FLOER THEORY FOR HAMILTONIAN PDE USING MODEL THEORY

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Abstract. Under natural restrictions it is known that a nonlinear Schrödinger equation is a Hamiltonian PDE which defines a symplectic flow on a symplectic Hilbert space preserving the Hilbert norm. When the potential is one-periodic in time and after passing to the projectivization, it makes sense to ask whether the natural analogue of the Arnold conjecture holds. After employing methods from non-standard model theory, we show how Hamiltonian Floer theory can be used to prove the existence of infinitely many fixed points of the time-one flow of nonlinear Schrödinger equations of convolution type.

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1. Hamiltonian partial differential equations

Nonlinear Schrödinger equations play a very important role in mathematical physics. In contrast to the well-known linear Schrödinger equation describing the time evolution of the quantum wave function of a single particle, nonlinear Schrödinger equations describe multi-particle...
systems, where the nonlinearity models the interaction between different particles. An example of a nonlinear Schrödinger equation is the so-called Gross-Pitaevskii equation

\[ i\partial_t u = -\Delta u + c|u|^2u + V(t, x)u, \]

which plays an important role in the theory of Bose-Einstein condensates. Here \( u = u(t, x) \in \mathbb{C} \) is a complex-valued function depending on space and time, \( \partial_t \) is the derivative with respect to the time \( t \in \mathbb{R} \), \( \Delta \) denotes the Laplace operator with respect to the spatial coordinate \( x \), \( V(t, x) \) is a time-dependent exterior potential and \( c \in \mathbb{R} \) is a scalar whose sign depends on whether the particles are attracting or repelling each other.

Nonlinear Schrödinger equations are important examples of Hamiltonian partial differential equations, where we refer to [9] for definitions, statements and further references. This means that they can be written in the form \( \partial_t u = X_H^t(u) \), where the Hamiltonian vector field \( X_H^t \) is determined by the choice of a (time-dependent) Hamiltonian function \( H = H_t \) and a linear symplectic form \( \omega \). Here a bilinear form \( \omega : \mathbb{H} \times \mathbb{H} \to \mathbb{R} \) on a real Hilbert space \( \mathbb{H} \) is called symplectic if it is anti-symmetric and nondegenerate in the sense that the induced linear mapping \( i_{\omega} : \mathbb{H} \to \mathbb{H}^* \) is an isomorphism. As in the finite-dimensional case it can be shown that for any symplectic form \( \omega \) there exists a complex structure \( J_0 \) on \( \mathbb{H} \) such that \( \langle \cdot, \cdot \rangle = \omega(\cdot, J_0 \cdot) \).

In the case of nonlinear Schrödinger equations on the circle \( S^1 = \mathbb{R} / 2\pi \mathbb{Z} \), one chooses the Hilbert space \( \mathbb{H} = L^2(S^1, \mathbb{C}) \) of square-integrable complex-valued functions on the circle. The symplectic form \( \omega = \langle \cdot, \cdot \rangle \) is given by pairing the standard complex structure \( J_0 = i \) with the standard real inner product on \( L^2(S^1, \mathbb{R}^2) \). In order to stress the relation with the finite-dimensional case of \( \mathbb{R}^{2n} = \mathbb{C}^n \), note that, using the Fourier series expansion

\[ u(x) = \sum_{n=-\infty}^{\infty} \hat{u}(n) \cdot \exp(inx) = \sum_{n=0}^{\infty} q_n \exp(inx) + p_n \exp(-inx), \]

the symplectic Hilbert space \( L^2(S^1, \mathbb{C}) \) can be identified with the space \( l^2(\mathbb{C}) \) of square-summable complex-valued series \( \hat{u} : \mathbb{Z} \to \mathbb{C} \), equipped with the symplectic form \( \omega = \langle \cdot, \cdot \rangle = \sum_{n=0}^{\infty} dp_n \wedge dq_n \). The corresponding Hamiltonian function is of the form

\[ H_t(u) = \int_0^{2\pi} \frac{|\hat{u}|^2}{2} dx + F_t(u) \text{ with } F_t(u) = \int_0^{2\pi} \frac{1}{2} f(|u|^2, x, t) dx, \]

where \( f \) is a smooth, real-valued function on \( \mathbb{R}^+ \times S^1 \times \mathbb{R} \). Note that the Gross-Pitaveskii equation is recovered by setting
f(|u|^2, x, t) := c/2 \cdot |u|^4 + V(t, x) \cdot |u|^2.

2. Nonlinear Schrödinger equations of convolution type

While the symplectic form ω is nondegenerate on $L^2(S^1, \mathbb{C})$, the Hamiltonian $H_t$ is only well-defined and smooth on its dense subspace $H^{1,2}(S^1, \mathbb{C})$. While this is immediately apparent for the first summand as it involves the first derivative, observe that even the Hamiltonian $F_t$ modelling the nonlinearity is not defined on all of $\mathbb{H}$ when the resulting Schrödinger equation is truly nonlinear. This in turn immediately leads to problems with the existence of the corresponding Hamiltonian flow $\phi_t$, describing the time-evolution of solutions of the nonlinear Schrödinger equation.

We start with the case of the free nonlinear Schrödinger equation, that is, when the nonlinearity $f$ is equal to zero. Although the Hamiltonian vector field $X^0(u) = i\Delta u$ of the resulting Hamiltonian given by $H^0$ is only defined on $H^{2,2}(S^1, \mathbb{C})$, it is easy to see that the resulting linear flow given by

$$\phi^0_t(u) = \exp(it\Delta)(u) = \sum_{k=0}^{\infty} (it)^k \cdot \Delta^k(u)$$

extends to the full symplectic Hilbert space $\mathbb{H} = L^2(S^1, \mathbb{C})$. In order to see this, observe that, after applying the Fourier transform, the linear flow, still denoted by $\phi^0_t$, is given by

$$\phi^0_t(\hat{u})(n) = \exp(+i\tau^2) \cdot \hat{u}(n) \text{ for all } n \in \mathbb{N},$$

that is, in every frequency it multiplies the Fourier coefficient by a complex number of norm one.

On the other hand, if the Hamiltonian $F_t$ describing the nonlinearity is only densely defined, it is typically a very hard problem to establish the existence of a corresponding Hamiltonian flow $\phi^F_t$ on the full phase space $\mathbb{H}$. The problem is that the flow on $\mathbb{H}$ is no longer the unique solution to an ordinary differential equation given by the Hamiltonian vector field $X^F_t$. In order to circumvent problems arising from missing regularity in the nonlinear term, in this paper we restrict ourselves to nonlinear Schrödinger equations of convolution type, see [9]. Instead of considering density functions of the form $f(|u|^2, x, t)$, from now on we consider density functions of the form $f(|u * \psi|^2, x, t)$, where $\psi \in C^\infty(S^1, \mathbb{R})$ is some fixed smoothing kernel and $f$ still denotes a smooth, real-valued function on $\mathbb{R}^+ \times S^1 \times \mathbb{R}$. Redefining

$$F_t(u) := \int_0^{2\pi} \frac{1}{2} f(|u * \psi|^2, x, t) \, dx,$$
the comparison with the above example and a short computation show that the resulting nonlinear Schrödinger equations are given by

\[ i\partial_t u = -\Delta u + \partial_t f(|u*\psi|^2, x, t)(u*\psi)*\psi, \]

where \( \partial_t f \) means derivative with respect to the first coordinate.

Nonlinear Schrödinger equations of convolution type describe multiparticle systems with nonlocal interaction. We collect two important observations about these equations in the following

**Lemma 2.1.** For every nonlinear Schrödinger equation of convolution type the resulting flow \( \phi_t \) is defined on the full Hilbert space \( \mathbb{H} = L^2(S^1, \mathbb{C}) \). Furthermore it preserves the Hilbert space norm and hence descends to a symplectic flow on the projective Hilbert space \( \mathbb{P}(\mathbb{H}) \) equipped with the Fubini-Study form.

**Proof.** Since the convolution of a function \( u \in L^2(S^1, \mathbb{C}) \) with the smooth function \( \psi \) ensures that the resulting function is smooth, i.e., \( u*\psi \in C^\infty(S^1, \mathbb{C}) \), it immediately follows that resulting Hamiltonian \( F_t \) is well-defined and smooth on the full symplectic Hilbert space. In particular, the existence of a smooth Hamiltonian flow \( \phi^H_t \) on all of \( \mathbb{H} \) is ensured by the fact that it is defined by an ordinary differential equation on \( \mathbb{H} \). Since the same argument still holds true after replacing \( F_t \) by \( G_t := F_t \circ \phi^0_{-t} \) with the linear symplectic flow \( \phi^0_t \) on \( \mathbb{H} \), we see that the flow \( \phi_t := \phi^H_t \) of the full Hamiltonian \( H_t = H^0 + F_t \) given by \( \phi_t = \phi_t^H = \phi_t^I \circ \phi^0_t \) is also defined and smooth on the full symplectic Hilbert space \( \mathbb{H} = L^2(S^1, \mathbb{C}) \).

In order to prove that the flow \( \phi_t \) preserves the \( L^2 \)-norm, we first observe that this obviously holds true for the flow \( \phi^0_t \) in the free case, and we hence only need to show that it holds true for the flow of \( F_t \) (and hence of \( G_t \)). For this it suffices to show that, at every point \( u \in \mathbb{H} \), the symplectic gradient \( X^F(u) = X^F_t(u) = i\partial_t f(|u*\psi|^2, x, t)(u*\psi)*\psi \in \mathbb{H} \) is perpendicular to \( u \) with respect to the real inner product on \( \mathbb{H} = L^2(S^1, \mathbb{R}^2) \). First observe that the statement is immediately clear if \( u \) is truely real (that is, \( u(x) \in \mathbb{R} \subset \mathbb{C} \) for all \( x \in S^1 \)) or truely imaginary, as in this case \( X^F(u) \) is truely imaginary, or truely real, respectively. For the general case write \( u = u_R + iu_I \) and \( X^F(u) = X^F_R(u) + iX^F_I(u) \) with real-valued functions \( u_R, u_I, X^F_R, X^F_I \). In this case we can use the compatibility of the real \( L^2 \)-inner product with the product and convolution of functions to show that

\[ \langle u_R, \partial_t f(|u*\psi|^2, x, t)(u_I*\psi)*\psi \rangle = \langle u_I, \partial_t f(|u*\psi|^2, x, t)(u_R*\psi)*\psi \rangle \]

which in turn proves that \( \langle u, X^F(u) \rangle = \langle u_I, X^F_R(u) \rangle - \langle u_R, X^F_I(u) \rangle = 0 \).
In what follows we view the projective Hilbert space as quotient of the unit sphere $S(H)$ in $H$ by the action of $U(1) = S^1$, $\mathbb{P}(H) = S(H)/S^1$. Note that studying the (nonlinear) Schrödinger equation on $\mathbb{P}(H)$ in place of $H$ is also natural from the view point of quantum physics, in the sense that two quantum states sitting on the same projective line represent the same physical reality.

Before we can state the main theorem, we however first need to introduce a small technical assumption, which will play the role of a nondegeneracy condition for the Hamiltonian in infinite dimensions. Note that for the time-one flow map $\phi^1_0$ of the free Schrödinger equation it holds that its sequence of complex eigenvalues given by $\exp(im^2)$, $m \in \mathbb{Z}$, has $z = 1$ (and every other point on the unit circle) as an accumulation point. In order to avoid resulting problems with the lack of nondegeneracy of the free Schrödinger equation, we restrict ourselves to smoothing kernels $\psi$ which are admissible in the following sense: Denoting by $\hat{\psi} : \mathbb{Z} \rightarrow \mathbb{C}$ the Fourier transform of $\psi : S^1 \rightarrow \mathbb{R}$, we require that there exists some $\delta > 0$ such that $\hat{\psi}(m) = 0$ whenever $|\exp(im^2) - 1| < \delta$ for all frequencies $m \in \mathbb{Z}$. We emphasize that the threshold $\delta > 0$ can be chosen arbitrarily small and that every finite-dimensional nonlinearity, that is, where $\text{supp}(\hat{\psi}) \subset \{-\ell, +\ell\}$ for some $\ell \in \mathbb{N}$, is clearly admissible. On the other hand, for every $\delta < 2$ it follows by the same arguments that the spectrum of allowed frequencies $M_\delta := \{m \in \mathbb{Z} : |\exp(im^2) - 1| \geq \delta\}$ is unbounded in both directions.

3. Statement of the main theorem

From now on let us assume that the nonlinear term in the Schrödinger equation is one-periodic in time, that is, $f(|u*\psi|^2, x, t + 1) = f(|u*\psi|^2, x, t)$. Then it follows that every nonlinear Schrödinger equation defines a flow $\phi_t$ on the projective Hilbert space $\mathbb{P}(H)$ where the underlying Hamiltonian is one-periodic in time, $H_{t+1} = H^0 + F_{t+1} = H^0 + F_t = H_t$.

In the case of time-one-periodic smooth Hamiltonians on finite-dimensional projective spaces $\mathbb{C}P^n = \mathbb{P}(\mathbb{C}^{n+1})$, the degenerate version of the famous Arnold conjecture, proven in [17], asserts that the time-one map of the resulting Hamiltonian flow has at least $n + 1$ fixed points, provided that the so-called Hofer norm

$$|||H||| := \int_0^1 (\max H_t - \min H_t) \, dt$$

of the time-periodic Hamiltonian $H_t$ is strictly smaller than $\pi/2$, that is, $1/2$ of the minimal area of a non-constant holomorphic sphere in $\mathbb{C}P^n$. 
Viewing the Hamiltonian flow \( \phi_t \) on \( \mathbb{P}(\mathbb{H}) \) defined by the nonlinear Schrödinger equation of convolution type as an infinite-dimensional generalization, it is natural to ask whether an analogue of the Arnold conjecture also holds in this infinite dimensional context, establishing the existence of infinitely many fixed points of the time-one map. While the Hofer norm of \( H_t = H^0 + F_t \) is obviously infinite, note that in the case of the free Schrödinger equation with \( F_t = 0 \) it is easy to see that, after passing to the projectivization, the time-one flow map \( \phi^0 = \phi^0_t \) on \( \mathbb{P}(\mathbb{H}) \) has infinitely many different fixed points \( u^0_n \) given by the complex oscillations,

\[
u^0_n : S^1 \to \mathbb{C}, \quad u^0_n(x) = \exp(inx) \quad \text{for every } n \in \mathbb{N}.
\]

From now on let \( F_t, G_t = F_t \circ \phi^0_t \) and \( \phi_t = \phi^G_t \circ \phi^0_t \) denote the corresponding functions and flows on the projective Hilbert space \( \mathbb{P}(\mathbb{H}) = \mathbb{S}(\mathbb{H})/S^1 \). In this paper we prove the following infinite-dimensional version of Floer’s work on the Arnold conjecture.

**Theorem 3.1.** Assume that, after descending to the projectivization, the Hofer norm \( |||F||| \) of the Hamiltonian defining the nonlinearity is smaller than \( \pi/4 \) and that the underlying smoothing kernel is admissible. Then the time-one flow map \( \phi = \phi_1 \) on the projective Hilbert space \( \mathbb{P}(L^2(S^1, \mathbb{C})) \) of the corresponding nonlinear Schrödinger equation of convolution type has infinitely many different fixed points \( u^1_n \in \mathbb{P}(\mathbb{H}) \), \( n \in \mathbb{N} \). Furthermore, for every \( n \in \mathbb{N} \) there exists a Floer strip connecting \( u^1_n \) with \( u^0_n \).

In order to see that the condition on the Hofer norm of the nonlinearity is not too restrictive, we remark that we show below that the Hofer norm is indeed always finite for nonlinearities of convolution type, so that the condition can always be fulfilled after rescaling the function \( f \). Furthermore, we do not claim that the bound on the Hofer norm is sharp in any sense; indeed we only want to stress the fact that this is not a perturbative result in the spirit of KAM theory.

We first explain the last statement about the existence of a Floer strip connecting the fixed point \( u^0_n \) of the free Schrödinger equation with a fixed point \( u^1_n \) of the given Schrödinger equation of convolution type, which we view as a path \( u^1_s : [0, 1] \to \mathbb{P}(\mathbb{H}) \) with \( u^1_s(1) = \phi^0(\bar{u}^1_n(0)) \) and \( \partial_t u^1_n = X^G_t(\bar{u}^1_n) \). With this we mean a smooth map \( \tilde{u} : \mathbb{R} \times [0, 1] \to \mathbb{P}(\mathbb{H}) \) with \( \tilde{u}(\cdot, 1) = \phi^0(\tilde{u}(\cdot, 0)) \) satisfying the Floer equation

\[
0 = \partial_t \tilde{u} + \varphi(s) \cdot \nabla G_t(\tilde{u}),
\]

where \( \bar{\partial} = \partial_s + i \partial_t \) denotes the standard Cauchy-Riemann operator and \( \varphi \) is a smooth cut-off function with \( \varphi(s) = 0 \) for \( s \leq -1 \) and \( \varphi(s) = 1 \) for \( s \geq 0 \). It connects \( u^0_n \) and \( u^1_n \) in the sense that \( \tilde{u}(s, \cdot) \to u^0_n \) as \( s \to -\infty \) and there exists a sequence \( (s_k) \) of positive real number converging to \( +\infty \) such that \( \tilde{u}(s_k, \cdot) \to u^1_n \) as \( k \to \infty \). The latter
weaker asymptotic condition is a consequence of the fact that we do not want to assume that the nonlinearity is generic in the sense that all orbits are isolated.

In order to see that there are indeed infinitely many different fixed points \( u_{n} \), we will use bounds for their symplectic action. Note that, in the case of the free Schrödinger equation, the fixed points \( u_{0} \) of the time-one map \( \phi^{0} \) are distinguished by their symplectic action, defined in \( \mathbb{R} \). By choosing \( u_{0} = 1 \) as reference fixed point and choosing for each \( u_{0} \) a holomorphic strip \( \tilde{u}_{0} : \mathbb{R} \times [0, 1] \rightarrow \mathbb{P}(\mathbb{H}) \) with \( \tilde{u}_{0} (1, \cdot) = \phi^{0} (\tilde{u}_{0} (0, \cdot)) \), \( \tilde{u}_{0} (s, \cdot) \rightarrow u_{0}^{0} \) as \( s \rightarrow +\infty \) and \( \tilde{u}_{0} (s, \cdot) \rightarrow u_{0}^{0} \) as \( s \rightarrow -\infty \) given by a Morse gradient flow line of \( H^{0} \), it is easy to see that the symplectic action \( A(u_{0}^{0}) \) is given by

\[
A(u_{0}^{0}) := \int_{0}^{1} \tilde{u}_{0}^{*} \omega = -H^{0}(u_{0}^{0}) = \frac{n^{2}}{2},
\]

in particular, it grows quadratically with \( n \in \mathbb{N} \). Note that here we use the translation between Hamiltonian Floer theory and Floer for general symplectomorphisms, building on the fact that \( H^{0} \) restricts to a smooth function on every \( \mathbb{C}P^{2n} \subset \mathbb{P}(\mathbb{H}) \) and every gradient flow line connecting \( u_{0}^{0} \) and \( u_{0}^{0} \) indeed stays inside the corresponding finite-dimensional projective subspace. Using that the Hofer norm \( |||F||| \) of the nonlinearity is smaller than \( 1/4 \) of the minimal energy of a holomorphic sphere, we show below that the symplectic action of \( u_{m} \) and \( u_{n} \) is well-defined and has to be different for different \( m, n > 4 \).

Finally, in order to see that we cannot expect that the Arnold conjecture generalizes immediately to infinite-dimensional manifold, consider the smooth Hamiltonian \( L(u) = \int_{0}^{2\pi} V(x)|u|^{2}/2 \), whose linear time-one map on \( \mathbb{H} \) is given by \( (\phi^{1}(u))(x) = \exp(iV(x)) \cdot u(x) \). Here it is immediate to check that, for a general choice of the function \( V : S^{1} \rightarrow \mathbb{R} \), for every solution \( u \) one has \( u(x) = 0 \) almost everywhere, resulting in the fact that its time-one map on \( \mathbb{P}(\mathbb{H}) \) has no fixed points at all.

4. Case of finite-dimensional nonlinearities

Indeed, like for the linear Schrödinger equation, in our proof the appearance of the Laplace term in the nonlinear Schrödinger equations turns out to be essential to find infinitely many fixed points.

Forgetting for the moment that we are working in the setting of infinite-dimensional symplectic manifolds, one is naturally tempted to employ Gromov-Floer’s theory of holomorphic curves to establish the existence of infinitely many fixed points for the case of general nonlinear Schrödinger equations of convolution type, by studying appropriate moduli spaces of pseudo-holomorphic curves. In analogy to Gromov’s
existence proof of symplectic fixed points in [7], the idea would be to study moduli spaces of (modified) Floer strips \((\tilde{u}, T)\), where \(T \in \mathbb{R}_0^+\) is some non-negative real number and \(\tilde{u}\) denotes a smooth map \(\tilde{u} : \mathbb{R} \times [0, 1] \to \mathbb{P}(\mathbb{H})\) with \(\tilde{u}(s, t) \to u_n^0\) as \(s \to \pm \infty\), satisfying the periodicity condition \(\tilde{u}(\cdot, 1) = \phi^0(\tilde{u}(\cdot, 0))\) and the perturbed Cauchy-Riemann equation

\[
\bar{\partial}_t G \tilde{u} = \bar{\partial} \tilde{u} + \varphi_T(s) \cdot \nabla G_t(\tilde{u}) = 0,
\]

where \(\varphi_T : \mathbb{R} \to [0, 1]\) now denotes a family of smooth cut-off functions with compact support with \(\varphi_T(s) = 1\) for \(s \in [-T, +T]\).

Assuming that, as in the case of finite-dimensional projective spaces, one could compactify the above moduli space by just adding broken holomorphic strips corresponding to the case that \(T\) converges to \(+\infty\), one could immediately establish the existence of a fixed point of the time-one map \(\phi = \phi_1\), that is, a path \(u_1^1 : [0, 1] \to \mathbb{P}(\mathbb{H})\) satisfying \(u_1^1(1) = \phi^0(u_1^1(0))\) and \(\partial_t u_1^1 = X_1^G(u)\). Note that for the underlying compactness statement we would employ the uniform energy bound, guaranteed by the bound on the Hofer norm of \(F\). Still assuming that everything carries over naturally to the infinite-dimensional setup, the bound for the Hofer norm could indeed be used to prove that we obtain infinitely many different fixed point of \(\phi\) this way, using that the symplectic action of \(u_n^0\) agrees with \(-H^0(u_n^0) = n^2/2\) and hence grows (in norm) quadratically with \(n\).

Before we give more details on the strategy of our proof, in particular, explain how to deal with the problem of infinite dimensions, we first restrict ourselves to the case of finite-dimensional nonlinearities. With this we mean that the support of the Fourier transform \(\hat{\psi} : \mathbb{Z} \to \mathbb{C}\) of the smoothing kernel \(\psi\) is finite, that is, \(\text{supp}(\hat{\psi}) \subset \{-\ell, +\ell\}\) for some natural number \(\ell\). Identifying the symplectic Hilbert space \(\mathbb{H}\) with \(\ell^2(\mathbb{C})\) using the Fourier transform, it follows that \(G\) just depends on its value after applying the projection \(\pi_\ell : \mathbb{H} \to \mathbb{C}^{2\ell+1}\) onto the finite-dimensional symplectic subspace, \(G = G^\ell := G \circ \pi_\ell\); in particular, at every point the gradients \(\nabla G_t\) and \(X_t^G\) are vectors in \(\mathbb{C}^{2\ell+1} \subset \mathbb{H}\). We claim that in this case the existence of the Floer strip (and hence also of the fixed point of the time-one flow) follows from finite-dimensional Hamiltonian Floer theory - together with Liouville’s theorem.

Indeed, fixing \(n \leq \ell\) and assuming for the moment that the map \(\tilde{u}\) exists, let \(\tilde{u}^\ell = \pi_\ell \circ \tilde{u} : \mathbb{R} \times [0, 1] \to \mathbb{C}^{2\ell}\) denote the image of \(\tilde{u} : \mathbb{R} \times [0, 1] \to \mathbb{P}(\mathbb{H})\) under the projection from \(\mathbb{P}(\mathbb{H})\) to \(\mathbb{C}^{2\ell}\). Then it follows that \(\tilde{u}^\ell(s, t) \to u_n^0\) as \(s \to \pm \infty\), \(\tilde{u}^\ell\) satisfies the periodicity
Hamiltonian Floer theory. We claim that, in the case when \( n \) of \( \tilde{R} \) equation \( \tilde{u} \) that is, \( u_n^0 \in \mathbb{CP}^{2\ell} \), we indeed have that \( \tilde{u} \) has image in \( \mathbb{CP}^{2\ell} \subset \mathbb{P}(\mathbb{H}) \) and hence \( \hat{u} = \tilde{u}^\ell \).

Indeed, we first observe that, since \( \tilde{u} = u_n^0 \in \mathbb{CP}^{2n} \subset \mathbb{CP}^{2\ell} \) for \( T = 0 \), we may assume that the Floer strip \( \hat{u} \) sits in a tubular neighborhood of \( \mathbb{CP}^{2\ell} \) in \( \mathbb{P}(\mathbb{H}) \), possibly after passing to \( 0 < T' < T \). It follows that we can write \( \hat{u} \) as a pair of maps,

\[
\hat{u} = (\hat{u}^\perp, \hat{u}^\parallel) \colon [0, 1] \to (\mathbb{H} / \mathbb{C}^{2\ell+1}) \times \mathbb{CP}^{2\ell},
\]

where \( \hat{u}^\perp \) remembers the normal component. Here \( u_n^0 \) shall be viewed as a section in the pull-back \( (u^\parallel)^*N \to \mathbb{R} \times [0, 1] \) of the normal bundle of \( \mathbb{CP}^{2\ell} \subset \mathbb{P}(\mathbb{H}) \) which is unitarily trivial even after applying the natural identifications. With the latter we mean the natural identification of the fibre over \( (s, 1) \) (i.e., the fibre over \( (s, 1) = \phi_1^0(\hat{u}^\parallel(s, 0)) \) of \( N \)) with the fibre over \( (s, 0) \) using \( \phi_1^0 \) for all \( s \in \mathbb{R} \) and, after compactifying, of the fibre over \( (+\infty, t) \) with the fibre over \( (-\infty, t) \) for all \( t \in [0, 1] \), using that \( \tilde{u}^\parallel(s, t) \to u_n^0 \) as \( s \to \pm \infty \). In order to see that the bundle still remains trivial, observe that finite-dimensional Floer theory provides us with a homotopy from \( \hat{u} \) to the constant strip \( u_n^0 \).

Now the important observation is that, since \( G = G^\ell := G \circ \pi_\ell \), the projection of \( \nabla G_t \) to \( \mathbb{H} / \mathbb{C}^{2\ell+1} \) vanishes. This however implies that the perpendicular component \( \hat{u}^\perp \) is indeed truly holomorphic, that is, solves the unperturbed Cauchy-Riemann equation \( \partial\bar{\partial}\hat{u}^\parallel = 0 \). Since \( \hat{u}^\parallel_0(s, t) \to 0 \) for \( s \to \pm \infty \) as \( u_n^0 \in \mathbb{CP}^{2\ell} \), we can employ Liouville’s theorem to show that we indeed have \( \hat{u}^\perp = 0 \), that is, \( \hat{u} = \hat{u}^\parallel \). Note that, instead of referring to Liouville’s theorem, the result can be viewed as a consequence of the minimal surface property of pseudo-holomorphic curves. Indeed, using the technical lemma \ref{lem:technical-lemma}, proven below, the result also follows from the fact the \( L^2 \)-norm of \( \partial_s\hat{u}^\parallel \) is zero.

It follows that, although we first allowed the Floer strip \( \hat{u} \) to live in the infinite-dimensional manifold \( \mathbb{P}(\mathbb{H}) \), the finite-dimensionality of the nonlinearity ensures that it actually lives in the finite-dimensional submanifold \( \mathbb{CP}^{2\ell} \). Along the same lines using Liouville’s theorem or the minimal surface property, it is immediate to see that, in the case of \( n > \ell \), the Floer strip is constant and the fixed point \( u_n^0 \in \mathbb{CP}^{2n} \), \( u_n^0(x) = \exp(inx) \) of the free Schrödinger equation thus agrees with the corresponding fixed point \( u_n^1 \) of the nonlinear Schrödinger equation with convolution term. Furthermore, the same argument shows that
for \( n \in M_\delta \) the resulting fixed point \( u^1_n \) of the Schrödinger equation with nonlinearity cannot agree with any of trivial fixed points \( u^0_m \) for \( m \in \mathbb{Z} \) with \( \hat{\psi}(m) = 0 \). In particular, when the nonlinearity is not zero, there is always a nontrivial fixed point.

5. Strategy for the general infinite-dimensional case

While we will show below that, in the case of nonlinearities of convolution type, the Hofer norm of \( F \) is still finite, it is quite apparent that the underlying theory of pseudo-holomorphic curves does not instantly carry over from finite to infinite dimensions. Apart from problems with establishing the Fredholm property of the relevant nonlinear Cauchy-Riemann operators, the non-compactness of the target manifold leads to the fact that Gromov’s compactness theorem does not naturally generalize from finite projective spaces to \( \mathbb{P}(\mathbb{H}) \). While the work of Sukhov and Tumanov in [18] already illustrates the great challenges that arise when one wants to generalize Gromov’s theory of pseudo-holomorphic curves to infinite dimensions, in this paper we make use of a return ticket from infinite to finite dimensions: We will show how to prove the existence of the relevant pseudo-holomorphic curves in the infinite-dimensional setting using methods from model theory of mathematical logic.

Indeed it is well-known, see e.g. [10], [11] and [8], that there exist so-called non-standard models of mathematics in which there exists an extension of the notion of finiteness: There exist new so-called unlimited *-real (and *-natural) numbers which are greater than all standard real (and natural) numbers; in an analogous way there exist infinitesimal numbers, whose moduli are smaller than any positive standard real number. These *-real numbers can be introduced, using the axiom of choice, as equivalence classes of sequences of real numbers, where the standard numbers are included as constant sequences, while sequences converging to \( \pm \infty \) or 0 are examples of unlimited and infinitesimal numbers, respectively. In this paper we will use the resulting surprising fact that there exists a *-finite-dimensional symplectic space \( \mathbb{F} \), i.e., a finite-dimensional symplectic space in the sense of the non-standard model, which contains the infinite-dimensional Hilbert space \( \mathbb{H} \) as a subspace. Furthermore, after passing to the projectivizations \( \mathbb{P}(\mathbb{H}) \subset \mathbb{P}(\mathbb{F}) \), the smooth infinite-dimensional Hamiltonian flows \( \phi_t = \phi^\mathbb{H}_t \) can be represented by a *-finite-dimensional Hamiltonian flow \( \phi^\mathbb{F}_t \).

While the existence of ideal elements such as \( \mathbb{F} \) is established using the so-called saturation principle, the so-called transfer principle ensures that every statement that holds in finite dimensions and can
be formulated in first-order logic has an analogue in the *-finite-dimensional setting. In particular, applied to the Arnold conjecture for Hamiltonian flows on finite-dimensional projective spaces, this immediately proves that the *-finite-dimensional Hamiltonian flow map $\phi_1^F$ has fixed points, as the dimension of $\mathbb{F}$ (in the sense of the new model) is greater than any standard natural number. While it seems that this immediately proves the existence of fixed points of $\phi_1^\mathbb{H}$ on $\mathbb{P}(\mathbb{H})$, we emphasize that, without further reasoning, we do not know whether any of these fixed points actually sits in the infinite-dimensional projective Hilbert space, as the inclusion $\mathbb{P}(\mathbb{H}) \subset \mathbb{P}(\mathbb{F})$ is always proper.

Instead we will employ that, for every fixed point $u_n^0 \in \mathbb{P}(\mathbb{H}) \subset \mathbb{P}(\mathbb{F})$ of the free nonlinear Schrödinger equation and every positive *-real number $T$, the transfer principle indeed provides us with the existence of a Floer strip $\tilde{u}$ in $\mathbb{P}(\mathbb{F})$ converging asymptotically towards $u_n^0$ and satisfying the perturbed Cauchy-Riemann equation for the given $T$. In contrast to the case of fixed points, we can now employ a non-standard version of the minimal surface argument used for the finite-dimensional case to prove that the image of $\tilde{u}$ has to lie in an infinitesimal neighborhood of $\mathbb{P}(\mathbb{H})$. For this we are going to use that nonlinear Schrödinger equations of convolution type can be uniformly approximated by finite-dimensional Hamiltonian flows, together with the fact that the given fixed points $u_n^0$ already sit in $\mathbb{P}(\mathbb{H}) \subset \mathbb{P}(\mathbb{F})$.

In order to finish the proof, it suffices to choose some unlimited $T$ and use that, by the finite-dimensional approximability, the Hofer norm of the Hamiltonian $F$ modelling the nonlinearity is finite. Using this one can show that there exists some point in the image of the Floer strip such that the distance between this point and its image under $\phi_1^F$ is infinitesimal, i.e., smaller than any positive real number. This in turn proves the existence of a fixed point $u_n^1$ of $\phi_1^\mathbb{H}$, that is, a time-one-periodic solution of the given nonlinear Schrödinger equation of convolution type. In order to see that one obtains infinitely many different fixed points this way, we use, employing again the transfer principle, that the symplectic actions of the fixed points $u_n^0$ grow quadratically, together with the bound on the Hofer norm of $F$.

Summarizing we hence use that the existence of the fixed point and the corresponding Floer strip is guaranteed in the abstract set $\mathbb{P}(\mathbb{F})$ and we then show afterwards, using further properties, that both indeed sit in the correct subset $\mathbb{P}(\mathbb{H}) \subset \mathbb{P}(\mathbb{F})$. Very informally speaking, viewing $\mathbb{P}(\mathbb{F})$ as an enlargement of $\mathbb{P}(\mathbb{H})$ obtained by dropping any request on convergence, the existence follows from the proof in finite dimensions by allowing every number to be infinite, while convergence is only shown...
a posteriori. It might be instructive to compare our approach to the problem of finding zeroes of a polynomial in \( \mathbb{R} \): While the existence of a zero is always guaranteed in the abstract set \( \mathbb{C} \) containing \( \mathbb{R} \), one might be able to prove a posteriori that this zero has to be in \( \mathbb{R} \), by using further properties of the given polynomial.

### 6. \(*\)-Finite-dimensional representations of infinite-dimensional flows

In this section we prove the existence of the \(*\)-finite-dimensional symplectic vector space \( \mathcal{F} \) that contains the symplectic Hilbert space \( \mathcal{H} = L^2(S^1, \mathbb{C}) \cong \ell^2(\mathbb{C}) \) and of the \(*\)-finite-dimensional Hamiltonian flows that represent the infinite-dimensional Hamiltonian flows introduced earlier.

Although the appendix gives a basic introduction to non-standard model theory with all details, we start with a quick summary of the main ideas that are needed to follow our arguments. Nonetheless we ask the reader to consult the appendix below for precise statements and further details and examples.

It is the fundamental idea of non-standard model theory to enlarge every standard set \( A \) to a non-standard set \( *A \) which contains additional new ideal elements. This means that we have a map \( * : V \to W \), where \( V \) is a set of standard sets, called standard model, and \( W \) is the corresponding set of non-standard sets, called the non-standard model. More precisely, the standard and the non-standard model come with a filtration, \( V = (V_n)_{n \in \mathbb{N}} \) and \( W = (W_n)_{n \in \mathbb{N}} \), and the map \( * \) respects this filtration. In the appendix we show how to define \( V = (V_n)_{n \in \mathbb{N}} \) in such a way that it contains all sets that are needed for our proof. On the other hand, we also show how to prove the existence of the corresponding non-standard model \( W = (W_n)_{n \in \mathbb{N}} \) together with a transfer map \( * : V \to W \) respecting the filtration.

All applications of non-standard model theory rely on the following two principles:

- **Transfer principle**: If a theorem holds in the standard model \( V \), then the same theorem holds in the non-standard model \( W \), after replacing the elements from \( V \) by their images in \( W \) under \( * \). Informally speaking, this means that every mathematical statement about standard sets also holds for their non-standard extensions.

- **Saturation principle**: If \( (A_i)_{i \in I} \) is a collection of sets in \( W \) (and \( I \) is a set in \( V \)) satisfying \( A_{i_1} \cap \ldots \cap A_{i_n} \neq \emptyset \) for all
\[ i_1, \ldots, i_n \in I, \ n \in \mathbb{N} \ (\text{finite intersection property}), \] then also the common intersection of all \( A_i, \ i \in I \) is non-empty, 
\[ \bigcap_{i \in I} A_i \neq \emptyset. \]

Denoting, as before, by \( *a \in W \) the image of \( a \in V \) under \( * \), the transfer principle immediately implies that \( * : V \to W \) is indeed an embedding, since \( a \neq b \) in \( V \) implies that \( *a \neq *b \) in \( W \). Furthermore we have \( *\{a_1, \ldots, a_n\} = \{*a_1, \ldots, *a_n\} \) for all finite sets. On the other hand, if \( A \) is a set in \( V \) with infinitely many elements, then it easily follows from the saturation principle that \( *A \) is strictly larger than \( \overline{A} := \{a : a \in A\} \). In the latter case it even holds that \( A \) is not even a set in the new model \( W \). In particular, this applies to our original Hilbert space \( H := \overline{H} \), viewed as a subset of \( *H \). Indeed one can show that every non-standard model is necessarily not full, that is, there exist subsets of sets in \( W \) which do not belong to \( W \) itself. While this destroys all obvious logical paradoxes, on the positive side the transfer principle implies that every definable subset (subset of elements of a set in \( W \) which fulfill a sentence in the language of \( W \)) still belongs to \( W \), so the lack of fullness does not cause problems either.

While it is a classical result in non-standard model theory and easy to prove that every non-standard natural number \( N \) in \( *\mathbb{N} \setminus \mathbb{N} \) is unlimited in the sense that \( N > n \) for all standard natural numbers \( n \in \mathbb{N} \), the non-standard extension \( *\mathbb{R} \) of \( \mathbb{R} \) contains, among other elements, infinitesimal numbers \( \epsilon \approx 0 \) with the defining property that \( \epsilon < r \) for all positive standard real numbers \( r \in \mathbb{R}^+ \). More precisely, it is classically known, see proposition 11.11 in the appendix, that every \( *\)-real number \( R \in \mathbb{R} \) is either unlimited or infinitesimally close to a unique standard real number, that is, there exists a unique \( r \in \mathbb{R} \) such that \( R - r \approx 0 \) or \( R \approx r \). In the latter case \( \circ R := r \) is called the standard part of the limited \( *\)-real number \( R \).

Along the same lines, we now prove

**Proposition 6.1.** There exists a \( *\)-finite-dimensional unitary subspace \( F \) of \( *H \) which contains the infinite-dimensional space \( H \) as a subspace, 
\[ H \subset F \subset *H. \]

**Proof.** For a similar result we refer to [13]. The proof of this surprising fact is, like the proof of the existence of infinitely large numbers, just an immediate consequence of the saturation principle for non-standard models: For every element \( u \in H \) let \( A_u \) denote the set of \( *\)-finite-dimensional unitary subspaces \( F \) of \( *H \) such that \( u \in F \). Since for every finite collection \( u_1, \ldots, u_n \in H \) there exists a finite-dimensional unitary subspace containing \( u_i, \ i = 1, \ldots, n \), by saturation it follows
that there must exist a *-finite-dimensional unitary subspace, denoted by $F$, in the common intersection of all $A_u, u \in \mathbb{H}$. □

Using the Fourier transform in order to identify $\mathbb{H} = L^2(S^1, \mathbb{C})$ with $\ell^2(\mathbb{C})$, it follows that $\mathbb{F}$, without loss of generality, can be identified with $\ast \mathbb{C}^{2N+1}$ for some $N \in \ast \mathbb{N} \setminus \mathbb{N}$. Note that here we implicitly use that every *-finite-dimensional unitary subspace of $\ast \ell^2(\mathbb{C})$ is a unitary subspace of $\ast \mathbb{C}^{2N+1}$ for some *-natural number $N$. The unlimited natural number $\dim F := 2N + 1 \in \ast \mathbb{N} \setminus \mathbb{N}$ denotes the complex dimension of $F$ in the non-standard model. Note that the canonical complete unitary basis $(e_n)_{n \in \mathbb{Z}}$ of $\mathbb{H} = L^2(S^1, \mathbb{C})$ given by $e_n(x) = \exp(inx)$ naturally extends to a unitary basis $(e_n)_{n=-N}^{+N}$ of $\mathbb{F} = \ast \mathbb{C}^{2N+1}$ where still $e_n(x) = \exp(inx)$ for all unlimited $n \in \ast \mathbb{Z} \setminus \mathbb{Z}$. In particular, note that every $v \in \mathbb{F}$ can be written as a *-finite sum,

$$v = \sum_{n=-N}^{+N} \hat{v}(n) \cdot e_n \in \mathbb{F},$$

where $\hat{v}(n) := \langle v, e_n \rangle$ denotes the $n$th Fourier coefficient of $v \in \mathbb{F}$ for the *-integer number $-N \leq n \leq N$. Note that the *-extensions on $\ast \mathbb{H}$ of the symplectic form, the inner product and the complex structure on $\mathbb{H}$ restrict to a symplectic form, inner product and complex structure on $\mathbb{F}$ which extend the corresponding structures on $\mathbb{H} \subset \mathbb{F}$ by corollary 11.9. Furthermore, by transfer, the square of Euclidean norm $|\cdot|_F$ on the *-finite-dimensional space $\mathbb{F}$ is given by the *-finite sum $\sum_{n=-N}^{+N} |\hat{v}(n)|^2$.

Recall from above that every limited *-real number $r \in \ast \mathbb{R}$ is near-standard in the sense that there exists a standard real number $s \in \mathbb{R}$ with $r \approx s$, called the standard part $\circ r := s$ of $r$. Replacing the inclusion $\mathbb{H} \subset \ast \mathbb{R}$ of standard elements in a non-standard set by the inclusion $\mathbb{H} \subset \mathbb{F}$, we are lead to the following

**Definition 6.2.** An element $u \in \mathbb{F}$ is called

i) limited if its Euclidean norm $|u|_F \in \ast \mathbb{R}$ is limited in the sense of proposition 11.11,

ii) near-standard if there exists $v \in \mathbb{H}$ with $u \approx v$ and we call $v$ the standard part of $u$ and write $\circ u := v$.

Note that here we say that $u \approx v$ for two elements $u, v \in \mathbb{F}$ if their metric distance is infinitesimal, i.e., $|u - v|_F \approx 0$.

Of course, it is important to have a non-standard characterization of all near-standard points in $\mathbb{F}$. It is given by the following
Definition, note that, if \( \sum_{n=-N}^{K+1} |\hat{v}(n)|^2 \approx 0 \). In other words, if and only if \( v \) is infinitesimally close to all subspaces \( \mathbb{C}^{2K+1} \subset F \) of unlimited dimension \( K \in *\mathbb{N} \setminus \mathbb{N} \).

**Proof.** Note that, if \( v \in F \) is limited, then for all \( n \in *\mathbb{N} \) the coefficients \( \hat{v}(n) = \langle v, e_n \rangle \) must be limited and hence near-standard. Now, by definition, \( v \in F \) is near-standard if and only if there exists \( u \in \mathbb{H} \) with \( v \approx u \). Since this implies that \( \langle u, e_n \rangle = \hat{v}(v, e_n) \) for all \( n \in \mathbb{N} \), it follows that \( v \) is near-standard if and only if the sequence \( \sum_{n=-K}^{K} |\langle v, e_n \rangle|^2 \) converges as \( K \to \infty \). On the other hand, by proposition [11, 14] the latter is equivalent to requiring that for every unlimited \( K \in *\mathbb{N} \setminus \mathbb{N} \) the unlimited sum \( \sum_{n=-K}^{K} |\langle v, e_n \rangle|^2 \) is near-standard and its standard part is independent of \( K \in *\mathbb{N} \setminus \mathbb{N} \). But the latter is equivalent to requiring that \( \sum_{n=-N}^{K-1} |\langle v, e_n \rangle|^2 + \sum_{n=K+1}^{+N} |\langle v, e_n \rangle|^2 \approx 0 \) for all unlimited \( K \leq N \).

After showing that the infinite-dimensional symplectic Hilbert space \( \mathbb{H} \) is contained in a symplectic vector space which is finite-dimensional in the sense of the new model, it remains to be shown that the infinite-dimensional symplectic flow \( \phi_t^H := \phi_t \) defined by the nonlinear Schrödinger equation can be represented by a \(*\)-finite-dimensional symplectic flow \( \phi_t^F \) on \( F \). More precisely, from now on we will restrict ourselves to the time-one maps \( \phi_t^H \) and \( \phi_t^F \) on the projectivizations \( \mathbb{P}(\mathbb{H}) \subset \mathbb{P}(F) \), which we view as quotients \( \mathbb{P}(\mathbb{H}) = \mathcal{S}(\mathbb{H})/U(1) \) and \( \mathbb{P}(F) = \mathcal{S}(F)/U(1) \), where \( \mathcal{S}(\mathbb{H}) = \{ v \in \mathbb{H} : |v|_\mathbb{H} = 1 \} \), \( \mathcal{S}(F) = \{ v \in F : |v|_F = 1 \} \) and \( U(1) = \{ z \in \mathbb{C} : |z| = 1 \} \). With the latter we mean that for every near-standard \( v \in \mathbb{P}(F) \) we have that \( \phi_t^H(\circ v) = \circ(\phi_t^F(v)) \); in particular, on \( \mathbb{P}(\mathbb{H}) \subset \mathbb{P}(F) \) the time-one flows agree up to an error which is smaller than any positive real number, \( \phi_t^H \approx \phi_t^F \).

Before we turn to the general case, we start with the case of the free Schrödinger equation. In this case recall that, after identifying \( \mathbb{H} = L^2(S^1, \mathbb{C}) \) with \( \ell^2(\mathbb{C}) \) using the Fourier transform, the symplectic flow map \( \phi_t^{0,\mathbb{H}} = \phi_t^0 = \exp(i\Delta) \) maps \( \hat{v} \in \ell^2(\mathbb{C}) \) to \( \phi_t^{0,\mathbb{H}}(\hat{v}) \in \ell^2(\mathbb{C}) \) with \( \langle \phi_t^{0,\mathbb{H}}(\hat{v})(n) = \exp(in^2) \cdot \hat{v}(n) \) for all \( n \in \mathbb{Z} \). In particular, \( \phi_t^{0,\mathbb{H}} \) naturally restricts to finite-dimensional symplectic maps on finite-dimensional complex projective spaces. Now it immediately follows from the transfer principle that \( \phi_t^{0,*\mathbb{H}} := *(\phi_t^{0,\mathbb{H}}) \), the *-image of \( \phi_t^{0,\mathbb{H}} \), is a linear symplectomorphism of \(*\mathbb{H} \) which restricts to a \(*\)-finite-dimensional linear symplectomorphism \( \phi_t^{0,F} \) on \( F \subset *\mathbb{H} \). On
the other hand, since $\phi_t^{0,*H}$ extends the linear symplectomorphism $\phi_t^{0,R}$ on $H \subset *H$ by corollary 11.9 it follows that we indeed have $\phi_t^{0,R} = \phi_t^{0,H}$ on $P(H) \subset P(F)$.

In the general case of nonlinear Schrödinger equations of convolution type note that the flow $\phi_t^0$ has to be replaced by $\phi_t = \phi_t^G \circ \phi_t^0$ with $\phi_t^G$ being the flow of the smooth Hamiltonian

$$G_t = F_t \circ \phi_{-t}^0$$

with $F_t(u) = \int_0^{2\pi} f(|u * \psi|^2, x, t) \, dx$.

In this case everything furthermore relies on a finite-dimensional approximation result of the flow of $G_t$ ($F_t$), where we now crucially make use of the special form of the nonlinearity. To this end, define for the given convolution kernel $\psi$ with Fourier series expansion $\psi(x) = \sum_{n=-\infty}^{+\infty} \hat{\psi}(n) \exp(inx)$ for each $k \in \mathbb{N}$ the approximating kernel $\psi^k(x) = \sum_{n=-k}^{k} \hat{\psi}(n) \exp(inx)$ and define the resulting sequence of Hamiltonians $G_t^k (F_t^k)$ by

$$G_t^k := F_t^k \circ \phi_{-t}^0$$

with $F_t^k(u) := \frac{1}{2} \int_0^{2\pi} f(|u * \psi^k|^2, x, t) \, dx$

for all $k \in \mathbb{N}$.

**Lemma 6.4.** For each $k \in \mathbb{N}$ the symplectic flow $\phi_t^{G,k}$ of the time-dependent Hamiltonian $G_t^k$ on $P(H)$ restricts to a finite-dimensional symplectic flow on a $2k$-dimensional projective subspace of $P(H)$. The sequence of time-dependent Hamiltonians $G_t^k$ converges uniformly with all derivatives to the original Hamiltonian $G_t$ as $k \to \infty$. In particular, the same holds true for the symplectic time-one maps $\phi_t^{G,k}$ and $\phi_t^G$, and the Hofer norm $|||G_t|||$ of $G_t : P(H) \to \mathbb{R}$ is finite.

**Proof.** Based on the fact that the flow $\phi_t^0$ naturally restricts to finite-dimensional flows, for the first statement it suffices to observe that the symplectic gradient $X_t^{F,k}$ of $F_t^k : H \to \mathbb{R}$ given by

$$X_t^{F,k}(u) = \partial_t f(|u * \psi^k|^2, x, t)(u * \psi^k) * \psi^k$$

has vanishing Fourier coefficients, $X_t^{F,k}(u)(n) = 0$, for $|n| > k$. Since the supremum norm of $u * \psi - u * \psi^k$ can be bounded by

$$||u * \psi - u * \psi^k|| \leq ||u||_2 \cdot ||\psi - \psi^k||_2,$$

it follows from $||\psi - \psi^k||_2 \to 0$ as $k \to \infty$ and $||u||_2 = 1$ for all $u \in S(H)$ that $u * \psi^k \to u * \psi$ uniformly as $k \to \infty$. On the other hand, since $f$ is assumed to be smooth, it immediately follows that $F_t^k(u) \to F_t(u)$ and hence $G_t^k(u) \to G_t(u)$ as $k \to \infty$, uniformly with all derivatives. □

In order to see that this lemma immediately leads to the existence of $^*$-finite-dimensional symplectic flow $\phi_t^F$, we first observe that, by corollary 11.9 the $^*$-image of the sequence $(G_t^k)_{k \in \mathbb{N}}$ provides us with a
By choosing $k_{11.14}$, it follows from the above lemma that for all unlimited $K \in \ast N \setminus N$ we have $G^K := \ast G^K \approx \ast G$ on $\ast \mathbb{P}(\mathbb{H}) = \mathbb{P}(\ast \mathbb{H})$, including all derivatives. By choosing $K = N$ for $\dim F = 2N + 1$, using the transfer principle (in particular its consequences $11.14$ and $11.9$), the above lemma implies the proof of the following

**Proposition 6.5.** The time-one flow $\phi_1^{G,F} := \phi_1^{G,N}$ of $\ast G^N$ restricts to a $\ast$-finite-dimensional symplectic flow on $\mathbb{P}(F) = \ast \mathbb{C}F^{2N}$ with $|||G - \ast G^K||| \approx 0$ and hence with limited Hofer norm $|||\ast G^K||| \approx |||G|||$, which on $\mathbb{P}(\mathbb{H}) \subset \mathbb{P}(F)$ agrees with $\phi_1^{G,H} = \phi_1^G$ up to an infinitesimal error. After composing $\phi_1^{H} := \phi_1^{G,F} \circ \phi_1^{0,F}$ we hence find that, on $\mathbb{P}(\mathbb{H}) \subset \mathbb{P}(F)$, $\phi_1^{F}$ agrees with the infinite-dimensional symplectic flow $\phi_1^{H} = \phi_1^{G,H} \circ \phi_0^{H}$ of the nonlinear Schrödinger equation of convolution type up to an infinitesimal error. In particular, we have $\phi_1^{H}(v) = o(\phi_1^{F}(v))$ for all near-standard $v \in \mathbb{P}(F)$.

Indeed, while the first statement is an immediate consequence of the transfer principle, the statement about the Hofer norm follows from proposition $11.14$. On the other hand, since proposition $11.14$ further implies that $\phi_1^{G,F} = \phi_1^{G,N} \approx \phi_1^{G} = \ast (\phi_1^{G,H})$, corollary $11.9$ indeed proves that $\phi_1^{G,F} \approx \phi_1^{G,H}$ on $\mathbb{P}(\mathbb{H}) \subset \mathbb{P}(F)$.

### 7. Floer strips in complex projective spaces

After showing that the infinite-dimensional symplectic flow map $\phi_1^{H}$ on $\mathbb{P}(\mathbb{H})$ can be represented, without loss of information, by a symplectic flow $\phi_1^{F}$ on a complex projective space $\mathbb{P}(F)$ which is finite-dimensional in the sense of the non-standard model, we show how the existence of fixed points of $\phi_1^{H}$ can be proven by applying the transfer principle from non-standard model theory to classical results in Floer theory in finite dimensions. Since $\phi_1^{H}(v) = o(\phi_1^{F}(v))$ for all near-standard $v \in \mathbb{P}(F)$ by proposition $10.5$ it is sufficient to establish the existence of points $v \in \mathbb{P}(F)$ with $\phi_1^{F}(v) \approx v$ which additionally are near-standard. Since the $\ast$-finite-dimensional flow $\phi_1^{F}$ can be treated like a finite-dimensional symplectic flow by the transfer principle, in this section we collect classical results from finite-dimensional Floer theory that will be used later on to prove the existence of near-standard points $v \in \mathbb{P}(F)$ with $v \approx \phi_1^{F}(v)$.

From now on let us assume that the Hofer norm $|||F|||$ of $F_t$ on $\mathbb{P}(\mathbb{H})$ is strictly smaller than $\pi/4$. Since $G^k \to G$ uniformly as $k \to \infty$, note that it follows from $|||G||| = |||F||| < \pi/4$ that there exists some $k_0 \in \mathbb{N}$ such that $|||G_k||| < \pi/4$ for all $k \geq k_0$. Furthermore, for $T \in \mathbb{R}^+ \cup \{0\}$
Proposition 7.1. Let \( k \in \mathbb{N} \) with \( k \geq k_0 \) and \( T \in \mathbb{R}^+ \cup \{0\} \). Then for every \( n \leq k \) there exists a smooth map \( \tilde{u} = \tilde{u}_n = \tilde{u}_{n,T} : \mathbb{R} \times [0,1] \rightarrow \mathbb{C}\mathbb{P}^{2k} \) satisfying the periodicity condition \( \tilde{u}(\cdot,1) = \phi^n_0(\tilde{u}(\cdot,0)) \), the asymptotic condition \( \tilde{u}(s,\cdot) \rightarrow u^n_0 \) as \( s \rightarrow \pm \infty \), and the perturbed Cauchy-Riemann equation

\[
0 = \partial_T^2 \tilde{u} = \partial_T \tilde{u} + \varphi_T(s) \cdot \nabla G^k_1(\tilde{u}).
\]

Furthermore, for the resulting families of maps \( \tilde{u} = \tilde{u}_n \) we have

i) The energy \( E(\tilde{u}_n) \) defined by

\[
E(\tilde{u}_n) := \int_{-\infty}^{+\infty} \int_0^1 \frac{1}{2} \left( |\partial_s \tilde{u}_n|^2 + |\partial_t \tilde{u}_n - \varphi_T(s) \cdot X^{G,k}(\tilde{u}_n)|^2 \right) \, dt \, ds
\]

is bounded by \( 2 \| |G^k| | < \pi/2 \); in particular, there exists some \( s_0 \in [-T,+T] \) such that the path \( \tilde{u}_n(s_0,\cdot) := \tilde{u}_{n,T}(s_0,\cdot) \) satisfies

\[
\int_0^1 |\partial_t \tilde{u}_n(s_0,t) - X^{G,k}_1(\tilde{u}_n(s_0,t))|^2 \, dt \leq \frac{\pi}{4T}.
\]

ii) For every \( s \in [-T,+T] \) the symplectic action \( \mathcal{A}^n(\tilde{u}_n(s)) \) of the path \( \tilde{u}_n(s) = \tilde{u}_n(s,\cdot) : [0,1] \rightarrow \mathbb{C}\mathbb{P}^{2k} \), defined as

\[
\int (\tilde{u}_0,\# \tilde{u}_n|(-\infty,s) \times [0,1])^* \omega + \varphi_T(s) \cdot \int_0^1 G^{k}_1(\tilde{u}(s,t)) \, dt,
\]

satisfies the inequality

\[
|\mathcal{A}^n(\tilde{u}_n(s)) - \frac{n^2}{2}| \leq 2 \cdot \| |G^k| | < \pi/2.
\]

iii) The given definition of the symplectic action is independent of \( n \), that is, if \( \tilde{u}_m(s) = \tilde{u}_n(s') : [0,1] \rightarrow \mathbb{C}\mathbb{P}^{2k} \) for \( m,n \leq k \) and \( s, s' \in [-T,+T] \), then \( \mathcal{A}^m(\tilde{u}_m(s)) = \mathcal{A}^n(\tilde{u}_n(s')) \); in particular, \( \tilde{u}_m(s) \neq \tilde{u}_n(s') \) for different \( m,n > 4 \).

Proof. For the proof one observes that, for every \( k \in \mathbb{N} \), \( M := \mathbb{C}\mathbb{P}^{2k} \) is a closed symplectic manifold where the energy of a holomorphic sphere is bounded from below by \( \pi \). Furthermore, the linear symplectomorphism \( \phi := \phi^n_0 \) as well as the smooth Hamiltonian \( G^k : \mathbb{P}(\mathbb{H}) \rightarrow \mathbb{R} \) on \( \mathbb{P}(\mathbb{H}) \) restrict to a linear (and hence smooth) symplectomorphism and smooth Hamiltonian function on \( \mathbb{C}\mathbb{P}^{2k} \), respectively, where \( \mathbb{C}\mathbb{P}^{2k} \) is viewed as the set of equivalence classes \([v]\) of elements \( v = \sum_{n=-\infty}^{+\infty} \hat{v}(n) \cdot e_n \in \mathbb{H} \) with \( \hat{v}(n) = 0 \) for all \( |n| > k \). Since, for every \( n \leq k \), \( p_n := u^n_0 \in \mathbb{P}(\mathbb{H}) \) given by \( u^n_0(x) = \exp(inx) \) is a fixed point of \( \phi \) in \( \mathbb{C}\mathbb{P}^{2n} \subset M \subset \mathbb{P}(\mathbb{H}) \), it follows that the existence of the map \( \tilde{u} = \tilde{u}_{n,T} \) for every \( T \in \mathbb{R}^+ \cup \{0\} \) can be deduced using the
properties of the moduli space of Floer strips \((\tilde{u}, T)\) established in [3] and the references therein.

First we emphasize that an easy computation shows that the standard complex structure indeed satisfies the periodicity condition \(\phi_* i = i\) required in [3]. Assuming for the moment that transversality for the Cauchy-Riemann operator \(\bar{\partial}_G\) given by \(\bar{\partial}_G(\tilde{u}, T) = \bar{\partial}_G \tilde{u}\), first observe that it is shown in [3] that the moduli space of tuples \((\tilde{u}, T)\) is a one-dimensional manifold. Since for \(T = 0\) the constant strip \(\tilde{u}_{k,0}(s,t) = p_n\) staying over the fixed point \(p_n\) is the unique solution, the existence of a Floer strip \(\tilde{u}_{k,T}^n\) for all \(T \in \mathbb{R}^+\) follows from the Gromov-Floer compactness result. Here we emphasize that bubbling-off of holomorphic spheres is excluded due to fact that the energy \(E(\tilde{u})\) is bounded from above by twice the Hofer norm of the Hamiltonian \(G^k\), see [12], and the Hofer norm of \(G^k\) is smaller than \(1/2\) the minimal energy of a holomorphic sphere in \(\mathbb{C}P^{2k}\). Finally, since the Cauchy-Riemann operator \(\bar{\partial}_G^k\) cannot be expected to be transversal, one first has to approximate \(i\) by a family of \(t\)-dependent almost complex structures \(J^\nu\), \(t \in [0, 1]\) with \(J^\nu_1 = \phi_* J^\nu_0\) in the sense that \(J^\nu_1 \to J^\nu_0 = i\) as \(\nu \to 0\). Then the Gromov-Floer compactness result in [3] can be used again to deduce the existence of Floer strips for \(\nu = 0\) from the existence of Floer strips for \(\nu \neq 0\) for all \(T \in \mathbb{R}^+\). In particular, we emphasize that one does not need more elaborate technology like Kuranishi structures or polyfolds to establish the desired properties of the moduli space.

Concerning the additional statements, observe that the bound on \(E(\tilde{u})\) in \(i)\) has already been used to exclude bubbling-off for compactness and it can be found in [12]. It relies on the fact that, since \(\bar{\partial}_G^k \tilde{u} = 0\), the energy of \(u\) is given by

\[
E(\tilde{u}) = \int_{-\infty}^{+\infty} \int_0^1 \phi_T(s) \langle \nabla G^k_\tau(\tilde{u}), \partial_s \tilde{u} \rangle \ dt \ ds.
\]

Furthermore, the existence of \(s \in [-T, +T]\) with the desired property is an immediate consequence. On the other hand, the bound on the action in \(ii)\) follows from the same proof as used to establish the energy bound. For the definition of the symplectic action following [3] we use that, after concatenation with the strip \(\tilde{u}_{0n} : \mathbb{R} \times [0, 1] \to \mathbb{C}P^{2k}\) used in the definition of the symplectic action \(A(u^0_{0n})\), the Floer strip \(\tilde{u} = \tilde{u}_n\) can be used to connect \(\tilde{u}_n(s)\) with \(u^0_{0n}\). Then we just additionally need to recall that \(A(u^0_{0n})\) agrees with minus the value of the Hamiltonian \(H^0\) underlying \(\phi = \phi^1_1\) at \(u^0_{0n}\),

\[
A(u^0_{0n}) = -H^0(u^0_{0n}) = \frac{n^2}{2}.
\]
For iii) let us assume that \( \tilde{u}_m(s) = \tilde{u}_n(s') \). Then one has to show that the concatenated strips \( \tilde{u}_n \# \tilde{u}_m \) and \( \tilde{u}_n \# \tilde{u}_m \) are homotopic, that is, the homotopical difference still has to be represented by the canonical projection of the Floer strip onto \( \mathbb{CP}^{2k} \subset \mathbb{C}P^{2k} \), we claim that we can again write \( \tilde{u} \) as a pair of maps,\[
\tilde{u} = (\tilde{u}_\perp, \tilde{u}^\ell) : \mathbb{R} \times [0,1] \to \mathbb{C}^{2k-2\ell} \times \mathbb{C}P^{2\ell},
\]
where \( \tilde{u}_\perp \) remembers the normal component. More precisely, \( \tilde{u}_\perp \) shall be viewed as a section in the pull-back \( (\tilde{u}^\ell, N) \to \mathbb{R} \times [0,1] \) of the normal bundle of \( \mathbb{CP}^{2\ell} \subset \mathbb{C}P^{2k} \) which is unitarily trivial even after applying the natural identifications, see the discussion in the section about finite-dimensional nonlinearities.

**Lemma 7.2.** Assume that the image of \( \tilde{u} = \tilde{u}^k_{n,T} \) sits in a tubular neighborhood of \( \mathbb{CP}^{2\ell} \subset \mathbb{C}P^{2k} \) for some \( n \leq \ell \leq k \) and write \( \tilde{u} = (\tilde{u}_\perp, \tilde{u}^\ell) : \mathbb{R} \times [0,1] \to \mathbb{C}^{2k-2\ell} \times \mathbb{C}P^{2\ell} \). Then the \( L^2 \)-norm of \( \partial_t \tilde{u}^\ell \) is bounded by \( 2 \| \| G^k - G^\ell \| \| \).

**Proof.** The proof builds on lemma 8.1.6, remark 8.1.7 and the proof of theorem 9.1.1 in [12]. Introducing the energies \( E(\tilde{u}), E(\tilde{u}^\ell), E(\tilde{u}^\ell_{\perp}) \) to be the \( L^2 \)-norms of the corresponding partial derivatives \( \partial_s \tilde{u}, \partial_s \tilde{u}^\ell \) and \( \partial_t \tilde{u}^\ell_{\perp} \), we clearly have \( E(\tilde{u}) = E(\tilde{u}^\ell) + E(\tilde{u}^\ell_{\perp}) \). On the other hand, following lemma 8.1.6 in [12], we know that
\[
E(\tilde{u}) = \int \tilde{u}^* \omega + \int R_G(\tilde{u}) \, ds \wedge dt,
\]
\[
E(\tilde{u}^\ell) \geq \int (\tilde{u}^\ell)^* \omega + \int R_G(\tilde{u}^\ell) \, ds \wedge dt,
\]
with \( R_G \) denoting the corresponding Hamiltonian curvature form in the sense of ([12], 8.1). Note that the first summands in both (in)equalities are indeed zero due to homotopical reasons and in the second case we
indeed just expect an inequality, as \( \tilde{u}^\ell \) itself does not satisfy the Floer equation. On the other hand, it is easy to see from the definition of the curvature that

\[
R_{G^k}(\tilde{u}^\ell) = R_{G^\ell}(\tilde{u}^\ell) = R_{G^\ell}(\tilde{u}).
\]

Since \( R_{G^k} - R_{G^\ell} = R_{G^k - G^\ell} \), we summarizing obtain

\[
E(\tilde{u}^\perp) = E(\tilde{u}) - E(\tilde{u}^\ell) \leq \int R_{G^k}(\tilde{u}) \, ds \wedge dt - \int R_{G^k}(\tilde{u}^\ell) \, ds \wedge dt.
\]

Following remark 8.1.7 and the proof of 9.1.1 in [12], we know that the last expression can be bounded by the Hofer norm of the Hamiltonian curvature of \( G^k - G^\ell \), which itself agrees with \( 2 |||G^k - G^\ell||| \).

In the case of finite-dimensional nonlinearities note that, instead of using Liouville’s theorem, we can alternatively employ the above lemma to prove that \( E(\tilde{u}^\perp) = 0 \) which in turn again immediately implies \( \tilde{u}^\perp = 0 \).

8. Floer strips and *-finite-dimensional representations

After showing how the infinite-dimensional flow defined by the nonlinear Schrödinger equation can be represented by symplectic flow which is finite-dimensional in the sense of the non-standard model, and collecting the key results from finite-dimensional Floer theory, we now show how the latter results can be elegantly used to prove our main theorem. The key tool in order to achieve this is the afore-mentioned transfer principle of non-standard model theory which states that every *-finite-dimensional object can be treated like a finite-dimensional object. In particular, by applying the transfer principle to proposition 7.1 without any extra work we can immediately establish the existence of Floer strips in the *-finite-dimensional projective space \( \mathbb{P}(F) = *\mathbb{C}P^{2N} \) for the Hamiltonian \( G^F_t := *G^N_t : \mathbb{P}(F) \to *\mathbb{R} \).

**Corollary 8.1.** For every \( T \in *\mathbb{R}^+ \cup \{0\} \) and every \( n \in \mathbb{N} \) there exists a *-smooth map \( \tilde{u} = \tilde{u}_n = \tilde{u}^F_{n,T} : *\mathbb{R} \times *[0,1] \to *\mathbb{C}P^{2N} = \mathbb{P}(F) \) satisfying the periodicity condition \( \tilde{u}(s,1) = \phi^F_{t}(\tilde{u}(s',0)) \), the asymptotic condition \( \tilde{u}(s',1) \to u_0^0 \) as \( s \to \pm \infty \), and the perturbed Cauchy-Riemann equation

\[
0 = \tilde{\partial}^F_T \tilde{u} = \tilde{\partial} \tilde{u} + \phi_T(s) \cdot \nabla G^F_t(u)
\]

in the non-standard sense. Furthermore, we have

i) The energy \( E(\tilde{u}_n) \) is bounded by \( 2 |||G^F||| < \pi/2 \), so that, if \( T \in *\mathbb{R}^+ \cup \{0\} \) is chosen to be an unlimited positive *-real number, there exists some \( s_0 \in [-T, +T] \) such that \( \phi^F_{t}(\tilde{u}_n(s_0,0)) \approx \tilde{u}_n(s_0,0) \).
ii) The symplectic action $A(\tilde{u}(s))$ of the path $\tilde{u}(s) = \tilde{u}^F_{n,T}(s, \cdot)$ is well-defined and satisfies the inequality

$$|A(\tilde{u}(s)) - \frac{n^2}{2}| < \pi/2.$$ 

In particular, for different $m, n > 4$ we know that $\tilde{u}_m(s)$ and $\tilde{u}_n(s')$ are not infinitesimally close for all $s, s' \in [-T, +T]$.

Proof. As mentioned above, the existence of the Floer strips $\tilde{u} : \ast \mathbb{R} \times \ast [0, 1] \to \mathbb{P}(\mathbb{F})$ in the $\ast$-finite-dimensional projective space follows immediately from the transfer principle. Note that, since proposition 7.1 holds for every standard natural number $k$ greater than a fixed natural number $k_0$, it follows that the transferred result holds for every $\ast$-natural number $K \geq k_0$, in particular, for the unlimited $\ast$-natural number $N \in \ast \mathbb{N} \setminus \mathbb{N}$. Furthermore note that every standard natural number $n \in \mathbb{N}$ is smaller than the unlimited number $N$ and the quotient $\pi/4T$ is infinitesimal for every unlimited $\ast$-real number $T$, so that we have, after using the transfer of the Schwarz inequality,

$$\int_0^1 |\partial_t \tilde{u}(s_0, t) - X^G_F(\tilde{u}(s_0, t))| \, dt \approx 0$$

and hence $\phi^F_1(\tilde{u}(s_0, 0)) \approx \tilde{u}(s_0, 0)$.

Remark 8.2. We would like to emphasize that for our existence proof of fixed points of the infinite-dimensional flow map $\phi_1 = \phi^H_1$ it will not be important to understand the precise meaning of differentiation and integration in the non-standard model. In particular, we a priori need to assume that the derivatives of $\tilde{u}$ have unlimited norm, so that, even when the image of $\tilde{u}$ is near-standard, the resulting standard map $\circ \tilde{u} : \mathbb{R} \times [0, 1] \to \mathbb{P}(\mathbb{H})$ will, a priori, not be smooth. While we show in the upcoming section that the Floer strip $\tilde{u}$ is indeed near-standard and use this to prove the existence of infinitely many fixed points, in the last section we show that $\circ \tilde{u} : \mathbb{R} \times [0, 1] \to \mathbb{P}(\mathbb{H})$ is indeed smooth in the standard sense, by employing a bubbling-off argument and elliptic regularity. For the last statement in the proof and for whatever follows we indeed only need to know that every statement, e.g. an inequality, that holds in standard mathematics has an analogue in non-standard mathematics by the transfer principle, together with the computation rules for infinitesimals summarized in proposition 11.13.

In order to establish the existence of a fixed point of the infinite-dimensional symplectic flow $\phi^H_1$ given by the nonlinear Schrödinger equation, it suffices to show that $\tilde{u}^F_{n,T}(s_0, 0) \in \mathbb{P}(\mathbb{F})$ from corollary 8.1 ii) is near-standard, as this implies that the corresponding standard point $\circ \tilde{u}_{n,T}(s_0, 0) \in \mathbb{P}(\mathbb{H})$ is a fixed point of $\phi^H_1$. This is where crucially make use of the fact that, for every $n \in \mathbb{N}$, the transfer principle does not only provide us with a candidate for the fixed point for the flow.
\(\phi^H_1\) of the nonlinear Schrödinger equation, but also with a Floer strip connecting it with the fixed point \(u^0_n\) of \(\phi^F_1\).

9. FLOER STRIPS ARE NEAR-STANDARD AND EXISTENCE OF FIXED POINTS

In order to establish the existence of a fixed point of \(\phi^H_1\), recall that it is now sufficient to prove that the map \(\tilde{u}\) is near-standard in the sense that every point in the image of \(\tilde{u} = \tilde{u}^F_{n,T} : *\mathbb{R} \times [0,1] \rightarrow \mathbb{P}(\mathbb{F})\) is near-standard in the sense of definition 6.2. This is the content of Proposition 9.1. For every \(n \in \mathbb{N}\) and every \(T \in *\mathbb{R}^+ \cup \{0\}\) the map \(\tilde{u} = \tilde{u}^F_{n,T} : *\mathbb{R} \times [0,1] \rightarrow \mathbb{P}(\mathbb{F})\) is near-standard. In particular, by choosing \(T\) to be unlimited, for every \(n \in \mathbb{N}\) there exists a fixed point \(u^1_n := \varphi(\tilde{u}^F_{n,T}(s_0,0)) \in \mathbb{P}(\mathbb{H})\) of the time-one flow \(\phi^H_1\) of the given nonlinear Schrödinger equation of convolution type.

For the proof we essentially use that, very informally speaking, \(G_i^F : \mathbb{P}(\mathbb{F}) \rightarrow *\mathbb{R}\) agrees up to an infinitesimal error with \(G_i = G^H_i\) in a tubular neighborhood of \(\mathbb{P}(\mathbb{H}) \subset \mathbb{P}(\mathbb{F})\). Together with the minimal surface property of holomorphic curves used for the finite-dimensional case, this again implies that also the Floer strip \(\tilde{u} : *\mathbb{R} \times [0,1] \rightarrow *\mathbb{C}P^{2N} = \mathbb{P}(\mathbb{F})\) has to be infinitesimally close to \(\mathbb{P}(\mathbb{H})\) and hence near-standard.

**Proof.** In order to make this idea precise we use the characterization of near-standardness in proposition 6.3. That is, we need to prove that every point in the image of \(\tilde{u} = \tilde{u}^F_{n,T} : *\mathbb{R} \times [0,1] \rightarrow *\mathbb{C}P^{2N} = \mathbb{P}(\mathbb{F})\) is infinitesimally close to \(*\mathbb{C}P^{2K} \subset *\mathbb{C}P^{2N} = \mathbb{P}(\mathbb{F})\) for every unlimited \(K \in *\mathbb{N}\setminus\mathbb{N}\). Note that we do not need to prove that the points on the Floer strip are limited, as every point in \(\mathbb{P}(\mathbb{F}) = S(\mathbb{F})/\sim\) is limited, see remark 11.12. Furthermore we now crucially make use of the technical lemma 7.2.

Let us fix \(n \in \mathbb{N}\) and let us assume to the contrary that there exists some \(T \in *\mathbb{R}^+\) with map \(\tilde{u} = \tilde{u}^F_{n,T} : *\mathbb{R} \times [0,1] \rightarrow *\mathbb{C}P^{2N} = \mathbb{P}(\mathbb{F})\) such that \(\tilde{u}\) is not near-standard. Since for \(T = 0\) the image of \(\tilde{u}\) is a point in \(*\mathbb{C}P^{2n} \subset \mathbb{P}(\mathbb{F})\), for every unlimited \(K \in *\mathbb{N}\setminus\mathbb{N}\) we may assume without loss of generality that the image of \(\tilde{u}\) sits in a tubular neighborhood of \(*\mathbb{C}P^{2K} \subset *\mathbb{C}P^{2N} = \mathbb{P}(\mathbb{F})\); if not then we just have to pass to some \(0 < \bar{T} < T\). In particular, as discussed before lemma 7.2 we may write \(\tilde{u}\) as a pair of maps

\[(\tilde{u}^K_\perp, \tilde{u}^K) : *\mathbb{R} \times [0,1] \rightarrow *\mathbb{C}^{2N-2K} \times *\mathbb{C}P^{2K}\]

for every unlimited \(K \in *\mathbb{N}\setminus\mathbb{N}\). Using proposition 6.3 it suffices to prove that for every unlimited \(K \in *\mathbb{N}\setminus\mathbb{N}\) with \(K \leq N\) we have
\( \tilde{u}^K_\perp \approx 0 \), that is, \( \tilde{u}^K_\perp(s, t) \approx 0 \) for all \( (s, t) \in \ast \mathbb{R} \times \ast [0, 1] \).

In order to prove this, observe first that by proposition [6, 5] we know that
\[
|||G^N - G^K||| \leq |||G - G^K||| + |||G - G^N||| \approx 0
\]
as \( K \) and \( N \) are unlimited. After applying the transfer principle to lemma [12] this however immediately implies that the \( L^2 \)-norm of \( \partial_s \hat{u}^K_\perp \) is infinitesimal. On the other hand, after employing the a priori estimate for pseudo-holomorphic maps in lemma 4.3.1 in [12], we see that indeed \( \partial_s \hat{u}^K_\perp(s, t) \approx 0 \) for all points \( (s, t) \in \ast \mathbb{R} \times \ast [0, 1] \).

It remains to be shown that \( \partial_s \hat{u}^K_\perp \approx 0 \) indeed implies \( \hat{u}^K_\perp \approx 0 \). To this end observe first that, by \( \partial_t^G \hat{u} = 0 \), we know that \( \partial_s \hat{u}^K_\perp \approx 0 \) implies that \( \partial_t \hat{u}^K_\perp \approx (X^{G, \bar{F}}(\hat{u}))^K_\perp \), where \( (X^{G, \bar{F}}(\hat{u}))^K_\perp \) denotes the projection of the symplectic gradient \( X^{G, \bar{F}}(\hat{u}) \) of \( G^\perp \) to \( \ast \mathbb{C}^{2N-2K} \). Since we again have \( (X^{G, \bar{F}}(\hat{u}))^K_\perp = (X^{G, \perp}_t(\hat{u}))^K_\perp \approx (X^{G, K}_t(\hat{u}))^K_\perp = 0 \) by proposition [6, 5] it follows that
\[
\hat{u}^K_\perp(s, 0) \approx \hat{u}^K_\perp(s, 1) = \phi^0_1(\hat{u}^K_\perp(s, 0))
\]
for all \( s \in \ast \mathbb{R} \), where the last equality uses that the linear symplectomorphism \( \phi^0_1 = \phi^{0, \bar{F}}_1 \) of the free Schrödinger equation respects the splitting in \( \ast \mathbb{C}^{2N-2K} \times \ast \mathbb{CP}^{2K} \subset \mathbb{P}(\mathbb{F}) \).

In order to finish the proof, it just needs to be shown that \( \phi^0_1(\hat{u}^K_\perp(s, t)) \approx \hat{u}^K_\perp(s, t) \) indeed implies that \( \hat{u}^K_\perp(s, t) \approx 0 \) for all \( (s, t) \in \ast \mathbb{R} \times \ast [0, 1] \). Abbreviating \( v = \hat{u}^K_\perp(s, 0) \in \ast \mathbb{C}^{2N-2K} \), we claim that \( \phi^0_1(v) \approx v \) implies \( v \approx 0 \). In order to see this, we crucially use that the smoothing kernel \( \psi \) is chosen to be admissible in the sense that
\[
\text{supp}(\hat{\psi}) \subset M = M_\delta = \{ m \in \mathbb{Z} : \exp(|m|^2) - 1 \geq \delta \}
\]
for some standard \( \delta > 0 \). Indeed, using that the Fourier coefficients of \( \phi^0_1(v) - v \) are given by \( \exp(|m|^2) - 1 \cdot \hat{\psi}(m) \) for every \( 2K < m \leq 2N \), note that \( \phi^0_1(v) \approx v \) implies that
\[
\sum_{m=-N}^{-K-1} |\exp(im^2) - 1|^2 \cdot |\hat{\psi}(m)|^2 + \sum_{m=K+1}^{N} |\exp(im^2) - 1|^2 \cdot |\hat{\psi}(m)|^2 \approx 0.
\]
Now denoting \( \hat{M} = \ast M \cap (\{-N, -K - 1\} \cup \{K + 1, \ldots, N\}) \), it follows from the transfer principle that \( |\exp(im^2) - 1| \geq \delta \) for all \( m \in \hat{M} \), which in turn implies
\[
\sum_{m \in \hat{M}} |\hat{\psi}(m)|^2 \leq \delta^{-1} \cdot \sum_{m \in \hat{M}} |\exp(im^2) - 1|^2 \cdot |\hat{\psi}(m)|^2 \approx 0.
\]
In order to conclude that \( v \approx 0 \), it suffices to observe that we already know that \( \hat{\psi}(m) = 0 \) for all \( m \notin \hat{M} \) as \( \hat{\psi}(m) = 0 \). Indeed this follows
by the same arguments that we used in the case of finite-dimensional nonlinearities to prove that the Floer strips live in finite-dimensional projective spaces, namely using Liouville’s theorem or the minimal surface property. □

10. Limited derivatives and existence of Floer strips

After establishing the existence of fixed points of the time-one flow map of the nonlinear Schrödinger equation, in this last section we want to additionally use our results to prove the existence of Floer strips in the infinite-dimensional projective Hilbert space. Indeed we have already established that, for every \( n \in \mathbb{N} \) and every \( T \in * \mathbb{R}^+ \), the map \( \tilde{u} = \tilde{u}_{n,T} : * \mathbb{R} \times *[0,1] \to \mathbb{P}(\mathcal{F}) \) is near-standard, so that we can apply the standard part map to obtain a standard map \( \circ \tilde{u} : \mathbb{R} \times [0,1] \to \mathbb{P}(\mathcal{H}) \subset \mathbb{P}(\mathcal{F}) \). On the other hand, note that the asymptotic condition and the first derivatives appearing in the Cauchy-Riemann equation are only to be understood in the non-standard sense. In particular, the map \( \circ \tilde{u} : \mathbb{R} \times [0,1] \to \mathbb{P}(\mathcal{H}) \) can in general not be assumed to be smooth in the standard sense, since the derivatives of the \(*\)-smooth map \( \tilde{u} \) a priori could have unlimited norm at every point. Applying the transfer principle to well-known fundamental properties of finite-dimensional holomorphic curves, we however can indeed prove that all derivatives are limited.

Lemma 10.1. For every \( n \in \mathbb{N} \) and \( T \in * \mathbb{R}^+ \cup \{0\} \) and every \( \ell \in \mathbb{N} \) the \( \ell \)-th derivative of the map \( \tilde{u} = \tilde{u}_{n,T} : * \mathbb{R} \times *[0,1] \to \mathbb{P}(\mathcal{F}) \) has a limited norm at every point \( (s,t) \in * \mathbb{R} \times *[0,1] \).

Proof. For the proof we use a non-standard version of the classical bubbling-off argument from ([12], chapter 4) together with elliptic regularity from ([12], appendix B). Apart from the fact that the a priori estimate for bubbling, the elliptic estimate used to bound higher Sobolev norms as well as the Sobolev embedding theorems have analogues for non-standard maps to the \(*\)-finite-dimensional projective space \( \mathbb{P}(\mathcal{F}) \) by the transfer principle, the crucial observation for the proof is that the constants appearing in the used inequalities are still limited numbers.

Step 1: Limitedness of \( du \) using bubbling-off

Note that, while it follows from the transfer principle that the map \( \tilde{u} = \tilde{u}_{n,T} : * \mathbb{R} \times *[0,1] \to \mathbb{P}(\mathcal{F}) \) is smooth and converging to the fixed point \( u_{n,T}^0 \), in particular, the supremum norm of its first derivative is finite, this statement clearly only holds in the non-standard sense. In particular, we in general need to expect that this supremum norm is an unlimited \(*\)-real number. In order to show that the supremum norm of \( d\tilde{u} \) is indeed limited, we essentially use that the energy \( E(\tilde{u}) \)
of $\tilde{u}$ is strictly smaller than the minimal energy of a holomorphic sphere in $\mathbb{P}(\mathbb{F})$.

To the contrary, assume that $\|d\tilde{u}\|_{\infty} = \max\{|d\tilde{u}(z)| : z \in \ast \mathbb{R} \times \ast [0,1]\} = C$ is an unlimited $\ast$-real number and choose $z_0 \in \ast S^2$ such that $|d\tilde{u}(z_0)| = C$. Note that, due to the asymptotic condition, the maximum must be attained in the interior of the strip and we assume without loss of generality that $z_0 = (0,1/2)$. As in the classical bubbling-off proof we define $\tilde{v} : \ast B^2(C/4) \to \mathbb{P}(\mathbb{F})$ by $\tilde{v}(z) := \tilde{u}(z/C + z_0)$, such that $|d\tilde{v}(0)| = 1$ and $|d\tilde{v}(z)| \leq 1$ for all $z \in \ast B^2(C/4)$. Because $C \in \ast \mathbb{R}^+$ was assumed to be unlimited, note that $\ast B^2(C/4) \subset \ast \mathbb{C}$ is a disk of unlimited radius; in particular, it contains the full complex plane $\mathbb{C}$ as a subset.

The crucial ingredient now is to use the transfer of the \textit{a priori estimate} from ([12], lemma 4.3.1). First, by corollary 8.1 we know that the energy of $v$ we have

$$\int_{\ast B^2(C/4)} |d\tilde{v}(z)|^2 = E(\tilde{v}) \leq E(\tilde{u}) < \pi.$$ 

On the other hand, since $\pi$ is the minimal energy of a holomorphic sphere in every finite-dimensional complex projective space, after applying the transfer principle we can employ the \textit{a priori estimate} to conclude that

$$|d\tilde{v}(0)|^2 \leq \frac{8}{\pi \cdot C} \cdot \int_{\ast B^2(C/4)} |d\tilde{v}(z)|^2 \approx 0,$$

contradicting $|d\tilde{v}(0)| = 1$.

\textit{Step 2: Bounds for the higher derivatives using elliptic regularity}

Since regularity is a local property, let us consider the restriction of $\tilde{u}$ to a ball $\ast B^2(1/4)$ of finite radius which is located arbitrarily inside the strip $\ast \mathbb{R} \times \ast [0,1]$. Since the supremum norm of $d\tilde{u}$ is limited, it immediately follows from the limited area of $\ast B^2(1/4)$ that the $(1,p)$-norm of the restriction of $\tilde{u}$ is again a limited number. Here and below all norms are understood in the non-standard sense.

As in the proof of ([12], theorem B.4.1) we get from the elliptic estimates that also the $(k,p)$-norm of $\tilde{u}$ is limited for every standard natural number $k$. For the induction step observe that the composition $\nabla G^p \circ \tilde{u}$ has a limited $(k-1,p)$-norm if this holds for $\tilde{u}$ and for every standard $\ell$ the $\ell$th derivative of $G^p$ has a limited supremum norm. Apart from the fact that the derivatives of $G^p$ are limited by proposition 6.5 we use that the constants appearing in ([12], proposition B.4.9) are indeed independent of the dimension of the
underlying complex projective space, so that the elliptic estimate for holomorphic curves in $\mathbb{P}(\mathbb{F})$ holds with the same constant; in particular, the constant is again a limited number.

Finally, after transferring the Sobolev embedding theorem, we get that the supremum norm of the $\ell$th derivative of $\tilde{u}$ is limited for every standard natural number $\ell$. Here we again observe that the appearing constant in ([12], section B.1.1) is independent of the dimension of the target manifold and hence is the same for *-finite-dimensional target spaces by transfer. □

Using the limitedness of the non-standard derivatives of $\tilde{u} = \tilde{u}_{n,T}^\# : \ast \mathbb{R} \times [0,1] \to \mathbb{P}(\mathbb{F})$ we can now establish the existence of Floer strips in infinite dimensions. More precisely, we prove

**Proposition 10.2.** Distinguishing whether $T \in \ast \mathbb{R}^+$ is limited or unlimited, we get the following two results:

i) If $T$ is limited, then $\tilde{u}^H = \tilde{u}_{n,T}^H : \mathbb{R} \times [0,1] \to \mathbb{P}(\mathbb{H})$ is smooth in the standard sense, satisfies the Floer equation $0 = \partial \tilde{u}^H + \varphi_T(s) \cdot \nabla G_t(\tilde{u}^H)$, the boundary condition $\phi^H_1(\tilde{u}^H(\cdot,0)) = \tilde{u}^H(\cdot,1)$, and the asymptotic condition $\tilde{u}^H(s,\cdot) \to u_0^0$ as $s \to \pm\infty$.

ii) If $T$ is unlimited, then the shifted map $\bar{u}^H = \bar{u}_{n,T}^H : \mathbb{R} \times [0,1] \to \mathbb{P}(\mathbb{H})$ is smooth, meets the boundary condition $\phi^H_1(\bar{u}^H(\cdot,0)) = \bar{u}^H(\cdot,1)$ and satisfies the Floer equation $0 = \partial \bar{u}^H + \varphi_\infty(s) \cdot \nabla G_t(\bar{u}^H)$, where $\varphi_\infty : \mathbb{R} \to [0,1]$ is a smooth cut-off function with $\varphi_\infty(s) = 0$ for $s < -1$ and $\varphi_\infty(s) = 1$ for $s > 0$. Furthermore, while we again have $\bar{u}^H(s,\cdot) \to u_0^0$ as $s \to -\infty$, towards $+\infty$ we only know that there exists a sequence $(s_k)$ of positive real numbers such that $\tilde{u}^H(s_k,\cdot)$ converges to a fixed point of the time-one flow map $\tilde{\phi}_1^\#$ of the given nonlinear Schrödinger of convolution type.

**Proof.** Since the norms of the first derivatives of $\tilde{u} = \tilde{u}_{n,T}^\# : \ast \mathbb{R} \times [0,1] \to \mathbb{P}(\mathbb{F})$ have a limited uniform bound, we immediately get that $\tilde{u}$ is Lipschitz continuous with a limited Lipschitz constant. But with this it is immediate to see that, after applying the standard part map, the resulting map $\circ \tilde{u} : \mathbb{R} \times [0,1] \to \mathbb{P}(\mathbb{H})$ is Lipschitz continuous in the standard sense, where the corresponding Lipschitz constant for $\circ \tilde{u}$ is just the standard part of the Lipschitz constant for $\tilde{u}$. But this in turn immediately shows that $\circ \tilde{u}$ is indeed continuous in the standard sense.

Along the same lines we can even prove that $\circ \tilde{u}$ is smooth in the standard sense. In the same way as the limitedness of the first derivative of $\tilde{u}$ was used to prove continuity of $\circ \tilde{u}$, the latter
result proves that $\circ \tilde{u}$ is $\ell$-times continuously differentiable for all standard $\ell \in \mathbb{N}$, i.e., smooth in the standard sense. In particular, it is immediate to see that the $\ell$th derivative of $\tilde{u}$ at $(s, t) \in \mathbb{R} \times [0, 1]$ agrees, up to an infinitesimal error, with the $\ell$th derivative of $\circ \tilde{u}$ at $\circ(s, t) = (\circ s, \circ t) \in \mathbb{R} \times [0, 1]$.

While the boundary conditions are immediately clear, it still remains to show that $\tilde{u}^\mathbb{H}$ satisfies the corresponding Floer equation and meets the given asymptotic conditions.

We start with the case that $T$ is limited. In order to see that $\bar{\partial} \tilde{u} + \varphi_T(s) \cdot \nabla G_t^\mathbb{H}(\tilde{u}) = 0$ implies that $\circ \tilde{u} : \mathbb{R} \times [0, 1] \rightarrow \mathbb{P}(\mathbb{H})$ satisfies $\bar{\partial} \circ \tilde{u} + \varphi_T(s) \cdot \nabla G_t^\mathbb{H}(\circ \tilde{u}) = 0$, it suffices to use that the restriction of the first derivative of $G_t^\mathbb{H}$ to $\mathbb{P}(\mathbb{H}) \subset \mathbb{P}(\mathbb{F})$ agrees with the first derivative of $G_t = G_t^\mathbb{H}$ up to an infinitesimal error. In the case when $T$ is unlimited, note that the shifted map $\tilde{u}(-T, \cdot)$ satisfies the Floer equation $\bar{\partial} \tilde{u} + \varphi_T(s-T) \cdot \nabla G_t^\mathbb{H}(\tilde{u}) = 0$. Since $\varphi_T(-T)$ has support in $[-1, 2T + 1] \subset \mathbb{R}$ with $\varphi_T(s-T) = 1$ for $s \in [0, 2T]$, it follows that, after the restriction to $\mathbb{R} \subset \mathbb{R}$, $\varphi_T(-T)$ has support in $[-1, +\infty)$ with $\varphi_T(s) = 1$ for all $s \geq 0$.

For the asymptotic condition, note that in both cases, due to the finiteness of energy of $\tilde{u}^\mathbb{H}$, we know that $\partial_s \tilde{u}^\mathbb{H}$ converges to 0 as $s \rightarrow \pm \infty$. In order to show that this implies asymptotic convergence in the standard sense, it remains to employ nondegeneracy properties of the asymptotic orbits. Noting again that we may forget about frequencies $m \in \mathbb{Z}$ with $\hat{\psi}(m) = 0$ due to Liouville’s theorem, note that $u_n^0 \in \mathbb{P}(\mathbb{H})$ indeed can be treated as a nondegenerate orbit as the eigenvalues of the resulting restriction of $\phi_1^0$ are bounded away from one by the threshold $\delta > 0$. Note that here we crucially make use of the admissibility of $\psi$.

While this proves the asymptotic behavior in the case when $T$ is limited, in the case when $T$ is unlimited we still need to prove the asymptotic convergence towards $+\infty$. Since for $s > 0$ the corresponding Floer strip satisfies the Cauchy-Riemann equation with a Hamiltonian term, $0 = \partial_s \tilde{u}^\mathbb{H} + \nabla G_t(\tilde{u}^\mathbb{H})$, we now expect to get convergence towards a fixed point of $\phi_1 = \phi_1^G \circ \phi_1^0$. While we can no longer expect to get the desired nondegeneracy by simply forgetting all frequencies with $\hat{\psi}(m) = 0$, the important observation is now that this form of nondegeneracy still holds up to finite dimensions. Indeed, as $\hat{\psi}(m) \rightarrow 0$ as $m \rightarrow \infty$, it follows that there exists a finite-dimensional subspace $\mathbb{C}^d \subset \mathbb{H} / \mathbb{C}$ such that the eigenvalues of the restriction of the linearization of $\phi_1$ to the complement $(\mathbb{C}^d) \perp \subset \mathbb{H} / \mathbb{C}$ are bounded away from one by $\delta/2$. While this implies that $\tilde{u}^\mathbb{H}(\cdot, \cdot)$ indeed converges as $s \rightarrow +\infty$ in all but finitely
many dimensions, the existence of the sequence \((s_k)\) and of the asymptotic orbit can now be established as in the finite-dimensional case, see e.g. [17], using Gromov compactness.

\[\square\]

11. Appendix: Non-standard Model Theory

In this section we provide an outline of all the background and relevant definitions and statements about nonstandard analysis that the reader needs to know in order to follow the rest of the paper. Here we describe the original model-theoretic approach of Robinson ([15]), outlined in the excellent expositions [10], [11] as well as in [8], to which we refer and which shall also be consulted for more details and background.

Believing in the axiom of choice it is well-known, see e.g. ([11], theorem 2.9.10), that there exist non-standard models of mathematics in which, on one side, one can do the same mathematics as before (transfer principle) but, on the other side, all sets behave like compact sets (saturation principle). The idea is to successively introduce new ideal objects such as infinitely small and large numbers. The proof of existence of the resulting polysatured model is then performed in complete analogy to the proof of the statement that every field has an algebraic closure, by employing the axiom of choice.

A model of mathematics \(V\) of a family of sets which is rich enough in order to do all the mathematics that one has in mind. Since for existence proof of non-standard models it is crucial that \(V\) is still a set in the sense of set theory, there are (abstract) sets which are not in \(V\). Below we show how to define such a set \(V\) which contains all mathematical entities that we need for our proof. For most of the upcoming definitions and theorems on the general background on model theory we refer the reader to [11] as well as [8]. The first definition is taken from the appendix in ([11], section 2.9).

**Definition 11.1.** A sequence \(V = (V_n)_{n \in \mathbb{N}}\) of hierarchically ordered sets \(V_n, n \in \mathbb{N}\) is called a model if the elements in \(V_n\) are sets formed from the elements in \(V_0, \ldots, V_{n-1}\), i.e., \(V_n \subset \mathcal{P}(V_0 \cup \ldots \cup V_{n-1})\) and \(V_0\), called the set of urelements, does not contains elements from higher sets, i.e., \(V_0 \cap \bigcup_{n \geq 1} V_n = \emptyset\).

By choosing the model \(V = (V_n)_{n \in \mathbb{N}}\) large enough, one can ensure that the model contains all mathematical entities that one wants to work with. Apart from assuming that every subset formed from elements in \(V_0, \ldots, V_{n-1}\) is in \(V_n\), below we show explicitly that for our proof it turns out to be sufficient to take the real numbers as urelements, i.e., \(V_0 = \mathbb{R}\).
Definition 11.2. We call $V = (V_n)_{n \in \mathbb{N}}$ the standard model if the urelements are the real numbers, $V_0 = \mathbb{R}$, and the model is full in the sense that $V_n = \mathfrak{P}(V_0 \cup \ldots \cup V_{n-1})$.

In what follows, let $V = (V_n)_{n \in \mathbb{N}}$ denote the standard model. As discussed in ([III], 2.9), it follows that

$$V(\mathbb{R}) = \bigcup_{n=0}^{\infty} V_n(\mathbb{R}) \text{ with } V_n(\mathbb{R}) = \mathbb{R} \cup V_n$$

for all $n \in \mathbb{N}$.

is the superstructure over the real numbers in the sense of ([III], definition 2.1.1) and ([S], definition 15.4). Note that, for $n > 1$, we have for every full model that $V_{n-1} \subset V_n$.

Since in analysis one considers sets of functions which themselves can be viewed as sets built from the real numbers, the superstructure over the real numbers contains all mathematical entities that one needs to do analysis, see ([S], section 15B). In particular we claim

Proposition 11.3. The standard model $V = (V_n)_{n \in \mathbb{N}}$ contains (isomorphic copies of) all mathematical entities that we need in order to formulate and prove our non-squeezing theorem.

Proof. Instead of trying to give a proof listing all mathematical entities that will ever occur, we rather give the recipe and discuss the most important examples. This said, the proof of our proposition mostly relies on the following observation:

If $a$ and $b$ are sets in $V_n$, then every function $f : a \to b$ is an element of $V_{n+2}$ and every set of functions $f : a \to b$ is an element of $V_{n+3}$.

For this it suffices to observe that, in set theory, a function $f : a \to b$ is identified with the subset $\{(x, f(x)) : x \in a\}$ of $a \times b$ and for each $x \in a$, $y \in b$ the tuple $(x, y)$ is defined as the set $\{x, \{x, y\}\}$.

In order to see that all mathematical entities that we need are in the standard model, let us give the most relevant examples.

- First, since $V_0 = \mathbb{R}$, it follows that $\mathbb{R}$ as well as all its subsets like $\mathbb{N}$, $\mathbb{Z}$ are elements in $V_1$.
- Since tuples $(r_1, \ldots, r_n)$ of real numbers can be written as sets of tuples $\{(r_1, 1), \ldots, (r_n, n)\}$, we see that they are elements in $V_3$ and hence $\mathbb{R}^{2n} \cong \mathbb{C}^n$ belongs to $V_4$. The same holds true for the subsets $B^{2n}(r)$ and $Z^{2n}(R)$ and all other subsets.
- By identifying each separable real Hilbert space $\mathbb{H}$ with the space $\ell^2 = \ell^2(\mathbb{C})$ of square-summable series of complex numbers and using that every series of complex numbers is given by a function $f : \mathbb{N} \to \mathbb{C}$, it follows that all elements in $\mathbb{H}$ are
elements in $V_5$ and $\mathbb{H}$ as well as all its subsets are elements in $V_6$.

- Every Hamiltonian function $H$ and its symplectic gradient $X^H$ as well as every linear complex structure belong to $V_8$.
- Since the dual spaces of the linear spaces living in $V_6$ consists of functions, they hence belong to $V_9$.
- A linear symplectic form is an element in $V_{11}$. For this it does not matter whether we view it as a linear map from the linear space to its dual space ($\omega : \mathbb{H} \to \mathbb{H}^*$) or as a bilinear map on the linear space ($\omega : \mathbb{R}^{2n} \to \mathbb{R}^{2n} \to \mathbb{R}$, using that $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \in V_7$).
- An almost complex structure $J$, i.e., a complex structure on the tangent bundle, is a function from the space to the set of linear complex structures. Since the latter is an element of $V_7$, it follows that $J$ is again an element of $V_9$.
- Since the space of almost complex structures belongs to $V_{10}$, it follows that every one-parameter family $J : t \mapsto J_t$ of almost complex structures is an element of $V_{12}$.
- Every $J$-holomorphic map $\tilde{u}$ is a map between sets belonging to $V_6$ and hence belongs to $V_8$. Note that, although the defining almost complex structure only appears in $V_n$ for $n \geq 9$, the moduli space is a subset of the set of all maps and hence, by fullness of the model, is also an element in $V_7$.

In order to show that Floer’s existence result of symplectic fixed points and pseudo-holomorphic strips indeed continues to hold in infinite dimensions, we will use that, by abstract model theory, his statement also holds in the non-standard model which we are going to discuss below. To make the underlying transfer principle precise, we quickly recall all the necessary background from first-order predicate logic that is needed.

The idea is that, just like all mathematical entities that we need are contained in the standard model $V = (V_n)_{n \in \mathbb{N}}$, all statements that we will transfer can be formalized in first-order logic, that is, they are sentences in the language $\mathcal{L}_V$ for our standard model $V$. In the same way as the details in the precise definition of models are not ultimatively important in order to understand the strategy of our proof, we continue to recall all needed foundations from logic for the sake of completeness of the exposition. For the following definitions we continue to refer to the appendix in ([11], section 2.9) as well as ([8], section 15B).

**Definition 11.4.** The alphabet of the language $\mathcal{L}_V$ of the model $V = (V_n)_{n \in \mathbb{N}}$ consists of the logical symbols $\lor$, $\land$, $\exists$, $=, \in$, a countable number of variables, the elements in $V = \bigcup_{n \in \mathbb{N}} V_n$ as parameters, and auxiliary symbols like parentheses.
Definition 11.5. A sentence in the language \( \mathcal{L}_V \) of the model \( V = (V_n)_{n \in \mathbb{N}} \) is build inductively from the following rules:

i) If \( a, b \in V_{<\infty} \), then \( a \in b \) and \( a = b \) are sentences in \( \mathcal{L}_V \).

ii) If \( A \) and \( B \) are sentences in \( \mathcal{L}_V \), then \( A \lor B \) and \( \neg A \) are sentences in \( \mathcal{L}_V \).

iii) Let \( A \) be a sentence in \( \mathcal{L}_V \) and \( a, b \in V_{<\infty} \) are parameters in \( \mathcal{L}_V \). If \( x \) is a variable not occurring in \( A \), then \( \exists x \in aA_b(x) \) is a sentence in \( \mathcal{L}_V \), where \( A_b(x) \) is obtained from \( A \) by replacing each occurrence of the parameter \( b \) in \( A \) by the variable \( x \).

Every \( A(x) = A_b(x) \) as in part iii) with a free variable \( x \) is called a formula in \( \mathcal{L}_V \). Furthermore, for every parameter \( a \in V_{<\infty} \), by \( A(x)(a) \) we denote the new sentence in \( \mathcal{L}_V \) obtained by replacing the variable \( x \) by the parameter \( a \).

Whether a sentence \( A \) holds true in the model \( V \), written \( V \models A \), is decided using the usual interpretation for sentences in set theory, see ([11], 2.9), ([8], 15B).

Using the axiom of choice one can prove that there exists a so-called non-standard model in which the same mathematics hold true but in which every set from \( V \) can be viewed as a compact set. More precisely, after reformulating ([11], theorem 2.9.10), we have the following

Theorem 11.6. Given the standard model \( V = (V_n)_{n \in \mathbb{N}} \) there exists a corresponding non-standard model \( W = (W_n)_{n \in \mathbb{N}} \), together with an embedding \( * : V_{<\infty} \to W_{<\infty} \) respecting the filtration, i.e. \( \ast_n : V_n \to W_n \), satisfying the following two important principles.

- **Transfer principle:** If a sentence \( A \) holds in the language \( \mathcal{L}_V \) of the model \( V \), \( V \models A \), then the corresponding sentence \( \ast A \), obtained by replacing the parameters from \( V \) by their images in \( W \) under \( * \), holds in the language \( \mathcal{L}_W \) of the model \( W \), \( W \models \ast A \).

- **Saturation principle:** If \( (a_i)_{i \in I} \) is a collection of sets in \( W \), indexed by a set \( I \) in \( V \), and satisfying \( a_{i_1} \cap \ldots \cap a_{i_n} \neq \emptyset \) for all \( i_1, \ldots, i_n \in b, n \in \mathbb{N} \) (finite intersection property), then also the common intersection of all \( a_i, i \in I \) is non-empty, \( \bigcap_{i \in I} a_i \neq \emptyset \).

Proof. Since in the references the theorem is not precisely stated in the above form, let us quickly describe how it can be deduced from [11]. In ([11], theorem 2.9.10) it is claimed that there exists a so-called monomorphism from the superstructure \( V(\mathbb{R}) \) over \( \mathbb{R} \) into the superstructure \( V(\ast \mathbb{R}) \) over the set \( \ast \mathbb{R} \) of non-standard real numbers. The latter are defined explicitly as equivalence classes of sequences of
real numbers using the axiom of choice in \((\text{III}, \text{definition 1.2.3})\). Note that by \((\text{III}, \text{definition 2.4.3 and remark 2.4.4})\) the property of the map \(* : V(\mathbb{R}) \to V(*)\mathbb{R}\) being a monomorphism is equivalent to the transfer principle, in particular, the latter indeed implies that \(*\) respects the filtration. On the other hand, the fact that the formulation of the saturation principle given here is equivalent to the definition in \((\text{III}, \text{definition 2.9.1})\) is proven in \((\text{III}, \text{theorem 2.9.4})\), noticing that, by the definition of the cardinal number \(\kappa^+\) appearing in \((\text{III}, \text{theorem 2.9.10})\), every set in \(V\) is \(\kappa^+\)-small. Since the saturation property is only assumed when all sets \(a_i, i \in V\) are internal in the sense of \((\text{III}, \text{definition 2.8.1})\), that is, when they are elements in the \(*\)-image \(V_n(\mathbb{R}) \subset V_n(*)\mathbb{R}\) of the set \(V_n(\mathbb{R}) \in V_{n+1}(\mathbb{R})\), we follow the strategy in the appendix of \((\text{III}, \text{section 2.9})\) and define the non-standard model \(W = (W_n)_{n \in \mathbb{N}}\) by setting \(W_n := V_n\) for all \(n \in \mathbb{N}\). In particular, every set in the non-standard model \(W = (W_n)_{n \in \mathbb{N}}\) is internal. □

In what follows we follow the usual conventions and write \(*a := *((a))\) for every set \(a \in V_{<\infty}\setminus V_0\) and identify \(a := *(a)\) for every urelement \(a \in V_0 = \mathbb{R}\).

**Definition 11.7.** A set \(a\) is called

i) internal if \(a \in W_{<\infty}\),

ii) standard if \(a = *b := *((b)) \in W_{<\infty}\) for some \(b \in V_{<\infty}\).

iii) external if \(a\) is not internal.

We start with some immediate consequences of the transfer principle, see \((\text{III}, \text{proposition 2.4.6})\).

**Proposition 11.8.** Let \(a, b\) be sets in \(V_{<\infty}\). Then we have

i) \(a = b\) if and only if \(*a = *b\),

ii) \(a \in b\) if and only if \(*a \in *b\),

iii) \(a \subset b\) if and only if \(*a \subset *b\),

iv) \(f : a \to b\) if and only if \(*f : *a \to *b\).

These in turn lead to the following

**Corollary 11.9.** It follows

i) \(* : V_{<\infty} \to W_{<\infty}\) is an embedding.

ii) For every set \(b \in V_{<\infty}\) we have that \(*[b] := \{*a : a \in b\} \subset *b\).

iii) For every function \(f : a \to b\) we have that \(*f : *a \to *b\) is an extension of \(f\) in the sense that for all \(c \in a\) we have \(*f(c) = (*f)(*c) \in *b\).

**Examples:**

i) Since \(+\) is a function from \(\mathbb{R} \times \mathbb{R}\) to \(\mathbb{R}\), it follows that \(*+\) is a function from \(*\mathbb{R} \times *\mathbb{R}\) to \(*\mathbb{R}\) with \(*r + *s = *(r + s)\) for all \(a, b \in \mathbb{R}\).
ii) Since the symplectic form $\omega$ on $\mathbb{H}$ is a map from $\mathbb{H} \times \mathbb{H}$ to $\mathbb{R}$, its $*$-image $^*\omega$ is a map from $^*\mathbb{H} \times ^*\mathbb{H}$ to $^*\mathbb{R}$ which agrees with $\omega$ on $\mathbb{H} \times \mathbb{H} \subset ^*\mathbb{H} \times ^*\mathbb{H}$. Analogous statements hold true for the inner product $\langle \cdot, \cdot \rangle$ and the complex structure $J_0$ on $\mathbb{H}$.

iii) Since, for all $n \in \mathbb{N}$ and all $k \in \mathbb{N}$, we know that $\sum_{i=1}^k$ is a function from $(\mathbb{R}^n)^k$ to $\mathbb{R}^n$, it follows that, now even for all $n \in ^*\mathbb{N}$ and all $k \in ^*\mathbb{N}$, $^*\sum_{i=1}^k$ is a function from $^*(\mathbb{R}^n)^k$ to $^*\mathbb{R}^n$ with $^*\sum_{i=1}^k r_i = ^*\sum_{i=1}^{k-1} r_i + r_k$ for all $k \in ^*\mathbb{N}$.

iv) Since every sequence $s = (s_n)_{n \in \mathbb{N}}$ of natural numbers is a function from $\mathbb{N}$ to $\mathbb{R}$, it follows that its $*$-image $^*s$ is a function from $^*\mathbb{N}$ to $^*\mathbb{R}$ with $^*s_n = ^*(s_n)$ for all $n \in \mathbb{N}$.

We make the following

**Convention:** If no confusion is likely to arise, we make the convention to identify each standard set $b \in V_{<\infty}$ with $^*[b] \subset ^*b \in W_{<\infty}$. In particular, we have $\mathbb{R} \subset ^*\mathbb{R}$ and $\mathbb{N} \subset ^*\mathbb{N}$.

Indeed it is true that the saturation principle implies that the non-standard model $W = (W_n)_{n \in \mathbb{N}}$ is (much) larger than the standard model $V = (V_n)_{n \in \mathbb{N}}$. For the next statement we refer to ([11, proposition 2.4.6]) and ([11], proposition 2.9.7).

**Proposition 11.10.** We have the following dichotomy:

i) If $b \in V_{<\infty}$ has finitely many elements, then its $*$-image $^*b \in W_{<\infty}$ consists of the $*$-images of its elements,

$$^*\{a_1, \ldots, a_n\} = \{^*a_1, \ldots, ^*a_n\}.$$  

ii) If $b \in V_{<\infty}$ has infinitely many elements, then its $*$-image $^*b \in W_{<\infty}$ contains $b$ as a proper subset,

$$^*[b] = \{^*a : a \in b\} \subsetneq ^*b.$$  

In particular, it follows from ii) that $*: V_{<\infty} \to W_{<\infty}$ is a proper embedding.

**Proof.** While the part i) follows from the transfer principle after observing that the equality $b = \{a_1, \ldots, a_n\}$ can be encoded into the sentence $a \in b \iff a = a_1 \lor \ldots \lor a = a_n$ in $L_V$, for part ii) consider the collection of sets $(a_i)_{i \in b}$ given by $a_i := ^*b \setminus \{i\}$ for $i \in b$. While it is easy to see that they have the finite intersection property, $a_{i_1} \cap \ldots \cap a_{i_n} \neq \emptyset$ for all $i_1, \ldots, i_n \in b$, $n \in \mathbb{N}$, every element in $\bigcap_{i \in b} a_i \neq \emptyset$ is an element of $^*b \setminus b$. Note that, while in part i) the finite intersection property fails, in part ii) the transfer principle cannot be applied as the corresponding sentence would have infinite length, which is forbidden. $\square$
In particular, one can show that \( \ast \mathbb{R} \), the set of \( \ast \)-real (or hyperreal or non-standard real) numbers, contains infinitesimals as well as numbers which are greater than any real number.

**Proposition 11.11.** The saturation principle implies the existence of the following ideal objects.

i) There exist \( r \in \ast \mathbb{R} \setminus \{0\} \) such that \( |r| < 1/n \) for every standard natural number \( n \in \mathbb{N} \). Any such \( r \in \ast \mathbb{R} \) (including \( r = 0 \)) is called infinitesimal and we write \( r \approx 0 \).

ii) There exist \( r \in \ast \mathbb{R} \) such that \( |r| > n \) for every standard natural number \( n \in \mathbb{N} \). Any such \( r \in \ast \mathbb{R} \) is called unlimited. Any \( r \in \ast \mathbb{R} \) which is not unlimited is called limited.

iii) A number \( r \in \ast \mathbb{R} \) is limited if and only if it is near-standard in the sense that there exists a standard real number \( s \in \mathbb{R} \) with \( r - s \approx 0 \). For every near-standard \( r \in \ast \mathbb{R} \) we call \( \circ r := s \in \mathbb{R} \) the standard part of \( r \).

iv) Any limited \( n \in \ast \mathbb{N} \) is standard.

**Proof.** For the definitions we refer to ([11], definitions 1.2.7 and 1.6.9). Since the existence of infinitesimal and unlimited numbers is the key reason why to care about non-standard analysis, let us give the short proof: Define for every \( n \in \mathbb{N} \) the sets \( a_n := \{ r \in \ast \mathbb{R} : 0 < |r| < 1/n \} \) and \( b_n := \{ r \in \ast \mathbb{R} : |r| > n \} \). Since the corresponding collections of sets obviously have the finite intersection property, we find that \( \bigcap_{n \in \mathbb{N}} a_n \) and \( \bigcap_{n \in \mathbb{N}} b_n \) are non-empty and any element in these sets has the desired properties. For the third part we refer to ([11], proposition 1.6.11). Part iv) follows from the observation that there are only finitely many natural numbers smaller than a given one, so the \( \ast \)-image of the corresponding set does not contain any new elements. \( \square \)

**Remark 11.12.** Along the same lines we have:

i) Similar statements clearly hold when \( \ast \mathbb{R} \) is replaced by \( \ast \mathbb{R}^n \) for some standard \( n \in \mathbb{N} \). In particular, for every limited \( r > 0 \) every point on \( \ast S^{n-1}(r) \subset \ast \mathbb{R}^n \) is near-standard, i.e., \( \ast S^{n-1}(r) \) is obtained from \( S^{n-1}(r) \) by adding points which are infinitesimally close.

ii) In the same way as \( \ast \mathbb{R} \) contains much more elements than \( \mathbb{R} \) itself, the non-standard extension \( \ast \mathbb{H} \) of \( \mathbb{H} \) is a much larger space than \( \mathbb{H} \) itself.

In ([11], theorems 1.6.8 and 1.6.15) it is shown that limited and infinitesimal numbers furthermore have the following nice closure properties.

**Proposition 11.13.** We have

i) Finite sums, differences and products of limited numbers are limited.

ii) Finite sums, differences and products of infinitesimal numbers are infinitesimal.
iii) The product of an infinitesimal number with a limited number is still infinitesimal.
iv) The standard part of a sum, difference or product of two limited numbers is the sum, difference or product of their standard parts.

One of the main benefits of non-standard analysis is that the clumpy $\epsilon$-formalism can be avoided by introducing infinitesimals and unlimited $\ast$-natural numbers. For the following proposition we refer to ([11], theorem 1.7.1).

**Proposition 11.14.** A sequence $(s_n)_{n \in \mathbb{N}}$ converges to zero, $s_n \to 0$, as $n \to \infty$ if and only if $s_N := \ast s_N \approx 0$ for all unlimited $N \in \ast \mathbb{N} \setminus \mathbb{N}$.

Since convergence in metric spaces is defined by requiring that the distance between points converges to zero, the above result immediately generalizes to all metric spaces.

Apart from showing that the non-standard model contains infinitely-large numbers, the saturation principle immediately leads to the following, even more surprising fact, see ([11], theorem 2.9.2).

**Proposition 11.15.** For every standard set $b \in V_{<\infty}$ there exists a non-standard set $c \in W_{<\infty}$, which contains all elements of $a$, i.e., $a \in b$ implies $\ast a \in c$, and which is $\ast$-finite in the sense that there is a bijection from $c$ to an internal set $\{n \in \ast \mathbb{N} : n \leq N\}$ for some $N \in \ast \mathbb{N}$.

Since a subset of a finite set in the standard model $V = (V_n)_{n \in \mathbb{N}}$ is again a finite set, this seems to lead to an obvious logical contradiction. However, since the transfer principle only applies to subsets of $\ast$-finite sets which belong to the non-standard model, i.e., are internal themselves, the logical paradoxon is resolved in the following

**Proposition 11.16.** For every infinite set $b \in V_{<\infty}$, the corresponding proper subset $b = \ast [b] = \{\ast a : a \in b\}$ of $\ast b$ is external. For example, $\mathbb{R}$ and $\mathbb{N}$ are external. In particular, the non-standard model is not full, $W_n \subseteq \mathcal{P}(W_0 \cup \ldots \cup W_{n-1})$.

For the short proof we refer to ([11], proposition 2.9.6).

**Remark 11.17.** In an analogous way we will prove in the next subsection that every infinite-dimensional (separable) Hilbert space $\mathbb{H}$ is contained in a $\ast$-finite-dimensional Euclidean vector space $\mathbb{F}$ of some unlimited but $\ast$-finite dimension $N \in \ast \mathbb{N} \setminus \mathbb{N}$. The infinite-dimensional Hilbert space $\mathbb{H}$ is not a $\ast$-finite-dimensional Euclidean vector space itself, but is only contained in some space which behaves as if it were finite-dimensional.

The fact that every internal set containing an infinite standard set as a subset must be strictly larger leads to the so-called spillover principles, see ([11], theorem 2.8.12)
Proposition 11.18. Let $b$ denote an internal subset of $\ast \mathbb{R}$. If $b$ contains all standard real numbers $r \in \mathbb{R}$ (greater than a given real number $R \in \mathbb{R}$) then $b$ must contain an unlimited real number.

While these results are satisfactory from the theoretical point of view, for practical purposes it is rather important to know which subsets of an internal set in the non-standard model are still internal themselves, so that statements can be proven for them by applying the transfer principle. Since in applications one is almost exclusively interested in subsets which can be defined by requiring that their elements have a specific property, the following positive result originally due to Keisler, ([8], theorem 15.14), see also ([11], theorem 2.8.4), is sufficient for all our purposes.

Proposition 11.19. (Internal Definition Principle) Every definable set belongs to the non-standard model $W = (W_n)_{n \in \mathbb{N}}$, that is, for every formula $A(x)$ in $L_W$ the set \{a \in W_{<\infty} : W \models A(x)(a)\} is internal.

In particular, every finite subset of an internal set is internal.

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