Moment equations of neutrinos in supernova

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Abstract

We derive a series of moment equations describing the motion and flavor transformation of neutrinos in supernova. We find a particular series of moments of neutrino density matrix in supernova. The emission angle distribution of neutrinos is described by this series of moments. We expand the equation of neutrinos using these moments and obtain moment equations. We find that these moments have very good property of convergence and the infinite series of equations can be truncated to equations with a small set of moments. Using a small set of moment equations the required computational power is reduced by about two orders of magnitude compared to that in multi-angle simulation. The study on non-linear flavor transformation of neutrinos is substantially simplified using these equations. Two flavor system of neutrinos is also considered and new equations describing the flavor polarization vectors of neutrinos are found.

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1 Introduction

It is realized that neutrino flavor transformation in supernova is a very complicated problem. In addition to the vacuum oscillation effect and the effect of neutrino refraction with ordinary matter (Mikheyev-Smirnov-Wolfenstein effect), it is well known that neutrino-neutrino refraction can also be important in supernova. Neutrino self-interaction can be important in flavor oscillation when neutrino density is sufficiently large. In core collapse supernova neutrino density can be so large above the neutrino sphere that the non-linear flavor oscillation caused by neutrino-neutrino refraction can dominate neutrino flavor transformation.

Researches on non-linear neutrino oscillation in supernova have been done by many groups. Many interesting phenomena caused by neutrino self-interaction, such as synchronized oscillation, bipolar oscillation and spectral split, have been found. Detailed numerical analysis and qualitative analysis have been done to understand these phenomena. An incomplete list of researches on this subject is [1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21]. Although the development in this field is very fast there are still many problems not answered and the present situation of research is not satisfactory.
One problem hard to explore is how neutrino flavor transformation depends on the emission angle distribution of neutrinos on the neutrino sphere. It is known that if all neutrinos are emitted in radial direction neutrino self-interaction vanishes and non-linear flavor transformation disappears. Angular distribution of neutrino emission is essential in non-linear flavor transformation and a complete analysis of neutrino flavor transformation has to take it into account. However in the present formulation of the problem it is hard to study the effect of emission angle distribution of neutrinos. Previous researches on this problem use many angle bins in numerical simulation. It is very complicated and does not give us insight to physics in it.

In the multi-angle simulation flavor evolutions of neutrinos on trajectories of all angle bins should be studied. It is found that if $L_\nu = 10^{51}\ \text{erg/s}$ more than 500 angle bins are required in order to make the simulation converge. Even more angle bins are required for larger neutrino luminosity. Note that the energy range is also made discrete in numerical works. It is found that a complete numerical computation requires following the evolution governed by more than a million equations. It is terribly complicated and makes us hard to understand the physics in it. Another problem in multi-angle simulation is that exact spherical symmetry is essential in the simulation. It is unable to imagine how to simulate neutrino flavor transformation without the spherical symmetry in this approach. A better formulation of the problem is required to make it easier to solve numerically and to be understood qualitatively.

The present scheme to study neutrino evolution is not only too complicated in practice but also conceptually not necessary. It is not necessary to know the detailed evolution of wave function of neutrino in every trajectory. We just need the flux, flavor content, angular distribution and energy distribution of neutrinos at given time and given position in space. It is enough using these information to study the flavor conversion rates of neutrinos as functions of radius $r$. It is also sufficient using these information to compute the energy deposit of neutrinos to plasma environment. Effect of the emission angle distribution of neutrinos can be carefully taken into account in moment expansion of neutrino distribution in phase space. The whole set of equations can be tremendously reduced if appropriate set of moments are found.

In this article we present a new formulation describing the transport and flavor evolution of neutrinos above the neutrino sphere in core-collapse supernova. Instead of considering the neutrino flavor evolution on trajectories we study the density matrix of neutrino at given time and position. We do a moment expansion of the Liouville equation of neutrino using a series of moments of density matrix. The problem of neutrino flavor transformation in supernova is tremendously simplified by observing that this series of moments have very good property of convergence. Numerical computation on neutrino flavor transformation is greatly simplified using this series of moment equations. This series of moment equations is obtained when we examine the problem using spherical coordinate and find a series of functions which have very good convergence property. It is shown that the emission angle distribution of neutrinos can be systematically studied.

In section 2 we rewrite the Liouville equation using spherical coordinate. In section 3...
we simplify the equation of motion with the assumption of spherical symmetry. In section 4 we introduce a series of moments of neutrino density matrix and motivate the use of them by examining their property of convergence. In section 5 we expand the Liouville equation of neutrino and work out a series of moment equations. In section 6 we discuss stationary approximation and further simplify the equations. In section 7 we discuss the truncation of the moment equations. In section 8 we present some numerical analysis. In section 9 we work in two flavor system and derive equations governing the pendulum motion of neutrinos in flavor space. We conclude in section 10.

2 Equation in spherical coordinate

In this section we derive the equation of motion of neutrinos in spherical coordinate. The neutron star core of the core-collapse supernova is located at the origin of the coordinate system.

Consider the density matrix $\rho_{\vec{p}}(t, \vec{x})$ of neutrino with a given momentum $\vec{p}$ at position $\vec{x}$ and at time $t$. $\rho_{\vec{p}}(t, \vec{x})$ is a $3 \times 3$ matrix when considering three flavors of neutrinos and is a $2 \times 2$ matrix when considering two flavors of neutrinos. For anti-neutrino we express the density matrix as $\bar{\rho}_{\vec{p}}$. The evolution of neutrino flavor is described by the Liouville equation [22, 23]:

$$\frac{d\rho_{\vec{p}}}{dt} = -i[H_0 + \sqrt{2}G_FL + \sqrt{2}G_FD, \rho_{\vec{p}}],$$

(1)

where $D$ describes neutrino self-interaction and is expressed as

$$D = \int \frac{d^3q}{(2\pi)^3} (\rho_{\vec{q}} - \bar{\rho}_{\vec{q}})(1 - \cos \theta_{\vec{p}\vec{q}}),$$

(2)

and $H_0$ is the Hamiltonian for vacuum oscillation, $L = diag\{n_e, n_\mu, n_\tau\}$ in the flavor base is the matter term given by charged lepton number densities $n_{e,\mu,\tau}$. $G_F$ is the Fermi
constant. The right-hand side of Eq. (1) is a commutator of the effective Hamiltonian and the density matrix. For anti-neutrino the equation is similar except replacing $H_0$ by $-H_0$; $H_0 \rightarrow -H_0$.

The right-handed side of Eq. (1) governs the quantum evolution of neutrino flavor. It does not include the effect of streaming of relativistic neutrinos. Effect of the motion of neutrinos is implicitly given in the left-hand side of Eq. (1) which can be written as

$$\frac{d\rho_{\vec{p}}}{dt} = \frac{\partial \rho_{\vec{p}}}{\partial t} + \frac{dx^i}{dt} \frac{\partial \rho_{\vec{p}}}{\partial x^i} + \frac{dp^i}{dt} \frac{\partial \rho_{\vec{p}}}{\partial p^i}. \quad (3)$$

The second term in Eq. (3) is caused by the streaming of neutrino in space and the third term is caused by the time dependence of neutrino momentum. For free streaming neutrino the third term is zero in Cartesian coordinate. It does not vanish in spherical coordinate: $\vec{x} = (r, \theta, \varphi)$. Fig. 1 demonstrates that $\theta_p$, the angle of neutrino direction intersecting with the radial direction, changes as neutrino propagates. To figure out the left-hand side of Eq. (1) we need to know $\frac{dx^i}{dt}$ and $\frac{dp^i}{dt}$.

The spherical coordinate system is shown in Fig. 2. At position $\vec{x}$ we introduce a local coordinate system spanned by three orthogonal unit vectors: $\hat{r}$, $\hat{\theta}$ and $\hat{\varphi}$:

$$\hat{r} = \sin \theta \cos \varphi \hat{x} + \sin \theta \sin \varphi \hat{y} + \cos \theta \hat{z}, \quad (4)$$

$$\hat{\theta} = \cos \theta \cos \varphi \hat{x} + \cos \theta \sin \varphi \hat{y} - \sin \theta \hat{z}, \quad (5)$$

$$\hat{\varphi} = -\sin \varphi \hat{x} + \cos \varphi \hat{y}, \quad (6)$$

where $\hat{x}$, $\hat{y}$ and $\hat{z}$ are three orthogonal unit vectors in Cartesian coordinate. They are shown in Fig. 2. The momentum $\vec{p}$ of a neutrino at position $\vec{x}$ can be projected to $\hat{r}$, $\hat{\theta}$ and $\hat{\varphi}$ directions. $\theta_p$ and $\varphi_p$ are defined as:

$$\hat{p} \cdot \hat{r} = \cos \theta_p, \quad \hat{p} \cdot \hat{\theta} = \sin \theta_p \cos \varphi_p, \quad \hat{p} \cdot \hat{\varphi} = \sin \theta_p \sin \varphi_p, \quad (7)$$

where $\hat{p} = \vec{p}/|\vec{p}|$ is a unit vector. In Appendix we compute the projection of momentum to three directions. Using Eq. (86) we get

$$\frac{dr}{dt} = \frac{p^r}{p^0} = \cos \theta_p, \quad \quad (8)$$

$$\frac{d\theta}{dt} = \frac{p^\theta}{p^0} = \frac{1}{r \sin \theta_p \cos \varphi_p}, \quad \quad (9)$$

$$\frac{d\varphi}{dt} = \frac{p^\varphi}{p^0} = \frac{1}{r \sin \theta} \sin \theta_p \sin \varphi_p, \quad \quad (10)$$

where $p^0$ is the zero component of momentum.
\[ \frac{dp^i}{dt} \text{ is given by the geodesic equation:} \]
\[ \frac{dp^i}{dt} = -\Gamma^i_{jk} p^j p^k / p^0, \tag{11} \]

where \( \Gamma^i_{jk} \) is the Christoffel symbol and summation over repeated indices is assumed. Equations of \( p^r, p^\theta \) and \( p^\phi \) are computed in the Appendix. It is shown that \( |\vec{p}| \), the magnitude of neutrino momentum, is a constant. This is an expected result when gravitational potential is neglected. Hence in Eq. (3) we just need to include effect of angular dependence of neutrinos. Writing \( \rho_p(t, \vec{x}, \vec{p}) = \rho_p(t, \vec{x}, |\vec{p}|, \cos \theta_p, \varphi_p) \) (similarly for \( \bar{\rho}_p \)) we just need to include \( d\cos \theta_p/dt \) and \( d\varphi_p/dt \) in Eq. (3). They are given in Eqs. (87) and (88) in Appendix. To summarize we arrive at

\[ \frac{d\rho_{\vec{p}}}{dt} = \frac{\partial \rho_{\vec{p}}}{\partial t} + \cos \theta_p \frac{\partial \rho_{\vec{p}}}{\partial r} + \frac{1}{r} \sin \theta_p \cos \varphi_p \frac{\partial \rho_{\vec{p}}}{\partial \theta} + \frac{1}{r \sin \theta} \sin \theta_p \sin \varphi_p \frac{\partial \rho_{\vec{p}}}{\partial \varphi} \]
\[ + \frac{1}{r} (1 - \cos^2 \theta_p) \frac{\partial \rho_{\vec{p}}}{\partial \cos \theta_p} - \frac{1}{r \ctg \theta} \sin \theta_p \sin \varphi_p \frac{\partial \rho_{\vec{p}}}{\partial \varphi_p}. \tag{12} \]

### 3 Equation with spherical symmetry

In the remaining part of the present article we assume that the system in core-collapse supernova is spherically symmetric and the angular distribution of neutrino momentum
at position $\vec{x} = (r, \theta, \varphi)$ is cylindrically symmetric, that is

$$\frac{\partial L}{\partial \theta} = \frac{\partial L}{\partial \varphi} = 0,$$

(13)

$$\frac{\partial \rho_{\vec{p}}}{\partial \theta} = \frac{\partial \rho_{\vec{p}}}{\partial \varphi} = 0, \quad \frac{\partial \rho_{\vec{p}}}{\partial \varphi_p} = 0.$$  

(14)

We obtain

$$\frac{\partial \rho_{\vec{p}}}{\partial t} + \cos \theta_p \frac{\partial \rho_{\vec{p}}}{\partial r} + \frac{1}{r} (1 - \cos^2 \theta_p) \frac{\partial \rho_{\vec{p}}}{\partial \cos \theta_p} = -i[H_0 + \sqrt{2} G_F (L + D), \rho_{\vec{p}}].$$

(15)

Alternatively we introduce

$$\mu_p = 1 - \cos \theta_p.$$  

(16)

Eq. (15) becomes

$$\frac{\partial \rho_{\vec{p}}}{\partial t} + (1 - \mu_p) \frac{\partial \rho_{\vec{p}}}{\partial r} - \frac{1}{r} \mu_p (2 - \mu_p) \frac{\partial \rho_{\vec{p}}}{\partial \mu_p} = -i[H_0 + \sqrt{2} G_F (L + D), \rho_{\vec{p}}].$$

(17)

$\rho_{\vec{p}} = \rho_{\vec{p}}(t, r, |\vec{p}|, \mu_p)$ is the density matrix at time $t$ and radius $r$. $|\vec{p}|$ and $\mu_p$ are variables used for describing distribution of neutrinos in momentum space. $\mu_p$ takes value in the range $[0, 2]$.

We note that for a realistic distribution of supernova neutrinos the density matrix $\rho_{\vec{p}}$ should be a smooth function of $|\vec{p}|$ and $\mu_p$. Moreover, $\rho_{\vec{p}}$ should decrease to zero as $\mu_p \to 1$, i.e. as $\theta_p \to \pi/2$. This is because no neutrinos inside the neutrino sphere can be emitted in the direction with $\theta_{p0} \geq \pi/2$ on the neutrino sphere where $\theta_{p0}$ is the emission angle on the neutrino sphere as shown in Fig. 1. Neutrinos moving out of the supernova have $\theta_p < \pi/2$ for $r \geq r_0$. This is expressed by the following condition for the density matrix

$$\rho_{\vec{p}}(t, r, |\vec{p}|, \mu_p) = 0, \quad \text{for } \mu_p \geq 1 \text{ and } r \geq r_0.$$  

(18)

This condition will be used when we do moment expansion of equation.

A simple application of Eq. (15) is for the total intensity of neutrino

$$f_{\vec{p}} = Tr[\rho_{\vec{p}}],$$

(19)

where $Tr$ is the trace of matrix. The right-hand side of Eq. (14) is a commutator of matrices and its trace is zero. We find

$$\frac{\partial f_{\vec{p}}}{\partial t} + \cos \theta_p \frac{\partial f_{\vec{p}}}{\partial r} + \frac{1}{r} (1 - \cos^2 \theta_p) \frac{\partial f_{\vec{p}}}{\partial \cos \theta_p} = 0.$$  

(20)

$^*\theta_{p0} = \pi/2$ at $r = r_0$ means that it is a trajectory passing through the surface of the sphere. No neutrino can be emitted in this direction on the neutrino sphere.
We integrate Eq. (20) by \( \int d\Omega \vec{p} \) which is over all solid angle. Since the integration limit does not depend on \( r \), we can change the order of \( \frac{\partial}{\partial r} \) and the integration symbol \( \int d\Omega \vec{p} \).

After integration by part using condition Eq. (18) we find

\[
\frac{\partial f_0^p}{\partial t} + \frac{\partial (f_0^p \beta)}{\partial r} + \frac{2}{r} f_0^p \beta = 0,
\]

where

\[
f_0^p = \int d\Omega \vec{p} f\vec{p}, \quad f_0^p \beta = \int d\Omega \vec{p} \cos \theta f\vec{p}.
\]

The simplest case is that neutrinos are all emitted in radial direction at neutrino sphere. Hence \( \beta = 1 \) and Eq. (21) is simplified as

\[
\frac{\partial f_0^p}{\partial t} + \frac{1}{r^2} \frac{\partial (r^2 f_0^p)}{\partial r} = 0.
\]

The solution of this equation is

\[
f_0^p(t, r) = \frac{r_0^2}{r^2} f_0^p(t - (r - r_0), r_0).
\]

\( r_0 \) is the radius of neutrino sphere. The meaning of this solution is obvious. Intensity of neutrino at time \( t \) and radius \( r \) is directly related to the neutrino intensity on neutrino sphere at time \( t - (r - r_0) \). Intensity is suppressed by the geometric factor \( \frac{r_0^2}{r^2} \), as expected in a spherically symmetric system.

An interesting observation based on Eq. (24) is that if emission of neutrino on neutrino sphere (at \( r = r_0 \)) has very weak dependence on time, \( \frac{\partial f_0^p}{\partial t} = 0 \) is obtained for all \( r > r_0 \). It means Eq. (24) gives a stationary distribution of neutrino density in space. We denote the approximation \( \frac{\partial f_0^p}{\partial t} = 0 \) as the stationary approximation. It is very useful for simplifying the moment equations. We come back to this approximation later.

## 4 Moments of density matrix

In this section we examine in detail the geometric characteristics of neutrino transport in supernova. We introduce moments of density matrix and check their convergence properties. We motivate moment expansion of Eq. (17).

Consider a neutrino emitted at neutrino sphere at position \( \vec{x}_0 = (r_0, \theta_0, \varphi_0) \) with an emission angle \( \theta_{0p} \) intersecting with the radial direction. It is shown in Fig. 1. At time \( t \) neutrino arrives at position \( \vec{x} = (r, \theta, \varphi) \) with an angle \( \theta_p \) intersecting with the radial direction \( \hat{r} \). From Fig. 1 one can see that

\[
\sin \theta_p = \frac{r_0}{r} \sin \theta_{0p}.
\]

\(^1\)More discussion on this point is given in section 5.
From this we get
\[
\cos \theta_p = \sqrt{1 - \frac{r_0^2}{r^2} \sin^2 \theta_{p0}}
\] (26)

Neutrinos are more peaked in radial direction if \( r \) is larger. We can see that at radius \( r \) the condition Eq. (18) says that
\[
\rho_p = 0, \text{ for } \sin \theta_p \geq \frac{r_0}{r} \text{ or } \cos \theta_p \leq \sqrt{1 - \frac{r_0^2}{r^2}}.
\] (27)

Alternatively Eq. (27) is re-expressed as
\[
\rho_p = 0, \text{ for } \mu_p \geq S(r) \text{ at radius } r,
\] (28)
where \( S(r) \) is the geometric scaling factor:
\[
S(r) = 1 - \sqrt{1 - \frac{r_0^2}{r^2}} = \frac{r_0^2/r^2}{1 + \sqrt{1 - \frac{r_0^2}{r^2}}}.
\] (29)

\( S(r) \) scales as \( r^{-2} \).

We introduce \( k \)th moment of the density matrix:
\[
\rho_k(t, r, p = |\vec{p}|) = \int d\Omega_p \mu_p^k \rho_p(t, r), \ k = 0, 1, 2 \cdots.
\] (30)

The integration is over all solid angle of neutrinos. It’s easy to see that \( \rho_k \) converge very fast as neutrinos propagate out of supernova. Note that \( \int_{-1}^1 d\cos \theta_p = \int_0^2 d\mu_p \) and using (28) we find
\[
\rho_k(t, r, p) = \int_0^2 d\mu_p \int_0^{2\pi} d\varphi_p \mu_p^k \rho_p = \int_0^{S(r)} d\mu_p \int_0^{2\pi} d\varphi_p \mu_p^k \rho_p.
\] (31)

Because \( \rho_p = 0 \) for \( \mu_p \geq S(r) \) the integration over \( \mu_p \) is effectively restricted to the range \([0, S]\). \( \mu_p^k \) factor in the integration gives a strong suppression. The larger the number \( k \) is, the stronger the suppression is. From Eq. (31) we note that \( \rho_k \) is suppressed roughly by \( S^{k+1} \):
\[
\rho_k \propto S^{k+1}.
\] (32)

\( \rho_0 \) scales as \( S(r) \), i.e. approximately \( r^{-2} \), if neglecting effect of flavor conversion. This is the geometric factor shown in Eq. (24) for the total intensity. Comparing with \( \rho_0 \) \( \rho_k \) is further suppressed by the factor \( \mu_p^k \) in the integration and scales as \( S^{k+1}(r) \), i.e. approximately \( r^{-2(k+1)} \). Hence at large \( r \) higher moments can be safely neglected. It’s clear that this set of moments has very good convergence property. If we use this set of moments to expand Eq. (17) we should have a series of equations which has very good convergence property.

Moments given in Eq. (30) are very nice quantities supernova offered to us in studying flavor conversion of neutrinos. In section 5 we will use this series of moments to expand Eq. (17).
5 Moment expansion of the equation of motion

In this section we derive moment equations describing the transport and flavor transformation of neutrinos in supernova. First, we examine the right-hand side of Eq. (17) and express \( D \), the self-interaction term, in terms of \( \rho_k \). Notice that Eq. (14) says

\[
\int d^3 q \, \rho_{\vec{q}} (1 - \cos \theta_{\vec{q}}) = \int d^3 q \, \rho_{\vec{q}} (1 - \cos \theta_q \cos \theta_p),
\]

(33)

where \( \theta_q \) and \( \theta_p \) are angles of \( \vec{q} \) and \( \vec{p} \) separately. Using

\[
1 - \cos \theta_q \cos \theta_p = (1 - \cos \theta_q) + (1 - \cos \theta_p) - (1 - \cos \theta_q)(1 - \cos \theta_p),
\]

(34)

we can get

\[
D = (n_0 - \bar{n}_0) \mu_p + (n_1 - \bar{n}_1)(1 - \mu_p),
\]

(35)

where

\[
n_0(t, r) = \int \frac{d^3 q}{(2\pi)^3} \rho_{\vec{q}} = \int \frac{d|\vec{q}|}{(2\pi)^3} |\vec{q}|^2 \rho_0(t, r, |\vec{q}|).
\]

(36)

\[
n_1(t, r) = \int \frac{d^3 q}{(2\pi)^3} (1 - \cos \theta_q) \rho_{\vec{q}} = \int \frac{d|\vec{q}|}{(2\pi)^3} |\vec{q}|^2 \rho_1(t, r, |\vec{q}|).
\]

(37)

Similarly for \( \bar{n} \) and \( \bar{\rho} \).

We integrate Eq. (17) using \(
\int d\Omega_{\vec{p}} \, \mu_p^k = \int_0^{2\pi} d\varphi_p \int_0^2 d\mu_p \, \mu_p^k
\)

which is over all solid angle of neutrinos. Since the integration limit does not depend on \( r \) we find that

\[
\int d\Omega_{\vec{p}} \, \mu_p^k \frac{\partial}{\partial r} \rho_{\vec{p}} = \frac{\partial}{\partial r} (\int d\Omega_{\vec{p}} \, \mu_p^k \, \rho_{\vec{p}}).
\]

We do integration by part using the condition Eq. (18) which is universal for all \( r \geq r_0 \). We get

\[
\frac{\partial \rho_k}{\partial t} + \frac{\partial(\rho_k - \rho_{k+1})}{\partial r} + \frac{1}{r}[2(k + 1)\rho_k - (k + 2)\rho_{k+1}] = -i[H_A, \rho_k] - i[H_B, \rho_{k+1}],
\]

(38)

where

\[
H_A = H_0 + \sqrt{2}G_F(L + D_1), \quad H_B = \sqrt{2}G_F D_0,
\]

(39)

and

\[
D_0 = n_0 - \bar{n}_0 - (n_1 - \bar{n}_1), \quad D_1 = n_1 - \bar{n}_1.
\]

\[\text{‡}\] Since \( \rho_{\vec{p}} \) vanishes in some region effectively there is an integration limit which depends on \( r \) as shown in Eq. (31). This effective limit does not change the fact that the differential and integral operators can be interchanged. It can be checked as follows. \( \int d\mu_p \mu_p^k \frac{\partial}{\partial \mu_p} \rho_{\vec{p}} = \int d\mu_p \, \mu_p^k \frac{\partial}{\partial \mu_p} \rho_{\vec{p}}|_{\mu_p = S(r)} = \int d\mu_p \mu_p^k \frac{\partial}{\partial \mu_p} \rho_{\vec{p}} \). In the last step \( \rho_{\vec{p}}(r, \mu_p = S(r)) = 0 \) from Eq. (28) has been used.
In the remaining part of the article we will also use
\[ H_C = H_A + H_B \] (41)

Eq. (38) can be rewritten as
\[ \frac{\partial \rho_k}{\partial t} + \frac{1}{r^{2(k+1)}} \frac{\partial}{\partial r} (r^{2(k+1)} \rho_k) - \frac{1}{r^{k+2}} \frac{\partial}{\partial r} (r^{k+2} \rho_{k+1}) = -i[H_A, \rho_k] - i[H_B, \rho_{k+1}]. \] (42)

For \( k = 0 \) and \( k = 1 \) Eq. (42) gives
\[ \frac{\partial \rho_0}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho_0) - \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho_1) = -i[H_A, \rho_0] - i[H_B, \rho_1], \] (43)
\[ \frac{\partial \rho_1}{\partial t} + \frac{1}{r^4} \frac{\partial}{\partial r} (r^4 \rho_1) - \frac{1}{r^3} \frac{\partial}{\partial r} (r^3 \rho_2) = -i[H_A, \rho_1] - i[H_B, \rho_2]. \] (44)

If neglecting other terms the first two terms in left-hand side of Eq. (43) make \( \rho_0 \) scales as \( r^{-2} \) which is the correct geometric factor as shown in Eq. (32). The presence of \( \rho_1 \) says that the average velocity of neutrino gas is smaller than the speed of light. It modifies the scaling behavior of \( \rho_0 \). The first two terms in the left-hand side of Eq. (44) give the geometric scaling factor \( r^{-4} \) as observed in Eq. (32). The presence of \( \rho_2 \) in the left-hand side of Eqs. (44) modifies the scaling behavior of \( \rho_1 \). Similarly, the left-hand side of Eq. (38) makes \( \rho_k \) scale as \( r^{-2(k+1)} \) if neglecting \( \rho_{k+1} \) and the presence of \( \rho_{k+1} \) modifies the scaling behavior.

Eq. (38) systematically takes into account the effect of the angular distribution of neutrino emission. The effect is encoded in moments introduced in the present article. In particular, the scaling law of the strength of the effective Hamiltonian is modified when including higher moments \( \rho_k \) with \( k \geq 2 \). Furthermore, this series of moment equations has very good property of convergence and is better for studying flavor transformation of neutrino in supernova, for studying non-linear flavor transformation in particular. This virtue will help us to truncate the infinite series moment equations and will be further discussed in section 7.

An important point when studying effect of self-interaction in flavor conversion is that the minimal set of equations should include equations for \( \rho_0 \) and \( \rho_1 \), Eqs. (43) and (44), because if neutrinos are all emitted in radial direction, that is \( \rho_k = 0 \) for \( k > 0 \), neutrino self-interaction vanishes.

6 Stationary approximation

In solving the equations of neutrino evolution in supernova we assume the stationary approximation:
\[ \frac{\partial \rho_k}{\partial t} = 0. \] (45)
This approximation is assumed by noticing that the neutrino self-interaction is important in the region of radius less than hundreds kilometers in which neutrino density is sufficiently large. Neutrino travels through this region in $\sim 10^{-3}$ s. This time scale is much shorter than the time scale of neutrino emission in supernova which is around 10s. Therefore, we can assume that neutrino luminosity and energy do not change in the time scale as short as around $\sim 10^{-3}$ s. Furthermore, we can also assume that the distribution of ordinary matter in supernova does not change in such a short time scale. This assumption is valid if the disturbance of ordinary matter propagates with a speed much smaller than the speed of light.

An example supporting this assumption is given by examining neutrino intensity given in Eq. (24). According to the formula it is easy to see that if $\frac{\partial f}{\partial t} = 0$ is assumed at the neutrino sphere where $r = r_0$, $\frac{\partial f}{\partial t} = 0$ is satisfied at $r > r_0$.

We introduce $\rho'_k = z^{2(k+1)} \rho_k$, $\bar{\rho}'_k = z^{2(k+1)} \bar{\rho}_k$  

\[ \rho'_k = z^{2(k+1)} \rho_k, \quad \bar{\rho}'_k = z^{2(k+1)} \bar{\rho}_k \tag{46} \]

where

\[ z = \frac{r}{r_0}, \quad \tag{47} \]

$\rho'_k$ and $\bar{\rho}'_k$ correctly take into account the geometric scaling factor.

With the assumption of stationary approximation for the density matrix we get equation

\[ \frac{d\rho'_k}{dr} = z^k \frac{d}{dr} (z^{-(k+2)} \rho'_{k+1}) - i[H_A, \rho'_k] - i[H_B, z^{-2} \rho'_{k+1}], \quad k = 0, 1 \cdots \tag{48} \]

Using Eq. (48) repeatedly we can get

\[ \frac{d\rho'_k}{dr} = -i[H_A, \rho'_k] + z^{-n+1} \frac{d}{dr} (z^{-(k+n+1)} \rho'_{k+n}) - i[H_B, z^{-2n} \rho'_{k+n}] 
- r_0^{-1} \sum_{i=1}^{n-1} (k+i+1) z^{-(2i+1)} \rho'_{k+i} - i[H_C, \sum_{i=1}^{n-1} z^{-2i} \rho'_{k+i}], \tag{49} \]

where $n \geq 2$. For $n = 1$ the second line in Eq. (49) disappears and the equation reduces to Eq. (48). Equation of $\bar{\rho}'$ can be similarly obtained for anti-neutrinos.

7 Truncation of moment equations

In practical computation we have to truncate the infinite series of moment equations. If magnitude of higher moments are much smaller than $\rho_{0,1}$ we can assume that $\rho_k = 0$ for $k > N$ where $N$ is an integral. In this approximation we get a set of $2(N+1)$
equations for $\rho_k(\hat{\rho}_k)$ or $\rho'_k(\hat{\rho}'_k)$ where $k = 0, 1, \cdots, N$. We denote this approximation as $P_N$ approximation. In $P_N$ approximation we get

$$\frac{d\rho'_k}{dr} = -r^{-1}Q^1_k \cdot i[H_A, \rho'_k] - i[H_C, Q^2_k],$$

(50)

where $k = 0, 1, \cdots, N$ and

$$Q^1_k = z_k^2 \sum_{l=k+1}^{N} (l + 1)z^{-(2l+1)}\rho'_l, \quad Q^2_k = z_k^2 \sum_{l=k+1}^{N} z^{-2l}\rho'_l.$$

(51)

$Q^1_N = 0$. Similarly we have equations for $\hat{\rho}'_k$.

An appropriate choice of truncation depends both on the initial condition and on the property of convergence of these moments. It is interesting to note that higher moments are naturally suppressed at the neutrino sphere. For example,

$$\rho_k(t, r_0) = \frac{1}{k + 1} \rho_0(t, r_0)$$

(52)

is obtained using Eq. (50) if neutrino is uniformly emitted with respect to the emission angle $\theta_{p0}$. $\boxed{\rho_{10}}$ is about 10 times smaller than $\rho_0$. For a practical distribution which may have sharp falloff at an angle close to $\theta_{p0} = \pi/2 (\mu_{p0} = 1)$, higher moments are further suppressed. As an example, suppose the distribution is uniform in the region $0 \leq \mu_{p0} \leq 0.9$ and drops to zero rapidly at $\mu_{p0} = 0.9$ we find that

$$\rho_{10}(t, r_0) \approx \frac{1}{11} 0.9^{10} \rho_0(t, r_0) \approx 3\% \rho_0(t, r_0),$$

(53)

$$\rho_{10}(t, r_0) \approx \frac{2}{11} 0.9^9 \rho_1(t, r_0) \approx 7\% \rho_1(t, r_0)$$

(54)

$\rho_{10}$ has magnitude smaller than $\rho_0$ and $\rho_1$, the major quantities in the evolution problem.

Other examples of distribution can also be checked. Considering distribution proportional to $1 - \mu_{p0}$ or $\mu_{p0}(1 - \mu_{p0})$ which are zero at $\mu_{p0} = 1$ we find for these two examples

$$\rho_k(t, r_0) = \frac{2}{(k + 1)(k + 2)} \rho_0(t, r_0),$$

(55)

$$\rho_k(t, r_0) = \frac{6}{(k + 1)(k + 2)} \rho_1(t, r_0).$$

(56)

As noted previously, for realistic distribution of neutrinos $\rho_{\hat{p}}(r_0)$ should decrease to zero as $\theta_{p0} \to \pi/2$. Uniform distribution can be taken as an approximation to a distribution which is uniform in the range $0 \leq \mu_{p0} \leq 1 - \epsilon$ and decreases to zero rapidly in the range $1 - \epsilon \leq \mu_{p0} \leq 1$ where $\epsilon \ll 1$ is taken as a small positive number. This is a good approximation for not too large $k$. 
or
\[
\rho_k(t, r_0) = \frac{6}{(k + 2)(k + 3)} \rho_0(t, r_0),
\]
(57)
\[
\rho_k(t, r_0) = \frac{12}{(k + 2)(k + 3)} \rho_1(t, r_0).
\]
(58)

We find that moments of order larger than 10 are suppressed in these models of neutrino emission. The precise number of moments required in study depends on the model of emission angle distribution of neutrinos on neutrino sphere.

The idea that the series of moment equations can be truncated to a small set of equations is further supported by examining the geometric scaling property of higher moments. For example, according to geometric considerations \(\rho_2\) and \(\rho_3\) are suppressed by factors about \(1/3^2\) and about \(1/3^4\) compared to \(\rho_1\) at \(r = 3r_0\) (\(\sim 30\) km) if assuming \(\rho_k \sim \rho_0\) on the neutrino sphere. A numerical study presented in section 3 shows that \(P_5\) approximation is already quite good in the model described by Eq. (55). More numerical analysis on the truncation of moment equations will be presented in other publications.

We expect that a \(P_{10}\) approximation is probably enough to describe many phenomena in the transport and flavor transformation of supernova neutrinos.

The numerical analysis is substantially simplified when using Eq. (49) in \(P_N\) approximation when \(N\) is at most as large as around ten. In \(P_N\) approximation the required computational power is proportional to numbers of equations: \(2(N + 1)N_e\) where \(N_e\) is the number of energy bins. In contrast, in the multi-angle simulation the number of equations in evolution is \(6N_eN_\theta\) where \(N_\theta\) is the number of angle bins and the factor 6 is the number of all types of neutrinos and anti-neutrinos. It is a huge amount of equations by noticing that \(N_e, N_\theta \geq 500\) for \(L_\nu = 10^{51}\) erg/s \([4]\). More angle bins are required for larger neutrino luminosity in multi-angle simulation. The approach using moment equations reduces the computational power by about two orders of magnitude when \(N\) is around ten.

8 Numerical result

In this section we show some numerical analysis on the non-linear flavor transformation of neutrinos. For simplicity we neglect matter effect in our analysis.

The truncated set of Equations (50) should be carefully treated in numerical analysis. Three terms in the right hand side of Eq. (50) have different physics. The first term is caused by the diffusion of neutrinos and gives correction to the scaling law of moments. For example, \(Q_0^1\) modifies the scaling of the total flux. The second term leaves \(Tr[\rho_k^2], Tr[(\rho_k^3)^2]\) and \(Tr[(\rho_k^3)^3]\) etc invariant. It is responsible for the unitary flavor evolution of moments. The third term does not change \(Tr[\rho_k^2]\) but modifies \(Tr[(\rho_k^3)^2], Tr[(\rho_k^3)^3]\) etc. This term gives non-unitary evolution of neutrino flavor. The appearance of the non-unitary term in moment equations makes it hard to do numerical computations.
In the present analysis we simplify the numerical computation and search a self-consistent solution to the moment equations (50). We assume that the evolution is dominated by the first and the second term in the right hand side of Eq. (50). We search for solution of the following form

$$\rho_k' = \rho_k^0 + \Delta \rho_k'$$ \hspace{1cm} (59)

In this approximation $Q_{k}^{1,2}$ and $H_{A,C}$ are all expressed using $\rho_k^0$:

$$Q_{k}^{1,2} = Q_{k}^{1,2}(\rho_k^0), \quad H_{A,C} = H_{A,C}(\rho_k^0).$$ \hspace{1cm} (60)

$\rho_k^0$ satisfies

$$\frac{d\rho_k^0}{dr} = -r_0^{-1}Q_1^k(\rho_k^0) - i[H_A, \rho_k^0]$$ \hspace{1cm} (61)

For a small step we get

$$\rho_k^0(r + \Delta r) = -\frac{\Delta r}{r_0}Q_1^k(\rho_k^0) + e^{-iH_A\Delta r} \rho_k^0(r)e^{iH_A\Delta r}$$ \hspace{1cm} (62)

$\Delta \rho_k'$ is considered as a small correction and its initial condition is set to zero. It is sourced by $\rho_k^0$:

$$\frac{d\Delta \rho_k'}{dr} = -i[H_A, \Delta \rho_k'] - i[H_C, Q_2^k(\rho_k^0)].$$ \hspace{1cm} (63)

Eq. (63) can be re-written as

$$\frac{d\Delta \rho_k''}{dr} = -i[H'_C, Q_2^k(\rho_k''^0)],$$ \hspace{1cm} (64)

where

$$\Delta \rho_k'' = M_A^\dagger \Delta \rho_k'M_A, \quad \rho_k''^0 = M_A^\dagger \rho_k^0'M_A, \quad H'_C = M_A^\dagger H_CM_A.$$ \hspace{1cm} (65)

$M_A = M_A(\rho_k^0)$ is the evolution matrix given by Hamiltonian $H_A(\rho_k^0)$:

$$\frac{dM_A}{dr} = -iH_AM_A.$$ \hspace{1cm} (66)

Initial condition $M_A = 1$ is set. For a small step Eq. (64) gives

$$\Delta \rho_k''(r + \Delta r) - \Delta \rho_k''(r) = e^{-iH_C\Delta r}Q_2^k e^{iH_C\Delta r} - Q_2^k.$$ \hspace{1cm} (67)

We expect $\Delta \rho_k'$ receive small contributions from $\rho_k^0$ and its effect on the final result is negligible. Including $\Delta \rho_k'$ in $Q_{k}^{1,2}$ and $H_{A,C}$ introduces even smaller corrections. In
Figure 3: (color online) Fraction of $\nu_e$, $n_{\nu_e}/(n_{\nu_e} + n_{\nu_x})$, versus radius $r$ in two neutrino system. Two lines are for results with and without $\Delta \rho'_k$ correction separately. $|\Delta m^2_{31}| = 3 \times 10^{-3}$eV$^2$, $\sin^2 2\theta_{13} = 0.01$.

Numerical analysis the consistency of this approximation should be checked. That is, including $\Delta \rho'_k$ should not give much modification to $\rho'_k$.

This approximation is supported by the result of numerical computation. A result of numerical computation using this approximation is shown in Fig. 3. It is shown for two neutrino system of $(\nu_e, \nu_x)$ with inverted mass hierarchy and for emission angle distribution described by Eq. (55). We choose $L_{\nu_e} = L_{\bar{\nu}_e} = L_{\nu_x} = L_{\bar{\nu}_x} = 3 \times 10^{51}$ erg/s. The initial energy spectrum of neutrino is given by the Fermi-Dirac distribution

$$f_{\nu}(E) = \frac{1}{F_2 T_{\nu} e^{x - \mu_{\nu_e}} + 1}, \quad (68)$$

where $x = E/T_{\nu}$ and $F_2$ is the normalization factor. Parameters of four types of neutrinos are chosen as: $T_{\nu_e} = 2.76$ MeV, $T_{\bar{\nu}_e} = 4.01$ MeV, $T_{\nu_x} = T_{\bar{\nu}_x} = 6.26$ MeV. $\mu_{\nu_e} = \mu_{\bar{\nu}_e} = \mu_{\nu_x} = \mu_{\bar{\nu}_x} = 3$. In Fig. 3 one can see clearly the synchronized oscillation and the transition to bipolar oscillation.

We show both result with $\Delta \rho'_k$ corrections and the result without $\Delta \rho'_k$ corrections. As can be seen these two results agree very well. $\Delta \rho'_k$ corrections give negligible corrections to the neutrino evolution. This can be understood by noticing that in the small $r$ region it is dominated by synchronized oscillation where all $\rho'_k$ point to the same direction in flavor space and they commute with the effective Hamiltonian $H_C$. Hence effect of the third term in Eq. (50) does not change synchronized oscillation and is not important in small $r$ region. In the large $r$ region the magnitude of $Q^2_k$ is suppressed by $r_0^2/r^2$ and is again not important. This result shows that in this approximation we can neglect the third term in Eq. (50) and use the following equation in our analysis

$$\frac{d\rho'_k}{dr} = -r_0^{-1}Q^1_k - i[H_A, \rho'_k], \quad (69)$$
9 Equations in two flavor system

In two flavor system of neutrinos the density matrix can be expressed in terms of the unit matrix and the Pauli matrices. It’s much easier to do analytic study in two flavor system. In this section we re-write the moment equations for two flavor system.

For two flavors of neutrinos we write

\[
H_0 = \frac{1}{2} (\omega_0 + \omega E \vec{B} \cdot \vec{\sigma}), \quad L = \frac{1}{2} (L^0 + \vec{L} \cdot \vec{\sigma}),
\]

\[
E^2 \rho_k^{'}(E) = \frac{1}{2} (P_k^0 + \vec{P}_k \cdot \vec{\sigma}), \quad E^2 \tilde{\rho}_k^{'}(E) = \frac{1}{2} (P_k^{00} + \vec{P}_k^0 \cdot \vec{\sigma}).
\]

where \( E \) is the energy of neutrino and \( \vec{\sigma} = (\sigma^1, \sigma^2, \sigma^3) \) are Pauli matrices.

Equations are obtained when using commutation relation of the Pauli matrices. Neglecting moments of \( k > 1 \) and the third term in the right hand side of Eq. \((50)\) we get
in the mass base of neutrino

\[
\frac{d \vec{P}_0}{dr} = -2r_0^{-1} z^{-2} \vec{P}_1 + [\omega_E \vec{B} + \sqrt{2} G_F (\vec{L} + z^{-4} \vec{N}_1)] \times \vec{P}_0,
\]

(72)

\[
\frac{d \vec{P}_0'}{dr} = -2r_0^{-1} z^{-2} \vec{P}_1' + [-\omega_E \vec{B} + \sqrt{2} G_F (\vec{L} + r^{-4} \vec{N}_1)] \times \vec{P}_0',
\]

(73)

\[
\frac{d \vec{P}_1}{dr} = (\omega_E \vec{B} + \sqrt{2} G_F \vec{L} + \sqrt{2} G_F z^{-4} \vec{N}_1) \times \vec{P}_1,
\]

(74)

\[
\frac{d \vec{P}_1'}{dr} = (-\omega_E \vec{B} + \sqrt{2} G_F \vec{L} + \sqrt{2} G_F z^{-4} \vec{N}_1) \times \vec{P}_1'.
\]

(75)

where \( \omega_E = \Delta m^2 / 2E \) and

\[
\vec{N}_1 = \int \frac{dE}{(2\pi)^3} (\vec{P}_1 - \vec{P}_1').
\]

(76)

Eqs. (72), (73), (74) and (75) form a closed set of equations.

Eqs. (72), (73), (74) and (75) are equations we obtained for describing the pendulum motion of neutrinos in flavor space. Eqs. (74) and (75) are the equations widely used in two flavor analysis in literature [5]. An improvement of Eqs (74) and (75) is that the dependence on the radius \( r \) is explicitly derived. Correction to the scaling law of the Hamiltonian \( z^{-4} \vec{N} \) can be included when higher moments are included. This is another virtue using moment equations.

We note that an important piece of this set of equations, Eqs. (72) and (73), are missed in the literature. Actually what have been missed are more relevant to physical observables. Neutrino densities are described by \( P_0^0(P_0^0) \) and \( P_0^3(P_0^3) \). The dynamics of non-linear flavor transformation is controlled by \( \vec{P}_1 \) and \( \vec{P}_1' \) which are not real observables. Predictions of these two new equations on the flavor transformation should be explored.

It should be reminded that Eqs. (74), (75), (72) and (73) may not be reliably used in precise numerical analysis since moments of order \( k > 1 \) have all been neglected when deriving them. They can be used in qualitative analysis. We note that including the third term in the right hand side of Eq. (50) leads to extra term in pendulum equation of neutrinos.

10 Conclusion

In summary we derive a series of moment equations describing the transport and flavor transformation of neutrinos above neutrino sphere in core-collapse supernova. We examine the system of neutrinos in supernova using spherical coordinate and find a particular set of moments of the density matrix of neutrino. We expand the Liouville equation of neutrinos
using this series of moments and obtain the moment equations. We work with assumptions that the system is spherically symmetric and stationary in time scale around $10^{-3}s$. We arrive at a truncated set of moment equations (50). After a further approximation we get (69). Rates of neutrino flavor transformation can be solved as functions of radius $r$ using this set of truncated moment equations. These equations will be very helpful to future researches on supernova neutrinos [24].

We study geometric scaling properties of these moments in the expanding system of supernova and find that they have very good property of convergence. Higher moments $\rho_k(k > 1)$ converge to zero much faster than lower moments $\rho_{0,1}$. Furthermore, we check initial conditions of moments and find that magnitudes of higher moments are naturally smaller than $\rho_{0,1}$ on the neutrino sphere. Based on these observations we argue that the infinite series of moment equations can be safely truncated to be a small set of equations with $\rho_k(k \lesssim 10)$. Numerical analysis shows that the results in $P_N$ approximation indeed converge for $N \lesssim 10$. In Fig. 3 and Fig. 4 we show some numerical analysis and find patterns of synchronized oscillation and bipolar oscillation.

The truncated set of moment equations have a number of good properties. First, the truncated set of equations tremendously simplify numerical analysis of the problem. Previous numerical works use many angle bins and energy bins to simulate the neutrino evolution. In multi-angle simulation evolutions of more than 500 trajectories have to be followed. Together with more than 500 energy bins and an extra factor 6 (number of total types of neutrinos and anti-neutrinos), more than a million equations have to be solved in this kind of simulation. It is ultra-complicated. The approach presented in this article uses a small set of truncated moments rather than many angle bins and is much simpler than the multi-angle simulation. Using a small set of truncated moment equations the required computational power in numerical analysis is reduced by two orders of magnitude compared to that in multi-angle simulation.

Second, the moment equations systematically take into account the effect of the angular distribution of neutrino emission. As noted in Introduction, angular distribution of neutrino emission is essential for the effect of neutrino self-interaction to play an important in neutrino flavor transformation. The approach presented in the present article introduces moments of density matrix of neutrinos to describe the angular distribution of neutrinos. The equation of zeroth moment naturally includes the effect of higher moments on the evolution of neutrino intensity. The equations of higher moments systematically take into account the evolution of angular distribution of neutrino. In particular, modification to the scaling of the strength of the effective Hamiltonian can be included when including moments $\rho_k$ with $k \geq 2$. The dependence of this modification on the emission angle distribution of neutrinos can be systematically studied. In contrast, previous researches use fixed scaling law (basically $r^{-4}$ law) for the strength of the effective Hamiltonian. This set of moment equations provides a powerful tool towards a complete understanding to the effect of the emission angle distribution in flavor transformation of supernova neutrinos. More analysis on the effect of angular distribution in neutrino oscillation will be presented in other publications.
We also consider two flavor system of neutrinos. Using the truncated set of equations for zeroth and first moments we derive equations describing the pendulum motion of neutrinos in flavor space. In addition to equations given in literature, we find new equations which are important and are more relevant to physical observable.

There are still many problems. More detailed works should be done to check carefully the evolution of higher moments and their effect in convergence of moment equations. More numerical works should be done to analyze in detail the effect of higher moments on phenomena such as synchronized oscillation and bipolar oscillation, spectral split and adiabaticity of neutrino evolution, etc. One important remaining problem is how to extend formulation to the case without the assumption of spherical symmetry. Answer to this problem will enable us to understand the impact of anisotropic disturbance of ordinary matter on neutrino flavor transformation. To answer this question we need to examine Eq. (12) in more detail. Subtle problems in quantum evolution of moment equations should be investigated before we can answer this question. These problems will be explored in future works.

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Appendix

Neglecting the gravitational potential, the invariant distance in spherical coordinate \( x^i = (r, \theta, \varphi) \) is given by

\[
ds^2 = dt^2 - g_{ij} dx^i dx^j,
\]

where \( g_{ij} \) is the metric

\[
g_{ij} = \text{diag}\{1, r^2, r^2 \sin^2 \theta\}.
\]

The Christoffel symbol \( \Gamma^i_{jk} \) is computed using the metric \( g_{ij} \):

\[
\Gamma^i_{jk} = \frac{1}{2} g^{il} \left( \frac{\partial g_{lk}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^l} \right).
\]

We get

\[
\begin{align*}
\Gamma^r_{\theta \theta} &= -r, & \Gamma^r_{\varphi \varphi} &= -r \sin^2 \theta, & \Gamma^\theta_{r \theta} &= \Gamma^\theta_{\theta r} = r^{-1}, \\
\Gamma^\theta_{\varphi \varphi} &= -\sin \theta \cos \theta, & \Gamma^\varphi_{r \varphi} &= \Gamma^\varphi_{\varphi r} = r^{-1}, & \Gamma^\varphi_{\theta \varphi} &= \Gamma^\varphi_{\varphi \theta} = \cot \theta.
\end{align*}
\]

All other \( \Gamma \)'s are zero.

\footnote{Note added: After submission of this article I know from A. Mirizzi that an earlier attempt to do moment expansion of the equation of supernova neutrinos is made in Ref. [25].}
Using Eq. (11) we find

\[
\frac{dp_r}{dt} = \frac{1}{r} \frac{r^2 (p^\theta)^2 + r^2 \sin^2 \theta (p^\phi)^2}{p^0}, \tag{82}
\]

\[
\frac{dp^\theta}{dt} = -\frac{2}{r} \frac{p_r p^\theta}{p^0} + \sin \theta \cos \theta \frac{(p^\phi)^2}{p^0}, \tag{83}
\]

\[
\frac{dp^\phi}{dt} = -\frac{2}{r} \frac{p_r p^\phi}{p^0} - 2 \text{ctg} \theta \frac{p^\theta p^\phi}{p^0}, \tag{84}
\]

It's easy to verify that

\[
\frac{d}{dt} [(p^r)^2 + r^2 (p^\theta)^2 + r^2 \sin^2 \theta (p^\phi)^2] = 0. \tag{85}
\]

This is just the statement that $|\vec{p}| = \sqrt{g_{ij} p^i p^j} = \sqrt{(p^r)^2 + r^2 (p^\theta)^2 + r^2 \sin^2 \theta (p^\phi)^2}$, the magnitude of the momentum, is a constant. It equals to $p^0$ for neutrino with tiny masses. $p^r$, $\sqrt{g_{\theta\theta}} p^\theta = r p^\theta$ and $\sqrt{g_{\phi\phi}} p^\phi = r \sin \theta p^\phi$ are the momenta projected to $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$ directions separately. Hence we can write

\[
p^r = \cos \theta_p \ p^0, \ p^\theta = \frac{1}{r} \sin \theta_p \cos \varphi_p \ p^0, \ p^\phi = \frac{1}{r \sin \theta} \sin \theta_p \sin \varphi_p \ p^0. \tag{86}
\]

Using Eq. (82) and (83) we get

\[
\frac{d \cos \theta_p}{dt} = \frac{1}{r} (1 - \cos^2 \theta_p), \tag{87}
\]

\[
\frac{d \varphi_p}{dt} = -\frac{1}{r \text{ctg} \theta} \sin \theta_p \sin \varphi_p. \tag{88}
\]

The consequence of Eq. (87) is that $\cos \theta_p$ approaches to 1. This is consistent with the physical picture that as neutrinos go out of the supernova their motion gets closer to the radial direction.

**References**

[1] J. T. Pantaleone, Neutrino oscillations at high densities, Phys. Lett. B287, 128 (1992).

[2] R. F. Sawyer, Speed-up of neutrino transformations in a supernova environment, Phys. Rev. D72, 045003(2005)[hep-ph/0503013].

[3] H. Duan, G. M. Fuller and Y. Z. Qian, Collective Neutrino Flavor Transformation In Supernovae, Phys. Rev. D74, 123004 (2006) arXiv:astro-ph/0511275.
[4] H. Duan, G. M. Fuller, J. Carlson and Y. Z. Qian, Simulation of coherent non-linear neutrino flavor transformation in the supernova environment. I: Correlated neutrino trajectories, Phys. Rev. D74, 105014(2006)[arXiv:astro-ph/0606616].

[5] S. Hannestad, G. G. Raffelt, G. Sigl and Y. Y. Y. Wong, Self-induced conversion in dense neutrino gases: Pendulum in flavour space, Phys. Rev. D74, 105010 (2006) [Erratum-ibid. D 76, 029901(2007)]arXiv:astro-ph/0608695.

[6] H. Duan, G. M. Fuller and J. Carlson, Simulating nonlinear neutrino flavor evolution, Comput. Sci. Disc. 1, 015007(2008) [arXiv:0803.3650].

[7] G. G. Raffelt and A. Y. Smirnov, Self-induced spectral splits in supernova neutrino fluxes, Phys. Rev. D76, 081301 (2007) [Erratum-ibid. D77, 029903 (2008) arXiv:0705.1830).

[8] H. Duan, G. M. Fuller, J. Carlson and Y. Q. Zhong, Neutrino Mass Hierarchy and Stepwise Spectral Swapping of Supernova Neutrino Flavors, Phys. Rev. Lett. 99, 241802 (2007) [arXiv:0707.0290].

[9] G. L. Fogli, E. Lisi, A. Marrone and A. Mirizzi, Collective neutrino flavor transitions in supernovae and the role of trajectory averaging, JCAP 0712, 010(2007) [arXiv:0707.1998).

[10] G. L. Fogli, E. Lisi, A. Marrone, A. Mirizzi and I. Tamborra, Low-energy spectral features of supernova (anti)neutrinos in inverted hierarchy, Phys. Rev. D78, 097301(2008) [arXiv:0808.0807].

[11] H. Duan, G. M. Fuller, J. Carlson and Y. Z. Qian, Analysis of Collective Neutrino Flavor Transformation in Supernovae, Phys. Rev. D75, 125005(2007) [arXiv:astro-ph/0703776].

[12] H. Duan, G. M. Fuller and Y. Z. Qian, A Simple Picture for Neutrino Flavor Transformation in Supernovae, Phys. Rev. D76, 085013(2007)[arXiv:0706.4293].

[13] G. G. Raffelt and A. Y. Smirnov, Adiabaticity and spectral splits in collective neutrino transformations, Phys. Rev. D76, 125008(2007) [arXiv:0709.4641].

[14] B. Dasgupta and A. Dighe, Collective three-flavor oscillations of supernova neutrinos, Phys. Rev. D77, 113002(2008) [arXiv:0712.3798].

[15] H. Duan, G. M. Fuller and Y. Z. Qian, Stepwise Spectral Swapping with Three Neutrino Flavors, Phys. Rev. D77, 085016(2008) [arXiv:0801.1363].

[16] B. Dasgupta, A. Dighe, A. Mirizzi and G. G. Raffelt, Spectral split in prompt supernova neutrino burst: Analytic three-flavor treatment, Phys. Rev. D77, 113007(2008) [arXiv:0801.1660].
[17] S. Chakraborty, S. Choubey, B. Dasgupta and K. Kar, Effect of Collective Flavor Oscillations on the Diffuse Supernova Neutrino Background, JCAP **0809**, 013(2008) [arXiv:0805.3131].

[18] J. Gava and C. Volpe, Collective neutrinos oscillation in matter and CP-violation, Phys. Rev. **D78**, 083007(2008) [arXiv:0807.3418].

[19] R. C. Schirato and G. M. Fuller, Connection between supernova shocks, flavor transformation, and the neutrino signal, [arXiv:astro-ph/0205390].

[20] H. Duan, G. M. Fuller, J. Carlson and Y. Z. Qian, Analysis of Collective Neutrino Flavor Transformation in Supernovae, Phys. Rev. **D75**, 125005(2007) [arXiv:astro-ph/0703776].

[21] S. Pastor, G. G. Raffelt and D. V. Semikoz, Physics of synchronized neutrino oscillations caused by self-interactions, Phys. Rev. **D65**, 053011 (2002) [arXiv:hep-ph/0109035].

[22] G. Sigl and G. Raffelt, General kinetic description of relativistic mixed neutrinos, Nucl. Phys. **B406**, 423 (1993).

[23] B. H. J. McKellar and M. J. Thomson, Oscillating doublet neutrinos in the early universe, Phys. Rev. **D49**, 2710 (1994).

[24] M. D. Kistler, H. Yuksel, S. Ando, J. F. Beacom and Y. Suzuki, Core-Collapse Astrophysics with a Five-Megaton Neutrino Detector, [arXiv:0810.1959].

[25] G. G. Raffelt and G. Sigl, Self-induced decoherence in dense neutrino gases, Phys. Rev. D **75**, 083002 (2007) [arXiv:hep-ph/0701182].