Quantifying entanglement with probabilities

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We propose a new approach to the problem of defining the degree of entanglement between two particles in a pure state with Hilbert spaces of arbitrary finite dimensions. The central idea is that entanglement gives rise to correlations between the particles that do not occur in separable states. We individuate the contributions of these correlations to the joint and the conditional probabilities of local measurement outcomes. We use these probabilities to define the measure of entanglement. Our measure turns out to be proportional to the so-called 2-entropy and therefore satisfies the property required for any measure of entanglement. We conclude with an outlook on the problem of extending our approach to the case of multipartite systems and mixed states.

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I. INTRODUCTION

Entanglement is the most surprising nonclassical property of composite quantum systems [1]. In 1935 Einstein, Podolsky and Rosen (EPR) pointed out a conflict between their concepts of realism and completeness of quantum mechanics when two particles are in an entangled state [2]. Several years later Bell revisited the EPR problem considering the case of particles with Hilbert spaces of finite dimensions and found his celebrated inequalities [3] that show that entangled particles can possess correlations that cannot be explained within a local realistic theory. Such correlations display the essential quantum nature of the composite system. Many years later Greenberger, Horne and Zeilinger [4] extended Bell’s arguments to three–particle systems.

In the last decade there has been a revival of interest in entanglement as a consequence of the birth of quantum information theory [5–7]. It has been shown that entangled pairs are a more powerful resource than separable, i.e., non entangled, pairs in a number of applications, such as quantum cryptography [8], dense coding [9], teleportation [10] and investigations of quantum channels [11,12].

The superior potentiality of entangled states has raised the question “How much are two (or more) particles entangled?”, since pairs with a high degree of entanglement should be a better resource than less entangled ones. Many proposals for a measure of entanglement have appeared in the last years [13] but, in spite of our improved understanding of the problem, there is no universally accepted measure for the most general case of N particles in a mixed state and it is even argued by some authors that a unique definition does not exist.

The case of two particles A and B in a pure state |ψAB⟩ has been thoroughly investigated. Among the measures of entanglement that have been proposed for bipartite pure states the entropy of entanglement

\[ S(\rho_{AB}) \equiv -\text{Tr}_A [\rho_A \ln \rho_A] , \]

where \( \rho_A \equiv \text{Tr}_B (|\psi_{AB}\rangle\langle\psi_{AB}|) \), is the most popular since it has a clear physical meaning: It is equal to the ratio \( m/n \) of the number \( m \) of maximally entangled states that can be extracted from \( n > m \) identical pure states, in the asymptotic limit \( n \to \infty \) [14]. In fact, some authors claim that the entropy of entanglement is the only measure of entanglement for pure states [15] and that the measure for mixed states should contain it. However, there is no universal agreement on this point [16–18]. Moreover, the entropy of entanglement fails to be a measure of entanglement for mixed states [19].

In pursuing a measure of entanglement for mixed states and/or multipartite systems one can follow different approaches. One can look for measures that contain the entropy of entanglement Eq. (1) as a special case or examine the simplest case of pure states from a different point of view in order to get some hint for the solution of the general problem. In the latter approach, that we follow here, one can obtain measures different from the entropy of entanglement, though equivalent to it.

In the present paper we present a new approach, similar in spirit to Bell’s inequalities, to define the degree of entanglement of two particles A and B, possessed by two observers Alice and Bob, respectively, that are in a pure state |ψAB⟩. The Hilbert spaces \( \mathcal{H}_A \) and \( \mathcal{H}_B \) of the two particles have arbitrary and finite dimensions. Since several measures of entanglement for pure states have been already presented by different authors, it seems opportune to motivate our proposal. For this purpose, we quote some remarks about the entangled state

\[ |\psi_{AB}^{(+)\rangle} = \frac{1}{\sqrt{2}} (|\uparrow_A\downarrow_B\rangle + |\downarrow_A\uparrow_B\rangle) \]  

for two spin–1/2 particles that appear in the literature as an introduction into the basic feature of entanglement (emphases are ours): “If Alice performs a measurement of spin . . . she can predict with certainty . . . neither of the two qubits carries a definite value, but . . . as soon as one of the qubits is subject to a measurement, . . . the other one will be immediately found to carry the opposite value”. [17] Einstein, Podolsky and Rosen also say “If . . . we can predict with certainty
the value of a physical quantity..." [2], although they comment a different entangled state.

All these remarks concern (a) measurements of physical quantities, (b) results of conditional or joint measurements and (c) properties of the state \( | \psi_{AB}^{(+)} \rangle \), from which we obtain the statistics of measurements. In order to define a measure of entanglement, it is highly desirable to express these qualitative considerations touching the heart of entanglement in a quantitative form. This can be done when one considers that the properties of the state \( | \psi_{AB}^{(+)} \rangle \) discussed above are a consequence of the correlations that are (exclusively) due to entanglement. These correlations appear in the conditional and joint probabilities of measurement outcomes, as we shall show.

The outline of the paper is as follows: In Sec. II we define entangled and separable states and review the properties that a measure of entanglement must fulfill. We also remind two mathematical properties, the Schmidt decomposition and the definition of majorization, and a remarkable connection between them that has been recently found by Nielsen [24]. In Sec. III we examine two two-level particles. We show that entanglement has consequences on the conditional probabilities or, equivalently, on the joint probabilities that given values are obtained in local measurements on the two particles. The properties of these probabilities in entangled pairs result radically different than in separable pairs. We use this difference to define the measure of entanglement. We extend then our measure to \( N \)-level systems. Our measure results equal, up to a normalizing factor, to the 2–Rényi entropy [23]. Our approach can be extended without changes to two particles with different numbers \( N \) and \( N' \) of levels. Our findings can thus be considered as a physical motivation in favour of the 2–entropy as a measure of entanglement. In Sec. IV, after summarizing our findings, we comment the problems of defining the degree of entanglement in multipartite systems and in mixed states and generalizing our approach to these cases.

II. PROLEGOMENA

Two particles in a pure state are said to be separable if their state \( | \psi_{AB} \rangle \) can be decomposed into the tensor product

\[
| \psi_{AB} \rangle = | \phi_A \rangle \otimes | \phi_B \rangle
\]

of two one–particle states \( | \phi_A \rangle \) and \( | \phi_B \rangle \). In the more general case of mixed states described by a density matrix \( \rho_{AB} \) the particles are said to be separable if one can write

\[
\rho_{AB} = \sum_i p_i \rho_A^i \otimes \rho_B^i,
\]

where \( \rho_A^i \) (\( \rho_A^i \)) are density matrices for particle A (B), \( \sum_i p_i = 1 \) and \( p_i \geq 0 \) for any \( i \). If the particles are not separable, they are said to be entangled.

What conditions must the measure of entanglement \( E(\rho) \) (\( E(\psi) \) for pure states) satisfy? Clearly we want (i) \( E(\rho) = 0 \) if and only if the particles are separable. Moreover, \( E(\rho) \) must be positive, so with the introduction of appropriate normalization factors one can also require \( 0 \leq E(\rho) \leq 1 \) for any state \( \rho \), though this is not compulsory.

The second requirement is that (ii) \( E(\rho) \) must be invariant under the action of local unitary operations \( U_A \) and \( U_B \), i.e., \( E(\rho) = E(U_A U_B \rho U_B^\dagger U_A^\dagger) \). In other words, the amount of entanglement cannot depend on the choice of basis.

A more subtle property comes from the observation that local general measurements (LGM), i.e., measurements performed by Alice and Bob on each separated particle cannot increase on average the amount of entanglement between A and B, not even when the measurements on A and B are correlated after exchange of classical information between Alice and Bob [16]. Indeed, entanglement is a nonlocal property of quantum systems and therefore it can only deteriorate or remain constant when the system is locally probed. Local general measurements are described by sets of operators \( \{ A_i \}, \{ B_i \} \), acting on particles A and B, respectively, that satisfy the completeness relations \( \sum_i A_i^\dagger A_i = 1 \) and \( \sum_i B_i^\dagger B_i = 1 \). After an LGM assisted by classical communication (CC) has been performed, the pair goes into the state

\[
\rho_k = A_k B_k \rho A_k^\dagger B_k^\dagger
\]

with probability \( p_k \). Therefore (iii) entanglement monotonicity [7]

\[
E(\rho) \geq \sum_k p_k E(\rho_k)
\]

for the degree of entanglement \( E \) is required, which ensures that on average LGM+CC do not increase entanglement.

The three conditions (i)–(iii) presented in [24] are the most frequent conditions reported in the literature. It has also been argued that condition (iii) of monotonicity alone entails conditions (i) and (ii) [17]. Nonetheless, we keep the conditions (i) and (ii) since their meaning is easy to understand and they are often easy to verify, so that they constitute a first test for any definition of \( E \).

We recall now three mathematical properties concerning the pure states \( | \psi_{AB} \rangle \). The first is the Schmidt decomposition [23]. Assuming \( \text{dim} \mathcal{H}_A = N \leq N' = \text{dim} \mathcal{H}_B \) it is always possible to define a basis \( \{ | i_A \rangle \}, \{ | i_B \rangle \} \) in each Hilbert space such that

\[
| \psi_{AB} \rangle = \sum_{i=1}^N \sqrt{\lambda_i} | i_A \rangle \otimes | i_B \rangle,
\]
where the Schmidt parameters $\lambda_i$ are the eigenvalues of the single-particle reduced density matrices $\rho_A \equiv \text{Tr}_B(\rho_{AB})$ or $\rho_B \equiv \text{Tr}_A(\rho_{AB})$ obtained after tracing over the other particle. Since the Schmidt parameters $\lambda_i$ are invariant under a change of basis, they are good ingredients for any measure of entanglement [24].

The second property comes from majorization theory [24]. Given two real $N$-dimensional normalized vectors $x \equiv (x_1, \ldots, x_N)$ and $y \equiv (y_1, \ldots, y_N)$ with their components in decreasing order $x_1 \geq \cdots \geq x_N$, $y_1 \geq \cdots \geq y_N$, the vector $x$ is said to be majorized by $y$, written $x < y$, if

$$\sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i$$

(8)

for any $k = 1, \ldots, N$. The functions that preserve majorization are called Schur convex [18] whereas those that reverse majorization are said to be Schur concave. In particular, the function $f_q(x) = \sum_{i=1}^N x_i^q$ is Schur convex for any $q \geq 1$ and therefore $1 - \sum_{i=1}^N x_i^q$ is Schur concave.

The third and last property we recall has been recently demonstrated by Nielsen [24]. If the pure state $| \psi_{AB} \rangle$ is transformed into another pure state $| \phi_{AB} \rangle$ by means of LGM + CC, the Schmidt parameters $\lambda(\phi) \equiv (\lambda_1(\phi), \ldots, \lambda_N(\phi))$ of the state $| \phi_{AB} \rangle$ majorize the analogous parameters $\lambda(\psi) \equiv (\lambda_1(\psi), \ldots, \lambda_N(\psi))$ of the initial state $| \psi_{AB} \rangle$. As a consequence, all entanglement measures expressed in terms of the Schmidt parameters that are also Schur concave satisfy the entanglement monotonicity condition Eq. (8).

### III. PROBABILITIES AND MEASURE OF ENTANGLEMENT

We first discuss the case of two two-level systems in a pure state with great detail. We move then to the more general case of two $N$-level systems in a pure state. Since our starting point is the Schmidt decomposition our results can be immediately extended to the case when the Hilbert spaces of the two particles have different dimensions.

#### A. $2 \times 2$ systems

When the particles A and B have only two levels $|1\rangle$ and $|2\rangle$ the Schmidt decomposition for any pure state $|\psi_{AB}^{(2)}\rangle$ reads

$$|\psi_{AB}^{(2)}\rangle = \sqrt{\lambda} \ |1_A1_B\rangle + \sqrt{1-\lambda} \ |2_A2_B\rangle,$$

(9)

where $0 \leq \lambda \leq 1$ is the only parameter. Although the Schmidt decomposition can be not uniquely defined when $\lambda = 1/2$ [25], the Schmidt coefficients $\sqrt{\lambda}$ and $\sqrt{1-\lambda}$ are unique for a given state $|\psi_{AB}\rangle$.

When Alice (Bob) performs a measurement that projects $|\psi_{AB}^{(2)}\rangle$ on the basis states, she (he) finds her (his) particle in state $|1\rangle$ with probability

$$P(1_A) = P(1_B) = \lambda$$

(10)

and in state $|2\rangle$ with probability

$$P(2_A) = P(2_B) = 1 - \lambda.$$  

(11)

From these measurements alone Alice and Bob cannot conclude whether their particles are entangled or not, since they would obtain the same results Eq. (10) and Eq. (11) for simple measurement probabilities with the separable pair

$$|\psi_{AB}^{(2)}\rangle_{\text{sep}} \equiv \left(\sqrt{\lambda} \ |1_A\rangle + \sqrt{1-\lambda} \ |2_A\rangle \right) \otimes \left(\sqrt{\lambda} \ |1_B\rangle + \sqrt{1-\lambda} \ |2_B\rangle \right).$$

(12)

However, they can obtain hints about entanglement when they exchange information about their measurements.

Entanglement manifests itself in the conditional probabilities $P(n_A \mid m_B)$ of finding particle A in the state $|n\rangle$ after that particle B has been found in the state $|m\rangle$, and in the joint probabilities $P(n_A,m_B)$ of finding particle A in state $|n\rangle$ and particle B in state $|m\rangle$. Indeed, from the state Eq. (12) we find

$$P(1_A \mid 1_B) = 1 = P(2_A \mid 2_B),$$

$$P(1_A \mid 2_B) = 0 = P(2_A \mid 1_B).$$

(13)

For the separable state Eq. (12) it results $P(n_A \mid m_B) = P(n_A)$, since measurements on particle B have no influence on the state of particle A and cannot change the statistics of measurements performed on A. Therefore, the inequality $P(n_A \mid m_B) \neq P(n_A)$ is a consequence of entanglement. Analogous considerations can be done for the joint probabilities $P(n_A,m_B)$. For the separable state Eq. (12) the joint probability factorizes into the product of simple probabilities $P(n_A,m_B) = P(n_A)P(m_B)$, since the results of local measurements on separable pairs are independent events, whereas for the entangled state Eq. (12) we find

$$P(1_A,1_B) = \lambda \neq P(1_A)P(1_B) = \lambda^2,$$

$$P(2_A,2_B) = 1 - \lambda \neq P(2_A)P(2_B) = (1-\lambda)^2,$$

$$P(1_A,2_B) = 0 \neq P(1_A)P(2_B) = \lambda(1-\lambda),$$

$$P(2_A,1_B) = 0 \neq P(2_A)P(1_B) = \lambda(1-\lambda).$$

These considerations on the conditional and joint probabilities can be merged together and used to quantify the degree of entanglement when we consider that the difference

$$| P(n_A \mid m_B) - P(n_A) \mid P(m_B) $$

$$= | P(n_A,m_B) - P(n_A)P(m_B) |$$

(14)
points out the quantum correlations between measurements on the two particles. We stress again that these correlations are solely due to entanglement. The differences in Eq. (14) can be seen as the differences of conditional or joint probabilities of the entangled state (11) and the separable state (12) that give the same simple probabilities.

We are now ready to define the measure of the degree of entanglement $E$ of state Eq. (14) as

$$E(\psi_{AB}^{(2)}) \equiv \frac{2}{N} \sum_{n=1}^{N} \sum_{m=1}^{N} | P(n_A, m_B) - P(n_A)P(m_B) |,$$  \hspace{1cm} (15)

where $n$ and $m$ denote states of the Schmidt basis. Simple calculations give

$$E(\psi_{AB}^{(2)}) = | \lambda - \lambda^2 | + | -\lambda(1 - \lambda) | + | -\lambda(1 - \lambda) | + | 1 - \lambda - (1 - \lambda)^2 | = 4\lambda(1 - \lambda) \equiv E(\lambda) \hspace{1cm} (16)$$

This expression is the same result given by other two measures of entanglement, the Bures metric (11) and the 2-entropy and therefore satisfies the conditions (i)–(iii).

**B. $N \times N$ systems**

Our definition Eq. (13) for the measure of entanglement can be easily extended to particles with more than two levels. When the particles have $N$ levels $| 1 \rangle, | 2 \rangle, \ldots, | N \rangle$, the Schmidt decomposition of their state $| \psi_{AB}^{(N)} \rangle$ reads

$$| \psi_{AB}^{(N)} \rangle = \sum_{i=1}^{N} \sqrt{\lambda_i} | i_Ai_B \rangle,$$  \hspace{1cm} (17)

where the Schmidt parameters $\lambda_i$ satisfy the normalization condition $\sum_{i=1}^{N} \lambda_i = 1$.

In analogy to Eq. (15) we define the measure of the degree of entanglement $E$ of the state $| \psi_{AB}^{(N)} \rangle$, Eq. (17) as

$$E(\psi_{AB}^{(N)}) \equiv \frac{N}{2(N - 1)} \times \sum_{n,m=1}^{N} | P(n_A, m_B) - P(n_A)P(m_B) | \hspace{1cm} (18)$$

where $n$, and $m$ again denote states of the Schmidt basis and a normalization factor $N/[2(N - 1)]$ has been introduced in order to ensure $0 \leq E \leq 1$. This normalization factor reduces to 1 for $N = 1$, so the measure Eq. (18) contains the measure Eq. (15) for two–level particles as a particular case.

After simple calculations we arrive at

$$E(\psi_{AB}^{(N)}) = \frac{N}{N - 1} \left[ 1 - \sum_{i=1}^{N} \lambda_i^2 \right] \equiv E(\lambda_1, \ldots, \lambda_N).$$  \hspace{1cm} (19)

This expression is proportional to the 2–entropy (22) and is therefore a good measure of entanglement.

Indeed, condition (i) is clearly satisfied. Since $\lambda_i \leq 1$, $E$ is equal to 0 if and only if $\sum_{i=1}^{N} \lambda_i^2 = 1$. Because of the constraint $\sum_{i=1}^{N} \lambda_i = 1$, this occurs if one and only one Schmidt parameter $\lambda_i$ is nonvanishing and equal to 1. This corresponds to the disentangled state

$$| \psi_{AB}^{(N)} \rangle = | i_A \rangle \otimes | i_B \rangle.$$  \hspace{1cm} (20)

The highest degree of entanglement $E = 1$ occurs when $\lambda_1 = \ldots = \lambda_N = 1/N$. Indeed, the function

$$f(\lambda_1, \ldots, \lambda_N) = E(\lambda_1, \ldots, \lambda_N) + \mu \left( \sum_{i=1}^{N} \lambda_i - 1 \right)$$  \hspace{1cm} (21)

where $\mu$ is a Lagrange multiplier has a unique maximum at $\lambda_1 = \ldots = \lambda_N = 1/N$.

Condition (ii) is also satisfied since the Schmidt parameters are invariant under a change of basis (13). Finally, condition (iii) is also satisfied because the expression Eq. (19) is Schur concave and thus entanglement monotone.

**IV. DISCUSSION AND CONCLUSIONS**

We have examined the peculiar features of an entangled pure state of a bipartite system. Entanglement is the origin of correlations between the two particles of the system and manifests itself in the properties of the conditional and joint probabilities of measurement outcomes that are defined with the help of the Schmidt decomposition. These properties can be used to define a measure of entanglement, that results to be essentially equal to the 2–entropy.

An important question is how can our approach be extended to (a) multipartite systems and (b) mixed states, where the Schmidt decomposition does not always exist. At first sight these seem to be two radically different cases.

Let us first consider the case of a multipartite system composed of three particles A, B, and C, in a pure state $| \psi_{ABC} \rangle$ of a Hilbert space of finite dimension. In our opinion, one has to be extremely careful in formulating the correct question. We believe that the questions “What is the degree of entanglement of $| \psi_{ABC} \rangle$?” and “How much is A entangled with B+C in $| \psi_{ABC} \rangle$?” are fundamentally different and only the latter makes clear sense. Indeed, if we divide the system into the two subsystems A and B+C we can still use our measure to define the degree of entanglement of A with B+C. In general, the measure $E_{A,B+C}$ of entanglement between A and B+C is different from the measure $E_{B,A+C}$ of entanglement between B and A+C. It is thus not clear if a measure of entanglement of the whole system can ever be defined. Only if the three–partite system has a Schmidt decomposition we have $E_{B,A+C} = E_{B,A+C} = E_{B,A+C}$
and we can safely take this quantity as a measure of entanglement of the whole system.

One can also ask ‘What is the degree of entanglement between A and B in $|\psi_{ABC}\rangle$?’ One could be tempted to say that such a question is not well posed. After all, the lesson of EPR’s argument is that entanglement is a non-local, i.e., non–individual property of a quantum system and that none of the components of the system can be neglected, contrarily to what the last question above does. However, if one takes the density matrix $\rho_{ABC}$ and traces over the particle C, one ends up with the density matrix $\rho_{AB}$ of the two particles A and B. In this way the problem of measuring the entanglement between two particles of a multiparticle system in a pure state seems to reduce to the problem of measuring the entanglement between two particles in a mixed state. Although the two physical systems (two particles of a three–partite system in a pure state vs. two particles in a mixed state) are conceptually and physically different, we cannot distinguish between them if they are described by the same density matrix and the third particle C is not accessible to us.

We are thus led to the problem of extending our approach to mixed states. It seems that there are two major problems. We have already mentioned that the Schmidt decomposition does not exist in general for mixed states [27]. The second problem is that in mixed states also classical correlations occur [28]. One must then be able to separate the contribution of classical correlations to the joint and conditional probabilities from the quantum correlations that have their origin in entanglement. These problems are currently under investigation.

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