On $SW$-minimal models 
and $N=1$ supersymmetric 
Quantum Toda-field theories

Steffen Mallwitz†

Abstract

Integrable $N=1$ supersymmetric Toda-field theories are determined by a contragredient simple Super-Lie-Algebra (SSLA) with purely fermionic lowering and raising operators. For the SSLA’s $Osp(3|2)$ and $D(2|1;\alpha)$ we construct explicitly the higher spin conserved currents and obtain free field representations of the super $W$-algebras $SW(3/2,2)$ and $SW(3/2,3/2,2)$. In constructing the corresponding series of minimal models using covariant vertex operators, we find a necessary restriction on the Cartan matrix of the SSLA, also for the general case. Within this framework, this restriction claims that there be a minimum of one non-vanishing element on the diagonal of the Cartan matrix. This condition is without parallel in bosonic conformal field theory. As a consequence only two series of SSLA’s yield minimal models, namely $Osp(2n|2n-1)$ and $Osp(2n|2n+1)$. Subsequently some general aspects of degenerate representations of $SW$-algebras, notably the fusion rules, are investigated. As an application we discuss minimal models of $SW(3/2,2)$, which were constructed with independent methods, in this framework. Covariant formulation is used throughout this paper.

Address: BONN-TH-94-04 
Physikalisches Institut, Nußallee 12 
53115 Bonn 
Germany 
e-mail: steffen@avzw01.physik.uni-bonn.de

†

Present address: 
Max-Planck-Institut für Physik komplexer Systeme, Bayreuther Str.40, 01187 Dresden
1. Introduction

The analysis of conformal quantum field theories in two dimensions [1] by means of free field representations of the corresponding symmetry algebras is considered to be a standard procedure today. It gives insight into fusion rules and degenerate representations as well as into minimal models. In this way most known results concerning the Virasoro and Super-Virasoro algebra can easily be reproduced [2-6]. Extensions of the Virasoro algebra by additional simple fields are commonly called $W$-algebras, for a detailed review see [7].

For the case of the $A_n$-based $W_n$-algebras, formulas for the conformal dimensions of the primary fields contained in a rational model can be derived in essentially the same manner [8]. The generalization to arbitrary Casimir algebras can be found in [9]. In connection with classical Toda-field theories free field representations emerge as a very natural tool and yield in conjunction with the Miura transformation a means for the explicit construction of higher spin conserved currents. In the case of a quantum theory the Miura transformation is not at all obvious [10], though explicit calculations still enable one to write down the conserved quantities.

In this paper this is performed for the case of two $N=1$ supersymmetric Toda-field theories, namely for the ones based on $Osp(3|2)$ and $D(2|1;\alpha)$. We find by a simple general calculation strong evidence for a general rule indicating when one might expect not to encounter any minimal models: One simply has to check whether the Cartan matrix of the corresponding simple Super-Lie-Algebra (SSLA) has non-vanishing diagonal elements. If there is none, it is not possible to construct minimal models in the standard way. As a consequence we find that only two series of SSLA’s would be capable of generating minimal models.

In the case of $Osp(3|2)$ the Cartan matrix has one non-vanishing diagonal element and we demonstrate the applicability of our approach to all currently known minimal models of $SW(3/2, 2)$, which is the simplest non-trivial $SW$-algebra and therefore compares to the bosonic $W_3$-algebra. Subsequently we derive a set of rules for determining the fields involved in a minimal model. However, this set of rules is possibly not complete.

We would like to mention that many of the calculations that are presented below, could only be performed with the help of a specially implemented symbolic calculation package. This includes the computation of $N=1$ covariant operator product expansions as well as the reduction of normal ordered expressions to a standard form, confer also [11].

The organization of the paper is as follows: After having set the stage in section two, we briefly review supersymmetric Toda-field theories in section three, so that the results of the free field constructions can be given in section four. In section five the basic relations of $N=1$ representation theory are derived using the vertex operator approach. These are then used to study the case of $Osp(3|2)$ or $SW(3/2, 2)$ in some detail.

\[1\] This symbolic calculation package OPESUSY for Mathematica\textsuperscript{TM} is available via anonymous FTP from the host avzw02.physik.uni-bonn.de.
2. Notations, a short account on superspace

This section serves to fix our notation. For details we refer the reader to the general literature on the subject of supersymmetry as well as to refs. [4, 5] and references therein. We work on superspace $S$, the extension of the two-dimensional real vector space $\mathbb{R}^2$ with either euclidean or minkowskian metric by a two-dimensional Grassmann space $G^2$ spanned by two real Majorana spinors $\theta_\alpha$. Functions $\Phi(z, \theta) = \Phi(Z)$ defined on $S$ are called superfields. Expanding in $\theta$ they take the following form

$$\Phi(x, \theta) = \phi(x) + \bar{\theta} \psi(x) + \frac{1}{2} \bar{\theta} \theta \xi(x),$$

(2.1)

where $\bar{\theta}$ denotes the dirac conjugate of $\theta$. Taking $S$ to be euclidean we define $\theta = \theta_1 + \theta_2$, $z = x_1 + x_2$, $Z = (z, \theta)$, $D = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial z}$,

(2.2)

as well as the corresponding bared quantities, e.g. $\bar{\theta} = \theta_1 - \theta_2$ which bear a minus sign; and the super distance

$$Z_{ij}^n = \begin{cases} (z_1 - z_2 - \theta_1 \theta_2)^n, & \text{for } n \in \mathbb{Z}; \\ (\theta_1 - \theta_2)(z_1 - z_2 - \theta_1 \theta_2)^{(n-1)/2}, & \text{for } n \in \mathbb{Z} + \frac{1}{2}. \end{cases}$$

(2.3)

Now we comment briefly on superconformal quantum field theories in two dimensions [4, 5]. Their fields transform covariantly under the action of the infinitely generated conformal group and respect in addition $N=1$ supersymmetry. For a moment we consider chiral fields, i.e. they depend only on the unbared coordinates. The supersymmetric version of the energy momentum tensor is a superfield $T = \frac{1}{2}G + \theta \tilde{T}$ where the component fields are the “ordinary” bosonic energy momentum tensor $\tilde{T}$ and the so-called supercurrent $G$. The operator product expansion (OPE) of $T$ with itself gives the Super-Virasoro algebra

$$T(Z_1) \circ T(Z_2) = \frac{c/6}{Z_{12}^3} + \frac{3/2}{Z_{12}^{3/2}} T(Z_2) + \frac{1/2}{Z_{12}} T'(Z_2) + \frac{T''(Z_2)}{Z_{12}^{1/2}} + \text{reg.}.$$ 

(2.4)

Fields $\Phi_i$ with well-defined superconformal dimension are called primary fields, iff they satisfy

$$T(Z_1) \circ \Phi_i(Z_2) = \frac{\Delta_i}{Z_{12}^{3/2}} \Phi_i(Z_2) + \frac{1/2}{Z_{12}} \Phi'_i(Z_2) + \Phi''_i(Z_2) + \text{reg.}.$$ 

(2.5)

The primary fields are a subclass of the quasiprimary fields, which we do not define here [12]. Note that $T$ is a quasiprimary field. A finite set of simple, (quasi)primary superfields including $T$ closing in itself with the OPE as the binary associative operation is called a $SW$-algebra. “Closure” means that the fields on the right hand side of the OPE’s may be the fields itself, their Super-derivatives or normal ordered products thereof. Normal ordered products of superfields are given by the normal ordering prescription for the component fields. For more details on $SW$-algebras see e.g. refs. [7, 12, 13]. In section four we construct free field representations of certain $SW$-algebras from non-trivial conserved currents of supersymmetric Toda-field theories.
In this section we introduce Toda-field theories and some of their basic properties. For some relevant articles on this subject see ref. [15-25].

Let $K_{ij}$ be the Cartan matrix of a simple Super-Lie-Algebra $A$ with $r = \text{rank}(A)$ simple roots $\alpha_i$, fundamental weights $\mu_i$ and $\Phi$ a scalar superfield defined on $S$. In the basis of the simple roots $\alpha_i$ $\Phi$ has $r$ component fields $\Phi_i$. The symmetric bilinear form on weight space is denoted by $"\cdot"$. Then the theory is defined on $S$ by the action

$$S[\Phi] = \int d\bar{Z}dZ \left( \frac{1}{2} D\Phi \cdot D\Phi + \frac{1}{\beta^2} \sum_{i \in \mathcal{F}} \exp(\beta \alpha_i \cdot \Phi) + \frac{1}{\beta^2} \theta \theta \sum_{i \in \mathcal{B}} \exp(\beta \alpha_i \cdot \Phi) \right), \quad (3.1)$$

with $\beta$ being a coupling constant. The sum over the bosonic roots $\mathcal{B}$ spoils the supersymmetric covariance, whence we restrict ourselves to SSLA’s allowing for a system of fermionic simple roots $\mathcal{F}$ only. We find a classification of these in refs. [19, 28]. For completeness we list them here again: $sl(n + 1|n)$, $Osp(2n + 1|2n)$, $Osp(2n - 1|2n)$, $Osp(2n|2n - 2)$, $Osp(2n|2n)$ and $D(2|1; \alpha)$. In preserving supersymmetry we also find that the action is now invariant under superconformal transformations [19]. The equations of motion which follow from (3.1) are

$$\frac{\delta S}{\delta \Phi_j} = 0 \iff \bar{D}D\Phi = \exp(\sum_i \beta \alpha_i \cdot \Phi) \quad . \quad (3.2)$$

We will use them to construct the conserved higher spin fields which are then shown to form a $SW$-algebra. In the rather complex case of $D(2|1; \alpha)$ this is more straightforward than making a general ansatz for a field with given spin-$n$ and subsequently fixing the coefficients via equations obtained from the given OPE’s [13] of the $SW$-algebra.

In any decent quantum field theory the energy momentum tensor is a conserved quantity. As can be seen from a short computation, the quantity

$$T = -\frac{1}{2} \left( \sum_{i,j=1}^r K_{ij} N(D^2\Phi_i D\Phi_j) - \sum_{i=1}^r \left( \frac{1}{\beta} + \beta K_{ii} \right) D^3\Phi_i \right)$$

$$= -\frac{1}{2} N(D^2\Phi \cdot D\Phi) + \frac{1}{\beta} \rho_q D^3\Phi \quad , \quad (3.3)$$

$$\rho_q = \frac{1}{2} \sum_{i=1}^r \left( 1 + \beta^2 \alpha_i^2 \right) \mu_i \quad ,$$

is always conserved. ($\rho_q$ is the Weyl vector plus a “quantum correction”, $N$ denotes the normal ordered product.) This can be seen from the equations of motion eq.(3.2) and the
rules in appendix A. Now we address the question of how to obtain the conserved fields of higher spin, since that is what we are naturally interested in in order to obtain a free field representation of the corresponding $SW$-algebra. The existence of these fields is not trivial and related to the integrability of the theory. The conformal spin content of the conserved fields can be obtained from ref. [22].

For explicit calculations we introduce the spin one-half field $\Psi_i = D \Phi_i$, which serves as a “basis” for a spin-$n$ ansatz. After a “light-cone-like” quantization [21] on a quantization surface $\bar{Z} = const.$, the OPE of a “free” field is

$$
\Psi_i(Z_1) \circ \Psi_j(Z_2) = -\frac{K_{ij}^{-1}}{Z_{12}} + \text{reg.},
$$

where the constant $\bar{Z}$-component has been omitted.

A conserved current of spin higher than that of the energy momentum tensor is obtained by a general ansatz for a chiral spin-$n_2$ field of the following form

$$
W = \sum_{i_1,\ldots,i_r,p(n)} \alpha_{i_1,\ldots,i_r,p(n)} N(D^{n_1} \Phi_{i_1}, N(D^{n_2} \Phi_{i_1}, N(\ldots, D^{n_r} \Phi_{i_r})))
$$

where $p_r(n)$ denotes the monotonously ordered partitions of $n$ into $r$ integers. Note that eq.(3.5) is an expression in $\Psi_i$. The coefficients are partially fixed by the condition that the application of $\bar{D}$ annihilates it, i.e. it is conserved under the Toda dynamics. In this calculation one commutes the $\bar{D}$ with the $D$’s until the equation of motion eq.(3.2) is applicable. The derivatives of the exponentials are then evaluated and the resulting terms, which are in general normal ordered products, are rearranged to extract equations for the coefficients $\alpha_{p(n)}$. This process was performed with the help of the aforementioned symbolic calculation program. In the resulting expression no explicit $\bar{Z}$ dependence is retained.

By virtue of eq.(3.4) currents in terms of the $\Psi_i$ can be multiplied and may eventually give rise to a $SW$-algebra. Classically eq.(3.4) is converted to a Poisson bracket and the closure of the algebra of conserved currents to a classical $SW$-algebra is a direct consequence of the integrability of the Super Toda theory. This integrability on the other hand stems from the underlying SSLA-structure, see also refs. [15, 19]. Of course in the quantum case too, integrability ensures the closure to a quantum $SW$-algebra, but this is far less obvious and so far an unsolved question for general SSLA’s. Besides integrability we are mainly interested in the free field representation of the $SW$-algebra in order to investigate rational theories thereof.
4. Free field construction for $SW(3/2, 2)$ and $SW(3/2, 3/2, 2)$

Starting from the conserved currents of the Toda theory in question we construct the simple fields of the corresponding $SW$-algebra. The $SW(3/2, 2)$ was first constructed in [26]. It has been, together with the $SW(3/2, 3/2, 2)$, presented using covariant OPE’s for the first time in [13] and we rely heavily on this kind of representation. Finding a conserved current, we expect it to be an additional simple, primary field of a $SW$-algebra, though it is necessary to take its primary projection, first. Since all equations we used so far are linear, the normalizations of the fields are chosen in accordance with the non-linear OPE’s of the $SW$-algebra as given in [13].

$Osp(3|2)$

Our first example is $Osp(3|2)$-Toda-field theory. The Cartan matrix is given by

$$
\begin{pmatrix}
1 & -1 \\
-1 & 0
\end{pmatrix},
$$

so that the energy momentum tensor according to eq. (3.3) is

$$
T = -\frac{1}{2} (N(\Psi'_1, \Psi_1) - N(\Psi'_1, \Psi_2) - N(\Psi'_2, \Psi_1)) - \left(\frac{1}{\beta} + \beta\right) \Psi''_1 - \frac{1}{\beta} \Psi''_1,
$$

where “′” stands for a covariant derivative $D$. The associated Super-Virasoro algebra bears the central charge

$$
c = -(3 + \frac{9}{\beta^2}).
$$

Note that the sign of $c$ is quite formal at this point, since we have not determined the interesting range of $\beta$, yet. The most general ansatz for a chiral spin-2 field in two fields $\Psi_i$ is

$$
\tilde{W} = \sum_{i\leq j=1}^2 m_{ij} N(\Psi'_i, \Psi'_j) + \sum_{ij=1}^2 n_{ij} N(\Psi''_i, \Psi_j) + \sum_{ijk=1}^2 n_{ijk} N(\Psi'_i, N(\Psi_j, \Psi_k)) + \sum_{i=1}^2 m_{i} \Psi''''_i,
$$

where for the moment we keep the $\tilde{Z}$ dependance. Requiring it to be conserved under the Toda dynamics, i.e. $\tilde{D} \tilde{W} = 0$, yields

$$
n_{22} = \frac{N}{2\beta}, \quad n_{12} = (\frac{1}{\beta} + \beta) N, \quad m_2 = (\frac{1}{2} + \frac{1}{\beta^2}) N, \quad n_{112} = \frac{-N}{2}, \quad n_{212} = \frac{N}{2},
$$

with the normalization $N$ still being a free parameter. Since we want $\tilde{W}$ eventually to be the additional primary spin-2 field of $SW(3/2, 2)$, we introduce its primary projection

$$
W = \tilde{W} + a DT,
$$

$$
(4.6)
$$
where \( a \) evaluates to \( a = -2 \mathcal{N}(2 + \beta^2)/(3\beta) \). With the choice of \( \mathcal{N} \) as
\[
\mathcal{N} = \pm \frac{3\beta}{\sqrt{5(2 - \beta)(2 + \beta)(1 + 2\beta^2)}} \quad (4.7)
\]
the expressions for \( T \) and \( W \) are completely compatible with the following OPE, which is cited from [13], so that we obtained a free field representation of \( SW(3/2, 2) \).
\[
W(Z_1) \circ W(Z_2) = \frac{c}{Z_{12}^4} - \frac{\frac{6}{5} T(Z_2)}{Z_{12}^{5/2}} + \frac{C_1 W(Z_2) - \frac{2}{3} T'(Z_2)}{Z_{12}^2} + \frac{\frac{1}{2} C_1 W'(Z_2) - \frac{4}{5} T''(Z_2)}{Z_{12}^{3/2}} + \frac{\frac{1}{2} C_1 W''(Z_2) - \frac{1}{5} T'''(Z_2)}{Z_{12}} + \frac{C_2 N(T', T)(Z_2) + C_3 N(W, T)(Z_2) + \left( \frac{3}{10} C_1 - \frac{1}{5} C_3 \right) W''(Z_2) - \left( \frac{1}{3} C_2 + \frac{3}{10} \right) T'''(Z_2)}{Z_{12}^{1/2}} + \text{reg.},
\]
where the couplings are given by
\[
C_1^2 = \frac{4}{5} \frac{(5c + 6)^2}{(4c + 21)(c - 15)}, \quad C_2 = \frac{108}{5} \frac{1}{(4c + 21)}, \quad C_3 = \pm \frac{108}{\sqrt{5}} \frac{1}{\sqrt{(4c + 21)(c - 15)}}. \quad (4.8)
\]
The free field representation for \( SW(3/2, 2) \) was first calculated in a non-covariant component formulation in [14].

\[D(2|1; \alpha)\]

In this case three free fields are needed. The Cartan matrix which belongs to a purely fermionic root system is
\[
\begin{pmatrix}
0 & 1 & -1 - \alpha \\
1 & 0 & \alpha \\
-1 - \alpha & \alpha & 0
\end{pmatrix}, \quad (4.10)
\]
so that the energy momentum tensor takes the following form
\[
T = -\frac{1}{2} \left( N(\Psi_2', \Psi_1) + (-1 - \alpha) N(\Psi_3', \Psi_1) \right) + N(\Psi_1, \Psi_2') + \alpha N(\Psi'_3, \Psi_2) + (-1 - \alpha) N(\Psi'_1, \Psi_3) + \alpha N(\Psi'_2, \Psi_3) - \frac{1}{\beta} \left( \Psi''_1 + \Psi''_2 + \Psi'''_3 \right). \quad (4.11)
\]
Again we have a Super-Virasoro algebra with central charge
\[
c = \frac{3}{2} \left( 3 + \frac{4}{\beta^2} \left( 1 + \frac{1}{\alpha(1 + \alpha)} \right) \right). \quad (4.12)
\]
For the spin-3/2 and spin-2 field we make the following most general ansatz in three fields \(\Psi_i\)

\[
M = \left( \sum_{ij=1}^{3} M_{ij} N(\Psi'_i, \Psi'_j) + M_{123} N(\Psi_1, N(\Psi_2, \Psi_3)) + \sum_{i=1}^{3} M_i \Psi''_i \right),
\]

\[
U = \left( \sum_{i\leq j=1}^{3} U_{ij} N(\Psi'_i, \Psi'_j) + \sum_{ij=1}^{3} V_{ij} N(\Psi''_i, \Psi''_j) + \sum_{i=1}^{3} U_i \Psi''''_i \right) \quad (4.13)
\]

\[
+ \sum_{ijk=1}^{3} U_{ijk} N(\Psi'_i, N(\Psi'_j, \Psi'_k)) \right).
\]

Evaluating \(\bar{D} M = 0\) as described in section three yields

\[
M_1 = M_{31}/(\beta(1 + \alpha)), \quad M_{12} = -\beta(M_{32} + M_{31} + M_{32}\alpha), \quad M_{11} = 0,
\]

\[
M_{12} = M_{32}/\alpha, \quad M_{13} = -M_{31} - M_{32}(1 + \alpha)/\alpha, \quad M_{22} = 0,
\]

\[
M_2 = -M_{32}/(\beta\alpha), \quad M_{23} = M_{32} + M_{32}\alpha + M_{31}\alpha^2/(1 + \alpha), \quad M_{33} = 0,
\]

\[
M_{21} = -M_{31}/(1 + \alpha), \quad M_3 = -M_{32}(1 + \alpha)/(\beta\alpha) - M_{31}\alpha/(\beta(1 + \alpha)),
\]

which contains two free parameters as expected, since \(\bar{D} M = 0\) is linear and \(T\) is another solution independent of the primary field \(M\). The condition of primarity fixes \(M_{31}\) to be

\[
M_{31} = \frac{M_{32}(1 + \alpha)^2 (2 + \alpha)(2 + \beta^2\alpha)}{(1 - \alpha)\alpha^2 (-2 + \beta^2 + \beta^2\alpha)}. \quad (4.15)
\]

For the spin-2-field the solutions are much more elaborate and since they are not very elucidating here, they are presented in Appendix B. Now, having obtained solutions for the conserved spin-3/2 and spin-2 fields we investigate if their normalizations comply with the nonlinear OPE’s of ref. [13], e.g. the OPE of \(M\) with itself

\[
M(Z_1) \circ M(Z_2) = \frac{c/6}{Z_{12}^{3/2}} + \frac{\frac{3}{2} T(Z_2) + C_2 M(Z_2)}{Z_{12}^{3/2}} + \frac{\frac{1}{3} T'(Z_2) + \frac{1}{3} C_2 M'(Z_2) + C_3 U(Z_2)}{Z_{12}^{1/2}} + \text{reg.},
\]

which results in

\[
M_{32}^2 = \frac{-(-1 + \alpha)^2 \alpha^2 (-2 + \beta^2 + \beta^2\alpha)}{6 (2 + \beta^2)(1 + \alpha)^2 (2 + \beta^2\alpha)},
\]

\[
C_2 M_{32} = \frac{-2 (-1 + \alpha)^2 (2 + 5\alpha + 2\alpha^2)}{3 (2 + \beta^2)(1 + \alpha)^2 (2 + \beta^2\alpha)}, \quad (4.17)
\]

\[
C_3 V_{23} = \frac{-(-4 + 4\alpha + 3\beta^2\alpha + 4\alpha^2 + 3\beta^2\alpha^2)^2}{18 (1 + \alpha)(2 + \beta^2\alpha)(-2 + \beta^2 + \beta^2\alpha)}.
\]
Analogously
\[
M(Z_1) \circ U(Z_2) = \frac{C_4 M(Z_2) + C_5 U(Z_2) + \frac{1}{3} C_4 M'(Z_2) + \frac{1}{4} C_5 U'(Z_2) + \frac{1}{3} C_4 M''(Z_2)}{Z_{12}^{3/2}} \bigg/ \frac{1}{4} M(Z_2) + \frac{1}{2} C_5 U(Z_2) + \frac{1}{6} C_4 M''(Z_2) \bigg/ Z_{12}^{1/2} + \text{reg.}
\]

yields
\[
M_{32} V_{23} = -\frac{(2 + \beta^2) \alpha M_{32} C_6}{6 \beta^2},
\]
\[
M_{32} C_4 = \frac{M_{32} C_6 (4 + 4 \alpha + 3 \beta^2 \alpha + 4 \alpha^2 + 3 \beta^2 \alpha^2)}{6 \beta^2 \alpha (1 + \alpha)},
\]
\[
V_{23} C_5 = -\frac{4 (2 + \beta^2) M_{32} C_6 (2 + 5 \alpha + 2 \alpha^2)}{9 \beta^2 \alpha (-2 + \beta^2 + \beta^2 \alpha)},
\]

and last but not least
\[
U(Z_1) \circ U(Z_2) = \frac{-\frac{e}{2} M(Z_2) - \frac{6}{5} T(Z_2)}{Z_{12}^{3/2}} + \frac{C_8 U(Z_2) + \frac{1}{3} C_7 M' - \frac{2}{5} T'(Z_2)}{Z_{12}^{1/2}} + \frac{C_8/2 U''(Z_2) + 2/3 C_7 M'''(Z_2) - 4/5 T''(Z_2)}{Z_{12}^{3/2}} + \frac{C_8/2 U''(Z_2) + 1/6 C_7 M'''(Z_2) - 1/5 T'''(Z_2)}{Z_{12}^{1/2}} + \frac{C_{10} (2N(T, T')(Z_2) - N(T', T)(Z_2) - 1/4 T''')(Z_2))}{Z_{12}^{1/2}} + \frac{C_{10} (N(M, M')(Z_2)) + C_{11} N(T, M')(Z_2) + C_{12} N(T, U)(Z_2))}{Z_{12}^{1/2}} + \text{reg.}
\]

\[
V_{23}^2 = \frac{\alpha^2 (2 + \beta^2) (4 + 4 \alpha + 4 \alpha^2 + 3 \alpha \beta^2 + 3 \alpha^2 \beta^2)}{30 (4 - 2 \beta^2 - \alpha \beta^4 - \alpha^2 \beta^4)},
\]
\[
M_{32} C_7 = \frac{4 (-1 + \alpha)^2 (2 + 5 \alpha + 2 \alpha^2)}{15 (1 + \alpha)^2 (2 + \beta^2) (2 + \alpha \beta^2)},
\]
\[
V_{23} C_8 = \frac{(4 + 4 \alpha + 4 \alpha^2 + 3 \alpha \beta^2 + 3 \alpha^2 \beta^2) (10 + 10 \alpha + 10 \alpha^2 + 3 \alpha \beta^2 + 3 \alpha^2 \beta^2)}{45 (1 + \alpha) (-4 + 2 \beta^2 + \alpha \beta^4 + \alpha^2 \beta^4)}.
\]

The preceding calculations were again performed with the help of the symbolic calculation program. The coupling constants were first calculated in [13] and inserting their values in eqs. (4.17), (4.19) and (4.21) gives a consistent solution for $V_{23}$ and $M_{32}$. On the other hand the same set of equations leads to exactly the coupling constants as in [13], so that in some sense the structure of the $SW$-algebra, which we assumed as known so far, is
recovered in this fashion. We omitted the equations which stem from the terms of order $Z_{12}^{-1/2}$ in eq.(4.20), since they are too lengthy, but they do not change the picture. In the next chapter it is shown that this free field representation for $SW(3/2, 3/2, 2)$ does not support the construction of the corresponding series of minimal models.

5. Representation Theory

General considerations

The irreducible representations of the (Super-)Virasoro algebra are well-known, and the extension of this to the $A_n$-series of bosonic $W$-algebras has been achieved by Fateev and Lykyanov [8], simply by applying free field techniques to the root systems of $A_n$. Much more elaborate are the techniques used in [9], but the results cover all simple, classical Lie algebras. We will argue below that an application of the methods of ref. [8] using covariant vertex operators to the case of $N=1$ supersymmetry is possible, although we encounter some qualitatively new features. In the following we will restrict ourselves mainly to the Neveu-Schwarz-sector.

Let $W^{(k)}$ denote the simple (quasi)primary fields of the $SW(3/2, 2)$- and $SW(3/2, 3/2, 2)$-algebra, where $k \in \{T, W\}$ or $k \in \{T, M, U\}$, respectively. The modes of the $W^{(k)}$ are given by

$$W^{(k)} = \sum_{n \in \mathbb{Z}} W^{(k)}_n Z^{-n/2-\Delta_T(W^{(k)})} .$$

The power of $Z$ is obtained from eq.(2.3) by setting $Z_2$ equal to zero. The Verma module $V_{\Delta, c}$ is freely generated by the action of the modes $W^{(k)}_n$ with $n < 0$ on $|\Delta\rangle$, a highest weight vector satisfying

$$W^{(k)}_n |\Delta\rangle = 0 , \quad \forall n > 0 \quad \text{and} \quad W^{(k)}_0 |\Delta\rangle = \Delta_k |\Delta\rangle .$$

A state $|\xi_N\rangle$ is a null state in $V_{\Delta}$ on level $N$, iff $|\xi_N\rangle$ is a highest weight vector and $W^{(T)}_0 |\xi_N\rangle = (\Delta_T + N) |\xi_N\rangle$. Highest weight states are generated by applying primary fields at the origin to the vacuum $|0\rangle$. The vertex operator $V_\eta = \exp(\eta \cdot \Phi)$ is always a primary field with superconformal dimension

$$\Delta_T(V_\eta) = -\frac{1}{2} \eta \cdot \eta + \frac{1}{\beta} \eta \cdot \rho_q = \Delta_T(\eta) .$$

Here $\Phi$ is a free scalar superfield with conformal dimension zero and $\eta$ is called the charge of the vertex operator. We continue by introducing the notions of background charge and conjugation of a charge. The background charge $\eta_0$ satisfies the relations

$$\Delta_k(\eta) = \Delta_k(\eta_0 - \eta) , \quad \forall k$$

(5.4)
and is related to $\rho_q$ by $\eta_0 = \frac{2}{p} \rho_q$. The conjugation is the involutive operation $\bar{\eta} = \eta_0 - \eta$ which preserves the superconformal dimension of $V_\eta$. Next we introduce the so-called screening operators $Q_i$, which are closely connected with the BRST-charge of refs \cite{3, 6}. By definition they commute with all simple (quasi-)primary fields $W^{(k)}$ of the SW-algebra: $[W^{(k)}, Q_i] = 0$. This induces $\Delta_T(Q_i) = 0$, but the only two local superfields with conformal dimension zero are equivalent to the identity. Therefore we assume that the $Q_i$ are non-local and can be represented in the following form

$$Q_i = \oint C V_{\tilde{\eta}_i},$$

where the $\oint$ stands for $\frac{1}{2\pi i} \oint dz d\theta$. Then the OPE of a field $W^{(k)}$ with $V_{\tilde{\eta}_i}$ is the total covariant derivative with respect to $Z_2$ of some Laurent-polynomial in $Z_{12}$

$$W^{(k)}(Z_1) \circ V_{\tilde{\eta}_i}(Z_2) = D_{Z_2}(\ldots).$$

For $k = T$ this is equivalent to $V_{\tilde{\eta}_i}$ having superconformal dimension one half. Now, if $|\Delta(\eta)\rangle$ is a highest weight state, so is $|\xi(\eta)\rangle = Q_i |\Delta(\eta)\rangle$, since

$$W^{(k)} \frac{\Delta(T)}{2} |\Delta(\eta)\rangle = Q_i W^{(k)} |\Delta(\eta)\rangle = 0 \quad \forall k, \forall n > 0.$$  \hspace{1cm} (5.7)

Since the computations involve the free field $\Phi$, we evaluate this expression further (see also A.3),

$$|\xi(\eta)\rangle = Q_i |\Delta(\eta)\rangle$$

$$= \oint \exp(\tilde{\eta}_i \cdot \Phi(Z)) \circ \exp(\eta \cdot \Phi(0)) |0\rangle$$

$$= \oint Z^{-\tilde{\eta}_i \cdot \eta} \exp(\tilde{\eta}_i \cdot \Phi(Z) + \eta \cdot \Phi(0)) |0\rangle$$

$$= \left( \frac{1}{m_i!} \right) D_{Z_i}^m \left( \exp(\tilde{\eta}_i \cdot \Phi(Z) + \eta \cdot \Phi(0)) |0\rangle \right)|_{Z = 0},$$

which is well-defined (and non-zero), iff

$$\tilde{\eta}_i \cdot \eta = \frac{m_i + 1}{2}, \quad m_i \in \mathbb{N}_0.$$  \hspace{1cm} (5.9)

$|\xi(\eta)\rangle$ is a highest weight state and contained in a Verma module (the one of $|\Delta(\eta + \eta)\rangle$), so it must be a null state. For general states of the form

$$|\xi_{\tilde{\eta}, \ldots, \eta}(\eta)\rangle = \oint C_{m_n} \ldots \oint C_{m_1} V_{\tilde{\eta}_i} \ldots V_{\tilde{\eta}_i} |\Delta(\eta_0 - \eta - l_i \tilde{\eta}_i)\rangle,$$

i.e. $Q_i$ is applied $l_i$ times (with appropriately chosen contours $C_{m_i}$), the corresponding consistency condition reads

$$\tilde{\eta}_i^2 \frac{l_i - 1}{2} + \tilde{\eta}_i \cdot (\eta_0 - \eta - l_i \tilde{\eta}_i) = \frac{m_i + 1}{2}.$$  \hspace{1cm} (5.11)
We made use of \( \exp(\eta_1 \cdot \Phi(Z_1)) \circ \exp(\eta_2 \cdot \Phi(Z_2)) = Z_1^{-\eta_1 \cdot \eta_2} \exp(\eta_1 \cdot \Phi(Z_1) + \eta_2 \cdot \Phi(Z_2)) \), where \( \eta_1 \cdot \eta_2 \) takes integral values in the NS-sector, so that \( m_i \) is odd, while in the R-sector \( m_i \) is even. Using eq.(5.3) for \( \tilde{\eta}_i \) we obtain

\[
\tilde{\eta}_i \cdot \tilde{\eta} = \frac{1 - m_i}{2} + \eta_i^2 \frac{1 - l_i}{2} .
\]  

(5.12)

Now we comment briefly on the spectrum of solutions of eqs.(5.6). We rely on the (here unproven) fact that one part of the set of screening charges is always given by \( \tilde{\eta}_i = \beta \alpha_i \), \( 1 \leq i \leq r \). All other screening charges are assumed to be of the following form: \( \tilde{\eta}_i = \gamma_i \alpha_i \), \( r + 1 \leq i \leq r + s \), where \( \gamma_i \) will be in practice \( \frac{1}{\beta} \), see also [24, 25]. The “pairing” of screening charges (linear dependency) will happen exactly when \( \alpha_i^2 \neq 0 \), i.e. the Cartan matrix of the SSLA has non-vanishing elements on its diagonal.

Since the fundamental weights \( \mu_i \) form a basis dual to the \( \alpha_i \), i.e. \( \alpha_i \cdot \mu_j = \delta_{ij}, i,j \leq r \) the final form of the allowed charges of a vertex operator representing a highest weight of a degenerate theory is

\[
\tilde{\eta} = \sum_{i=1}^{s} \left( \beta \frac{1 - l_i}{2} \right) \mu_i + \sum_{i=1}^{r} \left( \frac{1}{\beta} \left( 1 - m_i \right) \right) \mu_i .
\]  

(5.13)

In generalization of a result of bosonic two-dimensional QFT, we need the “charge neutrality” condition

\[
\sum_{i} \eta_i = \eta_0 ,
\]  

(5.14)

for a \( n \)-point-function \( \langle 0| V_{\eta_1} \cdots V_{\eta_n} |0 \rangle \), which is necessarily fulfilled, if it does not vanish identically. This is equivalent to the restriction that it be independent of an ultraviolet regularization. By looking at three different ways of representing the three-point function of primary fields by the corresponding (conjugated) vertex operators, namely

\[
\langle 0| V_{\gamma} V_{\delta} V_{\epsilon} Q_{\tilde{\eta}_i} \cdots Q_{\tilde{\eta}_j} |0 \rangle \quad \langle 0| V_{\gamma} V_{\delta} V_{\epsilon} Q_{\tilde{\eta}_i} \cdots Q_{\tilde{\eta}_j} |0 \rangle \quad \langle 0| V_{\gamma} V_{\delta} V_{\epsilon} Q_{\tilde{\eta}_i} \cdots Q_{\tilde{\eta}_j} |0 \rangle ,
\]

(5.15)

one obtains with eq.(5.14) equations of the following form:

\[
\eta_0 - \gamma + \delta + \epsilon + \sum_{i} n_i \beta \alpha_i + \sum_{i} \tilde{n}_i \frac{1}{\beta} \alpha_i = \eta_0 ,
\]  

(5.16)

where \( n_i \) and \( \tilde{n}_i \) denote the respective number of screening charges. Now \( \gamma, \delta, \epsilon \) are, according to eq.(5.13), charges belonging to degenerate theories, e.g.

\[
\gamma = \sum_{i=1}^{s} \beta \hat{l}_{ci} \alpha_i + \sum_{i=1}^{r} \frac{1}{\beta} \hat{m}_{ci} \alpha_i ,
\]  

(5.17)

where we defined \( \hat{l}_{cj} = \left( \frac{1 - l_j}{2} \right) K^{-1}_{ij} \) and made use of \( \mu_i = K_{ij}^{-1} \alpha_j \). We introduced the additional subscript \( c \) to denote that the coefficient belongs to the charge \( \gamma \) and the same
was done for the $m_i$’s and also for the charges $\delta$ and $\epsilon$ where the subscripts are $d$, $e$, respectively. This leads to equations in the coefficients $l_i$ and $m_i$, namely

$$
\beta^2 \hat{l}_{ci} + \hat{m}_{ei} = \beta^2 \left( \hat{l}_{di} + \hat{l}_{ei} \right) + \hat{m}_{di} + \hat{m}_{ei} + n_i \beta^2 + \hat{n}_i.
$$

(5.18)

We solve a notational problem by setting $l_i = 1$ for $s + 1 \leq i \leq r + s$, since in general $s < r$. Of course, this procedure is to be repeated for the other three-point functions of (5.15), so that if we introduce the abbreviation $k_i = \beta^2 \hat{l}_i + \hat{m}_i$, we find a set of equations

$$
\begin{align*}
    k_{ci} &= (k_{di} + k_{ei}) + n_i \beta^2 + \hat{n}_i, \\
    k_{ci} &= (k_{di} - k_{ei}) - n_i \beta^2 - \hat{n}_i, \\
    k_{ci} &= (k_{ei} - k_{di}) - n_i \beta^2 - \hat{n}_i.
\end{align*}
$$

(5.19)

From the above remarks it is clear that $\hat{n}_i$ can be identically zero for some $i$, so that together with strictly positive $n_i$, the above set of equations restricts the possible values of $k_{ci}$. These restrictions form the desired fusion rules, which are a first step towards the exact fusion. At this point it is interesting to note that this set of fusion rules does not only deviate from the standard way of writing them [4], but they are also inequivalent for SW-algebras. For the Super-Virasoro-algebra our version is compatible with the standard one. The difference between the two is, that we do not impose relations by equating the coefficients of powers of $\beta$. In deriving minimal models within the free field framework, one takes advantage of the periodicity of the conformal dimensions $\Delta_k$ in the parameters $l_i$ and $m_i$. This can be expressed by a relation of the following kind

$$
p_i \tilde{n}_i + q_i \tilde{n}_{r+i} = 0.
$$

(5.20)

Of course we may run into the problem that $\tilde{n}_{r+i}$ does not exist for some $i$, that is we face a new situation, unknown in bosonic CFT. The possible range of values for $l_i$ and $m_i$ is usually determined by these numbers $p_i$ and $q_i$, so that the question arises how this is accomplished for the case of a “non-paired” screening charge $\tilde{n}_i$. We do not answer this question in this general context, but refer to the next section, where we derive some rules inductively.

Another approach to perform a free field construction is to impose that the SW-fields commute with the operators $\exp(\beta \alpha_i \Phi)$

$$
[W^{(k)}, \exp(\beta \alpha_i \Phi)] = 0.
$$

(5.21)

In this way it is explicit that the charges $\beta \alpha_i$ give rise to screening operators. In our approach we also obtain these solutions, but we will not always get their “partners” $\exp(\frac{1}{\beta} \alpha_i \Phi)$ as it is in the bosonic case, confer ref. [24, 25].
Partial results of this example were presented in ref. [18], but since the OPE given there differs from eq.(4.8), we performed an independent calculation. Nevertheless we obtain the same results for the screening charges as in ref. [18]. The superconformal dimension of a vertex operator \( V_\eta \) for this example is

\[
\Delta_T(V_\eta) = -\frac{1}{2} \eta \cdot \eta + \frac{1}{\beta} \eta \cdot \rho_q
\]

\[
= -\frac{\eta^2}{2} + \eta_1 \eta_2 + \frac{\eta_1 + \eta_2}{2 \beta} + \frac{\eta_1 \beta}{2} ,
\]

with \( \rho_q = -\frac{1}{2} (1, \beta^2 + 2) \).

The W-dimension is

\[
\Delta_W(V_\eta) = \mathcal{N} \left( \frac{3 \eta^2_2 \beta + \eta_2 (2 + \beta^2) + \eta^3_2 (4 \beta + 2 \beta^3)}{6 \beta^2} + \eta_1 (-4 - 6 \beta^2 - 2 \beta^4 + \eta_2 (-8 \beta - 4 \beta^3)) \right) .
\]

The background charge \( \eta_0 \) satisfies \( \Delta_T(V_{\eta_0}) = 0 \) and \( \Delta_W(V_{\eta_0}) = 0 \), which determines it to be

\[
\eta_0 = \left( -\frac{1}{\beta}, -\left( \frac{2}{\beta} + \beta \right) \right) = \frac{2}{\beta} \rho_q .
\]

It also satisfies \( \Delta_T(V_{\eta_0-\eta}) = \Delta_T(V_\eta) \) and \( \Delta_W(V_{\eta_0-\eta}) = \Delta_W(V_\eta) \) as was to be expected. For the screening charges \( \eta_i \) we evaluate

\[
W(Z_1) \circ V_{\tilde{\eta}_i}(Z_2) = D_{Z_2}(...) + \text{reg} ,
\]

and obtain three different solutions

\[
\tilde{\eta}_1 = \beta \alpha_1 , \quad \tilde{\eta}_2 = \beta \alpha_2 , \quad \tilde{\eta}_3 = \frac{1}{\beta} \alpha_1 ,
\]

which yield the correct superconformal dimension \( \Delta_T(V_{\tilde{\eta}_i}) = \frac{1}{2} \). This is quite surprising, since we expected four solutions at this stage. One immediate consequence of this is that eq.(5.20) involves only two of the three screening charges

\[
p \tilde{\eta}_1 + q \tilde{\eta}_3 = 0 \quad \iff \quad \frac{1}{\beta^2} = -\frac{p}{q} ,
\]
which then gives rise to the following formula for the central charge of minimal models

\[ c = -3 \left( 1 - \frac{p}{q} \right). \] (5.28)

Note the linear appearance of \( \frac{p}{q} \) and since \( c \leq 3 \) is necessary for minimal models [27] we find \( \frac{p}{q} \leq \frac{2}{3} \). From eqs. (5.17), (5.26) and (5.27) we find the parametrisation of superconformal dimensions

\[
\Delta_T(l_1, m_1, m_2) = \frac{-2q + 2m_2l_1q + 3p - m_3^2p - 2m_2m_1p}{8q} = \frac{c(p, q) - 3}{24} + \frac{n(rq - sp)}{8q}. \] (5.29)

\[ n = m_2, \ r = 2l_1, \ s = 2m_1 + m_2 \]

From this we observe \( \Delta_T(l_1 + p, m_1 + q, m_2) = \Delta_T(l_1, m_1, m_2) \), which actually reflects a symmetry of the charges of a minimal model as is explained by eq.(5.20). It can be combined with the symmetry of the weights \( \Delta_k(\eta) \) as in eq.(5.4), if we formally introduce the parameter \( l_2 \) corresponding to the (unphysical) screening charge \( \frac{1}{\beta} \eta_2 \). Then the above shift operation extends also to the pair \((l_2, m_2)\). By setting \( \alpha = \beta \lambda + \frac{1}{3} \lambda' \), the transformation

\[
\lambda \rightarrow \rho_q \lambda - \lambda',
\lambda' \rightarrow (q + 2) \rho_q - \lambda',
\] (5.30)

implies

\[
\alpha \rightarrow \beta \rho_q \lambda - \beta \lambda + \frac{1}{\beta} (q + 2) \rho_q - \frac{1}{\beta} \lambda'
= (\beta \rho_q + \frac{2}{\beta} q) \rho_q - \alpha + \frac{2}{\beta} \rho_q
= \eta_0 - \alpha,
\] (5.31)

which is equivalent to the index shifts

\[
l_1 \rightarrow (q - p - l_1 + 2), \quad l_2 \rightarrow (2 - p - l_2),
m_1 \rightarrow \left( \frac{(2 + q)q}{p} - q - m_1 \right), \quad m_2 \rightarrow (-q - m_2).
\] (5.32)

Of course \( l_2 = 1 \) was implicitly assumed so far and \( \frac{(2+q)}{p} \) must be an integral number which yields \( p \neq 2 \) if \( p \) and \( q \) are relatively prime integers.

So far there are four known SW(3/2, 2) minimal models [29, 30]:

| \((\Delta_T, \Delta_2)\) | \((0, 0)\) | \((\frac{1}{3}, -\frac{1}{25})\) | \((\frac{1}{3}, \frac{1}{25})\) | \((-\frac{1}{10}, 0)\) |
| \((l_1, m_1, m_2)\) | \((1, 1, 1)\) | \((2, 2, 1)\) | \((1, 1, 3)\) | \((1, 3, 1)\) |

Table 1: \((p, q) = (1, 5)\) \( c = -\frac{6}{5} \)
Table 2: \((p, q) = (-1, 6)\quad c = -\frac{9}{7}\)

| \((\Delta_T, \Delta_W)\) | (0, 0) | \((0, \frac{3}{28})\) | \((-\frac{1}{14}, \frac{1}{28})\) | \((-\frac{1}{14}, \frac{1}{14})\) | \((\frac{3}{7}, \frac{39}{28})\) | \((-\frac{1}{4}, -\frac{1}{8})\) |
| \((l_1, m_1, m_2)\) | (1, 1, 1) | (0, 1, 3) | (1, 3, 1) | (0, 3, 1) | (1, 1, 3) | (0, 1, 1) |

Table 3: \((p, q) = (1, 7)\quad c = -\frac{12}{7}\)

| \((\Delta_T, \Delta_W)\) | (0, 0) | \((\frac{7}{14}, \frac{1}{2})\) | \((\frac{3}{2}, \frac{51}{52})\) | \((-\frac{1}{16}, \frac{43}{116})\) | \((\frac{1}{16}, -\frac{1}{32})\) | \((-\frac{1}{4}, -\frac{1}{32})\) |
| \((l_1, m_1, m_2)\) | (1, 1, 1) | (1, 3, 3) | (1, 1, 5) | (0, 1, 3) | (1, 3, 1) | (0, 1, 1) |

Table 4: \((p, q) = (-1, 8)\quad c = -\frac{33}{8}\)

| \((\Delta_T, \Delta_W)\) | (0, 0) | \((\frac{7}{14}, \frac{1}{13})\) | \((\frac{11}{16}, \frac{187}{416})\) | \((-\frac{3}{16}, \frac{3}{416})\) | \((\frac{11}{16}, \frac{3}{52})\) | \((\frac{1}{8}, -\frac{1}{13})\) |
| \((l_1, m_1, m_2)\) | (0, 5, 1) | (0, 1, 5) | (1, 1, 3) | (0, 3, 1) | (0, 3, 3) | (1, 5, 1) |

Of course it is arbitrary up to now, whether we place the minus sign on \(q\) or \(p\). Since the parameters \(l_1\) and \(m_1\) are periodic in \((p, q)\), we choose the smallest non-negative value for them. As can be seen from the above examples, we have to distinguish two cases, namely \(q\) odd or even. Let us empirically deduce some “selection rules”. In the case \(q\) odd the sum of \(l_1\) and \(m_1\) is always even, which is also true for \(l_2 + m_2\), if \(l_2\) in accordance with the previous remarks is set equal to one. This can be understood in context with eq.(5.11). Furthermore we have \(\sum_i m_i + l_i \leq p + q\) and \(1 \leq l_i \leq m_i\) (which is void for \(i = 2\)) and after closer inspection one finds that this already determines all possible fields, except for \((p, q) = (1, 7)\). Here this rule predicts a field with parametrization (3,3,1) and dimensions \((\frac{3}{7}, \pm \frac{69}{28})\), which is not observed. There are two possibilities to interpret this. Either the above rules are not complete yet, i.e. they should be complemented by a rule stating for example that \(l_1\) and \(m_1\) cannot be the same odd number other than one, or we encounter a special case where the (3,3,1)-field belongs in principle to the multiplet, but drops out of the fusion, because the respective fusion coefficients are zero. The latter could not be clarified within the present approach. Nevertheless, for \(p\) positive the selection rules are very similar to the ones of the \(N = 1\) Super-Virasoro algebra.

For the case \(q\) even, since \((l_1, m_1)\) is defined only modulo \((p, q)\), \(l_1 + m_1\) is not necessarily even any more, though \(l_2 + m_2\) is. Also we find a pairing in the sense that if there is a field of the form \((1, m_1, m_2)\) there is one of the form \((0, m_1, m_2)\) and vice versa, which predicts

\[ a \quad \text{In tables 2 - 4 we divided } \Delta_W \text{ by } C_1 \]
an even number of fields. This seems to be the only allowed value of \( l_1 \), while the range of \( m_1 \) and \( m_2 \) is restricted by \( 1 \leq m_1 + m_2 \leq p + q \) with \( m_1 \) and \( m_2 \) odd. Again, this is sufficient to parameterize all fields in the above example, but it is of course no proof for higher \( q \). After all it is not clear if we chose the optimal parametrization of the conformal dimensions, though our choice is motivated by the construction of the null fields.

From the above minimal models one can also empirically deduce the effective central charge: \( c_{eff} = c - 24 \Delta_{min} \), namely: \( c_{eff} = 3 - \frac{9}{q} \), for \( c = c(\pm 1, q) \). This is equivalent to

\[
\Delta_{min} = \begin{cases} 
\frac{1}{12} c, & \text{for } c = c(1, q), \; q \text{ odd}; \\
-\frac{1}{4}, & \text{for } c = c(-1, q), \; q \text{ even}. 
\end{cases} 
\tag{5.33}
\]

The next minimal model would be expected at \( c = -2 \) with \( (p, q) = (1, 9) \). Unfortunately the explicit construction of this model is today beyond the computational possibilities. Apart from the above relations an explanation of the fact why \( |p| = 1 \) and why the sign changes with \( q \) being even or odd is still due.

It is not difficult to generalize eq.(5.32) to the Ramond-sector, since the two spin fields which have to be introduced then contribute twice the conformal dimension of \( \frac{1}{16} \). Indeed we can parametrize the \( \Delta_T \)-values of the above models for the R-sector [29, 30], but since we do not know the OPE of a spin-field with the free field \( \Phi \) we cannot compute their respective \( W \)-dimensions. We give the parametrizations of the two minimal models for which the Ramond-sector is known.

| \( \Delta_T \) | \( \frac{3}{4} \) | \( \frac{3}{20} \) | \( -\frac{1}{20} \) |
|---|---|---|---|
| \( (l_1, m_1, m_2) \) | \( (2, 1, 2) \) | \( (1, 2, 2) \) | \( (x, y, 0) \) |

Table 5: \( (p, q) = (1, 5) \) \( c = -\frac{6}{5} \)

| \( \Delta_T \) | \( \frac{7}{48} \) | \( \frac{59}{48} \) | \( -\frac{5}{48} \) | \( \frac{1}{16} \) | \( \frac{9}{16} \) | \( -\frac{3}{16} \) |
|---|---|---|---|---|---|---|
| \( (l_1, m_1, m_2) \) | \( (0, 0, 4) \) | \( (2, 4, 2) \) | \( (0, 0, 2) \) | \( (0, 2, 2) \) | \( (0, 0, 6) \) | \( (x, y, 0) \) |

Table 6: \( (p, q) = (-1, 6) \) \( c = -\frac{9}{2} \)

For \( q \) odd the selection rule \( l_i + m_i \) odd and \( \sum_i m_i + l_i \leq p + q \) seems to work. This would be complementary to the NS-sector, except that \( 1 \leq l_i \leq m_i \) is not satisfied. This leads in general to more fields in the R-sector.

For \( q \) even we find even values for \( l_i \) and \( m_i \), but the selection rules are not very clear. Common to the two models is that the field of dimension \( \Delta_T = c/24 \) is parameterized by \( (x, y, 0) \), where \( x \) and \( y \) are arbitrary, as long as \( m_2 \) is equal to zero.

Now, let us have a closer look at the fusion of the minimal models at \( c = -6/5 \) and \( c = -9/2 \). The model at \( c = -6/5 \) is a Super Virasoro minimal model and therefore its fusion is known exactly [31]. The non-zero fusion coefficients are all equal to one.
Table 7: Fusion at $c = -\frac{6}{5}$

How can this diagram be explained in our approach? First of all we cannot reproduce the fusion coefficients themselves, since all we demand is that the respective three point function does not vanish, i.e. we find necessary conditions for the appearance of a particular field in the fusion, see also eq.(5.14). To make the calculations explicit, we compute all data that we need in order to apply eq.(5.18) to the case of $c = -\frac{6}{5}$. The equations under consideration are

$$
\beta^2 \hat{l}_{ci} + \hat{m}_{ci} = \beta^2 \left( \hat{l}_{di} + \hat{l}_{ei} \right) + \hat{m}_{di} + \hat{m}_{ei} + n_i \beta^2 + \hat{n}_i,
$$

as well as

$$
\beta^2 \hat{l}_{ci} + \hat{m}_{ci} = \beta^2 \left( \hat{l}_{di} - \hat{l}_{ei} \right) + (\hat{m}_{di} - \hat{m}_{ei}) - n_i \beta^2 - \hat{n}_i
$$

and

$$
\beta^2 \hat{l}_{ci} + \hat{m}_{ci} = \beta^2 \left( \hat{l}_{ei} - \hat{l}_{di} \right) + (\hat{m}_{ei} - \hat{m}_{di}) - n_i \beta^2 - \hat{n}_i.
$$

It is important to realize that $\hat{n}_2$ is always zero and $n_1, \hat{n}_1, n_2$ are non-negative. In table 8 we give all necessary data:

Table 8: Representations of fields at $c = -\frac{6}{5}$

Here $\beta^2 = -5$ and for example the fusion $\Phi_1 \times \Phi_i = \Phi_i$ is understood if one takes $n_1 = n_2 = \hat{n}_2 = 0$ and the unbared representation of the identity for the first two equations, and the bared one for the last equation. The representation for the field $\Phi_i$ is arbitrary, except
that for the last equation the representation of the RHS is the bared of the one of the LHS (recall that the conjugation is involutive). The pattern of the choice of representatives and the numbers of screening charges involved for the first equation can be determined from the following table with the understanding that the given values are sometimes but by no means always unique:

\[
\begin{array}{|c|c|c|c|c|}
\hline
(\Delta_T, \Delta_W) & (0, 0) & \left( \frac{1}{5}, -\frac{1}{25} \right) & \left( \frac{1}{5}, \frac{1}{25} \right) & \left( -\frac{1}{10}, 0 \right) \\
\hline
(0, 0) & (u, u, u, 0, 0) & (u, u, u, 0, 0) & (u, u, u, 0, 0) & (u, u, u, 0, 0) \\
\hline
\left( \frac{1}{5}, -\frac{1}{25} \right) & (u, u, u, 0, 0) & (u, b, b, 0, 0, 0) & (b, a, a, 0, 0, 1) & (b, u, u, 0, 2, 1) \\
& (u, u, u, 0, 1) & (u, b, b, 0, 1, 1) & (b, u, u, 0, 1, 1) & (b, u, u, 0, 1, 1) \\
\hline
\left( \frac{1}{5}, \frac{1}{25} \right) & (u, u, u, 0, 0) & (b, a, a, 0, 0, 1) & (u, b, b, 0, 1, 1) & (b, u, u, 0, 2, 1) \\
& (u, u, u, 0, 1) & (u, b, b, 0, 1, 1) & (b, u, u, 0, 1, 1) & (b, u, u, 0, 1, 1) \\
\hline
\left( -\frac{1}{10}, 0 \right) & (u, u, u, 0, 0) & (b, u, u, 0, 2, 1) & (b, u, u, 0, 2, 1) & (u, b, b, 0, 0, 0) \\
& (u, u, u, 0, 1) & (u, u, u, 0, 1) & (u, u, u, 0, 1) & (u, u, u, 0, 1) \\
\hline
\end{array}
\]

Table 9: Fusion pattern at \( c = -\frac{6}{5} \)

Here an entry like \((b, u, u, 0, 2, 1)\) means that the conjugated (bared) charge \(\epsilon\) is fused with the unbared \(\delta\) and results into an unbared \(\gamma\). The last three entries denote the respective number of screening charges \((n_1, \tilde{n}_1, n_2)\). Multiple entries are ordered with respect to their appearance in table 7.

As a result all non-zero fusion coefficients of table 7 can be reproduced. On the other hand fusion coefficients equal to zero are not always predicted, which is a common failure of the free field approach. One may ask if there exists trivially a solution for \((n_1, \tilde{n}_1, n_2)\), but as a counterexample it may serve that e.g. the fusion of \(\Phi_2\) with itself forbids the appearance of \(\Phi_3\) and \(\Phi_4\), as can easily be checked. The whole argument involving the three point functions relies on the fact that the two-point function has \(\langle 0| V_{\gamma} V_{\gamma} |0 \rangle\) as its only non-zero representation. So in \(\langle 0| V_{\gamma} V_{\gamma} V_\delta Q_{\tilde{n}_1} \cdots Q_{\tilde{n}_j} |0 \rangle\) the fusion of \(V_{\delta}\) and \(V_{\epsilon}\) together with the screening charges is supposed to yield \(V_{\gamma}\). On the other hand in \(\langle 0| V_{\gamma} V_{\delta} V_\epsilon Q_{\tilde{n}_1} \cdots Q_{\tilde{n}_j} |0 \rangle\) the fusion of \(V_{\delta}\) and \(V_{\epsilon}\) yields \(V_{\gamma}\) and analogously for the third three point function. In this way we understand the pattern of solutions for the second and third equation given above. The number of screening charges stays the same as in table 7. A new aspect of all this is that starting from eq.\(5.18\) one cannot split it into two equations by collecting terms with equal powers of \(\beta\), which can be done for the \(A_n\)-algebras for example.

Next we inspect the other known fusion a little closer [32].
Table 10: Fusion at $c = \frac{-12}{7}$

| $(\Delta T, \Delta W)$ | (0, 0) | $(0, \frac{3}{22})$ | $(-\frac{1}{12}, \frac{1}{24})$ | $(-\frac{1}{6}, \frac{1}{66})$ | $(\frac{3}{4}, \frac{39}{88})$ | $(-\frac{1}{4}, -\frac{1}{88})$ |
|------------------------|--------|-------------------|---------------------|-------------------|-------------------|-------------------|
| $(0, 0)$               | $\Phi_1$ | $\Phi_2$ | $\Phi_3$ | $\Phi_4$ | $\Phi_5$ | $\Phi_6$ |
| $(0, \frac{3}{22})$   | $\Phi_2$ | $\Phi_1, \Phi_2, \Phi_4$ | $\Phi_3, \Phi_6$ | $\Phi_2, \Phi_4$ | $\Phi_6$ | $\Phi_3, \Phi_5, \Phi_6$ |
| $(-\frac{1}{12}, \frac{1}{24})$ | $\Phi_3$ | $\Phi_3, \Phi_6$ | $\Phi_1, \Phi_2, \Phi_4$ | $\Phi_3, \Phi_5, \Phi_6$ | $\Phi_4$ | $\Phi_2, \Phi_4$ |
| $(-\frac{1}{6}, \frac{1}{66})$ | $\Phi_4$ | $\Phi_2, \Phi_4$ | $\Phi_3, \Phi_5, \Phi_6$ | $\Phi_1, \Phi_2, \Phi_4$ | $\Phi_3$ | $\Phi_3, \Phi_6$ |
| $(\frac{3}{4}, \frac{39}{88})$ | $\Phi_5$ | $\Phi_6$ | $\Phi_4$ | $\Phi_3$ | $\Phi_1$ | $\Phi_2$ |
| $(-\frac{1}{4}, -\frac{1}{88})$ | $\Phi_6$ | $\Phi_3, \Phi_5, \Phi_6$ | $\Phi_2, \Phi_4$ | $\Phi_3, \Phi_6$ | $\Phi_2$ | $\Phi_1, \Phi_2, \Phi_4$ |

This fusion can also be checked with our method, and we find agreement. An interesting aspect of this model is the $Z_2$-symmetry which is respected by the fusion, namely the fields group into even $\{\Phi_1, \Phi_2, \Phi_4\}$ and odd $\{\Phi_3, \Phi_5, \Phi_6\}$ multiplets. This $Z_2$-symmetry is most probably common to all models with even $q$. For reasons of brevity we do not reproduce the analogues of the tables 8 and 9 here.

$\text{SW}(3/2, 3/2, 2)$

In order to repeat the previous analysis for the case of $D(2|1; \alpha)$ one has to look for all possible solutions of eqs. (5.6). We did not find any solutions for the screening charges other than

$$\tilde{\eta}_i = \beta \alpha_i ,$$

(5.34)
even when we looked for ones not generic in the parameter $\alpha$. This is interesting if one takes into account that $D(2|1; \alpha)$ is isomorphic to $Osp(4|2)$ for $\alpha = 1$ or $-2$. The immediate consequence of this fact is that there are no linearly dependent screening charges, meaning that eq.(5.20) is trivial. As a result, one can not build minimal models.

In order to gain more insight into this phenomenon we would like to mention two further attempts to construction rational conformal field theories for the $SW$ algebra based on $D(2|1, \alpha)$ [33].

It is well known that rational conformal field theories exist only for rational values of $c$. Since $c$ for the $SW$-algebra based on $D(2|1, \alpha)$ is a continuous functions of $\alpha$ (see (4.12)) this implies a quantization of $\alpha$ for rational models. One obvious trial for such a quantization is the choice $C_8 = 0$ in eq.(4.21). Because of the presence of the additional
parameter $\beta$ the central charge is not yet fixed by this choice. With this choice the
two solutions for $SW(3/2, 3/2, 2)$ found in [13] coincide. Therefore one would expect
to be able to write this $SW$-algebra as the direct sum of the $SW(3/2, 2)$ discussed at
the beginning of section 4 and a super Virasoro algebra because this is possible for the
second solution to $SW(3/2, 3/2, 2)$ at generic $c$ [13]. Vanishing self coupling constant $C_8$
implies that the central charge of this $SW(3/2, 2)$ is equal to $-\frac{6}{5}$ which leads to a minimal
model. This gives rise to the hope that with an appropriate choice of the central charge of
the additional Super Virasoro algebra involved one can construct minimal models of the
$SW$ algebra based on $D(2|1, \alpha)$ in this manner. Unfortunately, the change of basis from
$SW(3/2, 2) + SW(3/2)$ to a $SW(3/2, 3/2, 2)$ becomes singular precisely for vanishing self
coupling constant (compare also [27]). Therefore, the construction explained above does
not even give rise to a consistent $W$ algebra and definitely not to rational conformal field
theories.

In all known examples rational conformal field theories are encoded in null fields appearing
for particular values of the central charge. This was also attempted along the lines of [29]
for the $SW$-algebra under consideration using a non-covariant formulation. Looking for
null fields at scale dimension 4 (the lowest candidate dimension) one finds null fields if one
of the following relations is satisfied:

$$C_8^2 = -\frac{(10c - 27)^2c}{486(4c + 21)}, \quad C_8^2 = \frac{2(10c - 17)^2c}{27(10c - 7)}. \quad (5.35)$$

In the first case one finds two null fields, in the second case even three (at scale dimension
4). Let $u$ be the eigenvalue of the upper component of the dimension $\frac{3}{2}$ generator in the
representation, $v$ the eigenvalue of the lower component of the dimension 2 generator. Then all 8 conditions that were evaluated are satisfied for

$$8c^2u - 10c^2v - 180cu + 27cv - 27c + 567h - 2268u = 0 \tag{5.36}$$

if one considers the case $C_8^2 = -\frac{(10c - 27)^2c}{486(4c + 21)}$. In particular it is not even possible to fix $c$
and definitely not $h$. We should mention that because of technical difficulties one possible
type of conditions was neglected. On the one hand, odds are low that this would yield
enough supplementary conditions. On the other hand, all conditions factorize nicely with
the same linear factor which so far happened precisely if the model was not rational but
only degenerate. Note furthermore that from (5.36) we indeed expect at most one relation
in a degenerate model.

In the case $C_8^2 = \frac{2(10c - 17)^2c}{27(10c - 7)}$ the corresponding conditions are so complicated that it is
difficult to derive insight from them and therefore we omit any details. Note that also in
this case it is not possible to fix $c$ as it would be necessary for a rational model.

Finally we would like to notice that with the choice $C_8 = 0$ the determinant of all fields
at scale dimension 4 is proportional to some power of $c$ which is in accordance with the
above remarks.
In summary, all three arguments we would have expected to reveal rational models for our $SW(3/2, 3/2, 2)$ have failed to do so. This is a strong hint that the $SW$-algebra based on $D(2|1, \alpha)$ does indeed not have any rational models. In part of the argumentation the parameter $\alpha$ played an important role. Therefore it is not completely clear if the reason for the absence of minimal models is this parameter $\alpha$ or the purely fermionic root system. Our general considerations at the beginning of this section indicate, however, that the underlying reason most probably is the purely fermionic root system, i.e. the vanishing of all diagonal elements of the Cartan matrix.

May be it is noteworthy that exactly these minimal models were the original motivation of this work, since it was an interesting perspective to study the appearance of the free SSLA parameter $\alpha$ in the formulas.

## Conclusions

On behalf of the preceding considerations we argue that the existence of minimal models can be established from free field realizations only if the Cartan matrix of the underlying SSLA has non-zero entries on its diagonal. From ref. [19] we see that this is only the case for $Osp(2n + 1|2n)$ and $Osp(2n - 1|2n)$. In general, series of minimal models cannot be built, and the cause of this can be traced back to eq.(5.11). It may serve as a starting point, since in case $\alpha_i^2 \neq 0$ the $l_i$ will be meaningless, i.e. we lose one parameter to describe degenerate theories. In the literature we find the term “not completely degenerate”, if there are not sufficiently many null fields present to obtain differential equations which then determine all $n$-point functions. Furthermore, we observe from the explicit examples of $Osp(3|2)$ and $D(2|1; \alpha)$ that for $\alpha_i^2 = 0$ we do not find a screening charge of the form $\frac{1}{\beta} \alpha_i$, while $\beta \alpha_i$ is always a solution. These two screening charges (if they both exist) eventually give rise to the same equation (5.12) (with the values of $l_i$ and $m_i$ exchanged), so that for $\alpha_i^2 = 0$ this kind of “pairing” does not take place. In a nutshell $\alpha_i^2 = 0$ implies no pairing and no minimal model “in $\alpha_i$-direction”. From $Osp(3|2)$ we see that we need at least one $\alpha_i^2 \neq 0$ in order to build minimal models, but then the other charges are still involved (otherwise it would reduce to a Super-Virasoro minimal model). It is interesting to note in this context that the vanishing of all diagonal elements of $K_{ij}$ leads to an energy momentum tensor which does not differ in its form from the classical one, see eq.(3.3). So one may speculate that the system is in some way “too classical” to generate minimal models, though the expressions for the conserved currents differ from the corresponding classical ones. Another approach to understand the absence of minimal models could be, that supersymmetry is too restrictive to admit minimal models. Because if one allows at the level of Toda Theories for bosonic simple roots (which spoil supersymmetry) they have in general non-zero entries on the corresponding Cartan matrices. Most probably one can still generate $\tilde{W}$-algebras (corresponding to an integrable field theory), so that these may then admit minimal models.
The above discussion leads us to the conclusion that for the minimal models of $SW(3/2, 2)$ the conformal dimensions are almost known, in the sense that they can be parameterized by eq.(5.29), but the selection rules for the allowed parameters $l_1$, $m_1$ and $m_2$ have not been clarified completely, yet. In particular the case of negative $p$ is not really understood. A closely related question is why $p$ is of the form $p = (-1)^q$, and whether it is correct to attribute the negative sign to $p$, though it is favored to by the restriction of $l_1$ to one and zero in that case.

It is important to realize, that all the results can be easily generalized to the $SW$-algebras belonging to $Osp(2n + 1|2n)$. The central charge for minimal models is $c = \frac{3}{2}r + \frac{12}{\beta^2}q^2$. The conformal dimensions of multiplets are given by $\Delta_T = -\frac{1}{2}\beta^2 \eta^2 - \eta \cdot \rho_q$. Here $\eta$ is of course again determined by eq.(5.13). The same thing applies to the fusion rules.

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Appendix A: Rules for the computation of covariant OPE’s and NOP’s.

Basic for the implementation of the package ”OPESUSY” which computes $N=1$ supersymmetric OPE’s and reduces normal ordered products to a standard form are the following rules. The notation is set by

$$A(Z_1) \circ B(Z_2) = \sum_{n=0}^{\max} \frac{[AB]_n(Z_2)}{Z_{12}^{n/2}}, \quad (A.1)$$

where the power of $Z_{12}$ was defined in eq.(2.3). The definition of the integral over Grassmann numbers is

$$\int d\theta = 0, \quad \int d\theta \theta = 1, \quad (A.2)$$

So the supersymmetric generalization of the Cauchy theorem is

$$\frac{1}{2\pi i} \oint_{C_{z_2}} dz_1 d\theta_1 \frac{\Phi(Z_1)}{Z_{12}^{n/2}} = \frac{1}{2\pi i} \oint_{C_{z_2}} dz_1 d\theta_1 Z_{12}^{-n/2} \Phi(Z_1) = \frac{1}{[n-1]!} D^{n-1} \Phi(Z_2). \quad (A.3)$$

Note the hereby defined order for a fraction, in case $\Phi$ and $Z_{12}^{n/2}$ are both fermionic. Subsequently a complete set of rules is given, meaning that it is sufficient to compute arbitrary OPE’s and reduce normal ordered products to standard order, provided some “basic” OPE’s are given

$$[BA]_q = (-1)^{AB} \sum_{n=0}^{\max} \frac{(-1)^{\frac{n+q}{2}}}{[\frac{n}{2}]!} D^n [AB]_n \quad (A.4)$$

The important special case of two identical fermionic fields is given by:

$$[AA]_0 = -\frac{1}{2} \sum_{n=1}^{\max} \frac{(-1)^n}{n!} D^{2n} [AA]_{2n} \quad (A.5)$$

The analogue to Wick’s theorem is:

$$[A[BC]_0]_q = (-1)^{AB} (-1)^{\frac{2q}{2} \Delta_B} [B[AC]_q]_0 + [[AB]_q C]_0$$

$$+ \sum_{q-n \text{ odd}}^{q-2} \frac{[\frac{q-1}{2}]!}{[\frac{2(q-n-1)}{2}]! [\frac{q-n}{2}]!} [[AB]_{n+1} C]_{q-n-1} \quad (A.6)$$

$$[A[BC]_0]_0 = (-1)^{AB} [B[AC]_0]_0 - \sum_{n=1}^{\max} \frac{(-1)^n}{n!} [D^{2n} [AB]_{2n} C]_0$$

$$[A[AC]_0]_0 = -\frac{1}{2} \sum_{n=1}^{\max} \frac{(-1)^n}{[\frac{n}{2}]!} [D^{2n} [AA]_{2n} C]_0, \text{ A fermionic.}$$
B. Solutions for the spin-2 field

Although the solution given below belongs to a linear system of equations, it took a few hours of spark2 cpu-time to obtain it. It reads

\[
\begin{align*}
U_1 &= (V_{23}(\beta^2 + \beta^2\alpha) + V_{31}(4\alpha - \beta^2\alpha^2) + V_{32}(-\beta^2 - 2\beta^2\alpha - \beta^2\alpha^2))/(4\beta\alpha(1 + \alpha)), \\
U_{11} &= (V_{23}(-1 - \alpha) + V_{31}\alpha^2 + V_{32}(1 + 2\alpha + \alpha^2))/(4\alpha(1 + \alpha)), \\
U_{112} &= (\beta V_{31}\alpha^2 + V_{23}(-\beta - \beta\alpha) + V_{32}(\beta + 2\beta\alpha + \beta\alpha^2))/(2\alpha(1 + \alpha)), \\
U_{113} &= (-\beta V_{31}\alpha^2 + V_{23}(\beta + \beta\alpha) + V_{32}(-\beta - 2\beta\alpha - \beta\alpha^2))/(2\alpha), \\
U_{12} &= (V_{23}(1 - \beta^2 + \alpha - \beta^2\alpha) + V_{31}(-2\alpha - \alpha^2 + \beta^2\alpha^2) + \\
&+ V_{32}(1 + 2\beta^2\alpha - \alpha^2 + \beta^2\alpha^2))/(\alpha(1 + \alpha)), \\
U_{123} &= (V_{23}(-\beta - \beta\alpha) + V_{31}(2\beta\alpha - \beta\alpha^2) + V_{32}(-\beta - \beta\alpha - \beta\alpha^2))/(2(1 + \alpha)), \\
U_{13} &= (V_{23}(-1 - 3\alpha - 2\alpha^2 + \beta^2(1 + 2\alpha + \alpha^2)) + \\
&+ V_{31}(2\alpha + \alpha^2 - \beta^2\alpha^2 - \beta^2\alpha^3))/(\alpha(1 + \alpha)), \\
U_2 &= (-\beta V_{31}\alpha^2 + V_{23}\beta^2(1 + \alpha) + V_{32}(-4 - \beta^2 - 4\alpha + 2\beta^2\alpha - \beta^2\alpha^2))/(4\beta(\alpha + \alpha^2)), \\
U_{212} &= (-\beta V_{31}\alpha^2 + V_{23}(\beta + \beta\alpha) + V_{32}(-\beta - 2\beta\alpha - \beta\alpha^2))/(2\alpha(1 + \alpha)), \\
U_{213} &= (\beta V_{31}\alpha^2 + V_{23}(\beta + \beta\alpha) + V_{32}(-\beta + \beta\alpha^2))/(2\alpha), \\
U_{22} &= (V_{23}(-1 - \alpha) + V_{31}\alpha^2 + V_{32}(1 + 2\alpha + \alpha^2))/(4\alpha(1 + \alpha)), \\
U_{223} &= (\beta V_{31}\alpha^2 + V_{23}(-\beta - \beta\alpha) + V_{32}(\beta + 2\beta\alpha + \beta\alpha^2))/(2(1 + \alpha)), \\
U_3 &= (-\beta^2 V_{31}\alpha^2 + V_{23}(-4 + \beta^2 - 4\alpha + \beta^2\alpha) + V_{32}\beta^2(-1 - 2\alpha - \alpha^2))/(4\alpha(1 + \alpha)), \\
U_{312} &= (-\beta V_{31}\alpha^2 + V_{23}(-\beta - 3\beta\alpha - \beta\alpha^2) + V_{32}(\beta + 2\beta\alpha + \beta\alpha^2))/(2\alpha(1 + \alpha)), \\
U_{313} &= (\beta V_{31}\alpha^2 + V_{23}(-\beta + \beta\alpha) + V_{32}(\beta + 2\beta\alpha + \beta\alpha^2))/(2\alpha), \\
U_{323} &= (-\beta V_{31}\alpha^2 + V_{23}(\beta + \beta\alpha) + V_{32}(-\beta - 2\beta\alpha - \beta\alpha^2))/(2(1 + \alpha)), \\
U_{33} &= (V_{23}(-1 - \alpha) + V_{31}\alpha^2 + V_{32}(1 + 2\alpha + \alpha^2))/(4\alpha(1 + \alpha)), \\
V_{11} &= 0, V_{12} = V_{32}/\alpha, \\
V_{13} &= - V_{23}(\beta + \beta\alpha)/(\beta\alpha), \\
V_{21} &= -\beta V_{31}/(\beta + \beta\alpha), \\
V_{22} &= 0, \\
V_{33} &= 0.
\end{align*}
\]

The solutions contain three free parameters corresponding to an admixture of the covariant derivatives of $T$ and $M$ and the yet undetermined normalization of a the new spin-2 field $U$. The condition of primarity again fixes these three free parameters up to one

\[
\begin{align*}
V_{31} &= -(1 + \alpha)(2 + 2^2\alpha) V_{23}/(2\alpha^2 + \beta^2\alpha^2), \\
V_{32} &= -(2 + \beta^2 + \beta^2\alpha) V_{23}/((2 + \beta^2)(1 + \alpha))
\end{align*}
\]
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