Excitonic insulator [1–4], predicted in the 1960s, is an unconventional insulating state formed by the condensation of excitons (bound electron-hole pairs) stemming from the Coulomb interaction between electrons and holes in the conduction and valence bands, respectively. An excitonic insulator is analogous to a Bardeen-Cooper-Schrieffer superconductor that results from the condensation of electronic excitons (bound electron-hole pairs) stemming from the Coulomb interaction between electrons and holes in the conduction and valence bands, respectively. An excitonic insulator is analogous to a Bardeen-Cooper-Schrieffer superconductor that results from the condensation of electron Cooper pairs developed near the Fermi surface. Candidate materials for excitonic insulators previously studied include quantum well bilayers [5], quantum Hall bilayers [6, 7], Ta$_2$NiSe$_5$ [8–13], and 1T-TiSe$_2$ [14–17]. After the discovery of topological insulators, excitonic insulators with topologically non-trivial properties have been investigated intensively [18–33]. Particularly, a few representative topological exciton condensates, including the time-reversal invariant $s$-wave topological exciton condensate and the time-reversal breaking topological exciton condensate with $p$-wave pairing, have been predicted in HgTe/CdTe [23] and InAs/GaSb [24] quantum wells, respectively. Importantly, the evidence for the existence of a topological excitonic insulator state has been reported experimentally in InAs/GaSb quantum wells [34, 35].

Recently, the concept of topological insulators was generalized, and a novel topological phase of matter dubbed higher-order topological insulator [36–42] was established. Compared to the well-known topological insulators, higher-order topological insulators exhibit an unusual form of bulk-boundary correspondence. For instance, a second-order topological insulator in two dimensions exhibits topological gapless boundary states at its zero-dimensional boundary corners, in contrast to a conventional two-dimensional (2D) first-order topological insulator which features topologically protected gapless states at its one-dimensional edge. So far, higher-order topology has been explored in various physical systems [43–94], and particularly Majorana zero-energy corner modes [64–66] were found in superconducting systems. Yet, the topological exciton condensates known to date belong to the first-order topological phases, and the study of higher-order topology in exciton condensates is still lacking. Given the similarity of excitonic insulators to superconductors, it is highly desirable to explore higher-order topology of exciton condensates in a realistic system.

In this work, we demonstrate that topological corner states, which are considered as a smoking-gun signature for 2D second-order topological insulators, can be realized in the bilayer quantum spin Hall insulators with $s$-wave interlayer excitonic pairings. The bilayer quantum spin Hall insulators can be constructed by two coupled HgTe/CdTe quantum wells [95] shown in Fig. 1, where the Dirac mass can be tuned by varying the thickness of the central HgTe layers in the quantum wells. For the bilayer system with the negative Dirac mass, four topological excitonic corner states (ECs) are generated by applying an in-plane Zeeman field $h_z$. Electrons and holes (marked by blue and red spheres) residing on two opposing layers are spatially separated by an insulator layer. The excitonic gap $\Delta_X$ couples electrons with holes to form an excitonic insulator through the interlayer Coulomb interaction.

FIG. 1. Schematic illustration of the two coupled HgTe quantum wells with gating (electric bias $V$) under the applied in-plane Zeeman field $h_z$. Electrons and holes (marked by blue and red spheres) residing on two opposing layers are spatially separated by an insulator layer. The excitonic gap $\Delta_X$ couples electrons with holes to form an excitonic insulator through the interlayer Coulomb interaction.
achieved in the bilayer quantum spin Hall insulators by tuning the bias voltage and in-plane Zeeman field. The low-energy effective Hamiltonian of the gated bilayer quantum spin Hall insulators with an applied Zeeman field in the momentum space is given by

\[
H_{\text{QSH}}(k) = M(k)\sigma_z + A(k_{x}\sigma_x s_z + k_{y}\sigma_y) - \frac{V}{2}\tau_z + \mathbf{h} \cdot \mathbf{s},
\]

where the basis is \(\{\downarrow, \uparrow\} = \{\sigma_{d\alpha}\sigma_z \sigma_{d\beta}, \sigma_{d\beta}\sigma_z \sigma_{d\alpha}\}\), \(\alpha\) and \(\beta\) are different orbital degrees of freedom with opposite parity, \(\uparrow\) and \(\downarrow\) represent electron spin, and \(l = 1, 2\) is the layer index. \(s_{x,y,z}, \sigma_{x,y,z}\) and \(\tau_{x,y,z}\) are the Pauli matrices acting on the spin, orbital and layer degrees of freedom, respectively. \(\tau_0, \sigma_0\) and \(s_0\) are the \(2 \times 2\) identity matrices. \(M(k) = M - B(k_x^2 + k_y^2)\), where the Dirac mass parameter \(M\) determines the topological insulator phase, and \(V\) is the bias potential. \(h\) denotes the applied in-plane Zeeman field. When \(V = 0\), this Hamiltonian is exactly two copies of the Bernevig-Hughes-Zhang (BHZ) model [96] that describes the HgTe/CdTe quantum wells. The topologically nontrivial phase of the BHZ model on a square lattice exists when \(0 < M/(2B/a^2) < 2\) with the lattice constant \(a\). We have assumed the spatial separation between these two layers to be sufficiently large so that the single-particle tunneling between layers can be neglected. Note that, the model parameters are dependent on the thickness of the quantum wells. In subsequent calculations, the following parameters remain unchanged. \(A = 275\ \text{meV} \cdot \text{nm}\), \(B = -1300\ \text{meV} \cdot \text{nm}^2\), and the lattice constant is \(a = 20\ \text{nm}\) [23]. The results remain valid when the parameters vary. For our purpose, we set the Dirac mass \(M = -3\ \text{meV}\) in this section. In this case, each layer has inverted bands and contributes a Kramers pair of helical gapless edge states as shown in Fig. 2(a).

By turning on the electric bias \(V\), an electron Fermi surface and a hole Fermi surface are created on layer 1 and layer 2, respectively. Coherent exciton condensation can be induced by the interlayer Coulomb interaction. Throughout this paper, we focus on the time-reversal invariant s-wave exciton pairings which should be the leading order from the mean-field decomposition of the screened interlayer Coulomb interaction. In general, the s-wave excitonic order parameters could provide an energy gain to the system, which can be expressed as \(H_X = \Delta_X \tau_j \sigma_l s_k\) with \(\Delta_X > 0\) the pairing strength (The sign change in \(\Delta_X\) doesn’t affect the following results) and the subscripts \(i, j, k = 0, x, y, z\). The excitonic order parameters are considered to be momentum independent thanks to the short-range interaction. There are four relevant interlayer excitonic order parameters preserving time-reversal symmetry, which are proportional to \(\tau_0 \sigma_x s_0, \tau_0 \sigma_y s_0, \tau_0 \sigma_z s_0\), and \(\tau_0 \sigma_x s_y\) [23]. We present more details of these four excitonic order parameters in Ref. [97]. Among these order parameters, it was found that the \(\tau_0 \sigma_x s_0\) and \(\tau_0 \sigma_y s_z\) type orders can open a topological energy gap in the semimetallic bilayer HgTe/CdTe quantum wells, i.e., \(M = 0\), resulting in a helical topological excitonic insulator characterized by the \(\mathbb{Z}_2\) topological invariant, while the \(\tau_0 \sigma_x s_y\) and \(\tau_0 \sigma_y s_x\) type pairings only lead to a topologically trivial energy gap in the case of \(M = 0\) [23].

Next, we consider the case of \(M < 0\) with the \(\tau_0 \sigma_x s_y\) type interlayer excitonic order. The \(\tau_0 \sigma_x s_y\) order will give rise to the similar results, which is not discussed here. In the presence of the \(\tau_0 \sigma_x s_y\) type order, the two pairs of helical edge states are not stable and gapped out as depicted in Fig. 2(b). When an in-plane Zeeman field \(h_x\) is applied along the \(x\)-direction, we can see that the quasiparticle edge gap along the \(k_z\)-direction closes at the critical field \(h_x^c\) as shown in Fig. 2(c) and reopens as \(h_x\) increases [see Fig. 2(d)]. Whereas, during this process, we verified that the edge gap along the \(k_y\)-direction doesn’t show the closing-and-reopening behavior but only has a slight change in its amplitude. When \(h_x > h_x^c\), two distinct types of edge gaps are formed along the \(x\) and \(y\) directions, then we calculate the energy spectrum for a finite-sized square sample as shown in Fig. 2(e). We observe four zero-energy boundary-obstructed midgap states, which are located at the four corners of the square sample by measuring the probability density [as shown in Fig. 2(f)].

Next, we discuss the above observed midgap states are actually the topologically protected ECs. To unveil their topological property, we calculate the edge polarization by using the Wilson loop operators [38, 41, 50]. For a ribbon geometry with \(N_x\) unit cells in the \(y\)-direction and \(N_{\text{orb}}\) degrees of freedom per unit cell, we express the Wilson loop operator \(W_{x,k_x}\) on a path along the \(k_x\)-direction as \(W_{x,k_x} = \prod_{k_x} \cdots [F_{x,k_x + \Delta k_x} F_{x,k_x}]^m = \left\langle u_{k_x + \Delta k_x}^m | u_{k_x}^m \right\rangle \) with the step \(\Delta k_x = 2\pi / N_x\), and \(| u_{k_x}^m \rangle \) denotes the occupied Bloch functions with \(n = 1, \ldots, N_{\text{occ}}\). \(N_{\text{occ}} = N_{\text{orb}} N_y / 2\) is the number of occupied bands. The Wilson loop operator \(W_{x,k_x}\) sat-
isfies the following eigenvalue equation

$$W_{x,k_x} \nu_{x,k_x}^j = e^{i2\pi v_x^j} \nu_{x,k_x}^j,$$

where $j = 1, ..., N_{\text{occ}}$. We can define the Wannier Hamiltonian $H_{W_x}(k_x)$ as $W_{x,k_x} = e^{iH_{W_x}(k_x)}$, of which the eigenvalues $2\pi v_x^j$ correspond to the Wannier spectrum. The tangential polarization as a function of $R_y$ is given as $[38, 41, 50]$

$$p_x(R_y) = \sum_{n=1}^{N_{\text{occ}} \times N_y} \rho^j(R_y)\nu_x^j,$$

where $\rho^j(R_y) = \frac{i}{N_y} \sum_{k_x,\alpha, n} |u_{k_x}^n|^{R_y,\alpha}[\nu_x^{j,n}]^2$ is the probability density. $|u_{k_x}^n|^{R_y,\alpha}$ with $\alpha = 1, ..., N_{\text{orb}}, R_y = 1, ..., N_y$ represents the components of the occupied states, and $[\nu_x^{j,n}]^n$ is the $n$-th component of $|\nu_x^{j,n}=1\rangle$. The edge polarization at the $y$-normal edge is defined by $p_x^{\text{edge,y}} = \sum_{R_y=1}^{N_y} p_x(R_y)$. To fix the sign of the polarization, we add a perturbation term $\delta\tau_y \sigma_x y$ in our calculations. Similarly, we can derive $\nu_y$ and $p_y^{\text{edge,x}}$. We plot the Wannier spectra $\nu_x$ and $\nu_y$ as a function of $h_x$ in Figs. 3(a) and 3(b), respectively. When the in-plane Zeeman field is greater than the critical field $h_X$, the ECs appear. Correspondingly, the Wannier spectrum $\nu_x$ has a pair of values pinned at $1/2$, resulting in half quantized edge polarization $p_x^{\text{edge,y}}$ shown in Fig. 3(c). In contrast, $p_y^{\text{edge,x}}$ remains vanishing even for $h_x > h_X$. It indicates that the ECs originate from the quantized edge polarization $p_x^{\text{edge,y}}$.

**Edge theory.**—To provide an intuitive picture to the appearance of ECs, we construct the edge theory [64] to analyze the topological mass on each edge. For simplicity, we focus on the $V = 0$ case since the bias has no contribution to the formation of edge mass. The low-energy Hamiltonian of the exciton condensate around the $\Gamma$ point reads

$$H(k) = A(k_x \sigma_x s_z + k_y \sigma_y) + [M - B(k_x^2 + k_y^2)] \sigma_z + \Delta \tau_y \sigma_x s_x + h_x s_x.$$  

(4)

We firstly consider a semi-infinite geometry occupying the space $x \geq 0$ for edge I as marked in Fig. 2(f). In the spirit of $\mathbf{k} \cdot \mathbf{p}$ theory, we replace $k_x \rightarrow -i\partial_x$ and divide the Hamiltonian into $H = H_0(-i\partial_x) + H_p(k_y)$, in which

$$H_0(-i\partial_x) = -iA \sigma_x s_z \partial_x + (M + B \partial_x^2) \sigma_z,$$  

$$H_p(k_y) = Ak_y \sigma_y + \Delta \tau_y \sigma_x s_x.$$  

(5)

where all the $k_z^2$-terms have been omitted, and $h_x = 0$ for edge I. So we can solve $H_0$ first, and regard $H_p$ as a perturbation, which is justified when the exciton gap is small comparing to the energy gap. The eigenvalue equation $H_0|\psi_x^j(x)\rangle = E_\alpha|\psi_x^j(x)\rangle$ can be solved under the boundary condition $\psi_x^j(0) = \psi_x^j(+\infty) = 0$. A straightforward calculation gives four degenerate solutions with $E_\alpha = 0$, whose eigenstates can be written in the following form

$$\psi_\alpha(x) = N_x \sin(\kappa_x x) e^{-\kappa_x x} e^{ik_y y} \chi_\alpha,$$  

(6)

where $\alpha = 1, ..., 4$, and the normalization constant $N_x = 2\sqrt{\kappa_x/\kappa_1}$ with $\kappa_1 = \sqrt{(4BM - A^2)/4B^2}$ and $\kappa_2 = A/2B$. The eigenvectors $\chi_\alpha$ are determined by $\sigma_y s_z \chi_\alpha = -\chi_\alpha$. Here we choose

$$\chi_1 = |\sigma_y = -1\rangle \otimes |\uparrow\rangle \otimes |\tau_z = +1\rangle,$$  

$$\chi_2 = |\sigma_y = +1\rangle \otimes |\downarrow\rangle \otimes |\tau_z = +1\rangle,$$  

$$\chi_3 = |\sigma_y = -1\rangle \otimes |\uparrow\rangle \otimes |\tau_z = -1\rangle,$$  

$$\chi_4 = |\sigma_y = +1\rangle \otimes |\downarrow\rangle \otimes |\tau_z = -1\rangle.$$  

(7)

In this basis set, the matrix elements of the perturbation $H_p(k_y)$ are represented as

$$H_{1,\alpha\beta} = \int_0^{\infty} dx \psi_\alpha^\dagger(x) H_p(k_y) \psi\beta(x),$$  

(8)

which can be written in a more compact form

$$H_1 = -A k_y s_z - \Delta \tau_y s_x.$$  

(9)

FIG. 3. (a), (b) Wannier spectra $\nu_x$ and $\nu_y$ versus $h_x$. (c) Edge polarization $p_x^{\text{edge,y}}$ and $p_y^{\text{edge,x}}$ along y-normal and x-normal edges, respectively. In (a)-(c), $V = 1$ meV and $\Delta_X = 2$ meV are used. (d) Phase diagram for topological ECs in the $\tau_y \sigma_x s_x$-type exciton condensation on the plane formed by $\Delta_X$ and $h_x$. The red color region marked by NEI stands for the normal excitonic insulator, the orange region marked by ECS denotes the region that supports topological ECs, and the blue region means a normal excitonic metal. For $\Delta_X = 0$ and $h_x > |V|/2$, the edge states are gapped along the $x$-direction but remain gapless along the $y$-direction, which marked by the solid magenta line. The dashed lines are phase boundaries determined by the topological condition for ECs $\frac{1}{2} \sqrt{|V|^2 + 4 \Delta_X^2} < h_x < \frac{1}{2} \sqrt{(2M + |V|)^2 + 4 \Delta_X^2}$. (See [97] for the details of the derivations of phase boundary conditions), which agree well with the numerical results.
Similarly, for edges II, III and IV, we obtain
\[
H_{II} = -A k_x s_x - \Delta_X \tau_y s_x + h_x x \sigma_z, \\
H_{III} = A k_y s_z + \Delta_X \tau_y s_y, \\
H_{IV} = -A k_x s_x - \Delta_X \tau_y s_x + h_x x \sigma_z.
\] (10)

To be more clear, we introduce a unitary transformation \(U = \frac{1}{\sqrt{2}} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} \), then the edge Hamiltonians become
\[
\tilde{H}_{I} = -A k_y s_x + \Delta_X \tau_x s_y, \\
\tilde{H}_{II} = -A k_x s_x + \Delta_X \tau_x s_x + h_x x \sigma_z, \\
\tilde{H}_{III} = A k_y s_z + \Delta_X \tau_x s_y, \\
\tilde{H}_{IV} = A k_x s_z + \Delta_X \tau_x s_z + h_x x \sigma_z.
\] (11)

Now all the edge Hamiltonians are block-diagonal. The effective masses of edges I and III are \(M_I = M_{III} = \Delta_X\), while the effective masses of edges II and IV in the two blocks are \(M_{II} = M_{IV} = \Delta_X + h_x, \Delta_X - h_x\). Therefore, \(h_x > \Delta_X\) is the topological criteria to realize the topological ECs when \(V = 0\). Consequently, the effective edge masses of two adjacent boundaries have different sign, so mass domain walls appear at the intersection of these boundaries, which results in zero-energy excitonic modes according to the Jackiw-Rebbi theory [98].

In Fig. 3(d), we present the phase diagram of the bilayer system with the \(\tau_x \sigma_z s_z\)-type exciton condensate which is subjected to the in-plane Zeeman field \(h_x\). The topological excitonic corner state phase (ECS) occupies the regime between the normal excitonic insulator phase (NEI) and the normal excitonic metal phase (NEM). Here, the phase boundaries are the basis vectors acting on the orbit, spin and layer subspaces, respectively. Due to the conserving of \(C_{2x}\) symmetry, we can adopt the mirror winding number [99] along this line to characterize the topological properties of this type of ECs. In this line, \(H(k_x, k_y = 0)\) is invariant under the operation of \(C_{2x}\). The discrete version of \(H(k_x, 0)\) has the following form
\[
H(k_x, 0) = \frac{A}{a} \sin(k_x a) \sigma_x s_z + M(k_x, 0) \sigma_z \\
+ h_x x \sigma_z - \frac{V}{2} \tau_x + \Delta_X \tau_x \sigma_z,
\] (12)

where \(M(k_x, 0) = M - B[2 - 2 \cos(k_x a)]/a^2\). \(C_{2x}\) has two fourfold degenerate eigenvalues of \(\pm 1\), the eigenvectors of \(\pm 1\) are \(|\alpha\rangle \otimes (|\uparrow\rangle \pm |\downarrow\rangle) \otimes |l = 2\rangle, 1/\sqrt{2} |\beta\rangle \otimes (|\uparrow\rangle \mp |\downarrow\rangle) \otimes |l = 2\rangle, 1/\sqrt{2} |\alpha\rangle \otimes (|\downarrow\rangle \pm |\uparrow\rangle) \otimes |l = 1\rangle, 1/\sqrt{2} |\beta\rangle \otimes (|\downarrow\rangle \mp |\uparrow\rangle) \otimes |l = 1\rangle, \) where \(|\alpha\rangle (|\beta\rangle, |\downarrow\rangle (|\uparrow\rangle)\) and \(|l = 1, 2\rangle (|2\rangle)\) are the basis vectors acting on the orbit, spin and layer subspaces, respectively. Due to the conserving of \(C_{2x}\), we project \(H(k_x, 0)\) into the two subspaces corresponding to \(C_{2x} = \pm 1\), i.e., \(H(k_x, 0) = H_+(k_x, 0) \oplus H_-(k_x, 0)\). And the block Hamiltonians read
\[
H_\pm(k_x, 0) = [M(k_x, 0) \pm h_x] \sigma_z - \frac{V}{2} \tau_x \\
- \frac{A}{a} \sin(k_x a) \sigma_x + \Delta_X \tau_x \sigma_z.
\] (13)

Along the line \(k_y = 0\), we consider the Wilson loop operator \(W_{\pm, k_x}\), then the mirror winding number \(\nu_\pm\) can be evaluated
by [99]

\[ \nu_\pm = \frac{1}{i\pi} \log(\det|W_{\pm,k_y}|) \mod 2. \]  

(14)

When the ECs emerge, the mirror winding number shows that \( \nu_+ = \nu_- = 1 \).

Now let us discuss the excitonic order induced nodal phase in the bilayer system. In the case of the \( \tau_x,\sigma_z,s_0 \)-type exciton pairing, a nodal phase with Weyl nodes along the \( k_y \)-axis emerges. The excitonic nodal phase hosts flat band edge states as shown in Fig. 4(b). In order to characterize the topological properties of nodal phase, we use the Wilson loop method to calculate the bulk polarization of the system. By treating \( k_y \) as a parameter, the Hamiltonian is effectively reduced to a one-dimensional Hamiltonian \( H_{k_y}(k_x) \). For fixed \( k_y \), considering the Wilson loop operator in the \( x \)-direction \( W_{x,k_y} \), the Wannier center \( \nu_x^j \) is obtained by the following equation

\[ W_{x,k_y} \left| \nu_x^j, k_y \right> = e^{i2\pi\nu_x^j} \left| \nu_x^j, k_y \right>. \]  

(15)

Then, the bulk polarization can be defined as \( p = \sum_j \nu_x^j \mod 1 \) for a given \( k_y \). In Fig. 4(c), we plot the calculated bulk polarization as a function of \( k_y \). We can see that the polarization is quantized to 1/2 between two nodes and vanishes at other \( k_y \). Therefore, the topology of the nodal phase can be captured by the \( k_y \)-dependent polarization.

Finally, the phase diagram for \( \tau_x,\sigma_z,s_0 \)-type exciton condensate on the plane of \( \Delta_x \) and \( h_x \) is shown in Fig. 4(d). By numerically observing the gap closing of the bulk, we define the phase boundaries. Analytically, the boundary between the ECS and the excitonic nodal phase (ENP) is determined by \( \Delta_x = -M + \sqrt{\Delta_x^2 + |V|^2/4} \), while the boundary between the NEI and ENP is defined by \( h_x = M - \sqrt{\Delta_x^2 + |V|^2/4} \) [97].

**Conclusion and discussion.**—In this work, we identified two distinct types of ECs in the gated bilayer quantum spin Hall insulator model with \( s \)-wave exciton pairings in the presence of the in-plane Zeeman field. Experimentally, the ECs can be detected by Scanning Tunneling Microscope measurements. ECs manifest themselves as in-plane Zeeman field dependent zero-bias peaks in differential conductance (See Section V in [97] for more details). The different patterns of the two types of ECs, in turn, could be used to determine the excitonic pairing of the excitonic insulator in experiments. We also found an excitonic nodal phase with the flat-band edge states in this system. Considering these exotic topological phases, our work will stimulate more investigations on higher-order topology and topological nodal phases in exciton condensates.

Different from superconducting pairings, excitonic pairings don’t have to possess particle-hole symmetry. Therefore, ECs can appear at the finite energy, which is in contrast to Majorana corner states. In this paper, the ECs are pinned to zero energy as we use a particle-hole symmetric model. Removing particle-hole symmetry, we can still expect midgap ECs, but they will be shifted to the finite energy.

Additionally, we mainly focus on the corner states created in the time-reversal invariant singlet \( s \)-wave exciton condensates hereinbefore. In this case, an in-plane Zeeman field is necessary to create the ECs. Whereas, we would like to point out that Kramers pairs of ECs could be generated in this bilayer system without applying a Zeeman field when time-reversal invariant \( d \)-wave exciton pairings are formed.

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Supplemental Material to: “Topological excitonic corner states and nodal phase in bilayer quantum spin Hall insulators”

In this Supplemental Material, we give more details of the four excitonic order parameters in the main text used to study topological excitonic corner states. Subsequently, we show the different behaviors of the in-plane Zeeman fields along the $x$-direction and $y$-direction in producing the excitonic corner states for excitonic pairings $\Delta x \tau_y \sigma_x s_x$ and $\Delta x \tau_y \sigma_x s_y$. Besides, we also present the details of the derivation of the phase boundary conditions. Finally, we provide the local density of states plots, which can be used to detect the corner states in Scanning Tunning Microscope (STM) probe.

I. TIME-IN Variant EXCITONIC ORDER PARAMETERS

The low-energy effective Hamiltonian of the gated bilayer quantum spin Hall insulator model in the momentum space is given by

$$ H_{\text{QSH}}(k) = M(k) \tau_0 \sigma_x s_0 + Ak_x \tau_0 \sigma_x s_z + Ak_y \tau_0 \sigma_y s_0 - \frac{V}{2} \tau_z \sigma_0 s_0, \quad (S1) $$

where $s_{x,y,z}$, $\sigma_{x,y,z}$ and $\tau_{x,y,z}$ are Pauli matrices acting on the spin, orbital and layer degrees of freedom, respectively. $\tau_0$, $\sigma_0$ and $s_0$ are the $2 \times 2$ identity matrices. The Hamiltonian $H_{\text{QSH}}(k)$ preserves time-reversal symmetry, $T H_{\text{QSH}}(k) T^{-1} = H_{\text{QSH}}(-k)$ with $T = i \tau_0 \sigma_0 s_0 K$ (where $K$ is the complex conjugation). Meanwhile, $H_{\text{QSH}}(k)$ also has inversion symmetry with the inversion operator $P = \tau_x \sigma_z$. We focus on the uniform momentum-independent $s$-wave-like excitonic order parameters induced by interlayer Coulomb interaction. In general, the excitonic pairing term could be written as $H_X = \Delta X \tau_i \sigma_j s_k$ with $\Delta X$ the uniform pairing strength and the subscripts $i,j,k = 0, x, y, z$. There are in total 16 interlayer excitonic order parameters preserving time-reversal symmetry, which are represented as

$$ \tau_x \sigma_0 s_0, \tau_x \sigma_x s_0, \tau_x \sigma_y s_x, \tau_x \sigma_y s_y, $$

$$ \tau_y \sigma_0 s_z, \tau_y \sigma_y s_z, \tau_y \sigma_0 s_x, \tau_y \sigma_0 s_y, \tau_y \sigma_x s_z, \tau_y \sigma_x s_y, \tau_y \sigma_y s_z, \tau_y \sigma_y s_z. \quad (S2) $$

Among these order parameters, $\tau_x \sigma_x s_0$-type and $\tau_y \sigma_x s_z$-type commute with the mass term and anticommute with the second and third terms in Eq. (S1), hence they renormalize the Dirac mass. These two order parameters can lead to the time-invariant topological excitonic insulator with helical edge states. Note that, these two order parameters break inversion symmetry. The other relevant order parameters are $\tau_x \sigma_x s_0$-type and $\tau_y \sigma_y s_z$-type, which anticommute with $H_{\text{QSH}}(k)$. In the case of negative Dirac mass, the helical edge states of the quantum spin Hall insulator couple together to open an energy gap in the presence of $\tau_0 \sigma_z s_x$-type and $\tau_y \sigma_z s_y$-type order parameters. Applying in-plane Zeeman fields, these two classes of excitonic order parameters, including $\tau_x \sigma_0 s_0, \tau_0 \sigma_x s_z, \tau_0 \sigma_z s_x, \tau_y \sigma_x s_y, \tau_0 \sigma_y s_z$, and $\tau_y \sigma_z s_y$, can give rise to topological excitonic corner states. Therefore, we focus on these two classes excitonic order parameters in the main text.

II. COMMENTS ON THE IN-PLANE ZEEMAN FIELD

In this Section, by using the effective edge theory, we demonstrate the different behaviors of the in-plane Zeeman fields along the $x$ and $y$ directions in producing corner states for the excitonic pairings $\Delta X \tau_y \sigma_x s_x$ and $\Delta X \tau_y \sigma_x s_y$. For our purpose, we consider an in-plane Zeeman field with both $x$ and $y$ components. Then we construct the edge theory to analyze the effective mass on each edge. For simplicity, we ignore the bias term as it doesn’t contribute to the effective edge mass. First, let us consider the case of $\Delta X \tau_y \sigma_x s_x$-type pairing, then low-energy Hamiltonian around the $\Gamma$ point reads

$$ H(k) = A(k_x \sigma_x s_x + k_y \sigma_y) + [M - B(k_x^2 + k_y^2)]\sigma_z + \Delta X \tau_y \sigma_x s_x + h_x s_x + h_y s_y. \quad (S3) $$

We firstly consider a semi-infinite geometry occupying the space $x \geq 0$ for edge I as marked in Fig. 1(f) of the main text. In the spirit of $\mathbf{k} \cdot \mathbf{p}$ theory, we replace $k_x \rightarrow -i \partial_x$ and separate the Hamiltonian into $H = H_0(-i \partial_x) + H_p(k_y)$, in which

| $H_0(-i \partial_x)$ | $H_p(k_y)$ |
|-------------------|------------|
| $-iA \sigma_x s_x \partial_x + (M + B \partial_x^2) \sigma_z$ | $Ak_y \sigma_y + \Delta X \tau_y \sigma_x s_x + h_x s_x + h_y s_y$. |

(S4)
where all the $k_y^2$-terms have been neglected. Hence we can solve $H_0$ first, and treat $H_p$ as a perturbation. The eigenvalue equation $H_0\psi_{\alpha}(x) = E_{\alpha}\psi_{\alpha}(x)$ can be solved under the boundary condition $\psi_{\alpha}(0) = \psi_{\alpha}(+\infty) = 0$. A straightforward calculation gives four degenerate solutions with $E_{\alpha} = 0$, whose eigenstates can be written in the following form

$$\psi_{\alpha}(x) = N_x \sin(\kappa_1 x)e^{-\kappa_2 x}e^{i k_0 y} \chi_{\alpha}, \quad (S5)$$

where $\alpha = 1, \ldots, 4$, and the normalization constant $N_x = 2\sqrt{\kappa_2 (\kappa_1^2 + \kappa_2^2)}/\kappa_1^2$ with $\kappa_1 = \sqrt{(4BM - A^2)/4B^2}$ and $\kappa_2 = -A/2B$. The eigenvectors $\chi_{\alpha}$ are determined by $\sigma_y s_2 \chi_{\alpha} = -\chi_{\alpha}$. Here we choose

$$\chi_1 = |\sigma_y = -1, \top\rangle \otimes |\tau_z = +1\rangle,$$

$$\chi_2 = |\sigma_y = +1, \top\rangle \otimes |\tau_z = +1\rangle,$$

$$\chi_3 = |\sigma_y = -1, \top\rangle \otimes |\tau_z = -1\rangle,$$

$$\chi_4 = |\sigma_y = +1, \top\rangle \otimes |\tau_z = -1\rangle. \quad (S6)$$

In this basis set, the matrix elements of the perturbation $H_p(k_y)$ are represented as

$$H_{1,\alpha\beta}(k_y) = \int_0^{+\infty} dx \psi_{\alpha}^* H_p(k_y) \psi_{\beta}(x), \quad (S7)$$

which can be written in a more compact form

$$H_1 = -A k_y \tau_0 s_z - \Delta_X \tau_y s_y. \quad (S8)$$

For edge II, the separated Hamiltonians are

$$H_0 (-i \partial_y) = -i A \tau_0 \sigma_y s_0 \partial_y + (M + B \partial_y^2) \tau_0 \sigma_z s_0,$$

$$H_p (k_x) = A k_x \tau_0 \sigma_x s_z + \Delta_X \tau_y \sigma_x s_x + h_x s_x + h_y s_y. \quad (S9)$$

We choose the basis

$$\chi_1 = |\sigma_x = -1, \top\rangle \otimes |\tau_z = +1\rangle,$$

$$\chi_2 = |\sigma_x = +1, \top\rangle \otimes |\tau_z = +1\rangle,$$

$$\chi_3 = |\sigma_x = -1, \top\rangle \otimes |\tau_z = -1\rangle,$$

$$\chi_4 = |\sigma_x = +1, \top\rangle \otimes |\tau_z = -1\rangle. \quad (S10)$$

which satisfies $\tau_0 \sigma_x s_0 \xi_\alpha = -\xi_\alpha$. In this basis, we have

$$H_{II} = -A k_x \tau_0 s_z - \Delta_X \tau_y s_x + h_x \tau_0 s_x + h_y \tau_0 s_y. \quad (S11)$$

Similarly, for edges III and IV, we obtain

$$H_{III} = A k_y \tau_0 s_z - \Delta_X \tau_y s_y, \quad (S12)$$

$$H_{IV} = A k_x \tau_0 s_z - \Delta_X \tau_y s_x + h_x \tau_0 s_x + h_y \tau_0 s_y. \quad (S13)$$

When $\Delta_X = 0$, we can see that both the $x$-component and $y$-component of the in-plane Zeeman field can only open an energy gap on edges II and IV, while the edge states on edges I and III are not affected by the in-plane Zeeman field. In contrast, the excitonic pairing $\Delta_X \tau_y \sigma_x s_x$ produces a uniform gap for edge states.

To be more clear, we introduce a unitary transformation $U = \frac{1}{\sqrt{2}} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$, then the edge Hamiltonians become

$$\bar{H}_1 = -A k_y \tau_0 s_z + \Delta_X \tau_z s_y,$$

$$\bar{H}_{II} = -A k_x \tau_0 s_z + \Delta_X \tau_z s_x + h_x \tau_0 s_x + h_y \tau_0 s_y,$$

$$\bar{H}_{III} = A k_y \tau_0 s_z + \Delta_X \tau_z s_y,$$

$$\bar{H}_{IV} = A k_x \tau_0 s_z + \Delta_X \tau_z s_x + h_x \tau_0 s_x + h_y \tau_0 s_y. \quad (S13)$$

The eigenvalues are obtained by the diagonalizing the edge Hamiltonians.
\[ E_{\text{I}}, E_{\text{III}} = -\sqrt{A^2k_y^2 + \Delta_X^2} \pm \sqrt{A^2k_y^2 + \Delta_X^2}, \]
\[ E_{\text{II}}, E_{\text{IV}} = -\sqrt{A^2k_x^2 + \Delta_X^2} \pm \sqrt{A^2k_x^2 + \Delta_X^2} \pm \sqrt{A^2k_y^2 + \Delta_X^2 + h_x^2 + h_y^2}, \]

Apparently, for the \( \tau_y \sigma_x s_x \)-type pairing, only the \( h_x \)-component of the in-plane Zeeman field can drive a phase transition on edges II and IV in which the energy gap opened by the excitonic paring closes and reopens by increasing \( h_x \). As stated in the main text, this phase transition is the key to realize the excitonic corner states. In the case of \( \tau_y \sigma_x s_y \)-type excitonic pairing, however, \( h_y \) is responsible for the closing and reopening of the energy gap on edges II and IV.

### III. Phase Transition Point in the Case of \( \tau_y \sigma_x s_x \)-Type Pairing

In this section, we give the specific derivation process of the analytical expression of the phase transition points in Fig. 3(d). Here we first discuss the phase transition in the case of \( \tau_y \sigma_x s_x \)-type pairing. The Hamiltonian \( \mathcal{H}(k) \) of k-space is,

\[ \mathcal{H}(k) = [M - B(k_x^2 + k_y^2)] \tau_0 \sigma_z s_0 + A(k_x \tau_0 \sigma_x s_z + k_y \tau_0 \sigma_y s_0) - \frac{V}{2} \tau_z + \Delta_X \tau_y \sigma_x s_x + h_x \tau_0 \sigma_0 s_x, \quad (S15) \]

where the basis is \( c_0^\dagger = (c_{1\sigma_x \uparrow}^\dagger, c_{1\sigma_x \downarrow}^\dagger, c_{1\sigma_y \uparrow}^\dagger, c_{1\sigma_y \downarrow}^\dagger, c_{2\sigma_x \uparrow}^\dagger, c_{2\sigma_x \downarrow}^\dagger, c_{2\sigma_y \uparrow}^\dagger, c_{2\sigma_y \downarrow}^\dagger) \), \( \alpha \) and \( \beta \) are different orbital degrees of freedom with opposite parity, \( \uparrow \) and \( \downarrow \) represent electron spin, and 1, 2 are the layer indices. Since the energy gap of the bilayer HgTe quantum wells is at the \( \Gamma(k_x = 0, k_y = 0) \) point, we diagonalize \( \mathcal{H}(k) \) and obtain eight eigenvalues at \( \Gamma' \)

\[ \mathcal{E} = \pm h_x \pm \frac{1}{2} \sqrt{(2M \pm V)^2 + 4\Delta_X^2}, \quad (S16) \]

In the case of \( \tau_y \sigma_x s_x \)-type pairing, we set \( M < 0 \), therefore, when \( h_x = \frac{1}{2} \sqrt{(2M + |V|)^2 + 4\Delta_X^2} \), the bulk energy gap is closed, and when \( h_x > \frac{1}{2} \sqrt{(2M + |V|)^2 + 4\Delta_X^2} \), the system exhibits a metallic state. Meanwhile, the excitonic corner states are formed by gapping out the helical edge states, thus we can use the effective Hamiltonian obtained by the edge theory to determine the regime of the corner states. The effective Hamiltonians with bias are as follows:

\[ \tilde{\mathcal{H}}_{\text{I}} = -Ak_y \tau_0 s_z + \frac{V}{2} \tau_x s_0 + \Delta_X \tau_z s_y, \]
\[ \tilde{\mathcal{H}}_{\text{II}} = -Ak_x \tau_0 s_z + \frac{V}{2} \tau_x s_0 + \Delta_X \tau_z s_x + h_x \tau_0 s_x, \]
\[ \tilde{\mathcal{H}}_{\text{III}} = Ak_y \tau_0 s_z - \frac{V}{2} \tau_x s_0 + \Delta_X \tau_z s_y, \]
\[ \tilde{\mathcal{H}}_{\text{IV}} = Ak_x \tau_0 s_z - \frac{V}{2} \tau_x s_0 + \Delta_X \tau_z s_x + h_x \tau_0 s_x. \quad (S17) \]

The eigenvalues are obtained by the diagonalizing the edge Hamiltonians

\[ \mathcal{E}_1, \mathcal{E}_{\text{III}} = \pm \frac{1}{2} \sqrt{(2Ak_y \pm V)^2 + 4\Delta_X^2}, \]
\[ \mathcal{E}_{\text{II}}, \mathcal{E}_{\text{IV}} = \pm \frac{1}{2} \sqrt{4\Delta_X^2 + V^2 \pm \frac{1}{4} \sqrt{A^2V^2k_x^2 + (V^2 + 4\Delta_X^2)h_x^2 + 4A^2k_y^2 + 4h_y^2}}, \quad (S18) \]

When \( k_x = 0 \) and \( k_y = 0 \),

\[ \mathcal{E}_1, \mathcal{E}_{\text{III}} = \pm \frac{1}{2} \sqrt{V^2 + 4\Delta_X^2}, \]
\[ \mathcal{E}_{\text{II}}, \mathcal{E}_{\text{IV}} = \pm \sqrt{\frac{V^2}{4} + \Delta_X^2 \pm h_x)^2}. \quad (S19) \]
Obviously, when \( V, \Delta_X \neq 0 \), the energy gap between edge I and III will always exist. However, when \( h_x = \frac{1}{2} \sqrt{V^2 + 4\Delta_X^2} \), the energy gap of edge II and IV will close, and the energy gap will reopen and corner states emerge when \( h_x \) continues to increase.

In summary, when \( h_x < \frac{1}{2} \sqrt{V^2 + 4\Delta_X^2} \), the edge states of quantum spin Hall insulators are gapped by excitonic pairing \( \Delta_X \). At this time, the system is in the normal excitonic insulator (NEI) phase. When \( h_x > \frac{1}{2} \sqrt{(2M + |V|)^2 + 4\Delta_X^2} \), the bulk energy gap is closed and the system behaves as normal excitonic metal (NEM). Only when \( \frac{1}{2} \sqrt{V^2 + 4\Delta_X^2} < h_x < \frac{1}{2} \sqrt{(2M + |V|)^2 + 4\Delta_X^2} \), we can observe excitonic corner states (ECS).

IV. PHASE TRANSITION POINT IN THE CASE OF \( \tau_0 \sigma_z s_0 \)-TYPE PAIRING

In this section, we give the specific derivation process of the analytical expression of the phase boundary of Fig. 4(d) in the main text. Here we consider the case where the excitonic pairing of the system is \( \tau_0 \sigma_z s_0 \)-type, and we have \( M > 0 \). The Hamiltonian of the system

\[
H(k) = [M - B(k_x^2 + k_y^2)]\tau_0 \sigma_z s_0 + A(k_x\tau_0 \sigma_x s_x + k_y\tau_0 \sigma_y s_0) - \frac{V}{2} \tau_z + \Delta_X \tau_x \sigma_z s_0 + h_x\tau_0 \sigma_0 s_x.
\]

By diagonalization, the eigenvalues of the Hamiltonian at \( \Gamma \) point are given by

\[
E = \pm h_x \pm M \pm \frac{1}{2} \sqrt{V^2 + 4\Delta_X^2},
\]

Obviously, when \( h_x = -M + \frac{1}{2} \sqrt{V^2 + 4\Delta_X^2} \) or \( h_x = M - \frac{1}{2} \sqrt{V^2 + 4\Delta_X^2} \), the bulk energy gap of the system will be closed. This is exactly the two dashed lines denoting the phase boundary marked in Fig. 4(d) of the main text. In addition, we can confirm the regime between the two dashed lines are excitonic corner states through the mirror winding number.

V. LOCAL DENSITY OF STATES AND STM DETECTION

![FIG. S1](image.png)

FIG. S1. (a) The local density of states at the corner \( x = 1, y = 1 \) for different in-plane Zeeman field strength. (b) The local density of states at the edge \( x = 1, y = L_y/2 \) for different in-plane Zeeman field strength. In all plots, we choose \( M = -3 \) meV, \( V = 1 \) meV, \( \Delta_X = 2 \) meV. The size of the square-shaped sample is \( L_x \times L_y = 100 \times 100 \).

Experimentally, we predict that Scanning Tunnelling Microscope (STM) probes can be used to detect the existence of corner modes. In this section, we focus on the experimental suggestion on the excitonic corner states in the case of \( \tau_0 \sigma_z s_0 \)-type pairing. We know that STM serves as a probe of the local density of states (LDOS) of the sample. We assume that the STM probe detects layer 1 of the system (our calculations show that there is no essential difference between detecting layer 1 and layer 2). The local density of states is defined as \( \rho(E, l=1, x) = \sum_i \delta(E - E_i) \left( |\Psi_{E_i,\sigma=x}\rangle|^2 + |\Psi_{E_i,\sigma=\uparrow}\rangle|^2 + |\Psi_{E_i,\sigma=\downarrow}\rangle|^2 \right) \), where \( E_i \) is the \( i \)th eigenvalue, \( l = 1, 2, \sigma = \alpha, \beta \) and \( s = \uparrow, \downarrow \) are layer, orbital and spin degrees of freedom, respectively, \( \Psi_{E_i,\sigma=x} \) are the corresponding components of the eigenstate of the system. For comparison, we calculated the LDOS at the corner \( x = 1, y = 1 \) and the edge \( x = 1, y = L_y/2 \) of a square sample [the same as the sample in Fig. 2(f) in the main text].

Figures S1(a) and S1(b) illustrate the LDOS at the corner \( x = 1, y = 1 \) and the edge \( x = 1, y = L_y/2 \) with respect to energy \( E \) for different in-plane Zeeman field strength \( h_x \). Obviously, when \( h_x > h_x^c \approx 2.06 \), we see that zero energy peaks develop...
in the LDOS at the sample corner. In contrast, the U-shape LDOS of the edge retain even for \( h_x > h^c_x \). Therefore, the corner states in this system can be determined by the Zeeman field dependent zero bias peaks of different conductance \((dI/dV)\) in STM measurements. Note that, the peak splitting for \( h_x = 2.1 \) is because of the size effect that causes the overlap of wavefunctions of corner states. The corner states become more localized and the splitting vanishes as \( h_x \) increases.