The Luttinger liquid in superlattice structures: atomic gases, quantum dots and the classical Ising chain

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Abstract
We study the physical properties of a Luttinger liquid in a superlattice that is characterized by alternating two tunneling parameters. Using the bosonization approach, we describe the corresponding Hubbard model by the equivalent Tomonaga–Luttinger model. We analyze the spin–charge separation and transport properties of the superlattice system. We suggest that cold Fermi gases trapped in a bichromatic optical lattice and coupled quantum dots offer the opportunity to measure these effects in a convenient manner. We also study the classical Ising chain with two tunneling parameters. We find that the classical two-point correlator decreases as the difference between the two tunneling parameters increases.

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(Some figures in this article are in colour only in the electronic version.)

1. Introduction

Quantum many-body systems of one-dimensional (1D) interacting fermions have been attracting a great deal of interest for more than five decades. Contrary to what happens in two and three dimensions, these systems cannot be described by the Landau theory of normal Fermi liquids. The appropriate paradigm for 1D interacting fermions is instead provided by the Luttinger liquid concept introduced by Haldane [1]. A distinctive feature of the Luttinger liquid is that its low-energy excitations are collective oscillations of the charge or the spin density, as opposed to individual quasi-particles that carry both charge and spin. This leads immediately to the phenomenon of spin–charge separation, i.e. the fact that the low-energy spin and charge excitations of 1D interacting fermions are completely decoupled and propagate with different velocities. Despite the firm theoretical basis on which the Luttinger liquid theory rests, there has been precious little compelling experimental evidence suggesting that real 1D electron gases are anything but Fermi liquids. In recent years, it has become possible to fabricate single-channel quantum wires, but unwanted impurity causes backscattering and localization, thus destroying the Luttinger liquid phase. Fortunately, there is another experimental system that is expected to exhibit Luttinger liquid behavior and does not suffer from complications associated with impurities—namely 1D quantum Fermi gases in optical lattices.

In this context, it has now become possible to trap ultra-cold quantum gases in quasi-1D optical lattices. Most of the theoretical works have been on the 1D Bose gas, but more recently, Recati et al [2] studied 1D quantum gases of fermionic atoms in optical lattices using the Luttinger liquid approach, while Polini and Vignale [3] studied the spin–drag and spin–charge separation of the atomic Fermi–Dirac gas in a 1D optical lattice. Through superposition of optical lattices with different periods, it is now possible to generate more sophisticated periodic potentials characterized by a richer spatial modulation, the so-called optical superlattices [4]. An important and exciting application of optical superlattices is quantum computation [5]. The physics of 1D optical superlattices has been a subject of research in recent years, including fractional filling Mott insulator (MI) domains [6], dark [7] and gap solitons [8], the Mott–Peierls transition [9].
non-mean-field effects [10], the phase diagram in two-color superlattices [11], Bloch–Zener and dipole oscillations [12], collective oscillations [13] and the Bloch and Bogoliubov spectrum [14]. In section 3, we discuss the influence of such a kind of superlattice structure on the spin–charge separation and the compressibility of the Fermi gas in an optical superlattice.

Often referred to as artificial atoms, semiconductor quantum dots (QDs) offer an unprecedented possibility of constructing at will and exploring situations ranging from practically a single atom to a fully solid-state many-body systems. The nanofabrication possibilities of tailoring structures to desired geometries and specifications, and controlling the number and mobility of electrons confined within a region of space, make these structures unique tools for studying transport properties. Quantum transport in arrays of tunnel-coupled QDs has attracted much attention over the last few years [15–20]. The controllable quantum properties of the electron in such systems open the possibility of applying them to schemes of quantum computers [21]. In section 4, we study the conductivity of an electron that experiences an asymmetric tunneling when going to the left and right in 1D tunnel-coupled QDs.

The spin-1/2 Ising chain is considered the prototypical system for quantum phase transitions [22]. However, very little is known in the literature about its classical counterpart. In section 5, we discuss the classical Ising model in the absence of an external field when the exchange energy between site \(j\) and \(j + 1\) is not the same as \(j\) and \(j - 1\). With the development of molecular beam epitaxy, it is now possible to envisage a superlattice in which the exchange constant varies from layer to layer. Very often, one finds interesting properties of these systems [23]. Magnetic excitations in superlattices have been studied in numerous works (see [24] for a brief review). Hinchey and Mills [25] investigated a superlattice structure with alternating ferromagnetic and antiferromagnetic layers. A common feature that connects the three systems studied in this work is asymmetric tunneling.

Some interesting theoretical studies on Luttinger liquid superlattices have already been performed [26, 27]. These investigated the effective spin and charge velocities, the compressibility, correlation functions and the dc conductivity of Luttinger liquid superlattices, modeled by a repeated pattern of interacting and free Luttinger liquids. When the interacting subsystem is a Hubbard chain, the system exhibits a phase diagram with two metallic and two compressible insulating phases. In this paper, we show that coherent control over the transport property can be achieved.

### 2. The Hubbard Hamiltonian and the equivalent Tomonaga–Luttinger model

Our aim is to study strongly correlated systems in one spatial dimension. These are typically systems of interacting electrons, but we are interested in cold Fermi gases as well. The prototypical interacting electron system is the Hubbard model (HM). This is the lattice model whose Hamiltonian \((H_{	ext{HM}})\) in one dimension is

\[
H_{\text{HM}} = -
\sum_{j,\sigma} J_{\sigma} \left( c_{j+1,\sigma}^{\dagger} c_{j,\sigma} + c_{j,\sigma}^{\dagger} c_{j+1,\sigma} \right) + U \sum_{j} n_{j,\uparrow} n_{j,\downarrow}.
\]

The first term describes the hopping process, in which an electron can move from one site to the next with site-dependent amplitudes \(J_{\sigma}\) that take two distinct values, namely \(J_{0} + (-1)^{j} \Delta_{0}\) (depending on whether \(j\) is even or odd). Here \(\Delta_{0} < J_{0}\) is a small deviation from \(J_{0}\). The hopping process preserves the spin projection \(\sigma\). For cold Fermi gases, such kinds of hopping terms can be created by the superposition of two optical lattices of different periodicity (as described in section 3), whereas for the electron system, coupled quantum wells with appropriate voltages can generate such a hopping term (as described in section 4). The second term describes the local Coulomb repulsion \((U > 0)\) between opposite spin electrons residing on the same site. For cold Fermi gases, this would be the two-body interaction as discussed in the next section. Here, \(\hat{c}_{j,\sigma}\) is the annihilation operator for a fermion in the \(j\)th site with spin \(\sigma\). Also, \(n_{j,\sigma} = c_{j,\sigma}^{\dagger} c_{j,\sigma}\) is the fermionic number operator. In the next section, we will show that the Hamiltonian for the cold Fermi gas in a 1D bichromatic optical lattice can be reduced to the Fermi–Hubbard Hamiltonian (equation (1)). In this section, we will derive the equivalent bosonization Hamiltonian of the Fermi–Hubbard Hamiltonian. The bosonization technique is a powerful tool for studying the spectrum of low-lying excitations and correlation functions of 1D systems. Let us first look at the non-interacting limit \((U = 0)\). In this case, the Hamiltonian can be easily diagonalized by means of Fourier transformation. We define

\[
c_{j,\sigma} = \frac{1}{\sqrt{L}} \sum_{k} \left( c_{k,\sigma}^{\dagger} - i(-1)^{k} e^{i\pi k L} c_{k,\sigma} \right) e^{2i(kd)}.
\]

Here \(c_{k,\sigma}^{\dagger} - i(-1)^{k} e^{i\pi k L} c_{k,\sigma}\) is acting as collective Fourier space \(j\)-dependent operator. Also, \(L\) is the number of lattice sites with periodic boundary conditions \(c_{j+L,\sigma} = c_{j,\sigma}\). As the fermions move from one well to the next, they acquire an additional phase, which depends on the height of the barrier. As the height alternates, the phase also alternates. This picture is conveniently represented by the \(j\)-dependent inverse Fourier transform of equation (2). Substituting equation (2) into the non-interacting Fermi–Hubbard Hamiltonian, we have

\[
H = -2J_{0} \cos 2kd \left( c_{k,\sigma}^{\dagger} c_{-k,\sigma}^{\dagger} - c_{k,\sigma} c_{-k,\sigma} \right) - \Delta_{0} \sin 2kd \left( c_{k,\sigma}^{\dagger} c_{k,\sigma}^{\dagger} + c_{-k,\sigma} c_{-k,\sigma} \right).
\]

Finally, \(H\) can be brought to diagonal form by defining operators

\[
c_{k,\sigma}^{\dagger} = \frac{1}{\sqrt{L}} \sum_{j} \left( \alpha_{k} f_{j,\sigma}^{\dagger} + \beta_{k} f_{j,\sigma} \right)
\]

and

\[
c_{k,\sigma} = \frac{1}{\sqrt{L}} \sum_{j} \left( \alpha_{k} f_{j,\sigma}^{\dagger} + \beta_{k} f_{j,\sigma} \right),
\]

where \(|\alpha_{k}|^{2} + |\beta_{k}|^{2} = 1\) and \(\epsilon_{k} = \sqrt{4J_{0}^{2} \cos^{2} 2kd + \Delta_{0}^{2} \sin^{2} 2kd}\). This yields

\[
H = -\sum_{k,\sigma} \epsilon_{k} f_{k,\sigma}^{\dagger} f_{k,\sigma}.
\]
The ground state for $N$ fermions corresponds to filling up all the states from the lowest energy up, until the $N$ lowest-energy orbitals are filled up (taking into account spin degeneracy). The highest occupied level is the Fermi level, its energy is the Fermi energy $E_F$ and its wave vector is the Fermi wave vector $k_F$. The relationship between $N$ and $k_F$ is given as $N = 2k_F L / \pi$ or $n = N / L = 2k_F / \pi$. When we take into account interactions and if $U \ll J_F$ (perturbative region), it is natural to assume that only low-energy states will be affected. This is reasonable within second-order perturbation theory. We now introduce in the usual way the right movers (around $+k_F$) and left movers (around $-k_F$). We then have two linearized spectra around the two Fermi points $\epsilon(k) = v_F(k - k_F)$ (right-moving branch) and $\epsilon(k) = -v_F(k + k_F)$ (left-moving branch), according to the sign of the velocities. The relationship between this spectrum and the lattice one is given by

$$v_F = \frac{\partial \epsilon(k)}{\partial k} \bigg|_{k=k_F} = \frac{-d \sin (4k_Fd) (4J_0^2 - \Delta_0^2)}{\hbar \sqrt{(4J_0^2 \cos^2 2k_Fd + \Delta_0^2 \sin^2 2k_Fd)}}. \quad (7)$$

Note that we effectively restrict ourselves to low energies. Corresponding to the right- and left-moving fermions, we can introduce fermion annihilation (creation) operators $c_{\nu,\sigma}$ ($c_{\nu,\sigma}^\dagger$), where $\nu = R, L$ and the respective density fluctuation operators $\rho_{\nu,\sigma}(q) = \sum_k c_{\nu,\sigma}^\dagger (k + q) c_{\nu,\sigma}^\dagger (k)$. Note that $\rho_{\nu,\sigma}(-q) = \rho_{\nu,\sigma}^\dagger (q)$. These particle–hole excitations can be written in terms of bosonic creation and annihilation operators:

$$\rho_{\nu,\sigma}^\dagger (q) = \begin{cases} \sqrt{\frac{Lq_\nu}{2\pi}} b_{\nu,\sigma}^\dagger & \text{if } q > 0, \\ \sqrt{\frac{Lq_\nu}{2\pi}} b_{\nu,\sigma} & \text{if } q < 0. \end{cases} \quad (8)$$

The normally ordered number operator is defined as

$$\hat{N}_{\nu,\nu} = \sum_k c_{\nu,\nu}^\dagger (k) c_{\nu,\nu} (k) \quad \nu = R, L \quad (9)$$

and

$$\hat{N}_{\nu,\nu,\nu,\nu} = \frac{1}{\sqrt{2}} \left( \hat{N}_{\nu,\nu} \pm \hat{N}_{\nu,\nu} \right). \quad (10)$$

Here, $\ldots$ indicates the normal ordering of the operators. As part of the bosonization process, we also introduce boson field operators:

$$\phi_{R,L,\nu} = \frac{i}{\sqrt{L}} \sum_{q > 0} \frac{1}{\sqrt{q}} e^{-iq/2} (e^{i\alpha x} b_{R,\nu,\sigma}^\dagger - e^{i\alpha x} b_{L,\nu,\sigma}^\dagger), \quad (11)$$

$$\phi_{\nu,\nu,\nu,\nu} = \frac{1}{\sqrt{2}} (\phi_{\nu,\nu} (x) \pm \phi_{\nu,\nu} (x)), \quad (12)$$

$$b_{\nu,\nu,\nu,\nu} = \frac{1}{\sqrt{2}} \left( b_{\nu,\nu}^\dagger \pm b_{\nu,\nu}^\dagger \right). \quad (13)$$

The bosonized Hamiltonian is written as

$$H = H_0 + H_{\text{int}}, \quad (14)$$

where

$$H_0 = \hbar \sum_{\nu} \left\{ \frac{v_F}{2} \int_{-L/2}^{L/2} dx \sum_{\nu} : (\partial_x \phi_{\nu,\nu})^2 + \frac{\pi v_F}{L} \sum_{\nu} \hat{N}_{\nu,\nu}^2 : \right\}, \quad (15)$$

and

$$H_{\text{int}} = \hbar \sum_{\nu} \left\{ \frac{1}{4} \left[ \frac{g_{4k}}{2} \left( \hat{N}_{\nu,\nu}^2 + \hat{N}_{\nu,\nu}^2 \right) + g_{2\lambda} \hat{N}_{\nu,\nu} \hat{N}_{\nu,\nu} \right] \right\}$$

$$+ \hbar \sum_{\lambda = \perp} \int_{-L/2}^{L/2} dx \sum_{\nu} : \left[ \frac{g_{4k}}{2} \sum_{\nu} : (\partial_x \phi_{\nu,\nu})^2 : \right.$$ \n
$$- g_{2\lambda} \left( \phi_{\nu,\nu}^\dagger (\phi_{\nu,\nu}) \right) \left( \phi_{\nu,\nu}^\dagger (\phi_{\nu,\nu}) \right) : \right\}. \quad (16)$$

Here, $g_{4\lambda,c,s} = \pm g_{s,\perp}$, $i = 2, 4$. Note that $g_{s,\perp}$ is the intraspecies interaction, which is zero for cold fermionic atoms. $g_{s,\perp} = U d$ is the interspecies interaction for cold fermionic atoms. Here $d$ is the lattice spacing. $g_{2\lambda}$ is the strength of the forward scattering between particles belonging to different branches and with different spin states, whereas $g_{4\lambda}$ is the strength of the forward scattering between particles belonging to the same branches and with different spin states. We now have two decoupled sectors corresponding to charge and spin excitations. The Hamiltonian in equation (16) can be diagonalized by the following Bogoliubov transformation:

$$\tilde{d}_{\nu,\lambda}^\dagger = \cosh \gamma \tilde{b}_{\nu,\lambda}^R + \sinh \gamma \tilde{b}_{\nu,\lambda}^L, \quad (17)$$

$$\tilde{d}_{\nu,\lambda}^L = \sinh \gamma \tilde{b}_{\nu,\lambda}^R + \cosh \gamma \tilde{b}_{\nu,\lambda}^L, \quad (18)$$

where tanh $2\gamma = \lambda = g_{2\lambda}/2v_F g_{s,\perp}$. Finally, the diagonalized Hamiltonian is

$$H = \hbar \sum_{\mu} u_{\mu} \sum_{q > 0} \tilde{d}_{\nu,\lambda}^\dagger \tilde{d}_{\nu,\lambda} + \frac{\hbar \pi}{2L} \left( \nu_{\lambda,\nu} \hat{N}_{\nu,\nu} + \nu_{\lambda,\nu} \hat{J}_{\nu,\nu} \right), \quad (19)$$

where

$$\hat{N}_{\nu,\nu} = \hat{N}_{\nu,\nu} + \hat{N}_{\nu,\nu} \quad (20)$$

$$\hat{J}_{\nu,\nu} = \hat{N}_{\nu,\nu} - \hat{N}_{\nu,\nu} \quad (21)$$

$$\nu_{\lambda,\nu} = u_{\lambda} \quad (22)$$

$$\nu_{\lambda,\nu} = u_{\lambda} g_{s,\perp} \quad (23)$$

$$g_{s,\perp} = \sqrt{\frac{2\pi v_F + g_{4\lambda} - g_{2\lambda}}{2\pi v_F + g_{4\lambda} + g_{2\lambda}}}, \quad (24)$$

$$u_{\lambda} = \sqrt{\left( 1 + \frac{g_{4\lambda}}{2\pi v_F} \right)^2 - \left( \frac{g_{2\lambda}}{2\pi v_F} \right)^2}. \quad (25)$$

Here, $\lambda = c, s$. In the above discussions, we have neglected the backscattering and Umklapp terms. For the case of cold Fermi gases, this is justified if the optical lattice depth is not large. We now consider in the next two sections two specific systems, namely cold fermionic gases in optical superlattices and a system of coupled QDs, and we study some properties as a function of $\alpha = \Delta_0 / 2J_0$. 

3
3. The HM for a cold Fermi gas in a bichromatic optical lattice

We consider an elongated cigar-shaped (quasi-1D) dilute gas of fermionic atoms of mass \( m \) with two internal ground states, \( |\sigma = \uparrow, \downarrow\rangle \), representing a spin-1/2 system. We will assume that the two internal levels are equally populated, i.e. \( N_\uparrow = N_\downarrow \). The atoms are cooled below the Fermi-degeneracy temperature \( k_B T_F \sim N \hbar \omega_0 \), where \( N = N_\uparrow + N_\downarrow \) is the total number of particles. The condition for a quasi-1D system is a tight transverse harmonic trapping with frequency \( \omega_\perp \) exceeding the characteristic energy scale of the longitudinal motion. In this way the transverse degrees of freedom are frozen. Because of quantum degeneracy, the longitudinal motion has all the energy levels up to the Fermi energy \( \epsilon_F \approx k_B T_F \) filled. Typical values of \( \omega_\perp \) and \( \omega_0 \) (frequency of longitudinal confinement) are in the range of \( 2\pi (300-400) \) Hz and \( 2\pi (2-10) \) Hz, respectively. Thus, we require the total number of particles to be restricted by \( N < \omega_\perp / \omega_0 \), which is typically of the order of a few hundreds. Because of Pauli’s principle, at low temperature, only s-wave collisions between atoms in different internal states are allowed. Therefore, all the relevant interactions are characterized by intercomponent scattering length \( a \). The strength of the effective 1D interaction is \( g = 2 \pi \hbar^2 / a m l_\perp^2 \), where \( l_\perp = \sqrt{\hbar / m \omega_\perp} \) is the harmonic oscillator transverse length and \( a < l_\perp \).

Thus, the system is described by the following 1D Hamiltonian:

\[
H = \sum_\sigma \int dx \psi_\sigma^\dagger(x) \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V_{\text{ext}}(x) \right) \psi_\sigma \\
+ g \int dx \psi_\uparrow^\dagger(x) \psi_\uparrow^\dagger(x) \psi_\downarrow (x) \psi_\downarrow (x),
\]

where \( \psi_\sigma \) is the 1D field operator for atoms in state \( \sigma \), and \( g \) is the two-body interaction energy. The external potential \( V_{\text{ext}}(x) = V_1(x) + V_2(x) \) includes both the longitudinal confinement \( V_1(x) = \frac{1}{2} m \omega_\perp^2 x^2 \) and the two-color optical lattice potential \( V_2(x) = V_1 \cos^2 \frac{2\pi}{d_1} x + V_2 \cos^2 \frac{2\pi}{d_2} x \). Here \( d_1 \) and \( d_2 > (d_1) \) are, respectively, the primary and secondary lattice constants. \( V_1 \) and \( V_2 \) are the respective amplitudes. The secondary lattice acts as a perturbation and hence \( V_1 > V_2 \).

We will take the particular case \( d_2 = 2d_1 = 2d \) as shown in figure 1. In addition, we will consider the case when the optical lattice dominates over the harmonic potential. We expand the atomic field operators in the lowest-band Wannier basis

\[
\psi_\sigma = \sum_j W(x - x_j) \phi_{j,\sigma},
\]

where \( W(x - x_j) \) is the Wannier function centered at the \( j \)th site. Substituting equation (27) into (26) and retaining only the nearest-neighbor terms, we obtain the Fermi–Hubbard Hamiltonian \( H_{\text{FH}} \), as in section 2, as

\[
H_{\text{FH}} = - \sum_{j,\sigma} J_j (\phi_{j+1,\sigma}^\dagger \phi_{j,\sigma}^\dagger + \phi_{j,\sigma}^\dagger \phi_{j+1,\sigma}) + U \sum_j \hat{n}_{j,\uparrow} \hat{n}_{j,\downarrow}.
\]

Here, the on-site energies are taken to be zero. \( J_j \) is the site-dependent tunneling and takes two distinct values, namely \( J_0 + (-1)^j \frac{\Delta}{2} \) (depending on whether \( j \) is odd or even). The strength of the effective on-site interaction energy is \( U = g \int dx |W(x)|^4 \).

The simple form of the Hamiltonian (19) makes the calculation of some physical properties rather straightforward. One important quantity, the compressibility \( \kappa \) of the gas, is written as

\[
\kappa = \frac{\kappa_0}{\kappa_0} \frac{\partial^2 g \rho_c}{\partial \alpha^2}.
\]

Here, \( \kappa_0 \) is the compressibility of the non-interacting gas. The density of the gas is approximated as homogeneous. This is true if the trapping potential along the optical lattice axis is very weak and also the depth of the optical lattice is small. In systems with repulsive interactions, the optical lattice reduces the compressibility of the system, since the effect of repulsion is enhanced by the squeezing of the condensate wave function in each well. \( u_c/\rho_c \) fixes the energy needed to change the particle density. From figure 2, we note that the compressibility decreases with the parameter \( \alpha \). This means that with increasing strength of the secondary lattice, the energy required to change the particle density increases.

Figure 3 shows a plot of the spin velocity \( u_s \) and charge velocity \( u_c \) as a function of the quasi-momentum \( (kFd) \) and the parameter \( \alpha \). As the condensate moves across the Brillouin zone, the difference between the spin and charge velocity increases and is maximum at \( \pi / 4 \). This suggests an effective mechanism to observe the spin–charge separation. One can move the condensate across the Brillouin zone by accelerating the condensate in the optical lattice. On the other hand, the spin–charge separation decreases with increasing \( \alpha \). This is perhaps due to the fact that as the strength of the secondary lattice increases, the condensate becomes more localized.

At this point, it would be appropriate to discuss the model and the results of [26, 27] and compare them with our model and results. The ordinary one-band HM [28]
describes the movement of indistinguishable fermions on the lattice characterized by a single hopping parameter, generally independent of the electron spin. In the opposite limit, the Falicov–Kimball model (FKM) [29] can be viewed as a modification of the HM, where one kind of fermion is frozen, while the other kind is allowed to move over the lattice. The asymmetric Hubbard model (AHM) described in [27] is an intermediate situation between HM and FKM, where the tunneling parameter is spin dependent or mass dependent (as in the case of two species fermions in an optical lattice). They found that Mott-insulating and metallic regions coexist and when the difference between the fermion masses increases, the metallic phase disappears. In this paper, we describe a situation intermediate between the HM and AHM. Our model is asymmetric in the sense of asymmetric lattice and the hopping parameter is still spin (mass) independent. On the other hand, the same authors studied a different kind of Luttinger liquid superlattices [26], consisting of a repeated pattern of two different Luttinger liquids with different lengths perfectly connected. They found that the spin and charge velocities depend on the ratio of the lengths of the two types of Luttinger liquids. This is equivalent to the result we found, since the tunneling parameter would depend on the length of the Luttinger liquid. Another important similarity we note is the prediction of energy gap (see the expression for $\epsilon_k$), reflecting the superlattice structure. Such a band structure and energy gap was also predicted in [26].

### 4. Linear array of tunnel-coupled QDs

We consider electron transport in a linear array of nearly identical QDs that are electrostatically defined in a 2D electron gas by means of metallic gates on top of a semiconductor heterostructure (GaAs/AlGaAs) as shown in figure 4. The heterostructure contains a 2D electron gas located 470 Å beneath the surface with sheet density $3.7 \times 10^{11}$ cm$^{-2}$ and mobility $5 \times 10^{5}$ cm$^{2}$ V$^{-1}$ s$^{-1}$ [20]. The device can be fabricated with Schottky gates using electron-beam lithography and chrome/gold evaporation on the heterostructure surface. The lithographic size of each QD is $0.5 \times 0.8 \mu$m$^2$. The device is wired with independently tunable gate voltages: one for each tunnel barrier. The individual tunneling rates are determined by the voltages applied to the gates defining the corresponding interdot tunneling barriers. An important advantage of QD described above is that each element of the device is controlled by separate gates and can be individually tested and adjusted [20]. Thus, appropriate gate voltages have to be applied to reproduce the Hamiltonian of equation (1).

An appropriate quantity to study in this system is the conductance $G$ as a function of $U/2J_0$. In deriving $G$, we follow [30]. To begin with, we need the expression for the current in terms of the boson operators as

$$J(\epsilon) = -\frac{eV_F}{L} \sum_q e^{i\epsilon_q} (\rho^R_{q,\sigma} - \rho^L_{q,\sigma}).$$  \hspace{1cm} (30)

The corresponding conductance is calculated from

$$G(x', x) = \int_0^\beta d\lambda \int_0^{\inf} dt < J(x', -i\hbar \lambda)J(x, t)> e^{i\eta},$$  \hspace{1cm} (31)

where $\eta$ is an infinitesimally small positive quantity. The above expression for $G(x', x)$ follows from the definition of the response function. The $\langle \ldots \rangle$ means the thermal average and $\beta = 1/K_B T$ ($K_B$ is the Boltzmann constant and $T$ is the temperature). Performing the integrals over $t$ and $\lambda$, taking the limit $\eta \rightarrow +0$ and using the expressions for $\alpha_k$, $\beta_k$ and $\epsilon_k$ from section 2, we obtain the conductance $G$ as

$$G = G_0 / \sqrt{1 + \gamma},$$  \hspace{1cm} (32)

where $G_0 = 2e^2/h$ and $\gamma = 2U/(\pi \hbar V_F)$. In figure 5, we plot the dimensionless conductivity ($G/G_0$) for repulsive interaction versus the parameter $\alpha = \Delta q/2J_0$. We note that as $\alpha$ increases, the conductivity decreases, which again is an indication of localization of the electrons.
5. The classical Ising chain

It is known that the quantum transitions in the quantum Ising model in $d$ dimensions is intimately connected to finite temperature phase transitions in the classical Ising model in $D = d + 1$ dimensions. The $D = 1$ and $N = 1$ classical Ising model does not show any phase transition but has regions where the correlation length becomes very large, and the properties of these regions are very similar to those in the vicinity of the phase transition points in higher dimensions. Here, we will consider the $D = 1$ and $N = 1$ classical spin ferromagnet, more commonly known as the ferromagnetic Ising chain. This chain has the partition function

$$Z = \sum_{\sigma^z_i} \exp(-H),$$

where $\sigma^z_i$ are Ising spins on sites $i$ of a chain, which take the values $\pm 1$, and $H$ is given by

$$H = -\sum_{i=1}^{M} (J_0 - (-1)^i \Delta_0/2) \sigma^z_i \sigma^z_{i+1} - H \sum_{i=1}^{M} \sigma^z_i.$$  

(34)

Here $M$ is the total number of Ising spins and $h$ is the external magnetic field. We will assume periodic boundary conditions, therefore $\sigma^z_{M+1} = \sigma^z_1$. Now following Ising, we write $Z$ as a trace over a matrix product with one matrix for every site,

$$Z = \sum_{\sigma^z_i} \prod_{i=1}^{M} T_{1,i}(\sigma^z_i, \sigma^z_{i+1}) T_2(\sigma^z_i),$$

(35)

where $T_{1,i}(\sigma^z_i, \sigma^z_{i+1}) = \left( e^{(J_0 - (-1)^i \Delta_0/2)}, e^{-(J_0 - (-1)^i \Delta_0/2)} \right)$, and $T_2(\sigma^z_i) = \left( e^h, 0 \right)$.

$T_{1,i}$ will have different values depending on whether $i$ is even or odd. The matrix $T_1 T_2$ is identified as the transfer matrix of the Ising chain. Let us now define a matrix $T_3 = T_{1,i=\text{even}} T_{1,i=\text{odd}}^{-1}$. In the limit $M \to \infty$, one can show that

$$Z = \text{Tr}[T_1^2 T_2^2 T_3]^{M/2} = \epsilon_1^{M/2} + \epsilon_2^{M/2},$$

(38)

where $\epsilon_1$ and $\epsilon_2$ are the eigenvalues of $T_1^2 T_2^2 T_3$. For the case $h = 0$ (no magnetic field), $T_2 = 1$. The eigenvalues are found to be

$$\epsilon_1 = \frac{2 \sinh(J_0)}{\sinh(2J_0 + \Delta_0) + \sinh(\Delta_0)},$$

$$\epsilon_2 = \frac{2 \sinh(J_0)}{\sinh(2J_0 + \Delta_0) - \sinh(\Delta_0)}.$$  

(39)

(40)

The eigenvalue $\epsilon_1$ increases with increasing $\Delta_0$ and saturates at 1.2 at $\Delta_0 = 2 J_0$, while the other eigenvalue $\epsilon_2$ goes to zero at $\Delta_0 = 2 J_0$.

Now we calculate the correlation function exactly. For simplicity, we consider the case of zero external field ($h = 0$) and describe the two-point correlator

$$\langle \sigma^z_i \sigma^z_j \rangle = \frac{1}{Z} \sum_{\sigma^z_i} e^{(-H)\sigma^z_i \sigma^z_j}.$$  

(41)

In the limit of an infinite chain, the two-point correlator in terms of continuous variables is derived as

$$C(\tau, 0) = \langle \sigma(\tau) \sigma(0) \rangle e^{-|\tau|/\xi},$$

(42)

where the correlation length $\xi$ is written as

$$\frac{1}{\xi} = \frac{1}{a} \left\{ \ln[\coth(J_0)] + \frac{2}{a} \ln \left[ \frac{\sinh(2J_0) + \sinh(\Delta_0)}{\sinh(2J_0) - \sinh(\Delta_0)} \right] \right\}.$$  

(43)

Here, $\tau = j a$, where $a$ is the lattice spacing. Figure 6 displays the two-point correlator as a function of lattice site $j$ for three values of $\Delta_0/2 J_0 = 0.01, 0.25, 0.5$. Clearly, we see that as $\Delta_0/2 J_0$ increases the correlation decays faster, indicating that the spins are getting localized.

6. Conclusions

In conclusion, we have studied three 1D superlattice systems (characterized by two tunneling parameters), namely atomic gases in a 1D superlattice, a linear 1D array of QDs and the 1D classical Ising chain. In particular for atomic gases, we found that as the difference between the two tunneling...
parameters increases, the difference between the spin and charge velocities decreases. This is attributed to the increasing localization of the atoms in the wells of the optical lattice. On the other hand, for the case of a linear array of QDs, as the difference between the two tunneling parameters increases, the conductance decreases, which is attributed to the pinning of the electrons. For the classical Ising chain, the two-point correlator decreases with increasing strength of the superlattice, which is attributed to the localization of the spins. In general, we conclude that atoms, electrons and spins are comparatively more localized in a superlattice structure. This study demonstrates that by tuning the two tunneling parameters, one can coherently control the transport properties of a superlattice structure.

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