A COHOMOLOGICAL BUNDLE THEORY FOR KOHANOV’S CUBE CONSTRUCTION

MIHAIL HURMUZOV

Abstract. The construction of the Khovanov homology of links motivates an interest in decorated Boolean lattices. Placing this work in the context of a bundle theory of presheaves on small categories, we produce, for a certain set of naturally occurring cases, a Leray-Serre type spectral sequence relating the bundle to the cohomology of the total sheaf.

INTRODUCTION

In [2] Everitt and Turner develop a bundle theory for coloured posets. In their development, coloured posets act as a generalisation of Khovanov’s ‘cube’ construction in his celebrated paper on the categorification of the Jones polynomial [5]. The notion of homology for coloured posets differs from the usual definition of sheaf homology, so it is desirable to rebuild the theory in a more versatile form. In this paper, we do away with coloured posets and move to the full generality of presheaves of modules on small categories. Additionally, we rework the arguments from the more natural cohomological point of view. The main result is as follows:

Main Theorem. Let $\xi : B \to \text{Sh}$ be a bundle of sheaves with $B$ a recursively admissible finite poset, and $(E, F)$ the associated total sheaf. Then there is a spectral sequence

$$E_2^{p,q} = H^p(B, \mathcal{H}_{fib}^q(\xi)) \Rightarrow H^*(E, F).$$

Bundles of sheaves and the total sheaf are defined in §1 while recursively admissible posets are defined in §5.

We proceed as follows. In §1 we define a category $\text{Sh}$ of (pre)sheaves on small categories that features morphisms between objects reminiscent of the induced maps in [1], p.4. A bundle is then just a small category decorated with objects and morphisms of $\text{Sh}$. We also give a way to ‘glue up’ the elements of $\text{Sh}$ in a bundle into a total sheaf. The main aim of the paper is to understand the relationship between this total sheaf and the bundle. In §2 we describe explicitly the cochain modules giving rise to the sheaf cohomology of a sheaf. This gives the concrete tools needed to establish the quasi-isomorphisms in the main theorem. Next, the general construction of a spectral sequence from a bicomplex provides the first step of the overarching argument – a spectral sequence, constructed from the bundle, that converges to a particular cohomology. The rest of the argument...

2020 Mathematics Subject Classification. Primary 55N30; Secondary 55T05, 05E45.

Key words and phrases. Sheaf cohomology, spectral sequence, bundle of sheaves, Khovanov homology.

The author would like to thank B. Everitt for suggesting the direction for this work and for numerous fruitful conversations thereafter.
is establishing that this particular cohomology coincides (in some naturally occurring cases) with
the usual sheaf cohomology of the total sheaf.

The chain map $\varphi$ that will witness this coincidence is defined in §4. Similarly to [2], it involves
signed combinations of traversals of a grid, determined by a pair of sequences in the base small
category and in one of the small categories over an object of the base. The cohomological viewpoint
here necessitates our $\varphi$ goes ‘the opposite way’ to that in [2]. The next section collects some technical
tools and the definition of a recursively admissible poset – the naturally occurring case in which the
main theorem holds. The bulk of the work is in §6 and §7 with the establishment of two explicit
quasi-isomorphisms, giving rise to two long exact sequence in the cohomologies of the total complex
and of the sheaf. This is done by careful manipulation of spectral sequences and morphisms between
them. The last section §8 collects the results into a proof of the main theorem.

1. The category $\mathbf{Sh}$

Unless noted otherwise, $R$ is a commutative ring with 1.

We define a category $\mathbf{Sh}$ of sheaves on small categories. An object $(P, F)$ of $\mathbf{Sh}$ consists of a
small category $P$ and a contravariant functor $F : P \to RMod$. A morphism $\gamma : (P, F) \to (Q, G)$
is a pair of maps $(\gamma_1, \gamma_2)$, where $\gamma_1 : Q \to P$ is a covariant functor and $\gamma_2 : F\gamma_1 \to G$ is a natural
transformation:

\[ P \xrightarrow{\gamma_1} Q \xrightarrow{\gamma_2} RMod \]

\[ \begin{array}{c}
\xymatrix{ F(\gamma_1(x)) & F(\gamma_1(y)) \\
G(x) & G(y) }
\end{array} \]

The composition of two morphisms $\gamma : (P, F) \to (Q, G)$ and $\delta : (Q, G) \to (R, H)$ is then
$(\gamma_1 \circ \delta_1, \delta_2 \circ \gamma_2) : (P, F) \to (R, H)$.

**Definition 1.1.** Let $B$ be a small category. A bundle of sheaves with base $B$ is a contravariant
functor $\xi : B \to \mathbf{Sh}$.

**Example 1.2.** A constant sheaf $\Delta A : P \to RMod$ is a sheaf where $\Delta A(x) = A$ for all $x \in P$ and
$\Delta A(x \to y) = \text{id}_A$ for all arrows $x \to y$.

A constant bundle $\xi = (B \times (P, F))$ is a bundle of sheaves where we have $\xi(x) = (P, F)$ for all
$x \in B$ and $\xi(x \to y) = \text{id}_{(P,F)}$ for all arrows $x \to y$.

For clarity, if $\xi$ is a bundle of sheaves with base $B$ and $x \in B$, then we will use $E_x$ for the small
category that is the first coordinate of $\xi(x)$. Similarly, $F_x$ is the second coordinate of $\xi(x)$. Also,
if $y \in E_x$, then $\pi(y) = x$. Finally, we use $\xi_1(x_1 \to x_2)$ for the first coordinate of the $\mathbf{Sh}$-morphism
$\xi(x_1 \to x_2)$ (instead of $\xi(x_1 \to x_2)_1$), similarly $\xi_2(x_1 \to x_2)$ for the second.

**Definition 1.3.** Let $B$ be a small category and $\xi$ a bundle of sheaves with base $B$. The associated
total sheaf $F : E \to RMod$ is defined as follows:
• As a small category,
\[ \text{Obj}(E) = \bigsqcup_{x \in B} \text{Obj}(E_x). \]

The simple arrows of \( E \) are of two types. There is an arrow \( y_1 \to y_2 \) in \( E \) if
(a) \( y_1, y_2 \in E_x \) for some \( x \in B \) and \( y_1 \to y_2 \) is an arrow in \( E_x \);
(b) \( y_1 \in E_{x_1}, y_2 \in E_{x_2} \) for some distinct \( x_1, x_2 \in B \), \( x_1 \to x_2 \) is an arrow in \( B \), and we have
\[ \xi_1(x_1 \to x_2)(y_1) = y_2. \]

The set of all arrows of \( E \) is the smallest set containing the simple arrows that is closed under composition, where arrows of type a) from \( E_x \) respect the composition in \( E_x \), and arrows of type b) (and identity arrows) respect the composition in \( B \). Additionally, we impose the commutativity of squares: if \( x_1 \to x_2 \) is an arrow in \( B \) and \( y_1 \to y_2 \) is an arrow in \( E_{x_1} \), then the below square commutes in \( E \):
\[
\begin{array}{c}
  y_2 \\
  \downarrow \\
  y_1 \\
  \downarrow \\
  x_1 \\
\end{array}
\quad \gamma(y_2) \\
\quad \gamma(y_1) \\
\quad x_2
\]

• As a sheaf, \( F \) sends an object \( y \in E \) with \( \pi(y) = x \) to \( F_x(y) \). Arrows \( y_1 \to y_2 \) of type a) in some \( E_x \) are sent to the map \( F_x(y_1 \to y_2) \); arrows \( y_1 \to y_2 \) of type b) with \( \pi(y_1) = x_1, \pi(y_2) = x_2 \) are sent to \( \xi_2(x_1 \to x_2)(y_1) \). Composition arrows are sent to the appropriate composition of the above maps.

**Proposition 1.4.** The pair \((E, F)\) above is an object of \( \text{Sh} \).

**Proof.** It is easily checked that \( E \) is a small category. Furthermore, since the action of \( F \) on composition arrows is defined as the composition of actions on simple arrows, functoriality of \( F \) follows from the functoriality of \( \xi \) and \( F_x \) for all \( x \in B \). \( \square \)

**Proposition 1.5.** For any composition arrow \( \sigma \) in \( E \), there is a type a) arrow \( \delta \) and a type b) arrow \( \varepsilon \) with \( \sigma = \delta \circ \varepsilon \).

**Proof.** Since compositions of arrows of type a) or b) are still arrows of the same type, a composition arrow in \( E \) is an alternating sequence of arrows of type a) and b). If \( \sigma \) starts with a type a) arrow and ends with a type b), then the reader can convince themselves of the conclusion of the proposition by considering the commutativity of squares in Definition 1.3 and the diagram in Figure 1. \( \square \)

## 2. Cohomology Modules of \((P, F)\)

A sheaf \( F \) on \( P \) acts as a covariant functor on the poset of simplices of the nerve \( NP \) of \( P \). For \( \sigma = x_0 \to x_1 \to \cdots \to x_i, \tau = x_{i_0} \to \cdots \to x_{i_k} \), we set
\[
F(\sigma) = F(x_0),
\]
\[
F(\tau \subseteq \sigma) = F(x_0 \to x_{i_0}) = F(x_{i_0}) \to F(x_0),
\]
where the arrow \( x_0 \to x_{i_0} \) is given by the appropriate composition of arrows in \( \sigma \).
The module for the $k$-cochains ($k \geq 0$) is

$$S^k(P, F) = \prod_{\sigma} F(\sigma),$$

where the product ranges over all $k$-simplices $\sigma$. For $k < 0$, $S^k(P, F) = 0$.

The differential $d^k : S^{k-1}(P, F) \to S^k(P, F)$ for $k > 0$ is given by

$$d^k|_{\sigma} = \sum_{j=0}^{k} (-1)^j F(\sigma_j \subseteq \sigma)(u|_{\sigma_j}),$$

where $\sigma = x_0 \to x_1 \to \cdots \to x_k$, $u \in S^k(P, F)$, and

$$\sigma_j = x_0 \to \cdots \to \hat{x}_j \to \cdots \to x_k.$$

For $k \leq 0$, $d^k = 0$. It is easily seen that $S^*(P, F)$ is a chain complex.

Given a morphism $\gamma : (P, F) \to (Q, G)$, there is an induced map

$$\gamma^* : S^*(P, F) \to S^*(Q, G)$$

defined by

$$\gamma^* u|_{\sigma} = \gamma_{2x_0}(u|_{\gamma_1(\sigma)}).$$

Lemma 2.1. The induced map $\gamma^*$ is a well-defined chain map.

Proof. This follows from an easy calculation, using the naturality of $\gamma_2$. \qed

We have thus defined a covariant functor

$$S^* : \text{Sh} \to \text{Ch}_R,$$

from pairs of small categories and sheaves to chain complexes over $R$, i.e. for each $q \in \mathbb{Z}$ we have $S^q : (P, F) \mapsto S^q(P, F)$. Now define the homology of a small category $P$ with coefficients in a sheaf $F$ to be

$$H^n(P, F) = H^n(S^*(P, F)).$$
Since homology is a functor from chain complexes to graded $R$-modules, we therefore have a covariant functor

$$H^* : \text{Sh} \to \text{Gr}_R \text{Mod}.$$  

In particular, for each $q \in \mathbb{Z}$ we have a covariant functor

$$H^q : \text{Sh} \to R\text{Mod}.$$  

Given a bundle $\xi : B \to \text{Sh}$, the above gives us two sheaves on $B$. For any $q \in \mathbb{Z}$ the $q$-cochain sheaf of $B$ is the sheaf $S^q : B \to R\text{Mod}$, i.e. the composition

$$B \xrightarrow{\xi} \text{Sh} \xrightarrow{S^q} R\text{Mod}.$$  

Similarly, the homology of the fibres sheaf of $B$ is the sheaf $\mathcal{H}^q_{\text{fib}} : B \to R\text{Mod}$, i.e. the composition

$$B \xrightarrow{\xi} \text{Sh} \xrightarrow{S^*} \text{Ch}_R \xrightarrow{\mathcal{H}^q} R\text{Mod}.$$  

Explicitly, if $x \in B$, then $\mathcal{H}^q_{\text{fib}}(x) = H^q(E_x, F_x)$.

3. The bicomplex $K^{p,q}$

We want to construct a bicomplex $K^{p,q}$ by taking the $p$-cochain sheaf of $(B, S^q)$.

Let $\xi : B \to \text{Sh}$ be a bundle of sheaves and suppose $x \to y$ is an arrow in $B$. We have the commutative square

$$\begin{array}{ccc}
S^{q-1}(E_x, F_x) & \xrightarrow{d} & S^q(E_x, F_x) \\
\downarrow & & \downarrow \\
S^{q-1}(E_y, F_y) & \xrightarrow{d} & S^q(E_y, F_y)
\end{array}$$

where the vertical maps are the chain map from Lemma 2.1 induced by $\xi(x \to y)$. In particular, the differential $d$ induces a morphism $\gamma : (B, S^{q-1}) \to (B, S^q)$, where $\gamma_1$ is the identity functor and $\gamma_2$ are the differentials at each object of $B$. This gives us the induced map

$$\gamma^* : S^*(B, S^{q-1}) \to S^*(B, S^q).$$

Applying this for all $q \in \mathbb{Z}$ we get the commutative grid

$$\begin{array}{ccc}
\cdots & \xrightarrow{d} & S^p(B, S^q) \\
\downarrow & & \downarrow \\
\cdots & \xrightarrow{d} & S^p(B, S^{q-1}) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}$$
To make the squares anti-commute instead we apply the usual ‘Jedi sign trick’, i.e. we include a factor of $-1$ in each other horizontal map. This gives us the desired bicomplex $K^{p,q}$. Explicitly, we have

$$K^{p,q} = S^p(B, S^q);$$

if we denote

$$\sigma = x_0 \to \ldots \to x_p \in NB \text{ and } \tau = y_0 \to \ldots \to y_q \in NE_{x_0},$$

then the vertical differential is $d^v : S^p(B, S^{q-1}) \to S^p(B, S^q)$, defined by

$$d^v(u)_{|\sigma,\tau} = F_{x_0}(y_0 \to y_1)(u_{|\sigma,\tau_0}) + \sum_{j=1}^q (-1)^j u_{|\sigma,\tau_j}$$

and the horizontal differential is $d^h : S^{p-1}(B, S^q) \to S^p(B, S^q)$, defined by

$$d^h(u)_{|\sigma,\tau} = (-1)^{p+q} (\gamma_{2y_0}(u_{|\sigma_0,\gamma_1(\tau)}) + \sum_{i=1}^p (-1)^i u_{|\sigma_j,\tau}),$$

where $\xi(x_0 \to x_1) = \gamma$.

We can place the modules $K^{p,q}$ on the $E_0$ page of a spectral sequence and use the vertical maps as the differentials on that page. We can further use the quotients of the horizontal maps for the differentials on the $E_1$ page.

**Proposition 3.1.** The $E_2$ page of the spectral sequence defined above has

$$E_2^{p,q} = H^p(B, \mathcal{H}^q_{fib}(\xi)).$$

**Proof.** Consider the following diagram

$$
\begin{array}{ccc}
\xi & \xrightarrow{S^*} & (B, S^*) \\
 & \downarrow \downarrow & \downarrow \downarrow \\
 & S^p(B, S^*) & H^*(S^p(B, S^*)) \\
 & \downarrow H^* & \downarrow S^p(B, H^*_{fib}(\xi)) \\
 & (B, \mathcal{H}^*_{fib}(\xi)) & S^p(B, \mathcal{H}^*_{fib}(\xi)) \\
\end{array}
$$

The top path is how we get the modules in a given column on the $E_1$ page – we take vertical homology of a column in $E_0$. On the other hand, taking horizontal homology of rows formed by $S^p(B, H^*_{fib}(\xi))$ clearly gives the required modules $H^p(B, \mathcal{H}^q_{fib}(\xi))$. It is then enough to show that the two graded modules at the ends of the two paths are equal for each $p \in \mathbb{Z}$. This follows directly from cohomology commuting with the direct product. \qed

Now, there is a total complex $(T^*, d)$ associated to $K^{p,q}$, where

$$T^n(E, F) := \bigoplus_{p+q=n} K^{p,q},$$

with $d = d^h + d^v$. Then, from the general construction of a spectral sequence from a bicomplex and from the above proposition, we have the following proposition.
Figure 2. An example of a traversal of a $(\sigma, \tau)$ grid with a 3-long $\sigma$ and a 2-long $\tau$.

**Proposition 3.2.** If $\xi : B \to \text{Sh}$ is a bundle of sheaves with total sheaf $(E, F)$, then there is a spectral sequence

$$E_2^{p,q} = H^p(B, \mathcal{H}_f^q) \Rightarrow H^*(\mathcal{T}^*(E, F)).$$

4. Grid traversals

We want to define a chain map $\varphi : S^*(E, F) \to \mathcal{T}^*(E, F)$. To do that, to each pair $(\sigma, \tau)$ (where $\sigma \in NB, \tau \in NE_{x_0}$) we will associate a (signed) combination of all traversals of a particular grid in $E$.

To form this grid, we lay out $\sigma$ and $\tau$ and complete the grid using the morphisms $\xi(x_i \to x_{i+1})$:

\[
\begin{array}{c}
\vdots \\
y_0.q \to y_1.q \to \cdots \to y_p.q \\
\uparrow & \uparrow & \uparrow \\
\vdots & \vdots & \vdots \\
y_{0,0} \to y_{1,0} \to \cdots \to y_{p,0} \\
\end{array}
\]

\[
\sigma = x_0 \to x_1 \to \cdots \to x_p,
\]

where $y_{0,j} = y_j$ and $y_{i+1,j} = \xi_1(x_j \to x_{i+1})(y_{i,j})$.

If $\sigma = x_0 \to \cdots \to x_p \in NB$, $\tau = y_0 \to \cdots \to y_q \in NE_{x_0}$, then a grid traversal $z \in NE$ of the grid of $(\sigma, \tau)$ is a $p + q$-long chain of arrows in the grid. In particular, each arrow in $z$ is either $\xi_1(x_0 \to x_i)(y_j \to y_{j+1})$ or $y_{i,j} \to \xi_1(x_i \to x_{i+1})(y_{i,j})$. Note that these correspond to type a) and type b) in Definition 1.3.

For each grid traversal $z$ of the grid of $(\sigma, \tau)$ (with $\sigma$ and $\tau$ as above), define

$$m(z) = \sum_{y_{i,j} \to y_{i,j+1} \in z} p - i.$$ 

Intuitively, $m(z)$ counts how many squares in the grid are below and to the right of $z$. Finally, define $\iota(q) = \left\lceil \frac{q}{2} \right\rceil = \min \left\{ n \in \mathbb{Z} \mid n \geq \frac{q}{2} \right\}$.
We can now define the chain map we are interested in. If \( u \in S^*(E, F) \), we have
\[
\varphi(u)|_{\sigma, \tau} = (-1)^i(q) \sum_z (-1)^{m(z)} u|_z,
\]
where the sum is taken over all traversals \( z \) of the grid of \((\sigma, \tau)\).

**Proposition 4.1.** The map \( \varphi : S^*(E, F) \rightarrow T^*(E, F) \) defined above is a chain map.

**Proof.** We need to show that
\[
\varphi ds|_{\sigma, \tau} = d\varphi u|_{\sigma, \tau}
\]
for all appropriate \( d, \sigma, \) and \( \tau \).

For the rest of this proof we allow a slight abuse of notation – in cases where the head of a chain of arrows is deleted, we will write \( u|_{\sigma_0} \) instead of \( F(x_0 \rightarrow x_1)(u|_{\sigma_0}) \).

If \( \sigma = x_0 \rightarrow \cdots \rightarrow x_p \) and \( \tau = y_0 \rightarrow \cdots \rightarrow y_q \), writing out the various formulae gives
\[
\varphi ds|_{\sigma, \tau} = (-1)^i(q) \sum_z (-1)^{m(z)} \sum_{i=0}^{p+q} (-1)^i s_{z_i},
\]
\[
d\varphi u|_{\sigma, \tau} = \sum_{r=0}^{p} (-1)^r \varphi u|_{\sigma_r, \tau} + (-1)^{p+q} \sum_{t} \varphi u|_{\sigma, \tau_t}
\]
\[
= \sum_{r=0}^{p} \sum_{\tilde{z}} (-1)^{r+m(\tilde{z})+i(q)} u|_{\tilde{z}} + \sum_{t=0}^{q} \sum_{\hat{z}} (-1)^{p+q+t+m(\hat{z})+(q-1)} u|_{\hat{z}},
\]
where \( z \) traverses \((\sigma, \tau)\), \( \tilde{z} \) traverses \((\sigma_r, \tau)\), and \( \hat{z} \) traverses \((\sigma, \tau_t)\).

Now, *a priori* there are more summands in \( \varphi ds|_{\sigma, \tau} \). The extra summands arise from deleting the corners of traversals:

\[
z : \quad \begin{array}{c}
\rightarrow \\
\end{array} \quad \sim \quad \begin{array}{c}
\rightarrow \\
\end{array} \quad \sim \quad \begin{array}{c}
\rightarrow \\
\end{array} \quad z_i : \quad \begin{array}{c}
\rightarrow \\
\end{array}
\]

But all of these corners come in pairs – a lower-right and an upper-left. The difference in squares below and to the right in the grid for these paired traversals is exactly one, and so \( m(z) \) is of the opposite parity. Thus the summands corresponding to paired corner-cuts cancel out in \( \varphi ds|_{\sigma, \tau} \).

We are left with two cases – when \( z_i \) is shortened along a vertical stretch and when it is shortened along a horizontal stretch.

**Case 1:** Suppose \( z_i \) is shortened along a vertical stretch of \( z \):

\[
z : \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \sim \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \sim \quad \begin{array}{c}
\uparrow \\
\end{array} \quad z_i : \quad \begin{array}{c}
\uparrow \\
\end{array}
\]

\[
z : \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \sim \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \sim \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \begin{array}{c}
\uparrow \\
\end{array}
\]

\[
z : \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \sim \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \sim \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \begin{array}{c}
\uparrow \\
\end{array}
\]

\[
z : \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \sim \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \sim \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \begin{array}{c}
\uparrow \\
\end{array}
\]

\[
z : \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \sim \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \sim \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \begin{array}{c}
\uparrow \\
\end{array}
\]

\[
z : \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \sim \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \sim \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \begin{array}{c}
\uparrow \\
\end{array}
\]

\[
z : \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \sim \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \sim \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \begin{array}{c}
\uparrow \\
\end{array}
\]

\[
z : \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \sim \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \sim \quad \begin{array}{c}
\uparrow \\
\end{array} \quad \begin{array}{c}
\uparrow \\
\end{array}
\]
The traversal $\hat{z}$ matching $z_i$ in $d\varphi u|_{\sigma,\tau}$ appears when $\tau$ is shortened at $t$. The coefficient of $z_i$ is $(-1)^{\nu(q)+m(z)+i}$ and the coefficient of the matching traversal is $(-1)^{p+q+m(\hat{z})+\nu(q-1)}$. There are $p-r$ squares in the grid to the right of any of the arrows pictured. This means that

$$m(z) = m(\hat{z}) + p - r.$$ 

Also note that $i = t + r$. We have

$$\nu(q) + m(z) + i + p + q + t + m(\hat{z}) + \nu(q-1) =$$
$$= \nu(q) + \nu(q-1) + 2m(\hat{z}) + 2p + i + r + t \equiv$$
$$\equiv \nu(q) + \nu(q-1) + q + 2i$$
$$\equiv 0, \mod 2$$

thus the two coefficients are the same.

Case 2: Suppose $z_i$ is shortened along a horizontal stretch of $z$:

The traversal $\tilde{z}$ matching $z_i$ in $d\varphi u|_{\sigma,\tau}$ appears when $\sigma$ is shortened at $r$. The coefficient of $z_i$ is $(-1)^{\nu(q)+m(z)+i}$ and the coefficient of the matching traversal is $(-1)^{r+m(\hat{z})+\nu(q)}$. There are $t$ squares in the grid below any of the arrows pictured. This means that

$$m(z) = m(\hat{z}) + t.$$ 

Again note that $i = t + r$. We have

$$\nu(q) + m(z) + i + r + m(\hat{z}) + \nu(q) =$$
$$= 2\nu(q) + 2m(\hat{z}) + i + r + t \equiv$$
$$\equiv 2i$$
$$\equiv 0, \mod 2$$

thus the two coefficients are the same. $\square$

5. Technical Tools

The particular proof presented here depends on restricting the small categories to finite posets, i.e. small categories with a finite set of objects, where the arrows $\to$ form a partial order $\leq$ with a unique arrow $x \to y$ for each $x \leq y$. If $B$ is a poset and $x, y \in B$, we say that $y$ covers $x$ (denoted $x < y$) if, whenever $z \in B$ is such that $x \leq z \leq y$, we have $z = x$ or $z = y$. We also say that a poset $B$ has a 0 if $B$ has a unique minimal element $0 \in B$. Denote

$$B(x) := \{z \in B \mid x \leq z\} \text{ and } \overline{B}(x) := B \setminus B(x).$$

Note that both $B(x)$ and $\overline{B}(x)$ inherit the poset structure of $B$, so are subposets of $B$. Further note that if $\xi : B \to \text{Sh}$ is a bundle of sheaves and for each $x \in B$ the small category $E_x$ is a poset, then the small category $E$ of the total sheaf $(E, F)$ associated to $\xi$ is also a poset.

We will also require a certain technical condition on our posets.
Definition 5.1. A poset $B$ is called admissible if $B$ has a 0 and there is an $x \succ 0$ such that for any $y \in B(x)$ the set

$$B(x, y) := \{ z \in B(x) \mid y \leq z \}$$

has a unique minimum.

A poset $B$ is called recursively admissible if $B$ has a 0 and either

- $B$ is boolean of rank 1, or
- $B$ is admissible for some $x \succ 0$ and both $B(x)$ and $\overline{B}(x)$ are recursively admissible.

Now, if we have a bundle of sheaves $\xi : B \to \mathbf{Sh}$ and a subcategory $C$ of $B$, we can restrict the bundle $\xi$ to $C$ to obtain another bundle $\xi_C : C \to \mathbf{Sh}$ with total sheaf $(E_C, F)$. Note that we use $(E_x, F_x)$ for the sheaf $\xi(x)$ when $x$ is an object of $B$, which coincides with $(E_C, F)$ when $C$ is the subcategory of $B$ consisting only of $x$ and its identity arrow.

Lemma 5.2. Let $B$ be an admissible poset for some $x \succ 0$ and $\xi : B \to \mathbf{Sh}$ be a bundle of sheaves with total sheaf $(E, F)$. Assume further that for each $i \in B$ the small category $E_i$ is a poset. Then for all $y \in E_{\overline{B}(x)}$, the subposet

$$\{ z \in E_{B(x)} \mid y \leq z \}$$

has a unique minimal element.

Proof. Since $y \in E_{\overline{B}(x)}$, $y$ is an element of a particular $E_i$ for some $i \in B(x)$. By the admissibility of $B$, that means that the poset $B(x, i)$ has a unique minimum, say $j$. Then $i \leq j$ and there is an arrow $i \to j$ in $B$. Denote the sheaf morphism given by this arrow as $\gamma$. By the construction of the total sheaf, we have that $y \leq \gamma_1(y)$.

Suppose $y \leq z$ for some $z \in E_{B(x)}$ and suppose $z \in E_k, k \in B(x)$. Then by Proposition 1.5 we have a $z_0 \in E_k$ with $y \leq z_0 \leq z$ and an arrow $i \to k$ giving rise to a sheaf morphism $\gamma'$. Since $j$ is the minimal element of $B(x, i)$, we have that $j \leq k$. But there is a unique arrow $i \to k$, so $\gamma'_1$ factors through $E_j$ and the sheaf morphism given by $j \to k$ maps $\gamma_1(y)$ to $z_0$. But this means that $\gamma_1(y) \leq z_0 \leq z$, therefore $\gamma_1(y)$ is the minimum of the set $\{ z \in E_{B(x)} \mid y \leq z \}$. 

$\square$
For constant bundles over certain posets we have a calculation of the cohomology of the total complex. We will need the Mapping Lemma for spectral sequences.

**Definition 5.3** ([4], p.320). A morphism \( f : E \to E' \) of spectral sequences is a collection of maps \( f_{r}^{p,q} : E_{r}^{p,q} \to E'_{r}^{p,q} \) (for sufficiently large \( r \)) with \( d_{r}f_{r} = f_{r}d'_{r} \) such that \( f_{r+1} \) is the map induced by \( f_{r}^{p,q} \).

**Definition 5.4** ([4], p.321). A spectral sequence \( E \) is bounded below if for each degree \( n \) there is an integer \( s = s(n) \) such that \( E_{p,q}^{0} = 0 \) when \( p < s \) and \( p + q = n \).

**Lemma 5.5** (Mapping Lemma, [4], p.321). Let \( f : E \to E' \) be a morphism of spectral sequences and suppose for some \( r \) that \( f_{r}^{p,q} : E_{r}^{p,q} \to E'_{r}^{p,q} \) is an isomorphism for each \( p \) and \( q \). Then \( f_{s}^{p,q} : E_{s}^{p,q} \to E'_{s}^{p,q} \) is also an isomorphism for each \( p \) and \( q \) when \( s \geq r \). If \( E \) and \( E' \) are bounded below, then \( f_{\infty}^{p,q} : E_{\infty}^{p,q} \to E'_{\infty}^{p,q} \) is also an isomorphism.

**Proposition 5.6.** Suppose \( B \) is a poset, \( x \in B \) is a unique minimum, and \( (P, F) \) is a sheaf. If \( \xi = (B \times (P, F)) \) is a constant bundle (recall Example 1.2), then there is a chain map \( \alpha : S^{*}(P, F) \to T^{*}(E, F) \) such that
\[
H^{*}(P, F) \overset{\alpha^{*}}{\cong} H^{*}T^{*}(E, F),
\]
where \( (E, F) \) is the total sheaf associated with \( \xi \) and \( \alpha^{*} \) is the map \( \alpha \) induces on cohomology.

**Proof.** First consider the constant sheaf \( (P, A) \), where \( P \) is a poset with a unique minimum. Denote by \( u_{a} \) the constant element of \( S^{0}(P, A) \) with \( u_{a}|_{x} = a \in A \) for all \( x \in P \). Recall that
\[
H^{n}(P, A) \cong \begin{cases} A, & \text{if } n = 0, \\ 0, & \text{otherwise.} \end{cases}
\]
It is easy to check that the map \( \beta^{n} : H^{*}(P, A) \to S^{*}(P, A) \) given by
\[
\beta^{n}(u) = \begin{cases} u_{a}, & \text{if } n = 0 \text{ and } u = a, \\ 0, & \text{otherwise} \end{cases}
\]
is a quasi-isomorphism. Note that the map \( -\beta^{*} \) is also a quasi-isomorphism.

Returning to the case of the constant bundle \( \xi = (B \times (P, F)) \), define \( \alpha : S^{n}(P, F) \to T^{*}(E, F) \) by
\[
\alpha u|_{\sigma,\tau} = \begin{cases} (-1)^{n}u|_{\tau}, & \text{if } \text{length}(\sigma) = 0, \\ 0, & \text{otherwise.} \end{cases}
\]
A straightforward calculation shows that \( \alpha \) is a chain map.

We define two bicomplexes \( C_{1}^{*,*} \) and \( C_{2}^{*,*} \). First define
\[
C_{1}^{p,q} = \begin{cases} S^{q}(P, F), & \text{if } p = 0, \\ 0, & \text{otherwise} \end{cases}
\]
and let $d_{C_1}^h = 0$, $d_{C_1}^v = 0$ on the non-zero columns, and $d_{C_1}^v = d_{S^* (P, F)}$ on the 0-th column.

\[
\begin{array}{c}
C_{1}^{*,*} : \\
\downarrow \\
0 \rightarrow S^1 (P, F) \rightarrow 0 \rightarrow \cdots
\end{array}
\begin{array}{c}
\downarrow \\
0 \rightarrow S^0 (P, F) \rightarrow 0 \rightarrow \cdots
\end{array}
\]

Now define

\[C_2^{p,q} = S^p (B, S^q)\]

and let the vertical and horizontal maps be as for the bicomplex construction in §3.

\[
\begin{array}{c}
\cdots \rightarrow S^p (B, S^{q+1}) \rightarrow S^{p+1} (B, S^{q+1}) \rightarrow \cdots \\
\uparrow \\
\cdots \rightarrow S^p (B, S^q) \rightarrow S^{p+1} (B, S^q) \rightarrow \cdots
\end{array}
\]

We want to show that $\alpha$ induces a morphism of these two bicomplexes. To that effect, we need three facts.

(a) First, it is clear that $\alpha(S^q (P, F)) \subseteq S^0 (B, S^q)$.

(b) Second, we need $\alpha$ to induce a chain map on the vertical complexes. This is clear for $p \neq 0$.

Consider the diagram

\[
\begin{array}{c}
\cdots \rightarrow S^q (P, F) \xrightarrow{d} S^{q+1} (P, F) \rightarrow \cdots \\
\downarrow \alpha \\
\cdots \rightarrow S^0 (B, S^q) \xrightarrow{d^v} S^{1} (B, S^{q+1}) \rightarrow \cdots
\end{array}
\]

We want to show that $d^v \alpha = \alpha d$. Let $u \in S^0 (B, S^{q+1})$, $x \in B$, $y_0 \leq \cdots \leq y_{q+1} \in P$.

\[
d^v \alpha u |_{x, y_0 \leq \cdots \leq y_{q+1}} = \sum_{i=0}^{q+1} \alpha u |_{x, y_0 \leq \cdots \leq y_i \leq \cdots \leq y_{q+1}}
\]

\[
= \sum_{i=0}^{q+1} u |_{x, y_0 \leq \cdots \leq y_i \leq \cdots \leq y_{q+1}}
\]

\[
= d u |_{y_0 \leq \cdots \leq y_{q+1}}
\]

\[
= \alpha d u |_{x, y_0 \leq \cdots \leq y_{q+1}}.
\]
Therefore $\alpha$ induces a chain map on vertical complexes.

(c) Finally, we need $\alpha$ to induce chain maps on horizontal complexes. Consider the diagram

$$
\cdots \to 0 \to S^q(P, F) \to 0 \to \cdots
$$

$$
\downarrow \alpha \quad \downarrow 0
$$

$$
\cdots \to 0 \to S^0(B, S^q) \to S^1(B, S^q) \to \cdots
$$

If we denote $S^q(P, F) = A$, this is just an instance of the map $\beta$.

Now consider the two spectral sequences $E$ and $E'$ associated to the bicomplexes $C_1^{*,*}$ and $C_2^{*,*}$, respectively. The map $\alpha$, as a morphism of bicomplexes, acts as a morphism $E \to E'$ of spectral sequences. Note also that both $E$ and $E'$ are bounded below. The first pages of $E$ and $E'$ are as follows.

$$
\cdots \to H^1(P, F) \to 0 \to \cdots
$$

$E_1$:

$$
\cdots \to H^0(P, F) \to 0 \to \cdots
$$

$$
\cdots \to S^0(B, \mathcal{H}^1_{fib}) \to S^1(B, \mathcal{H}^1_{fib}) \to \cdots
$$

$E'_1$:

$$
\cdots \to S^0(B, \mathcal{H}^0_{fib}) \to S^1(B, \mathcal{H}^0_{fib}) \to \cdots
$$

As in the case of a constant bundle, the induced maps $\alpha$ are quasi-isomorphisms on the horizontal complexes. This means that $\alpha$ induces isomorphisms on the second pages of $E$ and $E'$. By the Mapping Lemma (Lemma 5.5), we have that

$$
\alpha : E_{\infty}^{p,q} \cong E'_{\infty}^{p,q}.
$$

By the above, the construction of the total complex of a bicomplex, and Proposition 3.2, we can conclude that

$$
\alpha : H^*(P, F) \xrightarrow{\cong} H^*T^*(E, F). \quad \square
$$

We will need to build spectral sequences around filtrations. A (cohomological) filtration $J$ of a chain complex $C^*$ is a family of subcomplexes $J^pC^*$ such that for each $n$

$$
\cdots \supseteq J^p-1C^n \supseteq J^pC^n \supseteq J^{p+1}C^n \supseteq \cdots
$$

If $J_1$ and $J_2$ are filtrations of $C_1^*$ and $C_2^*$, respectively, a morphism $\alpha : (J_1, C_1) \to (J_2, C_2)$ of filtered complexes is a chain map $\alpha : C_1^* \to C_2^*$ with $\alpha(J_1^pC_1^n) \subseteq J_2^pC_2^n$ for each $p$ and $n$, that induces chain maps $J_1^pC_1^n \to J_2^pC_2^n$ for each $p$.

A filtration $J$ of $C^*$ is convergent below if $\bigcup_p J^pC^n = C^n$ for all $n$ and is bounded above if for each $n$ there is an integer $s = s(n)$ such that $J^sC^n = 0$. 

Lemma 5.7 (Mapping Lemma for filtrations, [4], p.331). Let \( J_1, J_2 \) be convergent below and bounded above filtrations of \( C_1^*, C_2^* \), respectively. If a morphism of filtrations \( \alpha : (J_1, C_1) \to (J_2, C_2) \) is such that for some \( t \) the induced map

\[
\alpha^t : E_t(J_1, C_1) \cong E_t(J_2, C_2)
\]

is an isomorphism, then \( \alpha \) induces an isomorphism \( H^*(C_1) \to H^*(C_2) \).

6. LONG EXACT SEQUENCE IN THE COHOMOLOGY OF THE TOTAL COMPLEX

Theorem 6.1. Let \( \xi : B \to \text{Sh} \) be a bundle of sheaves with \( B \) an admissible poset for \( x > 0 \) and \( E_y \) a poset for each \( y \in B \). Then there is a long exact sequence

\[
\cdots \to H^{n-1} \mathcal{T}^*(E_{\overline{U}(x)}, F) \to H^n \mathcal{T}^*(E, F) \to H^n \mathcal{T}^*(E_B(x), F) \oplus H^n \mathcal{T}^*(E_{\overline{B}(x)}, F) \to \cdots
\]

We will need to leverage the admissibility condition in the theorem to establish the connection between the total complex of the whole sheaf and those of the two smaller parts, determined by the element \( x > 0 \). Let \( \xi : B \to \text{Sh} \) be a bundle of sheaves with \( B \) an admissible poset for \( x > 0 \) and \( E_y \) a poset for each \( y \in B \).

Where possible, we will use \( x \)'s to refer to objects in \( \overline{B}(x) \) and \( z \)'s to refer to objects of \( B(x) \). We can write down explicitly what \( \mathcal{T}^n(E, F), \mathcal{T}^n(E_B(x), F), \) and \( \mathcal{T}^n(E_{\overline{B}(x)}, F) \) are:

\[
\mathcal{T}^n(E, F) = \bigoplus_{p+q=n} \prod_{x_0 \leq \cdots \leq x_p \in B} \prod_{y_0 \leq \cdots \leq y_q \in E_{x_0}} F_{x_0}(y_0).
\]

\[
\mathcal{T}^n(E_B(x), F) = \bigoplus_{p+q=n} \prod_{x_0 \leq \cdots \leq x_p \in B(x)} \prod_{y_0 \leq \cdots \leq y_q \in E_{x_0}} F_{x_0}(y_0).
\]

\[
\mathcal{T}^n(E_{\overline{B}(x)}, F) = \bigoplus_{p+q=n} \prod_{x_0 \leq \cdots \leq x_p \in \overline{B}(x)} \prod_{y_0 \leq \cdots \leq y_q \in E_{x_0}} F_{x_0}(y_0).
\]

Define the quotient map

\[
\rho : \mathcal{T}^n(E, F) \to \mathcal{T}^n(E_B(x), F) \oplus \mathcal{T}^n(E_{\overline{B}(x)}, F)
\]

by deleting any coordinate corresponding to a sequence \( x_0 \leq \cdots \leq x_p \in B \) that has objects in both \( B(x) \) and \( \overline{B}(x) \). Explicitly, if \( u \in \mathcal{T}^{p+q}(E, F), \sigma = x_0 \leq \cdots \leq x_p \in B(x) \) or \( \overline{B}(x) \), and \( \tau = y_0 \leq \cdots \leq y_q \in E_{x_0} \), then

\[
\rho u|_{\sigma, \tau} = u|_{\sigma, \tau}.
\]

The reader can convince themselves that \( \rho \) is a chain map. It is also clearly surjective, so we have an SES

\[
0 \to D^* \to \mathcal{T}^*(E, F) \to \mathcal{T}^*(E_B(x), F) \oplus \mathcal{T}^*(E_{\overline{B}(x)}, F) \to 0,
\]

for a particular chain complex \( D^* \).

We describe \( D^* \) explicitly:

\[
D^n = \bigoplus_{p+q=n} \prod_{x_0 \leq \cdots \leq x_p \in B(x)} \prod_{y_0 \leq \cdots \leq y_q \in E_{x_0}} F_{x_0}(y_0),
\]

where \( x_0 \in \overline{B}(x), x_p \in B(x) \).
We can rewrite $D^*$ to pay attention to how many of the $x$'s are in $\overline{B}(x)$ and how many are in $B(x)$:

$$D^n = \bigoplus_{s+j+q=n} \prod_{x_0 \leq \cdots \leq x_s \leq z_0 \leq \cdots \leq z_j} \prod_{y_0 \leq \cdots \leq y_q \in E_{x_0}} F_{x_0}(y_0),$$

where $x_i \in \overline{B}(x)$, $z_i \in B(x)$, $s \geq 0$, $j \geq 1$.

**Proposition 6.2.** Let $\xi : B \to \text{Sh}$ be a bundle of sheaves with $B$ an admissible poset for $x > 0$ and $E_y$ a poset for each $y \in B$. If $D^*$ is as above, there is a chain map

$$\alpha_1 : T^{n-1}(E_{\overline{B}(x)}, F) \to D^n,$$

that induces an isomorphism on cohomology.

**Proof.** In an attempt to keep the notation less cluttered, denote

$$K^n = T^{n-1}(E_{\overline{B}(x)}, F).$$

We define the chain map $\alpha_1 : K^n \to D^n$, which will extend to a morphism of filtered complexes. By showing that $\alpha_1$ induces isomorphisms on the first pages of the two spectral sequences associated to the two filtrations, we conclude that it is a quasi-isomorphism.

Let $\sigma = x_0 \leq \cdots \leq x_s \leq z_0 \leq \cdots \leq z_j$ be a sequence in $B$ with $x_i \in \overline{B}(x)$, $z_i \in B(x)$, $s \geq 0$, $j \geq 1$. Denote $\sigma' = x_0 \leq \cdots \leq x_s$. Also let $\tau = y_0 \leq \cdots \leq y_q$ be a sequence in $E_{x_0}$. Now if $s+j+q = n$, we define $\alpha : K^n \to D^n$ by

$$\alpha_1 u|_{\sigma, \tau} = \begin{cases} (-1)^q u|_{\sigma', \tau} & \text{if } j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Intuitively, $\alpha_1$ acts like the map $\alpha$ in Proposition 5.7 on the portion of $D^*$ that matches $T^*(E_{\overline{B}(x)}, F)$. The proof that $\alpha_1$ is a chain map is a direct calculation that is left to the reader.

Now we define the filtrations of $D^*$ and $K^*$:

$$J^p D^n = \{ u \in D^n \mid u|_{\sigma, \tau} \neq 0 \Rightarrow \sigma = x_0 \leq \cdots \leq x_s \leq z_0 \leq \cdots \leq z_{j-1} \text{ with } s \geq p \},$$

$$J^p K^n = \{ u \in K^n \mid u|_{\sigma, \tau} \neq 0 \Rightarrow \sigma = x_0 \leq \cdots \leq x_s \text{ with } s \geq p \}.$$  

We want to use the Mapping Lemma 5.7 for these two filtrations, so the next step is establishing all the assumptions of the lemma. We prove them for $J$ with the arguments for $J'$ being analogous. ($J$ is a filtration): It is clear from the definition of $J$ that $J^p D^n \geq J^{p+1} D^n$ for each $p$ and $n$. Remains to show that $J^p D^*$ is a complex for each $p$. Let $u \in J^p D^n$ and $\sigma = x_0 \leq \cdots \leq x_s \leq z_0 \leq \cdots \leq z_{j-1}$ with $s < p$. Then for any sequence $\tau \in E_{x_0}$ (of appropriate length $q$) we have

$$du|_{\sigma, \tau} = \sum_{i=0}^s (-1)^i u|_{\sigma, \tau} + (-1)^{s+1} \sum_{k=0}^{j-1} (-1)^k u|_{\sigma_{s+k}, \tau} + \sum_{l=0}^{q} (-1)^l u|_{\sigma, \tau_l}.$$  

The summands in the first sum correspond to $x$-sequences of length $s-1 < p$, while the summands in the other two sums correspond to $x$-sequences of length $s < p$. All those coordinates are 0 in $u \in J^p D^n$, so $d$ induces a differential on $J^p D^*$.

($J$ is convergent below): Observe that $J^p D^n = D^n$, since $D^n$ does not have any coordinates corresponding to sequences in $B$ not containing elements of $\overline{B}(x)$.

($J$ is bounded above): Observe that $J^n D^n = 0$, since we need $s + j + q = n$ and $j \geq 1$.  

(α₁ is a morphism of filtrations): Let \( u \in J^{p}K^n \), \( \sigma = x₀ ≤ \cdots ≤ x_s ≤ z \), and \( \tau = y₀ ≤ \cdots ≤ y_q \). First suppose \( s+q+1 \neq n \). The potentially non-zero coordinates of \( \alpha₁u|_{\sigma,\tau} \) correspond to sequences of combined length \( s+q \neq n-1 \), so they are also 0. Now suppose \( s < p \). Again, the potentially non-zero coordinates of \( \alpha₁u|_{\sigma,\tau} \) correspond to \( x \)-sequences of length \( s < p \), so are also 0. Thus \( \alpha₁(J^{p}K^n) \subseteq J^{p}D^n \).

To see that \( \alpha₁ \) induces chain maps \( J^{p}K^n \to J^{p}D^n \) for every \( p \) note that we already know that \( dα₁ = α₁d \) and that \( \alpha₁(J^{p}K^n) \subseteq J^{p}D^n \).

Let \( E, E' \) be the spectral sequences associated to the filtrations \( J, J' \), respectively. We have

\[
E^{p,q}_0 = \frac{J^{p}D^{p+q}}{J^{p+1}D^{p+q}} = \{ u \in D^{p+q} \mid u|_{\sigma,\tau} \neq 0 \Rightarrow \sigma = x₀ ≤ \cdots ≤ x_p ≤ z₀ ≤ \cdots ≤ z_{j-1} \},
\]

\[
E^{p,q}_0 = \frac{J^{p}K^{p+q}}{J^{p+1}K^{p+q}} = \{ u \in K^n \mid u|_{\sigma,\tau} \neq 0 \Rightarrow \sigma = x₀ ≤ \cdots ≤ x_p \}.
\]

The vertical differentials in \( E_0 \) are given by

\[
du|_{x₀ ≤ \cdots ≤ x_p, z₀ ≤ \cdots ≤ z_{j-1}, y₀ ≤ \cdots ≤ y_{q-j}} = (-1)^{p+1} \sum_{i=0}^{j-1} (-1)^i u|_{x₀ ≤ \cdots ≤ x_p, z₀ ≤ \cdots ≤ z_{i}, z_{i+1} ≤ \cdots ≤ z_{j-1}, y₀ ≤ \cdots ≤ y_{q-j} + \sum_{l=0}^{q-j} (-1)^{p+q} \sum_{z \leq \cdots \leq y_j} (-1)^{q-j} u|_{x₀ ≤ \cdots ≤ x_p, z₀ ≤ \cdots ≤ z_{j-1}, y₀ ≤ \cdots ≤ y_{q-j}}
\]

and the vertical differentials in \( E'_0 \) are given by

\[
du|_{x₀ ≤ \cdots ≤ x_p, y₀ ≤ \cdots ≤ y_q} = (-1)^{p+q} \sum_{l=0}^{q} (-1)^l u|_{x₀ ≤ \cdots ≤ x_p, y₀ ≤ \cdots ≤ y_q}
\]

Using the notation from Definition 5.3, we can thus rewrite

\[
E^{p,*}_0 = \prod_{x₀ ≤ \cdots ≤ x_p} (-1)^{p+1} T^{s-1}(B(x, x_p) \times (E_{x₀}, F_{x₀}))
\]

and

\[
E^{p,*}_0 = \prod_{x₀ ≤ \cdots ≤ x_p} (-1)^{p+q} S^{s-1}(E_{x₀}, F_{x₀}.
\]

Now note that \( \alpha₁ \) acts as the product over all \( p \)-long \( x \)-sequences in \( \overline{B}(x) \) of the maps in Proposition 5.6 since \( B \) is an admissible poset and thus the subposet \( B(x, x_p) \) has a unique minimum. This means that \( \alpha₁ : E^{p,*}_0 \to E^{p,*}_0 \) is a quasi-isomorphism and thus

\[
E^{p,q}_1 = H^p(E^{p,*}_0) \cong H^p(E^{p,*}_0) = E^{p,q}_0.
\]

The Mapping Lemma 5.7 then implies that

\[
H^{n-1} T^{s}(E^{p,*}_0, F) \cong \alpha₁ T^{s}(E^{p,*}_0, F) \cong H^{n}(D^{s}).
\]

We can now easily complete the proof of the theorem, headlined in this section.
Proof of Theorem 6.1. We have the SES from before
\[ 0 \to D^* \to T^*(E, F) \to T^n(E_B(x), F) \oplus T^n(E_{B(x)}, F) \to 0, \]
from which we get a long exact sequence in homology
\[
\cdots \to H^{n-1}T^*(E_B(x), F) \oplus H^{n-1}T^*(E_{B(x)}, F) \to H^n D^* \to \\
H^n T^*(E, F) \to H^n T^*(E_B(x), F) \oplus H^n T^*(E_{B(x)}, F) \to \\
H^{n+1} D^* \to \cdots
\]
Replacing the occurrences of \( H^n D^* \) with \( H^{n-1}T^*(E_B(x), F) \) and the maps around those occurrences with the appropriate compositions with \( \alpha^*_1 \) and \( \alpha^*_1 \) gives the required long exact sequence. \( \square \)

7. LONG EXACT SEQUENCE IN SHEAF COHOMOLOGY

Theorem 7.1. Let \( \xi : B \to \text{Sh} \) be a bundle of sheaves with \( B \) an admissible poset for \( x > 0 \) and \( E_y \) a poset for each \( y \in B \). Then there is a long exact sequence
\[
\cdots \to H^{n-1}(E^*_B(x), F) \to H^n(E, F) \to H^n(E_B(x), F) \oplus H^n(E_{B(x)}, F) \to \cdots
\]
We now repeat this procedure for the cochain complex of the total sheaf \((E, F)\). Where possible, we will use \( x \)'s to refer to objects in \( E_B(x) \) and \( z \)'s to refer to objects of \( E_B(x) \). We can write down explicitly what \( S^n(E, F), S^n(E_B(x), F), \) and \( S^n(E_{B(x)}, F) \) are:
\[
S^n(E, F) = \prod_{x_0 \leq \cdots \leq x_n \in E} F(x_0)
\]
\[
S^n(E_B(x), F) = \prod_{x_0 \leq \cdots \leq x_n \in E_B(x)} F(x_0)
\]
\[
S^n(E_{B(x)}, F) = \prod_{x_0 \leq \cdots \leq x_n \in E_{B(x)}} F(x_0)
\]
Define the quotient map
\[
\rho : S^n(E, F) \to S^n(E_B(x), F) \oplus S^n(E_{B(x)}, F)
\]
by deleting any coordinate corresponding to a sequence \( x_0 \leq \cdots \leq x_n \) in \( E \) that has objects in both \( E_B(x) \) and \( E_{B(x)} \). This is a chain map by an analogous argument to the one for the quotient before Proposition 6.2.

The map \( \rho \) is clearly surjective, so we have an SES
\[
0 \to M^* \to S^*(E, F) \to S^n(E_B(x), F) \oplus S^n(E_{B(x)}, F) \to 0,
\]
for a particular chain complex \( M^* \).

We describe \( M^* \) explicitly:
\[
M^n = \prod_{x_0 \leq \cdots \leq x_n} F(x_0),
\]
where \( x_0 \in E_{B(x)}, x_n \in E_B(x) \).
We can rewrite $M^*$ to pay attention to how many of the $x$'s are in $E_B(x)$ and how many are in $E_B(x)$:

$$M^n = \prod_{x_0 \leq \cdots \leq x_p \leq z_0 \leq \cdots \leq z_{n-p-1}} F(x_0),$$

where $x_i \in E_B(x), z_i \in E_B(x), p \geq 0, n - p \geq 1$.

**Proposition 7.2.** Let $\xi : B \to \text{Sh}$ be a bundle of sheaves with $B$ an admissible poset for $x > 0$ and $E_y$ a poset for each $y \in B$. If $M^*$ is as above, there is a chain map

$$\alpha_2 : S^{n-1}(E_B(x), F) \to M^n,$$

that induces an isomorphism on cohomology.

**Proof.** We define a filtration $J$ of $M^*$:

$$J^p M^n = \{ u \in M^n \mid u|_{\sigma} \neq 0 \Rightarrow \sigma = x_0 \leq \cdots \leq x_s \leq z_0 \leq \cdots \leq z_{n-s-1}, \text{ with } s \geq p \}.$$

The proof that this is a filtration is analogous to the proofs of the filtrations from Proposition 6.2.

Let $E$ be the spectral sequence associated to the filtration $J$ of $M$. We have

$$E_{p+1}^{0,q} = \frac{J^p M^{p+q}}{J^{p+1} M^{p+q}} = \{ u \in M^n \mid u|_{\sigma} \neq 0 \Rightarrow \sigma = x_0 \leq \cdots \leq x_p \leq z_0 \leq \cdots \leq z_{q-1} \}.$$

The vertical differentials in $E_0$ are given by

$$du|_{x_0 \leq \cdots \leq x_p \leq z_0 \leq \cdots \leq z_{q-1}} = (-1)^{p+1} \sum_{i=0}^{q-1} (-1)^i u|_{x_0 \leq \cdots \leq \hat{x}_i \leq \cdots \leq x_p \leq z_0 \leq \cdots \leq z_{q-1}}.$$  

We can thus write

$$E_0^{p,q} = \prod_{x_0 \leq \cdots \leq x_p} (-1)^{p+1} S^{q-1}(\{ z \in E_B(x) \mid z \geq x_p \}, \Delta F(x_0)).$$

But the $S$ complex on the right is of a constant sheaf. By Lemma 5.2 the underlying poset has a unique minimum, so

$$E_1^{p,q} = H^q E_0^{p,q} = \left\{ \begin{array}{ll} \prod_{x_0 \leq \cdots \leq x_p} (-1)^{p+1} F(x_0) & \text{if } q = 1, \\
0 & \text{otherwise.} \end{array} \right.$$  

So on the $E_1$ page we have a single row

$$\cdots \to (-1)^n S^{n-1}(E_B(x), F) \to (-1)^{n+1} S^n(E_B(x), F) \to \cdots.$$

The differential on this page is induced by the differential

$$du|_{x_0 \leq \cdots \leq x_p \leq z_0 \leq \cdots \leq z_{q-1}} = \sum_{i=0}^{p} (-1)^i u|_{x_0 \leq \cdots \leq \hat{x}_i \leq \cdots \leq x_p \leq z_0 \leq \cdots \leq z_{q-1}}.$$
which, since it keeps the \( z \)-sequence constant, induces the following differential on the above row on the \( E_1 \) page:

\[
du|_{x_0 \leq \cdots \leq x_p} = \sum_{i=0}^{p} (-1)^i u|_{x_0 \leq \cdots \leq z_i \leq \cdots \leq x_p}.
\]

Since \( d \circ (-d) = -d \circ d = 0 \), \( \ker(-d) = \ker d \), and \( \text{im}(-d) = \text{im} d \), we have that the \( E_2 \) page is

\[
E_2^{p,q} \cong \begin{cases} H^{p+q-1}(EB(x), F) & \text{if } q = 1, \\ 0 & \text{otherwise.} \end{cases}
\]

Then \( E_2^{p,q} \cong E_\infty^{p,q} \) and so

\[
E \Rightarrow H^{n-1}(EB(x), F) \cong M^n.
\]

In particular, this isomorphism is witnessed by a similar quasi-isomorphism to that in Proposition 6.2, namely \( \alpha_2 : S^{n-1}(EB(x), F) \to M^n \) defined by

\[
\alpha_2 u|_{x_0 \leq \cdots \leq x_n} = \begin{cases} 
0 & \text{if } x_n \in E_B(x), x_n \in E_B(x), \\
\sum_{z} (-1)^{m(z)} u|_{z}, & \text{if } x_n \in E_B(x), x_n \leq x_n-1 \in E_B(x), x_n \in E_B(x).
\end{cases}
\]

We can now, again, easily prove the headlined theorem.

**Proof of Theorem 7.1.** We have the SES from before

\[
0 \to M^* \to S^*(E, F) \to S^*(EB(x), F) \oplus S^*(EB(x), F) \to 0,
\]

from which we get a long exact sequence in homology

\[
\cdots \to H^{n-1}(EB(x), F) \oplus H^{n-1}(EB(x), F) \to H^n M^* \to \\
\to H^n(E, F) \to H^n(EB(x), F) \oplus H^n(EB(x), F) \to \\
\to H^{n+1}M^* \to \cdots
\]

Replacing the occurrences of \( H^n M^* \) with \( H^{n-1}(EB(x), F) \) and the maps around those occurrences with the appropriate compositions with \( \alpha_2^* \) and \( \alpha_2^{* -1} \) gives the required long exact sequence. \( \square \)

8. The bicomplex and the total sheaf

We have all the necessary prerequisites to prove the main theorem.

**Theorem 8.1.** Let \( \xi : B \to \text{Sh} \) be a bundle of sheaves with \( B \) a recursively admissible finite poset, and \( (E, F) \) the associated total sheaf. Then there is a spectral sequence

\[
E_2^{p,q} = H^p(B, H^q_f(\xi)) \Rightarrow H^*(E, F).
\]

**Proof.** Proposition 3.2 gives us

\[
E_2^{p,q} = H^p(B, H^q_f(\xi)) \Rightarrow H^*(E, F),
\]

so it is enough to show that \( H^* T^*(E, F) = H^* (E, F) \). We will do this by induction on the cardinality of \( B \). Recall the chain map from \( \Phi : S^*(E, F) \to T^*(E, F) : \)

\[
\Phi u|_{\sigma, \tau} = (-1)^{t(q)} \sum z (-1)^{m(z)} u|_{z},
\]
where the sum is taken over all traversals \( z \) of the grid of \((\sigma, \tau)\). We have two short exact sequences from Theorems 6.2 and 7.2. The map \( \varphi \) gives a morphism of these short exact sequences

\[
0 \longrightarrow M^n \xrightarrow{\varepsilon} S^n(E, F) \xrightarrow{\pi} S^n(E_{\overline{B}(x)}, F) \oplus S^n(E_B(x), F) \longrightarrow 0
\]

\[
0 \longrightarrow D^n \xrightarrow{\varepsilon} T^n(E, F) \xrightarrow{\pi} T^n(E_{\overline{B}(x)}, F) \oplus T^n(E_B(x), F) \longrightarrow 0
\]

where the maps \( \varepsilon \) are the injections and the maps \( \pi \) the projections of the respective modules. The map \( \varphi' \) is the quotient of \( \varphi \) on the quotient spaces \( M^n \) and \( D^n \). A trivial calculation shows the commutativity of the two squares.

The naturality of the homology functor then gives a morphism of long exact sequences, which contains the commutative diagram in Figure 3.

Recall from Propositions 6.2 and 7.2 the quasi-isomorphisms

\[
\alpha_1 : T^{n-1}(E_{\overline{B}(x)}, F) \to D^n \quad \text{and} \quad \alpha_2 : S^{n-1}(E_{\overline{B}(x)}, F) \to M^n.
\]
Claim. The following diagram commutes
\[
\begin{array}{ccc}
S^{n-1}(E_{B(x)}, F) & \xrightarrow{\alpha_2} & M^n \\
\varphi \downarrow & & \downarrow \varphi' \\
T^{n-1}(E_{B(x)}, F) & \xrightarrow{\alpha_1} & D^n
\end{array}
\]

Proof of claim. Let \( u \in S^{n-1}(E_{B(x)}, F) \). Suppose
\[
\sigma = x_0 \leq \cdots \leq x_s \leq z_0 \leq \cdots \leq z_{j-1}, \tau = y_0 \leq \cdots \leq y_q
\]
with \( s + j + q = n \). If \( j > 1 \), it is clear that
\[
\alpha_1 \varphi u|_{\sigma, \tau} = 0 = \varphi' \alpha_2 u|_{\sigma, \tau},
\]
since each summand of \( \varphi' \alpha_2 u|_{\sigma, \tau} \) is 0 under \( \alpha_2 \).

If \( j = 1 \), let \( \sigma' = x_0 \leq \cdots \leq x_s \). Then we have
\[
\varphi' \alpha_2 u|_{\sigma, \tau} = (-1)^{\ell(q)} \sum_{z'}(-1)^{m(z')} \alpha_2 u|_{z'},
\]
where the sum is taken over the traversals \( z' \) of \( (\sigma, \tau) \).

Pick a traversal \( z' \) of \( (\sigma, \tau) \). We zoom in on the top right of the grid of \( (\sigma, \tau) \).

\[
\cdots \quad y'_1 \rightarrow y'_2 \\
\quad \uparrow \\
\quad \vdots
\]

Note that \( y'_0, y'_2 \in E_{z_0} \). If \( z' \) passes through \( y'_0 \), then \( \alpha_2 u|_{z'} = 0 \). If \( z' \) passes through \( y'_1 \), then \( \alpha_2 u|_{z'} = u|_{z} \), for a particular traversal \( z \) of \( (\sigma', \tau) \). Moreover, in this second case there are exactly \( q \) many squares in the rightmost column that are in the count for \( m(z') \), so \( m(z') = q + m(z) \). Therefore we have
\[
\varphi' \alpha_2 u|_{\sigma, \tau} = (-1)^{\ell(q)} \sum_{z'}(-1)^{m(z')} \alpha_2 u|_{z'} = (-1)^{\ell(q)} \sum_{z}(-1)^{m(z)+q} u|_{z}
\]
\[
= (-1)^q(-1)^{\ell(q)} \sum_{z}(-1)^{m(z)} u|_{z} = (-1)^q \varphi u|_{\sigma', \tau} = \alpha_1 \varphi u|_{\sigma, \tau}. \quad \square
\]

We can then form the augmented commutative diagram in Figure 4.

The two columns are exact since, by Propositions 6.2 and 7.2, the maps \( \alpha_1^* \) and \( \alpha_2^* \) are isomorphisms. The squares commute by the commutativity of the diagram from the morphism of long exact sequences and the claim.

We finish the proof by induction on the cardinality of \( B \). If \( |B| = 1 \), then
\[
T^n(E, F) = S^0(B, S^n) = \prod_{x \in B} S^n(E_x, F) = S^n(E, F),
\]
and \( \varphi = (-1)^{\ell(q)} \text{id} \), so it is a quasi-isomorphism.
Figure 4. Augmented commutative diagram, where the instances of $H^*M^*$ and $H^*D^*$ are replaced.

If $\varphi : S^n(E, F) \to T^n(E, F)$ is a quasi-isomorphism for $|B| < i$, then for $|B| = i$ we can form the above commutative diagram. Each row other than the middle one contains an instance of the inductive hypothesis, since $|E_B(x)|, |E_B(x)| < |B|$ and $B$ is recursively admissible. Therefore, by the Five Lemma, the middle row is an isomorphism and thus $\varphi$ is a quasi-isomorphism. This completes the induction and the proof of the theorem.

□

References

[1] Brent Everitt and Paul Turner, *Cellular cohomology of posets with local coefficients*, J. Algebra 439 (2015), 134–158. MR3373367

[2] , *Bundles of coloured posets and a Leray-Serre spectral sequence for Khovanov homology*, Trans. Amer. Math. Soc. 364 (2012), no. 6, 3137–3158. MR2888240

[3] , *Homology of coloured posets: a generalisation of Khovanov’s cube construction*, J. Algebra 322 (2009), no. 2, 429–448. MR2529096

[4] Saunders MacLane, *Homology*, 1st ed., Springer-Verlag Berlin Heidelberg, 1995.

[5] Mikhail Khovanov, *A categorification of the Jones polynomial*, Duke Math. J. 101 (2000), no. 3, 359–426. MR1740682
Department of Mathematics, University of York, York YO10 5DD, England
E-mail address: mh1937@york.ac.uk