Multi-Higgs-Doublet Models and Singular Alignment

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We consider a 4-Higgs-doublet model in which each Higgs doublet gives mass to one of the fermion sets \{m_t\}, \{m_b, m_{\tau}, m_c\}, \{m_u, m_s\}, and \{m_d, m_{\mu}, m_e\}. The sets have the feature that within each of them the masses are similar. Our model explains the mass hierarchies of the sets by hierarchies of the vacuum expectation values of the Higgs doublets associated to them. All Yukawa couplings are therefore of order one. Neutrino masses are generated by a type-I seesaw mechanism with PeV-scale singlet neutrinos. To avoid the appearance of tree-level flavor changing neutral currents, we assume there are, for each fermion a Higgs doublet is introduced, see also [4]. The mass hierarchies are explained by hierarchies of vacuum expectation values of the individual Higgs doublets: \(m_f \simeq v_f\), where \(v_f\) is the vacuum expectation value of the Higgs that is responsible for the fermion mass.

In the following, we assume Yukawa couplings to be order one, except for the top quark, \(y_t \ll 1\). In the following, we assume Yukawa couplings to be order one, \(y_f \ll 1\), and try to understand the fermion mass patterns through a theory with multiple Higgs doublets. The most extreme approach along this line would be the "private Higgs" scenario, in which among other things, for each fermion a Higgs doublet is introduced [2,3], see also [4]. The mass hierarchies are explained by hierarchies of vacuum expectation values of the individual Higgs doublets: \(m_f \simeq v_f\), where \(v_f\) is the vacuum expectation value of the Higgs that is responsible for the fermion mass.

In general, in a model with \(N\) Higgs doublets, \(\Phi_i (i = 1, 2, \ldots, N)\), where each of their neutral components acquires a vacuum expectation value (vev), \((\Phi_i^0) = v_i e^{i\theta_i}\), a relation among these vacua is satisfied:

\[
\sum_{i=1}^{N} v_i^2 = v_{EW}^2. \tag{1}
\]

Here \(v_{EW} \simeq 174\) GeV, \(v_i \geq 0\), and all doublets share the same hypercharge \(Y = \frac{1}{2}\).

Now, if we consider that each single Higgs is fully responsible for the mass of one single fermion (where \(N\) should equal the number of fermions in the theory), then the previous relation is modified to

\[
\sum_{i=1}^{N} \frac{m_i^2}{y_i^2} = v_{EW}^2. \tag{2}
\]

Once the neutrino mass ordering and hierarchy is determined, it is likely that additional questions will arise.

I. INTRODUCTION

An understanding of fermion masses and mixing is still lacking. In particular, the mass values display unexplained patterns and hierarchies; this is the case when one considers the three generations as well as the species.

\begin{align*}
\text{Intergeneration} & \quad \text{Intergeneration} \\
\text{Generation} & \quad \text{Generation} \\
\text{EW} \sim m_t & \gg m_c \gg m_u \\
& \vee \wedge \vee \\
& \vee \vee \vee \\
& \downarrow m_{\nu(2)} \, ? \, m_{\nu(2)} \, ? \, m_{\nu(1)}(3) \\
\end{align*}

We can summarize the situation by asking the following questions:

- Why is the top quark mass the only fermion mass of the order of the electroweak (EW) scale, \(m_t \simeq v_{EW}\) with \(v_{EW} \simeq 174\) GeV?

- Why do all charged fermions satisfy the hierarchy, \(m_3 \gg m_2 \gg m_1\)?

- Why do the down-type quarks and charged leptons have similar masses, \(m_d \sim m_e\) \((d_{1,2,3} = d, s, b, e_{1,2,3} = e, \mu, \tau)\)?

- Why are for the first generation the masses (except for neutrinos) closer to each other than for the other two generations, \(m_d \sim m_u \sim m_e\) versus \(m_s \sim m_{\mu}\) and \(m_t \gg m_b \sim m_{\tau}\)?

- What could the interspecies hierarchy, e.g. \(m_t \gg m_b > m_{\tau} \gg m_{3}\), be telling us?

- Why are neutrino masses much smaller than the charged fermions? \(\nu \sim 10^{-7}\) GeV?

This is commonly referred to as the problem of mass [1]. Part of the mystery lies in the contrast of expecting Yukawa couplings to be order one, \(y_f = \mathcal{O}(1)\), whereas the observed values with a single Higgs doublet are much smaller than 1, except for the top quark, \(y_f \ll 1\). In the following, we assume Yukawa couplings to be order one, \(y_f = \mathcal{O}(1)\), and try to understand the fermion mass patterns through theories with multiple Higgs doublets. The most extreme approach along this line would be the "private Higgs" scenario, in which among other things, for each fermion a Higgs doublet is introduced [2,3], see also [4]. The mass hierarchies are explained by hierarchies of vacuum expectation values of the individual Higgs doublets: \(m_f \simeq v_f\), where \(v_f\) is the vacuum expectation value of the Higgs that is responsible for the fermion mass.

In general, in a model with \(N\) Higgs doublets, \(\Phi_i (i = 1, 2, \ldots, N)\), where each of their neutral components acquires a vacuum expectation value (vev), \((\Phi_i^0) = v_i e^{i\theta_i}\), therefore of order one. Neutrino masses are generated by a type-I seesaw mechanism with PeV-scale singlet neutrinos. To avoid the appearance of tree-level flavor changing neutral currents, we assume there are, for each fermion a Higgs doublet is introduced, see also [4]. The mass hierarchies are explained by hierarchies of vacuum expectation values of the individual Higgs doublets: \(m_f \simeq v_f\), where \(v_f\) is the vacuum expectation value of the Higgs that is responsible for the fermion mass.

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Furthermore, if we consider that Yukawa couplings should be order one numbers, \( y_i = \mathcal{O}(1) \), we could approximately say that, to good approximation
\[
\sum_{i=1}^{N} m_i^2 \approx v_{\text{EW}}^2 .
\] (3)

In the case of the Standard Model (SM), with \( N = 12 \) fermions, the previous equation is fulfilled. We will call this relation the mass-vacuum relation. An amusing possibility from this relation is that if all \( N \) doublets have the same vev, one would have \( N \) fermions with mass of about \( 174/\sqrt{N} \) GeV, which would be about 50 GeV for 12 fermions. If two doublets have vev \( v_{\text{EW}}/\sqrt{2} \) and the rest a vanishing vev, then there would be two fermions with mass \( v_{\text{EW}}/\sqrt{2} \approx 123 \) GeV. In turn, if only one doublet has a vev, there is only one fermion with mass \( v_{\text{EW}} \). Forcing the mass-vacuum relation to be fulfilled and assuming that only one Higgs acquires a vev leaves hardly any mass for the other fermions and explains the top quark’s dominance. Moreover, this same argument could help us to understand why neutrinos are so light when assumed as Dirac fermions.

The particle content in the main scenario discussed in this paper is smaller than that for a private Higgs-like scenario. Our observation is that the fermion masses can be grouped into four different sets: \( \{m_t, m_b, m_{c}, m_{\tau} \} \), \( \{m_{\mu}, m_{s}, m_{e} \} \), and \( \{m_d, m_u, m_c \} \). In each set the masses are quite similar and can in fact be explained by similar \( \mathcal{O}(1) \) Yukawa couplings to an individual Higgs doublet \( \Phi_{t,b,\mu,d} \). Such a 4-Higgs-Doublet Model has to the best of our knowledge not been considered before. We find several attractive and testable features of the model, and demonstrate that it is not in conflict with measured Higgs couplings and other tests. Our model traces the hierarchy of the mass values of the different fermion sets to hierarchies of vevs of their respective Higgs doublets. We show that the smaller vevs can be induced by the larger vevs, and the hierarchy among them arises because the four vevs are protected by different symmetries.

The main problem in multi-Higgs doublet models is of course the presence of flavor changing neutral currents (FCNC). Theories which through the use of symmetries naturally avoid those FCNC are said to possess Natural Flavor Conservation (NFC). Options to evade FCNC are said to possess Natural Flavor Conservation (NFC). Theories which through the use of symmetries include, next to arranging the additional scalar particles to be very heavy, suppressing dangerous Yukawa couplings, separating the Yukawa matrices such that only one scalar doublet couples to a given right-handed fermion field \( \Phi \), or Yukawa alignment \( \Phi_{(1)} \), in which the different Yukawa matrices are proportional to each other. As a proof of principle that FCNC can be entirely avoided in our setup, we assume here another solution. We note that if the Yukawa matrices are proportional to any of the rank-one matrices that appear in the singular value decomposition of the fermion mass matrices, FCNC are absent. We denote this as "singular alignment".

The paper is organized as follows: In Section \( \text{II} \) we present singular alignment and discuss some of its features. The model with four Higgs doublets to explain the masses of the individual sets \( \{m_t, m_b, m_{c}, m_{\tau} \} \), \( \{m_{\mu}, m_{s}, m_{e} \} \), and \( \{m_d, m_u, m_c \} \) is presented and analyzed in Sec. \( \text{III} \). Conclusions are presented in Sec. \( \text{IV} \) and some technical details are delegated to Appendices.

### II. SINGULAR ALIGNMENT

In general, having multiple Higgs doublets coupling to fermions with the same electric charge will produce tree-level FCNC, which are experimentally strongly constrained. Three main possibilities to overcome this problem have typically been studied: (i) assume "dangerous" Yukawa couplings to be sufficiently suppressed at tree-level \( \Phi_{(2)} \); (ii) assume the corresponding Yukawa matrices of each type of fermion (up-type quarks, down-type quarks and charged leptons) to be proportional to the mass matrix \( \Phi_{(3)} \); (iii) impose an adequate symmetry such that each fermion type couples exactly to one of the doublets \( \Phi_{(4)} \). In the following, we comment only on the last two possibilities and introduce singular alignment.

Let us start from the most general case for a Yukawa Lagrangian in a NHDM,
\[
-\mathcal{L}_Y \supset \sum_{a=1}^{N} F_L Y_a^T f_R \Phi_a + \text{H.c.} ,
\] (4)

where \( F_L \) and \( f_R \) are three dimensional vectors in family space and transform as a doublet and as a singlet under \( SU(2)_L \), respectively. The \( N \) Higgs doublets acquire a vev, \( v_a = \langle \Phi_a \rangle \). In general, Yukawa couplings will couple all fermions to all Higgses. Therefore, the most general form of a mass matrix is
\[
M = v_1 Y_1 + v_2 Y_2 + \cdots + v_N Y_N .
\] (5)

Each Yukawa matrix, \( Y_a \), is a \( 3 \times 3 \), arbitrary, and complex matrix with rank 3. The appearance of tree-level FCNC is automatic within this setup as diagonalization of the mass matrices does not mean, in general, simultaneous diagonalization of the individual Yukawa matrices. However, to avoid introducing dangerous tree-level FCNC the following can be done:

1. **NFC theories:** Adequate symmetries are imposed in such a way that each of the three charged fermions will
only couple to a single Higgs [8,9], i.e. for each fermion type holds

\[ M = v_k Y_k , \]  

(6)

where no sum over \( k \) is intended. In this case diagonalization of the l.h.s. means diagonalization of the r.h.s. For \( N \) Higgs doublets, the easiest way to achieve this is via a symmetry of the form

\[ Z_2^{(1)} \times Z_2^{(2)} \times \cdots \times Z_2^{(\ell)} , \]

(7)

where in order for this symmetry to be realizable \( \ell = N - 1 \) should hold. Realizable symmetries are a set of allowed discrete symmetries of the scalar potential which have no accidental larger groups that could give rise, for example, to massless Goldstone bosons [12].

Now, before turning to the next possibility, let us comment on the Singular Value Decomposition (SVD) of a mass matrix:

\[ M = L^\dagger \Sigma R . \]

(8)

Here \( L \) and \( R \) are unitary matrices which rotate independently the left- and right-handed fermion fields and \( \Sigma = \text{diag}(m_1, m_2, m_3) \) with \( m_i > 0 \). Realize that the SVD may also be written as a sum of three rank 1 matrices,

\[ M = \sum_i m_i L_i^\dagger P_i R , \]

(9)

where \( P_i \) are three projector operators, \( P_i^2 = P_i \) and \( \sum_i P_i = 1_{3\times3} \), which have the form

\[ P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \]

\[ P_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} . \]

(10)

In the following, we will denote each rank 1 matrix appearing in the SVD by

\[ \Delta_i = L_i^\dagger P_i R , \]

and call it singular matrix.

\textbf{b. Yukawa Alignment:} As each Yukawa term in Eq. (5) is a rank 3 matrix, a second possibility to avoid FCNC, is to assume that each of them is proportional to \( \eta_i \) appearing in Eq. (15) equal to zero.

Hence, all fermion masses are independent linear combinations of the different vevs and all Higgs doublets can be responsible for giving mass to all fermions. In practice, models may also lead to Yukawa matrices with ranks less than 3. In this case the singular alignment can still hold and the only new difference would be to have some of the constants \( \eta_i, \Omega_i, \Lambda_i \) appearing in Eq. (15) equal to zero.

In short, singular alignment is the very strong Ansatz of choosing Yukawa matrices to be related to the rank 1 matrices appearing in the SVD. Through this alignment, no tree-level FCNC appear for any number of Higgs doublets. Let us consider now some explicit examples.

\textbf{A. The Two-Fermion Family Case}

We assume \( N \) Higgs doublets for two generations of charged fermions. In this case, the mass matrix is

\[ m = v_1 y_1 + \cdots + v_N y_N . \]

(18)

Diagonalization of the l.h.s. means diagonalization of the r.h.s. This is understandable as each Yukawa matrix is rank 3 and thus if related to the singular matrices should be composed of the three independent singular matrices. Furthermore, one has the constraint

\[ \sum_{j=1}^N \zeta_j = 1 . \]

(14)

c. Singular Alignment: A more general scenario is that in which each Yukawa matrix is given by a linear combination of the singular matrices, i.e.

\[ Y_i = (\eta_i \Delta_1 + \Omega_i \Delta_2 + \Lambda_i \Delta_3 ) . \]

(15)

Appendix A gives a straightforward proof of the absence of FCNC in case the Yukawa matrices take this form. Comparing with the full mass matrix, which can be written as

\[ M = m_1 \Delta_1 + m_2 \Delta_2 + m_3 \Delta_3 , \]

(16)

we identify

\[ m_1 = \sum_i \eta_i v_i, \quad m_2 = \sum_j \Omega_j v_j, \quad m_3 = \sum_k \Lambda_k v_k . \]

(17)

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been already transformed and we have explicitly written the most general expression for a unitary matrix in two dimensions. The two singular matrices are
\[
\Delta_1 = e^{-i\beta_3} \begin{pmatrix} c_\alpha e^{-i\beta_1} & 0 \\ s_\alpha e^{i\beta_2} & 0 \end{pmatrix}, \\
\Delta_2 = e^{-i\beta_3} \begin{pmatrix} 0 & -s_\alpha e^{-i\beta_2} \\ 0 & c_\alpha e^{i\beta_1} \end{pmatrix}.
\] (20)

Singly aligning our Yukawa matrices in flavor space means
\[
y_i = \eta_i \Delta_1 + \Omega_i \Delta_2,
\] (21)
which leads to
\[
m = \sum_i \eta_i v_i \Delta_1 + \sum_i \Omega_i v_i \Delta_2.
\] (22)

We identify the masses as
\[
m_1 = \sum_i \eta_i v_i \quad \text{and} \quad m_2 = \sum_i \Omega_i v_i.
\] (23)

Regarding FCNC, note that in the mass basis we have
\[
m = Lm = m_1 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + m_2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\] (24)

Hence, no FCNC are introduced in this model of \(N > 1\) Higgs doublets that couple to all fermions. Also, if matrices of lower rank are obtained through the use of convenient symmetries, then our general expressions in Eq. (23) will still hold but with some of the parameters \(\eta_i\) or \(\Omega_i\) vanishing.

### B. The Three-Fermion Family Case

The next example deals with three generations and three Higgs doublets. To be singularly aligned, each rank 1 Yukawa matrix
\[
Y_1 = \eta_1 \begin{pmatrix} a_1 & 0 & 0 \\ a_2 & 0 & 0 \\ a_3 & 0 & 0 \end{pmatrix}, \quad Y_2 = \Omega_2 \begin{pmatrix} 0 & b_1 & 0 \\ 0 & b_2 & 0 \\ 0 & b_3 & 0 \end{pmatrix},
\] (25)

\[
Y_3 = \Lambda_3 \begin{pmatrix} 0 & 0 & c_1 \\ 0 & 0 & c_2 \\ 0 & 0 & c_3 \end{pmatrix},
\]

should be seen as a column vector satisfying unitarity conditions (recall Eq. (15)):
\[
\langle a | a \rangle = 1, \quad \langle b | b \rangle = 1, \quad \langle c | c \rangle = 1, \\
\langle a | b \rangle = 0, \quad \langle a | c \rangle = 0, \quad \langle b | c \rangle = 0.
\] (26)

Here we have denoted \((a_1, a_2, a_3)^T \equiv |a\rangle\) and similarly for the other columns. Notice that the Yukawa couplings should not enter into these expressions. A practical way to implement all these conditions is to make use of an explicit parametrization of a unitary matrix. Then, a singularly aligned mass matrix could take the form
\[
M = e^{i\varphi} T \begin{pmatrix} v_1 \eta_1 c_\alpha c_\gamma & v_2 \Omega_2 s_\alpha c_\gamma & v_3 \Lambda_3 s_\gamma e^{-i\chi} \\ -v_1 \eta_1 (s_\alpha c_\beta + c_\alpha s_\beta s_\gamma e^{i\chi}) & v_2 \Omega_2 (c_\alpha c_\beta - s_\alpha s_\beta s_\gamma e^{i\chi}) & v_3 \Lambda_3 s_\beta c_\gamma \\ v_1 \eta_1 (s_\alpha c_\beta - c_\alpha s_\beta s_\gamma e^{i\chi}) & -v_2 \Omega_2 (c_\alpha c_\beta + s_\alpha s_\beta s_\gamma e^{i\chi}) & v_3 \Lambda_3 c_\beta c_\gamma \end{pmatrix} Q.
\] (27)

### C. Hierarchical Fermion Masses

A shared feature among all the charged fermions is that their masses are hierarchical,
\[
m_1 \ll m_2 \ll m_3.
\] (28)

To theoretically understand this in a \(\text{NHDM}\) with singular alignment, see Eq. (17), one must understand under what conditions this property gets always realized. We are not interested in any fine-tuned scenario where through adequate values for the set of parameters \(\{\eta, \Omega, \Lambda\}\) we generate hierarchical masses, we are assuming that \(\eta, \Omega, \Lambda = \mathcal{O}(1)\). Furthermore, we are actually interested in the minimal number of scalar doublets necessary to explain all the observed patterns in the fermion masses. For the moment, notice that one possibility is to couple a single Higgs to each different flavor with the
same electric charge. In this case we have
\[ m_1 = \eta_1 v_1, \quad m_2 = \Omega_2 v_2, \quad m_3 = \Lambda_3 v_3. \]  

(29)

It is obvious then that the only way to achieve hierarchical masses with $\mathcal{O}(1)$ parameters is through hierarchical vevs, i.e.
\[ v_3 \gg v_2 \gg v_1. \]  

(30)

This fact is connected to the mass-vacuum relation.

The maximal setup, if neutrinos are assumed as Dirac particles, would require 12 Higgs doublets. However, this large number of scalars can be significantly reduced if one notices that among the different masses there are majorly 4 (5) mass scales, where the (5) corresponds to Dirac neutrino masses. This is what we will deal with in Sec. III. In case of Majorana neutrinos there are four possibilities depending on from which Higgs doublet the Dirac mass matrix of the type-I seesaw mechanism stems. We will come back to this point later. Of course, neutrino mass could also be independent of the Higgs doublets.

**III. THE MINIMAL SETUP: A 4HDM**

Now we discuss a 4HDM which takes into account that among the measured fermion masses four different sets can be identified: \{\(m_t, m_b, m_\tau, m_\mu\), \(m_d, m_s, m_\tau\)\} and \(m_u, m_c, m_\mu\) and \(m_d, m_s, m_\tau\). Within each set the masses are within one order of magnitude. This fact is depicted in Figure 1. We will introduce four Higgs doublets \(\Phi_t, \Phi_b, \Phi_\mu\) and \(\Phi_d\), which are responsible for the masses in their respective set. The corresponding mass-vacuum-like relation in analogy to Eq. (3) would take the form
\[ v_t^2 + v_b^2 + v_\mu^2 + v_d^2 = v_{EW}^2. \]  

(33)

The model can be constructed by first imposing fields to transform under the symmetry $Z_2 \times Z_2^\prime \times Z_2^\prime$, as shown in Table 1. The Yukawa Lagrangian implied by the charge

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D. Beyond Singular Alignment

If a small amount of flavor violation via neutral mediators is permitted, then a less restrictive venue can be obtained through the following conditions: (i) the third Yukawa matrix for all fermion species is the only rank 1 matrix and proportional to the third singular matrix,
\[ Y_3 = \Lambda_3 \Delta_3; \]  

(31)

(ii) the first and second Yukawa matrices are no longer proportional to the singular matrices, so they may in general produce FCNC; (iii) however, to produce a hierarchy between the first and second generation, the second Yukawa matrix should be at most rank 2 and have no contributions to the first family masses; (iv) the first Yukawa matrix can be rank 3, 2 or 1. In other words, the three Yukawa matrices should imply the sequential symmetry breaking chain
\[ U(3)^3 \xrightarrow{Y_{f,3}} U(2)^3 \xrightarrow{Y_{f,2}} U(1)^3 \xrightarrow{Y_{f,1}} U(1)_F, \]  

(32)

where $F$ might either be baryon or lepton number. The introduction of flavor violation as allowed by the two lightest families means no risk as this set of flavor transitions will be sequentially suppressed by the approximately conserved symmetries at each step.
SCALAR SECTOR

\[ \begin{array}{c|cccc}
\Phi_1 & \Phi_2 & \Phi_3 & \Phi_4 \\
\hline
Z_{u_1} & + & - & - & - \\
Z_{d_1} & - & + & - & - \\
Z_{u_3} & + & + & - & - \\
Z_{d_3} & + & + & + & - \\
\end{array} \]

QUARK SECTOR

\[
\begin{array}{cccccccc}
Q_{uL} & Q_{dL} & Q_{uR} & u_{3R} & u_{1R} & d_{3R} & d_{2R} & d_{1R} \\
\hline
Z_{u_1} & + & + & + & + & + & + & + \\
Z_{d_1} & + & - & - & + & - & + & - \\
Z_{u_3} & + & + & - & + & - & + & - \\
Z_{d_3} & + & + & + & - & + & - & + \\
\end{array}
\]

LEPTON SECTOR

\[
\begin{array}{cccccccc}
E_{uL} & E_{dL} & E_{eL} & n_{3R} & n_{2R} & n_{1R} & e_{3R} & e_{2R} & e_{1R} \\
\hline
Z_{u_1} & + & + & + & + & + & + & + & + \\
Z_{d_1} & + & - & - & + & - & + & - & + \\
Z_{u_3} & + & + & - & + & - & + & - & + \\
Z_{d_3} & + & + & + & - & + & - & + & - \\
\end{array}
\]

TABLE I. Charge assignment under the discrete flavour symmetry group \( Z_2 \times Z'_2 \times Z''_2 \) for the different scalar and fermion fields in the 4HDM with rank one Yukawa matrices. This extension to the SM comprises 3 right-handed neutrinos and 3 new Higgs fields.

The way in which we have employed the charge assignment to couple fermions with Higgs doublets has given us a model where all Yukawa matrices for the charged fermions are rank 1. For example, in the up-quark sector we have

\[ M_{up} = v_d Y_u + v_h Y_c + v_t Y_t, \]

with

\[ Y_u = \begin{pmatrix}
y^u_1 & 0 & 0 \\
y^u_2 & 0 & 0 \\
y^u_3 & 0 & 0 \\
\end{pmatrix}, \quad Y_c = \begin{pmatrix}
y^c_1 & 0 & 0 \\
y^c_2 & 0 & 0 \\
y^c_3 & 0 & 0 \\
\end{pmatrix}, \quad Y_t = \begin{pmatrix}
0 & y^t_2 & 0 \\
0 & y^t_3 & 0 \\
0 & y^t_1 & 0 \\
\end{pmatrix}. \]

Similar expressions can be given for the other fermion species. Notice we are employing a conventional notation for the Yukawa couplings, \( y^f_i \), in order to distinguish at this point generic Yukawa matrices from those which have been singularly aligned.

Now, to singularly align these matrices, we demand that each column should be given by a single singular matrix (in order to have a hierarchy of masses with order one Yukawa couplings, cf. Section II C), in our up-type example this means:

\[ Y_u = \eta_u \Delta_u, \quad Y_c = \Omega \Delta_c, \quad Y_t = \Lambda_t \Delta_t. \]

The explicit form of these singular matrices was given in Section II B; they correspond to one of the three columns in Eq. (27). We can also write them as \( \Delta_{u,c,t} = \tilde{L}^1 P_{1,2,3} \tilde{R} \), see the discussion around Eq. (11).

The model presented here arranges that a certain Higgs doublet will couple to a given set of fermions, even if they possess different electric charge. All corresponding Yukawa matrices will already be rank 1. Through the special requirement that Yukawa matrices should be singularly aligned in flavor space, as discussed in Sec. II, it is possible to avoid flavor violation at tree-level.

The model allows to reproduce fermion mixing, as shown in Appendix B. Neutrino masses are generated via the type-I seesaw mechanism. We have associated the three right-handed neutrinos to the Higgs doublet \( \Phi_i \). This implies that via the type-I seesaw mechanism the heavy neutrino mass scale \( \mathcal{M} \) should be around PeV, where we have assumed that \( m_D \simeq \langle \Phi_d^0 \rangle \simeq \mathcal{O}(10 \text{ MeV}) \) and \( m_\nu \simeq m_D^2/\mathcal{M} \simeq 0.1 \text{ eV} \).

### A. The Scalar Potential

The most general, renormalizable and gauge invariant scalar potential of the model is \( V = V_0 + V_{\text{soft}} \), where

\[ V_0 = \sum_a \frac{\mu_a^2}{2} (\Phi_a^0 \Phi_a) + \lambda_a (\Phi_a^0 \Phi_a)^2 \]

\[ + \sum_{a \neq b} X_{ab} (\Phi_a^0 \Phi_b)^2 \]

\[ + \sum_{a \neq b} Y_{ab} (\Phi_a^0 \Phi_b)^2 \]

\[ + \sum_{a \neq b} Z_{ab} \left( (\Phi_a^0 \Phi_b)^2 + (\Phi_b^0 \Phi_a)^2 \right). \]

Here \( a, b = t, b, \mu, d \), and for the sake of simplicity we are assuming all couplings to be real. The term \( V_0 \) is invariant under \( Z_2 \times Z'_2 \times Z''_2 \), whereas \( V_{\text{soft}} \) includes different soft-breaking terms (\( V_{\text{soft}} \ll V_0 \)), see below.

In order to generate a hierarchy among the vevs we choose the particular case where

\[ \mu_t^2 < 0 \quad \text{and} \quad \mu_{b,\mu,d}^2 > 0, \]

such that the only Higgs acquiring a vev is \( \Phi_t \):

\[ \left. \frac{\partial V_0}{\partial \Phi_t} \right|_{\Phi_t^0} = 0 \quad \Rightarrow \quad v_t = \sqrt{-\frac{\mu_t^2}{\lambda_t}}. \]

We are following the convention \( \langle \Phi_t^0 \rangle = v_t/\sqrt{2} \). As \( \Phi_t \) has no charge under
any of the three Abelian symmetries, see Table I its vev preserves the symmetry. Equivalently, the symmetries are protecting the other scalars from acquiring a vev. Thereafter, through the following subset of soft-breaking terms,

\[ V_{\text{soft}} \supset \mu_{tb}^2 \left( \Phi_1^\dagger \Phi_b + \Phi_b^\dagger \Phi_1 \right) + \mu_{\mu}^2 \left( \Phi_2^\dagger \Phi_\mu + \Phi_\mu^\dagger \Phi_2 \right) + \mu_{\mu d}^2 \left( \Phi_\mu^\dagger \Phi_d + \Phi_d^\dagger \Phi_\mu \right), \]

(40)

where \( \mu_{ab}^2 \ll v_1^2, \mu_\mu^2 \), we induce vevs for the other three Higgs doublets. To be more specific, the particular choice of soft-breaking terms is motivated by the fact that each of them will only break a particular piece of the whole symmetry. That is, \( (\Phi_1^\dagger \Phi_b + \Phi_b^\dagger \Phi_1) \), \( (\Phi_b^\dagger \Phi_\mu + \Phi_\mu^\dagger \Phi_b) \), and \( (\Phi_\mu^\dagger \Phi_d + \Phi_d^\dagger \Phi_\mu) \) only break \( Z_3, Z_2', \) and \( Z_2'' \), correspondingly. Therefore, once the EW symmetry is spontaneously broken, the first soft-breaking term will induce a vev to \( \Phi_b \) which in return will induce a vev to \( \Phi_d \) until finally reaching \( \Phi_\mu \). It is possible to show that within this limit the minimization conditions are satisfied if the vevs are given as

\[ v_b \simeq \frac{-v_1 \mu_{tb}^2}{(X Y Z)_{11} v_1^2 + \mu_b^2}, \]

(41)

\[ v_\mu \simeq \frac{-v_1 \mu_{\mu}^2}{(X Y Z)_{11} v_1^2 + \mu_\mu^2}, \]

(42)

\[ v_d \simeq \frac{-v_1 \mu_{\mu d}^2}{(X Y Z)_{11} v_1^2 + \mu_d^2}, \]

(43)

together with Eq. (39) and where \((X Y Z)_{ab} = X_{ab} + Y_{ab} + Z_{ab} \) and \( \mu_{ab}^2 < 0 \). By virtue of this choice, the vevs are naturally small and obey the desired hierarchy

\[ v_1^2 \gg v_b^2 \gg v_\mu^2 \gg v_d^2. \]

(44)

For example, with \( v_1 \simeq 174 \text{ GeV}, \mu_{b,\mu,d} \sim 200 \text{ GeV}, |\mu_{(tb),(b\mu),(d\mu)}| \sim 35 \text{ GeV} \) one finds \( v_b \sim 1 \text{ GeV}, v_\mu \sim 0.1 \text{ GeV} \) and \( v_d \sim 0.001 \text{ GeV} \).

B. Fermionic Couplings to the SM-like Higgs

The introduction of the soft breaking terms (Eq. (10)) in the Higgs potential will produce a small mixing among the four Higgs doublets. For the moment, let us focus on the neutral scalars. We assume all parameters in the scalar potential to be real. Through this choice we consider it to be CP-symmetric. Hence, no admixture between the real and imaginary components of the neutral fields is allowed as they have definite CP quantum numbers. To compute their couplings to all fermions we start from the Yukawa Lagrangian in the mass basis which is written as

\[ -\mathcal{L}_Y = y_t t \left( v_1 + \frac{\phi_1}{\sqrt{2}} \right) + \sum_{f = t, \tau, c} y_f \bar{f} f \left( v_b + \frac{\phi_b}{\sqrt{2}} \right) + \sum_{f = \mu, s} y_f \bar{f} f \left( v_\mu + \frac{\phi_\mu}{\sqrt{2}} \right) + \sum_{f = d, u, c} y_f \bar{f} f \left( v_d + \frac{\phi_d}{\sqrt{2}} \right), \]

(45)

where we have changed our notation \( \{\eta, \Omega, \Lambda\} \) to the conventional one, \( y_f \). We can bring the CP-even scalar sector to its mass basis via

\[ \begin{pmatrix} \phi_1 \\ \phi_b \\ \phi_\mu \\ \phi_d \end{pmatrix} = \mathcal{R} \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \end{pmatrix}, \]

(46)

where \( \mathcal{R} \) is an orthogonal matrix, \( \mathcal{R}^T \mathcal{R} = \mathcal{R} \mathcal{R}^T = 1_{4 \times 4} \), and \( h_0 \) is the lightest state with a mass of \( m_{h_0} \simeq 125 \text{ GeV} \). Now, in order to find out how fermions couple to the SM-like Higgs, \( h_0 \), we substitute \( \phi_\kappa = \mathcal{R}_{1\kappa} h_0 \) in Eq. (45) to obtain

\[ -\mathcal{L}_Y \supset \sum_{f} \frac{m_f}{(246 \text{ GeV})} \xi^f_h \bar{f} f h_0. \]

(47)

We can define the following four classes of fermion-scalar couplings:

\[ \xi^f_h = \begin{pmatrix} R_{11} \\ R_{12} \\ R_{13} \\ \sqrt{1 - \sum_j R_{1j}^2} \end{pmatrix}, \]

\[ \xi^{b,\tau,c}_{h} = \begin{pmatrix} R_{12} \\ \frac{R_{12}}{\sin \alpha_1 \cos \alpha_2} \\ \frac{R_{13}}{\sin \alpha_1 \sin \alpha_2 \cos \alpha_3} \\ \frac{\sqrt{1 - \sum_j R_{1j}^2}}{\sin \alpha_1 \sin \alpha_2 \sin \alpha_3} \end{pmatrix}, \]

\[ \xi^{d,u,c}_{h} = \begin{pmatrix} \frac{R_{11}}{\sin \alpha_1} \\ \frac{R_{12}}{\cos \alpha_1} \\ \frac{R_{13}}{\cos \alpha_3} \\ \frac{\sqrt{1 - \sum_j R_{1j}^2}}{\sin \alpha_1 \sin \alpha_2 \sin \alpha_3} \end{pmatrix}. \]

(48)

The angles \( \alpha_i \) in these relations are

\[ \sin \alpha_1 = \sqrt{\frac{v_1^2 + v_\mu^2 + v_d^2}{v_1^2 + v_b^2 + v_\mu^2 + v_d^2}}, \]

\[ \sin \alpha_2 = \sqrt{\frac{v_\mu^2 + v_d^2}{v_1^2 + v_b^2 + v_\mu^2 + v_d^2}}, \]

\[ \sin \alpha_3 = \sqrt{\frac{v_d^2}{v_1^2 + v_\mu^2 + v_d^2}}. \]

(49)

(50)

(51)

We note an attractive and testable feature of the model, namely that the couplings between fermions and the SM-like Higgs are modified in the same way for each set. That is, the couplings of the sets \( \{m_t\}, \{m_b, m_\tau, m_c\}, \)
FIG. 2. The red dots are predictions for the modified couplings of fermions in the sets \( \{m_t, m_s, m_c\} \), \( \{m_b, m_\tau, m_\mu\} \), and \( \{m_d, m_u, m_e\} \) to the SM-like scalar. The four different benchmark scenarios are defined in Table II. The vertical dashed line corresponds to the SM expectation. The black star is the central value of the measurement with green (yellow) bands being the 1\(\sigma\) (2\(\sigma\)) ranges. For simplicity the measured couplings of the Higgs to the bottom quark and tau lepton have been merged here into a single one, \(\kappa_{b,\tau} = 0.92 \pm 0.10\).

\[ \{m_\mu, m_s\} \text{ and } \{m_d, m_u, m_e\} \text{ are changed with respect to the SM-case by the same amount for each set, see Fig. 2. The coupling to the top quark is always essentially SM-like, } \xi^t = 1. \text{ This is understood } \]

\[ \text{because } R_{11} \text{ and } \cos \alpha_1 \text{ are both very close to 1, which is caused by the vev hierarchy } v_t \gg v_{b,\mu,s}. \]

Notice that, even though the mixing \( R_{ik} \) in all cases is proportional to the soft-breaking parameters, the implied smallness in \( |R_{ik}| \) may be compensated by \( \alpha_i \ll 1 \), and therefore, in general, \( \xi^f_h \) should not be expected to be small. In fact, within this scenario we can have four different possibilities: (i) hyper-couplings with \( \xi^f_h > 1 \), aligned couplings with \( \xi^f_h = 1 \), hypo-couplings with \( \xi^f_h < 1 \), and a mixture of any of these (the \( \xi^f_h \) can even be negative). One has to confront the couplings in this model with present measurements of Higgs couplings. We adopt the following numbers from combined fits of data taken at \( \sqrt{s} = 13 \text{ TeV} \) [13, 14, 15]:

\[
\begin{align*}
\kappa_Z &= -0.87^{+0.08}_{-0.08}, & \kappa_W &= -1.00^{+0.09}_{-0.09}, \\
\kappa_t &= 1.02^{+0.19}_{-0.15}, & \kappa_\tau &= 0.93^{+0.13}_{-0.13}, \\
\kappa_b &= 0.91^{+0.17}_{-0.16}, & \kappa_\mu &= 0.72^{+0.50}_{-0.72}.
\end{align*}
\]

No useful information about the couplings to first and second generation fermions exist, except for the muon, where the uncertainties are nevertheless very large. In our case \( \kappa_{Z,W} \) can be reproduced as in any multi-Higgs doublet model. The values of \( \kappa_{t,\tau,b} \) need to be compared with our \( \xi^f_h \), which is what the plots in Fig. 2 do for the four benchmark scenarios to be discussed next.

C. Numerical Examples

A thorough analysis of the Higgs potential is beyond the scope of this work, nevertheless, we will present four numerical benchmark scenarios. They obey the following conditions and constraints:

- **Bounded from below conditions:**
  \[ \lambda_{t,b,\mu,d} \geq 0, \quad X_{ab} \geq -\sqrt{\lambda_a \lambda_b}, \]
  \[ 0 < \lambda_a \lesssim 2, \quad -4 \lesssim (XYZ)_{ab} \lesssim 2, \]
  \[ |X_{ab}| \lesssim 3, \quad |Y_{ab}| \lesssim 3, \quad |Z_{ab}| \lesssim 3, \]

where again \( (XYZ)_{ab} = X_{ab} + Y_{ab} + Z_{ab} \). We have numerically extracted these relations via the K-matrix formalism [14, 15] as done in [16, 17].

Since the experimentally allowed range for the muon coupling is quite large we do not include it in the plots.
TABLE III. Outputs for each of the four numerical benchmark scenarios. Scalar masses are given in GeV.

|          | \(m_{\eta} \) | \(m_{\eta} \) | \(m_{\eta} \) | \(m_{\eta} \) | \(m_{\eta} \) | \(m_{\eta} \) | \(m_{\eta} \) | \(M_{H}^{\pm} \) | \(M_{H}^{\pm} \) | \(M_{H}^{\pm} \) |
|----------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| Aligned  | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 |
| Hypo     | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 |
| Hyper    | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 |
| Mixed    | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 | 154 404 340 270 |

TABLE II. Four sets of numerical values giving rise to four different benchmark scenarios. Scalar masses are in GeV while the rest have no units. All parameters have been assumed to be real.

- **Vacuum stability:**
  \[
  X_{ta} > -\frac{\mu_{a}^{2}}{v_{t}^{2}}, \quad (X + Y + Z)_{\mu} > -\frac{\mu_{\mu}^{2}}{v_{t}^{2}},
  \]

where \( (a = b, \mu, d) \). This set of conditions was computed from the requirement that the squared mass matrices for the charged scalars and pseudo-scalars should be positive definite, for further details see Appendix [C].

- **Contributions to the \( \rho \) parameter:**
  \[
  \Delta \rho = 0.0005 \pm 0.0005 \ (\pm 0.0009),
  \]

that is, it should be consistent with the maximum allowed deviation from the SM-expectation [18].

- **Charged Higgs masses above the lower bound [20]:**
  \[
  80 \text{ GeV} \lesssim M^{\pm}_{k}.
  \]

- **Recently, a search for a Higgs-like particle, \( \phi \), decaying into a pair of bottom quarks with at least one additional bottom in proton-proton collision was reported [21].** The following mass range was excluded with 95% confidence level:

  \[
  100 \text{ GeV} < m_{\phi} < 300 \text{ GeV}.
  \]

While not directly comparable with our scenario, our benchmark points nevertheless obey this constraint.

IV. CONCLUSIONS

Within the SM the huge hierarchy of Yukawa couplings remains a puzzle. In this regard we used the fact that the observed fermion masses indicate that the following sets have similar Yukawa couplings: \( \{m_{t}\}, \{m_{b}, m_{c}, m_{c}\}, \{m_{d}, m_{u}, m_{e}\} \).

We have shown that a 4HDM can be constructed that explains this feature. Each set of fermions has its own Higgs doublet. Their vevs are hierarchical which explains the mass hierarchy of the sets. In the model a flavor symmetry was introduced to generate rank 1 Yukawa matrices. Soft breaking was included in the potential, which makes it possible to induce the smaller vevs by the larger ones, where each smaller vev corresponds to a different broken symmetry, and is thus protected by it. All Yukawa couplings take on "natural" values of order 1. In the model neutrino masses are generated via a type-I seesaw mechanism with a Dirac neutrino mass matrix of order of the down-quark mass scale, hence the right-handed singlet Majorana neutrinos are of PeV-scale. We have demonstrated that fermions of a given set couple to the SM-like Higgs with the same modified factor. In this regard, the clearest signal for this kind of models is to investigate their coupling to the
SM-like Higgs and determine if they are grouped. The top quark couples to the SM-like Higgs essentially with the same strength as in the SM. Benchmark scenarios with definite predictions for those couplings as well as for scalar masses were provided.

Multi-Higgs doublet models face of course problems with FCNC. By singularly aligning the Yukawa matrices we have shown explicitly that those can be evaded. This alignment assumes that the Yukawa matrices are related to the rank 1 matrices that appear in the singular value decomposition of the mass matrices. In this manner, it is in general, not only in our model, possible to avoid FCNC while simultaneously coupling several Higgs doublets to an individual given fermion.

The model as well as aspects of singular alignment allow for several follow-up studies regarding both model building and phenomenology.

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Appendix A: Proof of FCNC Disappearance in the singular Basis

Consider a given fermion type coupled to \( N \) different scalar doublets. Its mass matrix would be given by

\[
M = v_1 Y_1 + v_2 Y_2 + \cdots + v_N Y_N, \tag{A1}
\]

where we have assumed that each scalar doublet acquires a vev. On the other hand, the SVD of the mass matrix is

\[
M = L^\dagger \text{diag}(m_1, m_2, m_3) R, \tag{A2}
\]

where \( L \) and \( R \) are unitary transformations acting independently on the left- and right-handed fields.

Using Dirac notation, the SVD can be rewritten as

\[
M = \sum_i m_i |\ell_i\rangle \langle r_i|, \tag{A3}
\]

Singular alignment requires assuming each Yukawa matrix to be related to the rank 1 matrices, \(|\ell_i\rangle \langle r_i|\), of the SVD. In general, we can express the Yukawa matrices as a linear combination of the rank one singular matrices

\[
Y_i = \eta_i |\ell_1\rangle \langle r_1| + \Omega_i |\ell_2\rangle \langle r_2| + \Lambda_i |\ell_3\rangle \langle r_3|, \tag{A4}
\]

where the parameters \( \{\eta, \Omega, \Lambda\} \) are real.

In the mass basis, each Yukawa matrix would take the form,

\[
LY_i R^\dagger = \begin{pmatrix}
\eta & 0 & 0 \\
0 & \Omega & 0 \\
0 & 0 & \Lambda_i
\end{pmatrix}. \tag{A5}
\]

Therefore, through singularly aligning we have avoided the appearance of dangerous tree-level FCNC.

For last, notice that after substitution of the previous relation in Eq. (A1) we obtain

\[
m_1 = \sum_j v_j \eta_j, \quad m_2 = \sum_j v_j \Omega_j, \quad m_3 = \sum_j v_j \Lambda_j. \tag{A6}
\]

Appendix B: Numerical Example for Quark Mixing

The following singular matrices allow us to reproduce exactly the observed mixing in the quark sector as recently reported in the PDG 2018 \[13\]:

\[
\begin{align*}
\Delta_1^u &= \begin{pmatrix} 0.117482 & 0 & 0 \\ 0.984047 e^{+0.4851} & 0 & 0 \\ 0.133604 e^{-1.27973} & 0 & 0 \end{pmatrix}, & \Delta_2^u &= \begin{pmatrix} 0 & 0.0236066 & 0 \\ 0 & 0.135386 e^{-3.89058} & 0 \\ 0 & 0.990512 e^{+0.0259661} & 0 \end{pmatrix}, & \Delta_3^u &= \begin{pmatrix} 0 & 0 & 0.992794 e^{-0.511327} \\ 0 & 0 & 0.115423 e^{i\pi} \\ 0 & 0 & 0.0321998 e^{i\pi} \end{pmatrix}, \\
\Delta_1^d &= \begin{pmatrix} 0.109076 & 0 & 0 \\ 0.989786 e^{+3.00176} & 0 & 0 \\ 0.0917962 e^{+3.52235} & 0 & 0 \end{pmatrix}, & \Delta_2^d &= \begin{pmatrix} 0 & 0.0365871 & 0 \\ 0 & 0.0886041 e^{+2.59211} & 0 \\ 0 & 0.995395 e^{+6.27169} & 0 \end{pmatrix}, & \Delta_3^d &= \begin{pmatrix} 0 & 0 & 0.99336 e^{-6.13171} \\ 0 & 0 & 0.111685 \\ 0 & 0 & 0.0276181 e^{i\pi} \end{pmatrix},
\end{align*} \tag{B1}
\]

where the implied mixing matrix is

\[
|V_{\text{CKM}}^{\text{th}}| = \begin{pmatrix} 0.97445 & 0.22458 & 0.00364 \\ 0.22442 & 0.97358 & 0.04217 \\ 0.00897 & 0.04137 & 0.999104 \end{pmatrix}, \tag{B3}
\]

with a Jarlskog invariant of

\[
J_q^{\text{th}} = 3.18 \times 10^{-5}. \tag{B4}
\]
Appendix C: Scalar Mass Matrices

In this section we discuss the scalar mass matrices. With four Higgs doublets there are 4 physical CP-even scalars, 3 pseudoscalars and 3 pairs of charged Higgses. Computation of the scalar mass matrices in the limit

\[
\{v_t^2, \mu_b^2, \mu_\mu^2, \mu_d^2\} \gg \{\mu_{tb}, \mu_{b\mu}, \mu_{\mu d}, v_b^2, v_\mu^2, v_d^2\}\]  

leads to

\[
M_{CP-even}^2 \approx \begin{pmatrix}
2v_t^2\lambda_t & 2v_t v_b (XY)_{tb} + \mu_{tb}^2 & 2v_t v_\mu (XY)_{tb} + \mu_\mu^2 & 2v_t v_d (XY)_{tb} + \mu_d^2 \\
2v_t v_b (XY)_{tb} + \mu_{tb}^2 & v_t^2(XY)_{tb} + \mu_{tb}^2 & v_t^2(XY)_{bm} + \mu_{b\mu}^2 & v_t^2(XY)_{bd} + \mu_{bd}^2 \\
2v_t v_\mu (XY)_{bm} + \mu_\mu^2 & v_t^2(XY)_{bm} + \mu_\mu^2 & v_t^2(XY)_{pm} + \mu_\mu^2 & v_t^2(XY)_{pd} + \mu_{pd}^2 \\
2v_t v_d (XY)_{bd} + \mu_d^2 & v_t^2(XY)_{bd} + \mu_d^2 & v_t^2(XY)_{pd} + \mu_{pd}^2 & v_t^2(XY)_{pd} + \mu_d^2
\end{pmatrix},
\]

\[
M_{CP-odd}^2 \approx \begin{pmatrix}
v_t^2(X + Y - Z)_{tb} + \mu_b^2 & 0 & 0 & 0 \\
0 & v_t^2(X + Y - Z)_{tb} + \mu_\mu^2 & 0 & 0 \\
0 & 0 & v_t^2(X + Y - Z)_{td} + \mu_d^2 & 0 \\
0 & 0 & 0 & v_t^2(X + Y - Z)_{td} + \mu_d^2
\end{pmatrix},
\]

\[
M_{charged}^2 \approx \begin{pmatrix}
v_t^2 X_{tb} + \mu_b^2 & 0 & 0 & 0 \\
0 & v_t^2 X_{tb} + \mu_b^2 & 0 & 0 \\
0 & 0 & v_t^2 X_{td} + \mu_d^2 & 0 \\
0 & 0 & 0 & v_t^2 X_{td} + \mu_d^2
\end{pmatrix}.
\]

Three necessary but not sufficient conditions may help us produce positive definite squared mass sub-matrices. Given an Hermitian matrix \(A\)

(i) all diagonal elements should be positive:

\[|A|_{ii} > 0.\]  

(ii) the sum of any pair of diagonal entries should satisfy

\[|A|_{ii} + |A|_{jj} > 2|\Re(|A|_{ij})|.\]  

(iii) the largest element should lie on the diagonal.

By virtue of those conditions one may easily derive from the charged scalar matrix that:

\[X_{tb} > -\frac{\mu_{tb}^2}{v_t^2}, \quad X_{t\mu} > -\frac{\mu_{\mu}^2}{v_t^2}, \quad X_{td} > -\frac{\mu_{pd}^2}{v_t^2}.\]  

For the pseudo-scalar matrix:

\[(X + Y - Z)_{ta} > -\frac{\mu_a^2}{v_t^2}, \quad (a = b, \mu, d).\]  

We have not neglected here the off-diagonal contributions to the \(CP\)-even scalar matrix, as even though they are very small, they can still influence the Higgs-fermion couplings as already previously discussed.

Appendix D: Constrains from the \(\rho\) Parameter

The one-loop level contribution to the \(\rho\) parameter from a theory with \(N\) Higgs doublets has been calculated in...
Ref. [19] and is expressed as

\[ \Delta \rho = \frac{1}{32 \pi^2 v_{EW}^2} \left[ \sum_{i=1}^{2N} \sum_{j=2}^{2N} \left| (O^i S)_{ij} \right|^2 F \left( m_i^+, m_j^0 \right) \right. \]

\[ \left. - \sum_{i=2}^{2N-1} \sum_{j=i+1}^{2N} \left| (S^i S)_{ij} \right|^2 F \left( m_i^0, m_j^0 \right) \right] \]

\[ + 3 \sum_{i=2}^{2N} \left| (S^i S)_{ii} \right|^2 \left[ F \left( M_Z, m_i^0 \right) - F \left( M_W, m_i^0 \right) \right] \]

\[ - 3 \left[ F \left( M_Z, m_{h_0}^0 \right) - F \left( M_W, m_{h_0}^0 \right) \right] \]  

\[ \text{D1} \]

where

\[ F(x, y) \equiv \begin{cases} \frac{x^2 y^2}{x^2 + y^2} - \frac{x^2 y^2}{\pi x y \ln \frac{x^2}{y^2}}, & x \neq y \\ 0, & x = y \end{cases} \]  

(D2)

and the matrices \( O \) and \( S = R \oplus R' \) are the orthogonal matrices responsible for diagonalizing the mass matrices for the charged, \( CP \)-even and -odd scalars, respectively. The function \( F(x, y) \) is a positive function, symmetrical under the interchange of its arguments, and vanishing if and only if the arguments are equal. The behavior of this function has an interesting property, as it grows linearly with \( \max(x, y) \), that is, quadratically with the heaviest-scalar mass, when that mass becomes very large. As long as the difference in the scalar masses is small, \( \delta \lesssim 200 \text{ GeV} \), the maximum value of this function lies within the 3\( \sigma \) deviation in \( \Delta \rho \), as shown in Fig. [3] even if the masses become very heavy, \( M > 300 \text{ GeV} \). This can be seen from Taylor expanding the function, \( \delta \ll x \),

\[ F(x, x + \delta) = \delta^2 \left[ \frac{2}{3} - \frac{\delta^2}{30x^2} \right] + O(\delta^4). \]  

(D3)

Moreover, realize that these contributions will get further suppressed by the factors coming from the off-diagonal matrix elements in the product of the orthogonal matrices.

In the limit in which we are working, mass matrices can be considered to a very good degree of accuracy to be diagonal, therefore, the maximum amount of contributions in this model will take the form

\[ \Delta \rho \simeq \frac{1}{32 \pi^2 v_{EW}^2} \sum_{i=1}^{3} \left[ F \left( m_i^+, m_{h,i}^0 \right) + F \left( m_i^-, m_{h,i}^0 \right) \right] \]

\[ + 0.00017. \]  

(D4)