A Provably Efficient Algorithm for Linear Markov Decision Process with Low Switching Cost

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Abstract

Many real-world applications, such as those in medical domains, recommendation systems, etc, can be formulated as large state space reinforcement learning problems with only a small budget of the number of policy changes, i.e., low switching cost. This paper focuses on the linear Markov Decision Process (MDP) recently studied in Yang and Wang [2019a], Jin et al. [2019] where the linear function approximation is used for generalization on the large state space. We present the first algorithm for linear MDP with a low switching cost. Our algorithm achieves an $\tilde{O}(\sqrt{d^3 H^4 K})$ regret bound with a near-optimal $O(d H \log K)$ global switching cost where $d$ is the feature dimension, $H$ is the planning horizon and $K$ is the number of episodes the agent plays. Our regret bound matches the best existing polynomial algorithm by Jin et al. [2019] and our switching cost is exponentially smaller than theirs. When specialized to tabular MDP, our switching cost bound improves those in Bai et al. [2019], Zhang et al. [2020b]. We complement our positive result with an $\Omega(d H / \log d)$ global switching cost lower bound for any no-regret algorithm.

1 Introduction

Reinforcement learning (RL) is often used for modeling real-world sequential decision-making problems such as medical applications [Mahmud et al., 2018, Istepanian et al., 2009], personalized recommendation [Zheng et al., 2018, Zhao et al., 2018], hardware placements [Mirhoseini et al., 2017], database optimization [Krishnan et al., 2018], etc. For these applications, oftentimes it is desirable to restrict the agent from adjusting its policy frequently. For instance, in medical domains, changing a policy requires a thorough approval process by experts [Lei et al., 2012, Almirall et al., 2013, 2014]; for large-scale software and hardware systems, changing a policy requires to change the physical environment significantly [Mirhoseini et al.]

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Formally, we would like our RL algorithm admits a low switching cost. In this setting, the agent is only allowed to switch its policy for at most $N$ times, where $N$ is much smaller than the total number of rounds played.

The problem of designing provably efficient RL algorithm with low switching cost was first studied in Bai et al. [2019] where authors proposed a Q-learning algorithm with upper confidence bound (UCB) bonus for tabular RL problems. Their algorithm achieves both a low regret and a low switching cost. More detailed discussions are included in the related work part. However, two major problems remain open in the field:

1. **Large state space**: Bai et al. [2019] studied tabular RL setting and their bounds scale polynomially with the number of states. Many aforementioned applications have a large state space. In medical domains, the state space can be all possible combinations of features that describe a patient. In a large scale system, every configuration is one state. For these problems, we need to use function approximation to generalize across states and keep the switching cost low at the same time.

2. **Global switching cost**: Bai et al. [2019] studied the local switching cost, the sum of number of changes of the policy on the state in the episode. However, arguably in most applications, we are more interested in the global switching cost, which is the number of policy changes. For example, for medical domains, the cost of changing the entire policy is similar to that of changing a component of the policy (the decision on a specific state). Furthermore, for RL problems with a large state space, it is natural to study the global switching cost because the local switching cost necessarily scales with the number of states, which is already large. On the hand, the global switching cost need not to scale with the number of states, and thus it is a more meaningful quantity to characterize.

In this paper we tackle these two problems head-on in the linear Markov Decision Process (MDP) recently studied in Yang and Wang [2019a], Jin et al. [2019], in which the linear function is used for generalization across states. Our contributions are summarized below.

- We present the first provably efficient algorithm for linear MDP with low switching cost. Our algorithm enjoys $O\left(\sqrt{dH^4K}\right)$ regret and $O\left(dH\log K\right)$ global switching cost where $d$ is the feature dimension, $H$ is the planning horizon and $K$ is the number of episodes the agent plays. The regret bound matches the best existing polynomial algorithm by Jin et al. [2019] and the switching cost is significantly lower. Furthermore, since tabular MDP is a special case of linear MDP, our result directly implies an $O\left(SAH\log K\right)$ global switching cost bound where $S$ is the number of states and $A$ is the number of actions.

- We provide an $\Omega\left(dH/\log d\right)$ global switching cost lower bound for Linear MDP. To our knowledge, no previous work provides global switching cost lower bound for MDP, let alone linear MDP. For comparison, Bai et al. [2019] derived a local switching cost lower bound but it only implies an $\Omega\left(A\right)$ global switching cost lower bound.

### 2 Related Work

Here we discuss related theoretical works. There is a long line of works studying the sample complexity of tabular reinforcement learning [Kearns and Singh, 1999, Kakade, 2003, Singh and Yed, 1994, Azar et al., 2013, Sidford et al., 2018d, Agarwal et al., 2019, Zanette et al., 2019b, Li et al., 2020, Azar et al., 2017, Dann and Brunskill, 2015, Dann et al., 2017, 2019, Jin et al., 2018, Strehl et al., 2006, Zhang et al., 2020b, Simchowitz and Jamieson, 2019, Zanette and Brunskill, 2019, Dong et al., 2019, Wang et al., 2020a, Zhang et al., 2020b].
We use $\|\cdot\|$ throughout our paper, we consider the episodic Markov decision model $P_k$ periodically. At the beginning of an episode $H$ taking place, which we use to denote the state $x$, action $A$, and universal constant factors, and $\leq 3.2$ Markov Decision Process means the matrix $B$. Bai et al. [2019] provided an $O$ better regret than the one in Jin et al. [2019], but it is not computationally efficient.\[2019\], which is a polynomial time algorithm. Recently, Zanette et al. [2020] gave an algorithm which has a polynomial time algorithm. Currently, Zanette et al. [2020] actually only requires a low inherent Bellman error condition, which is weaker than the linear MDP assumption.

However, for real-world problems, the state space is often large, so we need to use function approximation. Developing provably efficient algorithms for large state space RL problems is a hot topic recently. Wen and Roy, [2013], Li et al. [2011], Du et al. [2019a, 202016, Jin et al. [2018]. The state-of-the-art analysis shows that one can obtain $O(\sqrt{HSAK})$ regret\[1\] and this is tight. Dann and Brunskill, 2015, Osband and Roy, 2016, Jin et al., 2018.

For a matrix $A$, we let $\det(A)$ denote its determinant. For two symmetric matrices, $A$ and $B$, $A \preceq B$ means the matrix $B - A$ is positive semidefinite. We use the standard $O(\cdot)$ and $\Omega(\cdot)$ notations to hide universal constant factors, and $\tilde{O}$, and $\tilde{\Omega}$ notations to hide logarithmic factors.

3 Preliminaries

3.1 Notations

We use $\|\cdot\|$ to denote the standard Euclidean norm. Given a positive integer $N$, we let $[N] = \{1, 2, \ldots, N\}$. For a matrix $A$, we use $\det(A)$ to denote its determinant. For two symmetric matrices, $A$ and $B$, $A \preceq B$ means the matrix $B - A$ is positive semidefinite. We use the standard $O(\cdot)$ and $\Omega(\cdot)$ notations to hide universal constant factors, and $\tilde{O}$, and $\tilde{\Omega}$ notations to hide logarithmic factors.

3.2 Markov Decision Process

Throughout our paper, we consider the episodic Markov decision model $(S, A, H, P, r)$. In this model, $S$ and $A$ denote the set of states and actions, respectively. All the episodes have the same number of transitions taking place, which we use $H \in N$ to denote. $P = \{P_h\}$ is the set of transition probability measures. Hence $P_h(x'|x, a)$ means the transition probability of taking action $a$ at step $h \in [H] = \{1, 2, \ldots, H\}$ on state $x$ to the state $x'$. $r$ is a collection of reward functions $r_h : S \times A \rightarrow [0, 1]$ for each step in an episode.

The dynamics of the episodic MDP can be view as the interaction of an agent with the environment periodically. At the beginning of an episode $k$, an arbitrary state $x_k^1 \in S$ is selected by the environment, and

\[\text{This regret bound applies to the setting where the transition probabilities can be different at each level and the reward at each level is bounded by 1.}\"
the agent is then in step 1. At each step $h$ in this episode, based on the current state $x_h \in S$ and the history information, the agent needs to decide which action to take. After action $a_h \in A$ is chosen, the environment will give the reward for the step $r_h(x_h, a_h)$ and move the agent to the next state $x_{h+1} \in S$. The episode automatically ends when the agent reaches the step $H + 1$. In other words, the agent will take at most $H$ actions and receive corresponding rewards in each episode.

To clarify the choice of actions for the agent in the episode, we define the policy function $\pi : S \times \mathcal{H} \to \mathcal{A}$. Namely, $\pi(x, h)$ is the action taken on state $x$ at step $h$ by the agent. We use $Q$-function to evaluate the long-term value for the action $a$ and subsequent decisions. The $Q$-function is defined as follows:

$$Q^\pi_h(x, a) := r_h(x, a) + \mathbb{E} \left[ \sum_{i=h+1}^H r_i(x_i, \pi(x_i, i)) \mid x_h = x, a_h = a \right].$$

In addition, we define the value function $V^\pi_h : S \to \mathbb{R}$ for the policy $\pi$ via the following formula:

$$V^\pi_h(x) := \mathbb{E} \left[ \sum_{i=h}^H r_i(x_i, \pi(x_i, i)) \mid x_h = x \right].$$

The $Q$-function and $V$-function obey the following Bellman equation: for any policy $\pi$,

$$Q^\pi_h(x, a) = (r_h + \mathbb{P}_h V^\pi_{h+1})(x, a), \quad V^\pi_h(x) = Q^\pi_h(x, \pi_h(x)), \quad \text{and} \quad V^\pi_{H+1}(x) = 0,$$

where

$$[\mathbb{P}_h V_{h+1}](x, a) := \mathbb{E}_{x' \sim \mathbb{P}_h(\cdot|x, a)} V_{h+1}(x').$$

We denote $V^*_h(x) = \sup_\pi V^\pi_h(x)$, $Q^*_h(x, a) = \sup_\pi Q^\pi_h(x, \pi(x))$ as the optimal value and $Q$-functions. The Bellman equation also holds for $V^*_h$ and $Q^*_h$ with respect to the optimal policy $\pi^*$.

Suppose an agent is allowed to interact with the MDP for $K$ episodes and plays policy $\pi_k$ at episode $k \in [K]$. We use regret to measure the performance of its algorithm, which is the difference of the value of the optimal policy and the policy adopted by the agent.

$$\text{Regret}(K) = \sum_{k=1}^K \left[ V^*_1(x^k_1) - V^*_1(x^k_1) \right]$$

### 3.3 Linear Markov Decision Process

The focus of our study is the linear MDP model [Yang and Wang, 2019a, Jin et al., 2019]. Linearity here represents that the transition probability and the reward are linear functions given the feature map. Formally, there exists a map from the state-action space to the feature space, namely $\phi : S \times A \to \mathbb{R}^d$ and a measure $\mu_h$ for $h \in [H]$ such that $\forall (x, a) \in S \times A$

$$\mathbb{P}_h(\cdot|x, a) = \langle \phi(x, a), \mu_h(\cdot) \rangle, \quad \text{and} \quad r_h(x, a) = \langle \phi(x, a), \theta_h \rangle.$$

We further assume that $\|\phi(x, a)\| \leq 1, \forall x \in S$, $\|\mu_h(x)\| \leq \sqrt{d}$, and $\|\theta_h\| \leq \sqrt{d}$. Linear MDP model is a strict generalization of the standard tabular RL model with $d = |S||A|$. For each $(s, a) \in S \times A$, we can let $\phi(s, a) = e_{(s, a)}$ be the canonical basis in $\mathbb{R}^d$. Then we can just define $\langle e_{(s, a)}, \mu_h(\cdot) \rangle = \mathbb{P}_h(\cdot | s, a)$ and $\langle e_{(s, a)}, \theta_h \rangle = r_h(s, a)$. See [Yang and Wang, 2019a, Jin et al. 2019] for more examples.
3.4 Switching Cost

The concept of switching cost is used to quantify the adaptability of reinforcement learning algorithms. The main focus of our work is the global switching cost, which counts the number of policy changes in the running of the algorithm in $K$ episodes, namely:

$$N_{\text{switch}}^{\text{gl}} \triangleq \sum_{k=1}^{K-1} \mathbb{I}\{\pi_k \neq \pi_{k+1}\}$$ (4)

The focus of Bai et al. [2019] is local switching cost:

$$N_{\text{switch}}^{\text{loc}} \triangleq \sum_{k=1}^{K-1} \left| \{ (h, x) \in [H] \times S : \pi_k^h(x) \neq \pi_{k+1}^h(x) \} \right|$$ (5)

Technically, we always have

$$N_{\text{switch}}^{\text{gl}} \leq N_{\text{switch}}^{\text{loc}} \leq |S| H N_{\text{switch}}^{\text{gl}}.$$

One crucial reason to use the global switching cost is that the definition of local switching cost is based on the number of states, which can be infinite in the linear MDP model, so the global switching cost is more meaningful quantity to study. Lastly, we emphasize that in the study of the switching cost, we only consider deterministic policies. Note that, by the Bellman optimality equation, there exist at least one optimal policy that is deterministic.

4 Algorithm and Result

In this section, we describe our main positive result. We first describe our approach, which is listed in Algorithm 1. Our algorithm has two crucial components: a $Q$-function estimation step and a policy update step. This estimation step largely follows the method in Jin et al. [2019]. To achieve low-switching cost, the planning step is novel. We use the determinant of the feature covariance matrix to guard the change of the policies.

More formally, in line 5- line 7, we use UCB to obtain an optimistic estimate of the optimal $Q$-function. In Line 5 we define $\Lambda_k^h$ to be the empirical covariance matrix based on all features collected at level $h$, in which a small regularization term $\lambda I$ is added to avoid degeneracy. In Line 6 we use least-square to estimate the linear coefficient, where we construct labels as $r_h(x, a, \tau) + \max_a Q_{k+1}^h(x, \tau, a)$, following the Bellman Equation. In Line 7 we define our optimistic estimate of $Q$-function as the summation of a linear function $(w_k^h)^\top \phi(\cdot, \cdot)$ and a bonus term $\beta [\phi(\cdot, \cdot)^\top (\Lambda_k^h)^{-1} \phi(\cdot, \cdot)]^{1/2}$. The bonus term ensures our estimate is optimistic (cf. Equation(5)). We also clips the value to $H$ if it is too large. We refer readers to Jin et al. [2019] for more intuitions about this estimation procedure.

For policy update, the policy at the $k$-th episode in Jin et al. [2019] is just to choose the action that maximizes the optimistic estimate $\tilde{Q}^k_h$. Since $\tilde{Q}^k_h$ is changing at every episode, the policy changes at every episode as well. Therefore, the switching cost can be linear in the number of episode. Our main technique to reduce the switching cost is a new criteria to decide whether to update the policy. More specifically, note when executing the policy, we always choose the action according to $Q^k_h$ (cf. Equation (16)) and $Q^k_h$ is updated according to $\tilde{Q}^k_h$ in line 11 where $\tilde{Q}^k_h$ is a reference $Q$-function estimate which changes infrequently. Note that, as we will show shortly, $Q^k_h$ does not change frequently. We use $\hat{k}$ as a reference counter, which is updated only in line 9 when the criteria in line 8 is met.
Algorithm 1 Algorithm for Linear MDP with Low Global Switching Cost

1: **Input:** regularization parameter $\lambda > 0$.
2: Set $\hat{k} \leftarrow 1$.
3: for episode $k = 1, 2, \ldots, K$ do
   Q-function Estimation
   4: for step $h = H, \ldots, 1$ do
   5: $\Lambda^k_h \leftarrow \sum_{\tau=1}^{k-1} \Phi (x^\tau_h, a^\tau_h) \phi (x^\tau_h, a^\tau_h) + \lambda \cdot I$
   6: $w^k_h \leftarrow (\Lambda^k_h)^{-1} \sum_{\tau=1}^{k-1} \phi (x^\tau_h, a^\tau_h) \left[ r_h (x^\tau_h, a^\tau_h) + \max_a \tilde{Q}^k_{h+1} (x^\tau_{h+1}, a) \right]$
   7: $\tilde{Q}^k_h (\cdot, \cdot) \leftarrow \min \left\{ (w^k_h)^\top \phi (\cdot, \cdot) + \beta \left[ \phi (\cdot, \cdot) \right]^\top (\Lambda^k_h)^{-1} \phi (\cdot, \cdot) \right\}^{1/2}, H \}$
   Policy Update
   8: if $(\Lambda^k_h)^{-1} \neq 2(\Lambda^k_h)^{-1}$ then
   9: Set $\hat{k} \leftarrow k$
   10: end if
   11: Set $Q^k_h \leftarrow \tilde{Q}^k_h$
   end for
   Policy Execution
12: end for
13: Receive the initial state $x^1_k$.
14: for step $h = 1, 2, \ldots, H$ do
15: Take action $a^k_h \leftarrow \arg \max_a Q^k_h (x^k_h, a)$.
16: Observe $x^k_{h+1}$.
17: end for

Now we explain our proposed criteria. At a high-level, since the empirical co-variance matrix $\Lambda^k_h$ determines both our estimate of $Q$-function and the bonus, if it changes a significant amount, this means we already learned new information and we need to change the policy to achieve low regret. Note this step is computationally efficient because we just need to check the least eigenvalue of $2(\Lambda^k_h)^{-1} - (\Lambda^k_h)^{-1}$ is non-negative or not. Geometrically, $(\Lambda^k_h)^{-1} \neq 2(\Lambda^k_h)^{-1}$ represents that, at the $k$th episode, there exists one direction at which we have learned twice information as the information we learned at the reference episode $\hat{k}$. We will explain more technical reasons in the next section.

We now state the main theorem on the regret bound and the switching cost bound of our algorithm.

**Theorem 1** (Regret and Switching Cost of Algorithm 1 for Linear MDP). *In the linear MDP setting, there exists a constant $c > 0$ such that, for any $p \in (0, 1)$, if we set $\lambda = 1$ and $\beta = cdH \sqrt{v} \sqrt{K}$ with $v = \log(2dKH/p)$ in Algorithm 1 then with probability $1 - p$, the total regret is at most $O(2^dH^4K \log K)$. Furthermore, the global switching cost of the algorithm is bounded by $O(dH \log K)$.*

Theorem 1 suggests our algorithm achieves the desired regret and switching cost guarantees. In terms of the regret, our bound matches the one in Jin et al. [2019], but our algorithm has significantly lower switching cost ($O(dH \log K)$ vs. $K$). Recently, Zanette et al. [2020] gave an $\tilde{O} \left( \sqrt{d^2H^4K} \right)$ regret bound but their algorithm is not computationally efficient. An interesting open problem is to design an algorithm which enjoys a regret bound of $\tilde{O} \left( \sqrt{d^2H^4K} \right)$ and a switching cost bound like ours. As will be seen in Section 5, our switching cost bound is near-optimal up to logarithmic factors. We provide a proof sketch in Section 5 and defer the full proof to appendix.
Recall tabular MDP is special case of linear MDP. Using the observation that in the tabular setting, whenever Algorithm 1 changes the policy, it only change one state-action pair, we obtain the following result for the local switching cost.

**Corollary 1** (Regret and Switching Cost of Algorithm 1 for Tabular MDP). *In the tabular setting, there exists a constant \( c > 0 \) such that, for any \( p \in (0, 1) \), if we set \( \lambda = 1 \) and \( \beta = c d H \sqrt{\tau} \) with \( \tau = \log(2 S A K) / p \) in Algorithm 1, then with probability \( 1 - p \), the total regret is at most \( O(\sqrt{S^3 A^3 H^4 K^2}) \).

Furthermore, the local switching cost of the algorithm is bounded by \( O(S A H \log K) \).

We present our corollary in terms of the local switching cost in order to have a fair comparison with the results in Bai et al. [2019], Zhang et al. [2020b]. Recall the local switching cost is always an upper bound of the global switching cost, so our bound also holds for the global switching cost. The best existing result is by Zhang et al. [2020b] who designed an algorithm with \( O(\sqrt{S A H K}) \) regret and \( O(S A H K \log K) \) switching cost. Comparing with Zhang et al. [2020b], our regret bound is larger but our switching cost is lower than theirs.

### 5 Proof Sketch of Theorem 1

The proof consists of two parts: bounding the regret and bounding the global switching cost. Note minimizing the regret and the switching cost are conflict to each other because a small switching cost requires us *not* to use the most updated information which can incur higher regret. The main technical novelty is that our criteria can achieve the same order regret as the one in Jin et al. [2019] and at the same time reduce the switching cost significantly.

#### 5.1 Regret Analysis

Due to the delayed policy update, the establishment of the bound for regret may be more difficult than the previous algorithm in Jin et al. [2019], yet the steps are very similar.

We start our proof by decomposing the regret into the error induced by the estimation error from the delayed policy update. To simplify the notation, for any \( k \in [K] \), we let \( \tilde{k} \leq k \) represents the episode index we update the policy to the one used in the \( k \)-th episode. We have the following decomposition.

\[
\text{Regret}(K) \leq \sum_{k=1}^{K} \left[ \tilde{V}_{1}^{k}(x_{1}^{k}) - \tilde{V}_{1}^{\tilde{k}}(x_{1}^{k}) \right]
\]

By definition, this term represents the error from the estimation in episode \( \tilde{k} \) in state \( x_{1}^{k} \).

**Analysis of Error due to the Delayed Policy Update** First, as will be seen in the appendix, we can obtain a recursive formula such that it is sufficient to bound the term \( (\tilde{Q}_{h}^{k} - Q_{h}^{\tilde{k}})(x_{h}^{k}, a_{h}^{k}) \). With some error analysis, we can bound it by

\[
(\tilde{Q}_{h}^{k} - Q_{h}^{\tilde{k}})(x_{h}^{k}, a_{h}^{k}) \leq \Delta_{h}^{k}(x, a) + \tilde{\delta}_{h}^{k}
\]

Here \( \tilde{\delta}_{h}^{k} \) is a zero-mean martingale difference sequence, so we can use standard concentration inequalities to bound it. \( \Delta_{h}^{k}(x, a) \) represents the bonus term that satisfies

\[
|\Delta_{h}^{k}(x, a)| \leq \beta \sqrt{\phi(x, a)^{\top} (A_{h}^{k})^{-1} \phi(x, a)}.
\]
By our policy update criteria, we have the following simple yet crucial property:

\[ \phi^\top (\Lambda^{\hat{k}}_h)^{-1} \phi \leq 2\phi^\top (\Lambda^k_h)^{-1} \phi. \]

Therefore, although the error due to the delayed update has additional terms, these terms are at most of the same order as the error occurring during the estimation phase. Using this observation, we can essentially reuse the proof for bounding the error due to the estimation here.

### 5.2 Switching Cost Analysis

The analysis of switching cost is trickier. We employ a potential based analysis. The potential function is the logarithm of the determinant of the empirical covariance matrix. The following lemma shows it is upper bounded by \( O(d \log K) \).

**Lemma 1.** Let \( \phi_r \) are \( d \)-dimensional vectors satisfying \( \|\phi_r\| \leq 1 \). Let \( A = \sum_{\tau=1}^K \phi_r (\phi_r)^\top + \lambda \cdot I \). Then we have

\[ \log \det A = O(d \log K). \]

Now we consider our update rule. Recall we update our policy only if \( (\Lambda^k_h)^{-1} \not\preceq 2(\Lambda^{\hat{k}}_h)^{-1} \). The following lemma shows whenever this condition holds, the potential function must increase by a constant.

**Lemma 2.** Assume \( m \leq n \), \( A = \sum_{\tau=1}^m \phi_r (\phi_r)^\top + \lambda \cdot I \), \( B = \sum_{\tau=1}^n \phi_r (\phi_r)^\top + \lambda \cdot I \). Then if \( A^{-1} \not\preceq 2B^{-1} \), we have

\[ \log \det B \geq \log \det A + \log 2. \]

To bound the switching cost, we note the potential is upper bounded by \( O(d \log K) \) and every time we update the policy, the potential must increase by \( \log 2 \), so in total we at most update the policy \( O(d \log K) \) times. We believe our proof strategy may be useful in other problems as well.

### 6 Lower Bound

To complement our upper bound on linear MDP, we present the following lower bound on the global switching cost.

**Theorem 2.** For \( d \geq 100 \), let \( M \) be the class of linear MDPs defined in Section 3.3. For any algorithm that uses a deterministic policy at each episode, if its global switching cost \( N_{\text{switch}}^{\text{gl}} \leq \frac{dH}{100 \log d} \), we have

\[ \sup_{M \in \mathcal{M}} \mathbb{E}_{s_1} \left[ \sum_{k=1}^K V^*_1(s_1) - V^*_1(s_1) \right] \geq KH/4. \]

Theorem 2 states that for any algorithm that achieves sub-linear regret, it must have at least \( \Omega(dH/ \log d) \) global switching. This shows our upper bound on the global switching cost cannot be improved up to logarithmic factors. One interesting open problem is to further close this gap. We remark that Bai et al. [2019] derived an \( \Omega(S.A.H) \) local switching cost, which only implies an \( \Omega(A) \) global switching cost. The simple multi-armed bandit problem also has an \( \Omega(A) \) global switching cost lower bound. Theorem 2 is, to our knowledge, the first non-trivial global switching cost lower bound in RL.
6.1 Proof Sketch of Theorem \cite{2}

The full proof is deferred to the appendix, and here we give an outline of the proof. The strategy is to construct a class of hard MDPs and show for any algorithm without any prior knowledge about this class, it must suffer enough regret and switching cost. The difficult part is how to construct hard instances.

We consider environments similar to combination lock \cite{Kakade2003}. Figure 1 shows a simplified version of our constructed environment. The agent starts at state \( u \), and \( v, w \) are other two states. The reward at \( v \) is always 1 while the reward at other states is 0. In order to go to state \( v \), the agent needs to select a sequence of correct actions. In each episode, the agent stays at \( u \) if the previous action is correct and even if only one action is incorrect, the agent will go to state \( w \) and stay there till the episode ends.

We further construct states and features to encode this problem as a linear MDP. To ensure the transition and the reward is linear we also need to adjust states and action carefully for which some auxiliary states are needed as well. The \( \log d \) factor in the denominator comes from our modifications of the environment described above.

7 Conclusion and Future Works

In the view of the switching cost, we study the reinforcement learning algorithms in the linear Markov decision process setting. Based on the current polynomial switching-cost algorithm with small regret bound, we design a new algorithm that matches its regret bound, with significantly lower global switching cost. The regret bound of our algorithm is regret \( \tilde{O}(\sqrt{d^3H^4K}) \), with the global switching cost being \( O(dH \log K) \). This bound also implies a switching cost improvement over existing results of tabular MDP. Furthermore, by constructing a series of hard MDP instances, we are able to prove the lower bound for the switching cost is \( \Omega(dH/\log d) \) provided that the deterministic algorithm has a sub-linear regret. We now list some future directions.

Towards Optimal Switching Cost Bound Currently, there is a \( \log K \log d \) factor gap between our upper bound and lower bound. In particular, we believe the upper bound can be further improved to \( \log \log K \), as in the bandit setting, this is achievable \cite{Cesa-Bianchi2013}. For the lower bound, we believe the \( \log d \) factor is removable though we found this is a technically challenging problem. We believe obtaining the optimal switching cost bound will greatly broaden our understanding on this problem.

Optimal Regret Bound with Switching Cost \cite{Zanette2020} recently showed for linear MDP, it is possible to obtain an \( \tilde{O}(\sqrt{d^3H^3K}) \), which is optimal up to logarithmic factors. Their algorithm is substantially different from the one by \cite{Jin2019} and it is not computationally efficient. It is possible
to combine their analysis and ours to obtain an algorithm which has near-optimal regret and at the same time, has low switching cost. A more interesting problem is to make this algorithm computationally efficient.

**Low Switching Cost Algorithm for RL with General Function Approximation** Recently, there are many works trying to design provably efficient algorithms with general function approximation, beyond the linear function approximation scheme. These works are based on different assumptions [Wen and Roy, 2013, Jiang et al., 2017, Sun et al., 2018, Wang et al., 2020b, Ayoub et al., 2020]. It would be interesting to extend our analysis to these settings.

This work does not present any foreseeable negative societal consequence. From the positive side, the algorithm proposed in this paper can be potentially applied in medical domain, and hence benefit the society.

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8 Appendix

9 Upper Bound Proof

9.1 Basic properties of the LSVI algorithm

In this part, we list some of the important lemmas for the LSVI algorithm, most of which are proven in the previous literature [Jin et al. 2019]. These lemmas are very useful for proving the regret bound in our main theorem.

Lemma 3. (Lemma B.3, Jin et al. [2019]). Define $$P_h V_{h+1}(x,a) := \mathbb{E}_{x' \sim P_h(\cdot|x,a)} V_{h+1}(x')$$. Under the setting of Theorem 1, let $$c_\beta$$ be the constant in our definition of $$\beta$$ (i.e., $$\beta = c_\beta \cdot d H^\sqrt{\kappa}$$). There exists an absolute constant $$C$$ that is independent of $$c_\beta$$ such that for any fixed $$p \in [0,1]$$ if we let $$E$$ be the event that:

$$\forall (k,h) \in [K] \times [H]:$$

$$\left| \sum_{\tau=1}^{k-1} \phi^\tau_h \left( V_{h+1}^k(x_{h+1}^\tau) - P_h V_{h+1}^k(x_{h+1}^\tau, a_{h+1}^\tau) \right) \right| \leq C \cdot d H^\sqrt{\chi}$$

where $$\chi = \log \left( \frac{2(c_\beta+1) d T}{p} \right)$$, then $$\mathbb{P}(E) \geq 1 - p/2$$.

Remark. We use $$V$$ to denote the set of all the value functions in the form of

$$\tilde{V}(\cdot) = \min \left\{ \max_a \phi(\cdot,a) + \beta \sqrt{\phi(\cdot,a)^\top \Lambda^{-1} \phi(\cdot,a)}, H \right\}.$$ 

Clearly $$V$$ includes all possible value function we generate throughout the algorithm. We can construct a $$\varepsilon$$-covering of $$V$$ with respect to the distance $$\text{dist}(V, V') = \sup_x |V(x) - V'(x)|$$. In addition, we can prove that $$N_\varepsilon$$, the $$\varepsilon$$-covering number of $$V$$, can be bounded. Thus, by combining decomposition inequality we can derive this lemma.

Lemma 4. (Lemma B.4, Jin et al. [2019]) There exists an absolute constant $$c_\beta$$ such that for $$\beta = c_\beta \cdot d H^\sqrt{\chi}$$ where $$\chi = \log \left( 2(c_\beta+1) d T/p \right)$$, and for any fixed policy $$\pi$$, on the event $$E$$ defined in Lemma 3 we have for all $$(x,a,h,k) \in S \times A \times [H] \times [K]$$ that:

$$\left\langle \phi(x,a), w_h^k \right\rangle - Q_h^k(x,a) = \mathbb{P}_h \left( \tilde{V}_{h+1}^k - V_{h+1}^k \right)(x,a) + \Delta_h^k(x,a)$$

for some $$\Delta_h^k(x,a)$$ that satisfies $$|\Delta_h^k(x,a)| \leq \beta \sqrt{\phi(x,a)^\top \Lambda_h^{-1} \phi(x,a)}$$

Remark. It is noted that although the definition of $$\tilde{\phi}$$ is not the same as it in [Jin et al. 2019], the proof of Lemma 4 still holds. Then the following two lemmas can be easily derived by Lemma 4 and induction.

Lemma 5. (Lemma B.5 (UCB), Jin et al. [2019]). On the event $$E$$ defined in Lemma 3 we have

$$\tilde{Q}_h^k(x,a) \geq Q_h^*(x,a)$$

for all $$(x,a,h,k) \in S \times A \times [H] \times [K]$$
For any \((h, k) \in [H] \times [k]\), let \(\tilde{\delta}_h^k = \tilde{V}_h^k(x_h^k) - V_{\pi_h^k}^k(x_h^k)\) denote the errors of the estimated \(\tilde{V}_h^k\) relative to \(V_{\pi_h^k}^k\).

**Lemma 6.** (Lemma B.6 (Recursive Lemma), Jin et al. [2019]) Let \(\tilde{\zeta}_{h+1}^k = \mathbb{E} \left[ \delta_{h+1}^k | x_h^k, a_h^k \right] - \tilde{\delta}_{h+1}^k.\) Then on the event \(\mathcal{E}\) defined in Lemma 4, we have the following: for any \((h, k) \in [K] \times [H]\)

\[
\tilde{\delta}_h^k \leq \tilde{\delta}_{h+1}^k + \tilde{\zeta}_{h+1}^k + 2\beta \sqrt{\phi_h^k {}^\top (\Lambda_h^k)^{-1} \phi_h^k}
\]

**Lemma 7.** (Lemma D.2, Jin et al. [2019]) Let \(\{\phi_t\}_{t \geq 0}\) be a bounded sequence in \(\mathbb{R}^d\) satisfying \(\sup_{t > 0} \|\phi_t\| \leq 1.\) Let \(\Lambda_0 \in \mathbb{R}^{d \times d}\) be a positive definite matrix. For any \(t \geq 0\), we define \(\Lambda_t = \Lambda_0 + \sum_{j=1}^t \phi_j {}^\top \phi_j.\) Then, if the smallest eigenvalue of \(\Lambda_0\) satisfies \(\lambda_{\min}(\Lambda_0) \geq 1\), we have

\[
\log \left[ \frac{\det (\Lambda_t)}{\det (\Lambda_0)} \right] \leq \sum_{j=1}^t \phi_j {}^\top \Lambda_{j-1}^{-1} \phi_j \leq 2 \log \left[ \frac{\det (\Lambda_t)}{\det (\Lambda_0)} \right]
\]

### 9.2 Decomposing and proving the regret bound

In this section, we show the decomposition of the regret bound via the following lemma:

**Lemma 8.** Let \(\tilde{\delta}_h^k\) be defined the same as that in the start of this part, then the following bound for the regret holds:

\[
\text{Regret}(K) \leq \sum_{k=1}^K \tilde{\delta}_h^k
\]

The proof of this lemma is straightforward: if we notice that the value function computed by the algorithm always estimates more than the true value, so the following equation holds:

\[
\text{Regret}(K) = \sum_{k=1}^K \left[ V_1^*_k(x_1^k) - V_{\pi_h^k}^k(x_1^k) \right] \\
\leq \sum_{k=1}^K \left[ \tilde{V}_h^k(x_1^k) - V_{\pi_h^k}^k(x_1^k) \right]
\]

### 9.3 Proof of the main theorem: the regret bound

In this part, we will prove the regret bound of the main theorem stated in the section 4.

Firstly, conditioning on the event \(\mathcal{E}\) defined in Lemma 4, we have:

\[
[\tilde{Q}_h^k - Q_{\pi_h^k}^k] (x, a) = \mathbb{P}_h \left( V_{h+1}^k - V_{h+1}^{\pi_h^k} \right) (x, a) + \Delta_h^k(x, a)
\]

Our update rule implies \(\pi_k = \pi_h^k, \tilde{Q}_h^k = Q_h^k\), so subtracting the previous equations, we have

\[
\left( \tilde{Q}_h^k - Q_{\pi_h^k}^k \right) (x_h^k, a_h^k) \leq \Delta_h^k(x, a) + \tilde{\delta}_h^k \leq \Delta_h^k(x, a) + \tilde{\zeta}_h + \tilde{\delta}_{h+1}
\]

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Since \( \{ \zeta_{kh} \}_{k \in [K]} \) are bounded martingale difference sequence (adapted to the history up to episode \( k-1 \)), by Azuma-Hoeffding inequality, we have, with probability at least \( 1 - p/2 \)

\[
\sum_{k=1}^{K} \sum_{h=1}^{H} \tilde{\zeta}_{kh} \leq \sqrt{HT}. \]

Let this event be \( \mathcal{E}' \).

Notice our update rule implies \( \phi_{kh} \mathcal{A}_{\tilde{\zeta}_{kh}}^{-1} \phi_{kh} \leq 2 \phi_{kh} \mathcal{A}_{\tilde{\zeta}_{kh}}^{-1} \phi_{kh} \), now we can use Lemma 3, 6, 8 to do the recursion for the regret bound. In summary, conditioning on \( \mathcal{E} \) and \( \mathcal{E}' \), the following inequalities hold:

\[
\text{Regret}(K) \leq \sum_{k=1}^{K} \tilde{\zeta}_{kh} + 3 \beta \sum_{k=1}^{K} \sum_{h=1}^{H} \sqrt{\phi_{kh} \mathcal{A}_{\tilde{\zeta}_{kh}}^{-1} \phi_{kh}} \leq \sqrt{HT} + 6H \sqrt{dK}. \]

The second part holds from the potential lemma and the Cauchy-Schwarz inequality.

### 9.4 Analysis for switching cost

**Proof of Lemma 1.** Assume \( \phi_{d} = (\alpha_1, \alpha_2, \cdots, \alpha_d)^T \). We know that \( \sum_{i=1}^{d} \alpha_i^2 \leq 1 \), thus \( |\alpha_i| \leq 1 \) for all \( i \in [d] \). Since \( \phi_{d} (\phi_{d}^T) = (\alpha_i \alpha_j)_{ij} \), the absolute value of each component of \( \phi_{d} (\phi_{d}^T) \) is no more than 1, and then the absolute value of each component of \( A_d \) is no more than \( K + \lambda \).

Use \( \bar{A}_d = (a_{ij}) \) to denote a \( d \)-dimensional matrix satisfying the feature above, i.e., \( |a_{ij}| \leq K + \lambda \). Clearly, if the 1st row and 1st column are deleted, the rest \( (d-1) \)-dimensional matrix is \( \bar{A}_{d-1} \). Thus

\[
|\det(A_d)| \leq \sum_{j=1}^{d} |a_{1j}| \cdot |\det(A_{d-1})| \leq d \cdot (K + \lambda) \cdot |\det(A_{d-1})| \leq d^d (K + \lambda)^d
\]

Hence, \( \log \det A_d = d \log d + d \log (K + \lambda) = O(d \log K) \). \( \square \)

To prove Lemma 2 we will use the following two linear algebraic facts.

**Fact 1 (Woodbury matrix identity).** For any PSD matrices \( A, \Delta \in \mathbb{R}^{d \times d} \), suppose \( A \) is invertible, then we have

\[
(A + \Delta)^{-1} = A^{-1} - A^{-1} \Delta^{1/2} (I + \Delta^{1/2} A^{-1} \Delta^{1/2})^{-1} \Delta^{1/2} A^{-1}.
\]

**Fact 2 (Matrix determinant lemma).** For any PSD matrices \( A, \Delta \in \mathbb{R}^{d \times d} \), suppose \( A \) is invertible, then

\[
\det(A + \Delta) = \det(I + \Delta^{1/2} A^{-1} \Delta^{1/2}) \cdot \det(A).
\]
By the matrix determinant lemma, we only need to show that
\[ \lambda_{\text{max}}(I + \Delta^{1/2}A^{-1}\Delta^{1/2}) \geq 2. \]
Since \( A^{-1} \neq 2B^{-1} \), it must be the case that, for some \( x \) with \( \|x\|_2 = 1 \), and
\[ x^\top (A^{-1} - 2B^{-1})x \geq 0. \]
Denote \( \Delta = B - A \geq 0 \). By Woodbury identity, we have
\[ x^\top (A^{-1} - 2B^{-1})x = x^\top (2A^{-1}\Delta^{1/2}(I + \Delta^{1/2}A^{-1}\Delta^{1/2})^{-1}\Delta^{1/2}A^{-1} - A^{-1})x \]
\[ \geq 0. \]
Let \( y = A^{-1/2}x \), we then have,
\[ 2y^\top A^{-1/2}\Delta^{1/2}(I + \Delta^{1/2}A^{-1}\Delta^{1/2})^{-1}\Delta^{1/2}A^{-1/2}y \]
\[ \geq \|y\|_2^2. \]
Hence,
\[ \lambda_{\text{max}}(A^{-1/2}\Delta^{1/2}(I + \Delta^{1/2}A^{-1}\Delta^{1/2})^{-1}\Delta^{1/2}A^{-1/2}) \geq 1/2. \]
Let us denote \( M = A^{-1/2}\Delta^{1/2} \), we have
\[ \lambda_{\text{max}}(M(I + M^\top M)^{-1}M^\top) \geq 1/2. \]
Let \( M = U\Sigma V^\top \) be the SVD decomposition of \( M \), where \( U \) and \( V \) are orthonormal and \( \Sigma \) is diagonal. Then we have,
\[ M(I + M^\top M)^{-1}M^\top = U\Sigma V^\top(I + V\Sigma^2 V^\top)^{-1}V\Sigma U^\top \]
\[ = U\Sigma(I + \Sigma^2)^{-1}\Sigma U^\top. \]
Note that \( \Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_d) \) is diagonal, we have
\[ \max_i \frac{\sigma_i^2}{1 + \sigma_i^2} \geq 1/2 \Rightarrow \max_i \sigma_i^2 \geq 1. \]
We additionally rewrite \( I + \Delta^{1/2}A^{-1}\Delta^{1/2} \) as
\[ I + \Delta^{1/2}A^{-1}\Delta^{1/2} = I + M^\top M = I + V\Sigma^2 V^\top. \]
Thus we have
\[ \lambda_{\text{max}}(I + \Delta^{1/2}A^{-1}\Delta^{1/2}) \geq 2 \]
as desired.

Combining the lemmas above, we are now ready to prove the switching cost bound in the theorem. \( \square \)

Proof of Theorem Let \( \{k_1, k_2, \ldots, k_{N_{\text{switch}}^{\text{gl}}} \} \) denote the \( \tilde{k} \) picked by the algorithm. From Lemma 2 we know \( \det \Lambda_{k_{i+1}} \geq 2 \det \Lambda_{k_i} \geq 2^{k_{i+1}} \det \Lambda_0 \). Hence, by combining Lemma 1 we have \( N_{\text{switch}}^{\text{gl}} \leq c_0 \log \det \Lambda_K = O(d \log K) \). \( \square \)
10 Analysis for Lower Bound

We will construct a set of linear MDPs with dimension $d = 4d_0$ and number of steps $H = 2H_0$ to prove theorem 2. These MDPs will have different action when facing different states, and we claim that an MDP with the number of action $|A|$ and the number of steps $H$ can be transformed to an MDP with the number of action 2 and the number of steps $H \log |A|$.

In fact, we can just add a binary tree with depth $\log |A|$ before each transformation, and each intermediate state has two possible action. Then the different leaves of the tree denote for the different chosen of action. Clearly the switching cost will not decrease during this operation.

10.1 Construction

For fixed $d_0$, let $e_i$ denote the vector $(0, 0, \cdots, 1, \cdots, 0)$ in $4d_0$-dimensional space, whose $i_\text{th}$ component is 1 and others are 0s. We first construct a set of linear MDPs $M_\star$ with dimension $4d_0$ as follows:

The state space $S$ is partitioned into 4 components, which we define as follows:

$$S_0 = \{ s_{h,i} | h \in [H], i \in [d_0] \}$$
$$U = \{ u \}$$
$$V = \{ v \}$$
$$W = \{ w \}$$

So we have $S = S_0 \cup U \cup V \cup W$. Intuitively, $S_0$ is the space where the agent usually explores. $u, v, w$ are three auxiliary states for “hiding” the rewards and normalizing the paths taken by the agent. We take $u$ as the initial state for each episode.

Now we consider the action space as follows:

$$A = \{ a_j | j \in [d_0] \}$$
$$\tilde{A} = \{ \tilde{a} \}$$

The agent can take any action in $A$ at state $u$, while there is only one feasible action $\tilde{a}$ for all other states. We use $\mathcal{A} = A \cup \tilde{A}$ to denote the whole action space and clearly the maximum number of feasible action in a particular state is $d_0$.

The last information the algorithm knows before exploration is the feature vectors of each state-action pair:

$$\phi(u, a_j) = e_j$$
$$\phi(s_{h,i}, \tilde{a}) = e_{2d_0+i}$$
$$\phi(v, \tilde{a}) = e_{3d_0}$$
$$\phi(w, \tilde{a}) = e_{4d_0}$$

We can easily verify that the agents cannot extract any useful information about the special action by these
feature vectors. More precisely, these feature vectors are orthonormal vectors, given constant $h$.

$$
\mu_{2h}(s_{h,i}) = e_i \\
\mu_{2h+1}(u) = e_{2d_0+i_h} \\
\mu_h(v) = e_{3d_0}, \quad h \neq 2h_\star + 1 \\
\mu_{2h_\star+1}(v) = e_{j_{h_\star}d_0+i_{h_\star}} \\
\mu_{2h}(w) = e_{4d_0} \\
\mu_{2h+1}(w) = \sum_{i \in [d_0]} \sum_{j=0,1} e_{jd_0+i} - e_{j_{h_\star}d_0+i_{h_\star}} + e_{4d_0}
$$

where $h_\star \sim \text{Unif}([H_0])$, $i_h \sim \text{Unif}[d_0]$ for all $h \in [h_\star]$ and other vectors are all $(0, 0, \ldots, 0)$. We can easily find that $v$ and $w$ are two sinks. As above shows, the agent starts at $u$ and the action $a_j$ leads to state $s_{1,j}$. If $j = i_h$, then the agent comes back to $u$ and then selects the next action, else the agent goes to $w$ and stays in $w$ forever. In other words, the agent will finally go to sink $v$ at step $2h_\star + 1$ along with the correct path $i_h$, or it will go to $w$ if taking any wrong action. We illustrate the construction in Figure 2. Note that denote state $u$ as $u_1, u_2, \ldots$, for the arrival of the $H$-th step at $u$. We do the same for $v$ and $w$.

The reward function is quite simple: the agent gets reward 1 only at state $v$:

$$
\theta_h = e_{3d_0}
$$

Clearly we have

$$
\mathbb{E}_{M_\star}[V_1^\pi(x_0)] = \mathbb{E}_{M_\star}\left[\sum_{h=1}^{H} 1\{x_h = v\}\right] \\
\mathbb{E}_{M_\star}[V_1^*(x_0)] = \mathbb{E}_{M_\star}[H - 2h_\star] = H_0
$$
10.2 Minimax Lower Bound

\[
\sup_{M \in \mathcal{M}} \mathbb{E}_{s_1, M} \left[ \sum_{k=1}^{K} V^*_1(s_1) - V^{\pi_k}_1(s_1) \right] \\
\geq \mathbb{E}_{M_0} \left[ \sum_{k=1}^{K} V^*_1(x_0) - V^{\pi_k}_1(x_0) \right] \\
= KH_0 - \sum_{k=1}^{K} \mathbb{E}_{M_0} [V^{\pi_k}_1(x_0)]
\]

It remains to upper bound \( \mathbb{E}_{M_0} [V^{\pi_k}_1(x_0)] \) for each \( k \).

For all \( k \geq 1 \), let

\[
N^{k}_{\text{switch}} = \sum_{j=1}^{k-1} 1\{\pi_j \neq \pi_{j+1}\}
\]

denote the switching cost at episode \( K \). Let

\[
S_* := \{s_{1,i_1}, s_{2,i_2}, \ldots, s_{h_v,i_{h_v}}\}
\]

be the correct path leading to state \( v \),

\[
S_k := \{\bar{s}_{h_1,i_1}, \bar{s}_{h_2,i_2}, \ldots, \bar{s}_{h_r,i_r}\}
\]

be the ordered set of the states \( s_{h,i} \) that have been reached throughout the execution of the algorithm,

\[
S^*_k := \{\bar{s}_{\tau,i_1}, \bar{s}_{\tau,i_2}, \ldots, \bar{s}_{\tau,i_{\tau}}\}
\]

be the states throughout the exploration of \( s_{\tau,i} \), i.e. \( S^*_k = S_k \cap \{s_{\tau,i} | i \in [d_0]\} \).

We begin by observing that \( r = \sum_{\tau=1}^{h_v} r_{\tau} \leq N^{k}_{\text{switch}} + H_0 + 1 \), i.e., after changing the policy for \( N^{k}_{\text{switch}} \) times, the algorithm can only explore at most \( N^{k}_{\text{switch}} \) states in \( S' \) except for the correct path. In fact, we know that if \( x_{2h} \in S' - S_* \) for some \( h \), then \( x_{2h+1} = w \) and so do the rest steps. Thus as long as the algorithm makes a mistake at some step, it can only explore one more state in \( S' - S_* \).

More precisely, if the algorithm has already known the structure of this MDP, clearly it still need to find the correct path \( S_* \) in order to achieve reward 1.

In this way,

\[
\mathbb{E}_{M_0} [V^{\pi_k}_1(x_0)] = \mathbb{E}_{M_0} \left[ \sum_{h=1}^{H} 1\{x_h = v\} \right] \\
\leq H \cdot \mathbb{E}_{M_0} [1\{x_H = v\}] \\
\leq H \cdot \mathbb{E}_{M_0} [\mathbb{P}(S_* \in S_k)] \\
\leq H/H_0 \cdot \sum_{h=1}^{H_0} \sum_{\tau=1}^{h_v} \mathbb{P}(s_{\tau,i_{\tau}} \in S^*_k) \\
= 2 \sum_{h=1}^{H_0} \sum_{\tau=1}^{h_v} \mathbb{P} \left( \bigcup_{j \geq 1} r_{\tau} \geq j, s_{\tau,i_{\tau}} \notin \{\bar{s}_{\tau,i_1}, \bar{s}_{\tau,i_2}, \ldots, \bar{s}_{\tau,i_{\tau-1}}\}, s_{\tau,i_{\tau}} = \bar{s}_{\tau,i_{\tau}} \right) \\
= 2 \sum_{h=1}^{H_0} \sum_{\tau=1}^{h_v} \mathbb{P}(r_{\tau} \geq j) \cdot \mathbb{P}(s_{\tau,i_{\tau}} \notin \{\bar{s}_{\tau,i_1}, \bar{s}_{\tau,i_2}, \ldots, \bar{s}_{\tau,i_{\tau-1}}\}, s_{\tau,i_{\tau}} = \bar{s}_{\tau,i_{\tau}} | r_{\tau} \geq j)
\]

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Now suppose that we know $r_\tau \geq j$. Noticing that $s_{\tau,i} \sim \text{Unif}(\{s_{\tau,i} | i \in [d_0]\})$, we have
\[
\mathbb{P}(s_{\tau,i} \notin \{\hat{s}_{\tau,i_1}, \hat{s}_{\tau,i_2}, \ldots, \hat{s}_{\tau,i_{j-1}}\}, s_{\tau,i} = \hat{s}_{\tau,i_j} | r_\tau \geq j) = \prod_{\gamma=1}^{j-1} \frac{d_0 - \gamma}{d_0 - \gamma + 1} \cdot \frac{1}{d_0 - j + 1} = \frac{1}{d_0}
\]
Substituting this into the preceding bound gives
\[
\mathbb{E}_{M_*}[V_{1}^{\pi_k}(x_0)] \leq 2 \sum_{h_*=1}^{H_0} \sum_{\tau=1}^{h_*} \sum_{j \geq 1} \mathbb{P}(r_\tau \geq j)/d_0
\]
\[
= \frac{2}{d_0} \sum_{h_*=1}^{H_0} \sum_{\tau=1}^{h_*} \mathbb{E}[r_\tau] = \frac{2}{d_0} \sum_{h_*=1}^{H_0} \mathbb{E}[r]
\]
\[
\leq \mathbb{E}[N_{\text{switch}}^h + H] \cdot 2/d_0 \leq \mathbb{E}[N_{\text{switch}}^g + H] \cdot 2/d_0 \leq H_0/2
\]
as $N_{\text{switch}}^g \leq dH/100$ almost surely and $d \geq 100$. And thus
\[
\sup_{M \in M} \mathbb{E}_{s_1,M} \left[ \sum_{k=1}^{K} V_1^*(s_1) - V_1^{\pi_k}(s_1) \right]
\]
\[
\geq KH_0 - \sum_{k=1}^{K} \mathbb{E}_{M_*}[V_1^{\pi_k}(x_0)]
\]
\[
\geq KH_0 - KH_0/2
\]
\[
= KH/4
\]

**Remark**  In fact, we can reduce states in $S_0$ and only reserve three states $\{u, v, w\}$ with similar structure: the agent needs to find the correct action set $\{a_{i_h}\}$ for $h_*$ steps. In this way, we can use identical $d$ actions in each states and thus we provide a tighter lower bound $\Omega(dH)$. 

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