Spin Glass approach to the Directed 2-distance Minimal Dominating Set problem

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Abstract
The directed L-distance Minimal Dominating Set problem has widely practical application in the computer science and communication network. Here we study this problem from the perspective of purely theoretical interest. We only gives the results of ER random graph and Regular Random graph, but this work can be extended to any kind of networks. We develop Spin Glass theory to study the directed 2-distance MDS problem. Firstly we found that the Belief Propagation algorithm does not converge when the inverse temperature is bigger than a threshold value on the both ER random network and Regular random network. Secondly the entropy density of the Replica Symmetric theory has the transition point at the finite inverse temperature on the Regular Random Graph when the degree is bigger than 4, and ER random graph when the node degree is bigger than 6.6, there is no entropy transition point (or $\beta = \infty$) in the other circumstance. Thirdly the results of the BP algorithm same with the Replica Symmetry theory and the results of the BPD algorithm better than the Greedy heuristic algorithm.

Keywords: directed 2-distance Minimal Dominating Set, Belief Propagation, ER random graph, Regular Random graph, Belief Propagation Decimation.

I Introduction
Consider a simple network $W$ formed by $N$ nodes and $M$ arcs (directed edges), each arc $(i, j)$ pointing from a parent vertex (predecessor) $i$ to a child vertex (successor) $j$. The arc density $\alpha$ is defined simply as $\alpha \equiv M/N$. There is one set $\gamma$, if any node of the network belong to this set or at least one parent neighbor or one quasi (2-distance) parent neighbor nodes belong to this set, then this set $\gamma$ called directed 2-distance Minimal Dominating Set (MDS) of the given network $W$. 
L-distance minimal dominating set (MDS) problem arises mainly from the designing of communication network in a real world. L-distance MDS has important application in several fields. For example, communication network[1], location of the servers[2] and copies of a distributed directory[3]. There are various kind of L-distance MDS problem, such as, Liar’s Dominating Set[4], extended dominating set [5], 2- distance Paired dominating set[6], [1,2]-Dominating set [7], (σ, ρ) dominating set [8], k-tuple Dominating Set [9]. The directed MDS has widely application in the biological networks, such as infectious disease spreading [10], genetic regulation [11,12], chemical reaction and metabolic regulation [13], it is also applied in the social, information and neuroscience[14-19]. If we naturally extend the directed regular MDS problem, we can get the corresponding directed 2-distance MDS problem, for example the power generation and transportation [20], it is a regular directed MDS problem. If we assume that the power station can transport the power within the 2-distance neighbor, then this power supply problem converted to directed 2-distance MDS problem. There are very few work on the directed 2-distance MDS problem, and they only consider some special case, for example Wang and Chang[21] study the unique minimum distance dominating set in directed split-stars using distributed algorithms, Banerjee et al[22] introduce directed d-hop MDS that in-degree equals to one.

In this paper we use spin glass theory to study this problem. Spin glass theory has widely application in the optimization problem, such as k-sat[23], vertex cover[24], feedback vertex set problem[25] and dominating set problem[26,27]. Recently we use spin glass theory to studied the Regular Minimal Dominating Set problem [28-31], we introduce Belief Propagation Decimation (BPD), Warning Propagation and Survey Propagation Decimation algorithms to get the Minimal Dominating Set, we find that our algorithms are very close to the optimal solution and the speed is very fast. This year we still use spin glass theory to study the undirected 2-distance MDS problem, and developing BPD algorithm and Greedy algorithm to calculate the size of 2-distance MDS. In this paper, we still use the statistical physics to study the directed 2-distance MDS, we found that entropy density of the directed network work like with undirected network, it indicates that the solution spaces of the directed and undirected network are connecting each other. We will study their solution space using one step replica symmetry breaking theory, we will compare our results with the more rigorous mathematical results on the spatial network. We still use three algorithm, respectively Population Dynamics, BPD and the Greedy heuristic algorithm, to calculate the directed 2-distance MDS, we find that the Population Dynamics and BPD results always better than Greedy heuristic algorithm on the both ER and RR random graph.

This paper organized as follows. In section 2, we introduce the Replica Symmetry (RS) theory for the directed 2-distance MDS problem and presenting the Belief Propagation equation and the corresponding thermodynamic quantities. In section 3, we introduce the BPD algorithm for the directed 2-distance MDS problem, deriving the Belief Propagation equation and marginal probability equation for the different vertex state condition. In section 4, we conclude and
summarize our results.

II Replica Symmetry

In this section we will introduce the mean field theory for the directed 2-Distance MDS problem. The partition function has very important role to our theory, all the calculation starts from partition function, usually it is not a simple task. We assume that every vertex interact with all the neighbors and quasi neighbors in the same time, so we put function node on the every vertex node of the given graph. Depend on the RS mean field theory of the statistical physics we can write the partition function $Z$ as

$$Z = \sum_{\xi} \prod_{i \in W} e^{-\beta\xi_i} \{ 1 - (1 - \delta_{c_i}) \prod_{j \in \partial_i^+} (1 - \delta_{c_j}) \theta([\sum_{j \in \partial_i^+} \delta_{c_j} - 2 + \sum_{j \in \partial_i^-} \delta_{c_j} + 2] - 1) \}$$

where $\xi \equiv (c_1, c_2, \ldots, c_n)$ denotes one of the $3^n$ configurations, $c_i = 0$ if node $i$ is occupied, but it forbidden any child neighbor nodes take in state $c_i = 2$. $c_i = 1$ denotes if node $i$ not be occupied but at least one father neighbor be occupied. $c_i = 2$ if node $i$ not be occupied and at least one father neighbor in the state $c_i = 1$, but it forbidden any father neighbor nodes take in state $c_i = 0$. $\beta$ is inverse temperature, $\partial_i^+$ denotes all the father neighbor nodes of node $i$, $\partial_i^-$ denotes all the child neighbor nodes of node $i$. The partition function therefore only takes into account all the directed 2-Distance MDS.

The RS mean field theory such as the Bethe-Peierls approximation[28] or partition function expansion[29] can solve the above spin glass model, these two theory give same results, in there we will derive the Belief Propagation equation using the Bethe-Peierls approximation theory. We set negative cavity message $p_{i \rightarrow j}^{(c_i, c_j)}$ on the every positive edge from the father nodes and positive cavity message $p_{i \rightarrow j}^{(c_i, c_j)}$ on the every negative edge from the child nodes, and these messages must satisfy following equations

$$p_{i \rightarrow j}^{(c_i, c_j)} = \frac{e^{-\beta\xi_i} \left[ \sum_{k \in \partial_i^+ \setminus j} \sum_{c_k \in A^+} p_k^{(c_k, c_i)} \right] - (1 - \delta_{c_i}) \theta([\sum_{k \in \partial_i^+ \setminus j} \delta_{c_k} - 2 + \sum_{k \in \partial_i^-} \delta_{c_k} + 2] - 1)}{\sum_{\epsilon_i, \epsilon_j} e^{-\beta\xi_i} \left[ \sum_{k \in \partial_i^+ \setminus j} \sum_{c_k \in A^+} p_k^{(\epsilon_k, c_i)} \right] - (1 - \delta_{c_i}) \theta([\sum_{k \in \partial_i^+ \setminus j} \delta_{c_k} - 2 + \sum_{k \in \partial_i^-} \delta_{c_k} + 2] - 1)}$$

$$p_{i \rightarrow j}^{(c_i, c_j)} = \frac{e^{-\beta\xi_i} \left[ \sum_{k \in \partial_i^- \setminus j} \sum_{c_k \in A^+} p_k^{(c_k, c_i)} \right] - (1 - \delta_{c_i}) \theta([\sum_{k \in \partial_i^- \setminus j} \delta_{c_k} - 2 + \sum_{k \in \partial_i^- \setminus j} \delta_{c_k} + 2] - 1)}{\sum_{\epsilon_i, \epsilon_j} e^{-\beta\xi_i} \left[ \sum_{k \in \partial_i^- \setminus j} \sum_{c_k \in A^+} p_k^{(\epsilon_k, c_i)} \right] - (1 - \delta_{c_i}) \theta([\sum_{k \in \partial_i^- \setminus j} \delta_{c_k} - 2 + \sum_{k \in \partial_i^- \setminus j} \delta_{c_k} + 2] - 1)}$$

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we can expand this equation as following

\[
p_{i\rightarrow j}^{(0,0)} = p_{i\rightarrow j}^{(0,1)} = p_{i\rightarrow j}^{(0,2)} = \frac{e^{-\beta} \prod_{k \in \partial^+ \setminus j} (p_{k \rightarrow i}^{(2,0)} + p_{k \rightarrow i}^{(1,0)} + p_{k \rightarrow i}^{(0,0)}) \prod_{k \in \partial^-} (p_{k \rightarrow i}^{(1,0)} + p_{k \rightarrow i}^{(0,0)})}{z_{i\rightarrow j}}
\]

\[p_{i\rightarrow j}^{(1,0)} = \frac{\prod_{k \in \partial^+ \setminus j} (p_{k \rightarrow i}^{(0,1)} + p_{k \rightarrow i}^{(1,1)} + p_{k \rightarrow i}^{(2,1)}) \prod_{k \in \partial^-} (p_{k \rightarrow i}^{(1,1)} + p_{k \rightarrow i}^{(2,1)} + p_{k \rightarrow i}^{(0,1)})}{z_{i\rightarrow j}} \]

\[p_{i\rightarrow j}^{(1,1)} = p_{i\rightarrow j}^{(1,2)} = \frac{\prod_{k \in \partial^+ \setminus j} (p_{k \rightarrow i}^{(2,1)} + p_{k \rightarrow i}^{(1,2)}) \prod_{k \in \partial^-} (p_{k \rightarrow i}^{(1,2)} + p_{k \rightarrow i}^{(2,2)} + p_{k \rightarrow i}^{(0,2)})}{z_{i\rightarrow j}} \]

\[z_{i\rightarrow j} = 3 * e^{-\beta} \prod_{k \in \partial^+ \setminus j} (p_{k \rightarrow i}^{(2,0)} + p_{k \rightarrow i}^{(1,0)} + p_{k \rightarrow i}^{(0,0)}) \prod_{k \in \partial^-} (p_{k \rightarrow i}^{(1,0)} + p_{k \rightarrow i}^{(0,0)})
+ [3 * \prod_{k \in \partial^+ \setminus j} (p_{k \rightarrow i}^{(0,1)} + p_{k \rightarrow i}^{(1,1)} + p_{k \rightarrow i}^{(2,1)}) - 2 * \prod_{k \in \partial^+ \setminus j} (p_{k \rightarrow i}^{(1,1)} + p_{k \rightarrow i}^{(2,1)})]
\times \prod_{k \in \partial^-} (p_{k \rightarrow i}^{(0,1)} + p_{k \rightarrow i}^{(1,1)} + p_{k \rightarrow i}^{(2,1)}) + [2 * \prod_{k \in \partial^+ \setminus j} (p_{k \rightarrow i}^{(1,2)} + p_{k \rightarrow i}^{(2,2)}) - \prod_{k \in \partial^+ \setminus j} (p_{k \rightarrow i}^{(2,2)})]
\times \prod_{k \in \partial^-} (p_{k \rightarrow i}^{(0,2)} + p_{k \rightarrow i}^{(1,2)} + p_{k \rightarrow i}^{(2,2)})
\]

\[p_{i\rightarrow j}^{(0,0)} = p_{i\rightarrow j}^{(0,1)} = p_{i\rightarrow j}^{(0,2)} = \frac{e^{-\beta} \prod_{k \in \partial^+} (p_{k \rightarrow i}^{(2,0)} + p_{k \rightarrow i}^{(1,0)} + p_{k \rightarrow i}^{(0,0)}) \prod_{k \in \partial^- \setminus j} (p_{k \rightarrow i}^{(1,0)} + p_{k \rightarrow i}^{(0,0)})}{z_{i\rightarrow j}} \]

\[p_{i\rightarrow j}^{(1,0)} = p_{i\rightarrow j}^{(1,1)} = p_{i\rightarrow j}^{(1,2)} = \frac{\prod_{k \in \partial^+} (p_{k \rightarrow i}^{(0,1)} + p_{k \rightarrow i}^{(1,1)} + p_{k \rightarrow i}^{(2,1)}) \prod_{k \in \partial^- \setminus j} (p_{k \rightarrow i}^{(1,1)} + p_{k \rightarrow i}^{(2,1)})}{z_{i\rightarrow j}} \]
the marginal probability \( p_i^{(2.0)} \) of node \( i \) is expressed as

\[
\begin{align*}
  p_i^{(2.0)} &= p_i^{(2.1)} = p_i^{(2.2)} = \\
  &\frac{\prod_{k \in \partial^{+} i} (p_k^{(1.2)} + p_k^{(2.2)}) - \prod_{k \in \partial^{+} i} p_k^{(2.2)} \prod_{k \in \partial^{-} i} (p_k^{(0.2)} + p_k^{(1.2)} + p_k^{(2.2)})}{z_{i \rightarrow j}}
\end{align*}
\] (12)

\[
\begin{align*}
  z_{i \rightarrow j} = 2 \cdot e^{-\beta} \sum_{p_{k \rightarrow i}} \prod_{k \in \partial^{+} i} (p_k^{(2.0)} + p_k^{(1.0)} + p_k^{(0.0)}) \prod_{k \in \partial^{-} i} (p_k^{(1.0)} + p_k^{(0.0)}) \\
  &+ 3 \sum_{p_{k \rightarrow i}} \prod_{k \in \partial^{+} i} (p_k^{(0.1)} + p_k^{(1.1)} + p_k^{(2.1)}) - \prod_{k \in \partial^{-} i} (p_k^{(1.1)} + p_k^{(2.1)}) \prod_{k \in \partial^{-} i} (p_k^{(0.1)} + p_k^{(1.1)} + p_k^{(2.1)}) \\
  &3 \sum_{p_{k \rightarrow i}} \prod_{k \in \partial^{+} i} (p_k^{(1.2)} + p_k^{(2.2)}) - \prod_{k \in \partial^{-} i} (p_k^{(2.2)}) \prod_{k \in \partial^{-} i} (p_k^{(0.2)} + p_k^{(1.2)} + p_k^{(2.2)})
\end{align*}
\] (13)

these two equations called Belief-Propagation (BP) equation. Where the Kronecker symbol \( \delta_m^n \) is 1 if \( m = n \) and 0 otherwise. The negative cavity message \( p_{i \rightarrow j}^{(c_i,c_j)} \) represents the joint probability that the father node \( i \) is in occupation state \( c_i \) and its adjacent child node \( j \) is in occupation state \( c_j \) when the constraint of node \( j \) is not considered. The positive cavity message \( p_{i \rightarrow j}^{(c_i,c_j)} \) represents the joint probability that the child node \( i \) is in occupation state \( c_i \) and its adjacent father node \( j \) is in occupation state \( c_j \) when the constraint of node \( j \) is not considered. If the node \( i \) in the state \( c_i = 0 \), it request the child neighbor nodes only take in the state \( c_k = 0 \) or \( c_k = 1 \), and the state \( c_k = 2 \) is forbidden, but the father neighbor nodes can take in the any state. If the node \( i \) in the state \( c_i = 1 \), it request the neighbor nodes can take any state \( c_k = 0 \), \( c_k = 1 \), \( c_k = 2 \), but at least one father neighbor must be occupied. If the node \( i \) in the state \( c_i = 2 \), it request the father neighbor nodes only take in the state \( c_k = 1 \) or \( c_k = 2 \), and at least one 2-Distance quasi father neighbor must be occupied, and the state \( c_k = 0 \) of the father neighbor nodes is forbidden, the child nodes can take in any state. The \( A^+ \) represents the set of possible father neighbor states, and the \( A^- \) represents the set of possible child node states.

we expand this equation as following

\[
\begin{align*}
  p_i^{(c_i)} &= e^{-\beta \delta_{c_i}^{(0)}} \sum_{p_{k \rightarrow i}^{(c_k,c_i)}} \prod_{k \in \partial^{+} i} \sum_{c_k \in A^+} p_{k \rightarrow i}^{(c_k,c_i)} (1 - \delta_{c_k}^{(0)}) \prod_{k \in \partial^{+} i} \sum_{c_k \geq c_i} p_{k \rightarrow i}^{(c_k,c_i)} \prod_{k \in \partial^{-} i} \sum_{c_k \in A^-} p_{k \rightarrow i}^{(c_k,c_i)}
\end{align*}
\] (14)

\[
\begin{align*}
  p_i^{(0)} = e^{-\beta} \prod_{k \in \partial^{+} i} (p_k^{(2.0)} + p_k^{(1.0)} + p_k^{(0.0)}) \prod_{k \in \partial^{-} i} (p_k^{(1.0)} + p_k^{(0.0)})
\end{align*}
\] (15)
\[ p_i^1 = \frac{\prod_{k \in \partial^+} \left( p_{k \rightarrow i}^{(0,1)} + p_{k \rightarrow i}^{(1,1)} + p_{k \rightarrow i}^{(2,1)} \right) - \prod_{k \in \partial^+} \left( p_{k \rightarrow i}^{(1,1)} + p_{k \rightarrow i}^{(2,1)} \right) \prod_{k \in \partial^-} \left( p_{k \rightarrow i}^{(0,1)} + p_{k \rightarrow i}^{(1,1)} + p_{k \rightarrow i}^{(2,1)} \right)}{z_i} \]  
\tag{16}

\[ p_i^2 = \frac{\prod_{k \in \partial^+} \left( p_{k \rightarrow i}^{(1,2)} + p_{k \rightarrow i}^{(2,2)} \right) - \prod_{k \in \partial^+} \left( p_{k \rightarrow i}^{(2,2)} \right) \prod_{k \in \partial^-} \left( p_{k \rightarrow i}^{(0,2)} + p_{k \rightarrow i}^{(1,2)} + p_{k \rightarrow i}^{(2,2)} \right)}{z_i} \]  
\tag{17}

\[ z_i = e^{-\beta} \prod_{k \in \partial^+} \left( p_{k \rightarrow i}^{(2,0)} + p_{k \rightarrow i}^{(1,0)} + p_{k \rightarrow i}^{(0,0)} \right) \prod_{k \in \partial^-} \left( p_{k \rightarrow i}^{(1,0)} + p_{k \rightarrow i}^{(0,0)} \right) \]
\[ + \prod_{k \in \partial^+} \left( p_{k \rightarrow i}^{(0,1)} + p_{k \rightarrow i}^{(1,1)} + p_{k \rightarrow i}^{(2,1)} \right) \prod_{k \in \partial^-} \left( p_{k \rightarrow i}^{(1,1)} + p_{k \rightarrow i}^{(2,1)} \right) \prod_{k \in \partial^-} \left( p_{k \rightarrow i}^{(0,2)} + p_{k \rightarrow i}^{(1,2)} + p_{k \rightarrow i}^{(2,2)} \right) \]  
\tag{18}

where the \( z_i \) is a normalization constant, we calculate the marginal probability using the converged messages of negative messages \( p_{k \rightarrow i}^{(c,c)} \) and positive message \( p_{k \rightarrow i}^{(c,c)} \). \( p_i^1 \) denotes the probability of the node \( i \) be covered. \( p_i^2 \) denotes the probability that the node \( i \) has at least one covered father neighbor. \( p_i^2 \) denotes the probability that the node \( i \) has at least one covered 2-distance quasi father neighbor.

finally the free energy could be calculated by mean field theory

\[ F_0 = \sum_{i=1}^{N} F_i - \sum_{(i,j)=1}^{M} F_{(i,j)} \]  
\tag{19}

where

\[ F_i = -\frac{1}{\beta} \ln \left[ \sum_{e_i} e^{-\beta \delta_i^0} \sum_{k \in \partial^+} p_{k \rightarrow i}^{(c,c)} - \delta_i^0 \sum_{k \in \partial^+} \sum_{c_k \geq e_i} p_{k \rightarrow i}^{(c,c)} \right] \prod_{k \in \partial^-} \sum_{c_k \in A^-} p_{k \rightarrow i}^{(c,c)} \]  
\tag{20}

\[ F_i = -\frac{1}{\beta} \ln \left[ e^{-\beta} \prod_{k \in \partial^+} \left( p_{k \rightarrow i}^{(2,0)} + p_{k \rightarrow i}^{(1,0)} + p_{k \rightarrow i}^{(0,0)} \right) \prod_{k \in \partial^-} \left( p_{k \rightarrow i}^{(1,0)} + p_{k \rightarrow i}^{(0,0)} \right) \right] \]
\[ + \prod_{k \in \partial^+} \left( p_{k \rightarrow i}^{(0,1)} + p_{k \rightarrow i}^{(1,1)} + p_{k \rightarrow i}^{(2,1)} \right) \prod_{k \in \partial^-} \left( p_{k \rightarrow i}^{(1,1)} + p_{k \rightarrow i}^{(2,1)} \right) \prod_{k \in \partial^-} \left( p_{k \rightarrow i}^{(0,2)} + p_{k \rightarrow i}^{(1,2)} + p_{k \rightarrow i}^{(2,2)} \right) \]  
\tag{21}
\[ F_{(i,j)} = -\frac{1}{\beta} \ln \left[ \sum_{c_i,c_j} p_{c_i \rightarrow c_j} p_{c_j \leftarrow c_i} \right] \]  

(22)

\[ F_{(i,j)} = -\frac{1}{\beta} \ln \left[ p_{i \rightarrow j}^{(0,0)} p_{j \leftarrow i}^{(0,0)} + p_{i \rightarrow j}^{(0,1)} p_{j \leftarrow i}^{(1,0)} + p_{i \rightarrow j}^{(1,0)} p_{j \leftarrow i}^{(0,1)} + p_{i \rightarrow j}^{(1,1)} p_{j \leftarrow i}^{(1,1)} 
+ p_{i \rightarrow j}^{(1,2)} p_{j \leftarrow i}^{(2,1)} + p_{i \rightarrow j}^{(2,0)} p_{j \leftarrow i}^{(0,2)} + p_{i \rightarrow j}^{(2,1)} p_{j \leftarrow i}^{(1,2)} + p_{i \rightarrow j}^{(2,2)} p_{j \leftarrow i}^{(2,2)} \right] \]  

(23)

Where the \( F_i \) denotes the free energy of function node \( i \), the \( F_{(i,j)} \) denotes the free energy of the edge \((i, j)\). We iterate the BP equation until it is converged to one stable point, and then calculate the mean free energy \( f \equiv F/N \) and the energy density \( \omega = 1/N \sum_i p_i^0 \) by equation (14) and (19). The entropy density calculates as \( s = \beta (\omega - f) \).

Figure 1: The RS and BP results for the directed 2-distance MDS problem on the ER random graph with mean connectivity \( c = 5 \) and \( N = 10000 \) using Belief Propagation and population dynamics. In the upper two and bottom left graphs, the x-axis denotes the inverse temperature \( \beta \), and the y-axis denotes the thermodynamic quantities. In the bottom right graph, the x-axis denotes the energy density and the y-axis denotes the entropy density.

From the Figure 1 we can see that the Belief Propagation equation can not converge when the inverse temperature bigger than 11.6 on the ER random graph which mean connectivity equals to five. Entropy density always is positive and the change rate is smaller and smaller with the inverse temperature, so the entropy density reach the transition point when the inverse temperature is very very big. Because of the Belief Propagation can not converge when the inverse
temperature bigger than some threshold value both on ER random graph and
RR random graph, so we get the ground state energy using the population
dynamics results. We use average value of the energy density when the inverse
temperature within ten to fifteen to determine the ground state energy.

![Figure 2: The RS and BP results for the directed 2-distance MDS problem
on the RR random graph with mean connectivity $c = 7$ and $N = 10^4$ using
Belief Propagation and population dynamics. In the upper two and bottom left
graphs, the $x$-axis denotes the inverse temperature $\beta$, and the $y$-axis denotes
the thermodynamic quantities. In the bottom right graph, the $x$-axis denotes
the energy density and the $y$-axis denotes the entropy density.]

From the Figure 2 we can see that the population dynamics equation still can
converge at the transition point of the Entropy density on the Regular Random
graph when vertex degree is greater than six, so we did not need to average over
the population dynamics results in this range. The difference of the entropy
density between ER random network and Regular Random network indicates
that the solution spaces of them have essential difference.

III Belief Propagation Decimation algorithm and
Greedy algorithm

In this work we use two algorithm to construct the solution of the given
graph, respectively, BPD algorithm and Greedy algorithm. Greedy algorithm
very fast, but it does not always guarantee good results such as BPD. BPD
algorithm not fast like Greedy algorithm, but it always gives good estimation
for the directed 2-distance MDS problem.
III.1 Belief Propagation Decimation

If a node $i$ is unobserved (it is empty and all the father neighbor and 2-distance quasi father neighbor nodes are not be occupied), the output message $p_{i \rightarrow j}$ on the arc $(i \rightarrow j)$ and the output message $p_{i \leftarrow j}$ on the arc $(i \leftarrow j)$ between node $i$ and node $j$ are updated according to Eq.(2,3). On the other hand, if node $i$ is empty but observed, and it has at least one occupied father neighbor node, namely $c_i = 1$, this node then presents no restriction to the states of all its unoccupied father neighbors. For such a node $i$, it has no opportunity to take $c_i = 2$, and the output message $p_{i \rightarrow j}$ or $p_{i \leftarrow j}$ on the link $(i,j)$ is then updated according to the following equations

\[
p_{i \rightarrow j}^{(c_i,c_j)} = \frac{e^{-\beta \delta_i} (1 - \delta_i^2) \prod_{k \in \partial^+} \sum_{c_k} \prod_{j \in \partial^+} \sum_{c_j \geq c_i} \prod_{k \in \partial^+} \sum_{c_k} \prod_{j \in \partial^+} \sum_{c_j} p_{k \rightarrow i} (c_k,c_j) p_{j \rightarrow i} (c_j,c_j)}{\sum_{c_i,c_j} e^{-\beta \delta_i} (1 - \delta_i^2) \prod_{k \in \partial^+} \sum_{c_k} \prod_{j \in \partial^+} \sum_{c_j \geq c_i} \prod_{k \in \partial^+} \sum_{c_k} \prod_{j \in \partial^+} \sum_{c_j} p_{k \rightarrow i} (c_k,c_j) p_{j \rightarrow i} (c_j,c_j)}
\]

we can expand this equation as following

\[
p_{i \rightarrow j}^{(0,0)} = p_{i \rightarrow j}^{(0,1)} = p_{i \rightarrow j}^{(0,2)} = \frac{e^{-\beta} \prod_{k \in \partial^+} \sum_{c_k} (p_{k \rightarrow i}^{(2,0)} + p_{k \rightarrow i}^{(1,0)} + p_{k \rightarrow i}^{(0,0)}) \prod_{k \in \partial^-} (p_{k \rightarrow i}^{(1,0)} + p_{k \rightarrow i}^{(0,0)})}{z_{i \rightarrow j}}
\]

\[
p_{i \rightarrow j}^{(1,0)} = p_{i \rightarrow j}^{(1,1)} = p_{i \rightarrow j}^{(1,2)} = \frac{\prod_{k \in \partial^+} \sum_{c_k} (p_{k \rightarrow i}^{(0,1)} + p_{k \rightarrow i}^{(1,1)} + p_{k \rightarrow i}^{(2,1)}) \prod_{k \in \partial^-} (p_{k \rightarrow i}^{(0,1)} + p_{k \rightarrow i}^{(1,1)} + p_{k \rightarrow i}^{(2,1)})}{z_{i \rightarrow j}}
\]

\[
p_{i \rightarrow j}^{(2,1)} = p_{i \rightarrow j}^{(2,2)} = 0
\]

\[
z_{i \rightarrow j} = 3 \cdot e^{-\beta} \prod_{k \in \partial^+} \sum_{c_k} (p_{k \rightarrow i}^{(2,0)} + p_{k \rightarrow i}^{(1,0)} + p_{k \rightarrow i}^{(0,0)}) \prod_{k \in \partial^-} (p_{k \rightarrow i}^{(1,0)} + p_{k \rightarrow i}^{(0,0)})
\]

\[
+ 3 \cdot \prod_{k \in \partial^+} \sum_{c_k} (p_{k \rightarrow i}^{(0,1)} + p_{k \rightarrow i}^{(1,1)} + p_{k \rightarrow i}^{(2,1)}) \prod_{k \in \partial^-} (p_{k \rightarrow i}^{(0,1)} + p_{k \rightarrow i}^{(1,1)} + p_{k \rightarrow i}^{(2,1)})
\]

\[
p_{i \rightarrow j}^{(0,0)} = p_{i \rightarrow j}^{(0,1)} = \frac{e^{-\beta} \prod_{k \in \partial^+} \sum_{c_k} (p_{k \rightarrow i}^{(2,0)} + p_{k \rightarrow i}^{(1,0)} + p_{k \rightarrow i}^{(0,0)}) \prod_{k \in \partial^-} (p_{k \rightarrow i}^{(1,0)} + p_{k \rightarrow i}^{(0,0)})}{z_{i \rightarrow j}}
\]

9
The marginal probability calculated by the node $j$ is covered, it sends message to the node $i$ as $p_{j ightarrow i}^{(0,0)} = p_{j ightarrow i}^{(0,1)} = 0.5$. It leads $p_{j ightarrow i}^{(0,1)} + p_{j ightarrow i}^{(1,1)} + p_{j ightarrow i}^{(2,1)} = p_{j ightarrow i}^{(0,1)} = p_{j ightarrow i}^{(1,1)} + p_{j ightarrow i}^{(2,1)},$ so the constraint of the node $i$ to all the other father neighbor nodes are automatically removed. The marginal probability calculated by the following equation

$$\frac{p_{i ightarrow j}^{(1,0)}}{p_{i ightarrow j}^{(1,1)}} = \frac{p_{i ightarrow j}^{(1,2)}}{p_{i ightarrow j}^{(2,1)}} = \frac{\prod_{k \in \partial^+} (p_{k ightarrow i}^{(0,0)} + p_{k ightarrow i}^{(1,0)} + p_{k ightarrow i}^{(2,1)}) \prod_{k \in \partial^- \setminus j} (p_{k ightarrow i}^{(0,0)} + p_{k ightarrow i}^{(1,1)} + p_{k ightarrow i}^{(2,1)})}{z_{i ightarrow j}}$$

(31)

$$p_{i ightarrow j}^{(2,0)} = p_{i ightarrow j}^{(0,1)} = p_{i ightarrow j}^{(2,2)} = 0$$

(32)

$$z_{i ightarrow j} = 2 * e^{-\beta} \prod_{k \in \partial^+} (p_{k ightarrow i}^{(2,0)} + p_{k ightarrow i}^{(1,0)} + p_{k ightarrow i}^{(0,0)}) \prod_{k \in \partial^- \setminus j} (p_{k ightarrow i}^{(1,1)} + p_{k ightarrow i}^{(0,0)})$$

(33)

$$+ 3 * \prod_{k \in \partial^- \setminus j} (p_{k ightarrow i}^{(0,1)} + p_{k ightarrow i}^{(1,1)} + p_{k ightarrow i}^{(2,1)}) \prod_{k \in \partial^- \setminus j} (p_{k ightarrow i}^{(0,1)} + p_{k ightarrow i}^{(1,1)} + p_{k ightarrow i}^{(2,1)})$$

$$z_{i ightarrow j} = 2 * e^{-\beta} \prod_{k \in \partial^+} (p_{k ightarrow i}^{(2,0)} + p_{k ightarrow i}^{(1,0)} + p_{k ightarrow i}^{(0,0)}) \prod_{k \in \partial^- \setminus j} (p_{k ightarrow i}^{(1,1)} + p_{k ightarrow i}^{(0,0)})$$

(33)

For the node $i(c_i = 1)$, there is at least one father neighbor node $j$ is covered, it sends message to the node $i$ as $p_{j ightarrow i}^{(0,0)} = p_{j ightarrow i}^{(0,1)} = 0.5$. It leads $p_{j ightarrow i}^{(0,1)} + p_{j ightarrow i}^{(1,1)} + p_{j ightarrow i}^{(2,1)} = p_{j ightarrow i}^{(0,1)} = p_{j ightarrow i}^{(1,1)} + p_{j ightarrow i}^{(2,1)},$ so the constraint of the node $i$ to all the other father neighbor nodes are automatically removed. The marginal probability calculated by the following equation

$$p_{i}^{c_i} = \frac{e^{-\beta \delta_{i}^0} (1 - \delta_{c}^i)}{\sum_{c_i} e^{-\beta \delta_{i}^0} (1 - \delta_{c}^i)} \prod_{k \in \partial^+ c_i} \prod_{k \in \partial^- c_k} \sum_{c_k} p_{k ightarrow i}^{(c_k,c_i)}$$

(34)

we expand this equation as following

$$p_i = \frac{e^{-\beta} \prod_{k \in \partial^+} (p_{k ightarrow i}^{(2,0)} + p_{k ightarrow i}^{(1,0)} + p_{k ightarrow i}^{(0,0)}) \prod_{k \in \partial^-} (p_{k ightarrow i}^{(1,1)} + p_{k ightarrow i}^{(0,0)})}{z_i}$$

(35)

$$p_i^1 = \frac{\prod_{k \in \partial^+} (p_{k ightarrow i}^{(0,1)} + p_{k ightarrow i}^{(1,1)} + p_{k ightarrow i}^{(2,1)}) \prod_{k \in \partial^-} (p_{k ightarrow i}^{(0,1)} + p_{k ightarrow i}^{(1,1)} + p_{k ightarrow i}^{(2,1)})}{z_i}$$

(36)

$$p_i^2 = 0$$

(37)

$$z_i = e^{-\beta} \prod_{k \in \partial^+} (p_{k ightarrow i}^{(2,0)} + p_{k ightarrow i}^{(1,0)} + p_{k ightarrow i}^{(0,0)}) \prod_{k \in \partial^-} (p_{k ightarrow i}^{(1,1)} + p_{k ightarrow i}^{(0,0)})$$

$$+ \prod_{k \in \partial^+} (p_{k ightarrow i}^{(0,1)} + p_{k ightarrow i}^{(1,1)} + p_{k ightarrow i}^{(2,1)}) \prod_{k \in \partial^-} (p_{k ightarrow i}^{(0,1)} + p_{k ightarrow i}^{(1,1)} + p_{k ightarrow i}^{(2,1)})$$

(38)

if node $i$ is empty but observed (it has no adjacent occupied father node, but it has one occupied 2-distance quasi father neighbor nodes), this node then
presents no restriction to the occupation states of all its unoccupied father neighbors. For such a node \( i \), the output message \( p_{i \rightarrow j} \) or \( p_{j \rightarrow i} \) on the link \((i, j)\) is then updated according to following equation

\[
p_{i \rightarrow j}^{(c_i,c_j)} = \frac{e^{-\beta \delta^0_{c_i}} \left( \prod_{j \in \partial^- j} \sum_{\epsilon_{c_i} \in A^+} p_{k \rightarrow i}^{(c_i,c_j)} \right) - (1 - \delta^0_{c_i} - \delta^2_{c_i})(\delta^1_{c_i} + \delta^1_{c_i}+1) \left( \prod_{j \in \partial^- j} \sum_{\epsilon_{c_i} \in A^+} p_{k \rightarrow i}^{(c_i,c_j)} \right) \prod_{j \in \partial^- j} \sum_{\epsilon_{c_i} \in A^+} p_{k \rightarrow i}^{(c_i,c_j)}}{\sum_{\epsilon_{c_i} \in A^+} \left[ \prod_{j \in \partial^- j} \sum_{\epsilon_{c_i} \in A^+} p_{k \rightarrow i}^{(c_i,c_j)} \right] - (1 - \delta^0_{c_i} - \delta^2_{c_i})(\delta^1_{c_i} + \delta^1_{c_i}+1) \left( \prod_{j \in \partial^- j} \sum_{\epsilon_{c_i} \in A^+} p_{k \rightarrow i}^{(c_i,c_j)} \right) \prod_{j \in \partial^- j} \sum_{\epsilon_{c_i} \in A^+} p_{k \rightarrow i}^{(c_i,c_j)}}
\]

we can expand this equation as following

\[
p_{i \rightarrow j}^{(0,0)} = p_{i \rightarrow j}^{(0,1)} = p_{i \rightarrow j}^{(0,2)} = \frac{e^{-\beta} \prod_{k \in \partial^- j} \left( p_{k \rightarrow i}^{(2,0)} + p_{k \rightarrow i}^{(1,0)} + p_{k \rightarrow i}^{(0,0)} \right) \prod_{k \in \partial^- j} \left( p_{k \rightarrow i}^{(1,0)} + p_{k \rightarrow i}^{(0,0)} \right)}{z_{i \rightarrow j}}
\]

\[
p_{i \rightarrow j}^{(1,0)} = \prod_{k \in \partial^- j} \left( p_{k \rightarrow i}^{(1,1)} + p_{k \rightarrow i}^{(2,1)} \right) \prod_{k \in \partial^- j} \left( p_{k \rightarrow i}^{(1,1)} + p_{k \rightarrow i}^{(2,1)} \right) \prod_{k \in \partial^- j} \left( p_{k \rightarrow i}^{(1,1)} + p_{k \rightarrow i}^{(2,1)} \right)
\]

\[
p_{i \rightarrow j}^{(1,1)} = p_{i \rightarrow j}^{(1,2)} = \prod_{k \in \partial^- j} \left( p_{k \rightarrow i}^{(1,2)} + p_{k \rightarrow i}^{(2,2)} \right) \prod_{k \in \partial^- j} \left( p_{k \rightarrow i}^{(1,2)} + p_{k \rightarrow i}^{(2,2)} \right) \prod_{k \in \partial^- j} \left( p_{k \rightarrow i}^{(1,2)} + p_{k \rightarrow i}^{(2,2)} \right)
\]

\[
p_{i \rightarrow j}^{(2,1)} = \prod_{k \in \partial^- j} \left( p_{k \rightarrow i}^{(2,2)} + p_{k \rightarrow i}^{(1,2)} \right) \prod_{k \in \partial^- j} \left( p_{k \rightarrow i}^{(2,2)} + p_{k \rightarrow i}^{(1,2)} \right) \prod_{k \in \partial^- j} \left( p_{k \rightarrow i}^{(2,2)} + p_{k \rightarrow i}^{(1,2)} \right)
\]

\[
p_{i \rightarrow j}^{(2,0)} = 0
\]
\[ z_{i\to j} = 3 \cdot e^{-\beta} \prod_{k \in \partial^+ \setminus j} (p^{(2,0)}_{k \to i} + p^{(1,0)}_{k \to i} + p^{(0,0)}_{k \to i}) \prod_{k \in \partial^-} (p^{(1,0)}_{k \to i} + p^{(0,0)}_{k \to i}) \\
+ 3 \cdot \prod_{k \in \partial^+ \setminus j} (p^{(0,1)}_{k \to i} + p^{(1,1)}_{k \to i} + p^{(2,1)}_{k \to i}) - 2 \cdot \prod_{k \in \partial^+ \setminus j} (p^{(1,1)}_{k \to i} + p^{(2,1)}_{k \to i}) \\
\times \prod_{k \in \partial^-} (p^{(0,1)}_{k \to i} + p^{(1,1)}_{k \to i} + p^{(2,1)}_{k \to i}) + 2 \cdot \prod_{k \in \partial^-} (p^{(1,2)}_{k \to i} + p^{(2,2)}_{k \to i}) \]
\[ (46) \]
\[ p^{(0,0)}_{i\to j} = p^{(0,1)}_{i\to j} = \frac{e^{-\beta} \prod_{k \in \partial^+} (p^{(2,0)}_{k \to i} + p^{(1,0)}_{k \to i} + p^{(0,0)}_{k \to i}) \prod_{k \in \partial^- \setminus j} (p^{(1,0)}_{k \to i} + p^{(0,0)}_{k \to i})}{z_{i\to j}} \]
\[ (47) \]
\[ p^{(1,0)}_{i\to j} = p^{(1,1)}_{i\to j} = p^{(1,2)}_{i\to j} = \frac{\left[ \prod_{k \in \partial^+} (p^{(0,1)}_{k \to i} + p^{(1,1)}_{k \to i} + p^{(2,1)}_{k \to i}) - \prod_{k \in \partial^+} (p^{(1,1)}_{k \to i} + p^{(2,1)}_{k \to i}) \right] \prod_{k \in \partial^- \setminus j} (p^{(0,1)}_{k \to i} + p^{(1,1)}_{k \to i} + p^{(2,1)}_{k \to i})}{z_{i\to j}} \]
\[ (48) \]
\[ p^{(2,0)}_{i\to j} = p^{(2,1)}_{i\to j} = p^{(2,2)}_{i\to j} = \frac{\prod_{k \in \partial^+} (p^{(1,2)}_{k \to i} + p^{(2,2)}_{k \to i}) \prod_{k \in \partial^- \setminus j} (p^{(0,2)}_{k \to i} + p^{(1,2)}_{k \to i} + p^{(2,2)}_{k \to i})}{z_{i\to j}} \]
\[ (49) \]
\[ z_{i\to j} = 2 \cdot e^{-\beta} \prod_{k \in \partial^+} (p^{(2,0)}_{k \to i} + p^{(1,0)}_{k \to i} + p^{(0,0)}_{k \to i}) \prod_{k \in \partial^-} (p^{(1,0)}_{k \to i} + p^{(0,0)}_{k \to i}) \\
+ 3 \left[ \prod_{k \in \partial^+} (p^{(0,1)}_{k \to i} + p^{(1,1)}_{k \to i} + p^{(2,1)}_{k \to i}) - \prod_{k \in \partial^+} (p^{(1,1)}_{k \to i} + p^{(2,1)}_{k \to i}) \right] \prod_{k \in \partial^- \setminus j} (p^{(0,1)}_{k \to i} + p^{(1,1)}_{k \to i} + p^{(2,1)}_{k \to i}) \\
3 \prod_{k \in \partial^+} (p^{(1,2)}_{k \to i} + p^{(2,2)}_{k \to i}) \prod_{k \in \partial^- \setminus j} (p^{(0,2)}_{k \to i} + p^{(1,2)}_{k \to i} + p^{(2,2)}_{k \to i}) \]
\[ (50) \]

For the node \(i(c_i = 2)\), there is at least one father neighbor node \(j\) that takes the state \(c_j = 1\), it sends message to the node \(i\) as \(p^{(2,1)}_{j\to i} = p^{(2,2)}_{j\to i} = 0\). It leads \(p^{(1,2)}_{j\to i} + p^{(2,2)}_{j\to i} = p^{(1,2)}_{j\to i}\), so the constraint of the node \(i\) to all the other father neighbor nodes are automatically removed. The marginal probability calculated by the following equation
we expand this equation as following
\[ p_i^0 = e^{-\beta} \prod_{k \in \partial^+} (p_{k \rightarrow i}^{(0,1)} + p_{k \rightarrow i}^{(1,0)} + p_{k \rightarrow i}^{(2,0)}) \prod_{k \notin \partial^+} (p_{k \rightarrow i}^{(0,0)} + p_{k \rightarrow i}^{(1,0)}) \]  
(52)

\[ p_i^1 = \frac{\prod_{k \in \partial^+} (p_{k \rightarrow i}^{(1,2)} + p_{k \rightarrow i}^{(2,1)}) \prod_{k \notin \partial^+} (p_{k \rightarrow i}^{(0,2)} + p_{k \rightarrow i}^{(1,2)} + p_{k \rightarrow i}^{(2,2)})}{z_i} \]  
(53)

\[ p_i^2 = \frac{\prod_{k \in \partial^+} (p_{k \rightarrow i}^{(2,0)} + p_{k \rightarrow i}^{(1,0)} + p_{k \rightarrow i}^{(0,0)}) \prod_{k \notin \partial^+} (p_{k \rightarrow i}^{(1,0)} + p_{k \rightarrow i}^{(2,0)})}{z_i} + \frac{\prod_{k \in \partial^+} (p_{k \rightarrow i}^{(0,1)} + p_{k \rightarrow i}^{(1,1)} + p_{k \rightarrow i}^{(2,1)}) \prod_{k \notin \partial^+} (p_{k \rightarrow i}^{(1,1)} + p_{k \rightarrow i}^{(2,1)})}{z_i} + \frac{\prod_{k \in \partial^+} (p_{k \rightarrow i}^{(1,2)} + p_{k \rightarrow i}^{(2,2)}) \prod_{k \notin \partial^+} (p_{k \rightarrow i}^{(0,2)} + p_{k \rightarrow i}^{(1,2)} + p_{k \rightarrow i}^{(2,2)})}{z_i} \]  
(54)

\[ z_i = e^{-\beta} \prod_{k \in \partial^+} (p_{k \rightarrow i}^{(2,0)} + p_{k \rightarrow i}^{(1,0)} + p_{k \rightarrow i}^{(0,0)}) \prod_{k \notin \partial^+} (p_{k \rightarrow i}^{(1,0)} + p_{k \rightarrow i}^{(2,0)}) + \prod_{k \in \partial^+} (p_{k \rightarrow i}^{(0,1)} + p_{k \rightarrow i}^{(1,1)} + p_{k \rightarrow i}^{(2,1)}) \prod_{k \notin \partial^+} (p_{k \rightarrow i}^{(1,1)} + p_{k \rightarrow i}^{(2,1)}) + \prod_{k \in \partial^+} (p_{k \rightarrow i}^{(1,2)} + p_{k \rightarrow i}^{(2,2)}) \prod_{k \notin \partial^+} (p_{k \rightarrow i}^{(0,2)} + p_{k \rightarrow i}^{(1,2)} + p_{k \rightarrow i}^{(2,2)}) \]  
(55)

We implement the BPD algorithm as following:

1. Input network \( W \), set all the nodes to be unobserved and set all the cavity message \( p_{i \rightarrow j}^{(c_i,c_j)} \) and \( p_{i \rightarrow j}^{(c_i,c_j)} \) to be uniform message. Set the inverse temperature \( \beta \) to a sufficiently large (depend on the at most convergence inverse temperature). Then iterating the BP equation until to converge one stable point. Finally we compute the occupation probability of each node \( i \) using Eq.(14).

2. Cover the small fraction (e.g., \( \gamma = 0.001 \)) of the unfixed nodes that having highest covering probabilities.

3. Update the state of all the uncovered nodes, such as, if node \( i \) is uncovered and have at least one father neighbor take in the state \( c_i = 0 \), then it takes in the state \( c_i = 1 \), and if node \( i \) is uncovered and have at least one father neighbor take in the state \( c_i = 1 \), then it takes in the state \( c_i = 2 \).

4. Fixing the observed nodes state, namely, if all the child neighbor nodes of the observed node \( c_k = 0 \) is taking in the state \( c_k = 0 \), \( c_k = 1 \), fixed \( c_k = 1 \) or fixed \( c_k = 2 \), then fixing the state of the node \( i \) as \( c_i = 1 \). If all the child
neighbor nodes of the observed node \( c_i = 2 \) is \( c_k = 0 \), \( c_k = 1 \), fixed \( c_k = 1 \) or fixed \( c_k = 2 \), then fixing the state of the node \( i \) as \( c_i = 2 \).

(5) If the network \( W \) still contains unobserved nodes, we then repeat operations (1)-(4) until all nodes are observed.

### III.2 Greedy

We can develop very simple greedy algorithm in the literature to solve the directed 2-distance MDS problem approximately, which is based on the concept of node general impact. The general impact of an unoccupied node \( i \) equals to sum of the impact of all the child neighbor nodes that not be occupied. The impact of an unoccupied node \( i \) equals to the number of child nodes that will be observed by occupying \( i \). Starting from an input network \( W \) with all the nodes unobserved, the greedy algorithm selects uniformly at random a node \( i \) from the subset of nodes with the highest general impact and fix its occupation state to \( c_i = 0 \), and then all the child neighbor nodes and the 2-Distance quasi child neighbor nodes of \( i \) be observed. Fixing the observed nodes state using the step (4) of the BPD implementation process. If there are still unobserved nodes in the network, the impact and general impact value for each of the unoccupied nodes is updated and the greedy occupying process is repeated until all the nodes are observed. This pure greedy algorithm is very easy to implement and very fast, and we find that it usually reach a true directed 2-distance MDS when the input network contains more edges.

The results of the Greedy for the ER random network and RR random network are compared with the results of the BPD algorithm in Figs. 3, 4. The BPD algorithm outperforms the Greedy algorithm, and it gives very close results with the RS theory on the both ER and RR random graph.

### IV Discussion

In this work, we proposed two heuristic algorithms (a Greedy-Impact local algorithm and a BPD message-passing algorithm) and presented a replica-symmetric mean field theory for solving the directed 2-distance MDS problem algorithmically and theoretically. We found that the BP and RS algorithm lead to an entropy transition in the both ER and RR network when the mean degree bigger than some threshold value, but it is not happen when the mean degree smaller than this threshold value(4 for RR network and 6.6 for the ER network). The reason for this result is that the solution space of the directed 2-distance MDS problem on the two networks has a structural transition, we will use one step replica symmetry breaking theory to study the solution space of the directed 2-distance MDS problem. Our numerical results shown in Figs. 3, 4 suggested that the Greedy algorithm and the BPD algorithm can construct near-optimal directed 2-distance MDS for random networks.

There are many theoretical work remaining to be studied. We will work on the one step replica symmetry breaking of the directed 2-distance MDS problem as
soon as possible. A more challenging and common problem in the dominating set is the directed connected dominating set problem, we will use spin glass theory [25] to study the directed minimal connected dominating set problem and the directed 2-distance minimal connected dominating set problem.

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Figure 4: The BPD, Greedy and RS results for the directed 2-Distance MDS problem on the RR random graph with the size of $N = 10^4$ nodes. The x-axis denotes the mean connectivity, and the y-axis denotes the energy density. Inverse temperature $\beta = 10.0$.

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