Some properties of accelerating observers in the Schwarzschild space

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It is well known that observers will be accelerated when they approach the planets. Thus, discussing the properties of accelerating observers in the Schwarzschild space is of sense. For the sake of simplicity, we can construct these observers’ world lines by comparing with the observers in the Hawking effect and Unruh effect, whose world lines are both hyperbolic curves in the appropriate coordinates. We do it after defining a certain special null hypersurface in the Kruskal coordinates. Our result shows that these accelerating observers defined in our paper can also detect the radiation with the analogy of the Unruh effect, though locally. Furthermore, we conjecture that, for any null hypersurface in any spacetime, the corresponding observers who can at least locally detect the radiation from it can be found.

Key words: accelerating observers, Schwarzschild space, Unruh effect, null hypersurface, radiation

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I. INTRODUCTION

Simply viewed from the astronomical exploration, it’s of sense to theoretically consider the properties of accelerating observers when approach the planets. However, the spacetime is an unity in general relativity [1], and the concept of accelerating in the universe is ambiguous. Since the outside spacetime of the planet can always be approximately viewed as the Schwarzschild spacetime, we can naturally define the concept of acceleration in terms of the second time derivatives of position in the usual Schwarzschild space with the killing vector ($\frac{\partial}{\partial t}$).

On the other hand, there have been many works considering the properties of accelerating observers in various of different spacetimes and conditions [2–4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]. And two significant representations of these works are the discovery of Hawking effect and Unruh effect [2, 5]. These two effects both conclude that the accelerating observers can detect the radiation as they are in the thermal bath. Moreover, the observers’ accelerations are uniform in the spacetime. In addition, both effects have another common property of their observers, which is that the observers’ world lines are all hyperbolic in the appropriate coordinates. Thus, for simplicity, we can construct our accelerating observers’ world lines in the Schwarzschild space by taking the analogy of these two effects. In the Kruskal coordinates, after defining a certain special null hypersurface, we can define the world lines of our accelerating observers to be the hyperbolic curves of which the asymptote is just the null hypersurface itself. And in the latter we can prove that the observers defined in this way truly accelerate in the Schwarzschild space. It should be emphasized that there are other constructions of the observers accelerating in the Schwarzschild space, but the properties of the observers may be different. For example, for the free drop observer and the static observer in the Schwarzschild space, the former corresponds to a geodesic observer and detects no radiation, while the latter can detect the well-known Hawking effect. In fact, the Hawking effect with the static observer can be viewed as a special case in our discussion. And with the very similar trick (defining the null hypersurface first), the authors in Ref[15] discussed various of properties of accelerating observers in the Minkowski spacetime.

The present work starts by defining a certain null hypersurface and its corresponding observers in the Kruskal coordinates. After discussing some properties of these observers, we will show that the event horizon with the static observers in the Hawking effect can be viewed as the special case in our discussion.

Then, another more significant property of these observers, detecting radiation, is discussed in section 3 by using the Damour- Ruffini method[13]. The radiation can be viewed as the analogy of the Unruh effect. However, because ($\frac{\partial}{\partial \eta}$) is not a killing vector, the thermal spectrum would be just local. That is, it is detected just by the observers near the horizon and it can’t be detected consistently with the observers in the infinity.

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Finally, in section 4, we will first show that these observers defined in this way truly accelerate in the Schwarzschild space, and then we give a brief conclusion and discussion. Note that, although the effective temperature is also very low to detect in the experiment as those in many other similar works, the theoretical discussion is still of sense. Particularly, we conjecture that, for any null hypersurface in any spacetime, the corresponding observers who can at least locally detect the radiation from it can be found. This conjecture can be seen to be partly supported in the Ref[15]. And its application can also be seen in Ref [16] where the authors defined the local temperature with this conjecture and obtained some very interesting results.

II. THE NULL HYPERSURFACE AND ITS CORRESPONDING OBSERVERS BY DEFINITION

The line element of the Schwarzschild spacetime in the Kruskal coordinates is given by

\[ ds^2 = \frac{32M^3}{r} e^{-\frac{r}{2M}} (-dT^2 + dX^2) + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \]  

where

\[ \left(\frac{r}{2M} - 1\right)e^{\frac{r}{2M}} = X^2 - T^2. \]

In the following, we restrict our discussions only in the region \( r > 2M \). Thus, we first define the null hypersurface in this region

\[ T = \pm(X - X_0). \]

where \( X_0 \) is a positive constant. It’s easily seen that, when \( X_0 = 0 \), the null hypersurface defined in (3) can become the usual event horizon of the Schwarzschild spacetime.

According to (3), the corresponding observer is

\[ (X - X_0)^2 - T^2 = e^{2a\xi_0}, \theta = \theta_0, \varphi = \varphi_0. \]

where \( a, \xi_0, \theta_0, \varphi_0 \) are also constants. In fact, a family of observers that we are interested in can be defined by choosing different \( \xi_0 \). It’s obvious that all the world lines of these observers are hyperbolic curves in the Kruskal coordinates. Furthermore, the null hypersurface defined in (3) can be viewed as their horizon, which can be easily obtained from the T-X diagram in Figure 1.

In fact, by these observers we can define a new coordinate system \( \{\eta, \xi, \theta, \phi\} \) that

\[ X = X_0 + e^{a\xi} \sinh \eta, T = e^{a\xi} \sinh \eta. \]

And the metric in this coordinates is

\[ ds^2 = \frac{32M^3}{r} e^{-\eta} e^{2a\xi} a^2(-d\eta^2 + d\xi^2) + r^2(d\theta^2 + \sin^2 \theta d\phi^2) = g_{00}d\eta^2 + g_{11}d\xi^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \]

where

\[ \left(\frac{r}{2M} - 1\right)e^{\frac{r}{2M}} = X^2_0 + 2e^{a\xi} X_0 \sinh \eta + e^{2a\xi}. \]

In this co-moving coordinate system, it’s obvious that observers labelled by different \( \xi \) are at rest, and the horizon is at \( \xi = -\infty \), which is just the null hypersurface defined in (3) and agrees with the former conclusion from Fig 1.

The tangent vector of the observer’s worldline \( (\xi = \xi_0) \) defined in (5) is
\[
\left( \frac{\partial}{\partial \eta} \right)^a = \frac{\partial x^\mu}{\partial \eta} \left( \frac{\partial}{\partial x^\mu} \right)^a = a[T(\frac{\partial}{\partial X})^a + (X - X_0)(\frac{\partial}{\partial T})^a]. \tag{8}
\]

Re-parameterize the world line with the proper time \(\tau\), we get

\[
\left( \frac{\partial}{\partial \eta} \right)^a \equiv \frac{d\tau}{d\eta} \left( \frac{\partial}{\partial \tau} \right)^a = \gamma(\frac{\partial}{\partial \tau})^a, \gamma \equiv \frac{d\tau}{d\eta}. \tag{9}
\]

where \(\frac{\partial}{\partial \tau}\) is the 4-velocity. Using the unitary property of the 4-velocity and the metric, we find

\[
\gamma^2 \equiv \left( \frac{d\tau}{d\eta} \right)^2 = a^2 e^{2a\xi_0} \frac{32M^3}{r} e^{-\frac{rM}{2\pi}}, \\
\Gamma^0_{00} = \Gamma^0_{11} = \Gamma^1_{01} = \frac{2(r + 2M)MT}{r^2 e^{\frac{rM}{2\pi}}}, \\
\Gamma^0_{01} = \Gamma^1_{00} = \Gamma^1_{11} = -\frac{2(r + 2M)MX}{r^2 e^{\frac{rM}{2\pi}}}. \tag{10}
\]

where the nonzero Christoffel coefficients are correlated with the Kruskal coordinates. Thus, the 4-velocity and the 4-acceleration are

\[
T^a = \left( \frac{\partial}{\partial \tau} \right)^a = \frac{1}{\gamma} \left( \frac{\partial}{\partial \tau} \right)^a = a \frac{T(\frac{\partial}{\partial X})^a + (X - X_0)(\frac{\partial}{\partial T})^a]{
\frac{e^{-2a\xi}}{32M^3 \frac{e^{-\frac{rM}{2\pi}}}B[T(\frac{\partial}{\partial T})^b + (X - X_0)(\frac{\partial}{\partial X})^b].} \tag{11}
\]

where

\[
B \equiv 1 - [(X - X_0)X_0 + e^{2a\xi_0}] \frac{2M(r + 2M)}{r^2} e^{-\frac{rM}{2\pi}}. \tag{12}
\]

And the proper acceleration can also be obtained

\[
A \equiv |A^a| = \frac{1}{e^{a\xi_0}} \left( \frac{32M^3}{r} e^{-\frac{rM}{2\pi}} \right)^{-1/2} B. \tag{13}
\]

From (8) to (13), we can draw a brief conclusion about the properties of these observers. First, these observers’ world lines are truly timelike such that they can be considered as detectors. Second, the proper acceleration changes along with observer’s proper time or \(\eta\), which is different from that of a static observer in the Schwarzschild space-time, which is a constant. Third, when the observer get closer to the horizon or in the limit \(\xi_0 \to -\infty\), the proper acceleration becomes positive infinite, which is a general property of the horizon. And this can also be understood that the proper acceleration of the light may be positive infinite (more details can be seen in the Ref[17]). Finally, we show that the event horizon with the static observers in the Hawking effect can be viewed as the special case in our discussion, if \(X_0\) vanishes. The above four properties are essential and can be easily obtained. In the following section, we will make more discussions about another interesting property of these observers-detecting radiation.

**III. RADIATION FROM THE NULL HYPERSURFACE**

In the above section, we have showed that the null hypersurface is just the horizon for its corresponding observers. Now we want to see whether these observers can detect the radiation or the thermal spectrum from the null hypersurface.

For the sake of simplicity, we only consider the real scalar field, and the Klein-Gordon equation in curved space-time is

\[
(\partial_\eta)^a = \partial x^\mu (\partial x^\mu)^a = a[T(\partial X)^a + (X - X_0)(\partial T)^a].
\]
\[
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial X^\mu} (\sqrt{-g} g^\nu_{\mu} \frac{\partial \Psi}{\partial X^\nu}) - \mu^2 \Psi = 0. \tag{14}
\]

Substitute the metric (6) into (14), we obtain
\[
\left[- \frac{\partial}{\partial \eta} (r^2 \frac{\partial \Psi}{\partial \eta}) + \frac{\partial}{\partial \xi} (r^2 \frac{\partial \Psi}{\partial \xi})\right] \cdot \frac{1}{g_{11}} + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial \Psi}{\partial \theta}) + \frac{1}{\sin^2 \theta} \frac{\partial^2 \Psi}{\partial \varphi^2} = \mu^2 r^2 \Psi. \tag{15}
\]

Usually, the scalar function \(\Psi\) fulfilling the covariant Klein-Gordon equation in the given spherically symmetrical metric can always be separated as
\[
\Psi(\eta, \xi, \theta, \varphi) = R(\eta, \xi) Y_{lm}(\theta, \varphi). \tag{16}
\]

where \(Y_{lm}(\theta, \varphi)\) is the usual spherical harmonics. Thus equation (15) can be simplified to a radial equation
\[
- \frac{\partial}{\partial \eta} \left( r^2 \frac{\partial R}{\partial \eta} \right) + \frac{\partial}{\partial \xi} \left( r^2 \frac{\partial R}{\partial \xi} \right) = \left[ \mu^2 r^2 R - l(l + 1) \right] g_{11}. \tag{17}
\]

In the usual quantum field theory, we can define the positive frequency with respect to \(\eta\) by solving equation (17) and then define the corresponding vacuum state. However, \((\frac{\partial}{\partial \eta})^a\) is not a killing vector, thus the vacuum state is not invariant, which leads to the fact that the following discussion about radiation from the null hypersurface or horizon is just local and only suitable for observers near the null hypersurface\(^{19}\).

Consider the asymptotic expression for the field near the horizon , and from (7) we obtain
\[
\frac{\partial r}{\partial \xi} = \frac{2ae^\omega X_{0ch}(\eta \xi)}{\sqrt{-g_{11}}} e^{i\omega \xi}, \quad \frac{\partial r}{\partial \eta} = \frac{2ae^\omega X_{0ch}(\eta \xi)}{\sqrt{-g_{11}}} e^{i\omega \xi}. \tag{18}
\]

Thus, when \(\xi = -\infty\), the location of the event horizon, (17) will be
\[
- \frac{\partial^2 R}{\partial \eta^2} + \frac{\partial^2 R}{\partial \xi^2} + 2r \frac{\partial R}{\partial \xi} \frac{\partial r}{\partial \eta} = 0. \tag{19}
\]

Obviously, there are two exact solutions
\[
R_{\text{in}}^\omega = e^{-i\omega (\eta + \xi)}, \quad R_{\text{out}}^\omega = \frac{C}{r} e^{-i\omega (\eta - \xi)} \quad (\text{here } C \text{ is the normalized factor}). \tag{20}
\]

which are just the two explicit asymptotic expression for the field near the horizon and can be viewed as the in-going wave and out-going wave. Because the radiation is local, in what follows we will use the Damour-Ruffini method to discuss the radiation from the horizon \[^{13}^{20}^{21}^{22}^{23}\], although there are many other approaches to prove the radiation \[^{4}^{11}^{14}\].

In order to clearly find out the analytic properties of the out-going wave and in-going wave, we use the advanced Eddington-Finkelstein coordinates which is defined by \(\nu = \eta + \xi\). And (20) becomes
\[
R_{\text{in}}^\nu = e^{-i\omega \nu}, \tag{21}
\]
\[
R_{\text{out}}^\nu = \frac{C}{r} e^{2i\omega \xi} e^{-i\omega \nu}. \tag{22}
\]

From (21) and (22), it’s easy to see that the out-going wave is not analytic in the horizon, while the in-going wave is.

Physically, the analytic in-going wave can be viewed as a classical particle locally detected by the observers near the horizon drops into the horizon to reach a negative-energy state. In the quantum description, this phenomenon allows an antiparticle to reach positive-energy states from the horizon by the tunneling effect. Thus, according to
Refs\[13, 18\], the prescription \( r \rightarrow r - i 0 \) will yield the unique continuation of Eq.(21) describing an antiparticle state. And it’s equivalent to analytical extension of the out-going wave through the lower complex plane. To get more insight of the analytical extension, we can introduce a new coordinates transformation

\[
\xi = \frac{1}{2a} \ln \rho. \tag{23}
\]

This transformation’s most advantage is that it will change the location of horizon from \( \xi = -\infty \) to \( \rho = 0 \). And it is analogy with the tortoise transformation in the Rindler spacetime\[19\]. After taking the transformation, the \( R_{\omega}^{\text{out}} \) is

\[
R_{\omega}^{\text{out}} = \frac{C}{r} e^{\omega/a} e^{-i \omega \xi} \quad (r > r_H). \tag{24}
\]

and the metric in (6) becomes

\[
d s^2 = \frac{32 M^3}{r} e^{-\frac{2\pi}{r} \rho a^2} (-d\eta^2 + d\xi^2) + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \tag{25}
\]

From (24) and (25), it is easy to see that if we exchange \( \rho \rightarrow |\rho| e^{-i \pi} \) through the lower complex plane, the analytic solution of \( R_{\omega} \) inside the horizon can be found to be

\[
R_{\omega}^{-\text{out}} = \frac{C}{r} e^{\pi \omega/a} e^{2i \omega \xi} e^{-i \omega \eta} \quad (r < r_H). \tag{26}
\]

By introducing the Heaviside function \( Y \),

\[
\wedge_{\omega}^{\text{out}} R_{\omega} = N_{\omega} [Y(r - r_H) R_{\omega}^{\text{out}} + Y(r_H - r) R_{\omega}^{-\text{out}}]. \tag{27}
\]

where \( N_{\omega} \) is a normalization factor. Eq.(27) is just the unique continuation of Eq.(21) describing an antiparticle state, and now it can also describe the splitting of \( \wedge_{\omega}^{\text{out}} R_{\omega} \) in a wave outgoing from the horizon and a wave falling into the horizon \[13\]. Using the relation for the Bosen particles

\[
< R_{\omega_1} , R_{\omega_2} > = -\delta(\omega_1 - \omega_2) \tag{28}
\]

we obtain

\[
N_{\omega}^2 = (e^{2\pi \omega/a} - 1)^{-1} \tag{29}
\]

Therefore, the temperature is

\[
T = \frac{a}{2\pi} \tag{30}
\]

which implicates that there is true radiation locally detected by the observers near the horizon. And the spectrum is also Planckian, though the effective temperature is obviously low which is the same as many other thermal effects by radiation. Note that \( a \) is a constant correlated with the proper acceleration of the observer \[19\].

IV. CONCLUSION AND DISCUSSION

In the above sections, we have defined some accelerating observers by defining the null hypersurface in the Kruskal coordinates at first. Some properties of these observers such as their 4-velocity, 4-acceleration and particularly the radiation effect are also obtained. However, as mentioned in the title, what we want to discuss are the properties of
accelerating observers in the Schwarzschild space. Thus, we should first check that the observers we defined are truly accelerating in the Schwarzschild space.

In section 2, we have proved that the observers defined in (4) are accelerating observers in the Schwarzschild spacetime. Considering the constant proper acceleration of the static observers in the Schwarzschild space, we can conclude qualitatively that the observers we defined must be accelerating observers in the Schwarzschild space. In fact, if we use the Schwarzschild coordinates \{t, r, \theta, \varphi\} to express the worldline in (4), we can obtain

\[
\left(\frac{r}{2M} - 1\right)^{1/2} e^{\frac{t}{4M}} = X_0 e^{\frac{t}{4M}} \sqrt{X_0^2 + e^{2a_0},}
\]

which shows that the observer defined in (4) is truly an observer accelerated in the space. And it can be viewed as a detector who is at first accelerated closely to the planet, and then gets far away from it (Figure 2). It’s obvious to find that as the same as that in many other similar works, the detected effective temperature in (29) is low to detect in the experiment. However, the theoretical discussion is still of sense, which can be seen in the following.

Property of detecting the radiation for accelerating observers is seemingly not surprising due to the analogy of Unruh effect. However, these observers accelerate in the space, not the spacetime, this may make some difference for the result. For example, though the free drop observer accelerates in the Schwarzschild space, it will detect zero radiation due to the fact that it’s a geodesic observer in the spacetime, while the static observer in the Schwarzschild space will detect the well-known Hawking effect. Of course, the key point of deciding whether there is radiation is due to the 4-acceleration of the observer. However, according to our proof, the more significant key point is whether there exists the corresponding null hypersurface which can be treated as the horizon for the observers. It’s true that, if there is a null hypersurface we can always find the corresponding observers, while the opposite argument may not be correct. Thus, we conjecture that, for any null hypersurface in any spacetime, we can find the corresponding observers who can detect (at least locally) the radiation from it. In fact, in Ref[15], the authors used the similar trick, defining the null hypersurface at first, and then discussed the properties of some accelerating observers in Minkowski spacetime. Their results can be viewed as a partial support to our conjecture. Particularly, in Ref[16], the authors qualitatively obtained the result of the conjecture from the equivalent principle of General Relativity. Moreover, they applied it to define the local temperature and obtained very interesting results.

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FIG. 1: (Color online) Null hypersurface and the corresponding observer by definition.

FIG. 2: (Color online) The behavior of the observer in the Schwarzschild coordinates.