Abstract

We prove existence of the scaling limit of the invasion percolation cluster (IPC) on a regular tree. The limit is a random real tree with a single end. The contour and height functions of the limit are described as certain diffusive stochastic processes.

This convergence allows us to recover and make precise certain asymptotic results for the IPC. In particular, we relate the limit of the rescaled level sets of the IPC to the local time of the scaled height function.

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1 Introduction

Invasion percolation on an infinite connected graph is a random growth model which is closely related to critical percolation, and is a prime example of self-organized criticality. It was introduced in the eighties by Wilkinson and Willemsen [14] and first studied on the regular tree by Nickel and Wilkinson [12]. The relation between invasion percolation and critical percolation has been studied by many authors (see for instance [5, 9]). More
recently, Angel, Goodman, den Hollander and Slade [2] have given a structural representation of the invasion percolation cluster on a regular tree, and used it to compute the scaling limits of various quantities related to the IPC such as the distribution of the number of invaded vertices at a given level of the tree.

Fixing a degree $\sigma \geq 2$, we consider $T = T_\sigma$: the rooted regular tree with index $\sigma$, i.e. the rooted tree where every vertex has $\sigma$ children. Invasion percolation on $T$ is defined as follows: Edges of $T$ are assigned weights which are i.i.d. and uniform on $[0,1]$. The invasion percolation cluster on $T$, denoted IPC, is grown inductively starting from a subgraph $I_0$ consisting of the root of $T$. At each step $I_{n+1}$ consists of $I_n$ together with the edge of minimal weight in the boundary of $I_n$. The invasion percolation cluster IPC is the limit $\bigcup I_n$.

We consider the infinite tree IPC as a metric space endowed with the shortest path metric, and consider its scaling limit in the sense of weak limits w.r.t. the Gromov-Hausdorff topology. The limit is a random $\mathbb{R}$-tree — a topological space with a unique rectifiable simple path between any two points. A useful way to describe such $\mathbb{R}$-trees is in terms of their contour or height functions (see subsection 4.5 below). Note that the IPC is infinite, so that we take a fixed object and only change the metric when taking the scaling limit.

**Theorem 1.1.** The IPC has a scaling limit w.r.t. local Gromov-Hausdorff topology, which is a random $\mathbb{R}$-tree.

A key point in our study is that the contour function (as well as height function and Łukaciewicz path) of an infinite tree do not generally encode the entire tree. If the various encodings of trees are applied to infinite trees they describe only the part of the tree to the left of the leftmost infinite branch. We present two ways to overcome this difficulty. Both are based on the fact (see [2]) that the IPC has a.s. a unique infinite branch. Following Aldous [1] we define a sin-tree to be an infinite one-ended tree (i.e. with a single infinite branch).

The first approach is to use the symmetry of the underlying graph $T$ and observe that the infinite branch of the IPC (called the backbone) is independent of the metric structure of the IPC. Thus for all purposes involving only the metric structure of the IPC we may as well assume (or condition) that the backbone is the rightmost branch of $T$. We denote by $\mathcal{R}$ the IPC under this condition. The various encodings of $\mathcal{R}$ encode the entire tree.

The second approach is to consider a pair of encodings, one for the part of the tree to the left of the backbone, and a second encoding the part to the
right of the backbone. This is done by considering also the encoding of the
reflected tree IPC. The reflection of a plane tree is defined to be the same
tree with the reversed order for the children of each vertex. The uniqueness
of the backbone implies that together the two encodings determine the entire
IPC.

In order to describe the limits we first define the process \( L(t) \) which is
the lower envelope of a Poisson process on \((\mathbb{R}^+)^2\). Given a Poisson process \( \mathcal{P} \) in the quarter plane, \( L(t) \) is defined by

\[
L(t) = \inf\{y : (x, y) \in \mathcal{P} \text{ and } x \leq t\}.
\]

Our other results describe the scaling limits of the various encodings of
the trees in terms of solutions of

\[
Y_t = B_t - \int_0^t L(-\underline{Y}_s) \, ds, \quad \mathcal{E}(L)
\]

where \( \underline{Y}_s = \inf_{0 \leq u \leq s} Y_u \) is the infimum process of \( Y \). The reason for the
notation is that we also consider solutions of equations \( \mathcal{E}(L/2) \) where in the
above, \( L \) is replaced by \( L/2 \). Note that by the scale invariance of the Poisson
process, \( kL(kt) \) has the same law as \( L(t) \). Hence the scaling of Brownian
motion implies that the solution \( Y \) has Brownian scaling as well.

We work primarily in the space \( C(\mathbb{R}^+, \mathbb{R}^+) \) of continuous functions from
\( \mathbb{R}^+ \) to itself with the topology of locally uniform convergence. We consider
three well known and closely related encodings of plane trees, namely the
Lukaciewicz path, and the contour and height functions (all are defined
in subsection 1.1.2 below). The three are closely related and indeed their
scaling limits are almost the same. The reason for the triplication is that
the contour function is the simplest and most direct encoding of a plane
tree, whereas the Lukaciewicz path turns out to be easier to deal with in
practice. The height function is a middle ground.

For the IPC conditioned on the backbone being on the right, we denote
its Lukaciewicz path (resp. height and contour functions) by \( V_\mathbb{R} \) (resp. \( H_\mathbb{R} \)
and \( C_\mathbb{R} \)). It is interesting that the scaling limit depends on \( \sigma \) only by a
multiplicative factor. The constant \( \gamma = \frac{\sigma - 1}{\sigma} \) will be used in giving this
factor.
Theorem 1.2. We have the weak limits in $\mathcal{C}(\mathbb{R}^+, \mathbb{R})$:

\begin{align*}
(k^{-1}V_R(k^2t))_{t \geq 0} &\rightarrow \left(\gamma^{1/2}(Y_t - \bar{Y}_t)\right)_{t \geq 0} \\
(k^{-1}H_R(k^2t))_{t \geq 0} &\rightarrow \left(\gamma^{-1/2}(2Y_t - 3\bar{Y}_t)\right)_{t \geq 0} \\
(k^{-1}C_R(2k^2t))_{t \geq 0} &\rightarrow \left(\gamma^{-1/2}(2Y_t - 3\bar{Y}_t)\right)_{t \geq 0}
\end{align*}

as $k \rightarrow \infty$, where $(Y_t)_{t \geq 0}$ solves $\mathcal{E}(L)$ (and is the same solution in all three limits).

To put this theorem into context, recall that the Lukaciewicz path of a critical Galton-Watson tree is an excursion of random walk with i.i.d. steps. From this it follows that the path of an infinite sequence of critical trees scales to Brownian motion. The height and contour functions of the sequence are easily expressed in terms of the Lukaciewicz path and, assuming the branching law has second moments, are seen to scale to reflected Brownian motion (cf Le Gall [11]). Duquesne and Le Gall generalized this approach in [7], and showed that the genealogical structure of a continuous-state branching process is similarly coded by a height process which can be expressed in terms of a Lévy process, and that this is also the limit of various Galton-Watson trees with heavy tails.

The case of sin-trees is considered by Duquesne [6] to study the scaling limit of the range of a random walk on a regular tree. His techniques suffice for analysis of the IIC, but the IPC requires additional ideas, the key difficulty being that the Lukaciewicz path is no longer a Markov process. The scaling limit of the IIC turns out to be an illustrative special case of our results, and we will describe its scaling limit as well (in Section 3.6).

For the unconditioned IPC we define its left part IPC$_G$ to be the sub-tree consisting of the backbone and all vertices to its left. The right part IPC$_D$ is defined as the left part of the reflected IPC. We can now define $V_G$ and $V_D$ to be respectively the Lukaciewicz paths for the left and right parts of the IPC, and similarly define $H_G, H_D, C_G, C_D$ (see also subsection 1.1.3 below).

Theorem 1.3. We have the weak limits in $\mathcal{C}(\mathbb{R}^+, \mathbb{R})$

\begin{align*}
(k^{-1}V_G(k^2t), V_D(k^2t))_{t \geq 0} &\rightarrow \gamma^{1/2} \left(\bar{Y}_t - \bar{Y}_t, \bar{Y}_t - \bar{Y}_t\right)_{t \geq 0} \\
(k^{-1}H_G(k^2t), H_D(k^2t))_{t \geq 0} &\rightarrow \gamma^{-1/2} \left(\bar{Y}_t - 2\bar{Y}_t, \bar{Y}_t - 2\bar{Y}_t\right)_{t \geq 0} \\
(k^{-1}C_G(2k^2t), C_D(2k^2t))_{t \geq 0} &\rightarrow \gamma^{-1/2} \left(\bar{Y}_t - 2\bar{Y}_t, \bar{Y}_t - 2\bar{Y}_t\right)_{t \geq 0}
\end{align*}
where \((Y_t)_{t \geq 0}\) and \(\widetilde{Y}_t\) \(t \geq 0\) are independent solutions of \(\mathcal{E}(L/2)\).

1.1 Background

1.1.1 Structure of the IPC

We now give a brief overview of the IPC structure theorem from [2], which is the basis for the present work. First of all, IPC contains a single infinite branch, called the backbone and denoted BB. The backbone is a uniformly random branch in the tree (in the natural sense). From the backbone emerge, at every height and on every edge away from the backbone, subcritical percolation clusters. This relates the IPC to the incipient infinite cluster (IIC), defined and discussed by Kesten [10] (see also [3]). The IIC consists of an infinite backbone from which emerge critical percolation clusters, hence it stochastically dominates the IPC. In particular, both IPC and IIC are sin-trees.

The subcritical percolation parameter of the percolation clusters attached to the backbone of the IPC increases towards the critical parameter \(p_c = \sigma^{-1}\) as one moves up along the backbone. This explains why the IPC and IIC resemble each other far above the root. However, the analysis of [2] shows that the convergence of the parameter of the attached clusters is slow enough that \(r\)-point functions and other measurable quantities such as level sizes possess different scaling limits.

All that remains is to describe the percolation parameter for each of the trees attached to the backbone. We will only recall part of the description here. These are given in terms of a certain Markov chain \(W_n\) with explicitly stated transition probabilities. The chain \(W_n\) is non-increasing and satisfies \(W_n \overset{n \to \infty}{\longrightarrow} p_c = \sigma^{-1}\). We then define \(\hat{W}_n = W_n \zeta(W_n)\), where \(\zeta(p)\) is the probability that the \(p\)-percolation cluster along a particular branch from the root of \(T\) is finite. The percolation parameter for the sub-trees attached to BB\(_n\) is \(\hat{W}_n\), which is a non-increasing sequence that converges a.s. to \(p_c\).

We will only be concerned with the scaling limit of \(\hat{W}_n\), which is the lower envelope process \(L(t)\) defined above. To be precise, writing \([x]\) for the integer part of \(x\), we have for any \(\varepsilon > 0\)

\[
\left( k \left( \sigma W_{[kt]} - 1 \right) \right)_{t \geq \varepsilon} \overset{k \to \infty}{\longrightarrow} \left( L(t) \right)_{t \geq \varepsilon}
\]

\[
\left( k \left( 1 - \sigma \hat{W}_{[kt]} \right) \right)_{t \geq \varepsilon} \overset{k \to \infty}{\longrightarrow} \left( L(t) \right)_{t \geq \varepsilon}
\]

(4)

The process \(L_t\) diverges as \(t \to 0\), which somewhat complicates the study of the IPC close to the root.
Note that setting $W_n \equiv p_c$ in the above description gives rise to the IIC on the one hand, while in the scaling limit $L$ is replaced by 0. This enables us to use a common framework for both clusters.

1.1.2 Encodings of finite trees

For completeness we include here the definition of the various tree encodings we are concerned with. We refer to Le Gall \[11\] for further details in the case of finite trees and to Duquesne \[6\] in the case of sin-trees discussed below.

A rooted plane tree $\theta$ (also called an ordered tree) is a tree with a description as follows. Vertices of $\theta$ belong to $\bigcup_{n \geq 0} \mathbb{N}^n$. By convention, $\emptyset \in \mathbb{N}^0$ is always a vertex of $\theta$ which is called the root. For a vertex $v \in \theta$, we let $k_v = k_v(\theta)$ be the number of children of $v$ and whenever $k_v = k > 0$, these children are denoted $v_1, \ldots, v_k$. In particular, the $i^{th}$ child of the root is simply $i$, and if $vi \in \theta$ then $\forall 1 \leq j < i$, $v_j \in \theta$ as well. Edges of $\theta$ are the edges $(v, vi)$ whenever $vi \in \theta$. Note that the set of edges of $\theta$ are determined by the set of vertices and vice-versa, which allows us to blur the distinction between a tree and its set of vertices. The $k^{th}$ generation of a tree contains every vertex $v \in \theta \cap \mathbb{N}^k$, so that the $0^{th}$ generation consists exactly of the root. Define $\# \theta$ to be the total number of vertices in $\theta$.

Let $(v_i)_{0 \leq i < \# \theta}$ be the vertices of $\theta$ listed in lexicographic order, so that $v_0 = \emptyset$. The Lukaciewicz path $V$ of $\theta$ is the continuous function $(V_t = V_t^\theta, t \in [0, \# \theta])$ defined as follows: For $n \in \{1, \ldots, \# \theta\}$

$$V_n = V_n^\theta := \sum_{i=0}^{n-1} (k_{v_i} - 1),$$

and between integers $V$ is interpolated linearly.

The values $V_n$ are also given by the following right description of the Lukaciewicz path. This description is simpler to visualize, though we do not know of a reference for it. For $v \in \theta$, consider the subtree $\theta^v \subset \theta$ formed by all the vertices which are smaller or equal to $v$ in the lexicographic order. Let $n(v, \theta)$ be the number of edges connecting vertices of $\theta^v$ with vertices of $\theta$.

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\[1\] In \[11, 7\], the Lukaciewicz path is defined as a piecewise constant, discontinuous function, but there the case when the scaling limit of this path is discontinuous is also treated. Note that only the values of $V_n, n \in \{1, \ldots, \# \theta\}$ are needed to recover the tree $\theta$. Moreover, in our case, $\sup_{i \geq 0} |V_{i+1} - V_i|$ is bounded by $\sigma$, so that the eventual scaling limit will be continuous. The advantage of our convention is that it allows us to consider locally uniform convergence of the rescaled Lukaciewicz paths in a space of continuous functions.
\( \theta \setminus \theta^v \). Then
\[ V(k) = n(v^k, \theta) - 1. \]

The reason we call this the right description is that \( n(v, \theta) \) is also the number of edges attached on the right side of the path from \( \emptyset \) to \( v \). It is straightforward to check that this description is consistent with other definitions.

The height function is the second encoding we wish to consider. We also define it to be a piecewise linear function with \( H(k) \) the height of \( v^k \) above the root. It is related to the Lukaciewicz path by
\[ H(n) = \#\{k < n : V_k = \min\{V_k, \ldots, V_n\}\}. \]

Finally, the contour function of \( \theta \) is obtained by considering a walker exploring \( \theta \) at constant unit speed, starting from the root at time 0, and going from left to right. Each edge is traversed twice (once on each side), so that the total time before returning to the root is \( 2(\#\theta - 1) \). The value \( C_\theta(t) \) of the contour function at time \( t \in [0, 2(\#\theta - 1)] \) is the distance between the walker and the root at time \( t \).

It is straightforward to check that the Lukaciewicz path, height function and contour function each uniquely determine — and hence represent — any finite tree \( \theta \). Figure 1 illustrates these definitions, as they are easier to understand from a picture.

At times it is useful to encode a sequence of finite trees by a single function. This is done by concatenating the Lukaciewicz paths or height function of the trees of the sequence. Note that when coding a sequence of trees, jumping from one tree to the next corresponds to reaching a new integer infimum in the Lukaciewicz path, while it corresponds to a visit to 0 in the height process.

### 1.1.3 Encoding sin-trees

While the definitions of Lukaciewicz path, and height and contour functions extend immediately to infinite (discrete) trees, these paths generally no longer encode a unique infinite tree. For example, all the trees containing the infinite branch \( \{\emptyset, 1, 11, 111, \ldots\} \) would have the identity function for height function, so that equal paths correspond to distinct infinite trees. In fact, the only part of an infinite tree which one can recover from the the height and contour functions is the sub-tree that lies left of the left-most infinite branch. The Lukaciewicz path encodes additionally the degrees of vertices along the left-most infinite branch.

\(^2\)Again, in [11], the height function of a non-degenerate tree is discontinuous.
A tree, its Lukaciewicz path $V$, its height function $H$, and its contour function $C$.

Figure 1: A finite tree and its encodings.
However, if we restrict the encodings to the class of trees whose only infinite branch is the rightmost branch, then the three encodings still correspond to unique trees. In particular, observe that IPC\(_G\) and \(R\) are fully encoded by their Łukaciewicz paths (as well as by their height, or contour functions). That is the reason we begin our discussion with these conditioned objects.

Not surprisingly, it is possible to encode any sin-tree, such as the IIC and IPC, by using two coding paths, one for the part of the tree lying to the left of the backbone, and one for the part lying to its right. More precisely, suppose \(T\) is a sin-tree, and BB denotes its backbone. The left tree is defined as the set of all vertices on or to the left of the backbone:

\[ T_G := \bigcup_{v \in BB} T^v = \{ x \in T : \exists v \in BB, x \leq v \}. \]

We do not define the right-tree of \(T\) as the set of vertices which lie on or to the right of the backbone. Rather, in light of the way the encodings are defined, it is easier to work with the mirror-image of \(T\), denoted \(\overline{T}\) and defined as follows: Since a plane tree is a tree where the children of each vertex are ordered, \(\overline{T}\) may be defined as the same tree but with the reverse order on the children at each vertex. We then define

\[ T_D = (\overline{T})_G. \]

Obviously, only the rightmost branches of \(T_G, T_D\) are infinite, so the Łukaciewicz paths \(V_G, V_D\), of \(T_G, T_D\), do encode uniquely each of these two trees (and so do the height functions \(H_G, H_D\) and the contour functions \(C_G, C_D\)). Therefore, the pair of paths \((V_G, V_D)\) encodes \(T\) (and so do the pairs \((H_G, H_D), (C_G, C_D)\)). Note that \(H_G, C_G\) are also respectively the height and contour functions of \(T\) itself, while \(H_D, C_D\) are respectively the height and contour functions of \(\overline{T}\).

### 1.2 Overview

Let us try to give briefly, and heuristically, some intuition of why Theorem 1.2 holds. For \(t > 0\), the tree emerging from BB\([kt]\) is coded by the \([kt]\)th excursion of \(V\) above 0. Except for its first step, this excursion has the same transition probabilities as a random walk with drift \(\sigma \hat{W}_{[kt]} - 1\), which, by the convergence [4], is approximately \(-L(t)/k\). Additionally, by [2 Proposition 3.1] \(W_k\) is constant for long stretches of time. It is well known (see for instance [8, Theorem 2.2.1]) that a sequence of random walks with
drift $c/k$, suitably scaled, converges as $k \to \infty$ to a $c$-drifted Brownian motion. Thus we expect to find segments of drifted Brownian paths in our limit. According to the convergence (4), the drift is expressed in terms of the $L$-process. This is what the definition of $Y$ expresses.

Thus, the idea when dealing with either the conditioned or the unconditioned IPC is to cut these sin-trees into pieces (which we call segments) corresponding to stretches where $W$ is constant, and to look separately at the convergence of each piece.

In Section 3, we look at the convergence of the rescaled paths coding a sequence of such segments for well chosen, fixed values of the $W$-process. In fact, we consider slightly more general settings which allows us to treat the case of the IIC as well as the various flavours of the IPC.

In Section 4, we prove Theorem 1.2 by combining segments to form $\mathcal{R}$. To deal with the fact that $W$ is random and exploit the convergence (4), we use a coupling argument (see Subsection 4.3). We then prove that the segments fall into the family dealt with in Section 3. Because of the divergence of the $L$-process at the origin, we only perform the above for sub-trees above certain levels, and bound the resulting error separately.

Finally, in section 5, we apply our convergence results to establish asymptotics for level and volume estimates of the IPC, to recover and extend results of [2].

2 Solving $\mathcal{E}(L)$

Claim 2.1. Solutions to $\mathcal{E}(L), \mathcal{E}(L/2)$ are unique in law.

An interesting question is whether the solutions to $\mathcal{E}(L)$ are a.s. pathwise unique (i.e. strong uniqueness). For our purposes uniqueness in law suffices.

Proof. We prove this claim for equation $\mathcal{E}(L)$. The proof for equation $\mathcal{E}(L/2)$ is identical.

Let $Y$ be a solution of $\mathcal{E}(L)$. Since $L$ is positive, $Y_t \leq B_t$. Since $L$ is non-increasing, $\int_t^s L(-Y_s)ds \leq \int_t^s L(-B_s)ds$. For any fixed $\varepsilon > 0$, a.s. for all small enough $s$, $-B_s > s^{1/2-\varepsilon}$, while a.s. for all small enough $u$, $L(u) < u^{-(1+\varepsilon)}$. We deduce that almost surely $\lim_{t \to 0} \int_0^t L(-Y_s)ds = 0$. Thus any solution of $\mathcal{E}(L)$ is continuous.

Let us now consider two solutions $Y^1, Y^2$ of $\mathcal{E}(B, L)$ and fix $\varepsilon > 0$. Introduce

$$f^\varepsilon := \inf \{ t > 0 : L(t) < \varepsilon^{-1} \}$$
and
\[ t_0^\varepsilon := \inf\{t > 0 : -B_t > j^\varepsilon\}, \]
\[ t_1^\varepsilon := \inf\{t > 0 : -Y_1^t > j^\varepsilon\}, \]
\[ t_2^\varepsilon := \inf\{t > 0 : -Y_2^t > j^\varepsilon\}. \]

From the continuity of $Y^1, Y^2$ we have $Y^1(t_1^\varepsilon) = Y^2(t_2^\varepsilon) = -j^\varepsilon$. Moreover, we have a.s. $t_1^\varepsilon \vee t_2^\varepsilon \leq t_0^\varepsilon$, and therefore
\[ t_1^\varepsilon \vee t_2^\varepsilon \xrightarrow{\text{a.s.}} 0. \] (6)

Introduce a Brownian motion $\beta$ independent of $B$ and consider the (SDE)
\[ Z_t^\varepsilon = \beta_t - \int_0^t L(j^\varepsilon - Z_s^\varepsilon) \, ds \quad \mathcal{E}(\varepsilon, L) \]

By standard arguments $\mathcal{E}(\varepsilon, L)$ is pathwise exact.

We then define
\[ Y_1^{1,\varepsilon}_t = \begin{cases} Y_1^t & \text{if } t < t_1^\varepsilon \\ Y_1^t + Z_t^\varepsilon & \text{if } t \geq t_1^\varepsilon \end{cases} \]
\[ Y_2^{2,\varepsilon}_t = \begin{cases} Y_2^t & \text{if } t < t_2^\varepsilon \\ Y_2^t + Z_t^\varepsilon & \text{if } t \geq t_2^\varepsilon \end{cases} \]

Clearly, $Y_1^{1,\varepsilon}, Y_2^{2,\varepsilon}$ are a.s. continuous, and moreover, $Y_1^1$ and $Y_1^{1,\varepsilon}$ have the same distribution, and so do $Y_2^2$ and $Y_2^{2,\varepsilon}$. However, $(Y_i^{i,\varepsilon}(t_i^\varepsilon + t))_{t \geq 0}$ for $i = 1, 2$ have a.s. the same path. From this fact, the continuity of $Y_1^{1,\varepsilon}, Y_2^{2,\varepsilon}$ and (6), it follows that for any $F \in \mathcal{C}_b(\mathbb{C}(\mathbb{R}_+, \mathbb{R}), \mathbb{R})$
\[ |E[F(Y_1^1)] - E[F(Y_2^2)]| = |E[F(Y_1^{1,\varepsilon})] - E[F(Y_2^{2,\varepsilon})]| \]
goes to 0 as $\varepsilon$ goes to 0, which completes the proof. \qed

### 3 Scaling simple sin-trees and their segments

The goal of this section is to establish the convergence of the rescaled paths encoding suitable sequences of well chosen segments. In order to cover the separate cases at once, we will work in a slightly more general context than might seem necessary. We first look at a sequence of particular sin-trees $T^k$ for which the vertices adjacent to the backbone generate i.i.d. subcritical (or
critical) Galton-Watson trees. The law of such a tree is determined by the branching law on these Galton-Watson trees and the degrees along the backbone. If the degrees along the backbone do not behave too erratically and the percolation parameter scales correctly then the sequence of Lukaciewicz paths \( V^k \) has a scaling limit.

The results for the IIC follow directly. Also, we determine the scaling limits of the paths encoding a sequence of subtrees obtained by truncations at suitably vertices on the backbones of \( T^k \). These will be important intermediate results in the proofs of Theorems 1.2 and 1.3.

### 3.1 Notations

Throughout this section we fix for each \( k \in \mathbb{Z}_+ \) a parameter \( w_k \in [0, 1/\sigma] \), and denote by \( (\theta_n^k)_{n \in \mathbb{Z}_+} \) a sequence of i.i.d. subcritical Galton-Watson trees with branching law \( \text{Bin}(\sigma, w_k) \). For each \( k \) we also let \( Z_k \) be a sequence of random variables \( (Z_{k,n})_{n \geq 0} \) taking values in \( \mathbb{Z}_+ \).

**Definition 3.1.** The \((Z_k, \theta_k^k)\)-tree is the sin-tree defined as follows. The backbone BB is the rightmost branch. The vertex BB\(_i\) has \( 1 + Z_{k,i} \) children, including BB\(_{i+1}\). Let \( v_0, \ldots \) be all vertices adjacent to the backbone, in lexicographic order, and identify \( v_n \) with the root of the tree \( \theta_n^k \).

Thus the first \( Z_{k,0} \) of the \( \theta \)'s are attached to children of BB\(_0\), the next \( Z_{k,1} \) to children of BB\(_1\), and so on. We will use the notation \( T^k \) to designate the \((Z_k, \theta_k^k)\)-tree, and \( V^k \) for its Lukaciewicz path.

**Definition 3.2.** Let \( T \) be a sin-tree whose backbone is its rightmost branch. For \( i \in \mathbb{Z}_+ \), let BB\(_i\) be the vertex at height \( i \) on the backbone of \( T \). The \( i \)-truncation of \( T \) is the sub-tree

\[
T^i := \{ v \in T : v \leq \text{BB}_i \}
\]

Thus the \( i \)-truncation of a tree consists of the backbone up to BB\(_i\), and the sub-trees attached strictly below level \( i \). We denote by \( T^{k,i} \) the \( i \)-truncation of \( T^k \), and by \( V^{k,i} \) its Lukaciewicz path. We further define \( \tau^{(i)} \) as the time of the \((i+1)\)th return to 0 of \( V^k \). Observe then that \( V^{k,i} \) coincides with \( V^k \) up to the time \( \tau^{(i)} \), takes the value \(-1\) at \( \tau^{(i)} + 1 \), and terminates at that time.

It will be useful to study first the special case where \( Z_k \) is a sequence of i.i.d. binomial \( \text{Bin}(\sigma, w_k) \) random variables. Observe that in this case the subtrees attached to the backbone are i.i.d. Galton-Watson trees (with
branching law $\text{Bin}(\sigma, w_k)$. We use calligraphed letters for the various objects in this case. We denote the binomial variables $Z_{k,n}$, we write $T^k$ for the corresponding $(Z_k, \theta^k)$-tree, $T^{k,i}$ for its $i$-truncation, and $\nu^k, \nu^{k,i}$ for the corresponding Lukács paths.

In the perspective of proving our main results, we note another special distribution of the variables $Z_{k,n}$ that is of interest. If $Z_{k,n}$ are i.i.d. $\text{Bin}(\sigma-1, w_k)$, then the subtrees emerging from the backbone of the $(Z_k, \theta^k)$-tree are independent sub-critical percolation clusters with parameter $w_k$. In particular, for suitably chosen values of $w_k, n_k$, $T^{k,n}$ has the same law as a certain segment of $R$. On the other hand if $w_k \equiv \sigma^{-1}$, then the corresponding $(Z, \theta)$-tree is simply the IIC conditioned on its backbone being the rightmost branch of $T$, which we denote by IIC$_R$. We will see below that the unconditioned IIC, as well as segments of the unconditioned IPC can be treated in a similar way.

For our key lemmas we consider the following properties:

$$\mathcal{A} : \begin{cases} 
\text{For any } k, \text{ the variables } (Z_{k,n})_n \text{ are i.i.d. for some } C, \alpha > 0, \mathbb{E} Z_{k,n}^{1+\alpha} < C, \\
\text{for some } \eta, \mathbb{P}(Z_{k,n} > 0) > \eta, \\
\text{if } m_k = \mathbb{E}Z_{k,n} \text{ then } m = \lim m_k \text{ exists.}
\end{cases}$$

However, note that the proofs use only easier consequences of these properties. We need to avoid many consecutive 0’s among the $Z$’s, we need the maximal $Z$ to behave and we need the cumulative sum of the $Z$’s to grow linearly in various regimes.

### 3.2 Scaling of segments

**Proposition 3.3.** Let $Z_{k,n}$ be random variables such that $\mathcal{A}$ holds a.s., and assume $w_k \leq \sigma^{-1}$ satisfy $\lim_k k(1-\sigma w_k) = u$. Then, as $k \to \infty$, weakly in $\mathcal{C}(\mathbb{R}_+, \mathbb{R})$,

$$\left( \frac{1}{k} V^k_{[k^2t]} \right)_{t \geq 0} \xrightarrow{k \to \infty} (X_t)_{t \geq 0},$$

where $X_t = Y_t - Y_0$ and $Y_t = B_{\gamma t} - ut$ is a drifted Brownian motion.

Since our goal is to represent segments of the IPC as well-chosen $T^{k,i}$, we have to deduce from Proposition 3.3 some results for the coding paths of the truncated trees. The convergence will take place in the space of continuous stopped paths denoted $S$. An element $f \in S$ is given by a lifetime $\zeta(f) \geq 0$
and a continuous function $f$ on $[0, \zeta(f)]$. $S$ is a Polish space with metric

$$d(f, g) = |\zeta(f) - \zeta(g)| + \sup_{t \leq \zeta(f) \land \zeta(g)} \{|f(t) - g(t)|\}.$$  

It is clear from the right description of Lukaciewicz paths that the path of $T_{k,i}$ visits 0 exactly when reaching backbone vertices. In particular its length is $\tau(n_k)$, where $\tau(i)$ denotes the $i^{th}$ return to 0 of the path $V^k$. We shall use this to prove

**Corollary 3.4.** Assume the conditions of Proposition 3.3 are in force. Assume further that $0 < x = \lim n_k/k$. Then, weakly in $S$,

$$\left( \frac{1}{k} V^k_{n_k} \right)_{t \leq \tau(n_k)/k^2} \xrightarrow{k \to \infty} (X_t)_{t \leq \tau_{mx}}$$

where $X$ and $Y$ are as in Proposition 3.3 and $\tau_y$ is the stopping time $\inf\{t > 0 : Y_t = -y\}$.

It is then straightforward to deduce convergence of the height functions. Let $h^k$ (resp. $h^{k,i}$) denote the height function coding the tree $T_k$ (resp. $T_{k,i}$).

**Corollary 3.5.** Suppose the assumptions of Corollary 3.4 are in force. Then weakly in $C(\mathbb{R}_+, \mathbb{R})$,

$$\left( \frac{1}{k} h^k \right)_{t \geq 0} \xrightarrow{k \to \infty} \left( \frac{2}{\gamma} (Y_t - Y_{t-}) - \frac{1}{m} Y_t \right)_{t \geq 0}$$

Furthermore, weakly in $S$,

$$\left( \frac{1}{k} h^{k, n_k} \right)_{t \leq \tau(n_k)/k^2} \xrightarrow{k \to \infty} \left( \frac{2}{\gamma} (Y_t - Y_{t-}) - \frac{1}{m} Y_t \right)_{t \leq \tau_{mx}}$$

### 3.3 Proof of Proposition 3.3

We begin with the following Lemma, which relates the Lukaciewicz paths of a sequence of trees, and that of the tree consisting of a backbone to which the trees of the sequence are attached.

**Lemma 3.6.** Let $(\theta_n)_{n \geq 0}$ be a sequence of trees, and define the sin-tree $T$ to be the sin-tree with a backbone BB on the right, such that the root of $\theta_n$ is identified with BB. Let $U$ be the Lukaciewicz path coding the sequence $\theta$, and let $V$ be the Lukaciewicz path of $T$. Then

$$V_n = U_n + 1 - \underline{U}_{n-1},$$

where $\underline{U}$ is the infimum process of $U$ and by convention $\underline{U}_{-1} = 1$.  

14
Proof. The Lemma follows directly from the definition of Lukaciewicz paths. $U$ reaches a new infimum (and $U$ decreases) exactly when the process completes the exploration of a tree in the sequence. The increments of $V$ differ from the increments of $U$ only at vertices of the backbone of $T$, where the degree in $T$ is one more than the degree in $\theta_n$. 

We first establish the proposition in the special case introduced earlier, where $Z_k$ is a sequence of i.i.d. $\text{Bin}(\sigma, w_k)$ random variables. In this case, the sub-trees attached to the backbone of $T_k$ are a sequence of i.i.d. Galton-Watson trees with branching law having expectation $\sigma w_k$ (which tends to 1 as $k \to \infty$), and variance $\sigma w_k(1 - w_k)$ (which tends to $\gamma$ as $k \to \infty$).

The Lukaciewicz path $U_k$ of this sequence of Galton-Watson trees is a random walk with drift $\sigma w_k - 1$ and stepwise variance $\sigma w_k(1 - w_k)$. From a well known extension of Donsker’s invariance principle (see for instance [8, Theorem II.3.5]) it follows that

$$\left(\frac{1}{k} U_k(k^2 t)\right)_{t \geq 0} \xrightarrow{k \to \infty} (Y_t)_{t \geq 0}$$

weakly in $C(\mathbb{R}_+, \mathbb{R})$. It now follows from Lemma 3.6 that

$$\left(\frac{1}{k} V_k(k^2 t)\right)_{t \geq 0} \xrightarrow{k \to \infty} (X_t)_{t \geq 0}.$$  \hfill (11)

Having Proposition 3.3 for $Z_k$, we now extend it to other degree sequences. By the Skorokhod representation theorem, we may assume (by changing the probability space as needed) that (11) holds a.s.:

$$\left(\frac{1}{k} V_k(k^2 t)\right)_{t \geq 0} \xrightarrow{\text{a.s.}} (X_t)_{t \geq 0}.$$  \hfill (12)

By adding the sequences $Z_k$ to the probability space we get a coupling of $T^k$ and $T^k$. Note that we use the same sequences $\theta^k$ in both trees. This allows us to identify each vertex of $\theta^k$ with one vertex in each of $T^k, T^k$, giving also a partial correspondence between $T^k$ and $T^k$ (with the backbone vertices remaining unmatched).

It will be convenient to consider the sets of points

$$G^k := \{(i, V^k(i)), i \in \mathbb{Z}_+\}, \quad G^k := \{(i, V^k(i)), i \in \mathbb{Z}_+\},$$

which are the integer points in the graphs of $V^k, V^k$. To each vertex $v \in T^k$ corresponds a point $(x_v, y_v) \in G^k$ (and similarly $(x_v, y_v) \in G^k$ for $v \in T^k$).
From the right description of Lukaciewicz paths introduced in subsection 1.1.2 we see that
\[ G^k = \{(x_v, y_v) : v \in T^k \} = \left\{ (\#(T^k)^v, n(v, T^k) - 1) : v \in T^k \right\}, \]
\[ G^k = \{(x_v, y_v) : v \in T^k \} = \left\{ (\#(T^k)^v, n(v, T^k) - 1) : v \in T^k \right\}, \]
The next step is to show that these two sets are close to each other. Any \( v \in \theta^k_n \) is contained in both \( T^k \) and \( T^k \). We first show that \( x_v \approx x_v \) and \( y_v \approx y_v \) for such \( v \), and then show how to deal with the backbones.

Any tree \( \theta^k_n \) is attached by an edge to some vertex in the backbone of \( T^k \) and \( T^k \). For any vertex \( v \in \theta^k_n \) we denote the height of this vertex by \( l_v \) and \( \ell_v \) respectively:
\[ l_v = \text{sup} \{ t : \text{BB}_t < v \in T^k \}, \quad \ell_v = \text{sup} \{ t : \text{BB}_t < v \in T^k \}. \]

These values depend implicitly on \( k \). Note that \( l_v, \ell_v \) do not depend on which \( v \in \theta^k_n \) is chosen, hence by a slight abuse of notation, we also use \( l_n, \ell_n \) for the same values whenever \( v \in \theta^k_n \).

**Lemma 3.7.** Assume \( v \in \theta^k_n \). Then
\[ |x_v - x_v| = |l_v - \ell_v|, \]
\[ |y_v - y_v| \leq \sigma + Z_{k, \ell_n}. \]

**Proof.** We have
\[ x_v = \#(T^k)^v = \sum_{i < n} \#(\theta^k_i)^v + \#(\theta^k_n)^v + \ell_n, \]
and similarly
\[ x_v = \#(T^k)^v = \sum_{i < n} \#(\theta^k_i)^v + \#(\theta^k_n)^v + l_n. \]
The first claim follows.

For the second bound use \( y_v = n(v, T^k) - 1 \). There are \( n(v, \theta^k_n) \) edges connecting \( (T^k)^v \) to its complement inside \( \theta^k_n \), and at most \( Z_{k, l_n} \) edges connecting \( \text{BB}_l \) to the complement. Similarly, in \( T^k \) we have the same \( n(v, \theta^k_n) \) edges inside \( \theta^k_n \) and at most \( Z_{k, \ell_n} \leq \sigma \) edges connecting \( \text{BB}_\ell \) to the complement. It follows that the difference is at most \( \sigma + Z_{k, l_n} \).

Next we prepare to deal with the backbone. For a vertex \( v \in T^k \), define \( u \in T^k \) by
\[ u = \text{min} \left\{ u \in (T^k \setminus \text{BB}) : u \geq v \right\}. \]

16
If \( v \notin \text{BB} \) then \( u = v \). If \( v \) is on the backbone then \( u \) is the first child of \( v \), unless \( v \) has no children outside the backbone. Note that \( u \in \theta^k_n \) for some \( n \), so we may also consider \( u \) as a vertex of \( T^k \). Note also that \( v \to u \) is a non-decreasing map from \( T^k \) to \( T^k \).

**Lemma 3.8.** For a backbone vertex \( v \) in \( T^k \), define \( n \) by \( \theta^k_n < v < \theta^k_{n+1} \). Then

\[
|\mathbf{x}_v - \mathbf{x}_u| \leq 1 + l_{n+1} - l_n, \\
|\mathbf{y}_v - \mathbf{y}_u| \leq \sigma + Z_{k,l_{n+1}}.
\]

**Proof.** The only vertices between \( v \) and \( u \) in the lexicographic order are \( u \) and some of the backbone vertices with indices from \( l_n \) to \( l_{n+1} \), yielding the first bound.

Let \( w \in \text{BB} \) be \( u \)'s parent. If \( v \) has children apart from the next backbone vertex then \( w = v \) and \( u \) is \( v \)'s first child, so \( \mathbf{y}_u - \mathbf{y}_v = k_u - 1 \leq \sigma - 1 \). If \( v \) has no other children then \( \mathbf{y}_u - \mathbf{y}_v = (k_u - 1) + (k_w - 1) \leq \sigma + Z_{k,l_{n+1}}. \)

**Lemma 3.9.** Fix \( \varepsilon, A > 0 \) and let \( w \) be the \( \lfloor Ak^2 \rfloor \) vertex of \( T^k \). Then with high probability \( \ell_w, l_w \leq k^{1+\varepsilon} \).

**Proof.** Since each \( \theta^k_n \) is (slightly) sub-critical, we have \( \mathbb{P}(\#\theta^k_n > k^2) > c_1 k^{-1} \) for some \( c_1 > 0 \). Consider the first \( k^{1+\varepsilon} \) vertices along the backbone in \( T^k \). With overwhelming probability, the number of \( \theta \)'s attached to them is at least \( \eta k^{1+\varepsilon}/2 \). On this event, with overwhelming probability at least \( c_2 k^{\varepsilon} \) of these have size at least \( k^2 \), hence there are \( c_2 k^{2+\varepsilon} \gg Ak^2 \) vertices \( v \) with \( l_v \leq k^{1+\varepsilon} \) (and these include the first \( Ak^2 \) vertices in the tree). \( \ell_w \) is dealt with in the same way.

**Lemma 3.10.** Fix \( A > 0 \) and let \( w \) be the \( \lfloor Ak^2 \rfloor \)th vertex of \( T^k \). For \( \varepsilon > 0 \) small enough,

\[
\mathbb{P}\left( \sup_{v < w} |\mathbf{x}_v - \mathbf{x}_u| > 3k^{1+\varepsilon} \right) \xrightarrow{k \to \infty} 0,
\]

and

\[
\mathbb{P}\left( \max_{v < w} |\mathbf{y}_v - \mathbf{y}_u| > k^{1-\varepsilon} \right) \xrightarrow{k \to \infty} 0,
\]

**Proof.** For a vertex \( v \in \theta^k_n \) off the backbone we have \( u = v \) and

\[
|\mathbf{x}_v - \mathbf{x}_u| \leq |l_v - \ell_v| \leq l_v + \ell_v \leq l_w + \ell_w,
\]

and with high probability this is at most \( 2k^{1+\varepsilon} \). If \( v < w \) is in the backbone then we argue that \( |\mathbf{x}_v - \mathbf{x}_u| \ll k^{1+\varepsilon} \). To this end, note that \( l_{n+1} - l_n \) is
dominated by a geometric random variable with mean $1/\eta$ (since the $Z_{k,n}$’s are independent). Since only $n < Ak^2$ might be relevant to the initial part of the tree, this shows that with high probability $|x_v - x_u| < c \log k \ll k^{1+\varepsilon}$.

The bound on the $y$’s follows from the bounds on $|y_v - y_u|$. All that is needed is to show that with high probability $Z_{k,n} < k^{1-\varepsilon}$ for all $n < k^{1+\varepsilon}$, and this follows from assumption $\mathcal{A}$ and Markov’s inequality. \hfill \Box

We now finish the proof of Proposition 3.3. Because the path of $V^k$ is linearly interpolated between consecutive integers, and since for any $A > 0$ the paths of $X$ are a.s. uniformly continuous on $[0, A]$, the proposition will follow if we establish that for any $A, \varepsilon > 0$,

$$\mathbb{P} \left( \sup_{t \in [0, A]} \left| \frac{1}{k} V_{[k^2t]}^k - X_t \right| > \varepsilon \right) \longrightarrow 0. \quad (13)$$

Consider first $t$ such that $k^2t \in \mathbb{Z}_+$. Then there is some vertex $v \in T^k$ so that $x_v = k^2t$. Let $u \in T^k$ be as defined above, and suppose $k^2s = x_u$. Then (12) implies that $|k^{-1}y_u - X_s|$ is uniformly small. Lemma 3.10 implies that with high probability $|k^2s - k^2t| = |x_u - x_v| \leq 3k^{1+\varepsilon}$ for all such $v$. Thus $|s - t| \leq k^{-1+\varepsilon} \ll 1$. Since paths of $X$ are uniformly continuous we find $|X_s - X_t|$ is uniformly small, and so $|k^{-1}y_u - X_t|$ is uniformly small. Finally, Lemma 3.10 states that $|y_u - y_v| \leq C$, so the scaled vertical distance is also $o(1)$.

Next, assume $m < k^2t < m + 1$. Then $V^k(k^2t)$ lies between $V^k(m)$ and $V^k(m + 1)$. Since both of these are close to the corresponding values of $X$, and since $X$ is uniformly continuous (and the pertinent points differ by at most $k^{-2}$) we may interpolate to find that (13) holds for all $t < A$.

### 3.4 Proof of Corollaries 3.4 and 3.5

#### Proof of Corollary 3.4

By Proposition 3.3 the limit of $(\frac{1}{k} V_{[k^2t]}^k)_{t \leq \tau(n_k)}$ must take the form $(X_t)_{t \leq \tau}$, for some possibly random time $\tau$, and furthermore $X_\tau = 0$. We need to show that $\tau = \tau_{max} = \inf \{ t \geq 0 : -Y_t = mx \}$.

In the special case of the tree $T^k$ we note that the infimum process $U^k$ records the index of the last visited vertex along the backbone. Therefore $\tau(n_k)$ is the time at which $U^k$ first reaches $-n_k$, and by assumption $n_k \sim xk$. Using the a.s. convergence of $\frac{1}{k} U^k(k^2t)$ towards $Y_t$, along with the fact that for any fixed $x > 0, \varepsilon > 0$, one has a.s. $Y_{\tau_x - \varepsilon} > -x > Y_{\tau_x + \varepsilon}$, we deduce that a.s., $\tau(n_k)/k^2 \rightarrow \tau_x$. It then follows that

$$\left( \frac{1}{k} V_{[k^2t]}^k, t \leq (\tau(n_k) + 1)/k^2 \right) \overset{\text{a.s.}}{\longrightarrow} (X_t, t \leq \tau_x) .$$
Since, in this case, \( m_k = \sigma w_k \to m = 1 \), this implies the corollary for this special distribution.

The general case is then a consequence of excursion theory. Indeed \((-Y_t, t \geq 0)\) can be chosen to be the local time at its infimum of \( Y \) (see for instance [13, paragraph VI.8.55]), that is a local time at 0 of \( X \), since excursions of \( Y \) away from its infimum match those of \( X \) away from 0. However, if \( N_t^{(\varepsilon)} \) denotes the number of excursions of \( X \) away from 0 that are completed before \( t \) and reach level \( \varepsilon \), then \( \lim_{\varepsilon \to 0} \varepsilon N_t^{(\varepsilon), t \geq 0} \) is also a local time at 0 of \( X \), which means that it has to be proportional to \((-Y_t, t \geq 0)\) (cf for instance [4, section III.3(c) and Theorem VI.2.1]). In other words, there exists a constant \( c > 0 \) such that for any \( t \geq 0 \),

\[
\lim_{\varepsilon \to 0} \varepsilon N_t^{(\varepsilon)} = -cY_t.
\]

In the special case when \( Z_{k,n} = \text{Bin}(\sigma, w_k) \) we have already proven the corollary. In particular, the number \( N_t^{(\varepsilon)} \) of excursions of \((\frac{1}{k} V_{k,t}, t \leq \tau^{(n_k)})\) which reach level \( \varepsilon \) is such that, when letting \( k \to \infty \) and then \( \varepsilon \to 0 \), we have \( \varepsilon N_t^{(\varepsilon)} \to cx \).

Let \( N_t^{(\varepsilon)} \) be the number of excursions of \((\frac{1}{k} V_{k,t}, t \leq \tau^{(n_k)})\) which reach level \( \varepsilon \). It follows from Proposition 3.3 that, in distribution, \( N_t^{(\varepsilon)} \to N_t^\varepsilon \) as \( k \to \infty \).

However, by assumption \( \mathcal{A} \) we can use law of large numbers for the sequences \((Z_{k,n})_{n \in \mathbb{N}}\) along with the fact that \( m_k \to m \), to ensure that \( \varepsilon N_t^{(\varepsilon)} \sim m \varepsilon N_t^{(\varepsilon)} \). Therefore, letting first \( k \to \infty \), then \( \varepsilon \to 0 \) we find \( \varepsilon N_t^{(\varepsilon)} \to m \varepsilon x \).

From the fact that \( \tau^{(n_k)} \) are stopping times, we deduce that \( \tau \) itself is a stopping time. Since \( X_{r} = 0 \), for any \( s > 0 \), the local time at 0 of \( X \) (that is, \(-Y\)) increases on the interval \((\tau, \tau + s)\). It follows that for a certain real-valued random variable \( R \), \( \tau = \tau_R = \inf\{t \geq 0 : -Y_t = R\} \), and we deduce that in distribution, \( R = mx \), i.e. \( \tau = \tau_{mx} \).

Proof of Corollary 3.5. The relation between the height function and Lukaciewicz paths is well known; see e.g. [11, Theorem 2.3.1 and equation (1.7)]. Combining with Proposition 3.3 one finds that the height process of the sequence of trees emerging from the backbone of \( T_k \) converges when rescaled to the process

\[
\frac{2}{\gamma} (Y_t - \Sigma_t).
\]

Moreover, the difference between the height process of \( T_k \) and that of the sequence of trees emerging from the backbone of \( T_k \) is simply \(-\Sigma\). As in
the proof of Corollary 3.4, one has weakly in $C(\mathbb{R}_+, \mathbb{R})$,

$$
\left( -\frac{1}{k} U^k_{[k^2t]} \right)_{t \geq 0} \overset{k \to \infty}{\longrightarrow} \left( -\frac{1}{m} Y^\tau_t \right)_{t \geq 0}
$$

and (9) follows. The proof of (10) is similar. \hfill \Box

In fact, [7, Corollary 2.5.1] states the joint convergence of Lukaciewicz paths, height, and contour functions. It is thus easy to deduce a strengthening of Corollary 3.5 to get the joint convergence.

3.5 Two sided trees

The limit appearing in Proposition 3.3 retains very minimal information about the sequence $Z_k$. If two trees (or two sides of a tree) are constructed as above using independent $\theta$’s but dependent sequences of $Z$’s, the dependence between two sequences might disappear in the scaling limit. For $k \in \mathbb{Z}_+$, let $w_k \in [0, 1/\sigma]$, and denote by $(\theta^k_n)_{n \in \mathbb{Z}_+}, (\tilde{\theta}^k_n)_{n \in \mathbb{Z}_+}$ two independent sequences of i.i.d. subcritical Galton-Watson trees with branching law $\text{Bin}(\sigma, w_k)$. We let $Z_k, \tilde{Z}_k$ be two sequences of random variables taking values in $\mathbb{Z}_+$ such that the pairs $(Z_{k,n}, \tilde{Z}_{k,n})$ are independent for different $n$; however, we allow $Z_{k,n}$ and $\tilde{Z}_{k,n}$ to be correlated.

Let $T^k, \tilde{T}^k$ designate respectively the $(Z_k, \theta^k)$-tree, $(\tilde{Z}_k, \tilde{\theta}^k)$-tree as defined in Section 3.1. Let $V^k$, resp. $\tilde{V}^k$ denote their Lukaciewicz paths. We recall that $T_{n_k}^k, \tilde{T}_{n_k}^k$ are respectively the $n_k$-truncation of $T^k$, resp. $\tilde{T}^k$, and we denote by $V^{k,n_k}, \tilde{V}^{k,n_k}$ their respective Lukaciewicz paths.

**Proposition 3.11.** Suppose $w_k \leq \sigma^{-1}$ is such that $u = \lim_{k \to \infty} k(1 - \sigma w_k)$ exists, and assume that both sequences of variables $Z_{k,n}, \tilde{Z}_{k,n}$ satisfy assumption $A$. Then, as $k \to \infty$, weakly in $C(\mathbb{R}_+, \mathbb{R})$

$$
k^{-1} \left( V^k_{[k^2t]}, \tilde{V}^k_{[k^2t]} \right)_{t \geq 0} \overset{k \to \infty}{\longrightarrow} \left( X_t, \tilde{X}_t \right)_{t \geq 0}
$$

where the processes $X, \tilde{X}$ are two independent reflected Brownian motions with drift $-u$ and diffusion coefficient $\gamma$.

Moreover, if $n_k/k \to x > 0$, $m_k \to m$, $\tilde{m}_k \to \tilde{m}$ as $k \to \infty$, we have

$$
k^{-1} \left( V^{k,n_k}_{[k^2t]}, \tilde{V}^{k,n_k}_{[k^2t]} \right)_{t \leq \tau(n_k)/k^2} \overset{k \to \infty}{\longrightarrow} \left( X_t, \tilde{X}_t \right)_{t \leq \tau_{m_x}}.$$

20
The proof is almost identical to that of Proposition 3.3. When the sequences $Z_k, \tilde{Z}_k$ are independent with $\text{Bin}(\sigma, w_k)$ elements the result follows from Proposition 3.3. For general sequences, the coupling of Section 3.3 shows that the sides have the same joint scaling limit.

3.6 Scaling the IIC

It is known that the IIC is the result of setting $w_k = 1/\sigma$ in the above constructions. Specifically, let us first suppose that $Z$ is a sequence of i.i.d. $\text{Bin}(\sigma - 1, 1/\sigma)$, and $(\theta_n)_n$ is a sequence of i.i.d. $\text{Bin}(\sigma, 1/\sigma)$ Galton-Watson trees. Let $T$ be a $(Z, \theta)$-tree, then observe that $T$ has the same distribution as $\text{IIC}_R$.

The convergence of the rescaled Lukaciewicz path encoding this sin-tree to a time-changed reflected Brownian path is thus a special case of Proposition 3.3. The scaling limits of the height and contour functions follow from Corollary 3.5. We have $m = \gamma$, so both limits are $\frac{2}{\gamma} B_{\gamma t} - \frac{2}{\gamma} \bar{B}_{\gamma t}$.

For the unconditioned IIC, let $Y_n$ be i.i.d. uniform in $\{1, \ldots, \sigma\}$. Let $Z_n \sim \text{Bin}(Y_n - 1, 1/\sigma)$ and $\tilde{Z}_n \sim \text{Bin}(\sigma - Y_n, 1/\sigma)$, independent conditioned on $Y_n$ and independently of all other $n$. Moreover, suppose that $\theta, \tilde{\theta}$ are two independent sequences of i.i.d. $\text{Bin}(\sigma, 1/\sigma)$ Galton-Watson trees. Then, $T$ and $\tilde{T}$ are jointly distributed as $\text{IIC}_G$ and $\text{IIC}_D$.

Since in this case $m = \bar{m} = \gamma / 2$, from Proposition 3.11, we see that the rescaled Lukaciewicz paths encoding these two trees converge towards a pair of independent time-changed reflected Brownian motions. Corresponding results hold for the right/left height and contour functions of the IIC. For example, if $H_G, H_D$ are the left and right height functions of the IIC, then weakly in $C(\mathbb{R}_+, \mathbb{R}_2)$,

$$\left(\frac{1}{k} H_G(k^2 t), \frac{1}{k} H_D(k^2 t)\right)_{t \geq 0} \xrightarrow{k \to \infty} \left(\frac{2}{\gamma} (B_{\gamma t} - 2\bar{B}_{\gamma t}), \frac{2}{\gamma} (\bar{B}_{\gamma t} - 2\bar{B}_{\gamma t})\right)_{t \geq 0},$$

where $B$ and $\bar{B}$ are two independent Brownian motions. Note that the limit $\frac{2}{\gamma} (B_{\gamma t} - 2\bar{B}_{\gamma t})$ is a constant times a 3-dimensional Bessel process.

4 Bottom-up construction

4.1 Right grafting and concatenation

Definition 4.1. Given a finite plane tree, its rightmost-leaf is the maximal vertex in the lexicographic order; equivalently, it is the last vertex to be
reached by the contour process, and is the rightmost leaf of the sub-tree above the rightmost child of the root.

**Definition 4.2.** The right-grafting of a plane tree $S$ on a finite plane tree $T$, denoted $T \oplus S$ is the plane tree resulting from identifying the root of $S$ with the rightmost leaf of $T$. More precisely, let $v$ be the rightmost leaf of $T$. The tree $T \oplus S$ is given by its set of vertices $\{u, vw : u \in T \setminus \{v\}, w \in S\}$.

Note in particular that the vertices of $S$ have been relabeled in $T \oplus S$ through the mapping from $S$ to $T \oplus S$ which maps $w$ to $vw$.

**Definition 4.3.** The concatenation of two functions $V_i \in S$ with $V_2(0) = 0$, denoted $V = V_1 \oplus V_2$, is defined by

$$V(t) = \begin{cases} V_1(t) & t \leq x_1, \\ V_1(x_1) + V_2(t - x_1) & t \in [x_1, x_2]. \end{cases}$$

**Lemma 4.4.** If each $Y_i \in S$ attains its minimum at $\zeta(Y_i)$ then

$$\bigoplus(Y_i - Y_i) = \bigoplus Y_i - \bigoplus Y_i.$$

The following is straightforward to check, and may be used as an alternate definition of right-grafting.

**Lemma 4.5.** Let $R = T \oplus S$ be finite plane trees, and denote the Lukaciewicz path of $R$ (resp. $T, S$) by $V_R$ (resp. $V_T, V_S$). Let $V_T'$ be $V_T$ terminated at $\#T$ (i.e. without the final value of $-1$). Then $V_R = V_T' \oplus V_S$.

Consider a sin-tree $T$ in which the backbone is the rightmost path (i.e. the path that takes the rightmost child at each step). Given some increasing sequence $\{x_i\}$ of vertices along the backbone we cut the tree at these vertices: Let

$$\overline{T_i} := \{v \in T : x_i \leq v \leq x_{i+1}\}.$$

Thus $\overline{T_i}$ contains the segment of the backbone $[x_i, x_{i+1}]$ as well as all the sub-trees connected to any vertex of this segment except $x_{i+1}$. We let $T_i$ be $\overline{T_i}$ rerooted at $x_i$ (Formally, $T_i$ contains all $v$ with $x_iv \in \overline{T_i}$.) It is clear from the definitions that $T = \bigoplus_{i=0}^{\infty} T_i$. Note that apart from being increasing, the sequence $x_i$ is arbitrary.
4.2 Notations

In the rest of this section we consider both \( R \) and the unconditioned IPC. Our goal is to establish Theorems 1.2 and 1.3. In the next subsection we establish Theorem 1.2. We shall use \( \mathcal{V} \) to denote the Lukaciewicz path of \( R \).

We construct below a sequence of copies of \( R \) whose scaling limits converge. These will be indexed by \( k \), though the dependence on \( k \) will frequently be implicit. We denote by \( R^k \) the \( k \)th instance of \( R \) in the sequence. Note that the use of Skorokhod representation theorem will allow us to construct the sequence so that the Lukaciewicz paths converge almost surely, rather than just in distribution.

While \( R \) is close to critical away from the root, the segments close to the root behave differently and need to be dealt with separately. We let \( R^\beta \) (implicitly depending on \( k \)) be the subtrees above a certain vertex in the backbone (see below), and let \( \mathcal{V}^\beta \) denote its Lukaciewicz path. As \( \beta \to \infty \) the trees will get closer to the full trees. Lemma 4.8 below will show that \( \mathcal{V}^\beta \) is uniformly close to \( \mathcal{V} \) (recalling that both depend implicitly on \( k \)).

4.3 IPC structure and the coupling

In this section we prove Theorem 1.2. Recall the \( \hat{W} \)-process introduced in paragraph 1.1.1 and the convergence (4). The \( \hat{W} \)-process is constant for long stretches, giving rise to a partition of \( R \) into what we shall call segments. Each segment consists of an interval of the backbone along which \( \hat{W} \) is constant, together with all sub-trees attached to the interval. To be precise, define \( x_i \) inductively by

\[
x_i = \inf_{n \geq x_i} \{ \hat{W}_n > \hat{W}_{x_i} \}.
\]

With a slight abuse, we also let \( x_i \) designate the vertex along the backbone at height \( x_i \).

Since we have convergence in distribution of the \( \hat{W} \)'s we may couple the IPC's for different \( k \)'s so that the convergence holds a.s.. More precisely, let \( J \) be the set of jump times for \( \{ L(t) \} \). We may assume that a.s., \( \{ k^{-1} x_i^k \} \overset{k \to \infty}{\longrightarrow} J \) in the sense that there is a 1-to-1 mapping from the jump times of \( \hat{W}^k \) (in \( R^k \)) into \( J \) that eventually contains every point of \( \hat{J} \). Furthermore, we may assume that for any \( t \not\in J \) we have a.s. \( k^{-1} (1 - \sigma \hat{W}_{[kt]}^k) \overset{k \to \infty}{\longrightarrow} L(t) \).

The backbone is the union of the intervals \([x_i, x_{i+1}]\) for all \( i \geq 0 \), and the rest of the IPC consists of sub-critical percolation clusters attached to
each vertex of the backbone \( y \in [x_i, x_{i+1}) \). We can now write

\[
\mathcal{R} = \bigoplus_{i=0}^{\infty} R_i
\]

where \( R_i \) is the \([x_i, x_{i+1}]\) segment of \( \mathcal{R} \), rerooted at \( x_i \). \( R_i \) has a rightmost branch of length \( n_i := x_{i+1} - x_i \). The degrees along this branch are i.i.d. Bin(\( \sigma - 1, \hat{W}_{x_i} \)), and each child off the rightmost branch is the root of an independent Galton-Watson tree with branching law Bin(\( \sigma, \hat{W}_{x_i} \)). In what follows, we say that \( R_i \) is a \( \hat{W}_{x_i} \)-segment of length \( n_i \), and we observe that these segments fall into the family dealt with in section 3.

We may summarize the above in the following lemma:

**Lemma 4.6.** Suppose \( \hat{W} \) consists of values \( U_i \) repeated \( n_i \) times. Then \( R_i \) is distributed as a \( U_i \)-segment of length \( n_i \), and conditioned on \( \{U_i, n_i\} \) the trees \( \{R_i\} \) are independent.

A difficulty we must deal with is that in the scaling limit there is no first segment, but rather a doubly infinite sequence of segments. Furthermore, the initial segments are far from critical, and so need to be dealt with separately. This is related to the fact that the Poisson lower envelope process diverges near 0, and has no “first segment”. Because of this we restrict ourselves at first to a slightly truncated invasion percolation cluster. For any \( \beta > 0 \) we define \( x_0^\beta = \min\{x : \sigma \hat{W}_x > 1 - \beta / k\} \). Thus we consider the first vertex along the backbone for which \( \sigma \hat{W}_x > 1 - \beta / k \). Let \( \tilde{R}^\beta \) (depending implicitly on \( k \)) denote the subtree of \( \mathcal{R}^k \) above \( x_0^\beta \), \( R^\beta \) the relabeled version of \( \tilde{R}^\beta \). If \( \beta \) is large then \( \tilde{R}^\beta \) is almost the entire tree. For any fixed \( \beta \), as \( k \to \infty \) the branches of \( \mathcal{R}^\beta \) are all close to critical. As for the entire tree, we define \( x_{i+1}^\beta = \inf\{n > x_i^\beta : \hat{W}_n > \hat{W}_{x_i^\beta} \} \). Note that \( x_0^\beta = x_m \) for some \( m \) and that \( x_i^\beta = x_{m+i} \) for the same \( m \) and all \( i \).

If \( \beta \notin \{L(t)\} \) then \( \beta \) gives rise to a partial indexing of \( J \). Let

\[
j_0^\beta = \inf\{t > 0 : L(t) < \beta\}
\]

and \( j_i^\beta+1 \) the time of the first jump of \( L \) after \( j_i^\beta \). Under the coupling above we have the limits \( k^{-1}x_i^\beta \xrightarrow[k \to \infty]{} j_i^\beta \), and for \( y \in [x_i^\beta, x_{i+1}^\beta] \) we have that \( k(1 - \sigma \hat{W}_y) \xrightarrow[k \to \infty]{} L(j_i^\beta) \).

Denote by \( V_i^\beta \) (implicitly depending on \( k \)) the Lukaciewicz path corresponding to the \( i^{th} \) segment \( R_i^\beta \) in \( \mathcal{R}^\beta \). For any \( \beta, i \), the \( i^{th} \) segment has
associated percolation parameter $w_k$ satisfying $k(1 - \sigma w_k) \xrightarrow[k \to \infty]{} u$ for some value $u$ of $L$, and length $n^\beta_i$ satisfying $k^{-1}n^\beta_i \to x$ for some $x > 0$. By Corollary 3.4 we have the convergence in distribution
\[ (k^{-1}V^\beta_i(k^2t), 0 \leq t \leq \tau(n^\beta_i)) \xrightarrow[k \to \infty]{} (X_t, 0 \leq t \leq \tau_y) \] (14)
where $X_t = Y_t - Y_0$, and $Y_t$ solves
\[ dY_t = \sqrt{\gamma} dB_t - u dt. \]
As in the previous section, $\tau(n^\beta_i)$ denotes the lifetime of $V^\beta_i$ (that is, its $(n^\beta_i)$th return to 0) and $\tau_y$ is the hitting time of $-y$ by $Y$.

Because this convergence holds for all $\beta, i$, we may construct the coupling of the probability spaces so that the convergence in (14) is also almost sure, and this is the final constraint in our coupling.

**Lemma 4.7.** Fix $\beta > 0$. In the coupling described above we have, almost surely, the scaling limit
\[ k^{-1}V^\beta(k^2t) \xrightarrow[k \to \infty]{} X_t, \]
where $X_t = Y^\beta_t - Y^\beta_0$, and $Y^\beta$ solves
\[ Y^\beta_t = \sqrt{\gamma} B_t - \int_0^t L \left( j^\beta_s - \frac{1}{\gamma} Y^\beta_s \right) ds. \]

**Proof.** Solutions of the equation for $Y^\beta$ are a concatenation of segments. In each segment the drift is fixed, and each segment terminates when $Y^\beta$ reaches a certain threshold. The corresponding segments of $X$ exactly correspond to the scaling limit of the tree segments $R^\beta_i$.

Lemma 4.7 then follows from Lemma 4.4 and Lemma 4.5.

**Lemma 4.8.** Almost surely,
\[ (Y^\beta_t, t > 0) \xrightarrow[\beta \to \infty]{} Y_t \]
where $Y$ solves
\[ Y_t = \sqrt{\gamma} B_t - \int_0^t L \left( -\frac{1}{\gamma} Y_s \right) ds. \]
Proof. Consider the difference between the solutions for a pair $\beta < \beta'$. We have the relation

$$Y_{\beta'} = Z \oplus Y_{\beta},$$

where $Z$ is a solution of $Z_t = \sqrt{\gamma} B_t - \int_0^t L \left( j_{0}^{\beta'} - \frac{1}{2} Z_s \right) ds$, killed when $Z$ first reaches $\gamma (j_{0}^{\beta'} - j_{0}^{\beta})$. In particular $Z$ is a stochastic process with drift in $[-\beta', -\beta]$ (and quadratic variation $\gamma$). Thus to show that $Y_{\beta'}$ is close to $Y_{\beta}$, we need to show that $Z$ is small both horizontally and vertically, i.e. $\zeta(Z)$ is small, as is $\|Z\|_{\infty}$.

The vertical translation of $Y_{\beta}$ is $\sqrt{\gamma} k^{-1} (x_{0}^{\beta} - x_{0}^{\beta'})$, which is at most $k^{-1} x_{0}^{\beta}$. From \[2\] we know that this tends to 0 in probability as $\beta \to \infty$. This convergence is a.s. since $x_{0}^{\beta}$ is non-increasing in $\beta$.

The values of $Z$ are unlikely to be large, since $Z$ has a non-positive (in fact negative) drift and is killed when $Z$ reaches some negative level close to 0.

Finally, there is a horizontal translation of $Y_{\beta}$ in the concatenation. This translation is just the time at which $Z$ first reaches $\gamma (j_{0}^{\beta'} - j_{0}^{\beta})$, which is also small, uniformly in $\beta'$.

Theorem 1.2(1) is now a simple consequence of Lemmas 4.7 and 4.8. Indeed, the process $Y - \bar{Y}$ has the same law as the righthand side of (1), due to the scale invariance of solutions of $E(L)$. We shall note that in fact, $Y$ is the limit of the rescaled Lukaciewicz path coding the sequence of off-backbone trees.

The same argument using Corollary 3.5 instead of Corollary 3.4 gives the convergence of the height function.

Finally, convergence of contour functions is deduced from that of height functions by a routine argument (see for instance [11, section 1.6]).

4.4 The two-sided tree

For convenience we use the shorter notation $T$ to designate the IPC, and write $V$ for its Lukaciewicz path. To deal with $T$ recall the left and right trees $T_G$ and $T_D$ as introduced in section 1.1.3. They obviously both have the same distribution, but are not independent. As in the previous section we may cut these two trees into segments along which the $\hat{W}$-process is constant. More precisely,

$$T_G = \bigoplus_{i=0}^{\infty} T_G^i, \quad T_D = \bigoplus_{i=0}^{\infty} T_D^i,$$
where the distribution of $T^i_D, T^i_G$ can be made precise as follows.

Let $(\theta^i_n), (\bar{\theta}^i_n)$ be two independent sequences of independent Galton-Watson trees with branching law $\text{Bin}(\sigma, \bar{W}^i_{x_i})$. Let $Y_n, n \in \mathbb{Z}_+$ be independent uniform on $\{1, \ldots, \sigma\}$, and conditionally on $Y_n$, let $Z_n$ be $\text{Bin}(Y_n - 1, \bar{W}^i_{x_i})$ and $\bar{Z}_n$ be $\text{Bin}(\sigma - Y_n, \bar{W}^i_{x_i})$, where conditioned on the $Y$’s all are independent. Then $T^i_G$ and $T^i_D$ are distributed as the $n_i$-truncations of the $(Z, \theta^i)$-tree, resp. of the $(\bar{Z}, \bar{\theta}^i)$-tree (constructed as in Definition 3.1).

The rest of the proof of Theorem 1.3 is then almost identical to that of Theorem 1.2. In particular, to deal with the fact that the scaling limit has no first segment, introduce subtrees $T^\beta$ and consider left and right trees $T^\beta_G, T^\beta_D$. We then perform a similar coupling. The convergence for each sequence of segments then follows from the second part of Proposition 3.11. However, note that the value of the expected number of children of a vertex on the backbone is divided by 2 compared to the conditioned case. As a consequence, the limits of the rescaled coding paths of $T^\beta_G, T^\beta_R$ will be expressed in terms of solutions to the equation

$$Y^\beta_t = \sqrt{\gamma}B_t - \int_0^t L \left( j^\beta_u - \frac{2}{\gamma} Y^\beta_s \right) ds.$$ 

Further details are left to the reader.

4.5 Convergence of trees

In this section we discuss weak convergence of the trees as ordered metric spaces. We refer to [11] for background on the theory of continuous real trees.

Given any two continuous functions $C_G, C_D : \mathbb{R}_+ \to \mathbb{R}_+$ that satisfy $C_G(0) = C_D(0) = 0$ we can define the continuum tree with these functions as its left and right contour functions (Duquesne and Le Gall call these “height processes”, though they are mathematically closer to the contour function of a discrete tree). To do this, first define

$$C(t) = \begin{cases} C_G(-t) & t \leq 0 \\ C_D(t) & t \geq 0. \end{cases}$$

Next, define a distance on $\mathbb{R}$ by

$$d(s, t) = C(s) + C(t) - 2 \inf_{u \in \mathbb{I}(s,t)} C(u)$$

27
where for \( s < t \) we denote
\[
I(s, t) = \begin{cases} [s, t] & st \geq 0 \\ \mathbb{R} \setminus [s, t] & st < 0. \end{cases}
\]
(This second case is where we differ from the usual theory for compact trees.)

The continuum tree is now defined as \( \mathbb{R} / \sim \) where \( \sim \) is the equivalence relation \( s \sim t \iff d(s, t) = 0 \).

We may now define the continuum random sin-tree \( \mathcal{T}^{IPC} \) whose left and right contour processes are defined for \( t \in \mathbb{R} \) by
\[
C_G(t) := \gamma^{-1/2}(Y_t - 2\overline{Y}_t), \quad C_D(t) := \gamma^{-1/2}(\tilde{Y}_t - 2\overline{\tilde{Y}}_t).
\]
Recall that for \( x > 0 \) we defined earlier \( \tau_x := \inf\{u > 0 : Y_u = -x\} \), and define \( \tilde{\tau}_x \) similarly for \( \tilde{Y} \). For \( x > 0 \), we may consider the subtree \( \mathcal{T}_x^{IPC} \) whose left and right contour processes are defined by
\[
C_{xG}(t) := C_G(t \wedge \tau_{\gamma^{1/2}x}), \quad C_{xD}(t) := C_D(t \wedge \overline{\tilde{\tau}_{\gamma^{1/2}x}}).
\]

The tree \( \mathcal{T}_x^{IPC} \) is a.s. a compact real tree. It corresponds to the truncation of IPC at height \( x \) on the backbone, that is, it consists of all the vertices at height below \( x \) on the backbone and their descendants off the backbone.

For \( n_k \) such that \( n_k/k \to x \), consider the \( n_k \)-truncation of IPC as a continuous tree, and rescale it so that its edges have length \( 1/k \). We denote by \( \text{IPC}_x^k \) the rescaled tree.

As in [7], one can then show that for any \( x > 0 \), the third line in Theorem 1.3 implies convergence of \( \text{IPC}_x^k \) towards \( \mathcal{T}_x^{IPC} \) in the sense of weak convergence in the space of compact real trees equipped with the Gromov-Hausdorff distance. In particular this yields Theorem 1.1.

Clearly, a similar construction also holds in the case of the IIC.

## 5 Level estimates

The goal of this section is to apply our convergence results to establish asymptotics for level, volume estimates of the invasion percolation cluster. In [2], it was proved that the size of the \( n^{\text{th}} \) level of the IPC, rescaled by a factor \( n \), converges to a non-degenerate limit. Similarly, the volume up to level \( n \), rescaled by a factor \( n^2 \), converges to non-degenerate limit. The Laplace transforms of these limits were expressed as functions of the \( L \)-process. However formulas (1.20)–(1.23) of [2] do not provide insight into
the limiting variables. With our convergence theorem for height functions of \( \mathcal{R} \), we can express the limit in terms of the continuous limiting height function. In the case of the asymptotics of the levels, we also provide an alternative way of expressing the limit.

For \( x \in \mathbb{R}_+ \) we denote by \( C[x] \) the number of vertices of the IPC at height \([x]\). We let \( C[0, x] = \sum_{i=0}^{[x]} C[i] \) denote the number of vertices of the IPC up to height \([x]\).

For simplicity, we use the shorter notation \((H_t, t \geq 0) := (\gamma^{-1/2}(2Y_t - 3Y_t), t \geq 0)\), to denote the continuous limit of the rescaled version of \( H_R \) (see Theorem 1.2). In particular, observe that

\[
\lim_{n \to \infty} \frac{1}{n^2} C[0, an] = \int_0^\infty 1_{[0, a]} \left( \frac{1}{n} H_R \left( \frac{sn^2}{n} \right) \right) ds.
\]

We denote by \( l_t^a(H) \) the standard local time at level \( a \), up to time \( t \), of the semimartingale \( H \), that is (since \( H \) has quadratic variation \( 2/\gamma \)):

\[
l_t^a(H) = \frac{2}{\gamma} \lim_{\eta \to 0} \frac{1}{\eta} \int_0^t 1_{[a, a+\eta]}(H_s) ds.
\]

**Proposition 5.1.** Let \( a > 0 \). We have the distributional limits

\[
\frac{1}{n^2} C[0, an] \xrightarrow{n \to \infty} \int_0^\infty 1_{[0, a]}(H_s) ds. \tag{15}
\]

Furthermore,

\[
\frac{1}{n} C[an] \xrightarrow{n \to \infty} \frac{\gamma}{4} l_\infty^a(H). \tag{16}
\]

The limiting quantity in (16) can be expressed as a sum of independent contributions corresponding to distinct excursions of \( Y - Y_\). These contributions are, conditionally on the \( L \)-process, independent exponential random variables. For \( c > 0 \), let us denote by \( e(c) \) an exponential variable with parameter \( c \).

**Corollary 5.2.** Let \( S \) be a point process such that conditioned on the \( L \)-process, \( S \) is an inhomogeneous Poisson point process on \([0, a\sqrt{\gamma}]\), with intensity :

\[
\frac{2 L(s) \, ds}{\exp \left( \left( a\sqrt{\gamma} - s \right) L(s) \right) - 1}.
\]

Then, conditionally on \( L \), and in distribution,

\[
\frac{1}{n} C[an] \xrightarrow{n \to \infty} \frac{\sqrt{\gamma}}{2} \sum_{s \in S} e \left\{ \frac{L(s)}{1 - \exp \left( -a\sqrt{\gamma} - s L(s) \right)} \right\}, \tag{17}
\]

where the terms in the sum are independent.
From this representation and immediate properties of the $L$-process, it is straightforward to recover the representation of the asymptotic Laplace transform of level sizes, (1.21) of [2]. Also, as the proof of the Corollary will show, a.s. $S$ is finite, thus only a finite number of distinct values of $L$ contribute to the sum in (17).

Proof of Proposition 5.1. We start by proving (15). Our objective is the limit in distribution

$$\int_0^\infty 1_{[0,a]} \left( \frac{1}{n} H_{R} \left( sn^2 \right) \right) ds \xrightarrow{n \to \infty} \int_0^\infty 1_{[0,a]}(H_s) ds.$$

This almost follows from Theorem 1.2. The problem is that $\int_0^A 1_{[0,a]}(X_s) ds$ is not a continuous function of the process $X$, and this is for two reasons. First, because of the indicator function, and second, because the topology is uniform convergence on compacts and not on all of $\mathbb{R}$.

To overcome the second we argue that for any $\varepsilon$ there is an $A$ such that

$$\mathbb{P}\left( \int_A^\infty 1_{[0,a]} \left( \frac{1}{n} H_{R} \left( sn^2 \right) \right) ds \neq 0 \right) < \varepsilon.$$

Indeed, in order for the height function to visit $[0,na]$ after time $n^2A$ the total size of the $[na]$ sub-critical trees attached to the backbone up to height $[na]$ must be at least $[n^2A]$. This probability is small for $A$ sufficiently large, even if the trees are replaced by $[na]$ critical trees. Thus it suffices to prove that for every $A$

$$\int_0^A 1_{[0,a]} \left( \frac{1}{n} H_{R} \left( sn^2 \right) \right) ds \xrightarrow{\text{dist.}} n \to \infty \int_0^A 1_{[0,a]}(H_s) ds.$$

(18)

Next we deal with the discontinuity of $1_{[0,a]}$ by a standard argument. We may bound $f_\varepsilon \leq 1_{[0,a]} \leq g_\varepsilon$ where $f_\varepsilon, g_\varepsilon$ are continuous and coincide with $1_{[0,a]}$ outside of $[a - \varepsilon, a + \varepsilon]$. Define the operators

$$F_\varepsilon(X) = \int_0^A f_\varepsilon(X_s) ds, \quad G_\varepsilon(X) = \int_0^A g_\varepsilon(X_s) ds.$$

Then we have a sandwich

$$F_\varepsilon \left( \frac{1}{n} H_{R} \left( sn^2 \right) \right) \leq \int_0^A 1_{[0,a]} \left( \frac{1}{n} H_{R} \left( sn^2 \right) \right) ds \leq G_\varepsilon \left( \frac{1}{n} H_{R} \left( sn^2 \right) \right),$$

and similarly for $H_s$. By continuity of the operators

$$F_\varepsilon \left( \frac{1}{n} H_{R} \left( sn^2 \right) \right) \stackrel{\text{dist.}}{\xrightarrow{n \to \infty}} F_\varepsilon(H_s), \quad F_\varepsilon \left( \frac{1}{n} H_{R} \left( sn^2 \right) \right) \stackrel{\text{dist.}}{\xrightarrow{n \to \infty}} F_\varepsilon(H_s).$$
In the limit we have
\[
G_\varepsilon(H_s) - F_\varepsilon(H_s) \xrightarrow{\varepsilon \to 0} a.s. 0.
\]
and since \( G_\varepsilon - F_\varepsilon \) is continuous we also have for any \( \delta > 0 \)
\[
\lim_{\varepsilon \to 0} \lim_{n \to \infty} \mathbb{P}\left( G_\varepsilon\left( \frac{1}{n} H_{\mathcal{R}} \left( s n^2 \right) \right) - F_\varepsilon\left( \frac{1}{n} H_{\mathcal{R}} \left( s n^2 \right) \right) > \delta \right) = 0.
\]
Combining these bounds implies (18), and thus (15).

We now turn to the proof of (16). From (15), we know that for any \( \eta > 0 \),
\[
\frac{1}{\eta n^2} C[an, (a + \eta)n] \xrightarrow{\text{dist.}} \frac{1}{\eta} \int_0^\infty 1_{[a,a+\eta]}(H_s) ds.
\]
Thus, (16) will follow if we can prove that for any \( \eta > 0 \), in probability,
\[
\lim_{n \to \infty} \left| \frac{\eta n C[an] - C[an, (a + \eta)n]}{\eta n^2} \right| = 0.
\]
(19)

For a given vertex \( v \), let \( h_v \) denote the height of \( v \). If \( v \) is not on the backbone, we let \( \text{perc}(v) \) be the percolation parameter of the off-backbone percolation cluster to which \( v \) belongs. We now single out the vertex on the backbone at height \( [an] \), and group together vertices at height \( [an] \) which correspond to the same percolation parameter.

More precisely, if \( w_1, w_2, w_3, ..., w_N \) are the distinct values taken by the \( W \)-process up to time \( [na] \), we let
\[
C_n^{(w_i)} := \{ v \in \text{IPC} \setminus \text{BB} : h_v = [an], \text{perc}(v) = w_i \},
\]
so that
\[
\mathcal{C}[an] := \{ v \in \text{IPC} : h_v = [an] \} = \bigcup_{i=1}^{N_n} C_n^{(w_i)} \cup \text{BB}_{[an]}, \quad C[an] = \#\mathcal{C}[an].
\]
Moreover, any vertex between heights \( [an] \) and \( [(a + \eta)n] \) in the IPC descends from one of the vertices of \( \mathcal{C}[an] \). We let
\[
\mathcal{P}_n^{(w_i)} := \left\{ v \in \text{IPC} \setminus \text{BB} : [an] \leq h_v \leq (a + \eta)n, \exists w \in C_n^{(w_i)} \text{ s.t. } w \leq v \right\},
\]
\[
\mathcal{P}_n^{\text{BB}[an]} := \left\{ v \in \text{IPC} : [an] \leq h_v \leq (a + \eta)n, \text{BB}_{[an]} \leq v \right\}.
\]
In particular, $C_n^{(w)} \subset \mathcal{P}_n^{(w)}$ and vertices of the backbone between heights $[an]$ and $[(a + \eta)n]$ are contained in $\mathcal{P}_n^{BB[an]}$. Moreover,

$$\mathcal{C}[an, (a + \eta)n] := \{v \in IPC : [an] \leq h_v \leq (a + \eta)n\} = \mathcal{P}_n^{BB[an]} \cup \bigcup_{i=1}^{N_n} \mathcal{P}_n^{(w)}.$$

However, the number of distinct values of percolation parameters which one sees at height $[an]$ remains bounded with arbitrarily high probability.

**Claim 5.3.** For any $\epsilon > 0$, there is $A > 0$ such that, for any $n \in \mathbb{N}$,

$$\mathbb{P}\left[ \#\{i \in \{1, \ldots, N_n\} : |C_n^{(w)}| \neq 0\} > A \right] \leq \epsilon.$$

From [2, Proposition 3.1], the number of distinct values the $\hat{W}$-process takes between $[na]/2$ and $[na]$ is bounded, uniformly in $n$, with arbitrarily high probability. Furthermore, it is well known that with arbitrarily high probability, among $[na]/2$ critical Galton-Watson trees, the number which reach height $[na]/2$ is bounded, uniformly in $n$. It follows that the number of clusters rising from the backbone at heights $\{0, \ldots, [na]/2\}$ and which possess vertices at height $[na]$ is, with arbitrarily high probability, also bounded for all $n$. The claim follows.

**Claim 5.4.** For any $\eta > 0$, in probability,

$$\lim_{n \to \infty} \left| \frac{1}{\eta n^2} \mathcal{P}_n^{BB[an]} \right| = 0.$$

Fix $\eta$. We observe that $\mathcal{P}_n^{BB[an]}$ is bounded by the total progeny up to height $\eta n$, of $\eta n$ critical Galton-Watson trees. If $|B|$ denotes a reflected Brownian motion, and $l_t^0(|B|)$ its local time at 0 up to $t$, we then deduce from a convergence result for a sequence of such trees (cf formula (7) of [11]) that for any $\epsilon > 0$,

$$\limsup_{n \to \infty} \mathbb{P}\left[ \frac{1}{\eta n^2} \mathcal{P}_n^{BB[an]} > \epsilon \right] \leq \mathbb{P}\left[ \frac{1}{\eta} \inf\{t > 0 : l_t^0(|B|) > \eta\} > \epsilon \right],$$

and the claim follows from the fact that $(\inf\{t > 0 : l_t^0(|B|) > \eta\}, \eta \geq 0)$ is a half stable subordinator.

**Claim 5.5.** For any $t \in (0, a)$, $\eta > 0$, in probability,

$$\lim_{n \to \infty} \left| \frac{\mathcal{P}_n^{(\hat{W}[nt])}}{\eta n^2} - \frac{\#(C_n^{(\hat{W}[nt])})}{n} \right| = 0.$$
Fix $t, \eta$, and define $w_n := \hat{W}_{[nt]}$. We have

$$P \left[ \left\| \frac{P^{(w_n)}}{\eta n^2} - \frac{\#(C^{(w_n)})}{n} \right\| > \epsilon \right]$$

$$\leq P \left[ \#(C^{(w_n)}) > n\epsilon^{-2} \right] + P \left[ \left\| \frac{P^{(w_n)}}{\eta n^2} - \frac{\#(C^{(w_n)})}{n} \right\| > \epsilon , \#(C^{(w_n)}) < \epsilon^2 n \right]$$

$$+ \sum_{k=[\epsilon^2 n]}^{[\epsilon^{-2} n]} P(\#(C^{(w_n)}) = k) P \left[ \left\| \frac{P^{(w_n)}}{\eta n^2} - \frac{\#(C^{(w_n)})}{n} \right\| > \epsilon \mid \#(C^{(w_n)}) = k \right].$$

Using a comparison to critical trees as in the previous argument, the first two terms in the sum above go to 0 as $n \to \infty$. Furthermore, from [7, Corollary 2.5.1], we know that, conditionally on the processes $\hat{W}, L$, for any $u > 0$, the level sets of $[un]$ subcritical Galton-Watson trees with branching law $\text{Bin}(\sigma, w_n)$ converge to the local time process of a reflected drifted Brownian motion $(|X_s|, s \geq 0)$, with drift $L(t)$, stopped at $\tau_u$. Therefore, for any $u > 0$,

$$\lim_{n \to \infty} P \left[ \left\| \frac{P^{(w_n)}}{\eta n^2} - \frac{\#(C^{(w_n)})}{n} \right\| > \epsilon \mid \#(C^{(w_n)}) = [nu] \right]$$

$$= P \left[ \frac{1}{\eta} \int_0^{\tau_u} 1_{[0,\eta]}(|X_s|) ds - l^0_t(|X|) \right] > \epsilon ,$$

which for any $\epsilon > 0$, goes to 0 as $\eta \to 0$. Thus by dominated convergence,

$$\lim \limsup_{\eta \to 0} \sum_{k=[\epsilon^2 n]}^{[\epsilon^{-2} n]} P(\#(C^{(w_n)}) = k)

\cdot P \left[ \left\| \frac{P^{(w_n)}}{\eta n^2} - \frac{\#(C^{(w_n)})}{n} \right\| > \epsilon \mid \#(C^{(w_n)}) = k \right] = 0.$$

Claim 5.5 follows.

From our decompositions of $C[an, (a + \eta)n], C[an]$, and claims 5.3, 5.4, and 5.5, we now deduce (19). This implies (16), and completes the proof of Proposition 5.1.

Proof of Corollary 5.2. From (16), the corollary will be proven if we manage to express $\gamma_{\frac{1}{4}t^q(H)}$ as the righthand side of (17). Note that, if $t^\tau \left( \sqrt{\frac{n}{\gamma}} \right)$
denotes the local time up to time $t$ at level $x$ of

$$\sqrt{\gamma} \frac{H}{2} = Y_t - \frac{3}{2} Y_t,$$

then

$$\sqrt{\gamma} \frac{L_t^a}{4} (H) = \frac{\sqrt{\gamma}}{2} \sqrt{\frac{2}{\gamma}} a \left( \frac{\sqrt{\gamma}}{2} H \right),$$

so that we may as well express $\sqrt{\gamma} \frac{L_t^a}{4} \sqrt{\frac{2}{\gamma}} a \left( \frac{\sqrt{\gamma}}{2} H \right)$.

To reach this goal, it is convenient to decompose the path of $\sqrt{\gamma} \frac{H}{2}$ according to the excursions above the origin of $Y - Y$. Let us introduce a few notations. We let $\mathcal{F}(\mathbb{R}_+, \mathbb{R})$ denote the space of real-valued finite paths, so that excursions of $Y$ and of $Y - Y$ are elements of $\mathcal{F}(\mathbb{R}_+, \mathbb{R})$. For a path $e \in \mathcal{F}(\mathbb{R}_+, \mathbb{R})$, we define $\bar{e} := \sup_{s \geq 0} e(s)$, $\underline{e} := \inf_{s \geq 0} e(s)$. For $c \geq 0$, we let $N^{(-c)}$ denote the excursion measure of drifted Brownian motion with drift $-c$ away from the origin, and $n^{(-c)}$ that of reflected drifted Brownian motion with drift $-c$ above the origin (see for example \cite{13} chapter VI.8).

**Lemma 5.6.** For any $c > 0$, $a > 0$, we have

$$n^{(-c)} (\bar{e} > a) = \frac{2c}{\exp(2ca) - 1},$$  \hfill (20)

$$N^{(-c)} (\underline{e} < -a) = \frac{c}{1 - \exp(-2ca)}$$  \hfill (21)

This result is well known, and can be proven by using basic properties of drifted Brownian motion and excursion measures.

We are now going to determine the excursions of $Y - Y$ which give a non-zero contribution to $\frac{\sqrt{\gamma}}{4 \sqrt{\gamma}} L_t^a (H)$. We may and will choose $-Y$ to be the local time process at 0 of $Y - Y$. Using excursion theory (see for instance \cite{13} section VI.8.55), we know that for this normalization of local time, conditionally on the $L$-process, the excursions of $Y - Y$ form an inhomogeneous Poisson point process $\mathcal{P}$ in the space $\mathbb{R}_+ \times \mathcal{F}(\mathbb{R}_+, \mathbb{R}_+)$ with intensity $ds \times n^{(-L(s))}$.

For $b \geq 0$, let $\tau_b$ denote the hitting time of $b$ by $-Y$. Note that for any $s \geq \tau_b$, $-Y_s > b$, from the fact that drifted Brownian motion started at 0 instantaneously visits the negative half line. We therefore observe that the last visit to $\sqrt{\gamma} a$ by $\frac{\sqrt{\gamma}}{2} H$ is at time $\tau_{a/\sqrt{\gamma}}$. Hence, any point of $\mathcal{P}$ whose first coordinate is larger than $a/\sqrt{\gamma}$ corresponds to a part of the path of $H$ which
lies strictly above \( a \), and therefore can not contribute to \( l_\infty^a(H) \). Moreover, a
part of the path of \( \sqrt{\frac{e}{2}} H \) which corresponds to an excursion of \( Y - \bar{Y} \) starting
at a time \( s < \tau_a \sqrt{\gamma} \) will only reach height \( \frac{\sqrt{\gamma}}{2} a \) whenever the supremum of this
excursion is greater or equal than \( \frac{1}{2}(a \sqrt{\gamma} - \bar{Y}) \). Therefore, any excursion of
\( Y - \bar{Y} \) which gives a nonzero contribution to \( l_\infty^a(H) \) corresponds to a point of \( \mathcal{P} \) whose
first coordinate is some \( s \), such that \( s \leq a \sqrt{\gamma} \), and whose second coordinate is an excursion \( e \) such that \( \bar{e} \geq \frac{1}{2}(a \sqrt{\gamma} - s) \).
These considerations, along with properties of Poisson point processes,
lead to the following claim.

**Claim 5.7.** Conditionally on the \( L \)-process, the excursions of \( Y - \bar{Y} \) which
give a nonzero contribution to \( \frac{\sqrt{\gamma}}{4} l_\infty^a(H) = \frac{\sqrt{\gamma}}{2} l_\infty^a \left( \frac{\sqrt{\gamma}}{2} H \right) \) are points of a
Poisson point process \( \mathcal{P} \subset \mathcal{P} \) on \( \mathbb{R}_+ \times \mathcal{F}(\mathbb{R}_+, \mathbb{R}_+) \) with intensity
\[
1_{[0,a \sqrt{\gamma}]}(s) \begin{pmatrix} e \geq \frac{1}{2}(a \sqrt{\gamma} - s) \\ ds \times n^{-L}(\cdot) \end{pmatrix}
\]
The number of points of \( \mathcal{P} \) clearly is almost surely countable, so we may
write \( \mathcal{P} = (s_i, e_i)_{i \in \mathbb{Z}_+} \). In particular, by \( \{20\} \), \( (s_i, e_i)_{i \in \mathbb{Z}_+} \) are the points of the Poisson point process on \( [0, a \sqrt{\gamma}] \) introduced in Corollary \( 5.2 \).
Note that \( \{e_i, i \in \mathbb{Z}_+\} \) correspond obviously to distinct excursions of
\( Y - \bar{Y} \), so that their contributions to \( l_\infty^a \left( \frac{\sqrt{\gamma}}{2} H \right) \) are independent.

**Claim 5.8.** Conditionally given \( L \), for each \( i \in \mathbb{Z}_+ \) the contribution of the
excursion \( e_i \) to \( l_\infty^a \left( \frac{\sqrt{\gamma}}{2} H \right) \) is exponentially distributed with parameter
\[
N^{-L(s)} \left( e_i \leq \frac{1}{2}(-a \sqrt{\gamma} + s_i) \right).
\]
Fix \( i \in \mathbb{Z}_+ \), and condition on \( L \). Recall that \((s_i, e_i)\) is one of the points of the
Poisson process \( \mathcal{P} \), so that \( e_i \) is chosen according to the measure
\( n^{-L(s_i)}(\cdot, \bar{e} > \frac{1}{2}(a \sqrt{\gamma} - s_i)) \). Up to the time at which \( e_i \) reaches \( \frac{1}{2}(a \sqrt{\gamma} - s_i) \), \( e_i \) does not contribute to \( l_\infty^a \left( \frac{\sqrt{\gamma}}{2} H \right) \). From the Markov property
under \( n^{-L(s_i)}(\cdot, \bar{e} > \frac{1}{2}(a \sqrt{\gamma} - s_i)) \), the remaining part of \( e_i \) (after it has
reached \( \frac{1}{2}(a \sqrt{\gamma} - s_i) \)) follows the path of a drifted Brownian motion, with
drift \( -L(s_i) \), started at \( \frac{1}{2}(a \sqrt{\gamma} - s_i) \), and stopped when it gets to the origin.
Thus, the contribution of \( e_i \) to \( l_\infty^a \left( \frac{\sqrt{\gamma}}{2} H \right) \) is exactly the local time of this
stopped drifted Brownian motion at level \( \frac{1}{2}(a \sqrt{\gamma} - s_i) \). By shifting vertically,
it is also \( l_0^\infty(X) \), the total local time at the origin of \( X \), a drifted Brownian motion, with drift \(-L(s_i)\), started at the origin and stopped when reaching \( \frac{1}{2}(-a\sqrt{\gamma} + s_i) \). By excursion theory, if \( \mathcal{P}_i \) is a Poisson point process on \( \mathbb{R}_+ \times \mathcal{F}(\mathbb{R}_+, \mathbb{R}) \) with intensity \( ds \times N(-L(s_i)) \), then \( l_0^\infty(X) \) is the coordinate of the first point of \( \mathcal{P}_i \) which falls into the set

\[
\mathbb{R}_+ \times \left\{ e \in \mathcal{F}(\mathbb{R}_+, \mathbb{R}): e < \frac{1}{2}(-a\sqrt{\gamma} + s_i) \right\},
\]

Claim 5.8 follows.

From Lemma 5.6, Claim 5.7 (along with the remark which follows it), and Claim 5.8 we deduce Corollary 5.2.

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