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POISSON ALGEBRAS OF BLOCK-UPPER-TRIANGULAR BILINEAR FORMS AND BRAID GROUP ACTION

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Abstract. In this paper we study a quadratic Poisson algebra structure on the space of bilinear forms on \( \mathbb{C}^N \) with the property that for any \( n, m \in \mathbb{N} \) such that \( nm = N \), the restriction of the Poisson algebra to the space of bilinear forms with block-upper-triangular matrix composed from blocks of size \( m \times m \) is Poisson. We classify all central elements and characterise the Lie algebroid structure compatible with the Poisson algebra. We integrate this algebroid obtaining the corresponding groupoid of morphisms of block-upper-triangular bilinear forms. The groupoid elements automatically preserve the Poisson algebra. We then obtain the braid group action on the Poisson algebra as elementary generators within the groupoid. We discuss the affinisation and quantisation of this Poisson algebra, showing that in the case \( m = 1 \) the quantum affine algebra is the twisted \( q \)-Yangian for \( \mathfrak{g}_n \) and for \( m = 2 \) is the twisted \( q \)-Yangian for \( \mathfrak{sp}_{2n} \). We describe the quantum braid group action in these two examples and conjecture the form of this action for any \( m > 2 \). Finally, we give an \( R \)-matrix interpretation of our results and discuss the relation with Poisson–Lie groups.

1. Introduction

In this paper we consider bilinear forms on \( \mathbb{C}^N \) defined by

\[
(x, y) := x^T A y, \quad \forall x, y \in \mathbb{C}^N, \quad A \in GL_N(\mathbb{C}).
\]

By block-upper-triangular bilinear form we mean a bilinear form such that the defining matrix \( A \) is block–upper–triangular. In particular we use the following:

**Notation 1.1.** We let a block-upper-triangular (b.u.t.) matrix \( A \) to be an \((nm)\times(nm)\)-matrix composed from blocks \( k_{I,J} \), \( I, J = 1, \ldots, n \), of size \( m \times m \) with the block-upper-triangular structure: we impose the restrictions that \( k_{I,J} = 0 \) for \( I > J \) and \( \det k_{I,I} = 1 \) for all \( I = 1, \ldots, n \). We denote by \( A_{n,m} \subset GL_{nm} \) the set of all such block-upper-triangular matrices.

In [5] it was proved that for any number of blocks \( n \) and for any size of blocks \( m \) the following brackets on the matrix elements \( a_{i,j} \) of \( A \):

\[
\{a_{i,j}, a_{k,l}\} = (\text{sign}(j - l) + \text{sign}(i - k))a_{i,l}a_{k,j} + (\text{sign}(j - k) + 1)a_{j,l}a_{i,k} + (\text{sign}(i - l) - 1)a_{i,j}a_{k,i}
\]

(1.1)
define a Poisson bracket on \( A_{n,m} \).

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Note that the brackets (1.1) depend neither on the size of the $m \times m$ blocks nor on the number $n^2$ of blocks, so that the full space $GL_N(\mathbb{C})$ of non-singular $N \times N$ matrices, $N = nm$, admits this Poisson algebra (1.1). In Theorem 2.1 we show that the block-upper-triangular case is a Poisson reduction of the full algebra in $(GL_N(\mathbb{C}), \{\cdot, \cdot\})$.

In the case of one-dimensional blocks (i.e. upper triangular matrices with 1 on the diagonal) this algebra reduces to the Dubrovin–Ugaglia [9, 22] bracket appearing in Frobenius manifold theory and extensively studied by Bondal [1, 2]. Its quantisation is also known as Nelson–Regge algebra in 2+1-dimensional quantum gravity [19, 20], and as Fock–Rosly bracket [13] in Chern–Simons theory. It is also known (see [4] and [11]) that particular symplectic leaves of these algebras admit a cluster variable description thus providing a bridge to geometry. It is tempting to find such representations for the b.u.t. case. We expect that for generic $m$ this algebra may have some interesting meaning in these fields.

The affine version of the algebra (1.1) is defined in terms of the generating function:

\begin{equation}
G_{i,j}(\lambda) := G_{i,j}^{(0)} + \sum_{p=1}^{\infty} G_{i,j}^{(p)} \lambda^{-p},
\end{equation}

where $G_{i,j}^{(p)}$ denotes the entry $i, j$ of the matrix $G^{(p)}$ and we impose that $G^{(0)} := A$, our block-upper-triangular matrix, allowing the matrices $G^{(p)}$ to be arbitrary full-size matrices for $p > 0$. The Poisson brackets are postulated to be [5]

\begin{align}
\{G_{i,j}(\lambda), G_{k,l}(\mu)\} &= \left( \text{sign}(i-k) - \frac{\lambda + \mu}{\lambda - \mu} \right) G_{k,j}(\lambda) G_{i,l}(\mu) + \\
&+ \left( \text{sign}(j-l) + \frac{\lambda + \mu}{\lambda - \mu} \right) G_{k,j}(\mu) G_{i,l}(\lambda) + \\
&+ \left( \text{sign}(j-k) - \frac{1 + \lambda \mu}{1 - \lambda \mu} \right) G_{i,k}(\lambda) G_{j,i}(\mu) + \\
&+ \left( \text{sign}(i-l) + \frac{1 + \lambda \mu}{1 - \lambda \mu} \right) G_{i,k}(\lambda) G_{k,i}(\mu).
\end{align}

We call the index $p$ the level of the corresponding element; elements of $A$ are then called zero–level elements. Analogously to the case of (1.1), the algebra (1.3) is Poisson for any choice of the zero level $A \in \mathcal{A}_{n,m}$, for any $n, m$ such that $nm = N$.

In our paper [5] we related this affine extension (1.3) in the case $m = 1$ to the algebra $\mathcal{D}_n$ of geodesic functions on an annulus with $n \mathbb{Z}_2$ orbifold points and, simultaneously, to the algebra of monodromy data of a $n+1$-dimensional Frobenius manifold with one non-semi-simple point. Still in the case $m = 1$ this affine extension (1.3) turns out to be the semi-classical limit of the twisted $q$-Yangian for the $\mathfrak{s}_n$ algebra introduced by Molev, Ragoucy, and Sorba [18]. A first generalisation of the above algebras to block-upper-triangular matrix case was constructed by Molev and Ragoucy in [17], where they developed the twisted Yangian $Y_q(\mathfrak{sp}_{2n})$ for the $\mathfrak{sp}_{2n}$ algebra. In this construction, the zero-level algebra was block-lower-triangular (equivalent to block-upper-triangular by simple transposition) with $2 \times 2$ blocks and with the restriction that each diagonal $2 \times 2$-block have the unit determinant.
In the work by Molev and Ragoucy a full description of the braid group action on $Y_q'(sp_{2n})$ was still missing and this was in part the trigger to the present work.\footnote{We are particularly grateful to Alexander Molev for asking this question to us.}

Before explaining our results on the braid group action we need to illustrate the ones on the central elements. We characterise all central elements of the Poisson algebra (1.1) and of its affine extension (1.3). They are of two types: polynomial central elements and rational central elements; together they form a set of $n\left[\frac{m+1}{2}\right] + \left[\frac{nm}{2}\right]$ algebraically independent central elements (here we let $\lfloor \cdot \rfloor$ denote the integer part of a number).

In Theorem 3.2, we prove that the \textit{polynomial central elements} for the Poisson algebra (1.1) are given by the coefficients of $\lambda^{-k}$, $k = 0, 1, \ldots, \left[\frac{N+2}{2}\right]$, of the polynomial

$$\det(\mathbb{A} + \lambda^{-1}\mathbb{A}^T),$$

while for the affine Poisson algebra (1.3) they are generated by the formal series

$$\det(G(\lambda)).$$

The \textit{rational central elements} are the same for both Poisson algebras (1.1) and (1.3). They are defined by the bottom–left minors of the diagonal blocks of the zero level matrix $\mathbb{A}$, i.e. let $\mathbb{A} \in \mathcal{A}_{n,m}$, for each diagonal block $\mathbb{A}^{(I)} := \mathbb{A}_{I,I}$, $I = 1, \ldots, n$ take

$$M_d^{(I)} := \det\begin{pmatrix} a_{m-d+1,1}^{(I)} & \cdots & a_{m+d-1,d}^{(I)} \\ \vdots & \ddots & \vdots \\ a_{m,1}^{(I)} & \cdots & a_{m,d}^{(I)} \end{pmatrix},$$

where $a_{i,j}^{(I)}$ denotes the $i,j$-th entry of $\mathbb{A}^{(I)} := \mathbb{A}_{I,I}$, then in Theorem 3.10 we prove that for every $d = 0, \ldots, \left[\frac{m}{2}\right]$ and $I = 1, \ldots, n$ the quantities

$$b_d^{(I)} := \frac{M_d^{(I)}}{M_{m-d}^{(I)}}$$

are central elements for both Poisson algebras (1.1) and (1.3).

Having characterised all central elements, we are ready to produce the braid group action on $\mathcal{A}_{n,m}$. For this, we follow Bondal’s approach \cite{1, 2} which consists in introducing a suitable notion of groupoid of block-upper-triangular quadratic bilinear forms in such a way that the Poisson bracket on the base space $\mathcal{A}_{n,1}$ is given in terms of the anchor map associated to the corresponding Lie algebroid. In Bondal’s case, namely when $m = 1$, the Lie algebroid is isomorphic to the Lie algebroid on the cotangent bundle $T^*\mathcal{A}_{n,1}$. As soon as $m > 2$ this ceases to be true, making the integration of the Lie algebroid rather tricky. We solve this problem in Section 4 where we characterise this groupoid.\footnote{This groupoid is the natural phase space of the Poisson sigma model with target space $\mathcal{A}_{n,m}$ \cite{3}. In this case we expect to be able to integrate the constraint equation explicitly.}

The braid group generators are obtained as those elements in the groupoid which swap the blocks and satisfy the braid group relations. To be more precise, the braid group acting on $\mathcal{A}_{n,m}$ is $B_n$ in which each braid $\beta_{I,I+1}$, $I = 1, \ldots, n - 1$ acts by changes of coordinates on $\mathbb{C}^N$. This action can be presented in the adjoint matrix form (see formula (5.35) below) $\beta_{I,I+1}[\mathbb{A}] = \mathbb{B}_{I,I+1}^{T}\mathbb{A}\mathbb{B}_{I,I+1}$ with the matrix $\mathbb{B}_{I,I+1}$ having the block form (5.36).
The extended braid group transformations for the affine algebra (1.3) in the case where the zero-level matrix $\mathcal{A}$ has the block-upper-triangular form is given by the same $\beta_{1,1+1}[\mathcal{G}(\lambda)] = B_{1,1+1}\mathcal{G}(\lambda)B_{1,1+1}^T$ and we have one extra generator $\beta_{n,1}$ given by the formulas (5.40) and (5.41).

Since the braid group elements belong to the groupoid, they preserve our algebras (1.1) and (1.3).

Finally we provide a quantisation of the affine algebra (1.3) in terms of quantum reflection equation for any $m$ and give formulae for the quantum braid group action in the case $m = 1$ and $m = 2$. This leads to an interesting explicit relation between the Lie algebroid of infinitesimal morphism of the b.u.t. algebra (1.1) and its $R$-matrix structure. This allows us to establish a link to Poisson–Lie groups which will be further developed in subsequent work [6].

2. Poisson reductions of the algebras (1.1) and (1.3)

**Theorem 2.1.** The affine algebra (1.3) is an abstract infinite-dimensional Poisson algebra for any choice of the block-upper-triangular form of the zero level matrix $\mathcal{A} \in \mathcal{A}_{n,m}$. Analogously, the restriction of the brackets (1.1) on $GL_N(\mathbb{C})$ to the block-upper-triangular matrices $\mathcal{A}_{n,m}$ for any $n, m \in \mathbb{N}$ such that $nm = N$, is Poisson.

**Proof.** The proof of the Jacobi relations in Appendix A of [5] used only combinatorial properties and was independent on possible reductions. So, it remains only to prove the consistency of the reductions with the affine algebra (the consistency of the reductions of (1.1) is a trivial corollary). For this, let us calculate the bracket between elements of the zeroth and $k > 0$ levels. From (1.3), we have (one can obtain the formula below by taking a formal limit $\lambda \to \infty$)

$$\{a_{i,j}, G_{k,l}^{(p)}\} = \left(\text{sign}(i-k) - 1\right)a_{k,j}G_{i,l}^{(p)} + \left(\text{sign}(j-l) + 1\right)G_{k,j}^{(p)}a_{i,l} + \left(\text{sign}(j-k) + 1\right)a_{i,k}G_{j,l}^{(p)} + \left(\text{sign}(i-l) - 1\right)a_{i,j}G_{k,l}^{(p)}.$$  

(2.4)

The right-hand side is nonzero (due to combinations of sign-factors) only for $i \leq k$ and/or $l \leq j$ and/or $k \leq j$ and/or $i \leq l$. We now use the specific form of the reduction, namely the fact that if we impose $a_{i,j} = 0$, then $a_{s,j} = 0$ for all $s \geq i$ and $a_{i,t} = 0$ for all $t \leq j$. Therefore if $i \leq k$ then $a_{k,j}$ is zero and the term $(\text{sign}(i-k) - 1)a_{k,j}G_{i,l}^{(p)}$ does not contribute. Analogously, for $k \leq j$ we have that $a_{i,k} = 0$ and the term $(\text{sign}(j-k) + 1)a_{i,k}G_{j,l}^{(p)}$ does not contribute. The same happens if $l \leq j$ and/or $i \leq l$. This proves the consistency between our reduction and the algebra (1.3). □

**Remark 2.2.** A more general statement is true: let us consider block-upper-triangular matrices with blocks of different sizes, or in other words let us consider an arbitrary partition of $N$ (previously equal to $mn$) into $n$ positive integers, $N = m_1 + \cdots + m_n$, and let $\mathcal{A}_{I,J}$ be a matrix of size $m_I \times m_J$. All the constructions of this paper, including the Poisson restriction (Theorem 2.1), central elements, and the action of the groupoid of b.u.t. matrices can be straightforwardly generalised to this case as well except the (classical and quantum) braid-group action, which is apparently lost in the case of different block sizes.

If we consider even more general case in which we loose the block upper triangular form, and take the Poisson reduction depicted in Fig. 1 where all elements below...
2.1. Reduction to the symmetric matrices. The Poisson structure (1.1) restricts also to the space $\text{Sym}_N$ of symmetric $N \times N$ matrices.

**Proposition 2.3.** The restriction of the Poisson structure (1.1) to the space $\text{Sym}_N$ of symmetric matrices is Poissonian.

**Proof.** Let us consider the Poisson bracket of the combination $a_{i,j} - a_{j,i}$ with any element $a_{k,l}$:

$$\{a_{i,j} - a_{j,i}, a_{k,l}\} = \text{sign}(j - l)a_{k,j}(a_{i,l} - a_{l,i}) + \text{sign}(i - k)a_{i,l}(a_{k,j} - a_{j,k}) + \text{sign}(j - k)a_{j,l}(a_{i,k} - a_{k,i}) + \text{sign}(i - l)a_{k,i}(a_{j,l} - a_{l,j}) + a_{j,l}a_{i,k} - a_{l,j}a_{k,i} - a_{i,l}a_{j,k} + a_{i,j}a_{k,l}.$$

By imposing the condition $a_{r,s} = a_{s,r}$ for all $r, s$, the above expression is always 0. □

The reduced bracket on $\text{Sym}_N$ reads:

$$\{a_{i,j}, a_{k,l}\} = (\text{sign}(j - l) + \text{sign}(i - k))a_{i,l}a_{k,j} + (\text{sign}(j - k) + \text{sign}(i - l))a_{j,l}a_{k,i}$$

This Poisson structure on $\text{Sym}_N$ was already studied by Bondal [2].

**Remark 2.4.** Observe that on the contrary the affine algebra (1.3) is not compatible with the restriction $A \in \text{Sym}_N$. This can be easily seen by observing that $\{a_{i,j} - a_{j,i}, G_{k,l}^{(p)}\} \neq 0$ for $\lambda \in \text{Sym}_N$. We do not know whether an affine extension of (2.5) exists.

2.2. $k$-level reductions of the twisted Yangian extension.

**Notation 2.5.** We call the $k$-level reduction of the algebra (1.3) the mapping

$$G(\lambda) \mapsto \tilde{G}(\lambda) = \lambda^{-k}\tilde{G}^{(1)} + \lambda^{-k+1}\tilde{G}^{(k-1)} + \lambda^{-k}\tilde{G}^{(k)} T,$$

where

$$\tilde{G}^{(i)} = \left[\tilde{G}^{(k-i)}\right]^T$$

**Proof.** Let us consider the Poisson bracket of the combination $a_{i,j} - a_{j,i}$ with any element $a_{k,l}$:

$$\{a_{i,j} - a_{j,i}, a_{k,l}\} = \text{sign}(j - l)a_{k,j}(a_{i,l} - a_{l,i}) + \text{sign}(i - k)a_{i,l}(a_{k,j} - a_{j,k}) + \text{sign}(j - k)a_{j,l}(a_{i,k} - a_{k,i}) + \text{sign}(i - l)a_{k,i}(a_{j,l} - a_{l,j}) + a_{j,l}a_{i,k} - a_{l,j}a_{k,i} - a_{i,l}a_{j,k} + a_{i,j}a_{k,l}.$$
and \( \tilde{G}^{(i)} = G^{(i)} \) for \( i = 1, \ldots, (k - 1)/2 \) for odd \( k \) and for \( i = 1, \ldots, k/2 - 1 \) for even \( k \) and \( \tilde{G}^{(k/2)}_{i,j} = G^{(k/2)}_{i,j} \) for \( i \geq j \) for even \( k \), whereas the other entries of \( \tilde{G}^{(i)} \) for \( i \geq k/2 \) are defined by the symmetry condition (2.7).

**Theorem 2.6.** The mapping (2.6) defines a surjective homomorphism of the algebra (1.3) to the corresponding algebra of the elements \( A_{i,j} \) and \( \tilde{G}^{(i)}_{i,j}, l = 1, \ldots, k-1 \), for any \( k \in \mathbb{Z}_+ \).

### 3. Central elements

In this section, we construct all the central elements of the algebra of block-upper-triangular matrices (1.1) and of its twisted Yangian extension (1.3). In order to find them we first need to characterise a simple automorphism and a simple anti-automorphism of the Poisson algebra (1.1).

#### 3.1. (Anti)automorphisms of the Poisson algebra

Let \( N = nm \) denote the total size of the matrix \( A \). Then the transformation

\[
P[A] = \tilde{A}, \quad \tilde{a}_{i,j} = a_{N+1-j, N+1-i}
\]

is an antiautomorphism of the Poisson algebra (1.1), that is,

\[
\{\tilde{a}_{i,j}, \tilde{a}_{k,l}\} = -\{a_{i,j}, a_{k,l}\} \bigg|_{a \to \tilde{a}}.
\]

Besides it we have the scaling transformation, which obviously leaves invariant the algebra (1.1):

\[
a_{i,j} \mapsto e^{\phi_i + \phi_j} a_{i,j}, \quad \phi_i = \phi_{N+1-i},
\]

where we impose the restriction on \( \phi_i \) in order to ensure the transformation (3.10) to be consistent with the antiautomorphism (3.8). We also impose that \( \sum_{i=J}^{J+m} \phi_i = 0, J = 1, \ldots, n \), to ensure the preservation of the determinant condition \( \det A_{J,J} = 1 \) for any \( J \).

**Remark 3.1.** Note that the fact that the scaling transformation (3.10) is an automorphism of the algebra (1.1) allows to restrict this algebra on the projective space \( \mathbb{P}^{N^2-1} \). This fact is relevant due to the recent interest in the vanishing locus of quadratic Poisson algebras on projective spaces in algebraic geometry [14].

#### 3.2. Polynomial central elements

In this subsection, we construct a part of central elements that can be obtained by standard methods based on algebra symmetries as, say, in [1] or [16].

**Theorem 3.2.** The polynomial functions of the elements of the algebra (1.3) in the infinite-series expansion of \( \det G(\mu) \) in powers of \( \mu^{-1} \) are central elements of the affine algebra (1.3).

**Proof.** Although it follows from the more general statement by Molev and Ragoucy [17] on the central elements of the (quantum) Yangian algebra, we can easily verify it directly using that

\[
\{G_{i,j}(\lambda), \det G(\mu)\} = \sum_{k,l=1}^{nm} \{G_{i,j}(\lambda), G_{k,l}(\mu)\} G^{-1}_{k,l}(\mu)
\]

(the invertibility of \( A \) ensures the existence of the inverse matrix \( G^{-1}(\mu) \) at least as a formal series). We now substitute the bracket (1.3), and using the obvious
identities $\sum_{l=1}^{nm} G_{x,l}(\mu)G_{l,k}^{-1}(\mu) = \delta_{x,k}$ and $\sum_{k=1}^{nm} G_{k,x}(\mu)G_{l,k}^{-1}(\mu) = \delta_{l,x}$ for $x = i, j$, we obtain zero. □

Corollary 3.3. The coefficients in the $\lambda^{-1}$-expansion of $\det(\mathbb{A} + \lambda^{-1}\mathbb{A}^T)$ are central elements of the Poisson algebra (1.1) restricted to the block-upper-triangular matrices $\mathbb{A} \in \mathcal{A}_{n,m}$ for any choice of $n, m \in \mathbb{N}$ such that $nm = N$. They form a family of $\left[\frac{N^2}{2}\right]$ algebraically independent central elements.

Proof. We need the statement of Theorem 2.6 for $k = 1$:

Lemma 3.4. The mapping $G(\lambda) \mapsto \mathbb{A} + \lambda^{-1}\mathbb{A}^T$ defines a surjective homomorphism of the algebra (1.3) to the algebra (1.1).

Proof. The proof of this Lemma is obtained by a direct substitution of expression (3.11) into (1.3) using the algebra (1.1). □

Proof of Corollary 3.3. The proof that the coefficients in the $\lambda^{-1}$-expansion of $\det(\mathbb{A} + \lambda^{-1}\mathbb{A}^T)$ are central elements of the Poisson algebra (1.1) follows directly from Theorem 3.2. The fact that no more than $\left[\frac{N+1}{2}\right]$ of them are algebraically independent follows from the simple observation that

$$\det(\mathbb{A} + \lambda^{-1}\mathbb{A}^T) = \frac{c_0\lambda^N + c_1\lambda^{N-1} + \cdots + c_N}{\lambda^N},$$

where $c_{N-k} = c_k$ for all $k = 0, 1, \ldots, \left[\frac{N}{2}\right]$ and $c_0 = \det(\mathbb{A}_{11})\det(\mathbb{A}_{22}) \cdots \det(\mathbb{A}_{nn}) = 1$.

The fact that generically the coefficients $c_1, \ldots, c_{\left[\frac{N}{2}\right]}$ form a family of $\left[\frac{N^2}{2}\right]$ algebraically independent central elements was proved in [9] for the most reduced case $m = 1$. □

3.3. Rational central elements. The central elements in Corollary 3.3 do not exhaust all the central elements of the algebra of entries of $\mathbb{A}$. We also have rational central elements. To describe them we begin by considering the case of the nonrestricted Poisson algebra $(\text{GL}_N, \{\cdot, \cdot\})$ where $\{\cdot, \cdot\}$ is given by (1.1), and make the following

Generality assumption: All the minors $M_d$ of size $d \times d$ located at the lower-left corner are non-zero.

Theorem 3.5. Under the above generality assumption, the elements

$$\det M_{N-d}/\det M_d, \text{ for } d = 0, \ldots, \left[\frac{N-1}{2}\right],$$

are central for the affine algebra (1.3) and are algebraically independent in the nonrestricted case.

Proof. The proof is based on the following:

Lemma 3.6. For the nonrestricted $N \times N$ matrix $\mathbb{A}$ in our generality assumption, denoting $a_{k,l} = G^{(0)}_{k,l}$, we have the following commutation relations:

$$\{\det M_d, G^{(p)}_{k,l}\} = c^d_{k,l}G^{(p)}_{k,l} \det M_d \text{ for } p = 0, 1, \ldots,$$
where
\[ c^d_{k,l} = -\delta_{k+d>N} + \delta_{d+1>i} + \delta_{d+1>k} - \delta_{l+d>N} \]
where \( \delta_{i>j} = 1 \) for \( i > j \) and 0 otherwise.

**Proof.** Let us deal with the minor \( M_2 \) first. By the Leibnitz rule, we obtain four brackets, and using relation (2.4) we obtain four terms for each bracket. Grouping together terms with the same \( G^{(p)}_{r,i} \) entry we obtain
\[
\left\{ M_d, G^{(p)}_{k,l} \right\} = (\text{sign}(N - k) - 1)G^{(p)}_{N,l} \begin{vmatrix} a_{N-1,1} & a_{N-1,2} \\ a_{k,1} & a_{k,2} \end{vmatrix} + \\
+ (\text{sign}(2 - l) + 1)G^{(p)}_{k,2} \begin{vmatrix} a_{N-1,1} & a_{N-1,1} \\ a_{N,1} & a_{N,l} \end{vmatrix} + \\
+ (\text{sign}(2 - k) + 1)G^{(p)}_{2,l} \begin{vmatrix} a_{N-1,1} & a_{N-1,k} \\ a_{N,1} & a_{N,k} \end{vmatrix} + \\
+ (\text{sign}(N - l) - 1)G^{(p)}_{N,k} \begin{vmatrix} a_{N-1,1} & a_{N-1,2} \\ a_{l,1} & a_{l,2} \end{vmatrix} + \\
+ (\text{sign}(N - 1 - k) - 1)G^{(p)}_{N-1,l} \begin{vmatrix} a_{k,1} & a_{k,2} \\ a_{N,1} & a_{N,2} \end{vmatrix} + \\
+ (\text{sign}(1 - l) + 1)G^{(p)}_{1,1} \begin{vmatrix} a_{N-1,1} & a_{N-1,2} \\ a_{N,l} & a_{N,2} \end{vmatrix} + \\
+ (\text{sign}(1 - k) + 1)G^{(p)}_{1,l} \begin{vmatrix} a_{N-1,k} & a_{N-1,2} \\ a_{N,k} & a_{N,2} \end{vmatrix} + \\
+ (\text{sign}(N - 1 - l) - 1)G^{(p)}_{k,N-1} \begin{vmatrix} a_{l,1} & a_{l,2} \\ a_{N,1} & a_{N,2} \end{vmatrix}.
\]
and each term in this sum is nonzero only for one choice of either \( k \) or \( l \). For example consider the last term on the r.h.s.: the coefficient \( \text{sign}(N - 1 - l) - 1 \) is nonzero only for \( l = N - 1 \) or \( l = N \). However in the latter case the determinant \( \begin{vmatrix} a_{l,1} & a_{l,2} \\ a_{N,1} & a_{N,2} \end{vmatrix} \) becomes zero, so we may only choose \( l = N - 1 \). It easily follows that (3.12) and (3.13) are satisfied.

In the case of \( d > 2 \) the computation is very similar: the first and fifth term above are replaced by the sum of \( d \) determinants enumerated by the index \( i = N - d + 1, \ldots, N \) and such that in each of the corresponding matrices the \( i \)th row vector is replaced by the \( k \)th row vector multiplied by \( \text{sign}(i - k) - 1 \) \( G^{(p)}_{i,l} \). If \( i < k \), the corresponding determinant is zero (the matrix then contains two proportional row vectors), so the only nonzero contribution occurs when \( i = k \), which is possible only if \( k > n - d \), and this contribution is \( -c^d_{k,l} \) \( \det M_d \), which gives the first term in the r.h.s. of (3.13). Using the same reasonings we can deal with three other terms. Because \( \delta_{i<0} = 1 - \delta_{i+1>0} \) we easily obtain from (3.13) that \( c^d_{k,l} = c^{N-d}_{k,l} \), which completes the proof of the Lemma. □

The proof of the fact that the elements \( \det M_{N-d} / \det M_d \) for \( d = 0, \ldots, \left[ \frac{N-1}{2} \right] \) are central then follows by the Leibnitz rule for the Poisson bracket and by observing that \( c^d_{k,l} = c^{N-d}_{k,l} \) for all \( k, l, d \), so that \( \det M_d \) and \( \det M_{N-d} \) have exactly the same commutation relations with all of \( a_{k,l} \) and with all of \( g^{(p)}_{k,l} \) for \( p \geq 1 \) in the twisted Yangian case.
That these central elements are algebraically independent was proved by Bondal [2] already for the restriction of the algebra (1.1) to $\text{Sym}_N$.

We now formulate the general algebraic independence lemma valid in the non-restricted case.

Lemma 3.7. The set of algebraically independent central elements of the non-restricted Poisson algebra $(GL_N, \{\cdot, \cdot\})$ where $\{\cdot, \cdot\}$ is given by (1.1), comprises the coefficients $c_k$ of $\lambda^{-k}$, $k = 0, 1, \ldots, [N/2]$, of the expansion of $\det(A + \lambda^{-1}h^T)$ and the rational central elements $b_l = \det M_{N-l}/\det M_l$, $l = 1, \ldots, [(N-1)/2]$ provided all $\det M_l$, $l = 1, \ldots, [(N-1)/2]$ are nonzero.

Proof. We have already proved that these elements are central and that each set $\{c_k\}$ and $\{b_l\}$ is algebraically independent. Suppose we have a function

$$F(\{c_k\}, \{b_l\}) = 0$$

for all values of $a_{i,j}$.

Because any transformation (3.10) is an automorphism of the algebra (1.1), choosing $\phi^{(i)}_l = s_i(\delta_{l,i+1} - \delta_{l,i})$ for $i = 1, \ldots, [(N-1)/2]$ we obtain that all $c_k$ and all $b_l$ with $l \neq i$ remain invariant whereas $b_i \rightarrow b_i e^{2\pi i}$. This means that if $F(\{c_k\}, \{b_l\}) = 0$ for some nonzero $\{b_l\}$, then $F(\{c_k\}, \{b_l e^{2\pi i}\}) = 0$ for any choice of $\phi_l \in \mathbb{R}$. Hence, the function $F$ is actually independent of all of $b_l$ and we have that $F(\{c_k\})\equiv 0$. Because the set of $\{c_k\}$ is algebraically independent, we have that $F \equiv 0$, which completes the proof.

Adding $\det A$, which corresponds both to the rational central element with $d = 0$ and to the polynomial central element given by the coefficient of power 0, to the set we have $\left[\frac{N+1}{2}\right]$ central elements described by Theorem 3.5 and $\left[\frac{N}{2}\right]$ central elements from Corollary 3.3, so, in the general case of a nonrestricted algebra $(GL_N, \{\cdot, \cdot\})$ where $\{\cdot, \cdot\}$ is given by (1.1), we have exactly $N$ algebraically independent central elements.

Remark 3.8. Elementary, but lengthy calculations demonstrate that the highest Poisson leaf dimension is not less than $N^2 - N$. Here we only briefly outline the way of proving it. For this it suffices to consider the case where all $a_{i,j}$ with $i \neq j$ are $e$-small as compared to all $a_{i,i}$ and to retain only terms of order $O(e)$ in the Poisson relations (1.1) neglecting all the terms of order $e^2$. Introducing then the combination $b_{i,j} = a_{i,j} - a_{j,i}$ and retaining the elements $a_{i,j}$ with $i \geq j$ we observe that in the limit of small $e$, all the $b_{i,j}$ commute with all the $a_{i,j}$, so that the Poisson algebra splits in two sub-algebras, the $a_{i,j}$-algebra and the $b_{i,j}$-algebra.

The $b_{i,j}$-algebra becomes the small-$e$ limit of the Dubrovin–Ugaglia or Nelson–Regge algebra (4.25) and therefore its highest Poisson leaf dimension is $\frac{N(N-1)}{2} - \left[\frac{N}{2}\right]$. The $a_{i,j}$-algebra becomes the Dubrovin–Ugaglia or Nelson–Regge algebra to which we add the diagonal elements, and therefore its highest Poisson leaf dimension is $\frac{N(N+1)}{2} - \left[\frac{N+1}{2}\right]$, so the highest rank of the Poisson relations (1.1) will be not less than $N^2 - N$, as expected.

We are now going to formulate the theorem describing the rational central elements in the general case of the block-upper-triangular matrix $A$ and its possible Yangian extensions. We begin by fixing our notation.

Notation 3.9. For a block-upper-triangular matrix $A$ with $n$ blocks of sizes $m_i \times m_i$, $i = 1, \ldots, n$ on the diagonal, let $M^{(i)}_d$, $d = 0, \ldots, m_i$, $i = 1, \ldots, n$, be minors of
size \( d \times d \) located at lower-left corners of the corresponding diagonal blocks of the matrix \( A \).

**Theorem 3.10.** Provided \( \det M_{d}^{(i)} \) are nonzero, all the quotients

\[
b_{d}^{(i)} \equiv \det M_{m_{i}−d}^{(i)}/\det M_{d}^{(i)}, \quad d = 0, \ldots, [(m_{i}−1)/2], \quad i = 1, \ldots, n
\]

are central elements of both the algebra (1.1) and its Yangian extension (1.2)–(1.3), for any choice of the zero level \( A \in A_{n,m} \).

The central elements \( b_{d}^{(i)} \), \( d = 0, \ldots, [(m_{i}−1)/2], \) \( i = 1, \ldots, n \), and the coefficients \( c_{r} \) of \( \lambda^{−r} \) terms \( (r = 1, 2, \ldots) \) of the expansion of \( \det G(\lambda) \) constitute an algebraically independent complete set of central elements of the affine algebra (1.3) whose zero level \( A \) is restricted to the block-upper-triangular form \( A \in A_{n,m} \) for any choice of \( n, m \).

These central elements remain central for all the \( k \)-level reductions (2.6). In this case, the complete set of algebraically independent central elements comprises the same elements \( b_{d}^{(i)} \) as above and the elements \( c_{r} \) with \( r = 1, \ldots, [(Nk)/2] \).

In particular, the maximal dimension of the Poisson leaves for the algebra (1.1) on \( A_{n,m} \) is

\[
\frac{n(n+1)}{2}m^{2}−nm−s\left[\frac{n}{2}\right], \quad s = \begin{cases} 1 & \text{for } m \text{ odd,} \\ 0 & \text{for } m \text{ even} \end{cases}
\]

which is always even.

**Proof.** The proof of the fact that the quotients \( b_{d}^{(i)} \) for \( d = 0, \ldots, [(m_{i}−1)/2], \) and \( i = 1, \ldots, n \) are central elements is analogous to the proof of Theorem 3.5 with the only distinction that now some of the row or column vectors will be zero because of the Poissonian restrictions. The proof of algebraic independence is analogous to the proof of Lemma 3.7 in which we must generalise the automorphism (3.10) to the affine case by setting \( G(\lambda) \rightarrow G(\lambda)e^{\phi_{i}+\phi_{j}} \) for all \( p = 0, 1, \ldots \). The computation of the maximal dimension of the Poisson leaves for the algebra (1.1) on \( A_{n,m} \) follows from the fact that the affine space \( A_{n,m} \) has dimension \( \frac{n(n+1)}{2}m^{2}−n \) because we have \( \frac{n(n−1)}{2} \) off diagonal blocks with \( m^{2} \) elements each, and \( n \) diagonal blocks with \( m^{2}−1 \) elements each. We then need to subtract from this the number of algebraically central elements. These are \( b_{d}^{(i)} \) for \( d = 1, \ldots, [(m−1)/2] \), and \( i = 1, \ldots, n \), giving \( \left[\frac{m−1}{2}\right] \) \( n \) algebraically independent central elements, and \( c_{k} \), \( k = 1, \ldots, \left[\frac{m−1}{2}\right] \), giving another \( \left[\frac{m−1}{2}\right] \) algebraically independent central elements. \( \square \)

**Remark 3.11.** Note that the constructed central elements are of two, very different sorts. Those generated by \( \det(\mathbb{A}+\lambda^{-1}\mathbb{A}^{T}) \) are invariant under the transformation (3.10) whereas, providing all \( \det M_{d}^{(i)} \) are nonzero, we can use transformations (3.10) to set all the central elements \( \det M_{m_{i}−d}^{(i)}/\det M_{d}^{(i)} \) equal to \pm1 (in the case of real parameters \( \phi_{a} \)). Then, the group of (anti)automorphisms of the Poisson algebra (1.1) in the case of the block-upper-triangular matrices from Definition 1.1 is presumably generated by the braid group transformations (5.35) with \( I = 1, \ldots, n−1, \) by the anti-automorphism \( P \) from (3.8), and, possibly, by “inner” automorphisms \( \beta_{i} \) (in terminology of Molev and Ragoucy paper [17], where these automorphisms were constructed for the case \( m = 2 \), \( i = 1, \ldots, n \), that act nontrivially only inside the blocks \( A_{k,i}, A_{k,k} (n \geq k > i) \), and \( A_{l,i} (i > l \geq 1) \) and mutually commute. In the case of general \( m \), these automorphisms, if exist, must also commute mutually, and
to preserve the rational central elements, we expect they to preserve the structure of minors $M^{(1)}_i$, that is, they must correspond to a sort of transposition w.r.t. the antidiagonal of the matrix $A_{i,i}$. The problem of existence of these automorphisms is under investigation.

4. Groupoid of block upper triangular bilinear forms

In this section, we follow Bondal’s approach [1, 2] which consists in introducing a suitable notion of groupoid of block–upper–triangular quadratic bilinear forms in such a way that the Poisson bracket on the base space $A_{n,1}$ is given in terms of the anchor map associated to the corresponding Lie algebroid. In this approach the braid group elements are then obtained as elementary generators of this groupoid and automatically preserve the Poisson structure.

In Bondal’s case, namely when $m = 1$, the Lie algebroid is isomorphic to the Lie algebroid on the cotangent bundle $T^*A_{n,1}$. As soon as $m > 2$ this ceases to be true, making the integration of the Lie algebroid rather tricky. We solve this problem in this Section. Let us first recall Bondal’s construction.

4.1. The case of upper-triangular matrices with one on the diagonal. In this section we recall some key results from Bondal’s work [1, 2], or at least our interpretation of them.

Denote by $\mathcal{A} \subset GL_n(\mathbb{C})$ the set of all upper–triangular matrices $A$ with 1 on the diagonal. The Lie group $GL_n(\mathbb{C})$ acts on $\mathbb{C}^n$ in the usual way, thus acting on bilinear forms as

$$\forall A, B \in GL_n(\mathbb{C}), \quad A \mapsto BAB^T.$$ 

This action of $GL_n(\mathbb{C})$ does not preserve $\mathcal{A}$, however, for any element $A \in \mathcal{A}$, one can take the subset $\mathcal{M}_A \subset GL(\mathbb{C}^n)$ of elements that preserve the structure of $A$, or in other words

$$\mathcal{M}_A = \{ B \in GL(\mathbb{C}^n) \mid \forall A \mapsto BAB^T \in \mathcal{A} \}.$$ 

Let $(\mathcal{A}, \mathcal{M})$ where $\mathcal{M} = \bigcup_{A \in \mathcal{A}} \mathcal{M}_A$ be the set of pairs $(A, B)$ such that $A \in \mathcal{A}$ and $B \in \mathcal{M}_A$. The identity morphism is defined as

$$e = (A, 1),$$

the inverse as

$$i : (A, B) \rightarrow (B^T A, B^{-1}),$$

and the partial multiplication as

$$m \left( (B_1A^T B_1^T, B_2), (A, B_1) \right) = (A, B_2 B_1).$$

These rules define the structure of smooth algebraic groupoid on $(\mathcal{A}, \mathcal{M})$ [1]. A smooth groupoid naturally defines a Lie algebroid $(\mathcal{A}, \mathfrak{g})$, i.e. its infinitesimal version:

$$\mathfrak{g} := \bigcup_{A \in \mathcal{A}} \mathfrak{g}_A$$

where

$$\mathfrak{g}_A := \{ g \in \mathfrak{gl}_n(\mathbb{C}), \mid A + Ag + g^T A \in \mathcal{A} \}.$$ 

We denote by $D_A$ the anchor map

$$D_A : \mathfrak{g}_A \rightarrow T_A \mathcal{A}, \quad g \mapsto Ag + g^T A.$$
The Lie bracket on the space of sections $\Gamma(\mathfrak{g})$ is defined by
\[
[v_1, v_2]_\mathfrak{g} := [g_1, g_2] + \sum_{i,j} \frac{\partial v_2}{\partial a_{i,j}} (D_A g_1)_{i,j} - \frac{\partial v_1}{\partial a_{i,j}} (D_A g_2)_{i,j},
\]
where for $i = 1, 2$, $v_i \in \Gamma(\mathfrak{g})$ and we denoted by $g_i \in \mathfrak{g}_A$ the image of $A \in \mathfrak{A}$ under $v_i$. Here the first term in the right hand side is the usual matrix commutator.

The following Lemma is based on the fact that the tangent bundle $T\mathfrak{A}$ can be identified with the space of strictly upper triangular matrices, while the cotangent bundle $T^*\mathfrak{A}$ can be identified with the space of strictly lower triangular matrices by the Killing form, which is given simply by the trace in this case.

**Lemma 4.1.** [1] The map
\[
P_A : T^*_{\mathfrak{A}} \mathfrak{A} \to g_A \quad w \mapsto P^-_{1/2}(w_A) - P^+_{1/2}(w^T_A),
\]
where $P_{\pm,1/2}$ are the projection operators:
\[
P_{\pm,1/2} a_{i,j} := \frac{1}{2} \pm \text{sign}(j-i) a_{i,j}, \quad i, j = 1, \ldots, n,
\]
defines an isomorphism between the Lie algebroid $(\mathfrak{g}, D_A)$ and the Lie algebroid $(T^*\mathfrak{A}, D_A P_A)$.

The Poisson bi-vector $\Pi$ on $\mathfrak{A}$ is then obtained by the anchor map on the Lie algebroid $(T^*\mathfrak{A}, D_A P_A)$ (see Proposition 10.1.4 in [15]) as:
\[
\Pi : T^*_{\mathfrak{A}} \mathfrak{A} \times T^*_{\mathfrak{A}} \mathfrak{A} \to C^\infty(\mathfrak{A}) \quad (\omega_1, \omega_2) \mapsto \text{Tr} (\omega_1 D_A P_A(\omega_2)).
\]

In coordinates one can compute the Poisson bracket by
\[
\{a_{i,k}, a_{j,l}\} = \frac{1}{2} \pm \text{sign}(j-i) a_{i,j}, \quad i, j = 1, \ldots, n,
\]
defines an isomorphism between the Lie algebroid $(\mathfrak{g}, D_A)$ and the Lie algebroid $(T^*\mathfrak{A}, D_A P_A)$.

This gives rise to the Poisson bracket on $\mathfrak{A}$ given by the Dubrovin–Ugaglia bracket [9, 22]:
\[
\{a_{i,k}, a_{j,l}\} = 0, \quad \text{for } i < k < j < l, \text{ and } i < j < l < k,
\]
\[
\{a_{i,k}, a_{j,l}\} = 2 (a_{i,j} a_{k,l} - a_{i,l} a_{k,j}), \quad \text{for } i < j < k < l,
\]
\[
\{a_{i,k}, a_{k,l}\} = a_{i,l} a_{k,l} - 2 a_{i,l}, \quad \text{for } i < k < l,
\]
\[
\{a_{i,k}, a_{j,k}\} = -a_{i,k} a_{j,k} + 2 a_{i,j}, \quad \text{for } i < j < k,
\]
\[
\{a_{i,k}, a_{i,l}\} = -a_{i,k} a_{i,l} + a_{k,l}, \quad \text{for } i < k < l.
\]

**Remark 4.2.** Note that the Poisson structure (4.25) is equivalent to the one given by (1.1) by plugging in the restriction $\mathfrak{A} \in \mathfrak{A}$ on the right-hand side.

### 4.2. The groupoid in the general case

The key point in our construction is based on Remark 4.2: for any $n, m$ our algebra (1.1) is given by the same Poisson bi-vector $\Pi$ as in the case $m = 1$. So, due to equation (4.23), we must retain the same Lie algebroid structure on $T^* \mathfrak{A}_{n,m}$, in other words we keep the same anchor map $D_A P_A$. Let us be more precise.

The tangent bundle $T \mathfrak{A}_{n,m}$ is now identified with the set of block–upper–triangular matrices $\delta \mathfrak{A}$ such that the diagonal blocks satisfy:
\[
\text{tr } A_i^{-1} \delta A_i = 0,
\]
Analogously the cotangent bundle $T^*A_{n,m}$ is now identified with the set of block–
lower–triangular matrices $\omega$ such that the diagonal blocks satisfy:

(4.27) \[ \text{tr} A_{-1/2}^T \omega_{1,1} = 0, \]

The map $P_\omega$ is defined as above:

\[ P_\omega : \quad T^*_\omega A_{n,m} \rightarrow \text{Mat}_N(C) \]

and has a non-trivial kernel now. We now define the Lie algebroid as image of this
map:

\[ \mathfrak{g}_\omega := \text{Im} (P_\omega), \quad \mathfrak{g} := \cup_{\omega \in A_{n,m}} \mathfrak{g}_\omega. \]

**Lemma 4.3.** The bilinear form \((4.23)\) considered as a bilinear form on a vector
space $\mathbb{C}^{(nm)^2}$ in which we substitute arbitrary (commuting) vectors $\omega_1, \omega_2 \in \mathbb{C}^{(nm)^2}$
is skew-symmetric.

**Proof.** We first write the explicit expression for the bilinear form \((4.23)\):

\[ \text{Tr}[\omega_1 D_\omega (P_\omega (\omega_2))] = \text{Tr}[\omega_1 \omega P_{-1/2}(\omega_2 \omega) - \omega_1 P_{+1/2}(\omega T A T) + \omega \omega_1 P_{+1/2}(A T \omega_2) - \omega \omega_1 P_{-1/2}(A \omega_2)]. \]

(4.29)

The assertion of the lemma follows if by substituting

$\omega_1 = \omega_2 = \omega$

in \((4.29)\) we obtain zero for any $\omega \in \mathbb{C}^{(nm)^2}$. Using that

\[ \omega \omega = P_{+1/2}(\omega \omega) + P_{-1/2}(\omega \omega), \]

(4.30)

for the sum of the second and third terms in the r.h.s. of \((4.29)\) we obtain under
the trace sign the expression

\[ -P_{+1/2}(\omega A) P_{+1/2}(\omega T A T) - P_{-1/2}(\omega A) P_{+1/2}(\omega T A T) \]

\[ + P_{+1/2}(A \omega) P_{+1/2}(A T \omega_2) + P_{-1/2}(A \omega) P_{+1/2}(A T \omega_2). \]

Here the second term is the transposed fourth term (with the opposite sign), so
the sum of these two terms vanishes under the trace sign. In the first and third
terms, only the products of diagonal projections, $P_d$, contribute to the trace, and
we obtain under the trace sign the expression

\[ -\frac{1}{4} P_d(\omega \omega) P_d(\omega T A T) + \frac{1}{4} P_d(A \omega) P_d(A T \omega T) \]

\[ = -\frac{1}{4} P_d(\omega \omega) P_d(\omega T A T) + \frac{1}{4} P_d(A \omega) P_d(A T \omega T) \]

\[ = -\frac{1}{4} P_d(\omega \omega) P_d(\omega T A T) + \frac{1}{4} P_d(\omega \omega) P_d(\omega T A T) = 0, \]

where we have used that, for any two matrices $X$ and $Y$, $P_d(X) P_d(Y) = P_d(Y) P_d(X)$
and $P_d(X T) = P_d(X)$.

For the sum of the first and fourth terms in the r.h.s. of \((4.29)\) we obtain (using
the cyclicity property of the trace)

\[ \text{Tr}[\omega A P_{-1/2}(\omega \omega) - \omega A P_{+1/2}(\omega \omega)] \]

(4.31) \[ = \text{Tr}[-\omega A P_{+1/2}(\omega \omega) + \omega A P_{+1/2}(\omega \omega)] \]
the Lie bracket (4.20) is already proved in [2] in the case of a Lie algebroid.

Assume that there exists a Lie groupoid integrating the Lie algebroid defined by (4.20) is given by (4.19), and the sets of central elements coincide:

\[ \{ b^{(1)}_d(B\hat{A}B^T) \} = \{ b^{(1)}_d(\hat{A}) \}, d = 0, \ldots, \left[ \frac{n}{2} \right], I = 1, \ldots, n \}, \]

where

\[ b^{(1)}_d \equiv \det M^{(1)}_{m-d}/\det M^{(1)}_d, \]

are the rational central elements of our algebras (1.1) and (1.3).

**Proof.** Assume there exists a Lie groupoid integrating the Lie algebroid defined by (4.28). Let us prove that it must then preserve all central elements. Let \( f \) be any central element for the algebra (1.1). Its variation along any element of the groupoid is

\[ \delta f = \sum_{i,j} \frac{\partial f}{\partial a_{i,j}} \delta a_{i,j} = \sum_{i,j} \frac{\partial f}{\partial a_{i,j}} (D_A P_A(\omega))_{i,j} \]
for $\omega = \delta a_{i,j}$ (thanks to the Killing form). Using the definition (4.23) of the Poisson bi-vector $\Pi$, we have that

$$\delta f = \Pi(df, \delta a_{j,i}) = \{f, a_{i,j}\},$$

where the right hand side is zero for a central element. This shows that the Lie groupoid integrating the Lie algebroid defined by (4.28) must preserve all central elements and therefore it is defined by (4.33), (4.34).

We conclude this Section observing that the identity morphism, the inverse and the partial multiplication for the groupoid $M_{n,m}$ are still given by (4.16), (4.17) respectively.

**Remark 4.6.** The question whether or not there exists a smooth groupoid on $A_{n,m}$ such that its Lie algebroid structure is given by $(T^* A_{n,m}, D_A P_A)$ remains open. We plan to address this question in the future by using the results developed by Crainic and Fernandes in [7, 8]. However it is worth noticing that if rather than the whole Poisson manifold $A_{n,m}$, we restrict to a symplectic leaf, call it $L_{n,m}$, then one can repeat Bondal’s construction as the Lie algebroid $(T^* L_{n,m}, D_A P_A)$ is indeed isomorphic to the restriction of $(g,D_A,[\cdot,\cdot])$ to $L_{n,m}$.

5. **Braid-group transformations**

The braid-group transformations $\beta_{I,I+1}$, $I = 1, \ldots, n-1$, are transformations from the groupoid (4.33), (4.34) preserving the form of the matrix $A$, so by construction they must preserve the Poisson structure (1.1). They act on $A$ as follows:

$$\beta_{I,I+1}[A] = B_{I,I+1}AB_{I,I+1}^T \equiv \tilde{A},$$

where the matrix $B_{I,I+1}$ has the block form

$$B_{I,I+1} = \begin{bmatrix}
I & E & \cdots & E \\
O & \ddots & \ddots & \vdots \\
O & \ddots & \ddots & E \\
O & \ddots & \ddots & \ddots \\
\end{bmatrix},$$

as above, $E$ and $O$ are the respective $m \times m$ unit and zero matrices.

It is straightforward to verify that the transformation (5.35) preserves the form of the matrix $A$ with

$$\tilde{A}_{I,I} = \tilde{A}_{I+1,I+1}, \quad \tilde{A}_{I+1,I+1} = \tilde{A}_{I,I}, \quad \tilde{A}_{I,I+1} = \tilde{A}_{I+1,I}^T,$$

(5.37) $J < I$ : $\tilde{A}_{J,I} = \tilde{A}_{J,I} \tilde{A}_{I,I}^{-1} \tilde{A}_{I+1,I+1} - \tilde{A}_{J,I+1}, \quad \tilde{A}_{J+1,I+1} = \tilde{A}_{J,I} \tilde{A}_{I,I}^{-1} \tilde{A}_{I+1,I}$,

$J > I + 1$ : $\tilde{A}_{I,J} = \tilde{A}_{I,J} \tilde{A}_{J,J}^{-1} \tilde{A}_{I+1,I+1} - \tilde{A}_{I+1,J}, \quad \tilde{A}_{I+1,J} = \tilde{A}_{I,J} \tilde{A}_{I,I}^T \tilde{A}_{I,J}$

and with all other blocks retaining their form.

We have two theorems concerning the transformations (5.35), (5.36).

**Theorem 5.1.** The transformations (5.35), (5.36) are automorphisms of the Poisson structure (1.1) restricted to the block-upper-triangular matrices $A$ from Definition 1.1.
The statement follows from that the transformation (5.35), (5.36) is a transformation from the groupoid of block-upper-triangular matrices.

**Theorem 5.2.** The transformations (5.35), (5.36) satisfy the braid-group relation, (5.38) \( \beta_{I,I+1} \beta_{I+1,I+2} \beta_{I,I+1} [A] = \beta_{I+1,I+2} \beta_{I,I+1} \beta_{I+1,I+2} [A], \) \( I = 1, \ldots, n - 2. \)

We prove this theorem and the following proposition by the direct calculation.

**Proposition 5.3.** We have that \( (\beta_{n-1,n} \cdots \beta_{2,3} \beta_{1,2})^n [A] = \tilde{A}, \) where \( \tilde{A}_{I,J} = \left( A_{I,I} A_{I,I}^T \right)^{n-2} \tilde{A}_{I,J} \left( A_{I,I}^{-1} A_{I,I}^T \right)^{n-2} \) and, in particular, \( \tilde{A}_{I,I} = A_{I,I} \) \( \forall n. \)

**5.1. Extension of the braid group action to the twisted Yangian case.** As in the case of the standard twisted Yangian algebra (see [5]), we have the extension of the braid-group action in the case where the matrix \( A \) has the original block-upper-triangular form with all blocks having the same size \( m \times m. \)

**Proposition 5.4.** The extended braid group transformations for the algebra (1.3) in the case where the matrix \( A \) has the block-upper-triangular form described in Definition 1.1 admits the following matrix representation in terms of the matrix \( G(\lambda) \) (1.2):

\[
\beta_{I,I+1} [G(\lambda)] = B_{I,I+1} G(\lambda) B_{I,I+1}^T, \quad I = 1, \ldots, n - 1
\]

where the matrices \( B_{I,I+1} \) have the form (5.36).

**Theorem 5.5.** The transformations (5.39), (5.40) satisfy the braid-group relation, (5.42) \( \beta_{I,I+1} \beta_{I+1,I+2} \beta_{I,I+1} [G(\lambda)] = \beta_{I+1,I+2} \beta_{I,I+1} \beta_{I+1,I+2} [G(\lambda)], \) \( I = 1, \ldots, n \mod n. \)

**6. Quantisation**

The affine algebra (1.3) is the semiclassical limit of the quantum algebra generated by the matrix elements \( G_{ij}^{(p)}, \) \( i,j = 1, \ldots, N, \) \( p \in \mathbb{Z}_{\geq 0} \) subject to the defining relations:

\[
R(\lambda, \mu) G(\lambda) R(\lambda^{-1}, \mu)^T = G(\mu) R(\lambda^{-1}, \mu)^T G(\lambda) R(\lambda, \mu)
\]

where the apex \( T_1 \) indicates the transposition in space one and the R-matrix is given by

\[
R(\lambda, \mu) = (\lambda - \mu) \sum_{i \neq j} E_{ii} \otimes E_{jj} + (q^{-1} \lambda - q \mu) \sum_i E_{ii} \otimes E_{ii} +
\]
it is a solution of the Yang–Baxter equation.

In coordinates the quantum algebra relations are pretty cumbersome, let us present here the formula for the level 1 reduction of the quantum algebra, or in other words, for the quantum analogue of our algebra (1.1):

\[
q^\delta_{i,j} + \delta_{i,s}a_{j,t} - q^\delta_{i,t} + \delta_{j,s}a_{i,s} = (q - q^{-1})q^{\delta_{i,j}}(\delta_{t,s} - \delta_{i,j})a_{j,s}a_{i,t} + \\
(q - q^{-1})(q^{\delta_{i,j}}\delta_{t,s}a_{j,s} - q^{\delta_{j,s}}\delta_{i,j}a_{i,j}a_{s,t}) + \\
(q - q^{-1})^2\delta_{s,i}(\delta_{t,s} - \delta_{i,j})a_{j,i}a_{s,t},
\]

where \(\delta_{i,j} = 1\) for \(i > j\) and 0 otherwise. For \(m = 2\) this quantum algebra coincides with the twisted quantised enveloping algebra \(U^{tw}(\mathfrak{sp}_{2n})\) [21, 18].

The affine quantum algebra (6.43) coincides in the case of \(m = 1\) (\(m = 2\)) with the twisted \(q\)-Yangian \(Y_q^n(\mathfrak{so}_n)\) \((Y_q^n(\mathfrak{sp}_{2n}))\) for the orthogonal (symplectic) Lie algebra introduced in [18]. For \(m > 2\) this algebra has never been studied before to the best of our knowledge.

In the semiclassical limit the affine quantum algebra (6.43) gives rise to our affine algebra (1.3). We already calculated this semiclassical limit in the case \(m = 1\) in [5], for general \(m\) the computation is exactly the same.

In this paper, we construct the quantum braid-group action only in the case of the \(\mathfrak{sp}_{2n}\) case, but the main features of the technique must remain unchanged for both the general \(m \times m\)-b.u.t. case and for the affine algebras.

6.1. Quantum braid group action for the \(\mathfrak{sp}_{2n}\) case. In order to quantise the braid group action, we need to find the quantum analogues of the inverse, \(A_{I,J}^{-1}\), the transposed, \(A_{I,J}^T\), and the inverse-transposed, \(A_{I,J}^{-T}\), matrices for the diagonal blocks and also the transposed \(A_{I,J,I+1}\) for the off-diagonal blocks.

We first find the laws of quantum complex conjugation for all the entries \(a_{i,j}\) (these formulas are valid for all \(n\) and \(m\)). The main point is that the quantum complex conjugation needs to be an automorphism of the algebra (6.43).

From formulas (6.45), assuming all the diagonal entries \(a_{i,i}, i = 1, \ldots, mn\), to be self-adjoint operators, we obtain the laws of conjugation:

\[
a_{i,s}^* = a_{i,s}, \quad \left\{ \begin{array}{ll}
a_{j,i}^* = qa_{j,i} & \text{for } i < j, \\
a_{i,j}^* = qa_{i,j} + (1 - q^2)a_{j,i} & \text{for } i > j. \end{array} \right.
\]

The last formula implies that in the case where we restrict to the b.u.t. case, the lower-triangular \((i < j)\) matrix entries \(a_{j,i}\) not belonging to the diagonal blocks vanish, and we obtain merely that \(a_{i,j}^* = qa_{i,j}\) for all the entries of the matrices \(A_{I,J}\) with \(I < J\).

We then have the following prescription:

1) All the transposition operations are replaced by the Hermitian conjugations.

Note that the Hermitian conjugation of the diagonal blocks is different from the Hermitian conjugation of the off-diagonal ones. As an example, we present the result of the Hermitian conjugation for the block \(A_{1,1}\),

\[
\left( \begin{array}{cc}
a_{11} & a_{12} \\
a_{21} & a_{22} \end{array} \right)^\dagger = \left( \begin{array}{cc}
a_{11} & qa_{21} \\
a_{21} + (1 - q^2)a_{22} & a_{22} \end{array} \right),
\]
and

\[ A_{I,j} = q[A_{I,j}]^T \text{ for } J > I, \]
where the transposition is understood here and hereafter in the standard matrix sense (note however that the transpose of the product of two matrices in the quantum case is not given by the reverse order product of their transposed due to the noncommutativity of their entries, say, \( (A_{I,J}A_{K,L})^T \neq A_{K,L}^T A_{I,J}^T \)).

(2) The inverse \( A_{I,J}^{-1} \) is to be found from the operatorial identity \( A_{I,J}A_{I,J}^{-1} = E \), in which the order in which we multiply operators follows from that of the matrix multiplication. In the \( m = 2 \) case, the result is (we present it for \( A_{1,1} \), the generalisation to other \( 2 \times 2 \)-blocks \( A_{I,J} \) is obvious)

\[ (a_{11} a_{12} \ a_{21} a_{22})^{-1} = \frac{1}{a_{11}a_{22} - q^2a_{12}a_{21}} \begin{pmatrix} a_{22} & -a_{12} + (q - q^{-1})a_{21} \\ -aq_{12} & a_{11} \end{pmatrix} \]

Here the combination \( a_{11}a_{22} - q^2a_{12}a_{21} \) is the quantum determinant of the block \( A_{1,1} \); all these determinants are self-adjoint central elements of the quantum algebra \( \mathfrak{g}p_{2n} \).

(3) eventually, the inverse-transposed in the quantum case becomes \( [A_{I,J}^{-1}]^+ = [A_{I,J}]^{-1} \) where we use the expressions (6.49) and (6.47); the result is merely (we always assume that \( q := e^{-i\pi\hbar} \) and, therefore, \( q^* = q^{-1} \))

\[ (a_{11} a_{12} \ a_{21} a_{22})^{-1} = \frac{1}{a_{11}a_{22} - q^2a_{12}a_{21}} \begin{pmatrix} a_{22} & -q^{-1}a_{21} \\ -qa_{12} & a_{11} \end{pmatrix} \]

**Theorem 6.1.** For the b.u.t matrix \( \mathcal{A}^h \) from Definition 1.1 whose entries are operators subject to the conjugation law (6.46) the action of the quantum braid group has the adjoint matrix form

\[ \beta_{I,J+1}^h [\mathcal{A}^h] = B_{I,J+1}^h \mathcal{A}^h \left[ B_{I,J+1}^h \right]^+ \equiv \tilde{\mathcal{A}}^h, \]

where the matrix \( B_{I,J+1}^h \) has the block form (here and hereafter we assume that all matrix entries are operators and that these operators are multiplied in the order prescribed by the matrix multiplication)

\[ B_{I,J+1}^h = \begin{pmatrix} E & \cdots & \cdots & E \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ E & \cdots & \cdots & E \end{pmatrix}, \]

where \( a = b + 1 \) for all \( m \) and \( b = 0 \) for \( m = 1 \) and \( b = 1 \) for \( m = 2 \).

Before proving the theorem, we formulate the following lemma, which can be verified by a simple calculation.

**Lemma 6.2.** The two quantum braid-group relations

\[ \beta_{I,J+1}^h \beta_{I-1,J}^h \beta_{I,J+1}^h [\mathcal{A}^h] = \beta_{I-1,J}^h \beta_{I,J+1}^h \beta_{I-1,J}^h [\mathcal{A}^h], \quad I = 2, \ldots, n - 1, \]
Remark 6.3. The transformation $\beta^{n}_{-1,n}\beta^{n}_{-2,n-1}\cdots\beta^{n}_{2,3}\beta^{n}_{1,2})^{n}[A_{1,j}] = (A_{1,1}^{-1}A_{1,2}^{-1})^{n-2}A_{1,j}^{-1}(A_{2,1}^{-1}A_{2,j}^{-1})^{n-2}$, which are the quantum analogues of the respective relation (5.38) and Proposition 5.3, are satisfied for any choice of the parameters $a$ and $b$ in the formula (6.52).

**Proof of Theorem 6.1.** We need to prove that the quantum algebra (6.45) is preserved under the quantum action (6.51). It is enough to restrict to the case of $n = 3$ and concentrate on the action of $\beta^{n}_{1,2}$:

$$\beta^{h}_{1,2}(A^{h}) = \tilde{A}^{h} = \begin{pmatrix} A_{2,2} & q^{-a+b}A_{1,2}^{-1} & q^{-a}(A_{1,2}^{-1}A_{1,3}^{-1}A_{1,2} - A_{2,2}) \\ \bigcirc & A_{1,1} & q^{-b}(A_{1,1}^{-1}A_{1,3}^{-1}A_{1,1}) \\ \bigcirc & \bigcirc & A_{3,3} \end{pmatrix}.$$  

In order for the $(1, 2)$ matrix entry to have the same conjugation law (6.48) as the original operator $\tilde{A}_{1,2}$ we need that $-a + b = -1$, or $a = b = 1$ for all $m$. However, when considering the $(2, 3)$ matrix entry, we observe the explicit difference between cases where $m = 1$ and $m = 2$. If $m = 1$, then $\tilde{A}_{1,1} = 1$, and we just have $q^{-b}\tilde{A}_{1,3}$, so, in order to preserve the conjugation law we must merely set $b = 0$. But in the case of $2 \times 2$-matrices the situation becomes different: we need a rather nontrivial calculation, which involves repeated applications of the commutation relations (6.45), to demonstrate that

$$\left(\tilde{A}_{1,1}\tilde{A}_{1,2}^{-1}\tilde{A}_{1,3}\right)^{\dagger} = q^{-1}\left[\tilde{A}_{1,1}\tilde{A}_{1,2}^{-1}\tilde{A}_{1,3}\right]^{T}$$  

for $m = 2$.

So, we need to set $b = 1$ for $m = 2$ in order to preserve the conjugation law.

The proof of preserving the algebra is then a straightforward brute force computation in which one needs to check relations (6.45) entry by entry.

**Remark 6.3.** The transformation $\beta_{1,2}$ given by the formula (6.51) in the case of $2 \times 2$-block matrix $A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ \bigcirc & A_{2,2} \end{pmatrix}$ results in that $A_{1,1} \rightarrow A_{2,2}$, $A_{2,2} \rightarrow A_{1,1}$, and $A_{1,2} \rightarrow [A_{1,2}]^{T}$. It is easy to see from the algebra (6.43) that this transformation is an automorphism of the corresponding quantum algebra. This automorphism was stated as Theorem 4.8 in [17].

**Conjecture 6.4.** It is plausible that the braid-group action in the general case of $m \times m$-matrix blocks has the same form (6.51) and (6.52) with $a = b + 1$ and $b = m - 1$ where we have to determine the inverse matrix $A_{I,J}^{-1}$ from the operatorial equality $A_{I,J}^{-1}A_{I,J}^{-1} = \mathbb{1}$ and use the conjugation rules (6.46).

**Conjecture 6.5.** In the twisted Yangian case (6.43) for $m = 2$ we expect to extend the same quantum conjugation relations to all levels and to quantise the element $\beta^{h}_{n,1}$ defined by (5.41) to

$$\beta^{h}_{n,1}(A^{h}) = B^{h}_{n,1}(\lambda)A^{h}\left[B^{h}_{n,1}(\lambda^{-1})\right]^{\dagger},$$
where the matrix $B_{n,1}^b(\lambda)$ has the block form:

$$B_{n,1}^b(\lambda) = \begin{pmatrix} \mathbb{O} & \lambda q^{-b} \kappa_{n,n} \kappa_{n,n}^\dagger & \cdots & \lambda q^{-b} \kappa_{n,n} \kappa_{n,n}^\dagger \\ & \mathbb{E} & \cdots & \mathbb{E} \\ & \cdots & \mathbb{E} \\ & \cdots & \cdots & \cdots \end{pmatrix}$$

with $b = m - 1$.

6.2. \textit{R-matrix role in the classical case.} The classical Poisson bracket (1.1) was obtained by imposing

$$\{a_{i,j}, a_{k,l}\} = \frac{\partial}{\partial a_{i,j}} \wedge \frac{\partial}{\partial a_{k,l}} S,$$

where $S$ is the Poisson bi-vector computed on the one forms $da_{i,j}$ and $da_{k,l}$:

$$S = \text{Tr} (da_{i,j} D_A P_A (da_{k,l})).$$

In this subsection we want to show that (6.55) can also be written in $R$-matrix notation as:

$$\{A_1, A_2\} = - \left( A_2 A_1 r - r A_1 A_2 + A_2 r^T A_1 - A_1 r^T A_2 \right),$$

where $r$ is given by

$$r = \sum_i E_{ii} \otimes E_{ii} + 2 \sum_{i>j} E_{ij} \otimes E_{ji}.$$

Of course, we could just do a brute force computation to simply prove that the two formulae coincide. However, we prefer to show a more general proof which relies on the relation between the $P_A$ operator and the classical $R$-matrix.

Given any matrix $X$, let us define the following four operators $r_+, r_-:

$$r_+(X) := \frac{1}{2} \text{Tr} \left( r^T X \right), \quad r_-(X) := \frac{1}{2} \text{Tr} \left( r X \right),$$

then our matrix $g := P_A(w)$ is given by two different formulae according to which space we want it in:

$$\begin{align*}
\frac{1}{2} g &= r_+ \left( \frac{1}{2} w \kappa \right) - r_+ \left( \frac{1}{2} w^T \kappa^T \right), \\
2 g &= r_- \left( \frac{1}{2} w \kappa \right) - r_+ \left( \frac{1}{2} w^T \kappa^T \right),
\end{align*}$$

where $r^T$ is the transposition in both spaces.

Now, it is clear that in $R$-matrix notation $S$ is given by

$$S = \text{Tr} \left[ (dA)^T \kappa r_- (dA)^T \kappa \right] - (dA)^T \kappa r_+ (dA)^T \kappa^T + \kappa (dA)^T r_+ (dA)^T \kappa - \kappa (dA)^T r_- (dA)^T \kappa.$$
where we did not specify the spaces as any choice leads to the same result (up to sign). For example choosing space 2 we get:

\[
S = \frac{1}{2} \text{Tr} \left[ (d\mathbb{A})^T \mathbb{A} r^T (d\mathbb{A})^T - (d\mathbb{A})^T \mathbb{A} r^T (d\mathbb{A})^T - (d\mathbb{A})^T \mathbb{A} r_{T^1} \mathbb{A} (d\mathbb{A})^T \right],
\]

(6.60)

so that

\[
\frac{\partial}{\partial d\mathbb{A}} \wedge \frac{\partial}{\partial d\mathbb{A}} S = -2 \mathbb{A} r^T + r^T \mathbb{A} + \mathbb{A} r_{T^1} \mathbb{A} - \mathbb{A} r_{T^1} \mathbb{A} = - \left( 2 \mathbb{A} r - r \mathbb{A} - \mathbb{A} r_{T^1} \mathbb{A} + \mathbb{A} r_{T^1} \mathbb{A} \right),
\]

as we wanted.

Using the \( r \)-matrix representation, we can establish the relation between Poisson algebras of \( \mathbb{A} \) and the standard Poisson–Lie groups. The statement below is analogous to the \( T, \overline{T} \)-representation of the algebroid of upper-triangular matrices in \([16, 17, 18]\):

**Lemma 6.6.** For \( T, \overline{T} \in GL_N(\mathbb{C}) \) satisfying the Poisson–Lie group relations,

\[
\frac{1}{2} \{ T \otimes T \} = rTT - TTr,
\]

\[
\frac{1}{2} \{ T \otimes \overline{T} \} = rTT - \overline{T}Tr,
\]

\[
\frac{1}{2} \{ T \otimes \overline{T} \} = rTT - \overline{T}Tr,
\]

the quantity \( \mathbb{A} := \overline{T}T^T \) satisfies the Poisson algebra (6.56).

The proof is the direct calculation using the properties of the \( r \)-matrix that \( r + r_{T^1}T^2 = 2P \), where \( P \) is the permutation \( r \)-matrix, \( P = \sum_{i,j} E_{i,j} \otimes E_{j,i} \).

The authors are writing a second paper to show that the groupoid \( \mathcal{M}_{n,m} \) described in Theorem 4.5 is indeed a Poisson groupoid and that the Poisson brackets on the morphisms \( B \) are of Poisson Lie form \([6]\). This opens the door to a link with integrable systems: indeed in \([10]\) a one–to–one correspondence between Poisson–Lie group actions on integrable Poisson manifolds and twisted multiplicative Hamiltonian actions on symplectic groupoids was established. For an action of a Poisson–Lie group, as in our case, on a Poisson manifold \( \mathcal{A}_{n,m} \), it is possible to build an explicit description of the lifted Hamiltonian action on the symplectic groupoid \([10]\). This work is postponed because \( \mathcal{M}_{n,m} \) is a Poisson groupoid rather than symplectic.

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References

[1] A. Bondal, A symplectic groupoid of triangular bilinear forms and the braid groups, preprint IHES/M/00/02 (Jan. 2000); Izv. Math., 68 (2004) 659–708.
[2] A. Bondal, Symplectic groupoids related to Poisson–Lie groups, Tr. Mat. Inst. Steklova, 246 (2004) 43–63.
[3] A.S. Cattaneo and G. Felder, Poisson sigma models and symplectic groupoids, Quantization of singular symplectic quotients, Progr. Math., 198, Birkhuser, Basel, (2001) 61–93.
[4] L.O. Chekhov and V.V. Fock, Observables in 3d gravity and geodesic algebras. Czech. J. Phys. 50 (2000) 1201-1208.
[5] L. Chekhov, M. Mazzocco, Isomonodromic deformations and twisted Yangians arising in Teichmüler theory, Advances Math. 226(6) (2011) 4731-4775, arXiv:0909.5350.
[6] L. Chekhov, M. Mazzocco, work in progress, (2013)
[7] M. Crainic and R. Fernandes, Integrability of Lie brackets. Ann. of Math. 157 (2003), no. 2, 575–620.
[8] M. Crainic and R. Fernandes, Integrability of Poisson brackets, J. diff. geom. 66 (2004) 71–137.
[9] Dubrovin B., Geometry of 2D topological field theories, Integrable systems and quantum groups (Montecatini Terme, 1993), Lecture Notes in Math., 1620, Springer, Berlin, (1996) 120–348.
[10] R. Fernandes and D. Iglesias, Integrability of Poisson–Lie Group Actions, Lett. Math. Phys. (2009) 90:137–159.
[11] V. Fock and A. Goncharov, Moduli spaces of local systems and higher Teichmüller theory, Publ. Math. IHES, 103 No.1 (2006) 1–211.
[12] V. Fock and A. Marshakov, A note on quantum groups and relativistic Toda theory, Nucl. Phys. B. 56 (1997) 208–214.
[13] Fock, V. V. and Rosly, A. A., Moduli space of flat connections as a Poisson manifold, Advances in quantum field theory and statistical mechanics: 2nd Italian-Russian collaboration (Como, 1996), Internat. J. Modern Phys. B 11 (1997), no. 26-27, 3195–3266.
[14] Hitchin, N., Deformations of holomorphic Poisson manifolds, ArXiv:1105.4775 (2011).
[15] Mackenzie, Kirill, General Theory of Lie Groupoids and Lie Algebroids, LMS Lect. Note Series 213 (2005).
[16] Molev A., Yangians and classical Lie algebras. Mathematical Surveys and Monographs, 143, American Mathematical Society, Providence, RI, (2007).
[17] A. Molev, E. Ragoucy, Symmetries and invariants of twisted quantum algebras and associated Poisson algebras, Rev. Math. Phys., 20(2) (2008) 173–198.
[18] A. Molev, E. Ragoucy, P. Sorba, Coideal subalgebras in quantum affine algebras, Rev. Math. Phys., 15 (2003) 789–822.
[19] Nelson J.E., Regge T., Homotopy groups and (2+1)-dimensional quantum gravity, Nucl. Phys. B 328 (1989), 190–199.
[20] Nelson J.E., Regge T., Zertuche F., Homotopy groups and (2 + 1)-dimensional quantum de Sitter gravity, Nucl. Phys. B 339 (1990), 516–532.
[21] M. Noumi, Macdonald’s symmetric polynomials as zonal spherical functions on some quantum homogeneous spaces, Adv. Math. 123 (1996), no. 1:16–77.
[22] Ugaglia M., On a Poisson structure on the space of Stokes matrices, Int. Math. Res. Not. 1999 (1999), no. 9, 473–493.