Adding virtual measurements by PWM-induced signal injection

Dilshad Surroop\textsuperscript{1,2}, Pascal Combes\textsuperscript{2}, Philippe Martin\textsuperscript{1} and Pierre Rouchon\textsuperscript{1}

Abstract—We show that for PWM-operated devices, it is possible to benefit from signal injection without an external probing signal, by suitably using the excitation provided by the PWM itself. As in the usual signal injection framework conceptualized in [1], an extra “virtual measurement” can be made available for use in a control law, but without the practical drawbacks caused by an external signal.

I. INTRODUCTION

Signal injection is a control technique which consists in adding a fast-varying probing signal to the control input. This excitation creates a small ripple in the measurements, which contains useful information if properly decoded. The idea was introduced in [2], [3] for controlling electric motors at low velocity using only measurements of currents. It was later conceptualized in [1] as a way of producing “virtual measurements” that can be used to control the system, in particular to overcome observability degeneracies. Signal injection is a very effective method, see e.g. applications to electromechanical devices along these lines in [4], [5], but it comes at a price: the ripple it creates may in practice yield unpleasant acoustic noise and excite unmodeled dynamics, in particular in the very common situation where the device is fed by a Pulse Width Modulation (PWM) inverter; indeed, the frequency of the probing signal may not be as high as desired so as not to interfere with the PWM (typically, it can not exceed 500 Hz in an industrial drive with a 4 kHz-PWM frequency).

The goal of this paper is to demonstrate that for PWM-operated devices, it is possible to benefit from signal injection without an external probing signal, by suitably using the excitation provided by the PWM itself, as e.g. in [6]. More precisely, consider the Single-Input Single-Output system

\begin{align}
\dot{x} &= f(x) + g(x)u, \\
y &= h(x),
\end{align}

where \(u\) is the control input and \(y\) the measured output. We first show in section II that when the control is impressed through PWM, the dynamics can be written as

\[\dot{x} = f(x) + g(x)(u + s_0(u, t)),\]

with \(s_0\) 1-periodic and zero-mean in the second argument, i.e. \(s_0(u, \sigma + 1) = s_0(u, \sigma)\) and \(\int_0^1 s_0(u, \sigma) d\sigma = 0\) for all \(u, \varepsilon\) is the PWM period, hence assumed small. The difference with usual signal injection is that the probing signal \(s_0\) generated by the modulation process now depends not only on time, but also on the control input \(u\). This makes the situation more complicated, in particular because \(s_0\) can be discontinuous in both its arguments. Nevertheless, we show in section III that the second-order averaging analysis of [1] can be extended to this case. In the same way, we show in section V that the demodulation procedure of [1] can be adapted to make available the so-called virtual measurement

\[y_v := H_1(x) := \varepsilon h'(x)g(x),\]

in addition to the actual measurement \(y_a := H_0(x) := h(x)\). This extra signal is likely to simplify the design of a control law, as illustrated on a numerical example in section V.

Finally, we list some definitions used throughout the paper; \(S\) denotes a function of two variables, which is \(T\)-periodic in the second argument, i.e. \(S(v, \sigma + T) = S(v, \sigma)\) for all \(v\):

- the mean of \(S\) in the second argument is the function (of one variable) \(\overline{S}(v) := \frac{1}{T} \int_0^T S(v, \sigma) d\sigma\); \(S\) has zero mean in the second argument if \(\overline{S}\) is identically zero
- if \(S\) has zero mean in the second argument, its zero-mean primitive in the second argument is defined by \(S(v, \tau) := \int_0^\tau S(v, \sigma) d\sigma d\tau\)
- \(\mathcal{O}_\infty\) denotes the uniform “big O” symbol of analysis, namely \(f(z, \varepsilon) = \mathcal{O}_\infty(\varepsilon^p)\) if \(|f(z, \varepsilon)| \leq K \varepsilon^p\) for \(\varepsilon\) small enough, with \(K > 0\) independent of \(z\) and \(\varepsilon\).

II. PWM-INDUCED SIGNAL INJECTION

When the control input \(u\) in (1) is impressed through a PWM process with period \(\varepsilon\), the resulting dynamics reads

\[\dot{x} = f(x) + g(x)\mathcal{M}(u, \varepsilon),\]

with \(\mathcal{M}\) 1-periodic and mean \(u\) in the second argument; the detailed expression for \(\mathcal{M}\) is given below. Setting \(s_0(u, \sigma) := \mathcal{M}(u, \sigma) - u\) \(\mathcal{M}\) obviously takes the form (2), with \(s_0\) 1-periodic and zero-mean in the second argument.

Classical PWM with period \(\varepsilon\) and range \([-u_m, u_m]\) is obtained by comparing the input signal \(u\) to the \(\varepsilon\)-periodic sawtooth carrier defined by

\[c(t) := \begin{cases} u_m + 4w(\frac{t}{\varepsilon}) & \text{if } -\frac{u_m}{2} \leq w(\frac{t}{\varepsilon}) \leq 0 \text{ with } w(\frac{t}{\varepsilon}) \leq \frac{u_m}{2}; \\
-u_m - 4w(\frac{t}{\varepsilon}) & \text{if } 0 \leq w(\frac{t}{\varepsilon}) \leq \frac{u_m}{2}; \end{cases}\]

\textsuperscript{1} D. Surroop, P. Martin and P. Rouchon are with the Centre Automatique et Systèmes, MINES ParisTech, PSL Research University, Paris, France \{dilshad.surroop, philippe.martin, pierre.rouchon\}@mines-paristech.fr
\textsuperscript{2} D. Surroop and P. Combes are with Schneider Toshiba Inverter Europe, Pacy-sur-Eure, France pascal.combes@se.com
Finally, the induced zero-mean probing signal is
\[
s_0(u, \sigma) := M(u, \sigma) - u
\]
\[
= u_m - u + \text{sign}(\frac{u - u_m}{\epsilon} - w(\sigma)) + \text{sign}(\frac{u - u_m}{\epsilon} + w(\sigma)),
\]
and its zero-mean primitive in the second argument is
\[
s_1(u, \sigma) := (1 - \frac{u}{u_m}) w(\sigma) - \left| \frac{u - u_m}{\epsilon} - w(\sigma) \right| + \left| \frac{u - u_m}{\epsilon} + w(\sigma) \right|.
\]

**Remark 1:** As \( s_0 \) is only piecewise continuous, one might expect problems to define the “solutions” of (2). But as noted above, if the input \( u(t) \) of the PWM encoder varies slowly enough, its output \( u_{\text{pwm}}(t) = M(u(t), \frac{\epsilon}{2}) \) will have exactly two discontinuities per PWM period. Chattering is therefore excluded, which is enough to ensure the existence and uniqueness of the solutions of (2), see [7], without the need for the more general Filippov theory [8]. Of course, we assume (without loss of generality in practice) that \( f, g \) and \( h \) in (1) are smooth enough.

Notice also \( s_1 \) is continuous and piecewise \( C^1 \) in both its arguments. The regularity in the second argument was to be expected as \( s_1(u, \cdot) \) is a primitive of \( s_0(u, \cdot) \); on the other hand, the regularity in the first argument stems from the specific form of \( s_0 \).

### III. AVERAGING AND PWM-INDUCED INJECTION

Section III-A outlines the overall approach and states the main Theorem [1] which is proved in the somewhat technical Section III-B. As a matter of fact, the proof can be skipped without losing the main thread; suffice to say that if \( s_0 \) were Lipschitz in the first argument, the proof would essentially be an extension of the analysis by "standard" second-order averaging of [1], with more involved calculations.

**A. Main result**

Assume we have designed a suitable control law
\[
\begin{align*}
\bar{u} &= \alpha(\bar{\eta}, \bar{Y}, t) \\
\bar{\eta} &= \alpha(\bar{\eta}, \bar{Y}, t),
\end{align*}
\]
where \( \bar{\eta} \in \mathbb{R}^q \), for the system
\[
\begin{align*}
\dot{x} &= f(x) + g(x)\bar{u}, \\
\dot{Y} &= H(x) := \left( \epsilon h'(x)g(x) \right).
\end{align*}
\]

By “suitable”, we mean the resulting closed-loop system
\[
\begin{align}
\dot{\bar{x}} &= f(\bar{x}) + g(\bar{x})\alpha(\bar{\eta}, H(\bar{x}), t), \\
\dot{\bar{\eta}} &= \alpha(\bar{\eta}, H(\bar{x}), t)
\end{align}
\]
has the desired exponentially stable behavior. We have changed the notations of the variables with \( \tau \) to easily distinguish between the solutions of (4) and of (7) below. Of course, this describes an unrealistic situation:

- **PWM is not taken into account**
On the other hand, we will see in section IV that, thanks to Remark 1, \(H_0(\tau(t))\) and \(H_1(\tau(t))\) in (8c) may be as smooth as desired (the regularity is inherited from only \(f, g, h, \alpha, \alpha\)), on the other hand, \(s_1(u(t), \frac{t}{\varepsilon})\) is only continuous and piecewise \(C^1\). Nevertheless, this is enough to justify all the Taylor expansions performed in the paper.

B. Proof of Theorem 2

Because of the lack of regularity of \(s_0\), we must go back to the fundamentals of the second-order averaging theory presented in [9, chapter 2] (with slow time dependence [9, section 3.1]). We first introduce two ad hoc definitions.

**Definition 1:** A function \(\varphi(X, \sigma)\) is slowly-varying in average if there exists \(\lambda > 0\) such that for \(\varepsilon\) small enough,

\[
\int_0^T \left\| \varphi(p(\varepsilon\sigma)) + \varepsilon q(\sigma), \sigma - \varphi(p(\varepsilon\sigma)), \sigma \right\| d\sigma \leq \lambda T \varepsilon^k,
\]

where \(p, q\) are continuous with \(q\) bounded; \(a\) and \(T > 0\) are arbitrary constants. Notice that if \(\varphi\) is Lipschitz in the first variable then it is slowly-varying in average. The interest of this definition is that it is satisfied by \(s_0\).

**Definition 2:** A function \(\phi(X, \sigma)\) is \(O_\infty(\varepsilon^2)\) in average if there exists \(K > 0\) such that \(\|\int_0^T \phi(q(s), s) ds\| \leq K \varepsilon^2\sigma\) for all \(\sigma \geq 0\). Clearly, if \(\phi\) is \(O_\infty(\varepsilon^3)\) then it is \(O_\infty(\varepsilon^3)\) in average.

The proof of Theorem 2 follows the same steps as [9, chapter 2], but with weaker assumptions. We first rewrite (7) in the fast timescale \(\sigma := t/\varepsilon\) as

\[
\frac{dX}{d\sigma} = \varepsilon F(X, \sigma, \varepsilon\sigma).
\]

where \(X := (x, \eta)\) and

\[
F(X, \sigma, \tau) := \left( f(x) + g(x)M\left( \alpha(\sigma, \eta, H(x, \eta, \sigma, \tau, \tau) \right) \right).
\]

Notice \(F\) is 1-periodic in the second argument. Consider also the so-called averaged system

\[
\frac{d\bar{X}}{d\sigma} = \varepsilon \bar{F}(\bar{X}, \varepsilon\sigma).
\]

where \(\bar{F}\) is the mean of \(F\) in the second argument.

Define the near-identity transformation

\[
X = X + \varepsilon W(\bar{X}, \varepsilon, \sigma),
\]

where \(\bar{X} := (\bar{x}, \bar{\eta})\) and

\[
W(\bar{X}, \sigma, \tau) := \left( g(\bar{X})0 \right) s_1\left( \alpha(\bar{\eta}, H(\bar{x}, \bar{\eta}, \sigma, \tau, \tau) \right).
\]

Inverting (13) yields

\[
\bar{X} = X - \varepsilon W(X, \sigma, \varepsilon\sigma) + O_\infty(\varepsilon^2).
\]

By lemma 1, this transformation puts (11) into

\[
\frac{d\bar{X}}{d\sigma} = \varepsilon \bar{F}(\bar{X}, \varepsilon\sigma) + \varepsilon^2 \Phi(\bar{X}, \varepsilon\sigma) + \phi(\bar{X}, \varepsilon\sigma); \quad \Phi\text{ periodic and zero-mean in the second argument, and}
\]

[1] for slowly-varying in average, and \(\phi\) is \(O_\infty(\varepsilon^3)\) in average.

\[\begin{align*}
\eta & = \alpha(\eta, \tilde{Y}, t), \\
\eta & = \alpha(\eta, \tilde{Y}, t).
\end{align*}\]
By lemma 2 the solutions $\bar{X}(\sigma)$ and $\tilde{X}(\sigma)$ of (12) and (15), starting from the same initial conditions, satisfy

$$\tilde{X}(\sigma) = \bar{X}(\sigma) + O_{\infty}(\varepsilon^2).$$

As a consequence, the solution $X(\sigma)$ of (11) starting from $X_0 = \varepsilon W(X_0, 0, 0)$ are related by $X(\sigma) = \bar{X}(\sigma) + \varepsilon W(\bar{X}(\sigma), \varepsilon, \sigma) + O_{\infty}(\varepsilon^2)$, which is exactly (13a–13b). Inserting (13a) in $y = h(x)$ and Taylor expanding yields (14).

**Remark 3:** If $s_0$ were differentiable in the first variable, $\Phi$ would be Lipschitz and $\phi$ would be $O_{\infty}(\varepsilon^3)$ in (15), hence the averaging theory of [9] would directly apply.

**Remark 4:** In the sequel, we prove for simplicity only the estimation $\tilde{X}(\sigma) = \bar{X}(\sigma) + O_{\infty}(\varepsilon^2)$ on a timescale $1/\varepsilon$. The continuation to infinity follows from the exponential stability of (11), exactly as in [1, Appendix].

In the same way, lemma 2 is proved without slow-time dependence, the generalization being obvious as in [9, section 3.3].

**Lemma 1:** The transformation (13) puts (11) into (15), where $\Phi$ is periodic and zero-mean in the second argument, and slowly-varying in average, and $\phi$ is $O_{\infty}(\varepsilon^3)$ in average.

**Proof:** To determine the expression for $d\bar{X}/d\sigma$, the objective is to compute $dX/d\sigma$ as a function of $\bar{X}$ with two different methods. On the one hand we replace $X$ with its transformation (13) in the closed-loop system (11), and on the other hand we differentiate (13) with respect to $\sigma$.

We first compute $s_0(\alpha(\eta, \varepsilon), \bar{X}(x, \eta, \sigma, \varepsilon), \sigma)$ as a function of $\bar{X} = (\bar{x}, \bar{\eta})$. Exactly as in (10), with $(\bar{x}, \bar{\eta})$ replacing $(x, \eta)$, and (14) replacing (9), we have

$$\bar{\Pi}(x, \eta, \sigma, \varepsilon) = H(\bar{x}) + O_{\infty}(\varepsilon^2).$$

Therefore, by Taylor expansion

$$\alpha(\eta, \bar{\Pi}(x, \eta, \sigma, \varepsilon), \varepsilon, \sigma) = \alpha(\bar{\eta}, H(\bar{x}), \varepsilon, \sigma) + \varepsilon^2 K_\alpha(\bar{X}, \sigma),$$

with $K_\alpha$ bounded. The lack of regularity of $s_0$ prevents further Taylor expansion; nonetheless, we still can write

$$s_0(\alpha(\eta, \bar{\Pi}(x, \eta, \sigma, \varepsilon), \varepsilon, \sigma), \sigma) = s_0(\alpha(\bar{\eta}, H(\bar{x}), \varepsilon, \sigma) + \varepsilon^2 K_\alpha(\bar{X}, \sigma), \sigma).$$

Finally, inserting (13) into (11) and Taylor expanding, yields after tedious but straightforward computations,

$$\frac{dX}{d\sigma} = \varepsilon \bar{F}(\bar{x}, \sigma) + \epsilon G(\bar{X}) s_0^{\alpha_0}(\bar{\eta}) + \varepsilon G'(\bar{X}) G(\bar{X}) s_0^{\alpha_0}(\bar{\eta}) + \varepsilon^2 K_0(\bar{X}, \sigma) + O_{\infty}(\varepsilon^3);$$

we have introduced the following notations

$$(\tilde{\gamma}) := (\bar{X}, \sigma, \varepsilon)$$

$$s_0^\alpha(\bar{\eta}) := s_0(\alpha(\bar{\eta}, H(\bar{x}), \varepsilon, \sigma), \sigma)$$

$$s_0^{\alpha_0}(\bar{\eta}) := s_0(\alpha(\bar{\eta}, H(\bar{x}), \varepsilon, \sigma) + \varepsilon^2 K_0(\bar{X}, \sigma), \sigma)$$

$$\Delta s_0^\alpha(\bar{\eta}) := \Delta s_0(\bar{\eta})$$

$$G(\bar{X}) := \left( \begin{array}{c} g(x) \\ 0 \end{array} \right)$$

$$\Phi(X, \varepsilon) := \left( f(x) + g(x) \alpha(\eta, H(x), \varepsilon, \sigma) \right).$$

We now differentiate (13), which reads with the previous notations

$$X = \bar{X} + \varepsilon G(\bar{X}) s_0^\alpha(\bar{\eta}).$$

This yields

$$\frac{dX}{d\sigma} = \frac{d\bar{X}}{d\sigma} + \varepsilon G'(\bar{X}) \frac{d\bar{X}}{d\sigma} s_0^\alpha(\bar{\eta}) + \varepsilon G(\bar{X}) \partial_1 s_0^\alpha(\bar{\eta}) \frac{d\bar{X}}{d\sigma}$$

$$+ \varepsilon G(\bar{X}) s_0^{\alpha_0}(\bar{\eta}) + \varepsilon^2 G(\bar{X}) \partial_3 s_0^\alpha(\bar{\eta}),$$

since $\partial_2 s_0^\alpha = s_0^{\alpha_0}$. Now assume $\bar{X}$ satisfies

$$\frac{d\bar{X}}{d\sigma} = \varepsilon \bar{F}(\bar{x}, \sigma) + \epsilon G(\bar{X}) \Delta s_0^\alpha(\bar{\eta}) + O_{\infty}(\varepsilon^3),$$

where $\Psi(\tilde{\gamma})$ is yet to be computed. Inserting (18) into (17),

$$\frac{dX}{d\sigma} = \varepsilon \frac{\bar{F}(\bar{X}, \varepsilon, \sigma)}{\bar{F}(\bar{x}, \sigma)} + \epsilon G(\bar{X}) \Delta s_0^\alpha(\bar{\eta}) + O_{\infty}(\varepsilon^3, \varepsilon^3).$$

Next, equating (19) and (16), $\Psi$ satisfies

$$\Psi(\tilde{\gamma}) = \left[ \bar{F}(\bar{X}, \varepsilon, \sigma) s_0^\alpha(\bar{\eta}) + G'(\bar{X}) G(\bar{X}) s_0^{\alpha_0}(\bar{\eta}) \right]$$

$$- G(\bar{X}) \partial_1 s_0^\alpha(\bar{\eta}) \bar{F}(\bar{x}, \sigma) - G(\bar{X}) \partial_3 s_0^\alpha(\bar{\eta})$$

$$- G(\bar{X}) \partial_3 s_0^{\alpha_0}(\bar{\eta}) \Delta s_0^\alpha(\bar{\eta}).$$

This gives the expressions of $\Phi$ and $\phi$ in (15),

$$\phi(\tilde{\gamma}) := \left[ \bar{F}(\bar{X}, \varepsilon, \sigma) s_0^\alpha(\bar{\eta}) + G'(\bar{X}) G(\bar{X}) s_0^{\alpha_0}(\bar{\eta}) \right]$$

$$- G(\bar{X}) \partial_1 s_0^\alpha(\bar{\eta}) \bar{F}(\bar{x}, \sigma) - G(\bar{X}) \partial_3 s_0^\alpha(\bar{\eta}),$$

with

$$\Psi_1(\tilde{\gamma}) := -\epsilon G(\bar{X}) \partial_1 s_0^\alpha(\bar{\eta}) G(\bar{X}) \Delta s_0^\alpha(\bar{\eta}).$$

The last step is to check that $\Phi$ and $\phi$ satisfy the assumptions of the lemma. Since $s_0^\alpha$, $\partial_1 s_0^\alpha$ and $\partial_3 s_0^\alpha$ are periodic and zero-mean in the second argument, and slowly-varying
in average, so is $\Phi$. There remains to prove that $\phi = O_{\infty}(\varepsilon^3)$ in average. Since $\Delta s^0_0$ is slowly-varying in average, 
$$\int_0^s \|\Delta s^0_0(\bar{\tau}(s))\| \, ds \leq \lambda_0 \sigma \varepsilon^2.$$ 
with $\lambda_0 > 0$. $G$ being bounded by a constant $c_g$, this implies 
$$\left\| \int_0^s \varepsilon G(\bar{X}(s)) \Delta s^0_0(\bar{\tau}(s)) \, ds \right\| \leq c_g \lambda_0 \sigma \varepsilon^3.$$ 
Similarly, $\partial_1 s_1$ being bounded by $c_{11}$, $\Psi_1$ satisfies 
$$\left\| \int_0^s \varepsilon^2 \Psi_1(\bar{\tau}(s)) \, ds \right\| \leq c_{11}^2 \lambda_0 \sigma \varepsilon \varepsilon^3.$$ 
Summing the two previous inequalities yields 
$$\left\| \int_0^s \phi(\bar{\tau}(s)) \, ds \right\| \leq \lambda_0 \varepsilon (1 + c_{11} c_g \varepsilon) \sigma \varepsilon^3,$$ 
which concludes the proof.

**Lemma 2:** Let $\bar{X}(\sigma)$ and $\bar{X}(\sigma)$ be respectively the solutions of (12) and (15) starting at 0 from the same initial conditions. Then, for all $\sigma \geq 0$ 
$$\bar{X}(\sigma) = X(\sigma) + O_{\infty}(\varepsilon^2).$$ 
**Proof:** Let $E(\sigma) := \bar{X}(\sigma) - X(\sigma)$. Then, 
$$E(\sigma) = \int_0^\sigma \frac{d\bar{X}}{d\sigma}(s) - \frac{dX}{d\sigma}(s) \, ds$$ 
$$= \varepsilon \int_0^\sigma \left[ F(\bar{X}(s)) - F(X(s)) \right] \, ds$$ 
$$+ \varepsilon^2 \int_0^\sigma \Phi(\bar{\tau}(s)) \, ds + \int_0^\sigma \phi(\bar{\tau}(s)) \, ds.$$ 
As $F$ is Lipschitz with constant $\lambda_F$, 
$$\varepsilon \int_0^\sigma \left\| F(\bar{X}(s)) - F(X(s)) \right\| \, ds \leq \varepsilon \lambda_F \int_0^\sigma \|E(\sigma)\| \, ds.$$ 
On the other hand, there exists by lemma [3] $c_1$ such that 
$$\varepsilon^2 \left\| \int_0^\sigma \Phi(\bar{\tau}(s)) \, ds \right\| \leq c_1 \varepsilon^2.$$ 
Finally, as $\phi$ is $O_{\infty}(\varepsilon^3)$ in average, there exists $c_2$ such that 
$$\left\| \int_0^\sigma \phi(\bar{\tau}(s)) \, ds \right\| \leq c_2 \varepsilon^3 \sigma.$$ 
The summation of these estimations yields 
$$\|E(\sigma)\| \leq \varepsilon \lambda_F \int_0^\sigma \|E(\sigma)\| \, ds + c_1 \varepsilon^2 + c_2 \varepsilon^3 \sigma.$$ 
Then by Gronwall’s lemma [9, Lemma 1.3.3] 
$$\|E(\sigma)\| \leq \left( \frac{c_2}{\lambda_F} + c_1 \right) e^{\lambda_F \sigma} \varepsilon^2,$$ 
which means $\bar{X} = X + O_{\infty}(\varepsilon^2)$.

The following lemma is an extension of Besjes’ lemma [9, Lemma 2.8.2] when $\varphi$ is no longer Lipschitz, but only slowly-varying in average.

**Lemma 3:** Assume $\varphi(X, \sigma)$ is $T$-periodic and zero-mean in the second argument, bounded, and slowly-varying in average. Assume the solution $X(\sigma)$ of $\dot{X} = O_{\infty}(\varepsilon)$ is defined for $0 \leq \sigma \leq L/\varepsilon$. There exists $c_1 > 0$ such that 
$$\left\| \int_0^\sigma \varphi(X(s), \sigma) \, ds \right\| \leq c_1.$$ 
**Proof:** Along the lines of [9], we divide the interval $[0, t]$ in $m$ subintervals $[0, T], \ldots, [(m - 1)T, mT]$ and a remainder $[mT, t]$. By splitting the integral on these intervals, we write 
$$\int_0^\sigma \varphi(x(s), s) \, ds = \sum_{i=0}^{mT} \int_{(i-1)T}^{iT} \varphi(x((i-1)T), s) \, ds$$ 
$$+ \sum_{i=0}^{mT} \int_{(i-1)T}^{iT} \left[ \varphi(x(s), s) - \varphi(x((i-1)T), s) \right] \, ds$$ 
$$+ \int_{mT}^{t} \varphi(x(s), s) \, ds,$$ 
where each of the integral in the first sum are zero as $\varphi$ is periodic with zero mean. Since $\varphi$ is bounded, the remainder is also bounded by a constant $c_2 > 0$. Besides 
$$x(s) = x((i-1)T) + \int_{(i-1)T}^{s} \dot{x}(\tau) \, d\tau$$ 
$$= x((i-1)T) + \varepsilon \lambda T \varepsilon$$ 
with $q$ continuous and bounded. By hypothesis, there exists $\lambda > 0$ such that for $0 \leq i \leq m$, 
$$\int_{(i-1)T}^{iT} \|\varphi(x(s), s) - \varphi(x((i-1)T), s)\| \, ds \leq \lambda T \varepsilon$$ 
Therefore by summing the previous estimations, 
$$\left\| \int_0^\sigma \varphi(x(s), s) \, ds \right\| \leq m \lambda T \varepsilon + c_2,$$ 
with $mT \leq t \leq L/\varepsilon$, consequently $m \lambda T \varepsilon + c_2 \leq \lambda L + c_2$; which concludes the proof.

**IV. Demodulation**

From (8c), we can write the measured signal $y$ as 
$$y(t) = y_0(t) + y_1(t) s_1(u(t), \frac{\varepsilon}{\varepsilon}) + O_{\infty}(\varepsilon^2),$$ 
where the signal $u$ feeding the PWM encoder is known. The following result shows $y_0$ and $y_1$ can be estimated from $y$, for use in a control law as described in section [11-A].

**Theorem 2:** Consider the estimators $\hat{y}_0$ and $\hat{y}_1$ defined by 
$$\hat{y}_0(t) := \frac{3}{2} M(y(t)) - \frac{1}{2} M(y)(t - \varepsilon)$$ 
$$k_\Delta(t) := (y(t) - \hat{y}_0(t)) s_1(u(t), \frac{\varepsilon}{\varepsilon})$$ 
$$\hat{y}_1(t) := \frac{M(k_\Delta(t))}{\sigma_1^2(u(t))},$$ 
where $M : y \mapsto \varepsilon^{-1} \int_0^\varepsilon y(\tau) \, d\tau$ is the moving average operator, and $\sigma_1^2$ the mean of $s_1^2$ in the second argument (cf. end of section [1]). Then, 
$$\hat{y}_0(t) = y_0(t) + O_{\infty}(\varepsilon^2) \quad (21a)$$ 
$$\hat{y}_1(t) = y_1(t) + O_{\infty}(\varepsilon^2). \quad (21b)$$
Recall that by construction \( y_a(t) = O_\infty(\varepsilon) \), hence (21b) is essentially a first-order estimation; notice also that \( s_1^2(u(t)) \) is always non-zero when \( u(t) \) does not exceed the range of the PWM encoder.

**Proof:** Taylor expanding \( y_a, y_v, u \) and \( s_1 \) yields

\[
y_a(t - \tau) = y_a(t) - \tau \dot{y}_a(t) + O_\infty(\tau^2)
\]

\[
y_v(t - \tau) = y_v(t) + O_\infty(\varepsilon) O_\infty(\tau)
\]

\[
s_1(u(t - \tau), \sigma) = s_1(u(t) + O_\infty(\tau), \sigma) = s_1(u(t), \sigma) + O_\infty(\tau);
\]

in the second equation, we have used \( y_a(t) = O_\infty(\varepsilon) \). The moving average of \( y_a \) then reads

\[
M(y_a)(t) = \frac{1}{\varepsilon} \int_0^\varepsilon y_a(t - \tau) d\tau = \frac{1}{\varepsilon} \int_0^\varepsilon (y_a(t) - \tau \dot{y}_a(t) + O_\infty(\tau^2)) d\tau = y_a(t) - \frac{\varepsilon}{2} \dot{y}_a(t) + O_\infty(\varepsilon^2).
\]

A similar computation for \( k_v(t) := y_v(t)s_1(u(t), \varepsilon) \) yields

\[
M(k_v)(t) = \frac{1}{\varepsilon} \int_0^\varepsilon y_v(t - \tau)s_1(u(t - \tau), \frac{\tau - \varepsilon}{\varepsilon}) d\tau = y_v(t)\left(\frac{1}{\tau_1}(u(t)) + O_\infty(\varepsilon)\right) + O_\infty(\varepsilon^2) = O_\infty(\varepsilon^2),
\]

since \( s_1 \) is 1-periodic and zero mean in the second argument. Summing (22) and (23), we eventually find

\[
M(y)(t) = y_a(t) - \frac{\varepsilon}{2} \dot{y}_a(t) + O_\infty(\varepsilon^2).
\]

As a consequence, we get after another Taylor expansion

\[
\frac{3}{2} M(y)(t) - \frac{1}{2} M(y)(t - \varepsilon) = y_a(t) + O_\infty(\varepsilon^2),
\]

which is the desired estimation (21a).

On the other hand, (21a) implies

\[
k_\Delta(t) = y_v(t)s_1^2(u(t), \varepsilon) + O_\infty(\varepsilon^2).
\]

Proceeding as for \( M(k_v) \), we find

\[
M(k_\Delta)(t) = \frac{1}{\varepsilon} \int_0^\varepsilon y_v(t - \tau)s_1^2(u(t - \tau), \frac{\tau - \varepsilon}{\varepsilon}) d\tau = y_v(t)\left(\frac{1}{\tau_1}(u(t)) + O_\infty(\varepsilon)\right) + O_\infty(\varepsilon^2) = y_v(t)s_1^2(u(t)) + O_\infty(\varepsilon^2).
\]

Dividing by \( s_1^2(u(t)) \) yields the desired estimation (21b).

**V. NUMERICAL EXAMPLE**

We illustrate the interest of the approach on the system

\[
\begin{align*}
\dot{x}_1 & = x_2, \\
\dot{x}_2 & = x_3, \\
\dot{x}_3 & = u + d, \\
y & = x_2 + x_1 x_3,
\end{align*}
\]

where \( d \) is an unknown disturbance; \( u \) will be impressed through PWM with frequency 1kHz (i.e. \( \varepsilon = 10^{-3} \)) and range \([-20, 20]\). The objective is to control \( x_1 \), while rejecting the disturbance \( d \), with a response time of a few seconds.

We want to operate around equilibrium points, which are of the form \( (x_1^{eq}, 0; 0; -d^{eq}, d^{eq}) \), for \( x_1^{eq} \) and \( d^{eq} \) constant. Notice the observability degenerates at such points, which renders not trivial the design of a control law.

Nevertheless the PWM-induced signal injection makes available the virtual measurement

\[
y_v = \varepsilon \begin{pmatrix} x_3 & 1 & x_1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \varepsilon x_1,
\]

from which it is easy to design a suitable control law, without even using the actual input \( y_a = x_2 + x_1 x_3 \). The system being now fully linear, we use a classical controller-observer, with disturbance estimation to ensure an implicit integral effect. The observer is thus given by

\[
\begin{align*}
\dot{x}_1 & = \hat{x}_2 + l_1 \left( \frac{u\varepsilon}{\varepsilon} - \hat{x}_1 \right), \\
\dot{x}_2 & = \hat{x}_3 + l_2 \left( \frac{u\varepsilon}{\varepsilon} - \hat{x}_1 \right), \\
\dot{x}_3 & = u + \hat{d} + l_3 \left( \frac{u\varepsilon}{\varepsilon} - \hat{x}_1 \right), \\
\dot{\hat{d}} & = l_d \left( \frac{u\varepsilon}{\varepsilon} - \hat{x}_1 \right),
\end{align*}
\]

\[
\text{Fig. 3: States } x_1, x_2, x_3 \text{ with ideal and actual control laws.}
\]
Finally, this ideal control law is implemented as

\[
\begin{align*}
\dot{x}_1 &= -K\eta + 3x_1^\text{ref} + L\hat{y}_v \\
\dot{x}_2 &= M\eta + 5x_1^\text{ref} + L\hat{y}_v \\
\dot{y}_v &= M\eta + 5x_1^\text{ref} + L\hat{y}_v,
\end{align*}
\]

where \( M \) is the PWM function described in section III and \( \hat{y}_v \) is obtained by the demodulation process of section IV.

The test scenario is the following: at \( t = 0 \), the system start at rest at the origin; from \( t = 2 \), a disturbance \( d = -0.25 \) is applied to the system; at \( t = 14 \), a filtered unit step is applied to the reference \( x_1^\text{ref} \). In Fig. 3, the ideal control law (24), i.e. without PWM and assuming \( \hat{y}_v \) known, is compared to the true control law (25): the behavior of (25) is excellent, it is nearly impossible to distinguish the two situations on the responses of \( x_1 \) and \( x_2 \) as by (8a) the corresponding ripple is only \( O_{\infty}(\varepsilon^2) \); the ripple is visible on \( x_3 \), where it is \( O_{\infty}(\varepsilon) \).

The corresponding control signals \( u \) and \( u_{\text{pwm}} \) are displayed in Fig. 4 and the corresponding measured outputs in Fig. 5.

To investigate the sensitivity to measurement noise, the same test was carried out with band-limited white noise (power density \( 1 \times 10^{-3} \), sample time \( 1 \times 10^{-2} \)) added to \( y \). Even though the ripple in the measured output is buried in noise, see Fig. 6 the virtual output is correctly demodulated and the control law (25) still behaves very well.

Applications

We have presented a method to take advantage of the benefits of signal injection in PWM-fed systems without the need for an external probing signal. For simplicity, we have restricted to Single-Input Single-Output systems, but there are no essential difficulties to consider Multiple-Input Multiple-Output systems. Besides, though we have focused on classical PWM, the approach can readily be extended to arbitrary modulation processes, for instance multilevel PWM; in fact, the only requirements is that \( s_0 \) and \( s_1 \) meet the regularity assumptions discussed in remark 1.

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