HAMILTONIAN DYNAMICS ON LIE ALGEBROIDS, UNIMODULARITY AND PRESERVATION OF VOLUMES

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Abstract. In this paper we discuss the relation between the unimodularity of a Lie algebroid $\tau_A : A \to Q$ and the existence of invariant volume forms for the hamiltonian dynamics on the dual bundle $A^*$. The results obtained in this direction are applied to several hamiltonian systems on different examples of Lie algebroids.

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1. Introduction

It is well-known that Hamilton equations in Classical Mechanics may be written in an intrinsic way using the canonical symplectic structure $\omega_Q$ of the phase space of momenta $T^*Q$. In fact, if
$H$ is the hamiltonian energy, the sum of the kinetic and the potential energy, then the solutions of
the Hamilton equations for $H$ are the integral curves of the hamiltonian vector field $\mathcal{H}^H \to Q$ of $H$
with respect to $\omega_Q$. The flow of $\mathcal{H}^H \to Q$ preserves the symplectic form. Thus, one directly deduces
Liouville’s theorem: the flow of the hamiltonian vector field preserves the symplectic volume (see, for
instance, [1]).

However, for a hamiltonian system on a general Poisson manifold, not necessarily symplectic, the
flow of the hamiltonian vector field doesn’t preserves, in general, a volume form on the phase space.
We remark that the existence of invariant volume forms is interesting for reasons of integrability (see
[8] and the references therein). So, it is important to obtain necessary and sufficient conditions for a
hamiltonian system on a Poisson manifold admits an invariant volume. In this direction, a very nice
result was proved by Kozlov [7]: for a hamiltonian system of kinetic type on the dual space $g^\ast$ of a
Lie algebra $g$ the flow of the corresponding hamiltonian vector field preserves a volume form on $g^\ast$ if
and only if the Lie algebra $g$ is unimodular. Note that, in this case, the Poisson structure on $g^\ast$ is the
Lie-Poisson structure induced by the Lie algebra structure of $g$.

Hamiltonian systems on the dual space of a Lie algebra $g$ arise, in a natural way, as the reduction
of standard left-invariant hamiltonian systems with configuration space a Lie group. More generally,
hamiltonian systems on Poisson manifolds arise, in a natural way, as the reduction of standard
hamiltonian systems which are invariant under the action of a symmetry Lie group $G$. The typical
situation is the following one (see, for instance, [13]). The configuration space $Q$ of the mechanical
system is the total space of a principal $G$-bundle $p : Q \to Q/G$. So, the reduced phase space is a vector
bundle $T^*Q/G$ over $Q/G$ and the Poisson structure on it is linear. This implies that the dual bundle
$TQ^*G$ over $Q/G$ admits a Lie algebroid structure. In fact, the vector bundle $\tau_{TQ^*G} : TQ^*G \to Q/G$
is just the Atiyah algebroid associated with the principal $G$-bundle $p : Q \to Q/G$ (see [11]).

We recall that Lie algebroids are a natural generalization of tangent bundles and Lie algebras and
that there exists a one-to-one correspondence between Lie algebroid structures on a vector bundle
$\tau_A : A \to Q$ and linear Poisson structures on the dual bundle $A^*$ (see [2, 11]). Thus, if we have a Lie
algebroid structure on the vector bundle $\tau_A : A \to Q$ and a hamiltonian function $H$ of mechanical
type on $A^*$, we can consider the hamiltonian vector field of $H$ with respect to the linear Poisson
structure $\Pi_A^*$ on $A^*$ and the corresponding dynamical system on $A^*$. Using this procedure, one may
recover the standard Hamilton equations (when $A$ is the standard Lie algebroid $\tau_{TQ} : TQ \to Q$), the
Lie-Poisson equations (when $A$ is a Lie algebra as a Lie algebroid over a single point), the Hamilton-
Poincaré equations (when $A$ is the Atiyah algebroid associated with a principal $G$-bundle), the Lie
Poisson equations on the dual of a semidirect product of Lie algebras (when $A$ is an action Lie
algebroid over a real vector space),... (see [9, 16]).

So, a natural problem arise: to find necessary and sufficient conditions for a hamiltonian system
on the dual bundle to a Lie algebroid $\tau_A : A \to Q$ admits an invariant volume.

In this paper, we will obtain such conditions. For this purpose, we will use a geometrical object
associated with the Lie algebroid: the modular class of $\tau_A : A \to Q$.

The modular class of a Lie algebroid $A$ was introduced in [3] (see also [17]) as follows. We will
assume that $Q$ and the vector bundle are orientable and we fix a volume form $\nu$ on $Q$ and a section
$\Lambda \in \Gamma(\Lambda^n A)$ such that $\Lambda_q \neq 0$, for all $q \in Q$, where $n$ is the rank of $A$. Then, $\nu$ and $\Lambda$ induce a
volume form on $\nu \wedge \Lambda$ on $A^*$ and the modular section of $A$ with respect to $\nu$ and $\Lambda$ is a section of
$\tau_{A^*} : A^* \to Q$ whose vertical lift is just the modular vector field of $\Pi_{A^*}$ with respect to the volume
form $\nu \wedge \Lambda$. We remark that such a vector field is defined using the divergence of the hamiltonian vector fields on $A^*$ with respect to the volume $\nu \wedge \Lambda$. The modular section defines a cohomology class in the cohomology complex of $A$ with trivial coefficients. This cohomology class doesn’t depend on the volumes $\nu$ and $\Lambda$ and it is called the modular class of the Lie algebroid $A$. $A$ is said to be unimodular if its modular class vanishes. The standard Lie algebroid $\tau_{TQ}: TQ \to Q$ is unimodular and a Lie algebra $\mathfrak{g}$ is unimodular as a Lie algebroid over a single point if and only if it is unimodular in the classical sense, that is, the modular character of $\mathfrak{g}$ is zero (for more details, see [3] and Section [3.2]).

Using the modular class of the Lie algebroid $\tau_A: A \to Q$ we deduce the main results of the paper. In fact, if $H: A^* \to \mathbb{R}$ is a hamiltonian function of mechanical type, we prove the following facts:

- **$A$ is unimodular if and only if the hamiltonian dynamics preserves a volume form on $A^*$ of basic type.**

  We remark that a volume form $\Phi = e^\sigma \nu \wedge \Lambda$ on $A^*$, with $\sigma \in C^\infty(A^*)$, is of basic type if $\sigma$ is a basic function, that is, $\sigma \in C^\infty(Q)$. The previous result generalizes Liouville’s theorem (note that the standard Lie algebroid $\tau_{TQ}: TQ \to Q$ is unimodular).

- **If the potential energy is constant then $A$ is unimodular if and only if the hamiltonian dynamics preserves a volume form on $A^*$.**

  It is clear that if we apply this result to the particular case when $A$ is a Lie algebra we recover Kozlov’s theorem.

The paper is organized as follows. In Sections 2 and 3, we recall some definitions and results on Lie algebroids, hamiltonian dynamics and modular class which will be used in the rest of the paper. Section 4 contains the main results. In fact, in this section we discuss the relation between the unimodularity of a Lie algebroid $A$ and the existence of invariant volumes for the hamiltonian dynamics on $A^*$ (see Theorem 4.1 and Corollaries 4.2 and 4.3). In Section 5, we apply the results of Section 4 to several hamiltonian systems on different examples of Lie algebroids. The paper ends with our conclusions and a description of future research directions.

2. Lie algebroids

2.1. Definitions and notation. Let $\tau_A: A \to Q$ be a vector bundle of rank $n$ over a manifold $Q$ of dimension $m$. Denote by $\Gamma(\tau_A)$ the space of sections of the vector bundle $\tau_A: A \to Q$.

**Definition 2.1.** A Lie algebroid structure on the vector bundle $\tau_A: A \to Q$ is a Lie bracket $[\cdot, \cdot]_A: \Gamma(\tau_A) \times \Gamma(\tau_A) \to \Gamma(\tau_A)$ on the space $\Gamma(\tau_A)$ and a vector bundle morphism $\rho_A: A \to TQ$, the anchor map, such that if we also denote by $\rho_A: \Gamma(\tau_A) \to \mathfrak{X}(Q)$ the corresponding morphism of $C^\infty(Q)$-modules then

$$[X, fY]_A = f[X, Y]_A + \rho_A(X)(f)Y, \text{ for } X, Y \in \Gamma(\tau_A) \text{ and } f \in C^\infty(Q).$$

If $(\{\cdot, \cdot\}_A, \rho_A)$ is a Lie algebroid structure on the vector bundle $\tau_A: A \to Q$ it follows that

$$\rho_A[X, Y]_A = [\rho_A(X), \rho_A(Y)], \text{ for } X, Y \in \Gamma(\tau_A) \tag{2.1}$$

Moreover, if $(q^i)$ are local coordinates in an open subset $U$ of $Q$ and $\{e_\alpha\}$ is a basis of sections of the vector bundle $\tau_A^{-1}(U) \to U$, we have that

$$[e_\alpha, e_\beta]_A = C^\gamma_{\alpha\beta} e_\gamma, \quad \rho_A e_\alpha = \rho^i_\alpha \frac{\partial}{\partial q^i},$$
with $C_{\alpha\beta}^\gamma, \rho^i_\alpha \in C^\infty(U)$. The functions $C_{\alpha\beta}^\gamma, \rho^i_\alpha$ are called the local structure functions of the Lie algebroid with respect to the local coordinates $(q^i)$ and the basis $\{e_\alpha\}$. Using (2.1) and the fact that $[\cdot, \cdot]_A$ is a Lie bracket, we deduce that

$$\rho^i_\alpha \frac{\partial \rho^i_\beta}{\partial q^j} - \rho^i_\beta \frac{\partial \rho^i_\alpha}{\partial q^j} = \rho^i_\gamma C_{\alpha\beta}^\gamma$$  \hspace{1cm} (2.2)

and

$$\sum_{\text{cyclic}(\alpha, \beta, \gamma)} [\rho^i_\alpha \frac{\partial C_{\beta\gamma}^\nu}{\partial q^i} + C_{\alpha\mu}^\nu C_{\beta\gamma}^\mu] = 0.$$  \hspace{1cm} (2.3)

These equations are called the local structure equations of the Lie algebroid with respect to the local coordinates $(q^i)$ and the basis $\{e_\alpha\}$.

2.2. Examples. Next, we will exhibit some examples of Lie algebroids.

The Atiyah algebroid associated with a principal $G$-bundle. Let $p : Q \to Q/G$ be a principal $G$-bundle. Then, we may consider the tangent lift of the principal action of $G$ on $Q$ and, it is well-known that, the space of orbits of this action, $TQ/G$, is a real vector bundle over $Q/G$. The vector bundle projection $\tau_{TQ/G} : TQ/G \to Q/G$ is given by

$$\tau_{TQ/G}[v_q] = p(q), \quad \forall v_q \in T_q Q.$$  

Furthermore, the space of sections $\Gamma(\tau_{TQ/G})$ may be identified with the set of $G$-invariant vector fields on $Q$. Thus, using that the Lie bracket of two $G$-invariant vector fields on $Q$ also is $G$-invariant, we may define, in a natural way, a Lie bracket on the space $\Gamma(\tau_{TQ/G})$ $\big[\cdot, \cdot\big]_{TQ/G} : \Gamma(\tau_{TQ/G}) \times \Gamma(\tau_{TQ/G}) \to \Gamma(\tau_{TQ/G})$.

On the other hand, the anchor map $\rho_{TQ/G} : TQ/G \to T(Q/G)$ is given by

$$\rho_{TQ/G}[v_q] = (T_q p)(v_q), \quad \text{for } v_q \in T_q Q,$$

where $T_p : TQ \to T(Q/G)$ is the tangent map to the principal bundle projection $p : Q \to Q/G$.

The resultant Lie algebroid $(TQ/G, [\cdot, \cdot]_{TQ/G}, \rho_{TQ/G})$ is called the Atiyah algebroid associated with the principal $G$-bundle $p : Q \to Q/G$ (see [11]).

Note that if the Lie group $G$ is trivial and $p = id$ then the Atiyah algebroid may be identified with the standard Lie algebroid $\tau_{TQ} : TQ \to Q$.

Another interesting particular case is when the manifold $Q$ is the Lie group $G$. In this case, using that $TG$ is diffeomorphic to the product manifold $G \times \mathfrak{g}$ ($\mathfrak{g}$ being the Lie algebra of $G$), we deduce that the Atiyah algebroid $TG/G$ may be identified with $\mathfrak{g}$ (as a Lie algebroid over a single point).

Finally, if the principal $G$-bundle is trivial, that is,

$$Q = G \times M, \quad Q/G = M,$$

$p$ is the canonical projection on the second factor and the action of $G$ on $Q = G \times M$ is defined by

$$g(g', x) = (gg', x), \quad \text{for } g, g' \in G \text{ and } x \in M,$$
then it is easy to prove that the quotient vector bundle $\tau_{TQ/G} : TQ/G \to Q/G$ may be identified with the vector bundle $\tau_{g \times TM} : g \times TM \to M$. Under this identification, the Lie algebroid structure $(\llbracket \cdot, \cdot \rrbracket_{g \times TM}, \rho_{g \times TM})$ on $\tau_{g \times TM} : g \times TM \to M$ is given by
\[
\llbracket (\xi, X), (\eta, Y) \rrbracket_{g \times TM} = (\llbracket \xi, \eta \rrbracket_g, [X, Y]), \quad \rho_{g \times TM}(\xi, X) = X,
\]
for $\xi, \eta \in g$ and $X, Y \in \mathfrak{X}(M)$, $\llbracket \cdot, \cdot \rrbracket_g$ being the Lie bracket on $g$.

**The Lie algebroid associated with a left infinitesimal action.** Let $g$ be a real Lie algebra of finite dimension and $\Phi : g \to \mathfrak{X}(Q)$ be a left infinitesimal action of $g$ on $Q$, that is, $\Phi$ is a $\mathbb{R}$-linear map and
\[
\Phi([\xi, \eta]_g) = -[\Phi(\xi), \Phi(\eta)], \quad \text{for} \quad \xi, \eta \in g.
\]
Then, the trivial vector bundle $\tau_A : A = g \times Q \to Q$ is a Lie algebroid. In fact, the anchor map $\rho_A : A = g \times Q \to TQ$ is given by
\[
\rho_A(\xi, q) = -\Phi(\xi)(q), \quad \text{for} \quad (\xi, q) \in A = g \times Q.
\]
On the other hand, it is clear that the space of sections $\Gamma(\tau_A)$ may be identified with the set $C^\infty(Q, g)$ of smooth functions from $Q$ on $g$. Under this identification, the Lie bracket $\llbracket \cdot, \cdot \rrbracket_A : \Gamma(\tau_A) \times \Gamma(\tau_A) \to \Gamma(\tau_A)$ is defined by
\[
\llbracket \varphi, \psi \rrbracket_A(q) = [\varphi(q), \psi(q)]_g - \Phi(\varphi(q))(q)(\psi) + \Phi(\psi(q))(q)(\varphi),
\]
for $\varphi, \psi \in C^\infty(Q, g)$ and $q \in Q$.

The resultant Lie algebroid is called the Lie algebroid associated with the left infinitesimal action $\Phi$ (see [5]).

**2.3. The differential associated with a Lie algebroid.** Let $(\llbracket \cdot, \cdot \rrbracket_A, \rho_A)$ be a Lie algebroid structure on a vector bundle $\tau_A : A \to Q$ of rank $n$ and $\tau_{A^*} : A^* \to Q$ be the dual vector bundle to $\tau_A : A \to Q$. Then, one may introduce the corresponding differential as a $\mathbb{R}$-linear map $d^A : \Gamma(\Lambda^k \tau_{A^*}) \to \Gamma(\Lambda^{k+1} \tau_{A^*})$, $k \in \{0, \ldots, n-1\}$, given by
\[
(d^A \alpha)(X_0, X_1, \ldots, X_k) = \sum_{i=0}^k (-1)^i \rho_A(X_i)(\alpha(X_0, \ldots, \hat{X}_i, \ldots, X_k)) \\
+ \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j]_A, X_0, X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k)
\]
for $\alpha \in \Gamma(\Lambda^k \tau_{A^*})$ and $X_0, X_1, \ldots, X_k \in \Gamma(\tau_A)$.

We have that
\[
d^A(\alpha \wedge \beta) = d^A \alpha \wedge \beta + (-1)^k \alpha \wedge d^A \beta, \quad (d^A)^2 = 0,
\]
for $\alpha \in \Gamma(\Lambda^k \tau_{A^*})$ and $\beta \in \Gamma(\Lambda^r \tau_{A^*})$. Thus, we can consider the corresponding cohomology groups
\[
H^k A = \frac{\text{Ker} d^A}{\text{Im} d^A}, \quad \text{for} \quad k \in \{0, 1, \ldots, n\} \quad \text{(see [11]).}
\]
Moreover, if $(q^i)$ are local coordinates on an open subset $U$ of $Q$ and $\{e_\alpha\}$ is a basis of sections of the vector bundle $\tau_A^{-1}(U) \to U$, it follows that
\[
d^A q^i = \rho^i_\alpha e^\alpha, \quad d^A e^\gamma = -\frac{1}{2} C^\gamma_{\alpha \beta} e^\alpha \wedge e^\beta,
\]
where $\{e^\alpha\}$ is the dual basis of $\{e_\alpha\}$ and $\rho^i_\alpha$, $C^\gamma_{\alpha \beta}$ are the local structure functions of $A$. 

On the other hand, if \( X \in \Gamma(\tau_A) \) one may define the Lie derivative operator
\[
\mathcal{L}_X^A : \Gamma(\Lambda^k\tau_{A^*}) \to \Gamma(\Lambda^k\tau_{A^*})
\]
as follows
\[
\mathcal{L}_X^A = i_X \circ d^A + d^A \circ i_X,
\]
where \( i_X : \Gamma(\Lambda^k\tau_{A^*}) \to \Gamma(\Lambda^{k-1}\tau_{A^*}) \) is the contraction by \( X \), that is,
\[
(i_X\alpha)(X_1, \ldots, X_{k-1}) = \alpha(X, X_1, \ldots, X_{k-1}),
\]
for \( X_1, \ldots, X_{k-1} \in \Gamma(\tau_A) \).

2.4. The linear Poisson structure associated with a Lie algebroid. Let \( ([\cdot, \cdot]_A, \rho_A) \) be a Lie algebroid structure on the vector bundle \( \tau_A : A \to Q \). Then, on may define a \( \mathbb{R} \)-linear bracket of functions
\[
\{ \cdot, \cdot \}_A^* : C^\infty(A^*) \times C^\infty(A^*) \to C^\infty(A^*)
\]
which is characterized by the following conditions
\[
\{ \hat{X}, \hat{Y} \}_A^* = -[X, Y]_A, \quad \{ f \circ \tau_{A^*} , \hat{X} \}_A^* = \rho_A(X)(f) \circ \tau_{A^*},
\]
and
\[
\{ f \circ \tau_{A^*} , g \circ \tau_{A^*} \}_A^* = 0,
\]
for \( X, Y \in \Gamma(\tau_A) \) and \( f, g \in C^\infty(Q) \). Note that if \( X \in \Gamma(\tau_A) \) then \( \hat{X} : A^* \to \mathbb{R} \) is the linear function on \( A^* \) given by
\[
\hat{X}(\alpha) = \alpha(\tau_{A^*}^{-1}(\alpha)), \quad \text{for } \alpha \in A^*.
\]
\( \{ \cdot, \cdot \}_A^* \) is a linear Poisson structure on \( A^* \), that is,
(i) \( \{ \cdot, \cdot \}_A^* \) is a Lie bracket on \( C^\infty(A^*) \),
(ii) \( \{ \cdot, \cdot \}_A^* \) satisfies the Leibniz rule
\[
\{ \varphi \varphi', \psi \}_A^* = \varphi \{ \varphi', \psi \}_A^* + \varphi' \{ \varphi, \psi \}_A^*,
\]
for \( \varphi, \varphi', \psi \in C^\infty(A^*) \) and
(iii) \( \{ \cdot, \cdot \}_A^* \) is linear or, in other words, the bracket of two linear functions on \( A^* \) is a linear function.

We will denote by \( \Pi_A^* \) the corresponding linear Poisson 2-vector on \( A^* \) which is characterized by the following condition
\[
\Pi_A^*(d\varphi, d\psi) = \{ \varphi, \psi \}_A^*
\]
(for more details, see [2]).

If \( (q^i) \) are local coordinates on an open subset \( U \) of \( Q \) and \( \{ e_\alpha \} \) is a basis of sections of the vector bundle \( \tau_{A^{-1}}(U) \to U \) we have that
\[
\Pi_A^* = \rho^i_\alpha \frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_\alpha} - \frac{1}{2} C^\gamma_{\alpha\beta} p_\gamma \frac{\partial}{\partial p_\alpha} \wedge \frac{\partial}{\partial p_\beta}
\]
(2.5)
where \( (q^i, p_\alpha) \) are the corresponding local coordinates on \( A^* \) and \( \rho^i_\alpha, C^\gamma_{\alpha\beta} \) are the local structure functions of \( A \).

Next, we will describe the linear Poisson structure on the dual vector bundle of some examples of Lie algebroids.
3. Hamiltonian dynamics on Lie algebroids and modular sections

3.1. Hamiltonian dynamics on Lie algebroids. Let \( \mathcal{H}_{H}^{A^*} \) be a Lie algebroid structure on a vector bundle \( \tau_A : A \to Q \) and \( \Pi_A \) be the corresponding linear Poisson 2-vector on \( A^* \).

If \( H : A^* \to \mathbb{R} \) is a Hamiltonian function on \( A^* \) then one may consider the Hamiltonian vector field \( \mathcal{H}_{H}^{A^*} \) of \( H \) with respect to the Poisson structure \( \Pi_A \), that is,

\[
\mathcal{H}_{H}^{A^*}(F) = \{ F, H \}_{A^*} = \Pi_A(dF, dH), \quad \text{for } F \in C^\infty(A^*). \tag{3.1}
\]

The solutions of the Hamilton equations for \( H \) are just the integral curves of \( \mathcal{H}_{H}^{A^*} \).

From (2.5) and (3.1), we deduce that the local expression of \( \mathcal{H}_{H}^{A^*} \) is

\[
\mathcal{H}_{H}^{A^*} = \frac{\partial H}{\partial p_\alpha} \rho^i_\alpha \frac{\partial}{\partial q^i} - (\frac{\partial H}{\partial q^i} \rho^i_\alpha + \frac{\partial H}{\partial p_\beta} C^\gamma_{\alpha\beta} p_\gamma) \frac{\partial}{\partial p_\alpha}. \tag{3.2}
\]

Thus, the Hamilton equations are

\[
\frac{dq^i}{dt} = \frac{\partial H}{\partial \rho^i_\alpha}, \quad \frac{dp_\alpha}{dt} = -\left( \frac{\partial H}{\partial q^i} \rho^i_\alpha + \frac{\partial H}{\partial p_\beta} C^\gamma_{\alpha\beta} p_\gamma \right)
\]

(for more details, see [9]).

If \( A \) is the standard Lie algebroid \( \tau_{TQ} : TQ \to Q \) the Hamilton equations for \( H : T^*Q \to \mathbb{R} \) are

\[
\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i},
\]

that is, the standard Hamilton equations (see, for instance, [1]).
For a real Lie algebra \( \mathfrak{g} \) of finite dimension we have that the Hamilton equations for \( H : \mathfrak{g}^* \to \mathbb{R} \) are
\[
\frac{dp_\alpha}{dt} = -\frac{\partial H}{\partial q^i} \xi^{\alpha}_{\beta} p_\beta \tag{3.3}
\]
where \( \xi^{\alpha}_{\beta} \) are the structure constants of \( \mathfrak{g} \) with respect to a basis \( \{e_\alpha\} \). Note that (3.3) are just the Lie-Poisson equations for \( H \) (see, for instance, [13]).

If \( H : \mathfrak{g}^* \times T^*M \to \mathbb{R} \) is a hamiltonian function on the dual bundle of the Atiyah algebroid \( \tau_{\mathfrak{g} \times TM} : \mathfrak{g} \times TM \to M \) then the Hamilton equations for \( H \) are
\[
\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}, \quad \frac{dp_\alpha}{dt} = -\frac{\partial H}{\partial q^i} \xi^{\alpha}_{\beta} p_\beta, \tag{3.4}
\]
where \( (p_\alpha, q^i, p_i) \) are local coordinates on \( \mathfrak{g}^* \times T^*M \) and \( \xi^{\alpha}_{\beta} \) are the structure constants of \( \mathfrak{g} \) with respect to a basis of \( \mathfrak{g} \). (3.4) are just the Hamilton-Poincaré equations for \( H \) (see, for instance, [9]).

3.2. Modular class of a Lie algebroid. Let \( ([\cdot, \cdot]_A, \rho_A) \) be a Lie algebroid structure on a vector bundle \( \tau_A : A \to Q \) of rank \( n \) with base manifold \( Q \) of dimension \( m \).

We will assume that \( Q \) and the vector bundle \( \tau_A : A \to Q \) are orientable.

If \( \alpha \) is a section of \( \tau_{A^*} : A^* \to Q \) we will denote by \( \alpha^v \in \mathfrak{X}(A^*) \) the vertical lift of \( \alpha \). We recall that
\[
\alpha^v(\gamma_q) = \frac{d}{dt}_{t=0} (\gamma_q + t\alpha(q)), \text{ for } \gamma_q \in A^*_q.
\]

Lemma 3.1. Let \( \nu \) be a volume form on \( Q \) and \( \Lambda \) be a section of the vector bundle \( \tau_{\Lambda^n A} : \Lambda^n A \to Q \) such that \( \Lambda(q) \neq 0 \), for all \( q \in Q \). Then, there exists a unique volume form \( \nu \wedge \Lambda \) on the dual bundle \( A^* \) to the vector bundle \( \tau_A : A \to Q \) such that
\[
\nu \wedge \Lambda (\tilde{Z}_1, \ldots, \tilde{Z}_m, \alpha^v_1, \ldots, \alpha^v_n) = \nu(Z_1, \ldots, Z_m)\Lambda(\alpha_1, \ldots, \alpha_n), \tag{3.5}
\]
for \( \alpha_1, \ldots, \alpha_n \in \Gamma(\tau_{A^*}) \) and \( \tilde{Z}_1, \ldots, \tilde{Z}_m \) vector fields on \( A^* \) which are \( \tau_{A^*} \)-projectable on the vector fields \( Z_1, \ldots, Z_m \) on \( Q \).

Proof. We can choose local coordinates \( (q^i) \) on an open subset \( U \) of \( Q \) and a basis of sections \( \{e_\alpha\} \) of the vector bundle \( \tau_{A}^{-1}(U) \to U \) such that on \( U \)
\[
\nu = e^{\sigma_v} dq^1 \wedge \cdots \wedge dq^m, \quad \Lambda = e_1 \wedge \cdots \wedge e_n, \quad \text{with } \sigma_v \in C^\infty(U).
\]
If \( (q^i, p_\alpha) \) are the corresponding local coordinates on \( A^* \) then on the open subset \( \tau_{A^*}^{-1}(U) \) of \( A^* \) we consider the volume form
\[
(\nu \wedge \Lambda)_{\tau_{A^{-1}}(U)} = e^{\sigma_v} dq^1 \wedge \cdots \wedge dq^m \wedge dp_1 \wedge \cdots \wedge dp_n.
\]
A direct computation proves that \( (\nu \wedge \Lambda)_{\tau_{A^{-1}}(U)} \) satisfies (3.5).

On the other hand, if \( \Phi \) is a volume form on \( \tau_{A}^{-1}(U) \) such that (3.5) holds then it follows that
\[
\Phi = (\nu \wedge \Lambda)_{\tau_{A^{-1}}(U)}.
\]
This proves the result. \( \square \)
Suppose that $\nu$ is a volume form on $Q$ and that $\Lambda \in \Gamma(\tau_{\lambda^*A})$ satisfies $\Lambda(q) \neq 0$, for all $q \in Q$. Then, we can consider the modular vector field $M^{\nu \wedge \Lambda} \in \mathfrak{X}(A^*)$ of the linear Poisson structure on $A^*$ with respect to the volume form $\nu \wedge \Lambda$ on $A^*$. $M^{\nu \wedge \Lambda}$ is given by

$$M^{\nu \wedge \Lambda}(H) = \text{div}_{\nu \wedge \Lambda}(H^\nu_A), \quad \text{for } H \in C^\infty(A^*),$$

(3.6)

where $\text{div}_{\nu \wedge \Lambda}(H^\nu_A)$ is the divergence of the Hamiltonian vector field $H^\nu_A$ with respect to the volume form $\nu \wedge \Lambda$ (see [17]).

We have that $M^{(\nu \wedge \Lambda)}$ is the vertical lift of a section $M^{(\nu, \Lambda)} \in \Gamma(\tau_{A^*})$, that is,

$$M^{(\nu \wedge \Lambda)} = (M^{(\nu, \Lambda)})^\nu.$$

(3.7)

$M^{(\nu, \Lambda)}$ is the modular section of $A$ with respect to $\nu$ and $\Lambda$ (see [3, 17]).

Denote by $\Omega_{\lambda}$ the section of the vector bundle $\tau_{\lambda^*A^*:} \Lambda^*A^* \to Q$ characterized by

$$\Omega_{\lambda}(X_1, \ldots, X_n)\lambda = X_1 \wedge \cdots \wedge X_n, \quad \text{for } X_1, \ldots, X_n \in \Gamma(\tau_{\lambda^*}).$$

It follows that $\Omega_{\lambda}(q) \neq 0$, for all $q \in Q$. Moreover,

$$M^{(\nu, \Lambda)}(X) = \text{div}_{\nu}(\rho_A(X)) - \text{div}_{\Omega_{\lambda}}X,$$

(3.8)

where $\text{div}_{\Omega_{\lambda}}X$ is the divergence of $X$ with respect to $\Omega_{\lambda}$, that is,

$$\mathcal{L}_X^A \Omega_{\lambda} = (\text{div}_{\Omega_{\lambda}}X)\Omega_{\lambda}$$

(for more details, see [3, 17]).

If $(q^i)$ are local coordinates on an open subset $U$ of $Q$ and $\{e_\alpha\}$ is a basis of sections of the vector bundle $\tau_{\lambda^*}(U) \to U$ such that

$$\nu = e^\sigma_{ij} dq_i \wedge \cdots \wedge dq^m, \quad \Lambda = e_1 \wedge \cdots \wedge e_n$$

then, from (3.8), we deduce that

$$M^{(\nu, \Lambda)} = (C^\beta_{\alpha\beta} + \partial_i \rho^i_{\alpha} + \rho^i_{\alpha} \partial_j \sigma^\beta_{ij})e^\alpha.$$

(3.9)

We also have the following result

**Proposition 3.2.** [3]

(i) $M^{(\nu, \Lambda)}$ is a 1-cocycle, that is,

$$d^A M^{(\nu, \Lambda)} = 0.$$

(ii) The cohomology class of $M^{(\nu, \Lambda)}$ doesn’t depend on the chosen volumes $\nu$ and $\Lambda$. In fact,

$$M^{(e^\sigma\nu, e^\mu\Lambda)} = M^{(\nu, \Lambda)} + d^A(\sigma + \mu), \quad \text{for } \sigma, \mu \in C^\infty(Q).$$

(3.10)

The cohomology class of $M^{(\nu, \Lambda)}$ is called the modular class of $A$.

The Lie algebroid $\tau_A : A \to Q$ is said to be unimodular if its modular class is zero, that is, there exists a real $C^\infty$-function on $Q$, $\sigma : Q \to \mathbb{R}$, such that

$$M^{(\nu, \Lambda)} = -d^A \sigma.$$

If $(q^i)$ are local coordinates on an open subset $U$ of $Q$ and $\{e_\alpha\}$ is a basis of sections of the vector bundle $\tau_{\lambda^*}(U) \to U$ such that

$$\nu = e^\sigma_{ij} dq_i \wedge \cdots \wedge dq^m, \quad \Lambda = e_1 \wedge \cdots \wedge e_n$$

then, from (3.8), we deduce that

$$M^{(\nu, \Lambda)} = (C^\beta_{\alpha\beta} + \partial_i \rho^i_{\alpha} + \rho^i_{\alpha} \partial_j \sigma^\beta_{ij})e^\alpha.$$
then, using (3.9), we deduce that $M^{(\nu, \Lambda)} = -d^A \sigma$ if and only if
\[
C_{\alpha \beta}^\gamma + \frac{\partial \rho^\gamma_{\alpha}}{\partial q^i} + \rho^\gamma_{\alpha} \frac{\partial (\sigma^\nu_{\gamma} + \sigma)}{\partial q^i} = 0, \text{ for all } \alpha.
\]

Next, we will discuss the unimodularity of some example of Lie algebroids.

Let $A$ be the standard Lie algebroid $\tau_{TQ} : TQ \to Q$. Then, the linear Poisson structure on $\Pi T^* Q$ is induced by the canonical symplectic structure $\Omega_{T^* Q}$. In addition, it is well-known that the hamiltonian vector fields on $T^* Q$ preserve the symplectic volume. Thus, the Lie algebroid $\tau_{TQ} : TQ \to Q$ is unimodular.

On the other hand, if $g$ is a real Lie algebra of finite dimension then, from (3.9), it follows that the modular section of $g$ with respect to $\Lambda = e_1 \wedge \cdots \wedge e_n$, $\{e_a\}$ being a basis of $g$, is just the modular character of $g$ (see [3]). Therefore, our notion of a unimodular Lie algebra coincides with the classical definition of a unimodular Lie algebra.

Now, we consider an Atiyah algebroid $\tau_{g \times TM} : g \times TM \to M$, with $g$ a real Lie algebra of finite dimension and $M$ a smooth manifold (see Section 2.2). Let $\nu$ be a volume form on $M$ and $\{e_a\}$ be a basis of $g$. Denote by $\chi_\nu$ the $m$-vector on $M$ given by
\[
\chi_\nu(\alpha_1, \ldots, \alpha_m)\nu = \alpha_1 \wedge \cdots \wedge \alpha_m, \text{ for } \alpha_1, \ldots, \alpha_m \in \Omega^1(M).
\]
Then,
\[
\Lambda = \chi_\nu \wedge e_1 \wedge \cdots \wedge e_n \in \Gamma(\Lambda^n(g \times TM)) \text{ and } \Lambda(x) \neq 0, \text{ for all } x \in M.
\]
Moreover, from (3.8), it follows that
\[
M^{(\nu, \Lambda)}(X) = 0, \quad M^{(\nu, \Lambda)}(e_a) = M_g(e_a),
\]
for $X \in \mathfrak{X}(M)$, where $M_g$ is the modular character of $g$. Note that, in this case,
\[
\Omega_\Lambda = \nu \wedge e_1 \wedge \cdots \wedge e_n,
\]
$\{e^a\}$ being the dual basis to $\{e_a\}$.

Thus, the Lie algebroid $\tau_{g \times TM} : g \times TM \to M$ is unimodular if and only if the Lie algebra $g$ is unimodular.

Finally, suppose that $\Phi : g \to \mathfrak{X}(Q)$ is a left infinitesimal action of $g$ on the manifold $Q$ and that $\tau_{g \times Q} : g \times Q \to Q$ is the corresponding action Lie algebroid. Then, $\tau_{g \times Q} : g \times Q \to Q$ is unimodular if and only if there exists $\sigma \in C^\infty(Q)$ such that
\[
div_\nu \Phi(\xi) - \Phi(\xi)(\sigma) = M_g(\xi), \text{ for } \xi \in g,
\]
where $\nu$ is a volume form on $Q$ and $M_g \in g^*$ is the modular character of $g$. In particular, if $g$ is a unimodular Lie algebra and the infinitesimal action preserves a volume form on $Q$ then $\tau_{g \times Q} : g \times Q \to Q$ is a unimodular Lie algebroid.
4. Unimodularity and Preservation of Volumes

Let \([\cdot, \cdot]_A, \rho_A\) be a Lie algebroid structure on a vector bundle \(\tau_A : A \to Q\) of rank \(n\) with base manifold \(Q\) of dimension \(m\).

In this section we will assume that \(Q\) is orientable and that the vector bundle \(\tau_A : A \to Q\) also is orientable.

Now, suppose that \(G\) is a bundle metric on \(A\) and that \(V : Q \to \mathbb{R}\) is a real \(C^\infty\)-function on \(Q\). Then, we can consider the Hamiltonian energy \(H : A^* \to \mathbb{R}\) given by

\[
H(\alpha) = \frac{1}{2} G(\alpha, \alpha) + V(\tau_A^*(\alpha)), \quad \text{for } \alpha \in A^*.
\]  
(4.1)

In other words, \(H\) is the sum of the kinetic energy and the potential energy. We will denote by \(\Lambda^G \in \Gamma(\tau_{A^*} A)\) the volume induced by \(G\), that is, \(\Lambda^G\) is characterized by the following condition

\[
\Lambda^G(q)(e^1_q, \ldots, e^n_q) = 1, \quad \text{for all } q \in Q,
\]

where \(\{e^1_q, \ldots, e^n_q\}\) is an orthonormal basis of \(A^*_q\) with positive orientation.

We will fix a volume form \(\nu\) on \(M\). Then, if \(\Phi\) is a volume form on \(A^*\) we may suppose, without the loss of generality, that

\[
\Phi = e^{\tilde{\sigma}} \nu \wedge \Lambda^G, \quad \text{with } \tilde{\sigma} \in C^\infty(A^*).
\]

The volume \(\Phi\) is said to be of basic type if \(\tilde{\sigma}\) is a basic function, that is, there exists a real \(C^\infty\)-function \(\mu : Q \to \mathbb{R}\) such that

\[
\mu \circ \tau_{A^*} = \tilde{\sigma}.
\]

Note that this definition doesn’t depend on the chosen volume form \(\nu\). In fact, one may prove that \(\Phi\) is a volume form of basic type if and only if

\[
\mathcal{L}_{\alpha^*} \Phi = 0, \quad \text{for } \alpha \in \Gamma(\tau_{A^*}),
\]

\(\mathcal{L}\) being the Lie derivative operator.

Using the function \(\tilde{\sigma}\) one may define the following objects:

- A real \(C^\infty\)-function \(\sigma : Q \to \mathbb{R}\) given by

\[
\sigma(q) = \tilde{\sigma}(0(q)), \quad \text{for all } q \in Q,
\]

where \(0 : Q \to A^*\) is the zero section of \(\tau_{A^*} : A^* \to Q\).

- The vertical derivative of \(\tilde{\sigma}\)

\[
\mathbb{F}\tilde{\sigma} : A^* \to A
\]

defined by

\[
\beta_q(\mathbb{F}\tilde{\sigma}(\alpha_q)) = \frac{d}{dt}_{t=0} \tilde{\sigma}(\alpha_q + t\beta_q), \quad \text{for } \alpha_q, \beta_q \in A^*_q.
\]  
(4.2)

Now, we will prove the main result of this paper.

**Theorem 4.1.** Let \(H : A^* \to \mathbb{R}\) be the Hamiltonian energy induced by a bundle metric \(G\) on \(A\) and a potential energy \(V : Q \to \mathbb{R}\).
(i) If $A$ is unimodular then the Hamiltonian vector field $\mathcal{H}^{\Pi_A^*}_H$ of $H$ with respect to the linear Poisson structure $\Pi_A^*$ preserves a volume form on $A^*$ of basic type. In fact, if $\mathcal{M}^{(\nu,\Lambda^S)} = -d^A\sigma$, with $\sigma \in C^\infty(Q)$ and $\mathcal{M}^{(\nu,\Lambda^S)}$ the modular section of $A$ with respect to the volumes $\nu$ and $\Lambda^S$, we have that $\mathcal{H}^{\Pi_A^*}_H$ preserves the volume form on $A^*$

$$\Phi = e^\sigma \nu \wedge \Lambda^S.$$  

Thus, from (3.10), it follows that $M$.

This implies that $\mathcal{H}^{\Pi_A^*}_H$ preserves the volume form on $A^*$.

(ii) If the Hamiltonian vector field $\mathcal{H}^{\Pi_A^*}_H$ preserves a volume form $\Phi$ on $A^*$,

$$\Phi = e^\sigma \nu \wedge \Lambda^S,$$

then

$$(\mathcal{M}^{(e^\sigma \nu, \Lambda^S)})^\nu \circ 0 = (T \mathcal{F} \mathcal{S} \circ (d^A V)^\nu) \circ 0$$

where $\mathcal{M}^{(e^\sigma \nu, \Lambda^S)}$ is the modular section of $A$ with respect to the volumes $e^\sigma \nu$ and $\Lambda^S$, $v$ is the vertical lift, $T \mathcal{F} \mathcal{S} : TA^* \to TA$ is the tangent map to $\mathcal{F} : A^* \to A$ and $v_S : \Gamma(\tau_A) \to \Gamma(\tau_{A^*})$ is the isomorphism of $C^\infty(Q)$-modules induced by $S$.

Proof. (i) Since $A$ is unimodular, we deduce that there exists a real $C^\infty$-function $\mu$ on $Q$ such that $\mathcal{M}^{(\nu, \Lambda^S)} = -d^A \mu$.

Thus, from (3.10), it follows that

$$\mathcal{M}^{(e^\nu, \Lambda^S)} = 0.$$  

This implies that $\mathcal{M}^{(e^\nu \Lambda^S)} = (\mathcal{M}^{(e^\nu, \Lambda^S)})^\nu \circ 0$ (see (3.7)). In particular,

$$0 = \mathcal{M}^{(e^\nu \Lambda^S)}(H) = div(e^\nu \Lambda^S)(\mathcal{H}^{\Pi_A^*}_H).$$

Therefore, the Hamiltonian vector field $\mathcal{H}^{\Pi_A^*}_H$ preserves the volume form of basic type

$$\Phi = e^\nu \nu \wedge \Lambda^S.$$

(ii) Suppose that $(q^i)$ are local coordinates on $Q$ such that

$$\nu = e^{\nu_i} dq^1 \wedge \cdots \wedge dq^m$$

and that $\{e_1, \ldots, e_n\}$ is a local orthonormal basis of $\Gamma(A)$ with positive orientation. Then,

$$\Lambda^S = e_1 \wedge \cdots \wedge e_n.$$  

Moreover, if $(q^i, p_\alpha)$ are the corresponding local coordinates on $A^*$, we have that (see (4.1) and the proof of Lemma 3.1)

$$H(q^i, p_\alpha) = \frac{1}{2} \sum_\alpha (p_\alpha)^2 + V(q^i), \quad \Phi = e^\sigma e^{\nu_i} dq^1 \wedge \cdots \wedge dq^m \wedge dp_1 \wedge \cdots \wedge dp_n. \quad (4.3)$$

Thus, from (3.6), (3.7), (3.9) and (4.3), it follows that

$$0 = e^\sigma e^{\nu_i} \mathcal{H}^{\Pi_A^*}_H(\bar{\sigma}) dq^1 \wedge \cdots \wedge dq^m \wedge dp_1 \wedge \cdots \wedge dp_n$$

$$+ e^\sigma e^{\nu_i} \frac{\partial p_\alpha}{\partial q^i} + C_\alpha^\beta p_\beta + \rho_\alpha^\beta \frac{\partial \sigma^\nu_i}{\partial q^i} p_\alpha dq^1 \wedge \cdots \wedge dq^m \wedge dp_1 \wedge \cdots \wedge dp_n.$$  

Therefore, using (3.2) and (4.3), we deduce that

$$\rho_\alpha^i p_\alpha^\beta \frac{\partial \sigma^\nu_i}{\partial q^i} = (\frac{\partial V}{\partial q^i} \rho_\alpha + C_\alpha^\beta p_\beta p_\gamma \frac{\partial \sigma^\nu_i}{\partial p_\alpha}) + (\frac{\partial p_\alpha^\beta}{\partial q^i} + C_\alpha^\beta p_\beta + \rho_\alpha^\beta \frac{\partial \sigma^\nu_i}{\partial q^i}) p_\alpha = 0.$$
Now, if we take the derivative of the above expression with respect to the variable $p_\mu$, we obtain that

$$0 = \rho_\mu^i \frac{\partial \tilde{\sigma}}{\partial q^i} + \rho_\alpha^i p_\alpha \frac{\partial^2 \tilde{\sigma}}{\partial q^i \partial p_\mu} - \frac{\partial V}{\partial q^i} \rho_\mu^i \frac{\partial^2 \tilde{\sigma}}{\partial p_\alpha \partial p_\mu} - C_{\alpha \gamma}^\mu \frac{\partial \tilde{\sigma}}{\partial p_\alpha} - C_{\alpha \beta}^\mu \frac{\partial \tilde{\sigma}}{\partial p_\alpha} - C_{\gamma}^\mu \frac{\partial \rho_\mu^i}{\partial q^i} + C_{\mu \beta}^\alpha \frac{\partial^2 \tilde{\sigma}}{\partial p_\alpha \partial p_\mu} + \frac{\partial \rho_\mu^i}{\partial q^i} + C_{\mu \beta}^\alpha \frac{\partial^2 \tilde{\sigma}}{\partial p_\alpha \partial p_\mu}.$$ 

Consequently, along the zero section $0 : M \rightarrow A$, we have that

$$\rho_\mu^i \frac{\partial (\sigma_U^\nu + \sigma)}{\partial q^i} + \frac{\partial \rho_\mu^i}{\partial q^i} + C_{\mu \beta}^\alpha \frac{\partial \rho_\mu^i}{\partial q^i} + C_{\mu \beta}^\alpha \frac{\partial^2 \tilde{\sigma}}{\partial p_\alpha \partial p_\mu} = 0, \quad \text{for all } \mu. \tag{4.4}$$

On the other hand, from (4.2), it follows that

$$\mathbb{F} \tilde{\sigma}(q^i, p_\alpha) = (q^i, \frac{\partial \tilde{\sigma}}{\partial p_\alpha}).$$

This implies that

$$T \mathbb{F} \tilde{\sigma} \circ (d^A V)^\nu = (\frac{\partial \sigma}{\partial q^i}) \frac{\partial}{\partial p_\mu} \tag{4.5}.$$

In addition, since $\{e_\alpha\}$ is an orthonormal basis, we have that

$$b_\beta(e_\alpha) = e^\alpha, \quad \text{for all } \alpha.$$

Thus, from (4.9) and (3.10), we deduce that

$$b_\beta^{-1}(M(e^\alpha, \Lambda^\nu)) = (\rho_\mu^i \frac{\partial (\sigma + \sigma_U^\nu)}{\partial q^i} + \frac{\partial \rho_\mu^i}{\partial q^i} + C_{\mu \beta}^\alpha \frac{\partial \rho_\mu^i}{\partial q^i} + C_{\mu \beta}^\alpha \frac{\partial^2 \tilde{\sigma}}{\partial p_\alpha \partial p_\mu}). \tag{4.6}$$

Therefore, using (4.4), (4.5) and (4.6), we prove the result. \qed

Note that if $\tilde{\sigma}$ is a basic function then, from (4.5), it follows that

$$T \mathbb{F} \tilde{\sigma} \circ (d^A V)^\nu = 0.$$

Consequently, using Theorem 4.1, we obtain that

**Corollary 4.2.** Let $H : A^* \rightarrow \mathbb{R}$ be the hamiltonian energy induced by a bundle metric on $A$ and a potential energy $V : Q \rightarrow \mathbb{R}$ on $A$. Then, $A$ is unimodular if and only if the hamiltonian vector field $\mathcal{H}^M_{H^*}$ of $H$ preserves a volume form on $A^*$ of basic type. In fact, if $\mathcal{M}(e^\alpha, \Lambda^\nu) = -d^A \sigma$, with $\sigma \in C^\infty(Q)$ and $\mathcal{M}(e^\alpha, \Lambda^\nu)$ the modular section of $A$ with respect to the volumes $\nu$ and $\Lambda^\nu$, we have that $\mathcal{H}^M_{H^*}$ preserves the volume form on $A^*$

$$\Phi = e^\sigma \nu \wedge \Lambda^\nu.$$

From Theorem 4.1, we also obtain the following result

**Corollary 4.3.** Let $H : A^* \rightarrow \mathbb{R}$ be the kinetic energy induced by a bundle metric on $A$ (in this case the potential energy is constant). Then, $A$ is unimodular if and only if the hamiltonian vector field $\mathcal{H}^M_{H^*}$ of $H$ preserves a volume form on $A^*$. In fact, if $\mathcal{M}(e^\alpha, \Lambda^\nu) = -d^A \sigma$, with $\sigma \in C^\infty(Q)$ and $\mathcal{M}(e^\alpha, \Lambda^\nu)$ the modular section of $A$ with respect to the volumes $\nu$ and $\Lambda^\nu$, we have that $\mathcal{H}^M_{H^*}$ preserves the volume form on $A^*$

$$\Phi = e^\sigma \nu \wedge \Lambda^\nu.$$
5. Examples

5.1. Standard mechanical hamiltonian systems. Let $Q$ be a smooth manifold of dimension $m$ and $H : T^*Q \to \mathbb{R}$ be a hamiltonian function,

$$H(\alpha) = \frac{1}{2} \mathcal{G}(\alpha, \alpha) + V(\tau_{T^*Q}(\alpha)),$$

for $\alpha \in T^*Q$, with $\mathcal{G}$ a Riemannian metric on $Q$ and $V : Q \to \mathbb{R}$ the potential energy. In this case, our Lie algebroid is the standard one $\tau_{TQ} : TQ \to Q$. As we know (see Subsection [3.2], $\tau_{TQ} : TQ \to Q$ is a unimodular Lie algebroid and, thus, the hamiltonian vector field $\mathcal{H}_{H}^{T^*Q}$ preserves a volume form $\Phi$ on $T^*Q$ of basic type (see Theorem [4.1]). In fact, we may take $\Phi = \Omega^m_Q$, where $\Omega_Q$ is the canonical symplectic structure of $T^*Q$ (note that if $Z$ is an arbitrary hamiltonian vector field on $T^*Q$ then $\mathcal{L}_Z \Omega_Q = 0$).

5.2. Mechanical hamiltonian systems on the dual bundle to an Atiyah algebroid. Let $Q = G \times M$ be total space of a tricial principal $G$-bundle over a manifold $M$ of dimension $m$. Then, the Atiyah algebroid associated with the principal bundle is the vector bundle $\tau_{g \times TM} : g \times TM \to M$, where $g$ is the Lie algebra of $G$. Thus, if $\mathcal{G}$ is a bundle metric on $\tau_{g \times TM} : g \times TM \to M$ and $V : M \to \mathbb{R}$ is a real $C^\infty$-function on $M$, we may consider the hamiltonian function $H : g^* \times T^*M \to \mathbb{R}$ given by

$$H(\alpha, \beta) = \frac{1}{2} \mathcal{G}((\alpha, \beta), (\alpha, \beta)) + V(\tau_{T^*M}(\beta)),$$

for $(\alpha, \beta) \in g^* \times T^*M$.

Denote by $\mathcal{H}_{H}^{H_{g^* \times T^*M}}$ the hamiltonian vector field of $H$ with respect to the linear Poisson structure $\Pi_{g^* \times T^*M}$ on $g^* \times T^*M$. Then, using Corollary [4.2] (see also Subsection [3.2]), we deduce that the vector field $\mathcal{H}_{H}^{H_{g^* \times T^*M}}$ preserves a volume form $\Phi$ on $g^* \times T^*M$ of basic type if and only if $g$ is a unimodular Lie algebra. In fact, if $g$ is unimodular and $\{e_\alpha\}$ is a basis of $g$ then we may take

$$\Phi = dp_1 \wedge \cdots \wedge dp_n \wedge \Omega^n_M,$$

where $p_\alpha$ are the (global) coordinates on $g^*$ induced by the basis $\{e_\alpha\}$ and $\Omega_M$ is the canonical symplectic structure on $T^*M$.

Note that in the particular case when $M$ is a single point then the potential energy $V$ is a constant and the linear Poisson structure on $g^* \times T^*M \simeq g^*$ is just the Lie-Poisson structure of $g^*$. Thus, we recover a result which was proved by Kozlov [7]: the hamiltonian vector field $\mathcal{H}_{H}^{H_{g^*}}$ preserves a volume form on $g^*$ if and only if $g$ is unimodular.

A very simple example of a mechanical system on an Atiyah algebroid is the Elroy’s beanie: two planar rigid bodies attached at their centers of mass, moving freely in the plane (see [10, 15]). In this case:

- The manifold $M$ is $S^1$, and the Lie group $G = SE(2)$.
- The bundle metric $\mathcal{G}$ on $\tau_{SE(2) \times T^1} : SE(2) \times T^1 \simeq \mathbb{R}^3 \times (S^1 \times \mathbb{R}) \to S^1$ is given by

$$\mathcal{G}(\theta)(((\xi_1, \xi_2, \xi_3), t), ((\xi'_1, \xi'_2, \xi'_3), t')) = m(\xi_1 \xi'_1 + \xi_2 \xi'_2) + (I_1 + I_2)\xi_3 \xi'_3 + I_2 t' + I_2(\xi_3 t' + t_3'),$$

for $\theta \in S^1$, $(\xi_1, \xi_2, \xi_3), (\xi'_1, \xi'_2, \xi'_3) \in \mathfrak{se}(2) \simeq \mathbb{R}^3$ and $t, t' \in \mathbb{R}$, where $m, I_1$ and $I_2$ are constants.

Since $\mathfrak{se}(2)$ is a unimodular Lie algebra, we deduce that the hamiltonian dynamics on $\mathfrak{se}(2)^* \times T^*S^1 \simeq \mathbb{R}^3 \times (S^1 \times \mathbb{R})$ preserves the volume form

$$\Phi = dp_1 \wedge dp_2 \wedge dp_3 \wedge \psi \wedge dt,$$
where $\psi$ is the length element of $S^1$.

5.3. Mechanical hamiltonian systems on the dual bundle to an action Lie algebroid. Let $\Phi : \mathfrak{g} \to \mathfrak{X}(Q)$ be a left infinitesimal action of $\mathfrak{g}$ on a manifold $Q$ and $\tau_{\mathfrak{g} \times Q} : \mathfrak{g} \times Q \to Q$ be the corresponding action Lie algebroid. Suppose that $\langle \cdot, \cdot \rangle$ is a scalar product on $\mathfrak{g}$, that $V : Q \to \mathbb{R}$ is a real $C^\infty$-function on $Q$ and that $H : \mathfrak{g}^* \times Q \to \mathbb{R}$ is the corresponding hamiltonian function given by

$$H(\alpha, q) = \frac{1}{2} \langle \alpha, \alpha \rangle + V(q), \text{ for } \alpha \in \mathfrak{g}^* \text{ and } q \in Q.$$  

Denote by $\mathcal{H}_H^{\mathfrak{g}^* \times Q}$ the hamiltonian vector field of $H$ with respect to the linear Poisson structure $\Pi_{\mathfrak{g}^* \times Q}$ on $\mathfrak{g}^* \times Q$. Then, using Corollary 4.3 (see also Subsection 3.2) we deduce that the hamiltonian vector field $\mathcal{H}_H^{\mathfrak{g}^* \times Q}$ preserves a volume form $\Phi$ on $\mathfrak{g}^* \times Q$ of basic type if and only if there exists a real $C^\infty$-function $\sigma : Q \to \mathbb{R}$ such that

$$\text{div}_\nu \Phi(\xi) - \Phi(\xi)(\sigma) = M_\mathfrak{g}(\xi), \text{ for all } \xi \in \mathfrak{g},$$  

where $\nu$ is an arbitrary volume form on $Q$ and $M_\mathfrak{g}$ is the modular character of $\mathfrak{g}$. Moreover, if (5.1) holds and $\{e_\alpha\}$ is a basis of $\mathfrak{g}$ we may take

$$\Phi = e^{-\sigma} dp_1 \wedge \cdots \wedge dp_n \wedge \nu$$

where $(p_\alpha)$ are the global coordinates on $\mathfrak{g}^*$ induced by the basis $\{e_\alpha\}$.

Note that if the Lie algebra $\mathfrak{g}$ is unimodular (that is, $M_\mathfrak{g} = 0$) and the left infinitesimal action preserves the volume form $\nu$ (that is, $\text{div}_\nu \Phi(\xi) = 0$, for all $\xi \in \mathfrak{g}$) then (5.1) holds (we may take $\sigma = 0$). As a particular example we can consider an interesting mechanical system: the heavy-top (see [12]). In this case:

- $\mathfrak{g} = \mathfrak{so}(3) \simeq \mathbb{R}^3$ is the Lie algebra of the special orthogonal group $SO(3)$ and $M$ is the sphere $S^2$.
- $\Phi : \mathfrak{so}(3) \to S^2$ is the left infinitesimal action induced by the standard left action of the Lie group $SO(3)$ on $S^2$.
- The scalar product on $\mathfrak{g} = \mathfrak{so}(3)$ is given by

$$\langle \omega, \omega' \rangle = \omega \cdot I \omega,$$

$I$ being the inertia tensor of the top.
- The potential $V : S^2 \to \mathbb{R}$ is defined by

$$V(x) = mglx \cdot e, \text{ for } x \in S^2,$$

where $e$ is the unit vector from the fixed point to the center of mass and $m$, $g$ and $l$ are constants.

Note that the action of $SO(3)$ on $S^2$ preserves the symplectic volume $\nu$ of $S^2$ and that $\mathfrak{g} = \mathfrak{so}(3)$ is a unimodular Lie algebra. Thus, the hamiltonian dynamics preserves the volume $\Phi$ on $\mathfrak{g}^* \times S^2 \simeq \mathfrak{so}(3)^* \times S^2 \simeq \mathbb{R}^3 \times S^2$ given by

$$\Phi = dp_1 \wedge dp_2 \wedge dp_3 \wedge \nu.$$
6. Conclusions and outlook

We have proved that a hamiltonian system of mechanical type on the dual bundle to a Lie algebroid \( \mathcal{A} \) preserves a volume form on \( \mathcal{A}^* \) of basic type if and only if \( \mathcal{A} \) is unimodular. In addition, if the potential energy of the hamiltonian function is constant then we deduce that the hamiltonian dynamics preserves a volume form on \( \mathcal{A}^* \) (not necessarily of basic type) if and only if \( \mathcal{A} \) is unimodular. These results generalize Liouville’s theorem (see [1]) and a previous result for Lie-Poisson equations which was proved by Kozlov [7].

On the other hand, after we finished this paper, we obtained some examples of hamiltonian systems of mechanical type (with non-constant potential energy) on the dual bundle to a non-unimodular Lie algebroid which preserve a volume form of non-basic type. So, it would be interesting to discuss the existence of such volumes on non-unimodular Lie algebroids. This will be the subject of a separate publication.

Another goal we have proposed is to extend the results of this paper for non-holonomic mechanical systems on Lie algebroids (see [4]). Previous results in this direction for some particular classes of Lie algebroids have obtained by several authors (see [6, 7, 8, 18]).

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