SU(1, 1) group action and the corresponding Bloch equation

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Abstract
Discrete time dynamics on the SU(1, 1) group is studied. It is shown that a map acting in the dual space is the stroboscopic map for a SU(1, 1) Bloch equation. Exact solution of the map is used to elucidate the corresponding dynamics. It is shown that dynamics of the SU(1, 1) Bloch equation in the elliptic case bears close analogy to the SU(2) Bloch dynamics.

1 Introduction
Discrete-time dynamical systems can be formulated in terms of group actions to exploit the group structure and get a better understanding of the corresponding dynamics. This approach was used to study discrete-time dynamics on some groups, see [1] and references therein. On the other hand, structure of Kleinian groups is naturally studied in the setting of discrete-time dynamical systems, revealing in this way connections with fractals [2, 3, 4]. For example, the Shimizu-Leutbecher map is a typical tool to study group structure [3, 4], see also [2, 3, 7].

Recently, we have investigated a possibility of relating group actions with stroboscopic maps of ordinary differential equations [1]. More exactly, we have studied the following dynamical system on a Lie group $G$:

$$R_{N+1} = Q_N R_N R_{N-1} Q_{N-1}^{-1} R_{N-1}^{-1} Q_N^{-1}, \quad N = 1, 2, \ldots, \quad (1)$$

where $Q_N, R_N \in G$. A general solution of the map [1] has been constructed and it was demonstrated that for $G = SU(2)$ and $Q_N \equiv Q$ Eq. [1] is a stroboscopic map of the Bloch equation [1]. The latter result is generalized in the present paper for the case $G = SU(1, 1)$.

Let us note here that the SU(1, 1) Bloch equation finds important applications in quantum optics. More exactly, the group of squeezings is generated by the Lie algebra $su(1, 1)$, the geometry of group manifold is that of
Minkowski spaces and time evolution is described by the \( SU(1,1) \) Bloch equation \[8, 9, 10, 11\].

The paper is organized as follows. In the next Section discrete time dynamics, defined and solved in \[1\] for arbitrary group \( G \), is studied in the case \( G = SU(1,1) \). It is shown in Section 3 that the map (1), considered in the dual space, is the stroboscopic map for the \( SU(1,1) \) Bloch equation which is written in the elliptic, parabolic and hyperbolic cases. In Section 4 computations intended to elucidate dynamics of the \( SU(1,1) \) Bloch equation are presented for the elliptic case and analogy with the \( SU(2) \) Bloch equation is stressed. The obtained results are summarized in the last Section.

2 Discrete-time dynamics on the \( SU(1,1) \) group

Let us recall that the map (1) admits an exact solution:

\[
R_{2K} = S_{2K}S_{2K-1} \ldots S_2R_0S_2^{-1} \ldots S_{2K-1}S_{2K}^{-1},
\]

and similar equations can be written for \( R_{2K+1} \) [11].

We shall consider a special case \( Q_N \equiv Q \) in (1). In the case \( G = SU(1,1) \) the following parameterization is used [12]:

\[
R_N = \exp \left( i\frac{1}{2} \vec{\kappa} \cdot \vec{r}_N \right), \quad \vec{r}_N \cdot \vec{r}_N = \eta,
\]

\[
Q = \exp \left( i\frac{1}{2} \vec{\kappa} \cdot \vec{q} \right), \quad \vec{q} \cdot \vec{q} = \eta,
\]

where \( i^2 = -1 \), \( \eta \in \{+1, 0, -1\} \) and is fixed, \( \vec{\kappa} \equiv [i\sigma^1, i\sigma^2, \sigma^3] \) where \( \sigma^1, \sigma^2, \sigma^3 \) are the Pauli matrices and scalar products are defined as

\[
\vec{r} \cdot \vec{x} \equiv -r_1x_1 - r_2x_2 + r_3x_3, \quad \vec{q} \cdot \vec{x} \equiv -q_1x_1 - q_2x_2 + q_3x_3,
\]

The three cases \( \eta = +1, 0, -1 \) are referred to as elliptic, parabolic and hyperbolic, respectively. For an exposition of theory of the \( SU(1,1) \) group the reader can consult [12] [13].

We obtain from (2) the following solution:

\[
R_{2K} = Q^{2K} P^K R_0 P^{-K} Q^{-2K},
\]

where \( P \equiv Q^{-1} S_1 Q^{-1} = Q^{-1} R_1 Q R_0 \), \( K = 0, 1, 2, \ldots \). We still have to impose initial condition \( R_0 \) while \( R_1 \) is computed as \( R_1 = Q P R_0^{-1} Q^{-1} \).

Matrix \( P \) is parameterized in form

\[
P = \exp \left( i\frac{a}{2} \vec{\rho} \cdot \vec{p} \right), \quad \vec{p} \cdot \vec{p} = \eta,
\]
and equation (7) can be written as
\[ \vec{r} \cdot \vec{r}_{2K} = \exp \left( iK\alpha \vec{r} \cdot q \right) \exp \left( iK \frac{\beta}{2} \vec{r} \cdot \vec{p} \right) \vec{r}_0 \exp \left( -iK\alpha \vec{r} \cdot q \right), \]
and, after introducing new quantities, \( K\alpha = \theta, \lambda = \beta \alpha \), reads
\[ \vec{r} \left( \theta \right) = \exp \left( i\theta \vec{r} \cdot q \right) \exp \left( i\lambda \theta \vec{r} \cdot \vec{p} \right) \vec{r}_0 \exp \left( -i\lambda \theta \vec{r} \cdot q \right) \exp \left( -i\theta \vec{r} \cdot q \right), \]
where \( \vec{r} \left( \theta \right) \equiv \vec{r}_{2K} \).

### 2.1 Elliptic case

To obtain \( \vec{r} \left( \theta \right) \) from Eq. (10) we shall need to compute \( \vec{t} \left( \gamma \right) \) given by:
\[ \vec{r} \cdot \vec{t} \left( \gamma \right) = S \vec{r} \cdot \vec{t} \cdot S^{-1} = \exp \left( i\frac{\gamma}{2} \vec{r} \cdot \vec{s} \right) \vec{r} \cdot \vec{t} \cdot \exp \left( -i\frac{\gamma}{2} \vec{r} \cdot \vec{s} \right), \]
in the case \( \eta = 1 \). The exponential map is simplified as
\[ \exp \left( i\frac{\gamma}{2} \vec{r} \cdot \vec{s} \right) = \cos \left( \frac{\gamma}{2} \right) \vec{r} + i \sin \left( \frac{\gamma}{2} \right) \vec{r} \cdot \vec{s}, \quad \vec{s} \cdot \vec{s} = 1. \]

Now, due to properties of the Pauli matrices we obtain from (11):
\[ \vec{t} \left( \gamma \right) = \cos \left( \gamma \right) \vec{t} - \sin \left( \gamma \right) \vec{t} \times \vec{s} + \left( 1 - \cos \left( \gamma \right) \right) \left( \vec{t} \cdot \vec{s} \right) \vec{s}, \]
where
\[ \vec{x} \times \vec{y} \stackrel{df}{=} \left[ -x_2y_3 + x_3y_2, -x_3y_1 + x_1y_3, x_1y_2 - x_2y_1 \right]. \]

Using twice the equation (13) in (10) we get:
\[ \vec{r} \left( \theta \right) = \cos \left( 2\theta \right) \vec{r} \left( \theta \right) - \sin \left( 2\theta \right) \vec{t} \left( \theta \right) \times \vec{q} + \left( 1 - \cos \left( 2\theta \right) \right) \left( \vec{q} \cdot \vec{t} \left( \theta \right) \right) \vec{q}, \]
\[ \vec{t} \left( \theta \right) = \cos \left( 2\lambda \theta \right) \vec{t}_0 - \sin \left( 2\lambda \theta \right) \vec{r}_0 \times \vec{p} + \left( 1 - \cos \left( 2\lambda \theta \right) \right) \left( \vec{p} \cdot \vec{r}_0 \right) \vec{p}. \]

For growing \( \theta \) the vector \( \vec{r} \left( \theta \right) \) evolves on manifold \( \vec{r} \left( \theta \right) \cdot \vec{r} \left( \theta \right) = 1 \), i.e. on two-sheeted hyperboloid.

### 2.2 Parabolic case

In the case \( \eta = 0 \) the exponential map reduces to
\[ \exp \left( i\frac{\gamma}{2} \vec{r} \cdot \vec{s} \right) = 1 + i\frac{\gamma}{2} \vec{r} \cdot \vec{s}, \quad \vec{s} \cdot \vec{s} = 0. \]

Now, due to properties of the Pauli matrices we compute from (11):
\[ \vec{t} \left( \gamma \right) = \vec{t} - \gamma \vec{t} \times \vec{s} + \left( \vec{t} \cdot \vec{s} \right) \vec{s}. \]
Using twice the equation (17) in (10) we obtain:

\[
\vec{r}' (\theta) = \vec{t} (\theta) - 2\theta \vec{t}' (\theta) \times \vec{q} + \left( \vec{q} \cdot \vec{t}' (\theta) \right) \vec{q}, \\
\vec{t} (\theta) = \vec{r}_0 - 2\lambda \theta \vec{r}_0 \times \vec{p} + (\vec{p} \cdot \vec{r}_0) \vec{p}.
\]  

(18a)  

(18b)

The vector \( \vec{r}' (\theta) \) evolves on manifold \( \vec{r}' (\theta) \cdot \vec{r}' (\theta) = 0 \), i.e. on the Minkowski cone.

### 2.3 Hyperbolic case

In the case \( \eta = -1 \) we have

\[
\exp \left( i \frac{\gamma}{2} \vec{s} \cdot \vec{s} \right) = \cosh \left( \frac{\gamma}{2} \right) \mathbf{1} + i \sinh \left( \frac{\gamma}{2} \right) \vec{s} \cdot \vec{s}, \quad \vec{s} \cdot \vec{s} = -1.
\]

(19)

Due to properties of the Pauli matrices we obtain from (11):

\[
\vec{t} (\gamma) = \cosh (\gamma) \vec{t} - \sinh (\gamma) \vec{t} \times \vec{s} + (\cosh (\beta) - 1) \left( \vec{t} \cdot \vec{s} \right) \vec{s}.
\]

(20)

Using twice the equation (20) in (10) we get:

\[
\vec{r}' (\theta) = \cosh (2\theta) \vec{t} (\theta) - \sinh (2\theta) \vec{t} \times \vec{q} + (\cosh (2\theta) - 1) \left( \vec{q} \cdot \vec{t} (\theta) \right) \vec{q}, \\
\vec{t} (\theta) = \cosh (2\lambda \theta) \vec{r}_0 - \sinh (2\lambda \theta) \vec{r}_0 \times \vec{p} + (\cosh (2\lambda \theta) - 1) \left( \vec{p} \cdot \vec{r}_0 \right) \vec{p}.
\]

(21a)  

(21b)

The vector \( \vec{r}' (\theta) \) evolves on manifold \( \vec{r}' (\theta) \cdot \vec{r}' (\theta) = -1 \), i.e. on one-sheeted hyperboloid.

### 2.4 Symmetry and restrictions of dynamics

Dynamical system (1) for \( Q_N \equiv Q \) has continuous symmetry:

\[
R_N \rightarrow Q^\kappa R_N Q^{-\kappa}, \quad \forall \kappa \in \mathbb{R}.
\]

(22)

It can be thus expected that dynamics of the quantity \( \vec{r}' (\theta) \cdot \vec{q} \) should decouple from other degrees of freedom in (11). Indeed, it follows from (15) that

\[
\vec{r}' (\theta) \cdot \vec{q} = \vec{t}' (\theta) \cdot \vec{q}.
\]

(23)

Since in the elliptic or hyperbolic case \( \vec{t} \cdot \vec{t} = \vec{q} \cdot \vec{q} = \pm 1 \) it follows from the Schwartz inequality for the Minkowski metric that \( \left( \vec{t} \cdot \vec{q} \right)^2 \geq \left( \vec{t} \cdot \vec{t} \right) (\vec{q} \cdot \vec{q}) = 1 \). Now, for given \( \vec{p}, \vec{q} \) and \( \vec{r}_0 \) on the upper sheet of the hyperboloid in the elliptic case we have

\[
1 \leq A_1 \leq \vec{t} (\theta) \cdot \vec{q} \leq A_2,
\]

(24)
The constants $A_1, A_2$ depending on the parameters $\mathbf{p}, \mathbf{q}$ and the initial condition $\mathbf{r}_0$ can be computed from (15b) by elementary means

$$A_1, A_2 = c \mp \sqrt{b^2 + (a-c)^2}, \quad (25)$$

where

$$a = \mathbf{r}_0 \cdot \mathbf{q}, \quad b = (\mathbf{r}_0 \times \mathbf{p}) \cdot \mathbf{q}, \quad c = (\mathbf{p} \cdot \mathbf{r}_0)(\mathbf{p} \cdot \mathbf{q}). \quad (26)$$

It thus follows that the motion on the hyperboloid is bounded by two planes:

$$A_1 \leq \mathbf{r}(\theta) \cdot \mathbf{q} \leq A_2. \quad (27)$$

i.e. equation of a straight line.

3 The Bloch equation

The map $Q_N \equiv Q$, is the stroboscopic map of a differential equation which is conveniently derived from the form (10). Since $\alpha$ and $\beta$ are arbitrary we shall treat $\theta$ as a continuous variable. Differentiating Eq.(10) with respect to $\theta$ and using (28) we get

$$\frac{d}{d\theta} \mathbf{K} \cdot \mathbf{r}(\theta) = i [\mathbf{K} \cdot \mathbf{u}(\theta), \mathbf{K} \cdot \mathbf{r}(\theta)], \quad (28)$$

where $[A, B] \equiv AB - BA$ and

$$\mathbf{u}(\theta) = \mathbf{q} + \lambda \mathbf{p}(\theta), \quad (\lambda = \beta/2\alpha) \quad (29a)$$

$$\mathbf{K} \cdot \mathbf{p}(\theta) = \exp(i\theta \mathbf{K} \cdot \mathbf{q}) \mathbf{K} \cdot \mathbf{p} \exp(-i\theta \mathbf{K} \cdot \mathbf{q}). \quad (29b)$$

It follows that the sequence $\mathbf{r}_0, \mathbf{r}_2, \mathbf{r}_4, \ldots$, generated by $R_0, R_2, R_4, \ldots$, cf. Eq.(7), interpolates flow of Eq.(28). It turns out that, $\theta$ interpreted as time, Eq.(28) is the $SU(1, 1)$ Bloch equation, cf.[10].

Equations (28), (29) can be written in explicit form.

3.1 Elliptic case

Using Eqs. (11), (13) we get from (29b)

$$\mathbf{p}(\theta) = \cos(\gamma) \mathbf{p} - \sin(\gamma) \mathbf{p} \times \mathbf{q} + (1 - \cos(\gamma)) (\mathbf{p} \cdot \mathbf{q}) \mathbf{q}, \quad (30)$$

and using properties of the Pauli matrices we obtain from (28) the Bloch equation in the elliptic case:

$$\frac{d}{d\theta} \mathbf{r} = -2 \mathbf{r}(\theta) \times \mathbf{u}(\theta), \quad (31)$$

with $\mathbf{u}(\theta)$ and $\mathbf{p}(\theta)$ given by (29a) and (30), respectively.
3.2 Parabolic case

Applying Eqs. (11), (17) to (29b)

\[ \vec{p}(\theta) = \vec{p} - \gamma \vec{p} \times \vec{q} + (\vec{p} \cdot \vec{q}) \vec{q}, \quad (32) \]

and using properties of the Pauli matrices we obtain from (28) the Bloch equation in the parabolic case:

\[ \frac{d \vec{r}}{d\theta} = -2\vec{r}(\theta) \times \vec{u}(\theta), \quad (33) \]

with \( \vec{u}(\theta) \) and \( \vec{p}(\theta) \) given by (29a) and (32), respectively.

3.3 Hyperbolic case

Using Eqs. (11), (20) we obtain from (29b)

\[ \vec{p}(\theta) = \cosh(\gamma) \vec{p} - \sinh(\gamma) \vec{p} \times \vec{q} + (\cosh(\gamma) - 1) (\vec{p} \cdot \vec{q}) \vec{q}, \quad (34) \]

and using properties of the Pauli matrices we obtain from (28) the Bloch equation in the hyperbolic case:

\[ \frac{d \vec{r}}{d\theta} = -2\vec{r}(\theta) \times \vec{u}(\theta), \quad (35) \]

with \( \vec{u}(\theta) \) and \( \vec{p}(\theta) \) given by (29a) and (34), respectively.

4 Computational results

We have performed several computations for the Bloch equation (31) and the discrete-time dynamical system (1), parameterized as in Eqs. (4), (5), \( Q_N \equiv Q, \eta = 1 \). Exact solutions of the map (1) as well as of the Bloch equation (31) in the elliptic case are given by (7) and (15), respectively.

The solution (15), of discrete-time dynamical system (1) with \( Q, P \) given by (5), (8) has been plotted in Fig. 1 for \( \vec{q} = [0, 0, 1], \vec{p} = [1, 0, \sqrt{2}], \lambda = \beta/(2 \alpha) = 3 \) and the initial vector \( \vec{r}_0 = \left[ \frac{1}{2}, \frac{1}{2}, \frac{\sqrt{3}}{2} \right] \) on the upper sheet of the hyperboloid. The whole trajectory has three-fold symmetry with respect to the \( \vec{q} \) axis. Circles indicate parallels \( A_{1,2} \) given by (25) confining the dynamics.

In Fig. 2 dynamics of vectors \( \vec{r}_N \) obtained from (9) has been plotted for \( \alpha = 5, \beta = 20 (\lambda = \beta/(2 \alpha) = 2) \) and other parameters unchanged. We thus obtain thirty six points marked with dots. The solution (15) has been also plotted. The closed curve has two-fold symmetry with respect to the \( \vec{q} \) axis.
Fig. 1. Exact solution of the Bloch equation (31), $\lambda = 3$.

Fig. 2. Exact solution of the Bloch equation (31) (thin line) and discrete-time dynamical system (1) (dots), $\alpha = 5$, $\beta = 20$, $\lambda = 2$.

In Fig. 3 initial stage of dynamics of vectors $\overrightarrow{r}_N$ has been plotted for $\lambda = \beta / (2\alpha) = 1.025$, dot marking the initial vector $\overrightarrow{r}_N$. 

7
5 Summary and discussion

We have introduced in [1] a class of discrete-time invertible maps (1) on an arbitrary group $G$ for which the exact solution (2) of this map has been found. Maps of form (1), parameterized on a Lie group, generate points in the dual (parameter) space which sample a trajectory in this space arbitrarily densely. This curve can be generated forward as well as backward from a given initial condition. This suggests that the group action (1) may correspond to a flow of a differential equation. We have demonstrated in the present paper that for $G = SU(1,1)$ the map (1), $Q_N \equiv Q$, considered as dynamical system in the dual space, is a stroboscopic map of a $SU(1,1)$ Bloch equation. It should be noted that the $SU(1,1)$ Bloch equation (28) is formally analogous to the $SU(2)$ Bloch equation, cf. Eq. (4.21) in [1].

Exact solutions of the map constructed in the present paper, (15), (18), (21), lead to a better understanding of the corresponding Bloch equations (31), (33), (35). More exactly, symmetries and restrictions of dynamics have been found explicitly. It is interesting that dynamics of the $SU(1,1)$ Bloch equation in the elliptic case bears close analogy to dynamics of the $SU(2)$ Bloch equation, compare Figs. 1, 2, 3 from the present paper with analogous figures in [1].
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