Rigid particle revisited: extrinsic curvature yields the Dirac equation

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We reexamine the model of relativistic particle with higher-derivative term linear on the first extrinsic curvature (rigidity). The passage from classical to quantum theory requires a number of rather unexpected steps which we report here. We found that, contrary to common opinion, quantization of the model in terms of so(3,2)-algebra yields massive Dirac equation.

I. INTRODUCTION

At the end of eighties much attention has been paid to the study of relativistic mechanical models with Lagrangians depending on extrinsic curvatures of the world line. These studies were mostly inspired by the Polyakov’s papers on rigid strings [1] and Chern-Simons theories [2]. Initially these models were considered as toy models for the above mentioned field-theoretical ones, but very soon it was realized that they are of their own interest, and probably could be considered as mechanical models of relativistic spin (see, e.g. [3]-[10]). The first interest, and probably could be considered as mechanical models for the above mentioned field-theoretical ones, theories [2]. Initially these models were considered as toy Polyakov’s papers on rigid strings [1] and Chern-Simons world line. These studies were mostly inspired by the Lagrangians depending on extrinsic curvatures of the system of this sort has been suggested by Pisarski [3] as k

\[ S = \int d\tau \sqrt{-\dot{x}^2} \left[ -mc + c_1 k_1(\dot{x}, \ddot{x}) \right], \]  

(1)

where \( k_1 \) is the first extrinsic curvature of world line

\[ k_1 = \sqrt{\frac{\dot{x}^2 \dot{\ddot{x}}^2 - (\dot{x} \ddot{x})^2}{|\dot{x}|^3}}, \quad |\dot{x}| = \sqrt{-\dot{x}^2}. \]  

(2)

This system, as well as other three- and four-dimensional systems with the Lagrangians depending on extrinsic curvatures were investigated by many authors. It has been observed that when \( m = 0 \), it describes massless particle with the helicity \( c_1 \), while the case \( m \neq 0, c_1 \neq 0 \) implies a model with ten-dimensional phase space, which on the quantum level does not describe an elementary system (that is an irreducible representation of Poincaré group). Further studies of similar systems in arbitrary space-time [12], as well as on null-curves [8] result in the same conclusion: only the actions proportional to single extrinsic curvature yield, upon quantization, the massless irreducible representations of Poincaré group. It was also established that the mass term in (1) prohibits the constraint which could be classical analog of Klein-Gordon equation.

However, Klein-Gordon equation could follows from the Dirac one at the quantum level, so that it is not necessary to have an analog of the later equation at the classical level [11][12]. Besides, the conclusion on reducibility of quantum space of states has been made from analysis of constraints of classical theory. Here we use an ambiguity in the passage from classical to quantum theory which was unnoticed in previous works. This allows us to fix the second Casimir operator [9] of Poincare group.

In the present work, following this ideology, we analyze the action (1) with \( m \neq 0 \) and \( c_1 = \frac{\sqrt{3}}{2} \), and show that its canonical quantization leads to the Dirac equation with the mass

\[ M = \frac{\sqrt{3}}{2} m. \]  

(3)

Any other choice of \( c_1 \) turns out to be inconsistent with our quantization procedure, see below. So, in contrast with common opinion, quantization of rigid particle given by the action (1) results in the elementary system of spin one-half.

II. HAMILTONIAN FORMULATION

Consider the time-like curve \( x^\mu(\tau) \) (parameterized by arbitrary \( \tau \)) in four-dimensional Minkowski space \( \eta^{\mu\nu} = (-,+,+,+) \) \( \dot{x}^2 < 0 \).

Let us consider the action (1), denoting, for convenience, \( -mc = c_0 \). To be able to construct Hamiltonian formulation of the theory, we first use the Ostrogradsky method and represent our higher-derivative Lagrangian as a Lagrangian with first-order derivatives. That is we represent (1) in the form

\[ S = \int d\tau \left[ c_0 |\omega| + c_1 \sqrt{\frac{\omega N(\omega)\omega}{|\omega|}} + p(\dot{x} - \omega) \right], \]  

(4)

where we have introduced the projector

\[ N^{\mu\nu} = \eta^{\mu\nu} - \frac{\omega^\mu \omega^\nu}{\omega^2}: \quad \omega_\mu N^{\mu\nu} = 0, \quad N^2 = N. \]  

(5)

Let us construct Hamiltonian formulation of the model. We denote the conjugated momenta for \( \omega^\mu \) as \( p_\mu \), and the conjugated momenta for \( x^\mu \) by \( p_\mu \). Applying the standard machinery [13][14] we get that the system possesses

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two primary constraints
\[ \omega \pi = 0, \quad \omega^2 \pi^2 + c_1^2 = 0, \]  
and the Hamiltonian
\[ H = p\omega - c_0|\omega| + v_1(\omega^2 \pi^2 + c_1^2) + v_2 \omega \pi, \]
where \( v_1 \) are Lagrangian multipliers associated with the primary constraints. Note, that Eq. (6) together with theory as a model of spin one-half particle.

Computing the Dirac brackets (10) and impose the first-class constraints
\[ \{\omega^\mu, \pi^\nu\} = \eta^{\mu\nu}, \quad \{\omega^\mu, \omega^\nu\} = \frac{\omega^2}{\pi^2(p\omega)} \eta^{\mu\nu}, \]
\[ \{\mu^\mu, \pi^\nu\} = \eta^{\mu\nu} - \frac{p^\mu \omega^\nu}{p\omega}, \quad \{\pi^\mu, \pi^\nu\} = 0, \]
\[ \{\omega^\mu, x^\nu\} = -\frac{\omega^2}{\pi^2(p\omega)} \omega^\mu \pi^\nu, \quad \{\pi^\mu, x^\nu\} = \frac{1}{p\omega} \omega^\mu \pi^\nu. \]

So, we are ready to quantize our model. We could quantize the Dirac brackets (10) and impose the first-class constraints (8) as operator equations on quantum states, thus obtaining some equations for the wave function.

**III. QUANTIZATION**

The function \( v_2 \) was not determined through the Dirac procedure, and enters as arbitrary function into general solution to equations of motion (11) for \( x \) and \( \omega \). Dynamics of basic variables is ambiguous, which is reflected by invariance of the action (11) under local transformations studied in details in (12). So we pass from initial gauge non-invariant variables to the set of candidates for observables. Then we show that quantization of the set admits a reasonable interpretation of resulting quantum theory as a model of spin one-half particle.

Namely, we note that in the model there is the set of phase-space functions
\[ \hat{J}^\mu = 2c_1 K^\mu (\frac{\omega^\alpha}{|\omega|} + \frac{p^\rho}{c_0}), \]
\[ \hat{J}^{\mu\nu} = 2K^\mu (\frac{\omega^\alpha + (\eta^{\mu\rho} p^\sigma/c_0) \pi^\sigma}{|\omega|} \pi^\beta - (\mu \leftrightarrow \nu), \]
where
\[ K^\alpha = \delta^\alpha_\mu A p^\mu p_\alpha, \quad A = \frac{\sqrt{p^2 + c_1^2} - |\omega|}{p^2 \sqrt{p^2 + c_0^2}. \]

Their Dirac brackets generate, on the first-class constraint surface, the \( so(3,2) \)-algebra
\[ \{\hat{J}^{5\mu}, \hat{J}^{5\nu}\} = -2 \hat{J}^{\mu\nu}, \]
\[ \{\hat{J}^{\mu\nu}, \hat{J}^{5\alpha}\} = 2(\eta^{\mu\alpha} \hat{J}^{5\nu} - 2\eta^{\rho\nu} \hat{J}^{5\rho}), \]
\[ \{\hat{J}^{\mu\nu}, \hat{J}^{\alpha\beta}\} = 2(\eta^{\mu\alpha} \hat{J}^{\nu\beta} - \eta^{\rho\beta} \hat{J}^{\nu\rho} - \eta^{\rho\nu} \hat{J}^{\rho\beta} + \eta^{\mu\beta} \hat{J}^{\rho\alpha}). \]

To see that this is indeed \( so(3,2) \), let us denote \( \hat{J}^{AB} = (\hat{J}^{5\mu}, \hat{J}^{\mu\nu}) \) and introduce the five-dimensional metric \( \eta^{AB} = (-, +, +, +, -) \). Then the algebra (14) acquires the form
\[ 2(\eta^{AC} \hat{J}^{BD} - \eta^{AD} \hat{J}^{BC} - \eta^{BC} \hat{J}^{AD} + \eta^{BD} \hat{J}^{AC}), \]
which is just \( so(3,2) \)-algebra.

The first-class constraints (5) can be presented through the new variables as follows
\[ p_\mu \hat{J}^{5\mu} + 2c_1 \sqrt{p^2 + c_0^2} = 0, \]
\[ (\omega \pi)^2 = -c_1^2 + \frac{1}{p^2} \left( \frac{1}{4} S^\mu S_\mu - c_1^2 \right) \]
where \( S^\mu = \frac{1}{2} \epsilon^{\mu\alpha\beta\gamma} p_\alpha \hat{J}_{\beta\gamma} \) is the Pauli-Lubanski vector (then \( S^2 \) represents Casimir operator of Poincare group).

The initial coordinates \( x^\mu \) have non-zero Dirac brackets with \( \hat{J}^{AB} \). To improve this, we introduce the effective coordinates \( X^\mu = x^\mu + |w| \pi^\mu/c_0 \) commuting with \( \hat{J}^{AB} \), and take operators associated with \( X^\mu \), \( p_\mu \) in the standard form, \( \hat{X}^\mu = X^\mu, \hat{p}_\mu = -i\hbar \frac{\partial}{\partial X^\mu} \).

To quantize the spin variables, we look for operators with commutators obeying the Dirac-brackets algebra on the constraint surface, the equation (14). That is we adopt the rule \( [\cdot, \cdot] = i\hbar \{\cdot, \cdot\}_{DBA|CC|q \rightarrow \tilde{q}} \). This guarantees the correspondence principle: the operators (in the Heisenberg picture) and corresponding classical variables will obey the same equations of motion.

Choice of spin-sector operators is dictated by the constraint \( \omega \pi = 0 \) as follows. According to Eq. (14), this contains product of variables with non-vanishing Dirac brackets, so the corresponding quantum operator will contain product of non-commuting operators. Any
two (Lorentz-invariant) operators which we can associate with the constraint differ on a number. So, there is an ambiguity in the passage from classical to quantum theory

\[ \omega \Psi = 0 \rightarrow \hat{\omega} \Psi = c_2. \]  

We propose to fix the ordering constant to be \( c_2^2 = -c_1^2 \). Then quantum counterpart of equation (16) states that Casimir operator of the Poincare group has fixed value

\[ S^2 = 4c_1^2c_0^2 = 3\hbar^2m^2c^2, \]  

corresponding to spin one-half irreducible representation (Note that our state-space is picked out by unique equation (10). The only irreducible representation of Poincaré group of such a kind is the spin one-half representation. So, any choice of \( c_1 \) different from \( c_1 = \sqrt{\frac{2\omega}{\sqrt{3}m c}} \) would lead to inconsistent picture. Besides, similarly to string theory, other choice of \( c_2 \) would lead to appearance of negative norm states in the spectrum).

Hence we are forced to quantize the variables \( \hat{J}^{5\mu} \) and \( \hat{j}^{\mu
u} \) by \( \gamma \)-matrices, \( \hat{J}^{5\mu} \rightarrow \hat{h}_\gamma^{\mu}, \hat{j}^{\mu
u} \rightarrow \hat{h}_\gamma^{\mu
u} \), where

\[ \gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \]  

and

\[ \gamma^{\mu
u} = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu). \]  

They form a representation of \( so(3,2) \)

\[ \left[ \gamma^\mu, \gamma^\nu \right] = -2i\gamma^{\mu\nu}, \quad \left[ \gamma^{\mu\nu}, \gamma^\alpha \right] = 2i\left( \gamma^{\mu\alpha} \gamma^\nu - \gamma^{\nu\alpha} \gamma^\mu \right), \quad \left[ \gamma^\mu\gamma^\nu, \gamma^\alpha\gamma^\beta \right] = 2i\left( \gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta - \gamma^\mu\gamma^\nu\gamma^\beta\gamma^\alpha - \gamma^\nu\gamma^\mu\gamma^\alpha\gamma^\beta + \gamma^\nu\gamma^\mu\gamma^\beta\gamma^\alpha \right). \]  

The operators act on space of Dirac spinors \( \Psi_a, a = 1, 2, 3, 4 \).

Let us see the meaning of the first-class constraint (16). Its quantum counterpart reads

\[ \left( \gamma^\mu \hat{p}_\mu + \frac{2c_1}{\hbar} \sqrt{\hat{p}^2 + c_0^2} \right) \Psi = 0. \]  

Then we obtain, as a consequence, the following Klein-Gordon equation

\[ \left( \hat{p}^2 + \frac{4c_1^2}{\hbar^2} \right) \Psi = 0. \]  

Substitute (24) into (23), this gives the Dirac equation

\[ (\gamma^\mu \hat{p}_\mu - 2c_0 c_1 (\hbar^2 + 4c_1^2)^{-\frac{1}{2}}) \Psi = 0. \]  

Taking \( c_0 = -mc \) and \( c_1 = \sqrt{\frac{2\omega}{\sqrt{3}mc}} \), we arrive at the final result

\[ \left( \gamma^\mu \hat{p}_\mu + \frac{\sqrt{3}}{2} mc \right) \Psi = 0. \]  

In resume, canonical quantization of the action (H) in properly chosen variables (11), (12) leads to the theory of spin one-half particle. This observation contradicts with common opinion about role of higher-derivative models in the description of massive spinning particles. Many questions arise in this respect. Is it possible to introduce reasonable interaction (13) with electromagnetic and curved backgrounds? Could other models with Lagrangians depending on extrinsic curvatures (say, on torsion) describe massive and massless spinning particles on space of non-constant curvature? If so, of which kind, and of which spin? Could this revision change our opinion on the role of such systems in quantum optics (17) and polymer physics (18, 19)? Clarification of this questions should be subject of separate investigation.

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[9] A. Nersessian, E. Ramos, Phys. Lett. B 445 (1998) 123; Mod. Phys. Lett. A 14 (1999) 2033.
[10] R. Banerjee, P. Mukherjee, B. Paul, JHEP 1108 (2011) 085.
[11] A. A. Deriglazov, Ann. Phys. 327 (2012) 398; Phys. Lett. A 376 (2012) 309; ibid. A 377 (2012) 13.
[12] A. A. Deriglazov, B. F. Rizzuti, G. P. Z. Chauca, P. S. Castro, J. Math. Phys. 53 (2012) 122303.
[13] D. M. Gitman and I. V. Tyutin, Quantization of fields with constraints, Springer-Verlag, Berlin, 1990.
[14] A. A. Deriglazov, Classical mechanics, Hamiltonian and Lagrangian formalism, Springer-Verlag, 2010.
[15] A. Nersessian, Theor. Math. Phys. 126 (2001) 147.
[16] D. A. Aghamalyan J. Contemp. Phys. (Armenian Academy of Science) 43 (2008) 261.
[17] D. Aghamalyan and A. Nersessian, Yad. Fiz. 73 (2010) 268 [Phys. Atom. Nucl. 73 (2010) 247].
[18] S.B. Smith, Y. Cui and C. Bustamante, Science 271 (1996) 795.
[19] A. Feoli, V. V. Nesterenko, G. Scarpetta, Nucl. Phys. 705 (2005) 577.