THE RICCI-BOURGUIGNON FLOW ON HEISENBERG AND QUATERNION LIE GROUPS

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Abstract. In this paper, we study the Ricci-Bourguignon flow on higher dimensional classical Heisenberg nilpotent Lie groups and construct a solution of this flow on Heisenberg and quaternion nilpotent Lie groups. In the end, we investigate the deformation of spectrum and length spectrum on compact nilmanifolds obtained of Heisenberg and quaternion nilpotent Lie groups.

1. Introduction

Geometric flow is an evolution of a geometric structure under a differential equation associated with some curvature and it is an important topic in many branches of mathematics and physics. A geometric flow is related to dynamical systems in the infinite-dimensional space of all metrics on a given manifold.

Let $M$ be an $n$-dimensional manifold with a Riemannian metric $g_0$, the family $g(t)$ of Riemannian metrics on $M$ is called a Ricci-Bourguignon flow when it satisfies the equations

\begin{equation}
\frac{d}{dt}g(t) = -2Ric(g(t)) + 2\rho R(g(t))g(t) = -2(Ric - \rho R)g(t), \quad g(0) = g_0
\end{equation}

where $Ric$ is the Ricci tensor of $g(t)$, $R$ is the scalar curvature and $\rho$ is a real constant. In fact the Ricci-Bourguignon flow is a system of partial differential equations which was introduced by Bourguignon for the first time in 1981 (see [3]). For closed manifolds, short time existence and uniqueness for solution to the Ricci-Bourguignon flow on $[0, T]$ have been shown by Catino and et’al in [4] for $\rho < \frac{1}{2(n-1)}$. When $\rho = 0$, the Ricci-Bourguignon flow is the Ricci flow.

A Riemannian metric $g$ on $N$ is left invariant if the left translations $L_p$’s are isometries for all $p \in N$. We will use $\langle, \rangle$ to denote both the inner product on $N = T_eN$ and the corresponding left invariant metric on $N$. Let $Z$ be the center of $N$, we denote the orthogonal complement of $Z$ in $N$ by $\mathcal{V}$ and write $N = \mathcal{V} \oplus Z$. Define a linear transformation $j : Z \to SO(\mathcal{V})$ by $j(Z)X = \langle (adX)^*Z, Y \rangle$, for all $X, Y \in \mathcal{V}$. Equivalently, for each $Z \in Z$, $j(Z) : \mathcal{V} \to \mathcal{V}$ is the skew-symmetric linear transformation defined by

\begin{equation}
\langle (adX)^*Z, Y \rangle = \langle Z, (adX)Y \rangle,
\end{equation}

for all $X, Y \in \mathcal{V}$. Here $(adX)(Y) = [X, Y]$ for all $X, Y \in \mathcal{N}$, and $(adX)^*$ denotes the (metric) adjoint of $adX$. A 2-step nilpotent Lie algebra $\mathcal{N}$ is called of Heisenberg

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type if $j(Z)^2 = -|Z|^2 \text{Id}$ for all $Z \in \mathcal{Z}$, for instance, the classical Heisenberg Lie group $H_n$ and quaternion Lie group $Q_n$ with special metrics are of Heisenberg type (see [3, 6, 10]).

Lauret in [11], studied the Ricci soliton on homogeneous nilmanifolds and then Payne in [13, 14] investigated the Ricci flow and the soliton metrics on nilmanifolds and nilpotent Lie groups. Also, Williams in [15] funded the explicit solution for the Ricci flow on some nilpotent Lie groups for instance, the classical Heisenberg Lie group $H_n$ of dimension $(2n + 1)$. Author and Razavi in [1], studied the eigenvalue variations of Heisenberg and quaternion Lie groups under the Ricci flow and investigated the deformation of some characteristics of compact nilmanifolds $\Gamma \setminus N$ under the Ricci flow, where $N$ is a simply connected 2-step nilpotent Lie group with a left invariant metric and $\Gamma$ is a discrete cocompact subgroup of $N$, in particular Heisenberg and quaternion Lie groups.

Motivated by the above works, in this paper, we will investigate the Ricci-Bourguignon flow on higher dimensional classical Heisenberg and quaternion nilpotent Lie groups and find the deformation of spectrum and length spectrum of compact nilmanifold obtained of the Heisenberg and quaternion nilpotent Lie groups.

2. Preliminaries

2.1. Curvature of Lie groups. We recall some properties about the geometry of Lie groups with left-invariant metrics, and derive the formula for the Ricci tensor (see [2, 10, 12]). Suppose that $\langle \ldots \rangle$ is a left-invariant metric on a Lie group $N$, which is equivalent to an inner product on the Lie algebra $\mathcal{N}$. Let $\nabla$ denote the Levi-Civita connection for the metric, and let $X, Y, Z, W \in \mathcal{N}$. We shall need the following useful theorems and propositions of ([2]) about the Ricci tensor of a Lie group.

**Proposition 2.1.** Let $\langle \ldots \rangle$ be a left-invariant metric on a Lie group $N$ and $\nabla$ the connection for this metric, for $X, Y, Z, W \in \mathcal{N}$, we have

a) $\nabla_X Y = \frac{1}{2} \{ (adX)Y - (adX)^* Y - (adY)^* X \}$,

b) $\langle R(X,Y)Z,W \rangle = \langle \nabla_X Z, \nabla_Y W \rangle - \langle \nabla_Y Z, \nabla_X W \rangle - \langle \nabla_{[X,Y]} Z, W \rangle$.

Besides, the maps

\[ (X, Y) \mapsto (adX)Y, \quad (X, Y) \mapsto (adX)^* Y, \]

from $\mathcal{N} \times \mathcal{N}$ to $\mathcal{N}$ are bilinear maps. We define

\[ U : \mathcal{N} \times \mathcal{N} \to \mathcal{N} \]

\[ U(X,Y) = - \frac{1}{2} \{ (adX)^* Y + (adY)^* X \}, \]

which is bilinear and symmetric.

**Proposition 2.2.** The Riemannian curvature tensor on $N$ is given by

\[ 4 \langle R(X,Y)Z,W \rangle = 2 \langle [X,Y], [Z,W] \rangle + \langle [X,Z], [Y,W] \rangle - \langle [X,W], [Y,Z] \rangle - \langle [[X,Y], Z], W \rangle + \langle [[X,Y], W], Z \rangle - \langle [[W],[Y]], X \rangle + 4 \langle U(X,Z), U(Y,W) \rangle - 4 \langle U(X,W), U(Y,Z) \rangle. \]
The dual coframe is and the only nontrivial Lie bracket relation is:

\( \text{ade} \)

It yields:

**Corollary 2.3.**

a) \( a^k_{ij} = C^m_{il} g_{jm} g^{kl} \),

b) If \( \nabla_e e_j = \gamma^k_{ij} e_k \) then

\( \gamma^k_{ij} = \frac{1}{2} g^{kl} (C^m_{ij} g_{lm} - C^m_{il} g_{jm} - C^m_{jl} g_{im}) \).

c) The components of the Riemann curvature tensor satisfy

\[
4R_{ijkl} = 2C^p_{ij} C^q_{kl} g_{pq} + C^p_{ik} C^q_{jl} g_{pq} - C^p_{ij} C^q_{kl} g_{pq} - C^p_{ij} C^q_{kl} g_{pq} + C^p_{ij} C^q_{kl} g_{pq} - C^p_{ij} C^q_{kl} g_{pq}
\]

\( (a^p_{ij} + a^p_{ji}) (a^q_{kl} + a^q_{lk}) g_{pq} - (a^p_{ij} + a^p_{ji}) (a^q_{kl} + a^q_{lk}) g_{pq} \).

d) The components of the Ricci curvature tensor satisfy

\[
4R_{ij} = \left\{ \begin{array}{ll}
2C^p_{ik} C^q_{jm} g_{pq} + C^p_{ik} C^q_{jm} g_{pq} - C^p_{ik} C^q_{jm} g_{pq} - C^p_{ik} C^q_{jm} g_{pq} + C^p_{ik} C^q_{jm} g_{pq} - C^p_{ik} C^q_{jm} g_{pq} \\
C^p_{km} C^q_{ij} g_{pq} - C^p_{km} C^q_{ij} g_{pq} - C^p_{km} C^q_{ij} g_{pq} + C^p_{km} C^q_{ij} g_{pq} \end{array} \right.
\]

2.2. **Heisenberg Lie group.** We now recall the construction and properties of the higher-dimensional, classical Heisenberg Lie group. Let \( H_n \) be a \((2n+1)\)-dimensional Heisenberg Lie group. Let 

\[
\begin{align*}
x & = (x^1, \ldots, x^n) \\
y & = (x^{n+1}, \ldots, x^{2n}).
\end{align*}
\]

If \( q = (x, y, z) \in H_n \) and \( q = (x', y', z') \in H_n \) then the group multiplication is 

\[
(x, y, z) \circ (x', y', z') = (x + x', y + y', z + z' + x'y')
\]

where \( x, y' \) is the usual inner product of vectors \( x \in \mathbb{R}^n \) and \( y' \in \mathbb{R}^n \). With respect to this multiplication, we have the following frame of left invariant vector fields,

\[
e_i = \partial_i, \quad e_{n+i} = \partial_{n+i} + x^i \partial_{2n+1}, \quad e_{2n+1} = \partial_{2n+1}, \quad \text{for all } 1 \leq i \leq n,
\]

and the only nontrivial Lie bracket relation is

\[
[e_i, e_{n+i}] = e_{2n+1}, \quad \text{for all } 1 \leq i \leq n.
\]

The dual coframe is

\[
\theta^i = dx^i, \quad \theta^{n+i} = dx^{n+i}, \quad \theta^{2n+1} = dx^{2n+1}, \quad \text{for all } 1 \leq i \leq n.
\]
Set $V = \text{span}\{e_i, e_{n+1} | 1 \leq i \leq n\}$ and $Z = \text{span}\{e_{2n+1}\}$. With the above multiplication $V \cup Z$ is an orthonormal basis for $H_n$, then $H_n = V \oplus Z$ and the Heisenberg Lie group is of Heisenberg type.

2.3. The Ricci-Bourguignon flow on the Heisenberg Lie group. In this section, we study solutions to The Ricci-Bourguignon flow \(g(t)\) starting at some initial metric $g_0$ on Heisenberg Lie group. Any one-parameter family of left invariant metrics $g(t)$ on $H_n$ which is a solution of the Ricci-Bourguignon flow, can be written as

$$g(t) = g_{IJ}(t)g^I \otimes g^J.$$ 

In [15], Williams, using Propositions 2.1, 2.2 and Corollary 2.3 showed that the Ricci tensor of $H_n$ as follows:

$$
\begin{cases}
R_{ij}(t) = -\frac{1}{2} g^{i+n,j+n}(t) g_{NN}(t) + \frac{1}{2} g_{iN}(t) g_{jN}(t) \sum, & \text{if } 1 \leq i, j \leq n; \\
R_{ij+N}(t) = \frac{1}{2} g^{i+n,j}(t) g_{NN}(t) + \frac{1}{2} g_{iN}(t) g_{j+n,N}(t) \sum, & \text{if } 1 \leq i, j \leq n; \\
R_{i+N,j}(t) = \frac{1}{2} g^{i,j+n}(t) g_{NN}(t) + \frac{1}{2} g_{i+N}(t) g_{jN}(t) \sum, & \text{if } 1 \leq i, j \leq n; \\
R_{i+N,N}(t) = \frac{1}{2} g_{i+N,N}(t) g_{NN}(t) \sum, & \text{if } 1 \leq i \leq n; \\
R_{NN}(t) = \frac{1}{2} g_{NN}(t) \sum.
\end{cases}
$$

which

$$
\sum = \sum_{k,m=1}^{n} g^{km}(t) g^{k+n,m+n}(t) - \sum_{k=1}^{n} \sum_{m=n+1}^{2n} g^{km}(t) g^{k+n,m-n}(t).
$$

and $N = 2n + 1$. We assume that the Riemannian metric is diagonal. From now on, we only use single subscripts for the metric components: $g_1(t), ..., g_N(t)$. This implies that the Ricci tensor stays diagonal under the Ricci-Bourguignon flow, and the Ricci tensor as follows:

$$
\begin{cases}
R_{ij}(t) = \begin{cases}
-\frac{1}{2} g^{i+n,j+n}(t) g_{NN}(t) & \text{if } i = j, \\
0 & \text{if } i \neq j,
\end{cases} & \text{if } 1 \leq i, j \leq n; \\
R_{i,j+n}(t) = 0, & \text{if } 1 \leq i, j \leq n; \\
R_{i+N,j}(t) = 0, & \text{if } 1 \leq i \leq n; \\
R_{i+n,j+N}(t) = \begin{cases}
-\frac{1}{2} g^{i,j+n}(t) g_{NN}(t) & \text{if } i = j, \\
0 & \text{if } i \neq j,
\end{cases} & \text{if } 1 \leq i, j \leq n; \\
R_{i+N,N}(t) = 0, & \text{if } 1 \leq i \leq n; \\
R_{NN}(t) = \frac{1}{2} g_{NN}(t) \sum.
\end{cases}
$$

By direct computation we obtain the scalar curvature as follows:

$$R(t) = -\frac{1}{2} g_{NN} \sum.$$ 

Then the Ricci-Bourguignon flow equation on $H_n$ with a diagonal left-invariant metric $g_0$ has the following form

$$
\begin{align}
\frac{d}{dt} g_i(t) &= \frac{g_{NN}(t)}{g_{i+n}(t)} - \rho g_i(t) g_{NN}(t) \sum, & \text{if } 1 \leq i \leq n; \\
\frac{d}{dt} g_{i+n}(t) &= \frac{g_{NN}(t)}{g_{i+n}(t)} - \rho g_{i+n}(t) g_{NN}(t) \sum, & \text{if } 1 \leq i \leq n; \\
\frac{d}{dt} g_{NN}(t) &= -(1 + \rho) g_{NN}(t) \sum.
\end{align}
$$

(2.9)
Let \( g_1, g_2, \ldots, g_{2n}, g_N \) be a solution of the Ricci-Bourguignon flow. As diagonal components of a metric, they are positive function of \( t \).

**Theorem 2.4.** Consider the Heisenberg Lie group \( H_n \) with a diagonal left-invariant metric \( g_0 \). Let \( g(t) \) be a solution to the Ricci-Bourguignon flow with initial metric \( g_0 \), then

a: \( \frac{d}{dt} g_i(t) = 0 \), if \( 1 \leq i \leq n \);

b: \( \frac{d}{dt} \left( g_1(t) \cdots g_n(t) g_N(t) \right)^{\frac{1-n\rho}{1+n\rho}} = 0 \),

c: If \( \rho < 0 \) and \( G_N(t) = \int_0^t g_N(t) dt \) then \( \lim_{t \to +\infty} G_N(t) = +\infty \).

d: Moreover, if \( g_i(0) g_{i+n}(0) \) is a solution of the Ricci-Bourguignon flow, for \( 1 \leq i \leq n \) then a solution \( g(t) \) has the following form

\[
(2.10) \quad \begin{cases}
  g_j(t) = g_j(0) \left( 1 + bt \right)^{\frac{1-n\rho}{1+n\rho}}, & \text{if } 1 \leq j \leq 2n \\
  g_N(t) = g_N(0) \left( 1 + bt \right)^{\frac{n\rho}{1+n\rho}}
\end{cases}
\]

where \( b = (n+2-n\rho) \frac{g_N(0)}{g_N(0) g_{i+n}(0)} \).

**Proof.**

a: Using (2.9) and direct computing we have

\[
\frac{d}{dt} \frac{g_i(t)}{g_{i+n}(t)} = 0.
\]

b: By differential respect to variabla time \( t \) and using (2.9) we obtain

\[
\frac{d}{dt} \left( g_1(t) \cdots g_n(t) g_N(t) \right)^{\frac{1-n\rho}{1+n\rho}} = \left( \sum_{k=1}^n \frac{1}{g_k(t)} \frac{d g_k(t)}{dt} + \frac{1-n\rho}{1+n\rho} \frac{dg_N(t)}{dt} \right) g_1(t) \cdots g_n(t) g_N(t) \]

\[
= \left( \sum_{k=1}^n \left( \frac{g_N(t)}{g_k(t) g_{n+k}(t)} - \rho g_N(t) \sum_{j=1}^n g_j(t) \right) g_1(t) \cdots g_n(t) g_N(t) \right)^{\frac{1-n\rho}{1+n\rho}} = 0,
\]

the part a of Theorem implies that \( \frac{g_i(t)}{g_{i+n}(t)} \) is constant for \( 1 \leq i \leq n \), so we can set

\[
A_i = \frac{g_i(t)}{g_{i+n}(t)} = \frac{g_i(0)}{g_{i+n}(0)},
\]

therefore \( g_{i+n}(t) = g_A(t) \). Hence \( \frac{d}{dt} \left( g_1(t) \cdots g_n(t) g_N(t) \right)^{\frac{1-n\rho}{1+n\rho}} = 0 \) results that \( \frac{d}{dt} \left( g_{i+n}(t) \right)^{\frac{1-n\rho}{1+n\rho}} = 0 \). This completes the proof part b of the Theorem.

c: For \( \rho < 0 \) the equations (2.9) implies that \( g_j, \ 1 \leq j \leq 2n \) is an increasing function, so \( \sum \) is positive and decreasing. Since \( g_N(t) \) is positive then last equation in (2.9) yields

\[
\frac{d}{dt} g_N(t) = -(1+n\rho) g_N^2(t) \sum \geq -g_N^2(t) \sum \geq - \sum (0) g_N^2(t),
\]

which by direct computation results that

\[
g_N(t) \geq \frac{1}{\sum (0) t + g_N^{-1}(0)}.
\]
Consider

\[ Z = \text{span} \{ e_{2n+1} \}, \quad V = \text{span} \{ e_1, e_2, ..., e_{2n} \}, \quad H_n = V \oplus Z ,\]

where \( Z \) is the center of \( H_n \) and \( V \) is the orthogonal complement of \( Z \) in \( H_n \). If \( Z = e_{2n+1} \) then

\[ Z = e_{2n+1}, \quad j(Z)e_i = e_{n+i}, \quad j(Z)e_{n+i} = -e_i, \]

hence

\[ j(Z) = \begin{bmatrix} o & -I_n \\ I_n & o \end{bmatrix}, \quad (j(aZ))^2 = \begin{bmatrix} -a^2I_n & o \\ o & -a^2I_n \end{bmatrix}, \]

where \( I_n \) is an \( n \times n \) identity matrix and it yields to

\[ (j(aZ))^2 = -|aZ|^2 I_d, \]

therefore \( H_n \) with this structure is of Heisenberg type.

**Proposition 2.5.** Heisenberg type of Lie group \( H_n \) is not preserved along the Ricci-Bourguignon flow with solution \( (2.10) \) with additional condition \( g_{2n+1}(0) = g_i(0)g_{n+i}(0) \), for \( 1 \leq i \leq n \).

**Proof.** In \((H_n, g_t = <., .>_t)\), if we assume that \( j(Z)e_i = \sum_{k=1}^{2n+1} a_k(t)e_k, 1 \leq i \leq n \) then

\[ <Z, [e_i, e_j]>_t = <j(Z)e_i, e_j>_t = \sum_{k=1}^{2n+1} a_k(t) <e_k, e_j>_t = a_j(t) <e_j, e_j>_t, \]

therefore

\[ j(Z)e_i = \sum_{j=1}^{2n+1} <Z, [e_i, e_j]>_t e_j = \frac{|Z|^2}{g_{n+i}(t)} e_{n+i}, \]

in the same way it can be shown that

\[ j(Z)e_{n+i} = -\frac{|Z|^2}{g_{i}(t)} e_i, \quad 1 \leq i \leq n, \]

hence

\[ j(Z) = A \begin{bmatrix} o & -B_1 \\ B_2 & o \end{bmatrix}, \]

where \( B_1 = \text{diag}(\frac{1}{g_1(t)}, ..., \frac{1}{g_{2n}(t)}) \), \( B_2 = \text{diag}(\frac{1}{g_{2n+1}(t)}, ..., \frac{1}{g_{2n}(t)}) \), \( A = |Z|^2 \). Now for any real constant \( a \) we obtain

\[ j(aZ) = aA \begin{bmatrix} o & -B_1 \\ B_2 & o \end{bmatrix}, \quad (j(aZ))^2 = -a^2 A^2 \begin{bmatrix} D & o \\ o & D \end{bmatrix}, \]
where \( D = \text{diag}(\frac{1}{g_1(0)g_{n+1}(0)}, \ldots, \frac{1}{g_1(0)g_{2n}(0)}) \). But for \( 1 \leq i \leq n \) we have \( g_i(0)g_{n+i}(0) = g_{2n+1}(0) \), then (2.10) results that

\[
(j(Z))^2 = -\frac{1}{(n + 2 - n\rho)t + 1} |Z|^2 I_{2n}.
\]

So, Heisenberg type of \( H_n \) is not preserved under the evolution of the Ricci-Bourguignon flow.

**Definition 2.6.**

*Proof.* Let \( \mu(Z) \) denote the number of distinct eigenvalues of \( j(Z)^2 \) and \(-\theta_1(Z)^2, -\theta_2(Z)^2, \ldots, -\theta_\mu(Z)^2 \) denote the \( \mu \) distinct eigenvalues of \( j(Z)^2 \) with the assumption that \( 0 \leq \theta_1(Z) < \theta_2(Z) < \ldots < \theta_\mu(Z) \).

\( a) \) A two-step nilpotent metric Lie algebra \((N, <, >)\) is Heisenberg-like if 
\[ [j(Z)X_m, X_m] \in \text{span}_\mathbb{R} Z \]
for all \( Z \in Z \) and all \( X_m \in W_m(Z), m = 1, \ldots, \mu(Z) \), where \( W_m \) denotes the invariant subspace of \( j(Z) \) corresponding to \( \theta_m(Z), m = 1, \ldots, \mu(Z) \).

If \( \mathcal{N} \) is of Heisenberg type, then for all \( Z \in Z \) and \( X \in V \),

\[
[X, j(Z)X] = |X|^2 Z.
\]

If \( \mathcal{N} \) is of Heisenberg-like, then for all \( Z \in Z \) and every \( X_m \in W_m(Z), m = 1, \ldots, \mu(Z) \),

\[
[X_m, j(Z)X_m] = (\frac{\theta_m(Z)}{|Z|} |X_m|)^2 Z.
\]

Therefore we have with the above notation \( H_n \) under the evolution of the Ricci-Bourguignon flow from Heisenberg type convert to Heisenberg-like type.

### 2.4. Quaternion Lie groups.

We now recall the construction of the higher-dimensional, classical quaternion Lie groups. Let \( N = Q_n \) be a \((4n+3)\)-dimensional quaternion group. Let

\[
x = (x_{11}, x_{21}, \ldots, x_{n1}, x_{1n}, x_{2n}, \ldots, x_{4n}) ,
\]

\[
z = (z_1, z_2, z_3) .
\]

Assume that \( q = (x, z) \in N \) and \( q' = (x', z') \in N \). Multiplication on \( N \) is defined as follows:

\[
L_q(q') = L_{(x,z)}(x', z') = (x, z) o (x', z') = (x + x', z_1 + z_1' + \frac{1}{2}(M_1x, x'), z_2 + z_2' + \frac{1}{2}(M_2x, x') , z_3 + z_3' + \frac{1}{2}(M_3x, x')) ,
\]

where

\[
M_k = \begin{bmatrix}
A_k & O & \cdots & O \\
O & A_k & \cdots & \vdots \\
\vdots & \ddots & \ddots & O \\
O & \cdots & O & A_k
\end{bmatrix}
\]
for \( k = 1, 2, 3 \) and

\[
A_1 = \begin{bmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{bmatrix},
A_2 = \begin{bmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{bmatrix},
A_3 = \begin{bmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{bmatrix},
\]

\((M_k x, x')\) is the usual inner product of vectors \( M_k x \in \mathbb{R}^{4n} \) and \( x' \in \mathbb{R}^{4n} \). With respect to this multiplication, we have the following vector fields

\[
X_{hl} = \frac{\partial}{\partial x_{hl}} + \frac{1}{2} \left( x_{2l} \frac{\partial}{\partial z_1} - x_{4l} \frac{\partial}{\partial z_2} - x_{3l} \frac{\partial}{\partial z_3} \right),
X_{2l} = \frac{\partial}{\partial x_{2l}} + \frac{1}{2} \left( -x_{1l} \frac{\partial}{\partial z_1} - x_{3l} \frac{\partial}{\partial z_2} + x_{4l} \frac{\partial}{\partial z_3} \right),
X_{3l} = \frac{\partial}{\partial x_{3l}} + \frac{1}{2} \left( x_{4l} \frac{\partial}{\partial z_1} + x_{2l} \frac{\partial}{\partial z_2} + x_{1l} \frac{\partial}{\partial z_3} \right),
X_{4l} = \frac{\partial}{\partial x_{4l}} + \frac{1}{2} \left( -x_{3l} \frac{\partial}{\partial z_1} + x_{1l} \frac{\partial}{\partial z_2} + x_{2l} \frac{\partial}{\partial z_3} \right),
\]

(2.15)

\[
Z_m = \frac{\partial}{\partial z_m},
\]

(2.16)

for \( l = 1, 2, ..., n \) and \( m = 1, 2, 3 \). The nonzero Lie brackets of vector fields are

\[
[X_{1l}, X_{2l}] = -Z_1, \quad [X_{1l}, X_{3l}] = Z_3, \quad [X_{1l}, X_{4l}] = Z_2,
[X_{2l}, X_{3l}] = Z_2, \quad [X_{2l}, X_{4l}] = -Z_3, \quad [X_{3l}, X_{4l}] = -Z_1.
\]

(2.17)

Given the above definitions, \( Q_n \) is two-step nilpotent. Note the dual of the above vector fields are as follows

\[
dx_{kl}, \quad \theta_r = dz_r - \frac{1}{2} (M_r x, dx), \quad k = 1, 2, 3, 4, \quad 1 \leq l \leq n, \quad r = 1, 2, 3.
\]

Set

\[
\mathcal{V} = \text{span}\{X_{1l}, X_{2l}, X_{3l}, X_{4l} | 1 \leq l \leq n\}, \quad \mathcal{Z} = \text{span}\{Z_1, Z_2, Z_3\}.
\]

If we choose an inner product on \( Q_n \) such that \( \mathcal{V} \cup \mathcal{Z} \) is an orthonormal basis for \( Q_n \) then quaternion Lie group is of Heisenberg type.

2.5. The Ricci-Burguignon flow on quaternion Lie group. Assume that

\[
e_i = X_{1i}, \quad e_{n+i} = X_{2i}, \quad e_{2n+i} = X_{3i}, \quad e_{3n+i} = X_{4i},
\]

\[
e_{4n+r} = Z_r, \quad i = 1, 2, ..., n, \quad r = 1, 2, 3.
\]

Let \( g \) be diagonal and \( g_{aa} = g_{aa} \). With the above symbols and \([e_i, e_j] = C^{kl}_{ij} e_k\), and Propositions 2.1, 2.2, and corollary 2.3 we concluded that the Ricci tensor stay
diagonal under the Ricci-Bourguignon flow, also as follows:

\[
R_i = -\frac{1}{2} \left( \frac{g_{4n+1} + g_{4n+3} + g_{4n+2}}{g_{n+i}} \right),
\]

\[
R_{n+1} = -\frac{1}{2} \left( \frac{g_{4n+1} + g_{4n+2} + g_{4n+3}}{g_{i}} \right),
\]

\[
R_{2n+1} = -\frac{1}{2} \left( \frac{g_{4n+3} + g_{4n+2} + g_{4n+1}}{g_{i}} \right),
\]

\[
R_{3n+1} = -\frac{1}{2} \left( \frac{g_{4n+2} + g_{4n+3} + g_{4n+1}}{g_{n+i}} \right),
\]

\[
R_{4n+1} = \frac{g_{4n+1}}{2} \sum_{i=1}^{n} \left( \frac{1}{g_{i}} + \frac{1}{g_{2n+i}g_{3n+i}} \right),
\]

\[
R_{4n+2} = \frac{g_{4n+2}}{2} \sum_{i=1}^{n} \left( \frac{1}{g_{i}g_{3n+i}} + \frac{1}{g_{n+i}g_{2n+i}} \right),
\]

\[
R_{4n+3} = \frac{g_{4n+3}}{2} \sum_{i=1}^{n} \left( \frac{1}{g_{i}g_{2n+i}} + \frac{1}{g_{n+i}g_{3n+i}} \right),
\]

and

\[
R = -\frac{1}{2} \sum (t),
\]

where

\[
\sum (t) = g_{4n+1} \sum_{i=1}^{n} \left( \frac{1}{g_{i}g_{n+i}} + \frac{1}{g_{2n+i}g_{3n+i}} \right) + g_{4n+2} \sum_{i=1}^{n} \left( \frac{1}{g_{i}g_{3n+i}} + \frac{1}{g_{n+i}g_{2n+i}} \right)
\]

\[+ g_{4n+3} \sum_{i=1}^{n} \left( \frac{1}{g_{i}g_{2n+i}} + \frac{1}{g_{n+i}g_{3n+i}} \right).
\]

Therefore, the Ricci-Bourguignon flow equation, \( \frac{\partial g}{\partial t} = -2Ric + 2\rho Rg \), on \( Q_n \) has the form (2.18)

\[
\begin{align*}
\frac{\partial}{\partial t} g_i &= \frac{g_{4n+1}}{g_{n+i}} + \frac{g_{4n+3}}{g_{2n+i}} + \frac{g_{4n+2}}{g_{3n+i}} - \rho g_i \sum', \\
\frac{\partial}{\partial t} g_{n+i} &= \frac{g_{4n+1}}{g_{i}} + \frac{g_{4n+2}}{g_{2n+i}} + \frac{g_{4n+3}}{g_{3n+i}} - \rho g_{n+i} \sum', \\
\frac{\partial}{\partial t} g_{2n+i} &= \frac{g_{4n+3}}{g_{i}} + \frac{g_{4n+2}}{g_{n+i}} + \frac{g_{4n+1}}{g_{3n+i}} - \rho g_{2n+i} \sum', \\
\frac{\partial}{\partial t} g_{3n+i} &= \frac{g_{4n+1}}{g_{i}} + \frac{g_{4n+3}}{g_{n+i}} + \frac{g_{4n+2}}{g_{2n+i}} - \rho g_{3n+i} \sum', \\
\frac{\partial}{\partial t} g_{4n+1} &= -\left( \sum_{i=1}^{n} \frac{g_{4n+1}}{g_{i}g_{n+i}} + \sum_{i=1}^{n} \frac{g_{4n+3}}{g_{2n+i}g_{3n+i}} \right) - \rho g_{4n+1} \sum', \\
\frac{\partial}{\partial t} g_{4n+2} &= -\left( \sum_{i=1}^{n} \frac{g_{4n+2}}{g_{i}g_{n+i}} + \sum_{i=1}^{n} \frac{g_{4n+3}}{g_{2n+i}g_{3n+i}} \right) - \rho g_{4n+2} \sum', \\
\frac{\partial}{\partial t} g_{4n+3} &= -\left( \sum_{i=1}^{n} \frac{g_{4n+3}}{g_{i}g_{n+i}} + \sum_{i=1}^{n} \frac{g_{4n+2}}{g_{2n+i}g_{3n+i}} \right) - \rho g_{4n+3} \sum'.
\end{align*}
\]

**Theorem 2.7.** Consider the quaternion Lie group \( Q_n \) with a diagonal left-invariant metric \( g_0 \). Let \( g(t) \) be a solution to the Ricci-Bourguignon flow with initial metric \( g_0 \), then

\[
a: \frac{\partial}{\partial t} \left( g_1(t)g_2(t)...g_{4n}(t)(g_{4n+1}(t)g_{4n+2}(t)g_{4n+3}(t))^{\frac{2(1-2\rho)}{1+\rho}} \right) = 0,
\]
b: If $\rho < 0$ and $G_k(t) = \int_0^t g_k(t) \, dt$ for $k = 4n+1, \ldots, 4n+3$ then $\lim_{t \to +\infty} G_k(t) = +\infty$.

c: If moreover $g_j(0) = g_1(0)$, $g_{4n+1}(0) = g_{4n+2}(0) = g_{4n+3}(0)$, for $1 \leq j \leq 4n$ then a solution $g(t)$ has the following form

$$
\begin{align*}
(2.19) & \quad \begin{cases}
g_j(t) = g_1(0) \left(1 + ct\right) \frac{3(1-2\rho)}{3(1+3\rho) - \rho} & \text{if } 1 \leq j \leq 4n \\
g_{4n+k}(t) = g_{4n+1}(0) \left(1 + ct\right) \frac{3(1-2\rho)}{3(1+3\rho) - \rho} & \text{if } 1 \leq k \leq 3
\end{cases}
\end{align*}
$$

where $c = \frac{g_{2n+1}(0)}{g_1(0)} (6 + 2n - 6n\rho)$.

Proof. a: Assume that $G(t) = g_1(t)g_2(t)\ldots g_{4n}(t)(g_{4n+1}(t)g_{4n+2}(t)g_{4n+3}(t))^{\frac{2(1-2\rho)}{3(1+3\rho) - \rho}}$, then we get

$$
\frac{d}{dt} G(t) = \left(\sum_{i=1}^{4n} \frac{1}{g_i} \frac{dg_i}{dt} + \frac{2(1-2n\rho)}{1+3\rho} \sum_{k=1}^{3} \frac{1}{g_{4n+k}} \frac{dg_{4n+k}}{dt}\right) G(t)
$$

$$
= \sum_{i=1}^{4n} \frac{1}{g_i} \frac{dg_i}{dt} + \frac{1}{g_{n+i}} \frac{dg_{n+i}}{dt} + \frac{1}{g_{2n+i}} \frac{dg_{2n+i}}{dt} + \frac{1}{g_{3n+i}} \frac{dg_{3n+i}}{dt} G(t)
$$

$$
+ \frac{2(1-2n\rho)}{1+3\rho} G(t) \sum_{k=1}^{3} \frac{1}{g_{4n+k}} \frac{dg_{4n+k}}{dt}.
$$

Using (2.18), we have

$$
\frac{d}{dt} G(t) = \sum_{i=1}^{n} \left( \frac{g_{4n+1}}{g_i g_{n+i}} + \frac{g_{4n+3}}{g_i g_{2n+i}} + \frac{g_{4n+2}}{g_i g_{3n+i}} - \rho \sum' \right) G(t)
$$

$$
+ \sum_{i=1}^{n} \left( \frac{g_{4n+1}}{g_{n+i} g_i} + \frac{g_{4n+2}}{g_{n+i} g_{2n+i}} + \frac{g_{4n+3}}{g_{n+i} g_{3n+i}} - \rho \sum' \right) G(t)
$$

$$
+ \sum_{i=1}^{n} \left( \frac{g_{4n+3}}{g_{2n+i} g_i} + \frac{g_{4n+2}}{g_{2n+i} g_{n+i}} + \frac{g_{4n+1}}{g_{2n+i} g_{3n+i}} - \rho \sum' \right) G(t)
$$

$$
+ \sum_{i=1}^{n} \left( \frac{g_{4n+2}}{g_{3n+i} g_i} + \frac{g_{4n+3}}{g_{3n+i} g_{n+i}} + \frac{g_{4n+1}}{g_{3n+i} g_{2n+i}} - \rho \sum' \right) G(t)
$$

$$
+ \frac{2(1-2n\rho)}{1+3\rho} \left( -\sum_{i=1}^{n} \frac{g_{4n+1}}{g_i g_{n+i}} - \sum_{i=1}^{n} \frac{g_{4n+1}}{g_{2n+i} g_{3n+i}} - \rho \sum' \right) G(t)
$$

$$
+ \frac{2(1-2n\rho)}{1+3\rho} \left( -\sum_{i=1}^{n} \frac{g_{4n+2}}{g_i g_{n+i}} - \sum_{i=1}^{n} \frac{g_{4n+2}}{g_{n+i} g_{2n+i}} - \rho \sum' \right) G(t)
$$

$$
+ \frac{2(1-2n\rho)}{1+3\rho} \left( -\sum_{i=1}^{n} \frac{g_{4n+3}}{g_i g_{2n+i}} - \sum_{i=1}^{n} \frac{g_{4n+3}}{g_{n+i} g_{3n+i}} - \rho \sum' \right) G(t) = 0.
$$
Proof. The proof is similar to proof of Proposition 2.5. In the resolution of the Ricci-Bourguignon flow with solution (2.19) and additional condition $g_4n+1$ is not preserved under the evolution of the Ricci-Bourguignon flow with solution (2.19) and additional condition $g_4n+1(0) = g_{4n+1}(0)$.

Proposition 2.8. Heisenberg type of Lie group $Q_n$ is not preserved under the evolution of the Ricci-Bourguignon flow with solution (2.19) and additional condition $g_4n+1(0) = g_{4n+1}(0)$.

Proof. The proof is similar to proof of Proposition 2.5. In $(Q_n, g_t = <, >_l)$, we have

\[ j(Z_1)e_i = \frac{g_{4n+1}(t)}{g_{n+i}(t)} e_{n+i}, \quad j(Z_1)e_{n+i} = \frac{g_{4n+1}(t)}{g_i(t)} e_i, \quad j(Z_1)e_{2n+i} = -\frac{g_{4n+1}(t)}{g_{3n+i}(t)} e_{3n+i}, \]

\[ j(Z_2)e_i = \frac{g_{4n+2}(t)}{g_{2n+i}(t)} e_{2n+i}, \quad j(Z_2)e_{n+i} = \frac{g_{4n+2}(t)}{g_{3n+i}(t)} e_{3n+i}, \quad j(Z_2)e_{2n+i} = -\frac{g_{4n+2}(t)}{g_{n+i}(t)} e_i, \]

\[ j(Z_3)e_i = \frac{g_{4n+3}(t)}{g_{3n+i}(t)} e_{3n+i}, \quad j(Z_3)e_{n+i} = -\frac{g_{4n+3}(t)}{g_{i}(t)} e_i, \quad j(Z_3)e_{2n+i} = -\frac{g_{4n+3}(t)}{g_{n+i}(t)} e_{n+i}. \]
By Theorem 2.7 for \(1 \leq i \leq 4n\) and \(1 \leq k \leq 3\) we have \(g_i(t) = g_i(t)\) and \(g_{4n+k}(t) = g_{4n+1}(t)\). Therefore, if we set \(E = \frac{g_{4n+1}(t)}{g(t)}\), then

\[
j(Z_1) = E \begin{bmatrix}
o & I_n & 0 & 0 \\
-o & -I_n & 0 & 0 \\
o & 0 & I_n & 0 \\
o & -I_n & 0 & 0 \\
o & o & o & I_n \\
-o & I_n & 0 & 0 \\
o & o & I_n & 0 \\
o & -I_n & 0 & 0 \\
o & o & o & I_n \\
o & -I_n & 0 & 0 \\
o & o & o & I_n \\
o & -I_n & 0 & 0 \\
o & o & o & I_n \\
\end{bmatrix}
\]

\[
j(Z_2) = E \begin{bmatrix}
o & o & 0 & -I_n \\
o & o & -I_n & 0 \\
o & I_n & 0 & 0 \\
o & I_n & 0 & 0 \\
o & o & o & I_n \\
o & -I_n & 0 & 0 \\
o & I_n & 0 & 0 \\
o & -I_n & 0 & 0 \\
o & o & o & I_n \\
o & -I_n & 0 & 0 \\
o & o & o & I_n \\
o & -I_n & 0 & 0 \\
o & o & o & I_n \\
\end{bmatrix}
\]

hence for any real constants \(c_1, c_2\) and \(c_3\), we find that

\[
j(c_1Z_1 + c_2Z_2 + c_3Z_3) = E \begin{bmatrix}
o & c_1I_n & -c_3I_n & -c_2I_n \\
-c_1I_n & o & -c_2I_n & c_3I_n \\
c_3I_n & c_2I_n & o & c_1I_n \\
c_2I_n & -c_3I_n & -c_1I_n & o \\
\end{bmatrix}
\]

If \(Z = c_1Z_1 + c_2Z_2 + c_3Z_3\) then

\[
(j(Z))^2 = -E^2(c_1^2 + c_2^2 + c_3^2)I_{4n} = -E^2|Z|^2I_{4n+1}(t)I_{4n} = \frac{1}{(6 + 2n - 6\rho)t + 1}|Z|^2I_{4n}.
\]

Hence, it is not of Heisenberg type. \(\square\)

Remark 2.9. By the Definition 2.6, \(Q_n\) along the Ricci-Bourguignon flow from Heisenberg type convert to Heisenberg-like type

3. Deformation of marked length spectrum

Suppose that the Lie group \(N\) is \(H_n\) or \(Q_n\) and \(g(t)\) is the solution of the Ricci-Bourguignon flow in (2.10) and (2.11) respectively, with some conditions given in Proposition 2.5 and 2.8. Then for \(t = 0\) we have

\[
j(Z)^2 = -|Z|^2Id \quad \text{for all } Z \in Z,
\]

that is the group \(N\) is Heisenberg type. But if in Heisenberg Lie group \((H_n, g(t))\) suppose that \(\eta_t = \frac{1}{(n+2-6\rho)t+1}\), then in \((H_n, g(t))\) from the proof of Proposition 2.3 we have

\[
j(Z)^2 = -\eta_t|Z|^2Id \quad \text{for all } Z \in Z,
\]

Also if in the quaternion Lie group \((Q_n, g(t))\) we suppose that \(\zeta_t = \frac{1}{(6+2n-6\rho)t+1}\) then in \((Q_n, g(t))\) from the proof of Proposition 2.8 we obtain

\[
j(Z)^2 = -\zeta_t|Z|^2Id \quad \text{for all } Z \in Z.
\]

Let \(P_t = \eta_t \text{ or } \zeta_t\). For \(H_n\) or \(Q_n\) we have \(j(Z)^2 = -P_t|Z|^2Id\). Similarly the argument of [1], we conclude the following statements about the deformation of spectrum and length spectrum. The spectrum and the length spectrum have relationship with each other (see [7, 8, 9]). By easy computation we have

Proposition 3.1. Let \((N, <, >_t)\) is the Lie algebra of \(N\) where \(N\) is \(H_n\) or \(Q_n\).

Then we have

(1) \(< j(Z)X, j(Z^*)X >_t = P_t < Z, Z^* >_t < X, X >_t\) for all \(Z, Z^* \in Z\) and \(X \in \mathcal{V}\);

(2) \(< j(Z)X, j(Z)Y >_t = P_t < Z, Z >_t < X, Y >_t\) for all \(Z \in Z\) and \(X, Y \in \mathcal{V}\).
(3) \(|j(Z)X| = P_t^k|Z|_t|X|_t\) for all \(Z \in \mathcal{Z}\) and \(X \in \mathcal{V}\);
(4) \(|j(Z) \circ j(Z^*) + j(Z^*) \circ j(Z)| = -2P_t < Z, Z^* >_t\ Id\) for all \(Z, Z^* \in \mathcal{Z}\);
(5) \(|X, j(Z)X| = P_t < X, X >_t\ Z\) for all \(Z \in \mathcal{Z}\) and \(X \in \mathcal{V}\).

**Proposition 3.2.** Let \(\sigma(s, t) = \exp(X(s, t) + Z(s, t))\) be a curve in 2-step nilpotent Lie group with left invariant metric \((N, g(t))\) where \(N = H_n\) or \(Q_n\), such that \(\sigma(0, t) = e\) and \(\sigma'(0, t) = X_0(t) + Z_0(t)\), where \(X_0(t) \in \mathcal{V}(t)\), \(Z_0(t) \in \mathcal{Z}(t)\) and \(e\) is the identity in \(N\). Let \(g(t)\) is the solution of the Ricci-Bourguignon flow on \(H_n\) and \(Q_n\) in \((L, g)\) and \((L, g)\) respectively. Then

\[
\begin{pmatrix}
X(s, t) \\
Z(s, t)
\end{pmatrix} = \begin{pmatrix}
(coss\theta - 1)J^{-1}X_0(t) + \frac{\sin\theta}{\theta}X_0(t) \\
(s(1 + \frac{|X_0(t)|^2}{2|Z_0(t)|^2}) + \frac{\sin\theta}{\theta} \frac{|X_0(t)|^2}{2|Z_0(t)|^2}) Z_0(t)
\end{pmatrix}
\]

(3.1)

where \(J = j(Z_0(t))\), \(\theta = \sqrt{P_t}|Z_0(t)|\).

**Definition 3.3.** A nonidentity elements \(\varphi(t)\) of \((N, g(t))\) translates a unit speed geodesic \(\sigma(s, t)\) in \((N, g(t))\) by an amount \(\omega(t) > 0\) if \(\varphi(t) \cdot \sigma(s, t) = \sigma(s + \omega(t), t)\) for all \(s \in \mathbb{R}\). The amount \(\omega(t)\) is called a period of \(\varphi(t)\).

**Definition 3.4.** Let \(N\) be a simply connected, nilpotent Lie group with a left invariant metric, and let \(\Gamma \subseteq N\) be a discrete subgroup of \(N\). The group \(\Gamma\) is said to be a lattice in \(N\) if the quotient manifold \(\Gamma \setminus N\) obtained by letting \(\Gamma\) act on \(N\) by left translation is compact.

**Proposition 3.5.** Let \((N, g(t))\) be \((H_n, g(t))\) or \((Q_n, g(t))\), \(g(t)\) is the solution of the Ricci-Bourguignon flow in \((L, g)\) and \((L, g)\) respectively, and \(\Gamma\) be a discrete subgroup of \(N\). Let \(\varphi(t) \in \Gamma\) be a family of nonidentity elements of the center of \(N\), such that \(log\varphi(t) \in \mathcal{Z}\). Then \(\varphi(t) = \exp(V^*(t) + Z^*(t))\) has the following periods.

\[
\left\{|Z^*(t)|_t, \sqrt{(4\pi k)(|Z^*(t)|_t - \pi k)}; \text{where } k \text{ is an integer and } 1 \leq k \leq \frac{1}{2\pi}|Z^*(t)|_t\right\}
\]

**Proof.** Every unit speed geodesic of \(N\) is translated by some element \(\varphi(t)\) of \(N\) (see \([K]\) and \([E]1\) proves the proposition. \(\square\)

**Definition 3.6.** Let \(M\) be a compact Riemannian manifold. For each nontrivial free homotopy class \(C\) of closed curves in \(M\) we define \(l(C)\) to be the collection of all lengths of smoothly closed geodesics that belong to \(C\).

**Definition 3.7.** The length spectrum of a compact Riemannian manifold \(M\) is the collection of all ordered pairs \((l, m)\), where \(l\) is the length of a closed geodesic in \(M\) and \(m\) is the multiplicity of \(l\), i.e. \(m\) is the number of free homotopy classes \(C\) of closed curves in \(M\) that contain a closed geodesic of length \(l\).

**Lemma 3.8.** Let \(g(t)\) be find the solution of the Ricci-Bourguignon flow in \((L, g)\) and \((L, g)\). Then \((\Gamma \setminus H_n, g(t))\) and \((\Gamma \setminus H_n, g_0)\) have the same length spectrum, also \((\Gamma \setminus Q_n, g(t))\) and \((\Gamma \setminus Q_n, g_0)\) have the same length spectrum.

**Proof.** Let \((N, g(t))\) be \((H_n, g(t))\) or \((Q_n, g(t))\). If \(\varphi(t)\) belongs to a discrete group \(\Gamma \subseteq N\), then the periods of \(\varphi(t)\) are precisely the lengths of the closed geodesic in \(\Gamma \setminus N\) that belong to the free homotopy class of closed curves in \(\Gamma \setminus N\) determined by \(\varphi(t)\). Therefore a free homotopy class of closed curves in \(\Gamma \setminus N\) corresponds to a conjugate class of an element \(\varphi\) in \(\Gamma\) and the collection \(l(C)\) is then precisely the
set of periods of $\varphi$. For any nonidentity elements $\varphi(t) = \exp(V^*(t) + Z^*(t)) \in N$ that does not lie in the center of $N$, by Lemma 3.2 in [1] it has a unique period $\omega(t) = |V^*(t)|_t$. Therefore in Heisenberg Lie group $(H_n, g(t))$, if we suppose that $V^*(t) = \sum_{i=1}^n a_i e_i + b_i e_{n+i}$ for some $a_i, b_i \in \mathbb{R}$, then we obtain

$$|V^*(t)|_t^2 = g_1(t) \sum_{i=1}^n (a_i^2 + b_i^2) = (1 + bt)^{\frac{1-n\rho}{n+2-n\rho}} |V^*(t)|_0^2,$$

where $b = (n + 2 - n\rho) \frac{a_n(0)}{g_t(0) g_{n+1}(0)}$ and in quaternion Lie group we suppose that $V^*(t) = \sum_{i=1}^n a_i X_{1i} + b_i X_{2i} + c_i X_{3i} + d_i X_{4i}$ for some $a_i, b_i, c_i, d_i \in \mathbb{R}$, then

$$|V^*(t)|_t^2 = \sum_{i=1}^n a_i^2 |X_{1i}|_t^2 + b_i^2 |X_{2i}|_t^2 + c_i^2 |X_{3i}|_t^2 + d_i^2 |X_{4i}|_t^2 = (1 + ct)^{\frac{3(1-2n\rho)}{n+2-n\rho}} |V^*(t)|_0^2,$$

where $c = (6 + 2n - 6n\rho) \frac{a_n(0)}{g_t(0)}$. Let

$$W^*(t) = (1 + ct)^{\frac{3(1-2n\rho)}{n+2-n\rho}} V^*(t)$$

and $\psi(t) = \exp(W^*(t) + Z^*(t))$ then $|W^*(t)|_t = |V^*(t)|_0$ in $(Q_n, g(t))$. Similarly, if $W^*(t) = (1 + bt)^{\frac{n\rho + 1}{n\rho}} V^*(t)$ then in $(H_n, g(t))$ we have $|W^*(t)|_t = |V^*(t)|_0$. Hence the period of $\psi(t)$ is $\omega(t)$. Also, since for arbitrary nonidentity elements $\varphi(t) = \exp(V^*(t) + Z^*(t)) \in N$ which are in the center of $N$, we have the following periods.

$$\left\{ |Z^*(t)|_t, \sqrt{(4\pi k)(|Z^*(t)|_t - \pi k)} \right\};$$

where $k$ is an integer and $\{1 \leq k \leq \frac{1}{2\pi} |Z^*(t)|_t \}$. Therefore in Heisenberg Lie group $(H_n, g(t))$ we see that $Z^*(t) = ae_{2n+1}$ for some $a \in \mathbb{R}$. We obtain

$$|Z^*(t)|_t^2 = a^2 |e_{2n+1}|_t^2 = (1 + bt)^{\frac{n\rho + 1}{n\rho}} |Z^*(t)|_0^2$$

and in quaternion Lie group we suppose that $Z^*(t) = \sum_{i=1}^3 a_i Z_{4n+i}$ for some $a_i \in \mathbb{R}$, then

$$|Z^*(t)|_t^2 = \sum_{i=1}^3 a_i^2 |Z_{4n+i}|_t^2 = (1 + ct)^{\frac{n\rho + 3}{n\rho}} |Z^*(t)|_0^2.$$
Definition 3.10. Two Riemannian manifolds \((M_1, g_1)\) and \((M_2, g_2)\) are said to have \(C^k\)-conjugate geodesic flows if there is a \(C^k\) diffeomorphism \(F : S(M_1, g_1) \to S(M_2, g_2)\) between their unit tangent bundles that intertwines their geodesic flows i.e., \(F \circ G^s_{M_1} = G^s_{M_2} \circ F\) where \(G^s_{M_1}\) and \(G^s_{M_2}\) are geodesic flow of \(M_1\) and \(M_2\) respectively.

Definition 3.11. A compact Riemannian manifold \(M\) is said to be \(C^k\)-geodesically rigid within a given class \(\mathcal{M}\) of Riemannian manifolds if any Riemannian manifold \(M_1\) in \(\mathcal{M}\) whose geodesic flow is \(C^k\)-conjugate to that of \(M\) is isometric to \(M\).

Definition 3.12. The solution \(g(t)\) of the Ricci-Bourguignon flow with the initial condition \(g(0) = g_0\) is called a Ricci-Bourguignon soliton if there exist a smooth function \(u(t)\) and a 1-parameter family of diffeomorphisms \(\psi_t\) of \(M^n\) such that \(g(t) = u(t)\psi^*_t(g_0),\) \(u(0) = 1, \psi_0 = id_{M^n}\).

By similar to the proof of Theorems 3.1 and 3.2 in [1], we have the following lemma.

Lemma 3.13. a) The spectrum and marked length spectrum on a compact nilmanifold is preserved under the Ricci-Bourguignon soliton. 
b) The geodesically rigidity on compact nilmanifold of Heisenberg type is invariant under the Ricci-Bourguignon soliton.

References

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