Quantum Spacetimes and Finite N Effects in 4D Super Yang-Mills Theories

Pei-Ming Ho\(^1\), Sanjaye Ramgoolam\(^2\) and Radu Tatar\(^2\)

1 National Taiwan University,
Taipei 10764, Taiwan, R.O.C.

2 Brown University
Providence, RI 02912

pmho@phys.ntu.edu.tw, ramgosk@het.brown.edu, tatar@het.brown.edu

The truncation in the number of single-trace chiral primary operators of \(\mathcal{N} = 4\) SYM and its conjectured connection with gravity on quantum spacetimes are elaborated. The model of quantum spacetime we use is \(AdS_5^q \times S_5^q\) for \(q\) a root of unity. The quantum sphere is defined as a homogeneous space with manifest \(SU_q(3)\) symmetry, but as anticipated from the field theory correspondence, we show that there is a hidden \(SO_q(6)\) symmetry in the construction. We also study some properties of quantum space quotients as candidate models for the quantum spacetime relevant for some \(Z_n\) quiver quotients of the \(\mathcal{N} = 4\) theory which break SUSY to \(\mathcal{N} = 2\). We find various qualitative agreements between the proposed models and the properties of the corresponding finite \(N\) gauge theories.
1. Introduction and summary

The Maldacena duality \cite{1,2,3,4} gives a relation between type IIB string theory on $AdS_5 \times S^5$ and the $\mathcal{N} = 4$ superconformal four dimensional super Yang-Mills. The gauge group of the field theory is $SU(N)$ when the flux through $S^5$ is $N$. In the case of large $N$ and large effective coupling, Maldacena’s conjecture relates the corresponding field theory and the classical supergravity. Finite $N$ effects contain important information about the qualitative novelties of quantum gravity compared to classical gravity.

It was proposed in \cite{5} that the quantum corrections in the $AdS \times S$ background have the effect of deforming spacetime to a non-commutative manifold. The concrete model studied there was $AdS_3 \times S^3$ where the group structure of the manifold allowed a simple non-commutative candidate by using quantum groups. An important part of the evidence was a quantum group interpretation of the cutoff on single particle chiral primaries, first studied under the heading of “stringy exclusion principle” in \cite{6}.

Here we develop the same line of argument to understand analogous cutoffs in the spectrum of chiral primaries of $\mathcal{N} = 4$ super Yang-Mills. The cutoffs originate from the fact that the $U(N)$ invariants of the form $tr\Phi^l$, where $\Phi$ is a matrix in the adjoint representation, are not independent for all values of $l$ and the set of independent invariants truncates at $N$. Since the Yang-Mills theory has an $SO(6)$ global symmetry, we get an $SO(6)$ covariant cutoff on the chiral primaries. In the large $N$ limit, the chiral primaries are matched on the gravitational side with modes coming form KK reduction on $S^5$. To understand the cutoff at finite $N$, we postulate that the $S^5$ is deformed to a quantum sphere in the following way: $S^5$ has a description as a coset space $SU(3)/SU(2)$ which generalizes to the q-deformed case as $SU_q(3)/SU_q(2)$. This construction is shown to have $SO_q(6)$ hidden symmetry by mapping it to another construction of a q-sphere based on \cite{7}, thus explaining why KK reduction on it gives a truncated set of reps. of $SO(6)$.

We then discuss $\mathcal{N} = 2$ theories obtained by taking $Z_n$ quotients of $U(Nn)$ theories with $\mathcal{N} = 4$ supersymmetry, which are dual to gravity on $AdS_5 \times S^5/Z_n$. We discuss the cutoffs in the spectrum of chiral primaries in the quotient theories. The main point is that, if we ignore states coming from twisted sectors associated with non-trivial reps. of $Z_n$, the cutoffs occur at $Nn$ as in the parent theory. To find candidates for the quantum space dual of the quiver theory we identify appropriate automorphisms of the dual theory, which are used to quotient the quantum $S_q^5$ to give a space with $SU_q(2) \times U_q(1)$ symmetry. When twisted sectors are taken into account, the cutoff in some chiral primaries charged...
under the $U(1)$ happens not at $Nn$ but at $N$. We begin a discussion of the quantum space explanation of this change of cutoffs. This requires the description of the $S_q^5$ as an $S^1$ fibration over a q-deformed ball, which is acted upon by the quotient.

2. Truncation of generating chiral primary operators in $\mathcal{N} = 4$ super Yang-Mills

Consider the chiral primaries of this theory which are of the form:

$$C_{a_1a_2\ldots a_l} tr (\Phi^{a_1} \cdots \Phi^{a_l})$$

(2.1)

where the $C$ are traceless symmetric tensors of $SO(6)$. These symmetric tensors can be decomposed under $SU(3)$ and contain the symmetric rep. of $SU(3)$ corresponding to a Young tableau with one row of length $l$. The polynomials corresponding to this Young tableau are $C_{i_1,i_2\ldots i_l} tr (\Phi^{i_1} \Phi^{i_2} \cdots \Phi^{i_l})$, where the $C$ are symmetric, and the $i_k$ are indices running from 1 to 3 and the corresponding scalars are complex.

It is useful in the discussion of cutoffs to decompose the invariant polynomials in reps. of $SU(3)$. In order to obtain the decomposition of symmetric representations of $SO(6)$ into representations of $SU(3)$, we can use the isomorphism between $SO(6)$ and $SU(4)$ under which the vector representation of $SO(6)$ goes into the antisymmetric representation of $SU(4)$, and each symmetric traceless representation of $SO(6)$ goes into the $(0, k, 0)$ representation of $SU(4)$. The branching rules for the $(0, k, 0)$ representations of $SU(4)$ into representations of $SU(3)$ are, for example,

For $k = 1$ \quad $6 \rightarrow 3 \oplus \bar{3}$,

For $k = 2$ \quad $20 \rightarrow 6 \oplus \bar{6} \oplus 8$,  

(2.2)

For $k = 3$ \quad $50 \rightarrow 10 + 10 + 15 + \bar{15}$.

So the vector representation of $SO(6)$ gives a fundamental and an anti-fundamental representation corresponding to $tr (\Phi^i)$ and $tr (\Phi^* i)$ $(i=1,2,3)$, which include the chiral primary operator of dimension 1. The symmetric traceless representation $20$ gives the following operators: $tr (\Phi^{(i_1,j_1)} \Phi^{(i_2,j_2)})$ (the $6$ representation), $tr (\Phi^* (i_1,j_1) \Phi^* (i_2,j_2))$ (the $\bar{6}$ representation), $tr (\Phi^{(i_1,j_1)} \Phi^* (i_2,j_2))$ (the $8$ representation). These include the chiral primary operator of dimension 2. The representation $50$ of $SO(6)$ gives the following: $tr (\Phi^{(i_1,j_1) \Phi^* (i_2,j_2)})$ (the $10$ representation), its conjugate one (involving the complex conjugate fields and corresponding to the $\bar{10}$ representation), $tr (\Phi^{(i_1,j_1) \Phi^* (i_2,j_2) \Phi^* (i_3,j_3)})$ (the $15$ representation) and its complex
conjugate (the $\bar{15}$ representation). These contain the chiral primary of dimension 3. We can continue the discussion to show that each symmetric traceless rep. of $SO(6)$ contains a chiral primary belonging to a rep. of $SU(3)$ associated to a symmetric Young tableau.

The chiral primary operators are not all independent at finite $N$. Consider first the operators which look like $tr(\Phi^{N+1})$ when the gauge group is $U(N)$. If the generators of the Lie algebra are $T^a$, the chiral primary operator is written as $tr(\Phi_a T^a)^{N+1} = \Phi_1 \cdots \Phi_{N+1} tr(T^{a_1} \cdots T^{a_{N+1}})$. But $tr(T^{a_1} \cdots T^{a_{N+1}})$ is just the $C_{N+1}$ Casimir operator of $U(N)$ and we know that $U(N)$ has only $N$ independent Casimir operators so $C_{N+1}$ is not independent and can be written in terms of lower Casimir operators. Therefore, the conclusion is that $tr(\Phi^{N+1})$ can be written in terms of $tr(\Phi^N)$, $tr(\Phi^{N-1})$ and so on. Thus $tr(\Phi^{N+1})$ does not describe a single particle state. The conclusion is that for the group $U(N)$ we have a truncation on the chiral primary operators such that the highest power is $N$. For instance, $\Phi$ can be one of the three $\Phi_i$’s. The rest of the symmetric polynomials like $tr(\Phi_1^2 \Phi_2^2 \Phi_3^2)$ can also be decomposed since they can be obtained from the chiral primary by action of $SO(6)$. In fact for any given dimension, there is only one short representation, so that operators of the form

$$tr(F\Phi\Phi\cdots)$$

(2.3)

can also be shown to be decomposable as they are obtained from the chiral primary by action of the SUSY operators.

The result is that we have a truncation on the short representations for the gauge group $U(N)$, the maximal symmetric traceless representation of $SO(6)$ being the one with $N$ boxes in the Young tableau. This will allow us to identify a candidate quantum sphere relevant for the spacetime understanding of finite $N$ effects. After describing some preliminaries on quantum groups we will describe the relevant quantum space in section 3.3

3. Non-commutative spacetime

Natural non-commutative candidates for $AdS_5 \times S^5$ are obtained by deforming the coset structure of the spaces involved using quantum groups. We will be concerned with some detailed properties of the q-deformed $S^5$ in this paper, which are relevant for the truncation in KK modes. The unit five sphere is the space $\sum_{i=1}^{6} x_i^2 = 1$ where $x_i$ are coordinates in $\mathbb{R}^5$. $SO(6)$ acts transitively on the solutions of this equation. A point, say $(0,0,0,0,0,1)$ is left fixed by $SO(5)$. This allows an identification of the sphere with the
coset $SO(6)/SO(5)$. It is possible to consider $\mathbf{R}^6$ as a complex space $\mathbf{C}^3$ with coordinates $z_0 = x_1 + ix_2, z_1 = x_3 + ix_4, z_2 = x_5 + ix_6$ and the sphere becomes a surface $\sum_{j=0}^{2} |z_j|^2 = 1$ in $\mathbf{C}^3$. In this case the sphere is seen as a coset $SU(3)/SU(2)$. The latter coset space structure allows a simple quantum group generalization.

3.1. Preliminaries of quantum groups

The standard $q$-deformation of quantum groups is given in [7]. For a matrix $T^i_j$ of a quantum group, the commutation relations among the matrix elements are given by

$$\hat{R}_{12}T_1T_2 = T_1T_2\hat{R}_{12}. \quad (3.1)$$

This is a shorthand of the following

$$\hat{R}^{ij}_{kl}T^k_mT^l_n = T^i_kT^j_l\hat{R}^{kl}_{mn}. \quad (3.2)$$

The matrix elements $T^i_j$ in the fundamental rep. generate the algebra of functions on the quantum group. The matrix $T$ has an inverse $T^{-1}$ given by the antipode. The antipode is an automorphism $S$ of this algebra such that $S(T^i_j) = (T^{-1})^i_j$.

For $SL_q(N; \mathbf{C})$, the $\hat{R}$ matrix is given by

$$\hat{R}^{ij}_{kl} = \delta^i_k\delta^j_l(1 + (q - 1)\delta^{ij}) + \lambda\delta^i_k\delta^j_l\theta(j - i), \quad (3.3)$$

where $\theta(j - i) = 1$ if $j > i$ and $\theta(j - i) = 0$ otherwise. In [7] a $*$-anti-involution is given for $SL_q(N; \mathbf{C})$. With respect to this $*$-anti-involution, a real form of $SL_q(N; \mathbf{C})$ can be defined by $(T^i_j)^* = (T^{-1})^j_i$. For $q$ being a phase, the real form $SL_q(N)$ can be defined by $T^* = T$; and for $q$ being real, the real form $SU_q(N)$ can be defined. (We require that the $*$ of a complex number be its complex conjugation, so e.g. $q^* = q^{-1}$ if $q$ is a phase.) In this paper what we need is $SU_q(N)$, but there is no $*$-anti-involution for this purpose when $q$ is a phase. It turns out that the appropriate $*$-structure is an involution instead of anti-involution. (An involution does not reverse the ordering of a product and an anti-involution does.) Let

$$g^i_j = q^i\delta^j_i, \quad (3.4)$$

where $i, j = 0, 1, \cdots, N - 1$, then we define

$$T^\dagger = g^{-1}T^{-1}g, \quad (3.5)$$
where $T^\dagger$ is the transpose of $T^\ast$. One can check that this definition gives a $\ast$-involution. First, because $S(S(T)) = g^2 T g^{-2}$, $(T^\ast)^\ast = T$. Secondly, (3.1) is invariant under the action of $\ast$. To check this, one can use the following identities
\begin{align*}
\hat{R}_{kl}^{ij} &= \hat{R}_{ij}^{kl}, \quad (3.6) \\
\hat{R}(q^{-1})_{12} &= \hat{R}(q)_{21}^{-1}, \quad (3.7) \\
g_1^{-1} g_2^{-1} \hat{R}_{12} g_1 g_2 &= \hat{R}_{12}. \quad (3.8)
\end{align*}

3.2. The quantum sphere

To begin we define the quantum complex plane $C_q^N$ which has the symmetry group $SU_q(N)$ acting on it. The algebra of functions on $C_q^N$ is generated by the coordinates $z_i$, $i = 0, 1, \ldots, N - 1$, which satisfy the following commutation relations
\begin{equation}
z_1 z_2 = q^{-1} z_1 z_2 \hat{R}_{12}. \quad (3.9)
\end{equation}

More explicitly, it is
\begin{equation}
z_i z_j = q^{-1} z_k z_l \hat{R}_{kl}^{ij}. \quad (3.10)
\end{equation}

The coordinates $z$ transform under an $SU_q(N)$ matrix $T$ as
\begin{equation}
z \to z T. \quad (3.11)
\end{equation}

We let all $z_k$’s to commute with all $T^i_j$’s. Due to (3.1), the relations (3.9) is preserved by this transformation. (Note that $T^i_j$ commutes with $z_k$ for all $i, j, k$.) The complex conjugation of $z$ is defined as a $\ast$-involution. Let $\bar{z} = z^\ast$. The $\ast$ of (3.9) is
\begin{equation}
\bar{z}_1 \bar{z}_2 = q^{-1} \hat{R}_{21} \bar{z}_1 \bar{z}_2, \quad (3.12)
\end{equation}

which is covariant under the transformation $\bar{z} \to T^\dagger \bar{z}$, as the $\ast$ of (3.11). To complete the definition of the algebra on $C_q^N$, we also need to define how $z$ commutes with $\bar{z}$. Let
\begin{equation}
\bar{z}_1 z_1 = q^{-1} z_2 g_2 \hat{R}_{12} g_1^{-1} \bar{z}_2, \quad (3.13)
\end{equation}

which means
\begin{equation}
\bar{z}^i z_k = q^{-1+j-k} z_j \hat{R}_{kl}^{ij} \bar{z}_l. \quad (3.14)
\end{equation}
It is covariant under the action of $SU_q(N)$.

The coefficient of $q^{-1}$ on the right hand side of (3.13) is chosen such that the radius squared

$$r^2 = zg\bar{z}$$  \hspace{1cm} (3.15)$$
is a central element in the algebra. One can also check that $r^2$ is real: $(r^2)^* = r^2$. Since $r^2$ commutes with everything else, we can define a new algebra by the algebra of $z, \bar{z}$, modulo the condition $r^2 = 1$. This is the algebra of functions on the quantum sphere $S^2_qN-1$. It can be identified with the quantum sphere defined as $SU_q(N)/SU_q(N-1)$ \cite{8}. Explicitly, the commutation relations of $z, \bar{z}$ are

\begin{align*}
z_i z_j &= q z_j z_i, \quad i < j, \\
z_i \bar{z}^j &= q \bar{z}^j z_i, \quad i \neq j, \\
\bar{z}^i z^j &= q^{-1} \bar{z}^j z^i, \quad i < j, \\
z_i \bar{z}^i &= \bar{z}^i z_i - q^{-1} \lambda \sum_{j>i} q^{j-i} z_j \bar{z}^j. \hspace{1cm} (3.19)
\end{align*}

Another natural candidate for the q-deformed sphere is the $SO_q(N)$-covariant quantum Euclidean space $\mathbb{R}^N_q$ modulo the unit radius condition. The quantum group $SO_q(N; \mathbb{C})$ is defined by (3.1) with a different $\hat{R}$ matrix, which also has the properties (3.6) and (3.7). In addition, one has

$$C_1 C_2 \hat{R}_{12} C_1 C_2 = \hat{R}_{21},$$  \hspace{1cm} (3.20)$$
where $C^i_j = \delta^i_{N+1-j}$.

For $q$ being a phase, the real form $SO_q(n, n)$ or $SO_q(n, n+1)$ can be defined by $T^* = T$ for a $*$-anti-involution. For real $q$, the real form $SO_q(N; \mathbb{R})$ exists with respect to the $*$-anti-involution $T^* = GTG^{-1}$, where $G^i_j = q^i \delta^i_{N+1-j}$. We need $SO_q(N; \mathbb{R})$ for $q$ being a phase, so again we can only define it with respect to a $*$-involution. We find the appropriate $*$-involution to be given by

$$T^* = CTC.$$  \hspace{1cm} (3.21)$$
Incidentally, if one wants to define $SO_q(2n, 2m)$, we can generalize the above to $T^* = \hat{C}T\hat{C}^{-1}$, where $\hat{C}^i_j = \epsilon_i \delta^i_{N+1-j} (N = 2(n + m))$ with $m$ of the $\epsilon$’s equal to $-1$ and the rest to $1$. This includes $SO(4, 2)$ which is of interest in defining the deformation of the
AdS part. The ∗-involution in the corresponding universal enveloping algebra is recently given in [9], where some useful information about unitary representations when q is a root of unity is also given. In the following we will concentrate on $SO_q(N; \mathbb{R})$, which will be denoted as simply $SO_q(N)$.

The $SO_q(N)$-covariant algebra of functions on the quantum Euclidean space is defined by

$$x_1 x_2 = q^{-1} x_1 x_2 \hat{R}_{12} + \kappa R^2 G,$$

where $\kappa = (1 - q^{-2})/(1 + q^{N-2})$ and $R^2 = x^t G x$ is the radius squared. The transformation of $SO_q(N)$ on $x$ is $x \rightarrow x T$. The ∗-involution compatible with (3.21) is

$$x^* = x C.$$

Since we have demanded all algebras to have the ∗-involution, it follows that there is a symmetry of $q \rightarrow q^{-1}$ if we simultaneously take $x \rightarrow x^*$, $T \rightarrow T^*$ etc. It is therefore equivalent to say that we have $S^5_q$ or $S^5_{q^{-1}}$.

It can be explicitly checked that the algebra of $C^3_q$ is the same as the algebra of $R^6_q$ via the identification $z_i = x_{i+1}$ and $\bar{z}^i = x_{-i}$ for $i = 0, 1, 2$, and then $R^2 = (q^2 + q^{-2}) \ell^2$. Therefore the three definitions of $S^5_q$ are actually equivalent: $SU_q(3)/SU_q(2) = C^3_q/(r^2 = 1) = R^6_q/(R^2 = q^2 + q^{-2})$. In the first two models of $S^5_q$, the action of $SU_q(3)$ is manifest. In the third model the action of $SO_q(6)$ is manifest. This suggests that $SU_q(3)$ can be realized as a subgroup of $SO_q(6)$. This is indeed the case. Given a $3 \times 3$ $SU_q(3)$ matrix $t$, we can define a special $6 \times 6$ $SO_q(6)$ matrix $T$ by $T_{ij} = t_{ij}$ for $i, j = 1, 2, 3, T_{ij} = t^*_{(7-i)(7-j)}$ for $i, j = 4, 5, 6$, and all other elements $T_{ij} = 0$.

### 3.3. Quantum sphere for $U(N)$ $N = 4$ SYM

Now that we have established the $SO_q(6)$ symmetry of the quantum sphere, we use the fact that KK reduction on this space will give a family of reps. of $SO_q(6)$. In Sec. 3.5 we will show that $SO_q(6)$ can be identified with $SU_q(4)$. For $q$ being a root of unity, the reps. of $SU_q(4)$ contain indecomposable reps. which form an ideal under tensor products. After quotienting these out, one is left with a set of standard reps. with a truncation. The length of the first row of the Young tableau cannot exceed $k$ for $q = e^{\frac{i \pi}{N+4}}$. This fact is familiar from 2D WZW models [10,11] and has been studied in detail for $U_q SU(2)$ in [12,13]. This allows us to identify $k \sim N$ to get a quantum sphere which gives a Kaluza-Klein reduction which agrees with the spectrum of chiral primaries discussed in section 2.
3.4. $Z_n$ automorphisms and symmetries of quantum quotient spaces

We now discuss some sub-algebras relevant for the quantum space analog of the $\mathcal{N} = 2$ quotient theories which will be discussed in the next section. The quantum group $SU_q(3)$ has $SU_q(2)$ as a subgroup. Given a $2 \times 2$ $SU_q(2)$ matrix $t$, we can define a $3 \times 3$ $SU_q(3)$ matrix $T$ by $T_{ij} = t_{ij}$ for $i, j = 1, 2$, $T_{33} = 1$ and all other elements $T_{ij} = 0$. The group $Z_n$ mentioned in previous sections can then be embedded in $SU_q(2)$ as a diagonal matrix $\text{diag}(\omega, \omega^{-1})$, where $\omega^n = 1$.

Classically, $SO(6)$ has a maximal subgroup $SU(2) \times SU(2) \times U(1)$. This is also true for $SO_q(6)$. The presence of a $U_q(SU(2)) \times U_q(SU(2)) \times U_q(U(1))$ subalgebra in $U_q(SO(6))$ or $U_q(SU(4))$ is clear from the definition of these algebras using the q-analog of the Chevalley-Serre basis [14]. From the point of view of the algebra of functions on $SO_q(6)$ we can describe a $SU_q(2) \times SU_q(2) \times U_q(1)$ as follows. The first $SU_q(2)$ is

$$T_{mn} = \begin{pmatrix} \alpha & \beta & 0 & 0 & 0 & 0 \\ \gamma & \delta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \alpha & -\beta \\ 0 & 0 & 0 & 0 & -\gamma & \delta \end{pmatrix}. \tag{3.24}$$

The second $SU_q(2)$ is

$$T_{mn} = \begin{pmatrix} a & 0 & 0 & 0 & b & 0 \\ 0 & a & 0 & 0 & 0 & -b \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ c & 0 & 0 & 0 & d & 0 \\ 0 & -c & 0 & 0 & 0 & d \end{pmatrix}. \tag{3.25}$$

The $U_q(1)$ is

$$T_{mn} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \Lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \Lambda^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.26}$$

These $6 \times 6$ matrices corresponding to the three subgroups $SU_q(2)$, $U_q(1)$ and $SU_q(2)$ commute with one another, and they all satisfy the relation (3.1) for $SO_q(6)$. Note that in checking whether these matrices commute with one another, we should take any entry in a matrix to commute with any entry in another matrix. The reason for this is that if we take two matrices $T$ and $T'$ from a quantum group, for their product to satisfy the same
RTT relation (3.1) we should let all entries in $T$ to commute with all entries in $T'$. (The functions on two copies of the group manifold is given in a tensor product.)

The $Z_n$ symmetry we quotient by is a subalgebra of $U_q(1)$ which is embedded as $T_{ij} = T_{(i+4)(j+4)} = \text{diag}(\omega, \omega^{-1})$ for $i, j = 1, 2$, $T_{33} = T_{44} = 1$, and all other elements vanishing, for $|\omega| = 1$.

The commutant of the $Z_n$ action is $SU_q(2) \times U_q(1)$, with the same $q$ as before. This fact will be useful in a quantum space-time understanding of the relation between cutoffs on chiral primaries in an $\mathcal{N} = 4 U(Nn)$ theory and its $Z_n$ quotient.

3.5. $SU_q(4)$ symmetry

Since the universal enveloping algebra $U_q(G)$ for a classical group $G$ is completely determined by the Cartan matrix of $G$, and since the Cartan Matrix of $SU(4)$ is the same as that for $SO(6)$, $SO_q(6)$ is identical to $SU_q(4)$ (up to global differences which do not affect the general sub-group structure). Therefore $SO_q(6)$ has a subgroup $SU_q(3)$ which is manifest in the $SU_q(4)$ description.

While we arrived at the $SU_q(4) \sim SO_q(6)$ symmetry by explicitly mapping to an algebra which had the larger symmetry, we can also guess it by an indirect argument. A hint for the hidden $SU(4)_q$ comes from a consideration of KK reduction on $SU_q(3)/SU_q(2)$. Suppose we are dimensionally reducing a scalar on the coset $SU(3)/SU(2)$. We have to look for all reps. of $SU(3)$ which contain a scalar of $SU(2)$ [15]. We know that the reps. of $SU(3)$ we have are precisely such that they combine into reps. of $SU(4)$. To KK reduce on $SU_q(3)/SU_q(2)$, we $q$-deform the rule above and look for reps. of $SU_q(3)$ which contain the scalar of $SU_q(2)$. It is very plausible that the reps. still combine into reps. of $SU_q(4)$ since the structure of the reps. at roots of unity remains the same as long as we stay within the cutoff. This can be proved by using the generalization of the Gelfand-Zetlin bases (described for example in [16]), which exists because of some special properties of the branching rules in the sequence of subgroups $SU(2) \subset SU(3) \subset SU(4)$. The Gelfand-Zetlin bases have been generalized to roots of unity (discussed for example in [17]) so this should provide the proof that the desired $q$-generalization of the branching rules is correct.

The matching between $SU_q(4)$ and $SO_q(6)$ can also be illustrated as follows. There is an $SU_q(3)$ subgroup of $SU_q(4)$ which acts on $(z_1, z_2, z_3)$. Let us represent it as

$$
\begin{pmatrix}
  a & b & c & 0 \\
  d & e & f & 0 \\
  g & h & p & 0 \\
  0 & 0 & 0 & 1
\end{pmatrix}.
$$

(3.27)
This corresponds to the $SU_q(3)$ embedded in $SO_q(6)$ which is described in a previous section.

In addition to the $SU_q(2)$ subgroups in this $SU_q(3)$, there are three other ways of embedding $SU_q(2)$ in $SU_q(4)$:

\[
\begin{pmatrix}
a & 0 & 0 & b \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
c & 0 & 0 & d \\
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & a & 0 & b \\
0 & 0 & 1 & 0 \\
0 & c & 0 & d \\
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & a & b \\
0 & 0 & 0 & 1 \\
0 & 0 & c & d \\
\end{pmatrix},
\tag{3.28}
\]

where $a, b, c, d$ are the four functions which constitute an $SU_q(2)$ matrix in the fundamental representation. These three $SU_q(2)$ subgroups all commute with one another. They are characterized by their being commuting with different $SU_q(2)$ subgroups of the $SU_q(3)$ mentioned above.

It is now not difficult to guess what their correspondence in $SO_q(6)$ is. The corresponding $SU_q(2)$ matrices are:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & a & 0 & b & 0 \\
0 & 0 & a & 0 & -b \\
0 & c & 0 & d & 0 \\
0 & 0 & -c & 0 & d \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix},
\begin{pmatrix}
a & 0 & 0 & b & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & a & 0 & -b \\
0 & c & 0 & d & 0 \\
0 & 0 & -c & 0 & d \\
0 & 0 & 0 & 0 & 1 \\
\end{pmatrix},
\tag{3.29}
\]

and

\[
\begin{pmatrix}
a & 0 & 0 & 0 & b \\
0 & a & 0 & 0 & -b \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
c & 0 & 0 & d & 0 \\
0 & -c & 0 & 0 & d \\
\end{pmatrix}.
\tag{3.30}
\]

The matching above takes care of the 8 generators in the $SU_q(3)$ and 2 generators in each of the three $SU_q(2)$. Since there are a total of 15 generators in $SU_q(4)$ or $SO_q(6)$, there is still one generator left, the $U(1)$ generator which can be represented as

\[
\begin{pmatrix}
\Lambda & 0 & 0 & 0 \\
0 & \Lambda & 0 & 0 \\
0 & 0 & \Lambda & 0 \\
0 & 0 & 0 & \Lambda^{-3} \\
\end{pmatrix},
\begin{pmatrix}
\Lambda \mathbf{1}_{3 \times 3} & 0 \\
0 & \Lambda^{-1} \mathbf{1}_{3 \times 3} \\
\end{pmatrix}
\tag{3.31}
\]

in $SU_q(4)$ and $SO_q(6)$ respectively.
4. Chiral primary operators for $\mathcal{N} = 2$ quotient theories

4.1. Conformal field theory discussion

Maldacena’s conjecture has been extended to the case of orbifolds. In order to preserve the conformal symmetry, we need to keep the AdS part untouched and to act with orbifold groups only on $S^5$ [18,19,20,21].

An $\mathcal{N} = 2$ theory is obtained if we act with a $\mathbb{Z}_n$ group on two out of three complex fields, one of them being left unchanged. The $\mathbb{Z}_n$ quotienting is accompanied by a gauge transformation.

$$\begin{align*}
\Omega \Phi_1 \Omega^{-1} &= \omega \Phi_1, \\
\Omega \Phi_2 \Omega^{-1} &= \omega^{-1} \Phi_2, \\
\Omega \Phi_3 \Omega^{-1} &= \Phi_3, \\
\Omega D_A \Omega^{-1} &= D_A,
\end{align*}$$

where $D_A$ is the covariant derivative.

The $\Phi$’s are $Nn \times Nn$ matrices. $\Omega$ can be chosen to be $diag(1, \omega^{-1}, \omega^{-2} \ldots \omega^{-(n-1)})$. After taking the quotient the gauge group becomes $SU(2) \otimes U(1)$, with the surviving gauge fields being diagonal $N \times N$ blocks, and the bosonic matter content is:

$$\Phi_1 = \begin{pmatrix}
0 & Q^{(1)}_1 & 0 & 0 & 0 & \cdots \\
0 & 0 & Q^{(2)}_1 & 0 & 0 & \cdots \\
0 & 0 & 0 & Q^{(3)}_1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},$$

$$\Phi_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots \\
0 & Q^{(1)}_2 & 0 & 0 & \cdots \\
0 & 0 & Q^{(2)}_2 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},$$

$$\Phi_3 = \begin{pmatrix}
\hat{\Phi}^{(1)}_3 & 0 & 0 & 0 & \cdots \\
0 & \hat{\Phi}^{(2)}_3 & 0 & 0 & \cdots \\
0 & 0 & \hat{\Phi}^{(3)}_3 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},$$

where $Q^{(i)}_1$ are fields in the $(N_i, \bar{N}_{i+1})$ representation, and $Q^{(i)}_2$ are in the $(\bar{N}_i, N_{i+1})$ representation. The surviving global symmetry is $SU(2)_R \times U(1)_R \times Z_n$. The pair $(Q^{(i)}_1, (Q^{(i)}_2)^*)$ is a doublet of $SU(2)_R$ and uncharged under $U(1)_R$, while $\Phi^{(i)}_3$ are singlets under this
SU(2)$_R$ but have charge 1 under the $U(1)$. The SU(2)$_R$ has a $U(1)$ subgroup, under which the $Q^{(i)}_1$ and $Q^{(i)}_2$ have charge 1. The $Z_n$ acts as cyclic permutations on the $n$ factors of SU($N$) and on the $i$ index of $Q^{(i)}_1$, $Q^{(i)}_2$, $\Phi_3^i$. Geometrically, the symmetry SU(2) $\times$ $U(1)$ $\times$ $Z_n$ can be understood, by describing $S^3/Z_n$ as a fibration of $S^2 \times S^1$ over $S^2/Z_n$.

Out of the chiral primaries of the $\mathcal{N} = 4$ theory, those giving non-trivial operators are of the form $tr\Phi^i_3$, $tr(\Phi_1 \Phi_2)^i$ and $tr(\Phi_1 \Phi_2 \Phi_3^m)^i$. As in the discussion of cutoffs for the $U(Nn)$ theory we can write traces of powers of these fields in terms of products of traces when the power exceeds $Nn$. For example this leads to

$$tr(\Phi_3)^{Nn+1} = tr(\Phi_3)^{Nn} tr(\Phi_3) + \cdots,$$

where the $\cdots$ stands for other terms involving other splittings of the $(Nn + 1)$'th power. We can rewrite this as follows

$$tr(\Phi_3)^{Nn+1} = \sum_i tr(\dot{\Phi}_3^{(i)})^{Nn} \sum_j tr(\dot{\Phi}_3^{(j)}) + \cdots.$$  \hspace{1cm} (4.3)

Note that the splitting involves factors which are separately $Z_N$ invariant. The same holds for other operators, e.g. $tr(\Phi_1 \Phi_2)^{Nn+1}$.

Another kind of splitting occurs if we allow the factors to come from twisted sectors. This splitting will happen at a different value of the powers, i.e at $N + 1$ rather than $Nn + 1$. It uses the fact that

$$tr(\dot{\Phi}_3^{(i)})^{N+1} = tr(\dot{\Phi}_3^{(i)})^N tr(\dot{\Phi}_3^{(i)}) + \cdots,$$

and it leads to

$$tr(\Phi_3)^{N+1} = \sum_{i=1}^{N} tr(\dot{\Phi}_3^{(i)})^N tr(\dot{\Phi}_3^{(i)}) + \cdots.$$  \hspace{1cm} (4.5)

Now the factors are not separately invariant. They come from twisted sectors. Similar equations can be written for the other operators, for example,

$$tr(\Phi_1 \Phi_2)^{N+1} = \sum_{i=1}^{N} tr(\dot{\Phi}_1^{(i)} \dot{\Phi}_2^{(i)})^N tr(\dot{\Phi}_1^{(i)} \dot{\Phi}_2^{(i)}) + \cdots.$$  \hspace{1cm} (4.7)

However as discussed in [21] the factors appearing on the right hand side in this expression are not chiral primaries because they appear as derivatives of a superpotential.
4.2. Quantum space explanation of the cutoffs

We saw in the previous subsection that if we start with a gauge theory with gauge group $U(Nn)$ and take the $Z_n$ quotient, we get a gauge theory with a product of $U(N)$ gauge groups. Restricting attention to $Z_n$ invariant operators, i.e. the untwisted sector, we find that the cutoff stays at $Nn$. We expect on the gravity side that a discussion ignoring the twisted sectors can be simply reproduced by studying a quotient of the q-sphere. We found indeed in the previous section that after quotienting $S_5^q$ by a $Z_n$ element inside the $SO_q(6)$, we are left with a surviving $SU_q(2) \times U_q(1)$ with the same value of $q$ that we started with. This gives an explanation of the fact that the cutoff $\sim Nn$ stays at $\sim Nn$.

4.3. Comments on twisted sectors

The quantum space explanation of the behaviour of the cutoffs when we include the twisted sectors is much more intricate. This is only a preliminary discussion.

The twisted sector has to do with dimensional reduction on the singular cycles of $D^4/Z_n$ followed by the reduction on $S^1$ [21].

To understand the twisted sector chiral primaries from the gravity point of view in the large $N$ limit one starts with a description of $S^5/Z_n$ as $S^1$ fibred over $B_4/Z_n$ when $B_4$ is the 4-ball. Ten-dimensional gravity is reduced on the $B_4/Z_n$ and then the KK reduction on the $S^1$ is performed. The cohomology of the blown-up $B_4/Z_n$ space is used to determine the type of particles we get [21].

We can try to generalize this discussion to the case of $S_5^q/Z_n$. The twisted sector states are localized in the $Z_1, Z_2$ direction when the $Z_n$ quotient acts in these directions. The wavefunctions of the twisted sector states are arbitrary functions of the phase of $Z_3$ and localized at $Z_1 = Z_2 = 0$ in the case $q = 1$. A very intriguing property of the $S_5^q$ we described is that $Z_1 = Z_2 = 0$ is not compatible with the algebra (3.16) - (3.19). In a sense the origin of the ball is smoothed out. It would be very interesting if this result of noncommutativity provides an effective way to describe the resolution of the fixed point in a similar way as [22], where it was shown that the instanton moduli space can be resolved by deforming the base space into a quantum space. Another possibility is that we could also have chosen to work with a $Z_n$ action on $Z_2, Z_3$. In that case there is a circle over $Z_2 = Z_3 = 0$ on $S_5^q$. So to describe the twisted sector states, we need to extend the algebra $S_5^q/Z_n$ by adding certain delta functions multiplied with arbitrary powers of $Z_3$ (with unit norm). The results from the field theory suggest that this algebra should admit a
consistent truncation which restricts the $U(1)$ charge at order $N$, since we saw that with the cutoffs on single trace operators of the form $tr\Phi_{ij}$ happens at order $N$. It will be very interesting to see if the quantum space techniques can reproduce this field theory result. Some relevant techniques on quantum principal bundles may be found in [23,24] and on K-theory of quantum spaces in [25].

5. Conclusions

In this paper we have considered the AdS/CFT conjecture for finite $N$ conformal theories. As opposed to the large $N$ limit which relates the corresponding field theories and the classical supergravity, for finite $N$ we need to consider quantum gravity. We have taken the point of view that the finite $N$ effects can be captured by gravity on a q-deformed version of $AdS_5 \times S^5$ space time where $q$ is a root of unity along the lines of [3]. We have discussed mainly the deformation of $S^5$ into $S^5_q$ where $q$ is a root of unity. The KK reduction on this quantum sphere is truncated at $N$ if $q = e^{i\pi N/4}$. This agrees with the cutoff on the chiral primaries of finite $N$ conformal field theory obtained on the boundary of $AdS_5$.

We then discussed the $\mathcal{N} = 2$ quotient theories and the corresponding cutoff in the supergravity and field theory. By considering only the untwisted sectors the result is that the chiral primaries have the same cutoff as in the initial $\mathcal{N} = 4$ theory. This result could be explained by considering a quotient of the quantum sphere. We also derived results in field theories regarding the cutoffs when twisted sectors are taken into account. We began a discussion of the corresponding quantum space picture.

It would be interesting to extend the results of this paper for other orbifolds of $S^5$, for six dimensional field theories obtained on the boundary of $AdS_7$ or for three dimensional theories obtained on the boundary of $AdS_4$. Some evidence for non-commutative gravity in $AdS_7 \times S^4$ background has been obtained in [26] and connections to the quantum group approach will be interesting to explore.

Acknowledgements: We are happy to acknowledge fruitful discussions with Antal Jevicki, Gilad Lifschytz, and Vipul Periwal. P.M.H. was supported in part by the National Science Council, Taiwan, R.O.C., and he thanks C.-S. Chu for discussions and the hospitality of the University of Neuchâtel, Switzerland, where part of this work was done. The work of R.T. and S.R. was supported by DOE grant DE-FG02/19ER40688-(Task A). S.R would like to thank the National Taiwan University for hospitality while part of this work was done.
References

[1] J. Maldacena, *The Large N Limit of Superconformal Field Theories and Supergravity*, Adv.Theor.Math.Phys. 2 (1998) 231-252.

[2] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Gauge Theory Corellators from Non-Critical String Theory*, Phys.Lett. B428 (1998) 105-114.

[3] E. Witten, *Anti-de-Sitter space and holography*, hep-th-9802150.

[4] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, Y. Oz, *Large N Field Theories, String Theory and Gravity*, hep-th/9905111.

[5] A. Jevicki, S. Ramgoolam, *Non-commutative gravity from the AdS/CFT correspondence*, JHEP 9904 (1999) 032.

[6] J. Maldacena, A. Strominger, *AdS3 Black Holes and a Stringy Exclusion Principle*, JHEP 9812 (1998) 005.

[7] L. D. Faddeev, N. Yu. Reshetikhin, L. A. Takhtadzhyan, *Quantization of Lie groups and Lie algebras*, Leningrad Math. J. 1 (1990) 193.

[8] L. L. Vaksman and Ya. S. Soibelman *The Algebra of functions on the quantum group SU(n + 1) and odd-dimensional quantum spheres*, Leningrad Math. J. Vol. 2 (1991), No. 5.

[9] H. Steinacker, *Unitary Representations of Noncompact Quantum Groups at Roots of Unity*, math.QA/9907021; *Unitary Representations and BRST Structure of the Quantum Anti-de Sitter Group at Roots of Unity*, q-alg/9710016; *Quantum Groups, Roots of Unity and Particles on Quantized Anti-de Sitter Space*, hep-th/9705211; *Finite Dimensional Unitary Representations of Quantum Anti-de Sitter Groups at Roots of Unity*, q-alg/9611009.

[10] V. Pasquier, H. Saleur, *Common Structures Between Finite Systems and Conformal Field through Quantum Groups*, Nucl.Phys. B330 (1990) 523.

[11] P. Furlan, A. Ch. Gauche, V. B. Petkova, *Quantum Groups and Fusion Rule Multicities*, Nucl. Phys. B343 (1990) 205.

[12] G. Keller, *Fusion Rules of U_qSL(2, C), q^m = 1*, Lett. Math. Phys. 21: 273, 1991.

[13] N. Reshetikhin, V. G. Turaev, “Invariants of three manifolds via link polynomials and quantum groups”, Invent.Math.103:547-597,1991.

[14] M. Jimbo, A q-analog of U(gl(N+1) Hecke Algebras and the Yang-Baxter Equation, Lett. Math. Phys. 11 (1986), 247 - 252.

[15] A. Salam, J. Strathdee, *On Kaluza-Klein Theory*, Annals in Phys. 141, 316, 1982.

[16] A. O. Barut and R. Raczka, *Theory of Group Representations and Applications* PWN, 1980.

[17] V. Chari, A. Pressley, *A guide to quantum groups*, CUP, 1994.

[18] S. Kachru, E. Silverstein, *4d Conformal Field Theories and Strings on Orbifolds*, Phys.Rev.Lett. 80 (1998) 4855-4858.
[19] A. Lawrence, N. Nekrasov, C. Vafa, *On Conformal Theories in Four Dimensions*, Nucl.Phys. **B533** (1998) 199-209.

[20] Y. Oz, J. Terning, *Orbifolds of AdS$_5 \times $S$^5$ and 4d Conformal Field Theories*, Nucl.Phys. **B532** (1998) 163-180.

[21] S. Gukov, *Comments on N=2 AdS Orbifolds*, Phys.Lett. **B439** (1998) 23-28.

[22] N. Nekrasov, A. Schwarz, *Instantons on Noncommutative $\mathbb{R}^4$ and (2,0) Superconformal Six Dimensional Theory*, [hep-th/9802068](https://arxiv.org/abs/hep-th/9802068).

[23] C.-S. Chu, P.-M. Ho, H. Steinacker, *q-deformed Dirac Monopole With Arbitrary Charge*, Z.Phys. **C71** (1996) 171-177.

[24] T. Brzezinski and S. Majid, *Quantum Group Gauge Theory on Quantum Spaces* Commun.Math.Phys. **157** (1993) 591.

[25] A. Connes, *Non-commutative geometry*, Academic Press 1994

[26] M. Berkooz and H. Verlinde, *Matrix Theory, AdS/CFT and Higgs-Coulomb Equivalence*, [hep-th/9907100](https://arxiv.org/abs/hep-th/9907100).