INTRODUCTION

1.1. Finite groups. Yang–Mills theory [1] is the cornerstone of the standard model of particle physics, and its immense success poses an enormous challenge to any new theories that attempt to go beyond the standard model in any way. In a recent paper [2] dedicated to Yang, ’t Hooft suggests that on the Planck scale reality should be discrete, and that a discrete (and therefore finite) version of $U(1) \times SU(2) \times SU(3)$ will be required in order to describe the fundamental theory. He then says he has no idea which finite group to use. On the other hand, these finite groups have been well-known to mathematicians for more than a hundred years [3], and over a period of time I have examined all of them closely in the hope of being able to answer ’t Hooft’s question. A number of possibilities are discussed in [4, 5, 6, 7] but only the last of these cases survives this detailed scrutiny.

Once it is clear which group we have to use, it is much easier to build the appropriate model, with much less speculation, and much more confidence that this is the one that must work, if any of them does. The purpose of this paper, therefore, is to go back to the beginning, and explain from first principles what the group is, where it comes from, and what the implications are.

1.2. Clifford algebras. The concept of spin is central to much of algebra, geometry and quantum theory, and is constructed mathematically using Clifford algebras [8]. There has always been a dilemma as to whether the signature of spacetime is $(3, 1)$ or $(1, 3)$. Naively, it seems it should not matter, but unfortunately it does, because the two Clifford algebras are quite different. This is even true in the non-relativistic case, where the signatures $(3, 0)$ and $(0, 3)$ also give non-isomorphic Clifford algebras:

$$Cl(3, 0) \cong M_2(C), \quad Cl(0, 3) \cong H + H,$$
$$Cl(3, 1) \cong M_4(R), \quad Cl(1, 3) \cong M_2(H).$$

The standard approach to this problem is to ignore it, and work instead with the complex matrix algebra $M_4(C)$, known as the Dirac algebra. Since $M_4(C)$ contains both $M_4(R)$ and $M_2(H)$, it then becomes unnecessary to make a decision.
This procedure, however, only evades the problem, and does not solve it. There is a real question here of which of these Clifford algebras describes the transformation properties of spins of elementary particles in the real world. The answer depends on properties of neutrinos, such as whether they are Dirac or Majorana particles, so that it is not easy to give a definitive answer on experimental grounds. Nevertheless, the principle of unitarity would seem to imply that we require $M_2(H)$ to act on the Dirac spinors. Of course, this structure only allows us to implement half of the Dirac algebra, so it may not necessarily contain enough information to cover everything that is done with the Dirac algebra in the standard model. In particular, we have to choose between $\gamma_0$ and $\gamma_5$, because we cannot have both at the same time. Whichever we choose, its eigenvalues are 1 and $-1$, which means that it acts as a quaternionic reflection on the space $\mathbb{H}^2$ of Dirac spinors. It may therefore be useful to study the two-dimensional quaternionic reflection groups.

2. Finite group theory and fundamental particles

2.1. Quaternionic reflection groups. The Dirac matrices themselves generate a quaternionic reflection group, of order 32, in which there are ten reflections

$$\pm \gamma_0, \pm \gamma_0 \gamma_1, \pm \gamma_0 \gamma_2, \pm \gamma_0 \gamma_3, \pm \gamma_1 \gamma_2 \gamma_3.$$  

However, this group does not show any asymmetry between the two quaternion coordinates, so does not capture enough of the known structure of physical spin for our purposes. A complete classification of finite quaternionic reflection groups was obtained by Cohen [9]. The interesting cases are listed in [9, Table III], where the 2-dimensional examples are denoted $O_1$, $O_2$, $O_3$, $P_1$, $P_2$ and $P_3$.

Those of type $P$ all contain the above group of order 32 generated by the Dirac matrices $\gamma_\mu$, extended by a dihedral, alternating or symmetric group on 5 points. They were described in detail in [10], and the connection to Dirac matrices is pointed out in [7], but they also have a symmetry between left-handed and right-handed spinors. Those of type $O_2$ and $O_3$ were mentioned very speculatively in [4] as possibly having a connection to quantum theory, but again there is a left-right symmetry. Thus the only one of the six that exhibits an asymmetry between left-handed and right-handed spinors is $O_1$. It therefore seems likely that this is the one that will be most useful for applications in quantum theory.

As an abstract group, $O_1$ is isomorphic to the binary icosahedral group, of order 120. Possible uses of this group in quantum physics have been discussed at length in [7], but the quaternionic reflection group property renders much of that speculation redundant, and provides a much more direct route from the mathematics to the applications. In particular, there is a distinguished set of 120 Dirac spinors, namely the ‘roots’ which define the reflections, as well as a distinguished set of 120 matrices in the Dirac algebra. This enables us to define sets of elementary fermions and bosons, and to investigate the relationships between them in detail. Of course, it is not obvious at this stage that the elementary particles defined by this mathematical model actually match up to the elementary particles in the real world.

In order to define the required objects, we first need two quaternions $\omega$ and $\phi$ that satisfy the relations

$$\phi^2 = (\omega \phi)^2 = \omega + \omega^2 = -1.$$  

Then the group of scalars generated by $\omega$ and $\phi$ has order 12, and consists of all elements $\omega^a \phi^b$, where we can suppose that $a = 0, 1, 2$ and $b = 0, 1, 2, 3$. The 120
roots can then be taken as these 12 scalar multiples of the following 10 spinors of
(squared) norm 3, where again we can assume $c = 0, 1, 2$:  
(4)  
$$ (\omega - \omega^2, 0), \quad (\phi + 1)\omega^c, 1), \quad (\omega^c, (\phi - 1)\omega) \quad (\omega^c, (\phi - 1)\omega^2). $$

Each root $r$ then defines a reflection $R_r$ via the formula  
(5)  
$$ R_r : x \mapsto x - \frac{x \cdot r}{r \cdot r} (1 - \omega) r. $$

Notice that this formula does not change when we replace $r$ by $\pm \omega^a r$, but if we
replace $r$ by $\pm \phi r$, then the factor $(1 - \omega)$ becomes $-\phi(1 - \omega) = (1 - \omega^2)$. Hence
we obtain 20 reflections of order 3, coming in 10 inverse pairs. The remaining 100
elements of the group can all be written as products of two reflections.

In matrix notation, we can take generators

\[
\begin{align*}
  f & := \frac{\omega - \omega^2}{3} \begin{pmatrix} 1 & \omega^2 \phi - \omega \\ \omega^2 \phi - \omega & \omega \phi \end{pmatrix}, \\
  g & := \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix}, \quad h := \begin{pmatrix} \phi & 0 \\ 0 & \phi \end{pmatrix}
\end{align*}
\]

such that the subgroup of diagonal matrices is a maximal subgroup, of order 12,
generated by $g$ and $h$. These matrices satisfy the relations

\[ f^2 = (gh)^2 = h^2 = -1, \]

\[ g^3 = (fg)^3 = (fh)^3 = 1. \]

2.2. **Classification of particles.** Of particular interest for potential applications
is the fact that $g$ acts only on the first quaternion coordinate, which might therefore
correspond in some way to the ‘left-handed’ spinor in the standard model. Since
there is no corresponding symmetry acting on the right-handed spinor, this is a
natural way in which the mathematics could model the ‘chirality’ of the weak
interaction. In this way we can distinguish the (Dirac) neutrinos as the 12 multiples
of the left-handed spinor $(\omega - \omega^2, 0)$. This includes a factor of 3 for the three
generations, and a factor of 4 which is sufficient to distinguish both neutrinos from
antineutrinos, and spin up from spin down.

The other nine types of particles also come in 12 states each, and can presumably
be identified with the three generations of electrons, and six flavours of quarks, in
some way. This gives us 72 quark states altogether, which is the same number
as in the standard model, if we distinguish six flavours, three colours and three
anti-colours, and spin up/down. Each can also be split into left-handed and right-
handed parts, but these are not particle states in the same sense. Finally we have
36 electron states, which is three times as many as in the standard model. Hence
this model can distinguish six spin states for each electron, compared to the simple
pair of spin up/down in the standard model.

In other words, there is a ‘hidden variable’ that takes one of three values, and
that is not in the standard models of quantum mechanics and particle physics.
This does not contradict Bell’s Theorem [11, 12], which only forbids continuous
local hidden variables. Two entangled electrons can therefore share this hidden
quantum number, and use it to determine the binary spin state as measured in any
particular experiment. This is extra information that is available to the electrons,
but is unknown to quantum mechanics, and may be sufficient to obviate the need for
any hypothetical superluminal transfer of information between entangled particles.
Presumably this hidden quantum number can also be carried by photons, in order to explain the properties of entangled photon polarisations, but for that we need a full classification of the bosonic fields. For bosons, then, we need to map from $\omega$ and $\phi$ in $SU(2)$ into the orthogonal group $SO(3)$, where we obtain a group of order 6, that is the rotation symmetry group of a triangular antiprism. In other words, a photon in this model has six intrinsic polarisation states rather than just two, again acquiring just enough additional information to explain the observed properties of entangled photons without superluminal transfer of information.

2.3. **Quantum fields.** Each of the fundamental fermions generates a reflection, that can be written as a $2 \times 2$ quaternion matrix, embedded in the Dirac algebra, and therefore has an interpretation as a quantum field \[13\] of some kind. There are 20 such matrices, and together they generate a 16-dimensional algebra. The neutrinos, for example, generate a 3-dimensional algebra with basis \(1, g, g^2\). This is a commutative algebra, isomorphic to \(\mathbb{R} \oplus \mathbb{C}\) with the following generators:

\[
\begin{align*}
(1 + g + g^2)/3 &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \\
(2 - g - g^2)/3 &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \\
g - g^2 &= \begin{pmatrix} \theta & 0 \\ 0 & -\theta \end{pmatrix}
\end{align*}
\]

where $\theta = \omega - \omega^2$ is pure imaginary, with $\theta^2 = -3$. We can also adjoin $h$ to this algebra, in order to incorporate the spin of the neutrinos, and hence obtain a 6-dimensional algebra isomorphic to $\mathbb{C} \oplus \mathbb{H}$.

To see this isomorphism, adjoin the generators

\[
\begin{align*}
 h(1 + g + g^2)/3 &= \begin{pmatrix} 0 & 0 \\ 0 & \phi \end{pmatrix}, \\
h(2 - g - g^2)/3 &= \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix}, \\
h(g - g^2) &= \begin{pmatrix} \phi \theta & 0 \\ 0 & 0 \end{pmatrix}.
\end{align*}
\]

The corresponding Lie algebra \[13\] is

\[
gl(1, \mathbb{C}) + gl(1, \mathbb{H}) = gl(1, \mathbb{R}) + u(1) + gl(1, \mathbb{R}) + su(2),
\]

which we would like to interpret in relation to the gauge group $SU(2)_L \times U(1)_Y$. If this is a viable interpretation, then $u(1)_Y$ must be generated by $h(1 + g + g^2)/3$ acting on the right-handed spinor only, while $su(2)_L$ is generated by

\[
h(2 - g - g^2)/3, \quad g - g^2, \quad h(g - g^2),
\]

acting on the left-handed spinor only.

Certainly this would show us the symmetry-breaking of the electroweak gauge group $SU(2) \times U(1)$ quite clearly. It also shows that the standard formulation involves taking quantum superpositions of the discrete particles implied by the finite group. To put this another way, the discrete particles that are actually observed (the $Z$ and $W$ bosons and the photon) do not fit neatly into the Lie algebras of the standard model. Indeed, $u(1)_{em}$ is most naturally defined to be generated by $h$, so that $h$ itself can be identified as an abstract photon, divorced from spacetime. If so, then the symmetries imply that $hg$ and $gh$ are also photons, which gives us the extra photon states already conjectured as being required to explain the experimental properties of the measurement of photon polarisation. This leaves us with 1 to denote the $Z$ boson, and $g, g^2$ for the $W^+$ and $W^-$ bosons.
However, this interpretation of photons is not consistent with the standard model, which would require $h, hg, gh$ to be ‘primordial’ massless weak bosons, so that the photons must go somewhere else. It is reasonable to assume that the massless bosons correspond to the elements of order 4 in the group, of which there are 15 up to sign. Any choice of signs gives us a basis for the adjoint representation of $SL(2, \mathbb{H})$, and the symmetry group $\langle g, h \rangle$ should describe the division into electrodynamics and the weak and strong forces.

With respect to $\langle g, h \rangle$ there are three orbits of lengths 3 + 6 + 6 on the 15 pairs $\pm x$ of elements of order 4 in the full group:

$$h, gh, hg, f, f^g, f^g, f^h, f^g h, f^g f, f^g f^2 h, f^g f^2 h, f^g f^2 h^2, f^g f^2 h^2.$$  

(12)

The first orbit consists of elements inside the subgroup, while the second orbit consists of elements in the two copies of the quaternion group $Q_8$ that are normalised by $g$. The third orbit consists of three pairs, with the property that the product of the two elements in each pair lies in the original group. A possible interpretation is that the second orbit consists of photons, with conjugation by $h$ corresponding to a change of polarisation. This would force the third orbit to consist of coloured gluons, and the first orbit of colourless gluons, extended from two to three dimensions in order to avoid the need for quantum superposition.

2.4. Elements of odd order. Let us now look at some of the other elements of the group, that represent massive bosons. We have seen that $g$ and $g^2$ might represent the $W^+$ and $W^-$ bosons, which suggests that the other elements of order 3 might represent massive bosons related to the strong force and/or electromagnetism. Up to inversion, the elements of order 3 fall into orbits of lengths 1 + 3 + 6 under the symmetry group $\langle g, h \rangle$. Conjugation by $g$ should be a weak force process that acts on fermions to pair the neutrino with the electron, the proton with the neutron, and so on, so approximately preserves the mass, while appearing to mix up the charges. Presumably it has a similar effect on bosons. Conjugation by $h$ on the other hand preserves the mass exactly, but appears to negate the charge on the bosons. Similarly, inversion also appears to fix the mass and negate the charge.

This suggests the splitting 3 + 6 is related to a splitting of massive strong force mediators into pions and kaons. The former case then consists of the six elements

$$fh, (fh)^g, (fh)^g, hf, (hf)^g, (hf)^g,$$  

(13)

which gives two mutually inverse states for each of the three pions. This doubling up of states is required in order to avoid having to express the neutral pion as a quantum superposition of $\bar{u}u$ and $\bar{d}d$. The states are distinguished (in theory) by which of the two colourless gluons is used to glue the quark and anti-quark together. Since these gluons cannot be directly observed, this model gives the correct number of pions in total. Both operations of conjugation by $h$, and inversion, act to negate the charge. In the other case, therefore, we should expect to see no more than six experimentally distinguishable particles among the 12 group elements.

These group elements are

$$g^f, g^f g, g^f g^2, g^{f h}, g^{f h g}, g^{f h g^2 h}.$$  

(14)
and their inverses. The mathematics allows either two charged and four neutral particles here, or four charged and two neutral, but experiment certainly favours the former. Our assumptions allow at most three distinct mass values, with four particle states each. It seems more likely that there are only two distinct masses, giving two charged and four neutral kaons, as in the standard model, but without the need for quantum superposition. There may, of course, be other possible interpretations.

The remaining elements of the group have order 5 modulo sign, and form 6 cyclic groups, with generators

\[ gfh, fhg, ghfg, \; ghf, hfg, gfhg. \] (15)

In the case of an element of 5, however, we should not expect its mass to be related in any simple way to the mass of its square, so that it would be reasonable to expect four distinct charged pairs and four masses of neutral bosons. This means we have enough room for all the remaining charged pseudoscalar mesons, of types \( D^\pm, D_s^\pm, B^\pm \) and \( B_s^\pm \), since the top quark does not hadronise.

There does not seem to be enough room for all the neutral pseudoscalar mesons, however, and perhaps the various types of eta meson must be explained in a different way. Then we have three neutral types \( D^0, B^0 \) and \( B_s^0 \), which leaves room for \( Z^0 \) to take the fourth available mass value, so that it can have spin 1 and the identity element of the group then has spin 0 and can be allocated to the Higgs boson. Alternatively, perhaps \( h \) is not constrained to fix the mass of neutral particles in the way that it seems to be constrained to fix the mass of charged particles. In that case, the neutral mesons may be distributed in quite a different way from the way I have suggested.

To summarise, in this context of a potential unification of electromagnetism with both the weak and strong forces, as well as a potential unification of the leptons and quarks, the price of separating the algebra \( M_4(\mathbb{C}) \) into two separate parts \( M_2(\mathbb{H}) \) and \( M_4(\mathbb{R}) \), with significantly different physical interpretations, seems a price worth paying. This also gives us a useful opportunity to re-write the relationship between the quantum ‘quaternionic world’ of \( M_2(\mathbb{H}) \) and the relativistic ‘real world’ of \( M_4(\mathbb{R}) \). In order to investigate this possibility further, we need to study the representation theory \[15\] of the binary icosahedral group in more detail.

3. Representation theory and effective field theory

3.1. Irreducible representations. It is well-known \[16, 17\] that the binary icosahedral group is a subgroup of \( SU(2) \), and therefore of the real quaternions \[18\]. A complete list of its elements in quaternionic notation is given in \[10\], reprinted in \[19\], from which we can easily read off suitable generators as

\[ f \mapsto i, \; g \mapsto (-1 + i + j + k)/2, \; h \mapsto i - \sigma j - \tau k, \] (16)

where \( \tau = (1 + \sqrt{5})/2 \) and \( \sigma = (1 - \sqrt{5})/2 \). All the irreducible representations can then be labelled by the ‘spin’ of the corresponding representations of \( SU(2) \), but at this stage we do not know what interpretation to give to this copy of \( SU(2) \), so we do not know what type of spin or isospin this might be. Let us therefore call it hyperspin, and define the above representation to have hyperspin \( 1/2 \).
Then the representations with hyperspin up to $7/2$ include all the irreducibles, labelled here with their complex dimensions:

| Hyperspin Decomposition | Hyperspin Decomposition |
|-------------------------|-------------------------|
| 0                       | 1                       |
| 1                       | $3a$                    |
| 2                       | $5$                     |
| 3                       | $3b + 4a$               |
| $1/2$                   | $2a$                    |
| $3/2$                   | $4b$                    |
| $5/2$                   | $6$                     |
| $7/2$                   | $2b + 6$                |

(17)

The representation $4b$ is the quaternionic reflection representation described above, and $2b$ can be obtained from $2a$ by changing the sign of $\sqrt{5}$.

The other representations can be obtained from the tensor products:

- $2a \otimes 2a = 1 + 3a$
- $2a \otimes 2b = 4a$
- $2b \otimes 2b = 1 + 3b$
- $2b \otimes 3a = 6$
- $2b \otimes 4b = 3b + 5$

(18)

The integer hyperspin representations are all real, so generate real matrix algebras

$$R + M_3(R) + M_3(R) + M_4(R) + M_5(R),$$

(19)

while the half-integer hyperspin representations are all quaternionic, so generate quaternionic matrix algebras

$$H + H + M_2(H) + M_3(H).$$

(20)

In particular, there is an obvious copy of $M_4(R)$ with which to attempt to reproduce the other half of the Dirac algebra, acting on $4a$.

In order to do this, we will need to understand how the finite group acts on this matrix algebra. This action is described by the tensor product

$$4a \otimes 4a = 1 + 3a + 3b + 4a + 5.$$  

(21)

Notice that this is equivalent to the Dirac algebra in the form

$$4b \otimes 4b = 1 + 3a + 3b + 4a + 5.$$  

(22)

In the standard model, this is a complex tensor product, and therefore a complex representation, but here it is a quaternionic tensor product, and therefore a real representation.

This remarkable coincidence implies that at the quantum level, that is at the level of the finite symmetries of the elementary particles, there is an exact equivalence between the $4 \times 4$ real matrices and the $2 \times 2$ quaternion matrices. But this equivalence is only an equivalence of *representations*, not an equivalence of *algebras*. Hence we can add the matrices together with impunity, but we must be very careful about multiplying them together. That is, we can either multiply matrices in $M_2(H)$, as is done in the Feynman calculus in the standard model of particle physics, or multiply matrices in $M_4(R)$, as is done in general relativity, but we cannot do both at the same time. In particular, we will need to pay particular attention to the differences between multiplication in the subgroup $SO(3, 1)$ of $M_4(R)$ and multiplication in the subgroup $SL(2, \mathbb{C})$ of $M_2(H)$.
3.2. **Gauge groups.** Gauge groups are used everywhere in physics to describe coordinate transformations in the mathematical theory that have no effect on the physical phenomena. These include the Lorentz group $SO(3,1)$ and the group $GL(4,\mathbb{R})$ of general covariance, and the Dirac group $SL(2,\mathbb{C})$ (often also called the Lorentz group), but elsewhere in particle physics only unitary groups are used. The underlying mathematical (as opposed to physical) reason for this is that representations of finite groups are always unitary. However, there are really three types of representations of finite groups, that is real (orthogonal), complex (unitary) and quaternionic (also called symplectic or pseudoreal). In the case of the binary icosahedral group, only orthogonal and symplectic representations arise, and there are no unitary groups.

The real representations have orthogonal gauge groups

$$SO(3) \times SO(3) \times SO(4) \times SO(5) \tag{23}$$

and the quaternionic representations have symplectic gauge groups

$$Sp(1) \times Sp(1) \times Sp(2) \times Sp(3) \tag{24}$$

Since there are natural two-to-one maps from $Sp(1) \times Sp(1)$ onto $SO(4)$, and from $SO(4)$ onto $SO(3) \times SO(3)$, and from $Sp(2)$ onto $SO(5)$, it is only necessary to use the symplectic groups. In the standard model, the quaternionic structure of $4b$ that defines the three generations of fermions is not used, so that only the (non-relativistic) spin group $SU(2)$ remains. The same seems to be true for $6$, reducing the gauge group to $SU(3)$, and for one of $2a$ or $2b$, reducing the gauge group to $U(1)$. In total, this reduces the gauge group from 37 dimensions to 15, leaving 22 arbitrary parameters behind, split roughly into three types as

$$2 + (1 + 3 + 3) + (1 + 6 + 6) \tag{25}$$

Hence the model provides a plausible mathematical source for the unexplained parameters of the standard model. This is of course not the same as providing a plausible physical explanation, which is likely to be a great deal more difficult. At the same time, the model provides a selection of extra gauge groups with which to gauge these parameters. As it stands, there is some ambiguity as to whether to allocate weak $SU(2)$ to a subgroup of $Sp(2)$ acting on $4b$, and non-relativistic spin to $Sp(1)$ acting on $2a$, or vice versa. It is not yet clear which version will give a better match to the real world, so we need to keep an open mind on this question.

Either way, electro-weak mixing in the standard model identifies $2a + 2b$ with $4b$, which is incompatible with the action of any of the triplet symmetries. One of these is the generation symmetry, so that the standard model Dirac algebra can only deal with one generation at a time. Another is the triplet symmetry of spin states, so that the standard model Dirac algebra can only distinguish two spin states rather than six, and therefore cannot explain the experimental properties of entangled spin states. It seems clear, therefore, that distinguishing two completely different types of ‘Dirac spinor’, in $2a + 2b$ and $4b$, will be necessary in order to go beyond the standard model. Indeed, there is a real ambiguity in the standard model as to whether the Dirac spinor should be acted on by $Spin(1,3)$, in which case it must lie in $4b$, or by $Spin(4)$, in which case it must lie in $2a + 2b$. Furthermore, the Dirac equation, essential for the quantum mechanical definition of mass, cannot be written on either $4b$ or $2a + 2b$ alone, because it requires a complex scalar that commutes with the Dirac matrices.
To make the gauge groups of the standard model, therefore, we start with the spin group $Sp(1)$ acting on a non-relativistic spinor in $2a$, and the weak $SU(2)_L$ as a subgroup of $Sp(2)$ acting on elementary particle labels in $4b$, or possibly $vice versa$. We also have a copy of $U(1)$ acting on $2b$, and thereby also defining complex structures on

$$2b \otimes 3a = 6,$$

$$2b \otimes 3b = 2b + 4b,$$

(26)

so that we obtain three copies of $U(1)$ with various possible interpretations. However, these three copies are really all the same, so that although we might interpret one copy as $U(1)_Y$ acting on $2b$, and therefore commuting with $SU(2)_L$ acting on $4b$, what we really have is a copy of $U(1)$ that also acts on $4b$. But the latter is not guaranteed to commute with $SU(2)_L$, and most likely it does not, so that it breaks the symmetry of $SU(2)_L$ and acquires a new interpretation as $U(1)_{em}$.

In order to go beyond the standard model and incorporate the triplet symmetries, it is necessary and sufficient to extend from $U(1)$ to $SU(2)$ acting on $2b$. This now provides a quaternionic structure on $2b$, which then transfers to a quaternionic structure on $4b$ and $6$, with which we can construct the required larger gauge groups $Sp(2)$ and $Sp(3)$. By extending the gauge group $U(1)_Y \times SU(2)_L \times SU(3)_c$ of the standard model to $Sp(1) \times Sp(2) \times Sp(3)$ we extend QED and the weak and strong forces to include all three generations. Of course, this is not in any sense a Grand Unified Theory, it is merely a suggestion for how the gauge groups of such a theory might be derived from a few basic physical principles. Building the theory itself is a much more difficult undertaking.

3.3. Effective field theories. If 't Hooft [20] is right, and a discrete model along the lines I have sketched is the fundamental design of the universe at the Planck scale, then at larger scales much of the discrete structure appears to us to be continuous, and the continuous gauge groups form the basis for the effective field theories that are used in practice. The simplest way to derive the effective field theories from the discrete model would seem to be to use $4b$ for the Dirac spinor, acted on by $Spin(1,3) = SL(2,\mathbb{C})$ contained in the quaternionic part of the Dirac algebra, generated by $\gamma_\mu$, for $\mu = 0, 1, 2, 3$.

The standard model then converts $Spin(1,3)$ acting on $4b$ to $Spin(4)$ acting on $2a + 2b$ by the expedient of multiplying the ‘time’ coordinate by $i$, thereby replacing $\gamma_0$ by $\gamma_5$, and splitting the spinor into two chiral pieces. At the same time, the triplet symmetries of the finite group are destroyed, so that flavour and colour symmetries have to be incorporated ‘by hand’. The physical nature of this chirality comes from the natural map from $Spin(4)$ acting on $2a + 2b$ onto $SO(4)$ acting on $2a \otimes 2b = 4a$. Finally, we convert back to $SO(3,1)$ by converting again between real and imaginary time.

In effective field theories, it is assumed that this process of getting from $Spin(1,3)$ to $SO(1,3)$ is given by a natural mathematical two-to-one map, but if it is obtained in the way I have described then there is nothing natural about it. Of the three necessary steps, only the second is natural, and both the other two involve choices. In 't Hooft’s terminology, these choices are made not by God, but by us. For example, in the last step, God’s computer program works with a finite subgroup of $SO(4)$, and knows nothing of $SO(3,1)$.
In order to clarify the issues, we can separate the two choices to some extent by supposing the identification of $2a + 2b$ with $4b$ to have already been done, using the Dirac equation, so that electrodynamics can be expressed in terms of

$$(2a + 2b) \otimes (2a + 2b) = 1 + 4a + 3a + 3b + 4a + 1.$$  

(27)

This allows us to implement the usual interpretation of the Dirac algebra, in which $4a$ is a vector representation of $SO(3,1)$, so can be interpreted as 4-momentum and/or spacetime position as appropriate. The other half of the algebra consists of $3a + 3b$ standing for the electromagnetic field, plus two scalars for mass and charge. But in the fundamental theory, one copy of $4a + 1$, representing 4-momentum and mass, is replaced by 5, as is necessary in order to allow the weak interaction to change the mass of elementary particles without changing the total energy.

In particular, we need to choose a copy of $SO(3,1)$ in order to define mass, without which our effective field theories do not work. But God’s computer program does not have a copy of $SO(3,1)$, and therefore does not have a concept of mass. Therefore, mass cannot be a fundamental physical concept, but must be emergent. In other words, mass is not a cause, but an effect. In particular, mass cannot be the ultimate cause of gravity, however much it may seem like that to us. Perhaps this is why it has proved so difficult to quantise gravity: have we misunderstood its cause?

I emphasise that this conclusion is not as ridiculous as it sounds: it is a necessary consequence of following ’t Hooft’s line of reasoning to its logical conclusion.

4. Consequences for mass and gravity

4.1. The measurement of mass. If we accept this conclusion, then it becomes much easier to understand such mysteries as why the electron mass is so small, and why the proton and neutron masses are so close to each other. These facts then cease to be fundamental properties of the universe, but properties that exist only in the eye of the beholder. This on its own does not explain the facts, but it does show that we have been looking in the wrong place for the answer. We need to look more closely into our own eyes. If the electron/proton/neutron mass ratios have no fundamental meaning, but only a practical meaning in effective field theory, then the near equality of proton and neutron masses can be put down to pure coincidence and nothing more.

Moreover, the mass ratios that we use in our effective field theories depend on our choice of $SO(3,1)$, which in practice is determined by our assumption that the laboratory frame of reference is near enough inertial that it doesn’t matter. But the laboratory frame of reference is not inertial, so that the actual copy of $SO(3,1)$ that we use is crucially dependent on such accidents as the relative lengths of the day, the month and the year, the angle of tilt of the Earth’s axis, the eccentricity of the orbits of the Earth and the Moon, and many other factors. That does not mean that the masses change when any of these parameters changes, because we are free to choose a ‘standard’ copy of $SO(3,1)$ that only approximately describes the laboratory frame of reference, so that the practical variability can be moved into the identification of $SU(2) \times SU(2)$ with $SL(2,\mathbb{C})$ instead. In order to investigate whether this choice of $SO(3,1)$ actually matters or not, we need to look at the history of this choice, and see whether it has left its imprint on the parameters of the standard model, in the form of suspicious coincidences.
To demonstrate that such coincidences do exist, consider the coincidence between the neutron/proton mass ratio
\[
m(n)/m(p) \approx 1.001378
\]
and the following formula based on the average number of days in a year:
\[
1 + 1/2 \times 365.24 \approx 1.001369.
\]
Clearly this is a pure coincidence, without physical meaning. To claim otherwise would be absurd. If we look at the electron as well, we have
\[
m(e)/m(p) \approx .000544617
\]
and considering the angle of tilt of the Earth’s axis to be around 23.44° we have the following formula:
\[
\sin(23.44°)/2 \times 365.24 \approx .000544558.
\]
Clearly this is a pure coincidence, without physical meaning. To claim otherwise would be absurd.

In any case, the angle of tilt of the Earth’s axis varies considerably, but the electron/proton mass ratio does not. We can ask what angle makes the coincidence exact? We need
\[
\sin \theta \approx 2 \times 365.24 \times .000544617 \approx .3978318
\]
\[
\theta \approx 23.442704° \approx 23°26'33.7''.
\]
Now we can check the astronomical almanac [21] to find when this value of the angle of tilt was attained. This has happened only three times since the end of the last ice age, in August 1957, June 1963 and March 1973. Is it a coincidence that this is the time that the standard model of particle physics was being built? Or is this real evidence that we chose the electron/proton mass ratio, not God?

The fault, dear Brutus, lies not in the stars, but in ourselves, . . .

If we accept ’t Hooft’s argument, then God’s computer program that runs the universe does not contain the electron/proton mass ratio. But our effective field theories that describe how we see the universe could not work without this mass ratio. Hence we are forced by our desire to have effective field theories to pick a value for it. But God does not care what value we pick, so we can (effectively) pick any value we like. The fact that we (effectively) picked the current value in the 1973 CODATA revision of fundamental physical constants [22], and have not materially changed it since, does not of itself give this particular value any fundamental physical meaning. I emphasise that this argument does not in any sense ‘explain’ the electron/proton mass ratio, but only suggests that an explanation might eventually be found in a quantum theory of gravity that mixes with the other forces along the lines indicated above.

An effective field theory can be built with any choice of this parameter, and the job of CODATA is to fix a complete consistent set of parameters that work, not to determine a ‘correct’ or ‘universal’ value of any single parameter. It is noticeable, for example, that the coincidence of the neutron/proton mass ratio cannot be made exact by fixing a date. This implies that the process of determining this mass ratio was historically more complicated than that of determining the electron/proton mass ratio, so that guessing a simple empirical formula does not work.
4.2. **The equivalence principle.** The above discussion strongly suggests that not all definitions of mass are equivalent. This question is usually framed in Newtonian rather than relativistic form, that is as the question of equivalence or otherwise of inertial mass defined by

\[ m := F/a \]  

and (active) gravitational mass, defined by

\[ M := ar^2/G \]

on the assumption (strongly supported by near-Earth experiment) that passive gravitational mass is the same as inertial mass. Whilst it is true that there is no definitive demonstration that active gravitational mass is different from inertial mass, there are some experimental anomalies, such as inconsistent measurements of \( G \), that might be interpreted as hints in that direction [23, 24, 25].

In this paper, however, it is the equivalence of two relativistic definitions of mass that is called into question instead. The first is the Einstein mass, defined in the special theory of relativity by

\[ M := \sqrt{E^2/c^4 - p^2/c^2}, \]

so that it transforms as a scalar under the Lorentz group \( SO(3, 1) \). This mass is used in Einstein’s theory of gravity (general relativity), so is analogous to, but not necessarily equal to, the Newtonian active gravitational mass. The second is the Dirac mass, defined by the Dirac equation

\[ m \psi = \frac{i\hbar}{c} \gamma^\mu \partial_\mu \psi, \]

so that it transforms as a scalar under the group \( SL(2, C) \). Since the Dirac mass is used in quantum mechanics, on which all mechanical forces ultimately depend, this is analogous to, and perhaps equal to, the Newtonian inertial mass. But it should be noted that the question of equivalence of Einstein and Dirac masses is not the same question as the equivalence of Newtonian inertial and gravitational masses.

The model proposed in this paper permits the Einstein and Dirac masses to be locally equivalent, but does not permit them to be globally equivalent. In fact, it is not necessary to invoke this model in order to obtain a mathematical demonstration that these two types of mass cannot be globally equivalent. This conclusion follows indeed from the general covariance of general relativity, which extends the Lorentz group \( SO(3, 1) \) to a group \( SL(4, R) \). If the Dirac and Einstein masses were equivalent, we would need to extend the group \( SL(2, C) \) to a double cover of \( SL(4, R) \) acting on the Dirac spinor. But no such group exists.

This implies that the local equivalence of Einstein and Dirac masses can only be assumed in a region of spacetime in which the gravitational field, and/or the non-inertial motion of the experiment, is sufficiently uniform. A number of experiments and observations in which this condition is not met show significant anomalies, which support the conclusion that the Einstein and Dirac masses are not the same in these circumstances. In particular, detailed observations of galaxy rotation curves show that the equivalence of Einstein and Dirac masses breaks down on a galactic scale [26]. The flyby anomaly [27] may similarly be evidence of a breakdown, in circumstances where the relative motion of the satellite and the laboratory is highly non-inertial.
If we suppose that the Einstein mass is defined by a copy of $SO(3,1)$ acting on $4a$, representing 4-momentum, and we suppose that classical and relativistic physics arises from the quantum effects described by the action of the finite group, then we must identify the tensors

$$\Lambda^2(4a) = 3a + 3b,$$

$$S^2(4a) = 1 + 4a + 5$$

for the finite group with the corresponding tensors for $SO(3,1)$, that is the representations with spin $(1,0) + (0,1)$ and $(0,0) + (1,1)$ respectively. Hence the Einstein mass is represented in $1$, and the 5-momentum (or mass-momentum-energy) lies in $1 + 4a$, which is equivalent to the permutation representation of the (binary) icosahedral group on the five cubes inscribed in the dodecahedron. There is then no consistent action of $SO(3,1)$ on $5$, although one can impose an action that is equivalent to the action on $1 + 4a$, if one breaks enough symmetry. The latter action is, however, inconsistent with general relativity.

If we now also suppose that the Dirac mass is defined by a copy of $SL(2,\mathbb{C})$ acting on $4b$, then we can transfer the action of the compact part $SU(2)$ to an action of $SO(3)$ on $5$ via the standard map from the spin group to the orthogonal group. We can also transfer the action to the tensors

$$\Lambda^2(4b) = 1 + 5,$$

$$S^2(4b) = 3a + 3b + 4a,$$

which allows us to extend the action to $SO(5,1)$ on $1 + 5$, but does not allow us any splitting of $3a + 3b + 4a$ into irreducibles, so does not give us any natural action on $3a + 3b$. In particular, the Dirac mass is represented inside $5$, which allows the weak interaction to change the Dirac mass, while leaving the Einstein mass unchanged.

Hence we see again, more explicitly, that identifying the Dirac mass with the Einstein mass has the effect of identifying the representations $1 + 4a$ and $5$. This can only be achieved by breaking the symmetry of the binary icosahedral group to a subgroup without elements of order 3. Such subgroups lie either in the quaternion group $Q_8$ or the dicyclic group of order 20 (corresponding to the rotations of the icosahedron that fix an axis between two opposite vertices).

On the other hand, if we allow ourselves to treat the Dirac and Einstein masses separately, then we see the difference in the difference between the permutation representation $1 + 4a$ and the monomial representation $5$ of the icosahedral group. These two representations can be written by mapping the generators $f, g, h$ to the following:

$$\begin{align*}
(1,2)(3,4), & \quad (1,2,3), \quad (1,2)(4,5); \\
(\omega 1, 2)(3, \omega 4), & \quad (1, \omega 2, 3)(4, \omega^2 4, \omega 4)(5, \omega 5, \omega^2 5), \quad (1, \omega 2)(\omega 4, 5),
\end{align*}$$

where $\omega$ is a primitive cube root of unity. In particular, writing $5$ as a monomial representation requires the extension from real to complex numbers. This is perhaps another reason why the standard model requires a complex rather than real Dirac algebra. However, by identifying these two symmetry groups, we obliterate the complex numbers in the action of $g$, and thereby obliterate the generation symmetry from the model.
As a final remark, we should note that the definition of mass in the early years of the 20th century was clearly the Einstein mass, while the definition of mass in particle physics in the early 21st century is clearly the Dirac mass. There must therefore have been a changeover period, during which these two distinct types of mass were calibrated against each other. As I have already indicated, this changeover period appears to have effectively ended in 1973, and begun probably in the late 1950s. Before this time, mass measurements were simply not accurate enough for it to be necessary to distinguish Einstein mass from Dirac mass.

4.3. Implications for quantum gravity. Attempts to quantise gravity directly, by quantising the gauge group $GL(4, \mathbb{R})$, have been unsuccessful [28, 29]. The mathematical reason for this is that the finite symmetry groups cannot reach into the non-compact part of the group, which has the effect of making the theory non-renormalizable. For a Yang–Mills theory of gravity, it is essential for the gauge group to be compact, although it is not essential for it to be complex unitary, rather than real orthogonal or quaternion symplectic. In order to obtain a Yang–Mills theory from the discrete model proposed here, it is necessary and sufficient to take the gauge group to be

$$SO(4) \times SO(5)$$

acting on $4a + 5$, since $SO(4)$ also acts as $SO(3) \times SO(3)$ on $3a + 3b$.

This group is a quotient of the symplectic group

$$Sp(1) \times Sp(1) \times Sp(2) \cong Spin(3) \times Spin(3) \times Spin(5)$$

$$\cong Spin(4) \times Spin(5)$$

acting on $2a + 2b + 4b$. In particular, the mixing of $2a + 2b$ with $4b$ that occurs in the standard model carries with it a mixing of $4a$ with $5$. Note also the similarity between this gauge group and the Pati–Salam [30] gauge group

$$SU(2)_L \times SU(2)_R \times SU(4).$$

The differences are of two kinds: one is the mathematical restriction from the 15-dimensional $SU(4)$ to the 10-dimensional subgroup $Sp(2)$, while the other is a significant difference in physical interpretation.

Now the adjoint representation of our proposed gauge group consists of

$$\Lambda^2(4a) = 3a + 3b$$

$$\Lambda^2(5) = 3a + 3b + 4a.$$  

Two important things to note here are that, first, the model allows a mixing between the two copies of $3a + 3b$, and second, that $\Lambda^2(5)$ differs from what we would expect from general relativity, that is

$$S^2(4a) = 1 + 4a + 5.$$  

The first property allows us to implement the required mixing between $4a$ and $5$ at the gauge group level, in order to describe the standard model electroweak mixing of $2a + 2b$ and $4b$ at the spinor level. It also permits a unification of the electromagnetic field in one copy of $3a + 3b$ with a 3-dimensional Newtonian gravitational field and a 3-dimensional gravito-magnetic field in the other copy of $3a + 3b$. In other words, it permits, and probably requires, a ‘mixing’ between gravity and electromagnetism at the level of effective quantum field theories.
The second property shows the difference between the permutation and monomial representations of the icosahedral group on six points. The six points here are the axes of an icosahedron joining opposite vertices, and the difference between the representations is simply whether one regards a reversed axis as being the same as, or the negative of, the original axis. We may label these axes so that \( f, g, h \) act as 
\[
(1, 2)(3, 4)(5, -5)(6, -6), \quad (1, 3, 5)(2, -4, 6), \quad (3, 5)(2, 6)(1, -1)(4, -4)
\]
respectively, with or without the signs, as appropriate. The proposed model therefore inserts these signs, that do not appear in general relativity.

A remarkable consequence of the insertion of these signs is that the hyperspin 2 representation \( 5 \) disappears from quantum gravity, and with it, apparently, the requirement for a spin 2 graviton. Instead we have the hyperspin 1 and hyperspin 3 representations, usually identified as spin \((1, 0)\), spin \((0, 1)\) and spin \((1/2, 1/2)\). The first two are normally interpreted, in the context of quantised long-range forces, as spin 1 photons, while the last would normally be interpreted as spacetime, or 4-momentum. Here we need it to consist of particles, with zero Einstein mass, but possibly non-zero Dirac mass. The only reasonable interpretation then seems to be as spin 1/2 neutrinos and anti-neutrinos. This permits a quantisation of Euclidean 4-momentum, and in particular gives the neutrinos a non-zero Dirac mass, which they need in order to explain neutrino oscillations, together with a zero Einstein mass, which they need in order to quantise gravity.

However, it is clear that this cannot be the whole story of quantum gravity, because the representation \( 5 \) contains real physical information that we have not used. In particular, we have taken no account of neutrino oscillations, which require us to extend from 4 degrees of freedom, representing momentum and a non-zero Dirac mass, to 9, representing momenta for three generations, combined with a zero Einstein mass. Mathematically, this can be achieved by extending Dirac neutrinos described by \( 2a \otimes 2b \) to Einstein neutrinos defined by 
\[
3a \otimes 3b = 4a + 5,
\]
but what this means physically is not at all clear.

5. Symmetry-breaking and the standard model

5.1. The standard model. As I have shown, the standard model of particle physics is based on an identification of two types of Dirac spinors, in the representations \( 2a + 2b \) and \( 4b \), that breaks the symmetry of the finite group. There are thus three different types of 'square' of the Dirac spinor, that describe three different ways of looking at the effective fields:
\[
(2a + 2b) \otimes (2a + 2b) = 1 + 4a + 3a + 3b + 4a + 1,
\]
\[
(2a + 2b) \otimes 4b = 5 + 3a + 3b + 5,
\]
\[
4b \otimes 4b = 1 + 4a + 3a + 3b + 5.
\]
At scales much greater than the Planck scale, all these representations can be interpreted as (effectively continuous) fields. But the interpretation may differ according to whether the scale is nuclear, atomic, classical electromagnetic, or classical gravitational. All three versions of the square of the spinor contain \( 3a + 3b \), which is usually interpreted as the electromagnetic field on all scales.
At the fundamental (Planck scale) level, the last version describes the quantum fields that are predicted by the model presented in this paper. At the gravitational level, these are essentially the same fields that are predicted by Einstein’s general theory of relativity, although usually interpreted somewhat differently. The first version describes the atomic scale, and the standard implementation of QED, while the second describes the nuclear scale, and the weak and strong forces. Electroweak unification is obtained by mixing together the first two versions, so that one copy (only) of $1 + 4a$ is mixed with $5$, via the identification of
\[
2a \otimes (2a + 2b) = 1 + 3a + 4a, \\
2a \otimes 4b = 3a + 5.
\]
(48)

This procedure gives the standard model the complete set of effective fields in $4b \otimes 4b$ that are needed in order to give a complete description of physics at the atomic and nuclear scales.

But in order to describe the necessary symmetries on $5$, the standard model interprets $3a + 5$ and/or $3b + 5$ as the adjoint representation of $SU(3)$, representing the strong force. This procedure unavoidably mixes the strong force with the electroweak forces which also use $3a$ and/or $3b$. In other words, the standard model tries to quantise the effective fields separately, but finds that this is impossible, since the quantised fields are elements of the finite group, that act on all the effective fields simultaneously, whereas the gauge groups each act on a single field. It is this fundamental distinction between the ways the finite group and the gauge groups embed in the group algebra that is the mathematical reason why the standard model is forced to incorporate some very messy ‘mixing’ between the gauge groups, in order to express the relationship between the gauge fields and the underlying discrete structure.

5.2. Space and time. In order to use this model to describe the dynamics of the universe it is necessary to break the symmetry of the spacetime representation $4a$ to separate space from time. The obvious way to do this is to restrict the finite symmetry group to the binary tetrahedral group, generated by $f$ and $g$. But all that is required mathematically is for $4a$ to become equivalent to either $1 + 3a$ or $1 + 3b$, for which it is necessary and sufficient for the subgroup to contain no elements of order $5$. Hence the subgroup generated by $g$ and $h$ is another possibility, as is the subgroup generated by $f$ and $h$. Of course, these three options give three quite different definitions of time, and the last two also distinguish one dimension of space from the other two.

Hence only the first possibility could be useful for the purpose of defining time relative to an isotropic 3-dimensional physical space. The others may be useful in circumstances where the symmetry of space is broken, for example by a current, or a gravitational field. In such circumstances, the physical experience of time is known to be different, and is described by special and general relativity respectively.

Now the group algebra of $\langle f, g \rangle$ is isomorphic to
\[
R + C + M_3(R) + H + M_2(C),
\]
(49)
with compact subgroup
\[
U(1) \times SO(3) \times SU(2) \times U(2).
\]
(50)
The corresponding real representations have dimensions 1, 2, 3, 4 and 4 respectively. The quaternionic 4, say \(4a\), is the restriction of both \(2a\) and \(2b\), while \(4b\) restricts as two copies of the other 4, say \(4b\). The quaternionic tensor square of \(4a\) is \(1 + 3\), which provides a natural map from \(SU(2)\) onto \(SO(3)\). The representation \(4b\) has two possible complex structures, related by complex conjugation, so has various different ‘tensor squares’. The complex tensor product of the two versions is a complexified copy of \(1 + 3\), that is \(1 + 1 + 3 + 3\) as a real representation, while the complex tensor square of either version is \(2 + 3 + 3\) as a real representation. This implies that there are two natural maps from \(U(2)\) onto \(U(1) \times SO(3)\), differing by complex conjugation on the scalars.

In other words, this gives us a toy model with gauge group \(SU(2) \times U(2)\), in which the \(SU(2)\) factor has a natural interpretation as the spin group, and \(U(2)\) as the gauge group of electroweak theory. However, the natural map from \(U(2)\) onto the rotation group \(SO(3)\) is somewhat unexpected here. Taken at face value, it relates the symmetry-breaking of the weak interaction to a symmetry-breaking of space itself. Whilst this is logically possible, and could, for example, be obtained physically by a mixing between the weak interaction and quantum gravity, it is not part of the standard model.

The latter attempts to remove any dependence of the weak interaction on the gravitational environment by mixing \(U(2)\) with \(SU(2)\) so that the image of \(U(2)\) in \(SO(3)\) is cancelled out by the image of \(SU(2)\) in \(SO(3)\). Since the direction of spin cannot be measured, this mathematical procedure cannot be detected experimentally, so that its physical validity can be neither proved nor disproved. From a mathematical point of view, however, this procedure is invalid, and converts what might have been an exact universal model into an approximate local model.

In this approximate local model, however, if we ignore the ‘mixed’ copy of the gauge group, then we effectively identify \(SU(2)\) acting on \(4a\) with \(SU(2)\) acting on \(4b\), so that we can extend this copy of \(SU(2)\) to \(SL(2, \mathbb{C})\) acting on \(4b\). Then in order to separate the left-handed and right-handed spin \(1/2\) representations of \(SL(2, \mathbb{C})\) we must complexify the representation, and hence work with a complex version of

\[
4b \otimes 4b = 1 + 1 + 3 + 3 + 3 + 3 + 2
\]

as the Dirac algebra. It is then possible to impose the standard structure of the Dirac algebra as a representation of \(SL(2, \mathbb{C})\), but this structure is not determined by the finite group, and is therefore not determined by the fundamental properties of the universe, but by our choice of experiments and our choice of models.

Moreover, by reducing the gauge group from \(U(2)\) to \(U(1)\), generated in the Feynman calculus by \((1 - \gamma_5)i\), we have removed the possibility of explaining the symmetry-breaking of the weak interaction in terms of the natural map from \(U(2)\) onto the rotation group \(SO(3)\) of space. The finite group model, however, implies that the electroweak mixing angle must be visible as a parameter of the non-inertial motion of the laboratory, that determines the choice of homomorphism between \(SL(2, \mathbb{C})\) and \(SO(3, 1)\) in the standard model. Moreover, there are really three copies of \(U(1)\) being mixed together here, so that there are two distinct physical angles that combine to make the electroweak mixing angle. Both are physical angles in space that affect the quantum gravitational field in which the experiments take place.
They certainly cannot be angles of latitude or longitude, but must be angles that remain the same for all experiments on the Earth. They could however be angles that determine tidal effects of gravity, such as the angle of tilt of the Earth’s axis, and of the Moon’s orbit, relative to the ecliptic. These angles are approximately 23.44° and 5.14° respectively, and their sum, 28.58° is indeed very close to the various different values of the electroweak mixing angle that are measured in different experiments. This coincidence is not exactly a prediction of the model, since there are many things I have not taken into account in this toy model. Nevertheless, it is striking, and suggests that it might be fruitful to examine this model more closely.

6. Conclusion

In this paper I have argued that there is essentially only one algebraic structure in which it is conceivable to build a mathematical model of quantum mechanics that is consistent with the most fundamental physical principles that are required by experiment, including locality, unitarity, gauge invariance and renormalisability. The structure of quaternion reflections leads naturally to both first and second quantisation, which appear to go beyond the standard model only by tripling the numbers of particle states for electrons and photons, and reinterpreting the colourless gluons as quantum superpositions of the ‘primordial’ massless weak bosons. Apart from this, no new particles or new forces are required.

I have built only the basic algebraic structures, and have not put the physics on top. There is therefore no guarantee that a model of this type can include the standard model, and no guarantee that it will be useful. Nevertheless it contains some important ingredients that will be necessary in any theory that seeks to go beyond the standard model in a meaningful way, including

- a mathematical definition of the elementary fermions that includes all three generations,
- a mathematical process whereby the elementary fermions generate quantum fields,
- a mathematical definition of massless mediators that includes photons and gluons,
- a mathematical definition of massive mediators that includes all pairs of charged intermediate vector bosons and pseudoscalar mesons, and
- an embedding of the gauge groups of the standard model into larger gauge groups that take account of the three generations.

In addition, it contains

- enough quantum states to explain the experimental measurement of electron spin and photon polarisation without the need for superluminal transfer of information, and
- a mathematical isomorphism between the quantum and classical fields.

This last suggests a division of both into dimensions $3+6+6$, identified as the God-given fundamental gravitational, electromagnetic and colour fields respectively. But in passing from the fundamental theory to the effective field theories we pass from the structure of the group itself to the structure of the group algebra, converted into a Lie algebra, which divides the fields instead as $3+3+4+5$. The relationship between the group elements that parametrise $3+6+6$ and the group representations that parametrise $3+3+4+5$ is quite complicated.
In particular, to see the relationship it is necessary to break the symmetry and mix the effective fields \(3 + 3 + 4 + 5\) together in complicated ways in order to obtain the fundamental quantum fields in \(3 + 6 + 6\). All this symmetry-breaking and mixing has nothing whatever to do with the fundamental theory, and has only to do with our choice of effective field theories. Mainstream opinion is that our effective field theories are fundamental, but by examining the data, I provide evidence that they may not be, and that we may have a lot more choice than is generally realised, for parameters to use in constructing effective field theories. Chief amongst these choices is our choice of the concept of mass, specifically our choice to identify the Einstein mass with the Dirac mass.

Finally, let me remark that this work does not solve the mathematical Yang–Mills mass gap problem as described in detail in [31]. It does however potentially solve the underlying physical problem, by modelling the discrete nature of mass. As such, it may render any potential solution of the Clay Millennium problem largely irrelevant for the understanding of the physical universe.

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