A SPECTRAL CURVE FOR THE GENERATION OF BIPARTITE MAPS IN TOPOLOGICAL RECURSION

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Abstract. We derive an efficient way to obtain generating functions of bipartite maps of arbitrary genus and boundary length using a spectral curve as initial data for the framework of topological recursion. Based on an earlier result of Chapuy and Fang counting these maps and having a structural proximity to topological recursion, we deduce the corresponding spectral curve which has a strong relation to the spectral curve giving rise to generating functions of ordinary maps.

1. Introduction and Main Result

The enumeration of maps has a long history, in which the techniques and tools became more and more efficient and the classes of maps more and more sophisticated: In his Census of Planar Maps, William Tutte achieved groundbreaking progress in the 1960’s [Tut63]. Bender and Canfield then left the realm of planar maps in the 1980’s and also took an embedding into higher-genus surfaces into consideration [BC88]. In the 2000’s, the branch of mathematical physics established a powerful and efficient universal procedure to reach all topological sectors in a recursive way: Topological recursion (TR) of Chekhov, Eynard and Orantin [EO07, CEO06] built a bridge between enumerative and complex geometry (and, based on the work [Kon92], bridges to intersection theory and integrable hierarchies, which we will neglect here) and thus covered numerous, seemingly disconnected areas of mathematical fields, by one universal recursion procedure.

Topological recursion possesses the initial data $(\Sigma, x, y, B)$, where $x : \Sigma \to \Sigma_0$ is a ramified covering of Riemann surfaces, $\omega_{0,1} = y \, dx$ is a meromorphic differential 1-form on $\Sigma$ regular at the ramification points and $\omega_{0,2} = B$ a symmetric bilinear differential form on $\Sigma \times \Sigma$ with double pole on the diagonal and no residue. From this initial data, TR computes recursively in the negative Euler characteristic $-\chi = 2g + n - 2$ an infinite sequence of symmetric meromorphic $n$-forms $\omega_{g,n}$ on $\Sigma^n$ with poles only at the ramification points for $-\chi > 0$. The precise formula and more details are given in Ch. [2]. For specific choices of the initial data $(\Sigma, x, y, B)$, the meromorphic $n$-forms are encoding some enumerative problems.

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The prime example of this framework was the recursive computation of generating functions counting objects known in the literature as ordinary maps (a very readable derivation can be found in [Eyn16]):

**Theorem 1.1 (Eyn16).** The spectral curve \((\mathbb{P}^1, x_{ord}, y_{ord}, \frac{dz_1 dz_2}{(z_1 - z_2)^2})\) with

\[
x_{ord}(z) = \gamma \left( z + \frac{1}{z} \right) \quad y_{ord}(z) = \sum_{k=0}^{d-1} u_{2k+1} z^{2k+1}
\]

where

\[
\gamma^2 = 1 + \sum_{k \geq 1} t_{2k} \binom{2k - 1}{k} \gamma^{2k} \quad u_{2k+1} = \gamma \left( \delta_{k,0} - \sum_{j \geq k+1} t_{2j} \binom{2j - 1}{j + k} \gamma^{2j-2} \right)
\]

computes via TR (see formula (2.1)) generating functions for the enumeration of ordinary maps with \(n\) marked faces of even boundary lengths. The faces have even degrees up to \(2d\), where a face of degree \(2k\) is weighted by \(t_{2k}\).

The theorem includes in general also faces of odd degree, but for later purposes, we want to state it in this form.

Several more classes of maps, e.g. subsets of the ordinary maps, were then discovered to be governed by TR, as ciliated and fully simple maps [BGF20, BCGF21].

In this letter, we will focus on another subset of maps, the bipartite maps containing only those ordinary maps of even face degrees, for which the corresponding maps have vertices in black and white such that no monochromatic edge occurs. A bipartite map is called rooted, if one edge is distinguished and oriented. This rooted edge (also called marked edge) conventionally has its origin in a white vertex (the root vertex). Rooting an edge creates a boundary of a certain even length \(2l_k\) following the face to the right of the rooted edge. Several edges can be rooted such that the roots do not correspond to the same boundary. Bipartite maps already showed up in the context of TR, namely in [CF16] in which the authors were motivated by TR and established a recursive formula sharing many characteristics with the TR. However, the aim of their work is rather to prove rationality statements about bipartite maps and is thus written more in a combinatorist’s language. All these statements are a direct consequence of TR. Their recursion and its proof are mainly built on ideas of TR, but no spectral curve was provided. Thus, the relation of their work to complex geometry will be established in the following Chapter 2. We will deduce:

**Theorem 1.2.** The spectral curve \((\mathbb{P}^1, x_{bip}, y_{bip}, \frac{dz_1 dz_2}{(z_1 - z_2)^2})\) with

\[
x_{bip}(z) = \gamma^2 \left( z + \frac{1}{z} \right) + 2\gamma^2 \quad y_{bip}(z) = \sum_{k=0}^{d-1} u_{2k+1} z^{k+1} / \gamma(1 + z) \]

\[\text{1 A more formal, but less illustrative definition for bipartite maps can be found in [CF16]}\]
computes via TR generating functions for the enumeration of bipartite maps with
n marked faces (or rooted edges) of even boundary lengths. The faces have even
degrees up to \(2d\), where a face of degree \(2k\) is weighted by \(t^{2k}\). We have the
following relation to Thm. 1.1:

\[
x_{\text{bip}}(z^2) = x_{\text{ord}}(z)^2 \quad y_{\text{bip}}(z^2) = \frac{y_{\text{ord}}(z)}{x_{\text{ord}}(z)}
\]

In order to avoid misunderstandings, we would like to mention that an uncon-
ventional definition of bipartite maps, deviating from the one in this paper, is
given in [Eyn16] and coincides just for genus zero and one boundary.

Given these two spectral curves for ordinary and bipartite maps, the machinery
of TR gives rise to generating functions as follows: Let \(\mathcal{T}_{2l_1, \ldots, 2l_n}^{(g)}\) denote the gener-
ating function of bipartite maps with a natural embedding into a genus-g surface
with \(n\) boundaries of length \(2l_1, \ldots, 2l_n\) (\(n\)-fold rooted bipartite maps) and in the
same manner \(\mathcal{T}_{2l_1, \ldots, 2l_n}^{(g)}\) for ordinary maps with faces of even degree. Note that in
particular \(2n - 1\) \(\mathcal{T}_{2l_1, \ldots, 2l_n}^{(0)} = \mathcal{T}_{2l_1, \ldots, 2l_n}^{(0)}\) holds for genus \(g = 0\), however not for
\(g > 0\), where only a small subset of ordinary maps are still bipartite. The prefactor
\(2n - 1\) has an easy combinatorial explanation: As described earlier, the root vertex
is by convention a white one, the black-white coloring of the vertices is completely
determined by fixing a root. Ignoring the colouring, as is it is done for ordinary
maps, an other boundary can have twice the number of labellings. Inductively,
this gives rise to \(2n - 1\) distinct graphs, if \(n\) faces are marked, as ordinary maps.

Define the correlators \(W\) and \(\tilde{W}\) as

\[
W_n^{(g)}(x_{\text{ord}, 1}, \ldots, x_{\text{ord}, n}) = \sum_{l_1, \ldots, l_n} \frac{\mathcal{T}_{2l_1, \ldots, 2l_n}^{(g)}}{x_{\text{ord}, 1}^{2l_1 + 1} \cdots x_{\text{ord}, n}^{2l_n + 1}}
\]

\[
\tilde{W}_n^{(g)}(x_{\text{bip}, 1}, \ldots, x_{\text{bip}, n}) = \sum_{l_1, \ldots, l_n} \frac{\mathcal{T}_{2l_1, \ldots, 2l_n}^{(g)}}{x_{\text{bip}, 1}^{l_1 + 1} \cdots x_{\text{bip}, n}^{l_n + 1}}
\]

from which the generating functions can be read off as a simple residue operation,
e.g. for bipartite maps:

\[
\mathcal{T}_{2l_1, \ldots, 2l_n}^{(g)} = (-1)^n \text{Res}_{x_{\text{bip}, 1} = \ldots = x_{\text{bip}, n} = \infty} x_{\text{bip}}^{l_1} \cdots x_{\text{bip}}^{l_n} W_n^{(g)}(x_{\text{bip}, 1}, \ldots, x_{\text{bip}, n}) dx_{\text{bip}, 1} \cdots dx_{\text{bip}, n}
\]

The crucial connection to the infinite sequence of meromorphic \(n\)-forms \(\omega_{g,n}\) generated by TR is the following identification for \(2g + n - 2 > 0\)

\[
\omega_{g,n}(z_1, \ldots, z_n) = \tilde{W}_n^{(g)}(x_{\text{bip}}(z_1), \ldots, x_{\text{bip}}(z_n)) dx_{\text{bip}}(z_1) \cdots dx_{\text{bip}}(z_n).
\]

For the stable topologies \(2g + n - 2 \leq 0\), the situation is a bit subtle.

From Thm. 1.2, we deduce the following equivalent representation of the gen-
erating functions of bipartite maps, building the bridge to TR:
Corollary 1.3. Let $\omega_{g,n}$ be the correlators of TR generated by \((2.1)\) with \((x_{\text{bip}}, y_{\text{bip}})\) given by Theorem 1.2. Then $\tilde{T}^{(g)}_{2l_1, \ldots, 2l_n}$ can be achieved as follows:

$$\tilde{T}^{(g)}_{2l_1, \ldots, 2l_n} = (-1)^n \text{Res}_{z_1, \ldots, z_n \to \infty} x_{\text{bip}}(z_1)^{l_1} \cdots x_{\text{bip}}(z_n)^{l_n} \omega_{g,n}(z_1, \ldots, z_n)$$

Analogously, generating functions for ordinary maps are obtained from the spectral curve of Theorem 1.1 (see [Eyn16] for more details). This spectral curve, together with the recursion formula of [CF16, Thm. 3.9], will be the basis for the proof of Theorem 1.2 by direct identification.

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2. Proof and Discussion

2.1. Reminder of previous results. First, we briefly recapitulate the procedure of topological recursion. Starting with the initial data, the spectral curve $(\Sigma, x, y, B)$, TR constructs recursively in $2g + n - 2$ an infinite sequence of meromorphic $n$-forms $\omega_{g,n}$, starting with

$$\omega_{0,1}(z) = y(z) \, dx(z) \quad \omega_{0,2}(z_1, z_2) = B(z_1, z_2),$$

via the following residue formula:

$$\omega_{g,n+1}(I, z) = \sum_{\beta_i \to \beta_i} \text{Res}_{q \to \beta_i} K_i(z, q) \left( \omega_{g-1,n+2}(I, q, \sigma_i(q)) + \sum_{g_1+g_2=g \atop I_1 \cup I_2 = I} \omega_{g_1, |I_1|+1}(I_1, q) \omega_{g_2, |I_2|+1}(I_2, \sigma_i(q)) \right).$$

Here $I = \{z_1, \ldots, z_n\}$ is a collection of $n$ variables $z_j$, the sum is over the ramification points $\beta_i$ of $x$ defined by $dx(\beta_i) = 0$. The kernel $K_i(z, q)$ is defined in the vicinity of $\beta_i$ by $K_i(z, q) = \frac{1}{2} \int_{z, \beta_i(q)} B(z, q')$, where $\sigma_i \neq \text{id}$ is the local Galois involution $x(q) = x(\sigma_i(q))$ near $\beta_i$, and $\beta_i$ as a fixed point.

A TR-like formula to recursively generate correlators for bipartite maps was found in the aforementioned paper.
Theorem 2.1 ([CF16]). Let \( x(z) = \frac{z}{(1+z+\gamma z^2)^2} \). A correlators function \( U_\gamma(x(z)) = \sum_{l=1}^{\infty} \tilde{T}_{2l}^{(g)} x^l \), \( g > 1 \), can be recursively obtained in the following way:

\[
U_\gamma(x(z)) = \frac{1}{P(z)} \text{Res}_{q \rightarrow \pm \frac{\gamma}{1+\gamma}} P(q) \frac{x(q)}{z-q} Y(q) \left( U_{\gamma-1}(q) + \sum_{g_1+g_2=g} U_{g_1}(q) U_{g_2}(q) \right)
\]

with \( P(q) = \frac{1-\gamma^2 q}{1+\gamma q} \), \( Y(q) \) see below. \( U_\gamma^{(2)} = \sum_{i_1, i_2=1}^{\infty} \tilde{T}_{2i_1, 2i_2} \gamma x^{i_1+i_2} \).

2.2. Proof of the spectral curve. The heuristic deduction of \( (x_{\text{bip}}, y_{\text{bip}}) \) that finally turns Thm. 2.1 into TR works as follows:

- \( x_{\text{bip}} \): The work of Chapuy and Fang mainly relies on two important variable transformations. The first is the definition of \( \gamma^2 \), arising already for ordinary maps and earlier works of Bender and Canfield [BC88]. The second, \( x(z) = \frac{z}{(1+z+\gamma z^2)^2} \) will determine \( x_{\text{bip}} \). Thm. 2.1 creates generating functions as a series in positive powers of \( x \). Sending \( z \rightarrow \frac{1}{z} \) and then taking the reciprocal of \( x \) gives the correct curve ramified covering. We confirm this with the relation \( x_{\text{bip}}(z^2) = x_{\text{ord}}^2(z) \) together with a comparison of the correlators \( W \) and \( W \), up to a global factor of \( \frac{1}{x_{\text{bip}}} \) on which we comment later - this factor becomes decisive for the geometry of the spectral curve.

- \( y_{\text{bip}} \): Analogously to ordinary maps, the expression \( y_{\text{bip}}(z) - y_{\text{bip}}(\sigma(z)) \) can be directly read off from the kernel representation of the Tutte equation for the disk. This kernel \( Y(z) = y_{\text{bip}}(1/z) - y_{\text{bip}}(z) \) is already given in Prop. 3.3 in [CF16] and shows up in the recursion formula Thm. 2.1 as well. After changing the variables as for \( x_{\text{bip}} \), we can extract from [CF16, Chap. 5.1] a suitable expression for \( Y(z) \cdot x_{\text{bip}}(z) \):

\[
Y(z) \cdot x_{\text{bip}}(z) = \gamma^2 \left( \frac{1+z)^2}{z} - (1+z) \left[ 2 - \sum_{k=1}^{d} t_{2k} \gamma^{2k} \left( \sum_{l=1}^{k-1} z^l \binom{2k-1}{k+l} - \sum_{l=-k}^{0} z^l \binom{2k-1}{k+l} \right) \right] \right).
\]

Inserting the implicit equation of \( \gamma^2 \) from Theorem 1.1 in the first term cancels partially the terms for \( l = -1, 0 \). After some further lengthy but trivial algebra, the expression can be ordered in positive and negative powers of \( z \), where the positive powers give

\[
y_{\text{bip}}(z) \cdot x_{\text{bip}}(z) = 1 + z - (1+z) \sum_{k=1}^{d-1} \sum_{j \geq k+1} t_{2j} \binom{2j-1}{j+k} \gamma^{2j} z^k.
\]

Finally, the definition of \( u_{2k+1} \) in terms of \( t_{2j} \) yields the desired identifications shown in Theorem 1.2.
2.3. Discussion of the result. Of particular interest is the somewhat unusual geometry of the spectral curve for bipartite maps. Its branch cut goes from \( x_{\text{bip}}(1) = a = 4\gamma^{2} \) to \( x_{\text{bip}}(-1) = b = 0 \). We naturally have the same Zhukovsky parametrisation as for \( x_{\text{ord}}(z) \):

\[
a + b = a - b \left( z + \frac{1}{z} \right) \quad \text{and} \quad \sqrt{(x-a)(x-b)} = \gamma^{2} \left( z - \frac{1}{z} \right) \tag{2.4}
\]

However, the branch point at \( 0 = x(\beta_{2}) \) corresponding to the ramification point \( \beta_{2} = -1 \) affects the pole structure of all \( \omega_{g,n} \) - the highest degree of the poles is different for the two ramification points. Due to the fact that \( y_{\text{bip}} \) is irregular at the ramification point \( \beta_{2} = -1 \) (whereas as required \( \omega_{0,1} = y \, dx \) is still regular), the maximum order of poles \( \frac{1}{(z+1)^{k}} \) is reduced by two in comparison to the poles \( \frac{1}{(z-1)^{l}} \). However, this does not change the fact that one can generate symmetric \( n \)-forms from \( (x_{\text{bip}}, y_{\text{bip}}) \). This is an interesting deviation from the most spectral curves. Despite the uncommon pole distribution at the ramification points, the universal symmetry under the Galois involution naturally holds:

\[
\frac{\omega_{g,n}(z, z_{I})}{dx(z)} + \frac{\omega_{g,n}(\frac{1}{z}, z_{I})}{dx(\frac{1}{z})} = 0 \quad \forall 2g + n - 2 > 0
\]

For illustrative purposes, we give \( \omega_{1,1} \) as an example and set \( \tilde{y}_{\text{bip}}(z) = \frac{1}{7} \sum_{k=0}^{d-1} u_{2k+1} z^{k+1} \)

\[
\omega_{1,1}(z) = \frac{1}{16\gamma^{2}(1+z)^{2} y_{\text{bip}}'(-1)} - \frac{1}{16\gamma^{2}(z-1)^{4} y_{\text{bip}}'(1)} - \frac{3 y_{\text{bip}}(1) + 3 y_{\text{bip}}'(1) + y_{\text{bip}}''(1)}{96\gamma^{2}(z-1)^{2} y_{\text{bip}}'(1)} + \frac{3 y_{\text{bip}}(1) + 3 y_{\text{bip}}'(1) + y_{\text{bip}}''(1)}{96\gamma^{2}(z-1)^{2} y_{\text{bip}}'(1)}
\]

Finally, we want to collect some interesting open questions: It is known \cite{BGF20, BCGF21} that the exchange of \( x_{\text{ord}} \) and \( y_{\text{ord}} \) gives rise to generating functions of fully simple maps. Does any sort of exchange of \( x_{\text{bip}} \) and \( y_{\text{bip}} \) have a comparable strong implication? Another question arises from the matrix models as realisations of those various types of maps. As known from the classical literature, bipartite maps arise from the complex matrix model, having a structural equivalence to the Hermitian 2-matrix model \cite{EF06}. This model is (for certain boundary structures) already solved by TR. What is the relation between the two distinct spectral curves? A final question is dedicated only to quadrangulations. In \cite{BHW20} the quartic Kontsevich model (QKM) was shown to be solvable in terms of correlators \( \omega_{g,n} \) that follow an extension of TR. In this so-called blobbed topological recursion (BTR; general framework developed in \cite{BST17}), the \( \omega_{g,n} \) split into parts with poles at the ramification points (polar part) and with poles somewhere else (holomorphic part). In \cite{BH21} it was stated that the pure (normalised) TR results contributing to \( \omega_{g,n} \) of the QKM generate bipartite (rooted)
quadrangulations, whereas ordinary quadrangulations are generated when taking the complete BTR into account. Understanding this different approach to the partition functions of the complex and hermitian 1-matrix model from the beginning will be an interesting challenge for the future.

2.4. Example of quadrangulations. In order to underpin the correctness of our spectral curve, let us only allow for $t_4 \neq 0$ and $n = l = 1$, yielding (with $u_1 = \frac{1}{\gamma}$ and $u_3 = -t_4\gamma^3$):

$$x_{\text{bip}}(z) = \gamma^2 \left( z + \frac{1}{z} \right) + 2\gamma^2, \quad y_{\text{bip}}(z) = \frac{z - t_4\gamma^2 z^2}{1 + z}, \quad \gamma^2 = \frac{1 - \sqrt{1 - 12t_4}}{6t_4}.$$ 

The expansions in $t_4$ by computer algebra can be found in Tab. 1. Bipartite rooted quadrangulations are in particular interesting, since Tutte’s famous bijection [Tut63] relates them to rooted ordinary maps for faces of any (not only even) degree.

| Order $t_4^k$ | $\tilde{T}_2^{(0)}$ | $\tilde{T}_2^{(1)}$ | $\tilde{T}_2^{(2)}$ | $\tilde{T}_2^{(0)}$ | $\tilde{T}_2^{(1)}$ | $\tilde{T}_2^{(2)}$ |
|--------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $(t_4)^0$    | 1               | 0               | 0               | 1               | 0               | 0               |
| $(t_4)^1$    | 2               | 0               | 0               | 2               | 1               | 0               |
| $(t_4)^2$    | 9               | 1               | 0               | 9               | 15              | 45              |
| $(t_4)^3$    | 54              | 20              | 0               | 54              | 198             | 2007            |
| $(t_4)^4$    | 378             | 307             | 21              | 378             | 2511            | 56646           |
| $(t_4)^5$    | 2916            | 4280            | 966             | 2916            | 31266           | 1290087         |

Table 1. These numbers are generated by Thm 1.2 and Thm. 1.1 together with Cor. 1.3 and coincide with [BC88] and with OEIS no. A006300 ($g = 1$) and no. A006301 ($g = 2$) for $\tilde{T}_2^{(g)}$.

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