Research Article

Nonexistence Results for Some Classes of Nonlinear Fractional Differential Inequalities

Mohamed Jleli and Bessem Samet

Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia

Correspondence should be addressed to Bessem Samet; bsamet@ksu.edu.sa

Received 23 September 2020; Accepted 16 October 2020; Published 28 October 2020

Academic Editor: Simone Secchi

Copyright © 2020 Mohamed Jleli and Bessem Samet. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We study the nonexistence of global solutions for new classes of nonlinear fractional differential inequalities. Namely, sufficient conditions are provided so that the considered problems admit no global solutions. The proofs of our results are based on the test function method and some integral estimates.

1. Introduction

We first consider the problem

\[
\begin{align*}
\quad \quad & C D_0^{1+\alpha} u(t) + C D_0^{1+\beta} u(t) \geq \left| C D_0^\gamma u(t) \right|^p, \quad t > 0, \\
& \left( u(0), u'(0) \right) = (u_0, u_1),
\end{align*}
\]

where \( p > 1, \alpha, \beta, \gamma \in (0, 1), \quad C D_0^\kappa, \kappa \in \{1 + \alpha, 1 + \beta, \gamma\} \) is the Caputo fractional derivative of order \( \kappa, u_0 \in \mathbb{R}, \) and \( u_1 \geq 0. \) Namely, we are interested in providing sufficient conditions for which problem (1) admits no global solution. Next, we study the same question for the inhomogeneous problem

\[
\begin{align*}
\quad \quad & \left| C D_0^{1+\alpha} u(t) \right| + \left| C D_0^{1+\beta} u(t) \right| \geq \left| C D_0^\gamma u(t) \right|^p + f(t), \quad t > 0, \\
& \left( u(0), u'(0) \right) = (u_0, u_1),
\end{align*}
\]

(2)

where \( p, q > 1, \alpha, \beta, \gamma \in (0, 1), u_0 \in \mathbb{R}, u_1 \geq 0, f \in L^1_{\text{loc}}([0,\infty)), \)

\( f \geq 0, \) and \( f \neq 0. \)

Due to the importance of fractional calculus in applications (see e.g. [1–5]), in the past few decades, there has been a growing interest in the study of fractional differential equations. In particular, from the theoretical point of view, the existence of solutions for different classes of fractional differential equations was investigated in many contributions (see e.g. [6–12] and the references therein).

For the issue of nonexistence of solutions for fractional differential equations and inequalities, we refer to [13–22] and the references therein. In particular, in [17], Laskri and Tatar studied the problem

\[
\begin{align*}
\quad \quad & D_0^\alpha u(t) \geq t^q |y(t)|^p, \quad t > 0, \\
& I_0^{1-\alpha} u(t) \big|_{t=0} = b,
\end{align*}
\]

(3)

where \( p > 1, 0 < \alpha < 1, \gamma > -\alpha, \) and \( b \geq 0, D_0^\alpha \) is the Riemann-Liouville fractional derivative of order \( \alpha, \) and \( I_0^{1-\alpha} \) is the left-sided Riemann-Liouville fractional integral of order \( 1 - \alpha. \) It was shown that, if \( p \leq (\gamma + 1/1 - \alpha), \) then problem (3) does not admit nontrivial global solution. In [16], Kassim et al. studied the problem

\[
\begin{align*}
\quad \quad & C D_0^{\alpha} u(t) + C D_0^\beta u(t) \geq t^q |y(t)|^p, \quad t > 0, \\
& u^{(i)}(0) = b, \quad i = 0, 1, \cdots, n - 1,
\end{align*}
\]

(4)
where $p > 1$, $n \geq 1$ is an integer, $n - 1 < \beta \leq \alpha < n$, and $b_i \geq 0$. It was shown that, if

$$p(1 - \beta) - 1 < \gamma < p - 1,$$  \hspace{1cm} (5)

then problem (4) does not admit nontrivial global solution. In [15], Furati and Kirane investigated the system of nonlinear fractional differential equations

$$\begin{cases}
u'(t) + C_{D^\beta_0}^\gamma u(t) = |v(t)|^\gamma, & t > 0, \\
\nu'(t) + C_{D^\beta_0}^\gamma \nu(t) = |u(t)|^\gamma, & t > 0,
\end{cases}$$  \hspace{1cm} (6)

subject to the initial conditions

$$(u(0), \nu(0)) = (u_0, \nu_0),$$  \hspace{1cm} (7)

where $0 < \alpha, \beta < 1, p, q > 1$, and $u_0, \nu_0 > 0$. It was shown that, if

$$1 - \frac{1}{pq} \leq \max \left\{ \frac{\beta}{p}, \frac{\alpha}{q} \right\},$$  \hspace{1cm} (8)

then solutions to system (6) subject to (7) blow up in a finite time.

For the issue of nonexistence of global solutions for fractional in time evolution equations, we refer to [6, 23–25] and the references therein.

On the other hand, to the best of our knowledge, the nonexistence of global solutions for problems of types (1) and (2) was not yet investigated.

Before stating our main results, let us mention what we mean by global solutions to problems (1) and (2).

**Definition 1.** A function $u \in AC^2([0, \infty))$ is said to be a global solution to problem (1), if $u$ satisfies

$$C_{D^\beta_0}^\alpha u(t) + C_{D^\beta_0}^\beta u(t) \geq C_{D^\beta_0}^\gamma u(t)|^\gamma,$$  \hspace{1cm} (9)

for almost every where $t > 0$, and

$$\left( u(0), u'(0) \right) = (u_0, u_1).$$  \hspace{1cm} (10)

**Definition 2.** A function $u \in AC^2([0, \infty))$ is said to be a global solution to problem (2), if $u$ satisfies

$$C_{D^\beta_0}^\alpha u(t) + C_{D^\beta_0}^\beta u(t) \geq C_{D^\beta_0}^\gamma u(t)|^\gamma + f(t),$$  \hspace{1cm} (11)

for almost every where $t > 0$, and

$$\left( u(0), u'(0) \right) = (u_0, u_1).$$  \hspace{1cm} (12)

We first consider problem (1). We discuss separately the cases $u_1 > 0$ and $u_1 = 0$.

**Theorem 3.** Let $\alpha, \beta, \gamma \in (0, 1)$, and $u_0 \in \mathbb{R}$. If $u_1 > 0$, then for all $p > 1$, problem (1) admits no global solution.

**Theorem 4.** Let $\alpha, \beta, \gamma \in (0, 1)$, $\alpha \leq \beta$, $u_0 \in \mathbb{R}$, and $u_1 = 0$.

(i) If $\gamma \leq \alpha$, then for all $p > 1$, the only global solution to problem (1) is $u \equiv u_0$

(ii) If $\gamma > \alpha$, then for all

$$1 < p < \frac{1}{\gamma - \alpha},$$  \hspace{1cm} (13)

the only global solution to problem (1) is $u \equiv u_0$.

Next, we consider problem (2).

**Theorem 5.** Let $\alpha, \beta, \gamma \in (0, 1)$, $u_0 \in \mathbb{R}$, $f \in L^1_{\text{loc}}([0, \infty))$, $f \geq 0$, and $f \equiv 0$. If $u_1 > 0$, then for all $p > 1$, problem (2) admits no global solution.

**Theorem 6.** Let $p > 1$, $\alpha, \beta, \gamma \in (0, 1)$, $\alpha \leq \beta$, $u_0 \in \mathbb{R}$, $u_1 = 0$, $f \in L^1_{\text{loc}}([0, \infty))$, $f \geq 0$, and $f \equiv 0$. If

$$\limsup_{T \to +\infty} \frac{1}{T^{\frac{\gamma}{p} + 1}} \int_0^T f(t) \, dt = +\infty,$$  \hspace{1cm} (14)

then problem (2) admits no global solution.

We discuss below some special cases of Theorem 6.

**Corollary 7.** Let $\alpha, \beta, \gamma \in (0, 1)$, $\alpha \leq \beta$, $u_0 \in \mathbb{R}$, and $u_1 = 0$. Let

$$f(t) = e^{at}, \hspace{1cm} t > 0,$$  \hspace{1cm} (15)

where $a \in \mathbb{R}$ and $a \neq 0$.

(i) If $a > 0$, then for all $p > 1$, problem (2) admits no global solution

(ii) If $a < 0$ and $\gamma \leq \alpha$, then for all $p > 1$, problem (2) admits no global solution

(iii) If $a < 0$ and $\gamma > \alpha$, then for all

$$1 < p < \frac{1}{\gamma - \alpha},$$  \hspace{1cm} (16)

problem (2) admits no global solution.

**Corollary 8.** Let $\alpha, \beta, \gamma \in (0, 1)$, $\alpha \leq \beta$, $u_0 \in \mathbb{R}$, and $u_1 = 0$. Let

$$f(t) = t^\sigma, \hspace{1cm} t > 0,$$  \hspace{1cm} (17)

where $\sigma > -1$. 
(i) If $\sigma \geq 0$, then for all $p > 1$, problem (2) admits no global solution

(ii) Let $-1 < \sigma < 0$

(a) If $\gamma \leq \alpha$, then for all $p > 1$, problem (2) admits no global solution

(b) If $\gamma > \alpha$ and $\gamma - 1 \leq \sigma < 0$, then for all $p > 1$, problem (2) admits no global solution

(c) If $\gamma > \alpha$ and $-1 < \sigma < \gamma - \alpha - 1$, then for all

$$1 < p < \frac{\sigma}{\gamma - \alpha + 1},$$

problem (2) admits no global solution.

The rest of the paper is organized as follows. In Section 2, we recall briefly some standard notions on fractional calculus and prove some properties. Section 3 is devoted to the Proofs of Theorems 3, 4, 5, and 6 and Corollaries 7 and 8.

2. Some Preliminaries

We denote by $AC([0,\infty))$ the space of absolutely continuous functions on $[0,\infty)$. Given an integer $n \geq 2$, we denote by $AC^n([0,\infty))$ the space of functions $f$ which have continuous derivatives up to order $n-1$ on $[0,\infty)$ such that $f^{(n-1)} \in AC([0,\infty))$. Here, $f^{(n-1)}$ denotes the derivative of order $n-1$ of $f$.

Let $T > 0$ be fixed. Given $\rho > 0$ and $f \in L^1(0,T)$, the left-sided Riemann-Liouville fractional integral of order $\rho$ of $f$ is defined by

$$I_0^\rho f(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-s)^{\rho-1} f(s) \, ds,$$  \hspace{1cm} (19)

for almost everywhere $0 \leq t \leq T$. Here, $\Gamma$ denotes the Gamma function. The right-sided Riemann-Liouville fractional integral of order $\rho$ of $f$ is defined by

$$I_T^\rho f(t) = \frac{1}{\Gamma(\rho)} \int_t^T (s-t)^{\rho-1} f(s) \, ds,$$  \hspace{1cm} (20)

for almost everywhere $0 \leq t \leq T$. Notice that, if $f \in C([0,T])$, then $I_0^\rho f$ is defined for all $0 < t \leq T$. Moreover, one has $\lim_{s \to 0^+} (I_0^\rho f)(t) = 0$. Similarly, if $f \in C([0,T])$, then $I_T^\rho f$ is defined for all $0 \leq t < T$. Moreover, one has $\lim_{t \to T^-} (I_T^\rho f)(t) = 0$.

Lemma 9 (see [5]). Let $\rho, \kappa > 0$ and $f \in L^1(0,T)$, where $1 \leq \tau < \infty$. Then

$$I_0^\rho(I_0^\kappa f)(t) = I_0^\rho(I_0^{\rho + \kappa} f)(t) = (I_0^{\rho \kappa} f)(t),$$  \hspace{1cm} (21)

for almost everywhere $0 \leq t \leq T$.

Lemma 10 (see [5]). Let $\rho > 0, \tau, \mu \geq 1$, and $(1/\tau) + (1/\mu) \leq 1 + \rho \tau \mu$. Let $f$ be in $L^\tau(0,T)$ and $g$ be in $L^\mu(0,T)$, then

$$\int_0^T (I_0^\rho f(t) g(t) \, dt = \int_0^T (I_0^\rho g(t) f(t)) \, dt.$$  \hspace{1cm} (22)

Let $-1 < \rho < n$ and $f \in AC^n([0,\infty))$, where $n \geq 1$ is an integer. The (left-sided) Caputo fractional derivative of order $\rho$ of $f$ is defined by

$$^\rho D_0^\rho f(t) = \left( t_0^\rho f^{(n)}(t) \right),$$  \hspace{1cm} (23)

for almost everywhere $t > 0$. Here, for $n = 1$, $AC^1([0,\infty)) = AC([0,\infty))$.

For $\lambda > 1$ ($\lambda$ is large enough), we define the function

$$\xi(t) = T^{-\lambda}(T-t)^{\lambda}, \hspace{1cm} 0 \leq t \leq T.$$  \hspace{1cm} (24)

Lemma 11. Let $\rho > 0$ and $0 < \kappa < 1$. Then

$$I_T^\rho \xi(t) = \frac{\Gamma(\lambda + 1)}{\Gamma(\rho + \lambda + 1)} T^{-\lambda}(T-t)^{\lambda \rho}, \hspace{1cm} 0 \leq t \leq T,$$  \hspace{1cm} (25)

$$I_T^\rho \xi'(t) = \frac{-\Gamma(\lambda + 1)}{\Gamma(\rho + \lambda)} T^{-\lambda}(T-t)^{\lambda \rho - 1}, \hspace{1cm} 0 \leq t \leq T,$$  \hspace{1cm} (26)

$$I_T^\rho \xi'^{(\kappa)}(t) = \frac{\Gamma(\lambda + 1)}{\Gamma(\rho + \lambda - \kappa)} T^{-\lambda}(T-t)^{\lambda \rho - \kappa}, \hspace{1cm} 0 \leq t \leq T,$$  \hspace{1cm} (27)

$$I_T^\rho \left[(I_T^\rho \xi)^{(\kappa)}\right]'(t) = \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + \rho - \kappa)} T^{-\lambda}(T-t)^{\lambda \rho - \kappa - 1}, \hspace{1cm} 0 \leq t \leq T.$$  \hspace{1cm} (28)

Proof. We prove only (25). Namely, differentiating (25), (26) follows. Similarly, differentiating (26), (27) follows. Moreover, taking $\rho = 1 - \kappa$ in (27) and using a similar calculation as in the proof of (25), (28) follows.

For $t \in [0,T)$, one has

$$I_T^\rho \xi(t) = \frac{T^{-\lambda}}{\Gamma(\rho)} \int_t^T (s-t)^{\rho-1}(T-s)^{\lambda} \, ds$$

$$= \frac{T^{-\lambda}}{\Gamma(\rho)} \int_0^T (s-t)^{\rho-1}((T-t) - (s-t))^{\lambda} \, ds$$

$$= \frac{T^{-\lambda}}{\Gamma(\rho)} \int_0^T (s-t)^{\rho-1} \left(1 - \frac{s-t}{T-t}\right)^{\lambda} \, ds.$$  \hspace{1cm} (29)
Using the change of variable $z = (s - t)/T - t$, one obtains

\[
\left( I_1^\beta \xi \right)(t) = \frac{T^{-1/(q+1)}}{\Gamma(p)} \int_0^1 e^{-(1-z)\rho} \, dz
\]

(30)

\[
= \frac{T^{-1/(q+1)}}{\Gamma(p)} B(\rho, \alpha + 1),
\]

where $B(\cdot, \cdot)$ is the beta function. Using the property

\[
B(a, b) = \frac{\Gamma(a) \Gamma(b)}{\Gamma(a + b)}, \quad a, b > 0,
\]

(31)

one obtains (25).

3. Proofs

The proofs of our results are based on the test function method (see e.g. [26]) and some integral estimates.

**Proof of Theorem 3.** Let us suppose that $u \in AC^2([0, \infty))$ is a global solution to (1). For $T > 0$, multiplying the differential inequality in (1) by $\xi$, where $\xi$ is the function defined by (24), and integrating over $(0, T)$, one obtains

\[
\int_0^T |C_{D_0^a} u(t)|^p \xi(t) \, dt \leq \int_0^T C_{D_0^a} u(t) \xi(t) \, dt + \int_0^T C_{D_0^a} \xi(t) \, dt.
\]

(32)

Without restriction of the generality, we may suppose that

\[
0 < \alpha \leq \beta < 1.
\]

(33)

On the other hand, using Lemma 10, one obtains

\[
\int_0^T C_{D_0^a} u(t) \xi(t) \, dt = \int_0^T \left( I_0^0 u' \right)(t) \xi(t) \, dt
\]

(34)

Using an integration by parts, the initial conditions, (26) and Lemma 10, it holds that

\[
\int_0^T C_{D_0^a} u(t) \xi(t) \, dt = -u_1 \left( I_1^\beta \xi \right)(0) - \int_0^T C_{D_0^a} u(t) \xi(t) \, dt.
\]

(35)

On the other hand, by Lemma 9 and using the initial conditions, one obtains

\[
u'(t) = (u(t) - u_0)' = \left( \int_0^t u'(s) \, ds \right)' = \left( I_0^0 u' \right)'(t)
\]

(36)

Therefore, by (35), one obtains

\[
\int_0^T C_{D_0^a} u(t) \xi(t) \, dt = -u_1 \left( I_1^\beta \xi \right)(0)
\]

\[
- \int_0^T \left[ I_1^1 (I_1^\beta u') \right](t) \left( I_1^\beta \xi \right)'(t) \, dt.
\]

(37)

Using an integration by parts, the initial conditions, (26) and Lemma 10, it holds that

\[
\int_0^T C_{D_0^a} u(t) \xi(t) \, dt = -u_1 \left( I_1^\beta \xi \right)(0) - \int_0^T C_{D_0^a} u(t) \xi(t) \, dt
\]

(38)

Similarly, one has

\[
\int_0^T C_{D_0^a} u(t) \xi(t) \, dt = -u_1 \left( I_1^\beta \xi \right)(0)
\]

(39)

Next, using (32), (38), and (39), one obtains

\[
\int_0^T |C_{D_0^a} u(t)|^p \xi(t) \, dt + u_1 \left( I_1^\beta \xi \right)(0) + \int_0^T C_{D_0^a} u(t) \xi(t) \, dt 
\]

\[
\leq \int_0^T |C_{D_0^a} u(t)| t \left( I_1^\beta \xi \right)'(t) \, dt + \int_0^T C_{D_0^a} u(t) \left( I_1^\beta \xi \right)'(t) \, dt.
\]

(40)

On the other hand, using $\varepsilon$-Young inequality with $0 < \varepsilon < (1/2)$, one obtains

\[
\int_0^T |C_{D_0^a} u(t)| t \left( I_1^\beta \xi \right)'(t) \, dt \leq \varepsilon \int_0^T |C_{D_0^a} u(t)|^p \xi(t) \, dt
\]

\[
+ C(\varepsilon, p) \int_0^T \xi(t) \left( I_1^1 (I_1^\beta \xi)' \right)(t) \, dt
\]

(41)
where $C(\epsilon, p)$ is a positive real number that depends only on $\epsilon$ and $p$. Similarly, one has

$$
\int_0^T |C D_0^\alpha u(t)| ||(H_T^{1-\beta})\xi(t)||^p \xi(t) dt \leq \epsilon \int_0^T |C D_0^\alpha u(t)||((H_T^{1-\beta})\xi(t))^{(p-1)} dt + C(\epsilon, p) \int_0^T \xi(t)^{1/(p-1)} \left[ (H_T^{1-\beta})\xi(t) \right]^{(p-1)} dt.
$$

(42)

Hence, it follows from (40), (41), and (42) that

$$
(1 - 2\epsilon) \int_0^T |C D_0^\alpha u(t)||((H_T^{1-\beta})\xi(t))^{(p-1)} dt + u_1 \left( (H_T^{1-\alpha})\xi(0) + (H_T^{1-\beta})\xi(0) \right)
\leq C(\epsilon, p) \left( \int_0^T \xi(t)^{1/(p-1)} \left[ (H_T^{1-\alpha})\xi(t) \right]^{(p-1)} dt + \int_0^T \xi(t)^{1/(p-1)} \left[ (H_T^{1-\beta})\xi(t) \right]^{(p-1)} dt \right).
$$

(43)

Since $0 < \epsilon < (1/2)$, one deduces from (43) that

$$
u_1 \left( (H_T^{1-\alpha})\xi(0) + (H_T^{1-\beta})\xi(0) \right)
\leq C(\epsilon, p) \left( \int_0^T \xi(t)^{1/(p-1)} \left[ (H_T^{1-\alpha})\xi(t) \right]^{(p-1)} dt + \int_0^T \xi(t)^{1/(p-1)} \left[ (H_T^{1-\beta})\xi(t) \right]^{(p-1)} dt \right).
$$

(44)

On the other hand, by (25), one has

$$
(H_T^{1-\alpha})\xi(0) = \frac{\Gamma(\lambda + 1)}{\Gamma(2 + \lambda - \alpha)} T^{1-\alpha},
$$

(45)

and

$$
(H_T^{1-\beta})\xi(0) = \frac{\Gamma(\lambda + 1)}{\Gamma(2 + \lambda - \beta)} T^{1-\beta},
$$

(46)

which yield

$$
u_1 \left( (H_T^{1-\alpha})\xi(0) + (H_T^{1-\beta})\xi(0) \right) = \Gamma(\lambda + 1) u_1 T^{1-\alpha} \left( \frac{1}{\Gamma(2 + \lambda - \alpha)} + \frac{T^{1-\beta}}{\Gamma(2 + \lambda - \beta)} \right).
$$

(47)

Since $u_1 > 0$, one deduces that

$$
u_1 \left( (H_T^{1-\alpha})\xi(0) + (H_T^{1-\beta})\xi(0) \right) \geq C_1 u_1 T^{1-\alpha},
$$

(48)

where $C_1 = (\Gamma(\lambda + 1)/\Gamma(2 + \lambda - \alpha)) > 0$. Next, using (28) with $\rho = \gamma$ and $\kappa = \alpha$, one obtains

$$
\int_0^T \xi(t)^{1/(p-1)} \left[ (H_T^{1-\alpha})\xi(t) \right]^{(p-1)} dt
= \left( \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + \gamma - \alpha)} \right)^{(p-1)} T^{1-\alpha} \int_0^T (t - 1)^{\lambda + \gamma - \alpha} dt
$$

(49)

which yields

$$
\int_0^T \xi(t)^{1/(p-1)} \left[ (H_T^{1-\alpha})\xi(t) \right]^{(p-1)} dt = C_2 T^{\lambda + \gamma - \alpha (p-1) + 1},
$$

(50)

where

$$
C_2 = \left( \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + \gamma - \alpha)} \right)^{(p-1)} \int_0^1 (1 - s)^{\lambda + \gamma - \alpha (p-1)} ds > 0.
$$

(51)

Similarly, one has

$$
\int_0^T \xi(t)^{1/(p-1)} \left[ (H_T^{1-\beta})\xi(t) \right]^{(p-1)} dt = C_3 T^{\lambda + \gamma - \beta (p-1) + 1},
$$

(52)

where

$$
C_3 = \left( \frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + \gamma - \beta)} \right)^{(p-1)} \int_0^1 (1 - s)^{\lambda + \gamma - \beta (p-1)} ds > 0.
$$

(53)

Therefore, it follows from (44), (48), (50), and (52) that

$$
C_1 u_1 T^{1-\alpha} \leq C(\epsilon, p) \left( C_2 T^{\lambda + \gamma - \alpha (p-1)} + C_3 T^{\lambda + \gamma - \beta (p-1) + 1} \right),
$$

(54)

which yields

$$
u_1 \leq \frac{C(\epsilon, p)}{C_1} T^{\alpha (p-1)/(p-1)} \left( C_2 + C_3 T^{\alpha (p-1)/(p-1)} \right), \quad T > 0.
$$

(55)

Notice that for all $p > 1$, one has

$$
\left\{ \begin{array}{l}
\alpha + \frac{(y - 1 - \alpha)p}{p - 1} < 0, \\
\frac{\alpha - \beta}{p - 1} \leq 0 \text{ (from (33))}
\end{array} \right.
$$

(56)
Hence, using (56) and passing to the limit as \( T \to +\infty \) in (55), one obtains \( u_t \leq 0 \), which contradicts the fact that \( u_t \geq 0 \). Therefore, one deduces that for all \( p > 1 \), problem (1) admits no global solution.

**Proof of Theorem 4.** Let \( u_t = 0 \). First, one observes that in this case \( u \equiv u_t \) is a global solution to (1). Suppose now that \( u \in AC^2([0,\infty)) \) is a global solution to (1). Taking \( u_t = 0 \) in (43), one obtains

\[
\int_0^T \left[ CD_0^\alpha u(t) \right]^p dt \\
\leq C' (\epsilon, p) \left( \int_0^T \xi(t)^{(1-\alpha)(p-1)} \left[ \left( I_T^{t-\alpha} \xi \right)^{\gamma} \right] (t) \right)^{(p/(p-1))} dt \\
+ \int_0^T \xi(t)^{-1/(p-1)} \left[ \left( I_T^{t-\alpha} \xi \right)^{\gamma} \right] (t) \right)^{(p/(p-1))} dt,
\]

(57)

for all \( T > 0 \), where \( C' (\epsilon, p) = (C(\epsilon, p)/1 - 2\epsilon) \) and \( 0 < \epsilon < (1/2) \). Next, using (24) and the estimates (50) and (52), one obtains

\[
\int_0^T \left[ CD_0^\alpha u(t) \right]^p dt \\
\leq C' (\epsilon, p) T^{(\alpha-\beta)/(1-\alpha)} \left( C_2 + C_3 T^{((\alpha-\beta)/p)(1-\alpha)} \right).
\]

(58)

Notice that since \( \alpha \leq \beta \), one has

\[
\frac{(\alpha - \beta)p}{p - 1} \leq 0. \quad \text{(59)}
\]

Moreover, if \( \gamma \leq \alpha \), then

\[
\frac{(\gamma - 1 - \alpha)p}{p - 1} + 1 < 0. \quad \text{(60)}
\]

Hence, passing to the infimum limit as \( T \to +\infty \) in (58) and using Fatou’s lemma, one obtains

\[
\int_0^\infty \left[ CD_0^\alpha u(t) \right]^p dt = 0,
\]

(61)

which yields

\[
CD_0^\alpha u(t) = 0,
\]

(62)

for almost everywhere \( t > 0 \). Then, using the initial conditions and Lemma 9, one deduces that

\[
I_0^\alpha \left( CD_0^\alpha u \right) (t) = I_0^\alpha \left( I_0^{t-\alpha} u \right) (t) = \left( I_0^\alpha u \right)' (t) = u(t) - u_0 = 0,
\]

(63)

for almost everywhere \( t > 0 \). Since \( u \) is continuous \((u \in AC^2([0,\infty))) \), it holds that \( u(t) = u_0 \) for all \( t \geq 0 \). This proves part (i) of Theorem 4.

Suppose now that \( \gamma > \alpha \). In this case, if \( 1 < p < (1/\gamma - \alpha) \), then (60) holds. Hence, proceeding as above, one obtains \( u(t) = u_0 \) for all \( t \geq 0 \), which proves part (ii) of Theorem 4.

**Proof of Theorem 5.** It is sufficient to observe that any global solution to problem (2) is a global solution to problem (1). Hence, using Theorem 3, one deduces that problem (2) admits no global solution.

**Proof of Theorem 6.** Let us suppose that \( u \in AC^2([0,\infty)) \) is a global solution to problem (2). Proceeding as in the Proof of Theorem 4 and using that \( u_t = 0 \), one obtains

\[
\int_0^T f(t)^\gamma dt \leq C \left( \int_0^T \xi(t)^{(1-\alpha)(p-1)} \left[ \left( I_T^{t-\alpha} \xi \right)^{\gamma} \right] (t) \right)^{(p/(p-1))} dt \\
+ \int_0^T \xi(t)^{-1/(p-1)} \left[ \left( I_T^{t-\alpha} \xi \right)^{\gamma} \right] (t) \right)^{(p/(p-1))} dt,
\]

(64)

for all \( T > 0 \), where \( C > 0 \) is a constant (independent on \( T \)). On the other hand, by (24), one has

\[
\int_0^T f(t)^\gamma dt = \int_0^T T^{-\lambda} (T - t)^{\lambda} f(t) dt \geq 2^{\lambda} \int_0^{T/2} f(t) dt.
\]

(65)

Moreover, by (50) and (52), one has

\[
\int_0^T \xi(t)^{-1/(p-1)} \left[ \left( I_T^{t-\alpha} \xi \right)^{\gamma} \right] (t) \right)^{(p/(p-1))} dt \\
+ \int_0^T \xi(t)^{-1/(p-1)} \left[ \left( I_T^{t-\alpha} \xi \right)^{\gamma} \right] (t) \right)^{(p/(p-1))} dt \\
\leq C_2 T^{((1-\alpha)p)/(1-\alpha)^2} + C_3 T^{((1-\alpha)p)/(1-\alpha)^2} + C_4 T^{((1-\alpha)p)/(1-\alpha)^2} + C_5 T^{((1-\alpha)p)/(1-\alpha)^2}.
\]

(66)

Next, it follows from (64), (65), and (66) that

\[
2^{\lambda} \int_0^{T/2} f(t) dt \leq CT^{((1-\alpha)p)/(1-\alpha)^2} \left( C_2 + C_3 T^{((1-\alpha)p)/(1-\alpha)^2} \right),
\]

(67)

which yields

\[
T^{((1-\alpha)p)/(1-\alpha)^2} \int_0^{T/2} f(t) dt \leq 2^{\lambda} C \left( C_2 + C_3 T^{((1-\alpha)p)/(1-\alpha)^2} \right).
\]

(68)

Finally, passing to the supremum limit as \( T \to +\infty \) in (68), using (14) and the fact \( \alpha \leq \beta \), a contradiction follows.
Proof of Corollary 7. For all $a \neq 0$, one has
\[ \int_0^{T/2} f(t) dt = \frac{1}{a} \left( e^{aT/2} - 1 \right). \]  
(69)

If $a > 0$, then
\[ \int_0^{T/2} f(t) dt \sim \frac{1}{a} e^{aT/2}, \quad \text{as } T \longrightarrow +\infty, \]  
(70)

which yields
\[ \lim_{T \to +\infty} T^{((a-\gamma)p+1)/(p-1))} \int_0^{T/2} f(t) dt = \lim_{T \to +\infty} \frac{1}{a} T^{((a-\gamma)p+1)/(p-1))} e^{aT/2} = +\infty. \]  
(71)

Hence, by Theorem 6, one deduces that for all $p > 1$, problem (2) admits no global solution, which proves part (i).

If $a < 0$, then
\[ \int_0^{T/2} f(t) dt \sim \frac{1}{a}, \quad \text{as } T \longrightarrow +\infty, \]  
(72)

which yields
\[ \lim_{T \to +\infty} T^{((a-\gamma)p+1)/(p-1))} \int_0^{T/2} f(t) dt = \lim_{T \to +\infty} \frac{1}{a} T^{((a-\gamma)p+1)/(p-1))} = +\infty. \]  
(73)

Therefore, if $\alpha \geq \gamma$, one has
\[ (\alpha - \gamma)p + 1 > 0, \quad p > 1, \]  
(74)

which yields
\[ \lim_{T \to +\infty} T^{((a-\gamma)p+1)/(p-1))} \int_0^{T/2} f(t) dt = +\infty, \quad p > 1. \]  
(75)

Hence, by Theorem 6, one deduces that for all $p > 1$, problem (2) admits no global solution, which proves part (ii). On the other hand, if $\gamma > \alpha$, one has
\[ (\alpha - \gamma)p + 1 > 0, \quad 1 < p < \frac{1}{\gamma - \alpha}, \]  
(76)

which yields
\[ \lim_{T \to +\infty} T^{((a-\gamma)p+1)/(p-1))} \int_0^{T/2} f(t) dt = +\infty, \quad 1 < p < \frac{1}{\gamma - \alpha}. \]  
(77)

Hence, by Theorem 6, one deduces that for all $1 < p < (1/\gamma - \alpha)$, problem (2) admits no global solution, which proves part (iii).

Proof of Corollary 8. For all $\sigma > -1$, one has
\[ \int_0^{T/2} f(t) dt = \frac{1}{2^{\sigma+1}(\sigma+1)} \left( T^{\sigma+1} - 2^{\sigma+1} \right). \]  
(78)

which yields
\[ \int_0^{T/2} f(t) dt \sim \frac{T^{\sigma+1}}{2^{\sigma+1}(\sigma+1)}, \quad \text{as } T \longrightarrow +\infty. \]  
(79)

Hence,
\[ \lim_{T \to +\infty} T^{((a-\gamma)p+1)/(p-1))} \int_0^{T/2} f(t) dt = \lim_{T \to +\infty} \frac{1}{2^{\sigma+1}(\sigma+1)} T^{((a-\gamma)p+1)/(p-1)) + \sigma+1}. \]  
(80)

Notice that
\[ \frac{(\alpha - \gamma)p + 1}{p - 1} + \sigma + 1 > 0 \iff (\alpha + 1 - \gamma + \sigma)p > \sigma. \]  
(81)

Hence, by Theorem 6, one deduces that for all $p > 1$ satisfying
\[ (\alpha + 1 - \gamma + \sigma)p > \sigma, \]  
(82)

problem (2) admits no global solution.

Consider the case $\sigma \geq 0$. In this case, for all $p > 1$, one has
\[ (\alpha + 1 - \gamma + \sigma)p > \sigma p \geq \sigma. \]  
(83)

Then (82) is satisfied for all $p > 1$. Hence, one deduces that for all $p > 1$, problem (2) admits no global solution, which proves part (i).

Suppose now that $-1 < \sigma < 0$. If $\gamma \leq \alpha$, then for all $p > 1$, one has
\[ (\alpha + 1 - \gamma + \sigma)p \geq (\sigma + 1)p > 0 > \sigma. \]  
(84)

Then, (82) is satisfied for all $p > 1$. Hence, one deduces that for all $p > 1$, problem (2) admits no global solution, which proves part (ii)(a). On the other hand, if $\gamma > \alpha$ and $-1 < \gamma - \alpha - 1 < \sigma < 0$, then
\[ (\alpha + 1 - \gamma + \sigma)p \geq 0 > \sigma. \]  
(85)

Hence, (82) is satisfied for all $p > 1$. Therefore, for all $p > 1$, problem (2) admits no global solution, which proves part (ii)(b). Finally, if $\gamma > \alpha$ and $-1 < \sigma < \gamma - \alpha - 1$, then for all $p > 1$, (82) is equivalent to
\[ 1 < p < \frac{\sigma}{\alpha + 1 - \gamma + \sigma}. \]  
(86)

Hence, for all $p$ satisfying the above condition, problem (2) admits no global solution, which proves part (ii)(c).
Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors’ Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Acknowledgments

The authors would like to extend their sincere appreciation to the Deanship of Scientific Research at King Saud University for its funding of this research through the Research Group Project No. RGP-1435-034.

References

[1] R. L. Bagley and P. J. Torvik, "Fractional calculus in the transient analysis of viscoelastically damped structures," AIAA Journal, vol. 23, no. 6, pp. 918–925, 1985.
[2] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.
[3] J. A. T. Machado, A. M. S. F. Galhano, and J. J. Trujillo, "On development of fractional calculus during the last fifty years," Scientometrics, vol. 98, no. 1, pp. 577–582, 2014.
[4] Y. A. Rossikhin and M. V. Shitikova, "Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids," Applied Mechanics Reviews, vol. 50, no. 1, pp. 15–67, 1997.
[5] S. G. Samko, A. A. Kilbas, and O. I. Marichev, Fractional Integrals and Derivatives: Theory and Applications, Gordon and Breach, Yverdon, Switzerland, 1993.
[6] H. Aydi, M. Jleli, and B. Samet, "On positive solutions for a fractional thermostat model with a convex-concave source term via $(\psi\phi)$-Caputo fractional derivative," Mediterranean Journal of Mathematics, vol. 17, no. 1, p. 16, 2020.
[7] Z. Bai, "On positive solutions of a nonlocal fractional boundary value problem," Nonlinear Analysis, vol. 72, no. 2, pp. 916–924, 2010.
[8] T. Chen, W. Liu, and Z. Hu, "A boundary value problem for fractional differential equation with –Laplace operator at resonance," Nonlinear Analysis, vol. 75, no. 6, pp. 3210–3217, 2012.
[9] J. Jiang and L. Liu, "Existence of solutions for a sequential fractional differential system with coupled boundary conditions," Boundary Value Problems, vol. 2016, no. 1, 2016.
[10] M. Jleli and B. Samet, "Existence of positive solutions to an arbitrary order fractional differential equation via a mixed monotone operator method," Nonlinear Analysis: Modelling and Control, vol. 20, no. 3, pp. 367–376, 2015.
[11] X. Liu, M. Jia, and W. Ge, "The method of lower and upper solutions for mixed fractional four-point boundary value problem with $p$-Laplacian operator," Applied Mathematics Letters, vol. 65, pp. 56–62, 2017.
[12] Y. Zou and G. He, "On the uniqueness of solutions for a class of fractional differential equations," Applied Mathematics Letters, vol. 74, pp. 68–73, 2017.
[13] A. Alsaeedi, B. Ahmad, M. B. M. Kirane, F. S. K. al Musali, and F. Alzahrani, "Blowing-up solutions for a nonlinear time-fractional system," Bulletin of Mathematical Sciences, vol. 7, no. 2, pp. 201–210, 2017.
[14] K. M. Furati, M. D. Kassim, and N.-E. Tatar, "Non-existence of global solutions for a differential equation involving Hilfer fractional derivative," Electronic Journal of Differential Equations, vol. 2013, no. 235, pp. 1–10, 2013.
[15] K. M. Furati and M. Kirane, "Necessary conditions for the existence of global solutions to systems of fractional differential equations," Fractional Calculus and Applied Analysis, vol. 11, no. 3, pp. 281–298, 2008.
[16] M. D. Kassim, K. M. Furati, and N.-E. Tatar, "Nonexistence of global solutions for a fractional differential problem," Journal of Computational and Applied Mathematics, vol. 314, pp. 61–68, 2017.
[17] Y. Laskri and N.-E. Tatar, "The critical exponent for an ordinary fractional differential problem," Computers & Mathematics with Applications, vol. 59, no. 3, pp. 1266–1270, 2010.
[18] A. Mennouni and A. Youkana, "Finite time blow-up of solutions for a nonlinear system of fractional differential equations," Electronic Journal of Differential Equations, vol. 152, pp. 1–15, 2017.
[19] B. Samet, "Nonexistence of global solutions for a class of sequential fractional differential inequalities," The European Physical Journal Special Topics, vol. 226, no. 16-18, pp. 3513–3524, 2017.
[20] J. R. Wang, M. Feckan, and Y. Zhou, "Nonexistence of periodic solutions and asymptotically periodic solutions for fractional differential equations," Communications in Nonlinear Science and Numerical Simulation, vol. 18, no. 2, pp. 246–256, 2013.
[21] X. Zhang, L. Liu, Y. Wu, and Y. Cui, "New result on the critical exponent for solution of an ordinary fractional differential problem," Journal of Function Spaces, vol. 2017, Article ID 3976469, 4 pages, 2017.
[22] X. Zhao, C. Chai, and W. Ge, "Existence and nonexistence results for a class of fractional boundary value problems," Journal of Applied Mathematics and Computing, vol. 41, no. 1-2, pp. 17–31, 2013.
[23] M. Jleli and B. Samet, "Finite time blow-up for a nonlocal in time nonlinear heat equation in an exterior domain," Applied Mathematics Letters, vol. 99, article 105985, 2020.
[24] M. Kirane, Y. Laskri, and N.-E. Tatar, "Critical exponents of Fujita type for certain evolution equations and systems with spatio-temporal fractional derivatives," Journal of Mathematical Analysis and Applications, vol. 312, no. 2, pp. 488–501, 2005.
[25] M. Kirane, M. Medved, and N.-E. Tatar, "On the nonexistence of blowing-up solutions to a fractional functional differential equations," Georgian Mathematical Journal, vol. 19, no. 1, pp. 127–144, 2012.
[26] E. Mitidieri and S. I. Pohozaev, "A priori estimates and blow-up of solutions to nonlinear partial differential equations and inequalities," Proceedings of the Steklov Institute of Mathematics, vol. 234, pp. 1–383, 2001.