POSITIVE MULTIVARIATE CONTINUOUS-TIME
AUTOREGRESSIVE MOVING-AVERAGE PROCESSES

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Abstract. In this article we study multivariate continuous-time autoregressive moving-average (MCARMA) processes with values in convex cones. More specifically, we introduce matrix-valued MCARMA processes with Lévy noise and present necessary and sufficient conditions for processes from this class to be cone valued. We derive specific hands-on conditions in the following two cases: First, for classical MCARMA on $\mathbb{R}^d$ with values in the positive orthant $\mathbb{R}^d_+$. Second, for MCARMA processes on real square matrices taking values in the cone of symmetric and positive semi-definite matrices. Both cases are relevant for applications and we give several examples of positivity ensuring parameter specifications. In addition to the above, we discuss the capability of positive semi-definite MCARMA processes to model the spot covariance process in multivariate stochastic volatility models. We justify the relevance of MCARMA based stochastic volatility models by an exemplary analysis of the second order structure of positive semi-definite well-balanced Ornstein-Uhlenbeck based models.

1. Introduction

Multivariate continuous-time autoregressive moving-average processes are the continuous time versions of the classical discrete-time VARMA models and have been studied thoroughly over the last two decades, see [34, 43, 17, 23, 30]. Similarly to their univariate analogs, the CARMA processes, MCARMA processes can be interpreted as the solution of a higher-order stochastic differential equation

$$D^p X_t + \tilde{A}_1 D^{p-1} X_t + \ldots + \tilde{A}_p X_t = \tilde{C}_0 D^{q+1} L_t + \tilde{C}_2 D^q L_t + \ldots + \tilde{C}_q D L_t, \quad (1)$$

where $D = \frac{d}{dt}$, $(\tilde{A}_i)_{i=1,\ldots,p}$ and $(\tilde{C}_j)_{j=0,\ldots,q}$ for $q, p \in \mathbb{N}$ are two families of linear operators and $(L_t)_{t \in \mathbb{R}}$ denotes a multivariate Lévy process. Naturally, equation (1) asks for a rigorous definition as the paths of Lévy processes are in general not differentiable. Heuristically, however, we can interpret (1) as the continuous-time version of a (V)ARMA difference equation and we therefore expect that similar key features governed by the autoregressive and moving-average structure of the defining equation find their counterpart in the continuous-time setting. The most notable feature of the (M)CARMA class is its flexible short memory structure. In general, short memory refers to an exponentially fast decaying auto-covariance function. However, the MCARMA class exhibits much more nuanced auto-covariance behavior in the short-time lags and allows, e.g., for non-monotone or sub-exponentially decaying configurations (although no polynomial or linear decaying ones such as processes with long memory admit). The memory effect is observed in many time series in applications and explains the popularity of modeling with (M)CARMA processes in subjects ranging from finance over meteorology to natural science and engineering, see e.g. [28, 35, 45, 13].

Key words and phrases. Multivariate CARMA processes, positive MCARMA, positive semi-definite processes, multivariate stochastic volatility.

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In many applications, where (M)CARMA models are employed, a crucial model feature is positivity, e.g. in modeling wind speed [13, 10], the velocity field in turbulence [4] or volatility in finance [6, 45, 14, 9]. It is therefore of great importance to understand the capability of (M)CARMA processes to model phenomena with positive states. In the univariate case positive CARMA processes were studied in [47, 46, 16, 14, 11, 36]. In particular, the authors in [47] give a set of necessary and/or sufficient parameter conditions such that CARMA processes of general order driven by a Lévy subordinator assume only non-negative values.

With the present article we contribute to the growing literature on MCARMA processes by studying positivity of multivariate CARMA processes. More precisely, we introduce the class of matrix-valued MCARMA processes which extends the classical $\mathbb{R}^{d}$-valued MCARMA processes and present necessary and sufficient conditions for matrix-valued MCARMA processes to be cone valued. The particularly interesting cases of $\mathbb{R}^{+}_{d}$, the positive orthant in $\mathbb{R}^{d}$, and $\mathbb{S}^{+}_{d}$, the cone of symmetric positive semi-definite $d \times d$ matrices, are included in our analysis. In fact, due to their relevance in applications, these two main examples coin our terminology of positive MCARMA (instead of, e.g. cone-valued MCARMA).

The starting point of our analysis will be the formulation of linear continuous-time state space models on the space of real $n \times m$-matrices. It is well known that $\mathbb{R}^{d}$-valued MCARMA processes are characterized by certain configurations of continuous-time state space models, see e.g. [34, 17, 43]. Linear continuous-time state space models are essentially given by an Ornstein-Uhlenbeck (OU) type process on the Cartesian product of the particular state space, in our case the space of all $n \times m$-matrices, and a linear output operator mapping the values of the higher-dimensional Ornstein-Uhlenbeck type process into the state space again. We show that for certain specifications of such models the vectorization of stationary output processes are equivalent to MCARMA processes as they were introduced in [34]. In this sense the matrix-valued state space models give rise to the novel class of matrix-valued MCARMA processes. The details are given in Section 2 below. Since in some applications one is interested in non-stable systems, we explicitly include the class of non-stable state space models in our analysis. Non-stable models often correspond to so called non-causal MCARMA processes, i.e. MCARMA processes that are not adapted to the natural filtration. A careful distinction between the stable and non-stable case is justified, since the stability conditions may interact with the imposed positivity constraints.

Once we set up the class of matrix-valued MCARMA processes we study its positivity. In particular, we are looking for necessary and sufficient parameter conditions such that a matrix-valued MCARMA process driven by a multivariate cone-valued Lévy process takes values solely in this cone. As noted above, in the univariate case the positivity of CARMA processes is well-studied and the relevance for applications is widely recognized. In the multivariate setting, however, positivity of MCARMA processes has not yet been studied in a systematical way. Partial results exist in the recent work [36], where the authors derive conditions ensuring the positivity of (univariate) CARMA processes. The authors made the claim that some parts of their results could be extended to the multivariate CARMA case, however, only few information about this extension are provided. Our Theorem 3.14 below supports this claim to some extent.

The main motivation for studying positivity of matrix-valued MCARMA processes comes from multivariate stochastic volatility modeling, see e.g. [7, 32, 38, 9, 21]. A multivariate stochastic volatility model consists of a $d$-dimensional (logarithmic) price process and a spot covariance process that modulates the multivariate noise of the former and takes values in the cone of symmetric and positive semi-definite...
we study the cone in-
We refer to \[ \text{semi-definite matrices and we present a detailed analysis of it in Section} \]




Lastly, we want to mention that also in a discrete-time setting studying autore-
gressive matrix-valued models is an active area of research, see e.g. [49, 19]. Many articles on (M)CARMA models deal with their connection to the discrete-time setting, e.g. studying high frequency sampling for MCARMA in [23, 30] or parameter estimation of the driving noise from discrete observations in [43]. Our findings on the equivalence of the vectorized matrix-valued MCARMA and the classical MCARMA suggests that analogous connections to the discrete-time setting continue to hold in the matrix-valued case and the good accessibility of the classical MCARMA is maintained accordingly.

1.1. Layout of the article. This article is structured as follows: In Section 2 we define a class of continuous-time state space models on matrices and show the equivalence with the classical MCARMA class. In Section 3 we study the cone invariance of matrix- and vector-valued MCARMA processes with a particular focus on the positive orthant in \( \mathbb{R}_d \) and the cone of symmetric positive semi-definite matrices. Lastly, in Section 4 we propose an MCARMA based multivariate stochastic volatility model and demonstrate its capability to capture memory effects by an exemplary analysis of positive semi-definite well-balanced OU processes.

1.2. Notation. By \( \mathbb{N} \) we denote the set of all integers and we set \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). For a complex number \( z = a + ib \in \mathbb{C} \) we denote its real part \( a \) by \( \Re(z) \) and its imaginary part \( b \) by \( \Im(z) \). For \( d \in \mathbb{N} \), we denote by \( \mathbb{R}_d \) the \( d \)-dimensional Euclidean space equipped with the standard inner-product \( (\cdot, \cdot)_d \). The closed positive orthant in \( \mathbb{R}_d \) will be denoted by \( \mathbb{R}_d^+ \) and the standard basis of \( \mathbb{R}_d \) is denoted by \( \{e_1, e_2, \ldots, e_d\} \).

Matrices. Let \( n, m \in \mathbb{N} \) and let \( \mathbb{K} \) denote either a field or a ring. Then we denote by \( \mathbb{M}_{n,m}(\mathbb{K}) \) the set of all \( n \times m \) matrices over \( \mathbb{K} \). If \( n = m \) we write \( \mathbb{M}_n(\mathbb{K}) \) and if \( \mathbb{K} = \mathbb{R} \) we simplify to \( \mathbb{M}_{n,m} \). If \( m = 1 \) we have \( \mathbb{M}_{n,1} = \mathbb{R}_n \) and use the latter notation. The \( p \)-times Cartesian product of \( \mathbb{M}_{n,m} \) will be denoted by \( \mathbb{M}_{n,m}^p \), which is just equivalent to \( \mathbb{M}_{pn,m} \), but we use the former notation as it is more suggestive. In the case where \( \mathbb{K} = \mathbb{R}[\lambda] \) is the polynomial ring over \( \mathbb{R} \) the set \( \mathbb{M}_n(\mathbb{R}[\lambda]) \) denotes the space of all matrix polynomials with coefficients in \( \mathbb{M}_n \). We refer to [25] for a comprehensive analysis of matrix polynomials. We denote the transpose of a matrix \( A \in \mathbb{M}_{n,m} \) by \( A^\top \), which is an element in \( \mathbb{M}_{m,n} \), and write \( \mathbb{S}_d \) for the subspace in \( \mathbb{M}_n \) consisting of all symmetric \( n \times n \) matrices, i.e. all \( A \in \mathbb{M}_n \) such that \( A^\top = A \). The set of all symmetric positive semi-definite \( n \times n \)-matrices will be denoted by \( \mathbb{S}_n^+ \), i.e. \( \mathbb{S}_n^+ = \{A \in \mathbb{S}_n : (Ax, x)_n \geq 0, \forall x \in \mathbb{R}_n\} \).

If necessary, we can express real \( n \times m \)-matrices in a component-wise notation
by $A = (a_{i,j})_{1 \leq i \leq n, 1 \leq j \leq m}$ for $a_{i,j} \in \mathbb{R}$. For all $n, m \in \mathbb{N}$ we can identify $M_{n,m}$ with $\mathbb{R}^{nm}$ through the vectorization operator vec: $M_{n,m} \rightarrow \mathbb{R}^{nm}$ which transforms a matrix into a vector by stacking the columns below each other. Similarly, we denote by vec: $\mathbb{S}_n \rightarrow \mathbb{R}^{n(n+1)/2}$ the operator that stacks only the lower triangular part of a symmetric matrix below another. On $M_{n,m}$ we consider the inner-product $(\langle \cdot, \cdot \rangle_{nm})$ given by $(A, B)_{nm} = (\text{vec}(A), \text{vec}(B))_{nm}$ and denote the induced norm by $\|\cdot\|_{nm}$. Let $n_1, n_2, m_1, m_2 \in \mathbb{N}$, then for $A \in M_{n_1,m_1}$ and $B \in M_{n_2,m_2}$ we denote the Kronecker product of $A$ and $B$ by $A \otimes B \in M_{n_1n_2,m_1m_2}$. We denote the Hadamard product of two matrices $A, B \in M_{n,m}$ by $A \odot B$ and let $I_{n,m} = (1_{i,j})_{1 \leq i \leq n, 1 \leq j \leq m}$ stand for the matrix in $M_{n,m}$ which is equal to one in every component and $0_{n,m}$ denotes the $n \times m$-zero matrix. If $n = m$ we write $I_n := I_{n,n}$, $0_n := 0_{n,n}$ and denote the identity matrix by $I_n$.

**Linear operators on matrices.** If we denote the algebra of all linear operators from $M_{n_1,m_1}$ to $M_{n_2,m_2}$ by $\mathcal{L}(M_{n_1,m_1}, M_{n_2,m_2})$. If $n_1 = n_2 = n$ and $m_1 = m_2 = m$ we write $\mathcal{L}(M_{n,m})$ and if $m = 1$, it is well known that $\mathcal{L}(M_{n,1}) = \mathcal{L}(\mathbb{R}^n) \cong \mathbb{R}^n$ and we use the latter notation. If $n, m \in \mathbb{N}$ are greater than one, we will denote elements of $\mathcal{L}(M_{n,m})$ by bold face letters, e.g. $A \in \mathcal{L}(M_{n,m})$ versus $A \in M_{n,m}$ and we reserve the calligraphic letters, e.g. $\mathcal{A}$, for linear operators mapping from or to $(M_{n,m})^p$ for some integer $p > 1$. Since in the sequel we will always make sure that there is no confusion regarding the matrix space that we are operating in, we denote the identity operator in $\mathcal{L}(M_{n,m})$ simply by $I$ and will only index $I$, when we speak of the identity in $\mathcal{L}(M_{n,m})^p$, in which case we write $I_p$. For every $A \in \mathcal{L}(M_{n,m})$ we denote its spectrum by $\sigma(A)$, which, as we work in finite-dimensions, is just the set of eigenvalues of $A$. Moreover, we denote the spectral bound of $A$ by $\tau(A)$, i.e. $\tau(A) := \max \{\Re(\lambda): \lambda \in \sigma(A)\}$.

**Matrix-valued Lévy processes.** Let $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions and let $(L_t)_{t \geq 0}$ be an $\mathbb{R}^{nm}$-valued Lévy process defined on this probability basis, see [41] for a comprehensive analysis of multivariate Lévy processes. Since we can always identify $M_{n,m}$ with $\mathbb{R}^{nm}$ the class of multivariate Lévy processes easily extends to matrix-valued Lévy processes. We recall that the Lévy characteristic exponent at $z \in M_{n,m}$ is given by

$$\psi_L(z) = i\langle \gamma_L, z \rangle_{nm} - \frac{1}{2} \langle Q_1 z, z \rangle_{nm} + \int_{M_{n,m}} (\psi(\xi, z))_{nm} - 1 - i\langle \chi(\xi), z \rangle_{nm} \nu_L(d\xi),$$

(2)

where $\gamma_L \in M_{n,m}$ denotes the drift of $L$, $Q_1$ is the covariance operator of the continuous part of the Lévy process, $\chi(\xi) := \xi I_{1 \leq \|\xi\|_{nm} \leq 1}(\xi)$ and $\nu_L: \mathcal{B}(M_{n,m}) \rightarrow \mathbb{R}^+$ is the Lévy measure. We call a Lévy process $(L_t)_{t \geq 0}$ integrable, if $\mathbb{E}[\|L_t\|_{nm}] < \infty$ for all $t \geq 0$ and square-integrable whenever $\mathbb{E}[\|L_t\|_{nm}^2] < \infty$ for all $t \geq 0$. For a square-integrable Lévy process $L$ with characteristic exponent (2) the mean of $L_1$ is denoted by $\mu_L$ and we have $\mu_L = (\gamma_L + \int_{M_{n,m}} \xi\nu_L(d\xi))$. Moreover, we denote by $Q \in \mathcal{L}(M_{n,m})$ the covariance operator of $L_1$, which is given by $Q = Q_1 + \int_{M_{n,m}} \xi \otimes \nu_L(d\xi)$. For any Lévy process $(L_t^1)_{t \geq 0}$ defined on the positive real line $\mathbb{R}^+$, we can choose a second, independent and identically distributed, Lévy process $(L_t^2)_{t \geq 0}$ to define a two-sided Lévy process $(L_t)_{t \in \mathbb{R}}$ by

$$L_t := I_{\mathbb{R}^+} (t) L_t^1 - I_{\mathbb{R}^+} (t) L_t^2 I_{-\mathbb{R}^+},$$

where $L_{t-} = \lim_{s \uparrow t} L_s$ for all $t \geq 0$. Throughout this article we use the conventional and intuitive notation of stochastic integration with respect to matrix-valued integrators analogous to e.g. [7, 9].
2. Linear continuous-time state space models and matrix-valued MCARMA processes

Throughout this section we fix \( m, n \in \mathbb{N} \) and let \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) denote a filtered probability space satisfying the usual conditions. Moreover, we assume that \((\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})\) is rich enough to carry a \( \mathbb{M}_{n,m} \)-valued two-sided Lévy process \( L = (L_t)_{t \in \mathbb{R}} \). We begin this section by introducing a very general class of linear continuous-time state space models defined on real \( n \times m \)-matrices:

**Definition 2.1.** Let \( p \in \mathbb{N} \) and let the tuple \((A, B, C, L)\) consist of a state transition operator \( A \in \mathcal{L}((\mathbb{M}_{n,m})^p)\), an input operator \( B \in \mathcal{L}(\mathbb{M}_{n,m}, (\mathbb{M}_{n,m})^p)\), an output operator \( C \in \mathcal{L}((\mathbb{M}_{n,m})^p, \mathbb{M}_{n,m})\) and a \( \mathbb{M}_{n,m} \)-valued two-sided Lévy process \( L = (L_t)_{t \in \mathbb{R}} \). A continuous-time linear state space model on \( \mathbb{M}_{n,m} \), associated with the parameter set \((A, B, C, L)\), consists of a state-space equation given by

\[
dZ_t = AZ_t \, dt + B \, dL_t, \quad t \in \mathbb{R},
\]

and an observation equation given by

\[
X_t = CZ_t, \quad t \in \mathbb{R}.
\]

We call the \((\mathbb{M}_{n,m})^p\)-valued process \((Z_t)_{t \in \mathbb{R}}\) the state process and the \( \mathbb{M}_{n,m} \)-valued process \((X_t)_{t \in \mathbb{R}}\) the output process of the continuous-time linear state space model (associated with \((A, B, C, L)\)).

Continuous-time linear state space models have been rigorously studied in the deterministic and stochastic control literature over many decades, see e.g. [48, 2, 22]. We note that if \( m = 1 \) (that means for the state space \( \mathbb{R}_n = \mathbb{M}_{n,1} \)) our definition of a linear continuous-time state space model coincides with the one in [43, Definition 3.1].

We note that the state process \((Z_t)_{t \in \mathbb{R}}\) of a linear continuous-time state space model is simply a Lévy driven Ornstein-Uhlenbeck type process on the space \((\mathbb{M}_{n,m})^p\) and a solution to (3) is given by the following variation-of-constant formula:

\[
Z_t = e^{(t-s)A}Z_s + \int_s^t e^{(t-u)A}B \, dL_u, \quad s < t \in \mathbb{R}.
\]

In general, a solution \((Z_t)_{t \in \mathbb{R}}\) to (3) is not unique. If for some \( s \in \mathbb{R} \) we are given an \( \mathcal{F}_s \)-measurable random variable \( Z_s \), then \((Z_t)_{t \geq s}\), given by (5), is the unique solution to (3) on \([s, \infty)\) adapted to the filtration \( (\mathcal{F}_t)_{t \geq s}\). If the spectral bound of the transition operator \( A \) is strictly negative, i.e. \( \tau(A) < 0 \), then it follows from [42, 20] that there exists a unique stationary solution to (3) if and only if \( \mathbb{E}[\log(||L_1||_{nm})] < \infty \). In this case the unique stationary solution \((Z_t)_{t \in \mathbb{R}}\) is adapted and given by

\[
Z_t = \int_{-\infty}^t e^{(t-s)A}B \, dL_s, \quad t \in \mathbb{R}.
\]

**Remark 2.2** (Uniqueness, stationarity and adaptedness). Note that in case of \( \tau(A) \geq 0 \), there could still, under certain conditions, exist a (unique) stationary solution to (3) on \( \mathbb{R} \), see also Proposition 2.9 below. However, it might happen that the stationary solution is not adapted to the natural filtration \( \mathbb{F} \), since \( Z_t \) possibly depends on the generated sigma-algebra \( \sigma(L_s: s > t) \). A concrete example of a stationary output process with \( \tau(A) \geq 0 \) is given in Section 4.3. If from a modeling perspective adaptedness to the natural filtration is required, then we shall either assume \( \tau(A) < 0 \), for which the existence of a unique \( \mathbb{F} \)-adapted solution is known by the reasoning above. Or in case of \( \tau(A) \geq 0 \), there may exist many solutions, but only one for every fixed \( \mathcal{F}_s \)-measurable initial condition, which also happens to be \( \mathbb{F} \)-adapted, but possibly non-stationary.
In the sequel we often distinguish between the two cases in Remark 2.2 which we call the \textit{stable} (where \( \tau(A) < 0 \)) and \textit{non-stable} (where \( \tau(A) \geq 0 \)) case (following the usual nomenclature it is actually the \textit{exponentially stable} and \textit{non-exponentially stable} case). We find it appropriate to give a precise definition to prevent confusion:

\textbf{Definition 2.3.} We call a continuous-time state space model associated with the parameter set \((A, B, C, L)\) \textit{stable}, whenever \( \tau(A) < 0 \). If moreover, \( L \) has finite log-moments, i.e. \( \mathbb{E}[\log(||L||_{n,m})] < \infty \), then whenever we refer to the state process \((Z_t)_{t \in \mathbb{R}}\) we mean the unique stationary solution given by (6). In this case we call \((Z_t)_{t \in \mathbb{R}}\) the \textit{stable state process} and \((X_t)_{t \in \mathbb{R}}\) the \textit{stable output process}. In case of \( \tau(A) \geq 0 \), we call the state space model \textit{non-stable} and refer to \((Z_t)_{t \in \mathbb{R}}\) in (5) simply as a state process. If uniqueness is required, we may fix an initial value \( Z_s \) at \( s \in \mathbb{R} \) for some \( \mathcal{F}_s \)-measurable measurable random variable \( Z_s \).

Given a state process \((Z_t)_{t \in \mathbb{R}}\) we now shift our focus to the output process \((X_t)_{t \in \mathbb{R}}\) defined in (4). In the next proposition we summarize some well known and easy to check properties of \((X_t)_{t \in \mathbb{R}}\).

\textbf{Proposition 2.4.} Let \((A, B, C, L)\) be as in Definition 2.1 and let \( \psi_L \) be the characteristic exponent of \( L \) given by (2). Then the process \((X_t)_{t \in \mathbb{R}}\) in (4) satisfies

\[ X_t = C e^{(t-s)A} Z_s + \int_s^t C e^{(t-u)A} B dL_u, \quad s < t \in \mathbb{R}, \]  

and for every \( x \in M_{n,m} \) and \( s \leq t \) we have

\[ \mathbb{E} \left[ e^{i(M_{(X_t,x)_{nm}} | \mathcal{F}_s)} \right] = \exp \left( i \langle C e^{(t-s)A} Z_s, x \rangle_{nm} + \int_s^t \psi_L (B^* e^{(t-u)A} C^* x) \, du \right). \]  

If \( L \) is integrable, then the conditional mean of \( X_t \) is finite and given by

\[ \mathbb{E}[X_t | \mathcal{F}_s] = C e^{(t-s)A} Z_s + \int_0^{t-s} C e^{uA} B \mu_u \, du, \quad s < t \in \mathbb{R}. \]

If moreover \( L \) has finite log-moments and \((X_t)_{t \in \mathbb{R}}\) is a \textit{stable output process}, then it is stationary and given by

\[ X_t = \int_{-\infty}^t C e^{(t-s)A} B dL_s, \quad t \in \mathbb{R}, \]  

and if in addition \( L \) is integrable, then the mean of \((X_t)_{t \in \mathbb{R}}\) is given by

\[ \mathbb{E}[X_t] = -C A^{-1} B \mu_L, \quad t \in \mathbb{R}. \]

From (10) we see that the dynamics of a \textit{stable output process} \((X_t)_{t \in \mathbb{R}}\) are only governed by the parameters \( A, B \) and \( C \) and the Lévy process \( L \). More specifically, the dynamics depend solely on the action of the kernel function \( g: \mathbb{R}^+ \to \mathcal{L}(M_{n,m}) \) given by \( g(t) := C e^{tA} B \) applied to the increments of \((L_t)_{t \in \mathbb{R}}\). In contrast, we see that in the non-stable case, the dynamics of \((X_t)_{t \in \mathbb{R}}\) also depend on the initial value \( Z_s \) for \( s < t \) through the term \( C e^{(t-s)A} Z_s \). As we will see, this has some consequences for the techniques available to study positivity of output processes in Section 3.

\textbf{2.1. Controller canonical form and matrix-valued MCARMA processes.}

In this section we introduce a more particular form of linear continuous-time state space models on \( M_{n,m} \) which in the discrete-time control literature is often called the \textit{controller canonical form}. This controller canonical form will prove itself useful for two reasons: First, it will allow us to interpret the output processes of certain continuous-time state space models as a linear transformation of MCARMA processes as they were introduced in the seminal work [34]. Second, this form is particularly convenient to study positivity questions in the next section.
Let \( p \in \mathbb{N} \) and denote by \( 0 \) the null operator in \( \mathcal{L}(M_{n,m}) \), i.e. \( 0 \) maps every \( x \in M_{n,m} \) to the null matrix \( 0_{n,m} \). Moreover, let \( A_1, A_2, \ldots, A_p \in \mathcal{L}(M_{n,m}) \) and define the state transition operator \( A_p: (M_{n,m})^p \to (M_{n,m})^p \) as follows:

\[
A_p := \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
& & \ddots & \ddots & \vdots \\
& & & \ddots & 0 \\
0 & \cdots & \cdots & 0 & 1
\end{bmatrix} \quad (12)
\]

where for every \( x = (x_1, \ldots, x_p)^T \in (M_{n,m})^p \) we understand that

\[
A_p(x) = (x_2, \ldots, x_p, \sum_{i=1}^{p} A_{p-i+1}(x_i))^T \in (M_{n,m})^p.
\]

Moreover, let \( q \in \mathbb{N}_0 \) such that \( q < p \) and let \( C_0, C_1, \ldots, C_{p-1} \in \mathcal{L}(M_{n,m}) \) with \( C_i = 0 \) for every \( q + 1 \leq i \leq p - 1 \). We then define the output operator \( C_q: (M_{n,m})^p \to M_{n,m} \) by

\[
C_q := [C_0, C_1, \ldots, C_{p-1}],
\]

where \( C_q \) is to be understood as follows: For \( x = (x_1, \ldots, x_p)^T \in (M_{n,m})^p \) we have

\[
C_q(x) = \sum_{i=1}^{q+1} C_{i-1}(x_i).
\]

Finally, we define the input operator \( E_p \in \mathcal{L}(M_{n,m}, (M_{n,m})^p) \) by

\[
E_p := e_p \otimes I,
\]

which for every \( x \in M_{n,m} \) is defined as \( E_p(x) = (0_{n,m}, \ldots, 0_{n,m}, x)^T \in (M_{n,m})^p \).

With this specification of \( (A_p, E_p, C_q, L) \) the state process \( (Z_t)_{t \in \mathbb{R}} \) becomes

\[
Z_t = e^{(t-s)A_p} Z_s + \int_s^t e^{(t-u)A_p} E_p \, du, \quad s < t \in \mathbb{R},
\]

and the output process

\[
X_t = C_q e^{(t-s)A_p} Z_s + \int_s^t C_q e^{(t-u)A_p} E_p \, du, \quad s < t \in \mathbb{R},
\]

and the analogous formulas (6) and (10) hold in the stable case. Note that the transition matrix \( A_p \) can be viewed as the companion block operator matrix of the following operator polynomial \( P \) with coefficients in \( \mathcal{L}(M_{n,m}) \):

\[
P(\lambda) := I\lambda^p - A_1\lambda^{p-1} - A_2\lambda^{p-2} - \cdots - A_p, \quad \lambda \in \mathbb{C}.
\]

In the same spirit we introduce the operator polynomial \( Q \) which is associated with the output operator \( C_q \) and given by

\[
Q(\lambda) := C_0 + C_1\lambda + C_2\lambda^2 + \cdots + C_q\lambda^q, \quad \lambda \in \mathbb{C}.
\]

Now, recall \( \text{vec}: M_{n,m} \to \mathbb{R}^{nm} \) being the linear isometric isomorphism that maps every matrix \( A \in M_{n,m} \) to the vector of its columns by stacking the columns below each other. If \( A \in \mathcal{L}(M_{n,m}) \) we denote by \( A^\text{vec} \) the matrix representation of \( A \) given by \( A^\text{vec} := \text{vec} \circ A \circ \text{vec}^{-1} \). Note that \( A^\text{vec} \in \mathcal{L}(\mathbb{R}^{nm}) \simeq M_{nm} \) and by identification we consider \( A^\text{vec} \) as a \( nm \times nm \)-matrix. Moreover, we denote by \( K^{(n,m)} \in M_{nm} \) the commutation matrix, which is the unique matrix in \( M_{nm} \) such that for every \( A \in M_{n,m} \) we have \( K^{(n,m)} \text{vec}(A) = \text{vec}(A^T) \). We denote the inverse of \( K^{(n,m)} \) by \( K^{-(n,m)} \), which happens to be the transpose of \( K^{(n,m)} \) as well.
Hence for every process \((X_t)_{t \in \mathbb{R}}\) we obtain an \(\mathbb{R}_{nm}\)-valued process \((\text{vec}(X_t))_{t \in \mathbb{R}}\). By linearity, we see that the process \((\text{vec}(X_t))_{t \in \mathbb{R}}\) can again be considered as an output process of a certain continuous-time linear state space model on \(\mathbb{R}_{nm}\). In the following proposition we show more, namely that the controller canonical form in (12)-(14) transforms, under the vec-transformation, into a controller canonical-like form of an \(\mathbb{R}_{nm}\)-valued state space model.

**Proposition 2.5.** Let \(p \in \mathbb{N}\) and \(q \in \mathbb{N}_0\) such that \(q < p\) and \((\mathcal{A}_p, E_p, C_p, L)\) be as in (12)-(14) and \(L\) a two-sided Lévy process on \(\mathcal{M}_{n,m}\). Moreover, let the state process \((Z_t)_{t \in \mathbb{R}}\) be given by (15) and the output process \((X_t)_{t \in \mathbb{R}}\) be as in (16). We set \(L^{\text{vec}} := \text{vec}(L)\) and define the state transition matrix \(\mathcal{A}_p^{\text{vec}} \in \mathcal{M}_{pn,mn}\) by

\[
\mathcal{A}_p^{\text{vec}} := \begin{pmatrix}
0_{nm} & K^{-(n,m)} & 0_{nm} & \cdots & 0_{nm} \\
0_{nm} & 0_{nm} & K^{-(n,m)} & \cdots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0_{nm} & \cdots & \cdots & 0_{nm} & K^{-(n,m)} \\
\mathcal{A}_p & \mathcal{A}_p^{\text{vec}} & \ldots & \ldots & \mathcal{A}_p^{\text{vec}}
\end{pmatrix},
\]

where \(\mathcal{A}_i^{\text{vec}} := \mathcal{A}_i^{\text{vec}} \circ K^{-(n,m)} \in \mathcal{M}_{nm}\) for every \(i = 1, \ldots, p\) and \(\mathcal{A}_i^{\text{vec}}\) denotes the matrix representation of \(\mathcal{A}_i\). Moreover, we define the output matrix \(C_q^{\text{vec}} \in \mathcal{M}_{mn,pmn}\) by

\[
C_q^{\text{vec}} := \begin{pmatrix}
C_0^{\text{vec}} & C_1^{\text{vec}} & \cdots & C_{p-1}^{\text{vec}}
\end{pmatrix},
\]

where \(C_j^{\text{vec}} = C_j^{\text{vec}} \circ K^{-(n,m)} \in \mathcal{M}_{mn}\) for every \(j = 0, \ldots, p-1\) and \(C_j^{\text{vec}}\) denotes the matrix representation of \(C_j\). Lastly, we define the input matrix \(E_p^{\text{vec}} \in \mathcal{M}_{pmn,nnm}\) by

\[
E_p^{\text{vec}} := e_p \otimes K^{-(n,m)}.
\]

Then \((\text{vec}(X_t))_{t \in \mathbb{R}}\) is the output process of a \(\mathbb{R}_{nm}\)-valued continuous-time state space model associated with \((\mathcal{A}_p^{\text{vec}}, C_q^{\text{vec}}, E_p^{\text{vec}}, L^{\text{vec}})\) and such that

\[
d\text{vec}(Z_t^\top) = \mathcal{A}_p^{\text{vec}} \text{vec}(Z_t^\top) dt + E_p^{\text{vec}} \text{d}L_t^{\text{vec}}, \quad t \in \mathbb{R},
\]

\[
\text{vec}(X_t) = \mathcal{C}_q^{\text{vec}} \text{vec}(Z_t^\top), \quad t \in \mathbb{R}.
\]

**Proof.** For every \(t \in \mathbb{R}\) write \(Z_t = (Z_t^{(1)}, Z_t^{(2)}, \ldots, Z_t^{(p)})^\top \in (\mathcal{M}_{n,m})^p\) where for every \(i = 1, \ldots, p\) we denote by \(Z_t^{(i)}\) the \(i\)-th \(n \times m\)-block matrix component of \(Z_t\). Given the state process \((Z_t)_{t \in \mathbb{R}}\) and output process \((X_t)_{t \in \mathbb{R}}\) we show that \((\text{vec}(X_t))_{t \in \mathbb{R}}\) solves (23) with \((\text{vec}(Z_t^\top))_{t \in \mathbb{R}}\) being a solution to (22). By definition we have \(X_t = C_q(Z_t) = \sum_{i=1}^{q+1} C_{i-1}(Z_t^{(i)})\) and hence by linearity of the vec-operation we see that

\[
\text{vec}(X_t) = \text{vec}(C_q(Z_t)) = \sum_{i=1}^{q+1} \text{vec}(C_{i-1}(Z_t^{(i)})) = \sum_{i=1}^{q+1} C_{i-1}^{\text{vec}} \text{vec}(Z_t^{(i)}), \quad t \in \mathbb{R}.
\]

As before let \(K^{(pn,m)}\) be the commutation matrix such that \(K^{(pn,m)} \text{vec}(F) = \text{vec}(F^\top)\) for every \(F \in (\mathcal{M}_{n,m})^p\). Note further that \(\text{vec}(Z_t^{(i)}) = K^{-(n,m)} \text{vec}(Z_t^{(i)}\top)\) and for every \(i = 1, 2, \ldots, p\) we can write

\[
\text{vec}(Z_t^{(i)}\top) = (e_i^\top \otimes I_{nm}) \text{vec}(Z_t^\top).
\]

Hence for every \(i = 1, 2, \ldots, p\) and \(t \in \mathbb{R}\) we have

\[
\text{vec}(Z_t^{(i)}) = K^{-(n,m)}(e_i^\top \otimes I_{nm})K^{(pn,m)} \text{vec}(Z_t).
\]
We thus continue with the term vec($Z_t$) appearing on the right-hand side of (25). Note that by linearity, inserting (15) into vec($Z_t$) yields

$$\text{vec}(Z_t) = \text{vec}(e^{(t-s)A_p}Z_s) + \int_s^t \text{vec}(e^{(t-u)A_p} E_p dL_u), \quad s < t \in \mathbb{R}. \quad (26)$$

We know that $t \mapsto e^{A_p t}v_0$ is the unique solution to the linear equation $\frac{d}{dt}v(t) = A_p v(t)$ with $v(0) = v_0$. Moreover, we see that $\frac{d}{dt} \text{vec}(v(t)) = A_p^{\text{vec}} \text{vec}(v(t))$ with $\text{vec}(v(0)) = \text{vec}(v_0)$ is uniquely solved by $\text{vec}(v(t)) = e^{A_p^{\text{vec}} t} \text{vec}(v_0)$. Hence by linearity, we conclude that $(e^{tA_p})^{\text{vec}} = e^{tA_p^{\text{vec}}}$ must hold for all $t \geq 0$. Therefore, the right-hand side in (26) coincides with

$$e^{(t-s)A_p^{\text{vec}}} \text{vec}(Z_s) + \int_s^t e^{(t-u)A_p^{\text{vec}}} \text{vec}(E_p dL_u), \quad s < t \in \mathbb{R}. \quad (27)$$

Hence by using the $K^{(p,n,m)}$-commutation matrix again, we see that for all real $s < t$ we have

$$\text{vec}(Z_t) = e^{(t-s)A_p^{\text{vec}}} K^{-(p,n,m)} \text{vec}(Z_s^\top) + \int_s^t e^{(t-u)A_p^{\text{vec}}} K^{-(p,n,m)} \text{vec}((E_p dL_u)^\top),$$

and hence by (25)

$$\text{vec}(Z_t^{(i)}) = K^{-(n,m)}(e_i^\top \otimes I_{nm})(K^{(p,n,m)} e^{(t-s)A_p^{\text{vec}}} K^{-(p,n,m)} \text{vec}(Z_s^\top))$$

$$+ \int_s^t K^{(p,n,m)} e^{(t-u)A_p^{\text{vec}}} K^{-(p,n,m)} \text{vec}((E_p dL_u)^\top), \quad s < t \in \mathbb{R}. \quad (28)$$

Now note that

$$K^{(p,n,m)} e^{tA_p^{\text{vec}}} K^{-(p,n,m)} = e^{tK^{(p,n,m)} A_p^{\text{vec}} K^{-(p,n,m)}}, \quad \text{for all } t \geq 0.$$

Next, we show that $K^{(p,n,m)} A_p^{\text{vec}} K^{-(p,n,m)} = \hat{A}_p^{\text{vec}}$. Let $F_p = (F_1, F_2, \ldots, F_p)^T \in (\mathbb{R}_{n,m})^p$ then

$$K^{(p,n,m)} A_p^{\text{vec}} K^{-(p,n,m)} \text{vec}(F_p^\top) = K^{(p,n,m)} \text{vec} \left( (F_2, F_2, \ldots, F_p, \sum_{i=1}^p A_{p+1-i}(F_i))^\top \right)$$

$$= \text{vec} \left( (F_2^\top, F_2^\top, \ldots, F_p^\top, \sum_{i=1}^p A_{p+1-i}(F_i)^\top) \right)$$

$$= (\text{vec}(F_2^\top), \ldots, \text{vec}(F_p^\top), \sum_{i=1}^p \text{vec}(A_{p+1-i}(F_i)))^\top$$

$$= \hat{A}^{\text{vec}}(\text{vec}(F_p^\top)),$$

where in the last equation we used that $K^{-(n,m)} \text{vec}(F_i^\top) = \text{vec}(F_i)$ and

$$\sum_{i=1}^p \text{vec}(A_{p+1-i}(F_i)^\top) = \sum_{i=1}^p A_{p+1-i}^{\text{vec}} K^{-(n,m)} \text{vec}(F_i) = \sum_{i=1}^p \hat{A}_{p+1-i}^{\text{vec}} \text{vec}(F_i).$$

This and vec($(E_p dL_u)^\top) = \hat{E}_p^{\text{vec}} d \text{vec}(L_u^\top)$ inserted into (28) imply

$$\text{vec}(Z_t^{(i)}) = K^{-(n,m)}(e_i^\top \otimes I_{nm})(e^{(t-s)\hat{A}_p^{\text{vec}}} \text{vec}(Z_s^\top))$$

$$+ K^{-(n,m)}(e_i^\top \otimes I_{nm}) \int_s^t e^{(t-u)\hat{A}_p^{\text{vec}}} \hat{E}_p^{\text{vec}} d \text{vec}(L_u),$$

which finally inserted back into (24) proves (22) and (23). \qed
Remark 2.6. i) Note that in order to obtain the correct autoregressive structure in (22), we have to take the transpose of the state process $Z_t$ for $t \in \mathbb{R}$. That means we first stack the columns of the first block matrix entry $Z^{(1)}_t$ below each other, then below this real vector of length $nm$ we append the stacked columns of $Z^{(2)}_t$ and so on until we finally obtain the vector \( \text{vec}(Z_t) = (\text{vec}(Z^{(1)}_t), \text{vec}(Z^{(2)}_t), \ldots, \text{vec}(Z^{(p)})) \in \mathbb{R}^{pnm} \).

ii) In case of $M_{n,1} = \mathbb{R}_n$ the vec-operator is the identity and also \( \text{vec}(Z_t) = \text{vec}(Z_t) \), i.e. in this case Proposition 2.5 is trivial. Note further that whenever the state process \((Z_t)_{t \in \mathbb{R}}\) takes values in \((\mathbb{S}_d)^p\) driven by a Levy process \(L\) with values in \(\mathbb{S}_d\), then \(K^{(d,d)} = \mathbb{I}_{d^2}\) and we see that \(\hat{A}_i = A^\text{vec}_i\), \(\hat{C}_j = C^\text{vec}_j\) and \(\hat{P}_p = P^\text{vec}_p\) for \(i = 1, \ldots, p\) and \(j = 0, \ldots, q\) and the representations in (19)-(21) become considerably simpler. Moreover, in this case we could replace vec by the vech operation.

iii) The controller canonical form for \(\mathbb{R}_d\)-valued MCARMA processes was already used in [17] for estimating the parameters of the driving Lévy process \((L_t)_{t \in \mathbb{R}}\). Proposition 2.5 suggests that the results of [17] straightforwardly extend to the matrix-valued case.

From the representation in (22) and (23) we can read off the following second order structure for the output process \((X_t)_{t \in \mathbb{R}}\), see also [7, 24]:

**Proposition 2.7.** Let \(p \in \mathbb{N}\) and \(q \in \mathbb{N}_0\) such that \(q < p\) and \((A^\text{vec}_p, E^\text{vec}_p, C^\text{vec}_q, L^\text{vec})\) be as in (19)-(21) with \(L\) being square-integrable and where we denote the covariance operator of \((L_t^\text{vec})_{t \in \mathbb{R}}\) by \(Q^\text{vec}\). Then the absolute second moment of the process \((\text{vec}(X_t))_{t \in \mathbb{R}}\), given by (23), exists and we have

\[
\text{Var}[\text{vec}(X_t) | \mathcal{F}_s] = \hat{\Sigma}^\text{vec}_q \Sigma_{t,s}^\text{vec}(\hat{C}_q^\text{vec})^\top, \quad s < t \in \mathbb{R},
\]

where

\[
\Sigma_{t,s}^\text{vec} := \int_s^t e^{uA_p} Q^\text{vec}(\hat{C}_q^\text{vec})^\top e^{uA_p^\top} du.
\]

Moreover, for every \(h \geq 0\) the auto-covariance of \((\text{vec}(X_t))_{t \in \mathbb{R}}\) satisfies

\[
\text{Cov}[\text{vec}(X_{t+h}), \text{vec}(X_t) | \mathcal{F}_s] = \hat{\Sigma}^\text{vec}_q e^{hA_p} \Sigma_{t,s}^\text{vec}(\hat{C}_q^\text{vec})^\top, \quad s < t \in \mathbb{R}, \quad h \geq 0.
\]

If in addition \((X_t)_{t \in \mathbb{R}}\) is stable and given by (10), then

\[
\text{Var}[\text{vec}(X_t)] = \hat{\Sigma}^\text{vec}_q \Sigma_{\infty}^\text{vec}(\hat{C}_q^\text{vec})^\top, \quad \forall t \in \mathbb{R},
\]

where

\[
\Sigma_{\infty}^\text{vec} := \int_0^\infty e^{uA_p} Q^\text{vec}(\hat{C}_q^\text{vec})^\top e^{uA_p^\top} du, \quad \text{and the auto-covariance is}
\]

\[
\text{Cov}[\text{vec}(X_{t+h}), \text{vec}(X_t)] = \hat{\Sigma}^\text{vec}_q e^{hA_p} \Sigma_{\infty}^\text{vec}(\hat{C}_q^\text{vec})^\top, \quad t \in \mathbb{R}, \quad h \geq 0.
\]

Proposition 2.5 tells us that the process \((\text{vec}(X_t))_{t \in \mathbb{R}}\) can be interpreted as the output process of an \(\mathbb{R}^{nm}\)-valued continuous-time state space model in a controller canonical form entrywise composited with the commutation matrix \(K^{-(n,m)}\). It is well known that every \(\mathbb{R}^{nm}\)-valued MCARMA process possesses a state space representation. Conversely, the result in [43, Theorem 3.3] describes precisely those state space specifications such that the associated output process gives rise to a (causal) MCARMA process. We introduce the following notion: We call the function \(H : \mathbb{C} \to \mathcal{L}(\mathbb{M}_{n,m})\) given by

\[
H(\lambda) := C(\lambda I_p - A)^{-1}B, \quad \lambda \in \mathbb{C},
\]

the **transfer function** of the continuous-time linear state space model associated with \((A, B, C, L)\). For the transfer function \(H\) associated with \((\hat{A}_p, \hat{E}_p, C^\text{vec}_q, L^\text{vec})\) as in Proposition 2.5 we have the following:
Lemma 2.8. Let $p \in \mathbb{N}$ and $q \in \mathbb{N}_0$ such that $q < p$ and let $(\hat{A}_p^{\text{vec}}, \hat{E}_p^{\text{vec}}, \hat{C}_q^{\text{vec}}, L^{\text{vec}})$ be as in Proposition 2.5. Then for every $\lambda \in \mathbb{C}$ we have

$$\hat{C}_q^{\text{vec}} (\hat{A}_{pmn}^{\text{vec}} - \hat{A}_p^{\text{vec}})^{-1} \hat{E}_p^{\text{vec}} = \hat{Q}(\lambda) \hat{P}(\lambda)^{-1},$$ \hspace{1cm} (34)

where $\hat{Q}, \hat{P} \in M_{nm}(\mathbb{R}[\lambda])$ for $\lambda \in \mathbb{C}$ are given by

$$\hat{Q}(\lambda) := \hat{C}_0^{\text{vec}} + \hat{C}_1^{\text{vec}} (K^{(n,m)}p) + \hat{C}_2^{\text{vec}} (K^{(n,m)}p)^2 + \ldots + \hat{C}_q^{\text{vec}} (K^{(n,m)}p)^q,$$ \hspace{1cm} (35)

and

$$\hat{P}(\lambda) := (K^{(n,m)}p)^p - \hat{A}_1^{\text{vec}} (K^{(n,m)}p)^{p-1} - \hat{A}_2^{\text{vec}} (K^{(n,m)}p)^{p-2} - \ldots - \hat{A}_p^{\text{vec}}.$$ \hspace{1cm} (36)

Proof. Let $\lambda \in \mathbb{C}$ and $F := (F_1, F_2, \ldots, F_p)^T \in \mathcal{L}(\mathbb{R}_{nm}, \mathbb{R}_{pmn})$ with $F_i \in \mathbb{R}_{nm}$ for all $i = 1, \ldots, p$ and such that $F(x) = (F_1 x, F_2 x, \ldots, F_p x)^T$ for all $x \in \mathbb{R}_{nm}$. We solve the matrix equation $(\lambda I_{pmn} - \hat{A}_p^{\text{vec}}) F = \hat{E}_p^{\text{vec}}$, where the left-hand side equals

$$(\lambda F_1 - K^{-(n,m)} F_2, \ldots, \lambda F_{p-1} - K^{-(n,m)} F_p, \lambda F_p - \hat{A}_p^{\text{vec}} F_1 - \ldots - \hat{A}_1^{\text{vec}} F_p)^T.$$ \hspace{1cm} (37)

Setting this equal to $(0_{nm}, \ldots, 0_{nm}, K^{-(n,m)})^T$ and solving for $F$ yields

$$F_i = (\lambda K^{(n,m)})^{-(p-i)} F_p, \quad \text{for } i = 1, \ldots, p - 1,$$ \hspace{1cm} (37)

and inserting this into the last equation we see that the term

$$\lambda F_p - \sum_{i=1}^p \hat{A}_i^{\text{vec}} F_{p-i} = \left( \lambda^p (K^{(n,m)}p)^{p-1} - \sum_{i=1}^p \hat{A}_i^{\text{vec}} (\lambda K^{(n,m)}p)^{p-i} \right) (\lambda K^{(n,m)}p)^{1-p} F_p,$$ \hspace{1cm} (38)

must coincide with $K^{-(n,m)}$, which by definition of $\hat{P}(\lambda)$ is equivalent to $F_p = (\lambda K^{(n,m)}p)^{-(p-1)} \hat{P}(\lambda)^{-1}$. Hence by (37) we see that $F_i = (\lambda K^{(n,m)}p)^{-(p-i)} \hat{P}(\lambda)^{-1}$, i.e.

$$F = (\hat{P}(\lambda)^{-1}, \lambda K^{(n,m)}p \hat{P}(\lambda)^{-1}, \ldots, (\lambda K^{(n,m)}p)^{-(p-1)} \hat{P}(\lambda)^{-1})^T.$$ \hspace{1cm} (39)

From this and the definition of $\hat{Q}$ we conclude that

$$\hat{C}_q^{\text{vec}} (\lambda I_{pmn} - \hat{A}_p^{\text{vec}})^{-1} \hat{E}_p^{\text{vec}} = \hat{C}_q^{\text{vec}} F = \hat{Q}(\lambda) \hat{P}(\lambda)^{-1}, \quad \lambda \in \mathbb{C},$$ \hspace{1cm} (39)

which proves (34). \hfill \Box

The following proposition is the key result for the definition of matrix-valued MCARMA processes. We relegated the proof to the Appendix A.

Proposition 2.9. Let $p \in \mathbb{N}$ and $q \in \mathbb{N}_0$ such that $q < p$ and $(\hat{A}_p^{\text{vec}}, \hat{E}_p^{\text{vec}}, \hat{C}_q^{\text{vec}}, L^{\text{vec}})$ be as in Proposition 2.5. Moreover, let $\hat{Q}(\lambda)$ and $\hat{P}(\lambda)$ be given by (35) and (36), respectively. Then there exist two matrix polynomials $\hat{Q}, \hat{P} \in M_{nm}(\mathbb{R}[\lambda])$ such that

$$\hat{Q}(\lambda) = \hat{C}_0 \lambda^q + \hat{C}_1 \lambda^{q-1} + \ldots + \hat{C}_q, \quad \lambda \in \mathbb{C},$$ \hspace{1cm} (38)

with $\hat{C}_j \in \mathbb{R}_{nm}$ for $j = 0, \ldots, q$ and

$$\hat{P}(\lambda) = \|nm\lambda^p - \hat{A}_1 \lambda^{p-1} - \ldots - \hat{A}_p, \quad \lambda \in \mathbb{C},$$ \hspace{1cm} (39)

with $\hat{A}_i \in \mathbb{R}_{nm}$ for $i = 1, \ldots, p$ satisfying

$$\hat{P}(\lambda)^{-1} \hat{Q}(\lambda) = \hat{Q}(\lambda) \hat{P}(\lambda)^{-1} \quad \text{for all } \lambda \in \mathbb{C},$$ \hspace{1cm} (40)

and $\det(\hat{P}(\lambda)) = 0$ if and only if $\det(\hat{P}(\lambda)) = 0$. If moreover, $E[\log(\|L_1\|_{nm})] < \infty$ and

$$\lambda \in \mathbb{C}: \det(\hat{P}(\lambda)) = 0 \subseteq \mathbb{R} \setminus \{0\} + i\mathbb{R},$$ \hspace{1cm} (41)

then equation (23) has a stationary solution, unique in law, given by

$$\text{vec}(X_t) = \int_{-\infty}^{\infty} g(t-s) \text{d}L^{\text{vec}}_s, \quad t \in \mathbb{R},$$ \hspace{1cm} (42)

where $g(t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi t} \hat{P}(i\xi)^{-1} \hat{Q}(i\xi) \text{d}\xi$ for all $t \in \mathbb{R}$. 

Following [34, Theorem 3.22], every \( \mathbb{R}_{nm} \)-valued MCARMA process \((Y_t)_{t \in \mathbb{R}}\) with moving-average polynomial matrix \( \hat{Q} \in \mathbb{M}_{nm}(\mathbb{R}[\lambda]) \), autoregressive polynomial matrix \( \hat{P} \in \mathbb{M}_{nm}(\mathbb{R}[\lambda]) \) and an input Lévy process \( L \) on \( \mathbb{R}_{nm} \) is given by

\[
Y_t = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\xi(t-s)} \hat{P}(i\xi)^{-1} \hat{Q}(i\xi) \, d\xi \, d\hat{L}_s, \quad t \in \mathbb{R},
\]

where \( \hat{P} \) satisfies \( \{ \lambda \in \mathbb{C} : \det(\hat{P}(\lambda)) = 0 \} \subseteq \mathbb{R} \setminus \{0\} + i\mathbb{R} \). It thus follows from Proposition 2.9, that the unique stationary process \((\text{vec}(X_t))_{t \geq 0} \) in (42) is an \( \mathbb{R}_{nm} \)-valued MCARMA process with moving-average polynomial matrix \( \hat{Q} \), autoregressive polynomial matrix \( \hat{P} \) and Lévy noise \( \mathcal{L}^{\text{nc}} \). Moreover, by [34, Remark 3.19] it can be interpreted as the solution of the higher order stochastic differential equation (1) with \( L \) replaced by \( \mathcal{L}^{\text{nc}} \). This justifies the following definition of a matrix-valued multivariate continuous-time autoregressive moving-average process (by means of transformed \( \mathbb{R}_{nm} \)-valued MCARMA processes):

**Definition 2.10.** Let \( p \in \mathbb{N} \) and \( q \in \mathbb{N}_0 \) such that \( q < p \) and \((A_p, E_p, C_q, L)\) be as in (12)-(14). If moreover \( \mathbb{E}[\log(||L_1||_{nm})] < \infty \) and (41) is satisfied, then we call the unique output process \((X_t)_{t \in \mathbb{R}}\) in (16) for which \((\text{vec}(X_t))_{t \in \mathbb{R}}\) is the stationary solution to (23), an \( \mathbb{M}_{nm} \)-valued continuous-time autoregressive moving-average (MCARMA) process of order \((p, q)\). In the special case where \( q = 0 \) and \( C_0 = I \), i.e. \( C_0 = [1, 0, \ldots, 0] \), we call \((X_t)_{t \in \mathbb{R}}\) an \( \mathbb{M}_{nm} \)-valued MCAR process of order \( p \) instead. Moreover, if in the situation above we have \( \tau(A_p) < 0 \), or equivalently

\[
\left\{ \lambda \in \mathbb{C} : \det(\hat{P}(\lambda)) = 0 \right\} \subset (-\infty, 0) + i\mathbb{R},
\]

then we say that \((X_t)_{t \in \mathbb{R}}\) is a causal MCAR(MA) process of order \((p, q)\) (resp. \( p \)). Otherwise, if \( \tau(A_p) \geq 0 \) we say that \((X_t)_{t \in \mathbb{R}}\) is non-causal.

**Remark 2.11.**

i) By Definition 2.10 we see that every causal \( \mathbb{M}_{nm} \)-valued MCARMA process is the output process of a stable linear state space model. In contrast, non-causal MCARMA processes correspond to non-stable linear state space models. Moreover, it can be seen from the representation (42) (see also the decomposition (80) in the proof of Proposition 2.9), that non-causal MCARMA processes are not adapted to the natural filtration \( \mathcal{F} \), since for \( t \in \mathbb{R}, X_t \) depends on \( \sigma(L_s : s > t) \).

ii) We want to emphasize here that the vectorized versions \( \hat{P} \) and \( \hat{Q} \) of the operator polynomials \( P \) and \( Q \) in (17) and (18) can in general not be interpreted as the moving-average, respectively, autoregressive polynomials of the MCARMA process. Instead, the polynomials \( \hat{Q} \) and \( \hat{P} \) in (40) can be naturally interpreted as such, i.e. by means of equation (1). If, however, \( \hat{P} \) and \( \hat{Q} \) commute, then \( \hat{P}(\lambda) = \hat{P}(\lambda) \) and \( \hat{Q}(\lambda) = \hat{Q}(\lambda) \) for all \( \lambda \in \mathbb{C} \).

3. **Positivity of MCARMA processes**

In this section we study positivity of \( \mathbb{M}_{nm} \)-valued MCARMA processes of order \((p, q)\), as they were defined in Definition 2.10 above. Recall that we use the term *positive* as a synonym for *cone valued* as our main examples of multivariate cones, \( \mathbb{R}^+_d \) and \( \mathbb{S}^+_d \), are both termed *positive*. This section is divided into the case of causal MCARMA in Section 3.2 and the non-causal MCARMA case in Section 3.3. A careful distinction between the two cases is justified as the positivity constraints may interact with the stability conditions. Our main results are Theorem 3.5 and Theorem 3.14 below which establish sufficient and/or necessary conditions for the positivity of \( \mathbb{M}_{nm} \)-valued causal and non-causal MCARMA processes.
3.1. Positive operators and cone valued Lévy processes. Before we study the positivity of $\mathbb{M}_{n,m}$-valued MCARMA processes in the next two sections, we recall some additional preliminaries concerning convex cones, (quasi)-positive operators and increasing Lévy processes. For $n, m \in \mathbb{N}$ we consider the inner product space $(\mathbb{M}_{n,m}, \langle \cdot, \cdot \rangle_{n,m})$ and assume that $\mathbb{M}_{n,m}$ is equipped with a convex cone $C$, i.e. $C \subseteq \mathbb{M}_{n,m}$ such that $C + C \subseteq C$, $\lambda C \subseteq C$ for all $\lambda \in \mathbb{R}^+$ and $C \cap (-C) = \{0_{n,m}\}$. Moreover, we write $\pi \subseteq \lambda C$ for the partial-ordering on $\mathbb{M}_{n,m}$ induced by $C$, i.e. for $x, y \in \mathbb{M}_{n,m}$: $x \leq_C y$ if and only if $y - x \in C$.

Positive and quasi-positive operators. We denote by $\pi(C) \subseteq \mathcal{L}(\mathbb{M}_{n,m})$ the set of all linear operators leaving the cone $C$ invariant, i.e.

$$\pi(C) = \{ A \in \mathcal{L}(\mathbb{M}_{n,m}) : A(u) \geq_C 0 \text{ for all } u \geq_C 0 \}.$$ 

We call elements in $\pi(C)$ positive operators on $\mathbb{M}_{n,m}$. Note that the set $\pi(C)$ is a convex algebra cone, that means it is a convex cone such that $B_1, B_2 \in \pi(C)$ implies $B_1B_2 \in \pi(C)$. We denote by $\pi \leq_C$ the partial ordering on $\pi(C)$ induced by $\pi(C)$. Moreover, we call an element $A \in \mathcal{L}(\mathbb{M}_{n,m})$ quasi-positive, if $\exp(At)$ exists for all $t \geq 0$, where $\exp(At)$ denotes the operator exponential of $At$. It is well known that $A$ is quasi-positive if and only if for all $u, v \in C$ with $\langle u, v \rangle_{n,m} = 0$ we have $\langle Au, v \rangle_{n,m} \geq 0$, see e.g. [26]. The following two cases are our main examples for convex cones in $\mathbb{M}_{n,m}$:

a) For $m = 1$ and $n = d$ we have $(\mathbb{R}_d, \langle \cdot, \cdot \rangle) = (\mathbb{M}_{d,1}, \langle \cdot, \cdot \rangle_{d,1})$. On $\mathbb{R}_d$ we consider the positive orthant $C = \mathbb{R}_d^+$, which is a convex cone and we denote its induced partial ordering by $\leq_d$. It is well known that the cone $\pi(\mathbb{R}_d^+) \subseteq \mathbb{M}_d$ of $\mathbb{R}_d^+$-preserving linear maps (matrices) is given by the set of all positive matrices (more precisely non-negative matrices), i.e.

$$\pi(\mathbb{R}_d^+) = \{ (a_{i,j})_{1 \leq i, j \leq d} \in \mathbb{M}_d : a_{i,j} \geq 0 \ \forall i, j = 1, \ldots, d \}.$$ 

It is also well-known that a matrix $A = (a_{i,j})_{1 \leq i, j \leq d} \in \mathbb{M}_d$ is quasi-positive (sometimes called cross-positive) if and only if $a_{i,j} \geq 0$ for all $i, j = 1, \ldots, d$ such that $i \neq j$, i.e. all off-diagonal elements are non-negative and the diagonal ones can be arbitrary, see e.g. [26].

b) In case of $n = m = d$ for some $d \in \mathbb{N}$ we consider the space of real $d \times d$-matrices $\mathbb{M}_d$. On $\mathbb{M}_d$ we consider the convex cone of all symmetric positive-semi definite matrices $C = \mathbb{S}_d^+$ and denote the induced partial ordering by $\leq_{\mathbb{S}_d^+}$. As far as we know there is no analogous characterization of the set $\pi(\mathbb{S}_d^+)$ known. Partial results were achieved in this direction, see e.g. [33] for some related results in the theory of linear preserver problems.

Multivariate Lévy processes on cones. Let $L = (L_t)_{t \in \mathbb{R}}$ denote a two-sided Lévy process on $\mathbb{M}_{n,m}$ defined on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and let $C$ be a convex cone in $\mathbb{M}_{n,m}$. We define a positive Lévy process as follows:

Definition 3.1. We call a two-sided Lévy process $(L_t)_{t \in \mathbb{R}}$ on $\mathbb{M}_{n,m}$ $C$-positive, if it is $C$-valued or, equivalently, if $L_t - L_s \in C$ for all $t, s \in \mathbb{R}$ such that $s < t$.

The characteristic exponent (2) of a $C$-positive Lévy process is given by

$$\varphi_L(z) = i\langle \gamma_L, z \rangle_{n,m} + \int_C (e^{i\langle \xi, z \rangle_{n,m}} - 1 - i\langle \chi(\xi), z \rangle_{n,m}) \nu_L(d\xi), \quad z \in \mathbb{M}_{n,m},$$

where the drift $\gamma_L$ is positive, i.e. $\gamma_L \in C$ and the Lévy measure $\nu_L$ is concentrated on $C \cap \{0_{n,m}\}$, hence jumps, small or large, of the Lévy process are positive. Moreover, note that compared to (2) the diffusion part vanishes, i.e. a positive Lévy process is of pure-jump type. We refer to [5] for more information on matrix-valued positive Lévy processes.
3.2. Positive causal MCARMA processes. Throughout this section we fix \( n, m \in \mathbb{N} \) and let \( C \) be a cone in \( \mathbb{M}_{n,m} \). Moreover, we assume that \( L = (L_t)_{t \in \mathbb{R}} \) is a \( C \)-valued two-sided Lévy process defined on the filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}, F)\) such that \( \mathbb{E}[\log(\|L\|_{n,m})] < \infty \). We let \( A_p, E_p \) and \( C_q \) be as in (12)-(14) and assume that \( \tau(A_p) < 0 \). This of course implies that the stable output process \((X_t)_{t \in \mathbb{R}}\), associated with \((A_p, E_p, C_q, L)\), is a causal \( \mathbb{M}_{n,m} \)-valued Lévy process.

\[ X_t = \int_{-\infty}^{t} C_q e^{(t-s)A_p} E_p \, dL_s, \quad t \in \mathbb{R}. \]  

We begin with a small lemma on the connection between quasi-positive operators and their spectral bound:

**Lemma 3.2.** Let \( V \) be a linear space and \( C \subseteq V \) a convex cone. Then for every quasi-positive \( A \in \mathcal{L}(V) \) with \( \tau(A) < 0 \) we have \( -A^{-1} \geq 0 \).

**Proof.** It follows from the spectral mapping theorem that for every \( A \in \mathcal{L}(V) \) with \( \tau(A) < 0 \) we have \( \tau(\exp(A)) < 1 \). It thus follows that \( e^{At} \to 0 \) as \( t \to \infty \) and therefore we see that

\[ -A^{-1} = \int_0^\infty e^{As} \, ds. \]

Hence, whenever \( A \) is quasi-positive, i.e. \( \exp(As) \geq 0 \) for every \( s \geq 0 \), we have \( \int_0^\infty e^{As} \, ds \geq 0 \) and consequently \( -A^{-1} \geq 0 \). \( \square \)

From the representation (43) we see that \( X_t \in C \) for every \( t \in \mathbb{R} \), whenever the Lévy process \((L_t)_{t \in \mathbb{R}}\) is \( C \)-valued and \( g(s) = C_q e^{sA_p} E_p \in \pi(C) \) holds true for every \( s \geq 0 \). In the following lemma we prove a particular form of the Laplace transform of the kernel function \( g \). The main part of the proof, the computation of the transfer function, is similar to the matrix representation case in Lemma 2.8.

**Lemma 3.3.** The Laplace transform \( \varphi: \mathbb{R}^+ \to \mathcal{L}(\mathbb{M}_{n,m}) \) of the kernel \( g(s) = C_q e^{sA_p} E_p \) exists and is given by

\[ \varphi(\lambda) = Q(\lambda)P(\lambda)^{-1}, \quad \lambda \geq 0. \]  

**Proof.** Since \( \tau(A_p) < 0 \) we see that for \( \lambda \geq 0 \) the resolvent \( R(\lambda, A_p) = (\lambda I - A_p)^{-1} \) is given by the Laplace transform of the matrix semigroup \( e^{\lambda A_p} \), i.e. \( R(\lambda, A_p) = \int_0^\infty e^{-\lambda s} e^{sA_p} \, ds \). We thus compute

\[ \varphi(\lambda) = \int_0^\infty e^{-\lambda s} g(s) \, ds = C_q \left( \int_0^\infty e^{-\lambda s} e^{sA_p} \, ds \right) E_p = C_q R(\lambda, A_p) E_p. \]

For \( \lambda \in \mathbb{C} \) we compute the term \( R(\lambda, A_p)E_p \) as follows: let \( F_p := [F_1, F_2, \ldots, F_p]^\top \in \mathcal{L}(\mathbb{M}_{n,m})^p \) with \( F_i \in \mathcal{L}(\mathbb{M}_{n,m}) \) and such that \( F_p(x) = (F_1x, F_2x, \ldots, F_p x)^\top \). Moreover, set \( A := [A_p, A_{p-1}, \ldots, A_1] \) and consider \((\lambda I - A_p)F_p = E_p\), which is equivalent to

\[ \begin{bmatrix} \lambda F_1 - F_2, \lambda F_2 - F_3, \ldots, \lambda F_{p-1} - F_p, \lambda F_p & -AF_p \end{bmatrix} = [0, \ldots, I]. \]

Solving for \( F_p \) yields \( F_1 = \lambda^{-1} F_2, F_2 = \lambda^{-1} F_3, \ldots, F_{p-1} = \lambda^{-1} F_p \) and thus for every \( i = 1, \ldots, p - 1 \) we have \( F_i = \lambda^{-(p-i)} F_p \). Moreover, the last equation reads as

\[ \begin{bmatrix} \lambda^p I - A_p - A_{p-1} \lambda - \ldots - A_1 \lambda^{p-1} \end{bmatrix} \lambda^{-(p-1)} F_p = I, \]

and hence by definition of \( P(\lambda) \) we see that \( F_p = \lambda^{p-1} P(\lambda)^{-1} \) and therefore \( F_i = \lambda^{-(p-i)} P(\lambda)^{-1} \) for \( i = 1, \ldots, p - 1 \). In vector notation this means

\[ F_p = [I \circ P(\lambda)^{-1}, \lambda I \circ P(\lambda)^{-1}, \ldots, \lambda^{p-1} I \circ P(\lambda)^{-1}] = (1, \lambda, \ldots, \lambda^{p-1}) \otimes P(\lambda)^{-1}. \]
Thus inserting $F_p = R(\lambda, A_p)E_p$ back into (45) and by definition of $Q$ in (18) yields
$$C_qF_p = C_0P(\lambda)^{-1} + \lambda C_1P(\lambda)^{-1} + \ldots + \lambda^{p-1}C_{p-1}P(\lambda)^{-1} = Q(\lambda)P(\lambda)^{-1}. \tag{46}$$

Next, we introduce the fundamental property of the Laplace transforms of the kernel function $s \mapsto g(s)$ that will ensure the positivity of the associated causal MCARMA processes. The following definition is adapted from [1, Definition 5.4].

**Definition 3.4.** We call a function $f: \mathbb{R}^+ \to \mathcal{L}(\mathcal{M}_{n,m})$ completely monotone with respect to $\pi(C)$, if $f$ is infinitely often differentiable and $(-1)^nf^{(n)}(\lambda) \geq 0$ for all $\lambda > 0$ and $n \in \mathbb{N}_0$.

The following proposition is our main result on the positivity of matrix-valued causal MCARMA processes:

**Theorem 3.5.** Let $(X_t)_{t \in \mathbb{R}}$ be a $\mathcal{M}_{n,m}$-valued causal MCARMA process of order $(p,q)$ given by (43). Moreover, let $P(\lambda)$ and $Q(\lambda)$ be the operator polynomials in (17) and (18), respectively. Then the following holds true:

i) $(X_t)_{t \in \mathbb{R}}$ is $C$-valued if and only if the map $\lambda \mapsto Q(\lambda)P(\lambda)^{-1}$ is completely monotone with respect to $\pi(C)$.

ii) If $\lambda \mapsto Q(\lambda)$ is completely monotone with respect to $\pi(C)$ and $P(\lambda)$ can be decomposed into linear factors as follows:

$$P(\lambda) = \prod_{i=1}^p(\lambda I - \hat{A}_i), \tag{46}$$

where for all $i = 1, \ldots, p$ the operator $\hat{A}_i \in \mathcal{L}(\mathcal{M}_{n,m})$ is quasi-positive and $\tau(\hat{A}_i) < 0$, then $(X_t)_{t \in \mathbb{R}}$ is $C$-valued.

**Proof.** By Lemma 3.3 the Laplace transform $\phi$ of the kernel $g(s) = C_qe^{sA_p}E_p$ is given by the operator rational function $\lambda \mapsto Q(\lambda)P(\lambda)^{-1}$. By a vector valued version of Bernstein’s theorem, see [1, Theorem 5.5], and Lemma 3.3 it follows that $s \mapsto g(s)$ is positive. Indeed, by Bernstein’s theorem we have $g(s) \in \pi(C)$ for all $s \geq 0$ if and only if its Laplace transform is completely monotone with respect to $\pi(C)$, but since its Laplace transform is given by $\lambda \mapsto Q(\lambda)P(\lambda)^{-1}$ Theorem 3.5 i) follows.

For the second statement Theorem 3.5 ii), note that from part i) it follows that the MCARMA process $(X_t)_{t \geq 0}$ is $C$-valued if and only if $\lambda \mapsto Q(\lambda)P(\lambda)^{-1}$ is completely monotone with respect to $\pi(C)$. Note further that for every $i = 1, \ldots, p$ the resolvent $R(\lambda, \hat{A}_i)$ exists for every $\lambda > 0$ and is completely monotone with respect to $\pi(C)$. Indeed, by assumption $\hat{A}_i$ is quasi-positive and $\tau(\hat{A}_i) < 0$. This implies that also $-\lambda I + \hat{A}_i$ is quasi-positive and $\tau(-\lambda I + \hat{A}_i) < 0$ for every $\lambda \geq 0$. Indeed, note that the operator $-\lambda I$ is always quasi-positive for $\lambda > 0$, since whenever $(u,v)_{n,m} = 0$ we have $(-\lambda Iu,v)_{n,m} = -\lambda(u,v)_{n,m} = 0$. By an application of Lemma 3.2 it follows that $(\lambda I - \hat{A}_i)^{-1} \succeq 0$. Moreover, for every $n \in \mathbb{N}$ we have

$$(-1)^n \frac{d^n}{d\lambda^n}(\lambda I - \hat{A}_i)^{-1} = (\lambda I - \hat{A}_i)^{-(1+n)}, \tag{47}$$

where the right-hand side of (47) is again positive, since $\pi(C)$ is an algebra cone. It thus follows that for every $i = 1, \ldots, p$ the linear factor $(\lambda I - \hat{A}_i)^{-1}$ in the decomposition (46) is completely monotone and by assumption $Q$ is completely monotone as well. Now the assertion follows since the product of completely monotone functions is again completely monotone (use the general Leibniz’s rule here) and hence Theorem 3.5 ii) follows from part i).
Remark 3.6. i) In the univariate case an analog to Theorem 3.5 i) was shown in [47, Theorem 2]. Here we extend the result to the multivariate setting. However, it is to be noted that the result can be extended even beyond finite-dimensions. In fact, also for certain Hilbert-valued CARMA processes as introduced in [12] a similar characterization can be shown. ii) The factorization of operator polynomials into the form (46) is well studied in the literature, see, e.g. [39] and in particular for matrix polynomials [25]. For instance, a sufficient criteria for $P(\lambda)$ to admit a factorization of the form (46) is that the transition operator $A_p$ is diagonalizable. If an operator polynomial is factorizable, then the operators $\hat{A}_i$ can be computed by iterated operator division and the additional positivity conditions can be checked thereafter. The strength of Theorem 3.5 ii), however, lays in the fact that it provides us with a simple method to construct positive stationary MCARMA processes by choosing suitable operators $\hat{A}_i$ for $i = 1, \ldots, p$. This is explained in the following example by means of a second order MCARMA process:

Example 3.7. Let $n = m = d$ for some $d \in \mathbb{N}$, $p = 2$ and $Q(\lambda) = I$. Moreover, let $\hat{A}_1, \hat{A}_2 \in \mathcal{L}(\mathbb{M}_d)$ be quasi-positive with $\tau(\hat{A}_1), \tau(\hat{A}_2) < 0$. For example, we could choose $\hat{A}_i, x := \hat{A}_i x + x \hat{A}_i^*$ for some matrix $A_i \in \mathbb{M}_d$ with $\tau(A_i) < 0$ for $i = 1, 2$. Note that in this case for $i = 1, 2$ we have $\sigma(\hat{A}_i) = \sigma(A_i) + \sigma(A_i)$, see [40]. If we set $A_1 := \hat{A}_1 + \hat{A}_2$ and $A_2 := \hat{A}_1 \hat{A}_2$, we see that

$$P(\lambda) = (\lambda I - \hat{A}_1)(\lambda I - \hat{A}_2) = \lambda^2 I - \lambda A_1 - A_2.$$ Hence following Theorem 3.5 ii) and since $\tau(A_2) = \tau(A_2)$, this specification gives rise to a $C$-positive causal MCAR process of order $p = 2$, whenever $\tau(A_2) = \tau(\hat{A}_1 \hat{A}_2) < 0$.

Note that the conditions in Theorem 3.5 ii) are not necessary for causal MCARMA processes to be positive. Indeed, in some situations we can check the conditions of Theorem 3.5 i) directly as illustrated by the following example:

Example 3.8. Let $p = 2$, $C_0 \in \pi(C)$ and $A \in \mathcal{L}(\mathbb{M}_{n,m})$ be invertible and such that $-A^2$ is quasi-positive. If we define

$$A_2 := \begin{bmatrix} 0 & I \\ -A^2 & 0 \end{bmatrix}, \quad \text{and} \quad C_0 := [C_0, 0],$$

then $\tau(A_2) = \tau(-A^2) < 0$ and the associated operator polynomials are $P(\lambda) = \lambda^2 I + A^2$ and $Q(\lambda) = C_0$. We thus see that $\lambda \mapsto Q(\lambda)P(\lambda)^{-1}$ is completely monotone, although no factorization of the form in (46) is available.

In the following Proposition 3.9 we extend another sufficient positivity criteria, known in the univariate case in [47, Theorem 1 e)], to $\mathbb{R}_d$-valued MCARMA processes. In order to this, we adapt the proof of [3, Theorem 1] to $\pi(\mathbb{R}_d^+)$-valued rational functions. Unsurprisingly, this multivariate version is more involved and requires some additional and rather technical assumptions.

Proposition 3.9. Let $p \in \mathbb{N}$ and $q \in \mathbb{N}_0$ such that $q < p$ and let $(X_t)_{t \geq 0}$ be an $\mathbb{R}_d$-valued causal MCARMA given by (43) with parameters $(A_p, P_p, C_q, L)$. Let $P(\lambda)$ and $Q(\lambda)$ denote the matrix polynomials in (17) and (18), respectively, and
assume that \( \{ \hat{C}_i \in \mathbb{M}_d : i = 1, \ldots, q \} \) and \( \{ \hat{A}_i \in \mathbb{M}_d : i = 1, \ldots, p \} \) are two commutative families of positive matrices such that

\[
Q(\lambda) = \prod_{i=0}^{q} (\lambda I - \hat{C}_i) \quad \text{and} \quad P(\lambda) = \prod_{i=1}^{p} (\lambda I - \hat{A}_i).
\]

Moreover, assume that for every \( \lambda > 0 \) we have \( Q(\lambda)P(\lambda)^{-1} = P(\lambda)^{-1}Q(\lambda) \), the matrix logarithms \( \log(P(\lambda)) \) and \( \log(Q(\lambda)) \) exist in \( \mathbb{M}_d \) and for every \( i = 0, \ldots, q \) there exist \( \hat{l}^{(i,1)}, \hat{l}^{(i,2)}, \ldots, \hat{l}^{(i,q+1)} \in \pi(\mathbb{R}_d^+) \) such that for any \( n \in \mathbb{N} \) and \( j = 0, \ldots, q \) we have \( \hat{l}^{(i,j+1)} \leq \hat{l}^{(i,j+1)} \).

\[
\hat{C}_i \leq \sum_{j=0}^{q} \hat{l}^{(i,j+1)} \odot \hat{A}_{j+1} \quad \text{and} \quad \sum_{i=0}^{q} \hat{l}^{(i,j+1)} = \mathbb{I}_d.
\]

If \( p > q + 1 \), we assume in addition that \( \tau(\hat{A}_i) < 0 \) for all \( i = q + 2, \ldots, p \). Then \( (X_t)_{t \geq 0} \) is an \( \mathbb{R}_d^+ \)-valued causal MCARMA process of order \( (p, q) \) with associated moving average polynomial \( Q \) and autoregressive polynomial \( P \).

**Proof.** Following Theorem 3.5 i), it suffices to show that \( \lambda \mapsto Q(\lambda)P(\lambda)^{-1} \) is completely monotone on \( \mathbb{R}_+ \) with respect to the cone \( \pi(\mathbb{R}_d^+) \). By (48) we have

\[
Q(\lambda)P(\lambda)^{-1} = \left( \prod_{i=0}^{q} (\lambda I - \hat{C}_i) \right) \left( \prod_{i=1}^{p} (\lambda I - \hat{A}_i) \right)^{-1}, \quad \forall \lambda > 0.
\]

Since the multiplication of monotone functions is again completely monotone it suffices to show the complete monotonicity of \( \lambda \mapsto Q(\lambda)\hat{P}(\lambda)^{-1} \) for

\[
\hat{P}(\lambda) = \prod_{i=1}^{q} (\lambda I - \hat{A}_i), \quad \lambda \in \mathbb{C},
\]

since by Theorem 3.5 ii) we conclude that the maps \( (\lambda I - \hat{A}_i)^{-1} \) are completely monotone for all \( i = q + 2, \ldots, p \). Moreover, it suffices to prove that the map \( \lambda \mapsto \log \left( Q(\lambda)\hat{P}(\lambda)^{-1} \right) \) is completely monotone, since for all \( \lambda > 0 \) we have

\[
Q(\lambda)\hat{P}(\lambda)^{-1} = \exp \left( \log(Q(\lambda))\hat{P}(\lambda)^{-1} \right) = \sum_{n=1}^{\infty} \frac{(\log(Q(\lambda))\hat{P}(\lambda)^{-1})^n}{n!}.
\]

By assumption we know that the matrix logarithms of \( Q(\lambda) \) and \( \hat{P}(\lambda)^{-1} \) exist for every \( \lambda > 0 \) and moreover the matrices \( \hat{C}_0, \ldots, \hat{C}_q \), the matrices \( \hat{A}_1, \ldots, \hat{A}_{q+1} \) and the matrix rational function \( Q(\lambda) \) and \( \hat{P}(\lambda)^{-1} \) mutually commute, thus we find:

\[
\log \left( Q(\lambda)\hat{P}(\lambda)^{-1} \right) = \log \left( Q(\lambda) \right) - \log \left( \hat{P}(\lambda) \right)
\]

\[
= \sum_{j=0}^{q} \log(\lambda I - \hat{C}_j) - \sum_{i=1}^{q+1} \log(\lambda I - \hat{A}_i)
\]

\[
= \int_{0}^{\infty} e^{-\lambda s} s^{-1} \sum_{i=0}^{q} (e^{s\hat{A}_{i+1}} - e^{s\hat{C}_i}) \, ds.
\]

From (51) we see that is suffices to show that \( \sum_{i=0}^{q} (e^{s\hat{A}_{i+1}} - e^{s\hat{C}_i}) \geq 0 \) holds for all \( s \geq 0 \). For this note that the matrix exponential is monotone with respect to \( \pi(\mathbb{R}_d^+) \), i.e. for \( G_1, G_2 \in \mathbb{M}_d \) such that \( G_1 \preceq G_2 \) we have \( e^{G_1} \preceq e^{G_2} \). This can be seen from the definition of the matrix exponential and since monomials are monotone with respect to \( \pi(\mathbb{R}_d^+) \). Note further that the matrices \( \hat{l}^{(i,j+1)} \) are in \( \pi(\mathbb{R}_d^+) \), i.e. entrywise non-negative, and the same holds true for \( \hat{C}_i \) for \( i = 1, \ldots, q + 1 \) and
$j = 0, \ldots, q$. Thus by Lemma B.1 we have $(l^{(i,j)}}{j+1})^n \preceq (l^{(i,j+1)})^n \preceq (C_i)^n$ for every $n \in \mathbb{N}$ and hence for $i = 1, \ldots, q$ and $j = 0, \ldots, q$ we see that

$$e^{l^{(i,j+1)} \otimes \hat{A}_i} = \sum_{n \in \mathbb{N}} \frac{(l^{(i,j+1)} \otimes \hat{A}_i)^n}{n!} \preceq \sum_{n \in \mathbb{N}} \frac{(l^{(i,j+1)} \otimes \hat{A}_i)^n}{n!} \preceq \sum_{n \in \mathbb{N}} \frac{(l^{(i,j+1)} \otimes \hat{A}_i)^n}{n!} \preceq \sum_{n \in \mathbb{N}} \frac{(l^{(i,j+1)} \otimes \hat{A}_i)^n}{n!}. \tag{52}$$

This together with assumption (49) and the convexity of the matrix exponential imply

$$\sum_{i=0}^{q} e^{sC_i} \preceq \sum_{i=0}^{q} e^{s} \sum_{j=0}^{q} (l^{(i,j+1)} \otimes A_{j+1}) \preceq \sum_{i=0}^{q} \sum_{j=0}^{q} (l^{(i,j+1)}) \otimes e^{sA_{j+1}} \preceq \sum_{j=0}^{q} \left( \sum_{i=0}^{q} (l^{(i,j+1)}) \right) \otimes e^{sA_{j+1}} \preceq \sum_{j=0}^{q} e^{sA_{j+1}}, \quad \forall s \geq 0.$$

Hence, from (51) it follows that log $(Q(\lambda)\hat{P}(\lambda)^{-1})$ is given by the Laplace transform of the $\pi(R^2_k)$-valued kernel $s \mapsto s^{-1} \sum_{i=0}^{q}(e^{sA_{i+1}} - e^{sC_{i}})$, which by Bernstein’s theorem yields the complete monotonicity of log $(Q(\lambda)\hat{P}(\lambda)^{-1})$ and hence following the reasoning above we conclude by Theorem 3.5 i) that $(X_t)_{t \in \mathbb{R}}$ is $R^2_k$-valued. That $Q$ and $P$ are the moving-average, respectively, autoregressive polynomials of $(X_t)_{t \in \mathbb{R}}$ then follows from the commutativity of $Q(\lambda)$ and $P(\lambda)$ and Remark 2.11.

**Remark 3.10.**

i) Following [27, Theorem 6.4.15 c)] the (real) logarithm of the matrix $P(\lambda)$ exists if and only if $P(\lambda)$ is non-singular and has an even number of Jordan blocks of each size for every negative eigenvalue.

ii) Note that the technical assumption of Proposition 3.9 is best understood when departing from the condition

$$\sum_{i=0}^{q} \hat{A}_{i+1} \succeq \sum_{i=0}^{q} \hat{C}_i. \tag{53}$$

Indeed, note that if (53) holds for all $1 \leq k, n \leq d$ we have $\sum_{i=0}^{q}(\hat{A}_{i+1})_{k,n} \succeq \sum_{i=0}^{q}(\hat{C}_i)_{k,n}$. Following the Hardy-Littlewood rearrangement inequality and Hall’s marriage theorem, see also [3, Equation 3] and references therein, we see that for all $i = 0, \ldots, q$ and $k, n = 1, \ldots, d$ there exist $(l^{(i,j+1)}_{k,n})_{j=0,\ldots,q}$ with $0 \leq l^{(i,j+1)}_{k,n} \leq 1$ such that $(\hat{C}_i)_{k,n} \leq \sum_{j=0}^{q} l^{(i,j+1)}_{k,n}(\hat{A}_{j+1})_{k,n}$ and $\sum_{i=0}^{q} l^{(i,j)}_{k,n} = 1$ for every $1 \leq k, n \leq d$. Hence, setting $l^{(i,j)} = (l^{(i,j)}_{k,n})_{1 \leq k, n \leq d}$, we see that the conditions in Proposition 3.9 are met if we assume (53) together with $(l^{(i,j)})^n \preceq \sum_{i=0}^{q} (e^{s\hat{A}_{i+1}} - e^{s\hat{C}_i}) \succeq 0$ for all $s \geq 0$ is sufficient and could replace the condition in (49).
3.3. Positive non-stable output processes. In this section we study positive non-stable output and non-causal MCARMA processes. As before we assume that $C$ is a cone in $M_{n,m}$ for $n,m \in \mathbb{N}$ and let $L = (L_t)_{t \in \mathbb{R}}$ be a $C$-valued two-sided Lévy process defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E} [\log norm ||L||_{n,m}] < \infty$. Moreover, we let $A_p, E_p$ and $C_q$ be as in (12)-(14) and assume throughout this section that $\tau(A_p) \geq 0$. Analogous to the stable case in the previous section, we are interested in sufficient and necessary conditions for the positivity of non-stable output processes. Recall that every non-stable output process $(X_t)_{t \in \mathbb{R}}$, associated with $(A_p, E_p, C_q, L)$, has the representation

$$X_t = C_q e^{(t-s)A_p} Z_s + \int_s^t C_q e^{(t-u)A_p} E_p dL_u, \quad s < t.$$

(54)

If, in addition to the above, condition (41) is satisfied, then there exists a unique stationary solution to (54) which is the associated non-causal MCARMA process. However, in contrast to the causal case, a non-causal MCARMA process does not admit the representation (54). Indeed, following the proof of Proposition 2.9, we see that the stationary representation of $(\vec{v}(X_t))_{t \in \mathbb{R}}$ in (42), and hence also of $(X_t)_{t \in \mathbb{R}}$, is considerably more complicated due to the kernel $g_2$ in decomposition (83), which in the non-causal case does not vanish. We therefore assess the positivity of non-stable output processes $(X_t)_{t \in \mathbb{R}}$ given by (54) as follows: We look for conditions on the model parameters $A_1, \ldots, A_p$ and $C_0, \ldots, C_q$ such that for every $s < t$ the process $(X_t)_{t \geq s}$ is $C$-valued whenever $Z_s \in C^p$.

Remark 3.11. i) In the deterministic control literature a similar type of positivity is often referred to as internal positivity, see, e.g. [22]: A state space model is called internal positive, if for every non-negative initial value $Z_s$ and every non-negative input process (in our case $(L_t)_{t \in \mathbb{R}}$) the output process $(X_t)_{t \geq s}$ assumes only positive values.

ii) Again we may assume that $Z_s$ is $F_s$-measurable, such that the process $(X_t)_{t \geq s}$ is adapted to the natural filtration. In this case, however, $(X_t)_{t \geq s}$ is not necessarily an MCARMA process anymore. Moreover, even if a unique stationary solution (MCARMA process) exists, conditions such that $(X_t)_{t \geq s}$ is positive for all positive initial values $Z_s \in C^p$ are in general neither sufficient nor necessary for the positivity of the MCARMA process.

Let $J \subseteq \{1, \ldots, p\}$ and denote by $C^{J,p}$ the wedge\(^1\) in $(M_{n,m})^p$ consisting of the cone $C$ in the $J$-th Cartesian coordinates (and $M_{n,m}$ otherwise). In particular for $J = \{1, \ldots, p\}$ we have $C^{J,p} = C^p$ and $C^{(1),p} = C \times M_{n,m} \times \cdots \times M_{n,m}$. To ensure the positivity of the term $C_q e^{(t-s)A_p} Z_s$ in (54) for every $Z_s \in C^p$, we see that for all $t > s$ the operator $C_q e^{(t-s)A_p}$ must map $C^p$ into $C$. Note that in the case of a MCAR process, i.e. where $C_0 = [1, 0, \ldots, 0]$, one would think that $e^{sA_p}$ has to map $C^p$ onto $C^{(1),p}$ only, as the output operator $C_q$ projects down onto the first block matrix component anyway. However, we have the following:

Lemma 3.12. Let $(A_p, E_p, C_q, L)$ be as above and let $(X_t)_{t \geq s}$ be the associated non-stable output process in (54). Then the following holds true:

i) If $C_j \in \pi(C)$ for all $j = 1, \ldots, q$ and $A_p$ is quasi-positive with respect to $C^p$, then $(X_t)_{t \geq s}$ is $C$-valued for all initial values $Z_s \in C^p$.

ii) If $A_p$ is quasi-positive with respect to $C_J^p$ then $J = \{1, 2, \ldots, p\}$.

iii) If $\mathbb{E} |L_s| = 0$ and $C = C_j^{-1}(C)$ for all $j = 0, \ldots, q$. Then the converse of part i) is also true.

\(^1\)A wedge $W \subseteq M_{n,m}$ is a convex cone without the property $W \cap (-W) = \{0_{n,m}\}$. 

Proof. We first prove part i). Suppose $A_p$ is quasi-positive with respect to $C_p$, then by definition $e^{tA_p}z \in C_p$ for every $z \in C_p$ and $t \geq 0$. Hence for $Z_s \in C_p$ we have $e^{(t-s)A_p}Z_s \in C_p$ and by definition of a $C$-positive Lévy process, we have $E_p(L_s - L_{s'} \in C_p$ for all $s > s'$. This implies that $\int_s^t e^{(t-s)A_p}E_p dL_s \in C_p$ for all $t \geq s$. If moreover $C_J \in \pi(C)$ for all $j = 1, \ldots, q$ then $C_q(e^{(t-s)A_p}Z_s) \in C$ and $C_q(\int_s^t e^{(t-s)A_p}E_p dL_s) \in C$, which according to $\text{(54)}$ yields $X_t \in C$ for all $t \geq s$.

Next we show that the particular form of $A_p$ implies that it can only be quasi-positive on $(M_{n,m})^p$ with respect to the wedge $C^{J,p}$ if $J = \{1, 2, \ldots, p\}$. Recall that quasi-positivity of $A_p$ with respect to $C^{J,p}$ can be equivalently characterized by the property that $\langle \langle A_p x, u \rangle \rangle_p \geq 0$, whenever $\langle \langle x, u \rangle \rangle_p = 0$ with $x, u \in C^{J,p}$ where we denote the inner-product on $(M_{n,m})^p$ by $\langle \langle \cdot, \cdot \rangle \rangle_p$. Therefore, let $x, u \in C^{J,p}$ and note that

$$\langle \langle A_p x, u \rangle \rangle_p = \langle \langle (x_1, x_3, \ldots, x_p, \sum_{i=1}^p A_i(x_i))^T, u \rangle \rangle_p = \sum_{j=1}^{p-1} \langle x_{j+1}, u_j \rangle_{n,m} + \sum_{i=1}^p \langle A_{p+1-j}(x_i), u_p \rangle_{n,m}. \quad (55)$$

We set $J^c = \{1, 2, \ldots, p\} \setminus J$ and suppose that $J^c$ is not empty. If $p \in J$, let $x, u \in C^{J,p}$ be such that $u_i = 0$ for every $j \in J$ and $(x_i, u_i)_{n,m} = 0$ for $i \in J^c$, then clearly $\langle \langle x, u \rangle \rangle_p = 0$, but from $(55)$ we see that $\langle \langle A_p x, u \rangle \rangle_p = \sum_{j \in J^c} \langle x_{j+1}, u_j \rangle_{n,m}$ which can be negative since we only assumed that $u_j \in M_{n,m}$ and $x_{j+1}$ can be chosen arbitrary as long as $x_{j+1}, u_j$ is nonzero in $J^c$. The case $p \in J^c$ then follows by a similar argument and we see that $J^c$ has to be empty, otherwise $A_p$ can not be quasi-positive with respect to $C^{J,p}$, which implies that we must have $J = \{1, 2, \ldots, p\}$.

Lastly, for the necessary direction in part iii) we assume that $\mathbb{E}[L_1] = 0$ and suppose that $X_t \in C$ for all $t \geq s$ and $Z_s \in C_p$. Then in particular $X_s = e^{(t-s)A_p}Z_s = \sum_{i=1}^{q+1} C_i Z_s^{(i)} \in C$ for all $Z_s = (Z_s^{(1)}, Z_s^{(2)}, \ldots, Z_s^{(p)}) \in C_p$. Thus, let $z \in C$ arbitrary and set $Z_1 = e_j \otimes z$ for $j = 1, \ldots, q$ and note that $Z_1 \in C_p$. By assumption we must have $X_t = C_{J^c-1}(z) \in C$ for all $j = 1, \ldots, q$ and since $z \in C$ was arbitrary, we conclude that $C_{J^c-1} \in \pi(C)$ for all $j = 1, \ldots, q$. Next, we show that $A_p$ must be quasi-positive with respect to the cone $C_p$. For this let $Z_s \in C_p$ be deterministic, but arbitrary. Since $\mathbb{E}[L_1] = 0$, there exists a $\omega \in \Omega$ and $t' > s$ such that $L_t = 0$ for all $s < t < t'$ and hence $\int_s^t e^{(t-s)A_p}E_p dL_u(\omega) = 0$ for all $s < t < t'$. Therefore by $(54)$ we see that $X_t(\omega) = e^{(t-s)A_p}Z_s(\omega) \in C$ for all $s < t < t'$ and every deterministic $Z_s \in C_p$. Since $C_j \in \pi(C)$ and by assumption $C = C_{J^{-1}}(C)$ for all $j = 0, \ldots, q$, we see that for every $s < t < t'$ there exist a $J \subseteq \{1, \ldots, p\}$ such that $\{1, \ldots, q\} \subseteq J$ and $e^{(t-s)A_p}Z_s \in C^{J,p}$. But since this holds for every $s < t < t'$, we find in every neighborhood of this interval infinitely many time points $(t_j)_{j \in \mathbb{N}}$ such that $e^{(t_j-s)A_p}Z_s \in C^{J,p}$ holds for the same $J$. From this we conclude that already $e^{tA_p}(C^{J,p}) \subseteq C^{J,p}$ must hold for all $t \geq 0$ and some set $J \subseteq \{1, \ldots, p\}$ (first prove this for rational time points and then extend to irrational by continuity). This, however, by definition means that $A_p$ is quasi-positive with respect to $C^{J,p}$ and thus we conclude from part ii) that already $J = \{1, 2, \ldots, p\}$ must hold, i.e. that $A_p$ must be quasi-positive with respect to $C_p$.\hfill $\square$

**Remark 3.13.** The condition $\mathbb{E}[L_1] = 0$ in Lemma 3.12 iii) implies that the drift of the Lévy process $L$ vanishes, i.e. $\gamma L + \int_{M_{n,m}\cap \{\xi: ||\xi||_{n,m} \geq 1\}} \xi \nu_L(d\xi) = 0$.

The following theorem is our main result on the positivity of non-stable output processes on $M_{n,m}$. Under an extra condition on the stationary distribution of the
non-stable output process, it also provides a sufficient condition for the positivity of non-causal MCARMA processes.

**Theorem 3.14.** Let \( p \in \mathbb{N} \) and \( q \in \mathbb{N}_0 \) with \( q < p \). For \( s < t \in \mathbb{R} \), let \( (X_t)_{t \geq s} \) be the non-stable output process in (54), associated with \((A_p, E_p, C_q, L)\) such that \( \tau(A_p) \geq 0 \). Then the following holds true:

i) If \( A_1 \in \mathcal{L}(\mathbb{M}_{n,m}) \) is quasi-positive and \( A_i, C_i \in \pi(C) \) for \( i = 2, \ldots, p \) and \( j = 0, \ldots, q \), then \((X_t)_{t \geq s}\) is \( C \)-valued for every initial value \( Z_s \in C^p \).

ii) If \( E[L_1] = 0 \) and \( C = C_j^{-1}(C) \) for all \( j = 0, \ldots, q \). Then \((X_t)_{t \geq s}\) is \( C \)-valued for every initial value \( Z_s \in C^p \) if and only if \( A_1 \in \mathcal{L}(\mathbb{M}_{n,m}) \) is quasi-positive and \( A_i, C_i \in \pi(C) \) for \( i = 2, \ldots, p \) and \( j = 0, \ldots, q \).

iii) If condition (41) is satisfied and the stationary distribution of \((Z_t)_{t \in \mathbb{R}}\) is quasi-positive with respect to \( C^p \), then the conditions in part i) are sufficient for the non-causal MCARMA process \((X_t)_{t \in \mathbb{R}}\) to be \( C \)-valued.

**Proof.** We begin with the proof of part i). By Lemma 3.12 i) it is enough to show that \( A_i, \pi(C) \) and \( A_1 \) is quasi-positive with respect to \( C \) implies that \( A_p \) is quasi-positive with respect to \( C_p \). Indeed, let \( x = (x_1, x_2, \ldots, x_p)^T \in C^p \) and \( u = (u_1, u_2, \ldots, u_p)^T \in C^p \) such that \( \langle (x, u) \rangle_p = 0 \) we then want to show that \( \langle (A_p x, u) \rangle_p \geq 0 \). As before we have

\[
\langle (A_p x, u) \rangle = \sum_{j=1}^{p-1} \langle x_{j+1}, u_j \rangle_{n,m} + \sum_{i=1}^{p} \langle A_{p-i+1}(x_i), u_i \rangle_{n,m}.
\]

and since \( x_{j+1}, u_j \in C \) for all \( j = 1, \ldots, p-1 \), we see that \( \langle x_{j+1}, u_j \rangle_{n,m} \geq 0 \) and hence the first sum in (56) is non-negative. Moreover, we see that \( \langle A_{p-i+1}(x_i), u_i \rangle \geq 0 \) for \( i = 1, \ldots, p-1 \) by assumption that \( A_1(C) \subseteq C \) for \( i = 2, \ldots, p \) and thus the remaining term of the second sum is \( \langle A_1(x_p), u_p \rangle_{n,m} \). By assumption we have \( \langle (x, u) \rangle = \sum_{j=1}^{p} \langle x_j, u_j \rangle_{n,m} = 0 \), and in particular \( \langle x_p, u_p \rangle_{n,m} = 0 \), which by the quasi-positivity of \( A_1 \) implies \( \langle A_1(x_p), u_p \rangle_{n,m} \geq 0 \). This implies that \( \langle (A_p x, u) \rangle_p \geq 0 \), which proves the quasi-positivity of \( A_p \) with respect to \( C_p \).

The second assertion is a consequence of Lemma 3.12 iii) and it is left to prove that the quasi-positivity of \( A_p \) with respect to \( C^p \) implies that \( A_1 \) is quasi-positive and \( A_i, \pi(C) \) for \( i = 2, \ldots, p \). For this, suppose that \( A_p \) is quasi-positive with respect to \( C^p \), then for every \( x, u \in C^p \) with \( \langle (x, u) \rangle_p = 0 \) we find that the term in (56) is non-negative. If we let \( u_p, x_1 \in C \) be such that \( \langle x_1, u_p \rangle_{n,m} = 0 \) and if we set \( u = (0, \ldots, 0, u_p)^T \) and \( x = (x_1, 0, \ldots, 0)^T \), then we observe that \( \langle (x, u) \rangle_p = 0 \) and (56) reduces to \( \langle A_1(x_p), u_p \rangle_{n,m} \geq 0 \). Since \( x_1 \) and \( u_p \) satisfy \( \langle x_1, u_p \rangle_{n,m} = 0 \) but are otherwise arbitrary, it follows that \( A_1 \) is quasi-positive. Moreover, we see that \( A_i, \pi(C) \) for \( i = 2, \ldots, p \) follows by analogous arguments. The third assertion follows immediately from part i) and formula (54), since by assumption the stationary state is in \( C^p \).

\[ \square \]

**Remark 3.15.** Note that the positivity condition in Theorem 3.14 i) only applies in the non-stable case, i.e. for \( \tau(A_p) \geq 0 \). Indeed, suppose that \( A_p \) is quasi-positive with respect to \( C^p \) and \( \tau(A_p) < 0 \), then by an application of Lemma 3.2 to the cone \( C^p \) implies that the inverse \( A_p^{-1} \) exists and must map positive vectors into
negatives, i.e. $A_p^{-1}C_p \subseteq -C_p$. However, the inverse of $A_p$ is explicitly known as

$$A_p^{-1} = \begin{bmatrix} A_{p-1}^{-1} & A_{p-2}^{-1} & \cdots & A_1^{-1} \\ I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix},$$

(57)

and obviously never satisfies $A_p^{-1}C_p \subseteq -C_p$. Conversely, note that if $A_i$ for $i = 1, \ldots, p$ satisfies the condition on the transition matrix for the case of $q < p$ such that $(\sigma(A_i))_{p \times p}$ is $\sigma(A_i)$-valued non-stable output processes. In the following two corollaries we concretize the positivity criteria in Theorem 3.16.

**Corollary 3.16.** Assume that $m = 1$ and $n = d$ for some $d \in \mathbb{N}$ and let $p \in \mathbb{N}$, $q \in \mathbb{N}_0$ such that $q < p$. Moreover, let $(L_t)_{t \geq 0}$ be an $\mathbb{R}^d_+$-valued diffusion process and $s \leq t$ in $\mathbb{R}$ we denote by $(X_t)_{t \geq s}$ the $\mathbb{R}^d_+$-valued non-stable output process associated with $(A_p, E_p, C_q, L)$ such that $\tau(A_p) \geq 0$ and $(A_i)_{i=1,\ldots,p}$ and $(C_j)_{j=0,\ldots,q}$ satisfy the following conditions:

i) For every $i = 2, \ldots, p$ we have $A_i = (a^{(i)}_{k,n})_{1 \leq k,n \leq d}$ such that $a^{(i)}_{k,n} \geq 0$ for all $1 \leq k, n \leq d$;

ii) $A_1 = (a^{(1)}_{k,n})_{1 \leq k,n \leq d}$ is such that $a^{(1)}_{k,n} \geq 0$ for all $1 \leq k, n \leq d$ with $k \neq n$;

iii) For every $j = 0, \ldots, q$ we have $C_j = (c^{(j)}_{k,n})_{1 \leq k,n \leq d}$ such that $c^{(j)}_{k,n} \geq 0$ for all $1 \leq k, n \leq d$.

Then $(X_t)_{t \geq s}$ is $\mathbb{R}^d_+$-valued.

Following Corollary 3.16 the condition on the transition matrix $A_p \in \mathbb{M}_{pd}$ can be visualized as follows:

$$A_p^{\text{vec}} = \begin{pmatrix} 0_d & I_d & 0_d & \cdots & 0_d \\ 0_d & I_d & \cdots & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \cdots & \cdots & 0_d & I_d \\ a^{(1)}_{1,1} \cdots a^{(1)}_{1,d} & a^{(2)}_{1,1} \cdots a^{(2)}_{1,d} & \cdots & \cdots & a^{(p)}_{1,1} \cdots a^{(p)}_{1,d} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a^{(1)}_{d,1} \cdots a^{(1)}_{d,d} & a^{(2)}_{d,1} \cdots a^{(2)}_{d,d} & \cdots & \cdots & a^{(p)}_{d,1} \cdots a^{(p)}_{d,d} \\ a^{(1)}_{i,j} \geq 0, \forall i,j & a^{(2)}_{i,j} \geq 0, \forall i,j & \cdots & \cdots & a^{(p)}_{i,j} \geq 0, \forall i,j \text{ s.t. } i \neq j \end{pmatrix}$$

We obtain an analogous result for $\mathbb{S}^d_+$-valued non-stable output processes:

**Corollary 3.17.** Assume that $m = n = d$ for some $d \in \mathbb{N}$ and let $p \in \mathbb{N}$, $q \in \mathbb{N}_0$ such that $q < p$. For $s < t$ in $\mathbb{R}$ we denote by $(X_t)_{t \geq s}$ the $\mathbb{M}_d$-valued non-stable output process associated with $(A_p, E_p, C_q, L)$ such that $\tau(A_p) \geq 0$ and $(A_i)_{i=1,\ldots,p}$ and $(C_j)_{j=0,\ldots,q}$ satisfy the following conditions:

i) For every $i = 2, \ldots, p$ there exists $a_i \in \mathbb{M}_d$ such that $A_i(x) = a_i x a_i^*$ for every $x \in \mathbb{S}_d$;
ii) There exists an $a_1 \in \mathbb{M}_d$ such that $A_1(x) = a_1 x + x a_1^*$ for every $x \in \mathbb{S}_d$;

iii) For every $j = 0, \ldots, q$ there exists $c_j \in \mathbb{M}_d$ such that $C_j(x) = c_j x c_j^*$ for every $x \in \mathbb{S}_d$.

Then $(X_t)_{t \geq 0}$ is $\mathbb{S}_d^+$-valued. Moreover, the state space representation of $(\text{vec}(X_t))_{t \geq 0}$ in (22)-(23) holds with state transition operator given by

$$
A_p^\text{vec} := 
\begin{bmatrix}
0_{d^2} & I_{d^2} & 0_{d^2} & \cdots & 0_{d^2} \\
0_{d^2} & 0_{d^2} & I_{d^2} & \cdots & 0_{d^2} \\
& & & \ddots & \vdots \\
& & & \ddots & 0_{d^2} \\
a_p \otimes a_p & a_{p-1} \otimes a_{p-1} & \cdots & 0_{d^2} & I_d \otimes a_1 + a_1 \otimes I_d
\end{bmatrix},
$$

the output operator is $C_q^\text{vec} = [c_0 \otimes c_0, c_1 \otimes c_1, \ldots, c_q \otimes c_q, 0_{d^2}, \ldots, 0_{d^2}]$ and the input operator is $E_p^\text{vec} = e_p \otimes I_d$.

Proof. This follows from Theorem 3.14 and the fact that maps of the form $x \mapsto axa^*$ for $a \in \mathbb{M}_d$ are in $\pi(\mathbb{S}_d^+)$ and maps of the form $x \mapsto ax + xa^*$ are quasi-positive. Moreover, note that

$$
A_1^\text{vec}(x) = (I_d \otimes a_1 + a_1 \otimes I_d) \text{vec}(x)
$$

and $A_i^\text{vec}(x) = a_i \otimes a_i \text{vec}(x)$ for $i = 2, \ldots, p$ and analogously for the operators $C_j$ for $j = 0, \ldots, q$.

4. MCARMA based stochastic covariance models

In this section we study multivariate stochastic volatility models based on symmetric and positive semi-definite MCARMA processes. We begin this section with a brief introduction to general stochastic covariance modeling and we recall some results on the second order structure of the return and covariance processes in Section 4.1. Subsequently, in Section 4.2, we discuss the multivariate Barndorff-Nielsen and Shepard (BNS) volatility model from [37] as a particular example for an MCAR based stochastic covariance model of order one. We motivate the use of higher-order MCARMA based covariance models to capture certain short-lag memory effects in realized variance and cross-covariance time-series. In Section 4.3, we demonstrate the gained flexibility in the second order structure when using higher-order MCARMA based stochastic covariance models through an exemplary analysis of positive semi-definite well-balanced Ornstein-Uhlenbeck processes.

4.1. Stochastic covariance models and their second order structure. Let $d \in \mathbb{N}$. We call an $\mathbb{R}_d \times \mathbb{S}_d^+$-valued process $(Y_t, X_t)_{t \geq 0}$ a stochastic covariance model on $\mathbb{R}_d$, whenever it consists of a $d$-dimensional logarithmic asset price process $(Y_t)_{t \geq 0}$ given by a stochastic differential equation of the form

$$
\begin{cases}
\text{d}Y_t = (\alpha + X_t \beta) \text{d}t + X_t^{1/2} \text{d}W_t, & t > 0, \\
Y_0 = y \in \mathbb{R}_d,
\end{cases}
$$

(58)

where $\alpha \in \mathbb{R}_d$ is the drift, $\beta \in \mathbb{R}_d$ the risk-premia, $(W_t)_{t \geq 0}$ denotes a $\mathbb{R}_d$-valued (standard) Brownian motion and the stochastic process $(X_t)_{t \geq 0}$ is called the spot covariance process, see also [9, 37]. The spot covariance process $(X_t)_{t \geq 0}$ is assumed to be integrable with respect to $(W_t)_{t \geq 0}$ and $\mathbb{S}_d^+$-valued such that for all $t \geq 0$ the matrix square-root $X_t^{1/2}$ exists and the stochastic integral in (58) is well-defined. The main challenge in stochastic covariance modeling is the appropriate specification of the spot covariance process $(X_t)_{t \geq 0}$ such that the model $(Y_t, X_t)_{t \geq 0}$
represents the stylized facts of financial data, while being sufficiently tractable for, e.g. simulations, statistical inference or option pricing. In contrast to the logarithmic price process \((Y_t)_{t \geq 0}\), the spot covariance process \((X_t)_{t \geq 0}\) is not directly observable in markets. However, it can be measured indirectly from the realized covariance of the (squared) return process as follows: First, assume that \((X_t)_{t \geq 0}\) is square-integrable and stationary and for any real positive number \(\Delta\) we define the discrete-time process \((Y^\Delta_n)_{n \in \mathbb{N}}\) by

\[
Y^\Delta_n := Y_{n\Delta} - Y_{(n-1)\Delta} = \int_{(n-1)\Delta}^{n\Delta} (\mu + X_t\beta) dt + \int_{(n-1)\Delta}^{n\Delta} X_t^{1/2} dW_t, \quad n \in \mathbb{N}.
\]

The process \((Y^\Delta_n)_{n \in \mathbb{N}}\) is the sequence of logarithmic returns over the intervals \([(n-1)\Delta, n\Delta]\) and it can be seen that for every \(n \in \mathbb{N}\) the random variable \(Y^\Delta_n\) given \(X^\Delta\) is normal distributed. More precisely, for every \(n \in \mathbb{N}\) we have

\[
Y^\Delta_n | X^\Delta_n \sim \mathcal{N}(\mu\Delta + X^\Delta_n \beta, X^\Delta_n) \quad \text{with} \quad X^\Delta_n := \int_{(n-1)\Delta}^{n\Delta} X_t dt = X_{n\Delta}^+ - X_{(n-1)\Delta}^+,
\]

where \(X_t^+ := \int_0^t X_s ds\) for \(t \geq 0\), see also [6]. We define the auto-covariance function \(\text{acov}_X : \mathbb{R}_{+} \to \mathcal{L}(\mathbb{M}_d)\) of the process \((X_t)_{t \geq 0}\) at \(t \geq 0\) by \(\text{acov}_X(h) := \text{Cov}[X_{t+h}, X_t]\) where \(h \geq 0\); Analogously, we define \(\text{acov}_X^\Delta\) and \(\text{acov}_Y^\Delta\) for the processes \((Y^\Delta_n)_{n \in \mathbb{N}}\) and \((X^\Delta_n)_{n \in \mathbb{N}}\), respectively, in which case we restrict to \(n, h \in \mathbb{N}\). Following [37], we observe that the second order structure of the spot covariance process \((X^\Delta_n)_{n \geq 0}\), respectively its discrete difference process \((X^\Delta_n)_{n \in \mathbb{N}}\), is inherited by the second order structure of the squared logarithmic return process \((Y^\Delta_n)^\top)_{n \in \mathbb{N}}\) such that

\[
\text{acov}_{Y^\Delta_n}^\top(h) = \text{acov}_{X^\Delta_n}^\top(h), \quad \text{for all} \quad h, n \in \mathbb{N}.
\]

Moreover, we recall from [37, Theorem 3.2] that for general square-integrable stationary spot covariance processes \((X_t)_{t \geq 0}\) we have

\[
\text{acov}_{X^\Delta_n}^\top(h) = r^{++}(h\Delta + \Delta) - 2r^{++}(h\Delta) + r^{++}(h\Delta - \Delta), \quad \forall h, n \in \mathbb{N},
\]

where \(r^{++} : \mathbb{R}_{+} \to \mathcal{L}(\mathbb{M}_d)\) is given by

\[
r^{++}(t) := \int_0^t \int_0^s \text{acov}_{X_n}(u) du ds, \quad t \geq 0.
\]

As we noted before, a prominent feature observed in many realized (co)variance time-series is the memory effect, which means that the observed auto-covariances of the (squared) returns exhibits sub-exponential decay or even non-monotone configurations, see, e.g. [14, 15] and the reference therein. To capture the memory effect in the short-time lags, we propose to model the spot covariance process \((X_t)_{t \geq 0}\) in (58) by \(\mathbb{S}^+_d\)-valued higher-order MCARMA processes as they were introduced in this work. In the univariate case the idea to model the spot variance process in stochastic volatility models by positive (higher-order) CARMA processes goes back to [14]. To demonstrate the potential of MCARMA based stochastic covariance models we derive the auto-covariance in (59) and (60) for the following two cases: First, for the multivariate BNS model in Section 4.2, which was studied in [37]. Secondly, for a stochastic covariance model based on a positive semi-definite well-balanced OU processes introduced in Section 4.3.

4.2. Positive semi-definite Ornstein-Uhlenbeck type processes. Stochastic covariance models based on positive semi-definite OU type processes were studied extensively in [7, 38, 37]. The class of matrix-valued OU processes is included in the class of matrix-valued MCARMA processes as they form the class of MCAR processes of order one, see Definition 2.10. We show that the positivity criteria
in Theorem 3.5 in case of a causal MCAR process of order one coincides with the well-known (sufficient) criteria for symmetric and positive semi-definite OU processes in [7]. Subsequently, we then recall the second order structure of OU based stochastic covariance models from [37]. Let \( p = 1, q = 0 \) and set \( A_1 = -A \) for some \( A \in \mathcal{L}(\mathbb{M}_d) \) such that (41) is satisfied and let \( C_q = \mathbf{I} \). Moreover, let \( L \) be a two-sided Lévy process on \( \mathbb{M}_d \) and assume that \( \mathbb{E}[\log(||L||_d)] < \infty \). We denote the MCAR process associated with the state space representation \((A_1, I, I, L)\) by \((X_t)_{t \in \mathbb{R}}\). This process is an Ornstein-Uhlenbeck type process on \( \mathbb{M}_d \) and has the following representation:

\[
X_t = e^{-(t-s)A}Z_s + \int_s^t e^{-(t-u)A}dL_u, \quad s < t \in \mathbb{R}.
\]

If we further assume that \( \tau(-A) < 0 \), then \( X_t = \int_{-\infty}^t e^{-(t-s)A}dL_s \) is the unique stationary OU process adapted to the natural filtration of \((L_t)_{t \in \mathbb{R}}\). It follows from Theorem 3.5 ii), that whenever \((L_t)_{t \in \mathbb{R}}\) is \( S^+_{d,\tau} \)-increasing, a sufficient condition for \((X_t)_{t \in \mathbb{R}}\) to be \( S^+_{d,\tau} \)-valued is the quasi-positivity of the operator \(-A\) with respect to \( S^+_{d,\tau} \). By recalling the definition of quasi-positivity this means \( e^{-tA}(S^+_{d,\tau}) \subseteq S^+_{d,\tau} \) for all \( t \geq 0 \) and we see that this criteria coincides with the usual sufficient condition for OU processes to be symmetric and positive semi-definite, see [7, Proposition 4.1].

In the following we recall some statistical properties of \( S^+_{d,\tau} \)-valued stationary OU process that follow from Propositions 2.4 and 2.7 and [7, Proposition 4.7]: For all \( t \geq 0 \) the mean of \( X_t \) is given by

\[
\mathbb{E}[X_t] = A^{-1}\mu_L = A^{-1} \begin{pmatrix} \gamma_L + \int_{S^+_{d,\tau}} \tau(||\xi||_d > 1) \xi \nu^L(d\xi) \end{pmatrix}
\]

and the auto-covariance of \((X_t)_{t \geq 0}\) at \( t \geq 0 \) is given by

\[
\text{Cov}[\text{vec}(X_t), \text{vec}(X_{t+h})] = e^{-hA^{\text{vec}}}D^{-1}Q^{\text{vec}}, \quad h \geq 0,
\]

where \( Q^{\text{vec}} = \text{vec} \circ Q \circ \text{vec}^{-1} \) and \( D \in \mathcal{L}(\mathbb{M}_d) \) is given by

\[
D(X) = A^{\text{vec}}X + X(A^{\text{vec}})^\top \quad \text{with} \quad A^{\text{vec}} = \text{vec} \circ A \circ \text{vec}^{-1}.
\]

Moreover, we have

\[
\int_0^t X_s \, ds = -A^{-1}(X_t - X_0 - L_t), \quad t \in \mathbb{R},
\]

and

\[
r^{++}(t) = \left((A^{\text{vec}})^{-2} \left(e^{-A^{\text{vec}}t} - \mathbb{I}_d^2\right) + (A^{\text{vec}})^{-1}t\right)D^{-1}Q^{\text{vec}}, \quad t \geq 0,
\]

which by (60) and (59) for every \( n \in \mathbb{N} \) yield

\[
\text{acov}_{Y^{\Delta}_{\mathcal{A}}(Y^{\Delta}_{\mathcal{A}})^\top}(h) = e^{-A^{\text{vec}}\Delta(h-1)}(A^{\text{vec}})^{-2}\left(\mathbb{I}_d^2 - e^{-A^{\text{vec}}\Delta}\right)^2D^{-1}Q^{\text{vec}}, \quad h \in \mathbb{N}.
\]

4.3. Positive semi-definite well-balanced Ornstein-Uhlenbeck processes.

In this section we introduce \textit{matrix-valued well-balanced OU processes} that extend the univariate well-balanced OU processes from [44] to \( S_{d,\tau} \)-valued processes. The authors in [44] proposed positive well-balanced OU processes as a model for the spot variance process in stochastic volatility models. Here, we extend this idea to the multivariate setting by studying \( S^+_{d,\tau} \)-valued well-balanced OU processes and show that this class is well-suited to model the spot covariance process in multivariate stochastic covariance models. Moreover, we show that stochastic covariance models based on well-balanced OU processes exhibit auto-covariance functions for the squared logarithmic returns \( Y^{\Delta}_{\mathcal{A}} \) that have slower decay compared to the multivariate BNS model.
Let \( A \in \mathcal{L}(S_d) \) with \( \tau(-A) < 0 \) and define \( A_2 \in \mathcal{L}((M_d)^2), E_2 \in \mathcal{L}(M_d,(M_d)^2) \) and \( C_0 \in \mathcal{L}(M_d^2,M_d) \) by

\[
A_2 := \begin{bmatrix} 0 & 1 \\ A^2 & 0 \end{bmatrix}, \quad E_2 := \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad C_0 := [-2A,0]. \tag{68}
\]

Moreover, let \((L_t)_{t \in \mathbb{R}}\) be a square-integrable \( S_d^+ \)-increasing Lévy process with expectation \( \mu_L \in S_d^+ \) and covariance operator \( Q \in \mathcal{L}(S_d) \). Let \((X_t)_{t \geq 0}\) denote the output process of the state space model associated with \((A_2, E_2, C_0, L)\) and initial value \(Z_0\), where \(Z_0\) is a random element in \((S_d)^2\) (not necessarily \(F_0\)-measurable).

The output process \((X_t)_{t \geq 0}\) is given by

\[
X_t = C_0 e^{tA_2}Z_0 + \int_0^t C_0 e^{(t-s)A_2}E_2 \, dL_s, \quad t \geq 0, \tag{69}
\]

and we note that since \(\sigma(A_2) = \sigma(A^2) \subseteq \mathbb{R}^+ \setminus \{0\} + i\mathbb{R}\) the process \((X_t)_{t \geq 0}\) is non-stable, but (41) is satisfied and thus following Proposition 2.9, there exist a unique stationary solution to (69).

In the following proposition we define positive semi-definite well-balanced OU processes as the unique stationary and positive semi-definite solution to (69). Moreover, we specify the stationary distribution and present sufficient positivity conditions for \((X_t)_{t \geq 0}\).

**Proposition 4.1.** Let \((A_2, E_2, C_0, L)\) be as above and in addition assume that \(-A\) is quasi-positive. Let \((X_t)_{t \geq 0}\) be as in (69) with \(Z_0 = (Z_0^{(1)}, Z_0^{(2)})\) given by \(Z_0^{(1)} = -A^{-1}\left(\frac{\pi + \pi_1}{2}\right)\) and \(Z_0^{(2)} = \frac{\pi_1 - \pi_2}{2}\) where \(\pi_1 := \int_0^\infty e^{sA} \, dL_s\) and \(\pi_2 := \int_0^\infty e^{-sA} \, dL_s\).

Then \((X_t)_{t \geq 0}\) is stationary, \(S_d^+\)-valued and can be represented as

\[
X_t = \int_{-\infty}^t e^{-(t-s)A} \, dL_s + \int_t^\infty e^{-(s-t)A} \, dL_s, \quad t \geq 0. \tag{70}
\]

Moreover, for all \(t \geq 0\) we have

\[
\mathbb{E}[X_t] = 2A^{-1}\mu_L, \tag{71}
\]

and the auto-covariance of \((X_t)_{t \geq 0}\) at \(t \geq 0\) is given by

\[
\text{Cov}[\text{vec}(X_{t+h}), \text{vec}(X_t)] = e^{-hA^\text{vec}} \left(e^{h\hat{D}} - I\right)\hat{D}^{-1}Q^\text{vec} + 2e^{-hA^\text{vec}}D^{-1}Q^\text{vec}, \quad h \geq 0, \tag{72}
\]

where \(\hat{D} \in \mathcal{L}(S_d)\) is given by \(\hat{D}(X) = A^\text{vec}X - X(A^\text{vec})^T\), \(D\) is as in (64) and \(A^\text{vec} := \text{vec} \circ A \circ \text{vec}^{-1}\). We call the process \((X_t)_{t \geq 0}\), given by (70), a positive semi-definite well-balanced OU process.

**Proof.** From the particular anti-diagonal form of \(A_2\) we see that for every \(k \in \mathbb{N}\) the following holds true:

\[
A_2^{2k} = \begin{bmatrix} (A^2)^k & 0 \\ 0 & (A^2)^k \end{bmatrix} \quad \text{and} \quad A_2^{2k+1} = \begin{bmatrix} 0 & (A^2)^k \\ (A^2)^{k+1} & 0 \end{bmatrix},
\]

which gives

\[
e^{tA_2} = \sum_{k=0}^\infty \frac{(tA_2)^{2k}}{(2k)!} + \sum_{k=0}^\infty \frac{(tA_2)^{2k+1}}{(2k+1)!} = \begin{bmatrix} \cosh(tA) & \sinh(tA)A^{-1} \\ \sinh(tA)A & \cosh(tA) \end{bmatrix}, \quad t \geq 0,
\]
where \( \cosh(tA) \equiv \sum_{k=0}^{\infty} \frac{(tA)^{2k}}{(2k)!} \) and \( \sinh(tA) \equiv \sum_{k=0}^{\infty} \frac{(tA)^{2k+1}}{(2k+1)!} \). Note further, that \( \sinh(A) \), \( \cosh(A) \) and \( A \) all commute mutually. Hence by (69) we obtain

\[
X_t = [-2A, 0] \begin{pmatrix} \cosh(tA)Z_0^{(1)} + \sinh(tA)A^{-1}Z_0^{(2)} \\ \sinh(tA)A Z_0^{(1)} + \cosh(tA)Z_0^{(2)} \end{pmatrix} \\
+ \int_0^t [-2A, 0] \begin{pmatrix} \sinh((t-s)A)A^{-1} dL_s \\ \cosh((t-s)A) dL_s \end{pmatrix} \\
= -2A \cosh(tA)Z_0^{(1)} - 2 \sinh(A)Z_0^{(2)} - 2 \int_0^t \sinh((t-s)A) dL_s \\
= -A e^{tA}Z_0^{(1)} - A e^{-tA}Z_0^{(1)} + X_t^{(1)} - X_t^{(2)},
\]

where in the last line (73) we used that \( \cosh(tA) = \frac{1}{2}(\exp(tA) + \exp(-tA)) \) and \( \sinh(tA) = \frac{1}{2}(\exp(tA) - \exp(-tA)) \) and set \( X_t^{(1)} := e^{-tA}Z_0^{(2)} + \int_0^t e^{(t-s)A} dL_s \), for \( t \geq 0 \), as well as \( X_t^{(2)} := e^{tA}Z_0^{(2)} + \int_0^t e^{(t-s)A} dL_s \). Now, recall \( \pi_1 = \int_{-\infty}^0 e^{tA} dL_s \) and \( \pi_2 = \int_{-\infty}^\infty e^{-tA} dL_s \), respectively, and hence inserting the initial state \( Z_0^{(1)} = -A^{-1} \frac{1}{2}(\pi_1 + \pi_2) \) and \( Z_0^{(2)} = \frac{1}{2}(\pi_1 - \pi_2) \) into (73) yields

\[
X_t = e^{tA} \pi_2 + \int_0^t e^{-(t-s)A} dL_s + e^{-tA} \pi_1 - \int_0^t e^{(t-s)A} dL_s \\
= \int_{-\infty}^\infty e^{(t-s)A} dL_s + \int_0^t e^{-(t-s)A} dL_s + \int_{-\infty}^0 e^{-(t-s)A} dL_s - \int_0^t e^{(t-s)A} dL_s \\
= \int_{-\infty}^0 e^{-(t-s)A} dL_s + \int_0^\infty e^{-A(s-t)} dL_s,
\]

which proves representation (70). We set \( \bar{X}_t^{(1)} = \int_{-\infty}^\infty e^{-A(t-s)} dL_s \) and \( \bar{X}_t^{(2)} = \int_{-\infty}^\infty e^{-A(s-t)} dL_s \). By assumption \(-A\) is quasi-positive and hence it follows from the derivations in Section 4.2 that both \( \bar{X}_t^{(1)} \) and \( \bar{X}_t^{(2)} \) are positive semi-definite for all \( t \geq 0 \). Thus we proved that \( (X_t)_{t \geq 0} \) is stationary, positive semi-definite and possesses the representation (70). We continue with the computation of the expectation and auto-covariance. It is easy to see that for all \( t \geq 0 \) we have

\[
E[\bar{X}_t^{(1)}] = E[\bar{X}_t^{(2)}] = A^{-1} E[L],
\]

which implies (71). It is left to prove that the auto-covariance of \( (X_t)_{t \geq 0} \) at \( t \geq 0 \) satisfies (72). For this, we recall that for all \( t, h \geq 0 \) we have

\[
\text{Cov}[\text{vec}(X_{t+h}), \text{vec}(X_t)] = E[\text{vec}(X_{t+h}) \text{vec}(X_t)^\intercal] - E[\text{vec}(X_{t+h})] E[\text{vec}(X_t)^\intercal],
\]

where according to (71) and linearity of the expectation we see that \( E[\text{vec}(X_{t+h})] = 2(Avec)^{-1} E[\text{vec}(L_1)] \) and \( E[\text{vec}(X_t)^\intercal] = 2E[\text{vec}(L_1)^\intercal] (Avec)^{-1} \). Thus by (70), we are left with the terms

\[
E[\text{vec}(\bar{X}_{t+h}^{(i)}) \text{vec}(\bar{X}_t^{(j)})^\intercal], \quad i, j = 1, 2.
\]

Note that for \( i = j = 1 \) this term is the auto-covariance of the stationary OU type process \( (X_t^{(1)})_{t \geq 0} \) adjusted by the following outer-square of the expectation:

\[
E[\text{vec}^{(1)} X_{t+h}^1] E[\text{vec}^{(1)} X_t^1] = (Avec)^{-1} E[\text{vec}(L_1)] E[\text{vec}(L_1)^\intercal] (Avec)^{-1}. \]

Note that following (63), the auto-covariance of the process \( (\bar{X}_t^{(1)})_{t \geq 0} \) is given by

\[
\text{Cov}[\text{vec}^{(1)} X_{t+h}^1, \text{vec}^{(1)} X_t^1] = e^{-hAvec} D^{-1} Q_{vec}, \quad h \geq 0.
\]
Similarly, for \( i = j = 2 \) a straightforward computation shows that the auto-covariance of \( (\tilde{X}^{(2)}_t)_{t \geq 0} \) is the same as the auto-covariance of \( (\text{vec}(\tilde{X}^{(1)}_t))_{t \geq 0} \). For \( i = 2 \) and \( j = 1 \), we see that by the properties of the two-sided Lévy process we have

\[
\text{Cov} \left[ \text{vec}(\tilde{X}^{(2)}_{t+h}), \text{vec}(\tilde{X}^{(1)}_t) \right] = 0.
\]

Hence we are left with the last term in (75), that is \( i = 1 \) and \( j = 2 \). Note first that for every \( h \geq 0 \) we have

\[
\text{vec}(\tilde{X}^{(1)}_{t+h}) \text{vec}(\tilde{X}^{(2)}_t)^\tau = \int_{-\infty}^{t+h} e^{-(t+h-u)A^\text{vec}} \text{d vec}(L_u) \int_t^\infty \text{d vec}(L_s)^\tau e^{-(s-t)(A^\text{vec})^\tau}.
\]

Hence by using the independent increments property of \((L_t)_{t \in \mathbb{R}}\) and by a change of variable, we see that

\[
\mathbb{E} \left[ \text{vec}(\tilde{X}^{(1)}_{t+h}) \text{vec}(\tilde{X}^{(2)}_t)^\tau \right] = \int_{-\infty}^{h} \int_0^h e^{A^\text{vec}sQ^\text{vec}} e^{-s(A^\text{vec})^\tau} ds
\]

\[
= e^{-A^\text{vec}h} \int_0^h e^{A^\text{vec}sQ^\text{vec}} e^{-s(A^\text{vec})^\tau} ds
\]

\[
+ (A^\text{vec})^{-1} \mathbb{E} \left[ \text{vec}(L_1) \right] \mathbb{E} \left[ \text{vec}(L_1)^\tau \right] (A^\text{vec})^{-1}
\]

\[
= e^{-hA^\text{vec}} \int_0^h e^{s\tilde{D}Q^\text{vec}} ds
\]

\[
+ (A^\text{vec})^{-1} \mathbb{E} \left[ \text{vec}(L_1) \right] \mathbb{E} \left[ \text{vec}(L_1)^\tau \right] (A^\text{vec})^{-1}
\]

The integral in the last line can be computed as

\[
\int_0^h e^{s\tilde{D}Q^\text{vec}} ds = D^{-1}(e^{h\tilde{D}} - I)Q^\text{vec}.
\]

Hence by collecting all the terms in (75) and since \( D, \tilde{D} \) and \( \exp(A) \) mutually commute, we obtain (72). \( \square \)

**Remark 4.2.** Following our terminology from Definition 2.10, we see that positive semi-definite well-balanced OU processes are non-causal MCARMA processes of order \((2, 0)\). Proposition 4.1 shows that the positivity criteria for non-stable state space models from Theorem 3.14 is indeed not necessary. As noted before, this happens as for stationary processes (such as the well-balanced OU) it would suffice to ensure the positivity only for its stationary states and not, as in Theorem 3.14, for all positive initial values \( Z_0 \). Moreover, note that the stationary distribution does not have to be supported on the positive cone \((S^+_d)^p\) for the output process \((X_t)_{t \geq 0}\) to be positive.

### 4.4. Second order structure of positive semi-definite well-balanced OU based stochastic covariance models

In this section we study the second order structure of stochastic covariance models with spot covariance process modeled by a positive semi-definite well-balanced OU process. In particular, we compare the obtained auto-covariance of the square (logarithmic)-return process with the corresponding auto-covariance in (67) of the multivariate BNS model. In the next lemma we first compute the function \( r^{++} \) from (66) for the positive semi-definite well-balanced OU process:
Lemma 4.3. Let $\mathbf{A} \in \mathcal{L}(\mathbb{R}^d)$ be such that $-\mathbf{A}$ is quasi-positive and $\tau(-\mathbf{A}) < 0$ and denote by $(X_t)_{t \geq 0}$ the associated positive semi-definite well-balanced OU process as in Proposition 4.1. For every $t \geq 0$ we have:

$$r^{++}(t) = \left((A^{\text{vec}})^{-2}e^{-tA^{\text{vec}}} (\mathbf{G}^e(D-I)) - \mathcal{D}_1 t + \mathcal{D}_2\right) \hat{D}^{-1} Q^{\text{vec}} + 2r^{++}_{\text{OU}}(t), \quad (76)$$

where $r^{++}_{\text{OU}}(t)$ is as in (66), $\mathcal{G} \in \mathcal{L}(\mathbb{R}^d)$ is defined by $\mathcal{G}(x) := (A^{\text{vec}})^2 x ((A^{\text{vec}})^T)^{-2}$ and $\mathcal{D}_i \in \mathcal{L}(\mathbb{R}^d)$ for $i = 1, 2$ is given by $\mathcal{D}_i(\mathbf{x}) := (A^{\text{vec}})^{-1} x - x ((A^{\text{vec}})^T)^{-1}$.

Proof. Let $T \geq 0$, then by definition of $r^{++}$ in (61) we have to compute

$$r^{++}(t) = \int_0^t \int_0^s \text{acov}_{X_t}(u) \, du \, ds, \quad t \geq 0,$$

for $\text{acov}_{X_t}(u) = \text{Cov}[X_{t+h}, X_t]$ given by (72). By (63) we see that for all $u \geq 0$ the auto-covariance function $\text{acov}_{X_t}(u)$ is the sum $e^{-uA^{\text{vec}}} (e^{uD-I})D^{-1} Q^{\text{vec}}$ and two times the auto-covariance function of a classical OU type process. We thus obtain

$$r^{++}(t) = \int_0^t \int_0^s e^{-uA^{\text{vec}}} (e^{uD-I})D^{-1} Q^{\text{vec}} \, du \, ds + 2r^{++}_{\text{OU}}(t)$$

$$= \int_0^t \int_0^s -D^{-1} Q^{\text{vec}} ((A^{\text{vec}})^T)^{-1} (e^{-s(A^{\text{vec}})^T} - I_d) + (A^{\text{vec}})^{-1} (e^{-sA^{\text{vec}}} - I_d)D^{-1} Q^{\text{vec}} ds + 2r^{++}_{\text{OU}}(t)$$

$$= -D^{-1} Q^{\text{vec}} ((A^{\text{vec}})^T)^{-1} (e^{-t(A^{\text{vec}})^T} - I_d) - (A^{\text{vec}})^{-2} (e^{-tA^{\text{vec}}} - I_d)D^{-1} Q^{\text{vec}}$$

$$+ D^{-1} Q^{\text{vec}} ((A^{\text{vec}})^T)^{-1} t - (A^{\text{vec}})^{-1} D^{-1} Q^{\text{vec}} + 2r^{++}_{\text{OU}}(t)$$

$$= ((A^{\text{vec}})^{-2}e^{-tA^{\text{vec}}} (\mathbf{G}^e(D-I)) - \mathcal{D}_1 t + \mathcal{D}_2) \hat{D}^{-1} Q^{\text{vec}} + 2r^{++}_{\text{OU}}(t),$$

which proves (76). \(\square\)

From Lemma 4.3 and (60) we see that the auto-covariance function of the square (logarithmic)-returns (59) in the positive semi-definite well-balanced OU based stochastic covariance model is given by

$$\text{acov}_{Y^{\Delta}}(h) = e^{-A^{\text{vec}}\Delta(h-1)}(A^{\text{vec}})^{-2} (\mathbf{G}^e\Delta(h-1))D^{-1} I_d e^{-A^{\text{vec}}\Delta}D^{-1} Q^{\text{vec}}$$

$$+ 2 e^{-A^{\text{vec}}\Delta(h-1)}(A^{\text{vec}})^{-2} (I_d e^{-A^{\text{vec}}\Delta})^2 D^{-1} Q^{\text{vec}}, \quad h \in \mathbb{N}, \quad (77)$$

for every $n \in \mathbb{N}$ and $\Delta > 0$. Note that the term in the second line of (77) is simply two times the auto-covariance of the multivariate BNS model (compare with (67)). Thus, the interesting term is the first in (77). Indeed, assume for simplicity that $A^{\text{vec}}$ is symmetric. Then note that for every $\Delta(h-1) > 0$ we obtain the following expansion of the the non-linear term in the first line of (77):

$$(A^{\text{vec}})^{-2} (\mathbf{G}^e\Delta(h-1))D^{-1} = \Delta(h-1) + \frac{1}{2} D(h(h-1))^2 + O((\Delta(h-1))^3).$$

From this, we observe the following small-time asymptotics of the auto-covariance function $\text{acov}_{Y^{\Delta}}(h)$:

$$\text{acov}_{Y^{\Delta}}(h) \approx \Delta(h-1) e^{-A^{\text{vec}}\Delta(h-1)} Q^{\text{vec}} + O((\Delta(h-1))^2) e^{-A^{\text{vec}}\Delta(h-1)} Q^{\text{vec}},$$

for small $\Delta(h-1)$. In comparison with the auto-covariance function of the square (logarithmic)-returns in the multivariate BNS model, the positive semi-definite well-balanced OU based model allows for a slower decay and even for non-monotone configurations in the short-time lags. The attainable auto-covariance functions are also beyond the ones obtained from superposition of positive semi-definite OU processes, see [8]. This example demonstrates that the class of higher-order MCARMA based
stochastic covariance models indeed is a very flexible model class providing multi-
ple options to capture short-memory effects in observed financial and non-financial
data.

Appendix A. Proof of Proposition 2.9
The existence of \( \hat{Q}, \hat{P} \in M_{nm}(\mathbb{R}[\lambda]) \) such that (40) holds with \( \det(\hat{P}(\lambda)) = 0 \) if and only if \( \det(\hat{P}(\lambda)) = 0 \) follows immediately from [29, Lemma 6.3-8] and Lemma 2.8.

From (34) it then follows that \((\hat{P}, \hat{Q})\) is a left-matrix fraction description of the transfer function \( H(\lambda) = \tilde{E}_q^\text{vec}(\lambda I_{nm} - \tilde{A}_p^\text{vec})^{-1} \tilde{E}_p^\text{vec} \), i.e.

\[
\tilde{E}_q^\text{vec}(\lambda I_{nm} - \tilde{A}_p^\text{vec})^{-1} \tilde{E}_p^\text{vec} = \hat{P}(\lambda)^{-1} \hat{Q}(\lambda), \quad \forall \lambda \in \mathbb{C}.
\]  

(78)

We define the two kernels \( g_1, g_2 : \mathbb{R} \to \mathcal{L}(\mathbb{R}_{nm}, \mathbb{R}_{nmp}) \) by

\[
g_i(t) := \frac{1}{2\pi i} \int_{\rho_i} e^{\lambda t} (\lambda I_{nm} - \tilde{A}_p^\text{vec})^{-1} \tilde{E}_p^\text{vec} \, d\lambda, \quad t \in \mathbb{R} \text{ and } i = 1, 2,
\]  

(79)

where we integrate anti-clockwise over simple closed curves \( \rho_1 \) and \( \rho_2 \) in the left- and right-half plane of the complex field encircling the zeroes of the map \( \lambda \mapsto (\lambda I_{nm} - \tilde{A}_p^\text{vec})^{-1} \tilde{E}_p^\text{vec} \). More specifically, let \( \rho_1 \) be the rectangle in the left-half plane with width \( M \) and height \( 2R \) (with \( M \) and \( R \) large enough such that all eigenvalues in the left-half plane are encircled) and such that the line segment

\[
\rho_{IR} := \{ \lambda \in \mathbb{C} : \Re(\lambda) = 0 \text{ and } |\Im(\lambda)| \leq R \},
\]

forms an edge of \( \rho_1 \). We then define the curve \( \rho_2 \) as the reflection of \( \rho_1 \) over the imaginary axis. Moreover, for \( i = 1, 2 \) we denote by \( \hat{\rho}_i \) the curve \( \rho_i \) without the line segment on the imaginary axis.

The complex integrals in (79) are well defined, since by assumption (41) there is no singularity of \( \lambda \mapsto (\lambda I_{nm} - \tilde{A}_p^\text{vec})^{-1} \tilde{E}_p^\text{vec} \) \( = (I_{nm}, \lambda I_{nm}, \ldots, \lambda^{p-1} I_{nm}) \tilde{P}(\lambda)^{-1} \) on the imaginary axis. For every \( t \in \mathbb{R} \) we set

\[
Z_t := \int_{-\infty}^{t} g_1(t - u) \, dL_u - \int_{t}^{\infty} g_2(t - u) \, dL_u,
\]  

(80)

and show in the following that the integrals over the kernels \( g_1 \) and \( g_2 \) are well defined as the limit of integrals \( \lim_{T \to \infty} \int_{-T}^{T} g_1(t - u) \, dL_u \) and \( \lim_{T \to \infty} \int_{-T}^{T} g_2(t - u) \, dL_u \), respectively. Moreover, we show that \((Z_t)_{t \in \mathbb{R}}\) is the unique stationary solution of (22). First, note that for \( i = 1, 2 \) the kernel \( g_i \) satisfies the equation \( \frac{d}{dt} g_i(t) = \tilde{A}_p^\text{vec} g_i(t) \), which can be seen by similar arguments as in Lemma 2.8.

Indeed, note that we have

\[
\tilde{A}_p^\text{vec} g_i(t) = \frac{1}{2\pi i} \int_{\rho_i} e^{\lambda t} (\lambda I_{nm} - \tilde{A}_p^\text{vec})^{-1} \tilde{A}_p^\text{vec} \tilde{E}_p^\text{vec} \, d\lambda,
\]

and the term \((\lambda I_{nm} - \tilde{A}_p^\text{vec})^{-1} \tilde{A}_p^\text{vec} \tilde{E}_p^\text{vec}\) can be computed by solving the following linear matrix equation

\[
(\lambda I_{nm} - \tilde{A}_p^\text{vec})^{-1} F = \tilde{A}_p^\text{vec} \tilde{E}_p^\text{vec},
\]

for \( F := (F_1, F_2, \ldots, F_p)^\top \in \mathcal{L}(\mathbb{R}_{nm}, \mathbb{R}_{nmp}) \) with \( F_i \in M_{nm} \) for all \( i = 1, \ldots, p \).

Since \( \tilde{A}_p^\text{vec} \tilde{E}_p^\text{vec} = (0_{nm}, \ldots, 0_{nm}, K^{-(n,m)}, \tilde{A}_p^\text{vec}) \) we can argue similarly to the proof of Lemma 2.8 and obtain

\[
F = (\lambda^2 K^{(n,m)}, \ldots, \lambda^{p+1} (K^{(n,m)})^p I_{nm})^\top \tilde{P}(\lambda)^{-1} - (0_{nm}, \ldots, 0_{nm}, K^{(n,m)})^\top.
\]
Now, by setting $\rho_i$ for $i = 1, 2$, the integral over the latter term vanishes and hence for all $t \in \mathbb{R}$ we obtain
\[
\frac{d}{dt} g_i(t) = \frac{1}{2\pi i} \int_{\rho_i} e^{\lambda t} (\lambda I_{nm}, \lambda^2 K^{(n,m)}, \ldots, \lambda^{p+1} (K^{(n,m)})^p I_{nm})^T \tilde{P}(\lambda)^{-1} d\lambda \\
= \hat{A}_p^{vec} g_i(t), \quad i = 1, 2.
\]
Since the homogeneous linear equation $\frac{d}{dt} v(t) = \hat{A}_p^{vec} v(t)$ is uniquely solved by $e^{t \hat{A}_p^{vec}} v_0$, we see that $g_i(t) = e^{t \hat{A}_p^{vec}} g_i(0)$ for $i = 1, 2$ and $t \in \mathbb{R}$. This has the following consequences: First, if follows that there exist $K > 0$ and $\delta > 0$ such that for all $u \leq 0$ we have $\|g_1(-u)\|_{\mathcal{L}(\mathbb{M}_{n,m},(\mathbb{M}_{n,m})^p)} \leq K e^{-\delta |u|}$ and for all $u \geq 0$ we have $\|g_2(-u)\|_{\mathcal{L}(\mathbb{M}_{n,m},(\mathbb{M}_{n,m})^p)} \leq K e^{-\delta |u|}$. This together with $E [\log(\|L_1\|_{nm})] < \infty$ implies the existence of the integrals $\int_{-\infty}^{\infty} g_1(t-u) dL_u$ and $\int_{t}^{\infty} g_2(t-u) dL_u$, respectively, as limits of integrals over the intervals $(-T,T]$, resp. $[t,T)$, for $T \to \infty$, see also [20]. Next, we show that $(Z_t)_{t \in \mathbb{R}}$ is a solution of (22), where as a second consequence from $g_i(t) = e^{t \hat{A}_p^{vec}} g_i(0)$ for $i = 1, 2$ and $t \in \mathbb{R}$ we conclude that for every $s < t \in \mathbb{R}$ the following equality holds true:
\[
e^{(t-s) \hat{A}_p^{vec}} Z_s = e^{(t-s) \hat{A}_p^{vec}} \left( \int_{-\infty}^{s} g_1(s-u) dL_u - \int_{s}^{\infty} g_2(s-u) dL_u \right) \\
= e^{t \hat{A}_p^{vec}} \left( \int_{-\infty}^{s} g_1(-u) dL_u - \int_{s}^{\infty} g_2(-u) dL_u \right). \tag{81}
\]
Now, by setting $\rho = \rho_1 + \rho_2$, the spectral representation of the matrix exponential $e^{t \hat{A}_p^{vec}}$, see e.g. [31, Theorem 17.5], yields
\[
e^{t \hat{A}_p^{vec}} \tilde{E}_p^{vec} = \frac{1}{2\pi i} \int_{\rho} e^{\lambda t} (\lambda I_{nm} - \hat{A}_p^{vec})^{-1} \tilde{E}_p^{vec} d\lambda, \quad \forall t \in \mathbb{R},
\]
and hence for every $t \in \mathbb{R}$ we have
\[
\int_{s}^{t} e^{(t-u) \hat{A}_p^{vec}} \tilde{E}_p^{vec} dL_u = e^{t \hat{A}_p^{vec}} \left( \int_{s}^{t} g_1(-u) dL_u + \int_{s}^{t} g_2(-u) dL_u \right). \tag{82}
\]
By summing up (81) and (82) we obtain
\[
e^{t \hat{A}_p^{vec}} \left( \int_{-\infty}^{0} g_1(-u) dL_u - \int_{t}^{\infty} g_2(-u) dL_u \right) = Z_t,
\]
which proves that $(Z_t)_{t \in \mathbb{R}}$ is a solution of (22). Moreover, it is easy to see that $(Z_t)_{t \in \mathbb{R}}$ is also stationary and unique in law. Now, let us set $Y_t := \tilde{E}_p^{vec}(Z_t)$ and $h(t) := g_1(t) \mathbb{I}_{[0,\infty)}(t) - g_2(t) \mathbb{I}_{(-\infty,0)}(t)$ for all $t \in \mathbb{R}$, then
\[
Y_t = \int_{-\infty}^{\infty} \tilde{E}_q^{vec} h(t-u) dL_u = \int_{-\infty}^{\infty} \left( \frac{1}{2\pi i} \int_{\rho} e^{\lambda t} (\lambda I_{nm} - \hat{A}_p^{vec})^{-1} \tilde{E}_p^{vec} d\lambda \right) dL_u,
\]
which by (78) equals
\[
Y_t = \int_{-\infty}^{\infty} \left( \frac{1}{2\pi i} \int_{\rho} e^{\lambda(t-u)} \tilde{P}^{-1}(\lambda) \tilde{Q}(\lambda) d\lambda \right) dL_u, \quad t \in \mathbb{R}, \tag{83}
\]
and if
\[
\int_{-\infty}^{\infty} \left( \frac{1}{2\pi i} \int_{\rho} e^{\lambda(t-u)} \tilde{P}^{-1}(\lambda) \tilde{Q}(\lambda) d\lambda \right) = \int_{-\infty}^{\infty} g(t-u) dL_u, \quad t \in \mathbb{R}, \tag{84}
\]
then by uniqueness we conclude that $\text{vec}(X_t) = Y_t$ for $t \in \mathbb{R}$ and that the representation (42) holds true. It is therefore left to prove that the identity in (84) holds true for every $t \in \mathbb{R}$. In order to prove this, we set $K(z,t) := e^{-zt} \tilde{P}(z)^{-1} \tilde{Q}(z)$ and
use the integration paths from above, i.e. $\rho_1 = \hat{\rho}_1 \pm \rho_{1R}$. We then compute the left hand side in (84) as follows:

\[
\begin{align*}
    h(t) &= g_1(t) 1_{[0, \infty)}(t) - g_2(t) 1_{(-\infty, 0)}(t) \\
         &= \frac{1}{2\pi i} \left( \int_{\rho_1} K(z, t) \, dz \, 1_{[0, \infty)}(t) - \int_{\rho_2} K(z, t) \, dz \, 1_{(-\infty, 0)}(t) \right) \\
         &= \frac{1}{2\pi i} \left( \int_{\rho_1} K(z, t) \, dz \, 1_{[0, \infty)}(t) - \int_{\rho_2} K(z, t) \, dz \, 1_{(-\infty, 0)}(t) \right) \\
         &\quad + \frac{1}{2\pi i} \left( \int_{-R}^{R} K(i\xi, t) \, d\xi \, 1_{[0, \infty)}(t) + \int_{-R}^{R} K(i\xi, t) \, d\xi \, 1_{(-\infty, 0)}(t) \right) \\
         &= \frac{1}{2\pi} \int_{-R}^{R} K(i\xi, t) \, d\xi + \frac{1}{2\pi i} \left( \int_{\rho_1} K(z, t) \, dz \, 1_{[0, \infty)}(t) - \int_{\rho_2} K(z, t) \, dz \, 1_{(-\infty, 0)}(t) \right).
\end{align*}
\]

Now by letting $R \to \infty$, we see that for every $t \in \mathbb{R}$ the first term converges to $g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} K(i\xi, t) \, d\xi$ and it remains to show that the latter term converges to zero. For this, we first consider the term $\int_{\rho_1} K(z, t) \, dz \, 1_{[0, \infty)}(t)$ for $t \geq 0$ and note that the integral over $\rho_1$ can be split into three separate integrals: The first is

\[
\int_{-R-M}^{-iR-M} e^{\lambda} \tilde{P}(\lambda)^{-1} \tilde{Q}(\lambda) \, d\lambda = \int_{-R}^{R} e^{\iota \xi t} e^{-tM} \tilde{P}(-i\xi - M)^{-1} \tilde{Q}(-i\xi - M) \, d\xi,
\]

where the term $e^{-tM}$ in the integral dictates the convergence to zero as $M \to \infty$ for arbitrary $R$. The two other terms are given by

\[
\int_{iR}^{iR-M} e^{\lambda} \tilde{P}(\lambda)^{-1} \tilde{Q}(\lambda) \, d\lambda = - \int_{0}^{M} e^{(iR - \lambda)t} \tilde{P}(iR - \xi) \, d\xi,
\]

and

\[
\int_{-iR-M}^{-iR-M} e^{\lambda} \tilde{P}(\lambda)^{-1} \tilde{Q}(\lambda) \, d\lambda = - \int_{0}^{M} e^{(-iR + \lambda)t} \tilde{P}(-iR + \xi)^{-1} \tilde{Q}(-iR + \xi) \, d\xi,
\]

where in both cases the integrals exists for arbitrary large $M$ and since $p > q$ implies that $\tilde{P}(-iR + \xi)^{-1} \tilde{Q}(-iR + \xi) \to 0$ as $R \to \infty$, we conclude that both integrals converge to zero as $R \to \infty$. Similarly, for the integral over $\rho_2$ and all $t < 0$, we see that

\[
\begin{align*}
    \int_{iR+M}^{iR} e^{\lambda} \tilde{P}(\lambda)^{-1} \tilde{Q}(\lambda) \, d\lambda &= \int_{-R}^{R} e^{\iota \xi t} e^{tM} \tilde{P}(-i\xi - M)^{-1} \tilde{Q}(-i\xi - M) \, d\xi, \\
    \int_{-iR-M}^{-iR-M} e^{\lambda} \tilde{P}(\lambda)^{-1} \tilde{Q}(\lambda) \, d\lambda &= - \int_{0}^{M} e^{(iR - \lambda)t} \tilde{P}(iR - \xi) \, d\xi,
\end{align*}
\]

and we see that for every $M > 0$ and since $p > q$ the integrals converge to zero, whenever $R \to \infty$. Thus the only remaining term of $h(t)$ when expanding the
integration domain is \( g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} K(i\xi, t) \, d\xi \) which proves (84) and we conclude the assertions of Proposition 2.9.

Appendix B. Submultiplicativity property of the Hadamard product

**Lemma B.1.** Let \( d \in \mathbb{N} \) and \( A, B \in M_d \) be non-negative matrices, i.e. \( A, B \in \pi(R^d_+). \) Then for all \( n \in \mathbb{N} \) we have \((A \otimes B)^n \preceq A^n \otimes B^n.\)

**Proof.** We write \( A = (a_{ij})_{i,j=1,\ldots,d} \) and \( B = (b_{ij})_{i,j=1,\ldots,d} \) and prove the statement by induction over \( n \in \mathbb{N}. \) For \( n = 1 \) the statement is trivial. In case of \( n = 2, \) it follows from the non-negativity of \((a_{ij})_{i,j=1,\ldots,d} \) and \((b_{ij})_{i,j=1,\ldots,d} \) that for every \( i, j \in \{1, \ldots, d\} \) we have

\[
((A \otimes B)^2)_{i,j} = \sum_{k=1}^{d} a_{ik} b_{kj} a_{kj} \leq \left( \sum_{k=1}^{d} a_{ik} a_{jk} \right) \left( \sum_{k=1}^{d} b_{ik} b_{jk} \right) = ((A^2 \otimes B^2))_{i,j},
\]

which proves the statement for \( n = 2. \) Now, suppose that \((A \otimes B)^n \preceq A^n \otimes B^n \) for some \( n \in \mathbb{N}. \) We show that also \((A \otimes B)^{n+1} \preceq A^{n+1} \otimes B^{n+1} \) holds. Indeed, we have

\[
((A \otimes B)^{n+1})_{i,j} = ((A \otimes B)(A \otimes B)^n)_{i,j} \leq ((A \otimes B)A^n \otimes B^n)_{i,j},
\]

where we note that \(((A \otimes B)A^n \otimes B^n)_{i,j} = \sum_{k=1}^{d} a_{ik}(A^n)_{k,j} b_{nk}(B^n)_{k,j} \) and hence

\[
((A \otimes B)^{n+1})_{i,j} \leq \sum_{k=1}^{d} a_{ik}(A^n)_{k,j} b_{nk}(B^n)_{k,j} \leq \left( \sum_{k=1}^{d} a_{ik}(A^n)_{k,j} \right) \left( \sum_{k=1}^{d} b_{nk}(B^n)_{k,j} \right) = ((A^{n+1} \otimes B^{n+1}))_{i,j},
\]

which yields the desired inequality \((A \otimes B)^{n+1} \preceq A^{n+1} \otimes B^{n+1}. \) \( \Box \)

**References**

[1] ARENDT, W. Generators of positive semigroups and resolvent positive operators. Mathematisches Institut der Universität Tübingen., 1984.

[2] ÅSTRÖM, K. J. Introduction to stochastic control theory., reprint of the 1970 original ed. ed. Mineola, NY: Dover Publications, 2006.

[3] BALL, K. Completely monotonic rational functions and Hall’s marriage theorem. J. Comb. Theory, Ser. B 61, 1 (1994), 118–124.

[4] BARNDORFF-NIELSEN, O. E., BENTH, F. E., AND VERAAART, A. E. D. Ambit stochastics, vol. 88 of Probab. Theory Stoch. Model. Cham: Springer, 2018.

[5] BARNDORFF-NIELSEN, O. E., AND PÉREZ-ABREU, V. Matrix subordinators and related upsilon transformations. Theory Probab. Appl. 52, 1 (2008), 1–23.

[6] BARNDORFF-NIELSEN, O. E., AND SHEPHARD, N. Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. (With discussion). J. R. Stat. Soc., Ser. B, Stat. Methodol. 63, 2 (2001), 167–241.

[7] BARNDORFF-NIELSEN, O. E., AND STELZER, R. Positive-definite matrix processes of finite variation. Probab. Math. Stat. 27, 1 (2007), 3–43.

[8] BARNDORFF-NIELSEN, O. E., AND STELZER, R. Multivariate supOU processes. Ann. Appl. Probab. 21, 1 (2011), 140–182.

[9] BARNDORFF-NIELSEN, O. E., AND STELZER, R. The multivariate supOU stochastic volatility model. Math. Finance 23, 2 (2013), 275–296.

[10] BENTH, F. E., CHRISTENSEN, T. S., AND RØHDE, V. Multivariate continuous-time modeling of wind indexes and hedging of wind risk. Quant. Finance 21, 1 (2021), 165–183.

[11] BENTH, F. E., AND RØHDE, V. On non-negative modeling with CARMA processes. J. Math. Anal. Appl. 476, 1 (2019), 196–214.

[12] BENTH, F. E., AND SÜSS, A. Continuous-time autoregressive moving-average processes in Hilbert space. In Computation and combinatorics in dynamics, stochastics and control. The Abel symposium, Rosendal, Norway, August 16–19, 2016. Selected papers. Cham: Springer, 2018, pp. 297–320.
Benth, F. E., and Saltyte Benth, J. Dynamic pricing of wind futures. *Energy Economics* 31, 1 (2009), 16–24.

Brockwell, P., and Lindner, A. Integration of CARMA processes and spot volatility modelling. *J. Time Ser. Anal.* 34, 2 (2013), 156–167.

Brockwell, P. J. Recent results in the theory and applications of CARMA processes. *Ann. Inst. Stat. Math.* 66, 4 (2014), 647–685.

Brockwell, P., Davis, R. A., and Yang, Y. Estimation for non-negative Lévy-driven CARMA processes. *J. Bus. Econ. Stat.* 29, 2 (2011), 250–259.

Brockwell, P. J., and Scheffel, E. Parametric estimation of the driving Lévy process of multivariate CARMA processes from discrete observations. *J. Multivariate Anal.* 115 (2013), 217–251.

Brut, M.-F. Wishart processes. *J. Theor. Probab.* 4, 4 (1991), 725–751.

Chen, R., Xiao, H., and Yang, D. Autoregressive models for matrix-valued time series. *Journal of Econometrics* 222, 1, Part B (2021), 539–560. Annals Issue: Financial Econometrics in the Age of the Digital Economy.

Chojnowska-Michalik, A.

Cuchiero, C., Filipovic, D., Mayerhofer, E., and Teichmann, J.

Cuccherino, C., Filipovic, D., Mayerhofer, E., and Teichmann, J. Affine processes on positive semidefinite matrices. *Ann. Appl. Probab.* 21, 2 (2011), 397–463.

Farina, L., and Rinaldi, S. *Positive linear systems. Theory and applications.* New York, NY: Wiley, 2000.

Fasen, V. Dependence estimation for high-frequency sampled multivariate CARMA models. *Scand. J. Stat.* 1, 1 (2016), 292–320.

Friesen, M., and Karbach, S. Stationary covariance regime for affine stochastic covariance models in Hilbert spaces. Available at https://arxiv.org/abs/2203.14750.

Garibaldi, I., Lancaster, P., and Rodman, L. *Matrix polynomials.* New York, NY: Wiley, 1995.

Garibaldi, I., Lancaster, P., and Rodman, L. *Matrix polynomials.* Reprint of the 1982 original ed., vol. 58 of *Classics Appl. Math.* Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM), 2009.

Gohberg, I., Lancaster, P., and Rodman, L. *Matrix polynomials.* Reprint of the 1982 original ed., vol. 58 of *Classics Appl. Math.* Philadelphia, PA: Society for Industrial and Applied Mathematics (SIAM), 2009.

Horn, R. A., and Schlemm, E. Positive linear systems. *Theory and applications.* New York, NY: Springer-Verlag, 2000.

Horn, R. A., and Johnson, C. R. *Topics in matrix analysis.* Cambridge, MA: Cambridge University Press, 1991.

Horn, R. A., and Johnson, C. R. *Topics in matrix analysis.* Cambridge, MA: Cambridge University Press, 1991.

Horn, R. A., and Schlemm, E. Positive linear systems. *Theory and applications.* New York, NY: Springer-Verlag, 2000.

Horn, R. A., and Schlemm, E. Positive linear systems. *Theory and applications.* New York, NY: Springer-Verlag, 2000.

Hyvönen, V. A. On processes of Ornstein-Uhlenbeck type in Hilbert space. *Stochastics* 21 (1987), 251–286.

Kingman, J. F. C. *Poisson processes.* Oxford, UK: Oxford University Press, 1993.

Kailath, T. *Linear systems.* Prentice Hall Inform. System Sci. Ser. Prentice Hall, Englewood Cliffs, NJ, 1980.

Kevei, P. Asymptotic moving average representation of high-frequency sampled multivariate CARMA processes. *Ann. Inst. Stat. Math.* 70, 2 (2018), 467–487.

Kallio, M. Stationary covariance regime for affine stochastic covariance models in Hilbert spaces. Available at https://arxiv.org/abs/2203.14750.

Karatzas, I., and Shreve, S. *Brownian motion and stochastic calculus.* New York, NY: Springer-Verlag, 1991.

Kemp, C. D., and Kemp, N. A. *Mathematical structures for classical mechanics.* 2nd ed. Reading, MA: Addison-Wesley, 1995.

Kemp, C. D., and Kemp, N. A. *Mathematical structures for classical mechanics.* 2nd ed. Reading, MA: Addison-Wesley, 1995.

Kemp, C. D., and Kemp, N. A. *Mathematical structures for classical mechanics.* 2nd ed. Reading, MA: Addison-Wesley, 1995.

Kemp, C. D., and Kemp, N. A. *Mathematical structures for classical mechanics.* 2nd ed. Reading, MA: Addison-Wesley, 1995.

Kemp, C. D., and Kemp, N. A. *Mathematical structures for classical mechanics.* 2nd ed. Reading, MA: Addison-Wesley, 1995.

Kemp, C. D., and Kemp, N. A. *Mathematical structures for classical mechanics.* 2nd ed. Reading, MA: Addison-Wesley, 1995.

Kemp, C. D., and Kemp, N. A. *Mathematical structures for classical mechanics.* 2nd ed. Reading, MA: Addison-Wesley, 1995.
References

[42] Sato, K.-i., and Yamazato, M. Stationary processes of Ornstein-Uhlenbeck type. Probability theory and mathematical statistics, Proc. 4th USSR-Jap. Symp., Tbilisi/USSR 1982, Lect. Notes Math. 1021, 541-551 (1983), 1983.

[43] Schlemm, E., and Stelzer, R. Multivariate CARMA processes, continuous-time state space models and complete regularity of the innovations of the sampled processes. Bernoulli 18, 1 (2012), 46–63.

[44] Schlemm, E., and Stelzer, R. Multivariate CARMA processes, continuous-time state space models and complete regularity of the innovations of the sampled processes. Bernoulli 18, 1 (2012), 46–63.

[45] Schnurr, A., and Woerner, J. H. C. Well-balanced Lévy driven Ornstein-Uhlenbeck processes. Stat. Risk. Model. 28, 4 (2011), 343–357.

[46] Tsai, H., and Chan, K. A note on the non-negativity of continuous-time ARMA and GARCH processes. Statistics and Computing 19 (2009), 149–153.

[47] Tsai, H., and Chan, K. S. A note on non-negative continuous time processes. J. R. Stat. Soc., Ser. B, Stat. Methodol. 67, 4 (2005), 589–597.

[48] Zadeh, L. A., and Desoer, C. A. Linear system theory. The state space approach. New York, NY: McGraw-Hill, 1963.

[49] Zhou, J., Jiang, F., Zhu, K., and Li, W. Time series models for realized covariance matrices based on the matrix-F distribution. arXiv: Statistics Theory (2019).

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