Applications of a new proposal for solving the “problem of time” to some simple quantum cosmological models

Atsushi Higuchi
Institut für theoretische Physik, Universität Bern
Sidlerstrasse 5, CH-3012 Bern, Switzerland
and
Robert M. Wald
Enrico Fermi Institute and Department of Physics, University of Chicago
5640 S. Ellis Ave., Chicago, IL 60637-1433

July 26, 1994

Abstract

We apply a recent proposal for defining states and observables in quantum gravity to some simple models. First, we consider the toy model of a Klein-Gordon particle in an external potential in Minkowski spacetime and compare our proposal to the theory obtained by deparametrizing with respect to a choice of time slicing prior to quantization. We show explicitly that the dynamics defined by the deparametrization approach depends upon the choice of time slicing. On the other hand, our proposal automatically yields a well defined dynamics – manifestly independent of the choice of time slicing at intermediate times – but there is a “memory effect”: After the potential is turned off, the dynamics no longer returns to the standard, free particle dynamics. Next, we apply our proposal to the closed Robertson-Walker quantum cosmology with a homogeneous massless scalar field. We choose our time variable to be the size of the universe, so the only dynamical variable is the scalar field. It is shown that the resulting theory has the expected semi-classical behavior up to the classical turning point from expansion to contraction, i.e., given a classical solution which expands for much longer than the Planck time, there is a quantum state whose dynamical evolution closely approximates this classical solution during the expanding phase. However,
when the “time” takes a value larger than the classical maximum, the scalar field becomes “frozen” at the value it had when it entered the classically forbidden region. The Taub model, with and without a homogeneous scalar field, also is studied, and similar behavior is found. In an Appendix, we derive the form of the Wheeler-DeWitt equation on minisuperspace for the Bianchi models by performing a proper quantum reduction of the momentum constraints; this equation differs from the usual form of the Wheeler-DeWitt equation, which is obtained by solving the momentum constraints classically, prior to quantization.

PACS #: 04.60.+n, 03.70.+k, 04.20.Cv
1 Introduction

The “problem of time” refers to the difficulties in defining a Hilbert space of states, observables, and a nontrivial notion of dynamics in a theory where the spacetime metric is itself a dynamical variable, so no background metrical or causal structure is present. In the canonical approach to quantum gravity, this difficulty manifests itself directly by the fact that ordinary time evolution corresponds to a diffeomorphism (i.e., a gauge transformation), and the Hamiltonian is thereby constrained to vanish \[1\]. As a result, the “wavefunction of the universe” in quantum gravity does not depend on “time”. This necessitates a serious reconsideration of the structure and interpretation of the theory provided by a naive application of the usual canonical quantization prescription.

Recently one us proposed \[2\] a prescription for defining states, observables, and dynamics in quantum gravity by making use of the background metrical and causal structure which is available in superspace. The main purpose of this paper is to further elucidate this proposal by applying it to some simple models. In Sec. 2 we briefly outline the proposal in the context of quantum cosmology. In Sec. 3 we study a toy model of massless scalar particle in an external potential in Minkowski spacetime – which shares some of the key features of the quantum-cosmological models – for the purpose of comparing the proposal with the “reduced theory” obtained by deparametrization with respect to a choice of time variable. We show explicitly in this model that the dynamics defined by the deparametrization approach depends upon the choice of time variable. In contrast, the dynamics of our proposal is well defined, but there is a “memory effect”, wherein the dynamics does not return to that of standard free particle theory after the potential is turned off. In Sec. 4 we then turn to an analysis of some features of the proposal in the context of general homogeneous cosmologies with a scalar field. In Sec. 5 we apply the proposal to the closed Robertson-Walker cosmology with a homogeneous scalar field. We find that the semi-classical description is a good approximation right up to the classical turning point from expansion to contraction. We also find that after this stage, the scalar field – which is the only dynamical variable – is frozen, i.e., it does not evolve with “time”, which is chosen to be the size of the universe. In Sec. 6 we repeat the analysis of Sec. 5 for the Taub universe with and without a homogeneous scalar field and find qualitatively similar results. Finally, in Appendix A, we derive the Wheeler-DeWitt equation for the Bianchi models by performing a quantum (rather than classical) reduction of the momentum constraints.

2 The proposal

First we briefly review the proposal of Ref. \[2\] in the context of quantum cosmology. The dynamical variables in quantum cosmology are the homogeneous spatial metric \(h_{ab}\) and possible homogeneous matter fields together with the conjugate momenta of these variables. Let us denote the configuration variables (i.e., coordinates on minisuperspace) by \(q^A (A = 1, \ldots, N)\) and their conjugate momenta by \(\pi_A\). Then, as will be discussed in detail in Sec. \[4\] (see also Appendix A), for the models under consideration here, the classical Hamiltonian constraint takes the form

\[
H = G^{AB} \pi_A \pi_B + V(q) = 0. \tag{2.1}
\]

Here the supermetric, \(G_{AB}\), has Lorentz signature, \(- + \cdots +\), and, in our models, minisuperspace is globally hyperbolic in this metric. Furthermore, \(G_{AB}\) possesses a one-parameter group of timelike conformal isometries corresponding to scaling transformations of the spatial metric,
\[ h_{ab} \rightarrow e^{2\alpha} h_{ab}. \] (2.2)

(This one-parameter group of conformal isometries extends to full superspace \[3\], and, thus, is not an artifact of the simplicity of our models.) For convenience, we shall redefine \( G_{AB} \), if necessary, by a conformal transformation so as to make these conformal isometries be true isometries.

We take the quantum version of the Hamiltonian constraint – i.e., the Wheeler-DeWitt equation – to be a Klein-Gordon equation of the form

\[ \left[ -\nabla^A \nabla_A + \xi R + V(q) \right] \Phi = 0. \] (2.3)

Since \( G_{AB} \) is naturally defined only up to conformal equivalence class, it would appear most natural to choose \( \xi \) so that (2.3) is conformally invariant \[4\] (see, however, the discussion following Eq. (A9) of Appendix A).

To define a Hilbert space structure on an appropriate subspace of solutions, \( \Phi \), of this equation, we make use of the natural (real) symplectic product, \( \Omega \), given by

\[ \Omega(\Phi_1, \Phi_2) = \int_{\Sigma} \left[ \Phi_2 \nabla_A \Phi_1 - \Phi_1 \nabla_A \Phi_2 \right] d\Sigma^A, \] (2.4)

where \( \Sigma \) is any Cauchy surface. Note that \( \Omega \) is conformally invariant \[5\] provided that we scale the wavefunction \( \Phi \) as \( \Phi \rightarrow \lambda(q)^1-N/2\Phi \) under \( G_{AB} \rightarrow \lambda(q)^2 G_{AB} \). With our choice of supermetric, minisuperspace is static as a spacetime under the isometries defined by Eq. (2.2), so if \( V(q) \) were \( \alpha \)-independent, then the solutions that are positive frequency with respect to \( \alpha \) could be chosen to form the Hilbert space \[6\], with inner product

\[ \langle \Phi_1, \Phi_2 \rangle_{KG} = -i\Omega(\bar{\Phi}_1, \Phi_2). \] (2.5)

However, the potential \( V(q) \) depends on \( \alpha \) and, in fact, there is no symmetry of the super-Hamiltonian, \( H \), that can be used to define positive frequency solutions \[8\]. Nevertheless, we have \( V \rightarrow 0 \) as \( \alpha \rightarrow -\infty \). Thus, the Wheeler-DeWitt equation (2.3) possesses an asymptotic symmetry at “early times”. It was conjectured in Ref. \[2\] that – at least in a suitable class of homogeneous cosmologies – this asymptotic symmetry enables one to construct the Hilbert space, \( \mathcal{H} \), of solutions that are positive frequency with respect to \( \alpha \) in the limit \( \alpha \rightarrow -\infty \), with inner product (2.3). We shall discuss this issue further in Sec. 4.

Assuming that the Hilbert space, \( \mathcal{H} \), is obtained as we described, we complete our construction of a quantum theory corresponding to the classical super-Hamiltonian (2.1) by specifying the self-adjoint operators on \( \mathcal{H} \) which represent “position and momentum” observables at a given “instant of time”. We take the notion corresponding to an “instant of time” in this theory to be the specification of a Cauchy hypersurface, \( \Sigma \), in minisuperspace. There exists a standard prescription for the construction of the desired position and momentum operators on the Hilbert space \( L^2(\Sigma) \) (see, e.g., appendix C of Ref. \[8\]) but our inner product is not the \( L^2 \) inner product but the Klein-Gordon one. Nevertheless, any \( C^1 \) solution to the Klein-Gordon equation which lies in our one-particle Hilbert space \( \mathcal{H} \) is uniquely determined by its value on \( \Sigma \) \[2\], since, by Eqs. (2.4) and (2.5), the difference between two solutions in \( \mathcal{H} \) having the same restriction to \( \Sigma \) must have vanishing Klein-Gordon norm. As in Ref. \[8\], we assume that the subspace, \( \mathcal{D} \), of \( C^1 \) solutions in \( \mathcal{H} \) whose restriction to \( \Sigma \) lies in \( L^2(\Sigma) \) is dense both as a subspace of \( \mathcal{H} \) and as a subspace of \( L^2(\Sigma) \), and that if a sequence in \( \mathcal{D} \) converges in both \( \mathcal{H} \) and \( L^2(\Sigma) \), then its limit in \( \mathcal{H} \) is nonzero if and only if its limit in \( L^2(\Sigma) \) is nonzero. If we view the
By comparison with Eq. (2.6), we obtain as the evolution equation for Ψ
\[ \langle \Phi_1, \Phi_2 \rangle_{KG} = (B(\Phi_1|\Sigma), B(\Phi_2|\Sigma))_{L^2} , \] (2.6)
where \( B : L^2(\Sigma) \rightarrow L^2(\Sigma) \) is a positive self-adjoint operator. We then define the position and momentum observables at “time” \( \Sigma \)
\[ \langle \Phi_1|\mathcal{O}|\Phi_2 \rangle_{KG} \equiv (B(\Phi_1|\Sigma), \hat{\mathcal{O}} B(\Phi_2|\Sigma))_{L^2} , \] (2.7)
where \( \hat{\mathcal{O}} \) is the standard representation of the observable on \( L^2(\Sigma) \). Note that the operator \( B \)
defined by Eq. (2.6) depends, in general, on the entire “history” of the minisuperspace from the “asymptotic past” (i.e., \( \alpha \rightarrow -\infty \)) to “time” \( \Sigma \), so the definition of \( \mathcal{O} \) is nonlocal in “time” as well as “space”.

The time evolution of a state \( \Phi \in \mathcal{H} \) is, of course, given simply by Eq. (2.3), but the observables, \( \mathcal{O} \), are not represented in a simple manner on \( \mathcal{H} \). It is useful, therefore, to derive the evolution equation for the corresponding states \( \Psi \equiv B(\Phi|\Sigma) \) in \( L^2(\Sigma) \), where the observables, \( \hat{\mathcal{O}} \), take a simple form. Let \( \Sigma_t \) be a one-parameter family of Cauchy surfaces labeled by a time function \( t \), and let \( t^a \) be a time evolution vector field, satisfying \( t^a \nabla_a t = 1 \), which may be decomposed into a lapse and shift via
\[ t^a = N n^a + N^a , \] (2.8)
where \( n^a \) denotes the unit normal to \( \Sigma_t \), and where \( N^a \) is tangential to \( \Sigma_t \). As noted above, for any \( \Phi \in \mathcal{H} \) which is \( C^1 \), the quantity \( (n^a \nabla_a \Phi)|_{\Sigma_t} \) is uniquely determined by \( \Phi|_{\Sigma_t} \). Assuming that \( (n^a \nabla_a \Phi)|_{\Sigma_t} \) lies in \( L^2(\Sigma_t) \) for at least a dense (in \( L^2(\Sigma_t) \)) subspace of \( \Phi \in D \), we obtain, for each \( t \) an operator \( C_t : L^2(\Sigma_t) \rightarrow L^2(\Sigma_t) \) such that
\[ (n^a \nabla_a \Phi)|_{\Sigma_t} = -iC_t(\Phi|_{\Sigma_t}) . \] (2.9)

In the following we shall omit writing “|\( \Sigma_t \)” , i.e., it should be understood that, for example, \( C_t \Phi \) means \( C_t(\Phi|_{\Sigma_t}) \). It follows immediately from Eqs. (2.4) and (2.3) that for \( \Phi_1, \Phi_2 \) in the subspace of \( D \) on which \( C_t \) is defined, we have
\[ \langle \Phi_1, \Phi_2 \rangle_{KG} = i(\Phi_1, n^a \nabla_a \Phi_2)_{L^2(\Sigma_t)} - i(n^a \nabla_a \Phi_1, \Phi_2)_{L^2(\Sigma_t)} \\
= (\Phi_1, C_t \Phi_2)_{L^2(\Sigma_t)} + (C_t \Phi_1, \Phi_2)_{L^2(\Sigma_t)} \\
= (\Phi_1, (C_t + C_t^\dagger) \Phi_2)_{L^2(\Sigma_t)} . \] (2.10)

By comparison with Eq. (2.6), we obtain
\[ B_t = (C_t + C_t^\dagger)^{1/2} . \] (2.11)

By brute force substitution, we obtain as the evolution equation for \( \Psi_t \equiv B_t \Phi_t \)
\[ \frac{\partial \Psi_t}{\partial t} = \frac{\partial}{\partial t} (B_t \Phi_t) \\
= \frac{\partial B_t}{\partial t} \Phi_t + B_t \frac{\partial \Phi_t}{\partial t} \\
= \frac{\partial B_t}{\partial t} B_t^{-1} \Psi_t - iB_t NC_t B_t^{-1} \Psi_t + B_t N^a \nabla_a (B_t^{-1} \Psi_t) \\
= -iH_{eff} \Psi_t , \] (2.12)
where the effective Hamiltonian for evolution in $L^2(\Sigma_t)$ is given by

$$H_{\text{eff}} \equiv i\frac{\partial B_t}{\partial t}B_t^{-1} + B_tC_tC_t^{-1} + iB_tN^a\nabla_a B_t^{-1}. \quad (2.13)$$

Since $B_t$ is given in terms of $C_t$ by Eq. (2.11), the effective Hamiltonian is determined directly by $C_t$. Note that since the natural volume element on $\Sigma_t$ will, in general, be “time dependent” (i.e., not Lie-derived by $t^a$), the natural $L^2$ inner product on $\Sigma_t$ will similarly be time dependent, and $H_{\text{eff}}$ will not be self-adjoint. However, we could define a self-adjoint evolution by working with a fixed “coordinate” volume element on $\Sigma_t$ to define a fixed inner product for $L^2(\Sigma_t)$ or, equivalently, working with densitized versions of $\Psi_t$. We have chosen not to do so here since this would further complicate the formulas, and, in all of the applications in this paper, the natural volume element on $\Sigma_t$ will be Lie derived by our time evolution vector field $t^a$.

The above general relations simplify considerably for the quantum cosmological models we shall consider in this paper. First, we shall choose as our Cauchy surfaces which foliate mini-superspace the hypersurfaces orthogonal to a timelike Killing field $t^a$. We choose $t^a$ as the time evolution vector field, so we have $N^a = 0$ and the Cauchy surfaces are labeled by the time function $t$ such that $t^a\nabla_a t = 1$. Furthermore, in our models, we have $N = 1$. Most importantly, in our models the $t$-dependence is separable, so we may work with (un-normalizable) basis functions for $\mathcal{H}$ of the form

$$\Phi_{\omega\sigma}(t, q^i) = \frac{f\omega(t)}{\sqrt{2\omega}}S_{\omega\sigma}(q^i), \quad (2.14)$$

where $f\omega(t)$ is such that $f\omega(t)\to e^{-i\omega t}$ as $t \to -\infty$. Here $q^i$ are the configuration variables apart from $t$, and the label $\sigma$ represents the quantum numbers other than $\omega$. The normalization of $S_{\omega\sigma}(q^i)$ is chosen to be $(S_{\omega'\sigma'}, S_{\omega\sigma})_{L^2} = \delta(\omega' - \omega)\delta(\sigma', \sigma)$, where $\delta(\sigma', \sigma)$ is an appropriate delta-function for the label $\sigma$, so that $\langle \Phi_{\omega'\sigma'}, \Phi_{\omega\sigma} \rangle_{KG} = \delta(\omega' - \omega)\delta(\sigma', \sigma)$. Note that the conservation of the Klein-Gordon inner product together with the asymptotic form of $f\omega(t)$ as $t \to -\infty$ implies that

$$\bar{f}_\omega(t)f'_\omega(t) - f\omega(t)\bar{f}'_\omega(t) = -2i\omega, \quad (2.15)$$

where $f' \equiv df/dt$.

Differentiating Eq. (2.14) with respect to $t$, we obtain

$$\frac{\partial \Phi_{\omega\sigma}}{\partial t} = \frac{f'_\omega(t)}{\sqrt{2\omega}}S_{\omega\sigma}(q^i) = \frac{f'_\omega(t)}{f\omega(t)}\Phi_{\omega\sigma}. \quad (2.16)$$

Thus, by inspection, the operator $C_t$ defined by Eq. (2.9) is diagonal in this basis, and we have

$$C_t\Phi_{\omega\sigma} = i\frac{f'_\omega(t)}{f\omega(t)}\Phi_{\omega\sigma}. \quad (2.17)$$

Using Eq. (2.13), we obtain

$$B_t\Phi_{\omega\sigma} = (C_t + C_t^\dagger)^{1/2}\Phi_{\omega\sigma}$$

$$= \sqrt{2\omega}\frac{1}{|f\omega(t)|}\Phi_{\omega\sigma}. \quad (2.18)$$
Substitution of the above expressions for $C_t$ and $B_t$ together with $N = 1$ and $N^a = 0$ into Eq. (2.13) then yields

$$H_{\text{eff}} \Psi_{\omega \sigma} = \left[ i \frac{d}{dt} \left( \frac{1}{|f_{\omega}(t)|} \right) |f_{\omega}(t)| + i \frac{f'_{\omega}(t)}{f_{\omega}(t)} \right] \Psi_{\omega \sigma}$$

$$= \frac{\omega}{|f_{\omega}(t)|^2} \Psi_{\omega \sigma} ,$$

(2.19)

where Eq. (2.13) was used again in the last line.

3 Comparison with Deparametrization

An alternative scheme to the one presented in the previous section for constructing a quantum theory corresponding to a classical theory with super-Hamiltonian of the form (2.1) is to “deparametrize” the theory prior to quantization in the manner proposed by Arnowitt, Deser, and Misner [8]. In this procedure, we first choose a time function, $t$, whose level surfaces are Cauchy surfaces, $\Sigma_t$, and choose a time evolution vector field, $t^a$, satisfying $t^a \nabla_a t = 1$. We then solve Eq. (2.1) for the momentum, $\pi_t$, canonically conjugate to $t$, thereby expressing $\pi_t$ as a function of $t$ and the remaining dynamical variables ($q^i, \pi_i$). The Hilbert space of states at time $t$ is then taken to be $L^2(\Sigma_t)$, and the quantum operators corresponding to $q^i$ and $\pi_i$ at time $t$ are defined by the standard prescription for $L^2(\Sigma_t)$. Dynamical evolution then is defined by

$$\frac{d\Phi}{dt} = -iH_{\text{ADM}} \Phi ,$$

(3.1)

where the ADM Hamiltonian is given by

$$H_{\text{ADM}} \equiv -\pi_t(t, q^i, \pi_i) .$$

(3.2)

Classically, this Hamiltonian generates the dynamics of $\{q^i, \pi_i\}$, with respect to time $t$, as can be verified by using the original Hamilton’s equations with the super-Hamiltonian $H$.

Since the proposal of the previous section also can be viewed as a theory defined on the Hilbert spaces $L^2(\Sigma_t)$ with the standard definitions of position and momentum operators, it differs from the quantum theory obtained by the above deparametrization method only via the use of the Hamiltonian (2.13) rather than (2.12). In this section, we shall compare the two approaches for the “toy model” of a relativistic particle in four dimensions with a potential of compact spacetime support, whose super-Hamiltonian is given by

$$H = -p_0^2 + p^2 + \lambda V(x_0, x) = 0 .$$

(3.3)

We study this model to first order in a perturbation series in $\lambda$. We shall show explicitly in this model that – as expected (see, e.g., [3]) – the dynamical evolution defined by the deparametrization procedure depends upon the choice of time variable, $t$, used to perform the deparametrization. The quantum theory defined by the proposal of the previous section is manifestly independent of a choice of slicing, but we shall show explicitly the presence of a “memory effect”, which illustrates the nonlocal-in-time character of dynamical evolution.

First we solve the Klein-Gordon equation (2.3) for this model to lowest order in $\lambda$. We have

$$\left( -\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x^2} \right) \Phi = \lambda V(x_0, x) \Phi .$$

(3.4)
Consider the “unperturbed” wavefunction
\[ \Phi^{(0)} = \int \frac{d^3 p}{(2\pi)^3 2p} f(p) e^{-ipx_0 + ip \cdot x}, \] (3.5)
where \( p \equiv |p| \), and write the solution to (3.4) to first order in \( \lambda \) as
\[ \Phi = \Phi^{(0)} + \Phi^{(1)} \] (3.6)
with the boundary condition \( \Phi^{(1)} \rightarrow 0 \) for \( x_0 \rightarrow -\infty \). Note that the Klein-Gordon norm of \( \Phi^{(0)} \) is
\[ \langle \Phi^{(0)}, \Phi^{(0)} \rangle_{KG} = \int \frac{d^3 p}{(2\pi)^3 2p} |f(p)|^2. \] (3.7)

The first order in \( \lambda \) correction, \( \Phi^{(1)} \), to \( \Phi \) due to the presence of the potential is given by
\[ \Phi^{(1)}(x_0, x) = \lambda \int dx_0' d^3 x' G_{KG}(x_0, x; x_0', x') V(x_0', x') \Phi^{(0)}(x_0', x'), \] (3.8)
where \( G_{KG} \) denotes the retarded Green function for the Klein-Gordon operator,
\[ G_{KG}(x_0, x; x_0', x') = -i\theta(x_0 - x_0') \int \frac{d^3 p'}{(2\pi)^3 2p'} \left[ e^{-ip'(x_0 - x_0') + ip' \cdot (x - x')} - e^{ip'(x_0 - x_0') + ip' \cdot (x - x')} \right]. \] (3.9)
Let \( x_0^{(M)} \) denote the maximum value of \( x_0 \) such that \( V(x_0, x) \neq 0 \) for some \( x \). Then we have for \( x_0 > x_0^{(M)} \),
\[ \Phi^{(1)}(x_0, x) = \lambda \int dx_0' d^3 x' G_{KG}(x_0, x; x_0', x') V(x_0', x') \Phi^{(0)}(x_0', x') \\
= -i\lambda \int \frac{d^3 p}{(2\pi)^3 2p} \frac{d^3 p'}{(2\pi)^3 2p'} \left[ \hat{V}(p' - p, p' - p) f(p) e^{-ip'x_0 + ip' \cdot x} \\
- \hat{V}(-p' - p, p' - p) f(p) e^{ip'x_0 + ip' \cdot x} \right], \] (3.10)
where
\[ \hat{V}(p_0, p) \equiv \int dx_0 d^3 x V(x_0, x) e^{ip_0x_0 - ip \cdot x}. \] (3.11)

Consider now the quantum theory obtained by deparametrizing with respect to the time variable \( t = x_0 \). Then we have
\[ \pi_t^2 = p^2 + \lambda V(t, x). \] (3.12)
The first point to note is that if \( V \) becomes negative, a real solution, \( \pi_t \), of Eq. (3.12) will not exist for all values of \( t, x \) and \( p \), and the quantization prescription will break down. In order to avoid this difficulty, we restrict consideration to potentials which are everywhere non-negative. (Note that, in contrast, the proposal given in the previous section does not require any such restriction on \( V \).) If \( V \) is non-negative, we have
\[ H_{ADM} = -\pi_t = \sqrt{p^2 + \lambda V(t, x)}. \] (3.13)

One of the nice features of this model is that the definition of \( H_{ADM} \) as a quantum operator is unambiguous: no “factor ordering” ambiguities occur for defining the operator \( p^2 + \lambda V \), and, by continuity, one must choose the positive square root of the operator in order to get agreement.
with the standard free particle theory at $\lambda = 0$. Thus, the dynamical evolution equation in this approach is

$$i \frac{\partial}{\partial x_0} \Psi_{ADM} = \left[ -\frac{\partial^2}{\partial x^2} + \lambda V(x_0, x) \right]^{1/2} \Psi_{ADM}. \quad (3.14)$$

Now consider the solution to Eq. (3.14) to first order in $\lambda$ of the form

$$\Psi_{ADM} = \Psi_{ADM}^{(0)} + \Psi_{ADM}^{(1)} \quad (3.15)$$

with $\Psi_{ADM}^{(1)} \to 0$ for $x_0 \to -\infty$, where

$$\Psi_{ADM}^{(0)} = \int \frac{d^3p}{(2\pi)^3 \sqrt{2p}} f(p)e^{-ipx_0 + ip \cdot x}. \quad (3.16)$$

Note that $\Psi_{ADM}^{(0)}$ corresponds precisely to $\Phi^{(0)}$, Eq. (3.5); the difference in normalization arises from the fact that $\Psi_{ADM}^{(0)}$ is normalized via the $L^2$ inner product, whereas $\Phi^{(0)}$ was normalized via the Klein-Gordon inner product.

The square root operator in (3.14) acts on $e^{ip \cdot x}$ to order $\lambda$ as

$$\left[ -\frac{\partial^2}{\partial x^2} + \lambda V(x_0, x) \right]^{1/2} e^{ip\cdot x} \approx pe^{ip\cdot x} + \lambda \int \frac{d^3p'}{(2\pi)^3} \frac{\tilde{V}_s(x_0, p' - p)}{p + p'} e^{ip' \cdot x}, \quad (3.17)$$

where

$$\tilde{V}_s(x_0, p) \equiv \int d^3x V(x_0, x)e^{-ip\cdot x}. \quad (3.18)$$

One can readily verify that the square of this operator is indeed $-\partial^2/\partial x^2 + \lambda V(x_0, x) + O(\lambda^2)$. From Eqs. (3.14) and (3.17), we find that the first order correction, $\Psi_{ADM}^{(1)}$, to $\Psi_{ADM}$ due to the presence of $V$ satisfies

$$\left( i \frac{\partial}{\partial x_0} - \left[ -\frac{\partial^2}{\partial x^2} + \lambda V(x_0, x) \right]^{1/2} \right) \Psi_{ADM}^{(1)} = \lambda \int \frac{d^3p}{(2\pi)^3 \sqrt{2p}} \frac{\tilde{V}_s(x_0, p' - p)}{p + p'} f(p)e^{ip' \cdot x}. \quad (3.19)$$

Using the retarded Green function for the Schrödinger operator appearing on the left side of this equation

$$G_{ADM}(x_0, x; x'_0, x'') = -i\theta(x_0 - x'_0) \int \frac{d^3p}{(2\pi)^3} e^{-ip(x_0 - x'_0) + ip\cdot (x - x')}, \quad (3.20)$$

we find for $x_0 > x_0^{(M)}$

$$\Psi_{ADM}^{(1)} = -i\lambda \int \frac{d^3p}{(2\pi)^3 \sqrt{2p}} \frac{d^3p'}{(2\pi)^3} \frac{\tilde{V}(p' - p, p' - p)}{p + p'} f(p)e^{-ip'x_0 + ip' \cdot x}. \quad (3.21)$$

In the region $x_0 > x_0^{(M)}$, we can associate to $\Psi_{ADM}$ a positive frequency solution $\Phi_{ADM}$ to the free Klein-Gordon equation by the correspondence

$$\Phi_{ADM} = (-4\nabla^2)^{-1/4}\Psi_{ADM}. \quad (3.22)$$
Writing $\Phi_{ADM} = \Phi^{(0)}_{ADM} + \lambda \Phi^{(1)}_{ADM} + O(\lambda^2)$, we have

$$\begin{align*}
\Phi^{(0)}_{ADM} &= \Phi^{(0)}, \\
\Phi^{(1)}_{ADM} &= -i\lambda \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \tilde{V}(p' - p, p' - p) f(p)e^{-ip'x^0 + ip'x^3},
\end{align*}$$

(3.23)

where $\Phi^{(0)}$ was given by Eq. (3.5). Note that $\Phi^{(1)}_{ADM}$ differs from $\Phi^{(1)}$ in Eq. (2.14). Indeed, unlike $\Phi^{(1)}_{ADM}$, $\Phi^{(1)}$ is not even a purely positive frequency solution. However, we can consider a WKB expansion of $\Phi^{(1)}_{ADM}$ and $\Phi^{(1)}$. For $V$ smooth, the negative frequency part of $\Phi^{(1)}$ is “nonperturbative” in such an expansion, i.e., it does not appear to any finite order in $\hbar$. Furthermore, we have verified that to first order in $\hbar$, $\Phi^{(1)}_{ADM}$ and $\Phi^{(1)}$ agree. This result is consistent with Barvinsky’s [10] much more general arguments in the context of quantum field theory that “reduced phase space quantization” agrees with “Dirac quantization” at one-loop order. However, in our model, we find that $\Phi^{(1)}_{ADM}$ and $\Phi^{(1)}$ differ at order $\hbar^2$.

Now we consider a second deparametrization scheme obtained by using a boosted Lorentz frame, i.e., we choose $t = x_0^\beta = x_0^0 \cosh \beta + x_1 \sinh \beta$. The initial ADM wavefunction which corresponds to $\Psi^{(0)}_{ADM}$ given by (3.16) is

$$\Psi^{(0)}_{ADM} = \int \frac{d^3p}{(2\pi)^3} f(p-\beta)e^{-ipx^3 + ip^\beta},$$

(3.25)

where $x^\beta = (x_1 \cosh \beta + x_0 \sinh \beta, x_2, x_3)$ and $p_{--} = (p_1 \cosh \beta - p_0 \sinh \beta, p_2, p_3)$. If we start from this initial wavefunction, we obtain for the lowest order correction, $\Psi^{(1)}_{ADM}$, an expression of the form (3.23) with $x_0$ replaced by $x_0^\beta = x_0^0 \cosh \beta + x_1 \sinh \beta$, $x_1$ by $x_1^\beta = x_1^0 \cosh \beta + x_0 \sinh \beta$, $f(p)$ by $f(p-\beta)$ and $\tilde{V}(p' - p, p' - p)$ by $\tilde{V}(p'_{--} - p_{--}, p'_{--} - p_{--})$. The corresponding solution, $\Phi^{(1)}_{ADM}$, of the free Klein-Gordon equation at late times is given by

$$\Phi^{(1)}_{ADM} = -i\lambda \int \frac{d^3p}{(2\pi)^3} \frac{d^3p'}{(2\pi)^3} \frac{2\sqrt{p_\beta^2 p_{--}^2 \tilde{V}(p' - p, p' - p)}}{p_\beta + p'_{--}} f(p)e^{-ip'x^0 + ip'x^3},$$

(3.26)

where $p_{--} = (p_1 \cosh \beta + p_0 \sinh \beta, p_2, p_3)$ and $p_\beta = |p_{--}|$. This disagrees with the solution $\Phi^{(1)}_{ADM}$, Eq. (3.24), obtained with the original choice of time slicing. Hence, if we physically identify ADM wavefunctions on different slices in the region to the future of the potential if they correspond to the same Klein-Gordon wavefunction, we find that the final state obtained here is physically distinct from the one obtained with the original choice of time slicing. Thus, this example – which is not clouded by factor ordering ambiguities – explicitly shows the dependence of the deparametrization method upon the choice of time slicing. It is also possible to show slice dependence of the deparametrization method with fixed initial and final time slices in lowest order in perturbation in the deformation of slices with a natural factor ordering [11].

Next, we study the model (1.3) using the quantization procedure outlined in Sec. 2. We shall study the dynamics predicted by this procedure by expressing the theory in $L^2$ form as described at the end of Sec. 2, using the time slicing $t = x_0$, so that a direct comparison can be made with the deparametrization procedure. As we shall see, a “memory effect” is present: The dynamics defined by $H_{eff}$ does not return to standard, free particle dynamics after the potential has been turned off, i.e., the system has the “memory” that it had a nonzero potential in the past. Although this effect will not be relevant for the cosmological models that we will study in
this paper, it could be important, for example, in models in which the universe tunnels through a potential barrier in an early stage.

In order to calculate $H_{\text{eff}}$, we must calculate the operator $C_t$ defined by Eq. (2.13) above. Since by Eq. (3.10) the Klein-Gordon solutions in the region $x_0 > x_0^{(M)}$ which are asymptotically positive frequency in the past take the form (to first order in $\lambda$

$$\Phi = e^{-ipt+ip\cdot x} - i\lambda \int \frac{d^3p'}{(2\pi)^3 2p'} \left[ \tilde{V}(p' - p, p' - p)e^{-ip' t + ip' \cdot x} - \tilde{V}(-p' - p, p' - p)e^{ip' t + ip' \cdot x} \right],$$

we find for $x_0 > x_0^{(M)}$

$$C_t e^{ip\cdot x} = pe^{ip\cdot x} - i\lambda \int \frac{d^3p'}{(2\pi)^3} \tilde{V}(-p' - p, p' - p)e^{i(p+p')t + ip' \cdot x}.$$ (3.27)

It follows that to first order in $\lambda$ the operator $B_t = [C_t + C_t^\dagger]^{1/2}$ is given by

$$B_t e^{ip\cdot x} = \sqrt{2} pe^{ip\cdot x} + i\lambda \int \frac{d^3p'}{(2\pi)^3} \frac{\tilde{V}(p' + p, p' - p)e^{-i(p+p')t} - \tilde{V}(-p' - p, p' - p)e^{i(p+p')t}}{\sqrt{2p + \sqrt{2p} \pm \sqrt{2p + \sqrt{2p} \mp \sqrt{2p}}}} e^{ip' \cdot x}.$$ (3.29)

Since $N = 1$ and $N^a = 0$, for $x_0 > x_0^{(M)}$ we obtain from Eq. (2.13) the result

$$H_{\text{eff}} e^{ip\cdot x} = pe^{ip\cdot x} + i\lambda \int \frac{d^3p'}{(2\pi)^3} \left[ \frac{\sqrt{p} e^{-ip' t + ip' \cdot x}}{\sqrt{p} + \sqrt{p}} \tilde{V}(p' + p, p' - p)e^{-i(p+p')t + ip' \cdot x} 

- \frac{\sqrt{p} e^{-ip' t - ip' \cdot x}}{\sqrt{p} + \sqrt{p}} \tilde{V}(-p' - p, p' - p)e^{i(p+p')t + ip' \cdot x} \right].$$ (3.30)

It is readily apparent that $H_{\text{eff}} \neq H_{\text{ADM}}$. Furthermore, although $H_{\text{eff}}$ agrees with the standard, free particle Hamiltonian $H_0 = \sqrt{-\nabla^2}$ prior to time when the potential is “turned on”, we have $H_{\text{eff}} \neq H_0$ after the potential is “turned off”. This means that, in principle, by observing the dynamics of “free” particles (after the potential has been turned off), one could deduce that a potential had previously been present. This “memory” phenomenon can occur because the positive frequency condition in the asymptotic past enters the definition of the operator $C_t$ and, thereby, $H_{\text{eff}}$ - thus making these operators nonlocal in time.

Some insight into the nature of this memory effect can be obtained by considering the simpler case where the potential is purely a function of time, $t$ (and, thus, is not of compact spacetime support). Then the positive frequency solution which in the past (before the potential is turned on) behaves as $e^{-ipt+ip\cdot x}$ evolves in the future (after the potential is turned off) to a solution of the form

$$e^{-ipt+ip\cdot x} \rightarrow \alpha(p)e^{-ipt+ip\cdot x} + \beta(p)e^{ipt+ip\cdot x},$$ (3.31)

where $|\alpha(p)|^2 - |\beta(p)|^2 = 1$. Then, it can be readily seen that for $x_0 > x_0^{(M)}$ the effective Hamiltonian is

$$H_{\text{eff}} e^{ip\cdot x} = \frac{p}{|\alpha(p)e^{-ipt} + \beta(p)e^{ipt}|^2} e^{ip\cdot x},$$ (3.32)
i.e., after the potential has been turned off, the $L^2$ wavefunctions evolve as

$$\Psi_{L^2} = \frac{\alpha(p) e^{-ipt} + \beta(p) e^{ipt}}{|\alpha(p) e^{-ipt} + \beta(p) e^{ipt}|} e^{ipt}. \quad (3.33)$$

In this case, the phase acquired in the period $T = 2\pi/p$ is still $-2\pi$ since $|\alpha(p)| > |\beta(p)|$. This implies that for a wavepacket sharply peaked around $p$, the motion of the expectation value of the position operator, $x$, will simply oscillate about the standard free-particle value. However, more complicated dynamical behavior would occur in the model considered above where $V$ has compact spacetime support.

4 Homogeneous cosmologies

In this section, we will provide more details concerning the application of the proposal outlined in Sec. 2 to general homogeneous cosmological models with a scalar field [12]. The possible homogeneous cosmologies are comprised by the Bianchi models and the Kantowski-Sachs model (see, e.g., [13]). We first consider the Bianchi models, following the discussion and notation of Ref. [14]. In the Bianchi models, the spacetime manifold, $M$, is taken to be $\mathbb{R} \times G$, where $G$ is a three dimensional Lie group. (We parametrize $\mathbb{R}$ by the variable, $t$.) For any $k \in G$, we define a diffeomorphism $\psi_k$ -- called left translation by $k$ -- by $\psi_k(g) \equiv kg$ for all $g \in G$. A tensor field $T^{abc} \, _{abcd}$ is said to be left invariant if $\psi^*_k T^{abc} \, _{abcd} = T^{abc} \, _{abcd}$, where $\psi^*_k$ is the map on tensor fields induced by $\psi_k$. The spacetime metric on $M$ is taken to be of the form

$$g_{ab} = -\nabla_a t \nabla_b t + h_{ab}(t), \quad (4.1)$$

where the spatial metric, $h_{ab}$, on $G$ is left invariant, and thus can be expanded as

$$h_{ab} = \sum_{i,j=1}^3 h_{ij}(t)(\sigma^i)_a(\sigma^j)_b, \quad (4.2)$$

where $(\sigma^i)_a$ are left invariant one-forms that are independent of $t$.

The structure constant tensor field $C^c_{ab}$ on $G$ is defined by $[v, w]^c = C^c_{ab} v^a w^b$ for any two left invariant vector fields $v^a$ and $w^a$. It is known that $C^c_{ab}$ can be expressed as [15]

$$C^c_{ab} = M^{cd} \epsilon_{dab} + \delta^c_{[a} A_{b]}, \quad (4.3)$$

where $M^{ab}$ is a left invariant symmetric tensor on $G$, $A_a$ is a left invariant one-form and $\epsilon_{abc}$ is a left invariant three-form on $\Sigma$. The Jacobi identity implies that $M^{ab} A_b = 0$. The Lie algebra is said to be of class A if $A_a = 0$, and of class B otherwise. No consistent Hamiltonian formulation is known for the Bianchi models of class B [16]. In particular, the Hamiltonian constraint does not generate the dynamics of the system. We specialize to the models of class A for this reason.

The Lie algebras of class A are uniquely determined up to isomorphisms by the rank and the signature of the tensor $M^{ab}$, where the overall sign of $M^{ab}$ is irrelevant. Thus, one has the following six inequivalent cases:

$$(0, 0, 0), (+, 0, 0), (+, -, 0), (+, +, 0), (+, +, -), (+, +, +).$$

The corresponding Lie algebras are called the Bianchi types I, II, VI$_0$, VII$_0$, VIII and IX, respectively. The unique connected, simply connected Lie group $G$ for Bianchi type IX is
SU(2), which has topology \(S^3\). The corresponding simply connected Lie groups for the other Bianchi types are noncompact, and only the (trivial) Bianchi I algebra is also the Lie algebra of a compact Lie group (namely \(U(1) \times U(1) \times U(1)\)) \cite{17}.

In the compact case, we choose the configuration variable, \(h_{ab}\), for the gravitational degrees of freedom to be the spatial metric at the identity element, \(e\). The conjugate momentum variable \(\pi^{ab}\), is then the usual momentum density at \(e\) multiplied by the “volume of space”, \(\mathcal{V}\), as determined by the 3-form \(\epsilon_{abc}\) introduced in eq.(4.3) (so that \(\pi^{ab}\) may be viewed as the integral of the momentum density over space). Similarly, we choose the configuration variable for the scalar field to be the value of \(\phi\) at the identity element and its conjugate momentum, \(\pi_\phi\), is taken to be the usual momentum density multiplied by \(\mathcal{V}\). The momentum constraint reads

\[
P_d \equiv \pi^a \epsilon_{cab} \epsilon_{dab} = 0.
\]

The Hamiltonian constraint is \cite{18}

\[
H \equiv \mathcal{V}^{-1} h^{-1/2} \left( \pi^{ab} \pi_{ab} - \frac{1}{2} \pi^2 \right) + \mathcal{V} h^{-1/2} \left( M_{ab} M^{ab} - \frac{1}{2} M^2 \right) + \frac{1}{24 \mathcal{V}} h^{-1/2} \pi^2_\phi + \sqrt{24} \mathcal{V} h^{1/2} \nu(\phi) = 0,
\]

The factors for the scalar contribution have been chosen for later convenience. Equations (4.4) and (4.5) also hold in the noncompact case (with \(\mathcal{V}\) an arbitrary constant) after suitable (infinite) rescalings have been made on the dynamical variables and \(H\).

In Appendix A, we shall consider the quantum theory obtained by treating the constraints (4.4) and (4.5) on an equal footing. In particular, we shall derive the explicit form of the Wheeler-DeWitt equation for the Bianchi IX model which results from imposing (4.4) as an operator constraint on the wavefunction. However, in the body of this paper we shall consider the models obtained by first reducing the problem at the classical level – thereby eliminating the momentum constraints classically – and then imposing (4.5) as an operator constraint. We achieve this reduction by noting that the classical momentum constraint (4.4) is equivalent to the statement that at each time, \(t\), the quantities \(\pi_{ab}\) and \(M^a_b\) commute when viewed as linear maps acting on the tangent space, \(V_e\), at the identity element \(e \in G\). (Here, all indices are raised and lowered with \(h_{ab}\) and \(h^{ab}\).) This means that we can simultaneously diagonalize these linear maps in an orthonormal basis of \(h_{ab}\). The classical evolution equations then imply that \(\pi_{ab}\), \(M^{ab}\), and \(h_{ab}\) will remain diagonal in this basis for all time. Thus, classically, there is no loss of generality in restricting to the “diagonal case”, in which case the momentum constraints become trivial.

It is convenient to choose the diagonal basis \((\sigma^i)_a\) so that \(M_i \equiv M^{ab}(\sigma^i)_a(\sigma^i)_b\) takes the form

\[
(M_1, M_2, M_3) = (0, 0, 0), (+1, 0, 0), (+1, -1, 0), \\
(+1, +1, 0), (+1, +1, -1), (+1, +1, +1),
\]

respectively, in the six cases corresponding to the different possible choices of signature. We then write the basis components of \(h_{ab}\) and \(\pi_{ab}\) as \(h_{ij} \equiv \text{diag}(h_1, h_2, h_3)\), \(\pi^{ij} \equiv \text{diag}(\pi^1, \pi^2, \pi^3)\) and \(\tilde{h}_i \equiv h^{-1/3} h_i\). We define \(\alpha\), \(\beta_+\), and \(\beta_-\) by

\[
h_1 \equiv e^{2\alpha + 2(\beta_+ + \sqrt{3}\beta_-)},
\]

\[
h_2 \equiv e^{2\alpha + 2(\beta_+ - \sqrt{3}\beta_-)},
\]

\[
h_3 \equiv e^{2\alpha - 4\beta_+}.
\]
(Note that this definition of $\alpha$ is consistent with Eq. (2.2), i.e., the maps on minisuperspace defined by Eq. (2.2) correspond to a translation of the parameter $\alpha$ defined by the above equations.) In terms of these variables, the Hamiltonian constraint becomes
\[ H = \left( \frac{V}{\sqrt{24}} \right)^{1/2} e^{-3\alpha}(\tilde{K} + \tilde{P}) = 0, \] (4.9)
where
\[ \tilde{K} = -\pi_\alpha^2 + \pi_\beta^2 + \pi_\phi^2 \] (4.10)
\[ \tilde{P} = e^{4\alpha} \left[ \frac{1}{2} \left( M_1^2 \tilde{h}_1^2 + M_2^2 \tilde{h}_2^2 + M_3^2 \tilde{h}_3^2 \right) - M_2 M_3 \tilde{h}_1^{-1} - M_3 M_1 \tilde{h}_2^{-1} - M_1 M_2 \tilde{h}_3^{-1} \right] + e^{6\alpha} V_s(\phi), \] (4.11)
where $\pi_\alpha$, $\pi_\pm$ and $\pi_\phi$ are the conjugate momenta of $\alpha$, $\beta_\pm$ and $\phi$, respectively. We also have eliminated the factor of $24 V^2$ in $\tilde{P}$ by shifting $\alpha$ by $(1/2) \ln(\sqrt{24}V)$. The Wheeler-DeWitt equation corresponding to (4.9) then is simply a Klein-Gordon equation in flat spacetime with potential $\tilde{P}$, i.e.,
\[ \left[ \partial_\alpha^2 - \partial_\beta^2 - \partial_\phi^2 + \tilde{P} \right] \Phi = 0. \] (4.12)

In the Kantowski-Sachs model [19], space has the topology of $S^1 \times S^2$ acted upon by the isometry group $U(1) \times SO(3)$. After performing a similar, classical reduction of the momentum constraints, the spacetime metric of this model can be written in the form
\[ ds^2 = -dt^2 + e^{2[\alpha(t)+2\beta(t)]} d\psi^2 + e^{2[\alpha(t)-\beta(t)]} (d\theta^2 + \sin^2 \theta d\phi^2). \] (4.13)

With scalar field matter present, the rescaled super-Hamiltonian takes the form (after some redefinition of variables)
\[ H = -\pi_\alpha^2 + \pi_\beta^2 + \pi_\phi^2 - e^{4\alpha+2\beta} + e^{6\alpha} V_s(\phi) = 0, \] (4.14)
so the Wheeler-DeWitt equation is simply
\[ \left[ \partial_\alpha^2 - \partial_\beta^2 - \partial_\phi^2 + \tilde{P} \right] \Phi = 0, \] (4.15)
\[ \tilde{P} = -e^{4\alpha+2\beta} + e^{6\alpha} V_s(\phi). \] (4.16)

It is worth elucidating the origin of the differences occurring between the quantum theory obtained by the above classical reduction of the momentum constraints as compared with the method described in the Appendix A. For definiteness, we focus attention upon the cases of the vacuum Bianchi I and Bianchi IX models. In the Bianchi I case, we have $M^{ab} = 0$, so the momentum constraint is absent entirely. The above “classical reduction” to the diagonal case is closely analogous to formulating a quantum theory of a free, nonrelativistic particle moving in $\mathbb{R}^3$ by first noting that, classically, the particle moves in a straight line, so that, classically, we may choose our coordinates in $\mathbb{R}^3$ so that the particle moves only in the $x$-direction. We may then formulate a quantum theory equivalent to that of a free particle moving in $\mathbb{R}^1$. However, this reduced theory – analogous to the treatment of the Bianchi I model given above – is not equivalent to standard free particle theory in $\mathbb{R}^3$, which is the analog of the treatment of the Bianchi I model given in Appendix A. On the other hand, for the Bianchi IX case, the
momentum constraints are nontrivial, and the number of degrees of freedom are the same for
the above classically reduced theory as for the theory obtained via a quantum reduction of
the momentum constraints as given in Appendix A. A good analog of the Bianchi IX model is
a free, nonrelativistic particle in $\mathbb{R}^3$ with the imposition of the additional constraint that its
vector angular momentum vanishes. The analog of above classical reduction procedure would
be to note that, classically, the particle moves on a straight line through the origin, and then
to formulate a quantum theory equivalent to that of a free particle moving in $\mathbb{R}^1$. The analog
of the procedure given in Appendix A would be to solve the constraint by restricting attention
to wavefunctions in $\mathbb{R}^3$ which are spherically symmetric. This yields a theory which is not
unitarily equivalent to standard free particle theory in $\mathbb{R}^1$.

Clearly, the quantum treatment of the momentum constraints given in Appendix A is “more
correct” than the classical reduction of them given above. Nevertheless, in the body of this
paper we choose to work with the models obtained by the above classical reduction, since the
Wheeler-DeWitt equation is considerably simpler in this case – being a wave equation in a flat
rather than curved spacetime. In addition, most of the previous literature has considered the
Bianchi models obtained by the classical reduction, so our consideration of them here should
facilitate comparison with previous approaches. Since the models will be used only to elucidate
the qualitative features of the quantum theory proposed in Sec. 2 – rather than to attempt to
make realistic predictions about the actual universe – we see little disadvantage to working with
the simpler, classically reduced models.

We turn our attention, now, to an investigation of whether the following two conditions –
which are necessary and sufficient for the implementation of the proposal of Sec. 2 – are likely
to be satisfied in the above Bianchi and Kantowski-Sachs models: (1) For the construction of
the Hilbert space, $\mathcal{H}$, of states, it is necessary for the potential term in the Wheeler-DeWitt
equation to vanish sufficiently rapidly as $\alpha \to -\infty$ that the notion of “asymptotically positive
frequency solutions” is well defined. (2) For the construction of observables on a “time slice”
$\Sigma_t$, it is necessary that the space, $\mathcal{D}$, of $C^1$ solutions in $\mathcal{H}$ whose restriction to $\Sigma_t$ lies in $L^2(\Sigma_t)$
be dense in both $\mathcal{H}$ and $L^2(\Sigma_t)$; it also is necessary that any sequence in $\mathcal{D}$ which converges in
both $\mathcal{H}$ and $L^2(\Sigma_t)$ have a nonzero limit in $\mathcal{H}$ if and only if it has a nonzero limit in $L^2(\Sigma_t)$.

With regard to condition (1), it should be noted that in all of our models (as well as in full
quantum gravity), the potential terms vanish exponentially rapidly as $\alpha \to -\infty$. Nevertheless,
since the potential terms also may blow up exponentially at spatial infinity, it does not
automatically follow that these terms can be neglected at “early times”, in the manner needed
for the construction of $\mathcal{H}$.

The following criterion is very useful to consider for the analysis of condition (1): Consider
the vector space, $S$, of real solutions to the Wheeler-DeWitt equation with initial data of
compact support on Cauchy surfaces. Define an “energy inner product” on $S$ associated with
each Cauchy surface $\alpha = \text{const.}$ by

$$E_\alpha(\Phi_1, \Phi_2) \equiv \int_{\alpha=\text{const.}} d\Sigma^A \left[ \partial_A \Phi_1 \partial_B \Phi_2 + \frac{1}{2} \tilde{G}_{AB}(\partial_C \Phi_1 \partial^C \Phi_2 + \tilde{P}_1 \Phi_2) \right] \left( \partial/\partial \alpha \right)^B, \quad (4.17)$$

where $\tilde{G}_{AB}$ is the flat metric appearing in (4.12) and (4.15), so that the “energy norm” at “time”
$\alpha$ of a solution, $\Phi$, is just the integral over the surface at that value of $\alpha$ of the stress-energy
tensor of $\Phi$ contracted once with the unit normal and once with $\partial/\partial \alpha$. Then the following
condition is sufficient to enable the desired Hilbert space, $\mathcal{H}$, to be constructed:

Condition (1'): For all $\Phi_1, \Phi_2 \in S$ the limit

$$E(\Phi_1, \Phi_2) \equiv \lim_{\alpha \to -\infty} E_\alpha(\Phi_1, \Phi_2) \quad (4.18)$$
exists and satisfies
\[ E(\Phi_1, \Phi_1)E(\Phi_2, \Phi_2) \geq K|\Omega(\Phi_1, \Phi_2)|^2 \] (4.19)
for some constant \( K > 0 \), where \( \Omega \) was defined by Eq. (2.4).

That condition \((1')\) implies that the desired \( \mathcal{H} \) can be constructed follows immediately from the work of Chmielowski [21]. We define the inner product, \( \mu \), on \( S \) satisfying

\[ \mu(\Phi_1, \Phi_1) = \frac{1}{\text{LUB}} \frac{1}{\mu(\Phi_2, \Phi_2)} |\Omega(\Phi_1, \Phi_2)|^2 \] (4.20)

to be the inner product associated to \( E \) by the construction of Proposition 1 of [21]. A Hilbert space, \( \mathcal{H} \), of solutions then can be constructed from \( \mu \) as discussed in detail in, e.g., [22] and [23].

This Hilbert space has the interpretation of being comprised of the “asymptotically positive frequency solutions” for the same reason that the standard one-particle Hilbert space for stationary spacetimes \([6]\) has the interpretation of being comprised of positive frequency solutions.

In the vacuum case, we believe that condition \((1')\) holds for the Bianchi models I, II, VI\(_0\), and VII\(_0\) as well as for the Kantowski-Sachs model. However, it appears that condition \((1')\) will fail for the vacuum Bianchi VIII and IX models. Nevertheless, we believe that condition \((1')\) will hold in all of the homogeneous cosmological models (including VIII and IX) in which a scalar field, \( \phi \), is present, provided only that the potential, \( V(\phi) \), does not grow exponentially rapidly in \( \phi \).

Our arguments for these beliefs are based mainly upon the classical “particle” dynamics associated with the super-Hamiltonian (4.9), since the behavior of suitable “wavepackets” satisfying (4.12) should be similar to this classical particle dynamics in the limit \( \alpha \to -\infty \). Consider, first, the vacuum case. The spatial region in the \((\beta_+ - \beta_-)\)-plane where the terms

\[ e^{4\alpha}(-M_2 M_3 \hat{h}_1^{-1} - M_3 M_1 \hat{h}_2^{-1} - M_1 M_2 \hat{h}_3^{-1}) \]

\[ = -M_2 M_3 e^{4[\alpha-(1/2)(\beta_+ - \sqrt{3}\beta_-)]} - M_3 M_1 e^{4[\alpha-(1/2)(\beta_+ - \sqrt{3}\beta_-)]} - M_1 M_2 e^{4(\alpha+\beta_+)} \]
in \( \hat{P} \) of (4.11) are large “moves away” from the origin at the speed of light as \( \alpha \to -\infty \). Hence, for large enough \(-\alpha\), these terms should be negligible in both the particle and wave dynamics, and the relevant contribution from \( \hat{P} \) should be

\[ \hat{P} \approx \frac{1}{2} (M_1^2 \hat{h}_1^2 + M_2^2 \hat{h}_2^2 + M_3^2 \hat{h}_3^2) \]

\[ = \frac{M_1^2}{2} e^{4\alpha + 4\beta_+ + 4\sqrt{3}\beta_-} + \frac{M_2^2}{2} e^{4\alpha + 4\beta_+ - 4\sqrt{3}\beta_-} + \frac{M_3^2}{2} e^{4\alpha - 8\beta_+}. \] (4.21)

These terms are well approximated as \( \alpha \to -\infty \) by potential walls located, respectively, at \( \beta_+ + \sqrt{3}\beta_- = -\alpha \), \( \beta_+ - \sqrt{3}\beta_- = -\alpha \) and \( -2\beta_+ = -\alpha \) [24]. These walls surround a region in the shape of a triangle and recede from the origin at the half the speed of light. In Bianchi types I, II, VI\(_0\), and VII\(_0\), the rank of \( M^{\alpha\beta} \) is less than three, i.e., one or more of the \( M_{ij} \)'s are zero, so at least one of these walls will be missing. In that case, if we consider evolution backwards in \( \alpha \), a generic particle trajectory should bounce from the remaining walls at most a finite number of times, after which time it can be treated as a free particle. Similarly, wavepackets should, after a finite time, escape to a region where they satisfy the free Klein-Gordon equation to an excellent approximation. Condition \((1')\) should then be satisfied in the vacuum Bianchi I, II, VI\(_0\), and VII\(_0\) models. (Indeed, these arguments can be made rigorous for the vacuum Bianchi I and II models, since they can be solved explicitly.) Similar arguments apply (rigorously) to the vacuum Kantowski-Sachs model, where the region where the potential, \(-e^{4\alpha + 2\beta_+}\), is large recedes from the origin at twice the speed of light.
In Bianchi type VIII and IX models we have $M_1^2 = M_2^2 = M_3^2 = 1$, so all the “potential walls” are present. Since these walls recede only at half the speed of light – whereas in the classical dynamics, the “particle” in the minisuperspace always moves at the speed of light between “bounces” – an infinite number of “bounces” generically occurs as $\alpha \to -\infty$. Since energy is lost on each of the “bounces” the energy should asymptotically approach zero. Similarly, in the wave dynamics, it appears that $E_\alpha \to 0$ as $\alpha \to -\infty$, and condition (1') should fail to hold.

However, the situation improves considerably if scalar field matter is included, provided that $V(\phi)$ does not grow exponentially rapidly in $|\phi|$. In that case, the scalar potential term, $e^{6\alpha}V(\phi)$, should become negligible for large $-\alpha$, and the momentum $\pi_\phi$ should become approximately conserved. The classical dynamics as $\alpha \to -\infty$ then corresponds to that of a massive particle, with mass $\pi_\phi^2$. Although the potential walls still recede at only half the speed of light, on account of the mass term the classical particle now will slow down each time it bounces off a wall. Eventually, its velocity will drop below half the speed of light, at which point the particle will no longer see the potential walls. Thus, if $\pi_\phi \neq 0$, only a finite number of bounces will occur, and the classical dynamics will not be chaotic as $\alpha \to -\infty$, i.e., the presence of a scalar field is sufficient to eliminate the classical chaotic behavior of the Bianchi VIII and IX models [26]. Furthermore, the energy of classical solutions will be approximately conserved as $\alpha \to -\infty$. Similar behavior should occur for wavepackets, so it appears that condition (1') should hold. Thus, it seems likely that all of the homogeneous cosmological models with a scalar field will satisfy the first condition needed for the implementation of the proposal of Sec. 2. Note, however, that it is known that the classical chaotic behavior is not eliminated if one includes a homogeneous electromagnetic field rather than a scalar field in Bianchi type IX model [27], so it appears that a scalar field is a necessary ingredient for the construction of the Hilbert space in the Bianchi VIII and IX models.

As for condition (2) – which requires that there be a dense subset of $L^2$-normalizable functions in the Hilbert space – we see no reason to doubt its validity in the Bianchi models for the surfaces of constant $\alpha$. However, in the Kantowski-Sachs model the solutions to Eq. (4.17), in the vacuum case which can be obtained by separation of the variables $\tau = (2\alpha + \beta)/\sqrt{3}$, $\xi = (2\beta + \alpha)/\sqrt{3}$ turn out to grow faster than exponentially for large $\beta$. We expect condition (2) to hold on the surfaces of constant $\tau$, but the rapid growth of the solutions in $\beta$ suggests that condition (2) will fail for the surfaces of constant $\alpha$, although we have not been able to prove this. Similarly, in the Bianchi IX model (as well as models obtained by suppressing some of its degrees of freedom, such as those studied in the next two sections), it appears that condition (2) will fail for some choices of Cauchy surface other than the surfaces of constant $\alpha$.

5 Robertson-Walker universe with a scalar field

In this section we apply the quantization prescription of Sec. 2 to the closed Robertson-Walker cosmology with a homogeneous, free, massless scalar field. We shall choose the hypersurfaces of constant $\alpha$ as our time slices in minisuperspace.

The classical (rescaled) super-Hamiltonian can be obtained from that of the Bianchi IX model, Eq. (4.9), by setting $\pi_\pm = \beta_\pm = 0$, i.e.,

$$H_{RW} = -\pi_\phi^2 + \pi_\phi^2 - e^{4\alpha} = 0.$$ (5.1)

The trajectories in minisuperspace corresponding to the classical solutions of the equations of
motion for this super-Hamiltonian are given by

\[ \alpha = \frac{1}{2} \{ \ln |p| - \ln \cosh [2(\phi - \phi_0)] \}, \]

(5.2)

where \( \phi_0 \) and \( p (= \pi_\phi) \) are arbitrary real constants. These trajectories describe classical universes which start at a “big bang” singularity at \( \alpha \to -\infty \) (with \( |\phi| \to \infty \)), expand to the maximum size \( \alpha_M = (1/2) \ln |p| \) at \( \phi = \phi_0 \), and then recollapse in a symmetrical manner.

We have two main motivations for studying this model: (i) As noted above, all of the classical solutions recollapse. On the other hand, the proposal of Sec. 2 treats \( \alpha \) as a “time variable”, which can be prescribed arbitrarily to “set the conditions” for the other dynamical variables. Thus, in particular, we could choose a state in \( \mathcal{H} \) which, at \( \alpha \to -\infty \), corresponds closely to a classical trajectory (5.2), and ask about the behavior of \( \phi \) for \( \alpha \gg \alpha_M \). How does \( \phi \) behave in this classically forbidden region of minisuperspace? (ii) The classical trajectory (5.2) is everywhere spacelike in minisuperspace. On the other hand, the propagation defined by the Wheeler-DeWitt equation is entirely causal with respect to the light cones defined by the metric on minisuperspace. Thus, one might anticipate some difficulties in approximating a classical trajectory by a state in \( \mathcal{H} \) even during the “expanding phase” of the classical solution. Clearly, the states in \( \mathcal{H} \) do not behave classically for \( \alpha > \alpha_M \). How close can one get to \( \alpha_M \) before the universe begins to behave nonclassically?

As we now shall see, for wavepackets in \( \mathcal{H} \) corresponding to classical solutions which expand for much longer than the Planck time, the answer to question (i) is that for \( \alpha > \alpha_M \), \( \phi \) “freezes” at the value, \( \phi_0 \), corresponding to the classical value of \( \phi \) at maximum expansion, \( \alpha = \alpha_M \). The answer to question (ii) is that despite the spacelike character of the classical trajectory, suitably chosen wavepackets in \( \mathcal{H} \) accurately describe the expanding phase of the classical solution until \( \alpha \approx \alpha_M \).

The Wheeler-DeWitt equation for this model can be obtained by setting \( \pi_\alpha = -i \partial_\alpha \) and \( \pi_\phi = -i \partial_\phi \) in Eq. (5.1), i.e., we have

\[ H_{RW} \Phi = \left[ \frac{\partial^2}{\partial \alpha^2} - \frac{\partial^2}{\partial \phi^2} - e^{4\alpha} \right] \Phi(\alpha, \phi) = 0. \]

(5.3)

Note that the deparametrization approach of Sec. 3 cannot be employed here because the potential is negative, but the method of Sec. 3 can be used without difficulty.

Since the momentum \( \pi_\phi \) is conserved, the solutions to the equation \( H_{RW} \Phi = 0 \) can be written as \( \Phi_p(\alpha, \phi) \propto f_{\{p\}}(\alpha) e^{ip\phi} \), where the function \( f_{\{p\}}(\alpha) \) satisfies

\[ \left( \frac{d^2}{d\alpha^2} + p^2 - e^{4\alpha} \right) f_{\{p\}}(\alpha) = 0. \]

(5.4)

The two linearly independent solutions of this equation are \( I_{\pm\{p\}/2}(e^{2\alpha}/2) \), where

\[ I_\nu(z) = \sum_{k=0}^{\infty} \frac{1}{k! \Gamma(\nu + k + 1)} \left( \frac{z}{2} \right)^{\nu+2k}. \]

(5.5)

The asymptotic positive frequency condition requires that \( f_{\{p\}}(\alpha) \to e^{-i\{p\}^\alpha} \) for \( \alpha \to -\infty \). Thus, the solutions relevant for constructing wavepackets which lie in \( \mathcal{H} \) are

\[ f_{\{p\}}(\alpha) = 2^{-i\{p\} \Gamma(1 - i\{p\}/2)} I_{-i\{p\}/2}(e^{2\alpha}/2). \]

(5.6)
If we impose the δ-function normalization $\langle \Phi_p, \Phi_{p'} \rangle_{KG} = \delta(p - p')$, our “basis” for solutions in $\mathcal{H}$ is

$$\Phi_p(\alpha, \phi) = \frac{f_{|p|}(\alpha)}{\sqrt{2|p|}} e^{ip\phi} \sqrt{2\pi}. \quad (5.7)$$

The function $I_\nu(z)$ behaves for large $|z|$ as

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}}. \quad (5.8)$$

Hence, in the classically forbidden region $e^\alpha \gg |p|$, the function $f_{|p|}(\alpha)$ grows like $\exp(e^{2\alpha}/2 - \alpha)$. This behavior of $f_{|p|}(\alpha)$ can be understood as follows. Eq. (5.4) has the form of a time-independent Schrödinger equation,

$$\left[ -\frac{d^2}{d\alpha^2} + e^{4\alpha} \right] f_{|p|}(\alpha) = p^2 f_{|p|}(\alpha), \quad (5.9)$$

with the term $e^{4\alpha}$ acting as a potential barrier. At large negative $\alpha$ the solutions of this equation are oscillatory, whereas at large positive $\alpha$ the solutions are growing and decaying. If one were solving a scattering problem in Schrödinger quantum mechanics, one would demand that the solution decay at large positive $\alpha$, in which case one would find equal admixtures of the oscillating solutions, $e^{-i|p|\alpha}$ and $e^{+i|p|\alpha}$, as $\alpha \to -\infty$. In our case, however, we demand that the solution behave as $e^{-i|p|\alpha}$ as $\alpha \to -\infty$, in which case the growing solution is present (and dominates) as $\alpha \to +\infty$.

The dynamical behavior of $\Phi$ in the classically forbidden region $e^\alpha \gg |p|$ is most easily examined in the $L^2$-version of the theory described in Sec. 2. By Eq. (2.19), the effective Hamiltonian which governs the $\alpha$-evolution of the $L^2$-wavefunction, $\Psi_p(\alpha, \phi)$, corresponding to $\Phi_p(\alpha, \phi)$ is

$$H_{\text{eff}} \Psi_p(\alpha, \phi) = \frac{|p|}{|f_{|p|}(\alpha)|^2} \Psi_p(\alpha, \phi). \quad (5.10)$$

Since the function $f_{|p|}(\alpha)$ grows rapidly for large $\alpha$, the effective Hamiltonian approaches zero rapidly. This implies that, as claimed above, the dynamical evolution of $\phi$ is “frozen” in the classically forbidden region, i.e., the probability distribution for $\phi$ does not change with “time”, $\alpha$, when $e^{4\alpha} \gg p^2$.

In the region where $e^{4\alpha} \ll p^2$, the wavefunctions $\Phi_p(\alpha, \phi)$ satisfy the free Klein-Gordon equation to an excellent approximation. Hence, there is no difficulty in constructing wavepackets which closely follow the classical trajectory in this region. On the other hand, as we have just seen, highly nonclassical behavior occurs for $e^{4\alpha} \gg p^2$. It is of interest to examine the behavior of wavepackets in the classically allowed region, $e^{4\alpha} < p^2$, when $e^{4\alpha} \sim p^2$ in order to determine more precisely where the semiclassical behavior breaks down. To investigate this, we consider the WKB approximation [29] and analyze the conditions under which it is valid and gives evolution corresponding closely to the classical dynamics. Since the equation we are solving is equivalent to the Schrödinger equation (5.3), we see immediately that the WKB solution to order $\hbar$ which is positive frequency in the asymptotic past is

$$f_{|p|}^{WKB}(\alpha) = \sqrt{|p| k(\alpha, p)} \exp \left[ -\frac{i}{\hbar} \int_0^\alpha k(\alpha, p) d\alpha \right], \quad (5.11)$$

where

$$k(\alpha, p) \equiv \sqrt{p^2 - e^{4\alpha}}. \quad (5.12)$$
The corresponding $L^2$-wavefunction is

$$\Psi_p^{WKB}(\alpha, \phi) = \frac{1}{\sqrt{2\pi}} \exp \left[ -i \int_0^{\alpha} k(\alpha, p) d\alpha + ip\phi \right].$$

(5.13)

The criterion for the validity of the WKB approximation is given by

$$\frac{1}{2k(\alpha, p)} \left| \frac{\partial}{\partial \alpha} k(\alpha, p) \right| = \frac{e^{4\alpha}}{|p^2 - e^{4\alpha}|^{3/2}} \ll 1.$$  (5.14)

This inequality is satisfied if $p^2 - e^{4\alpha} \gg p^{4/3}$, which, in turn, is satisfied if $p \gg 1$ and $p^{1/2} - e^\alpha \gg p^{-1/6}$. Restoring the Planck length, $l_P$, we find $p \sim (l_M/l_P)^2$, where $l_M$ is the maximum classical radius of the universe. Hence, the WKB approximation is valid provided only that $l_M \gg l_P$ and $l_M - l(\alpha) \gg (l_P/l_M)^{1/3}l_P$, where $l(\alpha) \equiv e^\alpha l_P$. Thus, as long as the universe expands classically to a radius much larger than the Planck length, the WKB approximation is valid essentially up to the classical turning point.

When the WKB approximation holds, the effective Hamiltonian is given by Eqs. (5.10) and (5.11). On the other hand, the classical deparametrized Hamiltonian (see Sec. 3) with respect to the classical turning point.

The above conclusions can be verified numerically. Fig. 1 shows the quantity $d\phi/d\alpha \equiv \partial H_{eff}/\partial p$ as a function of $\alpha$ for $p = 4, 16, 64$ and 256. It is compared with the “true” classical trajectory, $\phi_{cl}$, given by Eq. (5.2). One can clearly see that the quantity $d\phi/d\alpha$ is approximated by $d\phi_{cl}/d\alpha$ better and better as $p$ increases. Note also that the freezing of dynamics occurs more and more sharply as $|p|$ increases. Fig. 2 shows the wavepacket which, for $\alpha \rightarrow -\infty$, takes the form

$$\Phi_{p, \Delta p} = \left( \frac{2\Delta p}{\sqrt{\pi}} \right)^{1/2} \exp \left\{ -2(\Delta p)^2 \left[ \phi + \frac{1}{2} \ln 2p_0 \right] - ip_0(\alpha - \phi) \right\}$$

(5.15)

with $p_0 = 64$ and $\Delta p = 10$. The shaded area represents the interval in $\phi$ for each $\alpha$ which contains 90% of the squared $L^2$-normalized wavefunction. This is compared with the classical solution $\alpha = (1/2)[\ln p_0 - \ln \cosh(2\phi)]$. This wavepacket indeed follows the classical trajectory quite closely up to the classical turning point at $(\phi, \alpha) = (0, (1/2)\ln p_0)$. Then it is seen to freeze near $\phi = 0$ for $\alpha > (1/2)\ln p_0$.

Finally, we comment briefly on some differences between the approach taken here and those taken by Hartle and Hawking and Vilenkin. Our approach defines an entire Hilbert space of states, $\mathcal{H}$, and no state vector, $\Phi \in \mathcal{H}$, is, in any sense “preferred”. On the other hand, both Hartle and Hawking and Vilenkin seek to single out a particular wavefunction (without attempting to define a Hilbert space of states) via the imposition of boundary conditions. Nevertheless, one may inquire as to whether the Hartle-Hawking or Vilenkin wavefunctions lie in our Hilbert space, $\mathcal{H}$. Even in the context of minisuperspace models, it is not clear how to implement, in a mathematically precise manner, the general principles which have been proposed to determine the Hartle-Hawking and Vilenkin boundary conditions. However, in the context of the simple model considered in this section, it seems likely that the Hartle-Hawking wavefunction is the solution to Eq. (5.3) for which $\Phi$ is independent of $\phi$ and $\Phi(\alpha) \rightarrow \text{const.} \neq 0$ for $\alpha \rightarrow -\infty$, i.e., $\Phi_{HH}(\alpha) = I_0(e^{2\alpha}/2)$. Similarly, the Vilenkin wavefunction should be the solution to Eq. (5.3) for which $\Phi$ is independent of $\phi$ and $\Phi(\alpha)$ decays for $\alpha \rightarrow +\infty$, i.e., $\Phi_V(\alpha) = K_0(e^{2\alpha}/2)$. In any case, since the field transformation $\phi \rightarrow \phi + \text{const.}$ is a symmetry of our model, it is clear that both the Hartle-Hawking and Vilenkin
wavefunctions (if they exist) should be independent of $\phi$. On the other hand, any wavefunction in our Hilbert space approaches a Klein-Gordon normalizable free-particle wavefunction for $\alpha \to -\infty$ and the average of $\pi_\beta^2$ is necessarily nonzero. Hence, neither the Hartle-Hawking nor Vilenkin wavefunctions lie in our Hilbert space.

### 6 The Taub model

In this section we study the Taub model [37], which can be obtained from the Bianchi IX model by letting $\beta_+ = \pi_+ = 0$. From (4.9) we see that the classical super-Hamiltonian for the Taub model is

$$H_T = -\pi_\alpha^2 + \pi_\beta^2 + e^{4\alpha}V(\beta_+) = 0,$$

(6.1)

where

$$V(\beta_+) \equiv \frac{1}{3} e^{-8\beta_+} - \frac{4}{3} e^{-2\beta_+}.$$  

(6.2)

As in the Robertson-Walker model of the previous section, all of the classical solutions start at an initial singularity at $\alpha \to -\infty$, expand to a maximum size, and then recollapse.

The super-Hamiltonian (6.1) is simplified by defining

$$\tau = (2\alpha - \beta_+)/\sqrt{3},$$

(6.3)

$$\xi = (2\beta_+ - \alpha)/\sqrt{3}$$

(6.4)

so that the new coordinates, $(\tau, \xi)$, on minisuperspace differ from $(\alpha, \beta_+)$ by a Lorentz boost.

In terms of the new coordinates, the Wheeler-DeWitt equation takes the form

$$H_T \Phi = \left[ \frac{\partial^2}{\partial \tau^2} - \frac{4}{3} e^{2\sqrt{3}\tau} \frac{\partial^2}{\partial \xi^2} + \frac{1}{3} e^{-4\sqrt{3}\xi} \right] \Phi = 0.$$  

(6.5)

This equation can be solved exactly [38] by separation of variables. The equations for $\xi$ and $\tau$ both take the form

$$\left[ \frac{d^2}{dx^2} - A e^{Bx} \right] F(x) = -\omega^2 F(x).$$

(6.6)

The general solution to this equation is

$$F(x) = Z_{2i\omega/B} \left( \frac{2\sqrt{A}}{B} e^{x/2} \right),$$

(6.7)

where $Z_\nu(z)$ is any modified Bessel function. The relevant solutions for $\mathcal{H}$ are those of the form

$$\Phi_\omega(\tau, \xi) = \frac{1}{\sqrt{2\omega}} f_\omega(\tau) S_\omega(\xi).$$

(6.8)

where $f_\omega(\tau) \to e^{-i\omega \tau}$ ($\omega > 0$) for $\tau \to -\infty$ and where $S_\omega(\xi)$ has nonsingular $\xi$-dependence. These conditions determine $f_\omega$ and $S_\omega$ to be

$$f_\omega(\tau) = 3^{-i\omega/\sqrt{3}} \Gamma(1 - i\omega/\sqrt{3}) I_{-i\omega/\sqrt{3}} \left( \frac{2}{3} e^{\sqrt{3}\tau} \right)$$

(6.9)

and

$$S_\omega(\xi) \equiv \sqrt{\frac{\omega \sinh \left[ \frac{\pi \omega}{2\sqrt{3}} \right]}{\sqrt{3} \pi^2}} K_{-i\omega/(2\sqrt{3})} \left( \frac{1}{6} e^{-2\sqrt{3}\xi} \right).$$

(6.10)
Here we have normalized $S_\omega$ by requiring

$$\int_{-\infty}^{+\infty} d\xi S_\omega(\xi) S_{\omega'}(\xi) = \delta(\omega - \omega')$$

(6.11)

and we have chosen the phase of $S_\omega(\xi)$ to make it real. With this normalization, we have $\langle \Phi_\omega, \Phi_{\omega'} \rangle_{KG} = \delta(\omega - \omega')$.

In this model, it is easiest to investigate dynamical evolution by choosing the “time slices” in minisuperspace to be the hypersurfaces of constant $\tau$ rather than $\alpha$. Again, we have $N^a = 0$ and we choose $N_a = 0$. The $L^2$-wavefunctions corresponding to $\Phi_\omega(\tau, \xi)$ are given by

$$\Psi_\omega(\tau, \xi) \equiv B_{\tau} \Phi_\omega(\tau, \xi) = \frac{f_\omega(\tau)}{|f_\omega(\tau)|} S_\omega(\xi).$$

(6.12)

and, by Eq. (2.19), the effective Hamiltonian is

$$H_{\text{eff}} \Psi_\omega(\tau, \xi) = \frac{\omega}{|f_\omega(\tau)|^2} \Psi_\omega(\tau, \xi).$$

(6.13)

Since $f_\omega(\tau)$ blows up for $\tau \to +\infty$, we find that – as in the case of the scalar field dynamics in the Robertson-Walker model – $H_{\text{eff}}$ rapidly goes to zero, and the dynamics of $\xi$ with respect to $\tau$-time becomes “frozen” in the classically forbidden region, $\frac{4}{3} e^{2\sqrt{3}\tau} > \omega^2$.

The WKB approximation can be considered for the $\tau$-dependence of the wavefunction with fixed $\omega$. As in the Robertson-Walker model of the previous section, the criterion for the validity of the WKB approximation is satisfied in the classically allowed region, $\frac{4}{3} e^{2\sqrt{3}\tau} < \omega^2$, until very close to the classical turning point, provided only that the universe expands to a radius much larger than the Planck length. Thus, the (un-normalizable) basis functions for $L^2$ states can be approximated by

$$\Psi_\omega(\tau, \xi) \approx \exp \left( -i \int^\tau d\tau \kappa(\tau, \omega) \right) S_\omega(\xi),$$

(6.14)

where

$$\kappa(\tau, \omega) \equiv \left( \omega^2 - \frac{4}{3} e^{2\sqrt{3}\tau} \right)^{1/2}. $$

(6.15)

Similar arguments apply to the function $S_\omega(\xi)$. In this case, the WKB approximation breaks down near the “spatial” classical turning point, i.e., for $e^{-4\sqrt{3}\xi} \approx 3\omega^2$. However, as long as $\omega^2 \gg 1$, the region where the WKB approximation fails is small. Thus, except in a small region near the (spatial) classical turning point, the function $S_\omega(\xi)$ can be approximated by a linear combination of the functions

$$S_{\omega \pm}^{\text{WKB}}(\xi) = \sqrt{\frac{\omega}{2\pi k(\xi, \omega)}} \exp \left[ \pm i \int^\xi d\xi k(\xi, \omega) \right],$$

(6.16)

where

$$k(\xi, \omega) \equiv \left( \omega^2 - \frac{1}{3} e^{-4\sqrt{3}\xi} \right)^{1/3}. $$

(6.17)

Combining this result and that for the $\tau$-dependent part [Eq. (6.14)], we conclude that a wavepacket can be made to follow the classical trajectory (except very near the point where the wavepacket bounces off the “wall” in $\xi$) up to the classical turning point from expansion to contraction.
Again, this conclusion can be verified numerically. Fig. 3 shows a wavepacket with the central value of \( \bar{\omega} \equiv \omega/\sqrt{3} = 32 \) and \( \Delta \bar{\omega} = 5 \) which is arranged in such a way that it bounces off the wall before reaching the maximum expansion. It clearly shows that the wavepacket can follow the classical trajectory up to the classical turning point in \( \tau \) and then freeze there.

The same analysis can be repeated for the Taub model with a homogeneous scalar field. The super-Hamiltonian of this model is given by

\[
H_{TS} = -\pi_\alpha^2 + \pi_{\beta_k}^2 + \pi_\phi^2 + e^{4\alpha} V(\beta_+) = 0.
\] (6.18)

One can readily solve the corresponding Wheeler-DeWitt equation by again introducing the variables \( \tau \) and \( \xi \), Eqs. (6.3) and (6.4). Since \( \pi_\phi \) is conserved, we obtain the basis solutions

\[
\Phi_{\omega p}(\tau, \xi, \phi) = \frac{1}{\sqrt{2 \eta(\omega, p)}} f_{\eta(\omega,p)}(\tau) S_\omega(\xi) e^{ip\phi} \sqrt{2\pi},
\] (6.19)

where \( \eta(\omega, p) \equiv \sqrt{\omega^2 + p^2} \). The corresponding \( L^2 \) basis functions, \( \Psi_{\omega p} \), are given by

\[
\Psi_{\omega p}(\tau, \xi, \phi) = \frac{f_{\eta(\omega, p)}(\tau)}{|f_{\eta(\omega, p)}(\tau)|} S_\omega(\xi) \frac{e^{ip\phi}}{\sqrt{2\pi}}.
\] (6.20)

and the effective Hamiltonian, \( H_{\text{eff}} \), is

\[
H_{\text{eff}} \Psi_{\omega p}(\tau, \xi, \phi) = \frac{\eta(\omega, p)}{|f_{\eta(\omega, p)}(\tau)|^2} \Psi_{\omega p}(\tau, \xi, \phi).
\] (6.21)

One advantage of this model is that there is more than one dynamical variable present, so that one could regard one of the variables – say, \( \phi \), since classically it evolves monotonically – to be a “physical clock”, against which the evolution of the other variable, \( \xi \), is to be measured. However, since \( f_{\eta(\omega, p)}(\tau) \to \infty \) as \( \tau \to \infty \), it is clear that all of the dynamical variables, including the “physical clock”, will “freeze” in the classically forbidden region. Thus, we see that the “stoppage of motion” predicted in this model actually would be physically unobservable; rather, this freezing would correspond, physically, to measurable time stopping at the maximum expansion of the universe.

It would be of interest to investigate whether the above “freezing” phenomenon persists for other choices of time slicing – particularly the \( \alpha \)-slicing – and, if so, which variables “freeze”. Our calculations above for the \( \tau \)-slicing were enormously simplified by the presence of an orthogonal basis of solutions in a separated form. Since the \( \tau \)-derivatives of these basis solutions are proportional to themselves, \( C_t \) is diagonal in this basis. Thus, one can easily obtain \( C_t \), and, thereby, calculate \( H_{\text{eff}} \). On the other hand, for the \( \alpha \)-slicing, although it is not difficult to calculate \( C_t \) directly from the definition (2.9), we do not thereby obtain simple expression for \( C_t \). For this reason, we have been unable to calculate \( H_{\text{eff}} \) and study the issue of “freezing” in the \( \alpha \)-slicing.

**Acknowledgments**

We thank A. Ashtekar, A.O. Barvinsky, B.K. Berger, R. Geroch, P. Hájíček, A. Hosoya, B.S. Kay, H. Kodama, K. Kuchař, J. Louko, C. Rovelli and R.S. Tate for useful discussions. This work was supported in part by Schweizerischer Nationalfonds and the US National Science Foundation under Grant No. PHY 92-20644.
Appendix A: An alternative quantization of class A Bianchi models

In Sec. 4 we eliminated the momentum constraint of the class A Bianchi models by noting that, classically, we can choose $h_{\alpha\beta}$ and $\pi^{\alpha\beta}$ to be diagonal. We then wrote down the Wheeler-DeWitt equation for the diagonal Bianchi metrics and found it to have the form of a Klein-Gordon equation in a 3-dimensional flat spacetime with a time-dependent potential. However, it is more natural not to impose the diagonal form of the metric at the outset, and to proceed by treating the Hamiltonian and momentum constraints on an equal footing. We employ this latter approach in this Appendix. For simplicity, we specialize to the vacuum models and will focus upon the Bianchi I and Bianchi IX cases. Many of the results here, including the algebra of the momentum constraints, have been obtained by Ashtekar and Samuel [40].

We consider, first, the form of the Wheeler-DeWitt equation for the Bianchi models when the restriction to diagonal metrics is not imposed. For each Bianchi Lie group, $G$, minisuperspace is comprised of the 6-dimensional manifold of left invariant metrics, $h_{ab}$, on $G$. The Hamiltonian constraint is

$$ H = G_{abcd} \pi^{ab} \pi^{cd} + V = 0, \quad (A1) $$

where the inverse supermetric is

$$ G_{abcd} = \frac{1}{2} h^{-1/2} (h_{ac} h_{bd} + h_{ad} h_{bc} - h_{ab} h_{cd}) \quad (A2) $$

and the determinant, $h$, of $h_{ab}$ is defined with respect to a fixed, left invariant volume element on $G$. It should be noted that the two lower index pairs $(ab)$ and $(cd)$ correspond, respectively, to the upper indices $A$ and $B$ in Eq. (2.1) of Sec. 2. Thus, the supermetric, $G_{AB}$, is

$$ G_{abcd} = \frac{1}{2} h^{1/2} \left( h^{ac} h^{bd} + h^{ad} h^{bc} - 2 h^{ab} h^{cd} \right). \quad (A3) $$

The geometry of $G_{AB}$ was studied in the classic work of DeWitt [41]. The signature of $G_{AB}$ is Lorentzian ($-++++$), and the transformations $h_{ab} \rightarrow e^{2\alpha} h_{ab}$ define timelike, hypersurface orthogonal, conformal isometries of $G_{AB}$. We may identify each of these orthogonal hypersurfaces with the 5-dimensional manifold, $M_C$, of left invariant conformal metrics $\tilde{h}_{ab}$ on $G$. The supermetric then takes the form,

$$ G_{AB} = e^{3\alpha} ( - 24 \nabla_A \alpha \nabla_B \alpha + H_{AB} ). \quad (A4) $$

The Riemannian metric, $H_{AB}$, on $M_C$ is invariant under the $SL(3, \mathbb{R})$ transformations

$$ h_{ab} \rightarrow N^c_a N^d_b h_{cd}, \quad (A5) $$

where $N^c_a$ satisfies $\det N = 1$. If we introduce arbitrary local coordinates $\zeta^\Lambda (\Lambda = 1 - 5)$ on $M_C$, then the coordinate components of $H_{AB}$ take the explicit form

$$ H_{\Lambda \Sigma} = \tilde{h}_{ab} \frac{\partial \tilde{h}_{bc}}{\partial \zeta^\Lambda} \frac{\partial \tilde{h}_{cd}}{\partial \zeta^\Sigma}, \quad (A6) $$

where $\tilde{h}^{ab}$ is the inverse of $\tilde{h}_{ab}$. The Ricci tensor and the scalar curvature of $H_{AB}$ are

$$ \bar{R}_{AB} = - \frac{3}{4} H_{AB}, \quad (A7) $$

$$ \bar{R} = - \frac{15}{4} \quad (A8) $$
Thus, in contrast to the diagonal case, the geometry of mini-superspace is now curved. Thus, the Wheeler-DeWitt equation (2.3) corresponding to (A1) now takes the form of a curved space wave equation

\[
\frac{\partial^2}{\partial \alpha^2} - 24D_A D^A - 90\xi + 24V \Phi = 0,
\]
(A9)

where \(D_A\) is the derivative operator associated with \(H_{AB}\).

For the Bianchi I model, no momentum constraints are present so the full quantum constraints will be imposed on \(\Phi\) by satisfying Eq. (A9) with \(V = 0\). Since Eq. (A9) is a curved space equation, it is easily seen that the theory obtained in this manner is not, in any sense, equivalent to the theory obtained by restricting to diagonal metrics as done in Sec. 4. (The reasons for this inequivalence were explained in that section.)

One further feature of Eq. (A9) in the Bianchi I model is worth noting. Since minisuperspace is six dimensional, the natural, conformally invariant version of the Wheeler-DeWitt equation is obtained by choosing \(\xi = 1/5\). However, the lower bound of the spectrum of \(-D_A D^A\) is 1/2 (see, e.g., Ref. [42], p. 49). Thus, the Wheeler-DeWitt equation with this choice of \(\xi\) is “tachyonic”, and there exist spatially well behaved solutions which grow exponentially as \(\alpha \to \pm \infty\). In particular, the conformally invariant choice of Wheeler-Dewitt equation does not allow the division of the space of solutions into asymptotic positive and negative frequency subspaces as \(\alpha \to -\infty\), i.e., our prescription for the construction of \(\mathcal{H}\) breaks down. However, this difficulty can be avoided by making a different choice of \(\xi\).

The classical momentum constraints

\[
-2D_a \pi^{ab} = 0
\]
(A10)

are nontrivial in all the other Bianchi models. (Here \(D_a\) denotes the derivative operator on \(G\) associated with the left invariant metric \(h_{ab}\).) In order to motivate a natural operator version of these constraints, we rewrite them as follows. First, if \(s^a\) is a left invariant vector field on \(G\), then by a direct computation using the formula for the connection in terms of the structure constant field \(C^c_{ab}\) (see, e.g., Ref. [43], p. 314), we obtain

\[
D_a s^c h_{bc} = C^c_{ba} s^a = 0.
\]

Hence, we find that the momentum constraint

\[
P_s \equiv -2s^c h_{bc} D_a \pi^{ab}
\]
(A11)

can be rewritten as

\[
P_s = -2D_a(s^c h_{bc} \pi^{ab}) + 2(D_a s_b) \pi^{ab}
\]
\[
= (\mathcal{L}_s h_{ab}) \pi^{ab},
\]
(A12)

since the first term on the right side of the first line of this equation is the divergence of a left invariant vector field and, thus, vanishes.

When we promote \(P_s\) to an operator, it is natural to choose the factor ordering (A12), since then the constraint \(P_s \Phi = 0\) implies that the wavefunction \(\Phi\) is invariant under the transformation \(\delta h_{ab} = \mathcal{L}_s h_{ab}\). (As is well known, an analogous choice and interpretation of the momentum constraint operators can be made in full quantum gravity [41], the only difference being that neither \(h_{ab}\) nor \(s^a\) are restricted to be left invariant.) Note that with this choice of \(P_s\), we have

\[
[h_{ab}, P_s] = i\mathcal{L}_s h_{ab},
\]
(A13)

\[
[\pi^{ab}, P_s] = i\mathcal{L}_s \pi^{ab},
\]
(A14)
and

\[ [P_s, P_t] = iP_{[s,t]} . \]  \quad (A15)

Let us study the imposition of the momentum constraints for the case of the Bianchi IX model in some detail. The Lie group, \( G \), of the Bianchi IX model is \( SU(2) \), and the inverse, \((M^{-1})_{ab}\), of the left invariant tensor field \( M^{ab} \) defined by Eq. (4.3) provides a positive definite metric on the 3-dimensional vector space, \( W \), of left invariant vector fields. The exponentiated version of the momentum constraints \( P_s \Phi = 0 \) imply that the allowed wavefunctions are invariant under the transformation

\[ h_{ab} \rightarrow O_a^c O_b^d h_{cd} , \]

where \( O_a^c \) is an arbitrary orthogonal linear map [with respect to \((M^{-1})_{ab}\)] on \( W \) with positive determinant.

It is convenient to introduce the following parametrization of minisuperspace. Since \( h_{ab} \) defines a positive self-adjoint map on \( W \) in the inner product \((M^{-1})_{ab}\), we may characterize each \( h_{ab} \) (i.e., point in minisuperspace) by its eigenvalues, \( \lambda_1 \geq \lambda_2 \geq \lambda_3 > 0 \), and the rotation, \( O \), which carries its triad of eigenvectors into some fixed, orthonormal basis. Note, however, that this parametrization becomes singular when degeneracy occurs in the eigenvalues of \( h_{ab} \).

The Hilbert space of states, \( \mathcal{H} \), is constructed from functions \( \Phi(\lambda_1, \lambda_2, \lambda_3, O) \) on minisuperspace which satisfy the momentum and Hamiltonian constraints. The momentum constraints are easily imposed, since they simply require \( \Phi \) to be independent of \( O \). The form of the Wheeler-DeWitt equation in these coordinates can be obtained as follows. On the manifold, \( M_C \), of conformal metrics (where \( \lambda_1 \lambda_2 \lambda_3 = 1 \), so only two of the \( \lambda_i \)’s are independent), we have from Eq. (A11)

\[
H_{\Lambda \Sigma} = \text{Tr} \left[ h^{-1} \frac{\partial h}{\partial \zeta^\Lambda} h^{-1} \frac{\partial h}{\partial \zeta^\Sigma} \right] = -\text{Tr} \left( \frac{\partial T}{\partial \zeta^\Lambda} \frac{\partial T^{-1}}{\partial \zeta^\Sigma} \right) - 2\text{Tr} \left[ O^{-1} \frac{\partial O}{\partial \zeta^\Lambda} O^{-1} \frac{\partial O}{\partial \zeta^\Sigma} - O^{-1} \frac{\partial O}{\partial \zeta^\Lambda} T O^{-1} \frac{\partial O}{\partial \zeta^\Sigma} T^{-1} \right] . \quad (A16)
\]

where \( T_{ab} \) is the diagonal matrix with eigenvalues \( \lambda_1, \lambda_2, \lambda_3 \). By writing

\[
O^{-1} dO = \begin{pmatrix}
0 & -X_3 & X_2 \\
X_3 & 0 & -X_1 \\
-X_2 & X_1 & 0
\end{pmatrix} , \quad (A17)
\]

we obtain \[42\]

\[
H_{\Lambda \Sigma} d\zeta^\Lambda d\zeta^\Sigma = \left( d\ln \lambda_1 \right)^2 + \left( d\ln \lambda_2 \right)^2 + \left( d\ln \lambda_3 \right)^2 + \lambda_1 (\lambda_2 - \lambda_3)^2 X_1^2 + \lambda_2 (\lambda_3 - \lambda_1)^2 X_2^2 + \lambda_3 (\lambda_1 - \lambda_2)^2 X_3^2 . \quad (A18)
\]

Although the momentum constraints require \( \Phi \) not to depend upon \( O \), the portion of the supermetric involving \((X_1, X_2, X_3)\) makes a \( \lambda \)-dependent contribution to the volume element on superspace, and this, in turn, nontrivially affects the form of Wheeler-DeWitt equation. Taking into account the condition \( \lambda_1 \lambda_2 \lambda_3 = 1 \) on \( M_C \), we see that this portion of the metric contributes to \( \sqrt{\Pi} \) the additional factor

\[
C = \prod_{i>j} (\lambda_i - \lambda_j) . \quad (A19)
\]
Writing \( \lambda_1 \equiv e^{2(\beta_+ + \sqrt{3}\beta_-)} \), \( \lambda_2 \equiv e^{2(\beta_+ - \sqrt{3}\beta_-)} \), and \( \lambda_3 \equiv e^{-4\beta_+} \), we obtain

\[
C(\beta_{\pm}) = 8 \left| \sinh 2\sqrt{3}\beta_- \sinh(3\beta_+ - \sqrt{3}\beta_-) \sinh(3\beta_+ + \sqrt{3}\beta_-) \right|. 
\]

(A20)

The Wheeler-DeWitt equation then becomes

\[
\left[ \frac{\partial^2}{\partial \alpha^2} - \frac{1}{C(\beta_{\pm})} \sum_{i=\pm} \frac{\partial}{\partial \beta_i} C(\beta_{\pm}) \frac{\partial}{\partial \beta_i} + 24V - 90\xi \right] \Phi = 0. 
\]

(A21)

The variables \( \beta_{\pm} \) are restricted by the condition \( \lambda_1 \geq \lambda_2 \geq \lambda_3 > 0 \), but we may drop this restriction by requiring, instead, the invariance of \( \Phi \) under the transformations of \( \beta_{\pm} \) corresponding to permutations of \( \lambda_i \) \( (i = 1, 2, 3) \). The kinetic term in the Wheeler-DeWitt equation can be converted into flat spacetime wave operator by rescaling \( \Phi \) as \( \tilde{\Phi} \equiv C(\beta_{\pm})^{-1/2} \Phi \). Then, we obtain

\[
\left[ \frac{\partial^2}{\partial \alpha^2} - \frac{\partial^2}{\partial \beta_+^2} - \frac{\partial^2}{\partial \beta_-^2} + 24\tilde{V} - 90\xi \right] \tilde{\Phi} = 0
\]

(A22)

with

\[
\tilde{V} \equiv V + \frac{1}{24C(\beta_{\pm})^{1/2}} \left[ \frac{\partial^2}{\partial \beta_+^2} + \frac{\partial^2}{\partial \beta_-^2} \right] C(\beta_{\pm})^{1/2}.
\]

(A23)

This differs substantially from the form of the Wheeler-DeWitt equation derived in Sec. 4.

References

[1] The Hamiltonian does not vanish for asymptotically-flat spacetimes, but the same problems arise when one evolves the system with respect to slices which do not move at spacelike infinity.

[2] R.M. Wald, Phys. Rev. D 48, R2377 (1993).

[3] K. Kuchař, J. Math. Phys. 22, 2640 (1981).

[4] C. Misner, in Magic without Magic: John Archibald Wheeler edited by J. Klauder (Freeman, San Francisco, 1972). See also J.J. Halliwell, Phys. Rev. D 38, 2468 (1988).

[5] P. Hájíček and K.V. Kuchař, Phys. Rev. D 41, 1091 (1990).

[6] A. Ashtekar and A. Magnon, Proc. R. Soc. London A346, 375 (1975); B.S. Kay, Commun. Math. Phys. 62, 55 (1978).

[7] A. Ashtekar Lectures on Non-Perturbative Canonical Gravity (World Scientific, Singapore, 1991).

[8] R. Arnowitt, S. Deser, and C.W. Misner, in Gravitation: An Introduction to Current Research, edited by L. Witten (Wiley, New York, 1962).

[9] K. Kuchař, Proc. 4th Canadian Conf. General Relativity and Relativistic Astrophysics, edited by G. Kunstatter, D. Vincent, and J. Williams (World Scientific, Singapore, 1992).
[10] A. O. Barvinsky, Class. Quantum Grav. 10, 1957 (1993).

[11] A. Higuchi, in preparation.

[12] Inclusion of a scalar field turns out to be essential in some models. We only consider the case with one scalar field for simplicity, but the number of scalar fields does not affect our discussion.

[13] M.A.H. MacCallum, in General Relativity: An Einstein Centenary Survey, edited by S.W. Hawking and W. Israel (Cambridge University, Cambridge, 1979).

[14] R.M. Wald, General Relativity (University of Chicago Press, Chicago, 1984).

[15] G.F.R. Ellis and M.A.H. MacCallum, Commun. Math. Phys. 12, 108 (1969).

[16] M.A.H. MacCallum and A.H. Taub, Commun. Math. Phys. 25, 173 (1972); M.P. Ryan, J. Math. Phys. 15, 812 (1972); G.E. Sneddon, J. Phys. A: Math. Gen. 9, 229 (1976).

[17] Compact spaces admitting “locally homogeneous” actions of Bianchi groups have been analyzed recently by T. Koike, M. Tanimoto and A. Hosoya, Compact homogeneous universes, Tokyo Institute of Technology preprint TIT/HEP-208/COSMO. gr-qc/9405052. J. Math. Phys., in press.

[18] R.M. Wald, Phys. Rev. D 28, R2118 (1983).

[19] R. Kantowski and R.K. Sachs, J. Math. Phys. 7, 443 (1966).

[20] For a related observation, see J.D. Romano and R.S. Tate, Class. Quantum Grav. 6, 1487 (1989); K. Schleich, ibid. 7, 1529 (1990).

[21] P. Chmielowski, Class. Quantum Grav. 11, 41 (1994).

[22] B.S. Kay and R.M. Wald, Phys. Rep. 207, 49 (1991).

[23] R.M. Wald Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics (University of Chicago Press, Chicago, 1994).

[24] The validity of this approximation has not been rigorously proven. For a numerical evidence in the classical dynamics, see B.K. Berger, Gen. Relativ. Gravit. 23, 1385 (1991); Phys. Rev. D 49, 1120 (1994).

[25] C.W. Misner, Phys. Rev. Lett. 22, 1071 (1969).

[26] For other examples of nonchaotic behavior of Bianchi VIII and XI models with additional degrees of freedom, see P. Halpern, Gen. Relativ. Gravit. 19, 73 (1987); H. Ishihara, Prog. Theor. Phys. 74, 1490 (1985).

[27] S.M. Waller, Phys. Rev. D 29, 176 (1984).
[28] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, New York, 1980).

[29] The WKB approximation for this equation (for the solutions which decay for $\alpha \to +\infty$) can be found in C. Kiefer, Phys. Rev. D 38, 1761 (1988).

[30] L.I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1955), p. 271.

[31] Wavepackets spread out in general as in the usual quantum mechanics [29]. However, even this spreading does not occur in this model due to the fact that it is approximately the 2-dimensional massless scalar field for large and negative $\alpha$.

[32] J.B. Hartle and S.W. Hawking, Phys. Rev. D 28, 2960 (1983).

[33] S.W. Hawking, Nucl. Phys. B239, 257 (1984).

[34] A. Vilenkin, Phys. Rev. D 33, 3560 (1986).

[35] A. Vilenkin, Phys. Rev. D 37, 888 (1988).

[36] A. Vilenkin, “Approaches to Quantum Cosmology”, to be published.

[37] For the quantization of this model via a different approach, see A. Ashtekar, R. Tate and C. Uggla, Int. J. Mod. Phys. D2, 15 (1993).

[38] S. Martinez and M. Ryan, in Relativity, Cosmology, Topological Mass and Supergravity, Proceedings of the Fourth Silarg Symposium, Caracas, Venezuela, 1982, edited by C. Aragone (World Scientific, Singapore, 1983).

[39] A. Higuchi, G.E.A. Matsas, and D. Sudarsky, Phys. Rev. D 46, 3450 (1992).

[40] A. Ashtekar and J. Samuel, Class. Quantum Grav. 8, 2191 (1991).

[41] B.S. DeWitt, Phys. Rev. 160, 1113 (1967).

[42] A. Terras, *Harmonic Analysis on Symmetric Spaces and Applications II* (Springer, Berlin, 1988).

[43] C.W. Misner, K.S. Thorne, and J.A. Wheeler, *Gravitation* (Freeman, San Francisco, 1972).
Figure captions

Figure 1: $d\phi/d\alpha \equiv \partial H_{\text{eff}}/\partial p$ for $p = 4, 16, 64, 256$. The dotted line is the corresponding classical value $[1 - e^{4\alpha / p^2}]^{-1/2}$.

Figure 2: The region swept by the interval of $\phi$ (for given $\alpha$) containing 90% of the squared $L^2$ wavefunction for a Gaussian wavepacket with $\langle p \rangle = 64$ and $\Delta p \equiv [(p^2) - \langle p \rangle^2]^{1/2} = 10$ in the Robertson-Walker model with a massless scalar field. The solid line is the corresponding classical solution.

Figure 3: The region swept by the interval of $\bar{\xi}$ containing 90% of the squared $L^2$ wavefunction for a Gaussian wavepacket with $\langle \tilde{\omega} \rangle = 32$ and $\Delta \tilde{\omega} = 5$ in the Taub model, where we have defined $\tilde{\omega} \equiv 3^{-1/2} \omega$, $\bar{\xi} \equiv 3^{1/2} \xi$, $\bar{\tau} \equiv 3^{1/2} \tau$. The solid line is the corresponding classical solution.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9407038v1
This figure "fig1-2.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9407038v1
This figure "fig1-3.png" is available in "png" format from:

http://arxiv.org/ps/gr-qc/9407038v1