Collective Property of Numbers and Its Mathematical Refutation

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Abstract

A number has the “collective” property if the number is the greatest lower bound of a bounded, strictly decreasing sequence on the real line. We prove that numbers with the collective property constitute an empty set.

The properties of numbers for counting, calculating, or measuring are conceivable without considering “all numbers”. In mainstream mathematics, “all numbers” or “numbers as a whole” is a notion formulated based on the notion of “set”, and a number also has some “collective” properties different from the properties for counting, calculating, and measuring. A specific collective property of a number is necessary to the definition of “bounded, strictly decreasing sequence”. To define “bounded, strictly decreasing sequence”, one has to define a number, which is not a term of the sequence, as the greatest lower bound of the sequence. The property of the number as the greatest lower bound may not be necessary when the number is used for counting, calculating or measuring. In the following, the collective property of a number means “the number is the greatest lower bound of a bounded, strictly decreasing sequence on the real line”.

Due to the failure to prove the consistency of any formal system involving numbers with the collective property [5], and also due to the fact that the definition of the greatest lower bound is circular and hence is not on a logically secure ground [6, 8], in mainstream mathematics, one has to accept the collective property based on a philosophical belief (realism or Platonism). As a sharp contrast, constructive mathematicians reject the collective property [1, 2, 3, 4, 8]. However, the rejection is also based on a philosophical belief (constructivism or intuitionism), perhaps for lack of a convincing way to argue against the collective property mathematically. By proving the following theorem, we present a mathematical refutation of the collective property. In other words, we prove that numbers with the collective property constitute an empty set and hence are devoid of any meaning in mathematics.

**Theorem 1** Denote by $E$ a bounded set consisting of all terms of an arbitrarily given, strictly decreasing sequence (with a greatest lower bound $a \notin E$) on the real line. The above definition of $E$ implies a contradiction.

**Proof:** For each point $x \in (-\infty, \infty)$, define

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\[ f(x) = \begin{cases} 
 1, & x \in E; \\
 0, & \text{otherwise.} 
\end{cases} \tag{1} \]

By (1), \( f(x) = 0 \) does not hold for all \( x \). Let \( P(x) \) represent "\( f(z) = 0 \) for each \( z \leq x \)." Let \( v \) be a real-valued variable with the following properties. (I) \( v \) assumes values greater than or equal to \( x \) with \( P(x) \) in the order from left to right as the points appear on the real line. (II) In the order mentioned in (I), the set of values of \( v \) may have a last element defined by a condition, and the value of \( v \) can be any point less than or equal to the last value. (III) If \( v \) does not have a last value, then the value of \( v \) can be any point on the real line. If \( v \) runs from any \( x_0 \) with \( P(x_0) \) towards the right, i.e., if \( v \) increases from \( x_0 \), then \( v \) reaches eventually a unique \( x_1 \) with \( f(x_1) = 1 \) and \( f(x) = 0 \) for each \( x < x_1 \). To see this, we define \( v|x' \to x'' \) to mean "if \( P(x') \) holds, then let \( v \) take on each \( x \) with \( x \geq x' \) as its value, and let \( x'' \) be the last value of \( v \) if \( f(v) = 1 \) where the value of \( v \) is \( x'' \)." Note the difference between \( f(v) = 1 \) as the condition to define the last value of \( v \) and \( f(x) = 1 \), where \( x \) is a point in \( E \).

Assume \( P(x') \) holds. Then \( v|x' \to x'' \) determines a nonempty subset of values of \( v \). Denying "\( x'' \) is the last value of \( v \)" amounts to negating "\( f(v) = 1 \) where the value of \( v \) is \( x'' \)" in the definition of \( v|x' \to x'' \). The negation asserts "\( f(v) = 0 \) where the value of \( v \) can be any \( x \geq x'' \)." This contradicts (1). Since \( x'' \) is a value of \( v \), no \( x \) with \( x < x'' \) is the last value of \( v \). So \( f(v) = 0 \) where the value of \( v \) can be any \( x < x'' \). In other words, \( x'' \) is unique. The uniqueness of \( x'' \) is independent of \( x' \) so long as \( P(x') \) holds. By the definition of \( v|x' \to x'' \), given \( P(x') \), if \( v|x' \to x'' \), and if \( v|x' \to y'' \), then \( x'' = y'' \). Consider two arbitrarily given points \( x' \) and \( y' \) with \( x' \neq y' \). \( P(x') \), \( P(y') \), \( v|x' \to x'' \), and \( v|y' \to y'' \). If \( x'' \neq y'' \), then without loss of generality, assume \( y' < x' < x'' < y'' \), which implies \( v|y' \to x'' \). This contradicts \( v|y' \to y'' \). So \( x'' = y'' \). Consequently, for all \( x \) with \( P(x) \), there is a unique \( x_1 \) with \( v|x \to x_1 \).

The above argument has a clear physical meaning. Negating the argument then leads to an absurdity. Let \( r \) represent the position of a point particle moving along the real line towards the right from an initial position \( x_0 \) with \( P(x_0) \). The particle moves at \( r \) if and only if \( P(r) \) holds, i.e., the speed of the particle at \( r \) is positive if and only if \( P(r) \) holds. Then the particle stops at \( x_1 \) with \( f(x_1) = 1 \) and never moves again. If this is not the case, then the particle either is motionless with a positive speed, or moves at a speed equal to \( 0 \). This violates the basic physical law and hence is absurd.

If \( x_1 \) does not exist, then \( v \) runs towards \( \infty \), which implies \( f(x) = 0 \) at each \( x \). This contradicts (1) as we have shown already. However, if \( x_1 \) exists, then \( x_1 = \min E \in E \), which contradicts the definition of \( E \). We see a contradiction in either case above. Q.E.D.

Clearly, if we detach the collective property from numbers, then any contradiction or absurdity caused by the collective property disappears. The collective property comes into being by the definition of the greatest lower bound of \( E \). The contradiction implied by the definition of \( E \) suggests that the definition of the greatest lower bound is problematic. This is indeed the case. The definition of the greatest lower bound is circular [6, 8], and violates Russell’s vicious circle principle: No totality can contain members defined in terms of the totality itself [7]. Definitions violating this principle are called non-predicative (or impredicative). According to Russell, non-predicative definitions cause contradictions, such as the well-known Russell’s paradox [6]. To avoid contradictions, Russell proposed the vicious circle principle. Now let us take a look at the definition of the greatest lower bound.

A real number \( y \) is a lower bound of an infinite set \( S \) of real numbers, if \( y \leq x \) for each \( x \) in \( S \). Let \( L \) be the set of all lower bounds of \( S \). The greatest lower bound \( l \) of \( S \) is a member of \( L \) with \( l \geq y \) for each \( y \) in \( L \). But \( l \) may not necessarily be in \( S \). Clearly, the definition of \( l \) is circular and non-predicative: To define \( l \), one has to define a totality \( L \) first, in which \( l \) itself is a member. The contradiction implied by \( E \) is directly due to the circular, non-predicative definition of the greatest lower bound, which attaches the collective property to a point \( a \notin E \), and hence excludes \( \min E \) from \( E \). In fact, the definition of
E requires one to exclude minE from E. Achieved by the circular, non-predicative definition of the greatest lower bound, such exclusion is logically invalid [6, 7, 8]. As shown below, the definition of the greatest lower bound of E is merely a result of an artificial choice rather than a logical necessity. The choice also causes an absurdity.

Denote by \( \overline{N}(x) \) the number of different values taken on by \( f \) on \((−∞, x] \), and \( N'(x) \) the number of different values taken on by \( f \) on \((−∞, x) \). Both \( \overline{N} \) and \( N' \) are non-decreasing functions on \((−∞, ∞) \). If \( f \) takes on only one value on \((−∞, x] \), then \( \overline{N}(x) = N'(x) = 1 \). For any \( x \in (−∞, ∞) \), \( \overline{N}(x) ≥ N'(x) \). Write \( \Delta N(x) = \overline{N}(x) - N'(x) \). So

\[
\overline{N}(x) = N'(x) + \Delta N(x)
\]

with

\[
\lim_{x \to ∞} \overline{N}(x) = 1.
\]

Since \( f \) has only two different values, \( \Delta N(x) \in \{0, 1\} \).

For each \( x \in (−∞, ∞) \), all eligible (i.e., consistent with (2) and (3)) combinations of \( N'(x) \), \( \max\{f(y) : y < x\}, \Delta N(x), \overline{N}(x) \), and \( \max\{f(y) : y ≥ x\} \) give the conditions \( A(x), B(x), C(x) \) and \( D(x) \) below, such that either \((−∞, ∞) = \{x : A(x)\} \) or \((−∞, ∞) = \{x : B(x)\} ∪ \{x : C(x)\} ∪ \{x : D(x)\} \).

| \( A(x) \) | \( N'(x) \) | \( \max\{f(y) : y < x\} \) | \( \Delta N(x) \) | \( \overline{N}(x) \) | \( \max\{f(y) : y ≥ x\} \) |
|---|---|---|---|---|---|
| \( B(x) \) | \( 1 \) | \( 0 \) | \( 0 \) | \( 1 \) | \( 0 \) |
| \( C(x) \) | \( 1 \) | \( 0 \) | \( 1 \) | \( 2 \) | \( 1 \) |
| \( D(x) \) | \( 2 \) | \( 1 \) | \( 0 \) | \( 2 \) | \( 1 \) |

If \( \{x : A(x)\} = (−∞, ∞) \), then \( \{x : B(x)\} = \{x : C(x)\} = \{x : D(x)\} = \emptyset \). In other words, 0 is the only value of \( f \). This is prohibited by (1). Negating \( \{x : A(x)\} = (−∞, ∞) \), we obtain \( \{x : B(x)\} ∪ \{x : C(x)\} ∪ \{x : D(x)\} = (−∞, ∞) \), where none of \( \{x : B(x)\}, \{x : C(x)\}, \) and \( \{x : D(x)\} \) is empty. As can be readily seen, \( \{x : B(x)\} = (−∞, x_1), \{x : C(x)\} = \{x_1\}, \) and \( \{x : D(x)\} = (x_1, ∞) \) with \( \Delta N(x_1) = 1 \).

Consider two arbitrarily given points \( x_0 < x' \) with \( \overline{N}(x') > \overline{N}(x_0) \). We say \( \overline{N} \) increases if there are points \( y < x \) such that \( \overline{N}(y) < \overline{N}(x) \). We say \( \overline{N} \) increases at \( x' \) if \( \overline{N}(x) > \overline{N}(y) \) for any \( y < x \). Since \( \overline{N} \) increases on \([x_0, x']\), \( \overline{N} \) increases at a unique \( x_1 ∈ [x_0, x'] \), which implies \( \Delta N(x_1) = 1 \). The increase in \( \overline{N} \) can then be expressed by

\[
2 = \overline{N}(x') = \overline{N}(x_0) + \int_{−∞}^{∞} δ(x - x_1)dx = 1 + 1
\]

where \( δ \) is the Dirac delta function.

In mainstream mathematics, one denies that the negation of \( \{x : A(x)\} = (−∞, ∞) \) is \( \{x : B(x)\} ∪ \{x : C(x)\} ∪ \{x : D(x)\} = (−∞, ∞) \) with nonempty \( \{x : B(x)\}, \{x : C(x)\}, \) and \( \{x : D(x)\} \) by choosing, without any logically justified reason, to negate \( \Delta N(x_1) = 1 \). The denial implies not only the definition of the greatest lower bound of \( E \) but also an absurdity: \( \overline{N} \) does not increase at any point on the real line and \( \overline{N} \) increases. Using the definition of the greatest lower bound of \( E \) to explain away the absurdity is logically invalid.

Actually, any formal definition or informal notion of even \( \{n : n = 1, 2, ⋅⋅⋅\} \) is circular and non-predicative [9, 10]. A formal definition or informal notion of \( \{n : n = 1, 2, ⋅⋅⋅\} \) is the basis of mainstream mathematics. Again, in mainstream mathematics, the conception of \( \{n : n = 1, 2, ⋅⋅⋅\} \) is not on a
logically secure ground, and the only reason for one to accept \( \{ n : n = 1, 2, \cdot \cdot \cdot \} \) is a philosophical belief [9].

Let \( P_1(E) \) stand for “\( E \) is a bounded, strictly decreasing sequence on the real line”, \( P_2(1/N) \) for “\( 1/N \) is \( \{1/n : n = 1, 2, \cdot \cdot \cdot \} \) or any infinite subset of \( \{1/n : n = 1, 2, \cdot \cdot \cdot \} \)”, and \( P_3(N) \) for “\( N \) is \( \{n : n = 1, 2, \cdot \cdot \cdot \} \) or any infinite subset of \( \{n : n = 1, 2, \cdot \cdot \cdot \} \)”. The following results are immediate from Theorem 1.

**Corollary 1** (i) \( \{ E : P_1(E) \} = \emptyset \), which implies (with \( E = \{1/n : n = 1, 2, \cdot \cdot \cdot \} \)) \( \{1/N : P_2(1/N) \} = \emptyset \), and (due to the correspondence between \( \{1/n : n = 1, 2, \cdot \cdot \cdot \} \) and \( \{n : n = 1, 2, \cdot \cdot \cdot \} \)) \( \{N : P_3(N) \} = \emptyset \).

(ii) \( \{ x : x \) is a number with the collective property \( \} = \emptyset \).

By Corollary 1, \( \{ n : n = 1, 2, \cdot \cdot \cdot \} \) is meaningless, though \( \{1, 2, \cdot \cdot \cdot , n \} \) is meaningful for any given positive integer \( n \). Also by Corollary 1, no number has the collective property. In other words, the definition of the greatest lower bound defines a concept without any instance. In particular, defining numbers based on the collective property, such as the definition of irrational numbers, actually defines nothing. Consequently, mathematical reasoning based on a number system with the collective property may yield concepts with an empty set of instances and hence may be misleading. Nevertheless, any number with a decimal expression of a finite length still possesses the properties for counting, calculating and measuring. As shown by constructive mathematics, numbers without the collective property are sufficient for scientific and engineering applications.

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