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LATTICE VIRASORO FROM LATTICE KAC-MOODY

We propose a new version of quantum Miura transformation on the lattice based on the lattice Kac-Moody algebra. In particular, we built Faddeev- Takhtadjan-Volkov lattice Virasoro algebra from lattice current algebra and discuss the possibility of existence of the lattice analogue of the Sugawara construction.

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0. Introduction

Lattice versions of the well-known continuous symmetries became of great interest last time because of their possible relation to some certain models of conformal matter coupled to 2D quantum gravity. The activity in this direction began with the work of Faddeev and Takhtajan [1] on the classical lattice analogue of Liouville model, where for the first time classical lattice version of the Virasoro algebra was obtained. It has been developed further in the context of studying the integrable hierarchies related to matrix models, which seem to be very relevant for the description of the minimal matter coupled to gravity [2]. At the same time, the zoo of the lattice symmetries is being expanded very fast last years, providing a considerable amount of relevant material both for physicists and mathematicians [3-5].

The aim of the present paper is to extend further the analogy between the lattice world and the continuous one, discussing a possible lattice analogue of the well-known Sugawara construction. In particular, starting from the lattice Kac-Moody (LKM) algebra [6-8], we obtain the lattice Virasoro-Faddev-Takhtadjan-Volkov (FTV) algebra [1] in both quantum and classical cases.

1. Lattice Kac-Moody algebra

In this section we give the basic definitions and notations concerning the LKM, following the work [8]. Let $R^\pm$ be the two solutions of the Yang-Baxter equation

$$R^\pm_{12}R^\pm_{13}R^\pm_{23} = R^\pm_{23}R^\pm_{13}R^\pm_{12}$$

associated with the standard representation of $U_q(sl(N))$:

$$R^+ = q^{1/2} \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & q^{-1} - q & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}, \quad R^- = q^{-1/2} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

LKM-algebra $\mathcal{K}_N$ on the lattice with $N$ sites is defined as a free algebra of matrix elements of the matrix

$$J(n) = \begin{pmatrix} J(n)_{11} & J(n)_{12} \\ J(n)_{21} & J(n)_{22} \end{pmatrix} \in \mathcal{K}_N \otimes \text{End}(\mathbb{C}^2)$$

such that

$$J(n)_{11}J(n)_{22} - q^{-1}J(n)_{21}J(n)_{12} = q^{1/2}$$

with the relations

$$J(n)_{1}J(n)_{2} = R^+J(n)_{2}J(n)_{1}R^-, \quad J(n)_{1}J(n+1)_{2} = J(n+1)_{2}(R^+)^{-1}J(n)_{1}, \quad J(n)_{1}J(m)_{2} = J(m)_{2}J(n)_{1} \text{ for } m \neq n - 1, n, n + 1$$

Lattice analogues of quantum vertex operators are defined via the relations [8]:

$$h(n+1) = J(n)h(n)$$

and form the representation of LKM as

$$J(n)_{1}R^-h(n)_{2} = h(n)_{2}J(n)_{1} \quad J(n)_{1}h(n+1)_{2} = R^+h(n+1)_{2}J(n)_{1}$$

Further on we consider an infinite lattice in order to avoid possible problems with boundary conditions.
In the classical limit \( q \to 1 \) one obtains the following Poisson brackets for the currents

\[
\{ J(n)_1, J(n)_2 \} = r^+ J(n)_1 J(n)_2 + J(n)_1 J(n)_2 r^- ,
\]
\[
\{ J(n)_1, J(n + 1)_2 \} = -J(n + 1)_{2r^+} J(n)_1 ,
\]
\[
\{ J(n)_1, J(n - 1)_2 \} = -J(n)_{1r^-} J(n - 1)_2 ,
\]

where classical \( r^\pm \)-matrices defined from the expansion of \( R^\pm \) at \( k \to \infty \) as \( R^\pm = 1 + \frac{k^2}{2k^2} r^\pm \) have the form:

\[
r^+ = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 4 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \quad r^- = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & -4 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

Classical continuous limit is defined as follows \( (x \equiv n\Delta) \)

\[
J_{n1}^{11} \to 1 - \Delta j_{11}(x) + \ldots , \quad J_{n2}^{12} \to -\Delta j_{12}(x) + \ldots ,
\]
\[
J_{n1}^{21} \to \Delta j_{21}(x) + \ldots , \quad J_{n2}^{22} \to 1 - \Delta j_{22}(x) + \ldots
\]

with the condition (4) transformed to

\[
j_{11}(x) + j_{22}(x) = 0
\]

The fields \( j \) form classical \( sl(2) \)-KM algebra

\[
\{ j_{11}(x), j_{21}(y) \} = 2j_{11}(x)\delta(x - y) \quad \{ j_{11}(x), j_{12}(y) \} = -2j_{12}(x)\delta(x - y) \\
\{ j_{11}(x), j_{21}(y) \} = 2\delta'(x - y) + 4j_{11}(x)\delta(x - y) \quad \{ j_{11}(x), j_{11}(y) \} = -\delta'(x - y)
\]

2. Lattice Virasoro from Lattice Kac-Moody.

Recall first the standard \( Lu(1) \) realization of the FTV algebra [1]. If one has \( Lu(1) \)-current with the exchange relations

\[
U_n U_{n+1} = q^2 U_{n+1} U_n
\]

then the quantum Miura transformation gives us the FTV-generators in the following form [3,4]

\[
\sigma_n = 1 + U_n^{-1} + U_{n+1} + q^{-1}U_{n+1}^{-1}U_{n+1}
\]

So defined lattice fields \( \sigma_n \) form quantum FTV algebra [3,4], which will be written below. Classical version of this construction starts from the Poisson brackets

\[
\{ u_n, u_m \} = u_n u_m (\delta_{n,m+1} - \delta_{m,n+1})
\]

The Miura transformation

\[
S_n = \frac{u_{n+1}}{(1 + u_n)(1 + u_{n+1})}
\]

leads to the FTV algebra

\[
\{ S_n, S_{n+1} \} = S_n S_{n+1} (1 - S_n - S_{n+1}) \quad \{ S_n, S_{n+2} \} = -S_n S_{n+1} S_{n+2}
\]
2.1 Quantum case.

In this section we give explicit formulas, expressing the generators of the quantum FTV algebra in terms of the LKM generators. Namely, define the operators

\[ F_n = J_n^{12}J_n^{21}, \quad M_n = \frac{J_n^{12}J_n^{21}}{J_n^{22}J_n^{11}} \]

Direct calculation provides us with the following quantum commutators

\[
[F_n, F_{n+1}] = (1 - q^{-2})F_{n+1}M_{n-1}^{-1}F_n
\]

\[
[M_n, M_{n+1}] = q^{3/2}(q^2 - 1)M_{n+1}^{-1}M_{n+1}
\]

in which the reader acquainted with the abovecited literature on the subject readily recognizes quantum FTV algebra after the identification

\[
\sigma_{2n} = F_n, \quad \sigma_{2n+1} = M_n
\]

Thus, some new type of quantum Miura transformation is defined, dealing not with lattice \(u(1)\) but \(sl(2)\) currents. Some final remarks about the representation of the algebra above should be made. Direct calculation show, that although the Jacobi identity is satisfied in the sector (\(\text{Vir, Vir, } h\), where \(h\) is quantum vertex defined in (6)-(7) no good relation like (6) exist for the fields \(F\) and \(M\). Nonetheless, one can easily check that LKM currents (3) form the quantum representation of the FTV algebra in \(F - M\)-realization

\[ F_nJ_{n+1}^{11} = J_{n+1}^{11}F_n - (1 - q^{-2})F_nM_{n-1}^{-1}F_{n-1}(F_{n-1} + q)^{-1}J_n^{11} \]

\[ F_nJ_{n+1}^{12} = J_{n+1}^{12}F_n - (1 - q^{-2})F_nM_{n-1}^{-1}J_n^{12} \]

\[ F_nJ_{n+1}^{21} = q^{3/2}(q^2 - 1)J_n^{21}F_n + q^{3/2}(q^2 - 1)J_n^{21} \]

\[ F_nJ_{n+1}^{11} = J_{n+1}^{11}F_n + (1 - q^{-2})J_n^{11}F_n + q^{-1}F_{n+1}M_n^{-1}F_n \]

\[ F_nJ_{n+1}^{21} = J_{n+1}^{21}F_n + (1 - q^{-2})J_n^{21}F_n + q^{-1}M_n \]

\[ M_nJ_{n+1}^{11} = q^{-2}J_{n+1}^{11}M_n - (1 - q^{-2})J_n^{11}F_n + q^{-1} \]

\[ M_nJ_{n+1}^{12} = q^{-2}J_{n+1}^{12}M_n - (1 - q^{-2})J_n^{12}M_n \]

\[ M_nJ_{n+1}^{21} = J_{n+1}^{21}M_n - q^{3/2}(q^2 - 1)F_n^{-1}M_n \]

\[ M_nJ_{n+1}^{11} = q^{2}J_{n+1}^{11}M_n + (q^2 - 1)J_n^{11}F_n + q^{-1} \]

\[ M_nJ_{n+1}^{12} = J_{n+1}^{12}M_n + q^{3/2}(q^2 - 1)M_nF_n^{-1}J_n^{12} \]

\[ M_nJ_{n+1}^{21} = q^{2}J_{n+1}^{21}M_n + (q^2 - 1)J_n^{21} \]

2.2 Classical limit.

In this paragraph we study the classical limit of the construction above. First of all, we give the Poisson brackets version of (14).

\[
\{F_n, F_{n+1}\} = F_nF_{n+1} \frac{1}{M_n}
\]

\[
\{M_n, M_{n+1}\} = M_nM_{n+1} \frac{1}{F_{n+1}}
\]

\[
\{F_n, M_n\} = F_nM_n(1 + \frac{1}{F_n} + \frac{1}{M_n})
\]

\[
\{F_{n+1}, M_n\} = -F_{n+1}M_n(1 + \frac{1}{F_{n+1}} + \frac{1}{M_n})
\]
what means that FTV generators themselves are defined as

\[ A_{2n} \equiv \frac{1}{F_n}, \quad A_{2n+1} \equiv \frac{1}{M_n} \]

and form the original FTV-algebra (12).

Using (8) we find in continuous classical limit

\[ F(x) = -T^{\text{root}}(x) = -j_{12}(x)j_{21}(x) \]

\[ A_n \rightarrow -\frac{1}{T^{\text{root}}(x)} + \ldots \] (18)

This allows us to interpret the abovementioned FTV algebra in \( F - M \) realization in terms of certain integrable model. Namely, consider the following pair of Poisson brackets:

\[ \{m_n, f_n\}_1 = m_n f_n, \quad \{m_n, f_{n+1}\}_1 = -m_n f_{n+1} \] (19)

\[ \{f_n, f_{n+1}\}_2 = -m_n f_n f_{n+1}, \quad \{m_n, f_{n+1}\}_2 = -m_n f_{n+1}(m_n + f_{n+1}) \]

\[ \{m_n, m_{n+1}\}_2 = -m_n m_{n+1} f_{n+1}, \quad \{f_n, m_{n+1}\}_2 = -f_n m_n (f_n + m_n) \] (20)

It should be mentioned that the first bracket may be interpreted as induced one via the following construction

\[ m_n = e^{\phi_n - \phi_{n+1}}, \quad f_n = e^{p_n} \]

where \( \phi_n \) and \( p_n \) have canonical bracket

\[ \{\phi_n, p_n\} = 1 \]

The variables \( F_n, M_n \) can be thought about as those inverse to \( f_n, m_n \) respectively. Then the algebra FTV (12) is obtained as the sum of the first and the second brackets introduced above. This pair of brackets generates by means of bihamiltonian procedure an infinite series of integrals in involution

\[ \mathcal{H}^{(1)} = \sum f_n + m_n \]

\[ \mathcal{H}^{(2)} = \sum \frac{f_n^2}{2} + f_n m_n + f_{n+1} m_n + \frac{m_n^2}{2} \] (21)

\[ \ldots \quad \ldots \quad \ldots \]

for the Hamiltonians

\[ \mathcal{H}^{(0)}_m = \sum_n \ln m_n \quad \text{or} \quad \mathcal{H}^{(0)}_f = \sum_n \ln f_n \]

Note that after substituting the explicit form of \( f_n \) and \( m_n \) in (21) and taking continuous limit via the rule (8) one obtains expressions for the hamiltonians, containing the inverse powers of the \( sl(2) \)-currents.

3. A hint for Sugawara construction: pro and contra.

In this section we discuss various aspects of the intriguing problem of building a lattice analogue of the Sugawara construction. First of all, the very concept of the lattice analogue seems to be not very well defined. In the absence of any more clear definition\# one could use the continuous limit of the FTV generator as a starting point. However, even in the most trivial case of \( Lu(1) \) Miura transformation (11) the continuous limit of the FTV generator (12) is not the Sugawara energy-momentum tensor \( T^{(1)}_{u(1) - \text{Sug}} = u^2(x) \) but some twisted one \( T(x) = u^2(x) + \alpha u'(x) \) where \( \alpha \neq 0 \).

\# Say geometrical one, like in continuous WZW model
To understand this specifically lattice difficulties one should recall that \textit{latticization} is a kind of \textit{q-deformation} [9]. General philosophy of quantization is in that generically \textit{splits the degeneracy}. In a wide sense it implies that some objects coinciding in continuous limit may become different on the lattice. For example, the well-known isomorphism between bosons and fermions in two dimensions dissapears after the "latticization". As a more advanced example one can consider the direct correspondence between \textit{twist} superconformal theories and topological ones. In continuous case this correspondence bases on existence of the \textit{twist} operation. However, no satisfactory analogue of such an operation exists to connect these two classes of the models on the lattice [10]. To our conjecture, the continuous Sugawara construction is an example of certain degeneracy. It is well-known, that any Sugawara energy-momentum tensor can be represented as a sum of mutually commuting parts, corresponding to the appropriate subalgebras and coset-spaces. This, in particular, takes place in the framework of the bosonization procedure or BRST quantization. Such a decomposition in the continuous limit is a relevant technical tool necessary for better understanding the structure of the theory. The situation is drastically different on the lattice. Given a number of mutually commuting FTV algebras there is no any local mapping of the type

\[ \mathcal{T} : \text{FTV}_1 \otimes \text{FTV}_2 \otimes \ldots \text{FTV}_N \rightarrow \text{FTV} \]

or in other words there is no any function of the form

\[ S_n = \mathcal{T}(S^{(1)}_{n+a_1}, \ldots, S^{(1)}_{n+a_k}, |S^{(2)}_{n+b_1}, \ldots, S^{(2)}_{n+b_2}| \ldots |S^{(N)}_{n+c_1}, \ldots, S^{(N)}_{n+c_N}) \]

where all the sets \{a_i\}, \{b_i\}, \ldots, \{c_i\} \text{ finite}. Thus the conformal properties of different subsystems cannot be unified. In the case of \textit{Lsl}(2) current algebra one can built three mutually commuting FTV algebras but due to the argument above there is no any FTV algebra for the whole system of the lattice currents. These FTV generators and their continuous limits on the classical level have the form

\[ S^{(1)}_n = \frac{u_{n+1}}{p_n p_{n+1}} \rightarrow I^2(x) + I'(x) \]
\[ S^{(2)}_n = \frac{u_{n+1}}{(1 + u_n)(1 + u_{n+1})} \rightarrow (j_{12} j_{21})(x) - I^2(x) - I'(x) \]
\[ S^{(3)}_n = \frac{h_{n+1}}{(1 + h_n)(1 + h_{n+1})} \rightarrow j_{22}(x)^2 + j_{22}'(x) \]

where

\[ u_n = \frac{F_{n-1}}{M_{n-1}} \]
\[ p_n = 1 - F_n + \frac{F_n}{M_n} \]
\[ h_n = j_n^{22} \]
\[ I(x) = (\log j_{21}(x))^3(x) \]

More generally it can be shown that for each LKM algebra there is a set of \textit{N} mutually commuting FTV algebras where \textit{N} is dimension of the LKM. In continuous limit the sum of all these FTV-fields turns out to be the (twisted) Sugawara energy-momentum tensor. Thus, the specific property of lattice conformal models is that the number of lattice fields necessary to describe their conformal properties is always equal to the number of the fields parametrizing the phase space of the model. Lattice WZW theory is a particular example of this situation. Another example advocating for the validity of this conjecture is from lattice \textit{W}-algebras, recently obtained by the present authors [5]. Namely, for any pair of integer numbers \( N_1 < N_2 \) one cannot extract any \textit{LW}_{N_1}-subalgebra from the \textit{LW}_{N_2}-algebra.

Of course, there is an alternative approach to the lattice conformal invariance based on the notion of \textit{L}-operator, probably more fruitful in the case of lattice WZW-model than the direct one, based on the analysis of FTV-algebras for constituent parts of the model. This approach is another part of the program in progress.
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