Phenomenological analysis of supersymmetric $\sigma$–models on
coset spaces $\text{SO}(10)/\text{U}(5)$ and $E_6/[\text{SO}(10) \times \text{U}(1)]$

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Abstract

We discuss some phenomenological aspects of gauged supersymmetric $\sigma$–models on homogeneous coset-spaces $E_6/[\text{SO}(10) \times \text{U}(1)]$ and $\text{SO}(10)/\text{U}(5)$ which are some of the most interesting for phenomenology. We investigate in detail the vacuum configurations of these models, and study the resulting consequences for supersymmetry breaking and breaking of the internal symmetry. Some supersymmetric minima for both models with gauged full isometry groups $E_6$ and $\text{SO}(10)$ are physically problematic as the Kähler metric becomes singular and hence the kinetic terms of the Goldstone boson multiplets vanish. This leads us to introduce recently proposed soft supersymmetry-breaking mass terms which displace the minimum away from the singular point. A non-singular Kähler metric breaks the linear subgroup $\text{SO}(10) \times \text{U}(1)$ of the $E_6$ model spontaneously. The particle spectrum of all these different models is computed.
1 Introduction

Non-linear supersymmetric $\sigma$-models based on homogeneous Kählerian cosets spaces $G/H$ may have applications to physics beyond the standard model. For example, supersymmetric extensions of a Grand Unified Theory (GUT) may be relevant for particle physics since they contain less parameters than the Minimal Supersymmetric Standard Model (MSSM). One of the original guidelines of the construction of GUT theories was renormalizability. However, as such GUT models are likely to be realized quite close to the Planck scale, renormalizability is not necessarily an issue as supergravity theories are non-renormalizable by themselves. Moreover, supergravity models often include non-linear coset models such as $SU(1, 1)/U(1)$ in $N = 4$. Therefore a GUT may be part of a supersymmetric non-linear sigma model based on a coset space $G/H$, with $H$ a subgroup of $G$.

For the construction of this kind of models the coset space $G/H$ must be a Kähler manifold [1, 2]. The chiral fermion content of supersymmetric $\sigma$-models based on homogeneous Kählerian cosets spaces is often anomalous. The presence of chiral anomalies in internal symmetries restricts the usefulness of these models for phenomenological applications. Therefore, anomalies have to be removed to allow for gauging the internal symmetries. This is achieved [3] by coupling additional chiral superfields (generically called matter superfields) carrying representations of the coset space $G/H$. An important question in the context of supersymmetric matter is how it can be coupled to supersymmetric $\sigma$-models on Kähler manifolds without spoiling the (possibly non-linear) invariance of the original theory. This is required for the cancellation of anomalies as shown in [5]. Using the general procedure of canceling anomalies by coupling additional chiral superfields, consistent supersymmetric $\sigma$-models on coset spaces, including among others the grassmannian models on $SU(N + M)/[SU(N) \times SU(M) \times U(1)]$, the orthogonal unitary coset models on manifolds $SO(2N)/U(N)$, as well as models on exceptional cosets like $E_6/[SO(10) \times U(1)]$, have been studied in great detail [3, 4, 5, 7].

Since $E_6$ and $SO(10)$ are promising unification groups, the coset spaces $E_6/[SO(10) \times U(1)]$ and $SO(10)/[SU(5) \times U(1)]$ are the most interesting for (direct) phenomenology. In the $E_6/[SO(10) \times U(1)]$ model, the fermion partners of the Goldstone bosons—the quasi-Goldstone fermions—have precisely the right quantum numbers to describe one family of quarks and leptons, including a right-handed neutrino. The model on $SO(10)/U(5)$ contains the $SU(5) \times U(1)$ fermionic field content of one generation of quarks and leptons, including a right-handed neutrino as well. Therefore, these models could have interesting phenomenological applications. In earlier studies of anomaly-free extension of supersymmetric $\sigma$-model on $SO(10)/U(5)$, it was found that upon gauging the full $SO(10)$ the $D$-term potential sometimes force the scalar fields to take vacuum expectation values for which the model becomes singular, in the sense that the kinetic energy terms of the Goldstone boson and quasi-Goldstone fields disappear in the vacuum state, and the space of physical degrees of freedom is reduced. In a recent paper [8] we have investigated singularities in field geometry, where the kinetic terms vanish, by studying a simple supersymmetric model based on the homogeneous space $CP^1$. We showed that the metric
singularities can be regularized by addition of a soft supersymmetry-breaking mass parameter.

The present paper is a first step in the analysis of the phenomenology of those models. In order to discuss various properties of the models, we first review the construction of the lagrangians on coset-spaces that are globally consistent. We describe how anomaly cancellation can be achieved in supersymmetric σ-model on $SO(10)/U(5)$ and $E_6/[SO(10) \times U(1)]$, by adding matter fields; we then discuss the several interesting gauge extended versions of these models, and the resulting mass spectra.

This paper is structured as follows. The main aspects of gauged supersymmetric σ–models on Kähler cosets with anomalies canceled by matter fields is reviewed in section 2. In section 3 we derive the mass sum rule for non-linear supersymmetric σ–models. These relations play an important role in constructing realistic supersymmetric gauge theories, containing the standard model. In section 4 we summarize the anomaly-free supersymmetric σ–model on $SO(10)/U(5)$ as described in [4]. We perform a quite general analysis of gauging the full $SO(10)$ group in subsection 4.1. We investigate in particular the existence of zeros of the potential, and show that the models with fully gauged $SO(10)$ are singular. In subsection 4.2 we extend the model with soft supersymmetry breaking mass terms which preserve the non-linear $SO(10)$. To complete the phenomenological analysis, we also consider the gauging of the linear subgroup $SU(5) \times U(1)$ of the $SO(10)$-spinor model in subsection 4.3. Because this subgroup contains an explicit $U(1)$ factor, we added a Fayet-Iliopoulos term with parameter $\xi$ and we investigate in particular the existence of zeros of the potential, for which the model is anomaly-free, with positive definite kinetic energy. Then we discuss a number of physical aspects of these models, like supersymmetry and internal symmetry breaking, and the resulting mass-spectrum. Section 5 is devoted to phenomenological analysis of $E_6/[SO(10) \times U(1)]$ model. We first summarize the results obtained in [10, 11, 3]. Section 5.1 discusses the gauging of internal symmetries in general. In section 5.1.1, we consider in some detail the gauging of the full non-linear $E_6$ symmetry. Like in the on $SO(10)/U(5)$, in one of the supersymmetric minima, we find that the $D$-term potential drives the scalar fields to a singular point of the kinetic terms. We show that the singular metric can also be regularized by the addition of a soft supersymmetry-breaking mass parameter. Gauging the linear subgroup $SO(10) \times U(1)$ gives consistent models, but only for special values of couplings constant and non-zero value of the Fayet-Iliopoulos term. Section 6 contains the conclusions.

2 Supersymmetric σ–models on Kähler manifolds

$N = 1$ globally supersymmetric lagrangians for non-linear σ-models in 4-D space time, are formulated in terms of chiral superfields $\Phi^\alpha = (z^\alpha, \psi^\alpha_L, H^\alpha)$, $\alpha = 1, \ldots, N$ the components of which are complex scalars $z^\alpha$, an auxiliary field $H^\alpha$ and a (left-
handed) chiral fermion $^1 \psi_L$. The action is defined by two functions of superfields: the real Kähler potential $K(\bar{\Phi}, \Phi)$, and the holomorphic superpotential $W(\Phi)$. The component lagrangian after eliminating the auxiliary fields is [1]

$$\mathcal{L}_{\text{chiral}} = -G_{\alpha \bar{\alpha}}(z, \bar{z}) \left( \frac{\partial^\mu z^\alpha}{\partial \mu} \bar{z}^\alpha + \bar{\psi}^\alpha_L \frac{\partial}{\partial \bar{z}^\alpha} \psi^\alpha_L \right) + \frac{\partial}{\partial \mu} \psi^\alpha_L \psi^\beta_L \psi^\delta_L$$

$$-G^{\alpha \bar{\alpha}} \bar{W}_\alpha W_{\alpha} + W_{\alpha \bar{\beta}}(z) \bar{\psi}^\beta_R \psi^\alpha_L + \bar{W}_\alpha \bar{\psi}^\beta_R \psi^\alpha_L. \quad (1)$$

In this expression, we have used the following notation for the metric, connection and curvature constructed from the Kähler potential $K$, respectively:

$$G_{\alpha \bar{\alpha}} = K_{\alpha \bar{\alpha}}, \quad \Gamma^\alpha_{\beta \gamma} = G^{\alpha \gamma} G_{\alpha \beta \gamma}, \quad R_{\alpha \bar{\beta} \gamma \delta} = G_{\alpha \beta \gamma} \bar{G}^{\alpha \beta \gamma} G_{\beta \gamma \delta}, \quad (2)$$

with $G^{\alpha \bar{\alpha}}$ the inverse of the metric $G_{\alpha \bar{\alpha}}$. The comma denotes differentiation with respect to $z^\alpha, \bar{z}^{\alpha}$, while the semicolon denotes a covariant derivative. Moreover, the Kähler covariant derivative of a chiral spinor and the left-right arrow above the covariant derivative are defined by

$$\bar{\nabla} \psi^\alpha_L = \theta^\alpha \psi^\alpha_L + \Gamma^\alpha_{\beta \gamma} \theta^\beta \bar{z}^\gamma \psi^\alpha_L, \quad \bar{\psi}^\alpha_L \nabla \psi^\alpha_L = \bar{\psi}^\alpha_L \gamma^\mu \psi^\alpha_L - \partial^\mu \bar{\psi}^\alpha_L \psi^\alpha_L. \quad (3)$$

In general, the Kähler metric may admits a set of holomorphic isometries $R^\alpha_i (z)$, $\bar{R}^{\bar{\alpha}}_i (\bar{z})$ ($i = 1, \ldots, n$), which are the solutions of the Killing equation

$$R_{i \alpha} + \bar{R}_{i \bar{\alpha}} = 0. \quad (4)$$

These isometries define infinitesimal symmetry transformations on the Kähler manifold $G/H$. In components the transformation rules read

$$\delta z^\alpha = \theta^i R^\alpha_i (z), \quad \delta \bar{z}^{\bar{\alpha}} = \theta^i \bar{R}^{\bar{\alpha}}_i (\bar{z}), \quad \delta \psi^\alpha_L = \theta^i R^\alpha_i (z) \psi^\beta_L, \quad \delta \bar{\psi}^\alpha_L = \theta^i \bar{R}^{\bar{\alpha}}_i (\bar{z}) \bar{\psi}^\beta_L, \quad (5)$$

with $\theta^i$ the parameters of the infinitesimal transformations. As a result, the isometries form a Lie algebra:

$$R^\beta_i R^\alpha_{j, \beta} = R^\beta_i R^\alpha_{j, \beta} - R^\beta_j R^\alpha_{i, \beta} = f_{ij}^k R^\alpha_k. \quad (6)$$

Thus, infinitesimal transformations (5) define a (generally non-linear) representation of some Lie group $G$, called the isometry group of the manifold. The $f_{ij}^k$ are structure constants of the algebra. A special feature of Kähler manifolds is that the isometries can locally be written as the gradient of some real scalar functions, the Killing potentials $M_i (z, \bar{z})$ [9, 10]:

$$R^\alpha_i = -i G^{\alpha \bar{\alpha}} M_{i \bar{\alpha}}, \quad \bar{R}^{\bar{\alpha}}_i = i G^{\alpha \bar{\alpha}} M_{i \alpha}. \quad (7)$$

From these equations, one sees that the Killing potentials $M_i$ are defined up to an integration constant $c_i$. It turns out that one can always choose these $c_i$ in such a

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$^1$Our conventions for chiral spinors are such, that $\gamma_5 \psi_L = + \psi_L$ and $\bar{\psi}_L \gamma_5 = - \bar{\psi}_L$; charge conjugations acts as $\psi_R = C \bar{\psi}_L$, where $\psi_L = i \psi_R \gamma_0$. 
way that the potentials $M_i$ transform in the adjoint representation of the isometry group:

$$\delta_i M_j = R^\alpha_i M_{j,\alpha} + \bar{R}_i^\alpha M_{j,\bar{\alpha}} = -i G_{\alpha\bar{\alpha}} \left( R^\alpha_i \bar{R}_j^\alpha - R_j^\alpha \bar{R}_i^\alpha \right) = f_{ij}^k M_k. \quad (8)$$

Under the transformation (5) the Kähler potential itself transforms as

$$\delta_i K = F_i(z) + \bar{F}_i(z). \quad (9)$$

Now it can be shown that the functions $F^i$, $\bar{F}^i$ defined by

$$F_i = K_{,\alpha} R^\alpha_i + i M_i, \quad \bar{F}_i = K_{,\bar{\alpha}} \bar{R}_i^\alpha - i M_i, \quad (10)$$

are holomorphic:

$$F_{i,\bar{\alpha}} = 0, \quad \bar{F}_{i,\alpha} = 0. \quad (11)$$

From the Lie-algebra (6) it follows that one can choose the transformations of the functions $F_i(Z)$ to have the property

$$\delta_i F_j - \delta_j F_i = f_{ij}^k F_k. \quad (12)$$

We now turn to the possibility of realizing the transformation (5) locally. This is possible only if the symmetries are non-anomalous. As it is well known [3, 6], such anomalies can be removed by coupling additional chiral fermions $\chi^A_L$ contained in other chiral superfields $\Psi^A = (a^A, \chi^A_L)$ carrying specific line-bundle representations of the group $G$. Then the complete superfield content $\Sigma^I = (\Phi^\alpha, \Psi^A)$ of the model is specified by a scalar superfield $\Phi^\alpha = (z^\alpha, \psi^\alpha_L, H^\alpha)$, which includes the complex coordinate $z^\alpha$ of this manifold $G/H$, and a set of matter superfields $\Psi^A = (a^A, \chi^A_L, F^A)$.

Once the anomalies have been canceled, the $G$ symmetry group can be gauged in a way that respects the supersymmetry. In summary, one first introduces a set of vector multiplets $V^i = (A^i_{\mu}, \lambda^i_L, D^i)$, where $A^i_{\mu}$ is a gauge field, $\lambda^i_L$ a gaugino and $D^i$ is an auxiliary complex scalar. This gives rise to introduction of the gauge covariant derivatives, accompanied by Yukawa and a D-term potential, defined in terms of the Killing potential $M(z, \bar{z})$ for the isometries group $G$. And finally, one introduces the kinetic terms for the vector multiplets. Then the full lagrangian of globally anomaly-free supersymmetric $\sigma$-models on Kähler manifolds, after eliminating the auxiliary fields $(H^\alpha, F^A, D^i)$ becomes

$$\mathcal{L} = -G_{IJ} \left( DZ^I \cdot DZ^J + \bar{\psi}^I_L \gamma^I \psi^I_L \right) - \left( G_{IJL} D_{\mu} Z^I - G_{IJL} D_{\mu} Z^J \right) \bar{\psi}^I_L D^L \gamma^L \psi^I_L$$

$$+ R_{IJL} \bar{\psi}^J_R \psi^J_R \bar{\psi}^J_R \psi^J_R - \frac{g^2}{2} (M_i + \xi_i)^2 + 2 \sqrt{2} g G_{IJ} \left( \bar{R}_{IJ} \bar{\chi}^i_R \bar{\chi}^i_R + R^i_{II} \bar{\chi}^i_L \bar{\chi}^i_L \right)$$

$$- \frac{1}{4} F^i \cdot F_i - \bar{\lambda}^i_R \gamma^I \lambda^i_R + W_{IJ} \bar{\psi}^I_L \psi^J_R + W_{II} \bar{\psi}^I_L \psi^I_R - G^{IJ} \bar{W}_{IJ} W_{IJ}. \quad (13)$$

Here we have added a Fayet-Iliopoulos term with parameter $\xi_i$ in case there is a commuting $U(1)$ vector multiplet and $M_i(z, \bar{z}; a, \bar{a})$ is an extended version of the
Killing potentials introduced in (7). Furthermore, the notation $Z^I = (z^{\alpha}, a^A)$ and $\psi^I_L = (\psi_L^{\alpha}, \chi_L^A)$, $I = (\alpha, A)$, denote the scalar and spinor components of the superfields $\Sigma^I$. The covariant derivatives contained the gauge fields and field strength tensor $F_{\mu\nu}$ are

$$
D_{\mu} Z^I = \partial_{\mu} Z^I - g A_\mu^i R_{i}^I, \quad D_{\mu} \psi^I_L = \partial_{\mu} \psi^I_L - g A_\mu^i R_{i,j}^I \psi^I_L,
$$

$$
D_{\mu} \lambda^i_R = \partial_{\mu} \lambda^i_R - g f^{ijk} A^j_\mu \lambda^k_R, \quad F_{\mu\nu} = \partial_{\mu} A^i_\nu - \partial_{\nu} A^i_\mu - g f^{ijk} A^j_\mu A^k_\nu. \quad (14)
$$

### 3 The mass formula

A very particular feature of a supersymmetric theories is the existence of a mass formula valid for all possible vacua with spontaneously broken supersymmetry and vacua preserving supersymmetry, relating the masses of all the fields present in the theory. This mass formula is very convenient when discussing realistic models. It is well known that a mass formula holds when supersymmetry is not broken: all states belonging to a given supermultiplet have the same mass. This result has for consequence the following sum rule. The supertrace of the mass matrices squared of all states:

$$
\text{STr } m^2 = \text{Tr} \left( m_0^2 + 3m_1^2 - 2m_2^2 \right), \quad (15)
$$

where $m_0^2$, $m_1^2$ and $m_2^2$ are respectively the mass matrices squared of spin–1, $\frac{1}{2}$ (four component spinors) and 0 (real scalars) states of the theory. For a supersymmetric multiplet of a mass $m$, STr $m^2$ is defined so that

$$
\text{STr } m^2 = \sum m^2 \text{ (number of bosons } - \text{ number of fermions) } = 0. \quad (16)
$$

However, the vanishing of the supertrace for a supersymmetric theory is much weaker than statement of the equality of all masses within a supermultiplet. Indeed a formula for STr $m^2$ can be generalized to arbitrary vacua, including those breaking supersymmetry [12]. The standard choice for vacuum configurations is to allow for constant values of Lorentz invariant fields. Thus only scalars $Z^I$ are allowed to have a non-zero vacuum expectation values (v.e.v.), denoted by $\langle Z^I \rangle$. For this configuration, the theory reduces to the scalar potential

$$
V = -\mathcal{L} \left( \partial_{\mu} \langle Z^I \rangle = \langle \psi^I_L \rangle = \langle \lambda^i_R \rangle = \langle A^i_\mu \rangle = 0 \right). \quad (17)
$$

In this section we derive the supertrace formula for supersymmetric non-linear $\sigma$-models described by (13) relevant for later applications. Since in the models we consider in this paper, the isometry group $G$ does not allow for an invariant trilinear superpotential $W(\Sigma)$, we will not consider here the contributions of $W(\Sigma)$ to the mass formula. For this reason, from now on we take $W(\Sigma) = 0$ (hence the terms involving $W(Z)$ in the full lagrangian (13) are absent). In order to calculate STr $m^2$, we need the explicit form of the three mass matrices in (15).
We first consider the mass matrix squared for a spin–1 particle. When the scalar fields $Z^I$ acquire a vacuum expectation value, some gauge bosons will become massive in general. From (13), the part of the lagrangian quadratic in spin–1 particles is

$$L_1 = -G_{IL} DZ^I \cdot DZ^I - \frac{1}{4} F^i \cdot F^i,$$

with the field strength and the covariant derivative defined in (14). Substituting the expressions for the field strength and the covariant derivative the lagrangian (18) becomes:

$$L_1 = -\frac{1}{2} \left[ (\partial_{\mu} A_i^\nu) (\partial^\mu A_i^\nu - \partial^\nu A_i^{\mu\nu}) - 2g^2 \langle R_i^I \tilde{R}_J^L G_{IL} \rangle A^i_{\mu} A^{\mu i} \right].$$

This expression means that the mass matrix (squared) of spin–1 particles is

$$(m_1^2)_{ij} = 2g^2 \langle R_i^I \tilde{R}_J^L G_{IL} \rangle.$$

From (20), the trace of the mass matrix squared for gauge fields $A^i_\mu$ is

$$3\text{Tr} m_1^2 = 6g^2 \langle R_i^I \tilde{R}_J^L G_{IL} \rangle = 6g^2 \langle G^I M_{i,I} M_{i,L} \rangle.$$

The last equality follows up on using (7).

Turning to the spin–$\frac{1}{2}$ mass matrix, we collect all the terms bilinear in fermionic fields in lagrangian (13) with possible vacuum expectation values $\langle Z^I \rangle$. They read

$$L_{\frac{1}{2}} = -2 \langle G_{IL} \rangle \bar{\psi}_L \vec{\partial} \psi_L^I - 2\bar{\lambda}^i_L \vec{\partial} \lambda^i_R + 2\sqrt{2g} \left( M_{i,I} \bar{\lambda}^i_R \psi_L^I - M_{i,L} \bar{\psi}_L^I \lambda^i_R \right) + \ldots ,$$

where the dots represent total derivatives terms that do not affect the action. The non-vanishing mass term can be written in a matrix form as

$$L_{\frac{1}{2}} = -2 \langle G_{IL} \rangle \bar{\psi}_L \vec{\partial} \psi_L^I + 2\bar{\lambda}^i_L \vec{\partial} \lambda^i_R + 2 \left( \bar{\psi}_L^I \lambda^i_R \right) M_F \left( \begin{array}{c} \lambda^i_R \\ \bar{\psi}_L^I \end{array} \right),$$

with the fermion mass matrix evaluated at the classical minimum of the potential $M_F = \left( \begin{array}{cc} 0 & -i\sqrt{2g} M_{i,L} \\ i\sqrt{2g} M_{i,I} & 0 \end{array} \right)$.

From this expression we obtain the mass matrix squared of spin–$\frac{1}{2}$ particles

$$\left( M^2_{\frac{1}{2}} \right)_i = \left( M_F M_F^\dagger \right)_i = \left( \begin{array}{cc} 2g^2 \langle M_{i,I} M_{i,I} \rangle & 0 \\ 0 & 2g^2 \langle G^I M_{i,L} M_{j,L} \rangle \end{array} \right).$$

This mass matrix has to be normalized such that the kinetic terms of the fermionic fields take the standard form

$$\mathcal{L}_{\text{Dirac}} = -2\bar{\chi}^I (\vec{\partial} - M_{IJ}) \chi^J.$$
This is achieved by multiplying the mass matrix (24) with the inverse metric $G^I_J$ and introduces the Dirac fermions as a combination of a left-handed chiral fermions $\psi^I_L$ and the right-handed gauginos $\lambda^i_R$. As a result, the trace of the mass matrix squared of spin-$\frac{1}{2}$ particles is then

$$\text{Tr } m^2_{\frac{1}{2}} = \text{Tr } M^2_{\frac{1}{2}} = 4g^2 \langle G^I_J M^i_{i,I} M^i_{i,J} \rangle. \quad (27)$$

The last thing we need is the scalar mass matrix (squared). The lagrangian has the form

$$L_0 = -G^I_I \partial Z^I \cdot \partial Z^L - V(Z, \bar{Z}). \quad (28)$$

By expanding the scalar potential $V(Z, \bar{Z})$ to second order in complex fluctuation $\tilde{Z}^I$ around the minimum $Z^I = \langle Z^I \rangle$, the bilinear terms are

$$L_0 = -\langle G^I_I \rangle \partial \tilde{Z}^I \cdot \partial \tilde{Z}^L + \langle V_{IJ} \rangle \tilde{Z}^I \tilde{Z}^L + \frac{1}{2} \langle V_{I,J} \rangle \tilde{Z}^I \tilde{Z}^J + \frac{1}{2} \langle V_{I,J,L} \rangle \tilde{Z}^I \tilde{Z}^L \tilde{Z}^J$$

$$= -\langle G^I_I \rangle \partial \tilde{Z}^I \cdot \partial \tilde{Z}^L - \frac{1}{2} \begin{pmatrix} \tilde{Z}^I & \tilde{Z}^L \end{pmatrix} M_0^2 \begin{pmatrix} \tilde{Z}^I \\ \tilde{Z}^L \end{pmatrix}, \quad (29)$$

with the spin–0 mass matrix squared $M_0^2$:

$$M_0^2 = \begin{pmatrix} \langle V_{IJ} \rangle & \langle V_{IJ} \rangle \\ \langle V_{IJ} \rangle & \langle V_{IJ} \rangle \end{pmatrix}. \quad (30)$$

In a similar fashion the bosonic mass eigenstates have to be normalized such that their kinetic lagrangian takes the standard form. This is achieved again by multiplying the mass matrix squared (30) with the inverse metric $G^I_I$:

$$\text{Tr } \tilde{M}_0^2 = 2 \langle G^I_I V_{IJ} \rangle. \quad (31)$$

From the scalar potential

$$V = \frac{g^2}{2} (M_i + \xi_i)^2 \quad (32)$$

obtained from our general lagrangian (13), one has

$$V_{IJ} = g^2 \left(M_{IJ} M_{iI} + (M_i + \xi_i) M_{iIJ} \right). \quad (33)$$

After substituting the second mixed derivative of the scalar potential (33) in (31) we obtain the trace of the spin-less mass matrix squared:

$$\text{Tr } m^2_0 = 2g^2 G^I_I \left(M_{IJ} M_{iI} + (M_i + \xi_i) M_{iIJ} \right). \quad (34)$$

Finally, collecting results (34), (21) and (27) leads to the general mass sum rule for non-abelian gauged supersymmetric non-linear sigma models without a superpotential:

$$\text{STr } m^2 = 2g^2 G^I_I (M_i + \xi_i) M_{iIJ}, \quad D^i = (M_i + \xi_i) \quad (35)$$
which is valid for arbitrary vacuum expectation values $\langle Z^I \rangle$.

The general mass sum rule for Yang-Mills theories with local supersymmetry, was derived by Cremmer, Ferrara, Girardello and van Proeyen [12]. It has also been derived in superspace by considering 1-loop divergences [13, 14, 15] in the (non–singular) field space

$$\text{STr} m^2 = 2iD_i R^{iI},I = 2iD_i \left[ R^{iI},I + R^{iJ} \Gamma^I_{JI} \right]. \quad (36)$$

The equivalence of this result (36) to ours (35) is rather easy to show using (7). Observe here, that the first term $R^{iI},I$ in (36) always vanishes in supersymmetric $\sigma$-models on Kähler cosets with anomalies canceled by matter as in models considered here (non-abelian gauged supersymmetric non-linear sigma models.)

Some comments are in order here about the formula (35). It has been derived on the assumption that the Kähler metric $G_{II}$ is invertible. However, in some cases as we will discuss in the following sections, the Kähler metric $G_{II}$ develops a zero mode in the minimum of the potential; and the analysis of the theory becomes complicated by the appearance of the infinities at the classical level. A particular solution to this problem is to shift the minimum of potential away from the position where the singularities occur by adding to the model extra terms which break supersymmetry explicitly. These new terms, which break supersymmetry without generating unwanted quadratic divergences are called soft breaking terms.

Explicit breaking of global supersymmetry has been discussed in [16]. Here we only focus on the scalar soft breaking mass term, relevant for later applications:

$$L_{\text{break}} = |\mu|^2 X(Z, \bar{Z}). \quad (37)$$

Here $X$ is real scalar which is invariant under the full set of the isometries $G$, and $\mu^2$ is real and nonzero.

After the addition of the soft breaking terms (37), the supertrace formula becomes

$$\text{STr} m^2 = 2g^2 G^{II} (\mathcal{M}_i + \xi_i) \mathcal{M}_i \mathcal{U} + 2\mu^2 G^{II} X \mathcal{U}. \quad (38)$$

4 Analysis of particle spectrum of $SO(10)/U(5)$–spinor model

From the point of view of unification the coset space $SO(10)/[SU(5) \times U(1)]$ is a very interesting for phenomenological applications as both $SO(10)$ and $SU(5)$ are often used GUT groups. However, a supersymmetric model built on the $SO(10)/[SU(5) \times U(1)]$ coset is not free of anomalies by itself as all the 10 anti-symmetric complex coordinates $z^{ij}$ and their chiral superpartners $\psi^{ij}_L (i, j = 1, \ldots, 5)$ of this manifold carry the same charges. To construct a consistent supersymmetric model on this coset one has to include the fermion partners of the coordinates in an anomaly-free representation. As $SU(5)$ representations are not anomaly free by themselves, we have to use the full $SO(10)$ representations for our additional matter coupling in
this case. This has been achieved in [4] by introducing a singlet 1 and completely
anti-symmetric tensor with 4 indices which is equivalent to to complete the set of
complex chiral superfields to form a 16 of SO(10). The anti-symmetric coordinates of
the coset are combined into a 10 of SU(5) with a unit U(1) charge. An anomaly free
representation is obtained using the branching of the 16. Indeed, its decomposition
under SU(5) reads
\[ 16 = 10(1) + \bar{5}(-3) + 1(5), \]  
where the numbers in parentheses denote the relative U(1) charges. Therefore,
the supersymmetric model on the coste SO(10)/U(5) is defined by three chiral
superfields \((\Phi^{ij}, \Psi_i, \Psi)\): the target manifold \(SO(10)/U(5)\) is parametrized by 10
anti-symmetric complex fields \(z^{ij}\) in a chiral superfield \(\Phi^{ij} = (z^{ij}, \psi^{ij}_L, H^{ij})\), to
which are added \(SU(5)\) vector and scalar matter multiplets denoted respectively by \(\Psi_i = (k_i, \omega_{L i}, B_i)\), and \(\Psi = (h, \varphi_L, F)\).

The complete Kähler potential of the model is
\[ K(z, \bar{z}; k; h) = \frac{1}{2f^2} K_\sigma(z, \bar{z}) + K_{\bar{5}} + K_{\bar{1}}, \]
with the submetric \(\chi^{-1} = 1 + f^2 z \bar{z}\) and \(e^{f^2 K_\sigma} = (\det \chi)^{-1}\). The dimensionfull constant \(f\) is introduced to assign correct physical dimensions to the scalar fields \((z, \bar{z})\).
The Kähler metric \(G_{IL}\) derived from this Kähler potential \(K\) possesses a set of holomorphic Killing vectors generating a non-linear representation of \(SO(10)\):
\[ \delta z = \frac{1}{f} x - u^T z - zu + f z x^\dagger z, \]
\[ \delta k = -k \left(-u^T + f z x^\dagger + \text{tr}(-u^T + f z x^\dagger)1\right), \]
Here \(u\) represents the parameters of the linear diagonal \(U(5)\) transformations, and
\((x, x^\dagger)\) are the complex parameters of the broken off-diagonal \(SO(10)\) transfor-
mations. It is readily checked that under the transformations (41) the Kähler potential \(K\) transforms as in eq. (9):
\[ \delta K = \text{tr}(f z x^\dagger - u^T) + \text{h.c.} = F(z) + \text{h.c.}. \]
This result guarantees the invariance of the metric, as expected if the the transfor-
mations (41) are isometries. Equivalently, one may check that the Killing vectors
(41) satisfy the Killing equation (4) with a metric of the form
\[ G_{IL} = \frac{\partial^2 K}{\partial Z^I \partial \bar{Z}^L} = \begin{pmatrix}
G_{\sigma z^{ij} \bar{z}_{kl}} & G_{z^{ij} \bar{k}_l} & G_{z^{ij} h} \\
G_{k_i z_{ij}} & G_{k_i \bar{k}_j} & 0 \\
G_{h \bar{z}_{ij}} & 0 & G_{h \bar{h}}
\end{pmatrix}. \]
4.1 Gauging of the full SO(10) isometries

In order for the chiral fermions \( (\psi_{L}^{ij}, \omega_{Li}, \varphi_{L}) \) to have a physical interpretation as describing a family quarks and leptons, in this section we introduce gauge interactions. In this case supersymmetry implies the addition of a potential from elimination of the auxiliary \( D^i \) fields by substitution for the Killing potentials \([17]\). We consider the case in which the full \( SO(10) \) isometry is gauged. We denote collectively the \( SO(10) \) gauge fields as \( A_{\mu} = (U_{\mu}, W_{\mu}^{\dagger}, W_{\mu}) \) with \( W_{\mu}^{\dagger} \) and \( W_{\mu} \) the gauge fields corresponding to the broken \( SO(10) \) transformations parametrized by \((x, x^\dagger)\) and with \( U_{\mu} \), the gauge field of the diagonal transformations parametrized by \( u \). This requires the introduction of covariant derivatives for the dynamical fields:

\[
D_{\mu} z = \partial_{\mu} z - g_{10} \left( \frac{1}{f} W_{\mu} - U_{\mu}^{T} z - U_{\mu} z + f z W_{\mu}^{\dagger} z \right),
\]

\[
D_{\mu} k = \partial_{\mu} k + g_{10} k \left( f W_{\mu}^{\dagger} z - U_{\mu}^{T} + \text{tr}(f W_{\mu}^{\dagger} z - U_{\mu}^{T}) \right),
\]

\[
D_{\mu} h = \partial_{\mu} h - 2 g_{10} \text{tr}(f W_{\mu}^{\dagger} z - U_{\mu}^{T}) h.
\] (44)

In the construction of these covariant derivatives we replaced the infinitesimal parameters \((x, x^\dagger)\) by gauge fields.

For the \( D \)–term scalar potential we need the \( SO(10) \) Killing potentials. The full Killing potential \( M \) generating the Killing vectors \((41)\) can be written as

\[
M(u, x^\dagger, x) = \text{tr} \left( u M_u + x^\dagger M_{x^\dagger} + x M_x \right),
\] (45)

with the \( U(5) \) Killing potentials \( M_u \), and the broken Killing potentials \( (M_{x}, M_{x^\dagger}) \) given by \([4]\)

\[
-i M_u = M \left( 1 - 2 f^2 \bar{z} \chi z \right) + e f^2 K_\sigma \left( k^T \bar{k} T - f^2 \bar{z} \bar{k} k z \right),
\]

\[
-i M_{x^\dagger} = f \bar{z} \chi M + f e f^2 K_\sigma \bar{z} \bar{k} k,
\]

\[
-i M_x = - f \chi z M - f e f^2 K_\sigma \bar{k} k z, \quad M = \frac{1}{2 f^2} - 2 |h|^2 e^{-2 f^2 K_\sigma} + e f^2 K_\sigma k \chi^{-1} \bar{k}.
\] (46)

Alternatively, the \( D \)–term potential arising from gauging of \( SU(5) \times U(1) \) including a Fayet-Iliopoulos parameter \( \xi \) is

\[
V = \frac{g_1^2}{10} \left( \xi - i M_Y \right)^2 + \frac{g_5^2}{2} \text{tr} \left( -i M_t \right)^2,
\] (47)

with \( g_1 \) and \( g_5 \) are the \( U(1) \) and \( SU(5) \) gauge couplings respectively. The \( U(1) \) Killing potential \( M_Y \) is defined as the trace of \( U(5) \) Killing potential \( M_u \) whereas the remaining \( SU(5) \) Killing potential \( M_t \) is defined as a traceless part of \( M_u \):

\[
M_t = M_u - \frac{1}{5} M_Y 1 \quad 1, \quad M_Y = \text{tr} M_u.
\] (48)

The case of fully gauged \( SO(10) \) is obtained by taking the coupling constants equal: \( g_1 = g_5 = g_{10} \), and the Fayet-Iliopoulos term to vanish: \( \xi = 0 \).
The coupling of the gauge multiplets to the supersymmetric non-linear $\sigma$–model on $SO(10)/U(5)$ has interesting consequences for the spectrum. It can induce spontaneous breaking of supersymmetry, and further spontaneous breaking of the internal symmetry. For example, if we gauge the full $SO(10)$, all the Goldstone bosons $(z, \bar{z})$ are absorbed by the vector bosons $(W^\dagger_{\mu}, W_\mu)$ which become massive. In this case we may choose to study the model in the unitary gauge $z = \bar{z} = 0$. However, it was found in [4] that in this gauge, the Kähler metric (43) develop zero-modes in the vacuum: the metric $G_{\sigma z\bar{z}kl}$ for the Goldstone bosons and their fermions vanishes.

To see this, we start from the scalar potential (47). As already stated, we choose the unitary gauge: $z = \bar{z} = 0$, $\xi = 0$, and $g_1 = g_5 = g_{10}$. Then the potential for the fully gauged $SO(10)$ model becomes

$$V_{\text{uni}} = \frac{g_{10}^2}{10} \left(10|h|^2 - \frac{5}{2f^2} - 6|k|^2\right)^2 + \frac{2}{5} g_{10}^2 \left(|k|^2\right)^2.$$  \hfill (49)

From this we see that we only have a supersymmetric minimum if

$$|k|^2 = 0, \quad |h|^2 = \frac{1}{4f^2}.$$  \hfill (50)

It can be seen immediately that this solution yields the vanishing of the Kähler metric:

$$G_{zz} = G_{\sigma (ij)}^{(kl)} = \delta^{[k}_{i} \delta^{l]}_{j} \left(\frac{1}{2f^2} - 2|h|^2 + |k|^2\right) + k^{(j} \delta^{l)}_{(i} \bar{k}_{j)} = 0.$$  \hfill (51)

In this case the kinetic terms of the Goldstone superfield components vanish, therefore, mass terms for the $SO(10)$ gauge fields $(W^\dagger_{\mu}, W_\mu)$ vanish as well. Moreover, the theory becomes strongly coupled, with some of the four–fermion interactions exploding, namely:

$$\mathcal{L}_{4–\text{ferm}} = R_{zzhh} \tilde{\psi}_R \psi_L \tilde{\varphi}_L \varphi_R + \text{perm.}$$  \hfill (52)

with the curvature components given by

$$R_{zzhh} = R^{(ij)}_{(kl) h\bar{h}} = -2f^2 \delta^{[k}_{i} \delta^{l]}_{j} \left(1 + 2|h|^2 \left(\frac{1}{2f^2} - 2|h|^2\right)^{-1}\right).$$  \hfill (53)

This may point to a restauration of the $SO(10)$ symmetry. Clearly, not all of the physics described by this model is yet understood.

### 4.2 Softly broken supersymmetry

To avoid the problem of vanishing of the Kähler metric, we shift the minimum of the potential away from the singular point by adding $SO(10)$-invariant soft supersymmetry breaking scalars mass terms

$$\Delta V = \mu_1^2 |h|^2 e^{-2f^2 K_\sigma} + \mu_2^2 e^{f^2 K_\sigma} k \chi^{-1} \bar{k}.$$  \hfill (54)
to the potential. As a result the minimum of the potential is shifted to a position
where the expectation value of the Kähler metric is not vanishing; and the scalar \( h \) gets a vacuum expectation value

\[
|k|^2 = 0, \quad |h|^2 = v^2 = \frac{1}{4f^2} - \frac{\mu_1^2}{20g_{10}^2}, \quad \mu_1^2 < \frac{5g_{10}^2}{f^2}, \quad (55)
\]

breaking the linear local \( U(1) \) subgroup. The corresponding \( U(1) \) vector becomes
massive; and the remaining vectors of \( SU(5) \) stay massless. In the fermionic sector,
two Dirac fermions are realized as a combination of the fermions of the chiral
multiplets with the gauginos.

We now present details of the above mass spectrum. Since in general \( SO(10) \) is
broken in the vacuum, the Goldstone bosons \( (\bar{z}, z) \) are absorbed in the longitudinal
component of the charged vector bosons, and we may choose the unitary gauge
\( \bar{z} = z = 0 \). In this gauge the Kähler metric in the minimum (55) is automatically
diagonal:

\[
G_{II} = \begin{pmatrix}
G_{\sigma}^{ij}(kl) & 0 & 0 \\
0 & G_j^i & 0 \\
0 & 0 & G_{hh}
\end{pmatrix} = \begin{pmatrix}
\delta_i^k \delta_j^l & \frac{\mu_1^2}{10g_{10}} & 0 & 0 \\
0 & \delta_j^i & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad (56)
\]

and all the \( z \) dependence is removed from the covariant derivatives (44). To calculate
the bosonic mass spectrum, we consider the bosonic part of the model, which up to
the kinetic terms for the gauge bosons is described by the action

\[
L_{\text{bos}} = -g_{10}^2 G_{\sigma}^{kl}(ij) \bar{W}^{(ij)} \cdot W_{(kl)} - D\bar{k}^i \cdot Dk_i - D\bar{h} \cdot Dh - V_{\text{full}}
- \frac{1}{4} \left[ \frac{1}{2} F^{ij}(W) \cdot F^{(ij)}(W) + F^i_j(U) \cdot F^i_j(U) \right] + \ldots, \quad (57)
\]

where \( V_{\text{full}} \) is given by \( V_{\text{uni}} \) eq. (49) and \( \Delta V \) eq. (54). In this expression the covariant
derivatives include only the \( U(5) \) gauge field. To identify the masses of the gauge
fields, we decompose the \( U(5) \) vector multiplet \( U^i_j = (U^i_{\mu j}, \Lambda^i_{R j}) \) into a \( U(1) \) and
\( SU(5) \) vector multiplets denoted respectively by \( A = (A_\mu, \lambda_R) \) and \( V^i_j = (V^i_{\mu j}, \Lambda^i_{R j}) \):

\[
V = U - \frac{1}{5} A 1_{5} \quad \text{tr}(V) = 0, \quad A = \text{tr}(U). \quad (58)
\]

It follows that the kinetic terms for the \( SU(5) \times U(1) \) gauge fields become

\[
-\frac{1}{4} \text{tr} F_{\mu \nu}^2(U) - \bar{\Lambda}^i_{R i} \not\partial \Lambda^i_{R j} = -\frac{1}{5} \left( \frac{1}{4} F_{\mu \nu}^2(A) + \bar{\lambda}_R \not\partial \lambda_R \right) + \frac{1}{4} \text{tr}[F_{\mu \nu}^2(V)]
- \bar{\Lambda}^i_{R i} \not\partial \Lambda^i_{R j}. \quad (59)
\]

Notice that the kinetic terms for the \( U(1) \) multiplet are not canonically normalized.
To obtain the standard normalization, we redefine the \( U(1) \) multiplet according to

\[
A \rightarrow \sqrt{5}(\tilde{A}_\mu, \tilde{\lambda}_R). \quad (60)
\]
With the redefined fields, the kinetic terms for the gauge fields become
\[
\mathcal{L}_{\text{gauge}} = -\frac{1}{4} \left[ \frac{1}{2} \bar{F}_{(ij)}(W) \cdot F^{(ij)}(W) + F^{2}_{\mu \nu}(\tilde{A}) + F^{2}_{j}(V) \cdot F^{j}(V) \right] - \bar{\lambda}_{R} \gamma^{\uparrow} \lambda_{R} - \frac{1}{2} \left( \frac{1}{2} \bar{\lambda}_{R}^{(ij)} \gamma^{\uparrow} \lambda_{(ij)R} + \frac{1}{2} \bar{\lambda}_{L}^{(ij)} \gamma^{\uparrow} \lambda_{(ij)L} \right) - \bar{\lambda}_{R}^{j} \gamma^{\uparrow} \lambda^{j}_{R} \tag{61}
\]
Apart from the scalar \( h \), the masses of the gauge fields can be read off easily from the lagrangian \( \mathcal{L}_{\text{bos}} \) given by eq. (57); they read:
\[
m_{W}^{2} = \frac{4}{f^{2}} g_{10}^{2} M_{0}, \quad m_{A}^{2} = 40 g_{10}^{2} v^{2}, \quad M_{0} = \left( \frac{1}{2} f^{2} - 2|v|^{2} \right) = \frac{\mu_{1}^{2}}{10 g_{10}^{2}} > 0. \tag{62}
\]
By expanding the potential \( V_{\text{full}} \) to second order in \( \rho \) and \( \tilde{k} \) with scalar \( \rho \) defined by
\[
h = (v + \frac{1}{\sqrt{2}} \rho) e^{\frac{1}{\sqrt{2}} x i \alpha}, \tag{63}
\]
around the absolute minimum (55) we find
\[
V_{\text{full}} = V_{\text{uni}} + \Delta V = \frac{1}{2} m_{\rho}^{2} \rho^{2} + m_{k}^{2} \tilde{k}^{2} + \ldots, \tag{64}
\]
with \( m_{\rho}^{2} = 40 g_{10}^{2} v^{2} \) and \( m_{k}^{2} = \frac{1}{f^{2}} (\frac{3 \mu_{1}^{2}}{8} + \mu_{2}^{2}) \).

Next we construct the fermionic mass terms. The quadratic part of the lagrangian is
\[
\mathcal{L}_{\text{ferm}} = -\bar{\lambda}_{R} \gamma^{\uparrow} \lambda_{R} - \frac{1}{4} \left( \bar{\lambda}_{R}^{(ij)} \gamma^{\uparrow} \lambda_{(ij)R} + \bar{\lambda}_{L}^{(ij)} \gamma^{\uparrow} \lambda_{(ij)L} \right) - \bar{\lambda}_{R}^{j} \gamma^{\uparrow} \lambda^{j}_{R} - G_{\sigma_{(ij)}}^{(kl)} \tilde{\psi}_{L}^{(ij)} \gamma^{\uparrow} \tilde{\psi}_{(kl)L} - \tilde{\omega}_{L}^{j} \gamma^{\uparrow} \omega_{lL} - \tilde{\varphi}_{L}^{j} \gamma^{\uparrow} \varphi_{L} + 2 \sqrt{2} g_{10} G_{\sigma_{(ij)}}^{(kl)} \left[ \frac{1}{f} \bar{\lambda}_{R}^{(ij)} \psi_{L(kl)} + \text{h.c.} \right] + 2 \sqrt{2} g_{10} \left[ 2 \sqrt{5} v \bar{\lambda}_{R} \varphi_{L} + \text{h.c.} \right] \tag{65}
\]
As a result, two Dirac fermions are formed by combining the quasi-Goldstone fermions \( \psi_{L}^{[ij]} \) and \( \varphi_{L} \) with the right-handed gauginos \( \lambda_{R}^{[ij]} \) and \( \bar{\lambda}_{R} \) according to:
\[
\Psi = \bar{\lambda}_{R} + \varphi_{L}, \quad \Lambda^{[ij]} = \sqrt{M_{0} \psi_{L}^{[ij]}} + \frac{1}{2} \lambda_{R}^{[ij]} \tag{66}
\]
In terms of these fields, the fermionic lagrangian becomes
\[
\mathcal{L}_{\text{ferm}} = -\bar{\lambda} \gamma^{\uparrow} (\bar{\psi} - m_{A}) \lambda - \bar{\psi} \gamma^{\uparrow} (\bar{\psi} - m_{\psi}) \psi - \bar{\lambda}_{R}^{j} \gamma^{\uparrow} \lambda^{j}_{R} - \frac{1}{4} \bar{\lambda}_{R}^{(ij)} \gamma^{\uparrow} \lambda_{(ij)L} - \bar{\omega}_{L}^{j} \gamma^{\uparrow} \omega_{iL}, \tag{67}
\]
with the masses \( m_{A} = \frac{\sqrt{2} \mu_{1}}{\sqrt{5}} \) and \( m_{\psi} = 2 g_{10} v \sqrt{10} \). The \( \tilde{5} \) of the left-handed chiral fermions \( \omega_{iL} \), the \( \overline{10} \) of the left-handed gaugino’s \( \lambda_{L}^{[ij]} \), and the Majorana fermions
fermions

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{mass} & m^2_\Psi & m^2_{\lambda^i_{Lj}} & m^2_{\lambda^i_{Rj}} & m^2_\omega \\
\hline
\text{value} & 40 g_{10}^2 v^2 & \frac{2\mu_1}{5 f^2} & 0 & 0 & 0 \\
\hline
\end{array}
\]

Table 1: Fully gauged $SO(10)$ mass spectrum in the presence of soft supersymmetry
breaking.

$\lambda^i_{Rj}$ that are the gauginos of the unbroken $SU(5)$ symmetry remain massless. Notice here that in the limit $\mu_1^2, \mu_2^2 \to 0$ and $g_{10} = g_1$, one gets the same massive multiplets in the model with only gauged linear subgroup $SU(5) \times U(1)$ (see table 3). The only difference is, that in the case of gauged linear subgroup $SU(5) \times U(1)$ there are 20 massless Goldstone bosons ($\tilde{z}, \tilde{\zeta}$), and their superpartners ($\psi_L, \bar{\psi}_L$); and no gauge bosons ($\bar{W}, W$) of the 20 broken generators of $SO(10)$. (We have observed a similar thing to happen also in $E_6/SO(10) \times U(1)$ model discussed in the following section.) From the massive spectrum of the theory as summarized in the table 1, we obtain the general supertrace formula (38)

\[
\text{STr} m^2 = m^2_\rho + 2m^2_k + 3m^2_A + 6m^2_{W} - 4m^2_{\Psi} - 4m^2_\Lambda = \frac{1}{f^2} \left( \frac{4}{5} \mu_1^2 + 2 \mu_2^2 \right).
\]

Of course, the present theory cannot be regarded as complete. On the one hand, extra fermions must be coupled to the lagrangian (40) to represent the other families of quarks and leptons. Therefore the model must consist of (at least) three copies of $10, 5$ and $1$ of $SU(5)$ representations in its spectrum, of which one of the $10$ are Goldstone bosons of the coset space. On the other hand, since one must require the remaining $SU(5)$ symmetry to break down at lower energy to $SU(3) \times SU(2)_L \times U(1)$, additional interactions are required. For example, we can add the $24$ representation of $SU(5)$ to break $SU(5)$ down to smaller symmetry group, which can still accommodate at least unbroken $SU(3) \times U(1)$. However, the symmetry breaking in $SU(5)$-GUT via the $24$ ($\Phi$) that acquires a v.e.v. of the form

\[
\langle \Phi \rangle = \text{diag} \left( v, v, -\frac{3}{2} v, -\frac{3}{2} v \right),
\]

is problematic. This is because the Higgs-doublets and Higgs-triplets, originating from the $5$ and $\overline{5}$ representations will naturally have almost the same effective mass. Now these masses should be very large in order to avoid proton decay but on the
other hand small, else the standard model Higgses are far too heavy. This inconsistency is called the doublet-triplet-splitting problem. A way out of this problem is provided by the Dimopoulos-Wilczek mechanism [20] as is discussed in ref. [19], and recently by Witten [21]. Such an analysis of including other families of quarks and leptons as well as additional interactions to break $SU(5)$ down to the standard model gauge group is outside the scope of this paper and requires further development. For the moment we are satisfied with the observation that it is at least possible to cure some of the difficulties mentioned above for the present model with the scalar particle content summarized in table 2 in principle.

### 4.3 Gauging of the linear subgroup $SU(5) \times U(1)$

As an alternative to gauging $SO(10)$, one can gauge only the linear subgroup $SU(5) \times U(1)$ instead. This explicitly breaks the non-linear global $SO(10)$. It is then allowed in principle to construct superpotentials which are invariant only under the local gauge symmetry. In addition, when gauging any group containing the $U(1)$ as a factor, the introduction of a Fayet-Iliopoulos term is allowed. It turns out, that the

| Dimension repr. | U(1) charges | Notation | Description of the type of fields |
|------------------|--------------|----------|----------------------------------|
|                  |              | $z^{ij}$ | $SO(10)/[SU(5) \times U(1)]$ coset coordinates |
| 10               | 1            | $k_i$    | Matter additions to 10 |
|                  | -3           | $x^{ij}$ | to complete the 16 |
| 5                | 5            | $h$      | Second family |
| 1                |              | $a$      | Third family |
| $\overline{5}$   |              | $v_i$    | |
| 1                | -3           | $n_i$    | |
| 24               | 0            | $s^{ij}$ | Higgs for breaking the $SU(5)$ group to the standard model |
| 5                | -2           | $c_i$    | Higgses for breaking the $G_{SM}$ group to the $SU(3) \times U(1)$ |
| 5                | 2            | $c^j$    | |

Table 2: The various $SU(5)$ representations used for our construction of a phenomenological model build around $SO(10)/[SU(5) \times U(1)]$. The first column gives the dimension of the representations, the second column their charges, the third column the notation we use for the scalar components of chiral multiplets. A brief description of what these fields are is given in the last column.
corresponding models are indeed well-behaved for a range of non-zero values of this parameter.

As the $SU(5) \times U(1)$ subgroup of $SO(10)$ symmetry is not broken in the original $\sigma$-model, the Killing vectors corresponding to these symmetries are linear in the fields. The gauge covariant derivatives are then the usual one:

$$D_\mu h = \partial_\mu h - 2\sqrt{5}g_1\tilde{A}_\mu h, \quad D_\mu k = \partial_\mu k + g_5(V^T_\mu + \sqrt{5}\tilde{A}_\mu)k,$$

$$D_\mu \varphi_L = \partial_\mu \varphi_L - 2\sqrt{5}g_1\tilde{A}_\mu \varphi_L, \quad D_\mu \omega_L = \partial_\mu \omega_L + g_5(V^T_\mu + \sqrt{5}\tilde{A}_\mu)\omega_L. \quad (70)$$

To determine the physical realization and the spectrum of the theory, we have to minimize the potential (47). This potential has absolute minimum at zero if

$$|z|^2 = |k|^2 = 0, \quad |h|^2 = \frac{1}{4f^2} + \frac{1}{10}\xi = v^2, \quad -\frac{5}{2f^2} \leq \xi < 0. \quad (71)$$

This solution is supersymmetric and spontaneously breaks $U(1)$, whilst $SU(5)$ is manifestly preserved. As a result, the $U(1)$ gauge field $\tilde{A}_\mu$ become massive with a mass $m^2_{\tilde{A}_\mu} = m^2_\rho$, the mass of the real scalar $\rho$ defined by (63). The remaining vectors $V_\mu$ of $SU(5)$ stay massless. Of the gauginos, the right-handed components of the $U(1)$ gauge multiplet $\tilde{\lambda}_R$ combine with the left-handed chiral fermions $\varphi_L$ to become massive Dirac fermions with the same mass as the gauge boson $\tilde{A}_\mu$. However, the Majorana fermions $\lambda^i_{Rj}$ that are the gauginos of unbroken $SU(5)$ symmetry stay massless.

To see how this result is obtained in more detail, first notice that the mass term of the $U(1)$ vector field is generated through the kinetic terms by the v.e.v. of $h$, and reads

$$m^2_A = 40g_1^2 v^2 \quad (72)$$

Next we construct the kinetic terms and potential for the real scalar $\rho$; it reads

$$\mathcal{L}(\rho) = -\frac{1}{2}\left[\partial \rho \cdot \partial \rho - 40g_1^2 v^2 \rho^2\right] + ... \quad (73)$$

Table 3: Supersymmetric gauged $SU(5) \times U(1)$ mass spectrum

| Scalars | Vectors | Fermions |
|---------|---------|----------|
| Mass $m^2_\rho$ | Mass $m^2_{\tilde{A}_\mu}$ | Mass $m^2_\Psi$ |
| Value $40g_1^2 v^2$ | 0 | 0 |
| Mass $m^2_\Psi$ | Mass $m^2_{\omega_{L}}$ | Mass $m^2_{\omega_{L}}$ |
| Value $40g_1^2 v^2$ | 0 | 0 |
| Mass $m^2_{\omega_{L}}$ | Mass $m^2_{\omega_{L}}$ | Mass $m^2_{\omega_{L}}$ |
| Value $40g_1^2 v^2$ | 0 | 0 |

- Scalars
- Vectors
- Fermions
with \( \rho \) defined by equation (63). We then find that \( \rho \) represents a real scalar of mass \( m_\rho^2 = m_\lambda^2 \), the vector boson mass. Finally, the kinetic and mass terms for the fermion fields are given by equation (65) with \( g_{10} = g_1 \), but without the gauginos of the 20 broken generator of \( SO(10) \) (hence the terms involving \( \bar{\lambda}_R^{(ij)} \) are absent.)

\[
\mathcal{L}_{\text{ferm}} = -\bar{\lambda}_R \not{\partial} \lambda_R - \bar{\lambda}_R^{(ij)} \not{\partial} \lambda_R^{(ij)} - G_{\sigma(ij)}^{(kl)} \bar{\psi}^{(ik)}_L \not{\partial} \psi_{(kl)L} - \bar{\omega}_L \not{\partial} \omega_L - \bar{\varphi}_L \not{\partial} \varphi_L + 2\sqrt{2}g_1 \left[ 2\sqrt{5}v \bar{\lambda}_R \varphi_L + \text{h.c.} \right].
\] (74)

It follows that the Dirac spinor \( \Psi = \bar{\lambda}_R + \varphi_L \) satisfies the massive Dirac equation

\[
(\not{\partial} + m_\Psi) \Psi = 0,
\] (75)

with \( m_\Psi^2 = m_\lambda^2 = m_\rho^2 \). This establishes the presence of a massive vector supermultiplet \((A_\mu, \rho, \Psi)\) with mass squared given in table 3.

We end this section by remarking that one can also consider gauging either the \( U(1) \) (\( g_5 = 0 \)) or \( SU(5) \) (\( g_1 = 0 \)) symmetry. In the first case when gauging only the \( U(1) \) symmetry, the minimum potential is at the same point as in the \( SU(5) \times U(1) \) gauging. Therefore the above discussion applies here and one gets the same spectrum with equal masses for the \( U(1) \) gauge multiplet. On the other hand, if only \( SU(5) \) is gauged, the potential reaches its minimum at \( z = k = 0 \). Then no supersymmetry breaking or internal symmetry breaking occurs and all particles in the theory are massless.

5 Analysis of particle spectrum of \( E_6/\text{SO}(10) \times U(1) \) model

We turn our attention in this section to another well known model with a phenomenologically interesting particle spectrum, defined by the homogeneous coset space \( E_6/\text{SO}(10) \times U(1) \) [10, 11]. The target manifold \( E_6/\text{SO}(10) \times U(1) \) is parametrized by 16 complex fields \( z^{\alpha} \) in a chiral superfield \( \Phi_\alpha = (z_\alpha, \psi_{L\alpha}, H_\alpha) \) (\( \alpha = 1, \ldots, 16 \)), transforming as a Weyl spinor under \( SO(10) \). Their chiral fermion superpartners have the quantum numbers of one full generation of quarks and leptons, including a right-handed neutrino. To cancel the \( U(1) \)-anomaly the model is extended to a complete 27 of \( E_6 \). According to the branching rule: 27 \( \rightarrow \) 16(1) + 10(−2) + 1(4), where the numbers in parentheses denote the relative \( U(1) \) weights. With this choice of matter content, the cancellation of chiral anomalies of the full \( E_6 \) isometry group is achieved [3] by introducing a superfield \( \Psi_m = (N_m, \chi_{Lm}) \) (\( m = 1, \ldots, 10 \)) which is equivalent to a 10 of \( SO(10) \) with \( U(1) \) charge -2; and finally a singlet \( \Lambda = (h, \chi_L) \) of \( SO(10) \), with \( U(1) \) charge +4.

The anomaly-free supersymmetric \( \sigma \)-model on \( E_6/[\text{SO}(10) \times U(1)] \), is defined by three chiral superfields \((\Phi_\alpha, \Psi_m, \Lambda)\) with Kähler potential given by

\[
\mathcal{K}(\Phi, \bar{\Phi}; \Psi, \bar{\Psi}; \Lambda, \bar{\Lambda}) = K_\sigma + e^{-6f^2K_\sigma}|h|^2 + g_{mn}\bar{N}_mN_ne^{6f^2K_\sigma}, \tag{76}
\]
with $K_{\sigma} = \bar{z} [Q^{-1} \ln(1 + Q)] z$, the $\sigma$-model Kähler potential. We have introduced a constant $f$ with the dimension $m^{-1}$, determining the scale of symmetry breaking $E_6 \to SO(10) \times U(1)$. The positive definite matrix $Q$ is defined as

$$Q_{\alpha \beta} = \frac{f^2}{4} M_{\alpha \gamma} \bar{z}^\gamma z_\delta, \quad M_{\alpha \beta} = 3 \delta^\alpha_\gamma \delta^\beta_\delta - \frac{1}{2} \Gamma^+_{\beta} \Gamma^+_{\alpha \gamma}. \quad (77)$$

Here $\Gamma^+_{\beta} = \Gamma_{\beta \gamma} \delta_\gamma^\alpha + \delta_\gamma^\alpha \Gamma^+_{\beta \gamma} - \frac{1}{2} \Gamma^+_{\beta \gamma} \delta_\gamma^\alpha \Gamma^+_{\beta \gamma}$, and $\Gamma^+_{\beta} = \Gamma_{\beta \gamma} \delta_\gamma^\alpha + \delta_\gamma^\alpha \Gamma^+_{\beta \gamma} - \frac{1}{2} \Gamma^+_{\beta \gamma} \delta_\gamma^\alpha \Gamma^+_{\beta \gamma}$.

The lagrangian constructed from the Kähler potential (76) is invariant under a set of holomorphic Killing vectors generating a non-linear representation of $E_6$:

$$\delta z_\alpha = i \frac{\theta}{2} \sqrt{3} z_\alpha - \frac{1}{4} \omega_{mn} (\Gamma^+_{mn} \cdot z)_\alpha + \frac{1}{2} \left[ \frac{i}{f} \epsilon_\beta \delta^\beta_\alpha - \frac{i f}{4} \bar{z}^{\gamma} M_{\alpha \beta} z_\gamma \right],$$

$$\delta h = 2i \left( 3 \theta - 3 f \bar{\epsilon} \cdot z \right) h,$$

$$\delta N_n = -i \sqrt{3} \theta N_n - \omega_{mn} N_m - i f \bar{\epsilon} \cdot (\Gamma^+_{mn} - 3 \delta^+_{mn}) \cdot z N_m \quad (79)$$

where $\delta^+_{mn} = \delta_{mn} \delta^+ + \omega_{mn}$, and $\theta$, $\omega_{mn}$ and $\epsilon_\alpha$, $\bar{\epsilon}_\alpha$ are the infinitesimal parameters of the $U(1)$, $SO(10)$ and broken $E_6$ generators respectively. The corresponding Killing potentials are

$$M_\iota = M_\iota E - \frac{1}{8} e^{6K_{\sigma}} M_{\iota \alpha} g_\iota T^\delta (C^{\beta} \Sigma_m)^{\alpha \gamma} (\Sigma_n C)^{\beta \delta} \bar{N}_m N_n, \quad (80)$$

with $E$ and the $\sigma$-model Killing potentials $M_\iota = (M_\theta, M^{(mn)}, \bar{M}^\beta, M_\beta)$ given by

$$M_\theta = \frac{1}{f^2 \sqrt{3}} - \frac{1}{2} \sqrt{3} \bar{z}^\alpha K_{\sigma \alpha}, \quad M^{mn} = - \frac{i}{2} \bar{z}^\alpha \Gamma^+_{mn} \beta K_{\sigma \gamma},$$

$$\bar{M}^\beta = - \frac{1}{f} K_{\sigma \beta}, \quad M_\beta = - \frac{1}{f} K_{\sigma \beta}, \quad E = 1 - 6 e^{-6K_{\sigma}} |h|^2 + 6 e^{6K_{\sigma}} g_{mn} \bar{N}_m N_n \quad (81)$$

Observe the presence of the constant term in the $U(1)$ Killing potential $M_\theta$ which is required to close the Lie algebra on the Killing potentials.

### 5.1 The gauged model

Apart from the pure supersymmetric $\sigma$–model determined by this Kähler potential (76), we consider models in which (part of) the isometries (79) are gauged. As the $E_6$ is broken, the Higgs mechanism operates as follows: the Goldstone bosons $(\bar{z}^\alpha, z^\alpha)$ are absorbed in the longitudinal component of the charged vector bosons, and if the full $E_6$ is gauged, we may choose the unitary gauge $\bar{z}^\alpha = z^\alpha = 0$. To analyze the
model in this gauge, we introduce the covariant derivatives for the dynamical fields. The expressions for gauge-covariant derivatives of the complex scalar and fermions fields read

\[ D_\mu z_\alpha = \partial_\mu z_\alpha - g \left( \frac{i}{2} \sqrt{3} z_\alpha A_\mu + \frac{1}{4} (\Gamma^+_{mn})_\alpha A_\mu^{(mn)} + \frac{1}{2} \left( \frac{i}{f} A_{\alpha\mu} - \frac{if}{4} \bar{A}_\mu^{\alpha\beta} M_\alpha^{\gamma\delta} z_\gamma z_\delta \right) \right), \]

\[ D_\mu h = \partial_\mu h - 2ig \left( \sqrt{3} A_\mu - 3f a_\mu z_\alpha \right) h, \]

\[ D_\mu N_n = \partial_\mu N_n + ig \left( A_\mu N_n + g A_\mu^{(mn)} N_m + if g \bar{A}_\mu \cdot (\Gamma^+_{mn} - 3\delta^+_{mn}) \cdot z N_m \right), \]

\[ D_\mu \psi_L = \partial_\mu \psi_L - g \left( \frac{i}{2} \sqrt{3} A_\mu \psi_L + \frac{1}{4} A_\mu^{(mn)} \psi_L - \frac{if}{4} \bar{A}_\mu^{\alpha\beta} M_\alpha^{\gamma\delta} z_\gamma \psi_L \delta \right), \]

\[ D_\mu \chi_L = \partial_\mu \chi_L - 2ig \left( \sqrt{3} A_\mu \chi_L - 3f g f \bar{A}_\mu (\psi_L h + \chi_L z_\alpha) \right), \]

\[ D_\mu \chi_L n = \partial_\mu \chi_L n + g \left( 2i \sqrt{3} A_\mu \chi_L n + A_\mu^{(mn)} \chi_L m + if \bar{A}_\mu \cdot (\Gamma^+_{mn} - 3\delta^+_{mn}) \cdot (\psi_L N_m + \chi_L z_\alpha) \right). \]

Here we have introduced the notation \((A_\mu^\alpha, \bar{A}_\mu^\alpha)\) for the 32 charged gauge fields corresponding to the broken \(E_6\) transformations; \(A_\mu^{(mn)}\) and \(A_\mu\) are the gauge fields for the remaining \(SO(10)\) and \(U(1)\) transformations respectively.

We have now to add the kinetic terms for the vector multiplets. They are of the canonical form

\[ \mathcal{L}_{\text{gauge}} = -\frac{1}{2} \left( \lambda_R^{\alpha} \not{D} \lambda_R^\alpha + \bar{\lambda}_L^{\alpha} \not{D} \bar{\lambda}_L^\alpha \right) - \frac{1}{2} \lambda^{(mn)}_R \not{D} \lambda^{(mn)}_R - \bar{\lambda}_R \not{D} \lambda_R - \frac{1}{4} \left( F_{\mu\nu}^2 + \frac{1}{2} F_{\mu\nu}^{(mn)2} + F_{\mu\nu}^\alpha F_{\mu\nu\alpha} \right) + \frac{1}{2} \left( \bar{D}^\alpha D_\alpha + \frac{1}{2} D^{(mn)2} + D^2 \right) \]

Here we have included a factor \(\frac{1}{2}\) to correct for double counting due to anti-symmetry of the indices \(mn\).

Next we couple the gaugino fields to the quasi-Goldstone \(\psi_L^\alpha\) and matter fermions
$$(\chi_L^m, \chi_L)$$ through the Yukawa coupling

$$\mathcal{L}_{\text{Yuk}} = 2\sqrt{2}g G_{z_a z^a} \left[ -\frac{i}{2} \sqrt{3 \tilde{z}^a} \lambda_R - \frac{1}{4} (\tilde{z}^a \cdot \Gamma^{m+n}_{mn}) \lambda_R(mn) + i f \tilde{z}^a \tilde{z}^b M_{\gamma}^{2 \alpha} \lambda_R^{\gamma} \right] \psi_{\beta} + 2\sqrt{2}g G_{N_a N^a} \left[ (i \sqrt{3} \tilde{N}_m \tilde{\lambda}_R - \tilde{N}_i \lambda_R(ml) + i f \tilde{N}_m \tilde{z} \cdot (\Gamma^{m+n}_{mn} - 3 \delta^{m+n}_{mn}) \cdot \lambda_R) \chi_L \right]$$

Here $(G_{z_a z^a}, G_{N_a N^a}, \ldots)$ are the second mixed derivatives of the Kähler metric $G_{i\bar{J}} = K_{i\bar{J}L}$, where $I = (z_a, N_n, h)$ and $L = (z^a, N^a, \bar{h})$.

Finally, elimination of the auxiliary fields $(D^\alpha, D^{mn})$ from (83) leads to the scalar potential

$$V_D = \frac{g^2}{2} \sum_i |M_i|^2 = \frac{g^2}{2} \left( M_\theta^2 + \frac{1}{2} M_{mn}^2 + M_\alpha M_\beta \right).$$

### 5.1.1 Gauging of the full $E_6$ symmetry

In this section, we discuss in some detail the gauging of the full non-linear $E_6$. In this case as already stated, we can choose to study the model in the unitary gauge in which all the Goldstone bosons vanish: $z^a = \bar{z}_\alpha = 0$. This implies that the broken Killing potentials $M_\beta$ and $M_\alpha$ vanish automatically, leaving us with $SO(10)$ and $U(1)$ Killing potentials $M_\theta$ and $M_{mn}$:

$$M_\theta = \frac{1}{f^2 \sqrt{3}} - 2 \sqrt{3} |h|^2 + \sqrt{3} \sum_m |N_m|^2, \quad M_{mn} = -i \left( \tilde{N}_m N_n - \tilde{N}_n N_m \right).$$

Then the full potential becomes

$$V_{\text{unitary}} = \frac{g^2}{2} \left( \frac{1}{f^2 \sqrt{3}} - 2 \sqrt{3} |h|^2 + \sqrt{3} \sum_m |N_m|^2 \right)^2 + \frac{g^2}{2} \sum_{m,n} |\tilde{N}_m N_n - \tilde{N}_n N_m|^2.$$.

Observe here that in the unitary gauge, the potential contains only the terms that one also gets in gauging $SO(10) \times U(1)$. Minimization of the potential leads to the following set of supersymmetric minima characterized by the equation

$$|	ilde{N}_m N_n - \tilde{N}_n N_m|^2 = 0, \quad |h|^2 = \frac{1}{6f^2} + \frac{1}{2} \sum_m |N_m|^2.$$
The value of the potential vanishes: $\langle V \rangle = 0$, hence it is the absolute minimum of the potential. From (88), it follows that $|h| \neq 0$ and the $U(1)$ gauge symmetry is always broken; a solution with $|N_m| = 0$ is possible, preserving $SO(10)$. However, solutions with $|N_m| \neq 0$ breaking $SO(10)$ are allowed, and expected in the next stage of the symmetry breaking. For example, $SO(10)$ broken solution can be chosen as

$$f \tilde{N}_m = (0 0 0 0 0 0 0 0 0 \quad v_{10}), \quad |h|^2 = |v_h|^2 = \frac{1}{6} f^2 + \frac{v_{10}^2}{2 f^2}. \quad (89)$$

Since the complex scalar $N_m$ gets a vacuum expectation value; this breaks the internal linear $SO(10)$ symmetry, leaving only $SO(9)$. This shows that for gauged $E_6$, supersymmetry is always preserved, and therefore, one expects the spectrum of physical states fall into supersymmetric multiplets with vanishing mass supertrace.

$$\text{STr} \, m^2 = 2 g^2 G^{IJ} \mathcal{M}_I \mathcal{M}_{I,J} = 0. \quad (90)$$

As we have gauge the full $E_6$ the standard linear Fayet–Iliopoulos term is of course absent.

5.1.2 Softly broken supersymmetry

In this subsection we discuss the particle spectrum of the theory at the minimum with $SO(10)$ invariant solution:

$$|N_m|^2 = 0, \quad |h|^2 = \frac{1}{6} f^2. \quad (91)$$

This shows that the internal symmetry $SO(10) \times U(1)$ is broken to $SO(10)$. However, this solution is not acceptable by itself, as it leads to the to the vanishing of the metric of the $\sigma$–model fields $G_{\alpha \beta} = 0$ (and hence the masses of the 32 $E_6$ gauge fields $A_\alpha^a$ vanish) To see that in more detail, we first recall that the Kähler metric derived from the Kähler potential (76) in the unitary gauge reduces to the form:

$$G_{IL} = \mathcal{K}_{IL} = \begin{pmatrix} \delta_{\alpha \beta} \left( \frac{1}{f^2} - 6|h|^2 + 18 |N_m|^2 \right) & -4 \tilde{N}_m N_n (\Gamma^+_{mn})_{\alpha \beta} & 0 & 0 \\ 0 & \delta_{mn} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (92)$$

It is not difficult to see that at the minimum (91) the Kähler metric of the $\sigma$-model fields in the upper-left coner of (92) vanishes; and the four-fermion term $R_{i\alpha \beta} \bar{\psi}_R^\alpha \psi_L^\beta \bar{\chi}_L \chi_R$ diverge, just like in the $SO(10)/U(5)$–spinor model. Clearly, in this domain the model no longer correctly describes the physics of the situation (i.e., the correct vacuum and the corresponding spectrum of small fluctuations). Therefore we add soft breaking terms to shift the minimum away from the singular point, as we discussed in section 4. These terms involve mass terms of the form (37) for scalar fields $(N_m, h)$. We include an $E_6$–invariant soft supersymmetry breaking scalar mass term for the singlet $h$ and the vector $N_m$:

$$V_{\text{soft}} = \mu_1^2 e^{-6K} |h|^2 + \mu_2^2 g_{mn} \tilde{N}_m N_n e^{6f^2 K}, \quad (93)$$
The full scalar potential with soft breaking term in the unitary gauge is then:

\[ V = V_{\text{unitary}} + \mu_1^2 |h|^2 + \mu_2^2 |N_m|^2. \] (94)

As the complex scalar transforms only under \( U(1) \), we choose the unitary gauge for the \( U(1) \) symmetry, which allow us to write

\[ h = \left(v + \frac{1}{\sqrt{2}} \rho \right) e^{\frac{1}{\sqrt{2}} i \kappa}, \] (95)

where \( \kappa \) is the longitudinal component of the massive gauge field \( A_\mu \). We now determine the mass spectrum of the theory. Expanding the potential (94) to second order in the fluctuations \( \rho \) and \( \tilde{N}_m \) around the minimum

\[ |N_m|^2 = 0, \quad |h|^2 = v^2 = \frac{1}{6f^2} - \frac{\mu_1^2}{12g^2} \mu_2^2 < \frac{g^2}{f^2}, \] (96)

the bosonic terms in the action then become in the unitary gauge

\[
L_{\text{bos}} = -\frac{1}{4} F^2_{\mu \nu}(\tilde{A}) - \frac{1}{4} F^\alpha_{\mu \nu} F^\alpha_{\mu \nu} - \frac{1}{8} F_{(mn)}^{(mn)2} - \frac{1}{2} \partial \rho \cdot \partial \rho - \partial N_m \cdot \partial \tilde{N}_m \\
- \text{m}_{A}^2 \text{A}^a \cdot A^a - \text{m}_{A_n}^2 \text{A}_n^a \text{A}^a - \frac{\text{m}_{A_{mn}}^2}{2} \text{A}_{\mu (mn)}^2 - \frac{\text{m}_{N_m}^2}{2} \text{N}_m^2 - \text{V}_0 + \ldots,
\] (97)

In this expression, the dots represent interactions of the abelian vector field with the scalar \( \rho \). In addition, we have absorbed the Goldstone mode \( \kappa \) in the abelian vector by redefining the \( U(1) \) gauge field \( A_\mu \):

\[ A_\mu \rightarrow \tilde{A}_\mu = A_\mu - \frac{1}{2 \sqrt{6gv}} \partial _\mu \kappa. \] (98)

The masses of the bosonic fields read:

\[ m_A^2 = m_\rho^2 = 24g^2 v^2, \quad m_A^2 = \frac{\mu_1^2}{4f^2}, \quad m_{A_{mn}}^2 = \frac{1}{f^2} \left( \frac{1}{2} \mu_1^2 + \mu_2^2 \right), \quad m_{A_{nm}}^2 = 0. \] (99)

As expected the gauge bosons \( A_{\mu [mn]} \) of the non-broken \( SO(10) \) symmetry remain massless.

Analyzing the kinetic and mass terms of the fermions

\[
L_{\text{ferm}} = -G_\alpha^\beta \bar{\psi}_L^\alpha \gamma^\mu \psi_{L\beta} - \chi_L^a \gamma^\mu \lambda_{L\alpha} - \frac{1}{2} \left( \bar{\chi}_L^a \gamma^\mu \lambda_{L\alpha} + \bar{\lambda}_L^a \gamma^\mu \lambda_{L\alpha} \right) \\
- \frac{1}{2} \bar{\lambda}_{R (mn)}^a \gamma^\mu \lambda_{R (mn)}^a - \bar{\lambda}_R^a \gamma^\mu \lambda_R^a + \sqrt{2} g G_\alpha^\beta i f \left( \bar{\psi}_L^\alpha \gamma^\mu \lambda_{R \beta} - \bar{\lambda}_R^a \psi_{L \beta} \right) \\
+ 4i \sqrt{6vg} \left( \bar{\chi}_L \lambda_R - \bar{\lambda}_R \chi_L \right),
\] (100)
fermions

|         | $m_{\psi_{\alpha}}^2$ | $m_{\Omega}^2$ | $m_{\chi_n}^2$ | $m_{\lambda_{La}}^2$ | $m_{\lambda_{mn}}^2$ |
|---------|-----------------------|----------------|---------------|----------------------|---------------------|
| value   | $\frac{\mu^2}{2f^2}$ | $24g^2v^2$     | 0             | 0                    | 0                   |

vectors

|         | $m_A^2$ | $m_{A_{[mn]}}^2$ | $m_{A_{\alpha}}^2$ |
|---------|---------|-----------------|-------------------|
| value   | $24g^2v^2$ | 0               | $\frac{\mu^2}{4f^2}$ |

scalars

|         | $m_{\rho}^2$ | $m_{\tilde{N}}^2$ |
|---------|--------------|-------------------|
| value   | $24g^2v^2$   | $\frac{1}{f^2}\left(\frac{1}{2}\mu_1^2 + \mu_2^2\right)$ |

Table 4: Fully gauged $E_6$ mass spectrum in the presence of soft supersymmetric breaking.

one realizes that two massive Dirac fermions can be formed by combining the fermions of the chiral multiplets with two gauginos:

$$\Psi_{\alpha} = \frac{1}{\sqrt{2}} \lambda_{R\alpha} - i\sqrt{2} \frac{\mu_1}{2g} \psi_{La}, \quad \Omega = \lambda_R - i\chi_L. \quad (101)$$

In terms of these fields, the expression (100) becomes

$$\mathcal{L}_{\text{term}} = -\bar{\Psi}^a \gamma^\mu \Psi_{\alpha} - \bar{\Omega} \gamma^\mu \Omega + \sqrt{2} \frac{\mu_1}{f} \bar{\Psi}^a \Psi_{\alpha} + 4\sqrt{6} v g \bar{\Omega} \Omega. \quad (102)$$

The masses of these spinors are:

$$m_{\psi}^2 = \frac{\mu_1^2}{2f^2}, \quad m_{\Omega}^2 = 24g^2v^2. \quad (103)$$

The 16 of the left-handed gaugino’s $\lambda_{La}$ and quasi-Goldstone fermions $\chi_{Ln}$ remain massless, together with the Majorana fermions $\lambda^{mn}$ that are gauginos of the unbroken $SO(10)$ symmetry. Therefore, in this model the gaugino components $\lambda_{La}$ are now to be identified with a family of quarks and leptons, rather than the quasi Goldstone fermions themselves. (We have observed a similar thing to happen also in the $SO(10)/U(5)$–spinor model discussed in section 4.) The complete spectrum of the theory is summarized in table 4.

The conclusions that can be drawn from the above analysis may be summarized as follows. Gauging of the full $E_6$ in the presence of soft supersymmetry breaking may lead to a possibly realistic description of the lightest family of quarks and leptons. To make it fully realistic three important problems must be solved [22]:

1. How to break down the remaining $SO(10)$ symmetry, as required by low-energy phenomenology.

2. It should be possible to include (at least) three generations of quarks and leptons.
3. There should be a source of large Majorana masses, so that the see-saw mechanism provides the explanation for the small neutrino masses.

| Dimension repr. | U(1) charges | Notation | Description of the type of fields |
|-----------------|--------------|----------|----------------------------------|
| 16              | 1            | \( z_{\alpha} \) | \( E_6/[SO(10) \times U(1)] \) coset coordinates |
| 10              | -2           | \( N^m \) | Matter additions to 16 |
| 1               | 4            | \( h \) | to complete the 27 |
| 16              | \( q \)     | \( x_{\alpha}^+ \) | Two generations |
| 16              | \(-q\)      | \( x_{\alpha}^- \) | |
| 45              | 0            | \( A^{mn} \) | Higgses for the unification |
| 54              | 0            | \( S^{mn} \) | symmetry breaking |
| 210             | 0            | \( Q^{mnpq} \) | symmetry breaking |
| 126             | \( r \)     | \( D^{mnqr} \) | Higgses for neutrino Majorana masses |
| 126             | \(-r\)      | \( E_{mnqr} \) | and symmetry breaking |

Table 5: The various \( SO(10) \) representations used for our construction of a phenomenological model build around \( E_6/[SO(10) \times U(1)] \). The first column gives the dimension of the representations, the second column their charges, the third column the notation we use for the scalar components of chiral multiplets. A brief description of what these fields are is given in the last column. The charges \( q, r \) will be fixed by dynamical considerations like \( SO(10) \times U(1) \) anomaly cancellations and the requirement that various Yukawa couplings can appear in the superpotential.

Like in the \( SO(10)/U(5) \)–spinor model, these problems may be solved by adding additional matter multiplets. Let us start with the second problem in the list above. The 16 can accommodate one generation of quarks and leptons including the right-handed neutrino. Therefore we need at least three copies of this representation to account for three families. It would be economical (as far as the field content is concerned) to use a 16 both as a representation of quarks and leptons and as the representation that leads to the symmetry breaking \( SO(10) \longrightarrow SU(5) \times U(1) \longrightarrow SU(3) \times SU(2)_L \times U(1) \). Therefore a possible solution to the this problem is provided by adding the two other fermion families as additional matter multiplets \( \Phi_{\alpha}^\pm = (x_{\alpha}^\pm, \psi_{\alpha L}^\pm) \) carrying opposite \( U(1) \) charges so that that the internal symmetry is free of anomalies.

The first problem above can be solved by introducing the \( SO(10) \) breaking Higgs multiplets \( A^{mn}, S^{mn} \) and \( Q^{mnpq} \) with \( U(1) \) charges taken to be zero. This is not strictly necessary but very convenient in the following. The fermionic partners of the coset coordinates \( z_{\alpha} \) form one family of quarks and leptons, the other two family multiplets have scalar components \( x_{\alpha}^\pm \). We make the charge convention such that
has positive charge \( q \geq 0 \). Finally, we have two additional Higgses \( E_{mnpqr} \) and \( D_{mnpqr} \) that may also be responsible for symmetry breaking, but in addition are also supposed to give rise to Majorana masses for the right-handed neutrinos. \( D \) has charge \( r \) and \( E \) is its charge conjugate. In addition to all this there should be at least a 10 that can produce the supersymmetric standard model Higgses after symmetry breaking down to the standard model group \( SU(3) \times SU(2)_L \times U(1) \).

### 5.2 Gauging of \( SO(10) \times U(1) \) symmetry

The gauging of the \( SO(10) \times U(1) \) symmetry instead of the full \( E_6 \) gives analogous, but not quite identical, results. Also in this case one finds the potential (87), but in general with different values \( g_1 \) and \( g_{10} \) for the coupling constants of \( SO(10) \) and \( U(1) \). Except for special values of the parameters, it has a minimum for the \( SO(10) \) invariant solution, with \( z_\alpha = 0 \); and again the metric becomes singular. One way to shift the minimum away from this point is by introducing soft breaking terms (93). Another option is to add an extra Fayet-Iliopoulos term as the gauge group possesses an explicit \( U(1) \) factor. In the first case, the fermionic mass term is given by the last line of (100). As a result there is now one massive Dirac fermion, from the combination of \( \chi_L \) with the same gaugino of the broken \( U(1) \) as before. The gauginos \( \lambda^{mn} \) that are left over remain unpaired, and hence massless. Furthermore, the chiral fermions \( \psi^\alpha_L \) and \( \chi^n_L \) remain massless. The complete spectrum can be read from the table 6.

In the second case, for special values of the coupling constants \( g_1 \) and \( g_{10} \), or the Fayet-Iliopoulos parameter \( \xi \), one can get different results. Since the \( SO(10) \) and \( U(1) \) coupling constants are independent, one may choose to gauge only \( SO(10) \) \( (g_1 = 0) \). In that case both supersymmetry and internal symmetry are preserved, and the particle spectrum of a model contains of a massless \( SO(10) \) gauge boson, just like in the usual supersymmetric \( SO(10) \) grand unified models.

| Scalars | mass | \( m^2_\rho \) | \( m^2_{\chi_m} \) | \( m^2_{\xi^\alpha} \) |
|---------|------|----------------|----------------|----------------|
| value   | \( 24g_1^2 \) | 0              | 0              |                |

| Vectors | mass | \( m^2_A \) | \( m^2_{A_{mn}} \) | 0              |
|---------|------|-------------|-----------------|----------------|
| value   | \( 24g_1^2v^2 \) | 0           |                |                |

| Fermions | mass | \( m^2_\chi^\alpha \) | \( m^2_{\psi\chi_{L\alpha}} \) | \( m^2_{\chi_{Lm}} \) |
|----------|------|-----------------------|-----------------|----------------|
| value    | \( 24g_1^2v^2 \) | 0                     | 0               |                |

Table 6: Soft supersymmetry breaking gauged \( SO(10) \times U(1) \) mass spectrum
6 Conclusions

The Kähler manifolds $E_6/[SO(10) \times U(1)]$ and $SO(10)/SU(5) \times U(1)$ hold some special interest in the context of non-linear supersymmetric $\sigma$-models, because $E_6$, $SO(10)$ and $SU(5)$ are realistic grand unification groups. It was shown [3, 4] that it is possible to construct anomaly free models around these coset-spaces that are globally consistent.

In this article, we have discussed in detail the phenomenological analysis of supersymmetric $\sigma$-models on homogeneous coset-spaces $E_6/[SO(10) \times U(1)]$ and $SO(10)/U(5)$. We have analyzed the possible vacuum configurations of these models. We have investigated in particular the existence of the zeros of the potential, for which the models are anomaly-free, with positive definite kinetic energy. The consequences of these physical requirements have been analyzed. We found that there exist supersymmetric minima for both these models when the full isometry groups $E_6$ and $SO(10)$ are gauged. The analysis is straightforward as one can employ the unitary gauge to put the Goldstone bosons to zero. In some cases, we find that the Kähler metric is singular: the kinetic energy of the would-be Goldstone modes and their fermionic partners vanishes in the vacuum. We showed by addition of soft supersymmetry-breaking mass parameters, that the minimum can be shifted away from the singular point.

The particle spectrum in the presence of soft supersymmetry-breaking mass parameters is computed. The gauge bosons corresponding to the broken $E_6$ as well as the $SO(10)$ become massive, thereby eliminating all the Goldstone scalars from the theory. In addition some of the left-handed quasi-Goldstone fermions become massive by combining with right-handed gauginos corresponding to the broken ($SO(10)$, $E_6$) generators. The left-handed of these gauginos components remain massless and have the same quantum number as the original quasi-Goldstone fermions. Therefore, they can represent a family of quarks and leptons, with additional right-handed neutrino.

Continuing our line of investigation of the particle spectrum of supersymmetric $\sigma$-models on $E_6/[SO(10) \times U(1)]$, and $SO(10) \times U(1)$, we have also studied the possibility of gauging (part of) the linear subgroups, i.e., $SO(10) \times U(1)$ and $U(5)$. In each of these models, we found that the properties of the model investigated depend to a certain extent on the value of parameters (gauge couplings, Fayet-Iliopoulos term) and the presence of extra families and Higgses. We have obtained all supersymmetric minima, of which some are physically problematic as the kinetic terms of the Goldstone multiplets either vanish or have negative values.

In spite of all these nice features, there is still a lot of work needed to improve and extend the anomaly-free supersymmetric $\sigma$-models on the coset spaces $SO(10)/U(5)$ and $E_6/[SO(10) \times U(1)]$ discussed here. For example, it would be interesting to study their particle spectrum in presence of extra families and Higgses. In tables 2 and 5 we have summarized the most general scalar field content we consider for the phenomenological promising models build around the coset spaces $SO(10)/U(5)$ and $E_6/[SO(10) \times U(1)]$. 
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References

[1] B. Zumino “Supersymmetry and Kähler manifolds” Phys. Lett. B87 (1979) 203.

[2] D. Freedman and L. Alvarez-Gaumé “Geometrical structure and ultraviolet finiteness in the supersymmetric sigma model” Commun. Math. Phys. 80 (1981) 433.

[3] S. Groot Nibbelink and J.W. van Holten “Matter coupling and anomaly cancellation in supersymmetric σ-models” Phys. Lett. B442 (1998) 185-191; [hep-th/9808147].

[4] S. Groot Nibbelink T.S. Nyawelo and J.W. van Holten “Construction and analysis of anomaly-free supersymmetric SO(2N)/U(N) σ-models” Nucl. Phys. B594 (2001) 441-476; [hep-th/0008097].

[5] J. W. van Holten “Matter coupling in supersymmetric sigma models” Nucl. Phys. B260 (1985), 125.

[6] S. Groot Nibbelink “Line bundles in supersymmetric coset models” Phys. Lett. B473 (2000) (258); [arXiv:hep-th/9910075].

[7] S. Groot Nibbelink and J. W. van Holten “Consistent sigma models in N = 1 supergravity” Nucl. Phys. B588 (2000) 57; [hep-th/9903006].

[8] T.S. Nyawelo, F. Riccioni, S. Groot Nibbelink and J.W. van Holten ”Singular supersymmetric sigma models” Nucl.Phys.B663 (2003) 60-78; [hep-th/0302135].

[9] J. Bagger and E. Witten “The gauge invariant supersymmetric non-linear sigma model” Phys. Lett. B118 (1982) 103-106.

[10] Y. Achiman S. Aoyama and J.W. van Holten “Gauged supersymmetric sigma models and $E_6/[SO(10) \times U(1)]$” Nucl. Phys. B285 (1985) 179.

[11] Y. Achiman S. Aoyama and J.W. van Holten ”The non-linear supersymmetric sigma models on $E_6/[SO(10) \times U(1)]$” Phys. Lett. B141 (1984) 64.

[12] E. Cremmer, S. Ferrara, L. Girardello, and A. V. Proeyen ”Yang-Mills theories with local supersymmetry: lagrangian, transformation laws and super-Higgs effect” Nucl. Phys. B212 (1983) 413.
[13] M. T. Grisaru, F. Riva and D. Zanon "The one loop effective potential in superspace" Nucl. Phys. B214 (1983) 465.

[14] S. J. Gates, M. T. Grisaru, M. Rocek and W. Siegel “Superspace or one thousand and one lessons in supersymmetry” Front. Phys. 58 (1983) 1-548; [hep-th/0108200].

[15] M. T. Grisaru, M. Rocek and A. Karlhede “The super-Higgs effect in superspace” Phys. Lett. B120 (1983) 110.

[16] L. Girardello and M. T. Grisaru "Soft breaking of supersymmetry," Nucl. Phys. B194 (1982) 65.

[17] J. Wess and J. Bagger, Supersymmetry and supergravity Princeton Series in Physics, Princeton University Press, 1992.

[18] J. Gomis and S. Weinberg "Are nonrenormalizable gauge theories renormalizable" Nucl. Phys. B 469 (1996) 473 [hep-th/9510087].

[19] K. S. Babu and S. M. Barr “Natural suppression of Higgsino mediated proton decay in supersymmetric SO(10) ” Phys. Rev. D48 (1993) 5354; [hep-ph/9306242].

[20] S. Dimopoulos and F. Wilczek, NSF-ITP-82-07 (1982, unpublished).

[21] E. Witten “Deconstruction, G(2) holonomy, and doublet triplet splitting” [hep-ph/0201018].

[22] R. N. Mohapatra “Supersymmetric grand unification” [hep-ph/9801235].