ON A LOCALIZED RIEMANNIAN PENROSE INEQUALITY

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Abstract. Let $\Omega$ be a compact, orientable, three dimensional Riemannian manifold with boundary with nonnegative scalar curvature. Suppose its boundary $\partial \Omega$ is the disjoint union of two pieces: $\Sigma_H$ and $\Sigma_O$, where $\Sigma_H$ consists of the unique closed minimal surfaces in $\Omega$ and $\Sigma_O$ is metrically a round sphere. We obtain an inequality relating the area of $\Sigma_H$ to the area and the total mean curvature of $\Sigma_O$. Such an $\Omega$ may be thought as a region, surrounding the outermost apparent horizons of black holes, in a time-symmetric slice of a space-time in the context of general relativity. The inequality we establish has close ties with the Riemannian Penrose Inequality, proved by Huisken and Ilmanen [9] and by Bray [5].

1. Introduction

Let $M$ be a complete, asymptotically flat 3-manifold with nonnegative scalar curvature. Suppose its boundary $\partial M$ consists of the outermost minimal surfaces in $M$. The Riemannian Penrose Inequality, first proved by Huisken and Ilmanen [9] for a connected $\partial M$, and then by Bray [5] for $\partial M$ with any number of components, states that

$$m_{\text{ADM}}(M) \geq \sqrt{\frac{A}{16\pi}},$$

where $m_{\text{ADM}}(M)$ is the ADM mass [1] of $M$ and $A$ is the area of $\partial M$. Furthermore, the equality holds if and only if $M$ is isometric to a spatial Schwarzschild manifold outside its horizon.

Motivated by the quasi-local mass question in general relativity (see [2], [6], [7], etc.), we would like to seek a localized statement of the above inequality [1]. To be precise, we are interested in a compact, orientable, 3-dimensional Riemannian manifold $\Omega$ with boundary. We call $\Omega$ a body surrounding horizons if its boundary $\partial \Omega$ is the disjoint union of two pieces: $\Sigma_O$ (the outer boundary) and $\Sigma_H$ (the horizon boundary), and $\Omega$ satisfies the following assumptions:

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(a) $\Sigma_O$ is topologically a 2-sphere.
(b) Each component of $\Sigma_H$ is a minimal surface in $\Omega$.
(c) There are no other closed minimal surfaces in $\Omega$.

Physically, $\Omega$ is to be thought as a finite region in a time-symmetric slice of a spacetime containing black holes and $\Sigma_H$ corresponds to the outermost apparent horizon of the black holes. In such a context, if the spacetime satisfies the dominant energy condition and $m_{QM}(\Sigma_O)$ represents some quantity (to be defined) which could measure the quasi-local mass of $\Sigma_O$, then one would expect

$$m_{QM}(\Sigma_O) \geq \sqrt{\frac{A}{16\pi}}. \quad (2)$$

In this paper, we are able to establish an equality of the above form for a special class of body surrounding horizons. Our main result is

**Theorem 1.** Let $\Omega$ be a body surrounding horizons whose outer boundary $\Sigma_O$ is metrically a round sphere. Suppose $\Omega$ has nonnegative scalar curvature and $\Sigma_O$ has positive mean curvature. Then

$$m(\Sigma_O) \geq \sqrt{\frac{|\Sigma_H|}{16\pi}}, \quad (3)$$

where $m(\Sigma_O)$ is defined by

$$m(\Sigma_O) = \sqrt{\frac{|\Sigma_O|}{16\pi}} \left[ 1 - \frac{1}{16\pi |\Sigma_O|} \left( \oint_{\Sigma_O} H \, d\sigma \right)^2 \right], \quad (4)$$

where $|\Sigma_H|, |\Sigma_O|$ are the area of $\Sigma_H, \Sigma_O$, $H$ is the mean curvature of $\Sigma_O$ (with respect to the outward normal) in $\Omega$, and $d\sigma$ is the surface measure of the induced metric. When equality holds, $\Sigma_O$ is a surface with constant mean curvature.

We remark that, assuming (3) in Theorem 1 holds in the first place, one can derive (1) in the Riemannian Penrose Inequality. That is because, by a result of Bray [4], to prove (1), one suffices to prove it for a special asymptotically flat manifold $M$ which, outside some compact set $K$, is isometric to a spatial Schwarzschild manifold near infinity. On such an $M$, let $\Omega$ be a compact region containing $K$ such that its outer boundary $\Sigma_O$ is a rotationally symmetric sphere in the Schwarzschild region. Applying Theorem 1 to such an $\Omega$ and observing that, in this case, the quantity $m(\Sigma_O)$ coincides with the Hawking quasi-local mass [8] of $\Sigma_O$, hence agrees with the ADM mass of $M$, we see that (3) implies (1). On the other hand, our proof of Theorem 1 does make critical use of (1). Therefore, (3) and (1) are equivalent.
Besides the Riemannian Penrose Inequality, Theorem 1 is also largely inspired by the following result of Shi and Tam [14]:

**Theorem 2.** (Shi-Tam) Let $\tilde{\Omega}$ be a compact, 3-dimensional Riemannian manifold with boundary with nonnegative scalar curvature. Suppose $\partial \tilde{\Omega}$ has finitely many components $\Sigma_i$ so that each $\Sigma_i$ has positive Gaussian curvature and positive mean curvature $H$ (with respect to the outward normal), then

$$\int_{\Sigma_i} H \, d\sigma \leq \int_{\Sigma_i} H_0 \, d\sigma,$$

where $H_0$ is the mean curvature of $\Sigma_i$ (with respect to the outward normal) when it is isometrically imbedded in $\mathbb{R}^3$. Furthermore, equality holds if and only if $\partial \tilde{\Omega}$ has only one component and $\tilde{\Omega}$ is isometric to a domain in $\mathbb{R}^3$.

Let $\tilde{\Omega}$ be given in Theorem 2. Suppose $\partial \tilde{\Omega}$ has a component $\Sigma$ which is isometric to a round sphere with area $4\pi R^2$, then

$$\int_{\Sigma} H_0 \, d\sigma = 8\pi R,$$

and (5) yields

$$\frac{1}{8\pi} \int_{\Sigma} H \, d\sigma \leq R.$$

Now suppose there is a closed minimal surfaces $\Sigma_h$ in $\tilde{\Omega}$ such that $\Sigma_h$ and $\Sigma$ bounds a region $\Omega$ which contains no other closed minimal surfaces in $\tilde{\Omega}$ (by minimizing area over surfaces homologous to $\Sigma$, such a $\Sigma_h$ always exists if $\partial \tilde{\Omega}$ has more than one components). Applying Theorem 1 to $\Omega$, we have

$$\sqrt{\frac{|\Sigma_h|}{16\pi}} \leq \sqrt{\frac{|\Sigma|}{16\pi}} \left[1 - \frac{1}{16\pi |\Sigma|} \left(\int_{\Sigma} H \, d\sigma\right)^2\right],$$

which can be equivalently written as

$$\frac{1}{8\pi} \int_{\Sigma} H \, d\sigma \leq \sqrt{R(R - R_h)},$$

where $R = \sqrt{\frac{|\Sigma|}{4\pi}}$ and $R_h = \sqrt{\frac{|\Sigma_h|}{4\pi}}$. Therefore, Theorem 1 may be viewed as a refinement of Theorem 2 in this special case to include the effect on $\Sigma$ by the closed minimal surface in $\tilde{\Omega}$ that lies “closest” to $\Sigma$.

In general relativity, Theorem 2 is a statement on the positivity of the Brown-York quasi-local mass $m_{BY}(\partial \tilde{\Omega})$ [6]. Using the technique of weak inverse mean curvature flow developed by Huisken and Ilmanen...
[15], Shi and Tam [15] further proved that $m_{BY}(\partial \tilde{\Omega})$ is bounded from below by the Hawking quasi-local mass $m_H(\partial \tilde{\Omega})$. Suggested by the quantity $m(\Sigma_0)$ in Theorem [1] we find some new geometric quantities associated to $\partial \tilde{\Omega}$, which are interestingly between $m_{BY}(\partial \tilde{\Omega})$ and $m_H(\partial \tilde{\Omega})$ (hence providing another proof of $m_{BY}(\partial \tilde{\Omega}) \geq m_H(\partial \tilde{\Omega})$.) We include this discussion at the end of the paper.

This paper is organized as follows. In Section 2, we review the approach of Shi and Tam in [14] since it plays a key role in our derivation of Theorem 1. The detailed proof of Theorem 1 is given in Section 3. In Section 3.1, we establish a partially generalized Shi-Tam monotonicity. In Section 3.2, we make use of the Riemannian Penrose Inequality. In Section 4, we give some discussion on quasi-local mass. In particular, we introduce two quantities motivated by Theorem 1 and compare them with the Brown-York quasi-local mass $m_{BY}(\Sigma)$ and the Hawking quasi-local mass $m_H(\Sigma)$.

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2. Review of Shi-Tam’s approach

In [14], Shi and Tam pioneered the idea of using results on asymptotically flat manifolds to study compact manifolds with boundary. We briefly review their approach in this section.

Let $\tilde{\Omega}$ be given in Theorem 2. For simplicity, we assume $\partial \tilde{\Omega}$ has only one component $\Sigma$. Since $\Sigma$ has positive Gaussian curvature, $\Sigma$ can be isometrically imbedded in $\mathbb{R}^3$ as a strictly convex surface [11]. On the region $E$ exterior to $\Sigma$, the Euclidean metric $g_0$ can be written as

$$g_0 = d\rho^2 + g_\rho,$$

where $g_\rho$ is the induced metric on each level set $\Sigma_\rho$ of the Euclidean distance function $\rho$ to $\Sigma$. Motivated by the quasi-spherical metric construction of Bartnik [3], Shi and Tam showed that there exists a positive function $u$ defined on $E$ such that the warped metric

$$g_u = u^2 d\rho^2 + g_\rho$$

has zero scalar curvature, is asymptotically flat and the mean curvature of $\Sigma$ in $(E, g_u)$ (with respect to the $\infty$-pointing normal) agrees with the mean curvature of $\Sigma$ in $\Omega$. Furthermore, as a key ingredient to prove their result, they showed that the quantity

$$\int_{\Sigma_\rho} (H_0 - H_u) \, d\sigma$$
is monotone non-increasing in $\rho$, and

$$\lim_{\rho \to \infty} \oint_{\Sigma_\rho} (H_0 - H_u) \, d\sigma = 8\pi m_{\text{ADM}}(g_u),$$

where $H_0$, $H_u$ are the mean curvature of $\Sigma_\rho$ with respect to $g_0$, $g_u$, and $m_{\text{ADM}}(g_u)$ is the ADM mass of $g_u$. Let $M$ be the Riemannian manifold obtained by gluing $(E, g^m_u)$ to $\tilde{\Omega}$ along $\Sigma$. The metric on $M$ is asymptotically flat, has nonnegative scalar curvature away from $\Sigma$, is Lipschitz near $\Sigma$, and the mean curvatures of $\Sigma$ computed in both sides of $\Sigma$ in $M$ (with respect to the $\infty$-pointing normal) are the same. By generalizing Witten’s spinor argument [16], Shi and Tam proved that the positive mass theorem [13] [16] remains valid on $M$ (see [10] for a non-spinor proof). Therefore,

$$\oint_{\Sigma} (H_0 - H_u) \, d\sigma \geq \lim_{\rho \to \infty} \oint_{\Sigma_\rho} (H_0 - H_u) \, d\sigma = 8\pi m_{\text{ADM}}(g_u) \geq 0,$$

with $\oint_{\Sigma} (H_0 - H) \, d\sigma = 0$ if and only if $H = H_0$ and $\tilde{\Omega}$ is isometric to a domain in $\mathbb{R}^3$.

3. Proof of Theorem

We are now in a position to prove Theorem 1. The basic idea is to deform the exterior region of a rotationally symmetric sphere in a spatial Schwarzschild manifold in a similar way as Shi and Tam did on $\mathbb{R}^3$, then attach it to a body surrounding horizons and apply the Riemannian Penrose Inequality to the gluing manifold. The key ingredient in our proof is the discovery of a new monotone quantity associated to the deformed metric. We divide the proof into two subsections.

3.1. A monotonicity property for quasi-spherical metrics on a Schwarzschild background. Consider part of a spatial Schwarzschild manifold

$$(M^m_0, g^m) = \left(S^2 \times [r_0, \infty), \frac{1}{1 - \frac{2m}{r}} dr^2 + r^2 d\sigma^2\right),$$

where $r_0$ is a constant chosen to satisfy $\left\{ \begin{array}{ll} r_0 > 2m, & \text{if } m \geq 0 \\ r_0 > 0, & \text{if } m < 0. \end{array} \right.$ Here $m$ is the ADM mass of the Schwarzschild metric $g^m$, $r$ is the radial coordinate on $[r_0, \infty)$, and $d\sigma^2$ denotes the standard metric on the unit sphere $S^2 \subset \mathbb{R}^3$.

Let $N$ be the positive function on $M^m_0$ defined by

$$N = \sqrt{1 - \frac{2m}{r}}.$$
In terms of $N$, $g^m$ takes the form
\[(17) \quad g^m = \frac{1}{N^2} dr^2 + r^2 d\sigma^2.\]

The next lemma follows directly from the existence theory established in \[14\] (see also \[3\]).

**Lemma 1.** Let $\Sigma_0$ be the boundary of $(M^m_{r_0}, g^m)$. Given any positive function $\phi$ on $\Sigma_0$, there exists a positive function $u$ on $M^m_{r_0}$ such that

(i) The metric
\[(18) \quad g_u^m = \left(\frac{u}{N}\right)^2 dr^2 + r^2 d\sigma^2\]
has zero scalar curvature and is asymptotically flat.

(ii) The mean curvature of $\Sigma_0$ (with respect to the $\infty$-pointing normal) in $(M^m_{r_0}, g_u^m)$ is equal to $\phi$.

(iii) The quotient $\frac{u}{N}$ has the asymptotic expansion
\[(19) \quad \frac{u}{N} = 1 + \frac{m_0}{r} + O\left(\frac{1}{r^2}\right) \text{ as } r \to \infty,\]
where $m_0$ is the ADM mass of $g_u^m$.

**Proof.** Consider a Euclidean background metric
\[(20) \quad ds^2 = dr^2 + r^2 d\sigma^2\]
on $M^m_{r_0} = S^2 \times [r_0, \infty)$. By Theorem 2.1 in \[14\], there is a unique positive function $v$ on $M^m_{r_0}$ such that
\[(21) \quad g_v = v^2 dr^2 + r^2 d\sigma^2\]
has zero scalar curvature, is asymptotically flat and the mean curvature of $\Sigma_0$ in $(M^m_{r_0}, g_v)$ is given by $\phi$. Furthermore, $v$ has an asymptotic expansion
\[(22) \quad v = 1 + \frac{m_0}{r} + O\left(\frac{1}{r^2}\right),\]
where $m_0$ is the ADM mass of $g_v$. Let $u = Nv$, Lemma 1 is proved. $\square$

We note that metrics of the form $v^2 dr^2 + r^2 d\sigma^2$ are called (shear free) quasi-spherical metrics \[3\]. By the formula (2.26) in \[3\] (or (1.10) in \[14\]), the differential equation satisfied by $v = uN^{-1}$ in Lemma 1 is
\[(23) \quad \frac{2}{r} \frac{\partial v}{\partial r} = \frac{v^2}{r^2} \Delta_{S^2} v + \frac{(v-v^3)}{r^2},\]
where $\Delta_{S^2}$ denotes the Laplacian operator of the metric $d\sigma^2$ on $S^2$. 
Proposition 1. Let $u, g_m^m, m_0$ be given in Lemma 1. Let $\Sigma_r$ be the radial coordinate sphere in $M^{m}_{r_0}$, i.e. $\Sigma_r = S^2 \times \{r\}$. Let $H_S, H_u$ be the mean curvature of $\Sigma_r$ with respect to the metric $g^m, g_m^m$. Then
\[
\oint_{\Sigma_r} N(H_S - H_u) \, d\sigma
\]
is monotone non-increasing in $r$. Furthermore,
\[
limit_{r \to \infty} \oint_{\Sigma_r} N(H_S - H_u) \, d\sigma = 8\pi(m_0 - m).
\]

Proof. We have $H_S = \frac{2}{r}N$ and $H_u = \frac{2}{r}v^{-1}$, where $v = uN^{-1}$. Hence
\[
\oint_{\Sigma_r} N(H_S - H_u) \, d\sigma = \oint_{\Sigma_r} \left( \frac{2}{r} \right) (N^2 - Nv^{-1}) \, d\sigma
\]
(25)
\[
= \oint_{S^2} 2r(N^2 - Nv^{-1}) \, d\omega,
\]
where $d\omega = r^{-2} d\sigma$ is the surface measure of $d\sigma$ on $S^2$. As $N^2 = 1 - \frac{2m}{r}$, we have
\[
\oint_{\Sigma_r} N(H_S - H_u) \, d\sigma = \oint_{S^2} (2r - 4m - 2rNv^{-1}) \, d\omega.
\]
(26)

Therefore,
\[
\frac{d}{dr} \oint_{\Sigma_r} N(H_S - H_u) \, d\sigma = \oint_{S^2} \left[ (2 - 2Nv^{-1}) - 2r \frac{\partial N}{\partial r} v^{-1} \right] \, d\omega
\]
\[
+ \oint_{S^2} 2rNv^{-2} \frac{\partial v}{\partial r} \, d\omega.
\]
(27)

By (23), we have
\[
v^{-2} \frac{\partial v}{\partial r} = \frac{1}{2r} \Delta_{S^2} v + \left( \frac{v^{-1} - v}{2r} \right).
\]
(28)

Thus the last term in (27) becomes
\[
\oint_{S^2} 2rNv^{-2} \frac{\partial v}{\partial r} \, d\omega = \oint_{S^2} N\Delta_{S^2} v \, d\omega + \oint_{S^2} N(v^{-1} - v) \, d\omega
\]
\[
= \oint_{S^2} N(v^{-1} - v) \, d\omega,
\]
(29)

where we have used the fact that $N$ is a constant on each $\Sigma_r$ and $\oint_{S^2} \Delta_{S^2} v \, d\omega = 0$. Hence the right side of (27) is given by
\[
\oint_{S^2} \left[ (2 - 2Nv^{-1}) - 2r \frac{\partial N}{\partial r} v^{-1} + N(v^{-1} - v) \right] \, d\omega.
\]
(30)
Replace $v$ by $uN^{-1}$, the integrand of (30) becomes

\[(31) \quad 2 - N^2 u^{-1} - 2r \frac{\partial N}{\partial r} Nu^{-1} - u.\]

By (16), we have

\[(32) \quad N^2 + 2r \frac{\partial N}{\partial r} = 1.\]

Therefore, it follows from (27), (30), (31) and (32) that

\[(33) \quad \frac{d}{dr} \oint_{\Sigma_r} N(H_S - H_u) \, d\sigma = - \oint_{S^2} u^{-1} (u - 1)^2 \, d\omega,\]

which proves that $\oint_{\Sigma_r} N(H_S - H_u) \, d\sigma$ is monotone non-increasing in $r$.

To evaluate $\lim_{r \to \infty} \oint_{\Sigma_r} N(H_S - H_u) \, d\sigma$, we have

\[(34) \quad N v^{-1} = 1 - \frac{(m_0 + m)}{r} + O \left( \frac{1}{r^2} \right)\]

by (16) and (22). Therefore, by (26) we have

\[(35) \quad \oint_{\Sigma_r} N(H_S - H_u) \, d\sigma = \oint_{S^2} 2(m_0 - m) \, d\omega + O(r^{-1}),\]

which implies

\[(36) \quad \lim_{r \to \infty} \oint_{\Sigma_r} N(H_S - H_u) \, d\sigma = 8\pi(m_0 - m).\]

Proposition 1 is proved. \hfill \square

### 3.2. Application of the Riemannian Penrose Inequality

In this section, we glue a body surrounding horizons, whose outer boundary is metrically a round sphere, to an asymptotically flat manifold $(\mathcal{M}^m_{\infty}, g^m_u)$ constructed in Lemma 1, and apply the Riemannian Penrose Inequality and Proposition 1 to prove Theorem 1.

We start with the following lemma.

**Lemma 2.** Let $\Omega$ be a body surrounding horizons. Suppose its outer boundary $\Sigma_O$ has positive mean curvature, then its horizon boundary $\Sigma_H$ strictly minimizes area among all closed surfaces in $\Omega$ that enclose $\Sigma_H$.

**Proof.** As $\Omega$ is compact and the mean curvature vector of $\Sigma_O$ points into $\Omega$, it follows from the standard geometric measure theory that there exist surfaces that minimize area among all closed surfaces in $\Omega$ that enclose $\Sigma_H$, furthermore none of the minimizers touches $\Sigma_O$. Let $\Sigma$ be any such a minimizer. By the Regularity Theorem 1.3 in [9], $\Sigma$ is a $C^{1,1}$ surface, and is $C^\infty$ where it does not touch $\Sigma_H$; moreover, the
mean curvature of $\Sigma$ is 0 on $\Sigma \setminus \Sigma_H$ and equals the mean curvature of $\Sigma_H$ $\mathcal{H}^2$-a.e. on $\Sigma \cap \Sigma_H$. Suppose $\Sigma$ is not identically $\Sigma_H$. As $\Sigma_H$ has zero mean curvature, the maximum principle implies that $\Sigma$ does not touch $\Sigma_H$. Hence, $\Sigma$ is a smooth closed minimal surface in the interior of $\Omega$, contradicting the assumption that $\Omega$ has no other closed minimal surfaces except $\Sigma_H$. Therefore, $\Sigma$ must be identically $\Sigma_H$. \hfill \Box

Let $\Omega$ be a body surrounding horizons given in Theorem 1. Let $R$ and $R_H$ be the area radii of $\Sigma_O$ and $\Sigma_H$, which are defined by

$$4\pi R^2 = |\Sigma_O| \quad \text{and} \quad 4\pi R_H^2 = |\Sigma_H|. \quad (37)$$

It follows from Lemma 2 that $R > R_H$. To proceed, we choose $(M^m, g^m)$ to be one-half of a spatial Schwarzschild manifold whose horizon has the same area as $\Sigma_H$, i.e.

$$(M^m, g^m) = \left( S^2 \times [R_H, \infty), \frac{1}{1 - \frac{2m}{r}} d\tau^2 + r^2 d\sigma^2 \right), \quad (38)$$

where $m$ is chosen to satisfy $2m = R_H$. As $R > R_H$, $\Sigma_O$ can be isometrically imbedded in $(M^m, g^m)$ as the coordinate sphere

$$\Sigma_R = \{ r = R \}. \quad (39)$$

Henceforth, we identify $\Sigma_O$ with $\Sigma_R$ through this isometric imbedding. Let $M^m_o$ denote the exterior of $\Sigma_O$ in $M^m$. By Lemma 1 and Proposition 1, there exists a metric

$$g_u^m = \left( \frac{u}{N} \right)^2 d\tau^2 + r^2 d\sigma^2 \quad (40)$$

on $M^m_o$ such that $g_u^m$ has zero scalar curvature, is asymptotically flat, and the mean curvature of $\Sigma_O$ (with respect to the $\infty$-pointing normal) in $(M^m_o, g_u^m)$ agrees with $H$, the mean curvature of $\Sigma_O$ in $\Omega$. Furthermore, the integral

$$\int_{\Sigma_r} N (H_S - H_u) \, d\sigma \quad (41)$$

is monotone non-increasing in $r$ and converges to $8\pi (m_0 - m)$ as $r \to \infty$, where $m_0$ is the ADM mass of $g_u^m$.

Now we attach this asymptotically flat manifold $(M_o^m, g_u^m)$ to the compact body $\Omega$ along $\Sigma_O$ to get a complete Riemannian manifold $M$ whose boundary is $\Sigma_H$. The resulting metric $g_M$ on $M$ satisfies the properties that it is asymptotically flat, has nonnegative scalar curvature away from $\Sigma_O$, is Lipschitz near $\Sigma_O$, and the mean curvatures of $\Sigma_O$ computed in both sides of $\Sigma_O$ in $M$ (with respect to the $\infty$-pointing normal) agree identically.
Lemma 3. The horizon boundary $\Sigma_H$ is strictly outer minimizing in $M$, i.e. $\Sigma_H$ strictly minimizes area among all closed surfaces in $M$ that enclose $\Sigma_H$.

Proof. By the construction of $g_u^m$, we know $(M^m_o, g_u^m)$ is foliated by $\{\Sigma_r\}_{r \geq R}$, where each $\Sigma_r$ has positive mean curvature. Let $\Sigma$ be a surface that minimizes area among surfaces in $M$ that encloses $\Sigma_H$ (such a minimizer exists as $M$ is asymptotically flat). We claim that $\Sigma \setminus \Omega$ must be empty, for otherwise $\Sigma \setminus \Omega$ would be a smooth, compact minimal surface in $(M^m_o, g_u^m)$ with boundary lying in $\Sigma_O$, and that would contradict the maximum principle. Therefore, $\Sigma \subset \Omega$. It then follows from Lemma 2 that $\Sigma = \Sigma_H$. \hfill $\square$

The next lemma is an application of the “corner smoothing” technique in [10].

Lemma 4. There exists a sequence of smooth asymptotically flat metrics $\{h_k\}$ defined on the background manifold of $M$ such that $\{h_k\}$ converges uniformly to $g_M$ in the $C^0$ topology, each $h_k$ has nonnegative scalar curvature, $\Sigma_H$ has zero mean curvature with respect to each $h_k$ (in fact $\Sigma_H$ can be made totally geodesic w.r.t $h_k$), and the ADM mass of $h_k$ converges to the ADM mass of $g_M$.

Proof. Let $M'$ be an exact copy of $M$. We glue $M$ and $M'$ along their common boundary $\Sigma_H$ to get a Riemannian manifold $\tilde{M}$ with two asymptotic ends. Let $g_{\tilde{M}}$ be the resulting metric on $\tilde{M}$ and let $\Sigma'_O$ be the copy of $\Sigma_O$ in $M'$. Denote by $\Sigma$ the union of $\Sigma_O$, $\Sigma_H$ and $\Sigma'_O$, we then know that the mean curvatures of $\Sigma$ computed in both sides of $\Sigma$ in $\tilde{M}$ (with respect to normal vectors pointing to the same end of $\tilde{M}$) agree. (At $\Sigma_O$ and $\Sigma'_O$, this is guaranteed by the construction of $g_{\tilde{M}}$, and at $\Sigma_H$, this is provided by the fact that $\Sigma_H$ has zero mean curvature.)

Apply Proposition 3.1 in [10] to $\tilde{M}$ at $\Sigma$, followed by a conformal deformation as described in Section 4.1 in [10]. we get a sequence of smooth asymptotically flat metrics $\{g_k\}$, defined on the background manifold of $\tilde{M}$, with nonnegative scalar curvature such that $\{g_k\}$ converges uniformly to $g_{\tilde{M}}$ in the $C^0$ topology and the ADM mass of $g_k$ converges to the ADM mass of $g_{\tilde{M}}$ on both ends of $\tilde{M}$. Furthermore, as $\tilde{M}$ has a reflection isometry (which maps a point $x \in M$ to its copy in $M'$), detailed checking of the construction in Section 3 in [10] shows that $\{g_k\}$ can be produced in such a way that each $g_k$ also has the same reflection isometry. (Precisely, this can be achieved by choosing the mollifier $\phi(t)$ in equation (8) in [10] and the cut-off function $\sigma(t)$ in equation (9) in [10] to be both even functions.) Therefore, if we let
Let \( M_k \) be the Riemannian manifold obtained by replacing the metric \( g_{,\mathcal{M}} \) by \( g_k \) on \( M \), then \( \Sigma_H \) remains a surface with zero mean curvature in \( M_k \) (in fact \( \Sigma_H \) is totally geodesic). Define \( h_k \) to be the restriction of \( g_k \) to the background manifold of \( M \), Lemma 3 is proved.

We continue with the proof of Theorem 1. Let \( \{h_k\} \) be the metric approximation of \( g_M \) provided in Lemma 4. Let \( M_k \) be the asymptotically flat manifold obtained by replacing the metric \( g_{,\mathcal{M}} \) on \( M \) by \( h_k \). For any surface \( \tilde{\Sigma} \) in \( M \), let \( |\tilde{\Sigma}|_{k}, |\tilde{\Sigma}| \) be the area of \( \tilde{\Sigma} \) w.r.t. the induced metric from \( h_k \), \( g_{,\mathcal{M}} \) respectively. We cannot apply the Riemannian Penrose Inequality directly to claim \( m_{ADM}(h_k) \geq \sqrt{\frac{|\Sigma_H|}{16\pi}} \). That is because we do not know if \( \Sigma_H \) remains to be the outermost minimal surface in \( M_k \). However, since \( \Sigma_H \) is a minimal surface in \( M_k \), we know the outermost minimal surface in \( M_k \), denoted by \( \Sigma_k \), exists and its area satisfies

\[
|\Sigma_k|_k = \inf\{|\tilde{\Sigma}|_k \mid \tilde{\Sigma} \in \mathcal{S}\}
\]

where \( \mathcal{S} \) is the set of closed surfaces \( \tilde{\Sigma} \) in \( M \) that enclose \( \Sigma_H \) (see [5], [9]). By the Riemannian Penrose Inequality (Theorem 1 in [5]), we have

\[
m_{ADM}(h_k) \geq \sqrt{\frac{|\Sigma_k|_k}{16\pi}}
\]

Let \( k \) approach infinity, we have

\[
\lim_{k \to \infty} m_{ADM}(h_k) = m_{ADM}(g_M),
\]

and

\[
\lim_{k \to \infty} |\Sigma_k|_k = \inf\{|\tilde{\Sigma}| \mid \tilde{\Sigma} \in \mathcal{S}\}
\]

where we have used (12) and the fact that \( \{h_k\} \) converges uniformly to \( g_M \) in the \( C^0 \) topology. By Lemma 3, we also have

\[
|\Sigma_H| = \inf\{|\tilde{\Sigma}| \mid \tilde{\Sigma} \in \mathcal{S}\}.
\]

Therefore, it follows from (43), (44), (45) and (46) that

\[
m_{ADM}(g_M) \geq \sqrt{\frac{|\Sigma_H|}{16\pi}}.
\]

To finish the proof of Theorem 1, we make use of the monotonicity of the integral

\[
\int_{\Sigma_r} N(H_s - H_u) \, d\sigma.
\]
By Proposition 1, we have
\[
\oint_{\Sigma_{O}} N(H_S - H_u) \, d\sigma \geq \lim_{r \to \infty} \oint_{\Sigma_{r}} N(H_S - H_u) \, d\sigma = 8\pi (m_0 - m).
\]
(49)

On the other hand, we know
\[
m_0 = m_{ADM}(g^m_u) = m_{ADM}(g_M),
\]
and
\[
m = \frac{1}{2} R = \sqrt{\frac{|\Sigma_H|}{16\pi}}.
\]
(51)

Therefore, it follows from (49), (50), (51) and (47) that
\[
\oint_{\Sigma_{O}} N(H_S - H_u) \, d\sigma \geq 0.
\]
(52)

Plug in \(H_S = \frac{2}{R} N, H_u = H \) and \(N = \sqrt{1 - \frac{R_H}{R}}, \) we then have
\[
8\pi R \sqrt{1 - \frac{R_H}{R}} \geq \oint_{\Sigma_{O}} H \, d\sigma.
\]
(53)

Direct computation shows that (53) is equivalent to (3). Hence, (3) is proved.

Finally, when the equality in (3) holds, we have
\[
\oint_{\Sigma_{r}} N(H_S - H_u) \, d\sigma = 0, \quad \forall \ r \geq R.
\]
(54)

By the derivative formula (33), \(u \) is identically 1 on \(M^m_m. \) Therefore, the metric \(g^m_u \) is indeed the Schwarzschild metric \(g^m. \) Since the mean curvature of \(\Sigma_O \) in \((M^m_m, g^m_u)\) was arranged to equal \(H, \) the mean curvature of \(\Sigma_O \) in \(\Omega, \) we conclude that \(H = \frac{2}{R} \left(1 - \frac{R_H}{R}\right)^{\frac{3}{2}}, \) which is a constant. Theorem 1 is proved.

Comparing to the equality case in Theorem 2, one would expect that the equality in (3) holds if and only if \(\Omega \) is isometric to a region, in a spatial Schwarzschild manifold, which is bounded by a rotationally symmetric sphere and the Schwarzschild horizon. We believe that this is true, but are not able to prove it at this stage. A confirmation of this expectation seems to require a good knowledge of the behavior of a sequence of asymptotically flat 3-manifolds with controlled \(C^0\)-geometry, on which the equality of the Riemannian Penrose Inequality is nearly satisfied. We leave this as an open question.
4. Some discussion

Let $\Sigma$ be an arbitrary closed 2-surface in a general 3-manifold $M$ with nonnegative scalar curvature. Consider the quantity

$$m(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left[ 1 - \frac{1}{16\pi|\Sigma|} \left( \oint_{\Sigma} H \right)^2 \right]$$

where $|\Sigma|$ is the area of $\Sigma$, $H$ is the mean curvature of $\Sigma$ in $M$ and we omit the surface measure $d\sigma$ in the integral. Theorem 1 suggests that, if $\Sigma$ is metrically a round sphere, $m(\Sigma)$ may potentially agree with a hidden definition of quasi-local mass of $\Sigma$. Such a speculation could be further strengthened by the resemblance between $m(\Sigma)$ and the Hawking quasi-local mass $m_H(\Sigma)$

$$m_H(\Sigma) = \sqrt{\frac{|\Sigma|}{16\pi}} \left[ 1 - \frac{1}{16\pi} \oint_{\Sigma} H^2 \right].$$

By Hölder’s inequality, we have

$$m(\Sigma) \geq m_H(\Sigma)$$

for any surface $\Sigma$. On the other hand, if $\Sigma$ is a closed convex surface in the Euclidean space $\mathbb{R}^3$, the classic Minkowski inequality implies that $m(\Sigma) \leq 0$ and $m(\Sigma) = 0$ if and only if $\Sigma$ is a round sphere in $\mathbb{R}^3$. Therefore, even though bigger than $m_H(\Sigma)$, $m(\Sigma)$ shares the same character as $m_H(\Sigma)$ that it is negative on most convex surfaces in $\mathbb{R}^3$.

In order to gain positivity and to maintain the same numerical value on metrically round spheres, we propose to modify $m(\Sigma)$ in a similar way as the Brown-York mass $m_{BY}(\Sigma)$ is defined. Recall that, for those $\Sigma$ with positive Gaussian curvature, $m_{BY}(\Sigma)$ is defined to be

$$m_{BY}(\Sigma) = \frac{1}{8\pi} \left( \oint_{\Sigma} H_0 \, d\sigma - \oint_{\Sigma} H \, d\sigma \right)$$

where $H_0$ is the mean curvature of $\Sigma$ when it is isometrically embedded in $\mathbb{R}^3$. Now suppose $\Sigma$ is metrically a round sphere, then

$$\left( \oint_{\Sigma} H_0 \right)^2 = 16\pi|\Sigma|. $$
In this case, we can re-write \( m(\Sigma) \) as either

\[
m(\Sigma) = \sqrt{\frac{1}{16\pi}} \left[ 1 - \left( \frac{f_{\Sigma} H}{f_{\Sigma} H_0} \right)^2 \right]
\]

or

\[
m(\Sigma) = \frac{1}{16\pi} \left( \oint_{\Sigma} H_0 \right) \left[ 1 - \left( \frac{f_{\Sigma} H}{f_{\Sigma} H_0} \right)^2 \right].
\]

This motivates us to consider the following two quantities:

**Definition 1.** For any \( \Sigma \) with positive Gaussian curvature, define

\[
m_1(\Sigma) = \sqrt{\frac{1}{16\pi}} \left[ 1 - \left( \frac{f_{\Sigma} H}{f_{\Sigma} H_0} \right)^2 \right],
\]

and

\[
m_2(\Sigma) = \frac{1}{16\pi} \left( \oint_{\Sigma} H_0 \right) \left[ 1 - \left( \frac{f_{\Sigma} H}{f_{\Sigma} H_0} \right)^2 \right],
\]

where \( H \) is the mean curvature of \( \Sigma \) in \( M \) and \( H_0 \) is the mean curvature of \( \Sigma \) when it is isometrically embedded in \( \mathbb{R}^3 \).

The following result compares \( m_H(\Sigma), m_1(\Sigma), m_2(\Sigma) \) and \( m_{BY}(\Sigma) \).

**Theorem 3.** Suppose \( \Sigma \) is a closed 2-surface with positive Gaussian curvature in a 3-manifold \( M \). Then

(i) \( m_1(\Sigma) \geq m_H(\Sigma) \), and equality holds if and only if \( \Sigma \) is metrically a round sphere and \( \Sigma \) has constant mean curvature.

(ii) \( m_{BY}(\Sigma) \geq m_2(\Sigma) \), and equality holds if and only if \( \oint_{\Sigma} H_0 \, d\sigma = \oint_{\Sigma} H \, d\sigma \).

(iii) Suppose \( \Sigma \) bounds a domain \( \Omega \) with nonnegative scalar curvature and the mean curvature of \( \Sigma \) in \( \Omega \) is positive, then

\[
m_2(\Sigma) \geq m_1(\Sigma) \geq 0.
\]

Moreover, \( m_1(\Sigma) = 0 \) if and only if \( \Omega \) is isometric to a domain in \( \mathbb{R}^3 \), and \( m_2(\Sigma) = m_1(\Sigma) \) if and only if either \( \Omega \) is isometric to a domain in \( \mathbb{R}^3 \) in which case \( m_2(\Sigma) = m_1(\Sigma) = 0 \) or \( \Sigma \) is metrically a round sphere.

**Proof.** (i) Let \( m(\Sigma) \) be defined as in (55). By the Minkowski inequality (58), we have \( m_1(\Sigma) \geq m(\Sigma) \). By (57), we have \( m(\Sigma) \geq m_H(\Sigma) \). Therefore, \( m_1(\Sigma) \geq m_H(\Sigma) \) and equality holds if and only if \( \Sigma \) is metrically a round sphere and the mean curvature of \( \Sigma \) in \( M \) is a constant.
(ii) This case is elementary. Let \( a = \oint_{\Sigma} H \) and \( b = \oint_{\Sigma} H_0 \). Then (ii) is equivalent to the inequality \((1 - \frac{a}{b})^2 \geq 0\).

(iii) By the result of Shi and Tam \([14]\), i.e. Theorem 2, we have

\[(65) \quad 1 - \left( \frac{\oint_{\Sigma} H}{\oint_{\Sigma} H_0} \right)^2 \geq 0\]

with equality holding if and only if \( \Omega \) is isometric to a domain in \( \mathbb{R}^3 \). (iii) now follows directly from (65) and the Minkowski inequality (58). □

Suppose \( \Omega \) is a compact 3-manifold with boundary with nonnegative scalar curvature and its boundary \( \partial \Omega \) has positive Gaussian curvature and positive mean curvature. Theorem 3 implies that

\[(66) \quad m_{BY}(\partial \Omega) \geq m_2(\partial \Omega) \geq m_1(\partial \Omega) \geq m_H(\partial \Omega)\]

with \( m_1(\partial \Omega) \geq 0 \) and \( m_{BY}(\partial \Omega) = m_H(\partial \Omega) \) if and only if \( \Omega \) is isometric to a round ball in \( \mathbb{R}^3 \). This provides a slight generalization of a previous result of Shi and Tam (Theorem 3.1 (b) in \([15]\)), which showed \( m_{BY}(\partial \Omega) \geq m_H(\partial \Omega) \).

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