A further generalization of random self-decomposability

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Abstract

The notion of random self-decomposability is generalized further. The notion is then extended to non-negative integer-valued distributions.

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1. Introduction

Recently the notion of random self-decomposability (RSD) has been introduced by Kozubowski and Podgórski \cite{4} generalizing SD. They showed that if a CF is RSD then it is both SD and geometrically infinitely divisible (GID). Satheesh and Sandhya \cite{9} generalized this notion to Harris-RSD (HRSD) and showed that if a CF is HRSD then it is both SD and Harris-ID (HID). With this nomenclature RSD is geometric-RSD (GRSD). Here we explore further generalizations of HRSD viz. NRSD and \( \varphi \)RSD, motivated by the elegant Proposition 2.3 in Kozubowski and Podgórski \cite{4}.

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We need the notion of $\mathcal{N}$-infinitely divisible ($\mathcal{NID}$) laws here. Let $\varphi$ be a Laplace transform (LT) that is also a standard solution to the Poincare equation, $\varphi(t) = P(\varphi(\theta t)), \theta \in \Theta$ where $P$ is a probability generating function (PGF) (see Gnedenko and Korolev, [3], p.140).

**Definition 1.1.** Let $\varphi$ be a standard solution to the Poincare equation and $N_\theta$ a positive integer-valued random variable (r.v.) having finite mean with PGF $P_\theta(s) = \varphi(\frac{1}{\theta} \varphi^{-1}(s)), \theta \in \Theta \subseteq (0, 1)$. A characteristic function (CF) $f(t)$ is $\mathcal{NID}$ if for each $\theta \in \Theta$ there exists a CF $f_\theta(t)$ that is independent of $N_\theta$ such that $f(t) = P_\theta(f_\theta(t))$, for all $t \in \mathbb{R}$.

**Theorem 1.1.** (Gnedenko and Korolev, 1996, Theorem 4.6.3 on p.147) Let $\varphi$ be a standard solution to the Poincare equation. A CF $f(t)$ is $\mathcal{NID}$ iff it admits the representation $f(t) = \varphi(- \log h(t))$ where $h(t)$ is a CF that is ID. $f(t)$ is $\mathcal{N}$ stable if $h(t)$ is stable (p.151, [3]).

In the next section we describe $\mathcal{N}_{RSD}$ laws and its discrete analogue in Section 3. In Section 4 we describe $\varphi RSD$ laws and its discrete analogue.

### 2. $\mathcal{N}_{RSD}$ distributions

**Definition 2.1.** A CF $f(t)$ is $\mathcal{N}_{RSD}$ if for each $c \in (0, 1]$ and each $\theta \in [0, 1)$

$$f_{c,\theta}(t) = f_c(t).f_\theta(ct) \quad (1)$$

is a CF, where $f_c(t)$ and $f_\theta(t)$ are given by

$$f_c(t) = \frac{f(t)}{f(ct)} \quad (2)$$

$$f_\theta(t) = \varphi(\theta \varphi^{-1}(f(t))) \quad (3)$$

$\varphi$ being a standard solution to the Poincare equation.
We now notice that the discussion leading to conceiving and proving Proposition 2.3 in Kozubowski and Podgórski [4] holds in this generalization as well. When \(c = 1\) equation (1) becomes

\[
f_{1,\theta}(t) = f_\theta(t) = \varphi\{\theta\varphi^{-1}(f(t))\}
\]

(4)

Or

\[
f(t) = \varphi\{\frac{1}{\theta}\varphi^{-1}(f_\theta(t))\}
\]

(5)

for each \(\theta \in [0, 1)\). That is \(f(t)\) is \(\mathcal{NID}\) and hence has no real zeroes. On the other hand since \(\varphi(0) = 1\), when \(\theta = 0\) equation (1) implies

\[
f_{c,\theta}(t) = f_c(t) = \frac{f(t)}{f(ct)}
\]

(6)

is a CF for each \(c \in (0, 1]\). That is \(f(t)\) is \(SD\).

Conversely, if \(f(t)\) is \(SD\) then for each \(c \in (0, 1]\) the function \(f_c(t)\) in (2) is a genuine CF and similarly if \(f(t)\) is \(\mathcal{NID}\) then for each \(\theta \in [0, 1)\) the function \(f_\theta(t)\) in (5) also is a genuine CF. Consequently (1) is a well defined CF.

**Remark 2.1** It may be noted that for the CF \(f(t)\) to be \(SD\) we only require that (a result due to Biggins and Shanbhag see Fosum [2]) (2) holds for all \(c\) in some left neighbourhood of 1. Thus we may simplify the requirement here as: A CF \(f(t)\) is \(\mathcal{NRSID}\) if for each \(c \in (a, 1]\), and each \(\theta \in [0, 1)\) (1) holds, where \(0 < a < 1\).

**Remark 2.2** In fact we may have apparently still weaker requirement in describing CFs that are \(\mathcal{NRSID}\) as follows. A CF \(f(t)\) is \(\mathcal{NRSID}\) if for each \(c \in (a, 1]\), and each \(\theta \in (0, 1)\) (1) holds, where \(0 < a < 1\). Now letting \(c \uparrow 1\) we have \(f(t)\) is \(\mathcal{NID}\). On the other hand letting \(\theta \downarrow 0\) we have \(f(t)\) is \(SD\) since \(\lim_{\theta \downarrow 0} f_\theta(t) = 1\), see e.g Gnedenko and Korolev [3], page 149.
Example 2.1 For the LT $\varphi(s) = (1+s)^{-\alpha}$, $\alpha > 0$, $\varphi(\varphi^{-1}(s)/p)$ is a PGF of a non-degenerate distribution only if $\alpha = \frac{1}{k}$, $k \geq 1$ integer, see Example 1 in Bunge [1] or Corollary 4.5 in Satheesh et al. [6]. This PGF is that of Harris distribution (Satheesh et al. [7]) and the corresponding $\mathcal{N}RS\mathcal{D}$ distribution is $HR\mathcal{D}$. When $k = 1$ above, we have $GR\mathcal{D}$ ($R\mathcal{D}$ distributions of Kozubowski and Podgórski [4]).

Example 2.2 Invoking Theorem 1.1 when $\varphi(s)$ is $SD$ and $\log h(t) = -\lambda |t|^\alpha$ we have, for each $c \in (a, 1]$ $f(t) = \varphi(|t|^\alpha) = \varphi(c|t|^\alpha).\varphi_c(|t|^\alpha).$ (7)

That is $f(t)$ is both $SD$ and $\mathcal{N}$-strictly stable. Thus we have a good collection of CFs that are both $SD$ and $HID$ and thus $HR\mathcal{D}$. Kozubowski and Podgórski [4] present examples of a variety of CFs $h(t)$ that are stable.

3. Discrete analogue of $\mathcal{N}RS\mathcal{D}$ distributions

Steutel and van Harn [10] had described discrete SD ($DSD$) distributions. Satheesh and Sandhya [9] have described $DHR\mathcal{D}$, discrete analogue of $HR\mathcal{D}$ distributions. We now introduce discrete $\mathcal{N}RS\mathcal{D}$ ($DNRS\mathcal{D}$) distributions.

Definition 3.1. (Satheesh et al. [7]) Let $\varphi$ be a standard solution to the Poincare equation and $N_\theta$ a positive integer-valued r.v. having finite mean with PGF $P_\theta(s) = \varphi(\frac{1}{\theta}\varphi^{-1}(s))$, $\theta \in \Theta \subseteq (0, 1)$. A PGF $P(s)$ is $DNID$ if for each $\theta \in \Theta$ there exists a PGF $Q_\theta(s)$ that is independent of $N_\theta$ such that $P(s) = P_\theta(Q_\theta(s))$, for all $|s| \leq 1$. 
Theorem 3.1. (Satheesh et al. [7]) Let \( \varphi \) be a standard solution to the Poincare equation. A PGF \( P(s) \) is DNID iff it admits the representation \( P(s) = \varphi(-\log R(s)) \) where \( R(s) \) is a PGF that is DID.

Definition 3.2. A PGF \( P(s) \) is DNRSD if for each \( c \in (0, 1] \) and each \( \theta \in [0, 1) \)

\[
P_{c,\theta}(s) = P_c(s)Q_\theta(1 - c + cs)
\]  

is a PGF, where \( P_c(s) \) and \( Q_\theta(s) \) are given by

\[
P_c(s) = \frac{P(s)}{P(1 - c + cs)}
\]
\[
Q_\theta(s) = \varphi\{\theta\varphi^{-1}(P(s))\},
\]

\( \varphi \) being a standard solution to the Poincare equation.

We may now proceed as in Section 2 describing the relation between DSD, DNID and DNRSD distributions. Further, remarks similar to Remarks 2.1 and 2.2 are relevant here also and Examples on the lines of Example 2.1 nad 2.2 can also be discussed.

4. \( \varphi \)RSD distributions

A further generalization of NRSD distributions is possible invoking the notion of \( \varphi \)ID law that generalizes \( \mathcal{N} \)ID laws, see Satheesh [5] and Satheesh et al. [7, 8] for its discrete analogue. We first describe the discrete case.

Definition 4.1. (Satheesh et al. [7]) Let \( \varphi \) be a LT. A PGF \( P(s) \) is D\( \varphi \)ID if there exists a sequence \( \{\theta_n\} \downarrow 0 \) as \( n \to \infty \) and a sequence of PGFs \( Q_n(s) \) such that

\[
P(s) = \lim_{n \to \infty} \varphi\left(1 - \frac{Q_n(s)}{\theta_n}\right).
\]
Theorem 4.1. (Satheesh et al. [8]) Let \( \{Q_\theta(s), \theta \in \Theta\} \) be a family of PGFs and \( \varphi \) a LT. Then
\[
\lim_{\theta \downarrow 0} \varphi\left(\frac{1 - Q_\theta(s)}{\theta}\right)
\]
exists and is \( D\varphi ID \) iff there exists a PGF \( R(s) \) that is DID such that
\[
\lim_{\theta \downarrow 0} \frac{1 - Q_\theta(s)}{\theta} = -\log R(s)
\]

Definition 4.2. A PGF \( P(s) \) is \( D\varphi RSD \) if for each \( c \in (a,1) \) and each \( \theta \in (0,b), 0 < a, b < 1 \)
\[
P_{c,\theta}(s) = P_c(s).Q_\theta(1 - c + cs)
\]
is a PGF, where \( P_c(s) \) and \( Q_\theta(s) \) are given by
\[
P_c(s) = \frac{P(s)}{P(1 - c + cs)}
\]
\[
Q_\theta(s) = 1 - \theta \varphi^{-1}(P(s)).
\]

The restriction of \( \alpha = \frac{1}{k}, k \geq 1 \) integer in Example 2.1 is not in this notion. We may now proceed as in Section 3 describing the relation between \( DSD, D\varphi ID \) and \( D\varphi RSD \) distributions. This has been possible since
\[
\lim_{\theta \downarrow 0} Q_\theta(t) = 1.
\]
The case of \( \varphi RSD \) follows on similar lines.

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