An I/O-efficient Generator for Massive Complex Networks with Explicit Community Structure

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Abstract

The LFR benchmark is a popular benchmark graph model used to evaluate community detection algorithms. We present the first external memory algorithm that is able to generate massive complex networks following the LFR model. Its most expensive component is the generation of random graphs with prescribed degree sequences which can be divided in two steps: They are first materialized as deterministic graphs using the Havel-Hakimi algorithm and then randomized. Our main contribution are \textit{HP-HH} and \textit{ES-TFP}, two I/O-efficient external memory algorithms for these two steps. In an experimental evaluation we demonstrate their performance: our implementation is able to generate graphs with more than 10 billion edges on a single machine, is competitive with a parallel massively distributed algorithm and on smaller instances faster than a state-of-the-art internal memory implementation.

1 Introduction

Complex networks, such as web graphs or social networks, are usually composed of communities, also called clusters, that are internally densely connected but externally sparse. Finding these clusters, which may be disjoint or overlapping, is a common task in network analysis. Since there are different approaches to formalize this fuzzy concept, a large number of algorithms trying to find meaningful clusters have been proposed \cite{8,10}. Synthetic benchmarks are commonly used for evaluating clustering algorithms because for most real-world networks it is not known which clusters they contain and which of them are actually detectable through structure \cite{2}. In the last years, the LFR benchmark \cite{12,13} has been established as a popular benchmark for evaluating clustering algorithms both for non-overlapping and for overlapping communities.

With the emergence of massive networks that cannot be handled in the main memory of a single computer, new clustering schemes have been proposed for advanced models of computation \cite{4}. Since such algorithms typically use hierarchical input representations, quality results of small benchmarks may not be generalizable to larger instances. Therefore, we propose a generator in the external memory (EM) model of computation that follows the LFR benchmark and is capable of producing graphs larger than internal memory. In this paper we concentrate on the non-overlapping version. However, the algorithms presented can also be extended to overlapping clusters.

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Similarly, the distributed CKB benchmark [5] is a step in the same direction. However, it considers overlapping clusters and uses a different model of communities. In contrast, our approach is more flexible as it can be used with all possible parameterizations of LFR.

In preliminary experiments, we identified the generation of random graphs with prescribed degree sequence as the main difficulty when moving the LFR benchmark into an EM setting — both in terms of algorithmic complexity and run time. For this, the fixed degree sequence model (FDSM), also known as edge-switching markov-chain algorithm, is used by the LFR algorithm (e.g. [17]). While faster algorithms like the erased Configuration Model exist, they do not exactly generate the given degree sequence. Especially with skewed degree distributions this leads to significantly different graphs [22].

1.1 Our contribution

Our main contribution is the first external memory version of FDSM which we present as follows: After defining our notation, we introduce the LFR benchmark in more detail and then focus on the FDSM: It consists of a) generating a (deterministic) graph from a prescribed degree sequence (Section 4) and b) randomizing this graph using random edge switches (Section 5). These steps are logically linked forming a pipeline which moves data from one algorithm to the next. Sections 6 and 7 briefly describe EM algorithms for the remaining steps of the LFR benchmark. We conclude with an experimental evaluation of the external memory version of the FDSM and demonstrate that it is faster than an existing internal memory implementation, scales well to large instances and can compete with a parallel distributed algorithm [3].

2 Preliminaries

Define \([k] \equiv \{1, \ldots, k\}\) for \(k \in \mathbb{N}_+\). A graph \(G = (V, E)\) has \(n = |V|\) sequentially numbered nodes \(V = [n]\) and \(m = |E|\) edges. Unless stated differently, graphs are assumed to be undirected and unweighted. We denote an undirected edge \([u, v] \in E\) to indicate that \(u < v\). A graph is called simple if it contains neither multi-edges nor self-loops. We denote a powerlaw distribution with exponent \(-\gamma\) between the limits \(a < b\) as \(\text{Pld}_{a,b}^{(\gamma)}\). Let \(X\) be an integer random variable drawn from \(\text{Pld}_{a,b}^{(\gamma)}\) then \(P[X=k] \propto k^{-\gamma}\) if \(a \leq k \leq b\) and \(P[X=k] = 0\) otherwise.

2.1 External-Memory Model

We consider the commonly accepted external-memory model by Aggarwal and Vitter [1]. It features a two-level memory hierarchy with fast internal memory (IM) holding up to \(M\) data items and a slow disk of unbounded size. The measure of performance of an algorithm is the required number of I/Os where each I/O moves a block of \(B\) consecutive items between memory levels. Reading or writing \(n\) contiguous items from or to disk requires \(\text{scan}(n) = \Theta(n/B)\) I/Os. Sorting \(n\) consecutive items triggers \(\text{sort}(n) = \Theta((n/B) \cdot \log_{M/B}(n/B))\) I/Os. For all realistic values of \(N, B\) and \(M\), \(\text{scan}(n) < \text{sort}(n) \ll n\). Sorting complexity constitutes a lower bound for most nontrivial EM tasks.

2.2 Time Forward Processing

Let \(A\) be an algorithm which performs discrete events over time (e.g., iterations of a loop) that produce values which are later reused by following events. \(A\)’s data dependencies can be modeled using a directed acyclic graph \(G = (V, E)\) where every node \(v \in V\) corresponds to an event [14]. Edge \((u, v) \in E\) indicates that \(u\)’s value will be required by \(v\).
By computing a solution, the algorithm traverses $G$ in topological order. For simplicity, we assume $G$ to already be ordered, i.e. $\forall (u,v) \in E$: $u < v$. Then, the Time Forward Processing (TFP) technique uses a minimum priority queue (PQ) to provide the means to transport data as implied by $G$: Every event $u$ receives messages sent to it by claiming and removing all items with priority $u$ from the PQ which inductively are head elements at that time. The event then computes its value $x_u$ and sends it to every successor $v$ by inserting $x_u$ into the PQ with priority $v$. Using an appropriate EM PQ [24], this incurs $O(\text{sort}(k))$ I/Os where $k$ is the number of messages sent.

3 The LFR Benchmark

The LFR benchmark [13] describes a random graph model featuring an explicit community structure, a powerlaw degree distribution, and a powerlaw community size distribution. While the original benchmark was conceived for unweighted undirected graphs and non-overlapping communities, a revised version [12] also introduces weighted and directed graphs with overlapping communities. We consider this first setting, but our approach can easily be extended to overlapping communities. The model features the following parameters:

- The number $n$ of nodes and their powerlaw degree distribution $\text{P}_{\text{LD}}^{(γ)}_{\text{d}_{\text{min}},\text{d}_{\text{max}}}$. A commonly used value of $γ$ is 2.
- The powerlaw distribution of the communities’ sizes $\text{P}_{\text{LD}}^{(β)}_{\text{s}_{\text{min}},\text{s}_{\text{max}}}$.
- A mixing parameter $μ$ that gives the ratio of neighbors of each node $u$ that shall be in a different community than $u$.

The revised generator [12] changes the original algorithm [13] even for the initial scenario of unweighted, undirected graphs and non-overlapping communities. We describe the modern approach which is also used in the author’s implementation [7]: Initially the degree sequence $π=(d_1, \ldots, d_n)$ and community sizes $S = (s_1, \ldots, s_C)$ with $\sum_{s=1}^{C} s_s = n$ are randomly sampled. The algorithm then assigns every node to a community at random such that the requested community sizes are realized. Further it has to satisfy that for each node $u$ the desired internal degree $d_u^{\text{in}} = (1-μ) \cdot d_u$ is strictly smaller than the size $s_s$ of its community $s$. Using FDSM, the algorithm generates an intra-community graph per community using the desired internal degrees and a global inter-community graph with the remaining degrees. Finally, rewiring steps are applied on the global inter-community graph such that no edges exist between two nodes of the same community in order not to violate the mixing parameter $μ$.

While for realistic parameters most intra-community graphs fit in main memory, we cannot assume the same for the global graph. Here, an external memory variant of FDSM is applicable which we implement using HP-HH and ES-TFP described in Sections 4 and 5.

4 Deterministic Edges from a Degree Sequence

In this chapter we consider an EM-variant of the well-known Havel-Hakimi scheme that takes a positive non-increasing degree sequence $π$ and if possible outputs a graph $G_π=(V,E)$ with $\text{deg}_{G_π}(v)=d_v \ \forall v \in V$. A sequence $π$ is called graphical if a graph $G_π$ matching $π$ exists. Havel [9] and Hakimi [11] gave an inductive characterization of graphical sequences which directly leads to a deterministic graph generator: Given $π$, connect the first node to $d_1$-many vertices by emitting edges to nodes $d_2, \ldots, d_{d_1+1}$ and decrement the remaining degree of each new neighbor yielding a new sequence $π'$. 

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Remove zero-entries, then order \( \pi' \) and keep track of the original position to be able to output the correct node indices. Finally recurse until no positive entries remain. After every iteration, the dimension of \( \pi \) is reduced by at least one which gives a linear bound on the number of rounds.

To fit into our pipeline, we modify this scheme to emit a lexicographically ordered edge list. Furthermore, since writing the edge list can be overlapped with its first read in ES-TFP, we only count internal I/Os and do not take the output itself into account. In this setup, the classical algorithm triggers \( O(\text{scan}(\mathcal{ES}}-\text{TFP}) \) I/Os for iteration \( i \) with \( d_{i+1} \geq d_i > M \). Additionally, for degrees which appear often in \( \pi \), it is non-trivial to keep the sequence ordered triggering additional accesses.

We propose \( \text{HP-HH} \), which in conjunction with the ordered random number generation detailed in Appendix A yields an edge list without any I/Os for all practical scenarios. The algorithm exploits the skewed degree distributions of scale-free networks which results in a small degree support \( \tilde{D}_\pi = \{|d_i : 1 \leq i \leq n|\} \). As indicated in Fig. 1, there are general networks with \( D_\pi = \Theta(n) \). Lemma 1 shows \( D_\pi = O(n^{1/\gamma}) \) with high probability for networks sampled from a powerlaw distribution with exponent \(-\gamma\), where typically \( \gamma = 2 \) is used in LFR:

**Lemma 1.** Let \( \pi \) be a degree sequence with \( n \) nodes sampled from \( \text{PLD}^{(\gamma)}_{1/n} \). Then, the number of unique degrees \( D_\pi = \{|d_i : 1 \leq i \leq n|\} \) is bound by \( O(n^{1/\gamma}) \) with high probability.

**Proof.** Consider random variables \( (X_1, \ldots, X_n) \) as an unordered degree sequence and fix \( 1 \leq j \leq n \). We first show that the total probability mass of degrees higher than \( n^{1/\gamma} \) is bounded by \( O(n^{1/\gamma-1}) \). We hence can argue that their rare occurrences do not affect \( D_\pi \) asymptotically.

\[
\mathbb{P}[X > n^{1/\gamma}] = \frac{\sum_{i=1}^{n-1} i^{-\gamma}}{\sum_{i=1}^{n} i^{-\gamma}} \leq \frac{\int_{n^{1/\gamma}}^{\infty} x^{-\gamma} \, dx}{\int_{1}^{\infty} x^{-\gamma} \, dx} = \frac{1}{\zeta(\gamma) \left( n^{1-\gamma} - (n-1)^{1-\gamma} \right)} = O(n^{1/\gamma-1}),
\]

where \( \zeta(\gamma) = \sum_{i=1}^{\infty} i^{-\gamma} \) is the Riemann zeta function which satisfies \( \zeta(\gamma) \geq 1 \) \( \forall \gamma \in \mathbb{R}, \gamma \geq 1 \). For (ii) we exploit the monotonicity of \( \sum_i i^{-\gamma} \) to bound it between the two integrals \( \int_{n^{1/\gamma}}^{b+1} x^{-\gamma} \, dx \). Hence, the \( n^{1/\gamma} \) lowest degrees take the remainder of the probability mass. However, despite their high probability, they are only a few and are covered by the proposed bound. It remains to bound the number of high degree occurrences to \( O(n^{1/\gamma}) \): Define boolean indicator variables \( Y_i \) with \( Y_i = 1 \) iff \( X_i > n^{1/\gamma} \). Observe that they behave like a Bernoulli trail \( Y_i \in B(p) \) with \( p = O(n^{1/\gamma-1}) \). Hence the expected number of high degrees is \( E[\sum_{i=1}^{n} Y_i] = \sum_{i=1}^{n} \mathbb{P}[X_i > n^{1/\gamma}] = O(n^{1/\gamma}) \). Chernoff’s inequality gives an exponentially decreasing bound on the tail distribution of the former sum which thus holds WHP.

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**Figure 1:** Materialization of the degree sequence \( \pi_k = (k, k, k-1, k-1, \ldots, 2, 2, 1, 1) \) with \( D_\pi = k = \Theta(n) \) which asymptotically maximizes \( \text{HP-HH} \)’s memory consumption.
collapses consecutive nodes with equal degrees into a group. The degree sequence is then represented by a list \( L \) of nodes in it and \( s \) the smallest node index of the consecutive sequence. The algorithm is built around the following invariants which hold at the beginning (and end) of every iteration and are initially satisfied due to the ordered input:

1. For each iteration \( i \), the number of degrees \( \pi_i \) of every iteration and are initially satisfied due to the ordered input.
2. The algorithm’s memory consumption is dominated by \( O(m + D) \) where the number of edges is given by \( m = \sum_{i=1}^{D} d_i \). Since \( n \) and \( D \) are bound by \( O(m) \), HP-HH is output-sensitive with a total runtime of \( O(m) \). The algorithm’s memory consumption is dominated by \( L \) which is guaranteed to be of size \( O(D) \) due to (I1). From Lemma 2 it follows that \( D \) increases by at most a constant. Therefore HP-HH has a total space complexity of \( O(D) \).

Since the vast majority of nodes have low degrees, it also follows that a sufficiently large random powerlaw degree sequence \( \pi \) contains at most very few nodes which cannot be materialized as requested. Therefore, HP-HH assumes by default any input to be graphical and greedily reduces single degrees if they cannot be met.

**Data structure.** Instead of maintaining the degree of every node individually, HP-HH collapses consecutive nodes with equal degrees into a group. The degree sequence is then represented by a list \( L = [(d_i, n_i, s_i)] \), where \( d_i \) denotes the group’s degree, \( n_i \) the number of nodes in it and \( s_i \) the smallest node index of the consecutive sequence. The algorithm is built around the following invariants which hold at the beginning (and end) of every iteration and are initially satisfied due to the ordered input \( \pi \):

1. There are \( D \) groups
2. \( d_i > d_{i+1} \ \forall 1 < i < D \)
3. \( s_i + n_i = s_{i+1} \ \forall 1 < i < D \).

**Algorithm.** HP-HH works in \( n \) iterations. Each time it extracts \( \pi \)’s first entry which has a largest remaining degree \( d \) (due to (I2)) and the smallest available vertex id \( u \) (because of (I3)). The extraction is achieved by incrementing the first group’s lowest id and decreasing its size. If the group becomes empty, it is removed from \( L \).

In the next step, the current node \( u \) is connected to its \( d \) requested neighbors which are identified using group-wise operations on \( L \): HP-HH scans through \( L \) starting at its head group and outputs edges to all vertices contained. This continues as long as a group’s size does not exceed the number \( d' \) of yet unsatisfied neighbors. If a group’s size \( b \) exceeds the number \( d' \) of remaining neighbors requested, it is split into two blocks of sizes \( (b-d') \) and \( d' \). Then, the second group can be consumed as described above. Since splitting implies that all neighbor requests get satisfied, it immediately stops the scan.

**Correctness.** Invariant (I3) is satisfied after every iteration since the algorithm never changes the relative order of groups, performs splitting in a compatible way and only deletes groups at \( L \)’s beginning and end. Since the first segment of a split remains untouched, it may have the same degree as its predecessor whose degree was decreased. This violation of (I2) is solved by merging the two adjacent groups, which is guaranteed to be legal due to (I3). The same check has to be applied to the last group consumed and its successor. Only these two corner cases may require merging, because the degrees of the all earlier blocks is evenly reduced which keeps their relative differences constant. During the execution, (I1) is a direct consequence of (I2) and (I3).

**Complexity.** HP-HH performs constant work for every group created, node processed and edge generated. Hence, its time complexity is \( O(n + m + D) \) where the number \( m \) of edges is given by \( m = \sum_{i=1}^{D} d_i \). Since \( n \) and \( D \) are bound by \( O(m) \), HP-HH is output-sensitive with a total runtime of \( O(m) \). The algorithm’s memory consumption is dominated by \( L \) which is guaranteed to be of size \( O(D) \) due to (I1). From Lemma 2 it follows that \( D \) increases by at most a constant. Therefore HP-HH has a total space complexity of \( O(D) \).
Lemma 2. Let \( L = [(d_i, n_i, s_i)]_{1 \leq i \leq D_\pi} \) be the group sequence obtained from the degree sequence \( \pi \). Further denote the state after the consumption of the first group, i.e. after \( n_1 \) iterations, as \( \pi' \). Then \( D_\pi' \leq D_\pi \).

Proof sketch. Since the processing of the first group will eventually delete it, it suffices to show that the number of remaining groups increases by at most 1. If no split takes place during the processing of the first group, the claim is trivially satisfied. Otherwise, let \( i \) be the index of the group that is split with \( d \) neighbor requests left for node \( u \). In case this is the only split, the claim directly follows and we can stop. Consider the next split which occurred for vertex \( v \) and observe that \( u < v \). Then one of two cases arises:

- Between the two splits, merging took place: In this case, the number of groups is at most at the value it was before the first split and the second one can be performed.
- No merging happened between the two splits: Observe that since \( u \) and \( v \) are contained in the same group, their initial degrees \( d_u = d_v \) were equal. But because \( u \) was processed before \( v \), the second node already received \((v-u)\) more neighbors before its iteration. Therefore, both vertices reached group \( i \) with \( d \) unsatisfied neighbor requests and the second split has to take place in one of the two groups generated with the first split. Thus after the second split, there are a total of three fragments which originate from the same block and had the same initial degree. As a result two of them have the same degree and are merged which guarantees that the number of groups does not increase further.

I/O complexity. In case of scale-free networks, the memory complexity is \( \mathcal{O}(n^{1/\gamma}) \) with high probability. As a result, \( L \) fits into main memory for all realistic values of \( M \) and \( n \). For virtually all practical purposes the algorithm is I/O-free. However, HP-HH can be directly used with an ordinary EM list because of the strictly sequential accesses as well as the well-behaved insertion- and deletion patterns discussed above. In this setting, the algorithm triggers \( \mathcal{O}(\text{scan}(D_\pi)) \) I/Os per iteration \(^{20}\) yielding a total I/O complexity of \( \mathcal{O}(n \text{ scan}(D_\pi)) \).

4.1 Improving HP-HH’s I/O-complexity for Unordered Outputs

Omitting the lexicographical order of the output, the following modification improves the bound to \( \mathcal{O}(D_\pi/M \text{ scan}(D_\pi)) \): Observe that after HP-HH processed group \( j \), groups \( 1, \ldots, j-1 \) remain untouched until the next iteration and \( j \) is only revisited in case group \( j+1 \) is split. This allows us to delay the execution in a given iteration and start a partial processing of the next vertex: Instead of scanning through \( L \) for every node individually, we introduce super-phases. The modified HP-HH moves the first \( \Theta(M) \) groups of \( L \) into a hot-buffer in internal memory and deletes them in external memory. Then, edges between vertices currently in the hot-buffer are generated based on the lazy processing scheme. Subsequently, all remaining requests from the hot-buffer are answered in an collective scan through \( L \).

5 I/O-efficient Edge Switching

ES-TFP is a central building block of our pipeline and used to randomize and rewire existing graphs. It applies a sequence of edge swaps \( S = [\sigma_s]_{1 \leq s \leq k} \) to a simple graph \( G = (V, E) \). The graph is represented by a lexicographically ordered edge list \( E_L = [e_i]_{1 \leq i \leq m} \) which stores for every ordered edge \([u, v] \in E \) only the pair \((u, v)\) and omits \((v, u)\). As illustrated in Fig. 4 a swap \( \sigma(a, b, d) \) is encoded by a direction bit \( d \) and the edge ids \( a \) and \( b \) (i.e. the position in \( E_L \)) of the edges that are supposed to be swapped.
Figure 3: A swap consists of two edge ids and a direction flag. The edge ids describe an induced subgraph (left) while the direction flag indicates how the incident nodes are shuffled.

\[ \sigma(a, b) = \begin{cases} \{(\alpha_1, \beta_1), (\alpha_2, \beta_2)\} & \text{if } d = \text{false} \\ \{(\alpha_1, \beta_1), (\alpha_2, \beta_2)\} & \text{if } d = \text{true} \end{cases} \]

where \( \alpha_1, \alpha_2 = e_a \) and \( \beta_1, \beta_2 = e_b \) are the edges at positions \( a \) and \( b \) in the edge list \( E_L \).

Figure 4: Data flow during an ES-TFP run. Communication between phases is implemented via an EM sorter, self-loops use a PQ based TFP technique. Brackets within a phase represent the type of elements iterated over. If multiple input streams are used, they are joined with this key.

We denote the switched edges as \( e_a^\sigma \) and \( e_b^\sigma \) which are given by \( (e_a^\sigma, e_b^\sigma) \equiv \sigma(a, b, d) \) as defined in Fig. 3. We assume that the swap’s constituents are drawn independently and uniformly at random. Thus, the sequence may contain illegal swaps that would introduce multi-edges or self-loops. Such illegal swaps must not be execute but are simply skipped. Therefore, to perform \( \sigma(a, b, d) \) the following tasks have to be addressed: (i) gather the nodes incident to edges \( e_a \) and \( e_b \), (ii) compute \( e_a^\sigma \) and \( e_b^\sigma \) and skip if a self-loop arises, (iii) verify that the graph remains simple, i.e. skip if \( e_a^\sigma \in E_L \) or \( e_b^\sigma \in E_L \), (iv) update the graph representation \( E_L \).

5.1 ES-TFP for Independent Swaps

As illustrated in Fig. 4 ES-TFP conceptually consists of six phases which communicate via sorters\(^1\). As a simplification, we assume for the moment that there exist no two swaps that share any source id or target edge. Thus, we currently only consider the communication shown at the top of the figure:

- **Request nodes and load nodes** — The goal of these two phases is to load every referenced edge: We iterate over the input sequence of swaps \( S \) and push for the \( s \)-th swap \( \sigma(a, b, d) \) two messages \( \text{edge}_\text{req}(a, s, 0) \) and \( \text{edge}_\text{req}(b, s, 1) \) into a sorter \( \text{EdgeReq} \). A message’s third entry encodes whether the request is for the first or second edge of a swap, which becomes relevant when we allow dependencies. ES-TFP then scans in parallel through the edge list \( E_L \) and the requests \( \text{EdgeReq} \) which are now sorted by edge ids. If there is a request \( \text{edge}_\text{req}(i, s, p) \) for an edge \( e_i = [u, v] \), the edge’s node pair is sent to the requesting swap by pushing message \( \text{edge}_\text{msg}(s, p, (u, v)) \) into sorter \( \text{EdgeMsg} \).

\(^1\) The term sorter implies the following access pattern: First all elements are generated in an arbitrary order and written into EM (if necessary). The sorted sequence can be exclusively accessed by scanning and only after the last element has been pushed. Rewinding to the beginning of the stream is allowed. The elements are sorted lexicographically in an ascending fashion.
In contrast to our previous assumption, multiple swaps may share source ids or target edges. In this case, ES-TFP scans through the sequence of swaps S and EdgeMsg in parallel. For the s-th swap \(\sigma(a, b, d)\), there are exactly two messages \(\text{edge}_{\text{msg}}(s, 0, e_a)\) and \(\text{edge}_{\text{msg}}(s, 1, e_b)\) in EdgeMsg. This information suffices to compute the switched edges \(e^s_a\) and \(e^s_b\) under the assumption that the swap is performed.

To avoid parallel edges, it remains to test whether the switched edges do already exist. This is done by pushing existence requests \(\text{exist}_{\text{req}}(e^s_a, s)\) and \(\text{exist}_{\text{req}}(e^s_b, s)\) in the sorter \(\text{ExistReq}\). Afterwards, a parallel scan through the edge list \(E_L\) and \(\text{ExistReq}\) is performed to answer the existence requests. Only if an edge \(e\) requested by swap id \(s\) is found, the message \(\text{exist}_{\text{msg}}(s, e)\) is pushed into the sorter \(\text{ExistMsg}\). Both phases hence incur a total of \(O(\text{sort}(k) + \text{scan}(m))\) I/Os.

- **Simulate swaps and load existence** — The two phases gather all information required to decided whether a swap is legal: ES-TFP scans through the sequence of swaps \(S\) and EdgeMsg in parallel. For the \(s\)-th swap \(\sigma(a, b, d)\), there are exactly two messages \(\text{edge}_{\text{msg}}(s, 0, e_a)\) and \(\text{edge}_{\text{msg}}(s, 1, e_b)\) in EdgeMsg. This information suffices to compute the switched edges \(e^s_a\) and \(e^s_b\) under the assumption that the swap is performed.

To avoid parallel edges, it remains to test whether the switched edges do already exist. This is done by pushing existence requests \(\text{exist}_{\text{req}}(e^s_a, s)\) and \(\text{exist}_{\text{req}}(e^s_b, s)\) in the sorter \(\text{ExistReq}\). Afterwards, a parallel scan through the edge list \(E_L\) and \(\text{ExistReq}\) is performed to answer the existence requests. Only if an edge \(e\) requested by swap id \(s\) is found, the message \(\text{exist}_{\text{msg}}(s, e)\) is pushed into the sorter \(\text{ExistMsg}\). Both phases hence incur a total of \(O(\text{sort}(k) + \text{scan}(m))\) I/Os.

- **Perform swaps**: We rewind the EdgeMsg sorter and scan through the sequence of swaps \(S\) and the sorters EdgeMsg and ExistMsg in parallel. As described in the simulation phase, ES-TFP computes the switched edges \(e^s_a\) and \(e^s_b\) from the original state \(e_a\) and \(e_b\). The swap is marked illegal if a switched edge is a self-loop or if an existence info is received via \(\text{ExistMsg}\). If \(\sigma\) is legal we push the switched edges \(e^s_a\) and \(e^s_b\) into the sorter \(\text{EdgeUpdates}\), otherwise we push the source edges \(e_a\) and \(e_b\). This phase requires \(O(\text{sort}(k))\) I/Os.

- **Update edge list**: As the original edge list \(E_L\) and the updated edges \(\text{EdgeUpdates}\) are sorted, merging them is cheap. During this process we skip all edges in \(E_L\) that are flagged invalid in the bit stream \(\text{InvalidEdge}\). This phase triggers \(O(\text{scan}(m))\) I/Os.

### 5.2 Dependencies Between Swaps

In contrast to our previous assumption, multiple swaps may share source ids or target edges. In this case, ES-TFP resolves such conflicts by producing the same result as if the sequence were evaluated swap by swap. The following modifications ensure correct operation for all possible dependencies:

- Consider the case that swap \(\sigma_s(a, b, d)\) changes the state of edges \(e_a\) and \(e_b\) to \(e'^s_a\) and \(e'^s_b\), respectively. Later, a swap \(\sigma_2\) inquires about the existence of either of the four edges which has obviously changed compared to the initial state.

  We extend the simulation phase in order to track these modifications: Here, we not only push messages \(\text{exist}_{\text{req}}(e'^s_a, s_1)\) and \(\text{exist}_{\text{req}}(e'^s_b, s_1)\) into sorter \(\text{ExistReq}\), but also report that the original edges may change. This is achieved using messages \(\text{exist}_{\text{req}}(e_a, s_1, \text{may\_change})\) and \(\text{exist}_{\text{req}}(e_b, s_1, \text{may\_change})\) pushed into the same sorter. If there are dependencies, multiple messages are received for the same edge \(e\) during the load existence phase. In this case, only the request of the first swap involved is answered as before. Also every swap \(\sigma_s\) is informed about its direct successor \(\sigma_{s_2}\) (if any) by pushing the message \(\text{exist}_{\text{succ}}(s_1, e, s_2)\) into the sorter \(\text{ExistSucc}\). This yields a linear dependency chain.

  During the perform swaps phase, ES-TFP performs the same steps as described earlier. The swap may receive a successor for every edge it sent an existence request
to and it informs each successor about the state of the appropriate edge after the swap is processed. This communication uses time forward processing on exist msg messages. None of the modifications above changes the ES-TFP’s asymptotic I/O complexity.

As an optimization, the longest tail of dependency chain that consists only of may_change requests is deleted during the load existence phase.

- Two swaps share an edge id. Consider two swaps \( \sigma_s(a_1, b_1, d_1) \) and \( \sigma_s(a_2, b_2, d_2) \) where \( s_2 > s_1 \) is the smallest index with \( \{a_1, b_1\} \cap \{a_2, b_2\} \neq \emptyset \). This issue is detected during the load nodes phase when at least messages edge_req\((x, s_1, p_1)\) and edge_reg\((x, s_2, p_2)\) arrive for the same edge id \( x \). In this case, we answer only the request of \( s_1 \) and build a dependency chain as described before using messages id_succ\((s_1, p_1, s_2, p_2)\) that are pushed into the sorter IdSucc.

During the simulation phase, ES-TFP cannot decide whether a swap is legal. Therefore, if \( \sigma_s \) receives a successor notification id_succ\((s_1, p_1, s_2, p_2)\) it sends the original edge state \( e \) as well as the updated edge \( e^{\sigma_1} \) to the \( p_2 \)-th slot of \( \sigma_{s_2} \) using a PQ. If a swap receives multiple edge states per slot, it simulates the swap for every possible combination in the ordinary fashion.

During the perform swaps phase, ES-TFP operates as described in the independent case: it computes the swapped edges and determines whether the swap has to be skipped. If a successor exists, the new state is not pushed into the EdgeUpdates sorter but forwarded to the successor in a TFP fashion. This way, every invalidated edge id receives exactly one update in EdgeUpdates and the merging remains correct.

Due to the second modification, ES-TFP’s complexity increases with the number of swaps that target the same edge id: Consider two edge indices \( a \) and \( b \) which are targeted by two swaps \( \sigma_1 \) and \( \sigma_2 \) not sharing any nodes. Later a third swap \( \sigma_3 = (a, b, d_3) \) is supposed to switch \( a \) and \( b \). Since it is not known during the simulation phase whether \( \sigma_1 \) and \( \sigma_2 \) are legal, \( \sigma_3 \) receives two states from each and computes swaps over all four combinations. Recursion yields an unlikely worst case with a state space of size \( \Theta(2^l) \) where \( l \) is the length of the dependency chain. However, it requires \( \Theta(2^l) \) independent swaps. Therefore the state size grows at most linearly in the number of swaps involved.

Let \( X_i \) be a random variable expressing the number of swaps edge id \( i \) is referenced by. Since every swap constitutes two independent Bernoulli trials towards id \( i \), \( X_i \) is binomially distributed with \( p = 1/m \), yielding an expected chain length of \( 2k/m \). Also for \( k = m/2 \) swaps, \( \max_{1 \leq i \leq s} X_i = O(ln(m)/lnln(m)) \) holds with high probability [13].

In order to keep the dependency chains short, ES-TFP splits the sequence of swaps \( S \) into \( R \approx \alpha \cdot k/m \) runs of equal size where \( \alpha \) is a small constant integer (refer to chapter 9.2 for details). For every run, the algorithm executes the six phases as described before. Each time the graph is updated, the mapping between an edge and its id may change. The switching probabilities, however, remain unaltered due to the initial assumption of uniformly distributed swaps.

### 6 Community Assignment

Consider a sequences of community sizes \( S = [s_s]_{1 \leq s \leq C} \) with \( n = \sum_{s=1}^{C} s_s \) and a sequence of intra-community degrees \( D = [d_i^m]_{1 \leq i \leq n} \). Let \( S \) and \( D \) be non-increasing and non-negative. For simplicity assume that the last community \( s_C = 0 \) is empty. The task is to find a random surjective assignment \( \chi: V \rightarrow [C] \), s.t. every community \( s \) is filled \( s_s = |\{v|v \in V \land \chi(v) = s\}| \) as requested and every node \( v \in V \) becomes member of a sufficiently large community \( \chi(v) \) with \( s_\chi(v) > d_v^m \). Define \( p_x = \min\{s|s \in [C], s_x < d_v^m\} \)}
as the community with smallest index that cannot host \( v \). Due to \( S \)'s monotonicity, node \( v \) can become member of exactly all communities \( s < p_v \) which we therefore call legal.

**Algorithm.** Maintain an array \( L[1:D] \) in IM and initialize all cells \( s \) with \( L[s] = (s, s_s) \). \( L \) stores the number of unallocated memberships for every community and will be updated in every iteration. Let \( l = 1 \) be the lowest legal index of \( L \), \( x = 0 \) be a counter of empty communities and define \( p_0 = 1 \). Scan through the node degree sequence \( D \). For every \( v \) do:

- Compute \( p_v \) by scanning through \( S[p_{v-1} \ldots C] \). Pick a random legal index \( l \leq s < p_v \) and read \((s, l_s) \leftarrow L[s] \). Repeat in case the picked entry is empty, i.e. \( l_s = 0 \). Otherwise, assign node \( v \) to community \( s \).
- Decrement the number \( l_s \) of nodes that can be assigned to \( s \) by one. In case the community \( s \) is now empty increment the counter \( x \) by one. If \((p_v - l)/2 < x \) then half of the legal communities are empty and we compact \( L[l : p_v] \): In a single scan, we move all non-empty communities towards index \( p_v \) and maintain their relative order. Subsequently, we update the lowest index \( l \leftarrow l + x \) and set \( x \leftarrow 0 \).

**Complexity.** For every assignment, the algorithm computes the subset of legal communities represented by \( p_v \). As a result of \( D \)'s order, the sequence of all splitters \( P = [p_1]_{v \leq n} \) is non-decreasing and thus can be computed in time \( O(n + C) \). Since at least half of the active communities are non-empty, the expected number of random picks per element is \( O(1) \). While there can be \( O(\log n) \) compaction phases, the total work is bound by \( O(n) \) due to decreasing problem sizes. As a result, the algorithm requires \( O(n + C) \) time and performs \( O(\text{scan}(n + C)) \) I/Os. The IM consumption of \( O(C + B) \) is dominated by the array \( L \) and two blocks of size \( B \) required for scanning.

The memory consumption can be reduced to \( O(D_\pi + B) \) by increasing the I/O complexity to \( O(\text{sort}(n)) \). Observe that for two nodes \( u \) and \( v \) we have \( p_u \neq p_v \Rightarrow d^u_\pi \neq d^v_\pi \) and thus the number of unique splitter values \( |\{p_v \mid v \in V\}| = O(D_\pi) \) is bounded by \( D_\pi \) where \( \pi \) the graph's degree sequence. Exploiting that fact, the memory consumption can be reduced to \( O(D_\pi + B) \) by increasing the I/O complexity to \( O(\text{sort}(n)) \): The algorithm then executes two phases where the first one is similar the one above: however, we start with an empty \( L \). Every time we compute a new large splitter value \( p_v \neq p_{v-1} \), we group all communities that just became legal together summing by their sizes and represent them by a single bin in \( L \). During this phase, every node \( v \) is assigned to a bin \( b_v \) by pushing a tuple \((b_v, r_v, v) \) into a sorter \( T \) where \( r_v \) is a random value from a sufficiently large interval. We then scan through the sorter \( T \) and the community sizes \( D \) in parallel and assign all received in \( T \) to the current community and advance once it is full.

### 7 Edge Rewiring

Edges in the global graph between nodes of the same community need to be rewired through edge swaps in order to not to violate the chosen mixing parameter \( \mu \). We perform edge swaps with a random partner as – if sufficiently many communities exist – it is unlikely that the nodes that are incident to the resulting edges also share a community.

From the community assignment step we obtain \( \chi \) as an ordered sequence \((u, s)\)-pairs that contains for each node \( u \) its community \( s \). Based on this, we annotate every edge with the communities of both incident vertices by twice scanning through the edge list – once sorted by source nodes, once by target entries. For each forbidden edge, a swap is generated by drawing a random partner edge id and swap direction. Subsequently, all swaps are executed in a single run of ES-TP. As a result, we can get the set of edges that were involved in the edge swaps. We repeat the same algorithm for determining the forbidden edges using these involved edges instead of the whole edge list as input since
all other edges are still allowed. If only very few forbidden edges remain, the algorithm should be aborted as having another round of swaps would be too expensive compared to the small error that is introduced by deleting the remaining forbidden edges.

**Complexity.** For each round, at most $O(m \log m)$ work and $O(sort(m))$ I/Os are needed for selecting the swaps. The number of swaps is at most $m$ but much smaller in practice. The number of rounds is usually small but depends highly on the community size distribution — the smaller the communities, the less likely are edges inside them.

### 8 Implementation

We implemented HP-HH and ES-TFP in C++ using the STXXL library which provides implementations of EM data structures, a parallel EM sorter, and an EM priority queue. Amongst others, we applied the following optimizations for ES-TFP:

- Most message types contain both a swap id and a flag selecting which one of the swap’s edges is targeted. We encoded both of them in a single integer by using all but the least significant bit (LSB) for the swap id and store the flag in the LSB. This significantly reduces the memory volume and yields a simpler comparison operator since the standard integer comparison already ensures the correct lexicographic ordering.

- Instead of storing and reading the sequence of swaps several times, we exploit the implementation’s pipeline structure and directly issue edge id requests for every swap arriving. Since this is the only time edge ids are read from a swap, only the remaining direction flag is stored in an efficient EM vector that uses one bit per flag and supports I/O-efficient writing and reading. This can be overlapped with an ongoing ES-TFP run.

- Instead of storing each edge in the sorted external edge list as a pair of nodes, we only store each source node once and then a sequence of target nodes. This still supports sequential scan and merge operations which are the only operations we need. This almost halves the I/O volume of scanning or updating the edge list.

- During the execution of several runs we can delay the updating of the edge list and combine it with the loading of the edges in the second phase of the next run. This reduces the number of scans per additional run from three to two.

- We use asynchronous stream adapters for tasks such as streaming from sorters or the generation of random numbers. These adapters run in parallel to preprocess and buffer portions of the stream in advance and hand them over to the main thread.

### 9 Experimental Results

All runtime benchmarks were conducted on the following system: Intel Xeon CPU E5-2630 v3 (8 cores, 16 threads, 2.40GHz), 64 GB 2133 MHz RAM, 6× Samsung 850 PRO SSD (1 TB) of which 3 are used, Linux 4.3.0, GCC 5.3.1, STXXL master branch (11 Dec 2015).

The number of repetitions per data point (with different random seeds) is denoted with $S$. Errorbars always correspond to the unbiased estimation of the standard deviation. If not stated differently, all experiments are based on the powerlaw degree distribution $P_{PL}(2)$ which yields an average degree of 100 and gives parameters observable in real social networks [23,24].
In chapter 4, we bound the internal memory consumption of HP-HH by showing that a sequence of \( n \) numbers randomly sampled from \( \text{PLD}(\gamma) \) contains only \( O\left(\frac{n}{\gamma}\right) \) unique values with high probability.

In order to support Lemma 1 and to estimate the hidden constants, samples of varying size between \( 10^3 \) and \( 10^8 \) are taken from distributions with exponents \( \gamma \in \{1, 2, 3\} \). For each the number of unique elements is computed and averaged over \( S \geq 4 \) runs with identical configurations but different random seeds. The results illustrate in Fig. 5 support the predictions with small constants. For the commonly used exponent 2, we find \( 1.38\sqrt{n} \) unique elements in a sequence of length \( n \).

### 9.1 HP-HH State Size

In chapter 4 we bound the internal memory consumption of HP-HH by showing that a sequence of \( n \) numbers randomly sampled from \( \text{PLD}(\gamma) \) contains only \( O\left(\frac{n}{\gamma}\right) \) unique values with high probability.

In order to support Lemma 1 and to estimate the hidden constants, samples of varying size between \( 10^3 \) and \( 10^8 \) are taken from distributions with exponents \( \gamma \in \{1, 2, 3\} \). For each the number of unique elements is computed and averaged over \( S \geq 4 \) runs with identical configurations but different random seeds. The results illustrate in Fig. 5 support the predictions with small constants. For the commonly used exponent 2, we find \( 1.38\sqrt{n} \) unique elements in a sequence of length \( n \).

### 9.2 Inter-Swap Dependencies

If multiple swaps target the same edge, ES-TFP simulates all possible states to be able to retrieve possibly conflicting edges. In section 5.2 we argued that the number of dependencies (and thus the state size) remains manageable if the sequence of swaps is split into sufficiently short runs. We found that for \( m \) edges and \( k \) swaps \( 8k/m \) runs minimize the run time for large instances. As indicated in Fig. 5 in this setting, 78.7\% of swaps do not receive additional edge configurations during the simulation phase and less than 0.4\% have to consider more than four additional states. Similarly 78.6\% of existence requests remain without dependencies.
9.3 Performance

We study the runtime of HP-HH and ES-TFP for different graph sizes with up to \(1.55 \times 10^{10}\) edges. Our implementation of HP-HH produces \(180 \pm 5\) million edges per second over a wide range of settings including all shown in Fig. 6. Here, we subsume the computation of the input degree sequence, the conversion into the HP-HH’s compact representation as well as the writing of the output to EM.

Figure 6 presents the runtime of ES-TFP as a function of input size. For ES-TFP, \(R=8\) runs and the same number of swaps \(k\) and edges \(m\) are used. Since increasing the number of swaps has a linear time dependency and does not require additional resources, extrapolating our results by scaling to a common numbers of swaps ranging between \(10m\) and \(20m\) is possible \([17, 25]\). By varying the number of SSDs between 1 to 6 for \(m = 2.5 \times 10^9\) edges, we found a minimal runtime for three drives with a speed-up of 1.16 times compared to the slowest setting with a single SSD.

For reference, Fig. 6 includes the behavior of the existing internal memory edge swap algorithm ES-VL based on the authors’ implementation \([25]\). Here we report only on the edge swapping process excluding any precomputation and I/Os. To achieve comparability, we removed connectivity tests, fixed a memory management issue and adopted the number of swaps to match previously described configurations. ES-TFP is faster than ES-VL for graph instances with \(m \geq 1.5 \times 10^8\) edges which still fit into main memory. We were not able to apply ES-VL to instance with \(m \geq 5 \times 10^8\) edges due to failing memory allocation.

Bhuiyan et al. propose a distributed edge switching algorithm and evaluate it on a 64 node cluster each equipped with two Intel Xeon E5-2670 2.60GHz 8-core processors and 64GB RAM \([3]\). The authors report to perform \(k = 1.15 \times 10^{11}\) swaps on a graph with \(m = 10^{10}\) generated in a preferential attachment process in less than 3 h. In Fig. 6 we include graphs 1.5 times larger. We generated a preferential attachment graph with \(15\) matching the aforementioned graph size and carried out \(10^{10}\) swaps with ES-TFP. Scaling to \(k = 1.15 \times 10^{11}\) swaps yields a slow down of 13.8 on a machine with 1/128 the number of comparable cores and 1/64 of internal memory.

10 Outlook and Conclusion

We propose the first I/O-efficient graph generator for the LFR benchmark focusing on FDSM, which is the most challenging step involved: HP-HH materializes a graph based on a prescribed powerlaw degree distribution with high probability without I/O for virtually all realistic parameters. Including the generation of the degree sequence and the writing of the output to disk, our implementation generates \(1.8 \times 10^8\) edges per second for graphs exceeding main memory. ES-TFP perturbs existing graphs with \(m\) edges based on \(k\) edge switches using \(O(k/m \cdot \text{sort}(m))\) I/Os for \(k = \Omega(m)\). We demonstrate that ES-TFP is faster than the internal memory implementation \([25]\) for large instances still fitting in main memory and scales well beyond the limited main memory. Compared to the distributed approach by \([3]\) on a cluster with 128 CPUs, ES-TFP exhibits a slow down of only 13.8 and hence poses a viable and cost-efficient alternative.

Currently, ES-TFP does not fully exploit the parallelism offered by modern machines, however, preliminary experiments indicate that an extension is possible: All steps described in section 5.1 can easily be parallelized. Further, in the spirit of \([10]\), a run can be split into smaller batches which then can be processed nearly independently.

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A Sampling an Ordered Uniformly Sequence

The materialization of a graph as introduced in section 4 requires as input an ordered degree sequence $\pi$. The degrees $\pi$ are commonly obtained using an inverse sampling technique which in turn requires an ordered sequence of $n$ numbers drawn uniformly at random. The generation of such a sequence in the EM model without I/Os was considered for discrete values by [19,26]. Here we discuss the sampling from a continuous distribution.

Consider $m \geq 1$ random variables $X_1, \ldots, X_m$ drawn uniformly at random from $[a, 1]$ where $0 \leq a < 1$ and let the smallest amongst them be $X^* = \min\{X_1, \ldots, X_m\}$. Define $F_{a,m}[k] \equiv \mathbb{P}[X^* \leq k]$ for $a \leq k \leq 1$. Observe that this cumulative probability is governed by

$$F_{a,m}[k] = 1 - \mathbb{P}[X^* > k] = 1 - \prod_{i=1}^{m} \mathbb{P}[X_i > k] = 1 - \left(\frac{k-a}{1-a}\right)^m.$$

Since $F_{a,m}[k]$ is injective its inverse exists and is given by $F^{-1}_{a,m}(u) = (1-u^{1/m})(1-a) + a$. Hence, the inverse transform technique is applicable and we can sample $X^* = F^{-1}_{a,m}(u)$ based on $u$ drawn uniformly at random from $[0; 1]$.

Lemma 3. Given a stream $y_1, \ldots, y_n$ of i.i.d. samples drawn uniformly at random over $[0; 1]$, a monotonous sequence $x_1 \leq x_2 \leq \ldots \leq x_n$ uniformly distributed over $[0; 1]$ can be constructed in an online fashion with a constant state size $\mathfrak{f}.$

Proof. For simplicity define $x_0 = 0$. Consider the case that elements $x_1, \ldots, x_i$ have been generated by consuming $y_1, \ldots, y_i$. Then, the next element $x_{i+1}$ is the smallest sample amongst the remaining $n - i$ items drawn uniformly at random from $[x_i; 1]$. It hence is governed by $F_{x_i,n-i}(x_{i+1})$ and can be sampled based on $y_{i+1}$. \qed

\footnote{Assuming every number is represented by a constant number of machine words}