A FOCAL SUBGROUP THEOREM FOR OUTER COMMUTATOR WORDS

CRISTINA ACCIARRI, GUSTAVO A. FERNÁNDEZ-ALCOBER, AND PAVEL SHUMYATSKY

Abstract. Let $G$ be a finite group of order $p^a m$, where $p$ is a prime and $m$ is not divisible by $p$, and let $P$ be a Sylow $p$-subgroup of $G$. If $w$ is an outer commutator word, we prove that $P \cap w(G)$ is generated by the intersection of $P$ with the set of $m$th powers of all values of $w$ in $G$.

Let $G$ be a finite group and $P$ a Sylow $p$-subgroup of $G$. The Focal Subgroup Theorem states that $P \cap G'$ is generated by the set of commutators $\{[g, z] : g \in G, z \in P, [g, z] \in P\}$. This was proved by Higman in 1953 [4]. Nowadays the proof of the theorem can be found in many standard books on group theory (for example, Rose’s book [7] or Gorenstein’s [3]).

One immediate corollary is that $P \cap G'$ can be generated by commutators lying in $P$. Of course, $G'$ is the verbal subgroup of $G$ corresponding to the group word $[x, y] = x^{-1}y^{-1}xy$. It is natural to ask the question on generation of Sylow subgroups for other words. More specifically, if $w$ is a group word we write $G_w$ for the set of values of $w$ in $G$ and $w(G)$ for the subgroup generated by $G_w$ (which is called the verbal subgroup of $w$ in $G$), and one is tempted to ask the following question.

**Question.** Given a Sylow $p$-subgroup $P$ of a finite group $G$, is it true that $P \cap w(G)$ can be generated by $w$-values lying in $P$, i.e., that $P \cap w(G) = \langle P \cap G_w \rangle$?

However considering the case where $G$ is the non-abelian group of order six, $w = x^3$ and $p = 3$ we quickly see that the answer to the above question is negative. Therefore we concentrate on the case where $w$ is a commutator word. Recall that a group word is commutator if the sum of the exponents of any indeterminate involved in it is zero. Thus, we deal with the question whether $P \cap w(G)$ can be generated by $w$-values whenever $w$ is a commutator word.

The main result of this paper is a contribution towards a positive answer to this question: we prove that if $w$ is an outer commutator word, then $P \cap w(G)$ can be generated by the powers of values of $w$ which lie in $P$. More precisely, we have the following result.

**Theorem A.** Let $G$ be a finite group of order $p^a m$, where $p$ is a prime and $m$ is not divisible by $p$, and let $P$ be a Sylow $p$-subgroup of $G$. If $w$ is
an outer commutator word, then $P \cap w(G)$ is generated by $m$th powers of $w$-values, i.e., $P \cap w(G) = \langle P \cap G_{w,m} \rangle$.

Recall that an outer commutator word is a word which is obtained by nesting commutators, but using always different indeterminates. Thus the word $[x_1, x_2], [x_3, x_4, x_5], x_6$ is an outer commutator while the Engel word $[x_1, x_2, x_2]$ is not. An important family of outer commutator words are the simple commutators $\gamma_i$, given by

$$\gamma_1 = x_1, \quad \gamma_i = [\gamma_{i-1}, x_i] = [x_1, \ldots, x_i], \text{ for } i \geq 2.$$  

The corresponding verbal subgroups $\gamma_i(G)$ are the terms of the lower central series of $G$. Another distinguished sequence of outer commutator words are the derived words $\delta_i$, on $2^i$ indeterminates, which are defined recursively by

$$\delta_0 = x_1, \quad \delta_i = [\delta_{i-1}(x_1, \ldots, x_{2^{i-1}}), \delta_{i-1}(x_{2^{i-1}+1}, \ldots, x_{2^i})], \text{ for } i \geq 1.$$  

Then $\delta_i(G) = G^{(i)}$, the $i$th derived subgroup of $G$.

Some of the ideas behind the proof of Theorem A were anticipated already in [3] where somewhat similar arguments, due to Guralnick, led to a result on generation of a Sylow $p$-subgroup of $G'$ for a finite group $G$ admitting a coprime group of automorphisms. Later the arguments were refined in [1]. In both papers [4] and [1] the results on generation of Sylow subgroups were used to reduce a problem about finite groups to the case of nilpotent groups. It is hoped that also our Theorem A will play a similar role in the subsequent projects.

Another important tool used in the proof of Theorem A is the famous result of Liebeck, O’Brien, Shalev and Tiep [6] that every element of a nonabelian simple group is a commutator. The result proved Ore’s conjecture thus solving a long-standing problem. In turn, the proof in [6] uses the classification of finite simple groups as well as many other sophisticated tools.

1. Preliminaries

If $X$ and $Y$ are two subsets of a group $G$, and $N$ is a normal subgroup of $G$, it is not always the case that $XN \cap YN = (X \cap Y)N$, i.e., that $\overline{X \cap Y} = \overline{X} \cap \overline{Y}$ in the quotient group $\overline{G} = G/N$. In our first lemma we have a situation in which this property holds, and which will be of importance in the sequel.

**Lemma 1.1.** Let $G$ be a finite group, and let $N$ be a normal subgroup of $G$. If $P$ is a Sylow $p$-subgroup of $G$ and $X$ is a normal subset of $G$ consisting of $p$-elements, then $XN \cap PN = (X \cap P)N$. In other words, if we use the bar notation in $G/N$, we have $\overline{X \cap P} = \overline{X} \cap \overline{P}$.

**Proof.** We only need to worry about the inclusion $\overline{X \cap P} \subseteq \overline{X} \cap \overline{P}$. So, given an element $g \in XN \cap PN$, we prove that $g \in xN$ for some $x \in X \cap P$. Since we have $g \in XN$, we may assume without loss of generality that $g \in X$, and in particular $g$ is a $p$-element. Since also $g \in PN$, there exists $z \in P$ such that $gN = zN$.

Put $H = \langle g \rangle N = \langle z \rangle N$, and observe that $H' \leq N$. Since $P \cap N$ is a Sylow $p$-subgroup of $N$ and $z \in P$, it follows that $P \cap H = \langle z \rangle (P \cap N)$ is a Sylow $p$-subgroup of $H$. Now, $g$ is a $p$-element of $H$, and so we get $g^h \in P \cap H$ for
some $h \in H$. If we put $x = g^h$ then $x \in X \cap P$, since $X$ is a normal subset of $G$, and $g = x^{h^{-1}} = x[x, h^{-1}] \in xH' \subseteq xN$, as desired. \hfill \square

The next lemma will be fundamental in the proof of Theorem A, since it will allow us to go up a series from 1 to $w(G)$ in which all quotients of two consecutive terms are verbal subgroups of a word all of whose values are also $w$-values.

**Lemma 1.2.** Let $G$ be a finite group, and let $P$ be a Sylow $p$-subgroup of $G$. Assume that $N \leq L$ are two normal subgroups of $G$, and use the bar notation in the quotient group $G/N$. Let $X$ be a normal subset of $G$ consisting of $p$-elements such that $\overline{P} \cap \overline{X} = (\overline{P} \cap \overline{X})$. Then $P \cap L = (P \cap X, P \cap N)$.

**Proof.** By Lemma 1.1, we have $P \cap L = \langle P \cap X \rangle$, and this implies that $PN \cap L = \langle P \cap X \rangle N$. By intersecting with $P$, we get

$$P \cap L = P \cap (PN \cap L) = P \cap \langle P \cap X \rangle N = \langle P \cap X \rangle (P \cap N),$$

where the last equality follows from Dedekind’s law. This proves the result. \hfill \square

We will also need the following lemma, of a different nature.

**Lemma 1.3.** Let $G$ be a finite group, and let $N$ be a minimal normal subgroup of $G$. If $N$ does not contain any non-trivial elements of $G_{\delta_i}$, where $i \geq 1$, then $[N, G^{(i-1)}] = 1$.

**Proof.** We argue by induction on $i$. If $i = 1$ then, since $N$ is normal in $G$ and does not contain any non-trivial commutators of elements of $G$, it follows that $[n, g] = 1$ for every $n \in N$ and $g \in G$. Thus $[N, G] = 1$, as desired.

Assume now that $i \geq 2$. The fact that $N$ is a minimal normal subgroup of $G$ implies that the subgroup $\langle N \cap G_{\delta_{i-1}} \rangle$ must be either equal to $N$ or the trivial subgroup. In the former case, we have $N = (N \cap G_{\delta_{i-1}})$ and so $[N, G^{(i-1)}]$ is generated by elements of the form $[a, b]$ where $a \in N \cap G_{\delta_{i-1}}$ and $b \in G_{\delta_{i-1}}$. In particular, each commutator $[a, b]$ belongs to $N \cap G_{\delta_{i}}$ and must be 1 by the hypothesis. Hence $[N, G^{(i-1)}] = 1$. If $N \cap G_{\delta_{i-1}} = 1$, then it follows from the induction hypothesis that $[N, G^{(i-2)}] = 1$, and the result holds. \hfill \square

We conclude this preliminary section by showing that Theorem A holds for every word under the assumption that the verbal subgroup $w(G)$ is nilpotent.

**Theorem 1.4.** Let $G$ be a finite group of order $p^a m$, where $p$ is a prime and $m$ is not divisible by $p$, and let $P$ be a Sylow $p$-subgroup of $G$. If $w$ is any word such that $w(G)$ is nilpotent, then

$$P \cap w(G) = \langle P \cap G_{w^m} \rangle.$$

**Proof.** By Bezout’s identity, there exist two integers $\lambda$ and $\mu$ such that $1 = \lambda p^a + \mu m$. If we put $u = w^{\lambda p^a}$ and $v = w^{\mu m}$, then for every $g \in G_{w}$ we have

$$g = (g^{p^a})^\lambda \cdot (g^{m})^\mu \in \langle G_u \rangle \cdot \langle G_v \rangle.$$
Hence
\begin{equation}
 w(G) = \langle G_u, G_v \rangle. 
\end{equation}

Note that all elements of $G_u$ have $p'$-order, and all elements of $G_v$ have $p$-power order. Since $w(G)$ is nilpotent, it follows that $\langle G_u \rangle$ is a $p'$-subgroup of $w(G)$, $\langle G_v \rangle$ is a $p$-subgroup, and $G_u$ and $G_v$ commute elementwise. As a consequence of this and (1), we get
\begin{equation}
 w(G) = \langle G_u \rangle \times \langle G_v \rangle, 
\end{equation}
and $\langle G_u \rangle$ and $\langle G_v \rangle$ are a Hall $p'$-subgroup and a Sylow $p$-subgroup of $w(G)$, respectively. We conclude that $P \cap w(G) = \langle G_v \rangle$, which proves the theorem. \qed

2. The proof of Theorem A

The first step in the proof of Theorem A is to verify it for $\delta_i$, which is done in Theorem 2.3 below. For this, we will rely on the result by Liebeck, O’Brien, Shalev and Tiep \cite{6} that proved Ore’s conjecture, according to which every element of a non-abelian simple group is a commutator, and a fortiori, also a value of $\delta_i$ for every $i$. We will also need the following result of Gaschütz (see page 191 of \cite{3}).

**Theorem 2.1.** Let $G$ be a finite group, and let $P$ be a Sylow $p$-subgroup of $G$. If $N$ is a normal abelian $p$-subgroup of $G$, then $N$ is complemented in $G$ if and only if $N$ is complemented in $P$.

In the proof of Theorem A for both $\delta_i$ and an arbitrary outer commutator word, we will argue by induction. Then it will be important to take into account the following remark.

**Remark 2.2.** Let $G$ be a group of order $p^a m$ for which we want to prove Theorem A in the case of a given word $w$. Assume that $K$ is a group whose order $p^a m^*$ is a divisor of $p^a m$ (for example, a subgroup or a quotient of $G$), and let $P^*$ be a Sylow $p$-subgroup of $K$. If Theorem A is known to hold for $K$ and $w$, then we have $P^* \cap w(K) = \langle P^* \cap K_{w^*} \rangle$. Since $m/m^*$ is a positive integer which is coprime to $p$, it follows that $P^* \cap w(K) = \langle (P^* \cap K_{w^*})^{m/m^*} \rangle$, and so also that $P^* \cap w(K) = \langle P^* \cap K_{w^*} \rangle$. In other words, in the statement of Theorem A for $K$, we can replace the power word $w^{m^*}$ corresponding to the order of $K$ with the word $w^m$, which corresponds to the order of $G$.

We can now proceed to the proof of Theorem A for $\delta_i$.

**Theorem 2.3.** Let $G$ be a finite group of order $p^a m$, where $p$ is a prime and $m$ is not divisible by $p$, and let $P$ be a Sylow $p$-subgroup of $G$. Then, for every $i \geq 0$, we have
\begin{equation}
 P \cap G^{(i)} = \langle P \cap G_{\delta^m} \rangle. 
\end{equation}

**Proof.** We argue by induction on the order of $G$. The result is obvious if either $i = 0$ or $G^{(i)} = 1$, so we assume that $i \geq 1$ and $G^{(i)} \neq 1$.

Let $N \neq 1$ be a normal subgroup of $G$ which is contained in $G^{(i)}$. Then the result holds in $G = G/N$, and we have $P \cap G^{(i)} = \langle P \cap G_{\delta^m} \rangle$. By applying Lemma 1.2, we get
\begin{equation}
 P \cap G^{(i)} = \langle P \cap G_{\delta^m}, P \cap N \rangle. 
\end{equation}
Now we assume that \( N \) is a minimal normal subgroup of \( G \), and we consider three different cases, depending on the structure of \( N \).

(i) \( N \) is a direct product of isomorphic non-abelian simple groups.

By the positive solution to Ore’s conjecture, we have \( N = N_{δ_i} \). Hence we get \( P \cap N \subseteq N_{δ_i} \), and since the map \( z \mapsto z^m \) is a bijection in \( P \cap N \), it follows that \( P \cap N \subseteq P \cap N_{δ_n} \). Now the result is immediate from (3).

(ii) \( N \cong C_q \times \cdots \times C_q \), where \( q \) is a prime different from \( p \).

In this case, \( P \cap N = 1 \) and the result obviously holds.

(iii) \( N \cong C_p \times \cdots \times C_p \).

In this case, we have \( N \leq P \) and so \( P \cap N = N \). Since \( \langle N \cap G_{δ_i} \rangle \) is a normal subgroup of \( G \) and \( N \) is a minimal normal subgroup, it follows that either \( \langle N \cap G_{δ_i} \rangle = N \) or \( N \cap G_{δ_i} = 1 \). In the former case, we have \( N = \langle (N \cap G_{δ_i})^m \rangle \), since \( N \) is a finite \( p \)-group, and so \( N = \langle N \cap G_{δ_i}^m \rangle \) and the theorem follows again from (3). So we are necessarily in the latter case, and then by Lemma [13] we have \( [N, G^{(i-1)}] = 1 \).

If \( G \) is not perfect, then the theorem holds by induction in \( G' \), and so \( P \cap G^{(i+1)} = P \cap (G')^{(i)} \) can be generated by values of \( δ_i^m \) lying in \( P \). If \( G^{(i+1)} \neq 1 \) then we can use (3) with \( G^{(i+1)} \) in the place of \( N \), and we are done. On the other hand, if \( G^{(i+1)} = 1 \) then \( G^{(i)} \) is abelian, and the result is a consequence of Theorem [14].

Thus we may assume that \( G \) is perfect. Then \( P \cap G^{(i)} = P \). Also \( [N, G] = [N, G^{(i-1)}] = 1 \), and \( N \) is central in \( G \). Being a minimal normal subgroup of \( G \), this implies that \( |N| = p \). If \( N \leq \Phi(P) \) then it follows from (3) that \( P = \langle P \cap G_{δ_i}^m \rangle \), as desired. Hence we may assume that \( N \) is not contained in a maximal subgroup \( M \) of \( P \). Since \( |N| = p \), it follows that \( M \) is a complement of \( N \) in \( P \). By Theorem [24] it follows that \( N \) has also a complement in \( G \), say \( H \). Since \( N \leq Z(G) \), we conclude that \( G = H \times N \), a contradiction with the fact that \( G \) is perfect. This completes the proof. □

We will deal with arbitrary outer commutator words using some concepts from the paper [2], where outer commutator words are represented by binary rooted trees in the following way: indeterminates are represented by an isolated vertex, and if \( w = [u, v] \) is the commutator of two outer commutator words \( u \) and \( v \), then the tree \( T_w \) of \( w \) is obtained by drawing the trees \( T_u \) and \( T_v \), and a new vertex (which will be the root of the new tree) which is then connected to the roots of \( T_u \) and \( T_v \). For example, the following are the trees for the words \( γ_4 \) and \( δ_3 \) (we also label every vertex with the outer commutator word which is represented by the tree appearing on top of that vertex):

![Figure 1](image.png)

**Figure 1.** The trees of the words \( γ_4 \) and \( δ_3 \).
Each of these trees has a visual height, which coincides with the largest distance from the root to another vertex of the tree. Observe that the full binary tree of height \( i \) corresponds to the derived word \( \delta_i \). The following two concepts, also introduced in [2], will be essential in our proof of Theorem A.

**Definition 2.4.** Let \( u \) and \( w \) be two outer commutator words. We say that \( u \) is an *extension* of \( w \) if the tree of \( u \) is an upward extension of the tree of \( w \). If \( u \neq w \), we say that \( u \) is a *proper extension* of \( w \).

An important remark is that, if \( u \) is an extension of \( w \), then \( G_u \subseteq G_w \).

**Definition 2.5.** If \( w \) is an outer commutator word whose tree has height \( i \), the *defect* of \( w \) is the number of vertices that need to be added to the tree of \( w \) in order to get the tree of \( \delta_i \). In other words, if the tree of \( w \) has \( V \) vertices, the defect of \( w \) is \( 2^{i+1} - 1 - V \).

Thus the only words of defect 0 are the derived words. Our proof of Theorem A also depends on the following result, which is implicit in the proof of Theorem B of [2].

**Theorem 2.6.** Let \( w = [u, v] \) be an outer commutator word of height \( i \), different from \( \delta_i \). Then at least one of the subgroups \([w(G), u(G)]\) and \([w(G), v(G)]\) is contained in a product of verbal subgroups corresponding to words which are proper extensions of \( w \) of height \( i \).

Let us now give the proof of Theorem A.

**Proof of Theorem A.** We argue by induction on the defect of the word \( w \). If the defect is 0, then \( w \) is a derived word, and the result is true by Theorem 2.3. Hence we may assume that the defect is positive. If the height of \( w \) is \( i \), let \( \Phi = \{ \varphi_1, \ldots, \varphi_r \} \) be the set of all outer commutator words of height \( i \) which are a proper extension of \( w \). Since every word in the set \( \Phi \) has smaller defect than \( w \), the theorem holds for all \( \varphi_i \).

Put \( N_0 = 1 \), \( N_i = \varphi_1(G) \ldots \varphi_i(G) \) for \( 1 \leq i \leq r \), and \( N = N_r \). Let us write \( w = [u, v] \), where \( u \) and \( v \) are outer commutator words. Since \([w(G), w(G)]\) is contained in both \([w(G), u(G)]\) and \([w(G), v(G)]\), it follows from Theorem 2.6 that \([w(G), w(G)] \leq N \). Thus if \( G = G/N \), the verbal subgroup \( w(G) \) is abelian, and so Theorem A holds in \( G \), according to Theorem 1.2. Hence \( P \cap w(G) = \langle P \cap G_{w^m} \rangle \), and by applying Lemma 1.2 we get \( P \cap w(G) = \langle P \cap G_{w^m}, P \cap N \rangle \).

Consequently, it suffices to show that \( P \cap N \) can be generated by values of \( w^m \). We see this by proving that \( P \cap N_i = \langle P \cap N_i \cap G_{w^m} \rangle \) for every \( i = 0, \ldots, r \), by induction on \( i \). There is nothing to prove if \( i = 0 \), so we assume that \( i \geq 1 \). If \( G_i = G/N_{i-1} \), we have \( G_i = \varphi_i(G) \). Since the theorem is true for \( \varphi_i \), it follows that \( P \cap G_i = \langle P \cap G_{\varphi_i^m} \rangle \). By Lemma 1.2, we get

\[
P \cap N_i = \langle P \cap G_{\varphi_i^m}, P \cap N_{i-1} \rangle.
\]

Observe that, since \( \varphi_i \) is an extension of \( w \), we have \( G_{\varphi_i^m} \subseteq G_{w^m} \). Since also \( G_{\varphi_i^m} \subseteq \varphi_i(G) \subseteq N_i \), we can further say that \( G_{\varphi_i^m} \subseteq N_i \cap G_{w^m} \). Hence

\[
P \cap N_i = \langle P \cap N_i \cap G_{w^m}, P \cap N_{i-1} \rangle,
\]

and the result follows from the induction hypothesis. \( \square \)
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Department of Mathematics, University of Brasilia, Brasilia-DF, 70910-900 Brazil

E-mail address: acciarricristina@yahoo.it

Matematika Saila, Euskal Herriko Unibertsitatea, 48080 Bilbao, Spain

E-mail address: gustavo.fernandez@ehu.es

Department of Mathematics, University of Brasilia, Brasilia-DF, 70910-900 Brazil

E-mail address: pavel@unb.br