Comment on: “On the Dirac oscillator subject to a Coulomb-type central potential induced by the Lorentz symmetry violation”.

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Abstract

We analyze recent results on a Dirac oscillator. We show that the truncation of the Frobenius series does not yield all the eigenvalues and eigenfunctions of the radial equation. For this reason the eigenvalues reported by the authors are useless and the prediction of allowed oscillator frequencies meaningless.

In a recent paper Vitória and Belich [6] investigated the relativistic oscillator model for spin-1/2 fermionic fields, known as the Dirac oscillator, in a background of breaking the Lorentz symmetry governed by a constant vector field inserted in the Dirac equation by non-minimal coupling. The authors proposed two possible scenarios of Lorentz symmetry violation which induce a Coulomb type potential. They obtained the relativistic energy profile for the Dirac oscillator finding that the frequency of the Dirac oscillator is determined by the quantum numbers of the system and the parameters that characterize the scenarios of Lorentz symmetry violation.

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By a suitable separation of variables the authors arrived at an eigenvalue equation for the radial part of the solution and applied the Frobenius method. Since the coefficients of the expansion satisfy a three-term recurrence relation the authors forced a truncation of the series in order to obtain exact eigenfunctions and eigenvalues. In this Comment we analyze the effect of the truncation method just mentioned on the physical conclusions drawn by the authors.

In the first case the authors derived the following eigenvalue equation
\[
\frac{d^2\psi_s}{d\eta^2} + \frac{1}{\eta} \frac{d\psi_s}{d\eta} - \frac{\nu_s^2}{\eta^2} \psi_s - \frac{\beta}{\eta} \psi_s - \eta^2 \psi_s + W \psi_s = 0
\]
where \(l = 0, \pm 1, \pm 2, \ldots, a\) is a constant, \(\lambda\) is a parameter associated to the linear electric charge distribution, \(\omega\) is the oscillator frequency, \(m\) a mass and \(E\) the energy. The authors simply stated that \(\hbar = c = 1\), although there are efficient and rigorous ways of obtaining suitable dimensionless equations [2]. From a truncation method that we discuss below the authors obtained the energies \(E_{1,l,s}\) in terms of an angular frequency \(\omega_{1,l,s}\) that depends on the quantum numbers.

In the second case the authors arrived at the following eigenvalue equation
\[
\frac{d^2\psi_s}{d\eta^2} + \frac{1}{\eta} \frac{d\psi_s}{d\eta} - \frac{\nu_s^2}{\eta^2} \psi_s + \frac{\tau}{\eta} \psi_s - \eta g \psi_s - \eta^2 \psi_s + W \psi_s = 0,
\]
where \(l = 0, \pm 1, \pm 2, \ldots, a\) is a constant, \(\lambda\) is a parameter associated to the linear electric charge distribution, \(\omega\) is the oscillator frequency, \(m\) a mass and \(E\) the energy. The authors simply stated that \(\hbar = c = 1\), although there are efficient and rigorous ways of obtaining suitable dimensionless equations [2]. From a truncation method that we discuss below the authors obtained the energies \(E_{1,l,s}\) in terms of an angular frequency \(\omega_{1,l,s}\) that depends on the quantum numbers.

In what follows we will show that the authors’ interpretation of the results obtained from the truncation method are meaningless from a physical point of view. In order to facilitate the discussion we just focus on the eigenvalue equation
\[
\psi''(x) + \frac{1}{x} \psi(x) - \frac{\gamma_s^2}{x^2} \psi(x) - \frac{a}{x} \psi(x) - bx \psi(x) - \frac{\tau}{x^2} \psi(x) + W \psi(x) = 0,
\]
where $\gamma$, $a$ and $b$ are real model parameters that have nothing to do with the parameter in equations (1) and (2)). This eigenvalue equation, which is a generalization of (1) and (2), has square-integrable solutions
\[
\int_0^\infty |\psi(x)|^2 x \, dx < \infty,
\] for particular values (allowed values) of the eigenvalue $W$. Since the behaviour at origin is determined by $\gamma^2/x^2$ and the behaviour at infinity by the harmonic term $x^2$ then we conclude that there are bound states for all $-\infty < a, b < \infty$. Therefore, the eigenvalues $W(a, b)$ are surfaces in the three-dimensional $abW$ space. They are continuous functions of $a$ and $b$ that satisfy the Hellmann-Feynman theorem
\[
\frac{\partial W}{\partial a} = \langle \frac{1}{x} \rangle > 0, \quad \frac{\partial W}{\partial b} = \langle x \rangle > 0.
\] (5)

In what follows we apply the Frobenius method to the eigenvalue equation (3) by means of the ansatz
\[
\psi(x) = x^s \exp \left( -\frac{b}{2} x - \frac{x^2}{2} \right) H(x), \quad H(x) = \sum_{j=0}^{\infty} c_j x^j, \quad s = |\gamma|.
\] (6)
The expansion coefficients $c_j$ satisfy the three-term recurrence relation
\[
c_{j+2} = A_j c_{j+1} + B_j c_j, \quad c_{-1} = 0, \quad c_0 = 1,
\]
\[
A_j = \frac{2a + b (2j + 2s + 3)}{2(j + 2)} \frac{j + 2 (s + 1)}{j + 2 (s + 1)}, \quad B_j = \frac{4(2j + 2s - W + 2) - b^2}{4(j + 2) [j + 2 (s + 1)]}.
\] (7)
The authors showed that $H(x)$ is solution to a biconfluent Heun equation but they did not use the properties of this equation and resorted to a truncation of the Frobenius series. If the truncation condition $c_{n+1} = c_{n+2} = 0$, $c_n \neq 0$, $n = 0, 1, \ldots$, has physically acceptable solutions for $a, b$ and $W$ then we obtain exact eigenfunctions because $c_j = 0$ for all $j > n$. This truncation condition is equivalent to $B_n = 0$, $c_{n+1} = 0$ or
\[
W_{s}^{(n)} = 2(n + s + 1) - \frac{b^2}{4}, \quad c_{n+1}(a, b) = 0,
\] (8)
where the second condition determines a relationship between the parameters $a$ and $b$. On setting $W = W_{s}^{(n)}$ the coefficient $B_j$ takes a simpler form:
\[
B_j = \frac{2(j-n)}{(j+2) [j + 2 (s + 1)]}.
\] (9)
The second condition $c_{n+1}(a, b) = 0$ is a polynomial equation for $a$ and $b$ of degree $n+1$ in every variable. From this equation one obtains either $a_s^{(n,i)}(b)$ or $b_s^{(n,i)}(a)$, $i = 1, 2, \ldots, n + 1$, and it can be proved that all the roots are real [4, 5]. For example, for $n = 0, 1, 2, 3$ we have

$$2a + b (2s + 1) = 0, \quad (10)$$

$$4a^2 + 8ab (s + 1) + b^2 (2s + 1) (2s + 3) - 8 (2s + 1) = 0, \quad (11)$$

$$8a^3 + 12a^2b (2s + 3) + 2ab^2 (12s^2 + 36s + 23) - 32a (4s + 3) + b^3 (2s + 1) (2s + 3) (2s + 5) - 16b (2s + 1) (4s + 7) = 0, \quad (12)$$

and

$$16a^4 + 64a^3b (s + 2) + 8a^2b^2 (12s^2 + 48s + 43) - 640a^2 (s + 1) + 16ab^3 (4s^3 + 24s^2 + 43s + 22) - 128ab (10s^2 + 30s + 17) + b^4 (2s + 1) (2s + 3) (2s + 5) (2s + 7) - 32b^2 (2s + 1) (10s^2 + 45s + 47) + 576 (2s + 1) (2s + 3) = 0, \quad (13)$$

respectively. Figure 1 shows part of the curves $a_0^{(3,i)}$, $i = 1, 2, 3, 4$.

It follows from the analysis above that the polynomial solutions can be written as

$$\psi_s^{(n,i)}(x) = x^s \exp \left( -\frac{b}{2} x - \frac{x^2}{2} \right) H_s^{(n,i)}(x), \quad \gamma = |\gamma|,$$

$$n = 0, 1, \ldots, i = 1, 2, \ldots, n + 1 (14)$$

Vitória and Belich [1] only showed eigenvalues for $n = 1$ and overlooked the multiplicity of roots for each value of $n$.

It is worth noticing that the truncation condition does not provide all the solutions that satisfy equation (4) but only polynomial functions of the form (14) for which the parameters $a$ and $b$ exhibit certain polynomial relations like those in equations (10-13). The reason is that this problem is not exactly solvable, as Vitória and Belich appear to believe, but quasi-exactly solvable or conditionally
solvable (see [4–7] and, in particular, the remarkable review [8] and references therein for more details).

It is revealing to compare the eigenvalues given by the truncation method with the actual eigenvalues $W_{j,s}(a,b)$, $j = 0, 1, \ldots$ of equation (3). Since this eigenvalue equation is not exactly solvable [5, 8] we should apply an approximate method. Here, we resort to the well known Ritz variational method that is known to yield upper bounds to all the eigenvalues [9] and, for simplicity, choose the non-orthogonal basis set of Gaussian functions $\{ \varphi_{j,s}(x) = x^s + j \exp \left( -\frac{x^2}{2} \right), \; j = 0, 1, \ldots \}$.

Figure 2 shows some eigenvalues $W_0^{(n)}(b = 1)$ given by the truncation condition (red points) and the lowest variational eigenvalues $W_{j,0}(a, 1)$ (blue lines). We clearly appreciate that the truncation condition (8) yields only some particular points of the curves $W_{j,0}(a, 1)$. Therefore, any conclusion drawn from $W_s^{(n)}$ is meaningless unless one is able to organize these eigenvalues properly [5–7]. Vitória and Belich [1] completely overlooked this fact. The reason is that these authors appear to believe that the only acceptable solutions to the eigenvalue equation are those with polynomial factors as in equation (14). The fact is that this kind of solutions already satisfy equation (4) but they are not the only ones. Notice that the variational method also yields the polynomial solutions as shown by the fact that the blue lines connect the red points in Figure 2. In order to make the meaning of the eigenvalues $W_s^{(n)}$ and the associated multiplicity of roots $i = 1, 2, \ldots, n+1$ clearer, Figure 2 shows an horizontal line (green, dashed) at $W = W_0^{(8)}$ that intersects the curves $W_{j,0}(a, 1)$ exactly at the red points. The most important conclusion of present analysis is that the occurrence of allowed oscillator frequencies are fabricated by Vitória and Belich by picking out some isolated eigenvalues $W_s^{(n)}$ for some particular curve $a_s^{(n,i)}(b)$. Since there are eigenvalues $W_{j,s}(a, b)$ for all real values of $a$ and $b$ then there are bound states for every positive value of $\omega$ in their equations (1) and (2).

Summarizing: the truncation of the Frobenius series does not yield all the bound states of the radial eigenvalue equation but only some particular states of the form (14). These exact solutions only occur for some relationships between the model parameters $a$ and $b$. For this reason the eigenvalues reported
by Vitória and Belich [1] are meaningless unless one is able to arrange them carefully as shown in present Figure [2] and earlier papers [5][7]. The allowed oscillator frequencies conjectured by those authors are a mere artifact of the truncation method because there are solutions to equations (1) and (2) for all positive values of $\omega$.

The criticisms outlined in this Comment also apply to other almost identical papers appeared in the same journal [10][12].

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Figure 1: Curves $d_0^{(3,i)}$, $i = 1, 2, 3, 4$

Figure 2: Eigenvalues $W_0^{(n)}(a,1)$ from the truncation condition (red points) and $W_{j,0}(a)$ obtained by means of the variational method (blue lines)