Existence of global solution to a 2-D Riemann problem for Euler equations with general equation of state

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Abstract

In this article, we study the gas expansion problem by turning a sharp corner into vacuum for the two-dimensional pseudo-steady compressible Euler equations with a convex equation of state. This problem can be considered as interaction of a centered simple wave with a planar rarefaction wave. In order to obtain the global existence of solution up to vacuum boundary of the corresponding two-dimensional Riemann problem, we consider several Goursat type boundary value problems for 2-D self-similar Euler equations and use the ideas of characteristic decomposition and bootstrap method. Further, we formulate two-dimensional modified shallow water equations newly and solve a dam-break type problem for them as an application of this work. Moreover, we also recover the results from the available literature for certain equation of states which provide a check that the results obtained in this article are actually correct.

Keywords: Gas expansion; Characteristic decomposition; 2-D Riemann problem; Isentropic Euler equations; Wave interactions; 2-D Modified shallow water equations

MSC: 35A01; 35B45; 35L50; 35L65; 35M30

1. Introduction

Mathematical theory of compressible flows in two-dimensions developed rapidly in the recent years. Supersonic flow around a sharp corner is one of the most important and well-studied elementary flows in the study of compressible flows. Courant and Friedrichs [7] noted that the supersonic flow around a bend or sharp corner is effected by simple waves (compression or expansion wave). Consider an infinitely long wedge $OB$ with a horizontal wall $AO$ which is straight up to a sharp corner $O$. Initially, a supersonic flow arrives with a constant velocity $(u_0, 0)$ and density $\rho_0$ along the ground wall and then suddenly expands to vacuum in the other region of the corner; see Figure 1. The turn of the gas from the peak $O$ is effected by a centered simple wave which starts interacting with a planar rarefaction wave. Therefore, the problem of gas expansion can be essentially considered as interaction of a centered rarefaction wave with planar rarefaction wave. The gas expansion problem through a sharp corner has been well-studied recently by a group of mathematicians. Sheng and You [27] studied this problem for isentropic Euler equations with polytropic gas and proved the existence of global solution in the entire interaction region. Chen et al. [6] extended their ideas to magnetogasdynamics system with polytropic gas and established an existence result for global solution. Recently, Lai and Sheng [16] also studied the polytropic gas expansion problem for 2-D Euler equations by turning the gas around a sharp corner into vacuum for different cases of the wall angles and obtained some beautiful results. An obvious question that may arise in a reader’s mind after the success of these works is that can one generalize these results to any arbitrary equation of state? Inspiring from this idea, in this work we are trying to generalize these results for any general convex equation of state. For simplicity we assume that the inclination angle of the wall $OB$ satisfies $\theta \in (-\pi/2, 0)$.
Let us consider the two-dimensional isentropic Euler equations of gas dynamics [23]

\[
\begin{align*}
\rho_t + (\rho u)_x + (\rho v)_y &= 0, \\
(\rho u)_t + (\rho u^2 + p)_x + (\rho uv)_y &= 0, \\
(\rho v)_t + (\rho uv)_x + (\rho v^2 + p)_y &= 0,
\end{align*}
\]

where \(\rho\) denotes the density, \(u\) and \(v\) denotes the flow velocity in the \(x\) and \(y\) direction, respectively, \(p = p(\tau)\) denotes the pressure of the gas and \(\tau = 1/\rho\) is the specific volume.

Cauchy problem for the system (1.1) is a complicated and challenging open problem. The two-dimensional Riemann problem is a particular kind of Cauchy problem which consists of constant initial data along any ray passing through origin. The expansion problems of a flow into vacuum are usually special cases of two-dimensional Riemann problem, which are concerned with interaction of planar rarefaction waves and/or centered rarefaction waves. The study of two-dimensional Riemann problems are significant in theoretical and numerical analysis and many engineering applications too; see [8, 11, 17, 30]. So the study in this article is of utter importance. We refer Figure 1 to impose an initial data on the system (1.1) of the form

\[
(u, v, \tau)(x, y, 0) = \begin{cases} 
(u_0, 0, \tau_0), & x < 0, y > 0, \\
\text{vacuum}, & x > 0, y \geq x \tan \theta,
\end{cases}
\]

where \(u_0, \tau_0\) are constants. Clearly (1.1) with the initial data (1.2) is a 2-D Riemann problem with a boundary. Our main objective in this article is to solve this problem for any arbitrary convex pressure.

Throughout the article we assume that the pressure \(p(\tau)\) satisfies the following properties:

\[
p'(\tau) < 0, p''(\tau) > 0 \text{ for } \tau > \tau_0,
\]

which is generally true for most of the cases of equation of states of physical relevance.

In the recent years, a lot of significant work has been done for the two-dimensional compressible Euler system as well as numerous other related models for a variety of 2-D Riemann problems; see viz. [1, 2, 4, 21, 23, 29, 32]. In particular for gas expansion problems through a sharp corner or wedge we refer reader to [12, 13, 14, 19, 20, 28, 31] and references cited therein.

One of the major difficulties in establishing the global existence of solution for the system (1.1) is that the system (1.1) may changes its type from hyperbolic to elliptic in the interaction domain and the type of the system is not a priori known. Since different types of partial differential equations involve different solving notions, we need to establish a priori estimate of the physical variables. We can not use the method of characteristics in elliptic domain, therefore, to skip the possible occurrence of bad case of mixed type, we need to use the ideas of characteristic decompositions and invariant regions [22] to maintain the hyperbolicity of the system (1.1) in the interaction domain.

The rest of the article is organized as follows. We reduce system (1.1) in the form of self-similar variables and obtain characteristic decompositions of density and characteristic angles in Section 2 which are helpful for
developing *a priori* estimates of physical variables. In Section 3, we provide expressions for planar rarefaction wave and centered rarefaction wave and establish the boundary data estimates to prove the existence of local solution. We construct the invariant regions for characteristic angles and obtain the $C^0$ and $C^1$ norm estimates of the physical variables in the interaction region in Section 4. Section 5 is devoted to prove the existence of global solution by extending the local solution up to the vacuum boundary by solving several Goursat problems locally in each extension step. Further, we discuss some particular cases of equation of states to discuss some relevant physical models as applications of this work in Section 6. In particular, we formulate, first time in the literature, two-dimensional modified shallow water equations to solve a dam-break type problem and also recover results of gas expansion problem for magnetogasdynamics system from available literature. In Section 7 we finally provide the concluding remarks and future scope of this work.

2. System in $(\xi, \eta)$ plane

It is easy to see that the system (1.1) and initial data (1.2) are invariant under the transformation $(t, x, y) \rightarrow (\alpha t, \alpha x, \alpha y)$ for $\alpha > 0$. Then we can reduce the Euler system (1.1) in self-similar co-ordinates $(\xi = x/t, \eta = y/t)$ as follows

\[
\begin{align*}
  (\rho U)_{\xi} + (\rho V)_{\eta} + 2\rho &= 0, \\
  UU_{\xi} + VU_{\eta} + \tau p_{\xi} + U &= 0, \\
  UV_{\xi} + VU_{\eta} + \tau p_{\eta} + V &= 0,
\end{align*}
\]

(2.1)

where $U = u - \xi$ and $V = v - \eta$ denotes the components of pseudo-velocity in $(\xi, \eta)$ plane.

Also, the initial data (1.2) now changes into

\[
(u, v, \tau)(\xi, \eta) = \begin{cases} 
  (u_0, 0, \tau_0), & \xi < 0, \eta > 0, \\
  \text{vacuum}, & \xi > 0, \eta \geq \xi \tan \theta, \xi^2 + \eta^2 \rightarrow \infty.
\end{cases}
\]

(2.2)

Assuming that the flow is irrotational, one can introduce a function $\phi(\xi, \eta)$ such that $U\phi_{\xi} + V\phi_{\eta} = q^2$, where $q = \sqrt{U^2 + V^2}$. The function $\phi$ is usually referred as potential function; see [20]. Further, using the last two equations of system (2.1) it is easy to get the pseudo-Bernoulli’s law of the form

\[
q^2/2 + \int_{\tau_0}^{\tau} \tau p'(\tau)d\tau + \phi = 0.
\]

(2.3)

For smooth solution, (2.1) can be reduced into a matrix form as follows

\[
\begin{bmatrix}
  u \\
  v
\end{bmatrix}_{\xi} + \begin{bmatrix}
  -2UV/c^2 - U^2 \\
  -U^2/c^2 - V^2
\end{bmatrix} \begin{bmatrix}
  u \\
  v
\end{bmatrix}_{\eta} = 0,
\]

(2.4)

where $c(\tau) = \sqrt{-\tau^2 p'(\tau)}$ is the speed of sound.

It is straightforward to see that the eigenvalues of the system (2.4) are $\lambda_{\pm} = \frac{UV \pm c\sqrt{U^2 + V^2 - c^2}}{U^2 - c^2}$ with corresponding left eigenvectors $l_{\pm} = (1, \lambda_{\mp})$. The expression of these eigenvalues shows that the system (2.4) is a mixed type system and changes its behaviour from hyperbolic to elliptic across sonic boundary and depends on the choice of pseudo-Mach number $M = \sqrt{\frac{U^2 + V^2}{c}}$. For $M > 1$(supersonic) system (2.4) is hyperbolic while for $M < 1$(subsonic) it is elliptic. Then we define the two families of wave characteristics as

\[
\frac{d\eta}{d\xi} = \lambda_{\pm}.
\]

(2.5)

Moreover, we obtain the characteristic equations by multiplying $l_{\pm}$ to the system (2.4) as

\[
\begin{cases}
  \bar{\partial}_{\pm} u + \lambda_{-\pm} \bar{\partial}_{\pm} v = 0, \\
  \bar{\partial}_{-\pm} u + \lambda_{+\pm} \bar{\partial}_{-\pm} v = 0.
\end{cases}
\]

(2.6)
where $\bar{\partial}_{\pm} = \bar{\partial}_\xi + \lambda_{\pm} \bar{\partial}_\eta$.

Now we define the concept of characteristic angles as in [20]. The $C_+$ characteristic angle $\alpha$ is defined as the angle between the $C_+$ characteristic direction and $\xi$-axis. In a similar manner one can define the $C_-$ characteristic angle $\beta$. It is trivial to see that the eigenvalues $\lambda_{\pm}$ satisfy $\tan \alpha = \lambda_+ + \frac{\alpha - \beta}{2}$, $\tan \beta = \lambda_- + \frac{\alpha - \beta}{2}$, where $\delta$ is the angle between $C_+(C_-)$ characteristic and pseudo-velocity vector $(U,V)$ and $\sigma$ is the angle between $(U,V)$ and $\xi$-axis; see Figure 3. Therefore, we have the relations of the form

$$\begin{align*}
\bar{\partial}_+ u &= \cos(\sigma \pm \delta) + \frac{\cos \sigma}{\sin \delta} \bar{\partial}_+ c + \frac{c \cos \alpha \bar{\partial}_+ \beta - c \cos \beta \bar{\partial}_+ \alpha}{2 \sin^2 \delta}, \\
\bar{\partial}_- v &= \sin(\sigma \pm \delta) + \frac{\sin \sigma}{\sin \delta} \bar{\partial}_- c + \frac{c \sin \alpha \bar{\partial}_- \beta - c \sin \beta \bar{\partial}_- \alpha}{2 \sin^2 \delta}.
\end{align*}$$

(2.7)

2.1. First order characteristic decompositions

In this subsection, we derive first order characteristic decompositions for pseudo-steady irrotaional flow. From (2.7) it is easy to see that

$$\begin{align*}
\bar{\partial}_+ u &= \cos(\sigma \pm \delta) + \frac{\cos \sigma}{\sin \delta} \bar{\partial}_+ c + \frac{c \cos \alpha \bar{\partial}_+ \beta - c \cos \beta \bar{\partial}_+ \alpha}{2 \sin^2 \delta}, \\
\bar{\partial}_- v &= \sin(\sigma \pm \delta) + \frac{\sin \sigma}{\sin \delta} \bar{\partial}_- c + \frac{c \sin \alpha \bar{\partial}_- \beta - c \sin \beta \bar{\partial}_- \alpha}{2 \sin^2 \delta}.
\end{align*}$$

(2.8)

Exploiting (2.8) in (2.6) yields

$$\begin{align*}
\bar{\partial}_+ c &= -\frac{\cos 2\delta}{\cot \delta} + \frac{c}{\sin 2\delta} (\bar{\partial}_+ \alpha - \cos 2\delta \bar{\partial}_+ \beta), \\
\bar{\partial}_- c &= -\frac{\cos 2\delta}{\cot \delta} + \frac{c}{\sin 2\delta} (\cos 2\delta \bar{\partial}_- \alpha - \bar{\partial}_- \beta).
\end{align*}$$

(2.9)

Now differentiating the pseudo-Bernoulli law (2.3) and using (2.8), we have

$$\left( \frac{1}{\sin^2 \delta} + \kappa(\tau) \right) \bar{\partial}_+ c = \cot \delta \left( \frac{c \bar{\partial}_+ \delta}{\sin^2 \delta} - 1 \right),$$

(2.10)

where $\kappa(\tau) = \frac{-2p'(\tau) + \tau p''(\tau)}{2p'(\tau) + \tau p''(\tau)}$. 

Figure 2: Characteristic angles and pseudo-flow directions
Using (2.10) in (2.9) leads to the decompositions of the form

\[
\begin{align*}
\partial_{\pm} \beta &= \Omega(\tau, \delta) \cos^2 \delta (c \partial_{\pm} \alpha - 2 \sin^2 \delta), \\
\partial_{\pm} \alpha &= \Omega(\tau, \delta) \cos^2 \delta (c \partial_{\pm} \beta + 2 \sin^2 \delta),
\end{align*}
\]  

(2.11)

where \( \Omega(\tau, \delta) = m(\tau) - \tan^2 \delta = \frac{\kappa(\tau) - 1}{\kappa(\tau) + 1} - \tan^2 \delta \). By exploiting (2.11) in (2.9) we can obtain

\[
\begin{align*}
\partial_{\pm} \alpha &= \frac{\Omega(\tau, \delta)}{2\mu^2(\tau)} \sin 2\delta \partial_{\pm} c = \frac{\tau^2 p''(\tau)}{4c} \Omega \sin 2\delta \partial_{\pm} \tau, \\
\partial_{\pm} \beta &= -\frac{\tan \delta}{\mu^2(\tau)} \partial_{\pm} c - 2 \sin^2 \delta = \frac{\tau^2 p''(\tau)}{2c} \tan \delta \partial_{\pm} \tau - 2 \sin^2 \delta, \\
\partial_{\pm} \alpha &= \frac{\tan \delta}{\mu^2(\tau)} \partial_{\pm} c + 2 \sin^2 \delta = -\frac{\tau^2 p''(\tau)}{2c} \tan \delta \partial_{\pm} \tau + 2 \sin^2 \delta, \\
\partial_{\pm} \beta &= \frac{\Omega(\tau, \delta)}{2\mu^2(\tau)} \sin 2\delta \partial_{\pm} c = -\frac{\tau^2 p''(\tau)}{4c} \Omega \sin 2\delta \partial_{\pm} \tau,
\end{align*}
\]  

(2.12)

where \( \mu^2(\tau) = \frac{1}{1 + \kappa(\tau)} \).

Using (2.12) in (2.8), we obtain the decompositions of velocity

\[
\begin{align*}
\partial_{\pm} u &= \mp \frac{c}{\tau} \sin (\sigma \mp \delta) \partial_{\pm} \tau, \\
\partial_{\pm} v &= \pm \frac{\tau}{\tau} \cos (\sigma \mp \delta) \partial_{\pm} \tau.
\end{align*}
\]  

(2.13)

2.2. Second order characteristic decompositions

In this subsection we derive characteristic decomposition forms for the variable \( \rho \) which are very important to develop a priori gradient estimates of solution in the interaction domain. We first use the following second order normalized commutator relation from [20].

**Proposition 2.1.** (Normalized commutator relation) For any \( I(\xi, \eta) \), we have

\[
\bar{\partial}_{\pm} \bar{\partial}_{\pm} I - \bar{\partial}_{\pm} \bar{\partial}_{\pm} I = \frac{1}{\sin 2\delta} \left[ (\cos 2\delta \bar{\partial}_{\pm} \beta - \bar{\partial}_{\pm} \alpha) \bar{\partial}_{\pm} I - (\bar{\partial}_{\pm} \beta - \cos 2\delta \bar{\partial}_{\pm} \alpha) \bar{\partial}_{\pm} I \right].
\]  

(2.14)

**Proposition 2.2.** For the variable \( \rho \), we have the following second order characteristic decompositions

\[
\begin{align*}
\partial_{\pm} \partial_{\pm} \rho &= \partial_{\pm} \rho \left[ \sin 2\delta + \frac{\tau^4 p''(\tau)}{4c \cos^2 \delta} (\bar{\partial}_{\pm} \rho + (f - 1) \bar{\partial}_{\pm} \rho) \right], \\
\partial_{\pm} \partial_{\pm} \rho &= \partial_{\pm} \rho \left[ \sin 2\delta + \frac{\tau^4 p''(\tau)}{4c \cos^2 \delta} (\bar{\partial}_{\pm} \rho + (f - 1) \bar{\partial}_{\pm} \rho) \right],
\end{align*}
\]  

(2.15)

where

\[
f = 2 \sin^2 \delta - \frac{8p'(\tau) \cos^4 \delta}{\tau p''(\tau)} > 0 \text{ as } p''(\tau) > 0 \text{ and } p'(\tau) < 0.
\]  

(2.16)

**Proof.** This decomposition was first derived in [13] (see also [14]). Here we only sketch the proof of this Proposition. We refer reader to [13] for more details.

We use the commutator relation (2.14) on \( u \) and use (2.13) to obtain

\[
(\sin \alpha + \sin \beta) \frac{1}{c} \frac{d(c\tau)}{d\rho} \bar{\partial}_{\pm} \rho \partial_{\pm} \rho + \sin \alpha \partial_{\pm} \partial_{\pm} \rho + \sin \beta \partial_{\pm} \partial_{\pm} \rho
\]

\[
= \frac{1}{\sin 2\delta} \left[ (\sin \alpha \bar{\partial}_{\pm} \alpha - \sin \alpha \cos 2\delta \bar{\partial}_{\pm} \beta - \cos \alpha \sin 2\delta \bar{\partial}_{\pm} \alpha) \bar{\partial}_{\pm} \rho \\
- (\sin \beta \bar{\partial}_{\pm} \beta - \sin \beta \cos 2\delta \bar{\partial}_{\pm} \alpha + \cos \beta \sin 2\delta \bar{\partial}_{\pm} \beta) \bar{\partial}_{\pm} \rho \right].
\]

Then one can apply the commutator relation on \( \rho \) and use the relations (2.12) to prove this Proposition. 

\[\blacksquare\]
Proposition 2.3. If $\mathcal{F}(\rho)$ is any given smooth function then we have the following characteristic decompositions

$$\begin{align*}
&c \partial_+ \left( \frac{\mathcal{F}(\rho) \partial_+ \rho}{\sin^2 \delta} \right) = \tau_p''(\tau) \partial_+ \rho \left[ \frac{\mathcal{F}(\rho) \partial_+ \rho}{\sin^2 \delta} + \frac{G \mathcal{F}(\rho) \partial_+ \rho}{\sin^2 \delta} \right], \\
&c \partial_- \left( \frac{\mathcal{F}(\rho) \partial_+ \rho}{\sin^2 \delta} \right) = \tau_p''(\tau) \partial_- \rho \left[ \frac{\mathcal{F}(\rho) \partial_+ \rho}{\sin^2 \delta} + \frac{G \mathcal{F}(\rho) \partial_+ \rho}{\sin^2 \delta} \right],
\end{align*}$$

where

$$G = 2 \sin^2 \delta - \frac{8p'(\tau) \cos^4 \delta}{\tau p''(\tau)} - \frac{4c \cos^2 \delta}{\mathcal{F}(\rho)} - 2 \cos^2 \delta + 2 \Omega \cos^4 \delta - 1.$$  

Proof. The proof of this Proposition can be obtained by an easy manipulation on Proposition 2.2, so we omit the details.

3. Interaction of planar and centered rarefaction waves

We solve the Riemann problem (1.1)-(1.2) by utilizing the characteristic decompositions (2.12) and (2.15) in this section. The expansion of gas through a bend corner can be considered as interaction of a two-dimensional centered rarefaction wave with a planar rarefaction wave. For this reason, we first define these waves for 2-D Euler equations.

3.1. Two-dimensional planar rarefaction wave

Let us consider the system (1.1) with the initial data

$$(u, v, \tau)(x, y, 0) = \begin{cases} (u_0, 0, \tau_0), & \mu x + \nu y < 0, \\
\text{vacuum}, & \mu x + \nu y > 0, \end{cases}$$

where $\mu^2 + \nu^2 = 1$.

Let us use a transformation of coordinates of the form $\hat{x} = \mu x + \nu y, \hat{y} = -\nu x + \mu y$. Accordingly, we denote $\hat{u} = \mu u + \nu v, \hat{v} = -\nu u + \mu v$. Then one can solve a 1-D Riemann problem to obtain the planar rarefaction wave solution of the form

$$(\hat{u}, \hat{v}, \tau)(\hat{x}, \hat{y}, t) = \begin{cases} (u_0, 0, \tau_0), & \hat{\xi} < \hat{\xi}_2, \\
(u_r, 0, \tau_r)(\hat{\xi}), & \hat{\xi}_2 \leq \hat{\xi} \leq \hat{\xi}_1, \\
\text{vacuum}, & \hat{\xi} > \hat{\xi}_1, \end{cases}$$

where $\hat{\xi} = \hat{x}/t, \hat{\xi}_1 = \lim_{\tau_r \to \infty} u_r(\tau_r)$ and $\hat{\xi}_2 = u_0 - c(\tau_0)$. The functions $u_r(\hat{\xi})$ and $\tau_r(\hat{\xi})$ can be implicitly determined by

$$u_r = u_0 - \int_{\tau_0}^{\tau_r} \frac{c}{\tau} d\tau (\tau_0 < \tau_r), \quad \hat{\xi} = u_r - c(\tau_r).$$

Therefore, solution of the problem (1.1) and (3.1) is

$$(u, v, \tau)(x, y, t) = \begin{cases} (u_0, 0, \tau_0), & \hat{\xi} < \hat{\xi}_2, \\
(\mu u_r, \nu u_r, \tau_r)(\hat{\xi}), & \hat{\xi}_2 \leq \hat{\xi} \leq \hat{\xi}_1, \\
\text{vacuum}, & \hat{\xi} > \hat{\xi}_1. \end{cases}$$
3.2. Centered rarefaction wave

Motivated by the steady flow in a sharp corner we are trying to construct the centered rarefaction wave for system (2.3) and (2.6) in this subsection. We first define the $C_+$ type centered rarefaction wave as in [27] as follows.

**Definition 3.1.** Let $\Psi(t)$ be an angular domain of the form

$$\Psi(t) = \{ (\xi, \eta) | \xi \in [0, t], \xi \tan \alpha_v \leq \eta \leq \xi \tan \alpha_0, [\alpha_v, \alpha_0) \subset (-\frac{\pi}{2}, \frac{\pi}{2}) \}. \quad (3.5)$$

Then a function $(u, v, \tau)(\xi, \eta)$ is called a $C_+$ type centered rarefaction wave solution of the system (2.3) and (2.6) with $O(0, 0)$ as the center point if it satisfies the following properties:

1. $(u, v, \tau)$ can be determined implicitly from the continuously differentiable functions $\eta = g(\xi, \alpha) := \xi \tan \alpha$ and $(u, v, \tau)(\xi, \eta) = (\hat{u}, \hat{v}, \hat{\xi})(\xi, \alpha)$ defined on a rectangular domain $\Upsilon(t) = \{ (\xi, \alpha) | \xi \in [0, t], \alpha_v \leq \alpha \leq \alpha_0 \}$. Further, for any $(\xi, \alpha) \in \Upsilon(t)/\{ 0 \}, g_\alpha(\xi, \alpha) > 0$ holds.
2. Function $(u, v, \tau)$ satisfies the system (2.3) and (2.6) on $\Upsilon(t)/\{ 0 \}$.
3. $\alpha = \alpha_0$ and $\alpha = \alpha_v$ are the $C_+$ characteristic angles when the flow arrives at the point $P$ and at the vacuum state, respectively and correspond to $\eta = \xi \tan \alpha_0$ and $\eta = \xi \tan \alpha_v$.

The function $(\hat{u}, \hat{v}, \hat{\xi}, \hat{\phi})(0, \alpha) := (\hat{u}, \hat{v}, \hat{\xi}, \hat{\phi})(\alpha)$ is called the principal part of this $C_+$ type centered wave and $\alpha_0 - \alpha_v$ is called amplitude of the centered wave. Then we have the following Lemma.

**Lemma 3.1.** If $(u, v, \tau)(\xi, \eta) = (\hat{u}, \hat{v}, \hat{\xi})(\xi, \alpha)$ and $\eta = \xi \tan \alpha, [\alpha_v, \alpha_0) \subset (-\frac{\pi}{2}, \frac{\pi}{2})$ is the $C_+$ type centered rarefaction wave solution of the system (2.3) and (2.6) in pseudo-supersonic domain, then the principal part $(\hat{u}, \hat{v}, \hat{\xi}, \hat{\phi})(\alpha)$ satisfy

$$\begin{align*}
\frac{d\hat{u}}{d\alpha} + \tan \alpha \frac{d\hat{v}}{d\alpha} &= 0, \quad \hat{\phi}(\alpha) = \text{const.}, \\
\frac{1}{2} \hat{u}^2(\alpha) + \hat{v}^2(\alpha) + \int_{\tau_0}^{\tau} \tau p'(\tau) d\tau &= \text{const.}, \\
\tan \alpha &= \frac{\hat{u}(\alpha)\hat{v}(\alpha) + \hat{c}(\alpha)\sqrt{\hat{u}^2(\alpha) + \hat{v}^2(\alpha) - \hat{c}^2(\alpha)}}{\hat{u}^2(\alpha) - \hat{c}^2(\alpha)}. \quad (3.6)
\end{align*}$$

**Proof.** We substitute $(u, v, \tau)(\xi, \eta) = (\hat{u}, \hat{v}, \hat{\xi})(\xi, \alpha)$ and $\eta = \xi \tan \alpha$ into pseudo-Bernoulli’s law (2.3) and the system (2.6) to obtain

$$\begin{align*}
\frac{\partial \hat{u}}{\partial \xi} + \tan \beta \frac{\partial \hat{v}}{\partial \xi} &= 0, \\
\xi \sec^2 \alpha \frac{\partial \hat{v}}{\partial \xi} + \frac{\partial \hat{u}}{\partial \alpha} + \tan \alpha \frac{\partial \hat{v}}{\partial \alpha} &= 0, \\
\frac{1}{2} \left( (\hat{u} - \xi)^2 + (\hat{v} - \eta)^2 \right) + \int_{\tau_0}^{\tau} \tau p'(\tau) d\tau + \hat{\phi} &= \text{const.} \quad (3.7)
\end{align*}$$

Figure 3: A $C_+$ type centered rarefaction wave in the $(\xi, \eta)$ and $(\xi, \alpha)$ plane
Now for the potential function $\phi$, we have $\phi_\xi = U$ and $\phi_\eta = V$. Then we have

$$\xi \sec^2 \alpha \frac{\partial \hat{\phi}}{\partial \xi} - \left( \tan \alpha \frac{\partial \hat{\phi}}{\partial \alpha} + \cot \sigma \frac{\partial \hat{\phi}}{\partial \alpha} \right) = 0.$$  \hspace{1cm} (3.8)

In view of (3.7), (3.8) and considering $\xi \to 0$, it is easy to obtain the required relations of the Lemma.

3.3. Interaction of rarefaction waves and existence of local solution

It is worth noting that when the gas expands into vacuum from the sharp corner, the planar rarefaction wave $R_p$ and the centered rarefaction wave $R_c$ start interacting from the point $P$ in the self-similar plane where $P = (u_0 - c_0, c_0 \sqrt{u_0 - c_0})$. Therefore, we draw the $C_+$ wave characteristic curve $PQ$ of $R_p$ and $C_-$ characteristic curve $PR$ of $R_c$ to denote the interaction region $\Omega$ in $(\xi, \eta)$ plane which is bounded by curves $PQ$, $PR$ and vacuum boundary $QR$.

The $C_+$ wave characteristic curve $PR$ can be expressed in the following form

$$\begin{cases}
\xi = u_0 - c - \int_{\tau_0}^{\tau} \frac{c}{\tau} d\tau, \\
\eta = \left\{ \frac{\tau e}{\tau_0 c_0} \left[ \left( \frac{2u_0 c_0^2}{u_0 + c_0} \right) - \tau_0 c_0 \left( \frac{c}{\tau} + \int_{\tau_0}^{\tau} \frac{2c}{\tau^2} d\tau \right) \right] \right\}^{1/2}.
\end{cases}$$  \hspace{1cm} (3.9)

The values of $(u, v, \tau)$ on the $C_-$ characteristic curve $PR$ and the $C_+$ characteristic curve $PQ$ can be obtained by $R_c$ and $R_p$, respectively. Then the boundary data on $PQ$ and $PR$ is of the form

$$\begin{cases}
(u, v, \tau)(\xi, \eta) = \begin{cases} 
(u_+, v_+, \tau_+)(\xi, \eta), & \text{on } PQ, \\
(u_-, v_-, \tau_-)(\xi, \eta), & \text{on } PR.
\end{cases}
\end{cases}$$  \hspace{1cm} (3.10)

Noting that $\lambda_- = \frac{V^2 - c^2}{UV + \sqrt{U^2 + V^2 - c^2}}$, we have $\lambda_- = -\infty$ or $\beta = -\frac{\pi}{2}$ on the curve $PQ$. Then from (2.12) it is easy to see that

$$\bar{\partial}_+ \beta = 0, \bar{\partial}_+ \rho = -\frac{2\sin 2\omega c \rho^4}{p''(\tau)} < 0, \bar{\partial}_+ \alpha > 0$$  \hspace{1cm} (3.11)

whenever $p''(\tau) > 0$.
Similarly, across negative characteristic curve $\overline{PR}$, we must have
\[ \tilde{\partial}_-\alpha < 0, \tilde{\partial}_-\rho < 0. \] (3.12)

Then we have the following Lemma:

**Lemma 3.2.** (Local solution) For any $\tau \in (\tau_0, \infty)$, the Goursat problem (2.1) and (3.10) admits a unique $C^1$ solution in the triangular domain $\Omega_+$ bounded by the $C_-$ characteristic curve $\overline{PR}$, $C_+$ characteristic curve $\overline{PQ}$ and the level curve $\tau = \tilde{\tau}$ connecting $Q'$ and $R'$ provided $\tilde{\tau}$ is sufficiently small. Moreover, this solution satisfies
\[ \tilde{\partial}_+\tau > 0. \] (3.13)

**Proof.** From [18], we know that for sufficiently small $\tau = \tilde{\tau}$, the Goursat problem (2.1) and (3.10) admits a unique $C^1$ solution in the domain closed by $\overline{PQ}$, $\overline{PR}$ and the level curve $\tau = \tilde{\tau}$. Further, using the boundary data (3.11) and (3.12) and the characteristic decomposition (2.15) we have $\tilde{\partial}_+\rho < 0$ or equivalently $\tilde{\partial}_+\tau > 0$. ■

4. **Hyperbolicity and a priori $C^0$ and $C^1$ norm estimates**

One of the main difficulties in the present problem is to control the hyperbolicity in the domain of determinacy while extending the local solution to the whole interaction region. Therefore we try to construct invariant regions of characteristic angles in this section.

Let us denote $\tilde{\delta}(\tau) = \tan^{-1}(\sqrt{m(\tau)}, \lim_{\tau \to \infty} \tilde{\delta}(\tau) = \tilde{\delta}^*, \psi(\tau) = \tilde{\delta}(\tau) - \tilde{\delta}(\tau_0) + \int_{\tau_0}^{\tau} |\tilde{\delta}'(\tau)|d\tau$,
\[ \chi(\tau) = \tilde{\delta}(\tau) - \tilde{\delta}(\tau_0) - \int_{\tau_0}^{\tau} |\tilde{\delta}'(\tau)|d\tau \]
and assume that
\[
\begin{cases}
  m(\tau) > 0, p'(\tau) < 0, p''(\tau) > 0, \quad \text{as } \tau > \tau_0, \\
  2\delta(\tau_0) + \chi(\tau) < \alpha_0 + \frac{\pi}{2} < 4\delta(\tau_0).
\end{cases}
\] (4.1)

Then we have the following Lemma:

**Lemma 4.1.** If the Goursat problem (2.1), (3.10) admits a $C^1$ solution in the domain $\Omega_+$ under the hypothesis (4.1) and $\tau = \tau_0$, $\tau = \tau_1$ and $\tau = \tau_2$ are the only points of local extrema of the curve $\delta(\tau)$ in $\tau \in [\tau_0, \tau_2]$ such that $\tau_2 < \tilde{\tau} \in (\tau_0, \infty)$ with $\delta(\tau_0) < \delta(\tau_1)$. Then there exists a positive constant $\epsilon_1$ such that
1. For any $\tau \in (\tau_0, \tau_1)$ if $m'(\tau) > 0$ then we have
\[
\begin{cases}
  \alpha \in \left(\frac{-\pi}{2} - \epsilon_1, 2\delta(\tau_0) + \alpha_0 + \epsilon_1 + 2(\delta(\tau) - \tilde{\delta}(\tau_0))\right) \\
  \beta \in \left(\frac{-\pi}{2} - \epsilon_1 - 2(\delta(\tau) - \tilde{\delta}(\tau_0)), \alpha_0 + \epsilon_1 - 2\tilde{\delta}(\tau_0)\right)
\end{cases}
\] (4.2)

for all $\tau \in [\tau_0, \tau_1]$ on $C_+$ characteristic $\overline{PQ_1}$ and $C_-$ characteristic $\overline{PR_1}$ where $Q_1$ and $R_1$ are the points corresponding to $\tau = \tau_1$ on the $C_+$ and $C_-$ characteristics $\overline{PQ}$ and $\overline{PR}$, respectively.
2. For any $\tau \in (\tau_1, \tau_2)$ if $m'(\tau) < 0$ then we have
\[
\begin{cases}
  \alpha \in \left(\frac{-\pi}{2} - \epsilon_1 + 2\delta(\tau_0) + 2(\delta(\tau) - \tilde{\delta}(\tau_1))\right) \\
  \beta \in \left(\frac{-\pi}{2} - \epsilon_1 + \alpha_0 + \epsilon_1 - 2\tilde{\delta}(\tau_0) - 2(\delta(\tau) - \tilde{\delta}(\tau_1))\right)
\end{cases}
\] (4.3)

for all $\tau \in (\tau_1, \tau_2)$ on $C_+$ characteristic $\overline{Q_1Q_2}$ and $C_-$ characteristic $\overline{R_1R_2}$ where $Q_2$ is a point on $\overline{PQ}$ while $R_2$ is a point on $\overline{PR}$ corresponding to $\tau = \tau_2$, respectively.
Case 1. Let us assume that for $\tau \in (\tau_0, \tau_1)$, $m'(\tau) > 0$ or equivalently $\delta'(\tau) > 0$. Then using the fact that $\partial_{\pm}\tau > 0$ in $\Omega_{\pm}$ we have $\partial_{\pm}\delta > 0$ for all $\tau \in (\tau_0, \tau_1)$. Also for $\delta'(\tau) > 0$, we have $\chi(\tau) = 0$. Then on the $C_+$ characteristic $PQ_1$, we must have

$$(\alpha, \beta)(P) = \left(\alpha_0, -\frac{\pi}{2}\right) \subset \left(-\frac{\pi}{2} - \epsilon_1 + 2\delta(\tau_0), \alpha_0 + \epsilon_1 + 2(\delta(\tau) - \delta(\tau_0))\right) \times \left(-\frac{\pi}{2} - \epsilon_1 - 2(\delta(\tau) - \delta(\tau_0)), \alpha_0 + \epsilon_1 - 2\delta(\tau_0)\right)$$

for a sufficiently small positive constant $\epsilon_1$.

If there exists a point $J$ on $PQ_1$ such that $\alpha(J) = -\frac{\pi}{2} - \epsilon_1 + 2\delta(\tau_0)$. Then we have

$$\delta(J) = \frac{\alpha - \beta}{2} = \frac{2\delta(\tau_0) - \epsilon_1}{2} < \bar{\delta}(\tau_0) \leq \delta(\tau).$$

Hence, we use (2.12) to obtain

$$c \tilde{\partial}_+\alpha(\tilde{\tau}) = \frac{\tau^2 p''(\tau)}{4c} (\tan^2 \delta(\tau) - \tan^2 \delta) \sin 2\delta \tilde{\partial}_+\tau > 0.$$ 

Again, if there exists a point $J$ on $PQ_1$ such that $\alpha(J) = \alpha_0 + \epsilon_1 + 2(\bar{\delta}(\tau) - \delta(\tau_0))$ then using the hypothesis (4.1) we have

$$\delta(J) = \frac{\alpha_0 + \epsilon_1 + 2(\bar{\delta}(\tau) - \delta(\tau_0)) + \pi/2}{2} > \bar{\delta}(\tau),$$

which means that

$$\bar{\partial}_+\alpha(J) < 0.$$ 

Combining the above results it is easy to see that $-\frac{\pi}{2} - \epsilon_1 + 2\bar{\delta}(\tau_0) < \alpha < \alpha_0 + \epsilon_1 + 2(\bar{\delta}(\tau) - \delta(\tau_0))$ for all $\tau \in [\tau_0, \tau_1]$. Hence we have proved the first part of Lemma for positive characteristic $PQ_1$.

Now for $C_-$ characteristic curve $PR_1$, we use the first order decomposition of $\beta$

$$c \tilde{\partial}_-\beta = -\frac{\tau^2 p''(\tau)}{4c} (\tan^2 \tilde{\delta}(\tau) - \tan^2 \delta) \sin 2\delta \tilde{\partial}_-\tau.$$ 

Then in view of $\delta(\tau_0) > \bar{\delta}(\tau_0)$ and $\bar{\partial}_-\tau > 0$, we have $\bar{\partial}_-\beta > 0$ at the point $P$ which together with the fact that $\bar{\partial}_-\alpha < 0$ implies that $\bar{\partial}_-\delta < 0$ at the point $P$ or in other words $\beta'(\alpha)|_P = \frac{\bar{\partial}_-\beta}{\bar{\partial}_-\alpha}|_P < 0$.

Now we consider the properties of the curve $\beta = \beta(\alpha)$ along $PR_1$. Let $l$ be the line $\alpha - \beta = 2\bar{\delta}(\tau_0)$ in $(\alpha, \beta)$ plane. It is easy to see that below the line $l$ we have $\delta(\tau) > \bar{\delta}(\tau)$ therefore $\bar{\partial}_-\beta > 0$ or $\beta'(\alpha) < 0$; see Figure 5. Further, if $S$ is the point of intersection of the two curves $\beta(\alpha)$ and $l$, then $\beta'(\alpha)|_S = 0$. It is worth noting from the fact $\bar{\partial}_-\alpha < 0$ that the two curves have only one intersection point. Moreover, $\beta = \beta(\alpha)$ must pass through $l$ if
Let $\tau \in \tau$ for the sake of brevity. $\alpha$ such that $\bar{\alpha}$ as $m$.

*Case 2.* Now let us consider the case $\overline{m'}(\tau) < 0$ or $\overline{\delta'}(\tau) < 0$ when $\tau \in (\tau_1, \tau_2)$. Then we are going to prove that

$$(\alpha, \beta) \in \left(\frac{-\pi}{2} - \epsilon_1 + 2\delta(\tau_0) + 2(\delta(\tau) - \delta(\tau_1)), \alpha_0 + \epsilon_1\right) \times \left(\frac{-\pi}{2} - \epsilon_1, \alpha_0 + \epsilon_1 - 2\delta(\tau_0) - 2(\delta(\tau) - \delta(\tau_1))\right) .$$

Again on $C_+$ characteristic $\overline{Q_1Q_2}$ starting from the point $Q_1$, we have $\overline{\delta}(\tau) = 0$ so that if there exists a point on $\overline{Q_1Q_2}$ such that $\alpha = -\frac{\pi}{2} - \epsilon_1 + 2\delta(\tau_0) + 2(\delta(\tau) - \delta(\tau_1))$, then

$$\delta = \frac{2\delta(\tau_0) + 2(\delta(\tau) - \delta(\tau_1)) - \epsilon_1}{2} < \delta(\tau)$$

as $\delta(\tau_0) < \delta(\tau_1)$ which implies that $\overline{\delta}(\alpha) > 0$.

On a similar ground we can prove that $\overline{\delta}(\alpha) < 0$ under the hypothesis (4.1) if there exists a point on $\overline{Q_1Q_2}$ such that $\alpha = \alpha_0 + \epsilon_1$.

For the $C_-$ characteristics $\overline{R_1R_2}$ starting from the point $R_1$, we have $\overline{\delta'}(\alpha)|_{R_1} > 0$. Therefore, a similar argument as in case 1 of this Lemma is enough to prove this part as well so we don’t repeat the arguments again for the sake of brevity.

We can use the Lemma [4.1] to obtain an invariant square for the characteristic angles in the following Proposition.

**Proposition 4.1.** Let $\tau = \tau_0, \tau = \tau_1$ and $\tau = \tau_2$ are the only points of local extrema of the curve $\overline{\delta}(\tau)$ in $\tau \in [\tau_0, \tau_2]$ such that $\tau_2 < \overline{\tau} \in (\tau_0, \infty)$ with $\overline{\delta}(\tau_0) < \overline{\delta}(\tau_1)$ then

1. For any $\tau \in (\tau_0, \tau_1)$ if $\overline{m'}(\tau) > 0$ then

$$\alpha \in \left(\frac{-\pi}{2} - \epsilon_1 + 2\delta(\tau_0), \alpha_0 + \epsilon_1 + 2(\delta(\tau) - \delta(\tau_0))\right)$$

$$\beta \in \left(\frac{-\pi}{2} - \epsilon_1 - 2(\delta(\tau) - \delta(\tau_0)), \alpha_0 + \epsilon_1 - 2\delta(\tau_0)\right) (4.5)$$

for all $\tau \in [\tau_0, \tau_1]$ in $\Omega_{\tau_1}$ where $\Omega_{\tau_1}$ is a closed region bounded by the curves $\overline{PQ_1}$, $\overline{PR_1}$ and $\overline{Q_1R_1}$; see Figure 6.
2. For any \( \tau \in (\tau_1, \tau_2) \) if \( m'(\tau) < 0 \) then

\[
\begin{align*}
\alpha &\in \left( -\frac{\pi}{2} - \epsilon_1 + 2\delta(\tau_0), \alpha_0 + \epsilon_1 + 2(\delta(\tau) - \delta(\tau_0)) \right) \\
\beta &\in \left( -\frac{\pi}{2} - \epsilon_1, \alpha_0 + \epsilon_1 - 2\delta(\tau_0) - 2(\delta(\tau) - \delta(\tau_1)) \right)
\end{align*}
\]

(4.6)

for all \( \tau \in (\tau_1, \tau_2) \) in \( \Omega_{\tau_2}/\Omega_{\tau_1} \) where \( \Omega_{\tau_2} \) is a closed region bounded by the curves \( PR_1, PR_2 \) and \( Q_2 R_2 \); see Figure 6.

Proof. Case 1. To prove the first part of Proposition we need to prove that for any arbitrary point \( E \) in \( \Omega_{\tau_1} \), \( (\alpha, \beta)(E) \in \Gamma(\tau) \), where

\[
\Gamma: \left( -\frac{\pi}{2} - \epsilon_1 + 2\delta(\tau_0), \alpha_0 + \epsilon_1 + 2(\delta(\tau) - \delta(\tau_0)) \right) \times \left( -\frac{\pi}{2} - \epsilon_1 - 2(\delta(\tau) - \delta(\tau_0)), \alpha_0 + \epsilon_1 - 2\delta(\tau_0) \right).
\]

Let us assume that \( C_+ \) characteristic curve passing through \( E \) intersects \( PR_1 \) at \( E_+ \) while the \( C_- \) characteristic curve passing through \( E \) intersects \( PR_1 \) at \( E_- \) and \( D_E \) is a closed domain bounded by characteristic curves \( PE_+, PE_-, ED_E \) and \( ED_E \); see Figure 6. Then we claim that if \( (\alpha, \beta) \in \Gamma(\tau) \) as \( (\xi, \eta) \in D_E / \{ E \} \) then \( (\alpha, \beta) \in \Gamma(\tau_E) \) for all \( (\xi, \eta) \in D_E \), where \( \tau_E \leq \tau \).

We now divide the boundary of \( \Gamma(\tau_E) \) into the following six parts (see Figure 7):

- \( \partial \Gamma_1 = \left\{ -\frac{\pi}{2} - \epsilon_1 + 2\delta(\tau_0) \leq \alpha \leq \alpha_0 + \epsilon_1 + 2(\delta(\tau_E) - \delta(\tau_0)), \beta = \alpha_0 + \epsilon_1 - 2\delta(\tau_0) \right\} \)
- \( \partial \Gamma_2 = \left\{ \alpha = -\frac{\pi}{2} - \epsilon_1 + 2\delta(\tau_0), -\frac{\pi}{2} - \epsilon_1 - 2(\delta(\tau_E) - \delta(\tau_0)) \leq \beta \leq \alpha_0 + \epsilon_1 - 2\delta(\tau_0) \right\} \)
- \( \partial \Gamma_3 = \left\{ -\frac{\pi}{2} - \epsilon_1 + 2\delta(\tau_0) \leq \alpha \leq \alpha_0 + \epsilon_1 + 2(\delta(\tau_E) - \delta(\tau_0)), \beta = -\frac{\pi}{2} - \epsilon_1 - 2(\delta(\tau_E) - \delta(\tau_0)) \right\} \)
- \( \partial \Gamma_4 = \left\{ \alpha = \alpha_0 + \epsilon_1 + 2(\delta(\tau_E) - \delta(\tau_0)), -\frac{\pi}{2} - \epsilon_1 - 2(\delta(\tau_E) - \delta(\tau_0)) \leq \beta \leq \alpha_0 + \epsilon_1 - 2\delta(\tau_0) \right\} \)
- \( \partial \Gamma_5 = \left\{ \alpha = \alpha_0 + \epsilon_1 + 2(\delta(\tau_E) - \delta(\tau_0)), \beta = \alpha_0 + \epsilon_1 - 2\delta(\tau_0) \right\} \)

Figure 7: Invariant square for the case \( m'(\tau) > 0 \)
\[ \partial \Gamma_6 = \{ \alpha = -\frac{\pi}{2} - \epsilon_1 + 2\delta(\tau_0), \beta = -\frac{\pi}{2} - \epsilon_1 - 2(\delta(\tau_E) - \delta(\tau_0)) \} \]

From Lemma 4.1 we have \((\alpha, \beta) \in \Gamma(\tau_E)\) on \(PE_+\) and \(PE_–\). Then if we assume that the proposition is not true then there must exists a point \(F \in D_E\) such that \((\alpha, \beta) \in \Gamma(\tau_E)\) for all \((\xi, \eta) \in D_E/\{F\}\) and \((\alpha, \beta)(\Gamma) \in \bigcup_{i=1}^6 \partial \Gamma_i\) where \(D_F\) can be defined in the same manner as \(D_E\).

Let us assume that \((\alpha, \beta)(F) \not\in \partial \Gamma_1\). Then using the hypothesis \((\alpha, \beta) \in \Gamma(\tau)\) as \((\xi, \eta) \in D_E/\{E\}\) and noting that \(\partial \Gamma_1 \subset \partial \Gamma(\tau)\), we obtain \(E = F\). Now since \(\delta < \delta(\tau_E)\) in \(\partial \Gamma_1\) so we have

\[ c\partial_–(F) < 0. \]

However, from \((\alpha, \beta) \in \Gamma(\tau_E)\) as \((\xi, \eta) \in D_E/\{F\}\) we must have \(\partial_–(F) \geq 0\), which yields a contradiction. Hence \((\alpha, \beta) \not\in \partial \Gamma_2\). Similarly, one can prove that \((\alpha, \beta) \not\in \partial \Gamma_2\).

Now let us assume that \((\alpha, \beta)(F) \in \partial \Gamma_3\). Then noting that \(\delta > \delta(E) \geq \delta(F)\) on \(\partial \Gamma_3\), we must have

\[ c\partial_–(F) > 0. \]

However, according to \((\alpha, \beta) \in \Gamma(\tau_E)\) as \((\xi, \eta) \in D_E/\{F\}\) we must have \(\partial_–(F) \leq 0\), which is a contradiction. Hence \((\alpha, \beta) \not\in \partial \Gamma_4\). Similarly, one can prove that \((\alpha, \beta) \not\in \partial \Gamma_4\).

Now if \((\alpha, \beta) \in \partial \Gamma_5\), then we define a function \(\hat{\alpha}\) on \(\hat{F}_+\) by the following relations

\[ \begin{cases} \begin{align*} c\hat{\partial}_+\hat{\alpha} = & \frac{\tau^2p''(\tau)}{4c} \left[ \tan^2 \delta(\tau_E) - \tan^2 \left( \frac{\alpha - \alpha_0 + 2\delta(\tau_0) - \epsilon_1}{2} \right) \right] \hat{\partial}_+\tau, \\ \hat{\alpha}(\hat{F}_+) = & \alpha(\hat{F}_+). \end{align*} \end{cases} (4.7) \]

According to hypothesis that \((\alpha, \beta) \in \Gamma(\tau_E)\) as \((\xi, \eta) \in D_E/\{F\}\), it is easy to see that \(\alpha(\hat{F}_+) = \hat{\alpha}(\hat{F}_+) < \alpha_0 + \epsilon + 2(\delta(\tau_E) - \delta(\tau_0))\). Then integrating \((4.7)\) along the positive characteristic curve \(\hat{F}_+\) from \(\hat{F}_+\) to \(\hat{F}_6\), we obtain

\[ \hat{\alpha}(\hat{F}) < \alpha_0 + \epsilon + 2(\delta(\tau_E) - \delta(\tau_0)). \]

Combining \((4.7)\) and \((\alpha, \beta) \in \Gamma(\tau_E)\) as \((\xi, \eta) \in D_E/\{F\}\), we must have

\[ \left[ \tan^2 \left( \frac{\alpha - \alpha_0 + 2\delta(\tau_0) - \epsilon_1}{2} \right) - \tan^2 \left( \frac{\alpha - \beta}{2} \right) \right] > 0 \text{ or } \hat{\partial}_+ \hat{\alpha} - \alpha > 0. \]

Therefore, we have \(\alpha(F) < \hat{\alpha}(\hat{F}) < \alpha_0 + \epsilon + 2(\delta(\tau_E) - \delta(\tau_0))\), which leads to a contradiction. Hence, \((\alpha, \beta) \not\in \partial \Gamma_5\). On a similar ground one can prove that \((\alpha, \beta) \not\in \partial \Gamma_6\).

Combining all these results it can be stated that there is no such point \(F \in D_E\), hence we have proved the first part of Proposition.

Case 2. Now we proceed to the proof of second part of Proposition. Let us assume that the curve \(\delta(\tau)\) be monotonically decreasing in \((\tau_1, \tau_2)\). Then we need to prove that \((\alpha, \beta)(G) \in \Gamma(\tau)\) for any \(G \in \Omega_{\tau_2}/\Omega_{\tau_1}\), where

\[ \Gamma'(\tau) : \left( -\frac{\pi}{2} - \epsilon_1 + 2\delta(\tau_0) + 2(\delta(\tau) - \delta(\tau_1)), \alpha_0 + \epsilon_1 \right) \times \left( -\frac{\pi}{2} - \epsilon_1, \alpha_0 + \epsilon_1 - 2\delta(\tau_0) - 2(\delta(\tau) - \delta(\tau_1)) \right). \]

Similar to first part we divide the boundary of \(\Gamma'(\tau_0)\) into the following six parts

\[ \partial \Gamma_1 = \left\{ -\frac{\pi}{2} - \epsilon_1 + 2\delta(\tau_0) + 2(\delta(\tau_G) - \delta(\tau_1)) \leq \alpha \leq \alpha_0 + \epsilon_1, \beta = \alpha_0 + \epsilon_1 - 2\delta(\tau_0) - 2(\delta(\tau_G) - \delta(\tau_1)) \right\}; \]
such that Similar to first part of proof let us assume that the Proposition is not true then there must exists a point \( H \) which is a contradiction since defined similar to \( D \).

Prove that leads to a contradiction since \( \bar{\tau} \) since \( \partial_{\alpha,\beta} \).

First let us suppose that \( (\alpha,\beta) \in \Gamma'(\tau_G) \) for all \( (\xi,\eta) \in \mathcal{D}_H/\{H\} \) and \( (\alpha,\beta)(H) \in \bigcup_{i=1}^{6} \partial \Gamma'_i \) where \( \mathcal{D}_G \) and \( \mathcal{D}_H \) can be defined similar to \( \mathcal{D}_E \).

First let us suppose that \( (\alpha,\beta)(H) \in \partial \Gamma'_1 \). Then since \( \delta(H) < \bar{\delta}(\tau_G) \leq \bar{\delta}(\tau_H) \), we have \( \bar{\delta}(\tau_H) < 0 \)

which is a contradiction since \( \bar{\delta}(\tau_H) \geq 0 \) as \( (\alpha,\beta) \in \Gamma'(\tau_G) \) for all \( (\xi,\eta) \in \mathcal{D}_H/\{H\} \). Therefore, \( (\alpha,\beta)(H) \notin \partial \Gamma'_1 \). Similarly, one can prove that \( (\alpha,\beta) \notin \partial \Gamma'_i \).

Now let us suppose that \( (\alpha,\beta)(H) \in \partial \Gamma'_3 \). Then by the assumption of Proposition we must have \( G = H \), since \( \partial \Gamma'_3 \subset \partial \Gamma'(\tau) \). Hence we have \( \bar{\delta}(\tau_H) = \bar{\delta}(\tau_G) < \delta(H) \). Consequently, we obtain that \( \bar{\delta}(\tau_H) > 0 \) which leads to a contradiction since \( \bar{\delta}(\tau_H) \leq 0 \) as \( (\alpha,\beta) \in \Gamma'(\tau_G) \) for all \( (\xi,\eta) \in \mathcal{D}_H/\{H\} \). Similarly, one can prove that \( (\alpha,\beta) \notin \partial \Gamma'_i \).

Now let us assume that \( (\alpha,\beta) \in \partial \Gamma'_6 \), then we define the function \( \hat{\alpha} \) which satisfies

\[
\begin{align*}
\begin{cases}
  c \bar{\delta} + \hat{\alpha} = \frac{x^2 p''(\tau)}{4c} \left[ \tan^2 \bar{\delta}(\tau_G) - \tan^2 \left( \frac{\hat{\alpha} + \frac{x}{2} + \epsilon_1}{2} \right) \right] \bar{\delta}(\tau), \\
  \hat{\alpha}(H_+) = \alpha(H_+),
\end{cases}
\end{align*}
\]

\[ (4.9) \]
on \(H_+\). Then since \((\alpha, \beta)(H_+) \in \Gamma'(\tau_G)\), we have

\[-\frac{\pi}{2} - \epsilon_1 + 2\delta(\tau_0) + 2(\delta(\tau) - \delta(\tau_1)) < \alpha(H_+) = \alpha(H_+).\]  

(4.10)

Then combining (2.12) and (4.9) we have

\[
\begin{cases}
c\bar{\partial}_+ (\alpha - \alpha) = \frac{\tau^2 p''(\tau)}{4c} \left[ \tan^2 \delta(\tau_G) - \tan^2 \delta(\tau) \right] \tilde{\partial}_+ \tau \\
- \frac{\tau^2 p''(\tau)}{4c} \left[ \tan^2 \left( \frac{\alpha - \beta}{2} \right) - \tan^2 \left( \frac{\alpha - \beta}{2} \right) \right] \tilde{\partial}_+ \tau,
\end{cases}
\]

(4.11)

Then noting that \(\delta(\tau) > \delta(\tau_G)\) on \(H_+\) and the hypothesis on \(H\), we can integrate above to obtain

\[-\frac{\pi}{2} - \epsilon_1 + 2\delta(\tau_0) + 2(\delta(\tau) - \delta(\tau_1)) < \alpha(H) < \alpha(H),\]

(4.12)

which yields a contradiction, hence \((\alpha, \beta) \notin \partial \Gamma'_6\). In a similar manner, one can prove that \((\alpha, \beta) \notin \partial \Gamma'_5\). Hence proved.

In view of the boundary data (3.11) and (3.12) we define

\[M_1(\tau) = \min \{ \partial_+ \rho \bar{\Omega}_G, \partial_- \rho \bar{\Omega}_R \}\]

and

\[M_2(\tau) = \frac{\rho^n M_1(\tau)}{\sin^2 \delta}.\]

(4.13)

Then we have the following Lemma.

**Lemma 4.2.** If the Goursat problem (2.1) admits a classical solution in \(\Omega_{\hat{\tau}}\) for some \(\hat{\tau} \in (\tau_0, \infty)\).

Then we have

\[(\partial_+ \rho, \partial_- \rho) \in (M_1(\tau), 0) \times (M_1(\tau), 0), \left( \frac{\rho^n \partial_+ \rho}{\sin^2 \delta}, \frac{\rho^n \partial_- \rho}{\sin^2 \delta} \right) \in (M_2(\tau), 0) \times (M_2(\tau), 0)\]

(4.14)

in \(\Omega_{\hat{\tau}}\).

**Proof.** We first prove that if \(0 < \delta < \pi/2\) in \(\Omega_{\hat{\tau}}\) then \((\partial_+ \rho, \partial_- \rho) \in (M_1(\tau), 0) \times (M_1(\tau), 0)\) in \(\Omega_{\hat{\tau}}\). We claim that if \((\partial_+ \rho, \partial_- \rho) \in (M_1(\tau), 0) \times (M_1(\tau), 0)\) for all points in \(\mathcal{D}_E/\{E\}\) then \((\partial_+ \rho, \partial_- \rho) \in (M_1(\tau), 0) \times (M_1(\tau), 0)\) at \(E\) where \(\mathcal{D}_E\) is same as defined in Proposition 4.1. If \(\partial_- \rho(E) = M_1(\tau)\) and \(\partial_+ \rho(E) \in [M_1(\tau), 0)\), then by \((\partial_+ \rho, \partial_- \rho) \in (M_1(\tau), 0) \times (M_1(\tau), 0)\) for all \(\mathcal{D}_E/\{E\}\) we have \(\partial_- \partial_+ \rho \leq 0\) at \(E\). However, by second equation of (2.15) at the point \(E\), we have

\[c\partial_- \partial_+ \rho > \frac{\tau^4 p''(\tau) f}{4c \cos^2 \delta} M_2^2(\tau) + M_1(\tau) \sin 2\delta > \frac{\tau^4 p''(\tau) \sin^2 \delta}{2c \cos^2 \delta} M_1^2(\tau) + M_1(\tau) \sin 2\delta > 0,
\]

which leads to a contradiction. Hence by a continuity argument we must have \((\partial_+ \rho, \partial_- \rho) \in (M_1(\tau), 0) \times (M_1(\tau), 0)\) in \(\Omega_{\hat{\tau}}\).

Now for the other part of Lemma we need to prove that if \(0 < \delta < \pi/2\) in \(\Omega_{\hat{\tau}}\) then \(\left( \frac{\rho^n \partial_+ \rho}{\sin^2 \delta} \right) \in (M_2(\tau), 0) \times (M_2(\tau), 0)\) for \(\tau \in [\tau_0, \infty)\) in \(\Omega_{\hat{\tau}}\). Similar to the first part of proof we assume a point \(E\) in \(\Omega_{\hat{\tau}}\) such that \(\left( \frac{\rho^n \partial_+ \rho}{\sin^2 \delta} \right) \in (M_2(\tau), 0) \times (M_2(\tau), 0)\) for all points in \(\mathcal{D}_E/\{E\}\) then \(\partial_- \left( \frac{\rho^n \partial_+ \rho}{\sin^2 \delta} \right) \leq 0\) at \(E\). However noting that \(\mathcal{G} > 0\) for sufficiently large \(n\) we have \(\partial_- \left( \frac{\rho^n \partial_+ \rho}{\sin^2 \delta} \right) > 0\) at \(E\), which leads to a contradiction. Thus we have \(\left( \frac{\rho^n \partial_+ \rho}{\sin^2 \delta} \right) > M_2(\tau)\). Hence we have proved the Lemma.
It is easy to see that the Lemma 4.1 and Proposition 4.1 can be extended to the whole interaction domain \( \Omega_\bar{\tau} \) for any \( \bar{\tau} \in [\tau_0, \infty) \) if the curve \( \delta(\tau) \) possesses countable number of points of extrema in \([\tau_0, \infty)\) and \( \lim_{\bar{\tau} \to \infty} \delta(\bar{\tau}) = \delta^* \) exists. Let \( \bigcup_{i \in \mathbb{N}} \tau_{i-1} \) is a countable collection of points of local extrema of the curve \( \delta(\tau) \) in \([\tau_0, \infty)\). Then we can repeat the process of Lemma 4.1 and Proposition 4.1 and use Lemma 4.2 to obtain the following Lemma.

**Lemma 4.3.** If the Goursat problem (2.1) and (3.10) admits a \( C^1 \) solution on \( \Omega_\bar{\tau} \) where \( \tau \in [\tau_0, \infty) \) and \( \bigcup_{i \in \mathbb{N}} \tau_{i-1} \) is a countable collection of points of local extrema of the curve \( \delta(\tau) \) in \([\tau_0, \infty)\). Then there exists a sufficiently small \( \epsilon_2 > 0 \) which depends only on \( \tau \) such that \( (\alpha, \beta) \) lies in the invariant region

\[
\Gamma^*: \left( -\frac{\pi}{2} - \epsilon_1 + 2\delta(\tau_0) + \chi_\infty, \alpha_0 + \epsilon_1 + \psi_\infty \right) \times \left( -\frac{\pi}{2} - \epsilon_1 - \psi_\infty, \alpha_0 + \epsilon_1 - 2\delta(\tau_0) - \chi_\infty \right) \cap \{ \alpha - \beta > \epsilon_2 \} \text{ in } \Omega_\bar{\tau}
\]

where

\[
\psi_\infty = \delta^* - \delta(\tau_0) + \int_{\tau_0}^{\infty} |\delta'(\tau)|d\tau = \left\{ \begin{array}{ll}
\sum_{i \in \mathbb{N}} 2(\delta(\tau_{2i-1}) - \delta(\tau_{2i-2})), & \delta(\tau_0) < \delta(\tau_1), \\
\sum_{i \in \mathbb{N}} 2(\delta(\tau_{2i-1}) - \delta(\tau_{2i-2})), & \delta(\tau_0) > \delta(\tau_1),
\end{array} \right.
\]

\[
\chi_\infty = \delta^* - \delta(\tau_0) - \int_{\tau_0}^{\infty} |\delta'(\tau)|d\tau = \left\{ \begin{array}{ll}
\sum_{i \in \mathbb{N}} 2(\delta(\tau_{2i-1}) - \delta(\tau_{2i-2})), & \delta(\tau_0) < \delta(\tau_1), \\
\sum_{i \in \mathbb{N}} 2(\delta(\tau_{2i-1}) - \delta(\tau_{2i-2})), & \delta(\tau_0) > \delta(\tau_1).
\end{array} \right.
\]

**Proof.** By repeating the steps of Proposition 4.1 for \( \tau = \tau_0, \tau_1, \tau_2, \ldots, \tau_n, \tau_{n+1}, \ldots \) it is easy to see that for all \( \tau \in [\tau_0, \infty) \) we have

\[
(\alpha, \beta) \in \left( -\frac{\pi}{2} - \epsilon_1 + 2\delta(\tau_0) + \chi_\infty, \alpha_0 + \epsilon_1 + \psi_\infty \right) \times \left( -\frac{\pi}{2} - \epsilon_1 - \psi_\infty, \alpha_0 + \epsilon_1 - 2\delta(\tau_0) - \chi_\infty \right)
\]

in \( \Omega_\bar{\tau} \).

Now we move forward to prove the other part of the Lemma. Now if \( 0 < \alpha - \beta = \epsilon_2 < \alpha_0 + \frac{\pi}{2} \) at some point in \( \Omega_\bar{\tau} \) then we see from (2.12) that

\[
c\tilde{\partial}_\pm \left( \alpha - \beta \right) = \frac{\sin^2 \delta}{2} \left[ 2 + \frac{p''(\tau)}{4c\rho^{1+n}} \left[ \tan \delta - \frac{\Omega \sin 2\delta}{2} \right] \left( \frac{p''(\tau)}{\sin^2 \delta} \right) \right],
\]

\[
> \frac{\sin^2 \delta}{2} \left[ 2 + \frac{p''(\tau)}{4c\rho^{1+n}} \left[ \tan \delta - \frac{\Omega \sin 2\delta}{2} \right] M_2(\tau) \right] > 0.
\]

Since \( (\alpha - \beta)(P) = \alpha_0 + \pi/2 > \epsilon_2 \) then we have \( \alpha - \beta > \epsilon_2 \) in \( \Omega_\bar{\tau} \). Hence proved.
The above Lemma can be generalized for a more general case of \( \delta(\tau) \) as follows:

**Lemma 4.4.** Assume that the Goursat problem (2.1), (3.10) admits a \( C^1 \) solution in the domain \( \Omega_\tau \) then the invariant region for \( (\alpha, \beta) \) is

\[
\left( -\frac{\pi}{2} - \epsilon_1 + 2\delta(\tau_0) + \chi(\tilde{\tau}), \alpha_0 + \epsilon_1 + \psi(\tilde{\tau}) \right) \times \left( -\frac{\pi}{2} - \epsilon_1 - \psi(\tilde{\tau}), \alpha_0 + \epsilon_1 - 2\delta(\tau_0) - \chi(\tilde{\tau}) \right) \cap \{ \alpha - \beta > \epsilon_2 \}.
\]

From the above Lemmas it follows that if the Goursat problem (2.1) and (3.10) admits a \( C^1 \) solution on \( \Omega_\tau \) for any \( \tilde{\tau} \in [\tau_0, \infty) \) then we have

\[
0 < \frac{\epsilon_0}{2} < \delta < \epsilon_1 + \psi_\infty < \frac{\pi}{2} \quad \text{on} \quad \Omega_\tau.
\]

**4.1. \( C^0 \) and \( C^1 \) norm estimates of the solution**

**Lemma 4.5.** (\( C^0 \) estimates) If the Goursat problem (2.1) and (3.10) admits a \( C^1 \) solution in \( \Omega_\tau \), where \( \tilde{\tau} \in [\tau_0, \infty) \). Then there exists a positive \( \mathcal{H}_0 \) depending on \( \tilde{\tau} \) such that

\[
\| (u, v, \rho) \|_{C^0(\Omega_\tau)} < \mathcal{H}_0.
\]

**Proof.** In view of the relations (2.7) and Lemma 4.4, this Lemma can be easily proved, so we omit the details. \( \square \)

**Lemma 4.6.** (\( C^1 \) estimates) Suppose that the Goursat problem (2.1) and (3.10) admits a \( C^1 \) solution in \( \Omega_\tau \) for some \( \tilde{\tau} \in (\tau_0, \infty) \). Then there exists a positive \( \mathcal{H}_1 \) depending on \( \tilde{\tau} \) such that

\[
\| (u, v, \rho) \|_{C^1(\Omega_\tau)} < \mathcal{H}_1.
\]

**Proof.** This result can be immediately obtained using (2.13), (4.17), Lemma 4.2-4.4-4.5 and the relations

\[
\partial_\xi = \frac{\sin \alpha \bar{\delta}_- - \sin \beta \bar{\delta}_+}{\sin 2\delta}, \quad \partial_\eta = \frac{\cos \beta \bar{\delta}_+ - \cos \alpha \bar{\delta}_-}{\sin 2\delta}.
\]

\( \square \)

**5. Existence of global solution**

First let us assume that the Goursat problem (2.1), (3.10) admits a unique \( C^1 \) solution in \( \Omega_\tau \) where \( \tilde{\tau} \in [\tau_0, \infty) \). Then along the level curves of specific volume \( \tau \), we have

\[
\left| \frac{d\xi}{d\eta} \right| = \left| \frac{\cos \beta \bar{\delta}_+ - \cos \alpha \bar{\delta}_-}{\sin \beta \bar{\delta}_+ - \sin \alpha \bar{\delta}_-} \right| < \frac{\cos \beta \bar{\delta}_+}{\sin \beta \bar{\delta}_+ - \sin \alpha \bar{\delta}_-} + \frac{\cos \alpha \bar{\delta}_-}{\sin \beta \bar{\delta}_+ - \sin \alpha \bar{\delta}_-} < 2 \quad \text{for} \quad \sin(\tilde{\tau}) = \frac{\epsilon_1 + \psi_\infty - \frac{\pi}{2}}{\epsilon_0 - \delta}.
\]

Then we are going to prove the existence of global solution by extending the local solution in this Section. For this purpose, we solve several local Goursat problems in each extension step. Let \( \bar{XZ} \) and \( \bar{XY} \) be positive(\( C^+ \)) and negative(\( C^- \)) characteristics in \( \Omega_\tau \), respectively, where \( Y' = (\xi_Y, \eta_Y) \) and \( Z' = (\xi_Z, \eta_Z) \) are two points lying on the level curve \( \tau = \tilde{\tau} \). Then, we prescribe the boundary data

\[
(u, v, \tau) = \begin{cases} \{ u|_{\bar{XZ}}, v|_{\bar{XZ}}, \tau|_{\bar{XZ}} \}, & \text{on} \quad \bar{XZ} \\ \{ u|_{\bar{XY}}, v|_{\bar{XY}}, \tau|_{\bar{XY}} \}, & \text{on} \quad \bar{XY} \end{cases}
\]

(5.2)

where \( (u|_{\bar{XY}}, v|_{\bar{XY}}, \tau|_{\bar{XY}}) \) and \( (u|_{\bar{XZ}}, v|_{\bar{XZ}}, \tau|_{\bar{XZ}}) \) are the values of \( (u, v, \tau) \) on \( \bar{XY} \) and \( \bar{XZ} \), respectively. Then we have the following Lemma.
**Lemma 5.1.** If \( \eta_Z - \eta_Y < \frac{-\sin(2\tilde{\delta}(\tau_0)) + \chi(\tilde{\tau}) - \epsilon_1}{2\tilde{\tau}M_1(\tilde{\tau})} \), then the Goursat problem (2.1), (5.2) admits a global \( C^1 \) solution in a domain bounded by \( XY, XZ, TY \) and \( TZ \) where \( TY \) is the \( C_+ \) characteristic passing through \( Y \) while \( TZ \) is the \( C_- \) characteristic passing through \( Z \). Moreover, this solution satisfies \( \tilde{\tau} > \frac{-\epsilon_1}{\epsilon_1} \).

**Proof.** Let \( K \) be any point in the quadrilateral domain bounded by \( XY, XZ, TY \) and \( TZ \) such that the \( C_+ \) characteristics passing through \( K \) intersects with \( XY \) at \( K_+ \) while \( C_- \) characteristics passing through \( K \) intersects with \( XZ \) at \( K_- \). Then using Lemma 4.2 we have along \( K K_- \)

\[
\frac{1}{\tau(K)} = \frac{1}{\tau(K_-)} + \int_{K_+} \partial_\tau \rho
\]

\[
> \frac{1}{\tau(K_-)} + M_1(\tau)(\eta_Y - \eta_Z)
\]

\[
> \frac{1}{2\tilde{\tau}}
\]

Hence, using (2.13) we obtain a \( C^1 \) norm estimate of the solution to the Goursat problem (2.1) and (3.10), so by the theory of global classical solution for quasilinear hyperbolic equations [13], the Lemma can be proved. 

**Theorem 5.1.** (Global solution and regularity of vacuum boundary) If \( 2\tilde{\delta}(\tau_0) + \chi(\tau) < \alpha_0 + \frac{\pi}{2} < 4\tilde{\delta}(\tau_0) \) then the Goursat problem (2.1) with the boundary data (3.10) admits a unique global \( C^1 \) solution in the region \( PQR \) where the curve \( QR \) is an interface between gas and vacuum connecting \( Q \) and \( R \), which is Lipschitz continuous.

**Proof.** Let \( Y_0 = Q', Y_1, Y_2, Y_3, \ldots, Y_n = R' \) be \( n + 1 \) different points on the level curve \( \tau = \tilde{\tau} \) such that

\[ 0 < \eta_{Y_{i+1}} - \eta_{Y_i} < \frac{-\sin(2\tilde{\delta}(\tau_0)) + \chi(\tilde{\tau}) - \epsilon_1}{2\tilde{\tau}M(\tilde{\tau})} \]

Then we draw a \( C_+ \) characteristic curve from the point \( Y_i \) which cuts the \( C_- \) characteristic curve passing through \( Y_{i+1} \) at a point \( X_i \) for \( i = 0, 1, 2, \ldots, n - 1 \). Therefore, noting the fact that \( \partial_\tau \tau > 0 \) it is easy to see that the level curve \( \tau = \tilde{\tau} \) is a non-characteristic curve which means that \( X_i \neq Y_i \) and \( X_i \neq Y_{i+1} \) for any \( i = 0, 1, 2, \ldots, n-1 \).

From the point \( X_i \), we build a small Goursat problem below the level curve \( \tau = \tilde{\tau} \) where \( \overline{XY_i} \) and \( \overline{X_iY_{i+1}} \) are the characteristic boundaries in the domain \( \Omega_\tau \). Therefore, we can use Lemma 5.1 to conclude that the Goursat problem (2.1), (5.2) with the characteristic boundaries \( \overline{XY_i} \) and \( \overline{X_iY_{i+1}} \) admits a \( C^1 \) solution in the quadrilateral domain bounded by \( X_iY_i, X_iY_{i+1}, Y_iZ_{i+1} \) and \( Y_{i+1}Z_{i+1} \) where \( Y_iZ_{i+1} \) is the \( C_- \) characteristic curve passing through \( Y_i \) and \( Y_{i+1}Z_{i+1} \) is the \( C_+ \) characteristic curve passing through \( Y_{i+1} \).
Assume that \( Z_0 = Q \) and \( Z_{n+1} = R \), then for each \( i = 0, 1, 2, \ldots, n \), there exists a \( \tau_i \in (\tau_0, \infty) \), such that the Goursat problem for (2.1) admits a unique \( C^1 \) solution in the domain closed by \( Y_iZ_i, Y_iZ_{i+1} \) and the level curve \( \tau(\xi, \eta) = \tau_i \) with \( Y_iZ_i \) and \( Y_iZ_{i+1} \) as the characteristic boundaries. If we denote \( \tau_e = \min\{\tau_0, \tau_1, \tau_2, \ldots, \tau_n, \tau(Z_1), \tau(Z_2), \ldots, \tau(Z_n)\} \) then utilizing the fact \( \partial_+ \tau > 0 \) we see that \( \tau_e > \tilde{\tau} \). Then, the solution is extended from \( \Omega_{\tilde{\tau}} \) to \( \Omega_{\tau_e} \). Repeating the above process, one can construct the global solution of Goursat problem in the whole domain \( PQR \).

Further, from (5.1) we see that family of level curves of specific volume are Lipschitz continuous, hence we can use Arzela-Ascoli’s theorem to conclude that the vacuum boundary \( QR \) is also Lipschitz continuous.

6. Applications

In this section, we first formulate two-dimensional modified shallow water equations which can be considered as a particular case of two-dimensional Euler equations with a special equation of state. Then we obtain a global solution to a dam-break type problem for the two-dimensional modified shallow water equations as one of the application of our work. Further, we consider some special equation of states and solve the gas expansion problem through a sharp corner for them while recovering some of the results from available literature.

6.1. Two-constant equation of state

Let us consider an equation of state of the form (9)

\[
p(\tau) = A_1 \tau^{\gamma_1} + B_1 \tau^{\gamma_2}
\]

where \( \gamma_1, \gamma_2, A_1 \) and \( B_1 \) are constants. Such kind of equation of states are relevant in many physical models, for example it can be taken as sum of the fluid and magnetic pressure in the ideal magnetogasdynamics where the magnetic field is orthogonal to the velocity vector (26). Also, it can be used to express two-dimensional modified shallow water equations (25) where gas expansion problem can be expressed as a dam-break type problem. One other example of such equation of state is extended Chaplygin gas which has been utilized in the recent past by many mathematicians and physicists as a candidate to explain the accelerated expansion of the universe (24). We consider the first two cases one by one in this subsection and solve the corresponding 2-D Riemann problem for them.

6.1.1. Modified shallow water equations in two-dimensions

As an application of gas expansion problem, we consider a dam-break type problem for two-dimensional modified shallow water equations and construct a global solution for dam-break type problem using the construction technique of this article. We first formulate two-dimensional modified shallow water equations using the ideas from (10). We consider isentropic incompressible Euler equations in three-dimensions for a free surface fluid of the form

\[
\begin{cases}
\nabla \cdot \vec{u} = 0, \\
\vec{u}_t + \vec{u} \cdot \nabla \vec{u} + \frac{1}{\rho} \nabla p + \vec{g} = 0,
\end{cases}
\]

where \( \vec{u} = (u, v, w) \) is the velocity of the fluid, \( p \) is the pressure and \( \vec{g} = (0, 0, g) \) is the gravitational acceleration. We apply the no penetration boundary condition at the lower boundary while kinematic boundary condition at the free surface so that

\[
\begin{cases}
w|_{z=0} = 0, \\
w|_{z=h} = \frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y}.
\end{cases}
\]

Further, we assume hydrostatic pressure field of the form

\[
p = p_0 + gh(h - z),
\]
where \( h(x, y) \) is the fluid depth and \( p_0 \) is the atmospheric pressure. Therefore, \( z \)-momentum equation from (6.2) implies that
\[
\frac{Dw}{Dt} = \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} = 0, \tag{6.5}
\]
which reduces the governing system as
\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= 0, \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial x} &= 0, \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial y} &= 0. \tag{6.6}
\end{align*}
\]
Finally we can integrate equations in (6.6) with the boundary conditions (6.3) to obtain
\[
\begin{align*}
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} \int_0^h udz + \frac{\partial}{\partial y} \int_0^h vdz &= 0, \\
\frac{\partial}{\partial t} \int_0^h u^* dz + \frac{\partial}{\partial y} \int_0^h v^* dz + \frac{\partial}{\partial y} \int_0^h (h \bar{u} \bar{v})_x + \frac{\partial}{\partial x} \int_0^h (h \bar{v} \bar{u})_y + R^1_x + R^2_y &= 0, \\
\frac{\partial}{\partial t} \int_0^h v^* dz + \frac{\partial}{\partial x} \int_0^h u^* dz + \frac{\partial}{\partial y} \int_0^h (h \bar{v} \bar{u})_y + \frac{\partial}{\partial x} \int_0^h (h \bar{u} \bar{v})_y + R^1_y + R^2_x &= 0. \tag{6.7}
\end{align*}
\]
Let us represent the horizontal velocities \( u \) and \( v \) as a sum of depth averaged velocity and its variation from mean velocity such that
\[
u = \bar{u} + u^*, \quad v = \bar{v} + v^* \tag{6.8}
u
\]
where \( \bar{u} = \frac{1}{h} \int_0^h udz \) and \( \int_0^h u^* dz = 0 \) while \( \bar{v} = \frac{1}{h} \int_0^h vdz \) and \( \int_0^h v^* dz = 0 \).

Then we can simplify the system (6.7) to obtain
\[
\begin{align*}
h_t + \bar{u}_x + (h \bar{v})_y &= 0, \\
(h \bar{u})_t + (h \bar{u}^2 + \frac{gh^2}{2})_x + (h \bar{v} \bar{v})_y + R^1_x + R^2_y &= 0, \\
(h \bar{v})_t + (h \bar{v} \bar{u})_x + (h \bar{v}^2 + \frac{gh^2}{2})_y + R^1_y + R^2_x &= 0. \tag{6.9}
\end{align*}
\]
where \( R^1 = \int_0^h u^* dz, \quad R^2 = \int_0^h u^* v^* dz, \quad R^1 = \int_0^h v^* dz \) and \( R^2 = \int_0^h \frac{u^* v^*}{dz} \) are the nonlinear terms describing the effect of advective transport of the impulse caused by velocity difference. For simplicity, we consider the case where \( u^* \approx v^* \approx 0 \) as in [10] and assume that the effect of the nonlinear term \( u^* v^* \) is negligible so that the modified shallow water equations in two-dimensions can be obtained of the following form
\[
\begin{align*}
h_t + (h \bar{u})_x + (h \bar{v})_y &= 0, \\
(h \bar{u})_t + (h \bar{u}^2 + \frac{gh^2}{2})_x + (h \bar{v} \bar{v})_y &= 0, \\
(h \bar{v})_t + (h \bar{v} \bar{u})_x + (h \bar{v}^2 + \frac{gh^2}{2})_y &= 0. \tag{6.10}
\end{align*}
\]
where \( k = kh_0/g \) is a reduced factor characterizing advective transport of the impulse.

It is worth noting that the above modified shallow water equations are comparable with Euler equations in two-dimensions with an equation of state of the form \( p(\tau) = \frac{A_1}{\tau} + \frac{B_1}{\tau^2} \) where \( A_1 = k, \ B_1 = g/2 \) and \( \tau = 1/h \).
Hence we directly calculate

\[
\begin{align*}
p(\tau) &= A_1\tau^{-1} + B_1\tau^{-2}, \\
p'(\tau) &= -A_1\tau^{-2} - 2B_1\tau^{-3} < 0, \\
p''(\tau) &= 2A_1\tau^{-3} + 6B_1\tau^{-4} > 0, \\
m(\tau) &= \frac{A_1\tau^{-2} + B_1\tau^{-3}}{A_1\tau^{-2} + 3B_1\tau^{-3}} > 0, \\
m'(\tau) &= \frac{2B_1A_1\tau^{-6}}{[A_1\tau^{-2} + 3B_1\tau^{-3}]^2} > 0.
\end{align*}
\]  

(6.11)

Hence, we can use the Lemma [4.1], Proposition [4.1] and Theorem [5.1] to obtain the following result for modified shallow water equations in two-dimensions.

**Theorem 6.1.** If \(2\tilde{\delta}(\tau_0) < \alpha_0 + \pi/2 < 4\tilde{\delta}(\tau_0)\) then the modified shallow water system \([6.10]\) with initial data \([1.2]\) admits a global classical \(C^1\) solution.

### 6.1.2. Isentropic magnetogasdynamics system in two-dimensions

Let us consider the following system of isentropic magnetogasdynamics [3]

\[
\begin{align*}
\rho_t + \nabla.(\rho\vec{u}) &= 0, \\
(\rho\vec{u})_t + \nabla.(\rho\vec{u} \otimes \vec{u} + p\mathbf{I}) - \mu(\nabla \times \vec{H}) \times \vec{H} &= 0, \\
\vec{H}_t - \nabla \times (\vec{u} \times \vec{H}) &= 0, \\
\nabla . \vec{H} &= 0,
\end{align*}
\]

(6.12)

where \(\rho\) is the fluid density, \(p = A_1\rho^\gamma\) is the fluid pressure \((A_1 > 0)\), \(\vec{u}\) is the fluid velocity and \(\vec{H}\) is magnetic field vector.

If we assume that the velocity vector and magnetic field vector are perpendicular to each other or in other words if \(\vec{u} = (u, v, 0)\) and \(\vec{H} = (0, 0, H)\) then we can reduce the system \((6.12)\) as follows (see [5] for a complete derivation)

\[
\begin{align*}
\rho_t + (\rho u)_x + (\rho v)_y &= 0, \\
(\rho u)_t + (\rho u^2 + p + \frac{\mu}{2}H^2)_x + (\rho uv)_y &= 0, \\
(\rho v)_t + (\rho uv)_x + (\rho v^2 + p + \frac{\mu}{2}H^2)_y &= 0, \\
H_t + (Hu)_x + (Hv)_y &= 0.
\end{align*}
\]

(6.13)

From first and last equation of system \((6.13)\) it is easy to see that \(H = \kappa_0\rho\) along streamlines of the flow which is usually referred as frozen law in the literature; see viz. [26]. Using this substitution in the governing system \((6.13)\) we can obtain a modified pressure of the form \(p(\tau) = A_1\tau^{-\gamma} + B_1\tau^{-2}\), where \(B_1 = \frac{\mu\kappa_0^2}{2}\). Then we have

\[
\begin{align*}
p(\tau) &= A_1\tau^{-\gamma} + B_1\tau^{-2}, \\
p'(\tau) &= -A_1\gamma\tau^{-\gamma-1} - 2B_1\tau^{-3} < 0, \\
p''(\tau) &= A_1\gamma(\gamma + 1)\tau^{-\gamma-2} + 6B_1\tau^{-4} > 0, \\
m(\tau) &= \frac{A_1\gamma(\gamma + 1)\tau^{-\gamma-1} + 2B_1\tau^{-3}}{A_1\gamma(\gamma + 1)\tau^{-\gamma-1} + 3B_1\tau^{-3}} > 0, \\
m'(\tau) &= \frac{8B_1A_1\gamma(\gamma - 2)^2\tau^{-\gamma-5}}{[A_1\gamma(\gamma + 1)\tau^{-\gamma-1} + 3B_1\tau^{-3}]^2} > 0.
\end{align*}
\]

(6.14)

For this system we again use Lemma [4.1], Proposition [4.1] and Theorem [5.1] to recover the following result from [6].

**Theorem 6.2.** If \(2\tilde{\delta}(\tau_0) < \alpha_0 + \pi/2 < 4\tilde{\delta}(\tau_0)\) then the initial boundary value problem \((6.13)\) and \((1.2)\) admits a global classical \(C^1\) solution.

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6.2. Van der Waals gas

Let us consider a van der Waals type equation of state of the form [13]

\[ p(\tau) = \frac{S_1}{(\tau - 1)^{1+1} - \frac{1}{\tau^2}}, \]

(6.15)

where \( S_1 \) is a positive constant in correspondence with the gas entropy and \( \gamma \) is the gas constant lying between 0 and 1. It is easy to see that for \( \tau > \tau_0 \), we have \( p'(\tau) < 0 \) and \( p''(\tau) > 0 \) for sufficiently large \( \tau > 4 \) and for any given \( S_1 \in (1/4, 81/256) \), when \( \gamma \) is sufficiently close to 0 we have that \( m(\tau) > 0 \) and \( m'(\tau) < 0 \); see [15] for more details. Therefore, we can use the second part of the Lemma [4.1] and Proposition [4.1] and Theorem [5.1] to obtain the following result.

**Theorem 6.3.** If \( 2\tilde{\delta}(\tau_0) + \chi(\tau) < \alpha_0 + \pi/2 < 4\tilde{\delta}(\tau_0) \) then the initial boundary value problem (1.1) and (1.2) admits a global classical \( C^1 \) solution for van der Waals gas.

7. Conclusions and future scope

Here we established the existence of a global solution to a 2-D Riemann problem for compressible Euler equations with a general convex equation of state. It is a well-known fact that the Euler equations in self-similar plane is a mixed type system so to maintain the hyperbolicity of Euler equations, we constructed the invariant regions of the characteristic variables using the characteristic decomposition method and hence obtained global solution to a gas expansion problem by extending the local solution into the whole interaction domain. Further, we formulated two-dimensional modified shallow water equations and obtained a global solution for a dambreak type problem as one of the application of this work. Also, we recovered some of the existence results to gas expansion problem for certain equation of states from available literature. It was worth noting that we considered convex equation of state and a special assumption on wall angle in this work but in future we would like to relax the restriction on equation of state and solve this problem for any arbitrary equation of state having more than one inflection point. Moreover, we also wish to discuss different wall angles which may include interactions of composite waves and shocks also.

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