A Variational Inference based Detection Method for Repetition Coded Generalized Spatial Modulation

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Abstract—In this paper, we consider a simple coding scheme for spatial modulation (SM), where the same set of active transmit antennas is repeatedly used over consecutive multiple transmissions. Based on a Gaussian approximation, an approximate maximum likelihood (ML) detection problem is formulated to detect the indices of active transmit antennas. We show that the solution to the approximate ML detection problem can achieve a full coding gain. Furthermore, we develop a low-complexity iterative algorithm to solve the problem with low complexity based on a well-known machine learning approach, i.e., variational inference. Simulation results show that the proposed algorithm can have a near ML performance. A salient feature of the proposed algorithm is that its complexity is independent of the number of active transmit antennas, whereas an exhaustive search for the ML problem requires a complexity that grows exponentially with the number of active transmit antennas.

Index Terms—Spatial Modulation; Index Modulation; Repeated Transmit Diversity; Variational Inference

I. INTRODUCTION

In wireless communications, multiple-input multiple-output (MIMO) systems play a crucial role in improving the spectral efficiency by exploiting multiple antennas [1], [2] and have been actively considered for cellular systems, e.g., long-term evolution (LTE) [3] and fifth generation (5G) systems [4].

There have been various transmission schemes over MIMO channels that have different advantages and characteristics due to the fundamental tradeoff between diversity and multiplexing gain of MIMO channels [5]. In [6], a simple but efficient approach to exploit the spatial multiplexing gain of MIMO channels is proposed without channel state information (CSI) at the transmitter. This approach, which is called spatial modulation (SM), considers multiple transmit antennas as additional constellation points and uses one transmit antenna at a time, which requires a single radio frequency (RF) chain. As a result, although there can be a number of transmit antennas, the transmitter’s cost can be low. In [7], the optimal detection for SM is studied with performance analysis. In [8], [9], it is shown that the complexity of the maximum likelihood (ML) detection involving joint detection of the transmit antenna index and of the transmitted symbol can be low if a square-quadrature amplitude modulation (QAM) or phase shift keying (PSK) is employed. With multiple RF chains, SM can be further generalized where more than one transmit antenna can be active at a time as an index modulation (IM) scheme in the space domain [10]–[13]. The resulting SM is called generalized SM.

The main contributions of the paper are two-fold. First, we propose a transmit coding scheme for SM to allow a reliable detection of IM bits and show that a full transmit coding gain can be achieved by deriving an approximate expression for the probability of index error. Secondly, in order to lower the computational complexity of the detection, we consider
a two-step approach and propose a low-complexity iterative algorithm for the first step based on variational inference. The resulting approach can also provide an approximate solution to the ML problem in [25], [26] with the complexity that grows linearly with the number of transmit antennas and is independent of the number of active transmit antennas. On the other hand, the complexity of the approaches in [25], [26] grows exponentially with the number of active antennas.

The rest of the paper is organized as follows. Section II presents a system model for SM with a transmit coding scheme. Based on a Gaussian approximation, we formulate an approximate ML problem to detect the indices of active transmit antennas in Section III. In Section IV, we apply variational inference to the approximate ML problem and develop an iterative algorithm. A performance analysis is carried out in Section V to find the probability of index error of the approximate ML problem. We present simulation results in Section VI and conclude the paper with some remarks in Section VII.

Notation: Matrices and vectors are denoted by upper- and lower-case boldface letters, respectively. The superscripts \( T \) and \( H \) denote the transpose and complex conjugate, respectively. The \( p \)-norm of a vector \( \mathbf{a} \) is denoted by \( ||\mathbf{a}||_p \) (If \( p = 2 \), the norm is denoted by \( ||\mathbf{a}|| \) without the subscript). The support of a vector is denoted by \( \text{supp}(x) \) (which is the index set of the non-zero elements of \( x \)). \( \mathbb{E}[] \) and \( \text{Var}(\cdot) \) denote the statistical expectation and variance, respectively. \( \mathcal{CN}(\mathbf{a}, \mathbf{R}) \) represents the distribution of circularly symmetric complex Gaussian (CSCG) random vectors with mean vector \( \mathbf{a} \) and covariance matrix \( \mathbf{R} \).

II. SYSTEM MODEL

Suppose that there are \( L \) transmit and \( N \) receive antennas and denote by \( \mathbf{H} \) the MIMO channel matrix of size \( N \times L \). Let \( s_t \) denote the signal to be transmitted through the \( t \)-th transmit antenna. The received signal vector is given by

\[
\mathbf{y} = \mathbf{H} \mathbf{s} + \mathbf{n},
\]

where \( \mathbf{s} = [s_1 \ldots s_L]^T \) is the symbol vector and \( \mathbf{n} \sim \mathcal{CN}(0, \mathbf{N}_0 \mathbf{I}) \) is the background noise. In SM [12], there are a few RF chains, say \( K \), and only \( K \) symbols are non-zero, while the other symbols are zero. For convenience, the transmit antennas corresponding to non-zero symbols are referred to as active transmit antennas and \( K \) is referred to as the number of active transmit antennas. Clearly, \( K \leq L \). In addition, let \( \mathcal{I} \) denote the set of the indices of active transmit antennas, while the indices of active transmit antennas are referred to as active indices for convenience. In addition, a non-zero symbol, \(|s_t| > 0\), is called data modulation (DM) symbol. Denote by \( \mathcal{S} \) the alphabet of DM symbols. The mean and variance of a non-zero DM symbol, i.e., \(|s| > 0\), are assumed to be \( 0 \) and \( \sigma_s^2 \), respectively. For equally likely DM symbols, we have \( \sigma_s^2 = \mathbb{E}[|s|^2] = \frac{1}{|\mathcal{S}|} \sum_{s \in \mathcal{S}} |s|^2 \). While DM symbols can represent information bits, which are referred to as DM bits, as in conventional modulation schemes, the index set of non-zero symbols or the indices of active transmit antennas can also represent information bits that are referred to IM bits. The number of IM bits is given by \( B_{\text{IM}} = \left\lfloor \log_2 \left( \frac{K}{l} \right) \right\rfloor \).

In general, provided that the set of active indices is correctly decided, the detection of DM symbols can be reliably performed using various MIMO detectors for a sufficiently small \( K \) [27], [28]. On the other hand, the detection performance of active indices or IM bits can be relatively poor (especially, when \( L > N \)) for low and moderate signal-to-noise ratio (SNR) [16]. To avoid this problem, we can consider a modification of SM with multiple symbol vectors based on the approach in [22]. Suppose that a slot consists of \( M \) consecutive symbol vectors, where \( M \) is referred to as the slot length. For each symbol vector in a slot, different DM bits are transmitted, while the IM bits are fixed. In this case, the \( m \)-th received signal vector can be given by

\[
\mathbf{y}_m = \mathbf{H} \mathbf{s}_m + \mathbf{n}_m, \quad m = 1, \ldots, M, \tag{2}
\]

where \( \mathbf{s}_m \) is the \( m \)-th symbol vector and \( \mathbf{n}_m \sim \mathcal{CN}(0, \mathbf{N}_0 \mathbf{I}) \) is the background noise. Since the same IM bits are transmitted, we have

\[
\mathcal{I} = \text{supp}(\mathbf{s}_m), \quad m = 1, \ldots, M. \tag{3}
\]

The resulting SM with repeated uses of the same set of active indices is referred to as repetition coding SM (RCSM) in this paper. The total number of bits per slot that can be transmitted by RCSM becomes

\[
B_{\text{SYM}} = K \log_2 |\mathcal{S}| + \frac{\log_2 \left( \frac{K}{l} \right)}{M}. \tag{4}
\]

Clearly, the spectral efficiency decreases with \( M \). However, we expect to have a higher coding gain for a large \( M \), which results in a more reliable detection for IM bits.

III. ML APPROACH TO INDEX DETECTION

In this section, we consider an ML detection approach for IM bits or active indices in RCSM. Throughout the paper, we assume that the receiver has perfect CSI.

Prior to the discussion of an ML approach to detect the set of active indices, we consider a simple low-complexity detection method based on the correlators. The detection of active indices is similar to the detection of the spreading codes that are used by multiple users in the context of code division multiple access (CDMA). Assuming that each column of \( \mathbf{H} \) as a spreading code, the energy of the correlator can be used to decide whether or not the corresponding transmit antenna is active. To this end, the energy of the \( l \)-th correlator can be defined as \( p_l = \sum_{m=1}^{M} |\mathbf{h}_l^T \mathbf{y}_m|^2 \), where \( \mathbf{h}_l \) denotes the \( l \)-th column vector of \( \mathbf{H} \). Denote by \( \hat{l}(k) \) the index corresponding to the \( k \)-th largest energy of the correlator output. Then, the set of active indices can be estimated as

\[
\hat{l}_{\text{corr}} = \{\hat{l}(1), \ldots, \hat{l}(K)\}. \tag{5}
\]

This approach could provide a reasonably good performance if the \( \mathbf{h}_l \)'s are orthogonal or \( N \gg K \). The computational complexity of the correlator based detector is \( O(LN) \).
If $N$ is not sufficiently large, the correlator based detector cannot provide a good performance. Thus, we may need to consider an optimal approach \([7], [12]\). For convenience, let

$$x_l = \begin{cases} 1, & \text{if } |s_{m,l}| > 0 \\ 0, & \text{if } |s_{m,l}| = 0, \end{cases}$$

where $s_{m,l}$ represents the $l$th element of $s_m$. Let $x = [x_1 \ldots x_L]^T$, which is referred to as the index vector. In addition, define

$$X_K = \{x \mid ||x||_0 = K, \ x_l \in \{0,1\}\}. \quad (6)$$

The ML detection problem can be formulated as

$$\{\hat{s}_m, x\} = \arg\max_{x \in X_K, \{s_m\} \in S^K} f(y \mid \{s_m\}, x)$$

$$= \arg\max_{x \in X_K} \max_{\{s_m\} \in S^K} f(y \mid \{s_m\}, x), \quad (7)$$

where $f(y \mid \{s_m\}, x)$ is the likelihood function of $\{s_m\}$ and $x$ for given $y = [y_1^T \ldots y_M^T]^T$ and $|s|x$ represents the subvector of $s$ obtained by taking the elements corresponding to the non-zero elements of $x$. Unfortunately, the complexity to solve (7) is prohibitively high. To lower the complexity, we may consider a suboptimal two-step approach. In the first step, we detect the support of $s_m$, i.e., $\tilde{I}$. Once an estimate of $\tilde{I}$ is available, the detection of the non-zero elements of $s_m$ are carried out in the second step. Since the second step is the conventional MIMO detection, throughout the paper, we mainly focus on the first step, which is referred to as the support or index detection.

For the first step, we can consider the ML detection of $\tilde{I}$ or $x$ for given $y$ as follows:

$$\hat{x} = \arg\max_{x \in X_K} f(y \mid x), \quad (8)$$

where

$$f(y \mid x) = \prod_{m=1}^{M} \sum_{s_m} f(y_m \mid s_m, x) \Pr(s_m \mid x). \quad (9)$$

The likelihood function of $x$ in (9) is a Gaussian mixture. Thus, the computational complexity to find the likelihood function of $x$ for given $y$ in (9) can be high due to the summation over $s_m$. To avoid this problem, we resort to an approximation.

For given $x$, $s_m$ can also be expressed as

$$s_m = D_m x, \quad (10)$$

where $D_m = \text{diag}(d_{m,1}, \ldots, d_{m,L})$ is a diagonal matrix. Clearly, $d_{m,l} = s_{m,l}$ if $x_l = 1$. Otherwise, $d_{m,l}$ can be any non-zero element in $\tilde{I}$. Let us employ an Gaussian approximation where the $d_{m,l}$’s are assumed to be CSCG random variables with zero-mean and variance $\sigma^2$. Furthermore, for convenience, we assume that $\sigma^2 = 1$, while $N_0$ becomes $\gamma^{-1}$, where $\gamma$ represents the SNR, which is given by $\gamma = E[|s|^2]/N_0$. Based on the Gaussian approximation, for given $x$ and $H$, we have

$$y_m \mid x \sim f(y_m \mid x) \approx f_{ga}(y_m \mid x) = CN(0, HR_x H^H + \gamma^{-1}I), \quad (11)$$

where

$$R_x = \text{diag}(x_1, \ldots, x_L). \quad (12)$$

Then, the index detection based on the approximate likelihood function in (11) can be carried out as follows:

$$\hat{x}_{ga} = \arg\max_{x \in X_K} \prod_{m=1}^{M} f_{ga}(y_m \mid x)$$

$$= \arg\min_{x \in X_K} \sum_{m=1}^{M} y_m^H V(x)y_m + M\phi(x), \quad (13)$$

where $V(x) = (HR_x H^H + \gamma^{-1}I)^{-1}$ and $\phi(x) = \ln \det(HR_x H^H + \gamma^{-1}I)$. The detection problem in (13) is also considered (with $M = 1$) in [25], [26].

In RCSM, we expect to have a coding gain of $M$ in detecting the set of active indices, $\tilde{I}$, or $x$. However, in (13), since the approximate ML formulation is considered to lower the complexity, it is not clear whether or not the detection based on (13) can exploit a full coding gain. Fortunately, as will be shown in Section V, the solution to (13) can achieve a full transmit coding gain, $M$.

Thanks to the Gaussian approximation, the complexity to find the likelihood for given $x$ can be low compared to that in (9). However, the approximate ML formulation in (13) requires a complexity that is proportional to $|X_K|$ if an exhaustive search is used. In the next section, we apply a well-known machine learning approach to find an approximate solution to (13) with low complexity.

### IV. Application of Variational Inference for Index Detection

In this section, we consider a variational inference approach \([23], [24]\) for the index detection to solve (13) with low complexity, where a soft-decision of $x$ (rather than a hard-decision) is considered.

#### A. Variational Inference based Detection

We assume that the elements of $x$ are independent random variables in (13), which are seen as the latent variables. In particular, we consider the following assumption.

**A0** The $x_l$’s are independent binary random variables and denote by $q_l$ the probability that $x_l = 1$, i.e., $q_l = \Pr(x_l = 1)$, which is referred to as the variational probability.

In the context of variational inference \([23], [24], [29]\), the estimation of $x$ becomes the following optimization problem:

$$q^* = \arg\min_{q \in \mathcal{Q}} \text{KL}(q \mid \text{Pr}(x \mid y)), \quad (14)$$

where $\mathcal{Q}$ represents the collection of all the possible distributions of $q = [q_1 \ldots q_L]^T$, $\text{Pr}(x \mid y)$ is the a posteriori probability of $x$, and $\text{KL}(\cdot)$ is the Kullback-Leibler (KL) divergence \([30]\). Then, from $q^*$, the estimate of $x$ can be found as

$$\hat{x}_l = \begin{cases} 1, & \text{if } q^*_l \text{ is one of the } K \text{ largest values of } q^* \\ 0, & \text{o.w.} \end{cases} \quad (15)$$
As shown in [23], the minimization of the KL divergence in [14] is equivalent to the maximization of the evidence lower bound (ELBO), which is given by
\[
\text{ELBO}(q) = \mathbb{E}[\ln f(y, x)] - \mathbb{E}[\ln \Pr(x)],
\]
where the expectation is carried out over \(x\) under the assumption of \(A0\). Let \(x_{-i} = [x_1 \ldots x_{i-1} x_{i+1} \ldots x_L]^T\). Then, for given \(x_{-i}\), it can be shown that
\[
q_i \propto \exp (\mathbb{E}_{-i}[\ln f(x_i | x_{-i}, y)]),
\]
where the expectation, denoted by \(\mathbb{E}_{-i}\), is carried out over \(x_{-i}\). The coordinate ascent variational inference (CAVI) algorithm [23, 24] is to update the coordinate ascent variational inference (CAVI) algorithm where the approximation is due to the Gaussian approximation where the expectation is carried out over \(A0\).

Lemma 1. Under the assumption of \(A0\), we have
\[
\begin{align*}
\mathbb{E}_{-i}[\ln \Pr(x_i | y, x_{-i})] &= \mathbb{E}_{-i}[\ln f(y | x_{-i}, x_i)] + C_1 \\
&= -\sum_{m=1}^M y_m^T \mathbb{E}_{-i}[V(x)] y_m - M \mathbb{E}_{-i}[\phi(x)] + C_2,
\end{align*}
\]
where the approximation is due to the Gaussian approximation as in [13], and \(C_1\) and \(C_2\) are constants.

As shown in [22], we need to obtain \(\mathbb{E}_{-i}[V(x)]\) and \(\mathbb{E}_{-i}[\phi(x)]\) as closed-form expressions to perform (17). Unfortunately, it is not easy to find closed-form expressions. However, we can find tight bounds in the following lemma.

**Lemma 1.** Under the assumption of \(A0\), we have
\[
\begin{align*}
\mathbb{E}_{-i}[V(x)] &\geq R_t(x_i)^{-1} \\
\mathbb{E}_{-i}[\phi(x)] &\leq \ln \det R_t(x_i),
\end{align*}
\]
where \(R_t(x_i) = h_i h_i^T x_i + \sum_{t \neq i} h_i h_i^T q_t + \gamma^{-1} I\) and \(A \succeq B\) implies that \(A - B\) is positive semidefinite. The inequalities in (20) are tight if \(q_i \to 0\) or \(1\) for \(t \in \{1, \ldots, l-1, l+1, \ldots, L\}\).

**Proof:** See Appendix [A].

For the CAVI algorithm with the approximations in (20), let
\[
R_t^{(i)}(x_i) = h_i h_i^T x_i + \sum_{t \neq i} h_i h_i^T q_t^{(i)} + \sum_{t > i} h_i h_i^T q_t^{(i-1)} + \gamma^{-1} I,
\]
where the superscript \((i)\) represents the \(i\)th iteration (i.e., \(i\) is used for the iteration index). Then, the term on the right-hand side (RHS) in (17) can be approximated by
\[
\begin{align*}
\chi_t^{(i)}(x_i) &= e^{-\sum_{m=1}^M y_m^T R_t^{(i)}(x_i) y_m - M \ln \det(R_t^{(i)}(x_i))}, \\
x_i &\in \{0, 1\},
\end{align*}
\]
which is an estimate of \(\mathbb{E}_{x_{-i}}[\ln f(x_i | x_{-i}, y)]\) in (17) with the approximations in (20). For the normalization, let \(\chi_t^{(0)} = \frac{\chi_t^{(i)}}{\chi_t^{(0)} + \chi_t^{(i)}}\). Then, the updating rule for the CAVI algorithm can be given by
\[
q_t^{(i)} = (1 - \mu) q_t^{(i-1)} + \mu \chi_t^{(i)}, \quad l = 1, \ldots, L,
\]
where \(\mu \in [0, 1]\) is the step-size. In each iteration, the updating is carried out in the ascending order, i.e., from \(l = 1\) to \(L\).

After a number of iterations, we can decide the set of active indices with the \(x_i\)'s corresponding to the \(K\) largest values of \(q_t^{(N_{\text{run}})}\), where \(N_{\text{run}}\) denotes the number of iterations of the CAVI algorithm. The corresponding set of estimated active indices is denoted by \(I_{\text{cavi}}\).

**B. Low-Complexity Updating Rules**

From (22), we can see that the computational complexity of the CAVI algorithm depends on the complexity to perform the matrix inversion (i.e., \(R_t^{(i)}(x_i)^{-1}\)) and the determinant (i.e., \(\det(R_t^{(i)}(x_i))\)). Since the size of \(R_t^{(i)}(x_i)\) is \(N \times N\), the complexity might be prohibitively high if \(N\) is large. However, from (21), we can see that
\[
R_{t+1}^{(i)}(x_{t+1}) = h_t h_t^H (x_{t+1} - q_t^{(i-1)}) + R_t^{(i)}(q_t^{(i)}),
\]

Thus, using the matrix inversion lemma [31], we have
\[
R_{t+1}^{(i)}(x_{t+1})^{-1} = R_t^{(i)}(q_t^{(i)})^{-1} - \nu_t^{(i)}(x_{t+1}) R_t^{(i)}(q_t^{(i)})^{-1} h_t h_t^H + R_t^{(i)}(q_t^{(i)})^{-1} h_t h_t^H R_t^{(i)}(q_t^{(i)})^{-1},
\]
for \(x_{t+1} \in \{0, 1\}\), where
\[
\nu_t^{(i)}(x_{t+1}) = \frac{x_{t+1} - q_t^{(i-1)}}{1 + (x_{t+1} - q_t^{(i-1)}) h_t h_t^H R_t^{(i)}(q_t^{(i)})^{-1} h_t^H}.
\]
Provided that \(R_t^{(i)}(q_t^{(i)})^{-1}\) is available, according to (25), the complexity to find \(R_t^{(i)}(x_{t+1})^{-1}\) is mainly due to \(R_t^{(i)}(q_t^{(i)})^{-1} h_t h_t^H\), which becomes \(O(N^2)\) in terms of the number of multiplications. In addition, per iteration, there might be \(L\) updates for \(\{q_t^{(i)}\}\). Thus, the computational complexity per iteration becomes \(O(LN^2)\).

Note that using the matrix determinant lemma [31], from (24), the term \(\ln \det(R_t^{(i)}(q_t^{(i)}))\) in (22) can be updated as follows:
\[
\ln \det(R_{t+1}^{(i)}(x_{t+1})) = \ln \det(R_t^{(i)}(q_t^{(i)})) + \ln(1 + (x_{t+1} - q_t^{(i-1)}) h_t h_t^H R_t^{(i)}(q_t^{(i)})^{-1} h_t^H).
\]
Since \(R_t^{(i)}(q_t^{(i)})^{-1} h_t h_t^H\) is available in finding \(R_{t+1}^{(i)}(x_{t+1})^{-1}\) as shown in (25), there is no significant additional computational complexity to update the term \(\ln \det(R_t^{(i)}(q_t^{(i)}))\). Thus, the computational complexity per iteration remains \(O(LN^2)\).
In summary, the computational complexity of the CAVI algorithm is independent of the number of active transmit antennas, $K$, while it grows linearly with the number of transmit antennas, $L$, and quadratically with the number of receive antennas, $N$. Note that the complexity of the ML approach in (13) or the approach in [25, 26] is $O(|X_K|) = O\left(\frac{L^K}{K!}\right)$, which grows exponentially with $K$ (for a fixed $L$). In particular, when $K \ll L$, we can show that $\frac{L^K}{K!} \approx \frac{L^K}{K}$. Although the complexity for a given $x$ can be an order of $O(K^3)$ as shown in [26], the overall complexity is $O(K^3L^K)$ for $K \ll L$. Thus, if $K \geq 3$, the complexity of the ML approach in [25, 26] or that in (13) is much higher than that of the CAVI algorithm in this paper, while the performance of the CAVI algorithm is worse than that in [25, 26], because it provides an approximate solution to the ML problem in (13) (or that in [25, 26]).

V. PERFORMANCE ANALYSIS

In this section, we consider the performance of the ML detection in (13) based on the pairwise error probability (PEP) analysis, which can result in an approximate probability of index error, i.e., the probability that the estimated set of active indices is not the same as the true set. Based on the analysis in this section, we are able not only to show that the solution to the approximate ML formulation in (13) can achieve a full coding gain, but also to compare the performance of the CAVI algorithm with the ML performance (which will be shown in Section VI).

For tractable analysis, we consider the following assumptions.

A1) For DM symbols, a constant modulus modulation scheme (e.g., 4-QAM) is employed, i.e., $|s_{m,l}| = 1, l \in I$.

A2) The elements of $H$ are independent and identically distributed (i.i.d.) with zero-mean and finite variance. In particular, we assume that $H_{n,l} \sim \mathcal{C}\mathcal{N}(0, 1/N)$.

Based on the PEP analysis [32], the probability of index error is upper-bounded as

$$P_{\text{pe}} = \Pr(\hat{\mathcal{I}} \neq \mathcal{I}) = \Pr(\hat{x} \neq x) \leq \sum_{x \in \mathcal{X}_K} \sum_{x' \in \mathcal{X}_K, x' \neq x} P(x \rightarrow x') \Pr(x) = \frac{1}{|\mathcal{X}_K|} \sum_{x \in \mathcal{X}_K} \sum_{x' \in \mathcal{X}_K, x' \neq x} P(x \rightarrow x'),$$

where $\hat{\mathcal{I}}$ represents the estimated index set and $P(x \rightarrow x')$ is the PEP, i.e., the probability that $x'$ is decoded when $x$ is the correct index vector (assuming that there are only two possible index vectors, $x$ and $x'$). To find the PEP, let $y = [y_1^H \ldots y_M^H]^H$, $\Delta = V(x) - V(x')$, and $L(x) = \ln \prod_{m=1}^{M} f(y_m | x)$. When the ML detection in (13) is considered, the PEP is given by

$$P(x \rightarrow x') = \Pr(L(x) < L(x') | x) = \Pr\left(\sum_m y_m^H \Delta y_m > d(x, x') | x\right),$$

where

$$d(x, x') = M(\phi(x') - \phi(x)).$$

For tractable analysis in finding the probability of index error, we only consider the case that the Hamming distance between $x$ and $x'$ is a minimum. Since $|\text{supp}(x)| = |\text{supp}(x')| = K$, the minimum Hamming distance between $x$ and $x'$ is 2 and

$$\text{supp}(e) = \{l_1, l_2\},$$

where $e = x' - x$ and $l_1 \neq l_2$, $l_1, l_2 \in \{1, \ldots, L\}$. For convenience, we assume that

$$l_1 = 1 \in \text{supp}(x) \text{ and } l_2 = 2 \in \text{supp}(x').$$

For given $x$, let $\mathcal{X}(x) = \{x' | d_H(x', x) = 2, x' \in \mathcal{X}_K\}$, where $d_H(x', x)$ represents the Hamming distance between $x'$ and $x$. Since the PEP associated with the two index vectors that have the Hamming distance larger than the minimum might be sufficiently low, the following approximation might be reasonable:

$$\sum_{x' \in \mathcal{X}_K, x' \neq x} P(x \rightarrow x') \approx \sum_{x' \in \mathcal{X}(x)} P(x \rightarrow x'),$$

and from (27), we can have the following approximate probability of index error:

$$P_{\text{pe}} \approx \frac{1}{|\mathcal{X}_K|} \sum_{x \in \mathcal{X}_K} \sum_{x' \in \mathcal{X}(x)} P(x \rightarrow x').$$

In general, since the conditional probability density function (pdf) of $y_m$ for given $x$ is a Gaussian mixture as shown in (9), it is difficult to obtain a closed-form expression for the PEP. Thus, with $l_1 = 1$, we may need to consider the following Gaussian approximation for the interference-plus-noise:

$$y_m = h_1 s_{m,1} + e_m, \quad s_{m,1} \in S, \quad m = 1, \ldots, M,$$

where $e_m \sim \mathcal{C}\mathcal{N}(0, D)$, which is the sum of the other active DM signals and the background noise. Here, $D = HH^H + \gamma^{-1}I$. The approximation in (33) can be reasonable if the sum of active DM signals is well approximated by a Gaussian random variable or the noise term in $e_m$ is dominant.

Lemma 2. Consider $x$ and $x'$ with a Hamming distance of 2. With (30) and (31), under the assumption of A1, let $V(x)^{-1} = D + h_1 h_1^H$ and $V(x')^{-1} = D + h_2 h_2^H$. Here, $H$ is the submatrix of $H$ with the column vectors corresponding to the indices of $\text{supp}(x) \cap \text{supp}(x')$. Then, for given $x$, based

1As shown in Section VI, the CAVI algorithm requires several iterations to converge (less than 10 iterations). Thus, its complexity is $O(N_{\text{run}} LN^2)$, where $N_{\text{run}} < 10$. Here, $N_{\text{run}}$ represents the number of iterations.

2In the high SNR regime, the PEP associated with a Hamming distance larger than the minimum Hamming distance becomes negligible compared to the PEP associated with the minimum Hamming distance. Thus, a good approximate probability of index error can be obtained by considering only the PEPs associated with the minimum Hamming distance.
on the Gaussian approximation for the interference-plus-noise in (33), an upper-bound on the conditional PEP is given by

\[
\Pr \left( \sum_{m=1}^{M} y_m^H \Delta y_m > d \mid h_1, h_2, D \right) \leq e^{-\lambda d (\kappa(\lambda) \exp(-\gamma_{\text{PEP}}(\lambda)))^M},
\]

where \( \lambda \geq 0 \) and

\[
\kappa(\lambda) = \frac{(\alpha_1 + 1)(\alpha_2 + 1)\theta_2(\lambda)}{\lambda(\alpha_1 + 1 + \ln(\lambda))},
\]

\[
\gamma_{\text{PEP}}(\lambda) = \alpha_1 - \frac{(\alpha_1 + 1)(\alpha_2 + 1)\theta_2(\lambda)}{\alpha_1 + 1 + \ln(\lambda)}.
\]

(35)

Here, \( \gamma_{\text{PEP}}(\lambda) \) represents the PEP-SNR (i.e., the effective SNR for PEP), and \( \alpha_1 = h_1^H D^{-1} h_1, \beta = h_2^H D^{-1} h_2, \) and \( \theta_1(\lambda) = \frac{\lambda}{1+\alpha-\alpha^2}. \)

Proof: See Appendix B.

In (35), we can observe that the coding gain in detecting \( \mathcal{I} \) is up to \( M \) due to \( M \) independent data symbols, \( \{s_{m,1}\} \). To see this clearly, we consider an example with \( K = 1 \) (i.e., only one antenna is active). In this case, \( y_m \) becomes

\[
y_m = h_l s_{m,l} + n_m, \quad l \in \{1, \ldots, L\}, \quad m = 1, \ldots, M,
\]

where \( h_l \in \mathcal{H} = \{h_1, \ldots, h_L\} \) becomes the signal and the \( s_{m,l} \)'s are \( M \) independent random weights (like the fading coefficients) multiplied to the signal, \( h_l \). Here, \( \mathcal{H} \) can be seen as the signal constellation. Thus, in detecting \( h_l \in \mathcal{H} \), the coding gain can be up to \( M \) as shown in (35).

With (30) and (31), since

\[
\phi(x) = \ln \det(D + h_1 h_1^H) \quad \text{and} \quad \phi(x') = \ln \det(D + h_2 h_2^H),
\]

from (29), we have

\[
d(x, x') = M \ln \left( 1 + \frac{\alpha_2}{1+\alpha_1} \right).
\]

(36)

Thus, with \( d = M \ln \left( 1 + \frac{\alpha_2}{1+\alpha_1} \right) \), the (average) PEP in (28) can be obtained by taking the expectation of the conditional PEP in (35) with respect to \( h_1, h_2, \) and \( D \). Unfortunately, this expectation cannot be carried out unless the joint pdf of \( \alpha_1 \) and \( \beta \) is available. Even if the joint pdf is available, a closed-form expression for the expectation may not be easy to find. Thus, we may only consider asymptotic cases.

As in (33) and (34), for a large \( N \) with fixed \( \eta \), we expect that \( \alpha_1 \) converges to a constant, i.e.,

\[
\alpha_1 \to \bar{\alpha} = \gamma_{\text{MMSE}}(\gamma, \eta),
\]

where \( \gamma_{\text{MMSE}}(\gamma, \eta) \) is the asymptotic SNR of the minimum mean squared error (MMSE) receiver. Under the assumption of A2, a closed form expression for \( \gamma_{\text{MMSE}}(\gamma, \eta) \) can be found in [35, Eq. (9)]. In this case, we have \( d(x, x') = 0 \) from (37).

Lemma 3. For a sufficiently large \( N \) with fixed \( \eta \), the asymptotic PEP-SNR is approximated as follows:

\[
\gamma_{\text{PEP}}(\lambda) \approx \tilde{\gamma}_{\text{PEP}}(\lambda) = \bar{\alpha} - \frac{\bar{\alpha} + 1}{\bar{\alpha} + 1 + \lambda} \left( \frac{\bar{\alpha} + 1 + \bar{\gamma}_{\text{MMSE}}(\lambda)}{N} \right),
\]

(39)

where

\[
\tilde{\omega} = \frac{\gamma_{\text{MMSE}}(2 + \gamma, \eta)}{2\gamma^2 + 1} \quad \text{and} \quad \bar{\theta}(\lambda) = \frac{\lambda}{1+\bar{\alpha} - \bar{\alpha}(\lambda)}.
\]

Proof: See Appendix C.

For a large \( N \) with fixed \( \eta \), the asymptotic PEP-SNR and the asymptotic \( \kappa(\lambda) \), which is denoted by \( \kappa(\lambda) \), are not dependent on the realizations of \( h_1, h_2, \) and \( D \). Thus, the asymptotic conditional PEP becomes the asymptotic PEP, which can be approximated as follows from (34):

\[
P(x \to x') \approx P_2(\lambda) = (\kappa(\lambda) \exp(-\gamma_{\text{PEP}}(\lambda)))^M.
\]

(40)

The parameter \( \lambda \) can be optimized for a tight bound as follows:

\[
\lambda^* = \text{argmax}_{0<\lambda<\bar{\lambda}} \gamma_{\text{PEP}}(\lambda) - \ln \kappa(\lambda),
\]

(41)

Consequently, in (42), we can observe that although (13) is an approximate ML formulation for the index detection (based on the Gaussian approximation), its solution can achieve a full coding gain in RCSM.

Note that (42) is an approximation and would be reasonable for a large system (i.e., a large \( N \) with fixed \( \eta \)). If \( N \) is not sufficiently large, the expectation of the conditional PEP in (35) with respect to \( h_1, h_2, \) and \( D \) is to be carried out to obtain the average PEP. Thus, (42) may not be reasonable for a small system (i.e., when \( N \) is small). In particular, as shown in (34), the conditional probability of index error is proportional to \( e^{-M\gamma_{\text{PEP}}} \). Thus, the average PEP becomes proportional to \( \mathbb{E}[e^{-M\gamma_{\text{PEP}}}] \), while we consider \( e^{-M\gamma_{\text{PEP}}} \) in (42), which might be correct in the asymptotic case where \( N \to \infty \) with fixed \( \eta \) as mentioned earlier. However, for a finite \( N \), (42) leads to an underestimate for a high SNR (with a finite \( N \)) due to Jensen’s inequality.

VI. SIMULATION RESULTS

In this section, we present simulation results to see the performance of the CAVI algorithm for the index detection in RCSM. For simulations, we consider the assumptions of A1 and A2 with 4-QAM (except for Fig. 8(b)). For comparisons, we consider the ML performance (i.e., the performance of the ML detection with Gaussian approximation in (13)) using simulation results as well as the approximate probability of index error in (42). We expect that the CAVI algorithm can provide a near-ML performance so that it can be used as a low-complexity approach to find an approximate ML solution. By comparing the ML performance with the approximate probability of index error in (42), we can also confirm that the coding gain of RCSM, which is \( M \). In addition, we consider the correlator based detector as a (very) low-complexity solution.
Since the CAVI algorithm is an iterative algorithm, its performance depends on the number of iterations. In Fig. 1 we present the variational probability, $q^{(i)}_l$, for each iteration when $N = L = 10$, $K = 3$, $M = 2$, $\gamma = 10$ dB, and $\langle \mu, N_{\text{run}} \rangle = (0.5, 12)$. The initial probabilities of $q^{(0)}_l$ are set to $1/L$. The set of active indices is $I = \{2, 6, 10\}$. It is shown that the CAVI algorithm can provide large variational probabilities for $l \in I$ after a certain number of iterations. As a result, we can expect the correct result for the index detection after convergence.

![Fig. 1. The convergence behavior of $q^{(i)}_l$ for the CAVI algorithm when $N = L = 10$, $K = 3$, $M = 2$, $\gamma = 10$ dB, and $\langle \mu, N_{\text{run}} \rangle = (0.5, 12)$. The initial variational probabilities of $q^{(0)}_l$ are set to $1/L$, which are represented by □ markers. The final variational probabilities are represented by ◦ markers.](image1)

Fig. 2 shows the probability of index error as a function of the step-size, $\mu$, when $N = 10$, $L = 20$, $K = 2$, $M = 4$, $\gamma = 10$ dB, and $N_{\text{run}} = 10$. We can see that the performance is improved by increasing $\mu$. However, when $\mu \geq 0.4$, there is no performance improvement. We also show the performance of the correlator based detector with the dashed line in Fig. 2. Clearly, the CAVI algorithm can perform better than the correlator based detector as it can provide an approximate ML solution (at the cost of a higher computational complexity than that of the correlator based detector).

The impact of the number of iterations, $N_{\text{run}}$, on the performance is shown in Fig. 3 when $N = 10$, $L = 20$, $K = 2$, $M = 4$, $\gamma = 10$ dB, and $\mu \in \{0.2, 0.5\}$. It is clearly shown that the performance can be improved by more iterations, while 10 iterations (i.e., $N_{\text{run}} = 10$) might be sufficient for convergence.

The performances of the correlator based detector, the ML detector with Gaussian approximation [25] (i.e., [13]), and the CAVI algorithm are shown for different values of SNR in Fig. 4. Since the ML detector and CAVI algorithm are to jointly detect active indices, it does not significantly suffer from the interference from the signals from the other active transmit antennas (at a high SNR). On the other hand, in the correlator based detector, although the noise is negligible (at a high SNR), its performance is still degraded by the interference. Thus, as shown in Fig. 4 the performance gap between the CAVI algorithm and the correlator based detector becomes wider as the SNR increases, while the performance of the CAVI algorithm is slightly degraded from that of the ML detector in [13]. We can also see that [42] might be reasonable for a large system (as shown in Fig. 4(b)), while at a high SNR [42] becomes an underestimate of the probability of index error as mentioned earlier.

As shown in Fig. 4, the slot length, $M$, becomes the coding gain. Thus, we expect that the probability of index error decreases exponentially with $M$, which can be confirmed by the simulation results in Fig. 5(a). That is, we can see that the probability of index error of the ML detector and the asymptotic PEP-bound in [42] (that are shown by the solid line with ◦ markers and the dashed line with □ markers, respectively) agree with each other in Fig. 5(a). We can also observe that the performance of the CAVI algorithm is slightly
worse than that of the ML detector, which means the CAVI algorithm can have a near ML performance (of the detection in (13)).

In Fig. 3 (b), we also present simulation results with 16-QAM. Compared with the performances with 4-QAM, the performances with 16-QAM are degraded in both the correlator based detector and the CAVI algorithm. From this, we expect to have a trade-off between the spectral efficiency and the detection performance (in terms of the probability of index error). Since the performance analysis in Section V is carried out under the assumption of A1 (e.g., 4-QAM), we need to extend the analysis with non-constant modulus modulation schemes to clearly quantify the performance degradation by 16-QAM, which might be a further research topic in the future.

Fig. 4 (a) shows the probabilities of index error of the correlator based detector, ML detector with Gaussian approximation in (17), and CAVI algorithm with \((\mu, N_{\text{run}}) = (0.5, 12)\) for various values of transmit antennas, \(K\), when \(N = 10, L = 20, K = 2\), and \(M = 4\); (b) \(N = 40, L = 20, K = 4\), and \(M = 2\).

In general, we expect that the performance of index detection can be improved by increasing the number of receive antennas, \(N\), which can be confirmed by the simulation results in Fig. 7. It is interesting to note that the performance gap between the CAVI algorithm and the correlator based detector becomes narrower as \(N\) increases. For fixed \(L\) and \(K\), when \(N\) increases, the interference can be better suppressed in the correlator based detector. However, if \(N\) is not too large, it might be necessary to carefully deal with the interference and any joint detection approach (e.g., the CAVI algorithm) might be required for a reasonable performance.

The computational complexity of the CAVI algorithm was discussed in Subsection IV-B where we show that the complexity order is \(O(N_{\text{run}}N^2L)\). The computational complexity of the ML detection in (13) is also shown for comparisons. As mentioned earlier, the ML detection in (13) with \(M = 1\) is considered in [25], [26]. To see the computational complexity, we use “tic” and “toc” commands in MATLAB and show the results in Fig. 8 as functions of key parameters when \((\mu, N_{\text{run}}) = (0.5, 10)\) and SNR = 10 dB. Since the complexity is independent of the number of active transmit antennas, \(K\), the computing time is almost invariant with respect to \(K\) in Fig. 8 (a), while we can observe that the computing time grows quadratically with the number of receive antennas, \(N\), and linearly with the number of transmit antennas, \(L\), in Fig. 8 (b) and (c), respectively. From Fig. 8 we can also confirm that due to the exhaustive search for the ML detection in (13), in general, the complexity of the ML detection is much higher than that of the CAVI algorithm (except that \(K\) is sufficiently
small, i.e., \( K \leq 2 \) as mentioned earlier). Thus, the CAVI algorithm can be seen as a low-complexity approach to the ML detection in (13) and [25], [26] with negligible performance degradation (as shown in Figs. 4 and 5).

VII. CONCLUDING REMARKS

In this paper, we considered a simple transmit coding scheme for SM, namely RCSM, and studied the index detection problem using the Gaussian approximation, which led to an approximate ML formulation for the index detection. Based on the derived approximate closed-form expression for the probability of index error, we showed that the solution to the approximate ML formulation can still have a full coding gain. Since an exhaustive search to find the solution to the ML problem required a prohibitively high complexity, which grows exponentially with the number of active transmit antennas, we applied a variational inference approach, which is a well-known machine learning approach, and derived an iterative algorithm. The resulting iterative algorithm has a low complexity thanks to the Gaussian approximation. In particular, it was shown that the complexity of the proposed algorithm is independent of the number of active transmit antennas. From simulation results and the approximate closed-form expression, we also saw that the proposed iterative algorithm can achieve a near ML performance, which demonstrates the benefit of machine learning algorithms to the signal detection in SM.
APPENDIX A

PROOF OF LEMMA 1

Consider the expectation of $V(x)$ over $x_t$, which is given by

\[
\mathbb{E}_t[V(x)] = \mathbb{E}_t \left[ (h_t h_t^H x_t + A_t)^{-1} \right] = A_t^{-1} - \mathbb{E}_t \left[ \frac{x_t}{1 + x_t h_t^H A_t^{-1} h_t} \right] A_t^{-1} h_t h_t^H A_t^{-1},
\]

where $A_t = \sum_{p=1}^{P} h_p h_p^H + \gamma^{-1}I$ and the second equality is due to the matrix inversion lemma \[31\]. Since $\Pr(x_t = 1) = q_t$ and $\Pr(x_t = 0) = 1 - q_t$, it can be shown that

\[
\mathbb{E}_t \left[ \frac{x_t}{1 + x_t h_t^H A_t^{-1} h_t} \right] = \frac{q_t}{1 + h_t^H A_t^{-1} h_t} \leq 1 + q_t h_t^H A_t^{-1} h_t,
\]

which results in $\mathbb{E}_t[V(x)] \geq (h_t h_t^H q_t + A_t)^{-1}$. Therefore, if the expectation is carried out over $t \in \{1, \ldots, l-1, l+1, \ldots, L\}$, we can have the following result:

\[
\mathbb{E}_t[V(x)] \geq \left( h_t h_t^H x_t + \sum_{i \neq t} h_i h_i^H q_i + \gamma^{-1}I \right)^{-1},
\]

which proves the first equation in (20).

According to the matrix determinant lemma \[31\], we have

\[
\mathbb{E}_t[\phi(x)] = \mathbb{E}_t \left[ \ln \det \left( \sum_{i=1}^{L} h_i h_i^H x_i + \gamma^{-1}I \right) \right] = \mathbb{E}_t \left[ \ln(1 + x_t h_t^H A_t^{-1} h_t) + \ln \det(A_t) \right]
\]

\[
\leq \ln(1 + q_t h_t^H A_t^{-1} h_t) + \ln \det(A_t)
\]

\[
= \ln \det \left( h_t h_t^H q_t + \sum_{i \neq t} h_i h_i^H x_i + \gamma^{-1}I \right) (45)
\]

where the inequality is due to Jensen’s inequality.

In order to see the inequalities in (20) are tight as $q_t \to 0$ or 1 for $t \in \{1, \ldots, l-1, l+1, \ldots, L\}$, consider (44). It can be shown that

\[
\frac{q_t}{1 + q_t h_t^H A_t^{-1} h_t} = \frac{q_t(1 - q_t d_t)}{1 + q_t d_t(1 + d_t)}.
\]

where $d_t = h_t^H A_t^{-1} h_t$. Thus, as $q_t$ approaches 0 or 1, the inequality in (44) becomes tight. As a result, the first inequality in (20) becomes tight. Similarly, we can also show that the inequality in (45) is tight if $q_t$ approaches 0 or 1. Thus, the second inequality in (20) becomes tight for $q_t \to 0$ or 1.

APPENDIX B

PROOF OF LEMMA 2

Using the Chernoff bound \[30\], it can be shown that

\[
\Pr \left( \sum_{m=1}^{M} y_m^H \Delta y_m > d \big| h_1, h_2, D \right) \leq e^{-\lambda d} \mathbb{E} \left[ e^{\lambda \sum_{m=1}^{M} y_m^H \Delta y_m} \big| h_1, h_2, D \right]
\]

\[
= e^{-\lambda d} \prod_{m=1}^{M} \mathbb{E} \left[ e^{\lambda y_m^H \Delta y_m} \big| h_1, h_2, D \right],
\]

where $\lambda > 0$. Let $\tilde{y}_m = s_{m,1}' y_m$. From (33), under the assumption of A1 and the Gaussian approximation for the interference-plus-noise in (33), we can show that $\tilde{y}_m = h_1 + s_{m,1}' e_m$, where $s_{m,1}' e_m \sim \mathcal{CN}(0, D)$. Then, since $y_m^H \Delta y_m = \tilde{y}_m^H \Delta \tilde{y}_m$, after some manipulations, we have

\[
E \left[ e^{\lambda \sum_{m=1}^{M} y_m^H \Delta y_m} \big| h_1, h_2, D \right] = \frac{1}{\pi^2 \det(D)} \int e^{\lambda \sum_{m=1}^{M} y_m^H \Delta y_m - (y_m - h_1)^H D^{-1} (y_m - h_1)} d\gamma_m
\]

\[
= \frac{\det(W^{-1})}{\det(D)} \exp \left( -h_1^H (D^{-1} - D^{-1} W^{-1} D^{-1}) h_1 \right),
\]

where $W = D^{-1} - \lambda \Delta$. Using the matrix inversion lemma, we can show that

\[
\Delta = D^{-1} (\tau_2 h_2 h_2^H - \tau_1 h_1 h_1^H) D^{-1},
\]

where $\tau_1 = \frac{1}{1 + \alpha_2}$. According to (47), $\kappa(\lambda)$ is given by

\[
\kappa(\lambda) = \frac{\det(W^{-1})}{\det(D)} \frac{1}{\det(1 - \lambda D \Delta)}.
\]

Using the matrix determinant lemma \[31\], we have

\[
\det(1 - \lambda D \Delta) = \det(B) (1 + \lambda \tau_1 h_1 h_1^H B^{-1} h_1),
\]

where $B = I - \lambda \tau_2 h_2 h_2^H$. Using the matrix determinant lemma, it can be shown that

\[
\det(B) = 1 - \lambda \tau_2 \alpha_2 = \frac{\lambda}{1 + \alpha_2}.\]

Applying the matrix inversion lemma, it can be shown that

\[
B^{-1} = I + \frac{\lambda \tau_2}{1 - \lambda \tau_2 \alpha_2} h_2 h_2^H D^{-1}.
\]

From this, we have

\[
\lambda \tau_1 h_1^H D^{-1} B^{-1} h_1 = \lambda \tau_1 \left( \alpha_1 + \alpha_2 (|\beta|^2) \right).
\]

Substituting (51) and (52) into (50), it can be shown that

\[
\det(I - \lambda D \Delta) = \frac{\lambda}{(\alpha_2 + 1) \theta_2(\lambda)} \left( 1 + \frac{\lambda (\alpha_1 + \alpha_2 (|\beta|^2))}{1 + \alpha_1} \right).
\]

Substituting (53) into (49), we can show the first equation in (55).

Since $W = D^{-1} (D - \lambda D \Delta D)^{-1}$, it can be shown that

\[
\Delta = D^{-1} - D^{-1} W^{-1} D^{-1} h_1 = \alpha_1 - h_1^H (D - \lambda D \Delta)^{-1} h_1.
\]

Substituting (45) into (54) and using the matrix inversion lemma, we have

\[
\alpha_1 = \frac{c_1}{1 + \lambda \tau_2 c_1},
\]

where

\[
c_1 = h_1^H (D - \lambda \tau_2 h_2 h_2^H)^{-1} h_1 = \alpha_1 + \frac{\lambda \tau_2}{1 - \lambda \tau_2 \alpha_2} |h_1^H D^{-1} h_2|^2 = \alpha_1 + \theta_2(\lambda) |\beta|^2.
\]
since \( \theta_2(\lambda) = \frac{\lambda_2 - \lambda_3}{1 - \lambda_2 \alpha_2} \). Substituting (56) into (55), we have

\[
\gamma_{\text{PEP}}(\lambda) = h_1^T (D^{-1} - D^{-1}H^{-1}D^{-1})h_1
= \frac{\alpha_1 + 1}{\alpha_1 + 1 + \lambda(\alpha_1 + \theta_2(\lambda)|\beta|^2)}.
\]

(57)

which is the second equation in (35).

APPENDIX C

PROOF OF LEMMA 3

In (35), for large \( N \) and \( L \) with fixed \( \eta = Q/N \), \( \alpha_1 \) and \( |\beta|^2 \) approach \( \bar{\alpha} = \mathbb{E}[\alpha_1] \) and \( \mathbb{E}[|\beta|^2] \), respectively. Thus, the asymptotic PEP-SNR can be obtained by replacing \( \alpha_1 \) and \( |\beta|^2 \) with \( \bar{\alpha} \), which is given in (38) and \( \mathbb{E}[|\beta|^2] \), respectively.

For asymptotic \( |\beta|^2 \), from (34), we have

\[
|\beta|^2 \approx \mathbb{E}[|\beta|^2]
= \mathbb{E}[h_1^T D^{-1}h_2 h_1] = \mathbb{E}[h_1^T D^{-1}E[h_2 h_2^T]D^{-1}h_1]
= \frac{1}{N} \mathbb{E}[h_1^T D^{-2}h_1] = \frac{1}{N} \left( \mathbb{E}[\text{tr}(D^{-2})] \right),
\]

(58)

It can be shown that

\[
D^2 = (\gamma^{-1}I + HH^H) = \gamma^{-2}I + 2\gamma^{-1}HH^H + HH^H = (2\gamma^{-1} + 1) \left( \gamma^{-2} + 2\gamma I + HH^H \right),
\]

where the approximation is due to \( HH^H \approx I \) for a large \( N \) and fixed \( \eta = \frac{Q}{N} \). Then, it can be shown that

\[
\frac{\mathbb{E}[\text{tr}(D^{-2})]}{N} \approx \frac{\mathbb{E} \left( \left( \gamma^{-2} + 2\gamma I + HH^H \right)^{-1} \right)}{N}
= \frac{\gamma_{\text{mmse}}(\gamma + \gamma, \eta)}{2\gamma^{-1} + 1} = \bar{\omega}.
\]

(59)

From this, we have \( \mathbb{E}[|\beta|^2] \approx \frac{\bar{\omega}}{N} \), which is substituted into the PEP-SNR expression in (35). Then, we can obtain (39).

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