PERIODIC SOLUTIONS OF \( p \)-LAPLACIAN EQUATIONS VIA ROTATION NUMBERS

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Abstract. We investigate the existence and multiplicity of periodic solutions of the \( p \)-Laplacian equation \((\phi_p(x'))' + f(t,x) = 0\). Both asymptotically linear and partially superlinear nonlinearities are studied, in absence of global existence and uniqueness conditions on the solutions of the associated Cauchy problems and the sign assumption on \( f \). We use a approach of rotation number in the \( p \)-polar coordinates transformation, together with the phase-plane analysis of the rotational properties of large solutions and a recent version of Poincaré-Birkhoff theorem for Hamiltonian systems, for obtaining multiplicity results of \( p \)-Laplacian equation in terms of the gap between the rotation numbers of referred piecewise \( p \)-linear systems at zero and infinity.

1. Introduction. We are interested in the existence and multiplicity of periodic solutions of the \( p \)-Laplacian equation
\[
(\phi_p(x'))' + f(t,x) = 0,
\]
where \( \phi_p : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) is defined by \( \phi_p(x) = \frac{|x|^{p-2}x}{p} \), \( p > 1 \), \( f(t,x) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) is continuous, \( 2\pi_p \)-periodic with respect to the first variable, where
\[
\pi_p = \frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)}.
\]
These \( p \)-Laplacian equations are the generalization of the second order differential equations where \( p = 2 \). It is not restrictive to assume that the period of \( f \) in \( t \) is \( 2\pi_p \). When \( f \) is \( T \)-periodic in \( t \), we can use a similar discussions on replacing \( 2\pi_p \)-period with \( T \)-period to obtain the existence and multiplicity of \( T \)-periodic solutions of the \( p \)-Laplacian equation (1.1).

For the results corresponding to the existence of periodic solutions of (1.1) based on topological degree theory and variational method one can refer to [1, 3, 10, 14, 15, 2020 Mathematics Subject Classification. Primary: 34C25, 34B15; Secondary: 34D15.

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18, 28] and the references therein. Furthermore, the Poincaré-Birkhoff theorem is also an effective tool for studying the existence and multiplicity of periodic solutions of (1.1). It is well known that the global existence of all solutions is generally a basic assumption for the applications of the Poincaré-Birkhoff theorem, even for $p = 2$ [6, 20, 21, 22, 23]. When $p = 2$, influenced by works in the papers [7, 16] and two classical papers [11, 12], Qian, Wang and Torres [24] recently developed a new approach to deal with the problems without both global existence of the associated Cauchy problems and the sign assumption on $f(t, x)$. Precisely, they [24] used a rotation number approach and the phase-plane analysis of spiral properties to obtain multiplicity results in terms of the gap between the rotation numbers of the referred piecewise linear systems, at zero and infinity. Particularly, [24] obtained multiplicity results with a piecewise linear setting, i.e.

$$(H_l')$$ there exist functions $a_{±} \in L^1([0, 2\pi])$ such that

$$a_{±}(t) \leq \lim \inf_{x \to \pm \infty} \frac{f(t, x)}{x} \quad \text{uniformly a.e. in } t \in [0, 2\pi].$$

$$(H_r')$$ there exist functions $b_{±} \in L^1([0, 2\pi])$ such that

$$b_{±}(t) \geq \lim \sup_{x \to 0^\pm} \frac{f(t, x)}{x} \quad \text{uniformly a.e. in } t \in [0, 2\pi].$$

When $p \neq 2$, the case is relatively more complex. Recently, by the use of a version of the Poincaré-Birkhoff theorem [5, 9, 25, 22], Yan and Zhang [27] utilized the difference between the rotation numbers of asymptotically $p$-linear equations of (1.1) at the origin and infinity to find precise multiple $2\pi_p$-periodic solutions of (1.1). Precisely, they [27] used the following assumption

$$(f_1)$$ there exist $\omega_0, v_0, \omega_\infty, v_\infty \in L^1(\mathbb{R}/2\pi_p\mathbb{Z})$ such that the following limits

$$\lim_{x \to 0^+} \frac{f(t, x)}{|x|^{p-2}x} = \omega_0(t), \quad \lim_{x \to 0^-} \frac{f(t, x)}{|x|^{p-2}x} = v_0(t)$$

and

$$\lim_{x \to +\infty} \frac{f(t, x)}{|x|^{p-2}x} = \omega_\infty(t), \quad \lim_{x \to -\infty} \frac{f(t, x)}{|x|^{p-2}x} = v_\infty(t)$$

exist and are uniform a.e. in $t \in [0, 2\pi_p]$.

Assumption $(f_1)$ guarantees that all solutions of Cauchy problem associated with (1.1) are defined globally on $[0, 2\pi_p]$. Besides the assumption $(f_1)$, the uniqueness of the Cauchy problem is assumed in [27]. We refer to [2, 17, 19] and the references therein for other related papers in the $p$-Laplacian context, which assume some conditions to guarantee that all solutions of the associated Cauchy problems exist uniquely on the whole $t$-axis. We also notice that in [24], the uniqueness of solutions for the Cauchy problems associated with the equivalent system

$$x' = -y, \quad y' = f(t, x)$$

of (1.1) ($p = 2$) is required. In particular, when $y' = f(t, x)$ changes the sign at $y = 0$, in order to describe the spiral property of the solution with a given initial value at $y = 0$, the authors of [24] used a series of the solutions of analytic systems to approximate the unique solution of the original systems (1.2) with given initial value (see, Lemma 3.1 in [24]).

When $p > 2$, the right-hand side term $-\phi_q(y)$ of the equivalent system

$$x' = -\phi_q(y), \quad y' = f(t, x)$$
of (1.1) is only Hölder continuous at \( y = 0 \). Then, the uniqueness for the associated Cauchy problem is not easy to check.

In this paper we provide an extension to the \( p \)-Laplacian case, without assuming uniqueness of solutions, of the results presented in [24] for the case \( p = 2 \), and of some results given in [27] for the \( p \)-Laplacian but with more restrictive assumptions.

To overcome the lack of uniqueness, we use a recent version of the Poincaré-Birkhoff theorem for Hamiltonian systems by Fonda and Ureña ([8], [7]). We also provide a different method to improve the phase-plane analysis of the spiral properties when \( y' = f(t,x) \) changes the sign at \( y = 0 \).

We will consider a general piecewise \( p \)-linear setting, i.e., we assume that

\[(H_{\infty}^p) \text{ there exist functions } a_{\pm} \in L^1(\mathbb{R}/2\pi_p\mathbb{Z}) \text{ such that} \]
\[a_{\pm}(t) \leq \liminf_{x \to \pm \infty} \frac{f(t,x)}{\phi_p(x)} \text{ uniformly a.e. in } t \in [0,2\pi_p], \]

\[(H_0^p) \text{ there exist functions } b_{\pm} \in L^1(\mathbb{R}/2\pi_p\mathbb{Z}) \text{ such that} \]
\[b_{\pm}(t) \geq \limsup_{x \to 0} \frac{f(t,x)}{\phi_p(x)} \text{ uniformly a.e. in } t \in [0,2\pi_p]. \]

The following is the primary result of this paper, where the usual notations, \( x^+ = \max\{x,0\} \) and \( x^- = \max\{-x,0\} \), are used.

**Theorem 1.1.** Suppose that (1.1) satisfies \((H_{\infty}^p)\) and \((H_0^p)\). Denote by \(\rho(a_{\infty})\) and \(\rho(b_0)\) the rotation numbers of the piecewise \( p \)-linear equations

\[(\phi_p(x'))' + a_+(t)\phi_p(x^+) + a_-(t)\phi_p(x^-) = 0 \]

and

\[(\phi_p(x'))' + b_+(t)\phi_p(x^+) + b_-(t)\phi_p(x^-) = 0, \]

respectively. If \(\rho(a_{\infty}) > \rho(b_0)\), then for any rational number \( j/m \in (\rho(b_0),\rho(a_{\infty}))\), equation (1.1) has at least two \( 2m\pi_p \)-periodic solutions. Furthermore, such \( 2m\pi_p \)-periodic solutions have exactly \( 2j \) zeros in \([t_0,t_0 + 2m\pi_p]\).

**Remark 1.** The definition of rotation number will be given in the next section, see also in [27]. Theorem 1.1 improves the corresponding result of [27]. Theorem 1.1 gives also an extension to the \( p \)-Laplacian case, without uniqueness assumption, of the results presented in [24] for the case \( p = 2 \).

The model example is a partially \( p \)-superlinear Laplacian equation, i.e., for (1.1), we assume that

\[(f_p) \quad f(t,x)/\phi_p(x) \geq l(t) \text{ for } |x| \gg 1 \text{ and } t \in [0,2\pi_p], \]

moreover,

\[\lim_{|x| \to +\infty} \frac{f(t,x)}{\phi_p(x)} = +\infty, \quad \text{uniformly for } t \in I \subset [0,2\pi_p], \]

where \( l(t) \in L^1(\mathbb{R}/2\pi_p\mathbb{Z}) \) and \( I \) is a set of positive measure.

We can apply Theorem 1.1 above to prove the following result, which extends the results of [24, Corollary 1.1] and the classical results of Jacobowitz [12] and Hartman [11] to the partially \( p \)-superlinear Laplacian equation.

**Corollary 1.** Assume that (1.1) satisfies \((H_0^p)\) and \((f_p)\), then for any \( m \in \mathbb{N} \), there are infinitely many \( 2m\pi_p \)-periodic solutions for equation (1.1).
The remaining sections of the paper are organized as follows. In Section 2, we introduce the fundamental notion of rotation number in the p-polar coordinates transformation and present certain auxiliary lemmas. In Section 3, we give the phase plane analysis of the spiral properties. Section 4 is devoted to the proofs of Theorem 1.1 and Corollary 1. The Appendix (Section 5) recalls, and further augments, the primary tools for the sake of clarity and coherency.

2. Definition and properties of the rotation numbers. Let $p \in (1, \infty)$, denote by $q \in (1, \infty)$ the conjugate number of $p$, that is $1/p + 1/q = 1$. Yan and Zhang [27] introduced the following $p$-polar coordinate transformation of the plane $\mathbb{R}^2$

$$x = r^{2/p}C_p(\theta), \quad y = r^{2/q}S_p(\theta),$$

(2.1)

where $(C_p(\cdot), S_p(\cdot))$ is the unique solution of

$$x' = -\phi_q(y), \quad y' = \phi_p(x),$$

satisfying $(x(0), y(0)) = (1, 0)$. We also refer [29] for the $p$-polar coordinate transformation of the plane $\mathbb{R}^2$.

Now, we recall the basic properties of $C_p(\theta)$ and $S_p(\theta)$ (see [27]), which is beneficial for our purposes.

Lemma 2.1. (1) Both $C_p(\theta)$ and $S_p(\theta)$ are $2\pi_p$-periodic,

(2) $C_p(\theta)$ is even in $\theta$ and $S_p(\theta)$ is odd in $\theta$,

(3) $C_p(\theta + \pi_p) = -C_p(\theta), \quad S_p(\theta + \pi_p) = -S_p(\theta)$,

(4) $C_p(\theta) = 0$ if and only if $\theta = \pi_p/2 + n\pi_p$, $n \in \mathbb{Z}$, and $S_p(\theta) = 0$ if and only if $\theta = n\pi_p$, $n \in \mathbb{Z}$,

(5) $dC_p(\theta)/d\theta = -\phi_q(S_p(\theta))$, and $dS_p(\theta)/d\theta = \phi_p(C_p(\theta))$, and

(6) \[|C_p(\theta)|^p/p + |S_p(\theta)|^q/q = 1/p.\]

Let $y = -\phi_p(x')$. Equation (1.1) can be rewritten as a planar system

$$x' = -\phi_q(y), \quad y' = f(t, x).$$

(2.2)

In the following, we will write the solution of (2.2) as $z(t) = (x(t), y(t)) \in \mathbb{R}^2$. If $z(s) \neq 0$ for every $s$ in interval $[t_0, t]$, we can define the $t$-rotation number of $z(t)$ as

$$\text{Rot}(z(t); [t_0, t]) = \frac{\theta(t) - \theta(t_0)}{2\pi_p}, \quad \forall t \in [t_0, t_0 + 2\pi_p],$$

where $\theta(t)$ is the argument function of $z(t)$ associated with the $p$-polar coordinates (2.1). Indeed, Rot$(z(t); [t_0, t])$ describes counter-clockwise rotations performed by $z(t)$ around the origin, in $(x, y)$ phase-plane and the time interval $[t_0, t]$. Moreover, for a positive integer $j$, Rot$(z(t); [t_0, t]) > j(<j)$ implies that $z(t)$ performs more (less) than $j$ counter-clockwise rotations around the origin in $(x, y)$ phase-plane. In the following, we write by Rot$^j(t; z) = \text{Rot}(z(t); [t_0, t])$ for short.

When (2.2) is a piecewise $p$-linear system

$$x' = -\phi_q(y), \quad y' = \omega_+(t)\phi_p(x^+) - \omega_-(t)\phi_p(x^-),$$

(2.3)

the argument function $\theta(t)$ satisfies

$$\theta'(t) = \omega_+(t)|C_p^+(\theta)|^p + \omega_-(t)|C_p^-(\theta)|^p + p|S_p(\theta)|^q/q := A(t; \theta; p),$$

(2.4)

where $\omega_\pm(t) \in L^1(\mathbb{R}/2\pi_p\mathbb{Z})$. Thus, $A(t; \theta; p)$ is independent of $r$. By applying Lemma 2.1, we have

$$|A(t; \theta; p)| \leq \max\{\omega_\pm(t), 1\} \quad \text{for all } t \in [t_0, t_0 + 2\pi_p].$$
Then, $\theta(t)$ is well defined for all $t \in [t_0, t_0 + 2\pi p]$. Furthermore, $A(t, \theta; p)$ is $2\pi p$-periodic in both $t$ and $\theta$. Thus, Eq. (2.4) is a differential equation on a torus. Therefore, we can define the rotation number corresponding to (2.4) as

$$\rho(\omega) = \lim_{t \to +\infty} \frac{\theta(t; t_0, \theta_0) - \theta_0}{t},$$

where $\theta(t; t_0, \theta_0)$ is the unique solution for (2.4) with the initial condition $\theta(t_0; t_0, \theta_0) = \theta_0$. Yan and Zhang [27] gave further details of $\rho(\omega)$.

For any $t \in [t_0, t_0 + 2\pi p]$, we write the $t$-rotation number of the solution $z(t)$ of (2.3) as $\text{Rot}^\omega(t; v)$, where $v = (1, \theta_0) \in \Gamma_0 = \{\xi = (r, \theta) : r = 1, \ \theta \in \mathbb{R}\}$, $\theta_0$ is the polar angle of $z(t_0)$ in the $p$-polar coordinates.

In analogy to Lemma 4.1 [27], we can prove the following relations for the rotation number of $\rho(\omega)$ and the $t$-rotation number of the solution of (2.3).

**Lemma 2.2.** Let $j/m$ be a rational number, then

(i) $\rho(\omega) < j/m \iff \max_{v \in \Gamma_0} \text{Rot}^\omega(t_0 + 2m\pi p; v) < j$;

(ii) $\rho(\omega) > j/m \iff \min_{v \in \Gamma_0} \text{Rot}^\omega(t_0 + 2m\pi p; v) > j$.

Further, we give the following comparison with the result (Lemma 2.3) associated with the $t$-rotation number of the solution of (2.2). The proof of Lemma 2.3 is given in the Appendix. Hereafter, we write $|z(t)|_p := \sqrt{|x|^p + |y|^q}/q$ for simplicity, where $z(t) = (x(t), y(t))$.

**Lemma 2.3.** Let $f(t, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a $L^1$-Carathéodory function and $2\pi p$-periodic in the first variable, and let $\omega_\pm(t) \in L^1(\mathbb{R}/2\pi p\mathbb{Z})$ be such that

$$(f_3) \quad \liminf_{x \to \pm \infty} \frac{f(t, x)}{\phi_p(x)} \geq \omega_\pm(t) \quad \text{uniformly a.e. in } t \in [t_0, t_0 + 2\pi p],$$

then for each $\varepsilon > 0$, there is $R_\varepsilon > 0$ such that for each solution $z(t)$ of (2.2) with $|z(t)|_p \geq R_\varepsilon$, for all $t \in [t_0, t_0 + 2\pi p]$, it follows that

$$\text{Rot}^f(t; z) \geq \text{Rot}^\omega(t; v) - \varepsilon, \quad \forall \ t \in [t_0, t_0 + 2\pi p], \ \text{and } v = (1, \theta_0),$$

where $\theta_0$ is the polar angle of $z(t_0)$ in the $p$-polar coordinates.

If

$$(f_4) \quad \limsup_{x \to -0^+} \frac{f(t, x)}{\phi_p(x)} \leq \omega_\pm(t) \quad \text{uniformly a.e. in } t \in [t_0, t_0 + 2\pi p],$$

then for each $\varepsilon > 0$, there is $r_\varepsilon > 0$ such that for each solution $z(t)$ of (2.2) with $0 < |z(t)|_p \leq r_\varepsilon$, for all $t \in [t_0, t_0 + 2\pi p]$, it follows that

$$\text{Rot}^f(t; z) \leq \text{Rot}^\omega(t; v) + \varepsilon, \quad \forall \ t \in [t_0, t_0 + 2\pi p], \ \text{and } v = (1, \theta_0),$$

where $\theta_0$ is the polar angle of $z(t_0)$ in the $p$-polar coordinates.

**Remark 2.** Similar to [4], Lemma 2.3 does not require the global continuability of the solutions on $[t_0, t_0 + 2\pi p]$. The claims of this lemma have to be considered only in regard to those solutions $z(t)$ of (2.2) defined on $[t_0, t_0 + 2\pi p]$ such that $z(t) \neq 0$ for all $t \in [t_0, t_0 + 2\pi p]$.

Furthermore, we write $\text{Rot}^{(2.2)}_m(z(t)) := \text{Rot}(z(t); [t_0, t_0 + 2m\pi p])$, where $z(t)$ is a solution of system (2.2). We can find the following relations of $\text{Rot}^{(2.2)}_m(z(t))$ and the rotation number $\rho(\omega)$ of (2.3).
Lemma 2.4. Let \( f(t, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be a \( L^1 \)-Carathéodory function \( 2\pi_p \)-periodic in the first variable.

(i) Assume that \( f(t, x) \) satisfies \((f_3)\) and \( j/m \) is a rational number. If \( \rho(\omega) > j/m \), then there exists \( R_\varepsilon > 0 \) such that every solution \( z(t) \) of \((2.2)\) with \( |z(t)|_p \geq R_\varepsilon \), for all \( t \in [t_0, t_0 + 2m\pi_p] \), satisfies \( \text{Rot}_{2m}^{(2)}(z(t)) > j \).

(ii) Assume that \( f(t, x) \) satisfies \((f_4)\) and \( j/m \) is a rational number. If \( \rho(\omega) < j/m \), then there exists \( r_\varepsilon > 0 \) such that every solution \( z(t) \) of \((2.2)\) with \( 0 < |z(t)|_p \leq r_\varepsilon \), for all \( t \in [t_0, t_0 + 2m\pi_p] \), satisfies \( \text{Rot}_{2m}^{(2)}(z(t)) < j \).

**Proof.** As the proofs of the statements are identical, we will only prove the claim (i). By using Lemma 2.2, we have

\[
\text{Rot}^\omega(t_0 + 2m\pi_p; v) > j,
\]
where \( v = (1, \theta_0) \) and \( \theta_0 \) is the polar angle of \( z(t_0) \) in the \( p \)-polar coordinates. Thus, we can choose a suitable \( \varepsilon > 0 \) such that

\[
\text{Rot}^\omega(t_0 + 2m\pi_p; v) - \varepsilon > j.
\]

From \((f_3)\) and Lemma 2.3, for such \( \varepsilon \), there exists \( R_\varepsilon > 0 \) such that for each solution \( z(t) \) of \((2.2)\) with \( |z(t)|_p \geq R_\varepsilon \), \( \forall t \in [t_0, t_0 + 2m\pi_p] \), it follows that

\[
\text{Rot}_{2m}^{(2)}(z(t)) = \text{Rot}^f(t_0 + 2m\pi_p; z) \geq \text{Rot}^\omega(t_0 + 2m\pi_p; v) - \varepsilon > j.
\]

This is the claim (i). \( \square \)

3. The phase plane analysis of the spiral properties. We will prove that the solutions of the system \((2.2)\) have a spiral property described by Lemma 3.1 under \((H^m_{\infty})\). Notice that for \((2.2)\), the solution starting from an initial point at \( y = 0 \) is not necessarily unique. So, in the part of dealing with the spiral property of the solution from \( y = 0 \), we could not follow the approximation method used in [24] which is based on the uniqueness of the solution with given initial point. We will make some improvements of phase-plane analysis of the spiral properties. We will replace the analysis of the solutions starting from a small neighborhood of \( y = 0 \) instead of the analysis of the solutions starting from \( y = 0 \).

In particular, we will find a priori estimates of a compact set \( E_\delta \), which can be used to show that large solution of \((2.2)\) from \( y = 0 \) only meet \( \{ y = \pm \delta \} \cap E_\delta \) finite times. Then, by use of the properties of \( E_\delta \) and energy functions \( v(x, y) \) and \( u(x, y) \), we can discuss the spiral properties of the solution starting from \( |y| < \delta \) recursively (see the proof of the Lemma 3.1, Case 3).

Hereafter, the \( p \)-polar coordinate \((r(t_0), \theta(t_0))\) of \( z(t_0) \) will be simply denoted by \((r_0, \theta_0)\). Note that the polar radius \( r(t) \) of \( z(t) \) satisfies \( r(t) = |z(t)|_p \). The following lemma is crucial here.

**Lemma 3.1.** Let \( f(t, x) \) satisfy \((H^m_{\infty})\). Then for any fixed \( m, N_0 \in \mathbb{N} \) and sufficiently large \( r_\star \), there exist two strictly monotonically increasing functions \( \xi_{N_0}^- (r) \), \( \xi_{N_0}^+ (r) : [r_\star, +\infty) \to \mathbb{R} \) such that

\[
\xi_{N_0}^\pm (r) \to +\infty \quad \iff \quad r \to +\infty.
\]

If a solution \((r(t), \theta(t))\) to the system \((2.2)\) satisfies \( r_0 \geq r_\star \), then either

\[
\xi_{N_0}^-(r_0) \leq r(t) \leq \xi_{N_0}^+(r_0), \quad \text{for } t \in [t_0, t_0 + 2m\pi_p];
\]
or there exists $t_{N_0} \in (t_0, t_0 + 2m\pi_p)$ such that
$$\theta(t_{N_0}) - \theta_0 = 2N_0\pi_p,$$
and
$$\xi_{N_0}(r_0) \leq r(t) \leq \xi_{N_0}(r_0), \quad \text{for } t \in [t_0, t_{N_0}].$$

Proof. For simplicity, we assume $m = 1$, $N_0 = 1$ and $t_0 = 0$. Let $z(t) = (x(t), y(t))$ be a solution of (2.2) satisfied $(x(0), y(0)) = (0, y_0)$ and $y_0 = (qr_0^2/p)^{1/q}$ large enough. We divide $\mathbb{R}^2$ into four regions $D_i$, $i = 1, 2, 3, 4$ as

$$D_1 = \{(x, y)|x \leq 0, y > 0\}; \quad D_2 = \{(x, y)|x < 0, y \leq 0\};$$
$$D_3 = \{(x, y)|x \geq 0, y < 0\}; \quad D_4 = \{(x, y)|x > 0, y \geq 0\}.$$ 

The proof will be divided into two steps.

Step 1. We first prove that, there exist $\xi_{N_0}(2)(r_0)$, with
$$\xi_{N_0}(2)(r_0) \rightarrow +\infty \iff r_0 \rightarrow +\infty,$$

such that either
$$z(t) \in D_1 \cup D_2, \quad \xi_{N_0}(2)(r_0) \leq r(t) \leq \xi_{N_0}(2)(r_0)$$

for $t \in [0, 2\pi_p]$, or there exists $t_2 \in (0, 2\pi_p)$ such that the latter inequality holds for $t \in [0, t_2)$ and $z(t)$ meet $x = 0$ entering into $D_3$ at $t = t_2$ (see Fig.1). Thereby, we consider the estimates of $z(t)$ as follows:

Case 1. Let $z(t) \in D_1$ for $t \in [0, t_1)$, where $t_1 \leq 2\pi_p$. We define an energy function
$$u(x, y) = \frac{|y|^q}{q} + F_+(x),$$
where $F_+(x) = \int_0^x f_+(s)ds$, $f_+(x) = \text{sgn}(x) \max\{|x|, \max_{t\in[0,2\pi_p]}|f(t, x)|\}$. Then $F_+(x) \rightarrow +\infty$ as $|x| \rightarrow +\infty$. Moreover,$$
\frac{d}{dt}u(x(t), y(t)) = |y|^{q-2}yy' + f_+(x)x' = |y|^{q-2}yf(t, x) - \phi_q(y)f_+(x)$$

**Figure 1.** The trajectories in regions $D_1$ and $D_2
\[|y|^q - 2y(f(t, x) - f_+(x)) \geq 0,\]

which implies that
\[u(x(t), y(t)) \geq u(0, y_0) = r_0^2/p, \quad \text{for } t \in [0, t_1).\]  

Then
\[r(t) \geq \xi_{\omega_0}(r_0), \quad \text{for } t \in [0, t_1),\]

where \(\xi_{\omega_0}(r_0) = \min\{\sqrt{|x|^p + p|y|^q/q}u(x, y) = r_0^2/p\} > 0\). If
\[|y|^q + F_+(x) = \frac{r_0^2}{p} \to +\infty \quad (r_0 \to +\infty),\]

then the terms \(|x|\) and \(|y|\) have at least one, which tends to \(+\infty\) as \(r_0 \to +\infty\). Thus,
\[\xi_{\omega_0}(r_0) \to +\infty \iff r_0 \to +\infty.\]  

On the other hand, we consider an energy function
\[v(x, y) = \frac{|y|^q}{q} + \frac{|x|^p}{p}.\]

From \(H^1_\infty\), there exist \(\varepsilon_0 \leq 1\) and \(M_{\varepsilon_0} > 0\), such that
\[f(t, x) = (\varepsilon_0 - a_- (t))|x|^{p-1} \leq \frac{|y|^q}{q} + \frac{|x|^p}{p}, \quad \text{for } t \in [0, t_1],\]

for \(x < -M_{\varepsilon_0}\), where \(a_- (t) \in L^1(\mathbb{R}/2\pi_p\mathbb{Z})\). From (3.2), for a sufficiently large \(r_0\), there exists \(t_1 \in (0, t_1)\) such that \(x(t_1) = -M_{\varepsilon_0}\). Then \(x(t) \in [-M_{\varepsilon_0}, 0] \text{ for } t \in [0, t_1]\) and \(x(t) < -M_{\varepsilon_0} \text{ for } t \in (t_1, t_1)\). When \(t \in [0, t_1]\), we have
\[|y'(t)| \leq K_{\varepsilon_0},\]

where \(K_{\varepsilon_0} = \max\{|f(t, x)| \mid t \in [0, 2\pi_p], x \in [-M_{\varepsilon_0}, 0]\}\). Thus
\[y_0 - 2K_{\varepsilon_0} \pi_p \leq y(t) \leq y_0 + 2K_{\varepsilon_0} \pi_p, \quad \text{for } t \in [0, t_1],\]

which implies that
\[r(t) \leq M_{\varepsilon_0}, \quad \text{for } t \in [0, t_1],\]

where \(M_{\varepsilon_0} = \max\{\sqrt{|x|^p + p|y|^q/q}x \in [-M_{\varepsilon_0}, 0], y \in [y_0 - 2K_{\varepsilon_0} \pi_p, y_0 + 2K_{\varepsilon_0} \pi_p]\}\). Recalling \(y_0 = (q_0^2/p)^1/q\), we have
\[M_{\varepsilon_0} \to +\infty \iff r_0 \to +\infty.\]  

When \(t \in (t_1, t_1)\), using (3.4) and the Young inequality, we have
\[
\frac{d}{dt} v(x(t), y(t)) = 2\frac{q}{p} |y|^q y' + \frac{p}{q} |x|^{p-2} xx' = 2\frac{q}{p} |y|^q y f(t, x) - \frac{p}{q} |x|^{p-2} x \phi_x(y) \leq |y|^q + \frac{p}{q} (|a_- (t)| + 1) \frac{|x|^{p-1} |y|^{q-1}}{q} \leq \frac{p}{q} + \frac{q}{p} (|a_- (t)| + 1) v(x(t), y(t)).
\]

Then, there exists \(c_1(t) := \frac{p}{q} + \frac{q}{p} (|a_- (t)| + 1) \in L^1(\mathbb{R}/2\pi_p\mathbb{Z})\) such that
\[\frac{d}{dt} v(x(t), y(t)) \leq c_1(t) v(x(t), y(t)),\]

which implies that
\[v(x(t), y(t)) \leq v(x(t_1), y(t_1)) c_1(t) - c_1(t) ds \leq v(-M_{\varepsilon_0}, y_0 + 2K_{\varepsilon_0} \pi_p) c_1(s) ds,\]
for $t \in [t_1, t_1]$.

Set
\[
\xi^+_t = \max \{ \sqrt{|x|^p + p|y|^q/q} v(x, y) = v(-M_{x_0}, y_0 + 2K_{x_0} \pi_p) e^C \} < +\infty,
\]
where $C = \int_0^{2\pi_p} c_1(s) ds < +\infty$. We can use an argument similar to (3.3), for obtaining $\xi^+_t \to +\infty$ as $r_0 \to +\infty$. Inequalities (3.2), (3.5) and (3.8) imply that
\[
\xi^+_{N_0(1)}(r_0) \leq r(t) \leq \xi^+_{N_0(1)}(r_0), \quad t \in [0, t_1),
\]
(3.9)
where $\xi^+_{N_0(1)}(r_0) = \max \{ \xi^+_t, M_{r_0} \}$.

If $t_1 = 2\pi_p$, the proof is completed. If $t_1 < 2\pi_p$, $z(t)$ will meet $y = 0$ entering into $D_2$ for the first time at $t = t_1$.

Note that we are not assuming the sign condition $\text{sgn}(x)f(t, x) > 0$ for sufficiently large $|x|$. Thus $z(t)$ could eventually return back to $D_1$ for sufficiently large $|x|$.

Let $t'_2 \in [t_1, t_2)$ be a time such that $y(t'_2) = 0$. We first consider $z(t)$ in the situation when $x(t) < -M_{x_0}$ for $t \geq t_1$. We will estimate $z(t)$ the following two cases (see Fig. 1):

Case 2. For case (a), we have $z(t) \in D_2$ and $x(t) < -M_{x_0}$ for $t \in [t'_2, t_2]$. $t_1' \in (t'_2, t_2]$.

We affirm that an estimate of $r(t)$ for $t \in [t'_2, t_2]$ only depends on $r(t'_2)$.

By using the similar argument for (3.7), we have
\[
\frac{d}{dt} v(x(t), y(t)) \geq -c_1(t)v(x(t), y(t)).
\]
(3.10)

It follows that
\[
v(x(t), y(t)) \geq v(x(t'_2), 0) e^{-\int_{t'_2}^{t_2} c_1(s) ds}, \quad \text{for } t \in [t'_2, t_2].
\]
(3.11)

On the other hand,
\[
\frac{d}{dt} u(x(t), y(t)) = |y|^{q-2} y f(t, x) - f_+(x) \leq 0, \quad \text{for } z(t) \in D_2.
\]
Then
\[
u(x(t), y(t)) \leq u(x(t'_2), 0) = F_+(x(t'_2)), \quad \text{for } t \in [t'_2, t_2].
\]
(3.12)
Inequalities (3.11) and (3.12) imply that
\[
\xi^-_{t_2} \leq r(t) \leq \xi^+_{t_2}, \quad \text{for } t \in [t'_2, t_2],
\]
(3.13)
where
\[
\xi^-_{t_2} = \min \{ \sqrt{|x|^p + p|y|^q/q} v(x, y) = |x(t'_2)|^p e^{-C}/q \}
\]
and
\[
\xi^+_{t_2} = \max \{ \sqrt{|x|^p + p|y|^q/q} u(x, y) = F_+(x(t'_2)) \}.
\]

Case 3. Now we discuss the estimates of case (b). Denote by $E_1$ the compact set
\[
\{(x, y)|x_{t_1}^- \leq -x \leq x_{t_1}^+, |y| \leq 1\},
\]
where
\[ x^\pm_{t_1} = v^\pm_{t_1} + 2\pi p \quad \text{and} \quad v(-v^\pm_{t_1}, 0) = v(-(\xi_N^{(1)}(r_0))^{2/p}, 0)e^{\pm 2C}. \]

Recalling \( \xi_N^{(1)}(r_0) \to +\infty \iff r_0 \to +\infty \), we have
\[ x_{t_1}^\pm \to +\infty \iff r_0 \to +\infty. \]

Let
\[ K_1 = \max_{(x,y) \in E_1} \left| \frac{\partial v}{\partial x}(x, y) \right|, \quad K_2 = \frac{C}{2\pi p} v(-x^\pm_{t_1}, 0), \tag{3.14} \]
and denote \( \lambda = K_1/K_2 \). Then there exists \( \varepsilon > 0 \) such that
\[ 1 - \varepsilon \lambda \Delta t \geq e^{-\Delta t} \quad \text{and} \quad 1 + \varepsilon \lambda \Delta t \leq e^{\Delta t}, \quad \text{for} \ \Delta t > 0. \tag{3.15} \]

For such \( \varepsilon > 0 \), we can find \( \delta \in (0, 1) \) is sufficiently small such that
\[ |y| < \delta \Rightarrow |x'| = |y|^{q-1} \leq \varepsilon. \]

We estimate the solution \( z(t) \) for \( t \in [t_1, t_2^*] \) in following cases.

**Case 3-(i).** If \( |y(t)| < \delta \) for all \( t \in [t_1, t_2^*] \), then \( |x'(t)| = |y(t)|^{q-1} \leq \varepsilon \) for all \( t \in [t_1, t_2^*] \). It follows that
\[ |x(t) - x(t_1)| \leq \varepsilon |t - t_1| \leq 2\varepsilon \pi p, \quad \text{for} \ t \in [t_1, t_2^*], \]
which implies that \(-x(t) \in [x_{t_1}^-, x_{t_1}^+] \) for \( t \in [t_1, t_2^*] \). Thus, \( z(t) \) is in \( E_\delta \) for \( t \in [t_1, t_2^*] \), where
\[ E_\delta = \{(x, y)|x_{t_1}^- \leq -x \leq x_{t_1}^+, |y| < \delta\}. \]

Therefore, we can find an uniform bound of \( r(t) \) for \( t \in [t_1, t_2^*] \).

**Case 3-(ii).** Let \( t_1^* \) be the first time such that
\[ y(t_1^*) = -\delta \quad \text{or} \quad y(t_1^*) = \delta. \]

Namely, \( |y(t)| < \delta \) for all \( t \in [t_1, t_1^*] \). By using a similar argument in Case 3-(i), we have
\[ z(t) \in E_\delta \quad \text{and} \quad |x(t) - x(t_1)| \leq \varepsilon |t - t_1|, \quad \text{for} \ t \in [t_1, t_1^*]. \]

From (3.14)-(3.15), we have
\[ v(x(t), 0) \geq v(x(t_1), 0) - K_1\varepsilon(t - t_1) = v(x(t_1), 0) \left( 1 - \frac{\varepsilon K_1}{v(x(t_1), 0)}(t - t_1) \right) \]
\[ \geq v(x(t_1), 0) \left( 1 - \varepsilon \lambda(t - t_1) \frac{C}{2\pi p} \right) \geq v(x(t_1), 0)e^{-\frac{C}{2\pi p}(t-t_1)} \tag{3.16} \]
for \( t \in [t_1, t_1^*] \).

On the other hand, by (3.14)-(3.15), we have
\[ v(x(t), 0) \leq v(x(t_1), 0) + K_1\varepsilon(t - t_1) = v(x(t_1), 0) \left( 1 + \frac{\varepsilon K_1}{v(x(t_1), 0)}(t - t_1) \right) \]
\[ \leq v(x(t_1), 0) \left( 1 + \varepsilon \lambda(t - t_1) \frac{C}{2\pi p} \right) \leq v(x(t_1), 0)e^{\frac{C}{2\pi p}(t-t_1)} \tag{3.17} \]
for \( t \in [t_1, t_1^*] \).

Combining (3.16) with (3.17), we have
\[ v(x(t_1), 0)e^{-\frac{C}{2\pi p}(t-t_1)} \leq v(x(t), 0) \leq v(x(t_1), 0)e^{\frac{C}{2\pi p}(t-t_1)} \tag{3.18} \]
for \( t \in [t_1, t_1^*] \).
If \( y(t^1_1) = -\delta \) and \( y(t) < 0 \) for all \( t \in [t^1_1, t^2_2] \), then \( z(t) \in D_2 \) and \( x(t) < -M_{c_0} \) for \( t \in [t^1_1, t^2_2] \). Thus, we can use an argument similar to Case 2, for obtaining an uniform bound of \( r(t) \) for \( t \in [t^1_1, t^2_2] \).

If not, there would be \( t''_1 \) such that \( y(t''_1) = 0 \). Therefore, we can find \( \tau'_1 \in [t^1_1, t^2_2] \) such that \( y(\tau'_1) = 0, \ |y(t)| < \delta \) for \( t \in [t^1_1, \tau'_1] \) and \( y(t) < 0 \) for \( t \in (\tau'_1, t''_1) \) (see Fig. 2). Similar to (3.11), we have

\[
v(x(t), y(t)) \geq v(x(\tau'_1), 0)e^{-\int_{\tau'_1}^{t''_1} c_1(s)ds} \quad \text{for} \quad t \in [\tau'_1, t''_1].
\] (3.19)

On the other hand, we can use a similar argument as (3.12) to find that

\[
u(x(t), y(t)) \leq u(x(\tau'_1), 0), \quad \text{for} \quad t \in [\tau'_1, t''_1].
\] (3.20)

If \( y(t^1_1) = \delta \) and \( y(t) > 0 \) for all \( t \in [t^1_1, t^2_2] \), then \( z(t) \in D_1 \) for \( t \in [t^1_1, t^2_2] \). Using an argument similar to Case 1, we can obtain the estimate of \( r(t) \) for \( t \in [t^1_1, t^2_2] \).

If not, there would be \( t''_1 \) such that \( y(t''_1) = 0 \). Furthermore, there exists \( \tau'_1 \in [t^1_1, t^2_2] \) such that \( y(\tau'_1) = 0, \ |y(t)| < \delta \) for \( t \in [t^1_1, \tau'_1] \) and \( y(t) > 0 \) for \( t \in (\tau'_1, t''_1) \) (see Fig. 3).

Similar to (3.1) and (3.8), we have

\[
u(x(t), y(t)) \geq v(x(\tau'_1), 0)e^{\int_{\tau'_1}^{t''_1} c_1(s)ds}
\] (3.21)

and

\[
u(x(t), y(t)) \leq u(x(\tau'_1), 0)e^{\int_{\tau'_1}^{t''_1} c_1(s)ds}
\] (3.22)

for \( t \in [\tau'_1, t''_1] \), respectively.

Notice that (3.16) and (3.17) imply that \( z(t) \in E_\delta \) for \( t \in [t^1_1, \tau'_1] \). Moreover, using (3.19), (3.20),(3.21) and (3.22), let

\[
w_{\tau'_1}^+ = \max\{\sqrt{|x|^p + p|y|^{q'/q}}(x, y) \in E_\delta \}, \quad w_{\tau'_1}^- = \min\{\sqrt{|x|^p + p|y|^{q'/q}}(x, y) \in E_\delta \},
\]

\[
w_{\tau'_1}^+ = \max\{\sqrt{|x|^p + p|y|^{q'/q}}u(x, y) \leq u(x^+_{\tau'_1}, 0)\}, \quad w_{\tau'_1}^- = \min\{\sqrt{|x|^p + p|y|^{q'/q}}u(x, y) \leq u(x^-_{\tau'_1}, 0)\},
\]

\[
w_{\tau'_1}^+ = \max\{\sqrt{|x|^p + p|y|^{q'/q}}v(x, y) \leq v(x^+_{\tau'_1}, 0)e^C\}, \quad w_{\tau'_1}^- = \min\{\sqrt{|x|^p + p|y|^{q'/q}}v(x, y) \geq v(x^-_{\tau'_1}, 0)e^{-C}\},
\]

\[
w_{\tau'_1}^+ = \max\{\sqrt{|x|^p + p|y|^{q'/q}}u(x, y) \geq u(x^+_{\tau'_1}, 0)\}, \quad w_{\tau'_1}^- = \min\{\sqrt{|x|^p + p|y|^{q'/q}}v(x, y) \geq v(x^-_{\tau'_1}, 0)e^{-C}\},
\]
Next, we will estimate the solution $r$ we can use an argument similar to Case 3-(i), for obtaining a uniform bound of $r$. To prove that there are finite times $t \in [t_1, t_2]$, we have

$$\xi^+_{t_i} = \max\{w^+_i, w^+_{v_i}, w^+_{u_i}\}, \quad \xi^-_{t_i} = \min\{w^-_i, w^-_{v_i}, w^-_{u_i}\},$$

we have

$$\xi^-_{t_i} \leq r(t) \leq \xi^+_{t_i}, \quad \text{for } t \in [t_1, t_2].$$

(3.23)

Furthermore, by the monotony increasing property of $v(x(t), 0)$ along the solution of the system in $D_1$ and the monotony decreasing property of $v(x(t), 0)$ along the solution of the system in $D_2$, we have

$$v(x(t_1''), 0) \leq v(x(t_1'), 0) \quad \text{or} \quad v(x(t_1''), 0) \geq v(x(t_1'), 0)$$

Combining (3.18), (3.19) or (3.18), (3.22), it follows that

$$v(x(t_1), 0)e^{-\int_{t_i}^{t_1} f_i(s)ds} = v(x(t_1'), 0)e^{\int_{t_i}^{t_1} f_i(s)ds} \leq v(x(t_1''), 0) \leq v(x(t_1), 0)e^{\int_{t_i}^{t_1} f_i(s)ds}. \quad (3.24)$$

Next, we will estimate the solution $z(t)$ for $t \in [t_1'', t_2'']$. If $|y(t)| < \delta$ for all $t \in [t_1'', t_2'']$, we can use an argument similar to Case 3-(i), for obtaining a uniform bound of $r(t)$ for $t \in [t_1'', t_2'']$. If not, the solution $z(t)$ would be similar to Case 3-(ii). We can prove that there are finite times

$$t_1' \leq t_1'' < t_1' < t_2'' < t_1'' \leq \cdots \leq t_1'' < t_1'' \leq t_1'' = t_1'' = t_1$$

in $[t_1, t_2'']$, with the similar properties as $t_1'$, $t_1''$ and $t_2''$. That is for $j=2, \cdots, k$, either

$$y(t_j^{j-1}) = -\delta, \quad y(t_j^{j-1}) = y(t_j') = 0,$$

$$|y(t)| < \delta, \quad \text{for } t \in [t_j^{j-1}, t_j^{j-1}) \quad \text{and} \quad y(t) < 0 \quad \text{for } t \in (t_j^{j-1}, t_j'),$$

or

$$y(t_j^{j-1}) = -\delta, \quad y(t_j^{j-1}) = y(t_j') = 0,$$

$$|y(t)| < \delta, \quad \text{for } t \in [t_j^{j-1}, t_j^{j-1}) \quad \text{and} \quad y(t) > 0 \quad \text{for } t \in (t_j^{j-1}, t_j').$$

In fact, since $z(t_1')$ and $z(t_1'')$ are on two different straight lines, and $z(t) \in E_\delta$ for all $t \in [t_1, t_2'']$, we have

$$\delta \leq d(z(t_1'), z(t_1'')) \leq |x(t_1') - x(t_1)| + |y(t_1'') - y(t_1)| \leq K_3(t_1'' - t_1),$$

Figure 3. Trajectory intersects $y = 0$ and $y = \delta$
where \( K_3 = \max_{t \in [0, 2\pi_p]} |x(t, y)| \). It follows that
\[
t_{1}'' - t_{1} - t_{1} \leq \frac{\delta}{K_3}, \tag{3.25}
\]
Similar to (3.25), we have
\[
t_{1}'' - t_{1}' - t_{1} \geq \frac{\delta}{K_3}, \quad \text{for } j = 3, \ldots, k. \tag{3.26}
\]
Inequalities (3.25) and (3.26) imply that \( k < \frac{2K_3\pi_p}{\delta} + 1 \).

Similar to (3.19), (3.20), (3.21) and (3.22), for \( j = 3, \ldots, k \), recursively, we have either
\[
v(x(t), y(t)) \geq v(x(\tau_{1}^{j-1}), 0)e^{-\int_{\tau_{1}^{j-1}}^{\tau_{1}^{j}} c(s)ds}
\]
and
\[
u(x(t), y(t)) \leq u(x(\tau_{1}^{j-1}), 0),
\]
for \( t \in [\tau_{1}^{j-1}, \tau_{1}^{j}] \), or
\[
u(x(t), y(t)) \geq u(x(\tau_{1}^{j-1}), 0)
\]
and
\[
u(x(t), y(t)) \leq v(x(\tau_{1}^{j-1}), 0)e^{\int_{\tau_{1}^{j-1}}^{\tau_{1}^{j}} c(s)ds},
\]
for \( t \in [\tau_{1}^{j-1}, \tau_{1}^{j}] \), \( j = 3, \ldots, k \).

Moreover, it follows that
\[
v(x(t_{1}), 0)e^{-\frac{C_1}{\pi_p}(t_{1}'' - t_{1})} \leq \int_{\tau_{1}^{j-1}}^{\tau_{1}^{j}} c(s)ds,
\]
for \( j = 3, \ldots, k \).

Using (3.27), similar to (3.16) and (3.17), it follows that \( z(t) \in E_\delta \) for \( t \in [\tau_{1}^{j-1}, \tau_{1}^{j}] \), \( j = 3, \ldots, k \).

Then, we can obtain the same estimation as (3.23), that is
\[
\xi_{t_{1}}^- \leq r(t) \leq \xi_{t_{1}}^+, \quad \text{for } t \in [\tau_{1}^{j-1}, \tau_{1}^{j}], \quad j = 3, \ldots, k. \tag{3.28}
\]
For \( t > t_k \), we have one of the following possibilities:
(i) \( z(t) \in D_1 \) for all \( t \in (t_k, t_1^*) \);
(ii) \( z(t) \in D_2 \) for all \( t \in (t_k, t_1^*) \);
(iii) \( |y(t)| < \delta \) for all \( t \in (t_k, t_2^*) \).

By (3.23), (3.28) and the discussions in Cases 1, 2 and 3-(i), there exist \( \xi_{t_{1}}\pm(r_0) \), with
\[
\xi_{t_{1}}^- (r_0) \to +\infty \iff r_0 \to +\infty, \tag{3.29}
\]
such that
\[
x(t) < -M_{\epsilon_0}, \quad \xi_{t_{1}}^- (r_0) \leq r(t) \leq \xi_{t_{1}}^+(r_0) \quad \text{for } t \in [t_1, t_2^*]. \tag{3.30}
\]
If there exists \( \tilde{t} \in (t_2^*, t_2) \) such that \( x(\tilde{t}) = -M_{\epsilon_0} \) and \( y(\tilde{t}) < 0 \), by (3.29) and (3.30), we have \( r(\tilde{t}) \geq \xi_{t_{1}}^- (r_0) \to +\infty \) as \( r_0 \to +\infty \). Then \( r(\tilde{t}) \) is sufficiently large. Note that \( z(t) \) cannot return back to \( D_1 \) for \( t \in [\tilde{t}, t_2) \) again, where \( x(t_2) = 0 \) or \( t_2 = 2\pi_p \).

Indeed, for \( t \in [\tilde{t}, t_2) \), we have \( x'(t) = -|y(t)|^{q-2}y(t) \geq 0 \) then \( -M_{\epsilon_0} \leq x(t) < 0 \) which implies that
\[
y(t) \leq y(\tilde{t}) + 2K_{\epsilon_0}\pi_p < 0.
\]
Then
\[
\sqrt{p}\|y(t) + 2K_{\epsilon_0}p|^{\frac{q}{p}}/q \leq r(t) \leq \sqrt{|M_{\epsilon_0}|^p + p}\|y(t) - 2K_{\epsilon_0}p|^{\frac{q}{p}}, \quad \text{for } t \in [\bar{t}, t_2],
\] (3.31)
where \( y(\bar{t}) = -[q(r^2(\bar{t}) - M_{\epsilon_0}^p)/p]^{1/q} \).

Therefore, by (3.9), (3.13), (3.30) and (3.31), there exist \( \xi_{N_0}^\pm(r_0) \), with
\[
\xi_{N_0}^\pm(r_0) \to +\infty \iff r_0 \to +\infty,
\]
such that either
\[
z(t) \in D_1 \cup D_2, \quad \xi_{N_0}^-(r_0) \leq r(t) \leq \xi_{N_0}^+(r_0)
\]
for \( t \in [0, 2\pi_p] \), or the latter inequality holds for \( t \in [0, t_2) \).

**Step 2.** Similarly, we can discuss the cases of \( z(t) \in D_3 \) and \( D_4 \). To conclude, we can find \( \xi_{N_0}^\pm(r_0) \) such that
\[
\xi_{N_0}^-(r_0) \leq r(t) \leq \xi_{N_0}^+(r_0), \quad \text{for } t \in [t_2, 2\pi_p].
\]
Set
\[
\xi_{N_0}^+(r_0) = \max\{\xi_{N_0}^{+(i)}(r_0), \ i = 2, 4\}
\]
and
\[
\xi_{N_0}^-(r_0) = \min\{\xi_{N_0}^{-(i)}(r_0), \ i = 2, 4\}.
\]
Then either
\[
\xi_{N_0}^-(r_0) \leq r(t) \leq \xi_{N_0}^+(r_0), \quad \text{for } t \in [0, 2\pi_p],
\]
or there exists \( \hat{t}_1 \in [0, 2\pi_p] \), such that \( z(t) \) intersects \( x = 0 \) at \( t = \hat{t}_1 \) and \( z(t) \) completes one counter-clockwise turn around the origin when \( t \in [0, \hat{t}_1] \). Moreover,
\[
\xi_{N_0}^-(r_0) \leq r(t) \leq \xi_{N_0}^+(r_0), \quad \text{for } t \in [0, \hat{t}_1].
\]
Finally, it is clear that \( \xi_{N_0}^\pm(r_0) \) can be chosen as strictly increasing functions. For any \( t_0 \in (0, 2\pi_p] \), \( N_0 > 1 \) and \( m > 1 \), the conclusion can be proved by similar arguments. \( \square \)

4. Modified Hamiltonian systems and the existence of periodic solutions.

Notice that assumptions \((H_i^I)\) and \((H_i^C)\) cannot guarantee that all solutions of Cauchy problems associated with \((2.2)\) are defined globally on \([0, 2\pi_p]\). Thus the Poincaré map may not be well-defined. Then, we consider a modified planar system of \((2.2)\). Define a Hamiltonian function
\[
H(t, x, y) = \frac{|y|^q}{q} + \frac{|x|^p}{p} + K(r^2) \left( F(t, x) - \frac{|x|^p}{p} \right),
\]
where \( F(t, x) = \int_0^x f(t, s) \, ds \), \( K(r^2) = K(|x|^p + p|y|^q/q) \in C^\infty(\mathbb{R}^+, \mathbb{R}) \) is a truncating function satisfying
\[
K(r^2) = \begin{cases} 
1, & r \leq \tilde{r}_1; \\
\text{smooth connection,} & \tilde{r}_1 < r < \tilde{r}_2; \\
0, & r \geq \tilde{r}_2,
\end{cases}
\]
By using the Young inequality, we have

where \( \tilde{c} \) exists.

Proof. When \( t \) \( z \)

The solution of (4.1) will also be simply denoted by \( z(t) \). It is easy to check the following lemma.

**Lemma 4.1.** Assume \((H_0^*)\). Then every solution \( z(t) \) of (4.1) exists globally for \( t \in \mathbb{R} \). If a solution \( z(t) \) of (4.1) satisfies \( z(t_0) \neq (0, 0) \), then \( z(t) \neq (0, 0) \) for \( t \in \mathbb{R} \).

Proof. When \( z(t) = (x(t), y(t)) \) is sufficiently large, it satisfies autonomous Hamiltonian system

\[
x' = -\phi_q(y), \quad y' = \phi_p(x).
\]

Then \( z(t) \) exists globally for \( t \in \mathbb{R} \).

When \( z(t) = (x(t), y(t)) \) is sufficiently small, it satisfies (2.2). From \((H_0^*)\), there exists \( c(t) \in L^1(\mathbb{R}/2\pi p\mathbb{Z}) \) and \( \delta > 0 \) such that

\[
|f(t, x)| \leq |c(t)||x|^{p-1}, \quad \text{for } |x| < \delta.
\]

Let

\[
V(x, y) = \frac{|x|^p}{p} + \frac{|y|^q}{q}.
\]

By using the Young inequality, we have

\[
\left| \frac{dV(x(t), y(t))}{dt} \right| = \left| |y|^{q-2}yf(t, x) - |x|^{p-2}x\phi_q(y) \right| \leq |y|^{q-1}(|f(t, x)| + |x|^{p-1}) \\
\leq (|c(t)| + 1) \max\{q/p, p/q\} V(x(t), y(t)) \leq \sigma(t)V(x(t), y(t)),
\]

where the positive function \( \sigma(t) \in L^1(\mathbb{R}/2\pi p\mathbb{Z}) \). For any given \( T > 0 \), denote by \( \sigma_T = \int_{t_0}^{t_0+T} \sigma(s)ds < +\infty \). We have

\[
V(z(t_0))e^{-\sigma_T} \leq V(x(t), y(t)) \leq V(z(t_0))e^{\sigma_T}, \quad \text{for } t \in [t_0 - T, t_0 + T],
\]

which implies that if \( z(t_0) \neq (0, 0) \), then \( z(t) \neq (0, 0) \) for \( t \in [t_0 - T, t_0 + T] \). From the arbitrary choice of \( T \), we have proved the lemma.

When \( z(t) \) does not attain the origin, \( p \)-polar coordinates

\[
x(t) = r^{2/p}(t)C_p[\theta(t)], \quad y(t) = r^{2/q}(t)S_p[\theta(t)],
\]

are well-defined. 

**Lemma 4.2.** For any \( t_2 > t_1 \), the nonzero solutions of (4.1) satisfy

\[
\theta(t_2) - \theta(t_1) > -\pi_p.
\]

Proof. If \( x = 0 \) and \( y \neq 0 \), then \( F(t, x) = 0 \). It follows that \( x'y = -|y|^q < 0 \) for \( x = 0 \) and \( y \neq 0 \). Then a nonzero solution of (4.1) performs the counter-clockwise rotations at \( y \)-axis in this case. If the trajectory moves from the positive (negative) \( y \)-axis to the negative (positive) \( y \)-axis, the angular function \( \theta(t) \) of \( x, y \) will increment \( \pi_p \). When the trajectory moves back and forth between the second and third (first and fourth) quadrants, the increment of \( \theta(t) \) will be more than \(-\pi_p \).

Thus, for any \( t_2 > t_1 \), we have \( \theta(t_2) - \theta(t_1) > -\pi_p \).
To complete the proof of Theorem 1.1, we apply a recent version of Poincaré-Birkhoff theorem for Hamiltonian systems (see [8, 7]). Precisely, let \( 0 < R_1 < R_2 \) and consider the annulus in \( \mathbb{R}^2 \) defined as \( \Omega = B_{R_2} \setminus B_{R_1} \), where \( B_{R_i} \) is an open ball of radius \( R_i \), \( i = 1, 2 \). Let \( z(t) \) be the solution of (4.1), \( |z(t)| = \sqrt{x^2(t) + y^2(t)} \), and define

\[
\text{rot}_m(z(t)) := \frac{\dot{\vartheta}(t_0 + 2m\pi p) - \vartheta(t_0)}{2\pi}
\]

with the standard polar coordinates

\[
x(t) = |z(t)| \cos \vartheta(t), \quad y(t) = |z(t)| \sin \vartheta(t).
\]

Indeed, \( \text{rot}_m(z(t)) \) describes counter-clockwise rotations performed by the solution \( z(t) \) of (4.1) around the origin, in \((x, y)\) phase-plane and the time interval \([t_0, t_0 + 2m\pi p]\). Moreover, for a positive integer \( j \), \( \text{rot}_m(z(t)) > j(< j) \) implies that \( z(t) \) performs more (less) than \( j \) counter-clockwise rotations around the origin in \((x, y)\) phase-plane.

**Remark 3.** Write \( \text{Rot}^{(4.1)}_m(z(t)) := \text{Rot}(z(t); [t_0, t_0+2m\pi p]) \), where \( z(t) \) is a solution of system (4.1). Then, \( \text{Rot}^{(4.1)}_m(z(t)) > j(< j) \) implies that \( z(t) \) performs more (less) than \( j \) counter-clockwise rotations around the origin in \((x, y)\) phase-plane. It follows that

\[
\text{Rot}^{(4.1)}_m(z(t)) > j (< j) \iff \text{rot}_m(z(t)) > j (< j). \tag{4.2}
\]

As a consequence of Theorem 5 in [7], we have

**Theorem 4.3.** Assume that every solution \( z(t) \) of (4.1), departing from \( z(t_0) \in \partial \Omega \), is defined on \([t_0, t_0 + 2m\pi p]\) and satisfies

\[
z(t) \neq (0, 0), \quad \text{for } t \in [t_0, t_0 + 2m\pi p].
\]

Assume moreover that there is a positive integer \( j \) such that

\[
\text{rot}_m(z(t)) < j, \quad \text{if } |z(t_0)| = R_1, \quad \text{and } \text{rot}_m(z(t)) > j, \quad \text{if } |z(t_0)| = R_2.
\]

Then, the Hamiltonian system has at least 2 distinct \( 2m\pi p \)-periodic solutions \( z^i(t) \) with \( z^i(t_0) \in \Omega \), such that \( \text{rot}_m(z^i(t)) = j, \ i = 1, 2 \).

**Proof.** For simplicity, we assume \( t_0 = 0 \). Recall that \( \text{rot}^i(z, [0, T]) \) in [7] (see, Page 2154) is the number of clockwise rotations performed by \( z_i = (x_i, y_i) \) around the origin, in the time interval \([0, T]\). When \( N = 1 \) and \( i = 1 \), we have \( \text{rot}_m(z(t)) = -\text{rot}^i(z, [0, T]) \) with \( T = 2m\pi p \). Then all assumptions of Theorem 5 in [7] are satisfied, with \( N = 1 \).

**Proof of Theorem 1.1.** We will divide the proof into the following four steps.

**Step 1.** Following Lemma 2.4, there exists \( r_\epsilon > 0 \) such that, every solution \( z(t) \) of (2.2) with \( 0 < |z(t)|_p \leq r_\epsilon \) for all \( t \in [t_0, t_0 + 2m\pi p] \) satisfies

\[
\text{Rot}^{(2.2)}_m(z(t)) < j. \tag{4.3}
\]

Similar to the proof of Lemma 4.1, we have

\[
V(z(t_0))e^{-\sigma m} \leq V(x(t), y(t)) \leq V(z(t_0))e^{\sigma m}, \quad \text{for } t \in [t_0, t_0 + 2m\pi p],
\]

where \( \sigma_m = \int_{t_0}^{t_0+2m\pi p} \sigma(s)ds \).

We can find \( R_1 > 0 \) is sufficiently small such that if \( |z(t_0)| = R_1 \) then

\[
E_m = \{(x, y)|V(z(t_0))e^{-\sigma m} \leq V(x, y) \leq V(z(t_0))e^{\sigma m}\} \subset \{|z(t)|_p \leq r_\epsilon \}.
\]
Therefore, every solution \( z(t) \) of \((2.2)\) is in \( E_m \) for \( t \in [t_0, t_0 + 2m\pi_p] \) when \( |z(t_0)| = R_1 \). It follows from (4.3) that
\[
\text{Rot}^{(2.2)}_m(z(t)) < j, \quad \text{if } |z(t_0)| = R_1. \tag{4.4}
\]
Let \( \tilde{\epsilon}_1 > r_\epsilon \). Then the system \((4.1)\) is equivalent to \((2.2)\). By (4.4), we have
\[
\text{Rot}^{(4.1)}_m(z(t)) < j, \quad \text{if } |z(t_0)| = R_1. \tag{4.5}
\]

**Step 2.** By \((H'_\infty)\), \( \rho(a_\infty) > j/m \) and Lemma 2.4, there exists \( R_\epsilon > r_\epsilon \) such that, for any solution \( z(t) \) of \((2.2)\) with \( |z(t)|_p > R_\epsilon, \forall \ t \in [t_0, t_0 + 2m\pi_p] \), it follows that
\[
\text{Rot}^{(2.2)}_m(z(t)) > j. \tag{4.6}
\]
Let
\[
\Gamma = \{ z : |z|_p = R_\infty \},
\]
where \( R_\infty = (\xi^{-1}_{j+1})(R_\epsilon) \), and
\[
R_2 = \max\{ \sqrt{x^2 + y^2} : (x, y) \in \Gamma \}.
\]
Let
\[
\tilde{\epsilon}_1 = R'_\infty = \xi^{-1}_{j+1}(R'_2),
\]
where \( R'_2 = \max\{ |z|_p : \sqrt{x^2 + y^2} = R_2 \} \). Note that if \( r \leq R'_\infty \), then the system \((4.1)\) is equivalent to \((2.2)\).

Next, consider a solution of \((4.1)\) with \( |z(t_0)| = R_2 \). When \( R_\epsilon \leq r(t) \leq R'_\infty \), \( \forall \ t \in [t_0, t_0 + 2m\pi_p] \), it follows from (4.6) that
\[
\text{Rot}^{(4.1)}_m(z(t)) > j, \quad \text{if } |z(t_0)| = R_2. \tag{4.7}
\]
If there exists \( t_1 \in (t_0, t_0 + 2m\pi_p) \) such that \( r(t_1) < R_\epsilon \), then there are \( t_*, t'_1 \in (t_0, t_1) \) such that \( z(t_*) \in \Gamma, r(t'_1) = R_\epsilon \) and \( r(t) \leq R_\infty \) for \( t \in [t_*, t'_1] \). By Lemma 3.1, we have
\[
\theta(t'_1) - \theta(t_*) = 2(j + 1)\pi_p.
\]
Furthermore, by applying Lemma 4.2, we can deduce that
\[
\theta(t_0 + 2m\pi_p) - \theta(t_*) = (\theta(t_0 + 2m\pi_p) - \theta(t'_1)) + (\theta(t'_1) - \theta(t_*)) + (\theta(t_*) - \theta(t_0))
\]
\[
> -\pi_p + 2(j + 1)\pi_p - \pi_p = 2j\pi_p.
\]
Then (4.7) holds. Finally, if there exists \( t_2 \in (t_0, t_0 + 2m\pi_p) \) such that \( r(t_2) > R'_\infty \), the validity (4.7) is proven by the same arguments as given above.

**Step 3.** Consider the annular \( \Omega = \overline{B}_{R_2} \setminus B_{R_1} \). Then from (4.5), (4.7) and (4.2), we have
\[
\text{rot}_m(z(t)) < j, \quad \text{if } |z(t_0)| = R_1, \quad \text{and } \quad \text{rot}_m(z(t)) > j, \quad \text{if } |z(t_0)| = R_2.
\]
By applying Theorem 4.3, we can find at least two distinct \( 2m\pi_p \)-periodic solutions \( z^1(t) \) and \( z^2(t) \), such that \( z^i(t_0) \in \Omega \) and
\[
\text{rot}_m(z^i(t)) = j, \quad \text{for } i = 1, 2,
\]
It follows that
\[
\text{Rot}^{(4.1)}_m(z^i(t)) = j, \quad \text{for } i = 1, 2. \tag{4.8}
\]

**Step 4.** We will show that \( z^i(t) \) are in fact in \( r \leq \tilde{\epsilon}_1, i = 1, 2 \). Namely, \( z^i(t) \) are two distinct \( 2m\pi_p \)-periodic solutions of \((2.2)\), \( i = 1, 2 \). Notice that \( |z^i(t_0)|_p \leq R'_{2} \). If there exists \( t_3 \in (t_0, t_0 + 2m\pi_p) \) such that \( |z^i(t_3)|_p > \tilde{\epsilon}_1 \), then we can find \( t_3' \in (t_0, t_3) \)
It follows that for $t \mid l$ such that $z^{1}(t_{3}')_{p} = \hat{r}_{1}$ and $|z^{1}(t)|_{p} \leq \hat{r}_{1}$ for $t \in [t_{0}, t_{3}']$. Denote by $\theta_{1}(t)$ the argument function of $z^{1}(t)$. By Lemma 3.1, we have

$$\theta_{1}(t_{3}') - \theta_{1}(t_{0}) = 2(j + 1)\pi_{p}.$$  

Furthermore, by using Lemma 4.2, we have

$$\theta_{1}(t_{0} + 2m\pi_{p}) - \theta_{1}(t_{0}) = \theta_{1}(t_{0} + 2m\pi_{p}) - \theta_{1}(t_{3}') + \theta_{1}(t_{3}') - \theta_{1}(t_{0}) > 2j\pi_{p},$$

which contradicts (4.8). Therefore, $|z^{1}(t)|_{p} \leq R'_{\infty} = \hat{r}_{1}$, $\forall t \in [t_{0}, t_{0} + 2m\pi_{p}]$, i.e., $z^{1}(t)$ is a $2m\pi_{p}$-periodic solution of (2.2) with precisely $2j$ zeros in $[t_{0}, t_{0} + 2m\pi_{p}]$. Using a similar argument as above, $z^{2}(t)$ is also a $2m\pi_{p}$-periodic solution of the same node structure. The proof of Theorem 1.1 is thus completed. \hfill \Box

Now we give the proof of the claim on the partially $p$-superlinear Laplacian equation in Corollary 1.

**Proof.** By $(f_{p})$, for sufficient large $n \in \mathbb{N}$, we have

$$\liminf_{|x| \to +\infty} \frac{f(t, x)}{\phi_{p}(x)} \geq a_{n}(t) \quad \text{uniformly a.e. in } t \in [0, 2\pi_{p}],$$

where $a_{n}(t) = n^{2}$ for $t \in I$ and $a_{n}(t) = l(t)$ for $t \in [0, 2\pi_{p}] \setminus I$.

Consider the following $p$-linear equation

$$x' = -\phi_{q}(y), \quad y' = a_{n}(t)\phi_{p}(x). \quad (4.9)$$

Using a general $p$-polar coordinate

$$x = \left(\frac{r}{n}\right)^{\frac{2}{p}} C_{p}(\varphi), \quad y = r^{\frac{2}{q}} S_{p}(\varphi),$$

we have

$$\varphi'(t) = \frac{n^{2}/p(x'y' - px'y'/q)}{n^{2}|x|^{p} + p|y|^{q}/q} = \frac{n^{2}/p(a_{n}(t)|x|^{p} + p|y|^{q}/q)}{n^{2}|x|^{p} + p|y|^{q}/q}. $$

It follows that for $t \in I$, $\varphi'(t) \geq n^{2}/p$ and for $t \in [0, 2\pi_{p}] \setminus I$,

$$\varphi'(t) \geq -n^{2}/p l^{-}(t)|x|^{p} = -n^{2}(1 - \frac{1}{p}) l^{-}(t)|x|^{p} \geq -\frac{1}{n^{2}/q} l^{-}(t) (n^{2}|x|^{p} + p|y|^{q}/q) \geq -\frac{l^{-}(t)}{n^{2}/q},$$

where $l^{-}(t) = \max\{0, -l(t)\}$. Thus, consider the solution $(x(t), y(t))$ of (4.9) satisfying $x(t_{0}) = 1, y(t_{0}) = 0$. Its general argument function $\varphi(t)$ satisfies

$$\varphi(t_{0} + 2\pi_{p}) - \varphi(t_{0}) \geq n^{2}/p \text{mes}(I) - \frac{\int_{0}^{2\pi_{p}} |l(t)|dt}{n^{2}/q} \to +\infty \quad (n \to \infty). \quad (4.10)$$

Write the $t$-rotation number of the solution $(x(t), y(t))$ as $\text{Rot}^{a_{n}}(t_{0} + 2\pi_{p}; v)$, where $v \in \Gamma_{0}$. From (4.10), we have $\text{Rot}^{a_{n}}(t_{0} + 2\pi_{p}; v) \to +\infty$ as $n \to \infty$. By Lemma 2.2, we have $\rho(a_{n}) \to +\infty$ as $n \to \infty$. Therefore, applying Theorem 1.1, the claim on the partially $p$-superlinear Laplacian equation in Corollary 1 is proved. \hfill \Box
5. **Appendix.** Here we recall and also introduce, some technical tools that have been used in the proof of the existence and multiplicity of periodic solutions of the \(p\)-Laplacian equations considered in the preceding sections.

The following basic properties associated with the piecewise \(p\)-linear systems are needed for the proof of Lemma 2.3. As in the proof of Lemma 2.3 in Zhang [29], we can find that \(t\)-rotation number \(\text{Rot}^\omega(t; v)\) of equation (2.4) depends upon \(\omega(t)\) as follows.

**Lemma 5.1.** Let \(\omega(t), \lambda(t) \in L^1(\mathbb{R}/2\pi_p\mathbb{Z})\) such that \(\lambda(t) \leq \omega(t)\) for a.e. \(t \in [t_0, t_0 + 2\pi_p]\). Then,

\[
\text{Rot}^\lambda(t; v) \leq \text{Rot}^\omega(t; v), \quad \forall t \in [t_0, t_0 + 2\pi_p], \quad \forall v \in \Gamma_0.
\]

The proof of Lemma 5.1 can be given in spirit of Lemma 2.3 in Zhang [29]. So we omit it.

**Lemma 5.2.** Let \(\lambda(t) \in L^1(\mathbb{R}/2\pi_p\mathbb{Z})\). For each \(\epsilon > 0\) there is \(\delta > 0\) such that, for each function \(\omega(t) \in L^1(\mathbb{R}/2\pi_p\mathbb{Z})\) satisfying

\[
0 \leq \omega(t) - \lambda(t) \leq \delta,
\]

for a.e. \(t \in [t_0, t_0 + 2\pi_p]\), it follows that

\[
\text{Rot}^\omega(t; v) - \text{Rot}^\lambda(t; v) \leq \epsilon, \quad \forall t \in [t_0, t_0 + 2\pi_p], \quad \forall v \in \Gamma_0.
\]

**Proof.** Fix \(\epsilon > 0\) and assume that \(\lambda(t)\) and \(\omega(t)\) satisfy (5.1) for some \(\delta > 0\).

According to Lemma 5.1, we have

\[
\text{Rot}^{\omega-\delta}(t; v) \leq \text{Rot}^\lambda(t; v), \quad \forall t \in [t_0, t_0 + 2\pi_p], \quad \forall v \in \Gamma_0,
\]

where \(\text{Rot}^{\omega-\delta}(t; v)\) is the \(t\)-rotation number of

\[
(\phi_p(x')')' + (\omega_+(t) - \delta)\phi_p(x^+) + (\omega_-(t) - \delta)\phi_p(x^-) = 0.
\]

Hence, it is necessary to check whether, for a suitable choice of \(\delta > 0\) (sufficiently small), it follows that

\[
\text{Rot}^{\omega-\delta}(t; v) \geq \text{Rot}^\omega(t; v) - \epsilon, \quad \forall t \in [t_0, t_0 + 2\pi_p], \quad \forall v \in \Gamma_0.
\]

If not, for each \(n \in \mathbb{N}\), there would be \(t_n \in [t_0, t_0 + 2\pi_p]\) and \(v_n \in \Gamma_0\) such that

\[
\text{Rot}^{\omega-\delta}(t_n; v_n) < \text{Rot}^\omega(t_n; v_n) - \epsilon.
\]

There is no loss of generality in assuming \(t_n \to \tau\ (n \to \infty), \ v_n = (1, \alpha_n)\) and \(v_n \to \bar{v} := (1, \alpha)(n \to \infty)\). As \(\text{Rot}^\omega(\cdot; \cdot)\) is continuous on \([t_0, t_0 + 2\pi_p] \times \Gamma_0\), by passing to the upper limit on both sides of (5.4), it follows that

\[
\limsup_{n \to \infty} \text{Rot}^{\omega-\delta}(t_n; v_n) \leq \text{Rot}^\omega(\tau; \bar{v}) - \epsilon.
\]

We write, \((r_n(t), \theta_n(t))\) and \((r(t), \theta(t))\) as the polar coordinates of \((x_n(t), y_n(t))\) and \((x(t), y(t))\), respectively, which are the solutions of \(p\)-Laplacian equations

\[
(\phi_p(x')')' + \left(\omega_+(t) - \frac{1}{n}\right)\phi_p(x^+) + \left(\omega_-(t) - \frac{1}{n}\right)\phi_p(x^-) = 0
\]

and

\[
(\phi_p(x')')' + \omega_+(t)\phi_p(x^+) + \omega_-(t)\phi_p(x^-) = 0,
\]

with \((x_n(t_0), y_n(t_0)) = (C_p(\alpha_n), S_p(\alpha_n))\) and \((x(t_0), y(t_0)) = (C_p(\alpha), S_p(\alpha))\).

From (5.6), we have

\[
\theta''_n(t) = \frac{p[S_p(\theta_n)]^q}{q} + \left(\omega_+(t) - \frac{1}{n}\right)|C_p^+(\theta_n)|^p + \left(\omega_-(t) - \frac{1}{n}\right)|C_p^-\theta_n|^p.
\]
Proof of Lemma 2.3. Fixed \( t \in [t_0, t_0 + 2\pi p] \), where \( M_2 \) is the uniform upper bound of \( |C_p(\theta_n)|^p \). Following from a result on differential inequalities \[13\], we have

\[
\text{Rot}^{\omega - \hat{\theta}}(t_n; v_n) = \frac{\theta_n(t_n) - \theta_n(t_0)}{2\pi p} = \frac{\theta_n(t_n) - \alpha_n}{2\pi p} \geq \frac{\varphi_n(t_n) - \alpha_n}{2\pi p},
\]

where \( \varphi_n(t_n) \) is the solution of

\[
\begin{cases}
\theta'(t) = \frac{p|S_p(\theta)|^q}{q} + \omega(t)|C_p^+(\theta)|^p + \omega(t)|C_p^-(\theta)|^p - \frac{M_2}{n}
\\
\theta(t_0) = \alpha_n.
\end{cases}
\]

Using the theorem of continuous dependence of the solutions to above equation, we find that, as \( n \to \infty \), \( \varphi_n(t_n) \to \overline{\theta}(t) \) uniformly on \([t_0, t_0 + 2\pi p]\), where \( \overline{\theta}(t) \) is the solution of

\[
\begin{cases}
\theta'(t) = \frac{p|S_p(\theta)|^q}{q} + \omega_1(t)|C_p^+(\theta)|^p + \omega_1(t)|C_p^-(\theta)|^p,
\\
\theta(t_0) = \alpha.
\end{cases}
\]

Particularly, we have \( \varphi_n(t_n) \to \overline{\theta}(\tau) \ (n \to \infty) \) by using the uniform convergence property. Thus, from (5.7)-(5.8) and the definition of \( \text{Rot}^{\omega}(\tau; \tilde{v}) \), we have

\[
\lim \inf_{n \to \infty} \text{Rot}^{\omega - \hat{\theta}}(t_n; v_n) \geq \lim_{n \to \infty} \frac{\varphi_n(t_n) - \alpha_n}{2\pi p} = \frac{\overline{\theta}(\tau) - \alpha}{2\pi p} = \text{Rot}^{\omega}(\tau; \tilde{v}).
\]

Finally, it follows from (5.5) that

\[
\text{Rot}^{\omega}(\tau; \tilde{v}) \leq \text{Rot}^{\omega}(\tau; \tilde{v}) - \varepsilon
\]

which is a contradiction. Hence the proof is completed.

\[\square\]

**Proof of Lemma 2.3.** Fixed \( \varepsilon > 0 \), by Lemma 5.2, there exists \( \delta > 0 \) such that

\[
\text{Rot}^{\omega}(t; v) - \text{Rot}^{\omega - \delta}(t; v) \leq \frac{\varepsilon}{2}, \quad \forall t \in [t_0, t_0 + 2\pi p], \quad \forall v \in \Gamma_0.
\]

(5.9)

By \( (f_3) \) and the \( L^1 \)-Carathéodory condition, for such a \( \delta \), we can find \( l = l_\delta \in L^1([t_0, t_0 + 2\pi p], \mathbb{R}^+) \) such that

\[
f(t, x)x \geq (\omega_1(t) - \delta)|x^+|^p + (\omega_1(t) - \delta)|x^-|^p - l(t), \quad \forall x \in \mathbb{R},
\]

(5.10)

for a.e. \( t \in [t_0, t_0 + 2\pi p] \).

Now assume, by contradiction, that the statement of (2.5) is not true. This implies that, for each \( n \in \mathbb{N} \), there exists a solution \( z_n(t) \) of (2.2) defined on \([t_0, t_0 + 2\pi p]\) with \( |z_n(t)|_p \geq n \) for all \( t \in [t_0, t_0 + 2\pi p] \) such that, for some \( t_n \in [t_0, t_0 + 2\pi p] \),

\[
\text{Rot}^f(t_n; z_n) < \text{Rot}^{\omega}(t_n; v_n) - \varepsilon,
\]

(5.11)

where \( v_n = (1, \alpha_n) \) and \( \alpha_n \) is the polar angle of \( z_n(t_0) \) in the \( p \)-polar coordinates. Without loss of generality we can assume, if \( n \to \infty \), that \( t_n \to \tau \in [t_0, t_0 + 2\pi p] \), and \( v_n \to \tilde{v} = (1, \alpha) \in \Gamma_0 \) with \( \alpha_n \to \alpha \). Note that \( \text{Rot}^{\omega}(\cdot; \cdot) \) is continuous on \([t_0, t_0 + 2\pi p] \times \Gamma_0 \). Passing to the upper limit on both sides of (5.11), we find

\[
\limsup_{n \to \infty} \text{Rot}^f(t_n; z_n) \leq \text{Rot}^{\omega}(\tau; \tilde{v}) - \varepsilon.
\]

(5.12)
Note that \((r_n(t), \theta_n(t))\) is the \(p\)-polar coordinates of \(z_n(t) = (x_n(t), y_n(t))\) and satisfies \(r_n(t) \geq n\). Then, from (2.2) and (5.10),
\[
\theta_n'(t) = \frac{q f(t, x_n) x_n + p y_n}{q r^2_n} = p \left( \frac{f(t, x_n) x_n + |S_p(\theta_n)|^q}{q r^2_n} \right)
\geq p \left( \frac{(\omega_+ (t) - \delta)|x_n^+|^p + (\omega_- (t) - \delta)|x_n^-|^p - l(t)}{q r^2_n} + \frac{|S_p(\theta_n)|^q}{q} \right)
\geq (\omega_+ (t) - \delta)|C^+_p(\theta_n)|^p + (\omega_- (t) - \delta)|C^-_p(\theta_n)|^p + \frac{p |S_p(\theta_n)|^q}{q} - \frac{l(t)}{n^2}
\]
holds for a.e. \(t \in [t_0, t_0 + 2 \pi p]\). Following a result on differential inequality \([13]\), we have
\[
\text{Rot}^f_\tau (t_n; z_n) = \frac{\theta_n(t_n) - \theta_n(t_0)}{2 \pi p} = \frac{\theta_n(t_n) - \alpha_n}{2 \pi p} \geq \frac{\vartheta_n(t_n) - \alpha_n}{2 \pi p},
\]
where \(\vartheta_n(t_n)\) is the solution of
\[
\begin{cases}
\theta'(t) = (\omega_+ (t) - \delta)|C^+_p(\theta)|^p + (\omega_- (t) - \delta)|C^-_p(\theta)|^p + \frac{p |S_p(\theta)|^q}{q} - \frac{l(t)}{n^2}, \\
\theta(t_0) = \alpha.
\end{cases}
\]
By the continuous dependence of the solutions, we can find that, as \(n \to \infty\), \(\vartheta_n(t) \to \overline{\theta}(t)\), uniformly on \(t \in [t_0, t_0 + 2 \pi p]\), where \(\overline{\theta}(t)\) is the solution of
\[
\begin{cases}
\theta'(t) = (\omega_+ (t) - \delta)|C^+_p(\theta)|^p + (\omega_- (t) - \delta)|C^-_p(\theta)|^p + \frac{p |S_p(\theta)|^q}{q} - \frac{l(t)}{n^2}, \\
\theta(t_0) = \alpha.
\end{cases}
\]
In particular, it follows from the uniform convergence that \(\vartheta_n(t_n) \to \overline{\theta}(\tau) \ (n \to \infty)\). Thus, from (5.13) and the definition of \(\text{Rot}^{\omega - \delta}_\tau (\tau; \bar{v})\), we have
\[
\liminf_{n \to \infty} \text{Rot}^f_\tau (t_n; z_n) \geq \lim_{n \to \infty} \frac{\theta_n(t_n) - \alpha_n}{2 \pi p} = \frac{\overline{\theta}(\tau) - \alpha}{2 \pi p} = \text{Rot}^{\omega - \delta}_\tau (\tau; \bar{v}),
\]
which recalls (5.12) to get
\[
\text{Rot}^{\omega}_\tau (\tau; \bar{v}) - \text{Rot}^{\omega - \delta}_\tau (\tau; \bar{v}) \geq \varepsilon.
\]
The above inequality is in contradiction with (5.9), and hence (2.5) holds.

The proof of the other statement is quite similar to the one given earlier for the first claim and hence is omitted.

\textbf{Remark 4.} The proofs of Lemmas 5.2 and 2.3 are written in the spirit of Lemmas 3.3 and 3.4 in \([4]\), respectively. However, a more delicate analysis is needed in our case.

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\textbf{REFERENCES}

[1] L. Boccardo, P. Drábek, D. Giachetti and M. Kuček, Generalization of Fredholm alternative for nonlinear differential operators, Nonlinear Anal., 10 (1986), 1083–1103.

[2] Z. Cheng and J. Ren, Existence of multiplicity harmonic and subharmonic solutions for second-order quasilinear equation via Poincaré-Birkhoff twist theorem, Math. Methods Appl. Sci., 40 (2017), 6801–6822.
M. Cuesta and J. Gossez, A variational approach to nonresonance with respect to the Fučík spectrum, *Nonlinear Anal.*, **19** (1992), 457–500.

F. Dalbono and F. Zanolin, Multiplicity results for asymptotically linear equations, using the rotation number approach, *Meditarr. J. Math.*, **4** (2007), 127–149.

W. Ding, A generalization of the Poincaré-Birkhoff theorem, *Proc. Amer. Math. Soc.*, **88** (1983), 341–346.

T. Ding and F. Zanolin, Periodic solutions of Duffing’s equations with superquadratic potential, *J. Differ. Equ.*, **97** (1992), 328–378.

A. Fonda and A. Sfecci, Periodic solutions of weakly coupled superlinear systems, *J. Differ. Equ.*, **260** (2016), 2150–2162.

A. Fonda and A. J. Ureña, A higher dimensional Poincaré-Birkhoff theorem for Hamiltonian flows, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **34** (2017), 679–698.

J. Franks, Generalizations of the Poincaré-Birkhoff theorem, *Ann. Math.*, **128** (1988), 139–151.

M. García-Huidobro, R. Manásevich and F. Zanolin, A Fredholm-like result for strongly nonlinear second order ODE’s, *J. Differ. Equ.*, **114** (1994), 132–167.

H. Jacobowitz, Periodic solutions of $x'' + f(t, x) = 0$ via the Poincaré-Birkhoff theorem, *J. Differ. Equ.*, **20** (1976), 37–52.

V. Lakshmikantham and S. Leela, *Differential and integral inequalities; theory and applications*, Academic Press, New York and London, 1969.

R. Manásevich and J. Mawhin, Periodic solutions for nonlinear systems with $p$-Laplacian like operators, *J. Differ. Equ.*, **145** (1998), 367–393.

R. Manásevich and F. Zanolin, Time-mappings and multiplicity of solutions for the one-dimensional $p$-Laplacian, *Nonlinear Anal.*, **21** (1993), 269–291.

A. Margheri, C. Rebelo and P. J. Torres, On the use of Morse index and rotation numbers for multiplicity results of resonant BVP’s, *J. Math. Anal. Appl.*, **413** (2014), 660–667.

X. Ming, S. Wu and J. Liu, Periodic solutions for the 1-dimensional $p$-Laplacian equation, *J. Math. Anal. Appl.*, **325** (2007), 879–888.

M. del Pino, M. Elgueta and R. Manásevich, A homotopic deformation along $p$ of a Leray-Schauder degree result and existence for $(|u|^{p-2}u') + f(t,u) = 0, u(0) = u(T) = 0$, $p > 1$, *J. Differ. Equ.*, **80** (1989), 1–13.

M. del Pino, R. Manásevich and A. E. Murúa, Existence and multiplicity of solutions with prescribed period for a second order quasilinear ODE, *Nonlinear Anal.*, **18** (1992), 79–92.

M. del Pino, R. Manásevich, Infinitely many $2\pi$-periodic solutions for a problem arising in nonlinear elasticity, *J. Differ. Equ.*, **103** (1993), 260–277.

D. Qian, Infinity of subharmonics for asymmetric Duffing equations with the Lazer-Leach-Dancer condition, *J. Differ. Equ.*, **171** (2001), 233–250.

D. Qian and P. J. Torres, Periodic motions of linear impact oscillators via the successor map, *SIAM J. Math. Anal.*, **36** (2005), 1707–1725.

D. Qian, L. Chen and X. Sun, Periodic solutions of superlinear impulsive differential equations: a geometric approach, *J. Differ. Equ.*, **258** (2015), 3088–3106.

D. Qian, P. J. Torres and P. Wang, Periodic solutions of second Order equations via rotation Numbers, *J. Differ. Equ.*, **266** (2019), 4746–4768.

C. Rebelo, A note on the Poincaré-Birkhoff fixed point theorem and periodic solutions of planar systems, *Nonlinear Anal.*, **29** (1997), 291–311.

H. Royden, P. Fitzpatrick, *Real Analysis*, 4th edition, Printice-Hall Inc, Boston, 2010.

P. Yan and M. Zhang, Rotation number, periodic Fucik spectrum and multiple periodic solutions, *Commun. Contemp. Math.*, **12** (2010), 437–455.

M. Zhang, Nonuniform nonresonance at the first eigenvalue of the $p$-Laplacian, *Nonlinear Anal.*, **29** (1997), 41–51.

M. Zhang, Nonuniform nonresonance of semilinear differential equations, *J. Differ. Equ.*, **166** (2000), 33–50.

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