ON CATEGORICAL APPROACH TO VERDIER DUALITY

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To the memory of my grandmother and my grand grandmother

ABSTRACT. In present paper we develop categorical formalism of Verdier duality.

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1. Introduction

1.1. Classical picture. To the scheme $X$ one can associate étale topos $X_{\text{ét}}$ with corresponding derived category $D^b_c(X, \Lambda)$ of constructible sheaves on $X$. Classical approach $[AGV3]$ to six functor formalism states that if we have morphism of schemes $f: X \to Y$ we have associated morphism of topoi $f: X_{\text{ét}} \to Y_{\text{ét}}$, with corresponding derived functors:

$$L^*(f^*): D^b_c(Y, \Lambda) \leftrightarrow D^b_c(X, \Lambda): R^*(f_*)$$

and $!$-operations:

$$R^!(f_!): D^b_c(Y, \Lambda) \leftrightarrow D^b_c(X, \Lambda): f^!.$$

Also we have tensor product $\otimes$ on category $D^b_c(X, \Lambda)$ and inner hom functor. These functors satisfy natural compatibilities, such as base change property $[\text{Gro2}]$.

Starting from such data, Verdier duality functor $D$ for category $D^b_c(X, \Lambda)$ can be defined $[\text{Gro2}]$:

$$D: D^b_c(X, \Lambda) \to D^b_c(X, \Lambda)^\circ$$

The same approach works well in setting of constructible sheaves on topological space $[KS]$, however, there are some situations when this approach fails. For example, when $X$ is a smooth scheme we can consider derived category $D^b(\text{Mod}(D_X))$ of right $D$-modules on $X$. In general classical six functor formalism for $D$-modules does not exists, but Verdier duality functor can be defined $[\text{Ber}]$. We present axiomatic approach to the problem of construction Verdier duality and six functors formalism.

1.2. Covariant duality. Object for which we propose construction of Verdier duality is the ringed $D$-topos $(E, \Lambda)$ in the sense of $[AGV2]$ ($D$-topos $(E, \Lambda)$ is the category bifibered in topoi, over category $D$, with a ringed object $\Lambda$). Our framework for construction of Verdier duality functor is so called cross functor $([\text{Del}])$. That is quadruple of 2-functors:

$$H_*, H^*: D \to \text{Cat}, \quad H^*, H^! : D^\circ \to \text{Cat},$$

where $H_*$ and $H^*$ and $H^!$ and $H^!$ are adjoint and isomorphic on objects. Moreover we have natural coherence relations like base change property (Definition $3.2.3$).

By the Grothendieck cross functor associated with the ringed $D$-topos $(E, \Lambda)$ we understand cross functor $(H_*, H^*, H^!, H^!)$, such that pseudo-functors $H_*$ and $H_!$ are isomorphic and bifibration, which corresponds to $(H_*, H^!)$ is equivalent to the category of modules over ringed $D$-topos $(E, \Lambda)$ (Definition $3.4.1$).

To every cross functor we can associate pair of pseudo-functors $(H^*, H^!)$, where $H^!$ is called Mackey $!*$-functor and $H_!$ is called Mackey $!*$-functor. These pseudo-functors acts from the category of correspondences $\text{cospan}(D)$ to the category $\text{Cat}$:

$$H_*, H^*: \text{cospan}(D) \to \text{Cat},$$

such that, when restricting to $D$ Mackey pseudo-functor $H^!$ (resp. $H_*$) is isomorphic to $H^*$ (resp. $H_*$) and when restricting to $D^\circ$ is isomorphic to $H^!$ (resp. $H^!$):

$$i^*(H^!): H_! \to H^! \quad i^*(H_*) \to H^! \quad j^*(H^!): H^* \to H^! \quad j^*(H^!) \to H^!.$$
where i and j are natural embedding functors:

\[ i: D \hookrightarrow \text{cospan}(D) \hookrightarrow D^\circ : j. \]

Denote by \( D(\Gamma(H^*)) \) and \( D^*(\Gamma(H^)) \) derived categories of sections of corresponding fibrations. Covariant Verdier duality functors \( V_{s\rightarrow t} \) and \( V_{t\rightarrow s} \) act between categories \( D(\Gamma(H^*)) \) and \( D^*(\Gamma(H^)) \):

\[ V_{s\rightarrow t}: D(\Gamma(H^*)) \hookrightarrow D(\Gamma(H^)) \quad V_{t\rightarrow s}, \]

We define covariant Verdier duality functors as composition of Kan extension to the category \( D(\Gamma(H^)) \) (resp. \( D(\Gamma(H^*)) \)) and restriction functor (Definition 3.4.10).

We mostly interested in subcategories \( D_{\text{cocart}}(\Gamma(H^*)) \) and \( D_{\text{cart}}(\Gamma(H^)) \) consisting of objects with cocartesian (resp. cartesian) cohomology and generally Verdier duality functors do not preserve these subcategories. To fix this we introduce additional pair of pseudo-functors \((H^{**}, H^0)\), (Definition 3.4.9 and Definition 3.4.10):

\[ H^0, H^{**}: \text{cospan}(D) \rightarrow \text{Cat}, \]

with properties:

\[ i^*(H^{**}) \sim \sim H^* \quad i^*(H^0) \sim \sim H^! \quad j^*(H^{**}) \sim \sim H^* \quad j^*(H^0) \sim \sim H^!, \]

where by \( \circ \) we denote the opposite pseudo-functor. These pseudo-functors are called Mackey \( ** \)-functor and Mackey \( !! \)-functor. Analogically to the definition of pair \((H^t, H^s)\), these pseudo-functors can be canonically associated with the Grothendieck cross functor. We define functors \( \Xi_* \) and \( \Xi! \) analogically to Verdier duality functors. Functor \( \Xi_* \) is defined as compositions of Kan extension functor to category \( D(\Gamma(H^*)) \) and restriction functor:

\[ \Xi_*: D(\Gamma(H^*)) \rightarrow D_{\text{cocart}}(\Gamma(H^*)), \]

And functor \( \Xi! \) is defined as compositions of Kan extension functor to category \( D(\Gamma(H^)) \) and restriction functor:

\[ \Xi!: D(\Gamma(H^)) \rightarrow D_{\text{cart}}(\Gamma(H^)), \]

Functors \( \Xi_* \) and \( \Xi! \) are adjoint to inclusion functors \( \Xi^*: D_{\text{cart}}(\Gamma(H^*)) \rightarrow D(\Gamma(H^*)) \)
and \( \Xi^!: D_{\text{cocart}}(\Gamma(H^)) \rightarrow D(\Gamma(H^))). \) Then we define Verdier duality functors:

\[ V_{s\rightarrow t}^\text{cart}: D_{\text{cocart}}(\Gamma(H^*)) \hookrightarrow D_{\text{cart}}(\Gamma(H^)) \quad V_{t\rightarrow s}^\text{cocart}, \]

As compositions of Verdier duality functors and functors \( \Xi_! \) and \( \Xi_*. \)

1.3. Contravariant duality. In order to define contravariant Verdier duality functors we introduce notion of Grothendieck cross functor with !-dualizing object as the cross functor \((H_*, H^*, H_!, H^!)\) with distinguished object \( w_D \) in \( D(\Gamma(H^)) \) (Definition 3.4.17). With every Grothendieck cross functor with !-dualizing object we can associate duality functors \( D_{t\rightarrow s}! \) and \( D_{s\rightarrow t} \) which are functors acting between categories \( D(\Gamma(H^)) \) and \( D(\Gamma(H^*)) \) (Definition 3.4.18):

\[ D_{s\rightarrow t}: D(\Gamma(H^*)) \hookrightarrow D(\Gamma(H^)): D_{t\rightarrow s}! \]

For Grothendieck cross functor \((H_*, H^*, H_!, H^!)\) with !-dualizing object \( w_D \) we can define contravariant Verdier duality functors \( D_{(E,A)} \) and \( D_{H^!} \) as composition of above duality functors with covariant Verdier duality functors (Definition 3.4.18):

\[ D_{(E,A)}: D_{\text{cocart}}(\Gamma((E,A))) \rightarrow D_{\text{cocart}}(\Gamma((E,A)))^\circ \]

\[ D_{H^!} : D_{\text{cart}}(\Gamma(H^!)) \rightarrow D_{\text{cart}}(\Gamma(H^!))^\circ \]
From classical point of view (where we consider category of sheaves on locally compact space or scheme in étale topology) Verdier duality functor \(V_{\text{full}}\) plays role of functor, which takes sheaf \(\mathcal{F}\) to corresponding cosheaf of compact support sections \([\text{Lur}]\). Functor \(D_{\text{full}}\) is the linear duality functor, which takes cosheaf to corresponding linear dual sheaf and functor \(D(\mathcal{E}, \mathcal{A})\) corresponds to original Verdier duality functor \([\text{Ver}]\) \([\text{Gro2}]\).

1.4. Mackey functors. Our definition of Mackey pseudo-functors can be considered as 2-categorical analog of the classical notion of Mackey functor. By a Mackey functor \([\text{Lin}]\) over category \(\mathcal{D}\) we understand functor \(M\) from the category cospan(\(\mathcal{D}\)) to the category of abelian groups \(\text{Ab}\) (more generally with values in the category of spectra \(\mathcal{S}\)). It is well known \([\text{Lin}]\) that one can consider Mackey functor \(M\) as a pair of functors \((i^*M, j^*M)\), which satisfies base-change property. It is interesting problem try to construct functor from colimit of functor \(i^*M\) to the limit of functor \(j^*M\):

\[
\begin{align*}
\text{1.4.} & \quad V_{\text{full}}^{\text{full}} : \text{colim}_{\mathcal{D}} i^*M & \rightarrow & \lim_{\mathcal{D}} j^*M \\
\end{align*}
\]

Examples of such morphisms appear in different contexts (See \([\text{GM}]\), \([\text{HL}]\) and \([\text{MJHR}]\)) and perhaps the simplest one is given by Norm map between homology and cohomology of finite group \(G\) with coefficient in representation \(L\):

\[
\text{Nm} : C_q(G, L) \rightarrow C_q(G, L)
\]

Covariant Verdier duality functors can be considered as categorical analog of morphism \(V_{\text{full}}^{\text{full}}\). By correspondence between higher Picard groupoids and connective spectra we will explain how our formalism can be applied to construct such morphisms (Subsection 3.7).

1.5. Six operations. Let \(f : (\mathcal{E}, \mathcal{A}) \rightarrow (\mathcal{E}', \mathcal{A}')\) be a morphism of \(\mathcal{D}\)-topoi, we have corresponding functors:

\[
f_* : D_{\text{cocl}}(\mathcal{E}, \mathcal{A}) \leftrightarrow D_{\text{cocl}}(\mathcal{E}', \mathcal{A}') : f^*,
\]

where \(f_*\) is defined as composition of derived pushforward functor \(R(f_*)\) and functor \(\Xi_*\). Note that functors \(f_*\) and \(f^*\) are adjoint in the standard way. We can also define \(!\)-operations:

\[
f! : D_{\text{cocl}}(\mathcal{E}) \leftrightarrow D_{\text{cocl}}(\mathcal{E}') : f!,
\]

by the rule:

\[
f! := D_{(\mathcal{E}, \mathcal{A})} \circ f_* \circ D(\mathcal{E}, \mathcal{A}), \quad f! := D(\mathcal{E}, \mathcal{A}) \circ f^* \circ D(\mathcal{E}, \mathcal{A}).
\]

We also have tensor product \(\otimes^*\), defined componentwise and inner hom functor \(\text{Hom}^*\), which make \(D_{\text{cocl}}(\mathcal{E}, \mathcal{A})\) into closed monoidal category. When \(\mathcal{D}\)-topos \((\mathcal{E}, \mathcal{A})\) is associated with Artin stack \(X\), our construction gives six functors formalism for \(X\) \([\text{LO}]\). I hope to elaborate on this elsewhere.

1.6. Applications. Apart from applications, which were already mentioned we have following. Consider diagram of topological spaces (schemes) \(X_\beta\), where \(\mathcal{J}\) is some category. We have natural \(\text{Top}^\circ\)-topos \(\text{Sh}_{\text{Top}}\), whose fiber over object \(X \in \text{Top}\) is given by category of sheaves on \(X\), with corresponding pseudo-functors

\[1\]
(H_{\text{top}}, H^*_{\text{top}}). We can restrict this bifibration to category \( J \), to obtain \( J \)-topos \( \text{Sh}_J^{\text{top}} \). In this situation triangulated category \( D_{\text{cocart}}(\Gamma(H_J)) \) is the category of so called \textit{admissible} \( * \)-sheaves on diagram \( X_J \). This category has natural t-structure, whose heart is the category of sheaves on the colimit of diagram \( X_J \). Triangulated category \( D_{\text{cart}}(\Gamma(H_J)) \) is the category of \textit{admissible} !-sheaves in the sense of Beilinson and Drinfeld [BD2]. By Proposition 3.5.3 category \( D_{\text{cart}}(\Gamma(H_J)) \) can be equipped with perverse t-structure (see also Remark 4.3.4). And category \( D(\Gamma(H_J)) \) is what we call derived category of !-sheaves on diagram \( X_J \). Category \( D_{\text{cart}}(\Gamma(H_J)) \) has universal tensor structure \( \otimes ! \), defined by componentwise !-product. With every morphism \( f: X_J \rightarrow Y_J \) of diagrams we associate functors:

\[ f!: D_{\text{cart}}(\Gamma(H_{X_J})) \leftrightarrow D_{\text{cart}}(\Gamma(H_{Y_J})): f! \]
defined analogically to the case of \(*\)-sheaves and we can also associate \(*\)-operations for !-sheaves:

\[ f*: D_{\text{cart}}(\Gamma(H_{X_J})) \leftrightarrow D_{\text{cart}}(\Gamma(H_{Y_J})): f* \]

In addition to above operations one can also define Kan extension functors. Let \( j: J \rightarrow J \) be a functor, then we have a pairs of adjoint functors:

\[ j!: D_{\text{cart}}(\Gamma(H_{X_J})) \leftrightarrow D_{\text{cart}}(\Gamma(H_{X_J})): j! \]

where \( j! \) is the restriction functor and \( j! \) is defined as composition of left Kan extension functor and functor \( \Xi_! \). When diagram \( X_J \) is associated with Ran prestack our constructions give Koszul duality for factorizable sheaves (chiral algebras, n-algebras). Perverse t-structure on !-sheaves is important in construction of Hopf algebras from factorizable sheaves and quantization of Lie bialgebras. When diagram \( X_J \) is associated with de Rham prestack our constructions give six operations for \( D \)-modules. I hope to elaborate on these examples elsewhere.

1.7. Content. In the Section 2 we recall basic facts about topoi and D-topoi, mostly following [AGV2], [AGV2] and [Beh].

Section 3 is a main part of this paper. First we recall definition of a cross functor, following [Del] and some definitions of morphisms in monoidal categories, which were introduced in [FHM]. Then we give definition of Grothendieck cross functor as well as definitions of Mackey pseudo-functors and Verdier duality functors. We also study properties of these functors. In Subsection 3.6 we construct interesting t-structures on categories \( D_{\text{cocart}}(\Gamma(E, A)) \) and \( D_{\text{cart}}(\Gamma(H_{X_J})) \). In Subsection 3.7 we construct six operations formalism for ringed D-topos \( (E, A) \) with Grothendieck cross functor. In Subsection 3.7 we explain how our definition of Mackey pseudo-functors is related to original definition of Mackey functors and how categorical Verdier duality can be useful to construct analogs of Norm morphism [GM] [HL]. In Section 4 we construct Grothendieck cross functor for diagrams of topological spaces (schemes) and define Verdier duality functors. We also construct six functors formalism for !-sheaves on diagrams of topological spaces.

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1.9. Preliminaries. For a category $C$ we denote by $C^\circ$ opposite category, for a morphism $f: x \to y$ in $C$ we denote by $f^\circ$ corresponding morphism in opposite category $C^\circ$. By $[n]$ we denote finite ordinal category $[n] := \{0 \to 1 \to \cdots \to n\}$. By $\Delta$ we denote simplex category. It is category whose objects are nonempty finite ordinals $[n]$ and order preserving maps between them. We have subcategory $\Delta_+ \subset \Delta$, with same objects and morphisms given by order preserving maps, which send maximal element to maximal. For every $n \in \mathbb{N}$ we denote by $\Delta_{\leq n}$ (resp. $\Delta_{+ \leq n}$) full subcategory of $\Delta$ (resp. $\Delta_+$) on the objects $[0], [1], \ldots, [n]$. For a category $C$ with fiber products we denote by $\text{span}(C)$ category of correspondences in $C$. Objects of this category are the same as in $C$ and morphisms $f: x \to y$ given by equivalence classes of "roofs" $x \leftarrow z \to y$, where $z$ is an object of $C$, such morphisms are denoted by triples $(f_i, z, f_j)$, two roofs $x \leftarrow z \to y$, and $x \leftarrow z' \to y$, are said to be equivalent if we have an isomorphism $z \cong z'$, which is compatible with maps of roofs. We have embedding of categories $C \hookrightarrow \text{span}(C) \hookrightarrow C^\circ$. For the dual category $C^\circ$ we will use a notation $\text{cospan}(C) := \text{span}(C^\circ)$. Let $E \to B$ be a fibration (resp. cofibration) in the sense of [Gro1], between categories $E$ and $B$. For every morphism $m: i \to j$ in $B$ we have a functor $m^*: E_j \to E_i$ (or $m_*: E_i \to E_j$), where $E_i$ is a fiber over $i$. For a $B$-functor $\phi: E \to E'$ between two fibrations (resp. cofibrations) we denote by $\phi_i: E_i \to E'_i$ restriction of a functor $\phi$ to a fiber over $i$. Category of $B$-functors will be denoted by $\text{Hom}_B(E, E')$. Category of sections of fibration (resp. cofibration) $E$ will be denoted by $\Gamma(E) := \text{Hom}_B(E, B)$, we also have a subcategory $\Gamma^\text{cart}(E)$ (resp. $\Gamma^\text{cocart}(E)$) of cartesian (resp. cocartesian) sections. Fibration (resp. cofibration) is called abelian if every fiber is an abelian category and corresponding functors between fibers are right exact. If $B^\circ$ is a monoidal category and $E$ is a symmetric monoidal category, such that the underlying tensor product functor in $E$ preserves colimits in each variable, then category of sections $\Gamma(E)$ can be endowed with a monoidal structure via Day convolution product, denoted by $\boxtimes$. For fibration (resp. cofibration) $E \to B$ we denote by $E^\circ \to B$ cartesian (resp. cocartesian) dual fibration (resp. cofibration).

Let $B \to D$ be an abelian fibration over $D$. We have an abelian category $\Gamma(B)$ of sections of the fibration $B \to D$. We can associate with it triangulated category $D^\circ(\Gamma(B))$ where $* = +, \emptyset$. Denote by $D^\text{cart}_*(\Gamma(B))$, where $* = +, \emptyset$, triangulated subcategory consisting of complexes with cartesian cohomology. If category $B$ was a cofibred over $D$ we denote by $D^\text{cocart}_*(\Gamma(B))$, where $* = +, \emptyset$, triangulated subcategory consisting of complexes with cocartesian cohomology.

2. Topoi

2.1. Notations. First of all we want to recollect some facts about topoi, following [AGV1]. Let $T$ and $T'$ be topoi. Then a morphism of topoi $\varphi: T \to T'$ is given by a pair of functors $\varphi = (\varphi_*, \varphi^*)$, where $\varphi_*: T \to T'$ is right adjoint to functor $\varphi^*: T' \to T$. Functor $\varphi^*$ must preserves finite limits. Morphism $\varphi: T \to T'$ of topoi is called an embedding if $\varphi_*$ is fully-faithful. Embedding $\varphi$ is called closed (resp. open) if essential image of $\varphi_*$ is closed (resp. open) subtopos of $T'$. Ringed topos $(T, \mathcal{O}_T)$ is a pair, where $T$ is a topos and $\mathcal{O}_T$ is a ring object in $T$. Morphism between ringed topoi $(T, \mathcal{O}_T)$ and $(T', \mathcal{O}_{T'})$ is a pair $\varphi = (\varphi, \theta)$, where $\varphi$ is a morphism of topoi and $\theta: \mathcal{O}_{T'} \to \varphi^*\mathcal{O}_T$ is a morphism of rings. With a ringed topos $(T, \mathcal{O}_T)$ we can associate an abelian category of modules over a topos $\text{Mod}(T, \mathcal{O}_T)$. This category will be complete and cocomplete. Also filtered
colimits are exact in $\text{Mod}(T, \mathcal{O}_T)$. Morphism of ringed topoi $(\varphi, \theta)$ is called a closed embedding if $\varphi$ is a closed embedding of topoi and morphism $\theta: \mathcal{O}_T \to \varphi_* \mathcal{O}_T$ is surjective. In this case functor $\varphi_*: \text{Mod}(T, \mathcal{O}_T) \to \text{Mod}(T', \mathcal{O}_{T'})$ is exact.

A stratification $\mathcal{S}$ of a topos $T$ is a finite number of locally closed subtopoi $i_s: T_s \to T$, where $s \in \mathcal{S}$, called strata, such that $T$ is a disjoint union of strata and closure of each stratum is an union of strata. Pair $(T, \mathcal{S})$ will be called a stratified topos. An object $\mathcal{K} \in T$ is called a locally constant sheaf if it’s locally isomorphic to a constant object associated with a finite set. An object $\mathcal{K}$ in a stratified topos $(T, \mathcal{S})$ called constructible with respect to a stratification $\mathcal{S}$ if each restriction $i_s^* \mathcal{K}$ is a locally constant sheaf. An object $\mathcal{K} \in \text{Mod}(T, \mathcal{O}_T)$ is a locally constant if it’s locally constant as a sheaf of sets. We will denote a weak Serre subcategory of modules of locally constant sheaves by $\text{Mod}_{\text{lcc}}(T, \mathcal{O}_T)$. We also have a weak Serre subcategory $\text{Mod}_{\text{c}}(T, \mathcal{O}_T)$ of modules of constructible sheaves.

2.2. D-topoi. We are going to recollect some facts about fibered topos, partially following [AGV2]. Let $D$ be a small category.

**Definition 2.2.1.** Category $E$ is called a $D$-topos if we have a bifibration $E \to D$, such that for every $i \in D$ corresponding fiber $E_i$ is a topos and for every morphism $m: i \to j$ we have $m^* = f_* m_* = f^*$, where $f = (f_*, f^*): E_j \to E_i$ is a morphism of topoi.

**Lemma 2.2.2.** Let $A$ be a bifibration over $J$, suppose that we have functor $f: I \to J$. We can define pullback of bifibration by the rule $I \times_J A$. We have forgetful functor:

$$f^*: \Gamma (A) \to \Gamma (I \times_J A)$$

If each fiber of bifibration $A$ is complete and cocomplete then functor $f^*$ posses a right adjoint $f_*$ (resp. a left adjoint $f_!$). Functor $f_*$ is called right Kan extension and functor $f_!$ is called left Kan extension.

**Proof.** See [AGV2] Exposé Vbis, Lemma 1.2.10.1. □

Let $f: D' \to D$ be a functor, we can define category $D' \times_D E$ which will be a $D'$-topos. Following lemma will be very useful to us:

**Corollary 2.2.3.** Let $E$ be a $D$-topos, a restriction functor $f^*$ has a right adjoint $f_*$ (resp. a left adjoint $f_!$).

Let $e_i: i \to D$, be an inclusion of an object $i$ as a category with a unique object. Then we a have functor $e_i^*: \Gamma (E) \to E_i$, called evaluation at point $i$. This functor has a left adjoint $e_i!$ (resp. a right adjoint $e_i^*$).

**Proposition 2.2.4.** Let $E$ be a $D$-topos, then category $\Gamma (E)$ and a subcategory of cocartesian sections $\Gamma_{\text{cocart}}(E)$ are topoi. We have a morphism of topoi

$$\Xi: \Gamma (E) \to \Gamma_{\text{cocart}}(E),$$

where $\Xi^*$ is an inclusion functor.

**Proof.** See [AGV2] Exposé Vbis, Proposition 1.2.12] for $\Gamma (E)$. The same argument can be applied to proof that $\Gamma_{\text{cocart}}(E)$ is a topos. Existence of a functor $\Xi_*$ follows from an adjoint functor theorem. □
Definition 2.2.5. Morphism $\Phi$ of $\mathcal{D}$-topoi $E$ and $E'$ is a couple $(\Phi_*, \Phi^*)$ of adjoint $\mathcal{D}$-functors $(\Phi_*$ is a right adjoint to $\Phi^*)$, such that for an every $i \in \mathcal{D}$ we have a morphism $(\Phi_{*i}, \Phi^*_{!})$ of topoi $E_i$ and $E'_i$.

Proposition 2.2.6. Morphism of $\mathcal{D}$-topoi $\Phi = (\Phi_*, \Phi^*): E \longrightarrow E'$, induces a morphism $(\Gamma(\Phi_*), \Gamma(\Phi^*)): \Gamma(E) \longrightarrow \Gamma(E')$ of associated total topoi.

Proof. See [AGV2] Exposé V bis, Proposition 1.2.15. 

Example 2.2.7. Let $(\Phi_*, \Phi^*)$ be a morphism of $\mathcal{D}$-topoi. We call it open (resp. closed) if for every $i \in \mathcal{D}$ a morphism of topoi $(\Phi_{*i}, \Phi^*_{!}): E_i \longrightarrow E'_i$ is open (resp. closed). In this case morphism $(\Gamma(\Phi_*), \Gamma(\Phi^*)): \Gamma(E) \longrightarrow \Gamma(E')$ will also be open (resp. closed).

Definition 2.2.8. Ringed $\mathcal{D}$-topos is a couple $(E, A)$, where $E$ is a $\mathcal{D}$-topos and $A$ is a ring object in monoidal category $\Gamma(E)$.

Ring object $A$ is defined as ring $A_i$ in $E_i$ for every $i$ and for every morphism $m: i \rightarrow j$ we have morphism of rings $A_j \rightarrow m_*(A_i)$.

Definition 2.2.9. Let $(E, A)$ and $(E', A')$ be ringed $\mathcal{D}$-topoi. Pair $(\Phi, \theta)$, where $\Phi: E \longrightarrow E'$ is a morphism of $\mathcal{D}$-topoi and $\theta: A' \longrightarrow \Gamma(\Phi_*)(A)$ is homomorphism of rings called Morphism of ringed topos $(E, A)$ and $(E', A')$.

Proposition 2.2.10. Morphism of ringed $\mathcal{D}$-topoi $(\Phi, \theta)$ induces a morphism of total ringed topos $(\Gamma(\Phi), \theta): (\Gamma(E), A) \longrightarrow (\Gamma(E'), A')$.

Proof. See [AGV2] Exposé V bis, Remark 1.3.3.

With a ringed $\mathcal{D}$-topos $(E, A)$ we can associate category $\textbf{Mod}(E, A)$, bifibered in abelian categories over a category $\mathcal{D}$, whose fiber over $i \in \mathcal{D}$ given by a category of modules $\text{Mod}(E_i, A_i)$ over ringed topos $(E_i, A_i)$. We denote abelian category of modules over total topos $\Gamma(E)$ by $\text{Mod}(\Gamma(E), A)$. This category is identified with a category $\textbf{Hom}_{\mathcal{D}}(\text{Mod}(E, A))$ of sections of an abelian bifibration $\text{Mod}(E, A)$.

Example 2.2.11. Let $(S, \mathcal{O}_S)$ be ringed topoi. We can associate with it ringed $\mathcal{D}$-topos $(S \times \mathcal{D}, \mathcal{O}_S)$ called constant ringed $\mathcal{D}$-topos. By definition $\text{Mod}(\Gamma(S \times \mathcal{D}), \mathcal{O}_S)$ is equivalent to a category of functors from $\mathcal{D}$ to $\text{Mod}(S \times \mathcal{D}, \mathcal{O}_S)$.

Morphism $(\Phi, \theta)$ of ringed $\mathcal{D}$-topoi $(E, A)$ and $(E', A')$ induces additive functors $\varphi_*: \text{Mod}(E, A) \longrightarrow \text{Mod}(E', A')$ and $\varphi^*: \text{Mod}(E', A') \longrightarrow \text{Mod}(E, A)$, called direct and inverse image functors. When functor $\varphi^*$ is exact we call morphism $(\Phi, \theta)$ flat.

Example 2.2.12. (i) For a ringed $\mathcal{D}$-topoi $(E, A)$ an augmentation is a map of $\mathcal{D}$-topoi $\theta: E \longrightarrow S \times \mathcal{D}$. Let $(S \times \mathcal{D}, \mathcal{O}_S)$ be a constant ringed $\mathcal{D}$-topos, we have exact functor $\epsilon^*: \text{Mod}(S, \mathcal{O}_S) \longrightarrow \text{Mod}(\Gamma(S \times \mathcal{D}), \mathcal{O}_S)$, which is defined by attaching to an object of $\text{Mod}(S, \mathcal{O}_S)$ a constant functor. Functor $\epsilon^*$ has a right adjoint denoted by $\epsilon_*$. Functor $\epsilon_*$ associates to an element of $\text{Mod}(\Gamma(S \times \mathcal{D}), \mathcal{O}_S)$ its limit over category $\mathcal{D}$. We have functors:

\[(8) \quad \tilde{\theta}^* := \Gamma(\theta^*) \circ \epsilon^*: \text{Mod}(S, \mathcal{O}_S) \longrightarrow \text{Mod}(\Gamma(E), A)\]

and

\[(9) \quad \tilde{\theta}_* := \epsilon_* \circ \Gamma(\theta_*) : \text{Mod}(\Gamma(E), A) \longrightarrow \text{Mod}(S, \mathcal{O}_S)\]
Image of functor $\theta^*$ lies in subcategory $\Gamma^{cocart}(D, \text{Mod}(E, A))$.

**Proposition 2.2.13.** Let $f : D' \to D$ be a functor and $(E, A)$ be a ringed $D$-topos. Restriction functor
\[
(10) \quad f^*: \text{Mod}(\Gamma(E), A) \to \text{Mod}(\Gamma(E \times_D D'), A \circ f)
\]
has a right and left adjoint functors denoted by $f_*$ and by $f^!$, respectively.

*Proof.* See [AGV2, Exposé Vbis, Proposition 1.3.7].

**Remark 2.2.14.** In the same spirit one can consider the same construction for bifibrations, where fibers are complete and cocomplete.

For the ringed topos $(T, \mathcal{O}_T)$ with stratification $S$ we denote by $D^c(T, \mathcal{O}_T)$ triangulated subcategory of $D(T, A)$, consisting of complexes $K$ such that $H^n(K^q) \in \text{Mod}^c(T, A)$ for all $n$. This category has natural $t$-structure, whose heart is equivalent to $\text{Mod}_c(T, A)$. We have natural triangulated comparison functor $D(\text{Mod}_c(T, A)) \to D^c(T, A)$.

**Definition 2.2.15.** Let $(E, A)$ be ringed the $D$-topos, a stratification $S$ on $(E, A)$ is a stratification $S_i$ on $E_i$ for every $i \in D$, such that for every $m : i \to j \in D$ functors
\[
(11) \quad f^*: \text{Mod}(E_i, A) \xleftarrow{\sim} \text{Mod}(E_j, A) : f_*
\]
where $f_* := m^*$ and $f^* := m^*$ preserves constructible subcategories.

Object $K \in \text{Mod}(\Gamma(E), A)$ is called constructible with respect to $S$ if for every $i \in D$ object $K_i$ is constructible. For a ringed $D$-topos $(E, A)$ we have Serre subcategory $\text{Mod}_c(E, A)$ of constructible objects of topos $\text{Mod}(\Gamma(E), A)$. By $D_c(\Gamma(E), A)$, we denote triangulated subcategory of $D(\Gamma(E), A)$, consisting of objects $K$, such that for every $i \in D$ complex $K_i$ has constructible cohomology. This category has obvious $t$-structure, whose heart is equivalent to $\text{Mod}_c(\Gamma(E), A)$. We also have triangulated subcategories $D^{cocart,c}(\Gamma(E), A)$ and $D^{cart,c}(\Gamma(E), A)$.

3. Cross functors and Verdier duality

3.1. Notations. Let $D$ and $C$ be a pair of 2-categories, we can consider the 2-category of 2-functors (pseudo-functors) between $D$ and $C$. When $C$ is the 2-category of small categories $\text{Cat}$, and $D$ is an ordinary category we have an equivalence between the 2-category of 2-functors $\text{Hom}(D^\circ, \text{Cat})$ and the 2-category of fibrations over $D$. This equivalence is given by the Grothendieck construction [Gro1]. We always denote fibration and corresponding pseudo-functor by the same symbol. Given a pseudo-functor $\Psi : D^\circ \to \text{Cat}$ we can consider it is colimit over category $D^\circ$. We have equivalence of categories:
\[
(12) \quad \text{colim}_{D^\circ} \Psi \xrightarrow{\sim} \Psi[S^{-1}],
\]
where $\Psi[S^{-1}]$ is the localization of total category of a fibration $\Psi$, with respect to the class $S$ of cocartesian morphisms [AGV2, Exposé VI, Section 6]. We can also consider limit of pseudo-functor $\Psi$, over category $D^\circ$. We have following equivalence:
\[
(13) \quad \text{lim}_{D^\circ} \Psi \xrightarrow{\sim} \Gamma(\text{cart}(\Psi)),
\]
where $\Gamma_{\text{cart}}(\Psi)$ is the category of cartesian sections of corresponding fibration. If fibers of fibration are cocomplete and for every morphism $m : i \to j \in D$ functor $m^* : \Psi_j \to \Psi_j$ commutes with colimits we have following canonical morphism:

$$\text{can}, \text{can}_* : \colim D \to \lim D$$

Note, that if we have a bifibration $\Phi := (\Psi, \Theta)$, where $\Psi$ is a fibration and $\Theta$ is cofibration over $D$, such that for every $i \in D$ fibers of $\Phi$ are cocomplete, then we have following equivalence:

$$\colim D \sim \lim D$$

With every 2-category $C$ one can associate 2-category $C^{1-\circ}$ with reversed 1-morphism, 2-category $C^{2-\circ}$ with reversed 2-morphism and 2-category $C^{12-\circ}$ with reversed 1-morphisms and 2-morphisms.

3.2. Cross functors. Here we are going to recollect some facts about cross functors, following Deligne and Voevodsky [Del]. Let $D$ be a category with the class $B$ of commutative squares in $D$ and $C$ is the 2-category.

**Definition 3.2.1.** Pair $(H_*, H_!)$, where $H_*$ and $H_!$ are 2-functors from $D$ to $C$, such that $H_*(X) = H_!(x) := H(x)$ for every $x \in D$, is called lower e-functor if for each square $Q$ in $B$

![Diagram](https://example.com/diagram.png)

we have 2-morphism:

$$e_Q : p_! f_* \to h_* g_*$$

satisfying following axioms:

(i) Compatibility with vertical and horizontal compositions. For vertical it means that following solid is commutative:

(ii) If both vertical or both horizontal morphisms are identities then the exchange morphism is an isomorphism.
Definition 3.2.2. Pair \((H^{\ast}, H^{!})\), where \(H^{\ast}\) and \(H^{!}\) are 2-functors from \(D^{\circ}\) to \(C\), is called upper \(e\)-functor, if \((H^{\ast}, H^{!})\), is a lower \(e\)-functor with values in \(C^{1-\circ}\). Square \(Q\) gives 2-morphism:

\[
e_{Q} : f^{\ast} p^{!} \to g^{!} h^{\ast}
\]

Definition 3.2.3. Pair \((H^{!}, H^{\ast})\), where \(H^{!}\) is functor from \(D\) to \(C\) and \(H^{\ast}\) is 2-functors from \(D^{\circ}\) to \(C\), such that \(H^{!}(X) = H^{\ast}(x) := H(x)\) for every \(x \in D\), is called \(e^{\ast}\)-contradirectional functors if for every \(Q\) we have a morphism:

\[
e_{Q} : p^{!} h^{\ast} \to g f^{\ast}
\]

satisfying same axioms as previous functors.

Definition 3.2.4. Pair \((H_{\ast}, H^{!})\), where \(H_{\ast}\) is functor from \(D\) to \(C\) and \(H^{!}\) is 2-functors from \(D^{\circ}\) to \(C\), such that \(H_{\ast}(x) = H^{!}(x) := H(x)\) for every \(x \in D\), is called \(e^{!}\)-contradirectional functors if for every \(Q\) we have a morphism:

\[
e_{Q} : g f^{!} \to h^{!} p_{\ast}
\]

satisfying same axioms as \(e^{\ast}\)-contradirectional functors.

Now we can give main definition from [Del]:

Definition 3.2.5. A cross-functor from \(D\) to \(C\), relative to class \(B\), is the following collection of data:

(i) an upper \(e\)-functor \((H^{\ast}, H^{!})\), and a lower \(e\)-functor \((H_{\ast}, H^{!})\), from \(D\) to \(C\), which are equal on objects, \(H_{\ast}(x) = H^{\ast}(x) = H^{!}(x) = H^{\ast}(x)\), for every \(x \in D\). For a morphism \(f : x \to y\) in \(D\) we will write \(f_{\ast}\) for \(H_{\ast}(f)\), \(f^{\ast}\) for \(H^{\ast}(f)\), \(f_{!}\) for \(H^{!}(f)\), and \(f^{!}\) for \(H^{!}(f)\),

(ii) For every \(f : x \to y\) we have standard adjunction between \(f_{\ast}\) and \(f^{\ast}\) and \(f_{!}\) and \(f^{!}\), i.e. \(f_{\ast}\) is right adjoint to \(f^{\ast}\) and \(f_{!}\) is right adjoint to \(f^{!}\).

Following axioms should hold

(a) Compatibility of adjunction with composition

(b) Given a square in \(B\)

\[
\begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\downarrow f & & \downarrow g \\
\bullet & \rightarrow & \bullet \\
\downarrow h & & \downarrow p
\end{array}
\]

The corresponding exchange morphisms:

\[
p_{!} \circ f_{\ast} \to h_{\ast} \circ g_{!}, \quad f^{\ast} \circ p^{!} \to g^{!} \circ h^{\ast}
\]

transmute by into morphism:

\[
p^{\ast} \circ h_{!} \to g_{!} \circ f^{\ast}, \quad g_{!} \circ f^{\ast} \to h^{\ast} \circ p_{!}
\]
3.3. Monoidal categories. Here we are going to recall some facts about morphism of monoidal categories following [FHM]. Let $M$ and $M'$ be a pair of closed symmetric monoidal categories and let $f^*: M' \rightarrow M$ be a strong monoidal functor with right adjoint $f_*: M \rightarrow M'$. We denote counit and unit of adjunction by $\epsilon: f^* \circ f_* F \rightarrow F$ and $\eta: F \rightarrow f_* \circ f^* F$. We have following evident

**Proposition 3.3.1.** Adjuncts of natural morphisms

\[(22)\quad f^* \circ (f_* F \otimes f_* G) \cong f^* \circ f_* F \otimes f^* \circ f_* G \xrightarrow{\epsilon \otimes \epsilon} F \otimes G \xrightarrow{\mathcal{U}_{M'}} \mathcal{U}_{M'}
\]

make $f_*$ into lax monoidal functor.

We have natural morphism:

\[(23)\quad f^* \circ \text{Hom}(F, G) \otimes f^* \mathcal{K} \cong f^* \circ (\text{Hom}(F, G) \otimes \mathcal{K}) \xrightarrow{f^* \mathcal{ev}} f^* G
\]

with corresponding adjoint morphism

\[(24)\quad \alpha: f^* \circ \text{Hom}(F, G) \rightarrow \text{Hom}(f^* F, f^* G)
\]

We also have obvious

**Proposition 3.3.2.** Adjunct of morphism

\[(25)\quad f^* \circ \text{Hom}(F, f_* G) \xrightarrow{f_* \text{Hom}(\epsilon, id)} f_* \circ \text{Hom}(f^* F, f_* G) \xrightarrow{\text{Hom}(id, \epsilon)} \text{Hom}(F, f_* G)
\]

is natural isomorphism:

\[(26)\quad \text{Hom}(F, f_* G) \cong f_* \text{Hom}(f^* F, G)
\]

We can also define another morphism:

\[(27)\quad \beta: f_* \circ \text{Hom}(F, G) \xrightarrow{f_* \text{Hom}(\epsilon, id)} f_* \circ \text{Hom}(f^* F, f_* G) \cong \text{Hom}(f_*, F, f_* G)
\]

**Definition 3.3.3.** Let $M$ and $M'$ pair of closed monoidal categories with adjoint functors $(f_*, f^*)$. If natural morphism

\[(28)\quad \pi: F \otimes f_* G \rightarrow f_* \circ f^* F \otimes f_* G \rightarrow f_* \circ (f^* F \otimes G)
\]

is an isomorphism we say that projection formula holds.

Assume that in addition to adjoint functors $(f_*, f^*)$ between closed monoidal categories $M$ and $M'$ we have another pair of adjoint functors $f_1: M \leftrightarrow M': f_1^*$, where $f_1^*$ is right adjoint to $f_1$. We denote counit and unit of adjunction by $\sigma: f_1 \circ f_1^* F \rightarrow F$ and $\xi: F \rightarrow f_1^* \circ f_1 F$. We have evident

**Proposition 3.3.4.** Suppose that we have isomorphism $f^* \cong f_1^*$, then adjunct of natural morphisms

\[(29)\quad F \otimes G \xrightarrow{\xi \otimes \xi} f_1^* \circ f_1 F \otimes f_1^* \circ f_1 G \cong f^* \circ (f_1 F \otimes f_1 G) \xrightarrow{f_1 \mathcal{U}_M \circ \mathcal{U}_{M'}}
\]

make $f_1$ into op-lax monoidal functor.

Consider following natural morphisms:

\[(30)\quad \gamma: f_* \circ \text{Hom}(F, f_1^* G) \rightarrow \text{Hom}(f_* F, G)
\]

\[(31)\quad \delta: \text{Hom}(f^* F, f_1^* G) \rightarrow f_1^* \circ \text{Hom}(f_* F, G)
\]

\[(32)\quad \hat{\pi}: F \otimes f_1 G \rightarrow f_1 \circ (f^* F \otimes G)
\]
Proposition 3.3.5. Suppose that we are given one of the maps \( \hat{\pi}, \gamma \) or \( \delta \) then it determines all others. Moreover if one of these maps is natural isomorphism then so the others too.

Proof. See [FHM, Proposition 2.4].

We can also consider maps (30), (31) and (32) with reverse directions:

\[
\bar{\gamma} : \text{Hom}(f^*F, G) \longrightarrow f_* \circ \text{Hom}(F, f^!G)
\]
\[
\bar{\delta} : f^! \circ \text{Hom}(F, G) \longrightarrow \text{Hom}(f^*F, f^!G)
\]
\[
\bar{\pi} : f^! \circ (f^*F \otimes G) \longrightarrow F \otimes f!G
\]

Proposition 3.3.6. Suppose that we are given one of the maps \( \bar{\pi}, \bar{\gamma} \) or \( \bar{\delta} \) then it determines all others. Moreover if one of these maps is natural isomorphism then so the others too.

Proof. See [FHM, Proposition 2.8].

Proposition 3.3.7. Suppose that we have natural isomorphism \( f^! \cong f_* \). Then we can take \( \hat{\pi} \) to be \( \pi \) and \( \gamma \) is a natural morphism

\[
f_* \circ \text{Hom}(\mathcal{F}, f^!G) \xrightarrow{\delta} \text{Hom}(f_*F, f_* \circ f^!G) \xrightarrow{\text{Hom(id, } \sigma)} \text{Hom}(f_*F, G)
\]

Morphism \( \delta \) is adjunct of following natural morphism:

\[
f_* \circ \text{Hom}(f^*F, f^!G) \cong \text{Hom}(f_*F, f_*f^!G) \xrightarrow{\text{Hom(id, } \sigma)} \text{Hom}(\mathcal{F}, G)
\]

Proof. See [FHM, Proposition 2.9].

Definition 3.3.8. Grothendieck context is a pair of closed symmetric monoidal categories \( M \) and \( N \) and a triple of functors \( (f^*, f_*, f^!) \), \( f_* : M \to N : f^* \), where functor \( f^* \) is left adjoint to \( f_* \) and \( f^! \) is right adjoint to \( f_* \). We also assume that \( f^* \) is a strong monoidal functor and projection formula holds.

Definition 3.3.9. For an object \( \mathcal{K} \in M \) we define object \( \mathcal{D}_\mathcal{K}F := \text{Hom}(\mathcal{E}, \mathcal{K}) \), the \( \mathcal{K} \)-twisted dual of \( \mathcal{F} \). We call \( \mathcal{F} \) a \( \mathcal{K} \)-reflexive if we have isomorphisms \( \delta \cong \mathcal{D}_\mathcal{K}D_\mathcal{K} \).

Remark 3.3.10. Note then if \( \mathcal{F} \) is dualizable then \( D_{\mathcal{K}}F = D_{\mathcal{U}}F \otimes \mathcal{K} \).

Let \( f : M \to N \) be a Grothendieck context, if we set \( f^*\mathcal{G} = \mathcal{K} \) for an object \( \mathcal{G} \in N \), then from isomorphisms \( \delta \) and \( \gamma \) we get:

\[
f_*D_{\mathcal{K}}F \cong D_{\mathcal{G}}f_*F \quad D_{\mathcal{K}}f^*\mathcal{Y} \cong f^!D_{\mathcal{G}}\mathcal{Y}
\]

Definition 3.3.11. A dualizing object for a full subcategory \( M_0 \) of \( M \) is an object \( \mathcal{K} \in M_0 \) such that if \( \mathcal{F} \in M_0 \), then \( D_{\mathcal{K}}\mathcal{F} \) is in \( M_0 \) and \( \mathcal{F} \) is \( \mathcal{K} \)-reflexive. Thus \( D_{\mathcal{K}} \) defines an auto-duality of the category \( M_0 \):

\[
D_{\mathcal{K}} : M_0 \xrightarrow{\sim} M_0
\]
3.4. Grothendieck context. We assume that in category $D$ colimits over diagram $x \leftarrow z \rightarrow y$ exits for all $x, y$ and $z$ in $D$. For the class $B$ of commutative squares in $D$ we take class of pushouts in category $D$. We assume further that category $D^\circ$ satisfies following generalized filtered condition:

**Assumption 1.** For every $x$ and $y$ in $D^\circ$ we have object $z \in D^\circ$ and arrows $x \rightarrow z \leftarrow y$, for every pair of parallel morphisms $x \Rightarrow y$ we have an object $w \in D^\circ$, such that $x \rightarrow w \leftarrow y$, and $\sigma \circ j \circ u = j \circ u$, where $\sigma$ is an automorphism of $w$.

Let $(E, A)$ be ringed $D$-topos we give following:

**Definition 3.4.1.** A Grothendieck cross functor for the $D$-topos $(E, A)$ is a cross functor $(H^\star, H^{\bullet}, H^!, H^\ast)$, such that bifibration which corresponds to 2-functors $(H^\star, H^{\bullet})$ is isomorphic to bifibration $\text{Mod}(E, A) \rightarrow D$ and moreover we have isomorphism of $!$-lower e-functor and $\star$-lower e-functor:

\[
H^! \simeq H^\ast
\]

**Remark 3.4.2.** We specified to the setting of Grothendieck cross functors for several reasons. In order to make our definitions reasonable in general case we need to work in the setting of categories with enhancement. However, this is not the most important issue. In general, contravariant Verdier duality can not be expressed in terms of Verdier duality functors, which act between fibers, as it remarked in [Gai]. Also definition of dualizing object becomes much more involved [PHM].

We have triangulated categories of sections $D(\Gamma(H^*))$ and $D(\Gamma(H^!))$, with full triangulated subcategories $D_{\text{cocart}}(\Gamma(H^*))$ and $D_{\text{cart}}(\Gamma(H^!))$, corresponding inclusion functor:

\[
\Xi^\star: D_{\text{cocart}}(\Gamma(H^*)) \rightarrow D(\Gamma(H^*))
\]

and corresponding inclusion functor in the $!$-case:

\[
\Xi^!: D_{\text{cart}}(\Gamma(H^!)) \rightarrow D(\Gamma(H^!))
\]

**Definition 3.4.3.** Let $(H^t, H_*)$ be a pair of 2-functor from 2-category cospan($D$) to Cat, such that we have isomorphisms:

\[
i^*(H^t) \sim \sim H_t, \quad i^*(H_*) \sim \sim H^t, \quad j^*(H^t) \sim \sim H^*, \quad j^*(H_*) \sim \sim H_.*
\]

Pseudo functor $H^t$ is called Mackey $\star$-functor associated with Grothendieck cross functor and pseudo functor $H_*$ is called Mackey $!$-functor.

**Remark 3.4.4.** One can develop our theory for cross functors with values in more general categories, see Subsection 3.7.

**Proposition 3.4.5.** For every Grothendieck cross functor $(H_*, H^t, H_t, H^!)$, associated with $D$-topos $(E, A)$ there exists unique Mackey pseudo-functors $(H^t, H_*)$.

**Proof.** See proofs of Proposition 3.4.2.

\[
\square
\]

**Definition 3.4.6.** Let $(H_*, H^t, H_t, H^!)$ be a Grothendieck cross functor associated with ringed $D$-topos $(E, A)$. We define pair of adjoint triangulated functors by the rule $\nabla_{\ast \rightarrow !} := i^* \circ R(j_*)$ and $\nabla_{! \rightarrow \ast} := j^* \circ L(i!)$:

\[
\nabla_{\ast \rightarrow !}: D(\Gamma(H^*)) \leftrightarrow D(\Gamma(H^!)): \nabla_{! \rightarrow \ast}.
\]

These functors are called covariant Verdier duality.
**Remark 3.4.7.** Verdier duality functors are well defined as derived functors. Indeed functor $\mathcal{V}_{\rightarrow}$ is well defined: filtered colimits in modules over topos are exact. Functor $\mathcal{V}_{\leftarrow}$ is well defined, since for every morphism $m: i \rightarrow j$ functor $f_\ast := m^\ast$ takes injective objects to injective and thus $f^!$ takes injective to injective.

Sometimes we denote morphisms in $\text{cospan}(D)$ by $h := (h_l, k, h_r): i \rightarrow j$. For morphism $h$ corresponding functor between fibers will be denoted by $h^{\ast}$ in the case of $!$-Mackey functor and by $h^\ast$ in the case of $*$-Mackey functor.

**Proposition 3.4.8.** Let $(H_\ast, H^\ast, H_l, H_r)$ be a Grothendieck cross functor, such that for every $m: i \rightarrow j$ in $D$, morphism $(m_\ast, m^\ast)$ is closed, then covariant Verdier duality functor $\mathcal{V}_{\leftarrow}$ (resp. $\mathcal{V}_{\rightarrow}$) takes values in subcategories with cartesian (resp. cocartesian) cohomology:

$$(45) \quad \mathcal{V}_{\leftarrow}: \text{D}_{\text{cocart}}(\Upsilon(H^\ast)) \leftrightarrow \text{D}_{\text{cart}}(\Upsilon(H^\ast)): \mathcal{V}_{\rightarrow}.$$

**Proof.** Let us consider case of functor $\mathcal{V}_{\rightarrow}$, case of functor $\mathcal{V}_{\leftarrow}$ can be treated analogically. Let $m^\circ: i \rightarrow j$ be a morphism in $D^\circ$ and $\mathcal{K} \in \text{D}_{\text{cocart}}(\Upsilon(E), A)$, then by definition we have the morphism:

$$(46) \quad \text{holim}_{p: i \rightarrow w} R(p^\}_w) \mathcal{K}_w \rightarrow R^\ast(f^\}_w) \circ \text{holim}_{p: j \rightarrow w} R(p^\}_w) \mathcal{K}_w,$$

Which is induced by morphism of corresponding abelian functors. Thus it is enough to show that diagram, underlying first limit is contained in diagram, underlying the second one. Let $h = (h_l, v, h_r): i \rightarrow x$ be any morphism in $\text{cospan}(D)$, then we also have morphism $h' = (h'_l, v, h'_r): j \rightarrow x$, with property $h'_l = m^\circ \circ h_l$. Then we have adjunction morphism $h^\ast : h_l \rightarrow m^\circ \circ m^\circ \circ h^\ast$, which is isomorphism, since $m^\circ = f_\ast$ is fully faithful. $\square$

**Definition 3.4.9.** Let $H^\ast_\ast$ be a pseudo functor from $\text{cospan}(D)$ to $\text{Cat}$ such that:

$$(47) \quad j^\ast(E^\ast_\ast) \xrightarrow{\sim} H^\circ_\circ, \quad i^\ast(E^\ast_\ast) \xrightarrow{\sim} H^\ast.$$

We call such pseudo functor Mackey $\ast\ast$-functor associated with Grothendieck cross functor $(H_\ast, H^\ast, H_l, H_r)$.

Assume, that for every morphism $m: i \rightarrow j$ in category the $D$, functor $m_\ast = f^\ast: \text{Mod}(E_i, A_i) \rightarrow \text{Mod}(E_j, A_j)$ commutes with limits and flat. We define functor

$$(48) \quad \Xi_\ast : D(\Upsilon(H^\ast)) \rightarrow D(\Upsilon(H^\ast)),$$

by the rule $\Xi_\ast = j^\ast \circ R(j_\ast)$.

**Definition 3.4.10.** Let $H^\ast_!$ be a pseudo-functor from category cospan($D$) such that:

$$(49) \quad i^\ast(H^\ast_!) \xrightarrow{\sim} H^\circ_!, \quad j(H^\ast_!) \xrightarrow{\sim} E^\!.$$

We call such pseudo functor Mackey $\ast!$-functor associated with Grothendieck cross functor $(H_\ast, H^\ast, H_l, H_r)$.

Assume, that for every morphism $m: i \rightarrow j$ in the category $D^\circ$, functor $m^\ast = f^\ast: \text{Mod}(E_i, A_i) \rightarrow \text{Mod}(E_j, A_j)$ commutes with colimits. We define functor

$$(50) \quad \Xi_! : D(\Upsilon(H^\ast)) \rightarrow D(\Upsilon(H^\ast)),$$

by the rule $\Xi_! = i^\ast \circ L(i_\ast)$. 
Remark 3.4.11. We need to mention what we understand by functor $\Xi$. Usually functor $f^!$ is only left exact and since then functor $g_\ast \circ f^!$ is also left exact and fibrations $H^!$ and $H^\circ$ are not abelian. Using some enhancement (For example see [BDJ, Subsection 7.4.], [DG, Subsection 1.6.], [Lur2]) we can define derived version of this fibrations and $i^! \circ L(i)$ are just composition of homotopy Kan extension and restriction functor. In the same way one can remove flatness restriction in the definition of functor $\Xi_\ast$. In some situations it is also useful to consider functors from opposite fibrations:

\begin{equation}
\Psi_\ast : D(\Gamma(H^\circ)) \to D_{\text{cocart}}(\Gamma(H^*))
\end{equation}

and

\begin{equation}
\Psi_! : D(\Gamma(H^\circ)) \to D_{\text{cart}}(\Gamma(H^!)).
\end{equation}

Proposition 3.4.12. For every Grothendieck cross functor $(H_\ast, H^!, H^\circ, H^\circ!)$ over $D$-topos $(E, A)$ there exist a unique pair of Mackey pseudo-functors $(H_\ast^{\circ\circ}, H^\circ!!)$.

Proof. See proof of Proposition 3.4.8. □

Proposition 3.4.13. Let $(H_\ast, H^*, H^\circ, H^\circ!)$ be a Grothendieck cross functor over ringed $D$-topos $(E, A)$. Functor $\Xi_\ast$ takes values in subcategory $D_{\text{cocart}}(\Gamma(H^*))$.

Moreover if for every $i \in D$ limits over coslice category $\text{I/cospan}(D)$ are exact, then triangulated subcategory:

\begin{equation}
\Xi^* : D_{\text{cocart}}(\Gamma(H^*)) \to D(\Gamma(H^*))
\end{equation}

is a right admissible with a right adjoint functor $\Xi_\ast$. Triangulated subcategory:

\begin{equation}
\Xi^! : D_{\text{cart}}(\Gamma(H^!)) \to D(\Gamma(H^!))
\end{equation}

is left admissible with left adjoint functor $\Xi_!$.

Proof. Analogical to Proposition 3.4.8. Adjunction follows from assumption on category $D^\circ$. □

Remark 3.4.14. Note that if ring object $A$ is cocartesian we have isomorphism of functors $R_0(\Xi_\ast) \cong \Xi_\ast$.

Remark 3.4.15. Denote by $\text{Sp}$ category of spectra, by definition it is stabilization of category of pointed topological spaces $\text{Top}_\ast$. That is category $\text{Sp}$ is the category of sections of bifibration over $\mathbb{N}$, whose fiber is given by category $\text{Top}_\ast$ and functors between fibers are loop space object $\Omega$ and suspension functor $\Sigma$. Category $\text{Sp}$ has full subcategory $\Omega\text{Sp}$, of $\Omega$-spectra that is subcategory of $\Omega$-cartesian sections. Functors $\Xi_\ast$ and $\Xi_!$ are analogous to functor which takes category of spectra $\text{Sp}$ to the category of $\Omega$-spectra $\Omega\text{Sp}$ [BF].

\begin{equation}
Q : \text{Sp} \to \Omega\text{Sp}, \quad Q(X)^n := \hocolim_{i \in \mathbb{N}} \Omega^i X^{n+i}.
\end{equation}

Definition 3.4.16. Let $(H_\ast, H^*, H^\circ, H^\circ!)$ be a Grothendieck cross functor associated with ringed $D$-topos $(E, A)$. We define covariant Verdier duality functors, which respects categories $D_{\text{cocart}}(\Gamma(H^*))$ and $D_{\text{cart}}(\Gamma(H^!))$:

\begin{equation}
\forall_{\ast !}^\text{cart} : D_{\text{cocart}}(\Gamma(H^*)) \leftrightarrow D_{\text{cart}}(\Gamma(H^!)): \forall_{|_{\ast \to \ast}}^\text{cocart}
\end{equation}

by the rules:

\begin{equation}
\forall_{\ast !}^\text{cart} := \Xi_! \circ \forall_{\ast \to \ast}, \quad \forall_{|_{\ast \to \ast}}^\text{cocart} := \Xi_\ast \circ \forall_{|_{\ast \to \ast}}.
\end{equation}
Grothendieck cross functor \((H_\star, H^\star, H_!, H^!\)) is called constructible if we have stratification \(S_i\) on \(E_i\) for every \(i\), such that \((E, A)\) is a stratified D topoi and functors \(\Xi_\star\) and \(\Xi_!\) and covariant Verder duality functors respect constructible subcategories:

\[
\forall \text{cart} : \text{D}(\Gamma^\star(H^\star)) \leftrightarrow \text{D}(\Gamma(H!)) : \forall \text{cocart}
\]

**Definition 3.4.17.** Let \((H_\star, H^\star, H_!, H^!\)) be a Grothendieck cross functor\(^2\), such that for every \(m : i \to j \in \text{D}\) morphism \(f = (f_\star, f^\star, f! : \text{D}(E_i, A_i) \to \text{D}(E_j, A_j))\), is a Grothendieck context and we have an object \(w_D \in \text{D}(\Gamma^!(E^!, A))\), then we call \((H_\star, H^\star, H_!, H^!, w_D)\) Grothendieck cross functor with dualizing object \(w_D\).

Grothendieck cross functor with dualizing object \((H_\star, H^\star, H_!, H^!, w_D)\) is called constructible if \((H_\star, H^\star, H_!, H^!, w_D)\) is a constructible Grothendieck cross functor and objects \(w_i\) are dualizing objects for the category \(\text{D}(\Gamma^!(E^!, A))\), where \(i \in \text{D}\).

**Definition 3.4.18.** Let \((H_\star, H^\star, H_!, H^!, w_D)\) be a Grothendieck cross functor with dualizing object, then we have triangulated functors called **duality**

\[
\text{D}_{! \to \star} : \text{D}(\Gamma^!(H^!)) \leftrightarrow \text{D}(\Gamma(H^*))^\circ : \text{D}_{\star \to !},
\]

defined for every \(i \in \text{D}\) by duality functors \(\text{D}_{w_i}\).

**Proposition 3.4.19.** Duality functor \(\text{D}_{! \to \star}\) respects subcategories with cocartesian and cartesian cohomology:

\[
\text{D}_{! \to \star} : \text{D}(\Gamma^!(H^!)) \leftrightarrow \text{D}(\Gamma(H^*))^\circ,
\]

moreover if Grothendieck cross functor was constructible then both duality functors \(\text{D}_{! \to \star}\) respects subcategories with cocartesian and cartesian cohomology:

\[
\text{D}_{! \to \star} : \text{D}(\Gamma^!(H^!)) \leftrightarrow \text{D}(\Gamma(H^*))^\circ : \text{D}_{\star \to !},
\]

and induce mutual inverse equivalence.

**Proof.** Statement obviously follows from isomorphisms (38). \(\square\)

**Corollary 3.4.20.** Let \((H_\star, H^\star, H_!, H^!, w_D)\) be a constructible Grothendieck cross functor with dualizing object. Then following diagram commute:

\[
\begin{align*}
\text{D}_c(\Gamma(H^!)) & \xrightarrow{\Xi!} \text{D}_{\text{cart}, c}(\Gamma(H!)) \\
\text{D}_{\text{cart}, c}(\Gamma(H^*)) & \xrightarrow{\Xi_*} \text{D}_c(\Gamma(H^!))^\circ
\end{align*}
\]

\[
\begin{align*}
\text{D}_c(\Gamma(H^!)) & \xrightarrow{\Xi!} \text{D}_{\text{cart}, c}(\Gamma(H!)) \\
\text{D}_{\text{cart}, c}(\Gamma(H^*)) & \xrightarrow{\Xi_*} \text{D}_c(\Gamma(H^!))^\circ
\end{align*}
\]

**Proof.** Follows from the fact that inner hom functor takes colimits to limits and isomorphisms (38). \(\square\)

\(^2\)What we really need here is pseudo-functors with values in category of monoidal categories
Proposition 3.4.21. Let \((H_\star, H^\star, H!, H^!, w_D)\) be a Grothendieck cross functor with dualizing object. Then following diagram commute:

\[
\begin{array}{ccc}
D_c(\Gamma(H^!)) & \xrightarrow{\forall_{l\rightarrow*}} & D_c(\Gamma(H^*)) \\
\downarrow & & \downarrow \\
D!(l\rightarrow*) & \xrightarrow{\forall_{*\rightarrow!}} & D*(l\rightarrow!)
\end{array}
\]

(63)

Proof. Follows from fact that inner hom functor takes colimits to limits and isomorphisms (38).

□

Remark 3.4.22. Suppose that we have constructible Grothendieck cross functor over \(D\). Using Corollary 3.4.20 we can define functor \(\Xi^\star\) by the rule \(\Xi^\star := D_D \circ \Xi \circ D_D\).

Definition 3.4.23. Let \((H_\star, H^\star, H!, H^!, w_D)\) be a constructible Grothendieck cross functor associated with ringed \(D\)-topos \((E, A)\). Then we have triangulated functors

\[
\begin{aligned}
\mathbb{D}_{(E, A)} : \mathbb{D}_{\text{c}}(\Gamma(E), A) & \rightarrow \mathbb{D}_{\text{c}}(\Gamma(E), A) \\
\mathbb{D}^r_{H!} : \mathbb{D}_{\text{c}}(\Gamma(H!)) & \rightarrow \mathbb{D}_{\text{c}}(\Gamma(H!))
\end{aligned}
\]

(64)

defined as \(\mathbb{D}_{(E, A)} := D_{l\rightarrow*} \circ \forall_{\text{c} \rightarrow*!}\), and triangulated functor

\[
\begin{aligned}
\mathbb{D}^r_{H!} : \mathbb{D}_{\text{c}}(\Gamma(H!)) & \rightarrow \mathbb{D}_{\text{c}}(\Gamma(H!)) \\
\mathbb{D}^r_{H!} & \circ \forall_{\text{r} \rightarrow*!}
\end{aligned}
\]

(65)

for Grothendieck cross functor with dualizing object \((H_\star, H^\star, H!, H^!, w_D)\).

Remark 3.4.24. It is also useful to consider so called full Verdier duality functor:

\[
\forall^\text{full}_{l\rightarrow*} : \lim_D H^i \longrightarrow \lim_D H^*
\]

(66)

This functor is defined as composition of canonical functor \(\text{can}\) and covariant Verdier duality \(\forall^\text{r} \rightarrow*!\).

3.5. \(t\)-structure. Let \((E, A)\) be the ringed \(D\)-topos with corresponding Grothendieck cross functor \((H_\star, H^\star, H!, H^!, w_D)\). Suppose that for every \(i \in D\) we have a \(t\)-structure \(D^i_{\leq 0}\) on category \(D(E_i, A_i)\), with corresponding truncation functors denoted by \(\sigma^i_{<0}\), such that for every \(m : i \rightarrow j \in D\) functor \(\hat{f}^i = m_{\rightarrow*}\) is left \(t\)-exact and functor \(f^* := m_{\rightarrow!}\) is right \(t\)-exact.

Lemma 3.5.1. Category \(D(\Gamma(H_i))\) is naturally \(t\)-category with corresponding \(t\)-structure:

\[
D^{\leq 0}(\Gamma(H_i)) := \{ X \in (\Gamma(H_i)) \mid X_i \in D_i^{\leq 0}\}
\]

(67)

Proof. It easy to see that \(D^{\leq 0}(\Gamma(H_i))[1] \subset D^{\leq 0}(\Gamma(H_i))\). To prove that above subcategory defines \(t\)-structure we need to introduce truncation functors. We define truncation functor \(\sigma^i_{<0}\) by componentwise truncation functors \(\sigma^i_{<0}\). This truncation functor is well defined, since for every \(m : i \rightarrow j\) functor \(f^i = m^*\) is left \(t\)-exact and functor \(f^* := m_{\rightarrow!}\) is right \(t\)-exact.

□

Suppose that for every \(i \in D\) category \(D^i_{\leq 0}\) (resp. \(D^i_{\geq 0}\)) contains homotopy limits (colimits) and homotopy limits and colimits are exact.
Definition 3.5.2. We introduce t-structures on categories $D_{\text{cocart}}(\Gamma(E), A)$ and $D_{\text{cart}}(\Gamma(H') )$ by following rules:

\begin{align*}
D_{\text{cocart}}^{\leq 0}(\Gamma(E), A) & := \{ K \in D_{\text{cocart}}(\Gamma(E), A) | \exists_i \in D_i^{\leq 0} \} \\
D_{\text{cart}}^{\geq 0}(\Gamma(H')) & := \{ K \in D_{\text{cart}}(\Gamma(H')) | \exists_i \in D_i^{\geq 0} \}
\end{align*}

with corresponding truncation functors:

\begin{align*}
\tau_{\leq 0} : D_{\text{cocart}}(\Gamma(E), A) & \rightarrow D_{\text{cocart}}^{\leq 0}(\Gamma(E), A), \\
\tau_{\geq 0} : D_{\text{cart}}(\Gamma(H')) & \rightarrow D_{\text{cart}}^{\geq 0}(\Gamma(H'))
\end{align*}

Proposition 3.5.3. Categories $D_{\text{cocart}}(\Gamma(E), A)$ and $D_{\text{cart}}(\Gamma(H'))$ with above data are indeed t-categories.

Proof. Let us stick to the case of category $D_{\text{cocart}}(\Gamma(E), A)$, another case can be proved analogically. It easy to see $D_{\text{cocart}}^{\leq 0}(\Gamma(E), A)[1] \subset D_{\text{cocart}}^{\leq 0}(\Gamma(E), A)$. The rest follows from adjunction property of functor $\Xi_\ast$ and Lemma 3.5.1 and exactness of limits.

Remark 3.5.4. Examples of above t-structures are given by t-structure on $\ast$-crystals and perverse t-structure on $!$-sheaves (Remark 4.3.4).

3.6. Grothendieck operations. Let $(E, A)$ be the the ringed D-topos with associated Grothendieck cross functor. Category $D_{\text{cocart}}(E, A)$ is the closed monoidal category with underlying tensor product:

\[ \otimes^\ast : D_{\text{cocart}}(E, A) \times D_{\text{cocart}}(E, A) \rightarrow D_{\text{cocart}}(E, A) \]

defined by the rule:

\[ (K \otimes^\ast F)_i := K_i \otimes^L F_i, \quad i \in D \]

and inner hom functor:

\[ \text{Hom}^\ast : D_{\text{cocart}}(E, A)^\circ \times D_{\text{cocart}}(E, A) \rightarrow D_{\text{cocart}}(E, A) \]

defined as composition:

\[ \text{Hom}^\ast := \Xi_\ast \circ \text{Hom}^\text{naive}, \quad \text{Hom}^\text{naive}(E, S)_i := \text{Hom}_{D(E_i, A_i)}(E_i, S_i) \]

Let $f : (\Gamma(E), A) \rightarrow (\Gamma(E'), A)$ be a morphism of topoi, then we have morphism between categories $D_{\text{cocart}}(E, A)$ and $D_{\text{cocart}}(E', A')$ given by pair of adjoint triangulated functors:

\[ f^\ast : D_{\text{cocart}}(E, A) \leftrightarrow D_{\text{cocart}}(E', A') : f^\ast \]

where functor $f^\ast$ is defined as composition $f^\ast = \Xi_\ast \circ R^\ast(f_\ast)$ and $f^\ast = L^\ast(f^\ast)$. We can also define $!$-operations:

\[ f_! : D_{\text{cocart}}(E, A) \leftrightarrow D_{\text{cocart}}(E', A') : f^! \]

By the rule:

\[ f_! := D_{(E', A')} \circ f_\ast \circ D_{(E, A)}, \quad f^! := D_{(E, A)} \circ f^\ast \circ D_{(E', A')} \].
3.7. **Digression: norm maps.** By *Picard groupoid* we understand symmetric monoidal category $\mathcal{P}$, with underlying tensor product $+_\mathcal{P}: \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P}$, such that $\mathcal{P}$ is the groupoids and for every object $a \in \mathcal{P}$ functor $x \mapsto a +_\mathcal{P} x$ defines autoequivalence of $\mathcal{P}$. Unit object will be denote by $0_\mathcal{P}$, it follows from axioms that for every object $x \in \mathcal{P}$, there exists unique inverse object $-x$. We say that Picard groupoid $\mathcal{P}$ is *strictly commutative*, if $\mathcal{P}$ is strictly commutative as monoidal category. We defined *homotopy groups of Picard groupoid* as follows: $\pi_0(\mathcal{P})$ is group of isomorphism classes of objects in $\mathcal{P}$ and $\pi_1 := \text{Aut}_\mathcal{P}(0)$.

By Pic we denote 2-category of Picard groupoids, that is the 2-category with objects Picard groupoid and 1-morphisms are given by tensor functors and 2-morphisms are given by natural transformations of monoidal functors. We have 2-subcategory of strict Picard groupoids, denoted by $\text{Pic}^{\text{strict}}$ and 2-subcategory of *discrete Picard groupoids* denoted by $\text{Pic}^{\text{disc}}$. Corresponding homotopy category of Picard groupoids will be denote by $\text{Pic}^\flat$. We have subcategories of strict and discrete Picard groupoids: $\text{Pic}^{\text{strict}}$ and $\text{Pic}^{\text{disc}}$ (these categories are defined as categories with objects given by strict Picard groupoids and morphisms are equivalence classes of monoidal functors).

Denote by $D[-1,0](\text{Ab})$ derived category of length 2 complexes. We have functor:

\[
\text{ch}: D[-1,0](\text{Ab}) \rightarrow \text{Pic}^{\text{strict}},
\]

which takes complex $A^{-1} \rightarrow A^0$ of abelian groups and associates with it strict Picard groupoid $\mathcal{P}$ generated by elements of $A^0$, $\text{Ob}(\mathcal{P}) := A^0$, with tensor structure $+_\mathcal{P}$ given by addition operation in $A^0$ and unit object given by zero in $A^0$. For every $c, b \in G$ set of morphisms defines as follows:

\[
\text{Hom}_\mathcal{P}(c, b) := \{ f \in A^{-1} \mid d(f) = b - c\}.
\]

We have following classical result:

**Lemma 3.7.1.** Functor $\text{ch}$ defines equivalence of categories.

**Proof.** See [AGV3, Exposé XVIII].

---

**Example 3.7.2.** Under above equivalence category abelian groups Ab i.e. subcategory of complexes which live in zero degree corresponds to subcategory of discrete Picard groupoids $\text{Pic}^{\text{disc}}$.

**Remark 3.7.3.** In [AGV3, Exposé XVIII] Deligne actually proves equivalence between 2-category Pic and 2-category $C[-1,0](\text{Ab})$,

Let $M$ be a *Mackey functor* $\text{Lin}$ from category $D$ to the category $\text{Ab}$ of abelian groups:

\[
M: \text{span}(D) \rightarrow \text{Ab}
\]

Category of all Mackey functors from the category $D$ to the category $\text{Ab}$ will be denoted by $\text{mk}_D(\text{Ab}) := \text{Hom}(\text{span}(D), \text{Ab})$.

Denote by $\text{mk}_D^\sharp$ category, whose objects are Mackey $\sharp$-pseudo-functors and morphism are equivalence classes of pseudo-natural transformations.

---

3What we call the Mackey functor in $\text{Lin}$ is called the $\mathcal{P}$-functor.
Proposition 3.7.4. Functor $\text{ch}$ induces functor:

\[
\text{ch} : \text{mk}_D(\text{Ab}) \to \text{mk}_D^{\star},
\]

which takes Mackey functor $M$ and associates to it Mackey $\star!$-pseudo-functors $H^{\star!}$ over $D^\circ$, with values in subcategory of discrete Picard groupoids (précamp de Picard).

\textbf{Proof.} See [AGV3, Exposé XVIII]. \hfill \Box

Denote by $\text{Rig}_{\text{cocart}}$ category with objects cocartesian rigid symmetric monoidal categories and equivalence classes of monoidal functor between them. We have functor:

\[
\text{grp} : \text{Rig}_{\text{cocart}} \to \text{Pic}^\flat,
\]

which takes underlying groupoid $C^\times$ of rigid symmetric monoidal category $C$. We have obvious:

\textbf{Proposition 3.7.5.} Functor $\text{grp}$ admits a left adjoint:

\[
\text{comp} : \text{Pic} \leftarrow \text{Rig}_{\text{cocart}}
\]

called completion functor.

Assume that category $D$ is finite and let $H^{\star!}$ be a Mackey functor, with values in discreet Picard groupoids. By $H^{\star!}_+$ we denote corresponding completion of Mackey functor. Since fibers of pseudo-functor $H^{\star!}_+$ have finite colimits we can apply categorical Verdier duality construction:

\[
\text{Verdier duality map}
\]

\textbf{Lemma 3.7.6.} Categories $\text{colim}_{D^\circ} j^* H^{\star!}_+$ and $\text{lim}_{D} i^* H^{\star!}_+$ are cocartesian rigid symmetric categories.

\textbf{Proof.} Tensor product can be defined fiberwise, since for every morphism $m$ in $D^\circ$ functor $m_!$ is the Picard functor. \hfill \Box

By previous Lemma Verdier duality functor $\text{V}^{\text{full}}_{\text{lim}_{D^\circ} j^*}$ induces morphism:

\[
\text{Verdier duality map}
\]

\textbf{Example 3.7.7.} Let $G$ be a finite group and $X = K(G, 1)$. We denote by $\Pi_1(X)$ fundamental groupoid of $X$. We have well known correspondence between category $\text{Rep}(G)$ of representations of $G$ and category of local systems of abelian groups on $X$ :

\[
\text{LC}(K(G, 1), \text{Ab}) \astrikerightarrow \text{Rep}(G)
\]

Let $M$ be a representation of $F$ and $L_M \in \text{LC}(K(G, 1))$ be the corresponding local system, then we can associate with it Mackey functor $i_*(L_M)$. Categorical Verdier duality construction supplies us with the morphism:

\[
\text{Verdier duality map}
\]
which is can be rewritten as:

\[ V_{\text{full}}^1 : M_G \rightarrow M^G, \quad v \mapsto \sum_{g \in G} gv \]

where space \( M_G \) is the space of coinvariants and \( M^G \) is the space of invariants of \( G \). Note, that in this example, Verdier duality \( V_{D,\text{full}}^1 \) is actually isomorphic to canonical functor can, since our base category \( G \) is groupoid.

It is natural try to extend above construction to spectral Mackey functors. To do it we need to switch our setting to the world of \( \infty \)-groupoids.

**Remark 3.7.8.** This remark does not contain rigorous statements or proofs and should be viewed as a sketch. We have stable category of spectra \( \mathcal{S}p \), which is monoidal category with smash product \( \wedge \) and unit given by sphere spectrum \( S \).

**Homotopy groups** of spectra \( X \) are defined as follows:

\[ \pi_i(X) := \colim_{i \in \mathbb{N}} \pi_{i+j}(X_j). \]

Denote by \( \mathcal{S}p^{\geq 0} \) subcategory of **connective spectra**. It is objects are given by spectra \( X \in \mathcal{S}p \), such that \( \pi_i(X) = 0 \), for all \( i < 0 \). Subcategory \( \mathcal{S}p^{\geq 0} \) with corresponding truncation functor:

\[ \tau_{\geq 0} : \mathcal{S}p \rightarrow \mathcal{S}p^{\geq 0}, \]

defines t-structure on \( \mathcal{S}p \), with heart given by category of abelian groups \( \mathbb{A}b \). We say that spectra \( X \in \mathcal{S}p^{\geq 0} \) has a **stable homotopy type** \( n \) if all homotopy groups vanishes:

\[ \pi_i(X) = 0 \quad i > n. \]

Recall following correspondence which is due to Grothendieck (see [Gro3, p.114] and [Dri]).

**Construction 1.** Functor \( \Pi_{\infty} \) takes topological space \( X \) to corresponding fundamental \( \infty \)-groupoid \( \Pi_{\infty}(X) \) and functor \( B \) takes \( \infty \)-groupoid \( \mathcal{G} \) to classifying space \( B\mathcal{G} \). This functors induces functors between categories of connective spectra and category Picard \( \infty \)-groupoids \( \infty \text{-Pic} \). Moreover these functors take stable n-types to Picard n-groupoids and vice versa. These functors are adjoint and induce equivalence of homotopy categories:

\[ \Pi_{\infty} : \mathcal{S}p^{\geq 0} \leftrightarrow \infty \text{-Pic} : B \]

**Remark 3.7.9.** This correspondence is a stable analog of Grothendieck homotopy hypothesis [Gro3].

Denote by \( C^{\leq 0}(\mathbb{A}b) \) category of cochain complexes of abelian groups, which live in non positive degree. And by \( \mathbb{A}b^\Delta := \text{Hom}(\Delta, \mathbb{A}b) \) we denoted category of simplicial abelian groups. We can consider \( C^{\leq 0}(\mathbb{A}b) \) as subcategory of \( \mathcal{S}p^{\geq 0} \) via following constructions:

**Proposition 3.7.10.** Normalization functor:

\[ \text{Norm} : \mathbb{A}b^\Delta \rightarrow C^{\leq 0}(\mathbb{A}b) \]

induces equivalence of homotopy categories.
This construction is called *Dold-Kan equivalence*. For any simplicial abelian group $A$ one can associate another abelian simplicial group $B_A$, called *classifying group of $A$*. Thus, to a simplicial abelian group $A$ we can associate spectra by following rule:

$$BA_n := \Omega B \circ \cdots \circ B A_n$$

**Corollary 3.7.11.** Category $\mathbb{C}^{\leq 0}(\text{Ab})$ of cochain complexes of abelian groups, which live in non positive degree can be considered as full subcategory of $\mathbb{S}^{p \geq 0}$.

**Remark 3.7.12.** Note that under above inclusion functor, truncation functors on complexes $\tau_i$ correspond to truncation functors on spectra.

**Corollary 3.7.13.** Under stable homotopy hypothesis category $\mathbb{C}^{\leq 0}(\text{Ab})$ corresponds to subcategory of strict Picard $\infty$-groupoids:

$$\mathbb{C}^{\leq 0}(\text{Ab}) \leftrightarrow \infty - \text{Pic}^\text{strict}$$

**Remark 3.7.14.** Lemma[3.7.1] can be considered as special case of this equivalence. Subcategory $\mathbb{C}^{[-1,0]}(\text{Ab})$ of complexes $A^{-1} \to A^0$ can be considered as subcategory of stable 1-types and therefore corresponds to subcategory of strict Picard groupoids $\text{Pic}^{\text{strict}}$.

**Example 3.7.15.** Abelian group $A$ correspond to *Eilenberg-Maclane spectrum* $HA$ and corresponding Picard groupoids is discrete Picard groupoid.

Let $M$ be a spectral Mackey functor, over category $D$, which takes values in connective spectra $\mathbb{S}^{p \geq 0}$. Then existence of morphism

$$1^\text{full}_{\text{!}_{\to \text{*}}}: \text{holim}_D i^* M \to \text{holim}_D j^* M$$

is equivalent to existence of morphism:

$$1^{\text{full}}_{\text{!}_{\to \text{*}}}: \text{holim}_D i^* M \to \tau_{\geq 0} j^* \text{holim}_D M$$

due to adjunction of $\tau_{\geq 0}$. Thus we can apply Construction[1] and then use categorical Verdier duality construction to obtain desired morphism. We have following homotopy baby example:

**Example 3.7.16.** Let $G$ be a finite group and $M$ is the representation of $G$. We can associate with it spectral Mackey functor on Kan complex of $G$. Thus categorical Verdier duality supplies us with the morphism:

$$1^{\text{full}}_{\text{!}_{\to \text{*}}}: \text{holim}_G \mathcal{L}_M \to \text{holim}_G \mathcal{L}_M,$$

which is given by norm morphism from previous example.

**Remark 3.7.17.** We can also try to extend this construction to the setting of general spectra. Category of spectra $\mathbb{S}$ is equivalent to stabilization of category of connective spectra $\mathbb{S}^{p \geq 0}$. That means that we have $\infty$-bifibration $\text{Stab}(\mathbb{S}^{p \geq 0})$ over $\mathbb{N}$, with fibers given by category $\mathbb{S}^{p \geq 0}$ and functors between fibers are given by loop space object $\Omega$ and suspension functor $\Sigma$. Under this equivalence spectrum $E \in \mathbb{S}$ goes to the sequence $\{E_i\}_{i \in \mathbb{N}}$:

$$f: \Sigma E_i \to E_{i+1}, \quad E_i := \Sigma \circ \tau_{\geq -i} E.$$
Functor in inverse direction is given by attaching to the sequence \( \{E_i\}_{i \in \mathbb{N}} \) following homotopy colimit:

\[
\{E_i\}_{i \in \mathbb{N}} \mapsto \hocolim_{i \in \mathbb{N}} \Sigma^i E_i
\]

Therefore if we want to construct Verdier duality morphism between Mackey functors with values in \( \delta \mathbf{p} \) it is enough to construct family of morphisms between connective spectra, compatible with functor \( \Omega \). In order to be compatible, we impose condition that \( D \) is finite category.

4. Sheaves on Diagrams

4.1. Notations. We denote by \( \text{Top} \) category with objects locally compact, locally completely paracompact topological spaces of finite cohomological dimension, morphisms given by continues maps of topological spaces. It is a monoidal category with underlying cartesian tensor product. With an object \( X \in \text{Top} \) we associate topos \( \text{Sh}(X) \) of sheaves of sets on \( X \). Let \( \underline{A} \in \text{Sh}(X) \) be a constant ring object, associated with ring \( A \) (we further assume that \( A \) is the field). Then we denote by \( \text{Sh}_A(X) \) abelian category of sheaves of \( \underline{A} \)-modules. Let \( f : X \to Y \) be a morphism in \( \text{Top} \), we have a morphism of topoi, which is given by pair of adjoint functors: \( f^! : \text{Sh}(X) \to \text{Sh}(Y) : f^* \). When we restrict these functors to subcategories of sheaves of \( \underline{A} \)-modules functor \( f^* \) is exact and functor \( f^! \) is left exact and moreover functor \( f^! \) commutes with colimits. We have associated derived functors:

\[
\mathbf{R}(f^!) : \mathbf{D}(X, A) \to \mathbf{D}(Y, A) : \mathbf{R}(f^*)
\]

where \( \mathbf{D}(X, A) \) is derived category of category \( \text{Sh}_A(X) \). We also have pair of adjoint \(!\)-functors

\[
\mathbf{R}(f!) : \mathbf{D}(X, A) \to \mathbf{D}(Y, A) : f!
\]

Functor \( f! \) commutes with filtered homotopy colimits. We have unital tensor product \( \otimes : \)

\[
\otimes : \mathbf{D}(X, A) \times \mathbf{D}(X, A) \to \mathbf{D}(X, A)
\]

and inner hom functor \( \mathbf{RHom} : \)

\[
\mathbf{RHom} : \mathbf{D}(X, A)^\circ \times \mathbf{D}(X, A) \to \mathbf{D}(X, A).
\]

These operations make \( \mathbf{D}(X, A) \) into closed monoidal category. Functor \( f^* \) is monoidal and moreover for closed embedding \( i : Z \to X \) functor \( i_* \) is monoidal. For every \( X \in \text{Top} \) we have distinguished morphism \( a_X : X \to \text{pt} \), where pt is a point. We denote by \( w_X := a_X^! \) A dualizing complex in \( \mathbf{D}(X, A) \). We define Verdier duality functor:

\[
\mathcal{D} : \mathbf{D}(X, A) \to \mathbf{D}(X, A)^\circ, \quad \mathcal{K} \mapsto \mathbf{RHom}(\mathcal{K}, w_X).
\]

We have biduality morphism \( \mathbf{id} : \mathbf{D}(X, A) \to \mathbf{D}(X, A)^\circ, \quad \mathcal{K} \mapsto \mathbf{RHom}(\mathcal{K}, w_X) \).

The closed embedding \( i : Z \to X \) of topological spaces we have isomorphism of functors \( i_* \cong i^! \) and triangulated functor \( i^! \) is isomorphic to right derived functor of functor which takes sheaf to subsheaf which sections have support in \( Z \). Functor \( i^! \) commutes with homotopy colimits. Category \( \mathbf{D}(X, A) \) carries another unital monoidal structure denoted by \( \otimes^! \) and defined by the rule: \( \mathcal{K} \otimes^! \mathcal{G} := \Delta^! \mathcal{K} \boxtimes \mathcal{G} \), where \( \Delta : X \to X \times X \). This tensor product is called \(!\)-tensor product, unit is given by \( w_X \). Verdier duality functor takes tensor product \( \otimes^! \) to ordinary tensor product.
Note that for a proper morphism \( f \) functor \( f^! \) is monoidal with respect to \(!\)-tensor product and for closed embedding \( i \) functor \( i_* \) is also monoidal with respect to \( \otimes \).

Let \( (X, S) \) be a pair, where \( X \in \text{Top} \) and \( S \) is the stratification on \( X \). We denote by \( D_c(X, S) \), full triangulated subcategory of \( D(X, A) \), whose object are complexes with \( S \)-constructible cohomology, objects of latter category are called cohomological \( S \)-constructible sheaves. Category \( D_c(X, S) \) is also closed monoidal.

Let \( f : (X, S) \to (Y, S') \) be a stratified map in \( \text{Top} \), then functors \((L(f^*), R(f_*)) \) and \((L(f_!), f^!) \) preserve constructible subcategory. Verdier duality functor \( \mathcal{D} \) also preserves this subcategory and moreover defines antiequivalence.

### 4.2. \( \ast \)-sheaves

We denote by \( \mathbf{Sh}_{\text{top}} \) category whose objects are pairs \((X, \mathcal{K})\), where \( X \in \text{Top} \) is a topological space and \( \mathcal{K} \) is a sheaf of sets on \( X \). Morphisms between \((X_1, \mathcal{K}_1)\) and \((X_2, \mathcal{K}_2)\) are given by pairs \((h, H)\), where \( h : X_2 \to X_1 \) is map of topological spaces and \( H : \mathcal{K}_2 \to h^* \mathcal{K}_1 \) is a morphism of sheaves. This category is monoidal with underlying tensor product \((X_1, \mathcal{K}_1) \otimes (X_2, \mathcal{K}_2) := (X_1 \times X_2, \mathcal{K}_1 \boxtimes \mathcal{K}_2)\).

We have \( \text{Top} \)-topos \( \mathbf{Sh}_{\text{top}} \), whose fiber over topological space \( X \) is the category \( \mathbf{Sh}(X) \). For the diagram \( X_\beta : \mathcal{J} \to \text{Top} \), whose morphisms are closed embeddings \( X_i \to X_j \) for every \( m : i \to j \in \mathcal{J} \) we associate category \( \mathbf{Sh}^3_{\text{top}} := \mathcal{J}^0 \times_{\text{Top}} \mathbf{Sh}_{\text{top}} \), which will be a \( \mathcal{J}^\circ \)-topos. We assume that \( X_\beta \) sends fiber products in \( \mathcal{J} \) to fiber products in \( \text{Top} \). We have corresponding total topos \( \Gamma(\mathbf{Sh}^3_{\text{top}}) \). Let us fix constant ring object \( A \) in \( \mathbf{Sh}^3_{\text{top}} \), thus we have ringed \( \mathcal{J}^\circ \)-topos \((\mathbf{Sh}^3_{\text{top}}, A)\).

**Definition 4.2.1.** Category of \( \ast \)-sheaves \( \mathbf{Sh}^\ast(X_\beta) \) on the diagram \( X_\beta \) is the category of modules \( \text{Mod}(\Gamma(\mathbf{Sh}^3_{\text{top}}), A) \).

Category \( \mathbf{Sh}^\ast(X_\beta) \) is an abelian category and we have corresponding derived category \( D(\Gamma(\mathbf{Sh}^3_{\text{top}}), A) \). We have obvious:

**Lemma 4.2.2.** Abelian bifibration \( \text{Mod}(\Gamma(\mathbf{Sh}^3_{\text{top}}), A) \to \mathcal{J}^0 \) defines a pair of pseudo-functors \((H^2_\beta, H^3_\beta)\) with the structure of contradirectional \( e^\ast \)-functor.

**Proof.** Evident.

We denote by \( \mathbf{Sh}_{\text{adm}}^\ast(X_\beta) \) subcategory of \( \mathbf{Sh}^\ast(X_\beta) \) cocartesian sections of bifibration \( \text{Mod}(\Gamma(\mathbf{Sh}^3_{\text{top}}), A) \).

**Definition 4.2.3.** We denote by \( D_{\text{cocart}}(\Gamma(H^2_\beta)) \) full triangulated subcategory of \( D(\Gamma(H^2_\beta)) \), whose cohomology are in category \( \mathbf{Sh}_{\text{adm}}^\ast(X_\beta) \). Objects of the category \( D_{\text{cocart}}(\Gamma(H^2_\beta)) \) are called admissible \( \ast \)-sheaves on diagram \( X_\beta \).

Note, that heart of t-category \( D_{\text{cocart}}(\Gamma(H^2_\beta)) \) (with respect to the standard t-structure) is equivalent to \( \mathbf{Sh}_{\text{adm}}^\ast(X_\beta) \). Suppose that for every \( i \in \mathcal{J} \) we have stratification \( S_i \) on \( X_i \) and for every morphism \( m : i \to j \in \mathcal{J} \) we have stratified map \( X_i \to X_j \). Then we have subcategories of cohomological constructible \( \ast \)-sheaves \( D_c(\Gamma(H^2_\beta)) \) and \( D_{\text{cocart}, c}(\Gamma(H^2_\beta)) \).

---

4When it wont lead to misunderstanding we suppress \( X_\beta \) from notation
4.3. !-sheaves. Let us consider following category. It is objects are pairs \((X, \mathcal{K})\) where \(X \in \text{Top}\) is the topological space and \(\mathcal{K}\) is the sheaf of \(A\)-modules on \(X\). Morphisms are given by following data. A morphism from \((X_1, \mathcal{K}_1)\) to \((X_2, \mathcal{K}_2)\) is pair \((f, \Phi)\), where \(f: X_1 \to X_2\) is a morphism of topological spaces and \(\Phi: f_*\mathcal{K}_1 \to \mathcal{K}_2\) is a morphism of sheaves. We denote this category by \(H^\text{top}\). This category will be monoidal with tensor product \((X_1, \mathcal{K}_1) \otimes (X_2, \mathcal{K}_2) := (X_1 \times X_2, \mathcal{K}_1 \boxtimes \mathcal{K}_2)\). We have cofibered category \(H^\text{top}\) over \(\text{Top}\). Fiber over \(X\) is given by category \(\text{Sh}_A(X)\). Let \(X_3\) be a diagram of topological spaces, we associate with it cofibered category \(H^3 := \mathcal{J} \times_{\text{Top}} H^\text{top}\) over \(\mathcal{J}\).

**Definition 4.3.1.** Category of !-sheaves \(\text{Sh}_!(X_3)\) on diagram \(X_3\) is the total category of cofibration \(H^3_! \to \mathcal{J}\).

Category \(\text{Sh}_!(X_3)\) is an abelian and we have corresponding derived category \(D(\underline{\mathcal{J}}(H^3_!))\). We have a bifibration over \(\mathcal{J}\) denoted by \((H^3_1, H^3_2)\):

**Lemma 4.3.2.** We have a pair of pseudo-functors \((H^3_1, H^3_2)\), which defines the contradirectional \(c^!\)-functor.

**Proof.** Evident.

We have subcategory \(\text{Sh}^\text{adm}_!(X_3)\) of cartesian sections of fibration \(H^3_1 \to \mathcal{J}\).

**Definition 4.3.3.** We denote by \(\text{D}_{\text{cart}}(\underline{\mathcal{J}}(H^3_!))\) full triangulated subcategory of \(\text{D}(\underline{\mathcal{J}}(H^3_!))\), whose cohomology are in category \(\text{Sh}^\text{adm}_!(X_3)\). Objects of the category \(\text{D}_{\text{cart}}(\underline{\mathcal{J}}(H^3_!))\) are called admissible !-sheaves on diagram \(X_3\).

Like in the case of \(\ast\)-sheaves we can also define subcategories of cohomological constructible !-sheaves \(\text{D}_{c}(\underline{\mathcal{J}}(H^3_!))\) and \(\text{D}_{\text{cart},c}(\underline{\mathcal{J}}(H^3_!))\).

**Remark 4.3.4.** Unlike to the case of admissible \(\ast\)-sheaves, category \(\text{D}_{\text{cart},c}(\underline{\mathcal{J}}(H^3_!))\) does not carry trivial \(t\)-structure, however if diagram \(X_3\) is equipped with perversity function, then by Proposition 4.5.3 category of admissible !-sheaves can be equipped with perverse \(t\)-structure (which can also be defined via Verdier duality functor \(V_{\text{Verd}}\), with corresponding heart given by so called category of perverse !-sheaves \(\text{Perv}^\text{!}(X_3, p)\). For example one can consider diagram \(A^{\mathbb{R}}\) (Example 4.5.6, see also Remark 4.3.3). Categories of !-sheaves and \(\ast\)-sheaves are canonically equivalent and carries two natural perverse \(t\)-structures \(p_{\text{min}}\) and \(p_{\text{min}}\). Note that category of factorizable sheaves, which live in the heart \(\text{Perv}^\text{!}(A^{\mathbb{R}}_\mathbb{R}, p_{\text{min}})\) is equivalent to the category of \(\mathbb{N}\)-graded associative algebras.

4.4. !\(\ast\)-sheaves. Let us define following category, objects of this category are pairs \((X, \mathcal{K})\), where \(X \in \text{Top}\) is the topological space and \(\mathcal{K}\) is the sheaf of \(A\)-modules on \(X\). Morphism between objects \((X_1, \mathcal{K}_1)\) and \((X_2, \mathcal{K}_2)\) is defined as pair \((\varphi, f)\), where map \(f\) is a morphism in \(\text{span}(\text{Top})\) which is given by triple \(f := (f_1, Z, f_2)\) and \(\varphi\) is a morphism of sheaves: \(f_1f^!\mathcal{K}_1 \to \mathcal{K}_2\). Composition is defined by taking fibered products and base change theorem. We denote this category by \(H^\text{top}\). With the diagram \(X_3: \mathcal{J} \to \text{Top}\), we can associate diagram \(\text{cospan}(X_3): \text{cospan}(\mathcal{J}^\circ) \to \text{span}(\text{Top})\). We have cofibered category \(H^3_! := \text{cospan}(\mathcal{J}^\circ) \times_{\text{span}(\text{Top})} H^\text{top}\) over \(\text{cospan}(\mathcal{J}^\circ)\).

**Definition 4.4.1.** Category of !\(\ast\)-sheaves \(\text{Sh}^\text{!}(X_3)\) on the diagram \(X_3\) is the total category of cofibration \(H^3_! \to \text{cospan}(\mathcal{J}^\circ)\).
This category is also abelian and derived category will be denoted by $D(\Gamma(H^d_\mathcal{J}))$. Note that we actually have a bifibration $(H^d_\mathcal{J}, H^c_\mathcal{J})$ over cospan($\beta^\circ$).

**Proposition 4.4.2.** Quadruple $(H^d_\mathcal{J}, H^c_\mathcal{J}, H^c_\mathcal{J}, H^d_\mathcal{J})$ is the Grothendieck cross functor, associated with ringed $\beta^\circ$-topos $(\textbf{Sh}_{\topos}^d, \mathcal{A})$. Corresponding Mackey $\star!$-functor is pseudo-functor $H^d_\mathcal{J}$ and Mackey $\star!$-functor is $H^c_\mathcal{J}$.

**Proof.** Evident. \hfill $\square$

We have associated covariant Verdier duality functors:

$$V_{\leftarrow !} : D_{\text{cocart}}(\Gamma(H^d_\mathcal{J})) \leftrightarrow D_{\text{cart}}(\Gamma(H^d_\mathcal{J})): V_{!\leftarrow !},$$

Let $w_{\beta} \in D_{\text{cart}}(H^d_\mathcal{J})$ be an object defined as dualizing complex $w_i \in D(X_i, A)$ for every $i \in \mathcal{J}$, which we call $!$-dualizing objects on diagram $X_\mathcal{J}$. We have evident corollary from Proposition 4.4.2.

**Corollary 4.4.3.** $(H^d_\mathcal{J}, H^c_\mathcal{J}, H^c_\mathcal{J}, H^d_\mathcal{J}, w_{\beta})$ is Grothendieck cross functor with dualizing object.

If we have stratification $S$ on diagram $X_\mathcal{J}$, then we have constructible Grothendieck cross functor with corresponding duality functors:

$$D_{\leftarrow !} : D_{\text{cocart}}(\Gamma(H^d_\mathcal{J})) \leftrightarrow D_{\text{cart}}(\Gamma(H^d_\mathcal{J})), D_{\leftarrow !},$$

and contravariant Verdier duality functor for $!$-sheaves:

$$D_{H^d_\mathcal{J}} : D_{\text{cart}}(\Gamma(H^d_\mathcal{J})) \rightarrow D_{\text{cart}}(\Gamma(H^d_\mathcal{J})).$$

**Remark 4.4.4.** Let $X \in \text{Top}$ be a topological space with the filtration:

$$X_1 \subset X_2 \subset \cdots \subset X_n = X,$$

where $X_i$ are closed subspaces. We can consider space $X$ with such filtration as diagram from finite category $[n] \in \text{Cat}$. With every object $X \in D^b(X, A)$ we can associate cohomological Postnikov system [BBD], which plays role of corresponding $\star$-sheaf and homological Postnikov system, which plays role of $!$-sheaf. Then contravariant Verdier duality functor is identity and functor $D_{H^d_\mathcal{J}}$ is given by usual Verdier duality, which interchanges these two Postnikov systems.

We can consider space $X$ as stratified space $(X, S)$. Suppose that stratified spaces $(X, S)$ is equipped with perversity function $p : S \rightarrow \mathbb{Z}$. Construction from Proposition 3.5.3 in this case defines usual perverse t-structure, whose heart is an abelian category of p-perverse sheaves $\text{Perv}(X, S)$.

In the same spirit we can define pseudo-functors $H^d_{\mathcal{J}_{\star}}$ and $H^c_{\mathcal{J}_{\star}}$ with corresponding functors $\Xi_{\star}$ and $\Xi_{\star}$.

4.5. **Operations on $!$-sheaves.** Let $f : X_\mathcal{J} \rightarrow Y_\mathcal{J}$, be a morphism of diagrams, then we have a morphism between $D_{\text{cart}}(\Gamma(H^d_{X_\mathcal{J}}))$ and $D_{\text{cart}}(\Gamma(H^d_{Y_\mathcal{J}}))$, given by the pair of adjoint $!$-functors:

$$f_\beta : D_{\text{cart}}(\Gamma(H^d_{X_\mathcal{J}})) \leftrightarrow D_{\text{cart}}(\Gamma(H^d_{Y_\mathcal{J}})): f^\beta_\beta,$$

where functor $f_\beta$ is defined as composition of the componentwise $!$-pushforward functor $\mathbb{R}(f_\beta) : D(X_i, A) \rightarrow D(Y_i, A), \ i \in \mathcal{J}$ and functor $\Xi_{\beta}$. Functor $f^\beta_\beta$ is defined as componentwise $!$-pullback functor $f^\beta_\beta(X_\mathcal{J}) := f^\beta_\beta(X_\mathcal{J})$, where $f^\beta_\beta : D(Y_i, A) \rightarrow D(X_i, A).$
For a morphism of diagrams \( f: X_\beta \longrightarrow Y_\beta \) we can also define \(*\)-operations for \(!\)-sheaves:

\[
(112) \quad f_*: D_{\text{cart}, c}(\Gamma(H^1_{X_\beta})) \leftrightarrow D_{\text{cart}, c}(\Gamma(H^1_{Y_\beta})): f^*
\]

by the rule:

\[
(113) \quad f_* := \mathcal{V}_\rightarrow! \circ \Xi \circ \mathcal{R}(f_{\ast}) \circ \mathcal{V}_\leftarrow, \quad f^* := \mathcal{V}_\rightarrow! \circ \mathcal{L}(f^*) \circ \mathcal{V}_\leftarrow.
\]

**Lemma 4.5.1.** On constructible subcategory we have isomorphisms of functors:

\[
(114) \quad f_* \sim \mathcal{D}_{H^1_{\beta}}, \quad f^* \sim \mathcal{D}_{H^1_{\beta}} \circ \mathcal{D}_{\rightarrow!} \circ \Xi \circ \mathcal{R}(f_{\ast}) \circ \mathcal{V}_\leftarrow.
\]

**Proof.** By commutativity property \([38]\) of \(*\)-operations and \(!\)-operations with duality we get:

\[
(115) \quad \mathcal{D}_{H^1_{\beta}} \circ \mathcal{D}_{\rightarrow!} \circ \Xi \circ \mathcal{R}(f_{\ast}) \circ \mathcal{V}_\leftarrow \sim \mathcal{V}_\rightarrow! \circ \mathcal{D}_{\rightarrow!} \circ \Xi \circ \mathcal{R}(f_{\ast}) \circ \mathcal{V}_\leftarrow.
\]

Then by Proposition \([34, 21]\) we have:

\[
(116) \quad \mathcal{D}_{H^1_{\beta}} \circ \mathcal{D}_{\rightarrow!} \circ \Xi \circ \mathcal{R}(f_{\ast}) \circ \mathcal{V}_\leftarrow \sim \mathcal{V}_\rightarrow! \circ \mathcal{D}_{\rightarrow!} \circ \Xi \circ \mathcal{R}(f_{\ast}) \circ \mathcal{V}_\leftarrow.
\]

Thus since biduality morphism is equivalence we obtain desired result. Case of \(*\)-pullback can be proved analogically. \( \square \)

**Example 4.5.2.** For diagrams \( X_\beta \) with subdiagram \( j: Z_\beta \hookrightarrow X_\beta \) important example of \(*\)-functors is given by hyperbolic restriction functor \( \Phi_{Z_\beta} \). It is defined as:

\[
(117) \quad \Phi_{Z_\beta} := a_{Z_\beta} \circ j^* : D_{\text{cart}, c}(\Gamma(H^1_{X_\beta})) \longrightarrow D(A),
\]

where \( a_{Z_\beta} : Z_\beta \longrightarrow \{\text{pt}\} \), is the canonical morphism to the point. In the setting of Ran prestack of \( A^1 \) hyperbolic restriction to the Ran prestack of \( A^1 \) plays important role in the construction of Hopf algebras from factorizable sheaves. In this case functor \( \Phi_{A^1} \) obeys usual properties, such as commutativity with Verdier duality and exactness with respect to middle perverse t-structure.

Additional operations are given by Kan extension functors:

**Definition 4.5.3.** Let \( j: X \longrightarrow Y \) be a functor. We define pair of adjoint functors:

\[
(118) \quad j^*: D_{\text{cart}, c}(\Gamma(H^1_{X})) \longrightarrow D_{\text{cart}, c}(\Gamma(H^1_{X})),
\]

where functor \( j^* \) is given by restriction along functor \( j \) and functor \( j^* \) is defined as composition \( \Xi \circ \mathcal{L}(j) \).

Category \( D_{\text{cart}, c}(\Gamma(H^1_{X})) \) carries symmetric monoidal product denoted by:

\[
(119) \quad \otimes^1: D_{\text{cart}, c}(\Gamma(H^1_{X})) \times D_{\text{cart}, c}(\Gamma(H^1_{X})) \longrightarrow D_{\text{cart}, c}(\Gamma(H^1_{X}))
\]

It is given by componentwise \(!\)-tensor product of sheaves. Note that this tensor product is unital and unit is given by \( w_{X^*_\beta} \).

**Remark 4.5.4.** If category \( Y \) is monoidal category, with tensor product \(*: Y \times Y \longrightarrow Y \) then we can define additional symmetric tensor structure on \( D_{\text{cart}}(H^1_{\beta}) \), by Day convolution product:

\[
(120) \quad \otimes^*: D_{\text{cart}}(\Gamma(H^1_{X})) \times D_{\text{cart}}(\Gamma(H^1_{X})) \longrightarrow D_{\text{cart}}(\Gamma(H^1_{X}))
\]

Examples of such tensor products are given by tensor structures on the Ran prestack, introduced in \([13, 12]\).
Let $J$ be a category, denote by $i: J \to J \times J$ canonical diagonal embedding functor. Let $X_J: J \to \text{Top}$ be a diagram. We have diagrams $iX_J$, which is defined as Kan extension along morphism $i$. Denote by $X_J \times X_J: J \times J \to \text{Top}$ diagram, which is defined as composition of product diagram $X_J \times X_J$, with cartesian monoidal structure on Top. Then by universal property of colimits we have following morphism of diagrams:

\[(121) \text{diag} : i! X_J \to X_J \times X_J,\]

Note that morphism $\text{diag}$ is closed embedding. Contravariant Verdier duality functor for $!$-sheaves enjoys following properties:

**Proposition 4.5.5.** (i) Space of morphisms between $F \boxtimes K$ and $\text{diag} w X_J$, is represented by contravariant Verdier duality functor:

\[(122) \text{Hom}_{D_{\text{cart}}(\Gamma(H^!_{S^\emptyset})))}(K \boxtimes \mathcal{G}, \text{diag} w X_J) = \text{Hom}_{D_{\text{cart}}(\Gamma(H^!_{S^\emptyset})))}(K, \mathcal{D} H^!_{S^\emptyset} \mathcal{G})\]

(ii) We have canonical biduality morphism:

\[(123) \text{Id}_{D_{\text{cart}}(\Gamma(H^!_{S^\emptyset})))} \to \mathcal{D} H^!_{S^\emptyset} \circ \mathcal{D} H^!_{S^\emptyset}\]

**Proof.** Evident. $\square$

**Example 4.5.6.** Let $S$ be a category of finite sets and surjective morphisms between them. Following $[BD2]$ with the topological spaces $X$ we associate diagram $X_S$. Denote by $S^\emptyset$ subcategory of finite sets and isomorphisms, with canonical inclusion functor $\text{inc}: S^\emptyset \to S$. We have homotopy left Kan extension functors, which includes in the following commutative diagram:

\[(124) \quad \text{D}_{\text{cart}}(\Gamma(H^!_{S^\emptyset}))) \quad \text{D}_{\text{cocart}}(\Gamma(H^!_{S^\emptyset}))) \quad \text{D}_{\text{cart}}(\Gamma(H^!_{S^\emptyset}))) \]

Functor $\text{inc}^\emptyset$ induces equivalence between category $D_{\text{cart}}(\Gamma(H^!_{S^\emptyset})))$ and subcategory of graded $!$-sheaves on Ran space $D^{gr}_{\text{cart}}(\Gamma(H^!_{S^\emptyset})))$, which consists of objects $\mathcal{K} \in D_{\text{cart}}(\Gamma(H^!_{S^\emptyset})))$ such that

\[(125) \quad \mathcal{K}_I \in D^{gr}(X^I) \quad \text{supp(gr}_n \mathcal{K}_I) \subset X^n \quad n < |I|\]

Let us consider diagrams $X_S$ equipped with diagonal stratification, suppose that we have perversity function $p$ on $X_S$. Commutativity of above diagrams implies that functor $\text{inc}^\emptyset$ is perverse $t$-exact, thus we define category of graded $p$-perverse $!$-sheaves on $X_S$ as image of functor $\text{inc}^\emptyset$:

\[(126) \quad \text{inc}^\emptyset: \text{Perv}^p(X_S^\emptyset, p) \sim \to \text{Perv}^{gr}_{\text{cart}}(X_S^\emptyset, p)\]

Under above equivalence category of graded $!$-sheaves on the Ran spaces of $X$ can be equipped with the new tensor structure, which is defined by the $!$-tensor product $[119]$ on $D_{\text{cart}}(\Gamma(H^!_{S^\emptyset})))$:

\[(127) \quad \circ: D^{gr}_{\text{cart}}(\Gamma(H^!_{S^\emptyset}))) \times D^{gr}_{\text{cart}}(\Gamma(H^!_{S^\emptyset}))) \to D^{gr}_{\text{cart}}(\Gamma(H^!_{S^\emptyset})))\]

\[^5\text{This property is taken as definition in } [GM] \]
This tensor structure is unital and unit is given by $\text{inc}_!(w_{S^0})$. In the case when $X = \mathbb{A}^\mathbb{R}$ and perversity function is given by minimal perversity function $p_{\text{min}}$ tensor product $\circ$ on the category $\text{Perv}_{\text{cart}}^{!\text{gr}}(X_S, p_{\text{min}})$ is exact and corresponds to the white product of quadratic algebras $[PP]$, and $\text{inc}_!(w_{S^0})$ is the free associative algebra on one variable. Note that contravariant Verdier duality induces anti-equivalence of category of graded perverse $!$-sheaves:

$$\mathbb{D}_{H_{S^0}^!} : \text{Perv}_{\text{cart}}^{!\text{gr}}(X_S, p_{\text{min}}) \xrightarrow{\sim} \text{Perv}_{\text{cart}}^{!\text{gr}}(X_S, p_{\text{max}})^\circ$$

Thus we can also define another unital tensor structure on $\text{Perv}_{\text{cart}}^{!\text{gr}}(X_S, p_{\text{min}})$:

$$\mathcal{K} \bullet \mathcal{E} := \mathbb{D}_{H_{S^0}^!}^{-1}(\mathbb{D}_{H_{S^0}^!} \mathcal{K} \cdot \mathbb{D}_{H_{S^0}^!} \mathcal{E})),$$

Product $\cdot$ is defined analogically to $\circ$, instead of $!$-product we use ordinary product of sheaves. In the case when $X = \mathbb{A}^\mathbb{R}$ tensor product $\bullet$ on the category $\text{Perv}_{\text{cart}}^{!\text{gr}}(X_{S^\natural}, p_{\text{min}})$ corresponds to the black product of quadratic algebras and the unit is given by $\text{inc}(j_!^!j_!^*w_{S^0})$, where for every finite set $I$, sheaf $j_!^!j_!^*w_{S^0}$ is defined as $j_!^!j_!^*w_{X^I}$, and $j_! : U \hookrightarrow X^I$ is open complement, transversal to the smallest diagonal $X \subset X^I$. Under correspondence between factorizable sheaves and graded algebras this sheaf corresponds to the algebra of dual numbers.

**Remark 4.5.7.** Generally in the case of $!$-sheaves we does not have a notion of inner hom (compare with Example 4.5.6), but contravariant Verderer enjoys hom like adjunction property [22]. On the another hand in the case of $\star$-sheaves we have notion of inner hom $\text{Hom}^\star$ (see [75]). Reader can compare this with the case of $\mathcal{D}$-modules on the scheme.


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