ON THE GL(3) KUZNETSOV FORMULA WITH APPLICATIONS TO
SYMMETRY TYPES OF FAMILIES OF L-FUNCTIONS

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Abstract. We present an explicit approach to the GL(3) Kuznetsov formula. As an application, for a restricted class of test functions, we obtain the low-lying zero densities for the following three families: cuspidal GL(3) Maass forms \( \phi \), the symmetric square family \( \text{sym}^2 \phi \) on GL(6), and the adjoint family \( \text{Ad} \phi \) on GL(8). Hence we can identify their symmetry types; they are: unitary, unitary, and symplectic, respectively.

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1. Introduction

1.1. Symmetry Types.

In [KS99], Katz and Sarnak introduced the notion of symmetry type for a family of \( L \)-functions. Since then there has been a slew of activity regarding the following problem: given a family of \( L \)-functions, determine its symmetry type. A common approach to this determination, as outlined in [Sar08], is to analyze the density of low-lying zeros in the specified family, for test functions whose Fourier transforms have restricted support. Such an analysis has been carried out in many places, including [ILS00, Roy01, Gul05, You06, HM07, AIL+11] for GL(2) and [DM06] for some GL(4) and GL(6) families; see also [DM09].
In all cases in the literature (going beyond \textit{GL}(1) or abelian methods), the analysis involves a version of the \textit{GL}(2) Petersson/Kuznetsov formula.

The purpose of this paper is to carry out a similar analysis, for the first time using the \textit{GL}(3) Kuznetsov formula. Assuming the generalized Riemann hypothesis (to interpret the low-lying zero densities) and the generalized Ramanujan conjectures (for ease of exposition), we will determine symmetry types for the families of

(1) cuspidal \textit{GL}(3, \mathbb{Z}) automorphic forms $\phi$,

(2) the \textit{GL}(6) symmetric square family $\text{sym}^2 \phi$, and

(3) the \textit{GL}(8) adjoint family $\text{Ad} \phi$.

We will show that the symmetry types are:

(1) unitary,

(2) unitary, and

(3) symplectic,

respectively.

The methods presented here are capable of wide generalization, in particular, it should be possible to determine the symmetry types of families associated to pairs of cuspidal automorphic representations on \textit{GL}(n) for any $n \geq 2$. We hope to return to this topic in a future publication.

To state our results more precisely, we need some background.

1.2. Hecke-Maass forms.

Let $G = \text{GL}(3, \mathbb{R})$ with maximal compact $K = O(3)$ and center $Z = \mathbb{R}^\times$, let $\mathfrak{h}^3 = G/(K \cdot Z)$ be the generalized upper half plane, and take the lattice $\Gamma := \text{GL}(3, \mathbb{Z})$ in $G$.

The algebra of $G$-invariant differential operators acts on $\mathcal{H} := L^2(\Gamma \backslash \mathfrak{h}^3)$. The Hecke-Maass forms $\phi_j$ constitute an orthogonal (Hecke normalized) basis for

$$\mathcal{H}_0 := \bigoplus_{j=1}^{\infty} \mathbb{C} \phi_j \subset \mathcal{H},$$

where $\mathcal{H}_0$ is the cuspidal subspace in the Langlands spectral decomposition [Gol06, Prop. 10.13.1]

$$\mathcal{H} = \mathbb{C} \mathbf{1} \oplus \mathcal{H}_0 \oplus \mathcal{H}_{\text{min}} \oplus \mathcal{H}_{\text{max}} \oplus \mathcal{H}_{\text{res}}.$$  

Here $\mathcal{H}_{\text{min}}$, $\mathcal{H}_{\text{max}}$, and $\mathcal{H}_{\text{res}}$ are, respectively, the spans of integrals of the minimal and maximal parabolic Eisenstein series, and the residual spectrum.

Let the Hecke-Maass form $\phi_j$ have spectral parameters $\nu^{(j)} := (\nu_1^{(j)}, \nu_2^{(j)})$. When discussing a fixed form $\phi$, we drop the superscripts $(j)$. Our normalization$^1$ is such that for a tempered form, $\nu_1$ and $\nu_2$ are purely imaginary.

$^1$Note that our normalization differs from that used in [Gol06] by $\nu_j \mapsto 1/3 + \nu_j$. 
It is convenient to also introduce the spectral parameters

$$
\nu_3 := \nu_1 + \nu_2,
$$

and

$$
\alpha_1 := \nu_1 + \nu_3, \quad \alpha_2 := -\nu_1 + \nu_2, \quad \alpha_3 := -\nu_2 - \nu_3.
$$

Writing $\lambda_\phi$ for the Laplace eigenvalue of $\phi$, we have

$$
\lambda_\phi = 1 - 3(\nu_1^2 + \nu_2^2 + \nu_3^2) = 1 - (\alpha_1^2 + \alpha_2^2 + \alpha_3^2).
$$

Weyl’s Law in this setting [Mil01] states that

$$
\# \{ \phi : \lambda_\phi < T^2 \} \sim c T^5,
$$
as $T \to \infty$, for some constant $c > 0$.

1.3. The GL(3) Kuznetsov Formula.

We will state and use the GL(3) Kuznetsov formula with some naturally occurring weights, defined as follows. For $j = 1, 2, 3, \ldots$, let

$$
\mathcal{L}_j := \text{Res}_{s=1} L(s, \phi_j \times \tilde{\phi}_j)
$$
be the residue at the edge of the critical strip of the $L$-function attached to $\phi_j \times \tilde{\phi}_j$; generically this is the value at $s = 1$ of $L(s, \text{Ad} \phi_j)$.

We introduce an absolute constant $R \geq 10$, which is needed for certain technical reasons, see the estimates in §4.2. For $T \gg 1$, we define

$$
h_{T,R}(\nu) := e^{(\alpha_1^2 + \alpha_2^2 + \alpha_3^2)/T^2} \left( \prod_{1 \leq j \leq 3} \Gamma \left( \frac{2+R+3\nu_j}{4} \right) \Gamma \left( \frac{2+R-3\nu_j}{4} \right) \right)^2 \left( \prod_{1 \leq j \leq 3} \Gamma \left( \frac{1+3\nu_j}{2} \right) \Gamma \left( \frac{1-3\nu_j}{2} \right) \right)^{1/2}. \tag{1.1}
$$

In fact, $R$ is needed to enable us later to pull contours in certain integrals with respect to the $\nu_j$’s without passing through poles of the numerator in (1.1). Note that $h_{T,R}(\nu) > 0$, and is essentially supported on $\lambda < T^2$, or $|\nu_1|, |\nu_2|, |\nu_3| \ll T$. In this range, one sees from Stirling’s formula that if $\nu$ is tempered, then

$$
h_{T,R}(\nu) \sim c_R \left( (1 + |\nu_1|)(1 + |\nu_2|)(1 + |\nu_3|) \right)^R, \tag{1.2}
$$
for some $c_R > 0$. The non-tempered forms constitute a zero density set [Mil01].

Let $A_j(n_1, n_2)$ denote the coefficients of $\phi_j$ in the Fourier-Whittaker expansion, see §2.3.
Theorem 1.3. With the above notation and assuming the Ramanujan conjecture at the infinite place, we have the “Weyl Law”, that for some $c > 0$,
\[
\sum_j h_{T,R}(\nu(j)) \frac{L_j}{L_j} \sim c T^{5+3R}.
\] (1.4)

Moreover for fixed $\varepsilon > 0$, $R \geq 10$, $n_1, n_2, m_1, m_2 \in \mathbb{Z}_{\geq 1}$, and $T \gg 1$, we have
\[
\sum_j A_j(m_1, m_2) A_j(n_1, n_2) h_{T,R}(\nu(j)) \frac{L_j}{L_j} = \begin{cases} 
\sum_j h_{T,R}(\nu(j)) + O_{R,\varepsilon} \left( T^{3+3R+\varepsilon} \left| m_1 m_2 n_1 n_2 \right|^2 \right), & \text{if } m_1 = n_1, m_2 = n_2, \\
O_{R,\varepsilon} \left( T^{3+3R+\varepsilon} \left| m_1 m_2 n_1 n_2 \right|^2 \right), & \text{otherwise}.
\end{cases}
\] (1.5)

Remark 1.6. In light of the asymptotic formula (1.2), the analytic weight $h_{T,R}$ can be removed with a modicum of effort; we have chosen to leave the weight for ease of exposition. The same is done for GL(2) in [AIL+11].

Remark 1.7. The weight $L_j$ is more subtle; it is shown in [Blo11, (1.4)] that
\[
C_{\nu(j)}^{-1} \ll L_j \ll C_{\nu(j)}^\varepsilon,
\]
where
\[
C\nu = (1 + |\nu_1|)(1 + |\nu_2|)(1 + |\nu_3|).
\]
Moreover if one assumes the functorial transfer predicting $\phi \times \tilde{\phi}$ is automorphic on GL(9), then using the non-existence of Siegel zeros for the corresponding $L$-function [HR95], one can improve the lower bound above to $C_{\nu}^{-\varepsilon}$. With this assumption, the weight can be removed completely, as in [Luo01], giving rise to a clean cut-off.

Remark 1.8. We have not made any attempt to obtain the best possible error terms in (1.5). In particular, we have made no use of stationary phase, nor have we even invoked Deligne’s bounds for Kloosterman sums (see e.g. [BFG88, Larsen’s appendix]). We tried to present as simple a method as we could, keeping in mind the eventual goal of generalizing these techniques to GL($n$) with $n \geq 2$.

Remark 1.9. A similar result is obtained in [Blo11]. Blomer first chooses a test function on the geometric side, and then executes a delicate analysis to obtain implications on the spectral side. In our approach, we choose the test function on the spectral side first, making the asymptotic formula (1.2) immediately visible. In a private communication, Blomer has informed us that from the methods in [Blo11], he can also obtain (1.4) and (1.5) (with a better error term) for a range of test functions.
1.4. Low-Lying Zeros.

For a Hecke-Maass form $\phi$ on $GL(3)$, let $\rho(\phi)$ be one of $$\rho(\phi) = \begin{cases} 
\phi \\
\text{sym}^2 \phi \\
\text{Ad} \phi,
\end{cases}$$
and let $L(s, \rho(\phi))$ be the corresponding $L$-function. Let $\alpha_1, \alpha_2, \alpha_3$ be the spectral parameters associated to $\phi$. If the Laplace eigenvalue $\lambda_\phi = 1 - (\alpha_1^2 + \alpha_2^2 + \alpha_3^2)$ is sufficiently large, then we define the analytic conductor $c_{\rho(\phi)}$ of $\rho(\phi)$ as follows.

\begin{equation}
c_{\rho(\phi)} = \begin{cases} 
\pi^{-3} \prod_{1 \leq k \leq 3} \frac{|\alpha_k|}{2}, & \text{if } \rho(\phi) = \phi, \\
\pi^{-5} \prod_{1 \leq j < k \leq 3} \frac{|\alpha_j + \alpha_k|}{2}, & \text{if } \rho(\phi) = \text{sym}^2 \phi, \\
\pi^{-9} \prod_{j=1}^{3} \prod_{k=1}^{3} \frac{|\alpha_j - \alpha_k|}{2}, & \text{if } \rho(\phi) = \phi \times \overline{\phi}.
\end{cases}
\end{equation}

Remark 1.11. Note that this is off by a constant from the more standard Iwaniec-Sarnak definition of “conductor,” for which see e.g. [IK04, p. 95]. The constants are crucial in our applications, see specifically (7.7), so we make our definition as above.

We are interested in the weighted average value, denoted $C_\rho$, of the conductor $c_{\rho(\phi)}$ with respect to the weighting function $h_{T,R}$ defined in (1.1). Then $C_\rho$ is defined by

$$\sum_j \log c_{\rho(\phi)} \frac{h_{T,R}(\nu(j))}{\mathcal{L}_j} \sim \log C_\rho \sum_j \frac{h_{T,R}(\nu(j))}{\mathcal{L}_j}, \quad (T \to \infty),$$

and satisfies

$$C_\rho \asymp \begin{cases} 
T^3 & \text{if } \rho(\phi) = \phi, \\
T^6 & \text{if } \rho(\phi) = \text{sym}^2 \phi, \\
T^6 & \text{if } \rho(\phi) = \text{Ad} \phi.
\end{cases}$$

The weighted average value of the conductor in a family is introduced to normalize the low-lying zeros for comparison between the different families and the different matrix ensembles.

Let $\psi$ be an even test function of Schwartz class on $\mathbb{R}$ and define the low-lying zeros sum

$$D(\rho(\phi); \psi) := \sum_\gamma \psi \left( \gamma \frac{\log C_\rho}{2\pi} \right),$$
where \( \gamma \) runs over the ordinates of nontrivial zeros of \( L(s, \rho(\phi)) \), counted with multiplicity. To interpret this as capturing the low-lying zeros, we must assume GRH for the corresponding \( L \)-functions. As \( \psi \) has rapid decay, this sum localizes to those \( \gamma \) which are within \( 1/\log C_\rho \) of the origin (corresponding to the central point \( s = 1/2 \) of the \( L \)-function).

**Theorem 1.13.** Assume the Fourier transform \( \hat{\psi} \) of \( \psi \) has support in \((-\delta, \delta)\), where

\[
\delta = \begin{cases} 
4/15, & \text{if } \rho(\phi) = \phi, \\
2/27, & \text{if } \rho(\phi) = \text{sym}^2 \phi, \\
2/27, & \text{if } \rho(\phi) = \text{Ad} \phi.
\end{cases}
\]

Assume the Ramanujan conjectures, and GRH for the corresponding \( L \)-functions. Then we have the asymptotic formula

\[
\frac{1}{\sum_j \frac{h_{T,R}(\nu(j))}{\mathcal{L}_j}} \cdot \sum_j D(\rho(\phi_j); \psi) \frac{h_{T,R}(\nu(j))}{\mathcal{L}_j} = \int_\mathbb{R} \psi(x) W_{\rho(\phi)}(x) dx + O \left( \frac{\log \log T}{\log T} \right),
\]

as \( T \to \infty \), with the limiting density function \( W \) above given by

\[
W_{\rho(\phi)}(x) = \begin{cases} 
1, & \text{if } \rho(\phi) = \phi, \\
1, & \text{if } \rho(\phi) = \text{sym}^2 \phi, \\
1 - \frac{\sin(2\pi x)}{2\pi x}, & \text{if } \rho(\phi) = \text{Ad} \phi.
\end{cases}
\]

That is, the family \( \rho(\phi) \) has symmetry type: unitary, unitary, and symplectic, respectively.

**Remark 1.16.** The exterior square \( L \)-function on \( GL(3) \) is the same as the contragredient \( L \)-function (see [BF90, JS90, Kon10]). So the symmetry type for the exterior square family is unitary.

**Remark 1.17.** Note that (1.15) is consistent with a recent conjecture by Shin and Templier [ST12].

**Remark 1.18.** As in Remark 1.8, the range of \( \delta \) above can also be improved, and is intimately tied to the error terms in (1.5).

**Remark 1.19.** The Ramanujan conjectures are assumed to make the exposition of Theorem 1.13 as simple as possible. They can easily be removed by decreasing the size of \( \delta \) in Theorem 1.13.

1.5. **Outline.**

The rest of the paper is organized as follows. In §2, we collect various preliminaries on automorphic forms on \( GL_3(\mathbb{Z}) \) (their Fourier development and \( L \)-functions), and the Kontorovich-Lebedev-Whittaker transform, as explicated by the authors in [GK11]. In §3, we collect the \( GL(3) \) Kuznetsov formula, explicating all the terms which appear.

The careful definition of the choice of test function is given in §4, where we also analyze its growth/decay properties; this is the most important and involved section. We note that, though the argument is a bit complicated (four-dimensional integrals of 12 Gamma factors...
in the numerator and 7 Gamma factors in the denominator), the analysis uses nothing more than Stirling’s asymptotics for the Gamma function. In §5, we input the estimates of §4 into the Kloosterman integrals appearing on the geometric side of the Kuznetsov formula, giving bounds for these, as well as estimating away the contribution from the Eisenstein spectrum. Combining all the above estimates, we prove Theorem 1.3 in §6.

Next we turn our attention to the application to low-lying zeros. In §7, we develop the Explicit Formula for the various $L$-functions of interest, and analyze the local Langlands-Satake parameters in §8. Having done so, we apply Theorem 1.3 to the low-lying zeros sum in §9 to prove Theorem 1.13.

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2. Preliminaries on Automorphic Forms on $GL_3(\mathbb{Z})$

2.1. Jacquet’s Whittaker Function.

Let
\[ h^3 := \text{GL}_3(\mathbb{R})/(O_3(\mathbb{R}) \times \mathbb{R}^\times) \]
denote the generalized upper half plane. For $z \in h^3$ we use Iwasawa coordinates:
\[
z = xy = \begin{pmatrix} 1 & x_2 & x_3 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]
where $x_1, x_2, x_3 \in \mathbb{R}$ and $y_1, y_2 > 0$. We will frequently abuse notation, not distinguishing between $y$ as above and $y = (y_1, y_2)$. Equip $h^3$ with the Haar measure
\[
dz = \frac{dx_1 dx_2 dx_3 dy_1 dy_2}{(y_1 y_2)^3}.
\]
With this measure, the group $\Gamma = \text{GL}_3(\mathbb{Z})$ is a lattice, that is, the quotient $\Gamma \setminus h^3$ has finite volume. In fact, the volume is
\[
\int_{\Gamma \setminus h^3} \dz = \frac{3\zeta(3)}{2\pi}.
\]

For the pair $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$, set
\[
\nu_3 := \nu_1 + \nu_2.
\]

Then we have the $I$-function, defined by
\[
I_{\nu}(z) = (y_1 y_2)^{1+\nu_3} y_1^{\nu_1} y_2^{\nu_2}, \quad (z = xy \in h^3).
\]

We now define Jacquet’s Whittaker function for $GL_3(\mathbb{R})$.

**Definition 2.1 (Whittaker function).** For $\nu \in \mathbb{C}^2$ and $z \in h^3$, set
\[
W_{\nu}^\pm(z) := \pi^{-3\nu_3} \prod_{j=1}^3 \Gamma \left( \frac{1 + 3\nu_j}{2} \right) 
\times \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} I_{\nu} \left( \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & u_3 \\ 1 & u_1 \end{pmatrix} \right) e(-u_1 \mp u_2) \, du_1 du_2 du_3.
\]

This function, originally defined for $\Re \nu_1, \Re \nu_2 \gg 1$, has analytic continuation to all $\nu \in \mathbb{C}^2$. For $z = y$, the value of $W_{\nu}^\pm(y)$ is independent of the sign, so we drop the $\pm$.

It is convenient to define the parameters $\alpha$, given in terms of $\nu$, by the following linear relation:
\[
\alpha_1 = 2\nu_1 + \nu_2, \quad \alpha_2 = -\nu_1 + \nu_2, \quad \alpha_3 = -\nu_1 - 2\nu_2.
\]

2Throughout we use the completed Whittaker function, in the terminology of [Gol06].
Then $\alpha_1 + \alpha_2 + \alpha_3 = 0$ and there is an action of the Weyl group which permutes the parameters $\alpha_1, \alpha_2, \alpha_3$. We say that a function of $\nu_1, \nu_2$ is symmetric under the action of the Weyl group if it is invariant under all reorders of the triple $(\alpha_1, \alpha_2, \alpha_3)$.

Consider the representation of the Whittaker function as a double inverse Mellin transform

$$W_{\nu}(y) = \frac{y_1 y_2 \pi^{3/2}}{(2\pi i)^2} \int_{(C_1)} \int_{(C_2)} \frac{\prod_{j=1}^{3} \Gamma \left( \frac{s_1 + \alpha_j}{2} \right) \Gamma \left( \frac{s_2 - \alpha_j}{2} \right)}{4 \pi^{s_1 + s_2} \Gamma \left( \frac{s_1 + s_2}{2} \right)} y_1^{-s_1} y_2^{-s_2} ds_1 ds_2,$$

(2.5)

for any $C_1, C_2 > 0$. Here we use the standard convention that for $C \in \mathbb{R}$, the symbol $(C)$ denotes the line $C + i\mathbb{R}$. Note that $W_{\nu}$ is symmetric under the action of the Weyl group.

For $s \in \mathbb{C}$, Stade’s formula [Sta02] gives

$$\int \int_{\mathbb{R}^2_+} W_{\nu}(y) W_\mu(y) (\det y)^s \frac{dy_1 dy_2}{(y_1 y_2)^3} = \frac{\pi^{3(1-s)}}{\Gamma \left( \frac{3s}{2} \right)} \prod_{1 \leq j,k \leq 3} \Gamma \left( \frac{s + \alpha_j + \beta_k}{2} \right),$$

(2.6)

where $\mu_3 = \mu_1 + \mu_2$ and $\beta_1, \beta_2, \beta_3$ are defined in terms of $\mu_1, \mu_2$ as in (2.4). The left side above is originally only defined for $\Re(s)$ sufficiently large; of course the right side gives its meromorphic continuation.

### 2.2. Kontorovich-Lebedev transform.

Next, we give the analogue of the Kontorovich-Lebedev transform for $GL(3)$, often referred to as the Lebedev-Whittaker transform [GK11, Wal92]. Let $f : \mathbb{R}^2_+ \rightarrow \mathbb{C}$ and define $f^\sharp : \mathbb{C}^2 \rightarrow \mathbb{C}$ by

$$f^\sharp(\nu) := \int \int_{\mathbb{R}^2_+} f(y) W_{\nu}(y) \frac{dy_1 dy_2}{(y_1 y_2)^3},$$

(2.7)

provided the integral converges absolutely. Then $f^\sharp$ is termed the Lebedev-Whittaker transform of $f$. Note that $f^\sharp$ inherits the property that it is symmetric under the action of the Weyl group.

The inverse transform is given as follows. Assuming $g$ is invariant under the action of the Weyl group and has sufficient decay, we define

$$g^\flat(y) := \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} g(\nu) W_{\nu}(y) \frac{d\nu_1 d\nu_2}{\prod_{j=1}^{3} \Gamma \left( \frac{3\nu_j}{2} \right) \Gamma \left( \frac{-3\nu_j}{2} \right)}.$$

(2.8)

A sufficient condition on the test functions above (see [GK11]) is that $g(\nu)$ have holomorphic extension to a strip $-\eta < |\Re(\nu_1)|, |\Re(\nu_2)| < \eta$ (for some $\eta > 0$) and in this strip
satisfy
\[ |g(\nu)| < \exp \left( -\frac{3\pi}{4} \sum_{k=1}^{3} |\nu_k| \right) \prod_{k=1}^{3} (1 + |\nu_k|)^{-10}. \] (2.9)

Then under these growth assumptions we have
\[ g = f' \iff f = g', \]
and the Parseval-type relation:
\[
\int_{\mathbb{R}^2_+} f_1(y) \overline{f_2(y)} \frac{dy_1 dy_2}{(y_1 y_2)^3} = \frac{1}{(\pi i)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1^*(\nu) \overline{f_2^*(\nu)} \frac{d\nu_1 d\nu_2}{\prod_{j=1}^{3} \Gamma \left( \frac{3\nu_j}{2} \right) \Gamma \left( -\frac{3\nu_j}{2} \right)}. \] (2.10)

2.3. Cusp Forms.

Take a Hecke-normalized basis of Maass cusp forms \( \{\phi_j\}_{j=1,2,3,\ldots} \) for \( \mathcal{H}_0 \), the cuspidal subspace of \( L^2(\Gamma \backslash \mathfrak{h}^3) \). The form \( \phi_j \) is of type \((1/3 + \nu_1^{(j)}, 1/3 + \nu_2^{(j)})\) where \( \nu^{(j)} = (\nu_1^{(j)}, \nu_2^{(j)}) \in \mathbb{C}^2 \) denote the spectral parameters. When speaking of a fixed Maass form \( \phi \), we drop the superscript \( (j) \). For tempered forms, the spectral parameters \( \nu_1 \) and \( \nu_2 \) are purely imaginary. Then with \( \nu_3 = \nu_1 + \nu_2 \), the Laplace eigenvalue \( \lambda_\phi \) is related to \( \nu \) by
\[ \lambda_\phi = 1 - 6(\nu_1^2 + \nu_2^2 + \nu_3^2). \]

Each such \( \phi \) has the Fourier-Whittaker development given by \([\text{Sha73}, \text{PŠ75}, \text{Gol06}]:\)
\[
\phi(z) = \sum_{\gamma \in U_2(\mathbb{Z}) \backslash SL_2(\mathbb{Z})} \sum_{k_1 \geq 1, k_2 \neq 0} A_\phi(k_1, k_2) \frac{W_{\nu}(\gamma)}{k_1 |k_2|^{3|k|}} \left( \begin{pmatrix} k_1 & k_2 \\ \gamma & 1 \end{pmatrix} \right) z, \] (2.11)
with the Hecke normalization \( A_\phi(1,1) = 1 \).

The \( L \)-function attached to \( \phi \) is given by
\[ L(s, \phi) := \sum_{n \geq 1} \frac{A(1, n)}{n^s}, \]
where \( A(1, n) = A_\phi(1, n) \), i.e., we have dropped the \( \phi \) from the notation. This constitutes a degree 3 \( L \)-function, which in completed form has Euler product
\[ \Lambda(s, \phi) := \prod_p L_p(s, \phi) \]
with local factors for \( p < \infty \) of type
\[
L_p(s, \phi) := \prod_{k=1}^{3} \left( 1 - \frac{\alpha_k(p)}{p^s} \right)^{-1} = \left( 1 - \frac{A(p,1)}{p^s} + \frac{A(1,p)}{p^{2s}} - \frac{1}{p^{3s}} \right)^{-1},
\]
and for $p = \infty$,

$$L_\infty(s, \phi) := \pi^{-3s} \prod_{k=1}^{3} \Gamma \left( \frac{s + \alpha_k}{2} \right)^{-1}.$$  

The Rankin-Selberg $L$-function is

$$L(s, \phi \times \tilde{\phi}) := \zeta(3s) \sum_{k_1, k_2 \mid A(k_1, k_2)} \frac{|A(k_1, k_2)|^2}{(k_1^2 k_2^2)^s}.$$  

This $L$-function has a pole at $s = 1$. Standard Rankin-Selberg theory, together with Stade’s formula (2.6) shows that the $j$-th Maass form $\phi_j$ has $L^2$ norm given by

$$\|\phi_j\|^2 = 6 \mathcal{L}_j \prod_{k=1}^{3} \Gamma \left( \frac{1 + 3\nu_k(j)}{2} \right) \Gamma \left( \frac{1 - 3\nu_k(j)}{2} \right),$$  

where

$$\mathcal{L}_j := \text{Res}_{s=1} L(s, \phi_j \times \tilde{\phi}_j).$$  

(2.13)
3. The GL(3) Kuznetsov Formula

The following equation is the GL(3) Kuznetsov formula, as compiled from [BFG88] and [Blo11]:

\[ C + E_{\text{min}} + E_{\text{max}} = \mathcal{M} + \mathcal{K} + \tilde{\mathcal{K}} + \tilde{\mathcal{K}}^\vee, \]

(3.1)

where each component is explicated below. Let \( p : \mathbb{R}_+^2 \to \mathbb{C} \) be a test function with suitable decay properties; a sufficient condition is that

\[ |p(y_1, y_2)| \ll (y_1 y_2)^{2+\varepsilon}, \]

(3.2)

as \( y_1, y_2 \to 0 \), and that \( p \) is otherwise bounded. Fix positive integers \( n_1, n_2, m_1, m_2 \).

The left hand side of (3.1), called the spectral side, consists of cuspidal and Eisenstein contributions. The cuspidal contribution is given by

\[ C = \sum_j A_j(m_1, m_2) A_j(n_1, n_2) \frac{|p^\ast(\nu_j^{(1)}, \nu_j^{(2)})|^2}{6 \mathcal{L}_j \prod_{k=1}^3 \Gamma \left( \frac{1+3\nu_k^{(j)}}{2} \right) \Gamma \left( \frac{1-3\nu_k^{(j)}}{2} \right)}, \]

(3.3)

where the sum on \( j \) is over cuspidal Hecke-Maass forms \( \phi_j \) on \( GL(3, \mathbb{R}) \). The minimal Eisenstein series contributes

\[ E_{\text{min}} = \frac{1}{(4\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} A_{\nu}(m_1, m_2) A_{\nu}(n_1, n_2) \frac{|p^\ast(\nu_1, \nu_2)|^2}{\prod_{k=1}^3 \zeta(1+3\nu_k) \Gamma \left( \frac{1+3\nu_k}{2} \right)^2} d\nu_1 d\nu_2, \]

where the minimal Eisenstein coefficients satisfy

\[ |A_{\nu}(n_1, n_2)| \ll_{\varepsilon} (n_1 n_2)^{\varepsilon}. \]

(3.4)

Lastly, the maximal Eisenstein contribution is

\[ E_{\text{max}} = \frac{c}{2\pi i} \sum_{j=1}^{\infty} \int_{-i\infty}^{i\infty} B_{\nu, r_j}(m_1, m_2) B_{\nu, r_j}(n_1, n_2) \frac{|p_{T, R}^\ast \left( \nu - \frac{ir_j}{3}, \frac{2ir_j}{3} \right)|^2}{\left| \Gamma \left( \frac{1+3\nu-ir_j}{2} \right) \Gamma \left( \frac{1+2ir_j}{2} \right) \Gamma \left( \frac{1+3\nu+ir_j}{2} \right) \right|^2} d\nu, \]

where \( c \) is an absolute constant, and \( \{u_j\} \) is a basis of Hecke-Maass forms for \( GL(2, \mathbb{Z}) \), each of eigenvalue \( 1/4 + r_j^2 \). The trivial bound for these Fourier coefficients is

\[ |B_{\nu, r_j}(n_1, n_2)| \ll_{\varepsilon} (n_1 n_2)^{1/2+\varepsilon}. \]

(3.5)

Note that the residual spectrum does not contribute, having only degenerate terms in its Fourier expansion.

For functions \( p, q : \mathbb{R}_+^2 \to \mathbb{C} \) let

\[ \langle p, q \rangle = \int \int_{\mathbb{R}_+^2} p(y_1, y_2) q(y_1, y_2) \frac{dy_1 dy_2}{(y_1 y_2)^3}. \]
Let $1_C$ denote the indicator function, which is 1 if the condition $C$ holds and 0 otherwise. The right-hand side of (3.1), called the arithmetic side of the Kuznetsov formula, consists of a main term and Kloosterman contributions given by

$$\mathcal{M} = 1_{\{n_1=m_1\}} \langle p, p \rangle,$$

$$\tilde{K} = \sum_{\epsilon=\pm1} \sum_{D_1 D_2} \frac{\tilde{S}(\epsilon m_1, n_1, n_2, D_1, D_2)}{D_1 D_2} \tilde{J}_\epsilon \left( \sqrt{\frac{n_1 n_2 m_1}{D_1 D_2}} \right),$$

$$\tilde{K}^\prime = \sum_{\epsilon=\pm1} \sum_{D_2 | D_1} \frac{\tilde{S}(\epsilon m_2, n_1, n_2, D_2, D_1)}{D_1 D_2} \tilde{J}_\epsilon \left( \sqrt{\frac{n_1 n_2 m_2}{D_1 D_2}} \right),$$

$$K = \sum_{\epsilon_1, \epsilon_2 = \pm1} \sum_{D_1 D_2} S(\epsilon_1 m_1, \epsilon_2 m_2, n_1, n_2, D_1, D_2) \frac{\tilde{J}_{\epsilon_1, \epsilon_2}}{D_1 D_2} \left( \sqrt{\frac{m_1 n_2 D_1}{D_2}}, \sqrt{\frac{m_2 n_1 D_2}{D_1}} \right).$$

Here $S, \tilde{S}, J, \tilde{J}$ are certain GL(3) Kloosterman sums and integrals corresponding to various elements of the Weyl group.

Let $e(x) := e^{2\pi ix}$. The Kloosterman sums are given explicitly by:

$$\tilde{S}(m_1, n_1, n_2, D_1, D_2) := 1_{D_1 | D_2} \sum_{C_1 \pmod{D_1}, C_2 \pmod{D_2}} e \left( \frac{m_1 C_1 + n_1 \bar{C}_1 C_2}{D_1} \right) e \left( \frac{n_2 \bar{C}_2}{D_2 / D_1} \right),$$

and

$$S(m_1, m_2, n_1, n_2, D_1, D_2) := \sum_{B_1, C_1 \pmod{D_1}} \sum_{B_2, C_2 \pmod{D_2}} \sum_{(B_1, C_1, D_1) = 1 = (C_2, D_2 / D_1)} e \left( \frac{m_1 B_1 + n_1 (Y_1 D_2 - Z_1 B_2)}{D_1} \right) e \left( \frac{m_2 B_2 + n_2 (Y_2 D_1 - Z_2 B_1)}{D_2} \right),$$

where $Y_1, Y_2, Z_1, Z_2$ are determined by

$$Y_1 B_1 + Z_1 C_1 \equiv 1 \pmod{D_1} \quad \text{and} \quad Y_2 B_2 + Z_2 C_2 \equiv 1 \pmod{D_2}.$$

The Kloosterman integrals are given by:

$$\tilde{J}_\epsilon(A) = A^{-2} \int_{\mathbb{R}_+^2} \int_{\mathbb{R}^2} p(A y_1, y_2) e(-\epsilon A x_1 y_1) p \left( y_2, \frac{\sqrt{1 + x_1^2 + x_2^2}}{1 + x_1^2} \right) \frac{A}{y_1 y_2} \cdot \frac{\sqrt{1 + x_1^2}}{1 + x_1^2 + x_2^2} \right)$$

$$\times e \left( y_2 \cdot \frac{x_1 x_2}{1 + x_1^2} + \frac{A}{y_1 y_2} \cdot \frac{x_2}{1 + x_1^2 + x_2^2} \right) dx_1 dx_2 dy_1 dy_2 / y_1 y_2^2.$$
and

\[ J_{\epsilon_1,\epsilon_2}(A_1, A_2) = (A_1 A_2)^{-2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^3} \int p(A_1 y_1, A_2 y_2) e(-\epsilon_1 A_1 x_1 y_1 - \epsilon_2 A_2 x_2 y_2) \]

\times p \left( \frac{A_2}{y_2}, \frac{\sqrt{(x_1 x_2 - x_3)^2 + x_1^2 + 1}}{x_3 + x_2 + 1}, \frac{A_1}{y_1}, \frac{\sqrt{x_3^2 + x_2^2 + 1}}{(x_1 x_2 - x_3)^2 + x_1^2 + 1} \right)

\times e \left( -\frac{A_2}{y_2}, \frac{x_1 x_3 + x_2}{x_3 + x_2 + 1} - \frac{A_1}{y_1}, \frac{x_2(x_1 x_2 - x_3) + x_1}{(x_1 x_2 - x_3)^2 + x_1^2 + 1} \right)

\times dx_1 dx_2 dx_3 \frac{dy_1 dy_2}{y_1 y_2}.
4. Choice of Test Function and Bounds

We now make a specific choice for the test function \( p(y_1, y_2) \). By Lebedev-Whittaker inversion (2.8), we can just as well choose the transform \( p^\sharp(\nu_1, \nu_2) \). Let \( R \geq 10 \) and \( T \gg 1 \). We define

\[
p^\sharp_{T,R}(\nu_1, \nu_2) := \sqrt{6} e^{\alpha_1^2 + \alpha_2^2 + \alpha_3^2} \prod_{1 \leq j \leq 3} \Gamma\left(\frac{2 + R + 3\nu_j}{4}\right) \Gamma\left(\frac{2 + R - 3\nu_j}{4}\right). \tag{4.1}
\]

This choice is motivated by the fact that we need \( p^\sharp = p^\sharp_{T,R} \) to be invariant under the action of the Weyl group, while also requiring cancellation of the exponential growth of the \( \Gamma \)-factors in the denominator on the right side of (3.3) (cuspidal contribution to the Kuznetsov formula). The variable \( R \geq 10 \) is introduced to obtain absolute convergence of the sum (3.3), and to pull certain contours without passing through poles, see (4.16). Note first that \( p^\sharp_{T,R} \) easily satisfies the requisite bounds (2.9) for Lebedev-Whittaker inversion. It will be shown below that the inverse transform \( p_{T,R} \) satisfies (3.2), see (4.19).

Observe then that the cuspidal contribution (3.3) becomes

\[
\mathcal{C} = \sum_j A_j(m_1, m_2) \overline{A_j(n_1, n_2)} h_{T,R}(\nu_1^{(j)}, \nu_2^{(j)}) \mathcal{L}_j,
\]

exactly as desired in (1.1).

4.1. Some Auxiliary Bounds.

We collect here some bounds coming from Stirling’s asymptotic formula:

\[
|\Gamma(s + it)| \sim \sqrt{2\pi} |t|^{s-\frac{1}{2}} e^{-\frac{\pi |t|}{2}}, \quad (t \to \pm \infty), \tag{4.2}
\]

for fixed \( s \in \mathbb{R} \).

There are three types of integrals which we will need to estimate \( p_{T,R} \). Throughout we have \( y_1, y_2 > 0, \ R \geq 10, \ T \gg 1 \).

The First Integral:

For any \( C_1, C_2 \in \mathbb{R} \setminus \{-2, -4, -6, \ldots \} \), let

\[
\mathcal{I}^{(1)}_{T,R}(C_1, C_2; y_1, y_2) := \int_{(0)} \int_{(0)} \int_{(C_2)} \int_{(C_1)} e^{-\frac{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}{2T^2}} \prod_{1 \leq j \leq 3} \Gamma\left(\frac{2 + R + 3\nu_j}{4}\right) \Gamma\left(\frac{2 + R - 3\nu_j}{4}\right) \prod_{j=1}^{3} \Gamma\left(\frac{3\nu_j}{2}\right) \Gamma\left(\frac{-3\nu_j}{2}\right) \prod_{j=1}^{3} \Gamma\left(\frac{s_1 - \alpha_j}{2}\right) \Gamma\left(\frac{s_2 + \alpha_j}{2}\right) \frac{1}{4\pi^{s_1 + s_2}} y_1^{1-s_1} y_2^{1-s_2} ds_1 ds_2 d\nu_1 d\nu_2. \tag{4.3}
\]
The Second Integral:

Similarly, for any \( \kappa_1, \kappa_2, C_1 \in \mathbb{R} \) (so that the integrand below doesn’t pass through poles of \( \Gamma \)), let

\[
\mathcal{I}_{T,R}^{(2)}(\kappa_1, \kappa_2, C_1; y_1, y_2) := \int_{(\kappa_2)} \int_{(\kappa_1)} e^{\frac{\alpha^2 + \alpha^2 + \alpha^2}{2\pi^2}} \prod_{1 \leq j \leq 3} \frac{\Gamma \left( \frac{2+R+3\nu_j}{4} \right) \Gamma \left( \frac{2+R-3\nu_j}{4} \right)}{\Gamma \left( \frac{3\nu_j}{2} \right) \Gamma \left( \frac{3\nu_j}{2} \right) \Gamma \left( -\frac{3\nu_j}{2} \right)} \times \frac{\Gamma \left( \frac{s_1 - \alpha_2}{2} \right) \Gamma \left( \frac{s_1 - \alpha_3}{2} \right)}{\Gamma \left( \frac{1}{2} \right)} y_1^{1-s_1} y_2^{1+\alpha_1} ds_1 d\nu_1 d\nu_2.
\]

The Third Integral:

Lastly, for any \( \kappa_1, \kappa_2 \in \mathbb{R} \) passing through no poles, let

\[
\mathcal{I}_{T,R}^{(3)}(\kappa_1, \kappa_2; y_1, y_2) := \int_{(\kappa_2)} \int_{(\kappa_1)} e^{\frac{\alpha^2 + \alpha^2 + \alpha^2}{2\pi^2}} \prod_{1 \leq j \leq 3} \frac{\Gamma \left( \frac{2+R+3\nu_j}{4} \right) \Gamma \left( \frac{2+R-3\nu_j}{4} \right)}{\Gamma \left( \frac{3\nu_j}{2} \right) \Gamma \left( \frac{3\nu_j}{2} \right) \Gamma \left( -\frac{3\nu_j}{2} \right)} \times y_1^{1-\alpha_2} y_2^{1+\alpha_1} d\nu_1 d\nu_2.
\]

Define \( \kappa'_1, \kappa'_2, \kappa'_3 \) to be related to \( \kappa \)’s in the same way that \( \alpha \)’s are related to \( \nu \)’s, that is,

\[
\kappa'_1 = 2\kappa_1 + \kappa_2, \quad \kappa'_2 = -\kappa_1 + \kappa_2, \quad \kappa'_3 = -\kappa_1 - 2\kappa_2.
\]

Theorem 4.5. Fix \( R \geq 10 \) and \( \varepsilon > 0 \). For any \( y_1, y_2 > 0 \) and \( T \gg 1 \), we have the bound

\[
|\mathcal{I}_{T,R}^{(1)}(C_1, C_2; y_1, y_2)| \ll_{\varepsilon,C_1,C_2,R} (y_1 y_2)^T^{9/2+3R/2} \left( \frac{y_1}{T} \right)^{-C_1} \left( \frac{y_2}{T} \right)^{-C_2} T^\varepsilon.
\]

Moreover,

\[
|\mathcal{I}_{T,R}^{(2)}(\kappa_1, \kappa_2, C_1; y_1, y_2)| \ll_{\varepsilon,\kappa_1,\kappa_2,\kappa_3,R} (y_1 y_2)^T^{4+3R/2} \left( \frac{y_1}{T} \right)^{-C_1} \left( \frac{y_2}{T} \right)^{-C_2} T^\varepsilon.
\]

And finally,

\[
|\mathcal{I}_{T,R}^{(3)}(\kappa_1, \kappa_2; y_1, y_2)| \ll_{\varepsilon,\kappa_1,\kappa_2,R} (y_1 y_2)^T^{7/2+3R/2} \left( \frac{y_1}{T} \right)^{-\kappa'_1} \left( \frac{y_2}{T} \right)^{\kappa'_1} T^\varepsilon.
\]

We give separate treatments of each statement.

Proof of (4.6).

Write \( \nu_j = it_j \) and \( s_j = C_j + iu_j \). The first exponential in the integrand gives arbitrary decay once \( |t_j| > T^{1+\varepsilon} \) for any \( \varepsilon \). Bringing the absolute values inside and applying Stirling’s asymptotic formula gives

\[
|\mathcal{I}_{T,R}^{(1)}(C_1, C_2; y_1, y_2)| \ll_{C_1,C_2,R,\varepsilon} y_1^{1-C_1} y_2^{1-C_2} \int_{|t_1|,|t_2| \leq T^{1+\varepsilon}} \int_{\mathbb{R}^2} \mathcal{P} \cdot \exp \left( \frac{\pi}{4} \cdot \mathcal{E} \right) du_1 du_2 dt_1 dt_2,
\]
where $E = E(t_1, t_2, u_1, u_2)$ is the exponential factor:

$$E = 3 \sum_{k=1}^{3} |t_k| - 3 \sum_{k=1}^{3} |\alpha_k - iu_1| - 3 \sum_{k=1}^{3} |\alpha_k + iu_2| + |u_1 + u_2|,$$

and $P = P_{C_1, C_2, R}(t_1, t_2, u_1, u_2)$ is the polynomial factor:

$$P = \left(\prod_{k=1}^{3} (1 + |t_k|)^{(R+2)/2}\right) \left(\prod_{k=1}^{3} (1 + |\alpha_k - iu_1|)^{(C_1-1)/2}\right) \times \left(\prod_{k=1}^{3} (1 + |\alpha_k + iu_2|)^{(C_2-1)/2}\right) \times \left(1 + |u_1 + u_2|\right)^{(1-C_1-C_2)/2}.$$

Note that we always have $E \leq 0$, with equality only when

$$t_2 - t_1 \leq u_1 \leq 2t_1 + t_2 \quad \text{and} \quad t_1 - t_2 \leq u_2 \leq t_1 + 2t_2$$

or

$$-t_1 - 2t_2 \leq u_1 < t_2 - t_1 \quad \text{and} \quad -2t_1 - t_2 \leq u_2 \leq t_1 - t_2.$$

Hence, there is arbitrary decay outside of this range. Both inequalities have the same contribution, so we only deal with the second.

Make a linear change variables

$$u_1 \mapsto u_1 - t_1 - 2t_2 \quad \text{and} \quad u_2 \mapsto u_2 - 2t_1 - t_2,$$

so the new range is

$$0 \leq u_1 < 3t_2 \quad \text{and} \quad 0 \leq u_2 \leq 3t_1,$$

and the $P$ factor becomes

$$P_1 := \left(\prod_{k=1}^{3} (1 + |t_k|) (1 + |t_2|) (1 + |t_1 + t_2|)^{(R+2)/2}\right) \times \left(\prod_{k=1}^{3} (1 + |u_1|) (1 + |3t_1 + 3t_2 - u_1|) (1 + |u_1 - 3t_2|)^{(C_1-1)/2}\right) \times \left(\prod_{k=1}^{3} (1 + |u_2|) (1 + |3t_1 + 3t_2 - u_2|) (1 + |u_2 - 3t_1|)^{(C_2-1)/2}\right) \times \left(1 + |-3t_1 - 3t_2 + u_1 + u_2|\right)^{(1-C_1-C_2)/2}.$$

The integral of $P_1$ over (4.9) in $u_1, u_2$ is bounded up to constant by

$$P_2 := (1 + |t_1|)^{(R+2)/2+C_2} (1 + |t_2|)^{(R+2)/2+C_1} (1 + |t_1 + t_2|)^{(R+1)/2}.$$

Integrating $P_2$ over the range $|t_j| < T^{1+\varepsilon}$ gives (4.6), as claimed. □

Next we give a
Proof of (4.7).

Again by Stirling’s formula, we have

\[ |\mathcal{I}^{(2)}_{T,R}(\kappa_1, \kappa_2, C_1; y_1, y_2)| \ll \kappa_1, \kappa_2, R, \varepsilon \quad y_1^{-\kappa_1} y_2^{1+\kappa_1} \int \int \mathcal{P} \cdot \exp \left( \frac{\pi}{4} \cdot \mathcal{E} \right) du_1 dt_1 dt_2, \]

where \( \mathcal{E} = \mathcal{E}(t_1, t_2, u_1) \) is now the exponential factor:

\[ \mathcal{E} = -|t_1 - t_2 + u_1| - |t_1 + 2t_2 + u_1| + 3|t_2|, \]

and \( \mathcal{P} = \mathcal{P}_{C_1,C_2,\kappa_1,\kappa_2,R}(t_1, t_2, u_1) \) is now the polynomial factor:

\[ \mathcal{P} = (1 + |t_2|)^{R+1} (1 + |t_1|)^{(3\kappa_1+R+1)/2} (1 + |t_1 + t_2|)^{(3\kappa_1-3\kappa_2+R+1)/2} \times (1 + |t_1 - t_2 + u_1|)^{(C_1+\kappa_1-\kappa_2-1)/2} (1 + |t_1 + 2t_2 + u_1|)^{(C_1+\kappa_1+2\kappa_2-1)/2}. \]

Note that we always have

\[ \mathcal{E} \leq 0, \]

with equality only when

\[-t_1 - 2t_2 \leq u_1 \leq t_2 - t_1,\]

so we may restrict the \( u_1 \) integral to this range.

Make a linear change variables

\[ u_1 \mapsto u_1 - t_1 - 2t_2, \]

so the new range is

\[ 0 \leq u_1 < 3t_2, \]

and the \( \mathcal{P} \) factor becomes

\[ \mathcal{P}_1 := (1 + |t_2|)^{(R+2)/2} (1 + |t_1|)^{(3\kappa_1+R+1)/2} (1 + |t_1 + t_2|)^{(3\kappa_1-3\kappa_2+R+1)/2} \times (|u_1| + 1)^{(C_1+\kappa_1+2\kappa_2-1)/2} (|u_1 - 3t_2| + 1)^{(C_1+\kappa_1-\kappa_2-1)/2}. \]

The integral of \( \mathcal{P}_1 \) over the \( u_1 \) range is bounded up to constant by

\[ \mathcal{P}_2 = (1 + |t_1|)^{(3\kappa_1+R+1)/2} (1 + |t_2|)^{(R+2C_1+2\kappa_1+\kappa_2+2)/2} \times (1 + |t_1 + t_2|)^{(3\kappa_1-3\kappa_2+R+1)/2}. \]

Integrating \( \mathcal{P}_2 \) over \( |t_j| < T^{1+\varepsilon} \) gives the claim. \( \square \)

Finally, we give a

Proof of (4.8).

As before, we have

\[ |\mathcal{I}^{(3)}_{T,R}(\kappa_1, \kappa_2; y_1, y_2)| \ll_{\kappa_1, \kappa_2, R, \varepsilon} y_1^{1-\kappa'_1} y_2^{1+\kappa'_1} \int \int \mathcal{P} dt_1 dt_2, \]

where \( \mathcal{P} = \mathcal{P}_{\kappa_1, \kappa_2, R}(t_1, t_2) \) is the polynomial factor

\[ \mathcal{P} = (1 + |t_1|)^{(3\kappa_1+R+1)/2} (1 + |t_2|)^{(3\kappa_2+R+1)/2} (1 + |t_1 + t_2|)^{(3\kappa_1-3\kappa_2+R+1)/2}. \]
(Note that the exponential terms exactly cancel.) Integrating \( P \) gives the claim.

4.2. **Estimating \( p_{T,R} \).**

We use the bounds of the previous section to give an estimate for \( p_{T,R} \). Among other things, we must verify that the inverse Lebedev-Whittaker transform \( p_{T,R} \) satisfies (3.2). This will follow from the bound (4.19).

By Lebedev inversion (2.8), we define

\[
p_{T,R}(y) := \frac{1}{(\pi i)^2} \int_0^\infty \int_0^\infty p_{T,R}^\nu(y) \frac{d\nu}{\prod_{j=1}^3 \Gamma \left( \frac{3\nu_j}{2} \right) \Gamma \left( \frac{-3\nu_j}{2} \right)}.
\]

(4.10)

Recall the double inverse Mellin transform formula for the Whittaker function (2.5), and that \( W_\nu(y) = W_{-\nu}(y) \) for \( \nu \) tempered.

Then putting (2.5) into (4.10) and comparing with (4.3) gives

\[
p_{T,R}(y) = \frac{\sqrt{6\pi^3}}{(\pi i)^2} \cdot J_{T,R}^{(1)}(C_1, C_2; y_1, y_2),
\]

(4.11)

and the equality holds for any \( C_1, C_2 > 0 \). An immediate application of (4.6) proves that for any \( y_1, y_2, C_1, C_2 > 0 \),

\[
p_{T,R}(y_1, y_2) \ll_{C_1, C_2, R, \varepsilon} y_1 y_2 T^{9/2+3R/2} \left( \frac{y_1}{T} \right)^{-C_1} \left( \frac{y_2}{T} \right)^{-C_2} T^\varepsilon.
\]

(4.12)

4.2.1. **Pull past one set of poles.**

The above bound is insufficient for our purposes, so we return to the definition of \( J_{T,R}^{(1)} \), and pull the \( s_2 \) integral from the vertical line (\( C_2 \)) with \( C_2 > 0 \) to the vertical line (\( -C_2 \)), with \( C_2 = -C_2 \), \( 0 < C_2 < 2 \). In so doing, we pass through simple poles at \( s_2 = -\alpha_1, -\alpha_2, -\alpha_3 \) (generically the \( \alpha_j \) are distinct). Then we can write

\[
p_{T,R} = \mathfrak{M} + \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3,
\]

(4.13)

where \( \mathfrak{M} \) is the remaining 4-dimensional integral (that is, a constant times \( J_{T,R}^{(1)}(C_1, -C_2; y_1, y_2) \)), and the \( \mathcal{R}_j \) are the 3-dimensional contributions from the residues at \( s_2 = -\alpha_j \). Note that \( \mathcal{R}_1 \) is exactly equal to a constant times \( J_{T,R}^{(2)}(0, 0, C_1; y_1, y_2) \). In this integral, we pull the \( \nu_1, \nu_2 \) integrals from the vertical lines with \( \Re \nu_j = 0 \) to the lines \((\kappa_1), (\kappa_2)\) respectively, so that it becomes \( J_{T,R}^{(2)}(\kappa_1, \kappa_2, C_1; y_1, y_2) \). To ensure that we haven’t passed any new poles, we require that the \( \kappa \)'s satisfy:

\[
|\kappa_j| < \frac{R + 2}{3},
\]

(4.14)

\[
\kappa'_2, \kappa'_3 < C_1.
\]

Recall that here, as always, the \( \kappa' \) are related to \( \kappa \) by (4.4).
The estimate (4.7) bounds \( R_1 \) by
\[
y_1 y_2 T^{4+3R/2} \left( \frac{y_1}{T} \right)^{-C_1} \left( \frac{y_2}{T} \right) ^{\kappa_1'} T^\varepsilon,
\]
whereas the term \( \mathfrak{M} \) is dominated using (4.6) by
\[
y_1 y_2 T^{9/2+3R/2} \left( \frac{y_1}{T} \right)^{-C_1} \left( \frac{y_2}{T} \right) ^{C_2} T^\varepsilon.
\]

To make these the same in \( y_2 \), we would like to take \( \kappa_1' = 2\kappa_1 + \kappa_2 \) as large as \( C_2 \), subject to
(4.14), which requires \(-\kappa_1 + \kappa_2 < C_1 \) and \(-\kappa_1 - 2\kappa_2 < C_2 \). This is easily achieved by, say, setting \( \kappa_2 = 0, \kappa_1 > 0 \); then we can take \( \kappa_1 \) as large as 1, so that \( \kappa_1' \) can be as large as 2.

We can take \( \kappa_1 \) as large as 1, as needed, since \( R = 10 \). So under these conditions, we have dominated \( R_1 \) by the bound we already have on \( \mathfrak{M} \). The same can be done with \( R_2 \) and \( R_3 \), by pulling \( \kappa \)'s to different ranges.

We have thus given our second intermediate bound: for any \( y_1, y_2, C_1 > 0 \), and any \( 0 < C_2 < 2 \), we have
\[
p_{T,R}(y_1, y_2) \ll_{C_1, C_2, R, \varepsilon} y_1 y_2 T^{9/2+3R/2} \left( \frac{y_1}{T} \right)^{-C_1} \left( \frac{y_2}{T} \right) ^{C_2} T^\varepsilon.
\]
(4.15)

By symmetry, we have the same result with the subscripts “1” and “2” reversed.

4.2.2. Pull past two sets of poles.

The above is still insufficient, so we return to (4.13). In the \( \mathfrak{M} \) integral, we now also pull the \( s_1 \) integral from \((C_1)\) with \( C_1 > 0 \) to \((-C_1)\), where \( C_1 = -C_1 \), with \( 0 < C_1 < 2 \), passing through poles at \( s_1 = \alpha_j \), giving
\[
\mathfrak{M} = \tilde{\mathfrak{M}} + \tilde{R}_1 + \tilde{R}_2 + \tilde{R}_3.
\]

The integral \( \tilde{\mathfrak{M}} \) is exactly equal to an absolute constant times \( T^{(1)}_{T,R}(-C_1, -C_2; y_1, y_2) \), and hence we apply (4.6), giving
\[
\tilde{\mathfrak{M}} \ll_{C_1, C_2, R, \varepsilon} y_1 y_2 T^{9/2+3R/2} \left( \frac{y_1}{T} \right) ^{C_1} \left( \frac{y_2}{T} \right) ^{C_2} T^\varepsilon.
\]

The integrals \( \tilde{R}_j \) are of the same form as \( T^{(2)}_{T,R}(0, 0, C_2; y_1, y_2) \), and after pulling to appropriate \( \kappa \)'s, we can dominate the \( \tilde{R}_j \) integrals by \( \tilde{\mathfrak{M}} \), exactly as before.

In the integral \( R_1 \), which is exactly equal to a constant times \( T^{(2)}_{T,R}(0, 0, C_1; y_1, y_2) \), we can pull the \( s_1 \) integral from \((C_1)\) to \((-C_1)\), passing through poles at \( s_1 = \alpha_2, \alpha_3 \). Hence we can write correspondingly
\[
R_1 = R'_1 + P_{1,2} + P_{1,3}.
\]
Here \( R'_1 \) is a triple integral, exactly equal to a constant times \( T^{(2)}_{T,R}(0, 0, -C_1; y_1, y_2) \), and \( P_{1,2} \) is a double integral, exactly equal to a constant times \( T^{(3)}_{T,R}(0, 0; y_1, y_2) \). The term \( P_{1,3} \) is similar to \( P_{1,2} \).
In the double integral $\mathcal{P}_{1,2}$, we can pull contours in $\nu_j$ to any $(\kappa_j)$ with
\[|\kappa_j| < (2 + R)/3,\] (4.16)
without passing new poles. Apply the estimate (4.8) to bound $\mathcal{P}_{1,2}$ by
\[y_1 y_2 T^{7/2 + 3R/2} \left( \frac{y_1}{T} \right)^{-\kappa_1'} \left( \frac{y_2}{T} \right)^{\kappa_2'} T^\varepsilon.\]

Elementary linear algebra shows from (4.4) that if we choose
\[\kappa_1 = -\frac{1}{3}(C_1 + C_2), \quad \kappa_2 = \frac{1}{3}(-C_1 + 2C_2),\]
then $-\kappa_1' = C_1$ and $\kappa_2' = C_2$. Since $0 < C_1, C_2 < 2$, the condition $R \geq 10$ is more than sufficient to ensure that (4.16) is satisfied. So the $\mathcal{P}_{1,2}$ contribution is dominated by that from $\mathfrak{M}$. The same (by a different pull in $\kappa'$s) holds for $\mathcal{P}_{1,3}$.

Lastly, consider the triple integral $\mathcal{R}_1' = \mathcal{I}_{T,R}^{(2)}(0,0,-C_1; y_1, y_2)$. Since $-C_1 = C_1 < 0$, the next poles in $\nu$ arise when $\kappa_2', \kappa_3' = 2 + C_1$. Hence we can pull the $\nu$ variables to any $\kappa_1, \kappa_2$, satisfying
\[|\kappa_j| < \frac{R + 2}{3}, \quad \kappa_2', \kappa_3' < 2 - C_1,\] (4.17)
without passing more poles. The estimate (4.7) bounds $\mathcal{R}_1'$ by
\[y_1 y_2 T^{4 + 3R/2} \left( \frac{y_1}{T} \right)^{C_1} \left( \frac{y_2}{T} \right)^{C_2'} T^\varepsilon.\]
Again, taking $\kappa_2 = 0$ and $\kappa_1 > 0$, the second inequalities in (4.17) are satisfied, and we can take $\kappa_1 = C_2/2 < 1$, so that $\kappa_1' = C_2$. There are no new constraints on $R$.

Hence we see that the contribution by $\mathcal{R}_1$ is dominated by that of $\mathfrak{M}$. The same holds for $\mathcal{R}_2$ and $\mathcal{R}_3$ by symmetry, and we have established the following crucial bound.

**Proposition 4.18.** Fix $R \geq 10$ and $\varepsilon > 0$. For any $y_1, y_2 > 0$, $T \gg 1$, and any $0 < C_1, C_2 < 2$, we have
\[p_{T,R}(y_1, y_2) \ll_{C_1, C_2, R, \varepsilon} y_1 y_2 T^{9/2 + 3R/2} \left( \frac{y_1}{T} \right)^{C_1} \left( \frac{y_2}{T} \right)^{C_2} T^\varepsilon.\] (4.19)
In particular, $p_{T,R}$ satisfies (3.2), as needed in the Kuznetsov formula.

**Remark 4.20.** One can pull further and analyze contributions from higher poles. In so doing, the power in $T$ increases, so that the residual contributions $\mathcal{R}$ dominate the contribution from $\mathfrak{M}$. It may still be possible to get further improvements from such an analysis, but the above is sufficient for our purposes, so we stop here.
5. Bounds for the Kloosterman and Eisenstein Contributions

Since we showed in the previous section that our choice $p_{T,R}$ of test function satisfies the requisite bound (3.2), we now invoke Kuznetsov’s formula with this choice, and estimate the resulting components.

5.1. Bounds for the Kloosterman integrals $\mathcal{J}$ and $\tilde{\mathcal{J}}$.

We shall apply the estimates obtained in the previous section to bound the Kloosterman integrals $\mathcal{J}$ and $\tilde{\mathcal{J}}$ defined in §3. We begin with an analysis of the more difficult case of $\mathcal{J}$. For $\epsilon_1, \epsilon_2 \in \pm 1$, recall $\mathcal{J}_{\epsilon_1, \epsilon_2}$ is given by

$$\mathcal{J}_{\epsilon_1, \epsilon_2}(A_1, A_2) := (A_1 A_2)^{-2} \int_{\mathbb{R}_+^2} \int_{\mathbb{R}^3} e(-\epsilon_1 A_1 x_1 y_1 - \epsilon_2 A_2 x_2 y_2) \left( \frac{A_2}{y_2} \frac{x_1 x_3 + x_2}{y_3 + x_3} + \frac{A_1}{y_1} \frac{x_2 (x_1 x_2 - x_3) + x_1}{(x_1 x_2 - x_3)^2 + x_1^2 + 1} \right) \times p_{T,R} \left( \frac{A_2}{y_2} \frac{\sqrt{(x_1 x_2 - x_3)^2 + x_1^2 + 1}}{x_3 + x_2^2 + 1}, \frac{A_1}{y_1} \frac{\sqrt{x_3^2 + x_2^2 + 1}}{(x_1 x_2 - x_3)^2 + x_1^2 + 1} \right) dy \ dy. \tag{5.1}$$

Here $dx = dx_1 dx_2 dx_3$.

We put absolute values inside the integral and note that the resulting bounds are then independent of $\epsilon_1, \epsilon_2$, so it is convenient to drop $\epsilon_1, \epsilon_2$ from the notation. Since $p = p_{T,R}$, it is convenient to recall the dependence of $\mathcal{J}$ on $T$ and $R$, so we write henceforth $\mathcal{J}_{T,R}(A_1, A_2)$, etc. We have

$$|\mathcal{J}_{T,R}(A_1, A_2)| \leq \frac{1}{(A_1 A_2)^2} \int_{\mathbb{R}_+^2} \int_{\mathbb{R}^3} |p_{T,R}(A_1 y_1, A_2 y_2)| \left| p_{T,R} \left( \frac{A_2}{y_2} \frac{\xi_1^{1/2}}{\xi_2}, \frac{A_1}{y_1} \frac{\xi_2^{1/2}}{\xi_1} \right) \right| dy_1 dy_2,$$

where

$$\xi_1 = 1 + x_1^2 + (x_1 x_2 - x_3)^2, \quad \xi_2 = 1 + x_2^2 + x_3^2.$$

For $i = 1, 2$, break the $y_i$ integrals according to $y_i > 1$ or $y_i < 1$; this gives

$$\mathcal{J}_{T,R} \leq \mathcal{J}_0 + \mathcal{J}_1 + \mathcal{J}_1' + \mathcal{J}_2,$$

where the $y$ integral in $\mathcal{J}_0$ has $y_1, y_2 < 1$, the $\mathcal{J}_1, \mathcal{J}_1'$ integrals have one bigger and one smaller, and $\mathcal{J}_2$ has $y_1, y_2 > 1$.

We first estimate $\mathcal{J}_0$. Apply the bound in (4.19) to the second appearance of $p_{T,R}$, choosing the largest possible values $C_1 = C_2 = 2 - \epsilon$:

$$\mathcal{J}_0 \ll \epsilon T^{9/2+3R/2+\epsilon} \left( \frac{A_1 A_2}{T^2} \right)^{2-\epsilon} \int_0^1 \int_0^1 \left| p_{T,R}(A_1 y_1, A_2 y_2) \right| \frac{A_1 A_2}{(y_1 y_2)^{3-\epsilon}} \int_{\mathbb{R}^3} \frac{1}{(\xi_1 \xi_2)^{3-\epsilon}} dy_1 dy_2,$$

$$\ll T^{1/2+3R/2+3\epsilon} (A_1 A_2)^{1-\epsilon} \int_0^1 \int_0^1 \left| p_{T,R}(A_1 y_1, A_2 y_2) \right| dy_1 dy_2.$$
since the $x$ integral converges absolutely. Now estimate the first $p_{T,R}$ again using (4.19), with $C_1 = C_2 = 2 - \varepsilon/2$:

$$J_0 \ll T^{1/2+3R+3\varepsilon} (A_1 A_2)^{1-\varepsilon} \int_0^1 \int_0^1 \int_0^1 A_1 A_2 y_1 y_2 T^{9/2+3R/2+\varepsilon} \left( \frac{A_1 A_2 y_1 y_2}{T^2} \right)^{2-\varepsilon/2} \frac{1}{(y_1 y_2)^{3-\varepsilon}} \frac{dy_1 dy_2}{y_1 y_2}$$

$$\ll T^{1+3R+\varepsilon} (A_1 A_2)^{4-\varepsilon} \int_0^1 \int_0^1 (y_1 y_2)^{\varepsilon/2} \frac{dy_1 dy_2}{y_1 y_2} \ll T^{1+3R+\varepsilon} (A_1 A_2)^{4-\varepsilon},$$

since the $y$-integral converges absolutely.

To bound $J_1, J_1'$, and $J_2$, we simply follow the same procedure as above with minimal changes to ensure convergence, as follows. For $J_2$, in the second application of (4.19), we choose $C_1 = C_2 = 2 - 2\varepsilon$, say, so the final $y$-integral converges absolutely. Similarly, for $J_1$ and $J_1'$, we do the same as before, except in the second application of (4.19), we choose $C_1 = 2 - \varepsilon/2, C_2 = 2 - 2\varepsilon$ (or vice versa), so that the $y$-integral converges absolutely. We have thus proved that

$$J_{T,R}(A_1, A_2) \ll_{R,\varepsilon} T^{1+3R+\varepsilon} (A_1 A_2)^{4-\varepsilon}.$$

Next, we want a similar bound for $\tilde{J}_{T,R}(A)$. Recall that we have

$$\tilde{J}_{T,R}(A) \leq A^{-2} \int_{R^2_+} \int_{R^2} \left| p_{T,R}(Ay_1, y_2) \right| \times \left| p_{T,R} \left( y_2 \cdot \frac{1 + x_1^2 + x_2^2}{1 + x_1^2}, \frac{A}{y_1 y_2} \cdot \frac{1 + x_1^2}{1 + x_1^2 + x_2^2} \right) \right| dx_1 dx_2 \frac{dy_1 dy_2}{y_1 y_2}.$$

Note that here the integral involves $dy_2/y_2^2$, whereas in $J$ the integral has $dy_2/y_2$. This will result in a weaker final bound.

As before, for $i = 1, 2$, break the $y_i$ integrals according to $y_i > 1$ or $y_i < 1$; this gives

$$\tilde{J}_{T,R} \leq \tilde{J}_0 + \tilde{J}_1 + \tilde{J}_1' + \tilde{J}_2,$$

where the $y$ integral in $\tilde{J}_0$ has $y_1, y_2 < 1$, the $\tilde{J}_1, \tilde{J}_1'$ integrals have one bigger and one smaller, and $\tilde{J}_2$ has $y_1, y_2 > 1$.

We first bound $\tilde{J}_0$. Set $\xi_1 = 1 + x_1^2$ and $\xi_2 = 1 + x_1^2 + x_2^2$. Replace the second $p_{T,R}$ by its bound in (4.19), with the choice $C_1 = \varepsilon$ and $C_2 = 2 - \varepsilon$:

$$\tilde{J}_0 \ll_{\varepsilon} \frac{T^{9/2+3R/2+\varepsilon}}{A^2} \int_0^1 \int_0^1 \int_{-\infty}^\infty \left| p_{T,R}(Ay_1, y_2) \right| \frac{A}{y_1 \sqrt{\xi_1 \xi_2}} \left( \frac{y_2 \sqrt{\xi_2}}{T \xi_1} \right)^{\varepsilon} \left( \frac{A \sqrt{\xi_1}}{y_1 y_2 \xi_2 T} \right)^{2-\varepsilon} dx_1 dx_2 \frac{dy_1 dy_2}{y_1 y_2^2}$$

$$\ll T^{5/2+3R/2+\varepsilon} A^{1-\varepsilon} \int_0^1 \int_0^1 \left| p_{T,R}(Ay_1, y_2) \right| \frac{dy_1 dy_2}{(y_1 y_2)^{3-\varepsilon}} \frac{dy_1 dy_2}{y_1 y_2},$$
since again the $x$ integral converges absolutely. Here we used that for fixed $x_1 > 1$ and $Z < -1/2$,
\[
\int_{\mathbb{R}} (1 + x_1^2 + x_2^2)^Z \, dx_2 \ll x_1^{1+2Z}.
\]

Now apply (4.19) to $p_{T,R}(Ay_1, y_2)$ in the above integral with $\mathcal{C}_1 = \mathcal{C}_2 = 2 - \varepsilon/2$. It follows that
\[
\tilde{J}_0 \ll T^{5/2+3R+\varepsilon} A_1^{1-\varepsilon} \int_0^1 \int_0^1 Ay_1 y_2 T^{9/2+3R/2+\varepsilon} \left( \frac{Ay_1 y_2}{T^2} \right)^{2-\varepsilon/2} (y_1 y_2)^{-3+\varepsilon} dy_1 dy_2.
\]
Since $\varepsilon > 0$, the above $y$-integral converges, and we obtain the bound
\[
\tilde{J}_0 \ll T^{3+3R+\varepsilon} A_1^{4-\varepsilon}.
\]

Then we bound $\tilde{J}_1, \tilde{J}_1', \text{and } \tilde{J}_2$, by the same procedure as above, with suitable modifications, as before. We record the previous computations in the following.

**Proposition 5.2.** Fix $R \geq 10$, and any small $\varepsilon > 0$.

For any $A_1, A_2 > 0$ and $T \gg 1$, we have
\[
J_{T,R}(A_1, A_2) \ll_{R,\varepsilon} T^{1+3R+\varepsilon} (A_1 A_2)^{4-\varepsilon}.
\]

For any $A > 0$ and $T \gg 1$, we have
\[
\tilde{J}_{T,R}(A) \ll_{R,\varepsilon} T^{3+3R+\varepsilon} A_1^{4-\varepsilon}.
\]

**5.2. Bounds for the Kloosterman contributions.**

We shall now apply the bounds in Proposition 5.2 to estimate the Kloosterman contributions $K, \tilde{K}$, and $\tilde{K}^\vee$ appearing in the geometric side of the GL(3) Kuznetsov formula (3.6). We need estimates for these contributions using our choice of test function (or its transform) given by (4.1).

Let us begin by bounding the long element Kloosterman contribution $K$. We only use the trivial bound for the Kloosterman sum:
\[
|S(m_1, n_1, m_2, n_2, D_1, D_2)| \ll \varepsilon (D_1 D_2)^{1+\varepsilon}.
\]

It immediately follows from Proposition 5.2 and the trivial bound (5.5) that
\[
K \ll \sum_{D_1=1}^\infty \sum_{D_2=1}^\infty |S(m_1, m_2, n_1, n_2, D_1, D_2)| \left| J_{T,R} \left( \frac{\sqrt{m_1 n_2 D_1}}{D_2}, \frac{\sqrt{m_2 n_1 D_2}}{D_1} \right) \right|
\]
\[
\ll_{R,\varepsilon} |m_1 m_2 n_1 n_2|^2 T^{1+3R+\varepsilon} \sum_{D_1=1}^\infty \sum_{D_2=1}^\infty |D_1 D_2|^\varepsilon-2
\]
\[
\ll |m_1 m_2 n_1 n_2|^2 T^{1+3R+\varepsilon}.
\]

(5.6)
Next, we obtain a similar proposition for the lower rank Kloosterman contributions $\tilde{\mathcal{K}}$ and $\tilde{\mathcal{K}}^\vee$. In this case, we have the trivial bound

$$\tilde{S}(m_1, n_1, n_2, D_1, D_2) \ll \varepsilon (D_1 D_2)^{1+\varepsilon}.$$ 

It immediately follows from Proposition 5.2 that

$$\tilde{\mathcal{K}} \ll \sum_{D_2=1}^{\infty} \sum_{m_2|D_2} |\tilde{S}(m_1, n_1, n_2, D_1, D_2)| \left| \tilde{J}_{T,R} \left( \sqrt{\frac{n_1 n_2 m_1}{D_1 D_2}} \right) \right|$$

$$\ll R, \varepsilon \left| m_1 n_1 n_2 \right|^2 T^{3+3R+\varepsilon} \sum_{D_2=1}^{\infty} \sum_{m_2|D_2} |D_1 D_2|^{2\varepsilon-2}$$

$$\ll |m_1 n_1 n_2|^2 T^{3+3R+\varepsilon}. \quad (5.7)$$

The same argument applies to $\tilde{\mathcal{K}}^\vee$. We have proved the following.

**Proposition 5.8.** Fix $R \geq 10$, $T \gg 1$, and $\varepsilon > 0$. Then the Kloosterman contributions $\mathcal{K}$, $\tilde{\mathcal{K}}$, and $\tilde{\mathcal{K}}^\vee$ in (3.6) satisfy the bounds:

$$\mathcal{K} \ll R, \varepsilon \left| m_1 m_2 n_1 n_2 \right|^2 T^{1+3R+\varepsilon},$$

$$\tilde{\mathcal{K}} \ll R, \varepsilon \left| m_1 n_1 n_2 \right|^2 T^{3+3R+\varepsilon},$$

$$\mathcal{K}^\vee \ll R, \varepsilon \left| m_2 n_2 n_1 \right|^2 T^{3+3R+\varepsilon}.$$

### 5.3. Bounds on the Continuous Spectrum.

Next we obtain bounds for the terms $\mathcal{E}_{\min}$, $\mathcal{E}_{\max}$ coming from the continuous spectrum in the Kuznetsov formula (3.1). We begin with the term coming from the minimal parabolic Eisenstein series:

$$\mathcal{E}_{\min} = \frac{1}{(4\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A_\nu(n_1, n_2) A_\mu(m_1, m_2) \frac{|p_{T,R}^\sharp(\nu_1, \nu_2)|^2}{\prod_{k=1}^{3} |\zeta(1 + 3\nu_k) \Gamma \left( \frac{1+3\nu_k}{2} \right)|^2} \ d\nu_1 d\nu_2, \quad (5.9)$$

where the minimal Eisenstein coefficients satisfy (3.4). Inserting the choice of test function (4.1) into (5.9) and using the de la Vallée Poussin bound (Prime Number Theorem)

$$|\zeta(1 + it)| \gg \frac{1}{\log(2 + |t|)};$$
we get from Stirling’s formula (4.2) that

\[
\mathcal{E}_{\text{min}} \ll \frac{1}{(4\pi i)^2} \int_{-iT^{1+\varepsilon}}^{iT^{1+\varepsilon}} \int_{-iT^{1+\varepsilon}}^{iT^{1+\varepsilon}} |m_1m_2n_1n_2|^\varepsilon \prod_{j=1}^{3} \left| \Gamma \left( \frac{2+R+3\nu_j}{4} \right) \Gamma \left( \frac{2+R-3\nu_j}{4} \right) \right|^2 \, d\nu_1d\nu_2
\]

\[
\ll T^{2+3R+\varepsilon} |m_1m_2n_1n_2|^\varepsilon.
\]

Next we consider the term coming from the maximal parabolic Eisenstein series:

\[
\mathcal{E}_{\text{max}} = \frac{c}{2\pi i} \sum_{j=1}^{\infty} \int_{-1-i\infty}^{i\infty} \frac{B_{\nu,j}(n_1, n_2) \overline{B_{\nu,j}(m_1, m_2)}}{L(1, \text{Ad} u_j)|L(1+3\nu, u_j)|^2} \left| \frac{\nu - i\nu_j}{3}, \frac{2i\nu_j}{3} \right|^2 \prod_{i=1}^{3} \left| \Gamma \left( 1 + \frac{3\nu - i\nu_j}{2} \right) \Gamma \left( 1 + \frac{3\nu + i\nu_j}{2} \right) \right|^2 \, d\nu,
\]

where \( c \) is an absolute constant, \( \{u_j\} \) is a basis of Hecke-Maass forms for \( \text{GL}(2, \mathbb{Z}) \) (each of eigenvalue \( 1/4 + \nu_j^2 \)), and the Fourier coefficients satisfy (3.5). Here we have the lower bounds

\[
L(1, \text{Ad} u_j) \gg (1 + |\nu_j|)^{-\varepsilon}, \quad L(1+3\nu, u_j) \gg (1 + |\nu| + |\nu_j|)^{-\varepsilon}.
\]

These lower bounds follow from [HL94, HR95, JS77] and [GLS04]. Combining the above lower bounds with Stirling’s formula (4.2), it follows that

\[
\mathcal{E}_{\text{max}} \ll |m_1m_2n_1n_2|^{1/2+\varepsilon} \sum_{r_j \ll T^{1+\varepsilon}} \int_{-iT^{1+\varepsilon}}^{iT^{1+\varepsilon}} \frac{d\nu}{L(1, \text{Ad} u_j)|L(1+3\nu, u_j)|^2} \left| \frac{\nu - i\nu_j}{3}, \frac{2i\nu_j}{3} \right|^2 \prod_{i=1}^{3} \left| \Gamma \left( 1 + \frac{3\nu - i\nu_j}{2} \right) \Gamma \left( 1 + \frac{3\nu + i\nu_j}{2} \right) \right|^2
\]

\[
\ll |m_1m_2n_1n_2|^{1/2+\varepsilon} T^{3+3R+\varepsilon},
\]

using Weyl’s Law for \( \text{GL}(2) \) and the Ramanujan conjectures at infinity (for \( \text{GL}(2) \)). In summary, we have proved the following.

**Proposition 5.12.** Fix \( R \geq 10 \) and \( \varepsilon > 0 \). For any \( T \gg 1 \), we have

\[
\mathcal{E}_{\text{min}} \ll R \varepsilon |m_1m_2n_1n_2|^\varepsilon T^{2+3R+\varepsilon}, \quad \mathcal{E}_{\text{max}} \ll R \varepsilon |m_1m_2n_1n_2|^{1/2+\varepsilon} T^{3+3R+\varepsilon}.
\]
6. Proof of Theorem 1.3

6.1. The Main Term.

We begin by computing an asymptotic formula for the main term \( \mathcal{M} \) in the Kuznetsov formula (3.1), (3.6). It follows from (2.10) and Stirling’s asymptotic formula (4.2) that the inner product in the main term (3.6) becomes

\[
\langle p_{T,R}, p_{T,R} \rangle = \frac{1}{(\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \left| p_{T,R}^*(\nu_1, \nu_2) \right|^2 \prod_{1 \leq j \leq 3} \frac{d\nu_1 d\nu_2}{\Gamma \left( \frac{3\nu_j}{2} \right) \Gamma \left( \frac{-3\nu_j}{2} \right)} \quad (6.1)
\]

\[
\sim c' \int_{|\nu_1|, |\nu_2| \leq T} \prod_{1 \leq j \leq 3} \Gamma \left( \frac{2+R+3\nu_j}{4} \right) \Gamma \left( \frac{2+R-3\nu_j}{4} \right) \Gamma \left( \frac{3\nu_j}{2} \right) \Gamma \left( \frac{-3\nu_j}{2} \right) \quad (6.2)
\]

\[
\sim c T^{5+3R},
\]

for some constants \( c, c' > 0 \). This gives the \( T^{5+3R} \) main term as claimed in (1.4).

6.2. Completion of the proof of Theorem 1.2.

It follows from the Kuznetsov formula (3.1), together with the choice of test function \( p = p_{T,R} \) (with \( p_{T,R}^\# \) given by (4.1)), that

\[
\mathcal{C} = \sum_j A_j(m_1, m_2) A_j(n_1, n_2) \frac{h_{T,R}(\nu_1^{(j)}, \nu_2^{(j)})}{\mathcal{L}_j}
\]

\[
= \mathcal{M} + \mathcal{K} + \tilde{\mathcal{K}} + \mathcal{E}_{\min} - \mathcal{E}_{\max},
\]

with \( \mathcal{M} = \{ m_1 = m_2 \} \langle p_{T,R}, p_{T,R} \rangle \).

Then Theorem 1.3 is an immediate consequence of the estimates

\[
\langle p_{T,R}, p_{T,R} \rangle \sim c T^{5+3R}
\]

\[
|\mathcal{E}_{\min}|, |\mathcal{E}_{\max}| \ll_{\varepsilon} |m_1 m_2 n_1 n_2|^{1/2+\varepsilon} \quad T^{3+3R+\varepsilon}
\]

\[
|\mathcal{K}| + |\tilde{\mathcal{K}}| + |\tilde{\mathcal{K}}'| \ll_{R,\varepsilon} |m_1 m_2 n_1 n_2|^{2} T^{3+3R+\varepsilon}
\]

given in (6.1), and Propositions 5.12, 5.8, respectively.
7. The Explicit Formula

For a Hecke-Maass form $\phi$ on $GL(3)$, let $\rho(\phi)$ be one of

$$
\rho(\phi) = \begin{cases}
\phi \\
\text{sym}^2 \phi \\
\text{Ad} \phi,
\end{cases}
$$

and let $L(s, \rho(\phi))$ be the corresponding $L$-function. Note that the Maass form dual to $\phi$ is just the complex conjugate $\bar{\phi}$. In view of the identity

$$
L(s, \text{Ad} \phi) = L(s, \phi \times \bar{\phi}) \zeta(s),
$$

it is easier to work with the Rankin-Selberg convolution of $\phi$ and $\bar{\phi}$ instead of the adjoint $L$-function.

Define

$$
\Lambda(s, \rho(\phi)) := \begin{cases}
\pi^{-\frac{5s}{2}} \prod_{k=1}^{3} \Gamma \left( \frac{s+\alpha_k}{2} \right)^{-1} \prod_{p} \left( 1 - \frac{\alpha_k(p)}{p^s} \right)^{-1}, & \text{if } \rho(\phi) = \phi, \\
\pi^{-\frac{5s}{2}} \prod_{1 \leq j \leq k \leq 3} \Gamma \left( \frac{s+\alpha_j+\alpha_k}{2} \right)^{-1} \prod_{p} \left( 1 - \frac{\alpha_j(p)\alpha_k(p)}{p^s} \right)^{-1}, & \text{if } \rho(\phi) = \text{sym}^2 \phi, \\
\pi^{-\frac{3s}{2}} \prod_{j=1}^{3} \prod_{k=1}^{3} \Gamma \left( \frac{s+\alpha_j-\alpha_k}{2} \right)^{-1} \prod_{p} \left( 1 - \frac{\alpha_j(p)\alpha_k(p)}{p^s} \right)^{-1}, & \text{if } \rho(\phi) = \phi \times \bar{\phi}.
\end{cases}
$$

Then, in all the above cases, we have the functional equation

$$
\Lambda(s, \rho(\phi)) = \Lambda(1 - s, \rho(\phi)),
$$

where $\tilde{\pi}$ is the contragredient representation of $\pi$; its $L$-function has Dirichlet coefficients which are complex conjugates of the original. This follows from [GJ72, BG92, JPSS83], respectively.

We shall use the functional equation for $\Lambda(s, \rho(\phi))$ to determine the so-called “explicit formula” relating zeros and poles of $\Lambda(s, \rho(\phi))$ with sums over prime power Fourier coefficients of $L(s, \rho(\phi))$.

Let $G$ be any holomorphic function in the region $-1 \leq \Re(s) \leq 2$ satisfying

$$
G(s) = G(1 - s), \quad |s^2G(s)| \ll 1.
$$

Let $\rho_i = \frac{1}{2} + i\gamma_i$ ($i = \pm 1, \pm 2, \ldots$) run over the zeros of $\Lambda(s, \rho(\phi))$ with corresponding multiplicity. As we have assumed GRH, the ordinates form a real increasing sequence

$$
\ldots \leq \gamma_{-2} \leq \gamma_{-1} \leq 0 \leq \gamma_1 \leq \gamma_2 \leq \ldots
$$
By the functional equation and standard shifts of contours, together with the fact (first proved by [BG92]) that $\frac{\Lambda'(s)}{\Lambda(s)}$ has at most simple poles at $s = 0, 1$, with residue

\[ r_{\rho(\phi)} = \begin{cases} 0 & \text{if } \rho(\phi) = \phi, \\ 0 & \text{if } \rho(\phi) = \text{sym}^2 \phi \text{ and } \phi \text{ not self dual}, \\ 1 & \text{if } \rho(\phi) = \phi \times \bar{\phi}, \end{cases} \]

we have

\[
\sum_{\rho} G(p) - r_{\rho(\phi)}(G(0) + G(1)) = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} G(s) \left[ \frac{\Lambda'(s)}{\Lambda(s)}(s, \rho(\phi)) + \frac{\Lambda'(s)}{\Lambda(s)}(s, \bar{\rho}(\phi)) \right] ds 
\]

For a finite prime $p < \infty$, the function $H_{\rho(\phi)}(p)$ is defined by

\[
H_{\rho(\phi)}(p) = \begin{cases} -\sum_{\ell=1}^{\infty} \left( \sum_{k=1}^{3} (\alpha_k(p)^\ell + \overline{\alpha_k(p)^\ell}) \right) F(p^\ell) \log p, & \text{if } \rho(\phi) = \phi, \\ -\sum_{\ell=1}^{\infty} \left( \sum_{1 \leq j \leq k \leq 3} (\alpha_j(p)^\ell \alpha_k(p)^\ell + \overline{\alpha_j(p)^\ell \alpha_k(p)^\ell}) \right) F(p^\ell) \log p, & \text{if } \rho(\phi) = \text{sym}^2 \phi, \\ -2\sum_{\ell=1}^{\infty} \left( \sum_{j=1}^{3} \sum_{k=1}^{3} \alpha_j(p)^\ell \overline{\alpha_k(p)^\ell} \right) F(p^\ell) \log p, & \text{if } \rho(\phi) = \phi \times \bar{\phi}. \end{cases}
\]

Here $F(y)$ is the inverse Mellin transform of $G(s)$,

\[
F(y) = \frac{1}{2\pi i} \int_{(1/2)} G(s) y^{-s} \, ds.
\]

For $p = \infty$, we have that $H_{\rho(\phi)}(\infty)$ equals

\[
\begin{cases} -3F(1) \log \pi + \frac{1}{4\pi i} \sum_{k=1}^{3} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \left( \Gamma' \left( \frac{s+\alpha_k}{2} \right) + \Gamma' \left( \frac{s-\alpha_k}{2} \right) \right) G(s) \, ds, & \text{if } \rho(\phi) = \phi, \\ 5F(1) \log \pi + \frac{1}{4\pi i} \sum_{1 \leq j \leq k \leq 3} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \left( \Gamma' \left( \frac{s+\alpha_j+\alpha_k}{2} \right) + \Gamma' \left( \frac{s-\alpha_j-\alpha_k}{2} \right) \right) G(s) \, ds, & \text{if } \rho(\phi) = \text{sym}^2 \phi, \\ -9F(1) \log \pi + \frac{1}{4\pi i} \sum_{j=1}^{3} \sum_{k=1}^{3} \int_{\frac{1}{2} - i\infty}^{\frac{1}{2} + i\infty} \Gamma' \left( \frac{s+i\alpha_j-\alpha_k}{2} \right) G(s) \, ds, & \text{if } \rho(\phi) = \phi \times \bar{\phi}. \end{cases}
\]
Fix an even test function $\psi$ of Schwartz class whose Fourier transform has compact support, and apply the above formulae making the choice

$$G(s) = \psi \left( (s - 1/2) \frac{\log C_\rho}{2\pi i} \right),$$

where we recall (see (1.12)) that $C_\rho$ is the weighted average value of the conductor of $L(s, \rho(\phi))$.

Then

$$F(y) = \frac{1}{\sqrt{y \log C_\rho}} \hat{\psi} \left( \frac{\log y}{\log C_\rho} \right).$$

It follows that

$$H_{\rho(\phi)}(\infty) = \begin{cases} 
-3\hat{\psi}(0) \frac{\log \pi}{\log C_\rho} + \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \frac{\pi i x}{\log C_\rho} \psi(x) dx, & \rho(\phi) = \phi, \\
-\frac{5\hat{\psi}(0) \log \pi}{\log C_\rho} + \frac{1}{2 \log C_\rho} \sum_{1 \leq j \leq k \leq 3} \int_{-\infty}^{\infty} \frac{\pi i x}{\log C_\rho} \psi(x) dx, & \rho(\phi) = \text{sym}^2 \phi, \\
-\frac{9\hat{\psi}(0) \log \pi}{\log C_\rho} + \frac{1}{\log C_\rho} \sum_{j=1}^{3} \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \frac{\pi i x}{\log C_\rho} \psi(x) dx, & \rho(\phi) = \phi \times \bar{\phi}.
\end{cases}$$

Recall that

$$D(\rho(\phi); \psi) := \sum_{\Lambda(\frac{1}{2} + i\gamma, \rho(\phi)) = 0} \psi \left( \gamma i \frac{\log C_\rho}{2\pi} \right). \quad (7.4)$$

Consequently, (7.2) becomes

$$D(\rho(\phi); \psi) = \begin{cases} 
B_{\rho(\phi)} - \sum_{p < \infty} \sum_{\ell \geq 1} \hat{\psi} \left( \frac{\ell \log p}{\log C_\rho} \right) \frac{\log p}{\log C_\rho} \sum_{k=1}^{3} \frac{\alpha_k(p)^\ell + \alpha_k(p)^\ell}{p^{\ell/2}}, & \rho(\phi) = \phi, \\
B_{\rho(\phi)} - \sum_{p < \infty} \sum_{\ell \geq 1} \hat{\psi} \left( \frac{\ell \log p}{\log C_\rho} \right) \frac{\log p}{\log C_\rho} \sum_{1 \leq k \leq 3} \frac{\alpha_j(p)^\ell \alpha_k(p)^\ell + \alpha_j(p)^\ell \alpha_k(p)^\ell}{p^{\ell/2}}, & \rho(\phi) = \text{sym}^2 \phi, \\
B_{\rho(\phi)} - 2 \sum_{p < \infty} \sum_{\ell \geq 1} \hat{\psi} \left( \frac{\ell \log p}{\log C_\rho} \right) \frac{\log p}{\log C_\rho} \sum_{j=1}^{3} \sum_{k=1}^{3} \frac{\alpha_j(p)^\ell \alpha_k(p)^\ell}{p^{\ell/2}}, & \rho(\phi) = \phi \times \bar{\phi}.
\end{cases} \quad (7.5)$$
with

\[ B_{\rho(\phi)} = 2r_{\rho(\phi)} \psi \left( \frac{\log C_\rho}{4\pi l} \right) + \frac{A_{\rho(\phi)}}{\log C_\rho}, \]  

(7.6)

and where

\[
A_{\rho(\phi)} = \begin{cases}
-3\tilde{\psi}(0) \log \pi + \frac{1}{2} \sum_{k=1}^{3} \int_{-\infty}^{\infty} \left( \frac{\pi ix}{\log C_\rho} + \frac{1+2\alpha_k}{4} \right) \\
\quad + \frac{\Gamma'}{\Gamma} \left( \frac{\pi ix}{\log C_\rho} + \frac{1-2\alpha_k}{4} \right) \psi(x) \, dx, \quad \rho(\phi) = \phi,
\end{cases}
\]

\[
\begin{cases}
-5\tilde{\psi}(0) \log \pi + \frac{1}{2} \sum_{1 \leq j, k \leq 3} \int_{-\infty}^{\infty} \left( \frac{\pi ix}{\log C_\rho} + \frac{1+2(\alpha_j+\alpha_k)}{4} \right) \\
\quad + \frac{\Gamma'}{\Gamma} \left( \frac{\pi ix}{\log C_\rho} + \frac{1-2(\alpha_j+\alpha_k)}{4} \right) \psi(x) \, dx, \quad \rho(\phi) = \text{sym}^2 \phi,
\end{cases}
\]

\[
-9\tilde{\psi}(0) \log \pi + \sum_{j=1}^{3} \sum_{k=1}^{3} \int_{-\infty}^{\infty} \left( \frac{\pi ix}{\log C_\rho} + \frac{1+2(\alpha_j-\alpha_k)}{4} \right) \psi(x) \, dx, \quad \rho(\phi) = \phi \times \bar{\phi}.
\]

Now, for \( \mathfrak{Re}(\alpha) = 0 \), we have

\[
\int_{-\infty}^{\infty} \frac{\Gamma'}{\Gamma} \left( \frac{\pi ix}{\log C_\rho} + \alpha + \frac{1}{4} \right) \psi(x) \, dx = \tilde{\psi}(0) \frac{\Gamma'}{\Gamma} \left( \alpha + \frac{1}{4} \right) + \mathcal{O} \left( \left| \alpha + 1/4 \right| \log C_\rho \right)^{-2}.
\]

If we combine this with the fact that \( \frac{\Gamma'}{\Gamma} \left( \alpha + \frac{1}{4} \right) = \log \alpha + \mathcal{O}(1) \) for \( |\alpha| \geq \frac{1}{4} \), it follows that

\[
\begin{cases}
\tilde{\psi}(0) \log \left( \pi^{-3} \cdot \prod_{1 \leq k \leq 3} \frac{|\alpha_k|}{2} \right) + \mathcal{O}(1), \quad \rho(\phi) = \phi, \\
A_{\rho(\phi)} = \begin{cases}
\tilde{\psi}(0) \log \left( \pi^{-5} \cdot \prod_{1 \leq j, k \leq 3} \frac{|\alpha_j+\alpha_k|}{2} \right) + \mathcal{O}(1), \quad \rho(\phi) = \text{sym}^2 \phi,
\end{cases}
\]

\[
\tilde{\psi}(0) \log \left( \pi^{-9} \cdot \prod_{j=1}^{3} \prod_{k=1}^{3} \frac{|\alpha_j-\alpha_k|}{2} \right) + \mathcal{O}(1), \quad \rho(\phi) = \phi \times \bar{\phi}.
\]
Thus, in all cases, we have
\[
A_{\rho(\phi)} = \hat{\psi}(0) \log c_{\rho(\phi)} + \mathcal{O}(1),
\]  
where the analytic conductor \(c_{\rho(\phi)}\) is given by (1.10).

We combine the above formula for \(A_{\rho(\phi)}\) with (7.5) and (7.6). The contribution to (7.5) from \(\ell \geq 3\) is negligible (using the Ramanujan bound \(|\alpha_j(p)| \leq 1\)), so we have
\[
D(\rho(\phi); \psi) = \hat{\psi}(0) - \sum_{j=1}^{2} \rho(\phi) + 2\rho(\phi) \psi \left( \frac{\log c_{\rho(\phi)}}{4\pi i} \right) + \mathcal{O}\left( \frac{1}{\log c_{\rho(\phi)}} \right),
\]  
where \(\sum_{j=1}^{2} \rho(\phi)\) is the contribution from \(\ell = 1, 2\), namely
\[
\sum_{j=1}^{2} \rho(\phi) = \begin{cases} 
- \sum_{p < \infty} \hat{\psi} \left( \frac{\log p}{\log c_{\rho(\phi)}} \right) \frac{\log p}{\log C_{\rho}} \sum_{k=1}^{3} \frac{\alpha_k(p) + \overline{\alpha_k(p)}}{p^{1/2}}, & \text{if } \rho(\phi) = \phi, \\
- \sum_{p < \infty} \hat{\psi} \left( \frac{\log p}{\log c_{\rho(\phi)}} \right) \frac{\log p}{\log C_{\rho}} \sum_{1 \leq j \leq k \leq 3} \frac{\alpha_j(p)\alpha_k(p) + \overline{\alpha_j(p)}\overline{\alpha_k(p)}}{p^{1/2}}, & \text{if } \rho(\phi) = \text{sym}^2 \phi, \\
- 2 \sum_{p < \infty} \hat{\psi} \left( \frac{2 \log p}{\log c_{\rho(\phi)}} \right) \frac{\log p}{\log C_{\rho}} \sum_{j=1}^{3} \sum_{k=1}^{3} \frac{\alpha_j(p)\alpha_k(p)}{p^{1/2}}, & \text{if } \rho(\phi) = \phi \times \overline{\phi},
\end{cases}
\]  
and
\[
\sum_{j=1}^{2} \rho(\phi) = \begin{cases} 
- \sum_{p < \infty} \hat{\psi} \left( \frac{2 \log p}{\log c_{\rho(\phi)}} \right) \frac{\log p}{\log C_{\rho}} \sum_{k=1}^{3} \frac{\alpha_k(p)^2 + \overline{\alpha_k(p)}^2}{p}, & \text{if } \rho(\phi) = \phi, \\
- \sum_{p < \infty} \hat{\psi} \left( \frac{2 \log p}{\log c_{\rho(\phi)}} \right) \frac{\log p}{\log C_{\rho}} \sum_{1 \leq j \leq k \leq 3} \frac{\alpha_j(p)^2\alpha_k(p)^2 + \overline{\alpha_j(p)}^2\overline{\alpha_k(p)}^2}{p}, & \text{if } \rho(\phi) = \text{sym}^2 \phi,
\end{cases}
\]  
\[
- 2 \sum_{p < \infty} \hat{\psi} \left( \frac{2 \log p}{\log c_{\rho(\phi)}} \right) \frac{\log p}{\log C_{\rho}} \sum_{j=1}^{3} \sum_{k=1}^{3} \frac{\alpha_j(p)^2\alpha_k(p)^2}{p}, & \text{if } \rho(\phi) = \phi \times \overline{\phi}.
\]
8. Local Analysis

Recall the Fourier expansion of $\phi$:

$$\phi(z) = \sum_{\gamma \in U_2(\mathbb{Z}) \setminus \text{SL}_2(\mathbb{Z})} \sum_{k_1 \geq 1, k_2 \neq 0} \frac{A_\phi(k_1, k_2)}{|k_1| k_2} W_{\nu}^{\text{sgn}(k_2)} \left( \begin{pmatrix} k_1 & k_2 \\ 1 \\ 1 \end{pmatrix} \gamma \right),$$

where $\phi$ is normalized so that $A_\phi(1,1) = 1$. Then the Fourier coefficients $A_\phi(k_1, k_2)$ satisfy the Hecke relations

$$A_\phi(n,1)A_\phi(k_1, k_2) = \sum_{d_0, d_1, d_2 = n \atop d_1|k_1, d_2|k_2} A_\phi \left( \frac{k_1 d_0}{d_1}, \frac{k_2 d_1}{d_2} \right),$$

as well as the conjugation relation

$$A(k_1, k_2) = A(k_2, k_1).$$

Furthermore, the L-function associated to $\phi$ is given by

$$L(s, \phi) = \prod_p L_p(s, \phi) = \sum_{n=1}^{\infty} \frac{A_\phi(1, n)}{n^s},$$

with local factor

$$L_p(s, \phi) := \prod_{j=1}^{3} \left( 1 - \frac{\alpha_j(p)}{p^s} \right)^{-1} = \sum_{m \geq 0} \frac{A_\phi(1, p^m)}{p^{ms}}.$$ 

It follows that

$$A_\phi(1, p^m) = \sum_{u+v+w=m} \alpha_1(p)^u \alpha_2(p)^v \alpha_3(p)^w.$$

In particular,

$$A_\phi(1, p) = \alpha_1(p) + \alpha_2(p) + \alpha_3(p),$$

and

$$A_\phi(1, p^2) = \alpha_1(p)^2 + \alpha_2(p)^2 + \alpha_3(p)^2 + \alpha_1(p)\alpha_2(p) + \alpha_2(p)\alpha_3(p) + \alpha_1(p)\alpha_3(p)$$

$$= \alpha_1(p)^2 + \alpha_2(p)^2 + \alpha_3(p)^2 + \sum_{m \geq 0} A_\phi(1, p).$$

Next, we analyze the local contribution at a prime $p$ occurring in the sum $\sum_{p \neq (\phi)}$ given in (7.9). There are 3 cases to consider. In the second and third case we apply the Hecke relations
(8.2) to remove the product of two Fourier coefficients:

\[
\sum_{k=1}^{3} \alpha_k(p) = \left| A_\phi(1, p) \right|
\]

\[
\sum_{1 \leq j \leq k \leq 3} \alpha_j(p)\alpha_k(p) = A_\phi(1, p)^2 - \alpha_1(p)\alpha_2(p) - \alpha_1(p)\alpha_3(p) - \alpha_2(p)\alpha_3(p)
\]

\[
= A_\phi(1, p)^2 - \left| A_\phi(1, p) \right| = A_\phi(1, p)^2 - A_\phi(p, 1)
\]

\[
= \left| A_\phi(1, p^2) \right|
\]

\[
\sum_{j=1}^{3} \sum_{k=1}^{3} \alpha_j(p)\alpha_k(p) = A_\phi(1, p)A_\phi(p, 1) = 1 + A_\phi(p, p).
\]

Putting the above identities into (7.9) yields

\[
\Sigma^1_{\rho(\phi)} = \begin{cases} 
- \sum_{p < \infty} \widehat{\psi} \left( \frac{\log p}{\log C_\rho} \right) \frac{\log p}{\log C_\rho} \frac{A_\phi(1, p) + A_\phi(p, 1)}{p^{1/2}}, & \text{if } \rho(\phi) = \phi, \\
- \sum_{p < \infty} \widehat{\psi} \left( \frac{\log p}{\log C_\rho} \right) \frac{\log p}{\log C_\rho} \frac{A_\phi(1, p^2) + A_\phi(p^2, 1)}{p^{1/2}}, & \text{if } \rho(\phi) = \text{sym}^2 \phi, \\
-2 \sum_{p < \infty} \widehat{\psi} \left( \frac{\log p}{\log C_\rho} \right) \frac{\log p}{\log C_\rho} \frac{1 + A_\phi(p, p)}{p^{1/2}}, & \text{if } \rho(\phi) = \phi \times \bar{\phi}, 
\end{cases}
\]

Next, we do the same for \( \Sigma^2_{\rho(\phi)} \) given in (7.10). As before, there are three cases to consider. We require the following Hecke relations obtained from (8.2):

\[
A_\phi(p, 1)^2 = A_\phi(p^2, 1) + A_\phi(1, p),
\]

\[
A_\phi(1, p^2)^2 = A_\phi(p^2, 1) + A_\phi(p, p^2) + A_\phi(1, p^4),
\]

\[
A_\phi(p, 1)A_\phi(1, p^2) = A_\phi(p, p^2) + A_\phi(1, p^3),
\]

\[
A_\phi(p^2, 1)A_\phi(1, p^2) = A_\phi(1, 1) + A_\phi(p, p) + A_\phi(p^2, p^2),
\]

\[
-A_\phi(p^2, 1)A_\phi(p, 1) = -A_\phi(p, p) - A_\phi(p^3, 1),
\]

\[
-A_\phi(1, p^2)A_\phi(1, p) = -A_\phi(p, p) - A_\phi(1, p^3),
\]

\[
A_\phi(p, 1)A_\phi(1, p) = 1 + A_\phi(p, p).
\]
It follows from the above Hecke relations that:

\[
\sum_{k=1}^{3} \alpha_k(p)^2 = \boxed{A_{\phi}(1, p^2) - A_{\phi}(p, 1)},
\]

\[
\sum_{1 \leq j \leq k \leq 3} \alpha_j(p)^2 \alpha_k(p)^2 = (A_{\phi}(1, p^2) - A_{\phi}(p, 1))^2 - A_{\phi}(p^2, 1) - A_{\phi}(p, 1)
\]
\[
= A_{\phi}(1, p^2)^2 - 2A_{\phi}(p, 1)A_{\phi}(1, p^2) + A_{\phi}(p, 1)^2 - A_{\phi}(p^2, 1) + A_{\phi}(p, 1)
\]
\[
= \boxed{A_{\phi}(p^2, 1) + A_{\phi}(1, p^4) - A_{\phi}(p, p^2) - 2A_{\phi}(1, p^3) + A_{\phi}(1, p) + A_{\phi}(p, 1)},
\]

\[
\sum_{j=1}^{3} \sum_{k=1}^{3} \alpha_j(p)^2 \alpha_k(p)^2 = (A_{\phi}(1, p^2) - A_{\phi}(p, 1)) (A_{\phi}(p^2, 1) - A_{\phi}(1, p))
\]
\[
= A_{\phi}(p^2, 1)A_{\phi}(1, p^2) - A_{\phi}(p, 1)A_{\phi}(p^2, 1) - A_{\phi}(1, p^2)A_{\phi}(1, p) + A_{\phi}(p, 1)A_{\phi}(1, p)
\]
\[
= \boxed{2 + A_{\phi}(p^2, p^2) - A_{\phi}(1, p^3) - A_{\phi}(p^3, 1)}.
\]

Putting the above identities into (7.10) yields

\[
\Sigma_{\rho(\phi)}^2 = \begin{cases} 
- \sum_{p < \infty} \psi \left( \frac{2 \log p}{\log C_p} \right) \log \frac{p}{C_p} \left( \frac{A_{\phi}(1, p^2) - A_{\phi}(p, 1) + A_{\phi}(p^2, 1) - A_{\phi}(1, p)}{p} \right), & \rho(\phi) = \phi, \\
- \sum_{p < \infty} \psi \left( \frac{2 \log p}{\log C_p} \right) \log \frac{p}{C_p} \left( \frac{A_{\phi}(p^2, 1) + A_{\phi}(1, p^4) - A_{\phi}(p, p^2) - 2A_{\phi}(1, p^3) + A_{\phi}(p, 1) - A_{\phi}(p, 1)}{p} \right), & \rho(\phi) = \text{sym}^2 \phi, \\
-2 \sum_{p < \infty} \psi \left( \frac{2 \log p}{\log C_p} \right) \log \frac{p}{C_p} \left( \frac{2 + A_{\phi}(p^2, p^2) - A_{\phi}(1, p^3) - A_{\phi}(p^3, 1)}{p} \right), & \rho(\phi) = \phi \times \bar{\phi}. 
\end{cases}
\]

(8.4)

**Remark 8.5.** It is much simpler to apply the Kuznetsov formula to the sums \( \Sigma_{\rho(\phi)}^1, \Sigma_{\rho(\phi)}^2 \), given in (8.3) and (8.4) since the product of Fourier coefficients has been removed by the use of the Hecke relations.
9. Proof of Theorem 1.13

By Parseval’s Theorem,
\[ \int_{\mathbb{R}} \psi(x) W_{\rho(\phi)}(x) \, dx = \int_{\mathbb{R}} \hat{\psi}(y) \hat{W}_{\rho(\phi)}(y) \, dy, \]
where \( \hat{\psi} \) is the Fourier transform of \( \psi \), and \( \hat{W}_{\rho(\phi)} \) can be explicitly computed from (1.15) as a distribution:
\[ \hat{W}_{\rho(\phi)}(y) = \begin{cases} \delta_0(y), & \text{if } \rho(\phi) = \phi \text{ or } \text{sym}^2 \phi, \\ \delta_0(y) - \frac{1}{2}, & |y| < 1, \\ \frac{1}{4}, & |y| = \pm 1, \\ 0, & |y| > 1. \end{cases} \]
As the support of \( \hat{\psi} \) will be restricted well inside \((-1, 1)\), it follows that
\[ \int_{\mathbb{R}} \hat{\psi}(y) \hat{W}_{\rho(\phi)}(y) \, dy = \begin{cases} \hat{\psi}(0), & \text{if } \rho(\phi) = \phi, \\ \hat{\psi}(0), & \text{if } \rho(\phi) = \text{sym}^2 \phi \\ \hat{\psi}(0) - \frac{1}{2} \psi(0), & \text{if } \rho(\phi) = \text{Ad} \phi. \end{cases} \] (9.1)
Recall the asymptotic formula (7.8) for the low-lying zeros sum:
\[ D(\rho(\phi); \psi) = \hat{\psi}(0) - \Sigma_{1,\rho(\phi)} - \Sigma_{2,\rho(\phi)} + 2r_{\rho(\phi)}\psi \left( \frac{\log c_{\rho(\phi)}}{4\pi i} \right) + O \left( \frac{1}{\log c_{\rho(\phi)}} \right), \] (9.2)
where by (7.1), we have
\[ r_{\rho(\phi)} = \begin{cases} 0, & \text{if } \rho(\phi) = \phi, \\ 0, & \text{if } \rho(\phi) = \text{sym}^2 \phi \text{ and } \phi \text{ not self dual}, \\ 1, & \text{if } \rho(\phi) = \phi \times \bar{\phi}, \end{cases} \]
and \( \Sigma_{1,\rho(\phi)}, \Sigma_{2,\rho(\phi)} \) are given by (8.3), and (8.4), respectively.
We will prove Theorem 1.13 where the limiting density \( \int_{\mathbb{R}} \hat{\psi}(y) \hat{W}(y) \, dy \) is in the form (9.1).
Note that \( \hat{\psi}(0) \) already appears in (9.2).

The case when \( \rho(\phi) = \phi \times \bar{\phi} \):

The main contribution to \( \Sigma_{1,\phi \times \bar{\phi}} + \Sigma_{2,\phi \times \bar{\phi}} \) comes from
\[ 2 \sum_p \frac{1}{\sqrt{p}} \hat{\psi} \left( \frac{\log p}{\log C_{\rho}} \right) \frac{\log p}{\log C_{\rho}} + 4 \sum_p \frac{1}{p} \hat{\psi} \left( \frac{2 \log p}{\log C_{\rho}} \right) \frac{\log p}{\log C_{\rho}}. \] (9.3)
This may be computed in two steps.
Step 1: We apply the explicit formula (7.2) to the function

\[ \Lambda(s) = \pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) \zeta(s). \]

Note that for \( \log C_\rho \) sufficiently large, we have

\[ \sum_{\Lambda(\rho) = 0} G(\rho) = \sum_{\Lambda(\frac{1}{2} + i\gamma) = 0} \psi \left( \frac{\gamma \log C_\rho}{2\pi} \right) = 0, \]

since there are no low-lying (with \( |\gamma| \ll 1/\log C_\rho \)) non-trivial zeros of the Riemann zeta function. Further \( G(0) + G(1) = 2\psi \left( \frac{\log C_\rho}{4\pi i} \right) \). It follows that the explicit formula for the Riemann zeta function takes the form:

\[
2\psi \left( \frac{\log C_\rho}{4\pi i} \right) = 2 \sum_{\ell \geq 1} \frac{1}{p^{\ell/2}} \cdot \widehat{\psi} \left( \frac{\ell \log p}{\log C_\rho} \right) \frac{\log p}{\log C_\rho} + \frac{\widehat{\psi}(0) \log \pi}{\log C_\rho} \frac{\log \pi}{2} \]
\[
- \frac{1}{\log C_\rho} \int_{-\infty}^{\infty} \frac{1}{\Gamma} \left( \frac{2\pi i x}{\log C_\rho} + \frac{1}{4} \right) \psi(x) \, dx
\]
\[
= 2 \sum_{\ell = 1}^{2} \frac{1}{p^{\ell/2}} \cdot \widehat{\psi} \left( \frac{\ell \log p}{\log C_\rho} \right) \frac{\log p}{\log C_\rho} + O \left( \frac{1}{\log C_\rho} \right).
\]

Note that the above accounts for a large part of (9.3) and almost cancels the term \( 2\psi \left( \frac{\log c_\rho(\phi)}{4\pi i} \right) \) in (9.2). In fact, we get

\[
D(\phi \times \tilde{\phi}; \psi) = \widehat{\psi}(0) - \sum_{\phi \times \phi} 2\psi \left( \frac{\log c_\phi(\phi)}{4\pi i} \right) - 2\psi \left( \frac{\log C_\rho}{4\pi i} \right)
- 2 \sum_{p} \frac{1}{p} \psi \left( \frac{2 \log p}{\log C_\rho} \right) \frac{\log p}{\log C_\rho} + O \left( \frac{1}{\log c_\rho(\phi)} \right).
\]

What is really happening here is that every Rankin-Selberg \( L \)-function \( L(s, \phi \times \tilde{\phi}) \) in the family is divisible by the Riemann zeta function which has a pole at \( s = 1 \). But the Riemann zeta function does not contribute low lying zeros, so the contribution from the pole at \( s = 1 \) is cancelled.
Step 2: In the second step we make use of the classical Riemann hypothesis which implies that $\Psi(x) = x + O\left(x^{\frac{1}{2}+\epsilon}\right)$. It follows that

$$2 \sum_p \frac{1}{p} \hat{\psi}\left(\frac{2 \log p}{\log C_\rho}\right) \frac{\log p}{\log C_\rho} = \frac{2}{\log C_\rho} \int_1^\infty \hat{\psi}\left(\frac{2 \log x}{\log C_\rho}\right) x^{-1} d\Psi(x)$$

$$= \frac{2}{\log C_\rho} \int_1^\infty \hat{\psi}\left(\frac{2 \log x}{\log C_\rho}\right) x^{-2} \left(x + O\left(x^{\frac{1}{2}+\epsilon}\right)\right) dx$$

$$- \frac{2}{\log C_\rho} \int_1^\infty x \frac{2}{x \log C_\rho} \hat{\psi}'\left(\frac{2 \log x}{\log C_\rho}\right) x^{-1} \left(x + O\left(x^{\frac{1}{2}+\epsilon}\right)\right) dx$$

$$= \int_0^\infty \hat{\psi}(u) du + O\left(\frac{1}{\log C_\rho}\right)$$

$$= \frac{1}{2} \psi(0) + O\left(\frac{1}{\log C_\rho}\right).$$

If we now combine the computation in Step 2 with equation (9.4), it follows that

$$D(\phi \times \bar{\phi}; \psi) = \hat{\psi}(0) - \frac{1}{2} \psi(0) + 2 \psi\left(\frac{\log c_{\phi \times \bar{\phi}}}{4\pi i}\right) - 2 \psi\left(\frac{\log C_\rho}{4\pi i}\right) + \Sigma^3_{\phi \times \bar{\phi}} + O\left(\frac{1}{\log c_{\phi \times \bar{\phi}}}\right),$$

(9.5)

where

$$\Sigma^3_{\phi \times \bar{\phi}} = 2 \sum_p \left[ \frac{A_\phi(p, p)}{\sqrt{p}} \hat{\psi}\left(\frac{\log p}{\log C_\rho}\right) \right. $$

$$+ \left. \left(\frac{A_\phi(p^2, p^2) - A_\phi(1, p^2) - A_\phi(p^2, 1)}{p}\right) \hat{\psi}\left(\frac{2 \log p}{\log C_\rho}\right) \right] \frac{\log p}{\log C_\rho}. $$

(9.6)

To prove Theorem 1.13 for the family of Rankin Selberg L-functions, we make use of (9.5) and the decay properties of $h_{T,R}$ to obtain

$$\sum_j D(\phi_j \times \phi_j; \psi) \frac{h_{T,R}(\nu(j))}{L_j} = \sum_j \left[ \hat{\psi}(0) - \frac{1}{2} \psi(0) + \Sigma^3_{\phi_j \times \bar{\phi}_j} + O\left(\frac{\log \log T}{\log T}\right) \right] \frac{h_{T,R}(\nu(j))}{L_j},$$

as we average over Maass forms $\phi_j$ ($j = 1, 2, \ldots$). The term $\log \log T/\log T$ arises after breaking the sum into two pieces corresponding to $\phi_j$ with conductor $c_{\phi_j} \ll T^3/\log T$ and
To evaluate the above sum, it remains to estimate

$$\sum_j \varphi_j \sum \frac{h_{T,R}(\nu^{(j)})}{L_j}.$$ 

Say the support of $\hat{\psi}$ is in $(-\delta, \delta)$. It immediately follows from (9.6) that

$$\sum_j \varphi_j \sum \frac{h_{T,R}(\nu^{(j)})}{L_j} = 2 \sum_j \left[ \sum_{p \ll T^{65}} \frac{A_{\phi}(p, p)}{p} \hat{\psi}(\frac{\log p}{\log C_\rho}) \log p \log C_\rho \right. + \left. \sum_{p \ll T^{65}} \left( A_{\phi}(p^2, p^2) - A_{\phi}(1, p^3) - A_{\phi}(p^3, 1) \right) \hat{\psi}(2 \log p p^{\frac{1}{2}} \log C_\rho) \log p p^{\frac{1}{2}} \log C_\rho \right].$$

To finish the estimation, we apply Theorem 1.3 to obtain

$$\sum_j \varphi_j \sum \frac{h_{T,R}(\nu^{(j)})}{L_j} \ll T^{3+3R+\epsilon} \left[ \sum_{p \ll T^{65}} \log p \log C_\rho + \sum_{p \ll T^{65}} p^{7} \log p \log C_\rho \right] \ll T^{3+3R+27\delta+\epsilon}.$$ 

So we need

$$\delta < \frac{2}{27}.$$ 

With this choice of $\delta$ we obtain

$$\sum_j D(\phi_j \times \phi_j; \psi) \frac{h_{T,R}(\nu^{(j)})}{L_j} = \left[ \psi(0) - \frac{1}{2} \psi(0) + O \left( \frac{\log \log T}{\log T} \right) \right] \cdot \sum_j \frac{h_{T,R}(\nu^{(j)})}{L_j},$$

as claimed.

**Remark 9.7.** The family of Rankin-Selberg L-functions for $GL(3)$ has the same symmetry type as the family of adjoint L-functions in view of the identity $L(s, \text{Ad} \phi) = \frac{L(s, \phi \times \overline{\phi})}{\zeta(s)}$, and the fact that $\zeta(s)$ has no low-lying zeros.

**The case when $\rho(\phi) = \text{sym}^2 \phi$:**

It immediately follows from (9.2), (8.3), and (8.4), that

$$D(\text{sym}^2 \phi; \psi) = \hat{\psi}(0) - \sum_{\text{sym}^2 \phi} + \sum_{\text{sym}^2 \phi} + 2 \text{sym}^2 \phi \psi \left( \frac{\log c_{\text{sym}^2 \phi}}{4\pi i} \right) + O \left( \frac{1}{\log c_{\text{sym}^2 \phi}} \right),$$

where

$$\sum_{\text{sym}^2 \phi} = - \sum_{p < \infty} \hat{\psi} \left( \frac{\log p}{\log C_\rho} \right) \log p \frac{A_{\phi}(1, p^2) + A_{\phi}(p^2, 1)}{p^{1/2}} \log C_\rho.$$
and

\[ \Sigma_{\text{sym}^2 \phi}^2 = - \sum_{p < \infty} \hat{\psi} \left( \frac{2 \log p}{\log C_\rho} \right) \left( \frac{\log p}{\log C_\rho} \right) \times \frac{A_\phi(p^2, 1) + A_\phi(1, p^4) - A_\phi(p, p^2) - 2A_\phi(1, p^3) + A_\phi(1, p) - A_\phi(p, 1)}{p} \]

\[ + \frac{A_\phi(1, p^2) + A_\phi(p^4, 1) - A_\phi(p^2, p) - 2A_\phi(p^3, 1) + A_\phi(p, 1) - A_\phi(1, p)}{p}. \]

In this case \( r_{\text{sym}^2 \phi} = 0 \) unless \( \phi \) is self dual, which happens only if \( \phi \) is a symmetric square lift from \( GL(2) \). This occurs for \( \asymp T^2 \) cases out of \( \asymp T^6 \), and hence contributes a negligible error term to the low-lying zeros sum. The method to estimate \( \Sigma_{\text{sym}^2 \phi}^1 \) and \( \Sigma_{\text{sym}^2 \phi}^2 \), using Theorem 1.3, is exactly the same as the method used above for the case of \( \rho(\phi) = \phi \times \bar{\phi} \). Because of the presence of \( A_\phi(1, p^2), A_\phi(p^2, 1) \) in \( \Sigma_{\text{sym}^2 \phi}^1 \) and \( A_\phi(1, p^4), A_\phi(p^4, 1) \) in \( \Sigma_{\text{sym}^2 \phi}^2 \), we obtain the same value \( \delta < \frac{2}{27} \) (which we obtained for the case \( \rho(\phi) = \phi \times \bar{\phi} \)) for the support of \( \hat{\psi} \).

It follows that

\[ \sum_j D(\text{sym}^2 \phi; \psi) \frac{h_{T,R}(\nu^{(j)})}{L_j} = \left[ \hat{\psi}(0) + O\left( \frac{\log \log T}{\log T} \right) \right] \sum_j \frac{h_{T,R}(\nu^{(j)})}{L_j}. \]

**The case when \( \rho(\phi) = \phi \):**

In this case, we have that the residue \( r_\phi = 0 \). It then follows from (9.2), (8.3), and (8.4), that

\[ D(\phi; \psi) = \hat{\psi}(0) - \Sigma_\phi^1 - \Sigma_\phi^2 + O\left( \frac{1}{\log c_\phi} \right) \]

where

\[ \Sigma_\phi^1 = -2 \sum_{p < \infty} \hat{\psi} \left( \frac{\log p}{\log C_\rho} \right) \left( \frac{\log p}{\log C_\rho} \right) \frac{A_\phi(1, p) + A_\phi(p, 1)}{p^{1/2}} \]

and

\[ \Sigma_\phi^2 = - \sum_{p < \infty} \hat{\psi} \left( \frac{2 \log p}{\log C_\rho} \right) \left( \frac{2 \log p}{\log C_\rho} \right) \frac{A_\phi(1, p^2) - A_\phi(p, 1) + A_\phi(p^2, 1) - A_\phi(1, p)}{p}. \]

Assume the support of \( \hat{\psi} \) is in \( (-\delta, \delta) \). As before, we have

\[ \sum_j D(\phi_j; \psi) \frac{h_{T,R}(\nu^{(j)})}{L_j} = \sum_j \left[ \hat{\psi}(0) - \Sigma_{\phi_j}^1 - \Sigma_{\phi_j}^2 + O\left( \frac{\log \log T}{\log T} \right) \right] \frac{h_{T,R}(\nu^{(j)})}{L_j}, \]
and invoking Theorem 1.3 again gives

\[
\sum_j \left( -\Sigma_1^{\phi_j} - \Sigma_2^{\phi_j} \right) \frac{h_{T,R}(\nu^{(j)})}{L_j} = \sum_j \left[ \sum_{p \leq T^{\delta}} \hat{\psi} \left( \frac{\log p}{\log C_{\rho}} \right) \frac{\log p}{\log C_{\rho}} \frac{A_{\phi}(1,p) + A_{\phi}(p,1)}{p^{1/2}} \right. \\
+ \left. \sum_{p \leq T^{3\delta/2}} \hat{\psi} \left( \frac{2 \log p}{\log C_{\rho}} \right) \frac{\log p}{\log C_{\rho}} \frac{A_{\phi}(1,p^2) - A_{\phi}(p,1) + A_{\phi}(p^2,1) - A_{\phi}(1,p)}{p} \right] \frac{h_{T,R}(\nu^{(j)})}{L_j}
\]

\[
\ll T^{3+3R+\epsilon} \left[ \sum_{p \leq T^{\delta}} p^{3/2} + \sum_{p \leq T^{3\delta/2}} p^{3} \right]
\]

\[
\ll T^{3+3R+\epsilon} T^{\frac{15}{2}\delta}.
\]

So in this case, we need \( \delta < \frac{4}{15} \).

This completes the proof of Theorem 1.13.

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