Taub–NUT Dyons in Heterotic String Theory.

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Abstract
Starting with the Taub–NUT solution to Einstein’s equations, together with a constant dilaton, a dyonic Taub–NUT solution of low energy heterotic string theory with non–trivial dilaton, axion and \( U(1) \) gauge fields is constructed by employing \( O(1, 1) \) transformations. The electromagnetic dual of this solution is constructed, using \( SL(2, \mathbb{R}) \) transformations. By an appropriate change to scaled variables, the extremal limit of the dual solution is shown to correspond to the low energy limit of an exact conformal field theory presented previously.

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1. Introduction.

Since it is hoped that string theory can provide a consistent theory of quantum gravity, there has been a great deal of interest in studying nontrivial gravitational solutions of the string equations of motion\[1\]. A great deal of activity was stimulated by Witten’s discovery\[2\] that an $SL(2, \mathbb{R})/U(1)$ conformal field theory describes a two-dimensional black hole. Unfortunately, no such exact conformal field theory solutions providing a complete description of a black hole in four dimensions (including the asymptotically flat regions) has been constructed. In a number of cases though, the exact solutions describing the throat geometry of a four-dimensional black hole carrying an extremal charge have been found\[3][4\]. Amongst the exact solutions constructed in ref.\[4\], the low energy limit of one solution had a time coordinate which was compact, and which formed a nontrivial fibre bundle over the spatial two-sphere. In that paper, it was conjectured that this conformal field theory should correspond to the throat limit of an extremal Taub–NUT dyon in heterotic string theory.

In the present paper, we confirm this conjecture by constructing the Taub–NUT dyons, which are solutions of the low-energy effective string equations. Our construction relies on the two remarkable “duality” symmetries of the effective string action, which provide a technique to generate new solutions. In particular, for heterotic strings in $d$-dimensions coupled to a gauge group of rank $p$, there is a possible symmetry group of $O(d, d + p)$. These transformations are related to the heralded string duality which relates large and small radius compactifications. In section 2, we employ these transformations to produce a Taub–NUT solution carrying nontrivial electric and magnetic charges from the vacuum Taub–NUT solution. In four dimensions, there is also an $SL(2, \mathbb{R})$ symmetry of the low–energy equations of motion. These transformations yield the strong–to–weak coupling duality of four–dimensional string theory. The electric and magnetic couplings of particles and backgrounds are also exchanged under these transformations. In section 3, this electromagnetic duality transformation is applied to the Taub–NUT dyon. In section 4, the extremal limit of these dyons is considered, and we explicitly show that the low energy background fields of the exact conformal field theory
considered in ref.[4] are recovered. Section 5 provides a brief discussion of our results.

First let us establish our conventions for the normalization of the fields. We write the four-dimensional low energy effective action for the heterotic string as

\[ I = \int d^4x \sqrt{-G} e^{-\Phi} \left( R(G) + (\nabla \Phi)^2 - \frac{1}{12} H^2 - \frac{1}{8} F^2 + \ldots \right) . \tag{1.1} \]

Here, the three form \( H \) has components

\[ H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu} - \omega(A)_{\mu\nu\rho}, \tag{1.2} \]

where

\[ \omega(A)_{\mu\nu\rho} = \frac{1}{4}(A_\mu F_{\nu\rho} + A_\nu F_{\rho\mu} + A_\rho F_{\mu\nu}) \tag{1.3} \]

is the Chern–Simons three form for the U(1) gauge field\(^1\). The U(1) field strength is \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \). In eq.(1.1), the action is written in terms of the string or sigma model metric \( G_{\mu\nu} \). It is also convenient to introduce the Einstein metric, \( g_{\mu\nu} = \exp(-\Phi) G_{\mu\nu} \), for which the effective action becomes

\[ I = \int d^4x \sqrt{-g} \left( R(g) - \frac{1}{2} (\nabla \Phi)^2 - \frac{1}{12} e^{-2\Phi} H^2 - \frac{1}{8} e^{-\Phi} F^2 + \ldots \right) . \tag{1.4} \]

Our notation will always be to write the Einstein metric with a lower case \( g \), and the string metric with an upper case \( G \). In displaying the line elements, \( ds_E^2 \) and \( ds_S^2 \) will denote that for the Einstein and string metrics, respectively. (No subscript will be used if the two metrics are the same.)

2. The dyonic solution via \( O(1,1) \)

The Taub–NUT solution\(^5\):

\[ ds^2 = -f_1(dt + 2l \cos \theta \, d\phi)^2 + f_1^{-1}dr^2 + (r^2 + l^2)(d\theta^2 + \sin^2 \theta \, d\phi^2) \tag{2.1} \]

\(^1\) We have dropped the contribution to \( H \) from the Lorentz Chern–Simons three form, as being higher order in the \( \alpha' \) expansion. Note also that we have rescaled the gauge fields from their standard normalization by a factor of \( 1/\sqrt{\alpha'} \). Therefore thinking of our background solutions as valid for small \( \alpha' \) is equivalent to thinking of them as carrying large (electric and magnetic) charges.
where

\[ f_1 = 1 - 2 \frac{Mr + l^2}{r^2 + l^2}, \]  

(2.2)

is a solution to the vacuum Einstein’s equations \( R_{\mu \nu} = 0 \). This metric \( (2.1) \) is invariant under four Killing symmetries[6]. The first of these is simply time translations. The remaining three act as \( SO(3) \) rotations on the angular coordinates, but also involve time translations to preserve the differential \( dt + 2l \cos \theta \, d\phi \). Thus the orbit of a point under these “rotations” is in fact three-dimensional (when \( l \neq 0 \)). Consistency requires \( t \) to have periodicity \( 8l\pi \) as can be seen by taking care to ensure that there are no conical curvature singularities at \( \theta = 0 \) or \( \pi \). Surfaces of constant radius then have the topology of a three-sphere, in which there is a Hopf fibration of the \( S^1 \) of time over the spatial \( S^2 \). The metric \( (2.1) \) is also singular at \( r = r_\pm = M \pm (M^2 + l^2)^{1/2} \), but a nonsingular extension across these null surfaces can be found just as at the event horizon of a black hole[6]. Unfortunately, the periodicity of the time coordinate prevents an interpretation of the Taub–NUT solution as a black hole. Of course if \( l = 0 \), one recovers the usual Schwarzschild black hole geometry, and \( t \) is not periodically identified. The interior region (i.e., \( r_- < r < r_+ \)) may be interpreted as a cosmological solution for a universe with the spatial topology \( S^3 \).

Combined with vanishing gauge and antisymmetric tensor fields, as well as a constant dilaton — we choose \( \Phi = 0 \) for simplicity — the Taub–NUT metric \( (2.1) \) is a solution of the low energy string equations to leading order in the \( \alpha' \) expansion. We wish to generalize this solution by introducing a nontrivial electromagnetic field. A charged Taub–NUT solution of the Einstein-Maxwell equations is known[7], but it will not provide a solution of the string equations because of the nontrivial dilaton and axion couplings to the gauge field in the effective action \( (1.1) \). These couplings though lead to a remarkable symmetry of the action, which allows us to produce a new dyonic Taub–NUT solution.

In refs.[8][9], it was shown that there exists an \( O(d - 1, 1) \otimes O(d - 1, 1) \) symmetry of the space of solutions of the low energy field equations when the solutions are independent of \( d \) of the spacetime coordinates. (Here, the time coordinate has been included, with a Minkowskian signature metric.) In ref.[10], the extension of these
results to the case of heterotic string theory was presented. For solutions which are independent of $d$ of the spacetime coordinates and for which the background gauge field lies in a subgroup commuting with $p$ of the $U(1)$ generators of the gauge group, there is an $O(d - 1, 1) \otimes O(d + p - 1, 1)$ symmetry of the field equations. In fact, there is a larger symmetry group $O(d, d + p)$, however those transformations which are not in the $O(d - 1, 1) \otimes O(d + p - 1, 1)$ subgroup correspond to pure gauge transformations[9]. The remaining transformations provide a powerful technique to generate new solutions, and by which to explore the space of solutions.

Regarding the solution (2.1) with $\Phi = 0$ as an uncharged solution to heterotic string theory, we supplement the four spacetime coordinates with one extra “chiral” coordinate, $X$, which allows a $U(1)$ gauge field background to be considered. Now the starting solution is independent of the $t$ and $\phi$ coordinates, as well as having no gauge fields, which therefore trivially commutes with the $U(1)$ carried by the $X$ direction. In general, there is the possibility of doing $O(1, 1) \otimes O(2, 1)$ transformations to generate more solutions. For the concerns here it is not of interest to include the possibility of boosts in the $\phi$ direction, as this will break the “rotation” symmetries of the Taub–NUT solution discussed above.

So an $O(1, 1)$ boost in the gauge–time directions will be performed. Using the notation of ref.[10], the metric (2.1) is used to form the $9 \times 9$ matrix $M$

$$
M = \begin{pmatrix}
K_T G^{-1} K_- & K_T G^{-1} K_+ & -K_T G^{-1} A \\
K_T G^{-1} K_- & K_T G^{-1} K_+ & -K_T G^{-1} A \\
-A T G^{-1} K_- & -A T G^{-1} K_+ & A T G^{-1} A
\end{pmatrix}
$$

(2.3)

where

$$(K_{\pm})_{\mu \nu} = -B_{\mu \nu} - G_{\mu \nu} - \frac{1}{4} A_{\mu} A_{\nu} \pm \eta_{\mu \nu}
$$

(2.4)

and $T$ denotes matrix transpose. The boost matrix for the gauge–time directions

$$
\Omega = \begin{pmatrix}
I_7 & 0 & 0 \\
0 & x & \sqrt{x^2 - 1} \\
0 & \sqrt{x^2 - 1} & x
\end{pmatrix}
$$

(2.5)

is defined, where $I_7$ is a $7 \times 7$ identity matrix, and $x^2 \geq 1$. The equations of motion are invariant under

$$
M \rightarrow M' = \Omega M \Omega^T.
$$

(2.6)
From the matrix $\mathcal{M}'$, the new metric $G'$, antisymmetric tensor and gauge fields may be extracted by following the definitions in [10]. The new dilaton field is given by
\[ \Phi' = \frac{-1}{2} \log \left( \frac{\det G}{\det G'} \right) \quad (2.7) \]
where $G$ refers to the old string metric. This gives a dyonic Taub–NUT solution to heterotic string theory:
\[ ds_2^2 = -\frac{f_1}{f_2} (dt + (x + 1)l \cos \theta d\phi)^2 + f_1^{-1} dr^2 + (r^2 + l^2)(d\theta^2 + \sin^2 \theta d\phi^2), \quad (2.8) \]
where
\[ f_1 = 1 - 2 \frac{Mr + l^2}{r^2 + l^2} \quad \text{and} \quad f_2 = 1 + (x - 1) \frac{Mr + l^2}{r^2 + l^2} \quad (2.9) \]
and
\[ B_{t\phi} = \frac{f_1}{f_2} (x - 1) l \cos \theta \]
\[ A_\phi = -2 \frac{f_1}{f_2} \sqrt{x^2 - 1} l \cos \theta \]
\[ A_t = \sqrt{x^2 - 1} \left( \frac{1 - f_1}{f_2} \right) \]
\[ e^{-\Phi} = f_2. \quad (2.10) \]
The line element for the Einstein metric will be given by $ds_E^2 = f_2 ds_S^2$. Applying the previous analysis of the Taub–NUT solution to the present metric, one is again lead to consider constant $r$ surfaces with topology $S^3$ in which $t$ is identified with a period $4(x + 1)l\pi$. With $l = 0$, one recovers an electrically charged black hole solution discussed in ref.[11].

Examining the asymptotic behavior of $g_{tt}$ in the Einstein metric, the mass $\mu$ of the solution is easily evaluated as
\[ \mu = \frac{x + 1}{2} M. \quad (2.11) \]
The non–zero components of the gauge field strength are $F_{rt}, F_{r\phi}$ and $F_{\theta\phi}$. One may be tempted to define the magnetic and electric charges by integrating $F$ and its dual over the spatial two–sphere at infinity. However because of the topology
of the constant \( r \) surfaces, there are in fact no nontrivial two spheres on which to integrate. Hence we can only define the magnetic and electric charges in terms of the asymptotic behavior of the electromagnetic fields by analogy to that in an asymptotically flat space-time (i.e., \( F_{tr} \simeq Q_E/r^2 \) and \( F_{\theta\phi} \simeq Q_M \sin \theta \)).

\[
Q_E = 2M\sqrt{x^2 - 1} \quad \text{and} \quad Q_M = 2l\sqrt{x^2 - 1}.
\] (2.12)

Note that the physical mass, and magnetic and electric charges of the solution are completely independent, as they are independent functions of the three parameters \( M, x \) and \( l \). By examining the asymptotic behavior of the dilaton and using the analogy with an asymptotically flat space-time, one may also define a non–vanishing dilaton charge from \( e^\Phi = 1 + \frac{D}{r} \), which yields \( D = -(x - 1)M \).

3. The electromagnetic dual solution via \( SL(2, \mathbb{R}) \)

In four dimensions, the string equations of motion possess another noteworthy symmetry which may be employed as a further solution generating technique. This is a stringy electromagnetic duality invariance[12]. This symmetry is also known as \( S \) duality since it relates solutions of strong and weak coupling.

To apply these transformations, one works in terms of the Einstein metric. Then the scalar axion field, \( a \), must be determined from

\[
H_{\mu\nu\rho} = -e^{2\Phi}\epsilon_{\mu\nu\rho\kappa}\nabla^\kappa a
\] (3.1)

where \( \epsilon_{\mu\nu\rho\kappa} \) is the completely antisymmetric tensor in four dimensions with \( \epsilon_{tr\theta\phi} = \sqrt{-g} \). Next define the complex scalar

\[
\lambda = a + ie^{-\Phi}
\] (3.2)

and the complex gauge field strengths

\[
(F_{\pm})_{\mu\nu} = F_{\mu\nu} \pm \frac{i}{2} \epsilon_{\mu\nu\rho\kappa} F^{\rho\kappa}.
\] (3.3)

Alternatively, the same charges would be determined by considering the motion of point-like electric or magnetic charges in the asymptotic region.
The equations of motion are invariant under the transformations\cite{12}

\[ T : (\lambda, F_+, F_-) \to (\lambda + c, F_+, F_-) \]
\[ S : (\lambda, F_+, F_-) \to (-\frac{1}{\lambda}, -\lambda F_+, -\bar{\lambda} F_-) \]

(3.4)

where the Einstein metric is left invariant under both transformations. In the above \( \bar{\lambda} \) is the complex conjugate of \( \lambda \), and \( c \) in a real constant\footnote{Instanton corrections\cite{12} fix \( c \) to be 1, and hence break the group to \( SL(2, \mathbb{Z}) \).}. Combined, these symmetry transformations generate the group \( SL(2, \mathbb{R}) \).

For the solution presented in the previous section, the possible non–zero components of \( H \) are \( H_{r\phi t} \) and \( H_{\theta\phi t} \). After some algebra the first is seen to vanish, yielding for the the axion (discarding an integration constant):

\[ a = (x - 1) \frac{r - M}{r^2 + l^2}. \]

(3.5)

Note that the fact that the axion has only radial dependence is consistent with the full solution retaining the time translation and rotation invariances of the original Taub–NUT solution.

After applying the \( S \) duality transformation, the new axion and dilaton fields are

\[ e^{-\hat{\Phi}} = \frac{f_2}{a^2 + f_2^2}, \quad \hat{a} = -\frac{a}{a^2 + f_2^2}. \]

(3.6)

Combining the new dilaton with the (invariant) Einstein metric gives the new string metric:

\[ ds_S^2 = -\frac{f_1}{f_2^2}(a^2 + f_2^2)d\xi^2 + \frac{(a^2 + f_2^2)}{f_1}dr^2 + (a^2 + f_2^2)(r^2 + l^2)(d\theta^2 + \sin^2 \theta d\phi^2), \]

(3.7)

where \( d\xi = dt + (x + 1)l \cos \theta d\phi \), and \( f_1 \) and \( f_2 \) are given in eq. (2.9).

The new gauge fields strengths are given by

\[ \hat{F}_{\mu\nu} = \frac{1}{2}e^{-\hat{\Phi}}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} - aF_{\mu\nu} \]

(3.8)

and one finds that the only only non–vanishing components of the \( U(1) \) field strength
are $\hat{F}_{rt}, \hat{F}_{r\phi}$ and $\hat{F}_{\theta\phi}$ from which gauge potentials may be derived:

$$\hat{A}_\phi = 2\sqrt{x^2 - 1} \cos \theta \left( \frac{Mr^2 + ((x-1)M^2 + (x+1)l^2)r - Ml^2}{r^2 + (x-1)Mr + xl^2} \right)$$

$$\hat{A}_t = \frac{2\sqrt{x^2 - 1}(r - M)}{r^2 + Mr(x - 1) + xl^2}. \quad (3.9)$$

Together with the new axion and dilaton:

$$\hat{a} = -(x - 1)l(r - M) \frac{(r + M(x - 1))^2 + x^2l^2}{(r + M(x - 1))^2 + x^2l^2}$$

$$e^{-\hat{\Phi}} = \frac{r^2 + Mr(x - 1) + xl^2}{(r + M(x - 1))^2 + x^2l^2} \quad (3.10)$$

the new antisymmetric tensor field may be deduced as:

$$\hat{B}_{\phi t} = l(x - 1) \cos \theta \left( \frac{(r - M)^2 + x(M^2 + l^2)}{r^2 + Mr(x - 1) + xl^2} \right). \quad (3.11)$$

It is straightforward to calculate the electric and magnetic charges as before:

$$\hat{Q}_E = 2\sqrt{x^2 - 1}$$

$$\hat{Q}_M = -2M\sqrt{x^2 - 1}. \quad (3.12)$$

Comparing this to the charges (2.12) of the original dyon solution, it is noteworthy that the roles of $M$ and $l$ are simply exchanged under duality. The mass of the new solution is still

$$\hat{\mu} = \mu = \frac{x + 1}{2} M \quad (3.13)$$

as the Einstein metric remains invariant. In this case, $l = 0$ yields the magnetically charged black hole solution of ref.[11].

4. The extremal limit and an exact solution.

Now for both solutions the charge to mass ratio is given by

$$\frac{Q_E^2 + Q_M^2}{\mu^2} = 8 \left( \frac{M^2 + l^2}{M^2} \right) \frac{x - 1}{x + 1}. \quad (4.1)$$
In the pure black hole case \( l = 0 \), the extremal limit is obtained when this ratio is maximized, that is when \( Q^2 = 8\mu^2 \), where \( Q = Q_E \) or \( Q_M \) depending upon whether the \((l = 0)\) solution is electric (section 2) or magnetic (section 3). (Note that our conventions differ from those of ref.[11].) This limit is achieved by taking the limit \( M \to 0 \) and \( x \to \infty \) while holding the quantity \( m = xM \) a constant. For non-zero \( l \), it is clear that the choice \( l \to 0 \) must be made while holding the quantity \( \lambda = xl \) finite. One way of seeing this is to simply require a sensible limit for the line element

\[
d\xi = dt + (1 + x)l \cos \theta \, d\phi.
\]

One of the interesting properties of charged string black hole metrics is that at extremality the spatial geometry approaches that of an infinite throat of constant radius for increasing distance from the black hole. This is because at extremality the string metric component \( G_{rr} \) develops a double pole and therefore a measure of the proper distance \( D \) to travel to the horizon at \( r_H \) is:

\[
D = \int_{r_0}^{r_1} \frac{dr}{r - r_H} = \log \frac{r_1 - r_H}{r_0 - r_H}
\]

which diverges as \( r_0 \) approaches \( r_H \). So the asymptotic throat geometry for constant time slices is a manifestation of the fact that the horizon is infinitely far away from any point at finite \( r \) in the outside region.

A careful way of approaching the extremal limit is to is to change variables so as to send the asymptotically flat region away to infinity and define scaled variables which capture the physics near the horizon. To that end it is useful to set \( r = f(\sigma) \) where \( f(\sigma) \) is determined only by the fact that a constant radius throat geometry is to be approached in the large \( \sigma \) limit and that \( f(\sigma) = 0 \) at extremality.

Using either metric, this amounts to the condition that

\[
R_T^2(d\sigma^2 + d\Omega^2)
\]

be the form of spatial metric, where \( R_T \) is the throat radius. This condition yields the first order differential equation:

\[
(f')^2 = (f^2 - 2Mf - l^2).
\]

This has a simple solution, (fixing an integration constant by requiring that \( f(\sigma) \to 0 \) at extremality):

\[
f(\sigma) = \sqrt{M^2 + l^2} \cosh \sigma + M.
\]
Recall that the limiting procedure is to take \( M, l \to 0 \) and \( x \to \infty \) while holding constant \( m = xM \) and \( \lambda = xl \). To calculate the correctly scaling physics in this limit, set \( m = xM \) and \( \lambda = xl \) together with \( M = \delta \) and \( l = \delta \frac{\lambda}{m} \) in all quantities. The extremal limit is now simply \( \delta \to 0 \). Quantities which survive the extremal limit will be those which scale with the correct powers of \( \delta \).

The difference between the two dual cases arises when the throat radius is calculated in the limit. In the case of (2.8) it collapses:

\[
R_T^2 = (f^2 + l^2) \to 0
\]  

(4.6)

while in the case of (3.7) it is non-zero:

\[
\hat{R}_T^2 = (a^2 + f_2^2)(f^2 + l^2) = f^2 + 2(x-1)M f + (x-1)^2M^2 + l^2x^2 \to m^2 + \lambda^2.
\]  

(4.7)

Similarly, the rest of the metric (2.8) vanishes in the limit while for the dual metric (3.7):

\[
G_{\xi\xi} = -\frac{(m^2 + \lambda^2)}{m^2} \cosh^2 \sigma - 1 \frac{\cosh \sigma - 1}{(\cosh \sigma + \Delta)^2},
\]  

(4.8)

where \( \Delta = \sqrt{\frac{m^2 + \lambda^2}{m^2}} \).

So the case of interest is clearly the dual solution of section 3, where the string metric remains finite and non-zero. After calculating the extremal limit of the rest of the background fields by similar methods, the final solution for the extremal limit of the dual solution of section 3 in scaled variables is (after rescaling \( t \) by \( 1/m \)):

\[
d\hat{s}_S^2 = (m^2 + \lambda^2) \left( d\sigma^2 - \frac{\cosh^2 \sigma - 1}{(\cosh \sigma + \Delta)^2} (dt^2 + 2\frac{\lambda}{m} \cos \theta d\phi)^2 + d\theta^2 + \sin^2 \theta \, d\phi^2 \right);
\]

\[
e^{-\hat{\Phi}} = \cosh \sigma + \Delta;
\]

\[
\hat{a} = -\frac{\lambda}{m} \cosh \sigma;
\]

\[
\hat{A}_\phi = 2m\Delta \cos \theta \frac{\Delta \cosh \sigma + 1}{\cosh \sigma + \Delta};
\]

\[
\hat{A}_t = \frac{2\lambda}{m} \cosh \sigma \frac{\cosh \sigma}{\cosh \sigma + \Delta};
\]

\[
\hat{B}_{\phi t} = \frac{\lambda \Delta \cos \theta}{\cosh \sigma + \Delta}.
\]  

(4.9)
Notice that there was a shift in the dilaton and (exponentiated) axion by an infinite constant in order to keep them finite in the limit. That this is an allowed shift is easily seen by looking at the low energy effective action written in terms of the string metric $G_{\mu\nu}$ and the scalar axion $a = e^\rho$. This is

$$I = \int d^4x \sqrt{-G} e^{-\Phi} \left( R(G) + (\nabla \Phi)^2 - \frac{1}{8} F^2 \right)$$

$$- \frac{1}{2} e^{2(\Phi + \rho)} (\nabla \rho)^2 - \frac{1}{16} e^{\Phi + \rho} \epsilon^{\mu\nu\sigma\kappa} F_{\mu\nu} F_{\sigma\kappa} \right).$$

This action (and the resulting equations of motion) will be invariant under combined constant shifts

$$\Phi \rightarrow \Phi + c \quad \rho \rightarrow \rho - c$$

or

$$e^{-\Phi} \rightarrow (e^{-c}) e^{-\Phi} = \hat{c} e^{-\Phi}$$

$$a \rightarrow (e^{-c}) a = \hat{c} a$$

so the same constant can always be absorbed away into the axion and dilaton. This is indeed what was done to get the finite forms above.

This extremal solution (4.9) is the same form as presented in ref.[4] as the low energy limit of an exact conformal field theory derived as a ‘heterotic coset’ model. This is a consistent combination of a $(0,2)$ supersymmetric $SL(2, \mathbb{R}) \times SU(2)$ Wess–Zumino–Witten (WZW) model with a $U(1) \times U(1)$ subgroup gauged, together with a heterotic arrangement of fermions\textsuperscript{4}. To match forms exactly it is necessary to make a trivial $U(1)$ gauge transformation. Matching physical quantities defined there we get

$$Q_A = \frac{m\hat{\lambda}\Delta}{2}, \quad Q_B = \frac{m\Delta^2}{2}$$

which satisfy:

$$\frac{Q_A^2}{\Delta^2 - 1} = \frac{Q_B^2}{\lambda^2 + 1} = \frac{Q_A Q_B}{\Delta\lambda}$$

\textsuperscript{4} In ref.[4], the gauge group was chosen as $U(1) \times U(1)$ where each factor contained identical theories. This is not an essential choice, and was merely the most symmetrical arrangement for the purposes of that paper. It is consistent to truncate to just one copy of the $U(1)$’s.
where $\hat{\lambda} = \lambda/m$ and $\Delta = \sqrt{1 + \hat{\lambda}^2}$ replace the $\lambda$ and $\delta$, respectively, of ref.[4]. To compare dilatons it is necessary to rescale $\hat{\Phi} \to 2\hat{\Phi}$. Equations (4.14) are the precisely the relations which arise from the anomaly cancelation conditions of ref.[4], ensuring that the model was conformally invariant.

Notice that the $\hat{\lambda} = 0$ limit corresponds to the extremal limit of the magnetically charged dilaton black hole of ref.[11], which as a background of heterotic string theory was shown to correspond to the low energy limit of an exact conformal field theory in the first of refs.[3]. This conformal field theory is a tensor product of an orbifold of an $SU(2)$ WZW and the solution of ref.[2]. This orbifold theory was later shown in ref.[4] to be equivalent to an example of the type of ‘heterotic coset’ introduced there, and is indeed a special case of the heterotic coset which gives rise to the extremal ‘Taub–NUT+throat’ solution (4.9).

5. Discussion.

In this paper, we have presented new nontrivial solutions of the low energy effective equations of motion in the heterotic string theory. These solutions correspond to dyonic Taub–NUT spaces. We have also confirmed the conjecture made in ref.[4] that the extremal limit of these Taub–NUT dyons corresponds to the conformal field theory constructed there. This agreement is not only in the string metric geometry, but also in the dilaton, axion and gauge fields, all at one–loop in $\alpha'$. The exact conformal field theory provides all of the higher order (in $\alpha'$) corrections and also non–perturbative data. It is also satisfying to find that the solutions naturally obey the constraints imposed on the sigma model fields to ensure the vanishing of anomalies in the gauged symmetries. The construction in ref.[4] actually introduces two background gauge fields, and we have set the charges associated with one of these fields to zero in order to make the comparison with the present solutions. A second background gauge field could be added by introducing a second “chiral” coordinate in section 2. Then in addition to the boost in the gauge-time directions made there, a $O(2)$ rotation amongst the “chiral” coordinates would introduce the desired second nontrivial gauge field.
Both of the families of solutions given in sect.’s 2 and 3 are dyonic carrying both magnetic and electric charges (although we remind the reader that these charges are not defined in a conventional way due to the topology of the solution). The feature which truly distinguishes the two types of solutions is the behavior of the scalar fields. Recall in sect. 2, the dilaton charge was determined to be \( D = -(x-1)M \) for the solutions there. From eq. (3.10), one finds a dilaton charge of \( \hat{D} = +(x-1)M \) for the solutions discussed in sect. 3. Hence the string coupling \( e^{\Phi} \) decreases in the central region of the solutions in sect. 2, while it increases for those in sect. 3. One may think of the throat of the latter as being region of strong coupling, while it is a region of weak coupling in the solutions of sect. 2. These different behaviors are of course expected since in general the \( S \) duality transformation which relates the two families transforms between strong and weak coupling. Note that the axion charge also changes sign under this transformation.

It would be a trivial matter to extend the family of solutions constructed here by combining the \( S \) and \( T \) transformations in sect. 3. Equivalently, an integration constant \( c \) could be retained in eq. (3.5). Eq.’s (3.10), (3.7) and (3.8) would then remain unchanged although their explicit form would be slightly more complicated. A further \( SL(2,\mathbb{R}) \) transformation would be required to rescale the dilaton to restore \( e^{-\Phi} \to 1 \) as \( r \to \infty \). Then one finds that the resulting electric and magnetic charge become

\[
Q_E = 2\sqrt{x^2-1} \frac{l+cM}{\sqrt{1+c^2}} \quad Q_M = -2\sqrt{x^2-1} \frac{M-cl}{\sqrt{1+c^2}}.
\]  

(5.1)

Hence the resulting solutions smoothly interpolate between the two families presented here as \( c \) varies from 0 to \( \infty \). Although either \( Q_E \) or \( Q_M \) can be set to zero with an appropriate choice of \( c \), the solution will still contain both nontrivial electric and magnetic fields.

In the discussion so far, we have assumed that the boost parameter \( x \geq 1 \), but \( x \leq -1 \) is also a valid range for this parameter. In particular, the choice \( x = -1 \) corresponds to the famous discrete duality transformation, which takes \( G_{tt} \to 1/G_{tt} \). For this particular choice, \( G_{t\phi} \) vanishes. Thus the nontrivial fibration of the time coordinate over the spatial two-sphere is lost, and the topology of surfaces of constant \( r \) is
simply $S^1 \times S^2$. This change of topology under a duality transformation was noted in a similar setting by ref.[13].

The geometry described by the original Taub–NUT metric (2.1) is free of any curvature singularities (when $l \neq 0$). The present dyonic solutions will contain curvature singularities at $\hat{r}_\pm = -\frac{x-1}{2}M \pm \left[\frac{(x-1)^2}{4}M^2 - x l^2\right]^{1/2}$ where $f_2 = 0$. Thus no such singularities actually arise when $l^2 > \frac{(x-1)^2}{4x}M^2$ (in which case $\hat{r}_\pm$ are complex). In this aspect, the new solutions are similar to the charged Taub–NUT solution of the Einstein-Maxwell equations[7], for which singularities arise when a critical value of the charge is exceeded. In the present solutions when $x > 1$ (and $M > 0$), the singularities will occur at negative values of the radial coordinate, and so are “hidden” by the horizon-like surface at $r_+ = M + (M^2 + l^2)^{1/2}$. In contrast for the solutions with $x < -1$ (and $M > 0$), one finds that $\hat{r}_+ \geq r_+$, and so in this case the singularities are not hidden.

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**Note Added**

After completion of the work described in this paper, ref.[14] appeared, in which the solution of sections 2 and 3 is displayed. That paper also notes that the extremal limit yields the ‘Taub–NUT+throat’ metric, corresponding to the exact conformal field theory of ref.[4].
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