Noncompact Coset Spaces in String Theory

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A brief overview of strings propagating on noncompact coset spaces $G/H$ is presented in terms of gauged WZW models. The role played by isometries in the existence of target space duality and by fixed points of the gauged transformations in the existence of singularities and horizons, is emphasized. A general classification of the spaces with a single time-like coordinate is presented. The spacetime geometry of a class of models, existing for every dimension and having cosmological and black hole-like interpretations, is discussed.

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There has been a large amount of effort dedicated to find possible connections between String Theory and Physics. At present, the only available approach in that direction is to study the structure and physical properties of the semiclassical string vacua given by classes of conformal field theories (CFT). For applications to Particle Physics, it is enough to assume a flat four-dimensional spacetime and parametrize the different models by an internal CFT restricted by some consistency conditions such as worldsheet modular invariance. In this way many classes of models have been studied and some, close to the Standard Model of Particle Physics give hope that strings may actually be related to the real world. To ask questions about gravitation in the context of String Theory, we have to relax the flat spacetime assumption and substitute it by a general noncompact CFT. These questions are of fundamental importance since the main motivation for studying string theories is to provide a consistent way of quantizing gravity. In particular it is of prime importance to study singularities of cosmology and black hole-type of geometries in the context of String Theory, since it is in those regimes that the standard field theory methods of General Relativity fail to apply. Since at the moment, coset models $G/H$ provide the most general way of constructing CFT’s, it is natural to approach these questions in terms of noncompact coset spaces, where the noncompactness is needed to describe lorentzian spacetimes.

The study of noncompact coset CFT’s was undertaken in [1] for $\text{SL}(2,\mathbb{R})/U(1)$ current algebra via the conventional GKO construction. The formalism was later generalized to any coset in [2]. Given a level $k$ Kac-Moody algebra for a noncompact group $G$,

\[ J_A(z) J_B(w) \sim -\frac{k \eta_{AB}}{(z-w)^2} + \frac{i f_{ABC} J_C(w)}{(z-w)} \]

(1)

(wher $g \eta^{AB} = f_{AC}^D f_{BD}^C$ is the Cartan metric and $g$ is the Coxeter number of $G$), the stress-energy tensor for a CFT based on $G$ is given by the Sugawara from

\[ T_G(z) = \frac{\eta^{AB} :J_A(z) J_B(z):}{(-k+g)} . \]

(2)

The corresponding central charge is $c_G = k \dim G/(k-g)$. For the coset $G/H$ with stress-energy tensor $T_{G/H} = T_G - T_H$, the central charge is $c_{G/H} = c_G - c_H$. The only changes from the compact case are the sign of $k$ and of course the use of noncompact structure constants $f_{AB}^C$. The spectrum and the corresponding elimination of negative norm states is not yet entirely understood for these models, and more progress is needed before we can properly treat the string vacua obtained from this approach.
In order to have a geometrical interpretation of these vacua, we need to have a Lagrangian formulation of these CFT’s which is provided by the WZW construction. The WZW action for a group $G$ with elements $g(z, \bar{z})$ in complex coordinates is [3],

$$L(g) = \frac{k}{4\pi} \int d^2z \text{tr}(g^{-1} \partial g g^{-1} \bar{\partial}g) - \frac{k}{12\pi} \int_B \text{tr}(g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg) , \quad (3)$$

where the boundary of $B$ is the 2D worldsheet. It is known that this action provides the conserved currents $g^{-1} \partial g$ and $\bar{\partial}gg^{-1}$ to satisfy a chiral algebra like (1) and then giving a CFT from (2) for the group $G$. For the coset $G/H$ [4] the standard way in nonlinear sigma models [5] is to eliminate the $H$ degrees of freedom by gauging the subgroup $H$ as a symmetry of (3). To promote the global $g \rightarrow h_L^{-1}g h_R$ invariance to a local $g \rightarrow h_L^{-1}(z)g h_R(\bar{z})$ invariance, we let $\partial g \rightarrow \partial g + A g$, and $\bar{\partial}g \rightarrow \bar{\partial}g - g \bar{A}$. The gauge fields transform as $A \rightarrow h_L^{-1}(A + \partial)h_L$ and $\bar{A} \rightarrow h_R^{-1} (\bar{A} + \partial)h_R$ (so that $Dg \rightarrow h_L^{-1} Dg h_R$ for $D$ equal to either holomorphic or anti-holomorphic covariant derivative). Vector gauge transformations correspond to $h_L = h_R$, and axial gauge transformations to $h_L = h_R^{-1}$. Substituting covariant derivatives in (3) gives the gauged action

$$L(g, A) = L(g) + \frac{k}{2\pi} \int d^2z \text{tr}(A \bar{\partial}gg^{-1} - \bar{A} g^{-1} \partial g - g^{-1}Ag \bar{A}) . \quad (4)$$

Under the infinitesimal transformations $h_L \approx 1 + \alpha$, $h_R = 1 + \beta$, we have $\delta A = \partial \alpha + [A, \alpha]$ and $\delta \bar{A} = \bar{\partial} \beta + [\bar{A}, \beta]$. The anomalous variation of the effective action is (see e.g. [3] for a review)

$$\delta W = \frac{k}{2\pi} (\alpha \bar{\partial}A + \beta \partial \bar{A}) . \quad (5)$$

The variation of the counterterm $\text{tr}A\bar{A}$, on the other hand, is

$$\delta (\text{tr}A\bar{A}) = \text{tr}(-\beta \bar{\partial}A - \alpha \partial \bar{A} + (\alpha - \beta) [\bar{A}, A]) . \quad (6)$$

For the abelian case, we see that (6) can compensate the variation (3) for either $\alpha = \pm \beta$ since the commutator term automatically vanishes. Thus both vector and axial-vector gauging are allowed. In the non-abelian case, only the vector gauging $\alpha = \beta$ is allowed. If we change sign $\bar{A} \rightarrow -\bar{A}$ for the axial gauged case (to give $A$ and $\bar{A}$ the same transformation properties), then the gauged action may be written

$$L(g, A) = L(g) + \frac{k}{2\pi} \int d^2z \text{tr}(A \bar{\partial}gg^{-1} \mp \bar{A} g^{-1} \partial g + A \bar{A} \mp g^{-1}Ag \bar{A}) , \quad (7)$$
where the upper and lower signs represent respectively vector \((g \to hgh^{-1})\) and axial-vector \((g \to hgh)\) gauging.

We now consider some naive properties of the geometry described by (7) in the large \(k\) limit. Writing \(A = A^a \sigma_a\) in terms of the generators \(\sigma_a\) of \(H\), and integrating out the components \(A^a\) classically gives the effective action

\[
L = L(g) \pm \frac{k}{2\pi} \int d^2z \, \text{tr}(\sigma_b g^{-1} \partial g) \text{tr}(\sigma_a \partial gg^{-1}) \Lambda_{ab}^{-1},
\]

with \(\Lambda_{ab} \equiv \text{tr}(\sigma_a \sigma_b \mp \sigma_a g \sigma_b g^{-1})\). Notice that singularities of \(\Lambda\) occur at least at fixed points of the gauge transformation \(g \to hgh^\mp1\). This is because for infinitesimal \(h \approx 1 + \alpha^a \sigma_a\), we see that a fixed point \(g\) satisfies \(\sigma_a g \mp g \sigma_a = 0\). Multiplying by \(g^{-1} \sigma_b\) and taking the trace, we see that \(\Lambda = 0\) at a fixed point.

From the transformation properties of the gauge fields and (8), we note that in the case of \(H\) abelian the ungauged axial or vector symmetry remains a global symmetry, i.e. an isometry of the spacetime geometry. In the non-abelian case, not even a global vestige of the ungauged symmetry remains. In the abelian case, this implies that a fixed point of the ungauged symmetry corresponds to a point with vanishing Killing vector. For lorentzian signature, the surface carried into the fixed point by the isometry will be a null surface (the norm of the Killing vector is conserved), in general nonsingular and hence a horizon. We see that fixed points of symmetry transformations generically give rise to metric singularities when the symmetry is gauged and to horizons when ungauged. This general property is the origin of the singularity/horizon duality of the 2D black hole of [7]. For the vector gauging, the metric can be written \(ds^2 = -da db/(1 - ab)\), and the fixed point of the vector transformation (the gauged symmetry) corresponds to \(ab = 1\), which is the singularity. The fixed point of the axial transformation (the isometry) is \(a = b = 0\) indicating that the invariant surface \(ab = 0\) is null, and provides the event horizon illustrated in fig. 1. For the axial gauging, the metric is identical (i.e. the geometry is self-dual) but the role of the fixed points is exchanged, implying the horizon/singularity duality pointed out in [8].

We can see that the gauged “\(G/H\)” WZW models considered here are not the usual left or right \(G/H\) coset spaces with standard coset metric, as considered in the mathematical literature [9] and in standard treatments of coset space nonlinear \(\sigma\)-models [5]. This is because we gauge \(g \to hgh^{\mp1}\) type symmetries rather than \(g \to gh\) or \(g \to hg\), and as well we include a Wess-Zumino term which can add a torsion piece to the metric. Gauging the
Fig. 1: The causal structure of the two dimensional black hole spacetime of \([7]\). Regions I, IV are asymptotic regions, regions II, III are inside the horizon, and regions V, VI are beyond the singularities.

\(H\) subgroup nonetheless eliminates the \(H\) degrees of freedom, and it is easily verified that the signature of the resulting metric is the same as that of the standard coset metric. It is therefore straightforward to impose the phenomenological restriction to spaces with only a single timelike coordinate \([10]\). The only subtlety is that the level \(k\) appears in front of the action. Positive \(k\), in our sign conventions, results in a metric whose compact generators correspond to timelike directions and noncompact generators to spacelike directions. For negative \(k\) (when allowed by unitarity), the roles of compact and noncompact generators are interchanged in the correspondence.

To classify all the coset CFT’s with a single timelike coordinate we consider first the case \(k\) positive and examine the difference

\[ N \equiv |G|_c - |H|_c , \]

for all possible cosets, where \(|G, H|_c\) denote the number of compact generators. To this end we employ the known classification \([3]\) of symmetric spaces \(G/H\) (where \(H\) is a maximal subgroup and \(G\) is simple). From this list we eliminate all cases with \(N > 1\), since for a given \(G\) modding out by smaller (non-maximal) subgroups increases the value of \(N\). For \(N = 1\), this leaves only the case \(SO(D - 1, 2)/SO(D - 1, 1)\) \((\mathbb{11})\). For \(N = 0\), which corresponds to maximal compact subgroup embeddings, we can identify the cases for which \(H\) has a \(U(1)\) factor, \(H = H' \times U(1)\) (hermitian symmetric spaces), so that \(G/H'\) has an additional compact generator, hence one timelike coordinate. These cases exhaust all possibilities in which \(G\) is a simple group. For \(k\) negative, we consider instead
the difference \(N = |G|_{nc} - |H|_{nc}\) of noncompact generators, and find that the only solution with \(N = 1\) is \(SO(D, 1)/SO(D - 1, 1)\).

For \(G\) a product of simple groups and \(U(1)\) factors, there are several possibilities to consider:

(i) \(G = G_1 \otimes G_2 \otimes G_3\) and \(H = H_1 \otimes H_2 \otimes H_3\) where \(G_1/H_1\) has \(N = 1\), \(G_2/H_2\) has \(N = 0\) and \(G_3/H_3\) is a (product of) compact cosets.

(ii) \(G = G' \otimes \mathbb{R}\) where \(G'/H\) has \(N = 0\). In this case \(\mathbb{R}\) provides the timelike coordinate. Other possibilities may be obtained by enlarging consideration from semisimple groups \(G\) to non-semisimple groups of potential relevance. For single timelike coordinates, we consider the simplest class of coset models with a single timelike coordinate and any number of spacelike coordinates, namely, \(SL(2, \mathbb{R}) \otimes SO(1, 1)^{D-2}/SO(1, 1)\) models. In order to find the metric in the large \(k\) limit, we employ the standard procedure in nonlinear \(\sigma\)-models, i.e. find a parametrization of the \(G\) group elements, impose a unitary type gauge on the fields in the \(\sigma\)-model action and then solve for the (non-propagating) \(H\)-gauge fields to derive the \(G/H\) worldsheet action. From that action we can read off the corresponding background fields. We first parametrize the group elements as 
\[g = \text{diag}(g_0, g_1, \ldots, g_{D-2}),\]
where
\[g_0 = \begin{pmatrix} a & u \\ -v & b \end{pmatrix} \quad \text{(with } ab + uv = 1)\]  
and
\[g_i = \begin{pmatrix} \cosh r_i & \sinh r_i \\ \sinh r_i & \cosh r_i \end{pmatrix} \quad \text{with } i = 1, \ldots, D - 2.\]  
We choose the embedding such that the generator of \(H = SO(1, 1)\) is 
\[\sigma = \text{diag}(s_0, \ldots, s_{D-2})\]  
where
\[s_0 = q_0 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and } s_i = q_i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\]  
with coefficients normalized to \(\sum_{i=0}^{D-2} q_i^2 = 1\).

Under the infinitesimal vector gauge transformations \(\delta g = \varepsilon(\sigma g - g\sigma)\), the parameters transform as \(\delta a = \delta b = \delta r_i = 0\), and \(\delta u = 2\varepsilon q_0 u, \delta v = -2\varepsilon q_0 v\). The choices \(u = \pm v\) thus fix the gauge completely. From \(\pm u^2 = 1 - ab\), we are left with the parameters \(a, b, r_i\) as the \(D\) spacetime coordinates. Substituting into (8), we find the action (for both gauge choices \(u = \pm v\)):
\[L = \frac{k_0}{2\pi} \int d^2 z \left( -\frac{\partial a \overline{\partial b} + \partial b \overline{\partial a}}{2(1 - ab)} + \sum_i \kappa_i \left( \delta_{ij} + \frac{\kappa_j \eta_i \eta_j}{1 - ab} \right) \partial r_i \overline{\partial r_j} \right.\]
\[\left. + \frac{\kappa_i \eta_i}{2(1 - ab)} \left( (b \partial a - a \partial b) \overline{\partial r_i} + \partial r_i (b \overline{\partial a} - a \overline{\partial b}) \right) \right)\]
where \( \kappa_i \equiv k_i / k_0 \) and \( \eta_i \equiv q_i / q_0 \).

This action can be identified with a \( \sigma \)-model action of the form

\[
S = \int d^2 z \left( G_{MN} + B_{MN} \right) \partial X^M \partial X^N
\]

(13)
to read off the background metric and antisymmetric tensor field (torsion). We see that (12) gives for \( D = 2 \) the (dual) black hole metric of [7] \((ds^2 = -da\,db/(1 - ab))\). For \( \kappa_i \to 0 \), it reduces as expected to the 2D black hole and for \( \eta_i \to 0 \) gives the 2D black hole times \( D - 2 \) flat coordinates, again as expected since in this limit \( H = SO(1,1) \) is completely embedded in \( SL(2,\mathbb{R}) \). Note that for any \( D \) there is no torsion, in particular the WZ term can be seen to be a total derivative for our choice of gauge. Furthermore we can observe that there are at least \( D - 2 \) isometries since the metric does not depend explicitly on the coordinates \( r_i \). Finally, note that the metric blows up only at the fixed point \( ab = 1 \) which is the point where \( \Lambda \) in (8) vanishes and the classical integration is not justified. The fixed point of the isometry \( g \to hgh \) is at \( ab = 0 \), which we expect to lead to a horizon.

To further analyze this metric, we change to coordinates in which it is diagonal. We consider (as in the 2D case) the regions bounded by the horizon and singularity (fig. 1):

\[
(i) \ 0 < ab < 1 , \quad (ii) \ ab < 0 , \quad (iii) \ ab > 1 .
\]

(14)

(i) corresponds to the interior regions II, III; (ii) to the asymptotic regions I, IV ; and (iii) to the additional regions V,VI.

In the interior regions (i), we can change to coordinates \( t, X_0, X_i \) by defining

\[
a = \sin t \ e^{(X_0 + mX_{D-2})} \\
b = \sin t \ e^{-(X_0 + mX_{D-2})} \\
r_i = N_{ij} X_j ,
\]

(15)

where the coefficients \( N_{ij} \) and \( m \) are given in [11]. In these coordinates the metric takes the diagonal form

\[
ds^2 = \frac{k_0}{2\pi} \left( -dt^2 + \tan^2 t \, dX_0^2 + \sum_{i=1}^{D-2} dX_i^2 \right) .
\]

(16)

Notice that this metric is a particular case of the cosmological solutions found in [12]. The remaining regions are described similarly. For the asymptotic regions (ii), we find

\[
ds^2 = \frac{k_0}{2\pi} \left( dR^2 - \tanh^2 R \ dX_0^2 + \sum_{i=1}^{D-2} dX_i^2 \right) .
\]

(17)
Finally, in the regions (iii) beyond the singularity the metric is

\[ ds^2 = \frac{k_0}{2\pi} \left( dR^2 - \coth^2 R \, dX_0^2 + \sum_{i=1}^{D-2} dX_i^2 \right). \]  

(18)

Using the symmetry \( a \to -a, \ b \to -b \), we identify the geometry (2D black hole) \( \otimes \mathbb{R}^{D-2} \). In particular the isometry generated by \( g \to hgh \) is now explicit (it is a linear combination of translation in \( X_0 \) and the \( X_i \)’s). We can see how the associated Killing vector changes signature on each boundary: it is timelike in (17) and (18) and spacelike in the region (16) in between. In (12), this was not explicit in the \( a, b, r \) coordinates. Although we have chosen a general embedding of \( H = SO(1, 1) \) in all of \( G \), the resulting geometry nonetheless coincides with the case \( \eta_i = 0 \), where \( SO(1, 1) \) was embedded only in \( SL(2, \mathbb{R}) \). This is as expected since the \( SO(1, 1) \) factors in \( G \) are abelian and therefore transform trivially under \( g \to hgh^{-1} \). The spacetime diagram for the relevant 2D geometry was shown in fig. 1.

We now consider the axial gauging. The \( D = 3 \) case has also been discussed in [13]. Under the infinitesimal gauge transformation \( \delta g = \varepsilon (\sigma g + g \sigma) \), we see that \( \delta u = \delta v = 0 \) and \( \delta a = 2\varepsilon q_0 a, \ \delta b = -2\varepsilon q_0 b, \ \delta r_i = 2\varepsilon q_i \). A simple choice that fixes the gauge completely is \( a = \pm b \). Using \( \pm a^2 = 1 - uv \) leaves \( u, v \), and \( r_i \) as the spacetime coordinates. Substituting into (8), and using the above gauge fixing gives the effective action

\[ L = \frac{k_0}{2\pi} \int q^2 \, z \left( \left( \kappa_i \delta_{ij} - \frac{\kappa_i \kappa_j \eta_i \eta_j}{1 - uv + \rho} \right) \partial r_i \partial r_j + \frac{(u \partial v - v \partial u)(u \partial v - v \partial u)}{4(1 - uv + \rho)} \right. \]

\[ - \frac{1}{2} (\partial u \partial v + \partial v \partial u) - \frac{(u \partial v + v \partial u)(u \partial v + v \partial u)}{4(1 - uv)} \]

\[ \left. - \frac{\kappa_i \eta_i}{2(1 - uv + \rho)} [(u \partial v - v \partial u) \partial r_i - \partial r_i (u \partial v - v \partial u)] \right). \]  

(19)

Where \( \rho^2 \equiv \sum \kappa_i \eta_i^2 \). From this expression we can make the following observations. First, unlike the vector gauging, there is nonvanishing torsion given by the term in square brackets in (19), even though the WZ term vanishes for the gauge choice made. We also can see that the metric has singularities at \( uv = 1 \), which in 2D is the fixed point of the axial transformation, and also at \( uv = 1 + \rho \), which is not a fixed point. Again the lines \( uv = 0 \) represent horizons, and the metric and torsion do not depend on the \( r_i \) variables so there are also the \( D - 2 \) isometries \( r_i \to r_i + \text{constant} \). As in the vector case, the \( D = 2 \) \( (\kappa_i = 0) \) limit reproduces the 2D black hole of [7]. Furthermore the \( \eta_i = 0 \) limit gives the
geometry (2D black hole)$\otimes\mathbb{R}^{D-2}$ (with vanishing torsion) as in the vector case, recovering the self-duality of those solutions.

The general case is more conveniently studied via variables that diagonalize the metric in different regions. We will consider the analog of the three regions $(i), (ii), (iii)$ of the vector case (14), but with $a, b$ exchanged for $u, v$.

It is straightforward to see that the same changes of variables made for the vector case also diagonalize this metric, but now for $m = 0$. For $(i) 0 < uv < 1$, we find

$$ds^2 = \frac{k_0}{2\pi} \left( -dt^2 + \frac{1}{(\rho^2 + 1)\tan^2 t + \rho^2} \left( dX_0^2 + \tan^2 t \, dX_{D-2}^2 + \sum_{l=1}^{D-3} dX_l^2 \right) \right),$$

(20)

and the antisymmetric tensor is

$$B_{X_0 X_{D-2}} = \left( (\rho^2 + 1)\tan^2 t + \rho^2 \right)^{-1}.$$  

(21)

With similar expressions for the other regions obtained from analytic continuations as in (16). Again, (20) and (21) admit a cosmological interpretation.

From these metrics we can compute the corresponding curvature scalar in each of the regions and find

$$R = B_{X_0 X_{D-2}} + \text{constant}.$$  

(22)

We see that $R$ blows up only in region $uv > 1$ at the hyperbola $uv = 1 + \rho^2$ which is the real singularity, whereas the surface $uv = 1$ is only a metric singularity where the signature of the metric changes. The latter is another horizon, in addition to $uv = 0$. The geometry is thus $(3D$ black string)$\otimes\mathbb{R}^{D-3}$ with nonvanishing torsion and an inner horizon. The 2D representation ($uv$ diagram) with the eight different regions separated by the horizons and singularity, is presented in fig. 2. It is not surprising that there is a trivial $\mathbb{R}^{D-3}$ crossed on since the high action of $SO(1, 1)$ only acts on one nontrivial linear combination of $SO(1, 1)$ generators of $G$. The expression for the dilaton can be found by considering the correct measure in the path integral, but it is technically simpler to find it by solving the background field equations to lowest order in $\alpha'$. The exact expressions can be found in [11].

The two different spacetime geometries, corresponding to the vector and axial gaugings of the $G/H$ WZW model, can be viewed as different modular invariant combinations of representations of the same holomorphic and anti-holomorphic chiral algebras. There are general arguments [14] that show that the vector and axial gaugings are dual, in the sense

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of having equal partition functions. The duality is similar to the familiar $r \to 1/r$ duality in $c = 1$ conformal field theory where two seemingly different theories are as well related by a changing the sign of the left (or holomorphic) currents $J = J_L$ with respect to the right (or anti-holomorphic) currents $\bar{J} = J_R$. The lorentzian $D = 2$ case is special since the same geometry (2D black hole) is obtained by either the vector or axial gauging, so we say that the model is self-dual. We now point out the sense in which geometries for $D \geq 2$ are dual, placing the vector/axial duality in a more generalized context.

In ref. [15], the $r \to 1/r$ duality of compactified string theories was generalized to any string background for which the worldsheet action has at least one isometry. The worldsheet action for the bosonic string (8) in a background with $N$ commuting isometries, can be written as

$$S = \frac{1}{4\pi\alpha'} \int d^2z \left( Q_{\mu
u}(X_\alpha) \partial X^\mu \partial X^\nu + Q_{\mu n}(X_\alpha) \partial X^\mu \partial X^n + Q_{n\mu}(X_\alpha) \partial X^n \partial X^\mu + Q_{mn}(X_\alpha) \partial X^m \partial X^n + \alpha' R^{(2)}(2) \Phi(X_\alpha) \right),$$

where $Q_{MN} \equiv G_{MN} + B_{MN}$ and lower case latin indices $m, n$ label the isometry directions. Since the Lagrangian (23) depends on $X_m$ only through their derivatives, we can describe it in terms of the first order variables $V^m$ and add an extra term to the action $\hat{X}_m(\partial \bar{V}^m - \bar{\partial} V^m)$ imposing the constraint $V^m = \partial X^m$. Integrating the Lagrange multipliers $\hat{X}_m$ in the above gives back (23). After partial integration and solving for $V^m$ and $\bar{V}^m$, we find
the dual action $S'$ which has the identical form as $S$ with the dual backgrounds given in terms of the original ones by

\begin{align}
Q'_{mn} &= Q^{-1}_{mn} \\
Q'_{\mu\nu} &= Q_{\mu\nu} - Q^{-1}_{mn} Q_{\nu} Q_{\mu m} \\
Q'_{n\mu} &= Q^{-1}_{nm} Q_{m\mu} \\
Q'_{\mu n} &= -Q^{-1}_{mn} Q_{\mu m}.
\end{align}

(24)

To preserve conformal invariance, it can be seen \cite{15,16} that the dilaton field has to transform as $\Phi' = \Phi - \log \sqrt{\det G_{mn}}$. Notice that equations (24) reduce to the usual duality transformations for the toroidal compactifications of \cite{17} in the limit $Q_{m\mu} = Q_{\mu m} = 0$. For the case of a single isometry ($m = n = 0$), we recover the explicit expressions of \cite{15}. This is to our knowledge the most general statement of duality in string theory. In particular we see that a space with no torsion ($Q_{m\mu} = Q_{\mu m}$) can be dual to a space with torsion ($Q'_m = -Q'_n$). Notice that for every isometry we do not have to go to the first order formalism, i.e. we can integrate the Lagrange multipliers $\tilde{X}_m$ for some of the fields and instead the $V^m$ for the remaining fields with isometries. This is the most general form of these duality transformations, and eq. (24) should be read with indices $m, n$ running over only the variables with isometries that have been dualized. The total duality group includes these transformations as well as shifts in the antisymmetric tensor field and has been argued \cite{18}, \cite{19} to be equivalent to $SO(N, N, \mathbb{Z})$.

In string theory, duality symmetry was originally discovered in toroidal compactifications and found to interchange winding and momentum states in the compactified theory. We have seen that the toroidal compactification is a particular case of a $\sigma$–model with isometries and thus has this symmetry manifest. The interchange of winding and momenta states realizes the duality symmetry in this particular background but is not necessarily a generic feature of duality, so we might expect duality even in backgrounds where winding modes are not present. A particular example is given by the 2D Lorentzian black hole.

In order to make a connection between duality in this formulation and the vector–axial duality in $G/H$ WZW models, we just have to identify the correct action of the vector (axial) isometry when gauging the axial (vector) transformation and apply (24) (see also \cite{13}.) It is straightforward to see that regions II of both geometries are interchanged. Region V of fig. 1 is mapped to region I of fig. 2 and in particular the singularity of the first is mapped to one horizon in the second. Also, region I of the vector gauging black hole
gets mapped to regions V and VII together of the axial gauging black hole. This has the interesting implication that a surface which in one geometry is perfectly regular \((ab = \rho^2)\) is mapped to the singularity in the other geometry \((uv = 1 + \rho^2)\). This goes even further than the black hole singularity/horizon duality of the 2D black holes \([8]\), since in that case the horizon is a better behaved region than the singularity but there remains nontrivial behavior such as the exchange of spacelike and timelike coordinates. In the present case it can be seen explicitly that string theory can deal with spacetimes that have singularities at the classical level, in the sense that there still exists a description of interactions, etc. for that region of spacetime by going to the dual geometry!
References

[1] L. Dixon, J. Lykken and M. Peskin, Nucl. Phys. B325 (1989) 325.
[2] I. Bars, Nucl. Phys. B334 (1990) 125.
[3] E. Witten, Comm. Math. Phys. 92 (1984) 455.;
   For a review see P. Goddard and D. Olive, Int. J. Mod. Phys. 1 (1986) 303.
[4] E. Witten, Nucl. Phys. B223 (1983) 422;
   K. Bardacki, E. Rabinovici and B. Saering, Nucl. Phys. B301 (1988) 151;
   D. Karabali and H. J. Schnitzer, Nucl. Phys. B329 (1990) 649, and references therein.
[5] For a review see, J. W. van Holten, Z. Phys. C27:57 (1985).
[6] L. Alvarez-Gaumé and P. Ginsparg, Ann. Phys. 161 (1985) 423.
[7] E. Witten, Phys. Rev. D44 (1991) 314.
[8] R. Dijkgraaf, E. Verlinde, and H. Verlinde, IAS preprint IASSNS-HEP-91/22;
   A. Giveon, Mod. Phys. Lett. A6 (1991) 2843.
[9] S. Helgason, “Differential Geometry, Lie Groups, and Symmetric spaces”,
   Academic Press (1978);
   R. Gilmore, “Lie Groups, Lie Algebras and Some of Their Applications”, Wiley (1974).
[10] I. Bars and D. Nemechansky, Nucl. Phys. B348 (1991) 89.
[11] P. Ginsparg and F. Quevedo, Los Alamos – Neuchâtel preprint LA-UR-92-640,
   NEIP92-001 (hepth@xxx/9202092).
[12] M. Muller, Nucl. Phys. B337 (1990) 37.
[13] J.H. Horne and G. Horowitz, University of California preprint UCSBTH-91-39
   (hepth@xxx/9108001);
   J. Horne, G. Horowitz and A. Steif, University of California preprint UCSBTH-91-53
   (hepth@xxx/9110065); G. Horowitz, these proceedings.
[14] E.B. Kiritsis, Mod. Phys. Lett. A6 (1991) 2871.
[15] T. Buscher, Phys. Lett. 194B (1987) 59; Phys. Lett. 201B (1988) 466.
[16] P. Ginsparg and C. Vafa, Nucl. Phys. B289 (1987) 414;
   E. Alvarez and M. Osorio, Phys. Rev. D40 (1989) 1150.
[17] A. Giveon, E. Rabinovici and G. Veneziano, Nucl. Phys. B322 (1989) 167;
   A. Shapere and F. Wilczek, Nucl. Phys. B320 (1989) 669.
[18] S. Cecotti, S. Ferrara and Girardello, Nucl. Phys. B308 (1988) 436.
[19] M. Duff, talk at Trieste summer 1989, Nucl. Phys. B335 (1990) 610.