Low-Energy Effective Action of N=2 Gauge Multiplet Induced by Hypermultiplet Matter

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Abstract

We study one-loop effective action of hypermultiplet theory coupled to external N=2 vector multiplet. We formulate this theory in N=1 superspace and develop a general approach to constructing derivative expansion of the effective action based on an operator symbol technique adapted to N=1 supersymmetric field models. The approach under consideration allows to investigate on a unique ground a general structure of effective action and obtain both N=2 superconformal invariant (non-holomorphic) corrections and anomaly (holomorphic) corrections. The leading low-energy contributions to effective action are found in explicit form.

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1 Introduction

Effective action containing the quantum corrections to classical action plays in quantum theory a role analogous to one of action functional in classical theory. Being found, the effective action allows to investigate a broad spectrum of quantum properties associated with off-shell behavior. Therefore it is not wonder that the effective action is one of the central objects of quantum field theory.

In field models possessing some global or gauge symmetries on a classical level the exact effective action contains full information concerning these symmetries in quantum theory or their violation. Presence of the symmetries imposes the rigid restrictions on a structure of the effective action and allows sometimes to fix it very significantly in terms of proper functionals invariant under the symmetries. The bright examples of such models are the extended supersymmetric field theories.

One of the main approaches to practical evaluating the effective action is momentum (derivative) expansion where the effective action is investigated in form of a series in derivatives of its functional arguments. Keeping the lowest terms of such an expansion leads to a notion of low-energy effective action which can be described by local effective lagrangian. Another approach is the known loop expansion where the leading (one-loop) contribution to effective action is given by functional determinant of some (pseudo)differential operator. Both these approaches are used very often together.

The paper under consideration is devoted to study a structure of low-energy effective action in hypermultiplet model coupled to external abelian N=2 vector multiplet. The various aspects of effective action in field models possessing N=2 supersymmetry attracted recently very much attention due to famous work by Seiberg and Witten [1] where exact low-energy effective action has been found in N=2 super Yang-Mills theory with gauge group SU(2) spontaneously broken down to U(1). It is turned out that just extended supersymmetry was one of the essential points allowing to establish general non-perturbative structure of the effective action. The result by Seiberg and Witten has later been generalized for various gauge groups and coupling to a matter (see [2] for modern review). Another remarkable result was obtained by Dine and Seiberg [3] and concerned an exact structure of part of low-energy effective action depending on N=2 superfield strengths in N=4 super Yang-Mills theory with gauge group SU(2) spontaneously broken down to its abelian subgroup. Such a theory can be treated as a specific N=2 supersymmetric model and its extended supersymmetry has played a crucial role in obtaining the exact form of low-energy effective action.

A remarkable feature of supersymmetric field models consists in the fact that the low-energy effective action can be written as a sum of two contributions. One of them is integral over full superspace and another one is integral over chiral subspace of general superspace (plus conjugate). Therefore the low-energy effective action in such models is described by two types of effective lagrangians: chiral and general or holomorphic and non-holomorphic. We point out that a possibility of holomorphic corrections for N=1 SUSY models was firstly demonstrated in papers [4] and for N=2 SUSY models in papers [5]. Non-holomorphic superfield effective lagrangian was constructed in [6] (see also general discussion in [7]).

We concentrate the attention on the N=2 SUSY models containing an interaction with vector multiplet. In this case a part of effective action depending only on a vector
multiplet fields is written in the form

$$\Gamma[\mathcal{W}] = \left( \int d^8 z \mathcal{F}(\mathcal{W}) + h.c. \right) + \int d^{12} z \mathcal{H}(\mathcal{W}, \bar{\mathcal{W}}) + \ldots$$  \hspace{1cm} (1)$$

where $\mathcal{W}$ is N=2 superfield strength $\mathbb{F}$, $z$ are the N=2 superspace coordinates, $d^8 z$ and $d^{12} z$ mean the chiral and general N=2 superspace measures. $\mathcal{F}(\mathcal{W})$ is called holomorphic effective potential and $\mathcal{H}(\mathcal{W}, \bar{\mathcal{W}})$ is called non-holomorphic effective potential. The dots mean the terms depending on covariant derivatives of the strengths. In arbitrary N=2 SUSY models the holomorphic effective action $\mathcal{F}(\mathcal{W})$ determines low-energy behavior and $\mathcal{H}(\mathcal{W}, \bar{\mathcal{W}})$ corresponds to next to leading corrections (see f.e. [9]). However in the theories possessing quantum superconformal invariance, for example in N=4 super Yang-Mills theory, holomorphic effective action is trivial (it is proportional to $\mathcal{W}^2$) and namely non-holomorphic effective potential forms leading contribution to low-energy effective action. Recent progress in finding the holomorphic and non-holomorphic contributions to effective action in various N=2,4 SUSY models is discussed in refs [10]-[18]. In particular, a manifestly N=2 supersymmetric approach to calculating holomorphic effective action was developed in refs [12] on the base of concept of harmonic superspace [22]. This approach includes also a general N=2 superfield background field method [14], [15]. Structure of leading low-energy contributions to effective action of N=4 SYM theory has been investigated in recent refs. [16], [17], [21]. It is worth to point out a significance of detailed study of N=4 SYM effective action for understanding classical supergravity/quantum gauge theory duality [19], [20].

At present, the holomorphic effective potential is well established. However a structure of non-holomorphic effective potential for arbitrary N=2 models is still unclear. The known solid result corresponds to the contribution of the form $(\mathcal{W}\mathcal{W})^2$ (see first paper in refs [12]). Situation changes drastically in N=4 SYM theory or in the models possessing quantum N=2 superconformal symmetry. Here due to this symmetry one can get one-loop effective action for constant field background in terms of so called N=2 superconformal invariants [18]. However the approach [18] can not be applied literally to arbitrary N=2 models since they are not N=2 superconformal invariant on quantum level.

The paper under consideration is just devoted to developing a general method for evaluating low-energy one-loop effective action in arbitrary N=2 supersymmetric models. Our purpose consists in construction of the derivative expansion of the effective action preserving N=1 supersymmetry and gauge invariance. Another N=1 supersymmetry in nonmanifest but as we will see the final result for leading contribution in constant field background can be interpreted in N=2 SUSY terms.

We evaluate the effective action in the hypermultiplet model coupled to external abelian N=2 vector multiplet using a realization of the model in terms of N=1 superspace. This model is simple enough and allows to illustrate all basic steps of general derivative expansion technique discussed earlier (see various implementations of this technique in [36]).

For calculating one-loop effective action of corresponding N=1 superfield theory we develop a general approach based on a technique of the operator symbols closely connected with mathematical theory of deformation quantization (see the references in subsection 3.2). A main dignity of such an approach is a possibility to reformulate a problem of evaluating the traces of the operators acting in superspace as problem of calculating some integrals of suitable superfields with specific non-commutative multiplication rule.
containing all quantum aspects of the initial problem. We demonstrate that this approach is very efficient for constructing derivative expansion of one-loop superfield effective action.

The paper is organized as follows. In Section 2 we describe the properties of the models and discuss a formal definition of the effective action. Section 3 is devoted to general structure of the effective action, a overview of a method of operator symbols which we apply for evaluating the effective action and the specific features of implementations of this methods to N=1 superfield theories. In Section 4 we carry out the calculations of the low-energy effective action for constant field background

\[ W| = \Phi = \text{Const}, \quad D^i_\alpha W| = \lambda^i_\alpha = \text{Const}, \]
\[ D^i_{(\alpha} D^j_{\beta)} W| = f_{\alpha\beta} = \text{Const}, \quad D^\alpha(D^i_\alpha) W| = 0. \]  

and obtain a general result including both (known earlier) holomorphic and non-holomorphic effective potentials within a single method. Section 5 is devoted to a summary of the results and the prospects. Appendices contain some details of the calculations.

## 2 Description of the Model

We consider the hypermultiplet model interacting with external abelian N=2 vector multiplet. Our purpose is to integrate over hypermultiplet fields and construct an effective action depending on the vector multiplet fields.

As well known the model under consideration can be formulated by different (on-shell equivalent) ways: in terms of component fields, in terms of N=1 superfields, in terms of (constrained) N=2 superfields and in terms of unconstrained harmonic and projective superfields. In our case for constructing the effective action we use the simplest realization of the hypermultiplet in terms of N=1 chiral superfields. Although such a realization does not preserve manifest N=2 supersymmetry it allows to apply an efficient and well developed technique of N=1 superfield quantum field theory (see f.e. [7]).

The action of the model in above realization of the hypermultiplet is written as a sum of external fields action \( S_0 \) and hypermultiplet action coupled to the external fields \( S \).

\[ S_0 = \frac{1}{4g^2} [\int d^6z \frac{1}{2} W^\alpha W_\alpha + \int d^8z \bar{\Phi} e^{-V} \Phi e^V], \]
\[ S = \int d^8z (Q_+ e^V Q_+ + Q_- e^{-V} Q_-) + i \int d^6z Q_- \Phi Q_+ + i \int d^6z Q_+ \bar{\Phi} Q_- + \int d^6z Q_+ \bar{\Phi} Q_- \]  

Here \( Q_+ \) and \( Q_- \) are two N=1 chiral superfields with opposite \( U(1) \) charges; \( V \) and \( \Phi \) are N=2 vector multiplet superfield and chiral superfield respectively, together they form N=2 vector multiplet and the gauge coupling is included in the field definitions. The actions \( S_0 \) and \( S \) are N=1 supersymmetric by construction. However they are invariant under hidden extra N=1 supersymmetry

\[ \delta \Phi = \epsilon^\alpha W_\alpha, \quad \delta \bar{\Phi} = \bar{\epsilon}^\alpha \bar{W}_\alpha, \quad \delta \nabla = \epsilon \Phi, \quad \delta \bar{\nabla} = -\bar{\epsilon} \bar{\Phi}, \]
\[ \delta W_\alpha = -\epsilon_\alpha \nabla^2 \Phi + i\bar{\epsilon}^\alpha \nabla a \Phi, \quad \delta \bar{W}_\dot{\alpha} = -\bar{\epsilon}_{\dot{\alpha}} \nabla^2 \bar{\Phi} + i\epsilon^\alpha \nabla a \Phi. \]

These transformations form together with manifest N=1 supersymmetry transformations the full set of N=2 supersymmetry transformations. Here \( W_\alpha \) is the strength corresponding to N=1 gauge superfield \( V \). Besides, we have introduced the covariant derivatives in
background field vector representation.
\[ \nabla_\alpha = e^{-\frac{\chi}{2}} D_\alpha e^{\frac{\chi}{2}}, \quad \bar{\nabla}_{\dot{\alpha}} = e^{\frac{\chi}{2}} \bar{D}_{\dot{\alpha}} e^{-\frac{\chi}{2}}, \]
which satisfy usual constrains
\[ \{ \nabla_\alpha, \nabla_{\dot{\alpha}} \} = i \nabla_{\alpha \dot{\alpha}}, \quad \{ \nabla_{\dot{\alpha}}, \nabla_{\beta \dot{\beta}} \} = \epsilon_{\dot{\alpha} \dot{\beta}} W_\beta, \ldots \]
and covariant chiral superfields
\[ \Phi_c = e^{\frac{\chi}{2}} \Phi e^{-\frac{\chi}{2}}, \quad \bar{\Phi}_c = e^{-\frac{\chi}{2}} \bar{\Phi} e^{\frac{\chi}{2}}, \]
subject to the constrains \( \nabla \Phi_c = \nabla \bar{\Phi}_c = 0 \). The details of background vector representation see in [24], [7].

The superfields \( W, \bar{W} \) and \( \Phi, \bar{\Phi} \) are the \( N=1 \) projections of \( N=2 \) gauge strengths \( W \) and \( \bar{W} \)
\[ W = \Phi + \eta^\alpha W_\alpha - \eta^2 \nabla^2 \bar{\Phi} + \frac{i}{2} \eta^\alpha \bar{\eta}^\dot{\alpha} \nabla_{\alpha \dot{\alpha}} \Phi + \frac{i}{2} \eta^2 \bar{\eta}^2 \nabla_{\alpha \dot{\alpha}} W^\alpha + \frac{1}{4} \eta^2 \bar{\eta}^2 \Box \Phi, \]
\[ \bar{W} = \bar{\Phi} + \bar{\eta}^\dot{\alpha} \bar{W}_{\dot{\alpha}} - \bar{\eta}^2 \nabla^2 \Phi + \frac{i}{2} \eta^{\dot{\alpha}} \bar{\eta}^{\alpha} \nabla_{\alpha \dot{\alpha}} \bar{\Phi} + \frac{i}{2} \eta^2 \bar{\eta}^2 \nabla_{\alpha \dot{\alpha}} \bar{W}^{\alpha} + \frac{1}{4} \eta^2 \bar{\eta}^2 \Box \bar{\Phi}. \]
These relations allow to link the forms of \( N=2 \) supersymmetric functionals written in terms \( N=1 \) and \( N=2 \) superfields.

It is worth to point out that the model under consideration is not only \( N=2 \) supersymmetric but it possesses two more classical symmetries. First, it is gauge invariant (see f.e. [24]) and second, it is \( N=2 \) superconformal invariant, the corresponding superconformal transformations are given in [25].

### 3 The EA and Derivative Expansion Method

#### 3.1 General Definition of Effective Action

We introduce the effective action \( \Gamma \) of the model under consideration by the standard way
\[ e^{i \Gamma} = \int DQ_+ DQ_- e^{i(S_0 + S)} \] (5)

Since the action \( S \) is quadratic in (quantum) hypermultiplet superfields \( Q_+ \) and \( Q_- \) the effective action \( \Gamma \) has the following structure
\[ \Gamma = S_0 + \Gamma_{(1)} \] (6)
where quantum correction \( \Gamma_{(1)} \) to classical action \( S_0 \) of the gauge multiplet induced by hypermultiplet is formally written as follows
\[ \Gamma_{(1)} = -\frac{i}{2} \ln \text{Det} \hat{H} = -\frac{i}{2} \text{Tr} \ln \hat{H}, \] (7)
Here \( \hat{H} \) is some differential operator associated with action \( S \) and acting in space of \( N=1 \) chiral and antichiral superfields \( Q_+ \) and \( Q_- \). Its explicit form will be presented below. Eq
expresses the formal path integral (5) in terms of formal functional determinant. To provide a sense to these formal relations we have to give an informal definition allowing to compute unambiguously the functional determinants of the differential operators acting in N=1 superspace.

As we already pointed out in Section 2 the model under consideration possesses by three classical symmetries: N=2 supersymmetry, gauge symmetry and N=2 superconformal symmetry. Supersymmetry and gauge invariance are not anomalous on quantum level but superconformal symmetry is expected to be broken down due to (one-loop) divergences containing some scale. Therefore the quantum correction Γ(1) is N=2 supersymmetric and gauge invariant functional and hence it have to depend only on N=2 strengths W and W̄ or on their N=1 projections W_α, W̄_α, Φ and Φ.

Taking into account an appearence of the superconformal anomaly one can write Γ(1) in the form

Γ(1) = Γ^0(1) + Γ^1(1)  \hspace{1cm} (8)

where Γ^0(1) is a functional generating superconformal anomaly (its N=2 superconformal variation is equal to anomalous current) and Γ^0(1) is superconformal invariant functional. Of course, decomposition (8) is not unique since one can add an arbitrary superconformal functional to Γ^0(1) and it still will generate the given superconformal anomaly. The main purpose of this paper is developing a technique for efficient evaluation of the functional Γ(1). We show that the decomposition (8) arises quite naturally in our approach and a role of the Γ^0(1) is played by the known Seiberg’s type holomorphic effective action.

Since the effective action is expressed in form of functional determinant of the (superfield) differential operator \(\hat{H}\) its calculation can be carried out on the base of Fock-Schwinger proper-time technique appropriately formulated in superspace (see the aspects of such a formulation in ref. [6], [7], [36])

The evaluation of the determinants of the (pseudo)differential operators always involves some kind of regularization. For actual computations of the effective action we use elegant \(\zeta\)-function regularization formulated directly in superspace. Within this regularization scheme the functional determinant \(\text{Det}_\zeta(\hat{H}) = \exp(-\zeta'_\hat{H}(0))\) is supersymmetric and gauge invariant. As a result the effective action looks like \(\Gamma(1) = -\frac{i}{2} \zeta'_\hat{H}(0)\) and \(\zeta\)-function is defined by the expression

\[
\zeta_\hat{H}(s) = \frac{1}{\Gamma(s)} \int_0^\infty \,dT \, T^{s-1} \text{Tr}(e^{-\frac{T}{s^2} \hat{H}}) = \frac{1}{\Gamma(s)} \int_0^\infty \,dT \, T^{s-1} \int dz \, K(T, \frac{T}{\mu^2}), \hspace{1cm} (9)
\]

where \(dz\) is an appropriate (super)space measure and \(\mu\) is a renormalization point, which is introduced to make \(T\) dimensionless. One can show that the dependence on the parameter \(\mu\) occurs only in those terms that correspond to divergences for other renormalization schemes (proper-time cut-off, dimensional regularization, etc.). The quantity \(K(T)\) is the coincidence limit of heat kernel which can be represented in form of Schwinger-DeWitt expansion over proper time \(T\)

\[
K(T) = \frac{1}{(4\pi T)^2} \sum_{f=0}^\infty a_f T^f. \hspace{1cm} (10)
\]

Here \(a_f\) are DeWitt-Seely coefficients which are the scalars constructed from the coefficients of the operator \(\hat{H}\). The representation (10) is used to isolate the infinities and
finite contributions in effective action by Schwinger’s method. Diverging at small $T$ terms correspond to $\mu$-dependence of the effective action. The series (10) automatically leads to the following expansion of function $\hat{\zeta}(s)$

$$\zeta_H(s) = \sum_f a_f \zeta_f(s).$$

This asymptotic expansion encodes information about short-distance behavior of the effective action in invariant terms.

The action $S$ can be written as

$$S = \int d^8z \left( Q_- e^{-\frac{V}{2}}, \bar{Q}_+ e^{\frac{V}{2}} \right) \left( \frac{\Phi e^{\frac{V^2}{2}}}{\Phi c \bar{\nabla}^2} \right) \left( e^{-\frac{V}{2}} Q_- \right) \left( e^{\frac{V}{2}} Q_+ \right)$$

where $d^8z = d^4x d^2\theta d^2\bar{\theta}$ with $z^M = (x^m, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$ be N=1 superspace coordinates. Let us rewrite the operator $\hat{H}$ associated with eq (12) in more convenient form. We would like to avoid explicit dependence on background gauge prepotential $V$. To do that we introduce covariantly chiral functional variation $\delta Q(z) = \bar{\nabla}^2 \delta(z - z')$, and define the operator $\hat{H}$ as second variation of the action (12) according to previous relation. It leads to the following manifestly N=1 supersymmetric and gauge invariant form of the operator

$$\hat{H} = \left( \bar{\nabla}^2 \nabla^2 \Phi c \bar{\nabla}^2 \right).$$

Equations (7, 9, 13) are considered here as the definition of the effective action $\Gamma$ given by formal path integral (5). It is important to point out that this definition does not appeal to calculating the path integral (5) via direct integration over unconstrained chiral superfields $Q_+$ and $Q_-$ in order to obtain the standard $1/p^2$ propagator (see f.e. [40], [23]). As well known such a scheme requires to introduce an infinite tower of ghosts which contributes to effective action. Our definition of effective action by means of Eqs (7,9,13) avoids making use of everything associated with these ghosts and looks like most simple from computational point of view.

As was recently shown some known tricks of evaluating the one-loop effective action based on factorization of the functional determinants can be ambiguous because of so called multiplicative anomaly (see f.e. [26]). In its essence the multiplicative anomaly is a violation of the equalities $\text{Det}(AB) = \text{Det}(A)\text{Det}(B) = \text{Det}(B)\text{Det}(A)$ for functional determinants of the formal infinite matrices. The matter is all these determinants need regularization and there no guarantee that above equalities always survive after regularization. Our definition of the effective action on the base of Eqs (7,9,13) do not appeal to any factorization triiks and therefore allows to avoid in principle the problem of possible multiplication anomalies.

**3.2 Star-Product Algebras of Function and Derivative Expansion of Effective Action**

The evaluation of the effective action is always based on the computing traces of some operator functions. Exact calculations of such traces is possible only for very specific cases when the eigenvalues and eigenfunctions of the operator under consideration are known, that rather exception then a rule.
Recently the proof of existence of a "formal trace" has been given in ref. [27] within a so called deformation quantization in the form very similar to the Schwinger-DeWitt expansion
\[ \text{Tr}(\hat{O}) = \frac{1}{n!\hbar^n} \int_X d\mu \left( O\omega^n + \hbar \tau_1(O) + \hbar^2 \tau_2(O) + \cdots \right), \]
where \( \tau_k \) are local expressions in \( O \). Here, under the deformation quantization we mean a formal deformation of commutative algebra of functions with \( \hbar \) as the deformation parameter on arbitrary symplectic manifold to the noncommutative algebra of quantum observables subject to Dirac’s correspondence principle only. Calculating the traces via equations (3, 7) closely related to the index theorem and naturally leads to star product algebras of functions (see recent ref. [28]). Therefore, the natural language for the trace handling is the star product algebras on functions.

A proposal to reformulate an analysis in the operator algebras on a symbol calculus language copying the operator product has been introduced by Berezin [29]. The basic role in this construction plays a concept of symbol \( \sigma(O) \) of the operator \( \hat{O} \). The symbol is a classical function of finite number of the variables \( \gamma^A \) associated with the operator \( \hat{O}(\gamma) \) ordered by certain manner (we are implying \( \sigma(\hat{\gamma}_A) = \gamma_A \)).

In order to set the stages for computations, let us briefly review some basic notions used and terminology. A symplectic manifold \( M^{2n} \) can be treated as a cotangent fiber bundle \( X = (M^{2n}, M^n, T^*_x M^n, \omega) \) with the base space \( M^n \), fiber \( T^*_x M^n \) and a fundamental symplectic two-form \( \omega \) in the form \( \omega = \frac{1}{2} \omega_{AB} d\gamma^A \wedge d\gamma^B \) where the local coordinates \( \gamma^A = (p_i, x^i) \), \( \gamma \in M^{2n} \), \( x \in M^n \), \( p \in T^*_x M^n \).

In particular, nondegenerate matrix \( \omega_{AB} \) is a constant matrix in the Darboux coordinates and \( \omega^{AB} \) defines the standard Poisson bracket \( \{f, g\}_{PB} = f \leftarrow A \omega^{AB} \rightarrow B g \) which is a noncommutative product in \( M^{2n} \) in contrast to a pointwise product. The dynamical behavior of the system is then controlled by a function \( H \) defined on a manifold through the vector field associated with \( \omega \) by means of its differential.

The starting point of the symbol \( \leftrightarrow \) operator correspondence is an associative algebra that defines a noncommutative space and can be described in terms of a set of operators \( \hat{\gamma}_A \) and relations
\[ [\hat{\gamma}_A, \hat{\gamma}_B] = i\hbar \omega_{AB}(\gamma). \] (14)
Among these relations the most known are: 1) canonical structure \( \omega_{AB} = \text{Const} \); 2) Lie-algebra structure \( \omega_{AB} = \omega_{CAB}^A \hat{\gamma}_C; 3) \) quantum space structure \( \omega_{AB} = \omega_{CAB}^D \hat{\gamma}_C \hat{\gamma}_D \). One considers the generators \( \hat{\gamma}_A \) as coordinates and let the algebraic structure is the formal power series in these coordinates modulo to the relations (14). It means that the power series whose elements reordered with the help of these relations are considered as equivalent. The known Heisenberg-Weyl algebra \( \{\hat{\gamma}_A\} = \{\hat{P}_i, \hat{Q}^j\} \) is defined by the commutation relation \( [\hat{Q}^j, \hat{P}_i] = i\hbar \delta^j_i \).

Let us introduce an operator family as a Fourier transform \( s \)-parameterized by a weight function \( w_s(u, v) \) of displacement operators since they form a complete operator basis
\[ \hat{\Omega}(p, q; s) = \int du dv \ e^{i(qv - up)}w_s(u, v)e^{i\hat{\gamma}(up + sv)}. \] (15)
Because of any operator obeying certain conditions can be expanded in terms of the complete operator basis we can present an operator \( \hat{A}(\hat{P}, \hat{Q}) \) in the enveloping Heisenberg
algebra in the following way

\[ A(\hat{P}, \hat{Q}) = \frac{1}{(2\pi)^n} \int d^npd^nq \ A_{-s}(p, q)\hat{\Omega}(p, q; s), \quad A_s(p, q) = \text{Tr}(\hat{A}\hat{\Omega}(p, q; s)), \]  

where coefficient \( A_s(p, q) \) is a smooth function on \( T^*M \) and is called \( s \)-symbol of the operator \( A(\hat{P}, \hat{Q}) \).

The construction mentioned above can be extended to a more general case. Actually, the concept of the deformation quantization is related to Weyl’s quantization procedure (i.e. \( s = 0 \)). In this procedure a classical observable \( A(\gamma) \), some square integrable function on phase space \( X = (\gamma, \omega) \), is one-to-one associated to a bounded operator \( \hat{A} \) in the Hilbert space by the Weyl mapping

\[ \hat{A} = \int_X d\mu(\gamma) \ A_{-s}(\gamma)\hat{\Omega}(\gamma; s). \]  

An inverse formula which maps an operator into its symbol by Wigner mapping, is given by the trace formula

\[ A(\gamma) = \text{Tr}(\hat{\Omega}(\gamma; -s)\hat{A}), \]  

Both formulae \([17, 18]\) are determined by the choice of the Stratonowich-Weyl kernel \( \hat{\Omega}(\gamma; s) \), which is the Hermitian operator family parameterized by \( s \) and constructed from the operators \( \hat{\gamma}_A \). The Stratonowich-Weyl kernel (or a quantizer and also a dequantizer) possesses by a number of properties (see for example ref. \([30]\)): \( \hat{\Omega} \) is injective; \( \hat{\Omega} \) is self-adjoint; unit trace \( \text{Tr}(\hat{\Omega}) = 1 \); covariance \( U(g)\hat{\Omega}(x)U(g^{-1}) = \hat{\Omega}(g \cdot x) \); traciality \( \text{Tr}(\hat{\Omega}(x)\hat{\Omega}(y)) = \delta(x, y) \). One can see from expressions \([13]\) and \([14]\) that in general a symbol can possess a parametric dependence on \( \hbar \) by formal power series.

Correspondence \([17]\) relates \( \hat{A} \) to \( A_{-s}(\gamma) \) via integration. In practical calculation it is also helpful to employ a differential form of this relation

\[ \hat{A} = A_{-s}(-i\partial_\gamma)\hat{\Omega}(\gamma; s)|_{\gamma=0} \]  

Various \( s \) related to various ordering prescriptions in the corresponding enveloping algebra. This means that we can choose several different rules of normal ordering for operator products. For instance, the Weyl ordering (totally symmetrized operator product) is often a preferred choice for physical applications since it treats self-adjoint operators \( \hat{P}, \hat{Q} \) symmetrically. This ordering prescription has specific features leading to a possibility to construct the real symbols for the operators (i.e. complex conjugation is an algebra anti-automorphism). Other ordering prescriptions convenient for practical calculations are the standard \( PQ \) (all \( \hat{P} \) are disposed from the left of all \( \hat{Q} \)) and antistandard \( QP \).

The behavior of physics system is described in terms of states and observables. Both of them are represented by a set of functions on some space \( X \). The space \( X \) is a set of points with some particular structure. All the information about \( X \), without any loss, can be retrieved from the algebra of the functions alone. Moreover, the existence of a such space \( X \) even may not be necessary, if to transfer all relevant information concerning a physical theory into the algebra of functions. This is well-known Gelfand-Naimark duality: i.e. every structure defined on \( X \) has a natural counterpart on the algebra of functions. Particularly, canonical \( \omega_{AB} \) is transferred into the Poisson bracket on the algebra of functions in accordance with the Dirac’s correspondence principle. The symbol of the non-commutative product of operators can be written as a non-local star product

\[ (A \star B)(\gamma) \leftrightarrow \hat{A}\hat{B}, \]
which for a constant Poisson structure is called Moyal product and might be treated for a particular case of Weyl-ordering prescription in integral and differential forms as follows

\[(A \star B)(p, q) = \int \frac{d\xi}{2\pi\hbar} \frac{d\eta}{2\pi\hbar} e^{\frac{\bar{h}}{\hbar}S} A(\xi, \eta) B(\xi', \eta'),\]

\[(A \star B)(p, q) = A(p, q)e^{\frac{i\bar{h}}{\hbar}\left(\frac{\partial}{\partial q} - \frac{\partial}{\partial p}\right)} B(p, q),\] (19)

where \(S = \det \begin{pmatrix} 1 & 1 & 1 \\ q & \xi & \xi' \\ p & \eta & \eta' \end{pmatrix} \).

Of course, the non-locality of a star product is a consequence of the ordinary quantum-mechanical non-locality.

In the phase space parameterized by coordinates \(\gamma\) we have an analogue of algebra (14) with the Moyal bracket

\[[A, B]_{MB} = A \star B - B \star A = i\hbar\{A, B\} + O(\hbar^2),\] (20)

So far the star product is defined only by the relations (14). Unfortunately, all these results are formal in the sense that they do not offer a receipt for the procedure of the star-product construction. The basic problem in attempt to generalize the exponentiation idea (13) to a non-constant Poisson structure is that \(\frac{\partial}{\partial q}, \frac{\partial}{\partial p}\) no longer commute with the \(\omega\). Nevertheless, recall that for \(\hat{A}, \hat{B}\) in some Lie algebra and for \(\exp(\hat{A}) \exp(\hat{B}) = \exp(\hat{C})\) the Campbell-Baker-Hausdorff formula allows to define \(\hat{C}\) as a formal series whose terms are elements in the Lie algebra generated by \(\hat{A}, \hat{B}\). The associativity of such a way defined star product is induced from the associativity of the group multiplication.

It is essential that the star product determines the higher \(O(\hbar^2)\) terms up to gauge equivalence, which amounts to linear redefinitions of the functions

\[A'(\gamma) = A(\gamma) + \hbar S_1(A) + \hbar^2 S_2(A) + \cdots = (S(\hbar)A)(\gamma),\]

with \(S_i\) being differential operators. Two star products related to each other by \(S\) so that

\[S(A \star B) = S(A) \star' S(B)\]

for all \(A, B\) may therefore be considered equivalent. Imposing associativity will constrain this operation and determine higher terms.

The multiplication kernel in the integral form and the local form of star operator \(U = U(\hat{\partial}_\gamma, \hat{\partial}_\gamma; \omega)\) can be found from the Stratonovich-Weyl kernel in principle. At least in some special cases the kernel satisfies the Schrödinger equation, where the role of time is played by the noncommutativity parameter \(\hbar\) and the role of a hamiltonian is played by the Poisson structure associated with the deformation [29]. In cases under interest the star operator \(U\) has an exponential form of a one-parameter group element, like for the flat case, i.e. \(U = \exp(i\hbar\Delta(\hat{\partial}_\gamma, \hat{\partial}_\gamma; \omega))\). It should be noted that both integral and differential versions of a star product allow for the quasiclassical expansion of the composition law for symbols in power series of the noncommutativity parameter \(\hbar\) of algebra (14).

As we have already pointed out in this section the symbol \(\leftrightarrow\) operator correspondence is a mathematical quantization problem finding of spectrum of the operators. The technique of operator symbols allows to reformulate this problem from operator language on a language of the functions defined on some (classical) phase space with specific non-commutative product rule (star product). From this point of view the star operator is nothing but a quantum object. This means that we have to solve a quantum problem (i.e.
find eigenvalues and eigenfunctions on a manifold) for a particular operator in order to
construct star operator exactly leaving aside problems of convergence and of construction
of the Hilbert space. This scheme is generalized to the quantization of any symplectic or
Poisson manifold and the problem of existence and classification up to equivalences formal
star product was solved by several authors (see for example [28]) who support the belief
that the formal deformation encloses the essential information of the quantum system.
The main direction, which has resulted in a simple geometrical construction based on the
Weyl algebras bundle, consists in an observation that each tangent space of a symplectic
manifold is a symplectic vector space, so it can be quantized by the usual Moyal-Weyl
procedure developed in [27] as a way of constructing a quantum exponential map
which always exists [34]. Any star product is gauge equivalent to

\[ (A \star B) = [(\exp_x A)(y) \ast_h (\exp_x B)(y)]|_{y=0}, \] (21)

where \( \exp_x : T_x M^n \to M \) is the exponential map, defined in a neighborhood of the origin,
corresponding to the connection \( \nabla \) and the \( \ast_h \) refers to standard Moyal-Weyl star product
on symplectic vector space \( T_x M^n \). Below, it will be presented several significant explicit
examples that demonstrate practical realization of Eq (21).

Star product formulae (19) are very unhandy for practical computations. But, since
\[ (A \star B)(p, q) = A(p - \frac{i}{\hbar} \frac{\partial}{\partial q_1}, q + \frac{i}{\hbar} \frac{\partial}{\partial p_1})B(p_1, q_1)|_{p_1=p, q_1=q} = A(p_n, q_n)B(p_n, q_n) \times 1, \] (22)

where quantities \( p_n = p - \frac{i}{\hbar} \frac{\partial}{\partial q} q_n = q + \frac{i}{\hbar} \frac{\partial}{\partial p} \) were introduced. They can be considered
as a right regular representation of generating operators \( \hat{P}, \hat{Q} \).

In the general case, we can rewrite expression (19) in a more symmetrical form

\[ (A \star B)(\gamma) = 1 \times e^{i\hbar \hat{\Delta}(\hat{p}, \hat{q})} A(\gamma) e^{-i\hbar \hat{\Delta}(\hat{p}, \hat{q})} B(\gamma) e^{i\hbar \hat{\Delta}(\hat{p}, \hat{q})} \times 1 = \]
\[ = \hat{A}_h(\gamma) \hat{B}_h(\gamma) \times 1 = A(\gamma_h)B(\gamma_h) \times 1 = \]
\[ = 1 \times \hat{A}_h(\gamma) \hat{B}_h(\gamma) = 1 \times A(\gamma_h)B(\gamma_h). \] (23)

Operators \( \hat{\gamma}_h, \hat{\gamma}_h \) are left and right regular representation operators \( \hat{\gamma} \) with commutation relations (14). The recipe

\[ A \to A_h = U^{-1} A U \] (24)

in the geometric quantization aspect is just a prequantization procedure for the phase-
space \( X = (\gamma, \omega) \). The relation between symbols and operators \( A_h \) has a very simple form

\[ A = \hat{A}_h \times 1 = 1 \times \hat{A}_h \] (25)

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This rule just means that all derivatives that act on nothing must be omitted.

For our purposes, the most important formula, which takes place in the framework of
the symbol calculus due to the special properties of the Stratonowich-Weyl kernel, is the
trace definition

\[
\text{Tr} \hat{A} = \int_X d\mu(\gamma) A(\gamma),
\]

where \( X = (\gamma, \omega) \) is the phase space with invariant measure \( d\mu \) and \( A(\gamma) \) is some symbol
of operator \( \hat{A} \). It shold be noted that the definition (26) is correct for any allowed choice
of the Stratonowich-Weyl kernel and for any ordering prescription in the enveloping algebra
(14). According to Eq (26) finding of trace of the operator \( \hat{A} \) is reduced to constructing
the corresponding symbol \( A(\gamma) \). For example

\[
\sigma(\exp \hat{A}) = \sum \frac{1}{n!} A \star A \star \ldots \star A
\]

is a symbol of an evolution operator of a some quantum-mechanical problem, and to find
the trace of corresponding operator one have to perform the integration of the symbol.
However, for the operators having intricate structure, obtaining its symbol via explicit
evaluation of Wigner’s mapping appear intractable. The special representation \( \gamma_h \) of
operators \( \hat{\gamma} \) appears to be more suitable.

### 3.3 Examples of Star Product

The last and the principal question is how to construct and compute a concrete star
operator \( U \) on a special phase space. Let us cite several well-known solutions of this
problem which demonstrate practical realization of formula (21) on special symplectic
structures. Let us consider some well-known examples of star product explicitly. As we
will see these examples allow to clarify a principal possibility introducing the star product
construction in superspace.

The first example we take from ref. [31]. In those paper, it was shown that the
deformation quantization yields to a noncommutative algebra of functions (20) for each
Poisson-Lie structure on the arbitrary symplectic manifold \( M \), both in the nondegenerate
and degenerate cases in the presence of the second-class constrains. In the nondegenerate
case, we take the Darboux coordinates in the initial phase space as a local model, whereas
in the degenerate case the same role of special coordinates is played by physical variables
on the constraint surface. In this case one must consider Dirac’s brackets as a classical
limit of brackets (20). For the wide range of commutation relations (14) the formal scheme
has been found and consists in replacement partial derivatives by covariant ones
\[ \partial \rightarrow \nabla \] in the Groenewold’s noncommutative star product (19), which is manifestly associative.
This scheme allows one to avoid direct reducing the dynamic on the curved shell, because
such a reduction usually breaks explicit covariance and space-time locality.

In the second example, motivated by the flat case \( T^*R^n \), authors [32] constructed
homogeneous star products of Weyl and standard ordered type on every cotangent bundle
\( T^*Q \) by means of the Fedosov procedure using a symplectic torsion-free connection. Their
result presents a surprisingly natural analogue of the operator \( U \), which takes the form
\( U = \exp(\frac{A}{2i}) \), where the second order differential operator \( \Delta \) is equal to

\[
(\Delta) = \frac{\partial^2}{\partial q^i \partial p_i} + \Gamma^i_{jl}(q) \frac{\partial}{\partial p_l} + p_k \Gamma^k_{ij}(q) \frac{\partial^2}{\partial p_i \partial p_j} + \alpha_i(q) \frac{\partial}{\partial p_i}.
\]
Here $\Gamma^i_{jk}$ are the Christoffel symbols of the connection $\nabla$ and $\alpha_i$ is a particular choice of a one-form on $Q$ such that $-d\alpha$ is equal to the trace of the curvature tensor. For the Levi-Chivita connection of a Riemannian metric $\alpha = 0$.

The last example (see ref. [33] and references therein) gives the solution of the problem of gauge invariance in the classical-quantum correspondence. Let us consider an abelian gauge theory. The most natural conjecture is to replace the gauge dependent canonical momentum $\hat{P}'$ entering definition (15) by the gauge invariant kinetic momentum $\hat{P} = \hat{P}' + A(\hat{Q})$. Obviously, the product rule for gauge invariant Weyl symbols will be different from the usual Moyal product. The algebra of operators (14) is

$$[\hat{Q}_i, \hat{Q}_j] = 0, \quad [\hat{Q}_i, \hat{P}_j] = i\hbar\delta^i_j, \quad [\hat{P}_i, \hat{P}_j] = i\hbar F_{ij}(\hat{Q})$$

(27)

The "magnetic" star product $\star_F$ corresponding to commutation relations (27) can be calculated by the formula

$$(A \star_F B)(q,p) = A(q,p)U_F B(q,p),$$

(28)

where star operator is

$$U_F = \exp\left(\frac{i}{\hbar}\phi(q, i\hbar \hat{P}_p, i\hbar \hat{Q}_p) + \frac{i\hbar}{2}(\hat{\partial}_q \hat{\partial}_p - \hat{\partial}_p \hat{\partial}_q)\right),$$

(29)

and the phase $\phi$ defined as

$$\phi(q, u_2, u_1) = \int_0^1 ds \int_0^s dt u_2 F(q + (s - \frac{1}{2})u_1 + (t - \frac{1}{2})u_2) u_1.$$

The first two terms of the magnetic star product expansion are

$$A \star_F B = AB - \frac{i\hbar}{2} \{A, B\}_F + O(\hbar^2),$$

where $\{, \}_F$ is the Poisson bracket corresponding to the symplectic form $\omega_F = \omega_0 + \frac{1}{2} F_{ij}(q)dq^i \wedge dq^j$, i.e.

$$\{A, B\}_F = \partial_p A \partial_\mu B - \partial_\mu A \partial_p B + F_{ij} \partial_p A \partial_\mu B.$$

The main object we are interested in is the star operator $U_F$ (29). Using transformation (24) of appropriate symbols one can easily obtain all objects that are needed for algebra (27). For a kind of $QP$ ordering prescription the star operator has the form $U = e^{i\partial_p \cdot \nabla}$ (see for example ref. [36]) and leads to the right regular representation of the kinetic momenta

$$\nabla^h_\mu = ip_\mu + i \int_0^1 d\tau \tau F_{\mu\nu}(x + i\tau \partial_\mu) \partial_\nu,$$

(30)

which is a the normal coordinate expansion over $\partial_p \in T^*M$. It should be noted that we have never used gauge fixing, but the second term in the right side of (30) is nothing but the vector potential in the Fock-Schwinger gauge "$y^\mu A_\mu(x + y) = 0$". This particular example shows us that we can use covariant quantities from the very beginning, which leads us to a modified star product. Summarizing the above-stated facts, it can be noticed that the well-known ad-hoc quantization rules on cotangent bundles are obtained by the deformation quantization in a very systematic way.
3.4 Application of Symbol Technique to Superfield Models

Let us discuss how the above technique can be immediately applied to supersymmetric field theories formulated in N=1 superspace. Considered in the previous section examples of star product operator for nontrivial symplectic manifolds with and general construction of quantum exponential map [21] allow us to give a good definition of Moyal-Weyl deformations of phase superspace.

First we observe that \( X = (M^n, T^*M, \omega) \) in (26) is a symplectic manifold. For each given point in the base manifold \( x \in M^n \) we have all possible tangents \( y = \partial_p \) lying in the tangent space \( TM \) which is a flat space. Therefore, using the horizontal lift of the derivative operator in the tangent bundle allows one to transfer immediately the flat definitions into covariant and gauge invariant ones even on a superspace, because the SUSY algebra is an example of a linear graded Poisson structure. The exact determination of this algebra that corresponds to the quantization of systems with both, bosonic and fermionic degrees of freedom was given in ref. [35]. In the previous section we saw the general structure of star operator. Common sense suggests us that the structure of a star operator on a superspace should match the considered examples with minimal changes related to a specificity of the supersymmetry algebra. In fact, a star operator \( U \) on flat phase superspace was first introduced in ref. [37]. Direct its calculation on superspace can be found in ref. [36].

Since practical purpose consists in finding the trace \( \text{Tr}(e^{\hat{H}(\gamma)}) \), we can summarize the steps needed:

- Construct star operator \( U \) using the commutation relation of basis operators \( \hat{\gamma} \) and calculate exactly or approximately a regular representation \( \gamma_h \).
- Replace all operators \( \hat{\gamma} \) in \( e^{\hat{H}(\gamma)} \) by their special representation \( \gamma_h \) and fulfil ordering (disentangling \( \partial_p \) to make them acting on nothing) in \( e^{\hat{H}(\gamma)} \) in order to find the symbol of \( \sigma(e^{\hat{H}(\gamma)}) \).
- Implement the integration \( \sigma(e^{\hat{H}(\gamma)}) \) over phase space with the measure \( d\mu(\gamma) \).

Let us note that for our purposes (finding the trace) we need to know neither ordering prescription nor the Stratonowich-Weyl kernel but algebra (14) only. Therefore, we always can choose the most preferable ordering prescription. All obtained formulae depend on the symplectic structure determined by the algebra and therefore, all results are gauge independent.

We consider now the supersymmetric gauge theories in N=1 superspace. As well known the basic objects of all such theories are chiral superfields of matter, superfield strengths \( W_\alpha \) and its conjugate and supercovariant derivatives satisfying the algebra

\[
\{\nabla_\alpha, \nabla_\beta\} = \epsilon^{\beta\gamma} W_\gamma, \\
\{\nabla_\alpha, \nabla_\beta\} = \epsilon_{\alpha\beta} W_\gamma, \\
[i\nabla_\alpha, i\nabla_\beta] = i\epsilon_{\alpha\beta} f_{\alpha\gamma} + i\epsilon_{\alpha\beta} f_{\alpha\gamma},
\]

(31) (32)

which along with relations

\[
\{\nabla_\alpha, \theta^\beta\} = \delta^\beta_\alpha, \quad \{\nabla_\alpha, \bar{\theta}^\beta\} = \delta^\beta_\alpha, \quad [\nabla_\alpha x^\beta] = \delta^\beta_\alpha \delta^\gamma_\gamma
\]

(33)

provide the obvious Poisson-Lie superalgebra inherited connection with flat torsion.
Let us apply previously described receipts to find right regular representation of superspace derivatives. Firstly, we introduce the following notations for symbols of flat derivative operators
\[ \sigma\left(\frac{\partial}{\partial \theta}\right) = \psi, \sigma\left(\frac{\partial}{\partial \bar{\theta}}\right) = \bar{\psi}, \sigma\left(-i \frac{\partial}{\partial x}\right) = p. \]  \hfill (34)

Then the covariant superspace derivatives will have the following symbols
\[ \sigma(\nabla_\alpha) = \left(\psi - \frac{1}{2} \theta p + \mathcal{A}\right)_\alpha, \sigma(\tilde{\nabla}_{\tilde{\alpha}}) = \left(\bar{\psi} - \frac{1}{2} \bar{\theta} \bar{p} + \tilde{\mathcal{A}}\right)_{\tilde{\alpha}}, \sigma(\tilde{\nabla}_{\tilde{\alpha} \tilde{\alpha}}) = (ip + \mathcal{A})_{\alpha \tilde{\alpha}} \]  \hfill (35)

where \( \mathcal{A} \) stands for the connection.

As it was mentioned above the algebra of operators \( [4] \) leads to a star product of their symbols. We transform symbols to the right regular representation in order to use the prescription \( [23] \) for the star product calculation. First of all the star operator \( U \) has to be constructed. The way to construct star operator \( U \) is to direct transfer the basic definitions \( [13, 16] \) and \( [13] \) in a phase superspace. It can be obtained also by analogy with examples considered because the supersymmetry algebra \( [31] \) is a particular case of \( [14] \).

For further use it is convenient to introduce two non-symmetric chiral forms for the star operator \( U \). These forms are associated with suitable heat kernels \( K^- \) and \( K^+ \) which will be discussed in more details in the Section \( [1] \). Let’s define the following star operators
\[ \text{for } K^- : \quad U = e^{-\bar{\partial}_\alpha \nabla_\alpha} e^{\frac{1}{2} (\partial_{\alpha \beta} \theta - \theta \partial_{\beta} \psi)} e^{-\partial_{\alpha} \nabla_{\alpha}} e^{-i \bar{\partial}_\alpha \bar{\psi}}, \]
\[ \text{for } K^+ : \quad U = e^{-\bar{\partial}_\alpha \nabla_\alpha} e^{\frac{1}{2} (\partial_{\alpha \beta} \theta - \theta \partial_{\beta} \psi)} e^{-\partial_{\alpha} \nabla_{\alpha}} e^{-i \bar{\partial}_\alpha \bar{\psi}}. \]  \hfill (36)

Such form of star operators can be obtained by general method described above. These forms of \( U \) correspond to some special ordering prescription for operator product. The presence of exponential \( e^{\frac{1}{2} (\partial_{\alpha \beta} \theta - \theta \partial_{\beta} \psi)} \) in \( (36) \) related to the specificity of supersymmetry algebra and serves to guarantee covariance. The different expressions for \( U \) \( [16] \) are stipulated by different choice of the phase superspace coordinates. Using operators \( [16] \) and transformation rule \( [24] \), one can find the right regular representation \( [31] \). The operators \( \gamma_h \) are written as power series in normal coordinate system of vector bundles, where role of coordinates in the tangent space \( y \) are played by right derivatives \( \partial \) with coefficients which are superspace derivatives of strength fields. For calculation of the heat kernel \( K^- \) we find
\[ \nabla^h_{\tilde{\alpha}} = \bar{\psi}_{\tilde{\alpha}}, \]
\[ \nabla^h_\alpha = \psi_\alpha - p_{\alpha \beta} \partial^\beta + \frac{i}{2} \partial^\alpha (\partial^\beta f_{\beta \alpha} + \partial^\beta f_{\beta \alpha}) + i \bar{\partial}^\beta W_{\beta} + i \partial^\beta \partial_\alpha W_{\beta} + i \bar{\partial}^\beta \partial_\alpha W_{\beta} + \ldots, \]  \hfill (37)
\[ i \nabla^h_{\alpha \tilde{\alpha}} = \{\nabla^h_\alpha, \nabla^h_{\tilde{\alpha}}\}, \]  \hfill (38)

where derivatives mean \( \partial^\tilde{\alpha} = \frac{\partial}{\partial \tilde{\psi}_{\tilde{\alpha}}}, \partial^\alpha = \frac{\partial}{\partial \psi_\alpha}, \partial^{\alpha \tilde{\alpha}} = \frac{\partial}{\partial p_{\alpha \tilde{\alpha}}} \) and the dots stand for the number of apparent higher derivative terms. For images of the material and gauge strength fields transformed by quantum exponential map \( [24] \) we keep the original notations
\[ \Phi^h = \Phi + \partial^\alpha \Psi_\alpha - \partial^2 F + \ldots, \]  \hfill (39)
\[ W^h_{\alpha} = W_{\alpha} + \partial^\beta f_{\beta \alpha} - i \partial_\alpha D' - i \partial^2 (\nabla^h_\alpha W_{\tilde{\alpha}}) + \ldots, \]
where superfields defined as $\Psi = \nabla \Phi, F = \nabla^2 \Phi$ and the definition of component fields is given in Appendix C.

The next stage of the trace exponential operator calculation is to rewrite all operators in (13) in the regular representation form (37, 39). The resulting $H_\hbar$ is used in expression (4). After implementation of the ordering procedure we have to perform an integration with the measure

$$d\mu = \frac{1}{(2\pi)^8} d^4p d^2\psi d^2\bar{\psi} d^8z.$$  (40)

The next sections demonstrates the outlined program in practice. Some of applying tricks allow us to calculate string $H \ast H \ast H \ldots$ star product nonperturbatively.

4 The Heat Kernels and Effective action

4.1 Splitting of the contributions

We begin now an application of general symbol technique to evaluating the effective action in the hypermultiplet model couple to external N=2 vector multiplet.

To find the functional trace (7) for model (3) we will use $\zeta$-representation (9) with kinetic operator (13). Accordingly to the program described in the previous section, the trace calculation consists in replacement of the differential operators and superfields by their regular representation (37), (39) and integration over measure (40). The details of this procedure are presented in the previous section.

Operator (13) includes covariant superspace derivatives of the fields $W$, $\bar{W}$ and $\Phi, \bar{\Phi}$. The expansion in powers of $\nabla \Phi$ determines the auxiliary field potential. We will not touch this problem in the present paper since it requires a special and independent investigation.

Background (2) implies that all derivatives of $\Phi$ vanish, so that we are left with (13) with constant fields $\Phi, \bar{\Phi}$, which play the role of the "mass" parameter. In this approximation the heat kernel $K(T/\mu^2)$ in (9) can be placed in a separate exponential like it was done in ref. [39] since diagonal and non-diagonal parts of the matrix (13) become commuting

$$e^{T\hat{H}} = \sum_{n=0}^{\infty} \frac{T^{2n}}{(2n)!} \begin{pmatrix} \Phi \Phi \nabla^2 \nabla^2 & 0 \\ 0 & \bar{\Phi} \bar{\bar{\Phi}} \nabla^2 \nabla^2 \end{pmatrix}^n \exp \left( \nabla^2 \nabla^2 \right).$$

Performing the two dimensional matrix trace, we obtain $\zeta_{\hat{H}} = \zeta_{\hat{H}}^+ + \zeta_{\hat{H}}^-$ with

$$\zeta_{\hat{H}}^-(s) = \int d^8 z d^8 p \int_0^\infty \frac{dT}{\Gamma(s)} T^{s-1} \sum_{n=0}^{\infty} \frac{T^{2n}}{(2n)!} \left( \frac{\Phi \bar{\Phi}}{\mu^2} \right)^n \frac{d^n}{dT^n} \exp \left( \frac{T}{\mu^2} \nabla^2 \nabla^2 \right) n.$$  (41)

and $\zeta_{\hat{H}}^+ = \zeta_{\hat{H}}^-(\nabla^2 \leftrightarrow \nabla^2)$.

The operators $\nabla^2 \nabla^2$ and $\nabla^2 \nabla^2$ are equivalent to chiral and antichiral D’Alambertians since they are acting on subspaces of chiral and antichiral quantum superfields in (13) and therefore one can use in (11) the following identities (see f.e. [24])

$$\nabla^2 \nabla^2 = \Box_+ = \Box - i\bar{W}^a \nabla_\alpha - \frac{i}{2}(\nabla \bar{W}),$$

$$\nabla^2 \nabla^2 = \Box_- = \Box - iW^a \nabla_\alpha - \frac{i}{2}(\nabla W)$$  (42)
For a general background the separation of the Hilbert space of the superfields on invariant subspaces for the supersymmetry algebra representations is a highly non-trivial problem. This problem can be solved by the observation that external field action $S_0$ in N=1 form includes kinetic terms for the $W$ and $\Phi$ fields, which involve the integration over chiral and whole superspace measure. This observation suggests an idea to present the quantum corrections in the same form. So, in order to separate out the contributions which renormalize each kinetic term separately in the action $S_0$ we present (41) as a sum, which correctly determines corresponding renormalizations. Expression (41) is naturally rewritten as a sum of two terms (as we will see later, this definition gives the correct coefficients in leading terms)

$$ζ^±_W = ζ^±_W + ζ^±_Φ.$$  (43)

The part $ζ^±_W$ will include the renormalization of the kinetic term $W^2$ in the classical action (3) and the part $ζ^±_Φ$ includes the renormalization of the kinetic term $Φ \bar{Φ}$. So, we have

$$ζ^−_Φ(s) = ∫ d^8z d^8p ∫_0^∞ \frac{dT}{Γ(s)} T^{s-1} ∑_{n=1}^{∞} \frac{T^{2n}}{(2n)!} (\frac{Φ \bar{Φ}}{μ^2})^n d^{n-1}T [exp(\frac{T}{μ^2} \Box_−) \nabla^2 \nabla^2],$$  (44)

$$ζ^−_WW(s) = ∫ d^6z d^8p ∫_0^∞ \frac{dT}{Γ(s)} T^{s-1} ∑_{n=0}^{∞} \frac{T^{2n}}{(2n)!} (\frac{Φ \bar{Φ}}{μ^2})^n d^{n}T [exp(\frac{T}{μ^2} \Box_−) \nabla^2 \nabla^2],$$  (45)

with $ζ^+ = ζ^−(\nabla^2 \leftrightarrow \bar{∇}^2, \Box_+ \leftrightarrow \Box_−)$.

At the next step we evaluate the heat kernels (9) for (45) and (44). Detailed analysis is given in the Appendix A. The results have the form

$$K^−_Φ(T) = K_{Sch}(T) \times \{1 + W^2 W^2 T^3 \sin(TG/2) \left(\frac{sin(TG/2)}{TG/2}\right)^2 (1 - \frac{3}{4} \left(\frac{sin(TG/2)}{TG/2}\right)^2 \times \right.$$

$$×(λ_2 T \coth λ_2 T + λ_1 T \cot λ_1 T))\},$$

$$K^−_WW(T) = K_{Sch}(T) T^2 W^2 \left(\frac{sin(TG/2)}{TG/2}\right)^2,$$  (47)

where $G = λ_1 + iλ_2$ and $λ_{1,2}$ are the electric and magnetic Maxwell superfields (eigenvalues $F_{ααββ} = ε_{αβ} f_{αβ} + ε_{αβ} f_{αβ}$) in a special coordinate basis related to the invariants $(λ_1 ± iλ_2) = -\frac{1}{2} (F^2 \pm F^* F)$. The other heat kernels might be obtained via a simple substitution $K^+ = K^−(G \leftrightarrow G, W \leftrightarrow W)$. The notation

$$K_{Sch} = i \frac{λ_1 T λ_2 T}{(4πT)^2 \sinh Tλ_2 \sin Tλ_1}$$  (48)

is used for known Schwinger kernel at coinciding points.

### 4.2 Asymptotic expansion of Effective Action

Let us consider the proper time expansion (10), (11) for the obtained kernels (47), (46) and investigate various terms of effective action.
4.2.1 Divergent contributions

First terms in the decomposition (10) for all heat kernels are divergent, so we will treat them one by one. As it is shown in the Appendix B, the first coefficient for $K_{\Phi\bar{\Phi}}$ corresponding to $f=0$ is

$$\zeta_0(s) = \frac{\Phi\bar{\Phi}}{2(4\pi)^2} \left( \frac{\Phi\bar{\Phi}}{4\mu^2} \right)^{-s} \left( \frac{\sqrt{\pi}}{2} \frac{\Gamma(1-s)}{\Gamma(3/2-s)} \right).$$

Implementation of $\zeta'(0)$ leads to the known Kählerian potential in EA (see e.g. [3])

$$(\Gamma_{\Phi\bar{\Phi}})_{\text{div}} = \int d^8z \frac{\Phi\bar{\Phi}}{32\pi^2} \left( 2 - \ln \frac{\Phi\bar{\Phi}}{\mu^2} \right). \quad (49)$$

which gives rise to the holomorphic Seiberg type effective potential $\bar{\Phi}\mathcal{F}'(\Phi)$. For the heat kernel $K_{WW}$ the first term corresponding to $f=2$ is

$$\zeta_2(s) = \frac{1}{2} \left( \frac{\Phi\bar{\Phi}}{4\mu^2} \right)^{-s}, \quad (50)$$

which gives the standard divergent and the holomorphic scale dependent contribution

$$(\Gamma_{WW})_{\text{div}} = -\frac{1}{2(4\pi)^2} \int d^6z W^2 \ln \frac{\Phi}{\mu}. \quad (51)$$

It is easy to see that divergent contributions (49) and (51) may be combined together to make the holomorphic part of the N=2 EA (11)

$$\mathcal{F}(\mathcal{W}) = -\frac{1}{2(4\pi)^2} \sum_{f} \frac{\Gamma(f-2)}{(\Phi\bar{\Phi})^{f-2}} a_f \mathcal{W}^2 \ln \frac{\mathcal{W}}{\mu^2}. \quad (52)$$

Equation (52) is well known Seiberg type low-energy effective potential for the model under consideration.

4.2.2 Finite contributions

Other terms in the heat kernel decomposition (10), (11) give finite contributions and correspond to the inverse mass decomposition form

$$(\Gamma_{WW})_{\text{fin}} = \frac{1}{2(4\pi)^2} \sum_{f} \frac{\Gamma(f-2)}{(\Phi\bar{\Phi})^{f-2}} a_f \mathcal{W}^2 \ln \frac{\mathcal{W}}{\mu^2}. \quad (53)$$

for the kernel $K_{WW}$ and

$$(\Gamma_{\Phi\bar{\Phi}})_{\text{fin}} = -\frac{1}{4(4\pi)^2} \sum_{f} \frac{\Gamma(f-1)}{(\Phi\bar{\Phi})^{f-1}} a_f \mathcal{W} \ln \frac{\mathcal{W}}{\mu^2}. \quad (54)$$

for the kernel $K_{\Phi\bar{\Phi}}$, where the coefficients $a_f$ are obtained from decomposition (10) for the corresponding kernels.
From (53) and (47) we can obtain

$$\left. \right|_{W_2} \left( \frac{W^2}{(\Phi \Phi)^2} \right) \zeta(t\bar{\Psi}, t\Psi),$$

(55)

where the function $\zeta(x, y)$ and quantities $t\bar{\Psi}, t\Psi$, which transform as scalars with respect to N=1 superconformal group, introduced in [18] were used

$$\zeta(x, y) = \frac{x^2(\cosh y - 1) - y^2(\cosh x - 1)}{x^2y^2(\cosh x - \cosh y)}.$$

Taking into account another kernel $K^+_WW$, we find the non-holomorphic form of the whole contribution

$$\left. \right|_{WW} \left( \frac{W^2}{(\Phi \Phi)^2} \right) \zeta(t\bar{\Psi}, t\Psi).$$

(56)

Equations (54) and (46) lead to the following result.

$$\left. \right|_{\Phi \Phi} \left( \frac{W^2}{(\Phi \Phi)^2} \right) \zeta(t\bar{\Psi}, t\Psi) -$$

$$- \frac{1}{12(4\pi)^2} \int d^8z \int_0^\infty dt e^{-t} W^2 \bar{W}^2 \lambda(t\bar{\Psi}, t\Psi) \tau(t\bar{\Psi}, t\Psi),$$

(57)

where

$$\lambda(x, y) = \frac{x^2 - y^2}{\cosh x - \cosh y},$$

$$\xi(x, y) = \frac{(\cosh x - 1 - x^2/2) - (\cosh y - 1 - y^2/2)}{\cosh x - \cosh y},$$

$$\tau(x, y) = 1 - \frac{3}{2} \frac{\cosh y - 1}{y^2} \left( \frac{y \sinh y - x \sinh x}{\cosh y - \cosh x} \right).$$

Another chiral kernel can be obtained by the replacement $\left. \right|_{\Phi \Phi} \left( \frac{W^2}{(\Phi \Phi)^2} \right) \zeta(t\bar{\Psi}, t\Psi)$. According to the expression (43), the whole set of finite contributions is obtained as a sum

$$\left. \right|_{\Phi \Phi} \left( \frac{W^2}{(\Phi \Phi)^2} \right) \zeta(t\bar{\Psi}, t\Psi).$$

(58)

Eq (58) is our final result.

As one can see the low-energy effective action (58) contains the contributions of two different types. First, the contributions of the (56)-type. These terms are analogous to ones given in [18] although they were obtained completely another method. Such terms can be rewritten in manifest N=2 superconformal manner using the proper N=2 superconformal invariant functional [18]. Another type of contributions has the structure (57), it is quite new and never been investigated before. The corresponding terms are manifestly N=1 superconformal invariant however, in fact, they are invariant under N=2 superconformal transformations (see the transformations in [23]). Technique of building the manifest N=2 superconformal invariants on the base of their N=1 projections was described in [18]. The crucial role is played by the superfields $\Psi^2$ and $\bar{\Psi}^2$ with simple transformation laws under N=2 superconformal group. We should also to introduce full
N=2 superspace measure. As a result one can restore manifest N=2 superconformal structure of the (57)-type contributions. However it is necessary to point out that in process of such restoration we have to use the derivatives of the N=1 superfields $W_\alpha$ and $\Phi$ a final expression will certainly contain the terms with higher derivatives of the N=2 strengths $W$ and conjugate which are next to leading in compare with terms kept in [18]. Thus the method used here allows in principle to go beyond the results [18] and obtain the new N=2 superconformal invariant contributions in derivative expansion of low-energy effective action.

5 Conclusion

Let us sum up the results of the paper. We have investigated a structure of induced effective action in hypermultiplet theory coupled to external N=2 vector multiplet realizing this theory in terms of N=1 superfields. Such an effective action is typical for N=1 chiral superfield models.

Within Schwinger proper-time method a calculation of the effective action is reduced to mathematical problem of evaluating the functional (super)trace of the exponent of superspace differential operator associated with second variational derivative of initial classical action. We have shown that this problem can be efficiently studied on the base of technique of operator symbols adapted in this paper to SUSY theories formulated in N=1 superspace. Use of such superfield operator symbol technique allowed to develop a general procedure of (supercovariant)derivative expansion of the effective action and calculate the leading contributions to low-energy effective action of the model under consideration.

We have found the low energy-effective action as a sum of N=2 superconformal invariant terms (56,57) constructed from superconformal blocks introduced in [18] and the holomorphic Seiberg-type terms violating N=2 superconformal symmetry. The result obtained is most general up to now low-energy induced effective action for hypermultiplet theory coupled to external N=2 vector multiplet.

The approach developed in this paper can be applied to various problems associated with calculating effective action in superfield theories. As the actual example we point out a general construction of derivative expansion of effective action in quantum superconformal invariant theories including N=4 super Yang-Mills model.

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Appendices

Appendix A. The Heat Kernel Calculation

Let us consider the heat kernel corresponding to expression (44). In the constant field background (2) it contains an abelian field strength and its scalar superpartners without higher derivatives.

The full integration measure (40) on phase superspace in (44) means that all nonzero contributions have to be proportional at least to $\bar{\psi}^2$. The regular representation $\Box^h$ in the exponent will contain $\partial_\alpha$, which will act on $\bar{\nabla}_h^2$. This action can only lower degree of $\bar{\psi}$, which is a symbol of $\bar{\nabla}_h^2$. So that, we should omit all $\partial_\alpha$ in $\Box^h$ and implement $d^2\bar{\psi}$ integration. The Laplace-type operator $\Box^h$ becomes more simple and we obtain

$$K^{-\Phi}_{\Phi}(T) = \int \frac{d^4p}{(2\pi)^4} d^2\bar{\psi} [e^T (\delta + \partial^\alpha \bar{\nabla}_h \bar{W}_\alpha - \partial^2 \bar{W}^2 + i\psi_\alpha W^\alpha + i\bar{\psi}_\alpha \partial^\beta f^\beta_{\beta}) \bar{\nabla}_h^2 ]_h ,$$

(A.1)

with

$$\hat{\Box}_h = \frac{1}{2} [\bar{\nabla}_h \bar{\nabla}_h ], \quad \bar{\nabla}_h = i\bar{\partial}_{\alpha} + \frac{1}{2} (\partial^\beta f_{\beta \alpha} + \partial^\beta_{\beta} f_{\beta \alpha} ).$$

Using the known operator identity

$$e^{A+B} = e^A \exp(\int_0^1 d\tau e^{-\tau A} B e^{\tau A} )$$

we disentangle of the Grassmanian momentum derivatives $\partial^\alpha$ to make them acting on nothing. Then we implement a trivial integration over $d^2\bar{\psi}$. The result is

$$K^{-\Phi}_{\Phi}(T) = \int \frac{d^4p}{(2\pi)^4} e^{T \hat{\Box}_h} \{1 + \frac{1}{3} T^3 \bar{W}^2 \bar{W}^2 - \frac{1}{8} T^4 \bar{W}^2 \bar{W}^2 B_\beta^\alpha B_\delta^\beta \bar{\nabla}_h (T) \bar{\nabla}_h (T) \} ,$$

(A.2)

here $\bar{W}_\alpha = W^\beta \bar{B}_\beta^\alpha$ and

$$B_\beta^\alpha = \left( \frac{e^{-iTf} - 1}{-iTf} \right)_\beta^\alpha, \quad \bar{B}_\beta^\alpha = \left( \frac{e^{iTf} - 1}{iTf} \right)_\beta^\alpha ,$$

(A.3)

$$\nabla_{\alpha \alpha} (T) = \frac{1}{T} \int_0^T d\tau (e^{-iTf})_\alpha^\delta \nabla_{\delta \delta} (e^{-iTf})_\delta^\alpha = \nabla_{\delta \delta} F_{\alpha \alpha} (T).$$

The last step is the calculation of the standard Schwinger heat kernel and moments $\langle \nabla_A \nabla_B \ldots \rangle$, i.e.

$$K(T)_{AB\ldots} = \int \frac{d^4p}{(2\pi)^4} e^{T \hat{\Box}_h} \nabla_A \nabla_B \ldots$$

This problem can be solved with aid of an elegant technique [39]. The method was used to determine the moments $\langle \nabla_A \nabla_B \ldots \rangle$ in terms of the Gaussian itself. To solve a differential equation

$$\frac{dK}{dT} = \frac{1}{2} K_{\alpha \alpha}$$

(A.4)

one can use the identity

$$0 = \int \frac{d^4p}{(2\pi)^4} \partial_p^\alpha (e^{T \hat{\Box}_h} \nabla_{\delta \delta}^\alpha )$$

on $d^2\bar{\psi}$
in order to write the expression for $K_{\alpha\alpha}^{\beta\dot{\beta}}$ in terms of $K(T)$

$$
K_{\alpha\alpha}^{\beta\dot{\beta}} = -K\mathcal{F}^{-1}_{\alpha\alpha}^{\beta\dot{\beta}}. 
$$

Now we obtain an explicit form $\mathcal{F}, \mathcal{F}^{-1}$ in terms of Maxwell’s invariants. Let us rewrite

$$
F_{\mu\nu} \text{ in the special reference frame }
$$

where $\lambda_1,2$ are related to the invariants $\mathcal{H}_\pm = \frac{1}{2}(F^2 \pm iF^*F) = (\lambda_1 \pm i\lambda_2)$². Now, $\mathcal{F}, \mathcal{F}^{-1}$ can be rewritten in the Pauli matrix basis $\sigma = (\sigma^1)_{\alpha}^{\beta}, \bar{\sigma} = (\sigma^1)_{\dot{\alpha}}^{\dot{\beta}}$ as

$$
\mathcal{F} = \frac{T}{2} \left( \frac{\sin \alpha}{\alpha} + \frac{\sin \beta}{\beta} \right) + \sigma \left( \frac{\cosh \alpha - 1}{\alpha} + i \frac{\cos \beta - 1}{\beta} \right) + \bar{\sigma} \left( \frac{\cosh \alpha - 1}{\alpha} - i \frac{\cos \beta - 1}{\beta} \right) + \sigma \bar{\sigma} \left( \frac{\sin \alpha - \sin \beta}{\alpha - \beta} \right)
$$

$$
\mathcal{F}^{-1} = \frac{1}{4T} (\alpha \coth \frac{\alpha}{2} + \beta \cot \frac{\beta}{2}) + \sigma (-\alpha + i \beta) + \bar{\sigma} (-\alpha - i \beta) + \sigma \bar{\sigma} (\alpha \coth \frac{\alpha}{2} - \beta \cot \frac{\beta}{2}),
$$

where $\alpha = 2T\lambda_2$, $\beta = 2T\lambda_1$.

In this notation the solution of equation (A.4), (A.5) is

$$
K_{\text{Sch}} = \frac{C}{\sinh T\lambda_2 \sin T\lambda_1}.
$$

By choosing the constant $C$ as $\frac{i\lambda_1\lambda_2}{(4\pi)^2}$ which corresponds to standard boundary condition for $K(T)$ to be reduced to the ordinary $\frac{1}{(4\pi T)^2}$, one finds the Schwinger result for $K(T)$ and supersymmetric ”corrections”

$$
K_{\phi\phi}^{-}(T) = K_{\text{Sch}}(T) \{ 1 + \frac{T^3}{3} W^2 \bar{W}^2 + \frac{T^4}{8} W^2 \bar{W}^2 B^\alpha B^\beta (\mathcal{F}^{-1})^{\alpha\dot{\alpha}}_{\beta\dot{\beta}} \}. 
$$

Direct computation of the traces over spinor indices leads to the full heat kernel

$$
K_{\phi\phi}^{-}(T) = \frac{i}{(4\pi T)^2 \sin \lambda_1 T} \sin \lambda_2 T \times
$$

$$
\times \left\{ 1 + \frac{W^2 \bar{W}^2 T^3}{3} \left( \frac{\sin(TG/2)}{TG/2} \right)^2 \left( 1 - \frac{3}{4} \left( \frac{\sin(TG/2)}{TG/2} \right)^2 \times
$$

$$
\times (\lambda_2 T \coth \lambda_2 T + \lambda_1 T \cot \lambda_1 T) \right\},
$$

where $G = \lambda_1 + i\lambda_2$.

The calculations of the heat kernel for (A.3) is more simple and lead to the expression

$$
K_{WW}^{-}(T) = \int \frac{d^4p}{(2\pi)^4} d^2\psi d^2\bar{\psi} [e^{T\phi} - \nabla^2]^h =
$$

$$
= \frac{i}{(4\pi T)^2 \sin \lambda_1 T} \sin \lambda_2 T \times
$$

$$
\times \left\{ T^2 \left( \frac{\sin(TG/2)}{TG/2} \right)^2 \left( \sin(TG/2) \right)^2 \right\}.
$$

As one can easy to see in order to obtain the kernels $K^+$, we should just replace $G \rightarrow \bar{G}$ in (A.8), (A.9).
Appendix B. Calculation of the $\zeta$-Functions

Subleading terms of $\Phi \bar{\Phi}$

Kernel (A.8) can be represented as a power series in proper time $T$, which starts with term $\frac{1}{4\pi T^2}$. Let us rewrite this term in the form

$$\frac{1}{T^2} = \int_0^\infty dz e^{-zT},$$

in order to remove derivatives $\frac{d^{n-1}}{dT^{n-1}}$ in (44). Consequently integrating by parts one can find the appropriate decomposition (11) with the coefficients $a_n$ which are defined by $K_{\Phi \bar{\Phi}}(T)$ given in form (10).

We investigate the first term in (11) independently because of its singularity. The simple manipulations lead to the form

$$\zeta_{(0)}^{\Phi \bar{\Phi}}(s) = \frac{\Phi \bar{\Phi}}{2(4\pi)^2} \frac{\Gamma(s+2)}{\Gamma(s)} \left(\frac{\Phi \bar{\Phi}}{\mu^2}\right)^{-s} \int_0^\infty dz z^{s-1} z F_2(1, \frac{s}{2} + 1, \frac{s}{2} + \frac{3}{2}; \frac{3}{2}; -z)$$

which is the Mellin transform of the generalized hypergeometric function $3F_2$.

Using an integral representation

$$pF_q((a_p), (b_q); z) = \frac{\Gamma(b_q)}{\Gamma(a_p)\Gamma(b_q-a_p)} \int_0^1 dt t^{a_p-1}(1-t)^{b_q-a_p-1} F_q-1((a_p-1), (b_q-1); tz)$$

and

$$\int_0^\infty dz z^{-p-1} F_1(a, b, c; -z) = \frac{\Gamma(a+p)\Gamma(b+p)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+p)\Gamma(-p)}$$

we find

$$\zeta_{(0)}^{\Phi \bar{\Phi}}(s) = \frac{\Phi \bar{\Phi}}{2(4\pi)^2} \frac{\Gamma(s+2)}{\Gamma(s)} \left(\frac{\Phi \bar{\Phi}}{4\mu^2}\right)^{-s} \left(\frac{\sqrt{\pi}}{2} \Gamma(1-s)\Gamma(1/2-s)\right).$$

This leads to the known Kählerian potential [3]

$$(\Gamma_{\Phi \bar{\Phi}})_{\text{div}} = \int d^8 z \frac{\Phi \bar{\Phi}}{32\pi^2} \left(2 - \ln \frac{\Phi \bar{\Phi}}{\mu^2}\right). \quad (B.1)$$

This result illustrates a correctness of the technique under consideration.

The following terms of the decomposition in $(T/\mu^2)^f$ are

$$\zeta_{(f)}^{\Phi \bar{\Phi}}(s) = \frac{1}{2(4\pi)^2} \frac{\Gamma(s+2+f)}{\Gamma(s)} \left(\frac{\Phi \bar{\Phi}}{\mu^2}\right)^{-s-f} \times$$

$$\int_0^\infty dz z^{s+f-1} F_3(1, \frac{s}{2} + 1, \frac{s}{2} + \frac{3}{2}; s + f + 2, \frac{3}{2}, 2, s + 2; -z) =$$

$$= \frac{1}{2(4\pi)^2} \frac{1}{(\Phi \bar{\Phi})^{-f-1}} \left(\frac{\Phi \bar{\Phi}}{\mu^2}\right)^{-s} \frac{\Gamma(1-f-s)\Gamma(f+s)\Gamma(2-s-2f)}{\Gamma(s)\Gamma(1-f)\Gamma(3-2s-2f)}. \quad (B.2)$$

Let us consider the terms with $f \neq 0$ in the $\zeta'(0)$. To eliminate the zero at $s = 0$ derivative $\frac{d}{ds}$ must act only on $1/\Gamma(s) \sim s$. It leads to decomposition

$$(\zeta_{(f)}^{\Phi \bar{\Phi}})'(0) = -\frac{1}{4(4\pi)^2} \frac{\Gamma(f-1)}{(\Phi \bar{\Phi})^{-f-1}}$$

(B.3)

where the inverse power of $\Phi \bar{\Phi}$ plays the role of the effective scale or mass.
Subleading terms for $W^2$

Let us investigate another kernel (A.9). Once again we present $K_{WW}^-(T)$ as a series (10), i.e.

$$K_{WW}^-(T) = \frac{1}{(4\pi T)^2} \sum_{f=2} \alpha_f T^f$$

and once again we obtain the Mellin transformation

$$\zeta_{(f)WW}(s) = \left(\frac{\Phi\bar{\Phi}}{4\mu^2}\right)^{-s} \left(\frac{\Phi}{4}\right)^{2-f} \frac{1}{\Gamma(s)} \sqrt{\pi} \frac{\Gamma(s + f - 2)\Gamma(-s - 2(f - 2))}{\Gamma(2 - f)\Gamma(-s - f + 5/2)}.$$ (B.5)

We obtain after the integration

$$\zeta_{(f)WW}(s) = \left(\frac{\Phi\bar{\Phi}}{4\mu^2}\right)^{-s} \left(\frac{\Phi}{4}\right)^{2-f} \frac{1}{\Gamma(s)} \sqrt{\pi} \frac{\Gamma(s + f - 2)\Gamma(-s - 2(f - 2))}{\Gamma(2 - f)\Gamma(-s - f + 5/2)}. $$ (B.6)

If $f \neq 2$ then the only contribution to $\zeta'(0)$ is given by $1/\Gamma(s)$ and the corresponding term is

$$\zeta_{(f)WW}'(0) = \frac{\Gamma(f - 2)}{2(\Phi\bar{\Phi})^{f-2}}.$$ (B.7)

The case $f = 2$ leads to the standard divergence

$$\zeta_{(2)WW}(s) = \frac{1}{2} \left(\frac{\Phi\bar{\Phi}}{4\mu^2}\right)^{-s}, \quad \zeta_{(2)WW}'(0) \sim -\frac{1}{2} \ln \frac{\Phi\bar{\Phi}}{4\mu^2}.$$ (B.8)

and we obtain the known holomorphic scale dependent contribution. The other terms in the decomposition (10) for the heat kernel $K_{WW}^+$ certainly depend on $\Phi, \nabla_{(a}\bar{W}_{b)}$ and we have to recover the full superspace measure in (43), i.e. $f d^8z = f d^6z \nabla^2$ that leads to expression (56).

**Appendix C. Components**

We use the following component structure of the N=1 superfields $W, \Phi$

$$W = \lambda_\alpha + \theta^\beta f_{\beta\alpha} - i\theta_\alpha D' + i\theta^2 \partial_{\alpha\dot{\alpha}} \bar{\lambda}^\alpha + \frac{i}{2} \theta^\beta \bar{\theta}^\dot{\beta} \partial_{\beta\dot{\beta}} \lambda_\alpha + \frac{1}{4} \theta^2 \bar{\theta}^2 \square \lambda_\alpha - \frac{i}{2} \theta^\beta \bar{\theta}^\dot{\beta} \partial_{\beta\dot{\beta}} (f_{\alpha\beta} - iC_{\alpha\beta}D'),$$

$$\Phi = \phi + \theta^\alpha \psi_\alpha + \theta^2 F + \frac{i}{2} \theta^\beta \bar{\theta}^\dot{\beta} \partial_{\beta\dot{\beta}} \phi - \frac{i}{2} \theta^2 \bar{\theta}^2 \partial_{\beta\dot{\beta}} \psi_\alpha + \frac{1}{4} \theta^2 \bar{\theta}^2 \square \phi.$$  

The known identities

$$\nabla^2 \bar{\nabla}^2 + \bar{\nabla}^2 \nabla^2 - \nabla \bar{\nabla}^2 \nabla = \square_+ = \square - iW^\alpha \bar{\nabla}_\alpha - \frac{i}{2}(\nabla \bar{W}),$$

$$\bar{\nabla}^2 \nabla^2 + \nabla^2 \bar{\nabla}^2 - \bar{\nabla} \nabla^2 \bar{\nabla} = \square_- = \square - iW^\alpha \nabla_\alpha - \frac{i}{2}(\nabla W)$$

are also exploited.
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