A process algebra for the Span(Graph) model of concurrency

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Abstract

In this note we define a process algebra TCP (Truly Concurrent Processes) which corresponds closely with the automata model of concurrency based on Span(RGraph), the category of spans of reflexive graphs. In TCP, each process has a fixed set of interfaces. Actions are allowed to occur simultaneously on all the interfaces of a process. Asynchrony is modelled by the use of silent actions. Communication is anonymous: communication between two processes $P$ and $Q$ is described by an operation which connects some of the ports of $P$ to some of the ports of $Q$; and a process can only communicate with other processes via its interfaces. The model is naturally equipped with a compositional semantics in terms of the operations in Span(RGraph) introduced in [5], and developed in [6, 7, 10].
1 An overview of TCP

The set of TCP expressions will be built out of a summation operation, a non-communicating parallel operation, a communicating parallel operation, and recursion.

Interfaces Each process expression will have associated to it a fixed number of interfaces, which we divide into the number of left interfaces and the number of right interfaces. If a process expression $P$ has $m$ left interfaces and $n$ right interfaces, we will write $m : P : n$. Processes can communicate with other processes only via their interfaces.

Actions We will assume that we are given a fixed action set $A$ which includes a silent action $\tau$. Actions are assumed to occur simultaneously on each interface of a process; that is, the set of process expressions $P$ with $m : P : n$ will form (the vertices of) a transition system labelled by the set $A^m \times A^n$. In particular, if $m : P : n$ then an element of $A^m \times A^n$ is thought of as an action that $P$ may be able to perform. If $\vec{a} = (a_1, \ldots, a_{m+n}) \in A^m \times A^n$, then we refer to the elements $a_i$ (for all $i \in [m+n]$) as component actions of the action $\vec{a}$. Processes are only able to communicate (that is, synchronize) with other processes via their interfaces.

Asynchrony and silent actions Asynchrony is modelled by the use of silent actions; for example, if $1 : P : 2$, then a transition $P \rightarrow P'$ labelled by $(a, (b, \tau))$ is interpreted as an action that $P$ can perform before turning into $P'$; and as one in which the component action $a$ occurs on the single left interface, the component action $b$ occurs on the first right interface and nothing occurs on the second right interface. Later we will see how the example of the dining philosophers is modelled by the use of silent actions.

Summation Summation will have the usual interpretation given to it in CCS. We note, however, that summations $\sum_{i \in I} \vec{a}_i P_i$ are only defined if all the $P_i$'s have the same number of left and the same number of right interfaces, say $m$ and $n$ respectively; and, in which case, all the $\vec{a}_i$'s must be elements of $A^m \times A^n$.

Non-communicating parallel If $m : P : n$ and $s : Q : t$ then we can form their non-communicating parallel $m + s : P \otimes Q : n + t$. The interpretation of $P \otimes Q$ is that the two processes $P$ and $Q$ are operating in parallel and independently; in particular, they may execute actions simultaneously. Recall that above it was mentioned that processes only synchronize with other processes via their interfaces. In forming the their non-communicating parallel we are not connecting any interfaces: notice that we have $m + s : P \otimes Q : n + t$.

Communicating parallel If $l : P : m$ and $m : Q : n$ then we can form their communicating parallel $l : P \star Q : n$. The interpretation of $P \star Q$ is that the two processes $P$ and $Q$ are operating in parallel, but where the right interfaces
of $P$ have been connected to the left interfaces of $Q$; that is, $P$ can execute an action $\vec{a}$ at the same time as $Q$ can execute an action $\vec{b}$ — but, for each $i \in [m]$, the component actions $a_{i+1}$ and $b_i$ must agree. Notice that the operation has the effect of hiding the common interfaces.

**Wires and more general communication** Communicating parallel does not appear to allow for general communication; for example, it seems that three processes cannot be made to synchronize on a common interface; or that two interfaces of the same process cannot be connected, as in feedback. General communication can be achieved by the use of constants — that is, there is a class of special process expressions which, together with the parallel operations, allow general communication. We call these constants wires. The presence of wires is one of the features of TCP which distinguishes it from other process algebras, since the definitions of wires require the simultaneity of actions on several interfaces. In such process algebras as CCS and CSP [4] general communication is achieved on top of broadcast communication (see [2]). Wires may be used to hide or duplicate interfaces.

**Recursion** If $(X_i)_{i \in I}$ is a distinct family of variables, and $(P_i)_{i \in I}$ are a family of expressions, we construct expressions

$$\text{fix}_j (X_i = P_i)_{i \in I} \quad (j \in I).$$

The reaction rule for recursion is essentially that for the fix operator defined in [8].

## 2 The construction of TCP expressions

In this section, the set $\mathcal{P}$ of TCP process expressions will be defined. This will be done by defining, for each pair $(m, n)$ of natural numbers, a set $\mathcal{P}_{m,n}$ of process expressions (which corresponds to the set of processes with $m$ left interfaces and $n$ right interfaces). The set $\mathcal{P}$ is then defined to be the disjoint union $\sum_{m,n} \mathcal{P}_{m,n}$.

We begin by supposing the following data is given.

- A set $A$ of actions which includes a specified element $\tau \in A$, which we call the silent action.
- For each pair of natural numbers $(m, n)$, an infinite set $\mathcal{V}_{m,n}$ of variable names such that, if $(m, n) \neq (s, t)$ then $\mathcal{V}_{m,n} \cap \mathcal{V}_{s,t} = \emptyset$.

A variable name $V \in \mathcal{V}_{m,n}$ will be used to denote a variable process with $m$ left interfaces and $n$ right interfaces.

The sets $\mathcal{P}_{m,n}$ of process expressions are jointly defined by the following rules. We write $m : P : n$ to mean $P \in \mathcal{P}_{m,n}$.

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• For all pairs \((m, n)\),
\[ \mathcal{V}_{m,n} \subseteq \mathcal{P}_{m,n}. \]

• For all pairs \((m, n)\) and finite sets \(I\), if \((P_i)_{i \in I}\) is a family of process expressions with \(m : P_i : n\) (for each \(i \in I\)), and \((\vec{a}_i)_{i \in I}\) is a family of actions with \(\vec{a}_i \in A^m \times A^n\) (for each \(i \in I\)), then
\[ m : \left( \sum_{i \in I} \vec{a}_i.P_i \right) : n \]

• For all quadruples \((m, n, s, t)\), if \(m : P : n\) and \(s : Q : t\), then
\[ m + s : (P \otimes Q) : n + t \]

• For all triples \((l, m, n)\), if \(l : P : m\) and \(m : Q : n\) then
\[ l : (P \star Q) : n \]

• For all finite sets \(I\) and families \((X_i)_{i \in I}\) of distinct variables with \(m_i : X_i : n_i\) (for each \(i \in I\)), then
\[ m_j : (\text{fix}_j (X_i = P_i)_{i \in I}) : n_j \]

**Wires** For each relation \(R \subseteq [m+n] \times [m+n]\), we define the wire \(m : W_R : n\) as follows. Let \(A_R = \{ (a_1, \ldots, a_{m+n}) \in A^m \times A^n \mid \text{if} \ (i, j) \in R \ \text{then} \ a_i = a_j \}\). Suppose \(m : V : n\) is a variable. Then \(W_R\) is the expression
\[ \text{fix} \ (V = ( \sum_{\vec{a} \in A_R} \vec{a}.V)) \]

### 3 Reaction Rules for TCP

For each pair \((m, n)\), we define a transition system \(T_{m,n}\) whose set of states is \(\mathcal{P}_{m,n}\), and which is labelled by \(A^m \times A^n\).

**Transitions out of a sum** For each \(j \in I\) there is a transition
\[ \vec{a}_j : \left( \sum_{i \in I} \vec{a}_i.P_i \right) \rightarrow P_i \]

That is, we have the rule

| Sum | \( \vec{a}_j : \left( \sum_{i \in I} \vec{a}_i.P_i \right) \rightarrow P_i \) |
Transitions out of a non-communicating parallel expression For each pair of transitions
\[ \vec{a} : Q \rightarrow Q', \quad \vec{b} : R \rightarrow R' \]
there is a transition \[ (\vec{a}, \vec{b}) : (Q \otimes R) \rightarrow (Q' \otimes R'). \]
That is, we have the rule

\[
\text{Par} \quad \frac{\vec{a} : Q \rightarrow Q', \quad \vec{b} : R \rightarrow R'}{(\vec{a}, \vec{b}) : (Q \otimes R) \rightarrow (Q' \otimes R')}
\]

Transitions out of a communicating parallel expression Suppose \( m : Q : l \) and \( l : R : n \). Then for each pair of transitions
\[ (\vec{a}, \vec{b}) : Q \rightarrow Q', \quad (\vec{b}, \vec{c}) : R \rightarrow R' \quad (\vec{b} = (b_1, b_2, \ldots, b_l)) \]
there is a transition \[ (\vec{a}, \vec{c}) : (Q \star R) \rightarrow (Q' \star R') \]
That is, we have the rule

\[
\text{ComPar} \quad \frac{(\vec{a}, \vec{b}) : Q \rightarrow Q', \quad (\vec{b}, \vec{c}) : R \rightarrow R}{(\vec{a}, \vec{c}) : (Q \star R) \rightarrow (Q' \star R')}
\]

Transitions out of a recursive expression If \( (X_i)_{i \in I} \) is a finite family of distinct variables with \( m_i : X_i : n_i \), and if \( (P_i)_{i \in I} \) is a finite family of process expressions with \( m_i : P_i : n_i \), and if \( P_k((X_j := \text{fix}_j(X_i = P_i))_{i \in I}) \) is the result of replacing in \( P_k \), for all \( j \in I \), all occurrences of \( X_j \) by \( \text{fix}_j(X_i = P_i)_{i \in I} \) then for each transition
\[ \vec{a} : P_k((X_j := \text{fix}_j(X_i = P_i))_{i \in I}) \rightarrow Q \]
there is a transition \[ \vec{a} : (\text{fix}_k(X_i = P_i))_{i \in I} \rightarrow Q \]
That is, we have the rule

\[
\text{Rec} \quad \frac{\vec{a} : P_k((X_j := \text{fix}_j(X_i = P_i))_{i \in I}) \rightarrow Q}{\vec{a} : \text{fix}_k(X_i = P_i))_{i \in I} \rightarrow Q}
\]

Example: Joining three processes with the diagonal wire Suppose \( R = \{(1,2), (1,3)\} \subseteq [1 + 2] \times [1 + 2] \). We call \( 1 : W_R : 2 \) the diagonal and denote
it $\Delta$. It may be used to duplicate an interface. Using the fact that in this case $A_R \cong A$, we can write $\Delta$ explicitly as

$$\text{fix} \left( V = \sum_{a \in A} (a, a, a).V \right)$$

Suppose $l : P : 1$, $1 : Q : m$ and $1 : R : n$ are process expressions. Then the expression

$$((P \ast \Delta) \ast (Q \otimes R))$$

is to be thought of as a system formed as follows: duplicate the right interface of the process $P$ and then connected it with the left interfaces of the non-communicating parallel of $Q$ and $R$. The result is that the right interface of the process $P$ has been joined to the two left interfaces of the processes $Q$ and $R$.

It is clear that to give a transition out of $((P \ast \Delta) \ast (Q \otimes R))$ is to give three transitions $\vec{a} : P \rightarrow P'$, $\vec{b} : Q \rightarrow Q'$, $\vec{c} : R \rightarrow R'$ such that $a_l + 1 = b_1 = c_1$.

**Example: The Dining Philosophers and Feedback**

In this example we give a process expression intended to model the example of the dining philosophers. The example also shows how wires can be used to construct feedback.

Let $A = \{\tau, l, u\}$. The symbol $l$ denotes the action lock and the symbol $u$ denotes the action unlock.

First, we define the wires needed to construct feedback. The identity wire $1 : \iota : 1$ is $\text{WR}$ where $R = \{(1, 2)\} \subseteq [1 + 1] \times [1 + 1]$. Explicitly, it is the expression

$$\text{fix} \left( V = \sum_{a \in A} (a, a).V \right)$$

The wire $2 : \epsilon : 0$ is $\text{WR}$ where $R = \{(1, 2)\} \subseteq [2 + 0] \times [2 + 0]$. The wire $0 : \eta : 2$ is $\text{WR}$ where $R = \{(1, 2)\} \subseteq [0 + 2] \times [0 + 2]$.

We now define expressions $Ph_i$ intended to model (the states of) a single dining philosopher. Suppose $P_0$, $P_1$, $P_2$ and $P_3$ are variables in $V_{1,1}$. Then $Ph_i$ is the expression

$$\text{fix}_i \left( P_0 = ((\tau, \tau).P_0 + (l, \tau).P_1), \right.$$
$$P_1 = ((\tau, \tau).P_1 + (\tau, 1).P_2),$$
$$P_2 = ((\tau, \tau).P_2 + (u, \tau).P_3),$$
$$P_3 = ((\tau, \tau).P_3 + (\tau, u).P_0) \right)$$

The expression $Fk_i$ intended to model (the states of) a single fork is defined as follows. Suppose $F_0$, $F_1$ and $F_2$ are variables in $V_{1,1}$. Then $Fk_i$ is the expression

$$\text{fix}_i \left( F_0 = ((\tau, \tau).F_0 + (l, \tau).F_1 + (\tau, 1).F_2), \right.$$
$$F_1 = ((\tau, \tau).F_1 + (u, \tau).F_0),$$
$$F_2 = ((\tau, \tau).F_2 + (\tau, u).F_0) \right)$$
The system of two dining philosophers (in its initial state) is modelled by the expression
\[ (\eta \star ((Ph_0 \star Fk_0) \star (Ph_0 \star Fk_0)) \otimes \iota) \star \epsilon \]
which we denote DinPhil. Note that 0 : DinPhil : 0. Also notice that the effect of the wires in this expression is to feedback the right interface of the rightmost fork to the left interface of the leftmost philosopher: that is, to force a transition of the rightmost fork to have the same label on its right interface as does a transition of the leftmost philosopher on its left interface.

We shall see shortly (corollary 1) that, as far as transitions out of an expression are concerned, the operations \( \star \) and \( \otimes \) are associative, and hence we may ignore bracketting for these operations, so we may for simplicity write
\[ \text{DinPhil} = \eta \star ((Ph_0 \star Fk_0 \star Ph_0 \star Fk_0) \otimes \iota) \star \epsilon. \]

There is a transition from this state to each of the following four states:
\[ \eta \star ((Ph_0 \star Fk_0 \star Ph_0 \star Fk_0) \otimes \iota) \star \epsilon \]
\[ \eta \star ((Ph_1 \star Fk_0 \star Ph_0 \star Fk_2) \otimes \iota) \star \epsilon \]
\[ \eta \star ((Ph_0 \star Fk_2 \star Ph_1 \star Fk_0) \otimes \iota) \star \epsilon \]
\[ \eta \star ((Ph_1 \star Fk_2 \star Ph_1 \star Fk_2) \otimes \iota) \star \epsilon \]

Note that these transitions have no labelling since the system has no interfaces. The first transition corresponds to each philosopher and fork executing silent actions (that is, actions labelled \((\tau, \tau)\)). The second transition corresponds to the leftmost philosopher synchronizing with the rightmost fork (which is the fork to this philosopher’s left), while the other philosopher and fork execute silent actions. (Note that in fact all the components execute actions in which they are forced to agree with the other components on the interfaces they share, but we only use the word ‘synchronize’ to refer to actions which are not silent.) The third transition has a similar interpretation, but with the roles of the two philosophers, and the two forks, swapped. The final transition corresponds to both philosophers picking up their left forks simultaneously. Such a transition is an instance of true concurrency, since two separate actions are able to occur simultaneously.

Note that there are no transitions out of the fourth state to another state: this is corresponds to the deadlock state where both philosophers starve. The reader can check that from the second and third states above there are paths back to the initial state.

4 Semantics

The semantics of a process expression \( m : P : n \) is the subtransition system of \( T_{m,n} \) that is reachable from the state \( P \). We denote it by \( \text{Sem}(P) \). We view it as a transition system labelled by \( A^m \times A^n \) with the initial state \( P \).
If $T$ is a transition system and $s$ is a state of $T$ then $\text{Reach}(T, s)$ denotes the subtransition system of $T$ reachable from $s$.

**Proposition 1** For each pair $(m, n)$, any finite transition system $T$ labelled by $A^m \times A^n$ and any state $s$ of $T$, there exists a process expression $m : P : n$ and an isomorphism of labelled transition systems $\theta : \text{Sem}(P) \cong \text{Reach}(T, s)$ such that $\theta(P) = s$.

As a hint toward the proof, notice that $\text{Sem}(P_0)$ has four states

$$P_{h0}, P_{h1}, P_{h2}, P_{h3}$$

and eight transitions, and the four non-silent transitions cycle though the four states. It is clear how to build a general finite transition system using recursion.

**The operations of Span(RGraph)** Suppose $T$ is a transition system with a labelling of its transitions $\lambda : T \rightarrow A^m \times A^n$. Let $\text{proj}_l : A^m \times A^n \rightarrow A^m$ and $\text{proj}_r : A^m \times A^n \rightarrow A^n$ be the obvious projection functions. For each transition $e$ of $T$, we call $\text{proj}_l(\lambda(e))$ the left labelling and $\text{proj}_r(\lambda(e))$ the right labelling of $e$. In this way the transition system $T$ yields a span of reflexive graphs, with the special property that between two vertices there is at most one edge with a given left and right labelling. We call such spans light spans.

Given a transition system $S$ labelled by $A^m \times A^n$ and a transition system $T$ labelled by $A^l \times A^f$, their free product $S \otimes T$ is the transition system labelled by $A^{m+s} \times A^{n+t}$ defined as follows: the states of $S \otimes T$ are pairs $(s, t)$ of states of $S$ and $T$; a transition $(s, t) \rightarrow (s', t')$ is a pair $(e : s \rightarrow s', f : t \rightarrow t')$ of transitions in $S$ and $T$; and the transition $(e, f)$ is labelled by $(\vec{a}, \vec{b}, \vec{c}, \vec{d}) \in A^{m+s} \times A^{n+t}$, where $(\vec{a}, \vec{c})$ is the labelling of $e$ and $(\vec{b}, \vec{d})$ is the labelling of $f$. This is the tensor product of $S$ and $T$ regarded as spans.

Given a transition system $S$ labelled by $A^l \times A^m$ and a transition system $T$ labelled by $A^r \times A^n$, their composition $S \star T$ is the transition system labelled by $A^{l+r} \times A^{n}$ defined as follows: the states of $S \star T$ are pairs $(s, t)$ of states of $S$ and $T$; further given a pair $(e : s \rightarrow s', f : t \rightarrow t')$ of transitions in $S$ and $T$ such that the right labelling of $e$ equals the left labelling of $f$ then there is a transition $(s, t) \rightarrow (s', t')$ labelled by $(\vec{a}, \vec{b}) \in A^l \times A^n$, where $\vec{a}$ is the left labelling of $e$ and $\vec{b}$ is the right labelling of $f$. This operation is the composition of $S$ and $T$ regarded as spans, but then made light by equating same-labelled edges between the same pair of vertices.

**Proposition 2** For any process expressions $P$ and $Q$, there is an isomorphism of labelled transition systems

$$\text{Sem}(P \otimes Q) \cong \text{Sem}(P) \otimes \text{Sem}(Q).$$

For any process expressions $l : P : m$ and $m : Q : n$, there is an isomorphism of labelled transition systems

$$\text{Sem}(P \star Q) \cong \text{Reach}(\text{Sem}(P) \star \text{Sem}(Q), (P, Q)).$$
We leave the proof to a fuller version of the paper.

**Corollary 1** There is a bijection between transitions out of \( P \star (Q \star R) \) and those out of \( (P \star Q) \star R \) which preserves the labelling. This is also true for the operation \( \otimes \), and similarly for wire expressions (formed from wires using \( \otimes \) and \( * \)) which are deducibly equal from the Frobenius and separable equations [1, 3, 9].

## 5 Further Remarks

Notice that the only constants in the algebra arise from the application of recursion. We have taken this point of view to make the comparison with other process algebras easier. However we might have described more simply a process algebra with given constants, and no recursion (avoiding thereby some questionable processes), in which we could have expressed such examples as the Dining Philosophers. First the wire components could be expresses in terms of a number of constant processes each with one state (see the constants of [9]). Two of these we have already mentioned, namely the diagonal and the identity. For example, the rule corresponding to the diagonal \( 1 : \Delta : 2 \) is

\[
\text{Diag} \quad (a, a, a) : \Delta \rightarrow \Delta
\]

To describe example like the Dining Philosophers we could take in addition the constant processes

\[Ph_0, \; Ph_1, \; Ph_2, \; Ph_3, \; Fk_1, \; Fk_2\]

and add rules specific to these processes. Then the same expression as above would describe a system of dining philosophers.

For further comments on the relation of TCP to other process algebras see [2].

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