{L_0}{L_c} \]  

Hence, the stress carried by the loading wave (according to a one dimensional linear elastic analysis) is proportional to \( L_0/L_c \) and is inversely proportional to the non-dimensional material parameter \( \epsilon_0 \). Hence, with material properties fixed, the stress carried by the loading wave increases with increasing relative bar size, \( L_0/L_c \) with the quasi-static limit emerging as \( L_0/L_c \to 0 \).

Thus, the boundary value problem is characterized by the non-dimensional material parameters, \( \epsilon_0, \nu, N \) and \( m \); the non-dimensional geometry parameter \( R_0/L_0 \); the non-dimensional rise time \( t_r \); and the non-dimensional parameters, \( L_0/L_c \) and \( \kappa \), that involve geometry, loading and material parameters.

To illustrate the scaling, let \( \rho^* = A \rho \), then with

\[ E^* = A^p E, \quad \sigma_0^* = A^p \sigma_0 \]  

the linear elastic wave speed scales as \( c_0^* = A^{(p-1)/2} c_0 \).

Taking \( V_1^* = A^{(p-1)/2} V_1 \), \( L_0^* = A^{(p-1)/2} L_0 \), and \( \epsilon_0^* = \dot{\epsilon}_0 \), the non-dimensional ratios \( L_0/L_c \), \( \epsilon_0 \) and \( \kappa \), as well as the characteristic time \( t_c \), are unchanged with this scaling. Thus, with this scaling the solution of the dynamic initial/boundary value problem (with fixed \( R_0/L_0 \)) coincides for both these sets of material and loading parameters.
3. Numerical Method and Results

The numerical method is basically the same as in Needleman (1991); Knoche and Needleman (1993). The discretization is based on linear displacement triangular elements arranged in quadrilaterals of four “crossed” triangles and the time integration of the discretized governing equations are integrated numerically by an explicit integration procedure, Belytschko et al. (1976), with a lumped mass matrix. The constitutive update is based on the rate tangent modulus method of Peirce et al. (1984).

Figure 1: The 12 × 96 quadrilateral finite element mesh used in the computations. Each quadrilateral consists of four “crossed” linear displacement triangles.

The fixed (non-dimensional) constitutive parameters are taken to be \( \epsilon_0 = 0.004 \), \( \nu = 0.3 \). The calculations are carried out for strain rate hardening exponents \( m = 0.01 \) and \( m = 0.05 \). In most calculations the strain hardening exponent is taken to be \( N = 0.01 \), which is nearly ideally plastic, but a few calculations employ \( N = 0.1 \) to assess the effect of strain hardening. The value of the (non-dimensional) rise time is taken to be \( t_r = 1.265 \times 10^{-3} \) in all calculations.

To give a perspective on what these non-dimensional parameter values could correspond to, one possibility is \( E = 200 \text{GPa}, \Sigma_0 = 800 \text{MPa}, c_0 = 5000 \text{m/s}, V_1/L_0 = 1000 \text{s}^{-1}, \dot{\epsilon}_0 = 1000 \text{s}^{-1} \) and \( T_r = 20 \times 10^{-6} \text{s} \). Also, with these values \( L_c = c_0/(V_1/L_0) = 5 \text{m} \) (close to the value in Knoche and Needleman (1993)).

The bar aspect ratio is fixed at \( L_0/R_0 = 4 \) and the bar has a notch as depicted in Fig. 1. The notch is a semi-circle of radius \( R_0/10 \) centered at \( z = 0 \), \( r = R_0 \). However, as can be seen in Fig. 1 there are only three nodal points on the notch surface so that the circular
shape is not faithfully represented. The notch mainly serves as an imperfection to trigger necking at \( z = 0 \).

We define the ratio of current cross section area to initial cross section area, \( A_r(z) \), by

\[
A_r(z) = \frac{\pi R_0^2(z)}{\pi [R_0(z) + u_r(R_0(z), z)]^2}
\]

(22)

A calculation is terminated when the maximum of \( A_r(z) \) reaches 2 (an area reduction of 1/2). In the calculations here, this occurs either for \( A_r(0) \) or \( A_r(L_0) \). If \( A_r(0) \) reaches 2 first then necking has occurred at the notch, if \( A_r(L_0) \) reaches 2 first then necking has occurred away from the notch and necking has ignored the presence of the notch. The value \( A_r = 2 \) is chosen arbitrarily. This value was chosen because neck development has clearly occurred when \( A_r = 2 \) yet the strains are not so large that the finite element grid in Fig. II has become significantly deformed. As will be seen subsequently, the value of necking strain is not generally very sensitive to the precise cut-off value chosen.

At the termination of the calculation we define the ratio

\[
R_A = \frac{A_r(0)}{A_r(L_0)}
\]

(23)

so that \( R_A > 1 \) implies necking at the notch, \( R_A < 1 \) implies necking away from the notch and \( R_A = 1 \) corresponds to simultaneous necking at the notch and away from the notch. Assuming a non-negative effective Poisson’s ratio, the possible range of values for \( R_A \) is \( 0.5 \leq R_A \leq 2 \).

3.1. Fixed \( \kappa \), varying \( L_0/L_c \)

In this section the value of \( \kappa \), Eq. (16), is fixed at \( \kappa = 1 \) and \( L_0/L_c \), Eq. (15), is varied.

Figure 2: Nominal stress-strain curves. (a) \( m = 0.01, N = 0.01 \). (b) \( m = 0.05, N = 0.01 \).
Figure 3: Evolution of $A_r(0)$ and $A_r(L_0)$, with $A_r(z)$ given by Eq. (22). (a) $m = 0.01, N = 0.01, L_0/L_c = 0.004$. (b) $m = 0.01, N = 0.01, L_0/L_c = 0.012$.

Fig. 2 shows nominal stress-strain curves for $m = 0.01$, Fig. 2a, and for $m = 0.05$, Fig. 2b. In both plots $N = 0.01$. The quantity $S$ is the nominal stress, the force per unit reference area, and the end displacement $U$ is $U = \int V dt$. For sufficiently small $L_0/L_c$ the response is essentially quasi-static and this is nearly the case for $L_0/L_c = 0.004$ in Fig. 2.

As $L_0/L_c$ increases from 0.004, inertia plays an increasing role which delays the onset of necking. The same overall trend occurs for both $m = 0.01$ and $m = 0.05$. In Fig. 2a where $m = 0.01$ the maximum necking strain occurs for $L_0/L_c = 0.008$. In Fig. 2b where $m = 0.05$ the maximum necking strain occurs for $L_0/L_c = 0.012$. Thus, increased strain rate sensitivity results in the maximum necking strain (as defined here) occurring for a larger value of $L_0/L_c$. For the larger values of $L_0/L_c$ wave effects come into play which is what leads to the reduction in necking strain as also seen by Knoche and Needleman (1993).

Fig. 3 shows the evolution of the area ratios $A_r(0)$ and $A_r(L_0)$ with strain, $U/L_0$, for two cases. (Recall that $z$ denotes a convected coordinate so that $z = L_0$ corresponds to the impact end in the current configuration.) In Fig. 3a, $L_0/L_c = 0.004$ and necking occurs at the notch cross section while in Fig. 3b, $L_0/L_c = 0.012$ necking occurs at the impact end. In both cases, the area reduction (recall that $A_r$ in Eq. (22) increases with increasing area reduction) eventually increases rapidly at one end and remains nearly constant at the opposite end. In Fig. 3a, the response is quasi-static like and the relative area changes at $z = 0$ and $z = L_0$ are nearly the same initially. On the other hand in Fig. 3b, where material inertia plays an important role, significant deformation occurs at $z = L_0$ before much plastic deformation takes place at $z = 0$. It can also be seen in Fig. 3 from the rapid increase in $A_r$ when necking occurs, that although the precise value of the necking strain depends on the cut-off value chosen ($A_r = 2$ here), the trends will not be sensitive to this value.

The values of $R_A$, defined in Eq. (23), for all values of $L_0/L_c$ considered is shown in Fig. 4. The reference line $R_A = 1$ corresponds to simultaneous necking at the two ends of
Figure 4: Ratio of the relative area reductions at $z = 0$ and $z = L_0$, $R_A = A_r(0)/A_r(L_0)$. Values greater than 1 correspond to necking at $z = 0$ while values less than 1 correspond to necking at $z = L_0$.

the region analyzed. For $m = 0.01$ necking occurs at the notch for $L_0/L_c$ less than about 0.0084 and occurs at the impact end for greater values of $L_0/L_c$. For $m = 0.05$ this transition occurs just about at $L_0/L_c = 0.01$. Increasing the strain hardening exponent from $N = 0.01$ to $N = 0.1$ leads to the transition from necking at the notch to necking away from the notch taking place at a much larger value of $L_0/L_c$. However, although strain hardening and/or strain rate hardening can strongly affect the size, i.e. the value of $L_0/L_c$, at which this transition in necking location occurs, the results in Fig. 4 show that it is inertia that drives the transition. With $L_c = 5m$ as for the dimensional values given in Section 3 the transition value of $L_0$ for $m = 0.01$, $N = 0.01$ is 0.044m while with $m = 0.05$ this increases to $\approx 0.05m$ and for $m = 0.01$, $N = 0.1$, the transition to necking away from the notch takes place for $L_0 \geq 0.078m$.

Fig. 5 shows contours of effective plastic strain, $\epsilon_p$, and hydrostatic stress, $\sigma_h = \Sigma_h/\sigma_0$, for a case where necking occurs at the notch. There are two concentrations of plastic strain, $\epsilon_p$, in Fig. 5a: one at the neck center and one that extends at about 45° from the notch root. The strain concentration that extends at the notch root does not occur in a naturally necked specimen. The distribution of $\sigma_h$ has a hydrostatic tension peak of about 0.9 at the neck center and hydrostatic compression away from the neck along the bar axis that reaches $-0.6$. The distributions of $\epsilon_p$ and $\sigma_h$ are very similar to those that would be obtained from a quasi-static analysis.

With $L_0/L_c = 0.012$ in Fig. 6 necking has occurred at the impact end. As can be seen in Fig. 6a, a strain concentration did initiate at the notch root and reach $\epsilon_p \approx 0.3$, but the deformation eventually concentrated at the impact end. The distribution of $\sigma_h$ in Fig. 6b shows evidence of the initial neck development at $z = 0$. There are three concentrations of hydrostatic tension: at the notch root, at the center of the bar at $z = 0$ and at the center of the bar at $z = L_0$. The maximum positive hydrostatic stress is in the neck that forms at
$z = L_0$, reaching 0.79. There is also a region of negative $\sigma_h$ along the axis as a consequence of the initial neck formation there. The distributions of plastic strain $\varepsilon_p$ and $\sigma_h$ in Fig. 6 are very different from what would be obtained from a quasi-static analysis.

The calculation with $m = 0.05$, $N = 0.01$ and $L_0/L_c = 0.010$ is unusual in that necking occurred nearly simultaneously at the notch plane ($z = 0$) and at the loaded end ($z = L_0/2$). Fig. 7a shows the evolution of the area ratios $A_r(0)$ and $A_r(L_0)$ with strain, $U/L_0$. Although $A_r(L_0)$ grows slightly faster from the beginning, the two grow at nearly the same rate and
when $A_r(L_0) = 2$, the area ratio at the $z = 0$ is $A_r(0) = 1.96$. The strain distribution in the vicinity of the notch in Fig. 7b is very similar to that in Fig. 5 with the strain concentration from the notch root emanating at about $45^\circ$. In Fig. 7b the plastic strain near the notch root exceeds 1.0 but this strain concentration is very localized. The value of $\varepsilon_p$ in the neck at $z = L_0$ is about 0.7 over a fairly large region. The hydrostatic stress distribution in Fig. 7c is nearly symmetrical consistent with the nearly equal necking at $z = 0$ and $z = L_0$.

To give an indication of the length scales involved, with, from Eq. (15),
\[ L_c = c_0/(V_1/L_0) = 5m \]
\[ L_0/L_c = 0.01 \] corresponds to 5cm so that the transition from necking at $z = 0$ to $z = L_0$ occurs for $L_0$ between about 4cm and 8cm.
3.2. Fixed $L_0/L_c$, varying $\kappa$

Figure 8: (a) Stress strain curves. (b) Evolution of $R_A$ defined in Eq. (23). For $L_0/L_c = 0.010$, $m = 0.05$ and various values of $\kappa$.

The calculations in this section are carried out for $L_0/L_c = 0.01$ and $m = 0.05$. In Fig. 4, where $\kappa = 1$, this is the case for which necking occurs nearly simultaneously at $z = 0$ and $z = L_0$. Fig. 8 shows nominal stress-strain curves and the evolution of $R_A$ with imposed strain, $U/L_0$, for $\kappa = 10$, 1 and 0.1. Larger values of $\kappa$, see Eq. (16), correspond to larger imposed strain rates relative to the material strain rate $\dot{\epsilon}_0$. 

13
Figure 9: (a) Distribution of effective plastic strain $\epsilon_p$ for $L_0/L_c = 0.010$, $m = 0.05$. (a) $\kappa = 10$ at $U/L_0 = 0.253$. (b) $\kappa = 0.1$ at $U/L_0 = 0.257$.

From Eq. (17), $t_c \dot{\epsilon} \rightarrow 0$ as $\kappa \rightarrow \infty$ and $t_c \dot{\epsilon}_0 \rightarrow \infty$ as $\kappa \rightarrow 0$. With $t_c \dot{\epsilon}_0$ the ratio of inertial and material time scales, this indicates that inertia dominates as $\kappa \rightarrow 0$ and more quasi-static type behavior occurs as $\kappa \rightarrow \infty$. With $\dot{\epsilon}_0$ non-zero and finite, this means that increased the imposed strain rate can correspond to more quasi-static like behavior, which may seem counter intuitive.

To understand this consider $\dot{\epsilon}_0$ fixed and $V_1/L_0$ varying. Increasing $\kappa$, Eq. (16), then implies increasing $V_1/L_0$, but since $L_0/L_c$ fixed, $V_1/c_0$ is fixed. For $V_1/L_0$ to increase with $V_1/c_0$ unchanged requires $L_0$ to decrease. Since with $L_0/L_c$ fixed, the stress carried by the loading wave is fixed, Eq. (20), this scenario corresponds to a smaller specimen with a fixed loading wave stress. Note also that varying $V_1/L_0$ with $\dot{\epsilon}_0$ fixed, implies that $t_c$ varies, see Eq. (10), so that to obtain the properly scaled initial/boundary value problem, $T_r$ must be varied so that $t_r = T_r/t_c$ remains fixed.

Fig. 8a shows stress-strain plots for $\kappa = 10$, 1 and 0.1. The calculation for $\kappa = 1$ is the one discussed in Section 3.1 where necking occurred nearly simultaneously at $z = 0$ and $z = L_0$. The stress strain curves for all three values of $\kappa$ are qualitatively similar with the stress levels varying as expected due to the material strain rate sensitivity. It is worth noting that the maximum strain to necking (i.e. to $A_r = 2$ at some cross section) occurs for the intermediate value $\kappa = 1$.

Curves of the ratio $A_r(0)/A_r(L_0)$ are shown in Fig. 8b. For $\kappa = 1$, $A_r(0)/A_r(L_0) \approx 1$ consistent with the nearly simultaneous necking at $z = 0$ and $z = L_0$. For the increased relative strain calculation, $\kappa = 10$, $A_r(0)/A_r(L_0) > 1$, indicating necking at $z = 0$, while for the lower strain rate, $\kappa = 0.10$, $A_r(0)/A_r(L_0) < 1$ indicating necking at $z = L_0$.

Contours of effective plastic strain, $\epsilon_p$, for $\kappa = 10$ and $\kappa = 0.1$ are shown in Fig. 9 showing neck development at $z = 0$ for the higher relative strain rate and at $z = L_0$ for the lower relative strain rate.
The value of $\kappa$ can be varied by keeping all material, geometrical and loading parameters fixed except for $\dot{\epsilon}_0$. In this situation, the results here show that the where necking occurs can be sensitive to the value used for $\dot{\epsilon}_0$. Thus, the predicted response can depend qualitatively, not only quantitatively, on the value of $\dot{\epsilon}_0$.

4. Discussion

With fixed material properties and bar geometry, the results show that with a fixed imposed strain rate, $V_1/L_0$, a transition from notch triggered quasi-static like necking to notch ignoring dynamic necking occurs with increasing bar size, $L_0/L_c$. Thus with $\kappa$ fixed, the effect of material inertia on necking location increases with increasing values of the imposed velocity $V_1$. On the other hand, with $L_0/L_c$ fixed (with fixed bar geometry and with fixed material properties) but with $\kappa$ varying, a similar transition takes place for a decreasing value of the imposed strain rate $V_1/L_0$. In both cases, the effect of material inertia increases with increasing bar size $L_0$.

The idea that there can be a critical imposed velocity under dynamic loading conditions beyond which the apparent ductility decreases dates back to the 1930s, see for example Mann (1936); von Karman (1942); Knoche and Needleman (1993); Klepaczko (2005); Vaz-Romero and Rodriguez-Martínez (2016). This critical velocity is associated with material softening and often attributed to adiabatic heating. In the circumstances analyzed here there is a critical value of $L_0/L_c$ beyond which the strain for necking decreases as seen in Fig. 2. Since, for fixed material properties, $L_0/L_c \propto V_1$, there is a critical imposed velocity beyond which the necking strain decreases. This non-monotonic behavior is a consequence of material inertia; there is no material softening in the formulation.

As seen in Fig. 8b the qualitative nature of the response can depend on value of the parameter $\kappa$ that couples the loading rate and the constitutive response through the material parameter $\dot{\epsilon}_0$, see Eq. (19). Thus, it is worth noting that, for a given value of the imposed strain rate, the predicted response can strongly depend on the specified value of $\dot{\epsilon}_0$.

The parameter $\kappa$, the ratio between time scales associated with the material and the loading, does not enter a rate independent formulation but does enter the formulation for any value of the rate hardening exponent $m \neq 0$. Presumably, the rate independent limit is somehow approached as $m \to 0$; one possibility is that the applied strain at which the responses separate in Fig. 8b approaches infinity as $m \to 0$. In any case, the dependence of the necking behavior on $\kappa$ as $m \to 0$ remains to be investigated. Note that even in the quasi-static limit, $\kappa$ can play a significant role, as seen by Vasoya et al. (2016) in a context different from the one considered here.

The non-dimensional ratio $L_0/L_c$, Eq. (2), does enter in the rate independent limit as does the characteristic time $t_c$, Eq. (10), which then only serves as a quantity for normalizing the time, $t = T/t_c$.

In the calculations here only one notch geometry was considered, a semi-circular notch of radius $R_0/10$. It is expected that the transition from necking at the notch cross section to necking at another cross section will depend on the depth of the notch and, probably to a lesser extent, on the shape of the notch. The results in Knoche and Needleman (1993)
show that such a transition occurs (in some case with necking taking place at an interme-
diate location) even for naturally necking bars with very small geometrical imperfections,
but the variation of the transition bar size with increasing notch depth remains to be in-
vestigated. Also, the material model used in the present calculations excludes any softening
mechanism, such as thermal softening and porosity induced softening, which were included
in the calculations in Knoche and Needleman (1993). The dependence of the necking loca-
tion transition seen here on such softening mechanisms, as well as on other aspects of the
constitutive characterization of the material remains to be explored.

Fineberg and co-workers, e.g. Kolvin et al. (2015), have used soft materials with slow
wave speeds to study dynamic fracture processes at low velocities in order to observe aspects
of the crack growth process that would be difficult or impossible to observe directly in hard
materials. The scaling properties embodied in the non-dimensional equations here suggest
that it may be possible also to do this to study dynamic plastic instabilities, provided of
course, that both the soft and hard materials can be characterized using the same elastic-
viscoplastic constitutive framework.

5. Conclusions

The governing equations for dynamic deformations of an elastic-viscoplastic notched bar
subject to tensile loading were presented in non-dimensional form. Axisymmetric calcula-
tions were carried out for geometrically identical bars having various sizes. Two key non-
dimensional groups were identified that contain a parameter characterizing the bar size. The
main focus was on variations of the values of these two parameters that can be regarded as
corresponding to variations in bar size, although other interpretations are possible and were
discussed. Attention was principally directed at the effect of material inertia on whether
necking developed at the notch cross section or whether necking ultimately occurred away
from the notch.

It was found that:

1. With a fixed imposed strain rate, the applied strain to necking (as defined here) does
not depend monotonically on size.

2. The transition from notch induced necking to notch ignoring necking depends on size
and is driven by material inertia.

3. One of the key non-dimensional groups that involves a measure of bar size is the ratio
of the imposed velocity to an elastic wave speed, the other is the ratio of the imposed
strain rate to the material characteristic strain rate. The second non-dimensional
group is absent for a rate independent solid.

4. With a fixed imposed strain rate and a fixed elastic wave speed, necking was notch
induced for sufficiently small values of the imposed velocity (smaller bar sizes).
5. With a fixed imposed velocity and a fixed material characteristic strain rate, necking was notch induced for sufficiently large values of the imposed strain rate (smaller bar sizes).

Thus, smaller may be stronger, see e.g. Fleck et al. (1994); Greer et al. (2005), but larger is more dynamic.

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