THE NAVIER-STOKES-CAHN-HILLIARD EQUATIONS FOR MILDLY COMPRESSIBLE BINARY FLUID MIXTURES

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Abstract. We study the well-posedness for the mildly compressible Navier-Stokes-Cahn-Hilliard system with non-constant viscosity and Landau potential in two and three dimensional domains.

1. Introduction and main results. In this article, we consider the Navier-Stokes-Cahn-Hilliard system for mildly compressible binary mixtures. In a bounded smooth domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, the model reads as follows

\[
\begin{aligned}
\frac{\partial}{\partial t} u + (u \cdot \nabla) u + \frac{1}{2} (\text{div} u) u - \text{div}(\nu(\phi) D u) + \nabla p &= \mu \nabla \phi, \\
\varepsilon \frac{\partial}{\partial t} p + \text{div} u &= 0, \\
\frac{\partial}{\partial t} \phi + u \cdot \nabla \phi &= \Delta \mu, \\
\mu &= -\Delta \phi + \Psi'(\phi),
\end{aligned}
\]

in $\Omega \times (0, T)$, \hspace{1cm} (1)

with boundary and initial conditions

\[
\begin{aligned}
\begin{cases}
    u = 0, \quad \partial_n \mu = \partial_n \phi = 0 & \text{on } \partial \Omega \times (0, T), \\
    u(\cdot, 0) = u_0, \quad p(\cdot, 0) = p_0, \quad \phi(\cdot, 0) = \phi_0, & \text{in } \Omega.
\end{cases}
\end{aligned}
\]

(2)

Here $u = u(x, t)$, $p = p(x, t)$, $\phi = \phi(x, t)$, $\mu = \mu(x, t)$ denote the averaged fluid velocity, the pressure, the difference of the fluids concentrations, and the chemical potential, respectively. The term $\nu(\phi)$ represents the viscosity of the mixture. The parameter $\varepsilon$ is a positive coefficient meant to be small, although we just assume here that $\varepsilon > 0$ (cf. [28]). The notation $Du$ stands for the symmetric gradient $\frac{1}{2}(\nabla u + (\nabla u)^T)$. The function $\Psi$ represents the free energy density given by the Landau potential $\Psi(\phi) = \frac{1}{4}(\phi^2 - 1)^2$.

Navier-Stokes-Cahn-Hilliard (NSCH) type systems are well-regarded and widely researched mathematical models which arise from the Diffuse Interface (DI) theory. The DI approach combines in a unified framework the concept of transition interface with the energy-based formalism from thermodynamics and statistical mechanics.

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In fluid mechanics the DI theory has been widely used in the last decades to describe the motion of binary fluid mixtures and their topological transitions under the effect of the surface tension. The main advantages of the DI formulation are twofold: DI models can be regarded as a regularization of the free boundary models with the purpose of approximating the limit problem as the interface thickness converges to zero, and DI methods provide a realistic description of complex fluids (e.g. polymers and gels). We refer the interested reader to the recent review article [20] and the references therein.

The original NSCH system, also called model H, was introduced in [22, 26] for homogeneous (constant density) and incompressible binary mixtures. The model H corresponds to the above system (1)-(2) with $\varepsilon = 0$. This system has been extensively studied in the last years. In the first contribution [9], existence of global weak solutions and existence and uniqueness of strong solutions (global if $d = 2$, local if $d = 3$) were proven for the model H with non-constant viscosity and Landau potential. Later on, existence of global weak solutions and existence and uniqueness of strong solutions with initial velocity $u_0 \in V_2^{1+r}(\Omega)$, with $r > 0$ if $d = 2$ and $r > \frac{1}{2}$ if $d = 3$, were shown in [1] for the model H with non-constant viscosity and physically relevant Flory-Huggins logarithmic potential\(^2\). More recently, the uniqueness of weak solutions and the existence and uniqueness of strong solutions with initial velocity $u_0 \in V_\sigma$ have been established in [21] for the model H with non-constant viscosity and Flory-Huggins logarithmic potential. Global strong solutions for the model H with mixed partial viscosity have been shown in [12] in two dimensions. The long-time behavior in terms of attractors for the model H with constant viscosity and Landau potential has been studied in [16, 17] in the two and three dimensional cases. It is also worth mentioning the analysis of the incompressible NSCH with boundary conditions that account for moving contact line slip velocity achieved in [18].

The development of NSCH systems for non-homogeneous binary mixtures has started with the seminal paper [25]. The authors derived two main classes of NSCH models: fully compressible and quasi-incompressible systems. In the former case, in addition to the velocity $u$ and the concentration $\phi$, the density $\rho$ is an unknown of the systems, whereas in the latter case the density $\rho$ is a function of the concentration $\phi$. For the compressible NSCH model with classical boundary conditions, the existence of global weak solutions was proven in [5]\(^3\). More recently, the existence of global weak solutions has been achieved in the case with generalized Navier boundary conditions for the velocity and dynamic boundary conditions for the concentration in [13]. The mathematical analysis for the quasi-incompressible NSCH system proposed in [25] has been established in two papers: in [2] the author

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\(^1\)The space $V_2^{1+r}(\Omega)$ is the interpolation space $(W_\sigma, V_\sigma)_{1-r}$ (see [24]), where $V_\sigma = \{ u \in V : \text{div } u = 0 \in \Omega \}$ and $W_\sigma = \{ u \in V_\sigma \cap H^2(\Omega) \}$.

\(^2\)The Flory-Huggins entropy potential reads as $\Psi(\phi) = \frac{\theta}{2}[(1 + \phi) \ln(1 + \phi) + (1 - \phi) \ln(1 - \phi)] - \frac{\theta \phi}{2} \phi^2$, with $0 < \theta < \theta_0$. It is derived in statistical mechanics through the mean field approximation. In this case the solutions are called physical since they satisfy the property $-1 \leq \phi \leq 1$ for almost every $(x, t) \in \Omega \times (0, T)$. The classical Landau potential can be seen as an approximation of the Flory-Huggins potential. Moreover, the existence of physical solutions is not guaranteed for the Landau potential.

\(^3\)It is worth mentioning that the authors considered a slightly simplification of the model proposed in [25]. They studied a total free energy of the form $E(\rho, \phi) = \int_\Omega \rho \Psi(\rho, \phi) + \frac{1}{2} | \nabla \phi |^2 \ dx$ instead of $E(\rho, \phi) = \int_\Omega \rho \Psi(\rho, \phi) + \frac{1}{2} | \nabla \phi |^2 \ dx$ as in [25]. The analysis in the latter case is still open.
shows existence of global weak solutions, in [3] the author proves the existence and uniqueness of strong solutions. Later on, a novel NSCH model for non-homogeneous binary mixtures satisfying the incompressibility constraint has been derived in [6]. The existence of global weak solutions for this system has been achieved in [4] for classical boundary conditions (2), and in [19] for moving contact line boundary conditions. Further generalizations of the NSCH model for non-homogeneous mixtures were proposed by [10], [15] and [20].

The NSCH system (1) for mildly compressible binary fluids can be regarded as an intermediate model between the incompressible Model H and the non-homogeneous NSCH systems mentioned above. The system (1) is obtained from the model H by the Artificial Compressibility Method. This is a classical numerical approximation technique for the Navier-Stokes equations to avoid finite element approximation functions satisfying the divergence free condition. More precisely, one assumes the state equation \( \rho = \rho_0 + \varepsilon p \) (\( \rho_0 \) is a constant) and perform a linearization of the equation of motion with respect to \( \varepsilon \). We refer the interested reader to [23, Ch. 19] and [28, Ch. III, Section 8] for the derivation and the mathematical analysis for the corresponding Navier-Stokes equations (see also [29]). Moreover, system (1)-(2) corresponds to the macroscopical system associated with the lattice Boltzmann equations recently employed in [30] to simulate mixing flow in a centrifugal contactor. The aim of this paper is to extend the mathematical analysis of the Model H to the NSCH system (1)-(2) with non-constant viscosity and Landau potential. We first prove the existence of global weak solutions in two and three dimensions. In dimension two, we are able to show that weak solutions are unique. In order to achieve this result, we generalize the technique used in [21] to the case with non divergence free velocity fields. Next, we address the existence of global semi-strong solutions in two and three dimensions. This corresponds to the case where the velocity \( u \) is a weak solution of the Navier-Stokes equation, whereas the concentration \( \phi \) (up to its total mass \( \overline{\phi} = \frac{1}{|\Omega|} \int_\Omega \phi \, dx \), cf. Theorem 1.2 below) belongs to \( D(A^\frac{1+s}{N} \Omega) \) for any \( s \in (0, 1] \). For such semi-strong solutions in two dimensions, we also provide a more direct proof of uniqueness than the argument for weak solutions. We notice that such a result on semi-strong solutions seems to be novel even for the model H. Lastly, we show the existence and uniqueness of strong solutions. More precisely, the velocity \( u \) is a strong solution of the Navier-Stokes equation (according to [28]), and the concentration \( \phi \) belongs to \( D(A^\frac{1+s}{N} \Omega) \). In particular, we are able to address the following cases: \( s \in (0, 1] \) if \( d = 2 \), \( s \in \left[ \frac{1}{3}, 1 \right] \) if \( d = 3 \). The case \( s \in (0, \frac{1}{3}] \) remains open in dimension three. Besides, we observe that the case \( s > 1 \) can be also achieved with minor difficulties. It is worth mentioning at this point that it does not seem to be possible to extend the present analysis to the case with Flory-Huggins logarithmic potential due to the lack of conservation of mass for \( \phi \) and of the lack of incompressibility constraint.

We are now ready to state the main results of this paper. First, we have

**Theorem 1.1.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^d \), \( d = 2, 3 \), with smooth boundary, and \( \varepsilon > 0 \). Assume that \( \nu = \nu(r) \in W^{1,\infty}(\mathbb{R}) \) is such that \( \nu(r) \geq \nu_* > 0 \) for all \( r \in \mathbb{R} \). Let the initial datum \((u_0, p_0, \phi_0)\) satisfy \( u_0 \in H, p_0 \in L^2(\Omega), \phi_0 \in H^1(\Omega) \). Then, there exists a weak solution \((u, p, \phi)\) to system (1)-(2) on the interval \([0, T]\) such that

\[
\begin{align*}
  u &\in L^\infty(0, T; H) \cap L^2(0, T; \mathbb{V}), \\
  \partial_t u &\in L^4(0, T; \mathbb{V}'), \quad \partial_t \phi \in L^2(0, T; H), \\
  \int_\Omega \overline{\phi} &\text{ is a constant.}
\end{align*}
\]
which satisfies

\[ \langle \partial_t u, v \rangle + (u \cdot \nabla u, v) + \frac{1}{2}((\text{div} u)u, v) + (\nu(\phi)Du, \nabla v) - (p, \text{div} v) = (\mu \nabla \phi, v), \]

\[ \langle \partial_t \phi, v \rangle + (u \cdot \nabla \phi, v) + (\nabla \mu, \nabla v) = 0, \]

for all \( v \in V \), \( v \in H^1(\Omega) \), and for almost every \( t \in (0, T) \), where

\[ \varepsilon \partial_t p + \text{div} u = 0, \quad \mu = -\Delta \phi + \Psi'(\phi) \quad \text{a.e.} \quad (x, t) \in \Omega \times (0, T). \]

The solution satisfies \( \partial_t \phi = 0 \) on \( \partial \Omega \times (0, T) \) and the energy inequality (equality if \( d = 2 \))

\[ E(u(t), p(t), \phi(t)) + \int_0^t \int_\Omega (\nu(\phi) |Du|^2 + |\nabla \mu|^2) \, dx \, dt \leq E(u_0, p_0, \phi_0), \quad \forall 0 \leq t \leq T. \]

Furthermore, the weak solution is unique in two dimensions.

Next, our second result concerns the existence of global semi-strong solutions.

**Theorem 1.2.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^d, \, d = 2, 3 \), with smooth boundary, and \( \varepsilon > 0 \). Assume that \( \nu = \nu(r) \in W^{1, \infty}(\mathbb{R}) \) is such that \( \nu(r) \geq \nu_* > 0 \) for all \( r \in \mathbb{R} \).

For any \( s \in (0, 1] \), suppose that the initial datum \( (u_0, p_0, \phi_0) \) satisfies

\( u_0 \in H, \quad p_0 \in L^2(\Omega), \quad \phi_0 - \overline{\phi}_0 \in D(A_N^{\frac{2}{N}}), \quad \overline{\phi}_0 \in \mathbb{R}. \) Then, there exists a semi-strong solution \( (u, p, \phi) \) to system (1)-(2) on the interval \([0, T]\) such that

\[ u \in L^\infty(0, T; H) \cap L^2(0, T; V), \quad \partial_t u \in L^\frac{d}{2}(0, T; V'), \]

\[ p \in L^\infty(0, T; L^2(\Omega)) \cap W^{1, 2}(0, T; L^2(\Omega)), \]

\[ \phi - \overline{\phi} \in L^\infty(0, T; D(A_N^{\frac{2}{N}})) \cap L^2(0, T; D(A_N^{\frac{2}{N}})) \cap W^{1, 2}(0, T; D(A_N^{\frac{2}{N}})), \]

which satisfies

\[ \langle \partial_t u, v \rangle + (u \cdot \nabla u, v) + \frac{1}{2}((\text{div} u)u, v) + (\nu(\phi)Du, \nabla v) - (p, \text{div} v) = (\mu \nabla \phi, v), \]

\[ \langle \partial_t \phi, v \rangle + (u \cdot \nabla \phi, v) + (\nabla \mu, \nabla v) = 0, \]

for all \( v \in V, \quad v \in D(A_N^{\frac{2}{N}}), \) for almost every \( t \in (0, T) \), where

\[ \varepsilon \partial_t p + \text{div} u = 0, \quad \mu = -\Delta \phi + \Psi'(\phi) \quad \text{a.e.} \quad (x, t) \in \Omega \times (0, T), \]

and \( \overline{\phi}(t) = \overline{\phi}_0 - \int_0^t u(\tau) \cdot \nabla \phi(\tau) \, d\tau \) for all \( t \in [0, T] \). Moreover, the semi-strong solution is unique in two dimensions.

Finally, we discuss the existence and uniqueness of strong solutions.

**Theorem 1.3.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^d, \, d = 2, 3 \), with smooth boundary, and \( \varepsilon > 0 \). Assume that \( \nu = \nu(r) \in W^{1, \infty}(\mathbb{R}) \) is such that \( \nu(r) \geq \nu_* > 0 \) for all \( r \in \mathbb{R} \). We have the following results:

(i) Let \( d = 2 \). For any \( s \in (0, 1] \), assume that the initial datum \( (u_0, p_0, \phi_0) \) satisfies

\( u_0 \in V, \quad p_0 \in H^1(\Omega), \quad \phi_0 - \overline{\phi}_0 \in D(A_N^{\frac{2}{N}}), \quad \overline{\phi}_0 \in \mathbb{R}. \) Then, there exists a unique strong solution \( (u, p, \phi) \) to system (1)-(2) on the interval \([0, T]\) such that

\[ u \in L^\infty(0, T; V) \cap L^2(0, T; H^2 \cap V) \cap W^{1, 2}(0, T; H), \]

\[ p \in L^\infty(0, T; H^1(\Omega)) \cap W^{1, 2}(0, T; H^1(\Omega)), \]
\[ \phi - \overline{\phi} \in L^\infty(0, T; D(A^{3/2}_N)) \cap L^2(0, T; D(A^{3/4}_N)) \cap W^{1,2}(0, T; D(A^{\theta/2}_N)). \]

(ii) Let \( d = 3 \). For any \( s \in \left[ \frac{1}{2}, 1 \right] \), assume that the initial datum \((u_0, p_0, \phi_0)\) satisfies \( u_0 \in V, \ p_0 \in H^1(\Omega), \ \phi_0 - \overline{\phi_0} \in D(A^{3/4}_N), \ \overline{\phi_0} \in \mathbb{R} \). Then, there exist \( T_0 > 0 \) and a unique strong solution \((u, p, \phi)\) to system (1)-(2) on the interval \([0, T_0]\) such that

\[
\begin{align*}
\phi - \overline{\phi} & \in L^\infty(0, T_0; V) \cap L^2(0, T_0; H^2) \cap W^{1,2}(0, T_0; H^1), \\
p & \in L^\infty(0, T_0; H^1(\Omega)) \cap W^{1,2}(0, T_0; H^1(\Omega)), \\
\phi - \overline{\phi} & \in L^\infty(0, T_0; D(A^{3/4}_N)) \cap L^2(0, T_0; D(A^{3/4}_N)) \cap W^{1,2}(0, T_0; D(A^{3/4}_N)).
\end{align*}
\]

In both cases equations (1), (1)_2 and (1)_4 hold a.e. \((x, t) \in \Omega \times (0, T)\), and

\[
\langle \partial_t \phi, v \rangle + \langle u \cdot \nabla \phi, v \rangle + \langle \nabla \mu, \nabla v \rangle = 0, \quad \forall v \in D(A^{3/4}_N),
\]

for almost every \( t \in (0, T) \).

**Plan of the paper.** In Section 2 we introduce the functional framework. In Section 3 we prove the existence of global weak solutions. Section 4 is devoted to the uniqueness of strong solutions in two dimensions. In Section 5 we show the uniqueness of weak solutions in two dimensions. In Section 6 we prove the existence and uniqueness of strong solutions.

2. **Functional setup.** Let \( X \) be a (real) Banach or Hilbert space with norm denoted by \( \| \cdot \|_X \). The boldface letter \( X \) denotes the vectorial space \( X^d \) (\( d = 2, 3 \) is the spatial dimension), which consists of vector-valued functions \( u = (u_1, \ldots, u_d) \) such that \( u_i \in X, \ i = 1, \ldots, d \), with norm \( \| \cdot \|_X \). Let \( \Omega \) be a bounded domain in \( \mathbb{R}^d \), \( d = 2, 3 \), with smooth boundary \( \partial \Omega \). We denote by \( W^{k,p}(\Omega), \ k \in \mathbb{N} \), the Sobolev space of functions in \( L^p(\Omega) \) with distributional derivatives of order less than or equal to \( k \) in \( L^p(\Omega) \) and by \( \| \cdot \|_{W^{k,p}(\Omega)} \) its norm. For \( k \in \mathbb{N} \), the Hilbert space \( W^{1,2}(\Omega) \) is denoted by \( H^k(\Omega) \) with norm \( \| \cdot \|_{H^k(\Omega)} \). We denote by \( H^1_0(\Omega) \) the closure of \( C^\infty(\Omega) \) in \( H^1(\Omega) \) and by \( H^{-1}(\Omega) \) its dual space. Inner product and norm in \( L^2(\Omega) \) are denoted by \( \langle \cdot, \cdot \rangle \) and \( \| \cdot \|_{L^2(\Omega)} \), respectively. The symbol \( \langle \cdot, \cdot \rangle \) will stand for the duality product between \( (H^1(\Omega))^' \) and \( H^1(\Omega) \). We denote by \( \overline{\pi} \) the average of \( u \) over \( \Omega \), that is \( \overline{\pi} = |\Omega|^{-1} (u, 1) \), for all \( u \in (H^1(\Omega))^' \). By the Poincaré inequality we recall that \( \| \nabla u \|^2 + |\pi|^2 \frac{1}{2} \) is a norm on \( H^1(\Omega) \) equivalent to the natural one. We use the notation \( H = L^2(\Omega) \) and \( V = H^1_0(\Omega) \). We recall the Korn inequality

\[
\| \nabla v \|_{L^2(\Omega)} \leq 2 \| Dv \|_{L^2(\Omega)} \quad \forall v \in V.
\]

We recall the Gagliardo-Nirenberg, Agmon and Brezis-Gallouet inequalities

\[
\begin{align*}
\| u \|_{L^p(\Omega)} & \leq C \| u \|_{L^2(\Omega)}^{\frac{3}{p}} \| u \|_{H^1(\Omega)}^{1 - \frac{3}{p}}, \quad \forall u \in H^1(\Omega), 2 < p < \infty, \text{ if } d = 2, \\
\| u \|_{L^p(\Omega)} & \leq C \| u \|_{L^2(\Omega)}^{\frac{p-2}{2p}} \| u \|_{H^1(\Omega)}^{\frac{2(p-2)}{p}}, \quad \forall u \in H^1(\Omega), 2 < p \leq 6, \text{ if } d = 3, \\
\| u \|_{L^\infty(\Omega)} & \leq C \| u \|_{L^2(\Omega)}^{\frac{1}{2}} \| u \|_{H^2(\Omega)}^{\frac{1}{2}}, \quad \forall u \in H^2(\Omega), \text{ if } d = 2, \\
\| u \|_{L^\infty(\Omega)} & \leq C \| u \|_{H^1(\Omega)}^{\frac{1}{2}} \| u \|_{H^2(\Omega)}^{\frac{1}{2}}, \quad \forall u \in H^2(\Omega), \text{ if } d = 3, \\
\| u \|_{L^\infty(\Omega)} & \leq C \| u \|_{H^1(\Omega)} \ln^\frac{1}{2} \left( e \frac{\| u \|_{H^2(\Omega)}}{\| u \|_{H^1(\Omega)}} \right), \quad \forall u \in H^2(\Omega), \text{ if } d = 2.
\end{align*}
\]
and the following $L^2$-estimate of the product of functions (see [21, Proposition C.1])

$$
\|uv\|_{L^2(\Omega)} \leq C\|u\|_{L^2(\Omega)}\|v\|_{H^1(\Omega)} \ln^{\frac{1}{2}} \left( \frac{\|u\|_{H^1(\Omega)}}{\|u\|_{L^2(\Omega)}} \right), \quad \forall u, v \in H^1(\Omega), \text{ if } d = 2. \tag{9}
$$

The Laplace operator $(-\Delta)$ associated with homogeneous Neumann boundary conditions is denoted by $A_N$. This is an unbounded operator on $L^2_0(\Omega) = \{ u \in L^2(\Omega) : \pi = 0 \}$ with domain $D(A_N) = \{ u \in H^2(\Omega) : \pi = 0, \partial_n u = 0 \text{ on } \partial \Omega \}$. The operator $A_N$ is positive on $L^2_0(\Omega)$ since

$$(A_N u, u) = \| \nabla u \|^2 \geq \frac{1}{C_\Omega} \| u \|^2, \quad \forall u \in D(A_N).$$

Here, we have used a generalized Poincaré inequality (see, e.g., [27, Chapter I, Section 1.4]). Notice that $A_N$ is symmetric since $(A_N u, v) = (u, A_N v)$, for all $u, v \in D(A_N)$. The range of $A_N$ is $L^2_0$, namely $R(A_N) = L^2_0$, which follows from the solvability and regularity properties of the Neumann problem. Then, $A_N$ is self-adjoint on $L^2_0$ with compact inverse. By the spectral theory, there exists a sequence of real and positive eigenvalues $\alpha_j$, $j \in \mathbb{N}$, associated with $A_N$ such that $\alpha_j \leq \alpha_{j+1}$ and $\lim_{j \to \infty} \alpha_j = \infty$. The related eigenfunctions $w_j \in D(A_N)$ solve $A_N w_j = \alpha_j w_j$ and form an orthonormal basis of $L^2_0(\Omega)$. We have the spectral decomposition

$$
u = \sum_{j=1}^{\infty} (u, w_j) w_j \quad u \in L^2_0(\Omega), \quad A_N u = \sum_{j=1}^{\infty} \alpha_j (u, w_j) w_j \quad u \in D(A_N).$$

In addition, regularity theory entails that $w_j \in H^{k+2}(\Omega)$ provided that, for example, $\Omega \in C^k$. For $s \in \mathbb{R}$, we define the fractional powers by

$$A_N^s u = \sum_{j=1}^{\infty} \alpha_j^s (u, w_j) w_j, \quad u \in D(A_N^s), \tag{10}$$

where, for $s > 0$,

$$D(A_N^s) = \{ u \in L^2_0(\Omega) : A_N^s u \in L^2_0(\Omega) \},$$

for $s < 0$, $D(A_N^s)$ is the completion of $L^2_0(\Omega)$ for the norm $(\sum_{j=1}^{\infty} \alpha_j^{2s} |(u, w_j)|^2)^{\frac{1}{2}}$. We recall that $D(A_N^*) = (D(A_N^{s*}))'$ for $s < 0$. The domain $D(A_N^s)$ is a Hilbert space associated with the inner product and norm

$$(u, v)_s = (A_N^s u, A_N^s v), \quad \| u \|_s^2 = \sum_{j=1}^{\infty} \alpha_j^{2s} |(u, w_j)|^2.$$

For $s > 0$, we now introduce the fractional Sobolev spaces $H^s(\Omega)$, $s = m + \sigma$ where $m \in \mathbb{N}$, $\sigma \in (0, 1)$, as the set of functions $u$ in $L^2(\Omega)$ such that

$$\| u \|_{H^s(\Omega)} = \left( \| u \|_{H^m(\Omega)}^2 + \sum_{|j|=m} \int_{\Omega} \int_{\Omega} \frac{|D^j u(x) - D^j u(y)|^2}{|x-y|^{d+2\sigma}} \, dx \, dy \right)^{\frac{1}{2}} < \infty.$$

We recall the following embedding results (see, e.g., [14])

$$
\begin{cases}
H^s(\Omega) \hookrightarrow L^p(\Omega), & \text{if } 2s < d, \quad p \leq \frac{2d}{d-2s}, \\
H^s(\Omega) \hookrightarrow L^p(\Omega), & \text{if } 2s = d, \quad 1 \leq p < \infty, \\
H^s(\Omega) \hookrightarrow C^{[s-\frac{d}{2}],s-\frac{d}{2}+[s-\frac{d}{2}]}(\Omega), & \text{if } 2s > d, \quad s - \frac{d}{2} \notin \mathbb{N}, \\
H^s(\Omega) \hookrightarrow C^{s-\frac{d}{2}-1,\lambda}(\Omega), & \text{if } 2s > d, \quad s - \frac{d}{2} \in \mathbb{N}, \forall \lambda < 1.
\end{cases} \tag{11}
$$
We report a precise characterization of the spaces $D(A^s_N)$ in terms of the usual spaces $H^s(\Omega)$ (see [24]). For $0 < s < 1$, we have

$$
\begin{align*}
D(A^s_N) &= \{ u \in H^{2s}(\Omega) : \pi = 0 \}, & \text{if } 0 < s < \frac{3}{4}, \\
D(A^s_N) &= \{ u \in H^3(\Omega) : \pi = 0, \int_{\Omega} |\nabla u|^2 \, dx \}, & \text{if } s = \frac{3}{4}, \\
D(A^s_N) &= \{ u \in H^{2s}(\Omega) : \pi = 0, \, \partial_n u = 0 \text{ on } \partial \Omega \}, & \text{if } \frac{3}{4} < s < 1,
\end{align*}
$$

(12)

where $d(x) = \text{dist}(x, \partial \Omega)$ and the condition on the boundary is interpreted in the sense of traces. Moreover, we have

$$\|u\|_{H^2(\Omega)} \leq C\|A^s_N(u - \pi)\|_{L^2(\Omega)} + |\pi|).$$

(13)

We also recall the classical results

$$\|u\|_{H^2(\Omega)} \leq C(\|A_N u\|_{L^2(\Omega)} + |\pi|), \quad \|u\|_{H^3(\Omega)} \leq C(\|A_N u\|_{H^1(\Omega)} + |\pi|).$$

(14)

Next we consider the Lamé-Navier system

$$
\begin{align*}
-\Delta u - \nabla \text{div} u &= f & \text{in } \Omega \\
u &= 0 & \text{on } \partial \Omega.
\end{align*}
$$

(15)

By the Lax-Milgram theorem, for any $f \in \mathbf{V}'$, there exists a unique weak solution $u \in \mathbf{V}$ to (15) such that $a(u, v) = (f, v)$ for all $v \in \mathbf{V}$, where $a(u, v) = (\nabla u, \nabla v) + (\text{div} u, \text{div} v)$. We report the following regularity result. We refer the reader to [8, Chapter 8].

**Lemma 2.1.** Let $f \in \mathbf{H}$. Then, $u \in H^2(\Omega) \cap \mathbf{V}$. Moreover, we have

$$\|u\|_{H^2(\Omega)} \leq CLN\|f\|_{L^2(\Omega)}.$$  

(16)

In light of Lemma 2.1, the operator $A_{LN} = -\Delta - \nabla \text{div}$ is an unbounded operator on $\mathbf{H}$ with domain $D(A_{LN}) = H^2(\Omega) \cap \mathbf{V}$. The operator is positive self-adjoint with compact inverse. By the spectral theory, there exists a sequence of real and positive eigenvalues $\beta_j$, $j \in \mathbb{N}$, associated with $A_{LN}$ such that $\beta_j \leq \beta_{j+1}$ and $\lim_{j \to \infty} \beta_j = \infty$. The related eigenfunctions $w_j \in D(A_{LN})$ solve $A_{LN} w_j = \beta_j w_j$ and form an orthonormal basis of $\mathbf{H}$.

Finally we report the Osgood lemma (see, e.g., [7]).

**Lemma 2.2.** Let $f$ be a measurable function from $[0, T]$ to $[0, a]$, $g \in L^1(0, T)$, and $W$ a continuous and nondecreasing function from $[0, a]$ to $\mathbb{R}^+$. Assume that, for some $c \geq 0$, we have

$$f(t) \leq c + \int_0^t g(s)W(f(s)) \, ds, \quad \text{for a.e. } t \in [0, T].$$

- If $c > 0$, then for almost every $t \in [0, T]$,

$$-M(f(t)) + M(c) \leq \int_0^T g(s) \, ds, \quad \text{where } M(s) = \int_s^a \frac{1}{W(s)} \, ds.$$  

- If $c = 0$ and $\int_0^a \frac{1}{W(s)} \, ds = \infty$, then $f(t) = 0$ for almost every $t \in [0, T]$.

3. **Proof of Theorem 1.1:** Existence of weak solutions. For clarity of presentation, we will perform a series of a priori estimates formally, keeping in mind that all results may be justified in the strictest mathematical sense through the Galerkin approximation presented at the end of this section.
3.1. **Energy estimates.** Multiplying the equation (1)_1 by \( \mathbf{u} \) and integrating over \( \Omega \), we obtain

\[
\frac{1}{2} \frac{d}{dt} \left\| \mathbf{u} \right\|^2_{L^2(\Omega)} + \left( (\mathbf{u} \cdot \nabla) \mathbf{u} + \frac{1}{2} (\text{div} \, \mathbf{u}) \mathbf{u} + (\nu(\phi) \mathbf{D} \mathbf{u}, \nabla \mathbf{u}) + (\nabla p, \mathbf{u}) \right) = (\mu \nabla \phi, \mathbf{u}).
\]

Multiplying the equation (1)_2 by \( p \) and integrating over \( \Omega \), we find

\[
\frac{\varepsilon}{2} \frac{d}{dt} \left\| p \right\|^2_{L^2(\Omega)} + (\text{div} \, \mathbf{u}, p) = 0.
\]

Multiplying the equation (1)_3 by \( \mu \), integrating over \( \Omega \), and using the boundary condition for \( \mu \), we obtain

\[
\frac{d}{dt} \left\{ \frac{1}{2} \left\| \mathbf{u} \right\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2} \left\| p \right\|^2_{L^2(\Omega)} + \int_{\Omega} \frac{1}{2} \left| \nabla \phi \right|^2 + \Psi(\phi) \, dx \right\} + \int_{\Omega} \nu(\phi) |\mathbf{D} \mathbf{u}|^2 \, dx + \left\| \nabla \mu \right\|^2_{L^2(\Omega)} = 0.
\]

Combining (17), (18) and (19), we find

\[
\frac{d}{dt} \left\{ \frac{1}{2} \left\| \mathbf{u} \right\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2} \left\| p \right\|^2_{L^2(\Omega)} + \int_{\Omega} \frac{1}{2} \left| \nabla \phi \right|^2 + \Psi(\phi) \, dx \right\} + \int_{\Omega} \nu(\phi) |\mathbf{D} \mathbf{u}|^2 \, dx + \left\| \nabla \mu \right\|^2_{L^2(\Omega)} = 0.
\]

Setting

\[
E(\mathbf{u}, p, \phi) = \frac{1}{2} \left\| \mathbf{u} \right\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2} \left\| p \right\|^2_{L^2(\Omega)} + \frac{1}{2} \left\| \nabla \phi \right\|^2_{L^2(\Omega)} + \int_{\Omega} \Psi(\phi) \, dx,
\]

we rewrite (20) as follows

\[
\frac{d}{dt} E + \int_{\Omega} \nu(\phi) |\mathbf{D} \mathbf{u}|^2 \, dx + \int_{\Omega} |\nabla \mu|^2 \, dx = 0.
\]

Integrating (21) over \([0, T]\), we infer that

\[
E(\mathbf{u}(T), p(T), \phi(T)) + \int_0^T \int_{\Omega} \left( \nu(\phi) |\mathbf{D} \mathbf{u}|^2 + |\nabla \mu|^2 \right) \, dx \, dt = E(\mathbf{u}_0, p_0, \phi_0).
\]

We will use throughout this section the notation \( E_0 \) to denote constants whose value depends on \( E(\mathbf{u}_0, p_0, \phi_0) \). By Young’s inequality, it is easily seen that \( \Psi(\phi) \geq \frac{1}{8} \phi^4 - \frac{1}{4} \). Thus, we deduce that

\[
\sup_{t \in [0, T]} \left\{ \frac{1}{2} \left\| \mathbf{u}(t) \right\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2} \left\| p(t) \right\|^2_{L^2(\Omega)} + \frac{1}{2} \left\| \nabla \phi(t) \right\|^2_{L^2(\Omega)} + \frac{1}{8} \left\| \phi(t) \right\|^4_{L^4(\Omega)} \right\} + \int_0^T \int_{\Omega} \left( \nu(\phi) |\mathbf{D} \mathbf{u}|^2 + |\nabla \mu|^2 \right) \, dx \, dt \leq E_0.
\]

Therefore, \( \mathbf{u}, \sqrt{\varepsilon} p, \nabla \phi \) belong to a bounded set in \( L^\infty(0, T; L^2(\Omega)) \). Also, we have that \( \phi \) belongs to a bounded set in \( L^\infty(0, T; L^4(\Omega)) \), which entails that \( \phi \in \)
By combining the above bounds, we easily infer that
\[ \int_0^T \| \nabla u(\tau) \|_{L^2(\Omega)}^2 \, d\tau \leq \frac{E_0}{\nu_*}. \]  
(24)
Thus, \( u \) belongs to a bounded set of \( L^2(0,T;\nabla) \).

3.2. Estimate of \( \phi \) in \( L^2(0,T;H^3(\Omega)) \). Taking the gradient of (14), multiplying the equation by \(-\nabla \Delta \phi\) and integrating over \( \Omega \), we find
\[ \| \nabla \Delta \phi \|_{L^2(\Omega)}^2 - \int_\Omega 3\nu^2 \nabla \cdot \nabla \phi \, dx = -\int_\Omega \nabla \mu \cdot \nabla \Delta \phi \, dx - \int_\Omega \nabla \phi \cdot \nabla \Delta \phi \, dx. \]
Exploiting integration by parts, we have
\[ \| \nabla \Delta \phi \|_{L^2(\Omega)}^2 + \int_\Omega 3\nu^2 (\Delta \phi)^2 \, dx + \int_\Omega 6\nu |\nabla \phi|^2 \Delta \phi \, dx = -\int_\Omega \nabla \mu \cdot \nabla \Delta \phi \, dx - \int_\Omega \nabla \phi \cdot \nabla \Delta \phi \, dx. \]
Thus, we obtain
\[ \| \nabla \Delta \phi \|_{L^2(\Omega)}^2 \leq \left| \int_\Omega 6\nu |\nabla \phi|^2 \Delta \phi \, dx + \int_\Omega \nabla \mu \cdot \nabla \Delta \phi \, dx + \int_\Omega \nabla \phi \cdot \nabla \Delta \phi \, dx \right|. \]
By using (14), we deduce that
\[ \| \phi \|_{H^3(\Omega)}^2 \leq C \left| \int_\Omega 6\nu |\nabla \phi|^2 \Delta \phi \, dx + \int_\Omega \nabla \mu \cdot \nabla \Delta \phi \, dx + \int_\Omega \nabla \phi \cdot \nabla \Delta \phi \, dx \right| + E_0. \]
By (5), (7), (23), and interpolation in Sobolev spaces, we find
\[ C \left| \int_\Omega 6\nu |\nabla \phi|^2 \Delta \phi \, dx \right| \leq C \| \phi \|_{L^\infty(\Omega)} \| \nabla \phi \|_{L^4(\Omega)}^2 \| \Delta \phi \|_{L^2(\Omega)} \]
\[ \leq C \| \phi \|_{H^1(\Omega)} \| \phi \|_{H^2(\Omega)}^2 \]
\[ \leq E_0 \| \phi \|_{H^3(\Omega)}^2 \leq \frac{1}{4} \| \phi \|_{H^3(\Omega)}^2 + E_0. \]
Also, we have
\[ C \left| \int_\Omega \nabla \mu \cdot \nabla \Delta \phi \, dx + \int_\Omega \nabla \phi \cdot \nabla \Delta \phi \, dx \right| \leq \frac{1}{4} \| \phi \|_{H^3(\Omega)}^2 + C \| \nabla \mu \|_{L^2(\Omega)}^2 + C \| \nabla \phi \|_{L^2(\Omega)}^2. \]
By combining the above bounds, we easily infer that
\[ \int_0^T \| \phi(\tau) \|_{H^3(\Omega)}^2 \, d\tau \leq E_0 \int_0^T \left( 1 + \| \nabla \mu(\tau) \|_{L^2(\Omega)}^2 \right) \, d\tau. \]
In turn, thanks to the above regularity of \( \mu \), this implies that \( \phi \in L^2(0,T;H^3(\Omega)) \).

3.3. Estimate of \( \partial_t \phi \) in \( L^2(0,T;H^1(\Omega)) \). For every \( v \in H^1(\Omega) \), we have
\[ \langle \partial_t \phi, v \rangle = (\nabla \cdot (u \cdot \nabla \phi), v) + (\nabla \mu \cdot \nabla v) \]
\[ \leq \| u \|_{L^2(\Omega)} \| \nabla \phi \|_{L^2(\Omega)} \| v \|_{L^4(\Omega)} + \| \nabla \mu \|_{L^2(\Omega)} \| \nabla v \|_{L^2(\Omega)} \]
\[ \leq (E_0 \| \nabla \phi \|_{L^2(\Omega)} + \| \nabla \mu \|_{L^2(\Omega)}) \| v \|_{H^1(\Omega)}. \]
Hence, we get \( \| \partial_t \phi \|_{H^1(\Omega)} \leq C(\| \phi \|_{W^{1,2}(\Omega)} + \| \nabla \mu \|_{L^2(\Omega)}) \). Integrating over \([0,T]\), it yields
\[ \int_0^T \| \partial_t \phi(\tau) \|_{H^1(\Omega)}^2 \, d\tau \leq C \int_0^T \left( \| \phi(\tau) \|_{W^{1,2}(\Omega)}^2 + \| \nabla \mu(\tau) \|_{L^2(\Omega)}^2 \right) \, d\tau. \]  
(25)
Thus, \( \partial_t \phi \) belongs to a bounded set of \( L^2(0, T; (H^1(\Omega))') \).

3.4. Estimate of \( \mu \) in \( L^2(0, T; H^1(\Omega)) \). Owing to the definition of \( \mu \) and the boundary conditions, we observe that \( \mu = \frac{1}{|\Omega|} \int_{\Omega} \Psi'(\phi) \, dx \). Therefore, since \( \phi \in L^\infty(0, T; H^1(\Omega)) \), we deduce that \( \mu \in L^\infty(0, T) \). By the Poincaré inequality together with (23), we find that \( \mu \) belongs to a bounded set of \( L^2(0, T; H^1(\Omega)) \).

3.5. Estimate of \( \partial_t u \) in \( L^2(0, T; V') \). From the equation (1), we have
\[
\langle \partial_t u, v \rangle = \langle \mu \nabla \phi, v \rangle - \langle \nu(\phi) D u, \nabla v \rangle + (p, \div v) - \langle \hat{B}(u, u), v \rangle \quad \forall v \in V,
\]
where
\[
\langle \hat{B}(u, v), w \rangle = \hat{b}(u, v, w), \quad \hat{b}(u, v, w) = \frac{1}{2}(b(u, v) - b(u, w, v)) \quad \forall u, v, w \in V,
\]
with \( b(u, v, w) = \langle (u \cdot \nabla) v, w \rangle \).

Thanks to (23), we have
\[
\langle \mu \nabla \phi, v \rangle \leq ||\mu||_{L^\infty(\Omega)} ||\nabla \phi||_{L^2(\Omega)} ||v||_{L^2(\Omega)}
\leq C ||\mu||_{H^1(\Omega)} ||v||_{V'}.
\]

Owing once more to (23), and using the assumption on \( \nu \), we find
\[
| - \langle \nu(\phi) D u, \nabla v \rangle + (p, \div v) | \leq C(||u||_V + ||p||_{L^2(\Omega)}) ||v||_{V'}.
\]

Also, we have
\[
||\hat{b}(u, u, w)|| \leq C ||u||_{L^2(\Omega)}^2 ||w||_{V'}.
\]

Hence, using (4) and (5), we obtain
\[
||\hat{B}(u, u)||_{V'} \leq C ||u||_V, \quad \text{if} \quad d = 2, \quad ||\hat{B}(u, u)||_{V'} \leq C ||u||_{V'}^2, \quad \text{if} \quad d = 3.
\]

Therefore, we have \( \hat{B}(u) \in L^2(0, T; V') \). Collecting the above estimates, we deduce that
\[
\mu \nabla \phi + \div (\nu(\phi) D u) - \div p - (u \cdot \nabla) u - \frac{1}{2} \div (\div u) \in L^2(0, T; V'),
\]
which, in turn, implies that \( \partial_t u \) belongs to a bounded set of \( L^2(0, T; V') \).

In the rest of this section we show the existence of weak solutions in the sense of Theorem 1.1. We apply the Galerkin procedure. We consider a basis of \( V \) given by the eigenfunction \( w_j \) of \( A_{L,N} \) (see Section 2), and a basis in \( H^1(\Omega) \) given by the eigenfunction \( w_j \) of \( A_{N} \) augmented by the constant function \( w_0 = 1 \). We consider
\[
V_m = \text{span}\{w_1, \ldots, w_m\}, \quad V_m = \text{span}\{w_0, \ldots, w_m\}.
\]

For each \( m \), we define an approximate solution
\[
\begin{align*}
\mathbf{u}_m(x, t) &= \sum_{i=1}^{m} g_{im}(t) w_i(x), \\
p_m(x, t) &= \sum_{i=1}^{m} h_{im}(t) w_i(x), \\
\phi_m(x, t) &= \sum_{i=0}^{m} k_{im}(t) w_i(x), \\
m_m(x, t) &= \sum_{i=0}^{m} l_{im}(t) w_i(x),
\end{align*}
\]
of the following system
\[
\begin{align*}
\langle \partial_t \mathbf{u}_m, v \rangle + \hat{b}(\mathbf{u}_m, \mathbf{u}_m, v) + & \langle \nu(\phi_m) D \mathbf{u}_m, \nabla v \rangle - (p_m, \div v) = \langle \mu_m \nabla \phi_m, v \rangle, \\
\varepsilon (\partial_t p_m, q) + \langle \div \mathbf{u}_m, q \rangle &= 0, \\
\langle \partial_t \phi_m, v \rangle + & \langle \mathbf{u}_m \cdot \nabla \phi_m, v \rangle + \langle \nabla \mu_m, \nabla v \rangle = 0,
\end{align*}
\]
where
\[
\begin{align*}
\hat{b}(u, v, w) &= \langle (u \cdot \nabla) v, w \rangle, \\
\hat{B}(u, v, w) &= \hat{b}(u, v, w).
\end{align*}
\]
for all $v \in V_m$, $q \in V_m$, and $v \in V_m$, and
\[ \mu_m = -\Delta \phi_m + P_m \Psi' (\phi_m), \] (31)
with the initial condition
\[ u_m(0) = P_m u_0, \quad p_m(0) = P_m p_0, \quad \phi_m(0) = P_m \phi_0, \]
where $P_m$ is the projection from $V$ to $V_m$, and $P_m$ is the projection from $H^1(\Omega)$ to $V_m$. Performing exactly the same calculations as in Section 3, we prove the following estimates:

- $u_m$ is bounded independently of $m$ in $L^\infty(0,T; H) \cap L^2(0,T; V)$,
- $\partial_t u_m$ is bounded independently of $m$ in $L^4(0,T; V')$,
- $p_m$ is bounded independently of $m$ in $L^\infty(0,T; L^2(\Omega)) \cap W^{1,2}(0,T; L^2(\Omega))$,
- $\phi_m$ is bounded independently of $m$ in $L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; H^3(\Omega))$,
- $\partial_t \phi_m$ is bounded independently of $m$ in $L^2(0,T; (H^1(\Omega))')$,
- $\mu_m$ is bounded independently of $m$ in $L^2(0,T; H^1(\Omega))$.

By the above estimates, and letting $m \to \infty$, we have the following convergent subsequences (still denoted by $m$ for simplicity)

- $u_m \rightharpoonup u$ in $L^\infty(0,T; H)$,
- $u_m \to u$ in $L^2(0,T; V)$,
- $\partial_t u_m \to \partial_t u$ in $L^4(0,T; V')$,
- $p_m \rightharpoonup p$ in $L^\infty(0,T; L^2(\Omega))$,
- $\phi_m \rightharpoonup \phi$ in $L^\infty(0,T; H^1(\Omega))$,
- $\phi_m \to \phi$ in $L^2(0,T; H^3(\Omega))$,
- $\partial_t \phi_m \to \partial_t \phi$ in $L^2(0,T; (H^1(\Omega))')$,
- $\mu_m \to \mu$ in $L^2(0,T; H^1(\Omega))$.

Besides, in light of the above bounds, the Aubin-Lions compactness lemma entails

- $u_m \to u$ in $L^2(0,T; L^p(\Omega))$ for $p \in [1,6)$ if $d = 3$ and for all $p \in [1,\infty)$ if $d = 2$. The above convergence results enable us to pass to the limit in the weak formulation (28)-(31). This is a rather standard argument and the details are left to the reader. Finally the validity of the energy inequality as stated in Theorem 1.1 can be also proven as in [1, 28].

4. **Proof of Theorem 1.1: Uniqueness of weak solutions.** In this section we prove the uniqueness of weak solutions to system (1)-(2) with non-constant viscosity in a bounded smooth domain $\Omega$ in $\mathbb{R}^2$. The argument relies on the estimate of the difference between two solutions in dual Sobolev spaces. This is inspired by the recent work [21]. However, due to the lack of the incompressibility constraint, we need to extend the technique to this case.

We consider two weak solutions $(u_1, p_1, \phi_1)$ and $(u_2, p_2, \phi_2)$ to system (1)-(2) corresponding to the initial conditions $(u_{01}, p_{01}, \phi_{01})$ and $(u_{02}, p_{02}, \phi_{02})$. We define
the difference of the two solutions \( u = u_1 - u_2 \), \( p = p_1 - p_2 \), and \( \phi = \phi_1 - \phi_2 \). They solve the system

\[
\begin{align*}
\langle \partial_t u, v \rangle + ((u_1 \cdot \nabla) u, v) + ((u \cdot \nabla) u_2, v) + \frac{1}{2}((\text{div} u_1) u, v) \\
+ \frac{1}{2}((\text{div} u) u_2, v) + (\nu(\phi_1) D u, \nabla v) + ((\nu(\phi_1) - \nu(\phi_2)) D u_2, \nabla v)
\end{align*}
\]

\( (32) \)

and

\[
\begin{align*}
\langle \partial_t \phi, v \rangle + (u_1 \cdot \nabla \phi, v) + (u \cdot \nabla \phi_2, v) + (\nabla \mu, \nabla v) = 0,
\end{align*}
\]

\( (33) \)

for any \( v \in V \) and \( v \in H^1(\Omega) \), where

\[
\varepsilon p_t + \text{div} u = 0, \quad \mu = -\Delta \phi + \Psi(\phi_1) - \Psi(\phi_2).
\]

\( (34) \)

We recall that the weak solutions satisfy the regularity \( (i = 1, 2) \)

\[
\begin{align*}
&\ u_i \in L^\infty(0, T; H) \cap L^2(0, T; V) \cap W^{1, 2}(0, T; V'), \\
&\ p_i \in L^\infty(0, T; L^2(\Omega)) \cap W^{1, 2}(0, T; L^2(\Omega)), \\
&\ \phi_i \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^3(\Omega)) \cap W^{1, 2}(0, T; (H^4(\Omega))').
\end{align*}
\]

\( (35) \) \( (36) \) \( (37) \)

Recalling also that

\[
\mu_i \nabla \phi_i = -\text{div}(\nabla \phi_i \otimes \nabla \phi_i) + \nabla \left( \Psi(\phi_i) + \frac{1}{2} |\nabla \phi_i|^2 \right), \quad i = 1, 2,
\]

using that \( \nabla f = \text{div}(f I) \), when \( f \) is a scalar function and \( I \) is the identity matrix, we can write

\[
\begin{align*}
\mu_1 \nabla \phi + \mu \nabla \phi_2 = \text{div} \left( -\nabla \phi_1 \otimes \nabla \phi - \nabla \phi \otimes \nabla \phi_2 + (\Psi(\phi_1) - \Psi(\phi_2)) I \right) \\
+ \text{div} \left( \frac{1}{2} (\nabla \phi_1 \cdot \nabla \phi + \nabla \phi \cdot \nabla \phi_2) I \right).
\end{align*}
\]

\( (38) \)

We take \( v = A_{L_N}^{-1} u \) in (32) and we obtain

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} a(A_{L_N}^{-1} u, A_{L_N}^{-1} u) + ((u_1 \cdot \nabla) u, A_{L_N}^{-1} u) + ((u \cdot \nabla) u_2, A_{L_N}^{-1} u) \\
+ \frac{1}{2}((\text{div} u_1) u, A_{L_N}^{-1} u) + ((\text{div} u) u_2, A_{L_N}^{-1} u) + (\nu(\phi_1) D u, \nabla A_{L_N}^{-1} u) \\
+ ((\nu(\phi_1) - \nu(\phi_2)) D u_2, \nabla A_{L_N}^{-1} u) - (p, \text{div} A_{L_N}^{-1} u)
\end{align*}
\]

\( (39) \)

\[
\begin{align*}
= (\Psi(\phi_1) - \Psi(\phi_2)) I + \frac{1}{2} (\nabla \phi_1 \cdot \nabla \phi + \nabla \phi \cdot \nabla \phi_2) I, \nabla A_{L_N}^{-1} u).
\end{align*}
\]

Here we have used that

\[
\langle \partial_t v, A_{L_N}^{-1} u \rangle = a(A_{L_N}^{-1} \partial_t v, A_{L_N}^{-1} u) = \frac{1}{2} \frac{d}{dt} a(A_{L_N}^{-1} v, A_{L_N}^{-1} u),
\]

\[
\frac{1}{2} \frac{d}{dt} a(A_{L_N}^{-1} v, A_{L_N}^{-1} v).
\]
Taking Collecting (39), (40), and (43) together, we find for all \( v \in L^2(0,T;V) \cap W^{1,2}(0,T;V') \), where \( a(\cdot,\cdot) \) is defined in Section 2. In order to recover a coercive term in (39), we write using integration by parts

\[
\int_{\Omega} \nu(\phi_1) D u : \nabla A_{L,N}^{-1} u \, dx = \int_{\Omega} \nu(\phi_1) \nabla u : D A_{L,N}^{-1} u \, dx
\]

\[
= -\int_{\Omega} u \cdot \nabla(\phi_1) D A_{L,N}^{-1} u \nabla \phi_1 \, dx - \int_{\Omega} \nu(\phi_1) u \cdot \text{div}(D A_{L,N}^{-1} u) \, dx
\]

\[
= -\int_{\Omega} u \cdot \nabla(\phi_1) D A_{L,N}^{-1} u \nabla \phi_1 \, dx
\]

\[
- \frac{1}{2} \int_{\Omega} \nu(\phi_1) u \cdot \left( \Delta A_{L,N}^{-1} u + \nabla \text{div} A_{L,N}^{-1} u \right) \, dx
\]

\[
= -\int_{\Omega} u \cdot \nabla(\phi_1) D A_{L,N}^{-1} u \nabla \phi_1 \, dx + \frac{1}{2} \int_{\Omega} \nu(\phi_1) |u|^2 \, dx.
\]

(40)

Notice that here we have used that \( \Delta A_{L,N}^{-1} u + \nabla \text{div} A_{L,N}^{-1} u = -u \) which follows from the definition of \( A_{L,N} \). Next, we define \( \bar{p} = p - \bar{p} \). We multiply (34) by \( A_{L,N}^{-1} \bar{p} \), where \( A_N \) is the operator associated with \( -\Delta \) with Neumann boundary conditions (see Section 2). Integrating over \( \Omega \), we find

\[
\frac{\varepsilon}{2} \frac{d}{dt} \| A_{L,N}^{-1} \bar{p} \|^2_{L^2(\Omega)} = -\int_{\Omega} A_{L,N}^{-1} \bar{p} \text{div} u \, dx = \int_{\Omega} \nabla A_{L,N}^{-1} \bar{p} \cdot u \, dx.
\]

(41)

We observe from (34) that

\[
\frac{\varepsilon}{\bar{c}} \frac{d}{dt} \bar{p} = -\int_{\Omega} \text{div} u \, dx = 0,
\]

(42)

which implies that \( \bar{p}(t) = \bar{p}_0 \) for all \( t \in [0,T] \). Thus, we have

\[
\frac{d}{dt} \left\{ \frac{\varepsilon}{2} \| \nabla A_{L,N}^{-1} \bar{p} \|^2_{L^2(\Omega)} + \frac{1}{2} \| \bar{p} \|^2 \right\} = \int_{\Omega} \nabla A_{L,N}^{-1} \bar{p} \cdot u \, dx.
\]

(43)

Collecting (39), (40), and (43) together, we find

\[
\frac{d}{dt} \left\{ \frac{1}{2} a(A_{L,N}^{-1} u, A_{L,N}^{-1} u) + \frac{\varepsilon}{2} \| \nabla A_{L,N}^{-1} \bar{p} \|^2_{L^2(\Omega)} + \frac{\varepsilon}{2} \| \bar{p} \|^2 \right\} + \frac{\nu}{2} \| u \|^2_{L^2(\Omega)}
\]

\[
+ (u_1 \cdot \nabla) u, A_{L,N}^{-1} u) + ((u \cdot \nabla) u_2, A_{L,N}^{-1} u) + \frac{1}{2} ((\text{div} u_1) u, A_{L,N}^{-1} u)
\]

\[
+ \frac{1}{2} ((\text{div} u) u_2, A_{L,N}^{-1} u) + ((\nu(\phi_1) - \nu(\phi_2)) D u_2, \nabla A_{L,N}^{-1} u)
\]

\[
\leq (-\nabla \phi_1 \otimes \nabla \phi - \nabla \phi \otimes \nabla \phi_2, \nabla A_{L,N}^{-1} u) + (\Psi(\phi_1) - \Psi(\phi_2)) \text{div}(A_{L,N}^{-1} u)
\]

\[
+ \frac{1}{2} ((\nabla \phi_1 \cdot \nabla \phi + \nabla \phi \cdot \nabla \phi_2) I, \nabla A_{L,N}^{-1} u) + (u, \nu(\phi_1) D A_{L,N}^{-1} u \nabla \phi_1)
\]

\[
+ (\bar{p}, \text{div}(A_{L,N}^{-1} u) + (\nabla A_{L,N}^{-1} \bar{p}, u).
\]

(44)

Taking \( r = 1 \) in (33), and using integration by parts, we see that

\[
\frac{d}{dt} \int_{\Omega} \phi \, dx + \int_{\Omega} u_1 \cdot \nabla \phi \, dx + \int_{\Omega} u \cdot \nabla \phi_2 \, dx = 0.
\]

Multiplying the above equation by \( \phi \), we find

\[
\frac{1}{2} \frac{d}{dt} \phi^2 = -\frac{1}{|\Omega|} \left( \int_{\Omega} u_1 \cdot \nabla \phi \, dx + \int_{\Omega} u \cdot \nabla \phi_2 \, dx \right) \phi.
\]

(45)
We now turn in finding a differential equality for a dual norm of \( \phi \). We define 
\[ \tilde{\phi} = \phi - \bar{\phi} \]
Taking \( v = A_N^{-1} \tilde{\phi} \) in (33), we have
\[
\frac{1}{2} \frac{d}{dt} \| \nabla A_N^{-1} \tilde{\phi} \|_{L^2(\Omega)}^2 + (u_1 \cdot \nabla \tilde{\phi}, A_N^{-1} \tilde{\phi}) + (u \cdot \nabla \phi, A_N^{-1} \tilde{\phi}) + (\nabla \mu, \nabla A_N^{-1} \tilde{\phi}) = 0.
\]
By using (34)_2, we observe that
\[
(\nabla \mu, \nabla A_N^{-1} \tilde{\phi}) = \int_{\Omega} (-\Delta \bar{\phi} + (\Psi'(\phi_1) - \Psi'(\phi_2)) \bar{\phi}) \, dx
\]
\[
= \int_{\Omega} |\nabla \phi|^2 \, dx + \int_{\Omega} (\phi_3^1 - \phi_3^2) \bar{\phi} \, dx - \int_{\Omega} \bar{\phi}^2 \, dx
\]
\[
= \int_{\Omega} |\nabla \phi|^2 \, dx + \int_{\Omega} (\phi_3^1 - \phi_3^2) \phi \, dx - \int_{\Omega} (\phi_1^3 - \phi_2^3) \bar{\phi} \, dx - \int_{\Omega} \bar{\phi}^2 \, dx.
\]
Combining the two equations above, we are lead to
\[
\frac{1}{2} \frac{d}{dt} \| \nabla A_N^{-1} \tilde{\phi} \|_{L^2(\Omega)}^2 + \int_{\Omega} |\nabla \phi|^2 \, dx + \int_{\Omega} (\phi_3^1 - \phi_3^2) \phi \, dx
\]
\[
= \int_{\Omega} -u_1 \cdot \nabla \bar{\phi} A_N^{-1} \bar{\phi} \, dx + \int_{\Omega} -u \cdot \nabla \phi_2 A_N^{-1} \bar{\phi} \, dx + \int_{\Omega} (\phi_3^1 - \phi_3^2) \bar{\phi} \, dx + \int_{\Omega} \bar{\phi}^2 \, dx.
\]
By adding (45) to (46), we obtain
\[
\frac{d}{dt} \left( \frac{1}{2} \| \nabla A_N^{-1} \tilde{\phi} \|_{L^2(\Omega)}^2 + \frac{1}{2} \phi_3^2 \right) + \int_{\Omega} |\nabla \phi|^2 \, dx + \int_{\Omega} (\phi_3^1 - \phi_3^2) \phi \, dx
\]
\[
= \int_{\Omega} -u_1 \cdot \nabla \bar{\phi} A_N^{-1} \bar{\phi} \, dx + \int_{\Omega} -u \cdot \nabla \phi_2 A_N^{-1} \bar{\phi} \, dx + \int_{\Omega} (\phi_3^1 - \phi_3^2) \bar{\phi} \, dx + \int_{\Omega} \bar{\phi}^2 \, dx
\]
\[
+ \int_{\Omega} \bar{\phi}^2 \, dx + \frac{1}{|\Omega|} \left( \int_{\Omega} -u_1 \cdot \nabla \phi \, dx \right) \bar{\phi} + \frac{1}{|\Omega|} \left( \int_{\Omega} -u \cdot \nabla \phi_2 \, dx \right) \bar{\phi}.
\]
By adding (44) and (47), we arrive at

\[
\frac{1}{2} \frac{d}{dt} \left\{ a(\mathbf{A}_{LN}^{-1} \mathbf{u}, \mathbf{A}_{LN}^{-1} \mathbf{u}) + \varepsilon \| \nabla \mathbf{A}_{N}^{-1} \mathbf{p} \|_{L^2(\Omega)}^2 + \varepsilon \| \mathbf{p} \|_{L^2(\Omega)}^2 + \| \nabla \mathbf{A}_{N}^{-1} \phi \|_{L^2(\Omega)}^2 + \phi^2 \right\} \\
+ \frac{\nu}{2} \| \mathbf{u} \|_{L^2(\Omega)}^2 + \| \nabla \phi \|_{L^2(\Omega)}^2 \\
\leq - \left( \int (\mathbf{u}_1 \cdot \nabla) \mathbf{u}, \mathbf{A}_{LN}^{-1} \mathbf{u} \right)_{l_1} - \left( \int (\mathbf{u} \cdot \nabla) \mathbf{u}_2, \mathbf{A}_{LN}^{-1} \mathbf{u} \right)_{l_2} - \frac{1}{2} \left( \int (\nabla \mathbf{u}_1) \mathbf{u}, \mathbf{A}_{LN}^{-1} \mathbf{u} \right)_{l_3} \\
- \frac{1}{2} \left( \int (\nabla \mathbf{u}_2) \mathbf{u}, \mathbf{A}_{LN}^{-1} \mathbf{u} \right)_{l_4} - \left( \int (\nu(\phi_1) - \nu(\phi_2)) \mathbf{D} \mathbf{u}_2, \nabla \mathbf{A}_{LN}^{-1} \mathbf{u} \right)_{l_5} \\
- \left( \int (p, \mathbf{D}) \mathbf{A}_{LN}^{-1} \mathbf{u} \right)_{l_6} - \left( \int (\nabla \phi_1 \otimes \nabla \phi + \nabla \phi \otimes \nabla \phi_2, \nabla \mathbf{A}_{LN}^{-1} \mathbf{u} \right)_{l_7} \\
+ \left( \int (\phi_1 - \phi_2) \mathbf{D} \mathbf{A}_{LN}^{-1} \mathbf{u} \right)_{l_8} + \frac{1}{2} \left( \int (\nabla \phi_1 \cdot \nabla \phi + \nabla \phi_2 \cdot \nabla \phi_2) \mathbf{I}, \nabla \mathbf{A}_{LN}^{-1} \mathbf{u} \right)_{l_9} \\
+ \left( \int (\mathbf{u}, \nu(\phi_1) \mathbf{D} \mathbf{A}_{LN}^{-1} \mathbf{u} \nabla \phi_1) + \frac{1}{2} \left( \int (\nabla \mathbf{A}_{LN}^{-1} \mathbf{u}) \phi \right)_{l_{10}} + \left( \int -\mathbf{u}_1 \cdot \nabla \mathbf{A}_{LN}^{-1} \mathbf{\bar{p}} \right)_{l_{11}} \right) \\
+ \left( \int -\mathbf{u} \cdot \nabla \mathbf{\phi}_2 \mathbf{A}_{LN}^{-1} \mathbf{\bar{p}} \right)_{l_{12}} + \left( \int (\phi_1^3 - \phi_2^3) \mathbf{\bar{p}} \right)_{l_{13}} + \left( \int \mathbf{\bar{p}}^2 \right)_{l_{14}} \\
+ \left( \int -\mathbf{u}_1 \cdot \nabla \phi \mathbf{dx} \right)_{l_{15}} + \left( \int -\mathbf{u} \cdot \nabla \phi_2 \mathbf{dx} \right)_{l_{16}} \right)_{l_{17}}.
\]

Here we have used that \( \int_{\Omega} (\phi_1^3 - \phi_2^3) \phi \mathbf{dx} \geq 0 \). We now proceed by estimating all the terms \( I_i \), for \( i = 1, \ldots, 17 \). Exploiting integration by parts, we observe that

\[
I_1 + I_3 = - \frac{1}{2} \int_{\Omega} \left( \int (\nabla \mathbf{u}_1) \mathbf{u} \cdot \mathbf{A}_{LN}^{-1} \mathbf{u} \mathbf{dx} - \int_{\Omega} (\mathbf{u}_1 \cdot \nabla) \mathbf{A}_{LN}^{-1} \mathbf{u} \cdot \mathbf{u} \mathbf{dx} \right)
\]

and

\[
I_2 + I_4 = \frac{1}{2} \int_{\Omega} \left( \int (\mathbf{u} \cdot \nabla) \mathbf{u}_2 \cdot \mathbf{A}_{LN}^{-1} \mathbf{u} \mathbf{dx} - \frac{1}{2} \int_{\Omega} (\mathbf{u} \cdot \nabla) \mathbf{A}_{LN}^{-1} \mathbf{u} \cdot \mathbf{u}_2 \mathbf{dx} \right).
\]

By using (8), (16), and (35), we obtain

\[
|J_1| \leq C \| \nabla \mathbf{u}_1 \|_{L^2(\Omega)} \| \mathbf{u} \|_{L^2(\Omega)} \| \mathbf{A}_{LN}^{-1} \mathbf{u} \|_{L^\infty(\Omega)} \\
\leq \nu \frac{C}{2} \| \mathbf{u} \|_{L^2(\Omega)}^2 + C \| \nabla \mathbf{u}_1 \|_{H^1(\Omega)} \| \mathbf{A}_{LN}^{-1} \mathbf{u} \|_{H^1(\Omega)} \ln \left( C \| \mathbf{A}_{LN}^{-1} \mathbf{u} \|_{H^1(\Omega)} \right) \leq \frac{\nu}{52} \| \mathbf{u} \|_{L^2(\Omega)}^2 + C \| \nabla \mathbf{u}_1 \|_{L^2(\Omega)} \| \nabla \mathbf{A}_{LN}^{-1} \mathbf{u} \|_{L^2(\Omega)} \ln \left( C \| \nabla \mathbf{A}_{LN}^{-1} \mathbf{u} \|_{L^2(\Omega)} \right)
\]
We obtain

\[ |J_2| \leq \|u\|_{L^4(\Omega)} \|\nabla A_{L_N}^{-1} u\|_{L^4(\Omega)} \|u\|_{L^2(\Omega)} \]

\[ \leq C\|u\|_{L^4(\Omega)} \|\nabla A_{L_N}^{-1} u\|_{L^2(\Omega)}^2 \]

\[ \leq \frac{\nu_1}{52} \|u\|_{L^2(\Omega)}^2 + C\|u\|_{L^4(\Omega)}^4. \quad (52) \]

Similarly, we obtain

\[ |J_3| \leq C\|u\|_{L^2(\Omega)} \|\nabla A_{L_N}^{-1} u\|_{L^4(\Omega)} \|u\|_{L^2(\Omega)} \]

\[ \leq \frac{\nu_2}{52} \|u\|_{L^2(\Omega)}^2 + C\|u\|_{L^4(\Omega)}^4. \quad (53) \]

and

\[ |J_4| \leq C\|u\|_{L^4(\Omega)} \|\nabla A_{L_N}^{-1} u\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \]

\[ \leq \frac{\nu_3}{52} \|u\|_{L^2(\Omega)}^2 + C\|u\|_{L^4(\Omega)}^4. \quad (54) \]

Since \( \nu'(s) \) is bounded, by using (9), (16), and (35), we deduce that

\[ |J_5| = \left| \int_{\Omega} -\left( \int_0^1 \nu'(\tau \phi_1 + (1 - \tau) \phi_2) d\tau \right) \phi D u_2 : \nabla A_{L_N}^{-1} u \, dx \right| \]

\[ \leq C\|\phi\|_{L^2(\Omega)} \|D u_2\|_{L^2(\Omega)} + C\|\phi\|_{L^4(\Omega)} \|\nabla A_{L_N}^{-1} u\|_{L^2(\Omega)} \]

\[ \leq C\|D u_2\|_{L^2(\Omega)} \|\phi\|_{L^4(\Omega)} \|\nabla A_{L_N}^{-1} u\|_{L^2(\Omega)} \]

\[ + C\|\phi\|_{L^2(\Omega)} \|\nabla A_{L_N}^{-1} u\|_{L^2(\Omega)} \]

\[ \leq \frac{1}{16} \|\nabla \phi\|_{L^2(\Omega)}^2 + C\|D u_2\|_{L^2(\Omega)} \|\nabla A_{L_N}^{-1} u\|_{L^2(\Omega)} \]

\[ \leq \frac{1}{16} \|\nabla \phi\|_{L^2(\Omega)}^2 + C\|\nabla A_{L_N}^{-1} u\|_{L^2(\Omega)}^2. \quad (55) \]

By duality, and Lemma 2.1, we have

\[ |I_6| = \|\langle \bar{\rho}, \text{div} A_{L_N}^{-1} u \rangle \| \]

\[ \leq \|\bar{\rho}\|_{H^2(\Omega))} \|\text{div} A_{L_N}^{-1} u\|_{H^2(\Omega)} \]

\[ \leq C\|\nabla A_{L_N}^{-1} \bar{\rho}\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \]

\[ \leq \frac{\nu_4}{52} \|u\|_{L^2(\Omega)}^2 + C\|\nabla A_{L_N}^{-1} \bar{\rho}\|_{L^2(\Omega)}^2. \quad (56) \]

Also, we find

\[ |I_7| \leq (\|\nabla \phi_1\|_{L^2(\Omega)} + \|\nabla \phi_2\|_{L^2(\Omega)}) \|\nabla \phi\|_{L^2(\Omega)} \|\nabla A_{L_N}^{-1} u\|_{L^2(\Omega)} \]

\[ \leq \frac{1}{16} \|\nabla \phi\|_{L^2(\Omega)}^2 + C(\|\nabla \phi_1\|_{L^2(\Omega)}^2 + \|\nabla \phi_2\|_{L^2(\Omega)}^2) \|\nabla A_{L_N}^{-1} u\|_{L^2(\Omega)}^2. \quad (57) \]

By Sobolev embedding and (37), we obtain

\[ |I_8| \leq \left| \int_{\Omega} \left( \int_0^1 \Psi'(\tau \phi_1 + (1 - \tau) \phi_2) d\tau \right) \phi I : \nabla A_{L_N}^{-1} u \, dx \right| \]

\[ \leq C(\|\Psi'(\phi_1)\|_{L^3(\Omega)} + \|\Psi'(\phi_2)\|_{L^3(\Omega)}) \|\phi\|_{L^6(\Omega)} \|\nabla A_{L_N}^{-1} u\|_{L^2(\Omega)} \]

\[ \leq \frac{1}{16} \|\nabla \phi\|_{L^2(\Omega)}^2 + C(\|\bar{\rho}\|^2 + \|\nabla A_{L_N}^{-1} u\|_{L^2(\Omega)}^2). \quad (58) \]
Similarly to (61), we deduce that
\[ |I_9| \leq \left( \| \nabla \phi_1 \|_{L^\infty(\Omega)} + \| \nabla \phi_2 \|_{L^\infty(\Omega)} \right) \| \nabla \phi \|_{L^2(\Omega)} \| \nabla A_{L_N}^{-1} u \|_{L^2(\Omega)} \]
\[ \leq \frac{1}{16} \| \nabla \phi \|_{L^2(\Omega)}^2 + C \left( \| \nabla \phi_1 \|_{L^\infty(\Omega)}^2 + \| \nabla \phi_2 \|_{L^\infty(\Omega)}^2 \right) \| \nabla A_{L_N}^{-1} u \|_{L^2(\Omega)}^2. \]

(59)

Since \( \nu' \) is bounded, we find
\[ |I_{10}| \leq C \| u \|_{L^2(\Omega)} \| \nabla A_{L_N}^{-1} u \|_{L^2(\Omega)} \| \nabla \phi_1 \|_{L^\infty(\Omega)} \]
\[ \leq \frac{1}{16} \| \nabla \phi \|_{L^2(\Omega)}^2 + C \| \nabla \phi_1 \|_{L^\infty(\Omega)} \| \nabla A_{L_N}^{-1} u \|_{L^2(\Omega)}^2. \]

(60)

Also, we have
\[ |I_{11}| \leq \| \nabla A_{N}^{-1} \tilde{\phi} \|_{L^2(\Omega)} \| u \|_{L^2(\Omega)} \]
\[ \leq \frac{\nu_s}{52} \| \tilde{u} \|_{L^2(\Omega)}^2 + C \| \nabla A_{N}^{-1} \tilde{\phi} \|_{L^2(\Omega)}^2. \]

(61)

By using the Sobolev embedding, we infer that
\[ |I_{12}| \leq \| u_1 \|_{L^2(\Omega)} \| \nabla \phi \|_{L^2(\Omega)} \| A_{N}^{-1} \tilde{\phi} \|_{L^6(\Omega)} \]
\[ \leq \frac{1}{16} \| \nabla \phi \|_{L^2(\Omega)}^2 + C \| u_1 \|_{L^2(\Omega)} \| \nabla A_{N}^{-1} \tilde{\phi} \|_{L^2(\Omega)}^2. \]

(62)

and
\[ |I_{13}| \leq \| u \|_{L^2(\Omega)} \| \nabla \phi_2 \|_{L^2(\Omega)} \| A_{N}^{-1} \tilde{\phi} \|_{L^6(\Omega)} \]
\[ \leq \frac{\nu_s}{52} \| u \|_{L^2(\Omega)}^2 + C \| \nabla \phi_2 \|_{L^2(\Omega)} \| \nabla A_{N}^{-1} \tilde{\phi} \|_{L^2(\Omega)}^2. \]

(63)

By using the Poincaré inequality and (37), we obtain
\[ |I_{14}| \leq \int_\Omega \int_0^1 3(\tau \phi_1 + (1 - \tau) \phi_2)^2 d\tau \| \phi \|_{L^\infty(\Omega)} \]
\[ \leq \frac{1}{16} \| \nabla \phi \|_{L^2(\Omega)}^2 + C \| \tilde{\phi} \|_{L^2(\Omega)}^2. \]

(64)

By interpolation, we have
\[ |I_{15}| \leq C \| \nabla A_{N}^{-1} \tilde{\phi} \|_{L^2(\Omega)} \| \nabla \phi \|_{L^2(\Omega)} \]
\[ \leq \frac{1}{16} \| \nabla \phi \|_{L^2(\Omega)}^2 + C \| \nabla A_{N}^{-1} \tilde{\phi} \|_{L^2(\Omega)}^2. \]

(65)

By using the Cauchy-Schwarz inequality, we find
\[ |I_{16}| \leq C \| u_1 \|_{L^2(\Omega)} \| \nabla \phi \|_{L^2(\Omega)} \]
\[ \leq \frac{1}{16} \| \nabla \phi \|_{L^2(\Omega)}^2 + C \| u_1 \|_{L^2(\Omega)} \| \tilde{\phi} \|_{L^2(\Omega)}^2. \]

(66)

and
\[ |I_{17}| \leq C \| u \|_{L^2(\Omega)} \| \nabla \phi_2 \|_{L^2(\Omega)} \]
\[ \leq \frac{\nu_s}{52} \| u \|_{L^2(\Omega)}^2 + C \| \nabla \phi_2 \|_{L^2(\Omega)} \| \tilde{\phi} \|_{L^2(\Omega)}^2. \]

(67)

Collecting together (48) and (51)-(67), we eventually arrive at the differential inequality
\[ \frac{d}{dt} H + \frac{\nu_s}{4} \int_\Omega |\tilde{u}|^2 \] 
\[ + \frac{1}{2} \int_\Omega |\nabla \phi|^2 \]
\[ \leq GH \ln \left( \frac{C}{H} \right), \]

(68)
where 
\[ H = \frac{1}{2} a(A_{LN}^{-1} u, A_{LN}^{-1} u) + \frac{\varepsilon}{2} \| \nabla A_N^{-1} \phi \|_{L^2(\Omega)}^2 + \frac{\varepsilon}{2} |p|^2 + \frac{1}{2} \| \nabla A_N^{-1} \phi \|_{L^2(\Omega)}^2 + \frac{1}{2} \bar{\phi}^2, \]
and
\[ G = C(1 + \| \text{div} u_1 \|_{L^2(\Omega)}^2 + \| u_1 \|_{L^4(\Omega)}^4 + \| \nabla u_2 \|_{L^2(\Omega)}^2 + \| u_2 \|_{L^2(\Omega)}^4 + \| \nabla \phi_1 \|_{L^2(\Omega)}^2 + \| \nabla \phi_2 \|_{L^2(\Omega)}^2). \]

We notice that we used once more in (68) that \( s \mapsto s \ln \left( \frac{C}{s} \right) \) is increasing. In light of the regularity results shown in (35) and (37), it is easily seen from (4) and (6) that
\[ G \leq C(1 + \| u_1 \|_{H^1(\Omega)}^2 + \| u_2 \|_{H^1(\Omega)}^2 + \| \phi_1 \|_{H^2(\Omega)}^2 + \| \phi_2 \|_{H^2(\Omega)}^2) := \tilde{G} \in L^1(0, T). \]

To conclude the proof, we are only left to apply the Osgood Lemma 2.2. We define
\[ W(s) = s \ln \left( \frac{C}{s} \right) \quad \text{and} \quad M(s) = \ln(\ln \left( \frac{C}{s} \right)). \]
Since \[ \int_0^1 \frac{1}{W(s)} \, ds = \infty, \]
we infer that \( H(t) \equiv 0, \) for all \( t > 0, \) which implies the uniqueness of weak solutions. In addition, we deduce that
\[ -\ln \left( \frac{C}{H(t)} \right) + \ln \left( \frac{C}{H(0)} \right) \leq \int_0^t \tilde{G}(s) \, ds. \]

Defining \( T_0 > 0 \) such that
\[ \ln \left( \frac{C}{H(0)} \right) \geq \int_0^t \tilde{G}(s) \, ds, \quad \forall \, t \in [0, T_0], \]
we are lead to the following estimate
\[ H(t) \leq C \left( \frac{H(0)}{C} \right) e^{\int_0^t \tilde{G}(s) \, ds}, \quad \forall \, t \in [0, T_0], \quad (69) \]
showing a continuous dependence of the solution on the initial data.

5. **Proof of Theorem 1.2: Analysis of semi-strong solutions.**

5.1. **Existence in 3D bounded domains.** As before we proceed by proving formal a priori estimates. This argument can be fully justified in the Galerkin scheme provided in Section 3. The details are left to the reader.

We consider the convective Cahn-Hilliard equation
\[ \partial_t \phi + u \cdot \nabla \phi = \Delta (\Delta \phi + \Psi'(\phi)) \quad \text{in} \quad \Omega \times (0, T), \quad (70) \]
which is equipped with boundary conditions \( \partial_n \phi = \partial_n u = 0 \) on \( \partial \Omega \times (0, T). \) Recalling the equation for the total mass
\[ \frac{d}{dt} \bar{\phi} = - \frac{1}{|\Omega|} \int_{\Omega} u \cdot \nabla \phi \, dx, \]
and using the boundary conditions, we rewrite (70) in functional form
\[ \partial_t (\phi - \bar{\phi}) + u \cdot \nabla \phi - u \cdot \nabla \bar{\phi} = - A_N (A_N (\phi - \bar{\phi}) + \Psi'(\phi) - \Psi'(\bar{\phi})), \quad (71) \]
where \( A_N \) is defined in Section 2. Let us set \( \tilde{\phi} = \phi - \bar{\phi}. \) We rewrite (71) as follows
\[ \partial_t \tilde{\phi} + A_N^2 \tilde{\phi} + u \cdot \nabla \phi - u \cdot \nabla \bar{\phi} = \Delta (\Psi'(\phi) - \Psi'(\bar{\phi})). \quad (72) \]
Thanks to Theorem 1.1, we assume that \( (u, \phi) \) fulfills the regularity
\[ \phi \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^3(\Omega)), \quad u \in L^\infty(0, T; H) \cap L^2(0, T; V). \quad (73) \]
For any \( s \in (0, 1] \), we multiply (72) by \( A_N^{1+s} \phi \) and integrate over \( \Omega \). We find
\[
\frac{1}{2} \frac{d}{ds} \| A_N^{1+s} \phi \|_{L^2(\Omega)}^2 + \| A_N^{1+s} \phi \|_{L^2(\Omega)}^2 = -(u \cdot \nabla \phi, A_N^{1+s} \phi) - (\Delta (\phi^3 - \phi), A_N^{1+s} \phi).
\]
Since \( s \in (0, 1] \) we observe that \( 1 + s \leq \frac{2 + s}{2} \). Then, by exploiting the regularity of \( u \) in (73), we estimate the first term on the right-hand side as follows
\[
\left| -(u \cdot \nabla \phi, A_N^{1+s} \phi) \right| \leq \| u \|_{L^2(\Omega)} \| \nabla \phi \|_{L^\infty(\Omega)} \| A_N^{1+s} \phi \|_{L^2(\Omega)} 
\leq C \| \nabla \phi \|_{L^\infty(\Omega)} \| A_N^{1+s} \phi \|_{L^2(\Omega)} 
\leq \frac{1}{4} \| A_N^{1+s} \phi \|_{L^2(\Omega)}^2 + C \| \nabla \phi \|_{L^\infty(\Omega)}^2.
\]
Next, for the second term on the right-hand side we have
\[
(\Delta (\phi^3 - \phi), A_N^{1+s} \phi)
\leq \| \Delta (\phi^3 - \phi) \|_{L^2(\Omega)} \| A_N^{1+s} \phi \|_{L^2(\Omega)}
\leq \| 6\phi \nabla \phi \|^2 + 3\phi^3 \Delta \phi - \Delta \phi \|_{L^2(\Omega)} \| A_N^{1+s} \phi \|_{L^2(\Omega)}
\leq C (\| \phi \|_{L^\infty(\Omega)} \| \nabla \phi \|^2_{L^2(\Omega)} + \| \phi \|^2_{L^\infty(\Omega)} \| \Delta \phi \|_{L^2(\Omega)} + \| \Delta \phi \|_{L^2(\Omega)} \| A_N^{1+s} \phi \|_{L^2(\Omega)}
\leq \frac{1}{2} \| A_N^{1+s} \phi \|_{L^2(\Omega)}^2 + C (\| \phi \|^2_{L^\infty(\Omega)} \| \nabla \phi \|^2_{L^4(\Omega)} + \| \phi \|^2_{L^\infty(\Omega)} \| \Delta \phi \|_{L^2(\Omega)} + \| \Delta \phi \|_{L^2(\Omega)}).
\]
Collecting the above estimates together, we arrive at
\[
\frac{d}{dt} \| A_N^{1+s} \phi \|_{L^2(\Omega)}^2 + \| A_N^{1+s} \phi \|_{L^2(\Omega)}^2 \leq K,
\]
where
\[
K = C \| \nabla \phi \|^2_{L^\infty(\Omega)} + C \| \phi \|^2_{L^\infty(\Omega)} \| \nabla \phi \|^2_{L^4(\Omega)} + C \| \phi \|^2_{L^\infty(\Omega)} \| \Delta \phi \|_{L^2(\Omega)} + C \| \Delta \phi \|_{L^2(\Omega)}.
\]
We are only left to show that \( K \in L^1(0, T) \). By using (73), the embedding \( H^3(\Omega) \hookrightarrow W^{1,\infty}(\Omega) \), and the Agmon and Ladyzhenskaya’s inequalities in three dimensions, we have \( K \leq C \| \phi \|^2_{L^2(\Omega)} \). Since \( \phi \in L^2(0, T; H^3(\Omega)) \), it easily follows that \( K \in L^1(0, T) \). Therefore, integrating in time, we deduce that
\[
\sup_{t \in [0, T]} \| A_N^{1+s} \phi(t) \|_{L^2(\Omega)}^2 + \int_0^T \| A_N^{1+s} \phi(t) \|_{L^2(\Omega)}^2 \, dt 
\leq \| A_N^{1+s} (\phi_0 - \bar{\phi}_0) \|_{L^2(\Omega)}^2 + \int_0^T K(t) \, dt.
\]
We infer that, for any \( s \in (0, 1] \),
\[
\phi - \bar{\phi} \in L^\infty(0, T; D(A^{1+s}_N)) \cap L^2(0, T; D(A^{1+s}_N)).
\]
Thus, by definition of \( A_N \) and the boundary condition \( \partial_\nu \Delta \phi = 0 \) on \( \partial \Omega \times (0, T) \), we have \( \Delta \phi \in L^\infty(0, T; D(A^{1+s}_N)) \cap L^2(0, T; D(A^{1+s}_N)) \). Besides, since \( \phi \in L^\infty(0, T; L^6(\Omega)) \), we deduce that \( \Psi'(\phi) \in L^\infty(0, T; L^2(\Omega)) \), which implies that \( \Psi'(\phi) - \Psi'(\bar{\phi}) \in L^\infty(0, T; D(A^{1+s}_N)) \). Also, by using the regularity \( \phi \in L^\infty(0, T; H^1(\Omega)) \) and the inequalities (5) and (7), we observe that
\[
\| \Psi'(\phi) - \Psi'(\bar{\phi}) \|_{D(A^{1+s}_N)} \leq C \| \Psi'(\phi) - \Psi'(\bar{\phi}) \|_{D(A_N)}
\]
\[ \leq C\|\Delta(\phi^3 - \phi)\|_{L^2(\Omega)} \]
\[ \leq C\|\phi\nabla\phi\|^2_{L^2(\Omega)} + C\|\phi^2\Delta\phi\|_{L^2(\Omega)} + C\|\Delta\phi\|_{L^2(\Omega)} \]
\[ \leq C\|\phi\|_{L^\infty(\Omega)}\|\nabla\phi\|^2_{L^2(\Omega)} + C(1 + \|\phi\|^2_{L^\infty(\Omega)})\|\Delta\phi\|_{L^2(\Omega)} \]
\[ \leq C(1 + \|\phi\|^2_{H^2(\Omega)}) \]
\[ \leq C(1 + \|\phi\|_{H^4(\Omega)}). \]

As a consequence, we deduce that

\[ \mu - \bar{\mu} \in L^\infty(0, T; D((A_N^{1+s})^2)) \cap L^2(0, T; D((A_N^{1+s})^4)), \]

which, in turn, entails that

\[ \partial_t(\phi - \bar{\phi}) \in L^2(0, T; D((A_N^{1+s})^4)). \]

5.2. Uniqueness in 2D bounded domains. The uniqueness of solutions constructed in the previous section in two dimensions is a consequence of Section 4.

We show here below a simpler proof which does not rely on estimates in dual spaces. Let us consider two solutions \((u_1, p_1, \phi_1)\) and \((u_2, p_2, \phi_2)\) originating from the same initial condition \((u_0, p_0, \phi_0)\). We define \(u = u_1 - u_2, \quad p = p_1 - p_2, \quad \phi = \phi_1 - \phi_2\). We have the system

\[ \langle \partial_t u, v \rangle + ((u_1 \cdot \nabla) u, v) + ((u \cdot \nabla) u_2, v) + \frac{1}{2}((\text{div } u_1) u, v) \]
\[ + \frac{1}{2}((\text{div } u) u_2, v) + (\nu(\phi_1)Du_1, \nabla v) + ((\nu(\phi_1) - \nu(\phi_2))Du_2, \nabla v) \]  
\[ - (p, \text{div } v) = (\mu_1 \nabla \phi, v) + (\mu \nabla \phi_2, v), \] (75)

and

\[ \langle \partial_t \phi, v \rangle + (u_1 \cdot \nabla \phi, v) + (u \cdot \nabla \phi_2, v) - (\nabla \mu, \nabla v) = 0, \] (76)

for any \(v \in V\) and \(v \in D((A_N^{1+s})^2)\), where

\[ \varepsilon p_t + \text{div } u = 0, \] (77)

\[ \mu = -\Delta \phi + \Psi'(\phi_1) - \Psi'(\phi_2), \] (78)

and

\[ \frac{d}{dt} \bar{\phi} = -\frac{1}{u_1 \cdot \nabla \phi} - \frac{u \cdot \nabla \phi_2}{u_2}. \] (79)

We take \(v = u\) in (75), and we multiply (77) by \(p\) and integrate it over \(\Omega\). Summing up the two equations, we find

\[ \frac{d}{dt}\left\{ \frac{1}{2}\|u\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\|p\|^2_{L^2(\Omega)} \right\} + ((u_1 \cdot \nabla) u, u) + ((u \cdot \nabla) u_2, u) \]
\[ + \frac{1}{2}((\text{div } u_1) u, u) + \frac{1}{2}((\text{div } u) u_2, u) + (\nu(\phi_1)Du_1, Du) \]
\[ + ((\nu(\phi_1) - \nu(\phi_2))Du_2, \nabla u) = (\mu_1 \nabla \phi, u) + (\mu \nabla \phi_2, u). \] (80)

Since \(A_N^{1+s}(\phi - \bar{\phi}) \in L^2(0, T; D((A_N^{1+s})^2))\), we can take \(v = A_N^{1+s}(\phi - \bar{\phi})\) in (76). We obtain

\[ \frac{1}{2}\frac{d}{dt}\|A_N^{1+s}(\phi - \bar{\phi})\|^2_{L^2(\Omega)} + \|A_N^{1+s}(\phi - \bar{\phi})\|^2_{L^2(\Omega)} \]
\[ + ((u_1 \cdot \nabla \phi, A_N^{1+s}(\phi - \bar{\phi})) + (u \cdot \nabla \phi_2, A_N^{1+s}(\phi - \bar{\phi})) \]
\[ + (A_N(\phi_1^3 - \phi_2^3 - \phi), A_N^{1+s}(\phi - \bar{\phi})) = 0. \]
By multiplying the equation for the total mass (79) by $\bar{\phi}$ and adding to the above equation, we find
\begin{equation}
\begin{aligned}
\frac{d}{dt} \left\{ \frac{1}{2} |A_N^{1+s}(\phi - \bar{\phi})|_{L^2(\Omega)}^2 + \frac{1}{2} \|\bar{\phi}\|^2 \right\} + \|A_N^{1+s}(\phi - \bar{\phi})\|^2_{L^2(\Omega)} \\
+ (u_1 \cdot \nabla \phi, A_N^{1+s}(\phi - \bar{\phi})) + (u \cdot \nabla \phi_2, A_N^{1+s}(\phi - \bar{\phi})) \\
+ (A_N(\phi_1^2 - \phi_2^2 - \phi), A_N^{1+s}(\phi - \bar{\phi})) = -(u_1 \cdot \nabla \phi + u \cdot \nabla \phi_2) \bar{\phi}.
\end{aligned}
\end{equation}

We recall that $\nu(r) \geq \nu_\ast > 0$ for all $r \in \mathbb{R}$. By adding (80) and (81), we arrive at
\begin{equation}
\begin{aligned}
\frac{d}{dt} \left\{ \frac{1}{2} \|u\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2} \|p\|^2_{L^2(\Omega)} + \frac{1}{2} \|A_N^{1+s}(\phi - \bar{\phi})\|^2_{L^2(\Omega)} + \frac{1}{2} \|\bar{\phi}\|^2 \right\} \\
+ \nu \|Du\|^2_{L^2(\Omega)} + \|A_N^{1+s}(\phi - \bar{\phi})\|^2_{L^2(\Omega)} \\
\leq - \left( (u_1 \cdot \nabla) u, u \right) - \left( (u \cdot \nabla) u_2, u \right) - \frac{1}{2} \left( (\text{div } u_1) u, u \right) - \frac{1}{2} \left( (\text{div } u_2) u_2, u \right) \\
- \left( (\nu(\phi_1) - \nu(\phi_2)) Du_2, \nabla u \right) + (\mu_1 \nabla \phi_1, u) + (\mu \nabla \phi_2, u) \\
- \left( u_1 \cdot \nabla \phi, A_N^{1+s}(\phi - \bar{\phi}) \right) - \left( u \cdot \nabla \phi_2, A_N^{1+s}(\phi - \bar{\phi}) \right) \\
- \left( A_N(\phi_1^2 - \phi_2^2 - \phi), A_N^{1+s}(\phi - \bar{\phi}) \right) - \left( u_1 \cdot \nabla \phi + u \cdot \nabla \phi_2 \right) \bar{\phi}.
\end{aligned}
\end{equation}

We observe that $R_1 + R_3 = 0$. Arguing as in Section 4, we find
\begin{equation}
|R_2 + R_4| \leq \frac{\nu_\ast}{4} \|Du\|^2_{L^2(\Omega)} + C \|u_2\|_{H^1(\Omega)}^2 \|u\|^2_{L^2(\Omega)}.
\end{equation}

Since $H^{1+s}(\Omega) \hookrightarrow L^\infty(\Omega)$ for $s > 0$, we obtain
\begin{equation}
|R_5| \leq C \|\phi\|_{L^\infty(\Omega)} \|Du_2\|_{L^2(\Omega)} \|Du\|_{L^2(\Omega)} \\
\leq C \|\phi\|_{H^{1+s}(\Omega)} \|Du_2\|_{L^2(\Omega)} \|Du\|_{L^2(\Omega)} \\
\leq \frac{\nu_\ast}{4} \|Du_2\|^2_{L^2(\Omega)} + C \|Du_2\|_{L^2(\Omega)} \|\phi\|^2_{H^{1+s}(\Omega)}.
\end{equation}

By using (38), the embeddings $H^q(\Omega) \hookrightarrow L^{\frac{2q}{q+1}}(\Omega)$, for $q \in (0, 1)$, and $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for any $p \in [1, \infty)$, we deduce that
\begin{equation}
|R_6 + R_7| \leq C \left( \|\nabla \phi_1\|_{L^\frac{2}{1+s}(\Omega)} + \|\nabla \phi_2\|_{L^\frac{2}{1+s}(\Omega)} \right) \|\nabla \phi\|_{L^{\frac{2q}{q+1}}(\Omega)} \|\nabla u\|_{L^2(\Omega)} \\
+ C \left( \|\nabla \phi_1\|_{L^2(\Omega)} + \|\nabla \phi_2\|_{L^2(\Omega)} \right) \|\phi\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)} \\
\leq C \left( 1 + \|\phi_1\|_{H^2(\Omega)} + \|\phi_2\|_{H^2(\Omega)} \right) \|\phi\|_{H^{1+s}(\Omega)} \|Du\|_{L^2(\Omega)} \\
\leq \frac{\nu_\ast}{4} \|Du\|^2_{L^2(\Omega)} + C \left( \|\phi_1\|_{H^2(\Omega)} + \|\phi_2\|_{H^2(\Omega)} \right) \|\phi\|^2_{H^{1+s}(\Omega)}.
\end{equation}

Taking $q = s$ for $s \in (0, 1)$ and $q < s$ for $s = 1$, we find
\begin{equation}
|R_6 + R_7| \leq \frac{\nu_\ast}{4} \|Du\|^2_{L^2(\Omega)} + C \left( \|\phi_1\|_{H^2(\Omega)} + \|\phi_2\|_{H^2(\Omega)} \right) \|\phi\|^2_{H^{1+s}(\Omega)}.
\end{equation}

Similarly, we have
\begin{equation}
|R_8| \leq \|u_1\|_{L^\frac{2}{1+s}(\Omega)} \|\nabla \phi\|_{L^{\frac{2q}{q+1}}(\Omega)} \|A_N^{1+s}(\phi - \bar{\phi})\|_{L^2(\Omega)} \\
\leq C \|u_1\|_{H^1(\Omega)} \|\nabla \phi\|_{H^s(\Omega)} \|A_N^{1+s}(\phi - \bar{\phi})\|_{L^2(\Omega)}
\end{equation}
Choosing $q$ as above, we obtain
\[ |R_8| \leq \frac{1}{16} \| A_N^{\frac{s}{2}} (\phi - \bar{\phi}) \|_{L^2(\Omega)}^2 + C \| u_1 \|_{H^1(\Omega)}^2 \| \phi \|_{H^{1+s}(\Omega)}^2. \]
Also, since $1 + s \leq \frac{3+s}{2}$ for $s \in (0,1]$, we find
\[
|R_6| \leq \| u \|_{L^2(\Omega)} \| \nabla \phi_2 \|_{L^\infty(\Omega)} \| A_N^{\frac{1+s}{2}} (\phi - \bar{\phi}) \|_{L^2(\Omega)}^2 \\
\leq C \| u \|_{L^2(\Omega)} \| \nabla \phi_2 \|_{L^\infty(\Omega)} \| A_N^{\frac{s}{2}} (\phi - \bar{\phi}) \|_{L^2(\Omega)}^2 \\
\leq \frac{1}{16} \| A_N^{\frac{s}{2}} (\phi - \bar{\phi}) \|_{L^2(\Omega)}^2 + C \| \nabla \phi_2 \|_{L^\infty(\Omega)}^2 \| u \|_{L^2(\Omega)}^2.
\]
We recall that $H^{1+s} \hookrightarrow L^\infty(\Omega)$ for any $s > 0$, $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for any $p \in [1, \infty)$, and $H^s(\Omega) \hookrightarrow L^{\frac{2}{s+2}}(\Omega)$ for $s \in (0,1)$. By using the regularity (74) and interpolation in the spaces $D(A_N^s)$, we infer, that, for $s \in (0,1)$,
\[
|R_{10}| \leq C \left( \| \phi_1 \|_{L^2(\Omega)}^2 + \| \phi_2 \|_{L^2(\Omega)}^2 \right) + \| \phi_1 \|_{H^s(\Omega)} \||\Delta \phi_1 - \Delta \phi_2\|_{L^2(\Omega)} + \| A_N^{\frac{s+1}{2}} (\phi - \bar{\phi}) \|_{L^2(\Omega)} \\
\leq C \left( \| \phi \|_{L^\infty(\Omega)} \| \nabla \phi_1 \|_{L^2(\Omega)} + \| \phi_2 \|_{L^\infty(\Omega)} \| \nabla \phi_1 \|_{L^2(\Omega)} \right) \| A_N^{\frac{s+1}{2}} (\phi - \bar{\phi}) \|_{L^2(\Omega)} \\
\leq C \left( \| \phi \|_{L^\infty(\Omega)} \| \phi_1 \|_{H^2(\Omega)} + \| \phi_2 \|_{H^2(\Omega)} \right) \| A_N^{\frac{s+1}{2}} (\phi - \bar{\phi}) \|_{L^2(\Omega)} \\
\leq C \left( 1 + \| \phi_1 \|_{H^2(\Omega)} + \| \phi_2 \|_{H^2(\Omega)} \right) \| A_N^{\frac{s+1}{2}} (\phi - \bar{\phi}) \|_{L^2(\Omega)} \\
\leq \frac{1}{16} \| A_N^{\frac{3+s}{2}} (\phi - \bar{\phi}) \|_{L^2(\Omega)}^2 + \left( 1 + \| \phi_1 \|_{H^2(\Omega)} + \| \phi_2 \|_{H^2(\Omega)} \right) \| \phi \|_{H^{1+s}(\Omega)}^2.
\]
The case $s = 1$ can be done with minor changes, and it is left to the reader. Finally, we have
\[
|R_{11}| \leq C (\| u \|_{L^2(\Omega)} \| \nabla \phi \|_{L^2(\Omega)} + \| \phi \|_{L^2(\Omega)} \| \nabla \phi_2 \|_{L^2(\Omega)} \| \bar{\phi} \|_{L^2(\Omega)}) \\
\leq C \| \phi \|_{L^2(\Omega)}^2 + C \| \bar{\phi} \|_{L^2(\Omega)}^2 + C \| u \|_{L^2(\Omega)}^2.
\]
Therefore, we deduce the following differential inequality
\[
\frac{d}{dt} H_1 + \frac{\nu}{4} \| Du \|_{L^2(\Omega)}^2 + \frac{1}{2} \| A_N^{\frac{s+1}{2}} (\phi - \bar{\phi}) \|_{L^2(\Omega)}^2 \leq G_1 H_1,
\]
where
\[
H_1 = \frac{1}{2} \| u \|_{L^2(\Omega)}^2 + \frac{\varepsilon}{2} \| p \|_{L^2(\Omega)}^2 + \frac{1}{2} \| A_N^{\frac{s+1}{2}} (\phi - \bar{\phi}) \|_{L^2(\Omega)}^2 + \frac{1}{2} \| \bar{\phi} \|_{L^2(\Omega)}^2,
\]
and
\[
G_1 = C \left( 1 + \| u_1 \|_{H^1(\Omega)} + \| u_2 \|_{H^1(\Omega)} + \| \phi_1 \|_{H^2(\Omega)} + \| \phi_2 \|_{H^2(\Omega)} + \| \nabla \phi_2 \|_{L^\infty(\Omega)} \right).
\]
Since $u_i \in L^2(0,T;V)$, $\phi_i \in L^2(0,T;H^2(\Omega) \cap W^{1,\infty}(\Omega))$, for $i = 1,2$, $G_1 \in L^1(0,T)$ and the uniqueness follows from the Gronwall lemma.
6. Proof of Theorem 1.3: Well-posedness of strong solutions. In this section we provide formal a priori estimates which are sufficient to infer the regularity of the strong solutions as stated in Theorem 1.3. The rigorous proof relies on repeating the same argument in the Galerkin scheme and passing to the limit as in Section 3. This part is left to the reader.

6.1. Global strong solutions in 2D bounded domains. For any \( s \in (0, 1] \), let us assume that we have a weak solution \((u, p, \phi)\) satisfying
\[
\begin{align*}
&u \in L^\infty(0, T; H) \cap L^2(0, T; V), \\
p \in L^\infty(0, T; L^2(\Omega)) \cap W^{1,2}(0, T; L^2(\Omega)), \\
&\phi - \overline{\phi} \in L^\infty(0, T; D(A^{\frac{1+s}2}N)) \cap L^2(0, T; D(A^{\frac{3+s}2}N)) \cap W^{1,2}(0, T; D(A^{\frac{s+1}2}N)), \\
&\mu - \overline{\mu} \in L^\infty(0, T; D(A^{\frac{1+s}2}N)) \cap L^2(0, T; D(A^{\frac{3+s}2}N)).
\end{align*}
\]

We multiply the Navier-Stokes equations by \(-\div(Du)\) and we integrate over \(\Omega\). We find
\[
\frac{1}{2} \frac{d}{dt} \|Du\|_{L^2(\Omega)}^2 - ((u \cdot \nabla)u, \div(Du)) - \frac{1}{2}((\div(Du))u, \div(Du)) + \int_\Omega \nu(\phi)|\div(Du)|^2 \, dx + \int_\Omega \nu'(\phi)Du \nabla \phi \cdot \div(Du) \, dx + (\div(Du)) = (\mu \nabla \phi, -\div(Du)).
\]
We take the gradient of (1), we multiply it by \(\nabla p\) and we integrate over \(\Omega\). We obtain
\[
\frac{\varepsilon}{2} \frac{d}{dt} \|\nabla p\|_{L^2(\Omega)}^2 + (\nabla \div u, \nabla p) + (\div u, p) = 0.\tag{83}
\]
In order to recover the full norm of \(p\) in \(H^1(\Omega)\), we multiply (1) by \(p\) and integrate over \(\Omega\). Collecting the two equations together, we find
\[
\frac{1}{2} \frac{d}{dt} \|Du\|_{L^2(\Omega)}^2 + \|p\|_{H^1(\Omega)}^2 + (\nabla \div u, \nabla p) + (\div u, p) = 0.\tag{83}
\]
Summing up the two equations (82) and (83), we deduce that
\[
\frac{1}{2} \frac{d}{dt} \left\{ \|Du\|_{L^2(\Omega)}^2 + \varepsilon \|p\|_{H^1(\Omega)}^2 \right\} + ((u \cdot \nabla)u, -\div(Du)) + \frac{1}{2}((\div(Du))u, -\div(Du)) + \int_\Omega \nu(\phi)|\div(Du)|^2 \, dx + \int_\Omega \nu'(\phi)Du \nabla \phi \cdot \div(Du) \, dx + (\div(Du)) = (\mu \nabla \phi, -\div(Du)).
\]
Since \(\nu(\cdot) \geq \nu_*\) for all \(\varepsilon \in \mathbb{R}\), by using Lemma 2.1, we infer that
\[
\frac{d}{dt} \left\{ \|Du\|_{L^2(\Omega)}^2 + \varepsilon \|p\|_{H^1(\Omega)}^2 \right\} + \frac{\nu_*}{2C_{LN}} \|u\|_{H^1(\Omega)}^2 \
\leq ((u \cdot \nabla)u, \div(Du)) + \frac{1}{2}((\div(Du))u, \div(Du)) - \int_\Omega \nu'(\phi)Du \nabla \phi \cdot \div(Du) \, dx + (\div(Du)) - (\nabla \div u, \nabla p) - (\div u, p) + (\mu \nabla \phi, -\div(Du)).
\]
By using (3), (4), and the regularity of \(u\), we have
\[
((u \cdot \nabla)u, \div(Du)) + \frac{1}{2}((\div(Du))u, \div(Du)) \leq C \|u\|_{L^4(\Omega)} \|u\|_{W^{1,4}(\Omega)} \| - \div(Du)\|_{L^2(\Omega)}
\]
Also, we find
\[ \leq C\|u\|_{H^1(\Omega)} \|u\|_{H^2(\Omega)} ^{3/2} \]
\[ \leq \frac{\nu_s}{16 C_L N} \|u\|_{H^2(\Omega)} ^2 + C\|Du\|_{L^2(\Omega)}. \]

By recalling that \( H^s(\Omega) \hookrightarrow L^{\frac{2}{1+s}}(\Omega), \) for any \( s \in (0,1), \) and by using (4) and the above regularity for \( \phi, \) we obtain for \( s \in (0,1) \)
\[ \left| \int_{\Omega} \nu'(\phi)Du \nabla \phi \cdot \text{div}(Du) \, dx \right| \leq C\|Du\|_{L^s(\Omega)} \|\nabla \phi\|_{L^{\frac{2}{1+s}}(\Omega)} \|\text{div}(Du)\|_{L^2(\Omega)} \]
\[ \leq C\|Du\|_{L^2(\Omega)} \|u\|_{H^2(\Omega)} ^{2-s} \|\phi\|_{H^{1+s}(\Omega)} \]
\[ \leq \frac{\nu_s}{16 C_L N} \|u\|_{H^2(\Omega)} ^2 + C\|Du\|_{L^2(\Omega)}. \]

For \( s = 1, \) by (4) and regularity for \( \phi, \) we simply have
\[ \left| \int_{\Omega} \nu'(\phi)Du \nabla \phi \cdot \text{div}(Du) \, dx \right| \leq C\|Du\|_{L^1(\Omega)} \|\nabla \phi\|_{L^2(\Omega)} \|\text{div}(Du)\|_{L^2(\Omega)} \]
\[ \leq C\|Du\|_{L^2(\Omega)} \|u\|_{H^2(\Omega)} ^3 \]
\[ \leq \frac{\nu_s}{16 C_L N} \|u\|_{H^2(\Omega)} ^2 + C\|Du\|_{L^2(\Omega)}. \]

Also, we find
\[ (\nabla p, \text{div}(Du)) - (\nabla \text{div}u, \nabla p) - (\text{div}u, p) \leq C\|p\|_{H^1(\Omega)} \|u\|_{H^2(\Omega)} \]
\[ \leq \frac{\nu_s}{16 C_L N} \|u\|_{H^2(\Omega)} ^2 + C\|p\|_{H^1(\Omega)}. \]

By Sobolev embedding, for \( s \in (0,1), \) we deduce that
\[ \left| (\mu \nabla \phi, -\text{div}(Du)) \right| \leq \|\mu\|_{L^s(\Omega)} \|\nabla \phi\|_{L^{\frac{2}{1+s}}(\Omega)} \|\text{div}(Du)\|_{L^2(\Omega)} \]
\[ \leq C\|\mu\|_{H^1(\Omega)} \|\phi\|_{H^{1+s}(\Omega)} \|u\|_{H^2(\Omega)} \]
\[ \leq \frac{\nu_s}{16 C_L N} \|u\|_{H^2(\Omega)} ^2 + C\|\mu\|_{H^1(\Omega)} ^2. \]

The case \( s = 1 \) can be done as above with minor changes. Therefore, by setting \( F = \frac{1}{2} \|Du\|_{L^2(\Omega)} ^2 + \frac{1}{2} \|p\|_{H^1(\Omega)} ^2, \) we find the differential inequality
\[ \frac{d}{dt} F + \frac{\nu_s}{4 C_L N} \|u\|_{H^2(\Omega)} ^2 \leq CF(1 + \|Du\|_{L^2(\Omega)} ^2) + C\|\mu\|_{H^1(\Omega)} ^2. \]

Since \( Du \in L^2(0,T;H) \) and \( \mu \in L^2(0,T;H^1(\Omega)), \) by using the Gronwall lemma, we obtain, for any \( T > 0, \) bounds of \( u \) and \( p \) in the following spaces
\[ u \in L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega) \cap V), \quad p \in L^\infty(0,T;H^1(\Omega)). \]

Finally, by using the equations (1)1 and (1)2, together with the proved regularity on the solution, we easily infer that
\[ \partial_t u \in L^2(0,T;H), \quad \partial_t p \in L^2(0,T;H^1(\Omega)). \]

### 6.2. Local strong solutions in 3D bounded domains

Let us assume that we have a weak solution \((u, p, \phi)\) satisfying, for an arbitrary \( s \in \left[ \frac{1}{3}, 1 \right] \)
\[ u \in L^\infty(0,T;H) \cap L^2(0,T;V), \]
\[ p \in L^\infty(0,T;L^2(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega)), \]
\[ \phi - \phi_0 \in L^\infty(0,T;D(A^\frac{1}{2} N)) \cap L^2(0,T;D(A^\frac{3}{2} N)) \cap W^{1,2}(0,T;D(A^\frac{1}{2} N)), \]
\[ \dot{\phi} - \dot{\phi}_0 \in L^\infty(0,T;D(A^\frac{1}{2} N)) \cap L^2(0,T;D(A^\frac{3}{2} N)) \cap W^{1,2}(0,T;D(A^\frac{1}{2} N)). \]
\[ \mu - \overline{\mu} \in L^\infty(0, T; D(A_{N}^{\frac{2}{3}})) \cap L^2(0, T; D(A_{N}^{\frac{4}{3}})). \]

Arguing as in Section 6.1, we have
\[
\frac{d}{dt} \left\{ \frac{1}{2} \| Du \|_{L^2(\Omega)}^2 + \frac{\varepsilon}{2} \| p \|_{H^1(\Omega)}^2 \right\} + \frac{\nu_s}{2C_{LN}} \| u \|_{H^2(\Omega)}^2
\leq ((u \cdot \nabla)u, \text{div}(Du)) + \frac{1}{2} \langle (\text{div} u), \text{div}(Du) \rangle - \int_{\Omega} \nu'(\phi)Du \nabla \phi : \text{div}(Du) \, dx
\]
\[ + (\nabla p, \text{div}(Du)) - (\nabla \text{div} u, \nabla p) - (\text{div} u, p) + (\mu \nabla \phi, -\text{div}(Du)). \]

By using the Ladyzhenskaya inequality, the Korn inequality (3), and the regularity of \( u \), we have
\[
((u \cdot \nabla)u, \text{div}(Du)) + \frac{1}{2} \langle (\text{div} u), \text{div}(Du) \rangle \leq C\| u \|_{L^p(\Omega)}\| u \|_{W^{1,p}(\Omega)} - \text{div}(Du)\|_{L^2(\Omega)}
\]
\[ \leq C\| u \|_{H^1(\Omega)}^2 \| u \|_{H^2(\Omega)}^2 \]
\[ \leq \frac{\nu_s}{16C_{LN}} \| u \|_{H^2(\Omega)}^2 + C\| Du \|_{L^2(\Omega)}^2. \]

By using (5) and Lemma 2.1, we obtain
\[
\left| \int_{\Omega} \nu'(\phi)Du \nabla \phi : \text{div}(Du) \, dx \right| \leq C\| Du \|_{L^3(\Omega)}\| \nabla \phi \|_{L^6(\Omega)}\| \text{div}(Du)\|_{L^2(\Omega)}
\]
\[ \leq C\| Du \|_{L^2(\Omega)}^2 \| \nabla(\phi - \overline{\phi}) \|_{L^4(\Omega)} \| u \|_{H^2(\Omega)}^2 \]
\[ \leq \frac{\nu_s}{16C_{LN}} \| u \|_{H^2(\Omega)}^2 + C\| \phi - \overline{\phi} \|_{H^2(\Omega)}^2 \| Du \|_{L^2(\Omega)}^2. \]

Also, we have
\[
(\nabla p, \text{div}(Du)) - (\nabla \text{div} u, \nabla p) - (\text{div} u, p) \leq C\| p \|_{H^1(\Omega)} \| u \|_{H^2(\Omega)}
\]
\[ \leq \frac{\nu_s}{16C_{LN}} \| u \|_{H^2(\Omega)}^2 + C\| p \|_{H^1(\Omega)}^2. \]

We now consider the case \( s \in \left[ \frac{3}{5}, \frac{1}{2} \right] \). According to (11), we find that \( H^{1+s}(\Omega) \hookrightarrow W^{1,\frac{4}{3+s}}(\Omega) \). Since \( \phi \in L^\infty(0, T; H^{1+s}(\Omega)) \), we have
\[
(\mu \nabla \phi, -\text{div}(Du)) \leq \| \mu \|_{L^{\frac{4}{3+s}}(\Omega)} \| \nabla \phi \|_{L^{\frac{3}{s}}(\Omega)} \| \text{div}(Du)\|_{L^2(\Omega)}
\]
\[ \leq C\| \mu \|_{L^{\frac{4}{3+s}}(\Omega)} \| \phi \|_{H^{1+s}(\Omega)} \| u \|_{H^2(\Omega)}
\]
\[ \leq \frac{\nu_s}{16C_{LN}} \| u \|_{H^2(\Omega)}^2 + C\| \mu \|_{L^{\frac{4}{3+s}}(\Omega)}^2. \]

Therefore, recalling that \( F = \frac{1}{2} \| Du \|_{L^2(\Omega)}^2 + \frac{\varepsilon}{2} \| p \|_{H^1(\Omega)}^2 \), we find the differential inequality
\[
\frac{d}{dt} F + \frac{\nu_s}{4C_{LN}} \| u \|_{H^2(\Omega)}^2
\]
\[ \leq C\| Du \|_{L^2(\Omega)}^6 + \| \phi - \overline{\phi} \|_{H^2(\Omega)}^6 \| Du \|_{L^2(\Omega)}^2 + C\| p \|_{H^1(\Omega)}^2 + C\| \mu \|_{H^1(\Omega)}^2
\]
\[ \leq CF^6 + C(1 + \| \phi - \overline{\phi} \|_{H^2(\Omega)}^6 + \| \mu \|_{L^{\frac{4}{3+s}}(\Omega)}^2). \]
By interpolation in the spaces $D(A^s_N)$ (cf. [24]) $D(A^{\frac{1+s}{2}}_N), D(A^{\frac{1+s}{2}}_N)) \hookrightarrow D(A_N)$. Thus, we have

$$\|\phi - \overline{\phi}\|_{H^2(\Omega)}^6 \leq C\|\phi - \overline{\phi}\|^6_{D(A_N)} \leq C\|\phi - \overline{\phi}\|^{3+3s}_{D(A^{\frac{1+s}{2}}_N)} \|\phi - \overline{\phi}\|^{3-3s}_{D(A^{\frac{1+s}{2}}_N)}.$$ 

Since $\phi - \overline{\phi} \in L^\infty(0; T; D(A^{\frac{1+s}{2}}_N)) \cap L^2(0; T; D(A^{\frac{1+s}{2}}_N))$ and $s \in [\frac{1}{3}, 1]$, we infer that $\|\phi - \overline{\phi}\|_{H^2(\Omega)} \leq C\|\phi - \overline{\phi}\|^6_{D(A^{\frac{1+s}{2}}_N)}$. Here we have used that (II.4.12) entails that there exists $s$ such that $D(A^{\frac{1+s}{2}}_N)$ if $s \in [\frac{1}{3}, 1]$. We define $\phi = 0$.

To summarize, for any $s \in [\frac{1}{3}, 1]$ we have

$$\frac{d}{dt} F + \frac{\nu_s}{4C_{LN}} \|u\|^2_{H^2(\Omega)} \leq C\|F\|^3 + G,$$

where $G \in L^1(0, T)$. A classical comparison argument for ODEs (e.g., [11, Lemma II.4.12]) entails that there exists $T_0 > 0$ such that

$$u \in L^\infty(0, T_0; V) \cap L^2(0, T_0; H^2(\Omega) \cap V), \quad p \in L^\infty(0, T_0; H^1(\Omega)).$$

### 6.3. Uniqueness of strong solutions in 3D bounded domains.

Let us consider two solutions $(u_1, p_1, \phi_1)$ and $(u_2, p_2, \phi_2)$ originating from the same initial condition $(u_0, p_0, \phi_0)$. We define $u = u_1 - u_2, p = p_1 - p_2, \phi = \phi_1 - \phi_2$. They solve

$$\partial_t u + (u_1 \cdot \nabla) u + (u_1 \cdot \nabla) u_2 + \frac{1}{2}(\text{div} u_1) u + \frac{1}{2}(\text{div} u) u_2 - \text{div} (\nu(\phi_1) D u) - \text{div} ((\nu(\phi_1) - \nu(\phi_2)) D u_2) + \nabla p = \mu_1 \nabla \phi + \mu \nabla \phi_2, \quad (84)$$

and

$$\varepsilon \partial_t v + \text{div} u = 0, \quad (85)$$

where $\mu = -\Delta \phi + \Psi'(\phi_1) - \Psi'(\phi_2)$ and $\frac{\partial}{\partial t} \phi = -u_1 \cdot \nabla \phi - u \cdot \nabla \phi_2$. Multiplying (84) by $u$ and integrating over $\Omega$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|u\|^2_{L^2(\Omega)} + (u \cdot \nabla) u_2, u + \frac{1}{2}((\text{div} u_2, u) + \nu(\phi_1) D u, D u)
+ ((\nu(\phi_1) - \nu(\phi_2)) D u_2, \nabla u) + (\nabla p, u) = (\mu_1 \nabla \phi, u) + (\mu \nabla \phi_2, u).$$

Here we have used that $(u_1 \cdot \nabla) u + \frac{1}{2}((\text{div} u_1) u_1, u) = 0$. Multiplying (85) by $p$ and integrating over $\Omega$, we have

$$\varepsilon \frac{d}{dt} \|p\|^2_{L^2(\Omega)} + (\text{div} u, p) = 0.$$
We observe that \( (\nabla p, u) + (\text{div } u, p) = 0 \) since \( u = 0 \) on \( \partial \Omega \). Then, adding the above two equations, we arrive at
\[
\begin{align*}
\frac{d}{dt} & \left\{ \frac{1}{2} \| u \|^2_{L^2(\Omega)} + \frac{\nu}{2} \| p \|^2_{L^2(\Omega)} \right\} + ((\nabla u) u_2, u) + \frac{1}{2} ((\text{div } u) u_2, u) \\
& + (\nu(\phi_1) D u, D u) + ((\nu(\phi_1) - \nu(\phi_2)) D u_2, \nabla u) \\
& = (\mu_1 \nabla \phi, u) + (\mu \nabla \phi_2, u).
\end{align*}
\] (87)

Taking \( v = -\Delta \phi \) in (86), and using the definition of \( \mu \), we deduce that
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} & \left\{ \frac{1}{2} \| \nabla \phi \|^2_{L^2(\Omega)} + \| \Delta \phi \|^2_{L^2(\Omega)} \right\} \\
& = (u_1 \cdot \nabla \phi, \Delta \phi) + (u \cdot \nabla \phi_2, \Delta \phi) + ((\Psi''(\phi_1)) \nabla \phi_1 - \Psi''(\phi_2) \nabla \phi_2, \nabla \Delta \phi) \\
& - (u_1 \cdot \nabla \phi + u \cdot \nabla \phi_2) \phi.
\end{align*}
\] (88)

By summing (87) and (88), we find
\[
\begin{align*}
\frac{d}{dt} & \left\{ \frac{1}{2} \| u \|^2_{L^2(\Omega)} + \frac{\nu}{2} \| p \|^2_{L^2(\Omega)} \right\} + \| \Delta \phi \|^2_{L^2(\Omega)} \\
& + \nu \| D u \|^2_{L^2(\Omega)} + \| \nabla \phi \|^2_{L^2(\Omega)} \\
& \leq - \left( \frac{(u \cdot \nabla) u_2, u}{} + \frac{1}{2} ((\text{div } u) u_2, u) - \frac{1}{2} ((\nu(\phi_1) - \nu(\phi_2)) D u_2, \nabla u) \\
& + (\mu_1 \nabla \phi, u) + (\mu \nabla \phi_2, u) + (u_1 \cdot \nabla \phi, \Delta \phi) + (u \cdot \nabla \phi_2, \Delta \phi) \\
& + ((\Psi''(\phi_1)) \nabla \phi_1 - \Psi''(\phi_2) \nabla \phi_2, \nabla \Delta \phi) - (u_1 \cdot \nabla \phi + u \cdot \nabla \phi_2) \phi \right).
\end{align*}
\]

Since \( H^2(\Omega) \rightarrow L^\infty(\Omega) \), we easily observe that
\[
\begin{align*}
|K_1 + K_2| & \leq C \| u_2 \|_{H^2(\Omega)} \| u \|_{L^2(\Omega)} \| D u \|_{L^2(\Omega)} \\
& \leq \frac{\nu}{8} \| D u \|^2_{L^2(\Omega)} + C \| u_2 \|^2_{H^2(\Omega)} \| u \|^2_{L^2(\Omega)}.
\end{align*}
\]

Since \( \nu' \) is bounded, we have
\[
\begin{align*}
|K_3| & \leq C \| \phi \|_{L^3(\Omega)} \| D u_2 \|_{L^3(\Omega)} \| \nabla u \|_{L^2(\Omega)} \\
& \leq \frac{\nu'}{8} \| D u \|^2_{L^2(\Omega)} + C \| u_2 \|^2_{H^2(\Omega)} (\| \nabla \phi \|^2_{L^2(\Omega)} + |\bar{\phi}|^2).
\end{align*}
\]

Owing to (38), we find
\[
\begin{align*}
K_4 & = \int_\Omega (\nabla \phi_1 \otimes \nabla \phi + \nabla \phi \otimes \nabla \phi_2) : \nabla u \ dx \\
& - \int_\Omega (\Psi(\phi_1) - \Psi(\phi_2)) I : \nabla u \ dx \\
& - \frac{1}{2} \int_\Omega (\nabla \phi_1 \cdot \nabla \phi + \nabla \phi_1 \cdot \nabla \phi_2) I : \nabla u \ dx.
\end{align*}
\]
We estimate the above terms as follows
\[
|K_4 + K_5| \leq C(\|\nabla \phi_1\|_{L^\infty(\Omega)} + \|\nabla \phi_2\|_{L^\infty(\Omega)})\|\nabla \phi\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}
\]
\[
\leq \frac{\nu_2}{8} \|Du\|_{L^2(\Omega)}^2 + C(\|\nabla \phi_1\|_{L^\infty(\Omega)}^2 + \|\nabla \phi_2\|_{L^\infty(\Omega)}^2)\|\nabla \phi\|_{L^2(\Omega)}^2 ,
\]
and
\[
|K_6| \leq \left| \int_{\Omega} \left( \int_0^1 \nabla' \left( \tau \phi_1 + (1 - \tau) \phi_2 \right) d\tau \right) \phi I : \nabla u \, dx \right|
\]
\[
\leq C(\|\nabla' (\phi_1)\|_{L^2(\Omega)} + \|\nabla' (\phi_2)\|_{L^2(\Omega)})\|\phi\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}
\]
\[
\leq \frac{\nu_2}{8} \|Du\|_{L^2(\Omega)}^2 + C(\|\nabla \phi\|_{L^2(\Omega)}^2 + |\bar{\phi}|^2).
\]
With (14) we obtain
\[
|K_5| \leq \|u_1\|_{L^\infty(\Omega)}\|\nabla \phi\|_{L^2(\Omega)}\|\Delta \phi\|_{L^2(\Omega)} + \|u_2\|_{L^2(\Omega)}\|\nabla \phi_2\|_{L^\infty(\Omega)}\|\Delta \phi\|_{L^2(\Omega)}
\]
\[
\leq \frac{1}{4} \|\nabla \Delta \phi\|_{L^2(\Omega)}^2 + C(\|u_1\|_{L^\infty(\Omega)}^2 + \|\nabla \phi_2\|_{L^\infty(\Omega)}^2)\|\Delta \phi\|_{L^2(\Omega)}^2.
\]
By using the definition of $\Psi$, the inequalities (5) and (7), and the regularity of strong solutions, we have
\[
|K_6| = \left| \int_{\Omega} \nabla'' (\phi_1) \nabla \phi \cdot \nabla \Delta \phi \, dx + \int_{\Omega} \left( \nabla'' (\phi_1) - \nabla'' (\phi_2) \right) \nabla \phi_2 \cdot \nabla \Delta \phi \, dx \right|
\]
\[
\leq \|\nabla'' (\phi_1)\|_{L^\infty(\Omega)}\|\nabla \phi\|_{L^2(\Omega)}\|\nabla \Delta \phi\|_{L^2(\Omega)}
\]
\[
+ C(\|\nabla'' (\phi_1)\|_{L^\infty(\Omega)} + \|\nabla'' (\phi_2)\|_{L^\infty(\Omega)})\|\phi\|_{L^2(\Omega)}\|\nabla \phi_2\|_{L^2(\Omega)}\|\nabla \Delta \phi\|_{L^2(\Omega)}
\]
\[
\leq \frac{1}{4} \|\nabla \Delta \phi\|_{L^2(\Omega)}^2 + C(\|\phi_1\|_{L^\infty(\Omega)}^2 + \|\phi_2\|_{L^\infty(\Omega)}^2)\|\nabla \phi_2\|_{L^2(\Omega)}\|\Delta \phi\|_{L^2(\Omega)}^2.
\]
By using the Cauchy-Schwarz inequality, we find
\[
|K_7| \leq C(\|u_1\|_{L^2(\Omega)}\|\nabla \phi\|_{L^2(\Omega)} + \|u_2\|_{L^2(\Omega)}\|\nabla \phi_2\|_{L^2(\Omega)})\|\phi\|_{L^2(\Omega)}
\]
\[
\leq C(\|u_1\|_{L^2(\Omega)}^2 + \|\nabla \phi_2\|_{L^2(\Omega)}^2 + |\bar{\phi}|^2).
\]
Collecting all together the above estimates, we find the differential inequality
\[
\frac{d}{dt} H_2 \leq G_2 H_2,
\]
where
\[
H_2 = \frac{1}{2} \|u\|_{L^2(\Omega)}^2 + \frac{\nu_2}{2} \|p\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla \phi\|_{L^2(\Omega)}^2 + \frac{1}{2} |\bar{\phi}|^2,
\]
and
\[
G_2 = C(\|u_1\|_{L^\infty(\Omega)}^2 + \|u_2\|_{H^2(\Omega)}^2 + \|\nabla \phi_1\|_{L^\infty(\Omega)}^2 + \|\nabla \phi_2\|_{L^\infty(\Omega)}^2 + \|\phi_1\|_{H^2(\Omega)}^2 + \|\phi_2\|_{H^2(\Omega)}^2).
\]
Since $G_2 \in L^1(0,T)$, the uniqueness of strong solutions follows from the Gronwall lemma.

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