Abstract

We show that the $E_{11}$ representation that contains the space-time translation generators also contains the rank two and five totally anti-symmetric representations of $A_{10}$. By studying the behaviour of these latter $A_{10}$ representations under $SL(32)$, which we argue is contained in the Cartan involution invariant sub-algebra of $E_{11}$, we find that the rank two and five totally anti-symmetric representations must be identified with the central charges of the eleven dimensional supersymmetry algebra.

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1 Introduction

It has been shown that the entire bosonic sector of eleven dimensional supergravity, can be formulated as non-linear realisation [1]. The algebra, denoted $G_{11}$, used for this non-linear realisation was not a Kac-Moody algebra. However, it has been suggested [2] that this theory can be reformulated, or extended, in such a way that it can be described as a non-linear realisation based on a Kac-Moody algebra. Assuming this to be the case, it was shown [2] that this Kac-Moody symmetry should contain very extended $E_8$, i.e. $E_{11}$. Substantial fragments of this symmetry, as well as other evidence for it, has been presented [2,3,4]. Analogous results were also found for IIA [2] and IIB supergravity [5] where the corresponding Kac-Moody algebras were also found to be $E_{11}$ in each case. An account of work on symmetries in dimensionally reduced supergravity and string theories is given in reference [15].

The algebra $G_{11}$ contained the usual translation generators whose role was to introduce space-time into the the theory, however, these generators did not play any role in subsequent discussions on $E_{11}$. In this paper we will explain how the translation generators can be incorporated into a non-linear realisation based on $E_{11}$. The translation generators carry one lower index and transform in the corresponding representation of $A_{10}$. Since $E_{11}$ contains $A_{10}$ we will enlarge this $A_{10}$ representation to one belonging to $E_{11}$. We will find that the translations occur in this $E_{11}$ representation together with a second rank and fifth rank anti-symmetric tensor representations of $A_{10}$ as well as an infinite number of other tensors.

The gravity sector of eleven dimensional supergravity arises as the non-linear realisation of the IGL(11) sub-algebra of $G_{11}$, the I being the translations. Taking the non-linear realisation of IGL(11) with the Lorentz algebra SO(1,10) as the local sub-algebra does not uniquely lead to gravity. However, if one takes the simultaneous non-linear realisation of IGL(11) and the conformal group SO(2,11) one finds to Einstein’s theory of gravity [6,1] essentially uniquely. In reference [1] the non-linear realisation based on $G_{11}$ used for eleven dimensional supergravity was combined with a simultaneous non-linear realisation of the conformal algebra and this did lead to the unique bosonic field equations of eleven dimensional supergravity, up to one undetermined constant [1]. Subsequent papers did not address the question of how this conformal algebra combined with $E_{11}$. However, it was argued in reference [1] that the presence of the fermionic extension of bosonic sector of eleven dimensional supergravity considered above implied that the full theory would possess an Osp(1/64). This algebra contains GL(32) which rotates the spinor index on the supercharges and so this must also be a symmetry of M theory. In fact, the algebra GL(32) had previously been proposed as a symmetry of M theory as part of it occurred as a symmetry of the fivebrane equations of motion [8]. It was explained [8] that the SL(32) symmetry was a brane rotating symmetry and was the natural generalisation of the local spin or Lorentz algebra when branes were present.

The local sub-algebra used in formulating the $E_{11}$ non-linear realisation was taken to be the one which is invariant under the Cartan involution. We will argue that SL(32) is indeed part of this local sub-algebra and we calculate the transformations of the second and fifth rank anti-symmetric tensor representations mentioned above under SL(32). We will find that these objects transform in such a way that they should be identified with
central charges of the eleven dimensional supersymmetry algebra.

2 $E_{11}$ at Low Levels

We invite the reader to draw the Dynkin diagram of $E_{11}$ by drawing ten dots in a horizontal line labeled from one to ten from left to right and connected by a single line. Then place another dot, labeled eleven, above the third node (labeled eight) from the right. We consider $E_{11}$ as a member of the class of Kac-Moody algebras discussed in section three of reference [7], namely an algebra whose Dynkin diagram possess at least one node such that deleting it leads to a finite dimensional semi-simple Lie algebra. If we delete node eleven in the Dynkin diagram of $E_{11}$, the remaining algebra is $A_{10}$. The preferred simple root is $\alpha_{11}$ and the simple roots of $A_{10}$ are $\alpha_i$, $i = 1, \ldots, 10$. We may write [7]

$$\alpha_{11} = -\lambda_8 + x$$

where $x$ is a vector in a space orthogonal to the roots of $A_{10}$ and $\lambda_i$ are the fundamental weight vectors of $A_{10}$. The simple roots have length squared two and so $x^2 = 2 - \lambda_8^2 = -\frac{2}{11}$.

A root $\alpha$ of $E_{11}$ can be written as

$$\alpha = l\alpha_{11} + \sum_i m_i\alpha_i = lx - l\lambda_8 + \sum_{jk} A_{jk}^f \lambda_k$$

where $A_{jk}^f$ is the Cartan matrix of $A_{10}$. We define the level, denoted $l$, [4] of the roots of $E_{11}$ to be the number of times the root $\alpha_{11}$ occurs in its decomposition into simple roots given in the equation above. The generators of $E_{11}$ can also be classified according to their level which is just the level of the root associated with the generator.

The $E_{11}$ algebra contains the generators $K^a_b$ at level 0 and the generators

$$R^{a_1 a_2 a_3}, R^{a_1 a_2 \ldots a_6}, R^{a_1 a_2 \ldots a_8, b}$$

at levels zero, 1, 2 and 3 respectively [2,4], as well as the generators

$$R_{a_1 a_2 a_3}, R_{a_1 a_2 \ldots a_6}, R_{a_1 a_2 \ldots a_8, b}$$

at levels -1, -2, -3. The generators of $E_{11}$ at higher levels are listed in references [4,16].

The corresponding Borel sub-algebra up to, and including, level 3 obeys the commutation relations [2]

$$[K^a_b, K^c_d] = \delta^c_d K^a_b - \delta^a_d K^c_b, \tag{2.5}$$

$$[K^a_b, R^{c_1 \ldots c_6}] = \delta^c_b R^{ac_2 \ldots c_6} + \ldots, \quad [K^a_b, R^{c_1 \ldots c_3}] = \delta^c_b R^{ac_2 c_3} + \ldots, \tag{2.6}$$

$$[K^a_b, R^{c_1 \ldots c_8, d}] = (\delta^c_b R^{ac_2 \ldots c_8, d} + \ldots) + \delta^d_b R^{c_1 \ldots c_8, a}, \tag{2.7}$$

$$[R^{c_1 \ldots c_3}, R^{c_4 \ldots c_6}] = 2R^{c_1 \ldots c_6}, \quad [R^{a_1 \ldots a_6}, R^{b_1 \ldots b_3}] = 3R^{a_1 \ldots a_6 [b_1 b_2, b_3]}, \tag{2.8}$$

where $+ \ldots$ means the appropriate anti-symmetrisation. The level 0 and negative level generators obey the relations

$$[K^a_b, R^{c_1 \ldots c_3}] = -\delta^{c_3}_b R_{bc_2 c_3} - \ldots, \quad [K^a_b, R^{c_1 \ldots c_6}] = -\delta^{c_6}_b R_{bc_2 \ldots c_6} - \ldots, \tag{2.9}$$
The Cartan involution is a linear operator and acts on the generators as local sub-algebra. As such, its generators do not lead to fields in the non-linear realisation. The Cartan involution plays an important part in the non-linear realisation of reference [2] as it is taken to be the level three. Finally, the commutation relations between the positive and negative generators of up to level three are given by

$$[K^a_b, R_{c_1...c_8,d}] = - (\delta^a_{c_1} R_{bc_2...c_8,d} + \cdots) - \delta^a_d R_{c_1...c_8,b}.$$  

$$[R_{c_1...c_3}, R_{c_4...c_6}] = 2R_{c_1...c_6}, \quad [R_{a_1...a_6}, R_{b_1...b_3}] = 3R_{a_1...a_6[b_1b_2b_3]},$$

where $D = \sum_b K^b_b$, $\delta^a_{b_1b_2} = \frac{1}{2} (\delta^a_{b_1b_2} - \delta^a_{b_2b_1}) = \delta^a_{b_1} \delta^a_{b_2}$ with similar formulae when more indices are involved.

The above commutators can be deduced, using the Serre relations and from the identification of the Chevalley generators of $E_{11}$ which are given by [2]

$$E_a = K^a_{a+1}, a = 1, \ldots, 10, \quad E_{11} = R^{91011}; \quad F_a = K^{a+1}_{a}, a = 1, \ldots, 10, \quad F_{11} = R_{91011}$$

$$H_a = K^a_a - K^{a+1}_{a+1}, a = 1, \ldots, 10, \quad H_{11} = \frac{2}{3} (K^9_{10} + K^{10}_{11} + K^{11}_{11}) - \frac{1}{3} (K^1_{1} + \ldots + K^8_{8}).$$

The sub-algebra which is invariant under the Cartan involution, namely

$$E_a \rightarrow -F_a, \quad F_a \rightarrow -E_a, \quad H_a \rightarrow -H_a,$$

plays an important part in the non-linear realisation of reference [2] as it is taken to be the local sub-algebra. As such, its generators do not lead to fields in the non-linear realisation. The Cartan involution is a linear operator and acts on the generators as

$$K^a_b \rightarrow -K^b_a, \quad R^{a_1a_2a_3} \rightarrow -R_{a_1a_2a_3}, \quad R^{a_1...a_6} \rightarrow R_{a_1...a_6}, \quad R^{a_1...a_8,b} \rightarrow -R_{a_1...a_8,b}$$

The sub-algebra invariant under the Cartan involution is generated by $E_a - F_a$ and at low levels it includes the generators

$$J_{ab} = K^c_{b} \eta_{ac} - K^c_{a} \eta_{bc}, \quad S_{a_1a_2a_3} = R^{b_1b_2b_3}_{a_1a_2a_3} \eta_{b_1a_1} \eta_{b_2a_2} \eta_{b_3a_3} - R_{a_1a_2a_3},$$

$$S_{a_1...a_6} = R^{b_1...b_6}_{a_1a_2a_3} \eta_{b_1a_1} \eta_{b_2a_2} \cdots \eta_{b_6a_6} + R_{a_1...a_6}$$

$$S_{a_1...a_8,c} = R^{b_1...b_8}_{a_1a_2a_3} \eta_{b_1a_1} \cdots \eta_{b_8a_8} \eta_{bc} - R_{a_1...a_8,c}.$$

The generators $J_{ab}$ are those of the Lorentz algebra SO(1,10) and their commutators with the other generators just express the fact that they belong to a representation of the Lorentz algebra. The $S_{a_1a_2a_3}$ and $S_{a_1...a_6}$ generators obey the commutators

$$[S^{a_1a_2a_3}_{b_1b_2b_3}, S_{b_1b_2b_3}] = -36 \delta^{[a_1a_2}_{b_1b_2} J^{a_3]}_{b_3] + 2S^{a_1a_2a_3}_{b_1b_2b_3} \delta^{b_1b_2b_3} J^{a_3]_{b_3]} = -3 \delta^{b_1b_2b_3} J^{a_1a_2a_3}_{[a_1a_2a_3]}$$

$$[S_{a_1a_2a_3}, S^{b_1...b_6}] = -2 \frac{6!}{3} \delta^{b_1b_2b_3} b^{a_1a_2a_3} \delta^{b_4b_5b_6} - 3S^{b_1...b_6}_{[a_1a_2a_3].}$$
3 Translations and an $E_{11}$ Representation

The space-time translation generators carry a single lower index and transform in the corresponding $A_{10}$ representation. This is equivalent to a tensor with ten upper anti-symmetrised indices which, in our conventions, is the representation with weight $\lambda_1$, or Dynkin index $p_1 = 1$, all other $p_i$’s vanishing. We wish to consider the representation of $E_{11}$ which contain the translation generators. The fundamental weights of $E_{11}$ are given by [7]

$$l_i = \lambda_i + \lambda_8 \lambda_1 \frac{x}{x^2}, \quad i = 1, \ldots, 10 \quad l_{11} = \frac{x}{x^2} \quad (3.1)$$

The $E_{11}$ representation with highest weight $l_1 = \lambda_1 - \frac{3}{2} x$ obviously contains the states in the $A_{10}$ representation $\lambda_1$. The highest weight state $|l_1\rangle$ can be thought of as being at level $-\frac{3}{2}$. It is straightforward to construct the root string associated with the action of the simple negative roots $F_a$ on this highest weight state. One finds

$$l_1, l_1 - \alpha_1, l_1 - \alpha_1 - \alpha_2, \ldots, l_1 - \alpha_1 - \ldots \alpha_8 - \alpha_{11}, \ldots \quad (3.2)$$

The last weight written explicitly is the first one in the string where $\alpha_{11}$ enters and it corresponds to the appearance of a new $A_{10}$ representation in the string. In fact,

$$l_1 - \alpha_1 - \ldots - \alpha_8 - \alpha_{11} = \lambda_9 - \frac{5}{2} x, \quad (3.3)$$

which contains the highest weight for the $A_{10}$ representation whose only non-vanishing Dynkin index is $p_9 = 1$ or a second rank anti-symmetric tensor. Continuing in this way we find that the representation $l_1$ contains the following $A_{10}$ representations

$$p_1 = 1, (\frac{-3}{2}); p_9 = 1, (\frac{-5}{2}); p_6 = 1, (\frac{-7}{2}); p_4 = 1, p_{10} = 1, (\frac{-9}{2}); \quad \text{and} \quad p_3 = 1, (\frac{-9}{2}); \ldots \quad (3.4)$$

all other $p_i$’s vanishing. The number in brackets is the corresponding level.

Consider any Lie algebra $g$ with a representation $u(A)$. By definition, the linear operators $u(A)$, for each element $A \in g$, obey the relation $u(A_1 A_2) = u(A_1) u(A_2)$. If the representation is carried by the states $|X_s\rangle$ it defines the matrices $u(A)|X_s\rangle = (c(A))_{s^t} |X_t\rangle$. Clearly, $c(A_1 A_2) = c(A_2) c(A_1)$ and so a matrix representation of $g$ is defined by $d(A) = c(A^t)$ where $^t$ is any operation that inverts the order of the factors in the Lie algebra. The relevant operation for us is to take $A^t = I^c I^l(A)$ where $I^c$ is the Cartan involution and $I^l$ is the operators which inverts the order of operators and $I^l(A) = -A$. The latter operator is just the operator in the algebra which corresponds to inversion of group elements.

In these circumstance we can define a semi-direct product algebra. We associate with each state in the representation $|X_s\rangle$ a generator $X_s$ and we extend the algebra $g$ to include the new generators by adopting the commutation relation

$$[X_s, A] = d(A)_{s^r} X_r, \quad A \in g \quad (3.5)$$
This is consistent with the Jacobi identities involving two elements of \( g \) and one \( X_s \). The commutator between two elements \( X_s \) and \( X_r \) must be chosen and it is consistent to choose it to vanish.

Carrying out this procedure for \( E_{11} \) and the representation \( l_1 \) we introduce the generators

\[
P^{a_1 \ldots a_{10}}, W^{a_1 a_2}, W^{a_1 \ldots a_5}, W^{a_1 \ldots a_7}, \ldots
\]  

(3.6)

at levels \( \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2} \) respectively. It is straightforward to find the generator associated with the root string of equation (3.2). The entries explicitly given correspond to \( P^{2,11}, P^{13,11}, P^{124,11} \) and \( P^{91011} \). We denote the semi-direct product of \( E_{11} \) with its \( l_1 \) representation by \( E_{11} \oplus_s L_1 \).

It is simpler to work with the more familiar generator

\[
P_a = \frac{1}{10!} \epsilon_{ba_1 \ldots a_{10}} P^{a_1 \ldots a_{10}}
\]  

(3.7)

The \( A_{10} \) generators act on the translations as

\[
[K^c_b, P^{a_1 \ldots a_{10}}] = 10 \delta^{[a_1}_b P^{c[a_2 \ldots a_{10}]} - \frac{1}{2} \delta^c_b P^{a_1 \ldots a_{10}}
\]  

(3.8)

The last term on the right-hand side of the above equation is required as a result of the relation \([H_{11}, P^{a_1 \ldots a_{10}}] = 0 \). The corresponding commutator involving \( P_a \) is

\[
[K^c_b, P_a] = -\delta^c_a P_b + \frac{1}{2} \delta^c_b P_a
\]  

(3.9)

The root string of equation (3.2) corresponds to the commutators

\[
[R^{a_1 a_2 a_3}, P_b] = 3 \delta^{[a_1}_b W^{a_2 a_3]}, \quad [R^{a_1 a_2 a_3}, W^{b_1 b_2} b_2] = W^{a_1 a_2 a_3 b_1 b_2},
\]  

(3.10)

\[
[R^{a_1 a_2 a_3}, W^{b_1 \ldots b_5}] = W^{b_1 \ldots b_5[a_1 a_2 a_3]}
\]  

(3.11)

which also normalise the new generators.

Using these relationships and the Jacobi identities one deduces that

\[
[R^{a_1 \ldots a_6}, P_b] = -3 \delta^{[a_1}_b W^{a_2 \ldots a_6]}
\]  

(3.12)

Using the Jacobi identities and equations (3.10) and (3.11) the commutators involving the negative generators \( R_{abc} \) is found to be given by

\[
[R^{a_1 a_2 a_3}, P_b] = 0, \quad [R^{a_1 a_2 a_3}, W^{b_1 b_2}] = 12 \delta^{b_1 b_2}_{a_1 a_2} P_{a_3}
\]  

(3.13)

\[
[R^{a_1 a_2 a_3}, W^{b_1 \ldots b_5}] = 120 \delta^{b_1 b_2 b_3}_{a_1 a_2 a_3} W^{b_4 b_5}
\]  

(3.14)

It will be instructive to consider the commutators of the Cartan invariant sub-algebra generator \( S_{abc} \) of \( E_{11} \) with the generators associated with the representation \( l_1 \). Using equations (3.10) and (3.11) and equations (3.13) and (3.14) we find that

\[
[S^{a_1 a_2 a_3}, P_b] = 3 \eta_{b[a_1} W^{a_2 a_3]}, \quad [S^{a_1 a_2 a_3}, W^{b_1 b_2}] = W^{a_1 a_2 a_3 b_1 b_2} - 12 \delta^{b_1 b_2}_{a_1 a_2} P_{a_3}
\]  

(3.15)
\[ [S_{a_1a_2a_3}, W^{b_1...b_5}] = W^{b_1...b_5}_{[a_1a_2a_3]} - 120\delta^{[b_1b_2b_3}_{a_1a_2a_3} W^{b_4b_5} \] (3.16)

It is interesting to examine what adding the generators corresponding to the \( l_1 \) representation means in terms of the weight lattice. The weight \( l_1 = -\frac{1}{2}(l + \bar{l}) \) in the notation of reference [7], section five. The root lattice of \( E_{11} \) is given by [7]

\[ \Lambda_{E_8} \oplus \Pi^{(1,1)} \oplus \{(n, n), n \in \mathbb{Z} \} \] (3.17)

Adding the \( l_1 \) representation corresponds to adding the vector \(-\frac{1}{2}(1, -1)\) to the last factor in the above decomposition and so one is working on the full weight lattice.

4 GL(32) and Central Charges

In this section we review and expand the results of references [8], [9] and [1]. The eleven dimensional supersymmetry algebra is of the form [10]

\[ \{Q_\alpha, Q_\beta\} = Z_{\alpha\beta} = (\gamma^a C^{-1} P_a + \gamma^{a_1a_2} C^{-1} Z_{a_1a_2} + \gamma^{a_1...a_5} C^{-1} Z_{a_1...a_5})_{\alpha\beta} \] (4.1)

\[ [Q_\alpha, Z_{\alpha\beta}] = 0, [Z_{\alpha\beta}, Z_{\gamma\delta}] = 0 \] (4.2)

This algebra admits GL(32) as an automorphism whose action is given by

\[ [Q_\alpha, T_{\gamma\delta}] = \delta_\alpha^\gamma \delta_{\beta}^\delta, [Z_{\alpha\beta}, T_{\gamma\delta}] = \delta_\alpha^\gamma Z_{\alpha\beta} + \delta_{\beta}^\delta Z_{\alpha\gamma} \] (4.3)

This automorphism was found to play a role in the symmetries of the fivebrane dynamics [8], [9] and later was shown to be a symmetry of M theory [1].

To gain a more familiar set of generators we may expand \( T_{\gamma\delta} \) out in terms of the elements of the enveloping Clifford algebra

\[ T_{\gamma\delta} = \sum_{p} \sum_{a_1...a_p} \frac{1}{32p!} (-1)^{\frac{p(p-1)}{2}} (\gamma^{a_1...a_p})_{\gamma} \delta T_{a_1...a_p} \text{ or } (\gamma^{a_1...a_p})_{\delta} \gamma T_{\gamma\delta} = T^{a_1...a_p}. \] (4.4)

The summation only goes over \( p = 0, 1, \ldots, 4, 6 \) since we may use the relation

\[ \gamma^{a_1...a_p} = \frac{1}{q!} (-1)^{q(p-1)} (-1)^{\frac{q(q+1)}{2} \epsilon_{a_1...a_p b_1...b_q}}, \] (4.5)

for \( p + q = 11 \). We then find that

\[ [Q_\alpha, T^{a_1...a_p}] = (\gamma^{a_1...a_p})_{\alpha}^\beta Q_\beta \] (4.6)

As such, the \( Q_\alpha \) carry a representation of GL(32) with \( T^{a_1...a_p} \) represented by \( \gamma^{a_1...a_p} \). The Lorentz generators are represented in the usual way by \( J_{ab} = \gamma_{ab} \).

Using the relation

\[ [\gamma^{a_1...a_n}, \gamma_{b_1...b_m}] = \sum_{s=0}^{n} \frac{n!m!}{s!(n-s)!(m-s)!} (-1)^{\frac{s(s+1)}{2}} (-1)^s n \]
valid for $m > n$, it straightforward to show, for example, that

$$[T^{a_1a_2a_3}, T_{b_1b_2b_3}] = -36\delta^{[a_1a_2}_{[b_1b_2} J^{a_3]}_{b_3]} + 2T^{a_1a_2a_3} b_1b_2b_3$$

$$[T^{a_1a_2a_3}, T_{b_1...b_6}] = +36\delta_{[b_1}^{[a_1} T^{a_2a_3]}_{b_2...b_6]} - 2.5!\delta^{a_1a_2a_3}_{b_1b_2b_3} T_{b_4b_5b_6}$$

Examining the other commutators one finds that SL(32) is generated by multiple commutators of $T^{a_1a_2a_3}$.

For the effect of SL(32) on the central charges was given in equation (4.3). which in terms of the new basis, becomes

$$[Z_{a\beta}, T^{a_1...a_p}] = (\gamma^{a_1...a_p}(\gamma^a P_a + \gamma^{a_1a_2} Z_{a_1a_2} + \gamma^{a_1...a_5} Z_{a_1...a_5}))\alpha\delta(C^{-1})\delta\beta$$

$$+(-1)^{p(p+1)}((\gamma^a P_a + \gamma^{a_1a_2} Z_{a_1a_2} + \gamma^{a_1...a_5} Z_{a_1...a_5})\gamma^{a_1...a_p})\alpha\delta(C^{-1})\delta\beta$$

As a result we find that [8]

$$[P_b, T_{a_1a_2a_3}] = -6\eta_{b[a_1} Z_{a_2a_3]}, [Z^{b_1b_2}, T_{a_1a_2a_3}] = -5!Z_{a_1a_2a_3} b_1b_2 + 6\delta_{[a_1a_2} b_{3]} P_{a_3}$$

$$[Z^{b_1...b_5}, T_{a_1a_2a_3}] = -\frac{1}{4}\epsilon_{b_1...b_5} c_1...c_5[a_1a_2} Z_{a_3]c_1...c_5} + 2\delta_{[a_1a_2a_3} b_{5]}^{b_1b_2b_3} Z_{b_1b_2b_3}$$

GL(32) first arose in the dynamics of the M theory fivebrane which was found [8] to possess an unexpected symmetry whose generator had three anti-symmetrised eleven dimensional space-time indices. This generators was shown to obey commutation relations that identified it with a generator in contracted version of SL(32) [8]. Furthermore, when the dynamics of the fivebrane was described by a non-linear realisation the third rank

world volume gauge field strength was found to be the Goldstone boson for a rank three generator in SL(32) which acted on the supercharges as in equation (4.3).

It was explained in reference [8], that SL(32) is the natural generalisation of spin(1,10), required for the point particles, to the situation when branes are present in M theory and it acts as a brane rotating symmetry. Spin(1,10) can be defined is just the group which acts by a transformation on $Q_\alpha$ that takes $(\gamma^a)_\alpha\beta P_a$ into on object of the same form. However, when branes are present this last expression is replaced by the arbitrary matrix $Z_{\alpha\beta}$ and so the natural symmetry to consider is SL(32).

Finally, in reference [1] it was argued that GL(32) was a symmetry of M theory. We recall the outline of the argument. Eleven dimensional supergravity is invariant under the supersymmetry algebra and it is also invariant under the eleven dimensional conformal group. The latter follows from the formulation of eleven dimensional supergravity as a simultaneous non-linear realisation of the conformal algebra and $G_{11}$ [1]. However, it has been known [10] for many years that the only algebra that contains the conformal algebra and the supersymmetry algebra is Osp(1/64) and so the result follows.

This invariance is consistent with the much earlier result of reference [11] which showed that eleven dimensional supergravity was invariant under SO(16) provided one took only
SO(1,2) × SO(8) as the Lorentz group. This SO(16) is contained in SL(32) in a straightforward way; we write the 32 component spinor in terms of its spin(1,2) × spin(8) decomposition and then the spin(16) acts on the indices associated with the latter.

Very recently SL(32) has been considered in the context of the holonomy of M theory [12],[13],[14]. Some of the results of these papers can be viewed as a consequence of the SL(32) symmetry found in [8,9,1] and some of the conjectures made in [13] were already proposed in [8,9,1] and indeed shown in [1].

5 E_{11} and the identification of the central charges

It was suggested [8] that the Lorentz algebra should be replaced by SL(32) in the context of M theory. As such, one might anticipate that this algebra should be contained in \( E_{11} \) and, in particular, one might expect that SL(32) should be contained in the corresponding local sub-algebra which is the Cartan involution invariant sub-algebra of \( E_{11} \). Such an identification can also be seen from another viewpoint. The world volume gauge field strength that occurs in the fivebrane equation of motion is the Goldstone boson for part of the SL(32) automorphism symmetry [9] and it is also the case that the third rank gauge field of eleven dimensional supergravity is the Goldstone boson associated with the generator \( R^{a_1a_2a_3} \) of \( E_{11} \) [1,2]. However, these two fields appear in the world volume dynamics in a linear combination due to their gauge symmetry and as such the generators of the two algebras should be related to each other.

The generators of the Cartan involution invariant sub-algebra of \( E_{11} \) are given, at low levels, in equations (2.17) to (2.19) and their commutation relations in equations (2.20) and (2.21). Comparing these with the analogous relations for the SL(32) automorphism algebra of the eleven dimensional supersymmetry algebra given in equations (4.8) and (4.9) we find the same commutation relations provided we identify \( T_{a_1a_2a_3} = S_{a_1a_2a_3}, \quad T_{b_1...b_6} = S_{b_1...b_6} \) and \( S_{b_1...b_6} [a_1a_2a_3] = \frac{1}{2} \epsilon^{a_3a_2a_1} b_{b_1...b_6} c_{1...4} T_{c_1...c_4} \)

The reason for the contraction appearing in reference [8] is because the symmetry of the fivebrane dynamics considered there involves the generator \( R^{b_1b_2b_3} \) rather than \( S_{a_1a_2a_3} \). It follows from the above arguments that, even though we have not introduced the supersymmetry supercharges themselves, the SL(32) generators identified in \( E_{11} \) in this paper are the ones that acts on the supercharges as in equation (4.3).

At first sight the rank of SL(32) is too large for it to be contained in \( E_{11} \). However, \( E_{11} \) does contain very large commuting sub-algebras, certainly much larger than eleven, and to identify SL(32) we have set to zero some generators in \( E_{11} \) and this step could also increase the number of generators allowed in a commuting sub-algebra. The only remaining SL(32) generator that is left to identify is \( T_{a} \) which occurs at the next level and it would be good to extend the calculation given in this paper to the next level and so identify this generator.

Given the above identification of SL(32) one can examine if its action on the central charges given in equations (4.11) and (4.12) plays a role in \( E_{11} \). Indeed, comparing these equations with equations (3.15) and (3.16) we find the same commutators provided we identify the central charges of the eleven dimensional supersymmetry algebra with the two and five rank anti-symmetric tensors that appear in the \( E_{11} \) representation with highest
weight $l_1$, in particular we take
\[ W^{a_1a_2} = 2Z^{a_1a_2}, \quad W^{a_1...a_5} = 2.5!Z^{a_1...a_5}, \quad W^{a_1...a_7,c} = 60\varepsilon^{a_1...a_7}d_1...d_4Z^{cd_1...d_4} \] (5.2)

Since the generator $P_a$ is in common to the supersymmetry algebra and the semi-direct product of $E_{11}$ with its $l_1$ representation, we are obliged to identify the two and five rank anti-symmetric tensors that appear in $E_{11} \oplus_s L_1$.

The conjectured $E_{11}$ symmetry of M theory was uncovered in reference [2] by examining the bosonic sector of eleven dimensional supergravity and it is encouraging that aspects of M theory found outside this sector, i.e. in the fermionic sector and in the five-brane dynamics lead to symmetries that are contained in semi-direct product of $E_{11}$ with its $l_1$ representation. This bodes well for the incorporation of supersymmetry and the rest of the conformal group into an extension of $E_{11} \oplus_s L_1$.

The realisation that $E_{11}$ places the central charges in the same symmetry multiplet as the translations generators indicates that the theory which is invariant under $E_{11}$ should involve the usual space-time and coordinates associated with the central charges. In particular, we should construct the non-linear realisation of $E_{11} \oplus_s L_1$ and so consider a group element of the form
\[
g = \exp(x^aP_a + x_{a_1a_2}W^{a_1a_2} + x_{a_1...a_5}W^{a_1...a_5} + \ldots) \exp(h_a^bK^a_b)\exp\left(\frac{A_{c_1...c_3}R^{c_1...c_3}}{3!} + \frac{A_{c_1...c_6}R^{c_1...c_6}}{6!}\right)\ldots \tag{5.3}
\]
where $\ldots$ stand for higher level generators in $l_1$ and the final $\ldots$ for terms containing higher level generators of $E_{11}$. The fields should then depend on $x^a, x_{a_1a_2}, x_{a_1...a_5}, \ldots$. This formulation should be able to describe point particles and branes on an equal footing as it encodes the brane rotating symmetries. Traditionally, non-linear realisation have been used to describe the low energy dynamics of a theory in which a symmetry is spontaneously broken, however, at high energies the same symmetry is expected to be linearly realised. It is to be expected that the same might occur here and that the fundamental theory may have its symmetries realised on different variables. This presumably should include the coordinates.

It would be interesting to examine the dimensional reduction of the above non-linear realisation and compare the result in nine dimensions with the T duality invariant effective action constructed in reference [17].

6 Solutions and Weyl Transformations

In a recent paper [15], the Cartan sub-algebra of $E_{11}$, that is group elements of the form $g = e^{g^m(x)}H_m$ where $H_m$ are the Cartan generators of $E_{11}$, was considered and the action of the $E_{11}$ Weyl transformations induced on the space-time fields $q^m$ was found. After relating these to the diagonal components of the metric, it was shown that the moduli space of an enlarged set of Kasner solutions carry a representation of these Weyl transformations [15]. In this section we generalise this work and consider group elements that include the space-time translation generators and higher level generators in the $l_1$ representation. We will find the action of the Weyl transformations also on the space-time
and other coordinates at low levels. As a result, we will be able to find the action of Weyl transformations on the solutions themselves and not just their moduli. Our discussion will be rather brief, but we will set out the general procedure and illustrate it in the context of the generalised Kasner solutions.

Restricting the $E_{11}$ part of the group element to be in the Cartan sub-algebra, the group element of equation (5.3) takes the form

$$g = \exp(x^\mu P_\mu + x_{a1}W^{a1} + x_{a2}W^{a2} + \ldots) \exp(q^m H_m)$$

$$= \exp(x^\mu P_\mu + x_{a1}W^{a1} + x_{a2}W^{a2} + \ldots) \exp(p_\alpha K_\alpha^a). \quad (6.1)$$

For $E_{11}$ the Cartan sub-algebra is expressed in terms of the generators $K_\alpha^a$ which belong to GL(11) [2]. A detailed explanation of the change from the $q^m$ to the $p_\alpha$ variables is given in section 2 of reference [15]. The theory resulting from the non-linear realisation involving such a restriction only possess a diagonal metric whose components are given by $g_{\mu\mu} = e^{2p_\alpha \eta_{\alpha\mu}}$.

Although the Cartan sub-algebra of a Kac-Moody algebra $g$ carries a representation of the Weyl group this is not the case for the representations of $g$. However, an extension of the Weyl group by a cyclic group does have an action on representations of $g$. The effect of this extension is to introduce various signs into the action of the Weyl group. In this first account we will omit the, possibly significant, effects of the centre and give the effect of Weyl transformations are only up to signs. The Weyl reflections, $S_\alpha_a$ corresponding to the simple roots $\alpha_a$ on the weights $w$ of a representation are given by $S_\alpha_a w = w - (\alpha_a, w)\alpha_a$. Carrying this out on the root string of equation (3.2) using the correspondence between the weights and the generators, we find that $S_\alpha_a, a = 1, \ldots, 10$ does not introduce any $\alpha_{11}$ and so they do not change the level of the generator on which they act. They act on the space-time translation generators as

$$S_\alpha_a(P_a) = P_{a+1}, \quad S_\alpha_a(P_{a+1}) = P_a, \quad S_\alpha_a(P_b) = P_b, \ b \neq a, a + 1 \quad (6.2)$$

The Weyl transformation $S_{11}$ can change the level as it can introduce a $\alpha_{11}$ in its reflection. Its action on the translations is given by

$$S_{11}(P_a) = P_a, \ a \neq 9, 10, 11,$$

$$S_{11}(P_9) = W^{1011}, \ S_{11}(P_{10}) = W^{911}, \ S_{11}(P_{11}) = W^{910}, \quad (6.3)$$

while on the second rank central charge we find that

$$S_{11}(W^{1011}) = P_9, \ S_{11}(W^{911}) = P_{10}, \ S_{11}(W^{910}) = P_{11}, \ S_{11}(W^{ab}) = W^{ab}, \ a \leq 8, \ b > 8. \quad (6.4)$$

$S_{11}$ takes the remaining components of the two-form central charge into the five-form central charge. Examining the group element of equation (6.1) we find that the Weyl transformations $S_\alpha_a, a = 1, \ldots, 10$ act on the space-time coordinates and variables $p_\alpha$ as

$$x^a \leftrightarrow x^{a+1}, \ p_a \leftrightarrow p_{a+1}. \quad (6.5)$$
It is important to remember that the $p_a$ depend on the space-time and other coordinates and the change in these must also be carried out when evaluating the transformed variable. The Weyl transformation $S_{11}$ mixes the space-time coordinates with the central charge coordinates, for example $x^c \rightarrow x^c$, $c = 1, \ldots, 8$, $x^9 \leftrightarrow x_{1011}$, $x^{10} \leftrightarrow x_{911}$ and $x^{11} \leftrightarrow x_{910}$.

The generalised Kasner solutions can be labeled by the space-time variable that the metric depends on. The $x^b$-Kasner solution has a metric of the form

$$g_{\mu\nu} = \eta_{\mu a} e^{2\tilde{p}_a x^b}$$

where $\tilde{p}_a$ are constants which must satisfy

$$\sum_{c, c \neq b} \tilde{p}_c = \tilde{p}_b, \quad \sum_{c, c \neq b} (\tilde{p}_c)^2 = (\tilde{p}_b)^2$$

in order to obey Einstein’s equations. The usual Kasner solution has $b = 0$ i.e. depends on a time variable.

Using equation (6.5) we can carry out the Weyl transformations for $S_a$, $a = 1, \ldots, 10$ on the generalised Kasner solutions and we find that the solutions are indeed exchanged under the action of these Weyl transformations. It takes the $x^b$-Kasner solution into the $x^b$-Kasner solution if $b \neq a, a + 1$ and it swops the $x^a$-Kasner solution with the $x^{a+1}$-Kasner solution. We observe that this includes the swopping of the temporal and spatial generalised Kasner solutions if $a = 1$.

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