Introduction.— The EE has been attracting much interest in many fields of physics [1–53], including quantum information theory, condensed matter physics [3–12], quantum field theories [13–19], and quantum gravity [49, 50]. The EE has been attracting much interest not only for quantifying the resources for quantum information tasks [20, 21] but also for analyzing physical properties such as the central charge [11, 12], topological order [9, 10], many-body localization [45–48], and the Bekenstein-Hawking entropy [22–26]. Recent experiments are those of systems with symmetry-breaking scatterings, e.g., by impurities, rough walls or phonons, which define a particular rest frame. Multiple scatterings by such scatterers make NESSs nontrivial, i.e., much different from boosts of equilibrium states.

These numerous works studied the EE of the ground states or other energy eigenstates, almost all of which are equilibrium states according to the eigenstate-thermalization hypothesis [40–42]. A natural question is: How does the EE behave in nonequilibrium states? For thermalization processes in isolated systems, the time evolution of the EE was studied in condensed matter [40–48], quantum field theories and quantum gravity [49, 50]. In these systems, $S_L \leq O(S^eq_L(E))$ throughout the evolution, where $S^eq_L(E)$ is the equilibrium entropy at energy $E$ of the system. For NESSs, which are fundamental states in nonequilibrium physics [61–71], their EE was studied for certain systems [51–53]. As in the case of thermalization processes, it was shown that $S_L = O(S^eq_L(E))$ or $S_L = O(\sum \nu S^\nu_L(T^\nu_{\text{res}}))$, where $T^\nu_{\text{res}}$ is the temperature of the $\nu$th reservoir. However, this is because the systems of these NESSs are invariant under spatial translation, and consequently the NESSs are basically the boosts of equilibrium states. By contrast, the NESSs observed in common experiments are those of systems with symmetry-breaking scatterings, e.g., by impurities, rough walls or phonons, which define a particular rest frame. Multiple scatterings by such scatterers make NESSs nontrivial, i.e., much different from boosts of equilibrium states.

In this paper, we study $S_L$ of NESSs in a one-dimensional mesoscopic conductor with impurities [71–78], which is a long QWR connected to two electron reservoirs with $T^\nu_{\text{res}} = 0$. The difference $\Delta \mu := \mu^+ - \mu^-$ of chemical potentials $\mu^\pm$ of the reservoirs induces a steady current $J$ in the QWR, and a NESS is realized. While $S^eq_L = O(\ln L)$ at equilibrium, we find, in nontrivial NESSs far from equilibrium (as defined by (5) below),

$$S_L = \eta(L)L|\Delta k_F| + O(\ln L) \quad \text{for} \quad 1 \ll L \leq L_C.$$  

(1)

Here, $|\Delta k_F|$ is the difference of the Fermi wavenumbers of the reservoirs, $L_C$ is the length of the QWR, and $\eta(L)$ is a function of $L$ with the following properties: When $1 \ll L \leq L_C$, (i) $\eta(L)$ is independent of $|\Delta k_F|$, (ii) gradually decreasing with increasing $L$, (iii) and

$$a \leq \eta(L) \lesssim 2a \quad \text{for} \quad 1 \ll L \leq L_C,$$  

(2)

where $a$ is a positive constant independent of $L$ or $\Delta k_F$. Since $S_L \geq \Delta k_F + O(\ln L)$, we call Eq. (1) the quasi volume law. Consequently, $S_L > O(\sum \nu S^\nu_L)$ in contrast to $S_L \leq O(\sum \nu S^\nu_L)$ in the previous cases [40–53]. Both far from equilibrium and multiple scatterings breaking the translational symmetry are necessary for this anomalous enhancement of $S_L$.

Setup.— We consider a long QWR (conductor) [71–78] connected to two electron reservoirs of zero temperature. Although real reservoirs are usually two-dimensional, the total system can be mapped to a one-dimensional system [76, 77]. If many-body interactions are negligible, its ef-
effective Hamiltonian is given by
\[
\hat{H}_{\text{tot}} := -\sum_x (\hat{c}_x^\dagger \hat{c}_{x+1} + \text{h.c.}) + \sum_{|x|\leq L_c/2} v_x \hat{c}_x^\dagger \hat{c}_x. \tag{3}
\]

Here, \(\hat{c}_x^\dagger, \hat{c}_x\) is the creation and annihilation operators of an electron at site \(x \in \mathbb{Z}\). In the QWR of length \(L_c\) centered at \(x = 0\) (Fig. 1), a Gaussian random potential \(v_x\) of impurities exists (with vanishing average \([79]\)). Its strength is characterized by the standard deviation \(W\) of \(v_x\).

As a boundary condition, we require that a single-particle state should be either a scattering state \(\varphi_k(x)\) (Fig. 1) with incoming wavenumber \(k\) \((-\pi < k \leq \pi)\) and energy \(\varepsilon_k = -2 \cos k\), or a bound state \(\varphi_b(x)\) with a quantum number \(b\) and energy \(\varepsilon_b\) (\(|\varepsilon_b| > 2\)). According to the standard model of mesoscopic conductors \([73, 87]\), the quantum state at zero temperature of the total system is a pure quantum state \(|\Psi_{\text{tot}}\rangle\) such that \(\varphi_k(x)\) with \(-k_F^+ \leq k \leq k_F^+\) and \(\varphi_b(x)\) with \(\varepsilon_b < -2\) are occupied by electrons \([79]\). Here, \(k_F^+(k_F^-)\) is the Fermi wavenumber of the left (right) reservoir, i.e.

\[
\varepsilon_k = \mu^\pm + \mu \mp \Delta \mu/2, \tag{4}
\]

where \(\mu^\pm = (\mu^+ + \mu^-)/2\). Without loss of generality, we assume \(\Delta \mu \geq 0\), and hence \(\Delta k_F := k_F^+ - k_F^- \geq 0\). A NESS is realized when \(\Delta \mu > 0\).

**Entropy.**—Assuming \(|\Psi_{\text{tot}}\rangle\), we explore its EE. We take a subsystem \(A := [-L/2, L/2]\) of length \(L\) at the center of the QWR (Fig. 1), and consider the von Neumann entropy \(S_L := -\text{Tr} \ln \hat{\rho}_L\) of the reduced density operator \(\hat{\rho}_L\) of A. Since \(|\Psi_{\text{tot}}\rangle\) is a pure quantum state, \(S_L\) is the EE that quantifies the entanglement between A and the rest of the system.

For each \(|\Psi_{\text{tot}}\rangle\), which is determined by \(k_F^\pm\) and impurities, we examine the \(L\) dependence of \(S_L\). We are most interested in \(S_L\) in the QWR (i.e. \(S_L\) for \(L \leq L_c\)), in which the quantum state in a NESS differs significantly from that of an equilibrium state.

**Nontrivial NESSs.**—From the electron-hole symmetry, we can limit ourselves, without loss of generality, to the lower half of the band, \(-2 \leq \varepsilon_k \leq 0\). Furthermore, since we are not interested in any specific effects of the band edge \(\varepsilon_k = -2\) or the band center \(\varepsilon_k = 0\), we take \(-1.7 \leq \mu \leq -0.8\).

We exclude the case of a short QWR, \(L_c \sim 1\), because a QWR is actually a quantum dot, for which we cannot discuss the \(L\) dependence of \(S_L\) for \(L \leq L_c\). We therefore study the case of \(L_c \gg 1\). We take \(L_c = 401\) in numerical calculations \([79]\).

Since we assume zero temperature, the dimension-less conductance \([73]\) \(G := (J/\Delta \mu)/(e^2/2\pi \hbar)\) (which is a nonlinear one; see below) is simply the average value

\[
G = \frac{1}{\Delta \mu} \int_{\mu^-}^{\mu^+} |t_k| d\varepsilon_k \text{ of the transmittance } |t_k|^2 \text{ in } \mu^- \leq \varepsilon_k \leq \mu^+. \quad \text{Obviously, } 0 \leq G \leq 1.
\]

\(L_c\) is finite and impurities are absent in reservoirs, the Anderson localization \([88, 91]\) is incomplete, and \(G\) takes various values depending on the localization length \(\xi\) (defined for the hypothetical case \(L_c = \infty\)), as follows.

When \(L_c \gg \xi\), the system would be almost an insulator, and \(G \approx 0\). Hence, almost no current would flow even when finite \(\Delta \mu\) is applied. On the other hand, when \(L_c \ll \xi\), the electrons would not suffer scatterings, and \(G \approx 1\). The NESS in this case would be almost a boost of an equilibrium state (even in the presence of interactions between electrons \([92]\)). In these cases, it is obvious that \(S_L\) scales as in equilibrium. We therefore focus on the intermediate regime where \(L_c\) is comparable to \(\xi\), for which \(G\) takes an intermediate value, such as \(0.3 \lesssim G \lesssim 0.7\). We call NESSs in this case **nontrivial**, and take \(L_c\) so as to satisfy this condition.

For such NESSs, multiple scatterings by impurities are crucial, and the wavefunctions \(\varphi_k(x)\) have complicated shapes as shown in Fig. 1. Since \(k\) takes a continuous value, there are infinitely many states, including those nearly localized in the QWR (such as the red one) and those penetrating into the QWR (such as the blue one). Consequently, \(|t_k|^2\) varies rapidly as a function of \(k\), as shown in Fig. S1 of \([79]\), where each peak indicates the resonant tunneling through a nearly-localized state.

**Far from equilibrium.**—Since \(-\pi < k \leq \pi\) and the number of resonant states \(\approx L_c\), the average distance \(\Delta k_{\text{peak}}\) between the peaks of \(|t_k|^2\) is roughly \(\Delta k_{\text{peak}} \approx 2\pi/L_c\). Since \(\Delta k_{F} \ll \Delta k_{\text{peak}}\), the current-voltage characteristic is linear, i.e. \(G\) is independent of \(\Delta \mu\) (while \(G\) depends on \(\overline{p}\)). In this regime, the NESS is close to equilibrium. When \(\Delta k_F\) is increased to \(\Delta k_F \sim \Delta k_{\text{peak}}\), \(G\) depends sensitively on \(\Delta \mu\) (and \(\overline{p}\)), as shown in Fig. S2 of \([79]\), because only a small number of peaks in \(k_F^\pm\) contribute to the conduction, and characters of the individual peaks are reflected. When \(\Delta k_F\) is further increased to \(\Delta k_F \gg \Delta k_{\text{peak}}\), the dependence on \(\Delta \mu\) (and \(\overline{p}\)) becomes weak because many peaks and dips of \(|t_k|^2\) contribute to the conduction. When \(\Delta k_F = O(1)\), such a regime is always achieved for sufficiently large \(L_c\) be-
cause $\Delta k_{\text{peak}} \simeq 2\pi/L_C$. We call this regime such that

$$\Delta k_F = O(1) \gg \Delta k_{\text{peak}} \quad (5)$$

far from equilibrium. We will show the quasi volume law for nontrivial NESSs far from equilibrium.

Relation to number and current. — Let $\delta N_L^2 := (\langle \hat{N}_L \rangle - \langle \hat{N}_L \rangle)^2$, which is the fluctuation of the particle number $N_L := \sum_{|x| \leq L/2} \delta \hat{c}_x$ in $A$, where $\langle \cdot \rangle := (\Psi_{\text{tot}} | \langle \cdot \rangle | \Psi_{\text{tot}})$. From inequality (S.17) of [79], it is sufficient to show the quasi-volume law for $\delta N_L^2$ instead of $S_L$. We shall analyze $\delta N_L^2$ to see the mechanism and the order of magnitude, and calculate $S_L$ numerically to see the magnitude, of the anomalous enhancement.

To calculate $\delta N_L^2$, we neglect small contributions from bound states, and use the identity [79]:

$$\delta N_L^2 = \int \int_{\Omega_{\Delta\mu}} dk_1 dk_2 R_W^1(k_1, k_2). \quad (6)$$

Here, the region of integral $\Omega_{\Delta\mu}$ is such that $k_1$ is occupied by an electron whereas $k_2$ is empty. Therefore, in the $k_1$-$k_2$ plane shown in Fig. 2, $\Omega_{\Delta\mu}$ shifts toward the direction of $+45^\circ$ with increasing $\Delta \mu$. $R_W^1$ is given by

$$R_W^1(k_1, k_2) := |\Delta J_L^q|^2/[16 \sin^2(p/2) \sin^2(q/2)]. \quad (7)$$

Here, $p := k_1 + k_2$ and $q := k_1 - k_2$, where small $|p|$ ($|q|$) corresponds to backward (forward) scattering processes. Moreover, $\Delta J_L^q := J_{k_1k_2}(L/2) - J_{k_1k_2}(-L/2)$, where $J_{k_1k_2}(x)$ is the $k$ representation of the current on the bond at $x+1/2$: $J_{k_1k_2}(x+1/2) = i[\varphi_{k_1}^*(x+1)\varphi_{k_2}(x) - \varphi_{k_1}(x)\varphi_{k_2}^*(x+1)]$. When $p = 0$ or $q = 0$, where the denominator of (7) vanishes, the numerator $|\Delta J_L^q|^2$ also vanishes because $J_{k_1\pm k_2}(x)$ is independent of $x$ from the conservation laws [79]. Consequently, $R_W^1$ is finite everywhere in $\Omega_{\Delta\mu}$. Identity (6) means that $\delta N_L^2$ in $A$ is determined by the net current flowing into $A$ through its edges, $x = \pm L/2$.

**FIG. 2.** The region $\Omega_{\Delta\mu}$ of integral for (a) an equilibrium state ($\Delta \mu = 0$), and (b) a NESS ($\Delta \mu > 0$).

Equilibrium without random potential. — Using Eq. (6), we first reproduce the previous result for the case of $W = \Delta \mu = 0$ [32, 33] in such a way that the derivation can be generalized to the case of NESSs. When $W = \Delta \mu = 0$, we have $\Omega_{\Delta\mu} = \Omega_0$ and $R_W^1 = R_L^1 = \sin^2(qL/2)/4\pi^2 \sin^2(q/2)$. Here, $R_L^1$ is independent of $p$ because of the translational symmetry for $W = 0$. Then, Eq. (6) reduces to the previous result [32, 33]. While $R_L^0$ vanishes at $|q| = 2\pi n/L$ ($n = 1, 2, 3, \ldots$), it has peaks between these points. The peak values are anomalously large, $R_L^0 = O(L^2)$, for $|q| = O(1/L)$. This is responsible for the $L$ dependence of $\delta N_L^2$. In fact, let us divide the $q$ integral into two portions: $|q| \leq \epsilon$ and $\epsilon < |q|$, where $\epsilon$ is an arbitrary small positive constant independent of $L$. The integral over the latter gives just an $O(1)$ term to $\delta N_L^2$ [79]. Hence, for $\Delta \mu = 0$, we obtain $4\pi^2 \delta N_L^2 = \int_{-\epsilon}^{\epsilon} |q| R_L^0(q) dq + O(1)$, after integrating over $k_1$. Here, as seen from Fig. 2(a), the factor $|q|$ represents the length of integration over $k_1$ [32, 33]. Since this integral gives an $O(\ln L)$ term [79], we have $\delta N_L^2 = O(\ln L)$. Hence, from inequality (S.17) of [79], we expect the logarithmic law $S_L^\text{eq} = O(\ln L)$ for $S_L$ at equilibrium, in consistency with [32, 33]. We have confirmed this behavior numerically (Fig. S3 of [79]).

Equilibrium with random potential. — When $W > 0$ and $\Delta \mu = 0$, a drastic change occurs in $R_W^1$ (whereas $\Omega_{\Delta\mu} = \Omega_0$ remains unchanged). Since impurities scatter electrons back and forth, the difference in $R_W^1$ between the forward-scattering part (small $|q|$) and the backward-scattering part (small $|p|$) is obscured. As a result, $R_W^1$ comes to have peaks of anomalously large heights of $O(L^2)$ not only at $|q| = O(1/L)$ but also at $|p| = O(1/L)$, as shown in Fig. S4 of [79]. Furthermore, as seen from Fig. 2(a), the length of integration over $k_1$ for small $|p|$ is the same as that for small $|q|$. Hence, they give $\int_{-\epsilon}^{\epsilon} |q| \bar{R}_L^1 dq$ and $\int_{-\epsilon}^{\epsilon} |p| \bar{R}_L^1 dp$, where $\bar{R}_L^1$ denotes the average of $R_L^1$ (which depends on $k_1$ unlike the case of $W = 0$) over the interval of the $k_1$ integral. Since both these integrals give $O(\ln L)$, we have $\delta N_L^2 = O(\ln L) + O(1)$, as in the case of $W = \Delta \mu = 0$. Moreover, the heights of the $O(L^2)$ peaks of $R_L^1$ decrease (increase) with increasing $W$ for small $|p|$ (small $|q|$), as shown in Fig. 5 of [79]. Consequently, the value of $\delta N_L^2$ should be almost independent of $W$. According to inequality (S.17) of [79], we expect these properties also for $S_L$: At equilibrium $\Delta \mu = 0$, $S_L$ obeys the logarithmic law $S_L^\text{eq} = O(\ln L)$ even when $W > 0$, and its value is not sensitive to $W$. This is confirmed and further discussed in [79].

NESS with random potential. — When $\Delta \mu > 0$, $\Omega_{\Delta\mu}$ changes from Fig. 2(a) to (b) (whereas $R_W^1$ remains unchanged from the equilibrium case with $W > 0$). Consequently, we have $\int_{-\epsilon}^{\epsilon} |q| \bar{R}_L^1 dq$ and $\int_{-\epsilon}^{\epsilon} |\Delta k_F - p| \bar{R}_L^1 dp$, after integration over $k_1$. Here, the factors $|q|$ and $|\Delta k_F - p|$ are the lengths of integration over $k_1$ on a constant-$q$ line and on a constant-$p$ line (solid oblique lines in Fig. 2(b)), respectively. Unlike the equilibrium case, the length does not vanish on the line of $p = 0$ (whereas the length vanishes on the $q = 0$ line). This produces a new term when far from equilibrium, as follows (whereas the $q$ integral gives $O(\ln L)$ as before).

By taking $\epsilon = O(1)$ as $0 < \epsilon < \Delta k_F$, we can rewrite the
p integral as \( \Delta k F \int_{-a}^{a} \tilde{R}_L^W \, dp + \int_{-\infty}^{0} |p| \tilde{R}_L^W \, dp - \int_{0}^{\infty} |p| \tilde{R}_L^W \, dp \). The last two terms give \( O(\ln L) \) as before (though expected to cancel out each other after the random average). By contrast, the integral \( \int_{-a}^{a} \tilde{R}_L^W \, dp \) basically gives \( O(L) \) because \( \tilde{R}_L^W \) has the \( O(L^2) \) peaks in an interval of \( O(1/L) \). Actually, there is a small correction. Backward scatterings become weaker with increasing \( L \), i.e. as the edges \( (x = \pm L/2) \) of A approach the edges \( (x = \pm L_C/2) \) of the QWR. Consequently, the peaks of \( \tilde{R}_L^W \) grow slowly than \( O(L^2) \) with increasing \( L \), as shown in Fig. S4 of \([79]\). We represent this effect by a gradually decreasing function \( \eta(L) \) as \( \int_{-a}^{a} \tilde{R}_L^W \, dp = \eta(L) \). (See below for the explicit form of \( \eta(L) \).)

From these results and inequality (S.17) of \([79]\), we obtain the quasi volume law \([7]\), for nontrivial NESSs far from equilibrium. Figure 3 shows that \( S_L \) increases dramatically with increasing \( \Delta \mu \). To confirm Eq. (1), we note that the \( O(\ln L) \) term in its r.h.s. should be relatively insensitive to \( \Delta k F \). Hence, by subtracting Eq. (1) at \( \Delta \mu \) from that at another \( \Delta \mu' \), where |\( \Delta \mu - \Delta \mu' \)\| \( \ll 1 \), we expect

\[
\eta(L) \approx |S_L(\Delta \mu) - S_L(\Delta \mu')|/(\Delta k F - \Delta k F') L, \tag{8}
\]

for \( 1 \ll L \leq L_C \). The quasi volume law holds if this quantity has properties (i)-(iii) mentioned earlier around inequality \([2]\). To confirm this, we plot the r.h.s of Eq. (5) for various values of \( (\Delta \mu, \Delta \mu') \) in Fig. 4. When \( (\Delta \mu, \Delta \mu') \) is small (not far from equilibrium), neither of (i)-(iii) is satisfied \([93]\). However, when \( (\Delta \mu, \Delta \mu') \) is large, all the properties are satisfied. In particular, property (iii) is satisfied with \( a \approx 0.1 \) \([79]\). We have thus confirmed the quasi volume law far from equilibrium.

**Scaling in reservoirs.**—When \( L \) is increased to \( L > L_C \), where region A exceeds the QWR, the peaks of \( \tilde{R}_L^W \) at small \(|p|\) cease to grow (because impurities are absent in the reservoir regions), as shown in Fig. S6 of \([79]\), whereas those at small \(|q|\) continue growing. Consequently, the logarithmic law is recovered with an offset value:

\[
S_L = \eta(L_C) L C |\Delta k F| + O(\ln L) \quad \text{for } L > L_C, \tag{9}
\]
as shown in Fig. S7 of \([79]\). Even when \( L < L_C \), if impurities are absent \((W = 0)\) then \( S_L \) obeys the logarithmic law for any \( \Delta \mu \), as mentioned earlier (see \([79]\) for details). These facts clearly show that the quasi volume law is peculiar to nontrivial NESSs.

**Summary and discussions.**—We have studied the size dependence of the EE in NESSs of a one-dimensional mesoscopic conductor driven by the chemical potential difference of two reservoirs of zero temperature. At equilibrium, the EE obeys the logarithmic law, \( S_L^{\text{eq}} = O(\ln L) \). In nontrivial NESSs far from equilibrium, \( S_L \) is enhanced anomalously, as shown in Fig. 3 to obey the quasi volume law \([1]\), as shown in Fig. 4. Consequently, \( S_L > O(S_L^{\text{eq}}) \) in contrast to \( S_L \leq O(S_L^{\text{eq}}) \) of NESSs in the previous works \([51, 53]\). This anomalous behavior arises from far from nonequilibrium and multiple scatterings by impurities, which make the NESS much different from a boost of an equilibrium state.

This finding is not only important as a fundamental property of \( S_L \) in NESSs, but also interesting in the following points. Let \( W_L \) be the number of microstates composing the state \( \rho_L \) of subsystem A. It is given by \( W_L \simeq \exp(S_L) \). At equilibrium, \( W_L \) increases only as a power law. In nontrivial NESSs far from equilibrium, by contrast, \( W_L \) is exponentially large, \( W_L \geq \exp[O(L)] \), because the quasi volume law yields \( S_L \geq a L |\Delta k F| + O(\ln L) \). This indicates the possibility that the typicality \([59, 94, 101]\) could hold even at zero temperature for such NESSs. Furthermore, we find that the quasi volume law is accompanied with anomalous enhancement of long-range correlations of all local observables (up to two-site operators) \([102]\).

Although we have obtained these results for a specific model, similar results may be obtained for other models in which multiple scatterings are important.

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Anomalous Enhancement of Entanglement Entropy in Nonequilibrium Steady States:
Supplemental Material

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We present the quantum state of the total system, our method of calculating $S_L$, a useful formula for particle number fluctuation, and the results for $\delta N^2_L$ and $S_L$.

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I. QUANTUM STATE OF THE TOTAL SYSTEM

A. Note on the model

In numerical computations, we take $v_x$’s in the Hamiltonian (3) as follows. After $v_x$’s are generated randomly, we replace each $v_x$ with

$$v_x - \sum_x v_x / L_C$$

(S.1)

to eliminate effects of a non-vanishing average value of $\sum_x v_x / L_C$; for example, when it is positive, electrons would tend to be pushed away from the QWR.

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B. Single-particle states

We can derive the single-particle Shrödinger equation as
\[ -\phi(x - 1) - \phi(x + 1) + v_x \phi(x) = \varepsilon \phi(x). \] (S.2)

Here, \( \varepsilon \) and \( \phi(x) \) are a single-particle energy and a single-particle wave function. We consider scattering states and bound states for \( \phi(x) \).

The scattering states \( \{ \phi_k(x) \}_k \) \((-\pi < k \leq \pi)\) satisfy the boundary condition
\[
\sqrt{2\pi} \phi_k(x) = \begin{cases} 
 e^{ikx} + r_k e^{-ikx} & (x \to -\infty) \\
 t_k e^{ikx} & (x \to \infty)
\end{cases}
\] (S.3)

From the unitarity and the time-reversal symmetry,
\[
\begin{align*}
|t_k|^2 + |r_k|^2 &= 1 \\
t_k &= t_{-k} \quad (S.4) \\
r_k t_{-k}^* + r_{-k}^* t_k &= 0.
\end{align*}
\]

We also have \( \varepsilon_k = -2\cos k \). The \( k \) dependence of transmittance \( |t_k|^2 \) is plotted in Fig. S1. Since \( L_C \gg 1 \), \( |t_k|^2 \) depends sensitively on \( k \).

**FIG. S1:** \( k \) dependence of transmittance \( |t_k|^2 \) for \( W = 0.08 \) and \( L_C = 401 \) (these parameters are common to Fig. 1 of the text). We plot only the region \( k > 0 \) because of the symmetry \( t_k = t_{-k} \).

On the other hand, the bound states \( \{ \phi_b(x) \}_b \) \((b = 1, 2, \ldots)\) satisfy \( |\phi_b(x)| \to 0 \) (as \( |x| \to \infty) \), and single-particle energy \( \varepsilon_b \) satisfies \( \varepsilon_b < -2 \) or \( \varepsilon_b > 2 \). When \( L_C = 401 \), the number of bound states is \( \approx 20 \) for \( W = 0.05 \) and \( \approx 25 \) for \( W = 0.08 \), depending on the realization of the random potential.

These wave functions are orthonormal and satisfy the closure relation;
\[
\sum_x \phi_b^*_k(x) \phi_{b_1}(x) = \delta(k_1 - k_2), \quad \sum_x \phi_{b_1}(x) \phi_{b_2}(x) = \delta_{b_1, b_2}, \quad \sum_x \phi_k^*(x) \phi_b(x) = 0, \] (S.5)
\[
\int_{-\pi}^{\pi} dk \phi_k^*(x) \phi_k(y) + \sum_b \phi_b(x)^* \phi_b(y) = \delta_{x,y} \quad (\forall x,y). \] (S.6)

We have utilized this closure relation to estimate numerical errors, which come mainly from the numerical calculation of the wavefunctions and the numerical integration. Since we study the nontrivial case where \( 0.3 \lesssim G \lesssim 0.7 \) and \( L_C \gg 1 \), the numerical errors (as confirmed by the closure relation) increase quickly with increasing \( L_C \) because of the high sensitivity of \( \phi_k(x) \) on \( k \). We therefore take \( L_C = 401 \) in numerical calculations, for which the closure relation is confirmed to hold very well for all the values of \( W \) employed in the calculations.
C. Many-body state

The many-body state \( |\Psi_{\text{tot}}\rangle \) of the total system is given by a single Slater determinant of single-particle states

\[
|\Psi_{\text{tot}}\rangle := \prod_{-k_F^{-} \leq k \leq k_F^{+}} c_k^\dagger \prod_{b, \epsilon_b < -2} c_b^\dagger |0\rangle,
\]

where \( c_k^\dagger \) is the creation operator of the scattering (bound) states. Therefore, the two-point correlations are given by

\[
\langle c_{k_1}^\dagger c_{k_2} \rangle = \delta(k_1 - k_2) \Theta(-k_F^{-} < k_1 < k_F^{+})
\]

\[
\langle c_{k}^\dagger c_{b} \rangle = 0
\]

\[
\langle c_{b_i}^\dagger c_{b_j} \rangle = \delta_{b_i, b_j} \Theta(\epsilon_{b_i} < -2),
\]

where \( \Theta \) is a step function and \( \langle \bullet \rangle := \langle \Psi_{\text{tot}} | \bullet | \Psi_{\text{tot}} \rangle \). All the other two-point correlations vanish because they are not gauge-invariant. Moreover, the Wick’s theorem holds:

\[
\langle c_{x_1}^\dagger c_{x_2}^\dagger c_{x_3} c_{x_4} \rangle = \langle c_{x_1}^\dagger c_{x_2} \rangle \langle c_{x_3}^\dagger c_{x_4} \rangle + \langle c_{x_1}^\dagger c_{x_4} \rangle \langle c_{x_2}^\dagger c_{x_3} \rangle , \text{ and similarly for other correlations.}
\]

More precisely, the many-body state should be defined not by Eq. (S.7), for which some cares are necessary in an infinite system, but by Eqs. (S.8)-(S.11), which have no problem even in an infinite system [1]. In fact, all our calculations have been done using only Eqs. (S.8)-(S.11).

From the expectation value of the current, we obtain the dimension-less nonlinear conductance \( G \) as shown in Fig. S2. It is seen that the linear response regime is limited to a very tiny region \( \Delta \mu \lesssim 0.005 \).

**FIG. S2:** \( \Delta \mu \) dependence of \( G \) for \( W = 0.08 \) and \( \bar{\mu} = -1.5 \). We plot \( G \) in the region \( 0 \leq \Delta \mu \leq 1.0 \). When \( \Delta \mu = 1.0 \), \( \bar{\mu} \) reaches the band edge \( \epsilon_k = -2.0 \).

II. RELATION BETWEEN \( S_L \) AND \( \delta N_L^2 \)

In this section, we give a relation between \( S_L \) and \( \delta N_L^2 \). We first give a method for efficiently calculating \( S_L \) and \( \delta N_L^2 \) in subsystem A, by generalizing the method of Refs. [2, 3]. We then derive an inequality between \( S_L \) and \( \delta N_L^2 \), by generalizing that of Refs. [4, 5].

1 In numerical computations, although wavenumbers are taken discretely with small intervals, we have confirmed that the results are independent of the lengths of the intervals.
A. Calculation of $S_L$ and $\delta N_L^2$

Since the reduced density operator $\rho_L$ in subsystem $A$ satisfies the Wick’s theorem, $S_L$ and $\delta N_L^2$ can be determined by the two-point correlations $\langle \hat{c}_y^\dagger \hat{c}_y \rangle$ for $x, y \in A$. We therefore introduce the $L \times L$ matrix

$$
\Lambda_{xy} := \langle \hat{c}_y^\dagger \hat{c}_y \rangle = \int_{-\infty}^{\infty} dk \varphi_k^L(x) \varphi_k(y) + \sum_{b \ (\epsilon_b < -2)} \phi_b(x)\phi_b(y).
$$

(S.12)

Since $\Lambda$ is an Hermitian matrix, it can be diagonalized by a unitary $L \times L$ matrix. We can obtain $S_L$ from $\Lambda$ by

$$
S_L = -\text{tr}[\Lambda \ln \Lambda + (1 - \Lambda) \ln (1 - \Lambda)],
$$

(S.13)

where $\text{tr}[\cdot]$ means the trace of $L \times L$ matrix. We can also obtain all observables in $A$ from $\Lambda$. For example, the fluctuation of particle number $\delta N_L^2$ can be calculated by $\delta N_L^2 = \text{tr}[\Lambda(1 - \Lambda)]$.

B. Relation between $S_L$ and $\delta N_L^2$

Let us define $h(x)$ by

$$
h(x) := -x \ln x - (1 - x) \ln (1 - x) \quad (0 \leq x \leq 1).
$$

(S.14)

We note the following inequality [4, 5]

$$
x(1 - x) \leq h(x) \leq c \ln \epsilon_1 x(1 - x),
$$

(S.15)

where $c$ is a constant independent of $\epsilon_1 (> 0)$, and moreover, for $0 < \epsilon_1 < \epsilon_0$ we may choose $c = 1 + o(\epsilon_0)$. Using this inequality, we can evaluate $S_L(= \text{tr}[h(\Lambda)])$ as

$$
\text{tr}[\Lambda(1 - \Lambda)] \leq S_L \leq c_1 L - \{c \ln \epsilon_1 \} \text{tr}[\Lambda(1 - \Lambda)].
$$

(S.16)

By taking $\epsilon_1 = 1/L$, we obtain this important inequality

$$
\delta N_L^2 \leq S_L \leq 1 + c(\ln L)\delta N_L^2.
$$

(S.17)

The coefficient $\ln L$ in its right-hand side is derived from the logarithmic divergence of $\frac{d h(x)}{dx}$ as $x \to 0$ or 1. Therefore, we expect that the left inequality in Eq. (S.17) is better than the right one, i.e. $S_L \simeq \delta N_L^2$.

III. FORMULA FOR PARTICLE NUMBER FLUCTUATION

Since region $A$ is an open system, the current flows through its edges and the particle number fluctuates. We here derive their relations, Eqs. (6) and (7).

A. Derivation of Eqs. (6) and (7)

The current $\hat{j}(x)$ which flows $x \to x + 1$ is given by $\hat{j}(x) = \frac{1}{\tau} \left( \hat{c}_x^\dagger \hat{c}_{x+1} - \hat{c}_{x+1}^\dagger \hat{c}_x \right)$. In order to represent it in the Heisenberg picture, we rewrite $\hat{j}(x)$ by the single-particle eigenstates composed of the scattering states (created by $\hat{c}_k^\dagger$) and the bound states (created by $\hat{c}_k$), and we have

$$
\hat{j}(x, t) = \int dk_1 \int dk_2 J_{k_1, k_2}(x + 1/2) \hat{c}_{k_1}^\dagger \hat{c}_{k_2} e^{i(\epsilon_{k_1} - \epsilon_{k_2}) t} + (\text{terms involving bound states}),
$$

(S.18)
where $J_{k_1,k_2}(x+1/2) := \frac{1}{2} (\phi^*_{k_1}(x)\phi_{k_2}(x+1) - \phi^*_{k_2}(x+1)\phi_{k_1}(x))$ ($J_{k_2,k_1}(x) = (J_{k_1,k_2}(x))^*$. Since bound states do not seem to give significant contribution, we drop terms involving them.

The equation of continuity reads

$$\frac{d}{dt}\Delta \tilde{N}_L(t) = \Delta \tilde{J}(t).$$

Here, $\Delta \tilde{N}_L(t)$ denotes the particle number in $A$, and $\tilde{J}(t) := \tilde{j}(x_+ - 1, t) - \tilde{j}(x_+, t)$ is the net current into $A$, where $x_\pm = \pm \frac{L-1}{2}$ are the edges of $A$, and $\Delta \tilde{N}_L(t) := \tilde{N}_L(t) - \langle \tilde{N}_L(t) \rangle$, $\Delta \tilde{J}(t) := \tilde{J}(t) - \langle \tilde{J}(t) \rangle$. By Fourier transformation, we obtain

$$-i\omega \Delta \tilde{N}_L(\omega) = \Delta \tilde{J}(\omega) \quad \left( \Delta \tilde{N}_L(\omega) := \frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{-i\omega t} \Delta \tilde{N}_L(t) \right).$$

We also define the spectral intensity of $J$ by

$$g_J(\omega) := \int dt e^{i\omega t} \langle \Delta \tilde{J}(0) \Delta \tilde{J}(t) \rangle_{\text{sym}},$$

where $\langle \Delta \tilde{X} \Delta \tilde{Y} \rangle_{\text{sym}} := \frac{\langle \tilde{X} \tilde{Y} + \tilde{Y} \tilde{X} \rangle}{2}$. We then have

$$\delta N_L^2 = \int d\omega \int d\omega' \langle \Delta \tilde{N}_L(\omega) \Delta \tilde{N}_L(\omega') \rangle_{\text{sym}} = \int d\omega \frac{2\pi g_J(\omega)}{\omega^2}.$$  

Using Wick's theorem, we also obtain

$$g_J(\omega) = 2\pi \int_{\Omega_{2n}} dk_1 dk_2 |J_{k_1,k_2}(x_+ - 1/2) - J_{k_1,k_2}(x_+ + 1/2)|^2 \delta(\omega + \varepsilon_{k_2} - \varepsilon_{k_1}).$$

From these relations, we obtain Eqs. (6) and (7) of the text.

### B. Proof of $\Delta J_L^{pq} = \Delta J_L^{po} = 0$

Using the Shrödinger equation (S.2), we can show that $J_{kk}(x+1/2) = J_{kk}(x-1/2)$ for all $x$. This leads to $J_{kk}(x) = \text{constant} (\text{current conservation})$, and therefore $\Delta J_L^{pq} = 0$ holds. $\Delta J_L^{po} = 0$ can be shown in a similar manner.

### IV. RESULTS FOR $\delta N_L^2$ AND $S_L$

#### A. Equilibrium without random potential

We first reproduce the previous result for the case of $W = \Delta \mu = 0$ [4, 6] in such a way that the derivation can be generalized to the case of NESSs.

When $W = 0$, we simply have $\phi_k(x) = e^{ikx}/\sqrt{2\pi}$, and hence $|\Delta J_L^{pq}|^2 = (4/\pi^2) \sin^2(p/2) \sin^2(qL/2)$. As a result, Eq. (6) reduces to the previous result [4, 6];

$$\delta N_L^2 = \frac{1}{4\pi^2} \int_{\Omega_{2n}} dk_1 dk_2 F_L(q) \quad (q := k_1 - k_2, \quad F_L = R_L^0),$$

where $F_L$ is the Fejér kernel $F_L(q) := \frac{\sin^2(qL/2)}{\sin^2(q/2)}$. $F_L$ vanishes at $|q| = 2\pi n/L$ ($n = 1, 2, \ldots$), and has peaks between these points. The peak height is $O(L^2)$ for $q = O(1/L)$.

To see the $L$ dependence of $\delta N_L^2$, let us divide the $q$ integral into two portions: $|q| \leq \epsilon$ and $\epsilon < |q|$, where $\epsilon$ is an arbitrary small positive constant independent of $L$. In the integral over the latter portion, we can approximate $F_L(q) \approx 1/2 \sin^2(q/2)$. Hence, $L$-dependent terms arise only from the integral over the former portion, $|q| \leq \epsilon$. Hence, considering also Fig. 2 of the text, we get

$$4\pi^2 \delta N_L^2 = \int_{-\epsilon}^{\epsilon} |q| F_L(q) dq + O(1) = 2 \int_{0}^{\epsilon} |q| F_L(q) dq + O(1).$$

(S.25)
We further divide the integral as
\[
\int_0^\epsilon |q|F_L(q)\,dq = \int_0^{2\pi/L} |q|F_L(q)\,dq + \int_\epsilon^{2\pi/L} |q|F_L(q)\,dq
\]
(S.26)
\[
= \int_0^{2\pi/L} |q|F_L(q)\,dq + O(1)
\]
(S.27)
\[
\approx \int_0^{\epsilon} |q|(1/2)\,(q^2/4)\,dq + O(1)
\]
(S.28)
\[
= 2\ln L + O(1).
\]
(S.29)

Therefore, \( \delta N_L^2 = O(\ln L) \). We thus expect \( S_L^{eq} = O(\ln L) \), which is confirmed numerically in Fig. S3.

![Graph showing the dependence of \( S_L \) on \( L \)](chart.png)

**FIG. S3:** \( L \) dependence of \( S_L \) at equilibrium (\( \Delta\mu = 0 \)), averaged over 20 samples (\( W = 0, 0.05 \)) and 10 samples (\( W = 0.08 \)), for \( \mu = -1.5 \). This may be a precursor of the Anderson localization [7], which cuts off the length scale by the localization length. In our case, the system is an open system in which a random potential exists only in a finite region, and hence the localization is incomplete.

### B. Equilibrium and NESSs with random potential

As described in the text, we can analyze the cases of equilibrium with random potential and NESSs with random potential, by generalizing the above argument. Since the derivation is described in the text, we here present the numerical results that are referred in the text. [We will also present a simple power-counting argument, by which the quasi volume law may be understood easily, in Sec. IV C.]

In the equilibrium case (\( \Delta\mu = 0 \)), \( S_L^{eq} \) obeys the logarithmic law \( S_L^{eq} = O(\ln L) \) for any \( W \), and its value is not sensitive to \( W \), as shown in Fig. S3. However, it is seen that \( S_L^{eq} \) decreases slightly with increasing \( W \). This may be a precursor of the Anderson localization [7], which cuts off the length scale by the localization length. In our case, the system is an open system in which a random potential exists only in a finite region, and hence the localization is incomplete.

The forward-scattering part (small \( |q| \)) and the backward-scattering part (small \( |p| \)) of \( R_L^W \), divided by \( L^2 \), are plotted in Figs. S4a and S4b, respectively. It is seen that \( R_L^W \) has peaks of anomalously large heights of \( O(L^2) \) not only at \( |q| = O(1/L) \) but also at \( |p| = O(1/L) \).

The \( W \) dependences of \( R_L^W \) for small \(|q| \) and for small \(|p| \) are plotted in Fig. S5, where, to smear \( k_1, k_2 \) dependences, we have plotted the integral of \( R_L^W \) over the two-dimensional square regions \( 0 \leq k_F \pm k_1 \leq 10\pi/L, 0 \leq k_2 - k_F \leq 10\pi/L \). It is seen that the heights of the \( O(L^2) \) peaks of \( R_L^W \) increase (decrease) with increasing \( W \) for small \(|p| \) (small \(|q| \)).

Figure S6a shows the forward-scattering part (small \(|q| \)) of \( R_L^W \) divided by \( L^2 \), for \( L > L_C \). It is seen that the peaks of \( R_L \) at small \(|q| \) continue to grow as \( \propto L^2 \) even when \( L \) exceeds \( L_C \).

By contrast, Fig. S6b shows the backward-scattering part (small \(|q| \)) of \( R_L^W \), not divided by \( L^2 \), for \( L > L_C \). It is seen that the peaks of \( R_L^W \) at small \(|p| \) cease to grow when \( L \) exceeds \( L_C \). This is because the random potential is absent in the reservoir regions. As a result, the logarithmic law for \( S_L \) is recovered with an offset value when \( L > L_C \), as shown in Fig. S7, where the numerical results for \( L > L_C \) are least-square fitted to Eq. (9) of the text, as indicated by the red curve.
(a) The forward-scattering part (small |q|).
(b) The backward-scattering part (small |p|).

FIG. S4: $L$ dependence of $R^W_L$ (small |q| or |p|), divided by $L^2$, for $W = 0.08$. $k_1$ is taken as $\varepsilon k_1 = \mu = -1.5$.

FIG. S5: The $W$ dependences of $R^W_L$ integrated over small two-dimensional square regions, which are $0 \leq k_F - k_1 \leq 10\pi/L, 0 \leq k_2 - k_F \leq 10\pi/L$ for small |q| (forward) and $0 \leq k_F + k_1 \leq 10\pi/L, 0 \leq k_2 - k_F \leq 10\pi/L$ for small |p| (backward), respectively. We take $\beta = -1.5$ ($k_F = \arccos \frac{3}{4}$) and $L = 201$.

FIG. S6: $L$ dependence of $R^W_L$ (small |q| or |p|), divided by $L^2$ and not divided by $L^2$ respectively, in the reservoir ($L > L_C$), for $W = 0.08$. $k_1$ is taken as $\varepsilon k_1 = \mu = -1.5$. 

FIG. S7: L dependence of $S_L$ for $W = 0.08$, $\bar{\mu} = -1.5$, and $\Delta \mu = 0.4$, in a wide range of $L$ including the reservoir region $L > L_C$. Red curve: The least-square fit of the numerical results for $L > L_C$ to Eq. (9) of the text.

C. Simple power-counting argument

We here present a simple power-counting argument by which the quasi volume law may be understood intuitively.

Regarding the integral $\int |q| \tilde{R}_L^W dq$ (and $\int |\eta| \tilde{R}_L^W d\eta$), its $L$ dependence is estimated as $O(1/L)$ (from the significant interval of integral) $\times$ $O(1/L)$ (from significant value of $|q|$) $\times$ $O(L^2)$ (from $\tilde{R}_L^W$) = $O(L^0)$, which implies $O(\ln L)$ because it is integral.

By contrast, the integral $\int \tilde{R}_L^W d\eta$ is estimated as $O(1/L)$ (from the significant interval of integral) $\times$ $O(L^2)$ (from $\tilde{R}_L^W$) = $O(L)$. We also note the fact that backward scatterings become weaker as the edges ($x = \pm L/2$) of $\Lambda$ approach the edges ($x = \pm L_C/2$) of the QWR. As a result, the peaks of $\tilde{R}_L^W$ at small $|p|$ grow slower than $O(L^2)$ with increasing $L$, as shown in Fig. S4b. That is, the heights of the peaks are actually $O(L^2)\eta(L)$. Taking account of this factor, we can estimate that $\int \tilde{R}_L^W d\eta = \eta(L)$. Collecting these estimates, we arrive at the quasi volume law (1) of the text.

D. NESS ($\Delta \mu \neq 0$) without random potential ($W = 0$)

When $W = 0$, $\tilde{R}_L^W$ reduces to the Fejér kernel $F_L$. Therefore, we obtain Eq. (S.24) again. Even though the region $\Omega_{\Delta \mu}$ of integral is shifted from the equilibrium one, the integral over small $|p|$ (corresponding to backward-scattering processes) gives at most $O(1)$ contribution because $\tilde{R}_L^W$ for small $|p|$ does not have peaks of $O(L^2)$ when $W = 0$. Hence, only the integral over small $|q|$ (forward-scattering processes) gives the $O(\ln L)$ contribution. Consequently, $S_L$ obeys the logarithmic law when $W = 0$ for any $\Delta \mu$.

This behavior of $S_L$ (and $\delta N_L^2$) at $\Delta \mu \neq 0$ can be proved analytically as follows. When $W = 0$,

$$\Lambda_{xy} = e^{i\Delta k_F(y-x)/2} \Lambda_{xy}^{eq}. \quad (S.30)$$

Here, $\tilde{k}_F := (k_F^+ + k_F^-)/2$, and $\Lambda^{eq}$ is $\Lambda$ in the equilibrium state ($\Delta k_F = 0$) whose Fermi wavenumber is $\tilde{k}_F$. $\Lambda$ has the same eigenvalues as $\Lambda^{eq}$ because $\Lambda$ can be rewritten as

$$\Lambda_{xy} = \sum_{\alpha,\beta} (U_{\alpha x}^* \Lambda_{xy}^{eq} U_{\alpha y}), \quad (S.31)$$

where $(U)_{xy} = e^{i\Delta k_F x/2} \delta_{xy}$ is an unitary and diagonal matrix. Hence, $\hat{\rho}_L$ has the same eigenvalues as that at equilibrium. Therefore, $S_L$ is independent of $\Delta k_F$ (if $\tilde{k}_F$ is fixed).

This result may also be understood from the fact that a NESS with $W = 0$ is essentially an equilibrium state that is observed from a moving inertial frame (even in the presence of interactions between electrons [8]).

From this result and the result for equilibrium with $W > 0$, it is concluded that both the random potential and far from equilibrium are necessary for the quasi volume law (1). Mathematically, this comes from the facts that $\tilde{R}_L^W$ at small $|p|$ can be as large as $O(L^2)$ only when $W > 0$, and that $|\Delta k_F| \int \tilde{R}_L^W d\eta$ obviously vanishes when $\Delta \mu = 0$. 

FIG. S8: This graph shows the data when far from equilibrium in Fig.4. We calculated $a_0$ and $a_1$ in the range of $50 \leq L \leq 400$. The errors of the fitting are at most 1%.

E. Linear fitting

Figure 4 suggests that $\eta(L)$ is almost linear in $L$ for nontrivial NESSs far from equilibrium. To confirm this, we fit the data by

$$\eta(L) = a_0 - a_1 \frac{L}{\mathcal{C}}, \quad \text{(S.32)}$$

where $a_0$ and $a_1$ are fitting parameters. Using the least square method, we obtain $(a_0, a_1) = (0.19, 0.096 \times (1 - 0.01)), (0.19, 0.096 \times (1 + 0.04)), (0.19 \times (1 + 0.05), 0.096 \times (1 - 0.02))$ for $(\Delta \mu, \Delta \mu') = (0.4, 0.35), (0.3, 0.25), (0.2, 0.15)$, respectively, as shown in Fig. S8. Hence, condition (2) of the text is satisfied well enough with $a \simeq 0.19 - 0.096 \simeq 0.1$.

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