Entanglement of internal and external angular momenta of a single atom

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We consider the exchange of spin and orbital angular momenta between a circularly polarized Laguerre-Gaussian beam of light and a single atom trapped in a two-dimensional harmonic potential. The radiation field is treated classically but the atomic center-of-mass motion is quantized. The spin and orbital angular momenta of the field are individually conserved upon absorption, and this results in the entanglement of the internal and external degrees of freedom of the atom. We suggest applications of this entanglement in quantum information processing.

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1. INTRODUCTION

The Laguerre-Gaussian (LG) laser modes are known to possess well-defined, discrete values of orbital angular momentum per unit energy [1]. The orbital angular momentum of the field is distinct from the spin angular momentum associated with the polarization state of the field. In the paraxial limit, the orbital angular momentum is polarization-independent [2] and arises solely from the azimuthal phase dependence of the field mode which gives rise to helical wavefronts.

The interaction of LG modes with atoms has been studied extensively in the classical limit of the atom as a point particle [3]. It has been shown that the atom experiences a torque from the radiation pressure force, which transfers the angular momentum from the laser beam to the atom. This effect has been indirectly observed in the nonlinear four-wave mixing of LG modes in a cold atomic sample [4]. There have also been proposals to use LG modes to create vortices in Bose-Einstein condensates [5,6], where the orbital angular momentum is transferred from the laser beam to the vortex trap state.

In this paper we consider the interaction of a circularly polarized LG mode with a single trapped atom whose center-of-mass (CM) motion is quantized. We take the trapping potential to be harmonic in two dimensions, where the angular momentum of the atom as a whole, it is convenient to introduce the operators

\[ \hat{\alpha}_x = \frac{1}{\sqrt{2}} \left( \hat{X} R_0 + \frac{\hat{P}_X}{R_0/\hbar} \right), \]
\[ \hat{\alpha}_y = \frac{1}{\sqrt{2}} \left( \hat{Y} R_0 + \frac{\hat{P}_Y}{R_0/\hbar} \right), \]

where \( R_0 = \sqrt{\hbar/m\nu} \) sets the scale for the radial size of the trap. Since we are interested in the angular momentum of the atom as a whole, it is convenient to introduce the operators

\[ \hat{\alpha}_\pm = \frac{1}{\sqrt{2}} \left( \hat{\alpha}_x \mp i\hat{\alpha}_y \right), \]

which serve to raise (\( \hat{\alpha}_+ \)) and lower (\( \hat{\alpha}_- \)) the angular momentum \( L_z \) along the trap axis. We can show that

\[ \hat{L}_z = \hbar \left( \hat{\alpha}_+ \hat{\alpha}_- - \hat{\alpha}_- \hat{\alpha}_+ \right), \]

entanglement between the internal and external states of the atom. Finally in section 3 we consider the relevance of this phenomenon for quantum information applications.

2. BASIS STATES

The Hamiltonian for a harmonic trapping potential in two dimensions is

\[ \hat{H}_{CM} = \frac{1}{2m} \left( \hat{P}_X^2 + \hat{P}_Y^2 \right) + \frac{1}{2} m\nu^2 (\hat{X}^2 + \hat{Y}^2), \]

where \( m \) is the mass of the atom and \( \nu \) is the radial trap frequency. We assume that the atom is tightly confined along the trap axis, \( \nu_z \gg \nu \). The Hamiltonian in Eq. (1) describes two independent one-dimensional harmonic oscillators along the Cartesian axes with annihilation operators

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are given by the wave functions $\chi_n = \langle \alpha_n^1, \alpha_+ \rangle$ and $\chi_{-n} = \langle \alpha_n^1, \alpha_- \rangle$ count the number of right and left circular quanta respectively. The Hamiltonian for the two-dimensional oscillator in terms of these operators is

$$H_{CM} = \hbar(\alpha_+^+ \alpha_+ + \alpha_-^+ \alpha_- + 1).$$

The center-of-mass eigenstates of the atom can be written in terms of the energy and angular momentum quantum numbers, $N = n_+ + n_-$ and $M = n_+ - n_-$. For a fixed value of $N \geq 0$, there are $N + 1$ degenerate angular momentum states for which $M = -N, -N + 2, \ldots, N$. The ground state and the first excited states of the trap in polar coordinates, $R = \sqrt{X^2 + Y^2}$ and $\Phi = \tan^{-1}(Y/X)$, are given by the wave functions

$$\chi_{0,0}(R, \Phi) = \frac{1}{R_0 \sqrt{\pi}} \exp\left( -\frac{R^2}{2R_0^2} \right),$$

$$\chi_{1,\pm 1}(R, \Phi) = \frac{1}{R_0 \sqrt{\pi}} \left( \frac{R}{R_0} \right) \exp\left( -\frac{R^2}{2R_0^2} \pm i\Phi \right).$$

The energy levels for these states and the corresponding transition operators are shown in figure 1. The amplitudes of $\chi_{1,\pm 1}$ are shaped in the form of a doughnut with a null at the center, and the azimuthal phase is determined by the angular momentum $M = \pm 1$. In general, the $\chi_{N,M}$ wave functions are given by Laguerre-Gaussian modes.

To describe the internal angular momentum of the atom, we need to introduce a basis of electronic states. In this paper we consider the hydrogenic circular states $|\ell, l, m\rangle$, which have the maximum angular momentum component $m\hbar$ along the $z$ axis for a given principal quantum number $n = m + 1$. Only neighboring circular states are coupled according to dipole selection rules and hence these states serve as good approximations to two-level systems. The internal angular momentum in this basis is given by

$$\hat{l}_z = \sum_{m=0}^{\infty} m \hbar |m\rangle \langle m|.$$

For $\Delta m = \pm 1$ transitions, the dipole moment of the atom in the circular-state basis can be written as

$$d_m = \frac{1}{2} \sum_{m} d_m [\langle x + iy \rangle \hat{\sigma}_m + \langle x - iy \rangle \hat{\sigma}_m^+] .$$

The transverse profile of the mode at the beam waist $w_0$ is given by

$$u_I(R, \Phi) = |l| R_0 w_0 \exp\left( -\frac{R^2}{2w_0^2} + i\Phi \right).$$

We are interested in the limit in which the size of the trapped atom $R_0$ is small compared to the radius of the LG mode. This affords a linearization of the interaction Hamiltonian analogous to the Lamb-Dicke limit in trapping and cooling. In this limit, we are justified in expanding the LG mode in powers of $R/w_0$,

$$u_I(R, \Phi) = |l| R_0 w_0 \exp(i\Phi) + \mathcal{O}\left[ (R/w_0)^{|l|+2} \right].$$

Keeping only the leading-order term in this expansion, we quantize the atomic center-of-mass position as follows. For $l = \pm |l|$,

$$\left( \frac{R}{w_0} \right) \exp(\pm i\Phi) = \hat{X} \pm i\hat{Y} = \eta(\hat{a}_+^+ + \hat{a}_-) .$$

FIG. 1. Ground and first excited states of the trap

FIG. 2. Atomic circular states $|m\rangle = |n = m + 1, m, m\rangle$
a point particle. When $\eta \ll 1$, we can treat the interaction Hamiltonian to lowest order in the center-of-mass position operators for the atom.

To couple neighboring circular states in the atom, we need a left-circularly polarized field. Using the truncated form of the LG mode $u_l$ in Eq. (12), we write the electric field on the trap plane for $l = \pm |l|$ as

$$E_l = \mathcal{E}_l r^{|l|}(\hat{a}_\pm + \hat{a}_\mp)(x + iy)\exp(-iw_\nu t) + \text{h.c.}, \quad (14)$$

where h.c. denotes hermitian conjugate. The coupling between the LG mode and the trapped atom is described by the $\mathbf{d} \cdot \mathbf{E}$ Hamiltonian. Using the dipole moment and the field vector from Eq. (10) and Eq. (14), we find that for $l = \pm |l|$, the interaction Hamiltonian is given by

$$\hat{H}_{\text{int}} = -\mathbf{d} \cdot \mathbf{E}$$

$$= -\frac{\hbar}{2} \sum_m \eta^{|l|}\Omega_{m,l}(\hat{a}_m^\dagger + \hat{a}_m)\hat{\sigma}_m^\dagger \exp(-i\omega t) + \text{h.c.}, \quad (15)$$

where $\Omega_{m,l} = 2d_m\mathcal{E}_l/\hbar$ is the Rabi frequency, and we have used the vector identities $(x \pm y) \cdot (x \pm y) = 0$ and $(x \pm y) \cdot (x \mp y) = 2$.

We now specialize to the case of a two-level system formed by two neighboring circular states $m$ and $m + 1$. In the interaction picture, the states evolve only according to $H_{\text{int}}$. The atomic operators evolve as $\hat{\sigma}_m(t) = \hat{\sigma}_m \exp[-i(\omega_{m+1} - \omega_m)t]$, where $\omega_m$ are the atomic frequencies of the states. Similarly, the center-of-mass operators evolve as $\hat{a}_\pm(t) = \hat{a}_\pm \exp(-i\omega t)$, where $\nu$ is the trap frequency. Consider the situation where the field is tuned to the $|l|^{\text{th}}$ sideband below the atomic resonance,

$$\omega = (\omega_{m+1} - \omega_m) - |l|\nu. \quad (16)$$

In the rotating-wave approximation, we ignore counter-rotating terms in the interaction Hamiltonian and are left with only the two circular states $m$ and $m + 1$ contributing to the sum in Eq. (15). Furthermore, if the field is sufficiently narrow in spectrum compared to the trap frequency, only the $|l|^{\text{th}}$ power of the operators $\hat{a}_\pm$ contribute to the interaction in Eq. (15), assuming $\eta^{|l|}\Omega_{m,l} \ll \nu$, and we can ignore the cross terms. The interaction Hamiltonian simplifies to

$$\hat{H}_{\text{int}} = -\frac{\hbar}{2} \eta^{|l|}\Omega_{m,l} \hat{a}_m^\dagger \hat{\sigma}_m^\dagger + \text{h.c.}. \quad (17)$$

To interpret the interaction in physical terms, recall that $\hat{a}_\pm^\dagger$ and $\hat{a}_\mp$ raise the center-of-mass angular momentum, while $\hat{a}_\pm$ and $\hat{a}_\mp$ lower it. This can be seen from Eq. (15). Similarly, $\hat{\sigma}_m$ ($\hat{\sigma}_m$) raises (lowers) the internal angular momentum of the atom in the two circular states. Thus the Hamiltonian in Eq. (17) clearly shows that the orbital angular momentum of the LG mode is transferred to the external angular momentum of the atom, while the spin angular momentum associated with circular polarization is transferred to the internal angular momentum of the atom. The spin and orbital components are separately conserved in the paraxial limit.

Choosing the orbital angular momentum of the LG mode to be positive or negative, $l = \pm |l|$, correlates the change in the internal and external angular momenta of the atom, $\Delta m$ and $\Delta M$. For example, when $l < 0$ the external angular momentum is raised whenever the internal angular momentum is lowered, and vice versa. The choice of field frequency in Eq. (14) corresponds to tuning to the $|l|^{\text{th}}$ sideband on the lower side of the atomic resonance. This choice governs the parity of the transitions between the center-of-mass states as shown in figure 3, and correlates the change in the energy and angular momentum of the trap, $\Delta N$ and $\Delta M$.

4. ENTANGLEMENT

We use Eq. (17) as the starting point for a discussion of quantum entanglement between the internal and external angular momenta of the atom in the trap. Consider $l = -1$, which gives the left-circularly polarized LG field a net angular momentum of zero. In this case, the change in internal and external angular momenta of the atom are equal in magnitude but opposite in sign. The time evolution operator in this case is given by

$$\hat{U}_{\text{int}}(t) = \exp(-i\hat{H}_{\text{int}}t/\hbar)$$

$$= \exp[-\frac{\eta^{|l|}}{2}(\Omega_{m,-1}\hat{a}_\mp\hat{\sigma}_m^\dagger + \Omega_{m,-1}\hat{a}_\mp^\dagger\hat{\sigma}_m)]. \quad (18)$$

Consider the action of this operator on the state $|m⟩|\chi_{0,0}\rangle$. Since the internal angular momentum of the atom can only be raised by $\hat{a}_\mp$, the external angular momentum has to be lowered by $\hat{a}_\mp$. However $|\chi_{0,0}\rangle$ is the lowest energy state of the trap and cannot be further reduced in energy. Hence the state $|m⟩|\chi_{0,0}\rangle$ does not evolve in time according to this interaction. This restriction does not apply to the state $|m+1⟩|\chi_{0,0}\rangle$, since the atom is in the higher angular momentum circular state to begin with, and we find Rabi oscillations between states $|m+1⟩|\chi_{0,0}\rangle$ and $|m⟩|\chi_{1,1}\rangle$. To summarize, we find that

$$\hat{U}_{\text{int}}(t)|m⟩|\chi_{0,0}\rangle = |m⟩|\chi_{0,0}\rangle, \quad (19)$$

$$\hat{U}_{\text{int}}(t)|m+1⟩|\chi_{0,0}\rangle = \cos(\eta\Omega t/2)|m+1⟩|\chi_{0,0}\rangle + \text{h.c.,}$$

$$+ i\text{e}^{i\phi} \sin(\eta\Omega t/2)|m⟩|\chi_{1,1}\rangle. \quad (20)$$

where we have defined $\Omega_{m,-1} = \Omega e^{i\phi}$. Equations (19) and (20) give the basic ingredients for quantum control of the selected internal and external states of the atom. When the trap is in the ground state and the atom is prepared in a coherent superposition of the circular states $m$ and $m + 1$, a $\pi$-pulse transfers this coherence to the center-of-mass state of the atom in the trap,

$$\hat{U}_{\text{int},\pi} : [c_m|m⟩ + c_{m+1}|m+1⟩] |\chi_{0,0}\rangle$$

$$\mapsto |m⟩ [c_m|\chi_{0,0}\rangle + c_{m+1}|\chi_{1,1}\rangle], \quad (21)$$

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where we have chosen $\phi = -\pi/2$. Alternatively, if the atom is in the upper circular state $m+1$ and the trap is in the ground state, a $\pi/2$-pulse creates maximal entanglement between the internal and external states,

$$
\hat{U}_{\text{int}, \pi/2} : |m+1\rangle|\chi_{0,0}\rangle \rightarrow \frac{1}{\sqrt{2}} \left[ |m+1\rangle|\chi_{0,0}\rangle + |m\rangle|\chi_{1,1}\rangle \right],
$$

(22)

where $\phi = -\pi/2$ again. We have to be in the adiabatic limit where the pulse length is long enough that the spectrum does not overlap neighboring trap states in energy.

Equation (22) is the main result of this paper, that we can in principle generate states of a single atom that are entangled in internal and external angular momenta using a circularly polarized LG mode. This is a new form of entanglement that relies on the conservation of angular momentum rather than energy. The two observables that are entangled are $L_z$ and $l_z$, defined in Eqs. (5) and (9) respectively.

The experimental difficulty is in measuring the quantized center-of-mass state of the atom in the trap. A direct observation of the trap state may be engineered as follows. When the atoms are released from an excited trap state $|\chi_{1,1}\rangle$, they escape with a net linear momentum in the azimuthal direction, which may be detected by time-of-flight measurements using a suitably positioned detector array. An indirect observation of the entanglement present in Eq. (22) is possible using a weak probe pulse resonant with the circular states $m+1$ and $m+2$. In this case, only the state $|m+1\rangle|\chi_{1,1}\rangle$ is affected by the pulse, and the absorption of a photon would distinguish this state from $|m+1\rangle|\chi_{0,0}\rangle$.

5. DISCUSSION

It is intriguing to consider the application of the ideas in this paper to quantum information processing. Rydberg circular states are extremely long-lived, with radiative lifetimes of the order of 10 milliseconds even for $n = 30$, increasing as $n^5$ for larger $n$. The two circular states $m$ and $m+1$ may be thought of as a qubit, and the interaction with the LG mode provides a controlled coupling to the second qubit formed by the ground and first excited state of the trap. In this context, Eqs. (19) and (21) allow arbitrary controlled unitary operations, where the internal state of the atom plays the role of the control qubit.

One possibility to scale up this scenario is to consider two or more atoms individually trapped and manipulated in this manner. A coupling between two atoms may be achieved by entangled photons in LG modes, as demonstrated recently in the parametric downconversion experiment of Ref. [4]. The trap states of each atom can play the role of an auxiliary or intermediary qubit that enables information processing in the internal states of the atom. Decoherence issues involved with trapping and cooling the atom to the center-of-mass ground state benefit from the weak coupling of neutral atoms to the environment.

Lastly, we highlight the benefits of going beyond two internal states (beyond qubits) in the atom. We chose the circular states because they made a good two-level system. However, there are $n^2$ angular momentum states in the atom for each principal quantum number $n$, all of which are degenerate in hydrogen. This allows for the possibility of simultaneous control of these states and entanglement with the trap states. Angular momentum entanglement is particularly suited for large-scale information processing in the atom.

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