Screening Currents Ward Identity

and

Integral Formulas for the WZNW Correlation Functions

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Abstract

We derive, based on the Wakimoto realization, the integral formulas for the WZNW correlation functions. The role of the “screening currents Ward identity” is demonstrated with explicit examples. We also give a more simple proof of a previous result.

† Talk given at, “Workshop on Development in String Theory and New Field Theories”, Kyoto Sept. 1991.
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1. Introduction

In the Wess-Zumino-Novikov-Witten (WZNW) models [1−3], the correlation function can be characterized as the solution of the Knizhnik-Zamolodchikov (KZ) equation. The integral representation for this correlation function has been studied in the last few years [5, 17−20].

It is known that this representation can be obtained from the free field realization of the Kac-Moody algebras (the Wakimoto realization).† This realization has been discussed by several authors [4−6], and its general properties has been clarified [7−10]. In this method, the correlation function is represented by some integral of the free fields correlation. But, for arbitrary algebras and representations, it is difficult to evaluate the free fields correlation except in some particular cases [8, 14]. This is entirely due to the complicated structure of the realization [11−13].

In a previous paper [21], we solved this problem and derived the integral representation for the WZNW models corresponding to arbitrary simple Lie algebras. By using the “screening currents Ward identity”, we obtained a exact result without treating the complicated explicit form of the Wakimoto realization.

This paper is a more complete version of [21], and contains some explicit examples in the case of $\widehat{sl}(3)$ and refined proofs.

This paper is arranged as follows. In section 2, we refer to the free field realization of finite dimensional simple Lie algebras. In section 3, the Wakimoto realization of the affine Kac-Moody algebras is introduced. Next, in section 4, we study some examples of the integral formulas for the WZNW correlation functions. We then present the systematic derivation of these integral formulas in section 5. This section contains the main results of the paper. Some of the proofs are given in the Appendices.

† In [18−20], they constructed this integral representation by another approach based on the generalized theory of hypergeometric type equations [16].
2. The Free Field Realization of Simple Lie Algebras

We start with recapitulating the results of the free field realization (differential realization) of simple Lie algebras.

§ 2.1. A finite dimensional simple Lie algebra \( g \) is defined by the following commutation relations for the Chevalley generators, \( e_\alpha, f_\alpha \), and \( h_i \) \((i = 1, \ldots, l = \text{rank} g)\)

\[
\begin{align*}
[h_i, h_j] &= 0, \quad [e_\alpha, f_\alpha] = \delta_{ij} h_j, \\
[h_i, e_\alpha] &= a_{ij} e_\alpha, \quad [h_i, f_\alpha] = -a_{ij} f_\alpha.
\end{align*}
\]  

(2.1)

where \( \alpha_i \) is the simple root, and \( a_{ij} \) is the Cartan matrix. The Cartan matrix is realized as \( a_{ij} = (\nu_i \cdot \alpha_j) \), where \( \nu_i = \frac{2}{\alpha_i} \alpha_i \) is the coroot of \( \alpha_i \), and \( (\cdot) \) is the symmetric bilinear form.†

The Verma module \( V_\lambda \) is generated by the highest weight vector \(|\lambda\rangle\) which satisfies \( e_\alpha |\lambda\rangle = 0 \), \( h_i |\lambda\rangle = \lambda_i |\lambda\rangle \). The Dual module \( V_\lambda^* \) is generated by \( \langle \lambda | \) such that \( \langle \lambda | f_\alpha = 0 \), \( \langle \lambda | h_i = \lambda_i \langle \lambda | \). The bilinear form \( V_\lambda^* \otimes V_\lambda \rightarrow \mathbb{C} \) defined by \( \langle \lambda | \lambda \rangle = 1 \) is called the Shapovalov form.‡

§ 2.2. The algebra \( g \) is realized on the polynomial ring \( \mathbb{C}[x^\alpha] \), with positive root \( \alpha \in \Delta_+ \), as (twisted first order) differential operators. The differential operators \( J(\frac{\partial}{\partial x}, x, \lambda)'s \) corresponding to the generators \( J's \) are defined by the following “right action”* 

\[
J(\frac{\partial}{\partial x}, x, \lambda) |\lambda\rangle \equiv |\lambda\rangle J, \quad (2.2)
\]

where \( Z \equiv \exp(\sum_\alpha x^\alpha e_\alpha) \). They are given by

\[
J(\frac{\partial}{\partial x}, x, \lambda) = \sum_{\alpha > 0} V^\alpha(x) \frac{\partial}{\partial x^\alpha} + \sum_{i=1}^l W^i(x) \lambda_i, \quad (2.3)
\]

† For the quantities in Cartan space \( h \), for example \( h, H, \lambda, \phi \), their components are defined by \( \lambda_i = (\nu_i \cdot \lambda) \) and \( \lambda = \sum_i \lambda_i \nu_i \) etc.

‡ Strictly speaking the Shapovalov form is a bilinear form on \( V_\lambda \otimes V_\lambda \).

* For a positive generator \( e_\alpha, E_\alpha(\frac{\partial}{\partial x}, x) \) can be defined as \( E_\alpha(\frac{\partial}{\partial x}, x) Z = Z e_\alpha \).
for some polynomials $V^\alpha(x)$ and $W^i(x) \in \mathbb{C}[x]$.

There is another type of differential operators $S_\alpha$ induced by “left action” of $e_\alpha$ that plays an important role in this paper. They are defined by

$$S_\alpha\left(\frac{\partial}{\partial x}, x\right) Z = -e_\alpha Z,$$

and with some polynomials $S^\beta_\alpha(x)$ as

$$S_\alpha\left(\frac{\partial}{\partial x}, x\right) = \sum_{\beta > 0} S^\beta_\alpha(x) \frac{\partial}{\partial x^\beta}.$$  \hspace{1cm} (2.5)

From the associativity of $\langle \lambda | e_\alpha Z J \rangle$, we have\footnote{The formula in [21] has a wrong sign.}

$$[E_\alpha, S_\beta] = 0,$$

$$[H_i, S_\alpha] = (\nu_i \cdot \alpha)S_\alpha,$$

$$[F_\alpha, S_\alpha] = (\nu_i \cdot \alpha_j) x^{\alpha_i}S_{\alpha_j} + \lambda_i\delta_{ij},$$ \hspace{1cm} (2.6)

Useful formulas and the proof of (2.6) are given in Appendix A.

**EXAMPLE.** The $sl(3)$ algebra, whose Cartan matrix is $a_{11} = a_{22} = 2$ and $a_{12} = a_{21} = -1$, is realized on the polynomial ring $\mathbb{C}[x^\alpha] \alpha = 1, 2, 3$, where $x^3$ is associated with $e_3 = [e_1, e_2]$. For example the differential operators $E_1$, $F_1$ and $H_1$ corresponding to the Chevalley generators $e_1$, $f_1$ and $h_1$ are

$$E_1 = \frac{\partial}{\partial x^1} - \frac{1}{2} x^2 \frac{\partial}{\partial x^3},$$

$$H_1 = -2x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} - x^3 \frac{\partial}{\partial x^3} + \lambda_1,$$

$$F_1 = -x^1 x^1 \frac{\partial}{\partial x^1} + \left(\frac{1}{2} x^1 x^2 - x^3\right) \frac{\partial}{\partial x^2} - \frac{1}{2} x^1 \left(\frac{1}{2} x^1 x^2 + x^3\right) \frac{\partial}{\partial x^3} + \lambda_1 x^1,$$

and $S_1$ is

$$S_1 = -\frac{\partial}{\partial x^1} - \frac{1}{2} x^2 \frac{\partial}{\partial x^3}.$$ \hspace{1cm} (2.7)

$E_2$, $F_2$, $H_2$ and $S_2$ are given by replacing $x^1$ with $x^2$, $x^3$ with $-x^3$ and $\lambda_1$ with $\lambda_2$.\footnote{The formula in [21] has a wrong sign.}
\( \S \) 2.3. The state \( |0\rangle \equiv 1 \in \mathbb{C}[x^\alpha] \) is the highest weight vector such that
\[
\frac{\partial}{\partial x^\alpha}|0\rangle = 0, \quad E_\alpha|0\rangle = 0, \quad H_\mu|0\rangle = \lambda_\mu|0\rangle.
\] (2.9)

For an ordered set \( I = \{\alpha_1, \ldots, \alpha_n\} \), the vectors \( P^I_\lambda|0\rangle = \prod_{\alpha_i \in I} F_{\alpha_i}|0\rangle \in \mathbb{C}[x^\alpha] \) form the basis of the descendants of \( |0\rangle \). From (2.2), this basis can be written as the following expectation values
\[
P^I_\lambda = \langle \lambda | Z \prod_{i=1}^n f_{\alpha_i} | \lambda \rangle.
\] (2.10)

**Example.** The first few examples for \( sl(3) \) are
\[
\begin{align*}
P^{(1)}_\lambda &= \lambda_1 x^1, \\
P^{(1,1)}_\lambda &= \lambda_1 (\lambda_1 - 1) x^1 x^1, \\
P^{(2,1)}_\lambda &= \lambda_1 ((\lambda_2 + \frac{1}{2}) x^1 x^2 + x^3), \\
P^{(1,1,1)}_\lambda &= \lambda_1 (\lambda_1 - 1) (\lambda_1 - 2) x^1 x^1 x^1, \\
P^{(2,1,1)}_\lambda &= \lambda_1 (\lambda_1 - 1) ((\lambda_2 + 1) x^1 x^2 + 2 x^3) x^1, \\
P^{(1,2,1)}_\lambda &= \lambda_1 (((\lambda_1 - \frac{1}{2}) (\lambda_2 + \frac{1}{2}) - \frac{1}{4}) x^1 x^2 + (\lambda_1 - \lambda_2 - 1) x^3) x^1, \\
P^{(1,1,2)}_\lambda &= \lambda_1 \lambda_2 (\lambda_1 x^1 x^2 - 2 x^3) x^1.
\end{align*}
\] (2.11)

Since we have the Serre relations, this basis is not linearly independent. For instance,
\[
P^{(1,1,2)}_\lambda - 2P^{(1,2,1)}_\lambda + P^{(2,1,1)}_\lambda = 0.
\] (2.12)

\( ^\dagger \) We sometimes use the over-simplified notation \( I = \{\alpha_1, \ldots, \alpha_n\} \) to denote an ordered set \( I \) of simple roots \( \alpha_i \), although they should be understood as \( \{\alpha_{i_1}, \ldots, \alpha_{i_n}\} \).

\( ^\ddagger \) In what follows, the operator factors in the product symbol \( \prod \) are assumed to be ordered in the sense that \( \prod_{i \in \{2,1,1\}} O_i = O_2 O_1 O_1 \neq O_1 O_1 O_2 \).
3. The Wakimoto Realization

Next we turn to the Wakimoto realization of the affine Kac-Moody algebras.

§ 3.1. The affine Kac-Moody algebra $\hat{g}$ associated with the Lie algebra $g$ is defined by the operator product expansions (OPE);

\begin{align*}
H_i(z)H_j(w) &= \frac{k}{(z-w)^2} (\nu_i \cdot \nu_j) + \cdots, \\
H_i(z)E_{\alpha_j}(w) &= \frac{1}{z-w} a_{ij} E_{\alpha_j}(w) + \cdots, \\
H_i(z)F_{\alpha_j}(w) &= -\frac{1}{z-w} a_{ij} F_{\alpha_j}(w) + \cdots, \\
E_{\alpha_i}(z)F_{\alpha_j}(w) &= \frac{k}{(z-w)^2} \frac{2}{\alpha_i^2} \delta_{ij} + \frac{1}{z-w} \delta_{ij} H_j(w) + \cdots,
\end{align*}

where $k$ is the level. The energy-momentum tensor $T_{Sug}(z)$ is given by the Sugawara construction;

\[
T_{Sug}(z) = (2\kappa)^{-1} \sum_{i=1}^{l} H_i(z) H^i(z) + \sum_{\alpha>0} \frac{\alpha^2}{2} (E_\alpha(z) F_\alpha(z) + F_\alpha(z) E_\alpha(z)) : ,
\]

where $\kappa = k + h$ and $h$ is the dual Coxeter number of $g$.

For each positive root $\alpha \in \Delta_+$, we introduce bosons $\beta_\alpha(z)$ and $\gamma^\alpha(z)$, with conformal weights 1 and 0, satisfying the canonical OPE

\[
\beta_\alpha(z) \gamma^\beta(w) = \frac{1}{z-w} \delta_\alpha^\beta + \cdots,
\]

We also introduce free bosons $\phi_i(z)$’s for $i = 1, \cdots, l$, with the OPE\(^\dagger\)

\[
\phi_i(z) \phi_j(w) = \kappa^{-1} (\nu_i \cdot \nu_j) \log(z-w) + \cdots,
\]

The Kac-Moody algebra (3.1) can be realized by the free fields (3.3) and (3.4).

\(^\dagger\) Our notation for $\phi(z)$ is different from the ordinary one. To get the ordinary $\phi(z)$, we must substitute $\phi(z) \rightarrow i \kappa^{-1/2} \phi(z)$.
PROPOSITION. The free field realization of the Kac-Moody currents $J(z)$ is given by \[ J(\beta, \gamma, \kappa \partial \phi)(z) =: \sum_{\alpha > 0} V_{\alpha}(\gamma) \beta_{\alpha} + \kappa \sum_{i=1}^{l} W^{i}(\gamma) \partial \phi_{i} + \sum_{\alpha > 0} A_{\alpha}(\gamma) \partial \gamma^{\alpha} : (z) \quad (3.5) \]

where $V^{\alpha}(\gamma(z))$ and $W^{i}(\gamma(z)) \in C[\gamma^{\alpha}(z)]$ are the operators corresponding to the polynomials $V^{\alpha}(x)$ and $W^{i}(x) \in C[x^{\alpha}]$ of (2.3); and $A_{\alpha}(\gamma(z))$ are the anomalous parts, due to the normal ordering, which vanish for $E_{\alpha}(z)$ and $H_{i}(z)$. For $F_{\alpha i}(z)$, this is given by

\[ A_{\alpha}(\gamma(z)) = \delta_{\alpha, \alpha} \left( \frac{2}{\alpha_{i}^{2}} k + \frac{h - \alpha_{i}^{2}}{\alpha_{i}^{2}} \right). \quad (3.6) \]

The energy-momentum tensor $T_{\text{Sug}}(z)$ is also realized as

\[ T_{\text{Sug}}(z) = \sum_{\alpha > 0} : \partial \gamma^{\alpha} \beta_{\alpha} : (z) + \sum_{i=1}^{l} \frac{\kappa}{2} \partial \phi_{i} \partial \phi^{i} - \rho_{i} \partial^{2} \phi^{i} : (z), \quad (3.7) \]

where $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$ is the half sum of positive roots.

§3.2. There exists one more important ingredient in the Wakimoto realization, that is the screening current. Define the screening current $s_{i}(z)$, corresponding to the simple root $\alpha_{i}$

\[ s_{i}(z) = S_{\alpha_{i}}(z)e^{-\alpha_{i} \cdot \phi(z)}, \quad (3.8) \]

\[ S_{\alpha}(z) = \sum_{\beta > 0} : S_{\alpha}^{\beta}(\gamma(z)) \beta_{\beta}(z) :, \]

where the polynomial $S_{\alpha}^{\beta}(x) \in C[x^{\alpha}]$ is the same as in (2.5).
Then we get

**PROPOSITION.** The screening current \( s_i(z) \) satisfies

\[
E_\alpha(z)s_j(w) = 0 + \cdots,
\]
\[
H_i(z)s_j(w) = 0 + \cdots,
\]
\[
F_{\alpha_i}(z)s_j(w) = -\kappa\delta_{ij} \frac{2}{\alpha_i^2} \frac{\partial}{\partial w} \left( \frac{1}{z-w} e^{-\alpha_j\phi(w)} \right) + \cdots, \tag{3.9}
\]
\[
T_{\text{sug}}(z)s_j(w) = \frac{\partial}{\partial w} \left( \frac{1}{z-w} s_j(w) \right) + \cdots.
\]

Note that the screening charge \( \int dt e^{-\alpha_i\phi(t)} S_\alpha(t) \) is well defined only for the simple roots \( \alpha_i \)'s, though \( S_\alpha(t) \) is well-defined for all the positive roots.

§ 3.3. Now we give a natural highest weight representation. For arbitrary weight vector \( \lambda \), the vertex operator \( e^{\lambda\phi(z)} \) satisfies the highest weight condition;

\[
E_\alpha(z)e^{\lambda\phi(w)} = 0 + \cdots,
\]
\[
H_i(z)e^{\lambda\phi(w)} = \lambda_i \frac{1}{z-w} e^{\lambda\phi(w)} + \cdots. \tag{3.10}
\]

It has the conformal weight \( \lambda \cdot (\lambda + 2\rho)/2\kappa \) with respect to the \( T_{\text{sug}}(z) \).

For an “ordered” set of simple roots, \( I = \{\alpha_1, \cdots, \alpha_n\} \), the fields

\[
e^{\lambda\phi(z)} P^I_\lambda(z) \equiv \int \prod_{k=1}^n \frac{dt_k}{2\pi i} F_{\alpha_k}(t_k)e^{\lambda\phi(z)} \tag{3.11}
\]

form the basis of the descendants of the highest weight vector \( e^{\lambda\phi(z)} \). Since the \( \partial\gamma(z) \) terms (3.5) give no singularity, they do not contribute to this integral. So the operator \( P^I_\lambda(z) \in \mathbb{C}[\gamma^{\alpha}(z)] \) can be constructed from a classical polynomial \( P^I_\lambda \in \mathbb{C}[x^{\alpha}] \) (2.11) by replacing \( x^{\alpha} \) with \( \gamma^{\alpha}(z) \).

Hence the vectors \( e^{\lambda\phi(z)} P_\lambda(z) \), with arbitrary polynomial \( P_\lambda(z) \in \mathbb{C}[\gamma^{\alpha}(z)] \), form the highest weight representation of the Wakimoto realization. †

† Here we only consider the \( L_0 \) ground states.
4. Some Examples of Integral Formulas

We study some examples of the integral formulas for the WZNW correlation functions, based on the Wakimoto realization. Here we use the $\beta\gamma$ OPE relation and the explicit form for $S_\alpha(t)$ and $P(z)$.

§ 4.1. A correlation function of the chiral primary fields is represented by the correlation of the screening charges as well as the vertex operators in the form

$$\int \prod_{i=1}^{m} dt_i \prod_{i=1}^{m} e^{-\alpha_i \cdot \phi(t_i)} S_\alpha(t_i) \prod_{a=1}^{n} e^{\lambda_a \cdot \phi(z_a)} P_a(z_a)).$$

(4.1)

Here $\alpha_1, ..., \alpha_m$ are simple roots that fill the gap between the highest weights of the incoming states $\lambda_1, ..., \lambda_n$ and the outgoing state $\lambda_\infty$, i.e.

$$\lambda_\infty = \sum_{a=1}^{n} \lambda_a - \sum_{i=1}^{m} \alpha_i.$$  (4.2)

In the set $\{\alpha_1, ..., \alpha_m\}$ or $\{\lambda_1, ..., \lambda_n\}$ some $\alpha$’s or $\lambda$’s may be repeated.

Calculation of the $\phi$ field correlation is given by the difference products

$$Q \equiv \langle \prod_{i=1}^{m} e^{-\alpha_i \cdot \phi(t_i)} S_\alpha(t_i) \prod_{a=1}^{n} e^{\lambda_a \cdot \phi(z_a)} \rangle$$

$$= \prod_{i<j}^{m} (t_i - t_j)^{-\alpha_i \cdot \alpha_j} \prod_{i=1}^{m} \prod_{a=1}^{n} (t_i - z_a)^{-\alpha_i \cdot \lambda_a} \prod_{a<b}^{n} (z_a - z_b)^{-\lambda_a \cdot \lambda_b}.$$  (4.3)

§ 4.2. Let us next calculate the $\beta\gamma$ correlation

$$\omega \equiv \langle \prod_{i=1}^{m} S_\alpha(t_i) \prod_{a=1}^{n} P_a(z_a) \rangle.$$  (4.4)

Since $S_\alpha(t)$ ($P_a(z)$) have conformal weight 1 (0), this form $\omega$ is a single valued meromorphic 1-form (function) on the sphere with respect to the $t$’s ($z$’s).
From the transformation and the singularity property (3.3), we obtain the following relation

\[ \langle \beta_\alpha(z) \prod_{i=1}^{n} \beta_{\alpha_i}(z_i) \prod_{j=1}^{m} \gamma^j(w_j) \rangle = \sum_{k=1}^{m} \delta_{\alpha_k}^\beta \langle \prod_{i=1}^{n} \beta_{\alpha_i}(z_i) \prod_{j \neq k}^{m} \gamma^j(w_j) \rangle. \]  

(4.5)

This is because both sides, which are global 1-form on the sphere with respect to \( z \), have the same poles and residues. The charge at infinity which compensates for the \( \beta \gamma \) current anomaly is included implicitly. From this relation we can in principle obtain \( \omega \).

**EXAMPLE.** Let us consider some examples of \( \widehat{sl}(3) \). Recall that \( S_1(z) =: -\beta_1 - \frac{1}{2} \gamma^2 \beta_3 : (z) \), \( S_2(z) =: -\beta_2 + \frac{1}{2} \gamma^1 \beta_3 : (z) \), and using (2.11), we have

\[ \langle S_1(t) P^{(1)}_\lambda(z) \rangle = \langle - (\beta_1 + \frac{1}{2} \gamma^2 \beta_3)(t) \lambda_1 \gamma^1(z) \rangle = -\lambda_1 \langle \beta_1(t) \gamma^1(z) \rangle = -\frac{\lambda_1}{t-z}, \]

(4.6)

\[ \langle S_1(t_1) S_2(t_2) P^{(1,2)}_\lambda(z) \rangle \]

\[ = \langle (\beta_1 + \frac{1}{2} \gamma^2 \beta_3)(t_1)(\beta_2 - \frac{1}{2} \gamma^1 \beta_3)(t_2) \lambda_2(\lambda_1 + \frac{1}{2} \gamma^1 \gamma^2 - \gamma^3)(z) \rangle \]

\[ = \lambda_2(\lambda_1 + \frac{1}{2})(\beta_1(t_1)\beta_2(t_2)\gamma^1\gamma^2(z)) + \frac{\lambda_2}{2} (\beta_1(t_1)\gamma^1 \beta_3(t_2) \gamma^3(z)) \]

\[ - \frac{\lambda_2}{2} (\gamma^2 \beta_3(t_1) \beta_2(t_2) \gamma^3(z)) \]

\[ = \frac{-\lambda_2(\lambda_1 + \frac{1}{2})}{(t_1-z)(t_2-z)} + \frac{\lambda_2/2}{(t_1-t_2)(t_2-z)} - \frac{\lambda_2/2}{(t_2-t_1)(t_1-z)} \]

\[ = \left( \frac{-1}{t_1-t_2} - \frac{\lambda_1}{t_1-z} \right) \left( - \frac{\lambda_2}{t_2-z} \right), \]

(4.7)
\[ \langle S_1(t_1)S_1(t_2)P^{(1,1)}_\lambda(z) \rangle \]
\[ = \langle (\beta_1 + \frac{1}{2} \gamma_2 \beta_3)(t_1)(\beta_1 + \frac{1}{2} \gamma_2 \beta_3)(t_2) \lambda_1(\lambda_1 - 1) \gamma_1 \gamma_1(z) \rangle \]
\[ = \lambda_1(\lambda_1 - 1) \langle \beta_1(t_1) \beta_1(t_2) \gamma_1 \gamma_1(z) \rangle \]
\[ = \frac{2\lambda_1(\lambda_1 - 1)}{(t_1 - z)(t_2 - z)}, \]
\[ = \left( \frac{2}{t_1 - t_2} - \frac{\lambda_1}{t_1 - z} \right) \left( - \frac{\lambda_1}{t_2 - z} \right) + \left( \frac{2}{t_2 - t_1} - \frac{\lambda_1}{t_2 - z} \right) \left( - \frac{\lambda_1}{t_1 - z} \right), \]

and
\[ \langle S_1(t_1)S_1(t_2)P^{(1,1)}_\lambda(z_1)P^{(1,1)}_\lambda(z_2) \rangle \]
\[ = \langle (\beta_1 + \frac{1}{2} \gamma_2 \beta_3)(t_1)(\beta_1 + \frac{1}{2} \gamma_2 \beta_3)(t_2) \lambda_1^1 \lambda_1^2 \gamma_1(z_1) \lambda_1^2 \gamma_1(z_2) \rangle \]
\[ = \lambda_1^1 \lambda_1^2 \langle \beta_1(t_1) \beta_1(t_2) \gamma_1(z_1) \gamma_1(z_2) \rangle \]
\[ = \left( \frac{\lambda_1^1}{t_1 - z_1} \right) \left( \frac{\lambda_1^2}{t_2 - z_2} \right) + \left( \frac{\lambda_1^1}{t_2 - z_1} \right) \left( \frac{\lambda_1^2}{t_1 - z_2} \right). \]

In the last line of each example, we used the fact that the \( \beta \gamma \) propagator \( \frac{1}{z - w} \) satisfies \( \frac{1}{t_1 - t_2} \frac{1}{t_2 - t_3} = \frac{1}{t_1 - t_3} \left\{ \frac{1}{t_2 - t_3} - \frac{1}{t_2 - t_1} \right\}. \)

5. Integral Formulas from the Wakimoto Realization

Now we present the systematic derivation of the integral formulas [21]. We find that \( \omega \) can be calculated without using the explicit form for \( S_\alpha(t) \) and \( P(z) \).

§ 5.1. At first we evaluate the correlation in the case of \( P_\lambda(z) \) being an arbitrary polynomial in \( C[\gamma^\alpha(z)] \). All we need are the following \( S_\alpha(t), P(z) \) OPE relations.

Since \( S_\alpha(t) \) is constructed as the left action of \( e_\alpha \), the \( \beta \gamma \) parts of the screening currents and the vertex operator satisfy

\[ S_\alpha(t_1)S_\beta(t_2) = \sum_{\gamma > 0} \frac{1}{t_1 - t_2} f_{\alpha \beta}^\gamma S_\gamma(t_2) + \cdots, \]
\[ S_\alpha(t)P(z) = \frac{1}{t - z} (S_\alpha P)(z) + \cdots, \]

where \( f_{\alpha \beta}^\gamma \) are the structure constants of \( n_+ \) and the operator \( (S_\alpha P)(z) \in C[\gamma^\alpha(z)] \)
corresponds to the polynomial \( (S_\alpha P) = \sum_\beta S_\alpha^\beta(x) \frac{\partial}{\partial x^\beta} P(x) \in C[x^\alpha]. \)

From this singularity and the transformation property, we obtain the “screening currents Ward identity”:

\[
\langle S_\alpha(t) S_\alpha_1(t_1) \cdots S_\alpha_m(t_m) P_1(z_1) \cdots P_n(z_n) \rangle \\
= \sum_{i=1}^m \frac{1}{t - t_i} f_{\alpha\alpha_i}^\beta \langle S_\alpha_1(t_1) \cdots S_\beta(t_i) \cdots S_\alpha_m(t_m) P_1(z_1) \cdots P_n(z_n) \rangle \tag{5.2}
\]

\[
+ \sum_{a=1}^n \frac{1}{t - z_a} \langle S_\alpha_1(t_1) \cdots S_\alpha_m(t_m) P_1(z_1) \cdots S_\alpha P_a(z_a) \cdots P_n(z_n) \rangle.
\]

This equation gives

**THEOREM I.** For an arbitrary \( P_a(z_a) \), the \( \beta\gamma \) correlation \( \omega \) is given by

\[
\omega = \sum_{\text{part}} \prod_{a=1}^n \langle \prod_{i \in I_a} S_\alpha_i(t_i) P_a(z_a) \rangle, \tag{5.3}
\]

\[
\langle S_\alpha_1(t_1) \cdots S_\alpha_m(t_m) P(z) \rangle = \sum_{\text{perm}} \frac{\langle (S_\alpha_1 \cdots S_\alpha_m P)(z) \rangle}{(t_1 - t_2)(t_2 - t_3) \cdots (t_m - z)}, \tag{5.4}
\]

where \( \sum_{\text{part}} \) stands for the summation over all the partition of \( I = \{1, 2, \cdots, m\} \) into \( n \) disjoint union \( I_1 \cup I_2 \cup \cdots \cup I_n \) and \( \sum_{\text{perm}} \) the summation over all the permutation of the elements of \( \{1, 2, \cdots, m\} \).

The proof is given in Appendix C. Theorem I corresponds to the 1st solution of Schechtman and Varchenko [20].
Next we derive the correlation in the case where \( P \) is given by (3.11), \( P_{\lambda}^{(\beta_1,\cdots,\beta_n)}(z) \). A key formula is the \( s_{\mu}(t) F_{\beta}(z) \) OPE relation (3.9), which gives a more simple proof of the previous result [21].

By definition (3.11), \( \beta \) insertion for \( P_{\lambda}^{(\beta_1,\cdots,\beta_n)}(z) \) is given by

\[
e^{\lambda \cdot \phi(z)} P_{\lambda}^{(\beta,\cdots,\beta_n)}(z) = \int \frac{dw}{2\pi i} F_{\beta}(w)e^{\lambda \cdot \phi(z)} P_{\lambda}^{(\beta_1,\cdots,\beta_n)}(z)
\]  

(5.5)

Using the \( s_{\alpha}(t) F_{\beta}(z) \) OPE relation (3.9), we get the following inductive formula

\[
\int \prod_{i=1}^{m} dt_i \langle \prod_{i=1}^{m} e^{-\alpha_i \cdot \phi(t_i)} e^{\lambda \cdot \phi(z)} \prod_{i=1}^{m} S_{\alpha_i}(t_i) P_{\lambda}^{(\beta,\cdots,\beta_n)}(z) \rangle = \int \prod_{i=1}^{m} dt_i \sum_{k=1}^{m} \kappa \delta_{\beta \alpha_k} \frac{2}{\alpha_k} \frac{\partial}{\partial t_k} \langle \prod_{i=1}^{m} e^{-\alpha_i \cdot \phi(t_i)} e^{\lambda \cdot \phi(z)} \prod_{i \neq k}^{m} S_{\alpha_i}(t_i) P_{\lambda}^{(\beta_1,\cdots,\beta_n)}(z) \rangle.
\]  

(5.6)

From (4.3), we can calculate this derivative

\[
\frac{\partial}{\partial t_k} \langle \prod_{i=1}^{m} e^{-\alpha_i \cdot \phi(t_i)} e^{\lambda \cdot \phi(z)} \rangle = \frac{1}{\kappa} \left( \sum_{i \neq k}^{m} \frac{\alpha_k \cdot \alpha_i}{t_k - t_i} - \frac{\alpha_k \cdot \lambda}{t_k - z} \right) \langle \prod_{i=1}^{m} e^{-\alpha_i \cdot \phi(t_i)} e^{\lambda \cdot \phi(z)} \rangle.
\]  

(5.7)

Then we obtain the following “currents Ward identity” including screening currents

\[
\langle \prod_{i=1}^{m} S_{\alpha_i}(t_i) P_{\lambda}^{(\beta,\cdots,\beta_n)}(z) \rangle = \sum_{k=1}^{m} \delta_{\beta \alpha_k} \frac{2}{\alpha_k} \left( \sum_{i \neq k}^{m} \frac{\alpha_k \cdot \alpha_i}{t_k - t_i} - \frac{\alpha_k \cdot \lambda}{t_k - z} \right) \langle \prod_{i \neq k}^{m} S_{\alpha_i}(t_i) P_{\lambda}^{(\beta_1,\cdots,\beta_n)}(z) \rangle.
\]  

(5.8)

Iterative use of this equation gives the expression for \( \omega \). Note that (5.4) vanishes unless \( \sum_{i=1}^{m} \alpha_i = \sum_{j=1}^{n} \beta_j \). Then we have
THEOREM II. For the basis vectors $P_{\lambda a}^{I a}(z_a)$, the $\beta \gamma$ correlation $\omega$ is given by

$$
\omega = \sigma \prod_{a=1}^{n} \langle \prod_{\alpha_i \in I_a} S_{\alpha_i}(t_i) P_{\lambda a}^{I a}(z_a) \rangle,
$$

(5.9)

$$
\langle \prod_{i=1}^{n} S_{\alpha_i}(t_i) P_{\lambda}^{(a_1, \ldots, a_n)}(z) \rangle = \sigma \prod_{k=1}^{n} \frac{2}{\alpha_k^2} \left( \sum_{l=k+1}^{n} \frac{\alpha_k \cdot \alpha_l}{t_k - t_l} - \frac{\alpha_k \cdot \lambda}{t_k - z} \right),
$$

(5.10)

where $\sigma$ is the symmetrization of the $t$'s associated with the same $\alpha_i$'s.†

The proof is given in Appendix C. Theorem II coincides with the 2nd solution of Schechtman and Varchenko [20]. Note that $\langle \prod_{i=1}^{n} S_{\alpha_i}(t_i) P_{\lambda}^{(a_1, \ldots, a_n)}(z) \rangle$ depends on the ordering of the $\alpha_i$'s of $P_{\lambda}^{(a_1, \ldots, a_n)}$, although it is symmetric with respect to $S_{\alpha_i}(t_i)$'s.

§ 5.3. Finally we give the condition for the contour of this correlation function.

The correlation function ( a solution of the KZ equation ) should satisfy the Ward identity for Kac-Moody current $J^a(z)$ such that

$$
\langle J^a(z) \prod_{i=1}^{n} V_{\lambda_i}(w_i) \rangle = \sum_{i=1}^{n} \frac{(J^a)_{\lambda_i}}{z - w_i} \langle \prod_{i=1}^{n} V_{\lambda_i}(w_i) \rangle,
$$

(5.11)

where $(J^a)_{\lambda_i}$ acts on the primary field $V_{\lambda_i}(w_i)$.

From (3.9), $E_{\alpha}(z)$ and $H_{i}(z)$ insertion satisfies this relation, but there are the following corrections for $F_{\alpha}(z)$

$$
\int \prod_{i=1}^{m} dt_i \langle F_{\alpha}(z) \prod_{i=1}^{m} e^{-\alpha_i \cdot \phi(t_i)} S_{\alpha_i}(t_i) \prod_{a=1}^{n} e^{\lambda a \cdot \phi(z_a)} P_{a}(z_a) \rangle
$$

$$
= - \int \prod_{i=1}^{m} dt_i \sum_{j=1}^{m} \kappa \delta_{\alpha}^a \frac{\partial}{\partial t_j} \langle \frac{1}{z - t_j} \prod_{i=1}^{m} e^{-\alpha_i \cdot \phi(t_i)} \prod_{i \neq j}^{m} S_{\alpha_i}(t_i) \prod_{a=1}^{n} e^{\lambda a \cdot \phi(z_a)} P_{a}(z_a) \rangle
$$

$$
+ \sum_{b=1}^{n} \frac{(F_{\alpha})_{\lambda_b}}{z - z_b} \int \prod_{i=1}^{m} dt_i \langle \prod_{i=1}^{m} e^{-\alpha_i \cdot \phi(t_i)} S_{\alpha_i}(t_i) \prod_{a=1}^{n} e^{\lambda a \cdot \phi(z_a)} P_{a}(z_a) \rangle.
$$

(5.12)

† However we must not overcount $\sigma$ in (5.9) and (5.10).
The paths should be chosen so that the total divergence terms vanish. For example, we can use the Dotsenko Fateev or Felder type contours [22, 23].

6. Conclusion and Discussion

In this paper, we have explicitly derived the correlation function of the chiral primary field for the WZNW models, based on the Wakimoto realization. In general, it is difficult to calculate this correlator by simply using the $\beta\gamma$ OPE relation (4.5) except for some particular cases. But we made it possible by finding two important relations, the "screening currents Ward identity" (5.2) and the "currents Ward identity" including the screening currents (5.8).

In addition, we wish to make two further comments.

i). In terms of the differential representation of the KZ equation, where the matrix $\Omega_{ab}$ is replaced by some differential operators, our result gives only the solution over polynomial $C[x^\alpha]$. But in the case where the level of the KM algebra is non-integer, it is natural that a non-polynomial solution appears. So the structure of the non-integer level solution should be analyzed. The dimension of the space of these solutions (i.e., fusion rules) is now under investigation [26].

ii). In the differential realization of the simple Lie algebras, besides the "right action" (2.2), there is also the "left action" $\hat{J}(\frac{\partial}{\partial x}, x, \lambda)$ defined by

$$\hat{J}(\frac{\partial}{\partial x}, x, \lambda) Z |\hat{\lambda}\rangle = -J Z |\hat{\lambda}\rangle. \quad (6.1)$$

where $Z \equiv \exp(\sum_\alpha x^\alpha e_\alpha)$, and $|\hat{\lambda}\rangle$ is the lowest weight vector such that $f_\alpha |\hat{\lambda}\rangle = 0$, $h_1 |\hat{\lambda}\rangle = \lambda_1 |\hat{\lambda}\rangle$. As mentioned in Appendix A.2, $J(\frac{\partial}{\partial x}, x, \lambda)$ and $\hat{J}(\frac{\partial}{\partial x}, x, \lambda)$ are related by replacing $x^\alpha$ with $-x^\alpha$ and $\lambda_a$ with $-\lambda_a$. Note that one of these operators is (2.5).

Similarly, there is another Wakimoto realization in addition to (3.5), which is given by replacing $\beta_\alpha$ with $-\beta_\alpha$, $\gamma_\alpha$ with $-\gamma_\alpha$ and $\phi_i$ with $-\phi_i$. Since one of
these left currents is the screening current (3.8), they would be related to the quantum groups [24, 25].

Acknowledgments

This work has been carried out in collaboration with Y. Yamada. I am grateful to him for help in many ways. I would like to thank A. Tsuchiya and the members of KEK theory group for valuable discussions. I would also like to thank to N. A. McDougall for a careful reading of the manuscript.

Appendix A

In this Appendix we present some useful formulas for the free fields realization, and the proof of (2.6).

§ A.1. We give some helpful relations.

**PROPOSITION.** Gauss decomposition of the right action is given by the following differential operators

\[
Z \exp(te_a) = \exp(tE_a(\frac{\partial}{\partial x}, x) + O(t^2)) Z,
\]

\[
Z \exp(th_i) = \exp(th_i) \exp(t\hat{H}_i(\frac{\partial}{\partial x}, x) + O(t^2)) Z, \tag{A.1}
\]

\[
Z \exp(tf_{\alpha_i}) = \exp(tf_{\alpha_i}) \exp(th_i x^{\alpha_i}) \exp(t\hat{F}_{\alpha_i}(\frac{\partial}{\partial x}, x) + O(t^2)) Z,
\]

where

\[
E_a(\frac{\partial}{\partial x}, x) = \sum_\beta \left( \delta_{\alpha}^\beta - \frac{1}{2} \sum_{\gamma>0} f_{\alpha\gamma}^\beta x^\gamma + O(x^2) \right) \frac{\partial}{\partial x^\beta},
\]

\[
\hat{H}_i(\frac{\partial}{\partial x}, x) = - \sum_{\beta>0} (\nu_i, \beta) x^\beta \frac{\partial}{\partial x^\beta}, \tag{A.2}
\]

\[
\hat{F}_{\alpha_i}(\frac{\partial}{\partial x}, x) = - \sum_\beta \left( \sum_{\gamma} f_{-\alpha_i\gamma}^\beta x^\gamma + \frac{1}{2} (\nu_i, \beta) x^\alpha x^\beta + O(x^3) \right) \frac{\partial}{\partial x^\beta}.
\]

† Our parametrization for \( Z \) is the same as [12]. Restricted to the \( sl(n) \) case, the parametrization of [9] takes a more simple form.
Proof. It is given by using the Baker-Campbell-Hausdorff formula

\[ e^X e^Y = \exp \left( X + Y + \frac{1}{2} [X, Y] + \frac{1}{12} ([X, [X, Y]] + [Y, [Y, X]]) + \cdots \right), \]

\[ e^{-Y} e^X e^Y = \exp \left( X + [X, Y] + O(Y^2) \right). \]  

(A.3)

The right action of \( e^\alpha \in \mathfrak{n}_+ \) is

\[
\begin{align*}
Z \exp(\alpha t) &= \exp \left( \sum_{\beta} x^\beta e_\beta + t(e_\alpha - \frac{1}{2} \sum_{\beta\gamma} f_{\alpha\beta\gamma} x^\beta e_\gamma + O(x^2)) + O(t^2) \right), \\
&= \exp \left( \sum_{\beta} x^\beta e_\beta + t \left( \frac{\partial}{\partial x^\alpha} - \frac{1}{2} \sum_{\beta\gamma} f_{\alpha\beta\gamma} x^\beta \frac{\partial}{\partial x^\gamma} + O(x^2) \right) \sum_{\beta} x^\beta e_\beta + O(t^2) \right) \\
&= \exp \left( t \left( \frac{\partial}{\partial x^\alpha} - \frac{1}{2} \sum_{\beta\gamma} f_{\alpha\beta\gamma} x^\beta \frac{\partial}{\partial x^\gamma} + O(x^2) \right) + O(t^2) \right) Z.
\end{align*}
\]  

(A.4)

For \( h_i \in \mathfrak{h} \), Gauss decomposition of the right action is

\[
\begin{align*}
Z \exp(th_i) &= \exp(th_i) \exp \left( \sum_{\beta} x^\beta e_\beta - t \sum_{\beta} (\nu_i \cdot \beta) x^\beta e_\beta + O(t^2) \right) \\
&= \exp(th_i) \exp \left( - t \sum_{\beta} (\nu_i \cdot \beta) x^\beta \frac{\partial}{\partial x^\beta} + O(t^2) \right) Z.
\end{align*}
\]  

(A.5)

For \( f_{\alpha i} \in \mathfrak{n}_- \), with simple root \( \alpha_i \)

\[
\begin{align*}
Z \exp(tf_{\alpha i}) &= \exp(tf_{\alpha i}) \exp \left( \sum_{\beta} x^\beta e_\beta + th_i x^{\alpha_i} - t \sum_{\beta\gamma} f_{\alpha i\beta\gamma} x^\beta e_\gamma + O(t^2) \right) \\
&= \exp(tf_{\alpha i}) \exp(th_i x^{\alpha_i}) \Theta,
\end{align*}
\]  

(A.6)

\[
\Theta = \exp \left( \sum_{\beta} x^\beta e_\beta - t \left( \sum_{\beta\gamma} f_{\alpha i\beta\gamma} x^\beta e_\gamma + \frac{1}{2} \sum_{\beta} (\nu_i \cdot \beta) x^{\alpha_i} x^\beta e_\beta + O(x^3) \right) + O(t^2) \right) \\
&= \exp \left( - t \left( \sum_{\beta\gamma} f_{\alpha i\beta\gamma} x^\beta \frac{\partial}{\partial x^\gamma} + \frac{1}{2} \sum_{\beta} (\nu_i \cdot \beta) x^{\alpha_i} x^\beta \frac{\partial}{\partial x^\beta} + O(x^3) \right) + O(t^2) \right) Z.
\]  

(A.7)

Q.E.D.
Moreover, if we act with the highest weight vector \( \langle \lambda | \) from the left in (A.1) and take the infinitesimal limit with respect to \( t \), then we get the differential operators \( E_\alpha, F_\alpha, \) and \( H_i \) corresponding to the Chevalley generators \( e_\alpha, f_\alpha, \) and \( h_i \)

\[
E_\alpha \left( \frac{\partial}{\partial x}, x, \lambda \right) = E_\alpha \left( \frac{\partial}{\partial x}, x \right), \\
H_i \left( \frac{\partial}{\partial x}, x, \lambda \right) = \hat{H}_i \left( \frac{\partial}{\partial x}, x \right) + \lambda_i, \\
F_\alpha i \left( \frac{\partial}{\partial x}, x, \lambda \right) = \hat{F}_\alpha i \left( \frac{\partial}{\partial x}, x \right) + \lambda_i x^{\alpha_i}.
\]

(A.8)

Finally, for the left action in (2.4) and (6.1), the Gauss decompositions (of inverse direction) are given by taking the inverse of the right action and reversing the signs of \( x^{\alpha_i}'s \) as follows

\[
\exp(-te_\alpha) Z = \exp(t E_\alpha(-\frac{\partial}{\partial x}, x) + O(t^2)) Z, \\
\exp(-th_i) Z = \exp(t \hat{H}_i(-\frac{\partial}{\partial x}, x) + O(t^2)) Z \exp(-th_i), \\
\exp(-tf_\alpha i) Z = \exp(t \hat{F}_\alpha i(-\frac{\partial}{\partial x}, x) + O(t^2)) Z \exp(th_i x^{\alpha_i}) \exp(-tf_\alpha i).
\]

(A.9)

For instance

\[
S_\alpha \left( \frac{\partial}{\partial x}, x \right) = E_\alpha \left( -\frac{\partial}{\partial x}, x \right).
\]

(A.10)

§ A.2. Next we derive (2.6), by using the associativity for the Gauss decomposition.

Proof of (2.6.a).

\[
( \exp(-se_\beta) Z ) \exp(te_\alpha) = \exp(s S_\beta + O(s^2)) \exp(t E_\alpha + O(t^2)) Z, \\
\exp(-se_\beta)( Z \exp(te_\alpha) ) = \exp(t E_\alpha + O(t^2)) \exp(s S_\beta + O(s^2)) Z.
\]

(A.11)

From the associativity, \( E_\alpha \) and \( S_\alpha \) commute. Q.E.D.
Proof of (2.6.b).

\[
(\exp(-se\alpha) Z) \exp(th_i) = \exp(sS\alpha + O(s^2)) \exp(t(h_i + \hat{H}_i) + O(t^2)) Z, \quad (A.12)
\]

\[
\exp(-se\alpha)(Z \exp(th_i)) = \exp(t(h_i + \hat{H}_i) + O(t^2)) \exp(sS\alpha + O(s^2))
\]
\[
\exp(-st(\nu_i \cdot \alpha)S\alpha + O(s^2) + O(t^2)) Z,
\]

where (A.13) comes from

\[
\exp(-se\alpha) \exp(th_i) = \exp(th_i) \exp(st(\nu_i \cdot \alpha)e\alpha + O(s^2) + O(t^2)) \exp(-se\alpha).
\]

(A.14)

If we act with the highest weight vector \langle \lambda \rangle from the left, then

\[
\exp(sS\alpha + O(s^2)) \exp(tH_i + O(t^2))\langle \lambda \rangle Z
\]
\[
= \exp(tH_i + O(t^2)) \exp(sS\alpha + O(s^2)) \exp(-st(\nu_i \cdot \alpha)S\alpha + O(s^2) + O(t^2))\langle \lambda \rangle Z.
\]

(A.15)

Q.E.D.

Proof of (2.6.c).

\[
(\exp(-se\alpha_j) Z) \exp(tf_{\alpha_i})
\]
\[
= \exp(sS\alpha_j + O(s^2)) \exp(t(f_{\alpha_i} + h_i x^{\alpha_i} + \hat{F}_{\alpha_i}) + O(t^2)) Z
\]

(A.16)

\[
\exp(-se\alpha_j)(Z \exp(tf_{\alpha_i}))
\]
\[
= \exp(t(f_{\alpha_i} + h_i x^{\alpha_i} + \hat{F}_{\alpha_i}) + O(t^2)) \exp(sS\alpha_j + O(s^2))
\]
\[
\exp(-st(\delta_{ij} h_i + (\nu_i \cdot \alpha_j) x^{\alpha_i} S\alpha_j) + O(s^2) + O(t^2)) Z,
\]

where (A.17) comes from

\[
\exp(-se\alpha_j) \exp(t(f_{\alpha_i} + h_i x^{\alpha_i}))
\]
\[
= \exp(t(f_{\alpha_i} + h_i x^{\alpha_i})) \exp(st(-\delta_{ij} h_i + (\nu_i \cdot \alpha_j)e\alpha_j x^{\alpha_i}) + O(s^2) + O(t^2)) \exp(-se\alpha_j).
\]

(A.18)
Then we get

\[ \exp(sS_{\alpha j} + O(s^2)) \exp(tF_{\alpha i} + O(t^2)) \langle \lambda | Z \rangle = \exp(tF_{\alpha i} + O(t^2)) \exp(sS_{\alpha j} + O(s^2)) \exp(-st(\delta_{ij} \lambda_i + (\nu_i \cdot \alpha_j) x^\alpha S_{\alpha j}) + O(s^2) + O(t^2)) \langle \lambda | Z \rangle. \]  

(A.19)

Q.E.D.

Appendix B

In this Appendix we explain another derivation of the “current Ward identity” including screening currents (5.8). The outline of this derivation is indicated in [21]. Although it is more complicated than that in the text, it shows the clear relation between the 1st and 2nd solution.

§ B.1. We start with evaluating \( \langle (S_{\alpha_1} \cdots S_{\alpha_m} P)(z) \rangle \) in (5.4), in the case where \( P \) is given in (3.11). We will show that this correlation can be calculated by using only finite dimensional algebra.

Note that only the c-number term of the operator \( (S_{\alpha_1} \cdots S_{\alpha_m} P)(z) \in \mathbb{C}[\gamma^\alpha(z)] \) has a nonzero contribution to the correlation \( \langle (S_{\alpha_1} \cdots S_{\alpha_m} P)(z) \rangle \). Moreover, the constant term of the operator \( (S_{\alpha_1} \cdots S_{\alpha_m} P)(z) \in \mathbb{C}[\gamma^\alpha(z)] \) is the same as that of the polynomial \( S_{\alpha_1} \cdots S_{\alpha_m} P \in \mathbb{C}[x^\alpha] \).

By definition of the differential representation (2.4) and (2.10),

\[ S_{\alpha_1} \cdots S_{\alpha_m} P = (-1)^m \langle \lambda | e_{\alpha_m} \cdots e_{\alpha_1} Z f_{\beta_1} \cdots f_{\beta_m} | \lambda \rangle. \]  

(B.1)

Since the constant term of this polynomial is given by the value at \( x^\alpha = 0 \), i.e. \( Z = 1 \), the correlator \( \langle (S_{\alpha_1} \cdots S_{\alpha_m} P)(z) \rangle \) is nothing but the “Shapovalov form”
The Shapovalov form enjoys the inductive formula

$$\langle \lambda | \prod_{i=m}^{1} e_{\alpha_i} f_{\beta_{j}} \prod_{j=1}^{n} f_{\beta_{j}} | \lambda \rangle = -\sum_{k=1}^{m} \delta_{\beta_k}^{\alpha_k} \frac{2}{\alpha_k^2} (\sum_{l=k}^{m} \alpha_k \cdot \alpha_l - \alpha_k \cdot \lambda) \langle \lambda | \prod_{i \neq k}^{1} e_{\alpha_i} \prod_{j=1}^{n} f_{\beta_{j}} | \lambda \rangle.$$ 

So we have the following property

$$\langle \lambda | e_{\alpha_m} \cdots [e_{\alpha_i}, e_{\alpha}] \cdots e_{\alpha_1} f_{\beta_{j}} \prod_{j=1}^{n} f_{\beta_{j}} | \lambda \rangle = -\delta_{\beta_k}^{\alpha_k} \frac{2}{\alpha_k^2} (\alpha \cdot \alpha_i) \langle \lambda | e_{\alpha_m} \cdots e_{\alpha_1} \prod_{j=1}^{n} f_{\beta_{j}} | \lambda \rangle + \cdots,$$ 

Here, and in the following, $+ \cdots$ means a sum of terms proportional to $\delta_{\beta_k}^{\alpha_k}$ for $k = 1, \cdots, q$.

Combining this Shapovalov form and Theorem I, we obtain

$$\langle S_{\alpha_1}(t_1) \cdots [S_{\alpha}, S_{\alpha_i}](t_i) \cdots S_{\alpha_n}(t_n) P_{\lambda}^{\{\beta_1, \cdots, \beta_n\}}(z) \rangle$$

$$= \delta_{\beta_k}^{\alpha_k} \frac{2}{\alpha_k^2} (\alpha \cdot \alpha_i) \langle \prod_{i=1}^{m} S_{\alpha_i}(t_i) P_{\lambda}^{\{\beta_1, \cdots, \beta_n\}}(z) \rangle + \cdots,$$ 

$$\langle S_{\alpha_1}(t_1) \cdots S_{\alpha_n}(t_n) (S_{\alpha} P_{\lambda}^{\{\beta_1, \cdots, \beta_n\}})(z) \rangle$$

$$= -\delta_{\beta_k}^{\alpha_k} \frac{2}{\alpha_k^2} (\alpha_l \cdot \lambda) \langle \prod_{i=1}^{m} S_{\alpha_i}(t_i) P_{\lambda}^{\{\beta_1, \cdots, \beta_n\}}(z) \rangle + \cdots,$$ 

where $[S_{\alpha}, S_{\beta}](t)$ means $\sum_{\gamma} f_{\alpha \beta \gamma} S_{\gamma}(t)$. 

(B.2)
§B.2. Proof of (5.8).

From (5.4) and (B.3), we can expand the left hand side by $\delta^\alpha_{\beta k}$ as follows

$$\langle \prod_{i=1}^{m} S_{\alpha_i}(t_i) P^\lambda_{\beta,\beta_1,\ldots,\beta_n}(z) \rangle = \sum_{k=1}^{m} \delta^\alpha_{\beta k} \Theta_k(t, z, \alpha, \beta).$$

(B.8)

Since $\{\alpha_1, \ldots, \alpha_n\}$ is arbitrary, the $\delta^\alpha_{\beta k}$’s constitute a linearly independent basis. So the function $\Theta_k(t, z, \alpha, \beta)$ is uniquely defined as a coefficient.

To obtain the $\Theta_1(t, z, \alpha, \beta)$, we use (5.2). Then,

$$(B.8) = \sum_{i=2}^{m} \frac{1}{t_1-t_i} \langle S_{\alpha_2}(t_2) \cdots [S_{\alpha_1}, S_{\alpha_1}](t_i) \cdots S_{\alpha_n}(t_n) P^\lambda_{\beta,\beta_1,\ldots,\beta_n}(z) \rangle$$

$$+ \frac{1}{t_1-z} \langle S_{\alpha_2}(t_2) \cdots S_{\alpha_n}(t_n) (S_{\alpha_1} P^\lambda_{\beta,\beta_1,\ldots,\beta_n})(z) \rangle.$$  

(B.9)

From (B.6) and (B.7), we have

$$(B.8) = \delta^\alpha_{\beta 1} 2 \alpha_1^2 \left( \sum_{l=2}^{m} \frac{\alpha_1 \cdot \alpha_l}{t_1-t_l} - \frac{\alpha_1 \cdot \lambda}{t_1-z} \right) \langle \prod_{i=2}^{m} S_{\alpha_i}(t_i) P^\lambda_{\beta_1,\ldots,\beta_n}(z) \rangle + \cdots.$$  

(B.10)

The 1st term of this gives the $\Theta_1(t, z, \alpha, \beta)$.

Since (B.8) is symmetric with respect to $S_{\alpha_i}(t_i)$’s, we get

$$\Theta_k(t, z, \alpha, \beta) = \frac{2}{\alpha_k^2} \left( \sum_{l \neq k}^{m} \frac{\alpha_k \cdot \alpha_l}{t_k-t_l} - \frac{\alpha_k \cdot \lambda}{t_k-z} \right) \langle \prod_{i \neq k}^{m} S_{\alpha_i}(t_i) P^\lambda_{\beta_1,\ldots,\beta_n}(z) \rangle.$$  

(B.11)

Q.E.D.
Appendix C

In this Appendix we give the proofs of Theorem I and Theorem II.

§ C.1. Proof of Theorem I from (5.2).  

Proof of (5.4). We prove (5.4) by induction on the number of \( S_\alpha(t) \), which we denote by \( n \). For \( n = 1 \), the assertion is valid. Assume that (5.4) holds for all \( n \leq m \). It is sufficient to show

\[
\langle S_\alpha(t)S_{\alpha_1}(t_1)\cdots S_{\alpha_m}(t_m)P(z) \rangle = \sum_{\text{perm}} \langle (S_\alpha S_{\alpha_1} \cdots S_{\alpha_m} P)(z) \rangle \]

Both sides are global 1-form on the sphere with respect to \( t \). On the right hand side, the residue at the pole \( z \) is

\[
\frac{1}{2\pi i} \int dt \sum_{\text{perm}} \frac{\langle (S_\alpha S_{\alpha_1} \cdots S_{\alpha_m} S_{\alpha_i} P)(z) \rangle}{(t-t_1)(t_1-t_2)\cdots(t_{i-1}-t)(t-t_{i+1})\cdots(t_m-z)}
\]

And the residue at the pole \( t_i \) is

\[
\frac{1}{2\pi i} \int dt \sum_{\text{perm}} \left( \frac{\langle (S_\alpha S_{\alpha_1} \cdots S_{\alpha_i} S_{\alpha} P)(z) \rangle}{(t_1-t_2)\cdots(t_{i-1}-t_i)(t_i-t)(t-t_{i+1})\cdots(t_m-z)} + \frac{\langle (S_\alpha S_{\alpha_1} \cdots S_{\alpha_m} P)(z) \rangle}{(t_1-t_2)\cdots(t_{i-1}-t_i)(t_i-t)(t-t_{i+1})\cdots(t_m-z)} \right)
\]

By the inductive hypothesis, the poles and the residues are the same on both sides of (C.2), so they must be equal. Q.E.D.
Proof of (5.3). This is also given by induction on the number of $S_{\alpha}(t)$ using simple combinatrics. For $n = 1$, (5.3) holds. Assume that the assertion is valid for all $n \leq m$. Let $\sum_{\text{part}(m)}$ be the summation over all the partition of $I = \{1, 2, \cdots, m\}$ into $n$ disjoint union $I_1^m \cup I_2^m \cup \cdots \cup I_n^m$, then we get from (5.2)

$$
\langle S_{\alpha_{m+1}}(t_{m+1}) \prod_{i=1}^{m} S_{\alpha_i}(t_i) \prod_{a=1}^{n} P_a(z_a) \rangle
\quad = \quad \sum_{\text{part}(m)} \sum_{a=1}^{n} \langle S_{\alpha_{m+1}}(t_{m+1}) \prod_{i \in I_a^m} S_{\alpha_i}(t_i) P_a(z_a) \rangle \prod_{b \neq a} \langle \prod_{i \in I_b^m} S_{\alpha_i}(t_i) P_b(z_b) \rangle
\quad = \quad \sum_{\text{part}(m+1)} \prod_{a=1}^{n} \langle \prod_{i \in I_a^{m+1}} S_{\alpha_i}(t_i) P_a(z_a) \rangle
$$

(C.4) Q.E.D.

§ C.2. Proof of Theorem II from (5.8).

Proof of (5.10). Let $I$ be an ordered set of $\alpha_i$’s. We can expand the left hand side of (5.10) by $\delta^I_{\{\alpha_1, \cdots, \alpha_n\}}$ with no ambiguity, as

$$
\langle \prod_{i=1}^{n} S_{\alpha_i}(t_i) P_{\lambda}^{(\alpha_1, \cdots, \alpha_n)}(z) \rangle = \sum_I \delta^I_{\{\alpha_1, \cdots, \alpha_n\}} \Theta_I(t, z, \alpha),
$$

(C.5)

From (5.8), $\Theta_{\{\alpha_1, \cdots, \alpha_n\}}(t, z, \alpha)$ is easy to find

$$
\Theta_{\{\alpha_1, \cdots, \alpha_n\}}(t, z, \alpha) = \prod_{k=1}^{n} \frac{2}{\alpha_k^2} \left( \sum_{l=k+1}^{n} \frac{\alpha_k \cdot \alpha_l}{t_k - t_l} - \frac{\alpha_k \cdot \lambda}{t_k - z} \right).
$$

(C.6)

By the symmetry with respect to $S_{\alpha_i}(t_i)$, we can get the other $\Theta_I(t, z, \alpha)$ similarly. Moreover $\sum_I \delta^I_{\{\alpha_1, \cdots, \alpha_n\}}$ is nothing but the symmetrization of the $t$’s associated with the same $\alpha_i$’s.

Q.E.D.

Proof of (5.9). It is easy; notice only that $\sum_{\text{pert}}$ is no more than the symmetrization of the $t$’s associated with the same $\alpha_i$’s belonging to different $I_a$.

Q.E.D.
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