A natural semantics for the pullback of fiber bundles of structures

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ABSTRACT
We remark that forcing on fiber bundles of structures of first order languages is not a compatible semantics with the pullback (of fiber bundles). Motivated by a combination of epistemology and geometry, we describe a semantics which behaves well with respect to the pullback. This new semantics uses parallel transport in its definition and allows to introduce two different types of extensions for the formulae: vertical and horizontal extensions.

KEYWORDS
fiber bundles, sheaves, semantics, forcing, Ehresmann connection, parallel transport, epistemology.

1. Introduction
Sheaves of structures on topological spaces correspond to the semantics of Intuitionism (see Caicedo (1995)), located in between of Kripke semantics and topoi logic. This is a paradigm of truth continuity (continuidad veritativa, according to Caicedo (1995)), which means that if a statement is true in a point therefore it continues being true in a neighborhood of that point. There are other similar approaches of sheaves of structures in several logics (e.g., Continuous Logic (Ochoa and Villaveces (2016))), where the key idea is still preserving the truth of statements in a neighborhood of a point. This idea was generalized to sheaves based on some special kind of lattices extending the lattice of open-sets of a topology (e.g., locales and quantales, Borceaux & van-der Bossche (1986); Johnstone (2002)), which still keeps some geometry behind them and correspond to variants of intuitionism and links topoi and quantum logic.

In this paper, we intend to study this idea on (smooth) fiber bundles. Our results appeared as a consequence of playing with a soft epistemological interpretation of Ehresmann connections (see Appendix A) which allows to distinguish, for a given proposition, the observer who claims it, the region of space (space–time) where the proposition is claimed, and the accuracy of the measurement on which the proposition is based.
holds (see Section 2).

Examples 5.1, 5.9 and 5.11 show that the pullback (see Appendix A) is incompatible with respect to classical forcing (see Remark 5.3). In geometry, the pullback is a very important operation between fiber bundles. For example, it classifies vector bundles over a given topological space $X$ (let say compact and Hausdorff), explicitly homotopy classes of continuous functions from $X$ to the Grassmannian correspond to isomorphic classes of vector bundles (see [Hatcher, 2003, Theorem 1.16]). It also describes elements of the K–theory of $X$ because they are the pullback of the canonical virtual class over the Fredholm operators according to Atiyah–Janich theorem (see [Atiyah, 1967, Theorem A1]).

Since the pullback of a fiber bundle is an important geometric operation and it is compatible with the notion of fiber bundles of structures (see Definition 3.1 and Proposition 3.9), it is natural to look for a semantics which is compatible with the pullback (in Remark 5.3 we are more explicit about what this compatibility means). To do so, we involve our epistemological interpretation of (Ehresmann) connections into the game (see Section 2), and we introduce the notion of parallel semantics, Definition 6.1. We find out that parallel semantics is compatible with the pullback (see Theorem 6.4). In this semantics, the continuity of the truth is defined via curves that play the role of observers moving in space (space–time). As expected from our epistemological motivation (see Section 2), parallel semantics allows to distinguish three new aspects associated to truth continuity: space-time stability (truth continuity a la Caicedo), preservation of truth of statements through the observer movement in space-time (during a time interval) and stability of the ”experimental measure” made by the observer, which we can think as related with the accuracy of the measurements done by the observer.

To our knowledge, the interaction which we use in this article between Differential Geometry and Mathematical Logic is novel. There have been interaction in other directions. For example, the interaction between Complex Geometry and Model Theory had been explored by multiple authors (see [Moosa and Pillay (2008) and references there in], more recently between Differential Geometry and Mathematical Logic (see [Heller & Krol (2016)].

In Section 2 we present the (epistemological) intuition that lead us to define the parallel semantics in Section 6. In Section 3 we define fiber bundles of structures and we establish its compatibility with the pullback (see Proposition 3.9). In Section 4, we define forcing on fiber bundles in the line of previous work on sheaves (see [Caicedo (1995)]). Section 5 illustrates through examples and propositions where in the complexity of the formula the pullback becomes incompatible with forcing (Remark 5.3), we prove that the forcing of a formula without free variables obtained as quantification of an atomic formula is compatible with the pullback (see Proposition 5.7). It is is Section 6 where the differential structure, through the connection, enters in the game defining what we call parallel semantics. In Section 7 using the notion of parallel sections associated to a connection (see Definition A.23), we write some results that show how forcing and parallel forcing can be related. The fact that the curvature of the connection is 0 plays an important role in these propositions, so we can say that the curvature is an obstruction to establish a relation between forcing and parallel forcing (see Corollary 7.8). In Section 8 we explain how
the connection on a fiber bundle allows us to define three different types of extension of a formula: the spatial, the horizontal and the vertical extension. Finally, since this article involves Mathematical Logic and Differential Geometry, which might be considered disconnected branches of mathematics, we include non exhaustive appendixes in both subjects at the end of this article.

Along this article, $M$ and $N$ will denote (smooth) manifolds, $\pi : A \to M$ will denote a fiber bundle and $\mathcal{L}$ will denote a first order language.

2. Epistemological motivation

One of the epistemological motivations of [Caicedo, 1995] for introducing forcing on sheaves is the fact that the subjects of propositions should be extended or variables. This extension or variation of the subjects of propositions is based on the (intuitive) idea that objects and situations in the world are presented to us as extended in space and time. According to this, for science or conversation, there is no point-wise subject of propositions, neither instant phenomenon, because subjects and phenomenons should occupy a detectable region of space and time. However this notion of extension of the subjects and phenomenons is not defined formally and is left as a soft intuitive motivation in [Caicedo, 1995].

In this article, we would like to distinguish two extensions for the subjects of a proposition that were not considered in [Caicedo, 1995]. First, we heuristically think as subjects of propositions the measurements of observers and we distinguish for them what we call horizontal and vertical extensions. We think the horizontal extension as the region in space (or space–time) where a observer experiments and the vertical extension as the actual values of these experiments.

It turns out that using Differential Geometry (explicitly the notion of Ehresmann connection) on fiber bundles we can distinguish these three epistemological ingredients:

1. the observer who experiments,
2. the values of her/his experiments and
3. the region of space-time where the experiment is carried out.

Let $\pi : A \to X$ be a fiber bundle with fiber $F$ and let $\Phi$ be an Ehresmann connection (see Definition A.9) on $\pi : A \to X$.

| Geometry of fiber bundles | Epistemological interpretation |
|---------------------------|--------------------------------|
| The base space $X$.       | Space-time.                   |
| The fiber $A_x = \pi^{-1}(x)$. | Experimental measurements.   |
| The connection $\Phi$.    | Relates the values of the measurements. done at different points of space time. |
| Paths $\alpha : (-\varepsilon, \varepsilon) \to X$. | Observers moving in space–time. |
| Horizontal lifts $\tilde{\alpha} : (-\varepsilon, \varepsilon) \to A$. | Measurement $\tilde{\alpha}(t) \in A_{\alpha(t)}$ at the point $\alpha(t)$ of the space–time. |

Table 1.

In Table 1. we summarize the epistemological interpretation of geometric objects
associated to a connection. We interpret $X$ as space–time, the (smooth) paths $\alpha: (-\varepsilon, \varepsilon) \to X$ as observers moving in space–time; given a point $x$ in space–time $X$, we interpret the fiber $A_x := \pi^{-1}(x)$ as the possible experimental measurements, a observer situated at $x$ can make. The connection $\Phi$ is used to relate the values of the measurements done at different points of space time, i.e. if $x_0, x_1 \in X$, then $\Phi$ is used to relate $A_{x_0}$ and $A_{x_1}$. The relation between these fibers is obtained via the horizontal lifting of paths (see Appendix A). Let $\tilde{\alpha}: (-\varepsilon, \varepsilon) \to A$ be the horizontal lifting of a path $\alpha: (-\varepsilon, \varepsilon) \to X$, we interpret that, for the observer $\alpha$, the measurement $\tilde{\alpha}(t) \in A_{\alpha(t)}$ at the point $\alpha(t)$ of the space–time, is equivalent to the measurement $e_0 := \tilde{\alpha}(0)$ at $x_0 := \alpha(0)$.

This interpretation of the Ehresmann connection has been listened by the first author in discussions about fiber bundles but to our knowledge it is not explicitly written in text books or articles. This article mixes the notion of forcing with this soft epistemological interpretation of differential geometry on fiber bundles. As it can be seen, this makes our heuristic different to the one used by Caicedo in Caicedo (1995), even when our intuition can be considered a refinement of Caicedo’s point of view.

3. Fiber bundles of structures and their pullback

Let $\mathcal{L}$ be a first–order signature. We use the geometric background included in Appendix A.

**Definition 3.1.** (cf. [Caicedo, 1995, Definició 2.2]) A fiber bundle $\mathfrak{A}$ of $\mathcal{L}$–structures is a fiber bundle $\pi: A \xrightarrow{F} M$ such that for each $m \in M$ the fiber $A_m := \pi^{-1}(m)$ is the universe of an $\mathcal{L}$–structure $\mathfrak{A}_m$ such that

i) For each relational symbol $R \in \mathcal{L}$ of arity $k < \omega$, the set $R^{\mathfrak{A}} := \bigcup_{m \in M} \{m\} \times R^{\mathfrak{A}_m}$ is an open subset of the direct sum $\bigoplus_{i=1}^{k} A$ of fiber bundles (see Definition A.4).

ii) For each function symbol $f \in \mathcal{L}$ of arity $k < \omega$, the function $f^{\mathfrak{A}} : \bigoplus_{i=1}^{k} A \to A$ defined by $f^{\mathfrak{A}}(m, e) := f^{\mathfrak{A}_m}(e) \in A_{\alpha_{m}} (e \in A_{m})$ is a $C^\infty$–function.

iii) For each constant symbol $c \in \mathcal{L}$, the function $c^{\mathfrak{A}} : M \to A$ given by $m \mapsto c^{\mathfrak{A}_m}$ is a section of $A$.

We will denote this fiber bundle of structures by $\pi: \mathfrak{A} \xrightarrow{F} M$.

The following is a very known fact, which follows from the definition of fiber bundle.

**Fact 3.2.** For every smooth fiber bundle $\pi: A \to M$ and every $e \in A$, there exists a local section $s: M \to A$ such that $s(m) = e$.

We can define fiber bundles of $\mathcal{L}$–structures of regularity $C^k$ requiring that the sections involved in conditions ii) and iii) given above are $C^k$–sections.

**Remark 3.3.** Along the article we will identify $R^{\mathfrak{A}}$ as defined in item i) of Definition 3.1, with $\bigcup_{m \in M} R^{\mathfrak{A}_m}$.

The fiber bundle of structures are, via sheafification, examples of sheaves of structures (explained in Caicedo (1995)). In a similar way differential manifolds are examples of topological spaces but the differential structure allows to define concepts like tangent vectors, tangent bundle or de Rham complex. These concepts simply do not
exist on general topological spaces. In this article we refine logical or model theoretical concepts on sheaves using smooth fiber bundles, these refinements are not possible on general sheaves of structures without using extra structure.

**Example 3.4.** Vector bundles are fiber bundles whose fibers are vector spaces and whose trivializations are linear transformations. All vector bundles over a manifold $M$ are the pullback of the canonical bundle of a Grassmannian (see Madsen and Tornehave (1997)). Each vector bundle is an $\mathcal{L}$–fiber bundle of structures where $\mathcal{L}$ is the first order signature of $\mathbb{R}$-vector spaces $\{+,\cdot,\alpha : \alpha \in \mathbb{R}\}$. An important example of vector bundle is the tangent space $TM$ of a manifold $M$ (see Remark A.4).

**Example 3.5.** Principal bundles are very important in Gauge theory, these are fiber bundles whose fibers are groups and whose trivializations are morphism of groups. Each principal bundle is a $\mathcal{L}$–fiber bundle of structures where $\mathcal{L}$ is the first order language of groups $\{\cdot, e, (\cdot)^{-1}\}$.

**Example 3.6.** In relativity, space–time is modeled as a 4-dimensional manifold with a Lorentzian metric $g$. This metric defines the light cone $C := g^{-1}(0, \infty)$. The tangent space $TM$ and the light cone $C$ conform an example of an $\mathcal{L}$–fibre bundle of structures for the language of vector spaces and an unary relation symbol $\mathcal{L} := (+, \cdot, \alpha : \alpha \in \mathbb{C}, R)$. The symbols of sum $+$ and scalar product $\cdot$ are interpreted as the corresponding sum and scalar products defined in each fiber of $TM$, and $R^A := C$.

**Remark 3.7.** We observe that given $s_1, \cdots, s_r$ local sections of $A$ and $t(x_1, \cdots x_r)$ an $\mathcal{L}$-term, it is straightforward to see that the fiber bundle function of the fiber bundle $\pi : A \xrightarrow{F} M$ defined by $m \mapsto t(s_1(m), \cdots s_r(m))$ is in fact a smooth section.

**Definition 3.8.** A morphism of fiber bundles of structures $\pi_i : \mathfrak{A}_i \rightarrow M_i$, $i = 1, 2$ with associated fiber bundles $\pi_i : A_i \rightarrow M_i$ is a morphism of fiber bundles which also preserve the $\mathcal{L}$–structure over each fiber. More precisely, a morphism is a pair of smooth maps $(\Phi : A_1 \rightarrow A_2, \phi : M_1 \rightarrow M_2)$ such that the following diagram commutes

\[
\begin{array}{ccc}
A_1 & \xrightarrow{\Phi} & A_2 \\
\downarrow_{\pi_1} & & \downarrow_{\pi_2} \\
M_1 & \xrightarrow{\phi} & M_2
\end{array}
\]

and such that

i) For each relation symbol $R \in \mathcal{L}$ of arity $k$, if $R^A_m(a_1, \cdots, a_k)$ for $(a_1, \cdots, a_k) \in (A_1)_m$ implies $R^A_m(\Phi(a_1), \cdots, \Phi(a_k))$.

ii) For each function symbol $f \in \mathcal{L}$ of arity $k$, $\Phi(f^A_m(a_1, \cdots, a_k)) = f^{A_1}_m(\Phi(a_1), \cdots, \Phi(a_k))$.

iii) Each constant symbol $c \in \mathcal{L}$ satisfies $\Phi(c^A_m) = c^{A_1}_m$.

The following definition is key in this article, it shows how to define a fiber bundle of $\mathcal{L}$-structures on a pullback.

**Proposition 3.9.** Let $h : N \rightarrow M$ be a smooth function, where $M$ and $N$ are man-
ifolds. Then we can naturally define the pullback $\mathcal{B} := h^*(\mathfrak{A})$ of a fiber bundle of $\mathcal{L}$-structures where $\mathfrak{A}$ is a fiber bundle of $\mathcal{L}$-structures over $M$.

**Proof.** Along this proof, we will use the canonical isomorphisms given in Proposition A.18. Let $\pi : A \to M$ be the fiber bundle that underlies the fiber bundle of structures $\mathfrak{A}$. We will define a fiber bundle of structures $\mathcal{B}$ over the fiber bundle $h^*(A)$ as follows:

i) For each relational symbol $R \in \mathcal{L}$ of arity $k < \omega$, we define

$$R^\mathcal{B} := \{ (n, b_1, \ldots, b_k) \in N \times h^*(A) : (\hat{h}(b_1), \ldots, \hat{h}(b_k)) \in R_{h(n)}^\mathfrak{A} \}$$

where $\hat{h} : h^*(A) \to A$ is the function defined by $\hat{h}(n, a) = a$, as indicated below Definition A.11. Using the identification of Remark 3.3, we notice that $R^\mathcal{B} = \bigoplus_{i=1}^k h^*(A) \cap \hat{h}^{-1}(R^\mathfrak{A})$, which proves that $R^\mathcal{B}$ is an open set.

ii) For each function symbol $f \in \mathcal{L}$ of arity $k < \omega$, we define the function $f^\mathcal{B} : \bigoplus_{i=1}^k h^*(A) \to h^*(A)$ by

$$f^\mathcal{B}(n, (n, a_1), \ldots, (n, a_k)) := (n, f^{\mathcal{B}_{h(n)}}(n, (a_1, \ldots, a_k))).$$

By definition of the smooth structure of $\bigoplus_{i=1}^k h^*(A)$, $f^\mathcal{B}$ is a $C^\infty$-function.

iii) For each constant symbol $c \in \mathcal{L}$, we define the function $c^\mathcal{B} : N \to h^*(A)$ by $n \mapsto (n, c^{\mathcal{B}_{h(n)}})$, which is a section of $h^*(A)$.

$\square$ Prop. 6.9

### 4. Pointwise forcing and local modeling

Let $\mathfrak{A}$ be a fiber bundle of $\mathcal{L}$-structures. In this section we adapt the forcing of [Caicedo 1995] to fiber bundles. We recall that in [Caicedo 1995] sections of the fiber bundle $\pi : A \to M$ are thought as a kind of nouns of the $\mathcal{L}$-formulae as sentences. The main difference between forcing on sheaves (as explained in [Caicedo 1995]) and forcing on fiber bundles is that in our new setting we have to impose the locality of the truth for equality of terms, because fiber bundles do not have discrete topology in their fibers.

**Definition 4.1.** (cf. [Caicedo 1995, Definición 3.1]) Let $\mathfrak{A}$ be a fiber bundle of $\mathcal{L}$-structures and let $s_1, \ldots, s_r$ be local sections of the fiber bundle $\pi : A \to M$ defined on a point $m \in M$. We define recursively on $\mathcal{L}$-formulae, the notion of forcing on the point $m$, for the sections $s_1, \ldots, s_r$ defined on $m$, denoted by

$$\mathfrak{A} \models_m \varphi(s_1, \ldots, s_r),$$

as follows:

1) (atomic case) If $t_1(x_1, \ldots, x_r), \ldots, t_n(x_1, \ldots, x_r)$ are $\mathcal{L}$-terms,

   i) $\mathfrak{A} \models_m (t_1 = t_2)(s_1, \ldots, s_r)$ if there exists an open neighborhood $U \subseteq \text{dom}(s_1) \cap \ldots \cap \text{dom}(s_r)$ such that

   $$(s_1, \ldots, s_r) \in U \in \mathfrak{A} \implies t_1 \equiv t_2,$$

   where $\equiv$ stands for the equality of terms in the theory $\mathcal{L}$. For the sake of brevity, we do not distinguish between $A$ and $U$ in the formulas of $\mathcal{L}$.

   ii) $\mathfrak{A} \models_m \varphi(s_1, \ldots, s_r)$ if for every $U \subseteq \mathfrak{A}$, every $s \in U$, and every $\varphi(x_1, \ldots, x_r)$ there exists a neighborhood $V \subseteq U$ such that $s \in V \implies \varphi(s_1, \ldots, s_r)$.

   iii) $\mathfrak{A} \models_m \exists x \varphi(s_1, \ldots, s_r)$ if $\mathfrak{A} \models_m \varphi(s_1, \ldots, s_{r-1}, v)$ for some $v \in U$, where $U$ is a neighborhood of $s_1, \ldots, s_{r-1}$.

   iv) $\mathfrak{A} \models_m \forall x \varphi(s_1, \ldots, s_r)$ if $\mathfrak{A} \models_m \varphi(s_1, \ldots, s_{r-1}, c)$ for all constants $c \in U$, where $U$ is a neighborhood of $s_1, \ldots, s_{r-1}$.

   v) $\mathfrak{A} \models_m \neg \varphi(s_1, \ldots, s_r)$ if $\mathfrak{A} \not\models_m \varphi(s_1, \ldots, s_r)$.

   vi) $\mathfrak{A} \models_m \varphi(s_1, \ldots, s_r)$ if $\mathfrak{A} \models_m \varphi(s_1, \ldots, s_{r-1}, c)$ for all constants $c \in U$, where $U$ is a neighborhood of $s_1, \ldots, s_{r-1}$.

   vii) $\mathfrak{A} \models_m \exists x \varphi(s_1, \ldots, s_r)$ if $\mathfrak{A} \models_m \varphi(s_1, \ldots, s_{r-1}, v)$ for some $v \in U$, where $U$ is a neighborhood of $s_1, \ldots, s_{r-1}$.

   viii) $\mathfrak{A} \models_m \forall x \varphi(s_1, \ldots, s_r)$ if $\mathfrak{A} \models_m \varphi(s_1, \ldots, s_{r-1}, c)$ for all constants $c \in U$, where $U$ is a neighborhood of $s_1, \ldots, s_{r-1}$.

   ix) $\mathfrak{A} \models_m \neg \varphi(s_1, \ldots, s_r)$ if $\mathfrak{A} \not\models_m \varphi(s_1, \ldots, s_r)$.

   x) $\mathfrak{A} \models_m \varphi(s_1, \ldots, s_r)$ if $\mathfrak{A} \models_m \varphi(s_1, \ldots, s_{r-1}, c)$ for all constants $c \in U$, where $U$ is a neighborhood of $s_1, \ldots, s_{r-1}$.
\[ \cdots \cap \text{dom}(s_k) \subseteq M \text{ of } m \text{ such that for all } u \in U \]

\[ \mathcal{A}_u = (t^{3u}_1 = t^{3u}_2)(s_1(u), \ldots, s_r(u)) \]

for all \( u \in U \).

ii) If \( R \in \mathcal{L} \) is a relational symbol of arity \( r < \omega \), \( \mathcal{A} \models_m R(s_1, \ldots, s_r) \) if there exists an open neighborhood \( U \subseteq \text{dom}(s_1) \cap \cdots \cap \text{dom}(s_k) \subseteq M \) of \( m \) such that for all \( u \in U \)

\[ \mathcal{A}_u \models R^m(t_1, \ldots, t_r)(s_1(u), \ldots, s_r(u)). \]

2) \( \mathcal{A} \models_m (\varphi \land \psi)(s_1, s_2, \ldots, s_r) \) if \( \mathcal{A} \models_m \varphi(s_1, s_2, \ldots, s_r) \) and \( \mathcal{A} \models_m \psi(s_1, s_2, \ldots, s_r) \).

3) \( \mathcal{A} \models_m (\varphi \lor \psi)(s_1, s_2, \ldots, s_r) \) if \( \mathcal{A} \models_m \varphi(s_1, s_2, \ldots, s_r) \) or \( \mathcal{A} \models_m \psi(s_1, s_2, \ldots, s_r) \).

4) \( \mathcal{A} \models_m \neg \varphi(s_1, s_2, \ldots, s_r) \) if there exists an open neighborhood \( U \subseteq M \) of \( m \) such that for all \( u \in U \), \( \mathcal{A} \models_u \varphi(s_1, s_2, \ldots, s_r) \).

5) \( \mathcal{A} \models_m (\varphi \rightarrow \psi)(s_1, s_2, \ldots, s_r) \) if there exists an open neighborhood \( U \subseteq M \) of \( m \) such that for all \( u \in U \), \( \mathcal{A} \models_u \varphi(s_1, s_2, \ldots, s_r) \) implies \( \mathcal{A} \models_u \psi(s_1, s_2, \ldots, s_r) \).

6) \( \mathcal{A} \models_m \exists v \varphi(v, s_1, s_2, \ldots, s_r) \) if there exist a (local) section \( s \) defined on \( m \) such that \( \mathcal{A} \models_m \varphi(s, s_1, s_2, \ldots, s_r) \).

7) \( \mathcal{A} \models_m \forall v \varphi(v, s_1, s_2, \ldots, s_r) \) if there exists an open neighborhood \( U \subseteq M \) of \( m \) such that for any \( u \in U \) and any section \( s \) defined on \( u \), \( \mathcal{A} \models_u \varphi(s, s_1, s_2, \ldots, s_r) \).

When we say that an atomic formula of relation is forced in a tuple of sections, we are committing a slight abuse of notation, since formally speaking it should be forced in a section of the direct sum of the fiber bundle (see Definition \[\text{A.12}\]). To solve this abuse of notation we use the canonical isomorphism explained in iii) of Proposition \[\text{A.13}\]. The use of this type of identification is usual in geometry without explicitly mentioning the isomorphism, in this article we try to point out us much as possible the identification used.

In \[\text{Caicedo, 1993}\], Definición 3.1, it is required that atomic formulae are true at the point \( m \), in contrast to our requirement of being true in an open. It is so because for sheaves, \[\text{Caicedo, 1993}\], Lemma 2.2 guarantees that the lifting of sections for local homeomorphisms implies the stability or extension of atomic formulae involving equalities.

**Definition 4.2.** We say that the fiber bundle of \( \mathcal{L} \)-structures \( \mathcal{A} \) locally models the \( \mathcal{L} \)-formula \( \varphi \) around \( p \in M \) at the sections \( s_1, \ldots, s_n \) if there exists an open neighborhood \( \emptyset \neq U \subseteq \text{dom}(s_1) \cap \cdots \cap \text{dom}(s_k) \subseteq M \) of \( p \) such that for all \( u \in U \),

\[ \mathcal{A}_u \models \varphi(s_1(u), \ldots, s_n(u)). \]

The geometry of the fiber bundle of \( \mathcal{L} \)-structures makes relational formulae stable or extensive in the sense of the following lemma.

**Lemma 4.3.** (cf. \[\text{Caicedo, 1993}, \text{Lemma 2.2}\]) Let \( \varphi(x_1, \ldots, x_r) \) be a \( \mathcal{L} \)-first order formula that contains only the logic operators \( \vee, \wedge, \exists \) and atomic formulae without \( = \). Let \( s_1, \ldots, s_r \) be local sections of \( A \) defined around a fixed \( m \in M \). If \( \mathcal{A}_m \models \varphi(s_1(m), \ldots, s_r(m)) \), then there exists some open neighborhood \( V \) of \( m \) such that \( \mathcal{A}_v \models \varphi(s_1(v), \ldots, s_r(v)) \) for all \( v \in V \) (i.e. \( \mathcal{A} \) locally models the \( \mathcal{L} \)-formula \( \varphi \) around
For fiber bundles of \( \mathcal{L} \)-structures it is easy to provide examples that show that this kind of stability is lost for formulae with equality.

**Example 4.4.** Let \( \pi_x : \mathbb{R}^2 \to \mathbb{R} \) be a fiber bundle with fiber \( \mathbb{R} \) \((\pi_x(x,y) = x)\) and consider it as a fiber bundle of \( \mathcal{L} \)-structures \( \mathfrak{A} \) for \( \mathcal{L} := \{=\} \). Consider the sections \( s_1(x) = (x,x) \) and \( s_2(x) = (x,-x) \). For the formula \( \varphi(x,y) : (x = y) \) we have \( \mathfrak{A}_0 \models \varphi(s_1(0), s_2(0)) \), but locally \( \varphi(s_1, s_2) \) does not hold in \( \mathfrak{A} \) around 0.

The following theorem is valid by definition.

**Theorem 4.5.** (cf. [Caicedo, 1995, Teorema 3.1]) \( \mathfrak{A} \models_m \varphi(s_1, s_2, \ldots, s_n) \) if and only if there exists and open neighborhood \( U \) of \( m \) such that \( \mathfrak{A} \models_u \varphi(s_1(u), s_2(u), \ldots, s_n(u)) \) for all \( u \in U \).

Notice that pointwise modelling is not equivalent to local modelling, but pointwise forcing is in fact equivalent to local forcing.

The proof of the next theorem follows from Definition 4.1 and it is analogous to the proof of [Caicedo, 1995, Teorema 3.2].

**Theorem 4.6.** (cf. [Caicedo, 1995, Teorema 3.2]) Let \( \bar{s} := (s_1, \ldots, s_n) \) be a tuple of local sections defined on \( m \in M \). \( \mathfrak{A} \models_m \lnot \varphi(\bar{s}) \) if and only if there exists an open neighborhood \( U \) of \( m \) such that \( \{u \in U : \mathfrak{A} \models_u \varphi(\bar{s})\} \) is dense in \( U \).
5. Compatibility of formulae with the pullback of fiber bundles

In this section we give examples which illustrate the incompatibility of the pullback (of fiber bundles) and the pointwise forcing. We use the notion of incompatibility with the pullback intuitively but we believe that it would be interesting to formalize it (to see a synthesis of the results of this article around semantics incompatible with pullback see Remark 5.3). The notions of pullback of a fiber bundle and of a section were reviewed in Appendix A; the pullback of a fiber bundle of structures was explained in Proposition 3.9.

As explained in Section 2, one can think that a experimental measurement on a point $m$ in space–time $M$ is given by a tuple $(e_1,\ldots,e_n) \in A^n_m$. For any $L$-formula $\varphi$, if $A_m \models \varphi(e_1,\ldots,e_n)$, in order to be able of making experimental measures, one would expect that for any observer and what she or he measures, the $L$-formula $\varphi$ is extensive in time, i.e. $\varphi$ continues being true in some interval of time independently of the movement of the observer. One could wrongly think that this formally means that if $A \models_m \varphi(s_1,\ldots,s_k)$ then for all path (observer) $\sigma : (-1,1) \to M$ such that $\sigma(0) = m$, $\sigma^*(A) \models_m \varphi(s_1,\ldots,s_k)$ where $s_i(t) := (t,s_i(\sigma(t)))$ and $\sigma^*(A)$ is the pullback of $A$ as explained in Proposition 3.9. The following example shows that this is in fact wrong.

Example 5.1. Let $L$ be the first order language with a relational symbol $R$ of arity 1. Let us consider the fiber bundle of $L$–structures with underlying fiber bundle $\rho : \mathbb{R}^3 \to \mathbb{R}^2$ ($p(x,y,z) := (x,y)$) and $R^A := \mathbb{R}^3 - \{ t(1,0,0) : t \in \mathbb{R} \}$. Let $s(x,y) := (x,y,0)$ and $\sigma(t) = (t,0)$ ($t \in \mathbb{R}$). Using Theorem 4.6, $A \models_{(0,0)} \neg R(s)$ because $\{(x,y) \in \mathbb{R}^2 : s(x,y) \in R^A(x,y)\}$ is dense in $\mathbb{R}^2$. In the same line of reasoning, $\sigma^*A \models_0 \neg R(\sigma^*s)$ where $\sigma^*s(t) := (t,(s \circ \sigma)(t)) = (t,(t,0,0))$, because $\{ t \in \mathbb{R} : \sigma^*s(t) \in R^{\sigma^*A} \} = \emptyset$, so it is impossible to find an open neighborhood of 0 on which this set is dense.

Remark 5.2. The previous example can be adapted to sheaves. Let us endow $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ with the product topology considering $\mathbb{R}^2$ with the usual topology and $\mathbb{R}$ with the discrete topology. With this topology $\rho : \mathbb{R}^3 \to \mathbb{R}^2$ is a sheaf over $\mathbb{R}^2$ and one can observe that the fiber bundle of structures, the section $s$ and the path $\sigma$ of Example 5.1 provides an example that forcing is not compatible with the pullback on sheaves.

The incompatibility of the pullback points out that the pullback of forcing is a notion that depends of the observer (the path or function on which we are taking the pullback).

Remark 5.3. It would be interesting to define the notion of semantics compatible with a pullback in a more general way. For the moment, we have that the classic semantics and local modeling are compatible with the pullback in the sense of Proposition 5.6 and Corollary 5.6. The incompatibility of the pointwise forcing with the pullback means that there are formulae which are not compatible with the pullback (as it is proved by Examples 5.1, 5.2 and 5.11). In Proposition 5.7 we prove that the forcing of a sentence obtained as quantification of an atomic formula is compatible with the pullback. For parallel semantics (see Section 6) the compatibility of the pullback means that Theorem 6.4 holds.

Definition 5.4. Let $L$ be a signature. A $L$–formula $\varphi$ is compatible with pullbacks if for every fiber bundle of $L$–structures $A$ with underlying fiber bundle $\pi : A \to M$, and for every smooth function $f : N \to M$,
Proposition 5.5. Let $\mathfrak{A}$ be a fiber bundle of structure with underlying fiber bundle $\pi: \mathcal{A} \to M$ and let $f: N \to M$ be a smooth function between smooth manifolds. Suppose that $e_1, \ldots, e_k \in A_m$ and $\varphi(x_1, \ldots, x_k)$ is an $\mathcal{L}$-formula. Then, $\mathfrak{A}_m \models \varphi(e_1, \ldots, e_k)$ if and only if for all $n \in f^{-1}(m)$, $f^*\mathfrak{A}_n \models \varphi((e_1, n), \ldots, (e_k, n))$.

Proof. The proof follows by the definition of $\models$ and pullback of structures (see Proposition [3,9]). □

As a corollary of the previous proposition we have the following result.

Corollary 5.6. Let $\mathcal{L}$ be a first order signature. For local modeling, all the $\mathcal{L}$-formulae are compatible with the pullback; i.e. if $\mathfrak{A}$ is a fiber bundle of structures and $\mathfrak{A}$ locally models $\varphi$ at $p$ for the sections $s_1, \ldots, s_n$, then for all smooth $f: N \to M$ and for all $q \in f^{-1}([p])$, there exists a local section $R$ such that for all $u \in f^{-1}U$, $\mathfrak{A}_u \models \varphi(f_s(s_1(u)), \ldots, f_s(s_n(u))).$

Proof. Let us suppose that $\mathfrak{A}$ locally models $\varphi$ at $p$ for the sections $s_1, \ldots, s_n$. This means that there is a neighborhood $U$ of $p$ such that for all $u \in U$, $\mathfrak{A}_u \models \varphi(s_1(u), \ldots, s_n(u))$. Let $f: N \to M$ be a smooth function. Since the fibers of the pullback of the fiber bundle of structures $\mathfrak{A}$ are essentially the same fibers of $\mathfrak{A}$ (see Proposition [3,9]), for all $u \in f^{-1}U$, $f^*\mathfrak{A}_u \models \varphi(f^*s_1(u), \ldots, f^*s_n(u)).$ □

Proposition 5.7. Let $\mathfrak{A}$ be a fiber bundle of structures for the signature $\mathcal{L}$ with underlying fiber bundle $\pi: \mathcal{A} \to M$. Let $\varphi$ be an $\mathcal{L}$-sentence obtained as quantification of an atomic formula. If $\mathfrak{A} \models \varphi$ and $f: N \to M$ is a smooth function, then for all $n \in f^{-1}([m])$, $f^*\mathfrak{A} \models \varphi$.

Proof. Let $\varphi$ be an atomic $\mathcal{L}$-formula of the form $\varphi(x_1, \ldots, x_n) = h(x_1, \ldots, x_n)$ where $h$ and $g$ denote terms in the signature $\mathcal{L}$ with free variables $x_1, \ldots, x_n$. Let us suppose $\mathfrak{A} \models \varphi$ and $s(\mathfrak{A}) = (a_1, \ldots, a_n)$ and the fact that the definition of forcing $\models$ imply that, in fiber bundles, $\mathfrak{A} \models \forall x_1 \ldots \forall x_n \varphi(x_1, \ldots, x_n)$. Notice that for every $(a_1, \ldots, a_n) \in (\Psi_{i=1}^n A)_{m}$ there exists a local section $s$ of $\Psi_{i=1}^n A$ such that $s(m) = (a_1, \ldots, a_n)$. This fact and the definition of forcing $\models$ imply that, in fiber bundles, $\mathfrak{A} \models \forall x_1 \ldots \forall x_n (h = g)(x_1, \ldots, x_n)$ is equivalent to the fact that $h^\mathfrak{A} = g^\mathfrak{A}$ in an open subset of $\Psi_{i=1}^n A$ of the form $\pi^{-1}(U)$, where here $\pi$ is the projection $\pi: \Psi_{i=1}^n A \to M$ and $U$ is an open neighborhood of $m$ in $M$. Let us suppose that $U$ is such an open neighborhood of $m$ for the formula of equality $\varphi$. Then $h^\mathfrak{A}(\mathfrak{A}) = g^\mathfrak{A}(\mathfrak{A})$ in the open subset $\pi^{-1}(f^{-1}(U))$ of $f^*(\Psi_{i=1}^n A)$ where here $\pi$ is the projection $\pi: f^*(\Psi_{i=1}^n A) \to N$.

Now let us suppose that $\varphi$ is an atomic formula of relation, explicitly $\varphi(x_1, \ldots, x_n) \in R(x_1, \ldots, x_n)$ where $R$ is a symbol of relation in the signature $\mathcal{L}$. By Fact [3,2], there exists a local section $s: M \to \mathcal{A}$ such that $s(m) = e$. This implies that $\mathfrak{A} \models \forall x_1 \ldots \forall x_n \varphi(x_1, \ldots, x_n)$ is equivalent to the existence of an open neighborhood $U$ of $m$ in $M$ such that $R^\mathfrak{A} = \Psi_{i=1}^n A_u$ for all $u \in U$. We observe that if $R^\mathfrak{A}_u = \Psi_{i=1}^n A_u$ for all $u \in U$ then $R^\mathfrak{A}(\mathfrak{A}) = \Psi_{i=1}^n f^*(A)_v$ for $\mathfrak{A} \models \forall x_1 \ldots \forall x_n \varphi(x_1, \ldots, x_n)$ of $\mathfrak{A} \models \forall x_1 \ldots \forall x_n \varphi(x_1, \ldots, x_n)$ because $f^*(A)_v = \{f(v)\} \times A_f(v)$ and $R^\mathfrak{A}(\mathfrak{A}) = \{v\} \times R^\mathfrak{A}(\mathfrak{A})$. Then, for all $n \in f^{-1}(m)$, $f^*\mathfrak{A} \models \varphi$. □
Next proposition shows that compatible formulae with the pullback are closed under \(\land, \lor\) and \(\exists\).

**Proposition 5.8.** Let \(\mathfrak{A}\) be a fiber bundle of structures for the signature \(\mathcal{L}\) with underlying fiber bundle \(\pi : A \to M\). Let \(\varphi\) and \(\psi\) be formulae which are compatible with the pullback. Then, \(\varphi \land \psi, \varphi \lor \psi, \exists x \varphi\) are also compatible with pullbacks.

**Proof.** The proposition is straightforward for \(\land\) and \(\lor\). Let us suppose that \(\mathfrak{A} \models_p \exists x \varphi(s)\), for some fiber bundle of structures \(\mathfrak{A}\) with underlying fiber bundle \(\pi : A \to M\). Let \(f : N \to M\) be a smooth function. By Definition 4.1, \(\mathfrak{A} \models_p \exists x \varphi(x, s)\) means that there exists a (local) section \(r\) defined in \(p\), such that \(\mathfrak{A} \models_p \varphi(r, s)\). Since \(\varphi\) is compatible with the pullback, for all \(q \in f^{-1}\{p\}\), \(f^*\mathfrak{A} \models_q \varphi(f^*r, f^*s)\), which is equivalent to \(f^*\mathfrak{A} \models_q \exists x \varphi(x, f^*s)\).

**Example 5.9.** Let \(\mathcal{L}\) be a signature with two symbols of functions \(f\) and \(g\) of arity 1, and a symbol of constant 0. Let us consider a fiber bundle of \(\mathcal{L}\)-structures \(\mathfrak{A}\) with underlying fiber bundle \(\rho : \mathbb{R}^3 \to \mathbb{R}^2\) (\(\rho(x,y,z) := (x,y)\)) and such that
\[
f^{\mathfrak{A}}(x,y,z) := (x, y, 9 - x^2 - y^2), \quad g^{\mathfrak{A}}(x,y,z) = (x, y, 4 - x^2 - y^2)
\]
and \(0^{\mathfrak{A}(.,.,.):= (x,y,0)}\). Let us define \(\varphi : f(z) = 0, \psi : g(z) = 0\), and the section \(s(x,y) := (x,y,1)\) of the fiber bundle \(\rho : \mathbb{R}^3 \to \mathbb{R}^2\). We have that
\[
\mathfrak{A} \models_{(3,0)} (\varphi \to \psi)(s)
\]
because \(\mathfrak{A} \models_{(a,b)} \varphi(s)\) for all \((a,b) \in \mathbb{R}^2\). Let \(\sigma : \mathbb{R} \to \mathbb{R}^2\) be the function \(\sigma(t) := (3 \cos t, 3 \sin t)\). We have that
\[
f^{\sigma^*\mathfrak{A}}(t, 3 \cos t, 3 \sin t, z) = (t,0), \quad g^{\sigma^*\mathfrak{A}}(t, 3 \cos t, 3 \sin t, z) = (t, 4 - 9) = (t, -5)
\]
and \(0^{\sigma^*\mathfrak{A}(.,.,.,.):= (t,0}\). So, for all \(t \in \sigma^{-1}\{(3,0)\}\) = \(\{2n\pi : n \in \mathbb{Z}\}\), \(\sigma^*\mathfrak{A} \models_t \varphi(\sigma^* s)\) but \(\sigma^*\mathfrak{A} \models_t \psi(\sigma^* s)\), hence
\[
\sigma^*\mathfrak{A} \not\models_t (\varphi \to \psi)(\sigma^* s).
\]

**Proposition 5.10.** Let \(\mathfrak{A}\) be a fiber bundle of structures for the signature \(\mathcal{L}\) with underlying fiber bundle \(\pi : A \to M\). Suppose that \(R\) is a relational symbol and \(\varphi := \neg R(x_1, \cdots, x_n)\). Then, \(\mathfrak{A} \models_p \varphi(s)\) if and only if there exists an open neighborhood \(U\) of \(p\) such that:
\[
\{v \in U : \tilde{s}(v) \notin R^{\mathfrak{A}}_v\} = U.
\]

**Proof.** From Definition 4.1 ii) and 4), \(\mathfrak{A} \models_p \varphi(s)\) is equivalent to the existence of an
open neighborhood $U$ of $p$ such that
\[ \{ v \in U : \bar{s}(v) \notin \mathbb{R}^n \} \] is dense in $U$.

Since $\{ v \in U : \bar{s}(v) \notin \mathbb{R}^n \}$ is closed in $U$, because it is the preimage of the open $\bigoplus_{i=1}^n \mathbb{A} \setminus \mathbb{R}^n$ under $\bar{s}$ (which is a continuous function), we are done. \hfill \Box

Proposition 5.10

The following example shows a negation of an equality of terms which is incompatible with the pullback.

Example 5.11. Let $\mathcal{L}$ be a signature with two symbols of function $f$ and $g$ both of arity 1. Let us consider a fiber bundle of $\mathcal{L}$–structures $\mathbb{A}$ with underlying fiber bundle $\rho : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $\rho(x,y,z) := (x,y)$ and such that $f^{\mathbb{A}(x,y)}(z) = (x,2x)$, $g^{\mathbb{A}(x,y)}(z) = (x,3x)$. Notice that both $f^{\mathbb{A}(x,y)}$ and $g^{\mathbb{A}(x,y)}$ are constant functions. Let us define $\varphi : f(z) = g(z)$ and the section $s(x,y) := (x,y,1)$ of the fiber bundle $\rho : \mathbb{R}^3 \to \mathbb{R}^2$. We have that
\[ \mathbb{A} \models_{(0,0)} \neg \varphi(s) \]

because for all $(x,y) \in \mathbb{R}^2$, $\mathbb{A} \not \models_{(x,y)} \varphi(s)$. Consider the path $\sigma(t) := (0,t)$ $(t \in \mathbb{R})$, we have that
\[ \sigma^* \mathbb{A} \not \models_0 \neg \varphi(\sigma^*s), \]

because for all open neighborhood $U$ of 0 in $\mathbb{R}$ there is an $u \in U$ such that $\sigma^* \mathbb{A} \models_u \varphi(\sigma^*s)$.

The previous examples illustrate the incompatibility of the forcing with the pullbacks. In the epistemological interpretation of Caicedo (1993) we would say that the sentence $\varphi(x)$ is true in the extensive subject (the section) $s$ but there would be no interpretation of the incompatibility with the pullback. In terms of our epistemological interpretation (see Section 2, Table 1), we have a richer interpretation of all the geometric ingredients in the game:

i) The section $s$ can be thought as a way to \textit{horizontally translate the measurements} done by all observers (i.e. paths of the base space) at a point in the base space (space-time) at some instant of time. This interpretation of the direct image $s$ is justified since it could be (locally) the integral submanifold of a Ehresmann connection (thought as a distribution). We recall that we are thinking the parallel translation as the equivalence of measurements in different points of the base space (see Section 2).

ii) The path $\sigma$ can be thought as an observer moving in the base space.

iii) $\sigma^*(s)$ is the horizontal lift of $\sigma$ under the connection given in the previous item. We can interpret it in the following way: The measurement $\sigma^*(s)(t)$ done at the point $\sigma(t)$ of the base space would be equivalent to the measurement $\sigma^*(s)(0)$ done at the point $\sigma(0)$ of the base space. Moreover, all measurements $\sigma^*(s)(t)$ are equivalent to each other and we can think of them as the same measurement performed in different times and points in the base space.
iv) The sentence \( \varphi(x) \) is not true for the observers (paths) because even when we are evaluating the formula in the same measurement \( \sigma^*(s)(t) \), its forcing does not hold, in symbols \( \sigma^*(\mathfrak{A}) \not\models_0 \varphi(\sigma^*(s)) \).

6. Parallel forcing of a point

In this section we define parallel forcing, a semantics based in the epistemological motivation given in Section 2. From now on, we will work with a smooth fiber bundle of \( \mathcal{L} \)-structures \( \mathfrak{A} \) whose fiber bundle \( \pi: A \xrightarrow{F} M \) has a connected basis space \( M \). We suppose also that \( \pi: A \xrightarrow{F} M \) is endowed with a connection \( \Phi \) (see Appendix A.7).

**Definition 6.1.** Given \( e_1, \ldots, e_n \in A_m \) for a fixed \( m \in M \), an \( \mathcal{L} \)-formula \( \varphi(x_1, \ldots, x_n) \) is said to be \( (e_1, \ldots, e_n) \)-**parallel forced** if for all path \( \sigma: (-1, 1) \rightarrow M \) such that \( \sigma(0) = m \) we have \( \sigma^*(\mathfrak{A}) \models_0 \varphi(s_1, \ldots, s_n) \) where the section \( s_k \) is the \( \sigma^*(\Phi) \)-lift to the fiber bundle \( \sigma^*(\mathfrak{A}) \) of the identity path : \( (-1, 1) \rightarrow (-1, 1) \) such that \( s_k(0) = (0, e_k) \) (see Figure 1). We denote it by \( \mathfrak{A}^\Phi \models_{e_1, \ldots, e_n} \varphi(x_1, \ldots, x_n) \).

![Figure 1](image)

To explain better the sections \( s_k \) in the previous definition, we refer to Proposition A.19.

The requirement of two terms being locally equal for the forcing of an equality of terms (see part 1 i) of Definition 4.1 can be considered artificial. Contrasting, the next lemma makes it a consequence of the geometry when we are considering the equality of variables, in analogy to (Caicedo, 1995, Lemma 2.1) in the context of sheaves. See Proposition 7.7 for a generalization of this lemma for the equality of terms in general.

**Lemma 6.2.** (cf. (Caicedo, 1995, Lema 2.1)) If \( \tilde{\sigma}_1, \tilde{\sigma}_2 : (-\varepsilon, \varepsilon) \rightarrow A \) are parallel lifts of a curve \( \sigma : (-\varepsilon, \varepsilon) \rightarrow M \) associated to some connection \( \Phi \) of the fiber bundle \( \pi: A \xrightarrow{F} M \) and for some \( t_0 \in (-\varepsilon, \varepsilon) \), \( \tilde{\sigma}_1(t_0) = \tilde{\sigma}_2(t_0) \) then there exists a real interval \( (t_0 - \delta, t_0 + \delta) \) such that for all \( s \in (t_0 - \delta, t_0 + \delta) \), \( \tilde{\sigma}_1(s) = \tilde{\sigma}_2(s) \).
Proof. Recall that the parallel transport of any path \( \sigma : (-\varepsilon, \varepsilon) \to M \) is a solution of the equation

\[
\frac{d}{dt} \tilde{\sigma}(t) = d\pi^{-1}(\sigma'(t)),
\]

where we are considering \( d\pi \) as an isomorphism of the horizontal space \( HA \) (induced by \( \Phi \)) and \( TM \). This Lemma is a direct consequence of the theorem of existence and uniqueness of solutions of ordinary differential equations. \( \square \)

Parallel forcing is preserved under isomorphisms of fiber bundles with connections (see Definition [A.20]). It implies the following lemma which is our main tool to prove that parallel semantics (see Definition [6.1]) is in fact compatible with the pullback of any smooth function, not only with respect to paths.

**Lemma 6.3.** Let \( f : N \to M \) and \( g : M \to P \) be smooth functions \((M, N \text{ and } P \text{ manifolds})\), let \( \pi : A \to P \) be a fiber bundle. Suppose that \( \mathcal{A} \) is a fiber bundle of structures with underlying fiber bundle \( \pi : A \to P \) over the signature \( \mathcal{L} \). Then, for all formula \( \varphi \) in the signature \( \mathcal{L} \), and all sections \( s_1, \ldots, s_k \) of \( \pi : A \to P \), and all \( n_0 \in N \), we have:

\[
f^*(g^*\mathcal{A}) \models_{n_0} \varphi(s_1, \ldots, s_k) \text{ if and only if } (g \circ f)^*\mathcal{A} \models_{n_0} \varphi(\tilde{s}_1, \ldots, \tilde{s}_k),
\]

where \( \tilde{s}_i \) is the section of \((g \circ f)^*\mathcal{A}\) induced by the section \( s_i(n) = (n, f(n), T_i(n)) \) of \( f^*(g^*\mathcal{A}) \) through the canonical isomorphism between \( f^*(g^*\mathcal{A}) \) and \((g \circ f)^*\mathcal{A}\) defined in Proposition [A.18] specifically \( \tilde{s}_i(n) = (n, T_i(n)) \).

**Proof.** Doing induction on formulas one can observe that the forcing at \( n \in N \) of the formula \( \varphi \), on the fiber bundles of structures \((g \circ f)^*\mathcal{A} \text{ and } f^*(g^*\mathcal{A})\) for the sections \( s_1, \ldots, s_k \) and \( \tilde{s}_1, \ldots, \tilde{s}_k \) respectively, depends in the same way of the functions \( T_i : N \to A \).

\( \square \)

**Theorem 6.4.** Let \( f : N \to M \) be a smooth function and let \( \pi : \mathcal{A} \to M \) be a fiber bundle of structures. If \( e_1, \ldots, e_k \in A_m \) then,

\[
\text{If } \mathcal{A} \Phi \models_{e_1, \ldots, e_k} \varphi, \text{ then for all } n \in f^{-1}(m), f^*(\mathcal{A}) f^*(\Phi) \models_{(n, e_1), \ldots, (n, e_k)} \varphi.
\]

**Proof.** Let us suppose \( \mathcal{A} \Phi \models_{e_1, \ldots, e_k} \varphi \). Let \( \sigma : (-\varepsilon, \varepsilon) \to N \) be such that \( \sigma(0) = n \) for \( n \in f^{-1}[\{m\}] \). According to Lemma [6.3],

\[
(f \circ \sigma)^*\mathcal{A} \models_{0} \varphi(s_1, \ldots, s_k)
\]

if and only if \( \sigma^*(f^*\mathcal{A}) \models_{0} \varphi(\tilde{s}_1, \ldots, \tilde{s}_k) \),

for \( s_i \) the section induced by the \((f \circ \sigma)^*\Phi\)--horizontal lift of the identity of \( \mathbb{R} \) to the fiber bundle \((f \circ \sigma)^*\mathcal{A} \) such that \( s_i(0) = (0, e_i) \) and \( \tilde{s}_i \) is the section of \( \sigma^*(f^*\mathcal{A}) \) defined by \( \tilde{s}_i(t) = (t, \sigma(t), r_i(t)) \) where \( r_i(t) \) is the function such that \( s_i(t) = (t, r_i(t)) \). Using Propositions [A.17] and [A.18] we can see that \( \tilde{s}_i \) is equal to the \( \sigma^*(f^*\Phi)\)--horizontal lift of the identity of \( \mathbb{R} \) to the fiber bundle \( \sigma^*(f^*\mathcal{A}) \).

\( \square \)
7. Classic semantics, forcing and parallel forcing

In this section we illustrate differences and similarities between forcing, parallel forcing and the classical semantics. We continue also clarifying the epistemological motivation given in Section 2.

Examples 5.1, 5.9 and 5.11 distinguish forcing (Definition 4.4) from parallel forcing by proving that the former one is not compatible with pullbacks. We proved that parallel forcing is compatible with pullbacks in Theorem 6.3. The classic semantics (=) is also compatible with pullbacks (see Proposition 5.5) which also distinguishes it from parallel forcing. Next example shows a difference between classic semantics and parallel forcing.

Example 7.1. On the fiber bundle $\rho : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ ($\rho(x, y, z) = (x, y)$) we consider the connection $\Phi = \frac{dz}{dz}$ (see Example 1.27). Let $\mathcal{L}$ be a signature with a relational symbol $R$ of arity 1. Let us suppose that $\mathfrak{A}$ is a fiber bundle of structures with underlying fiber bundle $\rho : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ and that $R^\mathfrak{A} := \{(x, y, z) \in \mathbb{R}^3 : x - 2y + z \neq 0\}$, in other words in each fiber we have $R^\mathfrak{A}(x, y, z) = \{(x, y, z) : x = a, y = b, a - 2b + z \neq 0\}$. Then, we have $\mathfrak{A}(0, 0, 0) \models \neg R(0, 0, 0)$ but at $e_1 := (0, 0, 0) \in A(0, 0)$ we have that $\mathfrak{A} \models \neg R(x)$ because for $\sigma(t) = (t, t)$, the $\Phi$-horizontal lift $\tilde{\sigma}$ such that $\tilde{\sigma}(0) = (0, 0, 0)$ is given by $\tilde{\sigma}(t) = (t, t, 0)$, but for every neighborhood of $t = 0$, there exists an $s$ such that $\tilde{\sigma}(s)$ belong to $R^\mathfrak{A}$.

Next we study relations between the classic semantics, forcing and the parallel forcing with respect to atomic formulas of equality. We have the following characterization of parallel forcing for equality of terms.

Proposition 7.2. Let $\mathfrak{A}$ be a fiber bundle of structures and let $\Phi$ be a connection on $\pi : A \rightarrow M$, the underlying fiber bundle. Suppose that $f$ and $g$ are $\mathcal{L}$-terms. Then the following are equivalent:

i) $\mathfrak{A} \models^\Phi_{f, e_1, \ldots, e_l} (f = g)(x_1, \ldots, x_l)$, where $e_1, \ldots, e_l \in A_m$.

ii) For all $\sigma : (-\epsilon, \epsilon) \rightarrow M$ such that $\sigma(0) = m$, there exists a $\delta > 0$ such that $\mathfrak{A}_{\pi t}((f = g)(\sigma_1(t), \ldots, \sigma_l(t)))$ for $t \in (-\delta, \delta)$ where $\sigma_i$ is the $\Phi$-lift of $\sigma$ such that $\sigma_i(0) = e_i$.

iii) For all $\sigma : (-\epsilon, \epsilon) \rightarrow M$ such that $\sigma(0) = m$, there exists a $\delta > 0$ such that $\mathfrak{A}_{\pi t}((f = g)(\tilde{\sigma}(t)))$ for $t \in (-\delta, \delta)$, where $\tilde{\sigma}$ is the $\Phi^i_{t \in \mathcal{L}}$-lift of $\sigma$ such that $\tilde{\sigma}(0) = (e_1, \ldots, e_l)$.

Proof. The equivalence of i) and ii) follows from the observation that if $s_i$ denotes the $\sigma^*(\Phi)$-lift to $\sigma^*(A)$ of the identity $i : \mathbb{R} \rightarrow \mathbb{R}$, such that $s_i(0) = (0, e_i)$, we have $s_i(t) = (t, 0, e_i)$.

ii) and iii) are equivalent due to Proposition A.16.

The following Corollary says that the parallel forcing implies the classical pointwise semantics.

Corollary 7.3. Let $\mathfrak{A}$ be a fiber bundle of structures and let $\Phi$ be a connection on $\pi : A \rightarrow M$ the underlying fiber bundle. Let $f$ and $g$ be terms of the signature $\mathcal{L}$, and let $e_1, \ldots, e_n \in A_m$ for $m \in M$. If $\mathfrak{A} \models^\Phi_{f, e_1, \ldots, e_n} (f = g)(x_1, \ldots, x_n)$,
then $\mathfrak{A}_m \models (f = g)(e_1, \cdots, e_n)$.

Proof. This follows directly from the definition of $\models^{\Phi}(e_1, \cdots, e_n)$. □

Notice that we also have the converse of the previous Corollary for variables as $\mathcal{L}$-terms.

**Proposition 7.4.** Let $\mathfrak{A}$ be a fiber bundle of structures and let $\Phi$ be a connection on $\pi: A \rightarrow M$ the underlying fiber bundle. Suppose that $e_1, e_2 \in A_m$ for $m \in A$, then, $\mathfrak{A}_m \models (x = y)(e_1, e_2)$ if and only if $\mathfrak{A} \models^{\Phi}(e_1, e_2)(x = y)$.

Proof. It is a direct consequence of Lemma 6.2 and Corollary 7.3. □

In contrast with Proposition 7.4, next example shows that we do not have the previous equivalence for forcing (see Definition 4.1).

**Example 7.5.** Consider the fiber bundle $\rho: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ ($\rho(a, b, c) = (a, b)$). We have $\mathfrak{A}_{(0,0)} \models (x = y)(0, 0, 1), (0, 0, 1)$, but we can construct sections $s_1$ and $s_2$ for which $s_1(0, 0) = (0, 0, 1) = s_2(0, 0)$ and $\mathfrak{A}_{(0,0)} \not\models (x = y)(s_1, s_2)$. For instance take $s_1(x, y) = (x, y, 1 + x)$ and $s_2(x, y) = (x, y, 1 + y)$.

Next example shows that the converse of Corollary 7.3 is not true for equality of terms, i.e. there are equalities of terms which are true in a fiber but which are not parallel forced.

**Example 7.6.** Let $\rho: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be the fiber bundle with connection $\Phi$ of Example 7.4. Let $\mathcal{L}$ be a signature with symbols of function $f$ and $g$ of arity one. Let $\mathfrak{A}$ be a fiber bundle of structures on which $f^\mathfrak{A}(a, b, c) = (a, b, c + a)$ and $g^\mathfrak{A}(a, b, c) = (a, b, c + b)$. Clearly, $\mathfrak{A}_{(0,0)} \models (f(x) = g(x))(0, 0, 0)$. Let us consider the path $\sigma(t) := (t, 2t)$. Example 7.4 shows that $\tilde{\sigma}(t) = (t, 2t, 0)$, where $\tilde{\sigma}$ is the $\Phi$–horizontal lift of the path $\sigma$ such that $\tilde{\sigma}(0) = (0, 0, 0)$. If we denote by $p$ and $q$ the interpretation of the symbols of function $f$ and $g$ in $\sigma^*\mathfrak{A}$ (see Proposition 7.4), we have $f^\sigma^*\mathfrak{A}(t, (t, 2t, v)) := (t, t, 2t, v + t)$ and $g^\sigma^*\mathfrak{A}(t, (t, 2t, v)) := (t, t, 2t, v + 2t)$, then $\sigma^*\mathfrak{A} \not\models (f(x) = g(x))(s)$ where $s$ is $\sigma^*\Phi$–horizontal lift of the identity. Hence $\mathfrak{A} \not\models^{\Phi}(f(x) = g(x))$.

In terms of the epistemological motivation of Section 2, the previous example shows that the observation of equality of two terms at a point is not sufficiently stable, it depends on where the observer is located and it might depend of a very particular observer, the one who stays at the point where the observation is made. The formula $(f(x) = g(x))$ in Example 7.6 is not parallel forced because of this lack of stability and this dependence of the observer.

Next proposition proves that for fiber bundles with connections $\Phi$ of curvature $0$, $\Phi$–parallel forcing is equivalent to force on naturally associated sections. The main tool to prove this is that for connections of curvature $0$ (see Definition A.22) the horizontal bundle is an integrable distribution and hence it induces a local section (see Proposition A.24).

**Proposition 7.7.** Let $\mathfrak{A}$ be a fiber bundle of structures and let $\Phi$ be a connection on $\pi: A \rightarrow M$ of curvature $0$ on the underlying fiber bundle. Suppose that $f$ and $g$ are $\mathcal{L}$–
terms. Then \( \mathfrak{A} \models_{(e_1, \ldots, e_l)} (f = g)(x_1, \ldots, x_l) \) if and only if \( \mathfrak{A} \models_m (f = g)(s_1, \ldots, s_l) \) where \( s_i \) is a \( \Phi \)-parallel (local) sections of \( A \) such that \( s_i(m) = e_i \) defined on a sufficiently small open set of \( M \).

**Proof.** Since \( \Phi \) has curvature 0, we have for each \( e_i \in A_m \), there exists \( U_i \subset M \) an open neighborhood of \( m \) and \( s_i : U_i \rightarrow A \) such that \( s_i \) is a \( \Phi \)-lift of the inclusion \( j : U_i \rightarrow M \) and \( s_i(m) = e_i \) (see Proposition \[A,24\]). This is equivalent to note that each \( s_i \) is the integral subvariety of the horizontal bundle \( HA_\Phi \) at \( e_i \). Equation \[A3\] and Proposition \[7.2\] prove that if \( \mathfrak{A} \models_m (f = g)(s_1, \ldots, s_l) \), then \( \mathfrak{A} \models_{(e_1, \ldots, e_l)} (f = g)(x_1, \ldots, x_l) \).

We proceed by contradiction. Suppose that we can not find such neighborhood contained in \( \mathfrak{A} \models_{(e_1, \ldots, e_l)} (f = g)(x_1, \ldots, x_l) \). We use ii) of Proposition \[7.2\] to prove that there exists an open neighborhood \( U \subset M \) of \( m \) such that for all \( n \in U \),

\[
\mathfrak{A}_n \models (f = g)(s_1(n), \ldots, s_k(n)).
\]

We proceed by contradiction. Suppose that we can not find such neighborhood \( U \), then there exists a sequence \( (a_i)_{i \in \mathbb{N}} \subset M \) which converges to \( m \) (as fast as needed) and such that

\[
\mathfrak{A}_{a_i} \not\models (f = g)(s_1(a_i), \ldots, s_k(a_i)).
\]

By interpolation methods, there exists a path \( \sigma : (-\varepsilon, \varepsilon) \rightarrow M \) such that \( \{a_k : k \in \mathbb{N}\} \) is contained in \( \sigma(-\varepsilon, \varepsilon) \). Since by Equation \[A3\] the \( \Phi \)-lift at \( m \) is given by \( \tilde{s} = s \circ \sigma \), we have that Proposition \[7.2\]ii) does not hold, which is a contradiction. This finishes the proof.

**Corollary 7.8.** Let \( \mathfrak{A} \) be a fiber bundle of structures and let \( \Phi \) be a connection on \( \pi : A \rightarrow M \), the underlying fiber bundle. Suppose that the curvature of \( \Phi \) is 0. Then for every atomic formula \( \varphi(x_1, \ldots, x_k) \) we have that the following are equivalent:

i) \( \mathfrak{A} \models_{(e_1, \ldots, e_k)} \varphi(x_1, \ldots, x_k) \).

ii) There exist \( s_1, \ldots, s_k \) \( \Phi \)-parallel (local) sections of \( A \) whose domains contain \( m \) such that \( \mathfrak{A} \models_{m} \varphi(s_1, \ldots, s_k) \).

**Proof.** For equality of \( L \)-terms, it follows from Proposition \[7.7\]. If \( \varphi = R(f_1, \ldots, f_l)(x_1, \ldots, x_k) \) where each \( f_i \) is a \( L \)-term and \( \mathfrak{A} \models_{(e_1, \ldots, e_k)} \varphi(x_1, \ldots, x_k) \), and therefore \( \mathfrak{A}_m \models \varphi(e_1, \ldots, e_k) \), then \( (f_1^A, \ldots, f_l^A)(e_1, \ldots, e_k) \) belongs to \( R^A \) and there exists an open neighborhood \( \tilde{U} \) of \( (f_1^A, \ldots, f_l^A)(e_1, \ldots, e_k) \) in \( \Phi_{i=1}^A A \) such that \( \tilde{U} \subset R^A \). Let us denote \( U := (f_1^A, \ldots, f_l^A)^{-1}(\tilde{U}) \). \( U \) is open in \( \Phi_{i=1}^A A \), the domain of \( (f_1^A, \ldots, f_l^A) \). Hence for arbitrary local sections \( s_1, \ldots, s_k \) of \( A \) such that for all \( m \in \pi^{-1}(U) \subset M \), \( (s_1, \ldots, s_k)(m) \in U \), we have \( \mathfrak{A} \models_{m} \varphi(s_1, \ldots, s_k) \). From this fact follows the proposition.

Let us suppose that \( \mathfrak{A} \models_{m} \varphi(s_1, \ldots, s_k) \) where \( s_1, \ldots, s_k \) are \( \Phi \)-parallel (local) sections. Since \( \varphi \) is atomic, \( \mathfrak{A} \models_{m} \varphi(s_1, \ldots, s_k) \) is equivalent to the existence of an open neighborhood \( U \) of \( m \) in \( M \) such that \( \mathfrak{A}_u \models \varphi(s_1(u), \ldots, s_k(u)) \) for all \( u \in U \). This and Equation \[A3\] imply that \( \mathfrak{A}_{\sigma(t)} \models (\tilde{s}_1(t), \ldots, \tilde{s}_k(t)) \), and the corollary follows from the definition of parallel forcing.
Remark 7.9. Notice that the hypothesis of curvature 0 is needed in the previous Corollary [7,8] for existence of \( \Phi \)-parallel (local) sections (see Definition [A.23] and Proposition [A.24]).

Remark 7.10. Corollary [7,8] can be extended to formulae constructed from atomic formulas using disjunction and conjunction.

However the next example shows that forcing of a formula on parallel sections (see Definition [A.23]) is not equivalent to parallel forcing.

Example 7.11. Let \( \mathcal{L} \) be a signature with a unary symbol of relation \( R \) and let \( \mathfrak{A} \) be a fiber bundle of structures (see Definition [3.7]) with underling fiber bundle \( \pi: \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) \((\pi(x,y,z) = (x,y))\) and such that \( R^\mathfrak{A} := \mathbb{R}^3 - \{(x,y,z) \in \mathbb{R}^3 : x = y, z = 0 \}. \) Let \( \Phi := \frac{\partial}{\partial z} \otimes dz \) be a connection on \( \pi: \mathbb{R}^3 \rightarrow \mathbb{R}^2. \) The \( \Phi \)-parallel section at \((0,0,0)\) is \( s(x,y) = (x,y,0). \) We have

\[
\mathfrak{A} \models_{(0,0)} \neg \neg R(x).
\]

Let us define \( \sigma(t) = (t,t), \) then \( \tilde{\sigma}(t) = (t,t,0) \) by equation [A.3]. It is easy to check that \( \sigma^* \mathfrak{A} \models_0 \neg \neg R(\sigma^* s). \) This proves

\[
\mathfrak{A} \models_{(0,0,0)} \neg \neg R(x).
\]

8. Spatial, horizontal and vertical extensions

As explained in Section [2] an \( n \)-tuple \((a_1,\cdots,a_n) \in A_m^n \) can be interpreted as an experimental measurement at a point \( m \) of \( M \) interpreted as space–time. Intuitively, the sentences about these measurements should be extensive in space–time and to a certain level independent of the accuracy of the measurements. The horizontal and vertical bundle catch this distinction between a continuity of the truth depending of the space–time (the spatial or horizontal extension) and a continuity of the truth depending of the accuracy of the measurement (the accuracy extension or vertical extension), respectively.

Definition 8.1. (cf. [Caicedo, 1995, Sección II]) Let \( \mathfrak{A} \) be a fiber bundle of \( \mathcal{L} \)-structures and suppose that \( \pi: A \xrightarrow{F} M \) is endowed with a connection \( \Phi. \) We define the **spatial extension** of the sections \( s_1,\cdots,s_n \) of \( A \) for an \( \mathcal{L} \)-formula \( \varphi(x_1,\cdots,x_n) \) in an open subset \( U \) of \( M \) as follows:

\[
[[\varphi(s_1,\cdots,s_r)]]_U := \{u \in U : \mathfrak{A} \models_u \varphi(s_1,\cdots,s_r)\}.
\]

The next definition generalizes the notion of spatial extension to parallel semantics.

Definition 8.2. Let \( \mathfrak{A} \) be a fiber bundle of \( \mathcal{L} \)-structures and suppose that \( \pi: A \xrightarrow{F} M \) is endowed with a connection \( \Phi. \) We define the **horizontal extension** of \((e_1,\cdots,e_n) \in A_m^n \) for an \( \mathcal{L} \)-formula \( \varphi(x_1,\cdots,x_n) \) in an open subset \( U \) of \( M \) as follows:

\[
\Phi[[\varphi(e_1,\cdots,e_n)]]_U := \{u \in U : \text{there is a path } \sigma : [0,1] \rightarrow U \text{ such that } \sigma(0) = m, \sigma(1) = u \text{ and } \mathfrak{A} \models^\Phi_\sigma \varphi(x_1,\cdots,x_n) \text{ for all } t \in [0,1]\},
\]
where \( \tilde{\sigma} \) are the \( \Phi \)-lifts of \( \sigma \) such that \( \tilde{\sigma}(0) = e_i \).

**Example 8.3.** Consider the fiber bundle \( \pi : \mathbb{R}^2 \to \mathbb{R}^2 \) (with \( \pi(x,y) = x \)) with connections \( \Phi_1 := dy \otimes \frac{\partial}{\partial y} \) and \( \Phi_2 := (dx+dy) \otimes \frac{\partial}{\partial y} \). We observe that the vertical bundle of \( \pi \) is generated by \( \frac{\partial}{\partial y} \), the \( \Phi_1 \)-horizontal bundle is generated by \( \frac{\partial}{\partial x} \) and the \( \Phi_2 \)-horizontal bundle is generated by \( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \). The identification between the tangent space \( T \mathbb{R}^2 \) and the \( \Phi_2 \)-horizontal bundle is given by \( d\pi(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}) = \frac{\partial}{\partial x} \). The identification between the tangent space \( T \mathbb{R}^2 \) and the \( \Phi_1 \)-horizontal bundle is given by \( d\pi(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}) = \frac{\partial}{\partial x} \).

Given a smooth path \( \sigma : (-\epsilon, \epsilon) \to \mathbb{R}^2 \) such that \( \sigma(0) = 0 \), the \( \Phi_2 \)-horizontal lift of \( \sigma \) is the solution \( \tilde{\sigma} : (-\epsilon, \epsilon) \to \mathbb{R}^2 \) of the differential equation

\[
d\pi^{-1}(\sigma'(t) \frac{\partial}{\partial t}) = \tilde{\sigma}'(t) = (-\sigma'(t), \sigma'(t)) \quad \text{with initial condition } \tilde{\sigma}(0) = (0,0).
\]

Similarly, the \( \Phi_2 \)-horizontal lift of \( \sigma \) is the solution \( \tilde{\sigma} : (-\epsilon, \epsilon) \to \mathbb{R}^2 \) of the differential equation

\[
d\pi^{-1}(\sigma'(t) \frac{\partial}{\partial t}) = \tilde{\sigma}'(t) = (\sigma'(t), 0) \quad \text{with initial condition } \tilde{\sigma}(0) = (0,0).
\]

If the signature \( \mathcal{L} \) has a unary relation symbol \( R \), suppose that we have a fibre bundle of \( \mathcal{L} \)-structures over the fiber bundle \( \pi : \mathbb{R}^2 \to \mathbb{R}^2 \) (with \( \pi(x,y) = x \)), such that \( R^\mathbb{A} := B_1(0) \), the open unitary ball in \( \mathbb{R}^2 \). With the information given above, we can deduce that the horizontal extensions at \( 0 = (0,0) \) of \( \varphi(x) := R(x) \) associated to the connections \( \Phi_1 \) and \( \Phi_2 \) are

\[
\Phi_1[[\varphi(0)]]_{\mathbb{R}} = (-1,1) \quad \text{and} \quad \Phi_2[[\varphi(0)]]_{\mathbb{R}} = (-\sqrt{2}/2, \sqrt{2}/2).
\]

In the previous example we obtained that the horizontal extensions are open subsets of the base space. This is not always the case.

**Example 8.4.** Consider the fiber bundle \( \pi : \mathbb{R}^3 \to \mathbb{R}^2 \) (with \( \pi(x,y,z) = (x,y) \)) with connection \( \Phi := dz \otimes \frac{\partial}{\partial z} \). We observe the vertical bundle of \( \pi \) is generated by \( \frac{\partial}{\partial z} \). The \( \Phi \)-horizontal bundle is generated by \( \frac{\partial}{\partial x} \) and \( \frac{\partial}{\partial y} \). The identification between the tangent space \( T \mathbb{R}^2 \) and the \( \Phi \)-horizontal bundle is given by \( d\pi(\frac{\partial}{\partial x}) = \frac{\partial}{\partial x} \) and \( d\pi(\frac{\partial}{\partial y}) = \frac{\partial}{\partial y} \).

Given a smooth path \( \sigma : (-\epsilon, \epsilon) \to \mathbb{R}^2 \), \( \sigma(t) := (\sigma_1(t), \sigma_2(t)) \), such that \( \sigma(0) = (0,0) \), the \( \Phi \)-horizontal lift of \( \sigma \) is the solution \( \tilde{\sigma} : (-\epsilon, \epsilon) \to \mathbb{R}^3 \) of the differential equation

\[
d\pi^{-1}(\sigma'(t) \frac{\partial}{\partial t}) = d\pi^{-1}(\sigma_1'(t) \frac{\partial}{\partial t} + \sigma_2'(t) \frac{\partial}{\partial y}) = \tilde{\sigma}'(t) = (-\sigma_1'(t), \sigma_2'(t), 0) \quad \text{with initial condition } \tilde{\sigma}(0) = (0,0,0).
\]

If the signature \( \mathcal{L} \) has a unary relation symbol \( R \), then define \( \mathfrak{A} \) a fibre bundle of \( \mathcal{L} \)-structures such that \( R^\mathfrak{A} := \mathbb{R}^3 - \{t(1,0,0) : t \in \mathbb{R}\} \). We observe that \( \Phi[[-R(0,0,0)]]_{\mathbb{R}} = \{(t,0) : t \in \mathbb{R}\} \) which is not an open subset. Moreover, \( \Phi[[R(0,0,0)]]_{\mathbb{R}} \) is the empty set that indicates that there is not an inductive relation in formulas for the horizontal extension of formulas.

Next we define the vertical extension of a formula. As mentioned before, the intuition of the vertical extension is how much the validity of the formula depends on the accuracy of the experimental measurement.

**Definition 8.5.** Let \( \mathfrak{A} \) be a fiber bundle of \( \mathcal{L} \)-structures and suppose that \( \pi : A \to M \)
is endowed with a connection $\Phi$. We define the **vertical extension** of an $\mathcal{L}$-formula $\varphi(x_1,\ldots,x_n)$ at $(e_1,\ldots,e_n) \in (\oplus_{k=1}^n A)_m$ in an open subset $W$ of the fiber $(\oplus_{k=1}^n A)_m$ as follows:

$$
\Phi((\varphi(e_1,\ldots,e_n)))_W := \{(f_1,\ldots,f_n) \in (\oplus_{k=1}^n A)_m : \text{there is a path } \alpha : [0,1] \to W \text{ such that } \alpha(0) = (e_1,\ldots,e_n), \alpha(1) = (f_1,\ldots,f_n) \text{ and } \mathfrak{A}^\Phi \models_{\alpha(s)} \varphi(x_1,\ldots,x_n), \text{ where } s \in [0,1]\}
$$

**Proposition 8.6.** Let $\mathfrak{A}$ be a fiber bundle of $\mathcal{L}$-structures with underlying fiber bundle $\pi : A \xrightarrow{\pi} M$ whose fiber $F$ and base space $M$ are connected then, for all connection $\Phi$ the vertical extension of the $\mathcal{L}$-formula $\varphi(x,y) := (x = y)$ at $(e,e) \in (\oplus_{i=1}^2 A)_m$ is $\Phi((\varphi))_M := \{(f,f) : f \in A_m\}$ and the horizontal extension is $[[\varphi]]_M := M$.

**Proof.** Since $M$ is path–connected, for all $m' \in M$ there is smooth path $\sigma : [0,1] \to M$ such that $\sigma(0) = m$ and $\sigma(1) = m'$ and by uniqueness of the parallel lift we have $\mathfrak{A}^\Phi \models_{(\sigma(s),\sigma(s))} (x = y)$, hence horizontal extension of $(x = y)$ at $(e,e)$ is $M$.

Since $F$ is path–connected, given $f \in A_m$ there exists a path $\alpha : [0,1] \to A_m$ such that $\alpha(0) = e$ and $\alpha(1) = f$. The path $\beta(s) := (\alpha(s),\alpha(s))$ satisfies $\mathfrak{A}^\Phi \models_{\beta(s)} (x = y)$.

We believe that the formalization that we offer of horizontal and vertical extensions of a formula could help to clarify interactions between Geometry, Physics and Mathematical Logic.

**Appendix A. Geometric background**

In this section we indicate briefly the basics of Differential Geometry needed to understand this article.

**A.1. Fiber bundles and their connections**

Fiber bundles are spaces that *locally look like* Cartesian products. They provide a geometric formalization of the idea of continuous families of spaces all of which are diffeomorphic.

**Definition A.1.** A $C^k$–fiber bundle $(k \in \mathbb{N})$ consists of three $C^k$–manifolds $A$, $M$ and $F$ and a map $\pi : A \to M$ such that:

i) $\pi$ is surjective.

ii) For each $m \in M$ there exists an open neighborhood $U \subseteq M$ and a $C^k$–diffeomorphism $\psi : \pi^{-1}(U) \to U \times F$ such that the following diagram commutes

$$
\begin{array}{ccc}
\pi^{-1}(U) & \xrightarrow{\psi} & U \times F \\
\downarrow & & \downarrow \pi_1 \\
U & \xrightarrow{id} & U
\end{array}
$$

\[ ]^{\text{Prop. 8.6}}
We denote this fiber bundle by $\pi : A \overset{F}{\to} M$.

We will call $A$ the **total space**, $M$ the **base space** and $F$ the **standard fiber** of the fiber bundle $\pi : A \overset{F}{\to} M$. The functions $\psi$ in ii) are called the **local trivializations** of the fiber bundle $\pi : A \overset{F}{\to} M$. A $C^\infty$–fiber bundle is a $C^k$–fiber bundle for every $k \in \mathbb{N}$.

Fiber bundles have different notions of morphism which provide different categories. We will be interested in morphisms between fiber bundles over the same base.

**Definition A.2.** A **morphism of fiber bundles** between $\pi_1 : A_1 \to M$ and $\pi_1 : A_1 \to M$ is a smooth map $\Phi : A_1 \to A_2$ such that the following diagram commutes

$$
\begin{array}{ccc}
A_1 & \xrightarrow{\Phi} & A_2 \\
\downarrow{\pi_1} & & \downarrow{\pi_2} \\
M & \xrightarrow{i} & M
\end{array}
$$

where the map $i : M \to M$ is the identity.

Throughout this article, we assume that all the fiber bundles are $C^\infty$.

**Definition A.3.**

- A **global section** of a fiber bundle $\pi : A \overset{F}{\to} M$ is a $C^\infty$–function $s : M \to A$ such that $\pi \circ s(m) = m$ for all $m \in M$. The set of all global sections of $\pi : A \overset{F}{\to} M$ will be denoted by $\Gamma(A)$.

- A **local section** of a fiber bundle $\pi : A \overset{F}{\to} M$ is a $C^\infty$–function $s : U \to A$ defined
on a open neighborhood $U$ of $M$ such that $\pi \circ s(m) = m$ for all $m \in U$

**Remark A.4.** If $M$ is a manifold, it is a usual exercise in differential geometry to observe that $TM := \bigcup_{m \in M} T_m M$ (the disjoint union of all tangent spaces of $M$) is the total space of a fiber bundle with basis $M$ and fiber $\mathbb{R}^n$ where $n$ is the dimension of the manifold $M$ and $\pi : TM \to M$ is the natural projection. Special types of fiber bundles are the principal bundles and vector bundles.

**Notation A.5.** Let $f : M \to N$ be a smooth function between two manifolds $M$ and $N$. We will denote the derivative of $f$ in $m \in M$ by $df_m : T_m M \to T_{f(m)} N$.

Let $\pi : A \xrightarrow{\Phi} M$ be a fiber bundle. For each $a \in A$, the tangent space $T_a(A_m)$ of the fiber at $m := \pi(a)$ define a vector space of directions in $T_a A$ called vertical directions, more formally:

**Definition A.6.** The subvector bundle $VA := \text{Ker}(d\pi)$ of $TA$ is called the vertical bundle of the fiber bundle $\pi : A \xrightarrow{\Phi} M$.

We observe that $T_a(A_m) = (VA)_a$ where $a \in A_m$. To define horizontal directions we need a connection on $\pi : A \xrightarrow{\Phi} M$, that is basically a choice of a projection $\Phi_a : T_a A \to T_a A$ on $(VA)_a$ for each $a \in A_m$.

**Definition A.7.** A connection $\Phi$ for a fiber bundle $\pi : A \xrightarrow{\Phi} M$ is a smooth 1–form of $A$ with values in $VA$ such that for each $a \in A$, $\Phi_a^2 = \Phi_a$ and $\text{Im}(\Phi_a) = (VA)_a$.

A connection $\Phi$ belongs to $\Omega^1(A, VA)$ and $\Phi(a)$ can be thought as a linear map from $T_a A$ to $T_a A$ for each $a \in A$. A connection $\Phi$ induces a notion of **horizontal bundle** $HA$:

$$HA := HA := \text{Im}(I_{TA} - \Phi) \subseteq TA.$$  \hspace{1cm} (A1)

Moreover, for all $a \in A$, we have that $d\pi_a : (HA)_a \to T_{\pi(a)} M$ is a canonical isomorphism, and it is easy to see that $TA = VA \oplus HA$. In particular, the tangent directions of the total space $A$ are decomposed in horizontal and vertical directions.

The parallel transport of a curve $\sigma : (-r, r) \to M$ at a point $a \in A$ is the lift $\tilde{\sigma} : (-r, r) \to A$ of $\sigma$ (i.e $\pi \circ \tilde{\sigma} = \sigma$) whose velocity belongs to the horizontal direction and such that $\tilde{\sigma}(0) = a$. The following theorem formalizes this notion and guarantees that, for every connection $\Phi$ of $A$ and every curve $\sigma$ and point $a \in A$, there exists (locally) a unique parallel transport.

**Theorem A.8.** ([Kolár et al., 1993, Theorem 9.8]) Let $\pi : A \xrightarrow{\Phi} M$ be a fiber bundle with connection $\Phi$ and let $\sigma : (-r, r) \to M$ be a smooth curve such that $\sigma(0) = m$. Then, there exists a neighborhood $U$ of $A_m \times \{0\}$ in $A_m \times (-r, r)$ and a smooth function $\tilde{\sigma} : U \to A$ such that:

i) $\pi(\tilde{\sigma}(a,t)) = \sigma(t)$ for all $(a,t) \in U \subseteq A_m \times (-r, r)$ and $\tilde{\sigma}(a,0) = a$.

ii) $\Phi_\gamma (\frac{\partial}{\partial t} \tilde{\sigma}(a,t)) = 0$ for all $(a,t) \in U$.

iii) $U$ is maximal with respect to i) and ii).
With the notation of the previous theorem, let us recall that $d\pi_a$ is an isomorphism between the horizontal bundle $HA_a$ at $a$ of the connection $\Phi$ (see (A1)) and the tangent space $T_aA$. With this identification, condition ii) of Theorem A.8 can be written

$$d\pi_{a(t)}^{-1}(\sigma'(t)) = \tilde{\sigma}'(t).$$

(A2)

Using the notation of Theorem A.8.

**Definition A.9.**

i) Given $a \in A_m$ ($m \in M$), the function $t \mapsto \tilde{\sigma}(a,t)$ defined in the previous theorem is called parallel transport along the curve $\sigma$ of $a$ (associated to the connection $\Phi$).

ii) A connection $\Phi$ on $\pi : A \overset{F}{\longrightarrow} M$ is called a complete connection, if the parallel transport $\tilde{\sigma}$ along any smooth curve $\sigma : (-r,r) \rightarrow M$ is defined in all elements belonging to $A_{\sigma(0)} \times (-r,r)$. Also we call $t \mapsto \tilde{\sigma}(a,t)$ the horizontal lift of $\sigma$ at $a$.

Intuitively, parallel transport formalizes the notion of a movement on a configuration space that does not change the internal states (see Section 2). The notion of completeness of a connection is a technical condition which will simplify this presentation. Complete connections are also called Ehresmann connections. The following theorem allows us to consider a complete connection in any fiber bundle, which helps us to avoid technicalities. This is the reason because we assume completeness of all connections considered along this article.

**Theorem A.10.** (Kolář et al., 1993, Page 81) Each fiber bundle admits complete connections.

Next, we define the notion of pullback of a fibre bundle.

**Definition A.11.** Given a smooth function $f : N \rightarrow M$ the pullback of a fiber bundle $A$ is the fiber bundle $f^*(A)$ whose total space is

$$f^*(A) := \{(n,a) \in N \times A : f(n) = \pi(a)\}$$

with projection $\pi_{f^*A}(n,a) = n$, the topology inherited from $N \times A$ and differential structure naturally defined from the trivializations induced by the fiber bundle $A$.

Given a smooth function $f : N \rightarrow M$ there is a natural morphism of fiber bundles $\tilde{f} : f^*(A) \rightarrow A$ defined by $\tilde{f}(n,a) = a$ and illustrated by the following diagram.

\[
\begin{array}{ccc}
\tilde{f} & : & f^*(A) \longrightarrow A \\
\downarrow \pi_N & & \downarrow \pi \\
N & \longrightarrow & M \\
\end{array}
\]

Given a fiber bundle $\pi : A \overset{F}{\longrightarrow} M$ with fiber $F$ and $k \in \mathbb{N} \setminus \{0\}$, we define its $k$–direct sum (denoted by denote by $\oplus_k A$) as a fiber bundle over $M$ whose fiber at $m \in M$ is given by $A_m^k$. In contrast, the Cartesian power $A^k$ corresponds to the fiber bundle
over the Cartesian power \( M^k \) with the natural projection \( \pi \times \cdots \times \pi \).

**Definition A.12.** Let \( d : M \to \Pi_{i=1}^k M \) be the diagonal function defined by \( d(m) := (m, m, \cdots, m) \). The \( k \)-**direct sum** of a fiber bundle \( \pi : A \xrightarrow{F} M \), denoted by \( \oplus_{i=1}^k A \), is defined by

\[
\oplus_{i=1}^k A := d^*(A^k).
\]

From a categorical point of view, the direct sum defined in Definition A.12 is the product of the category of fiber bundles over a fixed basis. In the categories of vector spaces and vector bundles, the direct sum coincides with the product. We hope this justifies the abuse of terminology.

We can also pullback sections of fiber bundles to sections of (the pullbacked) fiber bundle.

**Definition A.13.** Given a fiber bundle \( \pi : A \to M \) and a function \( f : N \to M \), we define the **pullback of a section** \( s : U \to M \) as the section \( f^*s \) of \( f^*A \) defined by

\[
f^*s(n) := (n, s(f(n))).
\]

The next proposition claims that it is possible to pullback connections.

**Proposition A.14.** Let \( \Phi \) be a connection on the fiber bundle \( \pi : A \to M \) and let \( f : N \to M \) be a smooth function. \( f^*(\Phi) \) induces a connection on \( f^*(A) \).

**Proof.** \( f^*(\Phi) \) is a 1–form of \( f^*(A) \) with values in \( VA \), hence we can think of it as an homomorphism from \( TF^*(A) \) to \( VA \). Recall the diagram below Definition A.11 for the definition of \( \tilde{f} \). \( df \) induces an identification of the vertical bundle of the pullback \( Vf^*(A) \) and the vertical bundle of \( A \). Hence we can define a connection \( \varphi \) on \( f^*(A) \) by \( \varphi_{(n,a)} := df_{\pi^{-1}(n)} \circ f^*(\Phi)_{(n,a)} \). Using that \( f^*(\Phi)_{(n,a)} : T_{(n,a)}f^*(A) \to VA \) is given by \( \Phi_a \circ df_n \), we can observe that \( \varphi_{(n,a)} : T_{(n,a)}f^*(A) \to T_{(n,a)}f^*(A) \) is a projection on \( V_{(n,a)}f^*(A) \).

\( \square \)

**Proposition A.14**

With an slight abuse of notation, we will denote the connection \( \varphi \) in the previous proposition by \( f^*(\Phi) \).

**Corollary A.15.** A connection \( \Phi \) on \( \pi : A \xrightarrow{F} M \) induces a connection on the \( k \)-direct sum \( \oplus_{i=1}^k A \) that we will denote by \( \oplus_{i=1}^k \Phi \).

**Proof.** The connection \( \Phi \) on the fiber bundle \( \pi : A \to M \) induces the connection \( \Pi_{i=1}^k \Phi \) on the Cartesian product \( \Pi_{i=1}^k A \) (thought as a fiber bundle over \( \Pi_{i=1}^k M \)). This connection is defined by \( \Pi_{i=1}^k \Phi(e_1, \cdots, e_k) = \Phi_{e_1} \oplus \cdots \oplus \Phi_{e_k} \), where we are identifying the tangent space \( T\Pi_{i=1}^k A(e_1, \cdots, e_k) \) naturally with \( \oplus_{i=1}^k T_{e_i}A \). This identification induces a connection on \( \oplus_{i=1}^k A \) taking \( d^*(\Pi_{i=1}^k \Phi) \) (denoted by \( \oplus_{i=1}^k \Phi \) where \( d : M \to \Pi_{i=1}^k M \) is the diagonal map.

\( \square \)

**Corollary A.15**

**Proposition A.16.** Let \( \Phi \) be a connection on the fiber bundle \( \pi : A \to M \) and let \( \sigma : (-\varepsilon, \varepsilon) \to M \) be a path such that \( \sigma(0) = m \). Let \( (m, e_1, \cdots, e_k) \in \oplus_{i=1}^k A \) and let
us denote \( \tilde{\sigma} \) the \( \phi \)-lift of \( \sigma \) to the fiber bundle \( A \) such that \( \tilde{\sigma}(0) = e_i \). If \( \alpha \) is the \( \otimes_{i=1}^k \Phi \)-lift of \( \sigma \) to the fiber bundle \( \otimes_{i=1}^k A \) such that \( \alpha(0) = (m, e_1, \ldots, e_k) \), then \( \alpha(t) = (\sigma(t), \tilde{\sigma}_1(t), \ldots, \tilde{\sigma}_k(t)) \).

**Proof.** We observe that the inclusion of \( \otimes_{i=1}^k A \) in \( A \times \cdots \times A \) is an embedding. We can see that \( H_{\otimes_{i=1}^k \Phi} \) the horizontal bundle of \( \otimes_{i=1}^k A \) is equal to

\[
(H_{\otimes_{i=1}^k \Phi}(a_1, \ldots, a_k)) = \{(v_1, \ldots, v_k) \in T(A \times \cdots \times A)_{a_1, \ldots, a_k} : d\pi_{a_1}(v_1) = \cdots = d\pi_{a_k}(v_k)\},
\]

where \( \pi : A \to M \) denotes the projection of the fibre bundle \( A \). With this identification, for all \( t \in (-\varepsilon, \varepsilon) \), \( \alpha'(t) = (\tilde{\sigma}_1'(t), \ldots, \tilde{\sigma}_k'(t)) \) since \( \tilde{\sigma}_i'(t) = d\pi^{-1}(\sigma(t)) \) for \( d\pi^{-1} \) restricted from \( TM \) to the \( \Phi \)-horizontal bundle \( HA \). Since \( \alpha(0) = (\tilde{\sigma}_1(0), \ldots, \tilde{\sigma}_k(0)) \), we have proved the proposition. \( \square \) Proposition A.16

**Proposition A.17.** Let \( f : M \to N \) be a smooth function and let \( \pi : A \to N \) be a fiber bundle with fiber \( F \) and connection \( \Phi \). Suppose that \( \sigma : (-\varepsilon, \varepsilon) \to M \) is a smooth path such that \( \sigma(0) = m \) and let \( (m, e) \in f'(A)_m \subset M \times A \) be fixed. Let us denote by \( \tilde{\alpha} \) the \( \Phi \)-horizontal lift of \( \alpha := f \circ \sigma \) such that \( \tilde{\alpha}(0) = e \). Then \( \tilde{\sigma} \), the \( f^* \Phi \)-horizontal lift of \( \sigma \) such that \( \tilde{\sigma}(0) = (m, e) \), is equal to \( t \mapsto (\sigma(t), \tilde{\alpha}(t)) \).

**Proof.** Let us denote \( \beta(t) := (\sigma(t), \tilde{\alpha}(t)) \), clearly \( \pi_{f^*A} \beta = \sigma(t) \). We have \( \beta'(t) = (\sigma'(t), \tilde{\alpha}'(t)) \). We recall that \( f^* \Phi = df^{-1} \Phi f \) (see Proposition A.14). Then \( f^*(\Phi)\beta'(t) = df^{-1} \Phi f (\sigma'(t), \tilde{\alpha}'(t)) = df^{-1} \Phi (\sigma'(t)) = 0 \). \( \square \) Proposition A.17

Next proposition synthetizes some canonical equivalences of the pullback which are used along the article.

**Proposition A.18.** Let \( f : M \to N \) and \( g : N \to P \) be smooth functions and let \( \pi : A \to P \) be a fiber bundle with fiber \( F \). Then,

i) The fiber bundle \((g \circ f)^*(A)\) is canonically isomorphic to \( f^*(g^*(A)) \).

ii) The fiber bundle \( \otimes_{i=1}^k f^*(A) \) is canonically isomorphic \( f^*(\otimes_{i=1}^k A) \).

iii) Let \( U \) be an open subset of \( P \), and let us denote \( \Gamma(U, A) \) the sections of the fiber bundle \( A \) whose domain contains \( U \). We have that there is a canonical identification of \( \Gamma(U, \otimes_{i=1}^k A) \) and \( \Gamma(U, A)^k \).

**Proof.** Let us observe that \((g \circ f)^*(A) = \{(m, a) \in M \times A : (g \circ f)(m) = \pi(a)\}\) and

\[
f^*(g^*(A)) = \{(m, b) \in M \times g^*(A) : f(m) = \pi_{f^*(A)}(b)\} = \{(m, n, a) \in M \times N \times A : f(m) = n \text{ and } g(n) = \pi(a)\}.
\]

It is straightforward to see that the canonical isomorphism is given by \( (m, a) \mapsto (m, f(m), a) \).
We have that
\[ f^*(\oplus_{i=1}^k A) = \{(n, m, a_1, \ldots, a_k) : f(n) = m \text{ and } a_i \in A_{m_i}\} \]
and
\[ \oplus_{i=1}^k f^*(A) = \{(n, ((n, a_1), \ldots, (n, a_k)) \in N \times f^*(A)^k : a_i \in A \text{ and } \pi(a_i) = f(n)\}. \]

The isomorphism between \( f^*(\oplus_{i=1}^k A) \) and \( \oplus_{i=1}^k f^*(A) \) is given by \( (n, ((n, a_1), \ldots, (n, a_k))) \mapsto (n, f(n), a_1, \ldots, a_k) \).

The identification of \( \Gamma(U, \oplus_{i=1}^k A) \) and \( \Gamma(U, A)^k \) is given by \((s_1, \ldots, s_k) \mapsto \bar{s}\) where \( \bar{s}(p) = (p, s_1(p), \ldots, s_k(p)) \) for \( p \in P \). Given \( \bar{s} \in \Gamma(U, \oplus_{i=1}^k A) \), with \( \bar{s}(p) = (p, s_1(p), \ldots, s_k(p)) \), its inverse under this map is the \( k \)-tuple \((s_1, \ldots, s_k) \in \Gamma(U, A)^k \).

**Proposition A.19.** Let \( \pi : A \to N \) be a fiber bundle with fiber \( F \) with a connection \( \Phi \). Let \( \sigma : (-e, e) \to M \) be a path such that \( \sigma(0) = m \) and let us denote \( \alpha(t) := t \), the identity path on \( \mathbb{R} \). Then, we can naturally interpret the \( \sigma^* \Phi \)-horizontal lifts of \( \alpha \), as a section of the fiber bundle \( \alpha^* A \). Moreover, given \( e \in A_m \), if \( \tilde{\sigma} \) denotes the \( \Phi \)-horizontal lift of \( \sigma \) such that \( \tilde{\sigma}(0) = e \) and \( s \) is the \( \sigma^* \Phi \)-horizontal lift of \( \alpha \) such that \( s(0) = (0, e) \), then \( s(t) = (t, \tilde{\sigma}(t)) \).

**Proof.** \( s(t) = (t, \tilde{\sigma}(t)) \) is a direct consequence of Proposition A.17. Clearly, \( s(t) \) is a section of \( \sigma^* A \).

**Definition A.20.** Let \( \pi_1 : A_1 \to M \) and \( \pi_2 : A_2 \to M \) be fiber bundles with connections \( \Phi_1 \) and \( \Phi_2 \). We say that these fiber bundles are isomorphic if there is an isomorphism of fiber bundles
\[
\begin{array}{ccc}
A_1 & \xrightarrow{h} & A_2 \\
\downarrow{\pi_1} & & \downarrow{\pi_2} \\
M & \xrightarrow{i} & M
\end{array}
\]
which preserves the horizontal bundles induced by the connection, explicitly \( dh(HA_1) = HA_2 \).

Isomorphism of fiber bundles with connections send horizontal liftings in horizontal liftings.

As a consequence of the previous propositions, we have.

**Corollary A.21.** Let \( f : M \to N \) and \( g : N \to P \) be smooth functions and let \( \pi : A \to N \)
be a fiber bundle with fiber $F$ and connection $\Phi$. The following pairs of fiber bundles with connection are isomorphic,

1. $(f \circ g)^*A, (f \circ g)^*\Phi$ and $f^*(g^*A), f^*(g^*\Phi)$.
2. $(f^*(\oplus_{i=1}^k A_i), f^*(\oplus_{i=1}^k \Phi_i))$ and $(\oplus_{i=1}^k f^*A_i, \oplus_{i=1}^k f^*\Phi_i)$.

We will denote by

$$\chi := I_{TA} - \Phi$$

the projection on the horizontal bundle of a connection $\Phi$.

**Definition A.22.** Let $\Phi$ be a connection on the fiber bundle $\pi : A \xrightarrow{E} M$. The curvature of $\Phi$ is the two form $R$ with values in $VA$ defined by

$$R(X,Y) := \Phi([\chi(X), \chi(Y)])$$

where $X, Y$ are vector fields of $A$.

The curvature of the connection is an obstruction (via the Frobenius theorem, see [Warner (1983)](#)) to the integrability of the differential distribution $HA_\Phi$ on $A$.

**Definition A.23.** Given $\Phi$ a connection on $\pi : A \xrightarrow{E} M$, we will say that a local section $s : M \rightarrow A$ is $\Phi$–parallel if the horizontal bundle $HA_\Phi$ restricted to $U$ is equal to the image of $ds$ restricted to $TU$.

We observe that $\Phi$–parallel sections exist only when the connection $\Phi$ has curvature 0.

**Proposition A.24.** Let $\Phi$ be a connection on the fiber bundle $\pi : A \xrightarrow{E} M$ whose curvature $R$ is 0. Then, for every $m \in M$, and every $e \in A_m$, there is a neighborhood $U$ of $m$ and a local section $s : U \rightarrow A$ such that $s$ is a $\Phi$–parallel section and $s(m) = e$.

**Proof.** Let us take $m \in M$ and $e \in A_m$. Since the curvature of $\Phi$ is zero, the horizontal bundle $HA_\Phi$ of $\Phi$ is integrable. The theorem of Frobenius (see [Warner (1983)](#)) implies that there exists an integral submanifold $N$ of $M$ which contains $a_i.e.$ a submanifold $N$ such that for all $n \in N$, $T_n N = (HA_\Phi)_n$. If we take $f := \pi|_N$ then $df : (HA_\Phi)_a \rightarrow T_{\pi(a)} M$ is an isomorphism. The inverse function theorem implies that there exists an open neighborhood $U$ of $m$ such that $f^{-1} : U \rightarrow N$ exists. We can take $s \in f^{-1}$.

**Remark A.25.** As a general principle, we work in this article around small open neighborhoods of the point $m \in M$. Hence, the section whose existence is claimed by Proposition A.24 is unique in the sense that, if $s_1$ and $s_2$ with connected domains $U_1 \subset M$ and $U_2 \subset M$ respectively, are parallel sections such that $s_1(m) = s_2(m) = e$, then $s_1 = s_2$ in the open neighborhood of $m U_1 \cap U_2$.

Each section $s$ of the previous proposition is a local section of $A$ and for all path $\sigma : (-\varepsilon, \varepsilon) \rightarrow M$ such that $\sigma(0) = m$, we have

$$\tilde{\sigma}(t) = s \circ \sigma(t) \quad \text{(A3)}$$
where $\tilde{\sigma}$ is the $\Phi$–lift of a path $\sigma$ such that $\tilde{\sigma}(0) = e$ (see (A2)).

**Remark A.26.** On $\mathbb{R}^n$ there is a natural identification of the tangent space of $\mathbb{R}^n$ $T\mathbb{R}^n$ and $\mathbb{R}^n \times \mathbb{R}^n$, specifically we identify $T_{(p_1,\ldots,p_n)} \mathbb{R}^n$ with $\mathbb{R}^n$ using the linear map $(a_1,\ldots,a_n) \mapsto \sum_{i=1}^{n} a_i \partial_{x_i}$. 

Many examples of this article will be related with the following basic fiber bundle with connection.

**Example A.27.** Consider the fiber bundle $\rho: \mathbb{R}^3 \to \mathbb{R}^2$ with $\rho(x,y,z) := (x,y)$ whose fiber is $\mathbb{R}$. Let us define $\Phi = dz \otimes \partial_z$. Using the identification of Remark A.26, we can write $\Phi_{(x,y,z)^t} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. One can check that $HA_{(x,y,z)^t} = \text{Span}\{\partial_x|_{(x,y,z)^t}, \partial_y|_{(x,y,z)^t}\}$. With the identification of Remark A.26, $HA_{(x,y,z)^t} = \text{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$. The isomorphism of (A2) is given by $d\pi^{-1}(a\partial_x + b\partial_y) = a\partial_x + b\partial_y$.

Let us denote $\sigma(t) := (mt, nt)$, for fixed $m,n$ in $\mathbb{N}$, with this considerations, it is straightforward to see that $\tilde{\sigma}$ the $\Phi$–horizontal lift of $\sigma$ such that $\tilde{\sigma}(0) = (0,0,a)$ is $\tilde{\sigma}(t) = (mt, nt, a)$.

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