INEQUALITIES FOR EIGENVALUES OF THE WEIGHTED HODGE LAPLACIAN

DAGUANG CHEN* AND YINGYING ZHANG

Abstract. In this paper, we obtain "universal" inequalities for eigenvalues of the weighted Hodge Laplacian on a compact self-shrinker of Euclidean space. These inequalities generalize the Yang-type and Levitin-Parnovski inequalities for eigenvalues of the Laplacian and Laplacian. From the recursion formula of Cheng and Yang [12], the Yang-type inequality for eigenvalues of the weighted Hodge Laplacian are optimal in the sense of the order of eigenvalues.

1. Introduction

Let $M^m$ be an $m$-dimensional complete Riemannian manifold and $\Omega$ be a bounded domain in $M^m$. The Dirichlet eigenvalue problem of Laplacian is given by

$$\begin{cases}
\Delta u = -\lambda u, & \text{in } \Omega \\
u = 0, & \text{on } \partial \Omega.
\end{cases} \quad (1.1)$$

It is well known that the spectrum of this problem is real and discrete:

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \rightarrow \infty,$$

where each $\lambda_i$ has finite multiplicity which is repeated according to its multiplicity.

The main developments were obtained by Payne, Pólya and Weinberger [32], Hile and Protter [24] and Yang [36]. In 1956, Payne, Pólya and Weinberger [32] proved that

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{mk} \sum_{i=1}^{k} \lambda_i. \quad (1.2)$$

In 1980, Hile and Protter [24] improved (1.2) to

$$\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \geq \frac{mk}{4}. \quad (1.3)$$

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In 1991, Yang (see [36] and more recently [11]) obtained a very sharp inequality
\[
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)(\lambda_{k+1} - (1 + \frac{4}{m})\lambda_i) \leq 0.
\] (1.4)

There has been much work dedicated to extending and strengthening the classical inequalities of Payne-Pólya-Weinberger, Hile-Prottter and Yang. When \(M^m\) is an \(m\)-dimensional compact manifold, there are similar results about the eigenvalue estimates for the Laplacian (see, e.g. [10, 11, 17, 28, 31, 37]). For the compact Riemannian manifolds isometrically immersed in an Euclidean space or a sphere, J. M. Lee [27] proved Hile-Prottter type bounds for eigenvalues for Hodge Laplacian on \(p\)-forms. In 2002, B. Colbois [15] derived a Payne-Pólya-Weinberger type inequality for the rough Laplacian. In [25], S. Ilias and O. Makhoul obtained inequalities for the eigenvalues of the Hodge Laplacian.

In 1991, N. Anghel [1] obtained the analogous estimate of (1.2) for the Dirac operator. In 2009, the Yang-type inequality (1.4) was extended to the eigenvalues of Dirac operator by the first author in [8].

In [19], Harrell gave an abstract algebraic argument involving operators, their commutators and traces, which generalize the original PPW arguments. These algebraic ideas were developed in different contexts to produce many new universal eigenvalues inequalities (see [5, 20–23, 30]).

In present paper, making use of a theorem of Ashbaugh and Hermi [5], we obtain the Yang-type inequality for higher order eigenvalues of the weighted Hodge Laplacian for submanifolds in Euclidean space.

**Theorem 1.1.** Let \(x : (M^m, g) \rightarrow (\mathbb{R}^n, \text{can})\) be a compact self-shrinker, \(\Delta_{p,x} = \Delta_H + \frac{1}{2} \mathcal{L}_{|x|^2}\) (see below (2.9)) be the weighted Hodge Laplacian acting on \(p\)-forms over \(M^m\). Assume that \(\{\lambda_i^{(p)}\}_{i=1}^{\infty}\) are the eigenvalues of \(\Delta_{p,x}\) and \(\{\varphi_i\}_{i=1}^{\infty}\) is a corresponding orthonormal basis of \(p\)-eigenforms. We have, for any \(p \in \{0, 1, \ldots, m\}\),
\[
m \sum_{i=1}^{k} \left(\lambda_{k+1}^{(p)} - \lambda_i^{(p)}\right)^2 \leq \sum_{i=1}^{k} \left(\lambda_{k+1}^{(p)} - \lambda_i^{(p)}\right) \left(4\lambda_i^{(p)} + 2m - \int_{M^m} |x|^2 |\varphi_i|^2 e^{-\frac{|x|^2}{2}} \text{dvol}ight. \left. - 4 \int_{M^m} \langle \text{Ric} \varphi_i, \varphi_i \rangle e^{-\frac{|x|^2}{2}} \text{dvol} + 4 \int_{M^m} \langle \nabla |x|^2 \varphi_i, \varphi_i \rangle e^{-\frac{|x|^2}{2}} \text{dvol} \right)
\] (1.5)

where \(\{e_i\}_{i=1}^{m}\) is a local orthonormal basis of \(TM^m\) with respect to the induced metric \(g\) and \(\text{Ric} = -\omega^i \wedge \iota(e_j)R(e_i, e_j)\) (see also [2.8]) is the curvature operator acting on \(p\)-forms.
Remark 1.1. When \( p = 0 \), i.e., \( \lambda_i := \lambda_i^{(0)} \) are the eigenvalues of the operator \( \mathcal{L} = \Delta_{0,x} = \Delta + \langle x, \cdot \rangle \) acting on scalar functions, we have

\[
m \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left( 4\lambda_i + 2m - \int_{M^m} |x|^2 |\bar{\varphi}_i|^2 e^{-\frac{|x|^2}{4}} \,dvol \right) \leq \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left( 4\lambda_i + 2m - \min_{M^n} |x|^2 \right),
\]

which is Theorem 1.1 in [13]. Therefore, Theorem 1.1 generalizes eigenvalue estimates from the operator \( \mathcal{L} \) to the weighted Hodge Laplacian \( \Delta_{p,x} \).

Remark 1.2. If \( |x| = c, (c > 0) \), the manifold \( M^m \) is a submanifold of sphere \( S^{n-1}(\frac{1}{c}) \) in Euclidean space \( \mathbb{R}^n \). Furthermore, the weighted Hodge Laplacian \( \Delta_{p,x} \) is reduced to the ordinary one.

For a compact self-shrinker (see (2.4)) in Euclidean space, we have

**Corollary 1.1.** Let \( x: (M^m, g) \to (\mathbb{R}^n, \text{can}) \) be a compact self-shrinker, \( H, h \) be the second fundamental form and the mean curvature of the immersion \( x \), respectively. We have, \( p \in \{1, \ldots, m\}, \)

\[
\frac{1}{m} \sum_{i=1}^{k} (\lambda_{k+1}^{(p)} - \lambda_i^{(p)})^2 \\
\leq 4 \sum_{i=1}^{k} \left( \lambda_{k+1}^{(p)} - \lambda_i^{(p)} \right) \left[ \lambda_i^{(p)} + \frac{m}{2} + 1 \right] \\
\left. + \int_{M^m} \left( p|H||h| - \Phi(H, h) - \frac{1}{4} |x|^2 \right) |\bar{\varphi}_i|^2 e^{-\frac{|x|^2}{4}} \,dvol \right] \leq 4 \sum_{i=1}^{k} \left( \lambda_{k+1}^{(p)} - \lambda_i^{(p)} \right) \left[ \lambda_i^{(p)} + \frac{m}{2} + 1 \right] \\
+ \max_{M^m} \left( p|H||h| - \Phi(H, h) - \frac{1}{4} |x|^2 \right),
\]

where \( \Phi(H, h) \) is a function depending on the second fundamental form \( h \) and the mean curvature \( H \) defined in (3.10).

From Theorem 1.1, we can obtain the spectral gaps of the consecutive eigenvalues of the weighted Hodge Laplacian \( \Delta_{p,x} \).
Corollary 1.2. Under the same assumption in Corollary [1.1], we have
\[
\lambda_{k+1}^{(p)} - \lambda_k^{(p)} \leq 2 \left[ \frac{1}{m} \sum_{i=1}^k \lambda_i^{(p)} + \frac{2}{m} \max_{M^m} \left( p |H||h| - \Phi(H,h) - \frac{1}{4} |x|^2 \right) \right]^2 - \left( 1 + \frac{4}{m} \right) \frac{1}{k} \sum_{j=1}^k \left( \lambda_j^{(p)} - \frac{1}{k} \sum_{i=1}^k \lambda_i^{(p)} \right)^{\frac{2}{k}} \frac{2}{k}.
\]

For the lower order eigenvalues of (1.1), in 1956, Payne, Pólya and Weinberger [32] proved that for \( \Omega \subset \mathbb{R}^2 \),
\[
\lambda_2 + \lambda_3 \leq 6 \lambda_1,
\]
which was extended to domains \( \Omega \subset \mathbb{R}^m \) in [35] (or see Section 3.2 of [2])
\[
\sum_{i=1}^m (\lambda_{i+1} - \lambda_i) \leq 4 \lambda_1.
\]

There are also a variety of extensions of results of this type, for examples, see [2,7–11,34]. Recently, S. Ilias and O. Makhoul [26] obtained the universal inequality for eigenvalues of the Hodge Laplacian.

In the second part of this paper, by using an algebraic identity deduced by Levitin and Parnovski [30], we can obtain

**Theorem 1.2.** Let \( x : (M^m,g) \rightarrow (\mathbb{R}^n,can) \) be a compact self-shrinker and \( \Delta_{p,x} \) be the weighted Hodge Laplacian defined acting on \( p \)-forms over \( M^m \). Assume that \( \{\lambda_i^{(p)}\}_{i=1}^\infty \) are the eigenvalues of \( \Delta_{p,x} \) and \( \{\varphi_i\}_{i=1}^\infty \) is a corresponding orthonormal basis of \( p \)-eigenforms. We have, for any \( p \in \{0,1,\ldots,m\} \),
\[
\sum_{i=1}^m (\lambda_{i+t}^{(p)} - \lambda_i^{(p)}) \leq 4 \lambda_i^{(p)} + 2m - \int_{M^m} |x|^2 |\varphi_i|^2 e^{-\frac{|x|^2}{2}} dvol
- 4 \int_{M^m} \langle \text{Ric}(\varphi_i), \varphi_i \rangle e^{-\frac{|x|^2}{2}} dvol \tag{1.8}
+ 4 \int_{M^m} \langle \nabla (\nabla \frac{|x|^2}{2}) \varphi_i, \varphi_i \rangle e^{-\frac{|x|^2}{2}} dvol.
\]

**Remark 1.3.** When \( p = 0 \), i.e., \( \lambda_i = \lambda_i^{(0)} \) is the eigenvalues of the operator \( \mathcal{L} = \Delta_{0,x} = \Delta + \langle x, \cdot \rangle \) acting on scalar functions, we have
\[
\sum_{i=1}^m (\lambda_{i+t} - \lambda_i) \leq 4 \lambda_i + 2m - \int_{M^m} |x|^2 |\varphi_i|^2 e^{-\frac{|x|^2}{2}} dvol
\leq 4 \lambda_i + 2m - \min_{M^m} |x|^2.
\tag{1.9}
\]

Since \( i \) is arbitrary, (1.9) is more general than Proposition 4.1 in [13].
Corollary 1.3. Let \( x : (M^m, g) \rightarrow (\mathbb{R}^n, \text{can}) \) be a self-shrinker, \( H, h \) be the second fundamental form and the mean curvature of the immersion \( x \), respectively. Assume that \( \{\lambda_i(p)\}_{i=1}^\infty \) are the eigenvalues of \( \Delta_{p,x} \) and \( \{\varphi_i\}_{i=1}^\infty \) is a corresponding orthonormal basis of \( p \)-eigenforms. We obtain, for \( p \in \{1, \ldots, m\} \),

\[
\sum_{l=1}^m (\lambda_{i+l}(p) - \lambda_i(p)) \leq 4\lambda_i(p) + 2m + 4 + 4 \int_{M^m} \left( p|H||h| - \Phi(H, h) - \frac{1}{4}|x|^2 \right) |\varphi_i|^2 e^{-\frac{|x|^2}{4}} \, d\text{vol} \\
\leq 4\lambda_i(p) + 2m + 4 + \max_{M^m} \left( p|H||h| - \Phi(H, h) - \frac{1}{4}|x|^2 \right) .
\]

(1.10)

Furthermore, from the recursion formula of Cheng and Yang [12], we can obtain an upper bound for eigenvalue \( \lambda_k^{(p)} \):

Corollary 1.4. Let \( M^m \) be an \( m \)-dimensional compact self-shrinker in \( \mathbb{R}^n \). Then, eigenvalues of the weighted Hodge Laplacian \( \Delta_{p,x} \) satisfy, for any \( k \geq 1 \),

\[
\mu_{k+1} \leq C_0(m) \mu_k \frac{1}{m} \mu_1
\]

where \( C_0(m) \leq 1 + \frac{4}{m} \) is a constant and \( \mu_i = \lambda_i^{(p)} + \frac{m}{2} + 1 + \max_{M^m} \left( p|H||h| - \Phi(H, h) - \frac{1}{4}|x|^2 \right) \).

This paper is organized as follows: In Section 2, we present some formulas for submanifolds in Euclidean space, the definitions of the weighted Hodge Laplacian. In Section 3, in order to prove main theorems, we derive several lemmas for differential forms. In Section 4 and Section 5, we give the proofs of Theorem 1.1 and 1.2.

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2. Preliminaries

2.1. Submanifold in Euclidean space and self-shrinker. Let \( x : M^m \rightarrow \mathbb{R}^n \) be an \( m \)-dimensional submanifold of \( n \)-dimensional Euclidean space \( \mathbb{R}^n \). Let \( \{e_1, \ldots, e_m\} \) be a local orthonormal basis of \( TM^m \) with respect to the induced metric, and \( \{\omega^1, \ldots, \omega^m\} \) be their dual 1-forms. Let \( \{e_{m+1}, \ldots, e_n\} \) be the local orthonormal unit normal vector fields. In this paper we make the following conventions on the range of indices:

\[
1 \leq i, j, k \leq m; \quad m + 1 \leq \alpha, \beta, \gamma \leq n.
\]
Then we have the following structure equations (see [8, 13])

\[ dx = \omega^i e_i, \quad \omega^\alpha = 0, \]
\[ de_i = \omega^j e_j + \omega^\alpha e_\alpha, \quad \omega_i^\alpha = h_{ij}^\alpha \omega^j, \]
\[ de_\alpha = \omega^j e_j + \omega^\beta e_\beta, \]

where \( h_{ij}^\alpha \) denote the components of the second fundamental form of \( M^m \).

We denote by
\[
|h|^2 = \sum_{\alpha,i,j} (h_{ij}^\alpha)^2,
\]
the norm square of the second fundamental form, \( H = \sum_{\alpha} H^\alpha e_\alpha = \sum_{\alpha} (\sum_i h_{ii}^\alpha) e_\alpha \) the mean curvature vector field over \( M^m \).

One can deduce that, pointwise on \( M^m \),
\[
\sum_{A=1}^n |\nabla x^A|^2 = m, \tag{2.2}
\]

and
\[
\frac{1}{2} |x|_{ij}^2 = \frac{1}{2} \sum_{A=1}^n (x^A)^2, \quad \langle h_{ij}^\alpha e_\alpha, x \rangle + \delta_{ij}. \tag{2.3}
\]

The submanifold \( M^m \) is called a self-shrinker [16] if it satisfies the quasilinear elliptic system:
\[
H = -x^\perp, \tag{2.4}
\]

where \( H \) denotes the mean curvature vector field of the immersion and \( \perp \) is the projection onto the normal bundle of \( M^m \).

### 2.2. Differential forms and the weighted Hodge Laplacian

Let \((M^m, g)\) be an \( m \)-dimensional compact Riemannian manifold. For any two \( p \)-forms \( \varphi \) and \( \psi \), we let \( \varphi_{i_1 \ldots i_p} = \varphi(e_{i_1}, \ldots, e_{i_p}) \) and \( \psi_{i_1 \ldots i_p} = \psi(e_{i_1}, \ldots, e_{i_p}) \) denote the components of \( \varphi \) and \( \psi \), with respect to a local orthonormal frame \( \{e_i\}_{i=1}^m \). Their pointwise inner product with respect to Riemannian metric \( g \) is given by
\[
\langle \varphi, \psi \rangle = \sum_{1 \leq i_1 < \cdots < i_p \leq m} \varphi_{i_1 \ldots i_p} \psi_{i_1 \ldots i_p} = \frac{1}{p!} \sum_{1 \leq i_1, \ldots, i_p \leq m} \varphi_{i_1 \ldots i_p} \psi_{i_1 \ldots i_p}.
\]

We denote by \( \Delta_p \) the Hodge Laplacian acting on \( p \)-forms
\[
\Delta_p = (d \delta + \delta d), \tag{2.5}
\]
where \( d \) is the exterior derivative acting on \( p \)-forms and \( \delta \) is the adjoint of \( d \) with respect to Riemannian measure \( d\text{vol} \).

In [6, 33], the operator \( (2.5) \) is generalized to the weighted Hodge Laplacian acting on differential forms. Let \( f \in C^\infty(M^m, \mathbb{R}) \) be a smooth function
defined on $M^m$. When the Riemannian measure is changed from being $d\text{vol}$ to $e^{-f}d\text{vol}$, it is natural to define the weighted Hodge Laplacian by

$$\Delta_{p,f} = d\delta' + \delta'd$$

(2.6)

where $\delta'=e^f\delta e^{-f}$, which is the adjoint operator of the exterior derivative $d$ with respect to Riemannian measure $e^{-f}d\text{vol}$.

For the weighted Hodge Laplacian, we have the following Bochner-Weitzenböck type formula [33]

$$\Delta_{p,f} = \Delta_p + L\nabla f = \nabla^*\nabla - \omega^j \wedge \iota(e_j)R(e_i,e_j) + L\nabla f$$

(2.7)

where $L$ is the Lie derivative, $\iota(X)$ for $X \in \Gamma(TM^m)$ is inner product acting on forms, $\nabla X$ acting on form $\varphi$ is given by

$$\nabla X \varphi = X^j \omega^j \wedge \iota(e_j) \varphi$$

and

$$\mathcal{Ric} = -\omega^j \wedge \iota(e_j)R(e_i,e_j).$$

(2.8)

With respect to the measure $e^{-f}d\text{vol}$, the spectrum of $\Delta_{p,f}$ consists of a nondecreasing, unbounded sequence of eigenvalues with finite multiplicities

$$\text{Spec}(\Delta_{p,f}) = \{0 \leq \lambda_1^{(p)} \leq \lambda_2^{(p)} \leq \lambda_3^{(p)} \leq \cdots \leq \lambda_k^{(p)} \leq \cdots \}.$$ 

Let $x = (x^1, \cdots, x^n) : M^m \rightarrow \mathbb{R}^n$ be an $m$-dimensional submanifold of $\mathbb{R}^n$. In this article, we will consider the operator (2.6) over $M^m$, for $f = e^{\frac{|x|^2}{2}}$,

$$\Delta_{p,x} = d\delta' + \delta'd. \quad (2.9)$$

For $\Delta_{p,x}$ acting on the scalar functions, the operator (2.9) is given by

$$\mathcal{L} = \Delta + \langle x, \cdot \rangle = -\frac{|x|^2}{2} \text{div} \left( e^{-\frac{|x|^2}{2}} d \right) = \delta'd = \Delta_{0,x}. \quad (2.10)$$

where $\Delta$ is the positive operator. If $M^m$ is a self-shrinker, we have

$$\mathcal{L} x^A = x^A, \quad A = 1, \cdots, n. \quad (2.11)$$

3. Some lemmas

In order to prove our main theorems, we will derive some lemmas in this section.

By the direct calculations, we have

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The Laplacian operator is different in [16] with a minus sign.
Lemma 3.1. For $f, u \in C^\infty(M, \mathbb{R})$ and $\varphi \in \bigwedge^p(T^*M^m)$, we have
\[
\mathcal{L}_{\nabla f}(u\varphi) = g(\nabla f, \nabla u)\varphi + u\mathcal{L}_{\nabla f}\varphi. \tag{3.1}
\]
\[
[\Delta_{p,f}, u]\varphi = [\Delta_p, u]\varphi + [\mathcal{L}_{\nabla f}, u]\varphi \tag{3.2}
\]
\[
\delta_f(u\varphi) = -i(\nabla u)\varphi + u\delta_f\varphi \tag{3.3}
\]
where $[\Delta_{p,f}, u]\varphi = \Delta_{p,f}(u\varphi) - u\Delta_{p,f}\varphi$.

Lemma 3.2. Assuming that $T_{ij}$ is a symmetric 2-tensor, we have, for any $p$-form $\varphi$,
\[
T_{ij}\omega^i \wedge i(e_j)\varphi = \frac{1}{p!} \sum_{i_1, \ldots, i_p} (T\varphi)_{i_1 \ldots i_p} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \tag{3.4}
\]
\[
= \frac{1}{(p-1)!} \sum_{i_1, \ldots, i_p} \left( \sum_j T_{ji_1} \varphi_{ji_2 \ldots i_p} \right) \omega^{i_1} \wedge \cdots \wedge \omega^{i_p},
\]
and
\[
\left\langle \sum_{i, j=1}^m T_{ij}\omega^i \wedge i(e_j)\varphi, \varphi \right\rangle = \frac{1}{(p-1)!} \sum_{j, i_1, \ldots, i_p} T_{ji_1} \varphi_{ji_2 \ldots i_p} \varphi_{i_1 \ldots i_p} \tag{3.5}
\]
where $(T\varphi)_{i_1 \ldots i_p} = \sum_{j} \sum_{k=1}^p T_{ji_k} \varphi_{i_1 \ldots j \ldots i_p}.$

Proof. Assuming that $\varphi = \frac{1}{p!} \sum_{i_1, \ldots, i_p} \varphi_{i_1 \ldots i_p} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}$, then we get
\[
T_{ij}\omega^i \wedge i(e_j)\varphi
\]
\[
= \frac{1}{p!} \sum_{i, j=1}^m T_{ij}\omega^i \wedge i(e_j) \left( \sum_{i_1, \ldots, i_p} \varphi_{i_1 \ldots i_p} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p} \right)
\]
\[
= \frac{1}{p!} \sum_{i, j=1}^m \sum_{i_1, \ldots, i_p} (-1)^{k-1} T_{ij} \delta^i_j \varphi_{i_1 \ldots i_p} \omega^i \wedge \omega^{i_1} \wedge \cdots \wedge \hat{\omega}^k \wedge \cdots \wedge \omega^{i_p}
\]
\[
= \frac{1}{p!} \sum_{i, i_1, \ldots, i_k, \ldots, i_p} (-1)^{k-1} T_{ij} \varphi_{ij_1 \ldots j_i} \omega^i \wedge \omega^{i_1} \wedge \cdots \wedge \hat{\omega}^k \wedge \cdots \wedge \omega^{i_p}
\]
\[
= \frac{1}{p!} \sum_{i_1, \ldots, i_p} (T\varphi)_{i_1 \ldots i_p} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}
\]
\[
= \frac{1}{(p-1)!} \sum_{i_1, \ldots, i_p} T_{ji_1} \varphi_{ji_2 \ldots i_p} \omega^{i_1} \wedge \cdots \wedge \omega^{i_p}.
\]
Therefore, we obtain
\[
\langle \sum_{i,j=1}^{m} T_{ij} \omega^i \wedge \iota(e_j) \varphi, \varphi \rangle = \frac{1}{p!} \sum_{i_1, \ldots, i_p} (T \varphi)_{i_1 \ldots i_p} \varphi_{i_1 \ldots i_p} = \frac{1}{(p-1)!} \sum_{j,i_1, \ldots, i_p} T_{ji} \varphi_{j i_2 \ldots i_p} \varphi_{i_1 \ldots i_p}.
\]

\[\square\]

**Lemma 3.3.** Under the same assumptions in Lemma 3.2, then we have

\[
| \sum_{i,j,i_2 \ldots i_p} T_{ij} \varphi_{i i_2 \ldots i_p} \varphi_{j i_2 \ldots i_p} | \leq p! |T| \varphi^2 \tag{3.6}
\]

and

\[
\langle \sum_{i,j=1}^{m} T_{ij} \omega^i \wedge \iota(e_j) \varphi, \varphi \rangle \leq p |T| \varphi^2 \tag{3.7}
\]

where \(|T| = (\sum_{i,j} T_{ij}^2)^{\frac{1}{2}}\).

**Proof.**

\[
| \sum_{i,j,i_2 \ldots i_p} T_{ij} \varphi_{i i_2 \ldots i_p} \varphi_{j i_2 \ldots i_p} | = \sum_{i_2, \ldots, i_p} \left( \sum_{j} \left( \sum_{i} T_{ij} \varphi_{i i_2 \ldots i_p} \right)^2 \sum_{k} \varphi_{k i_2 \ldots i_p}^2 \right)^{\frac{1}{2}} 
\]

\[
\leq \sum_{i_2, \ldots, i_p} \left( \sum_{j} \left( \sum_{i} T_{ij}^2 \varphi_{i i_2 \ldots i_p} \right) \sum_{k} \varphi_{k i_2 \ldots i_p}^2 \right)^{\frac{1}{2}} 
\]

\[
\leq \sum_{i_2, \ldots, i_p} \left( \sum_{j} \sum_{i} T_{ij}^2 \sum_{l} \varphi_{i l_2 \ldots i_p}^2 \sum_{k} \varphi_{k i_2 \ldots i_p}^2 \right)^{\frac{1}{2}} 
\]

\[
= \left( \sum_{i,j} T_{ij}^2 \right)^{\frac{1}{2}} \sum_{k,i_2, \ldots, i_p} \varphi_{k i_2 \ldots i_p}^2 
\]

\[
= |T| \sum_{i,i_2, \ldots, i_p} \varphi_{i i_2 \ldots i_p}^2 
\]

\[
= p! |T| \varphi^2. \tag{3.8}
\]

\[\square\]

**Lemma 3.4.** Assume that \(x : M^m \longrightarrow \mathbb{R}^n\) is a compact self-shrinker, \(H, h\) are the second fundamental form and the mean curvature of the immersion
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x, respectively. We have, for any p-form ϕ, p ∈ {1, . . . , m}
\begin{align*}
\int_{M^m} \langle \nabla (\nabla \frac{|x|^2}{2}) \rangle \varphi, \varphi \rangle e^{-\frac{|x|^2}{2}} d\text{vol} \\
\leq \int_{M^m} |\varphi|^2 e^{-\frac{|x|^2}{2}} d\text{vol} + p \int_{M^m} |H||h||\varphi|^2 e^{-\frac{|x|^2}{2}} d\text{vol} \\
\leq \int_{M^m} |\varphi|^2 e^{-\frac{|x|^2}{2}} d\text{vol} + p \max_{M^m} |H||h|. 
\end{align*}
(3.8)

Proof. From (2.3), and taking \( T_{ij} = \langle h^\alpha_{ij} e_\alpha, x \rangle \) in (3.7), we obtain
\begin{align*}
\int_{M^m} \langle \nabla (\nabla \frac{|x|^2}{2}) \rangle \varphi, \varphi \rangle e^{-\frac{|x|^2}{2}} d\text{vol} \\
= \int_{M^m} \langle \langle h^\alpha_{ij} e_\alpha, x \rangle \omega^j \wedge \iota(e_j) \varphi, \varphi \rangle e^{-\frac{|x|^2}{2}} d\text{vol} + \int_{M^m} |\varphi|^2 e^{-\frac{|x|^2}{2}} d\text{vol} \\
\leq \int_{M^m} |\varphi|^2 e^{-\frac{|x|^2}{2}} d\text{vol} + p \int_{M^m} \left( \sum_{i,j} \langle h^\alpha_{ij} e_\alpha, x \rangle^2 \right)^{\frac{1}{2}} |\varphi|^2 e^{-\frac{|x|^2}{2}} d\text{vol} \\
= \int_{M^m} |\varphi|^2 e^{-\frac{|x|^2}{2}} d\text{vol} + p \int_{M^m} \left( \sum_{i,j} (H^\alpha h^\alpha_{ij})^2 \right)^{\frac{1}{4}} |\varphi|^2 e^{-\frac{|x|^2}{2}} d\text{vol} \\
\leq \int_{M^m} |\varphi|^2 e^{-\frac{|x|^2}{2}} d\text{vol} + p \int_{M^m} |H||h||\varphi|^2 e^{-\frac{|x|^2}{2}} d\text{vol} \\
\leq \int_{M^m} |\varphi|^2 e^{-\frac{|x|^2}{2}} d\text{vol} + p \max_{M^m} |H||h|. 
\end{align*}

Combining Proposition 4.1 in [18] and Theorem 1.1 in [29], we obtain the estimate of \( \text{Ric} \) (2.8) acting on p-forms. (c.f. Theorem 3.2 of [25])

Lemma 3.5.
\begin{align*}
\langle \text{Ric}(\varphi), \varphi \rangle \geq \Phi(h, H)|\varphi|^2, \quad \varphi \in \wedge^p (T^* M^m), 
\end{align*}
(3.9)
where
\begin{align*}
\Phi(h, H) = \begin{cases} 
-p^2 \left[ \left( \frac{m-5}{4} \right) |H|^2 + |h|^2 - \frac{1}{4m^2} \left( \sqrt{m-1} (m-2) |H| \right. \\
\left. - 2 \sqrt{m|h|^2 - |H|^2} \right)^2 \right] - \frac{1}{2} \sqrt{p(p-1)} \left( |H|^2 + |h|^2 \right) \end{cases}. 
\end{align*}
(3.10)

4. INEQUALITIES FOR EIGENVALUES

In order to obtain the extrinsic bounds of higher order eigenvalues of the weighted Hodge Laplacian, we firstly introduce the abstract formula derived by Ashbaugh and Hermi [5].

Let \( \mathcal{H} \) be a complex Hilbert space with inner product \( (, ) \), \( \mathcal{A} : \mathcal{D} \subset \mathcal{H} \rightarrow \mathcal{H} \) a self-adjoint operator defined on a dense domain \( \mathcal{D} \) that is bounded below.
and has a discrete spectrum
\[ \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots. \]
Let \( \{B_k : A(D) \rightarrow \mathcal{S}\}_{k=1}^N \) be a collection of symmetric operators leaving \( D \) invariant and \( \{\varphi_i, \lambda_i\}_{i=1}^\infty \) be the spectral resolution of \( A \). Moreover, \( \{\varphi_i\}_{i=1}^\infty \) consisting of the orthonormal basis w.r.t. inner product \( (,)_D \) for \( \mathcal{S} \) is assumed. Define the commutator \([A, B]\) and the norm \( \|\varphi\| \) by, respectively
\[
[A, B] = AB - BA, \quad \|\varphi\|^2 = (\varphi, \varphi).
\]
Based on commutator algebra and the Rayleigh-Ritz principle, M.S. Ashbaugh and L. Hermi [5] obtained

**Theorem 4.1.** The eigenvalues \( \lambda_i \) of the operator \( A \) satisfy the Yang-type inequality
\[
\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \rho_i \leq \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)\Lambda_i \tag{4.1}
\]
where \( \rho_i, \Lambda \) are defined by, respectively,
\[
\rho_i = \sum_{k=1}^N \langle [A, B_k]\varphi_i, B_k\varphi_i \rangle,
\]
\[
\Lambda_i = \sum_{k=1}^N \| [A, B_k]\varphi_i \|^2.
\]
Applying Theorem 4.1 to the weighted Hodge Laplacian \( \Delta_{p,f} \), we have

**Lemma 4.1.** Let \((M^m, g)\) be an \( m \)-dimensional Riemannian manifold with Riemannian measure \( e^{-f}d\text{vol} \) and \( u \) be a smooth function defined on \( M^m \).
For the eigenvalues \( \{\lambda_i^{(p)}\}_{i=1}^\infty \) of the weighted Hodge Laplacian \( \Delta_{p,f} \) acting on \( p \)-forms, we have, \( p \in \{0, 1, \ldots, m\} \),
\[
\sum_{i=1}^k (\lambda_{k+1}^{(p)} - \lambda_i^{(p)})^2 \int_{M^m} |\nabla u^i|^2 |\varphi_i|^2 e^{-f}d\text{vol} \leq \sum_{i=1}^k (\lambda_{k+1}^{(p)} - \lambda_i^{(p)}) \int_{M^m} \left( (\Delta_{0,f} u^i)^2 |\varphi_i|^2 + 4|\nabla u^i|^2 \varphi_i^2 \right) \tag{4.2}
\]
\[
- 4\langle \Delta_{0,f} u^i \varphi_i, \nabla u^i \varphi_i \rangle e^{-f}d\text{vol}
\]
where \( \{\varphi_i\}_{i=1}^\infty \) is a corresponding orthonormal basis of \( p \)-eigenforms, i.e.
\[
\int_{M^m} \langle \varphi_i, \varphi_j \rangle e^{-f}d\text{vol} = \delta_{ij}.
\]
Proof. It is easy to check that $A = \Delta_{p,f}$ and $B = u \in C^\infty(M^m, \mathbb{R})$ satisfy the conditions in Theorem 4.1. Therefore, by the estimate of (4.1), we have

$$\sum_{i=1}^{k} \left( \lambda^{(p)}_{k+1} - \lambda^{(p)}_{i} \right)^{2} \int_{M} \langle [\Delta_{p,f}, u] \varphi_i, u \varphi_i \rangle e^{-f} \text{dvol} \leq \sum_{i=1}^{k} \left( \lambda^{(p)}_{k+1} - \lambda^{(p)}_{i} \right) \| [\Delta_{p,f}, u] \varphi_i \|^{2},$$

(4.3)

where $\| [\Delta_{p,f}, u] \varphi_i \|^{2} = \int_{M^m} \langle [\Delta_{p,f}, u] \varphi_i, [\Delta_{p,f}, u] \varphi_i \rangle e^{-f} \text{dvol}$. By direct calculations, we have

$$[\Delta_{p,f}, u] \varphi_i = [\Delta_{p} + \mathcal{L}_{\nabla f}, u] \varphi_i = [\Delta_{p}, u] \varphi_i + \mathcal{L}_{\nabla f}, u] \varphi_i.$$  

(4.4)

From (3.1), we obtain

$$[\mathcal{L}_{\nabla f}, u] \varphi_i = g(\nabla f, \nabla u) \varphi_i.$$  

(4.5)

By (2.7), we have

$$[\Delta_{p}, u] \varphi_i = [\nabla^* \nabla, u] \varphi_i = \Delta u \varphi_i - 2\nabla u \varphi_i.$$  

(4.6)

Therefore, from (4.4) to (4.6) we get

$$[\Delta_{p,f}, u] \varphi_i = \Delta_{0,f} u \varphi_i - 2\nabla u \varphi_i.$$  

(4.7)

From (4.7), we have

$$\int_{M^m} \langle [\Delta_{p,f}, u] \varphi_i, u \varphi_i \rangle e^{-f} \text{dvol} = \int_{M^m} \langle \Delta_{0,f} u \varphi_i - 2\nabla u \varphi_i, u \varphi_i \rangle e^{-f} \text{dvol}.$$  

By integration by parts, we have

$$2 \int_{M^m} \langle \nabla \nabla u \varphi_i, u \varphi_i \rangle e^{-f} \text{dvol} = \frac{1}{2} \int_{M^m} \langle |\varphi_i|^2, \nabla u^2 \rangle e^{-f} \text{dvol}$$

$$= \int_{M^m} \langle u \Delta_{0,f} u - |\nabla u|^2 \rangle |\varphi_i|^2 e^{-f} \text{dvol}.$$  

Finally, we obtain

$$\int_{M^m} \langle [\Delta_{p,f}, u] \varphi_i, u \varphi_i \rangle e^{-f} \text{dvol} = \int_{M^m} |\nabla u|^2 |\varphi_i|^2 e^{-f} \text{dvol}. $$  

(4.8)

On the other hand, using (4.7), we get

$$\| [\Delta_{p,f}, u] \varphi_i \|^{2} = \int_{M^m} \left( (\Delta_{0,f} u)^2 |\varphi_i|^2 + 4|\nabla \nabla u \varphi_i|^2 ight) e^{-f} \text{dvol}.$$  

(4.9)
Inserting (4.8) and (4.9) into (4.3), we obtain (4.2).

Proof of Theorem 1.1. Letting $f = \frac{1}{2}|x|^2$ and therefore $\Delta_{p,x} = \Delta_{p,f}$, substituting $u = x^A, A = 1, \cdots, n$, the $p^{th}$ component of the isometric immersion $x = (x^1, \cdots, x^n) : M^m \rightarrow \mathbb{R}^n$ in (4.2), and taking summation on $p$ from 1 to $n$, we have

$$\sum_{A=1}^n \sum_{i=1}^k (\lambda_k^{(p)} - \lambda_i^{(p)})^2 \int_{M^m} |\nabla x^A|^2 |\varphi_i|^2 e^{-\frac{|x|^2}{2}} \text{dvol}$$

$$\leq \sum_{i=1}^k (\lambda_k^{(p)} - \lambda_i^{(p)}) \int_{M^m} \sum_{A=1}^n (|\mathcal{L} x^A|^2 |\varphi_i|^2 + 4|\nabla_{x^A} \varphi_i|^2 - 4\langle \mathcal{L} x^A, \nabla_{x^A} \varphi_i \rangle) e^{-\frac{|x|^2}{2}} \text{dvol},$$

where $\mathcal{L}$ is the weighted Hodge Laplacian acting on functions given by (2.10).

From (2.2), we obtain

$$\int_{M^m} \sum_{p=1}^n |\nabla x^A|^2 |\varphi_i|^2 e^{-\frac{|x|^2}{2}} \text{dvol} = m \int_{M^m} |\varphi_i|^2 e^{-\frac{|x|^2}{2}} \text{dvol} = m.$$  

(4.11)

From (2.11) and (4.9), we get

$$\sum_{A=1}^n \|\Delta_{p,x^A} \varphi_i\|^2$$

$$= \sum_{A=1}^n \int_{M^m} \left( |\mathcal{L} x^A|^2 |\varphi_i|^2 + 4|\nabla_{x^A} \varphi_i|^2 - 4\langle \mathcal{L} x^A, \nabla_{x^A} \varphi_i \rangle \right) e^{-\frac{|x|^2}{2}} \text{dvol}$$

$$= \sum_{A=1}^n \int_{M^m} \left( x^A |\varphi_i|^2 + 4|\nabla_{x^A} \varphi_i|^2 - 4\langle x^A \varphi_i, \nabla_{x^A} \varphi_i \rangle \right) e^{-\frac{|x|^2}{2}} \text{dvol}.$$  

(4.12)

Since $M^m$ is a compact self-shrinker, by integration by parts and (2.3), we have

$$4 \sum_{A=1}^n \int_{M^m} \langle x^A \varphi_i, \nabla_{x^A} \varphi_i \rangle e^{-\frac{|x|^2}{2}} \text{dvol} = - \int_{M^m} 2(m - |x|^2)|\varphi_i|^2 e^{-\frac{|x|^2}{2}} \text{dvol}$$

Since $\sum_{A=1}^n |\nabla_{x^A} \varphi_i|^2 = |\nabla \varphi|^2$, we have

$$\sum_{A=1}^n \|\Delta_{p,x^A} \varphi_i\|^2 = 2m - \int_{M^m} |x|^2 |\varphi_i|^2 e^{-\frac{|x|^2}{2}} \text{dvol} + 4 \int_{M^m} |\nabla \varphi|^2 e^{-\frac{|x|^2}{2}} \text{dvol}.$$  

(4.13)
By integration by parts, from (2.7), (3.8) and (3.9), we have

\[
\int_{M^m} |\nabla \varphi_i|^2 e^{-\frac{|x|^2}{2}} \, d\text{vol} = \int_{M^m} \langle \nabla^{\ast} \nabla \varphi_i, \varphi_i \rangle e^{-\frac{|x|^2}{2}} \, d\text{vol}
\]

\[
= \int_{M^m} \langle (\Delta_{p,x} - \text{Ric} \nabla \frac{|x|^2}{2}) \varphi_i, \varphi_i \rangle e^{-\frac{|x|^2}{2}} \, d\text{vol}
\]

\[
= \lambda_i^{(p)} - \int_{M^m} \langle \text{Ric} \varphi_i, \varphi_i \rangle e^{-\frac{|x|^2}{2}} \, d\text{vol}
\]

\[
+ \int_{M^m} \langle \nabla \frac{|x|^2}{2} \varphi_i, \varphi_i \rangle e^{-\frac{|x|^2}{2}} \, d\text{vol},
\]

where $\nabla^{\ast}$ is the adjoint operator of $\nabla$ with respect to the Riemannian measure $e^{-\frac{|x|^2}{2}} \, d\text{vol}$. Therefore, we obtain

\[
\sum_{A=1}^{n} \| \Delta_{p,x} x^A |\varphi_i| \|^2 = 4\lambda_i^{(p)} + 2m
\]

\[
+ 4 \int_{M^m} \langle \nabla \frac{|x|^2}{2} \varphi_i, \varphi_i \rangle e^{-\frac{|x|^2}{2}} \, d\text{vol}
\]

\[
- 4 \int_{M^m} \langle \text{Ric} \varphi_i, \varphi_i \rangle e^{-\frac{|x|^2}{2}} \, d\text{vol}
\]

\[
- \int_{M^m} |x|^2 |\varphi_i|^2 e^{-\frac{|x|^2}{2}} \, d\text{vol},
\]

which completes the proof of Theorem 1.1. \qed
Proof of Corollary 1.1. From (4.10), (3.8) and (3.9), we have

\[ m\sum_{i=1}^{k} (\lambda_{k+1}^{(p)} - \lambda_{i}^{(p)})^2 \leq \sum_{i=1}^{k} (\lambda_{k+1}^{(p)} - \lambda_{i}^{(p)}) (4\lambda_{i}^{(p)} + 2m + 4 + 4p \int_{M^m} |H||h|\varphi_i|^2 e^{-\frac{|x|^2}{4}} \, dvol - \int_{M^m} |x|^2 |\varphi_i|^2 e^{-\frac{|x|^2}{4}} \, dvol - 4 \int_{M^m} (\nabla \varphi_i, \varphi_i) e^{-\frac{|x|^2}{4}} \, dvol) \]

\[ \leq 4 \sum_{i=1}^{k} (\lambda_{k+1}^{(p)} - \lambda_{i}^{(p)}) \left[ \lambda_{i}^{(p)} + \frac{m}{2} + 1 + \int_{M^m} (p|H||h| - \Phi(H, h) - \frac{1}{4}|x|^2) |\varphi_i|^2 e^{-\frac{|x|^2}{4}} \, dvol \right] \]

\[ \leq 4 \sum_{i=1}^{k} (\lambda_{k+1}^{(p)} - \lambda_{i}^{(p)}) \left[ \lambda_{i}^{(p)} + \frac{m}{2} + 1 + \max_{M^m} (p|H||h| - \Phi(H, h) - \frac{1}{4}|x|^2) \right]. \]

\[ \square \]

5. Generalization of the Levitin–Parnovski inequality

In this section, we will give the proof of Theorem 1.2 by similar argument in [26]. Firstly, we recall the following algebraic identity obtained by Levitin and Parnovski (see identity 2.2 of Theorem 2.2 in [30]).

Lemma 5.1. Let \( L \) and \( G \) be two self-adjoint operators with domains \( D_L \) and \( D_G \) contained in a same Hilbert space and such that \( G(D_L) \subseteq D_L \subseteq D_G \). Let \( \lambda_j \) and \( u_j \) be the eigenvalues and orthonormal eigenvectors of \( L \). Then, for each \( j \),

\[ \sum_{k=1}^{\infty} \frac{|\langle [L, G]u_j, u_k \rangle|^2}{\lambda_k - \lambda_j} = -\frac{1}{2} \langle [L, G]u_j, u_j \rangle_{L^2} \]  \hspace{1cm} (5.1)

(The summation is over all \( k \) and is correctly defined even when \( \lambda_k = \lambda_j \) because in this case \( \langle [L, G]u_j, u_k \rangle = 0 \).

Proof of Theorem 1.2. By applying Lemma 5.1 with \( L = \Delta_{p,x} \) and \( G = x^A \), where \( x^A \) is one of the components of the isometric immersion \( x \), we have

\[ \sum_{k=1}^{\infty} \frac{|\langle [\Delta_{p,x}, x^A]u_j, u_k \rangle|^2}{\lambda_k - \lambda_j} = -\frac{1}{2} \langle [\Delta_{p,x}, x^A]u_j, u_j \rangle_{L^2}. \]  \hspace{1cm} (5.2)
From (4.7), we have

\[ [(\Delta_{p,x}, x^A), x^A] \psi_i = \sum_{\lambda, m} \psi_i^{(\lambda, m)} = \Delta_{p,x} x^A \psi_i - x^A (\Delta_{p,x} x^A) \psi_i \]

\[ = 2\nabla_{x^A} (x^A \psi_i) - 2\nabla_{x^A} (x^A \psi_i) \]

\[ = -2\nabla_{x^A} (x^A \psi_i) + 2x^A \nabla_{x^A} \psi_i \]

\[ = -2|\nabla x^A|^2 \psi_i, \]

hence

\[ -\frac{1}{2} \int_{M^m} \langle [(\Delta_{p,x}, x^A), x^A] \psi_i, \psi_i \rangle e^{-\frac{|x|^2}{2}} d\text{vol} = \int_{M^m} |\nabla x^A|^2 |\psi_i|^2 e^{-\frac{|x|^2}{2}} d\text{vol}. \]

From (5.2), we have

\[ \int_{M^m} |\nabla x^A|^2 |\psi_i|^2 e^{-\frac{|x|^2}{2}} d\text{vol} \]

\[ = \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{(p)} - \lambda_1^{(p)}} \left( \int_{M^m} \langle [(\Delta_{p,x}, x^A) \psi_i, \psi_k \rangle e^{-\frac{|x|^2}{2}} d\text{vol} \right)^2. \tag{5.3} \]

For a fixed \( i \), from the Gram-Schmidt orthogonalization, we can find the coordinate system \( \{x^A\}_{A=1}^n \) in Euclidean space \( \mathbb{R}^n \) such that the matrix

\[ \left( \int_{M^m} \langle [(\Delta_{p,x}, x^A) \psi_i, \psi_{i+k} \rangle e^{-\frac{|x|^2}{2}} d\text{vol} \right)_{1 \leq k, A \leq n} \]

is a real upper triangular matrix. That is,

\[ \int_{M^m} \langle [(\Delta_{p,x}, x^A) \psi_i, \psi_{i+k} \rangle e^{-\frac{|x|^2}{2}} d\text{vol} = 0, \quad 1 \leq k < A \leq n. \tag{5.4} \]

By (5.4), we can estimate the right hand side of (5.3) in the following

\[ \sum_{k=1}^{\infty} \frac{1}{\lambda_k^{(p)} - \lambda_1^{(p)}} \left( \int_{M^m} \langle [(\Delta_{p,x}, x^A) \psi_i, \psi_k \rangle e^{-\frac{|x|^2}{2}} d\text{vol} \right)^2 \]

\[ = \sum_{k=i+1}^{\infty} \frac{1}{\lambda_k^{(p)} - \lambda_1^{(p)}} \left( \int_{M^m} \langle [(\Delta_{p,x}, x^A) \psi_i, \psi_k \rangle e^{-\frac{|x|^2}{2}} d\text{vol} \right)^2 \]

\[ + \sum_{k=i+1}^{\infty} \frac{1}{\lambda_k^{(p)} - \lambda_1^{(p)}} \left( \int_{M^m} \langle [(\Delta_{p,x}, x^A) \psi_i, \psi_k \rangle e^{-\frac{|x|^2}{2}} d\text{vol} \right)^2 \]

\[ \leq \sum_{k=i+1}^{\infty} \frac{1}{\lambda_k^{(p)} - \lambda_1^{(p)}} \left( \int_{M^m} \langle [(\Delta_{p,x}, x^A) \psi_i, \psi_k \rangle e^{-\frac{|x|^2}{2}} d\text{vol} \right)^2 \]

\[ \leq \frac{1}{\lambda_1^{(p)}} \sum_{k=1}^{\infty} \left( \int_{M^m} \langle [(\Delta_{p,x}, x^A) \psi_i, \psi_k \rangle e^{-\frac{|x|^2}{2}} d\text{vol} \right)^2 \]

\[ \leq \frac{1}{\lambda_1^{(p)}} \int_{M^m} \langle [(\Delta_{p,x}, x^A) \psi_i, \psi_k \rangle e^{-\frac{|x|^2}{2}} d\text{vol} \]

where Parceval’s identity is used in the last equality.
Taking summation on $A$ from 1 to $n$, from (5.3), (4.13), (4.14) and (2.7), we have
\[
\sum_{A=1}^{n} (\lambda_{1+A}^{(p)} - \lambda_{1}^{(p)}) \int_{M^n} |\nabla x^A|^2 |\varphi_i|^2 e^{-\frac{|x|^2}{2}} \, d\text{vol} \\
\leq \sum_{A=1}^{n} \int_{M^n} (\Delta_{p,x} x^A) \varphi_i|^2 e^{-\frac{|x|^2}{2}} \, d\text{vol} \\
= 2m - \int_{M^n} |x|^2 |\varphi_i|^2 e^{-\frac{|x|^2}{2}} \, d\text{vol} + 4 \int_{M^n} |\nabla \varphi_i|^2 e^{-\frac{|x|^2}{2}} \, d\text{vol} \\
= 4\lambda_i^{(p)} + 2m - \int_{M^n} |x|^2 |\varphi_i|^2 e^{-\frac{|x|^2}{2}} \, d\text{vol} \\
- 4 \int_{M^n} \langle \text{Ric} \varphi_i, \varphi_i \rangle e^{-\frac{|x|^2}{2}} \, d\text{vol} \\
+ 4 \int_{M^n} \langle \nabla (\frac{|x|^2}{2}) \varphi_i, \varphi_i \rangle e^{-\frac{|x|^2}{2}} \, d\text{vol}. 
\] (5.5)

Since $M^n$ is isometrically immersed in $\mathbb{R}^n$, it is easy to check
\[
\sum_{A=1}^{n} \lambda_{1+A}^{(p)} |\nabla x^A|^2 \geq \sum_{i=1}^{m} \lambda_i^{(p)}. 
\] (5.6)

Therefore, we have
\[
\sum_{i=1}^{m} (\lambda_{1+i}^{(p)} - \lambda_{1}^{(p)}) \leq 4\lambda_i^{(p)} + 2m - \int_{M^n} |x|^2 |\varphi_i|^2 e^{-\frac{|x|^2}{2}} \, d\text{vol} \\
- 4 \int_{M^n} \langle \text{Ric} \varphi_i, \varphi_i \rangle e^{-\frac{|x|^2}{2}} \, d\text{vol} \\
+ 4 \int_{M^n} \langle \nabla (\frac{|x|^2}{2}) \varphi_i, \varphi_i \rangle e^{-\frac{|x|^2}{2}} \, d\text{vol}. 
\]

Proof of Corollary 1.3. The proof of Corollary 1.3 follows directly from (5.8) and (3.9). □

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