The Strassen Invariance Principle for Certain Non-stationary Markov-Feller Chains

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Abstract

We propose certain conditions which are sufficient for the functional law of the iterated logarithm (the Strassen invariance principle) for some general class of non-stationary Markov-Feller chains. This class may be briefly specified by the following two properties: firstly, the transition operator of the chain under consideration enjoys a non-linear Lyapunov-type condition, and secondly, there exists an appropriate Markovian coupling whose transition probability function can be decomposed into two parts, one of which is contractive and dominant in some sense. The construction of such a coupling derives from the paper of M. Hairer (Probab. Theory Related Fields, 124(3):345–380, 2002). Our criterion may serve as a useful tool in verifying the functional law of the iterated logarithm for certain random dynamical systems, developed eg. in molecular biology. In the final part of the paper we present an example application of our main theorem to the mathematical model describing stochastic dynamics of gene expression.

Keywords: Markov chain, random dynamical system, invariant measure, law of the iterated logarithm, asymptotic coupling

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Introduction

The law of iterated logarithm (LIL) can be viewed as a refinement of the the strong law of large numbers (SLLN). It improves the order rate, following from the SLLN, of the partial sums of the sample paths of the examined sequence of random variables, and even gives the proportionality constant. To be more precise, it provides the precise values of the limit superior and the limit inferior of the almost all sequences formed by the properly scaled sums of these sample paths. Moreover, the LIL gives an interesting illustration of the difference between almost sure and distributional statements, such as the central limit theorem (CLT).

The functional version of the LIL (now usually called the Strassen invariance principle) was first proven for independent random variables by V. Strassen (cf. \cite{21}). Then, the assertion was established for certain square integrable martingales (see eg. \cite{9,10}) and also for particular classes of Markov chains, including stationary processes (cf. \cite{14,23}), as well as non-stationary ones. The results for the latter concern, for instance, positive Harris
recurrent Markov chains which are assumed to be uniformly ergodic in the total variation norm (cf. [18]) and the Markov-Feller chains enjoying the exponential mixing property in the Wasserstein metric (see [1]).

Here, we also prove a version of the Strassen invariance principle for a quite general class of non-stationary Markov-Feller chains. However, on the contrary to [1], we do not require any form of continuous dependence of the distributions of the given Markov chain on the initial conditions. A priori, we even do not demand the exponential-mixing type property (defined as in [8]). Instead, we propose a set of conditions, relatively easy to verify, which yield the desired assertion. The motivation to establish such a result derives from our research on certain random dynamical systems, developed mainly in molecular biology (see eg. the models for gene expression investigated in [11, 3, 17] or the model for cell cycle discussed in [16, 22]), to which we were not able to apply [1, Theorem 1] directly.

The class of Markov-Feller chains for which we obtain our main result (Theorem 3.7), that is the Strassen invariance principle for the LIL, can be characterized briefly by the following two properties. Firstly, the transition operator of the chain under consideration enjoys a non-linear Lyapunov-type condition. Secondly, there exists an appropriate Markovian coupling whose transition function can be decomposed into two parts, one of which is contractive and dominant in some sense. The construction of such a coupling is described in details eg. in [8, 13, 20, 22]. Theorem 3.7 is formulated in the same spirit as [13, Theorem 2.1] and [1, Theorem 2.1], which were both applied to a particular discrete-time Markov dynamical system (cf. [3]) in order to verify its exponential ergodicity (in the context of weak convergence of probability measures) and the CLT, respectively. Theorem 3.7 can be used to establish the functional LIL for such a system (cf. Theorem 2.1). The analysis is analogous to the one presented in [1]. The aforementioned stochastic system has interesting biological interpretations. First of all, it can be viewed as the chain given by the post-jump locations of some piecewise-deterministic Markov process, which occurs in a simple model of gene expression (cf. [3, 17]). On the other hand, a special case of the above-mentioned model provides a mathematical framework for modelling the concentration of the compounds involved in the gene autoregulation at times of transcriptional bursts (for details, see [11]). The second example indicates the importance of considering a non-locally compact space as the state space in the abstract framework. Finally, let us indicate that Theorem 3.7 might be also useful for verifying the Strassen invariance principle in other models, like eg. the one investigated in [5], where its application to a Poisson-driven stochastic differential equation is described.

Some proof techniques are adapted from [1, 12], which both pertain to the martingale results by C.C. Heyde and D.J. Scott [10].

The paper is organised as follows. In Section 1 we gather notation and definitions used throughout the paper. Mainly, we relate to the general theory of Markov chains, discussed more widely eg. in [18,19], and, in particular, we introduce the notion of Markovian coupling. In Section 2 we quote some auxiliary results established in [13, 1], while in Section 3 we formulate and prove our main result, namely a version of the Strassen invariance principle for the LIL (see Theorem 3.7). At the beginning of this section we also present a few general observations concerning martingales, whose proofs are contained in Appendix. Finally, in Section 4 we apply our main result to the Markov chain given by the post-jump locations
of the piecewise-deterministic Markov process considered in [3].

1 Preliminaries

In the beginning, we shall introduce some notation and recall certain general definitions, as well as basic facts, useful in our further analysis.

Let us write \( \mathbb{R}_+ = [0, \infty) \) and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) with \( \mathbb{N} \) standing for the set of all positive integers. For any point \( x \) and any set \( A \), the symbols \( \delta_x \) and \( \mathbbm{1}_A \) will denote the Dirac measure at \( x \) and the indicator function of \( A \), respectively.

We will consider a complete separable metric space \((X, \varrho)\) endowed with the \( \sigma \)-field \( \mathcal{B}_X \) of its Borel subsets. By \( B_b(X) \) we will denote the space of all bounded measurable functions \( f : X \to \mathbb{R} \) equipped with the supremum norm \( \|f\|_{\infty} = \sup_{x \in X} |f(x)| \), while by \( C_b(X) \) and \( Lip_b(X) \) we will denote the subspaces of \( B_b(X) \) consisting of all continuous and all Lipschitz continuous functions, respectively. In the paper we shall also refer to the space \( B_b(X) \) of functions \( f : X \to \mathbb{R} \) which are Borel measurable and bounded below.

In what follows, we will write \( \mathcal{M}_{fin}(X) \) and \( \mathcal{M}_1(X) \) for the spaces of finite and probability Borel measures on \( X \), respectively. We shall also introduce the space

\[
\mathcal{M}_{1,r}(X) = \left\{ \mu \in \mathcal{M}_1(X) : \int_X V^r(x) \mu(dx) < \infty \right\} \quad \text{for} \quad r > 0
\]

and any given Lyapunov function \( V : X \to [0, \infty) \), that is, a function which is continuous, bounded on bounded sets, and, in the case of unbounded \( X \), satisfies \( \lim_{r(x) \to \infty} V(x) = \infty \) for some fixed point \( x \in X \). For brevity, for any \( f \in B_b(X) \) and any signed Borel measure \( \mu \) on \( X \), we will write \( (f, \mu) \) for \( \int_X f(x) \mu(dx) \). As usual, \( \text{supp} \mu \) will denote the support of \( \mu \in \mathcal{M}_{fin}(X) \).

To evaluate the distance between probability measures, we will use the so-called Fortet-Mourier distance (see eg. [15]), defined as follows:

\[
d_{FM}(\mu_1, \mu_2) = \sup \{ |(f, \mu_1 - \mu_2)| : f \in Lip_{FM}(X) \} \quad \text{for} \quad \mu_1, \mu_2 \in \mathcal{M}_1(X),
\]

where

\[
Lip_{FM}(X) = \{ f \in C_b(X) : \|f\|_{BL} \leq 1 \}
\]

with \( \|f\|_{BL} = \max\{\|f\|_{Lip}, \|f\|_{\infty}\} \), and \( |f|_{Lip} \) standing for the minimal Lipschitz constant of \( f \). It should be noted that for the metric space \((X, \varrho)\), which is assumed to be complete and separable, the convergence in \( d_{FM} \) is equivalent to the weak convergence of probability measures. Moreover, upon this assumption, the space \((\mathcal{M}_1(X), d_{FM})\) is complete (see [7] for the proofs of both these facts).

A mapping \( \Pi : X \times \mathcal{B}_X \to [0, 1] \) is called a (sub)stochastic kernel if \( \Pi(\cdot, A) : X \to [0, 1] \) is a Borel measurable map for any fixed \( A \in \mathcal{B}_X \), and \( \Pi(x, \cdot) : \mathcal{B}_X \to [0, 1] \) is a (sub)probability Borel measure for any fixed \( x \in X \). Any stochastic kernel naturally induces a Markov operator \( P : \mathcal{M}_{fin}(X) \to \mathcal{M}_{fin}(X) \) and its dual operator \( U : B_b(X) \to B_b(X) \), which are
Let us introduce the probabilities \( P_c \), where \( x \) setting, for every \( \Omega := \mathcal{M}_{\text{fin}}(X) \), and define the higher-dimensional distributions \( P \) and there exists \( q \in (0, 1) \) such that

\[
d_{FM}(P^n \mu, \mu_s) \leq q^n c(\mu) \quad \text{for any} \quad \mu \in \mathcal{M}_{1,1}(X), \ n \in \mathbb{N},
\]

where \( c(\mu) \) is a constant depending only on \( \mu \).

Given a (sub)stochastic kernel \( \Pi \), we can define the \( n \)-th step kernels \( \Pi^n \), \( n \in \mathbb{N}_0 \), by setting, for every \( x \in X \) and any \( B \in \mathcal{B}_X \),

\[
\Pi^0(x, B) = \delta_x(B), \quad \Pi^1(x, B) = \Pi(x, B), \quad \Pi^n(x, B) = \int_X \Pi(y, B) \Pi^{n-1}(x, dy) \quad \text{for} \quad n \geq 2.
\]

Let us introduce the probabilities \( \mathbb{P}^n_x \) of the form

\[
\mathbb{P}^n_x(\cdot) = \Pi^n(x, \cdot) \quad \text{for} \quad x \in X,
\]

and define the higher-dimensional distributions \( \mathbb{P}^1_{x,...,n} \) on \( X^n, \ x \in X \), as follows: provided that \( \mathbb{P}^1_{x,...,k} \) on \( X^k \) have already been defined for every \( k < n \), the probability measure \( \mathbb{P}^1_{x,...,n} \) is given as the unique measure which satisfies

\[
\mathbb{P}^1_{x,...,n}(A \times B) = \int_A \mathbb{P}^1_{w_{n-1}}(B) \mathbb{P}^1_{x,...,n-1}(dw_1 \times \ldots \times dw_{n-1}), \quad A \in \mathcal{B}_{X^{n-1}}, \ B \in \mathcal{B}_X.
\]

Suppose that \( (\phi_n)_{n \in \mathbb{N}_0} \) is a time-homogeneous \( X \)-valued Markov chain, defined on a probability space \( (\Omega, \mathcal{A}, \mathbb{P}) \). We say that \( (\phi_n)_{n \in \mathbb{N}_0} \) has the one-step transition law determined by a stochastic kernel \( \Pi \), if

\[
\Pi(x, A) = \mathbb{P}(\phi_{n+1} \in A | \phi_n = x) \quad \text{for} \quad x \in X, \ A \in \mathcal{B}_X, \ n \in \mathbb{N}_0.
\]

Let us consider \( \Omega := X^{\mathbb{N}_0} \) with the product topology, and let \( (\phi_n)_{n \in \mathbb{N}_0} \) denote the sequence of mappings acting from \( \Omega \) to \( X \) given by \( \phi_n(\omega) = x_n \) for \( \omega = (x_0, x_1, \ldots) \in \Omega \). Then, according to [18, Theorem 3.4.1], for any \( \mu \in \mathcal{M}_1(X) \), and any stochastic kernel \( \Pi : X \times \mathcal{B}_X \to [0, 1] \),
there exists a probability measure $\mathbb{P}_\mu \in \mathcal{M}_1(\Omega)$ such that, for every $n \in \mathbb{N}$,

$$\mathbb{P}_\mu(B_0 \times \ldots \times B_n \times X \times X \ldots) = \int_{B_0} \mathbb{P}_x^{1 \ldots n}(B_1 \times \ldots \times B_n) \mu(dx), \quad B_0, \ldots, B_n \in \mathcal{B}_X,$$

(1.6)

where $\mathbb{P}_x^{1 \ldots n}$ are defined by (1.3), (1.4). In particular, $(\phi_n)_{n \in \mathbb{N}_0}$ is then a time-homogeneous Markov chain on the probability space $(\Omega, \mathcal{B}_\Omega, \mathbb{P}_\mu)$ with transition probability function $\Pi$ and initial distribution $\mu$. Clearly, $\mathbb{P}_\mu(B)$ is then the probability of the event $\{(\phi_n)_{n \in \mathbb{N}_0} \in B\}$ for $B \in \mathcal{B}_\Omega$. The Markov chain defined according to the above scheme will be further called a canonical Markov chain. By convention, we will write $\mathbb{P}_x(B) = \mathbb{P}_\mu(B|\phi_0 = x)$ for $B \in \mathcal{B}_\Omega$. The expected values corresponding to $\mathbb{P}_x, \mathbb{P}_\mu \in \mathcal{M}_1(\Omega)$ will be denoted by $\mathbb{E}_x, \mathbb{E}_\mu$, respectively.

A time-homogeneous Markov chain $(\phi_n^{(1)}, \phi_n^{(2)})_{n \in \mathbb{N}_0}$ evolving on $X^2$ (endowed with the product topology) is said to be a Markovian coupling of some stochastic kernel $\Pi$ whenever its transition law $C : X^2 \times \mathcal{B}_{X^2} \to [0, 1]$ satisfies

$$C(x, y, A \times X) = \Pi(x, A) \quad \text{and} \quad C(x, y, X \times A) = \Pi(y, A) \quad \text{for any} \quad x, y \in X, \ A \in \mathcal{B}_X.$$  

Conventionally, the kernel $C$ itself is often called a coupling of $\Pi$, too.

Appealing to certain well-known results, already mentioned above, we can consider the canonical Markov chain $(\phi_n^{(1)}, \phi_n^{(2)})_{n \in \mathbb{N}_0}$ with transition law $C$ and an arbitrarily fixed initial distribution $\alpha \in \mathcal{M}_1(X^2)$. Such a chain is defined on $((X^2)^{\mathbb{N}_0}, \mathcal{B}_{(X^2)^{\mathbb{N}_0}}, \mathbb{C}_\alpha)$, where $\mathbb{C}_\alpha \in \mathcal{M}_1((X^2)^{\mathbb{N}_0})$ satisfies the appropriate condition corresponding to (1.6). We will further write $\mathbb{C}_{x,y} = \mathbb{C}_\alpha(\{(\phi_0^{(1)}, \phi_0^{(2)}) = (x, y)\})$ for any $(x, y) \in X^2$. The expected value corresponding to the measure $\mathbb{C}_{x,y}$ will be denoted by $\mathbb{E}_{x,y}$.

Let us also indicate that, for any transition probability function $\Pi$ and any substochastic kernel $Q : X^2 \times \mathcal{B}_{X^2} \to [0, 1]$ satisfying

$$Q(x, y, B \times X) \leq \Pi(x, B) \quad \text{and} \quad Q(x, y, X \times B) \leq \Pi(y, B) \quad \text{for} \quad x, y \in X, \ B \in \mathcal{B}_X, \ (1.7)$$

there exists a substochastic kernel $R : X^2 \times \mathcal{B}_{X^2} \to [0, 1]$ such that $C = Q + R$ is a Markovian coupling of $\Pi$ (see eg. [3, 13, 22] for the explicit formula of $R$).

2 Conditions Sufficient for the Exponential Ergodicity

Consider a transition probability function $\Pi : X \times \mathcal{B}_X \to [0, 1]$, and let $P, U$ be the operators given by (1.1), (1.2), respectively. We assume what follows:

(B0) The Markov operator $P$ has the Feller property.

(B1) There exist a Lyapunov function $V : X \to [0, \infty)$ and constants $a \in (0, 1)$ and $b \in (0, \infty)$ such that

$$\langle V, P\mu \rangle \leq a \langle V, \mu \rangle + b \quad \text{for every} \quad \mu \in \mathcal{M}_1^V(\Omega).$$
Further, we require the existence of a substochastic kernel $Q : X^2 \times \mathcal{B}_{X^2} \to [0,1]$ which satisfies (1.7) and, for some $F \subset X^2$, enjoys the following conditions:

(B2) There exists $\delta \in (0,1)$ such that
\[
\text{supp } Q(x,y,\cdot) \subset F \quad \text{and} \quad \int_{X^2} q(u,v) Q(x,y,du \times dv) \leq \delta \tilde{q}(x,y) \quad \text{for} \quad (x,y) \in F.
\]

(B3) Letting $U(r) = \{(u,v) \in F : q(u,v) \leq r \}$, $r > 0$, we have
\[
\inf_{(x,y) \in F} Q(x,y,U(\delta \tilde{q}(x,y))) > 0.
\]

(B4) There exist constants $\beta \in (0,1]$ and $c_\beta > 0$ such that
\[
Q(x,y,X^2) \geq 1 - c_\beta \tilde{q}(x,y) \quad \text{for every} \quad (x,y) \in F.
\]

(B5) There exists a Markovian coupling $(\phi_n^{(1)}, \phi_n^{(2)})_{n \in \mathbb{N}_0}$ of $\Pi$ with transition law $C \geq Q$ such that for some $\Gamma > 0$ we can choose $\gamma \in (0,1)$ and $c_\gamma > 0$ for which
\[
\mathbb{E}_{x,y}(\gamma^{-\rho}) \leq c_\gamma, \quad \text{whenever} \quad V(x) + V(y) < 4b(1-a)^{-1},
\]
where
\[
\rho = \inf \left\{ n \in \mathbb{N} : (\phi_n^{(1)}, \phi_n^{(2)}) \in F \text{ and } V\left(\phi_n^{(1)}\right) + V\left(\phi_n^{(2)}\right) < \Gamma \right\}. \quad (2.1)
\]

Below we quote two results to which we refer many times in the present paper. They are proven in [13] and [4], respectively.

**Theorem 2.1** ([13, Theorem 2.1]). Suppose that conditions (B0)-(B5) hold with some $Q$ satisfying (1.7) and $F \subset X^2$. Then, $P$ possesses a unique invariant measure $\mu_* \in \mathcal{M}_1(X)$ such that $\mu_* \in \mathcal{M}_{V,1}(X)$, where $V$ is the Lyapunov function determined by (B1). Moreover, there exist constants $q \in (0,1)$, $c > 0$ such that
\[
d_{FM}(P^n \mu, \mu_*) \leq cq^n (1 + \langle V, \mu \rangle + \langle V, \mu_* \rangle) \quad \text{for any} \quad \mu \in \mathcal{M}_{V,1}(X), \ n \in \mathbb{N}_0.
\]

**Lemma 2.2** ([4, Lemma 2.3]). Under the assumptions of Theorem 2.1, there exist $q \in (0,1)$ and $c > 0$ such that
\[
\mathbb{E}_{x,y} \left| g\left(\phi_n^{(1)}\right) - g\left(\phi_n^{(2)}\right) \right| \leq c \|g\|_{BL} q^n (1 + V(x) + V(y)) \quad (2.2)
\]
for all $(x,y) \in X^2$, $g \in \text{Lip}_b(X)$ and $n \in \mathbb{N}_0$, where the coupling $(\Phi_n^{(1)}, \Phi_n^{(2)})_{n \in \mathbb{N}_0}$ is determined by (B3).

## 3 Criteria on the Invariance Principle for the LIL

The section is divided into two parts. The first one contains a few general observations concerning martingales defined on the path space of a given Markov chain, while the sec-
ond one presents a criterion on the Strassen invariance principle for the LIL for certain non-stationary Markov-Feller chains. The proof techniques that we use are mainly based on [11,12].

Let $V : X \to [0, \infty)$ be the Lyapunov function given by $V(x) = g(x, \bar{x})$ for every $x \in X$ and some arbitrarily fixed $\bar{x} \in X$. In the analysis that follows condition (B1) will be considered with this particular function $V$.

### 3.1 Auxiliary Results

Consider an arbitrary stochastic kernel $\Pi : X \times B_X \to [0, 1]$, and the corresponding operators $P$ and $U$ given by (1.1) and (1.2), respectively. We assume that there exists a unique invariant measure $\mu_* \in M_1(X)$ for $P$ such that $(P^n\mu)_{n \in \mathbb{N}}$ converges weakly to $\mu_*$ for every $\mu \in M_1(X)$, as $n \to \infty$. Now, let $(\phi_n)_{n \in \mathbb{N}_0}$ be the canonical Markov chain on $(\Omega, \mathcal{B}_\Omega, \mathbb{P}_\mu)$ with initial measure $\mu \in M_1(X)$ and transition law $\Pi$ (cf. Section 1). By $(\mathcal{F}_n)_{n \in \mathbb{N}_0}$ we denote the natural filtration of $(\phi_n)_{n \in \mathbb{N}_0}$.

Let $T : \Omega \to \Omega$ stand for the shift operator, that is $T(x_0, x_1, \ldots) = (x_1, x_2, \ldots)$, and let $(m_n)_{n \in \mathbb{N}_0}$ be a martingale with respect to $(\mathcal{F}_n)_{n \in \mathbb{N}_0}$ such that $z_n = z_1 \circ T^{n-1}$ for $n \in \mathbb{N}$, where $z_n = m_n - m_{n-1}$ and $m_0 = 0$. Further, assume that

$$\sigma^2 := \mathbb{E}_{\mu_*}(z_1^2) \in (0, \infty), \quad (3.1)$$

and define

$$h_n^2(\mu) = \mathbb{E}_{\mu}(m_n^2) \quad \text{for} \quad n \in \mathbb{N}_0, \quad \mu \in M_1(X).$$

Now, let $\Sigma_T$ denote the $\sigma$-algebra of the sets that are invariant with respect to $T$, i.e.

$$\Sigma_T = \{ A \in \mathcal{F} : \mathbb{1}_{T^{-1}(A)} = \mathbb{1}_A \mathbb{P}_{\mu_*}\text{-a.s.} \}. \quad (3.2)$$

One can prove that $\mathbb{P}_{\mu_*}$ is the unique probability measure which is invariant under $T$, i.e.

$$\mathbb{P}_{\mu_*}(T^{-1}(A)) = \mathbb{P}_{\mu_*}(A) \quad \text{for any} \quad A \in \mathcal{F}.$$ 

In particular, $\mathbb{P}_{\mu_*}$ is an ergodic measure, i.e. $\mathbb{P}_{\mu_*}(A) \in \{0, 1\}$ for all $A \in \Sigma_T$. The SLLN for stationary sequences (cf. [11]) then implies the following statement.

**Lemma 3.1** ([13] Theorem 17.1.2). If $Z : \Omega \to X$ is a $\mathbb{P}_{\mu_*}$-integrable random variable, then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{l=0}^{n-1} Z \circ T^l = \mathbb{E}_{\mu_*}(Z|\Sigma_T) = \mathbb{E}_{\mu_*}(Z) \quad \mathbb{P}_{\mu_*}\text{-a.s.}$$

Let $(\phi_n^{(1)}, \phi_n^{(2)})_{n \in \mathbb{N}_0}$ be an arbitrary Markovian coupling of $\Pi$ on some properly constructed probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \mathbb{C})$, and let $\mathbb{E}_{x,y}$ denote an expected value with respect to $\mathbb{C}_{x,y} = \mathbb{C}(\phi_0^{(1)} = x, \phi_0^{(2)} = y)$. For every random variable $Z : \bar{\Omega} \to X$, let us consider $Z^{(i)} : \bar{\Omega} \to X$ given by

$$Z^{(i)}(\omega) = Z(\phi_0^{(i)}(\omega), \phi_1^{(i)}(\omega), \ldots) \quad \text{for} \quad \omega \in \bar{\Omega} \quad \text{and} \quad i \in \{1, 2\}. \quad (3.2)$$
In what follows, we formulate a few lemmas, whose proofs are given in Appendix.

**Lemma 3.2.** Suppose that
\[
\sum_{n=1}^{\infty} E_{x,y} |z_n^{(1)} - z_n^{(2)}| < \infty \quad \text{for all } x, y \in X. \tag{3.3}
\]
Then the functions
\[
f_{\inf}(x) := E_x \left( \left\lfloor \liminf_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} (z_l^2 \wedge m) - c \right\rfloor \wedge 1 \right),
\]
\[
f_{\sup}(x) := E_x \left( \left\lfloor \limsup_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} (z_l^2 \wedge m) - c \right\rfloor \wedge 1 \right) \tag{3.4}
\]
are constant (and, in particular, continuous) for any \(m \in \mathbb{N} \cup \{\infty\}\) and \(c \geq 0\). By convention, we write \(x \wedge m = x\) for \(m = \infty\) and every \(x \in \mathbb{R}\).

**Lemma 3.3.** Suppose that the functions \(f_{\inf}\) and \(f_{\sup}\), given by (3.4), are continuous for all \(m \in \mathbb{N} \cup \{\infty\}\) and any \(c \geq 0\). Then, for every \(\mu \in M_1(X)\) and any \(m \in \mathbb{N} \cup \{\infty\}\), we have
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} (z_l^2 \wedge m) = E_{\mu^*} (z_1^2 \wedge m) \quad \mathbb{P}_\mu\text{-a.s.}
\]
In particular, for \(m = \infty\), we obtain
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} z_l^2 = \sigma^2 \quad \mathbb{P}_\mu\text{-a.s.},
\]
where \(\sigma^2\) is defined by (3.1).

**Lemma 3.4.** Suppose that condition (3.3) holds, and that, for some \(\mu \in M_1(X)\), there exists \(r \in (0, 2)\) such that
\[
\sup_{n \in \mathbb{N}} E_{\mu} |z_n|^{2+r} < \infty. \tag{3.5}
\]
Then
\[
\lim_{n \to \infty} h_n^2(\mu) = \sigma^2, \tag{3.6}
\]
and also
\[
\lim_{n \to \infty} \frac{1}{h_n^2(\mu)} \sum_{l=1}^{n} z_l^2 = 1 \quad \mathbb{P}_\mu\text{-a.s.} \tag{3.7}
\]

**Lemma 3.5.** Assume condition (3.3), and suppose that (3.5) holds for some \(r \in (0, 2)\) and some \(\mu \in M_1(X)\). Then, there exists \(N \in \mathbb{N}\) such that \(h_n(\mu) > 0\) for all \(n \geq N\), and the
following statements hold:

\[ \sum_{n=N}^{\infty} h_n^{-1}\mathbb{E}_\mu \left( z_n^4 \mathbb{1}_{\{|z_n|<\upsilon h_n(\mu)\}} \right) < \infty \quad \text{for every} \quad \upsilon > 0, \quad (3.8) \]

\[ \sum_{n=N}^{\infty} h_n^{-1}\mathbb{E}_\mu \left( |z_n| \mathbb{1}_{\{|z_n|\geq \vartheta h_n(\mu)\}} \right) < \infty \quad \text{for every} \quad \vartheta > 0. \quad (3.9) \]

3.2 The Invariance Principle for the LIL for Certain Markov Chains

Define \( C \) as a Banach space of all real-valued continuous functions on \([0, 1]\) with the supremum norm. By \( K \) we denote the subspace of \( C \) consisting of all absolutely continuous functions \( f \) such that \( f(0) = 0 \) and \( \int_0^1 (f'(t))^2 \, dt \leq 1 \). Further, let \( (\phi_n)_{n \in \mathbb{N}_0} \) be an \( X \)-valued time-homogeneous Markov chain with initial distribution \( \mu \in \mathcal{M}_1(X) \) and transition law \( \Pi \), generating the Markov operator \( P \) and its dual operator \( U \) (according to the formulas \( (1.1) \) and \( (1.2) \)). Assume that conditions \( (B_0)-(B_1) \) are satisfied.

Further, suppose that there exists a substochastic kernel \( Q : X^2 \times \mathcal{B}_X \to [0, 1] \) enjoying \( (1.7) \), and such that conditions \( (B_2)-(B_5) \) hold for some \( F \subset X^2 \). Then it follows from Theorem 2.1 that \( P \) possesses a unique invariant measure \( \mu^* \in \mathcal{M}_1(X) \) such that \( \mu^* \in \mathcal{M}_1^{\varrho(\cdot, \bar{x})} \) and that \( (P^n \nu)_{n \in \mathbb{N}_0} \) converges weakly to \( \mu^* \) for every \( \nu \in \mathcal{M}_1(X) \).

Let \( g \in \text{Lip}_b(X) \), and define \( \bar{g} = g - \langle g, \mu^* \rangle \). Note that \( \langle \bar{g}, \mu^* \rangle = 0 \), and observe that Theorem 2.1 implies that there exist some \( q \in (0, 1) \) and some \( c > 0 \) such that

\[ \langle \bar{g}, P^i \delta_x \rangle = \langle \bar{g}, P^i \delta_x \rangle - \langle \bar{g}, \mu^* \rangle \leq \|ar{g}\|_{BL} d_{FM}(P^i \delta_x, \mu^*) \]

\[ \quad \leq c \|ar{g}\|_{BL} q^i (1 + \varrho(x, \bar{x}) + \langle \varrho(\cdot, \bar{x}), \mu^* \rangle) \leq \tilde{c} \|ar{g}\|_{BL} q^i (1 + \varrho(x, \bar{x})) \quad \text{for every} \quad x \in X \quad \text{and any} \quad i \in \mathbb{N}, \quad (3.10) \]

where \( \tilde{c} := c(1 + \langle \varrho(\cdot, \bar{x}), \mu^* \rangle) \). It then follows that

\[ \sum_{i=0}^{\infty} |U^i \bar{g}(x)| = \sum_{i=0}^{\infty} \left| \langle \bar{g}, P^i \delta_x - \mu^* \rangle \right| \leq \frac{\tilde{c} \|ar{g}\|_{BL}}{1-q} (1 + \varrho(x, \bar{x})) , \quad (3.11) \]

and we can therefore define

\[ \chi(\bar{g})(x) = \sum_{i=0}^{\infty} U^i \bar{g}(x) \quad \text{for any} \quad x \in X. \quad (3.12) \]
Note that $\chi(\bar{g})$ has the following property:

$$
|\chi(\bar{g})(x) - \chi(\bar{g})(y)| \leq \sum_{i=0}^{\infty} |\langle \bar{g}, P^i \delta_x - P^i \delta_y \rangle| \leq \|\bar{g}\|_{BL} \sum_{i=0}^{\infty} d_{FM}(P^i \delta_x, P^i \delta_y)
$$

$$
\leq \|\bar{g}\|_{BL} \sum_{i=0}^{\infty} (d_{FM}(P^i \delta_x, \mu_*) + d_{FM}(P^i \delta_y, \mu_*))
$$

$$
\leq \frac{2\epsilon\|\bar{g}\|_{BL}}{1 - q} (1 + g(x, \bar{x}) + g(y, \bar{x})), \quad x, y \in X,
$$

for some $q \in (0, 1)$ and some $\epsilon > 0$, where the last inequality follows from (3.10).

Now, introduce

$$
M_0(\bar{g}) = 0, \quad M_n(\bar{g}) = \chi(\bar{g})(\phi_n) - \chi(\bar{g})(\phi_0) + \sum_{i=0}^{n-1} \bar{g}(\phi_i) \quad \text{for} \quad n \in \mathbb{N},
$$

and note that $(M_n(\bar{g}))_{n \in \mathbb{N}_0}$ is a martingale with respect to the natural filtration of $(\phi_n)_{n \in \mathbb{N}_0}$ (for the proof see eg. [12, Lemma 3]). For any $g \in Lip_b(X)$ we also define

$$
Z_n(\bar{g}) = M_n(\bar{g}) - M_{n-1}(\bar{g}) = \chi(\bar{g})(\phi_n) - \chi(\bar{g})(\phi_{n-1}) + \bar{g}(\phi_{n-1}) \quad \text{for} \quad n \in \mathbb{N},
$$

$$
\sigma^2(\bar{g}) = \mathbb{E}_{\mu_*} (Z_1^2(\bar{g})) ,
$$

$$
h_n^2(\mu)(\bar{g}) = \mathbb{E}_{\mu}(M_n^2(\bar{g})) \quad \text{for} \quad n \in \mathbb{N}_0.
$$

One can easily note that $Z_n(\bar{g}) = Z_1(\bar{g}) \circ T^{n-1}$ for $n \in \mathbb{N}$.

Further, let us consider the sequence of random variables $(r_n(\bar{g}))_{n \in \mathbb{N}_0}$ with values in $\mathcal{C}$, determined by

$$
r_n(\bar{g})(t) = \begin{cases} \sum_{i=0}^{k-1} \bar{g}(\phi_i) + (nt - k)\bar{g}(\phi_k) & \text{for} \quad n > e, \quad t \in (0, 1], \\ \frac{\sigma(\bar{g})\sqrt{2n \ln \ln n}}{\sigma(\bar{g})\sqrt{2n \ln \ln n}} & \text{for} \quad k \leq nt \leq k + 1, \quad k = 1, \ldots, n - 1; \\ 0 & \text{for} \quad n \leq e \quad \text{or} \quad t = 0. \end{cases}
$$

For any given function $g \in Lip_b(X)$, we say that the Markov chain $(g(\phi_n))_{n \in \mathbb{N}_0}$ satisfies the invariance principle for the LIL if $0 < \sigma^2(\bar{g}) < \infty$, the family $\{r_n(\bar{g}) : n \in \mathbb{N}_0\}$ is relatively compact in $\mathcal{C}$, and the set of its limit points coincides with $K \subset \mathbb{R}$, $\mathbb{P}_{\mu}$-a.s. Observe that, whenever the chain $(g(\phi_n))_{n \in \mathbb{N}_0}$ satisfies the invariance principle for the LIL, it also obeys the LIL itself. Indeed, if $0 < \sigma^2(\bar{g}) < \infty$, then for any $n > e$ we can define

$$
\hat{r}_n(\bar{g}) = r_n(\bar{g})(1) = \frac{\sum_{i=1}^{n} \bar{g}(\phi_i)}{\sigma(\bar{g})\sqrt{2n \ln \ln n}},
$$

which, due to the definition of $K$, satisfies

$$
\limsup_{n \to \infty} \hat{r}_n(\bar{g}) = 1 \quad \text{and} \quad \liminf_{n \to \infty} \hat{r}_n(\bar{g}) = -1 \quad \mathbb{P}_{\mu}$-a.s.

Our aim now is to provide a theorem which may prove to be useful for biologists and
physicists to study their models in terms of the LIL. Therefore it shall be formulated in the same spirit as Theorem 2.1 and [11 Theorem 3.2] (see [2]-[4] for the description of the possible applications of these theorems). While conditions (B0)-(B5) are sufficient for the Markov operator $P$ to be exponentially ergodic in $d_{FM}$, the Strassen invariance principle for the LIL is proven upon assuming additionnally

\begin{align*}
(B1^*) \text{ there exist } a^* \in (0, 1), b^* \in (0, \infty) \text{ such that } \\
\left\langle q^{2+r}(\cdot, \bar{x}), P\nu \right\rangle^{1/(2+r)} \leq a^* \left\langle q^{2+r}(\cdot, \bar{x}), \nu \right\rangle^{1/(2+r)} + b^* \text{ for any } \nu \in \mathcal{M}_{1,1+r}(X).
\end{align*}

Remark 3.6. Let us compare condition (B1') and (B1*). Condition (B1*) is of the same type, although it does not need to imply (B1). Consequently, in Theorem 3.7 we assume both (B1) and (B1*).

Theorem 3.7. Suppose that $(\phi_n)_{n \in \mathbb{N}_0}$ is an $X$-valued time-homogeneous Markov chain with transition law $\Pi$ and initial distribution $\mu$ such that $\mu \in \mathcal{M}_{1,2+r}(X)$ for some $r \in (0, 2)$. Let $P$ denote the Markov operator corresponding to $\Pi$. Further, assume that there exists a substochastic kernel $Q : X^2 \times \mathcal{B}_{X^2} \to [0, 1]$ satisfying (1.7), such that conditions (B0),(B5) hold for $P$ and $Q$ with some $F \subset X^2$. Then, for every non-constant $g \in \text{Lip}_0(X)$, the chain $(g(\phi_n))_{n \in \mathbb{N}_0}$ satisfies the Strassen invariance principle for the LIL.

Before we prove Theorem 3.7, we first need to state several auxiliary facts. Lemmas 3.8,3.10 established below, concern certain properties of $(Z_n(\bar{g}))_{n \in \mathbb{N}_0}$, given by (3.15), while Lemma 3.11 indicates mutual relations between $\sigma^2(\bar{g})$ and $h_n^2(\mu)(\bar{g})$, given by (3.16) and (3.17), respectively. Finally, Lemmas 3.12 and 3.13 allow us to ensure the functional LIL for the appropriate sequence of random variables, introduced later on in (3.19) (cf. [10 Theorem 1]).

Let $(\phi^1, \phi^2)_{n \in \mathbb{N}_0}$ be a coupling of $\Pi$ such that condition (B5) holds.

Lemma 3.8. Under the assumptions of Theorem 3.7 we have

\[ \sum_{n=1}^{\infty} \mathbb{E}_{x,y} \left| Z_n^{(1)}(\bar{g}) - Z_n^{(2)}(\bar{g}) \right| < \infty \text{ for } x, y \in X, \]

where $Z_n^{(1)}$ and $Z_n^{(2)}$ are defined according to the rule given in (3.2), applied for the above-specified coupling $(\phi^1, \phi^2)_{n \in \mathbb{N}_0}$.

Proof. Note that

\begin{equation}
\left| Z_n^{(1)}(\bar{g}) - Z_n^{(2)}(\bar{g}) \right| \leq \left| \chi(\bar{g}) \left( \phi_n^{(1)} \right) - \chi(\bar{g}) \left( \phi_n^{(2)} \right) \right| + \left| \chi(\bar{g}) \left( \phi_n^{(1)} \right) - \chi(\bar{g}) \left( \phi_{n-1}^{(1)} \right) \right| + 2\|\bar{g}\|_{\infty}.
\end{equation}
Further, putting $C^n_{x,y}(\cdot) := C^n(x,y,\cdot)$, we can deduce that

$$E_{x,y} \left| \chi(\bar{g}) \left( \phi_n^{(1)} \right) - \chi(\bar{g}) \left( \phi_n^{(2)} \right) \right| \leq \sum_{i=0}^{\infty} E_{x,y} \left| U^i \bar{g} \left( \phi_n^{(1)} \right) - U^i \bar{g} \left( \phi_n^{(2)} \right) \right|$$

$$= \sum_{i=0}^{\infty} \int_{X^2} |U^i \bar{g}(u_1) - U^i \bar{g}(u_2)| |C^n_{x,y}(du_1 \times du_2)|$$

$$= \sum_{i=0}^{\infty} \int_{X^2} |\langle \bar{g}, P^i \delta_{u_1} \rangle - \langle \bar{g}, P^i \delta_{u_2} \rangle| |C^n_{x,y}(du_1 \times du_2)|$$

$$\leq \sum_{i=0}^{\infty} \int_{X^2} \int_{X^2} |g(v_1) - g(v_2)| |C^i_{u_1,u_2}(dv_1 \times dv_2)| |C^n_{x,y}(du_1 \times du_2)|$$

$$= \sum_{i=0}^{\infty} \int_{X^2} |g(v_1) - g(v_2)| |C^{n+i}_{x,y}(dv_1 \times dv_2) = \sum_{i=n}^{\infty} E_{x,y} \left| g \left( \phi_i^{(2)} \right) - g \left( \phi_i^{(2)} \right) \right|.$$

Hence, applying Lemma 2.2, we infer that there exist $q \in (0,1)$ and $c > 0$ such that

$$E_{x,y} \left| \chi(\bar{g}) \left( \phi_n^{(1)} \right) - \chi(\bar{g}) \left( \phi_n^{(2)} \right) \right| \leq c\|\bar{g}\|_{BL} (1 + g(x,\bar{x}) + g(y,\bar{x})) \sum_{i=n}^{\infty} q^i$$

$$\leq c\|\bar{g}\|_{BL} q^n (1 - q)^{-1} (1 + g(x,\bar{x}) + g(y,\bar{x}))$$

for every $n \in \mathbb{N}$. Combining (3.19) with (3.20), finally gives

$$\sum_{n=1}^{\infty} E_{x,y} \left| Z_n^{(1)}(\bar{g}) - Z_n^{(2)}(\bar{g}) \right| < \infty,$$

which completes the proof.

\[ \square \]

**Lemma 3.9.** Under the assumptions of Theorem 5.4, for any $g \in \text{Lip}_b(X)$ and any $\mu \in \mathcal{M}^{(\cdot,\cdot)}_{1,2+r}(X)$, where $r \in (0,2)$ is determined by condition (B1*), we have

$$\sup_{n \in \mathbb{N}} E_{\mu} \left| Z_n(\bar{g}) \right|^{2+r} < \infty.$$

**Proof.** Let $n \in \mathbb{N}$ and $\mu \in \mathcal{M}^{(\cdot,\cdot)}_{1,2+r}(X)$. Note that, due to the Markov property, we have

$$E_{\mu} \left| Z_n(\bar{g}) \right|^{2+r} = E_{\mu} \left( E_{\mu} \left( |Z_1(\bar{g})|^{2+r} \circ T^{n-1} \mid \mathcal{F}_{n-1} \right) \right) = E_{\mu} \left( E_{\phi_{n-1}} \mid Z_1(\bar{g}) \right|^{2+r})$$

$$= \int_X E_u \mid Z_1(\bar{g}) \mid^{2+r} P^{n-1} \mu(du).$$

One can easily prove that, for $r \in (0,2)$, there exists some $p \in (2,\infty)$ such that

$$(\psi_1 + \psi_2)^{2+r} \leq p \left( \psi_1^{2+r} + \psi_2^{2+r} \right) \quad \text{for any} \quad \psi_1, \psi_2 \geq 0.$$  

(3.22)
Hence, due to the definition of $Z_n(\bar{g})$, we obtain
\[
\mathbb{E}_\mu |Z_n(\bar{g})|^{2+r} \leq p \int_X |\chi(\bar{g})(u)|^{2+r} P^n \mu(du) + p^2 \int_X |\chi(\bar{g})(u)|^{2+r} P^{n-1} \mu(du) \\
+ p^2 \int_X |\bar{g}(u)|^{2+r} P^{n-1} \mu(du),
\]
where the last term can be majorized by $p^2\|\bar{g}\|_{\infty}^{2+r}$. Then, according to [3.13], there exist $q \in (0, 1)$ and $\tilde{c} > 0$ such that, for all $n \in \mathbb{N}$,
\[
\int_X |\chi(\bar{g})(u)|^{2+r} P^n \mu(du) = \int_X |\chi(\bar{g})(u) - \chi(\bar{g})(\bar{x}) + \chi(\bar{g})(\bar{x})|^{2+r} P^n \mu(du) \\
\leq p |\chi(\bar{g})(\bar{x})|^{2+r} + p \int_X |\chi(\bar{g})(u) - \chi(\bar{g})(\bar{x})|^{2+r} P^n \mu(du) \\
\leq p |\chi(\bar{g})(\bar{x})|^{2+r} + p^2 \left(\frac{2\tilde{c}\|\bar{g}\|_{BL}}{1-q}\right)^{2+r} \left(1 + \langle \bar{g}^{2+r}(\cdot, \bar{x}), P^n \mu \rangle \right).
\]
Further, from (B1*) it follows that
\[
\langle \bar{g}^{2+r}(\cdot, \bar{x}), P^n \mu \rangle^{1/(2+r)} \leq a^* \langle \bar{g}^{2+r}(\cdot, \bar{x}), P^{n-1} \mu \rangle^{1/(2+r)} + b^* \\
\leq \ldots \leq (a^*)^n \langle \bar{g}^{2+r}(\cdot, \bar{x}), \mu \rangle^{1/(2+r)} + \frac{b^*}{1-a^*},
\]
which gives
\[
\langle \bar{g}^{2+r}(\cdot, \bar{x}), P^n \mu \rangle \leq \left( \langle \bar{g}^{2+r}(\cdot, \bar{x}), \mu \rangle^{1/(2+r)} + \frac{b^*}{1-a^*} \right)^{2+r} \text{ for all } n \in \mathbb{N}.
\]
Finally, recalling that $\mu \in \mathcal{M}_{0,2+r}(X)$, we obtain
\[
\sup_{n \in \mathbb{N}} \mathbb{E}_\mu |Z_n(\bar{g})|^{2+r} < p\tilde{c} + p^2\tilde{c} + p^2\|\bar{g}\|_{\infty}^{2+r}
\]
with
\[
\tilde{c} = p |\chi(\bar{g})(\bar{x})|^{2+r} + p^2 \left(\frac{2\tilde{c}\|\bar{g}\|_{BL}}{1-q}\right)^{2+r} \left(1 + \left( \langle \bar{g}^{2+r}(\cdot, \bar{x}), \mu \rangle^{1/(2+r)} + \frac{b^*}{1-a^*} \right)^{2+r} \right) < \infty.
\]
The proof of Lemma 3.9 is therefore completed. \hfill \Box

**Lemma 3.10.** Under the assumptions of Theorem 3.7, for any $g \in Lip_b(X)$, we have
\[
\sigma^2(\bar{g}) = \mathbb{E}_\mu Z_1^2(\bar{g}) < \infty.
\]

**Proof.** For every $k \in \mathbb{N}$, we define $\tilde{V}_k : X \to [0, k]$ by $\tilde{V}_k(x) = \min\{k, \bar{g}^{2+r}(x, \bar{x})\}$ for $x \in X$. Note that $\tilde{V}_k \in C_b(X)$ for all $k \in \mathbb{N}$. Hence, letting $\mu \in \mathcal{M}_{0,2+r}(X)$, and keeping in mind that $P^n \mu$ converges weakly to $\mu_*$, as $n \to \infty$, we have
\[
\langle \tilde{V}_k, \mu_* \rangle = \lim_{n \to \infty} \langle \tilde{V}_k, P^n \mu \rangle \text{ for every } k \in \mathbb{N}.
\]

\[\tag{3.25} \]
Observe that \((\tilde{V}_k)_{k \in \mathbb{N}}\) is a non-increasing sequence of non-negative functions satisfying 
\[
\lim_{k \to \infty} \tilde{V}_k(x) = \varrho^{2+r}(x, \bar{x})
\]
for any \(x \in X\). Therefore, using the Monotone Convergence Theorem, together with (3.25) and (3.23), we obtain
\[
\langle \varrho^{2+r}(-, \bar{x}), \mu^* \rangle = \lim_{k \to \infty} \langle \tilde{V}_k, \mu^* \rangle = \lim_{n \to \infty} \lim_{k \to \infty} \langle \tilde{V}_k, P^n \mu \rangle
\]
\[
\leq \limsup_{n \to \infty} \langle \varrho^{2+r}(-, \bar{x}), P^n \mu \rangle \leq \left( \frac{b^*}{1-a^*} \right)^{2+r}
\]
which implies that \(\mu^* \in \mathcal{M}_{1,2+r}(X)\).

Hence, according to Lemma 3.9 and the Hölder inequality, we in particular obtain
\[
\mathbb{E}_{\mu^*} Z_1^2(\bar{g}) < \infty,
\]
which completes the proof. \(\Box\)

**Lemma 3.11.** Under the assumptions of Theorem 3.7, for every \(g \in \text{Lip}_b(X)\), we have
\[
\lim_{n \to \infty} \frac{h_n^2(\mu, g)}{n} = \sigma^2(g),
\]
where \(h_n(\mu, g)\) and \(\sigma(g)\) are defined by (3.17) and (3.16), respectively.

**Proof.** The assertion follows from Lemma 3.4. Note that conditions (3.3) and (3.5) are provided by Lemmas 3.8 and 3.9, respectively. \(\Box\)

**Lemma 3.12.** Let \(g \in \text{Lip}_b(X)\). Under the assumptions of Theorem 3.7, \(h_n(\mu, g)\) and \((Z_n(\bar{g}))_{n \in \mathbb{N}}\), given by (3.14) and (3.15), respectively, are related with each other in the following way:
\[
\lim_{n \to \infty} \frac{1}{h_n^2(\mu, g)} \sum_{l=1}^n Z_l^2(\bar{g}) = 1 \quad \mathbb{P}_{\mu^*}-a.s.
\]

**Proof.** Lemmas 3.8 and 3.9 guarantee that \((Z_l(\bar{g}))_{l \in \mathbb{N}}\) satisfies the assumptions of Lemma 3.4, which in turn implies the assertion of this lemma. \(\Box\)

**Lemma 3.13.** Under the assumptions of Theorem 3.7, for any \(g \in \text{Lip}_b(X)\), we have
\[
\sum_{n=1}^{\infty} h_n^{-4}(\mu, g) \mathbb{E} \left( Z_n^4(\bar{g}) \mathbb{1}_{\{|Z_n(\bar{g})| < \vartheta h_n(\mu, g)\}} \right) < \infty \quad \text{for every} \quad \vartheta > 0,
\]
\[
\sum_{n=1}^{\infty} h_n^{-4}(\mu, g) \mathbb{E} \left( |Z_n(\bar{g})| \mathbb{1}_{\{|Z_n(\bar{g})| \geq \vartheta h_n(\mu, g)\}} \right) < \infty \quad \text{for every} \quad \vartheta > 0.
\]

**Proof.** Having in mind that condition (3.5) is provided by Lemma 3.9, we see that the claim follows directly from Lemma 3.5. \(\Box\)

**Proof of Theorem 3.7.** The existence of \(\mu^* \in \mathcal{M}_1(X)\), being a unique invariant measure of \(P\), follows from Theorem 2.1.

The proof proceeds in two steps.

**Step I.** Let \(g \in \text{Lip}_b(X)\) be an arbitrary non-constant function. First of all, we will show
that the sequence \((h_n(\mu)(\bar{g}))\) is strictly increasing for some sufficiently large \(N \in \mathbb{N}\), which equivalently means that \(E_\mu(Z_n^2(\bar{g})) > 0\) for \(n \geq N\), and, simultaneously, that \(\sigma^2(\bar{g}) > 0\). Since \(Z_n(\bar{g}) = Z_1(\bar{g}) \circ T^{n-1}\), we obtain

\[
E_\mu(Z_n^2(\bar{g})) = E_\mu(E_\mu(Z_2^2(\bar{g}) \circ T^{n-1}|\mathcal{F}_{n-1})) = E_\mu(E_{\phi_n^{-1}}(Z_1^2(\bar{g})))
= \int \Omega E_{\phi_n^{-1}}(\omega)(Z_1^2(\bar{g})) \mathbb{P}_\mu(d\omega) = \int_X E_x(Z_1^2(\bar{g})) P^{n-1} \mu(dx),
\] (3.29)

where the second equality follows from the Markov property. According to (3.15), we have

\[
E_x(Z_1^2(\bar{g})) = E_x((\chi(\bar{g})(\phi_1) - \chi(\bar{g})(\phi_0) + \bar{g}(\phi_0))^2)
= U\chi^2(\bar{g})(x) + \chi^2(\bar{g})(x) + \bar{g}^2(x) + 2\bar{g}(x)U\chi(\bar{g})(x) - 2\chi(\bar{g})(x)U\chi(\bar{g})(x) - 2\chi(\bar{g})(x)\bar{g}(x).
\] (3.30)

Note that \(\chi^2(\bar{g}) \in \bar{B}_b(X)\), and therefore we can apply to it the extension of the dual operator \(U\), given by (1.12). On the other hand, from conditions (3.11) and (B1) (with \(V(\cdot) = \bar{g}(\cdot, \bar{x})\)), it follows that \(\chi(\bar{g})\) is integrable with respect to \(P\delta_x\) for every \(x \in X\), and thus \(U\chi(\bar{g})(x) = \int_X \chi(\bar{g})(y)P\delta_x(dy)\) is well-defined for any \(x \in X\). Further, note that

\[
U\chi(\bar{g})(x) = \int_X \sum_{i=0}^{\infty} U^i \bar{g}(y) P\delta_x(dy) = \sum_{i=0}^{\infty} \int_X U^{i+1} \bar{g}(y) \delta_x(dy) = \chi(\bar{g})(x) - \bar{g}(x).
\] (3.31)

Now, combining (3.30) with (3.31), we obtain

\[
E_x(Z_1^2(\bar{g})) = U\chi^2(\bar{g})(x) + \chi^2(\bar{g})(x) + \bar{g}^2(x) + 2\bar{g}(x)\chi(\bar{g})(x) - \bar{g}(x)
- 2\chi(\bar{g})(x)\chi(\bar{g})(x) - \bar{g}(x) - 2\chi(\bar{g})(x)\bar{g}(x)
= U\chi^2(\bar{g})(x) - (\chi(\bar{g})(x) - \bar{g}(x))^2 = U\chi^2(\bar{g})(x) - (U\chi(\bar{g})(x))^2,
\] (3.32)

which implies that \(E_x(Z_1^2(\bar{g})) > 0\) if and only if \(U\chi^2(\bar{g})(x) - (U\chi(\bar{g})(x))^2 > 0\). Note that the weak inequality always holds due to the Cauchy–Schwarz inequality, and it can only be an equality in the case of \(\chi(\bar{g}) \equiv c\) for some \(c \in \mathbb{R}\). Hence, whenever \(\chi(\bar{g})\) is not a constant function, (3.32) and (3.29) imply that \(E_\mu(Z_n^2(\bar{g})) > 0\) for every \(n \in \mathbb{N}\). This in turn yields that \((h_n(\mu)(\bar{g}))_{n \in \mathbb{N}}\) is strictly increasing, and, in particular \(\sigma^2(\bar{g}) > 0\). On the other hand, if \(\chi(\bar{g}) \equiv c\), then \(Z_1(\bar{g}) = \bar{g}(\phi_0)\), and thus, due to (3.29), we see that

\[
E_\mu(Z_n^2(\bar{g})) = \langle \bar{g}^2, P^{n-1} \mu \rangle \quad \text{for} \quad n \in \mathbb{N}, \quad \mu \in \mathcal{M}_1(X).
\] (3.33)

Further, from Theorem 2.1 it follows that

\[
\lim_{n \to \infty} \langle \bar{g}^2, P^{n-1} \mu \rangle = \langle \bar{g}^2, \mu_+ \rangle \quad \text{for any} \quad \mu \in \mathcal{M}^{V}_{1,1}(X),
\] (3.34)

and, according to the Cauchy–Schwarz inequality, we have

\[
\langle \bar{g}^2, \mu_+ \rangle = \langle \bar{g}^2, \mu_+ \rangle - \langle g, \mu_+ \rangle^2 > 0,
\] (3.35)

since \(g\) is not constant. Consequently, (3.33)-(3.35) imply that, in the case of constant \(\chi(\bar{g})\),
the sequence \((h_n(\mu))_{n \geq N}\) is strictly increasing for some sufficiently large \(N \in \mathbb{N}\). Let us also observe that \(\sigma^2(\bar{g}) = \mathbb{E}_{\mu_*}(Z^2_1(\bar{g})) = \langle \bar{g}^2, \mu_* \rangle > 0\), as claimed.

Upon the above reasoning, we can assume, without loss of generality, that the sequence \((h_n(\mu)(\bar{g}))_{n \in \mathbb{N}_0}\) is strictly increasing, and therefore we are allowed to introduce

\[
\eta_n(\bar{g})(t) = \frac{M_k(\bar{g}) + (h^2_n(\mu)(\bar{g})t - h^2_k(\mu)(\bar{g})) (h^2_{k+1}(\mu)(\bar{g}) - h^2_k(\mu)(\bar{g}))}{\sigma(\bar{g})\sqrt{2n \ln n \ln n}}
\text{for } n > e, \ t \in (0, 1]
\]

\[
\eta_n(\bar{g})(t) = 0 \text{ for } n \leq e \text{ or } t = 0.
\]

Further, Lemmas 3.12 and 3.13 ensure conditions (3.26) and (3.27)-(3.28), respectively. Combining this with (3.37) and referring to [10, Theorem 1], we can conclude that \(\{\eta_n(\bar{g}) : n \in \mathbb{N}_0\}\) is relatively compact in \(C\), and that the set of its limit points coincides with \(\mathcal{K} \mathbb{P}_{\mu}\text{-a.s.}\).

Now, define

\[
\tilde{\eta}_n(\bar{g})(t) = \frac{M_k(\bar{g}) - (nt - k)Z_{k+1}(\bar{g})}{\sigma(\bar{g})\sqrt{2n \ln n}} \text{ for } n > e, \ t \in (0, 1],
\]

\[
\tilde{\eta}_n(\bar{g})(t) = 0 \text{ for } n \leq e \text{ or } t = 0.
\]

Let \(t \in (0, 1]\) and \(n > e\). If \(k \leq nt \leq k + 1\), then

\[
\frac{k\sigma^2(\bar{g})}{h^2_k(\mu)(\bar{g})} h^2_n(\mu)(\bar{g}) \leq \frac{n\sigma^2(\bar{g})}{h^2_n(\mu)(\bar{g})} th^2_n(\mu)(\bar{g}) \leq \frac{(k+1)\sigma^2(\bar{g})}{h^2_{k+1}(\mu)(\bar{g})} h^2_{k+1}(\mu)(\bar{g}).
\]

Referring to Lemma 3.11, we see that \(\lim_{n \to \infty} n\sigma^2(\bar{g})h^{-2}_n(\mu)(\bar{g}) = 1\), and hence, due to (3.39), for any \(\epsilon > 0\), we have

\[
\frac{1 - \epsilon}{1 + \epsilon} h^2_n(\mu)(\bar{g}) \leq th^2_n(\mu)(\bar{g}) \leq \frac{1 + \epsilon}{1 - \epsilon} h^2_{k+1}(\mu)(\bar{g})
\]

for sufficiently large \(n \in \mathbb{N}\) and \(k \in \mathbb{N}\) such that \(k \leq nt \leq k + 1\).

Let us now prove that, for a fixed \(t \in (0, 1]\), there exists a sequence \((t_n)_{n \in \mathbb{N}}\) of positive numbers such that

\[
\tilde{\eta}_n(\bar{g})(t) = \eta_n(\bar{g})(t_n) \text{ for any } n \in \mathbb{N}
\]

and

\[
\lim_{n \to \infty} t_n = t.
\]
Fix $n > e$, and let $k$ be such that $k \leq nt \leq k + 1$. According to definitions (3.36) and (3.38), we see that equality (3.41) is satisfied for

$$t_n = \frac{(nt - k) \left( h_{k+1}^2(\mu)(g) - h_n^2(\mu)(g) \right) + h_k^2(\mu)(g)}{h_n^2(\mu)(g)},$$

whenever

$$h_k^2(\mu)(g) \leq t_n h_n^2(\mu)(g) \leq h_{k+1}^2(\mu)(g).$$

On the other hand, (3.44) obviously holds, since $0 \leq nt - k \leq 1$. Moreover, for every $\epsilon$ and sufficiently large $n$, we have

$$t_n \in \left[ \frac{1 - \epsilon}{1 + \epsilon}, \frac{1 + \epsilon}{1 - \epsilon} \right], \text{ whenever } k \leq nt \leq k + 1.$$

Indeed, from (3.40) and (3.41) it follows that

$$t_n \in \left[ \frac{h_k^2(\mu)(g)}{h_n^2(\mu)(g)} \frac{h_{k+1}^2(\mu)(g)}{h_n^2(\mu)(g)} \right] \subset \left[ \frac{(1 - \epsilon)h_n^2(\mu)(g)}{(1 + \epsilon)h_{k+1}^2(\mu)(g)}, \frac{(1 + \epsilon)h_{k+1}^2(\mu)(g)}{(1 - \epsilon)h_k^2(\mu)(g)} \right] \text{ for } k \leq nt \leq k + 1,$$

and, according to Lemma 3.11

$$\frac{h_{k+1}^2(\mu)(g)}{h_k^2(\mu)(g)} = \frac{h_{k+1}^2(\mu)(g)}{k + 1} \frac{k + 1}{h_k^2(\mu)(g)} = \frac{k + 1}{h_k^2(\mu)(g)}$$

converges to 1, as $n$, and therefore also $k$, tends to infinity.

Summarizing, for any $t \in (0, 1]$ there exists a sequence $(t_n)_{n \in \mathbb{N}}$ of positive reals, which enjoys properties (3.41) and (3.42), and this finally implies that $\{\bar{\eta}_n(g) : n \in \mathbb{N}_0\}$ is relatively compact in $\mathcal{C}$, and the set of its limit points coincides with $\bar{K} \mathbb{P}_\mu$-a.s.

**Step II.** To complete the proof it suffices to show that

$$\limsup_{n \to \infty} \sup_{t \in [0,1]} |\bar{\eta}_n(g)(t) - r_n(g)(t)| = 0,$$

(3.45)

where $(r_n(g))_{n \in \mathbb{N}_0}$ is given by (3.18). Indeed, note that (3.45), together with the conclusion of Step I, implies that $(g(\phi_n))_{n \in \mathbb{N}_0}$ satisfies the invariance principle for the LIL.

In order to establish (3.45), fix an arbitrary $\bar{\epsilon} > 0$ and, for $k, n \in \mathbb{N}$, define the sets

$$A_{k,n} = \left\{ \frac{|M_k(g) - \sum_{i=0}^{k-1} g(\phi_i)|}{\sigma(g) \sqrt{n \ln \ln n}} \geq \bar{\epsilon}/2 \right\} \cup \left\{ \frac{|Z_{k+1}(g) - g(\phi_k)|}{\sigma(g) \sqrt{n \ln \ln n}} \geq \bar{\epsilon}/2 \right\}.$$

Note that, for every $r > 0$, there exists $p \in (2, \infty)$ such that (3.22) holds. Using this
property, as well as the Markov inequality, we obtain

$$
P_\mu\left(\frac{|M_k(\bar{g}) - \sum_{i=0}^{k-1} \bar{g}(\phi_i)|}{\sigma(\bar{g})\sqrt{n \ln \ln n}} \geq \bar{\epsilon}/2\right) = P_\mu\left(\frac{|\chi(\bar{g})(\phi_k) - \chi(\bar{g})(\phi_0)|}{\sigma(\bar{g})\sqrt{n \ln \ln n}} \geq \bar{\epsilon}/2\right) \leq \frac{(2/\bar{\epsilon})^{2+r}}{} \left(\frac{\text{E}_\mu|\chi(\bar{g})(\phi_k)|^{2+r} + \text{E}_\mu|\chi(\bar{g})(\phi_0)|^{2+r}}{(\sigma(\bar{g})\sqrt{n \ln \ln n})^{2+r}}\right).$$

From (3.13) we know that there exist \( q \in (0,1) \) and \( \bar{\epsilon} \in (0,\infty) \) such that

$$\text{E}_\mu |\chi(\bar{g})(\phi_k)|^{2+r} = \int_X |\chi(\bar{g})(u)|^{2+r} P_\mu(du) \leq p |\chi(\bar{g})(\bar{x})|^{2+r} + p \int_X |\chi(\bar{g})(u) - \chi(\bar{g})(\bar{x})|^{2+r} P_\mu(du) \leq p |\chi(\bar{g})(\bar{x})|^{2+r} + p^2 \left(\frac{2\bar{\epsilon}}{1-q}\right)^{2+r} \left(1 + \langle \phi^{2+r}(\cdot, \bar{x}), P_\mu \rangle\right), \quad k \in \mathbb{N}. \tag{3.46}$$

Then, after applying (B1*) we obtain

$$\text{E}_\mu |\chi(\bar{g})(\phi_k)|^{2+r} \leq p |\chi(\bar{g})(\bar{x})|^{2+r} + p^2 \left(\frac{2\bar{\epsilon}}{1-q}\right)^{2+r} \left(1 + \langle \phi^{2+r}(\cdot, \bar{x}), \mu \rangle^{1/(2+r)} + \frac{b^*}{1-a^*}\right),$$

and hence

$$\mathbb{P}_\mu\left(\frac{|M_k(\bar{g}) - \sum_{i=1}^{k} \bar{g}(\phi_i)|}{\sigma(\bar{g})\sqrt{n \ln \ln n}} \geq \bar{\epsilon}/2\right) \leq \frac{c_1}{(\sigma(\bar{g})\sqrt{n \ln \ln n})^{2+r}} \quad \text{for any } n, k \in \mathbb{N}, \tag{3.47}$$

where \( c_1 > 0 \) is some constant independent of \( n \) and \( k \). Similarly, we deduce that

$$\mathbb{P}_\mu\left(\frac{|Z_{k+1}(\bar{g}) - \bar{g}(\phi_k)|}{\sigma(\bar{g})\sqrt{n \ln \ln n}} \geq \bar{\epsilon}/2\right) \leq \frac{c_2}{(\sigma(\bar{g})\sqrt{n \ln \ln n})^{2+r}} \quad \text{for any } n, k \in \mathbb{N}, \tag{3.48}$$

where \( c_2 > 0 \) also does not depend on \( n \) and \( k \). Now, (3.47) and (3.48) imply the convergence of \( \sum_{n=1}^\infty \mathbb{P}_\mu(A_{k,n}) \) for every \( k \in \mathbb{N} \), and hence, from the Borel–Cantelli Lemma, it follows that \( \mathbb{P}_\mu(\cup_{n=1}^\infty \cap_{m=1}^\infty A'_{k,n}) = 1 \) for all \( k \in \mathbb{N} \). Let

$$\Omega_0 := \bigcap_{k=1}^\infty \bigcup_{m=1}^\infty \bigcap_{n=m}^\infty A'_{k,n}.$$

Obviously, \( \mathbb{P}(\Omega_0) = 1 \). Furthermore, it is easily seen that for each \( \omega \in \Omega_0 \) one can choose \( n_0 > \epsilon \) such that

$$\sup_{t \in [0,1]} \left|\frac{M_k(\bar{g}) - (nt - k)Z_{k+1}(\bar{g})}{\sigma(\bar{g})\sqrt{n \ln \ln n}} - \frac{\sum_{i=0}^{k-1} \bar{g}(\phi_i) + (nt - k)\bar{g}(\phi_k)}{\sigma(\bar{g})\sqrt{n \ln \ln n}}\right| < \bar{\epsilon}.$$
for every $n > n_0$ and any $k \in \{1, \ldots, n-1\}$ satisfying $k < nt \leq k + 1$. The proof is now completed, since $\varepsilon$ was chosen arbitrarily. 

4 An Example Application to a Gene Expression Model

Let $(H, \| \cdot \|)$ and $Y$ be a separable Banach space and a closed subset of this space, respectively. Further, for any $h \in H$ and any $r > 0$, let $B(h, r)$ denote an open ball in $H$ centered at $h$ and of radius $r$. We additionally consider a topological measure space $(\Theta, B(\Theta), \Delta)$ with a $\sigma$-finite Borel measure $\Delta$. With a slight abuse of notation, we will write $d\theta$ instead of $\Delta(d\theta)$ in the rest of the paper. Finally, fix $N \in \mathbb{N}$, and endow the space $I := \{1, \ldots, N\}$ with the metric $(k, l) \mapsto d(k, l)$ given by $d(k, l) = 1$ for $k \neq l$ and $d(k, l) = 0$ for $k = l$.

A random dynamical system $(Y(t))_{t \in \mathbb{R}_+}$, which is a research object here, evolves through random jumps on the space $Y$. The jumps occur at random time points $\tau_n$, $n \in \mathbb{N}$, which coincide with the jump times of a Poisson process with intensity $\lambda$. In the time intervals $[\tau_{n-1}, \tau_n)$, $n \in \mathbb{N}$, where $\tau_0 = 0$, the system is deterministically driven by a finite number of semiflows $S_i : \mathbb{R}_+ \times Y \to Y$, $i \in I$, which are assumed to be continuous with respect to each variable. The semiflows are switched at the jump times according to a matrix of continuous functions $\pi_{ij} : Y \to [0, 1]$, $i, j \in I$, which satisfy $\sum_{j \in I} \pi_{ij}(y) = 1$ for any $y \in Y$, $i \in I$. The above description can be formalized by writing

$$Y(t) = S_{\xi_n}(t - \tau_n, Y(\tau_n)) \quad \text{for} \quad t \in [\tau_n, \tau_{n+1}),$$

where $\xi_n$ is an $I$-valued random variable indicating the index of a semiflow chosen directly after the $n$th jump.

For $n \in \mathbb{N}$, the post-jump location $Y(\tau_n)$ is a result of a transformation of the state $Y(\tau_n^-)$ just before the jump, attained by a function randomly selected among all the possible ones $w_\theta : Y \to Y$, $\theta \in \Theta$, and adding a random disturbance $H_n$, which remains within an $\varepsilon$-neighbourhood of zero. Formally, we have $Y(\tau_n) = w_{\theta_n}(Y(\tau_n^-)) + H_n$.

It is required that all the maps $(y, \theta) \mapsto w_\theta(y)$ are continuous, and also that there exists $\varepsilon^* > 0$ for which

$$w_\theta(y) + h \in Y \quad \text{whenever} \quad h \in B(0, \varepsilon^*), \quad \theta \in \Theta, \quad y \in Y.$$  

Moreover, we assume that all the variables $H_n$ (with values in $H$) have a common distribution $\nu^\varepsilon \in \mathcal{M}_1(H)$, supported on a ball $B(0, \varepsilon)$ with $\varepsilon \in [0, \varepsilon^*]$, and that the probability of choosing $w_\theta$ (at the jump time $\tau_n$) is determined by the density $\theta \mapsto p(y, \theta)$ whenever $Y(\tau_n^-) = y$, where $p : Y \times \Theta \to [0, \infty)$ is a continuous function satisfying $\int_{\Theta} p(y, \theta) d\theta = 1$ for any $y \in Y$.

The main result of this section pertains to the sequence of random variables $(Y_n)_{n \in \mathbb{N}_0}$ given by the post-jump locations of $(Y(t))_{t \in \mathbb{R}_+}$, that is, $Y_n = Y(\tau_n)$ for $n \in \mathbb{N}_0$. Such a sequence can be defined on an appropriate probability space, say $(\Omega, \mathcal{F}, \mathbb{P})$, by

$$Y_{n+1} = w_{\theta_{n+1}}(S_{\xi_n}(\Delta \tau_{n+1}, Y_n)) + H_{n+1} \quad \text{for} \quad n \in \mathbb{N}_0. \quad (4.1)$$

The random variables appearing in (4.1), together with their distributions, are specified by
the following conditions:

(i) The distributions of \( Y_0 : \Omega \to Y \) and \( \xi_0 : \Omega \to I \) are arbitrarily fixed.

(ii) The sequence \( (\tau_n)_{n \in \mathbb{N}_0}, \) where \( \tau_n : \Omega \to [0, \infty) \) for \( n \in \mathbb{N}_0 \) and \( \tau_0 = 0 \), is strictly increasing and such that \( \tau_n \to \infty \), as \( n \to \infty \). Moreover, the increments \( \Delta \tau_{n+1} := \tau_{n+1} - \tau_n \) are mutually independent and have the common exponential distribution with intensity \( \lambda > 0 \).

(iii) The disturbances \( H_n : \Omega \to H, n \in \mathbb{N}, \) are identically distributed with \( \nu^\circ \).

(iv) The variables \( \theta_n : \Omega \to \Theta \) and \( \xi_n : \Omega \to I, n \in \mathbb{N}, \) are defined inductively as follows:

\[
\mathbb{P}(\theta_{n+1} \in D \mid S_{\xi_n}(\Delta \tau_{n+1}, Y_n) = y; W_n) = \int_D p(y, \theta) \, d\theta \quad \text{for} \quad D \in \mathcal{B}(\Theta), \, y \in Y, \, n \in \mathbb{N}_0,
\]

\[
\mathbb{P}(\xi_{n+1} = j \mid Y_{n+1} = y, \xi_n = i; W_n) = \pi_{ij}(y) \quad \text{for} \quad y \in Y, \, i, j \in I, \, n \in \mathbb{N}_0,
\]

where \( W_0 = (Y_0, \xi_0) \) and \( W_n = (W_0, H_1, \ldots, H_n, \tau_1, \ldots, \tau_n, \theta_1, \ldots, \theta_n, \xi_1, \ldots, \xi_n) \) for \( n \in \mathbb{N} \).

We also demand that, for any \( n \in \mathbb{N}_0 \), the variables \( \Delta \tau_{n+1}, H_{n+1}, \theta_{n+1} \) and \( \xi_{n+1} \) are (mutually) conditionally independent given \( W_n \), and that \( \Delta \tau_{n+1} \) and \( H_{n+1} \) are independent of \( W_n \).

Finally, we assume that there exist \( \bar{y} \in Y \), a function \( \mathcal{L} : Y \to \mathbb{R}_+ \) which is bounded on bounded sets, and constants \( \alpha \in \mathbb{R}, \, L, \, L_w, \, L_p, \, \delta_\pi, \, \delta_\nu > 0 \) such that

\[
LL_w + \alpha/\lambda < 1, \tag{4.2}
\]

and, for all \( i, i_1, i_2 \in I, \, y_1, y_2 \in Y, \, t \geq 0 \), the following conditions hold:

\[
\sup_{y \in Y} \int_0^\infty e^{-\lambda t} \int_\Theta \|w_\theta(S_i(t, \bar{y})) - \bar{y}\| p(S_i(t, y), \theta) \, d\theta \, dt < \infty, \tag{A1}
\]

\[
\|S_{i_1}(t, y_1) - S_{i_2}(t, y_2)\| \leq Le^{\alpha t} \|y_1 - y_2\| + t\mathcal{L}(y_2)d(i_1, i_2), \tag{A2}
\]

\[
\int_\Theta p(y_1, \theta) \|w_\theta(y_1) - w_\theta(y_2)\| \, d\theta \leq L_w \|y_1 - y_2\|, \tag{A3}
\]

\[
\sum_{j \in I} |\pi_{ij}(y_1) - \pi_{ij}(y_2)| \leq L_\pi \|y_1 - y_2\| \quad \text{and} \quad \int_\Theta \|p(y_1, \theta) - p(y_2, \theta)\| \, d\theta \leq L_p \|y_1 - y_2\|, \tag{A4}
\]

\[
\sum_{j \in I} \min\{\pi_{i_1, j}(y_1), \pi_{i_2, j}(y_2)\} \geq \delta_\pi \quad \text{and} \quad \int_{\Theta(y_1, y_2)} \min\{p(y_1, \theta), p(y_2, \theta)\} \, d\theta \geq \delta_\nu, \tag{A5}
\]

where \( \Theta(y_1, y_2) := \{\theta \in \Theta : \|w_\theta(y_1) - w_\theta(y_2)\| \leq L_w \|y_1 - y_2\|\} \). Hypotheses (A1)-(A5) are discussed eg. in [3, 5].

An easy computation shows that \( (Y_n, \xi_n)_{n \in \mathbb{N}_0} \) is a time-homogeneous Markov chain,
evolving on $X = Y \times I$, with transition law $\Pi_\varepsilon : X \times \mathcal{B}(X) \to [0, 1]$ given by

$$
\Pi_\varepsilon(y, i, A) = \int_0^\infty \lambda e^{-\lambda t} \int_\Theta p(S_i(t, y), \theta) \times \int_{B(0, \varepsilon)} \left( \sum_{j \in I} \mathbf{1}_A(w_\theta(S_i(t, y)) + h, j) \pi_{ij}(w_\theta(S_i(t, y)) + h) \right) \nu^\varepsilon(dh) \, d\theta \, dt
$$

(4.3)

for any $(y, i) \in X$ and any $A \in \mathcal{B}_X$.

From the proof of [3, Theorem 4.1] it follows that, if conditions (A1)-(A5) hold with constants satisfying (4.2), then the hypotheses of Theorem 2.1 are fulfilled with the Markov process $\Pi_\varepsilon$ generated by $\Pi_\varepsilon$, a suitable substochastic kernel $Q$. Consequently, $P_\varepsilon$ is then exponentially ergodic in $d_{FM}$ induced by the metric $\rho_\varepsilon : X \times X \to \mathbb{R}$, given by

$$
\rho_\varepsilon((y_1, i), (y_2, j)) = \|y_1 - y_2\| + \hat{c} d(i, j) \quad \text{for} \quad (y_1, i), (y_2, j) \in X,
$$

with a sufficiently large $\hat{c}$ (defined explicitly in [3]).

Willing to verify the Strassen invariance principle for the LIL, we strengthen conditions (A1) and (A3). Namely, we require that, for some $r \in (0, 2)$, there exist $\bar{y} \in Y$ and $L^*_w > 0$ such that

$$
\sup_{y \in Y} \int_0^\infty e^{-\lambda t} \int_\Theta \|w_\theta(S_i(t, \bar{y})) - \bar{y}\|^{2+r} p(S_i(t, y), \theta) \, d\theta \, dt < \infty \quad \text{for} \quad i \in I. \quad (A1^*)
$$

$$
\int_\Theta \|w_\theta(y_1) - w_\theta(y_2)\|^{2+r} p(y_1, \theta) \, d\theta \leq L^*_w \|y_1 - y_2\|^{2+r} \quad \text{for} \quad y_1, y_2 \in Y. \quad (A3^*)
$$

Due to the Hölder inequality, conditions $(A1^*)$, $(A3^*)$ imply $(A1)$ and $(A3)$, respectively.

Within this section we assume that $V : X \to [0, \infty)$ is the Lyapunov function given by

$$
V(y, i) = \|y - \bar{y}\| \quad \text{for every} \quad (y, i) \in X, \quad (4.4)
$$

where $\bar{y}$ is determined by $(A1^*)$.

**Theorem 4.1.** Let $(Y_n, \xi_n)_{n \in \mathbb{N}_0}$ be the Markov chain with transition law $\Pi_\varepsilon$, given by (4.3), and an arbitrary initial distribution $\mu \in \mathcal{M}_1(X)$. Further, assume that conditions (A1)-(A5) with $(A1)$ and $(A3)$ strengthened to $(A1^*)$ and $(A3^*)$, respectively, hold with

$$
L^{2+r} L^*_w + (2 + r)\alpha \lambda^{-1} < 1. \quad (4.5)
$$

Then, for every non-constant $g \in \text{Lip}_b(X)$, the chain $(g(Y_n, \xi_n))_{n \in \mathbb{N}_0}$ satisfies the invariance principle for the LIL, provided that $\mu \in \mathcal{M}_{1, 2+r}(X)$ for some $r > 0$ and $V$ given by (4.4).

**Proof.** First of all, we will show that (4.5) implies that (4.2) holds with $L_w = (L^*_w)^{1/(2+r)}$. To see this, suppose, conversely to (4.2), that $LL_w + \alpha/\lambda \geq 1$. Then, noting that $\alpha \lambda^{-1} < 1/(2 + r) < 1$, we obtain $(LL_w)^{2+r} \geq (1 - \alpha/\lambda)^{2+r}$, which due to the Bernoulli inequality, leads to the contraction with (4.5).
First of all, note that (4.5) obviously implies \( \alpha \lambda^{-1} < 1/(2 + r) < 1 \). Let \( L_w = (L_w^*)^1/(2+r) \), and suppose, conversely to (4.2), that \( LL_w + \alpha/\lambda \geq 1 \). This, however, yields \( (LL_w)^{2+r} \geq (1 - \alpha/\lambda)^{2+r} \), which, due to the Bernoulli inequality, leads to the contradiction with (4.2). We therefore obtain that inequality (4.5) implies (4.2) with \( L_w = (L_w^*)^1/(2+r) \).

Let \( P_\varepsilon \) be the Markov operator corresponding to \( \Pi_\varepsilon \). We shall use the criterion for the invariance principle for the LIL, stated as Theorem 3.7. Note that \( \mathcal{M}_1^{1,2+r}(X) \) for \( \bar{x} = (\bar{y}, \tilde{t}) \) with \( \tilde{t} \in I \) and \( \bar{y} \in Y \), determined by (A1'). Hence, we have \( \mu \in \mathcal{M}_1^{1,2+r}(X) \). Since conditions (B0)-(B5) have already been verified for \( P_\varepsilon \) and a suitable substochastic kernel \( Q \) in the proof of [3, Theorem 4.1] (cf. also [5]), the proof of Theorem 4.1 reduces to showing (B1'). We have

\[
\langle g_{c}^{2+r}(\cdot, \bar{x}), P_\varepsilon \mu \rangle = \int_{X} \int_{X} g_{c}^{2+r}((z, l), (\bar{y}, \tilde{t})) \Pi_\varepsilon(y, i, dz \times dl) \mu(dy \times di)
\]

\[
= \int_{X} \int_{0}^{\infty} \lambda e^{-\lambda} \int_{\Theta} p(S_i(t, y), \theta) \int_{B(0, \varepsilon)} \left( \sum_{j \in I} \left( \|w_\theta(S_i(t, y)) + h - \bar{y}\right)
+ cd(j, \tilde{t}) \right)^{2+r} \pi_{ij} (w_\theta(S_i(t, y)) + h) \nu^\varepsilon(dh) \, d\theta \, dt \, \mu(dy \times di).
\]

(4.6)

Now, introduce \( Z = X \times [0, \infty) \times \Theta \times H \times I \), and define \( \nu \in \mathcal{M}_1(Z) \) as follows:

\[
\nu(A) = \int_{X} \int_{0}^{\infty} \lambda e^{-\lambda} \int_{\Theta} p(S_i(t, y), \theta) \int_{B(0, \varepsilon)} \left( \sum_{j \in I} \mathbb{I}_A(y, i, t, \theta, h, j) \pi_{ij} (w_\theta(S_i(t, y)) + h) \right)
\times \nu^\varepsilon(dh) \, d\theta \, dt \, \mu(dy \times di)
\]

for \( A \in \mathcal{B}_Z \).

Let us further consider \( \varphi_0 : Z \to \mathbb{R} \) given by

\[
\varphi_0(y, i, t, \theta, h, j) = \|w_\theta(S_i(t, y)) + h - \bar{y}\| + cd(j, \tilde{t}).
\]

Note that \( \varphi_0 \) is a non-negative Borel measurable function, and that

\[
\varphi_0(y, i, t, \theta, h, j) \leq \|w_\theta(S_i(t, y)) - w_\theta(S_i(t, \bar{y}))\| + \|w_\theta(S_i(t, \bar{y})) - \bar{y}\| + \|h\| + cd(j, \tilde{t}).
\]

Hence, using the Minkowski inequality, we obtain

\[
\langle g_{c}^{2+r}(\cdot, \bar{x}), P_\varepsilon \mu \rangle^{1/(2+r)}
\]

\[
= \left( \int_{Z} \varphi_0^{2+r}(y, i, t, \theta, h, j) \nu(dy \times di \times dt \times d\theta \times dh \times dj) \right)^{1/(2+r)}
\]

\[
\leq \left( \int_{Z} \|w_\theta(S_i(t, y)) - w_\theta(S_i(t, \bar{y}))\|^{2+r} \nu(dy \times di \times dt \times d\theta \times dh \times dj) \right)^{1/(2+r)}
\]

\[
+ \left( \int_{Z} \|w_\theta(S_i(t, \bar{y})) - \bar{y}\|^{2+r} \nu(dy \times di \times dt \times d\theta \times dh \times dj) \right)^{1/(2+r)} + \varepsilon + e,
\]

(4.7)
where the second component on the right-hand side of the inequality is finite due to (A1∗). According to assumptions (A3∗) and (A2), we further have

\[
\int_Z \| w_\theta(S_i(t,y)) - w_\theta(S_i(t,\bar{y})) \|^2 + r \nu(dy \times di \times dt \times d\theta \times dh \times dj) 
\leq \int_X \int_0^\infty \lambda e^{-\lambda t} L^*_w \| S_i(t,y) - S_i(t,\bar{y}) \|^2 + r dt \mu(dy \times di)
\]

\[
\leq \int_X \int_0^\infty \lambda e^{-\lambda t} L^*_w L^{2+r} e^{(2+r)\lambda t} \| y - \bar{y} \|^2 + r dt \mu(dy \times di)
\]

\[
\leq \lambda L^*_w L^{2+r} \left( \int_0^\infty e^{-(\lambda-(2+r)\alpha)t} dt \right) \left( \int_X \| y - \bar{y} \|^2 + r \mu(dy \times di) \right)
\]

\[
\leq \frac{\lambda L^*_w L^{2+r}}{\lambda-(2+r)\alpha} \langle \varphi_{2+r}^c(\cdot, \bar{x}), \mu \rangle,
\]

where the last inequality follows from the fact that \((2+r)\alpha < \lambda\), which is provided by (4.5). Hence, referring to (4.7) and (4.8), we obtain condition (B1∗) with

\[
a^* := \frac{\lambda L^*_w L^{2+r}}{\lambda-(2+r)\alpha}
\]

and

\[
b^* := \sup_{y \in Y} \left| \int_0^\infty e^{-\lambda t} \| w_\theta(S_i(t,\bar{y})) - \bar{y} \|^2 + r p(S_i(t,y),\theta) \mu(dy \times di) \right|^{1/(2+r)} + \varepsilon_* + c < \infty.
\]

Moreover, due to condition (4.5), we see that \(a^* \in (0,1)\), which completes the proof. \qed

Appendix

Within the appendix, we present the proofs of lemmas from Section 3.1.

Proof of Lemma 3.2. Fix an arbitrary \(m \in \mathbb{N} \cup \{\infty\}\) and \(c \geq 0\). We shall give the proof for \(f_{m,c}^{inf}\). The reasoning for \(f_{m,c}^{sup}\) is analogous. Note that, for every \(n \in \mathbb{N}\) and every \(k \in \mathbb{N}\), we have

\[
\frac{1}{n} \sum_{l=k+1}^{n} (z_l^2 \wedge m) = \frac{1}{n} \sum_{l=1}^{n} (z_l^2 \wedge m) - \frac{1}{n} \sum_{l=1}^{k} (z_l^2 \wedge m)
\]

and therefore

\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{l=k+1}^{n} (z_l^2 \wedge m) = \liminf_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} (z_l^2 \wedge m)
\] (4.9)
Hence, for an arbitrarily fixed \( l_0 \in \mathbb{N} \setminus \{1\} \), we obtain

\[
\begin{align*}
\inf_{m,c} f_{m,c}(x) &= \mathbb{E}_x \left( \left\lfloor \lim_{n \to \infty} \left( \frac{1}{n} \sum_{l=l_0}^{n} (z_l^2 \wedge m) - c \right) \right\rfloor \wedge 1 \right) \\
&= \lim_{n \to \infty} \mathbb{E}_x \left( \left\lfloor \inf_{k \geq n} \left( \frac{1}{k} \sum_{l=l_0}^{k} (z_l^2 \wedge m) - c \right) \right\rfloor \wedge 1 \right) \\
&= \lim_{n \to \infty} \lim_{N \to \infty} \mathbb{E}_x \left( \left\lfloor \min_{k \in \{n,n+1,\ldots,n+N\}} \left( \frac{1}{k} \sum_{l=l_0}^{k} (z_l^2 \wedge m) - c \right) \right\rfloor \wedge 1 \right).
\end{align*}
\]

Consequently, defining \( H_{n,N} : X \to \mathbb{R} \) for \( n, N \in \mathbb{N} \) and \( n \geq l_0 \), by the formula

\[
H_{n,N}(x) = \mathbb{E}_x \left( \left\lfloor \min_{k \in \{n,n+1,\ldots,n+N\}} \left( \frac{1}{k} \sum_{l=l_0}^{k} (z_l^2 \wedge m) - c \right) \right\rfloor \wedge 1 \right),
\]

we get

\[
\inf_{m,c} f_{m,c}(x) = \lim_{n \to \infty} \lim_{N \to \infty} H_{n,N}(x) \quad \text{for every } x \in X. \tag{4.10}
\]

Let us now observe that, for any \( \alpha_i, \lambda \in \mathbb{R} \), where \( i \in I \) and \( I \) is a nonempty, finite set, we have

\[
\left| \min_{i \in I} \alpha_i - \lambda \right| \wedge 1 = \left| \min_{i \in I} (\alpha_i \wedge (1 + \lambda) - \lambda) \right| \wedge 1.
\]

This in turn implies

\[
\begin{align*}
H_{n,N}(x) &= \mathbb{E}_x \left( \left\lfloor \min_{k \in \{n,n+1,\ldots,n+N\}} \left( \frac{1}{k} \sum_{l=l_0}^{k} (z_l^2 \wedge m) \wedge (1 + c) - c \right) \right\rfloor \wedge 1 \right) \\
&= \mathbb{E}_x \left( \left\lfloor \min_{k \in \{n,n+1,\ldots,n+N\}} \frac{1}{k} \left( \left( \sum_{l=l_0}^{k} (z_l^2 \wedge m) \wedge k(1 + c) - c \right) \right) \wedge 1 \right) \right) \quad x \in X.
\end{align*}
\]

For every pair \((n, N)\) such that \( n, N \in \mathbb{N} \) and \( n \geq l_0 \), let us consider a random variable \( \Psi_{n,N} : \Omega \to X \) given by

\[
\Psi_{n,N} = \min_{k \in \{n,n+1,\ldots,n+N\}} \frac{1}{k} \left( \sum_{l=l_0}^{k} (z_l^2 \wedge m) \wedge k(1 + c) \right) - c.
\]

Then \( H_{n,N}(x) = \mathbb{E}_x(|\Psi_{n,N}| \wedge 1), \) \( x \in X, \) and hence

\[
\begin{align*}
|H_{n,N}(x) - H_{n,N}(y)| &\leq \mathbb{E}_{x,y} \left| (|\Psi_{n,N}^{(1)}| \wedge 1) - (|\Psi_{n,N}^{(2)}| \wedge 1) \right| \\
&\leq \mathbb{E}_{x,y} \left| \Psi_{n,N}^{(1)} - |\Psi_{n,N}^{(2)}| \right| \leq \mathbb{E}_{x,y} \left| \Psi_{n,N}^{(1)} - \Psi_{n,N}^{(2)} \right|, \tag{4.11}
\end{align*}
\]
where the second inequality is implied by the following general property:

$$|\alpha \land c - \lambda \land c| \leq |\alpha - \lambda| \quad \text{for any} \quad \alpha, \lambda, c \in \mathbb{R}_+.$$  \hspace{1cm} (4.12)

Now, since for any \(c \in \mathbb{R}\) and all \(\alpha_i, \lambda_i \in \mathbb{R}_+, \, i \in I\), where \(I\) is a nonempty finite set,

$$|\min_{i \in I} \alpha_i - \min_{i \in I} \lambda_i| \leq \max_{i \in I} |\alpha_i - \lambda_i|$$

and

$$\left| \left( \sum_{i \in I} \alpha_i \right) \land c - \left( \sum_{i \in I} \lambda_i \right) \land c \right| \leq \sum_{i \in I} |\alpha_i \land c - \lambda_i \land c|,$$

we see that

$$\left| \Psi_{n,N}^{(1)} - \Psi_{n,N}^{(2)} \right| \leq \max_{k \in \{n, n+1, \ldots, n+N\}} \frac{1}{k} \left| \left( \sum_{l=l_0}^{k} \left( z_l^{(1)} \right)^2 \land m \right) \land k(1 + c) - \left( \sum_{l=l_0}^{k} \left( z_l^{(2)} \right)^2 \land m \right) \land k(1 + c) \right|$$

$$\leq \max_{k \in \{n, n+1, \ldots, n+N\}} \frac{1}{k} \sum_{l=l_0}^{k} \left( z_l^{(1)} \right)^2 \land k(1 + c) - \left( z_l^{(2)} \right)^2 \land k(1 + c),$$

where the last inequality follows from (4.12). Further, applying the inequality

$$|\alpha_1^2 \land \lambda - \alpha_2^2 \land \lambda| \leq 2\sqrt{\lambda} |\alpha_1 - \alpha_2|, \quad \text{where} \quad \alpha_1, \alpha_2 \in \mathbb{R} \quad \text{and} \quad \lambda \in \mathbb{R}_+,$$

we obtain

$$\left| \Psi_{n,N}^{(1)} - \Psi_{n,N}^{(2)} \right| \leq \max_{k \in \{n, n+1, \ldots, n+N\}} \frac{1}{k} \left| 2k(1 + c) \sum_{l=l_0}^{k} \left| z_l^{(1)} - z_l^{(2)} \right| \right| = 2(1 + c) \sum_{l=l_0}^{n+N} \left| z_l^{(1)} - z_l^{(2)} \right|.$$  \hspace{1cm} (4.13)

From the above estimation and (4.11) it follows that

$$|H_{n,N}(x) - H_{n,N}(y)| \leq 2(1 + c) \sum_{l=l_0}^{n+N} \mathbb{E}_{x,y} \left| z_l^{(1)} - z_l^{(2)} \right| \quad \text{for} \quad x, y \in X, \, n, N \in \mathbb{N}, \, n \geq l_0.$$  \hspace{1cm} (4.13)

Hence, according to (4.10), we have

$$|f_{m,c}^{\inf}(x) - f_{m,c}^{\inf}(y)| \leq 2(1 + c) \sum_{l=l_0}^{\infty} \mathbb{E}_{x,y} \left| z_l^{(1)} - z_l^{(2)} \right| \quad \text{for} \quad x, y \in X.$$  \hspace{1cm} (4.13)

Finally, since \(l_0 \in \mathbb{N}\) was chosen arbitrarily, and, by assumption of this lemma,
\[ \sum_{l=1}^{\infty} \mathbb{E}_{x,y} \left| z_l^{(1)} - z_l^{(2)} \right| < \infty \] for any \( x, y \in X \), we can conclude that
\[ \left| f_{m,c}^\inf(x) - f_{m,c}^\inf(y) \right| = 0 \] for any \( x, y \in X \).

The proof is therefore completed.

\[ \square \]

**Proof of Lemma 3.3.** Fix an arbitrary \( m \in \mathbb{N} \cup \{ \infty \} \) and let \( c_m := \mathbb{E}_{\mu_*} (z_1^2 \wedge m) \). According to Lemma 3.1, we know that
\[ \lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} (z_l^2 \wedge m) = \lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} (z_l^2 \wedge m) \circ T^{l-1} = \mathbb{E}_{\mu_*} (z_1^2 \wedge m) = c_m \ \text{P}_{\mu_*}\text{-a.s.,} \]
which ensures
\[ \int_X f_{m,c_m}^\inf(x) \mu_*(dx) = \mathbb{E}_{\mu_*} \left( \lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} (z_l^2 \wedge m) - c_m \right) \wedge 1 = 0, \quad (4.14) \]
\[ \int_X f_{m,c_m}^\sup(x) \mu_*(dx) = \mathbb{E}_{\mu_*} \left( \lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} (z_l^2 \wedge m) - c_m \right) \wedge 1 = 0. \quad (4.15) \]

Referring to (4.13), and using the fact that \( z_l^2 \circ T^N = z_{l+N}^2 \), for any \( N \in \mathbb{N} \), we have
\[ \mathbb{E}_{\mu} \left( \lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} (z_l^2 \wedge m) - c_m \right) \wedge 1 = \mathbb{E}_{\mu} \left( \lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} (z_l^2 \wedge m) - c_m \right) \wedge 1 = \mathbb{E}_{\nu_N} \left( \lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} (z_l^2 \wedge m) - c_m \right) \wedge 1 = \int_X f_{m,c_m}^\inf(x) \ P^N \mu(dx). \]

Similar reasoning leads to
\[ \mathbb{E}_{\mu} \left( \lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} (z_l^2 \wedge m) - c_m \right) \wedge 1 = \int_X f_{m,c_m}^\sup(x) \ P^N \mu(dx) \quad \text{for any} \ N \in \mathbb{N}. \]

The functions \( f_{m,c_m}^\inf \) and \( f_{m,c_m}^\sup \) are obviously bounded, and, by the assumption of the lemma, they are also continuous. Therefore, using the fact that \( (P^n \mu)_{n \in \mathbb{N}} \) converges weakly to \( \mu_* \in \mathcal{M}_1(X) \), and applying identities (4.14), (4.15), we can deduce that
\[ \mathbb{E}_{\mu} \left( \lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} (z_l^2 \wedge m) - c_m \right) \wedge 1 = \lim_{N \to \infty} \int_X f_{m,c_m}^\inf(x) \ P^N \mu(dx) \]
\[ = \int_X f_{m,c_m}^\inf(x) \mu_*(dx) = 0, \]
and analogously
\[
\mathbb{E}_\mu \left( \limsup_{n \to \infty} \left( \frac{1}{n} \sum_{l=1}^{n} (z_l^2 \wedge m) \right) - c_m \right) = c_m \wedge 1
\]
According to Lemmas 3.2 and 3.3 we know that, for any \( n \in \mathbb{N} \),
\[
\limsup_{n \to \infty} \int_X f_{m,c_n}^n (x) \, P^N \mu(dx) = \int_X f_{m,c_n}^n (x) \, \mu_* (dx) = 0.
\]
Finally, we obtain
\[
\liminf_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} (z_l^2 \wedge m) = \limsup_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} (z_l^2 \wedge m) = c_m \quad \mathbb{P}_\mu \text{-a.s.}
\]
which ends the proof.

**Proof of Lemma 7.4.** According to Lemmas 3.2 and 3.3 we know that, for any \( m \in \mathbb{N} \),
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} (z_l^2 \wedge m) = \mathbb{E}_{\mu_*} (z_1^2 \wedge m) \quad \mathbb{P}_\mu \text{-a.s.}
\]
Moreover, we see that \( (n^{-1} \sum_{l=1}^{n} (z_l^2 \wedge m))_{n \in \mathbb{N}} \) is bounded by \( m \), and therefore, due to the Dominated Convergence Theorem, we can conclude that
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} \mathbb{E}_\mu (z_l^2 \wedge m) = \mathbb{E}_\mu \left( \lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} (z_l^2 \wedge m) \right) = \mathbb{E}_{\mu_*} (z_1^2 \wedge m) \quad \text{for } m \in \mathbb{N}. \tag{4.16}
\]
Further, note that
\[
\sup_{l \in \mathbb{N}} \mathbb{E}_\mu \left( z_l^2 \mathbb{1}_{\{z_l^2 \geq m\}} \right) = \sup_{l \in \mathbb{N}} \mathbb{E}_\mu \left( |z_l|^{2+r} (z_l^2)^{-r/2} \mathbb{1}_{\{z_l^2 \geq m\}} \right) \leq m^{-r/2} \sup_{l \in \mathbb{N}} \mathbb{E}_\mu |z_l|^{2+r} \quad \text{for } m \in \mathbb{N}.
\]
Hence, according to assumption (3.5), we infer that
\[
\sup_{l \in \mathbb{N}} \mathbb{E}_\mu \left( z_l^2 \mathbb{1}_{\{z_l^2 \geq m\}} \right) \to 0, \quad \text{as } m \to \infty. \tag{4.17}
\]
Now, observe that, for every \( l \in \mathbb{N} \) and any \( m \in \mathbb{N} \),
\[
\begin{align*}
|\mathbb{E}_\mu (z_l^2 \wedge m) - \mathbb{E}_\mu (z_l^2)| &\leq \left| \mathbb{E}_\mu (z_l^2 \wedge m) - \mathbb{E}_\mu (z_l^2 \mathbb{1}_{\{z_l^2 \geq m\}}) \right| + \mathbb{E}_\mu (z_l^2 \mathbb{1}_{\{z_l^2 \geq m\}}) \\
&= \left| \mathbb{E}_\mu (\mathbb{1}_{\{z_l^2 \geq m\}}) + \mathbb{E}_\mu (z_l^2 \mathbb{1}_{\{z_l^2 < m\}}) - \mathbb{E}_\mu (z_l^2 \mathbb{1}_{\{z_l^2 < m\}}) + \mathbb{E}_\mu (z_l^2 \mathbb{1}_{\{z_l^2 \geq m\}}) \right| + \mathbb{E}_\mu (z_l^2 \mathbb{1}_{\{z_l^2 \geq m\}}) \\
&= \mathbb{E}_\mu (\mathbb{1}_{\{z_l^2 \geq m\}}) + \mathbb{E}_\mu (z_l^2 \mathbb{1}_{\{z_l^2 \geq m\}}) \\
&\leq 2\mathbb{E}_\mu (z_l^2 \mathbb{1}_{\{z_l^2 \geq m\}}) \leq 2 \sum_{l \in \mathbb{N}} \mathbb{E}_\mu (z_l^2 \mathbb{1}_{\{z_l^2 \geq m\}}),
\end{align*}
\]
which gives, for any \( n \in \mathbb{N} \),
\[
\left| \frac{1}{n} \sum_{l=1}^{n} \mathbb{E}_{\mu} \left( z_{l}^2 \wedge m \right) - \frac{1}{n} \sum_{l=1}^{n} \mathbb{E}_{\mu} \left( z_{l}^2 \right) \right| \leq \frac{1}{n} \sum_{l=1}^{n} \left| \mathbb{E}_{\mu} \left( z_{l}^2 \wedge m \right) - \mathbb{E}_{\mu} \left( z_{l}^2 \right) \right| \\
\leq 2 \sup_{l \in \mathbb{N}} \mathbb{E}_{\mu} \left( z_{l}^2 \mathbb{1}_{\{z_{l}^2 \geq m\}} \right).
\]

Consequently, we can write
\[
\left| \mathbb{E}_{\mu_*} \left( z_{1}^2 \right) - \frac{1}{n} \sum_{l=1}^{n} \mathbb{E}_{\mu} \left( z_{l}^2 \right) \right| \leq \left| \mathbb{E}_{\mu_*} \left( z_{1}^2 \right) - \mathbb{E}_{\mu_*} \left( z_{1}^2 \wedge m \right) \right| + \left| \mathbb{E}_{\mu_*} \left( z_{1}^2 \wedge m \right) - \frac{1}{n} \sum_{l=1}^{n} \mathbb{E}_{\mu} \left( z_{l}^2 \wedge m \right) \right| \\
+ 2 \sup_{l \in \mathbb{N}} \mathbb{E}_{\mu} \left( z_{l}^2 \mathbb{1}_{\{z_{l}^2 \geq m\}} \right)
\]
for any \( m \in \mathbb{N} \), and hence, due to \((4.16)\), we have
\[
\limsup_{n \to \infty} \left| \frac{1}{n} \sum_{l=1}^{n} \mathbb{E}_{\mu} \left( z_{l}^2 \right) - \mathbb{E}_{\mu_*} \left( z_{1}^2 \right) \right| \leq 2 \sup_{l \in \mathbb{N}} \mathbb{E}_{\mu} \left( z_{l}^2 \mathbb{1}_{\{z_{l}^2 \geq m\}} \right) + \left| \mathbb{E}_{\mu_*} \left( z_{1}^2 \wedge m \right) - \mathbb{E}_{\mu_*} \left( z_{1}^2 \right) \right|
\]
for all \( m \in \mathbb{N} \). Now, we see that the right-hand side of the above inequality tends to zero, as \( m \to \infty \), which follows from \((4.17)\). Finally, by the orthogonality of martingale differences, we get
\[
\lim_{n \to \infty} \frac{h_{n}^2(\mu)}{n} = \lim_{n \to \infty} \frac{1}{n} \mathbb{E}_{\mu} \left( m_{n}^2 \right) = \lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} \mathbb{E}_{\mu} \left( z_{l}^2 \right) = \mathbb{E}_{\mu_*} \left( z_{1}^2 \right) = \sigma^2
\]
which completes the proof of \((3.6)\).

Now, in order to establish \((3.7)\), it is enough to observe that \((3.6)\) and Lemma \(3.3\) imply
\[
\lim_{n \to \infty} \frac{1}{h_{n}^2(\mu)} \sum_{l=1}^{n} z_{l}^2 = \frac{1}{\sigma^2} \lim_{n \to \infty} \frac{1}{n} \sum_{l=1}^{n} z_{l}^2 = 1.
\]

\[\square\]

**Proof of Lemma \(3.5\).** Let \( \upsilon, \vartheta > 0 \). From Lemma \(3.4\), we can deduce that \( h_{n}(\mu) > 0 \) for \( n \geq N \), where \( N \) is some sufficiently large constant. Now, since \( r \in (0, 2) \), we obtain, for \( n \geq N \),
\[
h_{n}^{-4}(\mu) \mathbb{E}_{\mu} \left( z_{n}^4 \mathbb{1}_{\{|z_{n}| < \upsilon h_{n}(\mu)\}} \right) \leq h_{n}^{-4}(\mu) \mathbb{E}_{\mu} \left( |z_{n}|^{2+r} \vartheta^{2-r} h_{n}^{2-r}(\mu) \right) \\
\leq \upsilon^{2-r} \left( \sup_{n \in \mathbb{N}} \mathbb{E}_{\mu} |z_{n}|^{2+r} \right) h_{n}^{-2-r}(\mu),
\]

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and similarly
\[ h_n^{-1}(\mu)\mathbb{E}_\mu \left( |z_n| \mathbb{1}_{\{|z_n| \geq \vartheta h_n(\mu)\}} \right) \leq h_n^{-1}(\mu)\mathbb{E}_\mu \left( \frac{|z_n|^{2+r}}{(\vartheta h_n(\mu))^{1+r}} \right) \]
\[ \leq \vartheta^{-1-r} \left( \sup_{n \in \mathbb{N}} \mathbb{E}_\mu |z_n|^{2+r} \right) h_n^{-2-r}(\mu). \]

Since \( \sup_{n \in \mathbb{N}} \mathbb{E}_\mu |z_n|^{2+r} < \infty \), and, due to Lemma 2.4, \( \sum_{n=N}^{\infty} h_n^{-2-r}(\mu) < \infty \), the proof is completed. \( \square \)

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References

[1] W. Bolt, A.A. Majewski, and T. Szarek. An invariance principle for the law of the iterated logarithm for some Markov chains. Studia Mathematica, 212:41–53, 2012.

[2] D. Czapla. A criterion on asymptotic stability for partially equicontinuous Markov operators. Stochastic Processes and their Applications, 128(11):3656–3678, 2018.

[3] D. Czapla, K. Horbacz, and H. Wojewódka. Ergodic properties of some piecewise-deterministic Markov process with application to gene expression modelling. Preprint at https://arxiv.org/abs/1707.06489 (2017), Unpublished results.

[4] D. Czapla, K. Horbacz, and H. Wojewódka. A useful version of the central limit theorem for a general class of Markov chains. Preprint at https://arxiv.org/abs/1804.09220 (2018), Unpublished results.

[5] D. Czapla and J. Kubieniec. Exponential ergodicity of some Markov dynamical systems with application to a Poisson driven stochastic differential equation. Dynamical Systems, In press, doi:10.1080/14689367.2018.1485879, 2018.

[6] J.L. Doob. Stochastic Processes. John Wiley & Sons, New York, 1953.

[7] R.M. Dudley. Probabilities and metrics. Convergence of laws on metric spaces, with a view to statistical testing. Lecture Notes Series, No. 45, Matematisk Institut, Aarhus Universitet, Aarhus, 1976.

[8] M. Hairer. Exponential mixing properties of stochastic PDEs through asymptotic coupling. Probability Theory and Related Fields, 124(3):345–380, 2002.

[9] P. Hall and C. C. Heyde. Martingale Limit Theory and its Application. Elsevier, 1980.
[10] C. C. Heyde and D. J. Scott. Invariance principles for the law of the iterated logarithm for martingales and processes with stationary increments. The Annals of Probability, 1(3):428–436, 1973.

[11] S.C. Hille, K. Horbacz, and T. Szarek. Existence of a unique invariant measure for a class of equicontinuous Markov operators with application to a stochastic model for an autoregulated gene. Annales mathématiques Blaise Pascal, 23(2):171–217, 2016.

[12] S.C. Hille, K. Horbacz, T. Szarek, and H. Wojewódka. Law of the iterated logarithm for some Markov operators. Asymptotic Analysis, 97(1-2):91–112, 2016.

[13] R. Kapica and M. Ślęczka. Random iterations with place dependent probabilities. Preprint at https://arxiv.org/abs/1107.0707 (2012), Unpublished results.

[14] Wu L. Functional law of iterated logarithm for additive functionals of reversible Markov processes. Acta Mathematicae Applicatae Sinica, 16(2):149–161, 2000.

[15] A. Lasota. From fractals to stochastic differential equations, in: Chaos-the interplay between stochastic and deterministic behaviour. Lecture Notes in Phys. (Springer Verlag), 457:235–255, 1995.

[16] A. Lasota and M.C. Mackey. Cell division and the stability of cellular populations. J. Math. Biol., 38:241–261, 1999.

[17] M.C. Mackey, M. Tyran-Kamińska, and R. Yvinec. Dynamic behavior of stochastic gene expression models in the presence of bursting. SIAM Journal on Applied Mathematics, 73(5):1830–1852, 2013.

[18] S.P. Meyn and R.L. Tweedie. Markov chains and stochastic stability. Springer-Verlag, London, 1993.

[19] D. Revuz. Markov chains. North-Holland Elsevier, Amsterdam, 1975.

[20] M. Ślęczka. Exponential convergence for Markov systems. Ann. Math. Sil., 29:139–149, 2015.

[21] V. Strassen. An invariance principle for the law of the iterated logarithm. Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete, 3(3):211–226, 1964.

[22] H. Wojewódka. Exponential rate of convergence for some Markov operators. Statistics & Probability Letters, 83(10):2337–2347, 2013.

[23] O. Zhao and M. Woodroofe. Law of the iterated logarithm for stationary processes. The Annals of Probability, 36(1):127–142, 2008.