ON THE SPACE-LIKE ANALYTICITY IN THE EXTENSION PROBLEM FOR NONLOCAL PARABOLIC EQUATIONS

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Abstract. In this note we give an elementary proof of the space-like real analyticity of solutions to a degenerate evolution problem that arises in the study of fractional parabolic operators of the type \((\partial_t - \text{div}_x (B(x) \nabla_x))^s\), \(0 < s < 1\). Our primary interest is in the so-called extension variable. We show that weak solutions that are even in such variable, are in fact real-analytic in the totality of the space variables. As an application of this result we prove the weak unique continuation property for nonlocal parabolic operators of the type above, where \(B(x)\) is a uniformly elliptic matrix-valued function with real-analytic entries.

1. Introduction and Statement of the main result

We consider the space \(\mathbb{R}^{n+1}\) with generic variable \(X = (x, x_{n+1})\), where \(x \in \mathbb{R}^n, x_{n+1} \in \mathbb{R}\). We let \(|X| = \sqrt{|x|^2 + x_{n+1}^2}\) and denote by \(B_r = \{X \in \mathbb{R}^{n+1} | |X| < r\}\) the open ball of radius \(r\) centred at the origin. We also let \(R_{+}^{n+1} = \{X \in \mathbb{R}^{n+1} | x_{n+1} > 0\}\) and indicate the upper part of the open ball with \(B_+^r = B_r \cap R_{+}^{n+1}\). The symbol \(B_-^r\) will indicate the corresponding lower part of \(B_r\). The thin space \(\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}\) will be routinely identified with \(\mathbb{R}^n\), and we let \(B_r = B_r \cap \mathbb{R}^n\). We indicate with \(x \to A(x) = [a_{ij}(x)]\) a given \((n+1) \times (n+1)\) matrix-valued function of the form

\[
a_{ij}(x) = \sum_{i,j=1}^{n} b_{ij}(x) e_i \otimes e_j + e_{n+1} \otimes e_{n+1},
\]

where the \(b_{ij}\)’s are assumed symmetric, uniformly elliptic, independent of \(x_{n+1}\), and globally real-analytic in \(\mathbb{R}^n\). Moreover, we assume that the heat kernel of the parabolic operator \(\partial_t - \text{div}_x (B(x) \nabla_x)\) satisfies the stochastic completeness assumption \((2.1)\) below. From now on, we indicate with \(\text{div}\) and \(\nabla\) respectively the divergence and gradient with respect to the variable \(X \in \mathbb{R}^{n+1}\).

Given a function \(U(X, t)\) in \(\mathbb{R}^{n+1}_+ \times \mathbb{R}\), and a number \(a \in (-1, 1)\), we refer to \((2.3)\) below for the meaning of weighted normal derivative \(\partial^a_{x_{n+1}} U((x, 0), t)\) on the thin space \(\mathbb{R}^n \times \mathbb{R}\). The purpose of this note is to present an elementary proof of the following result.

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Theorem 1.1. Let \( a \in (-1, 1) \) and \( T > 0 \). Assume that the function \( U = U(X, t) \) is a weak solution to the following degenerate parabolic problem

\[
\begin{cases}
\mathcal{L}_a U \overset{\text{def}}{=} \text{div}(x_{n+1}^a A(x) \nabla U) - x_{n+1}^a U_t = 0 & \text{in } \mathbb{B}_3^+ \times (-1, T],
\partial_{x_{n+1}} U((x, 0), t) = 0 & \text{in } B_3 \times (-1, T].
\end{cases}
\]

Let \( \hat{U} \) denote the even reflection of \( U \) across \( \{x_{n+1} = 0\} \). Then \( \hat{U} \in C^\infty(\mathbb{B}_1 \times (0, T)) \) and, for every fixed \( t \in (0, T) \), the function \( X \to \hat{U}(X, t) \) is real-analytic in \( \mathbb{B}_1 \).

We note that if we replace the Neumann condition in (1.2) with the Dirichlet assumption \( U((x, 0), t) = 0 \) in \( B_3 \times (-1, T] \), then Theorem 1.1 ceases to hold. Take for instance \( B(x) \equiv I_n \) and \( U(X, t) = x_{n+1}^{1-a/2} \). As it is well-known by now, if \( s \in (0, 1) \) and \( a = 1 - 2s \), then (1.2) represents the parabolic counterpart of the Caffarelli-Silvestre extension problem for the fractional operator \( (\partial_t - \text{div}_x(B(x)\nabla_x))^s \). In view of this aspect, as a consequence of Theorem 1.1 we obtain the following.

Corollary 1.2. Let \( u(x, t) \) solve \( (\partial_t - \text{div}_x(B(x)\nabla_x))^s u = 0 \) in \( B_1 \times (-1, 0) \). Then \( x \to u(x, t) \) is real-analytic in \( B_1 \) for any fixed \( t \).

The reader should note that we have assumed global real-analyticity of \( B(x) \) only for the simplicity of exposition. We only need \( B(x) \) to be real-analytic in, say, \( B_3 \). Moreover, the stochastic completeness hypothesis (2.1) is easily ensured by e.g. smoothly extending \( B(x) \) to the whole of \( \mathbb{R}^n \) in such a way that \( B(x) \equiv I_n \) outside \( B_5 \).

Real-analyticity results for nonlocal operators have a long history that goes back to the seminal work of M. Riesz [19] for the fractional Laplacian, and especially Kotake-Narasimhan [12] for operators of the type \( L^s \), where \( L \) is a general elliptic operator of order \( 2m \). In the special case \( A(x) \equiv I_{n+1} \), and for time-independent solutions to (1.2), the real-analyticity has been recently proved in [10, Appendix B] using functional analytic tools and weak type estimates. Differently from [10], our approach is based on the fundamental solution of the extended operator in (1.2) computed in [8], which we use to obtain an explicit Green type representation of symmetric solutions. In order to circumvent a singularity in such representation, we use: (a) the space-like real analyticity of the fundamental solution to complexify the space variables; (b) a limiting argument. Part (b) involves careful estimates of the heat kernel in (2.6) below of the extended operator in (1.2), and this is the key novelty of our work. This general scheme, based on Green representation and complexifying the space variables, is inspired to an idea in [9] for the heat equation.
As an interesting application of Theorem 1.1 we obtain the following weak unique continuation property.

**Proposition 1.3.** Let \( B(x) \) be a uniformly elliptic matrix-valued function with real-analytic entries and let \( u \) be a solution to \( (\partial_t - \text{div}_x(B(x)\nabla_x))^n u = 0 \), such that \( u = 0 \) in \( B_1 \times (-1,0) \). Then \( u \equiv 0 \) in \( \mathbb{R}^n \times (-1,0) \).

The reader should note that the previous result displays a purely nonlocal phenomenon: the zero set of \( u \) propagates even in the region where the equation is not satisfied. Besides its own interest, Proposition 1.3 finds application to Runge type approximations for inverse problems, and in the special case when \( B(x) \equiv I_n \) it has been earlier obtained in [13, Prop. 5.5] by means of Carleman estimates. Our elementary approach shows that the use of Carleman estimates can be avoided.

The paper is organized as follows. In Section 2 we introduce some basic notations and gather some preliminary results that are relevant to our work. In Section 3 we prove Theorem 1.1 and Proposition 1.3.

### 2. Preliminaries

In this section we collect some known results that will be used in this note. Without loss of generality, we will assume that the uniformly parabolic operator \( \partial_t - \text{div}(B(x)\nabla_x) \) in \( \mathbb{R}^n \times \mathbb{R} \) has a globally defined fundamental \( p(x,x',t) \) that satisfies for every \( x \in \mathbb{R}^n \) and \( t > 0 \)

\[
P_t 1(x,t) = \int_{\mathbb{R}^n} p(x,x',t) dx' = 1.
\]

We will also assume without any restriction that the power series expansion centred at 0 of the matrix \( A(x) \) converges uniformly in \( B_8 \). Henceforth, following a standard use we will write \( z = x + iy, z' = x' + iy' \), etc. for points in \( \mathbb{C}^n \). We will routinely identify with \( x \in \mathbb{R}^n \) the point \( x + i0 \in \mathbb{C}^n \). In Section 3 we will also need to complexify the thick space \( \mathbb{R}^{n+1} \). Thus, we will denote by \( Z = X + iY, Z' = X' + iY', \) etc., points in \( \mathbb{C}^{n+1} \). Again, the point \( X = (x,x_{n+1}) \in \mathbb{R}^{n+1} \) will be routinely identified with \( X + i0 \in \mathbb{C}^{n+1} \). We then have the following result, which is [6, Theor. 8.1, p. 178]. We state it as a lemma since in this note we make multiple references to it.

**Lemma 2.1.** Let \( x' \in B_3 \). For \( t > 0 \) the fundamental solution \( p(x,x',t) \) can be analytically continued to a function \( p(z,x',t) \) defined in the complex ball \( \{ z = x + iy \in \mathbb{C}^n \mid |z - x'| < 5 \} \).
Moreover, there exist positive constants $C_0, c_0$ and $c_1$, depending on the ellipticity and the real-analytic character of the coefficient matrix $A(x)$, such that the following estimates hold

$$
\begin{aligned}
\left| p(z, x', t) \right| &\leq C_0 t^{-\frac{n+1}{2}} e^{-\frac{c_0 |x-x'|^2 - c_1 |z|^2}{4t}}, \\
|\nabla_x p(z, x', t)| &\leq C_0 t^{-\frac{n+1}{2}} e^{-\frac{c_0 |x-x'|^2 - c_1 |z|^2}{4t}}.
\end{aligned}
$$

(2.2)

With the notation in the opening of the note, given a function $U(X, t) = U((x, x_{n+1}), t)$ in $\mathbb{R}_+^{n+1} \times (0, \infty)$, and a number $a \in (-1, 1)$, we denote with $\partial^a_{x_{n+1}} U$ the weighted normal derivative

$$
\partial^a_{x_{n+1}} U((x, 0), t) \overset{\text{def}}{=} \lim_{x_{n+1} \to 0^+} x_{n+1}^{a} \partial_{x_{n+1}} U((x, x_{n+1}), t).
$$

(2.3)

The second-order degenerate parabolic differential operator $\mathcal{L}_a$ is defined as in (1.2) above. As we have mentioned in the introduction, if $s \in (0, 1)$ and $a = 1 - 2s$, then (1.2) represents the parabolic counterpart of the Caffarelli-Silvestre extension problem for the fractional operator $(\partial_t - \text{div}(B(x) \nabla_x))^s$. For some background on this problem the reader is referred to [11, 5, 18, 22, 3, 4, 16].

We next recall that it was proved in [8] that, given $\phi \in C^\infty_0(\mathbb{R}_+^{n+1})$, the solution of the Cauchy problem with Neumann condition

$$
\begin{aligned}
\mathcal{L}_a U &= 0 &\text{in } \mathbb{R}_+^{n+1} \times (0, \infty) \\
U(X, 0) &= \phi(X), &X \in \mathbb{R}_+^{n+1}, \\
\partial^a_{x_{n+1}} U(x, 0, t) &= 0 &x \in \mathbb{R}^n, \ t \in (0, \infty)
\end{aligned}
$$

(2.4)

is given by the formula

$$
\mathcal{P}_t^{(a)} \phi(X) \overset{\text{def}}{=} U(X, t) = \int_{\mathbb{R}_+^{n+1}} \phi(X') \mathcal{G}(X, X', t) (x'_{n+1})^a dX',
$$

(2.5)

where

$$
\mathcal{G}(X, X', t) = p(x, x', t) p^{(a)}(x_{n+1}, x'_{n+1}, t).
$$

(2.6)

In (2.6) we have indicated with $p(x, x', t)$ the fundamental solution of the operator $\partial_t - \text{div}(B(x) \nabla_x)$, and with $p^{(a)}(x_{n+1}, x'_{n+1}, t)$ that of the Bessel operator $\mathcal{B}_a = \partial^a_{x_{n+1}} + \frac{a}{x_{n+1}} \partial_{x_{n+1}}$ on $\mathbb{R}_+^{n+1} dx_{n+1}$ with Neumann boundary condition in $x_{n+1} = 0$ (reflected Brownian motion). Such function is given by the formula

$$
p^{(a)}(x_{n+1}, x'_{n+1}, t) = (2t)^{-\frac{n+a+1}{2}} \left( \frac{x_{n+1}x'_{n+1}}{2t} \right)^{1-a} I_{\frac{a}{2}} \left( \frac{x_{n+1}x'_{n+1}}{2t} \right) e^{-\frac{x_{n+1}^2 + (x'_{n+1})^2}{4t}}.
$$

(2.7)
In (2.7) we have denoted by $I_{\frac{a-1}{2}}$ the modified Bessel function of the first kind and order $\frac{a-1}{2}$ defined by the power series
\begin{equation}
I_{\frac{a-1}{2}}(w) = \sum_{k=0}^{\infty} \frac{(w/2)^{a+2k}}{\Gamma(k+1)\Gamma(k+1+(a-1)/2)}, \quad |w| < \infty, \quad |\arg w| < \pi.
\end{equation}

From the asymptotic behaviour of $I_{\frac{a-1}{2}}(w)$ near $w = 0$ and at infinity, one immediately obtains the following estimate for some $C(a), c(a) > 0$ (see e.g. [14, formulas (5.7.1) and (5.11.8)]),
\begin{equation}
|I_{\frac{a-1}{2}}(w)| \leq C(a)|w|^{\frac{a}{2}} \quad \text{if} \quad 0 < |w| \leq c(a), \quad I_{\frac{a-1}{2}}(w) \leq C(a)w^{-1/2}e^w \quad \text{if} \quad w \geq c(a).
\end{equation}

For future use we note explicitly that (2.6) and (2.7) imply that for every $x, x' \in \mathbb{R}^n$ and $t > 0$ one has
\begin{equation}
\lim_{x_{n+1} \to 0^+} x_{n+1}^{a} \partial_{x_{n+1}} \mathcal{G}((x, x_{n+1}), (x', 0), t) = 0.
\end{equation}

We emphasise that for every fixed $t > 0$ and $X' \in \mathbb{R}^{n+1}$ the function $X \to \mathcal{G}(X, X', t)$ is real-analytic in $\mathbb{R}^{n+1}$. This will play a crucial role in our proof of Theorem 1.1. It is also important to keep in mind that using (2.1), and [8, Propositions 2.3 and 2.4], we infer that (2.5) defines a stochastically complete semigroup, and therefore for every $X \in \mathbb{R}^{n+1}_+$ and $t > 0$ we have
\begin{equation}
\mathcal{P}^{(a)}_t 1(X) = \int_{\mathbb{R}^{n+1}_+} \mathcal{G}(X, X', t)(x_{n+1})^a dX' = 1,
\end{equation}
and also
\begin{equation}
\mathcal{P}^{(a)}_t \phi(X) \to \phi(X). \quad \text{as} \quad t \to 0^+.
\end{equation}

We close this section by stating a preliminary regularity result for (1.2) (we refer to [15, Chap. 4] for the relevant notion of parabolic Hölder spaces). In its statement we denote by $\tilde{U}$ the even extension of $U$ across $\{x_{n+1} = 0\}$, and for economy of notation we continue to denote with $\mathcal{L}_a$ the extension (2.13) of the operator to the whole space-time space $\mathbb{R}^{n+1} \times \mathbb{R}$. We explicitly remark that the following Lemma 2.2 only requires that the matrix-valued function $A(x)$ be continuous.

**Lemma 2.2.** Let $\tilde{U}$ be a weak solution to (1.2). Then the function $\tilde{U}$ solves
\begin{equation}
\mathcal{L}_a \tilde{U} \overset{\text{def}}{=} \text{div}(|x_{n+1}|^a A(x) \nabla \tilde{U}) - |x_{n+1}|^a \partial_t \tilde{U} = 0
\end{equation}
in $\mathbb{B}^3 \times (-1, T]$, and moreover $\tilde{U} \in H^{1+\alpha}(\mathbb{B}_2 \times (0, T])$ for all $\alpha > 0$.

**Proof.** The fact that $\tilde{U}$ solves (2.13) is standard. When $A(x) \equiv I_{n+1}$, it follows from [2, Lemma A.1] that $\tilde{U} \in H^{2+\beta}(\mathbb{B}_2 \times (0, T])$ for some $\beta > 0$. Applying compactness arguments as in the proof of Proposition A.3 in [2], we can conclude that $\tilde{U} \in H^{1+\alpha}(\mathbb{B}_2 \times (0, T])$ for any $\alpha > 0$. 

3. Proof of the main result

In this section we present the proof of Theorem 1.1. For notational convenience, throughout the proof we will respectively denote by $\tilde{U}, \tilde{V}$ the symmetric extensions across the thin set $\{x_{n+1} = 0\}$ of the relevant functions $U, V$. Also, in the ensuing computations we denote by $dS$ the surface measure on $\partial \mathbb{B}_2$.

Proof of Theorem 1.1. We consider the region $\mathbb{B}_2 \times (0, T]$. For a given $X \in \mathbb{B}_1$ and $\varepsilon > 0$, we let

$$V(X', t) = \mathcal{G}(X, X', T + \varepsilon - t).$$

Since the pole of $\mathcal{G}(X, X', T + \varepsilon - t)$ is at the point $(X, T + \varepsilon)$, the reflected function $\tilde{V}$ is smooth in $\mathbb{B}_2 \times (0, T)$, real-analytic in $X'$ for every fixed $t \in (0, T)$, and solves the following backward equation

$$\mathcal{L}_a^* \tilde{V} = \text{div}(|x_{n+1}'|^a A(x')\nabla \tilde{V}) + |x_{n+1}'|^a \partial_t \tilde{V} = 0.$$  

Let us notice that the equations (2.13) and (3.2), satisfied by $\tilde{U}$ and $\tilde{V}$ respectively, give in the cylinder $[\mathbb{B}_2 \cap \{|x_{n+1}'| > \delta\}] \times (0, T)$

$$0 = V.\mathcal{L}_a^* \tilde{V} - \tilde{U}.\mathcal{L}_a^* \tilde{V} = \text{div}(|x_{n+1}'|^a (\tilde{V}A(x')\nabla \tilde{U} - \tilde{U}A(x')\nabla \tilde{V}) - |x_{n+1}'|^a (\tilde{U} \tilde{V})).$$

Integrating by parts on such set, using the regularity result in Lemma 2.2 and the zero Neumann condition (2.10) satisfied by $\tilde{V}$, after letting $\delta \to 0$ we find

$$0 = \int_{\mathbb{B}_2} \tilde{V}(X', T)\tilde{U}(X', T)|x_{n+1}'|^a dX' - \int_{\mathbb{B}_2} \tilde{V}(X', 0)\tilde{U}(X', 0)|x_{n+1}'|^a dX'$$

$$- \frac{1}{2} \int_0^T dt \int_{\partial \mathbb{B}_2} \{ \tilde{V} \langle A \nabla \tilde{U}, X' \rangle - \tilde{U} \langle A \nabla \tilde{V}, X' \rangle \} |x_{n+1}'|^a dS(X').$$

If we now let $\varepsilon \to 0^+$, using (2.12) we have

$$\int_{\mathbb{B}_2} \tilde{V}(X', T)\tilde{U}(X', T)|x_{n+1}'|^a dX' \to 2\tilde{U}(X, T).$$

The factor 2 in the right-hand side of (3.4) is caused by the fact that, in view of (2.5), (2.11), each half of the integral on $\mathbb{B}_2$ tends to $\tilde{U}(X, T)$ in the limit as $\varepsilon \to 0^+$. Using (3.4) in (3.3) we thus obtain

$$2\tilde{U}(X, T) = \int_{\mathbb{B}_2} \mathcal{G}(X, X', T)\tilde{U}(X', 0)|x_{n+1}'|^a dX'$$

$$+ \frac{1}{2} \int_0^T dt \int_{\partial \mathbb{B}_2} \{ \mathcal{G}(X, X', T - t)\langle A(x')\nabla \tilde{U}(X', t), X' \rangle \}$$

□
\[-\tilde{U}(X', t)\langle A(x')\nabla \mathcal{G}(X, X', T - t), X'\rangle |x'_{n+1}|^a dS(X')\]

We note that (3.5) constitutes a Green representation for \(\tilde{U}(X, T)\). Since the functions \(\mathcal{G}, \nabla \mathcal{G}\) are \(C^\infty\) in \((X, T)\), (3.5) and a fairly standard limiting argument imply, in particular, that a local solution to (1.2) is \(C^\infty\) in \((X, T)\). However, the real-analyticity in \(X\) cannot similarly be obtained, and this is why we next resort to complexifying the function \(\mathcal{G}\). In so doing, however, the estimates in the imaginary direction in \(\mathbb{C}^{n+1}\) deteriorate and we need a more delicate analysis.

As we have mentioned already, from Lemma 2.1, the expression of \(\mathcal{G}\) in (2.6), and from (2.8), one sees that the symmetric extension of \(\mathcal{G}(X, X', t)\) is real-analytic in \(X\) and \(X'\) for \(t < T\) and \(|X - X'| < 5\). Furthermore, a computation and the identity

\[
\frac{d}{dw}[w^{-\nu} I_\nu(w)] = w^{-\nu} I_{\nu+1}(w),
\]

(see for instance [14, (5.7.9) on p.110]) allow to verify that jointly in \(X\) and \(X'\) variables,

\[
\langle A(x')\nabla \mathcal{G}(X, X', T - t), X'\rangle
\]
is also a real-analytic function, symmetric in the variables \(x_{n+1}\) and \(x'_{n+1}\). We now use the formula (3.5) to extend \(X \to \tilde{U}(X, T)\) to \(Z \to \tilde{U}(Z, t)\), where \(Z = X + iY\) ranges in an appropriate domain \(D \subset \mathbb{C}^{n+1}\). Using Lemma 2.1, (2.6) and (2.8), it is seen that the first integral in the right-hand side of (3.5) can be extended to an analytic function of \(Z\) for \(|Z| < 2\). The analyticity of the second integral is not obvious and we thus proceed with the more delicate arguments that follow. For a given \(\delta > 0\), we let

\[
F_\delta(Z) = \int_0^{T-\delta} dt \int_{\partial \overline{B}_2} \left\{ \mathcal{G}(Z, X', T - t)\langle A(x')\nabla \tilde{U}(X', t), X'\rangle - \tilde{U}(X', t)\langle A(x')\nabla \mathcal{G}(Z, X', T - t), X'\rangle \right\} |x'_{n+1}|^a dS(X')
\]

From Lemma 2.1, (2.6) and (2.8) the function \(F_\delta\) is analytic in the region \(\{|Z| < 3/2\}\). We intend to show that, for a suitably chosen number \(\varepsilon_0 > 0\), the holomorphic functions \(F_\delta\)'s are uniformly convergent as \(\delta \to 0\) in the region

\[
D(\varepsilon_0) \overset{def}{=} \{Z = X + iY \mid X \in \mathbb{B}_1 \text{ and } |Y| \leq \varepsilon_0\}.
\]

Since uniformly convergent sequences of holomorphic functions have holomorphic limits (see e.g. [17, Prop. 5 in Chap. 1]), we would infer that \(\lim_{\delta \to 0} F_\delta = F\) is holomorphic in \(D(\varepsilon_0)\). On the other
hand, we clearly have
\[ F(Z) = \int_0^T dt \int_{\partial B_2} \left\{ \mathcal{G}(Z, X', T - t) (A(x') \nabla \tilde{U}(X', t), X') - \tilde{U}(X', t) (A(x') \nabla \mathcal{G}(Z, X', T - t), X') \right\} |x'_{n+1}|^a dS(X'). \]

From the representation (3.5) we would thus conclude that \( X \to \tilde{U}(X, T) \) is real-analytic in \( B_1 \).

To prove the uniform convergence of the \( F_\delta \)'s in the appropriate region \( D(\varepsilon_0) \subset \mathbb{C}^{n+1} \), we proceed as follows. We write \( F_\delta(Z) = F^1_\delta(Z) - F^2_\delta(Z) \), where
\[
(3.8) \quad F^1_\delta(Z) = \int_0^T dt \int_{\partial B_2} \mathcal{G}(Z, X', T - t) (A(x') \nabla \tilde{U}(X', t), X') |x'_{n+1}|^a dS(X'),
\]
\[
(3.9) \quad F^2_\delta(Z) = \int_0^T dt \int_{\partial B_2} \tilde{U}(X', t) (A(x') \nabla \mathcal{G}(Z, X', T - t), X') |x'_{n+1}|^a dS(X'),
\]
and prove that \( F^k_\delta(Z), k = 1, 2 \) converge uniformly for \( Z \in D(\varepsilon_0) \). Since the arguments are essentially identical, we present details only for \( F^1_\delta(Z) \), confining ourselves to briefly indicate at the end the changes necessary to treat \( F^2_\delta(Z) \). We first note that (2.6) and (2.7) give
\[
(3.10) \quad \mathcal{G}(Z, X', T - t) = p(x + iy, x', T - t) p^{(a)}(x_{n+1} + iy_{n+1}, x_{n+1}', T - t)
\]
\[= p(x + iy, x', T - t) (2(T - t))^{-\frac{2a+1}{2}} e^{-\frac{(x_{n+1})^2 + x_{n+1}'^2 + y_{n+1}^2 + 2ix_{n+1}y_{n+1}}{4(T - t)}} \]
\[\times \left( \frac{x_{n+1}'(x_{n+1} + iy_{n+1})}{2(T - t)} \right)^{\frac{1-a}{2}} I_{\frac{a-1}{2}} \left( \frac{x_{n+1}'(x_{n+1} + iy_{n+1})}{2(T - t)} \right). \]

To establish the uniform convergence of \( F^1_\delta(Z) \) it will thus suffice to show that, for a sufficiently small choice of \( \varepsilon_0 > 0 \), the function \( \mathcal{G}(Z, X', T - t) \) is uniformly bounded as \( t \to T \) for \( Z \in D(\varepsilon_0) \).

For later use we notice that for \( Z = X + iY \in D(\varepsilon_0) \) we have \( |y_{n+1}| \leq |Y| \leq \varepsilon_0 \). Since for \( X' \in \partial B_2 \) we have \( |x'_{n+1}| \leq |X'| \leq 2 \), we thus have
\[
(3.11) \quad |x'_{n+1}y_{n+1}| \leq 2\varepsilon_0.
\]

Furthermore, in view of the symmetry in the \((n + 1)\)-th coordinate of \( \mathcal{G} \) and \( \langle A(x') \nabla \mathcal{G}, X' \rangle \), it suffices to consider points \( X' \in \partial \mathbb{B}_2 \cap \{x_{n+1} > 0\} \), and \( Z = X + iY \) such that \( X \in \overline{\mathbb{B}_1^+} \). We split the analysis into Cases (1) & (2), each of them composed of two subcases, (1a) & (1b), and (2a) & (2b). In the sequel \( c_0, c_1 \) and \( C(a), c(a) \) will respectively denote the constants in (2.2) of Lemma
2.1 and those in (2.9). Also, from the definition (2.8) of the modified Bessel function it is clear that

\[(3.12)\]

\[
\left| \left( \frac{x_{n+1}'(x_{n+1} + iy_{n+1})}{2(T-t)} \right)^{1+a} I_{-a} \left( \frac{x_{n+1}'(x_{n+1} + iy_{n+1})}{2(T-t)} \right) \right| \leq \left| \left( \frac{x_{n+1}'(x_{n+1} + iy_{n+1})}{2(T-t)} \right)^{1+a} I_{-a} \left( \frac{x_{n+1}'(x_{n+1} + iy_{n+1})}{2(T-t)} \right) \right|.
\]

**Case (1):** \((x_{n+1}')^2 + x_{n+1}^2 \leq \varepsilon_0\). If we assume that

\[(3.13)\]

\[\varepsilon_0 < 1/16,\]

we see that it must be

\[(3.14)\]

\[|x_{n+1} - x_{n+1}'| \leq |x_{n+1}| + |x_{n+1}'| \leq 2\varv_0 < 1/2.\]

Moreover, since \(X' \in \partial \mathbb{B}_2^+\) and \(X \in \mathbb{B}_2^+\), we have \(|X' - X| > 1\). Using this along with (3.14), by an application of the triangle inequality we deduce that the following holds

\[(3.15)\]

\[|x - x'| \geq |X - X'| - |x_{n+1} - x_{n+1}'| > 1/2.\]

We now distinguish two possibilities.

**Case (1a):** \(\frac{x_{n+1}'(x_{n+1} + iy_{n+1})}{2(T-t)} \leq c(a)\). In such case, by (3.12) and the first inequality in (2.9), we find

\[(3.16)\]

\[
\left| \left( \frac{x_{n+1}'(x_{n+1} + iy_{n+1})}{2(T-t)} \right)^{1+a} I_{-a} \left( \frac{x_{n+1}'(x_{n+1} + iy_{n+1})}{2(T-t)} \right) \right| \leq C(a).
\]

If we now use in (3.10) the first inequality in (2.2), (3.15), (3.16), and also the fact that in \(D(\varepsilon_0)\) we have \(|Y| \leq \varepsilon_0\), we find that the following estimate holds

\[|\mathcal{G}(Z, X', T - t)| \leq \frac{C(n, a)}{(T-t)^{(n+a+1)/2}} e^{-\frac{c_0/a - (c_1 + 1)\varepsilon_0^2}{2(T-t)}}.\]

If in addition to (3.13) we assume that

\[(3.17)\]

\[\varepsilon_0^2 < \frac{c_0}{8(c_1 + 1)},\]

we can thus guarantee that \(\mathcal{G}(Z, X', T - t)\) is uniformly bounded as \(t \to T\).

**Case (1b):** \(\frac{x_{n+1}'(x_{n+1} + iy_{n+1})}{2(T-t)} > c(a)\). In this case, (3.12) and the second inequality in (2.9) imply

\[(3.18)\]

\[
\left| \left( \frac{x_{n+1}'(x_{n+1} + iy_{n+1})}{2(T-t)} \right)^{1+a} I_{-a} \right| \leq C(a)c(a)^{1+a} \left| \left( \frac{x_{n+1}'(x_{n+1} + iy_{n+1})}{2(T-t)} \right)^{1+a} \right| \exp \left\{ \left| \frac{x_{n+1}'(x_{n+1} + iy_{n+1})}{2(T-t)} \right| \right\}
\]
where we have used the inequality (3.11). If we now use in (3.10) the first estimate in (2.2), (3.15), (3.18) and again \( |Y| \leq \varepsilon_0 \), we find

\[
|\mathcal{G}(Z, X', T - t)| \leq \frac{C(n, a)}{(T - t)^{\frac{n}{2} + 2}} e^{-\frac{\varepsilon_0 + \varepsilon_0 - \varepsilon_0}{4(T - t)}} \leq \frac{C(n, a)}{(T - t)^{\frac{n}{2} + 2}} e^{-\frac{\varepsilon_0}{4(T - t)}},
\]

provided that

\[
(3.19) \quad 4\varepsilon_0 + (c_1 + 1)\varepsilon_0^2 < \frac{c_0}{8}.
\]

Under the hypothesis (3.13) and (3.19), we again obtain the uniform boundedness of \( \mathcal{G}(Z, X', T - t) \) as \( t \to T \). We next consider the situation complementary to Case (1).

Case (2): \((x'_{n+1})^2 + x_{n+1}^2 > \varepsilon_0\). In this case also we distinguish two possibilities, Cases (2a) and (2b). Case (2a) will be further subdivided into two subcases, Case (2a)\(_1\) and (2a)\(_2\).

Case (2a): \(\frac{x'_{n+1}^2 + x_{n+1}^2}{2(T - t)} \leq c(a)\). As we have said, this case is further subdivided into two subcases.

Case (2a)\(_1\): \(\frac{x'_{n+1}^2 + x_{n+1}^2}{2(T - t)} \leq c(a)\). As in Case (1a), we again have the bound (3.16), which we use in (3.10) along with the first estimate in (2.2), the present assumptions that \((x'_{n+1})^2 + x_{n+1}^2 > \varepsilon_0\), and \(|Y| \leq \varepsilon_0\), to obtain (note that we have used \( |\exp( -\frac{\varepsilon_0}{4(T - t)})| = 1 \))

\[
|\mathcal{G}(Z, X', T - t)| \leq \frac{C(n, a)(n+a+1)}{(T - t)^{\frac{n}{2} + 2}} e^{-\frac{\varepsilon_0 - (c_1 + 1)\varepsilon_0}{4(T - t)}} \leq \frac{C(n, a)}{(T - t)^{\frac{n}{2} + 2}} e^{-\frac{\varepsilon_0}{8(T - t)}},
\]

provided \( \varepsilon_0 \) is such that

\[
(3.20) \quad 1 > 2(c_1 + 1)\varepsilon_0.
\]

Under (3.20) we thus have that \( \mathcal{G}(Z, X', T - t) \) is uniformly bounded as \( t \to T \).

Case (2a)\(_2\): \(\frac{x'_{n+1}^2 + x_{n+1}^2}{2(T - t)} > c(a)\). Similarly to Case (1b), we again have the inequality (3.18), which we use in (3.10), along with the first estimate in (2.2), and the already observed inequality \(|X' - X| > 1\), to find

\[
|\mathcal{G}(Z, X', T - t)| \leq \frac{C(n, a)}{(T - t)^{(n+2)/2}} e^{-\frac{\varepsilon_0|X - X'|^2 + |x_{n+1} - x'_{n+1}|^2 - 4\varepsilon_0 - (c_1 + 1)\varepsilon_0^2}{4(T - t)}} \leq \frac{C(n, a)}{(T - t)^{(n+2)/2}} e^{-\frac{\min(c_0, 1)|X - X'|^2 - 4\varepsilon_0 - (c_1 + 1)\varepsilon_0^2}{4(T - t)}} \leq \frac{C(n, a)}{(T - t)^{(n+2)/2}} e^{-\frac{\min(c_0, 1) - 4\varepsilon_0 - (c_1 + 1)\varepsilon_0^2}{4(T - t)}},
\]
\[
\leq \frac{C(n,a)}{(T-t)^{(n+2)/2}} e^{-\frac{7/8 \min(c_0,1)}{4(T-t)}} ,
\]
provided \( \varepsilon_0 \) satisfies
\[
\frac{\min(c_0,1)}{8} > 4\varepsilon_0 + (c_1 + 1)\varepsilon_0^2.
\]
Thus, also in this Case (2a) we find that \( \mathcal{G}(Z, X', T-t) \) is uniformly bounded as \( t \to T \). Combining this with the discussion in Case (2a), we infer that the same conclusion holds in Case (2a) provided that \( \varepsilon_0 \) is small enough.

**Case (2b)** \( \frac{x_{n+1}'}{2(T-t)} > c(a) \). Since this assumption obviously implies \( \left| \frac{x_{n+1}'(x_{n+1}+iy_{n+1})}{2(T-t)} \right| > c(a) \), we can repeat the arguments in Case (2a) and obtain a uniform bound for \( \mathcal{G}(Z, X, T-t) \) as \( t \to T \).

With all this being said, we now choose \( \varepsilon_0 > 0 \) so small that (3.13), (3.17), (3.19), (3.20) and (3.21) concurrently hold. This guarantees that the function \( F_\delta(Z) \) in (3.8) converges uniformly for \( Z \in D(\varepsilon_0) \) as \( \delta \to 0 \). For the uniform of convergence of \( F_\delta^2(Z) \), we use (3.6) to obtain a representation of
\[
\langle A(x')\nabla \mathcal{G}(Z, X', T-t), X' \rangle
\]
similar to that in (3.10) for \( \mathcal{G}(Z, X', T-t) \), but this time in terms of the modified Bessel functions \( I_{a+\frac{1}{2}} \) and \( I_{a-\frac{1}{2}} \). Once this is observed, we can repeat the arguments in Case (1) and (2) above, except that now we need to also use the second estimate in (2.2). Since at this point the reader can easily fill in the necessary details, we skip them altogether and just affirm that \( F_\delta(Z) \) converges uniformly in \( Z \in D(\varepsilon_0) \). In view of the discussion after (3.7), we conclude that \( X \to \tilde{U}(X, T) \) is real-analytic in \( \mathbb{B}_1 \) and this finishes the proof of the theorem.

\[\square\]

With Theorem 1.1 in hand, we now provide the

**Proof of Proposition 1.3.** Given \( u \) as in the statement of the proposition, let \( U \) be the solution of the corresponding extension problem (1.2), i.e.,
\[
\begin{cases}
\mathcal{L}_a U = 0 \text{ in } \{x_{n+1} > 0\} \\
U = \partial_{x_{n+1}}^a U = 0 \text{ on } \{x_{n+1} = 0\} \cap [B_1 \times (-1,0)],
\end{cases}
\]
with \( \mathcal{L}_a \) as in (1.2) and where the \( (n+1) \times (n+1) \) matrix valued function \( A(x) \) is of the form (1.1). From Theorem 1.1 we infer that the evenly reflected \( \tilde{U} \) is space-like real analytic in \( \mathbb{B}_1 \times (-1,0) \). Moreover, since \( U \) and \( \partial_{x_{n+1}}^a U \) both vanish on \( \{x_{n+1} = 0\} \), by an argument in [13, Lemma 5.1] (see also the proof of [3, Lemma 7.7]), which involves repeated differentiation in \( y \)-variable and a
bootstrap type argument, it follows that for every $t \in (-1, 0)$ the function $\tilde{U}(\cdot, t)$ vanishes to infinite order in the $x_{n+1}$ variable at every $(x_0, 0) \in B_1 \cap \{x_{n+1} = 0\}$ \(^1\). In view of the real-analyticity of $\tilde{U}(\cdot, t)$ in $B_1$ we conclude that $\tilde{U}(\cdot, t) \equiv 0$ in $B_1$ for every $t \in (-1, 0)$. We now note that away from $\{x_{n+1} = 0\}$, $\tilde{U}$ solves a uniformly parabolic PDE with smooth coefficients and vanishes identically in the $B_1^+ \times (-1, 0)$. We can thus appeal to [1, Theor. 1] to assert that $\tilde{U}$ vanishes to infinite order both in space and time at every $(X, t) \in B_1^+ \times (-1, 0)$. At this point, we can use the strong unique continuation result in [7, Theor. 1] to finally conclude that $U(X, t) = 0$ for $(X, t) \in \mathbb{R}^{n+1} \times (-1, 0)$. Letting $x_{n+1} = 0$, this implies $u(x, t) = U((x, 0), t) \equiv 0$ for $(x, t) \in \mathbb{R}^n \times (-1, 0)$. This completes the proof of the proposition.

\[ \square \]

References

[1] G. Alessandrini & S. Vessella, Remark on the strong unique continuation property for parabolic operators. Proc. Amer. Math. Soc. 2 (2004) 499-501.
[2] A. Banerjee, D. Danielli, N. Garofalo & A. Petrosyan, The regular free boundary in the thin obstacle problem for degenerate parabolic equations, Algebra i Analiz 32 (2020), no. 3, 84-126.
[3] A. Banerjee & N. Garofalo, Monotonicity of generalized frequencies and the strong unique continuation property for fractional parabolic equations, Adv. Math. 336 (2018), 149-241.
[4] A. Biswas & P. R. Stinga, Regularity estimates for nonlocal space-time Master equations in bounded domains, J. Evol. Equ. 21 (2021), 503-565.
[5] L. Caffarelli & L. Silvestre, An extension problem related to the fractional Laplacian. Comm. Partial Differential Equations 32 (2007), no. 7-9, 1245-1260.
[6] S. Eidelman, Parabolic systems, North-Holland Publishing company, 1969.
[7] L. Escauriaza & F. Fernandez, Unique continuation for parabolic operators. Ark. Mat. 41 (2003), no. 1, 35-60.
[8] N. Garofalo, Two classical properties of the Bessel quotient $I_{\nu+1}/I_{\nu}$ and their implications in pde’s. Advances in harmonic analysis and partial differential equations, 57-97, Contemp. Math., 748, Amer. Math. Soc., Providence, RI, 2020.
[9] F. John, Partial differential equations, Fourth edition. Applied Mathematical Sciences, 1. Springer-Verlag, New York, 1982. x+249 pp.
[10] S. Jeon & A. Petrosyan, Almost minimizers for certain fractional variational problems, Algebra i Analiz 32 (2020), no. 4, 166-199.
[11] B. F. Jones, Lipschitz spaces and the heat equation. J. Math. Mech. 18 (1968/69), 379-409.
[12] T. Kotake & M. Narasimhan, Regularity theorems for fractional powers of a linear elliptic operator, Bull. Soc. Math. France 90 (1962), 449-471.

\(^1\)It is worth mentioning here that, although in the cited works [3] and [13] only the case $B = I_n$ was treated, the above mentioned bootstrap argument works unchanged for a smooth $B(x)$. 

[13] R. Lai, Y. Lin & A. Ruland, *The Calderón problem for a space-time fractional parabolic equation*, SIAM J. Math. Anal. 52 (2020), no. 3, 2655-2688.

[14] N. N. Lebedev, *Special functions and their applications*. Revised edition, translated from the Russian and edited by R. A. Silverman. Unabridged and corrected republication. Dover Publications, Inc., New York, 1972.

[15] G. Lieberman, *Second order parabolic differential equations*, World Scientific Publishing Co., Inc., River Edge, NJ, 1996. xii+439 pp. ISBN: 981-02-2883-X.

[16] M. Litsgård, K. Nyström, *On local regularity estimates for fractional powers of parabolic operators with time-dependent measurable coefficients*, arXiv:2104.07313.

[17] R. Narasimhan, *Several complex variables*. Reprint of the 1971 original. Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1995. x+174 pp.

[18] K. Nyström & O. Sande, *Extension properties and boundary estimates for a fractional heat operator*, Nonlinear Analysis, 140 (2016), 29-37.

[19] M Riesz, *Integrales de Riemann-Liouville et potentiels*, Acta Sci. Math. (Szeged), 9(1-1):1-42, 1938.

[20] S. G. Samko, *Hypersingular integrals and their applications*. Analytical Methods and Special Functions, 5. Taylor & Francis Group, London, 2002. xviii+359 pp.

[21] C. H. Sampson, *A characterization of parabolic Lebesgue spaces*. Thesis (Ph.D.)-Rice University. 1968. 91 pp.

[22] P. R. Stinga & J. L. Torrea, *Regularity theory and extension problem for fractional nonlocal parabolic equations and the master equation*, SIAM J. Math. Anal. 49 (2017), 3893–3924.

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