THE LAZER-MCKENNA CONJECTURE FOR AN ANISOTROPIC PLANAR ELLIPTIC PROBLEM WITH EXPONENTIAL NEUMANN DATA

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ABSTRACT. Let \( \Omega \subset \mathbb{R}^2 \) be a bounded smooth domain, we study the following anisotropic elliptic problem

\[
\begin{aligned}
-\nabla (a(x) \nabla \upsilon) + a(x) \upsilon &= 0 \quad \text{in } \Omega, \\
\frac{\partial \upsilon}{\partial \nu} &= e^\upsilon - s \phi_1 - h(x) \quad \text{on } \partial \Omega,
\end{aligned}
\]

where \( \nu \) denotes the outer unit normal vector to \( \partial \Omega \), \( h \in C^{0,\alpha}(\partial \Omega) \) is given, \( a(x) \) is a smooth function over \( \Omega \) satisfying

\[
a_1 \leq a(x) \leq a_2
\]

for some constants \( 0 < a_1 < a_2 < +\infty \), \( \phi_1 \) is a positive first eigenfunction of the Steklov problem (see [2]):

\[
\begin{aligned}
-\nabla (a(x) \nabla \phi) + a(x) \phi &= 0 \quad \text{in } \Omega, \\
\frac{\partial \phi}{\partial \nu} &= \lambda \phi \quad \text{on } \partial \Omega.
\end{aligned}
\]

This paper is concerned with the analysis of solutions to the Neumann boundary value problem

\[
\begin{aligned}
-\nabla (a(x) \nabla \upsilon) + a(x) \upsilon &= 0 \quad \text{in } \Omega, \\
\frac{\partial \upsilon}{\partial \nu} &= e^\upsilon - s \phi_1 - h(x) \quad \text{on } \partial \Omega,
\end{aligned}
\]

as \( s \to +\infty \), where \( \Omega \subset \mathbb{R}^2 \) is a bounded smooth domain, \( \nu \) denotes the outer unit normal vector to \( \partial \Omega \), \( h \in C^{0,\alpha}(\partial \Omega) \) is given, \( a(x) \) is a smooth function over \( \Omega \) satisfying

\[
a_1 \leq a(x) \leq a_2
\]

for some constants \( 0 < a_1 < a_2 < +\infty \), \( \phi_1 \) is a positive first eigenfunction of the Steklov problem (see [2]):

\[
\begin{aligned}
-\nabla (a(x) \nabla \phi) + a(x) \phi &= 0 \quad \text{in } \Omega, \\
\frac{\partial \phi}{\partial \nu} &= \lambda \phi \quad \text{on } \partial \Omega.
\end{aligned}
\]
Obviously, if we denote its positive first eigenvalue as $\lambda_1$ and set $\rho(x) \in H^1(\Omega)$ as a unique solution of
\[
\begin{align*}
-\nabla(a(x)\nabla \rho) + a(x)\rho &= 0 \quad \text{in } \Omega, \\
\frac{\partial \rho}{\partial \nu} &= h(x) \quad \text{on } \partial \Omega,
\end{align*}
\]
then equation (1.1) is equivalent to solving for $u = v + \frac{\omega}{\lambda_1} \phi_1 + \rho$, the problem
\[
\begin{align*}
-\nabla(a(x)\nabla u) + a(x)u &= 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= k(x)e^{-t\phi_1}e^u \quad \text{on } \partial \Omega,
\end{align*}
\]
where $k(x) = e^{-\rho(x)}$ and $t = s/\lambda_1$.

This work directly arises from the study of the corresponding isotropic case of equation (1.1), namely the following elliptic Neumann problem
\[
\begin{align*}
-\Delta v + v &= 0 \quad \text{in } \Omega, \\
\frac{\partial v}{\partial \nu} &= e^v - s\phi_1 - h(x) \quad \text{on } \partial \Omega,
\end{align*}
\]
(1.5)
or its equivalent form
\[
\begin{align*}
-\Delta u + u &= 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= k(x)e^{-t\phi_1}e^u \quad \text{on } \partial \Omega,
\end{align*}
\]
(1.6)
where $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$ $(N \geq 2)$. Simulated by a conjecture in [13] raised by Lazer and McKenna for a Dirichlet problem with the same type of nonlinearity, it is natural to ask if problem (1.5) has an unbounded number of solutions as $s \to +\infty$. When $N = 2$, a recent result in [24] gives a positive answer to this question by constructing non-simple boundary bubbling solutions of equation (1.6) with the following asymptotic behaviors
\[
k(x)e^{-t\phi_1}e^{u_t} \to 2\pi \sum_{i=1}^l m_i \delta_{\xi_i}, \quad \text{and} \quad u_t = \sum_{i=1}^l m_i G_N(x,\xi_i) + o(1),
\]
with some integers $m_i > 1$, where $\xi_i$'s are boundary maxima of $\phi_1$ and $G_N(x,\xi)$ denotes the Green’s function of the Neumann problem
\[
\begin{align*}
-\Delta x G_N(x,\xi) + G_N(x,\xi) &= 0, \quad x \in \Omega, \\
\frac{\partial G_N}{\partial \nu_x}(x,\xi) &= 2\pi \delta_\xi(x), \quad x \in \partial \Omega.
\end{align*}
\]
It is quite surprising that this multiple boundary bubbling phenomenon is in opposition to a slightly modified version of equation (1.6)
\[
\begin{align*}
-\Delta u + u &= 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= \varepsilon e^u \quad \text{on } \partial \Omega,
\end{align*}
\]
(1.7)
as $\varepsilon \to 0^+$, where $\Omega \subset \mathbb{R}^2$ is a bounded smooth domain. Actually, the asymptotic behavior of families of solutions $u_\varepsilon$ of equation (1.7) with $\varepsilon \int_{\partial \Omega} e^{u_\varepsilon}$ bounded is well understood in [9]. It turns out that, up to subsequences, there is an integer $l \geq 1$, such that
\[
\lim_{\varepsilon \to 0^+} \varepsilon \int_{\partial \Omega} e^{u_\varepsilon} = 2\pi l.
\]
(1.8)
More precisely, \( u_\varepsilon \) makes \( l \) different points simple blow up on \( S = \{ \xi_1, \ldots, \xi_l \} \subseteq \partial \Omega \) such that
\[
\varepsilon u_\varepsilon^{\alpha} \to 2\pi \sum_{i=1}^l \delta_{\xi_i} \quad \text{and} \quad u_\varepsilon = \sum_{i=1}^l G_N(x, \xi_i) + o(1).
\]
The location of these boundary bubbling points can be characterized as critical points of a functional in terms of the Green’s function and its regular part. Conversely, solutions of equation (1.7) with this boundary bubbling behavior has also been constructed in [9]. Moreover, it is proven in [9] that, given any integer \( l \geq 1 \), equation (1.7) has at least two distinct families of solutions \( u_\varepsilon \) for which (1.8) holds, and the bubbles of these two solutions are located around two distinct critical points of a functional of \( l \) points of the boundary.

Equation (1.1) is seemingly similar to problem (1.5). Due to the fact that equation (1.1) is a natural generalization of (1.5), one may expect similar results in [24] hold. For this reason our idea here is to consider axially symmetric solutions of (1.5) and then its regular part is given by
\[
\begin{cases}
-\Delta_g G(x, y) + G(x, y) = 0, & x \in \Omega, \\
\frac{\partial G}{\partial t_x}(x, y) = 2\pi \delta_y(x), & x \in \partial \Omega,
\end{cases}
\]
(1.9)
then its regular part is given by
\[
H(x, y) = G(x, y) - 2\log \frac{1}{|x - y|}.
\]
(1.10)
In this way, for any \( x \in \partial \Omega \), \( H(x, \cdot) \in C^\alpha(\Omega) \cap C^2(\Omega \setminus \{x\}) \) for any \( 0 < \alpha < 1 \), and the corresponding Robin function \( x \mapsto H(x, x) \in C^1(\partial \Omega) \) (see [21]). Moreover, by the maximum principle, for any \( x \in \partial \Omega \), \( G(x, \cdot) > 0 \) over \( \Omega \).

Our main result can be stated as follows.
Theorem 1.1. Let $\Lambda$ be a subset of $\partial \Omega$ satisfying $\sup_{\partial \Lambda} a(x) \phi_1 < \sup_{\Lambda} a(x) \phi_1$ and $\sup_{\partial \Omega} \phi_1 < 2 \min_{\partial \Omega} \phi_1$. Then given any positive integer $m$, there exists $t_m > 0$ such that for any $t > t_m$, problem (1.4) has a solution $u_t$ satisfying

$$u_t(x) = \sum_{i=1}^{m} \left[ \log \frac{1}{|x - \xi_{i,t} - \varepsilon_{i,t} \mu_{i,t} \nu(\xi_{i,t})|} + H(x, \xi_{i,t}) \right] + o(1),$$

where $o(1) \to 0$, as $t \to +\infty$, uniformly on each compact subset of $\overline{\Omega} \setminus \{\xi_{i,t}, \ldots, \xi_{m,t}\}$, $\nu(\xi_{i,t})$ denotes the outer unit normal vector to $\partial \Omega$ at $\xi_{i,t}$, the parameters $\varepsilon_{i,t}$ and $\mu_{i,t}$ satisfy

$$\varepsilon_{i,t} = e^{-\tau(\xi_{i,t})}, \quad \frac{1}{C} \leq \mu_{i,t} \leq C t^{(m-1)(m^2+1)\epsilon_2},$$

for some $C > 0$, and $(\xi_{i,t}, \ldots, \xi_{m,t}) \in \Lambda^m$ satisfies

$$\text{dist}(\xi_{i,t}, S) \to 0 \quad \text{for all } i, \quad \text{and} \quad |\xi_{i,t} - \xi_{j,t}| > t^{-\frac{(m^2+1)\epsilon_2}{m^2}} \quad \forall i \neq j,$$

with $S = \{ x \in \Lambda | a(x) \phi_1(x) = \sup_{\Lambda} a(x) \phi_1 \}$.

The immediate consequence for problem (1.1) can be stated as follows.

Theorem 1.2. Let $m$ be a positive integer. Then for any $s$ sufficiently large, there exists a solution $v_s$ of problem (1.1) such that

$$\lim_{s \to +\infty} \int_{\partial \Omega} a(x) e^{v_s} = 2\pi \sum_{i=1}^{l} m_i a(\xi_i),$$

with positive integers $m_1, \ldots, m_l$ satisfying $m = m_1 + \cdots + m_l$, where $\xi_1, \ldots, \xi_l$ are some different maximum points of $a(x) \phi_1$ on $\partial \Omega$. More precisely, given any subset $\Lambda$ of $\partial \Omega$ satisfying $\sup_{\partial \Lambda} a(x) \phi_1 < \sup_{\Lambda} a(x) \phi_1$ and $\sup_{\partial \Omega} \phi_1 < 2 \min_{\partial \Omega} \phi_1$, and a sequence $s \to +\infty$, there is a subsequence and points $\xi_i \in \Lambda$ with $a(\xi_i) \phi_1(\xi_i) = \sup_{\Lambda} a(x) \phi_1$ such that as $s \to +\infty$,

$$a(x) e^{v_s} \to 2\pi \sum_{i=1}^{l} m_i a(\xi_i) \delta_{\xi_i} \quad \text{weakly in the sense of measure on } \partial \Omega,$$

for some positive integers $m_1, \ldots, m_l$ satisfying $m = m_1 + \cdots + m_l$.

From Theorems 1.1 and 1.2 we observe that if $a(x) \phi_1$ has an isolated local boundary maximum point $\xi_0$, then for any positive integer $m$ problem (1.4) has a family of solutions $u_t$ which exhibits an $m$-bubbles concentration at $\xi_0$, namely, $a(x) k(x) e^{-\tau \epsilon_0} e^{v_s} \to 2\pi ma(\xi_0) \delta_{\xi_0}$ and $u_t \to mG(x, \xi_0) + o(1)$.

It is important to remark about the analogy existing between our results and those known for the Ambrosetti-Prodi problem [1]:

$$\begin{cases}
-\Delta u = g(u) - s \phi_1 - h(x) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases} \quad (1.11)$$

as $s \to +\infty$, where $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$ ($N \geq 2$), $h \in C^{0,\alpha}(\overline{\Omega})$ is given, $\phi_1$ is a positive eigenfunction of $-\Delta$ with Dirichlet boundary condition corresponding to the first eigenvalue $\lambda_1$, and $g : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $-\infty \leq \alpha = \lim_{t \to -\infty} \frac{g(t)}{t} < \lim_{t \to +\infty} \frac{g(t)}{t} = \beta \leq +\infty$ and $(\alpha, \beta)$ contains some eigenvalues of $-\Delta$ subject to Dirichlet boundary condition. In the early 1980s Lazer
and McKenna conjectured that if \( \alpha < \lambda_1 < \beta = +\infty \) and \( g(t) \) does not grow too fast at infinity, problem \((1.11)\) has an unbounded number of solutions as \( s \to +\infty \) (see [13]). In fact, it is well known that the Lazer-McKenna conjecture holds true for problem \((1.11)\) with many different types of nonlinearities, which can be found in [10] for the exponential nonlinear case, in [17, 18, 19] for the asymptotically linear case, in [3, 8, 11] for the superlinear homogeneous case, in [4] for the superlinear nonhomogeneous case, in [6, 7] for the subcritical case, and in [5, 12, 15, 16, 22, 23] for the critical case. In particular when \( N = 2 \) and \( g(t) = e^t \), del Pino and Muñoz in [10] proved the Lazer-McKenna conjecture by constructing solutions of problem \((1.11)\) with the accumulation of arbitrarily many bubbles around maximum points of \( \phi_1 \) in the domain.

Let us point out that for problem \((1.5)\) in dimension \( N \geq 3 \), the symmetry of the domain allows us to focus only on equation \((1.1)\) and simplify the computations in a considerable way, but the multiplicity and concentration phenomena of solutions of equation \((1.1)\) that we obtain should be present in a more general domain. We strongly believe that the generalization of the Lazer-McKenna conjecture for problem \((1.5)\) in dimension \( N \geq 3 \) holds true even if we remove the symmetry restriction.

The proof of our results relies on a very well known Lyapunov-Schmidt reduction procedure. The same strategy has been applied in [21] to build solutions for the two-dimensional anisotropic Neumann problem

\[
\begin{align*}
-\nabla(a(x)\nabla u) + a(x)u &= 0 \quad \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= \varepsilon e^u \quad \text{on } \partial\Omega,
\end{align*}
\]

as \( \varepsilon \to 0^+ \), which exhibit multiple bubbling behavior around any strict local maximum point of the uniformly positive, smooth function \( a(x) \) on the boundary. However, in contrast with the result in [21], the location of bubbles in our present work is not been characterized as isolated local boundary maximum points of \( a(x) \), but those of \( a(x)\phi_1 \), which needs us to investigate deeply the effect of the interaction between anisotropic coefficient \( a(x) \) and first positive eigenfunction \( \phi_1 \) on the existence of boundary bubbling solutions. This is the delicate description during we carry out the whole reduction procedure to construct boundary bubbling solutions of equation \((1.1)\).

Throughout this paper, the letters \( c, C \) will always denote generic positive constants independent of \( t \), which could be changed from one line to another.

2. The approximation of the solution. In this section we construct an appropriate approximation for a solution of problem \((1.4)\) and compute the error of the corresponding scaling problem created by the choice of our approximation. The elements for our construction are based on the classification solutions of the following limit equation

\[
\begin{align*}
\Delta u &= 0 \quad \text{in } \mathbb{R}_+^2, \\
\frac{\partial u}{\partial \nu} &= e^u \quad \text{on } \partial\mathbb{R}_+^2, \\
\int_{\mathbb{R}_+^2} e^u &< +\infty,
\end{align*}
\]

where \( \mathbb{R}_+^2 \) denotes the upper half-plane \( \{(x_1, x_2) : x_2 > 0\} \) and \( \nu \) the outer unit normal to \( \partial\mathbb{R}_+^2 \), which are exactly given by a family of functions.
\[ \psi_{\mu, \tau}(x) = \psi_{\mu, \tau}(x_1, x_2) = \log \frac{2\mu}{(x_1 - \tau)^2 + (x_2 + \mu)^2} \]  

(2.2)

with parameters \( \tau \in \mathbb{R} \) and \( \mu > 0 \) (see [14, 20, 25]). Set

\[ \psi_{\mu}(x) := \psi_{\mu, \tau}(x)|_{\tau = 0} = \log \frac{2\mu}{x_1^2 + (x_2 + \mu)^2}. \]  

(2.3)

For notational convenience we fix a subset \( \Lambda \) of \( \partial \Omega \) as in the statement of Theorems 1.1 and 1.2, and further assume

\[ 2 \inf_{x \in \overline{\Lambda}} \phi_1(x) > \sup_{x \in \overline{\Lambda}} \phi_1(x) = 1. \]  

(2.4)

Set \( \xi = (\xi_1, \ldots, \xi_m), \xi_i \in \overline{\Lambda} \subset \partial \Omega \), define

\[ \mathcal{O}_t := \left\{ \xi = (\xi_1, \ldots, \xi_m) \in \overline{\Lambda}^m \mid |\xi_i - \xi_j| \geq \frac{1}{t}, a(\xi_i)\phi_1(\xi_i) \geq \sup_{x \in \overline{\Lambda}} a(x)\phi_1(x) - \frac{1}{\sqrt{t}}, \right. \]

\[ i, j = 1, \ldots, m, \ i \neq j \}, \]  

(2.5)

where \( \beta \) is given by

\[ \beta = \frac{(m^2 + 1)a_2}{2a_1}. \]  

(2.6)

Let us fix \( \xi \in \mathcal{O}_t \). Notice that the function

\[ u(x) = \psi_{\mu}(x/\varepsilon) - 2 \log \varepsilon = \log \frac{2\mu}{x_1^2 + (\varepsilon x_2 + \mu)^2} \]

satisfies

\[ \begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2_+, \\ \frac{\partial u}{\partial \nu} = \varepsilon e^u & \text{on } \partial \mathbb{R}^2_+. \end{cases} \]

Thus, for numbers \( \mu_i > 0, \ i = 1, \ldots, m \) that will be chosen later on, we define

\[ u_i(x) = \log \frac{2\mu_i}{k(\xi_i)|x - \xi_i - \varepsilon_i \mu_i \nu(\xi_i)|^2}, \]  

(2.7)

where

\[ \varepsilon_i = \varepsilon_i(t) = e^{-t\phi_1(\xi_i)}. \]  

(2.8)

We define the approximate solution of problem (1.4) as

\[ U(x) := \sum_{i=1}^{m} U_i(x) = \sum_{i=1}^{m} \left[ u_i(x) + H_i(x) \right], \]  

(2.9)

where \( H_i(x) \) is a correction term defined as the solution of

\[ \begin{cases} -\Delta a H_i + H_i = \nabla \log a(x) \nabla u_i - u_i & \text{in } \Omega, \\ \frac{\partial H_i}{\partial \nu} = \varepsilon_i k(\xi_i) e^{u_i} - \frac{\partial u_i}{\partial \nu} & \text{on } \partial \Omega. \end{cases} \]  

(2.10)

Then \( U_i(x) := u_i + H_i \) satisfies

\[ \begin{cases} -\Delta a U_i + U_i = 0 & \text{in } \Omega, \\ \frac{\partial U_i}{\partial \nu} = \varepsilon_i k(\xi_i) e^{u_i} & \text{on } \partial \Omega. \end{cases} \]  

(2.11)
Lemma 2.1. For any $0 < \alpha < 1$, $\xi = (\xi_1, \ldots, \xi_m) \in O_i$, then we have

$$H_i(x) = H(x, \xi_i) - \log 2\mu_i + \log k(\xi_i) + O(\varepsilon_i^\alpha \mu_i^2),$$

uniformly over $\overline{\Omega}$, where $H$ is the regular part of Green’s function defined in (1.10).

Proof. We first note that, on the boundary, we have

$$\frac{\partial H_i}{\partial \nu} = \varepsilon_i k(\xi_i) e^{u_i} - \frac{\partial u_i}{\partial \nu} = \frac{2\varepsilon_i \mu_i + 2(x - \xi_i - \varepsilon_i \mu_i \nu(\xi_i)) \cdot \nu(x)}{|x - \xi_i - \varepsilon_i \mu_i \nu(\xi_i)|^2}.$$

Then

$$\lim_{\varepsilon_i \mu_i \to 0} \frac{\partial H_i}{\partial \nu}(x) = 2 \frac{(x - \xi_i) \cdot \nu(x)}{|x - \xi_i|^2}, \quad \forall x \neq \xi_i.$$

The regular part of Green’s function $H(x, \xi_i)$ satisfies

$$\begin{cases}
-\Delta_a H(x, \xi_i) + H(x, \xi_i) = \nabla \log \alpha \nabla \log \frac{1}{|x - \xi_i|^2} - \log \frac{1}{|x - \xi_i|^2} & \text{in } \Omega, \\
\frac{\partial H_i}{\partial \nu}(x, \xi_i) = 2 \frac{(x - \xi_i) \cdot \nu(x)}{|x - \xi_i|^2} & \text{on } \partial \Omega.
\end{cases} \tag{2.13}$$

Hence if we set $z(x) = H_i(x) - H(x, \xi_i) + \log 2\mu_i - \log k(\xi_i)$, we have

$$\begin{cases}
-\Delta_a z + z = \log \frac{|x - \xi_i - \varepsilon_i \mu_i \nu(\xi_i)|^2}{|x - \xi_i|^2} - \nabla \log \alpha \nabla \log \frac{|x - \xi_i - \varepsilon_i \mu_i \nu(\xi_i)|^2}{|x - \xi_i|^2} & \text{in } \Omega, \\
\frac{\partial z}{\partial \nu} = \frac{\partial H_i}{\partial \nu} - 2 \frac{(x - \xi_i) \cdot \nu(x)}{|x - \xi_i|^2} & \text{on } \partial \Omega.
\end{cases} \tag{2.14}$$

From Lemma 3.1 of [9] and Appendix of [21], we can easily prove that for $1 < p < 2$,

$$\left\| \log \frac{|x - \xi_i - \varepsilon_i \mu_i \nu(\xi_i)|^2}{|x - \xi_i|^2} \right\|_{L^p(\Omega)} \leq C \varepsilon_i \mu_i,$$

and

$$\left\| \nabla \log \alpha \nabla \log \frac{|x - \xi_i - \varepsilon_i \mu_i \nu(\xi_i)|^2}{|x - \xi_i|^2} \right\|_{L^p(\Omega)} \leq C (\varepsilon_i \mu_i)^\frac{2-p}{p},$$

and for $p > 1$,

$$\left\| \frac{\partial H_i}{\partial \nu} - 2 \frac{(x - \xi_i) \cdot \nu(x)}{|x - \xi_i|^2} \right\|_{L^p(\partial \Omega)} \leq C (\varepsilon_i \mu_i)^\frac{1}{p}.$$

By $L^p$ theory,

$$\|z\|_{W^{1,\kappa, p}(\Omega)} \leq C \left( \| -\Delta_a z + z\|_{L^p(\Omega)} + \left\| \frac{\partial z}{\partial \nu} \right\|_{L^p(\partial \Omega)} \right) \leq C (\varepsilon_i \mu_i)^\frac{2-p}{p}$$

for $0 < \kappa < \frac{1}{p}$. By Morrey embedding we obtain

$$\|z\|_{C^{\gamma}(\overline{\Omega})} \leq C (\varepsilon_i \mu_i)^\frac{2-p}{p}$$

for $0 < \gamma < \frac{1}{2} + \frac{1}{p}$. This proves the result with $\alpha = \frac{2-p}{p}$. \qed

For our approximation to be more accurate, we hope that the remainder $U - u_i = H_i + \sum_{j \neq i} (u_j + H_j)$ becomes small near the point $\xi_i$, which can be achieved by the following precise choice of the parameter $\mu_i$:

$$\log 2\mu_i = \log k(\xi_i) + H(\xi_i, \xi_i) + \sum_{j \neq i}^m G(\xi_i, \xi_j). \tag{2.14}$$
We thus fix \( \mu_i \) \textit{a priori} as a function of \( \xi \) in \( \mathcal{O}_t \) and write \( \mu_i = \mu_i(\xi) \) for all \( i = 1, \ldots, m \). Since \( \xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_t \), there exists a constant \( C > 0 \) independent of \( t \) such that

\[
\frac{1}{C} \leq \mu_i \leq Ct^{2(m-1)\beta} \quad \text{and} \quad |\partial_{\xi_k} \log \mu_i| \leq Ct^\beta, \quad \forall \ i, k = 1, \ldots, m. \tag{2.15}
\]

Consider now the change of variables

\[
\omega(y) = u(\varepsilon y) - 2t \quad \forall \ y \in \overline{\Omega}_t,
\tag{2.16}
\]

with

\[
\varepsilon = \varepsilon(t) \equiv e^{-t} \quad \text{and} \quad \Omega_t = \varepsilon^{-1}\Omega,
\tag{2.17}
\]

then \( u(x) \) solves equation (1.4) if and only if \( \omega(y) \) satisfies

\[
\begin{cases}
-\Delta a(e\xi y)\omega + \varepsilon^2 \omega = -2t\varepsilon^2 & \text{in} \ \Omega_t, \\
\frac{\partial \omega}{\partial \nu} = q(y, t)e^{\omega} & \text{on} \ \partial\Omega_t,
\end{cases}
\tag{2.18}
\]

where

\[
-\Delta a(e\xi y)\omega = -\Delta \omega - \varepsilon \nabla \log a(e\xi y)\nabla \omega \quad \text{and} \quad q(y, t) = k(\varepsilon y) \exp \{ -t[\phi_1(\varepsilon y) - 1] \}. \tag{2.19}
\]

Let us write \( \xi'_i = \xi_i / \varepsilon \) and define the initial approximate solution of (2.18) as

\[
V(y) = U(\varepsilon y) - 2t,
\tag{2.20}
\]

where \( U \) is defined by (2.9). We will seek a solution of problem (2.18) in the form \( \omega = V + \phi \), where \( \phi \) will represent a lower order correction. Problem (2.18) can be stated as to find \( \phi \) a solution to

\[
\begin{cases}
-\Delta a(e\xi y)\phi + \varepsilon^2 \phi = 0 & \text{in} \ \Omega_t, \\
\frac{\partial \phi}{\partial \nu} - W\phi = R + N(\phi) & \text{on} \ \partial\Omega_t,
\end{cases}
\tag{2.21}
\]

where

\[
W(y) = q(y, t)e^{V(y)},
\tag{2.22}
\]

the “error term” is

\[
R(y) = W(y) - \frac{\partial V(y)}{\partial \nu},
\tag{2.23}
\]

and the “nonlinear term” is given by

\[
N(\phi) = q(y, t)e^{V(y)}(e^\phi - 1 - \phi).
\tag{2.24}
\]

Let us measure how well \( V(y) \) solves problem (2.18) so that the “error term” \( R(y) \) is sufficiently small for any \( y \in \partial\Omega_t \). Assume first \( |y - \xi'_i| \leq 1/(\varepsilon t^{2\beta}) \) for some index \( i \). Then we have

\[
\frac{\partial V(y)}{\partial \nu} = \varepsilon \sum_{j=1}^m \left[ \frac{\partial u_j}{\partial \nu} + \frac{\partial H_j}{\partial \nu} \right](x) = \varepsilon \sum_{j=1}^m \varepsilon_j k(\xi_j)e^{u_j}(x) = \sum_{j=1}^m \frac{2\varepsilon \varepsilon_j \mu_j}{\varepsilon y - \xi_j - \varepsilon_j \mu_j \nu(\xi_j)}^2 + \sum_{j \neq i} O(\varepsilon \varepsilon_j \mu_j t^{2\beta}),
\tag{2.25}
\]

and

\[
W(y) = \varepsilon^2 q(y, t) \exp \left\{ \sum_{j=1}^m [u_j(\varepsilon y) + H_j(\varepsilon y)] \right\}
\]
which, together with (2.23) and (2.25), implies that in this region

\[ \mu \]

where

\[ H \]

Hence for

\[ \|H\| \leq \epsilon \]

Using (2.12) and the fact that \( H(\cdot, x) \) is \( C^\alpha(\partial \Omega) \) for any \( x \) in \( \partial \Omega \) and \( \alpha \in (0, 1) \), we get

\[ H_j(\epsilon y) = H(\epsilon y, \xi_j) - \log 2 \mu_j + \log k(\xi_j) + O(\epsilon^\alpha_j \mu_j^\alpha) \]

\[ = H(\xi_j, \xi_j) - \log 2 \mu_j + \log k(\xi_j) + O(\epsilon^\alpha_j |y - \xi_j|^\alpha) + O(\epsilon^\alpha_j \mu_j^\alpha), \forall y \in \Omega. \]  

Hence for \( |y - \xi_j| \leq 1/(\epsilon t^{2\beta}) \),

\[ H_i(\epsilon y) + \sum_{j \neq i}^m \left[ \log \frac{2 \mu_j}{k(\xi_j)|y - \xi_j - \epsilon_j \mu_j \nu(\xi_j)|^2} + H_j(\epsilon y) \right] \]

\[ = H(\xi_i, \xi_i) - \log 2 \mu_i + \log k(\xi_i) \]

\[ + \sum_{j \neq i}^m \left[ \log \frac{1}{\xi_i - \xi_j^2} + H(\xi_i, \xi_j) + O\left( \frac{|y - \xi_i| + \epsilon_i \mu_i}{\xi_i - \xi_j} \right) \right] \]

\[ + O(\epsilon^\alpha_j |y - \xi_j|^\alpha) + \sum_{j=1}^m O(\epsilon^\alpha_j \mu_j^\alpha) \]

\[ = H(\xi_i, \xi_i) - \log 2 \mu_i + \log k(\xi_i) + \sum_{j \neq i}^m G(\xi_i, \xi_i) \]

\[ + O(\epsilon t^{2\beta} |y - \xi_j|) + O(\epsilon^\alpha_j |y - \xi_j|^\alpha) + \sum_{j=1}^m O(\epsilon^\alpha_j \mu_j^\alpha) \]

\[ = O(\epsilon t^{2\beta} |y - \xi_j|) + O(\epsilon^\alpha_j |y - \xi_j|^\alpha) + \sum_{j=1}^m O(\epsilon^\alpha_j \mu_j^\alpha), \]  

where the last equality is due to the choice of \( \mu_i \) in (2.14). Thus if \( |y - \xi_j| \leq 1/(\epsilon t^{2\beta}) \),

\[ W(y) = \frac{2 \gamma_i}{|y - \xi_i' - \gamma_i \nu(\xi_i')|^2} \left\{ 1 + O(\epsilon t^{2\beta} |y - \xi_i'|) + O(\epsilon^\alpha_j |y - \xi_j|^\alpha) + \sum_{j=1}^m O(\epsilon^\alpha_j \mu_j^\alpha) \right\}, \]  

which, together with (2.23) and (2.25), implies that in this region

\[ R(y) = \frac{2 \gamma_i}{|y - \xi_i' - \gamma_i \nu(\xi_i')|^2} \left\{ O(\epsilon t^{2\beta} |y - \xi_j|) + O(\epsilon^\alpha_j |y - \xi_j|^\alpha) \right\} \]
While if \( |y - \xi_0| > 1/(\varepsilon t^{2\beta}) \) for all \( i \), by (2.7) and (2.12) we obtain
\[
\frac{\partial V(y)}{\partial \nu} = \sum_{i=1}^{m} O(\varepsilon \varepsilon_i \mu_i t^{4\beta})
\]
and
\[
W(y) = O(\varepsilon^2 q(y,t) \exp \left\{ \sum_{i=1}^{m} G(\xi_i, \varepsilon y) \right\}) = O(\varepsilon t^{4m\beta} e^{-t\phi_1(\varepsilon y)}).
\]
Hence
\[
R(y) = O(\varepsilon t^{4m\beta} e^{-t\phi_1(\varepsilon y)}) + \sum_{i=1}^{m} O(\varepsilon \varepsilon_i \mu_i t^{4\beta}).
\]

3. Solvability of a linear problem. In this section we consider the solvability of the following linear problem: given \( h \in L^\infty(\partial \Omega_t) \) and points \( \xi = (\xi_1, \ldots, \xi_m) \in \Omega_t \), we find a function \( \phi_1 \), and scalars \( c_1, \ldots, c_m \), such that
\[
\begin{align*}
-\Delta a(y) \phi + \varepsilon^2 \phi &= 0 & \text{in } \Omega_t, \\
\frac{\partial \phi}{\partial \nu} - W \phi &= h + \frac{1}{a(\varepsilon y)} \sum_{i=1}^{m} c_i \chi_i Z_{1i} & \text{on } \partial \Omega_t, \\
\int_{\Omega_t} \chi_i Z_{1i} \phi &= 0 & \forall \ i = 1, \ldots, m,
\end{align*}
\]
where \( W = q(y,t)e^{V(y)} \) satisfies (2.29) and (2.32), and \( Z_{1i}, \chi_i \) are defined as follows: let
\[
Z_0(z) = 1 - 2\frac{z_2 + 1}{z_1^2 + (z_2 + 1)^2} \quad \text{and} \quad Z_1(z) = -2\frac{z_1}{z_1^2 + (z_2 + 1)^2}.
\]
It is well known (see [9]) that any bounded solution to
\[
\begin{align*}
\Delta \phi &= 0 & \text{in } \mathbb{R}^2_+ , \\
\frac{\partial \phi}{\partial \nu} - \frac{2}{z_1^2 + 1} \phi &= 0 & \text{on } \partial \mathbb{R}^2_+ ,
\end{align*}
\]
is a linear combination of \( Z_0 \) and \( Z_1 \). Given \( \xi \in \partial \Omega \), we consider a rotation map \( A_i : \mathbb{R}^2 \to \mathbb{R}^2 \) satisfying \( A_i \nu_\Omega(\xi_i) = \nu_{\Omega_2}(0) \). Let \( d > 0 \) be a small but fixed radius, depending only on the geometry of \( \Omega \), such that
\[
H_i : B_d(0) \cap A_i(\Omega - \{\xi_i\}) \to M \cap \mathbb{R}^2_,
\]
is a \( C^2 \) diffeomorphism satisfying \( H_i(B_d(0) \cap A_i(\partial \Omega - \{\xi_i\})) = M \cap \partial \mathbb{R}^2_+ \), where \( M \) is an open neighborhood of the origin. We can select \( H_i \) so that it preserves area. Then for any \( i = 1, \ldots, m, j = 0, 1, \) we define
\[
H_i^j(y) = \frac{1}{\varepsilon} H_i(A_i(\varepsilon y - \xi_i)) \quad \text{and} \quad Z_{ji}(y) = \frac{1}{\gamma_i} Z_j \left( \frac{1}{\gamma_i} H_i^j(y) \right).
\]
Next, we choose a large but fixed positive number \( R_0 \) and nonnegative smooth function \( \chi : \mathbb{R} \to \mathbb{R} \) such that \( 0 \leq \chi(r) \leq 1 \), \( \chi(r) = 1 \) for \( r \geq R_0 \) and \( \chi(r) = 0 \) for \( r < R_0 \). For \( i = 1, \ldots, m \), we define
\[
\chi_i(y) = \chi \left( \frac{1}{\gamma_i} |H_i^j(y)| \right).
\]
Equation (3.1) will be solved for \( h \in L^\infty(\partial\Omega_t) \), but we will be able to estimate the size of the solution by introducing the following norm:

\[
\|h\|_{*, \partial\Omega_t} = \sup_{y \in \partial\Omega_t} \left( \sum_{i=1}^{m} \frac{\gamma_i^\sigma}{(|y - \xi_i|^2 + \gamma_i)^{1+\sigma}} + \varepsilon \right)^{-1} h(y),
\]

where \( \sigma \) is a small but fixed positive number.

**Proposition 3.1.** Let \( m \) be a positive integer. Then there exist constants \( t_m > 1 \) and \( C > 0 \) such that for any \( t > t_m \), any points \( \xi = (\xi_1, \ldots, \xi_m) \in \Omega_t \) and any \( h \in L^\infty(\partial\Omega_t) \), there is a unique solution \( \phi \in L^\infty(\Omega_t) \), \( c_1, \ldots, c_m \in \mathbb{R} \) to problem (3.1). Moreover

\[
\|\phi\|_{L^\infty(\Omega_t)} \leq Ct\|h\|_{*, \partial\Omega_t} \quad \text{and} \quad |c_i| \leq C\|h\|_{*, \partial\Omega_t}, \quad i = 1, \ldots, m. \tag{3.7}
\]

We carry out the proof in the following steps. 

**Step 1:** Constructing a suitable barrier.

**Lemma 3.2.** There exist positive constants \( R_1 \) and \( C \), independent of \( t \), such that if \( t \) is sufficiently large, there exists \( \psi : \Omega_t \setminus \bigcup_{i=1}^{m} B_{R_1, \gamma_i} (\xi'_i) \to \mathbb{R} \), smooth and positive, satisfying

\[
\begin{cases}
-\Delta_a(\varepsilon y) \psi + \varepsilon^2 \psi \geq \sum_{i=1}^{m} \frac{\gamma_i^\sigma}{|y - \xi_i|^2 + \gamma_i^{1+\sigma}} + \varepsilon^2 & \text{in } \Omega_t \setminus \bigcup_{i=1}^{m} B_{R_1, \gamma_i} (\xi'_i), \\
\partial \psi \over \partial \nu - W \psi \geq \sum_{i=1}^{m} \frac{\gamma_i^\sigma}{|y - \xi_i|^2 + \gamma_i^{1+\sigma}} + \varepsilon & \text{on } \partial\Omega_t \setminus \bigcup_{i=1}^{m} B_{R_1, \gamma_i} (\xi'_i), \\
\psi \geq 1 & \text{on } \Omega_t \cap \bigcup_{i=1}^{m} \partial B_{R_1, \gamma_i} (\xi'_i).
\end{cases}
\]

Moreover, \( \psi \) has a uniform bound,

\[
0 < \psi \leq C \quad \text{in } \Omega_t \setminus \bigcup_{i=1}^{m} B_{R_1, \gamma_i} (\xi'_i). \tag{3.9}
\]

**Proof.** Let \( \eta_i \in C_0^\infty(\mathbb{R}^2) \) be such that \( 0 \leq \eta_i \leq 1, \eta_i \equiv 1 \) in \( \Omega_t \cap B_{d/(2\varepsilon)} (\xi'_i) \), \( \eta_i \equiv 0 \) in \( \Omega_t \setminus B_{d/\varepsilon}(\xi'_i) \), \( \nabla \eta_i \leq C\varepsilon \) in \( \Omega_t \), \( |\Delta \eta_i| \leq C\varepsilon^2 \) in \( \Omega_t \). Let \( \psi_0(y) = \psi_0(\varepsilon y) \), where \( \psi_0 \) is the solution to

\[
\begin{cases}
-\Delta_a \psi_0 + \psi_0 = 1 & \text{in } \Omega, \\
\partial \psi_0 \over \partial \nu = 1 & \text{on } \partial\Omega,
\end{cases}
\]

so that

\[
-\Delta_a(\varepsilon y) \psi_0 + \varepsilon^2 \psi_0 = \varepsilon^2 \quad \text{in } \Omega_t, \quad \text{and} \quad \partial \psi_0 \over \partial \nu = \varepsilon \quad \text{on } \partial\Omega_t.
\]

Clearly, \( \psi_0 \) is uniformly bounded in \( \Omega_t \). Take the function

\[
\psi = \sum_{i=1}^{m} \eta_i \left[ \gamma_i^\sigma \frac{(y - \xi'_i) \cdot \nu(\xi'_i)}{p^{1+\sigma}} + C \left( 1 - \frac{\gamma_i^\sigma}{p^{1+\sigma}} \right) \right] + C \psi_0,
\]

where \( r = |y - \xi'_i - \gamma_i \nu(\xi'_i)| \). Following the proof of Lemma 4.3 from [9], it is easily checked that \( \psi \) meets the required conditions.

**Step 2:** Transferring a linear equation. We first study the linear equation

\[
\begin{cases}
-\Delta_a(\varepsilon y) \phi + \varepsilon^2 \phi = f & \text{in } \Omega_t, \\
\partial \phi \over \partial \nu - W \phi = h & \text{on } \partial\Omega_t,
\end{cases}
\]
where we use \( \| \cdot \|_{*,\partial \Omega_t} \) to estimate \( h \in L^\infty(\partial \Omega_t) \), and for \( f \in L^\infty(\Omega_t) \) we introduce the following norm:

\[
\| f \|_{*,\Omega_t} = \sup_{y \in \Omega_t} \left( \sum_{i=1}^m \frac{\gamma_i^\sigma}{|y - \xi_i|^{2+\sigma} + \varepsilon^2} \right)^{-1} f(y). \tag{3.11}
\]

For solutions of (3.10) satisfying more orthogonality conditions than those in (3.1), we prove the following a priori estimate.

**Lemma 3.3.** There exist \( R_0 > 0 \) and \( t_m > 1 \) such that for any \( t > t_m \) and any solution \( \phi \) of (3.10) with the orthogonality conditions

\[
\int_{\Omega_t} \chi_i Z_{ji} \phi = 0 \quad \forall \ i = 1, \ldots, m, \ j = 0, 1, \tag{3.12}
\]

we have

\[
\| \phi \|_{L^\infty(\Omega_t)} \leq C (\| h \|_{*,\partial \Omega_t} + \| f \|_{*,\Omega_t}), \tag{3.13}
\]

where \( C > 0 \) is independent of \( t \).

**Proof.** Take \( R_0 = 2R_1, \ R_1 \) being the constant of Lemma 3.2. Thanks to the barrier \( \psi \) of that lemma, we have the following maximum principle: if \( \phi \in H^1(\Omega_t \setminus \bigcup_{i=1}^m B_{R_1}(\xi_i^\prime)) \) satisfies

\[
\begin{cases}
-\Delta_{\alpha(\varepsilon)} \phi + \varepsilon^2 \phi \geq 0 & \text{in } \Omega_t \setminus \bigcup_{i=1}^m B_{R_1}(\xi_i^\prime), \\
\frac{\partial \phi}{\partial \nu} - W \phi \geq 0 & \text{on } \partial \Omega_t \setminus \bigcup_{i=1}^m B_{R_1}(\xi_i^\prime), \\
\phi \geq 0 & \text{on } \Omega_t \cap (\bigcup_{i=1}^m \partial B_{R_1}(\xi_i^\prime)).
\end{cases}
\]

then \( \phi \geq 0 \) in \( \Omega_t \setminus \bigcup_{i=1}^m B_{R_1}(\xi_i^\prime) \), see Lemma 3.4 of [21] for a proof.

Let \( f, h \) be bounded and \( \phi \) a solution to (3.10) satisfying (3.12). Consider the "inner norm"

\[
\| \phi \|_i = \sup_{\Omega_t \cap (\bigcup_{i=1}^m B_{R_1}(\xi_i^\prime))} |\phi|,
\]

and take

\[
\hat{\phi} = C_1 \psi (\| \phi \|_i + \| h \|_{*,\partial \Omega_t} + \| f \|_{*,\Omega_t}).
\]

with \( C_1 \) a large constant independent of \( t \). By the above maximum principle we deduce that \( -\hat{\phi} \leq \phi \leq \hat{\phi} \) in \( \Omega_t \setminus \bigcup_{i=1}^m B_{R_1}(\xi_i^\prime) \). Since \( \psi \) is uniformly bounded, we get

\[
\| \phi \|_{L^\infty(\Omega_t)} \leq C (\| \phi \|_i + \| h \|_{*,\partial \Omega_t} + \| f \|_{*,\Omega_t}), \tag{3.14}
\]

for some constant \( C \) independent of \( \phi \) and \( t \).

We prove the lemma by contradiction. Assume that there are sequences of parameters \( t_n \to +\infty \), points \( \xi^n = (\xi_1^n, \ldots, \xi_m^n) \in \mathcal{O}_n \), functions \( \phi_n, \ W_n, \ f_n \) and \( h_n \) with \( \| \phi_n \|_{L^\infty(\Omega_n)} = 1, \ \| h_n \|_{*,\partial \Omega_n} \to 0, \ \| f_n \|_{*,\Omega_n} \to 0 \) such that for each \( n \), \( \phi_n \) solves equation (3.10) and satisfies the orthogonality conditions (3.12). From estimate (3.14) we have that there exists an index \( i \in \{1, \ldots, m\} \) such that \( \sup_{\partial B_{R_1}(\xi_i^\prime)} |\phi_n| \geq c > 0 \) for all \( n \). Set \( \hat{\phi}_i^n(z) = \phi_n((A_i^n)^{-1}z + (\xi_i^n)) \). By the expansion of \( W_n \) in (2.29) and elliptic regularity we deduce that \( \hat{\phi}_i^n \) converges uniformly over compact sets to a bounded nontrivial solution \( \hat{\phi}_i^\infty \) of equation (3.3), which implies \( \hat{\phi}_i^\infty \) is a linear combination of \( Z_0 \) and \( Z_1 \). On the other hand, taking the limit in the orthogonality conditions (3.12), we find that \( \int_{\mathbb{R}^2_x} \chi_{Z_j} \hat{\phi}_i^\infty = 0 \) for \( j = 0, 1 \). This contradicts the fact that \( \hat{\phi}_i^\infty \neq 0 \). \( \Box \)
Step 3: Establishing an a priori estimate for solutions to (3.10) with the orthogonality condition \( \int_{\Omega_i} \chi_i Z_{1i} \phi = 0 \) only.

**Lemma 3.4.** For \( t \) sufficiently large, if \( \phi \) is a solution of (3.10) and satisfies

\[
\int_{\Omega_i} \chi_i Z_{1i} \phi = 0 \quad \forall \ i = 1, \ldots, m, \tag{3.15}
\]

then

\[
\| \phi \|_{L^\infty(\Omega_t)} \leq Ct (\| h \|_{\ast, \partial \Omega_t} + \| f \|_{\ast, \Omega_t}), \tag{3.16}
\]

where \( C > 0 \) is independent of \( t \).

**Proof.** Let \( R > R_0 + 1 \) be large and fixed, \( \delta > 0 \) small but fixed. Denote for \( i = 1, \ldots, m, \)

\[
\hat{Z}_{0i}(y) = Z_{0i}(y) - \frac{1}{\gamma_i} + a_{0i} G(\xi_i, \varepsilon y), \tag{3.17}
\]

where

\[
a_{0i} \equiv \frac{1}{\gamma_i} \left[ H(\xi_i, \xi_i) - 2 \log(\varepsilon \gamma_i R) \right]. \tag{3.18}
\]

From assumption (2.4), estimate (2.15) and definitions (2.8), (2.17) and (2.27) we find

\[
C_1 | \log \varepsilon_i | \leq - \log(\varepsilon \gamma_i R) \leq C_2 | \log \varepsilon_i |, \tag{3.19}
\]

and

\[
\tilde{Z}_{0i}(y) = O \left( \frac{G(\varepsilon y, \xi_i)}{\gamma_i | \log \varepsilon_i |} \right). \tag{3.20}
\]

Let \( \eta_1 \) and \( \eta_2 \) be radial smooth cut-off functions in \( \mathbb{R}^2 \) such that

\[
0 \leq \eta_1 \leq 1; \quad | \nabla \eta_1 | \leq C \text{ in } \mathbb{R}^2; \quad \eta_1 \equiv 1 \text{ in } B_R(0); \quad \eta_1 \equiv 0 \text{ in } \mathbb{R}^2 \setminus B_{R+1}(0);
\]

\[
0 \leq \eta_2 \leq 1; \quad | \nabla \eta_2 | \leq C \text{ in } \mathbb{R}^2; \quad \eta_2 \equiv 1 \text{ in } B_{\frac{1}{4}}(0); \quad \eta_2 \equiv 0 \text{ in } \mathbb{R}^2 \setminus B_{\frac{1}{2}}(0).
\]

Set

\[
\eta_{1i}(y) = \eta_1 \left( \frac{1}{\gamma_i} | H_{\xi_i}(y) | \right), \quad \eta_{2i}(y) = \eta_2 \left( \varepsilon | H_{\xi_i}(y) | \right), \tag{3.21}
\]

and define the test function

\[
\tilde{Z}_{0i}(y) = \eta_{1i} Z_{0i} + (1 - \eta_{1i}) \eta_{2i} \tilde{Z}_{0i}. \tag{3.22}
\]

Given \( \phi \) satisfying (3.10) and (3.15), let

\[
\tilde{\phi} = \phi + \sum_{i=1}^m d_i \tilde{Z}_{0i} + \sum_{i=1}^m e_i \chi_i Z_{1i}. \tag{3.23}
\]

We will first prove the existence of \( d_i \) and \( e_i \) such that \( \tilde{\phi} \) satisfies the orthogonality condition

\[
\int_{\Omega_i} \chi_i Z_{1i} \tilde{\phi} = 0 \quad \forall \ i = 1, \ldots, m, \ j = 0, 1. \tag{3.24}
\]

Notice that the map \( H_{\xi_i}^z \) preserves area and \( \tilde{Z}_{0i} \) coincides with \( Z_{0i} \) in the region \( \{ y \in \Omega : | H_{\xi_i}(y) | \leq \gamma_i R \} \). Then \( \tilde{Z}_{0i} \) is orthogonal to \( \chi_i Z_{1i} \) for each \( i = 1, \ldots, m. \)

Using definition (3.23), orthogonality condition (3.24) for \( j = 1 \) and the fact that \( \chi_i \chi_k \equiv 0 \) if \( i \neq k \), we obtain

\[
e_i = - \int_{\Omega_i} \sum_{k \neq i} d_k \chi_i Z_{1i} \frac{\tilde{Z}_{0i}}{Z_{0i}} / \int_{\Omega_i} \chi_i^2 Z_{1i}^2, \quad i = 1, \ldots, m. \tag{3.25}
\]
By (3.20) and (3.22),
\[ \int_{\Omega_t} \chi_i Z_{1i} \bar{Z}_{0k} dy = O \left( \frac{\gamma_i \log t}{\gamma_k |\log \varepsilon_k|} \right), \quad \forall \, i \neq k, \]
and
\[ \int_{\Omega_t} \chi_i^2 Z_{1i}^2 dy = \int_{\mathbb{R}^2_+} \chi^2(|z|) Z_{1i}^2(z) dz = c > 0, \quad \forall \, i. \]
Then
\[ |e_i| \leq C \sum_{k \neq i}^m |d_k| \frac{\gamma_i \log t}{\gamma_k |\log \varepsilon_k|}. \tag{3.26} \]
We only need to show that \(d_i\) is well defined. Testing definition (3.23) against \(\chi_k Z_{0k}\), we obtain a system of \((d_1, \ldots, d_m)\),
\[ \sum_{i=1}^m d_i \int_{\Omega_t} \chi_k Z_{0k} \bar{Z}_{0i} = -\int_{\Omega_t} \chi_k Z_{0k} \phi, \quad \forall \, k = 1, \ldots, m. \tag{3.27} \]
Note that
\[ \int_{\Omega_t} \chi_k Z_{0k} \bar{Z}_{0i} dy = O \left( \frac{\gamma_k \log t}{\gamma_i |\log \varepsilon_i|} \right), \quad \forall \, i \neq k, \]
and
\[ \int_{\Omega_t} \chi_k Z_{0k} \bar{Z}_{0k} dy = \int_{\Omega_t} \chi_k Z_{0k}^2 dy = \int_{\mathbb{R}^2_+} \chi(|z|) Z_{0k}^2(z) dz = C > 0, \quad \forall \, k. \]
We denote \(H\) the coefficient matrix of system (3.27). By the above estimates, it is clear that \(P^{-1}HP\) is diagonally dominant and thus invertible, where \(P = \text{diag}(\gamma_1, \ldots, \gamma_m)\). Hence \(H\) is also invertible and \((d_1, \ldots, d_m)\) is well defined.

Estimate (3.16) is a direct consequence of the following three claims.

**Claim 1.** Let \(L = -\Delta_{\alpha(y)} + \varepsilon^2\), then for any \(i = 1, \ldots, m,\)
\[ \|L(\chi_i Z_{1i})\|_{*, \Omega_t} \leq C \frac{\gamma_i}{\gamma_i |\log \varepsilon_i|}, \quad \|L(\bar{Z}_{0i})\|_{*, \Omega_t} \leq \frac{C}{\gamma_i |\log \varepsilon_i|}. \tag{3.28} \]

**Claim 2.** Let \(B = \frac{\partial}{\partial n_y} - W\), then for any \(i = 1, \ldots, m,\)
\[ \|B(\chi_i Z_{1i})\|_{*, \partial \Omega_t} \leq C \frac{\gamma_i}{\gamma_i |\log \varepsilon_i|}, \quad \|B(\bar{Z}_{0i})\|_{*, \partial \Omega_t} \leq \frac{C \log t}{\gamma_i |\log \varepsilon_i|}. \tag{3.29} \]

**Claim 3.** For any \(i = 1, \ldots, m,\)
\[ |d_i| \leq C \gamma_i |\log \varepsilon_i| (\|h\|_{*, \partial \Omega_t} + \|f\|_{*, \Omega_t}), \quad |e_i| \leq C \gamma_i \log t (\|h\|_{*, \partial \Omega_t} + \|f\|_{*, \Omega_t}). \tag{3.30} \]

In fact, the definition of \(\bar{\phi}\) in (3.23) tells us
\[ \begin{cases} L(\bar{\phi}) = f + \sum_{i=1}^m d_i L(\bar{Z}_{0i}) + \sum_{i=1}^m e_i L(\chi_i Z_{1i}) & \text{in } \Omega_t, \\ B(\bar{\phi}) = h + \sum_{i=1}^m d_i B(\bar{Z}_{0i}) + \sum_{i=1}^m e_i B(\chi_i Z_{1i}) & \text{on } \partial \Omega_t. \end{cases} \tag{3.31} \]
Since (3.24) holds, by Lemma 3.3 we get
\[ \|\bar{\phi}\|_{L^\infty(\Omega_t)} \leq C \left\{ \|f\|_{*, \Omega_t} + \|h\|_{*, \partial \Omega_t} + \sum_{i=1}^m |d_i| \left( \|L(\bar{Z}_{0i})\|_{*, \Omega_t} + \|B(\bar{Z}_{0i})\|_{*, \partial \Omega_t} \right) + \sum_{i=1}^m |e_i| \left( \|L(\chi_i Z_{1i})\|_{*, \Omega_t} + \|B(\chi_i Z_{1i})\|_{*, \partial \Omega_t} \right) \right\}. \tag{3.32} \]
Furthermore, using the definition of $\tilde{\phi}$ again and the fact that
\begin{equation}
\| \chi_i Z_{1i} \|_{L^\infty(\Omega_i)} \leq \frac{C}{\gamma_i} \quad \text{and} \quad \| \tilde{Z}_{0i} \|_{L^\infty(\Omega_i)} \leq \frac{C}{\gamma_i}, \quad i = 1, \ldots, m, \quad (3.33)
\end{equation}
the Lemma 3.4 then follows from Claims 1-3 and estimate (3.32).

**Proof of Claim 1.** Since $H_i^e(\xi_i^0) = (0, 0)$ and $\nabla H_i^e(\xi_i^0) = A_i$, we have the expansions
\begin{equation}
- \Delta_y = -\Delta_{z_i} + O(\varepsilon |z_i|) \nabla_y^2 + O(\varepsilon) \nabla_y, \quad \nabla_y = A_i \nabla_{z_i} + O(\varepsilon |z_i|) \nabla_{z_i}, \quad (3.34)
\end{equation}
and
\begin{equation}
\frac{\partial}{\partial y_y} = -\frac{\partial}{\partial z_{i2}} + O(\varepsilon |z_i|) \nabla_{z_i}, \quad (3.35)
\end{equation}
where
\begin{equation}
\begin{aligned}
z_i := H_i^e(y) = & \frac{1}{\varepsilon} H_i(A_i(y - \xi_i)) = A_i(y - \xi_i^0) \{ 1 + O(\varepsilon A_i(y - \xi_i)) \}. \quad (3.36)
\end{aligned}
\end{equation}
In the region $|z_i| \leq \gamma_i(R_0 + 1)$,
\begin{equation}
\mathcal{L}(\chi_i Z_{1i}) = (-\Delta_y - \varepsilon \nabla \log a(\varepsilon y) \nabla_y + \varepsilon^2) \left[ \frac{1}{\gamma_i} (\chi Z_1) \left( \frac{z_i}{\gamma_i} \right) \right] = O \left( \frac{1}{\gamma_i} \right),
\end{equation}
and so $\| \mathcal{L}(\chi_i Z_{1i}) \|_{L^\infty, \Omega_i} = O \left( 1/\gamma_i \right)$. We prove now the second inequality in (3.28).

Consider four regions
\begin{equation}
\begin{aligned}
\Omega_1 &= (H_i^e)^{-1} \left( \{|z_i| \leq \gamma_i R \} \cap \mathbb{R}^2_+ \right), \\
\Omega_2 &= (H_i^e)^{-1} \left( \{|z_i| \leq \gamma_i(R + 1) \} \cap \mathbb{R}^2_+ \right), \\
\Omega_3 &= (H_i^e)^{-1} \left( \{|z_i| \leq \delta \varepsilon/4 \} \cap \mathbb{R}^2_+ \right), \\
\Omega_4 &= (H_i^e)^{-1} \left( \{|z_i| \leq \delta \varepsilon/3 \} \cap \mathbb{R}^2_+ \right).
\end{aligned}
\end{equation}

Note first that, by (1.9) and (3.17),
\begin{equation}
\mathcal{L}(Z_{0i} - \tilde{Z}_{0i}) = (-\Delta_{a(\varepsilon y)} + \varepsilon^2) \left[ \frac{1}{\gamma_i} - a_{0i} G(\xi_i, \varepsilon y) \right] = \frac{\varepsilon^2}{\gamma_i}. \quad (3.37)
\end{equation}
Recalling that $\Delta Z_0 = 0$, by (3.34) we get
\begin{equation}
\begin{aligned}
\mathcal{L}(Z_{0i}) &= (-\Delta_y - \varepsilon \nabla \log a(\varepsilon y) \nabla_y + \varepsilon^2) \left[ \frac{1}{\gamma_i} Z_0 \left( \frac{z_i}{\gamma_i} \right) \right] \\
&= \frac{\varepsilon^2}{\gamma_i} Z_0 \left( \frac{z_i}{\gamma_i} \right) + O \left( \frac{\varepsilon |z_i|}{\gamma_i} \right)^2 \nabla^2 \left[ Z_0 \left( \frac{z_i}{\gamma_i} \right) \right] + O \left( \frac{\varepsilon}{\gamma_i} \right) \nabla_{z_i} \left[ Z_0 \left( \frac{z_i}{\gamma_i} \right) \right]. \quad (3.38)
\end{aligned}
\end{equation}
In $\Omega_1$,
\begin{equation}
\mathcal{L}(\tilde{Z}_{0i}) = \mathcal{L}(Z_{0i}) = O \left( \frac{\varepsilon}{\gamma_i^2} \right). \quad (3.39)
\end{equation}
In $\Omega_2$,
\begin{equation}
\begin{aligned}
\mathcal{L}(\tilde{Z}_{0i}) &= \eta_{i1} \mathcal{L}(Z_{0i} - \tilde{Z}_{0i}) + \mathcal{L}(\tilde{Z}_{0i}) - 2 \nabla \eta_{i1} \nabla (Z_{0i} - \tilde{Z}_{0i}) \\
&- (Z_{0i} - \tilde{Z}_{0i}) \Delta_{a(\varepsilon y)} \eta_{i1} \\
&= (\eta_{i1} - 1) \mathcal{L}(Z_{0i} - \tilde{Z}_{0i}) + \mathcal{L}(Z_{0i}) - 2 \nabla \eta_{i1} \nabla (Z_{0i} - \tilde{Z}_{0i}) \\
&- (Z_{0i} - \tilde{Z}_{0i}) \Delta_{a(\varepsilon y)} \eta_{i1}. \quad (3.40)
\end{aligned}
\end{equation}
Note that in $\Omega_t$, by (3.17)-(3.18),
\[
Z_{0t} - \tilde{Z}_{0t} = \frac{1}{\gamma_i} - a_{0t} G(\xi_t, \varepsilon y) = a_{0t} \left[ 2 \log \frac{|y - \xi_t|}{\gamma_i R} + H(\xi_t, \varepsilon y) - H(\xi_t, \varepsilon y) \right],
\]
which, together with (3.19) and the fact that $H(\xi_t, \cdot)$ is $C^\alpha(\overline{\Omega}) \cap C^2(\overline{\Omega} \setminus \{\xi_t\})$ for any $\alpha \in (0, 1)$, implies
\[
|Z_{0t} - \tilde{Z}_{0t}| = O \left( \frac{1}{R \gamma_i |\log \varepsilon|} \right) \quad \text{and} \quad |\nabla (Z_{0t} - \tilde{Z}_{0t})| = O \left( \frac{1}{R \gamma_i^2 |\log \varepsilon|} \right).
\]
Moreover, $|\nabla \eta_{2t}| = O(1/\gamma_t)$ and $|\Delta_a(\varepsilon y) \eta_{2t}| = O(1/\gamma_t^2)$. Hence by (3.37)-(3.38) and (3.40)-(3.42),
\[
\left\| \mathcal{L}(\tilde{Z}_{0t}) \right\|_{L^\infty(\Omega_t)} = O \left( \frac{1}{R \gamma_i^3 |\log \varepsilon|} \right).
\]
In $\Omega$, by (3.22), (3.37) and (3.38),
\[
\mathcal{L}(\tilde{Z}_{0t}) = \mathcal{L}(\tilde{Z}_{0t}) = \mathcal{L}(Z_{0t}) - \frac{\varepsilon^2}{\gamma_i} = O \left( \frac{\varepsilon^2}{|z_i|^2} \right) + O \left( \frac{\varepsilon}{|z_i|} \right).
\]
Finally in $\Omega_t$, since $|\nabla \eta_{2t}| = O(\varepsilon/\delta)$, $|\Delta_a(\varepsilon y) \eta_{2t}| = O(\varepsilon^2/\delta^2)$,
\[
|\tilde{Z}_{0t}| = O \left( \frac{|\log \delta|}{\gamma_i |\log \varepsilon|} \right), \quad |\nabla \tilde{Z}_{0t}| = O \left( \frac{\varepsilon}{\gamma_i \delta |\log \varepsilon|} \right),
\]
and by (3.37)-(3.38),
\[
\mathcal{L}(\tilde{Z}_{0t}) = \mathcal{L}(Z_{0t}) - \frac{\varepsilon^2}{\gamma_i} = O \left( \varepsilon^3 \right),
\]
and hence
\[
\mathcal{L}(\tilde{Z}_{0t}) = \eta_{2t} \mathcal{L}(\tilde{Z}_{0t}) + 2 \nabla \eta_{2t} \nabla \tilde{Z}_{0t} + \tilde{Z}_{0t} \Delta_a(\varepsilon y) \eta_{2t} = O \left( \frac{\varepsilon^2 |\log \delta|}{\gamma_i \delta^2 |\log \varepsilon|} \right).
\]
Combining estimates (3.39), (3.43), (3.44) and (3.47), we arrive at
\[
\left\| \mathcal{L}(\tilde{Z}_{0t}) \right\|_{*, \Omega_t} = O \left( \frac{1}{\gamma_i |\log \varepsilon|} \right).
\]
Proof of Claim 2. Observe first that for any $|z_i| \leq \delta/\varepsilon^{2g}$ and $z_i \in \partial \mathbb{R}_+^2$, by (2.29), (3.35)-(3.36),
\[
\mathcal{B}(Z_{ji}) = \left( -\frac{\partial}{\partial z_{i,2}} - W + O(\varepsilon |z_i|) \nabla z_i \right) \left[ \frac{1}{\gamma_i} Z_j \left( \frac{z_i}{\gamma_i} \right) \right]
\]
\[
= \frac{2}{|z_i - \gamma_i \mu_{k_j} (0)|^2} \left[ O(\varepsilon t^2 |z_i|) + O(\varepsilon^a |z_i|^a) \right] + O \left( \frac{\varepsilon}{\gamma_i} \right),
\]
holds for all $|z_i| \leq \gamma_i (R_0 + 1)$ and $z_i \in \partial \mathbb{R}_+^2$. Hence by (3.6), we get
\[
\| \mathcal{B}(\chi_i Z_{1i}) \|_{*, \partial \Omega_t} = O(1/\gamma_i).
\]
\[
\square
\]
Let us prove the second inequality in (3.29). Consider four regions

\[ B_1 = (H^*_i)^{-1} \left( \{ |z_i| \leq \gamma_i R \} \cap \partial \mathbb{R}^2_+ \right), \]
\[ B_2 = (H^*_i)^{-1} \left( \{ \gamma_i R < |z_i| \leq \gamma_i (R + 1) \} \cap \partial \mathbb{R}^2_+ \right), \]
\[ B_3 = (H^*_i)^{-1} \left( \{ \gamma_i (R + 1) < |z_i| \leq \frac{\delta}{4 \varepsilon} \} \cap \partial \mathbb{R}^2_+ \right), \]
\[ B_4 = (H^*_i)^{-1} \left( \{ \frac{\delta}{4 \varepsilon} < |z_i| \leq \frac{\delta}{3 \varepsilon} \} \cap \partial \mathbb{R}^2_+ \right). \]

Note that

\[ B(\tilde{Z}_{0i}) = \eta_{1i} B(Z_{0i} - \tilde{Z}_{0i}) + \eta_{2i} B(Z_{0i} + \tilde{Z}_{0i}) + \frac{\partial \eta_{1i}}{\partial \nu_y} (Z_{0i} - \tilde{Z}_{0i}) + \frac{\partial \eta_{2i}}{\partial \nu_y} \tilde{Z}_{0i}. \tag{3.49} \]

On $B_1$, by (3.22) and (3.48),

\[ B(\tilde{Z}_{0i}) = B(Z_{0i}) = \sum_{l=1}^m O \left( \frac{\varepsilon_i \mu_i^0}{\gamma_i} \right). \tag{3.50} \]

On $B_2$, due to $\frac{\partial \eta_{1i}}{\partial \nu_y} (\tilde{z}_i) |_{z_i} = 0 = 0$, we have

\[ \frac{\partial \eta_{1i}}{\partial \nu_y} = - \frac{\partial}{\partial \tilde{z}_i} + O(\varepsilon |z_i|) \nabla |_{z_i} \left[ \eta_i \left( \frac{\tilde{z}_i}{\gamma_i} \right) \right] = O (\varepsilon), \]

and then, by (2.29), (3.22), (3.42) and (3.48),

\[ B(\tilde{Z}_{0i}) = B(Z_{0i}) + (1 - \eta_{1i}) W(Z_{0i} - \tilde{Z}_{0i}) + \frac{\partial \eta_{1i}}{\partial \nu_y} (Z_{0i} - \tilde{Z}_{0i}) = O \left( \frac{1}{R^3 \gamma_i^2 \log \varepsilon_i} \right). \tag{3.51} \]

On $B_3$, by (3.22),

\[ B(\tilde{Z}_{0i}) = B(Z_{0i}) = B(Z_{0i}) + W \left[ \frac{1}{\gamma_i} - a_{0i} G(\xi_i, \varepsilon y) \right]. \]

To estimate these two terms, we need to split $B_3$ into several subregions. We set

\[ B_{3,i} = (H^*_i)^{-1} \left( \{ \gamma_i (R + 1) \leq |z_i| \leq \frac{\delta}{4 \varepsilon} \} \cap \partial \mathbb{R}^2_+ \right), \]
\[ B_{3,k} = B_3 \cap (H^*_k)^{-1} \left( \{ \gamma_k \leq \frac{\delta}{4 \varepsilon} \} \cap \partial \mathbb{R}^2_+ \right), k \neq i, \text{ and } \bar{B}_3 = B_3 \setminus \bigcup_{l=1}^m B_{3,l}. \]

By (2.29), (3.32) and (3.48) we get

\[ B(Z_{0i}) = \begin{cases} \frac{2}{|z_i - \gamma_i \nu_{B_2}^0(0)|^2} \left[ O(\varepsilon t^\beta |z_i|) + O(\varepsilon^\alpha |z_i|^\alpha) + \sum_{l=1}^m O(\varepsilon_i \mu_i^0) \right] & \text{on } B_{3,i}, \\ O \left( \gamma_i^{-1} t^4 \varepsilon^\lambda e^{-t \phi(y)} \right) + O(\varepsilon^2 t^4 \beta) & \text{on } \bar{B}_3, \end{cases} \]

Moreover, by (3.18), (3.19) and (3.41),

\[ W \left[ \frac{1}{\gamma_i} - a_{0i} G(\xi_i, \varepsilon y) \right] = \begin{cases} \frac{2}{|z_i - \gamma_i \nu_{B_2}^0(0)|^2} O \left( \frac{\log |z_i| - \log \gamma_i R + \varepsilon^\alpha |z_i|^\alpha}{|\log \varepsilon_i|} \right) & \text{on } B_{3,i}, \\ O \left( \gamma_i^{-1} t^4 \varepsilon^\lambda e^{-t \phi(y)} \right) & \text{on } \bar{B}_3, \end{cases} \]

Hence on $B_{3,i} \cup \bar{B}_3$,

\[ B(\tilde{Z}_{0i}) = B(Z_{0i}) = O \left( \frac{\log |z_i| - \log \gamma_i R}{|z_i - \gamma_i \nu_{B_2}^0(0)|^2} + \frac{1}{|\log \varepsilon_i|} \right). \tag{3.52} \]
On $B_{3,k}$, $k \neq i$, by (2.29) and (3.20),
\[
B(\tilde{Z}_{0i}) = B(\tilde{Z}_{0i}) = \left( -\frac{\partial}{\partial z_{i;2}} + O(\varepsilon|z_i|)\nabla z_i \right) \left[ \frac{1}{\gamma_i} Z_0 \left( \frac{z_i}{\gamma_i} \right) \right] - W \tilde{Z}_{0i}
\]
\[
= O \left( \frac{2}{|z_i - \gamma_i \nu_{\beta_2}(0)|^2} \right) + O \left( \frac{2\gamma_k}{|z_k - \gamma_k \nu_{\beta_2}(0)|^2} \cdot \log t \right) + O (\varepsilon^2 t^{2\beta}) .
\]
Finally, on $B_4$, due to $|\tilde{Z}_{0i}| = O(\frac{\log t}{\gamma_i \log \varepsilon_i})$ and $\frac{\partial \eta_{2i}}{\partial y}$, by (3.22) we get
\[
B(\tilde{Z}_{0i}) = B(\eta_{2i} \tilde{Z}_{0i}) = \eta_{2i} B(\tilde{Z}_{0i}) + \frac{\partial \eta_{2i}}{\partial y} \tilde{Z}_{0i}
\]
\[
= \eta_{2i} \left( -\frac{\partial}{\partial z_{i;2}} + O(\varepsilon|z_i|)\nabla z_i \right) \left[ \frac{1}{\gamma_i} Z_0 \left( \frac{z_i}{\gamma_i} \right) \right] - W \eta_{2i} \tilde{Z}_{0i} + \frac{\partial \eta_{2i}}{\partial y} \tilde{Z}_{0i}
\]
\[
= O \left( \frac{\varepsilon |\log | \gamma_i \delta | \log \varepsilon_i} {\gamma_i \delta | \log \varepsilon_i} \right) .
\]
Putting estimates (3.50)-(3.54), we conclude
\[
\|B(\tilde{Z}_{0i})\|_{*, \partial \Omega_t} = O \left( \frac{\log t}{\gamma_i |\log \varepsilon_i|} \right) .
\]

**Proof of Claim 3.** Testing (3.31) against $a(\varepsilon y) \tilde{Z}_{0i}$ and using estimates (3.28)-(3.29) and (3.32)-(3.33), we obtain
\[
\sum_{k=1}^m d_k \left[ \int_{\Omega_t} a(\varepsilon y) L(\tilde{Z}_{0k}) \tilde{Z}_{0i} + \int_{\partial \Omega_t} a(\varepsilon y) B(\tilde{Z}_{0k}) \tilde{Z}_{0i} \right]
\]
\[
= -\sum_{k=1}^m c_k \left[ \int_{\Omega_t} a(\varepsilon y) \chi_k Z_{1k} L(\tilde{Z}_{0i}) + \int_{\partial \Omega_t} a(\varepsilon y) \chi_k Z_{1k} B(\tilde{Z}_{0i}) \right]
\]
\[
+ \int_{\Omega_t} a(\varepsilon y) \left[ \tilde{\phi}(\tilde{Z}_{0i}) - f \tilde{Z}_{0i} \right] + \int_{\partial \Omega_t} a(\varepsilon y) \left[ \tilde{\phi}(\tilde{Z}_{0i}) - h \tilde{Z}_{0i} \right]
\]
\[
\leq C \frac{1}{\gamma_i} (\|h\|_{*, \partial \Omega_t} + \|f\|_{*, \Omega_t}) + C \frac{\log t}{\gamma_i |\log \varepsilon_i|} \left[ \sum_{k=1}^m \frac{\log t}{\gamma_k |\log \varepsilon_k|} |d_k| + \sum_{k=1}^m \frac{1}{\gamma_k} |c_k| \right].
\]
Furthermore, using estimate (3.26) of $c_k$, we get
\[
|d_i| \int_{\Omega_t} a(\varepsilon y) L(\tilde{Z}_{0i}) \tilde{Z}_{0i} + \int_{\partial \Omega_t} a(\varepsilon y) B(\tilde{Z}_{0i}) \tilde{Z}_{0i} \leq C \frac{1}{\gamma_i} (\|h\|_{*, \partial \Omega_t} + \|f\|_{*, \Omega_t}) + C \frac{\log t}{\gamma_i |\log \varepsilon_i|} \left[ \sum_{k=1}^m \frac{|d_k| \log^2 t}{\gamma_k |\log \varepsilon_k| |\log \varepsilon_k|} \right]
\]
\[
+ C \sum_{k \neq i}^m |d_k| \left[ \int_{\Omega_t} a(\varepsilon y) L(\tilde{Z}_{0i}) \tilde{Z}_{0i} + \int_{\partial \Omega_t} a(\varepsilon y) B(\tilde{Z}_{0i}) \tilde{Z}_{0i} \right].
\]
\[ I = \sum_{i=1}^{4} \left[ \int_{\Omega_i} a(\varepsilon y) \mathcal{L}(\tilde{Z}_{0i}) \tilde{Z}_{0i} + \int_{B_i} a(\varepsilon y) B(\tilde{Z}_{0i}) \tilde{Z}_{0i} \right] \]
\[ = \sum_{i=1}^{4} (I_1 + I_2). \tag{3.56} \]

From (3.39), we have
\[ I_1 = \int_{\Omega_1} a(\varepsilon y) \mathcal{L}(Z_{0i}) Z_{0i} = \int_{\{ |z_i| \leq \gamma_i R \} \cap \mathbb{R}^2_+} \mathcal{O} \left( \frac{\varepsilon}{\gamma_i} \right) \frac{1}{\gamma_i} Z_{0i} \left( \frac{z_i}{\gamma_i} \right) = \mathcal{O} \left( \frac{\varepsilon}{\gamma_i} \right). \tag{3.57} \]

From (3.17)-(3.19) and (3.44), we deduce
\[ I_3 = \int_{\Omega_3} a(\varepsilon y) \mathcal{L}(\tilde{Z}_{0i}) \tilde{Z}_{0i} \]
\[ = \int_{\Omega_3} a(\varepsilon y) \mathcal{L}(\tilde{Z}_{0i}) \left\{ Z_{0i} - a_{0i} \left[ 2 \log \frac{|y - \xi_i|}{\gamma_i R} + \mathcal{O} (\varepsilon^a |y - \xi_i|)^a \right] \right\} \]
\[ = \int_{\{ \gamma_i (R+1) \leq |z_i| \leq \frac{\varepsilon}{\gamma_i} \} \cap \mathbb{R}^2_+} \mathcal{O} \left( \frac{\varepsilon}{|z_i|^2} + \frac{\varepsilon^2}{|z_i|} \right) \]
\[ \times \left\{ \frac{1}{\gamma_i} + \mathcal{O} \left( \frac{1}{|z_i|} + \log \frac{|z_i|}{\gamma_i R} + \mathcal{O} (\varepsilon^a |z_i|^a) \right) \right\} \]
\[ = \mathcal{O} \left( \frac{\varepsilon \log \varepsilon_i}{\gamma_i} \right). \tag{3.58} \]

From (3.45) and (3.47), we derive that
\[ I_4 = \int_{\Omega_4} a(\varepsilon y) \mathcal{L}(\tilde{Z}_{0i}) \eta_{0i} \tilde{Z}_{0i} = \int_{\{ \frac{\varepsilon}{\gamma_i} < |z_i| \leq \frac{\varepsilon}{\gamma_i} \} \cap \mathbb{R}^2_+} \mathcal{O} \left( \frac{\varepsilon^2 |\delta|^2}{\gamma_i^2 |\log \varepsilon_i|^2} \right) d z_i \]
\[ = \mathcal{O} \left( \frac{|\log \delta|^2}{\gamma_i^2 |\log \varepsilon_i|^2} \right). \tag{3.59} \]

Regarding the expression \( I_2 \), we readily have
\[ I_2 = - \int_{\Omega_2} a(\varepsilon y) \tilde{Z}_{0i} (Z_{0i} - \tilde{Z}_{0i}) \Delta a(\varepsilon y) \eta_{1i} - \int_{\Omega_2} 2a(\varepsilon y) \tilde{Z}_{0i} \nabla \eta_{1i} \nabla (Z_{0i} - \tilde{Z}_{0i}) \]
\[ + \int_{\Omega_2} a(\varepsilon y) \tilde{Z}_{0i} [\eta_{1i} \mathcal{L}(Z_{0i} - \tilde{Z}_{0i}) + \mathcal{L}(\tilde{Z}_{0i})]. \]

Integrating by parts the first term and using estimates (3.37)-(3.38) for the last term, we find
\[ I_2 = - \int_{\Omega_2} a(\varepsilon y) \tilde{Z}_{0i} \nabla \eta_{1i} \nabla (Z_{0i} - \tilde{Z}_{0i}) + \int_{\Omega_2} a(\varepsilon y) (Z_{0i} - \tilde{Z}_{0i})^2 |\nabla \eta_{1i}|^2 \]
\[ + \int_{\Omega_2} a(\varepsilon y) (Z_{0i} - \tilde{Z}_{0i}) \nabla \eta_{1i} \nabla \tilde{Z}_{0i} + \mathcal{O} \left( \frac{\varepsilon R}{\gamma_i} \right) \]
\[ = I_{21} + I_{22} + I_{23} + \mathcal{O} \left( \frac{\varepsilon R}{\gamma_i} \right). \tag{3.60} \]
From (3.2), (3.4), (3.34) and (3.42), we get $|\nabla \eta_{1i}| = O\left(\frac{1}{\gamma_i}\right)$ and $|\nabla \hat{Z}_{0i}| = O\left(\frac{1}{\gamma_i^{1/2}|\log \varepsilon_i|}\right)$ in $\Omega_2$. Furthermore,

$$I_{22} = O\left(\frac{1}{R^{3\gamma_i^2}|\log \varepsilon_i|}\right) \quad \text{and} \quad I_{23} = O\left(\frac{1}{R^{3\gamma_i^2}|\log \varepsilon_i|}\right).$$  \hspace{1cm} (3.61)

Note that $a(\varepsilon y) = a(\xi_i)\left[1 + O\left(\varepsilon |z_i|\right)\right]$ and $\hat{Z}_{0i} = Z_{0i}\left[1 + O\left(\frac{1}{R|\log \varepsilon_i|}\right)\right]$ in $\Omega_2$. By (3.19), (3.34), (3.36) and (3.41), we conclude

$$I_{21} = -\frac{a_0i}{\gamma_i^2} \int_{\{\gamma_i R < |z_i| \leq \gamma_i(R + 1)\} \cap \partial \Omega^2_i} \frac{2}{z_i} a(\varepsilon y) \eta_1^i \left(\frac{|z_i|}{\gamma_i}\right) Z_0 \left(\frac{z_i}{\gamma_i}\right) \left(1 + o(1)\right) dz_i = -\frac{2\pi a_0i}{\gamma_i^2} \int_R^{R + 1} a(\xi_i) \eta_1^i(r) \left[1 + O\left(\frac{1}{r}\right)\right] dr = \frac{\pi a(\xi_i)}{\gamma_i^2 |\log \varepsilon_i|} \left[1 + O\left(\frac{1}{R}\right)\right].$$  \hspace{1cm} (3.62)

On the other hand, by (3.50) we have

$$J_1 = \int_{B_1} a(\varepsilon y) B(Z_{0i}) Z_{0i} = \int_{\{|z_i| \leq \gamma_i R\} \cap \partial \Omega^2_i} \frac{1}{\gamma_i} Z_0 \left(\frac{z_i}{\gamma_i}\right) \sum_{l=1}^m O\left(\frac{\varepsilon^0 \mu_l^0}{\gamma_i^2}\right) dz_{i,1} = \sum_{l=1}^m O\left(\frac{\varepsilon^0 \mu_l^0}{\gamma_i^2}\right).$$ \hspace{1cm} (3.63)

By (3.42) and (3.51), we get

$$J_2 = \int_{B_2} a(\varepsilon y) B(\hat{Z}_{0i}) \left[Z_{0i} - (1 - \eta_{1i})(Z_{0i} - \hat{Z}_{0i})\right]$$

$$\quad = \int_{\{\gamma_i R < |z_i| \leq \gamma_i(R + 1)\} \cap \partial \Omega^2_i} O\left(\frac{1}{R^{3\gamma_i^2}|\log \varepsilon_i|}\right)$$

$$\quad \times \left[\frac{1}{\gamma_i} Z_0 \left(\frac{z_i}{\gamma_i}\right) + O\left(\frac{1}{R\gamma_i|\log \varepsilon_i|}\right)\right] dz_{i,1} = O\left(\frac{1}{R^{3\gamma_i^2}|\log \varepsilon_i|}\right).$$ \hspace{1cm} (3.64)

By (3.17), (3.20), (3.52) and (3.53), we derive that

$$J_3 = \int_{B_3, k \neq i} a(\varepsilon y) B(\hat{Z}_{0i}) \eta_{2i} \hat{Z}_{0i} + \sum_{k \neq i} \int_{B_3, k} a(\varepsilon y) B(\hat{Z}_{0i}) \hat{Z}_{0i} = O\left(\frac{1}{R^{3\gamma_i^2}|\log \varepsilon_i|}\right).$$ \hspace{1cm} (3.65)

By (3.54), we deduce

$$J_4 = \int_{B_4} a(\varepsilon y) B(\eta_{2i} \hat{Z}_{0i}) \eta_{2i} \hat{Z}_{0i} = \int_{\{\Delta \varepsilon i < |z_i| \leq \Delta \varepsilon i\} \cap \partial \Omega^2_i} O\left(\frac{\varepsilon \log \delta^2}{\gamma_i^2 |\log \varepsilon_i|^2}\right) dz_{i,1}$$

$$= O\left(\frac{|\log \delta^2}{\gamma_i^2 |\log \varepsilon_i|^2}\right).$$ \hspace{1cm} (3.66)

Combining all these estimates, we conclude that for $R$ and $t$ large enough, and $\delta$ small enough,

$$\int_{\Omega_i} a(\varepsilon y) L(\hat{Z}_{0i}) \hat{Z}_{0i} + \int_{\partial \Omega_i} a(\varepsilon y) B(\hat{Z}_{0i}) \hat{Z}_{0i} = \frac{\pi a(\xi_i)}{\gamma_i^2 |\log \varepsilon_i|} [1 + o(1)].$$ \hspace{1cm} (3.67)
According to (3.55), we just need consider \( \int_{\Omega_t} a(\varepsilon y) \mathcal{L}(\tilde{Z}_{0i}) \tilde{Z}_{0k} + \int_{\partial \Omega_t} a(\varepsilon y) \mathcal{B}(\tilde{Z}_{0i}) \tilde{Z}_{0k} \). Indeed, using the above estimates of \( \mathcal{L}(\tilde{Z}_{0i}), \mathcal{B}(\tilde{Z}_{0i}) \) and \( \tilde{Z}_{0k} \), we readily get

\[
\int_{\Omega_t} a(\varepsilon y) \mathcal{L}(\tilde{Z}_{0i}) \tilde{Z}_{0k} = O\left( \frac{\varepsilon \log t}{\gamma_k |\log \varepsilon|} \right),
\]

\[
\int_{\Omega_t} a(\varepsilon y) \mathcal{L}(\tilde{Z}_{0i}) \tilde{Z}_{0k} = O\left( \frac{\log t}{\gamma_i \gamma_k |\log \varepsilon_i||\log \varepsilon_k|} \right),
\]

\[
\int_{\Omega_t} a(\varepsilon y) \mathcal{L}(\tilde{Z}_{0i}) \tilde{Z}_{0k} = O\left( \frac{\varepsilon |\log \varepsilon_i|}{\gamma_k} \right),
\]

\[
\int_{\Omega_t} a(\varepsilon y) \mathcal{L}(\tilde{Z}_{0i}) \tilde{Z}_{0k} = O\left( \frac{|\log \delta|^2}{\gamma_i \gamma_k |\log \varepsilon_i||\log \varepsilon_k|} \right),
\]

and

\[
\int_{B_{1}} a(\varepsilon y) \mathcal{B}(\tilde{Z}_{0i}) \tilde{Z}_{0k} = \sum_{l=1}^{m} O\left( \frac{\varepsilon \mu^2 \log t}{\gamma_i \gamma_k |\log \varepsilon|} \right),
\]

\[
\int_{B_{2}} a(\varepsilon y) \mathcal{B}(\tilde{Z}_{0i}) \tilde{Z}_{0k} = O\left( \frac{\log t}{\gamma_i \gamma_k |\log \varepsilon_i||\log \varepsilon_k|} \right),
\]

\[
\int_{B_{3}} a(\varepsilon y) \mathcal{B}(\tilde{Z}_{0i}) \tilde{Z}_{0k} = O\left( \frac{|\log \delta|^2}{\gamma_i \gamma_k |\log \varepsilon_i||\log \varepsilon_k|} \right),
\]

\[
\int_{B_{1} \cup B_{3}} a(\varepsilon y) \mathcal{B}(\tilde{Z}_{0i}) \tilde{Z}_{0k} = O\left( \frac{\log t}{\gamma_i \gamma_k |\log \varepsilon_i||\log \varepsilon_k|} \right),
\]

\[
\int_{B_{1,i}} a(\varepsilon y) \mathcal{B}(\tilde{Z}_{0i}) \tilde{Z}_{0k} = O\left( \frac{\log^2 t}{\gamma_i \gamma_k |\log \varepsilon_i||\log \varepsilon_k|} \right) \text{ for any } l \neq i.
\]

As a consequence, from all above estimates we have that if \( k \neq i \),

\[
\int_{\Omega_t} a(\varepsilon y) \mathcal{L}(\tilde{Z}_{0i}) \tilde{Z}_{0k} + \int_{\partial \Omega_t} a(\varepsilon y) \mathcal{B}(\tilde{Z}_{0i}) \tilde{Z}_{0k} = O\left( \frac{\log^2 t}{\gamma_i \gamma_k |\log \varepsilon_i||\log \varepsilon_k|} \right). \tag{3.68}
\]

Substituting (3.67) and (3.68) into (3.55), we conclude

\[
\frac{|d_i|}{\gamma_i} \leq C |\log \varepsilon_i| (\|h\|_{\ast,\partial \Omega_t} + \|f\|_{\ast,\Omega_t}) + C \sum_{k=1}^{m} \frac{|d_k|}{\gamma_k} \frac{\log^2 t}{|\log \varepsilon_k|},
\]

and then, by (2.8), \( |d_i| \leq C \gamma_i |\log \varepsilon_i| (\|h\|_{\ast,\partial \Omega_t} + \|f\|_{\ast,\Omega_t}) \). Furthermore, by (3.26) we get

\[
|e_i| \leq C \gamma_i \log t (\|h\|_{\ast,\partial \Omega_t} + \|f\|_{\ast,\Omega_t}). \quad \square
\]

**Step 4:**

**Proof of Proposition 3.1.** To prove the solvability of problem (3.1) we consider first a related problem: given \( h \in L^\infty(\partial \Omega_t) \) and points \( \xi = (\xi_1, \ldots, \xi_m) \in \Omega_t \), we find a function \( \phi \in L^\infty(\Omega_t) \), and scalars \( f_1, \ldots, f_m \in \mathbb{R} \), such that

\[
\begin{cases}
-\Delta_{a(\varepsilon y)} \phi + \varepsilon^2 \phi = \frac{1}{a(\varepsilon y)} \sum_{i=1}^{m} f_i \chi_i Z_{1i} \quad \text{in } \Omega_t, \\
\frac{\partial \phi}{\partial \nu} - W \phi = h \quad \text{on } \partial \Omega_t, \\
\int_{\Omega_t} \chi_i Z_{1i} \phi = 0 \quad \forall \ i = 1, \ldots, m.
\end{cases} \tag{3.69}
\]
Let us first prove that for any \( \phi \in L^\infty(\Omega_t) \), \( f_1, \ldots, f_m \in \mathbb{R} \) solution to (3.69), the bound
\[
\|\phi\|_{L^\infty(\Omega_t)} \leq Ct\|h\|_{*, \partial\Omega_t}
\] (3.70)
holds. Indeed, from Lemma 3.4 and the fact that \( \|\chi_i Z_{1i}\|_{*, \Omega_t} \leq C\gamma_t \), we have
\[
\|\phi\|_{L^\infty(\Omega_t)} \leq Ct\left(\|h\|_{*, \partial\Omega_t} + \sum_{i=1}^m \gamma_i |f_i|\right),
\] (3.71)
and therefore it is enough to prove that \( |f_i| \leq C\gamma_t^{-1} \|h\|_{*, \partial\Omega_t} \).

Let \( \eta_{2i} \) be the cut-off function defined in (3.21). Testing equation (3.69) against \( a(\varepsilon y)\eta_{2i} Z_{1i} \), we find
\[
\sum_{k=1}^m f_k \int_{\Omega_t} \eta_{2i} Z_{1i} \chi_k Z_{1k} = \int_{\Omega_t} a(\varepsilon y) \phi \mathcal{L}(\eta_{2i} Z_{1i}) + \int_{\partial\Omega_t} a(\varepsilon y) \left[ \phi \mathcal{B}(\eta_{2i} Z_{1i}) - h\eta_{2i} Z_{1i} \right].
\] (3.72)
It is easy to see that \( \int_{\partial\Omega_t} a(\varepsilon y) h\eta_{2i} Z_{1i} = O \left( \gamma_t^{-1} \|h\|_{*, \partial\Omega_t} \right) \). By (3.34) and (3.36), we get
\[
\mathcal{L}(\eta_{2i} Z_{1i}) = (-\Delta z_i + O(\varepsilon|z_i|) \nabla z_i) + O(\varepsilon) \nabla z_i + O(\varepsilon|z_i|)
\]
\[
= O \left( \frac{\varepsilon}{\gamma_t + |z_i|^2} \right) + O \left( \frac{\varepsilon^2}{\gamma_t + |z_i|} \right) + O(\varepsilon^3). \]

Then
\[
\left| \int_{\Omega_t} a(\varepsilon y) \phi \mathcal{L}(\eta_{2i} Z_{1i}) \right| \leq C\varepsilon t \|\phi\|_{L^\infty(\Omega_t)}. \] (3.73)

By (3.35), we can compute
\[
\mathcal{B}(\eta_{2i} Z_{1i}) = \left( -\frac{\partial}{\partial z_{i2}} - W + O(\varepsilon|z_i|) \right) \left[ \eta_{2i} (\varepsilon z_i) \frac{1}{\gamma_t} Z_{1i} \left( \frac{z_i}{\gamma_t} \right) \right]
\]
\[
= \left( \frac{2\gamma_t}{|z_i - \gamma_t z_2(0)|^2} - W \right) \left[ \eta_{2i} (\varepsilon z_i) \frac{1}{\gamma_t} Z_{1i} \left( \frac{z_i}{\gamma_t} \right) \right] + O \left( \frac{\varepsilon}{\gamma_t + |z_i|} \right)
\]
\[
= \mathcal{B}_1 + O \left( \frac{\varepsilon}{\gamma_t + |z_i|} \right). \]

To estimate \( \mathcal{B}_1 \), we divide \( \text{supp}(\eta_{2i}) \cap \partial\Omega_t \) into some pieces:
\[
\mathcal{B}_{1k} = \text{supp}(\eta_{2i}) \cap (H_k^*)^{-1} \left( \{|z_k| \leq \delta/(\varepsilon t^{2\beta})\} \cap \partial\mathbb{R}^2_+ \right), \quad \forall \ k = 1, \ldots, m,
\]
\[
\mathcal{B}_2 = (\text{supp}(\eta_{2i}) \cap \partial\Omega_t) \setminus \bigcup_{k=1}^m \mathcal{B}_{1k}, \]
where \( \text{supp}(\eta_{2i}) \cap \partial\Omega_t = (H_k^*)^{-1} \left( \{|z_k| \leq \frac{\delta}{\varepsilon t^{2\beta}}\} \cap \partial\mathbb{R}^2_+ \right) \). Observe that, by (2.5) and (3.36),
\[
|z_i| \geq C|y - \xi'_k| \geq C(|\xi'_k - \xi_k| - |y - \xi'_k|) \geq C (|\xi'_k - \xi_k| - 2|z_k|)
\]
\[
\geq C \left( |\xi'_k - \xi_k| - \frac{2\delta}{\varepsilon t^{2\beta}} \right) \geq C \frac{1 - \frac{2\delta}{t^{2\beta}}}{\varepsilon t^{3\beta}}, \] (3.74)
uniformly on $\hat{B}_{1 k}, k \neq i$. From expansion (2.29) of $W$ we get, on $\hat{B}_{1 i}$,

$$B_i = \frac{2 \gamma_i}{|z_i - \gamma_i \nu_{k^2}(0)|^3} \left\{ O(\varepsilon t^3 |z_i|) + O(\varepsilon |z_i|^\alpha) + \sum_{j=1}^{m} O(\varepsilon_j^\alpha \mu_j^\alpha) \right\},$$

and on $\hat{B}_{1 k}, k \neq i$, by (3.74),

$$B_i = \left[ O\left(\frac{\gamma_i}{|z_i|^2}\right) + O\left(\frac{2 \gamma_k}{|z_k - \gamma_k \nu_{k^2}(0)|^2}\right) \right] O\left(\frac{1}{|z_i|}\right).$$

But on $\hat{B}_2$, owing to (2.32),

$$B_i = O\left(\varepsilon^2 \varepsilon_i \mu_i t^6 \beta \right) + O\left(\varepsilon^2 t^{(4m+2)\beta} e^{-t \phi_1(z)}\right).$$

Hence by (2.15), (2.17) and (2.27),

$$\int_{\partial \Omega_1} a(\varepsilon \gamma_i) \phi \beta(\eta_{2k} Z_{1k}) \leq C \frac{1}{\gamma_i} \max_k \{\varepsilon \gamma_k\} \|\phi\|_{L^\infty(\Omega_i)}.$$  \hfill (3.75)

Note that

$$\int_{\Omega_i} \eta_{2k} Z_{1k} \chi_k Z_{1k} = \left\{ \begin{array}{ll} \int_{\mathbb{R}_2^2} \chi(|z|) Z_k^2(z) dz & \text{if } k = i, \\ O\left(\gamma_k e^{t \beta}\right) & \text{if } k \neq i. \end{array} \right.$$  \hfill (3.76)

As a consequence, using the above estimates in (3.72), we obtain

$$|f_i| \leq C \left( \frac{1}{\gamma_i} \max_k \{\varepsilon \gamma_k\} \|\phi\|_{L^\infty(\Omega_i)} + \frac{1}{\gamma_i} \|h\|_{*, \partial \Omega_i} + \sum_{k \neq i} \gamma_k e^{t \beta} |f_k| \right),$$

and then

$$|f_i| \leq C \frac{1}{\gamma_i} \left( \max_k \{\varepsilon \gamma_k\} \|\phi\|_{L^\infty(\Omega_i)} + \|h\|_{*, \partial \Omega_i} \right).$$

Putting this estimate in (3.71), we conclude

$$|f_i| \leq C \frac{1}{\gamma_i} \|h\|_{*, \partial \Omega_i}.$$  \hfill (3.77)

Now consider the Hilbert space $K_\xi = \{\phi \in H^1(\Omega_i) : \int_{\Omega_i} \chi_i Z_{1k} \phi = 0, \forall i = 1, \ldots, m\}$ with the norm $\|\phi\|^2_{K_\xi} = \int_{\Omega_i} a(\varepsilon \gamma_i) (|\nabla \phi|^2 + \varepsilon^2 \phi^2)$. Equation (3.69) is equivalent to find $\phi \in K_\xi$, such that

$$\int_{\Omega_i} a(\varepsilon \gamma_i) (|\nabla \phi|^2 + \varepsilon^2 \phi^2) = \int_{\partial \Omega_i} a(\varepsilon \gamma_i) W \phi = \int_{\partial \Omega_i} a(\varepsilon \gamma_i) h \phi \quad \forall \psi \in K_\xi.$$

By Fredholm’s alternative this is equivalent to the uniqueness of solutions to this problem, which is guaranteed by (3.70).

To show solvability of problem (3.1), let $Y_i \in L^\infty(\Omega_i), c_{i1}, \ldots, c_{im} \in \mathbb{R}$ be the solution of equation (3.69) with $h = \frac{1}{a(\varepsilon \gamma_i)} \chi_i Z_{1k}$, that is,

$$\begin{cases} -\Delta a(\varepsilon \gamma_i) Y_i + \varepsilon^2 Y_i = \frac{1}{a(\varepsilon \gamma_i)} \sum_{k=1}^{m} c_{kk} \chi_k Z_{1k} \quad & \text{in } \Omega_i, \\ \partial Y_i / \partial \nu - W Y_i = \frac{1}{a(\varepsilon \gamma_i)} \chi_i Z_{1i} \quad & \text{on } \partial \Omega_i, \\ \int_{\Omega_i} \chi_k Z_{1k} Y_i = 0 \quad & \forall k = 1, \ldots, m. \end{cases}$$  \hfill (3.78)
By the above argument there exist a unique solution \( Y_t \in L^\infty(\Omega_t) \), and \( c_{ik} \in \mathbb{R} \), \( k = 1, \ldots, m \), to this equation such that
\[
\|Y_t\|_{L^\infty(\Omega_t)} \leq Ct \quad \text{and} \quad |c_{ik}| \leq \frac{C}{\gamma_k}. \tag{3.79}
\]
We claim that there exist a constant \( A \), independent of \( t \), such that
\[
c_{ik} = \frac{1}{\gamma_k} A \delta_{ik} + O \left( \frac{t}{\gamma_k} \max_i \{ (\varepsilon \gamma_l)^\alpha \} \right), \tag{3.80}
\]
where \( \delta_{ik} \) denotes Kronecker’s symbol. Indeed, testing equation (3.78) against \( a(\varepsilon y) \eta_{2k} Z_{1k} \) we find
\[
\int_{\Omega_t} a(\varepsilon y) Y_t \mathcal{L}(\eta_{2k} Z_{1k}) + \int_{\partial \Omega_t} a(\varepsilon y) Y_t \mathcal{B}(\eta_{2k} Z_{1k})
= c_{ik} \int_{\Omega_t} \chi k Z_{1k}^2 + \int_{\partial \Omega_t} \chi l \eta_{2k} Z_{1k} + \sum_{l \neq k} c_{il} \int_{\Omega_t} \eta_{2k} Z_{1l} \chi l Z_{1l}.
\]
This, combined with estimates (3.73), (3.75), (3.76) and (3.79), gives
\[
c_{ik} \int_{\mathbb{R}^n} \chi Z_{1k}^2 + \frac{1}{\gamma_k} \int_{\mathbb{R}^n} \chi Z_{1k}^2 = O \left( \frac{t}{\gamma_k} \max_i \{ (\varepsilon \gamma_l)^\alpha \} \right) + O (\varepsilon t^2),
\]
which implies the validity of expansion (3.80). Hence the matrix \( D \) with entries \( \gamma_k c_{ik} \) is invertible for large \( t \) and \( \|D^{-1}\| \leq C \) uniformly on \( t \).

Given \( h \in L^\infty(\partial \Omega_t) \) and points \( \xi = (\xi_1, \ldots, \xi_m) \in \Omega_t \), we have \( \phi_1 \in L^\infty(\Omega_t) \), \( f_1, \ldots, f_m \in \mathbb{R} \) as the solution of equation (3.69), and define
\[
\phi = \phi_1 + \sum_{i=1}^m c_i Y_i,
\]
where \( c_i \) satisfies \( \sum_{i=1}^m c_i c_{ik} = -f_k \) for any \( k = 1, \ldots, m \). Then \( \phi \) satisfies equation (3.1) and that by estimates (3.70) and (3.77), we have
\[
\|\phi\|_{L^\infty(\Omega_t)} \leq \|\phi_1\|_{L^\infty(\Omega_t)} + Ct \sum_{i=1}^m |c_i|
\leq Ct\|h\|_{L^\infty(\partial \Omega_t)} + Ct \sum_{k=1}^m \gamma_k |f_k| \leq Ct\|h\|_{L^\infty(\partial \Omega_t)}.
\]

**Remark 3.1.** A slight modification of the above proof also shows that for any \( h \in L^\infty(\partial \Omega_t) \) and \( f \in L^\infty(\Omega_t) \) the equation
\[
\begin{align*}
\mathcal{L}(\phi) = -\Delta_{a(\varepsilon y)} \phi + \varepsilon^2 \phi &= f \quad \text{in} \quad \Omega_t, \\
\mathcal{B}(\phi) = \frac{\partial \phi}{\partial \nu} - W \phi &= h + \frac{1}{a(\varepsilon y)} \sum_{i=1}^m c_i \chi_i Z_{1i} \quad \text{on} \quad \partial \Omega_t, \tag{3.81} \\
f_{ik} \chi_i Z_{1k} \phi &= 0 \quad \forall \quad i = 1, \ldots, m,
\end{align*}
\]
has a unique solution \( \phi, c_1, \ldots, c_m \) and that the following estimates hold:
\[
\|\phi\|_{L^\infty(\Omega_t)} \leq Ct (\|h\|_{L^\infty(\partial \Omega_t)} + |f|_{L^\infty(\Omega_t)}), \quad |c_i| \leq C \left( \|h\|_{L^\infty(\partial \Omega_t)} + |f|_{L^\infty(\Omega_t)} \right), \quad i = 1, \ldots, m. \tag{3.82}
\]

The result of Proposition 3.1 implies that the unique solution \( \phi = T(h) \) of (3.1) defines a continuous linear map from the Banach space \( C_* \) of all functions \( h \) in \( L^\infty \) for which \( \|h\|_{L^\infty(\partial \Omega_t)} < \infty \), into \( L^\infty \).
Lemma 3.5. The operator $T$ is differentiable with respect to the variables $\xi = (\xi_1, \ldots, \xi_m)$ in $\Omega$, precisely for any $k = 1, \ldots, m$,

$$
\|\partial_{\xi_k} T(h)\|_{L^\infty(\Omega_\xi)} \leq C t^2 \|h\|_{\ast, \partial \Omega_t}.
$$

(3.83)

Proof. Differentiating equation (3.1) with respect to $\xi_k$, formally $Z = \partial_{\xi_k} \phi$ should satisfy

$$
\begin{cases}
\mathcal{L}(Z) = 0 & \text{in } \Omega_t, \\
B(Z) = \phi \partial_{\xi_k} W + \frac{1}{a(x,y)} \sum_{i=1}^m c_i \partial_{\xi_k} (\chi_i Z_{1i}) + \bar{c}_i \chi_i Z_{1i} & \text{on } \partial \Omega_t,
\end{cases}
$$

with (still formally) $\bar{c}_i = \partial_{\xi_k} c_i$, and the orthogonality conditions become

$$
\int_{\partial \Omega_t} \chi_i Z_{11} Z = - \int_{\partial \Omega_t} \phi \partial_{\xi_k} (\chi_i Z_{1i}), \quad i = 1, \ldots, m.
$$

Let us consider the constants $b_i$ defined as

$$
b_i \int_{\Omega_t} \chi_i^2 |Z_{1i}|^2 = \int_{\Omega_t} \phi \partial_{\xi_k} (\chi_i Z_{1i}),
$$

and the functions

$$
a = \sum_{i=1}^m b_i \mathcal{L}(\chi_i Z_{1i}), \quad b = \phi \partial_{\xi_k} W + \sum_{i=1}^m b_i B(\chi_i Z_{1i}) + \frac{1}{a(x,y)} \sum_{i=1}^m c_i \partial_{\xi_k} (\chi_i Z_{1i}).
$$

We first estimate $\partial_{\xi_k} W$. Note that $\partial_{\xi_k} W = W \partial_{\xi_k} V$. By (2.29), (2.32) and (3.6) we clearly have that $\|W\|_{\ast, \partial \Omega_t} = O(1)$. Similar to the proof in Lemma 2.1, we can also derive that $\partial_{\xi_k} H_i(x) = O((\varepsilon \gamma_k)^{\alpha})$ uniformly over $\Omega$. Furthermore, by (2.9) and (2.20) we get

$$
\partial_{\xi_k} V(y) = -Z_{1k}(y) + O((\varepsilon \gamma_k)^{\alpha}).
$$

(3.84)

This, combined with the fact that $\frac{1}{\gamma_k} \leq C$ uniformly on $t$, gives

$$
\|\partial_{\xi_k} V\|_{L^\infty(\Omega_\xi)} = O(1) \quad \text{and} \quad \|\partial_{\xi_k} W\|_{\ast, \partial \Omega_t} = O(1).
$$

(3.85)

Next, by definitions (3.4)-(3.6), a straightforward computation shows that

$$
\|b\|_{\ast, \partial \Omega_t} \leq C t \|h\|_{\ast, \partial \Omega_t} \quad \text{and} \quad \|b\|_{\ast, \partial \Omega_t} \leq C t \|h\|_{\ast, \partial \Omega_t}.
$$

(3.86)

Define

$$
\bar{Z} = Z + \sum_{i=1}^m b_i \chi_i Z_{1i}.
$$

We then have

$$
\begin{cases}
\mathcal{L}(ar{Z}) = a & \text{in } \Omega_t, \\
B(\bar{Z}) = b + \frac{1}{a(x,y)} \sum_{i=1}^m \bar{c}_i \chi_i Z_{1i} & \text{on } \partial \Omega_t, \\
\int_{\Omega_t} \chi_i Z_{1i} \bar{Z} = 0 & \forall i = 1, \ldots, m.
\end{cases}
$$
The remark above implies this equation has a unique solution $\tilde{Z}$, $\tilde{c}_1, \ldots, \tilde{c}_m$, and thus $\partial_{\xi}^k T(h) = \tilde{Z} - \sum_{i=1}^m b_i \chi_i Z_{1i}$ is well defined. From (3.82) and (3.86) we conclude

$$\|\partial_{\xi}^k T(h)\|_{L^\infty(\Omega_t)} \leq \|\tilde{Z}\|_{L^\infty(\Omega_t)} + \sum_{i=1}^m \frac{C|b_i|}{\gamma_i}$$

$$\leq C (t\|a\|_{\ast, \Omega_t} + t\|b\|_{\ast, \partial\Omega_t} + \|\phi\|_{L^\infty(\Omega_t)}) \leq C t^2 \|h\|_{\ast, \partial\Omega_t}. \quad \square$$

4. **The intermediate nonlinear problem.** Consider the intermediate nonlinear problem:

$$\begin{aligned}
-\Delta a(\varepsilon y) \phi + \varepsilon^2 \partial_\nu \phi &= 0 \quad \text{in } \Omega_t, \\
\frac{\partial \phi}{\partial \nu} - W \phi &= R + N(\phi) + \frac{1}{a(\varepsilon y)} \sum_{i=1}^m c_i \chi_i Z_{1i} \quad \text{on } \partial \Omega_t, \\
\int_{\Omega_t} \chi_i Z_{1i} \phi &= 0 \quad \forall \ i = 1, \ldots, m,
\end{aligned} \tag{4.1}$$

where $W$ is as in (2.29) and (2.32), and $R$, $N(\phi)$ are given by (2.23) and (2.24), respectively. For this problem we have the following result.

**Proposition 4.1.** Let $m$ be a positive integer. Then there exist constants $t_m > 1$ and $C > 0$ such that for any $t > t_m$ and any points $\xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_t$, problem (4.1) admits a unique solution $\phi \in L^\infty(\Omega_t)$, and scalars $c_1, \ldots, c_m \in \mathbb{R}$, such that

$$\|\phi\|_{L^\infty(\Omega_t)} \leq Ct \max_i \{(\varepsilon \gamma_i)^{a}\}. \tag{4.2}$$

Furthermore, the map $\xi' \rightarrow \phi(\xi') \in C(\overline{\Omega}_t)$ is $C^1$, precisely for $k = 1, \ldots, m$,

$$\|\partial_{\xi}^k \phi\|_{L^\infty(\Omega_t)} \leq Ct^2 \max_i \{(\varepsilon \gamma_i)^{a}\}, \tag{4.3}$$

where $\xi' := (\xi_1', \ldots, \xi_m') = (\frac{1}{\varepsilon} \xi_1, \ldots, \frac{1}{\varepsilon} \xi_m)$.

**Proof.** Proposition 3.1 and Lemma 3.5 allow us to apply the contraction mapping principle and the implicit function theorem to find a unique solution for problem (4.1) satisfying (4.2) and (4.3). Since it is a standard procedure, we shall not present the detailed proof here, see Lemma 5.1 of [9] for a proof. But we just mention that here $\|R\|_{\ast, \partial \Omega_t} = O \left( \max_i \{ (\varepsilon \gamma_i)^{a}\} \right)$ and $\|\partial_{\xi}^k R\|_{\ast, \partial \Omega_t} = O \left( \max_i \{ (\varepsilon \gamma_i)^{a}\} \right)$ due to (2.30), (2.33) and (3.84).

5. **Variational reduction.** After problem (4.1) has been solved, we find a solution of problem (2.21) and hence to the original problem (1.4) if we find $\xi'$ such that the coefficient $c_i(\xi')$ in (4.1) satisfies

$$c_i(\xi') = 0 \quad \text{for all } i = 1, \ldots, m. \tag{5.1}$$

This problem is indeed variational: it is equivalent to finding critical points of a function of $\xi = \varepsilon \xi'$. Associated to (1.4), let us consider the energy functional $J_t$ given by

$$J_t(u) = \frac{1}{2} \int_{\Omega} a(x) (|\nabla u|^2 + u^2) - \int_{\partial \Omega} a(x) k(x) e^{-t \phi_1} e^u \quad \text{for } u \in H^1(\Omega), \tag{5.2}$$

and the finite-dimensional restriction

$$F_t(\xi) = J_t(U(\xi) + \tilde{\phi}(\xi)) \quad \forall \xi \in \mathcal{O}_t, \tag{5.3}$$
where $U$ is the function defined in (2.9) and $\hat{\phi}(\xi)(x) = \phi(\bar{x}, \hat{\xi})$, $x \in \Omega$, with $\phi = \phi_C$, the unique solution to problem (4.1) given by Proposition 4.1. Critical points of $F_t$ correspond to solutions of (5.1) for large $t$, as the following result states.

**Proposition 5.1.** $F_t : \mathcal{O}_t \to \mathbb{R}$ is of class $C^1$. Moreover, for all $t$ sufficiently large, if $D_t F_t(\xi) = 0$, then $\xi$ satisfies (5.1).

**Proof.** A direct consequence of the results obtained in Proposition 4.1 and the definition of function $U(\xi)$ is the fact that the map $\xi \mapsto F_t(\xi)$ is of class $C^1$. Let

$$I_t(\omega) = \frac{1}{2} \int_{\Omega_t} a(\varepsilon(\xi)) (|\nabla \omega|^2 + \varepsilon^2 \omega^2) - \int_{\partial \Omega_t} a(\varepsilon(\xi)) k(\varepsilon(\xi)) e^{-t(\phi_1(\varepsilon(\xi)) - 1)} e\omega. \quad (5.4)$$

Observe that

$$F_t(\xi) = I_t(U(\xi) + \hat{\phi}(\xi)) = I_t(V(\xi') + \phi_C'). \quad (5.5)$$

From (4.1) we derive that

$$\partial_{\xi_k} F_t(\xi) = \frac{1}{\varepsilon} D_I t(V(\xi') + \phi_C') \left[ \partial_{\xi_k} V(\xi') + \partial_{\xi_k} \phi_C' \right]$$

$$= \frac{1}{\varepsilon} \sum_{i=1}^m c_i(\xi') \int_{\partial \Omega_t} \chi_i Z_{1i} \left[ \partial_{\xi_k} V(\xi') + \partial_{\xi_k} \phi_C' \right].$$

Assume that $D_t F_t(\xi) = 0$. Then for all $k = 1, \ldots, m$,

$$\sum_{i=1}^m c_i(\xi') \int_{\partial \Omega_t} \chi_i Z_{1i} \left[ \partial_{\xi_k} V(\xi') + \partial_{\xi_k} \phi_C' \right] = 0.$$

Since $\|\partial_{\xi_k} \phi_C'\|_{L^\infty(\Omega_t)} = O(t^2 \max \{ (\varepsilon \gamma_i)^\alpha \})$ and $\partial_{\xi_k} V(\xi') = -Z_{1k} + O((\varepsilon \gamma_k)^\alpha)$, by (2.15) and (2.27) we get the validity of a system of equations of the form

$$\sum_{i=1}^m c_i(\xi') \int_{\partial \Omega_t} \chi_i Z_{1i} (-Z_{1k} + o(1)) = 0 \quad \text{for all } k = 1, \ldots, m. \quad (5.6)$$

Note that

$$\int_{\partial \Omega_t} \chi_i Z_{1i} Z_{1k} = \begin{cases} 1 & i = k, \\ \frac{1}{\kappa k} \int_{\partial \Omega_t^2} \chi(|z|) Z_i^2(z) dz & i \neq k, \end{cases} \quad \text{if } i = k,$$

$$O(\varepsilon \gamma_i^\alpha) \quad \text{if } i \neq k.$$

This implies that system (5.6) is strictly diagonal dominant, and thus $c_i(\xi') = 0$ for all $i = 1, \ldots, m$. \hfill $\square$

In order to find critical points of the function $F_t$, we need to give its expected closeness to the function $J_t(U)$, which will be applied in the next section.

**Proposition 5.2.** The following expansion holds

$$F_t(\xi) = J_t(U(\xi)) + \hat{\theta}_t(\xi), \quad (5.7)$$

where $|\hat{\theta}_t| + \|\nabla \hat{\theta}_t\| = o(1)$, uniformly on points $\xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_t$ as $t \to +\infty$.

**Proof.** Observe that, by (5.4)-(5.5),

$$F_t(\xi) - J_t(U(\xi)) = I_t(V(\xi') + \phi_C') - I_t(V(\xi')).$$

Using $DI_t[V + \phi_C'](\phi_C') = 0$, a Taylor expansion and an integration by parts give

$$F_t(\xi) - J_t(U(\xi)) = \int_0^1 D^2 I_t(V + \kappa \phi_C') \phi_C' \|2(1 - \kappa) d\kappa.$$
Proof. Observe that from (2.7) and (2.12) we get

\[ \expansion\] uniformly for points \( \xi \).

3. Expansion of the energy. With the choice of \( \xi \), the continuity in view of

\[ \max \{ (\epsilon \gamma_i)^a \} \]

Let us differentiate the representation (5.8) with respect to

\[ \partial \]

in the

\[ \| \phi \|_{\text{L}^\infty(\Omega_t)} \]

by (2.4), (2.8), (2.15) and (2.27) we get

\[ \partial \xi, \theta \] (6.1)

The continuity in \( \xi \) of all these expressions is inherited from that of \( \phi \) and its derivatives in \( \xi \) in the \( \text{L}^\infty \) norm.

6. Expansion of the energy.

Proposition 6.1. With the choice of \( \mu_i \)'s given by (2.14), the following asymptotic expansion holds

\[ \expansion \] uniformly for points \( \xi = (\xi_1, \ldots, \xi_m) \in \mathcal{O}_t \) and for \( t \) large enough.

Proof. Observe that

\[ \frac{1}{2} \int_{\partial \Omega} a(x) (|\nabla U|^2 + U^2) = \frac{1}{2} \int_{\partial \Omega} a(x)U \frac{\partial U}{\partial \nu} = \frac{1}{2} \sum_{i,j=1}^{m} \int_{\partial \Omega} a(x)U_j \frac{\partial U_i}{\partial \nu} \]

\[ \expansion \]

From (2.7) and (2.12) we get

\[ \expansion \]

so by (2.8), (2.15) and (2.27), we get

\[ \expansion \]

in view of

\[ \expansion \]
Changing variables $\varepsilon_i \mu_i z = A_i (x - \xi_i)$, we deduce
\[
\int_{\partial \Omega} \varepsilon_i k(\xi_i) a(x) e^{\nu_i}(u_j + H_j)
= \int_{\partial \Omega_{\varepsilon_i \mu_i}} \frac{2a (\xi_i + \varepsilon_i \mu_i A_i^{-1} z)}{|z - \nu_{R_i^2}(0)|^2} \left[ \log \frac{1}{|\xi_i - \xi_j + \varepsilon_i \mu_i A_i^{-1} z - \varepsilon_j \mu_j \nu(\xi)|^2} 
+ H(\xi_i + \varepsilon_i \mu_i A_i^{-1} z, \xi_j) + O(\varepsilon_i^\alpha \mu_i^\alpha) \right] \, dz,
\]
where $\partial \Omega_{\varepsilon_i \mu_i} = \frac{1}{\varepsilon_i \mu_i} A_i (\partial \Omega - \{\xi_i\})$. But
\[
\int_{\partial \Omega_{\varepsilon_i \mu_i}} a(\xi_i + \varepsilon_i \mu_i A_i^{-1} z) \frac{2}{|z - \nu_{R_i^2}(0)|^2} = 2 \pi a(\xi_i) + O(\varepsilon_i \mu_i),
\]
\[
\int_{\partial \Omega_{\varepsilon_i \mu_i}} a(\xi_i + \varepsilon_i \mu_i A_i^{-1} z) \frac{2}{|z - \nu_{R_i^2}(0)|^2} \log \frac{1}{|z - \nu_{R_i^2}(0)|^2} = - 4 \pi a(\xi_i) \log 2 + O(\varepsilon_i^\alpha \mu_i^\alpha),
\]
and
\[
\int_{\partial \Omega_{\varepsilon_i \mu_i}} a(\xi_i + \varepsilon_i \mu_i A_i^{-1} z) \frac{2}{|z - \nu_{R_i^2}(0)|^2} [H(\xi_i + \varepsilon_i \mu_i A_i^{-1} z, \xi_j) - H(\xi_i, \xi_j)]
= O(\varepsilon_i^\alpha \mu_i^\alpha).
\]

Then
\[
\int_{\partial \Omega} \varepsilon_i k(\xi_i) a(x) e^{\nu_i}(u_j + H_j)
= \begin{cases} 
2 \pi a(\xi_i) [H(\xi_i, \xi_i) - 2 \log(2\varepsilon_i \mu_i)] + O(\varepsilon_i^\alpha \mu_i^\alpha) & \forall \, i = j, \\
2 \pi a(\xi_i) G(\xi_i, \xi_j) + O(\varepsilon_i^\alpha \mu_i^\alpha + \varepsilon_j^\alpha \mu_j^\alpha) & \forall \, i \neq j.
\end{cases}
\]

Therefore,
\[
\frac{1}{2} \int_{\Omega} a(x) (|\nabla U|^2 + U^2)
= \pi \sum_{i=1}^m a(\xi_i) \left[ H(\xi_i, \xi_i) - 2 \log(2\varepsilon_i \mu_i) + \sum_{j \neq i}^m G(\xi_i, \xi_j) \right] + o(1). \quad (6.2)
\]

On the other hand, using (2.19), (2.20), (2.22) and the change of variables $x = \varepsilon y = e^{-t} y$, we get
\[
\int_{\partial \Omega} a(x) k(x) e^{-\phi_i(x)} e^{U(x)}
= \int_{\partial \Omega_{\varepsilon y}} a(\varepsilon y) k(\varepsilon y) e^{-t\phi_i(\varepsilon y) - 1} e^{U(\varepsilon y) - 2t} \, dy
= \int_{\partial \Omega_{\varepsilon y} \cup \bigcup_{i=1}^m B_{\frac{1}{\varepsilon y} \xi_i}} a(\varepsilon y) W dy + \sum_{i=1}^m \int_{\partial \Omega_{\varepsilon y} \cap B_{\frac{1}{\varepsilon y} \xi_i}} a(\varepsilon y) W dy.
\]

From (2.29) and (2.32) we obtain
\[
\int_{\partial \Omega_{\varepsilon y} \cup \bigcup_{i=1}^m B_{\frac{1}{\varepsilon y} \xi_i}} a(\varepsilon y) W dy
= \int_{\partial \Omega_{\varepsilon y} \cup \bigcup_{i=1}^m B_{\frac{1}{\varepsilon y} \xi_i}} \left( \varepsilon t^4 m^\beta e^{-t \phi_i(\varepsilon y)} \right) \, dy = o(1),
\]
for $\varepsilon y = e^{-t} y$. Therefore, we have
\[
\int_{\partial \Omega} a(x) k(x) e^{-\phi_i(x)} e^{U(x)}
= \int_{\partial \Omega_{\varepsilon y} \cup \bigcup_{i=1}^m B_{\frac{1}{\varepsilon y} \xi_i}} a(\varepsilon y) W dy + \sum_{i=1}^m \int_{\partial \Omega_{\varepsilon y} \cap B_{\frac{1}{\varepsilon y} \xi_i}} a(\varepsilon y) W dy.
\]
and
\[ \int_{\partial O_t \cap B_{1+\varepsilon}^a(\xi)} a(\varepsilon y) W dy = \int_{\partial O_t \cap B_{1+\varepsilon}^a(\xi)} \frac{2\gamma a(\varepsilon y)}{|y - \xi'| - \gamma \nu(\xi')^2} \left\{ 1 + O(\varepsilon t^{\beta} |y - \xi'| + \varepsilon^\alpha |y - \xi'|^{\alpha}) + o(1) \right\} dy = 2\pi a(\xi_t) + o(1). \]

Then
\[ \int_{\partial O} a(x) k(x) e^{-t\phi_1(x)} e^{U(x)} = O(1), \quad (6.3) \]

Hence by (5.2), (6.2) and (6.3), we derive that
\[ J_t(U(\xi)) = \pi \sum_{i=1}^{m} a(\xi_i) H(\xi_i, \xi_i) - 2 \log(2\varepsilon_i \mu_i) + \sum_{\substack{j \neq i}} G(\xi_i, \xi_j) + O(1), \]

which, together with the definition of \( \varepsilon_i \) in (2.8) and the choice of \( \mu_i \) in (2.14), implies (6.1) holds.

7. Proof of Theorem 1.1. We recall the definition of \( O_t \) in (2.5) and assume that \( \Lambda \) is a subset of \( \partial \Omega \) satisfying \( \sup_{\partial \Omega} a(x) \phi_1 < \sup_{\Lambda} a(x) \phi_1 \) and \( \sup_{\Lambda} \phi_1 < 2 \inf_{\Lambda} \phi_1 \). Then we have

**Lemma 7.1.** For any \( t > 1 \) sufficiently large, the following maximization problem
\[ \max_{(\xi_1, \ldots, \xi_m) \in \Omega_t} F_t(\xi_1, \ldots, \xi_m) \]

has a solution in the interior of \( O_t \).

**Proof.** Let \( \xi_t = (\xi_{1,t}, \ldots, \xi_{m,t}) \in \Omega_t \) be the maximizer of \( F_t \). We need to prove that \( \xi_t \) belongs to the interior of \( O_t \). First, we obtain a lower bound for \( F_t \) over \( \Omega_t \). Let us fix a point \( \bar{x} \in S = \{ x \in \Lambda | a(x) \phi_1(x) = \sup_{\Lambda} a(x) \phi_1 \} \). Around the point \( \bar{x} \in \partial \Omega \), we consider a small but fixed number \( d \) and a smooth change of variables
\[ H_\pi^t(y) = e^t H_\pi(e^{-t} y), \]

where \( H_\pi : B_d(\bar{x}) \to M \) is a diffeomorphism and \( M \) is an open neighborhood of the origin such that \( H_\pi(B_d(\bar{x}) \cap \Omega) = M \cap \mathbb{R}^{2+t} \) and \( H_\pi(B_d(\bar{x}) \cap \partial \Omega) = M \cap \partial \mathbb{R}^{2+t} \). For each \( i = 1, \ldots, m \), let
\[ \xi_i^0 = e^{-t} H_\pi^{-1} \left( \frac{e^t \xi_i^t}{\sqrt{t}} \right) \]

where \( \xi_i^t \in M \cap \partial \mathbb{R}^{2+t} \) satisfies \( |\xi_i^0 - \xi_{i+1}^0| = \rho \) for any \( \rho > 0 \) sufficiently small, fixed and independent \( t \). Using the expansion \( (H_\pi^{-1})^{-1}(y) = e^t \bar{x} + y + O(e^{-t}|y|) \), we find
\[ \xi_i^0 = \bar{x} + \frac{1}{\sqrt{t}} \xi_i^t + O \left( \frac{e^{-t}}{\sqrt{t}} |\xi_i^t| \right). \]

Obviously, \( \xi_i^0 = (\xi_{i,0}^0, \ldots, \xi_{m,0}^0) \in O_t \) because \( \beta > 1 \) and \( a(\xi_i^0) \phi_1(\xi_i^0) = a(\bar{x}) \phi_1(\bar{x}) + O(t^{-1}) \). From expansion (6.1) and Proposition 5.2, we obtain
\[ \max_{\xi \in \Omega_t} F_t(\xi) \geq J_t(U(\xi^0)) + \theta_t(\xi^0). \]
rest of the properties of $u$ in Proposition 4.1. Hence by the definition of $U_{\xi}$ problem (1.4) if
\[ \frac{\partial}{\partial t} u = g(t) \]
which contradicts to (7.1). In the second case, we have
\[ u \]
 guarantees the existence of such a critical point and therefore a solution
According to Proposition 5.1, Proof of Theorem 1.1.

\[ \text{max}_{\xi \in \overline{\Omega}} F(t) = F(t) \leq 2\pi t \left[ (m-1)a(\bar{x})\phi_1(\bar{x}) + a(\bar{x})\phi_1(\bar{x}) - \frac{1}{\sqrt{t}} \right] + O(\log t) \]
which contradicts to (7.1). In the second case, we have
\[ \text{max}_{\xi \in \overline{\Omega}} F(t) = F(t) \leq 2\pi t a(\bar{x})\phi_1(\bar{x}) - 2\pi a(\xi_{0,t}) \log t + O(1). \]
Comparing with (7.1), we derive that
\[ 2\pi a(\xi_{0,t}) \log t + O(1) \leq (m-1) \left[ a(\xi_1) + \cdots + a(\xi_m) \right] \log t + O(1), \]
which is impossible by the property of $a(x)$ in (1.2) and the choice of $\beta$ in (2.6).

**Proof of Theorem 1.1.** According to Proposition 5.1, $U(\xi_t) + \tilde{\phi}(\xi_t)$ is a solution to problem (1.4) if $\xi_t \in \overline{\Omega}$ is a critical point of $F_t$ defined in (5.3). Lemma 7.1 guarantees the existence of such a critical point and therefore a solution $u_t$ for problem (1.4), where $u_t = U(\xi_t) + \tilde{\phi}(\xi_t)$. Notice that $\|\tilde{\phi}(\xi_t)\|_{L^\infty(\Omega)} \to 0$ as predicted in Proposition 4.1. Hence by the definition of $U(\xi_t)$ in (2.9), we can easily get the rest of the properties of $u_t$.

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