Generalized Master Function Approach to
Quasi-Exactly Solvable Models

M. A. Jafarizadeh$^{a,b,c}$ *, S. J. Akhtarshenas $^{a,b,c}$ †

$^a$Department of Theoretical Physics and Astrophysics, Tabriz University, Tabriz 51664, Iran.

$^b$Institute for Studies in Theoretical Physics and Mathematics, Tehran 19395-1795, Iran.

$^c$Pure and Applied Science Research Center, Tabriz 51664, Iran.

November 21, 2018

*E-mail:jafarzadeh@ark.tabrizu.ac.ir

†E-mail:akhtarash@ark.tabrizu.ac.ir
Abstract

By introducing the generalized master function of order up to four together with corresponding
weight function, we have obtained all quasi-exactly solvable second order differential equations. It is
shown that these differential equations have solutions of polynomial type with factorization properties,
that is polynomial solutions $P_m(E)$ can be factorized in terms of polynomial $P_{n+1}(E)$ for $m \geq n + 1$. All
known quasi-exactly quantum solvable models can be obtained from these differential equations, where
roots of polynomial $P_{n+1}(E)$ are corresponding eigen-values.

Keywords: Quasi-Exact, Differential Equations, Orthogonal Polynomials, Factorization.

PACs Index: 03.65.Ge
1 INTRODUCTION

During the last decade a remarkable new class of quasi-exactly solvable spectral problems was introduced [1, 2, 3, 4, 5]. These occupy an intermediate position between exactly solvable and unsolvable models in the sense that exact solution in an algebraized form exists only for a part of the spectrum.

The usual approach to the analysis of quasi-exactly solvable systems is an algebraic one in which the operator is expressed as a non-linear combination of generators of a Lie algebra. Another recent development is the work of Bender-Dunne [6] where they have shown that the eigen-functions of a quasi-exactly solvable Schrödinger equation is the generating function for a set of orthogonal polynomials $P_m(E)$ in energy variable. It was further shown that, these polynomials satisfy the three-term recursion relation. Also, all polynomials beyond a critical polynomial $P_m(E)$ factorize into the product of polynomial $P_{n+1}(E)$ and another arbitrary polynomial.

In this paper we suggest a generalization of Bender-Dunne approach to all possible one-dimensional quasi-exactly second order differential equations.

For this purpose, the successful master function approach of references [7, 8] to exactly solvable models, is generalized to a master function of order up to four which gives all possible one-dimensional quasi-exactly solvable models, where Bender-Dunne model [6] and Heun differential equation [9] are among them.

The paper is organized as follows: In section II we show that we can generalize the usual quadratic master function to a master function of at most four order polynomials, then the most general quasi-exactly solvable differential operators related to generalized master function of degree $k = 3$ and $k = 4$ are given, respectively.

In section III, expanding their solutions in powers of $x$, we get 3-term and 4-term recursion relations among their coefficients, where Bender-Dunne factorization follows through imposing the quasi-exactly solvability conditions of section II. At the end of this section we list all possible related quasi-exactly solvable differential equations for $k = 3$ and $k = 4$ in Tables I and II, respectively.

Finally at section IV, we derive all possible one-dimensional quasi-exactly solvable quantum Hamiltonian
from the differential operators of section III, via prescription of references [7,8], where we have listed them at the end of section III, except for those which can given in terms of elliptic functions. Paper ends with a brief conclusion.

2 QUASI-EXACTLY SOLVABLE DIFFERENTIAL EQUATIONS ASSOCIATED WITH GENERALIZED MASTER FUNCTIONS

By generalizing master function of order up to two [7,8] to polynomial of order up to $k$, together with the non-negative weight function $W(x)$, defined at interval $(a,b)$ such that $\frac{1}{W(x)} \frac{d}{dx} (A(x)W(x))$ to be a polynomial of degree at most $(k-1)$, we can define the operator

$$L = \frac{1}{W(x)} \frac{d}{dx} \left( A(x)W(x) \frac{d}{dx} \right) + B(x), \quad (2-1)$$

where $B(x)$ is a polynomial of order up to $(k-2)$. The interval $(a,b)$ is chosen so that, we have $A(a)W(a) = A(b)W(b) = 0$.

It is straightforward to show that the above defined operator $L$ is a self adjoint linear operator which at most maps a given polynomial of order $m$ to another polynomial of order $(m+k-2)$. Now, by an appropriate choice of $B(x)$ and weight function $W(x)$, the operator $L$ can have an invariant subspace of polynomials of order up to $n$. Then by choosing the set of orthogonal polynomials $\{\phi_0(x), \phi_1(x), \ldots, \phi_n(x)\}$ defined in the interval $(a,b)$ with respect to the weight function $W(x)$:

$$\int_a^b \phi_m(x)\phi_n(x)W(x)dx = 0, \quad \text{for} \quad m = n \quad (2-2)$$

as the base, the matrix elements of the operator $L$ on this base will have the following block diagonal form:

$$L_{ij} = 0, \quad \text{if} \quad \{i \leq n \ \text{and} \ j \geq n + 1\} \ \text{or} \quad \{i \geq n + 1 \ \text{and} \ j \leq n\}. \quad (2-3)$$

Since, according to the well known theorem of orthogonal polynomials, $\phi_n(x)$ is orthogonal to any poly-
nominal of order up to $n - 1$, therefore, for matrix $L$ we get

$$L = \begin{bmatrix} M & 0 \\ 0 & N \end{bmatrix},$$

(2-4)

where $M$ is an $(n + 1) \times (n + 1)$ matrix with matrix elements

$$M_{ij} = \int_a^b dx W(x) \phi_i(x) L(x) \phi_j(x), \quad i, j = 0, 1, 2, \ldots, n,$$

(2-5)

and $N$ is an infinite matrix element defined as above with $i, j \geq n + 1$.

The block diagonal form of the operator $L$ indicates that by diagonalizing the $(n + 1) \times (n + 1)$ matrix $M$, we can find $(n + 1)$ eigen-values of the operator $L$ together with the related eigen-functions as linear functions of orthogonal polynomials $\{\phi_0(x), \phi_1(x), \cdots, \phi_n(x)\}$.

In order to determine the appropriate $B(x)$ and $W(x)$ for a given generalized master function $A(x)$, we Taylor expand those functions:

$$A(x) = \sum_{i=0}^k \frac{A^{(i)}(0)}{i!} x^i,$$

where $A^{(i)}(0) = \frac{d^i A(x)}{dx^i} \big|_{x=0}$

(2-6)

$$\frac{(A(x)W(x))'}{W(x)} = \sum_{i=0}^{k-1} \frac{\left(\frac{(AW)'}{W}\right)^{(i)}(0)}{i!} x^i,$$

where $\left(\frac{(AW)'}{W}\right)^{(i)}(0) = \frac{d^i}{dx^i} \left(\frac{(A(x)W(x))'}{W(x)}\right) \big|_{x=0}$

(2-7)

$$B(x) = \sum_{i=0}^{k-2} \frac{B^{(i)}(0)}{i!} x^i,$$

where $B^{(i)}(0) = \frac{d^i B(x)}{dx^i} \big|_{x=0}$.

(2-8)

Then, the existence of invariant subspace of the polynomials of order $n$ of the operator $L$ leads to the following linear equation between the coefficients of above Taylor expansions:

$$- \frac{A^{(i+2)}}{(i+2)!} \frac{1}{l(l-1)} - \frac{\left(\frac{(AW)'}{W}\right)^{(i+1)}}{(i+1)!} l + \frac{B^{(i)}}{i!} = 0,$$

(2-9)
where

\[
\begin{align*}
  l & = n, \quad \text{and} \quad i = 1, \quad 2, \quad \ldots, \quad k - 2 \\
  l & = n - 1, \quad \text{and} \quad i = 2, \quad 3, \quad \ldots, \quad k - 2 \\
  & \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
  l & = n - k + 4, \quad \text{and} \quad i = k - 3, \quad k - 2 \\
  l & = n - k + 3, \quad \text{and} \quad i = k - 2
\end{align*}
\]

The number of above equations, for a given value of \( k \), is \( \frac{(k-1)(k-2)}{2} \). If we are to determine only the unknown function \( B(x) \) without having any further constraint on the weight function \( W(x) \), then the above \( \frac{(k-1)(k-2)}{2} \) equations should be satisfied with \( k - 2 \) coefficients of Taylor expansion of \( B(x) \) as the only unknowns, since \( B^{(0)} \) can be absorbed in the eigen-spectrum operator \( L \). Therefore, we are left with \( k - 2 \) unknowns to be determined, where the compatibility of equations (2-9) require \( k = 3 \) at most. On the other hand, if we add the coefficients of Taylor expansions of \( A(x) \) and \( \frac{(AW)'(x)}{W(x)} \) to our list of unknowns, ( to be determined by solving equations (2-9) ), then their compatibility conditions require that:

\[
3(k - 1) \geq \frac{(k-1)(k-2)}{2},
\]

or \( k \leq 8 \), where further investigations show that we can have at most \( k = 4 \), since for \( k \geq 5 \) the coefficients \( A^{(k)}(0) \) and \( \left( \frac{(AW)'}{W} \right)^{(k-1)}(0) \) will vanish. Below we summarize the above-mentioned discussion for \( k = 3 \) and \( k = 4 \), separately.

### 2.1 \( k = 3 \)

In this case, \( B(x) \) is a second order polynomial where \( B^{(1)} \) can be determined by solving equation (2-9):

\[
B^{(1)} = \frac{n}{2} \left( \frac{A^{(3)}}{3}(n - 1) + \left( \frac{(AW)'}{W} \right)^{(2)} \right),
\]

which is the only unknown in this case.
Again, the solving of equations (2-9) leads to:

\[ B^{(1)} = \frac{n}{2} \left( \frac{A^{(3)}}{3} (n - 1) + \left( \frac{(AW)'}{W} \right)^{(2)} \right), \quad (2-12) \]

\[ B^{(2)} = -\frac{A^{(4)}}{12} n(n - 1), \quad (2-13) \]

and

\[ \left( \frac{(AW)'}{W} \right)^{(3)} = -\frac{A^{(4)}}{2} (n - 1). \quad (2-14) \]

Here, besides having constraint over second order polynomial \( B(x) \), we have to put further constraints on the weight function \( W(x) \) given in (2-14).

Definitely we can determine \( n + 1 \) eigen-spectrum of the operator \( L \), simply by diagonalizing the \( (n + 1) \times (n + 1) \) matrix \( M \), since it is a self-adjoint operator in Hilbert space of polynomials and it has a block diagonal form given in (2-4).

As we are going to see in the next section, we can determine its eigen-spectrum analytically, using some recursion relations.

### 3 RECURSION RELATIONS

In this section we show that the eigen-functions of the operator \( L \) are a generating function for a new set of polynomials \( P_m(E) \) where the eigen-function equation of the operator \( L \) leads to the recursion relation between these polynomials. Quasi-exact solvable constraints (2-9) will lead to their factorization, that is, \( P_{n+N+1}(E) = P_{n+1}(E)Q_N(E) \) for \( N \geq 0 \), where roots of polynomials \( P_{n+1}(E) \) turn out to be the eigen-values of the operator \( L \).

To achieve these results, first we expand \( \psi(x) \), the eigen-function of \( L \), as:

\[ \psi(x) = \sum_{m=0}^{\infty} P_m(E)x^m, \quad (3-15) \]
Master Function Approach to Quasi-Exact Models

where eigen-function equation:

\[ L\psi(x) = E\psi(x) \]  

(3-16)

can be expressed as:

\[-A(x) \sum_{m=2}^{\infty} m(m-1)P_m(E)x^{m-2} - \frac{(A(x)W(x))'}{W(x)} \sum_{m=1}^{\infty} mP_m(E)x^{m-1} \]

\[ + B(x) \sum_{m=0}^{\infty} P_m(E)x^m = E \sum_{m=0}^{\infty} P_m(E)x^m, \]  

(3-17)

which leads to the following recursion relations for the coefficients \( P_m(E) \):

\[
\left( A^{(1)}(m+1)(m+2) + \left( \frac{(AW)'}{W} \right)^{(0)}(m+2) \right) P_{m+2}(E) \\
+ \left( \frac{A^{(2)}}{2!} m(m+1) + \left( \frac{(AW)'}{W} \right)^{(1)}(m+1) + E \right) P_{m+1}(E) \\
+ \left( \frac{A^{(3)}}{3!} (m-1) + \frac{(AW)'}{W} \right) \frac{(m+2)}{2!} x - B^{(1)} \right) P_m(E) \\
+ \left( \frac{A^{(4)}}{4!} (m-1)(m-2) + \frac{(AW)'}{W} \right) \frac{(m+3)}{3!} m - B^{(2)} \right) P_{m-1}(E) = 0.
\]  

(3-18)

Below we investigate recursion relations thus obtained for \( k = 3 \) and \( k = 4 \), separately.

3.1 \( k = 3 \)

In this case the 4-term general recursion relation reduce to the following 3-term recursion relation:

\[
\left( A^{(1)}(m+1)(m+2) + \left( \frac{(AW)'}{W} \right)^{(0)}(m+2) \right) P_{m+2}(E) \\
+ \left( \frac{A^{(2)}}{2!} m(m+1) + \left( \frac{(AW)'}{W} \right)^{(1)}(m+1) + E \right) P_{m+1}(E) \\
+ \left( \frac{A^{(3)}}{3!} (m-1) + \frac{(AW)'}{W} \right) \frac{(m+2)}{2!} m - B^{(1)} \right) P_m(E) = 0.
\]  

(3-19)
In order to have finite eigen-spectrum, that is, quasi-integrable differential equation, the above recursion relation should be truncated for some value of \( m = n \), which is obviously possible by an appropriate choice of:

\[
B^{(1)} = \frac{n}{2} \left( \frac{A^{(3)}}{3} (n - 1) + \left( \frac{(AW)'}{W} \right)^{(2)} \right),
\]  

(3-20)

which is in agreement with the result of previous section given in (2-11).

Using the recursion relation (3-19), with \( B^{(1)} \) given in (3-20), we get a factorization of polynomial \( P_{n+N+1}(E) \) for \( N \geq 0 \) in terms of \( P_{n+1}(E) \) as follows:

\[
P_{n+N+1}(E) = P_{n+1}(E)Q_N(E) \quad N \geq 0
\]

(3-21)

where, by choosing the eigen-values \( E \) as roots of polynomial \( P_{n+1}(E) \), all polynomials of order higher than \( n \) will vanish.

In order to determine corresponding eigen-functions, it is sufficient to evaluate \( P_m(E_i) \) for \( m = 0, 1, 2 ... n \) with \( E_i \) as roots of \( P_{n+1}(E) \), then eigen-function \( \psi_i(x) \) corresponding to eigen-value \( E_i \) can be written as:

\[
\psi_i(x) = \sum_{m=0}^{n} P_m(E_i) x^m, \quad i = 0, 1 ... n.
\]

(3-22)

The above eigen-functions are polynomials of order \( n \), hence they can have at most \( n \) roots in the interval \((a, b)\), where, according to the well known oscillation and comparison theorem of second-order linear differential equation [10], these numbers order the eigen-values according to the number of roots of corresponding eigen-functions. Therefore, we can say that the eigen-values thus obtained are the first \( n + 1 \) eigen-values of the operator \( L \).

Using the recursion relations (3-19), we can evaluate the polynomials \( P_m(E) \) in terms of \( P_0(E) \), where we have chosen \( P_0(E) = 1 \). We have evaluated the first five polynomials which appear in the Appendix I.

As an illustration we give the results for \( A(x) = x \) and \( n = 3 \) with \( \alpha = 1, \beta = 0, \gamma = -1 \) which is equivalent to the Bender-Dunne model:

\[
P_1(E) = -\frac{1}{2}E.
\]
\[ P_2(E) = -1 + \frac{1}{12}E^2, \]
\[ P_3(E) = \frac{1}{4}E - \frac{1}{144}E^3, \]
\[ P_4(E) = \frac{1}{10} - \frac{1}{48}E^2 + \frac{1}{2880}E^4. \]

Obviously \( P_m(E) \) have the parity of \( m \).

By finding the 4-roots of \( P_4(E) \) we determine the corresponding four eigen-values:

\[ E_0 = -7.398556194, \]
\[ E_1 = -2.293766823, \]
\[ E_2 = 2.293766823, \]
\[ E_3 = 7.398556194. \]

Finally for the coefficient \( P_m(E_i) \) we get:

\[ P_0(E_0) = 1, \quad P_1(E_0) = 3.699278097, \quad P_2(E_0) = 3.561552813, \quad P_3(E_0) = -.962769686, \]
\[ P_0(E_1) = 1, \quad P_1(E_1) = 1.146883412, \quad P_2(E_1) = -.5615528135, \quad P_3(E_1) = -.4896337383, \]
\[ P_0(E_2) = 1, \quad P_1(E_2) = -1.146883412, \quad P_2(E_2) = -.5615528135, \quad P_3(E_2) = .4896337383, \]
\[ P_0(E_3) = 1, \quad P_1(E_3) = -3.699278097, \quad P_2(E_3) = 3.561552813, \quad P_3(E_3) = -.962769686. \]

Using the above coefficients we can determine the corresponding eigen-functions through formula (3-22).

In Table I we give all quasi-exactly solvable operators which can be obtained by choosing different generalized master function of order 3. This Table contains all possible models corresponding to different choice of \( A(x) \) up to translation and rescaling of variable \( x \). Also by choosing \( A(x) \) as a polynomial of up to second order with \( \gamma = 0 \) we lead to the exactly solvable models of references [7, 8].
| $A(x)$ | $W(x)$ | $L(x)$ |
|--------|--------|--------|
| $x$    | $x^\alpha e^{\beta x + \gamma x^2}$ | $-x \frac{d^2}{dx^2} - (\alpha + 1 + \beta x)$ |
|        | $0 \leq x < +\infty$ | $+2\gamma x^2 \frac{dx}{dx} + 2n\gamma x$ |
|        | $\alpha > -1, -\infty < \beta < +\infty, \gamma < 0$ | |
| $x^2$  | $x^\alpha e^{\beta/x + \gamma x}$ | $-x^2 \frac{d^2}{dx^2} + (\beta - (\alpha + 2)x)$ |
|        | $0 \leq x < +\infty$ | $-\gamma x^2 \frac{dx}{dx} + n\gamma x$ |
|        | $-\infty < \alpha < +\infty, \beta < 0, \gamma < 0$ | |
| $x(1-x)$ | $x^\alpha (1-x)^\beta e^{-\gamma x}$ | $x(x-1) \frac{d^2}{dx^2} + (-\alpha - 1)$ |
|        | $0 \leq x \leq 1$ | $(\alpha + \beta + \gamma + 2)x$ |
|        | $\alpha > -1, \beta > -1, -\infty < \gamma < +\infty$ | $-\gamma x^2 \frac{dx}{dx} + n\gamma x$ |
| $x^3$  | $x^\alpha e^{-\beta/x^2 - \gamma/x}$ | $-x^3 \frac{d^2}{dx^2} - (2\beta + \gamma x)$ |
|        | $0 \leq x < +\infty$ | $(\alpha + 3)x^2 \frac{dx}{dx} + n(n + \alpha + 2)x$ |
|        | $\alpha < -3, \beta > 0, -\infty < \gamma < +\infty$ | |
| $x^2(1-x)$ | $x^\alpha (1-x)^\beta e^{-\gamma/x}$ | $x^2(x-1) \frac{d^2}{dx^2} + (-\gamma + (\gamma - \alpha - 2)x)$ |
|        | $0 \leq x \leq 1$ | $(\alpha + \beta + 3)x^2 \frac{dx}{dx}$ |
|        | $-\infty < \alpha < +\infty, \beta > -1, \gamma > 0$ | $-n(n + \alpha + \beta + 2)x$ |
| $x(1+x^2)$ | $x^\alpha (1+x^2)^\beta e^{\gamma \tan^{-1}x}$ | $-x(1+x^2) \frac{d^2}{dx^2} - (\alpha + 1 + \gamma x)$ |
|        | $0 \leq x < +\infty$ | $(\alpha + 2\beta + 3)x^2 \frac{dx}{dx}$ |
|        | $\alpha > -1, \beta <- (\alpha + 3)/2, -\infty < \gamma < +\infty$ | $+n(n + \alpha + 2\beta + 2)x$ |
| $x(1-x)(a-x)$ | $x^\alpha (1-x)^\beta(a-x)^\gamma$ | $x(x-1)(a-x) \frac{d^2}{dx^2} + (-a(\alpha + 1))$ |
| $a > 1$ | $0 \leq x \leq 1$ | $+((a + 1)(\alpha + 2) + a\beta + \gamma)x - (\alpha + \beta)$ |
|        | $\alpha > -1, \beta > -1, -\infty < \gamma < +\infty$ | $+\gamma + 3)x^2 \frac{dx}{dx} + n(n + \alpha + \beta + \gamma + 2)x$ |

Table 1: Quasi-exactly differential operators obtained from generalized master function of order up to 3
3.2 \( k = 4 \)

Again in order to truncate the recursion relation (3-18) and to factorize polynomials \( P_{n+N+1}(E) \) in terms of \( P_{n+1}(E) \), we should have:

\[
B^{(1)} = \frac{n}{2} \left( \frac{A^{(3)}}{3} (n-1) + \left( \frac{(AW)'}{W} \right)^{(2)} \right),
\]

\[
\frac{B^{(2)}}{2!} = \frac{A^{(4)}}{4!} (n-1)(n-2) + \frac{\left( \frac{(AW)'}{W} \right)^{(3)}}{3!} n,
\]

and

\[
\frac{B^{(2)}}{2!} = \frac{A^{(4)}}{4!} n(n-1) + \frac{\left( \frac{(AW)'}{W} \right)^{(3)}}{3!} (n+1).
\]

Solving the above equations we get:

\[
B^{(2)} = -\frac{A^{(4)}}{12} n(n-1),
\]

and

\[
\left( \frac{(AW)'}{W} \right)^{(3)} = -\frac{A^{(4)}}{2} (n-1).
\]

The equations (3-23), (3-26) and (3-27) are the same equations which are required in the reduction of the operator \( L \) to its block diagonal form.

Again roots of polynomials \( P_{n+1}(E) \) will correspond to \( n+1 \) eigen-values of the differential operator \( L \) with eigen-functions which can be expressed in terms of \( P_m(E) \) for \( m \leq n \), where polynomials \( P_m(E) \) can be obtained from recursion relation by choosing \( P_0(E) = 1 \) and \( P_{-1}(E) = 0 \), where we have given the first four polynomials in Appendix II.

In Table II we list all quasi-exactly differential operators which can be obtained from the generalized master function of order up to four.

Tables I and II contain all quasi-exactly second order differential equations which can be obtained from Lie algebraic methods \[4\]. For example we get the Bender-Dunne model \[6\] for the choice of \( A(x) = x \) and \( \beta = 0 \), and similarly, we get the Heun differential operator (Fuxian equation with four regular singular point \[9\]) for the choice of \( A(x) = x(x - 1)(x - a) \).
| $A(x)$ | $W(x)$ | $L(x)$ |
|--------|--------|--------|
|        |        |        |
| $x^4$  | $x^n e^{\beta x^3 + \gamma / x^2 + \delta / x}$ | $-x^4 \frac{d^2}{dx^2} + (3\beta + 2\gamma x + \delta x) - (n+1)x^2$ |
|        | $0 \leq x < +\infty$ | $+\delta x^2 - (\alpha + 4)x^n \frac{d}{dx} - n\delta x - n(n-1)x^2$ |
|        | $\beta < 0, -\infty < \gamma < +\infty, -\infty < \delta < +\infty$ |        |
|        | $\alpha = -2(n+1)$ |        |
| $x^3(1-x)$ | $x^n (1-x) e^{-\gamma x^3 - \delta x^2}$ | $x^3 \frac{d^2}{dx^2} + (\gamma - \alpha - 3)x + (\alpha + 3\beta + 4)x^3 \frac{d}{dx}$ |
|        | $0 \leq x \leq +1$ | $+2n(\alpha + 4 + \beta + \gamma + \delta)x + n(n-1)x^2$ |
|        | $\beta > -1, -\infty < \gamma < +\infty, \delta > 0$ |        |
|        | $\alpha = -2(n+1) - \beta$ |        |
| $x^2(1 + x^2)$ | $x^n (1 + x^2) e^{\gamma x^3 + \delta x^2 / (1-x)}$ | $x^2 \frac{d^2}{dx^2} + 2ax - (a + 4 + \beta + \gamma + \delta)x + n((n+1)x^n \frac{d}{dx}$ |
|        | $0 \leq x \leq +1$ | $+2n(\alpha + 4 + \beta + \gamma + \delta)x + n(n-1)x^2$ |
|        | $\beta > -1, -\infty < \gamma < +\infty, \delta > 0$ |        |
|        | $\alpha = -2(n+1) - \beta - \gamma$ |        |
| $x^2(1-x)(a-x)$ | $x^n (1-x) e^{-\gamma x^3 + \delta x^2 / (1-x)}$ | $x^2 \frac{d^2}{dx^2} + (\gamma - (\alpha + 2\gamma + 2)x + n(n-1)x^2$ |
| $a > 1$ | $0 \leq x \leq +1$ | $+2n(\alpha + 4 + \beta + \gamma + \delta)x + n(n-1)x^2$ |
|        | $\beta > -1, -\infty < \gamma < +\infty, \delta > 0$ |        |
|        | $\alpha = -2(n+1) - \beta - \gamma$ |        |
| $x(a-x)(1+x^2)$ | $x^n (a-x) e^{\gamma x^3 + \delta x^2 / (1-x)}$ | $x^2 \frac{d^2}{dx^2} + (\gamma - (\alpha + 2\gamma + 2)x + n(n-1)x^2$ |
| $a > 0$ | $0 \leq x \leq a$ | $+2n(\alpha + 4 + \beta + \gamma + \delta)x + n(n-1)x^2$ |
|        | $-\infty < \gamma < +\infty, -\infty < \delta < +\infty$ |        |
|        | $-1 < \beta < -2n - 2\gamma$ |        |
|        | $\alpha = -2(n+1) - \beta - 2\gamma$ |        |
| $x(1-x)(a-x)(b-x)$ | $x^n (1-x) e^{\gamma x^3 + \delta x^2 / (1-x)}$ | $x^2 \frac{d^2}{dx^2} + (\gamma - (\alpha + 2\gamma + 2)x + n(n-1)x^2$ |
| $1 < a < b$ | $0 \leq x \leq +1$ | $+ab(\alpha + 1) + (2a + b + ab + ab(\alpha + \beta))x + n(n-1)x^2$ |
|        | $-1 < \beta < -2n - \gamma$ |        |
|        | $\alpha = -2(n+1) - \beta - 2\gamma$ |        |
4 QUASI-EXACTLY POTENTIAL ASSOCIATED WITH GENERALIZED MASTER FUNCTION

As in references [7, 8], writing:

\[ \psi(t) = A^{1/4}(x)W^{1/2}(x)\phi(x), \] (4-28)

with a change of variable \( \frac{dx}{dt} = \sqrt{A(x)} \), the eigen-value equation of the operator \( L \) reduces to the Schrödinger equation:

\[ H(t)\psi(t) = E\psi(t), \] (4-29)

with the same eigen-value \( E \) and \( \psi(t) \) given in (4-29), in terms of the eigen function of \( L \), where \( H(t) = -\frac{d^2}{dt^2} + V(t) \) is the similarity transformation of \( L(x) \) defined as:

\[ H(t) = A^{1/4}(x)W^{1/2}(x)L(x)A^{-1/4}(x)W^{-1/2}(x) \] (4-30)

with:

\[ V(t) = -\frac{3}{16} \frac{A^2(t)}{A^2(t)} - \frac{1}{4} \frac{W^2(t)}{W^2(t)} + \frac{1}{4} \frac{\dot{A}(t)\dot{W}(t)}{A(t)W(t)} + \frac{1}{4} \frac{\ddot{A}(t)}{A(t)} + \frac{1}{2} \frac{\ddot{W}(t)}{W(t)} + B(t). \] (4-31)

It is also straightforward to show that:

\[ \int dt\phi(t)H(t)\psi(t) = \int_a^b dxW(x)\psi(x)L(x)\psi(x). \] (4-32)

Hence block diagonalization of \( L \) leads to block-diagonalization of \( H \).

As an illustration we give below an example with \( A(x) = x^3 \), weight function \( W(x) = x^\alpha e^{\beta/x^2 - \gamma/x} \) and interval \([0, \infty)\), where \( \alpha < -3 \) and \( \beta, \gamma > 0 \).

From a change of variable \( dx/dt = \sqrt{A(x)} \), we get \( x(t) = 4/t^2 \), hence using equation (4-31) we have for potential \( V(t) \):

\[ V_n(t) = \frac{\gamma}{2}(\alpha + 1) + \left( \frac{15}{4} + \alpha^2 + 4n\alpha + 4\alpha + 4n^2 + 8n \right) \frac{1}{t^2} + \frac{1}{4} \left( \alpha \beta + \frac{1}{4} \gamma^2 \right) t^2 \]
\[ + \frac{\beta \gamma}{16} t^4 + \frac{\beta^2}{64} t^6 \] (4-33)
Below we give a list of quasi-exact solvable potentials, except for those potentials which can be expressed in terms of elliptic functions, since in this case we get rather long expressions for them:

\[ A(x) = x, \quad x(t) = \frac{t^2}{4} \]

\[ V_n(t) = \frac{\beta}{2} (\alpha + 1) + \left( \alpha^2 - \frac{1}{4} \right) \frac{1}{t^2} + \frac{1}{2} \left( \frac{\beta^2}{8} + \gamma \left(n + 1 + \frac{\alpha}{2}\right) \right) t^2 + \frac{\beta \gamma}{16} t^4 + \frac{\gamma^2}{64} t^6, \]

\[ A(x) = x^2, \quad x(t) = e^t \]

\[ V_n(t) = \frac{1}{4} (1 + \alpha^2 - 2\beta\gamma + 2\alpha - 2\alpha\beta e^{-t} + \beta^2 e^{-2t} + 2\gamma e^{2t} + 2 \left(2 \gamma + 2n\gamma + \alpha\gamma\right) e^t), \]

\[ A(x) = x(1 - x), \quad x(t) = \frac{1 + \sin(t)}{2} \]

\[ V_n(t) = \frac{1}{2} \left[ \left( n \gamma - \alpha \beta - \beta - \alpha + \frac{1}{2} \left( \beta \gamma - \alpha^2 - \beta^2 - \alpha \gamma - 1 \right) + \left( \alpha \gamma + \frac{\beta \gamma + n \gamma}{2} \right) \sin(t) \right) \right. \]

\[ \left. \frac{1}{2} \left( \alpha^2 + \beta^2 - \frac{1}{2} + \left( \beta^2 - \alpha^2 \right) \sin(t) \right) \frac{1}{\cos^2(t)} + \frac{\gamma^2}{16} \cos^2(t), \right] \]

\[ A(x) = x^3, \quad x(t) = \frac{4}{t^2} \]

\[ V_n(t) = \frac{\gamma}{2} (\alpha + 1) + \left( \frac{15}{4} + \alpha^2 + 4n\alpha + 4\alpha + 4n^2 + 8n \right) \frac{1}{t^2} + \frac{1}{4} \left( \alpha \beta + \frac{\gamma^2}{4} \right) t^2 \]

\[ + \frac{\beta \gamma}{16} t^4 + \frac{\gamma^2}{64} t^6, \]

\[ A(x) = x^2(1 - x), \quad x(t) = 1 - \tanh^2 \left( \frac{t}{2} \right) \]

\[ V_n(t) = \left( \frac{1}{\cosh^2(t) - 1} \right) \left[ - \left( 2n^2 + 2 + 2n\alpha + \frac{\alpha^2}{2} + 4n + \alpha \beta + 2\alpha + \frac{\alpha \gamma}{4} + 2n\beta + 2\beta \right) \cosh(t) \right. \]

\[ + \left. \frac{1}{2} \left( -\frac{\gamma^2}{4} - \frac{\alpha \gamma}{2} + \frac{1}{2} - \gamma + \alpha - \beta \gamma + \frac{\alpha^2}{2} \right) \cosh^2 + \frac{\alpha \gamma}{4} \cosh^3 + \frac{\gamma^2}{16} \cosh^4 \right]. \]
\[ + \left( \frac{\gamma}{2} + 4n + \alpha \beta + \frac{3 \alpha}{2} + \beta^2 + 2 \beta + \frac{3 \alpha^2}{4} + \frac{\alpha \gamma}{4} + \frac{\beta \gamma}{2} + \frac{\gamma^2}{16} + 2n\beta + 2n^2 + 2n\alpha \right) \),

\[ A(x) = x^4, \quad x(t) = \frac{1}{t} \]

\[ V_n(t) = \left( \frac{\delta^2}{4} + \gamma + 2n\gamma \right) + (\gamma \delta + 3n\beta + 3 \beta) t + \left( \frac{3\beta \delta}{2} + \gamma^2 \right) t^2 + 3 \beta \gamma t^3 + \frac{9\beta^2}{4}t^4, \]

\[ A(x) = x^3(1 - x), \quad x(t) = \frac{4}{4 + t^2} \]

\[ V_n(t) = - \left( \frac{\gamma}{2} + \delta + \beta \delta + \frac{\beta \gamma}{2} + n \gamma \right) + \left( \beta^2 - \frac{1}{4} \right) \frac{1}{t^2} + \frac{1}{2} \left( -n\delta + \frac{\delta^2}{2} - \delta + \frac{\gamma^2}{8} + \frac{\gamma \delta}{2} \right) t^2 + \frac{\delta}{8} \left( \frac{7}{2} + \delta \right) t^4 + \frac{\delta^2}{64} t^6, \]

\[ A(x) = x^2(1 + x^2), \quad x(t) = \frac{-1}{\sinh(t)} \]

\[ V_n(t) = \left( n + n^2 - \frac{\gamma \delta}{2} + \frac{1}{4} + 2n\beta + \beta + \beta^2 - (n\gamma + \gamma \beta) \sinh(t) \right)
+ \left( \frac{\delta^2}{4} + \beta \delta \sinh(t) - \beta^2 + \frac{1}{4} \right) \frac{1}{\cosh^2(t)} + \frac{\gamma^2}{4} \cosh^2(t), \]

\[ A(x) = x^2(1 - x^2), \quad x(t) = \frac{1}{\cosh(t)} \]

\[ V_n(t) = \left( \frac{1}{\cosh^2(t) - 1} \right) \left( \left( \frac{-\gamma}{2} - n - \frac{\beta}{2} + \frac{\gamma^2}{4} + \frac{\delta^2}{4} - \frac{\beta \delta}{2} + \frac{\gamma \delta}{2} - \frac{1}{2} - n\beta + \frac{\beta^2}{4} - n\gamma - n^2 \right)
+ \left( \frac{\beta \delta}{2} + n\beta - \frac{\gamma \delta}{2} + n\gamma + \frac{\gamma}{2} + \frac{1}{4} + \beta \gamma + \frac{\gamma \delta}{2} - \frac{\delta^2}{2} + n^2 + \frac{\beta^2}{4} + n + \frac{\gamma^2}{4} \right) \cosh(t)^2
+ \left( -\frac{\gamma \delta}{2} - n\delta - \delta - \frac{\gamma^2}{2} - \frac{\beta \delta}{2} + \frac{\beta^2}{2} \right) \cosh(t) + \left( \delta + n \delta + \frac{\beta \delta}{2} + \frac{\gamma \delta}{2} \right) \cosh^2(t) + \frac{\delta^2}{4} \cosh^4(t) \right), \]

\[ A(x) = x^2(1 - x)^2, \quad x(t) = \frac{e^t}{1 + e^t} \]
\[ V_n(t) = \left( \frac{1}{(e^{-t} + 1)^4} S_6 \right) \left( \frac{\delta^2}{4} S_4 + \delta \left( \frac{\beta}{2} - \frac{\delta}{2} \right) S_5 \right) \\
+ \left( \frac{1}{4} \gamma \frac{\delta}{2} + n^2 + \frac{\beta^2}{4} - n \gamma + \frac{\beta}{2} + n - 2 \beta \delta + \frac{3 \delta^2}{2} + n \beta \right) S_6 \\
- \left( -3 \beta \delta + \frac{3 \gamma}{2} - 3 n \gamma + \gamma + 4 n + \delta^2 - 2 \gamma \delta + \beta^2 + 1 + 4 n^2 + 2 \beta + 4 n \beta \right) S_7 \\
+ \left( 3 \beta + 6 n + \frac{3 \beta^2}{2} - 3 \gamma \frac{\delta}{2} + \frac{3}{4} \gamma + \frac{\delta^2}{4} + 6 n^2 - 2 \beta \delta + 4 \gamma + 6 n \beta + 2 \beta \gamma - 2 n \gamma \right) S_8 \\
+ \left( 4 n^2 - \frac{\beta \delta}{2} - 2 \gamma \delta + 3 \beta \gamma + 1 + 2 n \gamma + 4 n + \beta^2 + 2 \beta + \gamma^2 + 6 \gamma + 4 n \beta \right) S_9 \\
+ \left( \gamma \frac{\delta}{2} + \frac{3 \gamma^2}{2} + 4 \gamma + n + 2 \beta \gamma + 3 n \gamma + \frac{1}{4} + \frac{\beta^2}{4} + \frac{\beta}{2} + n \beta + n^2 \right) S_{10} \\
+ \left( n \gamma + \gamma^2 + \frac{\beta \gamma}{2} + \gamma \right) S_{11} + \frac{\gamma^2}{4} S_{12} \),

where \( S_k \) is defined as:

\[ S_k = \exp \left( -2 \left( \frac{k}{2} \gamma e^{-t} + \gamma + \delta \left( e^t + 1 \right) \right) \right), \quad k = 4, 5, \ldots, 12. \]

## 5 CONCLUSION

As we saw by introducing of master function \( A(x) \) as a polynomial of order at most four, we could obtain all quasi-exactly second order differential equations. It is shown that the eigen-equation relation \( L \Psi(x) = E \Psi(x) \) generates a set of polynomials \( P_m(E) \), where these polynomials satisfy 3-term and 4-term recursion relations for master function of at most three and four, respectively. Finally the quasi-exactly solvability leads to factorization of polynomials \( P_{n+N+1}(E) \) for \( N \geq 0 \) in terms of \( P_{n+1}(E) \), where by determining the roots of \( P_{n+1}(E) \) we can determine first \( n + 1 \) eigen-values of these quasi-exactly solvable differential equations.

## APPENDIX I

The First Five Polynomials \( P_n(E) \), For \( k = 3 \)

To abbreviate, we set \( F^{(i)} = (A W^{(i)}) \),
\[ P_1(E) = -\frac{E}{F^{(0)}}. \]

\[ P_2(E) = \frac{1}{2} \frac{E F^{(1)} + E^2 + B^{(1)} F^{(0)}}{F^{(0)} (A^{(1)} + F^{(0)})}, \]

\[ P_3(E) = \left( -A^{(2)} E F^{(1)} - A^{(2)} E^2 - A^{(2)} B^{(1)} F^{(0)} - 2 E F^{(1)} E^2 - 2 F^{(1)} B^{(1)} F^{(0)} - E^3 \right. \]
\[ \quad -3 E B^{(1)} F^{(0)} + E F^{(2)} A^{(1)} + E F^{(2)} F^{(0)} - 2 E B^{(1)} A^{(1)} \]
\[ \quad \left. \bigg/ \left( 6 F^{(0)} \left( A^{(1)} + F^{(0)} \right) \left( 2 A^{(1)} + F^{(0)} \right) \right) \right), \]

\[ P_4(E) = -\left( -3 A^{(2)} E^2 - 3 B^{(1)} F^{(0)}^2 - 6 B^{(1)} F^{(0)}^2 \cdot F^{(0)} A^{(1)} + 3 F^{(2)} B^{(1)} F^{(0)}^2 + A^{(3)} B^{(1)} F^{(0)}^2 \right. \]
\[ \quad + A^{(3)} E^2 F^{(0)} + 2 A^{(2)} E^2 A^{(1)} - 8 E^2 B^{(1)} A^{(1)} + 4 E^2 F^{(2)} F^{(0)} + 7 E^2 F^{(2)} A^{(1)} - 6 E^2 B^{(1)} F^{(0)} \]
\[ \quad -6 F^{(1)} B^{(1)} F^{(0)} - 13 A^{(2)} E F^{(1)} E^2 - 9 A^{(2)} E F^{(1)} E^2 - 3 A^{(2)} B^{(1)} F^{(0)} - 3 A^{(2)} E F^{(1)} \]
\[ \quad -4 A^{(2)} E^3 - 6 E F^{(1)} B^{(1)} F^{(0)} - 11 F^{(1)} E^2 - 6 F^{(1)} E^3 - 9 A^{(2)} F^{(1)} B^{(1)} F^{(0)} - 6 A^{(2)} E B^{(1)} A^{(1)} \]
\[ \left. + 3 A^{(2)} E F^{(2)} F^{(0)} + 3 A^{(2)} E F^{(2)} A^{(1)} - 10 A^{(2)} E B^{(1)} F^{(0)} - 12 F^{(1)} E B^{(1)} A^{(1)} + 6 F^{(1)} E F^{(2)} F^{(0)} \right. \]
\[ \quad + 9 F^{(1)} E F^{(2)} A^{(1)} - 14 F^{(1)} E B^{(1)} F^{(0)} + 2 A^{(3)} B^{(1)} F^{(0)} A^{(1)} + A^{(3)} E F^{(1)} F^{(0)} \]
\[ \quad + 2 A^{(3)} E F^{(1)} A^{(1)} + 6 F^{(2)} B^{(1)} F^{(0)} A^{(1)} - E^4 \right) \]
\[ \left. \bigg/ \left( 24 F^{(0)} \left( A^{(1)} + F^{(0)} \right) \left( 2 A^{(1)} + F^{(0)} \right) \left( 3 A^{(1)} + F^{(0)} \right) \right) \right), \]
\[ P_5(E) = \left( -46 E A^{(3)} B^{(1)} F^{(0)} A^{(1)} - 88 E F^{(2)} B^{(1)} F^{(0)} A^{(1)} + 16 A^{(3)} E F^{(2)} A^{(1)} F^{(0)} \right) \\
\]
\[ + 24 E F^{(2)}^2 A^{(1)} F^{(0)} - 48 F^{(2)} E B^{(1)} A^{(1)}^2 + 18 E F^{(2)}^2 A^{(1)}^2 + 6 E F^{(2)}^2 F^{(0)}^2 + 24 E B^{(1)}^2 A^{(1)}^2 \]
\[ + 12 A^{(3)} E F^{(2)} A^{(1)}^2 + 4 A^{(3)} E F^{(2)} F^{(0)}^2 - 24 A^{(3)} E B^{(1)} A^{(1)}^2 - 32 F^{(1)} A^{(3)} B^{(1)} F^{(0)} A^{(1)} \]
\[ - 60 F^{(1)} F^{(2)} B^{(1)} F^{(0)} A^{(1)} + 50 E B^{(1)} F^{(0)} A^{(1)} - 25 E F^{(2)} B^{(1)} F^{(0)}^2 - 13 A^{(3)} B^{(1)} F^{(0)}^2 \]
\[ - 46 F^{(1)} A^{(3)} E^2 A^{(1)} + 80 F^{(1)} E^2 B^{(1)} A^{(1)} - 40 F^{(1)} E^2 F^{(2)} F^{(0)} - 91 F^{(1)} E^2 F^{(2)} A^{(1)} \]
\[ + 50 F^{(1)} E^2 B^{(1)} F^{(0)} - 54 A^{(2)} F^{(1)} E F^{(2)} F^{(0)} - 84 A^{(2)} F^{(1)} E F^{(2)} A^{(1)} + 137 A^{(2)} F^{(1)} E B^{(1)} F^{(0)} \]
\[ - 24 A^{(2)} A^{(3)} B^{(1)} F^{(0)} A^{(1)} - 10 A^{(2)} A^{(3)} E F^{(1)} F^{(0)} - 24 A^{(2)} A^{(3)} E F^{(1)} A^{(1)} \]
\[ - 54 A^{(2)} F^{(2)} B^{(1)} F^{(0)} A^{(1)} + 20 F^{(1)} B^{(1)} F^{(0)} F^{(0)}^2 - 5 A^{(3)} E^3 F^{(0)} - 14 A^{(3)} E^3 A^{(1)} + 20 E^3 B^{(1)} A^{(1)} \]
\[ - 10 E^3 F^{(2)} F^{(0)} - 25 E^3 F^{(2)} A^{(1)} + 10 E^3 B^{(1)} F^{(0)} + 15 E B^{(1)}^2 F^{(0)}^2 + 66 A^{(2)} E^2 B^{(1)} A^{(1)} \]
\[ - 33 A^{(2)} E^2 F^{(2)} F^{(0)} - 63 A^{(2)} E^2 F^{(2)} A^{(1)} + 50 A^{(2)} E^2 B^{(1)} F^{(0)} + 72 A^{(2)} F^{(1)}^2 B^{(1)} F^{(0)} \]
\[ + 72 F^{(1)} E B^{(1)} A^{(1)} - 36 F^{(1)} E F^{(2)} F^{(0)} - 72 F^{(1)} E F^{(2)} A^{(1)} + 70 F^{(1)} E B^{(1)} F^{(0)} \]
\[ - 12 A^{(3)} E F^{(1)} F^{(0)} - 32 A^{(3)} E F^{(1)} A^{(1)} + 48 F^{(1)} B^{(1)} F^{(0)} A^{(1)} - 24 F^{(1)} F^{(2)} B^{(1)} F^{(0)}^2 \]
\[ - 12 F^{(1)} A^{(3)} B^{(1)} F^{(0)} F^{(0)} - 17 F^{(1)} A^{(3)} E^2 F^{(0)} - 24 A^{(2)} F^{(2)} B^{(1)} F^{(0)}^2 - 10 A^{(2)} A^{(2)} E^2 F^{(0)} \]
\[ - 24 A^{(2)} A^{(3)} E^2 A^{(1)} - 10 A^{(2)} A^{(3)} B^{(1)} F^{(0)}^2 + 108 A^{(2)} F^{(1)} E B^{(1)} A^{(1)} \]
\[ + 10 F^{(1)} E^4 + 18 A^{(2)} E^2 E^2 + 27 A^{(2)} E^3 + 10 A^{(2)} E^4 + 24 E F^{(1)} E^4 + 50 F^{(1)} E^2 \]
\[ + 35 F^{(1)} E^3 + E^5 + 93 A^{(2)} F^{(1)} E^2 + 66 A^{(2)} E F^{(1)} E^2 + 18 A^{(2)} B^{(1)} F^{(0)} + 18 A^{(2)} E F^{(1)} + 22 A^{(2)} B^{(1)} F^{(0)}^2 + 72 A^{(2)} E F^{(1)} A^{(1)} \]
\[ + 127 A^{(2)} F^{(1)} E^2 + 65 A^{(2)} F^{(1)} E^3 \]
\[ + 24 F^{(1)} E B^{(1)} F^{(0)} + 66 A^{(2)} F^{(1)} B^{(1)} F^{(0)} + 36 A^{(2)} E B^{(1)} A^{(1)} \]
\[ - 18 A^{(2)} E F^{(2)} F^{(0)} - 18 A^{(2)} E F^{(2)} A^{(1)} + 63 A^{(2)} E B^{(1)} F^{(0)} + 48 A^{(2)} B^{(1)} F^{(0)} A^{(1)} \]
\[ \left/ \left( 120 F^{(0)} \left( A^{(1)} + F^{(0)} \right) \left( 2 A^{(1)} + F^{(0)} \right) \left( 3 A^{(1)} + F^{(0)} \right) \left( 4 A^{(1)} + F^{(0)} \right) \right) \right. \]

**APPENDIX II**
The First Four Polynomials $P_n(E)$, For $k = 4$

$$P_1(E) = \frac{E}{F^{(0)}}.$$  

$$P_2(E) = -\frac{1}{2} \frac{EF^{(1)} + E^2 - B^{(1)} F^{(0)}}{F^{(0)} (A^{(1)} + F^{(0)})},$$

$$P_3(E) = \left( A^{(2)} EF^{(1)} + A^{(2)} E^2 - A^{(2)} B^{(1)} F^{(0)} + 2 EF^{(1)}^2 + 3 F^{(1)} E^2 - 2 F^{(1)} B^{(1)} F^{(0)} + E^3 + EB^{(1)} F^{(0)} - EF^{(2)} A^{(1)} - EF^{(2)} F^{(0)} + 2 EB^{(1)} A^{(1)} + B^{(2)} F^{(0)} A^{(1)} + B^{(2)} F^{(0)}^2 \right)$$

$$\left/ \left( 6 F^{(0)} \left( A^{(1)} + F^{(0)} \right) \left( 2 A^{(1)} + F^{(0)} \right) \right) \right.,$$

$$P_4(E) = - \left( 3 A^{(2)} B^{(2)} F^{(0)}^2 + A^{(3)} B^{(1)} F^{(0)}^2 + 9 A^{(2)} EF^{(1)}^2 - 3 A^{(2)} EF^{(2)} A^{(1)} + 2 A^{(2)} EB^{(1)} F^{(0)} - 9 A^{(2)} F^{(1)} B^{(1)} F^{(0)} - 2 EB^{(2)} F^{(0)}^2 + 3 A^{(2)} B^{(2)} F^{(0)} A^{(1)} + 6 A^{(2)} EB^{(1)} A^{(1)} - 3 A^{(2)} EF^{(2)} F^{(0)} - 9 F^{(1)} EF^{(2)} A^{(1)} + 4 F^{(1)} EB^{(1)} F^{(0)} - 3 B^{(1)} F^{(0)}^2 - 8 EB^{(2)} F^{(0)} A^{(1)} - 2 A^{(3)} EF^{(1)} A^{(1)} - A^{(3)} EF^{(1)} F^{(0)} + 2 A^{(3)} B^{(1)} F^{(0)} A^{(1)} + 6 F^{(2)} B^{(1)} F^{(0)} A^{(1)} + 3 EF^{(3)} A^{(1)} F^{(0)} - 6 F^{(1)} EF^{(2)} F^{(0)} + 3 A^{(3)} E^2 + 4 A^{(3)} E^3 + 6 EF^{(1)}^3 + 11 F^{(1)} E^2 + 6 F^{(1)} E^3 + 3 A^{(2)} EF^{(1)} - 3 A^{(3)}^2 B^{(1)} F^{(0)} + 13 A^{(2)} F^{(1)} E^2 - 6 F^{(1)} B^{(1)} F^{(0)} + 3 F^{(1)} B^{(2)} F^{(0)}^2 + 4 E^2 B^{(1)} F^{(0)} - 7 E^2 F^{(2)} A^{(1)} - 4 E^2 F^{(2)} F^{(0)} + 8 E^2 B^{(1)} A^{(1)} - 2 A^{(2)} E^2 A^{(1)} - A^{(2)} E^2 F^{(0)} + 3 F^{(2)} B^{(1)} F^{(0)}^2 - 6 B^{(1)} F^{(0)} A^{(1)} + 2 EF^{(3)} A^{(1)}^2 + EF^{(3)} F^{(0)}^2 - 6 EB^{(2)} A^{(1)}^2 + E^4 + 3 F^{(1)} B^{(2)} F^{(0)} A^{(1)} \right) \right/ \left( 6 F^{(0)} \left( A^{(1)} + F^{(0)} \right) \left( 2 A^{(1)} + F^{(0)} \right) \right),$$


\[ +12 F^{(1)} E B^{(1)} A^{(1)} \bigg/ \left( 24 F^{(0)} \left( A^{(1)} + F^{(0)} \right) \left( 2 A^{(1)} + F^{(0)} \right) \left( 3 A^{(1)} + F^{(0)} \right) \right) \]

ACKNOWLEDGEMENT

We wish to thank Dr. S. K. A. Seyed Yagoobi for his careful reading the article and for his constructive comments.

References

[1] A. V. Turbiner, *Commun. Math. Phys.* **118**, 467 (1988).

[2] M. A. Shifman and A. V. Turbiner, *Commun. Math. Phys.* **120**, (1989), 347.

[3] A. C. Ushveridze, *Sov. J. Part. Nucl.* **20**, 504 (1988).

[4] M. A. Shifman, *Int. J. Mod. Phys. A* **4** (1989), 2897.

[5] V. V. Ulyanov and O. B. Zalavskii, *Phys. Rep.* **216**, 179 (1992).

[6] C. M. Bender and G. V. Dunne, *J. Math. Phys.* **37** (1996), 6.

[7] M. A. Jafarizadeh and H. Fakhri, *Annals of Physics* **262** (1998), 260.

[8] M. A. Jafarizadeh and H. Fakhri, *Physics Letters A230* (1997), 157.

[9] Z. X. Wang and D. R. Guo, “Special Functions”, World Scientific Publishing Co. LTD, (1989).

[10] E. A. Coddington and N. Levinson, “Theory of Ordinary Differential Equations”, Tata McGraw-Hill Publishing Co. LTD, New Delhi, (1983).