Stochastic Performance Modeling for Practical Byzantine Fault Tolerance Consensus in Blockchain

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Abstract

The practical Byzantine fault tolerant (PBFT) consensus mechanism is one of the most basic consensus algorithms (or protocols) in blockchain technologies, thus its performance evaluation is an interesting and challenging topic due to a higher complexity of its consensus work in the peer-to-peer network. This paper describes a simple stochastic performance model of the PBFT consensus mechanism, which is refined as not only a queueing system with complicated service times but also a level-independent quasi-birth-and-death (QBD) process. From the level-independent QBD process, we apply the matrix-geometric solution to obtain a necessary and sufficient condition under which the PBFT consensus system is stable, and to be able to numerically compute the stationary probability vector of the QBD process. Thus we provide four useful performance measures of the PBFT consensus mechanism, and can numerically calculate the four performance measures. Finally, we use some numerical examples to verify the validity of our theoretical results, and show how the four performance measures are influenced by some key parameters of the PBFT consensus. By means of the theory of multi-dimensional Markov processes, we are optimistic that the methodology and results given in this paper are applicable in a wide range research of PBFT consensus mechanism and even other types of consensus mechanisms.

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1 Introduction

The past decade has witnessed rapid development and growing popularity of blockchain technologies. This has been attracting tremendous interests and enthusiasm from both research communities and industrial applications. The blockchain technologies were originated from a digital financial sector as a decentralized, immutable, auditable, account-ability ledger system in order to deal with daily transactional data. So far it has been envisioned as a powerful backbone/framework for decentralized data processing and data-driven autonomous organization in a peer-to-peer and open-access network. For blockchain technologies, readers may refer to books by Narayanan et al. [44], Bashir [5], Raj [54], Maleh et al. [39], Rehan and Rehmani [55] and Schar and Berentsen [58]; and survey papers by Fauziah et al. [17] for smart contracts, Sharma et al. [60] for cloud computing, Ekramifard et al. [14] for AI, Dai et al. [13] for IoT and Huang et al. [26].

The consensus mechanisms are always an important direction in the research of blockchain technologies. Up to now, there have been more than 50 different consensus mechanisms developed in blockchain technologies. We refer readers to recent survey papers by, for example, Cachin and Vukolić [7], Bano et al. [4], Natoli et al. [45], Chaudhry and Yousaf [12], Nguyen and Kim [48], Salimitari and Chatterjee [57], Wang et al. [66], Pahlajani et al. [51], Carrara et al. [9], Wan et al. [65], Xiao et al. [68], Ferdous et al. [18], Nijssse and Litchfield [49], Yao et al. [69] and Khamar and Patel [28].

The PBFT consensus mechanism is the most basic one of blockchain consensus mechanisms, and it plays a key role in extending, generalizing and developing new effective blockchain consensus mechanisms. A reliable computer system must be able to cope with the failure of one or more of its components, and a failure components can send conflicting information to different parts of the computer system. In this case, solving the type of failure and conflicting problems is called a Byzantine generals problem. See Lamport et al. [32], Lamport [31] and Martin and Alvisi [31]. Based on the Byzantine generals problem, Pease et al. [53] and Lamport [32] provided the Byzantine fault tolerant (BFT) consensus mechanism. To prevent malicious attacks and guarantee security of blockchain, Castro
and Liskov [10] proposed the PBFT consensus mechanism. Thereafter, some researchers further developed various different PBFT consensus mechanisms to effectively improve the performance of the PBFT consensus mechanism. Important examples include Castro and Liskov [11], Veronese et al. [64], Abraham et al. [1], Kiayias and Russell [29], Hao et al. [24], Gueta et al. [23], Malkhi et al. [40], Bravo et al. [6], Sakho et al. [56], Meshcheryakov et al. [42], and Alqahtani and Demirbas [2].

It is an interesting topic to provide stochastic models for performance evaluation of PBFT consensus mechanism. To our best knowledge, this paper is the first one to set up such a stochastic model for performance evaluation of PBFT consensus mechanisms. To this end, we first set up a queueing model with more complicated service times, which are due to a higher complexity of block validation. Then we express the PBFT queue as a level-independent QBD process, which is a two-dimensional Markov process. By using the matrix-geometric solution, we can provide a more detailed analysis for performance evaluation of PBFT consensus mechanism, in which four performance measures are provided. It is worthwhile to note that Li et al. [35, 36] are two closely related works which gave performance analysis of the Proof of Work (another more important consensus mechanism). Obviously, the PBFT consensus mechanism makes the queueing model, together with the level-independent QBD process, more complicated than that of the Proof of Work. Therefore, this paper, together with Li et al. [35, 36], can be regarded as some key research on the matrix-analytic method in performance evaluation of blockchain consensus mechanisms. Also, we believe that the matrix-analytic method can play a key role in finding a more complete solution of stochastic models in the study of blockchain technologies.

It is clear that the Markov processes and queueing theory play a key role in the study of blockchain technologies. Readers may refer to survey papers by, for example, Li et al. [36], Smetanin et al. [61], Fan et al. [15] and Huang et al. [26]. Based on this, it is necessary and useful to review some available literature in this area as follows:

(a) **Markov processes of blockchain systems:** Up to now, few papers have applied the Markov processes (or Markov chains) to analysis of blockchain systems. See Li et al. [36] for some early research. For some research well related to this paper, Göbel et al. [21] used a two-dimensional Markov process to provide a computational method for dealing with the influence of propagation delay on the blockchain evolution. Also, a new computational method was further developed by Javier and Fralix [27]. Li et al. [34] provided a new theoretical framework of pyramid markov processes for blockchain selfish mining, where
a new matrix-geometric solution is developed and is different from that used in Li et al. [35, 36].

Nayak et al. [46] constructed three different Markov processes to analyze the stubborn mining strategies and designed a provable security consensus protocol. Kiffer et al. [30] proposed a simple consistency method of blockchain protocols by using a Markov chain. Huang et al. [25] applied the Markov processes to performance evaluation of the consistency algorithm Raft in the blockchain network. Carlsten [8] used a Markov process to analyze the influence of transaction fees on the blockchain selfish mining. Bai et al. [34] used the Markov chains to discuss how the existence of multiple misbehaving pools influences the profitability of selfish mining. Li et al. [38] applied the Markov processes to provide performance and security analysis for the direct acyclic graph-based ledger for Internet of Things. Li et al. [37] employed the Markov processes to the block access control in wireless blockchain network.

(b) Queueing models of blockchain systems: Li et al. [36] provided an overview for queueing models of blockchain systems. From that time forward, some new literatures are further listed, such as Gopalan et al. [22], Papadis et al. [52], Geissler [20], Mišić et al. [43], Fang and Liu [16], Fralix [19], Varma and Maguluri [63] and Wilhelmi and Giupponi [67].

Based on the above analysis, the main contributions of this paper are summarized as:

1. This paper describes a simple stochastic performance model of the PBFT consensus mechanism, which is refined as not only a queueing system with complicated service times but also a level-independent QBD process.

2. By applying the matrix-geometric solution, this paper obtains a necessary and sufficient condition under which the PBFT consensus system is stable, and provides the stationary probability vector of the QBD process, which is used to set up and numerically compute four useful performance measures of the PBFT consensus mechanism.

3. Some numerical examples are used to verify the validity of our theoretical results, and to show how the four performance measures are influenced by some key parameters of the PBFT consensus mechanism. Although our Markov queueing model is simply designed for the PBFT consensus mechanism, our analytic method will open a series of potentially promising research in queueing theory and Markov processes of PBFT consensus mechanisms.
The rest of this paper is organized as follows. Section 2 describes a stochastic performance model of PBFT consensus mechanism. Section 3 expresses the stochastic performance model as a level-independent QBD process. By using the matrix-geometric solution, we obtain a necessary sufficient condition and the stationary probability vector of the QBD process, and provide four performance measures of the PBFT consensus mechanism. Section 4 uses some numerical examples to verify the validity of our theoretical results, and shows how the four performance measures are influenced by key system parameters. Some concluding remarks are given in Section 5.

2 A Stochastic Performance Model

In this section, we describe a simple stochastic performance model of PBFT consensus mechanism based on the operations structure of PBFT consensus mechanism, in which several key factors are assumed to be exponential distributions or Poisson processes. Also, we introduce some necessary notations used in our later study.

In the PBFT consensus mechanism, there are $N$ nodes in the blockchain network, and the number of Byzantine nodes is no more than $f$. Let $N = 3f + 1$. To set up a useful relationship between the PBFT consensus mechanism and a queueing model, we assume that there are always a lot of transactions in the transaction pool, thus a number of transactions is packed into a file package. These transaction packages are made one by one, and they are regarded as an arrival process at the client. See Figure 1 for more details.

Based on Figure 1, we provide a detailed description for the stochastic performance model of PBFT consensus mechanism as follows:

(1) The package arrival process

We assume that the arrivals of many transaction packages at the client is a Poisson process with arrival rate $\lambda$ for $\lambda > 0$.

(2) A major node and $N - 1$ slave nodes

A major node can be chosen from the $N$ nodes with equal probability. Once a node is the major node, all the other nodes become $N - 1$ slave nodes. On the other hand, once a block is pegged on the blockchain by using the PBFT consensus mechanism, the major node can receive a reward $c$ from the blockchain system.

(3) The block-generated and block-pegged process
Figure 1: A useful relationship between the PBFT consensus and a queueing model

Once a transaction package arrives at the client, it is immediately submitted to a chosen major node, and the major node also immediately submits the transaction package to each of \(N - 1\) slave nodes. In this situation, the major node and the \(N - 1\) slave nodes begin to deal with the transaction package through the following three stages of PBFT consensus mechanism: \textit{Prepare}, \textit{commit} and \textit{reply}, e.g., see Figure 1 for their details. Now, we further explain these three stages as follows:

\textit{Prepare}: Once one of the \(N - 1\) slaves receives the message of the transaction package from the major node, it verifies the message content to ensure that the message content has not been tampered with during transmission. After the message is validated correctly, the slave node immediately sends a preparation message to all other nodes except itself, that is, \(N - 2\) slave nodes and the major node. When \(2f + 1\) different nodes receive that the message is consistent with that of the major node, the next \textit{commit} stage will be
started immediately. In fact, the major node has the same work as that of the $N - 2$ slave nodes in this stage.

*Commit:* Once the *prepare* stage is finished, each node sends a *commit* message to other nodes except itself. After other nodes receive the *commit* message, they verify the message content. When the node has received $2f + 1$ *commit* messages including itself and the message content is consistent with that of the previous *prepare* stage, the *commit* message is verified and enters the next *reply* stage.

*Reply:* Each node sends a *reply* message to the client. If the client receives the same *reply* messages sent by $2f + 1$ different nodes, then it is regarded as that all the nodes (the major node and the $N - 1$ slave nodes) have reached a round of consensus. Therefore, the transaction package becomes a legitimate block which can be pegged on the blockchain.

For simplicity of analysis, we assume that for each of the $N$ nodes (the major node and the $N - 1$ slave nodes), the time duration of going through these three stages (*prepare*, *commit* and *reply*) is exponential with mean $1/\mu$ for $\mu > 0$.

(4) Independence

We assume that all the random variables defined above are independent of each other.

**Remark 1.** If the three stages (*prepare*, *commit* and *reply*) are assumed to be exponential with means $1/\mu_1$, $1/\mu_2$ and $1/\mu_3$, respectively, then the total time duration of going through the three stages is a generalized Erlang distribution of order 3. In this case, analysis of the stochastic performance model will be more complicated and is regarded as one of our future studies. Therefore, this paper uses a simple model to express a clear operations structure and four useful performance measures in the PBFT consensus mechanism.

## 3 A QBD process

In this section, we express the stochastic performance model as a level-independent QBD process, which is a two-dimensional continuous-time Markov process. By using the matrix-geometric solution, we obtain a necessary sufficient stable condition of the QBD process and its stationary probability vector. Based on this, we provide four performance measures of the PBFT consensus mechanism.

In the PBFT consensus mechanism, let $K(t)$ be the number of transaction packages which are waiting at the client and will be submitted to the major node one by one at time $t$. Let $M(t)$ be the number of nodes who have verified the validity of one transaction
package in the PBFT consensus mechanism at time $t$. It is easy to see that $K(t) \in \{0, 1, 2, \ldots \}$ and $M(t) \in \{0, 1, 2, \ldots , 2f\}$. Then $\{(K(t), M(t)) : t \geq 0\}$ is a two-dimensional continuous-time Markov process whose state transition relations are shown in Figure 2.

Figure 2: The state transition relations of the Markov process

From Figure 2, it is easy to see that the state space of the Markov process $\{(K(t), M(t)) : t \geq 0\}$ is given by

$$
\Omega = \{(k \text{ (red)}, m \text{ (blue)}) : k = 0, 1, 2, \ldots ; m = 0, 1, 2, \ldots , 2f\}
$$

$$
= \bigcup_{k=0}^{\infty} \text{Level } k
$$
where

\[ \text{Level } k = \{(k, 0), (k, 1), (k, 2), \ldots, (k, 2f)\} \].

Based on the state space \( \Omega = \bigcup_{k=0}^{\infty} \text{Level } k \), the infinitesimal generator of the Markov process \( \{(K(t), M(t)) : t \geq 0\} \) is written as

\[
Q = \begin{pmatrix}
B_1 & B_0 \\
B_2 & A_1 & A_0 \\
 & A_2 & A_1 & A_0 \\
 & & & & \ddots & \ddots & \ddots
\end{pmatrix},
\]

(1)

where

\[ B_1 = -\lambda, B_0 = \lambda(1, 0, 0, \ldots, 0), \]

\[ B_2 = (N - 2f)\mu(0, 0, \ldots, 0, 1)^T, \]

\[
A_1 = \begin{pmatrix}
-N\mu - \lambda & \lambda \\
-2f\mu & -(N - 2f + 1)\mu - \lambda \\
\ldots & \ldots & \ldots
\end{pmatrix},
\]

\[
A_0 = \begin{pmatrix}
\lambda \\
\lambda \\
\ldots
\end{pmatrix},
\]

\[
A_2 = \begin{pmatrix}
\lambda \\
\lambda \\
\ldots
\end{pmatrix}.
\]

The following theorem provides a necessary and sufficient condition under which the QBD process \( Q \) is stable. Based on this, we can obtain the system stability of the PBFT consensus mechanism.

**Theorem 1.** The QBD process \( Q \) is stable if and only if \( \rho = (\lambda/\mu) \sum_{k=0}^{2f} 1/(N - k) < 1 \).

Thus, the stochastic system of the PBFT consensus mechanism is stable if and only if \( \rho = (\lambda/\mu) \sum_{k=0}^{2f} 1/(N - k) < 1 \).
Proof: To discuss the stability of the level-independent QBD process \( Q \), we will apply the mean drift method given Neuts [47] or Li [33]. To this end, we write

\[ A = A_0 + A_1 + A_2 = \begin{pmatrix}
-N\mu & N\mu \\
-(N-1)\mu & (N-1)\mu \\
\vdots & \vdots \\
-(N-2f+1)\mu & (N-2f+1)\mu \\
(N-2f)\mu & -(N-2f)\mu
\end{pmatrix}.\]

It is clear that the Markov chain \( A \) is irreducible, aperiodic and positive recurrence due to the fact that its state space is finite and \( Ae = 0 \), where \( e \) is a column vector with all components ones. In this case, let \( \theta = (\theta_0, \theta_1, \ldots, \theta_{2f-1}, \theta_{2f}) \) be the stationary probability vector of the Markov chain \( A \). Therefore, the stationary probability vector \( \theta \) satisfies the system of linear equations: \( \theta A = 0 \) and \( \theta e = 1 \). Based on this, we obtain

\[ -N\theta_0 + (N-2f)\theta_{2f} = 0, \]
\[ (N-k)\theta_k - (N-k-1)\theta_{k+1} = 0, \quad 1 \leq k \leq 2f-1. \]

Hence, we get

\[ \theta_0 = \frac{1}{\sum_{k=0}^{2f} \frac{N}{N-k}} \]

and for \( 1 \leq k \leq 2f \)

\[ \theta_k = \frac{N}{N-k} \theta_0 = \frac{N}{N-k} \frac{1}{\sum_{k=0}^{2f} \frac{N}{N-k}}. \]

By using the mean drift method, it is easy to see that the level-independent QBD process \( Q \) is positive recurrent if and only if

\[ \theta A_0 e < \theta A_2 e. \]

It is easy to check that

\[ \theta A_0 e = \lambda \theta I e = \lambda, \]

where \( I \) is an identity matrix. At the same time, we have

\[ \theta A_2 e = \theta_{2f} (N-2f) \mu = \frac{\mu}{\sum_{k=0}^{2f} \frac{1}{N-k}}. \]
This gives
\[ \frac{\mu}{\sum_{k=0}^{2f} \frac{1}{N-k}} < \lambda, \]
that is,
\[ \rho = \frac{\lambda}{\mu} \frac{\sum_{k=0}^{2f} \frac{1}{N-k}} < 1. \]
This completes the proof. ■

In what follows we compute the stationary probability vector of the QBD process \( Q \). Based on this, we can set up four useful performance measures of the PBFT consensus mechanism.

If \( \rho = (\lambda/\mu) \sum_{k=0}^{2f} 1/(N-k) < 1 \), then the QBD process \( Q \) is irreducible and positive recurrent. In this case, Let \( \pi = (\pi_0, \pi_1, \pi_2, \pi_3, \ldots) \) be the stationary probability vector of the QBD process \( Q \), where
\[ \pi_0 = (\pi_{00}) \]
and for \( k = 1, 2, 3, \ldots, \)
\[ \pi_k = (\pi_{k0}, \pi_{k1}, \ldots, \pi_{k2f-1}, \pi_{k2f}). \]
Then the vector \( \pi \) uniquely satisfies the system of linear equations: \( \pi Q = 0 \) and \( \pi e = 1 \).

Note that the QBD process \( Q \) is level-independent, thus it follows from Chapter 3 of Neuts [47] that the stationary probability vector \( \pi \) is a matrix-geometric solution.

The following theorem is a direct result of the level-independent QBD process \( Q \) by using Chapter 3 of Neuts [47], thus we restate it but its proof is omitted here.

**Theorem 2.** If \( \rho = (\lambda/\mu) \sum_{k=0}^{2f} 1/(N-k) < 1 \), then the QBD process \( Q \) is irreducible and positive recurrent, and its stationary probability vector \( \pi = (\pi_0, \pi_1, \pi_2, \pi_3, \ldots) \) is given by
\[ \pi_k = \pi_1 R^{k-1}, \ k \geq 1, \]
where \( \pi_0 \) and \( \pi_1 \) can be uniquely determined by the following system of linear equations
\[ \pi_0 B_1 + \pi_1 B_2 = 0, \]
\[ \pi_0 B_0 + \pi_1 (A_1 + RA_2) = 0, \]
\[ \pi_0 + \pi_1 (I - R)^{-1} e = 1. \]
Note that the stationary probability vector \( \pi = (\pi_0, \pi_1, \pi_2, \pi_3, \ldots) \) in general has not an explicit expression, thus we mainly develop the numerical solution to the vector \( \pi \). To this end, it is easy to see from Chapter 3 of Neuts \[47\] that we first need to numerically compute the rate matrix \( R \), which is the minimal nonnegative solution to the following nonlinear matrix equation

\[
R^2A_2 + RA_1 + A_0 = 0.
\]

Based on this, the rate matrix \( R \) can be approximately calculated by an iterative algorithm as follows:

\[
R_0 = 0,
\]

and for \( n = 0, 1, 2, \ldots \),

\[
R_{n+1} = (R^2A_2 + A_0)(-A_1)^{-1}.
\]

For the matrix sequence \( \{R(n); n \geq 0\} \), Chapter 3 of Neuts \[47\] indicated that \( R(n) \uparrow R \) as \( n \to \infty \). For any sufficiently small positive number \( \varepsilon \) within the desired degree of accuracy set at \( 10^{-12} \), if there exists a positive integer \( n \) such that

\[
\|R(n+1) - R(n)\| = \max_{0 \leq i \leq 2f} \left\{ \sum_{j=0}^{2f} \left| R(n+1)_{i,j} - R(n)_{i,j} \right| \right\} < \varepsilon,
\]

then we take \( R = R(n) \). This gives an approximate solution of the stationary probability vector \( \pi \) by means of Theorem 2.

Once the stationary probability vector \( \pi \) is computed numerically, we can provide four useful performance measures of the PBFT consensus mechanism as follows:

(a) The average stationary number of transaction packages at the client

\[
E[K] = \sum_{k=1}^{\infty} k\pi_k e = \sum_{k=1}^{\infty} k\pi_k R^{k-1} e = \pi_1 (I - R)^{-2} e.
\]

(b) The average stationary number of nodes to have verified the validity of a transaction package

\[
E[M] = \sum_{k=1}^{\infty} \sum_{m=0}^{2f} m\pi_{k,m} = \sum_{k=1}^{\infty} \pi_k \phi = \pi_1 (I - R)^{-1} \phi.
\]
where $\phi = \begin{pmatrix} 0, 1, 2, \ldots, 2f - 1, 2f \end{pmatrix}^T$.

(c) The stationary block-pegged rate of the PBFT consensus mechanism

From Figure 1, it is seen that the block-generated and block-pegged processes are described as a queueing process, in which the inputs are a Poisson process but the service process is relatively complicated.

From Figure 2, we can observe that the block-pegged process is a Markovian arrival process (MAP) whose matrix representation $(C, D)$ of infinite sizes are given by

$$C = \begin{pmatrix}
  B_1 & B_0 \\
  A_1 & A_0 \\
  A_1 & A_0 \\
  \vdots & \vdots \\
  \end{pmatrix}, \quad D = \begin{pmatrix}
  0 & B_2 & 0 \\
  A_2 & 0 & \vdots \\
  \vdots & \vdots & \ddots \\
  \end{pmatrix}.$$

It is clear that $Q = C + D$.

If $\rho = (\lambda/\mu) \sum_{k=0}^{2f} 1/(N - k) < 1$, then the QBD process $Q$ is irreducible and positive recurrent, and the stationary block-pegged rate of the PBFT consensus mechanism is given by

$$\gamma = \pi D e = \pi_1 B_2 e + \left( \sum_{k=2}^{\infty} \pi_k \right) A_2 e$$

$$= \pi_1 B_2 e + \pi_1 R (I - R)^{-1} A_2 e.$$

(d) The stationary block-pegged reward per unit time of the major node

Note that the $N$ nodes are equal as a major node in the PBFT consensus mechanism, thus any one of them becomes the major node with probability $1/N$. It is easy to see that the stationary block-pegged total reward per unit time of the PBFT consensus mechanism is $\gamma c$, thus the stationary block-pegged reward per unit time of the major node is given by

$$\Upsilon = \frac{1}{N} \gamma c = \frac{1}{N} \left[ \pi_1 B_2 e + \pi_1 R (I - R)^{-1} A_2 e \right] c.$$

4 Numerical Analysis

In this section, we use two groups of numerical examples to verify the validity of our theoretical results, and to show how the four performance measures of the PBFT consensus mechanism depend on some key system parameters.
**Group one:** The number $f$ of Byzantine nodes, and the rate $\mu$ of going through these three stages (*prepare, commit* and *reply*).

The system parameters are taken as follows: $\lambda = 1$; $\mu \in (3, 9)$; $c = 12.5$BTC; $f = 50, 100, 320$; and using $N = 3f + 1$ leads to $N = 151, 301, 961$.

![Figure 3: $E[K]$ and $E[M]$ vs. $\mu$ and $f$](image)

From the left of Figure 3, it is seen that $E[K]$ decreases as $\mu$ increases, and it also decreases as $f$ (or $N$) increases. This shows that the PBFT consensus mechanism makes the block-pegged on blockchain faster either as $\mu$ increases or as $f$ (or $N$) increases. Such a numerical result as $\mu$ increases is intuitive from a real observation; while another result as $f$ (or $N$) increases has an interesting practical significance, i.e., we can speed up the block-pegged on blockchain through increasing the number $N$ of nodes in the blockchain network. Similarly, it is observed from the right of Figure 3 that $E[M]$ decreases as $\mu$ increases; but it increases as $f$ (or $N$) increases, this can easily be understood due to the increase of $N$.

From the left of Figure 4, it is seen that $\gamma$ increases as $\mu$ increases, and it also increases as $f$ (or $N$) increases. This shows that the PBFT consensus mechanism makes the block-pegged on blockchain faster either as $\mu$ increases or as $f$ (or $N$) increases. Such two numerical results are intuitive. Similarly, it is observed from the right of Figure 4 that $\Upsilon$ increases as $\mu$ increases; but it decreases as $f$ (or $N$) increases, this shows that a bigger number $N$ can increase the profit of each node.

**Group two:** The number $f$ of Byzantine nodes, and the arrival rate $\lambda$ of transaction packages at the client.

The system parameters are taken as follows: $\mu = 9$; $\lambda \in (1, 3)$; $c = 12.5$BTC; $f =$
Figure 4: $\gamma$ and $\Upsilon$ vs. $\mu$ and $f$

50, 100, 320; and using $N = 3f + 1$ leads to $N = 151, 301, 961$.

Figure 5: $E[K]$ and $E[M]$ vs. $\lambda$ and $f$

From the left of Figure 5, it is seen that $E[K]$ increases as $\lambda$ increases, but it decreases as $f$ (or $N$) increases. This shows that the PBFT consensus mechanism makes the block-pegged on blockchain faster either as $\lambda$ increases or as $f$ (or $N$) increases. Similarly, it is observed from the right of Figure 5 that $E[M]$ increases as $\lambda$ increases; and it also increases as $f$ (or $N$) increases, this can easily be explained by increasing the number $N$.

From the left of Figure 6, it is seen that $\gamma$ increases as $\lambda$ increases, and it also increases as $f$ (or $N$) increases. This shows that the PBFT consensus mechanism makes the block-pegged on blockchain faster either as $\mu$ increases or as $f$ (or $N$) increases. Similarly, it is observed from the right of Figure 6 that $\Upsilon$ increases as $\mu$ increases; but it decreases as $f$ (or $N$) increases, this shows that a bigger number $N$ can increase the profit of each node.
5 Concluding Remarks

In this paper, we describe a simple stochastic performance model of the PBFT consensus mechanism, and refine the stochastic performance model as not only a queueing system with complicated service times but also a level-independent QBD process. We apply the matrix-geometric solution to obtain a necessary and sufficient condition under which the PBFT consensus system is stable, and establish the stationary probability vector of the QBD process. Based on this, we provide four useful performance measures of the PBFT consensus mechanism, and can numerically compute each of them. Finally, we use some numerical examples to show how the four performance measures are influenced by some key parameters of the PBFT consensus mechanism. By means of the theory of multidimensional Markov processes, we are optimistic that the methodology and results given in this paper are applicable in a wide range research of PBFT consensus mechanism and even other types of consensus mechanisms.

Along this line, we will continue our future research on several interesting directions as follows:

— Let all the three stages (prepare, commit and reply) be differently exponential distributions with means $1/\mu_1$, $1/\mu_2$ and $1/\mu_3$, respectively. Note that the extended stochastic performance model is far more difficult than that of this paper due to some complicated phase-type calculation.

— Developing effective algorithms for more general stochastic performance model with the Markovian arrival process of transaction packages at the client, and let all the three
stages (prepare, commit and reply) be phase-type (PH) distributions with matrix representations \((\alpha_1, T_1), (\alpha_2, T_2)\) and \((\alpha_3, T_3)\), respectively.

— When the arrivals of transaction packages at the client are a renewal process, and/or some of the three stages (prepare, commit and reply) follow general probability distributions, an interesting future research is to focus on fluid and diffusion approximations of PBFT consensus mechanism.

— Setting up reward function with respect to cost structure, transaction fee, mining reward, security and so forth. It is very interesting in our future study to develop stochastic optimization, Markov decision processes and stochastic game models in the study of PBFT consensus mechanism and even other types of consensus mechanisms.

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