Inclusive weak decay rates of heavy hadrons

M.B. Voloshin
Theoretical Physics Institute, University of Minnesota, Minneapolis, MN 55455
and
Institute of Theoretical and Experimental Physics, Moscow, 117259

Expanded version of a contribution to the final report book of the Fermilab Workshop on B
Physics at Tevatron

Abstract

A compact review of the theory, including some recent developments, of inclusive weak decay rates of charmed and $b$ hadrons with an emphasis on predictions that can be tested in the forthcoming experiments.
1 Introduction

The dominant weak decays of hadrons containing a heavy quark, $c$ or $b$, are caused by the decay of the heavy quark. In the limit of a very large mass $m_Q$ of a heavy quark $Q$ the parton picture of the hadron decay should set in, where the inclusive decay rates of hadrons, containing $Q$, mesons ($Qar{q}$) and baryons ($Qqq$), are all the same and equal to the inclusive decay rate $\Gamma_{\text{parton}}(Q)$ of the heavy quark. Yet, the known inclusive decay rates $[1]$ are conspicuously different for different hadrons, especially for charmed hadrons, whose lifetimes span a range of more than one order of magnitude from the shortest $\tau(\Omega_c) = 0.064 \pm 0.020$ ps to the longest $\tau(D^+) = 1.057 \pm 0.015$ ps, while the differences of lifetime among $b$ hadrons are substantially smaller. The relation between the relative lifetime differences for charmed and $b$ hadrons reflects the fact that the dependence of the inclusive decay rates on the light quark-gluon ‘environment’ in a particular hadron is a pre-asymptotic effect in the parameter $m_Q$, which effect vanishes as an inverse power of $m_Q$ at large mass.

A theoretical framework for systematic description of the leading at $m_Q \to \infty$ term in the inclusive decay rate $\Gamma_{\text{parton}}(Q) \propto m_Q^5$ as well as of the terms relatively suppressed by inverse powers of $m_Q$ is provided $[2, 3, 4]$ by the operator product expansion (OPE) in $m_Q^{-1}$. Existing theoretical predictions for inclusive weak decay rates are in a reasonable agreement, within the expected range of uncertainty, with the data on lifetimes of charmed particles and with the so far available data on decays of $B$ mesons. The only outstanding piece of present experimental data is on the lifetime of the $\Lambda_b$ baryon: $\tau(\Lambda_b)/\tau(B_d) \approx 0.8$, for which ratio a theoretical prediction, given all the uncertainty involved, is unlikely to produce a number lower than 0.9. The number of available predictions for inclusive decay rates of charmed and $b$ hadrons is sufficiently large for future experimental studies to firmly establish the validity status of the OPE based theory of heavy hadron decays, and, in particular, to find out whether the present contradiction between the theory and the data on $\tau(\Lambda_b)/\tau(B_d)$ is a temporary difficulty, or an evidence of fundamental flaws in theoretical understanding.

It is a matter of common knowledge that application of OPE to decays of charmed and $b$ hadrons has potentially two caveats. One is that the OPE is used in the Minkowsky kinematical domain, and therefore relies on the assumption of quark-hadron duality at the energies involved in the corresponding decays. In other words, it is assumed that sufficiently many exclusive hadronic channels contribute to the inclusive rate, so that the accidentals of the low-energy resonance structure do not affect the total rates of the inclusive processes. Theoretical attempts at understanding the onset of the quark-hadron duality are so far
limited to model estimates \cite{5, 6}, not yet suitable for direct quantitative evaluation of possible deviation from duality in charm and \( b \) decays. This point presents the most fundamental uncertainty of the OPE based approach, and presently can only be clarified by confronting theoretical predictions with experimental data. The second possible caveat in applying the OPE technique to inclusive charm decays is that the mass of the charm quark, \( m_c \), may be insufficiently large for significant suppression of higher terms of the expansion in \( m_c^{-1} \). The relative lightness of the charm quark, however, accounts for a qualitative, and even semi-quantitative, agreement of the OPE based predictions with the observed large spread of the lifetimes of charmed hadrons: the nonperturbative effects, formally suppressed by \( m_c^{-2} \) and \( m_c^{-3} \) are comparable with the ‘leading’ parton term and describe the hierarchy of the lifetimes.

Another uncertainty of a technical nature arises from poor knowledge of matrix elements of certain quark operators over hadron, arising as terms in OPE. These can be estimated within theoretical models, with inevitable ensuing model dependence, or, where possible, extracted from the experimental data. With these reservations spelled out, we discuss here the OPE based description of inclusive weak decays of charm and \( b \) hadrons, with emphasis on specific experimentally testable predictions, and on the measurements, which would less rely on model dependence of the estimates of the matrix elements, thus allowing to probe the OPE predictions at a fundamental level.

2 OPE for inclusive weak decay rates

The optical theorem of the scattering theory relates the total decay rate \( \Gamma_H \) of a hadron \( H_Q \) containing a heavy quark \( Q \) to the imaginary part of the ‘forward scattering amplitude’. For the case of weak decays the latter amplitude is described by the following effective operator

\[
L_{\text{eff}} = 2 \text{Im} \left[ i \int d^4x e^{iqx} T \{L_W(x), L_W(0)\} \right],
\]

in terms of which the total decay rate is given by\cite{7}

\[
\Gamma_H = \langle H_Q | L_{\text{eff}} | H_Q \rangle.
\]

The correlator in equation (2) in general is a non-local operator. However at \( q^2 = m_Q^2 \) the dominating space-time intervals in the integral are of order \( m_Q^{-1} \) and one can expand the

\footnote{We use here the non-relativistic normalization for the heavy quark states: \( \langle Q | Q^\dagger Q | Q \rangle = 1 \).}
correlator in $x$, thus producing an expansion in inverse powers of $m_Q$. The leading term in this expansion describes the parton decay rate of the quark. For instance, the term in the non-leptonic weak Lagrangian $\sqrt{2} G_F V(\overline{q}_1L \gamma_\mu Q_L)(\overline{q}_2L \gamma_\mu q_3L)$ with $V$ being the appropriate combination of the CKM mixing factors, generates through eq.(1) the leading term in the effective Lagrangian

$$L_{\text{eff}, nl}^{(0)} = |V|^2 \frac{G_F^2 m_Q^5}{64 \pi^3} \eta_{nl} \left( \overline{Q}Q \right),$$

where $\eta_{nl}$ is the perturbative QCD radiative correction factor. This expression reproduces the well known formula for the inclusive non-leptonic decay rate of a heavy quark, associated with the underlying process $Q \rightarrow q_1 q_2 \overline{q}_3$, due to the relation $\langle H_Q | \overline{Q}Q | H_Q \rangle \approx \langle H_Q | Q^i Q | H_Q \rangle = 1$, which is valid up to corrections of order $m_Q^{-2}$. One also sees form this example, that in order to separate individual semi-inclusive decay channels, e.g. non-leptonic with specific flavor quantum numbers, or semi-leptonic, one should simply pick up the corresponding relevant part of the weak Lagrangian $L_W$, describing the underlying process, to include in the correlator $(1)$.

The general expression for first three terms in the OPE for $L_{\text{eff}}$ has the form

$$L_{\text{eff}} = L_{\text{eff}}^{(0)} + L_{\text{eff}}^{(2)} + L_{\text{eff}}^{(3)} = c^{(0)} \frac{G_F^2 m_Q^5}{64 \pi^3} \left( \overline{Q}Q \right) + c^{(2)} \frac{G_F^2 m_Q^3}{64 \pi^3} \left( \overline{Q} \sigma^{\mu\nu} G_{\mu\nu} Q \right) + \frac{G_F^2 m_Q^2}{4 \pi} \sum_i c_i^{(3)} (\overline{q}_i \Gamma_i q_i) (\overline{Q} \Gamma_i' Q),$$

where the superscripts denote the power of $m_Q^{-1}$ in the relative suppression of the corresponding term in the expansion with respect to the leading one, $G_{\mu\nu}$ is the gluon field tensor, $q_i$ stand for light quarks, $u, d, s$, and, finally, $\Gamma_i, \Gamma_i'$ denote spin and color structures of the four-quark operators. The coefficients $c^{(a)}$ depend on the specific part of the weak interaction Lagrangian $L_W$, describing the relevant underlying quark process.

One can notice the absence in the expansion $(4)$ of a term suppressed by just one power of $m_Q^{-1}$, due to non-existence of operators of suitable dimension. Thus the decay rates receive no correction of relative order $m_Q^{-1}$ in the limit of large $m_Q$, and the first pre-asymptotic corrections appear only in the order $m_Q^{-2}$.

The mechanisms giving rise to the three discussed terms in OPE are shown in Figure 1. The first, leading term corresponds to the parton decay, and does not depend on the light quark and gluon ‘environment’ of the heavy quark in a hadron. The second term describes the effect on the decay rate of the gluon field that a heavy quark ‘sees’ in a hadron. This term in fact is sensitive only to the chromomagnetic part of the gluon field, and contains the operator of the interaction of heavy quark chromomagnetic moment with the chromomagnetic field.
Thus this term depends on the spin of the heavy quark, but does not depend on the flavors of the light quarks or antiquarks. Therefore this effect does not split the inclusive decay rates within flavor SU(3) multiplets of heavy hadrons, but generally gives difference of the rates, say, between mesons and baryons. The dependence on the light quark flavor arises from the third term in the expansion (4) which explicitly contains light quark fields. Historically, this part is interpreted in terms of two mechanisms [2, 8, 9]: the weak scattering (WS) and the Pauli interference (PI). The WS corresponds to a cross-channel of the underlying decay, generically \( Q \rightarrow q_1 q_2 \overline{q}_3 \), where either the quark \( q_3 \) is a spectator in a baryon and can undergo a weak scattering off the heavy quark: \( q_3 Q \rightarrow q_1 q_2 \), or an antiquark in meson, say \( \overline{q}_1 \), weak-scatters (annihilates) in the process \( \overline{q}_1 Q \rightarrow q_2 \overline{q}_3 \). The Pauli interference effect arises when one of the final (anti)quarks in the decay of \( Q \) is identical to the spectator (anti)quark in the hadron, so that an interference of identical particles should be taken into account. The latter interference can be either constructive or destructive, depending on the relative spin-color arrangement of the (anti)quark produced in the decay and of the spectator one, thus the sign of the PI effect is found only as a result of specific dynamical calculation. In specific calculations, however, WS and PI arise from the same terms in OPE, depending on the hadron discussed, and technically there is no need to resort to the traditional terminology of WS and PI.

![Graphs](Figure 1: Graphs for three first terms in OPE for inclusive decay rates: the parton term, the chromomagnetic interaction, and the four-quark term.)
In what follows we discuss separately the effects of the three terms in the expansion (4) and their interpretation within the existing and future data.

3 The parton decay rate

The leading term in the OPE amounts to the perturbative expression for the decay rate of a heavy quark. In $b$ hadrons the contribution of the subsequent terms in OPE is at the level of few percent, so that the perturbative part can be confronted with the data in its own right. In particular, for the $B_d$ meson the higher terms in OPE contribute only about 1% of the total non-leptonic as well as of the semileptonic decay rate. Thus the data on these rates can be directly compared with the leading perturbative term in OPE.

The principal theoretical topic, associated with this term is the calculation of QCD radiative corrections, i.e. of the factor $\eta_{nl}$ in eq. (3) and of a similar factor, $\eta_{sl}$, for semileptonic decays. It should be noted, that even at this, perturbative, level there is a known long-standing problem between the existing data and the theory in that the current world average for the semileptonic branching ratio for the $B$ mesons, $B_{sl}(B) = 10.45 \pm 0.21\%$, is somewhat lower than the value $B_{sl}(B) \geq 11.5$ preferred from the present knowledge of theoretical QCD radiative corrections to the ratio of non-leptonic to semileptonic decay rates (see e.g. [14]). However, this apparent discrepancy may in fact be due to insufficient ‘depth’ of perturbative QCD calculation of the ratio $\eta_{nl}/\eta_{sl}$. In order to briefly elaborate on this point, we notice that the standard way of analyzing the perturbative radiative corrections in the nonleptonic decays is through the renormalization group (RG) summation of the leading log terms and the first next-to-leading terms [11, 12] in the parameter $L \equiv \ln(m_W/m_b)$. For the semileptonic decays the logarithmic dependence on $m_W/m_b$ is absent in all orders due to the weak current conservation at momenta larger than $m_b$, thus the correction is calculated by the standard perturbative technique, and a complete expression in the first order in $\alpha_s$ is available both for the total rate [13, 14] and for the lepton spectrum [15]. In reality however the parameter $L \approx 2.8$ is not large, and non-logarithmic terms may well compete with the logarithmic ones. This behavior is already seen from the known expression for the logarithmic terms: when expanded up to the order $\alpha_s^2$ the result of Ref. [16] for the rate of decays with single final charmed quark takes the form

$$\frac{\Gamma(b \to c\bar{u}d) + \Gamma(b \to c\bar{u}s)}{3 \Gamma(b \to c\bar{e}\nu)} = 1 + \frac{\alpha_s}{\pi} + \frac{\alpha_s^2}{\pi^2} \left[ 4 L^2 + \left( \frac{7}{6} + \frac{2}{3} c(m_c^2/m_b^2) \right) L \right] ,$$

where, in terms of notation of Ref. [16], $c(a) = c_{22}(a) - c_{12}(a)$. The behavior of the function...
c(a) is known explicitly \[16\] and is quite weak: \( c(0) = 19/2, \) \( c(1) = 6, \) and \( c(m_c^2/m_b^2) \approx 9.0 \) for the realistic mass ratio \( m_c/m_b \approx 0.3. \) One can see that the term with the single logarithm \( \mathcal{L} \) contributes about two thirds of that with \( \mathcal{L}^2 \) in the term quadratic in \( \alpha_s. \) Under such circumstances the RG summation of the terms with powers of \( \mathcal{L} \) does not look satisfactory for numerical estimates of the QCD effects, at least at the so far considered level of the first next-to-leading order terms, and the next-to-next-to-leading terms can be equally important as the two known ones, which would eliminate the existing impasse between the theory and the data on \( B_{sl}(B). \) One can present some arguments \[17\] that this is indeed the case for the \( b \) quark decay, although a complete calculation of these corrections is still unavailable.

### 4 Chromomagnetic and time dilation effects in decay rates

The corrections suppressed by two powers of \( m_Q^{-1} \) to inclusive decay rates arise from two sources \[7\]: the \( O(m_Q^{-2}) \) corrections to the matrix element of the leading operator, \( \langle \overline{Q} Q, \) and the second term in OPE \[\mathcal{H}\) containing the chromomagnetic interaction. The expression for the matrix element of the leading operator with the correction included is written in the form

\[
\langle H_Q | \overline{Q} Q | H_Q \rangle = 1 - \frac{\mu^2}{2 m_Q^2} \ldots,
\]

where \( \mu^2 \) is defined as

\[
\mu^2 = \langle H_Q | \overline{Q} \langle i \bar{D} \rangle^2 Q | H_Q \rangle,
\]

\[
\mu_g^2 = \langle H_Q | \overline{Q} \frac{1}{2} \sigma^{\mu \nu} G_{\mu \nu} Q | H_Q \rangle,
\]

with \( D \) being the QCD covariant derivative. The correction in equation (6) in fact corresponds to the time dilation factor \( m_Q/E_Q, \) for the heavy quark decaying inside a hadron, where it has energy \( E_Q, \) which energy is contributed by the kinetic part (\( \propto \mu^2 \)) and the chromomagnetic part (\( \propto \mu_g^2 \)). The second term in OPE describes the effect of the chromomagnetic interaction in the decay process, and is also expressed through \( \mu_g^2. \)

The explicit formulas for the decay rates, including the effects up to the order \( m_Q^{-2} \) are found in \[7\] and for decays of the \( b \) hadrons read as follows. For the semileptonic decay rate

\[
\Gamma_{sl}(H_b) = \frac{|V_{cb}|^2 G_F^2 m_b^5}{192 \pi^3} \langle H_b | \overline{b} b | H_b \rangle \left[ 1 + \frac{\mu_g^2}{m_b^2} \left( \frac{x}{2} \frac{d}{dx} - 2 \right) \right] \eta_{sl} I_0(x, 0, 0),
\]

where

\[
\eta_{sl} = \frac{1}{2} \frac{x}{2} \frac{d}{dx} - 2.
\]
and for the non-leptonic decay rate

\[ \Gamma_{nl}(H_b) = \frac{|V_{cb}|^2 G_F^2 m_b^5}{64 \pi^3} \langle H_b | \bar{b}b | H_b \rangle \left\{ \frac{1 + \frac{\mu_g^2}{m_b^2} \left( x \frac{d}{dx} - 2 \right)}{2} \right\} \eta_{nl} I(x) - 8 \eta_2 \frac{\mu_g^2}{m_b^2} I_2(x) \right\} . \]  

(9)

These formulas take into account only the dominant CKM mixing \( V_{cb} \) and neglect the small one, \( V_{ub} \). The following notation is also used: \( x = m_c/m_b \), \( I_0(x, y, z) \) stands for the kinematical suppression factor in a three-body weak decay due to masses of the final fermions. In particular,

\[ I_0(x, 0, 0) = (1 - x^4)(1 - 8 x^2 + x^4) - 24 x^4 \ln x , \]

(10)

\[ I_0(x, x, 0) = (1 - 14 x^2 - 2 x^4 - 12 x^6) \sqrt{1 - 4 x^2} + 24 (1 - x^4) \ln \frac{1 + \sqrt{1 - 4 x^2}}{1 - \sqrt{1 - 4 x^2}} . \]

Furthermore, \( I(x) = I_0(x, 0, 0) + I_0(x, x, 0) \), and

\[ I_2(x) = (1 - x^2)^3 + \left( 1 + \frac{1}{2} x^2 + 3 x^4 \right) \sqrt{1 - 4 x^2} - 3 x^2 (1 - 2 x^4) \ln \frac{1 + \sqrt{1 - 4 x^2}}{1 - \sqrt{1 - 4 x^2}} . \]

Finally, the QCD radiative correction factor \( \eta_2 \) in eq. (9) is known in the leading logarithmic approximation and is expressed in terms of the well known coefficients \( C_+ \) and \( C_- \) in the renormalization of the non-leptonic weak interaction: \( \eta_2 = \frac{C_+^2(m_b) - C_-^2(m_b))}{6} \) with

\[ C_- (\mu) = C_+^{-2} (\mu) = \left[ \frac{\alpha_s(\mu)}{\alpha_s(m_W)} \right]^{4/b} , \]

(11)

and \( b \) is the coefficient in the QCD beta function. The value of \( b \) relevant to \( b \) decays is \( b = 23/3 \).

Numerically, for \( x \approx 0.3 \), the expressions for the decay rates can be written as

\[ \Gamma_{sl}(H_b) = \Gamma_{sl}^{parton} \left( 1 - \frac{\mu_\pi^2(H_b) - \mu_\phi^2(H_b)}{2 m_b^2} - 2.6 \frac{\mu_\phi^2(H_b)}{m_b^2} \right) , \]

\[ \Gamma_{nl}(H_b) = \Gamma_{nl}^{parton} \left( 1 - \frac{\mu_\pi^2(H_b) - \mu_\phi^2(H_b)}{2 m_b^2} - 1.0 \frac{\mu_\phi^2(H_b)}{m_b^2} \right) , \]

(12)

where \( \Gamma^{parton} \) is the perturbation theory value of the corresponding decay rate of \( b \) quark.

The matrix elements \( \mu_\pi^2 \) and \( \mu_\phi^2 \) are related to the spectroscopic formula for a heavy hadron mass \( M \),

\[ M(H_Q) = m_Q + \bar{\Lambda}(H_Q) + \frac{\mu_\pi^2(H_Q) - \mu_\phi^2(H_Q)}{2 m_Q} + \ldots \]

(13)
Being combined with the spin counting for pseudoscalar and vector mesons, this formula allows to find the value of $\mu^2_g$ in pseudoscalar mesons from the mass splitting:

$$\mu^2_g(B) = \frac{3}{4} \left( M^2_{B^*} - M^2_B \right) \approx 0.36 \text{GeV}^2. \quad (14)$$

The value of $\mu^2_\pi$ for $B$ mesons is less certain. It is constrained by the inequality \[\mu^2_\pi(H_Q) \geq \mu^2_g(H_Q),\] and there are theoretical estimates from the QCD sum rules \[\mu^2_\pi(H_Q) = 0.54 \pm 0.12 \text{GeV}^2\] and from an analysis of spectroscopy of heavy hadrons \[\mu^2_\pi(B) = 0.3 \pm 0.2 \text{GeV}^2.\] In any event, the discussed corrections are rather small for $b$ hadrons, given that $\mu^2_g/m_b^2 \approx 0.015$. The largest, in relative terms, effect of these corrections in $B$ meson decays is on the semileptonic decay rate, where it amounts to 4 - 5 % suppression of the rate, which rate however is only a moderate fraction of the total width. In the dominant non-leptonic decay rate the effect is smaller, and, according to the formula $(12)$ amounts to about 1.5 – 2 %.

The effect of the $m_{Q}^{-2}$ corrections can be evaluated with a somewhat better certainty for the ratio of the decay rates of $\Lambda_b$ and $B$ mesons. This is due to the fact that $\mu^2_g(\Lambda_b) = 0$, since there is no correlation of the spin of the heavy quark in $\Lambda_b$ with the light component, having overall quantum numbers $J^{P} = 0^+$. Then, applying the formula $(12)$ to $B$ and $\Lambda_b$, we find for the ratio of the (dominant) non-leptonic decay rates:

$$\frac{\Gamma_{nl}(\Lambda_b)}{\Gamma_{nl}(B)} = 1 - \frac{\mu^2_\pi(\Lambda_b) - \mu^2_\pi(B)}{2 m_b^2} + 0.5 \frac{\mu^2_g(B)}{m_b^2}. \quad (15)$$

The difference of the kinetic terms, $\mu^2_\pi(\Lambda_b) - \mu^2_\pi(B)$, can be estimated from the mass formula:

$$\mu^2_\pi(\Lambda_b) - \mu^2_\pi(B) = \frac{2 m_b m_c}{m_b - m_c} \left[ \overline{M}(B) - \overline{M}(D) - M(\Lambda_b) + M(\Lambda_c) \right] = 0 \pm 0.04 \text{GeV}^2, \quad (16)$$

where $\overline{M}$ is the spin-averaged mass of the mesons, e.g. $\overline{M}(B) = (M(B) + 3M(B^*))/4$. The estimated difference of the kinetic terms is remarkably small. Thus the effect in the ratio of the decay rates essentially reduces to the chromomagnetic term, which is also rather small and accounts for less than 1% difference of the rates. For the ratio of the semileptonic decay rates the chromomagnetic term is approximately four times larger, but then the contribution of the semileptonic rates to the total width is rather small. Thus one concludes that the terms of order $m_{b}^{-2}$ in the OPE expansion for the decay rates can account only for about 1% difference of the lifetimes of $\Lambda_b$ and the $B$ mesons.

The significance of the $m_{Q}^{-2}$ terms is substantially different for the decay rates of charmed hadrons, where these effects suppress the inclusive decays of the $D$ mesons by about 40%
with respect to those of the charmed hyperons in a reasonable agreement with the observed pattern of the lifetimes.

It should be emphasized once again that the $m_Q^2$ effects do not depend on the flavors of the spectator quarks or antiquarks. Thus the explanation of the variety of the inclusive decay rates within the flavor SU(3) multiplets, observed for charmed hadrons and expected for the $b$ ones, has to be sought among the $m_Q^{-3}$ terms.

5 $L_{eff}^{(3)}$. Coefficients and operators

Although the third term in the expansion (4) is formally suppressed by an extra power of $m_Q^1$, its effects are comparable to, or even larger than the effects of the second term. This is due to the fact that the diagrams determining the third term (see Fig. 1) contain a two-body phase space, while the first two terms involve a three-body phase space. This brings in a numerical enhancement factor, typically $4\pi^2$. The enhanced numerical significance of the third term in OPE, generally, does not signal a poor convergence of the expansion in inverse heavy quark mass for decays of $b$, and even charmed, hadrons the numerical enhancement factor is a one time occurrence in the series, and there is no reason for similar ‘anomalous’ enhancement among the higher terms in the expansion.

Here we first present the expressions for the relevant parts of $L_{eff}^{(3)}$ for decays of $b$ and $c$ hadrons in the form of four-quark operators and then proceed to a discussion of hadronic matrix elements and the effects in specific inclusive decay rates. The consideration of the effects in decays of charmed hadrons is interesting in its own right, and leads to new predictions to be tested experimentally, and is also important for understanding the magnitude of the involved matrix elements using the existing data on charm decays.

We start with considering the term $L_{eff}^{(3)}$ in $b$ hadron non-leptonic decays, $L_{eff,nl}^{(3,b)}$, induced by the underlying processes $b \to c \pi d$, $b \to c \tau s$, $b \to c \pi s$, and $b \to c \tau d$. Unlike the case of three-body decay, the kinematical difference between the two-body states $c\bar{c}$ and $c\bar{u}$, involved in calculation of $L_{eff,nl}^{(3,b)}$ is of the order of $m_c^2/m_b^2 \approx 0.1$ and is rather small. At present level of accuracy in discussing this term in OPE, one can safely neglect the effect of finite charmed quark mass$^2$. In this approximation the expression for $L_{eff,nl}^{(3,b)}$ reads as

$$L_{eff,nl}^{(3,b)} = |V_{bc}|^2 \frac{G_F^2 m_b^2}{4\pi} \left\{ \tilde{C}_1 (\bar{b}\Gamma_{\mu}b)(\bar{u}\Gamma_{\mu}u) + \tilde{C}_2 (\bar{b}\Gamma_{\mu}b)(\bar{u}\Gamma_{\mu}u) + \right.$$

$^2$The full expression for a finite charmed quark mass can be found in [21]
\begin{eqnarray}
\tilde{C}_5 (\bar{b} \gamma_\mu b + \frac{2}{3} \bar{b} \gamma_\mu \gamma_5 b)(\bar{q} \Gamma_\mu q) + \tilde{C}_6 (\bar{b} \gamma_\mu b_k + \frac{2}{3} \bar{b} \gamma_\mu \gamma_5 b_k)(\bar{q}_k \Gamma_\mu q_i) + \\
\frac{1}{3} \tilde{\kappa}^{1/2} (\tilde{\kappa}^{-2/9} - 1) \left[ 2 (\tilde{C}_+^2 - \tilde{C}_-^2) (\bar{b} \Gamma_\mu t^a b) j^a_\mu - \\
(5\tilde{C}_+^2 + \tilde{C}_-^2 - 6 \tilde{C}_+ \tilde{C}_-) (\bar{b} \Gamma_\mu t^a b + \frac{2}{3} \bar{b} \gamma_\mu \gamma_5 t^a b) j^a_\mu \right],
\end{eqnarray}

where the notation \((\bar{q} \Gamma q) = (\bar{d} \Gamma d) + (\bar{s} \Gamma s)\) is used, the indices \(i, k\) are the color triplet ones, \(\Gamma_\mu = \gamma_\mu (1 - \gamma_5)\), and \(j^a_\mu = \bar{u} \gamma_\mu t^a u + \bar{d} \gamma_\mu t^a d + \bar{s} \gamma_\mu t^a s\) is the color current of the light quarks with \(t^a = \lambda^a/2\) being the generators of the color \(SU(3)\). The notation \(\tilde{C}_\pm\), is used as shorthand for the short-distance renormalization coefficients \(C_{\pm}(\mu)\) at \(\mu = m_b\):
\(\tilde{C}_\pm \equiv C_{\pm}(m_b)\). The expression \([17]\) is written in the leading logarithmic approximation for the QCD radiative effects in a low normalization point \(\mu\) such that \(\mu \ll m_b\) (but still, at least formally, \(\mu \gg \Lambda_{QCD}\)).

For such \(\mu\) there arises so called 'hybrid' renormalization \([22]\), depending on the factor \(\tilde{\kappa} = \alpha_s(\mu)/\alpha_s(m_b)\). The coefficients \(\tilde{C}_A\) with \(A = 1, \ldots, 6\) in eq.\((17)\) have the following explicit expressions in terms of \(\tilde{C}_\pm\) and \(\tilde{\kappa}\):

\begin{align}
\tilde{C}_1 &= \tilde{C}_+^2 + \tilde{C}_-^2 + \frac{1}{3} (1 - \kappa^{1/2})(\tilde{C}_+^2 - \tilde{C}_-^2), \\
\tilde{C}_2 &= \kappa^{1/2} (\tilde{C}_+^2 - \tilde{C}_-^2), \\
\tilde{C}_3 &= -\frac{1}{4} (\tilde{C}_+ - \tilde{C}_-) + \tilde{C}_+^2 + \frac{1}{3} (1 - \kappa^{1/2})(5\tilde{C}_+^2 + \tilde{C}_-^2 + 6\tilde{C}_+ \tilde{C}_-) \}
, \\
\tilde{C}_4 &= -\frac{1}{4} \kappa^{1/2} (5\tilde{C}_+^2 + \tilde{C}_-^2 + 6\tilde{C}_+ \tilde{C}_-) \}, \\
\tilde{C}_5 &= -\frac{1}{4} \kappa^{1/2} (5\tilde{C}_+^2 + \tilde{C}_-^2 + 6\tilde{C}_+ \tilde{C}_-) \}, \\
\tilde{C}_6 &= -\frac{1}{4} \kappa^{1/2} (5\tilde{C}_+^2 + \tilde{C}_-^2 - 6\tilde{C}_+ \tilde{C}_-) \}.
\end{align}

The expression for the CKM dominant semileptonic decays of \(b\) hadrons, associated with the elementary process \(b \to c \ell \nu\) does not look to be of an immediate interest. The reason is that this process is intrinsically symmetric under the flavor \(SU(3)\), and one expects no significant splitting of the semileptonic decay rates within \(SU(3)\) multiplets of the \(b\) hadrons. The only possible effect of this term, arising through a penguin-like mechanism can be in a small overall shift of semileptonic decay rates between \(B\) mesons and baryons. However, these effects are quite suppressed and are believed to be even smaller than the ones arising form the discussed \(m_b^{-2}\) terms.

For charm decays there is a larger, than for \(b\) hadrons, variety of effects associated with \(L_{eff}^{(3)}\), that can be studied experimentally, and we present here the relevant parts of the effective Lagrangian. For the CKM dominant non-leptonic decays of charm, originating
from the quark process $c \rightarrow s u \bar{d}$, the discussed term in OPE has the form

$$L_{eff, nl}^{(3, \Delta C=\Delta S)} = \cos^4 \theta_c \frac{G_F^2 m_c^2}{4\pi} \left\{ C_1 (\bar{c} \Gamma_{\mu c})(\bar{d} \Gamma_{\mu d}) + C_2 (\bar{c} \Gamma_{\mu d})(\bar{d} \Gamma_{\mu c}) + C_3 (\bar{c} \Gamma_{\mu c} + \frac{2}{3} \bar{c} \Gamma_{\mu \gamma_5 c})(\bar{s} \Gamma_{\mu s}) + C_4 (\bar{c} \Gamma_{\mu c} + \frac{2}{3} \bar{c} \Gamma_{\mu \gamma_5 c})(\bar{\sigma}_k \Gamma_{\mu s}) + C_5 (\bar{c} \Gamma_{\mu c} + \frac{2}{3} \bar{c} \Gamma_{\mu \gamma_5 c})(\bar{\pi} \Gamma_{\mu u}) + C_6 (\bar{c} \Gamma_{\mu c} + \frac{2}{3} \bar{c} \Gamma_{\mu \gamma_5 c})(\bar{\tau} \Gamma_{\mu u}) + \frac{1}{3} \kappa^{1/2} (\kappa^{-2/9} - 1) \left[ 2 (C_+^2 - C_0^2) (\bar{c} \Gamma_{\mu t^a c}) j_{\mu}^a - (5C_+^2 + C_0^2) (\bar{c} \Gamma_{\mu t^a c} + \frac{2}{3} \bar{c} \Gamma_{\mu \gamma_5 t^a c}) j_{\mu}^a \right] \right\} ,$$

(19)

where, $\theta_c$ is the Cabibbo angle, and the coefficients without the tilde are given by the same expressions as above for the $b$ decays (i.e. those with tilde) with the replacement $m_b \rightarrow m_c$. The part of the notation in the superscript $\Delta C = \Delta S$ points to the selection rule for the dominant CKM unsuppressed non-leptonic decays. One can rather realistically envisage however a future study of inclusive rates for the once CKM suppressed decays of charmed hadrons$^3$, satisfying the selection rule $\Delta S = 0$ and associated with the quark processes $c \rightarrow d u \bar{s}$ and $c \rightarrow d u \bar{d}$. The corresponding part of the effective Lagrangian for these processes reads as

$$L_{eff, sl}^{(3, \Delta C=\Delta S)} = \cos^2 \theta_c \sin^2 \theta_c \frac{G_F^2 m_c^2}{4\pi} \left\{ C_1 (\bar{c} \Gamma_{\mu c})(\bar{s} \Gamma_{\mu s}) + C_2 (\bar{c} \Gamma_{\mu s})(\bar{s} \Gamma_{\mu c}) + 2 C_3 (\bar{c} \Gamma_{\mu c} + \frac{2}{3} \bar{c} \Gamma_{\mu \gamma_5 c})(\bar{\pi} \Gamma_{\mu u}) + 2 C_6 (\bar{c} \Gamma_{\mu c} + \frac{2}{3} \bar{c} \Gamma_{\mu \gamma_5 c})(\bar{\tau} \Gamma_{\mu u}) + \frac{2}{3} \kappa^{1/2} (\kappa^{-2/9} - 1) \left[ 2 (C_+^2 - C_0^2) (\bar{c} \Gamma_{\mu t^a c}) j_{\mu}^a - (5C_+^2 + C_0^2) (\bar{c} \Gamma_{\mu t^a c} + \frac{2}{3} \bar{c} \Gamma_{\mu \gamma_5 t^a c}) j_{\mu}^a \right] \right\} ,$$

where again the notation $(\bar{c} \Gamma_{\mu q}) = (\bar{d} \Gamma d) + (\bar{s} \Gamma s)$ is used.

The semileptonic decays of charm, the CKM dominant, associated with $c \rightarrow s \ell \nu$, and the CKM suppressed, originating from $c \rightarrow s \ell \nu$, contribute to the semileptonic decay rate, which certainly can be measured experimentally. The expression for the part of the effective Lagrangian, describing the $m_Q^3$ terms in these decays is $[17, 24, 25]$

$$L_{eff, sl}^{(3)} = \frac{G_F^2 m_c^2}{12\pi} \left\{ \cos^2 \theta_c \left[ L_1 (\bar{c} \Gamma_{\mu c} + \frac{2}{3} \bar{c} \Gamma_{\mu \gamma_5 c})(\bar{s} \Gamma_{\mu s}) + L_2 (\bar{c} \Gamma_{\mu c} + \frac{2}{3} \bar{c} \Gamma_{\mu \gamma_5 c})(\bar{s} \Gamma_{\mu s}) \right] +$$

$^3$Even if the inclusive rate of these decays is not to be separated experimentally, they contribute about 10% of the total decay rate, and it is worthwhile to include their contribution in the balance of the total width.
\[
\sin^2 \theta_c \left[ L_1 (\overline{c} \Gamma_\mu c + \frac{2}{3} \overline{c} \gamma_\mu \gamma_5 c)(\overline{d} \Gamma_\mu d) + L_2 (\overline{d} \Gamma_\mu c_k + \frac{2}{3} \overline{d} \gamma_\mu \gamma_5 c_k)(\overline{d} \Gamma_\mu d_i) \right] - \\
2 \kappa^{1/2} (\kappa^{-2/9} - 1) (\overline{c} \Gamma_\mu t^a c + \frac{2}{3} \overline{c} \gamma_\mu \gamma_5 t^a c) j_\mu^a \right] ,
\]
with the coefficients \(L_1\) and \(L_2\) found as
\[
L_1 = (\kappa^{1/2} - 1), \quad L_2 = -3 \kappa^{1/2} .
\]

6 Effects of \(L_{eff}^{(3)}\) in mesons

The expressions for the terms in \(L_{eff}^{(3)}\) still leave us with the problem of evaluating the matrix elements of the four-quark operators over heavy hadrons in order to calculate the effects in the decay rates according to the formula (2). In doing so only few conclusions can be drawn in a reasonably model independent way, i.e. without resorting to evaluation of the matrix elements using specific ideas about the dynamics of quarks inside hadrons. The most straightforward prediction can in fact be found for \(b\) hadrons. Namely, one can notice that the operator (17) is symmetric under the flavor U spin (an SU(2) subgroup of the flavor SU(3), which mixes s and d quarks). This is a direct consequence of neglecting the small kinematical effect of the charmed quark mass. However the usual (in)accuracy of the flavor SU(3) symmetry is likely to be a more limiting factor for the accuracy of this symmetry, than the corrections of order \(m_c^2/m_b^2\). Modulo this reservation the immediate prediction from this symmetry is the degeneracy of inclusive decay rates within U spin doublets:
\[
\Gamma(B_d) = \Gamma(B_s) , \quad \Gamma(\Lambda_b) = \Gamma(\Xi_b^0) ,
\]
where \(\Gamma(B_s)\) stands for the average rate over the two eigenstates of the \(B_s - \overline{B_s}\) oscillations. The data on decay rates of the cascade hyperon \(\Xi_b^0\) are not yet available, while the currently measured lifetimes of \(B_d\) and \(B_s\) are within less than 2% from one another. Theoretically, the difference of the lifetimes, associated with possible violation of the SU(3) symmetry and with breaking of the U symmetry of the effective Lagrangian (17), is expected to not exceed about 1%.

For the non-vanishing matrix elements of four-quark operators over pseudoscalar mesons one traditionally starts with the factorization formula and parametrizes possible deviation from factorization in terms of ‘bag constants’. Within the normalization convention adopted here the relations used in this parametrization read as
\[
\langle P_{\eta \bar{q}} | (\bar{Q} \Gamma_\mu q) (\bar{q} \Gamma_\mu Q)| P_{\bar{q} \eta} \rangle = \frac{1}{2} f_p^2 M_P B ,
\]
\[ \langle P_{Q7}|(\overline{7}\Gamma_{\mu}Q)(\overline{7}\Gamma_{\mu}q)|P_{Q7}\rangle = \frac{1}{6} f_P^2 M_P \tilde{B}, \]  

(24)

where \( P_{Q7} \) stands for pseudoscalar meson made of \( Q \) and \( 7 \), \( f_P \) is the annihilation constant for the meson, and \( B \) and \( \tilde{B} \) are bag constants. The parameters \( B \) and \( \tilde{B} \) generally depend on the normalization point \( \mu \) for the operators, and this dependence is compensated by the \( \mu \) dependence of the coefficients in \( L_{eff}^{(3)} \), so that the results for the physical decay rate difference do not depend on \( \mu \). If the normalization point \( \mu \) is chosen at the heavy quark mass (i.e. \( \mu = m_b \) for \( B \) mesons, and \( \mu = m_c \) for \( D \) mesons) the predictions for the difference of total decay rates take a simple form in terms of the corresponding bag constants (generally different between \( B \) and \( D \)):

\[
\Gamma(B^\pm) - \Gamma(B^0) = |V_{cb}|^2 \frac{G_F^2 m_b^3 f_P^2}{8\pi} \left[ (\tilde{C}_+^2 - \tilde{C}_-^2) B(m_b) + \frac{1}{3} (\tilde{C}_+^2 + \tilde{C}_-^2) \tilde{B}(m_b) \right] \\
\approx -0.025 \left( \frac{f_P}{200 \text{ MeV}} \right)^2 \text{ps}^{-1}, \tag{25}
\]

\[
\Gamma(D^\pm) - \Gamma(D^0) = \cos^4 \theta_c \frac{G_F^2 m_c^3 f_D^2}{8\pi} \left[ (C_+^2 - C_-^2) B(m_c) + \frac{1}{3} (C_+^2 + C_-^2) \tilde{B}(m_c) \right] \\
\sim -0.8 \left( \frac{f_D}{200 \text{ MeV}} \right)^2 \text{ps}^{-1}, \tag{26}
\]

where the numerical values are written in the approximation of exact factorization: \( B = 1 \), and \( \tilde{B} = 1 \). It is seen from the numerical estimates that, even given all the theoretical uncertainties, the presented approach is in reasonable agreement with the data on the lifetimes of \( D \) and \( B \) mesons. In particular, this approach describes, at least qualitatively, the strong suppression of the decay rate of \( D^\pm \) mesons relative to \( D^0 \), the experimental observation of which has in fact triggered in early 80-s the theoretical study of preasymptotic in heavy quark mass effects in inclusive decay rates. For the \( B \) mesons the estimate (25) is also in a reasonable agreement with the current data for the discussed difference (\( -0.043\pm0.017 \text{ps}^{-1} \)).

7 Effects of \( L_{eff}^{(3)} \) in baryons

The weakly decaying heavy hyperons, containing either \( c \) or \( b \) quark are: \( \Lambda_Q \sim Qud, \Xi_Q^{(u)} \sim Qus, \Xi_Q^{(d)} \sim Qds, \) and \( \Omega_Q \sim Qss \). The first three baryons form an SU(3) (anti)triplet. The light diquark in all three is in the state with quantum numbers \( J^P = 0^+ \), so that there is no correlation of the spin of the heavy quark with the light component of the baryon.
On the contrary, in $\Omega_Q$ the two strange quarks form a $J^P = 1^+$ state, and a correlation between the spins of heavy and light quarks is present. The absence of spin correlation for the heavy quark in the triplet of hyperons somewhat reduces the number of independent four-quark operators, having nonvanishing diagonal matrix elements over these baryons. Indeed, the operators entering $L^{(3)}_{\text{eff}}$ contain both vector and axial bilinear forms for the heavy quarks. However the axial part requires a correlation of the heavy quark spin with that of a light quark, and is thus vanishing for the hyperons in the triplet. Therefore only the structures with vector currents are relevant for these hyperons. These structures are of the type $(\bar{c}\gamma_\mu c)(\bar{q}\gamma_\mu q)$ and $(\bar{c}_i\gamma_\mu c_k)(\bar{q}_k\gamma_\mu q_i)$ with $q$ being $d$, $s$ or $u$. The flavor SU(3) symmetry then allows to express, for each of the two color combinations, the matrix elements of three different operators, corresponding to three flavors of $q$, over the baryons in terms of only two combinations: flavor octet and flavor singlet. Thus all effects of $L^{(3)}_{\text{eff}}$ in the triplet of the baryons can be expressed in terms of four independent combinations of matrix elements. These can be chosen in the following way:

\[
x = \left\langle \frac{1}{2} (\bar{Q}\gamma_\mu Q) \left[ (\bar{u}\gamma_\mu u) - (\bar{s}\gamma_\mu s) \right] \right\rangle_{\Omega_Q - \Xi_Q^{(d)}}^{\Lambda_Q - \Xi_Q^{(s)}},
\]

\[
y = \left\langle \frac{1}{2} (\bar{Q}_i\gamma_\mu Q_k) \left[ (\bar{u}_k\gamma_\mu u_i) - (\bar{s}_k\gamma_\mu s_i) \right] \right\rangle_{\Omega_Q - \Xi_Q^{(d)}}^{\Lambda_Q - \Xi_Q^{(s)}},
\]

with the notation for the differences of the matrix elements: $\langle O \rangle_{A-B} = \langle A|O|A \rangle - \langle B|O|B \rangle$, for the flavor octet part and the matrix elements:

\[
x_s = \frac{1}{3} \langle H_Q | (\bar{Q}\gamma_\mu Q) \left( (\bar{u}\gamma_\mu u) + (\bar{d}\gamma_\mu d) + (\bar{s}\gamma_\mu s) \right) | H_Q \rangle
\]

\[
y_s = \frac{1}{3} \langle H_Q | (\bar{Q}_i\gamma_\mu Q_k) \left( (\bar{u}_k\gamma_\mu u_i) + (\bar{d}_k\gamma_\mu d_i) + (\bar{s}_k\gamma_\mu s_i) \right) | H_Q \rangle
\]

for the flavor singlet part, where $H_Q$ stands for any heavy hyperon in the (anti)triplet.

The initial, very approximate, theoretical estimates of the matrix elements \[^{[4]}\] were essentially based on a non-relativistic constituent quark model, where these matrix elements are proportional to the density of a light quark at the location of the heavy one, i.e. in terms of the wave function, proportional to $|\psi(0)|^2$. Using then the same picture for the matrix elements over pseudoscalar mesons, relating the quantity $|\psi(0)|^2$ to the annihilation constant $f_P$, and assuming that $|\psi(0)|^2$ is approximately the same in baryons as in mesons, one arrived at the estimate

\[
y = -x = x_s = -y_s \approx \frac{f_P^2 M_D}{12} \approx 0.006 \text{ GeV}^2,
\]
where the sign relation between $x$ and $y$ is inferred from the color antisymmetry of the constituent quark wave function for baryons. Since the constituent picture was believed to be valid at distances of the order of the hadron size, the estimate (29) was applied to the matrix elements in a low normalization point where $\alpha_s(\mu) \approx 1$. For the matrix elements of the operators, containing $s$ quarks over the $\Omega_Q$ hyperon, this picture predicts an enhancement factor due to the spin correlation:

$$\langle \Omega_Q | (\bar{Q} \Gamma_{\mu} Q)(\bar{s} \Gamma_{\mu} s) | \Omega_Q \rangle = -\langle \Omega_Q | (\bar{Q}_k \Gamma_{\mu} Q_k)(\bar{s}_k \Gamma_{\mu} s_k) | \Omega_Q \rangle = \frac{10}{3} y$$

(30)

Although these simple estimates allowed to correctly predict [4] the hierarchy of lifetimes of charmed hadrons prior to establishing this hierarchy experimentally, they fail to quantitatively predict the differences of lifetimes of charmed baryons. We shall see that the available data indicate that the color antisymmetry relation is badly broken, and the absolute value of the matrix elements is larger, than the naive estimate (29), especially for the quantity $x$.

It should be emphasized that in the heavy quark limit the matrix elements (27) and (28) do not depend on the flavor of the heavy quark, provided that the same normalization point $\mu$ is used. Therefore, applying the OPE formulas to both charmed and $b$ baryons, one can extract the values for the matrix elements from available data on charmed hadrons, and then make predictions for $b$ baryons, as well as for other inclusive decay rates, e.g. semileptonic, for charmed hyperons.

The only data available so far, which would allow to extract the matrix elements, are on the lifetimes of charmed hyperons. Therefore, one has to take into account several essential types of inclusive decay, at least those that contribute to the total decay rate at the level of about 10%. Here we first concentrate on the differences of the decay rates within the SU(3) triplet of the hyperons, which will allow us to extract the non-singlet quantities $x$ and $y$, and then discuss the SU(3) singlet shifts of the rates.

The differences of the dominant Cabibbo unsuppressed non-leptonic decay rates are given by

$$\delta_{1}^{nl,0} \equiv \Gamma_{\Delta S=\Delta C}(\Xi^{0}_c) - \Gamma_{\Delta S=\Delta C}(\Lambda_c) = \cos^4 \theta_c \frac{G^2_F m^2_c}{4\pi} [ (C_5 - C_3) x + (C_6 - C_4) y ] ,$$

$$\delta_{2}^{nl,0} \equiv \Gamma_{\Delta S=\Delta C}(\Lambda_c) - \Gamma_{\Delta S=\Delta C}(\Xi^{+}_c) = \cos^4 \theta_c \frac{G^2_F m^2_c}{4\pi} [ (C_3 - C_1) x + (C_4 - C_2) y ] .$$

(31)

The once Cabibbo suppressed decay rates of $\Lambda_c$ and $\Xi^{+}_c$ are equal, due to the $\Delta U = 0$ property of the corresponding effective Lagrangian $L_{eff,nl}^{(3,1)}$ (eq.(21)). Thus the only difference for these
decays in the baryon triplet is
\[ \delta_{nl}^{1} \equiv \Gamma_{D=0}^{nl}(\Xi_{c}^{0}) - \Gamma_{D=0}^{nl}(\Lambda_{c}) = \cos^{2} \theta_{c} \sin^{2} \theta_{c} \frac{G_{F}^{2} m_{c}^{2}}{4\pi} [(2 C_{5} - C_{1} - C_{3}) x + (2 C_{6} - C_{2} - C_{4}) y] . \] (32)

The dominant semileptonic decay rates are equal among the two \( \Xi_{c} \) baryons due to the isotopic spin property \( \Delta I = 0 \) of the corresponding interaction Lagrangian, thus there is only one non-trivial splitting for these decays:
\[ \delta_{nl}^{st,0} \equiv \Gamma_{D=\pm 1}^{nl}(\Xi_{c}^{0}) - \Gamma_{D=\pm 1}^{nl}(\Lambda_{c}) = -\cos^{2} \theta_{c} \frac{G_{F}^{2} m_{c}^{2}}{12\pi} [L_{1} x + L_{2} y] . \] (33)

Finally, the Cabibbo suppressed semileptonic decay rates are equal for \( \Lambda_{c} \) and \( \Xi_{c}^{0} \), due to the \( \Delta V = 0 \) property of the corresponding interaction. Thus the only difference for these is
\[ \delta_{nl}^{st,1} \equiv \Gamma_{D=0}^{nl}(\Lambda_{c}) - \Gamma_{D=0}^{nl}(\Xi_{c}^{\pm}) = -\sin^{2} \theta_{c} \frac{G_{F}^{2} m_{c}^{2}}{12\pi} [L_{1} x + L_{2} y] . \] (34)

Using the relations (31) - (34) one can find expressions for two differences of the measured total decay rates, \( \Delta_{1} = \Gamma(\Xi_{c}^{0}) - \Gamma(\Lambda_{c}) \) and \( \Delta_{2} = \Gamma(\Lambda_{c}) - \Gamma(\Xi_{c}^{\pm}) \), in terms of the quantities \( x \) and \( y \):
\[ \Delta_{1} = \delta_{nl}^{1} + \delta_{nl}^{2} + 2 \delta_{nl}^{0} = \frac{G_{F}^{2} m_{c}^{2}}{4\pi} \cos^{2} \theta \left\{ x \left( \cos^{2} \theta (C_{5} - C_{3}) + \sin^{2} \theta (2 C_{5} - C_{1} - C_{3}) - \frac{2}{3} L_{1} \right) + \\
y \left( \cos^{2} \theta (C_{6} - C_{4}) + \sin^{2} \theta (2 C_{6} - C_{2} - C_{4}) - \frac{2}{3} L_{2} \right) \right\} , \] (35)

and
\[ \Delta_{2} = \delta_{nl}^{1} - 2 \delta_{nl}^{0} + 2 \delta_{nl}^{1} = \frac{G_{F}^{2} m_{c}^{2}}{4\pi} \left\{ x \left( \cos^{4} \theta (C_{3} - C_{1}) + \frac{2}{3} (\cos^{2} \theta - \sin^{2} \theta) L_{1} \right) + \\
y \left( \cos^{4} \theta (C_{4} - C_{2}) + \frac{2}{3} (\cos^{2} \theta - \sin^{2} \theta) L_{2} \right) \right\} . \] (36)

By comparing these relations with the data, one can extract the values of \( x \) and \( y \). Using the current data for the total decay rates: \( \Gamma(\Lambda_{c}) = 4.85 \pm 0.28 \text{ps}^{-1} \), \( \Gamma(\Xi_{c}^{0}) = 10.2 \pm 2 \text{ps}^{-1} \), and the updated value \( \Gamma(\Xi_{c}^{\pm}) = 3.0 \pm 0.45 \text{ps}^{-1} \), we find for the \( \mu \) independent matrix element \( x \)
\[ x = -(0.04 \pm 0.01) \text{GeV}^{2} \left( \frac{1.4 \text{GeV}}{m_{c}} \right)^{2} , \] (37)
while the dependence of the thus extracted matrix element \( y \) on the normalization point \( \mu \) is shown in Fig. 2. \footnote{It should be noted that the curves at large values of \( \kappa, \kappa > \sim 3 \), are shown only for illustrative purpose. The coefficients in the OPE, leading to the equations (32)-(34), are purely perturbative. Thus, formally, they correspond to \( \alpha_{s}(\mu) \ll 1 \), i.e. to \( \kappa \ll 1/\alpha_{s}(m_{c}) \sim (3 - 4) \).}
Figure 2: The values of the extracted matrix elements $x$ and $y$ in $GeV^3$ vs. the normalization point parameter $\kappa = \alpha_s(\mu)/\alpha_s(m_c)$. The thick lines correspond to the central value of the data on lifetimes of charmed baryons, and the thin lines show the error corridors. The extracted values of $x$ and $y$ scale as $m_c^{-2}$ with the assumed mass of the charmed quark, and the plots are shown for $m_c = 1.4 GeV$.

Notably, the extracted values of $x$ and $y$ are in a drastic variance with the simplistic constituent model: the color antisymmetry relation, $x = -y$, does not hold at any reasonable $\mu$, and the absolute value of $x$ is substantially enhanced\footnote{A similar, although with a smaller enhancement, behavior of the matrix elements was observed in a recent preliminary lattice study \cite{27}.}

Once the non-singlet matrix elements are determined, they can be used for predicting differences of other inclusive decay rates within the triplet of charmed hyperons as well as for the $b$ baryons. Due to correlation of errors in $x$ and $y$ it makes more sense to express the predictions directly in terms of the total decay rates of the charmed hyperons. The thus arising relations between the rates do not depend on the normalization parameter $\mu$. In this way one finds \cite{28} for the difference of the Cabibbo dominant semileptonic decay rates between either of the $\Xi_c$ hyperons and $\Lambda_c$:

$$
\Gamma_{sl}(\Xi_c) - \Gamma_{sl}(\Lambda_c) \approx \delta_{sl,0} = 0.13 \Delta_1 - 0.065 \Delta_2 \approx 0.59 \pm 0.32 \text{ ps}^{-1} .
$$

When compared with the data on the total semileptonic decay rate of $\Lambda_c$, $\Gamma_{sl}(\Lambda_c) = 0.22 \pm 0.08 \text{ ps}^{-1}$, this prediction implies that the semileptonic decay rate of the charmed cascade hyperons can be 2–3 times larger than that of $\Lambda_c$.

The predictions found in a similar way for the inclusive Cabibbo suppressed decay rates
are \[28\]: for non-leptonic decays

\[
\delta^{nl,1} = 0.082 \Delta_1 + 0.054 \Delta_2 \approx 0.55 \pm 0.22 \text{ ps}^{-1}
\]  

(39)

and for the semileptonic ones

\[
\delta^{sl,1} = \tan^2 \theta_c \delta^{sl,0} \approx 0.030 \pm 0.016 \text{ ps}^{-1} .
\]  

(40)

For the only difference of the inclusive rates in the triplet of \(b\) baryons, \(\Gamma(\Lambda_b) - \Gamma(\Xi_b^-)\), one finds an expression in terms of \(x\) and \(y\), or alternatively, in terms of the differences \(\Delta_1\) and \(\Delta_2\) between the charmed hyperons,

\[
\Gamma(\Lambda_b) - \Gamma(\Xi_b^-) = \cos^2 \theta_c |V_{bc}|^2 \frac{G_F^2 m_c^2}{4\pi} \left[ (\tilde{C}_5 - \tilde{C}_1) x + (\tilde{C}_6 - \tilde{C}_2) y \right] \approx \]

\[
|V_{bc}|^2 \frac{m_b^2}{m_c^2} (0.85 \Delta_1 + 0.91 \Delta_2) \approx 0.015 \Delta_1 + 0.016 \Delta_2 \approx 0.11 \pm 0.03 \text{ ps}^{-1} .
\]  

(41)

When compared with the data on the total decay rate of \(\Lambda_b\) this result predicts about 14% longer lifetime of \(\Xi_b^-\) than that of \(\Lambda_b\).

The singlet matrix elements \(x_s\) and \(y_s\) (cf. eq.\([28]\)) are related to the shift of the average decay rate of the hyperons in the triplet:

\[
\mathcal{T}_Q = \frac{1}{3} \left( \Gamma(\Lambda_Q) + \Gamma(\Xi^1_Q) + \Gamma(\Xi^2_Q) \right) .
\]  

(42)

For the charmed baryons the shift of the dominant non-leptonic decay rate is given by \([29]\)

\[
\delta^{(3,0)}_{nl} \Gamma_c = \cos^4 \theta \frac{G_F^2 m_c^2}{8\pi} (C_+^2 + C_-^2) \kappa^{5/18} (x_s - 3y_s) ,
\]  

(43)

while for the \(b\) baryons the corresponding expression reads as

\[
\delta^{(3)} \mathcal{T}_b = |V_{bc}|^2 \frac{G_F^2 m_b^2}{8\pi} (\tilde{C}_+ - \tilde{C}_-)^2 \kappa^{5/18} (x_s - 3y_s) .
\]  

(44)

The combination \(x_s - 3y_s\) of the SU(3) singlet matrix elements cancels in the ratio of the shifts for \(b\) hyperons and the charmed ones:

\[
\frac{\delta^{(3)} \mathcal{T}_b}{\delta^{(3,0)} \Gamma_c} = \frac{|V_{bc}|^2 m_b^2}{\cos^4 \frac{m_c^2}{m_c^2} (C_+^2 + C_-^2)} \left[ \frac{\alpha_s(m_c)}{\alpha_s(m_b)} \right] \frac{5/18}{5/18} \delta^{(3,0)}_{nl} \Gamma_c \approx 0.0025 \delta^{(3,0)}_{nl} \Gamma_c .
\]  

(45)

(One can observe, with satisfaction, that the dependence on the unphysical parameter \(\mu\) cancels out, as it should.) This equation shows that relatively to the charmed baryons the shift of the decay rates in the \(b\) baryon triplet is greatly suppressed by the ratio \((\tilde{C}_+ - \tilde{C}_-)\).
\(\tilde{C}_-^2/(C_+^2 + C_-^2)\), which parametrically is of the second order in \(\alpha_s\), and numerically is only about 0.12.

An estimate of \(\delta^{(3)}\Gamma_b\) from eq. (45) in absolute terms depends on evaluating the average shift \(\delta_{nl}^{(3,0)}\Gamma_c\) for charmed baryons. The latter shift can be conservatively bounded from above by the average total decay rate of those baryons: \(\delta_{nl}^{(3,0)}\Gamma_c < \Gamma_c = 6.0 \pm 0.7 \text{ ps}^{-1}\), which then yields, using eq. (15), an upper bound \(\delta^{(3)}\Gamma_b < 0.015 \pm 0.002 \text{ ps}^{-1}\). More realistically, one should subtract from the total average width \(\Gamma_c\) the contribution of the ‘parton’ term, which can be estimated from the decay rate of \(D_0\) with account of the \(O(m_c^2)\) effects, as amounting to about 3 \(\text{ ps}^{-1}\). (One should also take into account the semileptonic contribution to the total decay rates, which however is quite small at this level of accuracy). Thus a realistic evaluation of \(\delta^{(3)}\Gamma_b\) does not exceed 0.01 \(\text{ ps}^{-1}\), which constitutes only about 1% of the total decay rate of \(\Lambda_b\). Thus the shift of the total decay rate of \(\Lambda_b\) due to the effects of \(L_{eff}^{(3)}\) is dominantly associated with the SU(3) non-singlet difference (11). The shift of the \(\Lambda_b\) decay rate with respect to the average width \(\Gamma_b\) due to the non-singlet operators is one third of the splitting (11), i.e. about 5%. Adding to this the 1% shift of the average width and another 1% difference from the meson decays due to the suppression of the latter by the \(m_b^{-2}\) chromomagnetic effects, one concludes that at the present level of theoretical understanding it looks impossible to explain a more than 10% enhancement of the total decay rate of \(\Lambda_b\) relative to \(B_d\), where an ample 3% margin is added for the uncertainties of higher order terms in OPE as well as for higher order QCD radiative effects in the discussed corrections. In other words, the expected pattern of the lifetimes of the \(b\) hyperons in the triplet, relative to \(B_d\), is

\[
\tau(\Xi_b^0) \approx \tau(\Lambda_b) < \tau(B_d) < \tau(\Xi^-_c),
\]

with the “best” theoretical estimate of the differences to be about 7% for each step of the inequality.

For the double strange hyperons \(\Omega_c\) and \(\Omega_b\) there is presently no better approach to evaluating the four-quark matrix elements, than the use of simplistic relations, like (30) based on constituent quark model. Such relations imply that the effects of the strange quark, WS and PI, in the \(\Omega_Q\) baryons are significantly enhanced over the same effects in the cascade hyperons. In charmed baryons a presence of strange spectator quark enhances the decay through positive interference with the quark emerging from the \(c \to s\) transition in the decay. For \(\Omega_c\) this implies a significant enhancement of the total decay rate (11), which is in perfect agreement with the data on the \(\Omega_c\) lifetime. Also a similar enhancement is expected for the semileptonic decay rate of \(\Omega_c\). In \(b\) baryons, on the contrary, the interference effect
for a spectator strange quark is negative. Thus the non-leptonic decay rate of $\Omega_b$ is expected to be suppressed, leaving $\Omega_b$ most probably the longest-living particle among the $b$ baryons.

8 Relation between spectator effects in baryons and the decays $\Xi_Q \rightarrow \Lambda_Q \pi$

Rather unexpectedly, the problem of four-quark matrix elements over heavy hyperons is related to decays of the type $\Xi_Q \rightarrow \Lambda_Q \pi$. The mass difference between the charmed cascade hyperons $\Xi_c$ and $\Lambda_c$ is about 180 MeV. The expected analogous mass splitting for the $b$ hyperons should be very close to this number, since in the heavy quark limit

$$M(\Xi_b) - M(\Lambda_b) = M(\Xi_c) - M(\Lambda_c) + O(m_c^{-2} - m_b^{-2}).$$  \hspace{1cm} (47)

Therefore in both cases are kinematically possible decays of the type $\Xi_Q \rightarrow \Lambda_Q \pi$, in which the heavy quark is not destroyed, and which are quite similar to decays of ordinary ‘light’ hyperons. Surprisingly, the rate of these decays for both $\Xi_c$ and $\Xi_b$ is not insignificantly small, but rather their branching fraction can reach a level of few per mill for $\Xi_c$ and of one percent or more for $\Xi_b$.[30]

The transitions $\Xi_Q \rightarrow \Lambda_Q \pi$ are induced by two underlying weak processes: the ‘spectator’ decay of a strange quark, $s \rightarrow u \pi d$, which does not involve the heavy quark, and the ‘non-spectator’ weak scattering (WS)

$$s c \rightarrow c d$$  \hspace{1cm} (48)

through the weak interaction of the $c \rightarrow d$ and $s \rightarrow c$ currents. One can also readily see that the WS mechanism contributes only to the decays $\Xi_c \rightarrow \Lambda_c \pi$ and is not present in the decays of the $b$ cascade hyperons. An important starting point in considering these transitions is that in the heavy quark limit the spin of the heavy quark completely decouples from the spin of the light component of the baryon, and that the latter light component in both the initial and final baryons forms a $J^P = 0^+$ state with quantum numbers of a diquark. Since the momentum transfer in the considered decays is small in comparison with the mass of the heavy quark the spin of the amplitudes with spin flip of the heavy quark, and thus of the baryon, are suppressed by $m_Q^{-1}$. In terms of the two possible partial waves in the decay $\Xi_Q \rightarrow \Lambda_Q \pi$, the $S$ and $P$, this implies that the $P$ wave is strongly suppressed and the decays are dominated by the $S$ wave.
According to the well known current algebra technique, the $S$ wave amplitudes of pion emission can be considered in the chiral limit at zero four-momentum of the pion, where they are described by the PCAC reduction formula (pole terms are absent in these processes):

$$\langle \Lambda_Q \pi_i(p=0) | H_W | \Xi_Q \rangle = \frac{\sqrt{2}}{f_\pi} \langle \Lambda_Q | \left[ Q^5_i, H_W \right] | \Xi_Q \rangle,$$

where $\pi_i$ is the pion triplet in the Cartesian notation, and $Q^5_i$ is the corresponding isotopic triplet of axial charges. The constant $f_\pi \approx 130 \text{MeV}$, normalized by the charged pion decay, is used here, hence the coefficient $\sqrt{2}$ in eq.(49). The Hamiltonian $H_W$ in eq.(49) is the non-leptonic strangeness-changing hamiltonian:

$$H_W = \sqrt{2} G_F \cos \theta_c \sin \theta_c \left\{ (C_+ + C_-) \left[ (\bar{\pi}_L \gamma_\mu s_L) (\bar{d}_L \gamma_\mu u_L) - (\bar{\tau}_L \gamma_\mu \bar{s}_L) (\bar{d}_L \gamma_\mu \bar{c}_L) \right] + (C_+ - C_-) \left[ (\bar{d}_L \gamma_\mu \bar{s}_L) (\bar{\pi}_L \gamma_\mu u_L) - (\bar{\tau}_L \gamma_\mu \bar{c}_L) (\bar{\pi}_L \gamma_\mu c_L) \right] \right\}.$$

In this formula the weak Hamiltonian is assumed to be normalized (in LLO) at $\mu = m_c$. The terms in the Hamiltonian (50) without the charmed quark fields describe the ‘spectator’ nonleptonic decay of the strange quark, while those with the $c$ quark correspond to the WS process (48).

It is straightforward to see from eq.(49) that in the PCAC limit the discussed decays should obey the $\Delta I = 1/2$ rule. Indeed, the commutator of the weak Hamiltonian with the axial charges transforms under the isotopic SU(2) in the same way as the Hamiltonian itself. In other words, the $\Delta I = 1/2$ part of $H_W$ after the commutation gives an $\Delta I = 1/2$ operator, while the $\Delta I = 3/2$ part after the commutation gives an $\Delta I = 3/2$ operator. The latter operator however cannot have a non vanishing matrix element between an isotopic singlet, $\Lambda_Q$, and an isotopic doublet, $\Xi_Q$. Thus the $\Delta I = 3/2$ part of $H_W$ gives no contribution to the $S$ wave amplitudes in the PCAC limit.

Once the isotopic properties of the decay amplitudes are fixed, one can concentrate on specific charge decay channels, e.g. $\Xi_c^- \to \Lambda_b \pi^-$ and $\Xi_c^0 \to \Lambda_c \pi^-$. An application of the PCAC relation (49) with the Hamiltonian from eq.(50) to these decays, gives the expressions for the amplitudes at $p = 0$ in terms of baryonic matrix elements of four-quark operators:

$$\langle \Lambda_b \pi^-(p = 0) | H_W | \Xi_b^- \rangle =$$

$$\frac{\sqrt{2}}{f_\pi} G_F \cos \theta_c \sin \theta_c \langle \Lambda_b | (C_+ + C_-) \left[ (\bar{\pi}_L \gamma_\mu s_L) (\bar{d}_L \gamma_\mu d_L) - (\bar{\tau}_L \gamma_\mu \bar{s}_L) (\bar{d}_L \gamma_\mu \bar{c}_L) \right] + (C_+ - C_-) \left[ (\bar{d}_L \gamma_\mu \bar{s}_L) (\bar{\pi}_L \gamma_\mu d_L) - (\bar{\tau}_L \gamma_\mu \bar{c}_L) (\bar{\pi}_L \gamma_\mu c_L) \right] | \Xi_b^- \rangle =$$

21
\[
\frac{\sqrt{2}}{f_\pi} G_F \cos \theta_c \sin \theta_c \langle \Lambda_b | C_- \left[ (\bar{u}_L \gamma_\mu s_L) (\bar{d}_L \gamma_\mu d_L) - (\bar{d}_L \gamma_\mu s_L) (\bar{u}_L \gamma_\mu d_L) \right] - \\
\frac{C_+}{3} \left[ (\bar{u}_L \gamma_\mu s_L) (\bar{d}_L \gamma_\mu d_L) + (\bar{d}_L \gamma_\mu s_L) (\bar{u}_L \gamma_\mu d_L) + 2 (\bar{u}_L \gamma_\mu s_L) (\bar{u}_L \gamma_\mu u_L) \right] \mid \Xi_b^- \rangle \, ,
\]
where in the last transition the operator structure with \( \Delta I = 3/2 \) giving a vanishing contribution is removed and only the structures with explicitly \( \Delta I = 1/2 \) are retained, and

\[
\langle \Lambda_c \pi^- (p = 0) \mid H_W \mid \Xi_c^0 \rangle = \langle \Lambda_b \pi^- (p = 0) \mid H_W \mid \Xi_b^- \rangle + \\
\frac{\sqrt{2}}{f_\pi} G_F \cos \theta_c \sin \theta_c \langle \Lambda_c \mid (C_+ + C_-) (\bar{c}_L \gamma_\mu s_L) (\bar{u}_L \gamma_\mu c_L) + \\
(C_+ - C_-) (\bar{u}_L \gamma_\mu s_L) (\bar{c}_L \gamma_\mu c_L) \mid \Xi_c^0 \rangle \, .
\]

In the latter formula the first term on the r.h.s. expresses the fact that in the heavy quark limit the ‘spectator’ amplitudes do not depend on the flavor or the mass of the heavy quark.

The rest of the expression (52) describes the ‘non-spectator’ contribution to the amplitude of the charmed hyperon decay. Using the flavor SU(3) symmetry the latter contribution can be related to the non-singlet matrix elements (27) (normalized at \( \mu = m_c \)) as

\[
\Delta A \equiv \langle \Lambda_c \pi^- (p = 0) \mid H_W \mid \Xi_c^0 \rangle - \langle \Lambda_b \pi^- (p = 0) \mid H_W \mid \Xi_b^- \rangle = \\
\frac{G_F \cos \theta_c \sin \theta_c}{2 \sqrt{2} f_\pi} \left[ (C_- - C_+) x - (C_+ + C_-) y \right] \, .
\]

Furthermore, with the help of the equations (33) and (30) relating the matrix elements \( x \) and \( y \) to the differences of the total decay widths within the triplet of charmed hyperons, one can eliminate \( x \) and \( y \) in favor of the measured width differences. The resulting expression has the form

\[
\Delta A \approx -\frac{\sqrt{2} \pi \cos \theta_c \sin \theta_c}{G_F m_c^2 f_\pi} \left[ 0.45 \left( \Gamma(\Xi_c^-) - \Gamma(\Lambda_c) \right) + 0.04 \left( \Gamma(\Lambda_c) - \Gamma(\Xi_c^+) \right) \right] = \\
-10^{-7} \left[ 0.97 \left( \Gamma(\Xi_c^0) - \Gamma(\Lambda_c) \right) + 0.09 \left( \Gamma(\Lambda_c) - \Gamma(\Xi_c^+) \right) \right] \left( \frac{1.4 GeV}{m_c} \right)^2 \, ps \, ,
\]
where, clearly, in the latter form the widths are assumed to be expressed in \( ps^{-1} \), and \( m_c = 1.4 GeV \) is used as a ‘reference’ value for the charmed quark mass. It is seen from eq.(54) that the evaluation of the difference of the amplitudes within the discussed approach is mostly sensitive to the difference of the decay rates of \( \Xi_c^0 \) and \( \Lambda_c \), with only very little sensitivity to the total decay width of \( \Xi_c^+ \). Using the current data the difference \( \Delta A \) is estimated as

\[
\Delta A = -(5.4 \pm 2) \times 10^{-7} \, ,
\]
with the uncertainty being dominated by the experimental error in the lifetime of $\Xi_c^0$. An amplitude $A$ of the magnitude, given by the central value in eq.(55) would produce a decay rate $\Gamma(\Xi_Q \to \Lambda_Q \pi) = |A|^2 p_\pi/(2\pi) \approx 0.9 \times 10^{10} \text{s}^{-1}$, which result can also be written in a form of triangle inequality

$$\sqrt{\Gamma(\Xi_\mu^+ \to \Lambda_\mu \pi^-)} + \sqrt{\Gamma(\Xi_\mu^0 \to \Lambda_\mu \pi^-)} \geq \sqrt{0.9 \times 10^{10} \text{s}^{-1}} . \quad (56)$$

Although at present it is not possible to evaluate in a reasonably model independent way the matrix element in eq.(51) for the ‘spectator’ decay amplitude, the inequality (56) shows that at least some of the discussed pion transitions should go at the level of 0.01 ps$^{-1}$, similar to the rates of analogous decays of ‘light’ hyperons.

9 Summary

We summarize here specific predictions for the inclusive decay rates, which can be argued with a certain degree of theoretical reliability, and which can be possibly experimentally tested in the nearest future.

$B$ mesons:

$$\tau(B_d)/\tau(B_s) = 1 \pm 0.01 .$$

Charmed hyperons:

$$\Gamma_{st}(\Xi_c) = (2 \sim 3) \Gamma_{st}(\Lambda_c) \quad \Gamma_{st}(\Omega_c) > \Gamma_{st}(\Xi_c) ,$$

$$\Gamma_{n}^{\Delta S=-1}(\Xi_c^+) \approx \Gamma_{n}^{\Delta S=-1}(\Lambda_c) ,$$

$$\Gamma_{n}^{\Delta S=-1}(\Xi_c^0) - \Gamma_{n}^{\Delta S=-1}(\Lambda_c) \approx 0.55 \pm 0.22 \text{ps}^{-1} .$$

$b$ hyperons:

$$\tau(\Xi_\mu^0) \approx \tau(\Lambda_b) < \tau(B_d) < \tau(\Xi_\mu^-) < \tau(\Omega_b) ,$$

$$\Gamma(\Lambda_b) - \Gamma(\Xi_\mu^-) \approx 0.11 \pm 0.03 \text{ps}^{-1} ,$$

$$0.9 < \frac{\tau(\Lambda_b)}{\tau(B_d)} < 1 .$$

Strangeness decays $\Xi_Q \to \Lambda_Q \pi$:

The $\Delta I = 1/2$ rule should hold in these decays, so that $\Gamma(\Xi_Q^{(d)} \to \Lambda_Q \pi^-) = 2 \Gamma(\Xi_Q^{(u)} \to \Lambda_Q \pi^0)$. The rates are constrained by the triangle inequality (56).
Acknowledgement
This work is supported in part by DOE under the grant number DE-FG02-94ER40823.

References

[1] Particle Data Group, Eur. Phys. J. C 3 (1998) 1.

[2] M.A. Shifman and M.B. Voloshin, (1981) unpublished, presented in the review V.A.Khoze and M.A. Shifman, Sov. Phys. Usp. 26 (1983) 387.

[3] M.A. Shifman and M.B. Voloshin, Sov. J. Nucl. Phys. 41 (1985) 120.

[4] M.A. Shifman and M.B. Voloshin, Sov. Phys. JETP 64 (1986) 698.

[5] B. Chibisov, R.D. Dikeman, M.A. Shifman, and N. Uraltsev, Int. J. Mod. Phys. A12 (1997) 2075.

[6] I. Bigi, M. Shifman, N. Uraltsev, and A. Vainshtein, Phys. Rev. D59 (1999) 054011.

[7] I.I. Bigi, N.G. Uraltsev, and A.I. Vainshtein, Phys. Lett. B293 (1992) 430; [E: B297 (1993) 477].

[8] B. Guberina, S. Nussinov, R. Peccei, and R. Rückl, Phys. Lett. B89 (1979) 11.

[9] N. Bilić, B. Guberina, and J. Trampetić, Nucl. Phys. B248 (1984) 261.

[10] I. Bigi, B. Blok, M.A. Shifman, and A. Vainshtein, Phys. Lett. B323 (1994) 408.

[11] G. Altarelli and S. Petrarca, Phys. Lett. B261 (1991) 303.

[12] E. Bagan, P. Ball, V.M. Braun, and P. Gosdzinsky, Phys. Lett. B342 (1995) 362.

[13] Q. Hokim and X.Y. Pham, Phys. Lett. B122 (1983) 297; Ann. Phys. (NY) 155 (1984) 202.

[14] Y. Nir, Phys. Lett. B221 (1989) 184.

[15] A. Czarnecki and M. Jezabek, Phys. Lett. B427 (1994) 3.

[16] E. Bagan, P. Ball, V.M. Braun and P. Gosdzinsky, Nucl. Phys. B432 (1994) 3.
[17] M.B. Voloshin, Int. J. Mod. Phys. A11 (1996) 4931.

[18] M.B. Voloshin, Surv. High En. Phys., 8 (1995) 27.

[19] P. Ball and V. Braun, Phys. Rev. D49 (1994) 2472.

[20] M. Neubert, Trieste Summer School Lectures, Report CLNS 00/1660, Jan. 2000; [hep-ph/001334].

[21] M. Neubert and C.T. Sachrajda, Nucl. Phys. B483 (1997) 339.

[22] M.A. Shifman and M.B. Voloshin, Sov. J. Nucl. Phys. 45 (1987) 292.

[23] M.B. Voloshin, Phys. Lett. B 385 (1996) 369.

[24] H.-Y. Cheng, Phys. Rev. D56 (1997) 2783.

[25] B. Guberina and B. Melić, Eur.Phys.J. C2 (1998) 697.

[26] Particle Data Group. 1999 WWW Update, [http://pdg.lbl.gov/1999/bxxx.html]

[27] UKQCD Collaboration: M. Di Pierro, C.T. Sachrajda, and C. Michael, Univ. of Southampton report SHEP 99-07, June 1999; [hep-lat/9906031].

[28] M.B. Voloshin, Phys. Rep. 320 (1999) 275.

[29] M.B. Voloshin, Phys.Rev.D61 (2000) 074026.

[30] M.B. Voloshin, Phys.Lett. B476 (2000) 297.