INSTANTONS VERSUS MONOPOLES

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Abstract. We review results of the last two years concerning caloron solutions of unit charge with non-trivial holonomy, revealing the constituent monopole nature of these instantons. For $SU(n)$ there are $n$ such BPS constituents. New is the presentation of the exact values for the Polyakov loop at the three constituent locations for the $SU(3)$ caloron with arbitrary holonomy. At these points two eigenvalues coincide, extending earlier results for $SU(2)$ to a situation more generic for general $SU(n)$.

1. Introduction

Calorons are finite temperature instanton solutions. They are defined on $\mathbb{R}^3 \times S^1$. Due to the periodic boundary conditions in the time direction, the Polyakov loop at spatial infinity (the so-called holonomy) can take on a non-trivial value (independent of directions)

\[ \mathcal{P}_\infty = \lim_{|\vec{x}| \to \infty} P(\vec{x}), \quad P(\vec{x}) = P \exp(\int_0^\beta A_0(t, \vec{x}) dt). \quad (1) \]

A non-trivial value reveals that the charge one $SU(n)$ caloron actually contains $n$ constituent BPS monopoles, whose masses are determined by the eigenvalues of the Polyakov loop

\[ \mathcal{P}_\infty^0 \equiv \exp(2\pi i \text{diag}(\mu_1, \mu_2, \ldots, \mu_n)), \quad \sum_{i=1}^n \mu_i = 0. \quad (2) \]

For defining the constituent masses it is important to note that one can choose a gauge in which $\mu_1 \leq \cdots \mu_n \leq \mu_{n+1} \equiv \mu_1 + 1$ (for a proof see [1]).

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This guarantees that the masses $8\pi^2 \nu_m / \beta$, with $\nu_m \equiv \mu_{m+1} - \mu_m$, add up to $8\pi^2 / \beta$ such that the action equals that of a charge one instanton.

The separation of the constituents, which are part of the moduli of the solution, can be chosen freely and for large separation the action density becomes static. This will be discussed in sect. 2, where we also discuss the localisation of the fermion zero-mode on one of the constituents. In sect. 3 we give some details for the explicit computation of the gauge field, not presented before for $SU(n)$. New will also be the study of the $SU(n)$ constituent monopole location in terms of the vanishing of the Higgs field, which in the present context is replaced by the Polyakov loop variable. In the core of the constituents the Polyakov loop generically will have two of its eigenvalues degenerate. For $SU(2)$ this implies that the Polyakov loop becomes $\pm I_2$, as was verified explicitly [2]. In sect. 4 we present for $SU(3)$ the simple result for the Polyakov loop at the three constituent locations. We also review the fact that in a suitable gauge only one of the constituents’ gauge fields is non-static, in accordance to the Taubes-winding [3] required to form out of monopoles a four dimensional gauge field configuration with non-trivial topological charge, as is discussed in some detail.

We conclude in sect. 5 with some comments concerning the fact that perhaps monopoles are more fundamental building blocks, since we know how to make instantons out of them. But we hasten to say the opposite point of view can be taken as well, since monopoles can be made out of an array of instantons. It more seems to imply they occur on equal footing, in accordance with the “democratic” vacuum model described in ref. [4].

2. Densities

Using the classical scale invariance we can always arrange $\beta = 1$, as will be assumed throughout. A remarkably simple formula for the $SU(n)$ action density exists [5]

$$\text{Tr} F_{\alpha\beta}^2(x) = \partial_\alpha^2 \partial_\beta^2 \log \psi(x), \quad \psi(x) = \frac{1}{2} \text{tr}(A_n \cdots A_1) - \cos(2\pi t),$$

$$A_m = \frac{1}{r_m} \begin{pmatrix} r_m & |\tilde{\rho}_{m+1}| \\ 0 & r_{m+1} \end{pmatrix} \begin{pmatrix} \cosh(2\pi \nu_m r_m) & \sinh(2\pi \nu_m r_m) \\ \sinh(2\pi \nu_m r_m) & \cosh(2\pi \nu_m r_m) \end{pmatrix},$$

with $r_m \equiv |\vec{x} - \vec{y}_m|$ and $\tilde{\rho}_m \equiv \vec{y}_m - \vec{y}_{m-1}$, where $\vec{y}_m$ is the location of the $m$th constituent monopole with a mass $8\pi^2 \nu_m$. Note that the index $m$ should be considered mod $n$, such that e.g. $r_{n+1} = r_1$ and $\vec{y}_{n+1} = \vec{y}_1$ (but note $\mu_{n+1} = 1 + \mu_1$). It is sufficient that only one constituent location is far separated from the others, to show that one can neglect the $\cos(2\pi t)$ term in $\psi(x)$, giving rise to a static action density in this limit [5, 6].

These generalised caloron solutions can be found [7, 5] using the Nahm transformation [8] and the Atiyah-Drinfeld-Hitchin-Manin (ADHM) con-
struction [9], related by a suitably defined Fourier transformation. Other
methods, relying more exclusively on the Nahm transformation, were de-
veloped as well [10].
The Nahm equation for charge one calorons reduces to an abelian prob-
lem on the circle, parametrised by $z \mod 1$,
\[
\frac{d}{dz} \hat{A}_j(z) = 2\pi i \sum_m \rho_m^j \delta(z - \mu_m),
\]
giving $\hat{A}_j(z) = 2\pi i y_m^j$, for $z \in [\mu_m, \mu_{m+1}]$. In the monopole literature $\hat{A}_j(z)$ is usually denoted by $T_j(z)$.
The basic ingredient in the construction of caloron solutio ns is a Green's
function, defined on the circle $z \in [0, 1]$, satisfying
\[
\left( \left( \frac{1}{2\pi i} \frac{d}{dz} - t \right)^2 + r^2(x; z) + \frac{1}{2\pi} \sum_m \delta(z - \mu_m)|\vec{\rho}_m| \right) \hat{f}_x(z, z') = \delta(z - z'),
\]
where $r^2(x; z) = r_m^2(x)$ for $z \in [\mu_m, \mu_{m+1}]$. This can be solved using a
similarity with the impurity scattering problem on the circle [5, 1], which
we present here for the case that $\mu_m \leq z' \leq z \leq \mu_{m+1}$ (extended to $z < z'$
by $\hat{f}_x(z', z) = \hat{f}_x^*(z, z')$)
\[
\hat{f}_x(z, z') = \frac{\pi e^{2\pi it(z-z')}}{r_m \psi} \left( e^{-2\pi it} \sinh (2\pi (z - z')r_m) + <v_m(z')|A_{m-1} \cdots A_1 A_n \cdots A_m|w_m(z)> \right),
\]
where the spinors $v_m$ and $w_m$ are defined by
\[
v_m^1(z) = -w_m^2(z) = \sinh (2\pi (z - \mu_m)r_m),
v_m^2(z) = w_m^1(z) = \cosh (2\pi (z - \mu_m)r_m).
\]
With the gauge field in the periodic gauge ($A_\alpha(t+1, \vec{x}) = A_\alpha(t, \vec{x})$) one
can look for the fermion zero-modes that satisfy the boundary condition
$\Psi_z(t + 1, \vec{x}) = \exp(2\pi iz)\Psi_z(t, \vec{x})$. To obtain the finite temperature fermion
zero-mode one puts $z = \frac{1}{2}$, whereas for the fermion zero-mode with periodic
boundary conditions one takes $z = 0$. For the density of these fermion
zero-modes a very simple result in terms of the Green’s function can be
derived [11, 1]
\[
|\Psi_z(x)|^2 = -(2\pi)^{-2} \partial^2_\alpha \hat{f}_x(z, z).
\]
From this it is easily seen that in case of well separated constituents the
zero-mode is localised only at $y_m$ for which $z \in [\mu_m, \mu_{m+1}]$. To be specific,
in this limit \( \hat{f}_x(z, z) = \pi \tanh(\pi r_m \nu_m)/r_m \) for \( SU(2) \), and more generally
\[
\hat{f}_x(z, z) = \frac{2\pi \sinh[2\pi(z - \mu_m)r_m]\sinh[2\pi(\mu_{m+1} - z)r_m]}{r_m \sinh[2\pi \nu_m r_m]}. \tag{9}
\]
We illustrate the localisation of the fermion zero-modes for a typical \( SU(3) \) caloron in figure 1.

Figure 1. The action density (top) for a \( SU(3) \) caloron at \( t = 0 \), for \( \beta = 1 \), on a logarithmic scale cut off at \( 1/(2e) \), with \( (\mu_1, \mu_2, \mu_3) = (-17, -2, 19)/60 \) shown in the plane defined by \( \vec{y}_1 = (-2, -2, 0), \vec{y}_2 = (0, 2, 0) \) and \( \vec{y}_3 = (2, -1, 0) \). The masses \( 8\pi^2 \nu_i \) are given by \( (\nu_1, \nu_2, \nu_3) = (0.25, 0.35, 0.4) \). On the bottom-left is shown the zero-mode density for fermions with anti-periodic boundary conditions in time and on the bottom-right for periodic boundary conditions, at equal logarithmic scales cut off below \( 1/e^5 \).

3. Gauge fields

To construct the gauge potential \( A_\alpha(x) \), it is instructive to summarise the ADHM formalism for \( SU(n) \) charge \( k \) instantons \[9\]. It employs a \( k \) dimensional vector \( \lambda = (\lambda_1, \ldots, \lambda_k) \), where \( \lambda^i \) is a two-component spinor in the \( \bar{n} \) representation of \( SU(n) \). Alternatively, \( \lambda \) can be seen as an \( n \times 2k \) complex matrix. In addition one has four complex hermitian \( k \times k \) matrices \( B_\alpha \), combined into a \( 2k \times 2k \) complex matrix \( B = \sigma_\alpha B_\alpha \), using the unit quaternions \( \sigma_\alpha = (1_2, i\tau) \) and \( \bar{\sigma}_\alpha = (1_2, -i\tau) \), where \( \tau_i \) are the Pauli matrices. Together \( \lambda \) and \( B \) constitute a \( (n + 2k) \times 2k \) dimensional matrix \( \Delta(x) \), to which is associated a complex \( (n + 2k) \times n \) dimensional normalised \( (v^\dagger(x)v(x) = I_n) \) zero-mode matrix \( v(x) \),
\[
\Delta^\dagger(x)v(x) \equiv \left( \lambda^\dagger, B^\dagger(x) \right) v(x) = 0, \quad B(x) \equiv B - xI_k. \tag{10}
\]
Here \( x \) denotes the quaternion \( x = x_\alpha \sigma_\alpha \). The gauge field is now given by \( A_\alpha(x) = v^\dagger(x)\partial_\alpha v(x) \). For this to be self-dual, \( \Delta(x) \) has to satisfy the
quadratic ADHM constraint,
\[ \Delta^\dagger(x) \Delta(x) = \sigma_0 f_x^{-1}, \]  
(11)

defining \( f_x \) as a hermitian \( k \times k \) Green’s function. The gauge field can conveniently be written as (cmp. \[12, 5\])
\[ A_\alpha(x) = \frac{1}{\lambda} \phi^\dagger(x) \lambda \hat{\eta}_{\alpha \beta} \partial^\beta f_x \lambda \phi^\dagger(x) + \frac{1}{\lambda} [\phi^\dagger(x), \partial_\alpha \phi^\dagger(x)]. \]  
(12)

where \( \phi(x) \) is a positive definite \( n \times n \) matrix, and \( \hat{\eta}_{\alpha \beta} \) and \( \eta_{\alpha \beta} \) are the (anti-)self-dual \( 't \) Hooft tensors,
\[ \phi(x) \equiv (1 - f_x \lambda^\dagger)^{-1}, \quad \eta_{\alpha \beta} = \eta^a_{\alpha \beta} \sigma_a \equiv \sigma_{[\alpha} \sigma_{\beta]}, \quad \hat{\eta}_{\alpha \beta} = \bar{\eta}^a_{\alpha \beta} \sigma_a \equiv \bar{\sigma}_{[\alpha} \sigma_{\beta]} \].
(13)

The charge one caloron with Polyakov loop \( P_0^\infty \) at infinity is built out of a periodic array of instantons, twisted by \( P_0^\infty \). This is implemented in the ADHM formalism by \( \lambda_{p+1} = P_0^\infty \lambda_p \), which implies \( \lambda^m_p = \exp(2\pi i p m)\zeta_m \), with \( \zeta \) constant, \( m \) the colour and \( p \) the “charge” index (spinor indices are implicit throughout), where \( \bar{\rho}_m \equiv -\pi \zeta m \zeta^\dagger m \). The phases of \( \zeta_m \) are related to global gauge transformations that leave \( P_0^\infty \) invariant, and define the “framing” of the caloron. We note that \( f_x(z, z^\dagger) \) is the Fourier transform of the infinite dimensional matrix \( f_x \) as it occurs in the ADHM construction. It can be shown \[13\] that \( \text{Tr} F^2_{\alpha \beta}(x) = -\partial^2_\alpha \partial^2_\beta \log \det f_x \). With the help of eq. (6) we find \( \partial_\alpha \log \det f_x = -\partial_\alpha \log \psi(x) \), see eq. (3). We also perform Fourier transformation to obtain
\[ \hat{\lambda}(z) = \sum_p e^{-2\pi i p z} \lambda_p, \quad \hat{\lambda}^m(z) = \delta(z - m) \zeta^m. \]  
(14)

This implies a remarkably simple result for the gauge field in the algebraic (or singular) gauge, \( A_\alpha(t + 1, \bar{x}) = P_0^\infty A_\alpha(\bar{x}) (P_0^\infty)^\dagger \),
\[ A_\alpha(x) = \frac{1}{\lambda} \phi^\dagger(x) C_\alpha(x) \phi^\dagger(x) + \frac{1}{\lambda} [\phi^{-1}(x), \partial_\alpha \phi^{-1}(x)], \]  
(15)

with
\[ C^m_{\alpha k}(x) \equiv \zeta_m \bar{\eta}_{\alpha \beta} \zeta^k_\beta \partial_\beta \hat{f}_x(\mu_m, \mu_k), \quad \phi^{-1}_{mk} = \delta_{mk} - \zeta_m \zeta^k_\beta \hat{f}_x(\mu_m, \mu_k). \]  
(16)

Note that \( \zeta^m_\alpha \zeta^m_\beta = (|\bar{\rho}_m| - \bar{\rho}_m \cdot \bar{\tau})/2\pi \) and that \( \sum_m \bar{\rho}_m = \bar{0} \) implies a constraint on \( \zeta \). In particular for \( SU(2) \) one finds \( \zeta_1 \zeta^1_2 = 0 \), and together with \( \hat{f}_x(\mu_1, \mu_1) = \hat{f}_x(\mu_2, \mu_2) \), \( \phi(x) \) is found to be a multiple of the identity, such that \( A_\alpha(x) = \frac{1}{\lambda} \phi(x) C_\alpha(x) \). The computation of \( C_\alpha(x) \) further simplifies when rotating \( \bar{\rho}_1 = -\bar{\rho}_2 = (0, 0, \pi \rho^2) \), which can be obtained from \( \zeta_1 = (1, 0) \rho \) and \( \zeta_2 = (0, 1) \rho \). With \( \chi \equiv \rho^2 \hat{f}_x(\mu_2, \mu_1) \) and
\( \phi(x) = (1 - \rho^2 \hat{f}_x(\mu_2, \mu_2))^{-1} \) this gives the result of ref. [7]

\[
A_\alpha = \frac{i}{2} \bar{\eta}_{\alpha\beta} \tau_3 \partial_\beta \log \phi + \frac{i}{2} \phi \text{Re} \left( (\bar{\eta}_{\alpha\beta} - i \eta_{\alpha\beta}^2) (\tau_1 + i \tau_2) \partial_\beta \chi \right).
\] (17)

Also for \( SU(3) \), \( \hat{f}_x(\mu_m, \mu_n) \) can be determined from the explicit result in eq. (6) (since two intervals on the circle, partitioned in three parts, always are neighbours). Choosing the three constituents to lie in the \( x-y \)-plane we can take \( \zeta_m = (|\vec{\rho}_m|, i\rho_{m}^2 - \rho_1^m) / \sqrt{2\pi|\vec{\rho}_m|} \). It is then a simple matter to compute the gauge field explicitly, albeit in a complicated form, due to the need to diagonalise \( \phi(x) \).

4. Polyakov loops and Taubes-winding

In the appendix of ref. [2] the following exact result was found for the \( SU(2) \) Polyakov loop along the line (taken along the \( z \)-axes) connecting the two constituents,

\[
\frac{1}{2} \text{Tr} P(0, 0, z) = -\cos(\nu_1 \pi + \frac{1}{2} \partial_z \text{acosh}[\frac{1}{2} \text{tr}(A_2 A_1)]). \quad (18)
\]

From this it is easily seen that \( P(\vec{x}) \) takes on each of the values \( \pm I_2 \) only once, with \( P(\vec{y}_1) = -I_2 \) and \( P(\vec{y}_2) = I_2 \) for well separated constituents. These are the equivalent to the conditions for the Higgs field to vanishing, an alternative way to specify the location of the constituents. When the constituents get nearer to each other, these "zeros" shift outwards (whereas the maxima of the energy density shift inwards), see figure 2.

![Figure 2](image-url)

Figure 2. Shift of the locations where \( P^2(\vec{x}) = I_2 \) as compared to the location of the constituent monopole centers \( \vec{y}_i \) for \( SU(2) \). Horizontally is plotted the distance \( d = \pi \rho^2 \) between the constituents and vertically the position of \( z_1 = \nu_2 d \) and \( z_2 = -\nu_1 d \), and the locations where \( P(0, 0, z) = I_2 \) \( (z > 0) \) and \( P(0, 0, z) = -I_2 \) \( (z < 0) \).
Note that the result for the Polyakov loop associated to the constituent at \( \vec{y}_1 \) is consistent with \( A_0 = 0 \), but for the constituent associated to \( \vec{y}_2 \) this only holds after applying a gauge transformation that is \textit{anti-periodic} in the time direction (since such a gauge transformation changes the sign of the Polyakov loop). This gauge transformation has an interesting relation to the so-called Taubes-winding [3], which is most easily understood by looking at the explicit expression for the gauge field. In the \textit{periodic gauge} one finds [7]

\[
A^\text{per}_\alpha(x) = \frac{i}{2} \eta^3_{\alpha\beta} \tau_3 \partial_\beta \log \phi + \pi i \nu_1 \tau_3 \delta_{\alpha,0} + \frac{i}{2} \phi \text{Re} \left( (\eta^1_{\alpha\beta} - i \eta^2_{\alpha\beta})(\tau_1 + i \tau_2)(\partial_\beta + 2 \pi i \nu_1 \delta_{\beta,0}) \tilde{\chi} \right),
\]

where \( \tilde{\chi} \) has the following expansion

\[
\tilde{\chi} \equiv e^{-2\pi i \nu_1} \chi = \frac{4\pi \rho^2}{(r_2 + r_1 + \pi \rho^2)^2} \left\{ r_2 e^{-2\pi r_2 \nu_2} e^{-2\pi it} + r_1 e^{-2\pi r_1 \nu_1} \right\},
\]

up to relative errors \( O(e^{-4\pi \min(r_1 \nu_1, r_2 \nu_2)}) \). For large \( \rho, \phi(x) \) becomes time independent, confirming the static nature of the configuration for large constituent separations. The time dependence of the constituent at \( \vec{y}_2 \) is a full (gauge) rotation - the \textit{Taubes-winding} - responsible for the topological charge of the otherwise time independent monopole pair [3], see figure 3. This gauge rotation is achieved by the \textit{anti-periodic} gauge transformation \( g(x) = \exp(-\pi i \tau_3) \), since \( g(x)(\tau_1 + i \tau_2)g^\dagger(x) = \exp(-2\pi it)(\tau_1 + i \tau_2) \). Also note that for the spherically symmetric Bogomol’nyi-Prasad-Sommerfield (BPS) monopole [14] a gauge rotation is equivalent to an ordinary rotation.

\[\text{Figure 3. The gauge field with unit topological charge is constructed from two oppositely charged monopoles by rotating one of them over one full rotation, while moving over one time-period.}\]
with mass proportional to the length \((\nu_m)\) of the interval. Taking \(|\vec{y}_m| \to \infty\) creates an infinite barrier for the interval \([\mu_n, \mu_{n+1}]\) and this leaves the interval \([\mu_1, \mu_n]\), allowing for the interpretation of an \(SU(n)\) monopole with \(\mu_i\) specifying the eigenvalues of the Higgs field at infinity, for which it is crucial the \(\mu_i\) add to zero. Indeed, in the periodic gauge \(A_0\) tends to a constant at spatial infinity.

This static monopole solution is composed out of \(n - 1\) basic BPS monopoles of mass \(\nu_m\), located at \(\vec{y}_m\), for \(m = 1, \ldots, n - 1\). We conclude that with our choice of parameters, always the field associated with the constituent at \(\vec{y}_m\) has to have a time-dependence to give rise to the topological charge of the caloron, and we conclude it is this one that carries the Taubes-winding. Note that our argument also demonstrates that for \(|\vec{y}_m| \to \infty\) with \(m \neq n\), one is left with a gauge field that cannot be time independent, even though the resulting action density is static \([6]\).

The question now arises, what is the equivalent of the zeros of the Higgs field for \(SU(n)\). To answer this question we remember that the constituents are basic BPS monopoles, which are obtained by embedding \(SU(2)\) in \(SU(n)\). The \(SU(2)\) subgroup relevant for this embedding is exactly determined by the unbroken \(SU(2)\) at the core of the constituent. The restoration of the \(SU(2)\) symmetry arises due to the degeneracy of two of the eigenvalues of the Higgs field, or of \(P(\vec{x})\) in our case (for \(n = 2\) this indeed implies a vanishing Higgs field, which has zero trace, or in our case it implies \(P(\vec{x}) = \pm I_2\) as its determinant is unity).

We denote by \(z_m\) the position associated to the the \(m\)-th constituent where two eigenvalues of the Polyakov loop coincide. Arranging by a global gauge rotation that \(P_\infty = P_\infty^0\) (see eq. (2)), we established numerically for \(SU(3)\) that with any choice of holonomy and constituent locations

\[
P_1 = P(\vec{z}_1) = \text{diag}(e^{-\pi i \mu_3}, e^{-\pi i \mu_3}, e^{2\pi i \mu_3}),
\]

\[
P_2 = P(\vec{z}_2) = \text{diag}(e^{2\pi i \mu_1}, e^{-\pi i \mu_1}, e^{-\pi i \mu_1}),
\]

\[
P_3 = P(\vec{z}_3) = \text{diag}(-e^{-\pi i \mu_2}, e^{2\pi i \mu_2}, -e^{-\pi i \mu_2}).
\]

For the caloron presented in figure 1 we find \(|z_m - \vec{y}_m| < 0.02\), and for well separated constituents \(z_m = \vec{y}_m\). The salient features are that \(P_m\) is determined by \(\mu_k\) not associated to the interval \([\mu_m, \mu_{m+1}]\) (on which \(A_j\) takes the value \(2\pi i y^j_m\), see eq. (4)). Furthermore, the extra minus signs occurring in \(P_3\) reflect the Taubes-winding, read-off from the gauge field in the core of the constituent. It will not be difficult to conjecture the values of \(P_n\) for general \(SU(n)\): \(P_1 = \text{diag}(e^{\pi i (\mu_1 + \mu_2)}, e^{\pi i (\mu_1 + \mu_2)}, e^{2\pi i \mu_3}, \ldots, e^{2\pi i \mu_n}, \ldots, \ldots, \), \(P_{n-1} = \text{diag}(e^{2\pi i \mu_1}, e^{2\pi i \mu_1}, e^{2\pi i \mu_{n-2}}, e^{2\pi i (\mu_{n-2} + \mu_1)}, e^{2\pi i (\mu_{n-1} + \mu_n)}),\)

\(P_n = \text{diag}(e^{\pi i (\mu_1 + \mu_n)}, e^{2\pi i \mu_2}, \ldots, e^{2\pi i \mu_{n-1}}, e^{-\pi i (\mu_1 + \mu_n)}).\) Note that \(P_n\) can also be written as \(P_n = \text{diag}(e^{\pi i (\mu_n + \mu_{n+1})}, e^{2\pi i \mu_2}, \ldots, e^{2\pi i \mu_{n-1}}, e^{\pi i (\mu_n + \mu_{n+1})}).\)
5. Discussion

As we have seen, an instanton has BPS monopoles as constituents, and the explicit results in eq. (19) also easily reveal the abelian limit. As long as all $\nu_m \neq 0$, the field outside the core of both monopoles is indeed that of two “BPS” Dirac monopoles (i.e. dyons). For $SU(2)$ outside the cores, i.e. assuming $r_m \nu_m \gg 1$ for all $m$, one has $\tilde{\chi} = 0$ and

$$\phi(x) = \frac{|\vec{x} - \vec{y}_1| + |\vec{x} - \vec{y}_2| + |\vec{y}_2 - \vec{y}_1|}{|\vec{x} - \vec{y}_1| + |\vec{x} - \vec{y}_2| - |\vec{y}_2 - \vec{y}_1|}. \quad (22)$$

We note that when neglecting the exponential corrections, $\phi^{-1}(x)$ vanishes on the line connecting the two constituents, and $\log \phi(x)$ is harmonic outside of this line. The singularity represents the return flux of the Dirac monopole pair, described in the abelian limit by a term proportional to $\partial_j^2 \log \phi(x)$ in the magnetic field

$$E_k = \frac{i}{2} \tau_3 \partial_k \partial_3 \log \phi, \quad B_k = \frac{i}{2} \tau_3 \left( \partial_k \partial_3 \log \phi - \delta_{k3} \partial_j^2 \log \phi \right). \quad (23)$$

In the full theory the return flux is absent (indeed $\phi^{-1}(x)$ has only an isolated zero at $x = 0$, corresponding to a gauge singularity [7]).

![Figure 4. A closed monopole line, rotating its frame when completing the circle. The topological charge is given by the net number of windings (here one) of the frame.](image)

As was already emphasised in ref. [4], to understand how in the abelian limit the topological charge can be recovered, one needs to preserve the Taubes-winding. This information is described by the framing of the core due to the charged components of the monopole field. Interestingly this describes a Hopf fibration [7], see figure 4. Recently the role of the Taubes-winding and Hopf fibration for retrieving topological charge was confirmed in great detail within the context of the abelian projection [17].

$^2$The Harrington-Shepard solution [15] with trivial holonomy (all $\mu_i = 0$), can be reinterpreted as a bound state of a massive and a massless constituent BPS monopole. Massless monopole constituents, giving rise to so-called non-abelian clouds, may play an important role in electric-magnetic duality as the dual of gluons [16]. These massless constituents are delocalised and have no obvious abelian limit.
We have also seen that the fermion zero-mode is localised on the constituent with the Taubes-winding and this makes it likely to conjecture that this holds for general monopole loops that support non-trivial Taubes-winding, which may be relevant to understanding chiral symmetry breaking in the context of monopole degrees of freedom. The question which field configurations are more important, is however a bit like the chicken and the egg problem. After all, we can now make instantons out of monopoles and monopoles out of instantons. Monopoles are typically used for describing the confining large distance behaviour. Recently it has, however, been pointed out there can be a hidden large size instanton component hitherto undetected [18]. Much remains to be done here, but the calorons have provided us interesting new avenues to follow.

Let us end by mentioning that the calorons with non-trivial holonomy have also been found on a finite lattice. For $SU(2)$ with twisted-boundary conditions remarkably it was found that when the twist is in the time direction, the constituent locations are free, but both constituents have equal mass, see figure 5. Whereas, when the twist is in the space direction, the constituent locations are maximally separated in the direction of the magnetic flux, but the mass ratios can be arbitrary [2]. This has been understood in the general context of the Nahm transformation for the torus, relating finite volume calorons to finite volume instantons [19]. The suggestion of ref. [7] to find these calorons on the lattice by freezing the links on the boundary of the lattice to enforce the proper holonomy has also been realised recently [20].
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