PARTIAL MODEL CATEGORIES
AND THEIR SIMPLICIAL NERVES

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Abstract. In this note we consider partial model categories, by which we mean relative categories that satisfy a weakened version of the model category axioms involving only the weak equivalences. More precisely, a partial model category will be a relative category that has the two out of six property and admits a 3-arrow calculus.

We then show that Charles Rezk’s result that the simplicial space obtained from a simplicial model category by taking a Reedy fibrant replacement of its simplicial nerve is a complete Segal space also holds for these partial model categories.

We also note that conversely every complete Segal space is Reedy equivalent to the simplicial nerve of a partial model category and in fact of a homotopi- cally full subcategory of a category of diagrams of simplicial sets.

1. Introduction

1.1. Background and motivation.

(i) Given a relative category \((C, U)\) (i.e. a pair consisting of a category \(C\) and a subcategory \(U\) which contains all the objects and of which the maps are called weak equivalences), one can form the localization of \(C\) with respect to \(U\), i.e. the category \(\text{Ho}(C, U)\) (called its homotopy category), obtained from \(C\) by formally inverting the weak equivalences.

However it was noted in [DK1] that one can also form a simplicial localization \(L(C, U)\) of \(C\) with respect to \(U\), which is a simplicial category (i.e. a category enriched in simplicial sets) which has the same objects as \(C\) and which has the property that the category obtained by replacing each simplicial set by its set of components is exactly \(\text{Ho}(C, U)\).

Moreover it was noted in [DK2, DK3] that

• if \(M\) is a simplicial model category and \(X\) and \(Y\) are respectively cofibrant and fibrant objects of \(M\), then the function complex \(M_\ast(X, Y)\) has the same homotopy type as the simplicial set \(L(M, W)(X, Y)\), where \(W \subset M\) denotes its category of weak equivalences.

A key step in the proof of this result was the observation that if

• a relative category \((C, U)\) with the two out of three property admits a 3-arrow calculus (which means that there exists subcategories \(U_c\) and \(U_f \subset U\) which have some of the properties of the categories of the trivial cofibrations and trivial fibrations in a model category)

then

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• for every two objects $C_1, C_2 \in C$, the homotopy type of $L(C, U)(C_1, C_2)$ admits a rather simple description in terms of 3-arrow zigzags

$$C_1 \leftarrow \cdot \rightarrow \cdot \leftarrow C_2$$

in which the outside maps are weak equivalences.

(ii) While [DK1, DK2, DK3] dealt with attempts to understand the function complexes in simplicial model categories, [DHKS] turned its attention to the homotopy limit and colimit functors. It was noted there that if

• a relative category $(C, U)$, instead of the usual two out of three property, had the stronger two out of six property that, for every three maps $r, s$ and $t \in C$ for which $sr$ and $ts$ were defined, if the two maps $sr$ and $ts$ were in $U$, then so were the other four maps $r$, $s$, $t$ and $tsr$,

then

• one could define homotopy limit and colimit functors which, if they existed, were homotopically unique, and give sufficient conditions for their existence and composability

and

• these sufficient conditions could be simplified if $(C, U)$ was saturated (i.e. a map in $C$ was a weak equivalence iff its image in $\text{Ho}(C, U)$ was an isomorphism)

while

• a sufficient condition for such saturation was the presence of a 3-arrow calculus (i).

(iii) All this suggests that the notion of a relative category which has the two out of six property and admits a 3-arrow calculus is a useful one and deserves further investigation. As these three conditions are essentially parts of the model category axioms which only involve the weak equivalences we will refer to categories with weak equivalences with these three properties as partial model categories.

1.2. Some examples of partial model categories. Some rather obvious examples of partial model categories are:

(i) Model categories, and in particular the relative category $S$ of simplicial sets.

(ii) If $(C, U)$ is a partial model category, then so are all its homotopically full relative categories, i.e. all relative categories of the form $(C', C' \cap U)$ where $C'$ is a full subcategory of $C$ with the property that, for every object $C' \in C'$, all objects of $C$ which are weakly equivalent to $C'$ are also in $C'$.

(iii) If $(C, U)$ is a partial model category, then so is, for every relative category $(A, X)$ the relative category $(C, U)^{A, X}$ of the relative functors $(A, X) \rightarrow (C, U)$.

Our aim in this note now is to prove the following.

1.3. A generalization of a result of Charles Rezk.

(i) Rezk’s result [R, 8.3] that “for every simplicial model category the Reedy fibrant replacement of its simplicial nerve is a complete Segal space” also holds for partial model categories.
Moreover, conversely, 

(ii) every complete Segal space is Reedy equivalent to the simplicial nerve of a partial model category, and in fact of a homotopically full relative subcategory of a category of diagrams of simplicial sets.

As if often the case with more general results, the proof of (i) is simpler than Rezk’s. In proving the Segal part of his result, i.e. showing that certain fibre products (which are iterated pullbacks) are homotopy fibre products (which are iterated homotopy pullbacks) he relied heavily on the simplicial structure of his model category. However it turns out that, in view of the Quillen Theorem B3 for homotopy pullbacks of [BK4, 8.2–4] this problem reduces to a rather straightforward calculation which only involves the partial model structure.

The proof of (ii) however requires a very different argument involving the following.

1.4. A relative Yoneda embedding and partial modelization. Given a relative category \((C, U)\) we construct a relative Yoneda embedding \((C, U) \rightarrow S^{C^{op}, U^{op}}\) and note that

- the essential image \(E_y\) of \(y\), i.e. the homotopically full relative subcategory of \(S^{C^{op}, U^{op}}\) (and hence of \(S^{C^{op}}\)) spanned by the image of \(y\), is a partial model category

and that

- the inclusion \((C, U) \rightarrow E_y\) is a DK-equivalence i.e. its simplicial localization is a weak equivalence of simplicial categories [Be] or equivalently [BK1, 1.8] its simplicial nerve is a Rezk (i.e. complete Segal) equivalence of simplicial spaces.

1.5. Organization of the paper. There are four more sections.

In §2 we introduce partial model categories and discuss a few immediate consequences of their definition. The notion of a 3-arrow calculus is slightly stronger than the one used in [DK2, DK3] and [DHKS] in that in addition to the functorial factorization we require that the trivial cofibration-like subcategory be closed under pushouts and that the trivial fibration-like subcategory be closed under pullbacks.

In §3 and §4 we then state and prove the results that were mentioned in 1.3(i) and 1.3(ii) respectively, while §5 deals with the partial modelization 1.4.

2. Partial model categories

In this section we introduce partial model categories and discuss a few immediate consequences of their definition.

2.1. Partial model categories. A partial model category will be a pair \((C, W)\) consisting of a category \(C\) and a subcategory \(W \subseteq C\) (the maps of which will be called weak equivalences) which, roughly speaking, satisfies those parts of the model category axioms (as for instance reformulated in [DHKS, 9.1]) which involve only the weak equivalences. More precisely we require that

a) \((C, W)\) be a relative category, that \(W\) contains all the objects of \(C\) (and hence also their identity maps),
b) \((C, W)\) has the **two out of six property** that, if \(r, s\) and \(t\) are maps in \(C\) such that the two compositions \(sr\) and \(ts\) exist and are in \(W\), then the four maps \(r, s, t\) and \(tsr\) are also in \(W\) (which together with other readily implies that \((C, W)\) has the two out of three property and that \(W\) contains all the isomorphisms.

c) \((C, W)\) admits a **3-arrow calculus**, i.e. there exists subcategories \(U, V \subset W\) which behave very much like the categories of the trivial cofibrations and the trivial fibrations in a model category in the sense that

(i) for every map \(u \in U\), its pushouts in \(C\) exist and are again in \(U\),

(ii) for every map \(v \in V\), its pullbacks in \(C\) exist and are again in \(V\), and

(iii) the maps \(w \in W\) admit a functorial factorization \(w = vu\) with \(u \in U\) and \(v \in V\) (which implies that \(U\) and \(V\) contain all the objects).

It should be noted that conditions (i) and (ii) are stronger than the ones that were used in [DK2] and [DHKS]. However we prefer them as they are cleaner and easier to work with and are likely to be usually automatically satisfied.

One then readily verifies that the following are

2.2. **Examples of partial model categories.**

(i) For every model category its underlying relative category is (of course) a partial model category.

(ii) Every homotopically full \((1.2(ii))\) relative subcategory of a partial model category is again a partial model category.

(iii) For every partial model category \((C, U)\) and relative category \((A, X)\), the relative functor category \((C, U)^{A, X}\) \((1.2(iii))\) is again a partial model category.

(iv) For every partial model category \((C, U)\) the relative category \((U, U)\) is a partial model category.

3. **A generalization of a result of Rezk**

3.1. **Saturation** [DHKS 36.4]. Every partial model category \((C, W)\) is saturated in the sense that a map of \(C\) is in \(W\) iff it goes to an isomorphism in the homotopy category \(\text{Ho}(C, W)\), i.e. the category obtained from \(C\) by “formally inverting” the weak equivalences.

Using this we will now show that a much stronger result of Rezk on simplicial model categories [R 8.3] also holds for partial model categories.

Before formulating this we first recall

3.2. **Rezk’s complete Segal model structure.** In [R]

a) Rezk constructed a “homotopy theory of homotopy theories” model structure on the category \(sS\) of simplicial spaces (i.e. bisimplicial sets) by means of an appropriate left Bousfield localization of the Reedy model structure, the fibrant objects of which he referred to as **complete Segal spaces**, and

b) described a Rezk (or simplicial) **nerve functor** \(N\) from the category \(\text{RelCat}\) of relative categories \((\mathcal{C}, \mathcal{W})\) and relative functors between them to \(sS\) which sends a relative category \((C, W)\) to the simplicial space which
in dimension $k \geq 0$ has as its $n$-simplices ($n \geq 0$) the commutative squares of the form

\[
\begin{array}{cccc}
  & c_1 & \cdots & c_k \\
\downarrow & & & \downarrow \\
\cdots & & & \cdots \\
\downarrow & & & \downarrow \\
w_1 & \cdots & \cdots & w_n \\
\end{array}
\]

in which the vertical maps are in $W$.

He then noted that

* for every simplicial model category $M$ one (and hence every) Reedy fibrant replacement of the simplicial space $NM$ is a complete Segal space.

Our aim thus is to prove the following.

3.3. A generalization of Rezk’s result. If $(C, W)$ is a partial model category (2.1), then one (and hence every) Reedy fibrant replacement of $N(C, W)$ is a complete Segal space.

Proof. The proof consists of two parts, a Segal part and a completion part.

To deal with the Segal part

(i) let for every integer $k \geq 0$, $A_k$ denote the category which has as its objects the sequences

\[
\begin{array}{cccc}
  & a_1 & \cdots & a_k \\
\downarrow & & & \downarrow \\
\cdots & & & \cdots \\
\downarrow & & & \downarrow \\
a_1' & \cdots & a_k' \\
\end{array}
\]

and as its maps the commutative diagrams of the form

\[
\begin{array}{cccc}
  & a_1 & \cdots & a_k \\
\downarrow & & & \downarrow \\
\cdots & & & \cdots \\
\downarrow & & & \downarrow \\
a_1' & \cdots & a_k' \\
\end{array}
\]

in which the vertical maps are in $W$.

Then we have to show that, for every integer $k \geq 2$, the pullback square

\[
\begin{array}{ccc}
  A_k & \rightarrow & A_{k-1} \\
  \downarrow & & \downarrow \\
  A_1 & \rightarrow & A_0 = W
\end{array}
\]

is a homotopy pullback square.

To do this

(ii) for every integer $k \geq 2$, denote by $B_k$ the category which has as its objects the zigzags

\[
\begin{array}{cccc}
  & b_1 & x & w & y & b_2 & \cdots & b_k \\
\downarrow & & & & & & & \downarrow \\
\cdots & & & & & & & \cdots \\
\downarrow & & & & & & & \downarrow \\
b_1' & x' & w' & y' & b_2' & \cdots & b_k' \\
\end{array}
\]

in $C$. 
that, for every integer \( k \) fact that, in view of 2.2(iv),

\[
\text{ir} \quad \text{in which all the unmarked arrows are identity maps, } w = v_1 u_1 \text{ with } u_1 \in U \text{ and } v_1 \in V (2.14) \text{ and the squares involving two } u \text{'s are pushout squares and those involving two } v \text{'s are pullback squares.}
\]

On \( A'_k \) this zigzag reduces to the zigzag

\[
\text{in which } x, y \text{ and } w \text{ are in } W \text{ and as its maps the commutative diagrams of the form}
\]

\[
\text{in } C
\]

\[
\text{in which the vertical maps are in } W, \text{ and}
\]

(iii) for every integer \( k \geq 2 \), denote by

\[
h_k : A_k \to B_k
\]

the monomorphism which between the first two maps inserts three identity maps, and denote by

\[
A'_k \subset B_k
\]

the image of \( A_k \) under \( h_k \).

In view of the Quillen Theorem B3 for homotopy pullbacks [BK1 8.2–4] and the fact that, in view of 2.2(iv) \( A_0 = W \) has property \( C_3 \), it then suffices to show that, for every integer \( k \geq 2 \), the inclusion \( i : A'_k \to B_k \) is a homotopy equivalence, i.e. that there exists a retraction \( r : B_k \to A'_k \) such that the compositions \( ir \) and \( ri \) are naturally weakly equivalent to the identity functor of \( B_k \) and \( A'_k \) respectively.

Such a retraction, together with a zigzag of natural weak equivalences connecting the functors \( ir \) and \( 1B_k \) can be obtained by means of the following (natural) commutative diagram in \( C \)

\[
in which all the unmarked arrows are identity maps, \( w = v_1 u_1 \) with \( u_1 \in U \) and \( v_1 \in V (2.14) \) and the squares involving two \( u \)’s are pushout squares and those involving two \( v \)’s are pullback squares.
\]
which does not completely lie inside $A_k'$. To remedy this, i.e. to get a natural weak equivalence connecting the top with the bottom inside $A_k'$ we note the existence of the zigzag

\[
\begin{array}{ccccccc}
\vdots & b_1 & b_2 & \ldots & b_k \\
\downarrow v_1 & u_1 & u_2 & \ldots & u_k \\
\downarrow v_1 & v_2 & v_3 & \ldots & v_k \\
\downarrow v_1 & b_1 & b_2 & \ldots & b_k \\
\end{array}
\]

in which the bottom row is obtained from the top row by pushing out along $v_1 u_1$ which is an identity map. Combining the bottom halves of the last two diagrams we now get two composable natural weak equivalences

\[
\begin{array}{ccccccc}
\vdots & b_1 & b_2 & \ldots & b_k \\
\downarrow v_1 & v_2 & v_3 & \ldots & v_k \\
\downarrow v_1 & b_1 & b_2 & \ldots & b_k \\
\end{array}
\]

of which the composition

\[
\begin{array}{ccccccc}
\vdots & b_1 & v & b_2 & \ldots & b_k \\
\downarrow v_1 & v & v & \ldots & v_k \\
\downarrow v_1 & b_1 & b_2 & \ldots & b_k \\
\end{array}
\]

yields the desired natural weak equivalence between $ri$ and $1_{A'}$.

It thus remains to deal with the completeness part of the proof. However this is essentially the same as Rezk’s proof of [R, 8.3] in view of the fact that the partial model category $(C, W)$ is saturated [5.1].

4. A converse of Rezk’s result

We now prove

4.1. A converse of Rezk’s result. Every complete Segal space is Reedy equivalent to the simplicial nerve of a partial model category and in fact of a homotopically full relative subcategory of a category of diagrams of simplicial sets.

The key to this is a partial modelization lemma which we will state in 4.3 but prove in [10] below. Its formulation requires the following.

4.2. A relative Yoneda embedding. Let $L^H$ denote the hammock localization of [DK2]. Given a relative category $(C, U)$, its relative Yoneda embedding will be the relative functor between relative categories

\[ y = y_{C, U} : (C, U) \rightarrow S^{C^\text{op}, U^\text{op}} \]

which sends each object $A \in C$ to the relative functor $yA : (C^\text{op}, U^\text{op}) \rightarrow S$ which sends each object $B \in C^\text{op}$ to the simplicial set $L^H(C, U)(B, A)$. 

4.3. The partial modelization. Given a relative category $(C, U)$,

(i) the essential image $E \eta$ of its Yoneda embedding (4.2) is a partial model category and in fact a homotopically full (1.2(ii)) relative subcategory of a category of diagrams of simplicial sets, and

(ii) the embedding $c: (C, U) \to E \eta$ is a DK-equivalence (1.4).

Using this we now can give

4.4. A proof of 4.1. First recall from [BK1, 5.3 and 4.4] the existence of

(i) an adjunction $K \xi: \text{RelCat} \leftrightarrow \text{sS}: N \xi$ of which the unit $\eta: 1 \to N \xi K \xi$ is a natural Reedy equivalence, and

(ii) a natural Reedy equivalence $\pi^*: N \to N \xi$ (3.2b)

and from [R, 7.2] that

(iii) every Reedy equivalence in sS is a Rezk equivalence (4.3(ii)) and every Rezk equivalence between two complete Segal spaces is a Reedy equivalence.

Given a complete Segal space $X$ one then can consider the zigzag

$$X \xrightarrow{\eta} N \xi K \xi X \xleftarrow{\pi^*} N K \xi X \xrightarrow{c} N E \eta K \xi X$$

in which, in view of (i) and (ii) above and (4.3(ii)), respectively, the first two maps are Reedy equivalences, while the third is a Rezk equivalence, and note that it follows from (iii) above and (3.3) that every Reedy fibrant replacement of the partial model category $E \eta K \xi X$ (4.3(i)) is Reedy equivalent to $X$.

5. A proof of the partial modelization lemma (4.3)

In preparation for the proof of lemma 4.3 (in 5.4 below) we first

• discuss in 5.1 relative simplicial categories and in particular relative partly simplicial ones in which the weak equivalences form an ordinary category, and

• review in 5.2 and 5.3 the notions of fully faithfulness and essential surjectivity and of essential image in the categories of categories, simplicial categories, relative categories and relative simplicial categories.

5.1. Relative (partly) simplicial categories. Let SCat denote the category of simplicial categories, i.e. categories enriched over simplicial sets, and let RelSCat denote the resulting category of relative simplicial categories, i.e. pairs consisting of a simplicial category and a sub-simplicial category (of which the maps are called weak equivalences) that contains all the objects. Then it turns out that, for our purposes here, it is convenient to work in the somewhat simpler

$$\text{RelP SCat} \subset \text{Rel SCat}$$

spanned by what we will call the relative partly simplicial categories, i.e. the objects of which the weak equivalences form an ordinary category.

A simplicial model category then can be considered as

• an object of RelCat consisting of the underlying model category and its weak equivalences

or as
• an object of \( \text{Rel} \mathcal{P} \text{SCat} \) consisting of the larger simplicially enriched model category and those same weak equivalences.

Moreover in the remainder of this paper we will consider the category \( S \) of simplicial sets only as an object of \( \text{Rel} \mathcal{P} \text{SCat} \).

An object \( L \in \text{SCat} \) thus gives rise to

• an object \( (S^L, \sim) \in \text{RelCat} \) in which \( S^L \) denotes the (model) category which has as objects the simplicial functors \( L \to S \) and as maps the natural transformations between them and \( \sim \) denotes the subcategory of the natural weak equivalences, and

• an object \( (S^L, \sim) \in \text{RelSCat} \) in which \( S^L \) denotes the simplicial (model) category of the simplicial functors \( L \to S \) ([DK2, 1.3(v)] and [GJ, IX, 1.4]) and \( \sim \) is as above.

We end with noting that similarly an object \( (L, Z) \in \text{Rel} \mathcal{P} \text{SCat} \) gives rise to

• an object \( (S^{L, Z}, \sim) \in \text{RelCat} \) which is the subobject of \( (S^L, \sim) \) spanned by the relative simplicial functors \( (L, Z) \to S \)

and that

(i) if \( Z \) is neglectible in \( L \) in the sense that every map in \( Z \) goes to an isomorphism in \( \text{Ho} L \), then \( (S^{L, Z}, \sim) = (S^L, \sim) \), and

(ii) for every object \( (C, U) \in \text{RelCat} \), the object \( (S^{C^{op}, U^{op}}, \sim) \in \text{RelCat} \) is exactly the same as the object \( S^{C^{op}, U^{op}} \) mentioned in [LZ].

5.2. Fully faithfulness and essential surjectivity. We will denote by \( L^H \) not only the functor \( \text{RelCat} \to \text{SCat} \) which sends each object to its hammock localization [DK2, 2.1], but also the functor \( \text{RelSCat} \to \text{SCat} \) which sends each object to the diagonal of the bisimplicial category obtained from it by dimensionwise application of the hammock localization [DK2, 2.5].

Then we recall the following.

A functor \( f : G \to H \) between categories (respectively, simplicial categories) is called fully faithful if, for every two objects \( G_1, G_2 \in G \), it induces an isomorphism (resp. weak equivalence) \( G(G_1, G_2) \to H(fG_1, fG_2) \), and similarly a relative functor \( f : (C, U) \to (D, V) \) between relative categories (resp. relative simplicial categories) is called fully faithful if, for every two objects \( C_1, C_2 \in C \), it induces a weak equivalence

\[
L^H(C_1, C_2) \to L^H(D, V)(fC_1, fC_2) \in S
\]

which implies that

(i) if \( f \) and \( g \) are (relative) functors such that \( gf \) is defined and \( g \) is fully faithful, then \( gf \) is fully faithful iff \( f \) is so.

A functor \( f : G \to H \) between categories (respectively, simplicial categories) is called essentially surjective if every object in \( H \) is isomorphic in \( H \) (resp. \( \text{Ho} H \)) to an object in the image of \( f \) (resp. \( \text{Ho} f \)), and similarly a relative functor \( f : (C, U) \to (D, V) \) between relative categories (resp. relative simplicial categories) is called essentially surjective if the induced functor

\[
L^H(f) : L^H(C, U) \to L^H(D, V)
\]

is so, which implies that

(ii) if \( f \) and \( g \) are (relative) functors such that \( gf \) is defined and \( f \) is essentially surjective, then \( gf \) is essentially surjective iff \( g \) is so.
Then

(iii) a map in $\bf{Cat}$ is an equivalence of categories iff it is fully faithful and essentially surjective, and

(iv) a map in $\bf{RelCat}, \bf{SCat}$ or $\bf{RelSCat}$ is a DK-equivalence iff it is fully faithful and essentially surjective.

### 5.3. Essential images.

The essential image $Ef$ of a functor $f: G \to H$ between categories (respectively, simplicial categories) is the full subcategory (resp. full simplicial subcategory) of $H$ spanned by the objects which are isomorphic in $H$ (resp. $\text{Ho}H$) to objects in the image of $f$ (resp. $\text{Ho}f$) and similarly the essential image $Ef$ of a relative functor $f: (C,U) \to (D,V)$ between relative categories (resp. relative simplicial categories) is defined by the pullback diagram

\[
\begin{array}{ccc}
Ef & \to & (D,V) \\
\downarrow & & \downarrow \\
LHf & \to & LH(D,V)
\end{array}
\]

which implies that

(i) the resulting maps

\[G \to Ef\quad\text{and}\quad(C,U) \to Ef\]

and

\[Ef \to H\quad\text{and}\quad Ef \to (D,V)\]

are respectively essentially surjective and fully faithful.

We end with noting that

(ii) the essential image defined in 1.4 is a special case of the ones defined above.

Now we are finally ready for

### 5.4. A proof of the relative modelization lemma (4.3).

It follows from 5.3(i) and (ii) that the map $e: (C,U) \to Ef$ is essentially surjective and it thus remains to prove that it is also fully faithful. To do this it suffices, in view of 5.3(ii) and 5.2(i), to show that, in the notation of 5.1(ii),

(i) the Yoneda embedding $y: (C,U) \to (SC^{op},U^{op},\sim)$ is fully faithful.

For this we note that $y$ admits a factorization

\[(C,U) \xrightarrow{y'} (SL^{H}(C^{op},U^{op}),\sim) \xrightarrow{5.1(i)} (SL^{H}(C^{op},U^{op}),U^{op},\sim) \xrightarrow{(c^{op})^{*}} (SC^{op},U^{op},\sim)\]

in which $y'$ sends each object $A \in C$ to the simplicial functor $L^{H}(C^{op},U^{op}) \to S$ which sends each object $B \in L^{H}(C^{op},U^{op})$ to $L^{H}(B,A) \in S$, and $c: (C^{op},U^{op}) \to L^{H}(C^{op},U^{op}),U^{op}$ is the obvious inclusion $[DK2]$ 3.1. The latter is a DK-equivalence $[BK1]$ 3.2 and hence $[DK2]$ 2.2 so is the map $(c^{op})^{*}$. Hence, in view of 5.2(iv) and 5.2(i) the condition (i) above is equivalent to condition

(ii) the map $y': (C,U) \to (SC^{op}(C^{op},U^{op}),\sim)$ is fully faithful.
To prove this we embed this map in the commutative diagram

\[
\begin{array}{ccc}
(C, U) & \xrightarrow{y'} & (S^L_H(C^{op}, U^{op}), \sim) \\
\downarrow c & & \downarrow \text{incl.} \\
(L^H(C, U), U) & \xrightarrow{r'} & (S_*^L(C^{op}, U^{op}), \sim)
\end{array}
\]

in which the map in the right is as in [5.1] and \( r' \) is induced by the simplicial Yoneda embedding of [DK4, 1.3(vi)]

\[ r : L^H(C, U) \longrightarrow S_*^L(C^{op}, U^{op}) \in \text{SCat} \]

which sends each object \( A \in L^H(C, U) \) to the simplicial functor \( L^H(C^{op}, U^{op}) \to S \) which sends each object \( B \in L^H(C^{op}, U^{op}) \) to \( L^H(B, A) \in S \). The map on the left is (see above) a DK-equivalence and so is, in view of [DK3, 4.8] the map on the right and hence, to prove (ii), it suffices to show that the bottom map is fully faithful.

For this we embed this map in the following diagram

\[
\begin{array}{ccc}
L^H(L^H(C, U), U) & \xrightarrow{L^H r'} & L^H(S_*^L(C^{op}, U^{op}), \sim) \\
\downarrow & & \downarrow \\
L^H(C, U) & \xrightarrow{r} & S_*^L(C^{op}, U^{op})
\end{array}
\]

in which the vertical maps are the obvious inclusions [DK2, 3.1]. As both categories of weak equivalences are neglectible [5.1(i)] it follows from [DK1, 6.4] that both vertical maps are DK-equivalences. Moreover it was noted in [DK4, 1.3(vi)] that the map \( r \) is fully faithful (and in fact so in the strong sense that the required weak equivalences are actually isomorphisms). All this implies that \( L^H r' \) is fully faithful and so is therefore the map \( r' \) itself.

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