DIFFERENTIAL GEOMETRIC FORMULATION OF THE 
CAUCHY NAVIER EQUATIONS

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Abstract. The paper presents a reformulation of some of the most basic entities and equations of linear elasticity – the stress and strain tensor, the Cauchy Navier equilibrium equations, material equations for linear isotropic bodies - in a modern differential geometric language using differential forms and lie derivatives. Similar steps have been done successfully in general relativity, quantum physics and electrodynamics and are of great use in those fields. In Elasticity Theory, however, such a modern differential geometric approach is much less common. Furthermore, existing reformulations demand a vast knowledge of differential geometry, including nonstandard entities such as vector valued differential forms and the like. This paper presents a less general but more easily accessible approach to using modern differential geometry in elasticity theory than those published up to now.

1. INTRODUCTION

Modern differential geometry often allows for a clearer, more geometric approach to solve physical problems than Gibbsian vector calculus or Ricci calculus (i.e. the index-based form) of Riemannian geometry. Even more importantly, equations stated in modern differential geometric terms are not bound to the use of special classes of coordinate systems, such as cartesian or orthonormal ones. It provides useful and powerful concepts such as differential forms and Lie derivatives and with them a generalization of vector calculus expressions like rotation, gradient and divergence and of various integral theorems like Gauss’ divergence theorem or Stoke’s integral theorem on planes to just one integral theorem. Furthermore, coordinate transformations in curvilinear coordinate systems are often simplified.

In general, it is desirable to express equations in a connection and metric tensor free manner, using only differential forms and Lie derivatives, thus simplifying coordinate system changes. Unfortunately it is not always possible to express vector or tensor equations as equations in such a way. One reason of this is the antisymmetric nature of differential forms of degree $\geq 2$. While e.g. the gradient of a scalar function and the rotation and divergence of a vector field can be expressed using differential forms, this is not possible with the gradient of a vector field. It is, however, possible to reformulate certain differential equations of vector fields as ones of differential forms.

In electrodynamics, differential forms are already a quite popular way to reformulate maxwell’s equations etc. In continuum mechanics, however, there
are only few steps towards using differential forms, most notably from J. E. Marsden [Mar94] and V.I. Arnold [Arn89]. In [Kan07], the stress and strain tensors are introduced as vector or covector valued differential forms of various degree. In that approach, the physical meaning of those tensors are clearer than in the traditional way of expressing them interchangeably as contravariant, covariant or mixed tensors. However, vector or covector valued differential forms are a nonstandard subject in differential geometry and usually do not simplify calculations.

For that reason, in this paper a path between traditional tensor analysis and the very general and mathematically rigorous approach of [Mar94] is taken, thereby restricting to linear elasticity. Due to their relevance, only static problems without body forces are considered.

Absolutely necessary to determine deformations inside a body is a suitable system of differential equations for displacements, stresses or strains, respectively. Since displacement fields can be considered as fundamental in the sense that stress or strain fields can be determined from them, the cauchy-navier displacement equations shall be reformulated in chapter (2). The resulting set of equations have only the displacement variables as unknowns, so displacement boundary conditions are easy to handle in this system of equations. In order to allow also traction boundary conditions and to relate stresses, strains and displacements, it is necessary to express the well-known stress and strain tensor of linear elasticity in the same modern differential geometric formalism. This will be done in chapter (3).

It shall be noted that throughout the paper it was assumed that manifolds have sufficiently smooth boundaries and boundary values are also sufficiently smooth.

2. Cauchy Navier’s Displacement Equations of Static Linear Elasticity

2.1. Basic Definitions. Problems in static linear elasticity consist of finding either stress fields that satisfy the equilibrium equation \( \nabla \cdot \sigma = 0 \) or, equivalently, displacements that fulfil the Cauchy Navier equation [Tim70] inside a suitable sub-manifold \( B \) of \( E^3 \):

\[
\mu \Delta \bar{u} + (\lambda + \mu) \nabla (\nabla \cdot \bar{u}) = 0
\]

(2.1)

In this article, we will regard the latter equation as fundamental because stresses and strains can easily be calculated from displacements. Also, there is no need for additional compatibility equations to be satisfied as is the case for the equilibrium equation for stresses or strains.

Displacements in elasticity theory are naturally tangent vector fields, e.g. \( \bar{u} : E^3 \supset B \rightarrow TB \). In order to make use of exterior calculus, a corresponding displacement covector field \( u : E^3 \rightarrow T^* E^3 \) is introduced, which is derived trivially from the displacement vector field by index lowering:

\[
u := \bar{u}^b = g(\bar{u})
\]

(2.2)

Here, \( g(\bar{u}) \) is the action of the covariant metric tensor on the vector \( \bar{u} \). The superscript \( b \) indicates the index lowering process or association of a covector field to a vector field, respectively. The inverse action is the index raising operation \( u^\# = g^{-1}(u) \).
2.2. Vector Differential Operators in Exterior Calculus.

**Lemma.** For the differential operators grad, div and rot, applied on vector or scalar fields, respectively, the following relationships are valid:

\[ \nabla f = df^\# \]

\[ \nabla \cdot \mathbf{u} = \ast d \ast \mathbf{u} \]

\[ \nabla \times \mathbf{u} = (\ast du)^\# \]

Proofs for these relations can be found for example in [Fla90].

2.3. Cauchy Navier Equation.

**Theorem 1.** Given a displacement vector field \( \mathbf{u} \) that satisfies the Cauchy Navier equations, then the corresponding covector field \( u \) satisfies

\[ (\lambda + 2\mu) d\delta u + \mu \delta du = 0 \]

Proof. The correspondence between the divergence or rotation of vector fields and exterior derivatives of differential forms is well known. Since, however, there is no equivalent of the gradient of a vector field in exterior calculus, the Laplace operator has to be restated using the well known formula

\[ \triangle \mathbf{u} = \nabla \cdot \nabla \mathbf{u} = \nabla (\nabla \cdot \mathbf{u}) - \nabla \times \nabla \times \mathbf{u} \]

With this, the Cauchy Navier equation appears in the equivalent form

\[ (\lambda + 2\mu) \nabla (\nabla \cdot \mathbf{u}) - \mu \nabla \times \nabla \times \mathbf{u} = 0 \]

Now, using 2.3 to 2.5 we can restate 2.8 in terms of the corresponding displacement one-form \( u = \mathbf{u}^b \):

\[ (\lambda + 2\mu) d \ast d \ast u - \mu \ast d \ast du = 0 \]

A somewhat more concise notation of 2.9 uses the codifferential operator, defined in \( E^3 \) as

\[ \delta : \Omega^n(E^3) \rightarrow \Omega^{n-1}(E^3) \]

\[ \omega \mapsto -\ast d \ast \omega \quad \text{if} \ \omega \in \Omega^1(E^3) \text{ or } \omega \in \Omega^3(E^3) \]

\[ -\ast d \ast \omega \quad \text{if} \ \omega \in \Omega^2(E^3) \]

With this we arrive at the proposed formula

\[ (\lambda + 2\mu) d\delta u + \mu \delta du = 0 \]
2.4. **Boundary Conditions.** The treatment of displacement boundary conditions are trivial: the displacement vector field on the boundary just has to be mapped to its corresponding covector field.

**Corollary.** Given a displacement vector field \( \bar{u} \) that satisfies the Cauchy Navier equations and assumes given values \( \bar{u} \mid \partial B \) on the boundary \( \partial B \), then the corresponding covector field \( u \) satisfying (2.6) assumes the values \( g(\bar{u}) \) on the same boundary.

Traction boundary conditions, however, are much harder to handle. It would be desirable to find a way to express traction forces or the elastic stress tensor in terms of the displacement covector field, resulting in boundary conditions that can be stated as a set of differential equations in the displacement covector field. This will be the content of the next and largest part of this paper.

3. **Traction Boundary Conditions**

3.1. **The Linear Strain Tensor.** According to [Mar94], the strain tensor \( \bar{\epsilon} \in T^0_2(E^3) \) of linear elasticity

\[
\bar{\epsilon} = \epsilon_{ij} e^i \otimes e^j = \frac{1}{2}(\nabla \bar{u} + \nabla \bar{u}^T)
\]

can also be written as the Lie derivative of the metric tensor with respect to the displacement vector field \( \bar{u} \):

\[
(3.1) \quad \bar{\epsilon} = \frac{1}{2} L_{\bar{u}} \bar{g}
\]

with the metric tensor \( \bar{g} = g_{ij} dx^i \otimes dx^j \). Beware that we need the displacement vector field here, not the corresponding one-form. This is unfortunate, because we now need to map one-form into vector field components and vice versa. However, in the case of orthonormal bases and physical vector components this is simple: the components are unchanged. Otherwise, index lowering or raising has to be used.

As an example, using a general canonical vector base, it shall be shown, that the formula above is equivalent to results of classical tensor analysis.

**Corollary.** The components of (3.1) satisfy

\[
(3.2) \quad \bar{\epsilon} = \epsilon_{ij} dx^i \otimes dx^j = \frac{1}{2}(u^k g_{ij,k} + g_{kj} u^k_i + g_{ik} u^k_j) dx^i \otimes dx^j
\]

**Proof.** This can be proven by direct calculation.

The Lie derivative adheres to the product rule for tensor products:

\[
(3.3) \quad L_{\bar{g}}(T \otimes S) = L_{\bar{g}}T \otimes S + T \otimes L_{\bar{g}}S
\]

This holds for tensor fields of any rank, and therefore especially for scalar functions and one-forms. Thus we have

\[
(3.4) \quad L_{\bar{u}}(g_{ij} dx^i \otimes dx^j) = (L_{\bar{u}}g_{ij}) dx^i \otimes dx^j + g_{ij} L_{\bar{u}}dx^i \otimes dx^j + g_{ij}dx^i \otimes L_{\bar{u}}dx^j
\]

Cartan’s „magic formula“ can be applied to calculate the lie derivatives of one-forms:

\[
(3.5) \quad L_{\bar{x}}\alpha = \bar{x} \cdot d\alpha + d(\bar{x} \cdot \alpha)
\]

Applied to the canonical basis covectors \( dx^i \) and afterwards to (3.4)
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\[ L_\mathbf{u}dx^i = du^i = u_k^i dx^k \]

\[ L_\mathbf{u}(g_{ij}dx^i \otimes dx^j) = u^k g_{ij,k}dx^i \otimes dx^j + g_{ij}u_k^i dx^i \otimes dx^j \]

\[ (3.6) \]

Thus we have

\[ \epsilon_{ij}dx^i \otimes dx^j = \frac{1}{2}(u^k g_{ij,k} + g_{kj}u^k_{i} + g_{ik}u^k_{j})dx^i \otimes dx^j \]

\[ (3.7) \]

\[ \epsilon_{ij} = \frac{1}{2}((g_{ik}u^k_{j} + g_{jk}u^k_{i} + g_{ik}u^k_{j} - g_{ik}u^k_{j} - g_{jk}u^k_{i} + g_{ij}u^k_{k})) \]

\[ (3.8) \]

\[ \epsilon_{ij} = \frac{1}{2}(g_{ik}u^k_{j} + g_{jk}u^k_{i} + g_{ij}u^k_{k}) \]

\[ (3.9) \]

A comparison shows that \( \tilde{\epsilon}_{ij} = \epsilon_{ij} \).

3.2. The Stress Tensor. For isotropic homogeneous bodies, the stress and strain tensor of linear elasticity are related as follows [Mar94]:

\[ \tilde{\sigma} = \lambda \epsilon \tilde{g} + 2\mu \epsilon \]

\[ (3.10) \]

Here, \( e = \nabla \cdot \mathbf{u} \) is the volume expansion under deformation. For a displacement one-form \( u \) we have \( e = -\delta u = * d * u \). Since \( \epsilon = \frac{1}{2}L_\mathbf{u} \tilde{g} \), we get:

\[ (3.11) \]

\[ \tilde{\sigma} = -\lambda \delta u \tilde{g} + \mu L_\mathbf{u} \tilde{g} \]

Beware that as before, unfortunately, we have to deal with the displacement one-form and vector field at the same time.
3.3. Traction Boundary Conditions.

Lemma. Given a traction vector field $\vec{t} : \partial B \supset \partial S \rightarrow TB$ on a part of $B$'s boundary. The corresponding traction covector field $t := \vec{t}^b$ then satisfies

\begin{equation}
(3.12) \quad t := \vec{t}^b = \sigma \cdot \vec{n} = \lambda e n + 2 \mu \epsilon \cdot \vec{n}.
\end{equation}

Proof. This is a trivial implication of Cauchy's stress theorem \hfill \Box

Theorem. Given a traction covector field $t : \partial B \supset \partial S \rightarrow T^*B$ and a displacement covector $u$ field satisfying 2.6. The thus given traction boundary conditions are satisfied, if the displacement covector field and its corresponding vector field $\vec{u}$ adheres also to the following equation on $\partial S$:

\begin{equation}
(3.13) \quad t = -\lambda \delta un + \mu (d(\vec{u} \cdot n) + \vec{u} \cdot dn + [\vec{n}, \vec{u}]^b)
\end{equation}

where $\vec{n}$ is a normal vector at a given point and $n = \vec{n}^b$ its corresponding one-form.

Proof. Since the Lie derivative is also a derivative with respect to scalar products $(L_g(T \cdot S) = L_gT \cdot S + T \cdot L_gS$ for any two tensors $T$ and $S$) we can write:

\begin{align*}
2 \epsilon \cdot \vec{n} & = L_g \vec{g} \cdot \vec{n} = L_g(\vec{g} \cdot \vec{n}) - \vec{g} \cdot L_g \vec{n} \\
& = L_g \vec{n} - [\vec{u}, \vec{n}]^b = d(\vec{u} \cdot n) + \vec{u} \cdot dn + [\vec{n}, \vec{u}]^b
\end{align*}

Now only the part $\lambda en$ is missing from (3.12) which is equal to $-\lambda \delta un$. \hfill \Box

Now let's assume we have a coordinate system on which one of the canonical base vectors equals the surface normal $\vec{n}$. In accordance with cylindrical or spherical coordinates, let's call this coordinate function $r$, the canonical base vector along this coordinate $\partial_r$ and the canonical base covector $dr$. Let's also assume that $\vec{n}$ (and thus $n$) are normalized, as well as $\partial_r$ and $dr$. In this case $\vec{n} = \partial_r$ and $n = dr$. Then we can reformulate 3.13 according to the following

Corollary. Given a coordinate system with a coordinate function $r$ satisfying the description above, a traction covector field $t : \partial B \supset \partial S \rightarrow T^*B$ and a displacement covector $u$ field satisfying 2.6. the latter have to adhere to the following equation

\begin{equation}
(3.14) \quad t = -(\lambda \delta u)dr + \mu (du^r + [\partial_r, \vec{u}]^b)
\end{equation}

Proof. We look at all the substitutions one after the other. $-\lambda \delta un = -(\lambda \delta u)dr$ just by the definition $n = dr$ given above.

$d(\vec{u} \cdot n) = du^r$ follows from the fact that $\vec{u} \cdot n = \vec{u} \cdot dr = u^r$.

$\vec{u} \cdot dn$ vanishes because $dn = ddr = 0$.

$[\vec{n}, \vec{u}]^b = [\partial_r, \vec{u}]^b$ also by definition. \hfill \Box
4. Conclusion

In the first part of this article we have derived Cauchy Navier’s equation of static equilibrium in linear elasticity using only differential forms instead of vector fields. The second part treats the harder problem of expressing strain and stress tensors in a more modern differential geometric way, and, finally, how to deal with traction boundary conditions.

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