GAMMA: A Mathematica package for performing $\Gamma$-matrix algebra and Fierz transformations in arbitrary dimensions

Ulf Gran

Department of Theoretical Physics
Chalmers University of Technology and Göteborg University
SE-412 96 Göteborg, Sweden

Abstract
We have developed a Mathematica package capable of performing $\Gamma$-matrix algebra in arbitrary (integer) dimensions. As an application we can compute Fierz transformations.
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This documentation is part of the GAMMA package, available for download at http://fy.chalmers.se/~gran/, developed for Mathematica version 4.0.

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If you find the GAMMA package useful for doing research, acknowledging the author’s work by proper reference to the package would be appreciated.
1 Introduction

The motivation to develop this package arose from calculations in eleven-dimensional supergravity [1, 2], which required an immense amount of both pure $\Gamma$-matrix algebra and Fierz transformations to be performed. The rule-based programming of Mathematica [3] is ideally suited for handling algebraic computations, compared to ordinary procedural programming. For a very nice comparison between rule-based and procedural programming, as well as a toy implementation of $\Gamma$-matrix algebra in four dimensions, see [4], which describes a Mathematica package performing $\Gamma$-matrix algebra in $(4-\epsilon)$ dimensions. This was the first Mathematica implementation of $\Gamma$-matrix algebra according to the ’t Hooft-Veltman scheme. There are quite a few programs handling $\Gamma$-matrices in four-dimensions [3, 4, 5, 7, 8, 9, 10, 11], intended to be used when computing Feynman diagrams. There is also a Mathematica package handling computations in $D = 10$, $N = 1$ supergravity, including $\Gamma$-matrix algebra, [12].

In section 2, we will introduce and explain the functions in the package concerning the algebraic manipulation of $\Gamma$-matrices. This will be done through simple examples. In section 3 we will turn to the functions concerning fierzing. A complete list of the functions in the package, including brief descriptions, can be found in appendix A. In appendix B we list the built-in Mathematica functions that are redefined by the package. GAMMA is available for download at http://fy.chalmers.se/~gran/.

2 $\Gamma$-matrix algebra

To load the package type

\begin{verbatim}
In[1]:=<<GAMMA.m
\end{verbatim}

and remember to include the path if the package is put in a directory where Mathematica does not automatically search. By default, the space-time dimension is set to 11 and the spinor dimension is set to 32. If you want to use any other values they must be specified before any calculations can be done. This is done through, e.g.,

\begin{verbatim}
In[1]:=SetDim[10]; SetSpinorDim[16];
\end{verbatim}

The spinor dimension is used only when calculating the trace of $\Gamma$-matrices. We now introduce the various objects that can be handled. Kronecker deltas, $\Gamma$-matrices, tensors and tensor-spinors are entered like

\begin{verbatim}
In[1]:=Delta[{a,b},{c,d}]
Out[1]:=\delta_{cd}^{ab}
\end{verbatim}
\textbf{In[2]}:=\text{GammaProd}[\{a,b\},\{c\}]
\text{Out}[2]=\Gamma^{ab} \cdot \Gamma^c

\text{In[3]}:=\text{Tensor}[X,\{a,b,c\}]
\text{Out}[3]=X_{abc}

\text{In[4]}:=\text{TensorSpinor}[Y,\{a,b,c\}]
\text{Out}[4]=\tilde{Y}_{abc}

The indices in each list in these expressions are assumed to be antisymmetric and will be put in a canonical order (defined by Mathematica’s sort algorithm, which therefore means alphabetical order), sometimes resulting in a sign in front of the canonically ordered expression. In order to discern tensors from tensor-spinors (since we do not use explicit spinor indices) we have added a tilde over the symbol designating the tensor-spinor. This indicator can be changed by, \textit{e.g.,}

\textbf{In[1]}:=\text{SetTensorSpinorInd}[\text{OverHat}]

in which case the Mathematica function \text{OverHat} is applied to the symbol designating the tensor-spinor. See the Mathematica documentation for a complete list of possible choices. It is also possible to change the formatting of the symbols designating a tensor, where the default function is

\textbf{In[1]}:=\text{SetTensorInd}[\text{Identity}]

Since all calculations are assumed to be performed in a Lorentz frame, we do not have to care about whether an index is upstairs or downstairs and contracted indices may both be upstairs or downstairs. Enforcing explicit antisymmetry in some indices in an expression like $\Gamma_{[a}X_{b]}$ is done by

\textbf{In[1]}:=\text{ASym}[\text{GammaProd}[\{a\}] \text{ Tensor}[X,\{b\}],\{a,b\}]
\text{Out}[1]=\frac{1}{2}(X_b \Gamma_a - X_a \Gamma_b)

Using the function \text{Sym[]} instead enforces symmetry in the specified indices.

We now turn to the functions concerning the manipulation of $\Gamma$-matrices. We have already seen how to write a product of $\Gamma$-matrices, if we want to expand a general product we write

\textbf{In[1]}:=\text{GammaExpand}[\text{GammaProd}[\{a,b\},\{c\},\{d\}]]
\text{Out}[1]=\delta_c^a \Gamma_a \cdot \Gamma_d - \delta_c^b \Gamma_b \cdot \Gamma_d + \Gamma_{abc} \cdot \Gamma_d
where the product of the first two $\Gamma$-matrices have been expanded. Since only the product of two $\Gamma$-matrices are expanded each time \texttt{GammaExpand[]} is applied, repeated application is sometimes needed. In the expansion we only use the algebra $\{\Gamma^a, \Gamma^b\} = 2\delta^a_b$ and therefore, as long as the Kronecker deltas have an equal number of upstairs and downstairs indices, there will be no difference between various signatures. In a specific expression, if we want a Kronecker delta to have, \textit{e.g.}, only upstairs (or downstairs) indices we get a trivial difference since using Minkowski signature the Kronecker delta becomes an $\eta$ while it stays a delta in the euclidean case. Therefore, with the understanding that a \texttt{DeltaProd[]} has an equal number of upstairs and downstairs indices, the expressions will look the same for all signatures.

Since we know that the expression we started with in this example above is antisymmetric in $a$ and $b$ we can simplify the result by putting these indices in canonical order by

$$\text{In[2]}:=\text{ACanonicalOrder}[%\{a,b\}]$$

$$\text{Out[2]}=-2\delta_a^c\Gamma_b^{\bullet \bullet}\Gamma_d + \Gamma_{abc}^{\bullet \bullet}\Gamma_d$$

remembering that that the expression is antisymmetric in $a$ and $b$. Even though the simplification in this simple case is not that significant the importance to use this function, when applicable, in more complicated cases cannot be underestimated. The function \texttt{SCanonicalOrder[]} puts indices in canonical order assuming symmetry instead. When indices are contracted between adjacent $\Gamma$-matrices it is more efficient to use

$$\text{In[1]}:=\text{GammaContract}[\text{GammaProd}[\{a,b,c\},\{b,c\}]]$$

$$\text{Out[1]}=-90\Gamma_a$$

which is specialized for this situation, than to use \texttt{GammaExpand[]}. To compute traces of expressions involving $\Gamma$-matrices we write

$$\text{In[1]}:=\text{GammaTrace}[\text{GammaProd}[\{a,c\},\{b,c\}]]$$

$$\text{Out[1]}=-320\delta_a^b$$

When using tensor-spinors, which are assumed to be irreducible and therefore $\Gamma$-traceless, it is sometimes needed to rearrange the $\Gamma$-matrices in order to be able to use the $\Gamma$-tracelessness to simplify an expression. We see in this example

\footnote{When tracing a $\Gamma$-matrix with the same number of indices as the spacetime dimension, when this number is odd, \texttt{GammaProd[]} is kept and the user can himself replace it with a Levi-Civita tensor density according to his conventions (note that \texttt{GammaProd[]} is \textit{not} multiplied by the spinor dimension).}

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that we have to use \texttt{GammaExtract[]} to be able to simplify the expression. What \texttt{GammaExtract[]} does is to extract one $\Gamma$-matrix at a time, having an index contracted to the tensor-spinor, from, in this case, the $\Gamma_{abc}$ term, in this way enabling the use of $\Gamma$-tracelessness, \textit{e.g.}, $\Gamma_c \tilde{\gamma}_c = 0$. Note that it is very important to use $\star\star$, \textit{i.e.}, \texttt{NonCommutativeMultiply[]}, in this example since there are more than one object carrying spinor indices. After some manipulations, one often ends up with expressions where contracted, \textit{i.e.}, dummy, indices are denoted by different symbols in otherwise identical expressions like in

\begin{verbatim}
In[1]:=GammaProd[{a,b,c}]\star\star TensorSpinor[Y,{c}]
Out[1]=\Gamma_{abc}\star\star \tilde{\gamma}_c

In[2]:=GammaExtract[%,{a,b}]
Out[2]=-2\Gamma_a\star\star \tilde{\gamma}_b
\end{verbatim}

In this case Mathematica can not tell that these terms are identical and we have to rename the dummy indices using some canonical set of indices. The is done by

\begin{verbatim}
In[2]:=RenameDummy[%]
Out[2]=2\Gamma_{d1}\tilde{\gamma}_{d1}
\end{verbatim}

The set of indices $\{d_1,d_2,\ldots\}$ are used to denote dummy indices and are therefore preferably avoided. It is possible to cut, paste and copy formatted expressions containing $\Gamma$-matrices and also to enter expressions as they look in the output form. Converting expressions to \texttt{\LaTeX} format is done by \texttt{TeXForm[expr]}.

\section{Fierz transformations}

We can also use \texttt{GAMMA} to perform Fierz transformations. This will be illustrated using a simple example in which the free vector indices of the $\Gamma$-matrices are contracted to a tensor. Start by making the definitions

\begin{verbatim}
structures={x1,x2,y1,y2,y3,z1,z2,z3};
ind=HoldForm[\frac{1}{4}\text{tr}[P\star\star X\star\star P\star\star X\star\star Q] + \frac{2}{3}\text{tr}[P\star\star X\star\star Q]};
x1={{a},{b},ind,{{},{}}};
x2={{c1},{c1,a,b},ind,{{},{}}};
\end{verbatim}
The list structures contains all the different terms we want to relate by fierzing. A term \( w_1 = \{\{a\}, \{b\}\}, \text{ind}, \{\{c\}, \{d\}\}\) corresponds to
\[
(\Gamma^a)_{\alpha\beta} (\Gamma^b)_{\gamma\delta} (\Gamma^c)_P (\Gamma^d)_Q.
\] (1)

Note that the terms in this example have two free vector indices, \( a \) and \( b \). To perform the Fierz transformation we contract two spinor indices of each term with a basis of \( \Gamma \)-matrices, \( \Gamma^{(1)} \), \( \Gamma^{(2)} \) and \( \Gamma^{(5)} \) for symmetric spinor indices and \( C \), \( \Gamma^{(3)} \) and \( \Gamma^{(4)} \) for antisymmetric spinor indices. In our case we are interested in three symmetric indices, \( (\alpha\beta\gamma)\delta \). Since we do use explicit spinor indices on the \( \Gamma \)-matrices we use the expression \( \text{ind} \) to specify the symmetry of the four spinor indices. In \( \text{ind} \), \( P \) will be replaced by the first \( \Gamma \)-matrix in a term and \( Q \) by the second. \( X \) will be replaced by the basis matrices and \( \text{tr} \) by the function \( \text{GammaTrace} \). It is then easy to see that \( \text{ind} \) corresponds to the symmetry specified above\(^3\). We are now ready to perform the Fierz transformation.

\[
\text{In[1]:=FierzSolve[Fierz[y1,y2,y3,structures,{a,b},S,MakeTensor]]}
\]
\[
\text{Out[1]=}
\begin{pmatrix}
-\frac{4}{5} & \frac{3}{5} & -\frac{1}{5} \\
\frac{9}{5} & -\frac{8}{5} & \frac{1}{5} \\
\frac{24}{5} & -\frac{48}{5} & \frac{11}{5} \\
-\frac{36}{5} & \frac{2}{5} & \frac{1}{5} \\
18 & -8 & 3
\end{pmatrix}
\]

where the first argument to \text{Fierz} is the terms that are chosen as the basis in which the other terms will be expressed. It is important to know exactly how many linearly independent terms there are and to get this information we recommend the lie algebra program \text{Lie} \[^{13}\]. The second argument is the list of terms, the third is a list of the free vector indices, the fourth is \( S \) since we are fierzing symmetric indices.

\(^3\)Contract a basis \( \Gamma \)-matrix, \( \Gamma^X \), with two of the symmetrized indices, \( (\Gamma^X)^{\alpha\beta}(\Gamma^P)^{\gamma\delta}_{\alpha\beta} = \frac{1}{4} \text{tr}(\Gamma^X \Gamma^P \Gamma^Q)_{\gamma\delta} + \frac{1}{2} (\Gamma^P \Gamma^X \Gamma^Q)_{\gamma\delta}, \) where the RHS is the expression to be used as \( \text{ind} \).
in this example (use A for antisymmetric indices). The last argument \texttt{MakeTensor} is used to indicate that the symmetry of the free indices is that of a tensor. In the output, each row contains the coefficients for a term in the chosen basis. In this case, e.g., \( z_2 = -\frac{36}{5} y_1 + \frac{2}{5} y_2 + \frac{1}{5} y_3 \).

When we Fierz tensor-spinors, care has to be taken in order to be able to use the \( \Gamma \)-tracelessness. If, e.g., the index \( b \) in the previous example is contracted to a vector-spinor having the spinor index \( \gamma \) we would have to use

\[
\text{ind} = \text{HoldForm}[\frac{1}{3} \text{tr}[P^{\cdots} X^{\cdots} Q^{\cdots}] \text{TensorSpinor}[Y, \{b\}] + \frac{2}{3} P^{\cdots} X^{\cdots} Q^{\cdots} \text{TensorSpinor}[Y, \{b\}]]
\]

The tensor-spinor is automatically removed after the simplification is complete when fierzing.

Fierz transformations like the one above where the free vector indices are contracted to an antisymmetric tensor are relatively simple to perform, it is harder to Fierz hook-tensors\(^4\). Even though GAMMA may not be able to directly calculate the Fierz transformation as shown above, it can still be of great use. We can try to use \texttt{Fierz[]} but if that fails we can use \texttt{FierzTerm[]} instead, which contracts the basis of \( \Gamma \)-matrices to a specific term, or list of terms, simplifies and then returns the result. The problem with hook-tensors is that a lot of Kronecker delta functions are generated and it is not easy to rewrite them on a canonical form allowing Mathematica to determine which terms are linearly independent.

The way to specify a hook-tensor term is a bit different from the tensor case. Let us look at a term arising when we want to Fierz a \((11000)\) representation, \((\Gamma_{c_1 c_2})_{\alpha \beta} (\Gamma^f_a)_{\gamma \delta} Z_{c_1 c_2, a} \), \(^2\) where we have one free vector index. This term is represented by

\[
f_{1} = \{\}, \{f\}, \{c_1, c_2\}, \{a\}, 2, \text{ind}, \{\}, \{\} \};
\]

Here, the first entry corresponds to the vector indices on the first \( \Gamma \)-matrix, not including indices contracted to the hook-tensor, and the second entry is defined in the same way but for the second \( \Gamma \)-matrix. The third entry is the antisymmetric hook-tensor indices, except those that sit on a Kronecker delta function (after removing the hook-tensor\(^5\)), and the fourth entry is the symmetric index on the hook-tensor. The fifth entry is the number of vector indices on the first \( \Gamma \)-matrix, \text{ind} again specifies the structure of the spinor indices and the last entry is the potential Kronecker delta function indices. When we remove the hook-tensor in \(^2\) we must enforce the hook-tensor symmetries on the \( \Gamma \)-matrix terms, this is done by the function

\footnote{A hook-tensor is denoted \( Z_{c_1 \cdots c_p, a} \) satisfying \( Z_{[c_1 \cdots c_n, a]} = 0 \) and \( Z_{c_1 \cdots c_{n-1} c, a} = 0 \).}

\footnote{For a term as in \(^2\) we have to introduce a Kronecker delta function when removing the hook-tensor if the free index \( f \) would sit on the hook-tensor instead of on a \( \Gamma \)-matrix.}
MakeHook[term,gmatrix], which at the same time contracts two spinor indices to the spinor indices on the Γ-matrix gmatrix as specified by ind. By writing

\[
\text{In}[1]:=\text{FierzTerm[}\{f1\},\{f\},S,\text{MakeHook}]\]

we generate the expressions obtained by contracting the symmetric Γ-matrix basis, in this case, to the hook structure specified by f1, treating the indices contained in the list of the second argument as antisymmetric. These expressions can then be studied “by hand” and in this way we can also simplify hook fierzes.

4 Summary

We have presented a Mathematica package capable of performing Γ-matrix algebra in arbitrary (integer) dimensions. The various functions have been introduced through examples and a complete list of all functions in the package, including brief descriptions, can be found in appendix \[A\]. As a specific application, we have considered Fierz transformations, which in general require a lot of Γ-matrix calculations. Note that some of the built-in functions in Mathematica have been altered, see appendix \[B\] for a complete list.

Comments, suggestions for improvements and reports regarding any discovered bugs are appreciated. When reporting bugs, please include a Mathematica notebook illustrating the bug in question. Reporting bugs in this way will result in the quickest possible bug-fixes. Please note that only the most basic error handling is implemented. Improved error handling invariably results in a slower program and in the choice between error handling or speed, we chose the latter. This unfortunately means that what might at first sight look like a bug quite often can be attributed to some error in the input.

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A List of commands

Here is a list of all the user-available functions in the GAMMA package together with brief descriptions.

\text{ACanonicalOrder[expr,list]} assumes that expr is antisymmetric in the indices given by list and puts them in canonical order, thus enabling additional simplification.

\text{ASym[expr,list]} explicitly antisymmetrizes expr in the indices contained in list.
\[
\text{Delta[\text{list1, list2}]} \text{ represents a Kronecker delta with the two groups of indices given by list1 and list2.}
\]

\[
\text{Fierz[\text{basislist, structures, asymlist, sym, function}]} \text{ performs the Fierz transformation of the terms specified in structures using the terms in the list basislist as basis elements. Antisymmetry is assumed in the indices contained in the list asymlist and sym (A or S) gives the symmetry of the spinor indices to be fierzed. Finally, function is applied to generate explicit } \Gamma \text{-matrix expressions given a term in structures.}
\]

\[
\text{FierzSolve[eqlist]} \text{ solves the list of equations generated by Fierz[].}
\]

\[
\text{FierzTerm[\text{termlist, asymlist, sym, function}]} \text{ as Fierz[] but does not perform the complete Fierz transformation. Instead it produces a list of expressions obtained by contracting two of the spinor indices, as specified by ind, with the basis of } \Gamma \text{-matrices specified by sym.}
\]

\[
\text{GammaContract[expr]} \text{ simplifies the expression expr containing contracted indices on adjacent } \Gamma \text{-matrices.}
\]

\[
\text{GammaExpand[expr]} \text{ expands the product of the first two } \Gamma \text{-matrices in expr.}
\]

\[
\text{GammaExtract[expr, list]} \text{ extracts } \Gamma \text{-matrices that contract indices on a tensor-spinor, assuming antisymmetry in the indices contained in list, in order to be able to invoke the } \Gamma \text{-tracelessness of the tensor-spinor.}
\]

\[
\text{GammaProd[\text{list1}, \ldots, \text{listn}]} \text{ represents the product of n } \Gamma \text{-matrices with indices given by list1 to listn.}
\]

\[
\text{GammaTrace[expr, list]} \text{ computes the } \Gamma \text{-trace of an expression expr containing } \Gamma \text{-matrices, enforcing antisymmetry in the indices contained in list.}
\]

\[
\text{MakeHook[term, basiselement]} \text{ makes an explicit } \Gamma \text{-matrix expression from term contracting the } \Gamma \text{-matrix basiselement with two of the spinor indices, as specified by ind.}
\]

\[
\text{MakeTensor[term, basiselement]} \text{ makes an explicit } \Gamma \text{-matrix expression from term contracting the } \Gamma \text{-matrix basiselement with two of the spinor indices, as specified by ind.}
\]

\[
\text{RenameDummy[expr]} \text{ renames dummy (contracted) indices in expr, thus enabling additional simplification.}
\]

\[
\text{SCanonicalOrder[expr, list]} \text{ assumes that expr is symmetric in the indices given by list and puts them in canonical order, thus enabling additional simplification.}
\]

\[
\text{SetDim[dim]} \text{ sets the space-time dimension to dim.}
\]

\[
\text{SetTensorInd[function]} \text{ sets the Mathematica function to be applied to the symbol designating a tensor to function.}
\]
SetTensorSpinorInd[function] sets the Mathematica function to be applied to the symbol designating a tensor-spinor, in order to discern it from a tensor, to function.

SetSpinorDim[dim] sets the spinor dimension to dim.

Sym[expr,list] explicitly symmetrizes expr in the indices contained in list.

Tensor[sym,list] represents a tensor, denoted by the symbol sym, with indices given by list.

TensorSpinor[sym,list] represents a tensor-spinor, denoted by the symbol sym, with (vector) indices given by list.

B Redefined Mathematica built-in functions

NonCommutativeMultiply[]: Redefining NonCommutativeMultiply[] is what enables us to use new, noncommutative, objects like Γ-matrixes and spinors in Mathematica. NonCommutativeMultiply[] can also be entered as **.

Power[]: In order to be able to interpret expressions entered in the output format, and also to be able to cut, copy and paste expressions, we have to modify Power[] so that e.g., $\Gamma^\mu$ is replaced by GammaProd[\mu].

Subscript[]: The same reason as for Power[] except that Subscript[] takes care of downstairs indices instead of upstairs ones.
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