The Massive Soft Anomalous Dimension Matrix at Two Loops

Alexander Mitov, George Sterman, Ilmo Sung
C.N. Yang Institute for Theoretical Physics, Stony Brook University, Stony Brook, New York 11794–3840, USA
(Dated: March 18, 2009)

We study two-loop anomalous dimension matrices in QCD and related gauge theories for products of Wilson lines coupled at a point. We verify by an analysis in Euclidean space that the contributions to these matrices from diagrams that link three massive Wilson lines do not vanish in general. We show, however, that for two-to-two processes the two-loop anomalous dimension matrix is diagonal in the same color-exchange basis as the one-loop matrix for arbitrary masses at absolute threshold and for scattering at ninety degrees in the center of mass. This result is important for applications of threshold resummation in heavy quark production.

The infrared structure of perturbative amplitudes is relevant for a variety of hard-scattering processes, including the production of heavy particles, whether charged under QCD or not, and of high-\(p_T\) jets. Infrared enhancements in these amplitudes are typically regularized by continuing to \(D\) dimensions, where they appear as poles in \(\varepsilon = 2 - D/2\). Infrared poles, of course, do not appear in physical predictions for infrared safe quantities, in which they cancel after an appropriate sum over final states. Nevertheless, the all-order structure of infrared poles is of interest for exact fixed-order calculations \(^1\), where the velocities \(\beta_i\) are lightlike for light partons, \(\beta_i^2 = 0\), and may be scaled to unit length for heavy particles, \(\beta_i^2 = 1\). Integrals in the exponent are carried out in \(D > 4\), using the \(D\)-dimensional running coupling, \(\alpha_s(\mu, \varepsilon)\). Equation \((3)\), along with corresponding expressions for the jet functions, determines the infrared pole structure to all orders in perturbation theory for processes involving the wide-angle scattering of any number of massless and massive partons \(^2\) \(^3\) \(^4\).

The determination of the anomalous dimension matrix for an arbitrary process with massless and massive partons is equivalent to the renormalization of a set of color tensors that link the corresponding product of Wilson lines at a point \(^8\) \(^9\). Each Wilson line follows the velocity \(\beta_i\) of the corresponding parton, without recoil, from this point to infinity, either from the initial state or into the final state. These composite operators mix under renormalization in general, leading to the matrix structure shown in Eq. \((3)\). The one-loop anomalous dimensions for gluons and for both massless and massive quarks have been known for some time \(^8\) \(^9\) \(^10\). At two loops, the matrices for any \(2 \rightarrow n\) process with massless lines satisfy the relation \(^11\)

\[
\Gamma_{\text{S}_i}^{(2)}(\beta_i) = \frac{K}{2} \Gamma_{\text{S}_i}^{(1)}(\beta_i),
\]

with \(\Gamma_{\text{S}_i}^{(1)}\) the coefficient of \((\alpha_s/\pi)^j\), and with \(K = C_A(67/18 - \zeta(2)) - 10T_Fn_F/9\). This is exactly the relation satisfied by the expansion of the cusp anomalous dimension \(^12\), which generates the leading, double poles in the elastic form factor \(^13\) \(^14\).
necessary analysis below, and show that when the non-zero, Eq. (4) no longer holds. A generalization of Eq. (4), however, given by Eq. (3) below, does hold for two-to-two processes for special momentum configurations.

The result (4) for massless partons is a consequence of the vanishing of the single poles of those two-loop “3E” diagrams in which color is exchanged coherently between three eikonal lines in the figures. The arguments of Ref. [11] do not, however, generalize directly to massive Wilson lines, with velocity vectors \( \beta_i \neq 0 \). While an analytic determination of \( I_3^{(2)} \) would, of course, be desirable, numerical determination is also of interest, and is certainly adequate to determine how far Eq. (4) generalizes to the production of massive particles. We provide the necessary analysis below, and show that when the \( \beta_i^2 \) are non-zero, Eq. (4) no longer holds. A generalization of Eq. (4), however, given by Eq. (3) below, does hold for two-to-two processes for special momentum configurations.

Much of our analysis will be carried out in position, rather than momentum space. In the following, we will take every parton as massive, and use the scale invariance of Wilson lines to set \( \beta_i^2 = 1 \). Because we are calculating renormalization constants, we can carry out our analysis in Euclidean space. Indeed, a numerical result in Euclidean space is adequate to establish that the matrix does not follow Eq. (4) in Minkowski space. Otherwise, analytic continuation through Wick rotation would imply that the same result would hold in Euclidean space as well.

We begin with the diagram, Fig. (1), in which three eikonal lines are coupled by gluons that are linked at a three-gluon coupling [11]. In the configuration space evaluation of this diagram, we must integrate the position of the three-gluon vertex over all space. The three propagators each have one end fixed at this vertex and the other end fixed at a point \( \lambda_i \beta_i \) along the ith Wilson line. Each parameter \( \lambda_i \) is integrated from the composite vertex to infinity. This diagram vanishes in Minkowski space for massless Wilson lines [11].

Suppressing color factors, we represent the 3E diagram Fig. (1) as

\[
F_{3g}(2)(\beta) = \int d^D x \prod_{i=1}^3 \int_0^\infty d\lambda_i V(x, \beta_i) .
\]

Here \( \beta = \{\beta_1, \beta_2, \beta_3\} \) denotes the set of three massive velocities of the lines to which the gluons attach, while the propagators and numerator factors of the integrand are given by a sum over six terms,

\[
V(x, \beta_i) = \sum_{i,j,k=1}^3 \epsilon_{ijk} \delta_{ij} (x, \beta_i) .
\]

Each of these terms involves the derivative of one of the propagators, according to the usual gauge theory rules for the three-vector coupling,

\[
v_{ijk}(x, \beta) = -i (g_{\mu})^4 \beta_i \cdot \beta_j \Delta(x - \lambda_j \beta_j) \Delta(x - \lambda_k \beta_k) \times \beta_k \cdot \partial_\mu \Delta(x - \lambda_i \beta_i) ,
\]

where \( \Delta \) represents the position-space scalar propagator,

\[
\Delta(x - \lambda_i \beta_i) = - \frac{\Gamma(1 - \epsilon)}{4\pi^{2-\epsilon}} \frac{1}{(x - \lambda_i \beta_i)^{2(1-\epsilon)}} .
\]

We work in Feynman gauge. The contribution of this (scaleless) diagram to the anomalous dimension matrix is found from the residue of its simple ultraviolet pole. We note that all diagrams found from products of Wilson lines are scaleless overall, and are defined by their renormalization constants [11].

At fixed \( x \), for massive eikonal the \( \lambda \) integrals in Eq. (5) are all finite in four dimensions. After these integrals are carried out, the \( \beta \)-dependence enters only through the combination

\[
\zeta_i \equiv \frac{\beta_i \cdot x}{\sqrt{x^2}} .
\]
and we can write
\[
F^{(2)}_{3g}(β_1, ε) = N(ε) \int d^D x \sum_{i,j,k=1}^3 ε_{ijk} γ_{ijk}(\sqrt{x^2}, ζ_1, ε),
\]

where \(N(ε)\) absorbs overall factors that are finite in the limit \(ε = 0\). To simplify our notation, in the following we normalize \(F_{3g}\) so that \(N(ε) = 1\). We recall that we have used the scale invariance of eikonal lines to set \(β_1^2 = 1\), and that \(I\) represents the set \(i, j, k\). Each term \(γ_{ijk}\) is now given by
\[
γ_{ijk}(\sqrt{x^2}, ζ_1, ε) = β_i · β_j f(x, β_j, ε) f(x, β_k, ε) × β_k · ∂f x, β_k, ε, \tag{11}
\]
where the functions \(f(x, β, ε)\) are simply the integrals of the \(x\)-dependent factors of the propagators,
\[
f(x, β, ε) = \int_0^∞ dλ \frac{1}{(x^2 - 2λβ · x + λ^2)^{1-ε}}. \tag{12}
\]
After a change of variables to \(N' \equiv λ/\sqrt{x^2}\), the dependence on the variables \(x^2\) and \(ζ_1\) factorizes,
\[
f(x, β, ε) = \frac{1}{\sqrt{x^2}} \int_0^∞ dλ' \frac{1}{(1 - 2λ'ζ_1 + λ'^2)^{1-ε}} \equiv \frac{1}{\sqrt{x^2}} g(ζ_1, ε). \tag{13}
\]
For the full expression, we also need the gradient of this function, which can be written as
\[
\partial^2 f x, β, ε = \frac{1}{\sqrt{x^2}} \left[(2ε - 1)x^i \sqrt{x^2} g(ζ_1, ε) + \left(-\frac{x^i ζ_1}{\sqrt{x^2}} + β_1 \right) \frac{∂g(ζ_1, ε)}{∂ζ_1}. \tag{14}\n\]
We note that this derivative is necessary to produce an overall \(x^{-4}\) fall-off at infinity and a singularity at \(x = 0\), corresponding to logarithmic infrared and ultraviolet behaviors.

We next substitute the expressions for the functions \(f\) in (13) and their gradients (14) into Eq. (11) for the terms \(γ_{ijk}\), to find
\[
γ_{ijk}(\sqrt{x^2}, ζ_1, ε) = g(ζ_2, ε)g(ζ_3, ε) \frac{β_1 · β_2}{\sqrt{x^2} 4-ε} \times \left[(2ε - 1)ζ_3 g(ζ_1, ε) + (-ζ_3ζ_1 + β_1 · β_3) \frac{∂g(ζ_1, ε)}{∂ζ_1}. \tag{15}\n\]
In this expression, the first term in the square brackets is symmetric in the pair \((i, j)\) and the third is symmetric in the pair \((j, k)\). The full nonvanishing contribution to Eq. (10) is thus simply
\[
F^{(2)}_{3g}(β_1, ε) = - \int d^D x \sum_{i,j,k=1}^3 ε_{ijk} ζ_3 ζ_1 \frac{β_i · β_j}{\sqrt{x^2} 4-ε} \times g(ζ_j, ε)g(ζ_k, ε) \frac{∂g(ζ_i, ε)}{∂ζ_i}. \tag{16}\n\]
Using the freedom to reintroduce dependence on the \(\sqrt{β_i}\) by demanding scale invariance, we can use this result in both Minkowski and Euclidean space to identify and isolate the ultraviolet pole. It is now straightforward to show two important results that follow from the antisymmetries built into Eq. (10).

First, working in Minkowski space, we can readily confirm the vanishing of \(F_{3g}\) for arbitrary massless \(β\). In this case, the function remains scale-invariant in the \(β_i\), although of course we cannot rescale by \(β_i^2 = 0\). Nevertheless, the explicit form of \(g(ζ, ε)\) is
\[
g(ζ, ε) = \frac{1}{2ε ζ} (β^2 = 0), \tag{17}\n\]
which, using \(ζ(dg/dζ) = -g\), immediately gives a vanishing integrand in Eq. (16) by antisymmetry. It is interesting to note that, unlike the discussion in momentum space in Ref. [11], this proof of the vanishing of the three-gluon diagram, \(F^{(2)}_{3g}\), Fig. 1a, does not require a change of variables.

In fact, the vanishing of Fig. 1a, extends to the case where only two of the three lines are massless [13]. Taking for definiteness \(β_1^2 = β_2^2 = 0\) with \(β_3^2 \neq 0\), and using Eq. (17) in Eq. (16), we find
\[
F^{(2)}_{3g}(β_1, ε) = - \int d^D x \frac{1}{\sqrt{x^2} 4-ε} \times g(ζ_3, ε)g(ζ_2, ε) \frac{∂g(ζ_1, ε)}{∂ζ_1} \left(\frac{β_1 · β_3}{ζ_1} - \frac{β_2 · β_3}{ζ_2}\right). \tag{18}\n\]
In this case, we follow [11] and make a change of variables, using the light-like directions \(β_{1,2}\) to define light-cone coordinates. To be specific, if we choose \(χ_1 = ζ_1/β_1 · β_3\) and \(χ_2 = ζ_2/β_2 · β_3\), we derive an integrand that is manifestly antisymmetric in \(χ_1\) and \(χ_2\). We note that the momentum space method of [11] also applies directly to the case when two of the three Wilson lines are massless, although this was not pointed out explicitly there. In both cases, the relevant change of variables exchanges two lightlike directions. This approach does not show that diagrams with a single massless line vanish identically, and indeed this seems unlikely, given that in this case there is only a single lightlike direction.

Our second main result is that for both massive and massless Wilson lines the function \(γ^{(2)}_{3g}(β_1 · β_2)\) vanishes when any pair of the invariants are equal, say, \(β_1 · β_2 =
\( \beta_1 \beta_3 \). We can show this by changing variables in Euclidean space from \( x_i \) to \( r = \sqrt{x^2} \) and the \( \zeta_i = \beta_i \cdot x_i / \sqrt{x^2} \).

A straightforward calculation gives a form in which the overall scalelessness of the diagram is manifest in the radial integral,

\[
F^{(2)}_{3g}(\beta_1, \varepsilon) = -\int_0^\infty \frac{dr}{r^{1-4\varepsilon}} \int d\zeta_1 d\zeta_2 d\zeta_3 \sqrt{K(\zeta)} \times \sum_{i,j,k=1}^3 \epsilon_{ijk} \kappa_i \beta_i : \beta_j g(\zeta_j, \varepsilon)g(\zeta_k, \varepsilon) \frac{\partial g(\zeta, \varepsilon)}{\partial \zeta_i}.
\]

(19)

Defining \( w_j \equiv \beta_1 \beta_i, \xi_j \equiv \zeta_j - w_j \zeta_1, \) and \( z_3 \equiv [\beta_2 \beta_3 - w_2 w_3] / \sqrt{1 - w_2^2} \), the explicit form of \( K(\zeta) \) for arbitrary \( w_2 \) and \( w_3 \) is

\[
K(\zeta) = -(1 - w_2^2)\xi_3^2 - (1 - w_2^2)\xi_1^2 + 2\xi_2\xi_3 z_3 \sqrt{1 - w_2^2} + (1 - \xi_1^2)(1 - w_2^2)(1 - z_3^2 - \xi_3^2).
\]

(20)

The function \( K(\zeta) \) is symmetric under the interchange of \( \zeta_2 \) and \( \zeta_3 \) when \( w_2 = w_3 \). We easily check that when \( w_2 = w_3 \) the remaining integrand in Eq. (19) is antisymmetric under the exchange of \( \zeta_2 \) and \( \zeta_3 \). The variables \( \zeta_2 \) and \( \zeta_3 \) are exchanged by a reflection in the two-sphere defined by any fixed value of \( \zeta_1 \) about the axis specified by the projection of \( \beta_2 + \beta_3 \) into this two-sphere. The integration region of \( \zeta_2 \) and \( \zeta_3 \) is therefore also symmetric. We conclude that the integral over \( \zeta_2 \) and \( \zeta_3 \) in Eq. (19) vanishes when \( \beta_1 \cdot \beta_2 = \beta_1 \cdot \beta_3 \). This relation holds as well in Minkowski space, which can be reached by analytic continuation at fixed ratios of the inner products \( \beta_1 \cdot \beta_2 / \beta_1 \cdot \beta_3 \).

We do not have here an analytic form for the residue of the UV pole of \( F^{(2)}_{3g} \) for generic values of the invariants \( \beta_i \cdot \beta_j \). A numerical analysis of Eq. (16), however, is particularly straightforward in Euclidean space. For this purpose, it is convenient to use \( D \)-dimensional polar coordinates, \( r, \Omega_{D-1} \). The single overall ultraviolet pole in the scaleless integral (16) appears at \( r = 0 \), and the remaining three angular integrals at \( \varepsilon = 0 \) determine the residue of the pole in MS renormalization. These can be carried out readily, using the elementary form of the function \( g(\zeta, 0) \) in Eq. (13),

\[
g(\zeta, 0) = i \pi - \arccos \zeta \sqrt{1 - \zeta^2}.
\]

(21)

In Fig. 2 we show a plot of the residue of the UV pole of \( F^{(2)}_{3g} \) for \( \beta_1 \cdot \beta_3 = 0.5 \) in the \( \beta_1 \cdot \beta_2 / \beta_1 \cdot \beta_3 \)-plane, suppressing the surface where the integral is negative for clarity. Notice the lines of zeros at \( \beta_1 \cdot \beta_2 = \beta_2 \cdot \beta_3 \) and at \( \beta_1 \cdot \beta_3 = 0.5 \) and \( \beta_2 \cdot \beta_3 = 0.5 \). The peak towards \( \beta_1 \cdot \beta_2 \to 1 \) reflects an additional singularity in the integrand when both velocities are parallel.

We now turn to the remaining 3E “double-exchange” diagrams, Fig. 1b,c. These diagrams are given by four \( \lambda \) integrations, two of which are along the \( \beta_j \) line. For example, Fig. 1b may be written in the notation introduced above as

\[
M_b = C_{1b} \overline{N}(\varepsilon) (\beta_1 \cdot \beta_j) (\beta_j \cdot \beta_k) \int_0^\infty d\lambda_{j,a} \int_0^{\lambda_{j,a}} d\lambda_{j,b} \times f(\lambda_{j,b} \beta_j, \beta_k, \varepsilon) f(\lambda_{j,b} \beta_j, \beta_k, \varepsilon), \tag{22}
\]

where the functions \( f \) are defined as in Eq. (13) with \( x = \lambda_{j,c} \beta_j \) and with \( c = a, b, \overline{N}(\varepsilon) \) absorbs overall factors that are finite in the limit \( \varepsilon = 0 \). In the following we set \( \overline{N}(\varepsilon) = 1 \). We have kept the overall color factor, represented by \( C_{1b} \). The variables \( \zeta_i = \beta_i \cdot x / \sqrt{x^2} \), are independent of the scale of \( x \) in Eq. (22), so that all dependence on the \( \lambda_{j,c} \) variable is in the overall factor of \( x^2 = \lambda_{j,c}^2 \), giving

\[
M_b = C_{1b} (\beta_1 \cdot \beta_j) (\beta_j \cdot \beta_k) \int_0^\infty d\lambda_{j,a} \int_0^{\lambda_{j,a}} d\lambda_{j,b} \times \frac{g(\beta_j \cdot \beta_k, \varepsilon)}{(\lambda_{j,a} \lambda_{j,b})^{1-2\varepsilon}}. \tag{23}
\]

The exchange diagrams contribute to two color structures, symmetric and antisymmetric with respect to the generators \( T_3 \) and \( T_6 \) on line \( \beta_j \) that are linked by gluon exchange to the lines \( \beta_1 \) and \( \beta_k \), respectively. Combining Fig. 1b and c, we thus write

\[
M_b(\beta_1, \varepsilon) + M_c(\beta_1, \varepsilon) = M^{(A)}_{b+c}(\beta_1, \varepsilon) + M^{(S)}_{b+c}(\beta_1, \varepsilon). \tag{24}
\]
where the difference in color factors in square brackets produces a commutator for the color matrices on the $\beta_j$ line. Clearly, the integrals in parenthesis are identical and cancel. This result applies to arbitrary masses for the Wilson lines. As in the massless case, only the symmetric contribution of the double exchange diagrams survives and contributes to the soft matrix. As described in Ref. [11], however, the symmetric contribution to the soft matrix at two loops is already generated by the exponential of the one-loop anomalous dimension in Eq. (3).

The reasoning above has a significant phenomenological application to $2 \to 2$ production processes. Using Wick rotation, the vanishing of 3E diagrams after one-loop renormalization when pairs of invariants are equal applies to $2 \to 2$ processes involving the production of pairs of heavy quarks from light quarks or gluons. In particular, we consider Wilson line velocities corresponding to momentum configurations with $t = (p_1 - p_3)^2 = (p_1 - p_2)^2 = u$ in Eq. (1) with $n = 2$ [10]. Note that because it is trivial to reintroduce $\beta^2_j$-dependence, the result applies as well to the limit where one or more line becomes massless.

Of special interest are the anomalous dimension matrices that enter pair production: the $2 \times 2$ matrix for $q\bar{q} \to Q\bar{Q}$ and the $3 \times 3$ matrix for $gg \to Q\bar{Q}$. At one loop, these matrices are diagonal in their $s$-channel singlet-octet bases at $u = t$ [9]. Specifically, the off-diagonal elements of these $\Gamma_{S}^{(1)}$ are all proportional to $\ln(u/t)$. One- and two-loop diagrams contributing to these anomalous dimension matrices are illustrated by Fig. 3, where the single eikonal lines represent incoming light partons (quark pairs or gluons) and the double lines heavy quark pairs. At $u = t$, the off-diagonal zeros in $\Gamma_{S}^{(1)}$ directly reflect cancellation between pairs of diagrams such as those in Fig. 3a and b.

To illustrate the pattern of cancellation, we continue the notation above and write the amplitude, $M_D$ corresponding to eikonal diagram $D$ as the product of a color factor and a velocity-dependent factor,

$$M_D^{(n)} = C_D F_D^{(n)}(\beta_t).$$

(26)

In these terms, the vanishing of matrix elements that describe the mixing of color singlet-octet tensors in $\Gamma_{S}^{(1)}$ follows from identities relating the two diagrams,

$$F_{2a}^{(1)}(\beta_t) = F_{2b}^{(1)}(\beta_t),$$

(27)

$$C_{2a} = -C_{2b},$$

(28)

which are easily verified at one loop for $u = t$. We have shown above that at two loops with $u = t$, 3E diagrams like Fig. 3c vanish independently of their color structure, leaving only 2E diagrams and the color-symmetric parts of 3E exchange diagrams at two loops. This suggests that at two loops the same pattern of cancellation may persist for $\Gamma_{S}^{(2)}$. This is indeed the case, as we now explain.

At one loop, the expansion of Eq. (3) in terms of one-loop anomalous dimensions can be thought of as the sum of contributions from each gluon-exchange diagram. Expanding the one-loop form of Eq. (3) to order $\alpha_s^2$, we generate all color-symmetric contributions of two-loop ladder diagrams, corresponding for example to the color-symmetric contributions of Fig. 3d and e above. The sum of all 2E diagrams is thus easily rewritten in terms of the expansion of Eq. (3) in terms of $\Gamma_{S}^{(1)}$, plus two-loop contributions that involve commutators of generators on one or more of the Wilson lines.

Examples of diagrams with antisymmetric color structure, which do not correspond to the expansion of the one-loop anomalous dimensions, are Figs. 3f and 3i and
The diagrams contribute only their antisymmetric combinations of color generators to $\Gamma^{(2)}_S$, and are thus proportional to $C_A$ times a one-loop color factor. Figure 3 is a schematic representation of the color structure of such diagrams and of their contributions to the two-loop anomalous dimension matrices.

We note that exactly as for the massless case in Ref. [11], all 2E diagrams are generated from an exponential of “webs” [14]. Webs are themselves 2E diagrams that are two-particle irreducible under cuts of the Wilson lines, starting with single-gluon exchange. The contributions of two-loop webs are precisely the antisymmetric color structures that we have just identified.

As we have observed, Eq. (28) holds for the color factors of Fig. 3a and b. The same relation then holds for the web contributions of Fig. 3i and e, because they are proportional to the same color structure as the one-loop diagrams, as illustrated in Fig. 3f. Therefore, whenever the identity for velocity factors, Eq. (27), holds at two loops, this pair of two-loop diagrams cancels. The 2E diagrams of Fig. 3a and b, however, can depend only on the invariants formed from the two eikonal velocities in question, so that at $u = t$,

$$ F_{2a}^{(2)}(\beta_t) = F_{2b}^{(2)}(\beta_t) \quad (u = t) . $$

(29)

This relation will hold as well for diagrams associated with crossed ladders, as in Fig. 4, and other diagrams with color-antisymmetric contributions, including those with self-energies of the exchanged gluon and vertex corrections on the massive eikonal. As a result, cancellations between pairs of 2E diagrams that occur at one loop recur at two. For 2 → 2 kinematic configurations with $u = t$, then, we have in place of Eq. (4) the relation

$$ \Gamma^{(2)}_{S_t}(\beta_t) = D(\beta_t) \Gamma^{(1)}_{S_t}(\beta_t) \quad (u = t) , $$

(30)

where $D(\beta_t)$ is a matrix that is diagonal in the $s$-channel singlet-octet basis. For massless two-loop cases, the matrix $D(\beta_t)$ is proportional to the identity, and we recover a special case of Eq. (4). More generally, however, the integrals of the 2E webs depend on masses. We will give explicit results elsewhere. In any case, the relation (30) applies for scattering at ninety degrees in the center of mass, including the limiting case of production at rest ($s = 4m_Q^2$), of particular relevance to threshold resummation for the total cross section for heavy quark production [18]. A subset of these diagrams have been analyzed very recently in [10].

For massless particles, as noted in [11] it is certainly natural to generalize the result Eq. (4) for light partons to all orders. This possibility has been explored recently in Refs. [20]. The result that we have found here, that Eq. (4) does not apply to arbitrary massive kinematics, suggests that the zero-mass case is indeed special. We believe this should encourage the search for a symmetry or principle underlying the relation, Eq. (4).

In summary, we have shown that for products of massive Wilson lines, one- and two-loop soft anomalous dimensions are generally not proportional. We have noted, however, that diagrams that link two massless with one massive line through the three-gluon coupling vanish by a variant of the reasoning for three massless lines. We have also shown that diagrams that link a single massless line with two massive lines cancel for the case of two-to-two production processes at threshold, and when $u = t$, that is, for production at ninety degrees in the center of mass system. For these momentum configurations, the two-loop anomalous dimension matrix is diagonal in the same color basis as the one loop, although they are not related by a simple proportionality as in the massless case. The color-antisymmetric parts of exchange diagrams linking three eikonal lines cancel independently of masses. We carried out a numerical evaluation of the three-gluon diagram in Euclidean space, but the qualitative conclusions of this paper extend to Minkowski space. Whatever the field-theoretic origin of the results described above for massive and massless partons, the renormalization of the composite operators that link Wilson lines will be relevant to the analysis of light-parton jets, heavy quarks and potential new, strongly interacting particles at the Tevatron and the LHC.

Acknowledgments

This work was supported by the National Science Foundation, grants PHY-0354776, PHY-0354822 and PHY-0653342. The work of AM is supported by a fellowship from the US LHC Theory Initiative through NSF grant 0653342. We thank Michal Czakon, Lance Dixon, Einan Gardi and Lorenzo Magnea for helpful exchanges.
arXiv:hep-ph/0612149. T. Becher and K. Melnikov, JHEP **0706**, 084 (2007) [arXiv:0704.3582 [hep-ph]].

[8] R. A. Brandt, F. Neri and M. a. Sato, Phys. Rev. D **24**, 879 (1981).

[9] N. Kidonakis and G. Sterman, Nucl. Phys. B **505**, 321 (1997) [arXiv:hep-ph/9705234]; N. Kidonakis, G. Oderda and G. Sterman, Nucl. Phys. B **531** (1998) 365 [hep-ph/9803241].

[10] Yu. L. Dokshitzer and G. Marchesini, JHEP **0601**, 007 (2006) [arXiv:hep-ph/0509078]; M. Sjodahl, JHEP **0812**, 083 (2008) [arXiv:0807.0555 [hep-ph]]; M. H. Seymour and M. Sjodahl, JHEP **0812**, 066 (2008) [arXiv:0810.5756 [hep-ph]].

[11] S. Mert Aybat, L. J. Dixon and G. Sterman, Phys. Rev. Lett. **97**, 072001 (2006) [arXiv:hep-ph/0606254]; S. Mert Aybat, L. J. Dixon and G. Sterman, Phys. Rev. D **74**, 074004 (2006) [arXiv:hep-ph/0607309].

[12] G. P. Korchemsky and A. V. Radyushkin, Sov. J. Nucl. Phys. **44**, 877 (1986) [Yad. Fiz. **44**, 1351 (1986)]; G. P. Korchemsky and A. V. Radyushkin, Phys. Lett. B **171**, 459 (1986).

[13] J. C. Collins and D. E. Soper, Nucl. Phys. B **193**, 381 (1981) [Err.-ibid. B **213**, 545 (1983)]; A. Sen, Phys. Rev. D **24** (1981) 3281.

[14] L. Magnea and G. Sterman, Phys. Rev. D **42**, 4222 (1990).

[15] This observation has also been made by E. Gardi, using a similar representation in z-space. We thank E. Gardi for this private communication.

[16] We observe that the relations $s = -t$ or $s = -u$ do not correspond to equal invariants in the sense above.

[17] G. Sterman, in *AIP Conference Proceedings Tallahassee, Perturbative Quantum Chromodynamics*, eds. D. W. Duke, J. F. Owens, New York, 1981, p. 22; J. G. M. Gatheral, Phys. Lett. B **133**, 90 (1983); J. Frenkel and J. C. Taylor, Nucl. Phys. B **246**, 231 (1984); C. F. Berger, hep-ph/0305076.

[18] R. Bonciani, S. Catani, M. L. Mangano and P. Nason, Nucl. Phys. B **529**, 424 (1998) [Erratum-ibid. B **803**, 234 (2008)] [arXiv:hep-ph/9801375]; M. Cacciari, S. Frixione, M. L. Mangano, P. Nason and G. Ridolfi, JHEP **0809**, 127 (2008) [arXiv:0804.2800 [hep-ph]]; S. Moch and P. Uwer, Phys. Rev. D **78** (2008) 034003 [arXiv:0804.1476 [hep-ph]]; M. Czakon and A. Mitov, arXiv:0812.0353 [hep-ph]; N. Kidonakis and R. Vogt, Phys. Rev. D **78** (2008) 074005 [arXiv:0805.3844 [hep-ph]].

[19] N. Kidonakis, arXiv:0903.2561 [hep-ph].

[20] T. Becher and M. Neubert, arXiv:0901.0722 [hep-ph], arXiv:0903.1126 [hep-ph]; E. Gardi and L. Magnea, arXiv:0901.1091 [hep-ph].