Quantum Computing and a Unified Approach to Fast Unitary Transforms

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ABSTRACT

A quantum computer directly manipulates information stored in the state of quantum mechanical systems. The available operations have many attractive features but also underly severe restrictions, which complicate the design of quantum algorithms. We present a divide-and-conquer approach to the design of various quantum algorithms. The class of algorithm includes many transforms which are well-known in classical signal processing applications. We show how fast quantum algorithms can be derived for the discrete Fourier transform, the Walsh-Hadamard transform, the Slant transform, and the Hartley transform. All these algorithms use at most $O(\log^2 N)$ operations to transform a state vector of a quantum computer of length $N$.

1. INTRODUCTION

Discrete orthogonal transforms and discrete unitary transforms have found various applications in signal, image, and video processing, in pattern recognition, in biocomputing, and in numerous other areas [1–11]. Well-known examples of such transforms include the discrete Fourier transform, the Walsh-Hadamard transform, the trigonometric transforms such as the Sine and Cosine transform, the Hartley transform, and the Slant transform. All these different transforms find applications in signal and image processing, because the great variety of signal classes occurring in practice cannot be handled by a single transform.

On a classical computer, the straightforward way to compute a discrete orthogonal transform of a signal vector of length $N$ takes in general $O(N^2)$ operations. An important aspect in many applications is to achieve the best possible computational efficiency. The examples mentioned above allow an evaluation with as few as $O(N \log N)$ operations or – in the case of the wavelet transforms – even with as little as $O(N)$ operations. In view of the trivial lower bound of $\Omega(N)$ operations for matrix-vector-products, we notice that these algorithms are optimal or nearly optimal.

The rules of the game change dramatically when the ultimate limit of computational integration is approached, that is, when information is stored in single atoms, photons, or other quantum mechanical systems. The operations manipulating the state of such a computer have to follow the dictum of quantum mechanics. However, this is not necessarily a limitation. A striking example of the potential speed-up of quantum computation over classical computation has been given by Shor in 1994. He showed that integers can be factored in polynomial time on a quantum computer. In contrast, there are no polynomial time algorithms known for this problem on a classical computer.

The quantum computing model does not provide a uniform speed-up for all computational tasks. In fact, there are a number of problems which do not allow any speed-up at all. For instance, it can be shown that a quantum computer searching a sorted database will not have any advantage over a classical computer. On the other hand, if we use our classical algorithms on a quantum computer, then it will simply perform the calculation in a similar manner to a classical computer. In order for a quantum computer to show its superiority one needs to design new algorithms which take advantage of quantum parallelism.

A quantum algorithm may be thought of as a discrete unitary transform which is followed by some I/O operations. This observation partially explains why signal transforms play a dominant role in numerous quantum algorithms.\textsuperscript{12–14} Another reason is that it is often possible to find extremely efficient quantum algorithms for
the discrete orthogonal transforms mentioned above. For instance, the discrete Fourier transform of length \( N = 2^n \) can be implemented with \( O(\log^2 N) \) operations on a quantum computer.

2. QUANTUM COMPUTING

The basic unit of information in classical computation is a bit, a system with two distinguishable states representing logical values 0 or 1. We mentioned in the introduction that a quantum computer will store such information in the states of a quantum mechanical system. Suppose that the system has two distinguishable states. We will denote these states by \(|0\rangle\) and \(|1\rangle\), where the notation reminds us that these states represent the logical values 0 and 1.

A potential candidate for the storage of a single bit is given by a spin-\( \tfrac{1}{2} \) particle, such as an electron, proton, or neutron. We can choose the state with the rotation vector pointing upward (spin-up) and the state with the rotation vector pointing downward (spin-down) to represent 0 and 1, respectively. However, we know from quantum mechanics that quantum system can be in a superposition of states. In the case of a spin-\( \tfrac{1}{2} \) particle, a superposition

\[
|\psi\rangle = a|0\rangle + b|1\rangle
\]

yields a state which rotates about a different axis. The coefficients \( a, b \) in this superposition are complex numbers, which determine this spin axis.

The consequent abstraction of the preceding example leads to the notion of a quantum bit, or shortly qubit, the basic unit of information in quantum computation. A quantum bit is given by a superposition of the states \(|0\rangle\) and \(|1\rangle\) such as

\[
|\psi\rangle = a|0\rangle + b|1\rangle, \quad a, b \in \mathbb{C}.
\]

The value of a quantum bit remains uncertain until it is measured. A measurement will collapse \(|\psi\rangle\) to either the state \(|0\rangle\) or to the state \(|1\rangle\). The coefficients \( a \) and \( b \) determine the probability of outcome of this measurement, namely

| Event | Probability |
|-------|-------------|
| \(|\psi\rangle\) collapses to \(|0\rangle\) | \( |a|^2 / (|a|^2 + |b|^2) \) |
| \(|\psi\rangle\) collapses to \(|1\rangle\) | \( |b|^2 / (|a|^2 + |b|^2) \) |

In either case, we will learn the outcome of the measurement. Since proportional states lead to the same measurement results, it is conventionally assumed that the state is normalized to length 1, i.e., it is assumed that \( |a|^2 + |b|^2 = 1 \) holds.

The measurement allows to implement a fair coin flip on a quantum computer. Indeed, preparing a quantum bit in the state \(|\psi\rangle = \sqrt{\tfrac{1}{2}}|0\rangle + \sqrt{\tfrac{1}{2}}|1\rangle\), and measuring the result yields either 0 or 1. According to the above rule, either event will occur exactly with probability 1/2. This example might suggest that computations on a quantum computer are indeterministic and maybe even somewhat fuzzy. However, this is not the case. We will see in a moment that all operations apart from measurements are completely deterministic. The only operations that might introduce some randomized behaviour are the measurements, which – as Penrose puts it – ‘magnify an event from the quantum level to the classical level’ [15, pp. 7-8].

We discuss now the deterministic operations on a quantum computer. We begin with the simplest case, the operations which manipulate the state of a single quantum bit. First of all, it should be noted that the states \(|0\rangle\) and \(|1\rangle\) can be understood as an orthonormal basis of the complex inner product space \( \mathbb{C}^2 \). It is customary to associate the base states \(|0\rangle\) and \(|1\rangle\) with the standard basis vectors \((1,0)^t\) and \((0,1)^t\), respectively. Therefore, a quantum bit in the state \(a|0\rangle + b|1\rangle\) is represented by the state vector

\[
\begin{pmatrix}
  a \\
  b
\end{pmatrix} = a \begin{pmatrix}
  1 \\
  0
\end{pmatrix} + b \begin{pmatrix}
  0 \\
  1
\end{pmatrix}.
\]

A deterministic operation has to realize a unitary evolution of the quantum state, following the rules of quantum mechanics. In other words, a single quantum bit operation is given by a unitary operator \( U : \mathbb{C}^2 \to \mathbb{C}^2 \) acting
The latter notation is often abbreviated to every single position of the state vector is manipulated. This is a striking example of quantum parallelism. We space doubles significant bit of (1). The resulting state is single qubit operation. Suppose that we apply a single qubit operation, say the Hadamard gate $| \rangle$, with the addition of a single quantum bit.

In more traditional mathematical notation, we can formulate this as the action of the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}. $$

The operation $X$ realizes a NOT operation, $X |0 \rangle = |1 \rangle$ and $X |1 \rangle = |0 \rangle$. The operation $Z$ implements a phase shift operation, $Z |0 \rangle = |0 \rangle$ and $Z |1 \rangle = -|1 \rangle$. An extremely useful operation is given by the Hadamard gate $H$, which is for instance used to create superpositions,

$$H |0 \rangle = \frac{1}{\sqrt{2}} |0 \rangle + \frac{1}{\sqrt{2}} |1 \rangle, \quad H |1 \rangle = \frac{1}{\sqrt{2}} |0 \rangle - \frac{1}{\sqrt{2}} |1 \rangle.$$ 

The Hadamard gate should be familiar to readers with a background in signal processing or coding theory. In the following, we will keep the notations for these gates without further notice.

The operations get more interesting in the case of multiple quantum bits. Quantum mechanics tells us that the state space of a combined quantum system is given by the tensor product of the state spaces of its parts. A remarkable consequence of this rule is that the state space of a system with $n$ quantum bits is given by the tensor product of $n$-fold tensor product. This simply means that the dimension of the state space doubles with the addition of a single quantum bit.

The state of a system with two quantum bits can thus be described by a vector $(a_{00}, a_{01}, a_{10}, a_{11})^t \in \mathbb{C}^4$ or, isomorphically, by the vector

$$| \psi \rangle = a_{00} |0 \rangle \otimes |0 \rangle + a_{01} |0 \rangle \otimes |1 \rangle + a_{10} |1 \rangle \otimes |0 \rangle + a_{11} |1 \rangle \otimes |1 \rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2. \quad (1)$$

The latter notation is often abbreviated to $a_{00} |00 \rangle + a_{01} |01 \rangle + a_{10} |10 \rangle + a_{11} |11 \rangle$. The label $x_1 x_0$ in the Dirac ket notation $|x_1 x_0 \rangle$ specifies a location in the quantum memory.

A dramatic consequence of the tensor product structure of the quantum memory can be illustrated with a single qubit operation. Suppose that we apply a single qubit operation, say the Hadamard gate $H$, on the least significant bit of (1). The resulting state is

$$| \psi' \rangle = a_{00} |0 \rangle \otimes H |0 \rangle + a_{01} |0 \rangle \otimes H |1 \rangle + a_{10} |1 \rangle \otimes H |0 \rangle + a_{11} |1 \rangle \otimes H |1 \rangle$$

$$= \frac{1}{\sqrt{2}} \left( (a_{00} + a_{01}) |0 \rangle \otimes |0 \rangle + (a_{00} - a_{01}) |0 \rangle \otimes |1 \rangle + (a_{10} + a_{11}) |1 \rangle \otimes |0 \rangle + (a_{10} - a_{11}) |1 \rangle \otimes |1 \rangle \right).$$

In more traditional mathematical notation, we can formulate this as the action of the matrix

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} a_{00} \\ a_{01} \\ a_{10} \\ a_{11} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} a_{00} + a_{01} \\ a_{00} - a_{01} \\ a_{10} + a_{11} \\ a_{10} - a_{11} \end{pmatrix}. $$

Therefore, the resulting operation is $(1 \otimes H) |\psi\rangle = |\psi'\rangle$. Although we act only on one quantum bit, we see every single position of the state vector is manipulated. This is a striking example of quantum parallelism. We observe that a butterfly structure, well-known from many signal processing algorithms, can be implemented with a single operation on a quantum computer.
Figure 2. Single qubit operation $U$. The left side shows the schematic for $U$ acting on the least significant qubit; this circuit implements the matrix $1 \otimes U$. The figure on the right shows the single qubit gate acting on the most significant qubit; this circuit implements the matrix $U \otimes 1$.

The direct generalization to arbitrary single qubit operations is shown in Figure 2. In general, a single qubit operation is specified by a unitary $2 \times 2$ matrix $U$, and the position of the target qubit to which $U$ is applied. Suppose that the target qubit position is $i$, then each state $|x_{n-1} \ldots x_{i+1}x_ix_{i-1} \ldots x_0\rangle$ is unconditionally transformed to $|x_{n-1} \ldots x_{i+1} \rangle \otimes U|x_i\rangle \otimes |x_{i-1} \ldots x_0\rangle$, where $x_k \in \{0,1\}$.

We can specify more elaborate gates, which allow to create an interaction between quantum bits. Let $C_0$ and $C_1$ be two disjoint sets of quantum bit positions, neither of which contains the target bit position $i$. A conditional $U$-operation maps the state $|x_{n-1} \ldots x_{i+1}x_ix_{i-1} \ldots x_0\rangle$ to the state $|x_{n-1} \ldots x_{i+1} \rangle \otimes U|x_i\rangle \otimes |x_{i-1} \ldots x_0\rangle$, in case $x_i = 0$ for all $i \in C_0$ and $x_j = 1$ for all $j \in C_1$. The state remains unchanged in all other cases. The set $C_0$ describes the set of zero-conditions and $C_1$ the set of one-conditions. In the schematics, we will use the symbol ◦ to denote a zero-condition and the symbol • to denote a one-condition. Figure 3 shows the simplest, but most important, conditional quantum gate – the controlled NOT operation.

Figure 3. The controlled NOT gate is a reversible XOR gate. The states $|00\rangle$ and $|01\rangle$ remain unchanged, since the most significant qubit must be 1. If the most significant bit is 1, then a NOT operation is applied to the least significant bit. Therefore, $|10\rangle$ is mapped to $|11\rangle$, and $|11\rangle$ is mapped to $|10\rangle$.

We can use controlled NOT gates to get an interaction between different quantum bits. For example, consider the circuit in Figure 4. This circuit swaps the states of the two quantum bits.

Figure 4: Circuit which swaps the state of two quantum bits.

Engineering controlled $U$ operations is in general a difficult task. We will refer to controlled NOT gates with a single control bit and to single qubit operations as elementary gates. Elementary quantum gates are available in all mature quantum computing technologies. It can be shown that it is possible to implement a general controlled $U$ operation with $O(\log N)$ elementary gates. We will always refer to elementary gates in gate counts, but we will use multiply controlled $U$ gates for the sake of brevity in circuit descriptions. There exist standard algorithms which transform these more general gates into a sequence of elementary gates.

3. DIVIDE-AND-CONQUER METHODS

We have seen that a number of powerful operations are available on a quantum computer. Suppose that we want to implement a unitary or orthogonal transform $U \in U(2^n)$ on a quantum computer. The goal will be to find an implementation of $U$ in terms of elementary quantum gates. Usually, our aim will be to find first a factorization of $U$ in terms of sparse structured unitary matrices $U_i$,

$$U = U_1U_2 \cdots U_k,$$

where, of course, $k$ should be small. The philosophy being that it is often very easy to derive quantum circuits for structured sparse matrices. For example, if we can find an implementation with few multiply controlled unitary gates for each factor $U_i$, then the overall circuit will be extremely efficient.
The success of this method depends of course very much on the availability suitable factorization of $U$. However, in the case orthogonal transforms used in signal processing, there are typically numerous classical algorithms available, which provide the suitable factorizations. It should be noted that, in principle, an exponential number of elementary gates might be needed to implement even a diagonal unitary matrix. Fortunately, we will see that most structured matrices occurring in practice have very efficient implementations. In fact, we will see that all the transforms of size $2^n \times 2^n$ discussed in the following can be implemented with merely $O(\log^2 2^n) = O(n^2)$ elementary quantum gates.

We present a simple – but novel – approach to derive such efficient implementations. This approach is based on a divide-and-conquer technique. Assume that we want to implement a family of unitary transforms $U_N$, where $N = 2^n$ denotes the length of the signal. Suppose further the family $U_N$ can be recursively generated by a recursive circuit construction, for instance, such as the one shown in Figure 5. We will give a generic construction for the family of precomputation circuits $\text{Pre}_N$ and the family of postcomputation circuits $\text{Post}_N$. This way, we obtain a fairly economic description of the algorithms.

![Figure 5. Recursive implementation of a family of quantum circuits $U_N$. If the preparation circuit $\text{Pre}_{N/2}$ and postcomputation circuits $\text{Post}_{N/2}$ have small complexity, then the overall circuit family will have an efficient implementation.]

Assume that a total of $P(N)$ elementary operations are necessary to implement the precomputation circuit $\text{Pre}_{N/2}$ and the postcomputation circuit $\text{Post}_{N/2}$. Then the overall number $T(N)$ of elementary operations can be estimated from the recurrence equation

$$T(N) = T(N/2) + P(N).$$

The number of operations $T(N)$ for the recursive implementation can be estimated as follows:

**Lemma.** If $P(N) \in \Theta(\log^p N)$, then $T(N) \in O(\log^{p+1} N)$.

## 4. FOURIER TRANSFORM

We will illustrate the general approach by way of some examples. Our first example is the discrete Fourier transform. A quantum algorithm implementing this transform found a most famous application in Shor’s integer factorization algorithm. Recall that the discrete Fourier transform $F_N$ of length $N = 2^n$ can be described by the matrix

$$F_N = \frac{1}{\sqrt{N}} \left( \begin{array}{cccc} \omega^{0j} & \omega^{1j} & \cdots & \omega^{(N-1)j} \\ \end{array} \right)_{j,k=0,\ldots,N-1},$$

where $\omega$ denotes a primitive $N$-th root of unity, $\omega = \exp(2\pi i/N)$. And $i$ denotes a square root of $-1$.

The main observation behind the fast quantum algorithm dates at least back to work by Danielson and Lanczos in 1942 (and is implicitly contained in numerous earlier works). They noticed that the matrix $F_N$ might be written as

$$F_N = \frac{1}{\sqrt{2}} P_N \left( \begin{array}{cc} F_{N/2} & F_{N/2} \\ F_{N/2} T_{N/2} & -F_{N/2} T_{N/2} \end{array} \right),$$

where $P_N$ denotes the permutation of rows given by $P_N |b x\rangle = |x b\rangle$ with $x$ an $n-1$-bit integer, and $b$ a single bit, and $T_{N/2} := \text{diag}(1, \omega, \omega^2, \ldots, \omega^{N/2-1})$ denotes the matrix of twiddle factors.
This observation allows to represent $F_N$ by the following product of matrices:

$$F_N = P_N \left( \begin{array}{cc} F_{N/2} & 0 \\ 0 & F_{N/2} \end{array} \right) \left( \begin{array}{cc} 1_{N/2} & 0 \\ 0 & T_{N/2} \end{array} \right) \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1_{N/2} & 1_{N/2} \\ 1_{N/2} & -1_{N/2} \end{array} \right)$$

$$= P_N (1_2 \otimes F_{N/2}) \left( \begin{array}{c} 1_{N/2} \\ T_{N/2} \end{array} \right) (F_2 \otimes 1_{N/2})$$

This factorization yields an outline of an implementation on a quantum computer. The overall structure is shown in Figure 6.

![Figure 6](image)

**Figure 6:** The recursive structure of the quantum Fourier transform.

It remains to detail the different steps in this implementation. The first step is a single qubit operation, implementing a butterfly structure. The next step is slightly more complicated. We observe that $T_{N/2}$ is a tensor product of diagonal matrices $D_j = \text{diag}(1, \omega^{2^j})$. Indeed,

$$T_{N/2} = D_{n-1} \otimes \cdots \otimes D_2 \otimes D_1.$$ 

Thus, $1_{N/2} \oplus T_{N/2}$ can be realized by controlled phase shift operations, see Figure 7 for an example. We then recurse to implement the Fourier transform of smaller size. The final permutation implements the cyclic rotation of the quantum wires.

![Figure 7](image)

**Figure 7:** Implementation of the twiddle matrix $1_8 \oplus T_8$.

The complexity of the quantum Fourier transform can be estimated as follows. If we denote by $R(N)$ the number of gates necessary to implement the DFT of length $N = 2^n$ on a quantum computer, then Figure 6 implies the recurrence relation

$$R(N) = R(N/2) + \Theta(\log N)$$

which leads to the estimate $R(N) = O(\log^2 N)$.

It should be noted that all permutations $P_N (1_2 \otimes P_{N/2}) \cdots (1_{N-2} \otimes P_4)$ at the end can be combined into a single permutation of quantum wires. The resulting permutation is the bit reversal, see Figure 8.

**Remark.** Another explanation of the discrete Fourier transform algorithm is contained in [17]. Note that the row permutations are mistaken in that article. An approximate version of the discrete Fourier transform has been proposed by Coppersmith, which saves some operations.
5. THE WALSH-HADAMARD TRANSFORM

The Walsh-Hadamard transform $W_N$ is maybe the simplest instance of the recursive approach. This transform is defined by the Hadamard gates $W_2 = H$ in the case of signals of length 2. For signals of larger length, the transform is defined by

$$ W_N = (1_2 \otimes W_{N/2})(H \otimes 1_{N/2}). $$

This yields the recursive implementation shown in Figure 9.

Since $P(N) = \Theta(1)$, the Lemma in Section 3 shows that the number of operations $T(N) \in O(\log N)$. It is of course trivial to see that in this case exactly $\log N$ operations are needed.

6. THE SLANT TRANSFORM

The Slant transform is used in image processing for the representation of images with many constant or uniformly changing gray levels. The transform has good energy compaction properties. It is used in Intel’s ‘Indeo’ video compression and in numerous still image compression algorithms.

The Slant transform $S_N$ is defined for signals of length $N = 2$ by the Hadamard matrix

$$ S_2 = H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, $$

and for signals of length $N = 2^k$, $N > 2$, by

$$ S_N = Q_N \begin{pmatrix} S_{N/2} & 0_{N/2} \\ 0_{N/2} & S_{N/2} \end{pmatrix}, $$

(2)

where $0_{N/2}$ denotes the all-zero matrix, and $Q_N$ is given by the matrix product

$$ Q_N = P_N^a (1_{N/2} \oplus \hat{Q}_N) (H \otimes 1_{N/2}) P_N^b. $$

(3)

The matrices in (3) are defined as follows (see also [19]): $1_{N/2}$ is the identity matrix, $H$ is the Hadamard matrix, and $P_N^a$ realizes the transposition $(1, N/2)$, that is,

$$ P_N^a |1\rangle = |N/2\rangle, \quad P_N^a |N/2\rangle = |1\rangle, \quad \text{and} \quad P_N^a |x\rangle = |x\rangle \text{ otherwise}. $$
The matrix $P^b_N$ is defined by $P^b_N |x⟩ = |x⟩$ for all $x$ except in the case $x = N/2 + 1$, where it yields the phase change $P^b_N |N/2 + 1⟩ = -|N/2 + 1⟩$. Finally

$$\hat{Q}_N = \begin{pmatrix} A_N & 0 \\ 0 & 1_{2^{N/2}} \end{pmatrix}, \quad A_N = \begin{pmatrix} a_N & b_N \\ -b_N & a_N \end{pmatrix},$$

where $a_N$ and $b_N$ are recursively defined by $a_2 = 1$ and

$$b_N = \frac{1}{\sqrt{1 + 4(a_{N/2})^2}} \quad \text{and} \quad a_N = 2b_Na_{N/2}.$$  

It is easy to check that $A_N$ is a unitary matrix.

The definition of the Slant transform suggests the following implementation. Equation (2) tells us that the input signal of a Slant transform of length $N$ is first processed by two Slant transforms of size $N/2$, followed by a circuit implementing $Q_N$. We can write equation (2) in the form

$$S_N = \hat{Q}_N \left( S_{N/2} \begin{pmatrix} 0_{N/2} \\ 0_{N/2} \\ S_{N/2} \end{pmatrix} \right) = \hat{Q}_N (1_2 \otimes S_{N/2}).$$

The tensor product structure $1_2 \otimes S_{N/2}$ is compatible with our decomposition into quantum bits. This means that a single copy of the circuit $S_{N/2}$ acting on the lower significant bits will realize this part. It remains to give an implementation for $Q_N$. Equation (3) describes $Q_N$ as a product of four sparse matrices, which are easy to implement. Indeed, the matrix $P^b_N$ is realized by conditionally exerting the phase gate $Z$. The matrix $H \otimes 1_{N/2}$ is implemented by a Hadamard gate $H$ acting on the most significant bit. A conditional application of $A_N$ implements the matrix $1_{N/2} \oplus \hat{Q}_N$. A conditional swap of the least and the most significant qubit realizes $P^a_N$, that is, three multiply controlled NOT gates implement $P^a_N$. The quantum circuit realizing this implementation is depicted in Figure 10.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure10.png}
\caption{Implementation of the Slant transform. The recursive step is realized by a single Slant transform of size $S_{N/2}$. The next three gates implement $P^b_N$, $H \otimes 1_{N/2}$, and $1_{N/2} \oplus \hat{Q}_N$, respectively. The last three gates implement $P^a_N$. Thus, the implementation of $Q_N$ totals five multiply controlled gates and one single qubit gate.}
\end{figure}

**Theorem 6.1.** The Slant transform of length $N = 2^k$ can be realized on a quantum computer with at most $O(\log^2 N)$ elementary operations (that is, controlled NOT gates and single qubit gates), assuming that additional workbits are available.

**Proof.** Recall that a multiply controlled gate can be expressed with at most $O(\log N)$ elementary operations as long as additional workbits are available. It follows from the Lemma in Section 3 that at most $O(\log^2 N)$ elementary operations are needed to implement the Slant transform. $\square$

### 7. THE HARTLEY TRANSFORM

The discrete Hartley transform $H_N$ is defined for signals of length $N = 2^n$ by the matrix

$$H_N = \frac{1}{\sqrt{N}} \left( \cos(2\pi k\ell) + \sin(2\pi k\ell) \right)_{k,\ell=0,...,N-1}.$$
The discrete Hartley transform is very popular in classical signal processing, since it requires only real arithmetic but has similar properties. In particular, there are classical algorithms available, which outperform the fastest Fourier transform algorithms. We derive a fast quantum algorithm for this transform, again based on a recursive divide-and-conquer algorithm. A fast algorithm for the discrete Hartley transform based on a completely different approach has been discussed by Klappenecker and Rötteler.\textsuperscript{17}

The Hartley transform can be recursively represented as\textsuperscript{20}

\[
H_N = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & 1_N/2 \\ 1_N/2 & -1_N/2 \end{array} \right) \left( \begin{array}{c} 1 \\ BC_N/2 \end{array} \right) \left( \begin{array}{cc} H_N/2 & H_N/2 \end{array} \right) Q_N
\]

where \(Q_N\) is the permutation \(Q_N |xb\rangle = |bx\rangle\), with \(b\) a single bit, separating the even indexed samples and the odd indexed samples; for instance, \(Q_8(x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7)^t = (x_0, x_2, x_4, x_6, x_1, x_3, x_5, x_7)^t\). The matrix \(BC_{N/2}\) is given by

\[
BC_{N/2} = \left( \begin{array}{c} 1 \\ CS_{N/2-1} \end{array} \right), \quad \text{with} \quad \left( \begin{array}{cccccccc} c_1^N & \cdots & c_{N/4-1}^N & s_{N/4-1}^N & s_{N/4-1}^N & \cdots & s_{N}^1 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{N}^1 & \cdots & s_{N/4-1}^N & c_{N/4-1}^N & c_{N/4-1}^N & \cdots & c_1^N \end{array} \right)
\]

The equation (4) leads to the implementation sketched in Figure 11.

![Figure 11: Recursive implementation of the Hartley transform.](image)

It remains to describe the implementation of \(BC_{N/2}\). It will be instructive to detail the action of the matrix \(BC_{N/2}\) on a state vector of \(n-1\) qubits. We will need a few notations first. Denote by \(|bx\rangle\) a state vector of \(n-1\) qubits, where \(b\) denotes a single bit and \(x\) an \(n-2\) bit integer. We denote by \(x'\) the two’s complement of \(x\). We mean by \(x = 0\) the number 0 and by \(1\) the number \(2^{n-2} - 1\), that is, \(1\) has all bits set and \(0\) has no bit set. Then the action of \(BC_{N/2}\) on \(|bx\rangle\) is given by

\[
BC_{N/2} |00\rangle = |00\rangle, \quad BC_{N/2} |0y\rangle = c_N^y |0y\rangle + s_N^y |1y'\rangle, \\
BC_{N/2} |01\rangle = |01\rangle, \quad BC_{N/2} |1y\rangle = s_N^y |0y'\rangle - c_N^y |1y\rangle,
\]

where \(s_N^k = \sin(2\pi k/N)\) and \(c_N^k = \cos(2\pi k/N)\).

We are now in the position to describe the implementation of \(BC_{N/2}\) shown in Figure 12. In the first step, the least \(n-2\) qubits are conditionally mapped to their two’s complement. More precisely, the input signal \(|bx\rangle\) is mapped to \(|bx'\rangle\) if \(b = 1\), and does not change otherwise. Thus, the circuit \(TC\) implements the involutory permutation corresponding to the two’s complement operation. This can be done with \(O(n)\) elementary gates, provided that sufficient workspace is available.\textsuperscript{21} In the next step, a sign change is done if \(b = 1\), that is, \(|1x\rangle \mapsto -|1x\rangle\), unless the input \(x\) was equal to zero, \(|10\rangle \mapsto |10\rangle\). The next step is a conditioned cascade of...
Figure 12: Implementation of the matrix $BC_{N/2}$.

rotations. The least significant bits determine the angle of the rotation on the $(n-1)$st most significant qubit. The $k$th qubits exerts a rotation,

$$R_{2^k} = \begin{pmatrix} \cos(2\pi 2^k/N) & -\sin(2\pi 2^k/N) \\ \sin(2\pi 2^k/N) & \cos(2\pi 2^k/N) \end{pmatrix},$$

on the most significant qubit. Finally, another two's complement circuit is conditionally applied to the state.

One readily checks that the implementation indeed maps $BC_{N/2} |00\rangle$ to $|00\rangle$ and $BC_{N/2} |10\rangle$ to $|10\rangle$. The input $|0x\rangle$ is mapped to $c_N^x |0x\rangle + s_N^x |1x\rangle$, as desired. Assume that the input is $|1x\rangle$ with $x \neq 0$. Then the state is changed to $|1x'\rangle$ by the circuit $TC$, and after that its sign is changed, which yields $-|1x'\rangle$. The rotations map this state to $s_N^x |0x'\rangle - c_N^x |1x'\rangle$. The final conditional two's complement operation yields the state $s_N^x |0x'\rangle - c_N^x |1x\rangle$, which is exactly what we want.

The initial permutation, the circuit $BC_{N/2}$ and the Hadamard gate in Figure 11 can be implemented with $\Theta(\log N)$ elementary gates. It is crucial that additional workbits are available, otherwise the complexity will increase to $\Theta(\log^2 N)$.

The Lemma in Section 3 then completes the proof of the following theorem:

**Theorem 7.1.** There exists a recursive implementation of the discrete Hadamard transform $H_N$ on a quantum computer with $O(\log^2 N)$ elementary gates (that is, controlled NOT gates and single qubit gates), assuming that additional workbits are available.

8. CONCLUSIONS

We have presented a new approach to the design of quantum algorithms. The method takes advantage of a divide-and-conquer approach. We have illustrated the method in the design of quantum algorithms for the Fourier, Walsh, Slant, and Hartley transforms. The same method can be applied to derive fast algorithms for various discrete Cosine transforms. It might seem surprising that divide-and-conquer methods have not been previously suggested in quantum computing (to the best of our knowledge). One reason might be that the quantum circuit model implements only straight-line programs. We defined recursions on top of that model, similar to macro expansions in many classical programming languages. The benefit is that many circuits can be specified in a very lucid way.

It should be emphasized that our divide-and-conquer approach is completely general. It can be applied to a much larger class of circuits, and is of course not restricted to signal processing applications. Moreover, it should be emphasized that many variations of this method are possible. We would like to encourage the reader to work out a few examples – quite often this is a simple exercise.

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