An Effective Method for Locating Nonlinear Components in Periodic Structures

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Abstract. An effective method is developed to detect the position of nonlinear components in periodic structures. The detection procedure requires exciting the nonlinear systems only once by using a sinusoidal-like input which has two frequency components with one frequency considerably smaller than the other. The effectiveness of this method is demonstrated by numerical study. As the position of a nonlinear component often corresponds to the location of defect in periodic structures, this new method is of great practical significance in fault diagnosis for mechanical and structural systems.

1. Introduction
Periodic structures are defined as structures consisting of identical substructures connected to each other in an identical manner. Real life systems which can be modelled as finite or infinite, one-dimensional or multi-dimensional periodic structures range from the simple structures like periodically supported beams and plates to building block [1]. The free and forced vibration and the modal analysis for linear periodic structures are of particular interests. Attention has also been paid to the study of nonlinear periodic structures [2]. In engineering practice, there are considerable periodic structures that behave nonlinearly just because one or a few components have nonlinear properties, and the nonlinear component is often the component where a fault or abnormal condition occurs. One of the well known examples is beam structures [3] with breathing cracks. Therefore it is of great significance to effectively detect the position of nonlinear components in a periodic structure. The detection of damage in large periodic structures had been studied by Zhu and Wu [4].

Based on the Volterra series theory, Lang et al [5] have put forward a new concept - Nonlinear Output Frequency Response Functions (NOFRFs), which are one-dimensional functions of frequency and allow the analysis of nonlinear systems to be implemented in a manner similar to the analysis of linear systems. In this paper, a novel method is derived based on the NOFRF concept to detect the position of the nonlinear component in periodic structures.

2. Nonlinear Output Frequency Response Functions

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The definition of NOFRFs is based on the Volterra series theory of nonlinear systems. Consider the class of nonlinear systems which are stable at zero equilibrium and which can be described in the neighborhood of the equilibrium by the Volterra series

\[ x(t) = \sum_{n=1}^{\infty} \cdots \int_{\mathbb{R}^n} h_n(\tau_1, \ldots, \tau_n) \prod_{i=1}^{n} u(t - \tau_i) d\tau_i \]  

where \( x(t) \) and \( u(t) \) are the output and input of the system, \( h_n(\tau_1, \ldots, \tau_n) \) is the \( n \)th order Volterra kernel, and \( N \) denotes the order of the Volterra series. The output frequency response of this class of nonlinear systems can be explicitly expressed [6] as

\[
\begin{cases}
X(\omega) = \sum_{n=1}^{\infty} X_n(j\omega) & \text{for } \forall \omega \\
X_n(j\omega) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} H_n(j\omega_1, \ldots, j\omega_n) \prod_{i=1}^{n} U(j\omega_i) d\sigma_{\omega n}
\end{cases}
\]

where

\[
H_n(j\omega_1, \ldots, j\omega_n) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} h_n(\tau_1, \ldots, \tau_n) e^{-j\omega_1 \tau_1 - \cdots - j\omega_n \tau_n} d\tau_1 \cdots d\tau_n
\]

is the \( n \)th order Generalised Frequency Response Function (GFRF) [6].

The new concept of the NOFRFs recently proposed by Lang and Billings [5] is defined as

\[
G_n(j\omega) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} H_n(j\omega_1, \ldots, j\omega_n) \prod_{i=1}^{n} U(j\omega_i) d\sigma_{\omega n}
\]

under the condition that

\[
U_n(j\omega) = \int_{\mathbb{R}^n} \prod_{i=1}^{n} U(j\omega_i) d\sigma_{\omega_n} \neq 0
\]

By introducing the NOFRFs \( G_n(j\omega), \ n = 1, \cdots N \), equation (2) can be written as

\[
X(j\omega) = \sum_{n=1}^{N} X_n(j\omega) = \sum_{n=1}^{N} G_n(j\omega) U_n(j\omega)
\]

which is similar to the description of the output frequency response for linear systems. The NOFRFs reflect a combined contribution of the system and the input to the system output frequency response behaviour.

![Figure 1. A multi-degree freedom oscillator with locally nonlinear component.](image-url)
3. NOFRFs of Nonlinear Periodic Structures

Consider the one-dimensional nonlinear periodic structures where the $L$th component is nonlinear, which had been used in [2], as shown in Fig 1.

Assume the restoring forces $S_{LS}(\Delta)$ and $S_{LD}(\Delta)$ of the $L$th spring and damper are the polynomial functions of the deformation $\Delta$ and $\dot{\Delta}$ respectively, e.g.,

$$S_{LS}(\Delta) = \sum_{i=1}^{P} r_i N^i, \quad S_{LD}(\Delta) = \sum_{i=1}^{P} w_i N^i$$

(7)

where $P$ is the degree of the polynomial. Without loss of generality, further assume $L \neq 1, n$. Denote

$$NonF = \sum_{i=1}^{P} w_i (x_{L+1} - x_i) + \sum_{i=1}^{P} r_i (x_{L+1} - x_i)$$

(8)

$$NF = \begin{bmatrix} NonF & NonF & \cdots & NonF \\ 0 & \cdots & 0 & NonF \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix}$$

(9)

Then the motion of the nonlinear oscillator in Figure 1 can be described in a matrix form as

$$M\ddot{x} + C\dot{x} + Kx = -NF + F(t)$$

(10)

where $M$ is the system mass matrix,

$$M = \begin{bmatrix} m_1 & 0 & \cdots & 0 \\ 0 & m_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & m_n \end{bmatrix}$$

and $C$ is the system damping matrix, $x = (x_1, \cdots, x_n)^\top$ is the displacement vector, and

$$F(t) = (0, \cdots, 0, u(t), 0, \cdots, 0)$$

(11)

is the external force vector acting on the $J$th mass of the oscillator.

The system described by equation (10) is a typical locally nonlinear periodic structure. The $L$th nonlinear component can lead the whole system to behave nonlinearly. In this case, the Volterra series can be used to describe the relationships between the displacements $x_i(t)$ ($i = 1, \cdots, n$) and the input force $u(t)$ as below

$$x_i(t) = \sum_{j=0}^{N} \sum_{j'=0}^{N} \cdots \sum_{j''=0}^{N} h_{i,j}(\tau_j, \cdots, \tau_1) \prod_{r=1}^{l} u(t - \tau_r) d\tau$$

(12)

where $h_{i,j}(\tau_j, \cdots, \tau_1)$ is the $j$th order Volterra kernel associated to the $i$th mass. In the frequency domain, the relationship (12) can be expressed as
\[
X_i(j\omega) = \sum_{j=1}^{N} X_{(i,j)}(j\omega) = \sum_{j=1}^{N} G_{(i,j)}(j\omega)U_j(j\omega) \quad (i=1,\ldots,n)
\]  

(13)

where \( G_{(i,j)}(j\omega) \) is the \( i \)th order NOFRF associated to the \( i \)th mass.

Without loss of generality, assume \( L<J \), as revealed in [7], for any two consecutive masses, the NOFRFs of system (10) satisfy the following relationships

\[
\lambda_{(i-1)}^{(i-1)}(j\omega) = \frac{G_{(i-1,i)}(j\omega)}{G_{(i,i-1)}(j\omega)} = \cdots = \lambda_{(n-1)}^{(n-1)}(j\omega) = \frac{G_{(n-1,n)}(j\omega)}{G_{(n,n-1)}(j\omega)} \quad (1 \leq i \leq n-1)
\]

(14)

\[
Z_{(i-1)}^{(i-1)}(j\omega) = \frac{G_{(i-1,i)}(j\omega)}{G_{(i,i-1)}(j\omega)} = \frac{G_{(i-1,i)}(j\omega)}{G_{(i,i-1)}(j\omega)} = \frac{G_{(i-1,i)}(j\omega)}{G_{(i,i-1)}(j\omega)} = \frac{G_{(i-1,i)}(j\omega)}{G_{(i,i-1)}(j\omega)} = \frac{G_{(i-1,i)}(j\omega)}{G_{(i,i-1)}(j\omega)} \quad (1 \leq i \leq L-2 \text{ or } J \leq i \leq n-1; \ 2 \leq Z \leq N)
\]

(15)

\[
Z_{(i-1)}^{(i-1)}(j\omega) = \frac{G_{(i-1,i)}(j\omega)}{G_{(i,i-1)}(j\omega)} = \frac{G_{(i-1,i)}(j\omega)}{G_{(i,i-1)}(j\omega)} = \frac{G_{(i-1,i)}(j\omega)}{G_{(i,i-1)}(j\omega)} = \frac{G_{(i-1,i)}(j\omega)}{G_{(i,i-1)}(j\omega)} = \frac{G_{(i-1,i)}(j\omega)}{G_{(i,i-1)}(j\omega)} \quad (L-1 \leq i \leq J-1, 2 \leq Z \leq N)
\]

(16)

Based on these relationships of the NOFRFs, a novel method can be developed to determine position of the nonlinear element in system (10).

4. The Nonlinear Component Position Detection Method

Consider a sinusoidal-like force input with two frequency components such that

\[
u(t) = A(\sin(\omega_r t) + \sin(\Delta\omega_r t))
\]

(17)

where \( A \) is the amplitude of the input, and \( \omega_r \) and \( \Delta\omega_r \) are the frequencies of the two components and satisfy \( \Delta\omega_r << \omega_r \). According to the reference [6], the frequency components of the \( D \)th order output \( X_{(i,D)}(j\omega) \) of the \( i \)th mass of system (10) can be determined as

\[
\Omega_D = \bigcup_{n=1}^{N} \Omega_D
\]

(18)

and the frequency components of the \( i \)th mass \( X_i(j\omega) \) can be determined as

\[
\Omega = \bigcup_{D=1}^{N} \Omega_D
\]

(19)

From equations (18) and (19), it can be deduced that the component \( \omega_F \) in the output response \( X_i(j\omega) \) is only contributed by the odd order outputs such as \( X_{(i,1)}(j\omega), X_{(i,3)}(j\omega), \ldots \) and \( X_{(i,2(n+1)\cdot1)}(j\omega) \) ([·] denotes the operator of taking the integer part). Therefore, the component \( \omega_F \) of \( x_i(t) \) \((i=1,\ldots,n)\) can be written as

\[
X_i(j\omega_r) = \sum_{k=1}^{[\sqrt{N}+1]} G_{(2k-1)}(j\omega_r)U_{2k-1}(j\omega_r) \quad (i=1,\ldots,n)
\]

(20)

For the masses on the left of the nonlinear spring or on the right of the input force, substituting equation (15) into (20) yields

\[
X_i(j\omega_r) = \lambda_{(i-1)}^{(i-1)}(j\omega_r) \sum_{k=1}^{[\sqrt{N}+1]} G_{(2k-1)}(j\omega_r)U_{2k-1}(j\omega_r) = \lambda_{(i-1)}^{(i-1)}(j\omega_r)X_{(i-1)}(j\omega_r) \quad (1 \leq i \leq L-2 \text{ or } J \leq i \leq n-1)
\]

(21)

Therefore,
\[ X_{i}(j\omega_r) = \frac{X_{i}(j\omega_r)}{X_{i}(j\omega_r)} \quad (1 \leq i \leq L - 2 \text{ or } J \leq i \leq n - 1) \] (22)

For the masses located between the nonlinear spring and the input force, substituting equation (16) into (20) yields,

\[ X_{i}(j\omega_r) = \dot{X}_{i}(j\omega_r)G_{i+1}(j\omega_r)U_{i}(j\omega_r) + \ddot{X}_{i}(j\omega_r) \sum_{k=1}^{[\lambda/2]} G_{i+1,2k+1}(j\omega_r)U_{2k+1}(j\omega_r) \]

\[ = \dot{X}_{i}(j\omega_r)X_{i}(j\omega_r) \quad (L - 1 \leq i \leq J - 1) \] (23)

Obviously,

\[ \dot{X}_{i}(j\omega_r) = \frac{X_{i}(j\omega_r)}{X_{i}(j\omega_r)} \quad (L - 1 \leq i \leq J - 1) \] (24)

In addition, from equations (18) and (19), it can be deduced that the component \( \omega_r + \Delta\omega_r \) in the output response \( X_{i}(j\omega) \) is only contributed by the even order outputs such as \( X_{i+2}(j\omega) \), \( X_{i+4}(j\omega) \), \ldots, and \( X_{i+(2n/2)}(j\omega) \). Therefore, the component \( \omega_r + \Delta\omega_r \) of \( X_{i}(j\omega) \) \( (i = 1, \ldots, n) \) can be written as

\[ X_{i}(j(\omega_r + \Delta\omega_r)) = \sum_{k=1}^{[\lambda/2]} G_{i+1,2k+1}(j(\omega_r + \Delta\omega_r))U_{2k+1}(j(\omega_r + \Delta\omega_r)) \quad (i = 1, \ldots, n) \] (25)

Substituting equation (14) into (25) yields

\[ X_{i}(j(\omega_r + \Delta\omega_r)) = \dot{X}_{i}(j(\omega_r + \Delta\omega_r)) \sum_{k=1}^{[\lambda/2]} G_{i+1,2k+1}(j(\omega_r + \Delta\omega_r))U_{2k+1}(j(\omega_r + \Delta\omega_r)) \]

\[ = \dot{X}_{i}(j(\omega_r + \Delta\omega_r))X_{i+1}(j(\omega_r + \Delta\omega_r)) \quad (i = 1, \ldots, n - 1) \] (26)

Consequently,

\[ \dot{X}_{i}(j(\omega_r + \Delta\omega_r)) = \frac{X_{i}(j(\omega_r + \Delta\omega_r))}{X_{i+1}(j(\omega_r + \Delta\omega_r))} \quad (i = 1, \ldots, n - 1) \] (27)

Similarly, it can be deduced that

\[ \dot{X}_{i}(j(\omega_r - \Delta\omega_r)) = \frac{X_{i}(j(\omega_r - \Delta\omega_r))}{X_{i+1}(j(\omega_r - \Delta\omega_r))} \quad (i = 1, \ldots, n - 1) \] (28)

As \( \dot{X}_{i}(j\omega) \) \( (i = 1, \ldots, n - 1) \) is a continuous function about \( \omega \) [12], in most cases, given \( \Delta\omega << \omega \) the following relationship is tenable

\[ \dot{X}_{i}(j\omega) \approx \frac{\dot{X}_{i}(j(\omega - \Delta\omega)) + \dot{X}_{i}(j(\omega + \Delta\omega))}{2} \quad (i = 1, \ldots, n - 1) \] (29)

Denote

\[ R_{i+1}^{\dot{X}_{i}} = \frac{X_{i}(j\omega_r)}{X_{i+1}(j\omega_r)} \quad \text{and} \quad R_{i+2}^{\dot{X}_{i}} = \frac{X_{i+1}(j\omega_r - \Delta\omega)}{X_{i+1}(j\omega_r - \Delta\omega)} \quad \text{and} \quad R_{i+2}^{\dot{X}_{i}} = \frac{X_{i+1}(j\omega_r + \Delta\omega)}{X_{i+1}(j\omega_r + \Delta\omega)} \quad \text{and} \quad R_{2i}^{\dot{X}_{i}} = \frac{R_{i+1}^{\dot{X}_{i}} + R_{i+2}^{\dot{X}_{i}}}{2} \]

From equations (22), (24), (27)–(29), it can be known that
\[ R_{i,j=1}^i \approx R_{2,j=1}^i \quad (1 \leq i \leq L - 2 \text{ or } J \leq i \leq n - 1) \] (30)

and

\[ R_{i,j=1}^i \neq R_{2,j=1}^i \quad (L - 1 \leq i \leq J - 1) \] (31)

The relationships given in (30) and (31) provide a simple way to detect the position of nonlinear components in the nonlinear system (10). Obviously, for all the masses where \( R_{i,j=1}^i \neq R_{2,j=1}^i \), the component on the right side of the furthest left mass is the nonlinear one. It is worth noting here that, if the force position is located on the \( L^{th} \) mass or on the left side of the nonlinear component, that is, \( J \leq L \), then the nonlinear component is the one on the right side of the furthest right mass which satisfies the relationship \( R_{i,j=1}^i \neq R_{2,j=1}^i \).

5. Numerical Study

In order to verify the nonlinear component position detection method, a damped 8-DOF oscillator whose fifth spring are nonlinear (\( L=5 \)) was used. The damping was assumed to be a proportional damping, e.g., \( C = \mu K \) in the system. The values of the system parameters are taken as \( m_1 = \ldots = m_n = 1 \), \( r_1 = k_1 = \ldots = k_s = 3.5531 \times 10^4 \), \( \mu = 0.01 \), \( r_s = 0.8 \times r_s, r_3 = 0.4 \times r_3 \), \( w_4 = \mu r_i, w_5 = 0.1 \mu^2 k_5, w_6 = 0. \)

![Figure 2. Fourier spectrum of the response of the first mass.](image)

| \( \omega_F \times 10^5 \) | \( \omega_F - \Delta \omega_F \times 10^5 \) | \( \omega_F + \Delta \omega_F \times 10^5 \) |
|-----------------|-----------------|-----------------|
| \( i=1 \) | 0.0475 - 0.0205i | 0.0406 + 0.0251i | -0.0357 - 0.0032i |
| \( i=2 \) | 0.0960 - 0.0071i | 0.0568 + 0.0665i | -0.0610 - 0.0296i |
| \( i=3 \) | 0.1255 + 0.0665i | 0.0207 + 0.1228i | -0.0529 + 0.0899i |
| \( i=4 \) | 0.0837 + 0.2003i | -0.0917 + 0.1590i | 0.0253 - 0.1648i |
| \( i=5 \) | -0.0785 + 0.3630i | 0.1179 - 0.1681i | -0.0301 + 0.1580i |
| \( i=6 \) | -0.4660 + 0.3624i | -0.0008 - 0.1217i | 0.0487 + 0.0741i |
| \( i=7 \) | 0.0045 + 0.3737i | -0.0562 - 0.0494i | 0.0607 + 0.0022i |
| \( i=8 \) | 0.2454 + 0.2964i | -0.0730 - 0.0028i | 0.0531 - 0.0360i |

The sinusoidal-like force is \( u(t) = \sin(80\pi t) + \sin(4\pi t) \) and is imposed at the sixth mass, that is \( J = 6 \). A fourth-order Runge-Kutta method is used to obtain the forced response of the system. Fig. 2 shows the Fourier spectrum of the response of the first mass where the components marked with text arrows are the three components that are going to be used in the nonlinear component detection procedure.
Table 2. The results of $R_{i1}^{i+1}$, $R_{i2}^{i+1}$, $R_{i2}^{i+1}$ and $R_{i1}^{i+1}$.

| $i$  | $R_{i1}^{i+1}$       | $R_{i2}^{i+1}$       | $R_{i1}^{i+1}$       | $R_{i1}^{i+1}$       |
|------|---------------------|---------------------|---------------------|---------------------|
| $i=$1| 1.7444 + 0.5606i    | 1.7678 + 0.6708i    | 1.7561 + 0.6157i    | 1.7569 + 0.6089i    |
| $i=$2| 1.2208 + 0.7325i    | 1.2808 + 0.8528i    | 1.2508 + 0.7927i    | 1.2482 + 0.7850i    |
| $i=$3| 1.1372 + 0.9385i    | 1.2383 + 1.0107i    | 1.1877 + 0.9746i    | 1.1814 + 0.9706i    |
| $i=$4| -1.1082 - 0.1021i   | -0.9644 - 0.0348i   | 1.0363 - 0.0685i    | 1.4032 + 0.9781i    |
| $i=$5| 0.4845 - 0.3469i    | 0.3960 - 0.3836i    | 0.4403 - 0.3653i    | 1.2189 + 1.0203i    |
| $i=$6| 0.4084 - 0.4595i    | 0.3967 - 0.5588i    | 0.4026 - 0.5091i    | 0.3826 - 0.5044i    |
| $i=$7| 0.7582 + 0.6153i    | 0.8515 - 0.6233i    | 0.8049 - 0.6193i    | 0.8009 - 0.6470i    |

Table 1 gives the values of the three interested frequency components of all the masses, and the calculated results of $R_{i1}^{i+1}$, $R_{i2}^{i+1}$, $R_{22}^{i+1}$ and $R_{i1}^{i+1}$ are given in Table 2. Obviously, $R_{i1}^{i+1} \approx R_{i2}^{i+1}$ at $i = (1, 2, 3, 6$ and 7) and $R_{i1}^{i+1} \neq R_{i2}^{i+1}$ at $i = (4, 5)$. According to the proposed method, it can be known that the component on the right side of the 4th mass is the nonlinear one, that is, the 5th spring component.

6. Conclusions and Remarks

Based on the properties of NOFRFs, a novel method has been developed to detect the position of the nonlinear component in periodic structures. The detection procedure requires exciting the nonlinear systems only once by using a sinusoidal-like input which has two frequency components with one frequency considerably smaller than the other. A numerical study has been performed to demonstrate the effectiveness of this method. The distinct advantage of this method is that it only needs the test data under one sinusoidal-like force, which can be readily carried out in practice. Since the positions of the unusual nonlinear components in periodic structures often correspond to the location of faults, the nonlinear component position detection method is of practical significance in the fault diagnosis for mechanical and structural systems.

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