Effective Degrees of Freedom at Chiral Restoration
and the Vector Manifestation in HLS theory

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Abstract

The question as to what the relevant effective degrees of freedom at the chiral phase tran-
sition are remains largely unanswered and must be addressed in confronting both terrestrial
and space laboratory observations purporting to probe matter under extreme conditions.
We address this question in terms of the vector susceptibility \(\chi_V\) (VSUS in short) and the
axial-vector susceptibility \(\chi_A\) (ASUS in short) at the temperature-induced chiral transition.
We consider two possible, albeit simplified, cases that are contrasting, one that is given by
the standard chiral theory where only the pions figure in the vicinity of the transition and the
other that is described by hidden local symmetry (HLS) theory with the Harada-Yamawaki
vector manifestation (VM) where nearly massless vector mesons also enter. We find that
while in the standard chiral theory, the pion velocity \(v_\pi\) proportional to the ratio of the
space component \(f_\pi^s\) of the pion decay constant over the time component \(f_\pi^t\) tends to zero
near chiral restoration with \(f_\pi^t \neq 0\), in the presence of the vector mesons with vanishing mass,
the result is drastically different: HLS with VM predicts that \(\chi_V\) automatically equals \(\chi_A\)
in consistency with chiral invariance and that \(v_\pi \sim 1\) with \(f_\pi^t \approx f_\pi^s \to 0\) as \(T \to T_c\). These
results are obtained in the leading order in power counting but we expect their qualitative
features to remain valid more generally in the chiral limit thanks to the VM point.

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1 Introduction

One of the most crucial questions to answer in the effort to understand chiral restoration in relativistic heavy-ion collisions as well as in dense medium as in compact stars is: What are the relevant *effective* degrees of freedom just before and after the phase transition? The standard scenario, generally accepted in the community, is that the only relevant excitations in the broken symmetry sector near the phase transition are the pions, the pseudo-Nambu-Goldstone modes of broken chiral symmetry, i.e., the standard chiral theory. However there is no a priori reason to exclude other scenarios. In fact, it has been argued by Harada and Yamawaki [1, 2] that the vector manifestation (VM) with the massless (in the chiral limit) vector mesons can correctly describe chiral restoration. In a recent attempt to understand some of the puzzling results coming out of relativistic heavy ion experiments at RHIC, Brown and Rho [3] invoked the VM scenario in which the “light” $\rho$ mesons play a crucial role: The vector mesons there are “relayed” via a Higgsing to the gluons in the QCD sector.

In this paper, we address the issue of what the relevant degrees of freedom can be at the chiral transition induced by high temperature and their possible implications on observables in heavy-ion physics. In doing this, we focus on the vector and axial-vector susceptibilities and the pion velocity very near the critical temperature $T_c$ using the result obtained in [4] who have shown that the VM holds at $T = T_c$. The issue of what happens at high density is discussed in [5].

As a way of introduction to the main objective of this paper, we begin by briefly summarizing the arguments and results obtained in the standard chiral theory scenario [6, 7].

Consider the vector isospin susceptibility (VSUS) $\chi_V$ (denoted by SS as $\chi_I$) and the axial-vector isospin susceptibility (ASUS) $\chi_A$ (denoted by SS as $\chi_{I5}$) defined in terms of the vector charge density $V_0^a(x)$ and the axial-vector charge density $A_0^a(x)$ by the Euclidean correlators:

$$\delta^{ab} \chi_V = \int_0^{1/T} d\tau \int d^3\vec{x} \langle V_0^a(\tau, \vec{x}) V_0^b(0, \vec{0}) \rangle_\beta, \quad (1.1)$$

$$\delta^{ab} \chi_A = \int_0^{1/T} d\tau \int d^3\vec{x} \langle A_0^a(\tau, \vec{x}) A_0^b(0, \vec{0}) \rangle_\beta \quad (1.2)$$

where $\langle \rangle_\beta$ denotes thermal average and

$$V_0^a \equiv \bar{\psi} \gamma^0 \frac{\tau^a}{2} \psi, \quad A_0^a \equiv \bar{\psi} \gamma^0 \gamma^5 \frac{\tau^a}{2} \psi \quad (1.3)$$

with the quark field $\psi$ and the $\tau^a$ Pauli matrix the generator of the flavor $SU(2)$. 
We are interested in these SUS’s near the critical temperature \( T = T_c \) at zero baryon density \( n = 0 \). In particular we would like to compute them “bottom-up” approaching \( T_c \) from below. In order to do this, we need to resort to effective field theory of QCD which requires identifying, in the premise of an EFT, all the relevant degrees of freedom.

Let us first assume as done by Son and Stephanov (SS) \([6, 7]\) that the only relevant effective degrees of freedom in heat bath are the pions, and that all other degrees of freedom can be integrated out with their effects incorporated into the coefficients of higher order terms in the effective Lagrangian. Here the basic assumption is that near chiral restoration, there is no instability in the channel of the degrees of freedom that have been integrated out. In this pion-only case, the appropriate effective Lagrangian for the axial correlators is the in-medium chiral Lagrangian dominated by the current algebra terms,\(^1\)

\[
L_{\text{eff}} = \frac{f_\pi^2}{4} \left( \text{Tr} \nabla_0 U \nabla_0 U^\dagger - v_\pi^2 \text{Tr} \partial_i U \partial_i U^\dagger \right) - \frac{1}{2} \langle \bar{\psi} \psi \rangle \text{Re} M^\dagger U + \cdots \tag{1.4}
\]

where \( v_\pi \) is the pion velocity, \( M \) is the mass matrix introduced as an external field, \( U \) is the chiral field and the covariant derivative \( \nabla_0 U \) is given by \( \nabla_0 U = \partial_0 U - i \frac{2}{3} \mu_A (\tau_3 U + U \tau_3) \) with \( \mu_A \) the axial isospin chemical potential. The ellipsis stands for higher order terms in spatial derivatives and covariant derivatives. \(^1\) Given the effective action described by (1.4) with possible non-local terms ignored, then the ASUS takes the simple form

\[
\chi_A = - \frac{\partial^2}{\partial \mu_A^2} L_{\text{eff}} |_{\mu_A=0} = f_\pi^2. \tag{1.5}
\]

The principal point to note here is that as long as the effective action is given by local terms (subsumed in the ellipsis) involving the \( U \) field, this is the whole story. There is no other contribution to the ASUS than the temporal component of the pion decay constant.

Next one assumes that at the chiral phase transition point \( T = T_c \), the restoration of chiral symmetry dictates the equality

\[
\chi_A = \chi_V. \tag{1.6}
\]

While there is no lattice information on \( \chi_A \), \( \chi_V \) has been measured as a function of temperature \([8, 9]\). In particular, it is established that

\(^1\)The notation here deviates a bit from that of SS. For example, it will turn out that the pion velocity will have the form \( v_\pi^2 = f_\pi^2/f_s^2 \) (see Eq. (5.14)) where \( f_\pi^2 \) (\( f_s^2 \)) is the temporal (spatial) component of the pion decay constant.
\[ \chi_V|_{T=T_c} \neq 0, \]  

which leads to the conclusion \[6, 7\] that

\[ f_\pi^I|_{T=T_c} \neq 0. \]  

On the other hand, it is expected and verified by lattice simulations that the space component of the pion decay constant \( f_\pi^s \) should vanish at \( T = T_c \). One therefore arrives at

\[ v_\pi^2 \sim f_\pi^s / f_\pi^t \to 0, \quad T \to T_c. \]  

This is the main conclusion of the standard chiral theory.

To check whether this prediction is firm, let us see what one obtains for the VSUS in the same effective field theory approach. The effective Lagrangian for calculating the vector correlators is of the same form as the ASUS, Eq. (1.4), except that the covariant derivative is now defined with the vector isospin chemical potential \( \mu_V \) as \( \nabla_0 U = \partial_0 U - \frac{1}{2} \mu_V (\tau_3 U - U \tau_3) \). Now if one assumes as done above for \( \chi_A \) that possible non-local terms can be dropped, then the SUS is given by

\[ \chi_V = - \frac{\partial^2}{\partial \mu_V^2} L_{eff}|_{\mu_V=0} \]  

which can be easily evaluated from the Lagrangian. One finds that

\[ \chi_V = 0 \]  

for all temperature. While it is expected to be zero at \( T = 0 \), the vanishing \( \chi_V \) for \( T \neq 0 \) is at variance with the lattice data at \( T = T_c \). \#2

We now turn to the main objective of this paper: the prediction by the vector manifestation (VM) \[1\]. Basically the same scenario was suggested some time ago in conjunction with Brown-Rho scaling \[10, 11\]. As will be shown in detail in the following sections, the VM requires that the vector mesons figure on the same footing with the pions as the relevant degrees of freedom as the chiral transition point is approached from below. The key reason for this conclusion is that the chiral transition coincides with the VM fixed point at which the vector meson mass must vanish in the chiral limit \[12\]. This means that the vector-meson degrees of freedom cannot be integrated out near chiral restoration.

\#2The reason for this defect is explained in terms of hydrodynamics by Son and Stephanov \[6, 7\].
Our principal results - which are basically different from the standard chiral theory scenario - can be summarized as follows. In the presence of the $\rho$-meson, the only approach that is consistent with chiral perturbation theory is the hidden local symmetry (HLS) with the VM fixed point \(^3\). The present analysis based on this theory predicts \(^4\)

$$f_\pi|_{T=T_c} = f_\pi^a|_{T=T_c} = 0 , \quad v_\pi|_{T=T_c} \lesssim 1$$

(1.12)

and

$$\chi_A|_{T=T_c} = \chi_V|_{T=T_c} = \frac{N_f^2 T_c^2}{6} ,$$

(1.13)

where we have included the normalization factor of $2N_f$. Note that the equality of $\chi_A$ and $\chi_V$ at $T = T_c$ is an output of the theory. This result is a direct consequence of the fact that the $\rho$ and $\pi$ enter on the same footing in the VM: At the VM fixed point, the longitudinal components of the vector mesons and the pions form a degenerate multiplet.

The rest of the paper is devoted to the derivation of the main results (1.12) and (1.13). In Section 2, hidden local symmetry (HLS) theory is briefly introduced. Section 3 describes how thermal two-point functions are calculated in the HLS theory. In Section 4 we write down the in-medium vector and axial-vector current correlators that are needed in what follows. Pion decay constants and pion velocity are computed in the given framework in Section 5. The susceptibilities are defined in Section 6 and computed for temperature $T \sim T_c$. The conclusion is given in Section 7. The Appendices contain explicit formulas used in the main text. A more extensive treatment of the material covered in this paper together with other issues of finite temperature effective field theory in the VM is found in Ref. [14].

## 2 Hidden Local Symmetry

In this Section, we briefly summarize the HLS model. Our discussion will be highly sketchy. For details, the readers are invited to the review [2]. As mentioned above, the HLS spin-1 field is assumed to be as relevant as the pion field near chiral restoration. The HLS model \([15, 16]\) \(^3\) It has been stressed in the literature (see, e.g., [13, 2]) – and is stressed again – that HLS is a bona-fide effective field theory of QCD only if the $\rho$-meson mass is considered as of the same chiral order as the pion mass. In HLS theory, this condition is naturally met by the $\rho$-meson mass near the chiral transition point, so chiral perturbation theory should be more effective in this regime. This point that underlines our arguments that follow justifies our one-loop calculation.

\(^4\)The reason for that the $v_\pi$ deviates from 1 will be explained below.
is based on the $G_{\text{global}} \times H_{\text{local}}$ symmetry, where $G = \text{SU}(N_f)_L \times \text{SU}(N_f)_R$ is the global chiral symmetry and $H = \text{SU}(N_f)_V$ is the HLS. The basic quantities are the HLS gauge field $V_\mu$ and two variables or “coordinates”

$$\xi_{L,R} = e^{i\sigma / F_\sigma} e^{\mp i\pi / F_\pi} , \quad (2.1)$$

where $\pi$ denotes the pseudoscalar Nambu-Goldstone (NG) boson and $\sigma$ the NG boson absorbed into the HLS gauge field $V_\mu$ (longitudinal $\rho$). $F_\pi$ and $F_\sigma$ are corresponding decay constants, and the parameter $a$ is defined as $a \equiv F_\sigma^2 / F_\pi^2$. The transformation property of $\xi_{L,R}$ is given by

$$\xi_{L,R}(x) \rightarrow h(x)\xi_{L,R}(x)g_{L,R}^\dagger , \quad (2.2)$$

where $h(x) \in H_{\text{local}}$ and $g_{L,R} \in G_{\text{global}}$. The covariant derivatives of $\xi_{L,R}$ are defined by

$$D_\mu \xi_L = \partial_\mu \xi_L - iV_\mu \xi_L + i\xi_L L_\mu ,$$

$$D_\mu \xi_R = \partial_\mu \xi_R - iV_\mu \xi_R + i\xi_R R_\mu , \quad (2.3)$$

where $L_\mu$ and $R_\mu$ denote the external gauge fields gauging the $G_{\text{global}}$ symmetry. From the above covariant derivatives two 1-forms are constructed as

$$\hat{\alpha}_\perp^\mu = (D_\mu \xi_R \cdot \xi_R^\dagger - D_\mu \xi_L \cdot \xi_L^\dagger)/(2i) ,$$

$$\hat{\alpha}_\parallel^\mu = (D_\mu \xi_R \cdot \xi_R^\dagger + D_\mu \xi_L \cdot \xi_L^\dagger)/(2i) . \quad (2.4)$$

It should be noticed that, as first pointed by Georgi in Ref. [13] and developed further in Refs. [17][18][2], the systematic chiral perturbation can be performed with including the vector meson loop in addition to the pion loop in the HLS. The expansion parameter is a ratio of the $\rho$ meson mass to the chiral symmetry breaking scale $\Lambda_{\chi}$, $m_\rho/\Lambda_{\chi}$, in addition to the ratio of the momentum $p$ to $\Lambda_{\chi}$, $p/\Lambda_{\chi}$, as used in the ordinary chiral perturbation theory. The counting scheme is made as in the ordinary chiral perturbation theory by assigning $O(p)$ to the HLS gauge coupling $g$ [13][17].

With the above counting scheme the Lagrangian at the leading order, counted as $O(p^2)$, is given by [15][16]

$$\mathcal{L} = F_\pi^2 \text{tr} [\hat{\alpha}_\perp^\mu \hat{\alpha}_\perp^\mu] + F_\sigma^2 \text{tr} [\hat{\alpha}_\parallel^\mu \hat{\alpha}_\parallel^\mu] - \frac{1}{2g^2} \text{tr} [V_{\mu\nu} V^{\mu\nu}] , \quad (2.5)$$

where $g$ is the HLS gauge coupling and $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - i[V_\mu, V_\nu]$ the gauge field strength. When the kinetic term of the gauge field is ignored in the low-energy region, the second term of
Eq. (2.5) vanishes by integrating out $V_\mu$ and only the first term remains. Then, the HLS model is reduced to the nonlinear sigma model based on $G/H$. The one-loop quantum corrections calculated from the leading order Lagrangian in Eq. (2.5) are counted as $O(p^4)$. The divergences generated at $O(p^4)$ are renormalized by the $O(p^4)$ terms, a complete list of which are given in Refs. [17, 2]. Here we show the terms of the $O(p^4)$ Lagrangian relevant to the present analysis [17, 15]:

$$L(4) = z_1 \text{tr}[\hat{V}_\mu \hat{V}^{\mu\nu}] + z_2 \text{tr}[\hat{A}_\mu \hat{A}^{\mu\nu}] + z_3 \text{tr}[\hat{V}_\mu V^{\mu\nu}],$$

(2.6)

where

$$\hat{A}_\mu = \frac{1}{2}[\xi_R R_{\mu\nu} \xi_R^\dagger - \xi_L L_{\mu\nu} \xi_L^\dagger],$$

$$\hat{V}_\mu = \frac{1}{2}[\xi_R R_{\mu\nu} \xi_R^\dagger + \xi_L L_{\mu\nu} \xi_L^\dagger],$$

(2.7)

with $R_{\mu\nu}$ and $L_{\mu\nu}$ being the field strengths of $R_\mu$ and $L_\mu$.

We should stress that we assume that we obtain the bare HLS Lagrangian by integrating out the quark and gluon degrees of freedom at the matching scale $\Lambda$, so that bare parameters of the above HLS Lagrangian such as $F_\pi$, defined at $\Lambda$ are determined through the Wilsonian matching between the HLS and the underlying QCD [18]: As we briefly review in Appendix A.1, bare parameters are determined through the Wilsonian matching conditions (A.6)–(A.8) obtained by matching the axial-vector and vector current correlators in the HLS with those in the operator product expansion (OPE). As was shown in Refs. [18, 2], these bare parameters are scaled down to the low energy region through the Wilsonian renormalization group equations (RGEs) to predict several physical quantities in remarkable agreement with experiment.

Let us extend the above HLS Lagrangian to the analysis in hot matter. We assume that we obtain the bare HLS Lagrangian by integrating out the quarks and gluons at the matching scale $\Lambda$ in the presence of medium, and then the bare parameters are determined by matching the HLS to the underlying QCD. As was shown in Ref. [4] and briefly reviewed in Appendix A.2, when we make the matching in the presence of hot matter, the bare parameters have the intrinsic temperature dependences. In general, the Lorentz non-scalar operators such as $\bar{q} \gamma_\mu D_\nu q$ exist in the form of the current correlators derived by the OPE [19]. However, as discussed in Appendix A.2, such Lorentz symmetry violating contributions caused from the Lorentz non-scalar operators are suppressed by, at least, a factor of $1/\Lambda^6$ compared with $1+\alpha_s/\pi$, and the Lorentz violating effects in the bare $\pi$ decay constant and the bare $\sigma$ (longitudinal $\rho$)
decay constant are small: At bare level, the difference between $F_{\pi,\text{bare}}^t$ and $F_{\pi,\text{bare}}^s$ as well as that between $F_{\sigma,\text{bare}}^t$ and $F_{\sigma,\text{bare}}^s$ is small. Furthermore, since we will study the physical pion decay constants and the vector and axial-vector susceptibilities only near the critical temperature, the Lorentz violating effect in the HLS gauge coupling, which may distinguish $g_T$ from $g_L$ (see Appendix A.2), is irrelevant due to the decoupling nature of the transverse $\rho$ near the critical temperature in the VM. Thus, here we use the Lagrangian (2.5) with Lorentz invariance even at non-zero temperature to calculate the quantum and hadronic thermal corrections. We should stress that the explicit Lorentz violation in medium – which is not negligible – is of course taken into account (see Appendices).

3 Two-Point Functions

We need to consider two-point functions involving the isovector vector and axial-vector currents. Note that, when we calculate the hadronic thermal corrections, we assigned $O(p)$ to the temperature $T$ as in the approach based on the ordinary chiral perturbation theory [20].

Let us calculate them to one-loop order as in Refs. [18, 2]. In calculating the loops, we adopt the background field gauge (see Refs. [18, 2] for details in HLS theory) and the imaginary time formalism (see, e.g., Ref. [22]). For convenience, we introduce the following Feynman integrals to calculate the one-loop hadronic thermal corrections and quantum corrections to the two-point functions:

$$A_0(M; T) \equiv T \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \frac{1}{M^2 - k^2},$$

$$B_0(p_0, \bar{p}; M_1, M_2; T) \equiv T \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \frac{1}{[M_1^2 - k^2][M_2^2 - (k - p)^2]},$$

$$B^{\mu\nu}(p_0, \bar{p}; M_1, M_2; T) \equiv T \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \frac{(2k - p)^\mu (2k - p)^\nu}{[M_1^2 - k^2][M_2^2 - (k - p)^2]},$$

where $\bar{p} \equiv |\bar{p}|$, and the 0th component of the loop momentum is taken as $k^0 = i2n\pi T$, while that of the external momentum is taken as $p^0 = i2n'\pi T$ ($n'$: integer). Using the standard formula (see, e.g., Ref. [22]), we can convert the Matsubara frequency sum into an integral over $k_0$ with $k_0$ taken as the zeroth component of a Minkowski four vector. Accordingly, the above functions are divided into two parts as

$$A_0(M; T) = A_0^{(\text{vac})}(M) + \tilde{A}_0(M; T),$$

#5 The same treatment within the framework of the HLS was done before in Refs. [21, 4].
Figure 1: Feynman diagrams contributing to the $\mathcal{A}_\mu \mathcal{A}_\nu$ two-point function. Here $\pi$ represents the quantum pion field and likewise for the others.

$$B_0(p_0, \bar{p}; M_1, M_2; T) = B_0^{(\text{vac})}(p_0, \bar{p}; M_1, M_2) + \bar{B}_0(p_0, \bar{p}; M_1, M_2; T),$$

$$B^{\mu\nu}(p_0, \bar{p}; M_1, M_2; T) = B^{(\text{vac})\mu\nu}(p_0, \bar{p}; M_1, M_2) + \bar{B}^{\mu\nu}(p_0, \bar{p}; M_1, M_2; T),$$

where $A_0^{(\text{vac})}$, $B_0^{(\text{vac})}$ and $B^{(\text{vac})\mu\nu}$ are given by replacing $T \sum_{n=\infty}^{\infty}$ with $\int \frac{dk_0}{2\pi i}$ in Eqs. (3.1)–(3.3), and $\bar{A}_0$, $\bar{B}_0$ and $\bar{B}^{\mu\nu}$ are defined by Eq. (3.4). In the present analysis, the forms of $A_0^{(\text{vac})}$, $B_0^{(\text{vac})}$ and $B^{(\text{vac})\mu\nu}$ are equivalent to the zero-temperature ones. Then, with $p_0$ taken as the 0th component of the Minkowski four vector, they have no explicit temperature dependence while the intrinsic dependence mentioned above remains. Therefore, the functions $A_0^{(\text{vac})}$, $B_0^{(\text{vac})}$ and $B^{(\text{vac})\mu\nu}$ represent quantum corrections. In $\bar{B}_0$ and $\bar{B}^{\mu\nu}$ one can perform the analytic continuation of $p_0$ to the Minkowski variable after integrating over $k_0$: Here $p_0$ is understood as $p_0 + i\epsilon$ ($\epsilon \to +0$) for the retarded function and $p_0 - i\epsilon$ for the advanced function. It should be noticed that the argument $T$ in the above functions refers to only the temperature dependence arising from the hadronic thermal effects and not to the intrinsic thermal effects included in the parameters of the Lagrangian.

Now, let us calculate the one-loop corrections to the two-point function of the axial-vector (background) field $\mathcal{A}^\mu$. This is obtained by the sum of one particle irreducible diagrams with two legs of the axial-vector background field $\mathcal{A}^\mu$. In Fig. 11 we show the Feynman diagrams contributing to the $\mathcal{A}_\mu \mathcal{A}_\nu$ two-point function at one-loop level. With the help of (3.4), one can express the one-loop corrections to the two-point function in a simple form. The corrections from $\rho$ and/or $\pi$ shown in Figs. 11(a)–(c) lead to the two-point function

$$\Pi_{\perp}^{(1\text{-loop})\mu\nu}(p_0, \bar{p}; T) = -N_f a M_\rho^2 g^{\mu\nu} B_0(p_0, \bar{p}; M_\rho, 0; T) + N_f \frac{a}{4} B^{\mu\nu}(p_0, \bar{p}; M_\rho, 0; T) + N_f (a - 1) g^{\mu\nu} A_0(0, T),$$

(3.5)
where $B_0$-term comes from Fig. $\Pi(a)$, and $B^{\mu\nu}$-term and $A_0$-term from Fig. $\Pi(b)$ and Fig. $\Pi(c)$, respectively. Combining the above loop corrections with the tree contribution given by

$$
\Pi^{(\text{tree})\mu\nu}(p_0, \vec{p}) = g^{\mu\nu} F_\pi^2 + 2z_2,_{\text{bare}}(g^{\mu\nu} p^2 - p^\mu p^\nu),
$$

we have the two-point function of $\overline{A}_\mu - A_\nu$ at one-loop level as

$$
\Pi^{\mu\nu}_\perp(p_0, \vec{p}; T) = \Pi^{(\text{tree})\mu\nu}_\perp(p_0, \vec{p}) + \Pi^{(1\text{-loop})\mu\nu}_\perp(p_0, \vec{p}; T).
$$

Analogously to Eq. (3.4), we split the two-point function into two parts as

$$
\Pi^{\mu\nu}_\perp(p_0, \vec{p}; T) = \Pi^{(\text{vac})\mu\nu}_\perp(p_0, \vec{p}) + \Pi^{\mu\nu}_\perp(p_0, \vec{p}; T),
$$

where $\Pi^{(\text{vac})\mu\nu}_\perp$ includes the quantum correction and the contribution at tree level in Eq. (3.6), and $\Pi^{\mu\nu}_\perp$ represents the hadronic thermal correction. Since the hadronic thermal correction $\Pi^{\mu\nu}_\perp$ has no divergences, the renormalization conditions for $F_\pi^2$ and $z_2$ can be determined from $\Pi^{(\text{vac})\mu\nu}_\perp$. For $F_\pi^2$ we adopt the “on-shell” renormalization condition:

$$
\Pi^{(\text{vac})\mu\nu}_\perp(p_0 = 0, \vec{p} = \vec{0}) = g^{\mu\nu} F_\pi^2(0).
$$

From this renormalization condition, we obtain the $g^{\mu\nu}$-part of $\Pi^{(\text{vac})\mu\nu}_\perp$ in the form [2]

$$
p_\mu p_\nu \Pi^{(\text{vac})\mu\nu}_\perp(p_0, \vec{p}) = p^2 \left[ F_\pi^2(0) + \tilde{\Pi}^S(p^2) \right],
$$

where $\tilde{\Pi}^S(p^2)$ is the finite renormalization contribution satisfying

$$
\tilde{\Pi}^S(p^2 = 0) = 0.
$$

For $z_2$ we adopt the renormalization condition that $\Pi^{(\text{vac})\mu\nu}_\perp$ be given by

$$
\Pi^{(\text{vac})\mu\nu}_\perp(p_0, \vec{p}) = g^{\mu\nu} \left[ F_\pi^2(0) + p^2 \tilde{\Pi}^S(p^2) \right] + (g^{\mu\nu} p^2 - p^\mu p^\nu) \left[ 2z_2(M_\rho) + \tilde{\Pi}^{LT}(p^2) \right],
$$

where $z_2(M_\rho)$ is renormalized at the scale $M_\rho$ and $\tilde{\Pi}^{LT}(p^2)$ is the finite renormalization subject to the condition

$$
\text{Re} \tilde{\Pi}^{LT}(p^2 = M_\rho^2) = 0.
$$

To distinguish the hadronic thermal correction to the pion decay constant from that to the parameter $z_2$, we decompose the two-point function of $\overline{A}_\mu - A_\nu$ into four components as

$$
\Pi^{\mu\nu}_\perp = u^\mu u^\nu \Pi^T_\perp + (g^{\mu\nu} - u^\mu u^\nu) \Pi^s_\perp + P^{\mu\nu}_L \Pi^L_\perp + P^{\mu\nu}_T \Pi^T_\perp,
$$
Figure 2: Feynman diagrams contributing to the $\nabla_\mu \nabla_\nu$ two-point function. Here $\pi$ represents the quantum pion field and likewise for the others.

where $P_L^{\mu
u}$ and $P_T^{\mu
u}$ are the polarization tensors defined by

$$P_T^{\mu
u} = g_\mu^i \left( \delta_{ij} - \frac{\vec{p}_i \vec{p}_j}{|\vec{p}|^2} \right) g_\nu^j = (g^{\mu\alpha} - u^\mu u^\alpha) \left( -g_{\alpha\beta} \frac{p^\alpha p^\beta}{p^2} \right) \left( g^{\beta\nu} - u^\beta u^\nu \right),$$

$$P_L^{\mu
u} = - \left( g^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) - P_T^{\mu
u}$$

$$= \left( g^{\mu\alpha} - \frac{p^\mu p^\alpha}{p^2} \right) u_\alpha \frac{p^2}{|\vec{p}|^2} u_\beta \left( g^{\beta\nu} - \frac{p^\beta p^\nu}{p^2} \right).$$  (3.15)

Similarly to the division in Eq. (3.8), it is convenient to divide each component into two parts as

$$\Pi_{t}^{\perp}(p_0, \vec{p}; T) = \Pi_{(\text{vac})}^{t}(p_0, \vec{p}) + \Pi_{t}^{\perp}(p_0, \vec{p}; T),$$  (3.16)

where $\Pi_{(\text{vac})}^{t}(p_0, \vec{p})$ includes the tree contribution plus the finite renormalization effect and $\Pi_{t}^{\perp}(p_0, \vec{p}; T)$ is the hadronic thermal contribution. With Eq. (3.12) the functions $\Pi_{(\text{vac})}^{t,s,L,T}$ can be written as

$$\Pi_{(\text{vac})}^{t}(p_0, \vec{p}) = \Pi_{(\text{vac})}^{s}(p_0, \vec{p}) = F_{\pi}^2(0) + \Pi_{s}^{T}(p^2),$$

$$\Pi_{(\text{vac})}^{L}(p_0, \vec{p}) = \Pi_{(\text{vac})}^{T}(p_0, \vec{p}) = -\nu^2 \left[ 2 z_2(M_\rho) + \Pi_{L}^{T}(p^2) \right].$$  (3.17)

The explicit forms of the hadronic thermal corrections $\Pi_{t,s,L,T}^{\perp}(p_0, \vec{p}; T)$ are summarized in Eqs. (B.1)–(B.4) in Appendix B.

We show the diagrams contributing to the $\nabla_\mu \nabla_\nu$ two-point function in Fig. 2. We adopt the on-shell renormalization condition similar to the above to obtain the resultant forms of four components of the two-point function as
\[ \Pi_{\parallel}^{(\text{vac})} \parallel (p_0, \bar{p}) = \Pi_{\parallel}^{(\text{vac})} (p_0, \bar{p}) = F_\sigma^2 (M_\rho) + \Pi_{\parallel}^S (p^2) , \]
\[ \Pi_{\parallel}^{(\text{vac})} L (p_0, \bar{p}) = \Pi_{\parallel}^{(\text{vac})} T (p_0, \bar{p}) = -p^2 \left[ 2z_1 (M_\rho) + \Pi_{\parallel}^{LT} (p^2) \right] , \]

where the parameters are renormalized at the scale \( M_\rho \) and the finite renormalization terms satisfy

\[ \text{Re} \, \Pi_{\parallel}^S (p^2 = M_\rho^2) = \text{Re} \, \Pi_{\parallel}^{LT} (p^2 = M_\rho^2) = 0 \]  

For the two-point functions of \( V_\mu V_\nu \), we adopt similar on-shell renormalization conditions. The resultant sums of the tree contributions and quantum corrections take the forms

\[ \Pi_{\parallel}^{(\text{vac})} \parallel (p_0, \bar{p}) = \Pi_{\parallel}^{(\text{vac})} (p_0, \bar{p}) = \Pi_{\parallel}^{(\text{vac})} (p_0, \bar{p}) \]
\[ = F_\sigma^2 (M_\rho) + \Pi_{\parallel}^S (p^2) = \Pi_{\parallel}^{(\text{vac})} \parallel (p_0, \bar{p}) = \Pi_{\parallel}^{(\text{vac})} (p_0, \bar{p}) , \]
\[ \Pi_{\parallel}^{(\text{vac})} L (p_0, \bar{p}) = \Pi_{\parallel}^{(\text{vac})} T (p_0, \bar{p}) = -p^2 \left[ 1 \over g^2 (M_\rho) + \Pi_{\parallel}^{LT} (p^2) \right] , \]
\[ \Pi_{\parallel}^{(\text{vac})} L (p_0, \bar{p}) = \Pi_{\parallel}^{(\text{vac})} T (p_0, \bar{p}) = -p^2 \left[ z_3 (M_\rho) + \Pi_{\parallel}^{LT} (p^2) \right] , \]

where, as in Eq. (3.19), the finite renormalization terms satisfy

\[ \text{Re} \, \Pi_{\parallel}^S (p^2 = M_\rho^2) = \text{Re} \, \Pi_{\parallel}^{LT} (p^2 = M_\rho^2) = \text{Re} \, \Pi_{\parallel}^{LT} (p^2 = M_\rho^2) = 0 \]  

The hadronic thermal corrections to the above two-point functions relevant to the present analysis are given in Eqs. (B.5) and (B.6) in Appendix B.

It should be noticed that the renormalized parameters have the intrinsic temperature dependences in addition to the dependence on the renormalization point. Then, the notations used above for the parameters renormalized at on-shell should be understood as the following abbreviated notations:

\[ F_\pi (0) \equiv F_\pi (\mu = 0; T) , \]
\[ F_\sigma (M_\rho) \equiv F_\sigma (\mu = M_\rho (T); T) , \]
\[ g (M_\rho) \equiv g (\mu = M_\rho (T); T) , \]
\[ z_{1,2,3} (M_\rho) \equiv z_{1,2,3} (\mu = M_\rho (T); T) , \]

where \( \mu \) is the renormalization point and the mass parameter \( M_\rho \) is determined from the on-shell condition:
\[ M_\rho^2 \equiv M_\rho^2(T) = g^2(\mu = M_\rho(T); T)F_\rho^2(\mu = M_\rho(T); T). \] (3.23)

In addition, the parameter \( a \) appearing in several expressions in Appendices are defined as

\[ a \equiv \frac{F_\rho^2(\mu = M_\rho(T); T)}{F_\pi^2(\mu = M_\rho(T); T)}. \] (3.24)

## 4 Current Correlators

We now turn to construct the axial-vector and vector current correlators from the two-point functions calculated in the previous section. The correlators are defined by

\[ G^{\mu\nu}_A(p_0 = i\omega_n, \vec{p}; T)\delta_{ab} = \int_0^{1/T} d\tau \int d^3 \vec{x} e^{-i(p_0 \tau + \omega_n \tau)} \left\langle J^\mu_{ba}(\tau, \vec{x}) J^\nu_5(0, \vec{0}) \right\rangle_\beta, \]

\[ G^{\mu\nu}_V(p_0 = i\omega_n, \vec{p}; T)\delta_{ab} = \int_0^{1/T} d\tau \int d^3 \vec{x} e^{-i(p_0 \tau + \omega_n \tau)} \left\langle J^\mu_5(\tau, \vec{x}) J^\nu_5(0, \vec{0}) \right\rangle_\beta, \] (4.1)

where \( J^\mu_{ba} \) and \( J^\mu_5 \) are, respectively, the axial-vector and vector currents, \( \omega_n = 2n\pi T \) is the Matsubara frequency, \( (a, b) = 1, \ldots, N_f^2 - 1 \) denotes the flavor index and \( \left\langle \right\rangle_\beta \) the thermal average. The correlators for Minkowski momentum are obtained by the analytic continuation of \( p_0 \).

For constructing the axial-vector current correlator \( G^{\mu\nu}_A(p_0, \vec{p}; T) \) from the \( \overline{A}_\mu - \overline{A}_\nu \) two-point function, it is convenient to take the unitary gauge of the background HLS and parameterize the background fields \( \overline{\xi}_L \) and \( \overline{\xi}_R \) as

\[ \overline{\xi}_L = e^{-\overline{\phi}}, \quad \overline{\xi}_R = e^{\overline{\phi}}, \quad \overline{\phi} = \overline{\phi}_a T_a. \] (4.2)

where \( \overline{\phi} \) denotes the background field corresponding to the pion field. In terms of this \( \overline{\phi} \), the background \( \overline{A}_\mu \) is expanded as

\[ \overline{A}_\mu = A_\mu + \partial_\mu \overline{\phi} + \cdots , \] (4.3)

where the ellipses stand for the terms that include two or more fields. Then, the axial-vector current correlator is

\[ G^{\mu\nu}_A = \frac{p_\mu p_\nu \Pi^{\mu\nu} \Pi_\perp \Pi^{\mu\nu} \Pi_\perp}{-p_\mu p_\nu \Pi^{\mu\nu} \Pi_\perp} + \Pi^{\mu\nu} \] (4.4)

where the first term comes from the \( \overline{\phi} \)-exchange and the second term from the direct \( A_\mu - A_\nu \) interaction. By using the decomposition in Eq. (3.14), this can be rewritten as
\[ G_A^{\mu\nu} = P_L^{\mu\nu} G_A^L + P_T^{\mu\nu} G_A^T, \quad (4.5) \]

where

\[ G_A^L = \frac{p^2 \Pi_1^e \Pi_1^s}{p_0^2 \Pi_1^e - p^2 \Pi_2^s} + \Pi_1^L, \quad (4.6) \]
\[ G_A^T = -\Pi_1^e + \Pi_1^T. \quad (4.7) \]

One can see from Eqs. (4.6) and (4.7) that the pion exchange contribution is included only in the longitudinal component \( G_A^L \).

To obtain the vector current correlator \( G_V \), we first consider the \( \overline{V} \) propagator. By using the fact that the inverse \( \overline{V} \) propagator \( i(D^{-1})^{\mu\nu} \) is equal to \( \Pi_V^{\mu\nu} \), the propagator for the field \( \overline{V} \) can be expressed as

\[-iD_V^{\mu\nu} = u^\mu u'^\nu D_V^\mu + (g^{\mu\nu} - u^\mu u'^\nu) D_V^\mu + P_L^{\mu\nu} D_V^L + P_T^{\mu\nu} D_V^T, \quad (4.8)\]

where

\[ D_V^L = \frac{p^2 (\Pi_V^e - \Pi_V^L)}{p_0^2 \Pi_V^e (\Pi_V^e - \Pi_V^L) - p^2 \Pi_V^s (\Pi_V^e - \Pi_V^L)}, \quad (4.9) \]
\[ D_V^e = \frac{p^2 (\Pi_V^s - \Pi_V^L)}{p_0^2 \Pi_V^e (\Pi_V^e - \Pi_V^L) - p^2 \Pi_V^s (\Pi_V^e - \Pi_V^L)}, \quad (4.10) \]
\[ D_V^L = \frac{-p^2 \Pi_V^L}{p_0^2 \Pi_V^e (\Pi_V^e - \Pi_V^L) - p^2 \Pi_V^s (\Pi_V^e - \Pi_V^L)}, \quad (4.11) \]
\[ D_V^T = D_V^e - \frac{1}{\Pi_V^e - \Pi_V^L}. \quad (4.12) \]

By using the above propagator \( D_V \) and two-point functions of \( \overline{\nabla}_{\mu}\nabla_\mu \) and \( \overline{\nabla}_{\mu}\nabla_\nu \), \( G_V \) can be put into the form

\[ G_V^{\mu\nu} = \Pi_{\nabla_{\mu}\nabla_\nu} D_{\nabla_{\mu}\nabla_\nu} + \Pi_{\parallel}^{\mu\nu}. \quad (4.13) \]

After a lengthy calculation, we obtain

\[ G_V^{\mu\nu} = u^\mu u'^\nu \left[ \frac{D_V^L}{\Pi_V^L} \left\{ \frac{p^2}{p_0^2 \Pi_V^e (\Pi_V^e - \Pi_V^L) - p^2 \Pi_V^s (\Pi_V^e - \Pi_V^L)} \right\} \right. \]
\[ - \frac{\Pi_V^L}{p^2} \left\{ -p_0^2 (\Pi_V^e - \Pi_V^L) + p^2 \left( \Pi_V^e \Pi_V^e - \Pi_V^s \Pi_V^s \right) \right\} \right] + \Pi_{\parallel}^L \]
\[ + (g^{\mu\nu} - u^\mu u'^\nu) \left[ \frac{D_V^L}{\Pi_V^L} \left\{ \frac{p^2}{p_0^2 \Pi_V^e (\Pi_V^e - \Pi_V^L) - p^2 \Pi_V^s (\Pi_V^e - \Pi_V^L)} \right\} \right. \]
\[ - \frac{\Pi_V^e}{p^2} \left\{ -p_0^2 (\Pi_V^e - \Pi_V^L) + p^2 \left( \Pi_V^e \Pi_V^e - \Pi_V^s \Pi_V^s \right) \right\} \right] + \Pi_{\parallel}^e \]
One might worry that the above form does not satisfy the current conservation \( p_\mu G_{\mu\nu}^{\mu\nu} = 0 \). However, since, as shown in Eq. (B.5), the conditions

\[
\Pi_t^V = -\Pi_t^L = \Pi_t^T,
\]

\[
\Pi_s^V = -\Pi_s^L = \Pi_s^T
\]  

are satisfied, Eq. (4.14) can be rewritten as

\[
G_{\mu\nu}^{\mu\nu} = P_{L}^{\mu\nu} \left[ \frac{D_L^L}{\Pi_V^L} \left\{ -\Pi_L^V \Pi_L^V \Pi_L^L + \Pi_L^V \left( \Pi_L^L \Pi_L^L + \Pi_L^L \Pi_L^L \right) \right\} - \frac{1}{p^2} \left( p_0^2 \Pi_L^V - \vec{p}^2 \Pi_L^V \right) \left( \Pi_L^V \right)^2 } + \Pi_L^L \right] + P_{T}^{\mu\nu} \left[ \frac{D_T^L}{\Pi_V^T} \left\{ \frac{p_0^2 \Pi_L^V}{p^2} \left( \Pi_L^V \Pi_L^V - \Pi_L^L \Pi_L^V \right) - \frac{\Pi_L^V}{p^2} \left( -p_0^2 \left( \Pi_L^V \Pi_L^V - \Pi_L^L \Pi_L^V \right) + \vec{p}^2 \Pi_L^V \left( \Pi_L^L - \Pi_L^T \right) \right) + \frac{\left( \Pi_L^V \Pi_L^V - \Pi_L^T \right)^2}{\Pi_L^V - \Pi_L^T} \right] + \Pi_L^T \right].
\]  

(4.14)

Now it is evident that the current is conserved since \( p_\mu P_L^{\mu\nu} = p_\mu P_T^{\mu\nu} = 0 \). In the present analysis, the equality \( \Pi_t^V = \Pi_t^L \) is seen to hold as shown in Eq. (B.5). This implies that \( \Pi_t^V = \Pi_t^L \) is also satisfied since the quantum corrections to \( \Pi_V^T \) and \( \Pi_V^T \) are equal to each other due to Lorentz invariance. Thus, \( G_{\mu\nu}^{\mu\nu} \) can be written as

\[
G_{\mu\nu}^{\mu\nu} = P_L^{\mu\nu} G_L^{\mu\nu} + P_T^{\mu\nu} G_T^{\mu\nu},
\]  

(4.17)

where

\[
G_L^{\mu\nu} = \frac{\Pi_L^V \left( \Pi_L^V + 2\Pi_L^L \right)}{\Pi_V^L - \Pi_L^T} + \Pi_L^L
\]  

(4.18)

\[
G_T^{\mu\nu} = \frac{\Pi_L^V \left( \Pi_L^T + 2\Pi_L^T \right)}{\Pi_V^T - \Pi_L^T} + \Pi_L^T.
\]  

(4.19)

Note that, in the above expressions, we have dropped the terms \( \left( \Pi_V^L \right)^2 \) and \( \left( \Pi_V^T \right)^2 \) since they are of higher order.
5 Pion Decay Constants and Pion Velocity

We now proceed to study the on-shell structure of the pion. For this we look at the pole of the longitudinal component $G_A^L$ in Eq. (1.6). Since both $\Pi^t_\perp$ and $\Pi^s_\perp$ have imaginary parts, we choose to determine the pion energy $E$ from the real part by solving the dispersion formula

$$0 = \left[ p_0^2 \text{Re} \Pi^t_\perp(p_0, \vec{p}; T) - \vec{p}^2 \text{Re} \Pi^s_\perp(p_0, \vec{p}; T) \right]_{p_0 = E} ,$$

(5.1)

where $\vec{p} \equiv |\vec{p}|$. As remarked in Section 3 in HLS at one-loop level, $\Pi^t_\perp(p_0, \vec{p}; T)$ and $\Pi^s_\perp(p_0, \vec{p}; T)$ are of the form

$$\Pi^t_\perp(p_0, \vec{p}; T) = F^2_\pi(0) + \tilde{\Pi}^t_\perp(p^2) + \Pi^t_\perp(p_0, \vec{p}; T) ,$$

$$\Pi^s_\perp(p_0, \vec{p}; T) = F^2_\pi(0) + \tilde{\Pi}^s_\perp(p^2) + \Pi^s_\perp(p_0, \vec{p}; T) ,$$

(5.2)

where $\tilde{\Pi}^t_\perp(p^2)$ is the finite renormalization contribution, and $\Pi^t_\perp(p_0, \vec{p}; T)$ and $\Pi^s_\perp(p_0, \vec{p}; T)$ are the hadronic thermal contributions. Substituting Eq. (5.2) into Eq. (5.1), we obtain

$$0 = \left( E^2 - \vec{p}^2 \right) \left[ F^2_\pi(0) + \text{Re} \tilde{\Pi}^s_\perp(p^2 = E^2 - \vec{p}^2) \right] + E^2 \text{Re} \tilde{\Pi}^t_\perp(E, \vec{p}; T) - \vec{p}^2 \text{Re} \tilde{\Pi}^s_\perp(E, \vec{p}; T) .$$

(5.3)

The pion velocity $v_\pi(\vec{p}) \equiv E/\vec{p}$ is then obtained by solving

$$v^2_\pi(\vec{p}) = \frac{F^2_\pi(0) + \text{Re} \tilde{\Pi}^s_\perp(\vec{p}, \vec{p}; T)}{F^2_\pi(0) + \text{Re} \tilde{\Pi}^t_\perp(\vec{p}, \vec{p}; T)} .$$

(5.4)

Here we replaced $E$ by $\vec{p}$ in the hadronic thermal terms $\tilde{\Pi}^t_\perp(E, \vec{p})$ and $\tilde{\Pi}^s_\perp(E, \vec{p})$ as well as in the finite renormalization contribution $\tilde{\Pi}^s_\perp(p^2 = E^2 - \vec{p}^2)$, since the difference is of higher order. [Note that $\tilde{\Pi}^s_\perp(p^2 = 0) = 0$.]

Next we determine the wave function renormalization of the pion field, which relates the background field $\phi$ to the pion field $\pi$ in the momentum space as

$$\phi = \pi/\tilde{F}(\vec{p}; T) .$$

(5.5)

We follow the analysis in Ref. 23 to obtain

$$\tilde{F}^2(\vec{p}; T) = \text{Re} \tilde{\Pi}^t_\perp(E, \vec{p}; T) = F^2_\pi(0) + \text{Re} \tilde{\Pi}^t_\perp(\vec{p}, \vec{p}; T) .$$

(5.6)

Using this wave function renormalization and the velocity in Eq. (5.4), we can rewrite the longitudinal part of the axial-vector current correlator as

$$G_A^L(p_0, \vec{p}) = \frac{p^2 \Pi^t_\perp(p_0, \vec{p}; T) \Pi^s_\perp(p_0, \vec{p}; T)/\tilde{F}^2(\vec{p}; T)}{-[p_0^2 - v^2_\pi(\vec{p})\vec{p}^2 + \Pi_\perp(p_0, \vec{p}; T)] + \Pi^L_\perp(p_0, \vec{p}; T) ,}$$

(5.7)
where the pion self energy $\Pi_\pi(p_0, \vec{p}; T)$ is given by

$$\Pi_\pi(p_0, \vec{p}; T) = \frac{1}{\text{Re} \Pi_\perp^t(E, \vec{p}; T)}$$

$$\times \left[ \rho_0^2 \left\{ \Pi_\perp^t(p_0, \vec{p}; T) - \text{Re} \Pi_\perp(E, \vec{p}; T) \right\} - \vec{p}^2 \left\{ \Pi_\perp^s(p_0, \vec{p}; T) - \text{Re} \Pi_\perp(E, \vec{p}; T) \right\} \right].$$  \hspace{1cm} (5.8)

Let us now define the pion decay constant. A natural procedure is to define the pion decay constant from the pole residue of the axial-vector current correlator. From Eq. (5.7), the pion decay constant introduced in Ref. [24]. Following their notation, let $f_\pi$ denote the decay constant associated with the temporal component of the axial-vector current and $f_\pi^s$ the one with the spatial component. In the present analysis, they can be read off from the coupling of the $\bar{\pi}$ field to the axial-vector external field $A_\mu$:

$$f_\pi^t(\vec{p}; T) \equiv \frac{\Pi_\perp^t(E, \vec{p}; T)}{F(\vec{p}; T)} = \frac{\Pi_\perp^t(0) + \Pi_\perp^t(\vec{p}, \vec{p}; T)}{F(\vec{p}; T)},$$  \hspace{1cm} (5.10)

$$f_\pi^s(\vec{p}; T) \equiv \frac{\Pi_\perp^s(E, \vec{p}; T)}{F(\vec{p}; T)} = \frac{\Pi_\perp^s(0) + \Pi_\perp^s(\vec{p}, \vec{p}; T)}{F(\vec{p}; T)}.$$  \hspace{1cm} (5.11)

Comparing Eqs. (5.10) and (5.11) with Eqs. (5.24), (5.26) and (5.29), we have [24, 23]

$$F(\vec{p}; T) = \text{Re} f_\pi^t(\vec{p}; T),$$  \hspace{1cm} (5.12)

$$f_\pi^2(\vec{p}; T) = f_\pi^t(\vec{p}; T) f_\pi^s(\vec{p}; T),$$  \hspace{1cm} (5.13)

$$v_\pi^2(\vec{p}) = \frac{\text{Re} f_\pi^s(\vec{p}; T)}{\text{Re} f_\pi^t(\vec{p}; T)}.$$  \hspace{1cm} (5.14)

We are now ready to investigate what happens to the above quantities when the critical temperature $T_c$ is approached. Due to the VM in hot matter [4], the parametric $\rho$ meson mass goes to zero ($M_\rho \rightarrow 0$) and the parameter $a$ approaches one ($a \rightarrow 1$), so we have [see Eq. (B.7)]

$$\bar{\Pi}_\perp^t(\vec{p}, \vec{p}; T) \xrightarrow{T \rightarrow T_c} - \frac{N_f}{2} \bar{J}_\perp^t(0; T_c) = - \frac{N_f}{24} T_c^2,$$

$$\bar{\Pi}_\perp^s(\vec{p}, \vec{p}; T) \xrightarrow{T \rightarrow T_c} - \frac{N_f}{2} \bar{J}_\perp^s(0; T_c) = - \frac{N_f}{24} T_c^2.$$  \hspace{1cm} (5.15)

Substituting these into the expression of the pion velocity in Eq. (5.4), we obtain
\[ \pi^2(\vec{p}) \xrightarrow{T \to T_c} 1. \]  

(5.16)

This is our first main result: in the framework of the VM, the pion velocity approaches 1 near the critical temperature \( T_c \), not 0 as in the case of the pion-only situation \[6, 7\].

From Eq. (5.15), we can evaluate the pion decay constant Eq. (5.9) at the critical temperature which comes out to be

\[ f_2^2(\vec{p}; T_c) = F_2^2(0) - \frac{N_f}{24} T_c^2. \]  

(5.17)

Since this \( f_\pi \) is the order parameter and should vanish at the critical temperature, the parameter \( F_\pi^2(0) \) at \( T = T_c \) is given at \( T_c \) as \[4\]

\[ F_\pi^2(0) \xrightarrow{T \to T_c} \frac{N_f}{24} T_c^2. \]  

(5.18)

Substituting Eq. (5.15) together with Eq. (5.18) into Eqs. (5.10) and (5.11), we conclude that both temporal and spatial pion decay constants vanish at the critical temperature \( T_c \):

\[ f_t^4(\vec{p}; T_c) = f_s^4(\vec{p}; T_c) = 0. \]  

(5.19)

This is our second main result. Note that while \( f_t^4(T_c) = 0 \), \( \chi_A(T_c) \) is non-zero in consistency with the lattice result. Here the HLS gauge boson plays an essential role.

### 6 Axial-Vector and Vector Susceptibilities

In terms of the quantities defined in the preceding sections, the axial-vector susceptibility \( \chi_A(T) \) and the vector susceptibility \( \chi_V(T) \) for non-singlet currents \( #8 \) are given by the 00-component of the axial-vector and vector current correlators in the static–low-momentum limit:

\[ \chi_A(T) = 2N_f \lim_{\vec{p} \to 0} \lim_{p_0 \to 0} \left[ G_{A}^{00}(p_0, \vec{p}; T) \right], \]

\[ \chi_V(T) = 2N_f \lim_{\vec{p} \to 0} \lim_{p_0 \to 0} \left[ G_{V}^{00}(p_0, \vec{p}; T) \right], \]  

(6.1)

where we have included the normalization factor of \( 2N_f \). Using the current correlators given in Eqs. (4.25) and (4.16) and noting that \( \lim_{p_0 \to 0} P_L^{00} = \lim_{p_0 \to 0} \frac{p^2}{2} = -1 \), we can express \( \chi_A(T) \) and \( \chi_V(T) \) as

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#6 Modulo small Lorentz-breaking correction mentioned above.

#7 These results differ from those obtained in a framework in which the only relevant degrees of freedom near chiral restoration are taken to be the pions \( [6, 7] \). We will explain how this comes about in the conclusion section.

#8 We will confine ourselves to non-singlet (that is, isovector) susceptibilities, so we won’t specify the isospin structure from here on.
\[ \chi_A(T) = -2N_f \lim_{\vec{p} \to 0, \vec{p}_0 \to 0} \left[ \Pi_L^V(p_0, \vec{p}; T) - \Pi_L^V(p_0, \vec{p}; T) \right], \]

\[ \chi_V(T) = -2N_f \lim_{\vec{p} \to 0, \vec{p}_0 \to 0} \left[ \frac{\Pi_V^V \left( \Pi_L^V + 2\Pi_L^V \right)}{\Pi_V^V - \Pi_L^V} + \Pi_L^V \right], \quad (6.2) \]

where for simplicity of notation, we have suppressed the argument \((p_0, \vec{p}; T)\) in the right-hand-side of the expression for \(\chi_V(T)\). In HLS theory at one-loop level, the susceptibilities read

\[ \chi_A(T) = 2N_f \left[ F^2_\pi(0) + \lim_{\vec{p} \to 0, \vec{p}_0 \to 0} \left\{ \Pi_L^A(p_0, \vec{p}; T) - \Pi_L^A(p_0, \vec{p}; T) \right\} \right], \]

\[ \chi_V(T) = -2N_f \lim_{\vec{p} \to 0, \vec{p}_0 \to 0} \left[ \frac{(a(0)F^2_\pi(0) + \Pi_V^V) \left( \Pi_L^V + 2\Pi_L^V \right)}{a(0)F^2_\pi(0) + \Pi_V^V - \Pi_L^V} + \Pi_L^V \right], \quad (6.3) \]

where the parameter \(a(0)\) is defined by

\[ a(0) = \frac{\Pi_V^{(\text{vac})}(p_0 = 0, \vec{p} = 0)}{F^2_\pi(0)} = \frac{\Pi_V^{(\text{vac})}(p_0 = 0, \vec{p} = 0)}{F^2_\pi(0)}. \quad (6.4) \]

In Ref. [IS], \(a(0)\) was defined by the ratio \(F^2_\sigma(M_\rho)/F^2_\pi(0)\) without taking into account the finite renormalization effect which depends on the details of the renormalization condition. In the present renormalization condition \((3.20)\) with Eq. \((3.21)\), the finite renormalization effect leads to

\[ \bar{\Pi}_V^S(p^2 = 0) = \frac{N_f}{(4\pi)^2} M_\rho^2 \left( 2 - \sqrt{3} \tan^{-1} \sqrt{3} \right), \quad (6.5) \]

and then \(a(0)\) reads

\[ a(0) = \frac{F^2_\pi(M_\rho)}{F^2_\pi(0)} + \frac{N_f}{(4\pi)^2} \frac{M_\rho^2}{F^2_\pi(0)} \left( 2 - \sqrt{3} \tan^{-1} \sqrt{3} \right). \quad (6.6) \]

It follows from the static–low-momentum limit of \((\bar{\Pi}_L^V - \Pi_L^V)\) given in Eq. \((6.8)\) that the axial-vector susceptibility \(\chi_A(T)\) takes the form

\[ \chi_A(T) = 2N_f \left[ F^2_\pi(0) - N_f \bar{J}_1^2(0; T) + N_f a \bar{J}_1^2(M_\rho; T) \right. \]

\[ \left. - N_f \frac{a}{M_\rho^2} \left\{ \bar{J}_2^{(1)}(M_\rho; T) - \bar{J}_2^{(1)}(0; T) \right\} \right]. \quad (6.7) \]

Near the critical temperature \((T \to T_c)\), we have \(M_\rho \to 0, a \to 1\) due to the intrinsic temperature dependence in the VM in hot matter [I]. Furthermore, from Eq. \((5.18)\), we see that the parameter \(F^2_\pi(0)\) approaches \(\frac{N_f}{24} T_c^2\) for \(T \to T_c\). Substituting these conditions into Eq. \((6.7)\) and noting that

\[ \lim_{M_\rho \to 0} \left[ - \frac{1}{M_\rho^2} \left\{ \bar{J}_2^{(1)}(M_\rho; T) - \bar{J}_2^{(1)}(0; T) \right\} \right] = \frac{1}{2} \bar{J}_1^2(0; T) = \frac{1}{24} T_c^2, \quad (6.8) \]
we obtain
\[ \chi_A(T_c) = \frac{N_f^2}{6} T_c^2. \]  
(6.9)

To obtain the vector susceptibility near the critical temperature, we first consider \(a(0)F_2^2(0) + \bar{\Pi}_V\) appearing in the numerator of the first term in the right-hand-side of Eq. (6.3). Using Eq. (B.10), we get for the static–low-momentum limit of \(a(0)F_2^2(0) + \bar{\Pi}_V\) as
\[
\lim_{\bar{p} \to 0} \lim_{p_0 \to 0} \left[ a(0)F_2^2(0) + \bar{\Pi}_V(p_0, \bar{p}; T) \right] = a(0)F_2^2(0) - \frac{N_f}{4} \left[ 2\tilde{J}_0^0(M_\rho; T) - \tilde{J}_1^2(M_\rho; T) + a^2 \tilde{J}_2^2(0; T) \right].
\]
(6.10)

From Eq. (6.6) we can see that \(a(0) \to 1\) as \(T \to T_c\) since \(F_2^2(M_\rho) \to F_2^2(0)\) and \(M_\rho \to 0\). Furthermore, \(F_2^2(0) \to \frac{N_f}{24} T_c^2\) as we have shown in Eq. (6.10). Then, the first term of Eq. (6.10) approaches \(\frac{N_f}{24} T_c^2\). The second term, on the other hand, approaches \(-\frac{N_f}{24} T_c^2\) as \(M_\rho \to 0\) and \(a \to 1\) for \(T \to T_c\). Thus, we have
\[
\lim_{\bar{p} \to 0} \lim_{p_0 \to 0} \left[ a(0)F_2^2(0) + \bar{\Pi}_V(p_0, \bar{p}; T) \right] \to 0.
\]
(6.11)

This implies that only the second term \(\Pi_L^L\) in the right-hand-side of Eq. (6.3) contributes to the vector susceptibility near the critical temperature. Thus, taking \(M_\rho \to 0\) and \(a \to 1\), in Eq. (6.10), we obtain
\[ \chi_V(T_c) = \frac{N_f^2}{6} T_c^2, \]  
(6.12)

which agrees with the axial-vector susceptibility in Eq. (6.9). This is a prediction, not an input condition, of the theory. For \(N_f = 2\), we have
\[ \chi_A(T_c) = \chi_V(T_c) = \frac{2}{3} T_c^2. \]
(6.13)

This is our third main result. The result \(\chi_V(T_c) = \frac{2}{3} T_c^2\) is consistent with the lattice result as interpreted in [9]. It is interesting to note that the RPA result obtained in [9] in NJL model in terms of a quasi-quark-quasi-antiquark bubble is reproduced quantitatively by the one-loop graphs in HLS with the VM.

It should be noticed that the pion pole effect does not contribute to the ASUS in Eq. (6.9) since the pion decay constant \(f_\pi^L\) vanishes at the critical temperature as we have shown in the previous section, and that the contribution to the ASUS comes from the non-pole contribution expressed in Fig. [11]. In three diagrams, the third diagram in Fig. [11]c) is proportional to \((1 - a)\)
and then it vanishes at the critical temperature due to the VM. Similarly, since the transverse \( \rho \) decouples at the critical point in the VM \[1,4\], the first diagram in Fig. 1(a) does not contribute. Thus, the above result for the ASUS in Eq. (6.9) comes from only the contribution generated via the (longitudinal) vector meson plus pion loop [see Fig. 1(b)]. \#9 Similarly, the vector meson pole effect to the VSUS vanishes at the critical temperature as shown in Eq. (6.11), and the contribution to the VSUS is generated via the pion loop [Fig. 2(c)] and the longitudinal vector meson loop [Fig. 2(b)]. \#10 Since the longitudinal vector meson becomes massless, degenerate with the pion as the chiral partner in the VM, loop contribution to the ASUS becomes identical to that to the VSUS. Thus, the massless vector meson predicted by the VM fixed point plays an essential role to obtain the above equality between the ASUS and the VSUS.

In the present analysis, our aim is to show the qualitative structure of the ASUS and the VSUS in the VM, i.e., the equality between them is predicted by the VM. In order to compare our qualitative results with the lattice result, we need to go beyond the one-loop approximation. We note here that there is a result from a hard thermal loop calculation which gives \( \chi_V(T_c) \approx 1.3T_c^2 \) [25]. However this result cannot be compared to ours for two reasons. First we need to sum higher loops in our formalism which may be done in random phase approximation as in [26]. Second, the perturbative QCD with a hard thermal loop approximation may not be valid in the temperature regime we are considering. Even at \( T \gg T_c \), the situation is not clear as pointed out in Ref. [27].

### 7 Summary and Remarks

The notion of the vector manifestation in chiral symmetry requires that the zero-mass vector mesons be present at the chiral phase transition. As discussed by Brown and Rho [3], the light vector mesons near the transition point “bottom-up” can be considered as Higgsed gluons in the sense of color-flavor locking in the broken chiral symmetry sector proposed by Berges and Wetterich [28, 29] and could figure in heavy-ion processes measured at RHIC energies. In this paper we are finding that in the VM, the vector mesons with vanishing masses at the chiral transition (in the chiral limit) can figure importantly in the vector and axial-vector susceptibilit-

\#9 Note that \( \tilde{\sigma} \) in Fig. 1(b) is the quantum field corresponding to the NG boson absorbed into the vector meson, i.e., the longitudinal vector meson.

\#10 Note that the contribution from Fig. 2(a) vanishes since the transverse \( \rho \) decouples and that the one from Fig. 2(d) also vanishes since it is proportional to \( (1 - a) \).
ties near the chiral transition point. The notable results are that the VM confirms explicitly the equality $\chi_V = \chi_A$ at $T_c$ and that both $f^l_\pi$ and $f^s_\pi$ vanish simultaneously with the pion velocity $v_\pi \sim 1$. These differ from the results expected in a scenario where only the pions are the relevant effective degrees of freedom.

The reason for that the $v_\pi$ deviates from 1 is due to the Lorentz-breaking term in the bare HLS Lagrangian at which the matching to QCD is made in a thermal bath. We find the deviation is small (this will be detailed in [30]). The small deviation from 1 is also found in dense skyrmion matter studied in [5].

If one assumes that the only light degrees of freedom near $T_c$ are the pions, then one can simply take the current algebra terms in the Lagrangian and the axial-vector susceptibility (ASUS) $\chi_A$ is uniquely given by the temporal component of the pion decay constant $f^t_\pi$ with the degrees of freedom that are integrated out renormalizing this constant. Then the unquestionable equality $\chi_V = \chi_A$ at $T_c$ together with the lattice result $\chi_V|_{T_C} \neq 0$ leads to the Son-Stephanov result on the pion velocity $v_\pi = 0$. There is however a caveat to this simple result and it is that the same reasoning fails when one computes explicitly the vector susceptibility (VSUS) using the same current algebra Lagrangian.

Positing that the vector mesons enter in the VM near $T_c$ circumvents this caveat and at the same time, makes a concrete prediction. In this framework, the ASUS is given by a term related to $f^t_\pi$ plus contributions from the vector-meson (i.e., the longitudinal component $\sigma$) loop. At $T_c$, the $f^t_\pi$ vanishes and what remains comes out precisely equal to the VSUS $\chi_V$ in which the $\sigma$ (longitudinal $\rho$) loop in $\chi_A$ is replaced by the pion loop. All these are perfectly understandable in terms of the VM in HLS.

If chiral symmetry restoration à la HLS/VM – but not the standard chiral theory one – is the valid scenario as is argued in [2], its confirmation would provide a valuable insight into some of the basic tenets of effective field theories as expounded in [2].

As was shown in, e.g., Ref. [10] [9], we expect that baryons become light very near the phase transition point. At least when matter density is involved, the light baryons must figure importantly. In [31], we have simulated density effects by introducing quasiquarks whose effective mass is expected to lie below the in-medium vector meson mass $m^*_\rho$. In [9], the in-medium vector mesons whose mass must be much higher than that of quasiquarks were integrated out and the SUS was then given by the RPA bubble of quasiquark-quasiantiquark excitations in NJL model. In this paper where the vector meson plays the key role, it is not clear that we are not double-
counting if we introduce both the vector meson and the quasiquark in HLS/VM theory. Our point of view here is that whatever fermionic degrees of freedom are to be implemented in the theory should be color-singlet and hence the relevant fermionic degrees of freedom must be baryons. Assuming that the baryons are always more massive in the heat bath, our results should then correspond to HLS/VM with the baryons integrated out. Though our results clearly indicate the dual nature of the fermionic RPA bubble [9] and the one-loop HLS/VM, how to consistently implement fermions in a calculation of the type we considered here is still an open issue. We intend to return to this in a later publication.

So far we have not addressed the properties of the quantities we have studied in this paper away from the critical point $T_c$. The "intrinsic dependence" crucial in our formalism is a difficult problem to solve away from the $T = 0$ and $T = T_c$ points. We have not yet formulated how to go about this problem. Confrontation with future lattice data as well as with RHIC data will obviously require these properties to be worked out. In the case of density, this problem has been approached from a different perspective in [32]. It is plausible that a similar approach can be developed for the temperature case.

A more comprehensive discussion of the materials covered in this paper as well as other issues of HLS-VM in hot bath near chiral restoration will be discussed in a future publication [14].

In the present analysis, we used the bare HLS Largangian with Lorentz invariance since the Wilsonian matching between the HLS and the underlying QCD showed that the Lorentz violating effects to the bare parameters of the HLS Lagrangian are small, as we discussed around the end of section 2 and Appendix A. The details of the inclusion of such small corrections at the bare level and quantum effects based on such a Lagrangian will be presented in future publications.

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Appendices

A Wilsonian Matching and Intrinsic Temperature Dependences

The Wilsonian matching was originally proposed at $T = 0$ to determine the bare parameters of the HLS by matching the HLS to the underlying QCD. In Ref. the Wilsonian matching was extended to non-zero temperature and it was shown that the parameters of the HLS Lagrangian have the intrinsic temperature dependences. In this appendix we first briefly review the Wilsonian matching proposed in Ref. at $T = 0$ to determine the bare parameters of the HLS Lagrangian by matching the HLS with the underlying QCD. (For details, see Ref. 2.) Then, we extend the Wilsonian matching to the analysis at non-zero temperature to determine the intrinsic temperature dependences of the bare parameters of the (bare) HLS Lagrangian needed in the present paper.

A.1 Wilsonian matching conditions at $T = 0$

The Wilsonian matching proposed in Ref. is done by matching the axial-vector and vector current correlators derived from the HLS with those by the operator product expansion (OPE) in QCD at the matching scale $\Lambda$. The axial-vector and vector current correlators in the OPE up until $O(1/Q^6)$ at $T = 0$ are expressed as

$$\Pi^{(\text{QCD})}_A(Q^2) = \frac{1}{8\pi^2} \left(\frac{N_c}{3}\right) \left[ -\left(1 + \frac{3(N_c^2 - 1)}{8N_c} \frac{\alpha_s}{\pi}\right) \ln \frac{Q^2}{\mu^2} \right. \left. + \frac{\pi^2}{N_c} \frac{(\alpha_s G_{\mu\nu} G^{\mu\nu})}{Q^4} + \frac{\pi^3}{N_c} \frac{96(N_c^2 - 1)}{N_c^2} \left(\frac{1}{2} + \frac{1}{3N_c}\right) \frac{\alpha_s \langle \bar{q}q \rangle}{Q^6} \right], \quad (A.1)$$

$$\Pi^{(\text{QCD})}_V(Q^2) = \frac{1}{8\pi^2} \left(\frac{N_c}{3}\right) \left[ -\left(1 + \frac{3(N_c^2 - 1)}{8N_c} \frac{\alpha_s}{\pi}\right) \ln \frac{Q^2}{\mu^2} \right. \left. + \frac{\pi^2}{N_c} \frac{(\alpha_s G_{\mu\nu} G^{\mu\nu})}{Q^4} - \frac{\pi^3}{N_c} \frac{96(N_c^2 - 1)}{N_c^2} \left(\frac{1}{2} - \frac{1}{3N_c}\right) \frac{\alpha_s \langle \bar{q}q \rangle}{Q^6} \right], \quad (A.2)$$

where $\mu$ is the renormalization scale of QCD and we wrote the $N_c$-dependences explicitly (see, e.g., Ref. 34). In the HLS the same correlators are well described by the tree contributions with including $O(p^4)$ terms when the momentum is around the matching scale, $Q^2 \sim \Lambda^2$:

$$\Pi^{(\text{HLS})}_A(Q^2) = \frac{F^2(\Lambda)}{Q^2} - 2z_2(\Lambda), \quad (A.3)$$

#11 For the validity of the expansion in the HLS the matching scale $\Lambda$ must be smaller than the chiral symmetry breaking scale $\Lambda_x$. 
\[ \Pi_V^{(HLS)}(Q^2) = \frac{F_\sigma^2(\Lambda)}{M_\rho^2(\Lambda) + Q^2} \left[ 1 - 2g^2(\Lambda)z_3(\Lambda) \right] - 2z_1(\Lambda), \]  
(A.4)

where we defined the bare \( \rho \) mass \( M_\rho(\Lambda) \) as

\[ M_\rho^2(\Lambda) \equiv g^2(\Lambda)F_\sigma^2(\Lambda). \]  
(A.5)

We require that current correlators in the HLS in Eqs. (A.3) and (A.4) can be matched with those in QCD in Eqs. (A.1) and (A.2). Of course, this matching cannot be made for any value of \( Q^2 \), since the \( Q^2 \)-dependences of the current correlators in the HLS are completely different from those in the OPE: In the HLS the derivative expansion (in positive power of \( Q \)) is used, and the expressions for the current correlators are valid in the low energy region. The OPE, on the other hand, is an asymptotic expansion (in negative power of \( Q \)), and it is valid in the high energy region. Since we calculate the current correlators in the HLS including the first non-leading order \( \mathcal{O}(p^4) \), we expect that we can match the correlators with those in the OPE up until the first derivative. Then we obtain the following Wilsonian matching conditions \[18, 2\]

\[ \frac{F_\pi^2(\Lambda)}{\Lambda^2} = \frac{1}{8\pi^2} \left( \frac{N_c}{3} \right) \left[ 1 + \frac{3(N_c^2 - 1)}{8N_c} \frac{\alpha_s}{\pi} + \frac{2\pi^2 \langle \frac{\alpha_s}{\pi} G_{\mu\nu}G^{\mu\nu} \rangle}{N_c^2} \frac{\Lambda^4}{\Lambda^4} \right] - \frac{2}{\Lambda^2} \frac{288\pi(N_c^2 - 1)}{N_c^3} \frac{\alpha_s}{3N_c} \frac{\langle \bar{q}q \rangle^2}{\Lambda^6}, \]  
(A.6)

\[ \frac{F_\sigma^2(\Lambda)}{\Lambda^2} \left( M_\rho^2(\Lambda) + \Lambda^2 \right)^2 = \frac{1}{8\pi^2} \left( \frac{N_c}{3} \right) \left[ 1 + \frac{3(N_c^2 - 1)}{8N_c} \frac{\alpha_s}{\pi} + \frac{2\pi^2 \langle \frac{\alpha_s}{\pi} G_{\mu\nu}G^{\mu\nu} \rangle}{N_c^2} \frac{\Lambda^4}{\Lambda^4} \right] - \frac{2}{\Lambda^2} \frac{288\pi(N_c^2 - 1)}{N_c^3} \frac{\alpha_s}{3N_c} \frac{\langle \bar{q}q \rangle^2}{\Lambda^6}, \]  
(A.7)

\[ \frac{F_\pi^2(\Lambda)}{\Lambda^2} - \frac{F_\sigma^2(\Lambda)}{M_\rho^2(\Lambda) + \Lambda^2} \left[ 1 - 2g^2(\Lambda)z_3(\Lambda) \right] - 2[z_2(\Lambda) - z_1(\Lambda)] = \frac{4\pi(N_c^2 - 1)}{N_c^2} \frac{\alpha_s}{\Lambda^6} \left( \frac{\langle \bar{q}q \rangle^2}{\Lambda^6} \right). \]  
(A.8)

The above three equations (A.8), (A.6) and (A.7) are the Wilsonian matching conditions proposed in Ref. \[18\]. These determine several bare parameters of the HLS without much ambiguity. Especially, the first condition (A.6) determines the ratio \( F_\pi(\Lambda)/\Lambda \) directly from QCD.\[^{12}\]

\[^{12}\]One might think that there appear corrections from \( \rho \) and/or \( \pi \) loops in the left-hand-sides of Eqs. (A.6) and (A.7). However, such corrections are of higher order in the present counting scheme, and thus we neglect them here at \( Q^2 \sim \Lambda^2 \). In the low-energy scale we incorporate the loop effects into the correlators.
A.2 “Intrinsic” temperature dependence of the bare parameters

Let us consider the extension of the above matching conditions to the analysis in hot matter. The quark condensate as well as the gluon condensate appearing in the right-hand-side (RHS) of the above matching conditions generally have the temperature dependence which is converted into the intrinsic temperature dependence of bare parameters through the matching conditions \[4\]. We should note that there is no longer Lorentz symmetry in hot matter, and the Lorentz non-scalar operators such as $\bar{q}\gamma_\mu D_\nu q$ may exist in the form of the current correlators derived by the OPE \[19\]. Such a contribution may generate Lorentz symmetry violating effects in the RHS of the above matching conditions, and accordingly, we may have to use the bare Lagrangian with the Lorentz symmetry breaking effects included as in Appendix A of Ref. \[31\]. However, we will see that we can use the Lorentz invariant form of the bare Lagrangian in Eq. (2.5) near the critical temperature as a good approximation as follows:

In the RHS of the matching condition in Eq. (A.6), the Lorentz symmetry violating contribution from the operators such as $\bar{q}\gamma_\mu D_\nu q$ are small compared with the main term of $1 + \alpha_s$. This implies that the Lorentz symmetry breaking effect in the left-hand-side of Eq. (A.6), which is expressed by the Lorentz symmetry violation in the bare $\pi$ decay constant, is also small: The difference between $F_{\pi,bare}^t$ and $F_{\pi,bare}^s$ is small compared with their own values, or equivalently, the bare $\pi$ velocity defined by $v_{\pi,bare}^2 \equiv F_{\pi,bare}^s/F_{\pi,bare}^t$ is close to one. As a result we can determine, in a good approximation, the bare $\pi$ decay constant through the matching condition in Eq. (A.6) with putting possible temperature dependence on the gluon and quark condensates \[4\]:

$$\frac{F_{\pi}^2(\Lambda; T)}{\Lambda^2} = \frac{1}{8\pi^2} \left[ 1 + \frac{\alpha_s}{\pi} + \frac{2\pi^2}{3} \frac{\langle \alpha_s G_{\mu\nu}G^{\mu\nu} \rangle_T}{\Lambda^4} + \pi^3 \frac{1480}{27} \frac{\alpha_s \langle \bar{q}q \rangle^2_T}{\Lambda^6} \right], \quad (A.9)$$

where we took $N_c = 3$. We should stress again that, through the above condition (A.9), the temperature dependence of the quark and gluon condensates determines the intrinsic temperature dependence of the bare $\pi$ decay constant $F_\pi(\Lambda; T)$.

Next, we consider the intrinsic temperature dependence of other parameters near the critical temperature. As was shown in Ref. \[31\] for the VM in dense matter, the equality between the vector and axial-vector current correlators in the HLS requires the following VM conditions for the bare parameters at the leading order, which should be valid also in hot matter:

$$a_{t,bare}^t \equiv \left( \frac{F_{t,\pi,bare}^t}{F_{t,\pi,bare}^s} \right)^2 \to 1, \quad a_{s,bare}^s \equiv \left( \frac{F_{s,\pi,bare}^s}{F_{s,\pi,bare}^t} \right)^2 \to 1, \quad (A.10)$$

$$g_T,bare \to 0, \quad g_L,bare \to 0, \quad \text{for } T \to T_c. \quad (A.11)$$
The VM conditions for the $a$ parameter in Eq. \eqref{eq:A10} together with the above result that the Lorentz symmetry violating effect between $F_{\pi,\text{bare}}^t$ and $F_{\pi,\text{bare}}^s$ is small already implies that the effect of Lorentz symmetry breaking between $F_{\sigma,\text{bare}}^t$ and $F_{\sigma,\text{bare}}^s$ is small: The bare velocity of $\sigma$ (longitudinal $\rho$) defined by $v_{\sigma,\text{bare}}^2 = F_{\sigma,\text{bare}}^s / F_{\sigma,\text{bare}}^t$ is close to one near the critical temperature determined from the intrinsic temperature dependence. On the other hand, the ratio $v_{T,\text{bare}} = g_{L,\text{bare}} / g_{T,\text{bare}}$, which we call the bare velocity of the transverse $\rho$, cannot be determined through the Wilsonian matching, since the transverse $\rho$ decouples near the critical point in the VM \cite{1, 2, 4}. However, this decoupling nature of the transverse $\rho$ near the critical temperature implies that it becomes irrelevant to the quantities studied in this paper. Thus, in the present analysis, we set $v_{T,\text{bare}} = 1$ for simplicity of the calculation, and show how the transverse $\rho$ decouples from the quantities we study in this paper.

\section{B Hadronic Thermal Corrections}

In this appendix we summarize the hadronic thermal corrections to the two-point functions of $A_\mu - A_\nu$, $V_\mu - V_\nu$, $V_\mu - V_\nu$ and $V_\mu - V_\nu$.

The four components of the hadronic thermal corrections to the two point function of $A_\mu - A_\nu$, $\Pi_\perp$, are expressed as

$$
\bar{\Pi}^t_\perp(p_0, \bar{p}; T) = N_f(a - 1)\bar{A}_0(0, T) - N_f a M_\rho^2 \bar{B}_0(p_0, \bar{p}; M_\rho, 0; T) \\
+ N_f \frac{a}{4} \bar{B}^t(p_0, \bar{p}; M_\rho, 0; T),
$$

$$
\bar{\Pi}^s_\perp(p_0, \bar{p}; T) = N_f(a - 1)\bar{A}_0(0, T) - N_f a M_\rho^2 \bar{B}_0(p_0, \bar{p}; M_\rho, 0; T) \\
+ N_f \frac{a}{4} \bar{B}^s(p_0, \bar{p}; M_\rho, 0; T),
$$

$$
\bar{\Pi}^L_\perp(p_0, \bar{p}; T) = N_f \frac{a}{4} \bar{B}^L(p_0, \bar{p}; M_\rho, 0; T),
$$

$$
\bar{\Pi}^T_\perp(p_0, \bar{p}; T) = N_f \frac{a}{4} \bar{B}^T(p_0, \bar{p}; M_\rho, 0; T),
$$

where the functions $\bar{A}_0$, $\bar{B}_0$, and so on are given in Appendix C.

The two components $\Pi^t$ and $\Pi^s$ of hadronic thermal corrections to the two-point functions of $V_\mu - V_\nu$, $\nabla_\mu - \nabla_\nu$ and $\nabla_\mu - \nabla_\nu$ are written as

$$
\bar{\Pi}^t_V(p_0, \bar{p}; T) = \bar{\Pi}^t_V(p_0, \bar{p}; T) \\
= \bar{\Pi}^t_\parallel(p_0, \bar{p}; T) = \bar{\Pi}^t_\parallel(p_0, \bar{p}; T) \\
= -\bar{\Pi}^t_\parallel(p_0, \bar{p}; T) = -\bar{\Pi}^t_\parallel(p_0, \bar{p}; T)
$$

27
\[=-N_f \frac{1}{4} \left[ \tilde{A}_0(M; \rho; T) + a^2 \tilde{B}_0(0; T) \right] - N_f M^2_p \tilde{B}_0(p_0, \bar{p}; M_\rho; M_\rho; T). \] (B.5)

Among the remaining components only \( \Pi^L_\parallel \) is relevant to the present analysis. This is given by \(^{\#13}\)

\[\Pi^L_\parallel(p_0, \bar{p}; T) = N_f \frac{1}{8} \tilde{B}^L(p_0, \bar{p}; M_\rho, M_\rho; T) + N_f \frac{(2 - a)^2}{8} \tilde{B}^L(p_0, \bar{p}; 0; T). \] (B.6)

For obtaining the pion decay constants and velocity in Section 5, we need the limit of \( p_0 = \bar{p} \) of \( \Pi^i_\perp \) and \( \Pi^a_\perp \) in Eqs. (3.1) and (3.2). With Eq. (C.7), \( \Pi^i_\perp \) and \( \Pi^a_\perp \) reduce to the following forms in the limit \( M_\rho \rightarrow 0 \) and \( a \rightarrow 1 \):

\[\Pi^i_\perp(p_0 = \bar{p} + i \epsilon, \bar{p}; T) \xrightarrow{M_\rho \rightarrow 0, a \rightarrow 1} - \frac{N_f}{2} \tilde{J}^2_1(0; T) = - \frac{N_f}{24} T^2,\]
\[\Pi^a_\perp(p_0 = \bar{p} + i \epsilon, \bar{p}; T) \xrightarrow{M_\rho \rightarrow 0, a \rightarrow 1} - \frac{N_f}{2} \tilde{J}^2_1(0; T) = - \frac{N_f}{24} T^2. \] (B.7)

In the static–low-momentum limits of the functions listed in Eq. (C.7), the \( (\Pi^i_\perp - \Pi^L_\perp) \) appearing in the axial-vector susceptibility becomes

\[\lim_{\bar{p} \rightarrow 0} \lim_{p_0 \rightarrow 0} \left[ \Pi^i_\perp(p_0, \bar{p}; T) - \Pi^L_\perp(p_0, \bar{p}; T) \right] = -N_f \tilde{J}^2_1(0; T) + N_f a \tilde{J}^2_1(M_\rho; T) - N_f \frac{a}{M^2_\rho} \left[ \tilde{J}^2_1(M_\rho; T) - \tilde{J}^2_1(0; T) \right]. \] (B.8)

For the functions appearing in the vector susceptibility relevant to the present analysis we have

\[\lim_{\bar{p} \rightarrow 0} \lim_{p_0 \rightarrow 0} \left[ \Pi^i_\perp(p_0, \bar{p}; T) \right] = -\frac{N_f}{4} \left[ 2 \tilde{J}^0_1(M_\rho; T) - \tilde{J}_1^2(M_\rho; T) + a^2 \tilde{J}^2_1(0; T) \right], \] (B.9)
\[\lim_{\bar{p} \rightarrow 0} \lim_{p_0 \rightarrow 0} \left[ \Pi^a_\perp(p_0, \bar{p}; T) \right] = -N_f \frac{1}{4} \left[ M^2_\rho \tilde{J}^0_1(M_\rho; T) + 2 \tilde{J}^2_1(M_\rho; T) \right] - N_f \frac{(2 - a)^2}{2} \tilde{J}^2_1(0; T). \] (B.10)

C Functions

In this appendix we list the explicit forms of the functions that figure in the hadronic thermal corrections, \( A_0, B_0 \) and \( B^{\mu \nu} \) in various limits relevant to the present analysis.

The function \( \tilde{A}_0(M; T) \) is expressed as

\[\tilde{A}_0(M; T) = \tilde{J}^2_1(M; T), \] (C.1)

where \( \tilde{J}^2_1(M; T) \) is defined by

\(^{\#13}\)The explicit forms of other components will be listed in Ref. [14].
\[ \tilde{J}^a_t(M;T) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega(k;M)/T - 1} \frac{|\vec{k}|^{n-2}}{[\omega(k;M)]^l} , \]  
\hspace{1cm} (C.2)

with \( l \) and \( n \) being integers and \( \omega(k;M) \equiv \sqrt{M^2 + |\vec{k}|^2} \). In the massless limit \( M = 0 \), the above integration can be performed analytically. Here we list the result relevant to the present analysis:

\[ \tilde{J}^2_t(0;T) = \tilde{J}^0_{-1}(0;T) = \frac{1}{12} T^2 . \]  
\hspace{1cm} (C.3)

It is convenient to decompose \( \bar{B}^{\mu\nu} \) into four components as done for \( \Pi_{\perp}^{\mu\nu} \) in Eq. \( (C.1) \):

\[ \bar{B}^{\mu\nu} = u^\mu u^\nu B^t + (g^{\mu\nu} - u^\mu u^\nu) B^s + P_L^\mu B^L + P_T^\mu B^T . \]  
\hspace{1cm} (C.4)

We note here that, by explicit computations, the following relations are satisfied:

\[ \bar{B}^t(p_0, \vec{p}; M, M; T) = \bar{B}^s(p_0, \vec{p}; M, M; T) = -2 \bar{A}_0(M;T) = -2 \bar{J}_1^2(M;T) . \]  
\hspace{1cm} (C.5)

To obtain the pion decay constants and velocity in Section 5 we need the limit of \( p_0 = \vec{p} \) of the functions in Eqs. \( (B.1) \) and \( (B.2) \). As for the functions \( M_\rho^2 \bar{B}_0, \bar{B}^t \) and \( \bar{B}^s \) appearing in Eqs. \( (B.1) \) and \( (B.2) \), we find that, in the limit of \( M_\rho \) going to zero, they reduce to

\[ M_\rho^2 \bar{B}_0(p_0 = \vec{p} + i\epsilon, \vec{p}; M_\rho, 0; T) \xrightarrow{M_\rho \to 0} 0 , \]

\[ \bar{B}^t(p_0 = \vec{p} + i\epsilon, \vec{p}; M_\rho, 0; T) \xrightarrow{M_\rho \to 0} -2 \bar{J}_1^2(0;T) = -\frac{1}{6} T^2 , \]

\[ \bar{B}^s(p_0 = \vec{p} + i\epsilon, \vec{p}; M_\rho, 0; T) \xrightarrow{M_\rho \to 0} -2 \bar{J}_1^2(0;T) = -\frac{1}{6} T^2 . \]  
\hspace{1cm} (C.6)

The static–low-momentum limits of the functions appearing in the corrections to the axial-vector and vector susceptibility are summarized as

\[ \lim_{\vec{p} \to 0} \lim_{p_0 \to 0} \left[ M_\rho^2 \bar{B}_0(p_0, \vec{p}; M_\rho, 0; T) \right] = -\bar{J}_1^2(M_\rho;T) + \tilde{J}^2_t(0;T) , \]

\[ \lim_{\vec{p} \to 0} \lim_{p_0 \to 0} \left[ \bar{B}^t(p_0, \vec{p}; M_\rho, 0; T) - \bar{B}^L(p_0, \vec{p}; M_\rho, 0; T) \right] \]

\[ = -\frac{4}{M_\rho^2} \left[ -\bar{J}^2_{-1}(M_\rho;T) + \tilde{J}^2_{-1}(0;T) \right] , \]

\[ \lim_{\vec{p} \to 0} \lim_{p_0 \to 0} \left[ M_\rho^2 \bar{B}_0(p_0, \vec{p}; M_\rho, M_\rho; T) \right] = \frac{1}{2} \left[ \tilde{J}^0_{-1}(M_\rho;T) - \tilde{J}^2_t(M_\rho;T) \right] , \]

\[ \lim_{\vec{p} \to 0} \lim_{p_0 \to 0} \left[ \bar{B}^L(p_0, \vec{p}; M_\rho, M_\rho; T) \right] = -2 M_\rho^2 \tilde{J}^1_t(M_\rho;T) - 4 \bar{J}_1^2(M_\rho;T) , \]

\[ \lim_{\vec{p} \to 0} \lim_{p_0 \to 0} \left[ \bar{B}^L(p_0, \vec{p}; 0, 0; T) \right] = -4 \bar{J}_1^2(0;T) . \]  
\hspace{1cm} (C.7)
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