OBSTRUCTIONS TO DEFORMING CURVES ON A 3-FOLD, III:
DEFORMATIONS OF CURVES LYING ON A K3 SURFACE

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Abstract. We study the deformations of a smooth curve $C$ on a smooth projective 3-fold $V$, assuming the presence of a smooth surface $S$ satisfying $C \subset S \subset V$. Generalizing a result of Mukai and Nasu, we give a new sufficient condition for a first order infinitesimal deformation of $C$ in $V$ to be primarily obstructed. In particular, when $V$ is Fano and $S$ is $K3$, we give a sufficient condition for $C$ to be (un)obstructed in $V$, in terms of $(-2)$-curves and elliptic curves on $S$. Applying this result, we prove that the Hilbert scheme $\text{Hilb}^{sc} V_4$ of smooth connected curves on a smooth quartic 3-fold $V_4 \subset \mathbb{P}^4$ contains infinitely many generically non-reduced irreducible components, which are variations of Mumford’s example for $\text{Hilb}^{sc} \mathbb{P}^3$.

1. Introduction

Let $V$ be a smooth projective 3-fold over an algebraically closed field $k$. This paper is a sequel to the preceding papers [17, 20], in which the embedded deformations of a smooth curve $C$ in $V$ have been studied under the presence of an intermediate smooth surface $S$ satisfying $C \subset S \subset V$. As is well known, first order (infinitesimal) deformations $\tilde{C} \subset V \times \text{Spec} \ k[t]/(t^2)$ of $C$ in $V$ are in one-to-one correspondence with global sections $\alpha$ of the normal bundle $N_{C/V}$ of $C$ in $V$. Then the obstruction $\text{ob}(\alpha)$ to lifting $\tilde{C}$ to a second order deformation $\tilde{\tilde{C}} \subset V \times \text{Spec} \ k[t]/(t^3)$ of $C$ in $V$ is contained in $H^1(C, N_{C/V})$, and $\text{ob}(\alpha)$ is computed as a cup product $\alpha \cup \alpha$ by the map

$$H^0(C, N_{C/V}) \times H^0(C, N_{C/V}) \xrightarrow{\cup} H^1(C, N_{C/V})$$

(cf. Theorem [2.1]). It is generally difficult to compute $\text{ob}(\alpha)$ directly. Mukai and Nasu [17] introduced the exterior components of $\alpha$ and $\text{ob}(\alpha)$, which are defined as the images of $\alpha$ and $\text{ob}(\alpha)$ in $H^i(C, N_{S/V}|_C)$ $(i = 0, 1)$ by the natural projection $\pi_{C/S} : N_{C/V} \to N_{S/V}|_C$, and denoted by $\pi_{C/S}(\alpha)$ and $\text{ob}_S(\alpha)$, respectively (cf. [2.3]). They gave a sufficient condition for $\text{ob}_S(\alpha)$ to be nonzero, which implies the non-liftability of $\tilde{C}$ to any $\tilde{\tilde{C}}$.

In this paper, we generalize their result and give a weaker condition for $\text{ob}_S(\alpha) \neq 0$. Let $\tilde{C}$ or $\alpha \in H^0(C, N_{C/V})$ be a first order deformation of $C$ in $V$, and $\pi_{C/S}(\alpha) \in H^0(C, N_{S/V}|_C)$ the exterior component of $\alpha$. Suppose that the image of $\pi_{C/S}(\alpha)$ in
\[ H^0(C, N_{S/V}(mE) |_{C}) \] lifts to a section
\[ \beta \in H^0(S, N_{S/V}(mE)) \]
for an integer \( m \geq 1 \) and an effective Cartier divisor \( E \) on \( S \). In other words, we have \( \pi_{C/S}(\alpha) = \beta |_{C} \) in \( H^0(C, N_{S/V}(mE) |_{C}) \). Here \( \beta \) is called an \textit{infinitesimal deformations with a pole} (along \( E \)) of \( S \) in \( V \) (cf. \[2, 4\]). The following is a generalization of \[17\] Theorem 2.2, in which, it was assumed that \( m = 1 \), \( E \) is smooth and irreducible with negative self-intersection number \( E^2 < 0 \) on \( S \), and furthermore, \( E \) was assumed to be a \((-1)\)-curve on \( S \) (i.e. \( E \simeq \mathbb{P} \) and \( E^2 = -1 \)) in its application.

\textbf{Theorem 1.1.} Let \( \tilde{C} \), \( \alpha \), \( \beta \) be as above. Suppose that the natural map \( H^1(S, \mathcal{O}_S(kE)) \to H^1(S, \mathcal{O}_S((k+1)E)) \) is injective for every integer \( k \geq 1 \). If the following conditions are satisfied, then the exterior component \( \mathrm{ob}_S(\alpha) \) of \( \mathrm{ob}(\alpha) \) is nonzero:

(a) the restriction map \( H^0(S, \Delta) \xrightarrow{|E|} H^0(E, \Delta |_{E}) \) is surjective for \( \Delta := C + K_V |_{S} - 2mE \), a divisor on \( S \), and

(b) we have
\[ m \partial_E(\beta |_{E}) \cup \beta |_{E} \neq 0 \quad \text{in} \quad H^1(E, N_{S/V}((2m+1)E-C) |_{E}), \]
where \( \beta |_{E} \in H^0(E, N_{S/V}(mE) |_{E}) \) is the principal part of \( \beta \) along the pole \( E \), and \( \partial_E \) is the coboundary map of the exact sequence
\[ [0 \to N_{E/S} \xrightarrow{|E|} N_{E/V} \xrightarrow{\pi_{E/S}} N_{S/V} |_{E} \to 0] \otimes_{\mathcal{O}_E} \mathcal{O}_E(mE). \]

The relation between \( \alpha \) and \( \beta |_{E} \) is explained with Figure 1 in \[3\].

Given a projective scheme \( V \), let \( \mathrm{Hilb}^{xc} V \) denote the Hilbert scheme of smooth connected curves in \( V \). Mumford \[18\] first proved that \( \mathrm{Hilb}^{xc} \mathbb{P}^3 \) contains a generically non-reduced (irreducible) component. Later, many examples of such non-reduced components of \( \mathrm{Hilb}^{xc} \mathbb{P}^3 \) were found in \[10, 3, 7, 5, 19\], etc. More recently, Mumford’s example was generalized in \[17\] and it was proved that for many uniruled 3-folds \( V \), \( \mathrm{Hilb}^{xc} V \) contains infinitely many generically non-reduced components. (See \[23\] for a different generalization.) In the construction of the components, \((-1)\)-curves \( E \subset V \) on a surface \( S \subset V \) play a very important role. In this paper, as an application we study the deformations of curves \( C \) on a smooth Fano 3-fold \( V \) when \( C \) is contained in a smooth \( K3 \) surface \( S \subset V \). On a \( K3 \) surface, \((-2)\)-curves \( E \) (i.e. \( E \simeq \mathbb{P}^1 \) and \( E^2 = -2 \)) and elliptic curves \( F \) (then \( F^2 = 0 \)) play a role very similar to that of a \((-1)\)-curve. If we have \( m = 1 \) in the exact sequence \( (1.1) \), then the sheaf homomorphism \( \pi_{E/S} \otimes_{\mathcal{O}_E} \mathcal{O}_E(E) \) tensored with \( \mathcal{O}_E(E) \) induces a map (called the “\( \pi \)-map” for \((E, S)\))
\[ \pi_{E/S}(E) : H^0(E, N_{E/V}(E)) \to H^0(E, N_{S/V}(E) |_{E}) \]
on the cohomology groups. From now on, we assume that \( \text{char}(k) = 0 \).
Theorem 1.2. Let $V$ be a smooth Fano 3-fold, $S \subset V$ a smooth $K3$ surface, and $C \subset S$ a smooth connected curve, and put $D := C + K_V|_S$ a divisor on $S$. Suppose that there exists a first order deformation $\tilde{S}$ of $S$ which does not contain any first order deformations $\tilde{C}$ of $C$.

1. If $D \geq 0$ and there exist no $(-2)$-curves and no elliptic curves on $S$, or more generally, if $H^1(S, D) = 0$, then $\text{Hilb}^{sc} V$ is nonsingular at $[C]$.

2. If $D \geq 0$, $D^2 \geq 0$ and there exists a $(-2)$-curve $E$ on $S$ such that $E.D = -2$ and $H^1(S, D - 3E) = 0$, then we have $h^1(S, D) = 1$. If moreover, $\pi_{E/S}(E)$ is not surjective, then $\text{Hilb}^{sc} V$ is singular at $[C]$.

3. If there exists an elliptic curve $F$ on $S$ such that $D \sim mF$ for an integer $m \geq 2$, then we have $h^1(S, D) = m - 1$. If moreover, $\pi_{E/S}(F)$ is not surjective, then $\text{Hilb}^{sc} V$ is singular at $[C]$.

Note that $H^1(S, D) \simeq H^1(S, N_{S/V}(-C))^\vee$ (cf. (2.16)). If $H^1(S, D) = 0$, then $C$ is unobstructed in $V$. By using Theorem 1.1 we partially prove that $C$ is obstructed in $V$ if $H^1(S, D) \neq 0$. Under the assumption of Theorem 1.2 the Hilbert-flag scheme $HFV$ of $V$ (cf. §2.5) is nonsingular at $(C, S)$ (cf. Lemma 2.10). Therefore $(C, S)$ belongs to a unique irreducible component $W_{C,S}$ of $HF^{sc} V$. The image $W_{C,S}$ of $W_{C,S}$ in $\text{Hilb}^{sc} V$ is called the $S$-maximal family of curves containing $C$ (cf. Definition 2.11). Then $W_{C,S}$ is of codimension $h^1(S, D)$ in the tangent space $H^0(C, N_{C/V})$ of $\text{Hilb}^{sc} V$ at $[C]$ (cf. (1.1)).

Corollary 1.3. In (1), (2), (3) of Theorem 1.2 we have furthermore that

(a) If $h^1(S, D) \leq 1$, then $W_{C,S}$ is an irreducible component of $\text{Hilb}^{sc} V_{\text{red}}$.

(b) $\text{Hilb}^{sc} V$ is generically smooth along $W_{C,S}$ if $h^1(S, D) = 0$, and generically non-reduced along $W_{C,S}$ if $h^1(S, D) = 1$.

(c) If $H^0(S, -D) = 0$, then $\dim [C] \text{Hilb}^{sc} V = (-K_V|_S)^2/2 + g(C) + 1$, where $g(C)$ is the genus of $C$.

The following is a simplification or a variation of Mumford’s example. (See Examples 5.8 and 5.12 for more examples.)

Example 1.4 (char $k = 0$). In the following examples, the closure $\overline{W}$ of $W$ is an irreducible component of $\text{Hilb}^{sc} V_{\text{red}}$ and $\text{Hilb}^{sc} V$ is generically non-reduced along $W$. We have $h^0(C, N_{C/V}) = \dim W + 1$ at the generic point $[C]$ of $W$.

1. Let $V$ be a smooth quartic 3-fold $V_4 \subset \mathbb{P}^4$, $E$ a smooth conic on $V$ with trivial normal bundle $N_{E/V} \simeq \mathcal{O}_E^2$, $S$ a smooth hyperplane section of $V$ containing $E$ and such that $\text{Pic} S = \mathbb{Z} h \oplus \mathbb{Z} E$, where $h \simeq \mathcal{O}_S(1)$. Then a general member $C$ of the complete linear system $|2h + 2E|$ on $S$ is a smooth connected curve of degree 12 and genus 13. Such curves $C$ are parametrised by a locally closed irreducible subset $W$ of $\text{Hilb}^{sc} V$ of dimension 16.
(2) Let $V = \mathbb{P}^3$ and let $F$ be a smooth plane cubic (elliptic) curve, $S$ a smooth quartic surface containing $F$. Then a general member $C$ of $|4h + 2F|$ ($h \sim \mathcal{O}_S(1)$) is a smooth connected curve of degree 22 and genus 57 on $S$. Such curves $C$ are parametrised by a locally closed irreducible subset $W$ of $\text{Hilb}^{sc}\mathbb{P}^3$ of dimension 90.

The organization of this paper is as follows. The proof of Theorem 1.1 heavily depends on the analysis of the singularity of polar $d$-maps. Given a projective scheme $V$ and its hypersurface $S \subset V$, there exists a so-called “Hilbert-Picard” morphism $\psi_S : \text{Hilb}^d V \rightarrow \text{Pic} S$ from the Hilbert scheme of effective Cartier divisors on $V$ to the Picard scheme of $S$, sending a hypersurface $S' \subset V$ to the invertible sheaf $\mathcal{O}_V(S')|_S$ on $S$ (cf. §2.2). The tangent map $d_S : H^0(S, N_{S/V}) \rightarrow H^1(S, \mathcal{O}_S)$ of $\psi_S$ at $[S]$ is called the $d$-map for $S \subset V$. In §2.4 we show that this map is extended into a version $d_S^* : H^0(S, N_{S/V}(mE)) \rightarrow H^1(S, \mathcal{O}_S((m + 1)E))$ with a pole along a divisor $E \geq 0$ on $S$. We also prove that for any $\beta \in H^0(S, N_{S/V}(mE))$ the restriction $d_S^*(\beta)|_E$ to $E$ of the image $d_S^*(\beta)$ coincides with the coboundary image $\partial_E(\beta|_E)$ of (1.1) up to constant (cf. Proposition 2.6). In §3 applying this result to a 3-fold $V$, we prove Theorem 1.1. In §4 we prove Theorem 1.2 and Corollary 1.3 as an application of Theorem 1.1. In §5.1 we study the Mori cone $\overline{\text{NE}}(S_4)$ of a smooth quartic surface $S_4$ of Picard number 2 (cf. Lemmas 5.1, 5.3). Applying the results on Mori cones, in §5.2 for curves $C$ lying on $S_4$, we study the deformations of $C$ in $\mathbb{P}^3$ and $C$ in a smooth quartic 3-fold $V_4 \subset \mathbb{P}^4$. In particular, we give a sufficient condition for $W_{C,S_4}$ to be a generically non-reduced (or generically smooth) component of $\text{Hilb}^{sc}\mathbb{P}^3$ and $\text{Hilb}^{sc}V_4$ (cf. Theorems 5.5, 5.10 and 5.13).

2. Preliminaries

We start by recalling some basic facts on the deformation theory of closed subschemes, as well as setting up Notations. We work over an algebraically closed field $k$ of characteristic $p \geq 0$. Let $V \subset \mathbb{P}^n$ be a closed subscheme of $\mathbb{P}^n$ with the embedding invertible sheaf $\mathcal{O}_V(1)$ on $V$, and $X \subset V$ a closed subscheme of $V$ with the Hilbert polynomial $P_X = \chi(\mathcal{O}_X(n))$. Then as is well known, the Hilbert scheme $\text{Hilb}_P V$ of $V$ parametrises all closed subscheme $X'$ of $V$ with $P_{X'} = P_X$ (cf. [6]). We denote by $\text{Hilb} V$ the (full) Hilbert scheme $\bigsqcup_P \text{Hilb}_P V$ of $V$. Let $\mathcal{I}_X$ and $N_{X/V} = (\mathcal{I}_X/\mathcal{I}_X^2)^\vee$ denote the ideal sheaf and the normal sheaf of $X$ in $V$, respectively. The symbol $[X]$ represents the point of $\text{Hilb} V$ corresponding to $X$. Then the tangent space of $\text{Hilb} V$ at $[X]$ is known to be isomorphic to the group $\text{Hom}(\mathcal{I}_X/\mathcal{O}_X)$ of sheaf homomorphisms from $\mathcal{I}_X$ to $\mathcal{O}_X$, which is isomorphic to $H^0(X, N_{X/V})$. Every obstruction to deforming $X$ in $V$ is contained in the group $\text{Ext}^1(\mathcal{I}_X, \mathcal{O}_X)$, and if $X$ is a locally complete intersection in $V$, then it is contained in a smaller subgroup $H^1(X, N_{X/V}) \subset \text{Ext}^1(\mathcal{I}_X, \mathcal{O}_X)$. If $H^1(X, N_{X/V}) = 0$, then $\text{Hilb} V$ is nonsingular at $[X]$ of dimension $h^0(X, N_{X/V})$. A first order (infinitesimal) deformation of $X$ in $V$ is a closed subscheme $X' \subset X \times \text{Spec} D$, flat over the ring $D = k[t]/(t^2)$ of dual numbers, with a central fiber $X_0' = X$. By the universal property of the Hilbert
scheme, there exists a one-to-one correspondence between the set of \(D\)-valued points \(\gamma: \text{Spec } D \rightarrow \text{Hilb } V\) sending 0 to \([X]\), and the set of the first order deformations of \(X\) in \(V\). By the infinitesimal lifting property of smoothness (cf. [8, Proposition 4.4, Chap. 1]), if there exists a first order deformation of \(X\) in \(V\) not liftable to a deformation over \(\text{Spec } k[t]/(t^n)\) for some \(n \geq 3\), then \(\text{Hilb } V\) is singular at \([X]\). We say \(X\) is \textit{unobstructed} (resp. \textit{obstructed}) in \(V\) if \(\text{Hilb } V\) is nonsingular (resp. singular) at \([X]\), and for an irreducible closed subset \(W\) of \(\text{Hilb } V\), we say \(\text{Hilb } V\) is \textit{generically smooth} (resp. \textit{generically non-reduced}) along \(W\) if \(\text{Hilb } V\) is nonsingular (resp. singular) at the generic point \(X_\eta\) of \(W\).

2.1. Primary obstructions. Let \(V\) be a (projective) scheme over \(k\), \(X\) a closed subscheme of \(V\), \(\alpha\) a global section of \(N_{X/V}\). We define a cup product \(\text{ob}(\alpha) \in \text{Ext}^1(I_X, \mathcal{O}_X)\) by

\[
\text{ob}(\alpha) := \alpha \cup e \cup \alpha,
\]

where \(e \in \text{Ext}^1(\mathcal{O}_X, I_X)\) is the extension class of the standard short exact sequence

\[
0 \rightarrow I_X \rightarrow \mathcal{O}_V \rightarrow \mathcal{O}_X \rightarrow 0
\]
on \(V\). Though the following fact is well-known to the experts, we give a proof for the reader’s convenience.

\textbf{Theorem 2.1} (cf. [2, 13]). Let \(\tilde{X}\) be a first order deformation of \(X\) in \(V\) corresponding to \(\alpha\). If \(X\) is a locally complete intersection in \(V\), then \(\tilde{X}\) lifts to a deformation \(\tilde{\tilde{X}}\) over \(k[t]/(t^3)\), if and only if \(\text{ob}(\alpha)\) is zero.

\textit{Proof.} First we fix some notations for the proof. Let \(\mathcal{U} := \{U_i \mid i \in I\}\) be an open affine covering of \(V\), \(R_i\) the coordinate ring of \(U_i\), \(I_i\) the defining ideal of \(X \cap U_i\) in \(U_i\). We take a covering \(\mathcal{U}\) such that for all \(i, j\), (i) the intersections \(U_{ij} := U_i \cap U_j\) are affine, and (ii) \(I_i\) are generated by \(m\) elements \(f_{i1}, \ldots, f_{im}\) in \(R_i\), where \(m\) denotes the codimension of \(X\) in \(V\). (Such covering exists by assumption.) Let \(R_{ij}\) be the coordinate ring of \(U_{ij}\). Then since \(I_i\) and \(I_j\) agree on the overlap \(U_{ij}\), there exists a \(m \times m\) matrix \(A_{ij}\) with entries in \(R_{ij}\) (i.e., \(A_{ij} \in M(m, R_{ij})\)) such that

\[
f_j = A_{ij} f_i, \quad \text{where } f_i := \begin{pmatrix} f_{i1} \\ \vdots \\ f_{im} \end{pmatrix}.
\]

Here and later, for a ring \(R\), we denote by \(M(m, R)\) the set of \(m \times m\) matrices with entries in \(R\). For an object \(o\) in \(V\), we denote by \(o\) the restriction of \(o\) to \(X\). For example, if \(u\) is a section of a sheaf \(\mathcal{F}\) on \(V\), \(\overline{u}\) denotes the image of \(u\) in \(\mathcal{F}|_X = \mathcal{F} \otimes_{\mathcal{O}_V} \mathcal{O}_X\). We note that the restriction \(\overline{A_{ij}}\) to \(X\) of \(A_{ij}\) represents the transition matrix of \(N_{X/V}\) over \(U_{ij}\).
Secondly we recall the correspondence between $\tilde{X}$ and $\alpha$. Since $X$ is a locally complete intersection in $V$, so is $\tilde{X}$ in $V \times \text{Spec } k[t]/(t^2)$ (cf. [8 §9]). Then for each $i$, the defining ideal $J_i$ of $\tilde{X}$ over $U_i \times \text{Spec } k[t]/(t^2)$ is generated by

$$f_{i1} + tu_{i1}, \ldots, f_{im} + tu_{im}$$

in $R_i[t]/(t^2)$ for some $u_{ik} \in R_i$ ($k = 1, \ldots, m$). Since $J_i$ and $J_j$ agree on $U_{ij} \times \text{Spec } k[t]/(t^2)$, there exists a matrix $B_{ij} \in M(m, R_{ij})$ such that

$$f_j + tu_j = (A_{ij} + tB_{ij})(f_i + tu_i), \quad \text{where } u_i := \begin{pmatrix} u_{i1} \\ \vdots \\ u_{im} \end{pmatrix}.$$

Comparing the coefficient of $t$, we have

(2.3) \quad \mathbf{u}_j = A_{ij} \mathbf{u}_i + B_{ij} f_i,$$

which implies that $\mathbf{v}_j = A_{ij} \mathbf{v}_i$ in $\mathcal{O}_{X_{ij}}^{\oplus m}$. Let $\alpha_i$ be the section of $N_{X/V} \simeq \text{Hom}(\mathcal{I}_X, \mathcal{O}_X)$ over $U_i$ sending each $f_{ik} \in I_i$ to $\overline{u}_{ik} \in R_i/I_i$ ($k = 1, \ldots, m$), respectively. Then by (2.2) and (2.3), the local sections $\alpha_i$ ($i \in I$) over $U_i$ agree on $U_{ij}$ for every $i, j$ and define a global section of $N_{X/V}$, which is nothing but $\alpha$. In the rest of the proof, for convenience, we write as $\alpha_i(f_i) = \overline{u}_i$ instead of writing $\alpha_i(f_{ik}) = \overline{u}_{ik}$ ($k = 1, \ldots, m$).

Now we consider liftings of $\tilde{X}$ to a second order deformation $\tilde{X}$ of $X$ in $V$ (over $k[t]/(t^3)$). If there exists such a $\tilde{X}$, then its defining ideal $K_i$ over $U_i \times \text{Spec } k[t]/(t^3)$ is generated by

$$f_{i1} + tu_{i1} + t^2v_{i1}, \ldots, f_{im} + tu_{im} + t^2v_{im}$$

in $R_i[t]/(t^3)$ for some $v_{ik} \in R_i$ ($k = 1, \ldots, m$). Then there exists a matrix $C_{ij} \in M(m, R_{ij})$ such that

$$f_j + tu_j + t^2v_j = (A_{ij} + tB_{ij} + t^2C_{ij})(f_i + tu_i + t^2v_i), \quad \text{where } \mathbf{v}_i := \begin{pmatrix} v_{i1} \\ \vdots \\ v_{im} \end{pmatrix},$$

which is equivalent to that

(2.4) \quad \overline{\mathbf{v}}_j - A_{ij} \overline{\mathbf{v}}_i = B_{ij} \overline{\mathbf{u}}_i$$

in $\mathcal{O}_{X_{ij}}^{\oplus m}$ by comparison of the coefficient of $t^2$. We see that $\tilde{X}$ is defined as a subscheme of $V \times k[t]/(t^3)$, flat over $k[t]/(t^3)$ if and only if we can solve the equation (2.4) for $\mathbf{v}_i$.

On the other hand, let us define a 1-cochain $\beta := \{\beta_{ij}\} \in C^1(\mathfrak{M}, N_{X/V})$, where $\beta_{ij}$ is the section of $N_{X/V} \simeq \text{Hom}(\mathcal{I}_X, \mathcal{O}_X)$ over $U_{ij}$ with

(2.5) \quad \beta_{ij}(f_i) = \overline{B_{ij} \mathbf{u}}_i.$$

Then (2.4) implies that $\beta$ is cohomologous to zero, since $\overline{A_{ij}}$ is the transition matrix of $N_{X/V}$ over $U_{ij}$. In fact, if we have (2.4), then $\beta$ is equal to the coboundary of the 0-cochain.
\[ \alpha' = \{ \alpha'_i \} \in C^0(\mathcal{U}, N_{X/V}) \] defined by \( \alpha'_i(f_i) = \nabla_i \). Thus for the proof, it suffices to prove the next claim.

**Claim 2.2.** The cohomology class in \( H^1(X, N_{X/V}) \) represented by \( \beta \) equals \( \text{ob}(\alpha) \).

**Proof of Claim.** The functor \( \text{Hom}(\mathcal{I}_X, \ast) \) induces a coboundary map \( \delta : \text{Hom}(\mathcal{I}_X, \mathcal{O}_X) \to \text{Ext}^1(\mathcal{I}_X, \mathcal{I}_X) \). We also deduce from (2.1) an exact sequence of Čech complexes

\[
0 \longrightarrow C^*(\mathcal{U}, \text{Hom}(\mathcal{I}_X, \mathcal{I}_X)) \longrightarrow C^*(\mathcal{U}, \text{Hom}(\mathcal{I}_X, \mathcal{O}_X)) \longrightarrow C^*(\mathcal{U}, \text{Hom}(\mathcal{I}_X, \mathcal{O}_X)) \longrightarrow 0.
\]

We compute the image \( \delta(\alpha)(= \alpha \cup e) \) of \( \alpha \) by a diagram chase. Let \( \alpha_i := \alpha|_{U_i} \) for \( i \in I \). Then as we see before, we have \( \alpha_i(f_i) = \overline{u_i} \). If we define a section \( \hat{\alpha}_i \) of \( \text{Hom}(\mathcal{I}_X, \mathcal{O}_V) \) over \( U_i \) by \( \hat{\alpha}_i(f_i) = u_i \), then \( \hat{\alpha}_i \) is a local lift of \( \alpha \) over \( U_i \). Since \( \alpha \) is globally defined, \( \delta(\alpha)_{ij} = \hat{\alpha}_j - \hat{\alpha}_i \) becomes a section of \( \text{Hom}(\mathcal{I}_X, \mathcal{I}_X) \) over \( U_{ij} \) for every \( i, j \). Then by (2.3), we have \( \delta(\alpha)_{ij}(f_i) = B_{ij}f_i \). Thus we have computed \( \delta(\alpha) \) as an element of \( H^1(V, \text{Hom}(\mathcal{I}_X, \mathcal{I}_X)) \subset \text{Ext}^1(\mathcal{I}_X, \mathcal{I}_X) \). Since \( \text{ob}(\alpha) = \alpha \cup e \cup \alpha = \delta(\alpha) \cup \alpha \), \( \text{ob}(\alpha) \) is represented by the 1-cocycle \( \{ \alpha_i \circ \delta(\alpha)_{ij} \} \) of \( N_{X/V} \). Therefore \( \text{ob}(\alpha) \) is contained in \( H^1(X, N_{X/V}) \subset \text{Ext}^1(\mathcal{I}_X, \mathcal{O}_X) \). Since we have

\[
\alpha_i \circ \delta(\alpha)_{ij}(f_i) = \alpha_i(B_{ij}f_i) = B_{ij}\alpha_i(f_i) = \overline{B_{ij}}u_i,
\]

we conclude that \( \alpha_i \circ \delta(\alpha)_{ij} = \beta_{ij} \) by (2.5). Thus we have proved the claim and have finished the proof of Theorem 2.1. \( \square \)

**Definition 2.3.** Here \( \text{ob}(\alpha) \) is called the **(primary) obstruction** for \( \alpha \) (or \( \bar{X} \)).

**2.2. Hypersurface case and d-map.** Let \( X \) be an effective Cartier divisor on \( V \), i.e., a closed subscheme of \( V \) whose ideal sheaf is locally generated by a single equation. We denote by \( \text{Hilb}^d V \) the Hilbert scheme of effective Cartier divisors on \( V \). There exists a natural morphism \( \varphi : \text{Hilb}^d V \to \text{Pic} V \) to the Picard scheme \( \text{Pic} V \) of \( V \), sending a divisor \( D \) on \( V \) to the invertible sheaf \( \mathcal{O}_V(D) \) associated to \( D \). We define a morphism \( \psi_X : \text{Hilb}^d V \to \text{Pic} X \) by the composition of \( \varphi \) with the morphism \( \text{Pic} V \xrightarrow{\varphi} \text{Pic} X \) defined by the restriction to \( X \). By definition, the tangent map \( d_X \) of \( \psi_X \) at \([X]\) is the composite

\[
d_X : H^0(X, N_{X/V}) \xrightarrow{\delta} H^1(V, \mathcal{O}_V) \xrightarrow{\psi_X} H^1(X, \mathcal{O}_X),
\]

where \( \delta \) is the coboundary map of the exact sequence \( 0 \to \mathcal{O}_V \to \mathcal{O}_V(X) \to N_{X/V} \to 0 \). We call \( d_X \) the **d-map** for \( X \). Let \( \bar{X} \) be a first order deformation of \( X \) in \( V \), corresponding to a global section \( \beta \) of \( N_{X/V} \). Then by [20, Lemma 2.9], the primary obstruction \( \text{ob}(\beta) \) of \( \bar{X} \) equals the cup product \( d_X(\beta) \cup \beta \) by the map \( H^1(X, \mathcal{O}_X) \times H^0(X, N_{X/V}) \xrightarrow{\cup} H^1(X, N_{X/V}) \).
2.3. Exterior components. We recall the definition of the exterior components (cf. [17, 20]), which is useful for computing the obstructions to deforming subschemes of codimension greater than 1. Let $X$ and $Y$ be two closed subschemes of $V$ such that $X \subset Y$, $\pi_{X/Y} : N_{X/V} \to N_{Y/V}|_X$ the natural projection. Then we have the induced maps $H^i(\pi_{X/Y}) : H^i(X, N_{X/V}) \to H^i(X, N_{Y/V}|_X)$ on the cohomology groups for $i = 0, 1$. The two images

$$\pi_{X/Y}(\alpha) := H^0(\pi_{X/Y})(\alpha) \quad \text{and} \quad \text{ob}_Y(\alpha) := H^1(\pi_{X/Y})(\text{ob}(\alpha))$$

are called the exterior component of $\alpha$ and $\text{ob}(\alpha)$, respectively. These objects respectively correspond to the deformation of $X$ in $V$ into the normal direction to $Y$ and its obstruction. Now we assume that $Y$ is an effective Cartier divisor on $V$, and $X$ is a locally complete intersection in $Y$. The $d$-map $d_X$ in (2.6) is generalized and defined for a pair $(X, Y)$. (In fact, we have $d_Y = d_{Y,Y}$.)

**Definition 2.4.** Let $\delta$ be the coboundary map of $[0 \to \mathcal{I}_X \to \mathcal{O}_V \to \mathcal{O}_X \to 0] \otimes \mathcal{O}_V(Y)$. The composition

$$d_{X,Y} : H^0(X, N_{Y/V}|_X) \xrightarrow{\delta} H^1(V, \mathcal{I}_X \otimes \mathcal{O}_V(Y)) \xrightarrow{\iota_X} H^1(X, N_{X/V}^\vee \otimes N_{Y/V}|_X)$$

of $\delta$ with the restriction map $|_X$ to $X$ is called the $d$-map for $(X, Y)$.

Then the two $d$-maps $d_{X,Y}$ and $d_Y$ are related by the following commutative diagram:

$$
\begin{array}{ccc}
H^0(Y, N_{Y/V}) & \xrightarrow{d_Y} & H^1(Y, \mathcal{O}_Y) \\
|_X \downarrow & & |_X \downarrow \\
H^1(X, \mathcal{O}_X) & \xrightarrow{H^1(\iota)|_X} & H^1(X, N_{X/V}^\vee \otimes N_{Y/V}|_X),
\end{array}
$$

where $\iota : \mathcal{O}_X \to N_{X/V}^\vee \otimes N_{Y/V}|_X$ is the sheaf homomorphism induced by $\pi_{X/Y}$.

**Lemma 2.5** (cf. [17, Lemma 2.3 and 2.4]). Let $\tilde{X}$ and $\tilde{Y}$ be first order deformations of $X$ and $Y$ in $V$, with the corresponding global sections $\alpha$ and $\beta$ of $N_{X/V}$ and $N_{Y/V}$, respectively. If we have $\pi_{X/Y}(\alpha) = \beta|_X$ in $H^0(X, N_{Y/V}|_X)$, then we have

$$\text{ob}_Y(\alpha) = d_{X,Y}(\pi_{X/Y}(\alpha)) \cup_1 \alpha = d_Y(\beta)|_X \cup_2 \pi_{X/Y}(\alpha)$$

in $H^1(X, N_{Y/V}|_X)$, where $\cup_1$ and $\cup_2$ are the cup product maps

$$H^1(X, N_{X/V}^\vee \otimes N_{Y/V}|_X) \times H^0(X, N_{X/V}) \xrightarrow{\cup_1} H^1(X, N_{Y/V}|_X),$$

$$H^1(X, \mathcal{O}_X) \times H^0(X, N_{Y/V}|_X) \xrightarrow{\cup_2} H^1(X, N_{Y/V}|_X)$$

respectively.
2.4. **Infinitesimal deformations with poles and polar $d$-maps.** In this section, we recall the theory of *infinitesimal deformations with poles*, which was introduced in [17]. Here we develop the study [17, §2.4] on the polar $d$-maps further. The infinitesimal deformations with poles are defined as rational sections of some sheaves admitting a pole along some divisor, and they are usually regarded as the deformations of the open objects complementary to the poles (cf. [20]).

Let $V$ be a projective scheme, $Y$ and $E$ effective Cartier divisors on $V$ and $Y$, respectively. Put $Y^\circ := \text{int} \ Y$ and $V^\circ := \text{int} \ V$ and let $\iota : Y^\circ \to Y$ be the open immersion. Since $\iota_* \mathcal{O}_Y$ contains $\mathcal{O}_Y(mE)$ as a subsheaf for any $m \geq 0$, there exist natural inclusions $\mathcal{O}_Y \subset \mathcal{O}_Y(E) \subset \cdots \subset \mathcal{O}_Y(mE) \subset \cdots \subset \iota_* \mathcal{O}_Y$ of sheaves on $Y$. Similarly, since $\mathcal{N}_{Y/V}(mE) \subset \iota_* \mathcal{N}_{Y^\circ/V^\circ}$ for any $m \geq 0$, we regard $H^0(Y, \mathcal{N}_{Y/V}(mE))$ as a subgroup of $H^0(Y^\circ, \mathcal{N}_{Y^\circ/V^\circ})$ by the natural injective map

$$H^0(Y, \mathcal{N}_{Y/V}(mE)) \hookrightarrow H^0(Y^\circ, \mathcal{N}_{Y^\circ/V^\circ}).$$

A rational section $\beta$ of $\mathcal{N}_{Y/V}$ admitting a pole along $E$, i.e.

$$\beta \in H^0(Y, \mathcal{N}_{Y/V}(mE))$$

for some integer $m \geq 1$ is called an *infinitesimal deformation of $Y$ with a pole* along $E$. Every infinitesimal deformation of $Y$ in $V$ with a pole induces a first order deformation of $Y^\circ$ in $V^\circ$ by the above injection.

Now we assume that the natural map

$$H^1(Y, \mathcal{O}_Y(mE)) \to H^1(Y, \mathcal{O}_Y((m+1)E))$$

is injective for any integer $m \geq 1$. Then since $V$ is projective, by the same argument as in [17, Lemma 2.5], the natural map

$$H^1(Y, \mathcal{O}_Y(mE)) \to H^1(Y^\circ, \mathcal{O}_{Y^\circ})$$

is injective. By this map, we regard $H^1(Y, \mathcal{O}_Y(mE))$ as a subgroup of $H^1(Y^\circ, \mathcal{O}_{Y^\circ})$. Given an invertible sheaf $L$ on $Y$, we identify an element of $H^1(Y, \mathcal{O}_Y(mE))$ as a first order deformation of the invertible sheaf $L^\circ := \iota_* L$ on $Y^\circ$, and call it an *infinitesimal deformation of $L$ with a pole* along $E$.

Let $m \geq 1$ be an integer and $\beta \in H^0(Y, \mathcal{N}_{Y/V}(mE))$ an infinitesimal deformation of $Y$ with a pole along $E$. Let $d_{Y^\circ} : H^0(Y^\circ, \mathcal{N}_{Y^\circ/V^\circ}) \to H^1(Y^\circ, \mathcal{O}_{Y^\circ})$ be the $d$-map (2.6) for $Y^\circ \subset V^\circ$. The following is a generalization of [17, Proposition 2.6], which enables us to compute the singularity of $d_{Y^\circ} (\beta) \in H^1(Y^\circ, \mathcal{O}_{Y^\circ})$ along the boundary $E$.

**Proposition 2.6.** Let $m \geq 1$ be an integer. Then

1. $d_{Y^\circ} (H^0(Y, \mathcal{N}_{Y/V}(mE))) \subset H^1(Y, \mathcal{O}_Y((m+1)E)).$
(2) Let \( d_Y \) be the restriction of \( d_{Y^\circ} \) to \( H^0(Y, N_{Y/V}(mE)) \), and let \( \partial_E \) be the coboundary map of (1.1). Then the diagram

\[
\begin{array}{c}
H^0(Y, N_{Y/V}(mE)) \xrightarrow{d_Y} H^1(Y, \mathcal{O}_Y((m+1)E)) \\
|_E \downarrow \quad \quad \quad \quad \quad \downarrow |_E \\
H^0(E, N_{Y/V}(mE)|_E) \xrightarrow{m\partial_E} H^1(E, \mathcal{O}_E((m+1)E))
\end{array}
\]

is commutative.

In other words, if \( Y \) is a hypersurface in \( V \), then every infinitesimal deformation of \( Y \subset V \) with a pole induces that of the invertible sheaf \( N_{Y/V} \). The principal part of \( d_{Y^\circ}(\beta) \) along \( E \) coincides with the coboundary \( \partial_E(\beta|_E) \) of the principal part \( \beta|_E \), up to constant.

Proof. The proof is similar to the one in [17], where \( Y \) is a surface by assumption. Let \( \mathcal{U} := \{ U_i \}_{i \in I} \) be an open affine covering of \( V \) and let \( x_i = y_i = 0 \) be the local equation of \( E \) over \( U_i \) such that \( y_i \) defines \( Y \) in \( U_i \). Through the proof, for a local section \( t \) of a sheaf \( \mathcal{F} \) on \( V \), \( \bar{t} \) denotes the restriction \( t|_Y \in \mathcal{F}|_Y \) for conventions. Let \( D_{x_i} \) and \( D_{y_i} \) denote the open affine subsets of \( U_i \) and \( U_i \cap Y \) defined by \( x_i \neq 0 \) and \( y_i \neq 0 \), respectively. Then \( \{ D_{x_i} \}_{i \in I} \) is an open affine covering of \( Y^\circ \), since \( D_{x_i} = D_{x_i} \cap Y = U_i \cap Y^\circ \).

Let \( \beta \) be a global section of \( N_{Y/V}(mE) \cong \mathcal{O}_Y(Y)(mE) \). Then the product \( x_i^m \beta \) is contained in \( H^0(U_i, \mathcal{O}_Y(Y)) \) and lifts to a section \( s_i \in \Gamma(U_i, \mathcal{O}_Y(Y)) \) since \( U_i \) is affine. In particular, \( \beta \) lifts to the section \( s_i := s_i'/x_i^m \) of \( \mathcal{O}_{V^\circ}(Y^\circ) \) over \( D_{x_i} \). Let \( \delta : H^0(Y^\circ, N_{Y^\circ/V^\circ}) \to H^1(Y^\circ, \mathcal{O}_{V^\circ}) \) be the coboundary map in the definition (2.6) of \( d_{Y^\circ} \). Then we have

\[
\delta(\beta)_{ij} = s_j - s_i = \frac{s_j'}{x_j^m} - \frac{s_i'}{x_i^m} \quad \text{in} \quad \Gamma(D_{x_i} \cap D_{x_j}, \mathcal{O}_{V^\circ}(Y^\circ))
\]

for every \( i, j \). Since \( \beta \) is a global section of \( N_{Y^\circ/V^\circ} \), \( \delta(\beta)_{ij} \) is contained in \( \Gamma(D_{x_i} \cap D_{x_j}, \mathcal{O}_{V^\circ}) \).

Now we put

\[
f_{ij} := x_i^m x_j^m \delta(\beta)_{ij} = x_i^m s_j' - x_j^m s_i'.
\]

Since \( x_i^m s_i = s_i' \in \Gamma(U_i, \mathcal{O}_V(Y)) \) for every \( i \), \( f_{ij} \) is a section of \( \mathcal{O}_{U_{ij}} \). Now we recall the relations between the local equations \( x_i, y_i \) of \( E \) over \( U_i \). Since the two ideals \( (x_i, y_i) \) and \( (x_j, y_j) \) agree on the overlap \( U_{ij} \), there exist elements \( b_{ij} \) and \( c_{ij} \) of \( \mathcal{O}_{U_{ij}} \) satisfying

\[
x_i = b_{ij} y_j + c_{ij} x_j.
\]

Then we have

\[
f_{ij} = (x_i^m - c_{ij} x_j^m) s_j' + x_j^m (c_{ij}^m s_j' - s_i')
\]

\[
= \sum_{k=0}^{m-1} x_i^{m-k} (c_{ij} x_j) x_i^k (x_i - c_{ij} x_j) s_j' + x_j^m (c_{ij}^m s_j' - s_i')
\]

\[
= \sum_{k=0}^{m-1} x_i^{m-k} (c_{ij} x_j) b_{ij} y_j s_j' + x_j^m (c_{ij}^m s_j' - s_i').
\]
Since $y_j \in \Gamma(U_j, \mathcal{O}_Y(-Y))$ and $s_j' \in \Gamma(U_j, \mathcal{O}_V(Y))$, $b_{ij}y_js_j'$ is a section of $\mathcal{O}_{U_{ij}}$, while $c_{ij}^m s_j' - s_i' \in \Gamma(U_{ij}, \mathcal{O}_V(Y))$ is also a section of $\mathcal{O}_{U_{ij}}$, because we have

$$c_{ij}^m s_j' - s_i' = c_{ij}m \bar{x}_j^m \beta - \bar{x}_i^m \beta = (c_{ij}^m \bar{x}_j^m - \bar{x}_i^m)\beta = 0$$

in $\Gamma(Y \cap U_{ij}, \mathcal{N}_{Y/V})$. Therefore, $f_{ij}$ is contained in $\Gamma(U_{ij}, \mathcal{I}_E^{m-1})$ by (2.9), and hence $\bar{f}_{ij}$ is contained in $\Gamma(Y \cap U_{ij}, \mathcal{O}_Y((m-1)E))$. This implies that

$$(2.10)\quad d_{Y^0}(\beta)_{ij} = (\delta(\beta)_{ij})_{Y^0} = \frac{\bar{f}_{ij}}{\bar{x}_i^m \bar{x}_j^m} \quad \text{in} \quad \Gamma(D_{xi} \cap D_{xj}, \mathcal{O}_{Y^0}),$$

is contained in $\Gamma(Y \cap U_{ij}, \mathcal{O}_Y((m+1)E))$. Thus we have proved (1).

Now we compute the image of $d_Y(\beta) = d_{Y^0}(\beta)$ by the restriction map $H^1(Y, \mathcal{O}_Y((m+1)E)) \to H^1(E, \mathcal{O}_E((m+1)E))$, regarding $\mathcal{O}_E((m+1)E)$ as the quotient sheaf $\mathcal{O}_Y((m+1)E)/\mathcal{O}_Y(mE)$. Since $\bar{s}_i = \bar{c}_{ij}x_j$ and $\beta = \bar{s}_j/\bar{x}_j^m$, it follows from (2.9) and (2.10) that

$$d_Y(\beta)_{ij} = \sum_{k=0}^{m-1} \left( \frac{c_{ij}x_j}{x_i} \right)^k \frac{b_{ij}y_j}{x_i} \bar{s}_j + \frac{c_{ij}^m s_j' - s_i'}{x_i^m} = m \frac{b_{ij}y_j}{x_i} \beta + \frac{c_{ij}^m s_j' - s_i'}{x_i^m}$$

in $\Gamma(Y \cap U_{ij}, \mathcal{O}_Y((m+1)E))$. Since $c_{ij}^m s_j' - s_i'$ a section of $\mathcal{O}_{U_{ij}}$, $c_{ij}^m s_j' - s_i'/x_i^m$ is contained in $\Gamma(Y \cap U_{ij}, \mathcal{O}_Y(mE))$. On the other hand, the restriction of the 1-cochain $\{b_{ij}y_j/x_i\}_{i,j \in I}$ to $E$ is a cocycle and represents the extension class $e' \in H^1(E, \mathcal{O}_E(-Y + E))$ of the exact sequence (1.1) (cf. [17]). Therefore $d_Y(\beta)|_E$ is equal to $me' \cup (\beta|_E) = m\partial E(\beta|_E)$, which implies (2).

We finish this section by giving a refinement of Proposition 2.4, which will be used in the proof of Theorem 3.3 Let $E_i$ $(1 \leq i \leq k)$ be irreducible Cartier divisors on $Y$. Suppose that $E_i$ are mutually disjoint, i.e., $E_i \cap E_j = \emptyset$ for all $i, j$. We suppose furthermore that for any two effective divisors $D, D'$ on $S$ with supports on $\bigcup_{i=1}^k E_i$, if $D \leq D'$, then the natural map $H^1(Y, \mathcal{O}_Y(D)) \to H^1(Y, \mathcal{O}_Y(D'))$ is injective. Then as we have seen before, for any such divisor $D$, $H^1(Y, \mathcal{O}_Y(D))$ is regarded as a subgroup of $H^1(Y^\circ, \mathcal{O}_{Y^\circ})$, where $Y^\circ := Y \setminus \bigcup_{i=1}^k E_i$. Let $E = \sum_{i=1}^k m_i E_i$ be an effective divisor on $Y$ with coefficients $m_i \geq 1$, and let $\beta \in H^0(Y, \mathcal{N}_{Y/V}(E))$. We put $V^\circ := V \setminus \bigcup_{i=1}^k E_i$ and denote by $d_{Y^\circ}$ the $d$-map (2.6) for $Y^\circ \subset V^\circ$. If $H^1(Y, \mathcal{N}_{Y/V}) = 0$, then by the following lemma, $\beta \in H^0(Y, \mathcal{N}_{Y/V}(E))$ is written as a $k$-linear combination $\sum_{i=1}^k c_i \beta_i$ of $\beta_i \in H^0(Y, \mathcal{N}_{Y/V}(m_i E_i))$.

**Lemma 2.7.** Let $L$ be an invertible sheaf on $Y$ with $H^1(Y, L) = 0$, and let $E, E'$ be two effective divisors on $Y$ whose supports are mutually disjoint. Then the natural map $H^0(Y, \mathcal{O}_Y(E) \oplus H^0(Y, \mathcal{O}_Y(E')) \to H^0(Y, \mathcal{O}_Y(E + E'))$ is surjective.

**Proof.** It follows from the exact sequence $[0 \to \mathcal{O}_Y \to \mathcal{O}_Y(E) \oplus \mathcal{O}_Y(E') \to \mathcal{O}_Y(E + E') \to 0] \otimes L$ on $Y$ of Koszul type. \qed
Since the $d$-map $d_{Y^0}$ is $k$-linear, we find $d_{Y^0}(\beta) = \sum_{i=1}^k c_i d_{Y^0}(\beta_i)$. Because for each $i$, $d_{Y^0}(\beta_i)$ is contained in $H^1(Y, \mathcal{O}_Y((m_i + 1)E_i))$ by Proposition 2.8, $d_{Y^0}(\beta)$ is contained in $H^1(Y, \mathcal{O}_Y(\sum_{i=1}^k (m_i + 1)E_i))$. Furthermore, since $E_i$'s are mutually disjoint, we have $d_{Y^0}(\beta_i)\big|_{E_i} = 0$ if $i \neq j$ and $d_{Y^0}(\beta_i)\big|_{E_i} = m_i \partial_{E_i}(\beta_i)\big|_{E_i}$ by the same proposition. Thus we conclude that

**Proposition 2.8.** Let $m_i \geq 1$ be integers. If $H^1(Y, N_{Y/V}) = 0$, then

1. $d_{Y^0}(H^0(Y, N_{Y/V}(\sum_{i=1}^k m_i E_i))) \subset H^1(Y, \mathcal{O}_Y(\sum_{i=1}^k (m_i + 1)E_i))$.
2. Let $d_Y$ be the restriction of $d_{Y^0}$ to $H^0(Y, N_{Y/V}(\sum_{i=1}^k m_i E_i))$. Then the diagram

$$
\begin{array}{ccc}
H^0(Y, N_{Y/V}(\sum_{i=1}^k m_i E_i)) & \xrightarrow{d_Y} & H^1(Y, \mathcal{O}_Y(\sum_{i=1}^k (m_i + 1)E_i)) \\
|E_i| & & |E_i| \\
H^0(E_i, N_{Y/V}(m_i E_i)|_{E_i}) & \xrightarrow{m_i \partial_{E_i}} & H^1(E_i, \mathcal{O}_{E_i}((m_i + 1)E_i))
\end{array}
$$

is commutative for any $i = 1, \ldots, k$.

2.5. **Hilbert-flag schemes.** In this section, we recall some basic results on Hilbert-flag schemes. For the construction (the existence), the local properties, etc., of the Hilbert-flag schemes, we refer to [9, 10, 8, 22].

Let $V$ be a projective scheme, and let $X, Y$ be two closed subschemes of $V$ such that $X \subset Y$, with the Hilbert polynomials $P, Q$, respectively. Then there exists a projective scheme $HF_{P,Q} V$, called the Hilbert-flag scheme of $V$, parametrising all pairs $(X', Y')$ of closed subschemes $X' \subset Y' \subset V$ with the Hilbert polynomials $P$ and $Q$, respectively. There exists a natural diagram of the Hilbert(-flag) schemes

$$
\begin{array}{ccc}
HF_{P,Q} V & \xrightarrow{pr_2} & \text{Hilb}_Q V \\
p_{r_1} & & \\
\text{Hilb}_P V
\end{array}
$$

(2.11)

where $pr_i$ ($i = 1, 2$) are the forgetful morphisms, i.e., the projections. We denote the tangent space of $HF V$ at $(X, Y)$ by $A^1(X, Y)$. Then there exists a Cartesian diagram

$$
\begin{array}{ccc}
A^1(X, Y) & \xrightarrow{p_2} & H^0(Y, N_{Y/V}) \\
p_1 & & \Box \\
H^0(X, N_{X/V}) & \xrightarrow{\pi_{X/Y}} & H^0(X, N_{Y/V}|_X)
\end{array}
$$

where $p_i$ is the tangent map of $pr_i$ ($i = 1, 2$), $\rho$ is the restriction map, and $\pi_{X/Y}$ is the projection. In what follows, we assume that $X$ and $Y$ are smooth and Hilb$V$ is nonsingular at $[Y]$. Let

$$
\partial_X : H^0(X, N_{Y/V}|_X) \to H^1(X, N_{X/Y})
$$
be the coboundary map of the exact sequence
\[ 0 \to N_{X/Y} \to N_{X/V} \xrightarrow{\pi_{X/V}} N_{Y/V} \big|_X \to 0 \]
on V and let \( \alpha_{X/Y} \) be the composition \( \partial_X \circ \rho \) of \( \rho \) with \( \partial_X \). Then since \( \text{Hilb} V \) is nonsingular at \([Y]\), every obstruction to deforming a pair \((X,Y)\) of subschemes \(X,Y\) with \(X \subset Y \subset V\) is contained in the group
\[ A^2(X,Y) := \text{coker} \alpha_{X/Y}, \]
and we have
\[ \text{dim} A^1(X,Y) - \text{dim} A^2(X,Y) \leq \dim_{(X,Y)} \text{HF} V \leq \text{dim} A^1(X,Y) \]
(c.f. \[9,\, \text{Theorem 1.3.2},\, \text{[10, \S 2]}\]). Thus \( A^2(X,Y) \) represents the obstruction space of \( \text{HF} V \) at \((X,Y)\). There exists an exact sequence
\[ 0 \to H^0(Y, \mathcal{I}_{X/Y} \otimes N_{Y/V}) \to A^1(X,Y) \to H^0(X, N_{X/V}) \]
\[ \xrightarrow{\text{coker} \rho} A^2(X,Y) \to H^1(X, N_{X/V}) \to H^1(X, N_{Y/V} \big|_X) \]
(\[2.14\])
of cohomology groups, which connects the tangent spaces and the obstruction spaces of Hilbert(-flag) schemes (see \[9,\, \text{[10]}\] for the proof). If we have \( H^i(Y, N_{Y/V}) = 0 \) for \( i = 1, 2 \), then we deduce from the exact sequence \([0 \to \mathcal{I}_{X/Y} \to \mathcal{O}_Y \to \mathcal{O}_X \to 0] \otimes N_{Y/V}\) the two isomorphisms \( \text{coker} \rho \simeq H^1(Y, \mathcal{I}_{X/Y} \otimes N_{Y/V}) \) and \( H^1(X, N_{Y/V} \big|_X) \simeq H^2(Y, \mathcal{I}_{X/Y} \otimes N_{Y/V}) \).

If \( \text{dim} X = 1 \) then the last map of (\[2.14\]) is surjective. Thus we obtain (3) of the next lemma.

**Lemma 2.9.**

(1) If \( \rho \) is surjective (c.f. (\[2.11\])), then \( \text{pr}_1 : \text{HF} V \to \text{Hilb} V \) is smooth at \((X,Y)\) (c.f. \[10, \text{Lemma A10}]\).

(2) If \( H^0(Y, \mathcal{I}_{X/Y} \otimes N_{Y/V}) = 0 \), then \( \text{pr}_1 \) is a local embedding at \((X,Y)\).

(3) If \( \text{dim} X = 1, \text{dim} Y = 2 \) and \( H^i(Y, N_{Y/V}) = 0 \) \((i = 1, 2)\), then we have
\[ \text{dim} A^1(X,Y) - \text{dim} A^2(X,Y) = \chi(X, N_{X/V}) + \chi(Y, \mathcal{I}_{X/Y} \otimes N_{Y/V}) \]
\[ = \chi(X, N_{X/Y}) + \chi(Y, N_{Y/V}). \]
\[ \text{(2.15)} \]

The number (\[2.15\]) represents the expected dimension of the Hilbert-flag scheme \( \text{HF} V \) at \((X,Y)\). If \( A^2(X,Y) = 0 \), then \( \text{HF} V \) is nonsingular at \((X,Y)\) by (\[2.13\]). If moreover \( H^1(Y, \mathcal{I}_{X/Y} \otimes N_{Y/V}) = 0 \), then so is \( \text{Hilb} V \) at \([X]\) by Lemma 2.9(1). The following lemma will be essentially used in the proof of Theorem 1.2 (c.f. \[\S 1\]).

**Lemma 2.10.** Let \( V \) be a smooth Fano 3-fold, \( S \) a smooth K3 surface contained in \( V \), \( C \) a smooth curve on \( S \). Then

(1) \( H^i(S, N_{S/V}) = 0 \) for all \( i \geq 1 \). In particular, \( \text{Hilb} V \) is nonsingular at \([S]\).

(2) We have an isomorphism
\[ H^i(S, \mathcal{I}_{C/S} \otimes N_{S/V}) \simeq H^i(S, -D) \]
\[ \text{for every integer} \ i, \text{where} \ D := C + K_V \big|_S \text{is a divisor on} \ S. \]
(3) Suppose that there exists a first order deformation \( \tilde{S} \) of \( S \) which does not contain any first order deformations \( \tilde{C} \) of \( C \). Then we have \( A^2(C, S) = 0 \). In particular, the Hilbert-flag scheme \( HFV \) is nonsingular at \((C, S)\) of dimension \((-K_V|_S)^2/2 + g(C) + 1\), where \( g(C) \) is the genus of \( C \).

Proof. Since \( K_S \) is trivial, by adjunction, we have \( N_{S/V} \simeq -K_V|_S \) and \( N_{C/S} \simeq K_C \). Then (1) follows from the ampleness of \(-K_V\), and (2) from \( I_{C/S} \otimes N_{S/V} \simeq N_{S/V}(-C) \simeq -K_V|_S - C = -D \). On the other hand, we have \( H^1(C, N_{C/S}) \simeq k \). Thus the obstruction group \( A^2(C, S) \) (cf. (2.12)) of \( HFV \) at \((C, S)\) is of dimension at most 1. For proving (3), let \( \beta \) be the global section of \( N_{S/V} \) corresponding to \( \tilde{S} \). Then \( \rho(\beta) \) is not contained in the image of \( \pi_{C/S} \), because the diagram (2.11) is Cartesian. Hence the map \( \alpha_{C/S} \) is nonzero and we conclude that \( A^2(C, S) = 0 \). By Lemma 2.9(3), \( \dim_{(C, S)}HFV = \dim A^1(C, S) = \chi(-K_V|_S) + \chi(K_C) = (-K_V|_S)^2/2 + g(C) + 1 \). □

The first projection \( pr_1 \) induces a morphism \( pr'_1 : HFV^{sc} \to \text{Hilb}^{sc}V \), where \( HFV^{sc} := pr_1^{-1}(\text{Hilb}^{sc}V) \). If \( X \) is a smooth connected curve and \( HFV \) is nonsingular at \((X, Y)\), then there exists a unique irreducible component \( W_{X,Y} \) of \( HFV^{sc} \) passing through \((X, Y)\).

**Definition 2.11.** The image \( W_{X,Y} \) of \( W_{X,Y} \) by \( pr'_1 \) is called the \( Y \)-maximal family of curves containing \( X \).

2.6. \( K3 \) surfaces and quartic surfaces. We recall some basic results on \( K3 \) surfaces and quartic surfaces.

**Lemma 2.12.** Let \( S \) be a smooth projective \( K3 \) surface, \( D \neq 0 \) an effective divisor on \( S \).

1. If \( D \) is nef, then the complete linear system \( |D| \) has a base point if and only if there exist curves \( E \) and \( F \) on \( S \) and an integer \( k \geq 2 \) such that \( D \sim E + kF \), \( E^2 = -2 \), \( F^2 = 0 \) and \( E \cdot F = 1 \).

2. Let \( D^2 \geq 0 \). Then \( H^1(S, D) \neq 0 \) if and only if (i) \( D \cdot \Delta \leq -2 \) for some divisor \( \Delta \geq 0 \) with \( \Delta^2 = -2 \), or (ii) \( D \sim kF \) for some nef and primitive divisor \( F \geq 0 \) with \( F^2 = 0 \) and an integer \( k \geq 2 \). (We have \( h^1(S, D) = k - 1 \) in (ii).)

Proof. (1) follows from [21, 2.7] and (2) from [12]. □

The following lemma will be used in §5 to show the existence of quartic surfaces of Picard number two containing a rational curve or an elliptic curve.

**Lemma 2.13** (Mori [16], see also [8, p.138]). We assume that \( \text{char} \ k = 0 \).

1. There exists a smooth curve \( C \) of degree \( d > 0 \) and genus \( g \geq 0 \) on a smooth quartic surface \( S \subset \mathbb{P}^3 \) if and only if (i) \( g = d^2/8 + 1 \), or (ii) \( g < d^2/8 \) and \( (d, g) \neq (5, 3) \).

2. If there exists a smooth quartic surface \( S_0 \) containing smooth curve \( C_0 \) of degree \( d \) and genus \( g \), then there exists a smooth quartic surface \( S_1 \) containing a smooth
Lemma 2.14. Let \( k \) and the class \((3.1) \rightarrow O\) sequence \( H \) is a curve, we have [2] trivial. Thus we have \([\sqrt{\nabla}]\).

Proof. \( h(b) \) is satisfied. Then we have an inequality \( P \rightarrow \pi \) on \( E \). As we see in the proof of Lemma 2.10, by assumption, \( \pi \) is a coherent sheaf \( A \) \( S \), and \( \pi \) is globally generated, or \( P \) is elliptic and there exists a first order deformation \( \tilde{S} \) of \( S \) not containing any first order deformation \( \tilde{E} \) of \( E \), then the \( \pi \)-map \( \pi_{E/S}(E) \) in [1,2] is not surjective.

In this section, we compute obstructions to deforming curves on a 3-fold, and prove Theorem 1.3 and its refinement Theorem 3.3.

Let \( C \subset S \subset V \) be a sequence of a curve \( C \), a surface \( S \), a 3-fold \( V \), \( E \) an effective Cartier divisor on \( S \). We assume that \( C \) and \( S \) are Cartier divisors on \( S \) and \( V \), respectively. Let \( Z := C \cap E \) be the scheme-theoretic intersection of \( C \) and \( E \). In this section, given a coherent sheaf \( \mathcal{F} \) on \( S \) (resp. \( C \)), integers \( i, m \geq 0 \), and a cohomology class \( * \) in \( H^i(S, \mathcal{F}) \) (resp. \( H^i(C, \mathcal{F}) \)), we denote by \( \pi_{(m)} \) the image of \( * \) in \( H^i(S, \mathcal{F}(mE)) \) (resp. \( H^i(C, \mathcal{F}(mZ)) \)). We define \( k_C \in \text{Ext}^1(\mathcal{O}_C, \mathcal{O}_S(-C)) \) as the extension class of the exact sequence

\[ 0 \rightarrow \mathcal{O}_S(-C) \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_C \rightarrow 0, \tag{3.1} \]

and the class \( k_E \) similarly.
Lemma 3.1. Let $L$ be an invertible sheaf on $S$ and $\gamma$ a global section of $L|_C = L \otimes_{\mathcal{O}_S} \mathcal{O}_C$.

(1) Let $m \geq 0$ be an integer. Then $\tau_{(m)} \in H^0(C, L(mE)|_C)$ lifts to a section $\beta \in H^0(S, L(mE))$ on $S$ if and only if $\tau_{(m)} \cup k_C = 0$ in $H^1(S, L(mE - C))$.

(2) If $\tau_{(m)}$ lifts to a section $\beta \in H^0(S, L(mE))$ for $m \geq 1$, then the principle part $\beta|_E$ of $\beta$ is contained in $H^0(E, L(mE - C)|_E)$, and hence $\beta$ is contained in $H^0(S, \mathcal{I}_{E/S} \otimes L(mE))$. Here $\beta|_E$ is nonzero if and only if $\beta \not\in H^0(S, L((m-1)E))$, equivalently, $\tau_{(m-1)} \cup k_C \neq 0$ in $H^1(S, L((m-1)E - C))$.

Proof. (1) follows from the short exact sequence $(3.1) \otimes L(mE)$, whose coboundary map coincides with the cup product map $\cup k_C : H^0(C, L(mE)|_C) \rightarrow H^1(S, L(mE - C))$ with $k_C$. (2) follows from a diagram chase on the commutative diagram

![Diagram](image)

of cohomology groups, which is exact both vertically and horizontally. \(\square\)

**Proof of Theorem 1.1.** We first recall the relation between $\alpha$ and $\partial_E(\beta|_E)$ by Figure 1. We use the same strategy as [17], in which the proof is separated into 3 steps. We follow

![Figure 1](image)

the same steps and prove $\text{ob}_S(\alpha)_{(m+1)} \neq 0$ in $H^1(C, N_{S/V}((m+1)E)|_C)$ instead of proving $\text{ob}_S(\alpha) \neq 0$ in $H^1(C, N_{S/V}|_C)$.

Let $\alpha$ be a global section of $N_{C/V}$, and $\pi_{C/S}(\alpha) \in H^0(C, N_{S/V}|_C)$ the exterior component of $\alpha$. Suppose that the image $\pi_{C/S}(\alpha)_{(m)}$ of $\pi_{C/S}(\alpha)$ in $H^0(C, N_{S/V}(mE)|_C)$ lifts to a section $\beta \in H^0(S, N_{S/V}(mE)) \setminus H^0(S, N_{S/V}((m-1)E))$ for some integer $m \geq 1$, i.e., an
infinitesimal deformation with a pole along \( E \). We need the relation between the two \( d \)-maps \( d_{C,S} \) and \( d_S \) (cf. Definition \ref{def:1}), allowing a pole along \( E \). The polar version of the diagram \((2.7)\) is the following partially commutative diagram:

\[
\begin{array}{ccc}
\beta \in H^0(S, N_{S/V}(mE)) & \xrightarrow{d_S} & H^1(S, \mathcal{O}_S((m+1)E)) \\
\downarrow & & \downarrow c \downarrow \\
H^0(C, N_{S/V}(mE)|_C) & \cup & H^1(C, N_{C/V} \otimes N_{S/V}((m+1)E)|_C),
\end{array}
\]

\( \gamma \in H^0(C, N_{S/V}|_C) \xrightarrow{d_{C,S}} H^1(C, N_{C/V} \otimes N_{S/V}|_C) \)

in which, the commutativity holds only for \( \gamma \in H^0(C, N_{S/V}|_C) \) which has a lift \( \beta \in H^0(S, N_{S/V}(mE)) \). More precisely, for such a pair \( \gamma \) and \( \beta \), we have

\( \overline{d_{C,S}(\gamma)}_{(m+1)} = H^1(1)(d_S(\beta)|_C). \)

**Step 1** \( \overline{\text{ob}_S(\alpha)}_{(m+1)} = d_S(\beta)|_C \cup \pi_{C/S}(\alpha) \) in \( H^1(C, N_{S/V}((m+1)E)|_C). \)

**Proof.** By Lemma \ref{lem:2.5} we have \( \overline{\text{ob}_S(\alpha)}_{(m+1)} = d_{C,S}(\pi_{C/S}(\alpha))_{(m+1)} \cup \alpha \). Then it follows from \((3.3)\) that \( d_{C,S}(\pi_{C/S}(\alpha))_{(m+1)} = H^1(1)(d_S(\beta)|_C). \) By the commutative diagram

\[
\begin{array}{ccc}
H^1(N_{C/V} \otimes N_{S/V}((m+1)E)) & \times & H^0(N_{C/V}) \\
\uparrow H^1(1) & & \downarrow \pi_{C/S} \\
H^1(\mathcal{O}_C((m+1)Z)) & \times & H^0(N_{S/V}|_C) \end{array}
\]

\( \cup \) \( \downarrow \) \( H^1(N_{S/V}((m+1)E)|_C) \)

we have the required equation. \( \square \)

Next we relate \( \text{ob}_S(\alpha) \) with a cohomology class on \( E \). Let \( k_C \) and \( k_E \) be the extension classes defined by \((3.1)\).

**Step 2** \( \overline{\text{ob}_S(\alpha)}_{(2m)} \cup k_C = (d_S(\beta)|_E \cup \beta|_E) \cup k_E \) in \( H^2(S, N_{S/V}(2mE - C)). \)

**Proof.** We note that for every integers \( i, n \geq 0 \) and for any coherent sheaf \( \mathcal{F} \) on \( S \), the map \( H^i(S, \mathcal{F}) \rightarrow H^i(S, \mathcal{F}(nE)), \star \mapsto \mathcal{F}(n) \), and the cup product maps are compatible. For example, the diagram

\[
\begin{array}{ccc}
H^0(C, N_{S/V}|_C) & \xrightarrow{d_S(\beta)|_C \cup} & H^1(C, N_{S/V}((m+1)E)|_C) \\
\downarrow & & \downarrow \\
H^0(C, N_{S/V}((m-1)E)|_C) & \xrightarrow{d_S(\beta)|_C \cup} & H^1(C, N_{S/V}(2mE)|_C)
\end{array}
\]

is commutative. Therefore, by Step 1 we have

\( \overline{\text{ob}_S(\alpha)}_{(2m)} = d_S(\beta)|_C \cup \pi_{C/S}(\alpha)_{(m-1)} = d_S(\beta)|_C \cup \pi_{C/S}(\alpha)_{(m-1)} \).
in $H^1(C, N_{S/V}(2mE)|_{C})$. Since there exists a commutative diagram
\begin{equation}
\begin{array}{ccc}
H^1(\mathcal{O}_C((m+1)Z)) & \times & H^0(N_{S/V}((m-1)E)|_{C}) \\
\uparrow_{|C} & & \uparrow_{\cup k_C} \\
H^1(\mathcal{O}_S((m+1)E)) & \times & H^1(N_{S/V}((m-1)E - C)) \quad \cup & H^2(N_{S/V}(2mE - C)),
\end{array}
\end{equation}
we have
\[
\overline{\text{obs}(\alpha)}_{(2m)} \cup k_C = (d_S(\beta)|_C \cup \overline{\pi_{C/S}(\alpha)}_{(m-1)}) \cup k_C = d_S(\beta) \cup (\overline{\pi_{C/S}(\alpha)}_{(m-1)} \cup k_C)
\]
in $H^2(S, N_{S/V}(2mE-C))$. Then by Lemma 3.1, $\beta$ is contained in the subgroup $H^0(S, \mathcal{I}_{Z/S} \otimes N_{S/V}(mE)) \subset H^0(S, N_{S/V}(mE))$, and its restriction $\beta|_C$ to $C$ is a global section of the invertible sheaf
\[
\mathcal{I}_{Z/S} \otimes N_{S/V}(mE)|_C \simeq \mathcal{O}_C(-Z) \otimes N_{S/V}(mE)|_C \simeq N_{S/V}((m-1)E)|_C
\]
on $C$ and we have $\beta|_C = \overline{\pi_{C/S}(\alpha)}_{(m-1)}$ by assumption. Therefore we obtain
\[
d_S(\beta) \cup (\overline{\pi_{C/S}(\alpha)}_{(m-1)} \cup k_C) = d_S(\beta) \cup (\beta|_C \cup k_C).
\]
Then Lemma 2.8 shows that we have $\beta|_C \cup k_C = \beta|_E \cup k_E$ in $H^1(S, N_{S/V}((m-1)E - C))$. Hence we have
\[
d_S(\beta) \cup (\beta|_C \cup k_C) = d_S(\beta) \cup (\beta|_E \cup k_E) = (d_S(\beta)|_E \cup \beta|_E) \cup k_E,
\]
where the last equality follows from the commutative diagram
\begin{equation}
\begin{array}{ccc}
H^1(\mathcal{O}_E((m+1)E)) & \times & H^0(N_{S/V}(mE - C)|_E) \\
\uparrow_{|E} & & \uparrow_{\cup k_E} \\
H^1(\mathcal{O}_S((m+1)E)) & \times & H^1(N_{S/V}((m-1)E - C)) \quad \cup & H^2(N_{S/V}(2mE - C)),
\end{array}
\end{equation}
similar to (3.4). Thus we obtain the equation required. \hfill \square

**Step 3** Let $\partial_E$ be the coboundary map of (1.1). Then by Proposition 2.6 (2), we have $d_S(\beta)|_E \cup \beta|_E = m\partial_E(\beta|_E) \cup \beta|_E$, which is nonzero by the assumption (b). Consider the coboundary map
\[
\cup k_E : H^1(E, N_{S/V}((2m+1)E - C)|_E) \longrightarrow H^2(S, N_{S/V}(2mE - C)),
\]
which appears in \((3.5)\). By the Serre duality, it is dual to the restriction map
\[
H^0(S, C + K_V|_S - 2mE) \xrightarrow{\beta} H^0(E, (C + K_V - 2mE)|_E),
\]
which is surjective by the assumption (a). Hence the coboundary map \(\cup k_E\) is injective. Therefore we obtain \(ds(\beta)|_E \cup \beta|_E \cup k_E \neq 0\) and hence by Step 2 we conclude that \(\text{ob}(\alpha)(2m) \neq 0\) in \(H^1(C, N_{S/V}(2mE)|_E)\), and hence we have finished the proof of Theorem \([1.1]\). \(\square\)

Let \(\pi_{E/S}(mE) : H^0(E, N_{E/V}(mE)) \to H^0(E, N_{S/V}(mE)|_E)\) be the map induced by \([1.1]\). If this map is not surjective, then the sections \(\gamma\) in its image span a proper linear subsystem
\[
\Lambda' := \{\text{div}(\gamma) \mid \gamma \in \text{im} \pi_{E/S}(mE)\}
\]
of the complete linear system \(\Lambda := |N_{S/V}(mE)|_E|\) on \(E\), where \(\text{div}(\gamma)\) denotes the divisor of zeros for \(\gamma\). The condition (b) in Theorem \([1.1]\) can be replaced with the following conditions (b1), (b2) and (b3) in the next corollary, which is more accessible in many situations.

**Corollary 3.2.** Let \(C \subset S \subset V\), \(E, \alpha, \beta, \Delta\) be as in Theorem \([1.1]\). Suppose that \(\beta|_E \neq 0\). If the following conditions are satisfied, then the exterior component \(\text{ob}(\alpha)\) of \(\text{ob}(\alpha)\) is nonzero.

(a) The restriction map \(H^0(S, \Delta) \xrightarrow{\beta} H^0(E, \Delta|_E)\) is surjective,

(b1) \(m\) is not divisible by the characteristic \(p\) of the ground field \(k\),

(b2) \(E\) is irreducible curve of arithmetic genus \(g(E)\) and \((\Delta, E) = 2g(E) - 2 - (m+1)E^2\).

(b3) \(Z := C \cap E\) is not a member of \(\Lambda'\).

**Proof.** It suffices to prove that the condition (b) of Theorem \([1.1]\) follows from the conditions (b1), (b2) and (b3) of this corollary. Since we have \(N_{S/V}(mE - C) \simeq -K_V|_S + K_S + mE - C = K_S - \Delta - mE\), there exists an isomorphism
\[
N_{S/V}(mE - C)|_E \simeq \mathcal{O}_E(K_E - \Delta - (m+1)E)
\]
of invertible sheaves on \(E\), whose degree is zero by (b2). By Lemma \([3.1]\), \(\beta|_E\) is a nonzero global section of \(N_{S/V}(mE - C)|_E\). Since \(E\) is irreducible, the invertible sheaves in \((3.6)\) are trivial. Hence as a section of \(N_{S/V}(mE)|_E\), the divisor \(\text{div}(\beta|_E)\) of zeros associated to \(\beta|_E\) coincides with \(Z\). It follows from (b3) that \(\partial_E(\beta|_E) \neq 0\) in \(H^1(E, \mathcal{O}_E((m+1)E))\). Since \(\beta|_E\) is a nonzero section of the trivial sheaf \(N_{S/V}(mE - C)|_E\), the cup product \(m\partial_E(\beta|_E) \cup \beta|_E\) is nonzero. \(\square\)

We finish this section by giving a refinement of Theorem \([1.1]\). Let \(E_i (1 \leq i \leq k)\) be irreducible curves on \(S\), which are mutually disjoint. We assume that for any two effective divisors \(D, D'\) on \(S\) with supports on \(\bigcup_{i=1}^k E_i\), if \(D \leq D'\) then the map \(H^1(S, \mathcal{O}_S(D)) \to H^1(S, \mathcal{O}_S(D'))\) is injective.
Theorem 3.3. Let $E = \sum_{i=1}^{k} m_i E_i$ be a divisor on $S$ with $m_i \geq 1$. Let $C$ or $\alpha \in H^0(C, N_{C/S})$ be a first order deformation of $C$. Suppose that $H^1(S, N_{S/V}) = 0$ and the image of the exterior component $\pi_{C/S}(\alpha)$ in $H^0(C, N_{S/V}(E)|_C)$ lifts to an infinitesimal deformation $\beta \in H^0(S, N_{S/V}(E))$ with poles along $E$. If there exists an integer $1 \leq i \leq k$ satisfying the following conditions, then the exterior component $\text{ob}_S(\alpha)$ of $\text{ob}(\alpha)$ is nonzero:

(a) Let $\Delta := C + K_V|_S - \sum_{j \neq i} (m_j + 1) E_j - 2m_i E_i$. Then the restriction map

$$H^0(S, \Delta) \to H^0(E_i, \Delta|_{E_i})$$

to $E_i$ is surjective, and

(b) Let $\beta|_{E_i} \in H^0(E_i, N_{S/V}(m_i E_i)|_{E_i})$ be the principal part of $\beta$ along $E_i$, and let $\partial|_{E_i}$ be the coboundary map of the exact sequence (1.1) for $E = E_i$. Then we have

$$m_i \partial(\beta|_{E_i}) \cup \beta|_{E_i} \neq 0 \quad \text{in} \quad H^1(E_i, N_{S/V}((2m_i + 1) E_i - C)|_{E_i}).$$

Proof. The proof is similar to that of Theorem 1.1. We follow the same steps in the proof. Let $d_S : H^0(S^o, N_{S^o/V^o}) \to H^1(S^o, \mathcal{O}_{S^o})$ be the $d$-map for $S^o \subset V^o$. Then the image $d_S(\beta)$ of $\beta \in H^0(S, N_{S/V}(E))$ is contained in $H^1(S, \mathcal{O}_S(E')) \subset H^1(S^o, \mathcal{O}_{S^o})$ by Proposition 2.8, where $E' := E + \sum_{i=1}^{k} E_i = \sum_{i=1}^{k} (m_i + 1) E_i$. There exists a partially commutative diagram similar to (3.2), which connects the two polar $d$-maps $d_{C,S}$ and $d_S$ with poles along $E$. By using this diagram, we find

$$d_{C,S}(\pi_{C/S}(\alpha)) = H^1(\iota)(d_S(\beta)|_C) \quad \text{in} \quad H^1(C, N_{C/V} \otimes N_{S/V}(E')|_C).$$

We will show that the image $\text{ob}_S(\alpha)$ of $\text{ob}(\alpha)$ in $H^1(C, N_{S/V}(E')|_C)$ is nonzero. Using the argument in Step 1 before, we get $\text{ob}_S(\alpha) = d_S(\beta)|_C \cup \pi_{C/S}(\alpha)$, where the cup product is taken by the map

$$H^1(C, \mathcal{O}_C(E')) \times H^0(C, N_{S/V}|_C) \to H^1(C, N_{S/V}(E')|_C).$$

We consider a cup product

$$\text{ob}_S(\alpha) \cup k_C \in H^2(S, N_{S/V}(E' - C))$$

of $\text{ob}_S(\alpha)$ with $k_C \in \text{Ext}^1(\mathcal{O}_C, \mathcal{O}_S(-C))$. There exists a commutative diagram

$$
\begin{array}{ccc}
H^1(\mathcal{O}_C(E')) & \times & H^0(N_{S/V}|_C) \\
\cup & & \cup \\
H^1(\mathcal{O}_C(E')) & \times & H^0(N_{S/V}((m_i - 1) E_i)|_C) \\
\cup & & \cup \\
H^1(\mathcal{O}_S(E')) & \times & H^1(N_{S/V}((m_i - 1) E_i - C)) \\
\cup & & \cup \\
H^1(\mathcal{O}_S(E')) & \times & H^1(N_{S/V}((m_i - 1) E_i - C))
\end{array}
$$

with

$$
\begin{array}{ccc}
H^1(N_{S/V}(E')|_C) & \to & H^1(N_{S/V}(E' + (m_i - 1) E_i)|_C) \\
\cup & & \cup \\
H^2(N_{S/V}(E' + (m_i - 1) E_i - C)) & \to & H^2(N_{S/V}(E' + (m_i - 1) E_i - C)).
\end{array}
$$
By using this diagram, we compute the image of \( \overline{\text{obs}}(\alpha) \cup k_C \) in \( H^2(S, N_{S/V}(E' + (m_i - 1)E_i - C)) \) as

\[
\overline{\text{obs}}(\alpha) \cup k_C = (d_S(\beta))_C \cup (\pi_{C/S}(\alpha))_C \cup k_C
\]

\[
= (d_S(\beta))_C \cup (\pi_{C/S}(\alpha))_C \cup k_C
\]

\[
= d_S(\beta) \cup (\pi_{C/S}(\alpha)) \cup k_C
\]

\[
= d_S(\beta) \cup (\beta)_C \cup k_C,
\]

where \( \pi_{C/S}(\alpha) \) is the image of \( \pi_{C/S}(\alpha) \) in \( H^0(C, N_{S/V}((m_i - 1)E_i)|_C) \). There exists a commutative diagram

\[
\begin{align*}
H^1(O_{E_i}(E')) 
\times H^1(N_{S/V}((m_i - 1)E_i - C))
\xrightarrow{\cup k_{E_i}}
H^2(N_{S/V}(E' + (m_i - 1)E_i - C)) \\
H^1(O_{E_i}((m_i + 1)E_i))
\times H^0(N_{S/V}(m_iE_i - C)|_{E_i})
\xrightarrow{\cup k_{E_i}}
H^1(N_{S/V}((2m_i + 1)E_i - C)|_{E_i}).
\end{align*}
\]

Then by [17] Lemma 2.8 and Proposition [28] 2), we compute as

\[
d_S(\beta) \cup (\beta)_C \cup k_{E_i}
= d_S(\beta) \cup (\beta)_C \cup k_{E_i}
= (d_S(\beta))_{E_i} \cup (\beta)_{E_i} \cup k_{E_i}
= (m_i\partial_{E_i}\beta)_{E_i} \cup (\beta)_C \cup k_{E_i},
\]

where \( m_i\partial_{E_i}\beta|_{E_i} \cup (\beta)_{E_i} \cup k_{E_i} \neq 0 \) by assumption. It follows from the assumption (a) that the cup product map

\[
H^1(E_i, N_{S/V}((2m_i + 1)E_i - C)|_{E_i}) \xrightarrow{\cup k_{E_i}} H^2(S, N_{S/V}(E' + (m_i - 1)E_i - C))
\]

with \( k_{E_i} \in \text{Ext}^1(O_{E_i}, O_S(-E_i)) \) is injective. Therefore, we have \( (m_i\partial_{E_i}\beta|_{E_i}) \cup (\beta)_{E_i} \cup k_{E_i} \neq 0 \) and thus we have completed the proof.

\[
\square
\]

4. Obstructions to deforming curves lying on a K3 surface

In this section, we prove Theorem [1.2] and Corollary [1.3]. In this and later sections, we assume that char \( k = 0 \). Let \( C \subset S \subset V \) be as in the theorem. Then by Lemma [2.10] the Hilbert-flag scheme \( HFV \) is nonsingular at \( (C, S) \), and moreover, \( A^2(C, S) = 0 \). Put \( D := C + KV|_S \), a divisor on \( S \). Then by the same lemma together with the Serre duality, we have \( H^i(S, N_{S/V}(-C)) \simeq H^i(S, -D) \simeq H^{2-i}(S, D)^\vee \) for any integer \( i \).

Proof of Theorem [1.2] (1) It is known that if there exist no \((-2)\)-curves and no elliptic curves on a smooth K3 surface \( X \), then every nonzero effective divisor on \( X \) is ample. (Then the effective cone \( \text{NE}(X) \) and the ample cone \( \text{Amp}(X) \) coincide.) Therefore we have \( H^1(S, D) = 0 \) by assumption. Then the restriction map \( \rho : H^0(S, N_{S/V}) \rightarrow H^0(C, N_{S/V}|_C) \) is surjective. Then by Lemma [2.9](1), Hilb \( V \) is nonsingular at \([C]\).
We show that the tangent map $p_1$ of $pr_1 : HF V \to \text{Hilb} V$ at $(C, S)$ is not surjective and its cokernel is of dimension 1. Since $D \geq 0$ and $D \neq 0$, we have $H^0(S, -D) = 0$. Therefore by (2.14), there exists an exact sequence

\[(4.1) 0 \rightarrow A^1(C, S) \xrightarrow{p_1} H^0(C, N_{C/V}) \rightarrow H^1(S, -D) \rightarrow 0.\]

**Claim 4.1.** $H^1(S, -D) \simeq k$ and $H^1(S, -D + E) = 0$

**Proof of Claim 4.1.** Since $D.E = -2$ and $E^2 = -2$, there exists an exact sequence

\[(4.2) 0 \rightarrow O_S(D - (l + 1)E) \rightarrow O_S(D - lE) \rightarrow O_{p_1}(2l - 2) \rightarrow 0\]

for every integer $l$. Since $H^1(\mathbb{P}^1, O_{p_1}(2l - 2)) = 0$ for $l \geq 1$ and $H^1(S, D - 3E) = 0$, it follows from this exact sequence that $H^1(S, D - lE) = 0$ for $l = 1, 2$. We prove $H^2(S, D - E) = 0$. In fact, if $H^2(S, D - E) \neq 0$, then by the Serre duality, $-D + E$ is effective and we have $(-D + E)^2 = D^2 + 2 \geq 2 > 0$. Then it follows from the signature theorem (cf. [1, IV.2, Thm. 2.14 and VIII.1]) that $0 \leq (D - D + E) = -D^2 - 2 < 0$ and hence we get a contradiction. Therefore by (4.2), we have $H^1(S, D) \simeq H^1(\mathbb{P}^1, O_{p_1}(-2)) \simeq k$. By the Serre duality, we have proved the claim.

Let $\alpha$ be a global section of $N_{C/V}$. It suffices to prove the next claim.

**Claim 4.2.** $\text{ob}_S(\alpha) \neq 0$ if $\alpha \notin \text{im} p_1$.

**Proof of Claim 4.2.** Let $\pi_{C/S}(\alpha) \in H^0(C, N_{S/V}|_C)$ be the exterior component of $\alpha$. There exists a commutative diagram

\[
\begin{array}{ccccccc}
0 & \longrightarrow & A^1(C, S) & \xrightarrow{p_1} & H^0(C, N_{C/V}) & \longrightarrow & H^1(S, -D) & \longrightarrow & 0 \\
\downarrow{p_2} & & \downarrow{\pi_{C/S}} & & \downarrow{=} & & \downarrow{=} & & \downarrow{=} \\
0 & \longrightarrow & H^0(S, N_{S/V}) & \xrightarrow{\rho} & H^0(C, N_{S/V}|_C) & \cup_{kC} & H^1(S, -D) & \longrightarrow & 0,
\end{array}
\]

where $k_C$ is the extension class of (3.1). By this diagram, we see that $\pi_{C/S}(\alpha)$ is not contained in $\text{im} \rho$, and hence $\pi_{C/S}(\alpha) \cup k_C \neq 0$ in $H^1(S, -D)$. On the other hand, since $H^1(S, -D + E) = 0$, we have $\pi_{C/S}(\alpha)|_E \cup k_E = 0$ and $\pi_{C/S}(\alpha)$ lifts to an infinitesimal deformation $\beta \in H^0(S, N_{S/V}(E))$ with a pole along $E$ by Lemma 3.11(1). Then by (2) of the same lemma, the principal part $\beta|_E$ of $\beta$ is a nonzero global section of $N_{S/V}(E)|_E$, and its divisor of zeros contains $Z := C \cap E$ (i.e. $Z \subset \text{div}(\beta|_E)$).

Now we verify that the four conditions (a), (b1), (b2) and (b3) of Corollary 3.2 are satisfied. Put $\Delta = C + K_V|_S - 2E$, a divisor on $S$. Since $H^1(S, \Delta - E) = H^1(S, D - 3E) = 0$, (a) follows from the exact sequence $0 \rightarrow O_S(\Delta - E) \rightarrow O_S(\Delta) \rightarrow O_E(\Delta) \rightarrow 0$. (b1) is clear. Since $E$ is a $(-2)$-curve, we compute that $\Delta.E = (D - 2E).E = 2 = 2g(E) - 2 - 2E^2$, and hence (b2) follows. Then by (3.6), this implies that $N_{S/V}(E - C)|_E$ is trivial and hence we have $Z = \text{div}(\beta|_E)$. Finally, for (b3), we show that $H^1(S, C - E) = 0$. In fact, we have $C - E = D - E - K_V|_S$. Since $H^1(S, D - E) = 0$, we have $(D - E.E') \geq -1$
for any \((-2)\)-curve \(E'\) on \(S\) by Lemma \(2.12(2)\), which implies that \(C - E\) is nef because \(-K_V|_S\) is ample. Then \(C - E\) is big by
\[
(C - E)^2 = (D - E)^2 + 2(D - E - K_V|_S) + (-K_V|_S)^2 > (D - E)^2 > 0.
\]
Therefore \(H^1(S, C - E) = 0\) by the Kodaira-Ramanujam vanishing theorem. Then the rational map
\[
|C| \to |O_E(C)|,
\]
\[
C' \mapsto Z = C' \cap E
\]
is dominant. By assumption, \(\Lambda' = \{\text{div}(\gamma) \mid \gamma \in \text{im } \pi_{E/S}(E)\}\) is a proper linear subsystem of \(\Lambda = |N_{S/V}(E)|_E\). Therefore, if necessary, by replacing \(C\) with a general member \(C'\) of \(|C|\), we may assume that \(Z = C \cap E\) is not contained in \(\Lambda'\). In fact, by the upper semicontinuity, if a general member \(C' \in |C|\) is obstructed, then so is \(C\). Hence (b3) follows. By Corollary 3.2 we have proved the claim.

(3) The proof is very similar to that of (2). Suppose that \(D \sim mF\) for \(m \geq 2\) and an elliptic curve \(F\). Then by Lemma 2.12(2), we have \(H^1(S, -D) \simeq k^{m-1}\). Thus the cokernel of the tangent map \(p_1\) of \(pr_1\) is nonzero. Since \(H^1(S, -D + F) \simeq k^{m-2}\), the kernel of the natural map \(H^1(S, -D) \to H^1(S, -D + F)\) is of dimension at least one. Hence by (11), there exists a global section \(\alpha\) of \(N_{C/V}\) whose exterior component \(\pi_{C/S}(\alpha)\) satisfies \(\pi_{C/S}(\alpha) \cup k_C \neq 0\) in \(H^1(S, -D)\), while \(\pi_{C/S}(\alpha)_{(1)} \cup k_C = 0\) in \(H^1(S, -D + F)\). We fix such a global section \(\alpha\) and prove that \(\text{ob}_{\alpha}(\alpha) \neq 0\). Again by Lemma 3.1, there exists an infinitesimal deformation \(\beta \in H^0(S, N_{S/V}(F))\) with a pole along \(F\) such that \(\beta|_C = \pi_{C/S}(\alpha)_{(1)}\) and \(\beta|_F\) is a nonzero global section of \(N_{S/V}(F)|_F\). Thus it suffices to verify the conditions (a), (b1), (b2) and (b3) of Corollary 3.2. (a) follows from \(\Delta = (m - 2)F\) and \(O_F(F) \simeq O_F\). (b1) is clear. (b2) follows from \(\Delta F = 2g(F) - 2 - 2F^2 = 0\). Since \(C - F = -K_V + (m - 1)F\) is ample, we have \(H^1(S, C - F) = 0\) by the Kodaira vanishing theorem. Hence (b3) follows. The rest of the proof is same as that of (2).

Proof of Corollary 1.3. Let \(W_{C,S}\) be the \(S\)-maximal family of curves containing \(C\). If \(H^1(S, D) = 0\), then by Lemma 2.9(1), \(pr_1\) is smooth. Since a smooth morphism is flat and a flat morphism maps a generic point onto a generic point, we have the conclusion of (a) (cf. [9 Corollary 1.3.5]). Suppose that \(h^1(S, D) = 1\). Then it follows from (11) that
\[
\dim W_{C,S} \leq \dim_{[C]} \text{Hilb}^s V \leq h^0(C, N_{C/V}) = \dim W_{C,S} + 1.
\]

Since \(C\) is obstructed by the theorem, we have \(\dim_{[C]} \text{Hilb}^s V = \dim W_{C,S}\). Therefore (a) follows. Since \(C\) is a generic member of \(W_{C,S}\), we obtain (b). If \(H^0(S, -D) = 0\), then by Lemma 2.9(2) and Lemma 2.10 we obtain \(\dim_{[C]} \text{Hilb}^s V = \dim W_{C,S} = (-K_V|_S)^2/2 + g + 1\). Thus we have completed the proof.

5. Non-reduced components of the Hilbert scheme

In this section, as an application, we study the deformations of curves lying on a smooth quartic surface \(S\) in \(\mathbb{P}^3\), or a smooth hyperplane section \(S\) of a smooth quartic
3-fold $V_4 \subset \mathbb{P}^3$ (assuming char $k = 0$). We give some examples of generically non-reduced components of the Hilbert schemes $\text{Hilb}^{sc} \mathbb{P}^3$ and $\text{Hilb}^{sc} V_4$ (cf. Examples 5.8 and 5.12). As is well known, $S$ is a K3 surface. Here we consider $S$ (i) of Picard number two $(\rho(S) = 2)$, (ii) containing a smooth curve $E$ not a complete intersection in $S$, and such that (iii) Pic $S$ is generated by $E$ and the class $h$ of hyperplane sections of $S$ (i.e., $\text{Pic} S \simeq \mathbb{Z} h \oplus \mathbb{Z} E$). Kleppe and Ottem [11] have studied the deformations of space curves $C \subset \mathbb{P}^3$ lying on such a quartic surface $S$ by assuming that $E$ is a line, or a conic. They have also produced examples of generically non-reduced components of $\text{Hilb}^{sc} \mathbb{P}^3$ by a different method (cf. Remark 5.6).

5.1. **Mori cone of quartic surfaces.** Let $S \subset \mathbb{P}^3$ be a smooth quartic surface. If $S$ is general, then we have $\rho(S) = 1$ and Pic $S$ is generated by $h = -\frac{1}{4}K_{\mathbb{P}^3}|_S$. Then every curve $C$ on $S$ is a complete intersection of $S$ with some other surface in $\mathbb{P}^3$, and hence $C$ is arithmetically Cohen-Macaulay. Thus we see that $C$ is unobstructed, thanks to a result of Ellingsrud [1].

First we consider a quartic surface $S$ containing a (smooth) rational curve $E$. Then by its genus, $E$ is not a complete intersection in $S$. It follows from Lemma 2.13 that for every integer $e \geq 1$, there exists a smooth quartic surface $S \subset \mathbb{P}^3$ containing a rational curve $E$ of degree $e$, and such that $\text{Pic} S \simeq \mathbb{Z} h \oplus \mathbb{Z} E$. Let $(S, E)$ be such a pair of a surface $S$ and a curve $E \simeq \mathbb{P}^1$. Then every divisor $D$ on $S$ is linearly equivalent to $xh - yE$ for some $x, y \in \mathbb{Z}$. Since we have $h^2 = 4, h.E = e$ and $E^2 = -2$, we compute the self intersection number of $D$ on $S$ as $D^2 = 4x^2 - 2exy - 2y^2$. Recall that for a projective surface $X$, the effective cone $\text{NE}(X)$ of $X$ is defined as $\text{NE}(X) = \{ \sum_{i=1}^n a_i[C_i] \mid C_i \text{ is an irreducible curve on } X \text{ and } a_i \in \mathbb{R}_{\geq 0} \}$ and the Mori cone $\overline{\text{NE}}(X)$ is defined as the closure of $\text{NE}(X)$ in $\text{NS}(X)_{\mathbb{R}}$. Kovács [14] proved that for every $K3$ surface $X$ with $\rho(X) = 2$, $\overline{\text{NE}}(X)$ has two extremal rays, which can be generated by the classes of two $(-2)$-curves, one $(-2)$-curve and an elliptic curve, two elliptic curves, or two non-effective classes $x_1, x_2$ with $x_1^2 = 0$. Applying Kovács’s result, we have the following lemma.

**Lemma 5.1.** Let $S$ be a smooth quartic surface, $E$ a smooth rational curve of degree $e \geq 2$ on $S$ such that $\text{Pic} S = \mathbb{Z} h \oplus \mathbb{Z} E$, $D$ a divisor on $S$. Then

1. If $D$ is nef and only if $E.D = 0$.
2. There exists a (unique) $(-2)$-curve $E'$ on $S$ such that $\overline{\text{NE}}(S) = \mathbb{R}_{\geq 0}[E] + \mathbb{R}_{\geq 0}[E']$.
3. $D$ is nef if and only if $D.E \geq 0$.
4. Suppose that $D \geq 0$ and $D \neq 0$. Then a general member $C$ of $|D|$ is a smooth connected curve if and only if $D$ is nef, $D = E$, or $D = E'$.
5. If (i) $D \geq 0$ and $D$ is nef or (ii) $D = E, E'$, then $H^1(S, D) = 0$.

**Proof.** Since $E$ spans one of the two extremal rays, for proving (1) and (2), it suffices to prove that there exist no elliptic curves on $S$. Suppose that there exists a nonzero
divisor $D \sim xh - yE$ on $S$ with $D^2 = -2(y^2 + exy - 2x^2) = 0$. Then the discriminant $d = (ex)^2 - 4(-2x^2) = (8 + e^2)x^2$ of this quadratic equation (with a variable $y$) equals a power of an integer. Since $D \not\sim 0$ in Pic $S$, we have $8 + e^2 = k^2$ for some integer $k \geq 1$. By solving this equation, we have $(k, e) = (3, 1)$, or $(9/2, 7/2)$, which are both impossible by assumption. Since the nef cone and the Mori cone are dual to each other, we have (3).

Remark 5.2. (1) If $e = 1$, then $\overline{\text{NE}}(S)$ is generated by the classes of a line $E$ and a smooth elliptic curve $F \sim h - E$ of degree 3, contained in a plane in $\mathbb{P}^3$ (cf. [11, Proposition 5.1]).

(2) Once $e \geq 2$ is given, the class $xh - yE$ of the curve $E' \simeq \mathbb{P}^1$ spanning the other ray of $\overline{\text{NE}}(S)$ can be explicitly computed as the minimal nonzero solution $(x, y)$ of the equation

$$y^2 + exy - 2x^2 = 1,$$

which is equivalent to $(E')^2 = -2$. If $e$ is even, i.e., $e = 2m$ for $m \geq 1$, then we have $E' = mh - E$. If $e$ is odd, then we see that $y$ is odd and $x$ is even ($x = 2x'$), and the above equation is reduced to the Pell equation $X^2 - (e^2 + 8)Y^2 = 1$, by putting $X := y + ex'$ and $Y := x'$. Thus the class $xh - yE$ of $E'$ in Pic $S$ is obtained as the minimal solution of this Pell equation and computed as in Table 1.

| $e$  | 2   | 3    | 4    | 5    | 6    | 7    | 8     | 9     | \ldots |
|------|-----|------|------|------|------|------|-------|-------|--------|
| $(x, y)$ | (1, 1) | (16, 9) | (2, 1) | (8, 3) | (3, 1) | (40, 11) | (4, 1) | (106000, 23001) | \ldots |
| $d(E')$ | 2   | 37   | 4    | 17   | 6    | 83   | 8     | 216991 | \ldots |

Table 1. Classes of $E'$

Secondly, we consider elliptic quartic surfaces. It follows from Lemma 2.13 that for every integer $e \geq 3$, there exists a smooth quartic surface $S \subset \mathbb{P}^3$ containing a smooth elliptic curve $F$ of degree $e$ such that Pic $S \simeq Z h \oplus Z F$.

Lemma 5.3. Let $S$ be a smooth quartic surface, $F$ a smooth elliptic curve of degree $e \geq 4$ on $S$ such that Pic $S = Z h \oplus Z F$, $D$ a divisor on $S$. Then

(1) $D^2 \neq -2$. In particular, there exists no $(-2)$-curve on $S$.

(2) There exists a smooth elliptic curve $F'$ on $S$ such that $\overline{\text{NE}}(S) = \mathbb{R}_{\geq 0}[F] + \mathbb{R}_{\geq 0}[F']$.

Moreover, the class of $F'$ in Pic $S$ equals $eh - 2F$ if $e$ is odd, and $(e/2)h - F$ otherwise.

(3) $D$ is nef if and only if $D \geq 0$. 
(4) Suppose that \( D \geq 0 \) and \( D \neq 0 \). Then the following are equivalent: (i) every general member \( C \) of \( |D| \) is a smooth connected curve, (ii) \( D \not\sim kF \) and \( D \not\sim kF' \) for \( k \geq 2 \), (iii) \( H^1(S, D) = 0 \)

Proof. If \( D \sim xh - yF \) with \( x, y \in \mathbb{Z} \), then we have \( D^2 = 2x(2x - ye) \) by \( h^2 = 4, h.F = e \) and \( F^2 = 0 \). The Diophantine equation \( x(2x - ye) = -1 \) on \( (x, y) \) has no solutions if \( e \neq 1, 3 \). Thus we obtain (1). Since \( F \) spans one of the extremal rays, by Kovács’s result, the other ray is spanned by the class of a smooth elliptic curve \( F' \) on \( S \). In fact, by solving the equation \( x(2x - ye) = 0 \), we see that if \( D^2 = 0 \) and \( D \) is not spanned by \( F \), then \( D \) is spanned by the (primitive) classes given in the lemma. Thus we obtain (2). (3) follows from (2), because \( \overline{\text{NE}}(S) \) is dual to itself. For (4), suppose that \( D \geq 0 \) and \( D \neq 0 \). Then \( |D| \) has no fixed component. Hence by [21] Prop. 2.6, if \( D^2 > 0 \), then every general member \( C \) of \( |D| \) is a smooth connected curve and \( h^1(S, D) = 0 \), and otherwise \( D \) is a multiple \( kF \) (\( k \geq 1 \)) of a smooth elliptic curve \( F \) on \( S \) and \( h^1(S, D) = k - 1 \). \( \square \)

Finally, we consider quartic surfaces without \((-2)\)-curves nor elliptic curves. Let \( S \) be a smooth quartic surface not containing any \((-2)\)-curves nor smooth elliptic curves. Then \( \overline{\text{NE}}(S) \) coincides with the ample cone \( \text{Amp}(S) \) of \( S \). In particular, \( H^1(S, D) \) vanishes for every effective divisor \( D \) on \( S \). If \( \rho(S) = 2 \), then \( \overline{\text{NE}}(S) = \mathbb{R}_{\geq 0}[x_1] + \mathbb{R}_{\geq 0}[x_2] \) for two non-effective classes \( x_i \) \((i = 1, 2)\) with \( x_i^2 = 0 \). We give an example of such a quartic surface.

Example 5.4. It follows from Lemma [21,3] that there exists a smooth quartic surface \( S \subset \mathbb{P}^3 \) and a smooth connected curve \( \Gamma \) on \( S \) of degree 6 and genus 2 such that \( \text{Pic}(S) = \mathbb{Z} h \oplus \mathbb{Z} \Gamma \). Then on \( S \) there exist no divisors \( D \) with \( D^2 = -2 \) and no divisors \( D \) with \( D^2 = 0 \) and \( D \not\sim 0 \). In fact, let \( D \sim xh - y\Gamma \) for \( x, y \in \mathbb{Z} \). Then we have \( D^2 = 4x^2 - 12xy + 2y^2 = (2x - 3y)^2 - 7y^2 \). We have \( D^2 \neq -2 \), because \(-2 \equiv 5 \) is not a quadratic residue modulo 7. If \( D \not\sim 0 \), then we have \( D^2 \neq 0 \) as well. Thus we conclude that there exist no \((-2)\)-curves and no smooth elliptic curves on \( S \).

5.2. Hilbert schemes of \( \mathbb{P}^3 \) and \( V_4 \). Let \( V \) be \( \mathbb{P}^3 \) or a smooth quartic 3-fold \( V_4 \subset \mathbb{P}^4 \), \( S \) a smooth quartic surface in \( \mathbb{P}^3 \) or a smooth hyperplane section of \( V_4 \). It is known that if \( S \) is general (in \( |\mathcal{O}_{\mathbb{P}^3}(4)| \) or in \( |\mathcal{O}_{V_4}(1)| \)), the Picard group of \( S \) is generated by the class \( h \) of hyperplane sections of \( S \) (see e.g. [15] for \( V = V_4 \)). Let \( C \) be a smooth connected curve of degree \( d \) (= \( C.h \)) and genus \( g \) in \( S \), not a complete intersection in \( S \). Then there exists a first order deformation \( \tilde{S} \) of \( S \) not containing any first order deformation \( \tilde{C} \) of \( C \). Then by Lemma [2,10] \( HF \) is nonsingular at \((C, S)\) of dimension \( a^1(C, S) = (-K_S|_S)\)\(s\)\(2 + g + 1 \). Since \( K_{\mathbb{P}^3}|_S = -4h \) and \( K_{V_4}|_S = -h \), the number \( a^1(C, S) \) equals \( g + 33 \) if \( V = \mathbb{P}^3 \), and \( g + 3 \) if \( V = V_4 \). Let \( W_{C,S} \subset \text{Hilb}^a V \) be the \( S \)-maximal family of curves containing \( C \) (cf. Definition [2,11]). Then if \( d > 16 \) (resp. \( d > 4 \)), then we have \( H^0(S, N_{S/V}(-C)) = 0 \) and hence \( \dim W_{C,S} = a^1(C, S) \). In what follows, we assume that \( d > 16 \) if \( V = \mathbb{P}^3 \) and \( d > 4 \) if \( V = V_4 \).
**Theorem 5.5.** Let $V = \mathbb{P}^3$ or $V = V_4$, and let $S, C$ be as above. Suppose that there exists a smooth rational curve $E$ of degree $e \geq 2$ on $S$ such that $\text{Pic} S = \mathbb{Z} h \oplus \mathbb{Z} E$. Let $E'$ be as in Lemma 5.1 and suppose that $D := C + K_V|_S$ is effective.

1. If $D$ is nef, or $D = E, E'$, then $W_{C,S}$ is a generically smooth component of $\text{Hilb}^{sc} V$. 
2. Suppose that $N_{E/V}$ is globally generated if $V = V_4$. If $D.E = -2$ and $D \neq E$, then $W_{C,S}$ is a generically non-reduced component of $\text{Hilb}^{sc} V$.

**Proof.** (1) follows from Corollary 1.3 and Lemma 5.1. We prove (2). For an invertible sheaf $L \sim xh + yE$ on $S$, we have $L.E = 0$ if and only if $ex - 2y = 0$. This implies that if $L.E = 0$, then the class of $L$ in $\text{Pic} S$ is spanned by the class $2h + eE$ if $e$ is odd, and the class $h + (e/2)E$ otherwise. By assumption, there exists an integer $k \geq 1$ such that $D$ is linearly equivalent to the class

$$\begin{cases} k(2h + eE) + E & \text{(if } e \text{ is odd)}, \\ k(h + (e/2)E) + E & \text{(otherwise)}. \end{cases}$$

Since $(D - E.E) = 0$, we have $D^2 = (D - E)^2 + E^2 = (D - E)^2 - 2 > 0$. We show that $H^1(S, D - 3E) = 0$. Suppose that $e$ is even and put $e = 2e'$ $(e' \geq 1)$. If $k \geq 2$ or $e' \geq 2$, then $D - 3E$ is nef and effective, because $(D - 3E.E) = 4$ and $(D - 3E.E') = k(h.E' + (ke' - 2)E.E') > 0$. If $k = 1$ and $e' = 1$, then $D - 3E = h - E = E'$. Thus we conclude that $H^1(S, D - 3E) = 0$ by Lemma 5.1. Similarly, we can show that $H^1(S, D - 3E) = 0$ if $e$ is odd. It follows from Lemma 2.14 that $\pi_{E/S}(E)$ is not surjective. Therefore, by Corollary 1.3 $W_{C,S}$ is a generically non-reduced component of $\text{Hilb}^{sc} V$.

In Theorem 5.5, if $C \sim ah + bE$ $(b \neq 0$ by assumption), then we have $d = 4a + be$ and $g = 2a^2 + abe - b^2 + 1$. Given a class $(a, b)$ of $C$ in $\text{Pic} S$, one can check whether $D$ is nef or not by using Lemma 5.1(3) together with the method to compute the class of $E'$ shown in Remark 5.2.

**Remark 5.6.** Kleppe and Ottem [11] have also studied the deformations of space curves lying on a smooth quartic surface. They have considered a smooth quartic surface $S \subset \mathbb{P}^3$ containing a line $E$, and such that $\text{Pic} S \simeq \mathbb{Z} h \oplus \mathbb{Z} E$, and a curve $C \subset S$ of degree $d > 16$ and genus $g$, not a complete intersection in $S$, satisfying $D := C - 4h \geq 0$. Then they have proved that if $D.E \geq -1$ or $D \sim E$, then $W_{C,S}$ becomes a generically smooth component of $\text{Hilb}^{sc} \mathbb{P}^3$, and if $D.E \leq -2$ (i.e., $D.E = -2, -3, \text{ or } -4$), $D \neq E$, $d \geq 21$ and $g > \min \{G(d, 5) - 1, d^2/10 + 21\}$, then $W_{C,S}$ becomes a generically non-reduced component of $\text{Hilb}^{sc} \mathbb{P}^3$. Here $G(d, 5)$ denotes the maximum genus of curves of degree $d$ not contained in a surface of degree 4. More recently, we have been informed by Kleppe that if $D.E \leq -2$, then the assumption on the genus $g$ is almost always satisfied (with several exceptions of the classes of $C$ in $\text{Pic} S \simeq \mathbb{Z}^2$). See [11] for the details.
Remark 5.7. For a space curve $C \subset \mathbb{P}^3$, $s(C)$ denotes the minimal degree of surfaces containing $C$. An irreducible closed subset $W$ of $\text{Hilb}^{sc}\mathbb{P}^3$ is called $s$-maximal, if a general member $C$ of $W$ is contained in a surface of degree $s(C) = s$ (i.e. $s(W) = s$), and if we have $s(W') > s(W)$ for any closed irreducible subset $W' \subset \text{Hilb}^{sc}\mathbb{P}^3$ strictly containing $W$ (cf. [10]). If $C \subset \mathbb{P}^3$ is contained in a surface $S \subset \mathbb{P}^3$ of degree $s = s(C)$, then every $s$-maximal subset is a $S$-maximal family, i.e. the image of some irreducible component of $\text{HF}_n^{sc}\mathbb{P}^3$ passing through $(C, S)$. It is known that if a very general curve $C$ of a 4-maximal family $W$ sits on a smooth quartic surface $S$ and $d(C) > 16$, then the Picard number is at most 2 (cf. [11, Remark 2.3]).

We give infinitely many examples of generically non-reduced components of $\text{Hilb}^{sc}V_4$, which contains Example 1.4(1) as a special case ($n = 2$).

Example 5.8. Let $V_4 \subset \mathbb{P}^4$ be a smooth quartic 3-fold, $E \simeq \mathbb{P}^1$ a conic on $V_4$ with trivial normal bundle $N_{E/V_4} \simeq \mathcal{O}_{\mathbb{P}^1}^2$, $S$ a smooth hyperplane section of $V_4$ containing $E$ and such that $\text{Pic} S = \mathbb{Z}h \oplus \mathbb{Z}E$. We consider a complete linear system $\Lambda_n := |n(h + E)|$ on $S$ for an integer $n \geq 2$. Then $\Lambda_n$ is base point free and a general member $C$ of $\Lambda_n$ is a smooth connected curve of degree $6n$ and genus $3n^2 + 1$. Let $W_n$ be the $S$-maximal family $W_{C,S}$ of curves containing $C$. Since $D.E = (C - h.E) = -2$ and $N_{E/V_4}$ is globally generated, $W_n$ becomes generically non-reduced components of $\text{Hilb}^{sc}V_4$ of dimension $3n^2 + 4$ by Theorem 5.5.

Remark 5.9. It was proved in [11] Theorem 1.3] that for every smooth cubic 3-fold $V_3 \subset \mathbb{P}^4$, the Hilbert scheme $\text{Hilb}^{sc}V_3$ contains a generically non-reduced component $W$ of dimension 16. Then every general member $C$ of $W$ is a smooth connected curve of degree 8 and genus 5, and for each $C$, there exist a smooth hyperplane section $S_3$ of $V_3$ and a line $E$ on $S_3$ satisfying the linear equivalence $C \sim 2h + 2E \sim -K_{V_3}|_{S_3} + 2E$. It was also proved that if $N_{E/V_3}$ is trivial (in other words, $E$ is a good line on $V_3$), then $C$ is primarily obstructed. We refer to [20] for a generalization to a smooth del Pezzo 3-fold $V_n$ of degree $n$.

We have a similar result for curves lying on an elliptic quartic surface. We have the next theorem as a consequence of Lemma 5.3 Lemma 2.14 and Corollary 1.3.

Theorem 5.10. Let $V = \mathbb{P}^3$ or $V = V_4$, and let $S, C$ be as above. Suppose that there exists a smooth elliptic curve $F'$ of degree $e \geq 4$ on $S$ such that $\text{Pic} S = \mathbb{Z}h \oplus \mathbb{Z}F'$. Let $F'$ be as in Lemma 5.3 and we assume that $D := C + K_{V}|_{S}$ is effective.

1. If $D \not\sim kF$ and $D \not\sim kF'$ for $k \geq 2$, then $W_{C,S}$ is a generically smooth component of $\text{Hilb}^{sc}V$.

2. If $D \sim 2F'$ or $D \sim 2F$, then $W_{C,S}$ is a generically non-reduced component of $\text{Hilb}^{sc}V$. 
Remark 5.11. In Theorem 5.10, we have

(1) If \( C \sim ah + bF \) (\( b \neq 0 \) by assumption), then \( d = 4a + be \) and \( g = 2a^2 + abe + 1 \).

(2) Theorem 1.2 shows that if \( D \sim mE \) for some integer \( m \geq 2 \) and an elliptic curve \( E \) on \( S \), then \( C \) is obstructed. In this case, we have \( h^1(S, D) = m - 1 \). However, Theorems 1.1 and 1.2 are not sufficient for proving that \( W_{C,S} \) is an irreducible component of \( (\text{Hilb}^{sc} \mathbb{P}^3)_{\text{red}} \) (or \( (\text{Hilb}^{sc} \mathbb{V}_4)_{\text{red}} \)) for \( m > 2 \).

Example 5.12. Let \( V = \mathbb{P}^3 \), \( S \) a smooth quartic surface containing a smooth elliptic curve \( F \). Then the complete linear system \( \Lambda := |4h + 2F| \) on \( S \) is base point free and every general member \( C \) of \( \Lambda \) is a smooth connected curve on \( S \). Then \( C \) is obstructed by Theorem 1.2 and moreover, \( W_{C,S} \) is a generically non-reduced component of \( \text{Hilb}^{sc} \mathbb{P}^3 \) by Corollary 1.3.

One can compare this example with Mumford’s example 18 of a generically non-reduced component \( W \) of \( \text{Hilb}^{sc} \mathbb{P}^3 \), whose general member \( C \) is contained in a smooth cubic surface \( S \) and \( C \sim 4h + 2E \) on \( S \) for a line \( E \subset S \).

Theorem 5.13. Let \( V = \mathbb{P}^3 \) or \( V = \mathbb{V}_4 \), and let \( S, C \) be as above. Suppose that there exist no \((-2)\)-curves nor elliptic curves on \( S \). If \( D := C + K_V |_S \) is effective, then \( W_{C,S} \) is a generically smooth component of \( \text{Hilb}^{sc} \mathbb{V} \).

See Example 5.4 for an example of a smooth quartic surface not containing any \((-2)\)-curves nor elliptic curves.

Acknowledgments

I should like to thank Prof. Eiichi Sato for asking me a useful question concerning the existence of a non-reduced component of the Hilbert scheme of a smooth Fano 3-fold of index one. I am grateful to Prof. Jan Oddvar Kleppe for very helpful comments and a discussion on non-reduced components of the Hilbert scheme of space curves. Also thanks to the referee for his/her helpful comments. This work was partially supported by JSPS Grant-in-Aid (C), No 25400048, and JSPS Grant-in-Aid (S), No 25220701.

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