Non-abelian representations of the slim dense near hexagons on 81 and 243 points

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Abstract

We prove that the near hexagon $Q(5, 2) \times \mathbb{L}_3$ has a non-abelian representation in the extra-special 2-group $2_{1+12}^+$ and that the near hexagon $Q(5, 2) \otimes Q(5, 2)$ has a non-abelian representation in the extra-special 2-group $2_{1+18}^-$. The description of the non-abelian representation of $Q(5, 2) \otimes Q(5, 2)$ makes use of a new combinatorial construction of this near hexagon.

Keywords: near hexagon, non-abelian representation, extra-special 2-group

MSC2000: 05B25

1 Introduction

Let $\mathcal{S} = (P, L)$ be a partial linear space with point set $P$ and line set $L$. We suppose that $\mathcal{S}$ is slim, i.e., that every line of $\mathcal{S}$ is incident with precisely three points. For distinct points $x, y \in P$, we write $x \sim y$ if they are collinear. In that case, we denote by $xy$ the unique line containing $x$ and $y$ and define $x * y$ by $xy = \{x, y, x \ast y\}$. For $x \in P$, we define $x^\perp := \{x\} \cup \{y \in P : y \sim x\}$. If $x, y \in P$, then $d(x, y)$ denotes the distance between $x$ and $y$ in the collinearity graph of $\mathcal{S}$.

A representation [9, p.525] of $\mathcal{S}$ is a pair $(R, \psi)$, where $R$ is a group and $\psi$ is a mapping from $P$ to the set of involutions of $R$, satisfying:

(R1) $R$ is generated by the image of $\psi$;

(R2) $\psi$ is one-one on each line $\{x, y, x \ast y\}$ of $\mathcal{S}$ and $\psi(x)\psi(y) = \psi(x \ast y)$. Notice that if $x \sim y$, then $\psi(x)$ and $\psi(y)$ necessarily commute by condition (R2). The group $R$ is called a representation group of $\mathcal{S}$. A representation
(R, ψ) of S is faithful if ψ is injective and is abelian or non-abelian according as R is abelian or not. Note that, in [9], ‘non-abelian representation’ means that ‘the representation group is not necessarily abelian’. Abelian representations are called embeddings in the literature. For an abelian representation, the representation group is an elementary abelian 2-group and hence can be considered as a vector space over the field F₂ with two elements. We refer to [8] and [12, Sections 1 and 2] for more on representations of partial linear spaces with p + 1 points per line, where p is a prime.

A finite 2-group G is called extra-special if its Frattini subgroup Φ(G), its commutator subgroup G′ = [G, G] and its center Z(G) coincide and have order 2. We refer to [5, Section 20, pp.78–79] or [6, Chapter 5, Section 5] for the properties of extra-special 2-groups which we will mention now. An extra-special 2-group is of order 2^{1+2m} for some integer m ≥ 1. Let D₈ and Q₈, respectively, denote the dihedral and the quaternion groups of order 8. A non-abelian 2-group of order 8 is extra-special and is isomorphic to either D₈ or Q₈. If G is an extra-special 2-group of order 2^{1+2m}, m ≥ 1, then the exponent of G is 4 and either G is a central product of m copies of D₈, or G is a central product of m − 1 copies of D₈ and one copy of Q₈. If the former (respectively, latter) case occurs, then the extra-special 2-group is denoted by 2^{1+2m} (respectively, 2^{1+2m}).

A partial linear space S = (P, L) is called a near polygon if for every point p and every line L, there exists a unique point on L nearest to p. If d is the maximal distance between two points of S, then the near polygon is also called a near 2d-gon. A near polygon is called dense if every line is incident with at least three points and if every two points at distance 2 have at least two common neighbours. By [1], there are up to isomorphism 11 slim dense near hexagons. The paper [13] initiated the study of the non-abelian representations of these dense near hexagons.

Suppose (R, ψ) is a non-abelian representation of a slim dense near hexagon. Then by [13] Proposition 4.1, p.205], (R, ψ) necessarily is faithful and for x, y ∈ P, [ψ(x), ψ(y)] ≠ 1 if and only if x and y are at maximal distance 3 from each other. If S is the (up to isomorphism) unique slim dense near hexagon on 81 points, which will be denoted by Q(5, 2) × L₃ in the sequel, then it was shown in [13, Theorem 1.6, p.199] that R necessarily is isomorphic to the extra-special 2-group 2^{1+12}. If S is the (up to isomorphism) unique slim dense near hexagon on 243 points, which will be denoted by Q(5, 2) ⊗ Q(5, 2) in the sequel, then it was shown in [13, Theorem 1.6, p.199], that R necessarily is isomorphic to the extra-special 2-group 2^{1+18}. The question whether such non-abelian representations exist remained however
unanswered in [13]. The following theorem, which is the main result of this paper, deals with these existence problems.

**Theorem 1.1.** (1) The slim dense near hexagon $Q(5, 2) \times \mathbb{L}_3$ has a non-abelian representation in the extra-special 2-group $2_{1+12}^{1+12}$.

(2) The slim dense near hexagon $Q(5, 2) \otimes Q(5, 2)$ has a non-abelian representation in the extra-special 2-group $2_{1+18}^{1+18}$.

The slim dense near hexagon $Q(5, 2) \otimes Q(5, 2)$ has many substructures isomorphic to $Q(5, 2) \times \mathbb{L}_3$. We will describe a non-abelian representation of $Q(5, 2) \times \mathbb{L}_3$ in Section 4. In Section 5, we will use this to construct a non-abelian representation of $Q(5, 2) \otimes Q(5, 2)$. To describe the non-abelian representation of $Q(5, 2) \otimes Q(5, 2)$, we make use of a model of $Q(5, 2) \otimes Q(5, 2)$ which we discuss in Sections 2 and 3.

**Remark.** Two other constructions of non-abelian representations of slim dense near polygons, in particular, of the slim dense near hexagons on 105 and 135 points, can be found in the paper [10].

2 The point-line geometry $S_\theta$

Near quadrangles are usually called generalized quadrangles (GQ’s). A GQ is said to be of order $(s, t)$ if every line is incident with precisely $s + 1$ points and if every point is incident with precisely $t + 1$ lines. Up to isomorphism, there exist unique GQ’s of order $(2, 2)$ and $(2, 4)$, see e.g. [11]. These GQ’s are denoted by $W(2)$ and $Q(5, 2)$, respectively. A spread of a point-line geometry is a set of lines partitioning its point set. A spread $S$ of $Q(5, 2)$ is called a spread of symmetry if for every line $L \in S$ and every two points $x_1, x_2 \in L$, there exists an automorphism of $Q(5, 2)$ fixing each line of $S$ and mapping $x_1$ to $x_2$. By [2, Section 7.1], $Q(5, 2)$ has up to isomorphism a unique spread of symmetry.

Now, suppose $S$ is a given spread of symmetry of $Q(5, 2)$. If $L_1$ and $L_2$ are two distinct lines of $S$ and if $G$ denotes the unique $(3 \times 3)$-subgrid of $Q(5, 2)$ containing $L_1$ and $L_2$, then the unique line $L_3$ of $G$ disjoint from $L_1$ and $L_2$ is also contained in $S$.

Suppose $\theta$ is a map from $S \times S$ to $\mathbb{Z}_3$ (the additive group of order three) satisfying the following property:
If $L_1, L_2, L_3$ are three lines of $S$ contained in a grid of $Q(5,2)$, then

$$\theta(L_1, L_2) + \theta(L_2, L_3) = \theta(L_1, L_3).$$

Notice that $\theta(L, L) = 0$ and $\theta(M, L) = -\theta(L, M)$ for all $L, M \in S$. With $\theta$, there is associated a point-line geometry $S_\theta$. The points of $S_\theta$ are of four types:

(P1) The points $x$ of $Q(5,2)$.
(P2) The symbols $\bar{x}$, where $x$ is a point of $Q(5,2)$.
(P3) The symbols $\bar{\bar{x}}$, where $x$ is a point of $Q(5,2)$.
(P4) The triples $(x, y, i)$, where $i \in \mathbb{Z}_3$ and $x, y$ are distinct collinear points of $Q(5,2)$ satisfying $xy \in S$.

The lines of $S_\theta$ are of nine types:

(L1) The lines $\{x, y, z\}$ of $Q(5,2)$.
(L2) The sets $\{\bar{x}, \bar{y}, \bar{z}\}$, where $\{x, y, z\}$ is a line of $Q(5,2)$.
(L3) The sets $\{\bar{x}, \bar{y}, \bar{z}\}$, where $\{x, y, z\}$ is a line of $Q(5,2)$.
(L4) The sets $\{x, \bar{x}, \bar{\bar{x}}\}$, where $x$ is a point of $Q(5,2)$.
(L5) The sets $\{a, (a, b, i), (a, c, i)\}$, where $i \in \mathbb{Z}_3$ and $\{a, b, c\} \in S$.
(L6) The sets $\{\bar{a}, (b, a, i), (c, a, i)\}$, where $i \in \mathbb{Z}_3$ and $\{a, b, c\} \in S$.
(L7) The sets $\{\bar{a}, (b, c, i), (c, b, i)\}$, where $i \in \mathbb{Z}_3$ and $\{a, b, c\} \in S$.
(L8) The sets $\{(a, b, i), (b, c, j), (c, a, k)\}$, where $\{i, j, k\} = \mathbb{Z}_3$ and $\{a, b, c\}$ is a line belonging to $S$.
(L9) The sets $\{(a, u, i), (b, v, j), (c, w, k)\}$, where $(i) \{a, b, c\}$ and $\{u, v, w\}$ are two disjoint lines of $Q(5,2)$; $(ii) d(a, u) = d(b, v) = d(c, w)$

$$= 1; (iii) au, bv, cw \in S; (iv) j = i + \theta(au, bv), k = i + \theta(au, cw).$$

Incidence is containment. One can easily show that $S_\theta$ is a partial linear space. In order to show that two distinct points of $S_\theta$ are contained in at most one line of Type $(L9)$, one has to make use of Property $(\ast)$.

## 3 An isomorphism $Q(5,2) \otimes Q(5,2) \cong S_\theta$

The aim of this section is to show that the slim dense near hexagon $Q(5,2) \otimes Q(5,2)$ is isomorphic to a point-line geometry $S_\theta$ for a suitable spread of symmetry $S$ of $Q(5,2)$ and a suitable map $\theta : S \times S \to \mathbb{Z}_3$ satisfying Property
We start with recalling some known properties of the near hexagon $Q(5, 2) \otimes Q(5, 2)$.

1. Every two points $x$ and $y$ of $Q(5, 2) \otimes Q(5, 2)$ are contained in a unique convex subspace of diameter 2, called a quad. The points and lines which are contained in a given quad define a GQ which is isomorphic to either the $(3 \times 3)$-grid or $Q(5, 2)$.

2. If $Q$ is a $Q(5, 2)$-quad and $x \notin Q$, then $x$ is collinear with a unique point $\pi_Q(x) \in Q$ and we denote by $R_Q(x)$ the unique point of $x \pi_Q(x)$ distinct from $x$ and $\pi_Q(x)$. If $x \in Q$, then we define $\pi_Q(x) = R_Q(x) := x$. The map $x \mapsto R_Q(x)$ defines an automorphism of $Q(5, 2) \otimes Q(5, 2)$. If $Q_1$ and $Q_2$ are two disjoint $Q(5, 2)$-quads, then the map $Q_1 \to Q_2; x \mapsto \pi_{Q_2}(x)$ defines an isomorphism between $Q_1$ and $Q_2$.

3. There exist two partitions $T_1$ and $T_2$ of the point set of $Q(5, 2) \otimes Q(5, 2)$ into $Q(5, 2)$-quads.

4. Every element of $T_1$ intersects every element of $T_2$ in a line. As a consequence, $S^\circ := \{Q_1 \cap Q_2 : Q_1 \in T_1 \text{ and } Q_2 \in T_2\}$ is a spread of $Q(5, 2) \otimes Q(5, 2)$.

5. For every $Q \in T_i$, $i \in \{1, 2\}$, the set $\{Q \cap R : R \in T_{3-i}\}$ is a spread of symmetry of $Q$.

6. Every line $L$ of $Q(5, 2) \otimes Q(5, 2)$ not belonging to $S^\circ$ is contained in a unique quad of $T_1 \cup T_2$.

Now, let $Q$ and $\overline{Q}$ be two disjoint $Q(5, 2)$-quads belonging to $T_1$ and put $\overline{\overline{Q}} := R_Q(\overline{Q}) = R_{\overline{Q}}(Q)$. For every point $x$ of $Q$, put $\overline{x} := \pi_{\overline{Q}}(x) \text{ and } \overline{\overline{x}} := \pi_{\overline{\overline{Q}}}(x)$.

Put $S = \{Q \cap Q_2 : Q_2 \in T_2\}$. Then $S$ is a spread of symmetry of $Q$. For every $L \in S$, let $R_L$ denote the unique element of $T_2$ containing $L$. Let $L^*$ denote a specific line of $S$ and put $R^* = R_{L^*}$. For every $L \in S$, $R_L \cap (Q \cup \overline{Q} \cup \overline{\overline{Q}})$ is a $(3 \times 3)$-subgrid $\sigma_L$ of $R_L$. This $(3 \times 3)$-subgrid $\sigma_L$ is contained in precisely three $W(2)$-subquadrangles of $R_L$. We denote by $W^0, W^1, W^2$ the three $W(2)$-subquadrangles of $R^*$ containing $R^* \cap (Q \cup \overline{Q} \cup \overline{\overline{Q}})$. For every $L \in S$ and $i \in \mathbb{Z}_3$, put $W^i_L := \pi_{R_L}(W^i)$.

For every $i \in \mathbb{Z}_3$, for every $L \in S$ and for all $x, y \in L$ with $x \neq y$, we denote by $(x, y, i)$ the unique point $\mu$ of $R_L \setminus (Q \cup \overline{Q} \cup \overline{\overline{Q}})$ such that $\pi_Q(\mu) = x, \pi_{\overline{Q}}(\mu) = \overline{y}$ and $\mu \in W^i_L$. The point $(x, y, i)$ is the unique point of $W^i_L$ collinear with $x$ and $\overline{y}$, but not contained in $\sigma_L$.

**Lemma 3.1.** Every point of $Q(5, 2) \otimes Q(5, 2)$ not contained in $Q \cup \overline{Q} \cup \overline{\overline{Q}}$ has received a unique label.
Proof. Let $\mu$ be a point of $Q(5, 2) \otimes Q(5, 2)$ not contained in $Q \cup \overline{Q} \cup \overline{Q}$, let $R$ denote the unique element of $T_2$ containing $\mu$ and put $L := R \cap Q$. Then $R = R_\ell$. There exists a unique $W(2)$-subquadrangle of $R$ containing $\mu$ and $\sigma_R$. Let $i \in \mathbb{Z}_3$ such that $\mu \in W_\ell^i$. Let $x$ and $y$ be the points of $L$ such that $x = \pi_Q(\mu)$ and $y = \pi_\ell(\mu)$. If $x = y$, then $\{x, y, \mu\}$ is a set of mutually collinear points, implying that $\mu = \bar{x}$, contradicting $\mu \notin \overline{Q}$. Hence $x \neq y$ and the point $\mu$ has label $(x, y, i)$. It is also clear that $\mu$ cannot be labeled in different ways. \hfill $\square$

We will now define a map $\theta : S \times S \to \mathbb{Z}_3$. For each ordered pair $(L_1, L_2)$ of lines of $S$, the map $R^* \to R^* ; x \mapsto \pi_{R^*} \circ \pi_{R L_2} \circ \pi_{R L_1}(x)$ determines an automorphism of $R^*$ fixing each line of the spread $\{R^* \cap Q_1 : Q_1 \in T_1\}$ of $R^*$. By [2] Theorem 4.1], such an automorphism either is trivial or acts on any line of the form $R^* \cap Q_1, Q_1 \in T_1$, as a cycle. Since every line $R^* \cap Q_1, Q_1 \in T_1 \setminus \{Q, \overline{Q}, \overline{Q}\}$, intersects each $W(2)$-subquadrangle $W^i, i \in \mathbb{Z}_3$, in a unique point, the map $R^* \to R^* ; x \mapsto \pi_{R^*} \circ \pi_{R L_2} \circ \pi_{R L_1}(x)$ is either trivial or permutes the elements of $\{W^0, W^1, W^2\}$ in one of the following ways:

\[ W^0 \to W^1 \to W^2 \to W^0, W^0 \to W^2 \to W^1 \to W^0. \]

Hence, there exists a unique $\theta(L_1, L_2) \in \mathbb{Z}_3$ such that

\[ \pi_{R^*} \circ \pi_{R L_2} \circ \pi_{R L_1}(W^i) = W^{i + \theta(L_1, L_2)} \]

for every $i \in \mathbb{Z}_3$.

**Lemma 3.2.** The following holds:

(i) For every $L \in S$, $\theta(L, L) = 0$.

(ii) For any two lines $L_1$ and $L_2$ of $S$, $\theta(L_2, L_1) = -\theta(L_1, L_2)$.

(iii) If $L_1, L_2, L_3$ are three lines of $S$ which are contained in a grid, then $\theta(L_1, L_2) + \theta(L_2, L_3) = \theta(L_1, L_3)$.

(iv) If $L_1, L_2, L_3$ are three lines of $S$ which are not contained in a grid, then $\theta(L_1, L_2) + \theta(L_2, L_3) \neq \theta(L_1, L_3)$.

Proof. (i) For every $i \in \mathbb{Z}_3$, we have $\pi_{R^*} \circ \pi_{R L} \circ \pi_{R L}(W^i) = \pi_{R^*} \circ \pi_{R L}(W^i) = W^i$. Hence, $\theta(L, L) = 0$.

(ii) If $\pi_{R^*} \circ \pi_{R L_2} \circ \pi_{R L_1}(W^i) = W^{i + \theta(L_1, L_2)}$ for every $i \in \mathbb{Z}_3$, then $W^i = \pi_{R^*} \circ \pi_{R L_1} \circ \pi_{R L_1}(W^{i + \theta(L_1, L_2)})$ for every $i \in \mathbb{Z}_3$. It follows that $\theta(L_2, L_1) = -\theta(L_1, L_2)$. 

6
(iii) Let \( L_1, L_2, L_3 \) be three lines of \( S \) which are contained in a grid. Then
\[
\pi_{R^*} \circ \pi_{RL_3} \circ \pi_{RL_1}(W^i) = \pi_{R^*} \circ \pi_{RL_3} \circ \pi_{RL_2} \circ \pi_{RL_1}(W^i) = \pi_{R^*} \circ \pi_{RL_3} \circ \pi_{RL_2} \circ \pi_{RL_1}(W^i),
\]
Hence, \( \theta(L_1, L_3, L_2) = \theta(L_1, L_2) + \theta(L_2, L_3) \).

(iv) Let \( L_1, L_2, L_3 \) be three lines of \( S \) which are not contained in a grid. Suppose that \( \theta(L_1, L_3) = \theta(L_1, L_2) + \theta(L_2, L_3) \). Then for every \( y \in R^* \),
\[
\pi_{R^*} \circ \pi_{RL_3} \circ \pi_{RL_1}(y) = (\pi_{R^*} \circ \pi_{RL_3} \circ \pi_{RL_2}) \circ (\pi_{R^*} \circ \pi_{RL_2} \circ \pi_{RL_1})(y),
\]
Hence, the map \( R_{L_3} \rightarrow R_{L_3} \) defined by \( x \mapsto \pi_{RL_3} \circ \pi_{RL_2} \circ \pi_{RL_1}(x) \) is the identity map on \( R_{L_3} \). This implies that the points \( x, \pi_{RL_1}(x), \pi_{RL_2}(x) \) are mutually collinear for every \( x \in R_{L_3} \), that is, \( \{x, \pi_{RL_1}(x), \pi_{RL_2}(x)\} \) is a line for every \( x \in R_{L_3} \). This contradicts the fact that \( L_1, L_2, L_3 \) are not contained in a grid. Hence, \( \theta(L_1, L_3) \neq \theta(L_1, L_2) + \theta(L_2, L_3) \).

\[\text{Proposition 3.3.}\quad Q(5, 2) \otimes Q(5, 2) \cong S_\theta, \text{ where } \theta \text{ is as defined above.}\]

\textit{Proof.} We must show that the set of lines of \( Q(5, 2) \otimes Q(5, 2) \) are in bijective correspondence with the sets of Type (L1), (L2), \ldots, (L9) defined in Section 2. Obviously,

- the set of lines of \( Q(5, 2) \otimes Q(5, 2) \) contained in \( Q \) correspond to the sets of Type (L1);
- the set of lines of \( Q(5, 2) \otimes Q(5, 2) \) contained in \( \overline{Q} \) correspond to the sets of Type (L2);
- the set of lines of \( Q(5, 2) \otimes Q(5, 2) \) contained in \( \overline{Q} \) correspond to the sets of Type (L3);
- the set of lines of \( Q(5, 2) \otimes Q(5, 2) \) meeting \( Q, \overline{Q} \) and \( \overline{Q} \) correspond to the sets of Type (L4).

Consider a line \( M \) of \( R_L \) which is not contained in \( \sigma_L \) and which intersects \( \sigma_L \) in a point \( a \in L \) of \( Q \). Put \( L = \{a, b, c\} \). There exists a unique \( W(2) \)-subquadrangle \( W_k \) containing \( \sigma_L \) and \( M \). One readily sees that the points of \( M \) have labels \( a, (a, b, i) \) and \( (a, c, i) \). So, \( M \) corresponds to a set of Type (L5). Conversely, every set of Type (L5) corresponds to a (necessarily unique) line of \( Q(5, 2) \otimes Q(5, 2) \).

Next, consider a line \( M \) of \( R_L \) which is not contained in \( \sigma_L \) and which intersects \( \sigma_L \) in a point \( \bar{a} \) of \( \overline{Q} \). Put \( L = \{a, b, c\} \). Then, there exists a unique \( W(2) \)-subquadrangle \( W_k \) containing \( \sigma_L \) and \( M \). One readily sees that the
points of $M$ have labels $\bar{a}$, $(b, a, i)$ and $(c, a, i)$. So, $M$ corresponds to a set of Type $(L6)$. Conversely, every set of Type $(L6)$ corresponds to a (necessarily unique) line of $Q(5, 2) \otimes Q(5, 2)$.

Now, consider a line $M$ of $R_L$ which is not contained in $\sigma_L$ and which intersects $\sigma_L$ in a point $\bar{a}$ of $Q$. Put $L = \{a, b, c\}$. Then, there exists a unique $W(2)$-subquadangle $W^i_L$ containing $\sigma_L$ and $M$. One readily sees that the points of $M$ have labels $\bar{a}$, $(b, c, i)$ and $(c, b, i)$. So, $M$ corresponds to a set of Type $(L7)$. Conversely, every set of Type $(L7)$ corresponds to a (necessarily unique) line of $Q(5, 2) \otimes Q(5, 2)$.

Consider next a line $M$ of $R_L$ which is disjoint from $\sigma_L$. Then $M$ intersects each $W^i_L$, $i \in \mathbb{Z}_3$, in a unique point. Put $L = \{a, b, c\}$. The labels of the points of $M$ are $(u, u', i)$, $(v, v', j)$, $(w, w', k)$, where $\{i, j, k\} = \{0, 1, 2\}$, $(u, v, w) = \pi_Q(M) = \{a, b, c\}$, $(u', v', w') = \pi_Q \circ \pi_Q(M) = \{a, b, c\}$, $u \neq u'$, $v \neq v'$, $w \neq w'$. It readily follows that $(u, u', i), (v, v', j), (w, w', k)$ is a set of Type $(L8)$. Conversely, one can readily verify that every set of Type $(L8)$ corresponds to a line of $Q(5, 2) \otimes Q(5, 2)$.

Finally, let $M$ be a line of $Q(5, 2) \otimes Q(5, 2)$ not belonging to $S^{\otimes}$ and contained in a quad of $T_1 \setminus \{Q, \overline{Q}, \overline{Q}\}$. With $M$, there corresponds a set of the form $\{(a, u, i), (b, v, j), (c, w, k)\}$. We have that $\{a, b, c\} = \pi_Q(M)$ is a line of $Q$ not belonging to $S$. Similarly, $(u, v, w) = \pi_Q \circ \pi_Q(M)$ is a line of $Q$ not belonging to $S$. Moreover, we have that $au, bv, cw \in S$ and $j = i + \theta(au, bv)$, $k = i + \theta(au, cw)$ by the definition of the map $\theta$. So, $M$ corresponds to a set of Type $(L9)$. Conversely, we show that every set $\{(a, u, i), (b, v, j), (c, w, k)\}$ of Type $(L9)$ corresponds to a line of $Q(5, 2) \otimes Q(5, 2)$ not belonging to $S^{\otimes}$ and contained in a quad of $T_1 \setminus \{Q, \overline{Q}, \overline{Q}\}$. Let $x$ denote the point of $Q(5, 2) \otimes Q(5, 2)$ corresponding to $(a, u, i)$, let $Q_1$ denote the unique element of $T_1$ containing $x$ and let $M = \pi_{Q_1}(\{a, b, c\})$. Then $M$ corresponds to a set of the form $\{(a, u, i), (b, *, *), (c, *, *)\}$. Since $v, w, j, k$ are uniquely determined by $a, u, i, b, c$, this set is equal to $\{(a, u, i), (b, v, j), (c, w, k)\}$.

By the above discussion, we indeed know that $Q(5, 2) \otimes Q(5, 2) \cong S_{\theta}$. ■

**Definitions.** (1) An admissible triple is a triple $\Sigma = (L, G, \Delta)$, where:

- $G$ is a nontrivial additive group whose order $s + 1$ is finite.
- $L$ is a linear space, different from a point, in which each line is incident with exactly $s + 1$ points. We denote the point set of $L$ by $P$.
- $\Delta$ is a map from $P \times P$ to $G$ such that the following holds for any three points $x, y$ and $z$ of $L$: $x, y$ and $z$ are collinear $\iff \Delta(x, y) + \Delta(y, z) = \Delta(x, z)$. 

8
Suppose $\Sigma_1 = (L_1, G_1, \Delta_1)$ and $\Sigma_2 = (L_2, G_2, \Delta_2)$ are two admissible triples, where $L_1$ and $L_2$ are not lines. Then $\Sigma_1$ and $\Sigma_2$ are called equivalent if there exists an isomorphism $\alpha$ from $L_1$ to $L_2$, an isomorphism $\beta$ from $G_1$ to $G_2$ and a map $f$ from the point set of $L_1$ to $G_1$ satisfying $\Delta_2(\alpha(x), \alpha(y)) = (f(x) + \Delta_1(x, y) - f(y))^\beta$ for all points $x$ and $y$ of $L_1$.

Let $\mathcal{L}_S$ denote the linear space whose points are the elements of $S$ and whose lines are the unordered triples of lines of $S$ which are contained in a grid, with incidence being containment. Then $\mathcal{L}_S$ is isomorphic to the affine plane $AG(2, 3)$ of order three. By Lemma 3.2, we know that $(\mathcal{L}_S, \mathbb{Z}_3, \theta)$ is an admissible triple.

**Proposition 3.4.** Let $\theta_1$ and $\theta_2$ be two maps from $S \times S$ to $\mathbb{Z}_3$ such that $\Sigma_1 = (\mathcal{L}_S, \mathbb{Z}_3, \theta_1)$ and $\Sigma_2 = (\mathcal{L}_S, \mathbb{Z}_3, \theta_2)$ are admissible triples. If $\Sigma_1$ and $\Sigma_2$ are equivalent, then $S_{\theta_1} \cong S_{\theta_2}$.

**Proof.** Since $\Sigma_1$ and $\Sigma_2$ are equivalent, there exists an automorphism $\alpha$ of $\mathcal{L}_S$, an automorphism $\beta$ of $\mathbb{Z}_3$ and a map $f$ from $S$ to $\mathbb{Z}_3$ satisfying $\theta_2(\alpha(x), \alpha(y)) = (f(x) + \theta_1(x, y) - f(y))^\beta$ for all points $x$ and $y$ of $\mathcal{L}_S$. There exists an automorphism $\phi$ of $Q$ such that $\alpha(L) = \phi(L)$ for every line $L$ of $S$, see e.g. [3, Section 3, Example 1]. One readily verifies that the map $x \mapsto x^\phi; \overline{x} \mapsto \overline{x}^\phi; (a, b, i) \mapsto (a^\phi, b^\phi, (i - f(ab))^\beta)$ defines an isomorphism between $S_{\theta_1}$ and $S_{\theta_2}$.

It is known that the affine plane $AG(2, 3)$ admits, up to equivalence, a unique admissible triple. [This follows, for instance, from [4, Theorem 2.1] and the fact that there exists a unique generalized quadrangle of order $(2, 4)$, namely $Q(5, 2)$, and a unique spread of symmetry in $Q(5, 2)$.] If we coordinatize $AG(2, 3)$ in the standard way, then an admissible triple can be obtained by putting $\Delta(\{x_1, y_1\}, (x_2, y_2)) := x_1y_2 - x_2y_1 \in \mathbb{Z}_3$.

### 4 A non-abelian representation of the near hexagon $Q(5, 2) \times \mathbb{L}_3$

The slim dense near hexagon $Q(5, 2) \times \mathbb{L}_3$ is obtained by taking three isomorphic copies of $Q(5, 2)$ and joining the corresponding points to form lines of size 3. In this section we prove that there exists a non-abelian representation of $Q(5, 2) \times \mathbb{L}_3$. 


Let $Q$ and $B$, respectively, be the point and line set of $Q(5,2)$. Set \( \overline{Q} = \{ \bar{x} : x \in Q \} \), \( \overline{Q} = \{ \bar{x} : x \in Q \} \), \( \overline{B} = \{ \{\bar{x},\bar{y},\bar{z} : \{x,y,z\} \in B \} \) and \( \overline{B} = \{\{\bar{x},\bar{y},\bar{z} : \{x,y,z\} \in B \} \). Then \( (\overline{Q},\overline{B}) \) and \( (\overline{Q},\overline{B}) \) are isomorphic to \( Q(5,2) \). The near hexagon \( Q(5,2) \times \mathbb{L}_3 \) is isomorphic to the geometry whose point set \( P \) is \( Q \cup Q \cup Q \) and whose line set \( L \) is \( B \cup B \cup B \cup \{x,\bar{x},\bar{z} : x \in Q \} \).

It is known that if \( Q(5,2) \times \mathbb{L}_3 \) admits a non-abelian representation, then the representation group must be the extra-special 2-group \( 2^{1+12} \). Let \( R = 2^{1+12} \) with \( R' = \{1,\lambda\} \). Set \( V = R/R' \). Consider \( V \) as a vector space over \( \mathbb{F}_2 \). The map \( f : V \times V \to \mathbb{F}_2 \) defined by

\[
f(xR',yR') = \begin{cases} 
0 & \text{if } [x,y] = 1 \\
1 & \text{if } [x,y] = \lambda 
\end{cases}
\]

for \( x, y \in R \), is a non-degenerate symplectic bilinear form on \( V \) [Theorem 20.4, p.78]. Write \( V \) as an orthogonal direct sum of six hyperbolic planes \( K_i \) (1 \( \leq i \leq 6 \)) in \( V \) and let \( H_i \) be the inverse image of \( K_i \) in \( R \) (under the canonical homomorphism \( R \to R/R' \)). Then each \( H_i \) is generated by two involutions \( x_i \) and \( y_i \) such that \([x_i,y_i] = \lambda \). Let \( M = \langle x_i : 1 \leq i \leq 6 \rangle \) and \( \overline{M} = \langle y_i : 1 \leq i \leq 6 \rangle \). Then \( M \) and \( \overline{M} \) are elementary abelian 2-subgroups of \( R \) each of order \( 2^6 \). Further, \( M, \overline{M} \) and \( Z(R) \) pairwise intersect trivially and \( R = M \overline{M} Z(R) \). Also, \( C_M(\overline{M}) \) and \( C_{\overline{M}}(M) \) are trivial.

We regard the points and lines of \( Q \) as the points and lines of a nonsingular elliptic quadric of the projective space \( \operatorname{PG}(M) \), where \( M \) is regarded as a 6-dimensional vector space over \( \mathbb{F}_2 \). Let \( (M,\tau) \) be the natural abelian representation of \( (Q,B) \) associated with this embedding of \( Q \) in \( \operatorname{PG}(M) \). For every point \( x \) of \( Q \), put \( m_x = \tau(x) \). There exists a unique non-degenerate symplectic bilinear form \( g \) on \( M \) such that \( m_x^\perp = \langle m_y : y \in x^\perp \rangle \) for every point \( x \) of \( Q \), see e.g. [7, Section 22.3]. Here, the following notational convention has been used: for every \( m \in M \), \( m^\perp \) denotes the set of all \( m' \in M \) for which \( g(m,m') = 0 \).

Now, let \( m \) be an arbitrary element of \( M \). If \( m = 1 \), then we define \( \overline{m} = 1 \). Suppose now that \( m \neq 1 \). Then \( m^\perp \) is maximal in \( M \), that is, of index 2 in \( M \). So, the centralizer of \( m^\perp \) in \( \overline{M} \) is a subgroup \( \langle \overline{m} \rangle \) of order 2. Since \( m^\perp \) is maximal in \( M \), \( \langle m^\perp, m' \rangle = M \) for every \( m' \in M \setminus m^\perp \). The triviality of \( C_M(\overline{M}) \) then implies that \( \langle m, m' \rangle = \lambda \) for every \( m' \in M \setminus m^\perp \).

We prove that the map \( M \to \overline{M} : m \mapsto \overline{m} \) is an isomorphism. This map is easily seen to be bijective. (Notice that \( C_M(\overline{m}) = m^\perp \).) So, it suffices to prove that \( \overline{m_1m_2} = \overline{m_1} \overline{m_2} \) for all \( m_1, m_2 \in M \). Clearly, this holds if \( 1 \in \{m_1, m_2\} \) or \( m_1 = m_2 \). So, we may suppose that \( m_1 \neq 1 \neq m_2 \neq m_1 \).
The set \( \{m_1, m_2, m_1m_2\} \) corresponds to a line of \( \text{PG}(M) \). So, for every \( m \in (m_1m_2)\perp \), \((\overline{m_1}, m], [\overline{m_2}, m]) \) is equal to either \((1,1)\) or \((\lambda, \lambda)\). Then
\[
[\overline{m_1} \overline{m_2}, m] = [\overline{m_1}, m][\overline{m_2}, m] = 1.
\]

The first equality holds since \( R \) has nilpotency class 2. Thus \( \overline{m_1} \overline{m_2} \in C_M((m_1m_2)\perp) = (\overline{m_1m_2}). \)
Since \( \overline{m_1} \overline{m_2} \neq 1 \), we have \( \overline{m_1} \overline{m_2} = \overline{m_1m_2} \).

We conclude that if we define \( \overline{\tau} : \overline{Q} \to \overline{M}; \bar{x} \mapsto \overline{m_x} \) for every \( x \in Q \), then \( (\overline{M}, \overline{\tau}) \) is a faithful abelian representation of \( (Q, B) \).

Now, let \( m \) be an arbitrary element of \( M \). If \( m = 1 \), then we define \( \overline{m} := 1 \). If \( m = m_x \) for some \( x \in Q \), then we define \( \overline{m} := m\overline{m} \). If \( m \neq 1 \) and \( m \neq m_x \), \( \forall x \in Q \), then we define \( \overline{m} := m\overline{m}\lambda \). Since \( m^2 = \overline{m}^2 = \lambda^2 = [m, \overline{m}] = 1 \), \( \overline{m} \) is an involution. We prove that the map \( m \mapsto \overline{m} \) defines an isomorphism between \( M \) and an elementary abelian 2-group \( \overline{M} \) of order \( 2^6 \). Since \( R = \overline{M}\overline{Z}(R) \), this map is injective and hence it suffices to prove that \( \overline{m_1m_2} = \overline{m_1} \overline{m_2} \) for all \( m_1, m_2 \in M \). Obviously, this holds if \( 1 \in \{m_1, m_2\} \) or \( m_1 = m_2 \). So, we may suppose that \( m_1 \neq 1 \neq m_2 \neq m_1 \). The set \( \{m_1, m_2, m_1m_2\} \) corresponds to a line of \( \text{PG}(M) \). Suppose \( 3 - N \) elements of \( \{m_1, m_2, m_1m_2\} \) correspond to points of \( Q \). Then \( m_1 \in m_2^\perp \) if and only if \( N \) is even\(^1\). So, \( [\overline{m_1}, \overline{m_2}] = \lambda^N \). If \( N' \) is the number of elements of \( \{m_1, m_2\} \) corresponding to points of \( Q \), then \( 2 - N' - N \in \{-1, 0\} \) and \( 2 - N' - N = 0 \) if and only if \( m_1m_2 \) corresponds to a point of \( Q \). Hence, \( \overline{m_1} \overline{m_2} = \overline{m_1} m_1 m_2 \overline{m_2} \lambda^{2 - N'} = m_1 m_2 \overline{m_1} \overline{m_2} \lambda^{2 - N' - N} = m_1 m_2 \overline{m_1} \overline{m_2} \lambda^{2 - N' - N} = m_1 m_2 \overline{m_1} \overline{m_2} \).

So, if we define \( \overline{\tau} : \overline{Q} \to \overline{M} \) by putting \( \overline{\tau}(\bar{x}) := \overline{m_x} = m_x \overline{m_x} \) for all \( x \in Q \), then \( (\overline{M}, \overline{\tau}) \) is a faithful abelian representation of \( (Q, B) \).

Now, define a map \( \psi : P \to R \) which coincides with \( \tau \) on \( Q \), \( \overline{\tau} \) on \( \overline{Q} \) and \( \overline{\tau} \) on \( \overline{Q} \). Since \( R = \langle M, \overline{M} \rangle \), \( R = \langle \psi(P) \rangle \). By construction, \( (R, \psi) \) also satisfies Property (R2) in the definition of representation. Hence, \( (R, \psi) \) is a non-abelian representation of \( Q(5, 2) \times L_3 \).

---

\(^1\)Perhaps the case \( N = 3 \) needs more explanation. Suppose \( N = 3 \) and \( m_1 \in m_2^\perp \). Then the hyperplane \( \pi \) of \( \text{PG}(M) \) corresponding to \( m_2^\perp \) intersects \( Q \) in a nonsingular parabolic quadric \( Q(4, 2) \) of \( \pi \). Since the point of \( \text{PG}(M) \) corresponding to \( m_2 \) is the kernel of \( Q(4, 2) \), the line of \( \text{PG}(M) \) corresponding to \( \{m_1, m_2, m_1m_2\} \subset \pi \) must meet \( Q(4, 2) \), in contradiction with \( N = 3 \).
5 A non-abelian representation of the near hexagon $Q(5, 2) \otimes Q(5, 2)$

In this section, we prove that the slim dense near hexagon $Q(5, 2) \otimes Q(5, 2)$ has a non-abelian representation. By Proposition 3.3, this is equivalent with showing that the partial linear space $S_\theta$ has a non-abelian representation, where $\theta$ is as defined in Section 3.

We continue with the notation introduced in Section 3. Let $M^*$ be a line of $S^\otimes$ contained in $R^*$ but distinct from $R^* \cap Q, R^* \cap \bar{Q}$ and $R^* \cap \bar{Q}$. Then $M^*$ intersects each $W^i, i \in \{1, 2, 3\}$, in a unique point. For every point $x$ of $L^*$, put $\epsilon(x) := i$ if the unique point of $M^*$ collinear with $x$ belongs to $W^i$.

Lemma 5.1. Let $L_1$ and $L_2$ be two distinct lines in $S$ and let $\alpha_i \in L_i, i \in \{1, 2\}$. Then $\alpha_1 \sim \alpha_2$ if and only if $\epsilon(\alpha_2) = \epsilon(\alpha_1) = \theta(L_1, L_2)$.

Proof. Let $\alpha'_2$ be the unique point of $L_2$ collinear with $\alpha_1$, let $x_1$ and $x_2$ be the unique points of $L^*$ nearest to $\alpha_1$ and $\alpha'_2$, respectively, and let $z_i, i \in \{1, 2\}$, denote the unique point of $M^*$ collinear with $x_i$. The automorphism $R^* \to R^*; x \mapsto \pi_{R^*} \circ \pi_{R^*_L} \circ \pi_{R^*_L}(x)$ of $R^*$ maps $x_1$ to $x_2$ and hence $z_1$ to $z_2$.

This implies that $W^{\epsilon(x_1) + \theta(L_1, L_2)} = W^{\epsilon(x_2)}$, i.e. $\theta(L_1, L_2) = \epsilon(x_2) - \epsilon(x_1) = \epsilon(\alpha'_2) - \epsilon(\alpha_1)$. Hence, $\alpha_1 \sim \alpha_2$ if and only if $\alpha_2 = \alpha'_2$, i.e. if and only if $\epsilon(\alpha_2) - \epsilon(\alpha_1) = \theta(L_1, L_2)$. \hfill \Box

Lemma 5.2. Let $N = 2^{1+6}$ with $N' = \{1, \lambda\}$ and let $I_2(N)$ be the set of involutions in $N$. Then there exists a map $\delta$ from $Q$ to $I_2(N)$ satisfying the following:

(i) $\delta$ is one-one.

(ii) For $x, y \in Q$, $[\delta(x), \delta(y)] = 1$ if and only if $y \in x^\perp$.

(iii) If $x, y \in Q$ with $x \sim y$, then

$$\delta(x \ast y) = \begin{cases} 
\delta(x)\delta(y) & \text{if } xy \in S \\
\delta(x)\delta(y)\lambda & \text{if } xy \notin S.
\end{cases}$$

(iv) The image of $\delta$ generates $N$. 

12
Proof. We use a model for the generalized quadrangle $Q \cong Q(5, 2)$ which is described in [11] Section 6.1, pp.101–102. Put $\Omega = \{1, 2, 3, 4, 5, 6\}$ and $\Omega' = \{1', 2', 3', 4', 5', 6'\}$. Let $E$ be the set of all 2-subsets of $\Omega$ and let $F$ be the set of all partitions of $\Omega$ in three 2-subsets of $\Omega$. Then the point set of $Q$ can be identified with the set $E \cup \Omega \cup \Omega'$ and the line set of $Q$ can be identified with the set $F \cup \{\{i, \{i, j\}, j' : 1 \leq i, j \leq 6, j \neq i\}$. Now, consider the following nine lines of $Q$:

$L_1 = \{\{1, 2\}, \{3, 4\}, \{5, 6\}\}; L_2 = \{\{1, 4\}, 1, 4'\}; L_3 = \{\{2, 6\}, 2, 6'\};$
$L_4 = \{\{1, 6\}, \{2, 4\}, \{3, 5\}\}; L_5 = \{\{1, 5\}, 1', 5\}; L_6 = \{\{2, 3\}, 2', 3\};$
$L_7 = \{\{1, 3\}, \{2, 5\}, \{4, 6\}\}; L_8 = \{\{3, 6\}, 3', 6\}; L_9 = \{\{4, 5\}, 4, 5'\}.$

These 9 lines are mutually disjoint and hence determine a spread $S'$ of $Q$. Any two distinct lines $L_i$ and $L_j$ of $S'$ are contained in a unique $(3 \times 3)$-subgrid and the unique line of this subgrid disjoint from $L_i$ and $L_j$ also belongs to $S'$.

A spread of $Q(5, 2)$ having this property is called regular. Since any regular spread of $Q(5, 2)$ is also a spread of symmetry [2 Section 7.1], and there exists up to isomorphism a unique spread of symmetry in $Q(5, 2)$, we may without loss of generality suppose that $S = S'$.

Put $N = \langle a, b \rangle \circ \langle c, d \rangle \circ Q_8$, where $a, b, c, d$ are involutions and $\langle a, b \rangle \cong \langle c, d \rangle \cong D_8$. So, $[a, b] = [c, d] = \lambda$. Take $Q_8 = \{1, \lambda, 1, j, k, i, j, k, k \lambda\}$, where $i^2 = j^2 = k^2 = \lambda, ij = k, jk = i, ki = j$ and $[i, j] = [j, k] = [k, i] = \lambda$. We define $\delta : Q \rightarrow I_2(N)$ as follows:

\[
\begin{align*}
\delta(\{1, 2\}) &= a, \delta(\{3, 4\}) = c, \delta(\{5, 6\}) = ac, \\
\delta(\{1, 4\}) &= abd, \delta(1) = cdj, \delta(4') = abck, \\
\delta(\{2, 6\}) &= abi, \delta(2) = acdk, \delta(6') = bcdj, \\
\delta(\{1, 6\}) &= b, \delta(\{2, 4\}) = bd, \delta(\{3, 5\}) = d, \\
\delta(\{1, 5\}) &= abc, \delta(1') = cdj, \delta(5) = abd, \\
\delta(\{2, 3\}) &= bcdj, \delta(2') = acdk, \delta(3') = abk, \\
\delta(\{1, 3\}) &= abcdj, \delta(\{2, 5\}) = bc, \delta(\{4, 6\}) = ad, \\
\delta(\{3, 6\}) &= abck, \delta(3') = ablj, \delta(6) = bck, \\
\delta(\{4, 5\}) &= cdj, \delta(4') = abcj, \delta(5') = abdkj.
\end{align*}
\]

Put $W = N/N'$. Suppose $\{x_1, x_2, \ldots, x_6\}$ is a set of 6 points of $Q$ such that the smallest subspace $[x_1, x_2, \ldots, x_6]$ of $Q$ containing $\{x_1, x_2, \ldots, x_6\}$ coincides with $Q$. If $\tau$ is an abelian representation of $Q$ in $W$, then by Property (R1) in the definition of representation, $W = \langle \tau(x_1), \ldots, \tau(x_6) \rangle$ and hence $\{\tau(x_1), \ldots, \tau(x_6)\}$ is a basis of $W$ (regarded as $\mathbb{F}_2$-vector space).

Conversely, if $\{w_1, \ldots, w_6\}$ is a basis of $W$, then the map $x_i \mapsto w_i, i \in$
\(\{1, \ldots, 6\}\), can be extended to a unique abelian representation \(\tau\) of \(Q\) in \(W\). (Since there exists an abelian representation of \(Q\) in \(W\), there must exist an abelian representation \(\tau\) for which \(\tau(x_i) = w_i\), \(i \in \{1, \ldots, 6\}\). The uniqueness of \(\tau\) follows from the fact that \(\tau(y_1 \ast y_2) = \tau(y_1)\tau(y_2)\) for any two distinct collinear points \(y_1\) and \(y_2\) of \(Q\).) Consider now the special case where \(x_1 = \{1, 2\},\ x_2 = \{3, 4\},\ x_3 = \{3, 5\},\ x_4 = \{1, 6\},\ x_5 = \{4, 5\},\ x_6 = 1,\ w_1 = aN',\ w_2 = cN',\ w_3 = dN',\ w_4 = bN',\ w_5 = cdN'\) and \(w_6 = cdjN'\). One indeed readily verifies that \([x_1, \ldots, x_6] = Q\) and that \(\{w_1, \ldots, w_6\}\) is a basis of \(\mathbb{F}_2\)-vector space \(W\). Let \(\delta'\) denote the unique abelian representation of \(Q\) in \(W\) for which \(\delta'(x_i) = w_i,\ i \in \{1, \ldots, 6\}\). Then, using the fact that \(\delta'(y_1 \ast y_2) = \delta'(y_1)\delta'(y_2)\) for any two distinct collinear points \(y_1\) and \(y_2\) of \(Q\), one can verify that \(\delta'(y) = \delta(y)N'\) for every \(y \in Q\). This implies that \(\delta(y_1 \ast y_2)\) is equal to either \(\delta(y_1)\delta(y_2)\) or \(\delta(y_1)\delta(y_2)\lambda\) for any two distinct collinear points \(y_1\) and \(y_2\) of \(Q\).

Clearly, the map \(\delta : Q \rightarrow I_2(N)\) satisfies the properties (i) and (iv) of the lemma. We will now prove that also property (ii) of the lemma is satisfied. So, if \(\{y_1, y_2\}\) is one of the 351 unordered pairs of distinct points of \(Q\), then we need to prove that \([\delta(y_1), \delta(y_2)] = 1\) if and only if \(y_1 \in y_2^\perp\). Since \(Q = [x_1, \ldots, x_6]\), it suffices to prove the following three statements:

(I) the above claim holds if \(\{y_1, y_2\} \subseteq \{x_1, \ldots, x_6\}\); (II) if \([\delta(y_1), \delta(y_2)] = 1\) for some distinct collinear points \(y_1\) and \(y_2\), then also \([\delta(y_1), \delta(y_1 \ast y_2)] = 1\); (III) if the above claim holds for unordered pairs \(\{y_1, y_2\}\) and \(\{y_1, y_3\}\) of points where \(y_2 \sim y_3\) and \(y_1 \notin y_2 y_3\), then it also holds for the unordered pair \(\{y_1, y_2 \ast y_3\}\). Statement (I) is easily verified by considering all 15 pairs \(\{x_i, x_j\}\) where \(i, j \in \{1, \ldots, 6\}\) with \(i \neq j\). As to Statement (II), notice that \([\delta(y_1), \delta(y_1 \ast y_2)] = \delta(y_1)\delta(y_1)\delta(y_2)\delta(y_2)\) which is in any case equal to 1. We now prove Statement (III). Since \(\delta(y_2 \ast y_3) = \delta(y_2)\delta(y_3)\) or \(\delta(y_2)\delta(y_3)\lambda\), we have \([\delta(y_1), \delta(y_2 \ast y_3)] = [\delta(y_1), \delta(y_2)][\delta(y_1), \delta(y_3)]\). If \(y_1\) is collinear with precisely one of \(y_2, y_3\), then \(y_1\) is not collinear with \(y_2 \ast y_3\) and \([\delta(y_1), \delta(y_2 \ast y_3)] = [\delta(y_1), \delta(y_2)][\delta(y_1), \delta(y_3)] = 1 \cdot \lambda = \lambda\). If \(y_1\) is collinear with \(y_2 \ast y_3\), then \([\delta(y_1), \delta(y_2 \ast y_3)] = [\delta(y_1), \delta(y_2)][\delta(y_1), \delta(y_3)] = \lambda \cdot \lambda = 1\). So, this proves Statement (III) and finishes the proof of property (ii) of the lemma.

Property (iii) of the lemma is verified by considering all 45 lines \(L\) of \(Q\) and an ordered pair \((x, y)\) of distinct points of \(L\). Notice that by property (ii) of the lemma, we only need to consider one ordered pair \((x, y)\) for each line \(L\) of \(Q\).

It is known that if the near hexagon \(Q(5, 2) \otimes Q(5, 2)\) admits a non-
abelian representation, then the representation group must be the extra-special 2-group $2^{1+18}_+$. \[13\] Theorem 1.6, p.199. We next construct a non-abelian representation of $S_6 \cong Q(5, 2) \otimes Q(5, 2)$ in the group $2^{1+18}_+$.

Let $R = 2^{1+18}_+$ with $R' = \{1, \lambda\}$. Write $R$ as a central product $R = M \circ N$, where $M = 2^{1+12}_+ \text{and } N = 2^{1+6}_+$. Let $Y = Q \cup \overline{Q} \cup \overline{Q}$. Then the subgeometry of $S_6$ whose point set is $Y$ together with the lines of types (L1) – (L4) is isomorphic to $Q(5, 2) \times \mathbb{L}_3$. Let $P$ be the point set of $S_6$ and let $\delta$ be a map from $Q$ to $I_2(4)$ satisfying the conditions of Lemma 5.2. We extend $\delta$ to the set $P \setminus Y$ using the map $\epsilon : Q \rightarrow \mathbb{Z}_3$ which we defined in the beginning of this section:

For $L_1 \in S$, distinct points $a, b \in L_1$ and $j \in \mathbb{Z}_3$, we define $\delta(a, b, j) := \delta(u)$, where $u$ is the unique point of $L_1$ with $\epsilon(u) = j$.

Now, fix a non-abelian representation $(M, \phi)$ of $Y$. Such a representation exists by Section 4. Let $\psi$ be the following map from $P$ to $R$:

- if $q \in Y$, then $\psi(q) := \phi(q)$;
- if $q = (a, b, i) \in P \setminus Y$, then $\psi(q) = \psi(a, b, i) := \phi(b)\phi(\bar{a})\delta(a, b, i)$.

We prove the following.

**Theorem 5.3.** $(R, \psi)$ is a non-abelian representation of $S_6$.

**Proof.** Since the image of $\phi$ generates $M$ and the image of $\delta$ generates $N$, we have $R = \langle \psi(P) \rangle$. For every line $L_1 \in S$ and distinct $a, b \in L_1$, we have $[\phi(a), \phi(b)] = 1$, since $a$ and $\bar{b}$ are in distance two from each other. This implies that $\psi(q)$ is an involution for every $q \in P$. We need to verify condition (R2) in the definition of representation. This is true for all lines of types (L1) – (L4), since they are also lines of $Y$ and $\psi$ coincides with $\phi$ on $Y$.

Let $\{a, (a, b, i), (a, c, i)\}$ be a line of type (L5). Since $\delta(a, b, i) = \delta(a, c, i)$, we have $\psi(a, b, i)\psi(a, c, i) = \phi(b)\phi(\bar{a})\phi(c)\phi(\bar{a}) = \phi(b)\phi(\bar{a}) = \phi(a) = \psi(a)$. Similar argument holds for lines of types (L6) and (L7).

Next, consider a line $\{(a, b, i), (b, c, j), (c, a, k)\}$ of type (L8). We have $\psi(a, b, i)\psi(b, c, j) = \phi(b)\phi(\bar{a})\phi(c)\phi(\bar{b})\delta(a, b, i)\delta(b, c, j)$. Since $\{i, j, k\} = \mathbb{Z}_3$, $\{\delta(a, b, i), \delta(b, c, j), \delta(c, a, k)\} = \{\delta(a), \delta(b), \delta(c)\}$. Since $\{a, b, c\} \in S$, Lemma 5.2(iii) implies that $\delta(a, b, i)\delta(b, c, j) = \delta(c, a, k)$. So, $\psi(a, b, i)\psi(b, c, j) = \phi(b)\phi(c)\phi(\bar{a})\phi(\bar{b})\delta(c, a, k) = \phi(a)\phi(\bar{c})\delta(c, a, k) = \psi(c, a, k)$. Notice that the second equality holds since $\{a, b, c\}$ and $\{\bar{a}, \bar{b}, \bar{c}\}$ are lines of $Y$. 
Finally, consider a line \( \{(a, u, i), (b, v, j), (c, w, k) \} \) of type \((L9)\). Here the lines \( au, bv, cw \) are in \( S \), \( j = i + \theta(au, bv) \) and \( k = i + \theta(au, cw) \). Let \( \delta(a, u, i) = \delta(\alpha) \), \( \delta(b, v, j) = \delta(\beta) \) and \( \delta(c, w, k) = \delta(\gamma) \), where \( \alpha \in au, \beta \in bv \) and \( \gamma \in cw \).

So \( \epsilon(\alpha) = i, \epsilon(\beta) = j \) and \( \epsilon(\gamma) = k \). Since \( \epsilon(\beta) - \epsilon(\alpha) = j - i = \theta(au, bv) \), Lemma 5.1 implies that \( \alpha \sim \beta \). Similarly, \( \alpha \sim \gamma \).

Thus \( \{\alpha, \beta, \gamma\} \) is a line of \( Q \) not contained in \( S \). Then by Lemma 5.2(iii), \( \delta(a, u, i)\delta(b, v, j) = \delta(\alpha)\delta(\beta) = \delta(\gamma)\lambda = \delta(c, w, k)\lambda \). So \( \psi(a, u, i)\psi(b, v, j) = \phi(u)\phi(\bar{a})\phi(v)\phi(\bar{b})\delta(c, w, k)\lambda \).

Since \( v \) and \( \bar{a} \) are in distance three from each other, \( [\phi(\bar{a}), \phi(v)] = \lambda \). So \( \phi(\bar{a})\phi(v) = \phi(v)\phi(\bar{a})[\phi(\bar{a}), \phi(v)] = \phi(v)\phi(\bar{a})\lambda \).

Then \( \psi(a, u, i)\psi(b, v, j) = \phi(u)\phi(v)\phi(\bar{a})\phi(\bar{b})\delta(c, w, k) = \phi(w)\phi(\bar{c})\delta(c, w, k) = \psi(c, w, k) \).

The second equality holds since \( \{\bar{a}, \bar{b}, \bar{c}\} \) and \( \{u, v, w\} \) are lines of \( Y \). This completes the proof.

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