ANALYTICITY OF POSITIVE SEMIGROUPS IS INHERITED UNDER DOMINATION

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Abstract. For positive $C_0$-semigroups $S$ and $T$ on a Banach lattice such that $S(t) \leq T(t)$ for all times $t$, we prove that analyticity of $T$ implies analyticity of $S$. This answers an open problem posed by Arendt in 2004.

Our proof is based on a spectral theoretic argument: we apply Perron–Frobenius theory to multiplication operators that are induced by $S$ and $T$ on a vector-valued function space.

In a problem posed in [9], Arendt asked whether analyticity of positive semigroups is inherited under domination. The purpose of this note is to give a positive answer to this question:

Theorem 1. Let $S, T$ be $C_0$-semigroups on a complex Banach lattice $E$ such that $0 \leq S(t) \leq T(t)$ for all $t \in [0, \infty)$. If $T$ is analytic, then so is $S$.

Throughout, we freely use the theory of Banach lattices (see e.g. [11]) and of $C_0$-semigroups (see e.g. [3]). Standard references for positive semigroups are [1, 2], and domination between semigroups is discussed in detail in [11 Section C-II.4].

We prove Theorem [11] at the end of the article, before Remark [11]. The proof is surprisingly easy if one employs two appropriate results from the literature and reformulates one of them in a slick way. We begin by stating those results in the following two propositions.

The first proposition is (a special case of) a result by Kato [7]; a simplified version of the proof of “(ii) $\Rightarrow$ (i)” can be found in [4, Lemma 2.1].

Proposition 2 (Kato). For a $C_0$-semigroup $T$ on a complex Banach space $X$ the following assertions are equivalent:

(i) The semigroup $T$ is analytic.

(ii) There exists a time $t_0 > 0$ and a complex number $\lambda$ of modulus 1 with the following property: there is a constant $K > 0$ such that

\[ \lambda \not\in \sigma(T(t)) \quad \text{and} \quad \| R(\lambda, T(t)) \| \leq K \]

for all $t \in (0, t_0]$.

Here, $\sigma(T(t))$ denotes the spectrum of $T(t)$ and $R(\lambda, T(t)) := (\lambda - T(t))^{-1}$ the resolvent of $T(t)$ at $\lambda$.

The second proposition is from the realm of Perro–Frobenius theory (i.e., spectral theory of positive operators). It was proved (under slightly more general conditions) by Räbiger and Wolff [10, Theorem 1.4]; their proof employs techniques which were introduced earlier by Lotz [8].

Proposition 3 (Räbiger–Wolff). Let $0 \leq S \leq T$ be bounded linear operators on a complex Banach lattice $E$ and assume that $T$ is power bounded, i.e. $\sup_{n \in \mathbb{N}_0} \| T^n \| < \infty$. If $\lambda \in \sigma(S)$ and $|\lambda| = 1$, then also $\lambda \in \sigma(T)$.

Date: May 2, 2022.

2020 Mathematics Subject Classification. 47D06; 47B65; 47A10.

Key words and phrases. Domination; positive semigroup; analytic semigroup; holomorphic semigroup; Perron–Frobenius theory; spectrum of positive operators.
In an attempt to prove Theorem \[1\] it seems natural to first try the following approach:

There is no loss of generality in assuming that the semigroup \(T\) is bounded, i.e. that we have \(\sup_{t \in [0, \infty)} \|T(t)\| < \infty\). Now let \(T\) be analytic and let \(t_0\) and \(\lambda\) be as in Proposition \[2\] ii). For each \(t \in (0, t_0]\) we then have \(\lambda \not\in \sigma(T(t))\). Since, by the boundedness of \(T\), \(T(t)\) is power-bounded, and \(0 \leq S(t) \leq T(t)\), it follows from Proposition \[3\] that \(\lambda \not\in \sigma(S(t))\). So the only difficulty is to show that the resolvent \(\mathcal{R}(\lambda, S(t))\) is uniformly bounded as \(t\) runs through \((0, t_0]\).

We may even assume that \(\|T(t)\| \to 0\) as \(t \to \infty\), which means that the spectral radius of \(T(t)\) is strictly less than 1 for each \(t > 0\). Then the Neumann series representation of the resolvent immediately yields the estimate

\[
|\mathcal{R}(\lambda, S(t))x| \leq \mathcal{R}(1, T(t)) \|x\|
\]

for all \(x \in E\). However, this does not give the desired boundedness since the spectral radius of \(T(t)\) will be close to 1 for small \(t\), and thus \(\mathcal{R}(1, T(t))\) cannot be expected to be bounded as \(t \downarrow 0\).

To solve this problem, we show now that property (ii) in Kato’s characterisation can be rephrased as a spectral property of a single operator that acts on a vector-valued function space. This reformulation is based on the following simple lemma for general families of operators.

**Lemma 4.** Let \(X\) be a (real or complex) Banach space, let \(I\) be a non-empty set and let \(T = (T_i)_{i \in I}\) be a norm bounded family of bounded linear operators on \(X\). Consider the operator

\[
\hat{T} : \ell^\infty(I; X) \to \ell^\infty(I; X)
\]

given by

\[
(\hat{T}f)(i) = T_i f(i)
\]

for each \(f \in \ell^\infty(I; X)\) and \(i \in I\).

Then \(\hat{T}\) is bijective if and only if each of the operators \(T_i\) is bijective and \(\sup_{i \in I} \|T_i^{-1}\| < \infty\).

This is certainly well-known among experts in operator theory, and related results can be found in different places in the literature, for instance in \[4\] Section 2 and \[5\] Section 2. We include the proof to demonstrate that it is particularly simple in the situation of Lemma \[4\].

**Proof of Lemma 4**

“\(\Rightarrow\)” This implication is obvious.

“\(\Leftarrow\)” Assume that \(\hat{T}\) is bijective. For every \(f \in \ell^\infty(I; X)\) and every \(i \in I\) we then have

\[
(\hat{T}^{-1} f)(i) = T_i^{-1} f(i)
\]

By substituting functions for \(f\) which are 0 at each but one position, we can thus see that every operator \(T_i\) is surjective. Similarly, we have \(f = T_i^{-1} f\) for each \(f \in \ell^\infty(I; X)\), and by again substituting vectors for \(f\) which are 0 at each but one position, we can see that every operator \(T_i\) is injective.

Since we now know that each \(T_i\) is bijective, Equation \[1\] implies

\[
T_i^{-1}(f(i)) = (\hat{T}^{-1} f)(i)
\]

for each \(f \in \ell^\infty(I; X)\) and each \(i \in I\). We once again substitute vectors for \(f\) that are 0 at each but one position, and thus see that \(\|T_i^{-1}\| \leq \|\hat{T}^{-1}\|\) for each \(i \in I\). \(\square\)
As a direct consequence of the lemma, we can reformulate Kato’s characterisation from Proposition 2. We use the following notation: for a \( C_0 \)-semigroup \( T \) on a Banach space \( X \) and for \( t_0 > 0 \) we define a bounded linear operator
\[
\hat{T}_{t_0} : \ell^\infty((0, t_0]; X) \to \ell^\infty((0, t_0]; X)
\]
by
\[
(\hat{T}_{t_0}f)(t) = T(t)f(t)
\]
for each \( f \in \ell^\infty((0, t_0]; X) \) and each \( t \in (0, t_0] \). With this notation, we now obtain immediately:

**Corollary 5.** For a \( C_0 \)-semigroup \( T \) on a complex Banach space \( X \) the following assertions are equivalent:

(i) The semigroup \( T \) is analytic.

(ii) There exists a time \( t_0 > 0 \) and a complex number \( \lambda \) of modulus 1 such that \( \lambda \not\in \sigma(\hat{T}_{t_0}) \).

With this formulation of Kato’s characterisation, the proof of our main result is very easy:

**Proof of Theorem 1.** After rescaling \( S \) and \( T \) we may, and shall, assume that \( T \) is bounded. Since \( T \) is analytic there exists, by Corollary 5, a time \( t_0 > 0 \) and a complex number \( \lambda \) of modulus 1 such that \( \lambda \not\in \sigma(\hat{T}_{t_0}) \), where we use the notation \( \hat{T}_{t_0} \) introduced before Corollary 5.

We use the same notation for \( S \) and thus have \( 0 \leq \hat{S}_{t_0} \leq \hat{T}_{t_0} \) (where \( \ell^\infty(I; E) \) is ordered pointwise). Since \( \hat{T}_{t_0} \) is power bounded (due to the boundedness of the semigroup \( T \)), Proposition 4 shows that \( \lambda \not\in \sigma(\hat{S}_{t_0}) \). Hence, again by Corollary 5 the semigroup \( S \) is analytic. \( \square \)

**Remark 6.** Theorem 1 does not remain true if we consider domination of non-positive semigroups \( S \), i.e., if we only assume
\[
(2) \quad |S(t)x| \leq T(t) |x|
\]
for all \( x \in E \) and \( t \in [0, \infty) \) instead of \( 0 \leq S(t) \leq T(t) \).

As a simple counterexample, let \( p \in [1, \infty) \) and consider the semigroup \( S \) on \( \ell^p \) given by
\[
(S(t)f)(n) = e^{i\pi n} f(n)
\]
for all \( f \in \ell^p \) and \( n \in \mathbb{N} \). This semigroup is clearly not analytic, but it satisfies the domination condition (2) for \( T(t) = \text{id} \).

However, it seems that this example cannot be directly adapted to obtain a counterexample over the real field. More generally speaking, the author does not know whether analyticity of a positive semigroup \( T \) together with (2) implies analyticity of \( S \) if \( S \) is not positive but leaves the real part of the underlying Banach lattice invariant.

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