DISCRETE MODELS OF THE SELF-DUAL AND ANTI-SELF-DUAL EQUATIONS

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Abstract. In the case of a gauge-invariant discrete model of Yang-Mills theory difference self-dual and anti-self-dual equations are constructed.

1. Introduction

In 4-dimensional non-abelian gauge theory the self-dual and anti-self-dual connections are the most important extrema of the Yang-Mills action. Consider a trivial bundle \( P = \mathbb{R}^4 \times G \), where \( G \) is some Lie group. We define a connection as some \( g \)-valued 1-form \( A \), where \( g \) is the Lie algebra of the group \( G \) [5]. Then the connection 1-form can be written as follows

\[
A = \sum_{a, \mu} A^a_\mu(x) \lambda_a dx^\mu,
\]

where \( \lambda_a \) is the basis of the Lie algebra \( g \). The curvature 2-form \( F \) of the connection \( A \) is given by

\[
F = dA + A \wedge A.
\]

We specialize straightaway to the choice \( G = SU(2) \), then \( g = su(2) \). We define the covariant exterior differentiation operator \( d_A \) by

\[
d_A \Omega = d\Omega + A \wedge \Omega + (-1)^{r+1} \Omega \wedge A,
\]

where \( \Omega \) is an arbitrary \( su(2) \)-valued \( r \)-form. Compare (2) and (3) we obtain the Bianchi identity

\[
d_A F = 0.
\]

The Yang-Mills action \( S \) can be conveniently expressed (see [5, p. 256]) in terms of the 2-forms \( F \) and \( *F \) as

\[
S = -\int_{\mathbb{R}^4} tr(F \wedge *F),
\]

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where $\ast$ is the adjoint operator (Hodge star operator). The Euler-Lagrange equations for the extrema of $S$ are

$$d_A \ast F = 0. \tag{5}$$

Equations (4), (5) are called the Yang-Mills equations [4]. These equations are non-linear coupled partial differential equations containing quadratic and cubic terms in $A$.

In more traditional form the Yang-Mills equations are expressed in terms of components of the connection $A$ and the curvature $F$ (see [2,3]). Let

$$A_\mu = \sum_\alpha A_\mu^\alpha(x) \lambda_\alpha$$

be the component of the connection 1-form (1). Then the components of the curvature form are given by

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} + \frac{\partial A_\mu}{\partial x^\nu} + [A_\mu, A_\nu],$$

where $[\cdot, \cdot]$ be the commutator of the algebra Lie $su(2)$. In local coordinates the covariant derivative $\nabla_j$ can be written

$$\nabla_j F_{\mu\nu} = \frac{\partial F_{\mu\nu}}{\partial x^j} + [A_j, F_{\mu\nu}].$$

Then we can write Equations (4), (5) as

$$\nabla_j F_{\mu\nu} + \nabla_\mu F_{\nu j} + \nabla_\nu F_{j\mu} = 0, \tag{6}$$

$$\sum_{\mu=1}^4 \frac{\partial F_{\mu\nu}}{\partial x^\mu} + [A_\mu, F_{\mu\nu}] = 0. \tag{7}$$

Note that Equations 7 are obtained in the case of Euclidean space $\mathbb{R}^4$.

The self-dual and anti-self-dual connections are solutions of the following nonlinear first order differential equations

$$F = \ast F, \quad F = - \ast F. \tag{8}$$

Equations (8) are called self-dual and anti-self-dual respectively. It is obviously that if one can find $A$ such that $F = \pm \ast F$, then the Yang-Mills equations (5) are automatically satisfied.

2. The discrete model in $\mathbb{R}^4$

In [6] the gauge invariant discrete model of the Yang-Mills equations is constructed in the case of the $n$-dimensional Euclidean space $\mathbb{R}^n$. Following [6], we consider a combinatorial model of $\mathbb{R}^4$ as a certain 4-dimensional complex $C(4)$. Let $K(4)$ be a dual complex of $C(4)$. The complex $K(4)$ is a 4-dimensional complex of cochains with
su(2)-valued coefficients. We define the discrete analogs of the connection 1-form $A$ and the curvature 2-form $F$ as follows cochains

$$A = \sum_{k} \sum_{i=1}^{4} A^i_k \epsilon^k_i, \quad F = \sum_{k} \sum_{i<j}^{4} F^{ij}_k \epsilon^{k}_{ij},$$

where $A^i_k, F^{ij}_k \in su(2), \; \epsilon^k_i, \; \epsilon^{k}_{ij}$ are 1-, 2-dimensional basis elements of $K(4)$ and $k = (k_1, k_2, k_3, k_4), \; k_i \in \mathbb{Z}$. We use the geometrical construction proposed by A. A. Dezin in [1] to define discrete analogs of the differential, the exterior multiplication and the Hodge star operator.

Let us introduce for convenient the shifts operator $\tau_i$ and $\sigma_i$ as

$$\tau_i k = (k_1, ... \tau_i k_i, ... k_4), \quad \sigma_i k = (k_1, ... \sigma_i k_i, ... k_4),$$

where $\tau_i k_i = k_i + 1$ and $\sigma_i k_i = k_i - 1, \; k_i \in \mathbb{Z}$. Similarly, we denote by $\tau_{ij}$ ($\sigma_{ij}$) the operator shifting to the right (to the left) two differ components of $k = (k_1, k_2, k_3, k_4)$.

For example, $\tau_{12} k = (\tau_1 k_1, \tau_2 k_2, k_3, k_4), \; \sigma_{14} k = (\sigma_1 k_1, k_2, k_3, \sigma_4 k_4)$.

If we use (2) and take the definitions of $d$ and $\wedge$ in discrete case [1,6], then we obtain

$$F^{ij}_k = \Delta_k A^i_{\tau_j k} - \Delta_j A^i_k + A^i_k A^j_{\tau_i k} - A^j_k A^i_{\tau_j k},$$

where $\Delta_k A^i_k = A^i_{\tau_i k} - A^i_k, \; i, j = 1, 2, 3, 4$. The metric adjoint operation $\ast$ acts on the 2-dimensional basis elements of $K(4)$ as follows

$$\ast \epsilon^k_{12} = \epsilon^k_{34}, \quad \ast \epsilon^k_{13} = -\epsilon^k_{23}, \quad \ast \epsilon^k_{14} = \epsilon^k_{24}, \quad \ast \epsilon^k_{23} = -\epsilon^k_{13}, \quad \ast \epsilon^k_{24} = \epsilon^k_{12}.$$ 

Then we obtain

$$\ast F = \sum_k \left( F^{34}_{\sigma_{14} k} \epsilon^k_{12} - F^{24}_{\sigma_{24} k} \epsilon^k_{13} + F^{23}_{\sigma_{23} k} \epsilon^k_{14} + F^{14}_{\sigma_{14} k} \epsilon^k_{23} - F^{13}_{\sigma_{13} k} \epsilon^k_{24} + F^{12}_{\sigma_{12} k} \epsilon^k_{34} \right).$$

Comparing the latter and (9) the discrete analog of the self-dual equation (the first equation of (8)) we can written as follows

$$F^{12}_k = F^{34}_{\sigma_{14} k}, \quad F^{13}_k = -F^{24}_{\sigma_{24} k}, \quad F^{14}_k = F^{23}_{\sigma_{23} k}, \quad F^{23}_k = F^{14}_{\sigma_{14} k}, \quad F^{24}_k = -F^{13}_{\sigma_{13} k}, \quad F^{34}_k = F^{12}_{\sigma_{12} k}$$

for all $k = (k_1, k_2, k_3, k_4), \; k_i \in \mathbb{Z}$. Using (10) Equations (12) can be rewritten in the following difference form:

$$\Delta_{k_1} A^2_k - \Delta_{k_2} A^1_k + A^1_k \cdot A^2_{\tau_1 k} - A^2_k \cdot A^1_{\tau_2 k} =$$

$$= \Delta_{k_3} A^3_{\sigma_{34} k} - \Delta_{k_4} A^3_{\sigma_{34} k} + A^3_{\sigma_{34} k} \cdot A^4_{\sigma_{43} k} - A^4_{\sigma_{34} k} \cdot A^3_{\sigma_{34} k}.$$
Then we have for all \( k \),
\[
\Delta_k A_k^2 - \Delta_{k_4} A_{k_4}^1 + A_k^1 \cdot A_{k_4}^3 - A_k^4 \cdot A_{k_4}^1 =
\]
\[
= -\Delta_k A_{\sigma_2 k}^4 + \Delta_{k_4} A_{\sigma_2 k}^2 - A_{\sigma_2 k}^2 \cdot A_{k_4}^4 + A_{\sigma_2 k}^4 \cdot A_{\sigma_2 k}^4,
\]
\[
\Delta_k A_k^4 - \Delta_{k_4} A_{k_4}^1 + A_k^1 \cdot A_{k_4}^4 - A_k^4 \cdot A_{k_4}^1 =
\]
\[
= \Delta_k A_{\sigma_1 k}^4 - \Delta_{k_4} A_{\sigma_1 k}^1 + A_{\sigma_1 k}^1 \cdot A_{k_4}^4 - A_{\sigma_1 k}^1 \cdot A_{k_4}^4,
\]
\[
= \Delta_k A_{\sigma_1 k}^1 - \Delta_{k_4} A_{\sigma_1 k}^1 + A_{\sigma_1 k}^1 \cdot A_{k_4}^3 - A_{\sigma_1 k}^1 \cdot A_{k_4}^3,
\]
\[
\Delta_k A_k^1 - \Delta_{k_4} A_{k_4}^1 + A_k^1 \cdot A_{k_4}^1 - A_k^4 \cdot A_{k_4}^1 =
\]
\[
= \Delta_k A_{\sigma_3 k}^1 - \Delta_{k_4} A_{\sigma_3 k}^1 + A_{\sigma_3 k}^1 \cdot A_{k_4}^3 - A_{\sigma_3 k}^1 \cdot A_{k_4}^3,
\]
In the same way we obtain the difference anti-self-dual equation. From Equations (12) we obtain at once
\[
F^{jr}_{\sigma_k} = F^{jr}_{\sigma_k}
\]
for all \( j < r, \ r = 2, 3, 4 \), where \( \sigma k = (\sigma k_1, \sigma k_2, \sigma k_3, \sigma k_4) \).

Note that Equations (13) also are satisfied in the case of the difference anti-self-dual equations.

**Proposition 1.** Let \( F \) be a solution of the discrete self-dual or anti-self-dual equations. Then we have
\[
** F = F.
\]

**Proof.** From (11) we have
\[
** F = \sum_k \left( F^{34}_{\sigma_3 k} \epsilon_{12} + F^{24}_{\sigma_2 k} \epsilon_{13} + F^{13}_{\sigma_1 k} \epsilon_{24} + F^{12}_{\sigma_1 k} \epsilon_{34} \right) =
\]
\[
= \sum_k \left( F^{34}_{\sigma_3 k} + F^{24}_{\sigma_2 k} + F^{13}_{\sigma_1 k} + F^{12}_{\sigma_1 k} \right)
\]
\[
= \sum_{i<j} \sum_{k=2}^4 \sum_i F^{ij}_{\sigma_k} \epsilon_{ij}.
\]
Comparing the latter and (13) we obtain (14). \( \square \)
It should be noted that in the case of continual Yang-Mills theory for \(\mathbb{R}^4\) with the usual Euclidean metric Equation (14) is satisfied automatically for an arbitrary 2-form. But in the formalism we use the operation \((*)^2\) is equivalent to a shift.

The difference analog of Equations (13) is given by

\[
\Delta_k A^r_k - \Delta_k A^i_k = A^r_k \cdot A^i_{\tau r k} - A^r_k \cdot A^i_{\tau r k} = \\
= \Delta_k A^r_{\sigma k} - \Delta_k A^i_{\sigma k} + A^r_{\sigma k} \cdot A^i_{\sigma \tau k} - A^r_{\sigma k} \cdot A^i_{\sigma \tau k},
\]

where \(\sigma \tau k = (\sigma k_1...k_j...\sigma k_4)\).

3. The discrete model in Minkowski space

Let a base space of the bundle \(P\) be Minkowski space, i.e. \(\mathbb{R}^4\) with the metric \(g_{\mu \nu} = \text{diag}(-+++).\) In Minkowski space we write Equations (8) as

\[
*F = \mp i F,
\]

where \(i^2 = -1\). Recall that \(F\) is \(g\)-valued, so therefore is \(*F\). Then we must have \(ig = g\) in obvious notation. However, this latter condition is not satisfied for the Lie algebras of any compact Lie groups \(G\). To study Equations (15) we must choose non-compact \(G\) such as \(SL(n, \mathbb{C})\) or \(GL(n, \mathbb{C})\) say. This is a serious restriction since in physics the gauge groups chosen are usually compact [5]. Let the gauge group be \(G = SL(2, \mathbb{C})\).

We suppose that a combinatorial model of Minkowski space has the same structure as \(C(4)\). A gauge-invariant discrete model of the Yang-Mills equations in Minkowski space is given in [7]. Now the dual complex \(K(4)\) is a complex of \(sl(2, \mathbb{C})\)-valued cochains (forms). Because discrete analogs of the differential and the exterior multiplication are not depended on a metric then they have the same form as in the case of Euclidean space. For more details on this point see [7]. However, to define a discrete analog of the \(*\) operation we must take into accounts the Lorentz metric structure on \(K(4)\). We denote by \(\bar{x}_\kappa, \bar{e}_\kappa, \kappa \in \mathbb{Z}\) the basis elements of the 1-dimensional complex \(K\) which are corresponded to the time coordinate of Minkowski space. It is convenient to write the basis elements of \(K(4) = K \otimes K \otimes K \otimes K\) in the form \(\bar{\mu}^\kappa \otimes s^k\), where \(\bar{\mu}^\kappa\) is either \(\bar{x}^\kappa\) or \(\bar{e}^\kappa\) and \(s^k\) is a basis element of \(K(3) = K \otimes K \otimes K\), \(k = (k_1, k_2, k_3)\), \(\kappa, k_j \in \mathbb{Z}\). Then we define the \(*\) operation on \(K(4)\) as follows

\[
\bar{\mu}^\kappa \otimes s^k \cup *(\bar{\mu}^\kappa \otimes s^k) = Q(\mu)\bar{e}^\kappa \otimes e^{k_1} \otimes e^{k_2} \otimes e^{k_3},
\]

where \(Q(\mu)\) is equal to +1 if \(\bar{\mu}^\kappa = \bar{x}^\kappa\) and to -1 if \(\bar{\mu}^\kappa = \bar{e}^\kappa\). To arbitrary forms the \(*\) operation is extended linearly. Using (16) we obtain

\[
*F = \sum_k \left( F_{\sigma 34}^{34} \varepsilon_{12}^k - F_{\sigma 24}^{24} \varepsilon_{13}^k + F_{\sigma 23}^{23} \varepsilon_{14}^k - \\
-F_{\sigma 14}^{14} \varepsilon_{23}^k + F_{\sigma 13}^{13} \varepsilon_{24}^k - F_{\sigma 12}^{12} \varepsilon_{34}^k \right),
\]

where \(F_{ij}^k \in sl(2, \mathbb{C})\). Combining (17) with (9) the discrete self-dual equation \(*F = iF\) can be written as follows

\[
F_{\sigma 34}^{34} = iF_{k}^{12}, \quad F_{\sigma 24}^{24} = iF_{k}^{13}, \quad F_{\sigma 23}^{23} = iF_{k}^{14}, \\
F_{\sigma 14}^{14} = iF_{k}^{23}, \quad F_{\sigma 13}^{13} = iF_{k}^{24}, \quad F_{\sigma 12}^{12} = iF_{k}^{34}.
\]
for all \( k = (k_1, k_2, k_3, k_4), \ k_r \in \mathbb{Z}, \ r = 1, 2, 3, 4. \) From the latter we obtain
\[
F_{\sigma k}^{34} = iF_{\sigma 12 k}^{12} = -i^2 F_{k}^{34} = F_{\sigma k}^{34}, \quad F_{\sigma k}^{24} = -iF_{\sigma 13 k}^{13} = -i^2 F_{k}^{24} = F_{\sigma k}^{24}
\]
and similarly for any other components \( F_{k}^{jr}, \ j < r. \) So we have Relations (13). Thus a solution of the discrete self-dual equations (18) satisfies Equations (13) as in the Euclidean case.

We can also rewrite (18) in the difference form
\[
\Delta k_3 A^3_{\sigma 34 k} - \Delta k_4 A^3_{\sigma 34 k} + A^3_{\sigma 34 k} \cdot A^4_{\sigma 4 k} - A^4_{\sigma 4 k} \cdot A^3_{\sigma 3 k} = \notag
= i(\Delta k_1 A^2_{k} - \Delta k_2 A^1_{k} + A^1_{k} \cdot A^2_{\tau_1 k} - A^2_{k} \cdot A^1_{\tau_2 k}),
\]
\[
-\Delta k_2 A^4_{\sigma 24 k} + \Delta k_4 A^4_{\sigma 24 k} - A^4_{\sigma 24 k} \cdot A^4_{\sigma 4 k} + A^4_{\sigma 4 k} \cdot A^2_{\sigma 2 k} = \notag
= i(\Delta k_1 A^3_{k} - \Delta k_3 A^1_{k} + A^1_{k} \cdot A^3_{\tau_1 k} - A^3_{k} \cdot A^1_{\tau_3 k}),
\]
\[
\Delta k_2 A^3_{\sigma 23 k} - \Delta k_3 A^2_{\sigma 23 k} + A^2_{\sigma 23 k} \cdot A^3_{\sigma 3 k} - A^3_{\sigma 23 k} \cdot A^2_{\sigma 2 k} = \notag
= i(\Delta k_1 A^4_{k} - \Delta k_4 A^1_{k} + A^1_{k} \cdot A^4_{\tau_1 k} - A^4_{k} \cdot A^1_{\tau_4 k}),
\]
\[
-\Delta k_1 A^4_{\sigma 14 k} + \Delta k_4 A^1_{\sigma 14 k} - A^1_{\sigma 14 k} \cdot A^4_{\sigma 4 k} + A^4_{\sigma 4 k} \cdot A^1_{\sigma 1 k} = \notag
= i(\Delta k_2 A^3_{k} - \Delta k_3 A^2_{k} + A^2_{k} \cdot A^3_{\tau_2 k} - A^3_{k} \cdot A^2_{\tau_3 k}),
\]
\[
\Delta k_1 A^3_{\sigma 13 k} - \Delta k_3 A^1_{\sigma 13 k} + A^1_{\sigma 13 k} \cdot A^3_{\sigma 3 k} - A^3_{\sigma 13 k} \cdot A^1_{\sigma 1 k} = \notag
= i(\Delta k_2 A^4_{k} - \Delta k_4 A^2_{k} + A^2_{k} \cdot A^4_{\tau_2 k} - A^4_{k} \cdot A^2_{\tau_4 k}),
\]
\[
-\Delta k_1 A^2_{\sigma 12 k} + \Delta k_4 A^1_{\sigma 12 k} - A^1_{\sigma 12 k} \cdot A^2_{\sigma 2 k} + A^2_{\sigma 2 k} \cdot A^1_{\sigma 1 k} = \notag
= i(\Delta k_3 A^4_{k} - \Delta k_4 A^3_{k} + A^3_{k} \cdot A^4_{\tau_3 k} - A^4_{k} \cdot A^3_{\tau_4 k}).
\]

In similar manner we obtain the difference anti-self-dual equations. Obviously an anti-self-dual solution satisfies Equations (13).

**Proposition 2.** Let for any \( \text{sl}(2, \mathbb{C}) \)-valued 2-form \( F \) Conditions (13) are satisfied. Then we have
\[
** F = - F.
\]

**Proof.** If components of any discrete 2-form \( F \) satisfy (13), then \( F \) is a solution of the discrete self-dual or anti-self-dual equations. Hence
\[
** F = * (\mp i F) = \mp i * F = (\mp i)^2 F = - F.
\]
\( \square \)

**Remark.** In the continual case the self-dual and anti-self-dual equations are written in the form (15) because we have \( ** F = - F \) for an arbitrary 2-form \( F \) in Minkowski space. In the discrete model case it is easy to check that in \( K(4) \) we have
\[
** F = - \sum_{k} \sum_{i<j} \sum_{j=2}^{4} F_{\sigma k}^{ij} \varepsilon_{ij}.
\]
Thus Equations (15) are satisfied only under Conditions (13).

**Theorem.** If there exist some $N = (N_1, N_2, N_3, N_4)$, $N_r \in \mathbb{Z}$ such that

$$F_{ij}^k = 0 \text{ for any } |k| \geq |N|,$$

(19)

then Equations (15) (or (8)) have only the trivial solution $F = 0$.

**Proof.** Since for any solution of Equations (15) (or (8)) we have Relations (13) then the assertion is obvious.

Let $g$ be a discrete 0-form

$$g = \sum_k g_k x^k,$$

where $x^k$ is the 0-dimensional basis element of $K(4)$ and $g_k \in SU(2)$ (or $g_k \in SL(2, \mathbb{C})$). The boundary condition (19) in terms of the connection components can be represented as: there is some discrete 0-form $g$ such that

$$A_{ij}^k = -(\Delta_{kj} g_k) g_k^{-1} \text{ for any } |k| \geq |N|.$$

It follows from Theorem 3 [6].

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