ASPECTS OF THE LEVI FORM

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Abstract. We discuss various analytical and geometrical aspects of the Levi form, which is associated with a CR manifold having any CR dimension and any CR codimension.

1. Introduction

The Levi form is a rather important geometric notion, which appears in a fundamental way in several complex variables, complex differential geometry, algebraic geometry, in certain aspects of partial differential equations, and in particular in the theory of Cauchy-Riemann structures on real manifolds (known for short as CR manifolds). Its role is to measure certain second order effects, which are of natural interest in those subjects. However it appears in various incarnations, and many different authors have used it in different ways, each employing their own peculiar notation, way of writing it, and their own understanding of its geometric significance.

This has often led to confusion, in which even experts are unable to easily decipher what some other researcher has written. This article is an attempt to rectify the situation, and especially to establish once and for all a good and consistent notation, which distinguishes among the various incarnations of the Levi form. We hope that in the future mathematicians will find it helpful and convenient to adopt our conventions.

For the convenience of the reader, and to make everything understandable to people not already familiar with the Levi form, we have begun with a discussion of almost complex manifolds, passing to complex manifolds, then to abstractly defined CR manifolds (or almost CR manifolds), and finally to the situation of locally CR embeddable CR manifolds. And along the way, we try to keep track of the various aspects of the Levi form which naturally appear and explain the connections among them.

We have also stressed the various geometric meanings, of which there are several. In the CR embedded case, we explain the connection of the Levi form with the second fundamental form with respect to the induced metric from the ambient space. Finally, for the important case of homogeneous CR
manifolds, we explain how the Levi form can be computed from the point of view of Lie algebra, and illustrate it with a couple of examples.

2. Almost complex manifolds and the Nijenhuis tensor

Let $M$ be a smooth (real) manifold of dimension $2n$. An almost complex structure on $M$ is a smooth assignment of a complex structure on each fiber of $TM$, i.e., a fiber preserving smooth map $J : TM \to TM$ which is linear on the fibers and satisfies $J^2 = -\text{Id}$. This map $J$ uniquely extends to a smooth vector bundle automorphism of the complexified tangent bundle $\mathbb{C}TM$ ($= \mathbb{C} \otimes TM$), which we denote by the same symbol $J$. The condition $J^2 = -\text{Id}$ implies that $J$ has eigenvalues $i$ and $-i$, and we get a decomposition

$$\mathbb{C}TM = T^{1,0}M \oplus T^{0,1}M$$

into the corresponding eigenspaces

$$T^{1,0}M = \{ X - iJX \mid X \in TM \}$$

and

$$T^{0,1}M = \{ X + iJX \mid X \in TM \}.$$

We also note that

$$T^{0,1}M = \overline{T^{1,0}M}.$$

We call $T^{1,0}M$ the $T^{1,0}$-bundle, and $T^{0,1}M$ the $T^{0,1}$-bundle of the almost complex structure $J$.

To simplify notation, in the following we will use the same symbol for both vector fields and tangent vectors, whenever we believe this will not cause confusion.

Equivalently, an almost complex structure can be defined by the assigning of its $T^{1,0}$-bundle. Indeed, for each smooth complex subbundle $T^{1,0}M$ of $\mathbb{C}TM$ satisfying (2.1) with $T^{0,1}M := \overline{T^{1,0}M}$, there is a unique almost complex structure with this $T^{1,0}$-bundle.

Since by (2.1) the spaces $T^{1,0}M$ do not contain nontrivial purely imaginary vectors, taking the real part is an $\mathbb{R}$-linear isomorphism $\mathcal{R}_+$ of $T^{1,0}M$ onto $T_+M$ and therefore we may define $J$ on $T_+M$ by requiring that $X - iJX \in T^{1,0}M$. The complex structure $J$ on $TM$ extends to an anti-involution $J^c$ on $\mathbb{C}TM$, which is multiplication by $i$ on $T^{1,0}M$ and $(-i)$ on $T^{0,1}M$. With the notation

$$\mathcal{R} : T^{1,0}M \to TM, \quad \mathcal{R}(X) = \frac{1}{2}(X + \overline{X}), \quad \forall X \in T^{1,0}M,$$

we have

$$JX = \mathcal{R}(i\mathcal{R}^{-1}(X)), \quad \forall X \in TM.$$
In the same way, the $T^{0,1}$-bundle can also be used to define the almost complex structure, by giving a smooth complex subbundle $T^{0,1}M$ of $\mathbb{C}TM$ satisfying (2.1) with $T^{1,0}M := T^{0,1}M$.

We refer the reader to [B] for more details.

A complex manifold $M$ admits an almost complex structure, which can be described by using its complex coordinate charts to locally define its $T^{1,0}$-bundle to be the span of

$$\frac{\partial}{\partial z_j} := \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \quad j = 1, \ldots, n,$$

for any set of local holomorphic coordinates $z_1 = x_1 + iy_1, \ldots, z_n = x_n + iy_n$, with underlying real coordinates $x_j, y_j (j = 1, \ldots, n)$. By the holomorphic chain rule, this is a good definition and we call this object the almost complex structure defined by the complex structure of $M$.

On the other hand, not every almost complex structure is defined by a complex structure. If $T^{1,0}M$ is the $T^{1,0}$-bundle of an almost complex structure defined by a complex structure, then the formal integrability condition (2.3)

$$[T^{0,1}M, T^{0,1}M] \subset T^{0,1}M$$

is satisfied. This means that if $X$ and $Y$ are smooth (real) vector fields, then the commutator of $X + iJX$ and $Y + iJY$ is of the same sort, i.e.

$$[X + iJX, Y + iJY] = Z + iJZ$$

for some smooth (real) vector field $Z$. One easily checks that this formal integrability condition is equivalent to the vanishing of the Nijenhuis tensor, defined in [NW] by

$$N(X, Y) = [X, Y] - [JX, JY] + J[JX, JY] + J[JX, Y]$$

for $X, Y \in \Gamma(M, TM)$. The Newlander-Nirenberg theorem [NN] asserts that formal integrability of an almost complex structure implies in fact integrability, i.e. there exists a complex structure which defines the almost complex structure. An improved version of this theorem requiring a minimal amount of smoothness of the almost complex structure can be found in [HT].

3. Embedded and abstract (almost) CR manifolds

Let $M$ be a smooth manifold of real dimension $2n+k$. An almost CR structure $(HM, J)$ of type $(n, k)$ on $M$ consists of the data of a smooth sub-bundle $HM \subseteq TM$ of fiber dimension $2n$ and a fiber preserving smooth vector bundle isomorphism $J : HM \to HM$ satisfying $J^2 = -\text{Id}$. An almost CR manifold of type $(n, k)$ is a smooth manifold endowed with an almost CR structure of type $(n, k)$. The number $n$ is called the CR dimension and $k$ the CR codimension of $M$. In the following we will avoid for simplicity, whenever possible, to explicitly mention regularity assumptions on the manifolds under consideration.
An almost complex structure on a real manifold $M$ of dimension $2n$ is therefore the same as an almost CR structure of CR codimension zero.

As in the almost complex case, an almost CR structure of type $(n,k)$ on $M$ can be equivalently defined by the datum of a complex subbundle $T^{1,0}M$ of complex fiber dimension $n$, of the complexified tangent bundle $\mathbb{C}TM$, satisfying

$$T^{1,0}M \cap T^{0,1}M = \{0\}, \quad \text{where } T^{0,1}M = \overline{T^{1,0}M},$$

(this is equivalent to (2.1) if $k = 0$). Since $T^{1,0}M$ does not contain purely imaginary vectors, the real parts of its vectors form a real subbundle $H^\perp M$ of fiber dimension $2n$ of $TM$ and $J$ is defined on $HM$ by requiring that $X-iJX \in T^{1,0}M$ for all $X \in HM$. We have $T^{1,0}M = \{X - iJX \mid X \in HM\}$, $T^{0,1}M = \{X + iJX \mid X \in HM\}$, and $T^{1,0}M \oplus T^{0,1}M = \mathbb{C}HM$, where $\mathbb{C}HM$ is the complexification of $HM$, i.e. the complex linear span of $HM$ in $\mathbb{C}TM$.

We call $T^{1,0}M$ the $T^{1,0}$-bundle, and $T^{0,1}M$ the $T^{0,1}$-bundle of the almost CR structure $J$.

An almost CR structure on $M$ is called a CR structure if the formal integrability condition

$$[T^{0,1}M, T^{0,1}M] \subseteq T^{0,1}M,$$

is satisfied. Equation (3.2) means that for $X, Y \in \Gamma(M, T^{1,0}M)$ their commutator $[X, Y]$ still belongs to $\Gamma(M, T^{1,0}M)$. Since

$$[X + iJX, Y + iJY] = [X, Y] - [JX, JY] + i([X, JY] + [JX, Y]),$$

for $X, Y \in \Gamma(M, HM)$, formal integrability can be reformulated in terms of real vector fields by the two conditions, the second one involving a Nijenhuis tensor on $HM$:

$$[X, Y] - [JX, JY] \in \Gamma(M, HM) \quad \text{(3.3)}$$

$$N(X, Y):= [X, Y] - [JX, JY] + J([X, JY] + [JX, Y]) = 0 \quad \forall X, Y \in \Gamma(M, HM). \quad \text{(3.4)}$$

Prime examples of CR structures of CR dimension $n$ are provided by the real submanifolds $M$ of a complex manifold $\mathfrak{X}$ for which

$$\dim_\mathbb{C}(\mathbb{C}T_xM \cap T^{1,0}_x\mathfrak{X}) = n \quad \text{for all } x \in M. \quad \text{(3.5)}$$

Note that $\dim_\mathbb{R} M \geq 2n$ and $\dim_\mathbb{C} \mathfrak{X} \geq \dim_\mathbb{R} M - n$.

It is easy to check that

$$T^{1,0}M := \{\mathbb{C}T_xM \cap T^{1,0}_x\mathfrak{X}\}_{x \in M}$$

is a smooth complex subbundle (of complex fiber dimension $n$) of $\mathbb{C}TM$ satisfying (3.1), (3.2) and therefore induces a CR structure $(HM, J_M)$ of type $(n,k)$ on $M$, where, with $\Re Z := \frac{Z + \overline{Z}}{2}$ for $Z \in T^{1,0}_x\mathfrak{X}$,

$$HM := \Re(T^{1,0}M) \quad \text{and} \quad J_M X := \Re(i\mathbb{R}^{-1}X), \quad \forall X \in H_xM, \ x \in M.$$
We say that this CR structure on \( M \) is \textit{induced} from the complex structure of \( \mathcal{X} \). If \( J : T\mathcal{X} \to T\mathcal{X} \) is the almost complex structure of \( \mathcal{X} \) induced by its complex structure, i.e.
\[
JX = \Re(\partial\partial^{-1}X) \quad \text{for all} \quad X \in T\mathcal{X},
\]
then \( J_M \) is the restriction of \( J \) to \( HM \). In this situation, we say that \( M \) is \textit{CR embedded} in \( \mathcal{X} \).

The embedding codimension \( \ell \) of \( M \) in \( \mathcal{X} \) is always greater or equal to the CR codimension \( k \) of \( M \) and \( \ell - k \) is even (the complex dimension of \( \mathcal{X} \) is \( m = n + \frac{1}{2}(k + \ell) \)).

For each point \( x_0 \) of \( M \) we can find an open neighbourhood \( U \) of \( x_0 \) in \( \mathcal{X} \) such that \( M \cap U \) is given by
\[
(3.6) \quad M \cap U = \{ x \in U \mid \rho_1(x) = 0, \ldots, \rho_{\ell}(x) = 0 \}
\]
with smooth real valued functions \( \rho_1, \ldots, \rho_\ell \) satisfying
\[
d\rho_1 \wedge \ldots \wedge d\rho_\ell \neq 0 \quad \text{on} \quad M \cap V.
\]

Then, for the \( T^{1,0} \) and \( T^{0,1} \)-bundles, we have, for \( x \in M \cap U \),
\[
(3.7) \quad T_x^{1,0} M = \{ Z \in T_x^{1,0} \mathcal{X} \mid \partial\rho_1(Z) = \ldots = \partial\rho_k(Z) = 0 \},
\]
\[
(3.8) \quad T_x^{0,1} M = \{ Z \in T_x^{0,1} \mathcal{X} \mid \bar{\partial}\rho_1(Z) = \ldots = \bar{\partial}\rho_{\ell}(Z) = 0 \}.
\]
In particular, \( M \) is generically embedded (near \( x_0 \)) if and only if
\[
\partial\rho_1 \wedge \ldots \wedge \partial\rho_\ell \neq 0
\]
on a neighbourhood of \( x_0 \) in \( V \).

The notion of \textit{embedded} CR manifolds leads us to briefly discuss CR maps and CR embeddings in a more general setting:

If \((M, HM, J)\) is a CR manifold and \( \mathcal{X} \) is a complex manifold, then \( M \) is \textit{CR embeddable} into \( \mathcal{X} \) if one can find a smooth embedding \( \Psi : M \to \mathcal{X} \) such that \( \Psi(M) \) is CR embedded in \( \mathcal{X} \) and its induced CR structure agrees with the pushforward by \( \Psi \) of the abstract CR structure on \( M \). This means that \( \Psi \) and \( \Psi^{-1} \) are CR maps, according to the definition below:

Let \( M_1 \) and \( M_2 \) be CR manifolds. A smooth map \( \Psi : M_1 \to M_2 \) is called CR if \( \Psi_*(T^{1,0}M_1) \subseteq T^{1,0}M_2 \). An equivalent definition, not involving the complexification of the differential, requires that
\[
(3.10) \quad \Psi_*(HM_1) \subseteq HM_2 \quad \text{and} \quad \Psi \circ J_1 = J_2 \circ \Psi_*
\]
(where we set \( J_h \) for the CR structure on \( M_h, h=1,2 \)). If, moreover, \( \Psi \) is a diffeomorphism and \( \Psi_*(T^{1,0}M_1) = T^{1,0}M_2 \), then \( \Psi \) is called a CR isomorphism.

A CR manifold is called \textit{locally embeddable} if it admits local CR embeddings into some complex Euclidean space. One can prove (see, e.g., [HTa1, p. 118] or [B, p. 187]) that each locally embeddable \( C^\infty \)-smooth CR manifold admits local CR embeddings whose images are generically embedded.
Real-analytic CR manifolds are always embeddable (see e.g. [AH72, AF79]), but, in contrast to the complex situation (when \( k = 0 \)), the formal integrability condition (3.2) is, for arbitrary \( C^\infty \)-smooth CR manifolds, not sufficient to provide local embedding into complex manifolds. The local CR embedding problem is in fact very difficult and will not be discussed in this survey.

4. The Levi form of (almost) CR manifolds

Let \( M \) be a smooth real manifold with an almost CR structure of type \((n, k)\). The Levi form of \( M \) at \( x \) is the Hermitian symmetric map
\[
\mathcal{L}_{x}^{1,0} : T_{x}^{1,0}M \times T_{x}^{1,0}M \rightarrow \mathbb{C}T_{x}M/\mathcal{C}H_{x}M
\]
defined by \( \mathcal{L}_{x}^{1,0}(Z_{x}, W_{x}) = \frac{1}{2i} \pi_{x}([\bar{Z}, W]_{x}) \). Here \( \pi_{x} \) is the canonical projection \( \pi_{x} : \mathbb{C}T_{x}M \rightarrow \mathcal{C}T_{x}M/\mathcal{C}H_{x}M \), and \( Z, W \) are smooth sections of \( T_{x}^{1,0}M \) extending \( Z_{x}, W_{x} \). The class of \([\bar{Z}, W]_{x}\) in \( \mathcal{C}T_{x}M/\mathcal{C}H_{x}M \) is in fact independent of the choice of the extensions. Indeed, let \( L_{1}, \ldots, L_{n} \) be a smooth basis of \( T_{x}^{1,0}M \) near \( x \). If \( Z', W' \) are smooth sections of \( T_{x}^{1,0}M \) with \( Z'_{x} = Z_{x} \) and \( W'_{x} = W_{x} \), then \( Z' = Z + \sum_{j} \alpha_{j} L_{j}, \ W' = W + \sum_{j} \beta_{j} L_{j} \), for smooth complex valued functions \( \alpha_{j}, \beta_{j} \) vanishing at 0. Then the value at \( x \) of
\[
[\bar{Z'}, W'] - [\bar{Z}, W] = \sum_{j,k} \bar{\alpha}_{j} \beta_{k} [\bar{L}_{j}, L_{k}] + \sum_{j} \bar{Z}(\beta_{j}) L_{j} - \sum_{j} W(\bar{\alpha}_{j}) \bar{L}_{j},
\]
is an element of \( T_{x}^{1,0}M + T_{x}^{0,1}M = \mathcal{C}H_{x}M \). The (fiberwise) linear projection
\[
\mathfrak{R} : \mathbb{C}TM \ni X \rightarrow \mathfrak{R}X = \frac{1}{2i}(X + \bar{X}) \in TM
\]
of \( \mathbb{C}TM \) onto \( TM \) along \( iTM \) induces by passing to the quotients a left inverse \( \mathfrak{R}^{-1} : \mathbb{C}TM/\mathcal{C}H \rightarrow TM/\mathcal{C}HM \), of the inclusion \( TM/\mathcal{C}HM \hookrightarrow \mathbb{C}TM/\mathcal{C}HM \) which factors \( TM \hookrightarrow \mathbb{C}TM \).

Let \( x \in M \). Since \( \mathfrak{R} \) is a real linear isomorphism from \( T_{x}^{1,0}M \) onto \( H_{x}M \), we have a well-defined real bilinear map \( \mathcal{L}_{x} : H_{x}M \times H_{x}M \rightarrow T_{x}M/H_{x}M \) making the following diagram commute:
\[
\begin{array}{ccc}
T_{x}^{1,0}M \times T_{x}^{1,0}M & \xrightarrow{\mathcal{L}_{x}^{1,0}} & \mathbb{C}T_{x}M/\mathcal{C}H_{x}M \\
\downarrow \mathfrak{R} & & \downarrow \mathfrak{R} \\
H_{x}M \times H_{x}M & \xrightarrow{\mathcal{L}_{x}} & T_{x}M/H_{x}M.
\end{array}
\]

For \( X, Y \in \Gamma(M, T_{x}^{1,0}M) \), we have, as \( (\mathfrak{R}|_{T_{x}^{1,0}M})^{-1}X_{x} = X_{x} + iJX_{x} \) and \( (\mathfrak{R}|_{T_{x}^{1,0}M})^{-1}Y_{x} = Y_{x} - iJY_{x} \),
\[
\mathcal{L}_{x}(X_{x}, Y_{x}) := \mathfrak{R} \left( \frac{1}{2i} \pi_{x}([X + iJX, Y - iJY]) \right)
= \mathfrak{R} \left( \frac{1}{2i} \pi_{x}([X, Y] + [JX, JY] + i([X, JY] - [X, JY])) \right)
= \frac{1}{2} \pi_{x}([JX, Y] - [X, JY]),
\]
where, to obtain the last equality, we used the facts that \( \pi_x \) is complex linear and that \( \hat{\mathcal{R}} \) projects \( C T_x M/CH_x M \) onto \( T_x M/H_x M \) along \( i(T M/H M) \). In particular, \( \mathcal{L}_x \) is symmetric and \( \mathcal{L}(JX_x, JY_x) = \mathcal{L}(X_x, Y_x) \).

If the almost CR structure satisfies (3.3), then
\[
[JX, Y] \equiv ([J^2 X, JY] = [X, JY]) \mod H M
\]
and hence
\[
(4.2) \quad \mathcal{L}_x(X_x, Y_x) = \pi_x([JX, Y]) = \pi_x([JY, X]).
\]

For the corresponding quadratic forms
\[
L_{1,0}^x(Z_x) := \mathcal{L}_{1,0}^x(Z_x, Z_x) \quad \text{and} \quad L_x(X_x) := \mathcal{L}_x(X_x, X_x),
\]
(4.1) yields the following commutative diagram:
\[
\begin{array}{ccc}
T_{1,0}^x M & \xrightarrow{\mathcal{L}_{1,0}^x} & T_x M/H_x M \\
\downarrow \mathcal{R} & & \downarrow \mathcal{L}_x \\
H_x M & \xrightarrow{L_x} & H_x M
\end{array}
\]

Almost CR structures satisfying (3.3) were already considered in [1] and are called partially integrable (see also [CS]).

The direction of \( \mathcal{L}_x(X) \) does not change when \( X \) is replaced by another tangent vector in the real span of \( \{X, JX\} \); indeed, if \( Y = aX + bJX \), then
\[
(4.4) \quad \mathcal{L}_x(Y) = (a^2 + b^2) \mathcal{L}_x(X).
\]

In many situations, it is useful to also consider scalar Levi forms (as considered e.g. in [AFN] and [HN]). For this, we need to introduce the characteristic conormal bundle, which is the annihilator
\[
H^0 M = \{ \xi \in T^* M \mid \xi(X) = 0 \ \forall X \in \Gamma(M, HM) \}
\]
of \( HM \) in \( T^* M \).

We define a family of scalar Levi forms parametrized by \( \xi \) in the characteristic conormal bundle as follows: Given \( \xi \in H^0 M \) and \( X_x, Y_x \in H_x M \), we choose smooth sections \( \tilde{\xi} \) of \( H^0 M \) and \( X, Y \) of \( HM \) extending \( \xi, X_x, Y_x \). By the invariant formula for the exterior derivative (see, e.g., [S p. 213]),
\[
d\tilde{\xi}(X, Y) = X(\tilde{\xi}(Y)) - Y(\tilde{\xi}(X)) - \tilde{\xi}([X, Y]).
\]

Since \( \tilde{\xi}(X) = \tilde{\xi}(Y) = 0 \), this implies
\[
d\tilde{\xi}(X_x, Y_x) = -\xi([X, Y]).
\]
Hence both sides depend only on \( \xi, X_x, Y_x \) and we define
\[
(4.5) \quad \mathcal{L}_x(\xi, X_x) = \xi([JX, X]) = d\tilde{\xi}(X, JX),
\]
on \( H_x M \). Note that \( \mathcal{L}_x(\xi, \cdot) \) is Hermitian for the complex structure \( J \) on \( H_x M \): i.e. \( \mathcal{L}_x(JX_x) = \mathcal{L}_x(\xi, X_x) \).
The image $\mathcal{R}_i$ of $\mathcal{L}_x$ is a real cone in $T_x M/H_x M$. Its dual cone
\[
\mathcal{R}_i^0 = \{ \xi \in H_x^0 M \mid \mathcal{L}_i(\xi, X) \geq 0, \forall X \in H_x M \}
\]
is a closed convex cone in $H_x^0 M$.

It is also useful to consider the characteristic conormal sphere bundle $S^0 M = (H_x^0 M \setminus \{0\})/\mathbb{R}^+$ whose typical fiber is the $(k-1)$-dimensional sphere. If $\sigma : H_x^0 M \setminus \{0\} \to S^0 M$ is the canonical projection, then $\sigma(\mathcal{R}_i^0 \setminus \{0\})$ is the total space of a bundle on $M$ which has semialgebraic fibers and consist of a finite number of closed connected components if $M$ itself is semialgebraic.

The scalar Levi forms $\mathcal{L}_i(\xi, X)$ enable us to microlocalize with respect to the Levi form $\mathcal{L}_i(X)$. A geometric version of this microlocalization is discussed in [HN1] §5.

When we think this would not cause confusion, we will drop the subscript “$x$” and simply write $\mathcal{L}$ for $\mathcal{L}_x$.

5. Levi form of an embedded CR manifold

Assume that $M$ is a $(2n+k)$-dimensional smooth real submanifold of an $m$-dimensional complex manifold $\tilde{x}$ (with $m \geq n+k$) and that (3.5) holds true, so that the complex structure of $\tilde{x}$ induces on $M$ a CR structure $(HM, J)$ of type $(n, k)$. Our first goal in this section is to express the Levi form in terms of defining functions for $M$. To simplify notation and better understand the invariant meaning of the construction, it is convenient to introduce the real operator

\[
d^* = i(\bar{\partial} - \partial), \quad \text{satisfying} \quad \partial = \frac{i}{2}(d + i d^*), \quad \bar{\partial} = \frac{i}{2}(d - i d^*), \quad 2i\partial \bar{\partial} = dd^*.
\]

By considering the local situation, we can assume that our CR manifold $M$, of type $(n, k)$, is given by

\[
(5.1) \quad M \cap U = \{ z \in U \mid \rho_1(z) = \ldots = \rho_\ell(z) = 0 \},
\]
with $\ell = 2m - 2n - k \geq k$, for a system $\rho_1, \ldots, \rho_\ell$ of real $C^\infty$ defining functions, defined on an open neighborhood $U$ of $x_0 \in M$ in $\tilde{x}$ and satisfying $d\rho_1 \wedge \ldots \wedge d\rho_\ell \neq 0$ on $M$. By (3.7), the vectors $X$ in $HM$ are characterised by $d^* \rho_j(X) = 0$ for $1 \leq j \leq \ell$ and therefore the pullbacks $d^* \rho_j|_{TM}, \ldots, d^* \rho_\ell|_{TM}$ of the $d^*$’s of the defining functions on $M$ span the characteristic bundle $H_0^0 M$. However they are not linearly independent if $\ell > k$.

For a section $X$ of $HM$, with $X = \Re(Z) = \frac{i}{2}(Z + \bar{Z})$ for $Z \in T^{1,0} M$, we have by (4.5)

\[
(5.2) \quad \mathcal{L}(d^* \rho_j, X) = dd^* \rho_j(X, JX) = 2i\partial \bar{\partial} \rho_j(X, JX) = \partial \bar{\partial} \rho_j(Z, \bar{Z}).
\]

Then (5.2) yields

\[
(5.3) \quad \mathcal{L}(d^* \rho_j, X) = \sum_{\mu = 1}^{m} \frac{\partial^2 \rho_j(x)}{\partial z_\mu \partial \bar{z}_\nu} Z_\mu \bar{Z}_\nu, \quad 1 \leq j \leq \ell.
\]
A $\xi$ in $H^0_x M$ can be written as a linear combination $\xi = \sum_{j=1}^\ell \xi_j d' \rho_j(x)$ with real coefficients and correspondingly

$$L_x(\xi, X) = \sum_{j=1}^\ell \sum_{\mu, \nu=1}^m \xi_j \frac{\partial^2 \rho_j(x)}{\partial \overline{z}_\mu \partial z_\nu} Z_\mu \overline{Z}_\nu.$$  

Note that, when the embedding is generic ($\ell=k$) the $d' \rho_j$’s are a basis of $H^0_x M$ and the $\xi_j$ are uniquely determined by $\xi$.

Formula (5.4) is actually the classical definition of the scalar Levi forms which was used for a long time by various authors (e.g. [AFN], [ChSh], [HTa2], [Hen]).

The Levi form yields important information on the geometry of the embedding $M \hookrightarrow \mathfrak{x}$. In fact its signature is related to mutual positions of $M$ and holomorphic balls: By a $p$-dimensional holomorphic ball we mean the image of a smooth embedding $F : B^n = \{z \in \mathbb{C}^n | |z| \leq 1\} \rightarrow \mathbb{C}^n$. Let us first consider the case $k=1$ (a piece of a real hypersurface in $\mathfrak{x}$). In this case $H^0_x M$ is one-dimensional, so it is generated by one characteristic conormal direction $\xi$. Assume that the scalar Levi form $L_\xi(\xi, \cdot)$ is nondegenerate and has $p$ positive and $q$ negative eigenvalues (hence $p+q=n$). This means that we can find a $p$-dimensional holomorphic ball lying on one side of $M$, touching $M$ only with its center at $x$, tangent to the $p$-dimensional positive eigenspace of $L_\xi(\xi, \cdot)$ and a $q$-dimensional holomorphic ball lying on the other side of $M$, also touching $M$ only with its center at $x$, tangent to the $q$-dimensional negative eigenspace of $L_\xi(\xi, \cdot)$ (see e.g. [AH, pp. 798-800]).

Vice versa, if $L(\xi, \cdot)$ has for all nonzero $\xi$ in $H^0_x M$ at least $q$ negative eigenvalues, all holomorphic balls of dimension $d \geq q + \frac{1}{2}(\ell + k)$ centered at $x$ have an intersection with $M$ of positive dimension.

Now let $k \geq 1$ be arbitrary, and let us assume that, for some $\xi \in H^0_x M$, the scalar Levi form $L_\xi(\xi, \cdot)$ has $p$ positive eigenvalues. Then one can find a holomorphic ball, of complex dimension $p + \frac{1}{2}(\ell + k) - 1$, touching $M$ only with its center at $x$, where it is tangent to a complex $p$-dimensional complex linear subspace of $H_x M$ on which $L_\xi(\xi, \cdot)$ is positive definite. Indeed, assuming as we can that $\xi = d' \rho(x)$ for a smooth real valued function vanishing on $M$, for a large real $c$ the set $\{c + \sum_{j=1}^\ell \rho_j^2 = 0\}$ defines near $x$ a smooth CR hypersurface having a Levi form with $p + \frac{1}{2}(\ell + k) - 1$ positive eigenvalues.

Next we show that, after introducing a Hermitian metric on $\mathfrak{x}$, the Levi form can be rewritten as a map with values in the normal space of $M$, as was suggested in [Her].
A Riemannian metric $g$ on $\mathcal{X}$ is Hermitian if the complex structure $J$ is an isometry on $T\mathcal{X}$. In particular

$$g(X, JX) = 0 \text{ and } g(JX, JX) = g(X, X), \quad \forall X \in T\mathcal{X}.$$  

This is equivalent to the fact that $g$ is the real part of a complex valued Hermitian symmetric scalar product $h$. The tensors $g$ and $h$ are related by

$$h_x(X, Y) = g_x(X, Y) - i g_x(JX, Y), \quad \forall x \in \mathcal{X}, \forall X, Y \in T_x\mathcal{X}.$$  

Let $NM$ be the normal bundle of $M$ in $\mathcal{X}$, whose fibre at $x \in M$ is the space $(T_x M)^\perp = \{X_x \in T_x \mathcal{X} \mid g(X_x, Y_x) = 0, \forall Y_x \in T_x M\}$ and $\pi_N: T M \rightarrow NM$ the orthogonal projection. For $X \in T_x \mathcal{X}$, we set $X^{NM} = \pi_N(X)$.  

Since $HM = TM \cap JT M$, we have

$$HM = \{X \in TM \mid \pi_N(JX) = 0\}$$  

and therefore we obtain a commutative diagram

$$\begin{array}{ccc}
TM & \xrightarrow{\pi_N \circ J} & NM \\
\downarrow{\pi_N} & & \downarrow{\pi} \\
TM/HM & & 
\end{array}$$

where the vertical arrow is projection onto the quotient and $\pi$ is injective. Note that $\pi$ is a linear isomorphism when the embedding is generic. By composing $L$ with $\pi$ we obtain a Hermitian symmetric quadratic form

$$L^{NM}(X) = \pi \circ L(X) = (J[\{JX, X\}]^{NM}, \quad \forall X \in HM,$$  

on $HM$ taking value in the normal bundle.

If $M$ is given locally by a system of local defining functions (5.1), we can use (5.2) to obtain a description of $L^{NM}$. Indeed, the gradient $\nabla f$ of a real valued smooth function $f$ is the real vector field characterised by $g(\nabla f, X) = df(X)$ for all $X \in \Gamma(\mathcal{X}, T\mathcal{X})$. Thus

$$L(d^f \rho_j, X) = d^f \rho_j(J[\{JX, X\}]) = -d \rho_j(J[\{JX, X\}]) = -g(\nabla \rho_j, J[\{JX, X\}])$$  

and (5.3) expresses the scalar product of $L^{NM}(X)$ with the gradient of the defining function $\rho_j$. The matrix $(g(\nabla \rho_j, \nabla \rho_i))$ is symmetric and positive and, if $(A_{j\ell})$ is its inverse, we obtain

$$L^{NM}(X) = -\sum_{j,\ell} A_{j\ell} L(d^e \rho_j, X) \cdot \nabla \rho_i.$$  

The defining functions $\rho_1, \ldots, \rho_\ell$ may be chosen in such a way that, at a point $x \in M$, their gradients form an orthonormal basis $\partial/\partial z^\mu$ of $N_x M$. Fix holomorphic coordinates $z_1, \ldots, z_m$ on $\mathcal{X}$ centered at $x$. Then, with $X_x = \Re Z_x$ for a $Z = \sum_{\mu=1}^{m} Z_\mu (\partial/\partial z^\mu) \in T_x^{1,0} M$, (5.7) yields

$$L^{NM}(X_x) = -\sum_{j=1}^{\ell} \left( \sum_{\mu,\nu=1}^{m} \frac{\partial^2 \rho_j(x)}{\partial z^\mu \partial \bar{z}^\nu} Z_\mu Z_\nu \right) \nabla \rho_j(x).$$

\[\text{1} \text{This can be achieved by applying the Gram-Schmidt orthonormalising process to the } \ell\text{-tuple} (\nabla \rho_1(x), \ldots, \nabla \rho_\ell(x)).\]
There is a unique linear connection $\nabla$ for which both the metric tensor $g$ and the complex structure $J$ are parallel (see e.g. [GH]). Let us assume moreover that $\mathcal{X}$ is Kähler, so that the Hermitian and the Levi-Civita connections coincide. Denote by $\nabla$ the covariant derivative on $\mathcal{X}$. If $X, Y$ are vector fields on $\mathcal{X}$ which are tangent to $M$, then taking the normal component

$$B_x(X, Y) = (\nabla_X Y)^{NM}, \quad \text{for } x \in M,$$

of the covariant derivative of $Y$ with respect to $X$ defines an $NM$-valued symmetric tensor on $TM$, which is called the second fundamental form of the embedding of $M$ into $\mathcal{X}$. Likewise, the tangential projection $\nabla^M X$ is the covariant derivative of the Levi-Civita connection on $M$ of $g|_M$. The Levi form $\mathcal{L}^{NM}$ can be expressed by using the second fundamental form (see [EHS] or [HT12]):

$$\mathcal{L}^{NM}_x(X) = B_x(X, X) + B_x(JX, JX)$$

Indeed, if $X$ is a real vector field on $\mathcal{X}$ whose restriction to $M$ takes values in $HM$, we obtain, in view of (5.5) and (5.6),

$$B_x(X, X) + B_x(JX, JX) = (\nabla_X X)^{NM} + (\nabla_{JX} X)^{NM}
= (\nabla_X X + J\nabla_{JX} X)^{NM} = (J\nabla_{JX} X - J\nabla_X X)^{NM}
= (J[X, X])^{NM} = \mathcal{L}^{NM}(X).$$

Since the hermitian connection has no torsion on the two-dimensional planes $\langle X, JX \rangle$ (see e.g. [KN]), the second fundamental form is meaningful on complex tangent lines also in the non-Kähler situation, (5.10) also applies in the more general case.

6. On the geometric meaning of the Levi form

In the remainder of this section, we want to make some more comments on the geometrical interpretation of the Levi form. Let us start by discussing its invariance under CR diffeomorphisms:

Let $(M_1, T^{1,0}M_1)$ and $(M_2, T^{1,0}M_2)$ be CR manifolds and $\Psi : M_1 \to M_2$ a CR map. By

it differential $\Psi_*$ factors to yield a map

$$[\Psi_*(x)] : T_x M_1/(HM_1)_x \to T_x M_2/(HM_2)_x,$$

between the quotient spaces and one can check that

$$[\Psi_*(x)] \circ \mathcal{L}^{M_1}_x = \mathcal{L}^{M_2}_{\Psi_*(x)} \circ \Psi_*(x).$$

In particular, the Levi form is invariant under CR diffeomorphisms.

When $M$ is locally generically CR embedded into $\mathbb{C}^{n+k}$, we can find lots of local biholomorphic mappings near a fixed point $x \in M$. All of them induce local CR diffeomorphisms of a neighborhood of $x$ in $M$ onto a generic CR submanifold of codimension $k$ sitting in another Euclidean space of the
same complex dimension \( n+k \). Even though the image of \( M \) under such a local CR diffeomorphism may look quite different geometrically, its Levi form is invariant. An illustration of this is the fact that a strictly pseudoconvex hypersurface may not be convex, but it is locally CR diffeomorphic to a hypersurface that is strictly convex, in the elementary sense. Here by a strictly pseudoconvex hypersurface we mean one whose Levi form is either positive or negative definite.

More generally, as follows from Sylvester’s “law of inertia” for Hermitian forms, it is the signature of the Levi form that is invariant; by the “signature” of any scalar Levi form, we mean its number of positive, zero and negative eigenvalues. It is therefore impossible to change the signature of the Levi form by making any ambient biholomorphic change of coordinates. Nonetheless a simple rescaling procedure permits to change the ”size” of a nonzero eigenvalue.

Using formula (5.10), one can relate the Levi form of \( M \) to the extrinsic curvature of \( M \), when \( M \) is CR embedded. To see this, we assume that \( M \) is embedded into \( \mathbb{C}^m \) near some point \( x \in M \). We will consider 2-dimensional ”ribbons” inside \( M \) constructed as follows: Let \( \ell \) be a complex line in \( \mathbb{C}^m \) which passes through \( x \) and is tangent to \( M \) at \( x \). The (local) image of the exponential map \( \exp : TM \to M \) restricted to \( \ell \) is then a little piece of a real 2-dimensional surface \( \Sigma \) passing through \( x \); actually \( \Sigma \) is obtained as the union of all geodesics starting from \( x \) in all directions of \( \ell \). Equivalently, \( \Sigma \) is locally given by the “slice” \( M \cap \text{span} \{ \ell, N_x M \} \).

Now assume that \( \ell = \text{span} \{ X, J X \} \), and \( X \in H_x M \) is of length one. Since \( N_x M \subset N_x \Sigma \), we then obtain from (5.10)

\[
\mathcal{L}^{NM}(X) = B(X, X) + B(JX, JX) = (B_{\Sigma}(X, X) + B_{\Sigma}(JX, JX))^{NM},
\]

where \( B_{\Sigma} \) is the second fundamental form of \( \Sigma \), whereas \( B \) is the second fundamental form of \( M \). From (6.1) we get the following geometric interpretation of the Levi form of \( M \): \( \mathcal{L}^{NM}(X) \) is twice the projection of the mean curvature vector of \( \Sigma \) onto \( N_x M \).

We would like to emphasize that this mean curvature vector depends on \( x \) and \( \ell \), but not on a particular choice of \( X \). Indeed, if \( \ell \) is spanned by \( Y \) and \( JY \), where \( Y = aX + bJX \) and \( a^2 + b^2 = 1 \), then we obtain the same mean curvature vector. In fact \( \mathcal{L}^{NM}(Y) = \mathcal{L}^{NM}(X) \), as follows from (4.4).

In contrast to the Gauss curvature of a surface (which is a metric invariant but may change under conformal maps), the mean curvature is known to be a conformal invariant: it is possible to turn a strictly convex surface looking like a skull-cap into a saddle while keeping the mean curvature, say positive.

**An Example: tube CR manifolds.** Let \( \Gamma \) be a smooth real manifold in \( \mathbb{R}^{n+k} \), of real dimension \( n \) and real codimension \( k \). Then \( M = \Gamma \times \mathbb{R}^{n+k} \) is a CR manifold of type \((n, k)\), generically CR embedded into \( \mathbb{C}^{n+k} = \mathbb{R}^{n+k} + i \mathbb{R}^{n+k} \). Indeed, if \( \text{pr} : M \to \Gamma \) denotes the canonical projection, then \( T^{1,0}_x M = \mathbb{R}^{n+k} \).
\[ \odot \otimes T_{pr(x)} \Gamma. \] Following [HK], \( M \) is called a tube CR manifold over the base \( \Gamma \).

From (5.10) we can deduce that the Levi form of \( M \) at \( x \) is the second fundamental form of \( \Gamma \) at \( pr(x) \): If we choose any real tangent vector \( X \) to \( \Gamma \) at \( pr(x) \), then \( X \) is also tangent to \( M \) at \( x \), and \( JX = iX \) points along the flat fiber of \( M \). Hence (5.10) implies

\[
\mathcal{L}^{NM}(X) = B_\ast(X, X) + B_\ast(JX, JX) = B_\ast(X, X) = (B^r_{pr(x)}(X, X))^M_N = (B^r_{pr(x)}(X, X))^T_N = B^r_{pr(x)}(X, X).
\]

Therefore, assuming \( X \) has length one, \( \mathcal{L}^{NM}(X) = B^r_{pr(x)}(X, X) \) is the normal curvature vector to \( \Gamma \) at \( pr(x) \) in the direction \( X \), and its length is the curvature of any curve in \( \Gamma \) passing through \( pr(x) \) having tangent vector \( X \) at \( pr(x) \).

A similar observation applies to so-called Reinhardt CR manifolds. They are of the form \( M = \exp (\Gamma + i\mathbb{R}^{n+k}) \), where \( \exp \) denotes the exponential map that sends \( z = (z_1, \ldots, z_{n+k}) \) to \( \exp z = (e^{z_1}, \ldots, e^{z_{n+k}}) \). Reinhardt CR manifolds are thus invariant under multiplication of each coordinate by a different factor of modulus one, whereas tube CR manifolds are invariant under translations in the pure imaginary direction.

7. The homogeneous case

Using a notation which is customary while dealing with homogeneous spaces (see e.g. [Hel]) we add the subscript “0” to indicate real objects, while complex object will be left unsubscripted.

For a smooth bundle \( pr : E \to M \) we denote by \( \Gamma(M, E) \) the space \( \{ f \in C^\infty(M, E) \mid prf = \text{id}_M \} \) of smooth sections of \( E \) over \( M \).

7.1. Homogeneous CR manifolds and CR algebras.

Homogeneous spaces. Let \( M_0 \) be a smooth real manifold and \( G_0 \) a real Lie group, with identity \( e \). A differentiable action of \( G_0 \) on \( M_0 \) is a \( C^\infty \) smooth map

\[
G_0 \times M_0 \ni (a, p) \to ap \in M_0, \quad \text{with} \quad \begin{cases} eap = p, \\ a(a_1 a_2)p = (a_1 a_2)p, \\ \forall a_1, a_2 \in G_0, \forall p \in M_0. \end{cases}
\]

To each element \( X \) of the Lie algebra \( g_0 \) of \( G_0 \) we associate the smooth vector field \( X^* \in \Gamma(M_0, TM) \) generating the flux \( (t, p) \to \exp(tX) \cdot p \) on \( M_0 \). The space of \( X^* \) for \( X \) in \( g_0 \) is a real Lie subalgebra of \( \Gamma(M_0, TM) \) for the commutation of vector fields. The correspondence \( X \to X^* \) is indeed an antihomomorphism of Lie algebras, i.e. is \( \mathbb{R} \)-linear and

\[ [X^*, Y^*] = -[X, Y]^*, \quad \forall X, Y \in g_0. \]

For each \( a \in G_0 \), the map \( L_a(p) = a \cdot p \) is a diffeomorphism of \( M_0 \), yielding by differentiation a diffeomorphism \( dL_a : TM_0 \to TM_0 \). We will write
for simplicity $a \cdot v$ instead of $dL_a(v)$ if $a \in G_0$ and $v \in TM_0$, noting that also the map

$$G_0 \times TM_0 \ni (a, v) \rightarrow a \cdot v \in TM_0$$

is a smooth action of $G_0$ on $TM_0$.

We say that $M_0$ is $G_0$-homogeneous when the action of $G_0$ is transitive, i.e. when $G_0 \cdot p = M_0$ for some, and hence for all, $p$ in $M_0$. When this is the case, $T_p M_0 = \{X_p^* | X \in g_0\}$ for all points $p$ of $M_0$ and therefore one can use vector fields $X^*$ to get local frames near any $p$ in $M_0$.

The stabilizer $Stab_{G_0}(p_0) = \{x \in G_0 | x \cdot p_0 = p_0\}$ of any point $p_0$ of $M_0$ is a closed Lie subgroup of $G_0$. Denote by $stab_{G_0}(p_0)$ its Lie algebra.

If the action (7.1) is transitive, then $M_0$ has a unique real analytic structure such that for each point $p_0$ the map

$$\pi : G_0 \ni a \rightarrow a \cdot p_0 \in M_0,$$

is a real analytic submersion and the injective quotient yields a real analytic diffeomorphism

$$\tilde{\pi} : G_0 / Stab_{G_0(p_0)} \rightarrow M_0$$

and, by differentiating, a linear isomorphism

$$d\tilde{\pi} : g_0 / stab_{G_0(p_0)} \rightarrow T_{p_0} M_0.$$

**Homogeneous CR Manifolds.** Let $M_0$ is a CR manifold with structure bundle $HM$ and partial complex structure $J$.

We say that a differentiable action (7.1) is CR if the diffeomorphisms $p \mapsto a \cdot p$ of $M_0$ are CR for all $a$ in $G_0$. This means that

$$a \cdot HM = HM \quad \text{and} \quad a \circ J = J \circ a.$$

This can be rephrased by using the complexification of the action of $G_0$ on $TM_0$, by requiring that either one holds true

$$a \cdot T^{1,0} M_0 = T^{1,0} M_0, \quad \text{or} \quad a \cdot T^{0,1} M_0 = T^{0,1} M_0, \quad \forall a \in G_0.$$

We say that $M_0$ is a $G_0$-homogeneous CR manifold if the action (7.1) of the Lie group $G_0$ on $M_0$ is transitive and CR.

Fix a base point $p_0$ in $M_0$ and consider the smooth submersion (7.2). The pullback

$$Q = \pi^*(\Gamma(M_0, T^{0,1} M_0)) = \{ \zeta \in \Gamma(G_0, CTG_0) | d\pi((\zeta_a)_a) \in T^{0,1}_{a \cdot p_0}, \forall a \in G_0 \}$$

of $\Gamma(M_0, T^{0,1} M_0)$ to $G_0$ is a formally integrable distribution of complex vector fields on $G_0$, which is invariant by left translations.

Let $g$ be the complex Lie algebra obtained by complexifying $g_0$ and set

$$\mathfrak{a} = \{ Z \in g \mid d\pi_{\cdot, a}(Z) \in T^{0,1}_{p_0} M_0 \}.$$

For the following statements we refer to [AMN1] [AMN2] [MN].

**Proposition 7.1.**

1. The subspace $\mathfrak{a}$ defined in (7.5) is a complex Lie subalgebra of $g$. 
(2) \( q \cap \bar{q} \) equals the complexification of \( \text{stab}_{\bar{g}_0}(p_0) \).

(3) A complex vector field \( \zeta \) on \( G_0 \) belongs to \( Q \) iff \( a^{-1} \zeta a \in q \) for all \( a \in G_0 \).

(4) Vice versa, any complex Lie subalgebra \( q \) of \( g \) with \( q \cap \bar{q} = \mathbb{C} \otimes_{\mathbb{R}} \text{stab}_{\bar{g}_0}(p_0) \), defines on a \( G_0 \)-homogeneous space \( M_0 \) a \( G_0 \)-homogeneous CR structure by setting
\[
T^{0,1}_{\bar{a}p_0} M_0 = a \cdot d\pi_e(q), \quad \forall a \in G_0.
\]

A \( G_0 \)-homogeneous CR structure on \( M_0 \) is determined by the datum of the antiholomorphic tangent space at a point. Thus, by Proposition 7.1, having fixed a point \( p_0 \), the \( G_0 \)-homogeneous CR structures on \( M_0 \) are parameterized by the complex Lie subalgebras \( q \) of the complexification \( g \) of \( g_0 \) for which
\[
q \cap g_0 = \text{stab}_{G_0}(p_0).
\]

A different choice of the base point \( p_0 \) changes \( \text{Stab}_{G_0}(p_0) \) and \( q \) by \( G_0 \)-conjugation.

These remarks motivate the definition in [MN] of CR-algebras as pairs \((g_0, q)\) consisting of a real Lie algebra \( g_0 \) and a complex Lie subalgebra \( q \) of \( g = \mathbb{C} \otimes_{\mathbb{R}} g_0 \).

Indeed this notion encodes the possible realizations of \( M_0 \) as a smooth CR submanifold of a homogeneous complex manifold. Assume that:

- \( G_0 \) admits a complexification \( G \),
- there is a closed complex Lie subgroup \( Q \) of \( G \) with Lie algebra \( q \) and \( Q \cap G_0 = \text{Stab}_{G_0}(p_0) \).

Then the inclusions \( G_0 \hookrightarrow G \) and \( \text{Stab}_{G_0}(p_0) \hookrightarrow Q \) yield a smooth immersion of \( M_0 \) into the complex homogeneous space \( M = G/Q \) and we have a commutative diagram
\[
\begin{array}{ccc}
G_0 & \longrightarrow & G \\
\downarrow & & \downarrow \\
M_0 & \longrightarrow & M.
\end{array}
\]

If we are only interested in the local setting, we can always assume that the Lie group \( G_0 \) is linear, thus admitting a complexification \( G \), and that \( \text{Stab}_{G_0}(p_0) \) is connected. However, the analytic Lie subgroup \( Q \) generated by \( q \) may not be closed, (virtual in the sense of [O]). The quotient \( G/Q \) can be thought in this case as a germ of smooth complex manifold providing a local embedding of \( M_0 \) near \( p_0 \).

### 7.2. The Levi form

For a \( G_0 \)-homogeneous CR manifold \( M_0 \) one can describe the Levi form at a point \( p_0 \) by utilizing the attached CR algebra \((g_0, q)\) and Lie brackets on the complexification \( g \) of \( g_0 \). In fact we will consider the sesquilinear form \( L^{1,0}_{p_0} \) on \( T^{1,0}_{p_0} M_0 \) and the corresponding scalar Levi forms, indexed by \( H^0_{p_0} M_0 \). With the submersion \( \pi \) of (7.2) we have indeed \( T^{1,0}_{p_0} M_0 = d\pi_e(q) \), where conjugation in \( g \) is taken with respect to the real
form \( g_0 \). The vector fields of the complex distribution \( \mathcal{Q} \) generated by the left-invariant vector fields \( Z^* \) with \( Z \in q \) are \( \pi \)-related to the elements of \( \Gamma(M_0, \mathcal{T}^{1,0}M_0) \). Hence we have (cf. e.g. [Hei] Ch.I, Prop.3.3)

\[
d\pi_{p_0}([Z_1, \bar{Z}_2]) = [X_1, \bar{X}_2]_{p_0}
\]

if \( Z_1, Z_2 \in q, X_1, X_2 \in \Gamma(M_0, \mathcal{T}^{1,0}M_0) \) and \( \pi_i(Z_i) = X_i, i = 1, 2 \). Hence one obtains the following.

**Proposition 7.2.** Let \( \text{pr} : \mathfrak{g} \to \mathfrak{g}/(q + \bar{q}) \) be the projection onto the quotient and \( \varpi \) the isomorphism of \( \mathfrak{g}/(q + \bar{q}) \) with \( \mathcal{C}T_{p_0}M_0/\mathcal{C}H_{p_0}M_0 \) defined by the isomorphism \( d\pi_e : \mathfrak{g} \to \mathcal{C}T_{p_0}M_0, \) with \( d\pi_e(q + \bar{q}) = \mathcal{C}H_{p_0}M_0 \).

Then the Levi form of \( M_0 \) at \( p_0 \) is characterized by the commutative diagram

\[
\begin{array}{ccc}
q \times q & \xrightarrow{(Z_1, \bar{Z}_2) \to [Z_1, \bar{Z}_2]} & \mathfrak{g} \\
& \downarrow{(Z_1, \bar{Z}_2) \to (d\pi_e(Z_1), d\pi_e(\bar{Z}_1))} & \downarrow{\varpi} \\
T_p^1M_0 \times T_{\bar{p}_0}^1M_0 & \xrightarrow{\mathcal{L}_{p_0}^{1,0}} & \mathcal{C}T_{p_0}M_0/\mathcal{C}H_{p_0}M_0
\end{array}
\]

where the top left are the Lie brackets in \( \mathfrak{g} \).

We illustrate by some examples the actual computation. We need to parameterise the quotient \( T_{p_0}M_0 \) by taking a linear complement of \( (q + \bar{q}) \cap g_0 \) in \( g_0 \). Then we will write the Levi form \( L_{p_0}^{1,0} \) as an \( n \times n \) Hermitian symmetric matrix (with \( n \) equal to the CR dimension) depending upon \( k \) real parameters (with \( k \) equal to the CR codimension). The Hermitian symmetric forms obtained by fixing the real values of the parameters are the *scalar* Levi forms.

**Example 7.3.** We consider the flag manifold \( M \) consisting of the pairs \( \ell_1 \subset \ell_2 \subset \mathbb{C}^6 \), which is a compact complex manifold of complex dimension 11, homogeneous space for the action of \( \text{SL}_6(\mathbb{C}) \). Fixing on \( M \) the base point \( p_0 = (\langle e_1 \rangle, \langle e_1, e_2, e_3, e_4 \rangle) \), its stabiliser \( \mathcal{Q} \) in \( \text{SL}_6(\mathbb{C}) \) is the complex parabolic subgroup with Lie algebra

\[
\mathfrak{q} = \left\{ \begin{array}{cccccccc}
z_{1,1} & z_{1,2} & z_{1,3} & z_{1,4} & z_{1,5} & z_{1,6} \\
0 & z_{2,2} & z_{2,3} & z_{2,4} & z_{2,5} & z_{2,6} \\
0 & z_{3,2} & z_{3,3} & z_{3,4} & z_{3,5} & z_{3,6} \\
0 & z_{4,2} & z_{4,3} & z_{4,4} & z_{4,5} & z_{4,6} \\
0 & 0 & 0 & z_{5,5} & z_{5,6} & \ \\
0 & 0 & 0 & z_{6,5} & z_{6,6} & \end{array} \right\} \text{ with } \sum_{i=1}^6 z_{i,j} = 0.
\]

Let us consider the Hermitian symmetric matrix

\[
B = \left( \begin{array}{cccccc}
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
\end{array} \right),
\]
of signature \((4, 2)\), to define \(\text{SU}(2, 4) = \{ x \in \text{SL}_6(\mathbb{C}) \mid x^* B x = B \} \). Then \((e_1)\) is
an isotropic line and \((e_1, e_2, e_3, e_4)\) a 4 plane on which the restriction of \(B\) has
minimal rank. Hence the \(\text{SU}(2, 4)\)-orbit \(M_0\) of \(p_0\) in \(M\) is the minimal one.

The Lie algebra of \(\text{SU}(2, 4)\) is characterized by

\[
\text{su}(2, 4) = \{ X \in \mathfrak{sl}_6(\mathbb{C}) \mid X^* B + BX = 0 \}.
\]

Taking into account that \(B^2 = I_6\), it is the subalgebra of fixed points of the
conjugation \(Z \rightarrow -BX^* B\). Using this conjugation we obtain

\[
\tilde{q} = \begin{pmatrix}
z_{1,1} & z_{1,2} & z_{1,3} & z_{1,4} & z_{1,5} & z_{1,6} \\
0 & 0 & z_{2,2} & z_{2,3} & z_{2,4} & z_{2,5} \\
0 & 0 & 0 & z_{3,3} & z_{3,4} & z_{3,5} \\
0 & 0 & 0 & 0 & z_{4,4} & z_{4,5} \\
0 & 0 & 0 & 0 & 0 & z_{5,5} \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

with \(\sum_{i=1}^6 z_{i,i} = 0\).

To compute the CR type and the Levi form of \(M_0\) it is convenient to start
from a linear complement of \(q\) in \(\mathfrak{sl}_6(\mathbb{C})\) and construct the tangent to \(M_0\) at
\(p_0\) by adding, for each coefficient, its \(\text{su}(2, 4)\)-conjugate.

We obtain a representation of the tangent space to \(M_0\) at \(p_0\) of the form

\[
m_0 : \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
z_1 & 0 & 0 & 0 & 0 & 0 \\
w_1 & -z_2 & 0 & 0 & 0 & 0 \\
w_2 & -z_3 & 0 & 0 & 0 & 0 \\
w_3 & it & 0 & 0 & 0 & 0 \\
it_1 & -\bar{w}_3 & -\bar{w}_1 & -\bar{w}_2 & -\bar{z}_1 & 0
\end{pmatrix}
\]

The \(z_i\)’s, which are the terms in the complement of \(q\) which by conjugation are
sent into \(-\bar{z}_i\) lying in \(q\), correspond to the analytic tangent and should be
therefore considered “complex” coordinates. The remaining terms, whose
conjugates stay in the complement of \(q\), are the “real” coordinates, whose
entries will parametrise the Levi forms. The number of \(z_i\)’s is the CR
dimension, twice the number of \(w_i\)’s plus the number or \(t_i\)’s is the CR codimension:
in this example, \(M_0\) is of type \((3, 8)\) (we have \(z_1, z_2, z_3\) and \(w_1, w_2, w_3, t_1, t_2\)).

The Levi form is \(3 \times 3\) and depends on \(3\) complex (related to \(w_1, w_2, w_3\)) and
two real (corresponding to \(it_1, it_2\)) parameters. It can be represented by the
matrix

\[
\begin{pmatrix}
0 & w_1 & w_2 \\
\bar{w}_1 & t_2 & 0 \\
\bar{w}_2 & 0 & t_2
\end{pmatrix}
\]
Let us explain the way it is computed and thus its meaning. The $T_{p_i}^{1,0}M_0$ part is represented in $\mathfrak{m}=\mathbb{C}\otimes m_0$ by the matrices

\[
Z = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
z_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & z_2 & z_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 
\end{pmatrix},
\]

the $T^0{1,0}M_0$ part by the matrices

\[
\bar{Z} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & -\bar{z}_2 & 0 & 0 & 0 \\
0 & -\bar{z}_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\bar{z}_1 
\end{pmatrix}.
\]

It is natural to take $Z_i$ to be the matrix corresponding to $z_i=1$, $z_j=0$ for $j\neq i$ to define a basis of $T_{p_i}^{1,0}M_0$. In an analogous way we can build a basis for a linear complement of $\mathbb{C}H_{p_i}M_0$ in $\mathfrak{m}$ by using the $w_i$'s and $t_j$'s.

The vector valued Levi form $L^{1,0}$ is obtained by computing

\[
i[Z, \bar{Z}]
\]

for matrices $Z, \bar{Z}$ of the form (**) and e.g. the $w_i$ in the entry $(1, 2)$ of (*) means that $L^{1,0}(Z_1, Z_2)$ is a vector in $\mathfrak{m}/(\mathfrak{m} \cap (q + \bar{q}))$ proportional to the projection of

\[
W_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 
\end{pmatrix}
\]

The entries $w_i$'s and $t_j$'s of (*) can also be taken as dual variables in $H_{p_i}^0M$. In this way (*) can be thought of as the scalar Levi form. We note that it is identically zero on the 3-plane $\{w_1 = 0, w_2 = 0, t_2 = 0\}$ and that all nondegenerate scalar Levi forms have at least one positive and one negative eigenvalue.

**Example 7.4.** We consider the flag $M$ of $\text{SO}_7(\mathbb{C})$, consisting of the projective lines contained in the quadric

\[
Q = \{[z] \in \mathbb{C}\mathbb{P}^6 \mid z_1z_7+z_2z_6+z_3z_5+z_4^2=0\}
\]

of $\mathbb{C}\mathbb{P}^6$. Take the matrix

\[
B = \begin{pmatrix}
0 & 0 & 1 \\
0 & I_5 & 0 \\
1 & 0 & 0 
\end{pmatrix}.
\]
Then
\[ \text{SO}(1, 6) = \{ x \in \text{SO}(7, \mathbb{C}) \mid x^* B x = B \}. \]

In this way, the \( \text{SO}(1, 6) \)-orbit \( M_0 \) through the line \( p_0 = \{ [z] \in \mathbb{C} P^6 \mid z \in \langle e_1, e_2 \rangle \cap Q \} \) is the minimal orbit of \( \text{SO}(1, 6) \) in \( M \).

We have
\[
q = \begin{pmatrix}
z_{1,1} & z_{1,2} & z_{1,3} & z_{1,4} & z_{1,5} & z_{1,6} & 0 \\
0 & 0 & z_{2,3} & z_{2,4} & z_{2,5} & 0 & z_{1,7} \\
0 & 0 & 0 & z_{3,4} & 0 & z_{3,6} & z_{3,7} \\
0 & 0 & 0 & z_{4,5} & z_{4,6} & z_{4,7} & 0 \\
0 & 0 & 0 & 0 & z_{5,6} & z_{5,7} & 0 \\
0 & 0 & 0 & 0 & 0 & z_{6,7} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & z_{7,7}
\end{pmatrix}
\]

Then, to describe \( T_{p_0} M_0 \), we construct the matrix of \( \mathfrak{so}(1, 6) \) which is obtained by adding to the matrices of the linear complement
\[
m = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
z_{3,1} & z_{3,2} & 0 & 0 & 0 & 0 & 0 \\
z_{4,1} & z_{4,2} & 0 & 0 & 0 & 0 & 0 \\
z_{5,1} & z_{5,2} & 0 & 0 & 0 & 0 & 0 \\
z_{6,1} & 0 & -z_{5,2} & -z_{4,2} & -z_{3,2} & 0 & 0 \\
0 & -z_{6,1} & -z_{5,1} & -z_{4,1} & -z_{3,1} & 0 & 0
\end{pmatrix}
\]
of \( q \) their conjugate with respect to the real form \( \mathfrak{so}(1, 6) \). We label by \( z_i \)'s the entries that fall out of \( m \) by the conjugation and by \( w_i \)'s and \( t_i \)'s the complex and real entries which remain in \( m \) after conjugation. We obtain in this way the matrices
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
z_4 & 0 & -z_1 & -z_2 & -z_3 & 0 & 0 \\
w & z_1 & 0 & 0 & 0 & z_3 & 0 \\
t & z_2 & 0 & 0 & 0 & z_2 & 0 \\
w & z_3 & 0 & 0 & 0 & z_1 & 0 \\
z_4 & 0 & -z_3 & -z_2 & -z_1 & 0 & 0 \\
0 & -z_4 & -w & -t & -w & -z_1 & 0
\end{pmatrix}
\]

Then \( M_0 \) is of type \((4, 3)\) and the matrix for its Levi form is
\[
\begin{pmatrix}
0 & 0 & 0 & w \\
0 & 0 & 0 & t \\
0 & 0 & 0 & w \\
w & t & w & 0
\end{pmatrix}, \quad \text{with } w \in \mathbb{C} \text{ and } t \in \mathbb{R}.
\]

In particular, all nonzero scalar Levi forms have signature \(+, -, 0, 0, 0, 0, 0, 0\).

As explained in [AMN1], for homogeneous CR manifolds which, as the examples discussed above, are orbits of real forms in complex flag manifolds, it is possible to compute the Levi form by combinatorics on root
systems, by using the attached cross marked Satake diagrams. More general examples can be found in [AMN3].

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