FORCING THEORY FOR TRANSVERSE TRAJECTORIES OF SURFACE HOMEOMORPHISMS

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Abstract. This paper studies homeomorphisms of surfaces isotopic to the identity by means of purely topological methods and Brouwer theory. The main development is a novel theory of orbit forcing using maximal isotopies and transverse foliations. This allows us to derive new proofs for some known results as well as some new applications, among which we note the following: we extend Franks and Handel’s classification of zero entropy maps of $S^2$ for non-wandering homeomorphisms; we show that if $f$ is a Hamiltonian homeomorphism of the annulus, then the rotation set of $f$ is either a singleton or it contains zero in the interior, proving a conjecture posed by Boyland; we show that there exist compact convex sets of the plane that are not the rotation set of some torus homeomorphisms, proving a first case of the Franks-Misiurewicz Conjecture; we extend a bounded deviation result relative to the rotation set to the general case of torus homeomorphisms.

1. Introduction

The goal of this article is to give a new orbit forcing theory for surface homeomorphisms. A classical example of forcing result is Sharkovski’s theorem: there exists an explicit total order \( \preceq \) on the set of natural integers such that every continuous transformation $f$ on $[0, 1]$ that contains a periodic orbit of period $m$ contains a periodic orbit of period $n$ if $n \preceq m$. More precisely if $f$ admits a periodic orbit which period is not a power of $2$, one can construct a Markov partition and codes orbits with a sub-shift of finite type. In particular one can prove that the topological entropy of $f$ is positive.

Using Nielsen-Thurston classification of surface homeomorphisms, a forcing theory on periodic orbits can be constructed for surface homeomorphisms. In case of homeomorphisms isotopic to the identity, the theory deals with the braid types associated to periodic orbits. Many interesting articles has been written on the subject, see for example [B] or [Mo] for survey articles. A more subtle theory (homotopic Brouwer theory) was introduced by M. Handel and developed by J. Franks and Handel (using in particular Nielsen-Thurston classification result relative to the fixed point set of a surface diffeomorphism) which appeared to be a very efficient tool to prove general results in two-dimensional dynamics.

We will be interested in this paper with surface homeomorphisms isotopic to the identity. An important notion is the notion of maximal isotopy which means an isotopy $I = (f_t)_{t \in [0, 1]}$ from identity to $f$ such that there is no fixed point of $f$ whose trajectory (for $I$) is contractible relative to the fixed point set of $I$. Such an isotopy admits transverse foliations, which are singular oriented foliations $\mathcal{F}$ whose singular set coincides with the fixed point set of $I$ and such that the trajectory $I(z)$ of a point $z$ that is not fixed by $I$, is homotopic (relative to the ends) to a path that is transverse to the foliation (which means that it locally crosses every leaf from the right to the left). This path $I_{\mathcal{F}}(z)$, the transverse trajectory, is uniquely defined up to a natural equivalence relation (meaning that the induced path...
Then every invariant measure supported on the knowledge of a finite family $I$ results will deal about transverse trajectories. The basic question can be formulated in this way: from $I$ in the space of leaves is unique. By concatenation, one can define for every integer $n \geq 1$ the paths $I^n(z) = \prod_{0 \leq k < n} I(f^k(z))$ and $I^n_2(z) = \prod_{0 \leq k < n} I(f^k(z))$. One can also define the whole trajectory $I^n_z(z) = \prod_{k \in \mathbb{Z}} I(f^k(z))$ and the whole transverse trajectory $I^n_{\mathcal{F}}(z) = \prod_{k \in \mathbb{Z}} I(f^k(z))$. Our forcing results will deal about transverse trajectories. The basic question can be formulated in this way: from the knowledge of a finite family $I^n_1(z_i), 1 \leq i \leq p$, can we deduce the existence of other paths $I^n_2(z_i)$? The starting result in this direction is the following (Proposition 12): if two paths intersect transversally (the precise definition will be given later in the article) then one can construct two other paths $I^n_1+q(z_2)$ and $I^n_2+q(z_4)$ by a natural change of direction at the intersection point.

Maximal isotopies and transverse foliations were already known to be other efficient tools for the dynamical study of surface homeomorphisms but they were usually used in a dual way: studying the dynamics of the foliation by the knowledge of the dynamics of the homeomorphism. What is done in the present article is a direct way: studying the transverse trajectories $I^n_{\mathcal{F}}(z)$ and using this study to give general results on surface homeomorphisms. A point that can be noticed is the fact that the field of applications of our method is the whole space of surface homeomorphisms isotopic to the identity. Consequently, we will be able to generalize some results already known for diffeomorphisms. There is no doubt that they are many similarities with Franks-Handel methods. Looking more carefully at the links between the two methods should be a project of high interest.

Let us display now the main applications. Our original goal was to prove the following boundedness displacement result:

**Theorem A.** We suppose that $M$ is a compact orientable surface furnished with a Riemannian structure. We endow the universal covering space $\tilde{M}$ with the lifted structure and denote by $d$ the induced distance. Let $f$ be a homeomorphism of $M$ isotopic to the identity and $\tilde{f}$ a lift to $\tilde{M}$ naturally defined by the isometry. Assume that there exists an open topological disk $U \subset M$ such that the fixed points set of $\tilde{f}$ projects into $U$. Then;

- either there exists $K > 0$ such that $d(\tilde{f}^n(\tilde{z}), \tilde{z}) \leq K$, for all $n \geq 0$ and all recurrent point $\tilde{z}$ of $\tilde{f}$;

- or there exists a nontrivial covering automorphism $T$ and $q > 0$ such that, for all $r/s \in (-1/q, 1/q)$, the map $\tilde{f}^r \circ T^{-s}$ has a fixed point. In particular, $f$ has non-contractible periodic points of arbitrarily large prime period.

Our idea was using arguments introduced by P. Dávalos in [D1], [D2] where leaves of a transverse foliation are pushed along the dynamics. Trying to conceptualize his arguments has been our first step and the result has been Proposition 12 mentioned above. With arguments close to the ones appearing in the proof of Theorem A we have been able to get the following results about annulus homeomorphisms. Here, $\mathcal{M}(f)$ is the set of invariant Borel probability measures $\mu$ of $f$, the set supp($\mu$) the support of $\mu$, the rotation number $\rho(\mu)$ the integral $\int_\mathcal{A} \varphi \, d\mu$, where $\varphi : \mathcal{A} \to \mathbb{R}$ is the map lifted by $\pi_1 \circ \tilde{f} - \pi_1$ (the map $\pi_1 : (x, y) \mapsto x$ being the first projection), the segment $\text{rot}(\tilde{f})$ the set of rotation numbers of invariant measures.

**Theorem B.** Let $f$ be a homeomorphism of $\mathcal{A} = \mathbb{T}^1 \times [0, 1]$ that is isotopic to the identity and $\tilde{f}$ a lift to $\mathbb{R} \times [0, 1]$. Suppose that $\text{rot}(\tilde{f})$ is a non trivial segment and that $\rho$ is an end of $\text{rot}(\tilde{f})$ that is rational. Define

$$\mathcal{M}_\rho = \{\mu \in \mathcal{M}(f), \ \text{rot}(\mu) = \rho\}, \ \ X_\rho = \bigcup_{\mu \in \mathcal{M}_\rho} \text{supp}(\mu).$$

Then every invariant measure supported on $X_\rho$ belongs to $\mathcal{M}_\rho$. 
We deduce immediately the following positive answer to a question of P. Boyland:

**Corollary C.** Let $f$ be a homeomorphism of $A$ that is isotopic to the identity and preserves a probability measure $\mu$ with full support. Let us fix a lift $\tilde{f}$. Suppose that $\text{rot}(\tilde{f})$ is a non trivial segment. The rotation number $\text{rot}(\mu)$ cannot be an end of $\text{rot}(\tilde{f})$ if this end is rational.

The method of studying transverse trajectories permits us to get other results for torus homeomorphisms. Here again $\mathcal{M}(f)$ is the set of invariant Borel probability measures $\mu$ of $f$, the set $\text{supp}(\mu)$ the support of $\mu$ and the rotation vector $\rho(\mu)$ the integral $\int_{\mathbb{T}^2} \varphi \, d\mu$, where $\varphi : \mathbb{T}^2 \to \mathbb{R}^2$ is the map lifted by $\tilde{f} - \text{Id}$. The set of rotation vectors of invariant measures $\text{rot}(\tilde{f})$ is a compact and convex subset of $\mathbb{R}^2$.

Nothing is known about the plane subsets that can be written as such a rotation set. The following result gives the first obstruction:

**Theorem D.** Let $f$ be a homeomorphism of $\mathbb{T}^2$ that is isotopic to the identity and $\tilde{f}$ a lift of $f$ to $\mathbb{R}^2$. The frontier of $\text{rot}(\tilde{f})$ does not contain a segment with irrational slope that contains a rational point in its interior.

It was previously conjectured by Franks and Misiurewicz in [FM] that a line segment $L$ could not be realized as a rotation set of a torus homeomorphism in the following conditions: (i) $L$ has irrational slope and a rational point in its interior, (ii) $L$ has rational slope but no rational points and (iii) $L$ has irrational slope and no rational points. While Theorem D implies the conjecture for case (i), A. Ávila has announced a counter-example for case (iii).

The second result is a boundedness result:

**Theorem E.** Let $f$ be a homeomorphism of $\mathbb{T}^2$ that is isotopic to the identity and $\tilde{f}$ a lift of $f$ to $\mathbb{R}^2$. If $\text{rot}(\tilde{f})$ has a non empty interior, then there exist a constant $L$ such that for every $z \in \mathbb{R}^2$ and every $n \geq 1$, one has $d(\tilde{f}^n(z) - z, n\text{rot}(\tilde{f})) \leq L$.

Note that by definition of the rotation set one knows that

$$\lim_{n \to +\infty} \frac{1}{n} \left( \max_{z \in \mathbb{R}^2} d(\tilde{f}^n(z) - z, n\text{rot}(\tilde{f})) \right) = 0$$

Theorem E clarifies the speed of convergence. It was already known for homeomorphisms in the special case of a polygon with rational vertices (see Dávalos [D2]) and for $C^1+\varepsilon$ diffeomorphisms (see Addas-Zanata [AZ]). As already noted in [AZ], we can deduce an interesting result about maximizing measures, which means measure $\mu \in \mathcal{M}(f)$ whose rotation vector belongs to the frontier of $\text{rot}(\tilde{f})$. The rotation number of such a measure belongs to at least one supporting line of $\text{rot}(\tilde{f})$. Such a line admits the equation $\psi(z) = \alpha(\psi)$ where $\psi$ is a non trivial linear form on $\mathbb{R}^2$ and

$$\alpha(\psi) = \max_{\mu \in \mathcal{M}(f)} \psi(\text{rot}(\mu)) = \max_{\mu \in \mathcal{M}(f)} \int_{\mathbb{T}^2} \psi \circ \varphi \, d\mu.$$

Set

$$\mathcal{M}_\psi = \{\mu \in \mathcal{M}(f) \, | \, \psi(\text{rot}(\mu)) = \alpha(\psi)\}, \quad X_\psi = \bigcup_{\mu \in \mathcal{M}_\psi} \text{supp}(\mu).$$

The following result, that can be easily deduced from Theorem 49 and Atkinson’s Lemma in Ergodic Theory (see [A]), tells us that the sets $X_\psi$ behave like the Mather sets of the Tonelli Lagrangian systems.
Proposition F. Let $f$ be a homeomorphism of $\mathbb{T}^2$ that is isotopic to the identity and $\tilde{f}$ a lift of $f$ to $\mathbb{R}^2$. Assume that $\text{rot}(\tilde{f})$ has a non empty interior. Then, every measure $\mu$ supported on $X_\psi$ belongs to $\mathcal{M}_\psi$. Moreover, if $z$ lifts a point of $X_\psi$, then for every $n \geq 1$, one has $|\psi(\tilde{f}^n(z)) - \psi(z) - n\beta(\psi)| \leq L\|\psi\|$, where $L$ is the constant given by Theorem E.

It admits as an immediate corollary the torus version of Boyland’s question:

Corollary G. Let $f$ be a homeomorphism of $\mathbb{T}^2$ that is isotopic to the identity, preserving a measure $\mu$ of full support, and $\tilde{f}$ a lift of $f$ to $\mathbb{R}^2$. Assume that $\text{rot}(\tilde{f})$ has a non empty interior. Then $\text{rot}(\mu)$ belongs to the interior of $\text{rot}(\tilde{f})$.

Theorem A has also another interesting consequence for torus homeomorphisms. Say a homeomorphism $f$ of $\mathbb{T}^2$ is Hamiltonian if it preserves a measure $\mu$ with full support and it has a lift $\tilde{f}$ (called the Hamiltonian lift of $f$) such that the rotation vector of $\mu$ is null.

Corollary H. Let $f$ be a Hamiltonian homeomorphism of $\mathbb{T}^2$ such that all its periodic points are contractible, and such that it fixed point set is contained in a topological disk. Then there exists $K > 0$ such that if $\tilde{f}$ is the Hamiltonian lift of $f$, then for every $z$ and every $n \geq 1$, one has $\|\tilde{f}^n(z) - z\| \leq K$.

The next result is due to Llibre and MacKay, see [LIM]. Its original proof uses Thurston-Nielsen theory of surface homeomorphisms, more precisely the authors prove that there exists a finite invariant set $X$ such that $f|_{\mathbb{T}^2\setminus X}$ is isotopic to a pseudo-Anosov map. We will give here an alternative proof by exhibiting $(n, \varepsilon)$ separated sets constructed with the help of transverse trajectories.

Theorem I. Let $f$ be a homeomorphism of $\mathbb{T}^2$ that is isotopic to the identity and $\tilde{f}$ a lift of $f$ to $\mathbb{R}^2$. If $\text{rot}(\tilde{f})$ has a non empty interior, then the topological entropy of $f$ is positive.

While writing our article we observed that our theory fits very well with other kinds of problems. For example we could give a short proof of the following slight improvement of a result due to Handel [H].

Theorem J. Let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be an orientation preserving homeomorphism such that the complement of the fixed point set is not an annulus. If $f$ is topologically transitive then the number of periodic points of period $n$ for some iterate of $f$ grows exponentially in $n$. Moreover, the entropy of $f$ is positive.

Let us finish with a last application. J. Franks and M. Handel recently gave a classification result for area preserving diffeomorphisms of $\mathbb{S}^2$ with entropy 0 (see [FH]). Their proofs are purely topological but the $C^1$ assumption is needed to use a Thurston-Nielsen type classification result relative to the fixed point set (existence of a normal form). We will give a new proof of the fundamental decomposition result (Theorem 1.2 of [FH]) which is the main building block in their structure theorem. In fact we will extend their result to the case of homeomorphisms and replace the area preserving assumption by the fact that every point is non wandering.

Theorem K. Let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be an orientation preserving homeomorphism such that $\Omega(f) = \mathbb{S}^2$ and $h(f) = 0$. Then there exists a family of pairwise disjoint invariant open sets $(A_\alpha)_{\alpha \in \mathcal{A}}$ whose union is dense such that:

i) each $A_\alpha$ is an open annulus;

ii) the sets $A_\alpha$ are the maximal fixed point free invariant open annuli;
iii) the α-limit and ω-limit sets of a point $z \not\in \bigcup_{\alpha \in A} A_{\alpha}$ belong to a connected component of the fixed point set $\text{fix}(f)$ of $f$;

iv) let $C$ be a connected component of the frontier of $A_{\alpha}$ in $\mathbb{S}^2 \setminus \text{fix}(f)$, then the connected components of $\text{fix}(f)$ that contain $\alpha(z)$ and $\omega(z)$ are independent of $z \in C$.

Let us explain now the plan of the article. In the first section we will introduce the definitions of many mathematical objects, including precise definitions of rotation vectors and rotation sets. The second section will be devoted to the study of transverse paths to a surface foliation. We will introduce the notion of a pair of equivalent paths, of a recurrent transverse path and of transverse intersection between two transverse paths. An important result, which will be very useful in the proofs of Theorems 3 and 4 is Proposition 2 which asserts that a transverse recurrent path to a singular foliation on $\mathbb{S}^2$ that has no transverse self-intersection is equivalent to the natural lift of a transverse simple loop (i.e. an adapted version of Poincaré-Bendixson theorem). We will recall the definition of maximal isotopies, transverse foliations and transverse trajectories in Section 3. We will state the fundamental result about transverse intersections of transverse trajectories (Proposition 12) and its immediate consequences. An important notion that will be introduced is the notion of linearly admissible transverse loop. To any transverse intersections of transverse trajectories (Proposition 12) and its immediate consequences. An important notion that will be introduced is the notion of linearly admissible transverse loop. To any periodic orbit is naturally associated such a loop. A realization result (Proposition 18) will give us sufficient conditions for a linearly admissible transverse loop to be associated to a periodic orbit. Section 4 will be devoted to the proofs of two important results: suppose that there exist two recurrent points of $f$ (possibly equal) whose whole transverse trajectories have a transverse intersection, Theorem 21 asserts that the number of periodic points of period $n$ for some iterate of $f$ grows exponentially in $n$, Theorem 23 asserts that the entropy of $f$ is positive. We will give the proofs of Theorem A, B and C in Section 5 while Section 6 will be entirely devoted to the proof of Theorem K. We will begin by stating a “local version” relative to a given maximal isotopy (Theorem 31). We will study torus homeomorphisms in Section 7 and will give there the proofs of Theorems 3, 5 and 7.

2. Notations

We will endow $\mathbb{R}^2$ with its usual scalar product $\langle \ , \ \rangle$ and its usual orientation. We will write $\| \ |$ for the associated norm. For every point $z \in \mathbb{R}^2$ and every set $X \subset \mathbb{R}^2$ we write $d(z,X) = \inf_{z' \in X} \| z - z' \|$. We denote by $\pi_1 : (x, y) \mapsto x$ and $\pi_2 : (x, y) \mapsto y$ the two projections. If $z = (x, y)$, we write $z^+ = (-y, x)$. The $r$-dimensional torus $\mathbb{T}^r$ will be denoted $\mathbb{T}^r$.

2.1. Paths, lines, loops. A path on a surface $M$ is a continuous map $\gamma : J \to M$ defined on an interval $J \subset \mathbb{R}$. In absence of ambiguity its image will also be called a path and denoted by $\gamma$. We will write $\gamma^{-1} : -J \to M$ the path defined by $\gamma^{-1}(t) = \gamma(-t)$. When saying that $\gamma$ is proper we implicitly will suppose that $J$ is open. A line is a injective and proper path $\lambda : J \to M$ defined on an open interval. If $M = \mathbb{R}^2$, the complement of a line $\lambda$ has two connected components, $R(\lambda)$ which is on the right of $\lambda$ and $L(\lambda)$ which is on its left. One can define naturally such sets when $M$ is a non connected surface with connected components homeomorphic to $\mathbb{R}^2$.

Let us suppose that $\lambda_0$ and $\lambda_1$ are two disjoint lines of $\mathbb{R}^2$. We will say that they are comparable if their right components are comparable for the inclusion. Note that $\lambda_0$ and $\lambda_1$ are not comparable if and only if $\lambda_0$ and $(\lambda_1)^{-1}$ are comparable.
Let us consider three lines $\lambda_0$, $\lambda_1$, $\lambda_2$ in $\mathbb{R}^2$. We will say that $\lambda_2$ is above $\lambda_1$ relatively to $\lambda_0$ (and $\lambda_1$ is below $\lambda_2$ relatively to $\lambda_0$) if:

- the three lines are pairwise disjoint;
- none of the lines separates the two others;
- if $\gamma_1$, $\gamma_2$ are two disjoints paths joining $z_1 = \lambda_0(t_1)$, $z_2 = \lambda_0(t_2)$ to $z'_1 \in \lambda_1$, $z'_2 = \lambda_2$ respectively, that do not meet the three lines but at the ends, then $t_2 > t_1$.

This notion does not depend on the orientation of $\lambda_1$ and $\lambda_2$ but depends of the orientation of $\lambda_0$. If $\lambda_0$ is fixed, note that we get in that way an anti-symmetric and transitive relation on every set of pairwise disjoint lines that are disjoint from $\lambda_0$.

A proper path $\gamma$ of $\mathbb{R}^2$ defines a dual function $\delta$ on its complement (defined up to an additive constant): for every $z$ and $z'$ in $\mathbb{R}^2 \setminus \gamma$, the difference $\delta(z') - \delta(z)$ is the algebraic intersection number $\gamma \wedge \gamma'$ where $\gamma'$ is any path from $z$ to $z'$. If $\gamma$ is a line, one can choose $\delta = \delta_\gamma$ to take the value $0$ on $R(\gamma)$ and $1$ on $L(\gamma)$.

Consider a unit vector $\rho \in \mathbb{R}^2$, $\|\rho\| = 1$. Say that a proper path $\gamma : \mathbb{R} \to \mathbb{R}^2$ is directed by $\rho$ if

- $\lim_{t \to \pm \infty} \|\gamma(t)\| = +\infty$;
- $\lim_{t \to \pm \infty} \gamma(t)/\|\gamma(t)\| = \pm \rho$.

Observe that if $\gamma$ is directed by $\rho$, then $\gamma^{-1}$ is directed by $-\rho$ and that for every $z \in \mathbb{R}^2$, the translated path $\gamma + z : t \mapsto \gamma(t) + z$ is directed by $\rho$. Among the connected components of $\mathbb{R}^2 \setminus \gamma$, two of them $R(\gamma)$ and $L(\gamma)$ are uniquely determined by the following: for every $z \in \mathbb{R}^2$, one has $z - s\rho^\perp \in R(\gamma)$ and $z + s\rho^\perp \in L(\gamma)$ if $s$ is large enough. In the case where $\gamma$ is a line, the definitions agree with the former ones. Note that two disjoint lines directed by $\rho$ are comparable.

Instead of looking at paths defined on a real interval we can look at paths defined on an abstract interval $J$ (which means an oriented one dimensional manifold homeomorphic to a real interval). If $\gamma : J \to M$ and $\gamma' : J' \to M$ are two paths, if $J$ has a right end $b$ (in the natural sense), if $J'$ has a left end $a'$ and if $\gamma(b) = \gamma'(a')$ we can concatenate the two paths and define the path $\gamma \gamma'$ defined on the interval $J'' = J \cup J'/b \sim a'$ coinciding with $\gamma$ on $J$ and $\gamma'$ on $J'$. One can define in a same way the concatenation $\prod_{t \in I} \gamma_t$ of paths indexed by an interval (finite or infinite) of $\mathbb{Z}$.

A path $\gamma : \mathbb{R} \to M$ such that $\gamma(t+1) = \gamma(t)$ for every $t \in \mathbb{R}$ lifts a continuous map $\Gamma : \mathbb{T}^1 \to M$. We will say that $\Gamma$ is a loop and $\gamma$ its natural lift. If $n \geq 1$, we denote $\Gamma^n$ the loop lifted by the path $t \mapsto \gamma(nt)$. Here again, if $M$ is oriented and $\Gamma$ homologous to zero, one can define a dual function $\delta$ defined up to an additive constant on $M \setminus \Gamma$: for every $z$ and $z'$ in $\mathbb{R}^2 \setminus \Gamma$, the difference $\delta(z') - \delta(z)$ is the algebraic intersection number $\Gamma \wedge \gamma'$ where $\gamma'$ is any path from $z$ to $z'$.

2.2. Rotations vectors. Let us recall the notion of rotation vector and rotation set for a homeomorphism of a closed manifold, introduced by Schwartzman [Sc] (see also [P]). Let $M$ be an oriented closed connected manifold and $I$ an identity isotopy on $M$, which means an isotopy $(f_t)_{t \in [0,1]}$ such that $f_0$ is the identity. If $\omega$ is a closed form on $M$, one can define the integral $\int_{I(t)} \omega$ on every trajectory $I(z) : z \mapsto f_t(z)$. Write $f_1 = f$ and denote $\mathcal{M}(f)$ the set of invariant Borel probability measures.
For every $\mu \in \mathcal{M}(f)$, the integral $\int_{\mathcal{M}} \left( \int_{f(z)} \omega \right) d\mu(z)$ vanishes when $\omega$ is exact. One deduces that $\omega \to \int_{\mathcal{M}} \left( \int_{f(z)} \omega \right) d\mu(z)$ defines a natural linear form on the first group of cohomology $H^1(M, \mathbb{R})$, and by duality, an element of the first group of homology $H_1(M, \mathbb{R})$, which is called the rotation vector of $\mu$. The set $\mathcal{M}(f)$ munished with the weak* topology being convex and compact and the map $\mu \mapsto \text{rot}(\mu)$ being affine, one deduces that the set $\text{rot}(I) = \{\text{rot}(\mu), \mu \in \mathcal{M}(f)\}$ is a convex compact subset of $H_1(M, \mathbb{R})$. If $M$ is a surface of genus greater than 1, rotation vectors (and the rotation set) are independent of the isotopy, depending only on $f$. If $M$ is a torus, it depends on a given lift of $f$. Let us clarify this case (see [MZ]). Let $f$ be a homeomorphism of $T^2$ that is isotopic to the identity and $\rho$ of an ergodic measure. Indeed the set of Borel probability measures of rotation vector $\alpha$ for the positive half-leaf and $\phi_+^\alpha$ for the negative one. One can define the $\omega$-limit set $\omega(\phi) = \bigcap_{t \in \phi} \overline{\phi}\phi z$ and the $\alpha$-limit set $\phi(\phi) = \bigcap_{t \in \alpha} \overline{\phi}\phi z$. Suppose that a point $z \in \phi$ has a trivialization
neighborhood such that each leaf of $\mathcal{F}$ contains no more than one local leaf, in which case every point of $\phi$ satisfies the same property. If furthermore $\phi$ is not closed, we will say that $\phi$ is wandering. Recall the following facts, in the case where $M = \mathbb{R}^2$ and $\mathcal{F}$ is non singular (see [HR]):

- every leaf of $\mathcal{F}$ is a wandering line;
- the space of leaves $\Sigma$, munished with the quotient topology, inherits a structure of connected and simply connected one-dimensional manifold;
- $\Sigma$ is Hausdorff if and only if $\mathcal{F}$ is trivial (which means that it is the image of the vertical foliation by a plane homeomorphism) or equivalency if all the leaves are comparable.

A path $\gamma : J \to M$ is positively transverse\footnote{in the whole text “transverse” will mean “positively transverse”} to $\mathcal{F}$, if its image does not meet the singular set and if for every $t_0 \in J$, there exists a (continuous) chart $h : W \to (0, 1)^2$ at $\gamma(t_0)$ compatible with the orientation and sending the restricted foliation $\mathcal{F}_W$ onto the vertical foliation oriented downward such that the map $\pi_1 \circ h \circ \gamma$ is increasing in a neighborhood of $t_0$. If $\gamma : J \to \text{dom}(\mathcal{F})$ is positively transverse to $\mathcal{F}$, every lift $\tilde{\gamma} : J \to \tilde{\text{dom}}(\mathcal{F})$ to the universal covering space of $\text{dom}(\mathcal{F})$ is transverse to the lifted (non singular) foliation $\tilde{\mathcal{F}}$.

Suppose first that $M = \mathbb{R}^2$ and that $\mathcal{F}$ is non singular. Say that two transverse paths $\gamma : J \to \mathbb{R}^2$ and $\gamma' : J' \to \mathbb{R}^2$ are equivalent for $\mathcal{F}$ if they satisfy the three following equivalent conditions:

- there exists an increasing homeomorphism $h : J \to J'$ such that, for every $t \in J$, one has $\phi_{\gamma'(h(t))} = \phi_{\gamma(t)}$;
- the paths $\gamma$ and $\gamma'$ meet the same leaves;
- the paths $\gamma$ and $\gamma'$ define the same path in the space $\Sigma$.

Moreover, if $J = [a, b]$ and $J' = [a', b']$ are two segments, these conditions are equivalent to this last one:

- one has $\phi_{\gamma(a)} = \phi_{\gamma'(a')}$ and $\phi_{\gamma(b)} = \phi_{\gamma'(b')}$,

and in that case, the leaves met by $\gamma$ are the leaves $\phi$ such that

$$R(\phi_{\gamma(a)}) \subset R(\phi) \subset R(\phi_{\gamma(b)}).$$

If the context is clear, we just say that the paths are equivalent and omit the dependence on $\mathcal{F}$.

Let us say that a transverse path $\gamma : J \to \mathbb{R}^2$ has a leaf on its right if, there exists $a < b$ in $J$ and a leaf $\phi$ in the plane $L(\phi_{\gamma(a)}) \cap R(\phi_{\gamma(b)})$ that lies in the left of $\gamma|_{(a,b)}$. Similarly, one can define the notion of having a leaf on its left.

All previous definitions can be naturally extended in case every connected component of $M$ is a plane and $\mathcal{F}$ is not singular. Let us return to the general case. Two transverse paths $\gamma : J \to \text{dom}(\mathcal{F})$ and $\gamma' : J' \to \text{dom}(\mathcal{F})$ are equivalent for $\mathcal{F}$ if they can be lifted to the universal covering space of the domain $\tilde{\text{dom}}(\mathcal{F})$ as paths that are equivalent for the lifted foliation $\tilde{\mathcal{F}}$. A transverse path has a leaf on its right if it can be lifted to $\tilde{\text{dom}}(\mathcal{F})$ as a path with a leaf of $\tilde{\mathcal{F}}$ on its right (in that case every lift has a leaf on its right). Here again, one can define the notion of having a leaf on its left. Note that if $\gamma$ and $\gamma'$ have no leaf on their right and $\gamma\gamma'$ is well defined, then $\gamma\gamma'$ has no leaf on its right.
Similarly, a loop $\Gamma : T^{1} \to \text{dom}(F)$ is called \textit{positively transverse} to $F$ if it is the case for its natural lift $\gamma : \Bbb{R} \to \text{dom}(F)$. It has a leaf on its right or its left if it is the case for $\gamma$. Two transverse loops $\Gamma : T^{1} \to \text{dom}(F)$ and $\Gamma' : T^{1} \to \text{dom}(F)$ are \textit{equivalent} if there exists two lifts $\tilde{\gamma} : \Bbb{R} \to \text{dom}(F)$ and $\tilde{\gamma}' : \Bbb{R} \to \text{dom}(F)$ of $\Gamma$ and $\Gamma'$ respectively, a covering automorphism $T$ and an orientation preserving homeomorphism $h : \Bbb{R} \to \Bbb{R}$, such that, for every $t \in \Bbb{R}$, one has

$$\phi_{\tilde{\gamma}'(h(t))} = \phi_{\tilde{\gamma}(t)}, \quad \tilde{\gamma}(t + 1) = T(\tilde{\gamma}(t)), \quad \tilde{\gamma}'(t + 1) = T(\tilde{\gamma}'(t)), \quad h(t + 1) = h(t) + 1.$$  

Of course $\Gamma^n$ and $\Gamma'^n$ are equivalent transverse loops, for every $n \geq 1$, if it is the case for $\Gamma$ and $\Gamma'$. A transverse loop $\Gamma$ will be called \textit{prime} if it is not equivalent to a loop $\Gamma^n$, $n \geq 2$. A transverse path $\gamma : \Bbb{R} \to M$ will be said to be \textit{positively recurrent} if for every segment $J \subset \Bbb{R}$ and every $t \in \Bbb{R}$ there exists a segment $J' \subset [t, +\infty)$ such that $\gamma|_{J'}$ is equivalent to $\gamma|_J$. It is \textit{negatively recurrent} if for every segment $J \subset \Bbb{R}$ and every $t \in \Bbb{R}$ there exists a segment $J' \subset (-\infty, t]$ such that $\gamma|_{J'}$ is equivalent to $\gamma|_J$. It is \textit{recurrent} if it is both positively and negatively recurrent.

We will use very often the following remarks. Suppose that $\Gamma$ is a transverse loop homologous to zero and $\delta$ a dual function, then $\delta$ decreases along each leaf with a jump at every intersection point. One deduces that every leaf met by $\Gamma$ is wandering. In particular, $\Gamma$ does not meet any set $\alpha(\phi)$ or $\omega(\phi)$, which implies that for every leaf $\phi$, there exist $z_-$ and $z_+$ on $\phi$ such that $\Gamma$ does not meet neither $\phi_{z_-}$ nor $\phi_{z_+}$. Writing $n_+$ and $n_-$ for the value taken by $\delta$ on $\phi_{z_-}$ and $\phi_{z_+}$ respectively, one deduces that $n_+ - n_-$ is the number of times that $\Gamma$ intersect $\phi$. Note that $n_+ - n_-$ is uniformly bounded. Indeed, the fact that every leaf that meets $\Gamma$ is wandering implies that $T^{1}$ can be covered by open intervals where $\Gamma$ is injective and does not meet any leaf more than once. By compactness $T^{1}$ can be covered by finitely many such intervals, which implies that there exists $N$ such that $\Gamma$ meets each leaf at most $N$ times. We have a same result for a multi-loop $\Gamma = \sum_{1 \leq i \leq p} \Gamma_i$ homologous to zero. In case where $M = \Bbb{R}^2$, we have the same results for a proper transverse path of $\Bbb{R}^2$ with finite valued dual function.

In case of an infinite valued dual function, everything is true but the finiteness condition about intersection with a given leaf. In particular a transverse line $\lambda$ meets every leaf at most once (because the dual function takes only two values) and one can define the sets $r(\gamma)$ and $l(\gamma)$, union of leaves included in $R(\lambda)$ and $L(\lambda)$ respectively. They do not depend on the choice of $\lambda$ in the equivalent class. Note that if the leaves of $F$ are uniformly bounded, every path equivalent to $\lambda$ is still a line. We have similar results for directed proper paths. If $\gamma$ is a proper path directed by a unit vector $\rho$, one can define the sets $r(\gamma)$ and $l(\gamma)$, union of leaves included in $R(\gamma)$ and $L(\gamma)$ respectively. They do not depend on the choice of $\gamma$ in the equivalent class. Moreover, if the leaves of $F$ are uniformly bounded, every path equivalent to $\gamma$ is still a path directed by $\rho$.

### 3.1. Non singular plane foliations

Let us suppose first that $M = \Bbb{R}^2$ and that $F$ is non singular.

Let $\gamma_1 : J_1 \to \Bbb{R}^2$ and $\gamma_2 : J_2 \to \Bbb{R}^2$ be two transverse paths. The set

$$X = \{(t_1, t_2) \in J_1 \times J_2 \mid \phi_{\gamma_1(t_1)} = \phi_{\gamma_2(t_2)}\},$$

if not empty, is an interval that projects injectively on $J_1$ and $J_2$ as does its closure. Moreover, for every $(t_1, t_2) \in \overline{X} \setminus X$, the leaves $\phi_{\gamma_1(t_1)}$ and $\phi_{\gamma_2(t_2)}$ are not separated in $\Sigma$. To be more precise, suppose that $J_1$ and $J_2$ are real intervals and that $\phi_{\gamma_1(t_1)} = \phi_{\gamma_2(t_2)}$. Set $J^-_1 = J_1 \cap (-\infty, t_1]$ and
As explained above, these properties remain true when \( J_1 \) and \( J_2 \) are replaced by smaller parameters, or there exist \( a_1 < t_1 \) and \( a_2 < t_2 \) such that:

- \( \gamma_1|_{(a_1,t_1)} \) and \( \gamma_2|_{(a_2,t_2)} \) are equivalent;
- \( \phi_{\gamma_1(a_1)} \subset L(\phi_{\gamma_2(a_2)}) \), \( \phi_{\gamma_2(a_2)} \subset L(\phi_{\gamma_1(a_1)}) \)
- \( \phi_{\gamma_1(a_1)} \) and \( \phi_{\gamma_2(a_2)} \) are not separated in \( \Sigma \).

Observe that the second property (but not the two other ones) is still satisfied when \( a_1, a_2 \) are replaced by smaller parameters. Note also that \( \phi_{\gamma_2(a_2)} \) is either above or below \( \phi_{\gamma_1(a_1)} \) relatively to \( \phi_{\gamma_1(t_1)} \) and that this property remains satisfied when \( a_1, a_2 \) are replaced by smaller parameters and \( t_1 \) by any parameter in \( (a_1,t_1) \). We have a similar situation on the eventual right end of \( X \).

Let \( \gamma_1 : J_1 \to \mathbb{R}^2 \) and \( \gamma_2 : J_2 \to \mathbb{R}^2 \) be two transverse paths such that \( \phi_{\gamma_1(t_1)} = \phi_{\gamma_2(t_2)} = \phi \). We will say that \( \gamma_1 \) and \( \gamma_2 \) \textit{intersect transversally and positively at} \( \phi \) (and \( \gamma_2 \) and \( \gamma_1 \) \textit{intersect transversally and negatively at} \( \phi \)) if there exist \( a_1, b_1 \) in \( J_1 \) satisfying \( a_1 < t_1 < b_1 \), and \( a_2, b_2 \) in \( J_2 \) satisfying \( a_2 < t_2 < b_2 \), such that:

- \( \phi_{\gamma_2(a_2)} \) is below \( \phi_{\gamma_1(a_1)} \) relatively to \( \phi \);
- \( \phi_{\gamma_2(b_2)} \) is above \( \phi_{\gamma_1(b_1)} \) relatively to \( \phi \).

See Figure 1.

As none of the leaves \( \phi, \phi_{\gamma_1(a_1)}, \phi_{\gamma_2(a_2)} \) separates the two others, one deduces that

\[ \phi_{\gamma_1(a_1)} \subset L(\phi_{\gamma_2(a_2)}), \quad \phi_{\gamma_2(a_2)} \subset L(\phi_{\gamma_1(a_1)}) \]

and similarly that

\[ \phi_{\gamma_1(b_1)} \subset R(\phi_{\gamma_2(b_2)}), \quad \phi_{\gamma_2(b_2)} \subset R(\phi_{\gamma_1(b_1)}). \]

**Figure 1.** Transversal paths. The tangency point is also a point of transversal intersection.

As explained above, these properties remain true when \( a_1, a_2 \) are replaced by smaller parameters, \( b_1, b_2 \) by larger parameters and \( \phi \) by any other leaf met by \( \gamma_1 \) and \( \gamma_2 \). Note that \( \gamma_1 \) and \( \gamma_2 \) have at least one intersection point and that one can find two transverse paths \( \gamma'_1, \gamma'_2 \) equivalent to \( \gamma_1, \gamma_2 \) respectively, such that \( \gamma'_1 \) and \( \gamma'_2 \) have a unique intersection point, located on \( \phi \), with a topologically transverse intersection. Note also that if \( \gamma_1 \) and \( \gamma_2 \) do not intersect transversally, one can find two transverse paths \( \gamma'_1, \gamma'_2 \) equivalent to \( \gamma_1, \gamma_2 \), respectively, with no intersection point.
3.2. Transverse intersection in the general case. Here again, the notion of transverse intersection can be naturally extended in case every connected component of \( M \) is a plane and \( \mathcal{F} \) is not singular. Let us return now to the general case of a singular foliation \( \mathcal{F} \) on a surface \( M \). Let \( \gamma_1 : J_1 \to M \) and \( \gamma_2 : J_2 \to M \) be two transverse paths that meet a common leaf \( \phi = \phi_{\gamma_1(t_1)} = \phi_{\gamma_2(t_2)} \). We will say that \( \gamma_1 \) and \( \gamma_2 \) intersect transversally at \( \phi \) if there exist paths \( \tilde{\gamma}_1 : J_1 \to \text{dom}(\mathcal{F}) \) and \( \tilde{\gamma}_2 : J_2 \to \text{dom}(\mathcal{F}) \), lifting \( \gamma_1 \) and \( \gamma_2 \), with a common leaf \( \tilde{\phi} = \phi_{\tilde{\gamma}_1(t_1)} = \phi_{\tilde{\gamma}_2(t_2)} \) that lifts \( \phi \), and intersecting transversally at \( \tilde{\phi} \). If \( \tilde{\phi} \) is closed the choices of \( \tilde{\gamma}_1 \) and \( \tilde{\gamma}_2 \) do not need to be unique. Here again, we can give a sign to the intersection. Note that there exist \( t_1' \) and \( t_2' \) such that \( \tilde{\gamma}_1(t_1') = \tilde{\gamma}_2(t_2') \): we will say that \( \gamma_1 \) and \( \gamma_2 \) intersect transversally at \( \gamma_1(t_1') = \gamma_2(t_2') \). In the case where \( \gamma_1 = \gamma_2 \) we will talk of self-intersection. A transverse path \( \gamma \) has a self-intersection if for every lift \( \tilde{\gamma} \) to the universal covering space of the domain, there exists a non trivial covering automorphism \( T \) such that \( \tilde{\gamma} \) and \( T(\tilde{\gamma}) \) have a transverse intersection.

Similarly, let \( \Gamma \) be a loop positively transverse to \( \mathcal{F} \) and \( \gamma \) its natural lift. If \( \gamma \) intersects transversally a transverse path \( \gamma' \) at a leaf \( \phi \), we will say that \( \Gamma \) and \( \gamma' \) intersect transversally at \( \phi \). Moreover if \( \gamma' \) is the natural lift of a transverse loop \( \Gamma' \) we will say that \( \Gamma \) and \( \Gamma' \) intersect transversally at \( \phi \). Here again we can talk of self-intersection.

As a conclusion, note that if two transverse paths have a transverse intersection, they both have a leaf on their right and a leaf on their left.

3.3. Some useful results. In this section, we will state different results that will be useful in the rest of the article. Observe that the finiteness condition for the next proposition is satisfied if every leaf of \( \mathcal{F} \) is wandering, or when \( M \) has genus 0.

**Proposition 1.** Let \( \mathcal{F} \) be an oriented singular foliation on a surface and \( (\Gamma_i)_{1\leq i\leq m} \) a family of prime transverse loops that are not pairwise equivalent. We suppose the leaves met by the loops \( \Gamma_i \) are never closed and that there exists an integer \( N \) such that no loop \( \Gamma_i \) meets a leaf more than \( N \) times. Then, for every \( i \in \{1, \ldots, m\} \), there exists a transverse loop \( \Gamma'_i \) equivalent to \( \Gamma_i \) such that:

i) \( \Gamma'_i \) and \( \Gamma'_j \) do not intersect if \( \Gamma_i \) and \( \Gamma_j \) have no transverse intersection;

ii) \( \Gamma'_i \) is simple if \( \Gamma_i \) has no self-intersection.

**Proof.** We have a natural (partial) order on \( \text{dom}(\mathcal{F}) \) defined as follows: write \( z \leq z' \) if \( \phi_z \) is not closed and \( z' \in \phi_z^+ \). One can suppose, without loss of generality, that the loops \( \Gamma_i \) are included in the same connected component \( W \) of \( \text{dom}(\mathcal{F}) \). One can lift \( \mathcal{F}|_W \) to an oriented foliation \( \tilde{\mathcal{F}} \) on the universal covering space \( \tilde{W} \) of \( W \). We will parametrize \( \Gamma_i \) by a copy \( \mathbb{T}^1_i \) of \( \mathbb{T}^1 \) and consider the \( \mathbb{T}^1_i \) as disjoint circles. We will munish the set \( \mathbb{T}_* = \bigcup_{1\leq i\leq m} \mathbb{T}^1_i \) with the natural topology generated by the open sets of the \( \mathbb{T}^1_i \). We get a continuous map \( \Gamma : \mathbb{T}_* \to W \) (a multi-loop) by setting \( \Gamma(t) = \Gamma_{\iota(t)}(t) \), where \( t \in \mathbb{T}^1_{\iota(t)} \). Suppose that \( t \neq t' \) and \( \phi_{\Gamma(t)} = \phi_{\Gamma(t')} \). One can lift the loops \( \Gamma_{\iota(t)} \) and \( \Gamma_{\iota(t')} \) to lines \( \tilde{\gamma}_{\iota(t)} : \mathbb{R} \to \tilde{W} \) and \( \tilde{\gamma}_{\iota(t')} : \mathbb{R} \to \tilde{W} \) transverse to \( \tilde{\mathcal{F}} \) such that \( \tilde{\phi}_{\tilde{\gamma}_{\iota(t)}(\tilde{t})} = \tilde{\phi}_{\tilde{\gamma}_{\iota(t')}(\tilde{t}')} = \tilde{\phi} \), where \( \tilde{t} \) and \( \tilde{t}' \) lift \( t \) and \( t' \) respectively. The fact that the loops are prime and pairwise not equivalent implies that \( \tilde{\gamma}_{\iota(t)}|_{[\tilde{t},+\infty)} \) and \( \tilde{\gamma}_{\iota(t')}|_{[\tilde{t}',+\infty)} \) are not equivalent and similarly that \( \tilde{\gamma}_{\iota(t)}|_{(-\infty,\tilde{t})} \) and \( \tilde{\gamma}_{\iota(t')}|_{(-\infty,\tilde{t}')} \) are not equivalent. This implies that, if \( |\tilde{t}| \) is sufficiently large, then either \( \tilde{\phi}_{\tilde{\gamma}_{\iota(t')}^{\tilde{t}'}(\tilde{t}')} \) is above \( \tilde{\phi}_{\tilde{\gamma}_{\iota(t)}^{\tilde{t}}(\tilde{t})} \) relatively to \( \tilde{\phi} \) or the inverse is true. Suppose that for \( \tilde{t}'' \) large enough, \( \tilde{\phi}_{\tilde{\gamma}_{\iota(t')}^{\tilde{t}'}(\tilde{t}'')} \) is above \( \tilde{\phi}_{\tilde{\gamma}_{\iota(t)}^{\tilde{t}}(\tilde{t}'')} \) relatively to
\(\tilde{\gamma}\) and \(\tilde{\phi}_{\gamma(t)}(\tilde{\gamma})\) is above \(\tilde{\phi}_{\gamma(t)}(\tilde{\gamma})\) relatively to \(\tilde{\gamma}\), this will be also the case by other choices of lifts and we will write \(t \prec t'\). Observe that \(t\) and \(t'\) are always comparable if \(i(t) \neq i(t')\) and \(\Gamma_{i(t)}\) and \(\Gamma_{i(t')}\) have no transverse intersection or \(i(t) = i(t')\) and \(\Gamma_{i(t)}\) has no self-transverse intersection.

We will say that \(t \in T_s\) is a good parameter of \(\Gamma\), if for every \(t' \in T_s\), one has

\(t \prec t' \Rightarrow \Gamma(t) < \Gamma(t')\).

To get the proposition it is sufficient to construct, for every \(i \in \{1, \ldots, m\}\), a transverse loop \(\Gamma_i\) equivalent to \(\Gamma_i\) such that the induced multi-loop \(\Gamma'\) has only good parameters. Let us define the order \(o(t)\) of \(t \in T_s\) to be the number of \(t' \in T_s\) such that \(t \prec t'\). Note that every parameter of order 0 is a good parameter. We will construct \(\Gamma'\) by induction, supposing that every parameter of order \(\leq r\) is good and constructing \(\Gamma'\) such that every parameter of order \(\leq r + 1\) is good. Note that for every \(s\), the set \(T_{<r}\) of parameters of order \(s\) is closed and the set \(T_{\geq r}\) of good parameters is open. The set \(T_{bad} = T_{<r+1} \setminus T_{\geq r}\) is closed and disjoint from \(T_{<r}\): it contains only parameters of order \(r + 1\). Let us fix an open neighborhood \(O\) of \(T_{bad}\) disjoint from \(T_{\geq r}\). By hypothesis, for every \(t \in T_{bad}\), one can find \(r\) points \(\theta_0(t), \ldots, \theta_{r-1}(t)\) in \(T_s\) such that \(t \prec \theta_i(t)\) for every \(i \in \{0, \ldots, r-1\}\) and among the \(\Gamma(\theta_i(t))\) a smallest one \(\Gamma(\theta_i(t))\). Each \(\theta_i(t)\) belongs to \(T_{<r}\) and therefore is disjoint from \(O\). Note that each function \(\theta_i\) can be chosen continuous in a neighborhood of a point \(t\), which implies that \(t \mapsto \Gamma(\theta_i(t))\) is continuous on \(T_{bad}\). It is possible to make a perturbation of \(\Gamma\) supported on \(O\) by sliding continuously each point \(\Gamma(t)\) on \(\phi_{\gamma(t)}(t)\) to obtain a transverse multi-loop \(\Gamma'\) equivalent to \(\Gamma\) such that \(\Gamma'(t) < \Gamma(\theta_i(t))\). Since for every \(i \in \{0, \ldots, r-1\}\), \(\theta_i(t) \in T_{<r}\), we have \(\Gamma(\theta_i(t)) = \Gamma'(\theta_i(t))\) and so \(\Gamma'(t) < \Gamma'(\theta(t))\). \(\Box\)

Let us continue with the following adapted version of Poincaré-Bendixson theorem.

**Proposition 2.** Let \(F\) be an oriented singular foliation on \(S^2\) and \(\gamma : \mathbb{R} \to S^2\) a transverse recurrent path. The following properties are equivalent:

i) \(\gamma\) has no transverse self-intersection;

ii) there exists a transverse simple loop \(\Gamma'\) such that \(\gamma\) is equivalent to the natural lift \(\gamma'\) of \(\Gamma'\);

iii) the set \(U = \bigcup_{t \in \mathbb{R}} \phi_{\gamma(t)}\) is an open annulus.

**Proof.** To prove that ii) implies iii), just note that a dual function of \(\Gamma'\) takes only two consecutive values, which implies that every leaf of \(\mathcal{F}\) meets \(\Gamma'\) at most once.

To prove that iii) implies i) it is sufficient to note that if \(\bigcup_{t \in \mathbb{R}} \phi_{\gamma(t)}\) is an annulus, each connected component of its preimage in the universal covering space of \(\text{dom}(\mathcal{F})\) is a saturated open set where the lifted foliation \(\mathcal{F}\) is trivial. This implies that \(\gamma\) has no transverse self-intersection.

It remains to prove that i) implies ii). The path \(\gamma\) being recurrent, one can find \(a < b\) such that \(\phi_{\gamma(a)} = \phi_{\gamma(b)}\). Replacing \(\gamma\) by an equivalent transverse path, one can suppose that \(\gamma(a) = \gamma(b)\). Let \(\Gamma\) be the loop naturally defined by the closed path \(\gamma_{|[a,b]}\). As explained previously, every leaf that meets \(\Gamma\) is wandering and consequently, if \(t\) and \(t'\) are sufficiently close, \(\phi_{\Gamma(t)} \neq \phi_{\Gamma(t')}\). Moreover, because \(\Gamma\) is positively transverse to \(\mathcal{F}\), one cannot find an increasing sequence \((a_n)_{n \geq 0}\) and a decreasing sequence \((b_n)_{n \geq 0}\) such that \(\phi_{\gamma(a_n)} = \phi_{\gamma(b_n)}\). So, there exist \(a \leq a' < b' \leq b\) such that \(t \mapsto \phi_{\gamma(t)}\) is injective on \([a', b']\), but for the equality \(\phi_{\gamma(a')} = \phi_{\gamma(b')}\). Replacing \(\gamma\) by an equivalent transverse path, one can suppose that \(\gamma(a') = \gamma(b')\). The set \(U = \bigcup_{t \in [a', b']} \phi_{\gamma(t)}\) is an open annulus and the loop \(\Gamma'\) naturally defined by the closed path \(\gamma_{|[a', b']}\) is a simple loop.
Let us prove now that $\gamma$ is equivalent to the natural lift $\gamma'$ of $\Gamma'$. Being recurrent, it cannot be equivalent to a proper sub-path of $\gamma'$ and so it is sufficient to prove that it is included in $U$. We will give a proof by contradiction. We denote the two connected components of the complement of $U$ as $X_1, X_2$. Suppose that there exists $t \in \mathbb{R}$ such that $\gamma(t) \notin U$. The path $\gamma$ being recurrent and the sets $X_i$ saturated, there exists $t' \in \mathbb{R}$ separated from $t$ by $[a', b']$ such that $\gamma(t')$ is in the same component $X_i$ than $\gamma(t)$. More precisely, one can find real numbers

$$t_1 < a'' \leq a' < b' \leq b'' < t_2$$

and an integer $k \geq 1$, uniquely determined such that

- $\gamma|_{[a'', b'']} \equiv \gamma|_{[a', b']}$;
- $\gamma|_{(t_1, a'')}$ and $\gamma|_{(b', t_2)}$ are included in $U$ but do not meet $\phi_\gamma(a')$;
- $\gamma(t_1)$ and $\gamma(t_2)$ do not belong to $U$.

Moreover, if $\gamma(t_2)$ does not belong to the same component $X_i$ than $\gamma(t_1)$, one can find real numbers $t_2 < t_3 < t_4$ uniquely determined such that

- $\gamma(t_4)$ belongs to the same component $X_i$ than $\gamma(t_1)$;
- $\gamma|_{[t_2, t_4]}$ does not meet this component,
- $\gamma|_{(t_3, t_4)}$ is included in $U$;
- $\gamma(t_3)$ does not belong to $U$.

Observe now that if $\gamma(t_1)$ and $\gamma(t_2)$ belong to the same component $X_i$, then $\gamma|_{[t_1, b'']} \equiv \gamma|_{[a', t_2]}$ intersect transversally at $\gamma(a'') = \gamma(b')$. Suppose now that $\gamma(t_1)$ and $\gamma(t_2)$ do not belong to the same component $X_i$. Fix $t \in (t_3, t_4)$. There exists $t' \in [a', b']$ such that $\phi_\gamma(t') = \phi_\gamma(t)$. Observe that $\gamma|_{[t_1, t_2]}$ and $\gamma|_{[t_3, t_4]}$ intersect transversally at $\gamma(t') = \gamma(t)$. \hfill $\square$

Remark 3. Note that the proof above tells us that if $\gamma$ is positively or negatively recurrent, there exists a transverse simple loop $\Gamma'$ such that $\gamma$ is equivalent to a sub-path of the natural lift $\gamma'$ of $\Gamma'$.

![Figure 2. Proof of Proposition 2.](image-url)
**Proposition 4.** Let $\mathcal{F}$ be an oriented singular foliation on $\mathbb{R}^2$ with uniformly bounded leaves and $\gamma$ be a transverse proper path. The following properties are equivalent:

i) $\gamma$ has no transverse self-intersection;

ii) $\gamma$ meets every leaf at most once;

iii) $\gamma$ is a line.

**Proof.** The fact that ii) implies iii) is obvious, as is the fact that iii) implies i). It remains to prove that i) implies ii). Let us suppose that $\phi_{\gamma(a)} = \phi_{\gamma(b)}$, where $a < b$. We will prove that $\gamma$ has a transverse self-intersection. Like in the proof of the previous proposition, replacing $\gamma$ by an equivalent transverse path, one can find $a \leq a' < b' \leq b$ such that $\gamma(a') = \gamma(b')$, such that $U = \bigcup_{t \in [a', b']} \phi_{\gamma(t)}$ is an open annulus and such that the loop $\Gamma'$ naturally defined by the closed path $\gamma|_{[a', b']}$ is a simple loop. Write $X_1$ for the unbounded connected component of $\mathbb{R}^2 \setminus \Gamma'$ and $X_2$ for the bounded one. The path $\gamma$ being proper, one can find real numbers $t_1 < a'' \leq a' < b' \leq b'' < t_2$ and an integer $k \geq 1$, uniquely determined such that

- $\gamma|_{[a''', b''']}_{[a', b']}$ is equivalent to $\gamma|_{[a', b']}_{[a', b']}_{[a', b']}_{[a', b']}$
- $\gamma|_{(t_1, a''')}$ and $\gamma|_{(b'', t_2)}$ are included in $U$ but do not meet $\phi_{\gamma(a''')}$
- $\gamma(t_1)$ and $\gamma(t_2)$ do not belong to $U$.

As seen in the proof of the previous proposition, if $\gamma(t_1)$ and $\gamma(t_2)$ belong to the same component $X_i$, then $\gamma|_{[t_1, b''']}$ and $\gamma|_{[a''', t_2]}$ intersect transversally at $\gamma(a''') = \gamma(b''')$. If $\gamma(t_1) \in X_1$ and $\gamma(t_2) \in X_2$, using the fact that $\gamma$ is proper, one can find real numbers $t_2 \leq t_3 < t_4$ uniquely determined such that

- $\gamma(t_4)$ belongs to $X_1$,
- $\gamma|_{[t_2, t_4]}$ does not meet $X_1$,
- $\gamma|_{[t_3, t_4]}$ is included in $U$,
- $\gamma(t_3)$ belongs to $X_2$.

As seen in the proof of the previous proposition, $\gamma|_{[t_1, t_2]}$ and $\gamma|_{[t_3, t_4]}$ intersect transversally. The case where $\gamma(t_1) \in X_2$ and $\gamma(t_2) \in X_1$ can be treated analogously. $\square$

Let us add another result describing paths with no transverse self-intersection:

**Proposition 5.** Let $\mathcal{F}$ be an oriented singular foliation on $\mathbb{R}^2$, $\gamma$ a transverse proper path and $\delta$ a dual function of $\gamma$. If $\gamma'$ is a transverse path that does not intersect $\gamma$ transversally, then $\delta$ takes a constant value on the union of the leaves met by $\gamma'$ but not by $\gamma$.

**Proof.** Let us suppose that $\gamma'$ meets two leaves $\phi_0$ and $\phi_1$, disjoint from $\gamma$ and such that $\delta$ does not take the same value on $\phi_0$ and on $\phi_1$. One can suppose that $\gamma'$ joins $\phi_0$ to $\phi_1$. Let $W$ be the connected component of $\text{dom}(\mathcal{F})$ that contains $\gamma$. Write $\widetilde{W}$ for the universal covering space of $W$ and $\widetilde{\mathcal{F}}$ for the lifted foliation. Every lift of $\gamma$ is a line. Fix a lift $\widetilde{\gamma}'$, it joins a leaf $\widetilde{\phi}_0$ that lifts $\phi_0$ to a leaf $\widetilde{\phi}_1$ that lifts $\phi_1$. By hypothesis, there exists a lift $\widetilde{\gamma}$ of $\gamma$ such that the dual function $\delta_{\widetilde{\gamma}}$ do not take the same value on $\phi_0$ and $\phi_1$. One can suppose that $\phi_0 \subset r(\widetilde{\gamma})$ and $\phi_1 \subset l(\widetilde{\gamma})$ for instance. The foliation $\widetilde{\mathcal{F}}$ is
not singular and the sets \( r(\tilde{\gamma}) \) and \( l(\tilde{\gamma}) \) are closed. Consequently, there exists a sub-path \( \tilde{\gamma}' \) of \( \tilde{\gamma}' \) that joins a leaf of \( r(\tilde{\gamma}) \) to a leaf of \( l(\tilde{\gamma}) \) and that is contained but the ends in the open set \( \tilde{U} \), union of leaves met by \( \tilde{\gamma} \). Observe now that \( \tilde{\gamma} \) and \( \tilde{\gamma}' \) meet transversally and positively. \( \square \)

We deduce immediately

**Corollary 6.** Let \( \mathcal{F} \) be an oriented singular foliation on \( \mathbb{R}^2 \), \( \gamma \) a transverse path that is either a line or a proper path directed by a unit vector \( \rho \) and \( \gamma' \) a transverse path. If \( \gamma \) and \( \gamma' \) do not intersect transversally, then \( \gamma' \) cannot meet both sets \( r(\gamma) \) and \( l(\gamma) \).

Given a transverse loop \( \Gamma \) with transverse self-intersection and its natural lift \( \gamma \), there exists some integer \( K \) for which \( \gamma|_{[0,K]} \) also has a transverse self-intersection. Let us finish this section with an estimate of the minimal such \( K \) when \( \Gamma \) is homologous to zero.

**Proposition 7.** Let \( \mathcal{F} \) be an oriented singular foliation on \( M \) and \( \Gamma : \mathbb{T}^1 \to M \) a transverse loop homologous to zero in \( M \) with a transverse self-intersection. If \( \gamma : \mathbb{R} \to M \) is the natural lift of \( \Gamma \), then \( \gamma|_{[0,2]} \) has a transverse self-intersection.

**Proof.** Choose a lift \( \tilde{\gamma} \) of \( \gamma \) to the universal covering space of \( \text{dom}(\mathcal{F}) \) and write \( T \) for the covering automorphism such that \( \tilde{\gamma}(t+1) = T(\tilde{\gamma})(t) \), for every \( t \in \mathbb{R} \). Since \( \gamma \) has a transverse self-intersection and is periodic of period 1, there exist a covering automorphism \( S \) and

\[
\begin{align*}
a_1 < t_1 < b_1, & \quad a_2 < t_2 < b_2,
\end{align*}
\]

such that

- \( \tilde{\gamma}|_{[a_1,b_1]} \) is equivalent to \( S(\tilde{\gamma})|_{[a_2,b_2]} \);
- \( \tilde{\gamma}|_{[a_1,b_1]} \) and \( S(\tilde{\gamma})|_{[a_2,b_2]} \) have a transverse intersection at \( \tilde{\gamma}(t_1) = S(\tilde{\gamma})(t_2) \),
- both \( a_1, a_2 \) belong to \([0,1)\).

We will show that \( b_1 = a_1 + 1 \) and \( b_2 = a_2 + 1 \), which implies that \( \gamma|_{[0,2]} \) has a transverse self-intersection. Assume for a contradiction that \( b_1 > a_1 + 1 \) (the case where \( b_2 > a_2 + 1 \) is treated similarly). Then we can find \( a_1', a_2', b_2' \) with

\[
\begin{align*}
a_1 < a_1' < t_1, & \quad a_2 < a_2' < t_2 < b_2' < b_2
\end{align*}
\]

such that \( \tilde{\gamma}|_{[a_1',a_1'+1]} \) is equivalent to \( S(\tilde{\gamma})|_{[a_2',b_2']} \).

Consider first the case where \( b_2' = a_2' + 1 \). In that situation there exists an increasing homeomorphism \( h : [a_1',a_1'+1] \to [a_2',a_2'+1] \), such that \( h(t_1) = t_2 \) and \( \phi_{\tilde{\gamma}(t_1)} = \phi_{S(\tilde{\gamma})(h(t_1))} \). This implies that

\[
T(\phi_{\tilde{\gamma}(a_1')}) = \phi_{S(\tilde{\gamma})(a_2'+1)} = STS^{-1}\phi_{\tilde{\gamma}(a_2')} = STS^{-1}\phi_{\tilde{\gamma}(a_1')}.
\]

In case \( STS^{-1} = T \), one can extend \( h \) to a homeomorphism of the real line that commutes with the translation \( t \mapsto t + 1 \) such that \( \phi_{\tilde{\gamma}(t)} = \phi_{S(\tilde{\gamma})(h(t))} \), for every \( t \in \mathbb{R} \). If \( K \) is large enough, then \([-K,K]\) contains \([a_1,b_1]\) and \([h(-K,K)]\) contains \([a_2,b_2]\). This contradicts the fact that \( \tilde{\gamma}|_{[a_1,b_1]} \) and \( S(\tilde{\gamma})|_{[a_2,b_2]} \) have a transverse intersection at \( \tilde{\gamma}(t_1) = S(\tilde{\gamma})(t_2) \). In case \( STS^{-1} \neq T \), the leaf \( \phi_{\tilde{\gamma}(a_1')} \) is invariant by the commutator \( T^{-1}STS^{-1} \) and so projects into a closed leaf of \( \mathcal{F} \) that is homological to zero in \( \text{dom}(\mathcal{F}) \), which means that it bounds a closed surface. This surface being foliated by a non singular
foliation, one gets a contradiction by Poincaré-Hopf formula. One also gets a contradiction by saying that this closed leaf has a non zero intersection number with the loop $\Gamma$.

Now assume that $b_2' < a_2' + 1$. Let $s \in (b_2', a_2' + 1)$ and consider $\phi_{\gamma(s)}$. As noted before, since $\Gamma$ is homologous to zero, it intersects every given leaf a finite number of times. Let $n$ be the number of times it intersects $\phi_{\gamma(s)}$. It is equal to the number of times $\gamma|_{[a_1', a_1' + 1]}$ or $\gamma|_{[a_2', a_2' + 1]}$ intersect $\phi_{\gamma(s)}$. On the other hand, since $\gamma|_{[a_2', b_2]}$ is equivalent to $\gamma|_{[a_1', a_1' + 1]}$, it must also intersect $\phi_{\gamma(s)}$ exactly $n$ times, and since $s \in (b_2', a_2' + 1)$, $\gamma|_{[a_2', a_2' + 1]}$ needs to intersect $\phi_{\gamma(s)}$ at least $n + 1$ times, a contradiction.

Finally, if $b_2' > a_2' + 1$, then $\gamma|_{[a_1', a_2' + 1]}$ is equivalent to $\gamma|_{[a_1', b_1']}$, for some $b_1' < a_1' + 1$ and the same reasoning as above may be applied. \hfill $\Box$

3.4. Transverse homology set.

For any loop $\Gamma$ on $M$, let us denote $[\Gamma] \in H_1(M, \mathbb{Z})$ its singular homology class. The transverse homology set of $\mathcal{F}$ is the smallest set $\mathrm{THS}(\mathcal{F})$ of $H_1(M, \mathbb{Z})$, that is stable by addition and contains all classes of loops positively transverse to $\mathcal{F}$. The following result will also be useful:

**Proposition 8.** Let $\mathcal{F}$ be a singular oriented foliation on $\mathbb{T}^2$ and $\bar{\mathcal{F}}$ its lift to $\mathbb{R}^2$. If one can find finitely many classes $\kappa_i \in \mathrm{THS}(\mathcal{F})$, $1 \leq i \leq r$, that linearly generate the whole homology of the torus and satisfy $\sum_{1 \leq i \leq r} \kappa_i = 0$, then the leaves of $\bar{\mathcal{F}}$ are uniformly bounded.

**Proof.** Decomposing each class $\kappa_i$ and taking out all the loops homologous to zero, one can suppose (changing $r$ is necessary) that for every $i \in \{1, \ldots, r\}$, there exists a transverse loop $\Gamma_i$ such that $[\Gamma_i] = \kappa_i$. The fact that the $\kappa_i$ linearly generate the whole homology of the torus implies that the multi-loop $\Gamma = \sum_{1 \leq i \leq r} \Gamma_i$ is connected (as a set) and that the connected components of its complement are simply connected. Moreover, these components are lifted in uniformly bounded simply connected domains of $\mathbb{R}^2$, let us say by a constant $K$. The multi-loop $\Gamma$ being homologous to zero induces a dual function $\delta$ on its complement. It has been explained before that $\delta$ decreases on each leaf of $\mathcal{F}$ and is bounded. Consequently, there exists an integer $N$ such that every leaf meets at most $N$ components. If one lifts it to $\mathbb{R}^2$, one find a path of diameter bounded by $NK$. \hfill $\Box$

4. Maximal isotopies, transverse foliations, admissible paths

Let us begin by recall the notion of maximal isotopy. Let $f$ be a homeomorphism isotopic to the identity on an oriented surface $M$. Denote $\mathcal{I}$ the set of identity isotopies reaching $f$, which means the set of paths joining the identity to $f$ in the space of homeomorphisms of $M$, furnished with the $C^0$ topology (defined by the uniform convergence of maps and their inverse on every compact set). We write $I(z) : t \mapsto f_t(z)$ for the trajectory of a point $z \in M$ along an isotopy $I = (f_t)_{t \in [0,1]} \in \mathcal{I}$, and more generally, for every $n \geq 1$ we define $I^n(z) = \prod_{0 \leq k < n} I(f^k(z))$ by concatenation. We will also use the following notations

$$I^N(z) = \prod_{0 \leq k < +\infty} I(f^k(z)), \quad I^{-N}(z) = \prod_{-\infty < k < 0} I(f^k(z)), \quad I^{\mathbb{Z}}(z) = \prod_{-\infty < k < +\infty} I(f^k(z)).$$

The last path will be called the whole trajectory of $z$. 
For every $I = (f_t)_{t \in [0,1]} \in \mathcal{I}$ write $\text{fix}(I) = \bigcap_{t \in [0,1]} \text{fix}(f_t)$ for the set of points whose trajectory is trivial. There is a natural pre-order on $\mathcal{I}$ defined as follows: write $I \leq I'$ if $\text{fix}(I) \subset \text{fix}(I')$ and if $I'$ is homotopic to $I$ relatively to $\text{fix}(I)$. It is easy to see that $I$ is maximal for this preorder if and only if there is no point $z \in \text{fix}(f) \setminus \text{fix}(I)$ such that $I(z)$ is a loop homotopic to zero in $M \setminus \text{fix}(I)$ or equivalently, if the restriction $I|_{M \setminus \text{fix}(I)}$ is lifted to the universal covering space as an identity isotopy whose time one map is fixed point free. An example of a maximal isotopy is the isotopy $I = (f_t)_{t \in [0,1]}$ naturally defined by restriction to $[0,1]$ from a flow $I = (f_t)_{t \in \mathbb{R}}$. Does there always exist a maximal isotopy? Is any isotopy smaller than a maximal one?

The answer to these questions is yes, but the proof due to F. Béguin, S. Crovisier and F. Le Roux is not published yet (see [BCL]). A positive answer to these questions in a weaker framework was previously proved by O. Jaulent [J]. Instead of looking at isotopies defined on the whole surface, we look at “singular isotopies” defined on an invariant open set $\text{dom}(I)$ whose complement is included in the fixed point set. Denote the set of such isotopies by $\mathcal{I}_{\text{sing}}$. There is a pre-order on $\mathcal{I}_{\text{sing}}$ defined as follows: write $I \leq I'$ if $\text{dom}(I') \subset \text{dom}(I)$ and if for every $z \in \text{dom}(I')$, the trajectories $I'(z)$ and $I(z)$ are homotopic in $\text{dom}(I)$. Here again, saying that $I$ is maximal means that there is no point $z \in \text{fix}(f) \cap \text{dom}(I)$ such that $I(z)$ is homotopic to zero in $\text{dom}(I)$.

An interesting class of singular isotopies consists of the hereditary ones, that satisfies the following condition (always true in the case where the fixed point set of $f$ is totally disconnected): for every open set $U$ containing $\text{dom}(I)$, there exists $I' \in \mathcal{I}_{\text{sing}}$ such that $I' \leq I$ and $\text{dom}(I') = U$. The two questions above have a positive answer in the set $\mathcal{I}_{\text{her}}$ of hereditary singular isotopies (see [J]). Note that for every $I \in \mathcal{I}$ the restriction $I|_{M \setminus \text{fix}(I)}$ belongs to $\mathcal{I}_{\text{her}}$. What is proved in [BCL] is that every hereditary singular isotopy is equivalent to a restriction $I|_{M \setminus \text{fix}(I)}$, $I \in \mathcal{I}$. In other words, there exists an isotopy defined on $\text{dom}(I)$ and homotopic to $I$ that extends to the whole manifold by fixing every point out of $\text{dom}(I)$. In what follows by saying that $I$ is maximal, we can refer to any of the two definitions. The sets $\text{fix}(I)$ and $\text{dom}(I)$ will be defined to be complementary in $M$.

Let $I = (f_t)_{t \in [0,1]}$ be a maximal isotopy. The fact that $I|_{\text{dom}(I)}$ is lifted to the universal covering space $\tilde{\text{dom}}(I)$ as an identity isotopy $\tilde{I} = (\tilde{f}_t)_{t \in [0,1]}$ whose time one map is fixed point free, implies that there exists a foliation $\mathcal{F}$ on $\text{dom}(I)$ which is invariant by the covering automorphisms and such that every leaf $\phi$ is a Brouwer line (see [LeC2]), which means that $\tilde{f}_1(L(\phi)) \subset L(\phi)$. This is equivalent to saying that $\mathcal{F}$ lifts a foliation $\mathcal{F}$ on $\text{dom}(I)$ that is transverse to $I$ in the following sense: for every $z \in \text{dom}(I)$ the trajectory $I(z)$ is homotopic to the ends to a path $\gamma$ positively transverse to $\mathcal{F}$. Choose a lift $\tilde{z}$ of $z$ and write $\tilde{\gamma}$ for the lift of $\gamma$ that joins $\tilde{z}$ to $\tilde{f}_1(\tilde{z})$. The leaves of the lifted foliation $\tilde{\mathcal{F}}$ met by $\tilde{\gamma}$ are the leaves $\phi$ such that $R(\phi_{\tilde{z}}) \subset R(\phi) \subset R(\phi_{\tilde{f}_1(\tilde{z})})$. In particular, the equivalence class of $\tilde{\gamma}$ is uniquely defined and every leaf met by $\tilde{\gamma}$ is met by $\tilde{f}_1(\tilde{z})$. We have a similar result in $M$, the equivalence class of $\gamma$ is uniquely defined and every leaf met by $\gamma$ is met by $I(z)$. We will write $I_{\mathcal{F}}(z)$ for this equivalence class and called it the transverse trajectory of $z$. This notation will also mean any choice of a path homotopic to $I(z)$ (relatively to the ends) and positively transverse to $\mathcal{F}$. Similarly, for every $n \geq 1$, one can define $I_{\mathcal{F}}^n(z) = \prod_{0 \leq k < n} I_{\mathcal{F}}(f^k(z))$ (or its equivalence class). We will also use the following notations (defined up to equivalence)

$$I_{\mathcal{F}}^k(z) = \prod_{0 \leq k < +\infty} I_{\mathcal{F}}(f^k(z)), \quad I_{\mathcal{F}}^{-k}(z) = \prod_{-\infty < k < 0} I_{\mathcal{F}}(f^k(z)), \quad I_{\mathcal{F}}^\mathbb{Z}(z) = \prod_{-\infty < k < +\infty} I_{\mathcal{F}}(f^k(z)).$$

The last object will be called the whole transverse trajectory of $z$. 
If $z$ is a periodic point of period $q$, the sequence $(I_{\phi}(f^k(z)))_{k \in \mathbb{Z}}$ can be chosen $q$-periodic and $I_{\tilde{\phi}}^n$ is the natural lift of a transverse loop $\Gamma$. We will say that $\Gamma$, whose equivalence class is uniquely defined, is associate to $z$. We can see $F$ as a singular foliation on $M$ whose singular set $\text{sing}(F)$ is equal to $\text{fix}(\Gamma)$ and whose domain $\text{dom}(F)$ is equal to $\text{dom}(I)$.

Let us state two results that will be useful later.

**Lemma 9.** Fix $z \in \text{dom}(I)$, $n \geq 1$ and parametrize $I_{\tilde{\phi}}^n(z)$ by $[0,1]$. For every $0 < a < b < 1$, there exists a neighborhood $V$ of $z$ such that, for every $z' \in V$, the path $I_{\tilde{\phi}}^n(z)|_{[a,b]}$ is a sub-path of $I_{\tilde{\phi}}^n(z')$. Moreover, there exists a neighborhood $W$ of $z$ such that, for every $z'$ and $z''$ in $W$, the path $I_{\tilde{\phi}}(z')$ is a sub-path of $I_{\tilde{\phi}}^{n+2}(f^{-1}(z''))$.

**Proof.** Keep the notations introduced above. Fix a lift $\tilde{z} \in \text{dom}(I)$ of $z$ and denote by $\phi$ and $\phi'$ the leaves of $\tilde{F}$ containing $I_{\tilde{\phi}}^n(\tilde{z})(a)$ and $I_{\tilde{\phi}}^n(\tilde{z})(b)$ respectively. One has

$$R(\phi_\delta) \subset R(\phi) \subset R(\phi') \subset R(\phi') \subset R(\phi_f^n(\tilde{z})).$$

If $V \subset \text{dom}(I)$ is a topological disk, small neighborhood of $z$, the lift $\tilde{V}$ that contains $\tilde{z}$ satisfies

$$\tilde{V} \subset R(\phi), \tilde{f}^n(\tilde{V}) \subset L(\phi').$$

Consequently, for every $z' \in V$, the path $I_{\tilde{\phi}}^n(z)|_{[a,b]}$ is a sub-path of $I_{\tilde{\phi}}^n(z')$. Let us prove the second assertion. One can find a leaf $\phi$ of the lifted foliation such that

$$R(\phi_f^{-1}(\tilde{z})) \subset R(\phi) \subset R(\phi) \subset R(\phi_\delta)$$

and a leaf $\phi'$ such that

$$R(\phi_f^n(\tilde{z})) \subset R(\phi') \subset R(\phi') \subset R(\phi_f^{n+1}(\tilde{z})).$$

If $W \subset \text{dom}(I)$ is a topological disk, small neighborhood of $z$, the lift $\tilde{W}$ that contains $\tilde{z}$ satisfies

$$\tilde{f}^{-1}(\tilde{W}) \subset R(\phi), \tilde{W} \subset L(\phi), \tilde{f}^n(\tilde{W}) \subset R(\phi'), \tilde{f}^{n+1}(\tilde{W}) \subset L(\phi').$$

Consequently, for every $z'$ and $z''$ in $W$, the path $I_{\tilde{\phi}}^n(z')$ is a sub-path of $I_{\tilde{\phi}}^{n+2}(f^{-1}(z''))$. \hfill $\Box$

An immediate consequence is the fact that if $z$ is positively or negatively recurrent, then $I_{\tilde{\phi}}^n(z)$ is positively or negatively recurrent respectively.

**Lemma 10.** Suppose that $\gamma : [a, b] \to \text{dom}(I)$ is a transverse path that has a leaf on its right and a leaf on its left. Then, there exists a compact set $K \subset \text{dom}(I)$ such that for every transverse trajectory $I_{\tilde{\phi}}^n(z)$ that contains a sub-path equivalent to $\gamma$, there exists $k \in \{0, \ldots, n - 1\}$ such that $f^k(z)$ belongs to $K$.

**Proof.** Lifting our path to the universal covering space of the domain, it is sufficient to prove the result in the case where $\text{dom}(I)$ is a plane.

Suppose that $\phi_0$ is on the right of $\gamma$ and $\phi_1$ on its left and write $W$ for the connected component of the complement of $\phi_0 \cup \phi_1$ that contains $\gamma$. Since $\phi_0$ and $\phi_1$ are Brouwer lines, every orbit that goes from $R(\phi_0(a))$ to $L(\phi_1(b))$ is contained in $W$. Let $\delta$ be a simple path that joins a point $z_0$ of $\phi_0$ to a point $z_1$ of $\phi_1$, that is contained in $W$ but the ends and that does not meet neither $R(\phi_0(a))$ nor $L(\phi_1(b))$.

Write $V_\delta$ for the connected component of $W \setminus \delta$ that contains $L(\phi_1(b))$. We will extend $\delta$ as a line as follows: If $W$ is contained in $R(\phi_0)$, choose a simple path $\alpha_0$ that joins $f^2(z_0)$ to $z_0$ and is contained in $L(\phi_0) \cap R(f^2(\phi_0))$ but the ends, and then set $\beta_0 = f^2(\phi_0)\alpha_0$. Otherwise, if $W$ is contained in
Let \( f \) implies that \( f \) has no leaf on its right and no leaf on its left. The order \( n \) has no restriction to any segment of \( I \). Otherwise, if \( W \) is contained in \( L(\phi_1) \), choose a simple path \( \alpha_1 \) that joins \( f^2(z_1) \) to \( z_1 \) and is contained in \( L(\phi_1) \cap R(f^2(\phi_1)) \) but the ends, and then set \( \beta_1 = f^2(\phi^+_{z_2}) \alpha_1 \). The path \( \lambda = \beta_0 \delta \beta_1^{-1} \) is a line.

The image of \( \lambda = \alpha_0 \delta \alpha_1^{-1} \) by \( f^{-1} \) is compact and the images of \( \beta_0 \) and \( \beta_1 \) by \( f^{-1} \) are disjoint from \( W \). So, one can find a simple path \( \delta' \) that joins a point \( z'_0 \) of \( \phi_0 \) to a point \( z'_1 \) of \( \phi_1 \), that is contained in \( W \) but the ends, that does not meet \( V_0 \) and such that the connected component \( V_a \) of \( W \setminus \delta \) that does not contain \( V_0 \) (and that meets \( \overline{R(\phi_{\gamma(a)})} \)), does not intersect \( f^{-1}(\lambda) \). This implies that \( f(V_a) \cap V_b = \emptyset \).

So, every orbit that goes from \( \overline{R(\phi_{\gamma(a)})} \) to \( L(\phi_{\gamma(b)}) \) has to leave \( V_a \cup V_b \), it must meet the compact set \( K = W \setminus (V_a \cup V_b) \).

We will say that a path \( \gamma : [a, b] \to \text{dom}(I) \), positively transverse to \( F \), is admissible of order \( n \) if it is equivalent to a path \( f^2(z) \), \( z \in \text{dom}(I) \). We will say that it is admissible of order \( \leq n \) is it can be extended to an admissible path of order \( n \). If \( \tilde{\gamma} : [a, b] \to \text{dom}(I) \) is a lift of \( \gamma \), the fact that \( \gamma \) is admissible of order \( n \) means that

\[
\text{admissible of order } \leq n \text{ means that } \quad \tilde{f}^n(\phi_{\gamma(a)}) \cap \phi_{\gamma(b)} \neq \emptyset,
\]

the fact that it is admissible of order \( \leq n \) means that

\[
\text{admissible of order } \leq n \text{ means that } \quad \tilde{f}^n(\overline{R(\phi_{\gamma(a)})}) \cap L(\phi_{\gamma(b)}) \neq \emptyset.
\]

More generally, we will say that a path \( \gamma : J \to \text{dom}(I) \) defined on an interval is admissible if its restriction to any segment of \( J \) is admissible. Similarly, we will say that a loop \( \Gamma \) is admissible if its natural lift is admissible.

Let us begin with a useful result which says that except in some particular trivial cases, there is no difference between being of order \( \leq n \) and being of order \( n \) (and so of being of order \( \leq n \) and being of order \( m \) for every \( m \geq n \)).

**Proposition 11.** Let \( \gamma : [a, b] \to \text{dom}(I) \) be a transverse path of order \( \leq n \) but not of order \( n \), then \( \gamma \) has no leaf on its right and no leaf on its left.

**Proof.** Lifting the path to the universal covering space of the domain, it is sufficient to prove the result in case where \( \text{dom}(I) \) is a plane. The fact that \( f^n(\phi_{\gamma(a)}) \cap \phi_{\gamma(b)} = \emptyset \) and \( f^n(\overline{R(\phi_{\gamma(a)})}) \cap L(\phi_{\gamma(b)}) \neq \emptyset \) implies that \( f^n(\overline{L(\phi_{\gamma(a)})}) \subset L(\phi_{\gamma(b)}) \) and \( f^{-n}(\overline{R(\phi_{\gamma(b)})}) \subset R(\phi_{\gamma(a)}) \). Suppose that there exists a leaf.
φ in $L(\phi_{\gamma(a)}) \cap R(\phi_{\gamma(b)})$ that does not meet γ. Recall that φ is a Brouwer line. One of the sets $R(\phi)$ or $L(\phi)$ is included in $L(\phi_{\gamma(a)}) \cap R(\phi_{\gamma(b)})$. It cannot be $R(\phi)$, because $f^{-n}(R(\phi))$ would be contained both in $R(\phi)$ and in $R(\phi_{\gamma(a)})$; it cannot be $L(\phi)$, because $f^{n}(L(\phi))$ would be contained both in $L(\phi)$ and in $L(\phi_{\gamma(b)})$. We have a contradiction. □

4.1. The fundamental proposition.

We will define an operation that permits to construct admissible paths from a pair of admissible paths.

**Proposition 12.** Suppose that $\gamma_1 : [a_1, b_1] \to M$ and $\gamma_2 : [a_2, b_2] \to M$ are transverse paths that intersect transversally at $\gamma_1(t_1) = \gamma_2(t_2)$. If $\gamma_1$ is admissible of order $n_1$ and $\gamma_2$ is admissible of order $n_2$, then $\gamma_1|_{[a_1, t_1]} \gamma_2|_{[t_2, b_2]}$ and $\gamma_2|_{[a_2, t_2]} \gamma_1|_{[t_1, b_1]}$ are admissible of order $n_1 + n_2$. Furthermore, either one of these paths is admissible of order $\min(n_1, n_2)$ or both paths are admissible of order $\max(n_1, n_2)$

**Proof.** By lifting to the universal covering space of the domain, it is sufficient to prove the result in the case where $M$ is a plane and $F$ is non-singular.

By Corollary ??, each path $\gamma_1$, $\gamma_2$, $\gamma_1|_{[a_1, t_1]} \gamma_2|_{[t_2, b_2]}$ and $\gamma_2|_{[a_2, t_2]} \gamma_1|_{[t_1, b_1]}$ will be admissible of order $m$ if it is admissible of order $\leq m$. Note first that for every integers $k_1$, $k_2$ in $\mathbb{Z}$, one has

$$f^{k_1}(R(\phi_{\gamma_1(a)})) \cap f^{k_2}(R(\phi_{\gamma_2(a)})) = f^{k_1}(L(\phi_{\gamma_1(b)})) \cap f^{k_2}(L(\phi_{\gamma_2(b)})) = \emptyset.$$  

For every $i \in \{1, 2\}$ define the sets

$$X_i = f^{-n_1}(R(\phi_{\gamma_1(a)})) \cup L(\phi_{\gamma_1(b)}), \quad Y_i = f^{-n_1}(L(\phi_{\gamma_1(b)})) \cup R(\phi_{\gamma_1(a)}),$$

which are connected according to the admissibility hypothesis.

If $\gamma_1|_{[a_1, t_1]} \gamma_2|_{[t_2, b_2]}$ is not admissible of order $n_1$, then $X_1 \cap L(\phi_{\gamma_2(b)}) = \emptyset$ and so $X_1$ separates $R(\phi_{\gamma_2(a)})$ and $L(\phi_{\gamma_2(b)})$. This implies that none of the sets $X_1 \cap X_2$ and $X_1 \cap Y_2$ is empty. The first property implies that $f^{-n_2}(R(\phi_{\gamma_2(a)})) \cap L(\phi_{\gamma_1(b)}) \neq \emptyset$, which means that $\gamma_2|_{[a_2, t_2]} \gamma_1|_{[t_1, b_1]}$ is admissible of order $n_2$. The second property implies that $f^{-n_2}(L(\phi_{\gamma_2(b)})) \cap f^{-n_1}(R(\phi_{\gamma_1(a)}))$, which means that $\gamma_1|_{[a_1, t_1]} \gamma_2|_{[t_2, b_2]}$ is admissible of order $n_1 + n_2$.

If $\gamma_1|_{[a_1, t_1]} \gamma_2|_{[t_2, b_2]}$ is not admissible of order $n_2$, then $Y_2 \cap R(\phi_{\gamma_1(a)}) = \emptyset$ and so $Y_2$ separates $R(\phi_{\gamma_1(a)})$ and $L(\phi_{\gamma_1(b)})$. This implies that none of the sets $Y_2 \cap Y_1$ and $Y_2 \cap X_1$ is empty. The first property implies that $\gamma_2|_{[a_2, t_2]} \gamma_1|_{[t_1, b_1]}$ is admissible of order $n_1$. The second one implies that $\gamma_1|_{[a_1, t_1]} \gamma_2|_{[t_2, b_2]}$ is admissible of order $n_1 + n_2$.

In conclusion, $\gamma_1|_{[a_1, t_1]} \gamma_2|_{[t_2, b_2]}$ is admissible of order $n_1 + n_2$ and if it is not admissible of order $\min(n_1, n_2)$ then $\gamma_2|_{[a_2, t_2]} \gamma_1|_{[t_1, b_1]}$ is admissible of order $\max(n_1, n_2)$. The paths $\gamma_1$ and $\gamma_2$ playing the same role, we get the proposition. □

One deduces immediately the following

**Corollary 13.** Let $\gamma_i : [a_i, b_i] \to M$, $1 \leq i \leq r$, be a family of $r \geq 2$ transverse paths. We suppose that for every $i \in \{1, \ldots, r\}$ there exist $s_i \in [a_i, b_i]$ and $t_i \in [a_i, b_i]$, such that:

i) $\gamma_i|_{[s_i, b_i]}$ and $\gamma_{i+1}|_{[s_{i+1}, t_{i+1}]}$ intersect transversally at $\gamma_i(t_i) = \gamma_{i+1}(s_{i+1})$ if $i < r$;

ii) one has $s_1 = a_1 < t_1 < b_1$, $a_r < s_r < t_r = b_r$ and $a_i < s_i < t_i < b_i$ if $1 < i < r$;
iii) $\gamma_i$ is admissible of order $n_i$.

Then $\prod_{1 \leq i \leq r} \gamma_i|_{[s_i, t_i]}$ is admissible of order $\sum_{1 \leq i \leq r} n_i$.

**Proof.** Here again, it is sufficient to prove the result when $M = \mathbb{R}^2$ and $\mathcal{F}$ is not singular. One must prove by induction on $q \in \{2, \ldots, r\}$ that

$$
\left( \prod_{1 \leq i < q} \gamma_i|_{[s_i, t_i]} \right) \gamma_q|_{[s_q, b_q]}
$$

is admissible of order $\sum_{1 \leq i \leq q} n_i$. The result for $q = 2$ is nothing but Proposition 12. Suppose that it is true for $q < r$ and let us prove it for $q + 1$. The paths

$$
\left( \prod_{1 \leq i < q} \gamma_i|_{[s_i, t_i]} \right) \gamma_q|_{[s_q, b_q]}
$$

and $\gamma_{q+1}$ intersect transversally at $\gamma_q(t_q) = \gamma_{q+1}(s_{q+1})$ because this is the case for the sub-paths $\gamma_q|_{[s_q, b_q]}$ and $\gamma_{q+1}|_{[s_{q+1}, t_{q+1}]}$. One deduces that

$$
\left( \prod_{1 \leq i \leq q} \gamma_i|_{[s_i, t_i]} \right) \gamma_{q+1}|_{[s_{q+1}, b_{q+1}]}
$$

is admissible of order $\sum_{1 \leq i \leq q+1} n_i$. \hfill \Box

The following result is more subtle. The transverse intersection property is stated on the paths $\gamma_i$ and not on sub-paths but the signs of intersection are the same.

**Corollary 14.** Let $\gamma_i : [a_i, b_i] \to M$, $1 \leq i \leq r$, be a family of $r \geq 2$ transverse paths. We suppose that for every $i \in \{1, \ldots, r\}$ there exist $s_i \in [a_i, b_i]$ and $t_i \in [a_i, b_i]$, such that:

i) $\gamma_i$ and $\gamma_{i+1}$ intersect transversally and positively at $\gamma_i(t_i) = \gamma_{i+1}(s_{i+1})$ if $i < r$;  

ii) one has $s_1 = a_1 < t_1 < b_1$, $a_r < s_r < t_r = b_r$ and $a_i < s_i < t_i < b_i$ if $1 < i < r$;  

iii) $\gamma_i$ is admissible of order $n_i$.
Then \( \prod_{1 \leq i \leq r} \gamma_i \mid_{[s_i,t_i]} \) is admissible of order \( \sum_{1 \leq i \leq r} n_i \).

**Proof.** Here again, it is sufficient to prove the result when \( M = \mathbb{R}^2 \) and \( F \) is not singular. Here again, one must prove by induction on \( q \) that

\[
\prod_{1 \leq i < q} \gamma_i \mid_{[s_i,t_i]} \gamma_q \mid_{[s_q,b_q]}
\]

is admissible of order \( \sum_{1 \leq i < q} n_i \) and here again, the case \( q = 2 \) is nothing but Proposition 12. Supposing that it is true for \( q < r \), one must prove that

\[
\prod_{1 \leq i < q} \gamma_i \mid_{[s_i,t_i]} \gamma_q \mid_{[s_q,b_q]}
\]

and \( \gamma_{q+1} \) intersect transversally and positively at \( \gamma_q(t_q) = \gamma_{q+1}(s_{q+1}) \). By hypothesis, one knows that \( \phi_{\gamma_{q+1}(b_{q+1})} \) is above \( \phi_{\gamma_q(b_q)} \) relatively to \( \phi_{\gamma_q(t_q)} \). It remains to prove that \( \phi_{\gamma_{q+1}(a_{q+1})} \) is below \( \phi_{\gamma_1(a_1)} \) relatively to \( \phi_{\gamma_q(t_q)} \). For every \( i \in \{1,\ldots,q-1\} \), the leaves \( \phi_{\gamma_i(a_i)} \) and \( \phi_{\gamma_{i+1}(a_{i+1})} \) belong to \( R(\phi_{\gamma_i(t_i)}) \) and \( \phi_{\gamma_{i+1}(t_{i+1})} \) is below \( \phi_{\gamma_i(a_i)} \) relatively to \( \phi_{\gamma_i(t_i)} \). So, each \( \phi_{\gamma_i(a_i)} \) belongs to \( R(\phi_{\gamma_q(t_q)}) \) and \( \phi_{\gamma_{i+1}(a_{i+1})} \) is below \( \phi_{\gamma_i(a_i)} \) relatively to \( \phi_{\gamma_q(t_q)} \). One deduces that \( \phi_{\gamma_{q+1}(a_{q+1})} \) is below \( \phi_{\gamma_1(a_1)} \) relatively to \( \phi_{\gamma_q(t_q)} \).

Let us finish by explaining the interest of this result in the case where an admissible transverse path has a transverse self-intersection

**Proposition 15.** Suppose that \( \gamma : [a,b] \to M \) is a transverse path admissible of order \( n \) and that \( \gamma \) intersects itself transversally at \( \gamma(s) = \gamma(t) \) where \( s < t \). Then \( \gamma \mid_{[a,s]} \gamma \mid_{[t,b]} \) is admissible of order \( n \) and \( \gamma \mid_{[a,s]} (\gamma \mid_{[s,t]})^q \gamma \mid_{[t,b]} \) is admissible of order \( qn \) for every \( q \geq 1 \).

**Proof.** Applying Corollary 14 to the family:

\[
\gamma_i = \gamma, \ s_i = s \text{ if } 1 < i < q, \ t_i = t \text{ if } 1 \leq i < q
\]

one knows that

\[
\gamma \mid_{[a,t]} (\gamma \mid_{[s,t]})^{q-2} \gamma \mid_{[s,b]} = \gamma \mid_{[a,s]} (\gamma \mid_{[s,t]})^q \gamma \mid_{[t,b]}
\]

is admissible of order \( qn \) for every \( q \geq 2 \). Moreover the induction argument and the last sentence of Proposition 12 tell us either that \( \gamma \mid_{[a,s]} \gamma \mid_{[t,b]} \) is admissible of order \( n \), or that \( \gamma \mid_{[a,s]} (\gamma \mid_{[s,t]})^q \gamma \mid_{[t,b]} \) is admissible of order \( n \) for every \( q \geq 1 \). To get the proposition, one must prove that the last case is impossible.

We do not lose any generality by supposing that \( \text{dom}(F) \) is connected. Denote \( \tilde{f} = (\tilde{f}_t)_{t \in [0,1]} \) the lifted identity isotopy on the universal covering space \( \tilde{\text{dom}}(F) \) of \( \text{dom}(F) \) and set \( \tilde{f} = \tilde{f}_1 \). Write \( \tilde{F} \) for the lifted transverse foliation. Fix a lift \( \tilde{\gamma} \) of \( \gamma \) and denote \( T \) the covering automorphism such that \( \tilde{\gamma}(t) = T(\tilde{\gamma}(s)) \). The quotient space \( \widetilde{\text{dom}}(F) = \tilde{\text{dom}}(F)/T \) is an annulus and one gets an identity isotopy \( \tilde{f} = (\tilde{f}_t)_{t \in [0,1]} \) on \( \widetilde{\text{dom}}(F) \) by projection, as a homeomorphism \( \tilde{f} = \tilde{f}_1 \) and a transverse foliation \( \tilde{F} \). The path \( \tilde{\gamma} \) projects onto a transverse path \( \gamma \). The path \( \gamma' = \prod_{k \in \mathbb{Z}} T(\gamma \mid_{[s,t]}) \) is a line that lifts a loop \( \tilde{\gamma}' \) of \( \tilde{\text{dom}}(F) \) transverse to \( \tilde{F} \). The union of leaves that meet \( \tilde{\gamma}' \) is a plane \( \tilde{U} \) that lifts an annulus \( \tilde{U} \) of \( \tilde{\text{dom}}(F) \). The fact that \( \gamma \) intersects itself transversally at \( \gamma(t) = \gamma(s) \) means that \( \gamma \) and \( T(\tilde{\gamma}) \) intersect transversally at \( \tilde{\gamma}(t) = T(\tilde{\gamma}(s)) \). One deduces the following:
- the paths $\gamma_{[a,s]}$ and $\gamma_{[t,b]}$ are not contained in $\widehat{U}$;

- if $a' \in [a,s]$ is the largest value such that $\gamma(a') \notin \widehat{U}$ and $b' \in (t,b]$ the smallest value such that $\gamma(b') \notin \widehat{U}$, then $\gamma(a')$ and $\gamma(b')$ are in the same connected component of $\text{dom}(F) \setminus \gamma$.

The fact that $\gamma|_{[a,s]} (\gamma|_{[s,t]})^{g} \gamma|_{[t,b]}$ is admissible of order $n$ implies that

$$\gamma|_{[a,s]} \prod_{0 \leq k < q} T^k(\gamma|_{[s,t]}) T^{q-1}(\gamma|_{[t,b]})$$

is admissible of order $n$ or equivalently that $\tilde{f}^n(\phi_{\gamma(a')}) \cap T^{q-1}(\phi_{\gamma(b')}) \neq \emptyset$. So one must prove that this cannot happen if $q$ is large enough.

There is no loss of generality by supposing that the leaves $\phi_{\gamma(a')}$ and $\phi_{\gamma(b')}$ are on the right of $\gamma$. The projected leaves $\phi_{\gamma(a')}$ and $\phi_{\gamma(b')}$ are lines contained in the boundary of $\widehat{U}$. One can compactify the annulus $\text{dom}(F)$ with a point $S$ at the end on the right of $\widehat{\Gamma}$ and a point $N$ at the end on the left of $\widehat{\Gamma}$. We know that the $\alpha$-limit and $\omega$-limit sets of $\phi_{\gamma(a')}$ and $\phi_{\gamma(b')}$ are reduced to $S$. Let us choose real parametrizations $t \mapsto \phi_{\gamma(a')}(t)$ and $t \mapsto \phi_{\gamma(b')}(t)$ of $\phi_{\gamma(a')}$ and $\phi_{\gamma(b')}$. The sets $T^k(\tilde{f}^n(\phi_{\gamma(a')})))$, $k \in \mathbb{Z}$, are pairwise disjoint and one can choose a simple path $\alpha$ joining $T(\tilde{f}^n(\phi_{\gamma(a')}(0)))$ to $\tilde{f}^n(\phi_{\gamma(a')}(0))$ and disjoint from $\tilde{f}^n(\phi_{\gamma(a')})$ but at its ends. One can extend $\alpha$ in $L(\phi_{\gamma(a')}) \cap T(L(\phi_{\gamma(a')}))$ to a simple path $\alpha'$ joining $T(\phi_{\gamma(a')}(0))$ to $\phi_{\gamma(a')}(0)$ and disjoint from $R(\phi_{\gamma(a')})$ and $T(R(\phi_{\gamma(a')}))$ but at its ends. The path $\alpha'' = T(\phi_{\gamma(a')}([(-\infty,0])) \alpha' \phi_{\gamma(a')}([0,\infty])$ is a line and $L(\alpha'')$ contains $T(f^n(\phi_{\gamma(a')}([(-\infty,0])))$ and $f^n(\phi_{\gamma(a')}([0,\infty])))$. The fact that the $\alpha$-limit and $\omega$-limit sets of $\phi_{\gamma(b')}$ are reduced to $S$ implies that there exists $K > 0$ such that for every $q \geq 0$, $\alpha'$ does not meet neither $T^q(\phi_{\gamma(b')}([(-\infty,\infty) )))$ nor $T^q(\phi_{\gamma(b')}([K,\infty]))$. One deduces that there exists $q_0$ such that for every $q \geq q_0$, $\alpha'$ does not meet $T^q(\phi_{\gamma(b')}))$. This implies that if $q \geq q_0$, then $T^q(\phi_{\gamma(b')}))$ does not meet $\alpha''$ and so is included in $R(\alpha'')$. In particular it cannot intersect neither $T(\tilde{f}^n(\phi_{\gamma(a')}([(-\infty,0])))$ nor $\tilde{f}^n(\phi_{\gamma(a')}([0,\infty])))$. For $q \geq q_0 + 1$, this implies that $T^q(\phi_{\gamma(b')}))$ does not meet $T(\tilde{f}^n(\phi_{\gamma(a')}))$. □

**Corollary 16.** Let $\gamma : [a,b] \to M$ be a transverse path admissible of order $n$. Then there exists $\gamma' : [a,b] \to M$ a transverse path, also admissible of order $n$, such that $\gamma'$ has no transverse self-intersections, and $\phi_{\gamma'(a)} = \phi_{\gamma(a)}$, $\phi_{\gamma'(b)} = \phi_{\gamma(b)}$
Proof. Note first that there exists a transverse path $\gamma' : [a, b] \to M$ equivalent to $\gamma$ with finitely many number of (not necessarily transverse) self-intersections. Indeed, choose for every $z$ on $\gamma$, a trivialization neighborhood $W_z$. Divide the interval in $n$ intervals $J_i = [a_i, b_i]$ of equal length and set $\gamma_i = \gamma|_{J_i}$, so that $\gamma = \prod_{1 \leq i \leq n} \gamma_i$. If $n$ is large enough, then for every $i$, the union of $\gamma_i$ and all paths $\gamma_j$ that meet $\gamma_i$ is contained in a set $W_z$. Let us begin by perturbing each $\gamma_i$ to find an equivalent path $\gamma'_i$, such that $\gamma'_i(b_i) = \gamma'_{i+1}(a_i)$, if $i < n$, and such that the $\gamma'_i(b_i)$ are all distinct. One can also suppose that for every $i$, the union of $\gamma'_i$ and all $\gamma'_j$ that meet $\gamma'_i$ is contained in a set $W_z$. Suppose that for every $i < i_0$ and every $j$, the paths $\gamma'_i$ and $\gamma'_j$ have finitely intersection points. One can perturb in an equivalent way each $\gamma'_j$ on $(a_j, b_j)$, $j > j_0$, such that it intersects $\gamma_{i_0}$ finitely many often, without changing the intersection points with $\gamma_i$ if $i < i_0$ and such that condition concerning the trivialization neighborhoods is still satisfied.

Let $\mathcal{G}$ be the collection of all transversal paths that are admissible of order $n$ whose initial leaf is $\phi_{\gamma(a)}$ and whose final leaf is $\phi_{\gamma(b)}$. Let $\gamma' : [a, b] \to M$ be a path in $\mathcal{G}$ that is minimal with regards to the number of self-intersections. Then $\gamma'$ has no transverse self-intersections. Indeed, if $\gamma'$ had a transverse self-intersection at $\gamma'(t) = \gamma(s)$ where $s < t$, by the previous proposition the path $\gamma'_{|[a, s]} \gamma'_{|[s, t]}$ would also be also contained in $\mathcal{G}$ and it would have a strictly smaller number of self-intersections. □

4.2. Realizability of transverse loops. If a loop $\Gamma$ is associated to a periodic point of period $q$, its satisfies the following (where $\gamma$ is its natural lift):

$$(P_q) : \text{ for every } n \geq 1, \gamma|_{[0, n]} \text{ is admissible of order } nq.$$ The following question is natural:

Let $\Gamma$ be a transverse loop that satisfies $(P_q)$. Is $\Gamma$ equivalent to a transverse loop associated to a periodic orbit of period $q$?

We will see that it many situations, it is the case. In such situations, $F$ will have infinitely many periodic orbits. More precisely, for every rational number $r/s \in (0, 1/q)$ written in an irreducible way, the loop $\Gamma^r$ will be equivalent to a loop associated to a periodic orbit of period $s$. In fact the weaker following property will be sufficient:

$$(Q_q) : \text{ there exist two sequences } (r_k)_{k \geq 0} \text{ and } (s_k)_{k \geq 0} \text{ of natural integers satisfying }$$

$$\lim_{k \to +\infty} r_k = \lim_{k \to +\infty} s_k = +\infty, \limsup_{k \to +\infty} r_k/s_k \geq 1/q$$

such that $\Gamma^{r_k}$ is admissible of order $\leq s_k$.

We will say that a transverse loop $\Gamma$ is linearly admissible of order $q$ if it satisfies $(Q_q)$ (note that every equivalent loop will satisfy the same condition).

Let us define now the natural covering associated to $\Gamma$ ( or to its natural lift $\gamma$) and introduce some useful notations. Denote $\tilde{I} = (\tilde{f_t})_{t \in [0, 1]}$ the lifted identity isotopy on the universal covering space $\text{dom}(\mathcal{F})$ and set $\tilde{f} = \tilde{f}_1$. Write $\tilde{\mathcal{F}}$ for the lifted transverse foliation. Fix a lift $\tilde{\gamma}$ of $\gamma$ and denote $T$ the covering automorphism such that $\tilde{\gamma}(t + 1) = T(\tilde{\gamma}(t))$ for every $t \in \mathbb{R}$. The path $\tilde{\gamma}$ is a line and the union of leaves that intersect it is a topological plane $\tilde{U}$. Moreover it projects onto the natural lift of a loop $\Gamma$ in the quotient space $\text{dom}(\mathcal{F}) = \text{dom}(\mathcal{F})/T$. One gets an identity isotopy $\tilde{I} = (\tilde{f_t})_{t \in [0, 1]}$
on $\hat{\text{dom}}(F)$ by projection, as a homeomorphism $\hat{f} = \hat{f}_1$ and a transverse foliation $\hat{F}$. The loop $\hat{\Gamma}$ is transverse to $\hat{F}$ and the union of leaves that intersect it is a topological annulus $\hat{U}$.

Before stating the realization result, let us recall the following lemma (for example, see [Lec2], Theorem 9.1, for a proof that uses maximal isotopies and transverse foliations).

**Lemma 17.** Let $J$ be a real interval, $f$ a homeomorphism of $T^1 \times J$ isotopic to the identity and $\tilde{f}$ a lift of $f$ to $\mathbb{R} \times J$. We suppose that:

1. every essential simple loop $\Gamma \subset T^1 \times J$ meets its image by $f$;
2. there exist two probability measures $\mu_1$ and $\mu_2$ with compact support, invariant by $f$, such that their rotation numbers (for $\tilde{f}$) satisfy $\text{rot}(\mu_1) < \text{rot}(\mu_2)$.

Then, for every $r/s \in [\text{rot}(\mu_1), \text{rot}(\mu_2)]$ written in an irreducible way, there exists a point $z \in \mathbb{R} \times J$ such that $\tilde{f}^s(z) = z + (r, 0)$.

**Proposition 18.** Let $\Gamma$ be a linearly admissible transverse loop of order $q$ that satisfies one of the three following conditions. Then for every rational number $r/s \in (0, 1/q]$ written in an irreducible way, $\Gamma^r$ is associated to a periodic orbit of period $s$.

1. The loop $\Gamma$ has a leaf on its left and a leaf on its right, and the annulus $\hat{U}$ does not contain a simple loop homotopic to $\hat{\Gamma}$ disjoint from its image by $\hat{f}$.
2. There exists both an admissible transverse path that intersects $\Gamma$ transversally and positively, and an admissible transverse path that intersects $\Gamma$ transversally and negatively.
3. The loop $\Gamma$ has a transverse self-intersection.

**Proof.** The condition iii) is stronger than ii) because $\Gamma$ intersects itself transversally positively and negatively. The condition ii) tells us that there is an admissible transverse path that intersects $\hat{\Gamma}$ transversally and positively, and an admissible transverse path that intersects $\hat{\Gamma}$ transversally and negatively. But this implies that i) is satisfied. It remains to prove the result under the assumption that i) is satisfied.

We do not lose any generality by supposing that $\text{dom}(F)$ is connected, which means that $\hat{\text{dom}}(F)$ is a plane and $\hat{\text{dom}}(F)$ an annulus. By assumption, we know that there exists a leaf on the left of $\hat{\gamma}$ and a leaf on its right. One can compactify $\hat{\text{dom}}(F)$ with a point $S$ at the end on the right of $\hat{\Gamma}$ and a point $N$ at the end on the left of $\hat{\Gamma}$. We will denote by $\hat{\text{dom}}(F)_{\text{sph}}$ this compactification and still write $\hat{f}$ for the extension that fixes the added points. The omega-limit set $\omega(\hat{\phi})$ in $\hat{\text{dom}}(F)_{\text{sph}}$ of a leaf $\hat{\phi} \subset \hat{U}$ does not depend on $\hat{\phi}$.

**Lemma 19.** The set $\omega(\hat{\phi})$ is reduced to $S$.

**Proof.** If not, $\omega(\hat{\phi})$ is either a closed leaf that bounds $\hat{U}$ or the union of $S$ and of leaves homoclinic to $S$ (which means that the two limit sets are reduced to $S$). In the first case, the closed leaf that bounds $\hat{U}$ is homotopic to $\hat{\Gamma}$ and disjoint from its image by $\hat{f}$. A simple loop included in $\hat{U}$ sufficiently close to it will satisfy the same properties. This contradicts the assumptions of the proposition.
Let us study now the second case. Choose a point \( \tilde{z}^* \in \omega(\tilde{\phi}) \setminus \{S \} \) and denote by \( \hat{\phi}^* \) the leaf that contains \( \tilde{z}^* \). One can suppose that \( \hat{\Gamma} \) is on the right of \( \hat{\phi}^* \). This is independent of the choice of \( \tilde{z}^* \) and in that case, the leaf \( \tilde{\phi} \) is on the right of \( \hat{\phi}^* \). Moreover, in this situation, we have the following result: for every neighborhood \( V \) of \( S \), there exists a neighborhood \( W \) of \( \omega(\tilde{\phi}) \) in \( \text{dom}(F)_{sph} \) such that \( \tilde{f}(W \setminus V) \cap \tilde{U} = \emptyset \). Let us consider a simple path \( \hat{\beta} \) joining a point \( \tilde{z}^* \in \tilde{U} \) to \( \tilde{z}^* \) positively transverse to \( \tilde{F} \), included (but the end \( \tilde{z}^* \)) in \( \tilde{U} \) and sufficiently small that its image by \( \tilde{f} \) will be included in the connected component of \( \text{dom}(F) \setminus \hat{\phi}^* \) that is on the left of \( \hat{\phi}^* \). The leaf \( \tilde{\phi} \) meets \( \hat{\beta} \) in a “monotone” sequence \( (\tilde{z}_n)_{n \geq 0} \), where \( \lim_{n \to +\infty} \tilde{z}_n = \tilde{z}^* \). More precisely, for every real parametrization of \( \tilde{\phi} \), one has \( \tilde{z}_n = \tilde{\phi}(t_n) \), where \( t_{n+1} > t_n \), and \( \lim_{n \to +\infty} t_n = +\infty \). Moreover, \( \tilde{z}_{n+1} \) is “closer” to \( \tilde{z}^* \) than \( \tilde{z}_n \) on \( \tilde{\phi} \). We will prove that if \( n \) is large enough, the simple loop \( \tilde{\Gamma}_n \) obtained by concatenating the segment \( \tilde{\alpha}_n \subset \tilde{\phi} \) joining \( \tilde{z}_n \) to \( \tilde{z}_{n+1} \) and the sub-path \( \tilde{\xi}_n \) of \( \hat{\beta}^{-1} \) joining \( \tilde{z}_{n+1} \) to \( \tilde{z}_n \) is disjoint from its image by \( \tilde{f} \). We will begin by extending \( \hat{\beta} \) in a simple proper path (with the same name) contained in \( \text{dom}(F) \setminus \omega(\tilde{\phi}) \) “joining” the end \( N \) to \( \tilde{z}^* \). One can find a neighborhood \( W' \) of \( \omega(\tilde{\phi}) \) in \( \text{dom}(F)_{sph} \) that intersects \( \beta \) only between \( \tilde{z}_0 \) and \( \tilde{z}^* \). If \( n \) is large enough, \( \tilde{\alpha}_n \) will be contained in \( W' \) and so will intersect \( \hat{\beta} \) only at the points \( \tilde{z}_n \) and \( \tilde{z}_{n+1} \). We will suppose \( n \) large enough to satisfy this property. Fix a lift \( \tilde{z}_0 \) of \( z_0 \), write \( \tilde{\beta}_0 \) for the lift of \( \beta \) that contains \( \tilde{z}_0 \), write \( \tilde{z}_n^* \) for its end and \( \tilde{\phi}_0 \) for the lift of \( \hat{\phi}^* \) that contains \( \tilde{z}_0^* \). For every \( n \geq 0 \) define

\[
\tilde{\beta}_n = T^{-n}(\tilde{z}_0), \quad \tilde{\beta}_n = T^{-n}(\tilde{z}_0), \quad \tilde{\phi}_n = T^{-n}(\tilde{\phi}_0).
\]

Write \( \tilde{z}_n \) for the lift of \( z_n \) that lies on \( \tilde{\beta}_n \), write \( \tilde{\xi}_n \) for the segment of \( \tilde{\beta}_n \) that joins \( \tilde{z}_n \) to \( \tilde{z}_{n+1} \) and \( \tilde{\alpha}_n \) for the lift of \( \tilde{\alpha}_n \) that joins \( \tilde{z}_n \) to \( \tilde{z}_{n+1} \). We will prove that the line

\[
\tilde{\lambda}_n = \tilde{\phi}_n^*[-\infty,0][\tilde{\beta}_n-1][\tilde{\beta}_n+1][0, +\infty]
\]

is a Brouwer line if \( n \) is large enough. Observe first that one has

\[
L(\tilde{\phi}_n^*/ \cup L(\tilde{\phi}_n^*) \subset L(\lambda_n) \subset L(\tilde{\phi}_n^*[[-\infty,0][\beta_n^*-1]) \cap L(\tilde{\phi}_n^*{1}[0, +\infty])
\]

then note that if \( K \) is large enough one has

\[
\tilde{f}^{-1}(\tilde{\phi}_n^*[[-\infty,-K]) \subset R(\tilde{\phi}_n^*[[-\infty,0][\beta_n^*-1]), \quad \tilde{f}^{-1}(\tilde{\phi}_n^*[0,K]) \subset R(\tilde{\phi}_n^*[0, +\infty])
\]

Let \( V \) be a neighborhood of \( S \) such that \( \tilde{f}(V) \cap (\tilde{\phi}_n^*[[-K,K]) = \emptyset \) and \( W \) a neighborhood of \( \omega(\tilde{\phi}) \) such that \( \tilde{f}(W \setminus V) \cap \tilde{U} = \emptyset \). If \( n \) is large enough, then \( \tilde{\Gamma}_n \) is included in \( W \). Let us prove that \( \tilde{\lambda}_n \) is a Brouwer line of \( \tilde{f} \) and then that \( \tilde{\Gamma}_n \) is disjoint from its image by \( \tilde{f} \). The leaves \( \tilde{\phi}_n \) and \( \tilde{\phi}_n^* \) being Brouwer lines of \( \tilde{f} \), one has

\[
\tilde{f}(\tilde{\phi}_n^*[-\infty,0]) \subset L(\tilde{\phi}_n^*) \subset L(\tilde{\lambda}_n), \quad \tilde{f}(\tilde{\phi}_n^*[0, +\infty]) \subset L(\tilde{\phi}_n^*) \subset L(\tilde{\lambda}_n).
\]

By hypothesis on \( \tilde{\beta}_n \), one knows that

\[
\tilde{f}(\tilde{\xi}_n) \subset L(\tilde{\phi}_n^*) \subset L(\tilde{\lambda}_n), \quad \tilde{f}(\tilde{\phi}_n^*[0, +\infty]) \subset L(\tilde{\phi}_n^*) \subset L(\tilde{\lambda}_n).
\]

The path \( \tilde{\alpha}_n \) being included in a leaf of \( \tilde{F} \) and each leaf being a Brouwer line of \( \tilde{f} \), one knows that

\[
\tilde{f}(\tilde{\alpha}_n) \cap \tilde{\alpha}_n = \emptyset.
\]

To prove that \( \tilde{\lambda}_n \) is a Brouwer line, it remains to prove that \( \tilde{f}(\tilde{\alpha}_n) \) does not meet any of the paths

\[
\tilde{\phi}_n^*[-\infty,0], \quad \tilde{\phi}_n^*[0, +\infty]), \quad \tilde{\xi}_n, \quad \tilde{\xi}_n^*.
\]
By hypothesis, one knows that \( \hat{f}(\hat{\alpha}_n) \) does not meet neither \( \hat{\phi}^*([-K, K]) \), nor \( \hat{\zeta}_0 \). Moreover, one knows that \( \hat{\alpha}_n \) does not meet neither \( \hat{f}^{-1}(\hat{\phi}_n^*([-\infty, K])) \), nor \( \hat{f}^{-1}(\hat{\phi}_{n+1}^*([K, +\infty))) \). So, we are done.

To prove that \( \hat{\Gamma}_n \) is disjoint from its image by \( \hat{f} \), one must prove that \( \hat{\Gamma}_n \) is lifted to a path that is disjoint from its image by \( \hat{f} \). This path will be included in the union of the images by the iterates of \( T \) of the path \( \hat{\zeta}_n^{-1}\hat{\alpha}_n\hat{\zeta}_{n+1} \). So it is sufficient to prove that the union of these translates is disjoint from its image by \( \hat{f} \). Observe now that every path \( T^k(\hat{\zeta}_n^{-1}\hat{\alpha}_n\hat{\zeta}_{n+1}) \), \( k \in \mathbb{Z} \), is disjoint from \( L(\hat{\lambda}_n) \), which implies that it is disjoint from \( \hat{f}(\hat{\zeta}_n^{-1}\hat{\alpha}_n\hat{\zeta}_{n+1}) \).

\[\qed\]

**Lemma 20.** There is no simple loop included in \( \hat{\text{dom}}(\hat{F}) \) homotopic to \( \hat{\Gamma} \) that is disjoint from its image by \( \hat{f} \).

**Proof.** Suppose that there exists a simple loop \( \hat{\Gamma}_0 \) included in \( \hat{\text{dom}}(\hat{F}) \) that is homotopic to \( \hat{\Gamma} \) and disjoint from its image by \( \hat{f} \). One can suppose for instance that \( \hat{f}(\hat{\Gamma}_0) \) is included in the component of \( \text{dom}(\hat{F})_{\text{sph}} \setminus \hat{\Gamma}_0 \) that contains \( N \), and orient \( \hat{\Gamma}_0 \) in such a way that this component, denoted by \( L(\hat{\Gamma}_0) \), is on the left of \( \hat{\Gamma}_0 \). The loop \( \hat{\Gamma}_0 \) meets finitely many leaves of \( \hat{F} \) homoclinic to \( S \) that are on the frontier of \( \hat{U} \). We denote them \( \hat{\phi}_i \), \( 1 \leq i \leq p \). Let us prove first that \( \hat{\Gamma} \) is on the left side of each \( \hat{\phi}_i \). Indeed, if \( \hat{\Gamma} \) is on the right side of \( \hat{\phi}_i \), writing \( \hat{\Gamma}_0 \) for the lift of \( \hat{\Gamma}_0 \) and \( \hat{\phi}_i \) for a lift of \( \hat{\phi}_i \), one finds a non-empty compact subset \( \hat{L}(\hat{\Gamma}_0) \cap \hat{L}(\hat{\phi}_i) \) of \( \hat{M} \) that is forward invariant by \( \hat{f} \). But such a set does not exist because \( \hat{f} \) is fixed point free.

Each loop \( \hat{\phi}_i \cup \{S\} \) bounds a Jordan domain \( \hat{L}_i \) of \( \hat{\text{dom}}(\hat{F})_{\text{sph}} \) that contains \( N \). By a classical result of Kerékjártó [K], one knows that the connected component of \( L(\hat{\Gamma}_0) \cap (\bigcup_{1 \leq i \leq p} \hat{L}_i) \) that contains \( N \) is a Jordan domain whose boundary \( \hat{\Gamma}_1 \) is a simple loop homotopic to \( \hat{\Gamma} \) in \( \text{dom}(\hat{F}) \), disjoint from its image by \( \hat{f} \), and included in \( \hat{U} \cup L(\Sigma) \). By intersecting \( \hat{\Gamma}_1 \) with the leaves of \( \hat{F} \) homoclinic to \( N \) that are on the frontier of \( \hat{U} \), one constructs similarly a simple loop \( \hat{\Gamma}_2 \) included in \( \hat{\text{dom}}(\hat{F}) \cap \hat{U} \) that is homotopic to \( \hat{\Gamma} \) in \( \hat{M} \) and disjoint from its image by \( \hat{f} \). It remains to approximate \( \hat{\Gamma}_2 \) by a simple loop included in \( \hat{U} \).

\[\qed\]
End of the proof of Proposition 18 One must prove that for every rational number \( r/s \in (0, 1/q] \) written in an irreducible way, there exists a point \( \tilde{z} \in \text{dom}(F) \) such that \( \tilde{f}^s(\tilde{z}) = T^r(\tilde{z}) \). Indeed, its orbit should be contained in \( \tilde{U} \) and it will project in \( \text{dom}(F) \) on a periodic point \( z \) of \( f \) of period \( s \) and \( \Gamma^r \) will be associated to \( z \).

Write \( \tilde{\phi}_0 \) for the leaf containing \( \gamma(0) \) and \( \phi_0 \) for its projection in \( \text{dom}(F) \). Using the analogous of Lemma 19 for \( \alpha \)-limit sets, one can suppose that \( \tilde{\phi}_0 \) is a line. Let us fix a leaf \( \tilde{\phi}_N \) homoclinic to \( N \) and a leaf \( \tilde{\phi}_S \) homoclinic to \( S \). Each of them is disjoint from all its images by the \( (\text{non trivial}) \) iterates of \( \tilde{f} \). By a result of Béguin, Crovisier, Le Roux (see [Ler], Proposition 2.3.3) one knows that there exists a compactification \( \text{dom}(F)_{\text{ann}} \) obtained by blowing up the two ends \( N \) and \( S \) replaced by circles \( \Sigma_N \) and \( \Sigma_S \) such that \( \tilde{f} \) extends to a homeomorphism \( \tilde{f}_{\text{ann}} \) that admits fixed points on each added circle with a rotation number equal to zero for the lift that extends \( \tilde{f} \). Moreover, one can suppose that each set \( \omega(\tilde{\phi}_N) \) and \( \omega(\tilde{\phi}_S) \) is reduced to a unique point on \( \Sigma_N \) and \( \Sigma_S \) respectively. One can join a point of \( \tilde{\phi}_N \) to a point of \( \tilde{\phi}_S \) by a segment disjoint from \( \tilde{\phi}_0 \). Consequently, one can construct a line \( \tilde{\lambda} \) in \( \text{dom}(F) \), disjoint from \( \tilde{\phi}_0 \), that admits a limit on each added circle. Write \( \text{dom}(F)_{\text{ann}} = \text{dom}(F) \cup \Sigma_N \cup \Sigma_S \) for the universal covering space of \( \text{dom}(F)_{\text{ann}} \) and keep the notation \( T \) for the natural covering automorphism. Write \( \tilde{\lambda} \) for the lift of \( \lambda \) located between \( \tilde{\phi}_0 \) and \( T(\tilde{\phi}_0) \). One can construct a continuous real function \( \tilde{g} \) on \( \text{dom}(F)_{\text{ann}} \) that satisfies \( \tilde{g}(T(\tilde{z})) = \tilde{g}(\tilde{z}) + 1 \) and vanishes on \( \tilde{\lambda} \). The function \( \tilde{h} = \tilde{g} \circ \tilde{f} - \tilde{g} \) is invariant by \( T \) and lifts a continuous function \( h : \text{dom}(F)_{\text{ann}} \to \mathbb{R} \). If \( \mu \) is a Borel probability measure invariant by \( \tilde{f} \), the quantity \( \int_{\text{dom}(F)_{\text{ann}}} h \, d\mu \) is the rotation number of the measure \( \mu \) for the lift \( \tilde{f}_{\text{ann}} \). Let us consider now the real function \( \tilde{g}_0 \) on \( \text{dom}(F)_{\text{ann}} \), that coincides with \( \tilde{g} \) on \( \Sigma_N \cup \Sigma_S \), that satisfies \( \tilde{g}_0(T(\tilde{z})) = \tilde{g}_0(\tilde{z}) + 1 \) and that vanishes on \( \tilde{\phi}_0 \) and at every point located between \( \tilde{\phi}_0 \) and \( T(\tilde{\phi}_0) \). Note that \( \tilde{g} - \tilde{g}_0 \) is uniformly bounded by a certain number \( K \) and invariant by \( T \). The property \((Q_q)\) satisfied by \( \Gamma \) tells us that for every \( k \geq 0 \), one can find a point \( \tilde{z}_k \in R(\tilde{\phi}_0) \) such that \( \tilde{f}^{r_k}(\tilde{z}_k) \in L(T^{s_k}(\tilde{\phi}_0)) \). Write \( \tilde{z}_k \) for its projection in \( \text{dom}(F) \). Observe that

\[
\tilde{g}_0(f^{s_k}(\tilde{z}_k)) - \tilde{g}_0(\tilde{z}_k) \geq r_k.
\]

By taking a subsequence, one can suppose that

\[
\lim_{k \to +\infty} \frac{1}{s_k} (\tilde{g}_0(f^{s_k}(\tilde{z}_k)) - \tilde{g}_0(\tilde{z}_k)) = \rho \in \left[\frac{1}{q}, +\infty\right]
\]
and so that

$$\left| \frac{1}{s_k} \sum_{i=0}^{s_k-1} \mathcal{h}(\tilde{f}^i(\tilde{z}_k)) - \rho \right| = \frac{1}{s_k} \left| \tilde{g}(f^{s_k}(\tilde{z}_k)) - \tilde{g}(\tilde{z}_k) \right| - \rho \leq \frac{1}{s_k} \left| \tilde{g}_0(f^{s_k}(\tilde{z}_k)) - \tilde{g}_0(\tilde{z}_k) \right| - \rho + \frac{2K}{s_k}.$$ 

Write $\delta_z$ for the Dirac measure at a point $\tilde{z} \in \text{dom}(\mathcal{F})_{\text{ann}}$ and choose a measure $\mu$ that is the limit of a subsequence of $\left( \frac{1}{s_k} \sum_{i=0}^{s_k-1} \delta_{\tilde{f}^i_{\text{ann}}(\tilde{z}_k)} \right)_{k \geq 0}$ for the weak* topology. One knows that $\mu$ is an invariant measure of rotation number $\rho$. As the rotation number induced on the boundary circles are equal to 0, one deduces that the rotation set $\text{rot}(\tilde{f}_{\text{ann}})$ contains $[0, \rho]$. The intersection property supposed in i) implies by Lemma 17 that for every rational number $r/s \in (0, 1/q)$ written in an irreducible way, there exists a point $\tilde{z} \in \text{dom}(\mathcal{F})_{\text{ann}}$ such that $\tilde{f}_{\text{ann}}(\tilde{z}) = T_r(\tilde{z})$. But this point does not belong to the boundary circles because the induced rotation numbers are equal to 0. So its belongs to $\text{dom}(\mathcal{F})$. $\square$

5. Two important results

In this section we give a sufficient condition for the exponential growth of periodic points of a surface homeomorphism. This condition will imply that the topological entropy is positive in the compact case. We will make use of these criteria later.

We assume here, as in the previous section, that $I = (f_t)_{t \in [0,1]}$ is a maximal isotopy from $\text{Id}$ to $f$ on an oriented surface $M$ and that $\mathcal{F}$ is a singular oriented foliation transverse to $I$.

5.1. Exponential growth of periodic points. The main result of this section is

**Theorem 21.** Let $\gamma_1, \gamma_2 : \mathbb{R} \to M$ be two admissible positively recurrent transverse paths (possibly equal) with a transverse intersection. Then the number of periodic points of period $n$ for some iterate of $f$ grows exponentially in $n$.

We will begin with the following:

**Lemma 22.** Let $\gamma_1, \gamma_2$ be two admissible positively recurrent transverse paths (possibly equal) with a transverse intersection, and let $I_1$ and $I_2$ be two real segments. Then there exists a linearly admissible transverse loop $\Gamma$ with a transverse self-intersection, such that $\gamma_1|_{I_1}$ and $\gamma_2|_{I_2}$ are equivalent to sub-paths of the natural lift of $\Gamma$.

**Proof.** Let $a_1, b_1, t_1, a_2, b_2, t_2$ be such that $I_1 \subset [a_1, b_1]$, $I_2 \subset [a_2, b_2]$ and such that $\gamma_1|_{[a_1, b_1]}$ intersects transversally $\gamma_2|_{[a_2, b_2]}$ at $\gamma_1(t_1) = \gamma_2(t_2)$. Since $\gamma_1$ is recurrent, we can find

$$b_1 < a'_1 < t'_1 < b'_1 < a''_1 < t''_1 < b''_1$$

such that $\gamma_1|_{[a_1, t_1]}$, $\gamma_1|_{[t'_1, t_1]}$, and $\gamma_1|_{[t''_1, t_1]}$ are equivalent, as are $\gamma_1|_{[t_1, b_1]}$, $\gamma_1|_{[t'_1, b_1]}$, and $\gamma_1|_{[t''_1, b_1]}$. Moreover, replacing $\gamma_1$ by an equivalent path, one can suppose that $\gamma_1(t_1) = \gamma_2(t'_1) = \gamma_2(t''_1)$. Since $\gamma_2$ is recurrent, we can also find

$$b_2 < a'_2 < t'_2 < b'_2 < a''_2 < t''_2 < b''_2$$

and replace $\gamma_2$ by an equivalent path such that a similar statement holds with the necessary changes. Note that this implies that $\gamma_1$ is transverse to $\gamma_2$ at both $\gamma_1(t''_1) = \gamma_2(t_2)$ and $\gamma_1(t_1) = \gamma_2(t'_2)$. Suppose
that $\gamma_1|_{[a_1, b'_1]}$ and $\gamma_2|_{[a_2, b'_2]}$ are admissible of order $\leq q$ and let us apply Corollary \[13\] to the families $(\gamma_i)_{1 \leq i \leq 2n}$, $(s_i)_{1 \leq i \leq 2n}$, $(t_i)_{1 \leq i \leq 2n}$ where

$$\gamma_{2j+1} = \gamma_1|_{[a_1, b'_1]}, \quad \gamma_{2j} = \gamma_2|_{[a_2, b'_2]}$$

and

$$s_{2j+1} = t_1 \quad \text{if} \quad j > 0, \quad s_{2j} = t_2, \quad t_{2j+1} = t'_1, \quad t_{2j} = t'_2 \quad \text{if} \quad j < n.$$

One deduces that for every $n \geq 1$,

$$\gamma_1|_{[a_1, t_1]} \left( \gamma_1|_{[t_1, t'_1]} \gamma_2|_{[t_2, t'_2]} \right)^n \gamma_2|_{[t'_2, b_2]}$$

is admissible of order $2nq$ and consequently that $(\gamma_1|_{[t_1, t'_1]} \gamma_2|_{[t_2, t'_2]})^n$ is admissible of order $\leq 2nq$. So, the closed path $\gamma' = \gamma_1|_{[t_1, t'_1]} \gamma_2|_{[t_2, t'_2]}$ defines a loop that is linearly admissible: it satisfies the condition $(Q_{2q})$ stated in the previous section. Furthermore, since both $\gamma_1|_{[a'_1, b'_1]}$ and $\gamma_2|_{[a'_2, b'_2]}$ are sub-paths of $\gamma'$, the induced loop has a transverse self-intersection.

\[\square\]

**Lemma 23.** If there exists a linearly admissible transverse loop $\Gamma$ with transverse self-intersection, then the number of periodic points of period $n$ for some iterate of $f$ grows exponentially in $n$.

**Proof.** Suppose that $\Gamma$ satisfies the condition $(Q_{q_0})$ and denote $\gamma$ its natural lift. By assumption, there exist $s$ and $t$ such that $\gamma$ has a transverse self-intersection at $\gamma(s) = \gamma(t)$. So, one can apply the realization result (Proposition \[18\]) and deduce that $\Gamma$ is associated to a fixed point of $f^{q_0}$. Modifying $\Gamma$ in its equivalence class if necessary, one can suppose that for every $r \geq 1$, the path $\gamma|_{[0, rq_0]}$ is admissible of order $r$. Adding the same positive integer to both $s$ and $t$, one can find a positive integer $K = rq_0$ such that $\gamma|_{[0, K]}$ has a transverse self-intersection at $\gamma(s) = \gamma(t)$ and one knows that $\gamma|_{[0, mK]}$ is admissible of order $mr$ for every $m \geq 1$. Set

$$\gamma_1 = \gamma|_{[s, t]}, \quad \gamma_2 = \gamma|_{[t, s + K]}.$$

**Lemma 24.** For every sequence $(\varepsilon_i)_{i \in \mathbb{N}} \in \{1, 2\}^\mathbb{N}$, every $n \geq 1$, and every $m \geq 1$ the path

$$\gamma|_{[0, s]} \prod_{0 \leq i < n} \gamma_{\varepsilon_i} \gamma|_{[t, mK]}$$

is admissible of order $(n + m)r$.

**Proof.** We will give a proof by induction on $n$.

Let us begin with the case where $n = 1$. If $\varepsilon_0 = 1$, we must prove that $\gamma|_{[0, s]} \gamma|_{[s, t]} \gamma|_{[t, mK]} = \gamma|_{[0, mK]}$ is admissible of order $(m + 1)r$, which is true by hypothesis (it is admissible of order $mr$). If $\varepsilon_0 = 2$, we must prove that

$$\gamma|_{[0, s]} \gamma|_{[s + K, t]} \gamma|_{[t, mK]} = \gamma|_{[0, s]} \gamma|_{[s + K, t + K, (m + 1)K]}$$

is admissible of order $(m + 1)r$. The path $\gamma|_{[0, (m + 1)K]}$ having a transverse self-intersection at $\gamma(t) = \gamma(s)$ and being admissible of order $\leq (m + 1)r$, one deduces by Proposition \[15\] that $\gamma|_{[0, s]} \gamma|_{[t, (m + 1)K]}$ is admissible of order $(m + 1)r$. This last path has a transverse self-intersection at $\gamma(t + K) = \gamma(s + K)$, applying Proposition \[15\] again, one deduces that $\gamma|_{[0, s]} \gamma|_{[t, s + K]} \gamma|_{[t + K, (m + 1)K]}$ is admissible of order $(m + 1)r$.

Suppose now the result proved for $n$. There is three cases to consider.
If the sequence \( (\varepsilon_i)_{0 \leq i \leq n} \) is constant equal to 1, we apply Corollary \textup{14} to the families \((\gamma_i)_{1 \leq i \leq n+1}, (s_i)_{1 \leq i \leq n+1}, (t_i)_{1 \leq i \leq n+1} \) where
\[
\gamma_i = \gamma|_{[0,K]} \quad \text{if} \quad i \leq n, \quad \gamma_{n+1} = \gamma|_{[0,mK]}
\]
and
\[
s_i = s \quad \text{if} \quad i > 0, \quad t_i = t \quad \text{if} \quad i \leq n
\]
One deduces that
\[
\gamma|_{[0,t]} \left( \gamma|_{[s,t]} \right)^{n-1} \gamma|_{[s,mK]} = \gamma|_{[0,s]} \left( \gamma|_{[s,t]} \right)^{n+1} \gamma|_{[t,mK]}
\]
is admissible of order \((m+n)r\) and so is admissible of order \((m+n+1)r\).

If there exists \( n' < n \) such that \( \varepsilon_{n'} = 2 \) and \( \varepsilon_i = 1 \) if \( i > n' \), we apply Corollary \textup{14} to the families
\[
(\gamma_i)_{1 \leq i \leq n-n'+1}, (s_i)_{1 \leq i \leq n-n'+1}, (t_i)_{1 \leq i \leq n-n'+1}
\]
where
\[
\gamma_0 = \gamma|_{[0,s]} \prod_{0 \leq i < n'} \gamma_i, \gamma|_{[t,K]} \quad \gamma_i = \gamma|_{[K,2K]} \quad \text{if} \quad 1 \leq i < n-n', \quad \gamma_{n-n'+1} = \gamma|_{[K,(m+1)K]}
\]
and
\[
s_i = s + K \quad \text{if} \quad i > 0, \quad t_i = t + K \quad \text{if} \quad i \leq n-n'
\]
The induction hypothesis tells us that \( \gamma_0 = \gamma|_{[0,s]} \prod_{0 \leq i < n'} \gamma_i, \gamma|_{[t,K]} \) is \((n'+2)r\) admissible, so
\[
\gamma|_{[0,s]} \prod_{0 \leq i < n'} \gamma_i, \gamma|_{[t,s+K]} \left( \gamma|_{[s+K,t+K]} \right)^{n-n'+1} \gamma|_{[t+K,(m+1)K]} = \gamma|_{[0,s]} \prod_{0 \leq i < n} \gamma_i, \gamma|_{[t,mK]}
\]
is admissible of order \((n'+2)r + (n-n'-1)r + mr = (m+n+1)r\).

If \( \varepsilon_n = 2 \), we must prove that
\[
\gamma|_{[0,s]} \prod_{0 \leq i < n} \gamma_i, \gamma|_{[t,s+K]} \gamma|_{[t,mK]} = \gamma|_{[0,s]} \prod_{0 \leq i < n} \gamma_i, \gamma|_{[t,s+K]} \gamma|_{[t+K,(m+1)K]}
\]
is admissible of order \((m+n+1)r\). The path \( \gamma|_{[K,(m+1)K]} \) having a transverse self-intersection at \( \gamma(t+K) = \gamma(s+K) \), the same is true when we extend this path on the left by adding \( \gamma|_{[t,K]} \prod_{0 \leq i < n} \gamma_i, \gamma|_{[t,K]} \). Moreover by induction hypothesis, one knows that
\[
\gamma|_{[0,s]} \prod_{0 \leq i < n} \gamma_i, \gamma|_{[t,K]} \gamma|_{[K,(m+1)K]} = \gamma|_{[0,s]} \prod_{0 \leq i < n} \gamma_i, \gamma|_{[t,K]} \gamma|_{[t+K,(m+1)K]}
\]
is admissible of order \((m+n+1)r\). Applying Proposition \textup{15} one deduces that
\[
\gamma|_{[0,s]} \prod_{0 \leq i < n} \gamma_i, \gamma|_{[t,s+K]} \gamma|_{[t+K,(m+1)K]} = \gamma|_{[0,s]} \prod_{0 \leq i < n} \gamma_i, \gamma|_{[t,K]}
\]
is admissible of order \((m+n+1)r\).

Let \( e = (\varepsilon_i)_{i \in \mathbb{N}} \in \{0,1\}^\mathbb{N} \) be a periodic word of period \( q \). The previous lemma tells us that for every \( n \), the path
\[
\gamma|_{[0,s]} \left( \prod_{0 \leq i < q} \gamma_i \right)^n \gamma|_{[t,K]}
\]
is admissible of order \((1+qn)r\) and consequently that \( \left( \prod_{0 \leq i < q} \gamma_i \right)^n \) is admissible of order \((1+qn)r\).

So, the closed path \( \prod_{0 \leq i < q} \gamma_i \) defines a loop \( \Gamma_e \) that is linearly admissible: it satisfies the condition \((Q_q)\). Note that \( \Gamma_e \) has a self-intersection if \( q \geq 2 \) (which means that \( e \) is not a constant sequence). In that case, the realization result (Proposition \textup{18}) can be applied. In particular, \( \Gamma_e \) is associated
to a fixed point of \( f^q \). The paths \( \Gamma_e \) and \( \Gamma_{e'} \) are not equivalent if \( e \) and \( e' \) are two periodic words corresponding to different orbits of the Bernoulli shift. So the associated periodic orbits are distinct. The number of periodic points of \( f^q \) of period \( q \geq 2 \) is at least equal to the number of periodic points of period \( q \) of the Bernoulli shift.

\[ \square \]

5.2. **Topological entropy.** By the previous result, it is natural to believe that the topological entropy is positive, in case \( M \) is compact. The next result asserts that this is the case:

**Theorem 25.** Let \( M \) be a compact surface, \( \gamma_1, \gamma_2 : \mathbb{R} \rightarrow M \) be two admissible positively recurrent transverse paths (possibly equal) with a transverse intersection. Then the topological entropy of \( f \) is positive.

**Remark 26.** As we will see in the proof, Theorem 25 will be stated even in case where \( M \) is not compact by proving that its Alexandrov extension has positive entropy. More precisely, write \( \text{dom}(I)_{\text{alex}} \) for the Alexandrov compactification of \( \text{dom}(I) \) if it is not compact, and \( f_{\text{alex}} \) for the extension of \( f|_{\text{dom}(I)} \) that fixes the point at infinity (otherwise set \( \text{dom}(I)_{\text{alex}} = \text{dom}(I) \) and \( f_{\text{alex}} = f|_{\text{dom}(I)} \) in what follows). Of course, \( f_{\text{alex}} \) is a factor of \( f \) and so \( h(f) \geq h(f_{\text{alex}}) \) if \( M \) is compact.

Theorem 25 will be the direct consequence of Lemma 22 and the following lemma:

**Lemma 27.** Let \( \Gamma \) be a transverse loop with a transverse self-intersection, and \( \gamma \) its natural lift. Assume that there exists integers \( K, r \) such that \( \gamma||[0,K]\) has a transverse self-intersection, and such that \( \gamma||[0,mK] \) is admissible of order \( mr \) for every \( m \geq 1 \). Then the topological entropy of \( f_{\text{alex}} \) is at least equal to \( \log 2/(2r) \).

**Proof.** Keeping the notation of Lemma 22, set \( \gamma_1 = \gamma||[s,t] \), \( \gamma_2 = \gamma||[t,K+s] \), and \( \gamma'_1 = \gamma_1 \gamma_2 \), \( \gamma'_2 = \gamma_2 \gamma_1 \). Note that \( \gamma'_2 = \gamma||[2K+2s] \) contains \( \gamma||[K,2K] \) as a sub-path and so has a transverse self-intersection. One deduces that \( \gamma'_2 \) has a leaf on its right and a leaf on its left, which implies that \( \gamma'_1 \) satisfies the same property. One proves similarly that \( \gamma'_1 \) has a leaf on its right and a leaf on its left. Note also that, by Lemma 24, for every sequence \( (\varepsilon_i)_{i \in \mathbb{N}} \in \{1,2\}^\mathbb{N} \), every \( n \geq 1 \), and every \( m \geq 1 \) the path

\[ \left( \prod_{0 \leq i < n} \gamma'_{\varepsilon_i} \right)^m \]

is admissible of order \( (1 + 2nm)r \).

By Lemma 10 there exists a neighborhood \( V_\infty \) of \( \infty \) such that for every \( n \geq 1 \) and every \( z \in \text{dom}(I) \cap \bigcap_{0 \leq k < n} f^{-k}(V_\infty) \) none of the paths \( \gamma'_1 \) and \( \gamma'_2 \) is equivalent to a sub-path of \( I^p_f(z) \). By Lemma 9, for every \( z \in \text{dom}(I) \), there exists a neighborhood \( V_z \) of \( z \) such that, for every \( z' \) and \( z'' \) in \( V_z \), the path \( I^p_f(z') \) is a sub-path of \( I^p_f(f^{-1}(z'')) \). For every \( p \geq 1 \) define \( V_{\infty,p} = \bigcap_{|{\vec{k}|} \leq p} f^{-k}(V_\infty) \) and consider the covering \( V_p \) of \( \text{dom}(I)_{\text{alex}} \) that consists of \( V_{\infty,p} \) and the \( V_z \), \( z \in \text{dom}(I) \). Lemma 27 is an immediate consequence of the following:

**Lemma 28.** The entropy of \( f_{\text{alex}} \) relative to the covering \( V_p \) is at least equal to \( \log 2/(2r) - \log 6/p \).

**Proof.** As seen in the proof of the previous theorem, for every \( n \geq 1 \) and every \( \varepsilon = (\varepsilon_i)_{0 \leq i < n} \in \{1,2\}^n \), the path \( \prod_{0 \leq i < n} \gamma'_{\varepsilon_i} \) is admissible of order \( \leq (2n + 1)r \). One can apply Corollary ?? and deduce that \( \prod_{0 \leq i < n} \gamma'_{\varepsilon_i} \) is admissible of order \( (2n + 1)r \): there exists a point \( z_\varepsilon \in \text{dom}(I) \) such that \( I^{(2n+1)r}_f(z_\varepsilon) \)
is equivalent to \( \prod_{0 \leq \epsilon < n} \gamma'_\epsilon \). We will prove that every open set of the covering \( \{ 0, \ldots, (2n + 1)r \} \) has at most \( 6^{(2n + 1)r} / p \) points \( z_e \). We deduce that every finite sub-covering of \( \{ 0, \ldots, (2n + 1)r \} \) has at least \( 2n / 6^{(2n + 1)r} \) open sets and so

\[
 h(f_{\text{alex}}, \mathcal{P}) \geq \lim_{n \to +\infty} \frac{1}{(2n + 1)r} \log(2^n / 6^{(2n + 1)r} / p) = \log 2 / (2r) - \log 6 / p.
\]

If \( K \) is a subset of \( \{ 0, \ldots, (2n + 1)r \} \) we write \( \lfloor K \rfloor_p \) for the set of integers \( k \in \{ 0, \ldots, (2n + 1)r \} \) such that there exists \( k' \in K \) satisfying \( |k - k'| \leq s \). We will also define the set \( \pi_0(K) \) of maximal intervals contained in \( K \). Let us consider an element \( \mathcal{W} = \cap_{0 \leq \epsilon < (2n + 1)r} f^{-k}(\mathcal{V}) \) of \( \mathcal{V}_p \) and define

\[
 K = \{ k \in \{ 0, \ldots, (2n + 1)r - 1 \}, \mathcal{V}^k = \mathcal{V}_{\infty,p} \}.
\]

Note that every interval \( \epsilon \in \pi_0([K]_p) \) has at least \( p \) elements and that for every \( z \in W \), the path \( \prod_{k \in I} f^k(z) \) does not contain neither \( \gamma'_1 \) nor \( \gamma'_2 \) as an equivalent sub-path (by definition of \( V_\infty \) and \( V_{\infty,p} \)). Suppose that \( z_e \in W \). Say that \( l \in \{ 0, \ldots, n - 1 \} \) meets \( k \in \{ 0, \ldots, (2n + 1)r - 1 \} \) if \( I^k_{(z)} \) is not a equivalent to a sub-path of \( \prod_{0 \leq \epsilon < l} \epsilon \) and \( \prod_{0 \leq \epsilon < l} \epsilon \) is not equivalent to a sub-path of \( I^k_{(z)} \). Write \( L_e(z_e) \) for the set of \( l \in \{ 0, \ldots, n - 1 \} \) that meet an element of \( \pi_0([K]_p) \). There are two possibilities: either \( L_e(z_e) \) is reduced to an element \( l \) and we will set \( \gamma'_l(z_e) = 'l \), or it contains two elements \( l, l + 1 \) and we will set \( \gamma'_l(z_e) = 'l, 'l+1 \). In any event, for each \( l \in \pi_0([K]_p) \) and each \( z_e \in W \), there are six different possibilities for \( \gamma'_l(z_e) \).

Note that \( L(z_e) = \bigcup_{l \in \pi_0([K]_p)} L_l(z_e) \) is the set of \( l \in \{ 0, \ldots, n - 1 \} \) that meets an element of \( [K]_p \) and that

\[
 2\pi_0([K]_p) \leq \lfloor 2 \pi_0([K]_p) \rfloor \leq 2(2n + 1)r / p.
\]

We will prove that if \( z_e \) and \( z_{e'} \) belong to \( W \) and if the families \( (\gamma'_l(z_e))_{l \in \pi_0([K]_p)} \) and \( (\gamma'_l(z_{e'}))_{l \in \pi_0([K]_p)} \) are equal, then \( e = e' \). Note that, since \( 2 \pi_0([K]_p) \leq (2n + 1)r / p \), there are no more than \( 6(2n + 1)r / p \) such families, and we get our result.

There exists a transverse path \( \gamma'_l(z_e) \) equivalent to \( \gamma'_l(z_{e'}) \) that contains all the \( f^k(z_e) \), \( k \in I \). Fix \( \epsilon \) and \( \epsilon' \), two consecutive maximal intervals of \( [K]_p \), and \( k \in \epsilon \), \( k' \in \epsilon' \). One gets a transverse path by following \( \gamma'_l(z_e) \) until reaching \( f^k(z_e) \), then \( I^{k-k}_f(f^k(z_e)) \) and finally \( \gamma'_l(z_e) \) starting from \( f^{k'}(z_e) \). Up to equivalence, this path does not depend on \( k \) and \( k' \). Note now that it remains unchanged if \( z_e \) is replaced by \( z_{e'} \). Indeed, by definition of the open sets \( \mathcal{V}_z \), \( z \in \text{dom}(I) \), if \( k \) is the largest integer in \( \epsilon \) and \( k' \) the smallest integer in \( \epsilon' \), then \( I^{k-k}_f(f^k(z_e')) \) is equivalent to a sub-path of \( I^{k-k}_f(f^k(z_{e'})) \) and \( I^{k-k}_f(f^k(z_{e'})) \) equivalent to a sub-path of \( I^{k-k+2}_f(f^{k+1}(z_e)) \). Consequently, there is the same number of integers \( l \) between \( L_e(z_e) \) and \( L_{e'}(z_{e'}) \) and between \( L_{e'}(z_e) \) and \( L_{e'}(z_{e'}) \). Moreover, writing \( L_e(z_e) < l < L_{e'}(z_{e}) \) for the first set and \( L_{e'}(z_{e'}) < l < L_{e'}(z_{e'}) \) for the second one, one deduces that the sequences \( (\epsilon)_{L_e(z_e)} < l < L_{e'}(z_{e}) \) and \( (\epsilon')_{L_{e'}(z_{e'})} < l < L_{e'}(z_{e'}) \) are equal. One has a similar result at the left of the first interval \( \epsilon \) and at the right of the last one. One deduces that the families \( (L_e(z_e))_{\epsilon \in \pi_0([K]_p)} \) and \( (L_{e'}(z_{e'}))_{\epsilon \in \pi_0([K]_p)} \) are equal and that \( e = e' \).

As a direct application of Lemma [27] we can obtain the following result, which is connected to the study of the minimal entropy of pure braids in \( S^2 \). There are sharper results with a larger lower bound for the entropy (see [34]), but they use very different techniques.
Theorem 29. Let $I = (f_t)_{t \in [0,1]}$ be a maximal isotopy on $S^2$ and assume there exists $z \in \text{dom}(I) \cap \text{fix}(f)$ be such that the loop naturally defined by $I(z)$ is not homotopic to a multiple of a simple loop in $\text{dom}(I)$. Then the entropy of $f$ is at least equal to $\log(2)/(4)$.

Proof. Let $\mathcal{F}$ be a foliation transverse to the isotopy. By hypothesis, the transverse loop $\Gamma$ associated to $z$ is not a multiple of a simple loop, so by Proposition 2 it has a transverse self-intersection. If $\gamma$ is the natural lift of $\Gamma$, then for all integers $K$, $\gamma \mid_{[0,K]}$ is admissible of order $K$. Furthermore, by Proposition 7, $\gamma \mid_{[0,2]}$ has a transverse self-intersection. The theorem then follows directly from Lemma 2.

5.3. Associated subshifts. Let us give a natural application of Corollaries 13 and 14 and the results of this section. Let $I = (f_t)_{t \in [0,1]}$ be a maximal isotopy from $\text{Id}$ to $f$ on an oriented surface $M$ and $\mathcal{F}$ a foliation transverse to $I$. Consider a transverse path $\gamma : [a, b] \to \text{dom}(\mathcal{F})$ with finitely many double points, none of them corresponding to an end of the path and no triple points (by a slight modification of the argument given in the proof of Corollary 10 one can show that every transverse path is equivalent to such a path). There exists real numbers $a < t_1 < \ldots < t_{2r} < b$ and a fixed point free involution $\sigma$ on $\{1, \ldots, 2r\}$ such that $\gamma(t_i) = \gamma(t_{\sigma(i)})$, for every $i \in \{1, \ldots, 2r\}$ and such that $\gamma$ is injective on the complement of $\{t_1, \ldots, t_{2r}\}$. Let us set $t_0 = a$ and $t_{2r+1} = b$ and define for every $i \in \{0, \ldots, 2r\}$ the path $\gamma_i = \gamma|_{[t_i, t_{i+1}]}$. Let us consider the incidence matrix $P \in M_{2r+1}(\mathbb{Z})$ (indexed by $\{0, \ldots, 2r\}$) such that $P_{i,j} = 1$ if $j = i + 1$ or $j = \sigma(i+1)$ and $0$ otherwise (in particular if $i = 2r$). Note that the first column and the last row only contain 0. For every $P$-admissible word $(i_s)_{1 \leq s \leq s_0}$, which means that $P_{i_s,i_s+1} = 1$ if $s < s_0$, the path $\prod_{1 \leq s \leq s_0} \gamma_i$, is transverse to $\mathcal{F}$. Note that every transverse path $\gamma : [a', b'] \to \text{dom}(\mathcal{F})$ whose image is contained in the image of $\gamma$ is a sub-path of such a path $\prod_{1 \leq s \leq s_0} \gamma_{i_s}$.

Suppose now that $\gamma$ is admissible of order $n$. Can we decide when a path $\prod_{1 \leq s \leq s_0} \gamma_i$, is admissible and what is its order? More precisely, do there exist other incidence matrices $P'$ smaller than $P$ (which means that $P'_{i,j} = 0$ if $P_{i,j} = 0$) such that $\prod_{1 \leq s \leq s_0} \gamma_i$, is admissible if $(i_s)_{1 \leq s \leq s_0}$ is $P'$-admissible? Corollaries 13 and 14 imply that the following three matrices satisfy this property:

- the matrix $P^{\text{strong}}_{i,j}$, where $P^{\text{strong}}_{i,j} = 1$ if and only if $j = i + 1$, or if $j = \sigma(i+1)$ and $\gamma_i \gamma_{i+1}$ and $\gamma_{j-1} \gamma_j$ have a transverse intersection at $\gamma(t_{i+1}) = \gamma(t_j)$;
- the matrix $P^{\text{left}}_{i,j}$, where $P^{\text{left}}_{i,j} = 1$ if and only if $j = i + 1$, or if $j = \sigma(i+1)$ and $\gamma$ has a transverse positive self-intersection at $\gamma(t_{i+1}) = \gamma(t_j)$;
- the matrix $P^{\text{right}}_{i,j}$, where $P^{\text{right}}_{i,j} = 1$ if and only if $j = i + 1$, or if $j = \sigma(i+1)$ and $\gamma$ has a negative self-intersection at $\gamma(t_{i+1}) = \gamma(t_j)$.

More precisely, if $P'$ is one of the three previous matrices, then for every $P'$-admissible word $(i_s)_{1 \leq s \leq s_0}$, the path $\prod_{1 \leq s \leq s_0} \gamma_i$, is admissible of order $kn$, where $k$ is the number of $s < s_0$ such that $i_{s+1} = \sigma(i_s)$. As explained in Proposition 15 its order can be less. One can adapt the proof of Theorem 25 to give a lower bound to the topological entropy of $f_{\text{alex}}$. For example it is at least equal to $1/n$ times the logarithm of the spectral radius of $P'$ if the paths $\gamma_i$ have a leaf on their right and a leaf on their left, otherwise one has to replace these paths by finite admissible words. One can adapt the proof of Theorem 21 to show that for every $P'$-admissible word $(i_s)_{1 \leq s \leq s_0}$ such that $i_1 = i_{s_0}$, the loop naturally defined by $\prod_{1 \leq s \leq s_0} \gamma_i$, is associated to a periodic orbit (except some exceptional cases).

Let us illustrate this procedure with four examples, where we start with an admissible path of order 1:
For the first example, see Figure 8, the admissibility matrices are

\[
P_{\text{strong}}^1 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
\begin{align*}
P_{\text{left}}^1 &= \begin{pmatrix}
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, &
P_{\text{right}}^1 &= \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\end{align*}
\]

The matrix \(P_{\text{strong}}^1\) does not tell us anything, the only admissible paths are sub-paths of \(\gamma = \gamma_0 \ldots \gamma_4\). The only interesting informations got from \(P_{\text{left}}^1\) and \(P_{\text{right}}^1\) respectively are the facts that the loops naturally defined by \(\gamma_2\gamma_3\) and \(\gamma_1\gamma_2\) are linearly admissible of order 2. Nevertheless the first loop has no leaf on its left while the second one has no leaf on its right. So, one cannot apply Proposition 18 and deduce that the loops are associated to periodic points of period 2. Note also that the spectral radius of \(P_{\text{left}}^1\) and \(P_{\text{right}}^1\) are equal to 1. In this example, one cannot deduce neither the positivity of entropy, nor the existence of periodic orbits.

For the second example, see Figure 9, the admissibility matrices are

\[
P_{\text{strong}}^2 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]
Figure 9. Example 2 - Leafs of the foliation are represented as dashed lines, while transverse paths are solid.

The matrix $P_{\text{left}}^2$ does not tell us anything. The matrix $P_{\text{right}}^2$ is nilpotent and the only admissible paths are $\gamma_0\gamma_4$, $\gamma_0\gamma_1\gamma_3\gamma_4$ and $\gamma$, all of them admissible of order 1 by Proposition 15. The matrix $P_{\text{left}}^2$ is much more interesting: its spectral radius, the real root of the polynomial $X^3 - X^2 - 1$, is larger than 1. The loop naturally defined by $\gamma_2$ is linearly admissible of order 1 but has no leaf on its left: one cannot deduce that it is associate to a fixed point. If $p \geq 1$, the loop naturally defined by $\gamma_1^p\gamma_2\gamma_3$ is linearly admissible of order $p$ and has leaf on its right and a leaf on its left. More precisely, it has a transverse self-intersection, so one can apply Proposition 18 and deduce that it is associate to a periodic point of period $p$. In particular, the loop defined by $\gamma_1\gamma_2\gamma_3$ is associate to fixed point: one can apply Theorem 29 and deduce that the entropy of $f$ is at least $\log 2/4$.

Figure 10. Example 3 - Leafs of the foliation are represented as dashed lines, while transverse paths are solid.

In third example, see figure 10 the trajectory is the same as in the first example but the foliation is different. The admissibility matrices are
The matrices $P_{\text{right}}^3$ and $P_{\text{left}}^3$ are the same as in the first example. Nevertheless, one can say more. Indeed the loops defined by $\gamma_2 \gamma_3$ and $\gamma_1 \gamma_2$, which are linearly admissible of order 2, now have a leaf on their left and a leaf on their right. They intersect transversally and negatively the paths $\gamma_0 \gamma_1$ and $\gamma_3 \gamma_4$ respectively but they do not interest transversally and positively a path drawn on $\gamma$. So, one cannot apply Proposition 18. However, if they are not associated to periodic points of period 2, they are homotopic in the domain to a simple loop that does not meet its image by $f$. In particular, if $\Omega(f) = S^2$, they must be associate to periodic points of period 2. The matrix $P_{\text{strong}}^3$ is much more interesting: its spectral radius is equal to $\sqrt{2}$. Every path defined by a word of length $n$ in the alphabet $\{\gamma_1 \gamma_2, \gamma_3 \gamma_2\}$ is admissible of order $2n$ and intersect $\gamma$ transversally. The proofs of Theorem 25 and Theorem 21 tell us that the topological entropy of $f$ is at least equal to $\log 2/2$, and that the number of fixed point of $f^{2n}$ in the domain is at least equal to $e^n$ if $\Omega(f) = S^2$.

In the fourth example, see Figure 11, the foliation is the same as in the first example but the trajectory is different. In particular, there are three points of self-intersection of $\gamma$, and all are transversal. The admissibility matrices are:
$P_{1}^{\text{strong}} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}$,

$P_{1}^{\text{left}} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}$,

$P_{1}^{\text{right}} = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}$.

By inspection of $P_{1}^{\text{left}}$ one assures the existence of a single admissible loop $\gamma_{4}\gamma_{5}$, while inspection of $P_{1}^{\text{right}}$ we see that both loops $\gamma_{2}$ and $\gamma_{2}\gamma_{3}\gamma_{4}\gamma_{1}$ are admissible and that the entropy of $f$ must be positive.

6. Applications for general surfaces

In this section we give two applications for homeomorphisms of compact oriented surfaces. The first one is a new proof of Handel’s result on transitive homeomorphisms of the sphere. The second application provides sufficient conditions for the existence of non-contractible periodic orbits, and has as consequences a problem posed by P. Boyland for the annulus.

6.1. Transitive maps of surfaces of genus 0. Let us begin by the following slight improvement of a result due to Handel [H].

**Theorem 30.** Let $f : S^2 \to S^2$ be an orientation preserving homeomorphism such that the complement of the fixed point set is not an annulus. If $f$ is topologically transitive then the number of periodic points of period $n$ for some iterate of $f$ grows exponentially in $n$. Moreover, the entropy of $f$ is positive.

**Proof.** Recall that, in our case, the transitivity implies the existence of a point $z$ whose $\omega$-limit set is the whole sphere. One knows that every connected component of $S^2 \setminus \text{fix}(f)$ is invariant (see [BK]). Since $f$ has a dense orbit, this complement must be connected. Moreover it cannot be a disk because $f$ has a dense orbit. Indeed Brouwer’s plane translation theorem implies that every fixed point free orientation preserving homeomorphism of the plane has only wandering points. One deduces that the fixed point set has at least three connected components. Choose three fixed points in different connected components and an isotopy $I'$ from identity to $f$ that fixes these three fixed points (this is always possible). Using Jaulent’s or Béguin-Crovisier-Le Roux’s theorem, one can find a maximal isotopy $I$ larger than $I'$. Let $\mathcal{F}$ be a foliation transverse to this isotopy. It has the same domain as $I$, and this domain is not an annulus because $I$ is larger than $I'$. The fact that $\omega(z) = S^2$ implies that $I_f(z)$ contains as a sub-path (up to equivalence) every admissible segment. In particular $I_f(z)$ is an admissible recurrent transverse path, that crosses all leaves of $\mathcal{F}$. Since $\text{dom}(\mathcal{F})$ is not a topological
annulus, this implies that \( \tilde{f}(z) \) has a transverse self-intersection by Proposition 2 and the result follows from Theorems 21 and 25.

6.2. Existence of non-contractible periodic orbits. A periodic point \( x \in M \) of period \( q \) is said to have a contractible orbit if \( f^q(x) \) naturally defines a homotopically trivial loop, otherwise it is said to be non-contractible. In this subsection we examine some conditions that ensure the existence of non-contractible periodic orbits of arbitrarily high period. Through this subsection we assume that \( \tilde{M} \) is the universal covering space of \( M \), that \( \tilde{F} \) is the lift of \( F \), that \( \tilde{I} = (\tilde{f}_t)_{t \in [0,1]} \) is the lifted identity isotopy and we write \( \tilde{f}_1 = \tilde{f} \).

The main technical result is the following proposition:

**Proposition 31.** Suppose that there exists an admissible recurrent path \( \tilde{\gamma} \) for \( \tilde{f} \), a leaf \( \tilde{\phi} \) of \( \tilde{F} \) and three distinct covering automorphisms \( T_i, 1 \leq i \leq 3 \), such that \( \tilde{\gamma} \) crosses each \( T_i(\tilde{\phi}) \). Then there exists \( q > 0 \) and a non trivial covering automorphism \( T = T_i \circ T_j^{-1} \) such that for all \( r/s \in (0,1/q] \), the maps \( \tilde{f}^s \circ T^{-r} \) and \( \tilde{f}^s \circ T^r \) have fixed points. In particular, \( f \) has non-contractible periodic points of arbitrarily large period.

**Proof.** By assumptions, we know that \( M \) is not simply connected and that \( \tilde{M} \) is a topological plane. Note that for every loop \( \tilde{\Gamma} \) that is transverse to \( \tilde{F} \), the dual function \( \delta_{\tilde{\Gamma}} \) that vanishes on the unbounded connected component of \( \tilde{M} \setminus \tilde{\Gamma} \) is the winding number of \( \tilde{\Gamma} \).

**Lemma 32.** The set of singular points \( \tilde{z} \) of \( \tilde{F} \) such that \( \delta_{\tilde{F}}(\tilde{z}) \neq 0 \) is a non empty compact subset \( \Sigma_{\tilde{F}} \) of \( \tilde{M} \). Furthermore \( \Sigma_{\tilde{F}} = \Sigma_{\tilde{F}'} \) if \( \tilde{\Gamma} \) and \( \tilde{\Gamma}' \) are equivalent transverse loops.

**Proof.** The fact that \( \Sigma_{\tilde{F}} = \Sigma_{\tilde{F}'} \), if \( \tilde{\Gamma} \) and \( \tilde{\Gamma}' \) are equivalent transverse loops is obvious as is the fact that \( \Sigma_{\tilde{F}} \) compact. To prove that this set is not empty, let us consider a leaf \( \tilde{\phi} \) that meets \( \tilde{\Gamma} \). As recalled in the first section, at least one the two following assertion is true:

- the set \( \alpha(\tilde{\phi}) \) is a non empty compact set and \( \delta_{\tilde{\Gamma}} \) takes a constant positive value on it;
- the set \( \omega(\tilde{\phi}) \) is a non empty compact set and \( \delta_{\tilde{\Gamma}} \) takes a constant negative value on it.

Suppose for instance that we are in the first situation. If \( \alpha(\tilde{\phi}) \) contains a singular point, we are done. If not, it is a closed leaf disjoint from \( \tilde{\Gamma} \). More precisely, \( \alpha(\tilde{\phi}) \) is contained in a bounded connected component of \( \tilde{M} \setminus \tilde{\Gamma} \) where \( \delta_{\tilde{\Gamma}} \) takes a constant negative value. This component contains the bounded component of the complement of \( \alpha(\tilde{\phi}) \) and so contains a singular point.

Let us prove first that there exists an admissible loop \( \tilde{\Gamma} \) that crosses each \( T_i(\tilde{\phi}) \). Let us suppose first that \( \tilde{\gamma} \) has a transverse self-intersection and choose \( a_1, b_1, a_2, b_2, t_1, t_2 \) be such that \( \tilde{\gamma}_1|_{[a_1, b_1]} \) intersects transversally \( \tilde{\gamma}_2|_{[a_2, b_2]} \) at \( \tilde{\gamma}_1(t_1) = \tilde{\gamma}_2(t_2) \) and such that \( \tilde{\gamma}_1|_{[a_1, b_1]} \) crosses each \( T_i(\tilde{\phi}) \). The construction done in the proof of Lemma 22 gives us such a loop \( \tilde{\Gamma} \). Let us suppose now that \( \tilde{\gamma} \) has no self-intersection. By Proposition 2 one knows that \( \tilde{\gamma} \) is equivalent to the natural lift of a simple loop \( \tilde{\Gamma} \) and this loop satifies the desired property.

Let us prove now that one can find at least two distinct loops among the \( T_i^{-1}(\tilde{\Gamma}) \) that have a transverse intersection. If not, by Proposition 1 one can find for every \( i \in \{1,2,3\} \) a transverse loop \( \Gamma'_i \) equivalent to \( T_i^{-1}(\tilde{\Gamma}) \) such that the \( \Gamma'_i \) are pairwise disjoint. The three functions \( \delta_{\Gamma'_i} \) are decreasing on the leaf \( \tilde{\phi} \).

This implies that there exists a point \( \tilde{z} \in \tilde{\phi} \) and two different indices \( i \) and \( j \) such that \( \delta_{\Gamma'_i}(\tilde{z}) \neq 0 \) and
\( \delta_{\Gamma_j}(\bar{z}) \neq 0 \). The fact that there exists a point where the two dual functions do not vanish tells us that one the loop, let us say \( \bar{\Gamma}'_j \), is included in a bounded connected component of the complement of the other one \( \bar{\Gamma}_j \), and that \( \bar{\Gamma}'_j \) is included in the unbounded connected component of the complement of \( \bar{\Gamma}' \). One deduces that \( \Sigma_{\Gamma'_i} \subset \Sigma_{\Gamma'_j} \). Setting \( T = T_j \circ (T_i)^{-1} \), one gets the inclusion \( T(\Sigma_{\Gamma'}) \subset \Sigma_{\Gamma} \), where \( \Sigma_{\Gamma} \) is a non empty compact set. We have found a contradiction because \( T \) is a non trivial covering automorphism.

We have proved that there exist \( i \neq j \) such that \( T_i^{-1}(\bar{\Gamma}) \) and \( T_j^{-1}(\bar{\Gamma}) \) intersect transversally. This implies that \( \bar{\Gamma} \) and \( T(\bar{\Gamma}) \) intersect transversally, where \( T = T_j \circ (T_i)^{-1} \). Let \( L > 0 \) be an integer sufficiently large so that, if \( \gamma \) is the natural lift of \( \Gamma \), then \( \gamma|[0,L] \) has a transverse intersection with \( T(\gamma)|[0,L] \) at \( \gamma(t) = T(\gamma)(s) \), with \( s < t \). Let \( q > 0 \) be such that \( \gamma|[-L,2L] \) is admissible of order \( q \). Then it follows from Corollaries \( 14 \) and \( 18 \) that, for any \( n > 1 \), the paths

\[
\prod_{i=0}^{n-1} T^i(\gamma|[s-L,t+s+L]) = \prod_{i=0}^{n-1} T^{-i}(\gamma|[t-L,s+t])
\]

are admissible of order \( nq \), and both have self-intersections. Therefore the paths \( \gamma|[s-L,t+s+L] \) and \( \gamma|[t-L,s+t] \) project onto closed paths of \( M \) and the two loops naturally defined have transverse self-intersection and are linearly admissible. So, one can deduce Proposition \( 31 \) from Proposition \( 18 \). \( \square \)

Let us state a first application of Proposition \( 31 \). In \( 1 \) conditions are given for a homeomorphism \( f \), isotopic to the identity, of a compact surface \( M \) to have only contractible periodic points. There it is shown, using Nielsen-Thurston theory, that for such \( f \), under a suitable condition on the size of its fixed point set, there exists an uniform bound on the diameter of the orbits of periodic points. The next theorem improves the main result of that note, by extending the uniform bound on the diameter of orbits from \( \tilde{f} \) periodic points to \( \tilde{f} \) recurrent points. Note that the hypothesis that the fixed point set of \( \tilde{f} \) project in a disk cannot be removed. There exists an example of a \( C^\infty \) diffeomorphism \( f \) of \( \mathbb{T}^2 \) preserving the Lebesgue measure and ergodic such that every periodic orbit of \( f \) is contractible, and such that almost all points in the lift have orbits unbounded in every direction (see \( KT1 \)).

**Theorem 33.** We suppose that \( M \) is compact and furnished with a Riemannian structure. We endow the universal covering space \( \tilde{M} \) with the lifted structure and denote by \( d \) the induced distance. Let \( f \) be a homeomorphism of \( M \) isotopic to the identity and \( \tilde{f} \) a lift to \( \tilde{M} \) naturally defined by the isotopy. Assume that there exists an open topological disk \( U \subset \tilde{M} \) such that the fixed points set of \( \tilde{f} \) projects into \( U \). Then;

- either there exists \( K > 0 \) such that \( d(\tilde{f}^n(\tilde{z}), \tilde{z}) \leq K \), for all \( n \geq 0 \) and all recurrent point \( \bar{z} \) of \( \bar{f} \);
- or there exists a nontrivial covering automorphism \( T \) and \( q > 0 \) such that, for all \( r/s \in (-1/q, 1/q) \), the map \( \tilde{f}^q \circ T^{-p} \) has a fixed point. In particular, \( f \) has non-contractible periodic points of arbitrarily large prime period.

**Proof.** Let \( I \) be a maximal isotopy homotopic to the given isotopy and \( \mathcal{F} \) be a transverse foliation. Write \( \bar{I} \) for the lifted identity isotopy to \( \tilde{M} \) and \( \bar{\mathcal{F}} \) for the lifted foliation. The theorem follows directly from the next lemma and Proposition \( 31 \).

**Lemma 34.** There exists \( K > 0 \) such that, for all \( \bar{z} \) in \( \tilde{M} \) and all \( n \geq 0 \), if \( d(\tilde{f}^n(\bar{z}), \bar{z}) \geq K \), then there exists a leaf \( \bar{\phi} \) and three distinct covering automorphisms \( T_i, 1 \leq i \leq 3 \), such that \( T^p_i(\bar{z}) \) crosses each \( T_i(\bar{\phi}) \).
Proof. Denote by $\pi : \tilde{M} \to M$ the covering projection. One can find a neighborhood $V \subset U$ of $\pi(\text{fix}(\tilde{f}))$ such that for every point $\tilde{z} \in \pi^{-1}(V)$, the points $\tilde{z}$ and $\tilde{f}(\tilde{z})$ belong to the same connected component of $\pi^{-1}(U)$. For reasons explained in the proof of Lemma 10, one knows that for every $z \in M \setminus V$, there exists a small open disk $O_z \subset \text{dom}(F)$ containing $z$ such that $F_x((\pi^{-1}(z'))$ crosses $\phi_z$ if $z' \in O_z$. By compactness of $M \setminus V$, one can cover this set by a finite family $(O_{z_i})_{1 \leq i \leq r}$. One constructs easily a partition $(X_{z_i})_{1 \leq i \leq r}$ of $M \setminus V$ such that $X_{z_i} \subset O_{z_i}$. We have a unique partition $(\tilde{X}_\alpha)_{\alpha \in A}$ of $\tilde{M}$ such that, either $\tilde{X}_\alpha$ is contained in a connected component of $\pi^{-1}(U)$ and projects onto $V$, or there exists $i \in \{1, \ldots, r\}$ such that $\tilde{X}_\alpha$ is contained in a connected component of $\pi^{-1}(O_{z_i})$ and projects onto $X_{z_i}$. Write $\alpha(\tilde{z}) = \alpha$ if $\tilde{z} \in \tilde{X}_\alpha$. Let us define
\[
K_0 = \max_{\tilde{z} \in \tilde{M}} d(\tilde{f}(\tilde{z}), \tilde{z}), \quad K_1 = \max_{\alpha \in A} \text{diam}(\tilde{X}_\alpha).
\]
Fix $\tilde{z} \in \tilde{M}$, $n \geq 1$ and define a sequence $n_0 < n_1 < \ldots < n_s$ in the following inductive way:
\[
n_0 = 0, \quad n_{j+1} = 1 + \sup\{k \in \{n_j, \ldots, n-1\} \mid |\alpha(f^k(\tilde{z})) = \alpha(f^{n_j}(\tilde{z}))|, \quad n_s = n.
\]
Note that $d(\tilde{f}^{n_j}(\tilde{z}), \tilde{f}^{n_{j+1}}(\tilde{z})) \leq K_0 + K_1$, if $j < s$, and that at least one the sets $\tilde{X}_{\alpha(f^{n_j}(\tilde{z}))}$ or $\tilde{X}_{\alpha(f^{n_{j+1}}(\tilde{z}))}$ do not project on $V$, if $j < s-1$. Fix $K > (6r+1)(K_0 + K_1)$. If $d(\tilde{f}^s(\tilde{z}), \tilde{z}) \geq K$, then $j \geq 6r+1$ and there exist at least 3r sets $\tilde{X}_{\alpha(f^{n_j}(\tilde{z}))}$, $0 < j < s$, that do not project on $V$. This implies that there exist three points $f^{n_j}(z)$ that belong to the same $X_{z_i}$. □

Another application of Proposition 31 is a solution to a question posed by P. Boyland in the annulus. We recall first Atkinson’s Lemma on ergodic theory, that will be very useful in this paper (see [A]):

**Proposition 35.** Let $(X, B, \mu)$ be a probability space, and let $T : X \to X$ be an ergodic automorphism. If $\varphi : X \to \mathbb{R}$ is an integrable map such that $\int \varphi d\mu = 0$, then for every $B \in \mathcal{B}$ and every $\varepsilon > 0$, one has
\[
\mu \left( \left\{ x \in B, \exists n \geq 0, \; T^n(x) \in B \text{ and } \left| \sum_{k=0}^{n-1} \varphi(T^k(x)) \right| < \varepsilon \right\} \right) = \mu(B).
\]

We have the following:

**Theorem 36.** Let $f$ be a homeomorphism of $\mathbb{H} = \mathbb{T}^1 \times [0, 1]$ that is isotopic to the identity and $\tilde{f}$ a lift to $\mathbb{R} \times [0, 1]$. Suppose that $\text{rot}(f)$ is a non trivial segment and that $\rho$ is an end of $\text{rot}(f)$ that is is rational. Define
\[
\mathcal{M}_\rho = \{\mu \in \mathcal{M}(f), \; \text{rot}(\mu) = \rho\}, \quad X_\rho = \bigcup_{\mu \in \mathcal{M}_\rho} \text{supp}(\mu).
\]
Then every invariant measure supported on $X_\rho$ belongs to $\mathcal{M}_\rho$.

**Proof.** Replacing $f$ by a power $f^a$, and $\tilde{f}$ by a lift $\tilde{f}^a \circ T^p$, one can assume that $\rho = 0$ and $\text{rot}(f) = [0, a]$, where $a > 0$. The fact that $0$ is extremal implies that for every $\mu \in \mathcal{M}_0$, each ergodic measure $\mu'$ that appears in the ergodic decomposition of $\mu$ also belongs to $\mathcal{M}_0$. Atkinson’s Lemma, with $T = f$ and $\varphi$ the map lifted by $\tilde{\varphi} : z \mapsto \pi_1(\tilde{f}(\tilde{z}) - \tilde{z})$, tells us that $\mu'$-almost every point of $\mathbb{H}$ is lifted to a recurrent point of $\tilde{f}$. The union of the supports of such ergodic measures being dense in $\text{supp}(\mu)$, one deduces that the recurrent set of $\tilde{f}$ is dense in $\pi^{-1}(X_0)$. Writing $f = (f_1, f_2)$, one can extend $f$ to a homeomorphism of $\mathbb{T}^1 \times \mathbb{R}$ such that $f(x, y) = (f_1(x, 1), y)$ if $y \geq 1$ and $f(x, y) = (f_1(x, 0), y)$ if $y \leq 0$ and still denote by $\tilde{f}$ the lift that extends the initial lift. Let $I$ be a maximal isotopy of $f$
is lifted to an identity isotopy $\tilde{I}$ reaching $\tilde{f}$. Let $F$ be a transverse foliation to $I$ and $\tilde{F}$ its lift to $\mathbb{R}^2$. If there exists an invariant measure supported on $X_0$ whose rotation number is positive, there exists a recurrent point $z$ of rotation number strictly larger than 0. Let us fix a lift $\tilde{z}$. The path $I^{n}_{\tilde{z}}(\tilde{z})$ meets infinitely many translated of $\varphi_t$. But $\tilde{z}$ can be approximated by a recurrent point $\tilde{z}'$ and $I^{n}_{\tilde{z}}(\tilde{z}')$ will meet at least three of them. The result now follows from Proposition [31]. Indeed the periodic points that are given by this proposition must belong to $A$, in contradiction with the fact that 0 is an end of $\text{rot}(f)$.

We deduce immediately the following positive answer to a question of P. Boyland:

**Corollary 37.** Let $f$ be a homeomorphism of $\mathcal{A}$ that is isotopic to the identity and preserves a probability measure $\mu$ with full support. Let us fix a lift $\tilde{f}$. Suppose that $\text{rot}(f)$ is a non trivial segment. The rotation number $\text{rot}(\mu)$ cannot be an end of $\text{rot}(f)$ if this end is rational.

7. **Entropy zero conservative homeomorphisms of the sphere**

We will prove in this section the improvement of the result of J. Franks and M. Handel stated in the introduction about area preserving diffeomorphisms of $S^2$ with entropy 0. Let us begin by introducing an important notion due to Franks and Handel: let $f : S^2 \to S^2$ be an orientation preserving homeomorphism, a point $z$ is **free disk recurrent** if there exist an integer $n > 1$ and a topological open disk $D$ containing $z$ and $f^n(z)$ such that $f(D) \cap D = \emptyset$. We will also need the notion of **heteroclinic point**, which means that its $\alpha$-limit and $\omega$-limit sets are included in connected subsets of $\text{fix}(f)$.

Let us begin by stating some easy but useful facts:

**Proposition 38.** One has the following results:

i) The set of free disk recurrent points is an invariant open set $\text{fdrec}(f)$;

ii) it contains every positively or negatively recurrent point outside $\text{fix}(f)$;

iii) every point in $S^2 \setminus \text{fdrec}(f)$ is heteroclinic;

iv) every periodic connected component of $\text{fdrec}(f)$ is fixed.

**Proof.** If $D$ is a free disk (which means that $f(D) \cap D = \emptyset$) that contains $z$ and $f^n(z)$, it contains $z'$ and $f^n(z')$ if $z'$ is close to $z$. Moreover $f^k(D)$ is a free disk that contains $f^k(z)$ and $f^{k+n}(z)$, for every $k \in \mathbb{Z}$. So i) is true.

For every $z \in S^2 \setminus \text{fix}(f)$, one can choose a free disk $D$ that contains $z$. If $z$ is positively recurrent, there exists $n > 1$ such that $f^n(z) \in D$. If $z$ is negatively recurrent, there exist $n > 1$ such that $f^{-n}(z) \in D$, which implies that $f^n(D)$ is a free disk that contains $z$ and $f^n(z)$. In both cases, $z$ belongs to $\text{fdrec}(f)$, which means that ii) is true.

It is sufficient to prove iii) for the $\omega$-limit set, the proof for the $\alpha$-limit set being similar. Let us prove first that $\omega(z) \subset \text{fix}(f)$ if $z \notin \text{fdrec}(f)$. Indeed, if $z' \in \omega(z) \setminus \text{fix}(f)$, one can choose a free disk $D$ containing $z'$ and two integers $n' > n$ such that $f^n(z)$ and $f^{n'}(z)$ belong to $D$. This implies that $f^{-n}(D)$ is a free disk that contains $z$ and $f^{-n}(z')$, which contradicts the fact that $z \notin \text{fdrec}(f)$.

To prove that $\omega(z)$ is included in a connected component of $\text{fix}(f)$, it is sufficient to prove that it is contained in a connected component of $O$, for every neighborhood $O$ of $\text{fix}(f)$. There exist another neighborhood $O' \subset O$ of $\text{fix}(f)$ such that for every $z \in O' \cap f^{-1}(O')$, the points $z$ and $f(z)$ belong
to the same connected component of $O$. There exists $N$ such that $f^n(z) \in O'$ for every $n \geq N$. This implies that the $f^n(z)$, $n \geq N$, belong to the same connected component of $O$.

It remains to prove iv). If $W$ is a connected component of $\text{fdrec}(f)$ of period $q > 1$, it is not a connected component of $\mathbb{S}^2 \setminus \text{fix}(f)$ (see [BK]) and so one can find a path $\alpha$ in $\mathbb{S}^2 \setminus \text{fix}(f)$ joining a point $z \in W$ to a point $z' \notin W$. Taking a sub-path if necessary, one can suppose that $\gamma$ is included in $W$ but the end $z'$ (which is in the frontier of $W$ and not fixed). Let us choose a path $\beta$ in $W$ joining $z$ to $f^q(z)$. It is a classical fact that there exists a simple path $\gamma$ joining $z'$ to $f^q(z')$ whose image is included in the image of $\alpha^{-1} \beta f^q(\alpha)$. The point $z'$ is not periodic because it is neither in $\text{fix}(f)$ nor in $\text{fdrec}(f)$ and so the points $z'$, $f(z')$, $f^q(z')$ and $f^{q+1}(z')$ are distinct (recall that $q > 1$). More precisely, the path $\gamma$ is free and so one can find a free disk that contains it, which contradicts the fact that $z'$ is not in $\text{fdrec}(f)$. \hfill $\square$

Suppose now that the set of positively recurrent points is dense. This is equivalent to say that $\Omega(f) = \mathbb{S}^2$ and in that case the set of positively recurrent points is a dense $G_\delta$ set, as is the set of recurrent points (this happens in the particular case of an area preserving homeomorphism). Note that, in this case, every connected component of $\text{fdrec}(f)$ is periodic and so is fixed. Write $(W_\alpha)_{\alpha \in A_f}$ for the family of connected components of $\text{fdrec}(f)$ and define $A_\alpha$ to be the interior in $\mathbb{S}^2 \setminus \text{fix}(f)$ of the closure of $W_\alpha$. Note that

$$A_\alpha = \mathbb{S}^2 \setminus \bigcup_{\alpha' \in A_f \setminus \{\alpha\}} A_{\alpha'} \cup \text{fix}(f)$$

because the recurrent points are dense in $\mathbb{S}^2$ and contained in $\text{fdrec}(f)$ if not fixed.

We will prove the following result, that extends Theorem 1.2 of [FH].

**Theorem 39.** Let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be an orientation preserving homeomorphism such that $\Omega(f) = \mathbb{S}^2$ and $h(f) = 0$. Then one has the following results:

i) each $A_\alpha$ is an open annulus;

ii) the sets $A_\alpha$ are the maximal fixed point free invariant open annuli;

iii) every point that is not in a $A_\alpha$ is heteroclinic;

iii) let $C$ be a connected component of the frontier of $A_\alpha$ in $\mathbb{S}^2 \setminus \text{fix}(f)$, then the connected components of $\text{fix}(f)$ that contain $\alpha(z)$ and $\omega(z)$ are independent of $z \in C$.

We will begin by stating a local version of this result, which means a version relative to a given maximal isotopy $I$. We denote $\hat{I}$ the lifted identity isotopy to the universal covering space $\text{dom}(I)$ of $\text{dom}(I)$ and $\tilde{f}$ the induced lift of $f|_{\text{dom}(I)}$. Let us say that a point $z \in \text{dom}(I)$ is free disk recurrent relative to $I$ or $I$ free disk recurrent if there exists an integer $n > 0$ and a topological open disk $D \subset \text{dom}(I)$ containing $z$ and $f^n(z)$, such that each lift to $\text{dom}(I)$ is disjoint from its image by $\tilde{f}$ (we will say that $D$ is $I$-free). We have the following local version of Proposition 38.

**Proposition 40.** One has the following results:

i) the set of $I$-free disk recurrent points is an invariant open set $\text{fdrec}(I)$;

ii) it contains every positively or negatively recurrent point in $\text{dom}(I)$;
iii) every point in $S^2 \setminus \text{fdrec}(I)$ is heteroclinic and its $\alpha$-limit and $\omega$-limit sets are included in connected subsets of $\text{fix}(I)$;

iv) every periodic connected component of $\text{fdrec}(I)$ is fixed and lifted to fixed subsets of $\bar{f}$.

Proof. Replacing free disks by $I$-free disks, one proves the three first assertions exactly like in the global situation. Similarly, one can prove that every periodic connected component of $\text{fdrec}(I)$ is fixed. Writing $\pi : \text{dom}(I) \to \text{dom}(I)$ for the universal covering projection, it remains to prove that the connected components of $\pi^{-1}(W)$ are fixed by $\bar{f}$, if $W$ is a fixed connected component of $\text{fdrec}(I)$. If they are not fixed, they are not connected components of $\text{dom}(I)$, which means that $W$ is not a connected component of $\text{dom}(I)$. So one can find a simple path $\alpha$ joining a point $z \in W$ to a point $z' \in \partial W \cap \text{dom}(I)$ and included in $W$ but the end $z'$, and then construct a simple path $\gamma$ joining $z'$ to $f^2(z')$ included in $W$ but the two ends. It will lift to a $\bar{f}$-free simple path and so one can find a $I$-free disk that contains $\gamma$, which contradicts the fact that $z'$ is not in $\text{fdrec}(I)$. ∎

Suppose now that $\Omega(f) = S^2$. Write $(W_\beta)_{\beta \in B_1}$ for the family of connected components of $\text{fdrec}(I)$ and define $A_\beta$ to be the interior in $\text{dom}(I)$ of the closure of $W_\beta$. One knows that the sets $W_\beta$, $A_\beta$ are fixed and lifted to fixed subsets of $\bar{f}$. Here again, one has

$$A_\beta = \text{dom}(I) \setminus \bigcup_{\beta' \in B_1 \setminus \{\beta\}} A_{\beta'}.$$ 

The local version of Theorem 39 is the following:

Theorem 41. Let $f : S^2 \to S^2$ be an orientation preserving homeomorphism such that $\Omega(f) = S^2$ and $h(f) = 0$, and $I$ a maximal isotopy. Then one has the following results:

i) each $A_\beta$ is an open annulus;

ii) the sets $A_\beta$ are the maximal invariant open annuli in $\text{dom}(I)$;

iii) every point that is not in an $A_\beta$ is heteroclinic and its $\alpha$-limit and $\omega$-limit sets are included in connected subsets of $\text{fix}(I)$;

iv) let $C$ be connected component of the frontier of $A_\beta$ in $\text{dom}(I)$, then the connected components of $\text{fix}(I)$ that contain $\alpha(z)$ and $\omega(z)$ are independent of $z \in C$.

Let us explain first why the local theorem implies the global one.

Proof of Theorem 39. Let us explain first why every fixed point free invariant annulus $A$ is contained in a $A_\alpha$, $\alpha \in A_f$. It is sufficient to prove that $\text{fdrec}(f) \cap A$ is connected. Indeed $\text{fdrec}(f) \cap A$ will be contained in a $W_\alpha$, $\alpha \in A_f$, and consequently $A$ will be contained in $A_\alpha$. Let $W$ be a connected component of $\text{fdrec}(f) \cap A$. Applying Proposition 38 to the end compactification of $A$, one knows that $W$ is fixed. If it is not essential, one gets an invariant open disk $D \subset A$ by filling $W$, which means adding the non essential components of its complement. The restriction of $f$ to $D$ having non wandering points, there would exist a fixed point in this disk, which is impossible. Suppose now that there exist at least two connected components. The complement in $A$ of the union of two such components has a unique connected component that is a compact subset of $A$ (and that is located “between” these components). This last set is invariant (by uniqueness) and contains points that are
not free disk recurrent. But one knows that the $\alpha$-limit and $\omega$-limit sets of such points are reduced to one of the ends of $A$. We have a contradiction.

Let us prove now that each $A_\alpha$, $\alpha \in \mathcal{A}_f$, is an annulus. For that it is sufficient to prove that it is contained in a fixed point free invariant annulus. Let us consider a sequence $z_i, i \geq 0$ which is dense in $\text{fix}(f)$ (sequence which will be finite if there are finitely many fixed points). Let us fix $A_\alpha$. Let $I_0$ be a maximal isotopy whose fixed point set contains $z_0, z_1, z_2$. The set $W_\alpha$ is connected and included in $\text{fdrec}(I_0)$ so it is contained in a connected component $W_{\beta_0}$, $\beta_0 \in \mathcal{B}_{I_0}$. One deduces that $A_\alpha \subset A_{\beta_0}$. If $A_{\beta_0}$ is fixed point free, we stop the process. If not, we consider the first $z_{k_1}$ that belongs to $A_{\beta_0}$ and consider a maximal isotopy $I_1$ of $f|A_{\beta_0}$ that fixes $z_{k_1}$. Similarly, there exists $\beta_1 \in \mathcal{B}_{I_1}$ such that $A_\alpha \subset A_{\beta_1}$. If $A_{\beta_1}$ is fixed point free, we stop the process. If not, we consider the first $z_{k_2}$ that belongs to $A_{\beta_1}$ and we continue. If the process stops, the last annulus will be fixed point free. If the process does not stop, $A_\alpha$ is contained in the interior of $\bigcap_{i \geq 0} A_{\beta_i}$. The connected component $W$ of the interior of $\bigcap_{i \geq 0} A_{\beta_i}$ that contains $A_\alpha$ is invariant. Moreover, it is fixed point free because it is open and because the sequence $(z_i), i \geq 0$ is dense in $\text{fix}(f)$ and away from $W$. Let us prove that for $i$ large enough, $A_{\beta_{i+1}}$ is essential in $A_{\beta_i}$ and that $W$ is an annulus. Let us suppose that $A_{\beta_{i+1}}$ is unessential in $A_{\beta_i}$ for infinitely many $i$. Consider a simple loop $\Gamma$ in $W$. It bounds a disk (uniquely determined) included in $A_{\beta_{i+1}}$, every time $A_{\beta_{i+1}}$ is unessential in $A_{\beta_i}$, which implies that it bounds a disk included in $\bigcap_{i \geq 0} A_{\beta_i}$, and so included in $W$. This means that $W$ is a disk, which contradicts the fact it is fixed point free. Suppose now that $A_{\beta_{i+1}}$ is essential in $A_{\beta_i}$ for every $i \geq i_0$. We can prove like above that every simple loop $\Gamma$ that is inessential in the $A_{\beta_i}, i \geq i_0$, bounds a disk in $W$. This implies that $W$ is an open annulus, that is essential in the $A_{\beta_i}, i \geq i_0$.

The assertion iii) is obvious because every free disk recurrent point is contained in a $W_\alpha$ and so in a $A_\alpha$. We will postpone the proof of iv) to the end of this section because we need a little bit more that was is said in the local theorem. □

Before proving Theorem 11, we will state a result relative to a couple $(I, \mathcal{F})$, where $\mathcal{F}$ is a transverse foliation. By Theorem 25 and the density of the set of recurrent points, one knows that two transverse trajectories never intersect transversally. In particular, there is no transverse trajectory with transverse self-intersection and by Proposition 2, every whole transverse trajectory of a recurrent point is equivalent to the natural lift of a transverse simple loop $\Gamma$. We denote by $\mathcal{G}_{I, \mathcal{F}}$ the set of such loops (well defined up to equivalence) and $\text{rec}(f)_I$ the set of recurrent points whose whole transverse trajectory is equivalent to the natural lift of $\Gamma$. Suppose that the whole transverse trajectory of a point $z \in \text{dom}(I)$ meets a leaf more than once. Every segment of $I^\mathcal{F}_z(z)$ is a sub-path of $I^\mathcal{F}_{z'}(z')$, if $z'$ is a recurrent point close to $z$. One deduces that the set of points whose whole transverse trajectory meets a leaf more than once, admits a partition $\bigcup_{\Gamma \in \mathcal{G}_{I, \mathcal{F}}} W_\Gamma$ in disjoint invariant open sets, where $z \in W_\Gamma$ if $I^\mathcal{F}_z(z)$ meets a leaf at least twice and is a sub-path of the natural lift of $\Gamma$. Define $A_\Gamma = \text{int}(\overline{W_\Gamma})$. Note that

$$A_\Gamma = \text{int}(\text{rec}(f)_\Gamma) = \text{dom}(I) \setminus \bigcup_{\Gamma' \in \mathcal{G}_{I, \mathcal{F}} \setminus \{\Gamma\}} A_{\Gamma'}.$$  

Recall that $U_\Gamma$ is the union of leaves that meet $\Gamma$.

**Proposition 42.** One has the following results:

i) the set $A_\Gamma$ is an essential open annulus of $U_\Gamma;$
ii) every point in dom(I) \ ∪_{Γ ∈ G_{I,F}} A_{Γ} is heteroclinic and its α-limit and ω-limit sets are included in connected subsets of fix(I);

iii) let C be a connected component of the frontier of A_{Γ} in dom(I), then the connected components of fix(I) that contain α(z) and ω(z) are independent of z ∈ C.

Proof. The assertion ii) is an immediate consequence of the following: if z′ ∈ dom(I) belongs to the α-limit or ω-limit set of z ∈ dom(I), then the whole transverse trajectory of z meets infinitely often the leaf φ_{z′} and so z belongs to ∪_{Γ ∈ G_{I,F}} W_{Γ}.

Let us prove i). Fix a lift \tilde{γ} of Γ in \text{dom}(I), write T for the covering auto-morphism such that \tilde{γ}(t + 1) = T(\tilde{γ}(t)), write \text{dom}(I) = \text{dom}(I)/T for the annular covering space associated to Γ. Denote by \tilde{π} : \text{dom}(I) → \text{dom}(I) the covering projection, by \tilde{f} the induced identity isotopy, by \tilde{f} the induced lift of f, by \tilde{F} the induced foliation. The line \tilde{γ} projects onto the natural lift of a loop \tilde{Γ}. The union of leaves that meet \tilde{Γ}, denoted by \tilde{U}_{Γ}, is the annular component of \tilde{π}^{-1}(U_{Γ}). One gets a sphere \text{dom}(I)_{\text{sph}} by adding the end N of \text{dom}(I) at the left of Γ and the end S at the right. The complement of \tilde{U}_{Γ} has two connected components l(\tilde{Γ}) \cup \{N\} and r(\tilde{Γ}) \cup \{S\}. Note that \tilde{Γ} is the unique simple loop (up to equivalence) that is transverse to \tilde{F}. Like in dom(I), transverse trajectories do not intersect transversally. The set of points that lift a recurrent point of \tilde{f} is dense. If the trajectory of such a point z meets a leaf at least twice, then \tilde{Γ}_{\tilde{f}}(z) is the natural lift of Γ. Denote rec(\tilde{f}) the set of such points. Otherwise \tilde{Γ}_{\tilde{f}}(z) meets either l(\tilde{Γ}) or r(\tilde{Γ}), the two situations being excluded. Denote rec(\tilde{f})_{N} and rec(\tilde{f})_{S} the set of points z that lift a recurrent point of \tilde{f} and such that \tilde{Γ}_{\tilde{f}}(z) meets l(\tilde{Γ}) and r(\tilde{Γ}) respectively. Note that the intersection of the complete transverse trajectory of z ∈ rec(\tilde{f})_{N} and U_{Γ}, when not empty is equivalent to \tilde{Γ}_{j}, where J is an open interval of T^{1}. In particular there exists \n ≥ 0 such that \tilde{Γ}_{\tilde{f}}(\tilde{Γ}^{n}(z)) ⊂ l(\tilde{Γ}) and \tilde{Γ}_{\tilde{f}}^{-n}(\tilde{Γ}^{-n}(z)) ⊂ l(\tilde{Γ}). Write \tilde{W}_{Γ} for the set of points such that \tilde{Γ}_{\tilde{f}}(z) meets a leaf at least twice, write \tilde{W}_{N} for the set of points z ∈ \text{dom}(I) such that \tilde{Γ}_{\tilde{f}}(z) meets l(\tilde{Γ}), write \tilde{W}_{S} for the set of points such that \tilde{Γ}_{\tilde{f}}(z) meets r(\tilde{Γ}). We get three disjoint invariant open sets, that contain rec(\tilde{f})_{Γ}, rec(\tilde{f})_{N}, rec(\tilde{f})_{S} respectively and whose union is dense. Note that the α-limit and ω-limit sets of a point z ∉ \tilde{U}_{Γ} are reduced to one of the ends. These ends are both equal to N if z ∈ rec(\tilde{f})_{N} and both equal to S if z ∈ rec(\tilde{f})_{S}. We will see later that they both are equal to N if z ∈ \tilde{W}_{N} and both equal to S if z ∈ \tilde{W}_{S}. Note also that

\[ W_{N} = \bigcup_{k \in \mathbb{Z}} f^{-k}(l(Γ)), \quad W_{S} = \bigcup_{k \in \mathbb{Z}} f^{-k}(r(Γ)). \]

Indeed, every leaf φ that is not in \tilde{U}_{Γ} bounds a disk disjoint from \tilde{U}_{Γ}. So, if \tilde{Γ}_{\tilde{f}}(z) meets φ, then one of the point z or \tilde{Γ}_{\tilde{f}}(z) is in l(Γ) if φ ⊂ l(Γ) or in r(Γ) if φ ⊂ r(Γ).

Observe that \tilde{W}_{Γ} projects homeomorphically on W_{Γ} and that A_{Γ} = \text{int}(\tilde{W}_{Γ}) = \text{dom}(I)_{\text{sph}} \setminus \tilde{W}_{N} \cup \tilde{W}_{S} projects homeomorphically on A_{Γ}. We want to prove that A_{Γ} is an annulus.

Lemma 43. There exists a leaf φ_{S} in \tilde{U}_{Γ} that does not meet \tilde{W}_{N}.

Proof. Recall that the intersection of the whole transverse trajectory of z ∈ rec(\tilde{f})_{N} and \tilde{U}_{Γ}, when not empty is equivalent to \tilde{Γ}_{J}, where J is an open interval of T^{1}. Consider the set J of such intervals. The fact that there are no transverse intersection tells us that these intervals do not overlap: if two intervals intersect, one of them contains the other one. One deduces that there exists t ∈ T^{1} that does not belong to any J. Indeed, by a compactness argument, if T^{1} can be covered by the intervals of J,
there exists \( r \geq 2 \) such that it can be covered by \( r \) such intervals but not less. By connectedness, at least two of the intervals intersect and one can lower the number \( r \). Set \( \phi_S = \phi_{\Gamma(t)} \). The set \( \text{rec}(f) \) being dense in \( \overline{W_N} \), the leaf \( \phi_S \) does not meet the whole transverse trajectories of points in \( \overline{W_N} \). In particular, it does not meet \( \overline{W_N} \).

**Lemma 44.** The set \( O_S \) of points whose whole transverse trajectory meets \( \phi_S \) is a connected essential open set.

**Proof.** Fix a lift \( \tilde{\phi} \) of \( \phi_S \) in \( \tilde{\text{dom}}(I) \). The set \( \tilde{O} \) of points whose trajectory meets \( \tilde{\phi} \) is equal to \( \bigcup_{k \in \mathbb{Z}} \tilde{f}^{-k} \left( L(\phi) \cap R(f(\phi)) \right) \), it is connected and simply connected. So its projection \( O_S \) is connected. Every lift of a point in \( \text{rec}(f) \) belongs to all the translated \( T^k(\tilde{O}) \), \( k \in \mathbb{Z} \). So the union of the translated is connected, which means that \( O_S \) is essential. \( \square \)

**Lemma 45.** The set \( \overline{W_N} \) does not contains \( S \) and for every \( z \in \overline{W_N} \), one has \( \alpha(z) = \omega(z) = \{ N \} \).

**Proof.** The set \( \overline{W_N} \) does not contain \( S \) because it does not intersect the essential open set \( O_S \). Moreover, one knows that the \( \alpha \)-limit and \( \omega \)-limit sets of points in \( \overline{W_N} \) are reduced to one of the ends. They both are equal to \( N \), because \( S \notin \overline{W_N} \). \( \square \)

Similarly, there exists a leaf \( \phi_N \) in \( U_\Gamma \) that does not meet \( \overline{W_S} \) and the set \( O_N \) of points whose whole transverse trajectory meets \( \phi_N \) is a connected essential open set. Moreover, \( N \notin \overline{W_S} \) and for every \( z \in \overline{W_S} \), one has \( \alpha(z) = \omega(z) = \{ S \} \). Consequently \( \overline{W_N} \) and \( \overline{W_S} \) do not intersect. Two points in \( O_S \cap O_N \) are not separated neither by \( \overline{W_N} \) nor by \( \overline{W_S} \), because \( O_S \) and \( O_N \) are connected and disjoint from \( \overline{W_N} \) and \( \overline{W_S} \) respectively. So they are not separated by \( \overline{W_S} \cup \overline{W_N} \) because \( \overline{W_S} \cap \overline{W_N} = \emptyset \). One deduces that \( O_S \cap O_N \) is contained in a connected component \( O \) of the complement of \( \overline{W_S} \cup \overline{W_N} \), which is nothing but \( A_\Gamma \). So we have

\[
W_\Gamma \subset O_S \cap O_N \subset O \subset A_\Gamma \subset \overline{W_\Gamma}.
\]

We deduce that the sets appearing in the inclusions have the same closure and that \( A_\Gamma \) is connected because \( O \subset A_\Gamma \subset \overline{O} \). To conclude that \( A_\Gamma \) is an essential annulus, it is sufficient to note that

\[
\overline{W_N} = \bigcup_{k \in \mathbb{Z}} f^{-k} (l(\Gamma) \cup \{ N \}), \quad \overline{W_S} = \bigcup_{k \in \mathbb{Z}} f^{-k} (r(\Gamma) \cup \{ S \}).
\]

are connected, they are the two connected components of the complement of \( A_\Gamma \).

It remains to prove iii). Note first that every leaf of \( F \) is met by a transverse simple loop and so is wandering. This implies that the \( \alpha \)-limit and \( \omega \)-limit sets of a leaf are included in two different connected components of \( \text{fix}(I) \). Let us fix \( \Gamma \in \mathcal{G}_{I,F} \). The complement of \( A_\Gamma \) has two connected components. One of them contains all singularities at the left of \( \Gamma \) and all leaves in \( l(\Gamma) \), denoted by \( L(A_\Gamma) \). One defines similarly \( R(A_\Gamma) \). Write \( \Xi \) for the union of intervals \( J \in \mathcal{J} \) defined above. A point \( t \in \mathbb{T}^1 \) belongs to \( \Xi \) if and only if there exists \( z \in \text{rec}(f) \cap L(A_\Gamma) \) whose whole transverse trajectory meets \( \phi_{\Gamma(t)} \) or equivalently, if there exists \( z \in L(A_\Gamma) \) whose whole transverse trajectory meets \( \phi_{\Gamma(t)} \).

Note that if \( C \) is a connected component of \( (\partial A_\Gamma \setminus \text{fix}(I)) \cap L(A_\Gamma) \), then the set

\[
J_C = \{ t \in \mathbb{T}^1, \ C \cap \phi_{\Gamma(t)} \neq \emptyset \}
\]

is an interval contained in \( \Xi \). Denote by \( (t_-, t_+) \) the connected component of \( \Xi \) that contains this interval. The assertion iii) is an immediate consequence of the following:
Lemma 46. The interval $J_C$ is equal to $(t_-, t_+)$. Moreover, for every $z \in C$, the connected components of $\text{fix}(I)$ that contain $\alpha(z)$ and $\omega(z)$ coincide with the connected components of $\text{fix}(I)$ that contain $\omega(\phi_{t_-})$ and $\omega(\phi_{t_+})$ respectively.

Proof. Fix $z \in C$. Every point $f^k(z)$ belongs to a leaf $\phi_{t_k}$, where $t_k \in \mathbb{T}$. By definition of $\Xi$, one knows that the whole transverse trajectory of $z$ never meets a leaf $\phi_{t}$, $t \notin \Xi$, and so $t \in (t_-, t_+)$ if $\phi_{t}$ meets this trajectory. In particular, the sequence $(t_k)_{k \in \mathbb{Z}}$ is an increasing sequence in $(t_-, t_+)$. We set $t'_- = \lim_{k \to -\infty} t_k$ and $t'_+ = \lim_{k \to +\infty} t_k$. We write $F'_+$ for the connected component of $\text{fix}(I)$ that contains $\omega(\phi_{t'_+})$. We will prove first that the connected component of $\text{fix}(I)$ that contains $\omega(z)$ is $F'_+$ and then that $t'_+ = t_+$. We can do the same for the $\alpha$-limit set. One knows that $\omega(z)$ is contained in $L(A_{t'}) \cap \text{fix}(I)$. So, there exists a sequence $(z'_k)_{k \geq 0}$ that “converges to” $F'_+$ (in the sense that every neighborhood of $F'_+$ contains $z'_k$ for $k$ sufficiently large) and such that $z'_k \notin \omega(z)$ for every $k \geq 0$. Let us prove now that every neighborhood of $F'_+$ contains the segment $\gamma_k$ of $\phi_{t_k}$ between $z'_k$ and $z_k$, for $k$ sufficiently large. If not, there exists a subsequence of $(\gamma_k)_{k \geq 0}$ that converges for the Hausdorff topology to a set that contains a point $z \notin \text{fix}(I)$. This point belongs to $l(I)$ and the leaf $\phi_z$ is met by a loop $\Gamma' \in G_{t', \mathcal{F}}$. For convenience choose the loop passing through $z$, so that we know that $z_k$ belongs to $L(\Gamma')$, for infinitely many $k$. One deduces that the connected component of $\text{fix}(I)$ that contains $\omega(z)$ belongs to $L(\Gamma')$. But this implies that it also belongs to $L(A_{t'})$. This connected component being included in the open disk $A_{t'} \cup L(A_{t'})$, every point $z_k$ belongs to this disk for $k$ large enough. This contradicts the fact that $z \in \partial A_{t'}$, because $A_{t'} \cup L(A_{t'})$ is in the interior of $L(A_{t'})$. It remains to prove that $t'_+ = t_+$. If $t'_+ < t_+$, then $\phi_{t'_+}$ is met by a loop $\Gamma' \in G_{t', \mathcal{F}}$ such that $A_{t'} \subset L(A_{t'})$ and we prove similarly that for $k$ large enough $z_k$ belongs to the open disk $A_{t'} \cup L(A_{t'})$ getting the same contradiction. \hfill \qed

Note that if $\Gamma$ and $\Gamma'$ are two distinct elements of $G_{t', \mathcal{F}}$, then $\Gamma$ is not freely homotopic to $\Gamma'$. Indeed, there exists a leaf $\phi \in U_{t'} \setminus U_{t'}$. The two sets $\alpha(\phi)$ and $\omega(\phi)$ are separated by $\Gamma$ but not by $\Gamma'$ which implies that these two loops are not freely homotopic. Let us explain now why the families $(\text{rec}(f)_{t'})_{t \in G_{t', \mathcal{F}}}$ and $(A_{t'})_{t \in G_{t', \mathcal{F}}}$ are independent of $\mathcal{F}$ (up to reindexation), they depend only on $I$. In particular, if $\mathcal{F}'$ is another foliation transverse to $I$, then every $\Gamma' \in G_{t', \mathcal{F}}$ is freely homotopic to a unique $\Gamma' \in G_{t', \mathcal{F}}$, and one has $\text{Rec}(f)_{\Gamma} = \text{Rec}(f)_{\Gamma'}$. Let $z$ be a recurrent point and $D \subset \text{dom}(I)$ an open disk containing $z$. For every couple of points $(z', z'')$ in $D$, choose a path $\gamma_{z', z''}$ in $D$ joining $z'$ to $z''$. Let $(n_k)_{k \geq 0}$ be an increasing sequence of integers such that $\lim_{k \to +\infty} f^{n_k}(z) = z$. For $k$ large enough, the path $\Gamma^{n_k}(z)\gamma_{f^{n_k}(z), z}$ defines a loop whose homotopy class is independent of the choices of $D$ and $\gamma_{f^{n_k}(z), z}$. If $z$ belongs to $\text{rec}(f)_{\Gamma}$, this class is a multiple of the class of $\Gamma$. This means that the family of classes of loops $\Gamma \in G_{t', \mathcal{F}}$ does not depend on $\mathcal{F}$, as the family of sets $\text{rec}(f)_{\Gamma}$. We will denote $(A_{k})_{k \in K_I}$ and $(\text{rec}(f)_{k})_{k \in K_I}$ our families indexed by homotopy classes. The important fact in the proof of Theorem 41 will be the fact that the families $(A_{k})_{k \in K_I}$ and $(A_{k})_{k \in K_I}$ are the same.

Proof of Theorem 41. The fact that every invariant annulus contained in $\text{dom}(I)$ is contained in an $A_{\beta}$, $\beta \in B_I$, can be proven exactly like in the global case. So, to prove Theorem 41 particularly the fact that every $A_{\beta}$ is an annulus, it is sufficient to prove that it is equal to a $A_{k}$, $k \in K_I$. Note that a $A_{k}$ is an invariant annulus contained in $\text{dom}(I)$ and so is contained in a $A_{\beta}$. If we prove that every $I$-free disk recurrent point is contained in a $A_{k}$, we will deduce that each $A_{\beta}$ is a union of $A_{k}$, which implies that it is equal to one $A_{k}$ because it is connected. We will prove in fact that for every $I$-free
disk recurrent point $z$, there exists a transverse foliation $\mathcal{F}$ such that $z$ belongs to a $W_\Gamma$, $\Gamma \in G_{I,F}$.

Let us give the reason. In the construction of transverse foliations we have the following: if $X$ is a finite set included in an $I$-free disk $D$, one can construct a transverse foliation such that $X$ is included in a leaf (see Proposition 47 at the end of this section). Consequently, if $D$ contains two points $z$ and $f^n(z)$, $n > 0$, one can construct a transverse foliation such that $z$ and $f^n(z)$ are on the same leaf, which implies that $z$ belongs to a $W_\Gamma$.

Proof of assertion iv) of Theorem 39. Fix $\alpha_0 \in A_f$. The assertion iv) is obviously true if the complement of $A_{\alpha_0}$ is the union of two fixed points. Let us prove it in case exactly one the connected components of the complement of $A_{\alpha_0}$ is a fixed point $z_0$. By assertion iii) there exists at least one connected component $X_1 \neq \{z_0\}$ of fix($f$) that meets the frontier of $A_{\alpha_0}$. If $\{z_0\}$ and $X_1$ are the only connected components of fix($f$), the result is also obviously true. If not, choose a third component $X_2$, then choose $z_1 \in X_1 \cap \partial(A_{\alpha_0})$ and $z_2 \in X_2$ and finally a maximal isotopy $I$ that fixes the points $z_0$, $z_1$ and $z_2$. We will prove that the connected component $A_{\beta_0}, \beta_0 \in B_I$, that contains $A_{\alpha_0}$ is reduced to $A_{\alpha_0}$. This will imply iv). Suppose that $A_{\beta_0}$ is not reduced to $A_{\alpha_0}$. In that case it contains other $A_\alpha, \alpha \in A_f$, and the union of such sets is dense in $A_{\beta_0}$ and contain all the recurrent points. The two ends of $A_{\beta_0}$ are adjacent to $A_{\alpha_0}$ because $I$ fixes $z_1$. This implies that $A_{\alpha_0}$ is the unique $A_\alpha$ that is essential in $A_{\beta_0}$. So, if $A_\alpha$ is included in $A_{\beta_0}$ and $\alpha \neq \alpha_0$, the union of $A_\alpha$ and of the connected component of its complement that are included in $A_{\beta_0}$ is an invariant open disk $D_\alpha \subset A_{\beta_0}$ disjoint from $A_{\alpha_0}$. Let us consider a transverse foliation $\mathcal{F}$ and the loop $\Gamma \in G_{I,F}$ such that $A_\Gamma = A_\beta$. We will work in the annular covering space, where $A_\Gamma$ is homeomorphic to $A_\Gamma$ and will write $D_\alpha \subset A_\Gamma$ for the disk corresponding to $D_\alpha$ and $A_{\alpha_0}$ for the annulus corresponding to $A_{\alpha_0}$.

The fact that $\{z_0\}$ is a connected component of $S^2 \setminus A_{\alpha_0}$ and that $I$ fixes $z_0$ and two other points in different components of the fixed point set implies that one of the sets $r(\tilde{\Gamma})$ or $l(\tilde{\Gamma})$ is empty and the other one is not. We will assume for instance that $r(\tilde{\Gamma}) = \emptyset$ and $l(\tilde{\Gamma}) \neq \emptyset$. We have seen in the proof of Proposition 18 that there exists a compactification $\operatorname{dom}(I)_{\operatorname{ann}}$ obtained by blowing up the end $N$ at the left of $\tilde{\Gamma}$ by a circle $\tilde{\Sigma}_N$ such that $\tilde{f}$ extends to a homeomorphism $\tilde{f}_{\operatorname{ann}}$ that admits fixed points on the added circle with a rotation number equal to zero for the lift $\tilde{f}_{\operatorname{ann}}$ that extends $\tilde{f}$. Note now that every recurrent point of $\tilde{f}$ that belongs to a $D_\alpha$ has a rotation number (for the lift $\tilde{f}$) and that this number is a positive integer because $D_\alpha$ is fixed and included in $A_\Gamma$. So, every periodic orbit whose rotation number is not an integer belongs to $A_{\alpha_0}$.

There are different ways to get a contradiction. Let us begin by the following one. The closure of $A_{\beta_0}$ in $\operatorname{dom}(I)_{\operatorname{ann}}$ is an invariant essential closed set that contains $A_{\alpha_0}$ and meets $\Sigma_N$. In particular it contains fixed points of rotation number 0 on $\Sigma_N$. Denote by $K$ the complement of $A_{\alpha_0}$ in the closure of $A_{\beta_0}$ in $\operatorname{dom}(I)_{\operatorname{ann}}$. It contains the fixed points located on $\Sigma_N$ and all the $D_\alpha$, which means that it contains fixed points of positive rotation number. It is an essential compact set, because $A_{\alpha_0}$ is an essential annulus which is a neighborhood of the end of $\operatorname{dom}(I)_{\operatorname{ann}}$. All points in $K$ being non wandering, one can apply a result of S. Matsumoto [Mm] saying that $K$ contains a periodic orbit of period $q$ and rotation number $p/q$ for every $p/q \in (0,1)$. But one knows that all such periodic points must belong to $A_{\alpha_0}$. We have a contradiction.

Let us give another explanation: we will need the following intersection property: every essential simple loop in $\operatorname{dom}(I)_{\operatorname{ann}}$ meets its image by $\tilde{f}_{\operatorname{ann}}$. The reason is very simple. Perturbing our loop, it is sufficient to prove that every essential simple loop in $\operatorname{dom}(I)$ meets its image by $\tilde{f}$. Such a loop
meets $A_{\beta_0}$ because the two ends of $\hat{\text{dom}}(f)$ are adjacent to $A_{\beta_0}$ and so contains a non wandering point (every point of $A_{\beta_0}$ is non wandering). This implies that the loop meets its image by $\hat{f}$.

Using the fact that the entropy of $f^2$ is zero, one can consider the family of annuli $(A_{\alpha'})_{\alpha' \in A(f^2)}$, and denote by $A_{\alpha''}$ the annulus of $A_{\alpha'}$ that corresponds to an annulus $A_{\alpha'}$ contained in $A_{\beta_0}$. Every periodic point $z$ of period 3 and rotation number 1/3 or 2/3 belongs to an annulus $A_{\alpha''}$ and this annulus is $\hat{f}^2$-invariant. It must be essential in $A_{\beta_0}$, otherwise the rotation number of $z$ should be a multiple of 1/2. But if it is essential, it must be $\hat{f}$-invariant, its $\hat{f}$-period cannot be 2. It is included in $A_{\alpha_0}$, otherwise it would be included in a non essential $A_{\beta}$. Being given such an essential annulus, note that the set of periodic points of period 3 and rotation 1/3 or 2/3 strictly above (which means on the same side as $\Sigma_N$) is compact. Indeed, the rotation number induced on the added circle is 0. One deduces that there are finitely many annuli $A_{\alpha''}$ that contains periodic points of period 3 and rotation number 1/3 or 2/3 above a given one and so there exists an upper essential annulus $A_{\beta_0}^\alpha$ that contains periodic points of period 3 and rotation number 1/3 or 2/3. If one adds the connected component of $\hat{\text{dom}}(I)_{\text{ann}} \setminus A_{\beta_0}^\alpha$ containing $\Sigma_N$ to $A_{\beta_0}^\alpha$, one gets an invariant semi-open annulus $A$ that contains $\Sigma_N$ and all disks $D_\alpha$. The restriction $\hat{f}_{\text{ann}}|_A$ satisfies the intersection property stated in Lemma 17 because $A$ is essential in $\hat{\text{dom}}(I)_{\text{ann}}$. The annulus $A$ contains a fixed point of rotation number 0 and a fixed point of positive rotation number, so, by Lemma [17], it contains at least one periodic orbit of period 3 and rotation number 1/3 and one periodic orbit of period 3 and rotation number 2/3. These two orbits must be included in $A_{\beta_0}^\alpha$ by definition of this set. But $\hat{f}|_{A_{\beta_0}^\alpha}$ satisfies the intersection property because $A_{\beta_0}^\alpha$ is essential in $\hat{\text{dom}}(I)_{\text{ann}}$. So $A_{\beta_0}^\alpha$ contains a periodic point of period 2 and rotation number 1/2, which is impossible.

In the case where none of the connected components of the complement of $A_{\alpha_0}$ is a fixed point, one can crush one of these components to a point and used what has been done in the new sphere. □

Let us add some comments on the boundary of the annuli $A_{\alpha}$.

Let $f : S^2 \to S^2$ be an orientation preserving homeomorphism such that $\Omega(f) = S^2$ and $h(f) = 0$. Suppose moreover that the fixed point set is totally disconnected. Every annulus $A_{\alpha}$, $\alpha \in A_f$, admits accessible fixed points on its boundary. More precisely, if $X$ is a connected component of $S^2 \setminus A_{\alpha}$, there exists a simple path $\gamma$ joining a point $z \in A_{\alpha}$ to a point $z' \in \text{Fix}(f) \cap X$ and contained in $A_{\alpha}$ but the end $z'$. Indeed, one can always suppose that the other connected component of $S^2 \setminus A_{\alpha}$ is reduced to a point $z_0$ and that $f$ has least three fixed points (otherwise the result is obvious). What has been done in the previous proof tells us that there exists a maximal isotopy $I$, a transverse foliation $\mathcal{F}$ and $\Gamma \in \mathcal{G}_{I,\mathcal{F}}$ such that $A_{\alpha} = A_{\Gamma}$. There exists a leaf $\phi \subset U_{\Gamma}$ that is not met by any transverse trajectory that intersects $X$. This leaf (or the inverse of the leaf) joins $z_0$ to a fixed point $z \in X$ and is contained in $A_{\Gamma}$.

Let $f : S^2 \to S^2$ be an orientation preserving homeomorphism such that $\Omega(f) = S^2$ and $h(f) = 0$. Let $I$ be a maximal isotopy and $\mathcal{F}$ a transverse foliation. Every annulus $A_{\Gamma}$, $\Gamma \in \mathcal{G}_{I,\mathcal{F}}$, that meets $\phi$ is such that the connected components of $\text{Fix}(I)$ that contains $\alpha(\phi)$ and $\omega(\phi)$ are separated by $A_{\Gamma}$. One deduces immediately that a point $z \in \text{dom}(I)$ belongs to the frontier of at most two annuli $A_{\Gamma}$, $\Gamma \in \mathcal{G}_{I,\mathcal{F}}$. Of course this means that a point $z \in \text{dom}(I)$ belongs to the frontier of at most two annuli $A_{\beta}$, $\beta \in B_f$, but it also implies that a point $z \not\in \text{fix}(f)$ belongs to the frontier of at most two annuli $A_{\alpha}$, $\alpha \in A_f$. Indeed, suppose that $z \not\in \text{fix}(f)$ belongs to the frontier of $A_{\alpha_i}$, $0 \leq i \leq 2$. If $X_i$ is the
connected component of $\mathbb{S}^2 \setminus A_{\alpha_i}$ that does not contain $z$, then the three sets $X_i$ are disjoint. Choose a fixed point $z_i$ in each $X_i$ (such a fixed point exists because $X_i \cup A_{\alpha_i}$ is an invariant disk and $A_{\alpha_i}$ has no fixed points). Choose an isotopy $I$ that fixes the $z_i$ and denote $A_{\beta_i}$, $\beta_i \in B_I$, the annulus that contains $A_{\alpha_i}$. Note that the three annuli $A_{\beta_i}$ are distinct and that $z$ belongs to their frontier. We have a contradiction.

We conclude this section by justifying a point used above in the proof of Theorem 41.

**Proposition 47.** Let $f : M \to M$ be a homeomorphism isotopic to the identity on a surface $M$ and $I$ a maximal isotopy. Let $X$ be finite set contained in a $I$-free disk. Then, there exists a transverse foliation $F$ such that $X$ is contained in a leaf of $F$.

**Proof.** The proof can be deduced immediately from the construction of transverse foliations, that we recall now (see [Lec2]). A brick decomposition $D = (V, E, B)$ on a surface is given by a one dimensional stratified set, the skeleton $\Sigma(D)$, with a zero-dimensional submanifold $V$ such that any vertex $v \in V$ is locally the extremity of exactly three edges $e \in E$. A brick $b \in B$ is the closure of a connected component of the complement of $\Sigma(D)$. Say that a brick decomposition $D = (V, E, B)$ on $\text{dom}(I)$ is $I$-free, if every brick is $I$-free, or equivalently, if its lifts to a brick decomposition $\tilde{D} = (\tilde{V}, \tilde{E}, \tilde{B})$ on the universal covering $\tilde{\text{dom}}(I)$, whose bricks are $\tilde{f}$-free, where $\tilde{f}$ is the natural lift defined by the lifted identity isotopy $\tilde{I}$. Say that $D$ is minimal if there is no $I$-free brick decomposition whose skeleton is strictly included in the skeleton of $D$.

Write $G$ for the group of automorphisms of the universal covering space. Using the classical Franks' lemma on free disk chains [F1], one constructs a natural order $\leq$ on $\tilde{B}$ that satisfies the following:

- it is $G$-invariant;
- if $\tilde{f}(\tilde{b})$ meets $\tilde{b}'$, then $\tilde{b}' \leq \tilde{b}$;
- two adjacent bricks are comparable.

One can define an orientation on $\Sigma(D)$ (inducing an orientation on $\Sigma(D)$) such that the brick on the left of an edge $\tilde{e} \in \tilde{E}$ is smaller than the brick on the right. Moreover, every vertex $\tilde{v} \in \tilde{V}$ is the ending point of at least one oriented edge and the starting point of at least one oriented edge (in other words there are no sinks and no sources). We have three possibilities for the bricks of $\tilde{B}$:

- it can be a closed disk with a sink and a source on the boundary (seen from inside);
- it can be homeomorphic to $[0, +\infty[ \times \mathbb{R}$ with a sink on the boundary and a source at infinity;
- it can be homeomorphic to $[0, +\infty[ \times \mathbb{R}$ with a source on the boundary and a sink at infinity;
- it can be homeomorphic to $[0, 1] \times \mathbb{R}$ with a sink and a source at infinity (in this case it can project on a closed annulus).

Let us state now the fundamental result, easy to prove in the case where $G$ is abelian and much more difficult in the case it is not (see Proposition 3.2 of [Lec2]): one can cover $\Sigma(\tilde{D})$ with a $G$-invariant family of Brouwer lines of $\tilde{f}$, such that two such lines never intersect transversally (in the natural sense).

Such family of lines inherits the natural order about lines defined at the beginning of this article. One can ‘complete’ this family to get a larger family, with the same properties, that possesses the topological properties of a lamination (in particular every line admits a compact and totally ordered
neighborhood). Then one can arbitrarily foliate each brick $b \in B$ such that, when lifted to a foliation on a brick $\tilde{b} \in \tilde{B}$, every leaves goes from the source to the sink. We obtain then, in a natural way, a decomposition of $\text{dom}(I)$ by a $G$-invariant family of Brouwer lines that do not intersect transversally, and that possesses the topological structure of a plane foliation (it is a non Hausdorff one dimensional manifold).

It remains to blow up each vertex, by a desingularization process (see [Lec1]) to obtained a $G$-invariant foliation by Brouwer lines.

To get the proposition, it remains to say the following: if $X$ is a finite set contained in a $I$-free disk $D$, making this disk a little bit smaller, one can suppose that the closure of $D$ is $I$-free, and then one can find a $I$-free brick decomposition $D$ such that $D$ is contained in the interior of brick of $D$. There exists a minimal brick decomposition $D'$ such that $\Sigma(D') \subset \Sigma(D)$. One deduces that $D$ is in the interior of a brick $b$ of $D'$. When one constructs the foliation inside $b$, one can always suppose that $X$ is contained in a leaf. The desingularization process taking place in a neighborhood of the set of vertices, $X$ is still contained in a leaf of our final foliation.

\[ \square \]

8. Applications to torus homeomorphims

In this section an element of $\mathbb{Z}^2$ will be called an integer and an element of $\mathbb{Q}^2$ a rational. Let us begin by stating the main results of this section.

**Theorem 48.** Let $f$ be a homeomorphism of $T^2$ that is isotopic to the identity and $\tilde{f}$ a lift of $f$ to $\mathbb{R}^2$. The frontier of $\text{rot}(\tilde{f})$ does not contain a segment with irrational slope that contains a rational point in its interior.

**Theorem 49.** Let $f$ be a homeomorphism of $T^2$ that is isotopic to the identity and $\tilde{f}$ a lift of $f$ to $\mathbb{R}^2$. If $\text{rot}(\tilde{f})$ has a non empty interior, then there exist a constant $L$ such that for every $z \in \mathbb{R}^2$ and every $n \geq 1$, one has $d(\tilde{f}^n(z) - z, \text{rot}(\tilde{f})) \leq L$.

**Theorem 50.** Let $f$ be a homeomorphism of $T^2$ that is isotopic to the identity and $\tilde{f}$ a lift of $f$ to $\mathbb{R}^2$. If $\text{rot}(\tilde{f})$ has a non empty interior, then the topological entropy of $f$ is positive.

Recall that Theorem 50 has been known for a long time and is due to Llibre and MacKay, see [LM] and that Theorem 49 was known for homeomorphisms in the special case of a polygon with rational vertices, see [D2] and for $C^{1+\epsilon}$ diffeomorphisms, see [A].

Let us begin by stating some consequences of these results. Let $f$ be a homeomorphism of $T^2$ that is isotopic to the identity and $\tilde{f}$ a lift of $f$ to $\mathbb{R}^2$. We suppose that $\text{rot}(\tilde{f})$ has non empty interior. For every non trivial linear form $\psi$ on $\mathbb{R}^2$, define

$$\alpha(\psi) = \max\{\psi(\text{rot}(\mu)) : \mu \in \mathcal{M}(f)\}.$$
The affine line of equation $\psi(z) = \alpha(\psi)$ is a supporting line of $\text{rot}(\tilde{f})$. Set
\[ \mathcal{M}_\psi = \{ \mu \in \mathcal{M}(f) : \psi(\text{rot}(\mu)) = \alpha(\psi) \}, \quad X_\psi = \bigcup_{\mu \in \mathcal{M}_\psi} \text{supp}(\mu). \]

As already noted in \[A\], we can deduce from Theorem 49 and Atkinson’s Lemma the following result.

**Proposition 51.** Every measure $\mu$ supported on $X_\psi$ belongs to $\mathcal{M}_\psi$. Moreover, if $z$ lifts a point of $X_\psi$, then for every $n \geq 1$, one has $|\psi(\tilde{f}^n(z)) - \psi(z) - n\alpha(\psi)| \leq L\|\psi\|$, where $L$ is the constant given by Theorem 49.

**Proof.** We will prove the second statement, it obviously implies the first one. Noting that the ergodic components of a measure $\mu \in \mathcal{M}_\psi$ belong to $\mathcal{M}_\psi$, it is sufficient to prove that for every ergodic measure $\mu \in \mathcal{M}_\psi$, there exists a set $A$ of measure 1 such that if $z$ lifts a point of $A$, then $|\psi(\tilde{f}^n(z)) - \psi(z) - n\alpha(\psi)| \leq L\|\psi\|$ for every $n \geq 1$. As seen before, Atkinson’s result applied to the function lifted by $\psi \circ \tilde{f} - \psi - \alpha(\psi)$ implies that there exists a set $A$ of measure 1 such that, for every point $z$ lifting a point of $A$, there exists a subsequence $(n_l)_{l \in \mathbb{N}}$ such that
\[ \lim_{l \to +\infty} \psi(\tilde{f}^{n_l}(z)) - \psi(z) - n_l\alpha(\psi) = 0. \]

By Theorem 49, one knows that for every $z \in \mathbb{R}^2$ and every $n \geq 1$, one has $\psi(\tilde{f}^n(z)) - \psi(z) - n\alpha(\psi) \leq L\|\psi\|$. It remains to prove that $\psi(\tilde{f}^n(z)) - \psi(z) - n\alpha(\psi) \geq -L\|\psi\|$ if $z$ lifts a point of $A$. If $n_l$ is greater than $n$ one can write
\[ \psi(\tilde{f}^{n_l}(z)) - \psi(z) - n_l\alpha(\psi) \]
\[ = \left( \psi(\tilde{f}^{n_l}(z)) - \psi(z) - n_l\alpha(\psi) \right) - \left( \psi(\tilde{f}^{n_l}(z)) - \psi(\tilde{f}^{n_l}(z)) - (n_l - n)\alpha(\psi) \right). \]
\[ \geq \psi(\tilde{f}^{n_l}(z)) - \psi(z) - n_l\alpha(\psi) - L\|\psi\| \]

Letting $l$ tend to $+\infty$, one gets our inequality. \[\square\]

Let us state two corollaries. The first one is the strong version of Boyland’s conjecture in the torus and its proof is immediate.

**Corollary 52.** Let $f$ be a homeomorphism of $\mathbb{T}^2$ that is isotopic to the identity, preserving a measure $\mu$ of full support, and $\tilde{f}$ a lift of $f$ to $\mathbb{R}^2$. Assume that $\text{rot}(\tilde{f})$ has a non empty interior. Then $\text{rot}(\mu)$ belongs to the interior of $\text{rot}(\tilde{f})$.

**Corollary 53.** Let $\rho$ be a vertex of $\text{rot}(\tilde{f})$, and set
\[ \mathcal{M}_\rho = \{ \mu \in \mathcal{M}(f) : \text{rot}(\mu) = \rho \}, \quad X_\rho = \bigcup_{\mu \in \mathcal{M}_\rho} \text{supp}(\mu). \]

There exists a constant $L_\rho$ such that if $z$ lifts a point of $X_\rho$, then for every $n \geq 1$, one has $d(\tilde{f}^n(z)) - z - n\rho) \leq L_\rho$.

**Proof.** One can find two forms $\psi$ and $\psi'$, linearly independent such that $\rho$ belongs to the supporting lines defined by these forms. Set $X_\rho = X_\psi \cap X_{\psi'}$ and apply Proposition 51. \[\square\]
Write $\partial \left( \text{rot}(\tilde{f}) \right)$ for the frontier of $\text{rot}(\tilde{f})$. Let us define now

\[ \mathcal{M}_\partial = \left\{ \mu \in \mathcal{M}(f), \text{rot}(\mu) \in \partial \left( \text{rot}(\tilde{f}) \right) \right\}, \quad X_\partial = \bigcup_{\mu \in \mathcal{M}_\partial} \text{supp}(\mu) = \bigcup_{\psi \neq 0} X_\psi. \]

Similarly, we have:

**Proposition 54.** Every ergodic measure $\mu$ supported on $X_\psi$ belongs to $\mathcal{M}_\partial$. Moreover, if $z$ lifts a point of $X_\partial$, then for every $n \geq 1$, one has $d \left( \tilde{f}^n(z) - z, n \partial \left( \text{rot}(\tilde{f}) \right) \right) \leq L$, where $L$ is the constant given by Theorem 49.

**Proof.** Here again, it is sufficient to prove the second statement and for that to prove that for every non trivial linear form $\psi$, for every $n \geq 1$, and for every point $z$ lifting a point of $X_\psi$, one has

\[ d \left( \tilde{f}^n(z) - z, n \partial \left( \text{rot}(\tilde{f}) \right) \right) \leq L. \]

The fact that

\[ |\psi(\tilde{f}^n(z)) - \psi(z) - n \beta(\psi)| \leq L \|\psi\| \]

implies that $d(\tilde{f}^n(z) - z, \Delta) \leq L$ where $\Delta$ is the affine line of equation $\psi(z) = n\alpha(z)$. So, if $f^n(z) - z$ does not belong to $n \text{rot}(f)$, one has

\[ d \left( \tilde{f}^n(z) - z, n \partial \left( \text{rot}(\tilde{f}) \right) \right) = d(\tilde{f}^n(z) - z, n \text{rot}(\tilde{f})) \leq L, \]

and if $\tilde{f}^n(z) - z$ belongs to $n \text{rot}(\tilde{f})$, one has

\[ d \left( \tilde{f}^n(z) - z, n \partial \left( \text{rot}(\tilde{f}) \right) \right) \leq d(\tilde{f}^n(z) - z, \Delta) \leq L. \]

□

Another application is a classification result about Hamiltonian homeomorphisms which in our setting will mean that there exists a measure in $\mathcal{M}(f)$ with total support such that $\text{rot}(\mu) = (0, 0)$. This is the case in particular if $f$ is the time one map of a time dependent Hamiltonian flow which is 1 periodic in time and $\tilde{f}$ the natural lift.

We will need the following result, which can be found in [KT2]:

**Proposition 55.** Let $f$ be a homeomorphism of $\mathbb{T}^2$ isotopic to the identity and $\tilde{f}$ a lift of $f$. If $(0, 0)$ is a vertex of $\text{rot}(\tilde{f})$ then, for any measure $\mu \in \mathcal{M}(f)$ such that $\text{rot}(\mu) = (0, 0)$, almost every point lifts to a recurrent point of $\tilde{f}$.

We have:

**Theorem 56.** Let $f$ be a Hamiltonian homeomorphism of $\mathbb{T}^2$ such that its fixed point set is contained in a topological disk, and let $\tilde{f}$ be its Hamiltonian lift. Then one of the following three holds:

- The set $\text{rot}(\tilde{f})$ has not empty interior: in that case the origin lies in its interior.
- The set $\text{rot}(\tilde{f})$ is a non trivial segment: in that case $\text{rot}(\tilde{f})$ generates a line with rational slope, the origin is not an end of $\text{rot}(\tilde{f})$, furthermore, there exists an invariant essential open annulus in $\mathbb{T}^2$.
- The set $\text{rot}(\tilde{f})$ is reduced to the origin: in that case, there exists $K > 0$ such that, for every $z \in \mathbb{R}^2$ and every $k \in \mathbb{Z}$, one has $\|f^k(z) - z\| \leq K$. 

Proof. Suppose first that rot(\(\tilde{f}\)) is reduced to the origin. The origin being a vertex, one knows by Proposition \[55\] that the recurrent set of \(\tilde{f}\) is dense in \(\mathbb{R}^2\). So the assertion comes from Theorem \[33\].

Suppose now that rot(\(\tilde{f}\)) is a non trivial segment. If the origin was an end of rot(\(\tilde{f}\)) its would be a vertex and we would have a contradiction, still from from Proposition \[55\] and Theorem \[33\]. The fact that rot(\(\tilde{f}\)) generates a line with rational slope is a consequence of Theorem \[48\]. The existence of an essential open annulus which is left invariant by the dynamics whenever rot(\(f\)) is a non trivial segment that generates a line with rational slope is the main result of [GKT].

The case where rot(\(\tilde{f}\)) has non empty interior is nothing but Corollary \[52\].

Here again, as in Theorem \[33\], the requirement that the fixed point set is contained in a topological disk cannot be removed. As a consequence, we obtain the following boundedness result for area preserving homeomorphism of the torus with restriction on its rotational behaviour:

**Corollary 57.** Let \(f\) be a Hamiltonian homeomorphism of \(\mathbb{T}^2\) such that all its periodic points are contractible, and such that its fixed point set is contained in a topological disk. Then there exists \(K > 0\) such that if \(\tilde{f}\) is the Hamiltonian lift of \(f\), then for every \(z \in \mathbb{R}^2\) and every \(k \in \mathbb{Z}\), one has \(\|f^k(z) - z\| \leq K\).

Before proving our three theorems, let us state some introductory results. In what follows, \(f\) is a homeomorphism of \(\mathbb{T}^2\) that is isotopic to the identity and \(\tilde{f}\) a lift of \(f\) to \(\mathbb{R}^2\). We consider a maximal isotopy \(I\) that is lifted to an identity isotopy \(\tilde{I}\) of \(\tilde{f}\) and a transverse foliation \(\mathcal{F}\), which is lifted to a foliation \(\tilde{\mathcal{F}}\) on \(\mathbb{R}^2\) transverse to \(\tilde{I}\).

**Proposition 58.** If \((0,0)\) is included in the interior of rot(\(\tilde{f}\)) or in the interior of a segment with irrational slope included in \(\partial \left(\text{rot}(\tilde{f})\right)\), then the leaves of \(\tilde{\mathcal{F}}\) are uniformly bounded.

**Proof.** Suppose first that \((0,0)\) is included in the interior of rot(\(\tilde{f}\)). One can find finitely many extremal points \(\rho_i\) of rot(\(\tilde{f}\)), \(1 \leq i \leq r\), that linearly generate the plane and positive numbers \(t_i\), \(1 \leq i \leq r\), such that:

\[
\sum_{1 \leq i \leq r} t_i = 1, \quad \sum_{1 \leq i \leq r} t_i \rho_i = (0,0).
\]

Each \(\rho_i\) is the rotation number of an ergodic measure \(\mu_i \in \mathcal{M}(f)\). Applying Poincaré Recurrence Theorem and Birkhoff Ergodic Theorem, one can find a recurrent point \(z_i\) of \(f\) having \(\rho_i\) as a rotation number. Fix a lift \(\tilde{z}_i\) of \(z_i\) and a small neighborhood \(\tilde{W}_i\) of \(\tilde{z}_i\) that trivializes \(\tilde{\mathcal{F}}\). One can find a subsequence \((\tilde{f}^{n_i}(z_i))_{i \geq 0}\) of \(\tilde{f}^{n}(z_i)_{n \geq 1}\) and a sequence \((p_i,t)_{i \geq 0}\) of integers such that \(\tilde{f}^{n_i}(\tilde{z}_i) \in \tilde{W}_i + p_i t\) and such that \(\lim_{i \to +\infty} p_i t / n_i = \rho_i\). One deduces that the transverse homological space \(\text{THS}(\mathcal{F})\) contains \(p_i, t\). If \(l\) is large enough, the \(p_i, t\) generate the plane and \((0,0)\) is contained in the interior of the polygonal defined by these points. By Proposition \[8\] we deduce that the leaves of \(\tilde{\mathcal{F}}\) are uniformly bounded.

Suppose now that \((0,0)\) belongs to the interior of a segment with irrational slope included in \(\partial \left(\text{rot}(\tilde{f})\right)\). If this segment \([\rho_1, \rho_2]\) is chosen maximal, then \(\rho_1\) and \(\rho_2\) are extremal points of rot(\(\tilde{f}\)) and respectively equal to the rotation number of ergodic measures \(\mu_1\) and \(\mu_2\) in \(\mathcal{M}(f)\). Let \(W_i \subset \mathbb{T}^2\) be a trivializing box of \(\mathcal{F}\) such that \(\mu_i(W_i) \neq 0\) and \(\tilde{W}_i \subset \mathbb{R}^2\) a lift of \(W_i\). The first return map \(\Phi_i : W_i \to W_i, z \mapsto f^{\tau_i}(z)\) (where \(\tau_i : W_i \to \mathbb{N}\)) is defined \(\mu_i\)-almost everywhere on \(W_i\) as the displacement function \(\xi_i : W_i \to \mathbb{Z}^2\),
where $\tilde{f}^{\tau_i}(z) \in \tilde{W}_i + \xi_i(z)$, if $z$ is the lift of $z$ that belongs to $\tilde{W}_i$. Let $\psi : \mathbb{R}^2 \to \mathbb{R}$ be a non trivial linear form that vanishes on our segment. Using Birkhoff’s theorem, one knows that $\mu_i$-almost every point $z$ has a rotation number $\rho_i$, and so

$$\lim_{n \to +\infty} \frac{\sum_{k=0}^{n-1} \xi_i(\Phi^k_i(z))}{\sum_{k=0}^{n-1} \tau_i(\Phi^k_i(z))} = \rho_i.$$ 

By Kac’s theorem, one knows that $\tau_i$ is $\mu_i$-integrable and satisfies $\int_{W_i} \tau_i \, d\mu_i = \mu_i(\bigcup_{k \in \mathbb{Z}} f^k(W_i)) \in (0, +\infty)$. One can note that $\xi_i/\tau_i$ is bounded, which implies that $\xi_i$ is $\mu_i$-integrable. Consequently, one has

$$\lim_{n \to +\infty} \frac{\sum_{k=0}^{n-1} \xi_i(\Phi^k_i(z))}{\sum_{k=0}^{n-1} \tau_i(\Phi^k_i(z))} = \int_{W_i} \xi_i \, d\mu_i,$$

which implies that

$$\int_{W_i} \xi_i \, d\mu_i = \left( \int_{W_i} \tau_i \, d\mu_i \right) \rho_i \neq 0$$

and

$$\int_{W_i} \psi \circ \xi_i \, d\mu_i = \psi \left( \int_{W_i} \xi_i \, d\mu_i \right) = 0.$$

Note that $\psi \circ \xi_i(z) \neq 0$ if $\xi_i(z) \neq 0$, because $\xi_i(z)$ is an integer and the kernel of $\psi$ is generated by a segment with irrational slope. We deduce that there exists $z_1, z'_1$ in $W_1$ such that

$$\psi \circ \xi_1(z_1) < 0 < \psi \circ \xi_1(z'_1).$$

Consequently, one can find $z''_1 \in W_1$, $z''_1 \in W_2$ and integers $n_1, n_2$ such that $(0, 0)$ is in the interior of the quadrilateral determined by

$$\xi_1(z_1), \xi_1(z'_1), \sum_{k=0}^{n_1-1} \xi_1(\Phi^k_1(z''_1)), \sum_{k=0}^{n_2-1} \xi_2(\Phi^k_2(z_2)),$$

because the last two points may be chosen arbitrarily close to $\rho_1$ and $\rho_2$. The set $\text{TMS}({\mathcal{F}})$ containing the integers

$$\xi_1(z_1), \xi_1(z'_1), \sum_{k=0}^{n_1-1} \xi_1(\Phi^k_1(z''_1)), \sum_{k=0}^{n_2-1} \xi_2(\Phi^k_2(z_2)),$$

one can apply Proposition 8 to conclude that the leaves of $\tilde{\mathcal{F}}$ are uniformly bounded. \hfill $\square$

**Remark 59.** As a corollary, one deduces that $\mathcal{F}$ is singular and that $\tilde{f}$ is not fixed point free. Applying this to $f^q - p$, for every rational $p/q \in \text{int(rot}(\tilde{f}))$, one deduces that there exists a point $z \in \mathbb{R}^2$ such that $\tilde{f}^q(z) = z + q$. This result was already well known, see [F2].

**Proposition 60.** We suppose that the leaves of $\tilde{\mathcal{F}}$ are uniformly bounded. If there exists an admissible transverse path $\gamma : [a, b] \to \text{dom}(\mathcal{F})$ of order $q$ and an integer $p \in \mathbb{Z}^2$ such that $\gamma$ and $\gamma + p$ intersect transversally at $\phi_{\gamma(t)} = \phi_{(\gamma + p)(s)}$, where $s < t$, then $p/q$ belongs to $\text{rot}(\tilde{f})$.

**Proof.** By Corollary 14 and Corollary 20, one deduces that for every $k \geq 2$ the path

$$\gamma_{[a, t]} \left( \prod_{0 < i < k-1} (\gamma + ip)[s, t] \right) (\gamma + (k-1)p)[s, t]$$

is admissible of order $kq$. This implies that there exists a point $z_k \in \phi_{\tilde{\gamma}(a)}$ such that $\tilde{f}^{kq}(z_k) \in \phi_{(\gamma + (k-1)p)(b)} = \phi_{\tilde{\gamma}(b)} + (k-1)p$. The fact that the leaves of $\tilde{\mathcal{F}}$ are uniformly bounded tells us that
there exists \( K \) such that for every \( k \geq 1 \), one has \( \| \tilde{f}^k q(z_k) - \tilde{z}_k - (k - 1)p\| \leq K \). Denote \( z_k \) the projection of \( \tilde{z}_k \) in \( T^2 \). Choose a measure \( \mu \) that is the limit of a subsequence of \( \{ \frac{1}{kq} \sum_{i=0}^{kq-1} \delta_{\tilde{f}^i(z)} \}_{k \geq 2} \) for the weak* topology. It is an invariant measure of \( f \) of rotation number \( p/q \) for \( \tilde{f} \).

Let us state the following improved version of Atkinson’s Lemma:

**Proposition 61.** Let \((X, B, \mu)\) be a probability space and \( T : X \to X \) an ergodic automorphism. If \( \varphi : X \to \mathbb{R} \) is an integrable map such that \( \int \varphi \, d\mu = 0 \), then for every \( B \in B \) and every \( \varepsilon > 0 \), one has

\[
\mu \left( \left\{ x \in B, \quad \exists n \geq 0, \quad T^n(x) \in B \quad \text{and} \quad 0 \leq \sum_{k=0}^{n-1} \varphi(T^k(x)) < \varepsilon \right\} \right) = \mu(B).
\]

**Proof.** Let us consider \( B \in B \) and set

\[
A = B \backslash \left\{ x \in B, \quad \exists n \geq 0, \quad T^n(x) \in B \quad \text{and} \quad 0 \leq \sum_{k=0}^{n-1} \varphi(T^k(x)) < \varepsilon \right\}.
\]

Atkinson’s result directly implies that there exists a set \( A' \subset A \) with \( \mu(A') = \mu(A) \) such that, for every point \( x \in A' \), there exists a subsequence \( (n_l)_{l \in \mathbb{N}} \) such that \( T^{n_l}(x) \in A \) and \( \lim_{l \to \infty} \sum_{k=0}^{n_l-1} \varphi(T^k(x)) = 0 \).

Assume, for a contradiction, that \( \mu(A) > 0 \). There exists some \( x \in A' \) and \( n_0 > 0 \) such that \( y = T^{n_0}(x) \in A \) and \( a = \sum_{k=0}^{n_0-1} \varphi(T^k(x)) \in (-\varepsilon, \varepsilon) \), and since \( x \in A \) we know that \( a < 0 \). Since \( x \in A' \) there exists some \( n_1 > n_0 \) such that \( T^{n_1}(x) \in A \) and \( a < \sum_{k=0}^{n_1-1} \varphi(T^k(x)) < \varepsilon + a \). This implies that \( T^{n_1-n_0}(y) \in A \) and that \( 0 < \sum_{k=0}^{n_1-n_0-1} \varphi(T^k(y)) < \varepsilon \), which is a contradiction since \( y \in A \) proving the claim.

**Proof of Theorem 48.** We will give a proof by contradiction. Replacing \( f \) by \( f^q \) and \( \tilde{f} \) by \( \tilde{f}^q - p \), where \( q \in \mathbb{N} \) and \( p \in \mathbb{Z}^2 \), we can suppose that the frontier of \( \text{rot}(\tilde{f}) \) contains a segment \([\rho_0, \rho_1]\) with irrational slope, that \((0,0)\) is in its interior and that \( \rho_0 \) and \( \rho_1 \) are extremal points of \( \text{rot}(\tilde{f}) \). We can suppose moreover than for every \( \rho \in \text{rot}(\tilde{f}) \), one has \( \langle \rho^\perp, \rho \rangle \leq 0 \leq \langle \rho^\perp, \rho \rangle \). We consider two ergodic measures \( \mu_0 \) and \( \mu_1 \) in \( \mathcal{M}(f) \) whose rotation numbers are \( \rho_0 \) and \( \rho_1 \) respectively. We know that there exists a point \( z_0 \in \mathbb{R}^2 \) such that \( \text{rot}(z_0) = \rho_0 \) and that projects on a recurrent point of \( f \). By Proposition 61 we have a stronger result:

**Lemma 62.** There exists a point \( z_0 \) such that for every \( \varepsilon \in \{-1, 1\} \) one can find a sequence \((q_l, p_l)_{l \geq 0}\) in \( \mathbb{Z}^2 \times \mathbb{N} \) satisfying:

\[
\lim_{l \to +\infty} q_l = +\infty, \quad \lim_{l \to +\infty} \tilde{f}^q(z_0) - z_0 - p_l = 0, \quad \varepsilon\langle \rho^\perp, p_l \rangle > 0
\]

and a sequence \((q'_l, p'_l)_{l \geq 0}\) in \( \mathbb{Z}^2 \times \mathbb{N} \) satisfying:

\[
\lim_{l \to +\infty} q'_l = +\infty, \quad \lim_{l \to +\infty} \tilde{f}^{-q}(z_0) - z_0 - p'_l = 0, \quad \varepsilon\langle \rho^\perp, p'_l \rangle > 0.
\]

**Proof.** Let \( W_0 \) be a small disk such that \( \mu_0(W_0) \neq 0 \) and \( \tilde{W}_0 \) a lift of \( W_0 \). Define the maps \( \tau_0 \) and \( \xi_0 \) like in Proposition 48. The measure \( \mu_0 \) being ergodic, one knows that \( \int_{W_0} \tau_0 \, d\mu_0 = 1 \) and that \( \int_{\tilde{W}_0} \xi_0 \, d\mu_0 = \rho_0 \). Let us define on \( W_0 \) the transformation \( T : z \mapsto f^{\tau_0(z)}(z) \) and the function \( \varphi : z \mapsto \varepsilon\langle \rho^\perp, \xi_0(z) \rangle \).
For each integer \( i \geq 1 \), let \((B_{i,j})_{1 \leq j \leq k_i}\) be a covering of \( W_0 \) by open sets with diameter smaller than \( 1/i \), and let us define

\[
C_{i,j} = \left\{ x \in B_{i,j} \cap W_0, \ \exists n \geq 0, \ T^n(x) \in B_{i,j} \cap W_0 \text{ and } 0 \leq \sum_{k=0}^{n-1} \varphi(T^k(x)) < 1/i \right\}
\]

By Proposition 61, one knows that, if \( C \) is the union of \( C_{i,j} \) and if \( C = \bigcap_{i \geq 1} C_i \), then \( \mu(C_i) = \mu(C) = \mu(W_0) \).

If \( z_0 \) belongs to \( C \), one can find an increasing integer sequence \((m_i)_{i \geq 0}\) such that

\[
\lim_{l \to +\infty} T^{m_l}(z_0) = z_0, \quad \lim_{l \to +\infty} \sum_{k=0}^{m_l-1} \varepsilon(\rho_0^k, \xi_0(T^k(z_0))) = 0, \quad \lim_{l \to +\infty} \sum_{k=0}^{m_l-1} \varepsilon(\rho_0^k, \xi_0(T^k(z_0))) \geq 0.
\]

Setting \( p_l = \sum_{k=0}^{m_l-1} \tau_0(T^k(z_0)) \) and \( q_l = \sum_{k=0}^{m_l-1} \varepsilon(\rho_0^k, \xi_0(T^k(z_0))) \), one gets the first assertion of the lemma, with a large inequality instead of a strict one. Noting that \( \lim_{l \to +\infty} \|p_l\| = +\infty \) and that the line generated by \( \rho_0 \) has irrational slope, one deduces that the inequality is strict. The second assertion can be proved analogously.

\[\square\]

Consider a maximal identity isotopy \( I \) of \( f \) that is lifted to an identity isotopy of \( \tilde{F} \) and a transverse foliation \( F \) of \( I \) lifted to a transverse foliation \( \tilde{F} \). One knows by Proposition 58 that the leaves of \( \tilde{F} \) are uniformly bounded. The fact that \( \text{rot}(z_0) = \rho_0 \) tells us that the whole trajectory \( \tilde{T}^\gamma(z_0) \) is a proper path directed by \( \rho_0 \). The fact that the leaves of \( \tilde{F} \) are uniformly bounded and that every leaf met by \( \tilde{T}^\gamma(z_0) \) is also met by \( \tilde{T}^\gamma(z_0) \) implies that \( \tilde{T}^\gamma(z_0) \) is a transverse proper path directed by \( \rho_0 \). We parametrize \( \tilde{T}^\gamma(z_0) \) in such a way that \( \tilde{T}^\gamma(z_0)|_{[t,t+1]} = \tilde{I}_{[t]}(\tilde{F}(z_0)) \). We consider sequences \((p_l,q_l)_{l \geq 0}\) and \((p'_l,q'_l)_{l \geq 0}\) given by the previous lemma (the sign \( \varepsilon \) has no importance at the beginning).

**Lemma 63.** For every closed segment \([a,b] \subset \mathbb{R}\) and every positive real numbers \( L, \varepsilon \), there exists \( p \in \mathbb{Z}^2 \) and a segment \([a',b'] \subset \mathbb{R}\) satisfying \( a' - b > L \) such that \( |\langle \rho_0^1, p \rangle| < \varepsilon \) and such that the paths \( \tilde{T}^\gamma(z_0)|_{[a,b]} \) and \( (\tilde{T}^\gamma(z_0) + p)|_{[a',b']} \) are equivalent. One has a similar result replacing the inequality \( a' - b > L \) by \( a - b' > L \).

**Proof.** Let us choose integers \( q \) and \( q' \) such that \([a,b] \subset (q,q')\). One can find \( l \) sufficiently large, such that \( q_l > q' - q + L \) and such that \( \tilde{f}^q(q_0(z_0)) - p_l \) is so close to \( \tilde{f}^q(z_0) \) that we can affirm that \( \tilde{T}^\gamma(z_0)|_{[a,b]} \) is equivalent to a path \( \tilde{T}^\gamma(z_0) - p_l)|_{[a',b']} \), where \([a',b'] \subset (q + q_l, q' + q_l)\). The version with the inequality \( a - b' > L \) can be proven similarly by using the sequences \((p'_l)_{l \geq 0}\) and \((q'_l)_{l \geq 0}\).

**Lemma 64.** There is no \( p \in \mathbb{Z}^2 \setminus \{0\} \) such that \( \tilde{T}^\gamma(z_0) \) and \( \tilde{T}^\gamma(z_0) + p \) intersect transversally.

**Proof.** Write \( \tilde{T}^\gamma(z_0) = \gamma_0 \) for convenience. Suppose that \( \gamma_0 \) and \( \gamma_0 + p \) intersect transversally at \( \phi = \phi_{\gamma_0(t)} = \phi_{(\gamma_0 + p + p')(s)} \). The leaves being uniformly bounded, one knows that \( \phi_{\gamma_0(t)} \neq \phi_{\gamma_0(t) + p} \) and so \( t \neq s \). Replacing \( p \) with \(-p\) if necessary, one can suppose that \( s < t \). By Proposition 60, there exists \( q \geq 1 \) such that \( q/2 \in \mathbb{Z} \) and consequently that \( |\rho_0^q,p| \leq 0 \). By assumption, the segment \([0,\rho_0]\) has irrational slope and \( p \neq 0 \), so one deduces that \( |\rho_0^q,p| < 0 \).

Lemma 63 tells us that we can find \( p' \in \mathbb{Z} \) such that \( |\rho_0^q,p + p'| < 0 \) and such that \( \gamma_0 \) and \( \gamma_0 + p + p' \) intersect transversally at \( \phi_{\gamma_0(t)} = \phi_{(\gamma_0 + p + p')(s')} \), where \( s' > t \), and so that \( \gamma_0 \) and \( \gamma_0 - p - p' \) intersect transversally at \( \phi_{\gamma_0(t)} = \phi_{(\gamma_0 + p + p')(s')} \).
intersect transversally at $\phi_{\gamma_0(s')} = \phi(\gamma_0-p-p'(t))$. We deduce that $\langle p_0^\perp, -p - p' \rangle < 0$. We have got a contradiction. 

\[\square\]

**Lemma 65.** The path $\tilde{T}_F^\perp(z_0)$ is a line

**Proof.** Here again, write $\tilde{T}_F^\perp(z_0) = \gamma_0$. If $\gamma_0$ is a not a line, by Proposition 4 one knows that there exist two segments $[a_0,b_0]$ and $[a_1,b_1]$ such that $\gamma_0|_{[a_0,b_0]}$ and $\gamma_0|_{[a_1,b_1]}$ intersect transversally. By Lemma 63, one deduces that there exist $p \in \mathbb{Z}^2 \setminus \{0\}$ and a segment $[a'_1,b'_1]$ such that $\gamma_0|_{[a_0,b_0]}$ and $(\gamma_0 + p)|_{[a'_1,b'_1]}$ intersect transversally. This contradicts Lemma 64.

Similarly, we can find a point $z_1$ of rotation number $\rho_1$ that projects on a recurrent point of $f$ and such that $\tilde{T}_F^\perp(z_1)$ is a line directed by $\rho_1$ that does not meet transversally its integer translated.

**Lemma 66.** The line $\tilde{T}_F^\perp(z_1)$ intersects transversally one of the translates of $\tilde{T}_F^\perp(z_0)$.

**Proof.** Let us prove by contradiction that $\gamma_1 = \tilde{T}_F^\perp(z_1)$ intersects transversally the translates of $\gamma_0 = \tilde{T}_F^\perp(z_0)$. If not, let us denote by $U_0$ the union of leaves that are met by $\gamma_0$. Its complement can be written $R(U_0) \cup L(U_0)$ where $R(U_0) = R(\gamma_0) \setminus U_0$ is the union of $r(\gamma_0)$ and of the set of singularities at the right of $\gamma_0$ and $L(U_0) = L(\gamma_0) \setminus U_0$ is the union of $l(\gamma_0)$ and of the set of singularities at the left of $\gamma_0$. If $\gamma_1$ and $\gamma_0$ do not intersect transversally, then by Proposition 4 one knows that either $\gamma_1 \cap R(U_0) = \emptyset$ or $\gamma_1 \cap L(U_0) = \emptyset$. Consequently, $\gamma_0$ and $\gamma_1$ cannot meet a same leaf. Indeed if $\phi$ is such a leaf, one knows by Proposition 4 that it is met once by $\gamma_0$ and $\gamma_1$. So, the $\alpha$-limit set of $\phi$ is contained in $L(U_0) \cap \overline{L(\gamma_1)}$ and the $\omega$-limit set is included in $R(U_0) \cap \overline{R(\gamma_1)}$, which is impossible.

So, if the conclusion of our lemma is not true, there exists a partition $\mathbb{Z}^2 = A^- \sqcup A^+$, where

$$p \in A^- \Leftrightarrow (r(\gamma_0) \cup U) + p \subset l(\gamma_1),$$

$$p \in A^+ \Leftrightarrow (l(\gamma_0) \cup U) + p \subset r(\gamma_1),$$

and this partition is a cut of the order on $\mathbb{Z}^2$ defined as follows

$$p \preceq p' \Leftrightarrow \langle p_0^\perp, p \rangle \leq \langle p_0^\perp, p' \rangle.$$  

Let us fix a leaf $\phi$ that intersects $\gamma_1$. By Lemma 63 one knows that there exists $p_0 \neq (0,0)$ such that $\phi$ intersects $\gamma_1 + p_0$. One deduces that $A^- + p_0 = A^- + A^+ + p_0 = A^+$, which of course is impossible.

\[\square\]

**End of the proof of Theorem 48.** Replacing $\gamma_0$ by a translate if necessary, we can always suppose that $\gamma_0$ and $\gamma_1$ intersect transversally at $\gamma_0(t_0) = \gamma_1(t_1)$ and we define $\gamma = \gamma_0|_{t_0, t_1} \cap \gamma_1|_{t_1, +\infty}$ which is a transverse proper path. There exist two segments $[a_0,b_0]$ and $[a_1,b_1]$ containing $t_0$ and $t_1$ respectively in their interior, such that $\gamma_0|_{[a_0,b_0]}$ and $\gamma_1|_{[a_1,b_1]}$ intersect transversally at $\gamma_0(t_0) = \gamma_1(t_1)$. Using Lemma 63 one can find $p_0$ and $p_1$ in $\mathbb{Z}^2$ distinct, and segments $[a'_0,b'_0] \subset (-\infty, t_0)$ and $[a'_1,b'_1] \subset (t_1, +\infty)$ such that $(\gamma_0 + p_0)|_{[a'_0,b'_0]}$ and $(\gamma_1 + p_1)|_{[a'_1,b'_1]}$ is equivalent to $\gamma_1|_{[a_1,b_1]}$. We deduce that there exists $t'_0 \in (a'_0,b'_0)$ and $t'_1 \in (a'_1 + t_0 - t_1, b'_1 + t_0 - t_1)$ such that $\gamma + p_0$ and $\gamma + p_1$ intersect transversally at $\phi = \phi(\gamma_0 + p_0)(t_0) = \phi(\gamma_1 + p_1)(t_1)$. So $\gamma$ and $\gamma + p_1 - p_0$ intersect transversally at $\phi - p_0 = \phi(\gamma_0)(t'_0) = \phi(\gamma_1 + p_1)(t'_1)$. Observing that $t'_0 < t'_1$, one deduces that there exists $q \geq 1$ such that

\[\square\]
In the proof, we will use the sup norm \( \| \|_\infty \) where \( \| (x_1, x_2) \|_\infty = \max(|x_1|, |x_2|) \) which will be more convenient that the Euclidean norm and will write \( d_{\infty}(z, X) = \inf_{z' \in X} \| z - z' \|_\infty \).

Replacing \( f \) by \( f^q \) and \( \tilde{f} \) by \( \tilde{f}^q - p \), where \( q \in \mathbb{N} \) and \( p \in \mathbb{Z}^2 \), we can suppose that \((0,0)\) is in the interior of \( \text{rot}(\tilde{f}) \).

Replacing \( f \) by \( f^q \) and \( \tilde{f} \) by \( \tilde{f}^q - p \), where \( q \in \mathbb{N} \) and \( p \in \mathbb{Z}^2 \), we can suppose that \((0,0)\) is in the interior of \( \text{rot}(\tilde{f}) \).

Here again, let us consider a maximal identity isotopy \( I \) of \( f \) that is lifted to an identity isotopy of \( \tilde{f} \) and a transverse foliation \( \mathcal{F} \) of \( I \) lifted to a transverse foliation \( \tilde{\mathcal{F}} \). One knows by Proposition 58 that the leaves of \( \tilde{\mathcal{F}} \) are uniformly bounded. The theorem is an immediate consequence of the following, where the direction \( D(\gamma) \) of a path \( \gamma : [a, b] \to \mathbb{R}^2 \) is defined as \( D(\gamma) = \gamma(b) - \gamma(a) \):

**Proposition 67.** There exists a constant \( L \) such that for every transverse admissible path \( \gamma \) of order \( n \), one has \( d_{\infty}(D(\gamma), n \text{rot}(f)) \leq L \).

We will begin by proving:

**Lemma 68.** There exist a transverse admissible path \( \gamma^* : [0, 3] \to \mathbb{R}^2 \), a real number \( K^* \) and an integer \( p^* \in \mathbb{Z}^2 \) such that:

- every transverse path \( \gamma \) whose diameter is larger than \( K^* \) intersects transversally an integer translate of \( \gamma^*_|[1,2] \);
- \( \gamma^*|[2,3] \) and \( \gamma^*|[0,1] + p^* \) intersect transversally.

**Proof.** Let us choose \( N \) large enough such (1/N, 0) and (0, 1/N) belong to the interior of \( \text{rot}(\tilde{f}) \). As previously remarked, there exists a point \( z_0 \) satisfying \( \tilde{f}^N(z_0) = z_0 + (1, 0) \) and a point \( z_1 \) satisfying \( \tilde{f}^N(z_1) = z_1 + (0, 1) \). The transverse trajectories \( \tilde{T}_\mathcal{F}^n(z_0) \) and \( \tilde{T}_\mathcal{F}^n(z_1) \) are parametrized, such that \( \tilde{T}_\mathcal{F}^n(z_0)(t + 1) = \tilde{T}_\mathcal{F}^n(z_0)(t) + (1, 0) \) and \( \tilde{T}_\mathcal{F}^n(z_1)(t + 1) = \tilde{T}_\mathcal{F}^n(z_1)(t) + (0, 1) \).

**SubLemma 69.** There exists a real number \( K \) such that if \( \gamma \) is a transverse path that does not intersect transversally \( \tilde{T}_\mathcal{F}^n(z_0) \), then either \( \pi_2(\gamma(t)) > -K \) or \( \pi_2(\gamma(t)) < K \) and if it does not intersect transversally \( \tilde{T}_\mathcal{F}^n(z_1) \), then either \( \pi_1(\gamma(t)) > -K \) or \( \pi_1(\gamma(t)) < K \).

**Proof.** There exists \( K_0 > 0 \) such that the diameter of each leaf of \( \mathcal{F} \) is bounded by \( K_0 \) and there exists \( K_0' > 0 \) such that \( \tilde{T}_\mathcal{F}^n(z_0) \subset \mathbb{R} \times (-K_0', K_0') \). Setting \( K = K_0 + K_0' \), note that every leaf that intersects \( \mathbb{R} \times (-\infty, K] \) belongs to \( r(\tilde{T}_\mathcal{F}^n(z_0)) \) and every leaf that intersects \( \mathbb{R} \times [K, +\infty) \) belongs to \( l(\tilde{T}_\mathcal{F}^n(z_0)) \). It remains to apply Corollary 6. We have a similar argument for \( \tilde{T}_\mathcal{F}^n(z_1) \).

Setting \( K^* = 2K + 1 \), one deduces immediately:

**Corollary 70.** If \( \gamma \) is a transverse path and if the diameter of \( \pi_2 \circ \gamma \) is larger than \( K^* \), there exists \( p_0 \in \mathbb{Z}^2 \) such that \( \gamma \) intersects transversally \( \tilde{T}_\mathcal{F}^n(z_0) + p_0 \) and if the diameter of \( \pi_1 \circ \gamma \) is larger than \( K^* \), there exists \( p_1 \in \mathbb{Z}^2 \) such that \( \gamma \) intersects transversally \( \tilde{T}_\mathcal{F}^n(z_1) + p_1 \).
In particular \( \gamma_0 = \overline{P}_{\gamma}(z_0) \) intersects \( \gamma_1 = \overline{P}_{\gamma}(z_1) \) transversally at a point \( \gamma_0(t_0) = \gamma_1(t_1) \). One can find an integer \( r > 0 \) such that \( \gamma_0|_{t_0-r,t_0+r} \) and \( \gamma_1|_{t_1-r,t_1+r} \) intersect transversally at \( \gamma_0(t_0) = \gamma_1(t_1) \). Let \( \gamma^* : [0,3] \to \mathbb{R}^2 \) be a path such that
- \( \gamma^*_{[0,1]} \) is a reparametrization of \( \gamma_0|_{t_0-(4r+2),t_0-(2r+2)} \);
- \( \gamma^*_{[1,2]} \) is a reparametrization of \( \gamma_0|_{t_0-(2r+2),t_0}|_{t_1-t_1+(2r+2)} \);
- \( \gamma^*_{[2,3]} \) is a reparametrization of \( \gamma_1|_{t_1+(2r+2),t_1+(4r+2)} \).

Let us prove that \( \gamma^* \) satisfies the proposition. Observe first that \( \gamma^* \) is admissible of order \((8r+4)N\) by Corollary [13] and that the paths \( \gamma^*_{[0,2]} \) and \( \gamma^*_{[1,3]} \) are admissible of order \((6r+2)N\). Note first that \( \gamma^*_{[2,3]} \) and \( \gamma^*_{[0,1]} + (3r+2,3r+2) \) intersect transversally. One can set \( p^* = (3r+2,3r+2) \). Let \( \gamma \) be a transverse path such that the diameter of \( \pi_2 \circ \gamma \) is larger than \( K^* \). By the sub-lemma, one knows that there exists \( p_0 \in \mathbb{R}^2 \) such that \( \gamma \) intersects transversally \( \gamma_0 + p_0 \). This means that there exists two real segments \( J \) and \( J_0 \) such that
- \( \gamma_J \) intersects transversally \( \gamma_0, J_0 + p_0 \);
- \( \gamma_{\text{int},J} \) and \( \gamma_0|_{\text{int},J_0} + p_0 \) are equivalent.

If the length of \( J_0 \) is smaller than \( 2r+1 \), then \( J_0 \) is included in an interval \([t_0-(2r+2),t_0+t_0]+l_0] \). This implies that \( \gamma \) intersects transversally \( \gamma^*_{[1,2]} + p_0 + (l_0,0) \). If the length is at least equal to \( 2r+1 \), then \( J_0 \) contains an interval \([t_0-r+t_0,r+t_0]+l_0] \). This implies that \( \gamma \) intersects transversally \( \gamma_1|_{t_1-r,t_1+r} + p_0 + (l_0,0) \) and so intersects transversally \( \gamma^*_{[1,2]} + p_0 + (l_0,-r) \). We get the same conclusion for a transverse path such that the diameter of \( \pi_1 \circ \gamma \) is larger than \( K^* \). □

**Proof of the Proposition [6]**. We denote by \( K^{**} \) the diameter of \( \gamma^* \) and by \( K^{***} \) the diameter of \( \text{rot}(f) \). Let \( \gamma : [a,b] \to \mathbb{R}^2 \) be a transverse path such that \( \|D(\gamma)\| > 2K^* \). One can find \( c, d \) in \((a,b)\) with \( c < d \) such that \( \|D(\gamma_{[a,c]})\| = \|D(\gamma_{[d,b]})\| = K^* \). There exists \( p \) and \( p' \) in \( \mathbb{Z}^2 \) such that
- \( \gamma_{[a,c]} \) and \( \gamma^*_{[1,2]} + p \) intersect transversally at \( \gamma(t) = \gamma^*(s) + p \);
- \( \gamma_{[d,b]} \) and \( \gamma^*_{[1,2]} + p' \) intersect transversally at \( \gamma(t') = \gamma^*(s') + p' \).

If \( \gamma \) is admissible of order \( n \), then the path
\[
\gamma' = (\gamma^*_{[0,s]} + p)\gamma_{[t,c]}(\gamma^*_{[s',3]} + p')
\]
is admissible of order \( n + (12r+4)N \) by Corollary [13] and one has
\[
\|D(\gamma') - D(\gamma)\| \leq 2K^* + 2K^{**}.
\]
Recall that \( \gamma^*_{[2,3]} \) intersects transversally \( \gamma^*_{[0,1]} + p^* \). One deduces that \( (\gamma^*_{[s',3]} + p') \) intersects transversally \( (\gamma^*_{[0,s]} + p) + p'' \), where \( p'' = p' - p + p^* \) and so that \( \gamma' \) intersects transversally \( \gamma' + p'' \). Proposition [60] tells us that \( p''/(n + (12r + 4)N) \) belongs to \( \text{rot}(f) \), which implies
\[
d(p'', n \text{rot}(f)) \leq (12r + 4)NK^{***}.
\]
Observe now that \( \|p'' - D(\gamma')\| \leq K^{**} \) and so \( \|p'' - D(\gamma)\| \leq 2K^* + 3K^{**} \). So, one gets
\[
d(D(\gamma), n \text{rot}(f)) \leq 2K^* + 3K^{**} + (12r + 4)NK^{***}.
\]
□
\textbf{Proof of Theorem 49.} Here again, using the fact that for every \( q \geq 1 \) and every \( p \in \mathbb{Z}^2 \), one has\( \text{rot}(f^q + p) = q \cdot \text{rot}(f) + p \), it is easy to see that it is sufficient to prove the result in the case where \((0,0)\) belongs to the interior of \( \text{rot}(f) \). Here again, let us consider a maximal identity isotopy \( I \) of \( f \) that is lifted to an identity isotopy of \( \bar{f} \) an a transverse foliation \( \bar{F} \) of \( I \) lifted to a transverse foliation \( \bar{F} \). We know that the leaves of \( \bar{F} \) are uniformly bounded. We can immediately deduce the theorem from what has been done in the previous proof and Theorem 25. Indeed, we know that there are two transverse loops associated to periodic points that have a transverse intersection. We will give a proof that does not use Theorem 25 by exhibiting separated sets.

Let us begin with the following lemma:

\textbf{Lemma 71.} \textit{There exists a constant} \( K \) \textit{such that for every point} \( z \in \text{dom}(F) \) \textit{and any} \( z' \) \textit{for which} \( \phi_{z'} \) \textit{intersects} \( I_{\bar{F}}(z) \), \textit{one has} \( d(z, z') \leq K \).

\textit{Proof.} There exists \( K' > 0 \) such that the diameter of each leaf of \( \bar{F} \) is bounded by \( K' \). Moreover, the set \( \bigcup_{z \in [0,1]} I_{\bar{F}}(z) \), being compact, is included in \([-K'',K''+1]^2\), where \( K'' > 0 \). The leaves that \( I_{\bar{F}}(z) \) intersects, are also intersected by \( I(z) \) (see the beginning of Section 3). One deduces that \( K = K' + K'' \) satisfies the conclusion of the lemma. \( \square \)

We consider the paths \( \gamma_0 = \bar{I}_{\bar{F}}(z_0) \) and \( \gamma_1 = \bar{I}_{\bar{F}}(z_1) \) defined in the proof of Theorem 49. We keep the same notations and set \( z^* = \gamma_0(t_0) = \gamma_1(t_1) \). Let us define

\[ K'''' = \max \left( \text{diam}(\gamma_0|_{t_0,t_0+mr]}, \text{diam}(\gamma_1|_{t_1,t_1+mr}] \right) \]

and choose an integer \( m \geq 1 \) such that \( mr \geq K'''' + 2K_0 + K + 1 \). Set

\[ \gamma_0' = \gamma_0|_{t_0,t_0+mr], \gamma_1' = \gamma_1|_{t_1,t_1+mr} \]

Fix \( n \) and for every \( e = (\varepsilon_1, \ldots, \varepsilon_n) \in \{0,1\}^n \) define

\[ \gamma'_e = \prod_{1 \leq i \leq n} (\gamma_{\varepsilon_i} + p_{i-1}) \]

where the sequence \((p_i)_{0 \leq i \leq n}\) satisfies \( k_0 = 0 \) and is defined inductively by the relation:

\[ p_{i+1} = \begin{cases} p_i + (mr,0) & \text{if } \varepsilon_i = 0, \\ p_i + (0,mr) & \text{if } \varepsilon_i = 1. \end{cases} \]

The path \( \gamma'_e \) is admissible of order \( l = nmrN \). More precisely, there exists a point \( z_e \in \phi_{z^*} \) such that \( \bar{f}^l(z_e) \in \phi_{z^*} + k_n \), and such that \( \gamma'_e = \bar{I}_{\bar{F}}(z_e) \).

\textbf{Lemma 4.8:} \textit{If} \( e \) \textit{and} \( e' \) \textit{are two different sequences in} \( \{0,1\}^n \), \textit{there exists} \( j \in \{0, \ldots, l-1\} \) \textit{such that} \( \|\bar{f}^j(z_e) - \bar{f}^j(z_{e'})\| \geq 1 \).

\textit{Proof.} Let consider the index \( i^* \) such that \( \varepsilon_{i^*} \neq \varepsilon'_{i^*} \) and \( \varepsilon_i = \varepsilon'_i \) if \( i \neq i^* \). The leaf \( \phi_{z^*} + p_{i^*} \) is intersected by \( \gamma'_e \) but not by \( \gamma'_{e'} \). More precisely \( d(\phi_{z^*} + p_{i^*},\gamma'_e) \geq mr - K' - K_0 \). Using Lemma 71, one deduces that there exists \( j \in \{0, \ldots, l\} \) such that \( d(\bar{f}^j(z_e),\phi_{z^*} + p_{i^*}) \leq K \). Moreover, one knows that \( d(\bar{f}^j(z_{e'}),\gamma'_{e'}) \leq K_0 \) because \( \gamma'_{e'} \) intersects \( \phi_{\bar{f}^j(z_{e'})} \). One deduces that

\[ \|\bar{f}^j(z_e) - \bar{f}^j(z_{e'})\| \geq mr - K' - 2K_0 - K \geq 1. \]
To finish the proof of the proposition, let us define on \( T^2 \) the distance
\[
  d(Z, Z') = \inf_{\pi(z) = Z, \pi(z') = Z'} \| z - z' \|
\]
where
\[
  \pi : \mathbb{R}^2 \to T^2, \quad z \mapsto z + Z^2
\]
is the projection. Note that for every \( \pi \) and every map \( \pi \)

\( \mathrm{JKLM} \):

\[ \begin{align*}
  \pi^{-1}(B(Z, 1/4)) &= \bigcup_{\pi(z) = Z} B(z, 1/4) \\
  \end{align*} \]

and every map \( \pi|_{B(z, 1/4)} \) is an isometry from \( B(z, 1/4) \) onto \( B(Z, 1/4) \).

Fix \( \varepsilon \in (0, 1/4) \) such that for every \( z, z' \) in \( \mathbb{R}^2 \), one has
\[
  \| z - z' \| < \varepsilon \Rightarrow \| \tilde{f}(z) - \tilde{f}(z') \| < 1/4.
\]

One deduces that two points \( Z \) and \( Z' \) such that \( d(f^j(Z), f^j(Z)) < \varepsilon \), for every \( j \in \{0, \ldots, l - 1\} \) are lifted by points \( z, z' \) such that \( \| \tilde{f}(z) - \tilde{f}(z') \| < \varepsilon \), for every \( j \in \{0, \ldots, l - 1\} \).

Consequently, the points \( z_{\varepsilon} \) project on a \((nmrN, \varepsilon)\)-separated set of cardinality \( 2^n \). One deduces that \( h(f) > \log 2/mrN \).

\[ \square \]

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