Light-front quantization is instant-time quantization

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Commutation or anticommutation relations quantized at equal instant time and commutation or anticommutation relations quantized at equal light-front time cannot be transformed into each other. While they would thus appear to describe different theories, we show that this is not in fact the case. In instant-time quantization unequal instant-time commutation or anticommutation relations are evaluated at equal light-front time they are identical to the equal light-front time commutation or anticommutation relations. Light-front quantization and instant-time quantization are thus the same and thus describe the same physics. However for fermions there is a caveat, as the light-front anticommutation relations involve projection operators acting on the fermion fields. Since projection operators are not invertible, while one can derive fermion light-front anticommutators starting from instant-time ones, one cannot derive instant-time ones starting from light-front ones.

I. INTRODUCTION

In quantum field theory various choices of quantization are considered. The most common choice is to take commutation relations of pairs of fields at equal instant time \( x^0 \) to be specific singular c-number functions. Thus for a free massless scalar field with action

\[
I_S = \int dx^0 dx^1 dx^2 dx^3 \frac{1}{2} \partial_\mu \phi \partial^\mu \phi
\]

\[
= \int dx^0 dx^1 dx^2 dx^3 \frac{1}{2} [\delta_0 \phi(\partial_0 \phi)^2 - (\partial_1 \phi)^2 - (\partial_2 \phi)^2 - (\partial_3 \phi)^2]
\]

for instance, one identifies a canonical conjugate (see e.g. [1] for a review) one introduces coordinates \( x^\pm = x^0 \pm x^3 \), a line element \( s^2 = x^+ x^- - (x^1)^2 - (x^2)^2 \) and a metric

\[
\begin{pmatrix}
0 & 1/2 & 0 & 0 \\
1/2 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

With the action then taking the form

\[
I_S = \int d^4 x (-g)^{1/2} \left[ \partial_+ \phi \partial^+ \phi + \partial_- \phi \partial^- \phi + \partial_1 \phi \partial^1 \phi + \partial_2 \phi \partial^2 \phi \right]
\]

\[
= \frac{1}{2} \int d^4 x dx^1 dx^2 dx^3 \frac{1}{2} [2 \partial_+ \phi \partial_- \phi + 2 \partial_- \phi \partial_+ \phi - (\partial_1 \phi)^2 - (\partial_2 \phi)^2],
\]

one identifies a canonical conjugate \((-g)^{-1/2} I_S / \delta \partial_+ \phi = \partial^+ \phi = 2 \partial_- \phi \), and quantizes the theory according to the equal light-front time \( x^+ \) commutation relation (see e.g. [2, 3] and references therein)

\[
[\phi(x^+ + x^1, x^2, x^-), 2 \partial_- \phi(x^+ + y^1, y^2, y^-)] = i \delta(x^1 - y^1) \delta(x^2 - y^2) \delta(x^- - y^-). \tag{5}
\]

As written, (5) is already conceptually different from (2) since the light-front conjugate is \( 2 \partial_- \phi \) and not \( 2 \partial_+ \phi \), i.e., not the derivative with respect to the light-time, while the instant-time conjugate \( \partial_0 \phi \) is the derivative with respect to the instant time. Since \( \phi(x^+ + x^1, x^2, x^-) \) and \( \partial_- \phi(x^+ + y^1, y^2, y^-) \) are not at the same \( x^- \), (5) can be integrated to

\[
[\phi(x^+ + x^1, x^2, x^-), \phi(x^+ + y^1, y^2, y^-)] = -\frac{i}{4} \delta(x^- - y^-) \delta(x^1 - y^1) \delta(x^2 - y^2).
\]
where $\epsilon(x) = \theta(x) - \theta(-x)$. Since the analog instant-time commutation relation is given by
\begin{equation}
[\phi(x^0, x^1, x^2, x^3), \phi(y^0, y^1, y^2, y^3)] = 0,
\end{equation}
instant-time and light-front time quantization appear to be quite different. Nonetheless, as shown in [2, 3] instant-time and light-front time matrix elements of operators such as $\Omega(T[\phi(x)\phi(y)]|\Omega)$ (as time ordered with $x^+$ or $x^-$) are actually equal, and in this sense the two quantization schemes are equivalent.

In the present paper we establish an equivalence between the two quantization schemes at the operator level itself without needing to take matrix elements. To this end we note that in instant-time quantization one can use the equal-time commutation relation given in (2) and the wave equation $\partial_\mu \partial^\mu \phi = 0$ associated with $I_S$ to make an on-shell Fock space expansion of $\phi$ of the form
\begin{equation}
\phi(x, x^0) = \int \frac{d^3p}{(2\pi)^3(2\hat{p})^{1/2}} [a(\hat{p})e^{-ipx^0 + i\vec{p}\cdot\vec{x}} + a^\dagger(\hat{p})e^{ipx^0 + i\vec{p}\cdot\vec{x}}],
\end{equation}
where the normalization of the creation and annihilation operator algebra, viz.
\begin{equation}
[a(\hat{p}), a^\dagger(\hat{q})] = \delta^3(\hat{p} - \hat{q}),
\end{equation}
is fixed from the normalization of the canonical commutator given in (2). Given (8) and (9) one can evaluate the unequal-time commutation relation between two free scalar fields, to obtain
\begin{equation}
i\Delta(x - y) = \frac{d^3p}{(2\pi)^3(2\hat{p})^{1/2}} \left[ a(\hat{p})a^\dagger(\hat{q})e^{-ip(x^0 + i\vec{p}\cdot\vec{x} + y^0 + i\vec{q}\cdot\vec{y})} + [a(\hat{p}), a^\dagger(\hat{q})]e^{ip(x^0 + i\vec{p}\cdot\vec{x} + y^0 + i\vec{q}\cdot\vec{y})} \right]
\end{equation}
We note that this commutator is a c-number, and not a q-number, with (2) following from (10) since
\begin{equation}
\frac{\partial}{\partial y^0} i\Delta(x - y)|_{x^0 = y^0} = i\delta(x^1 - y^1)\delta(x^2 - y^2)\delta(x^3 - y^3).
\end{equation}
Since the unequal time $i\Delta(x - y)$ is defined at all $x^\mu$ and $y^\mu$, it is equally defined at equal light-front time $x^+ = x^0 + x^3 = y^0 + y^3 = y^+$. It is the purpose of this paper to show that at equal light-front time (10) precisely coincides with (6). Since the form for the unequal instant-time commutator follows solely from the imposition of the equal-time commutator given in (2) and the wave equation $\partial_\mu \partial^\mu \phi = 0$ obeyed by the scalar field, the identification of (10) with (6) would then entail that equal light-front time quantization is a consequence solely of equal instant-time quantization, with the light-front formulation not requiring any independent quantization of its own. Rather, it is just a consequence of instant-time quantization. In this paper we will also obtain similar results for fermions and gauge bosons. Thus in all these cases light-front quantization is instant-time quantization.

II. EQUIVALENCE FOR SCALAR FIELDS

To show the equivalence of instant-time quantization and light-front quantization in the scalar field case we first rewrite (10) in a manifestly covariant form
\begin{equation}
i\Delta(x - y) = -\frac{i}{2\pi}(x^0 - y^0)\delta[(x^0 - y^0)^2 - (x^1 - y^1)^2 - (x^2 - y^2)^2 - (x^3 - y^3)^2].
\end{equation}
We now substitute $x^0 = (x^+ + x^-)/2$, $x^3 = (x^+ - x^-)/2$, $y^0 = (y^+ + y^-)/2$, $y^3 = (y^+ - y^-)/2$ and rewrite (12) as
\begin{equation}
i\Delta(x - y) = -\frac{i}{2\pi}(\frac{1}{2}(x^+ + x^- - y^+ - y^-)\delta[(x^+ - y^+)(x^- - y^-) - (x^1 - y^1)^2 - (x^2 - y^2)^2].
\end{equation}
Since \( \epsilon(x/2) = \epsilon(x) \) for any \( x \), at \( x^+ = y^+ \) (13) takes the form
\[
 i\Delta(x-y) \big|_{x^+ = y^+} = -\frac{i}{2\pi} \epsilon(x^- - y^-) \delta[(x^1 - y^1)^2 + (x^- - y^-)^2]. \tag{14}
\]
Then since \( \delta(a^2 + b^2) = \pi \delta(a) \delta(b) / 2 \) for any \( a \) and \( b \), we can rewrite (14) as
\[
 i\Delta(x-y) \big|_{x^+ = y^+} = [\phi(x^+, x^1, x^2, x^-), \phi(x^+, y^1, y^2, y^-)] = -\frac{i}{4} \epsilon(x^- - y^-) \delta(x^1 - y^1) \delta(x^2 - y^2). \tag{15}
\]
We recognize (15) as (6), with the equal light-front commutation relation (6) thus being derived starting from the unequal instant-time commutation relation (10). Since the unequal instant-time commutation relation (10) itself follows from the equal instant-time commutation relation (2), we see that the equal light-front time commutation relation (6) follows directly from the equal instant-time commutation relation (2) and does not need to be independently postulated.

### III. EQUIVALENCE FOR FERMION FIELDS

In instant-time quantization the free fermionic Dirac action is of the form
\[
 I_D = \int d^4 x \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi. \tag{16}
\]
The canonical conjugate of \( \psi \) is \( i\psi^\dagger \), and the canonical anticommutation relations are of the form
\[
 \{ \psi_\alpha(x^0, x^1, x^2, x^3), \psi_\beta^\dagger(x^0, y^1, y^2, y^3) \} = \delta_{\alpha\beta} \delta(x^1 - y^1) \delta(x^2 - y^2) \delta(x^3 - y^3),
\]
\[
 \{ \bar{\psi}_\alpha(x^0, x^1, x^2, x^3), \psi_\beta(x^0, y^1, y^2, y^3) \} = 0. \tag{17}
\]
When the fermion field obeys the Dirac equation \( i\gamma^\mu \partial_\mu - m \psi = 0 \), the on-shell Fock space expansion of the fermion field is of the form (see e.g. [4])
\[
 \psi(\vec{x}, x^0) = \sum_{s = \pm} \int \frac{d^3 p}{(2\pi)^{3/2}} \left( \frac{m}{E_p} \right)^{1/2} [b(\vec{p}, s) u(\vec{p}, s) e^{-ip \cdot x} + d^\dagger(\vec{p}) v(\vec{p}, s) e^{ip \cdot x}], \tag{18}
\]
where \( E_p = +[(p_1)^2 + (p_2)^2 + (p_3)^2]^{1/2} \), where \( s \) denotes the spin projection, where the Dirac spinors \( u(\vec{p}, s) \) and \( v(\vec{p}, s) \) obey \( (\slashed{p} - m) u(\vec{p}, s) = 0 \), \( (\slashed{p} + m) v(\vec{p}, s) = 0 \), and where the non-trivial creation and annihilation operator anticommutation relations are of the form
\[
 \{ b(\vec{p}, s), b^\dagger(\vec{q}, s') \} = \delta_{ss'} \delta^3(\vec{p} - \vec{q}), \quad \{ d(\vec{p}, s), d^\dagger(\vec{q}, s') \} = \delta_{ss'} \delta^3(\vec{p} - \vec{q}). \tag{19}
\]
With these relations the unequal time anticommutator is given by (see e.g. [4])
\[
 \{ \psi_\alpha(x^0, x^1, x^2, x^3), \psi_\beta(y^0, y^1, y^2, y^3) \} = \left[ i\gamma^\mu \frac{\partial}{\partial x^\mu} + m \right]_{\alpha\beta} \delta(x-y), \tag{20}
\]
where \( \Delta(x-y) \) is given in (10).

For the light-front case we set
\[
 \partial_0 = \frac{\partial}{\partial x^+} + \frac{\partial}{\partial x^-} = \partial_+ + \partial_-, \quad \partial_3 = \frac{\partial}{\partial x^+} - \frac{\partial}{\partial x^-} = \partial_+ - \partial_-, \tag{21}
\]
and obtain
\[
 \gamma^0 \partial_0 + \gamma^3 \partial_3 = (\gamma^0 + \gamma^3) \partial_+ + (\gamma^0 - \gamma^3) \partial_- = \gamma^+ \partial_+ + \gamma^- \partial_-, \tag{22}
\]
with (22) serving to define \( \gamma^\pm = \gamma^0 \pm \gamma^3 \). In terms of \( \gamma^+ \) and \( \gamma^- \) the Dirac action takes the form
\[
 I_D = \frac{1}{2} \int dx^+ dx^1 dx^2 dx^- \psi^\dagger(i\gamma^0 \gamma^+ \partial_+ + \gamma^- \partial_- + \gamma^1 \partial_1 + \gamma^2 \partial_2) - \gamma^0 m \psi. \tag{23}
\]
With this action the light-front time canonical conjugate of $\psi$ is $i\psi^\dagger \gamma^0 \gamma^+$.  

In the construction of the light-front fermion sector we find a rather sharp distinction with the instant-time fermion sector. First, unlike $\gamma^0$ and $\gamma^3$, which obey $(\gamma^0)^2 = 1$, $(\gamma^3)^2 = -1$, $\gamma^+$ and $\gamma^-$ obey $(\gamma^+)^2 = 0$, $(\gamma^-)^2 = 0$, to thus both be non-invertible divisors of zero. Secondly, the quantities

$$ \Lambda^+ = \frac{i}{2} \gamma^0 \gamma^+ = \frac{i}{2} (1 + \gamma^0 \gamma^3), \quad \Lambda^- = \frac{i}{2} \gamma^0 \gamma^- = \frac{i}{2} (1 - \gamma^0 \gamma^3) $$

(24)

obey

$$ \Lambda^+ + \Lambda^- = I, \quad (\Lambda^+)^2 = \Lambda^+ = [\Lambda^+]^\dagger, \quad (\Lambda^-)^2 = \Lambda^- = [\Lambda^-]^\dagger, \quad \Lambda^+ \Lambda^- = 0. $$

(25)

We recognize (25) as a projector algebra, with $\Lambda^+$ and $\Lambda^-$ thus being non-invertible projection operators. Given the projector algebra we identify $\psi(+) = \Lambda^+ \psi$, $\psi(-) = \Lambda^- \psi$ (respectively known as good and bad fermions in the light-front literature), and thus identify the conjugate of $\psi$ as $2i \psi^\dagger(+)$, where $\psi^\dagger(+) = [\psi^\dagger](+) = \psi^\dagger \Lambda^+ = [\Lambda^+ \psi]^\dagger = [\psi(+)]^\dagger$. Since the conjugate is a good fermion, in the anticommutator of $\psi$ with its conjugate only the good component of $\psi$ will contribute since $\Lambda^+ \Lambda^- = 0$, with the equal light-front time canonical anticommutator being found to be of the form (see e.g. [2])

$$ \{[\psi(+)]_\alpha (x^+, x^1, x^2, x^-), [\psi^\dagger(+)]_\beta (x^+, y^1, y^2, y^-) \} = \Lambda^+_{\alpha \beta} \delta (x^- - y^-) \delta (x^1 - y^1) \delta (x^2 - y^2). $$

(26)

In this construction the bad fermion $\psi(-)$ has no canonical conjugate and is thus not a dynamical variable. To understand this in more detail we manipulate the Dirac equation ($i \gamma^+ \partial_+ + i \gamma^- \partial_- + i \gamma^\dagger \partial_1 + i \gamma^2 \partial_2 - m) \psi = 0$.

We first multiply on the left by $\gamma^0$ to obtain

$$ 2i \partial_+ \psi(+) + 2i \partial_- \psi(-) + i \gamma^0 (\gamma^1 \partial_1 + \gamma^2 \partial_2) \psi - m \gamma^0 \psi = 0. $$

(27)

Next we multiply (27) by $\Lambda^-$ and also multiply it by $\Lambda^+$ to obtain

$$ 2i \partial_+ \psi(+) = [-i \gamma^0 (\gamma^1 \partial_1 + \gamma^2 \partial_2) + m \gamma^0] \psi(+), $$

$$ 2i \partial_- \psi(+), \psi(+) = [-i \gamma^0 (\gamma^1 \partial_1 + \gamma^2 \partial_2) + m \gamma^0] \psi(-). $$

(28)

Since the $\partial_- \psi(-)$ equation contains no time derivatives, $\psi(-)$ is thus a constrained variable, consistent with it having no conjugate. Since it is a constrained variable it does not appear in any fundamental anticommutation relation, though one could use (26) and (28) to construct a $\{\psi(-), \psi^\dagger(-)\}$ anticommutator. Through the use of the inverse propagator ($\partial_-^{-1}(x^-) = \epsilon(x^-)/2$) we can rewrite the $\partial_- \psi(-)$ equation in (28) as

$$ \psi(-)(x^+, x^1, x^2, x^-) = \frac{1}{4i} \int dy^- \epsilon(x^+ - y^+)[-i \gamma^0 (\gamma^1 \partial_1 + \gamma^2 \partial_2) + m \gamma^0] \psi(+)(x^+, x^1, x^2, y^-), $$

(29)

and recognize $\psi(-)$ as obeying a constraint condition that is nonlocal. It is because $\psi(-)$ obeys such a nonlocal constraint (one that is interaction dependent when interactions are involved) that it is known as a bad fermion.

Since only the good fermion is dynamical, if we start from the unequal instant-time relation (20) and transform it to light-front variables, we should only try to recover the good fermion anticommutator. To this end we thus multiply both sides of (20) by $\Lambda^+$ on both the right and the left. Noting that

$$ \Lambda^+ \gamma^0 \Lambda^+ = 0, \quad \Lambda^+ \gamma^1 \gamma^0 \Lambda^+ = 0, \quad \Lambda^+ \gamma^2 \gamma^0 \Lambda^+ = 0, \quad \Lambda^+ \gamma^+ \gamma^0 \Lambda^+ = 0, \quad \Lambda^+ \gamma^- \gamma^0 \Lambda^+ = 2 \Lambda^+, $$

(30)

from the right-hand side of (20) we obtain

$$ \Lambda^+_{\alpha \gamma} [\gamma_{\gamma \delta} (i \gamma^\mu \partial_\mu + m) \gamma^\gamma] i \Delta(x - y) \Lambda^0_{\delta \beta} = 2i \Lambda^+_{\alpha \beta} \partial_- i \Delta(x - y). $$

(31)

We now substitute $x^0 = (x^+ + x^-)/2, x^3 = (x^+ - x^-)/2, y^0 = (y^+ + y^-)/2, y^3 = (y^+ - y^-)/2$, and using (13) rewrite the right-hand side of (31) as

$$ 2i \Lambda^+_{\alpha \beta} \frac{\partial}{\partial x^-} i \Delta(x - y) = 2i \Lambda^+_{\alpha \beta} \frac{\partial}{\partial x^-} \left[ -i \frac{1}{2 \pi} \epsilon [\frac{1}{2} (x^+ + x^- - y^+ - y^-) \delta((x^+ + y^+)(x^- - y^-) - (x^1 - y^1)^2 - (x^2 - y^2)^2)] \right] $$

$$ = \frac{1}{\pi} \Lambda^+_{\alpha \beta} \delta \left[ \frac{1}{2} (x^+ + x^- - y^+ - y^-) \delta((x^+ + y^+)(x^- - y^-) - (x^1 - y^1)^2 - (x^2 - y^2)^2)] \right] $$

$$ + 2i \Lambda^+_{\alpha \beta} \left[ -i \frac{1}{2 \pi} \epsilon [\frac{1}{2} (x^+ + x^- - y^+ - y^-) \delta((x^+ + y^+)(x^- - y^-) - (x^1 - y^1)^2 - (x^2 - y^2)^2)] \right]. $$

(32)
At $x^+ = y^+$ (32) takes the form
\begin{equation}
2i \Lambda^+_{\alpha \beta} \frac{\partial}{\partial x^\gamma} i \Delta(x-y) \bigg|_{x^+ = y^+} = \Lambda^+_{\alpha \beta} \delta(x^- - y^-) \delta(x^1 - y^1) \delta(x^2 - y^2).
\end{equation}
Equating with the good fermion projection of the left-hand side of (20) thus yields
\begin{equation}
\Lambda^+_{\alpha \beta} \{ \psi_\gamma(x^+, x^1, x^2, x^-), \psi_\delta(x^+, y^1, y^2, y^-) \} \Lambda^+_{\delta \beta} = \{ \psi_+(x^+, x^1, x^2, x^-), \psi_+(x^+, y^1, y^2, y^-) \} = \Lambda^+_{\alpha \beta} \delta(x^- - y^-) \delta(x^1 - y^1) \delta(x^2 - y^2).
\end{equation}
We recognize (34) as the light-front relation (26).

Thus in analog to the scalar field case, in the fermion field case we can construct the equal light-front time anticommutator from the equal instant-time anticommutator. Now it might be thought that we have had to provide additional information in the fermion field case that we did not need to have to provide in the scalar field case, namely that we restrict to good fermions alone. However this information is actually implicit in our starting assumptions, namely the assumption that the fermion field obeys the Dirac equation and the assumption of an equal instant-time anticommutation relation for it. Specifically, when written in light-front coordinates the same Dirac equation breaks up into good and bad fermion sectors, with the bad fermion not being an independent dynamical degree of freedom but one that obeys the constraint given in (29). Consequently, only the $\Lambda^+$ projection of the unequal instant-time relation (20) is of relevance in the fermionic light-front case. Because $\Lambda^+$ is a projection operator we can go from (20) to (34). However, projectors are not invertible. Thus we cannot recover (20) starting from (34). In contrast, there is no impediment to going either way in the scalar field case, and in this sense equal instant-time quantization for fermions is more basic than equal light-front quantization for fermions. However, since we can go from (20) to (34), it follows that the two quantization procedures still lead to the same physics, with matrix elements of products of fermion fields in the two cases nonetheless being equal, just as we showed in [2, 3].

IV. EQUIVALENCE FOR GAUGE FIELDS

For our purposes here it is convenient to take the instant-time gauge field action $I_G$ to be of the gauge fixing form
\begin{equation}
I_G = \int d^4x \left[ -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} - \frac{1}{2} (\partial_\mu A_\nu)^2 \right] = \int d^4x \left[ -\frac{1}{2} \partial_\mu A_\nu \partial^\nu A_\mu \right],
\end{equation}
where $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $A_\mu$ is an Abelian gauge field. The presence of the $-\chi^2/2$ term where $\chi = \partial_\mu A_\mu$ causes $I_G$ to be neither gauge invariant nor equal to the gauge invariant Maxwell action $I_M = -\frac{1}{4} \int d^4x F_{\mu \nu} F^{\mu \nu}$. Variation of the $I_G$ action with respect to $A_\mu$ yields an equation of motion of the form
\begin{equation}
\partial_\nu \partial^\nu A_\mu = 0.
\end{equation}
The utility of using (35) is that the various components of $A_\mu$ are decoupled from each other in the equation of motion. Consequently, we can treat each component of $A_\mu$ as an independent degree of freedom, and apply the scalar field analysis given above to each one of them. In this formulation (36) entails that $\partial_\nu \partial^\nu \chi = 0$. If one imposes the subsidiary conditions $\chi(x^0 = 0) = 0$, $\partial_0 \chi(x^0 = 0) = 0$ at the initial time $x^0 = 0$, then since $\partial_\nu \partial^\nu \chi = 0$ is a second-order derivative equation it follows that the non-gauge-invariant $\chi$ is zero at all times.

Given (35) one can define instant-time canonical conjugates of the form $\Pi^\mu = \delta I_G / \delta \partial_0 A_\mu = -\partial^\mu A_\mu$. This then leads to equal instantaneous commutation relations of the form (see e.g. [5], and more recently [2])
\begin{align}
[A_\nu, \Pi^\mu] &= [A_\nu(x^0, x^1, x^2, x^3), -\partial^\mu A^\mu(x^0, y^1, y^2, y^3)] = -i \delta^\nu_\mu \delta(x^1 - y^1) \delta(x^2 - y^2) \delta(x^3 - y^3), \\
[A_\nu(x^0, x^1, x^2, x^3), \partial_0 A_\mu(x^0, y^1, y^2, y^3)] &= ig_{\mu \nu} \delta(x^1 - y^1) \delta(x^2 - y^2) \delta(x^3 - y^3),
\end{align}
and in analog to the scalar field case, to unequal instantaneous commutation relations of the form (see e.g. [5])
\begin{align}
[A_\nu(x^0, x^1, x^2, x^3), A_\mu(y^0, y^1, y^2, y^3)] &= ig_{\mu \nu} \Delta(x - y) \\
&= -\frac{i}{2\pi} g_{\mu \nu} \epsilon(x^0 - y^0) \delta[(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2],
\end{align}
where $g_{\mu \nu}$ is the instant-time metric and $\Delta(x - y)$ is the scalar field $\Delta(x - y)$ as given in (12).
Given (35) one can also define equal light-front time canonical conjugates of the form $\Pi^\mu = \delta I_G / \delta \partial_+ A^\mu = -\partial^+ A^\mu = -2\partial_- A^\mu$. This leads to equal light-front time commutation relations of a form analogous to (6), viz. \[ [A_\nu, \Pi^\mu] = [A_\nu(x^+, x^1, x^2, x^-), -2\partial_- A^\mu(x^+, y^1, y^2, y^-)] = -i\delta_\mu^\nu \delta(x^1 - y^1)\delta(x^2 - y^2)\delta(x^- - y^-), \]

\[ [A_\nu(x^+, x^1, x^2, x^-), \partial_- A_\mu(x^+, y^1, y^2, y^-)] = \frac{i}{2} g_{\mu\nu} \delta(x^1 - y^1)\delta(x^2 - y^2)\delta(x^- - y^-), \]

\[ [A_\nu(x^+, x^1, x^2, x^-), A_\mu(x^+, y^1, y^2, y^-)] = -\frac{i}{4} g_{\mu\nu} \epsilon(x^- - y^-)\delta(x^1 - y^1)\delta(x^2 - y^2), \]

where the light-front metric $g_{\mu\nu}$ is given (3). Thus, in analog to the scalar field case, the last expression in (39) follows directly from (38), with the instant-time metric transforming into the light-front metric. Moreover, since the instant-time initial conditions have forced $\chi$ to be zero at all $x^0$, $\chi$ is thus zero at all $x^+$, with the subsidiary condition being maintained in the light-front case. Thus for gauge fields we again see that light-front quantization is instant-time quantization.

In the non-Abelian Yang-Mills case one has a non-Abelian group with structure coefficients $f_{abc}$. One defines a tensor $G^a_{\mu\nu} = \partial_\nu A^a_\mu - \partial_\mu A^a_\nu + gf^{abc} A^b_\mu A^c_\nu$ where $g$ is the coupling constant. In analog to (35) one defines an action (see e.g. [6])

\[ I_{Y.M} = \int d^4x \left[ -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} - \frac{i}{8} \partial_\mu A^a_\nu \partial_\nu A^a_\mu + \partial_\mu \bar{c}_a \partial^\mu c_a + g f^{abc} A^b_\mu \partial_\mu \bar{c}_b c_c \right], \]

where $\xi$ is a constant and the $c_a$ are Fadeev-Popov ghost fields that one has to introduce in the non-Abelian case, viz. spin zero Grassmann fields that are quantized with anticommutation relations. Since the $g$-dependent terms in $I_{Y.M}$ involve products of either three or four fields they can be treated as part of the interaction. On setting $\xi = 1$ the relevant part of $I_{Y.M}$ for quantization, viz. the free part, is thus

\[ I_{Y.M} = \int d^4x \left[ -\frac{1}{4} \partial_\nu A^a_\mu \partial^\nu A^a_\mu + \partial_\mu c_a \partial^\mu c_a \right], \]

and leads to equations of motion of the form

\[ \partial_\nu \partial^\nu A^a_\mu = 0, \quad \partial_\mu \partial^\mu c_a = 0. \]

With both (41) and (42) being diagonal in both spacetime and group indices, the discussion thus parallels the Abelian and scalar field cases, with $A^a_\mu$ acting the same way as the Abelian $A_\mu$ and $c_a$ acting the same way as $\phi$. And in addition, with the perturbative instant-time gauge boson propagator being of the form $-ig_{\mu\nu} \delta_{ab}/(p^2 + i\epsilon)$, the perturbative instant-time gauge boson propagator transforms into the perturbative light-front gauge boson propagator. And in addition, since the perturbative ghost propagator is of the form $i\delta_{ab}/(p^2 + i\epsilon)$ the perturbative instant-time ghost propagator transforms into the perturbative light-front ghost propagator. Thus as with the Abelian case, in the non-Abelian case light-front quantization again is instant-time quantization.

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