On the impulsive implicit \(\Psi\)-Hilfer fractional differential equations with delay

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In this paper, we investigate the existence and uniqueness of solutions and derive the Ulam-Hyers-Mittag-Leffler stability results for impulsive implicit \(\Psi\)-Hilfer fractional differential equations with time delay. It is demonstrated that the Ulam-Hyers and generalized Ulam-Hyers stability are the specific cases of Ulam-Hyers-Mittag-Leffler stability. Extended version of the Gronwall inequality, abstract Gronwall lemma, and Picard operator theory are the primary devices in our investigation. We provide an example to illustrate the obtained results.

KEYWORDS
existence and uniqueness, fractional differential equations, fractional integral inequality, stability, \(\Psi\)-Hilfer derivative

MSC CLASSIFICATION
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1 | INTRODUCTION

The investigation of impulsive fractional differential equations (FDEs) is driven not only by a theoretical enthusiasm but also in light of its applications in displaying several real-world phenomena that appearing in the applied sciences. Impulsive FDEs assume a significant job in demonstrating the physical and evolution processes, which are experienced from sudden changes in their states. For the subtleties on the application of impulsive FDEs, see Benchohra and Lazreg\(^1\) and Ahmad et al\(^2\) and the references referred therein. Because of the various applications in the literature, a lot of consideration has been paid. To investigate the impulsive FDEs, see\(^3\)-\(^{12,40}\) and the references therein.

Wang et al\(^{40}\) have proposed diverse ideas of solution for impulsive FDEs and provided the criterion to derive existence and uniqueness results. Aside from this, many interesting work on impulsive FDEs can be found in the literature that deals with existence, uniqueness, data dependence, and stability of solutions; see for instance the works of Wang et al\(^{7,9}\), Feckan et al\(^{10}\), Benchohra and Slimani\(^{11}\), Mophou\(^{12}\) and the references given therein.

On the other hand, Nieto et al\(^{13}\) and Benchohra et al\(^{11,14}\) have initiated the study of implicit FDEs and got interesting outcomes about existence and the Ulam-type stabilities. In 2016, Kucche et al\(^{15}\) acquired existence results along with data dependence of solutions for implicit FDEs employing fractional integral inequality and the \(\epsilon\)-approximated solutions. Shah et al\(^{16}\) investigated existence and the Hyers-Ulam stability of solution for implicit impulsive FDEs.

The existence and uniqueness of solutions and the Ulam-Hyers-Mittag-Leffler (UHML) stability of different kinds of fractional differential and integral equations with time delay have been investigated in Eghbali et al\(^{17}\), Wang et al\(^{18}\) and Niazi et al\(^{19}\) by using Picard operator theory and abstract Gronwall lemma. Then again, there are many fascinating research papers involving Hilfer fractional derivative, which incorporates the Riemann-Liouville and Caputo fractional derivative as special cases.\(^{20,21}\) For more recent advancement in the theory of FDEs involving the Hilfer fractional derivative, one can see Ahmed and El-Boraï\(^{22,24}\) and the references referred to in that.
Very recently, Liu et al.\textsuperscript{25} considered $\Psi$-Hilfer FDEs and obtained existence, uniqueness, and UHML stability of solutions via the Picard operator theory and a generalized Gronwall inequality involving $\Psi$–Riemann-Liouville fractional integral. In 2019, Sousa et al.\textsuperscript{26} analyzed impulsive FDEs involving $\Psi$-Hilfer derivative.

Contemplating the works referenced above, we firmly feel to consider the impulsive FDEs with generalized fractional derivatives, viz, $\Psi$-Hilfer fractional derivative, which brings together several well-known fractional derivatives. Such an investigation surely contribute to the fractional calculus. Propelled by this reality and motivated by previous works,\textsuperscript{16-25} in the present paper, we study the nonlinear implicit impulsive $\psi$-Hilfer fractional differential equation ($\psi$-HFDE) with a time delay of the form:

\begin{equation}
\mathcal{H}_{\xi_0}^{\alpha, \beta, \Psi} u(t) = f\left(t,u(t), u(h(t)), \mathcal{H}_{\xi_0}^{\alpha, \beta, \Psi} u(t)\right), \quad t \in J = (0, b) - \{t_1, t_2, \ldots, t_p\},
\end{equation}

\begin{equation}
\Delta_{\xi_0}^{1-\rho, \Psi} u(t_k) = J_k(u(t_k^+)), \quad k = 1, 2, \ldots, p,
\end{equation}

\begin{equation}
\mathcal{I}_{\xi_0}^{1-\rho, \Psi} u(0) = u_0 \in \mathbb{R}, \quad \rho = \alpha + \beta - \alpha \beta,
\end{equation}

\begin{equation}
u(t) = \phi(t), \quad t \in [-r, 0],
\end{equation}

where $\Psi \in C^1(J, \mathbb{R})$ be an increasing function with $\Psi'(x) \neq 0$, for all $x \in J$, $\mathcal{H}_{\xi_0}^{\alpha, \beta, \Psi} (\cdot)$ is the $\Psi$-Hilfer fractional derivative of order $0 < \alpha < 1$ and type $\beta (0 \leq \beta \leq 1)$, $\Delta_{\xi_0}^{1-\rho, \Psi}$ is left sided $\Psi$-Riemann Liouville fractional integration operator, $0 = t_0 < t_1 < t_2 < \ldots < t_p < t_{p+1} = b$, $\Delta_{\xi_0}^{1-\rho, \Psi} u(t_k) = \mathcal{I}_{\xi_0}^{1-\rho, \Psi} u(t_k^+)$, $\mathcal{I}_{\xi_0}^{1-\rho, \Psi} u(t_k^+) = \lim_{t \to t_k^+} \mathcal{I}_{\xi_0}^{1-\rho, \Psi} u(t) = \lim_{t \to t_k^-} \mathcal{I}_{\xi_0}^{1-\rho, \Psi} u(t)$. The functions $f : J \times \mathbb{R}^3 \to \mathbb{R}$ and $J_k : \mathbb{R} \to \mathbb{R}$ are appropriate functions specified latter, $h \in C(J, [-r, b])$ is a continuous delay function such that $h(t) \leq t$, $t \in J$. Further, $\phi \in C := C([-r, 0], \mathbb{R})$—the space of all continuous functions from $[-r, 0]$ to $\mathbb{R}$, with supremum norm $\|\phi\|_C = \max_{t \in [-r, 0]} |\phi(t)|$.

Our main objective is to investigate the existence and uniqueness of solutions and examine the UHML stability of impulsive implicit $\Psi$-HFDE (1)-(4) with time delay. It is observed that the Ulam-Hyers and the generalized Ulam-Hyers stability for the problem (1)-(4) are obtained as particular cases of UHML stability results that we acquired. Our analysis is based on an extended version of the Gronwall inequality, abstract Gronwall lemma, and the Picard operator theory.

The investigation of implicit FDEs is perceived as a significant one.\textsuperscript{13,15,16,27-31} Further, looking towards the multidimensional utilization of impulsive FDES,\textsuperscript{1,2} it is important to consider the new class implicit FDEs with impulse condition that incorporates a wide class of impulsive FDEs as particular cases. In this sense, we considered a new class of implicit FDEs with impulse condition involving $\Psi$-Hilfer fractional derivative, which is a generalized fractional derivative operator and it incorporates the wide class of well-known fractional derivatives including the Caputo and Riemann-Liouville derivative. If we take $\Psi(t) = t$ and $\beta = 1$, then the problem (1)-(4) reduces to implicit impulsive FDEs with the Caputo fractional derivative. For $\Psi(t) = t$ and $\beta = 0$, the problem (1)-(4) reduces to implicit impulsive FDEs with the Riemann-Liouville fractional derivative. Aside from this, for different function $\Psi$ and parameter $\alpha, \beta$, the $\Psi$-Hilfer fractional derivative $\mathcal{H}_{\xi_0}^{\alpha, \beta, \Psi}$ includes well-known Katugampola derivative, Hadamard derivative, Hilfer derivative, Chen derivative, Prabhakar derivative, Erdyl-Kober derivative, Riesz derivative, Feller derivative, Weyl derivative, Cassar derivative, and so forth; for more details, see Sousa and Oliveira.\textsuperscript{32} Therefore, the studies which we have done for the impulsive implicit $\Psi$-HFDE (1)-(4) with time delay includes the study of impulsive implicit FDEs with time delay involving the derivatives listed in Sousa and Oliveira.\textsuperscript{32} Further, the results obtained in the present paper extends previous studies.\textsuperscript{16-25}

The paper is organized as follows. In Section 2, we collect definitions and some basic results related to the $\Psi$-Hilfer derivative. In Section 3, we derive the representation formula for the solution of the problem (1)-(4). Section 4 deals with the existence and uniqueness of solution and UHML stability for $\Psi$-HFDE (1)-(4). In Section 4, we give an example to illustrate the results we obtained.

2 | PRELIMINARIES

Throughout the paper, we assume that $\Psi \in C^1(J, \mathbb{R})$ be an increasing function with $\Psi'(x) \neq 0, x \in J$. 
Definition 1 \((33)\). Let \(\alpha > 0\) and \(f\) be an integrable function defined on \(J\). Then, \(\Psi\)-Riemann-Liouville fractional integrals of \(f\) is given by
\[
\mathcal{I}_0^\alpha \Psi f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t A_\alpha^\Psi(t,s)f(s)ds, \quad \text{where} \quad A_\alpha^\Psi(t,s) = \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha - 1}.
\]
Note that for \(\delta > 0\), we have \(\mathcal{I}_0^{\alpha+\delta} \Psi (\Psi(t) - \Psi(0))^{-1} = \frac{\Gamma(\alpha)}{\Gamma(\alpha+\delta)} (\Psi(t) - \Psi(0))^{\alpha+\delta - 1}\).

Definition 2 \((32)\). Let \(f \in C^1(J, \mathbb{R})\). The \(\Psi\)-Hilfer fractional derivative of a function \(f\) of order \(0 < \alpha < 1\) and type \(0 \leq \beta \leq 1\) is defined by
\[
H_{\alpha}^{\beta;\psi} f(t) = \mathcal{I}_0^{\alpha - 1 - \beta} \Psi \left( \frac{1}{\psi'(t)} \frac{d}{dt} \mathcal{I}_0^{\alpha - 1 - \beta} \Psi f(t) \right).
\]

Theorem 1 \((32)\). Let \(f \in C^1(J, \mathbb{R})\), \(0 < \alpha < 1\) and \(0 \leq \beta \leq 1\). Then
\(i\) \(H_{\alpha}^{\beta;\psi} f(t) = f(t) - R_\alpha^\psi(t,0) I_0^{1-\beta} \Psi f(0), \) where \(R_\alpha^\psi(t,0) = \frac{(\Psi(t) - \Psi(0))^{\alpha+1}}{\Gamma(\alpha+1)}\).
\(ii\) \(H_{\alpha}^{\beta;\psi} f(t) = f(t)\).

Consider the weighted space\(^{32}\) defined by
\[
C_{1-\rho} \Psi (J, \mathbb{R}) = \{ u : (0, b) \to \mathbb{R} : (\Psi(t) - \Psi(0))^{1-\rho} u(t) \in C(J, \mathbb{R}) \}, \quad 0 < \rho \leq 1.
\]
Define the weighted space of piecewise continuous functions as
\[
PC_{1-\rho} \Psi (J, \mathbb{R}) = \left\{ u : (0, b) \to \mathbb{R} : u \in C_{1-\rho} \Psi ((t_k, t_{k+1}], \mathbb{R}), \quad k = 0, 1, \ldots, p, \quad \mathcal{I}_0^{1-\rho} \Psi u(t_k^+) \right\}
\]
exists and \(\mathcal{I}_0^{1-\rho} \Psi u(t_k^-) = \mathcal{I}_0^{1-\rho} \Psi u(t_k) \) for \(k = 1, \ldots, p\).

Then, \(PC_{1-\rho} \Psi (J, \mathbb{R})\) is a Banach space with the norm \(\|u\|_{PC_{1-\rho} \Psi (J, \mathbb{R})} = \sup_{t \in J} |(\Psi(t) - \Psi(0))^{1-\rho} u(t)|\). Observe that for \(\rho = 1\), the space \(PC_{1-\rho} \Psi (J, \mathbb{R})\) reduces to \(PC(J, \mathbb{R})\), which is dealt with in previous studies.\(^{10,\text{11,34}}\)

Next, we introduce the space
\[
X_{C, \rho, \Psi} = \{ u : [-r, b] \to \mathbb{R} : u \in C \cap PC_{1-\rho} \Psi (J, \mathbb{R}) \},
\]
with the norm \(\|u\|_{X_{C, \rho, \Psi}} = \max \left\{ \|u\|_{C}, \|u\|_{PC_{1-\rho} \Psi (J, \mathbb{R})} \right\}\). One can verify that \((X_{C, \rho, \Psi}, \| \cdot \|_{X_{C, \rho, \Psi}})\) is a Banach space.

For \(\nu \in X_{C, \rho, \Psi}\) and \(\epsilon > 0\), consider the following inequalities
\[
\begin{cases}
H_0^{\alpha;\Psi} \nu(t) = f(t, \nu(t), \nu(h(t))), \quad H_0^{\alpha;\Psi} \nu(t) \\
|\Delta_0^{1-\rho;\psi} \nu(t_k) - J_k(\nu(t_k^-))| \leq \epsilon, \quad k = 1, \ldots, p.
\end{cases}
\]
where \(E_\alpha\) is the Mittag-Leffler function\(^ {15} \) defined by
\[
E_\alpha(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(n\alpha + 1)}, \quad z \in \mathbb{C}, \quad \text{Re}(\alpha) > 0.
\]

To examine the Ulam-Hyers-Mittag-Leffler (UHML) stability of the problem \((1)-(2)\), we adopt the definitions given by Wang et al.\(^ {18} \) and Liu et al.\(^ {25} \)

Definition 3. Equations \((1)-(2)\) are said to be UHML stable with respect to \(E_\alpha \left( \xi_f \Psi(\Psi(t) - \Psi(0))^{\rho} \right)\), if for \(\epsilon > 0\) there exists a constant \(C_{p,E_\alpha} > 0\) such that for every solution \(\nu \in X_{C, \rho, \Psi}\) of the inequality \((5)\), there is a unique solution
Let \( x \in \mathbb{X}_{\mathcal{C}, \rho, \psi} \) to the problem (1)-(4) satisfying

\[
\begin{aligned}
&|u(t) - v(t)| = 0, \quad t \in [-r, 0], \\
&|v(t) - \Psi(t)| \leq C_{\rho, t} e^{-C_{\rho}} E_a \left( \zeta_{f, \psi}(\Psi(t) - \Psi(0))^\alpha \right), \quad \zeta_{f, \psi} > 0, \quad t \in J.
\end{aligned}
\]

**Remark 1.** We say that \( v \in \mathbb{X}_{\mathcal{C}, \rho, \psi} \) is the solution of the inequality (5) if there exist a function \( \mathcal{E} \in \mathbb{X}_{\mathcal{C}, \rho, \psi} \) and a sequence \( \{ \mathcal{E}_k \} \), \( k = 1, \ldots, p \) (both depending on \( v \)) such that

1. \( |\mathcal{E}(t)| \leq e E_a \left( \zeta_{f, \psi}(\Psi(t) - \Psi(0))^\alpha \right), \quad t \in J, \quad |\mathcal{E}_k| \leq \epsilon, \quad k = 1, \ldots, p, \)
2. \( H_{0, \alpha}^{\beta, \psi} v(t) = f(t, v(t), v(h(t)) + H_{0, \alpha}^{\beta, \psi} v(t)), \quad t \in J, \)
3. \( \Delta^n_{0, \alpha}^{\beta, \psi} v(t_k) = t_l(t_k) + \mathcal{E}_k, \quad k = 1, \ldots, p. \)

**Definition 4 \((35)\).** Let \((\mathcal{X}, \rho)\) be a metric space. The operator \( \mathcal{T} : \mathcal{X} \to \mathcal{X} \) is a Picard operator if there exists \( x^\ast \in \mathcal{X} \) such that

1. \( F_T = \{ x^\ast \}, \) where \( F_T = \{ x \in \mathcal{X} : \mathcal{T}(x) = x \}; \)
2. the sequence \( \{ T^n(x_0) \}_{n \in \mathbb{N}} \) converges to \( x^\ast \) for all \( x_0 \in \mathcal{X}. \)

**Lemma 1 \((36)\).** Let \((\mathcal{X}, \rho, \leq)\) be an ordered metric space and let \( \mathcal{T} : \mathcal{X} \to \mathcal{X} \) be an increasing Picard operator with \( F_T = \{ x^\ast \}. \) Then, for any \( x \in \mathcal{X}, x \leq \mathcal{T}(x) \) implies \( x \leq x^\ast. \)

**Lemma 2 \((37)\).** Let \( U \in \mathcal{PC}_{1-\rho, \psi}(J, \mathbb{R}) \) satisfy the following inequality:

\[
U(t) \leq V(t) + g(t) \int_0^t \mathcal{A}_\psi(t, s) U(s) ds + \sum_{0 < i < l} \beta_k U(t^+_k), \quad t > a,
\]

where \( g \) is a continuous function, \( V \in \mathcal{PC}_{1-\rho, \psi}(J, \mathbb{R}) \) is non-negative, \( \beta_k > 0 \) for \( k = 1, \ldots, p, \) then we have

\[
U(t) \leq V(t) \prod_{i=1}^k \left( 1 + \beta_i E_a \left( g(t) \Gamma(\psi(t) - \psi(0))^\alpha \right) \right) E_a \left( g(t) \Gamma(\psi(t) - \psi(0))^\alpha \right), \quad t \in (t_k, t_{k+1}].
\]

## 3 | FORMULA OF SOLUTIONS

We need the following lemma to derive the equivalent fractional integral of the impulsive problem (1)-(4).

**Lemma 3. \((38)\).** Let \( 0 < \alpha < 1 \) and \( h : J \to \mathbb{R} \) be continuous. Then, for any \( b \in J, \) a function \( u \in C_{1-\rho, \psi}(J, \mathbb{R}) \) defined by

\[
\begin{aligned}
&u(t) = R^\psi(t, 0) \left\{ \left( 1^{1-\rho, \psi} u(b) - 1^{1-\rho+\psi, \psi} h(t) \right)_{t=b} \right\} + 1^{\psi, \psi} h(t), \\
&\left. \right|_{t=b}
\end{aligned}
\]

is the solution of the \( \psi \)-Hilfer fractional differential equation \( H_{0, \alpha}^{\beta, \psi} u(t) = h(t), \quad t \in J. \)

**Theorem 2.** A function \( u \in \mathbb{X}_{\mathcal{C}, \rho, \psi} \) is a solution of implicit impulsive (1)-(4), if and only if, it is a solution of the following fractional integral equation

\[
\begin{aligned}
&u(t) = \begin{cases}
\phi(t), \quad t \in [-r, 0], \\
R^\psi(t, 0) \left( u_0 + \sum_{0 < i < l} J_k(u(t^+_k)) \right) + 1^{\psi, \psi} g_u(t), \quad t \in J,
\end{cases}
\end{aligned}
\]

where

\[
g_u(t) = f(t, u(t), u(h(t)), g_u(t)).
\]
Proof. Assume that \( u \in X_C, \rho, \psi \) satisfies the implicit impulsive \( \psi \)-HFDE (1)-(4). If \( t \in [0, t_1] \), then

\[
\begin{align*}
H_0^{\alpha, \beta; \psi} u(t) &= f \left( t, u(t), u(h(t)) \right), \\
H_0^{\alpha, \beta; \psi} u(0) &= u_0.
\end{align*}
\]

Let \( H_0^{\alpha, \beta; \psi} u(t) = g_u(t) \). Then we have \( g_u(t) = f \left( t, u(t), u(h(t)) \right) \), and (8) becomes

\[
\begin{align*}
\left\{ \begin{array}{l}
H_0^{\alpha, \beta; \psi} u(t) = g_u(t), \\
\mathbb{I}_{0}^{1-\rho; \psi} u(0) = u_0.
\end{array} \right.
\]

Then, problem (9) is equivalent to the following fractional integral \( 39 \)

\[
\psi(t) = R_{\psi}^\beta (t, 0) \ u_0 + \mathbb{I}_{0}^{1-\rho; \psi} g_u(t), \quad t \in [0, t_1].
\]

Now, if \( t \in (t_1, t_2] \), then in the view of (8), we have

\[
H_0^{\alpha, \beta; \psi} u(t) = g_u(t), \quad t \in (t_1, t_2]
\]

with \( \mathbb{I}_{0}^{1-\rho; \psi} u(t_1^+) - \mathbb{I}_{0}^{1-\rho; \psi} u(t_1^-) = J_1(u(t_1^+)) \).

By Lemma 3, we have

\[
\begin{align*}
\psi(t) &= R_{\psi}^\beta (t, 0) \left\{ \mathbb{I}_{0}^{1-\rho; \psi} u(t_1^+) - \mathbb{I}_{0}^{1-\rho; \psi} g_u(t) \right\} + \mathbb{I}_{0}^{\rho; \psi} g_u(t) \\
&= R_{\psi}^\beta (t, 0) \left\{ \mathbb{I}_{0}^{1-\rho; \psi} u(t_1^-) + J_1(u(t_1^-)) - \mathbb{I}_{0}^{1-\rho; \psi} g_u(t) \right\} \\
&\quad + \mathbb{I}_{0}^{\rho; \psi} g_u(t), \quad t \in (t_1, t_2].
\end{align*}
\]

Now, from (10), we have \( \mathbb{I}_{0}^{1-\rho; \psi} u(t) = u_0 + \mathbb{I}_{0}^{1-\rho; \psi} g_u(t) \). This gives

\[
\begin{align*}
\mathbb{I}_{0}^{1-\rho; \psi} u(t_1^-) - \mathbb{I}_{0}^{1-\rho; \psi} g_u(t) \bigg|_{t=t_1} &= u_0.
\end{align*}
\]

Using (12) in (11), we obtain

\[
\psi(t) = R_{\psi}^\beta (t, 0) \left( u_0 + J_1(u(t_1^-)) \right) + \mathbb{I}_{0}^{\rho; \psi} g_u(t), \quad t \in (t_1, t_2].
\]

Continuing in this manner, we obtain

\[
\psi(t) = R_{\psi}^\beta (t, 0) \left( u_0 + \sum_{i=1}^{k} J_i(u(t_i^-)) \right) + \mathbb{I}_{0}^{\rho; \psi} g_u(t), \quad t \in (t_k, t_{k+1}], \quad k = 1, \ldots, p.
\]

From above, we obtain (7).

Conversely, let \( u \in X_C, \rho, \psi \) satisfies the fractional integral Equation 7. Then, for \( t \in J \), we have

\[
\psi(t) = R_{\psi}^\beta (t, 0) \left( u_0 + \sum_{0 \leq l < t} J_l(u(t_l^-)) \right) + \mathbb{I}_{0}^{\rho; \psi} g_u(t), \quad t \in J.
\]

Applying the \( \psi \)-Hilfer fractional derivative operator \( H_0^{\alpha, \beta; \psi} \) on both sides of the above equation and using the result, \( 39, \) Page 10 \( H_0^{\alpha, \beta; \psi} (\psi(t) - \psi(0))^{\rho-1} = 0, \quad 0 < \rho < 1 \) and Theorem 1, we obtain

\[
H_0^{\alpha, \beta; \psi} u(t) = g_u(t) = f \left( t, u(t), u(h(t)) \right), \quad t \in J.
\]
which is (1). Further, from (10), we have
\[ \| t \|^{1-\rho} \psi u(t) = u_0 + \int_0^t \| t \|^{1-\rho} \psi R_\alpha(t, 0) + \int_0^t \| t \|^{1-\rho} \psi g_u(t) = u_0 + \int_0^t \| t \|^{1-\rho+\alpha} \psi g_u(t), \]
which gives
\[ \| t \|^{1-\rho} \psi u(0) = u_0. \]

Now from Equation 14, for \( t \in (t_k, t_{k+1}] \), we have
\[ \| t \|^{1-\rho} \psi u(t) = \left\{ u_0 + \sum_{i=1}^k J_i(u(t_i^-)) \right\} \| t \|^{1-\rho} \psi R_\alpha(t, 0) + \int_0^t \| t \|^{1-\rho} \psi g_u(t) \]
\[ = u_0 + \sum_{i=1}^k J_i(u(t_i^-)) + \int_0^t \| t \|^{1-\rho+\alpha} \psi g_u(t). \]

(16)

Again, for \( t \in (t_{k-1}, t_k] \), we have
\[ \| t \|^{1-\rho} \psi u(t) = \left\{ u_0 + \sum_{i=1}^{k-1} J_i(u(t_i^-)) \right\} \| t \|^{1-\rho} \psi R_\alpha(t, 0) + \int_0^t \| t \|^{1-\rho} \psi g_u(t) \]
\[ = u_0 + \sum_{i=1}^{k-1} J_i(u(t_i^-)) + \int_0^t \| t \|^{1-\rho+\alpha} \psi g_u(t). \]

(17)

Therefore, from (16) to (17), we obtain
\[ \| t \|^{1-\rho} \psi u(t_k^+) - \| t \|^{1-\rho} \psi u(t_k^-) = \sum_{i=1}^k J_i(u(t_i^-)) - \sum_{i=1}^{k-1} J_i(u(t_i^-)) = J_k(u(t_k^-)). \]

(18)

This completes the proof.

4 | EXISTENCE, UNIQUENESS, AND UHML STABILITY

In this section, we derive the existence and uniqueness of solution to the problem (1)-(4). Further, the UHML stability of Equation 1 and (2) is investigated.

**Theorem 3.** Assume that

(H1) the function \( f : J \times \mathbb{R}^3 \to \mathbb{R} \) is continuous and there exists constant \( K > 0 \) and \( 0 < L_f < 1 \) satisfy the following condition:
\[ | f(t, u_1, u_2, u_3) - f(t, v_1, v_2, v_3) | \leq K (| \Psi(t) - \Psi(0) |^{1-\rho}) \sum_{i=1}^3 | u_i - v_i | + L_f | u_3 - v_3 |, \]
for \( t \in J \) and \( u_i, v_i \in \mathbb{R} \) for \( i = 1, 2, 3. \)

(H2) The functions \( J_k : \mathbb{R} \to \mathbb{R} \) (\( k = 1, \ldots, p \)) satisfy the condition
\[ | J_k(u(t_k^-)) - J_k(u(t_k^-)) | \leq L_{J,K} (| \Psi(t_k^-) - \Psi(0) |^{1-\rho}) | u(t_k^-) - v(t_k^-) |, \]
where \( u \in PC_{1-\rho} \psi (J, \mathbb{R}) \) and \( L_{J,K} > 0. \)

(H3) \( L = \frac{\sum_{i=1}^p \gamma_i}{\Gamma(\rho)} + \frac{2K(\psi(0)\psi(0))^{1-\rho+1}}{(1-L_f)^{\rho+1}} < 1. \)

Then,

1. problem (1)-(4) has unique solution in the space \( X_c, \rho, \psi; \)
2. the Eq. (1)-(2) is UHML stable.
Proof. Part-1: By Theorem 2, the equivalent fractional integral equation of problem (1)-(4) is given by Equation 7. Define the operator \( T : (X_C, \rho, \psi, \| \cdot \|_{X_C, \rho, \psi}) \to (X_C, \rho, \psi, \| \cdot \|_{X_C, \rho, \psi}) \) by

\[
T(u(t)) = \begin{cases} 
\phi(t), & t \in [-r, 0], \\
R_\psi^p(t, 0) \left( u_0 + \sum_{a < t_k < t} J_k(u(t_k^-)) \right) + \frac{\psi}{\Gamma(p)} g_a(t), & t \in J,
\end{cases}
\]

where

\[
g_a(t) = f(t, u(t), u(h(t)), g_a(t)).
\]

Then, the solution of (1)-(4) will be the fixed point of \( T \). In order to prove \( T \) is Picard operator, we prove that \( T \) is contraction mapping. Let any \( u, \tilde{u} \in X_C, \psi \). Then for any \( t \in [-r, 0] \),

\[
|T(u(t)) - T(\tilde{u}(t))| = 0 \Rightarrow \| T(u(t)) - T(\tilde{u}(t)) \|_C = 0.
\]

Further, for any \( t \in J \), by definition of \( T \), we have

\[
|T(u(t)) - T(\tilde{u}(t))| \leq R_\psi^p(t, 0) \sum_{a < t_k < t} \left| J_k(u(t_k^-)) - J_k(\tilde{u}(t_k^-)) \right| \\
+ \frac{1}{\Gamma(a)} \int_0^t A_\psi^a(t, s) |g_a(s) - g_a(s)| \, ds.
\]

Using (H1) and (20) for any \( t \in J \), we have

\[
|g_a(t) - g_a(t)| \leq K(\Psi(t) - \Psi(0))^{1-\rho} \left( |u(t) - \tilde{u}(t)| + |u(h(t)) - \tilde{u}(h(t))| \right) + L_f |g_a(t) - g_a(t)|.
\]

This implies that

\[
|g_a(t) - g_a(t)| \leq \frac{K(\Psi(t) - \Psi(0))^{1-\rho}}{1 - L_f} \left( |u(t) - \tilde{u}(t)| + |u(h(t)) - \tilde{u}(h(t))| \right).
\]

Making use of hypothesis (H2) and the inequality (23), (22) takes the form,

\[
(\Psi(t) - \Psi(0))^{1-\rho} |T(u(t)) - T(\tilde{u}(t))| \leq \frac{1}{\Gamma(\rho)} \sum_{a < t_k < t} L_{J_k} (\Psi(t_k^-) - \Psi(0))^{1-\rho} |u(t_k^-) - \tilde{u}(t_k^-)| \\
+ \frac{K (\Psi(t) - \Psi(0))^{1-\rho}}{(1 - L_f)\Gamma(a)} \int_0^t A_\psi^a(t, s) (\Psi(s) - \Psi(0))^{1-\rho} \\
\times \left( |u(s) - \tilde{u}(s)| + |u(h(s)) - \tilde{u}(h(s))| \right) \, ds \\
\leq \frac{1}{\Gamma(\rho)} \sum_{k=1}^p L_{J_k} \| u - \tilde{u} \|_{p_{C_{1-\rho}, \psi}(J, \mathbb{R})} \\
+ \frac{2K(\Psi(b) - \Psi(0))^{1-\rho}}{(1 - L_f)\Gamma(a + 1)} \| u - \tilde{u} \|_{p_{C_{1-\rho}, \psi}(J, \mathbb{R})} \int_0^t A_\psi^a(t, s) \, ds \\
\leq \left( \sum_{k=1}^p \frac{L_{J_k}}{\Gamma(\rho)} + \frac{2K(\Psi(b) - \Psi(0))^{1-\rho+a}}{(1 - L_f)\Gamma(a + 1)} \right) \| u - \tilde{u} \|_{p_{C_{1-\rho}, \psi}(J, \mathbb{R})}.
\]

Therefore,

\[
\| T(u(t)) - T(\tilde{u}(t)) \|_{p_{C_{1-\rho}, \psi}(J, \mathbb{R})} = \sup_{t \in J} (\Psi(t) - \Psi(0))^{1-\rho} (T(u(t)) - T(\tilde{u}(t))) \leq L \| u - \tilde{u} \|_{p_{C_{1-\rho}, \psi}(J, \mathbb{R})},
\]

(24)
From (21) and (24), we have

\[
\|T(u(t)) - T(\bar{u}(t))\|_{X_{\psi}} = \max \left\{ \|T(u(t)) - T(\bar{u}(t))\|_C, \|T(u(t)) - T(\bar{u}(t))\|_{PC_{\psi}} \right\} \\
\leq \mathcal{L} \max \left\{ 0, \|u - \bar{u}\|_{PC_{\psi}} \right\} \\
\leq \mathcal{L} \|u - \bar{u}\|_{X_{\psi}}.
\]

Since \( \mathcal{L} < 1 \), \( T \) is a contraction on \( X_{\psi} \), hence by Banach contraction principle, \( T \) has a unique fixed point in \( X_{\psi} \), which is the unique solution of (1)-(4).

Part-2: In this part, we prove that problems (1) - (2) are UHML stable. Let \( v \in X_{\psi} \) be a solution of the inequality (5). Then by the Theorem 2 and Remark 1, we have

\[
v(t) = R^\rho_{\psi}(t, 0) \left( t_{0}^{1-\rho} \psi v(0) + \sum_{0 < t_k < t} (J_k(v(t_k^-)) + E_k) \right) + \|_{0}^{\rho} \psi g_{\psi}(t) + \|_{0}^{\rho} \psi E, \quad t \in J,
\]

where \( g_{\psi}(t) = f(t, v(t), v(h(t)), g_{\psi}(t)) \).

Let \( u \in X_{\psi} \) be the unique solution of the problem

\[
\begin{aligned}
H_{0}^{\alpha}_0 \psi u(t) &= f(t, u(t), u(h(t)), H_{0}^{\alpha}_0 \psi u(t)), \quad t \in J - \{ t_1, t_2, \ldots, t_p \}, \\
\Delta_{0}^{1-\rho} \psi u(t_k) &= J_k(u(t_k^-)), \quad k = 1, \ldots, p, \\
\|_{0}^{1-\rho} \psi u(0) &= \|_{0}^{1-\rho} \psi u(0), \\
u(t) &= v(t), \quad t \in [-r, 0].
\end{aligned}
\]

Then, from the Equation 25 and in the view of Remark 1, for any \( t \in J \), we have

\[
\begin{aligned}
\left| v(t) - R^\rho_{\psi}(t, 0) \left( t_{0}^{1-\rho} \psi v(0) + \sum_{0 < t_k < t} J_k(v(t_k^-)) \right) - t_{0}^{\rho} \psi g_{\psi}(t) \right| \\
\leq R^\rho_{\psi}(t, 0) \sum_{k=1}^{p} |E_k| + \frac{e}{\Gamma(\alpha)} \int_{0}^{t} A^\alpha_{\psi}(t, s) |E(s)| \, ds \\
\leq me \, R^\rho_{\psi}(t, 0) + \frac{e}{\Gamma(\alpha)} \int_{0}^{t} \psi(s) \left( 1 - \psi(s) - \psi(0) \right)^{\alpha-1} |E(s)| \, ds \\
= me \, R^\rho_{\psi}(t, 0) + \frac{e}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha + 1)} \int_{0}^{t} (\psi(t) - \psi(0))^{n\alpha} \, ds \\
&\quad \times \int_{0}^{t} \psi(s) \left( 1 - \frac{\psi(s) - \psi(0)}{\psi(t) - \psi(0)} \right)^{\alpha-1} |E(s)| \, ds \\
= me \, R^\rho_{\psi}(t, 0) + \frac{e}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha + 1)} (\psi(t) - \psi(0))^{n\alpha} \int_{0}^{1} (1 - \theta)^{\alpha-1} \theta^{\alpha} \, d\theta \\
&\quad \text{letting} \quad \theta = \frac{\psi(s) - \psi(0)}{\psi(t) - \psi(0)} \quad \text{we have,} \quad \psi(s) \, ds = (\psi(t) - \psi(0)) \, d\theta \\
= me \, R^\rho_{\psi}(t, 0) + \frac{e}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{(\psi(t) - \psi(0))^{n\alpha}}{\Gamma(n\alpha + 1)} \frac{\Gamma(n\alpha + 1)}{\Gamma(n\alpha + 1)} \int_{0}^{1} (1 - \theta)^{\alpha-1} \theta^{\alpha} \, d\theta \\
= me \, R^\rho_{\psi}(t, 0) + e \, \int_{0}^{t} (\psi(t) - \psi(0))^{n\alpha} \, ds \\
\leq me \, R^\rho_{\psi}(t, 0) + e \, E_{\psi}(\psi(t) - \psi(0))^{n\alpha}.
\end{aligned}
\]
Now for $t \in [-r, 0]$, $|v(t) - u(t)| = 0$. Further, utilizing ($H_2$), (23) and (27), for any $t \in J$, we have

$$|v(t) - u(t)| \leq \left| v(t) - R^ζ_ρ(t, 0) \left( \mathbb{1}_{0^+}^1 \Psi v(0) + \sum_{0 < t_k < t} J_k(v(t_k^-)) \right) - \mathbb{1}_{0^+}^1 \Psi g_0(t) \right|$$

$$+ R^ζ_ρ(t, 0) \sum_{0 < t_k < t} \left| J_k(v(t_k^-)) - J_k(u(t_k^-)) \right| + \left| \mathbb{1}_{0^+}^1 \Psi g_0(t) - g_0(t) \right|$$

$$\leq \left( m e R^ζ_ρ(t, 0) + \epsilon E_1((\Psi(t) - \Psi(0))^ρ) \right)$$

$$+ R^ζ_ρ(t, 0) \sum_{0 < t_k < t} \mathcal{L} J_k(\Psi(t_k^-) - \Psi(0))^{1-ρ} \left| v(t_k^-) - u(t_k^-) \right|$$

$$+ \frac{K}{(1 - \mathcal{L}_f)^Γ(α)} \int_0^t \mathcal{A}^ζ_ρ(t, s) (Ψ(s) - Ψ(0))^{1-ρ}$$

$$× \{|v(s) - u(s)| + |v(h(s)) - u(h(s))|\} \ ds.$$

This gives

$$(Ψ(t) - Ψ(0))^{1-ρ}|v(t) - u(t)| \leq \left( \mathbb{1}_{0^+}^1 \Psi v(0) + \sum_{0 < t_k < t} J_k(v(t_k^-)) \right) - \mathbb{1}_{0^+}^1 \Psi g_0(t)$$

$$+ \frac{1}{Γ(ρ)} \sum_{0 < t_k < t} \mathcal{L} J_k(Ψ(t_k^-) - Ψ(0))^{1-ρ} \left| v(t_k^-) - u(t_k^-) \right|$$

$$+ \frac{K}{(1 - \mathcal{L}_f)^Γ(α)} \int_0^t \mathcal{A}^ζ_ρ(t, s) (Ψ(s) - Ψ(0))^{1-ρ}$$

$$× \{|v(s) - u(s)| + |v(h(s)) - u(h(s))|\} \ ds. \quad (28)$$

Next, we consider the Banach space $B = C([-r, b], \mathbb{R}_+)$ of all continuous functions $ζ : [-r, b] \rightarrow \mathbb{R}_+$ endowed with the supremum norm $\|ζ\|_B = \sup_{t \in [-r, b]} |ζ(t)|$. Define the operator $Q : B \rightarrow B$ by

$$(Qζ)(t) = \left\{ \begin{array}{ll}
0, & t \in [-r, 0], \\
\left( \frac{me}{Γ(ρ)} + \epsilon (Ψ(b) - Ψ(0))^{1-ρ} \right) E_1((Ψ(b) - Ψ(0))^ρ) + \frac{1}{Γ(ρ)} \sum_{0 < t_k < t} \mathcal{L} J_k (Ψ(t_k^-)) \\
+ \frac{K}{(1 - \mathcal{L}_f)^Γ(α)} \int_0^t \mathcal{A}^ζ_ρ(t, s) (ζ(s) + ζ(h(s))) \ ds, & t \in J.
\end{array} \right. \quad (29)$$

We prove that $Q$ is a Picard operator. Let $ζ, \tilde{ζ} \in B$. Then,

$$|Qζ(t) - Q\tilde{ζ}(t)| = 0, \ t \in [-r, 0]. \quad (30)$$

Now, for any $t \in J$,

$$|Qζ(t) - Q\tilde{ζ}(t)| \leq \frac{1}{Γ(ρ)} \sum_{0 < t_k < t} \mathcal{L}_J ζ(t_k^-) - ζ(t_k)$$

$$+ \frac{K}{(1 - \mathcal{L}_f)^Γ(α)} \left( \int_0^t \mathcal{A}^ζ_ρ(t, s) ζ(s) - ζ(s) \ ds \right)$$

$$+ \int_0^t \mathcal{A}^ζ_ρ(t, s) ζ(h(s)) - ζ(h(s))) \ ds)$$

$$\leq \left( \sum_{k=1}^N \mathcal{L}_J ζ(t_k^-) + \frac{2}{Γ(ρ)} (1 - \mathcal{L}_f)^Γ(α + 1) \right) \|ζ - ζ\|_B. \quad (31)$$
From (30) and (31), it follows that

\[\|Qz - Q\tilde{z}\|_B \leq \left( \sum_{k=1}^{p} \mathcal{L}_{J_k} + \frac{2K (\Psi(b) - \Psi(0))^{1-\rho + a}}{(1 - \mathcal{L}_f) \Gamma(\alpha + 1)} \right) \|z - \tilde{z}\|_B.\]

By \((H_3)\), \(\left( \sum_{k=1}^{p} \mathcal{L}_{J_k} + \frac{2K (\Psi(b) - \Psi(0))^{1-\rho + a}}{(1 - \mathcal{L}_f) \Gamma(\alpha + 1)} \right) < 1\), therefore, \(Q\) is contraction. By Banach contraction principle \(F_Q = \{z^*\}\), it follows that

\[z^*(t) = \left( \frac{me}{\Gamma(\rho)} + \epsilon (\Psi(b) - \Psi(0))^{1-\rho} E_a((\Psi(b) - \Psi(0))^{a}) \right) + \frac{1}{\Gamma(\rho)} \sum_{0 < t_k < t} \mathcal{L}_{J_k} z^*(t_k^-) + \frac{K (\Psi(b) - \Psi(0))^{1-\rho}}{(1 - \mathcal{L}_f) \Gamma(\alpha)} \int_0^t A_a^\rho(t, s) (z^*(s) + z^*(h(s))) \, ds.\]  

(32)

Next, we show that \(z^*\) is increasing. Let any \(t_1, t_2 \in [-r, b]\) with \(t_1 < t_2\). If \(t_1, t_2 \in [-r, 0]\) then \(z^*(t_2) - z^*(t_1) = 0\). Let \(0 < t_1 < t_2 \leq b\). Define \(M = \min_{\epsilon \in [0, b]} (z^*(s) + z^*(h(s)))\). Then,

\[z^*(t_2) - z^*(t_1) = \frac{1}{\Gamma(\rho)} \sum_{0 < t_k < t_2} \mathcal{L}_{J_k} z^*(t_k^-) - \frac{1}{\Gamma(\rho)} \sum_{0 < t_k < t_1} \mathcal{L}_{J_k} z^*(t_k^-) + \frac{K (\Psi(b) - \Psi(0))^{1-\rho}}{(1 - \mathcal{L}_f) \Gamma(\alpha)} \left( \int_0^{t_2} \Psi'(s)(\Psi(t_2) - \Psi(s))^{a-1} (z^*(s) + z^*(h(s))) \, ds - \int_0^{t_1} \Psi'(s)(\Psi(t_1) - \Psi(s))^{a-1} (z^*(s) + z^*(h(s))) \, ds \right) \geq \frac{1}{\Gamma(\rho)} \sum_{0 < t_k < t_2 - t_1} \mathcal{L}_{J_k} z^*(t_k^-) + \frac{K M (\Psi(b) - \Psi(0))^{1-\rho}}{(1 - \mathcal{L}_f) \Gamma(\alpha)} \left( \int_0^{t_2} \Psi'(s)(\Psi(t_2) - \Psi(s))^{a-1} \, ds - \int_0^{t_1} \Psi'(s)(\Psi(t_1) - \Psi(s))^{a-1} \, ds \right) = \frac{1}{\Gamma(\rho)} \sum_{0 < t_k < t_2 - t_1} \mathcal{L}_{J_k} z^*(t_k^-) + \frac{K M (\Psi(b) - \Psi(0))^{1-\rho}}{(1 - \mathcal{L}_f) \Gamma(\alpha)} ((\Psi(t_2) - \Psi(0))^{a} - (\Psi(t_1) - \Psi(0))^{a}) > 0.\]

This proves that \(z^*\) is an increasing operator. Since \(h(t) \leq t\), \(z^*(h(t)) \leq z^*(t), t \in [0, b]\). Therefore, (32) reduces to

\[z^*(t) \leq \epsilon \left( \frac{m}{\Gamma(\rho)} + (\Psi(b) - \Psi(0))^{1-\rho} E_a((\Psi(b) - \Psi(0))^{a}) \right) + \frac{1}{\Gamma(\rho)} \sum_{0 < t_k < t} \mathcal{L}_{J_k} z^*(t_k^-) + \frac{2K (\Psi(b) - \Psi(0))^{1-\rho}}{(1 - \mathcal{L}_f) \Gamma(\alpha)} \int_0^t A_a^\rho(t, s) z^*(s) \, ds, t \in [0, b].\]

By applying Lemma 2 to the above inequality with

\[U^*(t) = z^*(t), \quad V(t) = \epsilon \left( \frac{m}{\Gamma(\rho)} + (\Psi(b) - \Psi(0))^{1-\rho} E_a((\Psi(b) - \Psi(0))^{a}) \right), \quad g(t) = \frac{2K (\Psi(b) - \Psi(0))^{1-\rho}}{(1 - \mathcal{L}_f) \Gamma(\alpha)}, \quad \beta_k = \frac{\mathcal{L}_{J_k}}{\Gamma(\rho)}, \]

we get
Therefore, by Lemma 1 we obtain

\[
\begin{align*}
  z^*(t) & \leq c \left( \frac{m}{\Gamma(\rho)} + (\Psi(b) - \Psi(0))^{1-\rho} E_\alpha((\Psi(b) - \Psi(0))^\alpha) \right) \\
  & \times \left[ \prod_{i=1}^k \left( 1 + \frac{L_i}{\Gamma(\rho)} E_\alpha \left( \frac{2 K (\Psi(b) - \Psi(0))^{1-\rho}}{(1 - L_f) \Gamma(\alpha)} (\Psi(t_i) - \Psi(0))^\alpha \right) \right) \right] \\
  & \times E_\alpha \left( \frac{2 K (\Psi(b) - \Psi(0))^{1-\rho}}{(1 - L_f) \Gamma(\alpha)} (\Psi(t) - \Psi(0))^\alpha \right), \quad t \in J.
\end{align*}
\]

Therefore,

\[
  z^*(t) \leq c E_\alpha \left( \zeta_{\rho, \psi} (\Psi(t) - \Psi(0))^\alpha \right), \quad t \in J,
\]

where

\[
  L_{max}^J = \max\{L_1^J, L_2^J, \ldots, L_p^J\}, \quad \zeta_{\rho, \psi} = \frac{2 (\Psi(b) - \Psi(0))^{1-\rho}}{(1 - L_f)}
\]

\[
  C_{p, E_\alpha} = \left( \frac{m}{\Gamma(\rho)} + (\Psi(b) - \Psi(0))^{1-\rho} E_\alpha((\Psi(b) - \Psi(0))^\alpha) \right) \\
  \times \left\{ 1 + \frac{L_{max}^J}{\Gamma(\rho)} E_\alpha \left( \frac{2 K (\Psi(b) - \Psi(0))^{1-\rho+\alpha}}{(1 - L_f)} \right) \right\}^p.
\]

Note that for \( z(t) = (\Psi(t) - \Psi(0))^{1-\rho} |v(t) - u(t)| \) from (28) we have \( z \leq Q(z) \), where \( Q \) is an increasing Picard operator. Therefore by Lemma 1 we obtain \( z \leq z^* \). This fact in combination with (33) gives

\[
  (\Psi(t) - \Psi(0))^{1-\rho} |v(t) - u(t)| \leq c E_{p, E_\alpha} \left( \zeta_{\rho, \psi} (\Psi(t) - \Psi(0))^\alpha \right), \quad t \in J.
\]

Thus, we have proved that the problem (1)-(2) is UHML stable.

**Remark 2.** Since \( E_\alpha(\cdot) \) is increasing, the inequality (34) can be written as

\[
  (\Psi(t) - \Psi(0))^{1-\rho} |v(t) - u(t)| \leq c E_{p, E_\alpha} \left( \zeta_{\rho, \psi} (\Psi(b) - \Psi(0))^\alpha \right), \quad \text{for all} \quad t \in J.
\]

Further, \( |u(t) - v(t)| = 0, t \in [-r, 0] \). Therefore,

\[
  \|v - u\|_{PC_{[-r, 0]}(J, \mathbb{R})} \leq c C_f,
\]

where \( C_f = E_{p, E_\alpha} \left( \zeta_{\rho, \psi} (\Psi(b) - \Psi(0))^\alpha \right) \). Further, for \( t \in [-r, 0], |v(t) - u(t)| = 0 \), this implies that

\[
  \|v - u\|_{C_{[-r, 0]}} = 0.
\]

Thus, from (35) and (36) and by definition of \( \|\cdot\|_{X_{\rho, \psi}} \) we have, \( \|v - u\|_{X_{\rho, \psi}} \leq c C_f \). This proves that the problem (1)-(2) is Ulam-Hyers stable. Further by defining \( \theta(c) = c C_f \), we get generalized Ulam-Hyers stability.

**Remark 3.** The important aspect of the main results which we have proved in Theorem 3 is that it not only guarantee the existence and uniqueness of solution for the class of implicit impulsive \( \Psi \)-HFDE but it also gives the UHML stability. From Remark 2, it follows that the Ulam-Hyers and the generalized Ulam-Hyers stability are the particular cases of UHML stability.
5 | EXAMPLES

Example 1. Consider the following implicit impulsive \( \Psi \)-HFDE with delay

\[
\left\{
\begin{array}{l}
H_{\Delta}^{\beta, \rho} \Psi u(t) = \frac{(\Psi(t) - \Psi(0))^{1-\rho}}{50 e^{\rho (t-\Psi(0))}} + \frac{|w|}{15 (1 + |w|)} , \\
\Delta^{1-\rho} \Psi u(t) = \frac{(\Psi(t) - \Psi(0))^{1-\rho} |u(t)|}{7 (1 + |u(t)|)} , \\
u(0) = u_0 \in \mathbb{R} , \\
u(t) = 0 , \quad t \in [-1, 0] .
\end{array}
\right.
\]

(37)

Because of high non-linearity in the implicit impulsive \( \Psi \)-HFDE considered above, one cannot state whether the problem (37) has a solution or not. We verify that the nonlinear functions involved in problem (37) satisfies the assumptions of Theorem 3 to guarantee the existence of solution and UHML stability.

Define \( f : (0, 1] \times \mathbb{R}^3 \rightarrow \mathbb{R} \) by

\[
f(t, u, v, w) = \frac{(\Psi(t) - \Psi(0))^{1-\rho}}{50 e^{\rho (t-\Psi(0))}} (1 + |u| + |v|) + \frac{|w|}{15 (1 + |w|)}
\]

and \( J_1 : \mathbb{R} \rightarrow \mathbb{R} \) by

\[
J_1(u) = \frac{(\Psi(t) - \Psi(0))^{1-\rho} |u|}{7 (1 + |u|)} .
\]

Then \( f \) satisfies \( (H_1) \) indeed for \( u_i, v_i, w_i \in \mathbb{R} \), for \( i = 1, 2 \) and for \( t \in (0, 1] \), we have

\[
| f(t, u_1, v_1, w_1) - f(t, u_2, v_2, w_2) | \\
\leq \frac{(\Psi(t) - \Psi(0))^{1-\rho}}{50 e^{\rho (t-\Psi(0))}} \left| \frac{1}{1 + |u_1| + |v_1|} - \frac{1}{1 + |u_2| + |v_2|} \right| + \frac{1}{15} \left| \frac{|w_1|}{1 + |w_1|} - \frac{|w_2|}{1 + |w_2|} \right| .
\]

This implies that \( f \) satisfies the hypothesis \( (H_1) \) with \( L_f = \frac{1}{15} \) and \( K = \frac{1}{50} \). Further, for any \( u, v \in \mathbb{R} \),

\[
|J_1(u) - J_1(v)| = \frac{(\Psi(t) - \Psi(0))^{1-\rho}}{7} \left| \left| \frac{|u|}{1 + |u|} - \frac{|v|}{1 + |v|} \right| \right| \leq \frac{(\Psi(t) - \Psi(0))^{1-\rho}}{7} |u - v| .
\]

This shows that \( J_1 \) satisfy the assumptions \( (H_2) \) with \( L_{J_1} = \frac{1}{7} \). Thus, in the view of Theorem 3, the problem (37) has unique solution if the condition,

\[
L = \left( \frac{1}{7 \Gamma(\rho)} + \frac{3 (\Psi(1) - \Psi(0))^{1-\rho+\alpha}}{70 \Gamma(\alpha + 1)} \right) < 1
\]

(38)

is satisfied. In addition for every solution \( v \in \mathcal{V}_C, \rho, \psi \) of the inequality

\[
\left\{
\begin{array}{l}
H_{\Delta}^{\beta, \rho} \psi v(t) = \frac{(\Psi(t) - \Psi(0))^{1-\rho}}{50 e^{\rho (t-\Psi(0))}} - \frac{|w|}{15 (1 + |w|)} , \\
\Delta^{1-\rho} \psi v(t) = \frac{(\Psi(t) - \Psi(0))^{1-\rho} |v(t)|}{7 (1 + |v(t)|)} , \\
u(0) = u_0 \in \mathbb{R} , \\
u(t) = 0 , \quad t \in (0, 1] .
\end{array}
\right.
\]

(39)

there exists unique solution of problem (37) such that

\[
(\Psi(t) - \Psi(0))^{1-\rho} |v(t) - u(t)| \leq \epsilon C_{\psi, E} E_{\psi} (\zeta_{f, \psi} (\Psi(t) - \Psi(0))^{\alpha}) , \quad t \in (0, 1] .
\]
where
\[
C_{p,E_a} = \left( \frac{1}{\Gamma(\rho)} + (\Psi(1) - \Psi(0))^{1-\rho} E_a((\Psi(1) - \Psi(0))^\rho) \right) \left\{ 1 + \frac{1}{7 \Gamma(\rho)} E_a \left( \frac{3 (\Psi(1) - \Psi(0))^{1-\rho+a}}{70} \right) \right\},
\]
\[\zeta_f, \Psi = \frac{15 (\Psi(1) - \Psi(0))}{7}.\]

In particular, take \( \alpha = \frac{1}{2}, \beta = 1 \), then \( \rho = 1 \). Let \( \Psi(t) = t \) and \( h(t) = t - \frac{1}{2}, t \in [0, 1] \). Then problem (37) reduces to the following implicit impulsive Caputo FDE

\[
\left\{ \begin{aligned}
\frac{c\mathbb{D}_0^\frac{1}{2}}{50} u(t) &= \frac{1}{\sqrt{1 + |u(t)| + |u(t - \frac{1}{2})|}} + \frac{\left| \frac{c\mathbb{D}_0^\frac{1}{2}}{15} u(t) \right|}{1 + \left| \frac{c\mathbb{D}_0^\frac{1}{2}}{15} u(t) \right|}, \\
\Delta u(t) &= \frac{|u(t)|}{7(1 + |u(t)|)}, \\
u(t) &= 0, \quad t \in [-1, 0].
\end{aligned} \right. \tag{40}
\]

In this case, from (38), \( \mathcal{L} \approx 0.1912 < 1 \), and hence (40) has unique solution and the corresponding problem is UHML-stable.

Further, for \( \Psi(t) = t, \alpha = \frac{1}{2}, \beta = 0 \), we have \( \rho = \frac{1}{2} \) and the problem (37) reduces to implicit impulsive Riemann-Liouville FDE with delay. In this case, \( \mathcal{L} \approx 0.1013 < 1 \). Hence, the implicit impulsive Riemann-Liouville FDE has unique solution and the corresponding problem is UHML-stable.

6 CONCLUDING REMARKS

With the help of Lemma 3 obtained in, we have given the representation formula for the solution of impulsive implicit H-HFDE (1)-(4) and successfully determined the existence and uniqueness of solutions and UHML stability via fixed point theorem. The Picard operators and the generalized Gronwall inequality are employed to get the prime outcomes. Since H-Hilfer fractional derivative operator incorporates various well-known fractional derivative operators including Caputo and Riemann-Liouville derivative. The results gained in this paper additionally incorporate the investigation of impulsive implicit FDEs with the fractional derivatives noted in Sousa and Oliveira.

Taking note of the significance of non-instantaneous impulsive differential equations provided in previous studies, our next objective is to analyze the non-instantaneous impulsive implicit H-Hilfer FDEs with delay and the boundary conditions. An interesting fundamental work on fractional order differential switched systems with coupled nonlocal initial and impulsive conditions in Wang et al is under study and will be the point of our next investigation with H-Hilfer fractional derivative.

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CONFLICTS OF INTEREST

This work does not have any conflicts of interest.

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