Physics of Symmetry Protected Topological phases involving Higher Symmetries and their Applications

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We discuss physical constructions, and the boundary properties of various symmetry protected topological phases that involve 1-form symmetries, from one spatial dimension (1d) to four spatial dimensions (4d). For example, the prototype 3d boundary state of 4d SPT states involving 1-form symmetries can be either a gapless photon phase (quantum electrodynamics) or gapped topological order enriched by 1-form symmetries, namely the loop excitations of these topological orders carry nontrivial 1-form symmetry charges. This study also serves the purpose of diagnosing anomaly of 3d states of matter. Connection between SPT states with 1-form symmetries and condensed matter systems such as quantum dimer models at one lower dimension will also be discussed. Whether a quantum dimer model can have a trivial gapped phase or not depends on the nature of its corresponding bulk state in one higher dimension.

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I. INTRODUCTION

The symmetry protected topological (SPT) phases have greatly enriched our understanding of quantum states of matter. With certain symmetries, the boundary of these SPT states cannot be trivially gapped without degeneracy. Especially, many exotic states of matter can be realized at the 2d boundary of 3d bosonic SPT states. For example, exotic quantum critical points (QCP) in 2d with spatial symmetries (both on the square or triangular lattice) can be realized at the boundary of certain 3d SPT states, and the conjectured emergent symmetry of the deconfined QCP matches well with the bulk symmetry of the SPT state, sometimes these emergent symmetries are only revealed through certain dualities between (2 + 1)d quantum field theories. The analysis of the SPT state in the (d + 1)-dimensional bulk can also be used as a diagnose of the “Lieb-Schultz-Mattis theorem” in d-dimensional systems with spatial symmetries, i.e., whether or not the d-dimensional system can be gapped without degeneracy is related to the nature of the corresponding bulk state in one higher dimension.

In recent years it was realized that the very concept of symmetry can be generalized to higher dimensional objects rather than just point like operators. Examples of SPT states that involve these generalized symmetries were discussed in previous literatures. For example a classification of SPT states based on generalized cobordism theory was given in Ref. 28,30. In the current manuscript we focus on physical construction and boundary properties of a series of SPT states with generalized concepts of symmetries, from (1 + 1)-dimension to (4 + 1)-dimension. We do not seek for exactly soluble models, instead we will focus on general physical pictures of these states. For example, the prototype 4d (or (4 + 1)d) SPT state we will discuss can be constructed by “decorated Dirac monopole loop” picture, which is analogous to the flux attachment construction in 2d SPT state. And the prototype 3d boundary state of the 4d SPT state is a photon phase with various constraints of dynamics, quantum numbers, and statistics on the electric and magnetic charges. We assume that the gauge invariant objects/excitations, i.e. objects that do not couple to dynamical gauge field, are always bosonic. These include point particles and higher dimensional excitations such as loops.

The 1-form symmetry transformation acts on loop-like operators such as the Wilson loop or ’t Hooft loop of a dynamical gauge field. The existence of an electric 1-form symmetry demands that the electric charge of the gauge field is infinitely heavy. In condensed matter systems the quantum dimer model naturally fits this criterion. It is well-known that the quantum dimer model can be mapped to a lattice gauge field. In a quantum dimer model, every site of the lattice is connected to a fixed number of dimers, which implies that there is a background electric charge distribution, but no dynamical charge in the system. Hence the quantum dimer model naturally has a 1-form symmetry. The quantum dimer model on certain d-dimensional lattice may be mapped to the boundary of a (d + 1)-dimensional SPT state with 1-form symmetry in certain limit, and the spatial symmetries of the quantum dimer model is mapped to the onsite symmetry of the bulk SPT state. The analysis of the SPT state in the bulk has strong indications on the allowed phenomena of the quantum dimer model at d-dimension.

Due to the inevitable complexity of notations used in this manuscript, we will keep a self-consistent conventions of notations:

The N–form symmetry G will be labelled as $G^{(N)}$, such as U(1)$^{(1)}$, Z$^{(1)}$, etc. Ordinary 0-form symmetry will be labelled without superscript.

Gauge symmetries associated with dynamical gauge field will be labelled as u(1)$^{(1)}$, Z$^{(2)}$, etc. depending on
the nature of the gauge fields. A topological order which corresponds to a dynamical discrete gauge field will also be labelled as, for example, a $z_n$ topological order.

Gauge symmetries associated with background gauge fields will be labelled as $U(1)^{(1)}$, $Z^{(2)}_n$, etc.

Classifications of SPT states will be labelled as $Z$, $Z_n$, etc.

For space and space-time dimensions, for example, $3d$ space refers to three spatial dimensions; $(3 + 1)d$ refers to the space-time dimension, which is the same as $4D$ Euclidean space-time. Also, QED$_4$ refers to quantum electrodynamics in $(3 + 1)d$ or $4D$ space-time dimension.

For a QED$_4$, there are point like particles such as electric charge, and Dirac monopole. We label bosonic (fermionic) electric charges as $e_b$ ($e_f$), and bosonic (fermionic) Dirac monopoles as $m_b$ ($m_f$). Some of these point excitations have no dynamics (infinitely heavy) due to the 1-form symmetries, we will label these immobile point particles as $e_{ob}$, $e_{of}$, etc. A QED$_4$ with bosonic electric charge and fermionic Dirac monopole is labelled as “QED$_4$($e_b, m_f$)".

II. BUILDING BRICKS: 1d SPT STATE WITH 1-FORM SYMMETRIES

The simplest SPT state that involves a 1-form symmetry exists in 1d space or $(1 + 1)d$ space-time. 1d SPT state with a 1-form symmetry is analogous to an ordinary SPT state in 1d space. For a U(1)$^{(1)}$ 1-form symmetry, a SPT state in 1d simply corresponds to a state with integer electric flux through the system. Let us take a 1d chain with electric field operators defined on the links. Due to the Gauss law constraint, $\nabla_x \hat{e}(x) = 0$, the electric field operator $\hat{e}(x)$ takes a uniform integer eigenvalue on the entire chain (in a compact $u(1)$ lattice gauge theory, the electric field operator $\hat{e}(x)$ takes discrete integer value, while its conjugate operator $\hat{a}(x)$ is periodically defined), hence for a U(1)$^{(1)}$ 1-form symmetry, the classification of 1d SPT states is $Z$, which corresponds to different integer eigenvalues of $\hat{e}(x)$. It is analogous to the Z classification of a zero dimensional ordinary SPT state with U(1) symmetry.

The Hamiltonian of a 1d lattice U(1) gauge field is also very simple, for example:

$$H = \sum_x g (\hat{e}(x) - k)^2.$$  

(1)

Due to the Gauss law constraint, a Hamiltonian must be invariant under gauge transformations $\hat{a} \rightarrow \hat{a} + \nabla_x f(x)$, where $\hat{a}$ is the conjugate operator of $\hat{e}$. A local 1d Hamiltonian that involves $\hat{a}$ cannot be gauge invariant, hence a local gauge invariant Hamiltonian is only a function of $\hat{e}$. In Eq. 1, $k$ can take continuous values. When $k$ is half integer, the system is at the transition between two SPT states, and the ground state of the Hamiltonian is two-fold degenerate with $\hat{e}(x) = k \pm 1/2$, namely the transition is a level crossing between two eigenvalues of $\hat{e}(x)$. This transition should be viewed as a first order transition.

One can also couple the electric field to a background 2-form $\mathcal{U}(1)^{(2)}$ gauge field:

$$S = \int d\tau dx \, i f_{\mu\nu} B_{\mu\nu}$$  

(2)

In $(1 + 1)d$ the stress tensor of the $u(1)$ gauge field is just the electric field: $f_{00} = \hat{e}(x)$, and $B_{01} = -B_{10}$ is a Lagrange multiplier. Hence the $(1 + 1)d$ topological response theory for the SPT state is

$$S_{1d-\text{topo}} = \int (1+1)d \, i k B,$$  

(3)

which is a $(1 + 1)d$ Chern-Simons action of the 2-form gauge field $B$, and its level $k$ takes only integer values. For each integer level--$k$, the electric field (the 1-form symmetry charge)

$$e(x) = \frac{\delta S_{1d-\text{topo}}}{\delta B(x)} = k.$$  

(4)

The 1d SPT state with 1-form symmetries will be the building bricks for SPT states in higher dimensions. Suppose we break the $U(1)^{(1)}$ down to $Z_n^{(1)}$ symmetry, the topological response theory Eq. 3 still applies, but $B$ is now a 2-form $Z_n^{(2)}$ background gauge field. The classification of the SPT state will reduce to $Z_n$, which means that in Eq. 3 the integer $k + n = k$.

III. 4d SPT STATES WITH $G_1^{(1)} \times G_2^{(1)}$ SYMMETRY

A. Parent 4d SPT state with $U(1)^{(1)} \times U(1)^{(1)}$ symmetry

We now discuss SPT states in 4d space that involves 1-form symmetries. This discussion is useful for diagnosing anomalies of 3d states of matter, namely some 3d states of matter can only be realized at the boundary of a 4d SPT state. The parent SPT state that we will start with is the $(4+1)d$ state with the U(1)$^{(1)} \times$U(1)$^{(1)}$ 1-form symmetry. With two U(1)$^{(1)}$ 1-form symmetries, the system can couple to two background $U(1)^{(2)}$ 2-form gauge fields $B^1$ and $B^2$, and the response theory in $(4 + 1)d$ reads

$$S_{4d-\text{topo}} = \int (4+1)d \, \frac{i k}{4\pi} \epsilon_{IJ} B^I \wedge dB^J,$$  

(5)

where $\epsilon_{IJ} = \imath \sigma^\theta$. For each integer $k$, Eq. 5 is a different Chern-Simons theory, and the system should correspond to a different SPT state, hence these SPT states described by Eq. 5 have a $Z$ classification. The $(3 + 1)d$ boundary of this state is a QED$_4$ without dynamical electric or magnetic charge (Dirac monopole). This QED$_4$ has a U(1)$^{(1)} \times$U(1)$^{(1)}$ mixed ’t Hooft anomaly as was derived in previous literatures.$^{18,21,22}$
To construct this 4d SPT state, we can start with two $(4 + 1)d$ $u(1)$ gauge fields $\vec{a}^1$ and $\vec{a}^2$. These two gauge fields both have electric 1-form $U(1)^{(1)}$ symmetry, namely both gauge fields have no dynamical electric charges, i.e. the Gauss law constraint on the electric field is strictly enforced. This is equivalent to tuning the electric charges in the 4d bulk to be infinitely heavy. Both $u(1)$ gauge fields allow dynamical Dirac monopole loop/line defects in the 4d space. We will first discuss the cases where the charges of $\vec{a}^1$ and $\vec{a}^2$ are both bosons, otherwise $\vec{a}^1$ and $\vec{a}^2$ would be Spin$^C$ connections. Situations with fermionic gauge charges of $\vec{a}^1$ and $\vec{a}^2$ will be discussed later.

We use the analogue of the “flux attachment” (or “decorated defect”) construction of the SPT state which was used to construct 2d bosonic SPT states. In 2d space, a $U(1) \times U(1)$ SPT state (the parent state of many 2d SPT states) can be constructed by binding the vortex defect of one $U(1)$ symmetry with the charge of the other $U(1)$ symmetry, and condense the bound state, which drives the system into a gapped SPT phase. In 4d space, the analogue of the vortex defect of an ordinary $U(1)$ 0-form symmetry, is the Dirac monopole loop/line of a $u(1)$ gauge field. We decorate the Dirac monopole loop of $\vec{a}^2$ with the 1d SPT state defined with the 1-form symmetry associated with $\vec{a}^2$ with level $(+k)$ in Eq. 5 and condense/proliferate the decorated loops (Fig. 1). Once the bound state between the monopole loop of $\vec{a}^2$ and the $(+k)$ unit of electric flux of $\vec{a}^2$ is condensed, the monopole loop of $\vec{a}^2$ will be automatically bound with $(-k)$ unit of electric flux of $\vec{a}^1$.

Condensation of Dirac monopole loops would normally drive a $(4 + 1)d$ $u(1)$ gauge field to the gapped confined phase (the loop excitation is coupled to a dual dynamical 2-form gauge field, and the condensate is gapped due to the Higgs mechanism). But because the Dirac monopole loop is decorated with another SPT state with 1-form symmetry in our case, after the condensation of the decorated monopole loops, the phase in the 4d bulk is not an ordinary confined phase, it is actually a SPT phase described by Eq. 5. In fact, Eq. 5 directly implies that the 1-form symmetry charge (electric field) $\vec{e}^2(x)$, which is the variation $\delta S_{4d-topo}/(i dB_{(2)}^2)$, equals to the flux of $B^2$, which is attached to the monopole of $\vec{a}^1$.

The 3d boundary of the 4d SPT state is most naturally a $(3 + 1)d$ QED$_4$ with both magnetic and electric 1-form symmetries. The electric 1-form symmetry of the boundary QED$_4$ is inherited from the 1-form symmetry of $\vec{a}^2$ in the bulk, while the magnetic 1-form symmetry of the QED$_4$ corresponds to the electric 1-form symmetry of $\vec{a}^2$ in the bulk, because the Dirac monopole line of $\vec{a}^2$ in the 4d bulk is bound/decorated with the electric 1-form symmetry charge of $\vec{a}^2$. As we mentioned previously, we will first discuss the situation with bosonic point particles, hence in this QED$_4$ the infinitely heavy electric charge and Dirac monopoles are both bosons. We label this QED$_4$ as QED$_4\{\epsilon_{0k}, m_{0k}\}$. Even though these point particles have infinite mass, their statistics still matter, because their Wilson loops (or ‘t Hooft loops) still exist. If these point particles are fermions, the Wilson loop will need a framing structure, and the Wilson loop or ‘t Hooft loop with a twist will acquire a minus sign.

B. Descendant 4d SPT state with $U(1)^{(1)} \times Z_n^{(1)}$ symmetry

Now we break one of the $U(1)^{(1)}$ 1-form symmetry down to the $Z_n^{(1)}$ symmetry. The topological response theory remains unchanged from Eq. 5 although one of the background 2-form gauge fields will become a $Z_n^{(2)}$ background 2-form gauge field. The decorated monopole line construction discussed in the previous section still applies here. One key difference is that, because the 1d SPT phase with $Z_n^{(1)}$ 1-form symmetry has a $Z_n$ classification itself, the flux attachment or decorated defect construction mentioned in the previous subsection will naturally lead to a $Z_n$ classification of the 4d SPT state also. Namely, when $k = n$ in Eq. 5 this bulk SPT state will be trivialized, because the 1d SPT state decorated on the Dirac monopole line is trivial.

We can always start with the QED$_4$ as a candidate boundary state. Now since the magnetic 1-form symmetry is only $Z_n^{(1)}$, it means that there are dynamical Dirac monopoles with $n$-magnetic charges (Dirac monopole with $2\pi n$ flux quantum). As we mentioned before we first focus on the cases where the point excitations are bosons, then we can condense the $n$-magnetic charge at the 3d boundary without breaking any symmetry. The condensate of the $2\pi n$ Dirac monopole will drive the boundary into a $3d$ $z_n$ topological order.

An ordinary 3d $z_n$ topological order is the deconfined phase of a dynamical $z_n^{(1)}$ gauge field. In an ordinary 3d $z_n$ topological order, normally there are two types of excitations: a point particle which is the remnant of the $2\pi$ Dirac monopole; and also another line/loop excitation which is coupled to a $z_n^{(2)}$ 2-form gauge field. If
the loop excitation is condensed (proliferated in 4D Euclidean space), the $z_n$ topological order is trivialized, and the system becomes gapped and nondegenerate.

The dynamics of the loop excitation can be schematically described by the following Hamiltonian

$$H_{\text{loop}} = \sum_{\mathcal{C}} -t_{\mathcal{C}} \cos \left( \sum_{\mathcal{C}} \hat{c}_\mathcal{C} - \sum_{\mathcal{C}} \hat{b}_\mathcal{C} \right) + \cdots \quad (6)$$

In this equation, $\mathcal{C}$ represents certain loop configuration; $\hat{c}_\mathcal{C}$ is a link which is part of this loop, and $\mathcal{A}_\mathcal{C}$ is a membrane whose boundary is the loop $\mathcal{C}$ ($\partial \mathcal{A}_\mathcal{C} = \mathcal{C}$); $\hat{b}_\mathcal{C}$ is a plaquette that belongs to $\mathcal{A}_\mathcal{C}$. $\Psi^\dagger_i \sim \exp(i\hat{c}_\mathcal{C})$ is the creation operator of the loop segment on link $\mathcal{C}$, and $\hat{b}_\mathcal{C}$ is a 2-form gauge field defined on plaquette $\mathcal{C}$. The direction of the link and the unit plaquette can be absorbed into the definition of $\hat{c}$ and $\hat{b}$ and render them a 1-form and 2-form fields.

For an ordinary $z_n$ topological order, both $\hat{c}_\mathcal{C}$ and $\hat{b}_\mathcal{C}$ take eigenvalues $2\pi N/n$ with integer $N$. Hence the “condensation” of the loop excitation will not lead to degeneracy because of the existence of the $z_n^{(2)}$ 2-form gauge field $\hat{b}$. Or in other words, the condensation of the loop excitation will be fully “Higgsed” due to the coupling to the $z_n^{(2)}$ dynamical gauge field $\hat{b}$, and this Higgs phase is the confined phase of the $z_n^{(1)}$ gauge theory.

However, if the loop excitation carries a $U(1)^{(1)}$ 1-form charge, the situation would be very different. Now $\hat{c}_\mathcal{C}$ can take continuous values between 0 and $2\pi$. Condensing the loop would just drive the system back into a gapless photon phase. Physically because the loop excitation carries a $U(1)^{(1)}$ 1-form charge, condensing the loop excitations would lead to spontaneous $U(1)^{(1)}$ 1-form symmetry breaking, whose “Goldstone mode” is precisely the photon.

With the bulk response action Eq. 5, the loop excitation of 3d boundary carries charge quantum $k/n$ of the $U(1)^{(1)}$ 1-form symmetry. However, when $k = n$, the quantum number of the loop excitation can be screened by binding with unfractioanlized integer 1-form symmetry charge, hence the loop excitations become completely neutralized. Then when $k = n$ the neutralized loop excitation can proliferate and drive the boundary to a fully gapped and nondegenerate state, just like the case of an ordinary $z_n^{(1)}$ gauge theory. This argument again leads to a $Z_n$ classification.

C. Descendant 4d SPT state with $Z_q^{(1)} \times Z_n^{(1)}$ symmetry

We can further break the left $U(1)^{(1)}$ 1-form symmetry down to $Z_q^{(1)}$ from the previous example. Now in the condensate of the $2\pi n$ Dirac monopole, the loop excitation will carry $k/n$ unit of the $Z_q^{(1)}$ 1-form symmetry charge, and the loop excitation is coupled to a dual $z_n^{(2)}$ gauge field. Our interest is to ask when this 3d boundary can be fully gapped without degeneracy.

Let us start with the simple example with $k = 1$, $q = 3$, and $n = 2$. Following the discussion in the previous subsection, we consider the $z_2$ topological order after condensing the $4\pi$ Dirac monopole at the boundary $QED_4$ (The $2\pi n$ monopole has dynamics and can condense). There is a loop excitation of this $z_2$ topological order, which couples to a dual $z_2^{(2)}$ gauge field, and carries half charge of the $Z_3^{(1)}$ 1-form symmetry. Now consider a loop excitation whose creation operator is $P_C^{(1)}$:

$$P_C^{(1)} \sim \prod_{\mathcal{C}} \Psi_{\mathcal{C}}^\dagger \sim \exp(i\sum_{\mathcal{C}} \hat{c}_\mathcal{C}). \quad (7)$$

$P_C^{(1)}$ carries half charge under $Z_3^{(1)}$, and it also couples to a dual $z_2^{(2)}$ gauge field. Under both the $Z_3^{(1)}$ symmetry and the $z_2^{(2)}$ gauge symmetry, $C$ transforms as

$$Z_3^{(1)} : P_C^{(1)} \rightarrow e^{i \frac{2\pi}{3}} P_C^{(1)},$$

$$z_2^{(2)} \text{\ gauage} : P_C^{(1)} \rightarrow -P_C^{(1)}, \quad (8)$$

with integer $N$. One can check that by combining the loop operator $P_C$ with unfractioanlized integer 1-form charges, the $Z_q^{(1)}$ transformation can be completely cancelled by a $z_n^{(2)}$ gauge transformation. In other words the fractional $Z_q^{(1)}$ charge carried by the $P_C^{(1)}$ can be “neutralized” by binding a gauge invariant $Z_3^{(1)}$ charge, and the 3d boundary system can be driven into a trivial gapped phase by condensing this $Z_3^{(1)}$ neutral loop excitation.

The discussions above can be generalized to other $q$ and $n$. With $k = 1$ in Eq. 4 after condensing the $2\pi n$ monopole, the 3d boundary system is driven into a $z_n$ topological order whose loop excitation carries $1/n$ fractional $Z_q^{(1)}$ 1-form symmetry charge. Our interest is to check, when this fractional 1-form symmetry charge can be “neutralized” by integer 1-form symmetry charge, namely by binding integer 1-form symmetry charge the $Z_q^{(1)}$ transformation can be completely absorbed/cancelled by the dual $z_n^{(2)}$ gauge transformation.

Under a $Z_q^{(1)}$ transformation, the loop creation operator $P_C^{(1)}$ acquires phase angle $2\pi/(nq)$; after binding with $Q$ units of integer $Z_q^{(1)}$ charge, the loop would acquire phase angle $2\pi/(nq) + 2\pi Q/q$. Now we seek for a pair of integer $(Q, N)$ which satisfies the following equation:

$$\frac{1}{nq} + \frac{Q}{q} = \frac{N}{n}. \quad (9)$$

This would mean that the $Z_q^{(1)}$ transformation can be totally absorbed/cancelled by a gauge transformation. For $(q, n) = (3, 2)$ one can choose $(Q, N) = (1, 1)$. In general the question is equivalent to finding a pair of integers $(Q, N)$ that satisfies $Nq - Qn = 1$, which is only possible
when $q$ and $n$ are coprime. When $q$ and $n$ are not coprime, the loop quantum number can be fully neutralized when $k = \gcd(q, n)$. This implies a $\mathbb{Z}_{\gcd(q, n)}$ classification.

— More States

All the SPT states discussed so far have bosonic electric charge and Dirac monopoles at its boundary QED$_4$, namely the boundary of all the SPT states are QED$_4\{\epsilon_{0b}, m_{0b}\}$ states. Let us revisit the starting point of our bulk construction of Eq. 5. The two $u(1)$ gauge fields $\vec{a}^1$ and $\vec{a}^2$ can have either bosonic or fermionic electric charges with infinite mass in the bulk, which become the static electric charges and Dirac monopoles of the boundary QED$_4$. Hence logically there will also be QED$_4\{\epsilon_{0b}, m_{0f}\}$, QED$_4\{\epsilon_{af}, m_{0b}\}$, QED$_4\{\epsilon_{af}, m_{0f}\}$ states that we need to discuss. As we pointed out before, the statistics of static particles still affect the Wilson/'t Hooft loops. We defer discussions of these states to section V.

IV. 4d SPT STATE WITH U(1)$^{(1)} \times G$ SYMMETRY AND 3d QUANTUM DIMER MODEL

Here we consider 4d SPT states with both a U(1)$^{(1)}$ symmetry and an ordinary 0-form symmetry $G$. The decorated defect construction in the previous section can be generalized here: we start with one $(4+1)d$ $u(1)$ gauge field $\vec{a}$ with a 1-form electric symmetry, and decorate its Dirac monopole line with the 1d SPT state with symmetry $G$, then condense the monopole line in the bulk. A prototype 4d SPT state with such construction was discussed previously, whose $G$ symmetry is SO(3), and its topological response theory is.

$$S_{4d-\text{topo}} = i\pi \int_{(4+1)d} w_2[A^{SO(3)}] \cup \frac{dB}{2\pi},$$

where $A^{SO(3)}$ is the external 1-form SO(3) gauge field.

Generally speaking the discussion of 4d SPT state with 1-form symmetry has implications on properties of 3d systems with loop-like excitations. If in certain limit a 3d system with spatial symmetries can be mapped to the boundary of a 4d state with onsite symmetries, then whether or not the 4d bulk is a nontrivial SPT state has strong implication on whether the 3d system can be trivially gapped or not, i.e. the nature of the 4d bulk helps us prove a Lieb-Schultz-Mattis (LSM) theorem$^{35,36}$ of the 3d system. In recent years much progress has been made in understanding the LSM theorems for quantum spin systems using the anomaly analysis of its corresponding higher dimensional bulk states$^{37,38,39}$. In condensed matter theories the quantum dimer model is an example of systems with loop like excitations. Dimers are defined on the links of the lattice, and each site of the lattice is connected to a fixed number of dimers. Previous literature has shown that, the 3d quantum dimer model can be mapped to a QED$_4$ without dynamical electric charge$^{40}$, but its monopole can carry nontrivial quantum number under spatial group due to the Berry phase, and in particular, for the quantum dimer model on the cubic lattice, the monopole of the QED$_4$ carries a “spin-1/2” representation (projective representation) of an emergent SO(3) symmetry$^{40}$. Hence this quantum dimer model is analogous to the boundary of a 4d SPT state with symmetry U(1)$^{(1)} \times SO(3)$, and there should be a LSM theorem for this quantum dimer model.

This LSM theorem for the quantum dimer model is consistent with the LSM theorem for spin-1/2 systems on the cubic lattice. In Ref. 7 various quantum spin systems on the cubic lattice were considered. For example, a SU($N$) spin system on the cubic lattice with fundamental and antifundamental representations on the two sublattices of the cubic lattice has a LSM theorem for even integer $N$, but there is no LSM theorem for odd integer $N$, i.e. the quantum spin system described above with odd integer $N$ can have a featureless gapped ground state on the cubic lattice. However, a quantum dimer model on the cubic lattice could be the low energy effective description of all these systems, since two nearest neighbor AB sites can always form a dimer (spin singlet), regardless of even or odd integer $N$.

One simple extension of Eq. 11 is that, when we break SO(3) down to its subgroup U(1) $\times Z_2$, Eq. 11 reduces to

$$S_{4d-\text{topo}} = \frac{\Theta}{(2\pi)^2} \int_{(4+1)d} dB \wedge dA,$$

where $A$ is the background U(1) gauge field. The integral in Eq. 11 is quantized, hence $\Theta$ is periodically defined: $\Theta = \Theta + 2\pi$. Under the $Z_2$ subgroup of SO(3), $A$ changes sign, hence a symmetric response theory demands $\Theta = k\pi$ with integer $k$. Eq. 11 with $k = 1$ corresponds to the nontrivial 4d SPT phase.

Eq. 11 also describes the corresponding 4d bulk state if instead we consider a quantum dimer model defined on a 3d tetragonal lattice, here the U(1) symmetry is further reduced to a $Z_4$ symmetry, and the $Z_4$ corresponds to the rotation of the square lattice in each layer. In this case in the topological response theory Eq. 11 $A$ is a background $Z_4$ gauge field. Eq. 11 still describes a nontrivial 4d SPT state with 1-form symmetry.

The situation will be very different if we consider a quantum dimer model on a 3d bipartite lattice with an effective $Z_3 \times Z_2 = S^3$ symmetry. The $Z_3$ should correspond to a three fold rotation $C_3$ in the XY plane, and $Z_2$ is a $\pi$-rotation about the x-axis. Such quantum dimer models can potentially be mapped to the boundary of a 4d system with U(1)$^{(1)} \times S^3$ symmetry. But there is no 1d SPT state with the $S^3$ symmetry, hence the 4d bulk with the U(1)$^{(1)} \times S^3$ symmetry is also trivial as a descendant state of the SPT state described by Eq. 11. Hence there should be no LSM theorem for these quantum dimer models, i.e. these quantum dimer models can in general have a gapped ground state without degeneracy, unless this model has higher symmetries than the lattice itself.
V. OTHER 4d SPT STATES

With just a $U(1)^{(1)}$ symmetry, there is already a nontrivial 4d SPT phase, whose boundary is a QED$_4$ with a 1-form electric symmetry, and the Dirac monopole is a fermion (labelled as $m_f$). The unit electric charge (labelled as $e_{0b}$) is infinitely heavy at the boundary QED$_4$ due to the $U(1)^{(1)}$ symmetry. We label this boundary QED$_4$ as state QED$_4^0(e_{0b}, m_f)$. The bulk is a nontrivial SPT state, namely its boundary QED$_4$ cannot be trivially gapped. One can condense a Cooper pair of the fermionic Dirac monopole $m_f$, and drive the QED$_4$ to a “monopole superconductor”, which is also a $z_2$ topological order. The loop excitation of the $z_2$ topological order will carry a fractional half charge of the $U(1)^{(1)}$ 1-form symmetry, and hence cannot lead to a fully gapped and nondegenerate state after condensation for the reasons explained previously in this manuscript. Although the electric charges are infinitely heavy due to the 1-form symmetry, its statistics still matters to physical observables such as the Wilson loops of the QED$_4$. And in this QED$_4$ the infinitely heavy electric charge is a boson.

This state remains a nontrivial SPT after breaking the $U(1)^{(1)}$ down to $Z_2^n$ with even integer $n$, the cases with $n = 2, 4$ were discussed in Ref. [27]. But this state will be trivialized if $n$ is an odd integer. For odd integer $n$, in the monopole superconductor constructed above, the loop excitation carries half charge of the $Z_2^n$ 1-form symmetry, and it can be “neutralized” by binding un fraction alized 1-form symmetry charge, i.e. the $Z_2^n$ transformation on the loop excitation can be completely cancelled by the $z_2^{(2)}$ gauge transformation on the loop excitation, then the condensation of the neutralized loop can lead to a trivially gapped phase.

There is even a nontrivial bosonic SPT state in 4d space without any symmetry; its boundary is a QED$_4$ whose both electric charge and Dirac monopole (including their bound state dyon) are fermions. We label this QED$_4$ as QED$_4^0(e_f, m_f')$. We view the QED$_4^0(e_f, m_f')$ and QED$_4^0(e_f, m_f')$ as two root states, and by “gluing” these two QED$_4$ states together, another new state can be constructed. One can condense the bound state of the Dirac monopoles (labelled as $(m_f, m_f')$) of both QED$_4$ systems, then the gauge fields from both QED$_4$ will be identified due to the Higgs mechanism, and $e_{0b}$ and $e_f$ are both confined since they both have nontrivial statistics with the condensed bound state of monopoles. Although $e_{0b}$ is infinitely heavy, its confinement can still be defined by the behavior of Wilson loop of its gauge field. In the condensed phase of bound state $(m_f, m_f')$, the Wilson loop of each individual gauge field obeys the area law. But the bound state $(e_{0b}, -e_f')$, which has trivial mutual statistics with $(m_f, m_f')$, remains deconfined, though it is still infinitely heavy. This new QED state has infinitely heavy fermionic electric charge, and dynamical fermionic Dirac monopole. This new state is labelled as QED$_4^1(e_f, m_f)$. One can also exchange $e$ and $m$, and label the state as QED$_4^1(e_f, m_f)$, i.e. a state with dynamical fermionic gauge charge, but infinitely heavy fermionic Dirac monopole.

**Summary of 4d SPT states with 1-form symmetries:**

Let us reinvestigate the states discussed in the end of section II. As we briefly discussed there, besides the states QED$_4^0(e_{0b}, m_f)$, logically there should also be QED$_4^1(e_{0b}, m_f)$, QED$_4^1(e_f, m_f)$, QED$_4^1(e_f, m_f)$, which can all be boundary states of (4+1)d SPT bulk. It turns out that these states can be constructed by gluing states in section III and V. For example, starting with the state QED$_4^0(e_{0b}, m_f)$ discussed in section III (we label its gauge field as $\vec{a}$), one can combine it with the state QED$_4^1(e_{0b}', m_f')$ (with gauge field $\vec{a}'$) discussed in section V and consider the charge bound state $(e_{0b}, e_{0b}')$. This bound state carries zero total gauge charge of $\vec{a}$ and $\vec{a}'$. We assume that there is only one $U(1)^{(1)}$ 1-form symmetry, hence the charge bound state $(e_{0b}, -e_{0b}')$, which carries zero total gauge charge, is no longer necessarily infinitely heavy and can acquire dynamics and condense. Its condensate would render $\vec{a} = \vec{a}'$ through the Higgs mechanism, and in the condensate the monopole bound state $(m_{0b}, m_{0b}')$ remains deconfined, as it has trivial mutual statistics with $(e_{0b}, -e_{0b}')$. The final state is identical to state QED$_4^0(e_{0b}, m_f)$ discussed in section III. Following the same argument, through gluing QED$_4^1(e_{0b}, m_f)$ and state QED$_4^1(e_f', m_f')$ discussed in section V (by condensing the bound state $(m_{0f}, -m_{0f}')$), one can obtain another state QED$_4^1(e_f, m_f)$ discussed in section III.

The construction of all these states discussed so far can be summarized mathematically in a single unified topological response theory in the (4+1)d bulk:

$$ S_{4d\text{-topo}} = \int_{(4+1)d} \frac{i k_0}{2 \pi} dB^1 \wedge d B^2 + \frac{i k_1}{2} dB^1 \cup w_2 + \frac{i k_2}{2} d B^2 \cup w_2 + i \pi k_3 w_2 \cup w_3. $$

$w_2$ and $w_3$ are the second and third Stiefel-Whitney class of the space-time manifold. $k_0$ takes arbitrary integer values, while $k_1$, $k_2$ and $k_3$ only take value 0 and 1, since the Stiefel-Whitney class is defined mod 2. This topological response theory is equivalent to the discussion based on the cobordism theory in Ref. [27].

The classification of 4d SPT states discussed so far is summarized as follows:

$$ U(1)^{(1)} : \mathbb{Z}_2 \otimes \mathbb{Z}_2; $$

$$ Z^{(1)}_{n} : \mathbb{Z}_2 \otimes \mathbb{Z}_2 \otimes \text{gcd}(2, n); $$

$$ U(1)^{(1)} \times U(1)^{(1)} : \mathbb{Z} \otimes \mathbb{Z}_2^3; $$

$$ U(1)^{(1)} \times Z^{(1)}_n : \mathbb{Z} \otimes \mathbb{Z}_2^3 \otimes \text{gcd}(2, n); $$

$$ Z^{(1)}_q \times Z^{(1)}_n : \mathbb{Z}_{\text{gcd}(q, n)} \otimes \mathbb{Z}_{\text{gcd}(2, q)} \otimes \mathbb{Z}_{\text{gcd}(2, n)} \otimes \mathbb{Z}_2. $$

(13)
VI. 3d SPT STATE WITH $G_1^{(1)} \times G_2$ SYMMETRY

A. Parent 3d SPT state with $U(1)^{(1)} \times U(1)$ symmetry

The parent 3d SPT state we will consider, is a state with $U(1)^{(1)} \times U(1)$ symmetry. We can couple its symmetry currents to a background 2-form gauge field $B$, and a 1-form gauge field $A$. The response theory for this SPT state is

$$S_{3d-topo} = \frac{ik}{2\pi} B \wedge dA = \frac{ik}{2\pi} A \wedge dB.$$  \hspace{1cm} (14)

To construct such state, again one can rely on the decorated defect picture. We can start with a photon phase with an electric $U(1)^{(1)}$ 1-form symmetry, namely there is no dynamical electric charge, or equivalently the electric charge is infinitely heavy, but there are dynamical Dirac monopoles. Then we decorate the Dirac monopole with a zero dimensional bosonic SPT state with $U(1)$ symmetry, which is a bosonic charge with $U(1)$ symmetry. This zero dimensional bosonic SPT state has $Z$ classification, which correspond to states with integer charges of a boson with $U(1)$ symmetry. These states can also be equivalently constructed by decorating the vortex line of the $U(1)$ order parameter with a 1d SPT state with $U(1)^{(1)}$ 1-form symmetry, i.e. the building bricks discussed in section [11]

After condensing the decorated Dirac monopole, the 3d bulk of the system is driven into a fully gapped state without degeneracy. The 2d boundary of the system would most naturally be a QED$_3$ whose dynamical $u(1)$ gauge field $\vec{a}$ has no dynamical gauge charge, but its magnetic flux carries conserved $U(1)$ quantum number that couples to $A$. The QED$_3$ is a dual of the superfluid phase with spontaneous breaking of the $U(1)$ symmetry. And the assumption that there is no dynamical electric charge of gauge field $\vec{a}$ is equivalent to the statement that there is no dynamical vortex of the dual superfluid, hence the superfluid cannot be disordered by condensing the vortices.

B. Descendant 3d SPT state with $U(1)^{(1)} \times \mathbb{Z}_n$ symmetry

We can break the $U(1)$ 0-form symmetry coupled to $A$ in Eq. (14) down to a $\mathbb{Z}_n$ symmetry, now the entire symmetry becomes $U(1)^{(1)} \times \mathbb{Z}_n$. The topological response theory Eq. (14) still applies, but now $A$ becomes a $\mathbb{Z}_n^{(1)}$ background gauge field. The decorated defect construction in the previous case would lead to a $\mathbb{Z}_n$ classification, because the zero dimensional SPT state with $\mathbb{Z}_n$ symmetry decorated at the Dirac monopole has a $\mathbb{Z}_n$ classification.

This classification can be understood at the boundary as well. The $(2+1)d$ boundary is a QED$_3$ whose flux carries $k$ units of the $\mathbb{Z}_n$ quantum number, where $k$ is given in Eq. (14) With $k = n$, the flux of the QED$_3$ basically carries trivial quantum number, and the QED$_3$ can be driven into a trivial confined phase. This boundary state is similar to the quantum dimer model on a 2d bipartite lattice, such as the square lattice. The quantum dimer model can be mapped to a compact QED$_3$ with no electric charge (the quantum dimer constraint, i.e. every site is connected to precisely one dimer, is strictly enforced), but the flux of the compact QED$_3$ carries nontrivial lattice quantum number. The description of the quantum dimer model in terms of QED$_3$ is analogous to the boundary of the 3d SPT state with $U(1)^{(1)} \times \mathbb{Z}_4$ symmetry at $k = 1$. It is well-known that the confined phase of the quantum dimer model on the square lattice cannot be a trivial gapped phase, instead it must have ground degeneracy due to spontaneous breaking of lattice symmetry. But in the quantum dimer model because the $\mathbb{Z}_4$ symmetry is a non-on-site lattice symmetry, the quantum dimer model exists as a well defined system in 2d.

This effect is inherited from the LSM theorem for spin-1/2 systems on the square lattice. There is no LSM theorem for a spin-2 system on the square lattice, and a spin-2 system can be viewed as four copies of spin-1/2 systems glued together, or a system with four spin-1/2s in each unit cell. All these observations are consistent with the $\mathbb{Z}_4$ classification of the 3d SPT state with $U(1)^{(1)} \times \mathbb{Z}_4$ symmetry discussed in this section.

C. Descendant 3d SPT state with $\mathbb{Z}_q^{(1)} \times U(1)$ symmetry

Next we consider the 3d SPT states as descendant states of Eq. (14) with $\mathbb{Z}_q^{(1)} \times U(1)$ symmetry. Again we will first consider the cases where all the point particles in the bulk are bosons. When we break the $U(1)^{(1)}$ symmetry down to $\mathbb{Z}_q$, the 2d boundary is a QED$_3$ whose flux carries $U(1)$ quantum number, and there are dynamical $q$-fold electric charges. The boundary can only be driven to a $z_q$ topological order by condensing the $q$-fold electric charge. One of the point like anyons of this topological order is the remnant of the $2\pi/q$ flux of the QED$_3$, which carries $k/q$ charges of the $U(1)$ symmetry quantum number. When $k = q$ this anyon carries unfractonalized quantum number, hence can be neutralized by binding with gauge invariant integer charge of the $U(1)$ symmetry. This neutralized anyon is a self-boson, and after condensation it drives the boundary into a trivial gapped state. Hence this 3d SPT state should have a $\mathbb{Z}_q^{(1)}$ classification.

To facilitate further discussions let us also consider a different 3d bulk state with $U(1)$ global symmetry only. This is a QED$_4$ whose electric charge is fermion, and Dirac monopole is a boson (using the notations introduced before, this bulk state is QED$_4\{e_f, m_b\}$). Again one can bind the Dirac monopole with another boson that carries $U(1)$ quantum number, and condense the bound state in the 3d bulk. Then the bulk is gapped and
nondegenerate, while the 2d boundary is a QED$_3$ whose electric charge is a fermion, while the gauge flux carries U(1) quantum number. However, this 3d bulk is not a SPT state, since one can put the electric charge at the boundary in a 2d Chern insulator with Hall conductivity 1, then the 2d boundary is gapped without breaking any symmetry. This is consistent with the classification of ordinary SPT states without higher form symmetries. With only U(1) symmetry, there is no nontrivial SPT state in 3d. One needs another time-reversal symmetry to construct a 3d bosonic SPT state, since the boundary Chern insulator of the fermionic gauge charge as we constructed above necessarily breaks the time-reversal.

One can again glue the 2d boundary states in the previous two paragraphs together. Let us recall that the boundary of a nontrivial 3d SPT state with $Z_q^{(1)} \times U(1)$ symmetry is a QED$_3$ whose flux carries U(1) quantum number, and its bosonic electric charges are infinitely heavy; the boundary of the trivial state discussed in the last paragraph is a QED$_3$ whose flux also carries U(1) quantum number, and its electric charge is a fermion with nonzero dynamics. Once we couple the two 2d systems together, the tunnelling between the gauge fluxes between the two QED$_3$ will be turned on, which identifies the two gauge fields. Now the 2d boundary state is a QED$_3$ whose gauge flux still carries U(1) quantum number, but its static electric charge is a fermion. This state is not a new SPT state since it can be constructed by gluing the 2d boundaries of the two systems discussed above.

D. Descendant 3d SPT state with $Z_q^{(1)} \times Z_n$ symmetry

Finally we can break the U(1)$^{(1)}$ 1-form symmetry in Eq. 14 to $Z_q^{(1)}$. Again we can start with the QED$_3$ state at the (2 + 1)d boundary. In this case there are dynamical $q$-fold electric charge of the $u(1)$ gauge field, and the magnetic flux of the $u(1)$ gauge field still carries $Z_n$ quantum number. One can condense the charge $-q$ bound state, and drive the 2d boundary into a 2d $z_q$ topological order. In an ordinary 2d $z_q$ topological order, there are two sets of anyons. The $e$ anyon is a remnant of the unit charge excitation of the QED$_3$ before the condensation of the $q$-fold electric charge, and the $m$ anyon is a $2\pi/q$ flux quantum of the $u(1)$ gauge flux. Both $e$ and $m$ anyons are self-bosons, but have a mutual $2\pi/q$ statistical angle. In our current case, due to the $Z_q^{(1)}$ 1-form symmetry, the $e$ anyons are not dynamical, and a $m$ anyon carries a fractional quantum number $1/q$ of the $Z_n$ symmetry (assuming $k = 1$ in Eq. 14). Both $e$ and $m$ anyons are coupled to $z_q$ gauge fields. Following the arguments in section III we can demonstrate that when $q$ and $n$ are not coprime, the fractional quantum number of the $m$ anyon can always be “neutralized” by binding with integer charges of the $Z_n$ symmetry, in the sense that the $Z_n$ transformation on the decorated $m$ anyon can always be cancelled by a $z_q$ gauge transformation. When $q$ and $n$ are not coprime, the quantum number of the $m$ anyon can be neutralized when $k = \gcd(q, n)$. The neutralized $m$ anyon can condense and drive the 2d boundary to a trivial gapped state without degeneracy. Hence as a descendant state of Eq. 14, the classification of the 3d SPT state with $Z_q^{(1)} \times Z_n$ symmetry is $Z_{\gcd(q, n)}$.

Summary of 3d SPT states with 1-form symmetries:

Here we summarize the classification of 3d SPT states that are descendants of Eq. 14. If there are special SPT states that cannot be described by Eq. 14 such as some of the states discussed in Ref. 23-24 they are not included in this list.

$$U(1)^{(1)} \times U(1) : \mathbb{Z};$$
$$Z_q^{(1)} \times U(1) : \mathbb{Z}_q;$$
$$U(1)^{(1)} \times Z_n : \mathbb{Z}_n;$$
$$Z_q^{(1)} \times Z_n : \mathbb{Z}_{\gcd(q, n)}. \tag{15}$$

VII. 2d SPT STATE WITH $G_1^{(1)} \times Z_2^T$ SYMMETRY

Several different (2+1)d SPT states that involve 1-form symmetries can be described by the following topological response term:

$$S_{2d-\text{topo}} = \int_{(2+1)d} \frac{i\Theta}{2\pi} dB \tag{16}$$

In principle $\Theta$ can take arbitrary value, because $dB$ is gauge invariant. But some extra symmetry can pin $\Theta$ to a specific value, like the $\Theta$ term of the ordinary topological insulator and bosonic SPT states.

As an example of such states, we assume that the 2-form background gauge field $B$ is unchanged under time-reversal transformation, this means that the 1-form symmetry charge will change sign under time-reversal. This implies that the total symmetry of the system is a direct product between the 1-form symmetry and time-reversal. $\Theta$ is clearly defined periodically, namely $\Theta + 2\pi = \Theta$, hence the time-reversal invariant states correspond to $\Theta = \pi k$ with arbitrary integer $k$.

For even integer $k$, the (2 + 1)d topological response theory Eq. 16 reduces to a boundary topological term that is identical to the topological response theory with 1d SPT state with a 1-form symmetry (section II). This means that, for even integer $k$, the boundary corresponds to a well-defined 1d state, hence an even integer $k$ would correspond to a trivial state in (2 + 1)d. On the other hand, for odd integer $k$, the boundary is a “half” 1d SPT state with 1-form symmetry $G^{(1)}$. Then the (2 + 1)d bulk could be a SPT state.

As we mentioned before, due to the strict constraint $\nabla_x e(x) = 0$ for 1-form charge in one dimension, a 1d system with 1-form symmetry is analogous to a 0d system.
with ordinary 0-form symmetry. Then whether there is a $(2 + 1)d$ SPT state with $G^{(1)} \times Z^3_2$ symmetry can also be determined by the existence of projective representation of $G \times Z^3_2$. And there is a 2-dimensional projective representation of $U(1) \times Z^3_2$, but not for $U(1) \times Z^2_2$. Indeed, if the symmetry of the system is $G^{(1)} \times Z^3_2$, namely $B$ is odd under time-reversal, the $\Theta$ coefficient is unchanged under time-reversal, hence time-reversal will not pin $\Theta$ to any specific value.

To summarize our result in two spatial dimensions, there is a nontrivial 2d SPT state with $U(1)^{(1)} \times Z^3_2$ symmetry, and this state remains nontrivial when $U(1)^{(1)}$ is broken down to $Z^q_2$ with even integer $q$.

The decorated defect construction also applies in this scenario, which is analogous to what was discussed in Ref. 14 for ordinary SPT states. We can construct the SPT state with $k = 1$ in Eq. 16 by first creating a domain wall of time-reversal symmetry, then embed each domain wall with a 1d SPT state described by Eq. 16 and finally proliferate the domain walls. Besides construction from 1d SPT state, we can also obtain this 2d SPT state by reduction from higher dimensions. For example, starting with the 3d SPT state with $U(1)^{(1)} \times U(1)$ symmetry described by the response theory $\chi_1$ one can compactify one of the three spatial dimensions (the 3d space $R^3$ becomes $R^2 \otimes S^1$), and insert a $\pi$–flux of the 1-form gauge field $A$ through $S^1$. Then the response theory $\chi_1$ reduces to Eq. 16 with $k = 1$. This is the same procedure of dimensional reduction introduced in Ref. 14.

VIII. DISCUSSION

In this work we discussed the classification, construction, and boundary properties of SPT states involving higher symmetries, from one to four spatial dimensions. Our discussion is mostly based on physical arguments. As an application of our discussion, we make connection between the SPT states with 1-form symmetry to quantum dimer model at one lower dimension. Quantum dimer model with spatial symmetries can be mapped to the boundary of a bulk state with onsite symmetries. Some of the universal features of the quantum dimer model is dictated by the nature of the corresponding bulk state.

In this work we only discussed quantum dimer models on bipartite lattices, which can be mapped to a QED with $U(1)^{(1)}$ 1-form symmetry. It is well known that some other dimer models can be naturally mapped to a $Z_2$ gauge field, such as quantum dimer model on the triangular lattice. Then these models would be examples of systems with $Z_2^{(1)}$ 1-form symmetry, and they can also be potentially mapped to the boundary of one higher dimensions. Insights for these systems gained from higher dimensions will be studied in later works.

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