Preserving Differential Privacy in Adversarial Learning with Provable Robustness

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Abstract

In this paper, we aim to develop a novel mechanism to preserve differential privacy (DP) in adversarial learning for deep neural networks, with provable robustness to adversarial examples. We leverage the sequential composition theory in differential privacy, to establish a new connection between differential privacy preservation and provable robustness. To address the trade-off among model utility, privacy loss, and robustness, we design an original, differentially private, adversarial objective function, based on the post-processing property in differential privacy, to tighten the sensitivity of our model. Theoretical analysis and thorough evaluations show that our mechanism notably improves the robustness of DP deep neural networks.

1 Introduction

In spite of advantages in terms of application utility, the pervasiveness of machine learning exposes new vulnerabilities in software systems. Adversaries can conduct devastating attacks, in which deployed machine learning models can be used to reveal sensitive information in private training data [1–4], or to cause the models to misclassify, such as adversarial examples [5–7]. Efforts to address this issue typically seek one of three solutions: (1) Privacy-preserving models, which do not reveal sensitive information about the subjects involved in the training data [8, 9]; (2) Adversarial training algorithms, which augment training data to consist of benign examples and adversarial examples crafted during the training process, thereby increasing robustness [4, 10–12]; and (3) Provable robustness, in which the robustness against adversarial examples is guaranteed [13–15].

On one hand, private models, trained with existing privacy preserving mechanisms, are unshielded under adversarial examples. On the other hand, adversarial learning algorithms (with or without provable robustness) do not offer privacy protections to the training data. Aggressive adversaries can attack a deployed model by using both privacy model attacks and adversarial examples. That poses serious risks to machine learning-based systems in critical applications, e.g., face recognition, healthcare, etc. To be assured, a model must be private and resilient to both privacy model attacks and adversarial examples. There is an urgent demand to train such a secure model with a high utility. Unfortunately, learning such a model has not been studied before. It remains a largely open challenge.

In this paper, we propose a novel mechanism to: 1) preserve privacy of the training data in adversarial learning, 2) be provably and practically robust to adversarial examples, and 3) retain high model utility. Such a mechanism will greatly extend the applicability of machine learning, by fortifying the
models in both privacy and security aspects. This is a non-trivial task. Existing algorithms cannot be applied to address the trade-off among model utility, privacy loss, and robustness.

**Our Contributions.** We first focus on establishing a solid theoretical and practical connection among privacy preservation, adversarial learning, and provable robustness. We develop a novel mechanism, called **differentially private adversarial learning (DPAL)**, to preserve differential privacy in adversarial learning and to achieve provable robustness against adversarial examples. **Differential privacy (DP)** [16] is an elegant cryptography-based formulation of privacy in probabilistic terms, and provides a rigorous protection for an algorithm to avoid leaking personal information contained in its inputs. In our mechanism, privacy-preserving noise is injected into inputs and hidden layers to achieve DP in learning private model parameters. This noise will be used to derive a novel robustness bound, by leveraging the **sequential composition theory** in DP [17]. In our theoretical analysis, noise injected into different layers is considered as a sequence of defensive mechanisms, providing different levels of robustness. The generalized robustness condition will be a composition of these levels of robustness. To our knowledge, our result establishes the first connection between DP preservation to protect the original training data and provable robustness in deep learning.

Although our robustness condition helps the model to avoid misclassifying adversarial examples, it does not improve the model decision boundary. To tackle this, we incorporate ensemble adversarial learning into our mechanism to improve the decision boundary; thus limiting the number of robustness violations. First, we introduce a concept of DP adversarial examples, which are crafted under DP guarantees. We then address the trade-off between model utility and privacy loss, by designing a new DP adversarial objective function to tighten the global sensitivity; thus reducing the amount of noise injected into our function. Rigorous experiments conducted on MNIST and CIFAR-10 datasets [18, 19] show that, our mechanism notably enhances the robustness of DP deep neural networks.

## 2 Background and Problem Definition

In this section, we revisit adversarial learning, DP, and introduce our problem definition. Let $D$ be a database that contains $n$ tuples, each of which contains data $x \in [-1, 1]^d$ and a **ground-truth label** $y \in \mathbb{Z}_K$. Let us consider a classification task with $K$ possible categorical outcomes; i.e., the data label $y$ given $x \in D$ is assigned to only one of the $K$ categories. Each $y$ can be considered as a one-hot vector of $K$ categories $y = \{y_1, \ldots, y_K\}$. On input $x$ and parameters $\theta$, a model outputs class scores $f: \mathbb{R}^d \rightarrow \mathbb{R}^K$ that maps $d$-dimensional inputs $x$ to a vector of scores $f(x) = \{f_1(x), \ldots, f_K(x)\}$ s.t. $\forall k: f_k(x) \in [0, 1]$ and $\sum_{k=1}^{K} f_k(x) = 1$. The class with the highest score value is selected as the predicted label for the data tuple, denoted as $y(x) = \max_{k \in K} f_k(x)$. A loss function $L(f(x), y)$ presents the penalty for mismatching between the predicted values $f(x)$ and original values $y$. We briefly revisit DP and DP-preserving techniques in deep learning.

**Definition 1** $(\epsilon, \delta)$-DP [16]. A randomized algorithm $A$ fulfills $(\epsilon, \delta)$-DP, if for any two databases $D$ and $D'$ differing at most one tuple, and for all $O \subseteq \text{Range}(A)$, we have:

$$Pr[A(D) = O] \leq e^\epsilon Pr[A(D') = O] + \delta$$

(1)

Here, $\epsilon$ controls the amount by which the distributions induced by $D$ and $D'$ may differ, $\delta$ is a broken probability. DP also applies to general metrics $\rho(D, D') \leq 1$, including Hamming metric as in Definition 1 and $l_p$-norms [20]. DP-preserving algorithms in deep learning can be categorized into two lines: 1) introducing noise into gradients of parameters [8, 21], and 2) injecting noise into objective functions [22, 24].

**Adversarial Learning.** For some target model $f$ and inputs $(x, y_{true})$, i.e., $y_{true}$ is the true label of $x$, the adversary’s goal is to find an adversarial example $x_{adv} = x + \alpha$, where $\alpha$ is the perturbation introduced by the attacker, such that: (1) $x_{adv}$ and $x$ are close, and (2) the model misclassifies $x_{adv}$, i.e., $y(x_{adv}) \neq y(x)$. In this paper, we consider well-known classes of $l_p \in \{1, 2, \infty\}$-norm bounded attacks [5]. Let $l_p(\mu) = \{\alpha \in \mathbb{R}^d : ||\alpha||_p \leq \mu\}$ be the $l_p$-norm ball of radius $\mu$, one of the goals in adversarial learning is to minimize the risk over adversarial examples:

$$\theta^* = \arg\min_\theta \mathbb{E}_{(x, y_{true}) \sim D} \left[ \max_{||\alpha|| \leq \mu} L(f(x + \alpha, \theta), y_{true}) \right]$$

where a specific attack is used to approximate solutions to the inner maximization problem, and the outer minimization problem corresponds to training the model $f$ with parameters $\theta$ over these adversarial examples $x_{adv} = x + \alpha$. There are two basic attacks. The first one is a **single-step**
algorithm, in which only a single gradient computation is required. For instance, Fast Gradient Sign Method (FGSM) algorithm [5] finds an adversarial example by maximizing the loss function $L(f(x, \theta), y_{true})$. The second one is an iterative algorithm, in which multiple gradients are computed and updated. For instance, in [25] FGSM is applied multiple times with small steps, each of which has a size of $\mu/T_n$, where $T_n$ is the number of steps.

To improve the robustness of models, prior work focused on two directions: 1) Producing correct predictions on adversarial examples, while not compromising the accuracy on legitimate inputs [10–12, 26–30]; and 2) Detecting adversarial examples, without introducing too many false positives [31–35]. Adversarial training appears to hold the greatest promise for learning robust models [36]. One of the well-known algorithms was proposed by [37]. At every training step, new adversarial examples are generated and injected into batches containing both benign and adversarial examples. Recently, several algorithms [13] have been proposed to derive provable robustness, in which each prediction is guaranteed to be consistent under the perturbation $\alpha$, if a robustness condition is held.

**DP and Provable Robustness.** Given a benign example $x$, we focus on achieving a robustness condition to attacks of $l_p(\mu)$-norm, as follows:

$$\forall \alpha \in l_p(\mu): f_k(x + \alpha) > \max_{i \neq k} f_i(x + \alpha)$$  \hspace{1cm} (2)

where $k = y(x)$, indicating that a small perturbation $\alpha$ in the input does not change the predicted label $y(x)$. To achieve the robustness condition in Eq. (2), Lecuyer et al. [38] introduce an algorithm, called PixelDP. By considering an input $x$ (e.g., images) as databases in DP parlance, and individual features (e.g., pixels) as tuples in DP, PixelDP shows that randomizing the scoring function $f(x)$ to enforce DP on a small number of pixels in an image guarantees robustness of predictions against adversarial examples that can change up to that number of pixels. To achieve this goal, random noise $\sigma_r$ is injected into either input $x$ or some hidden layer. That results in the following $(\epsilon_r, \delta_r)$-PixelDP condition:

**Lemma 1 (\(\epsilon_r, \delta_r\))-PixelDP [38].** Given a randomized scoring function $f(x)$ satisfying $(\epsilon_r, \delta_r)$-PixelDP w.r.t. a $l_p$-norm metric, we have:

$$\forall k, \forall \alpha \in l_p(1): \mathbb{E}f_k(x) \leq \epsilon_r \mathbb{E}f(x + \alpha) + \delta_r$$  \hspace{1cm} (3)

where $\mathbb{E}f_k(x)$ is the expected value of $f_k(x)$, $\epsilon_r$ is a predefined budget, and $\delta_r$ is a broken probability.

At the prediction time, a certified robustness check is implemented for each prediction. A generalized robustness condition is proposed as follows:

$$\tilde{\mathbb{E}}_{ub} f_k(x) > e^{2\epsilon_r} \max_{i \neq k} \tilde{\mathbb{E}}_{ub} f_i(x) + (1 + e^{\epsilon_r}) \delta_r$$  \hspace{1cm} (4)

where $\tilde{\mathbb{E}}_{ub}$ and $\tilde{\mathbb{E}}_{ub}$ are the lower and upper bounds of the expected value $\tilde{\mathbb{E}}f(x) = \frac{1}{n} \sum_{n} f(x) \eta$, derived from the Monte Carlo estimation with an $\eta$-confidence, given $n$ is the number of invocations of $f(x)$ with independent draws in the noise $\sigma_r$. Passing the check for a given input guarantees that no perturbation exists up to $l_p(1)$-norm that causes the model to change its prediction. In other words, the classification model, based on $\tilde{\mathbb{E}}f(x)$, i.e., argmax $\tilde{\mathbb{E}}f_k(x)$, is provably robust to attacks of $l_p(1)$-norm on input $x$ with probability $\geq \eta$. PixelDP does not preserve DP in learning private parameters $\theta$ to protect the training data [38]. That is different from our goal.

**3 DPAL with Provable Robustness**

We introduce a new DP-preserving mechanism in Alg. 1 (Appendix A). Given a deep neural network $f$ with model parameters $\theta$ (Line 2), the network is trained by optimizing the loss function $L(\theta)$ over the training examples $x_i \in D$. Optimization algorithms, e.g., SGD, are applied on $T$ random training batches, each of which is a set of $n$ training examples in $\tilde{D}$ (Lines 4-16). Our network can be represented as: $f(x) = g(\alpha(x, \theta_1), \theta_2)$, where $\alpha(x, \theta_1)$ is feature representation learning model with $x$ as an input, and $\eta$ will take the output of $\alpha(x, \theta_1)$ and return the class scores $f(x)$. Our idea is to use auto-encoder to simultaneously learn DP parameters $\theta_1$, and to ensure that the output of $\alpha(x, \theta_1)$ is DP. The reasons we choose auto-encoder are: (1) It is easier to train given its small size, and (2) It can be reused for different predictive models. Figure 1 shows the structure of our mechanism. We use a data reconstruction function (cross-entropy), given a batch $B$ of the input $x_i$. 

Figure 1: An instance of DPAL.
where the affine transformation of \( x_i \) is: \( h_i = \theta^T_i x_i \); and \( \mathbf{h}_1 = \{ \theta^T_i x_i \}_{x_i \in B} \), with \( \mathbf{h}_1 \) is the hidden layer of \( a(x, \theta_1) \), and \( \bar{x}_i = \theta_1 h_i \) is the reconstruction of \( x_i \). First, we derive the 1st-order polynomial approximation of \( R_B(\theta_1) \) by applying Taylor Expansion [39], denoted as \( \hat{R}_B(\theta_1) \). Then, Functional Mechanism [40] is employed to inject noise into coefficients of the approximated function \( \hat{R}_B(\theta_1) \).

\[
\hat{R}_B(\theta_1) = \sum_{x_i \in B} \sum_{j=1}^d \sum_{\beta=0}^1 \sum_{r=0}^{1} \frac{F_{ij}^{(\beta)}(0)}{r!} (\theta_{ij} h_i)^j
\]

where \( F_{ij}(z) = x_{ij} \log(1 + e^{-z}), F_{2j}(z) = (1 - x_{ij}) \log(1 + e^z) \), we have that: \( \tilde{R}_B(\theta_1) = \sum_{x_i \in B} \sum_{j=1}^d \left[ \log 2 + \theta_{ij} \left( \frac{1}{2} - x_{ij} \right) h_i \right] \). In \( \tilde{R}_B(\theta_1) \), parameters \( \theta_{ij} \), which will be derived from the function optimization, need to be DP; they do not disclose information from the data \( x_{ij} \). To achieve that, \( \left( \frac{1}{2} - x_{ij} \right) h_i \) are considered the coefficients of the parameters \( \theta_{ij} \). Laplace noise \( \frac{1}{m} \text{Lap}(\frac{\Delta_R}{\epsilon_1}) \) is injected into \( \left( \frac{1}{2} - x_{ij} \right) h_i \), where \( \Delta_R \) is the sensitivity of \( \hat{R}_B(\theta_1) \), as follows (log 2 can be ignored):

\[
\tilde{R}_B(\theta_1) = \sum_{x_i \in B} \sum_{j=1}^d \theta_{ij} \left( \frac{1}{2} - x_{ij} \right) h_i + \frac{1}{m} \text{Lap}(\frac{\Delta_R}{\epsilon_1}) \right] = \sum_{x_i \in B} \left[ \sum_{j=1}^d \left( \frac{1}{2} \theta_{ij} \bar{h}_i - x_i \bar{x}_i \right) \right]
\]

To ensure that the computation of \( \bar{x}_i \) does not access the original data, we further inject Laplace noise \( \frac{1}{m} \text{Lap}(\frac{\Delta_R}{\epsilon_1}) \) into \( x_i \) (Line 6). The perturbed function now becomes:

\[
\tilde{R}_B(\theta_1) = \sum_{x_i \in B} \left[ \sum_{j=1}^d \left( \frac{1}{2} \theta_{ij} \bar{h}_i - \bar{x}_i \bar{x}_i \right) \right]
\]

where \( \bar{x}_i = x_i + \frac{1}{m} \text{Lap}(\frac{\Delta_R}{\epsilon_1}), h_i = \theta^T_i \bar{x}_i, \bar{h}_i = h_i + \frac{2}{m} \text{Lap}(\frac{\Delta_R}{\epsilon_1}) \), and \( \bar{x}_i = \theta_1 \bar{h}_i \) (Lines 3 and 6). \( h_i \) is bounded in \([-1, 1]\) to compute the global sensitivity \( \Delta_R \).

**Lemma 2** Denote \( \beta \) as the number of hidden neurons in \( \mathbf{h}_1 \). The global sensitivity of \( \tilde{R} \) over any two neighboring batches, \( B \) and \( B' \), is as follows: \( \Delta_R \leq d(\beta + 2) \).

**Lemma 3** Algorithm 1 preserves \( \epsilon_1 \)-DP in learning \( \theta_1 \).

Detailed proofs of all the lemmas can be found in Appendix. The output of \( a(x, \theta_1) \) is the perturbed affine transformation \( \mathbf{h}_{1,B} = \{ \theta^T_i \bar{x}_i + \frac{2}{m} \text{Lap}(\frac{\Delta_R}{\epsilon_1}) \}_{x_i \in B} \), which is \( (\epsilon_1/\gamma) \)-DP as shown in the following lemma, given \( \gamma = \frac{2\Delta_R}{m\|\theta_1\|_1} \), and \( \|\mathbf{h}_{1,B}\|_1 \) is the maximum 1-norm of \( \theta_1 \)'s columns [41].

**Lemma 4** Algorithm 2 preserves \( (\epsilon_1/\gamma) \)-DP in computing the affine transformation \( \mathbf{h}_{1,B} \).

After preserving DP in learning \( \theta_1 \) and \( \mathbf{h}_{1,B} \), without using additional information from the original data, the computations of \( g(a(x, \theta_1), \theta_2) \) is also \( (\epsilon_1/\gamma) \)-DP, i.e., thank to the post-processing property of DP. This is crucial to tighten the sensitivity of our adversarial objective function at the output layer.

**DP Adversarial Learning.** To integrate adversarial learning, we first draft adversarial examples \( x_j \) using benign examples, with an ensemble of attack algorithms \( A \) and a random perturbation budget \( \mu_j \in [0, 1] \) at each step \( t \) (Lines 9-13). Adversarial examples (crafted from the training data) are very similar to the original benign examples. That clearly poses privacy risks. Therefore, \( x_j \) is perturbed to ensure DP in the training procedure. The ensemble adversarial examples definitely enhances the robustness of our model. Second, we propose a novel DP adversarial objective function \( L_B(\theta_2) \), in which two loss functions, denoted as \( L \) for benign examples and \( \Upsilon \) for adversarial examples, are combined to optimize the parameters \( \theta_2 \). \( L_B(\theta_2) \) is defined as follows:

\[
L_B(\theta_2) = \frac{1}{m(1 + \xi)} \left( \sum_{x_i \in \mathbb{P}} L(f(x_i, \theta_2), y_i) + \xi \sum_{x_j \in \mathbb{P}^{adv}} \Upsilon(f(x_j^{adv}, \theta_2), y_j) \right)
\]

where \( \xi \) is a hyper-parameter, and \( x_j^{adv} \) is a DP adversarial example, crafted as follows:

\[
x_j^{adv} = [x_j + \mu \cdot \text{sign} (\nabla_x L(f(x_j, \theta), y_j))] + \frac{1}{m} \text{Lap}(\frac{\Delta_R}{\epsilon_1})
\]

where \( y_j \) is the class prediction result of \( f(x, \theta) \) to avoid label leaking of the benign examples \( x_j \) during the adversarial example crafting. Similar to benign examples \( x \), training the auto-encoder with adversarial examples \( x^{adv} \), i.e., \( a(x^{adv}, \theta_1) \), preserves \( \epsilon_1 \)-DP. It can be extended to iterative attacks as
\[ \pi^{\text{adv}}_{j,t} = x_j, \pi^{\text{adv}}_{j,t+1} = \pi^{\text{adv}}_{j,t} + \frac{\mu}{T_\mu} \cdot \text{sign}(\nabla_{\pi^{\text{adv}}_{j,t}} L(f(\pi^{\text{adv}}_{j,t}, \theta), y_{f,t})), \pi^{\text{adv}}_{j,t+1} = \pi^{\text{adv}}_{j,t} + \frac{1}{m} \text{Lap}(\Delta^R) \]

(10)

where \( y_{f,t} \) is the prediction result of \( f(\pi^{\text{adv}}_{j,t}, \theta) \).

Now we are ready to preserve DP in objective functions \( L(f(\pi_i, \theta), y_i) \) and \( \Upsilon(f(\pi^{\text{adv}}_{i}, \theta), y_j) \) in order to achieve DP \( \mathcal{T}_B(\theta_2) \) (Eq. [9]). Since the objective functions use the labels \( y_i \) and \( y_j \) given \( \pi_i \in \overline{B} \) and \( \pi^{\text{adv}}_{i} \in \overline{B}^{\text{adv}} \), we need to protect the labels at the output layer. Let us first present the objective function \( L \) for benign examples. Given \( h_{\pi_i} \) computed from the \( \pi_i \) through the network with \( W_e \) is the parameter at the last hidden layer \( h_e \). Cross-entropy function can be applied as follows:

\[ L_\pi(\theta_2) = -\sum_{k=1}^{K} \sum_{\pi_i} \left[ y_{ik} \log(1 + e^{-h_{\pi_i}W_{s_k}}) + (1 - y_{ik}) \log(1 + e^{-h_{\pi_i}W_{s_k}}) \right] \]

\[ \approx \sum_{k=1}^{K} \sum_{\pi_i} \left[ h_{\pi_i}W_{s_k} - (h_{\pi_i}W_{s_k})y_{ik} + \log(1 + e^{-h_{\pi_i}W_{s_k}}) \right] \]

Based on Taylor Expansion, the term \( \log(1 + e^{-h_{\pi_i}W_{s_k}}) \) can be approximated as a 2nd-order polynomial function:

\[ L_{2\pi}(\theta_2) \approx \sum_{k=1}^{K} \sum_{\pi_i} \left[ h_{\pi_i}W_{s_k} - (h_{\pi_i}W_{s_k})y_{ik} + \frac{1}{2} |h_{\pi_i}W_{s_k}|^2 + \frac{1}{8} |h_{\pi_i}W_{s_k}|^4 \right] = L_{1\pi}(\theta_2) - L_{2\pi}(\theta_2) \]

where \( L_{1\pi}(\theta_2) = \sum_{k=1}^{K} \sum_{\pi_i} \left[ |h_{\pi_i}W_{s_k}|^2 + \frac{1}{8} |h_{\pi_i}W_{s_k}|^4 \right] \), and \( L_{2\pi}(\theta_2) = \sum_{k=1}^{K} \sum_{\pi_i} \left( h_{\pi_i}y_{ik} \right) W_{s_k} \).

Based on the \textit{post-processing property of DP}, \( h_{\pi_i} = \{h_{\pi_i}\} \in \mathbb{E} \) is \((\epsilon_1/\gamma)\)-DP, since the computation of \( H_{\pi_i} \) is \((\epsilon_1/\gamma)\)-DP (Lemma 4). As a result, we have that: 1) The optimization of the function \( L_{1\pi}(\theta_2) \) does not disclose any information from the training data; and 2) \( \frac{\partial \mathcal{L}(\pi_{\theta_2}(\pi_2))}{\partial \mathcal{L}(\pi_{\theta_2}(\pi_2))} \leq e^{\gamma - 1} \) given any two neighboring batches \( \overline{B} \) and \( \overline{B}' \). Thus, to preserve DP in \( L_{2\pi}(\theta_2) \), we only need to preserve \( \epsilon_2 \)-DP in the function \( L_{2\pi}(\theta_2) \), which access the ground-truth label \( y_{ik} \). Given coefficients \( h_{\pi_i}y_{ik} \), the sensitivity \( \Delta_{L_2} \) of \( L_{2\pi}(\theta_2) \) is computed as:

**Lemma 5** Let \( \overline{B} \) and \( \overline{B}' \) be neighboring batches of benign examples, we have the following inequality:

\[ \Delta_{L_2} \leq 2|H_{\pi_i}| \text{, where } |H_{\pi_i}| \text{ is the number of hidden neurons in } h_{\pi_i}. \]

The sensitivity of our objective function is notably smaller than the state-of-the-art bound, which is \( K |H_{\pi_i}| + \frac{1}{4} |H_{\pi_i}|^2 \) \([24]\). This is crucial to improve our model utility under strong attacks, while providing the same level of DP protections. The perturbed functions are as follows:

\[ L_{2\pi}(\theta_2) = L_{1\pi}(\theta_2) - L_{2\pi}(\theta_2), \text{ where } L_{2\pi}(\theta_2) = \sum_{k=1}^{K} \sum_{\pi_i} \left( h_{\pi_i}y_{ik} + \frac{1}{m} \text{Lap}(\Delta_{L_2}) \right) W_{s_k} \]

(11)

**Lemma 6** Algorithm preserves \((\epsilon_1/\gamma + \epsilon_2)\)-differential privacy in the optimization of \( L_{2\pi}(\theta_2) \).

Since the \( \epsilon_1/\gamma \) budget, accumulated from the perturbation of the auto-encoder, is tiny in practice (i.e., \( \approx 1e^{-3} \)), the additional privacy budget used to preserve DP in the function \( L_{2\pi}(\theta_2) \) can be considered \( \epsilon_2 \). We apply the same technique to preserve \( \epsilon_2 \)-DP in the optimization of the function \( \Upsilon(f(\pi^{\text{adv}}_{i}, \theta), y_j) \) over the adversarial examples \( \pi^{\text{adv}}_{j} \in \overline{B}^{\text{adv}} \). Since the perturbed functions \( \overline{L} \) and \( \overline{\Upsilon} \) are always optimized given two disjoint batches \( \overline{B} \) and \( \overline{B}^{\text{adv}} \), the privacy budget used to preserve DP in the adversarial objective function \( \mathcal{T}_B(\theta_2) \) is \( \epsilon_2 \), following the \textit{parallel composition} property in DP \([17]\). The total budget to learn private parameters \( \theta = \{\theta_1, \theta_2\} = \arg \min_{\theta_1, \theta_2} \left( \mathcal{R}_B(\theta_1) + L_B(\theta_2) \right) \) is \( \epsilon_1 + \epsilon_2 \). Similar to other objective function-based approaches \([24, 40, 42]\), the optimization of our mechanism is repeated in \( T \) steps without using additional information from the original data. It only reads perturbed inputs and perturbed coefficients. Thus, the privacy budget consumption will not be accumulated at each training step.

**Provable Robustness.** Now, we establish the correlation between our mechanism and provable robustness. On one hand, to derive the provable robustness condition against adversarial examples \( x + \alpha \), i.e., \( \forall \alpha \in L_p(1) \), PixelDP mechanism randomizes the scoring function \( f(x) \) by injecting \textit{robustness noise} \( \sigma_r \) into either input \( x \) or a hidden layer, i.e., \( x' = x + \text{Lap}(\Delta^x) \) or \( h' = h + \text{Lap}(\Delta^h) \), where \( \Delta^x \) and \( \Delta^h \) are the sensitivities of \( x \) and \( h \) measuring how much \( x \) and \( h \) can be changed given
the perturbation $\alpha \in l_p(1)$ in the input $x$. Monte Carlo estimation of the expected values $\hat{E}f(x)$, $\hat{E}_{lb}f_k(x)$, and $\hat{E}_{ub}f_k(x)$ are used to derive the robustness in Eq. 4.

On the other hand, in our mechanism, the privacy noise $\sigma_p$ includes Laplace noise injected into both input $x$, i.e., $\frac{1}{m} Lap(\frac{\Delta_x}{\epsilon_x})$, and its affine transformation $h$, i.e., $\frac{1}{m} Lap(\frac{\Delta_h}{\epsilon_h})$. Note that the perturbation of $\tilde{Z}_{\mathbb{F}}(\theta_2)$ is equivalent to $\tilde{Z}_{\mathbb{F}}(\theta_2) = \sum_{i=1}^{K} \sum_{r=1}^{N} (h_{ri}lb_k W_{rk} + \frac{1}{m} Lap(\frac{\Delta_h}{\epsilon_h}))$. This helps us to avoid injecting the noise directly into the coefficients $h_{ri}lb_k$. The connection between our DP preservation and provable robustness lies in the correlation between the privacy noise $\sigma_p$ and the robustness noise $\sigma_r$. We can derive a provable robustness condition by projecting the privacy noise $\sigma_p$ on the scale of the robustness noise $\sigma_r$. Given the input $x$, let $\kappa = \frac{\Delta_x}{r \epsilon_x}$, in our mechanism we have that: $\pi = x + Lap(\kappa \Delta_x/\epsilon_x)$. By applying a group privacy size $\kappa$ [17,38], the scoring function $f(x)$ satisfies $\epsilon_r$-PixelDP given $\alpha \in l_p(\kappa)$, or equivalently is $\kappa \epsilon_r$-PixelDP given $\alpha \in l_p(1), \epsilon_r = 0$. By applying Lemma 1 we have

$$\forall k, \forall \alpha \in l_p(\kappa): E f_k(x) \leq e^{\epsilon_r} E f_k(x + \alpha), \text{ or } \forall k, \forall \alpha \in l_p(1): E f_k(x) \leq e^{(\kappa \epsilon_r)} E f_k(x + \alpha) \quad (12)$$

With that, we can achieve a provable robustness condition against $l_p(\kappa)$-norm attacks, as follows:

$$\hat{E}_{lb}f_k(x) > e^{2 \epsilon_r} \max_{i : i \neq k} \hat{E}_{ub}f_k(x) \quad (13)$$

with the probability $\geq \eta_x$-confidence, derived from the Monte Carlo estimation of $\hat{E}f(x)$. Our mechanism also perturbs $h$ (Eq. 6). Given $\varphi = \frac{\Delta_h}{m \epsilon_h} / \frac{\Delta_x}{\epsilon_x}$, further have $\hat{h} = h + Lap(\frac{\varphi \Delta_h}{\epsilon_h})$. Therefore, the scoring function $f(x)$ also satisfies $\epsilon_r$-PixelDP given the perturbation $\alpha \in l_p(\varphi)$. In addition to the robustness to the $l_p(\kappa)$-norm attacks, we can achieve an additional robustness bound in Eq. 13 against $l_p(\varphi)$-norm attacks. Similar to PixelDP, these robustness conditions can be achieved as randomization processes in the inference time. They can be considered as two independent, provable defensive mechanisms, sequentially applied against two $l_p$-norm attacks, i.e., $l_p(\kappa)$ and $l_p(\varphi)$.

One challenging question here is: “What is the general robustness condition, given $\kappa$ and $\varphi$?” Intuitively, our model is robust to attacks with $\alpha \in l_p(\kappa + \varphi)$. We leverage the theory of sequential composition in DP [17] to theoretically answer this question. Given $S$ independent mechanisms $\mathcal{M}_1, \ldots, \mathcal{M}_S$, whose privacy guarantees are $\epsilon_1, \ldots, \epsilon_S$-DP with $\alpha \in l_p(1)$, each mechanism $\mathcal{M}_s$, which takes the input $x$ and outputs the value of $f(x)$ with the Laplace noise only injected into the position $s$ (i.e., no randomization at any other position), is defined as: $\forall s \in [1,S], \mathcal{M}_s f(x) : \mathbb{R}^d \rightarrow f_s(x) \in \mathbb{R}^K$. We aim to derive a generalized robustness of any composition scoring function $f(\mathcal{M}_1, \ldots, \mathcal{M}_s | x)$ bounded in $[0,1]$, defined as follows:

$$f(\mathcal{M}_1, \ldots, \mathcal{M}_s | x) : \mathbb{R}^d \rightarrow \prod_{s=1}^{S} f_s(x) \in \mathbb{R}^K \quad (14)$$

Our setting closely follows the sequential composition in DP [17]. Therefore, we can prove that the expected value $E f(\mathcal{M}_1, \ldots, \mathcal{M}_S | x)$ is insensitive to small perturbations $\alpha \in l_p(1)$ in the input.

**Lemma 7** Given $S$ independent mechanisms $\mathcal{M}_1, \ldots, \mathcal{M}_S$, which are $\epsilon_1, \ldots, \epsilon_S$-DP wrt a $l_p$-norm metric, then the expected output value of any sequential function $f$ of them, i.e., $f(\mathcal{M}_1, \ldots, \mathcal{M}_S | x)$, with bounded output $f(\mathcal{M}_1, \ldots, \mathcal{M}_S | x) \in [0,1]$, meets the following property: $\forall \alpha \in l_p(1)$:

$$E f(\mathcal{M}_1, \ldots, \mathcal{M}_S | x) \leq e(\Sigma_{s=1}^{S} \epsilon_s) E f(\mathcal{M}_1, \ldots, \mathcal{M}_S | x + \alpha)$$

Given the expected value of our scoring function $f$ is not sensitive to small perturbations $\alpha \in l_p(1)$, we derive our sequential composition of robustness as follows:

**Theorem 1** (Composition of Robustness) Given $S$ independent mechanisms $\mathcal{M}_1, \ldots, \mathcal{M}_S$. Given any sequential function $f(\mathcal{M}_1, \ldots, \mathcal{M}_S | x)$, using notation from Lemma 7 and further let $\hat{E}_{lb}$ and $\hat{E}_{ub}$ are lower and upper bounds with an $\eta$-confidence, for the Monte Carlo estimation of $\hat{E}f(\mathcal{M}_1, \ldots, \mathcal{M}_S | x) = \frac{1}{n} \sum_{i=1}^{n} f(\mathcal{M}_1, \ldots, \mathcal{M}_S | x)_i$. For any input $x$, if $\exists k \in K$ so that

$$\hat{E}_{lb}f_k(\mathcal{M}_1, \ldots, \mathcal{M}_S | x) > e^{\eta} \max_{i : i \neq k} \hat{E}_{ub}f_i(\mathcal{M}_1, \ldots, \mathcal{M}_S | x), \quad (15)$$

then the predicted label $k = \arg \max_k \hat{E}f_k(\mathcal{M}_1, \ldots, \mathcal{M}_S | x)$, is robust to adversarial examples $x + \alpha, \forall \alpha \in l_p(1)$, with probability $\geq \eta$, by satisfying: $\hat{E}f_k(\mathcal{M}_1, \ldots, \mathcal{M}_S | x + \alpha) > \max_{i : i \neq k} \hat{E}f_i(\mathcal{M}_1, \ldots, \mathcal{M}_S | x + \alpha)$, which is the targeted robustness condition in Eq. 2.
We have carried out an extensive experiment on MNIST and CIFAR-10 datasets. We consider the which is different from a common setting with existing mechanisms? Our DPAL mechanism is evaluated in comparison with state-of-the-art mechanisms in: (1) DP-preserving algorithms in deep learning, i.e., DP-SGD [8], AdLM [24], and in (2) Provable robustness, i.e., PixelDP [38]. To preserve DP, DP-SGD injects random noise into gradients of parameters, while AdLM is a Functional Mechanism-based approach. PixelDP is one of the state-of-the-art mechanisms providing provable robustness using DP bounds. The baseline models share the same design in our experiment. Four attacks (white-box) were used, including FGSM [8], Momentum Iterative Method (MIM) [43], and MadryEtAl [44]. All the models share the same architecture consisting of 2 and 3 convolution layers, respectively for MNIST and CIFAR-10 datasets (detailed configurations are in Appendix J). We apply two accuracy metrics as follows:

\[
\text{conventional acc} = \frac{\sum_{i=1}^{[\text{test}]} \text{isCorrect}(x_i)}{[\text{test}]} \quad \text{and} \quad \text{certified acc} = \frac{\sum_{i=1}^{[\text{test}]} \text{isCorrect}(x_i) \& \text{isRobust}(x_i)}{[\text{test}]}
\]

where \([\text{test}]\) is the number of test cases, \(\text{isCorrect}()\) returns 1 if the model makes a correct prediction (otherwise, returns 0), and \(\text{isRobust}()\) returns 1 if the robustness size is larger than a given attack bound \(\mu_a\) (else, returns 0). It is important to note that \(x \in [-1,1]^d\) in our setting, which is different from a common setting \(x \in [0,1]^d\). Thus, the attack size \(\mu = 0.3\) in the setting of \(x \in [0,1]^d\) is equivalent to an attack size \(2\mu = 0.6\) in our setting. The reason of using \(x \in [-1,1]^d\) is to achieve better model utility, while retaining the same global sensitivities to preserve DP, compared with \(x \in [0,1]^d\). \(\epsilon = (\epsilon_1 + \epsilon_2)\) is used to indicate the DP budget used to protect the training data; meanwhile, \(\epsilon_r\) is the budget for robustness. \(\epsilon_r\) is set to be 1.0 in the training of our model.
Figure 2: Conventional accuracy on the MNIST dataset, under $l_{\infty}(\mu_a = 0.2)$ attacks. A full result is in Figure 5 (Appendix K).

Figure 3: Certified accuracy on the MNIST dataset. A full result is in Figure 8 (Appendix K).

MNIST. Figures 2 and 4 (Appendix K) illustrate the conventional accuracy of each model as a function of the privacy budget ($\epsilon_1 + \epsilon_2$) on the MNIST dataset under $l_{\infty}(\mu_a)$-norm attacks, with $\mu_a \in \{0.1, 0.2\}$. It is clear that our DPAL outperforms AdLM and DP-SGD, in all cases, i.e., $p < 3e - 10$ (2 tail t-test). AdLM has better accuracies compared with DP-SGD. However, there is no guarantee provided in AdLM. Thus, the accuracy of AdLM algorithm seems to show no effect against adversarial examples, when the privacy budget is varied. This is different given our DPAL model and the DP-SGD model, whose accuracies are proportional to the privacy budget. Note that when $\mu_a = 0.2$, which is a pretty strong attack, AdLM and DP-SGD become defenseless. By constrast, our model just shows a small drop in terms of accuracy, when $\mu_a$ is increased from 0.1 to 0.2, i.e., 1.76% in average across all attacks and privacy budgets. This illustrates that our mechanism notably enhances the robustness, by incorporating DP into adversarial learning in an ensemble approach.

Figure 3 illustrates the certified accuracy of each model as a function of the adversarial perturbation $\mu_a \in [0.05, 0.6]$. The privacy budget is set to 2.0, offering a reasonable privacy protection. Following [38], in PixelDP, the construction attack bound $\epsilon_r$ is set to 0.1, offering a pretty reasonable defense. It is clear that with small perturbations, i.e., $\mu_a \leq 0.2$, PixelDP achieves better certified accuracies under all attacks. This is reasonable, since PixelDP does not preserve DP to protect the training data, compared with our DPAL. Meanwhile, our model outperforms PixelDP when $\mu_a \geq 0.3$, indicating a stronger defense to more aggressive attacks. More importantly, our DPAL has a consistent certified accuracy to different attacks given different perturbation budgets, compared with PixelDP. In fact, when $\mu_a$ is increased from 0.05 to 0.6, our DPAL shows a small drop (9.88% in average, from 82.28%($\mu_a = 0.05$) to 72.40%($\mu_a = 0.6$)) compared with a huge drop of the PixelDP, i.e., from 94.19%($\mu_a = 0.05$) to 9.08%($\mu_a = 0.6$) in average under I-FGSM, MIM, and MadryEtAl attacks, and to 77.47%($\mu_a = 0.6$) under FGSM attack. This is a promising result.

Our key observations are as follows. (1) Incorporating ensemble adversarial learning into DP preservation does enhance the consistency, robustness, and accuracy of our model against different attack algorithms with different levels of perturbations. (2) Our DPAL model outperforms baseline algorithms, including both DP-preserving and non-private approaches, in terms of conventional accuracy and certified accuracy in most of the cases.

CIFAR-10. Results on CIFAR-10 dataset strengthen our observations. In Figures 6 and 7, it is clear that our DPAL outperforms baseline models in all cases, especially when the privacy budget is small ($< 6$), yielding strong privacy protections. DP-SGD achieves better performance, compared with AdLM. When the privacy budget is increased from 2 to 10, the conventional accuracy of our DPAL model increase from 44.1% to 48.77%, showing a 4.67% improvement in average. However, the
conventional accuracy of our model under adversarial example attacks is still low, i.e., 44.1% in average given the privacy budget at 2.0. This opens a long term research avenue to achieve better robustness under strong privacy guarantees. The accuracy of our model is also consistent given different attacks with different adversarial perturbations under varied DP protections. Figure 9 also shows that our DPAL model is more accurate than PixelDP (ε_r = 0.1) in terms of certified accuracy in all cases, with the privacy budget set to 4.0, offering a reasonable privacy protection.

5 Conclusion
In this paper, we established a connection among DP preservation to protect the training data, adversarial learning, and provable robustness. A sequential composition robustness theory was introduced to generalize robustness given any sequential and bounded function of independent defensive mechanisms. An original DP-preserving mechanism was designed to address the trade-off among model utility, privacy loss, and robustness by tightening the global sensitivity bound. Our model shows promising results.

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Algorithm 1 DPAL Mechanism

```plaintext
Input: Database $D$, loss function $L$, parameters $\theta$, batch size $m$, learning rate $\varphi_t$, privacy budgets: $\epsilon_1$ and $\epsilon_2$, robustness parameters: $\epsilon_r$, $\Delta^r$, and $\Delta^b$, adversarial attack size $\mu_a$, the number of invocations $n$, ensemble attacks $A$, a parameter $\psi$, and the size $|h_\pi|$ of $h_\pi$, and a parameter $\xi$

1. Initialize $\theta = \{\theta_1, \theta_2\}$ randomly
2. Construct a deep neural network $f$ with hidden layers $\{h_1, \ldots, h_r\}$, where $h_r$ is the last hidden layer
3. Draw random perturbation value $\chi_1 \leftarrow [\text{Lap}(\Delta^a)]^d$, $\chi_2 \leftarrow [\text{Lap}(\Delta^a)]^\beta$, $\chi_3 \leftarrow [\text{Lap}(\Delta^a)]^{|h_n|}$
4. for $t \in [T]$ do
   5. Take a random batch $B_t$ with the size $m$
   6. Perturb $\forall x_i \in B_t: x_i \leftarrow x_i + \frac{\chi_1}{m}; y_i \leftarrow y_i + \frac{\chi_2}{m}$
   7. $B \leftarrow \{x_i\}_{x_i \in B_t}$
   8. Ensemble DP Adversarial Examples:
   9. Draw random perturbation value $\mu_t \in (0, 1]$
10. $B_{\text{adv}} \leftarrow \emptyset$
11. for $i \in A$ do
12.     Take a random batch $B_a$ with the size $m/|A|$
13.     $\forall x_j \in B_a$: Craft $\pi_{adv}$ by using attack algorithm $A[i]$ with $l_\infty(\mu_t)$, $B_{\text{adv}} \leftarrow B_{\text{adv}} \cup \pi_{adv}$
14. Perturb $L_{B, \pi_{adv}}(\theta_2)$ with the noise $\frac{\chi_3}{m}$
15. Descend: $\theta_1 \leftarrow \theta_1 - \varphi_t \nabla \theta_1 L_{B=\pi_{adv}}(\theta_1); \theta_2 \leftarrow \theta_2 - \varphi_t \nabla \theta_2 L_{B=\pi_{adv}}(\theta_2)$
Output: ($\epsilon_1 + \epsilon_2$)-DP parameters $\theta = \{\theta_1, \theta_2\}$, robust model with an $\epsilon_r$ budget

Verified Inference: (an input $x$, attack size $\mu_a$)

17. Compute robustness size $(\kappa + \varphi)_{max}$ in Eq. 16 of $x$
18. if $(\kappa + \varphi)_{max} \geq \mu$ then
19. Return isRobust$(x) = True$, label $k$, $(\kappa + \varphi)_{max}$
20. else
21. Return isRobust$(x) = False$, label $k$, $(\kappa + \varphi)_{max}$
```

B Proof of Lemma 2

Inspired by the proof of Functional Mechanism [40], we have our proof as follows:

**Proof 1** Assume that $B$ and $B'$ differ in the last tuple, $x_m (x'_m)$. Then,

$$\Delta_R = \sum_{j=1}^{d} \left[ \left\| \sum_{x_i \in B} \frac{1}{2} h_i - \sum_{x'_i \in B'} \frac{1}{2} h'_i \right\|_1 + \left\| \sum_{x_i \in B} x_{ij} - \sum_{x'_i \in B'} x'_{ij} \right\|_1 \right]$$

$$\leq 2 \max_{x_i} \sum_{j=1}^{d} \left( \left\| \frac{1}{2} h_i \right\|_1 + \left\| x_{ij} \right\|_1 \right) \leq d(\beta + 2)$$

C Proof of Lemma 3

Inspired by the proof of Functional Mechanism [40], we have our proof as follows:

**Proof 2** Given $\chi_1$ drawn as a Laplace noise $[\text{Lap}(\Delta^a)]^d$ and $\chi_2$ drawn as a Laplace noise $[\text{Lap}(\Delta^a)]^\beta$, the perturbation of the coefficient $\phi \in \Phi = \{\frac{1}{2} h_i, x_i\}$, denoted as $\overline{\phi}$, can be rewritten
as follows:

for $\phi \in \{x_i\} : \bar{\phi} = \sum_{x_i \in B} (\phi_{x_i} + \frac{\chi_1}{m}) = \sum_{x_i \in B} \phi_{x_i} + \chi_1 = \sum_{x_i \in B} \phi_{x_i} + [\text{Lap}(\frac{\Delta_R}{\epsilon_1})]^d$

for $\phi \in \{\frac{1}{2} h_i\} : \bar{\phi} = \sum_{x_i \in B} \frac{1}{2} \left( h_i + \frac{\chi_2}{m} \right) = \sum_{x_i \in B} \left( \phi_{x_i} + \frac{\chi_2}{m} \right) = \sum_{x_i \in B} \phi_{x_i} + \chi_2 = \sum_{x_i \in B} \phi_{x_i} + [\text{Lap}(\frac{\Delta_R}{\epsilon_1})]^\beta$

we have

$$Pr(\mathcal{R}_B(\theta_1)) = \prod_{j=1}^{d} \prod_{\phi \in \Phi} \exp \left( - \frac{\epsilon_1 \|\sum_{x_i \in B} \phi_{x_i} - \bar{\phi}\|_1}{\Delta_R} \right)$$

$\Delta_R$ is set to $d(\beta + 2)$, we have that:

$$\frac{Pr(\mathcal{R}_B(\theta_1))}{Pr(\mathcal{R}_B'(\theta_1))} = \prod_{j=1}^{d} \prod_{\phi \in \Phi} \exp \left( \frac{\epsilon_1 \|\sum_{x_i \in B} \phi_{x_i} - \bar{\phi}\|_1}{\Delta_R} \right) \leq \prod_{j=1}^{d} \prod_{\phi \in \Phi} \exp \left( \frac{\epsilon_1}{\Delta_R} 2 \max_{x_i \in B} \|\phi_{x_i}\|_1 \right) \leq \exp(\frac{\epsilon_1 d(\beta + 2)}{\Delta_R})$$

Consequently, the computation of $\mathcal{R}_B(\theta_1)$ preserves $\epsilon_1$-DP in Alg. 2. In addition, the parameter optimization of $\mathcal{R}_B(\theta_1)$ only uses the perturbed data $\pi_1$ in the computations of $h_i$, $\pi_1$, and $\bar{\pi}_i$. Thus, the perturbed optimal parameters $\bar{\theta}_1$ is $\epsilon_1$-DP.

D Proof of Lemma 3

Proof 3 Regarding the computation of $h_{1B} = \{\pi_1^T \bar{\pi}_i\}_{x_i \in B}$, we can see that $h_i = \bar{\pi}_i^T \pi_i$ is a linear function of $x$. The sensitivity of a function $h$ is defined as the maximum change in output, that can be generated by a change in the input $[38]$. Therefore, the global sensitivity of $h_1$ can be computed as follows:

$$\Delta_{h_1} = \frac{\|\sum_{x_i \in B} \bar{\pi}_i \pi_i - \sum_{x_i' \in B'} \bar{\pi}_i' \pi_i'\|_1}{\|\sum_{x_i \in B} \bar{\pi}_i - \sum_{x_i' \in B'} \bar{\pi}_i'\|_1} \leq \max_{x_i \in B} \frac{\|\bar{\pi}_i\|_1}{\|\pi_i\|_1} \leq \|\bar{\pi}_{1,1}\|_1$$

following matrix norms $[44]$: $\|\bar{\pi}_i\|_{1,1}$ is the maximum 1-norm of $\bar{\theta}_1$'s columns. By injecting Laplace noise $\text{Lap}(\frac{\Delta_{h_1}}{\epsilon_1})$ into $h_{1B}$, i.e., $h_{1B} = \{\bar{\pi}_i^T \pi_i + \text{Lap}(\frac{\Delta_{h_1}}{\epsilon_1})\}_{x_i \in B}$, we can preserve $\epsilon_1$-DP in the computation of $h_{1B}$. Let us set $\Delta_{h_1} = \|\bar{\pi}_{1,1}\|_1$, $\gamma = \frac{\Delta_{h_1}}{m\Delta_{h_1}}$, and $\chi_2$ drawn as a Laplace noise $[\text{Lap}(\frac{\Delta_R}{\epsilon_1})]^\beta$, in our mechanism, the perturbed affine transformation $\bar{\pi}_{1B}$ is presented as:

$$\bar{h}_{1B} = \{\bar{\pi}_i^T \pi_i + \frac{\chi_2}{m}\}_{x_i \in B} = \{\bar{\pi}_i^T \pi_i + \frac{\chi_2}{m}[\text{Lap}(\frac{\Delta_R}{\epsilon_1})]^\beta\}_{x_i \in B}$$

$$= \{\bar{\pi}_i^T \pi_i + [\text{Lap}(\gamma \Delta_{h_1})]^\beta\}_{x_i \in B}$$

$$= \{\bar{\pi}_i^T \pi_i + [\text{Lap}(\frac{\Delta_{h_1}}{\epsilon_1})]^\beta\}_{x_i \in B}$$

This results in an $(\epsilon_1 \gamma)$-DP affine transformation $\bar{h}_{1B} = \{\bar{\pi}_i^T \pi_i + [\text{Lap}(\frac{\Delta_{h_1}}{\epsilon_1})]^\beta\}_{x_i \in B}$. Therefore, Lemma 3 does hold.
E Proof of Lemma 5

Proof 4 Assume that $\mathcal{B}$ and $\mathcal{B}'$ differ in the last tuple, and $\mathcal{B}_m$ ($\mathcal{B}_m'$) be the last tuple in $\mathcal{B}$ ($\mathcal{B}'$), we have that

$$\Delta_{\mathcal{L}2} = \sum_{k=1}^{K} \left\| \sum_{\pi_i \in \mathcal{B}} (h_{\pi iy_{ik}}) - \sum_{\pi_i' \in \mathcal{B}'} (h'_{\pi iy'_{ik}}) \right\|_1$$

$$= \sum_{k=1}^{K} \left\| h_{\pi my_{mk}} - h'_{\pi my'_{mk}} \right\|_1$$

Since $y_{mk}$ and $y'_{mk}$ are one-hot encoding, we have that $\Delta_{\mathcal{L}2} \leq 2 \max_{\pi_i} ||h_{\pi iy}||_1$. Given $h_{\pi iy} \in [-1, 1]$, we have

$$\Delta_{\mathcal{L}2} \leq 2|h_{\pi iy}|$$

Lemma 5 does hold.

F Proof of Lemma 6

Proof 5 Let $\mathcal{B}$ and $\mathcal{B}'$ be neighboring batches of benign examples, and $\chi_3$ drawn as Laplace noise $\text{Lap}(\frac{\Delta_{\mathcal{L}2}}{\epsilon_2})$, the perturbations of the coefficients $h_{\pi iy} \in [h_{\pi iy}]$ can be rewritten as:

$$\bar{h}_{\pi iy} = \sum_{\pi_i} (h_{\pi iy} + \frac{\chi_3}{m}) = \sum_{\pi_i} (h_{\pi iy} + \text{Lap}(\frac{\Delta_{\mathcal{L}2}}{\epsilon_2}))$$

Since all the coefficients are perturbed, and given $\Delta_{\mathcal{L}2} = 2|h_{\pi iy}|$, we have that

$$\frac{Pr(\mathcal{L}_\mathcal{B}'(\theta_2))}{Pr(\mathcal{L}_\mathcal{B}(\theta_2))} = \frac{Pr(\mathcal{L}_\mathcal{B}'(\theta_2))}{Pr(\mathcal{L}_\mathcal{B}(\theta_2))} \times \frac{Pr(\mathcal{L}_\mathcal{B}'(\theta_2))}{Pr(\mathcal{L}_\mathcal{B}(\theta_2))}$$

$$\leq e^{\gamma/\gamma} \sum_{k=1}^{K} \exp\left(\frac{-\epsilon_2 \| h_{\pi iy_{ik}} - \bar{h}_{\pi iy_{ik}} \|_1}{\Delta_{\mathcal{L}2}}\right)$$

$$\leq e^{\gamma/\gamma} \sum_{k=1}^{K} \exp\left(\frac{-\epsilon_2 \| h_{\pi iy_{ik}} - \sum_{\pi_i'} h'_{\pi iy'_{ik}} \|_1}{\Delta_{\mathcal{L}2}}\right)$$

$$\Delta_{\mathcal{L}2} \leq \sum_{\pi_i} (h_{\pi iy} + \frac{\chi_3}{m}) = \sum_{\pi_i} (h_{\pi iy} + \text{Lap}(\frac{\Delta_{\mathcal{L}2}}{\epsilon_2}))$$

The computation of $\mathcal{L}_\mathcal{B}'(\theta_2)$ preserves $(\epsilon_1/\gamma + \epsilon_2)$-differential privacy. The optimization of $\mathcal{L}_\mathcal{B}'(\theta_2)$ does not access additional information from the original input $x_i \in B$. Consequently, the optimal perturbed parameters $\bar{\theta}_2$ derived from $\mathcal{L}_\mathcal{B}'(\theta_2)$ are $(\epsilon_1/\gamma + \epsilon_2)$-DP.

G Proof of Lemma 7

Proof 6 Thanks to the sequential composition theory in DP [17], $f(M_1, \ldots, M_S|x)$ is $(\sum_s \epsilon_s)$-DP, since for any $O = \prod_{s=1}^{S} s \in \prod_{s=1}^{S} f^s(x)(\in \mathbb{R}^K)$, we have that

$$P(f(M_1, \ldots, M_S|x) = O) = \frac{P(M_1 f(x) = a_1) \ldots P(M_S f(x) = a_S)}{P(f(M_1, \ldots, M_S|x + \alpha) = O)} \leq \prod_{s=1}^{S} \exp(\epsilon_s) = e^{(\sum_{s=1}^{S} \epsilon_s)}$$

As a result, we have

$$P(f(M_1, \ldots, M_S|x) \leq e^{(\sum_s \epsilon_s)} P(f(M_1, \ldots, M_S|x + \alpha))$$
The sequential composition of the expected output is as:

\[
\mathbb{E}f(M_1, \ldots, M_S|x) = \int_0^1 P(f(M_1, \ldots, M_S|x > t) dt
\]

\[
\leq e^{\sum_i \epsilon_i} \int_0^1 P(f(M_1, \ldots, M_S|x + \alpha > t) dt
\]

\[
= e^{\sum_i \epsilon_i} \mathbb{E}f(M_1, \ldots, M_S|x + \alpha)
\]

Lemma 7 does hold.

H Proof of Theorem 1

Proof 7 \(\forall \alpha \in l_p(1)\), from Lemma 7 with probability \(\geq \eta\), we have that

\[
\hat{E}f_k(M_1, \ldots, M_S|x + \alpha) \geq \frac{\hat{E}f_k(M_1, \ldots, M_S|x)}{e^{\sum_i \epsilon_i}} 
\]

\[
\geq \frac{\hat{E}b_i(M_1, \ldots, M_S|x)}{e^{\sum_s \epsilon_s}} \tag{18}
\]

In addition, we also have

\(\forall i \neq k : \hat{E}f_i(M_1, \ldots, M_S|x + \alpha) \leq e^{\sum_s \epsilon_s} \hat{E}f_i(M_1, \ldots, M_S|x)\)

\[
\Rightarrow \forall i \neq k : \hat{E}f_i(M_1, \ldots, M_S|x + \alpha) \leq e^{\sum_i \epsilon_i} \max_{i: i \neq k} \hat{E}b_i(M_1, \ldots, M_S|x) \tag{19}
\]

Using the hypothesis (Eq. 13) and the first inequality (Eq. 18), we have that

\[\hat{E}f_k(M_1, \ldots, M_S|x + \alpha) > \frac{e^{2(kr + \phi)} \max_{i: i \neq k} \hat{E}b_i(M_1, \ldots, M_S|x)}{e^{\sum_i \epsilon_i}} \]

\[
> e^{\sum_s \epsilon_s} \max_{i: i \neq k} \hat{E}b_i(M_1, \ldots, M_S|x)
\]

Now, we apply the third inequality (Eq. 19), we have that

\(\forall i \neq k : \hat{E}f_k(M_1, \ldots, M_S|x + \alpha) > \hat{E}f_i(M_1, \ldots, M_S|x + \alpha)\)

\[
\Leftrightarrow \hat{E}f_k(M_1, \ldots, M_S|x + \alpha) > \max_{i: i \neq k} \hat{E}f_i(M_1, \ldots, M_S|x + \alpha)
\]

The Theorem 1 does hold.

I Proof of Proposition 1

Proof 8 \(\forall \alpha \in l_p(1)\), by applying Theorem 1, we have

\[
\hat{E}b_i(M_h, M_x|x) > e^{2(kr + \phi)} \max_{i: i \neq k} \hat{E}b_i(M_h, M_x|x)
\]

\[
> e^{2(kr + \phi)} \hat{E}b_i(M_h, M_x|x)
\]

Furthermore, by applying group privacy, we have that

\[
\forall \alpha \in l_p(\kappa + \phi) : \hat{E}b_i(M_h, M_x|x) > e^{2\kappa} \max_{i: i \neq k} \hat{E}b_i(M_h, M_x|x)
\]

By applying Proof 7, it is straight to have

\[
\forall \alpha \in l_p(\kappa + \phi) : \hat{E}f_k(M_h, M_x|x + \alpha) > \max_{i: i \neq k} \hat{E}f_k(M_h, M_x|x + \alpha)
\]

with probability \(\geq \eta\). Proposition 7 does hold.
J Monte Carlo Estimation of $\hat{E} f(x)$

Recall that the Monte Carlo estimation is applied to estimate the expected value $\hat{E} f(x) = \frac{1}{n} \sum_{i=1}^{n} f(x)_i$, where $n$ is the number of invocations of $f(x)$ with independent draws in the noise, i.e., $\frac{1}{m} Lap(0, \frac{\Delta \epsilon}{\epsilon_1})$ and $\frac{2}{m} Lap(0, \frac{\Delta \epsilon}{\epsilon_1})$ in our case. When $\epsilon_1$ is small (indicating a strong privacy protection), it causes a notably large distribution shift among independent draws of the Laplace noise.

In addition, let us denote a single draw in the noise as $\chi_1 = \frac{1}{m} Lap(0, \frac{\Delta \epsilon}{\epsilon_1})$ used to train the scoring function $f(x)$, the model converges to the point that the noise $\chi_1$ and $2\chi_1$ need to be correspondingly added into $x$ and $h$ in order to make correct predictions. $\chi_1$ can be approximated as $Lap(\chi_1, \varphi)$, where $\varphi \rightarrow 0$. It is clear to see that independent draws of the noise $\frac{1}{m} Lap(0, \frac{\Delta \epsilon}{\epsilon_1})$ have distribution shifts with the fixed noise $\chi_1 \approx Lap(\chi_1, \varphi)$. These distribution shifts can also be large, when noise is large. We have experienced that these distribution shifts in having independent draws of noise to estimate $\hat{E} f(x)$ can notably degrade the inference accuracy of the scoring function, when privacy budget $\epsilon_1$ is small resulting in a large amount of noise injected to provide strong privacy guarantees.

To address this problem, one solution is to increase the number of invocations of $f(x)$, i.e., $n$, to a huge number per prediction. This is impractical in real-world scenarios. We propose a novel way to draw independent noise following the distribution of $\chi_1 + \frac{1}{m} Lap(0, \frac{\Delta \epsilon}{\epsilon_1}/\psi)$ for the input $x$ and $2\chi_1 + \frac{2}{m} Lap(0, \frac{\Delta \epsilon}{\epsilon_1}/\psi)$ for the affine transformation $h$, where $\psi$ is a hyper-parameter to control the distribution shifts. This approach works well and does not affect the DP bounds and the provable robustness condition, since: (1) Our mechanism achieves both DP and provable robustness in the training process; and (2) It is clear that $\hat{E} f(x) = \frac{1}{n} \sum_{i=1}^{n} f(x)_i = \frac{1}{n} \sum_{i=1}^{n} g(a(x + \chi_1 + \frac{1}{m} Lap_n(0, \frac{\Delta \epsilon}{\epsilon_1}/\psi), \theta_1) + 2\chi_1 + \frac{2}{m} Lap_n(0, \frac{\Delta \epsilon}{\epsilon_1}/\psi), \theta_2)$, where $Lap_n(0, \frac{\Delta \epsilon}{\epsilon_1}/\psi)$ is the $n$-th draw of the noise. When $n \rightarrow \infty$, $\hat{E} f(x)$ will converge to $\frac{1}{n} \sum_{i=1}^{n} g(a(x + \chi_1, \theta_1) + 2\chi_1, \theta_2)$, which aligns well with the convergence point of the scoring function $f(x)$. Injecting $\chi_1$ and $2\chi_1$ to $x$ and $h$ during the estimation of $\hat{E} f(x)$ yields better performance, without affecting the DP bounds and the provable robustness condition.

K Supplemental Experimental Results

K.1 Model Configuration

The MNIST database of handwritten digits [18]. Each example is a 28 × 28 size gray-level image. The CIFAR-10 dataset consists of color images belonging to 10 classes, i.e., airplanes, dogs, etc. The dataset is splitted into 50,000 training samples and 10,000 test samples [19].

**MNIST:** We used two convolution layers (32 and 64 features). Each hidden neuron connects with a 5x5 unit patch. A fully-connected layer has 256 units. The batch size $m$ was set to 2,400, $\xi = 1.5$, $\psi = 2$. I-FGSM, MIM, and MadryEtAl were used to draft $l_\infty(\mu)$ adversarial examples in training, with $T_\mu = 10$. **CIFAR-10:** We used three convolution layers (128, 128, and 256 features). Each hidden neuron connects with a 3x3 unit patch in the first layer, and a 5x5 unit patch in other layers. One fully-connected layer has 256 neurons. The batch size $m$ was set to 1,800, $\xi = 1.5$, $\psi = 10$, and $T_\mu = 3$. The ensemble of attack algorithms $A$ includes I-FGSM, MIM, and MadryEtAl. We do use data augmentation, including random crop, random flip, and random constrast. The implementation of our mechanism is available in TensorFlow. The experiments were conducted on a single GPU, i.e., NVIDIA GTX TITAN X, 12 GB with 3,072 CUDA cores.

K.2 Complete Experimental Results
Figure 4: Conventional accuracy on the MNIST dataset, under $l_\infty(\mu_a = 0.1)$.

Figure 5: Conventional accuracy on the MNIST dataset, under $l_\infty(\mu_a = 0.2)$. 
Figure 6: Conventional accuracy on the CIFAR-10 dataset, given $l_\infty (\mu_a = 0.1)$.

Figure 7: Conventional accuracy on the CIFAR-10 dataset, given $l_\infty (\mu_a = 0.2)$. 
Figure 8: Certified accuracy on the MNIST dataset. The privacy budget is set to 2, offering a reasonable privacy protection.

Figure 9: Certified accuracy on the CIFAR-10 dataset. The privacy budget is set to 4, offering a reasonable privacy protection.