Subsonic critical velocity at finite temperature

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Based on the dielectric formalism in the generalized random phase approximation, we generalize the description of a Bose condensed gas to allow for a relative velocity between the superfluid and normal fluid. In this model, we determine the critical velocity dynamically as the transition point between stable and unstable dynamics. Unlike the zero temperature case, at finite temperature the relative critical velocity of a dilute Bose gas is lower than the sound velocity. This result illustrates one relevant difference that exists between a conserving and gapless approximation and other approaches.

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One of the most fascinating aspects of the observed Bose condensation phenomenon is the superfluidity property i.e. the ability of the superfluid fraction of the Bose gas to move without apparent friction. No dissipation occurs provided that the relative motion of the superfluid with an obstacle, like an impurity or a local potential or a wall, does not exceed the sound velocity. Otherwise, the Landau criterion predicts that a phonon can be emitted decreasing the velocity of the superfluid \( \bar{v} \). The situation is drastically different if the obstacle corresponds to the normal component of the gas. In principle, the Landau criterion cannot be applied to the normal fluid as a whole but to each of its individual thermal excitation. Unfortunately, some of them are highly energetic and display a relative velocity higher than the sound velocity. In order to explain the persistence of the relative motion between the normal and superfluid, a full microscopic description is necessary. A quantum kinetic equation explaining this persistence has been proposed in \( \bar{v} \) and has been subsequently extended in \( \bar{v} \). At low temperature i.e. \( k_B T < g n \) (where \( g = 4 \pi a/m \), \( a \) the scattering length and \( n \) the total gas density), this equation is an extension of the Beliaev approach for a non equilibrium Bose gas and predicts that the critical and sound velocity are identical. However, the extension of the Beliaev theory including the presence of a thermal component is not conserving \( \bar{v} \). In particular the conservation law for the total mass density is missing in the systematic calculation made in \( \bar{v} \) using the Keldysh formalism. On the contrary, at high temperatures \( (k_B T > g n) \), the conservation laws are fulfilled but at the price of having a gap in the energy spectrum rendering possible the frictionless motion \( \bar{v} \). Of course, by the Hugenholtz-Pines theorem, a gap is forbidden and cannot be used to explain the superfluidity phenomenon. As a consequence, a gapless and conserving approach is needed in order to improve the description in a large range of temperature. Recently, such an approach has been developed in both equilibrium \( \bar{v} \) and in non equilibrium \( \bar{v} \) using the generalized random phase approximation and has been successful to explain the low-lying excitation observed in a trapped Bose gas \( \bar{v} \).

In this letter, based on this more coherent approach, we delineate the region for which a relative velocity between the normal fluid and the superfluid is possible at finite temperature. Unlike for other obstacles, the transition region from metastability to instability, due to counterflow between normal and superfluid components, occurs for velocity lower than the sound velocity. The persistence is explained both by the equilibrium and non equilibrium approaches. In equilibrium, we introduce an \( \eta \)-ensemble for which the condensed fraction evolves with a velocity \( \bar{v}_s \). The study of the dynamical fluctuations around this equilibrium allows to establish a critical value of \( \bar{v}_s \) above which these become unstable. In non equilibrium, a previous work has shown that the supression of the binary collision process between condensed and non condensed atoms allows precisely this persistence \( \bar{v}_s \). Above the critical velocity, this collisionless regime disappears leading to a damping of the relative velocity. We will show that the equilibrium formalism provides identical results to the non equilibrium one, in particular for the dispersion relation of the collective excitations.

To start with, we consider a homogenous Bose condensed gas populated with atoms with mass \( m \) in a volume \( V \). For any momentum \( k \), \( n_k \) is the density of the condensate and \( n_k^{\text{nc}} \) the density of the normal cloud in the mode \( k \). We extend the formalism developed in \( \bar{v} \) to the case where the \( N_c = V n_c \) condensate particles are moving with a momentum \( k_s \). For that purpose, we break the \( U(1) \) symmetry introducing the \( \eta \)-ensemble by adding the term proportional to the field operator \( c_{k_s} \) to the total many body Hamiltonian \( H \). Thus, any thermal averaging is written as:

\[
\langle A \rangle = \text{Tr}(A e^{-\beta (H - n_k \delta c_k^\dagger \delta c_k)})/\text{Tr}(e^{-\beta (H - n_k \delta c_k^\dagger \delta c_k)}) \quad (1)
\]

In this way, the condensate wave function \( \langle c_{k_s} \rangle = \sqrt{N_c} \) is adjusted to move with a velocity \( v_s = k_s/m \). The field operator in the momentum space can be written as \( c_k = \delta c_k + \delta_k \delta k_s \sqrt{N_c} \).

The Green's functions describing the non condensed fraction can be written as usual \( \bar{v} \):

\[
G_{\alpha\beta}(k,\omega) = -\int d\tau e^{i\omega\tau} \langle T_{\tau} [\delta c_{k\alpha}(\tau)\delta c_{k\beta}^\dagger(0)] \rangle \quad (2)
\]
where $\delta q_{\alpha}$ is $\delta k$ for $\alpha = 1$ and $\delta \epsilon_{2k_a - k}^\pm$ for $\alpha = 2$. The only difference with the previous approach \cite{7} is that the propagator line of momentum $k$ is coupled to the one with momentum $2k_s - k$ (see Fig.1). The Green’s functions satisfy the Beliaev-Dyson relations:

$$G_{\alpha\beta} = G_{\alpha\beta}^0 + G_{\alpha\gamma}^0 \Sigma_{\gamma\delta} G_{\alpha\delta}$$

where $\Sigma_{\gamma\delta}(k, \omega)$ are the usual self-energy matrix and where the Green’s functions of the non interacting system in presence of the momentum $k_s$ is given by:

$$G_{\alpha\beta}^0(k, \omega) = \begin{pmatrix} \omega - \epsilon_k + \mu & 0 \\ -\omega - \epsilon_{2k_a - k} + \mu \end{pmatrix}$$

where $\epsilon_k = k^2/(2m)$. Solving \cite{8}, we obtain:

$$G_{\alpha\beta}(k, \omega) = \frac{1}{D} \begin{pmatrix} \omega + \epsilon_{2k_a - k} - \mu + \Sigma_{22} & \Sigma_{12} \\ \Sigma_{21} & \epsilon_k - \omega - \mu + \Sigma_{11} \end{pmatrix}$$

where

$$D = (\omega - q.k_s/m - A_-)^2 - (\epsilon_q + \epsilon_{k_s} - \mu + A_+)^2 + \Sigma_{12} \Sigma_{21}$$

$$A_\pm = (\Sigma_{11} \pm \Sigma_{22})/2$$

and $q = k - k_s$. The proper part of a quantity is the sum of only those of its diagrams which remain connected after cutting a single interaction line. All the unknown functions can be expressed in terms of regular (i.e. proper and one-particle irreducible) quantities, denoted by an upper index:

$$\chi_{nn}(q, \omega) = \chi_{nn}^{(r)}(q, \omega) + A_{\alpha}(q, \omega) G_{\alpha\beta}(k, \omega) A^{(r)}_{\beta}(q, \omega)$$

$$\tilde{G}_{\alpha\beta} = G_{\alpha\beta}^0 + G_{\alpha\gamma}^0 \Sigma_{\gamma\delta}^{\pm} \tilde{G}_{\alpha\delta}$$

$$\Sigma_{\alpha\beta}(k, \omega) = \Sigma_{\alpha\beta}^{(r)}(k, \omega) + \frac{g A^{(r)}_{\alpha}(q, \omega) A^{(r)}_{\beta}(q, \omega)}{1 - g \chi_{nn}^0(q, \omega)}$$

The main difference to \cite{7} is that the susceptibility function is expressed in terms of the momentum $q = k - k_s$ relative to the condensate momentum while the Green’s function is expressed in terms of the absolute momentum $k$. As a consequence, the momentum of the incoming and outgoing lines in the vertex function must be relabelled adequately as shown in Fig.(2).

Once the regular functions are known, any physical quantities can be calculated. In the random phase approximation, these functions are found to be:

$$\chi_{nn}^{(r)}(q, \omega) = \chi_{nn}^0(q, \omega)/(1 - g \chi_{nn}^0(q, \omega))$$

$$A_{\alpha}^0(q, \omega) = \sqrt{N_c}/(1 - g \chi_{nn}^0(q, \omega))$$

$$\Sigma_{\alpha\beta}^{(r)}(k, \omega) = g(2n - n_c) \delta_{\alpha\beta} - g^2 N_c \chi_{nn}^0(q, \omega)/(1 - g \chi_{nn}^0(q, \omega))$$

These are expressed in terms of the susceptibility function of the non interacting normal fluid:

$$\chi_{nn}^0(q, \omega) = \sum_k \frac{\rho'}{\omega - \frac{k.s - q}{m} - \frac{q^2}{2m}}$$

Below the condensation point, the chemical potential is determined through the tadpole condition or the Hugenholtz-Pines theorem $\mu = \epsilon_k + g(2n - n_c)$ and contains a kinetic energy term of the condensate particle $\epsilon_{k_s}$. The particle momentum distribution for a normal fluid at rest reads in the Hartree-Fock approximation as $n'_k = 1/[e^{\beta(\epsilon_k^2/2m^2 + 2n_m - \mu)} - 1]$. The equation of state is:

$$n = n_c + \frac{1}{\lambda_{th}^3} g_{th}(e^{\beta(\mu - 2g_m)})$$

where $g_n(x) = \sum_{j=1}^{\infty} x^j / j^n$ and $\lambda_{th} = \sqrt{2\pi/mk_BT}$ denotes the thermal wavelength. The critical temperature is given by $k_BT_c = 2\pi m (n/\zeta(3/2))^{2/3}$.

With all these building blocks modified in presence of $k_s$, we get the results:
\[ G_{11}(k, \omega) = \frac{(\omega - k_s q/m + \epsilon_q)[1 - 2g\chi_n^0(q, \omega)] + gn_c[1 + 2g\chi_n^0(q, \omega)]}{\Delta(q, \omega)} \]  
(17)

\[ G_{12}(k, \omega) = -\frac{gn_c[1 + 2g\chi_n^0(q, \omega)]}{\Delta(q, \omega)} \]  
(18)

\[ \chi_{nn}(q, \omega) = \frac{[\omega - k_s q/m]^2 - c_s^2]}{\Delta(q, \omega)} \chi_n^0(q, \omega) + 2n_c\epsilon_q[1 + 2g\chi_n^0(q, \omega)] \]  
(19)

\[ \Delta(q, \omega) = \frac{[\omega - k_s q/m]^2 - c_s^2}{[1 - 2g\chi_n^0(q, \omega)] - 2gn_c\epsilon_q[1 + 2g\chi_n^0(q, \omega)]} \]  
(20)

We are looking for the zeroes of \( \Delta(q, \omega) \) the dielectric function. The real part and imaginary part of the roots \( \omega \) correspond to the collective excitation and the Landau damping respectively and have been determined previously in the case of thermodynamic equilibrium [2, 3]. For low momentum excitations, we consider the limit \( \omega \to 0 \) and \( |q| \to 0 \) and introduce the complex velocity of sound by \( \omega/|q| = c \). The only difference is that now we must consider terms of the form \( k_s q/|q| \). If the wavevector \( q \) is parallel to the critical velocity, we find the following generalization:

\[ \frac{(c - v_s)^2 - c_B^2}{(c - v_s)^2 + c_B^2} = \frac{2g}{k_B T \lambda_{3h}} \chi_n \left( \frac{c}{(2k_B T/m)^{1/2}} \right) \]  
(21)

where

\[ \chi_n(s) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dt \frac{1}{s - t} e^{t^2 - s^2(2m/2g_A)^2} - 1 \]  
(22)

is the dimensionless response function defined in the upper half of the complex plane. The critical velocity is defined as the velocity beyond which the amplitude of the collective excitations grows exponentially. In other words, if the imaginary part of any solution \( c \) changes its sign from negative to positive, then a transition from metastability to instability occurs. The point at which this transition occurs corresponds to the situation where \( \text{Im} c = 0 \) in Eq. (21) and, consequently, the imaginary part in the r.h.s. of (21) must be equal to zero. This condition is satisfied provided that the imaginary part of Eq. (22) is zero which is the case only for \( \text{Re} c = 0 \). Thus we are left with a closed equation for the critical velocity \( v_s \) that can be solved exactly:

\[ v_s = c_B \sqrt{\frac{k_B T \lambda_{3h}^3 + 2g\chi_n(0)}{k_B T \lambda_{3h}^3 - 2g\chi_n(0)}} \]  
(23)

\[ v_s = c_B \sqrt{\frac{k_B T \lambda_{3h}^3 - 2g g_{1/2}(e^{\beta(v_s - g_{1/2})})}{k_B T \lambda_{3h}^3 + 2g g_{1/2}(e^{\beta(v_s - g_{1/2})})}} \]  
(24)

It is common in the literature (e.g. [1]) to determine the critical velocity from the necessary condition that the release energy of an excitation from the moving condensate becomes positive. In contrast, we use the stability condition on the dynamics of the collective excitation which is necessary and sufficient to determine the critical velocity and this requires the knowledge of the Landau damping. As a consequence, this different criterion is not based on the energetic comparison between two thermodynamic states alone but rather on dynamical notions.

The critical velocity is calculated as a function of the temperature using (23). The dependences are displayed in Fig.3 and 4 for \( gn/k_BT_{c} = 0.1 \) and 0.3 respectively. In both figures, the critical velocity is lower than the sound velocity for temperatures not too close to the transition point by an amount which can exceed 10 percents. This difference is due to the particular dependence of the self-energy matrix on the momentum and the frequency in the presence of depletion. This dependence is needed in order to calculate the sound velocity whereas only the static expressions are required for the critical velocity. For comparison, we have also plotted the sound velocity obtained in the Popov approximation \( c_P = \sqrt{gn_c/m} \) [1]. That approximation is gapless but not conserving and the self-energy matrix does not depend on \( k \) and \( \omega \).
Knowing that $\Sigma_{11} = 2gn$ and $\Sigma_{12} = gn_c$, the application of the previous reasoning to this approximation allows to deduce that the spectrum of collective excitations does not contain any imaginary part. Since no dynamical instability appears, only the condition of sign change in the energy spectrum allows to determine the critical velocity which in this case corresponds to the sound velocity. For $T/T_c > 0.8$, we notice about a factor 2 of difference between the critical values obtained in the two approaches. As a consequence, the requirement that an approximation should be conserving modifies drastically the prediction.

The influence of the relative velocity on the condensate population is also analyzed. Since the chemical potential depends on the relative velocity, the equation of state [10] shows that the condensate fraction decreases when the velocity increases. This result is plotted in Fig.5 where we compare the two situations without relative velocity and at critical velocity which show only a minor difference of about few percents.

In conclusion, we determine the critical velocity from the dynamics of the collective excitations using the gapless and conserving generalized random phase approximation. The calculated value displays relevant differences in comparison with the Popov approach and appears to be subsonic for a wide range of temperature.

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FIG. 4: Same as for Fig.3 but for $gn/k_BT_c = 0.3$

FIG. 5: Condensate fraction as a function of temperature for $gn/k_BT_c = 0.3$ at the critical velocity (full line) and in the absence of a relative velocity (dashed line).