Renormalization and dressing in quantum field theory

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Abstract

We illustrate the mass and charge renormalization procedures in quantum field theory using, as an example, a simple model of interacting electrons and photons. It is shown how addition of infinite renormalization counterterms to the Hamiltonian helps to obtain finite and accurate results for the $S$-matrix. In order to remove the ultraviolet divergences from the Hamiltonian, we apply the Greenberg-Schueber “dressing transformation” and the Glazek-Wilson “similarity renormalization”. The resulting “dressed particle” Hamiltonian is finite in all orders of the perturbation theory and yields accurate $S$-matrix and bound state energies. The bare and virtual particles are removed from the theory, and physical dressed particles interact via direct action-at-a-distance.

1 Introduction

Consistent unification of relativity and quantum mechanics remains an unsolved theoretical problem in spite of many efforts applied to its solution in the 20th century. The fundamental difference between relativistic and non-relativistic physics follows from the famous Einstein’s formula $E = mc^2$. This formula, in particular, implies that if a system of particles has sufficient energy $E$ of their relative motion, then this energy may be converted to the mass $m$ of newly created particles. Generally, there is no limit on how many particles can be created in collisions, so any realistic quantum mechanical description of high-energy systems should involve states with any number of particles from zero to infinity. The number of particles is not conserved during time evolution. The most familiar example of such a behavior is the emission and absorption of light (photons) in electrodynamics.
First attempts to describe relativistic quantum systems were undertaken immediately after creation of the formalism of quantum mechanics in 1920’s. A quantum theory of the electromagnetic field was constructed by quantization of the classical Maxwell electrodynamics. In lowest perturbation orders, this theory agreed well with experimental results. However, perturbative calculations for the $S$-matrix did not work in higher orders, in particular, due to ultraviolet divergences.

The way to calculate the $S$-matrix in QFT accurately in all orders was provided by the renormalized QED formulated by Feynman, Schwinger, and Tomonaga in the late 1940’s. However this approach created a host of other problems. According to the prevailing interpretation, the creation and annihilation operators present in the Hamiltonian and Lagrangian of QED correspond to bare particles having infinite masses and charges. However, the bare particles have never been directly observed in experiments. They are believed to be surrounded by clouds of virtual photons and electron-positron pairs, thus forming complex objects called dressed particles. The dressed particles are supposed to be the eigenstates of the full Hamiltonian. They have finite experimentally observable masses and charges. The problem is that the bare particle Hamiltonian of QFT is formally infinite. Although, these infinities cancel out when the $S$-matrix is calculated, the Hamiltonian is useless if one wants to calculate the time evolution or to find wavefunctions of bound states via diagonalization procedure.

Two lines of research were initiated to cope with these problems. The dressed particle approach was suggested by Greenberg and Schweber [1]. Their goal was to get rid of the bare particles and to express the entire formalism of QFT through observable dressed particles only. The operators for dressed particles were sought as unitary transforms of bare particle operators. From another direction, Glazek and Wilson [2] introduced the similarity renormalization formalism. The idea was to apply a unitary transformation to the Hamiltonian in order to ensure that interaction potentials rapidly fall off as functions of energy differences, and to guarantee that all loop integrals are convergent. The similarity and dressing transformations are strikingly similar. They can be combined into one unitary similarity-dressing transformation which achieves two goals at once: the theory can be expressed in terms of real dressed particles and the Hamiltonian can be made finite in all perturbation orders. At the same time, the accurate and well-tested $S$-matrix of the renormalized theory remains intact. This

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1Infinite quantities appear frequently in the renormalization theory. In order to avoid them in practical calculations, one often uses regularization, e.g., cutting off all integrals at large integration momenta. In a regularized theory all quantities become finite. However, the regularization is an artificial trick, and in order to get rigorous and accurate results one should take the infinite limit of the cutoff momentum. As will be discussed later in the paper, the goal of renormalization is to introduce certain cancellations between (finite) regularized quantities, so that when the integration cutoff is lifted the physically relevant parameters have well defined finite limits. Without indicating this explicitly in calculations, we will always assume that the regularization and the taking the cutoff to infinity steps were properly executed. A quantity will be called finite (infinite) if it has (does not have) a finite value in the limit of infinite cutoff.
A combined approach was developed in refs. [3, 4, 5] and dubbed the relativistic quantum dynamics, or RQD.

In this paper we will discuss the reasons why the renormalization difficulties (e.g., the absence of a well-defined Hamiltonian and unsatisfactory treatment of the time evolution) persist in current relativistic quantum field theories. To avoid unnecessary mathematical complications, we will be working with a simple model theory which, nevertheless, shares some important features with QED. In section 9 we will explain how the similarity-dressing transformation of this model theory leads to a well defined (perturbatively) finite Hamiltonian that can be used for all kinds of quantum mechanical calculations ($S$-matrix, bound states, time evolution, etc.) without the need for renormalization. In addition, the RQD formalism does not use the dubious notions of bare and virtual particles.

## 2 Bound states, time evolution, and scattering

In this paper we will be concerned with three types of phenomena that are normally studied in physical experiments: bound states, time evolution, and scattering. These three areas account for the most of experimental information available about fundamental particles and their interactions. The key theoretical quantity involved in quantum mechanical description of these phenomena is the Hamilton operator $H$. The energies $E_n$ and state vectors $|\Psi\rangle_n$ of bound states can be found as eigenvalues and eigenvectors of the Hamiltonian

$$H|\Psi\rangle_n = E_n|\Psi\rangle_n$$  \hspace{1cm} (1)

The development of the state vector $|\Psi\rangle$ from time $t'$ to time $t$ is described by the time evolution operator $\exp(iHt)$

$$|\Psi(t)\rangle = e^{iH(t-t')}|\Psi(t')\rangle$$  \hspace{1cm} (2)

If the eigenvalues $E_n$ and eigenvectors $|\Psi\rangle_n$ of the Hamiltonian are known and the initial state is represented as a sum (and/or integral) over the basis states

$$|\Psi(t')\rangle = \sum_n C_n(t')|\Psi\rangle_n$$  \hspace{1cm} (3)

then the time evolution can be calculated as

$$|\Psi(t)\rangle = \sum_n C_n(t')e^{iE_n(t-t')}|\Psi\rangle_n$$  \hspace{1cm} (4)
Unfortunately, in most cases, the full spectrum of the Hamiltonian is not known, and the time evolution is difficult to predict.

This difficulty, however, is not that disappointing, because experimental observations of the time evolution in subatomic world are even more difficult than calculations. Most experiments in high energy physics are performed by preparing free particles or their bound states (like hydrogen atoms or deuterons), bringing them into collision, and studying the properties of free particles or bound states leaving the region of collision. In these experiments, it is not possible to observe the time evolution during interaction: particle reactions occur almost instantaneously and one can only register the reactants and products which move freely before and after the collision. This gives a lucky break for theoreticians: In such situations the theory is not required to describe the detailed dynamics of particles during the short interval of collision. It is sufficient to provide a mapping of free states before interaction onto the free states after the interaction. To describe scattering experiments, one needs only the formula for the time evolution from the remote past $t' \ll 0$ to the distant future $t \gg 0$

$$e^{iH(t-t')} = e^{iH_0 t} S e^{-iH_0 t'}$$

where the $S$-operator is defined by

$$S = \lim_{t \to \infty} \lim_{t' \to -\infty} e^{-iH_0 t} e^{iH(t-t')} e^{iH_0 t'}$$

One can read the right hand side of eq. (6) from right to left as a sequence of three steps: (i) the non-interaction evolution of the system from time $t'$ in the past to 0; (ii) the sudden jump at $t = 0$ described by the $S$-operator; (iii) the free evolution from $t = 0$ to the future time $t$.

The most effective technique available for calculations of the $S$-operator is the perturbation theory which can be written in many equivalent forms.\(^2\) The Dyson time-ordered expansion provides the most economical expressions that can be encoded in familiar Feynman diagrams. However, for the discussion in this paper, we found more useful two other perturbative expressions which differ from the Dyson’s formula only by re-shuffling the terms. The “old-fashioned” formula for the $S$-operator is

$$S = 1 + i \int_{-\infty}^{+\infty} V(t) \, dt - \int_{-\infty}^{+\infty} V(t) \, dt \int_{-\infty}^{t} V(t') \, dt' + \ldots$$

where $V$ is the interaction part of the total Hamiltonian $H = H_0 + V$.

\(^2\)In this paper we are not discussing the complicated issue of the convergence of perturbative expansions. We will assume that all perturbative series do converge.
\[ V(t) = e^{-iH_0 t} V e^{iH_0 t} e^{-\epsilon |t|} \] (8)

and the factor \( e^{-\epsilon |t|} \) in the limit \( \epsilon \to 0 \) serves for adiabatic switching the interaction on and off. Operators with \( t \)-dependence determined by the free Hamiltonian \( H_0 \) as in eq. (8) will be called \textit{regular}. Using convenient symbols for \( t \)-integrals

\[
Y(t) \equiv \int_{-\infty}^{t} Y(t') dt'
\]

\[
\bar{Y}(t) \equiv \int_{-\infty}^{+\infty} Y(t') dt'
\]

formula (7) can be written compactly as

\[
S = 1 + \Sigma(t)
\]

where

\[
\Sigma(t) = iV(t) - V(t)\bar{V}(t) - iV(t)\bar{V}(t)V(t) + V(t)V(t)V(t)V(t) + \ldots
\]

(10)

Another equivalent expression for \( S \) was suggested by Magnus [6]

\[
S = e^{iF(t)}
\]

(11)

Here the Hermitian operator \( F(t) \) can be represented as a series of multiple commutators with \( t \)-integrations

\[
F(t) = V(t) - \frac{i}{2}[V(t), V(t)] - \frac{1}{6}[V(t), [V(t), V(t)]] - \frac{1}{6}[[V(t), V(t)], V(t)]
\]

+ \( \frac{i}{12} [V(t), [[V(t), V(t)], V(t)]] \)

+ \( \frac{i}{12} [[V(t), [V(t), V(t)]], V(t)] + \frac{i}{12}([[V(t), V(t)], [V(t), V(t)]]] + \ldots \)

(12)

One important advantage of this representation is that the \( S \)-operator (11) is manifestly unitary. It follows from equations (9) and (11) that operators \( \Sigma(t) \) and \( F(t) \) are related to each other.
\[ F(t) = -i \frac{d}{dt} \log(1 + \Sigma(t)) \]  

so finding \( F(t) \) or \( \Sigma(t) \) are equivalent tasks.

The \( S \)-operator and the Hamiltonian provide two different ways to describe dynamics. The \( S \)-operator represents only “integrated” time evolution from the remote past to the distant future. The knowledge of the \( S \)-operator is sufficient to calculate the scattering cross-sections as well as energies and lifetimes of stable and metastable bound states.\(^3\) However, in order to describe the time evolution and the wavefunctions of bound states the full interacting Hamiltonian \( H \) is required.

It can be shown \(^7\) that two Hamiltonians \( H \) and \( H' \) related to each other by a unitary transformation \( e^{i \Phi} \)

\[ H' = e^{i \Phi} H e^{-i \Phi} \]

yield the same scattering \( S' = S \) as long as condition

\[ \lim_{t \to \pm \infty} e^{-i H_0 t} \Phi e^{i H_0 t} = 0 \]  

is satisfied. Such Hamiltonians \( H \) and \( H' \) are called scattering-equivalent. The energy spectra of two scattering equivalent Hamiltonians are identical. However, the eigenvectors are different and, according to eq. (11), the corresponding descriptions of dynamics are different as well. Therefore scattering-equivalent theories may be not physically equivalent.

Calculations of bound states, time evolution and scattering is a routine practice in non-relativistic quantum mechanics. However, the situation is less certain in relativistic quantum field theories. As mentioned in Introduction, the Hamiltonian of renormalized QFT is infinite. Therefore, it is not immediately clear if the above formulas (1) - (13) remain valid for the high energy relativistic phenomena, and what modifications, if any, should be introduced in quantum theory to take into account the variable number of particles. In this paper we illustrate the difficulties encountered in quantum field theories by analyzing a simple model theory with variable number of particles. We will demonstrate that the Hamiltonian can be redefined so that there is no need for renormalization and usual quantum mechanical techniques remain applicable even in the relativistic case.

\(^3\)The latter two quantities are represented by positions of poles of the \( S \)-operator on the complex energy plane.
3 Model theory

Our model theory describes two kinds of particles. These are massive spinless fermions which will be called *electrons* and massless bosons with zero helicity, which will be called *photons*. Here we disregard the spin and polarization degrees of freedom as they are not so important for the discussion of renormalization. To allow for creation and annihilation of particles, the system is described in the Fock space which is built as a direct sum of sectors with various numbers of particles. For example, if we denote $|0\rangle$ the no-particle vacuum state, $\mathcal{H}_{el}$ the one-electron Hilbert space and $\mathcal{H}_{ph}$ the one-photon Hilbert space, then the Fock space can be written as an infinite direct sum

$$\mathcal{H} = |0\rangle \oplus \mathcal{H}_{el} \oplus \mathcal{H}_{ph} \oplus (\mathcal{H}_{el} \otimes \mathcal{H}_{ph}) \oplus (\mathcal{H}_{el} \otimes_{asym} \mathcal{H}_{el}) \oplus (\mathcal{H}_{ph} \otimes_{sym} \mathcal{H}_{ph}) \ldots$$ (15)

The anticommutation and commutation relations for particle creation and annihilation operators ($a_p^\dagger$, $a_p$ for electrons and $c_k^\dagger$, $c_k$ for photons, respectively) are, as usual,

$$\{a_p, a_{p'}^\dagger\} = \delta(p - p')$$ (16)

$$[c_k, c_{k'}^\dagger] = \delta(p - p')$$ (17)

$$\{a_p, a_{p'}\} = \{a_{p'}^\dagger, a_p^\dagger\} = 0$$ (18)

$$[c_k, c_{k'}] = [c_{k'}^\dagger, c_k^\dagger] = 0$$ (19)

$$[a_{p'}^\dagger, c_k] = [a_p^\dagger, c_k] = [a_p, c_{k'}] = [a_p^\dagger, c_{k'}] = 0.$$ (20)

The full Hamiltonian $H = H_0 + V_1$ is the sum of the free Hamiltonian

$$H_0 = \int dp \omega_p a_p^\dagger a_p + \int dk |k| c_k^\dagger c_k$$ (20)

(\text{where } \omega_p = \sqrt{p^2 + m^2} \text{ and } |k| \text{ are one-particle energies of electrons and photons, respectively}) \text{ and the interaction, which we choose in the following form}

$$V_1 = e(2\pi)^{-3/2} \int_{k \neq 0} \frac{dpdk}{\sqrt{|k|}} a_{p+k}^\dagger a_p c_k + e(2\pi)^{-3/2} \int_{k \neq 0} \frac{dpdk}{\sqrt{|k|}} a_p^\dagger a_{p-k} c_k$$ (21)
The coupling constant $e$ is the absolute value of the electron charge. Here and in what follows the *perturbation order* of an operator (= the power of the coupling constant $e$) is shown by the subscript. For example, the free Hamiltonian $H_0$ does not depend on $e$, so it is of zero perturbation order; $V_1$ is of the first perturbation order, etc. The number of electrons is conserved by the interaction (21), but the number of photons is not conserved. So, this theory is capable of describing important processes of the emission and absorption of photons.

In this paper we will discuss ultraviolet divergences associated with the interaction (21). However, we will skip completely the discussion of “infrared divergences” which are related to the zero mass of photons and singularities $|k|^{-1/2}$ in (21). The easiest way to avoid infrared problems in practical calculations is to assign a small non-zero mass to photons.

In section 6 we are going to calculate the scattering operator (9) with the above Hamiltonian. Before doing that, some remarks are in order. The operator $S$ is obtained as a sum of products (10) (or commutators (12)) of interactions $V_1(t) = e^{-iH_0t}V_1e^{iH_0t}$ with $t$-integrations. Each term in these expressions can be written as a normally ordered product of $N$ creation operators $\alpha^\dagger$ and $M$ annihilation operators $\alpha$.\footnote{Here symbols $\alpha^\dagger$ and $\alpha$ refer to generic creation and annihilation operators without specifying the type of the particle. The pair of integers $(N, M)$ will be referred to as the index of the term $V_{NM}$.}

$$V_{NM}(t) = \int [dq]D_{NM}[q]e^{itE_{NM}[q]}\delta(P_{NM}[q])\alpha^\dagger_{q_1'}\ldots\alpha^\dagger_{q_N'}\alpha_{q_1}\ldots\alpha_{q_M} \quad (22)$$

where integration is carried over momenta of all created and annihilated particles $[dq]$. The momentum conservation law is guaranteed by the delta function in (22) whose argument is the sum of momenta of created particles minus the sum of momenta of annihilated particles

$$P_{NM}(q_1', \ldots, q_N', q_1, \ldots, q_M) \equiv \sum_{i=1}^N q_i' - \sum_{j=1}^M q_j \quad (23)$$

Usually, we will perform explicit integration over one momentum $q_i'$ which removes the delta function and expresses $q_i'$ as a linear function of other momenta. The argument of the exponent in (22) contains the *energy function*

$$E_{NM}(q_1', \ldots, q_N', q_1, \ldots, q_M) \equiv \sum_{i=1}^N \omega q_i' - \sum_{j=1}^M \omega q_j \quad (24)$$

which is the difference of energies of particles created and annihilated by $V_{NM}$. $D_{NM}$ is a numerical *coefficient function*. 

8
Suppose that a term $V_{NM}$ has coefficient function $D_{NM}$, then we introduce a useful notation $V_{NM} \circ \zeta$ for the operator whose coefficient function $D'_{NM}$ is a product of $D_{NM}$ and a function $\zeta$ of the same arguments.

$$D'_{NM}[q] = D_{NM}[q]\zeta[q]$$

Then, a $t$-dependent regular term $V_{NM}(t)$ can be written as

$$V_{NM}(t) = e^{-iH_{0}t}V_{NM}e^{iH_{0}t} = V_{NM} \circ e^{-iE_{NM}t}$$

and its definite $t$-integral is

$$\tilde{V}_{NM}(t) \equiv \int_{-\infty}^{\infty} V_{NM}(t)dt = 2\pi V_{NM} \circ \delta(E_{NM})$$

Eq. (25) means that each term in $\tilde{V}_{NM}(t)$ is non-zero only on the hypersurface of solutions of the equation

$$E_{NM}(q'_{1}, \ldots, q'_{N}, q_{1}, \ldots, q_{M}) = 0$$

if such solutions exist. This hypersurface in the momentum space is called the energy shell of the term $V_{NM}(t)$. Note that the scattering operator (9) is non-trivial only on the energy shell, i.e., where the energy conservation condition holds.

### 4 Three types of operators in the Fock space.

In this section we would like to get a further insight into the nature of operators in the Fock space by dividing them into three groups depending on their index $(N, M)$. We will call these types of operators renorm, phys, and unphys.

**Renorm operators** have either index $(0,0)$ (a numerical constant $C$) or index $(1,1)$ in which case the same type of particle is created and annihilated. The most general form of a renorm operator in our theory is\(^5\)

\[^5\]Here we write just the operator structure of $R$ omitting all numerical factors, indices, integration and summation signs. Note also that the interaction operator (21) commutes with the operator of charge $Q = -e \int dp a_{p}^{\dagger}a_{p}$, so any product or commutator of terms derived from $V_{1}$ should also commute with $Q$. This is not true for terms like $a_{1}^{\dagger}c$ and $c^{\dagger}a$, so they are not allowed in the theory.
The free Hamiltonian \( H_0 \) is an example of a renorm operator. The class of renorm operators is characterized by the property that the energy function \( E \) is identically zero. So, renorm operators always have an non-empty energy shell where they do not vanish. Regular renorm operators do not depend on \( t \).

**Phys operators** have at least two creation operators and at least two destruction operators (index \((N, M)\) with \( N \geq 2 \) and \( M \geq 2 \)). In this case the energy shell is non-empty. For example, the energy shell for phys operator \( a_{\mathbf{p}+\mathbf{k}}^\dagger a_{\mathbf{q}-\mathbf{k}}^\dagger a_{\mathbf{p}} a_{\mathbf{q}} \) is given by the set of solutions of equation \( \omega_{\mathbf{p}+\mathbf{k}} + \omega_{\mathbf{q}-\mathbf{k}} = \omega_{\mathbf{p}} + \omega_{\mathbf{q}} \) which is not empty.

**Unphys operators** have index \((1, N \geq 2)\) or \((N \geq 2, 1)\). It can be shown that the energy shell is empty for unphys operators. For example, the interactions \( \omega_{\mathbf{p}} + |\mathbf{k}| = \omega_{\mathbf{p}+\mathbf{k}} \) does not have a solution, because there are no photon states with zero momentum.

Renorm, phys, and unphys operators exhaust all possibilities in our theory, therefore any regular operator \( V \) must have a unique decomposition

\[
V(t) = V^{\text{ren}}(t) + V^{\text{unp}}(t) + V^{\text{ph}}(t)
\]

The rules for calculations of commutators, derivatives and \( t \)-integrals with different operator types are summarized in Table 9.2. For example, the \( t \)-integrals of phys and unphys operators are given by formula

\[
\frac{d}{dt} V(t) \quad = \quad V(t) \circ \frac{i}{E_V}
\]

Table 1: Operations with regular operators in the Fock space. (Notation: P=phys, U=unphys, R=renorm, NR=non-regular.)

| Type of operator | \([A, P]\) | \([A, U]\) | \([A, R]\) | \( \frac{dA}{dt} \) | \( A \) | \( A \) |
|----------------|---------|---------|---------|----------|-----|-----|
| P              | P       | P+U     | P       | P        | P   | P   |
| U              | P+U     | P+U+R   | U       | U        | U   | 0   |
| R              | P       | U       | R       | 0        | NR  | \( \infty \) |
Figure 1: Diagram representation of two terms in the interaction operator $V_1$ (eq. (21)).

5 Diagrams in the model theory.

Our goal in this section is to introduce the diagram technique which greatly facilitates perturbative calculations of scattering operators (10) and (12). Let us graphically represent each term in the interaction operator (21) as a vertex (see Fig. 1). Each particle operator in (21) is represented as an oriented line or arrow. The line corresponding to the annihilation operator enters the vertex, and the line corresponding to the creation operator leaves the vertex. Electron lines are shown by full arcs and photon lines are shown by broken arrows. Each line is marked by the momentum label of the corresponding particle operator. Free ends of the electron lines are attached to the vertical “order bar” on the left hand side of the diagram. The order of these external lines (from bottom to top of the diagram) corresponds to the order of particle operators in the interaction term (from right to left). An additional numerical factor is indicated in the upper left corner of the diagram.

The $t$-integral $V_1(t)$ differs from $V_1(t)$ only by the factor $iE_{V_1}$ (see eq. (27)) which is represented in the diagram by drawing a box that crosses all external lines. A line entering (leaving) the box contributes its energy with the negative (positive) sign to the energy function $E_{V_1}$. The diagram representation of the integral

$$V_1(t) = \frac{ie}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}d\mathbf{k}}{\sqrt{|\mathbf{k}| \omega_p + |\mathbf{k}| - \omega_{p+k}}} \alpha^+_p \alpha^+_k \alpha_{p+k}$$

$$+ \frac{ie}{(2\pi)^{3/2}} \int \frac{d\mathbf{p}d\mathbf{k}}{\sqrt{|\mathbf{k}| \omega_p - |\mathbf{k}| - \omega_{p-k}}} \alpha^+_p \alpha_{p-k} \alpha_k$$

(28)

is shown in Fig. 2.
The product of two operators $AB$ is represented by simply placing diagram $B$ below diagram $A$ and attaching the external electron lines of both diagrams to the same order bar. For example, the diagram for the product of the second term in (21) (Fig. 1(b)) and the first term in (28) (Fig. 2(a))

\[ V_1 V_1 \propto (a_p^\dagger a_p - k c_k)(a_q^\dagger c_{k'}^\dagger a_{q+k'}) + \ldots \]  

is shown in Fig. 3(a). This product should be further converted to the normal form, i.e., all incoming lines should be positioned below the outgoing lines. Due to the relations (16) - (19) each exchange of positions of the electron particle operators (full external lines on the diagram) changes the total sign of the expression. Each permutation of annihilation and creation operators (incoming and outgoing lines) of similar particles creates an additional expression and a new diagram in which the swapped lines are joined together. Using these rules we first move the photon operators in (29) to the rightmost positions, move the operator $a_q^\dagger$ to the leftmost position, and add another term due to the anticommutator \( \{a_{p-k}, a_q^\dagger\} = \delta(q - p + k) \).

\[ V_1 V_1 \propto a_{p-k}^\dagger a_p^\dagger a_p a_{q+k} c_k c_{k'}^\dagger + \delta(q - p + k) a_p^\dagger a_{q+k} c_k c_{k'}^\dagger \]

\[ = a_q^\dagger a_{p-k}^\dagger a_p a_{q+k} c_k c_{k'}^\dagger + a_p^\dagger a_{p-k+k'} c_k c_{k'}^\dagger + \ldots \]  

This expression is represented by two diagrams 3(b) and 3(c). In the diagram 3(b) the electron line marked $q$ has been moved to the top of the order bar. In the diagram 3(c) the product $\delta(q - p + k)$ and the integration by $q$ are represented by merging or \textit{pairing} the incoming electron line carrying momentum $p - k$ with the outgoing
According to equation (17), the normal ordering of photon operators in 3(b) yields diagrams 3(d) and 3(e).\footnote{We relabel the external momenta in Fig. 3(e) for future convenience.} Diagrams 3(f) and 3(g) are obtained from 3(c) in a similar manner.

Using diagrams, with some practice, one can perform calculations of scattering operators (10) and (12) much easier than in the usual algebraic way. During these
diagram manipulations we, actually, do not need to keep momentum labels of lines. The algebraic expression of the result can be easily restored from an unlabeled diagram by following these steps:

(I) Assign a distinct momentum label to each external line, except one, whose momentum is obtained from the momentum conservation condition.

(II) Assign momentum labels to internal lines so that the momentum conservation law is satisfied at each vertex. If there are loops, one needs to introduce additional independent loop momenta.\(^7\)

(III) Read external lines from top to bottom of the order bar and write corresponding particle operators from left to right.

(IV) For each box, write a factor \(i(E_f - E_i)^{-1}\), where \(E_f\) is the sum of energies of particles going out of the box and \(E_i\) is the sum of energies of particles coming into the box.

(V) Write a factor \(e^{-iE_Yt}\), where \(E_Y\) is the energy function of the diagram which is the sum of energies of all outgoing external lines minus the sum of energies of all incoming external lines.

(VI) For each vertex introduce a factor \(\frac{e}{\sqrt{(2\pi)^3|k|}}\), where \(k\) is the momentum of the photon line attached to this vertex.

(VII) Integrate the obtained expression by all independent external and loop momenta.

6 Electron-electron scattering.

Let us now try to extract some physical information from the above theory. We will calculate low order terms in the perturbation expansion (10) for the \(S\)-operator.

\[
\Sigma_1(t) = iV_1(t) \\
\Sigma_2(t) = -(V_1(t)V_1(t))^{unp} - (V_1(t)V_1(t))^{ph} - (V_1(t)V_1(t))^{ren}
\]

To obtain the corresponding contributions to the \(S\)-operator we need to take \(t\)-integrals of these expressions

\[
S = 1 + \Sigma_1(t) + \Sigma_2(t) + \ldots
\]

\(^7\)see diagram 3(g) in which \(k\) is the loop momentum.
Note that the right hand side of (31) and the first term on the right hand side of (32) are unphys. Their energy shell is empty, so, according to Table 1, they do not contribute to the $S$-operator.

Operator $(V_1 V_1)^{ph}$ on the right hand side of eq. (32) has two terms corresponding to two types of scattering processes allowed in the 2nd perturbation order. The term of the type $a^\dagger c^\dagger ac$ (see, e.g., fig. 3(f)) annihilates an electron and a photon in the initial state and recreates them (with different momenta) in the final state. So, this term describes the electron-photon (Compton) scattering. In this paper we will focus on the other term which describes the electron-electron scattering. Let us consider in more detail the second-order contribution to this process (see fig. 3(e))

$$S_2[a^\dagger a^\dagger aa] = 2\pi \int dp dq dk \delta(\omega_{p-k} + \omega_{q+k} - \omega_q - \omega_p)D_2(p, q, k)a^\dagger_{p-k}a^\dagger_{q+k}a_qa_p$$

The coefficient function in (33) can be read from the diagram, according to the rules (I) - (VII),

$$D_2(p, q, k) = \frac{i e^2}{(2\pi)^3 |k| (|k| + \omega_{p-k} - \omega_p)}$$

In the non-relativistic approximation ($p, q, k \ll mc$), the coefficient function $D_2(p, q, k)$ has singularity $|k|^{-2}$ which is characteristic for scattering of two electrons interacting via repulsive Coulomb potential

$$\frac{e^2}{4\pi |r_1 - r_2|}.$$  

So, our model theory is quite realistic.

7 Mass renormalization.

Next consider the third term on the right hand side of eq. (32). It is given by the diagram in fig. 3(g). According to rules (I) - (VII) this diagram is represented by the expression

$$(V_1(t) V_1(t))^{ren} = \frac{i e^2}{(2\pi)^3} \int dp dk \frac{a_{p}^\dagger a_{p}}{(\omega_{p-k} - \omega_{p} + k)k}$$  

8 We do not need to consider unconnected diagrams, like fig. 3(d), because they describe two or more disjoint scattering processes.
There are serious problems with this term. First, the loop integral by $k$ is divergent because the integrand in (34) has asymptotic behavior $\propto k^{-2}$ at large $k$. However, even if the integral were convergent, the presence of a renorm contribution in the operator $\Sigma(t)$ is unacceptable, because, according to Table 1, the $t$-integral of any renorm term is infinite. So, the scattering phase in the second order $\Sigma_2$ is infinite, which is absurd.

Moreover, if we continued calculations (31) - (32) to higher perturbation orders we would find out that even phys terms in $\Sigma(t)$ become infinite due to divergent loop integrals, so the theory with the Hamiltonian (20) - (21) is seriously flawed.

In a consistent theory we must require that

$$\Sigma^{ren} = 0$$

(35)

Therefore, operator $\Sigma$ must be purely phys

$$\Sigma = \Sigma^{ph}$$

(36)

It was shown in ref. [1] that condition (36) is equivalent to requiring that the $S$-operator leaves the vacuum and one-particle states invariant

$$S|0\rangle = |0\rangle$$

(37)

$$Sa_p^\dagger|0\rangle = a_p^\dagger|0\rangle$$

(38)

$$Sc_k^\dagger|0\rangle = c_k^\dagger|0\rangle$$

(39)

This is the mass renormalization condition of the traditional renormalization theory. To satisfy this condition, we must modify the Hamiltonian (20) - (21) by adding certain unphys $U$ and renorm $R$ counterterms to the interaction operator $V_1$. In other words, we are saying that the original Hamiltonian

$$H = H_0 + V_1$$

(40)

is not correct, and the modified Hamiltonian with renormalization counterterms

$$H^c = H_0 + V^c = H_0 + V_1 + U + R$$

(41)

\footnote{It appears that in our model theory the renormalization is achieved by unphys and renorm counterterms only. In the general case, e.g., in QED, phyс counterterms should be added as well.}
better describes interactions between particles. To comply with eq. (36), we must choose the counterterms in such a way that operator

$$\Sigma^c(t) = iV^c(t) - V^c(t)V^c(t) + \ldots$$

(42)
does not contain renorm terms. From Table 1, it is clear that renorm terms in $\Sigma^c(t)$ may appear due to the presence of renorm and unphys terms in $V^c$ and their products. So, in order to satisfy eq. (35), there should be such a balance between unphysical and renorm terms in $V^c(t)$ that all renorm terms in $\Sigma^c(t)$ cancel out in all orders of the perturbation theory. The mass renormalization is achieved by adding renorm counterterms in even orders: $R_2, R_4$, etc. The charge renormalization procedure will be discussed in section 8. It requires addition of unphys counterterms in odd orders $U_3(t), U_5(t)$, etc. We take these considerations into account by writing the general expression for the Hamiltonian of the renormalized theory

$$H^c(t) = H_0 + V^c(t)$$

where

$$V^c(t) = V_1(t) + R_2 + U_3(t) + R_4 + \ldots$$

(43)

We obtain formulas for $\Sigma^c_i(t)$ ($i = 1, 2, 3, \ldots$) by inserting interaction (43) in (10) and collecting terms of equal order.

$$\Sigma^c_1(t) = iV_1(t)$$

(44)

$$\Sigma^c_2(t) = -V_1(t)V_1(t) + iR_2$$

(45)

$$\Sigma^c_3(t) = -iV_1(t)V_1(t)V_1(t) - R_2V_1(t) - V_1(t)R_2 + iU_3(t)$$

(46)

$$\Sigma^c_4(t) = \sigma_4(t) - U_3(t)V_1(t) - V_1(t)U_3(t) + iR_4$$

(47)

where we denoted

$$\sigma_4(t) = V_1(t)V_1(t)V_1(t)V_1(t) + iV_1(t)V_1(t)R_2$$

$$- iV_1(t)R_2V_1(t) - iR_2V_1(t)V_1(t),$$

10 Of course, we are looking for a $t$-independent Hamiltonian $H^c$. Although, at intermediate calculation steps it is convenient to keep all operators $t$-dependent, as in eq. (8), in the end we should set $t = 0.$
Now we go order-by-order and choose counterterms $R_2, R_4, \ldots$ so that renorm terms are eliminated from the left hand sides of eqs (44) - (47). The first-order term (44) is unphys, so there is no need for renormalization in the first order. To ensure that $\Sigma_2^c(t)$ does not have a renorm part we choose the counterterm

$$R_2 = -i(V_1(t)V_1(t))^{ren}$$

(see diagram 3(g)). With this choice, we can rewrite the contributions to the $S$-operator in the 4 lowest orders

$$\Sigma_1^c(t) = 0$$
$$\Sigma_2^c(t) = -(V_1(t)V_1(t))^{ph}$$
$$\Sigma_3^c(t) = -iV_1(t)(V_1(t)V_1(t))^{p+u}$$
$$\Sigma_4^c(t) = \sigma_4(t) - U_3(t)V_1(t) - V_1(t)U_3(t) + iR_4,$$

where

$$\sigma_4(t) = V_1(t)V_1(t)(V_1(t)V_1(t))^{p+u} + V_1(t)V_1(t)(V_1(t)V_1(t))^{ren}$$
$$- iV_1(t)R_2V_1(t) - iR_2V_1(t)V_1(t) - iV_1(t)V_1(t)R_2$$
$$= V_1(t)V_1(t)(V_1(t)V_1(t))^{p+u} - iV_1(t)R_2V_1(t)$$
$$- iR_2V_1(t)V_1(t)$$

and the superscript $p + u$ denotes the sum of phys and unphys terms. The term on the right hand side of (48) has odd number of particle operators, hence it is free of renorm parts (which have either zero or two particle operators). Therefore, no renorm counterterms should be added there. Just as we did in the 2nd order, we can choose a renorm counterterm $R_4$ which simply cancels all renorm terms which may be present in the first three terms on the right hand side of (49). With the above choices, the operator $\Sigma^c(t)$ does not contain renorm terms up to the 4th order, as required by the mass renormalization condition,\(^{11}\) and the expression for $\Sigma_4^c$ simplifies

\(^{11}\)Our analysis demonstrates that that mass renormalization is always necessary when interaction
\[
\Sigma_i^c(t) = -((V_1(t)V_1(t))^{p+u}(V_1(t)V_1(t))^{p+u})^{ph}
\]
\[
+ (V_1(t)V_1(t))^{p+u}(V_1(t)V_1(t))^{ph} - (U_3(t)V_1(t))^{ph} - (V_1(t)U_3(t))^{ph}
\]

(51)

Using the diagram technique we find that the \(a^\dagger a^\dagger aa\) part of the first two terms on the right hand side of eq. (51) is represented by 7 diagrams shown in Fig. 4. From diagram rules (I) - (VII), we obtain the contribution of diagrams 4(a) - 4(b) to the coefficient function \(D\) for the electron-electron scattering on the energy shell.

\[
D_4^{(a-b)}(p, q, k) = \frac{-ie^4}{(2\pi)^6(\omega_{p-k} - \omega_p + k)k} \int \frac{dh}{h} \left( \frac{1}{BC} + \frac{1}{EF} \right)
\]

(52)

where \(B = \omega_{p-h} - \omega_p + h\), \(C = \omega_{p-k} - \omega_{p-h-k} - h\), \(E = \omega_{q-h} - \omega_q + h\), and \(F = \omega_{q+k} - \omega_{q+k-h} - h\). Unfortunately, this contribution is infinite (at large values of \(h\) the integrand behaves as \(h^{-3}\)). Thus we conclude that the mass renormalization procedure described above has not removed all divergences. To solve this problem we need to perform the second renormalization step known as the charge renormalization procedure. This step is explained in the next section.

8 Charge renormalization

First note that the divergent terms (52) have a singularity \(k^{-2}\) at \(k \to 0\). As we know from section 6 such a singularity is responsible for the low-energy electron-electron scattering at large distances. From classical physics we also know that long-distance interactions between charged particles depend on \(e^2\) (in our language, they are of the second perturbation order) and they are accurately described by the 2nd order term (33). Non-zero terms \(D_4^{(a-b)}\) mean that the charge of the electron is modified by the interaction. Actually, this modification is infinite, but even finite values of the terms like (52) are inconsistent with the classical limit. So, we will postulate that in orders higher that 2nd, singular coefficient functions like (52) should not be present at all, whether they are infinite or finite. This is the charge renormalization condition. To eliminate the divergent contribution (52) we can choose unphys 3rd order counterterms contains unphys terms, like trilinear operators in eq. (21) and in more realistic theories, such as QED, QCD, and Standard Model. Products or commutators of such terms in (10) or (12) give rise to renorm terms in the \(S\)-operator which should be compensated by adding renorm terms to the Hamiltonian. The operator structure of these terms is the same as in \(H_0\) (20). This results in the difference of masses of bare and dressed particles, as discussed at the end of section 8.
Figure 4: Contributions with the operator structure $a^\dagger a^\dagger aa$ to the first two terms on the right hand side of eq. (51).
Figure 5: Charge renormalization counterterms $U_3(t)$.

$U_3$. Two diagrams describing these counterterms are shown in Fig. 5(a) and 5(b). As expected, they are infinite, and expressions

$$-U_3(t)V_1(t) - V_1(t)U_3(t)$$

in [19] exactly cancel unwanted infinite diagrams $4(a)$ and $4(b)$ on the energy shell.

Surviving diagrams $4(c) - 4(g)$ are the 4th order radiative corrections to the electron-electron scattering. On the energy shell they yield the following contribution to the coefficient function of $\Sigma(t)$

$$D_4^{(c-g)}(p, q, k) = -\frac{ie^4}{(2\pi)^6} \int \frac{dh}{h|h+k|} \frac{1}{A} \left( \frac{1}{BC} + \frac{1}{DC} + \frac{1}{EF} + \frac{1}{DG} + \frac{1}{EG} \right)$$

(53)

where

$$A = \omega_{q-h} - \omega_q + h$$
$$B = \omega_{p-k-h} - \omega_p + |h+k|$$
$$C = \omega_{q-h} + \omega_{p-k} + |h+k| - \omega_q - \omega_p$$
$$D = \omega_{p+h} + \omega_{q-k} - \omega_q - \omega_p$$
$$E = \omega_{q+k} - \omega_q + |h+k| + h$$
$$F = \omega_{q+k} + \omega_{p-h-k} + h - \omega_q - \omega_p$$
$$G = \omega_{q+k} + \omega_{p+h} + |h+k| - \omega_q - \omega_p$$
The integrand in (53) is proportional to $h^{-5}$, and the integral is convergent at large values of the loop momentum $h$. Therefore $\Sigma_i^c(t)$ is finite on the energy shell. So, in the renormalized theory the electron-electron scattering is represented by a finite $S$-operator which up to the 4th order is described by the coefficient function

$$D = D_2 + D_4 + \ldots$$

where $D_2$ is given by eq. (33) and $D_4$ is given by eq. (53).

The renormalization technique presented above has not been applied to realistic theories, such as QED, yet. However, the close analogy between renormalization issues encountered in our toy model and in QED allows us to speculate that similar renormalization steps can be repeated in all perturbation orders in QED, so that finite and accurate values for various scattering amplitudes can be obtained. Note that in our approach the counterterms in the Hamiltonian (11) have the same operator structure as the terms in the original Hamiltonian (40). In relativistic renormalizable theories, like QED, even stronger statements can be made: the mass renormalization counterterms differ from the one-particle energy terms in $H_0$ only by a constant (but infinite) factor, and the charge renormalization counterterms are equal to the interaction term $V$ in the original Hamiltonian multiplied by a constant infinite factor. This conclusion is true in all perturbation orders. Thus, if the original Hamiltonian depends on finite masses $m_1, m_2, \ldots$ and coupling constants (charges) $e_1, e_2, \ldots$

$$H(m_1, m_2, \ldots; e_1, e_2, \ldots)$$

Then the addition of renormalization counterterms is equivalent to simply modifying (making them infinite) the values of masses and charges. The Hamiltonian $H^c$ after renormalization has the same functional form as $H$ where parameters $m_1, m_2, \ldots$ and $e_1, e_2, \ldots$ substituted by renormalized (infinite) values $\tilde{m}_1, \tilde{m}_2, \ldots$ and $\tilde{e}_1, \tilde{e}_2, \ldots$

$$H^c = H(\tilde{m}_1, \tilde{m}_2, \ldots; \tilde{e}_1, \tilde{e}_2, \ldots)$$

The traditional approach offers the following physical interpretation of these results. The creation and annihilation operators present in the theory describe so-called bare particles having infinite masses and charges. The interaction between bare particles drastically change their properties. For example, bare electrons constantly emit and re-absorb virtual photons and electron-positron pairs. So, bare electrons are surrounded by a coat of virtual particles. The mass of this coat is infinite, thus compensating the infinite mass of the bare particle. The virtual particles in the coat shield the true

\[\overset{12}{\text{From this point of view the mass renormalization was discussed in ref. [8].}}\]
(infinite) charge of the bare electron. The resulting mass and charge of the dressed particle is finite and exactly equal to the parameters measured in experiment. In experiment, we never see the bare particles and their virtual coats, we only see the dressed particles.

The Hamiltonian $H^c$ is formally infinite, but these infinities cancel when the $S$-operator is calculated by formula (7), and accurate results are obtained for scattering amplitudes and energies of bound states. However, the infinities do not cancel when one tries to find the eigenvectors of the Hamiltonian $H^c$ or calculate the time evolution operator $\exp(iH^c t)$. Direct application of this formula would lead to unphysical results. For example, if we calculated the time evolution of the simplest one-electron state we would obtain that this state dissociates into a complex linear combination of states over time

$$e^{iH^c t}a^\dagger|0\rangle = (1 + iH^c t + \ldots)a^\dagger|0\rangle \propto (1 + it(a^\dagger a + c^\dagger c + a^\dagger c^\dagger a + a^\dagger a c) + \ldots)a^\dagger|0\rangle \propto a^\dagger|0\rangle + a^\dagger c^\dagger|0\rangle + a^\dagger c^\dagger c^\dagger|0\rangle + \ldots$$

Moreover the coefficients multiplying the terms in (56) are given by divergent integrals. Therefore, with the infinite Hamiltonian $H^c$ the basic formulas of quantum theory (1) and (2) become useless for practical calculations. However, in most cases this is not a cause of trouble. As discussed in section 2, most experiments in high energy physics are concerned either with bound state energies or with scattering cross-sections. To calculate these properties, the knowledge of the $S$-operator is sufficient and the renormalized theory works fine. For approximate description of the time evolution in low energy, e.g., atomic, systems, one can use heuristic approaches, such as those based on the Dirac-Fock Hamiltonian.

9 Construction of the dressed particle Hamiltonian

We see two fundamental problems with the traditional renormalization theory presented above. First, from the practical point of view, the absence of a well-defined finite Hamiltonian does not allow one to study the time evolution of interacting states. One can expect that with advancement of experimental tools, the time-dependent information from the region of interaction will soon become available. However, accurate analysis of this information is not possible with ill-defined Hamiltonians of renormalized quantum field theories. Second, from the theoretical point of view, the traditional

\footnote{The way to remove the ultraviolet divergences from interaction operators, or to “renormalize” the Hamiltonian, was suggested by Glazek and Wilson in their similarity renormalization approach \cite{2}. However, their Hamiltonians still have unphysical terms \cite{9}, and result in “instability” of vacuum and one-particle states similar to that shown in eq. (56).}
approach operates with notions of bare and virtual particles that are, in principle, non-observable. It is desirable to have a theory formulated directly in terms of dressed particles and their interactions without mentioning the non-observable bare and virtual particles at all. This idea was first realized in the dressed particle formalism by Greenberg and Schweber [1].

In the rest of this paper we will discuss the RQD approach [3, 4, 5] which combines the similarity renormalization and dressed particle ideas. RQD is capable to fix the both problems of the traditional renormalization formalism mentioned above. Let us first focus on the dressing part of RQD. There are two approaches to dressing with different interpretations but equivalent physical results. One approach [10] tries to find an explicit unitary (dressing) transformation connecting creation and annihilation operators of bare particles ($a^\dagger, a, c^\dagger$, and $c$) with creation and annihilation operators of dressed particles ($A^\dagger, A, C^\dagger$, and $C$) and find a function $f$ which expresses the Hamiltonian $H^c$ of the renormalized theory through dressed operators

$$H^c = f(A^\dagger, A, C^\dagger, C)$$  \hspace{1cm} (57)

Another approach [3, 4, 5] is to pretend that operators $a^\dagger, a, c^\dagger$, and $c$ already describe the dressed particles and try to find a new finite Hamiltonian $H^d$ whose functional dependence on bare particle operators is given by the same function $f$ which expresses the dependence of $H^c$ on the dressed operators

$$H^d = f(a^\dagger, a, c^\dagger, c)$$  \hspace{1cm} (58)

We will stick to the second approach in this paper. There are two ways to proceed. First, we can try to find $H^d$ by applying a unitary dressing transformation to $H^c$. This way was described in ref. [3]. In this paper we are choosing another route: we will simply fit the Hamiltonian $H^d$ to the finite scattering operator $S^c$ known from the renormalized theory.

There are three requirements that we want to satisfy when looking for the “dressed particle” Hamiltonian $H^d = H_0 + V^d = H_0 + V_2^d + V_3^d + V_4^d + \ldots$.

(A) $H^d$ is scattering-equivalent to $H^c$.

(B) $V^d$ is finite.

(C) $V^d$ is phys.

The condition (A) is understandable as we know that the $S$-operator of renormalized theories (QED, Standard Model) agrees well with experiments, and we would like
to preserve this agreement in a theory with the new Hamiltonian $H^d$. Using eqs. (11) and (12) we can write

$$-i \log S^c = F^c_2(t) + F^c_3(t) + F^c_4(t) + \ldots$$

$$-i \log S^d = V^d_2(t) + V^d_3(t) + V^d_4(t) - \frac{i}{2} [V^d_2(t), V^d_2(t)] + \ldots$$

Then we see that condition (A) implies the following infinite set of relations between $V^d_i(t)$ and $F^c_i(t)$ on the energy shell

$$V^d_2(t) = F^c_2(t)$$

$$V^d_3(t) = F^c_3(t)$$

$$V^d_4(t) = F^c_4(t) + \frac{i}{2} [V^d_2(t), V^d_2(t)]$$

$$V^d_i(t) = F^c_i(t) + Q_i(t), \ i > 4$$

where $Q_i(t)$ denotes a sum of multiple commutators of $V^d_j(t)$ from lower orders ($2 \leq j \leq i - 2$) with $t$-integrations. We have expressions for operators $\Sigma^c_2(t)$ and $\Sigma^c_4(t)$ in the renormalized theory (eqs. (33) and (53)). Clearly, we can perform similar calculations for operators $F^c_i(t)$ whose values on the energy shell are present on the right hand sides of eqs. (59) - (62). Alternatively, we can express $F^c_i(t)$ through $\Sigma^c_i(t)$ using relationship (13).

Operators $F^c_i(t)$ are, of course, finite. This immediately implies that $V^d_2(t)$ and $V^d_3(t)$ are finite on the energy shell. Moreover, from eqs. (13) and (30) it follows that $F^c_2(t)$ and $F^c_3(t)$ are phsy on the energy shell. Therefore, $V^d_2(t)$ and $V^d_3(t)$ can be also chosen phsy on the energy shell. This proves that conditions (B) and (C) are satisfied on the energy shell up to the 3rd order. However, eqs. (59) - (62) tell us nothing about the behavior of $V^d_2(t)$ and $V^d_3(t)$ off the energy shell. The same is true for interactions in higher orders: the interaction operators $V^d_i(t)$ off the energy shell remain undetermined by our condition (A). This freedom in choosing interactions simply means that different choices of the coefficient functions of $V^d_i(t)$ off the energy shell permit us to obtain different forms for the interaction operators $V^d_i(t)$.

The use of the Magnus expansion for deriving $V^d_i(t)$ is preferable to the old-fashioned or Dyson’s formulas, because the right hand sides of eqs. (59) - (62) are expressed through commutators, so they are manifestly Hermitian. In addition, this approach automatically generates cluster separable interactions $V^d_i(t)$ (see ref. [3]).

Note that in applications we are dealing primarily with interactions near the energy shell, because in most processes, excluding very short virtual events, the total energy stays unchanged at all times.
shell result in scattering equivalent Hamiltonians. Since our experimental knowledge about interactions in high energy physics is limited to their effects on scattering, we are allowed to freely choose the behavior of $V_i^d(t)$ off the energy shell without any danger to get into contradiction with experiment. This freedom is exactly what is needed to satisfy conditions (B) and (C) in all orders. As we will see shortly, it is important to choose the behavior of $V_i^d(t)$ such that their coefficient functions fall off rapidly when the arguments move away from the energy shell. For example, we can choose the coefficient functions being proportional to $\zeta_i = \exp(-\gamma E^2)$ where $\gamma$ is a positive constant and $E$ is the energy function. Then the electron-electron interaction in the second order takes the form

$$V_2^d(t) = -\frac{i}{2} [V_1^c(t), V_1^c(t)]^{ph} \circ \zeta_2$$

$$= \frac{e^2}{(2\pi)^3} \int d\mathbf{p} d\mathbf{q} d\mathbf{k} \frac{e^{-\gamma E^2} e^{-itE}}{\mathbf{k} |(\mathbf{k} + \omega_{\mathbf{p}-\mathbf{k}} - \omega_{\mathbf{q}} - \omega_{\mathbf{k}})|} a_{\mathbf{p}-\mathbf{k}}^\dagger a_{\mathbf{q}+\mathbf{k}}^\dagger a_{\mathbf{q}} a_{\mathbf{p}}$$

(63)

where $E = \omega_{\mathbf{p}-\mathbf{k}} + \omega_{\mathbf{q}+\mathbf{k}} - \omega_{\mathbf{q}} - \omega_{\mathbf{p}}$ is the energy function.

There are no terms of the type $a^\dagger a^\dagger a a$ in the 3rd order term $F_3^c(t)$. Therefore the lowest (4th) order radiative correction to the electron-electron interaction (63) should be obtained from eq. (61). As discussed above, the value of $F_4^c(t)$ on the energy shell is finite. Let us now consider the term

$$\frac{i}{2} [V_2^d(t), V_2^d(t)]$$

(64)

on the right hand side of eq. (61). First, we note that, according to Table 1, this commutator is phys. Second, as we agreed above, the coefficient functions of $V_2^d(t)$ fall off rapidly outside the energy shell. Without this condition, the loop integrals encountered in calculations of (63) could be divergent. However, in our case these integrals are convergent: when the loop integration momentum goes to infinity, the $V_2^d(t)$ factors in the integrand go away from the energy shell, i.e., rapidly fall off. This guarantees that interaction $V_4^d(t)$ is phys and finite on the energy shell. Its coefficient function off the energy shell should be again chosen to decay rapidly to ensure the convergence of loop integrals in higher order operators $Q_i(t)$, where $V_4^d(t)$

\[16\text{In the representation where } H_0 \text{ is diagonal, this condition is equivalent to bringing the matrix of the Hamiltonian to the band-diagonal form, which is the central idea of the similarity renormalization method [2].}\]

\[17\text{Note that we should be careful not to set } \gamma \text{ to infinity. In this case, the coefficient functions of interaction operators } V_i^d \text{ become non-differentiable. This means that interaction is not separable [11], and the scattering theory formalism from section 2 is no longer applicable.}\]
may contribute. These arguments can be repeated in higher orders, which proves that the dressed particle Hamiltonian \( H^d \) is free of ultraviolet divergences.

The contribution \((F_4^c)^{ph}\) on the right hand side of eq. (61) is well-defined near the energy shell, but this is not true for the contribution \(\frac{i}{2}[V_2^d,V_2^d]\). This commutator depends on the behavior of \(V_2^d\) everywhere in the momentum space. So, it substantially depends on our choice of \(\zeta_2\) off the energy shell. There is a great freedom in this choice which is reflected in the uncertainty of \(V_4^d\) even on the energy shell. It is interesting to note that although the off-shell behavior of the 2nd order interaction and the on-shell behavior of the 4th order interaction cannot be separately determined in our theory, they are connected to each other in such a way that the ambiguity of the interaction does not affect the \(S\)-matrix for the electron-electron scattering.

The above construction does not allow us to obtain full information about \(V^d\): The off-shell behavior of interactions is rather arbitrary, and the on-shell behavior can be determined only for lowest order terms. However, this uncertainty is perfectly understandable: It simply reflects the one-to-many correspondence between the \(S\)-operator and Hamiltonians (see section 2). It means that there is a class of finite phys interactions \(\{V^d\}\) all of which satisfy our requirements (A) - (C) and can be used for \(S\)-matrix calculations without encountering divergent integrals. In order to find the unique Hamiltonian correctly describing the dynamics of particles, the theoretical predictions should be compared with time-resolved experimental data. However, our model theory is not sufficiently accurate to be comparable with experiment, and further efforts in this direction require building the dressed particle formulation of the full-blown quantum electrodynamics [3, 5].

Table 2: Examples of interaction terms in the dressed particle Hamiltonian (65). Bold numbers in the third column indicate perturbation orders in which the operators can be unambiguously obtained near the energy shell as discussed in section 9.

| Operator        | Physical meaning                                      | Perturbation Orders |
|-----------------|-------------------------------------------------------|---------------------|
| \(a^\dagger a^\dagger aa\) | electron-electron                                    | 2, 4, 6, ...        |
| \(a^\dagger c^\dagger ac\) | electron-photon (Compton)                           | 2, 4, 6, ...        |
| \(a^\dagger a^\dagger aaa\) | 3-electron potential                                 | 4, 6, ...           |
| **Elastic potentials** |                                                       |                     |
| \(a^\dagger a^\dagger c^\dagger aa\) | bremsstrahlung in electron-electron collisions      | 3, 5, ...           |
| \(a^\dagger a^\dagger aac\) | photon absorption in electron-electron collisions   | 3, 5, ...           |

| **Inelastic potentials** | | |

In contrast to the original Hamiltonian (41), there seems to be no way to write the dressed particle Hamiltonian \(H^d\) in a closed form. From the derivation outlined above it is clear that in higher perturbation orders there are more and more terms with
increasing complexity in the interaction operator $V^d$.

\[
H^d = H_0 + V^d = H_0 + a^\dagger a a + a^\dagger c^\dagger ac + a^\dagger a^\dagger a a + a^\dagger a^\dagger a a a + a^\dagger a^\dagger c^\dagger a a + \ldots \tag{65}
\]

Some of them are shown in Table 2. However, all these high order terms directly reflect real interactions and processes observable in nature. For example, the term $a^\dagger a^\dagger c^\dagger a a$ (bremsstrahlung) describes creation of a photon in electron-electron collisions. In the language of classical electrodynamics, this can be interpreted as radiation due to the acceleration during interaction of charged particles and is often referred to as the radiation reaction force. The Hermitian-conjugated term $a^\dagger a^\dagger a a c$ describes absorption of a photon by a colliding pair of charged particles.

Note that in contrast to the traditional renormalization approach in which the mass renormalization condition \([55] \) was ensured by maintaining a balance between unphys and renorm terms in the interaction, in the RQD Hamiltonian these terms are absent altogether. The interaction operator $V^d$ is purely phys which guarantees that scattering operators $\Sigma^d$ and $F^d$ are phys too. The phys character of the interaction $V^d$ means that it yields zero when acting on the vacuum and one-particle states. Interaction $V^d$ acts only when there are two or more dressed particles present. This is equivalent to saying that self-interaction effects are not present in the dressed particle approach. Naturally, there should be no dressing in our theory, because we are working with particles which are already fully dressed.

**Conclusions**

In this paper, we considered a model theory which provides a simplified description of interactions between electrons and photons, similar to exact interactions in QED. We found that in order to get sensible results for the $S$-matrix, the theory should be renormalized, just as QED, by adding infinite counterterms to the Hamiltonian. We also demonstrated that the renormalized version of the theory can be reformulated entirely in terms of dressed particles and their interactions without affecting predictions of the theory about scattering cross-sections or bound state energies. The dressed particle Hamiltonian $H^d$ is finite, thus, the ultraviolet divergences are not present anymore. There are no bare and virtual particles in the dressed particle approach. This simplifies substantially the physical interpretation of the theory. From the operator form of the dressed particle Hamiltonian \([55] \) it is clear that dressed particles interact with each other via instantaneous forces that generally do not conserve the number of photons.
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References

[1] O. W. Greenberg and S. S. Schweber, *Clothed particle operators in simple models of quantum field theory*, Nuovo Cimento 8 (1958), 378.

[2] St. D. Glazek and K. G. Wilson, *Renormalization of Hamiltonians*, Phys. Rev. D 48 (1993), 5863.

[3] E. V. Stefanovich, *Quantum field theory without infinities*, Ann. Phys. (NY) 292 (2001), 139.

[4] E. V. Stefanovich, *Is Minkowski space-time compatible with quantum mechanics?* Found. Phys. 32 (2002), 673.

[5] E. V. Stefanovich, *Relativistic quantum dynamics* (Mountain View, 2004) [physics/0504062].

[6] W. Magnus, *On the exponential solution of differential equations for a linear operator*, Commun. Pure Appl. Math. 7 (1954), 649; P. Pechukas and J. C. Light, *On the exponential form of time-displacement operators in quantum mechanics*, J. Chem. Phys. 44 (1966), 3897.

[7] H. Ekstein, *Equivalent Hamiltonians in scattering theory*, Phys. Rev. 117 (1960), 1590.

[8] A. Krüger and W. Glöckle, *One-nucleon effective generators of the Poncaré group derived from a field theory: Mass renormalization*, Phys. Rev. C 60 (1999), 024004; M. I. Shirokov, *Mass renormalization using noncovariant perturbation theory*, preprint JINR P2-2000-277, Dubna (2000).

[9] St. D. Glazek, *Similarity renormalization group approach to boost invariant Hamiltonian dynamics*, hep-th/971288.

[10] A. V. Shebeko and M. I. Shirokov, *Unitary transformations in quantum field theory and bound states*, Fiz. Elem. Chast. Atom. Yadra 32 (2001), 31 [English translation in Phys. Part. Nucl. 32 (2001), 15]. [nucl-th/0102037]
[11] S. Weinberg, *The Quantum Theory of Fields*, Vol. 1 (University Press, Cambridge, 1995).