A syntactic proof of decidability for the logic of bunched implication BI

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Abstract. The logic of bunched implication BI provides a framework for reasoning about resource composition and forms the basis for an assertion language of separation logic which is used to reason about software programs. Propositional BI is obtained by freely combining propositional intuitionistic logic and multiplicative intuitionistic linear logic. It possesses an elegant proof theory: its bunched calculus combines the sequent calculi for these logics. Several natural extensions of BI have been shown as undecidable, e.g. Boolean BI which replaces intuitionistic logic with classical logic. This makes the decidability of BI, proved recently via an intricate semantical argument, particularly noteworthy. However, a syntactic proof of decidability has thus far proved elusive. We obtain such a proof here using a proof-theoretic argument. The proof is technically interesting, accessible as it uses the usual bunched calculus (it does not require any knowledge of the semantics of BI), yields an implementable decision procedure and implies an upper bound on the complexity of the logic.

1 Introduction

The logic of bunched implication BI [13,14] provides a logical framework expressive enough to reason about resource composition and systems modelling and forms the basis for an assertion language of separation logic [9] used to reason about software programs. Specifically, a resource-aware logic is used in order to reason about the heap and the other consumable resources to which a program has access [1]. More generally, since every action in the real world generates or consumes resources, an ability to reason about such actions is critical from an AI perspective [16], and BI provides a formal logical system for such analysis [10]. In this paper a constructive proof of decidability of propositional BI is obtained using its standard proof calculus (so a knowledge of the semantics is not required to check the details in constraint to the existing proof [6]) which in turn yields an implementable decision procedure that implies an upper bound on the logical complexity. These issues are particularly important given the application-oriented resource interpretations of this logic.

Informally speaking, the logic BI can be seen as a system of reasoning that supports distinct and simultaneous argumentation in both propositional intuitionistic logic Ip and multiplicative intuitionistic linear logic MILL. Thus the

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logical operators are the union of the logical operators of these logics. A proof calculus for $\text{BI}$ can be obtained in the sequent calculus formalism (this is the formalism introduced by Gentzen [7] who used it to give a consistency proof of arithmetic) in a particularly simple manner by combining the rules of the sequent calculi for these logics. The language of propositional intuitionistic logic $\text{Ip}$ consists of the additive connectives $\lor$, $\land$, $\rightarrow$ and constants $\top$, $\bot$. The sequent calculus for $\text{Ip}$ is built from sequents of the form $X \vdash A$ where $X$ is a semicolon-separated list of $\text{Ip}$ formulae and $A$ is an $\text{Ip}$ formula. Following from the interpretation of semicolon as $\land$, the semicolon is given commutative, associative, contraction and weakening properties. Meanwhile the language of multiplicative intuitionistic linear logic $\text{MILL}$ consists of the multiplicative connectives $\otimes$, $\multimap$ and constant $1$. The sequent calculus for $\text{MILL}$ is built from sequents of the form $X \vdash A$ where $X$ is a comma-separated list of $\text{MILL}$ formulae and $A$ is an $\text{MILL}$ formula. Following from the interpretation of comma as $\otimes$, the comma is given commutative and associative properties (but it does not have contraction and weakening properties).

The formulae of $\text{BI}$ are constructed from the logical connectives and constants of $\text{Ip}$ and $\text{MILL}$. The bunched calculus $\text{LBI}$ can be viewed as the minimal extension of the $\text{Ip}$ and $\text{MILL}$ sequent calculi in the sense that it is built from bunched sequents of the form $X \vdash A$ where $X$ is a $\text{BI}$ formula and $X$ is a structure built from commas and semicolons starting from $\text{BI}$ formulae (hence the name ‘bunch’). The rules of the $\text{LBI}$ calculus are essentially the union of the rules for the $\text{Ip}$ and $\text{MILL}$ sequent calculi; while the structural connectives comma and semicolon are retained in $\text{LBI}$ and preserve their distinct properties, the rules from the two calculi are ‘linked’ by the use of a single $\vdash$ symbol. The consequence is an elegant proof-theory for $\text{LBI}$ which implies directly that $\text{BI}$ is conservative over three important logics: $\text{Ip}$ (which has no multiplicative connectives), $\text{MILL}$ (no additive connectives) and (bounded) Distributive Commutative Full Lambek logic $\text{DFL}_e$ (no intuitionistic implication $\rightarrow$).

The only known proof of decidability of $\text{BI}$ is the semantic proof [8] which uses resource tableaux. The proof there is intricate and requires the development of a large semantic framework, including objects “[to reflect] the information that can be derived from a given set of assumptions” and to built countermodels. Indeed the authors observe: “The relationships identified between resources, labels, dependency graphs, proof-search and resource semantics are central in this study [to prove decidability and finite model property].”

An alternative and distinct method of showing the decidability of a logic is to prove that its proof calculus need examine only finitely many proof candidates—the number depending on the given formula—to determine if the formula is a theorem of the logic or not. This method is called syntactic because it relies on the syntactic calculus (which is a finitistic object in some sense) and not on any semantics of the logic. To facilitate such a proof, the proof calculus has to be well-behaved enough to start with e.g. a sequent calculus with the subformula property which restricts the logical complexity of formulae that may appear in a possible proof. Then the main challenge is to suitably restricting the set of proof
candidates. The simplicity and elegance of a sequent calculus can be deceiving here; the elegant Lambek calculus with contraction FLc was recently proved undecidable [9], while Kripke [11] famously showed that decidability holds when the exchange rule is added (FLec). The decidability of a number of different relevant logics has been shown [8,2] by argumentation on bunched calculi. Nevertheless, no syntactic proof of decidability has been forthcoming for BI thus far. A significant complication of BI compared to those logics is the presence of two implications → and ⊸ interpreted via the same structural connective ⊢.

In this work we obtain a syntactic proof of decidability for BI. The contribution of a syntactic proof, applying in particular to the proof we present here, is that it directly yields a decision procedure for the logic. Indeed the proof calculus and the decision procedure could be implemented to obtain an (automated) theorem prover for BI. Moreover an upper bound for the complexity of the logic is also implied (we are not aware of any existing complexity bound for BI). On the technical side, the weight function that we introduce on sequents is interesting and novel and illustrates the scope of argument available on these calculi. Our proof is direct—rather than via a detour through the semantics—and uses standard bunched calculi which, in our opinion, makes it more accessible and easy to check than the semantic proof.

One further advantage of a syntactic proof is that it might be adapted directly to ‘syntactically-related’ logics with interesting resource interpretations. Intuitionistic layered graph logic ILGL [4] (replace MILL with its non-commutative non-associative counterpart) was proposed recently as a logic for reasoning about layers e.g. the infrastructure and social layer in a transport network, the relationship between a security policy and the system architecture. It seems that a similar argument to the one given here can be used to obtain decidability.

2 Preliminaries

We assume a countable infinite set 𝒫 of propositional variables. A formula of BI is a finite term from the following grammar.

\[ A := p \in \mathcal{V} \mid \top \mid \bot \mid 1 \mid (A \lor A) \mid (A \land A) \mid (A \rightarrow A) \mid (A \otimes A) \mid (A \twoheadrightarrow A) \]

The set of formulae is denoted \( \mathcal{Fm} \). For new symbols \( \varnothing_m \) and \( \varnothing_a \) define \( \mathcal{Fm}^{\varnothing} := \mathcal{Fm} \cup \{ \varnothing_m, \varnothing_a \} \). A bunch is a finite term from the following grammar:

\[ X := A \in \mathcal{Fm} \mid \varnothing_a \mid \varnothing_m \mid (X, X) \mid (X; X) \]

Definition 1 (sequent, sequent rule, sequent calculus). A sequent (denoted \( X \vdash A \)) is an ordered pair where \( X \) is a bunch and \( A \) is a formula. The structure \( X \) is called the antecedent of the sequent.

A sequent rule is typically written as follows for \( n \geq 0 \).

\[
\frac{X_1 \vdash A_1 \quad \ldots \quad X_n \vdash A_n}{X_0 \vdash A_0}
\]
The sequents above the line are called the premises and the sequent below the line is called the conclusion. A rule with no premises is called an initial sequent. A sequent calculus consists of a set of sequent rules.

**Definition 2 (derivation).** A derivation in a sequent calculus is defined recursively in the usual way as an initial sequent or the object obtained by applying some sequent rule to a smaller derivation.

Viewing a derivation as a sequent-labelled tree in the usual way, so that the initial sequents are leaves, the *height* of a derivation is the number of sequents (i.e. nodes) along its longest branch (i.e. the longest path from root to leaf).

**Notation.** Any formula built using binary connectives can be viewed in a natural way as an ordered binary tree. We write $U, V \ (U; V)$ to mean a tree with root node comma (resp. semicolon) and children $U$ and $V$. We write $\Gamma = \Gamma[A]$ ($A \in \text{Fm}^\emptyset$) to indicate a specific occurrence of $A$ in $\Gamma$. Also $\Gamma[U, V]$ indicates that $\Gamma$ contains a comma node with two children $U$ and $V$. Similarly $\Gamma[U; V]$ indicates that $\Gamma$ contains a semicolon node with two children $U$ and $V$. Later we extend this notation to non-binary trees (see after Def. 6). The symbol $\equiv$ is used to denote syntactic equality.

**Definition 3 (LBI).** The sequent calculus **LBI** consists of the rules in Fig. 1 where the antecedent of every sequent is read as a bunch.

The calculus **LBI** is identical to the original calculus [14,6] except (1) for technical reasons (see above Eg. 4) we use the projective ($\land_\ell$) rule, and (2) we explicitly present the associativity, exchange and identity rules for comma and semicolon. In contrast, the original calculus uses the following rule:

$$
\frac{X \vdash A}{Y \vdash A} \quad \text{(E) } X \equiv Y
$$

The equivalence relation $\equiv$ is specified as (i) the commutative monoid equations for $\emptyset_m$ and “,”, (ii) the commutative monoid equations for $\emptyset_a$ and “;” and (iii) congruence: if $X \equiv Y$ then $\Gamma[X] \equiv \Gamma[Y]$.

Following the standard terminology, the occurrences of the formulae in the premise are called the *active formula(e)* of the rule. Meanwhile the formula in the conclusion is called the *principal formula* of the rule.

**Definition 4 (interpretation of a structure).** The interpretation $X^I$ of a structure $X$ is the formula obtained by reading each comma as $\otimes$, semicolon as $\land$, $\emptyset_m$ as $1$ and $\emptyset_a$ as $\top$.

Several different semantics for **BI** have been proposed [14,15,6], including the important resource semantics. Since the focus of this paper is on syntax, we refer the reader to the literature for the details. The algebraic semantics are given by the class of **BI**-algebras i.e. Heyting algebras which carry an additional ordered commutative monoid structure with binary operation $\otimes$, identity $1$ and linear implication $\rightarrow$ such that $x \otimes y \leq z$ if $x \leq y \rightarrow z$ ($\leq$ is the Heyting lattice order).

**Theorem 1 ([14]).** The bunched sequent $X \vdash A$ is **LBI**-derivable iff the inequality $X^I \leq A$ is valid on all **BI**-algebras.
can also absorb the structural constants via $∅$ separated lists do not contain multiple occurrences of the same element. We certainly restrict our attention to simplified bunches in $LBI$. In particular, bunches differing only in the multiplicities of elements in so e.g. $p, p, p$ is labelled by a bunch sequent built using subformulae from the root.

3 The weight of a sequent: motivating the definition

We will effectively generate candidate trees via backward proof search from a given bunch sequent root. The root is valid in $BI$ iff one of the candidate trees is a $LBI$-derivation. It suffices to find a $LBI$-derivation of minimal height so we can exclude candidate trees which contain multiple occurrences of the same sequent on a branch. $LBI$ has the subformula property so each node in a candidate tree is labelled by a bunch sequent built using subformulae from the root.

Recall that the comma does not have the contractive or weakening properties, so e.g. $p, p \vdash p \otimes p$ is $LBI$-derivable but $p \vdash p \otimes p$ and $p, p, p \vdash p \otimes p$ are not. In particular, bunches differing only in the multiplicities of elements in their comma-separated lists need to be considered as distinct. However we can certainly restrict our attention to simplified bunches in $LBI$ such that semicolon-separated lists do not contain multiple occurrences of the same element. We can also absorb the structural constants via $∅, X \mapsto X$ and $∅, X \mapsto X$ to

\[ C \vdash C \]
\[ C \in \text{Fm} \]
\[ \emptyset \vdash 1 \]
\[ (1r) \]
\[ \Gamma \vdash C \]
\[ (\perp) \]
\[ X \vdash \top \]
\[ (\top) \]

(A) Initial sequents, logical constants and proper structural rules:

(B) Rules simulating Pym’s (E) rule:

\[ \Gamma[(X, Y), Z] \vdash A \]
\[ \Gamma[X, (Y, Z)] \vdash A \]
\[ \Gamma[X, Y] \vdash A \]
\[ \Gamma[X, X] \vdash A \]
\[ \Gamma[X] \vdash A \]
\[ \Gamma[\emptyset] \vdash A \]
\[ \Gamma[\emptyset, \emptyset] \vdash A \]

(C) Additives:

\[ X; C \vdash D \]
\[ X \vdash C \rightarrow D \]
\[ (\rightarrow) \]
\[ (\rightarrow r) \]
\[ \vdash \Gamma[C_1 \land C_2] \vdash A \]
\[ (\land) \]
\[ \vdash \Gamma[C_1] \vdash A \]
\[ \Gamma[C_1 \rightarrow C_2] \vdash A \]
\[ (\land r) \]
\[ \vdash \Gamma[C_1] \vdash A \]
\[ \Gamma[C_1 \land C_2] \vdash A \]
\[ (\land l) \]
\[ \vdash \Gamma[D] \vdash A \]
\[ \Gamma[C_1 \rightarrow C_2] \vdash A \]
\[ (\land l) \]
\[ \vdash \Gamma[C] \vdash A \]
\[ \Gamma[C_1 \rightarrow C_2] \vdash A \]
\[ (\land l) \]

(D) Multiplicatives:

\[ X \vdash C \rightarrow D \]
\[ X \vdash C \rightarrow D \]
\[ (\rightarrow r) \]
\[ \vdash \Gamma[C_1 \land C_2] \vdash A \]
\[ \Gamma[C_1 \rightarrow C_2] \vdash A \]
\[ (\rightarrow r) \]
\[ \vdash \Gamma[D] \vdash A \]
\[ \Gamma[C] \vdash A \]
\[ X \vdash C \rightarrow D \]
\[ (\rightarrow l) \]
\[ \vdash \Gamma[C] \vdash A \]
\[ \Gamma[C_1 \rightarrow C_2] \vdash A \]
\[ (\rightarrow l) \]
\[ \vdash \Gamma[C_1 \land C_2] \vdash A \]
\[ \Gamma[C_1 \rightarrow C_2] \vdash A \]
\[ (\rightarrow l) \]
\[ \vdash \Gamma[C_1 \land C_2] \vdash A \]
\[ \Gamma[C_1 \rightarrow C_2] \vdash A \]
\[ (\rightarrow l) \]

Fig. 1. LBI, LBI’, LBI’’ calculi. The double lines denotes two rules, via the upwards and downward direction. In LBI, each antecedent is a bunch. In LBI’, each antecedent is a bunch’ and the rules (as-c), (as-sc), (ex-c) and (ex-sc) are deleted. Finally, the rule instances of LBI’’ are precisely the rule instances of LBI’ minus the rules in (B) and the (ctr) rule with the function $r$ applied to the premise and conclusion antecedents.
exclude arbitrarily long trivial nestings of \( \varnothing_m \) and \( \varnothing_n \). We also observe that there is no rule in \( \text{LBI} \) which can increase the length of comma-separated lists (i.e., the number of contiguous commas in a bunch) except via duplication using semicolon, but this is the contractive effect we have already abstracted away. In short: to show that there are only finitely many bunched sequent labels for a candidate tree, the challenge is to bound the length of comma-separated lists.

We want to devise a size function ‘weight’ on sequents such that (i) the weight is nonincreasing from conclusion to premise and (ii) the weight bounds the length of comma-separated lists and also implies that the set of sequents of weight less than a fixed value is finite and computable. The obvious candidate for the size of a sequent \( X \vdash A \) is \(|X| + |A|\) where \(|\cdot|\) extends the standard definition of the size of a formula to bunches. In particular, \(|U,V| = |U| + |V| + 1\) and \(|U;V| = |U| + |V| + 1\). However, then the contraction rule (ctr) in \( \text{LBI} \) would violate objective (i). The point is that the premise would have greater size than the conclusion because \( X;X \) would have greater size than \( X \). Solution: define \(|U;V| = \max\{|U|,|V|\}\). 

\textit{Count commas, take maximum over semicolons.} Now consider the following rule instances:

\[
\frac{Y \vdash C \quad D \vdash A}{Y;C \Rightarrow D \vdash A} \quad (\rightarrow l) \quad \frac{C;C \vdash A}{C \vdash A} \quad \frac{C \Rightarrow D, C \vdash D}{C \vdash D} \quad (\text{ctr}) \quad \frac{C \Rightarrow D}{C \Rightarrow D} \quad (\Rightarrow r)
\]

(Above left) In the conclusion of \((\rightarrow l)\) we would measure the size as \(\max\{|Y|,|C \Rightarrow D|\} + |A|\). The size of the left premise would be \(|Y| + |C|\). So if \(|Y|\) and \(|C|\) have similar size and are much larger than the other variables, then the conclusion size \((\approx |C|)\) would be smaller than the premise size \((\approx 2|C|)\). In response we must somehow preemptively \textit{take into account} \(|U| + |B|\) for every substructure \(U;B\) when \(B\) is a formula and take the maximum over these candidates.

However, if we count the substructure \(U;B\) as \(|U| + |B|\) then the \((\text{ctr})\) rule above centre causes difficulties once more. For (above centre) the premise size \((\approx 2|C|)\) is greater than the conclusion size \(|C| + |A|\) when \(|C| \gg |A|\). So we must preemptively \textit{take into account} \(|C| + |C|\) for every formula \(C\) occurring in the sequent. In this way we finally obtain a measure achieving objective (i).

As an aside, (above right) indicates why \(\max\{|X|,|A|\}\) is an inadequate measure for the size of a sequent \(X \vdash A\). Under this measure the premise would have size \(|C \Rightarrow D| + 1 + |C|\) which is greater than its conclusion size \(|C \Rightarrow D|\).

Summary: the set \(\text{CP}(X \vdash A)\) of \textit{critical pairs} (Def. 11) of \(X \vdash A\) consists of:

1. \(\{X\}\{A\}\)
2. \(\{B\}\{B\}\) for every formula \(B\) in \(X \vdash A\)
3. \(\{U\}\{B\}\) for \(U;B\) in \(X\).

The size of a critical pair \(\{U\}\{B\}\) is defined as \(|U| + |B|\). The \textit{weight} of \(X \vdash A\) is then the maximum of the sizes of the critical pairs in \(\text{CP}(X \vdash A)\) (Def. 13).

Notice that the maximum length of a comma-separated list in \(X\) cannot be greater than the weight of \(X \vdash A\) so we have achieved objective (ii).
4 A normal form for bunches

A bunch can be viewed as an ordered binary tree such that every leaf is in \( \text{fm}^\circ \) and every interior node is either a comma or a semicolon.

**Definition 5 (multiplicative, additive bunch).** A bunch is multiplicative (additive) if its head connective is comma (resp. semicolon).

We will first transform such an ordered binary tree into an unordered tree (not necessarily binary) such that every path from the root to the leaf alternates between commas and semicolons.

**Definition 6 (bunch').** A bunch' is a finite object defined recursively as

1. A single node from \( \text{fm}^\circ \) and no edges.
2. A node comma with 2 or more children; no child is a multiplicative bunch'.
3. A node semicolon with 2 or more children; no child is an additive bunch'.

**Notation.** \( U_1, \ldots, U_{n+2} (U_1: \ldots; U_{n+2}) \) for \( n \geq 0 \) denotes a tree with root node comma (resp. semicolon) and children \( U_i \). Also \( \Gamma[U_1, \ldots, U_{n+2}] \) denotes that \( \Gamma \) contains a node comma with children \( U_i \). Note that this comma may also have other children that are not explicitly named here (previously we considered binary trees so this was not a possibility). Define \( \Gamma[U_1; \ldots; U_{n+2}] \) analogously.

The following program takes as input a bunch \( X \) and returns \( X' \).

```plaintext
1 input X
2 if X ∈ \text{fm}^\circ then X' := X
3 else if X = U; V
4 case U' and V'
5 U_1, \ldots, U_{n+1} and V_1, \ldots, V_{m+1}: X' := U_1, \ldots, U_{n+1}, V_1, \ldots, V_{m+1}
6 U_1, \ldots, U_{n+1} and V_1; \ldots; V_{m+1}: X' := U_1, \ldots, U_{n+1}, V'
7 U_1; \ldots; U_{n+1} and V_1, \ldots, V_{m+1}: X' := U', V_1, \ldots, V_{m+1}
8 U_1; \ldots; U_{n+1} and V_1; \ldots; V_{m+1}: X' := U', V'
9 else if X = U; V then obtain U' and V'
10 case U' and V'
11 U_1, \ldots, U_{n+1} and V_1, \ldots, V_{m+1}: X' := U'; V'
12 U_1, \ldots, U_{n+1} and V_1; \ldots; V_{m+1}: X' := U'; V_1; \ldots; V_{m+1}
13 U_1; \ldots; U_{n+1} and V_1, \ldots, V_{m+1}: X' := U_1; \ldots; U_{n+1}; V'
14 U_1; \ldots; U_{n+1} and V_1; \ldots; V_{m+1}: X' := U_1; \ldots; U_{n+1}; V_1; \ldots; V_{m+1}
15 return X'
```

**Lemma 1.** Let \( X \) be a bunch. Then \( X' \) is a bunch'.

**Proof.** Induction on the size of \( X \). Base case: \( X ∈ \text{fm}^\circ \) so \( X \) is already a bunch'.

In the inductive case, suppose that \( X = U; V \) (the case of \( U; V \) is similar). Then the induction hypothesis tells us that \( U' \) and \( V' \) are bunch'. By inspection, lines 5–8 of the program ensure that \( X' \) is a bunch'. Case \( U; V \) is analogous. □
Example 1. $X = ((p \rightarrow q, \emptyset_a); (p \rightarrow q, \emptyset_a), ((p \rightarrow 1; \emptyset_a), (\emptyset_m, r \otimes s))$ is the bunch corresponding to the ordered binary tree below left. Then $X^*$ corresponds to the unordered tree below right which is clearly is a bunch$^*$. 

\begin{center}
\begin{tikzpicture}
  \node (root) at (0,0) {$?$};
  \node (left) at (-2,-1) {$p \rightarrow q$};
  \node (right) at (2,-1) {$p \rightarrow q$};
  \node (middle) at (0,-2) {$p \rightarrow 1$};
  \node (bottom) at (0,-3) {$\emptyset_a$};
  \draw (root) -- (left);
  \draw (root) -- (right);
  \draw (left) -- (middle);
  \draw (middle) -- (bottom);
  \draw (right) -- (middle);
  \draw (right) -- (bottom);
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}
  \node (root) at (0,0) {$?$};
  \node (left) at (-2,-1) {$p \rightarrow q$};
  \node (right) at (2,-1) {$p \rightarrow q$};
  \node (middle) at (0,-2) {$\emptyset_m$};
  \node (bottom) at (0,-3) {$r \otimes s$};
  \draw (root) -- (left);
  \draw (root) -- (right);
  \draw (left) -- (middle);
  \draw (middle) -- (bottom);
  \draw (right) -- (middle);
  \draw (right) -- (bottom);
\end{tikzpicture}
\end{center}

We can write $X^*$ as $((p \rightarrow q, \emptyset_a); (p \rightarrow q, \emptyset_a), ((p \rightarrow 1; \emptyset_a), (\emptyset_m, r \otimes s)))$.

Now we introduce a calculus on bunch$^*$ sequents.

**Definition 7 (LBI$^*$).** The sequent calculus LBI$^*$ consists of the rules in Fig.1 minus (as-c), (as-sc), (ex-c) and (ex-sc) where the antecedent of every sequent is read as a bunch$^*$.

The calculus LBI$^*$ is defined independently of LBI. Nevertheless it may be seen that the rule instances of LBI$^*$ are precisely those obtained by applying $^*$ to the antecedents of the premise(s) and conclusions of LBI rule instances.

Example 2. Some rule instances and derivation fragments in LBI and LBI$^*$.

In LBI (antecedent is bunch) In LBI$^*$ (antecedent is bunch$^*$)

\[
\frac{(p, (q, r)), s \vdash t}{p, (q, r) \vdash s \rightarrow t} \quad \frac{p, q, r, s \vdash t}{p, q, r \vdash s \rightarrow t} \quad \text{(-or)}
\]

\[
\frac{(p, (r, q)) \vdash (p \otimes r) \otimes q}{p, (r, q) \vdash (p \otimes r) \otimes q} \quad \frac{p, q, r \vdash (p \otimes r) \otimes q}{p, q, r \vdash (p \otimes r) \otimes q} \quad \text{(-l)}
\]

\[
\frac{p, q \vdash (p \otimes r) \otimes q}{p, q \vdash (p \otimes r) \otimes q} \quad \frac{p, q, r \vdash (p \otimes r) \otimes q}{p, q, r \vdash (p \otimes r) \otimes q} \quad \text{(-l)}
\]

As we would expect, LBI and LBI$^*$ are equally expressive:

**Lemma 2.** (i) If $X \vdash A$ is LBI-derivable then $X^* \vdash A$ is LBI$^*$-derivable. (ii) If $Y \vdash A$ is LBI$^*$-derivable and $X^* = Y$, then $X \vdash A$ is LBI-derivable.

**Proof.** (i) Induction on the height of the derivation of $X \vdash A$. Consider the last rule $\rho$ of the derivation and apply the induction hypothesis to its premise(s). If the last rule was (as-c), (as-sc), (ex-c) or (ex-sc) we have already obtained the required derivation. Otherwise reapply $\rho$ (this time in LBI$^*$).

(ii) Induction on the height of the derivation of $Y \vdash A$. Let $Y_1 \vdash A_1$ denote the premise of the last rule $\rho$ of the derivation (the argument is similar when $\rho$ is binary). Obtain the bunch $Z_1$ (ordered binary tree) from the bunch$^*$ $Y_1$ (unordered
tree) by interpreting the commas and semicolons with left-associative precedence i.e. $U_1, \ldots, U_{n+2}$ becomes $(((U_1, U_2), U_3), \ldots, U_{n+2})$. Then $Z_1^* = Y_1$ and $Y_1 \vDash A_1$ is LBI*-derivable so the induction hypothesis yields that $Z_1 \vDash A_1$ is LBI*-derivable. Since all active formulae of $\rho$ in $Y_1$ also appear in $Z_1$, use (as-c), (as-sc), (ex-c) and (ex-sc) as required on $Z_1 \vDash A_1$ in order to apply $\rho$ (this time in LBI) to obtain the conclusion $Z \vDash A$. Now $Z$ and $X$ differ only in parenthetical ordering, so $X \vDash A$ is derivable in LBI from $Z \vDash A$ using (as-c), (as-sc), (ex-c) and (ex-sc). □

Now we introduce a more nuanced notion of a bunch* which removes comma-separated $\emptyset_m$, semicolon-separated $\emptyset_a$ and semicolon-separated duplicates.

**Definition 8 (bunch*).** A bunch* is a finite object defined recursively as

1. A single node from $\text{Fm}^\emptyset$ and no edges.
2. A node comma with 2 or more children; no child is a multiplicative bunch*.
   Also no child is $\emptyset_m$.
3. A node semicolon with 2 or more children; no child is an additive bunch*.
   Also no child is $\emptyset_a$ and no two children are identical.

A bunch* sequent is a sequent whose antecedent is a bunch*. Clearly a bunch* is a bunch but the other direction does not hold in general. We define a function $r$ on bunch* to achieve this transformation. For $A \in \text{Fm}^\emptyset$ define $A' = A$. Otherwise:

\[
(U_1, \ldots, U_{n+2})' = \begin{cases} 
\emptyset_m & \text{if } U_i' = \emptyset_m \text{ for every } i \\
U_i', \ldots, U_{s+2}' & (s_1, \ldots, s_{t+2}) \text{ subseq. of } (1, \ldots, n + 2) \\
\emptyset_m, U_i', \ldots, U_{s+2}' & (s_1, \ldots, s_{t+2}) \text{ subseq. of } (1, \ldots, n + 2) \\
\emptyset_m, U_i', \ldots, U_{s+2}' & (s_1, \ldots, s_{t+2}) \text{ subseq. of } (1, \ldots, n + 2) \\
\end{cases}
\]

\[
(U_1; \ldots; U_{n+2})' = \begin{cases} 
\emptyset_a & \text{if } U_i' = \emptyset_a \text{ for every } i \\
U_i', \ldots, U_{s+2}' & (s_1, \ldots, s_{t+2}) \text{ subseq. of } (1, \ldots, n + 2) \\
\emptyset_a, U_i', \ldots, U_{s+2}' & (s_1, \ldots, s_{t+2}) \text{ subseq. of } (1, \ldots, n + 2) \\
\emptyset_a, U_i', \ldots, U_{s+2}' & (s_1, \ldots, s_{t+2}) \text{ subseq. of } (1, \ldots, n + 2) \\
\end{cases}
\]

For $j, k \in \{s_1, \ldots, s_{t+2}\}$: $U_j' = U_k'$ if $j = k$

**Note.** We implicitly use that $(X' \bowtie Y)' = (X' \bowtie Y)'$ where $\bowtie$ is comma/semicolon.

**Example 3.** Let $X^*$ be the bunch* obtained in E.g. 1. Then $X'^*$ is computed below left. The tree representation is given below right.

\[
= ((p \rightarrow q, \emptyset_a); (p \rightarrow q, \emptyset_a))', (p \rightarrow 1; \emptyset_a)', r \otimes s \\
= p \rightarrow q, \emptyset_a, p \rightarrow 1, r \otimes s
\]

**Lemma 3.** Let $Y$ be a bunch* and $X^* = Y$ for some bunch* $X$. Then there is a LBI*-derivation of $\Gamma[X] \vdash A$ from $\Gamma[Y] \vdash A$ for any $\Gamma[]$ and $A \in \text{Fm}$. 

Proof. Let \( \# r(X) \) denote the number of recursive calls of \( r \) (including the original function call) that witness \( X' = Y \). Argue by induction on \( \# r(X) \). If \( \# r(X) = 1 \) (base case) then \( X \) is already a bunch\(^r \) or \( X = \emptyset_m, \ldots, \emptyset_m \) or \( X = \emptyset_a, \ldots, \emptyset_a \). In the first case we already have the required derivation, in the remaining two cases we proceed in \( \text{LBI}^* \) as follows.

\[
\begin{align*}
\Gamma[\emptyset_m]| A & \quad \Gamma[\emptyset_a]| A \\
(\emptyset_m l) & \quad (\emptyset_a l)
\end{align*}
\]

Now suppose that \( \# r(X) = k + 1 \). From the definition of \( r \) we have (i) \( X = U_1, \ldots, U_{n+2} \) and \( X' = U_1', \ldots, U_{n+2}' \) or (ii) \( X = U_1, \ldots, U_{n+2} \) and \( X' = U_1', \ldots, U_{n+2}' \). Noting that \( \# r(U_i) \leq k \) for every \( i \), we proceed in \( \text{LBI}^* \):

\[
\begin{align*}
\Gamma[U_1', \ldots, U_{n+2}']| A & \quad \Gamma[U_1', \ldots, U_{n+2}']| A \\
(\emptyset_m l) \text{ rules} & \quad (\emptyset_a l) \text{ and (weak) rules}
\end{align*}
\]

Now we introduce a calculus on bunch\(^r \) sequents.

**Definition 9 (LBI\(^{r'} \)).** The rule instances of LBI\(^{r'} \) are precisely the rule instances from the LBI\(^* \) minus the rules in (B) and the (ctr) rule calculus with the function \( r \) applied to the antecedent of every premise and the conclusion.

More explicitly (see below): the first (third) column is a rule instance of LBI\(^{r'} \) iff the second (fourth) column is a rule instance of LBI\(^* \) \{rules in (B), (ctr)\}.

\[
\begin{array}{c|c|c|c}
X_1| A & X_1| A & X_1| A & X_1| A \\
X| A & X| A & X| A & X| A
\end{array}
\]

**Remarks.** LBI\(^{r'} \) is a calculus on bunch\(^r \) sequents. The use of bunch\(^r \) necessitates the use of the projective (\( \land \)) rule (i.e. a single auxiliary formula in the premise rather than two auxiliary formulae); otherwise it would not be possible to derive a formula \( C \land C \) in the antecedent.

**Example 4.** Some rule instances in LBI\(^* \) (below left) and LBI\(^{r'} \) (below right).

\[
\begin{align*}
\frac{\emptyset_m, C \vdash D}{\emptyset_m \vdash C \rightarrow D} & \quad \frac{C \vdash D}{\emptyset_m \vdash C \rightarrow D} \\
(\text{o-r}) & \quad (\text{o'}-r)
\end{align*}
\]

Further remarks. The LBI\(^{r'} \) rule instances in row 2 and 3 in the right column contains some implicit contractions. We emphasise that moving from LBI\(^* \) to LBI\(^{r'} \) (i.e. from bunch\(^r \) to bunch\(^r' \)) does not eliminate contraction. We delete (ctr) because contraction—or more precisely, the essential contraction that is
In words, the size of a multiplicative bunch \( \ast \emptyset \) (necessarily non-multiplicative and non-

one. The size of an additive bunch \( \ast \emptyset \) is now \textit{incorporated} into the data structure. (Analogous to passing from a calculus for intuitionistic logic built from \textit{multisets} to \textit{sets}).

Although the definition of \( \text{LBI}' \) may appear unwieldy due to its reliance on \( \text{LBI}^* \), the rule schemata for \( \text{LBI}'' \) can be stated independently: present each logical rule using rule schemata (parametrised by the path length \( n \) from the root of the bunch\( ^* \) to the principal formula) for each of the possible implicit contractions. The number of different rule schemata required is exponential in \( n \). We would also need to add \(-\ast \) (Eg. \[4\]) and its corresponding additive \((-\ast \)r\).

**Lemma 4.** (i) If \( X \vdash A \) is \( \text{LBI}^* \)-derivable then \( X' \vdash A \) is \( \text{LBI}' \)-derivable. (ii) If \( Y \vdash A \) is \( \text{LBI}'' \)-derivable and \( X' = Y \), then \( X \vdash A \) is \( \text{LBI}^* \)-derivable.

**Proof.** (i) Since the \( \text{LBI}' \) rule instances are exactly those that are obtained by applying \( r \) to the antecedents of the premise(s) and conclusions of rule instances of \( \text{LBI}^* \) minus the rules in (B) and the (ctr) rule, the result follows via a straightforward induction on the height of \( X \vdash A \).

(ii) Every rule instance of \( \text{LBI}' \) can be obtained from some rule instance of \( \text{LBI}^* \) by absorbing all identity structure constants and then contracting every \( X; X \) to \( X \). Because the rules \( (\emptyset, l) \), \( (\emptyset, l) \) and (ctr) witnessing these transformations are in \( \text{LBI}^* \), it follows that \( Y \vdash A \) is derivable in \( \text{LBI}^* \). By assumption, \( X' = Y \) and hence by Lem. \( 3 \) we have that \( X \vdash A \) is \( \text{LBI}^* \)-derivable.

**Definition 10 (bunch\( '^* \) size).** If \( \alpha \) is 1, \( \top, \bot \) or a propositional variable then the size \( |\alpha| = 1 \). If \( \alpha \) is \( C \otimes D \) where \( \otimes \in \{\land, \lor, \rightarrow, \land \} \) then \( |\alpha| = |C| + |D| + 1 \). Extend to a bunch\( '^* \) as follows: \( |\emptyset| = |\emptyset| = 0 \). Also

\[
|U_1, \ldots, U_{n+2}| = \sum_{i=1}^{n+2} |U_i| + n + 1 \quad U_i \text{ not multiplicative (headconn. not comma)}
\]

\[
|U_1; \ldots; U_{n+2}| = \max \{|U_i|\}_{i=1}^{n+2} \quad U_i \text{ not additive (headconn. not semicolon)}
\]

In words, the size of a multiplicative bunch\( '^* \) is the sum of the sizes of its children (necessarily non-multiplicative and non-\( \emptyset \)) plus the number of children minus one. The size of an additive bunch\( '^* \) is the maximum of the sizes of its children (necessarily non-additive, non-\( \emptyset \), non-duplicative).

**Example 5 (computing the size).**

\[
|p \land q, (p; (\emptyset, q, 1 \land r); p \rightarrow q)| =
\]

\[
= |p \land q| + |(p; (\emptyset, q, 1 \land r); p \rightarrow q)| + 1
\]

\[
= |p \land q| + \max \{|p|, |(\emptyset, q, 1 \land r)|, |p \rightarrow q|\} + 1
\]

\[
= |p \land q| + \max \{|p|, (|\emptyset| + |q| + |1 \land r| + 2), |p \rightarrow q|\} + 1
\]

\[
= 3 + \max\{1, 6, 3\} + 1 = 10
\]

5 Main result

The notion of \textit{critical pairs} were motivated in Sec. \[3\].
Definition 11 (critical pair). A critical pair (denoted \(\{U\}\{V\}\)) is an ordered pair where \(U\) and \(V\) are structures. The set \(\text{CP}(X \vdash A)\) of critical pairs of \(X \vdash A\) is the smallest set satisfying the following:

(i) \(\{X\}\{A\} \in \text{CP}(X \vdash A)\)
(ii) \(\{B\}\{B\} \in \text{CP}(X \vdash A)\) for every formula \(B\) in \(X \vdash A\)
(iii) Antecedent-critical pair: \(\{U\}\{B\} \in \text{CP}(X \vdash A)\) if \(U; B\) occurs in \(X\).

A critical pair derived by (iii) is called an antecedent critical pair because both elements of the pair necessarily occur in the antecedent, unlike in (i) and (ii). Also observe that in (iii) it is the case that \(U\) and \(B\) occur in \(X\) as distinct structures. In contrast, in (ii): a single occurrence of a formula is used twice.

Since each element of the critical pair is a substructure of the sequent antecedent, the set of critical pairs of a sequent is finite.

Definition 12 (cp size). The size of a critical pair \(\{U\}\{V\}\) is \(|U| + |V|\).

We write \(|\{U\}\{V\}|\) to denote the size of \(\{U\}\{V\}\).

Definition 13 (weight). The weight \(w(X \vdash A)\) of a bunch \(^r\) sequent \(X \vdash A\) is the maximum of the sizes of the critical pairs in \(\text{CP}(X \vdash A)\).

Crucially, the weight measure is non-increasing in backward proof search.

Lemma 5. For every \(LBI^r\) rule instance: premise weight \(\leq\) conclusion weight.

Proof. It suffices to show that for every critical pair in a premise, there is a critical pair in the conclusion of greater or equal size. For each rule in \(LBI^r\):

If the premise critical pair is type (i). The result is immediate by inspection.

If the premise critical pair is type (ii)—i.e. \(\{B\}\{B\}\) for some \(B\) in the premise. By the subformula property there must be some formula \(B'\) occurring in the conclusion such that \(|B| \leq |B'|\) and thus \(|\{B\}\{B\}| \leq |\{B'\}\{B'\}|\).

If the premise critical pair is type (iii)—i.e. a critical pair \(\{U\}\{B\}\) such that \(U; B\) occurs in the premise antecedent. If \(U; B\) also occurs in the conclusion of the rule then we are done. The only rules where this might not be the case are the rules where the premise contains a semicolon-separated formula that was not present in the conclusion. Let us consider the possible cases. Observe that the implicit contractions will not cause any difficulties because bunch \(^r\) size takes the maximum over semicolon-separated structures. Indeed, the effect of contraction was anticipated when we set about defining ‘critical pair’, see Section 3

\((\rightarrow r)\)  Here are the possibilities for the critical pair \(\{U\}\{B\}\) in the premise. Note: below right the overbrace over the premise denotes that \(U = U'; C\).

\[
\frac{X; U; B \vdash D}{X; U \vdash B \rightarrow D} (\rightarrow r) \quad \frac{U}{X; U'; B \vdash D} (\rightarrow r)
\]
Note: above left, the pair of underbraces are used to identify the critical pair in the conclusion with size $\geq |\{U\}|$. In detail: above left, $\{X; U\}\{B \to D\}$ is a critical pair in the conclusion (of type (i)). Moreover $|\{X; U\}\{B \to D\}| \geq |U| + |B| = |\{U\}|$. Above right, the critical pair in the conclusion that we should choose depends on the relative sizes of $U'$ and $C$. In particular, if $|U'| > |C|$ then $|\{U\}| = |U'| + |B|$ so choose the critical pair $\{U\}\{B\}$ in the conclusion. Else $|C| \geq |U'|$ and $|\{U\}| = |C| + |B|$, so choose the critical pair $\{X; U; B\}\{C \to D\}$ in the conclusion.

\((\forall l)\) We consider some rule instances below. The other cases are similar. First row left: if $|U'| > |B \lor D|$ then choose $\{U\}\{B \lor D\}$, else choose $\{B \lor D\}\{B \lor D\}$. First row right: if $|U'| > |C|$ then choose $\{U\}\{C \lor D\}$, else choose $\{C \lor D\}\{C \lor D\}$. Second row: choose $\{U\}\{B \lor D\}$ as indicated by the underbraces.

\[
\begin{align*}
\frac{\Gamma[U]; B \lor D; D \vdash A}{\Gamma[U]; B \lor D \vdash A} & \quad \frac{\Gamma[U]; B \lor D; D \vdash A}{\Gamma[U]; B \lor D \vdash A} & \quad \frac{\Gamma[U]; C \lor D; D \vdash A}{\Gamma[U]; C \lor D \vdash A} & \quad \frac{\Gamma[U]; D \lor C; D \vdash A}{\Gamma[U]; D \lor C \vdash A}
\end{align*}
\]

\((\forall l)\) Below left: choose $\{U\}\{B \land D\}$ as indicated by the braces. Below center if $|U'| > |C|$ then choose $\{U\}\{C \land D\}$, else choose $\{C \land D\}\{C \land D\}$. Below right, if $|U'| > |B \land D|$ then choose $\{U\}\{B \land D\}$, else choose $\{B \land D\}\{B \land D\}$.

\[
\begin{align*}
\frac{\Gamma[U]; B \land D \vdash A}{\Gamma[U]; B \land D \vdash A} & \quad \frac{\Gamma[U]; C \land D \vdash A}{\Gamma[U]; C \land D \vdash A} & \quad \frac{\Gamma[U]; B \land D; D \vdash A}{\Gamma[U]; B \land D \vdash A}
\end{align*}
\]

\((\to l)\) The non-trivial case is given below. Observe that any type (iii) critical pair in the left premise must also be a critical pair in the conclusion.

\[
\frac{Y; C \to D \vdash A}{\Gamma[Y; C \to D] \vdash A} & \quad \frac{\Gamma[C \to D; \vdash A}{\Gamma[C \to D] \vdash A}
\]

\((\text{weak})\) It suffices to observe that any type (iii) critical pair in the premise must also be a critical pair in the conclusion.

\[\square\]

**Definition 14 (height of bunch*).** The height $h(X)$ of a bunch* $X$ is the number of nodes minus 1 along its longest branch (longest path from root to leaf).

The “minus 1” is a technical device to avoid counting the leaf node. The reason is that the following lemma relies on the interior alternating comma and semicolon nodes along a bunch*, it is these which relate the height of a bunch* to its size.
Lemma 6. Let \( X \) be a bunch\(^r\). For \( n \geq 1 \): if \( h(X) \geq 2n \) then \( |X| \geq n \).

Proof. Induction on \( n \). The base case is \( n = 1 \). Then \( h(X) \geq 2 \) so \( X \) contains at least one comma node. It follows that \( |X| \geq 1 \). Next suppose that \( n > 1 \).

If \( X = U_1; \ldots ; U_{k+2} \), without loss of generality taking that \( U_1 \) has maximal height with respect to \( U_i \), we have \( h(U_1) \geq 2n - 1 \) and \( U_1 = V_1, \ldots , V_{i+2} \). Once again without loss of generality \( h(V_1) \geq 2n - 2 = 2(n - 1) \). From the induction hypothesis we have \( |V_1| \geq n - 1 \). Then \( |X| \geq |U_1| \geq 1 + |V_1| = n \) as required.

If \( X = U_1, \ldots , U_{k+2} \) then without loss of generality \( h(U_1) \geq 2n - 1 \) and \( U_1 = V_1, \ldots ; V_{i+2} \) and \( h(V_1) \geq 2n - 2 = 2(n - 1) \). From the induction hypothesis we have \( |V_1| \geq n - 1 \). Now \( |X| \geq 1 + |U_1| \geq 1 + |V_1| = n \) as required.

□

Example 6 (Comparing the height and size of a bunch\(^r\)). In Eg. \( \textcircled{5} \) we computed \( |p \land q, (p; (\varnothing_a, q, 1, r); p \rightarrow q)| = 10 \). Meanwhile \( h(p \land q, (p; (\varnothing_a, q, 1, r); p \rightarrow q)) = 3 \) (witnessed e.g. by the branch with leaf \( \varnothing_a \)) so Lem. \( \textcircled{6} \) holds. To see a boundary case of Lem. \( \textcircled{6} \) observe that \( |\varnothing_m; (\varnothing_a; (\varnothing_m; (\varnothing_a, (\varnothing_m; \varnothing_m))))| = 2 \). Meanwhile \( h(\varnothing_m; (\varnothing_a; (\varnothing_m; (\varnothing_a, (\varnothing_m; \varnothing_a)))) = 4 = 2 \cdot 2 \) so Lem. \( \textcircled{6} \) holds.

Backward proof search on a sequent \( s_0 \) is the repeated application of calculus rules backwards (from the conclusion to the premises) starting with \( s_0 \). Then a candidate tree with sequent-labelled nodes with root \( s_0 \) is obtained. Since there may be multiple rules that may be applied backwards to a sequent, in general, many different candidate trees will be obtained via backward proof search.

A candidate tree is said to have minimal height if no branch of the tree contains a repetition of the same sequent.

Lemma 7. The set of candidate trees of minimal height obtained via backward proof search on a bunch\(^r\) sequent \( s_0 \) is finite and computable.

Proof. It suffices to show that the set of bunch\(^r\) sequents that can appear in a candidate tree from \( s_0 \) is finite and computable, since each node in a candidate tree has at most two children and the minimal height prohibits repetitions on a branch. Let \( \text{sf}(s_0) \) denote the set of subformulae in \( s_0 \). Clearly the cardinality \( |\text{sf}(s_0)| \leq \sum s_0 \) where \( \sum s_0 \) denotes the sum of the sizes of all formulae in \( s_0 \).

Next define the set \( \Omega(h) \) of bunch\(^r\) containing only formulae from \( \text{sf}(s_0) \cup \{ \varnothing_m, \varnothing_a \} \) whose size is bounded by \( w(s_0) \) and whose height is bounded by \( h \).

\[ \Omega(h) = \{ X \mid A \in \text{fm} \text{ in } X \text{ implies } A \in \text{sf}(s_0), |X| \leq w(s_0) \text{ and } h(X) \leq h \} \]

Notice that \( \Omega(h) \subset \Omega(h') \) for \( h < h' \). First we argue by induction that for fixed \( h \), the number \(|\Omega(h)|\) of elements in \( \Omega(h) \) is finite and depends only on \( s_0 \).

For the base case, let us compute \(|\Omega(1)|\). If \( X \) has height 1 then \( X \in \text{fm} \). Then there are only \( |\text{sf}(s_0)| + 2 \) possibilities for \( X \).

Inductive case. Suppose that \( X \in \Omega(k+1) \setminus \Omega(k) \). Then \( X \) must have the form (i) \( U_1, \ldots , U_{n+2} \) (\( U_i \) is not multiplicative cf. Def. \( \textcircled{5} \)) or (ii) \( U_1; \ldots ; U_{n+2} \) (\( U_i \) is not additive) where each \( U_i \in \Omega(k) \) (\( 1 \leq i \leq n+2 \)). In case (i) we have \( |X| \geq n+1 \). Also since \( X \in \Omega(k+1) \) we have \( |X| \leq w(s_0) \). Therefore \( n+1 \leq w(s_0) \).
Moreover there are at most $|\Omega(k)| < \infty$ choices for each element of the comma-separated list, where this value depends only on $s_0$ by the induction hypothesis. It follows that the possibilities for $X$ are finite and computable from $s_0$. In case (ii), because $X$ is a bunch structure we have $U_i = U_j$ iff $i = j$. So the possibilities for $X$ are limited to the elements in $\mathcal{P}(\Omega(k)) \setminus \emptyset$ and thus the number of possibilities is bounded by $2^{\Omega(k)}$ ($\mathcal{P}$ is the powerset operator).

We have shown $\Omega(h)$ is finite for every $h$ depending only on $s_0$. Moreover for any bunch sequent $X \vdash A$ appearing in the backward proof search, it must be the case that $h(X) < 2(w(s_0) + 1)$ for otherwise we would have $|X| \geq w(s_0) + 1$ (Lem. 6) which would mean that $X \vdash A$ has weight greater than $w(s_0)$ and this is impossible by Lem. 5. It follows that $X \in \Omega(2w(s_0) + 2)$. Moreover we have that $A \in \text{sf}(s_0)$. Therefore the number of possible sequents that may appear in a candidate tree is bounded by $|\Omega(2w(s_0) + 2)| \cdot |\text{sf}(s_0)|$.

**Corollary 1.** BI is decidable.

**Proof.** By Lem. 2 and 4 it suffices to show that LBI is decidable. We may restrict our attention to minimal height derivations. The set of minimal height candidate trees whose root is the bunch sequent $s_0$ is finite and computable (Lem. 7). Then $s_0$ is derivable iff one of these candidate trees is a derivation. \(\square\)

### 6 Conclusions and future work

A similar argument cannot even be attempted for BBI (the logic is anyhow undecidable [12]) because no cutfree bunched calculus for BBI is known.

The bunch structure facilitates a concise definition of critical pairs and the height of a bunch. These arguments could have been made directly on a bunch by consideration of the ‘connected regions’ of commas/semicolons. E.g. the height of a bunch would be the maximum number of transitions between comma and semicolon regions counting from the root.

It is likely that the argument here can be used to prove the decidability of intuitionistic layered graph logic ILGL (i.e. $\otimes$-non-commutative, non-associative BI). While bunch is an unordered graph, for ILGL we must use the corresponding **ordered graph**. Another extension would be to add the contraction rule for $\otimes$ to BI. We are unaware yet of a resource-interpretation for this logic, but it would be technically interesting to see if the argument for FLec [11] could be used. General theorem provers based on terminating sequent calculi have been implemented e.g. see [5] and it would be interesting to extend such a system to LBI*. Given that finding a syntactic proof of decidability has already proved so vexing, we defer the calculation of the complexity upper bound for derivability arising from this decision procedure as future work.

Many different logics have been presented via cutfree sequent calculi (and also using notions of analyticity weaker-than-cutfree but seemingly powerful nonetheless), and via generalisations of the sequent formalism such as hypersequent and nested sequent calculi. However it is often unclear how to make use of these cutfree calculi to obtain a decision procedure due to difficulties in bounding the
backward proof search. This in turn is due to problematic interactions between certain rules in the calculus. Some techniques to control backward proof search include loop check, the simplifications obtained from the careful elimination of problematic rules such as the contraction rule, and the use of novel parameters (e.g. [11]) which capture specific aspects of the calculus. The weight measure in this paper belongs to the latter category. We believe that it is imperative to build a toolkit of methods to tackle the combinatorial problems that arise when we put the calculi to use. Not only will this lead to new backward proof search procedures and complexity bounds, such investigations will help us understand when and why an analytic calculus is a fundamentally different object to a non-analytic presentation of the logic (e.g. Hilbert calculus). We view this paper as work in this direction.

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