Research Article

Inversion of Fourier Transforms by Means of Scale-Frequency Series

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We report on inversion of the Fourier transform when the frequency variable can be scaled in a variety of different ways that improve the resolution of certain parts of the frequency domain. The corresponding inverse Fourier transform is shown to exist in the form of two dual scale-frequency series. Upon discretization of the continuous scale factor, this Fourier transform series inverse becomes a certain nonharmonic double series, a discretized scale-frequency (DSF) series. The DSF series is also demonstrated, theoretically and practically, to be rate-optimizable with respect to its two free parameters, when it satisfies, as an entropy maximizer, a pertaining recursive nonlinear programming problem incorporating the entropy-based uncertainty principle.

1. Introduction

We revisit the classical problem of recovering a Fourier transformable signal

\[
 f(t) = \frac{1}{2\pi} \int_{\mathcal{F}(R)} \tilde{f}(\mu) e^{i\mu t} d\mu = \mathcal{F}^{-1}\{\tilde{f}(\mu)\} \quad (1)
\]

in \( \mathcal{F}(R), R = (-\infty, \infty), \) from its Fourier transform; see, for example, [1, 2], image \( \tilde{f}(\mu). \) The space \( \mathcal{F} \) is the set of all real functions \( f \) for which the integral \( \tilde{f}(\mu) = \int_{R} f(t)e^{-i\mu t} dt \) exists.

Obviously, the function to be integrated in (1) is the product

\[
 \tilde{f}(\mu) e^{i\mu t} = \tilde{f}(\mu) \left[ \cos \mu t + i \sin \mu t \right], \quad t \in R, \quad (2)
\]

and when the variable \( |t| \to \infty \), the function \( e^{i\mu t} \) will oscillate extremely rapidly. Then in order to follow the variations of the product \( \tilde{f}(\mu)e^{i\mu t} \) meaningfully in a \( \mu \)-quadrature formula, see, for example, [3], for (1), there is a need for a large number of \( \mu \)-points, even for slowly varying \( \tilde{f}(\mu). \) Consequently, the faster the \( \tilde{f}(\mu) \) decreases as \( |\mu| \to \infty \), the more the tractable computationally the integral (1) will be, and the more accurate are its numerical evaluations.

Bandlimiting of signals has been a practical way for easing the previous problem. A fact that has so far been motivating the wide engineering interest in bandlimited signals, for which

\[
 \mathcal{F}\{f(t)\} = \begin{cases} \tilde{g}(\mu), & |\mu| \leq 2\pi B, \\ 0, & \text{otherwise} \end{cases}
\]

(3)

where \( B \) is the bandwidth of \( f(t) \) with an image \( \tilde{f}(\mu) \) having the \([-2\pi B, 2\pi B]\) compact support of \( \tilde{g}(\mu). \) Inversion of bandlimited signals is currently performed by projections onto convex sets (POCS) algorithms [4], which encompass the Gerchberg-Papoulis algorithm [5].

In this work we report on a novel robust semianalytical method for integration of (1) to recover a signal \( f(t) \in \mathcal{F}(R) \) that is not necessarily bandlimited. In this method, the frequency variable can be scaled in a variety of different ways that improve the resolution of certain parts of the frequency domain. The corresponding inverse Fourier transform is shown to exist in the form of two dual scale-frequency series. Upon discretization of the continuous scale factor, this Fourier transform series inverse becomes a certain nonharmonic [6] double series, a discretized scale-frequency (DSF) series. This DSF series is demonstrated, for the first time in this work, to be the proper framework for an entropy-maximizing inverse Fourier transformation. A fact
has emerged after theoretically illustrating that the DSF series is rate-optimizable with respect to the scale factor step and the frequency gap size. The robustness of this optimization (which is illustrated by a simple example) reduces to the satisfaction, via entropy-maximization, of an emerging recursive nonlinear programming problem, incorporating the entropy-based uncertainty principle.

2. Theory of the Continuous Scale-Frequency Series Inverse

Let \( Z \) be the set of integers, \( Q \) the set of rational numbers, and \( \mathbb{Q}' \) the set of irrational numbers. \( \lambda \in R \setminus \{ 0 \} \) will be a scale factor, while the frequency \( \gamma \) can either be \( m \in Z \) or \( \nu = \nu(m) = (m + \psi) \in Q \) or \( \mathbb{Q}' \) (depending on whether \( \psi \in Q \) or \( \mathbb{Q}' \)). The symbol \( \circ \) means either multiplication or division of \( \gamma \) by \( \lambda \), and \( \gamma' \) is the alternative (complementary) symbol to \( \gamma \); that is, if \( \gamma = m \), then \( \gamma' = \nu \) and vice versa. The signal \( f(t) \in \mathcal{F}(R) \) is assumed to be expandable in the variably scaled trigonometric basis:

\[
q_{\gamma\lambda}(t) = \{ \lambda^{-1} \left[ \cos (\gamma \circ \lambda) t + \sin (\gamma \circ \lambda) t \right] \}. \tag{4}
\]

The ensemble of \( \gamma \) with \( \circ \) in (4) can be in any of their possible combinations \( C'_2 = 6 \) combinations of 4 quantities, taken two at a time. Accordingly throughout this work we will assume, as in [7], any of

\[
q_{\gamma,m\lambda}(t) = \lambda^{-1} \left[ \cos \nu \lambda t + \sin m \lambda t \right]; \\
\circ = >: \gamma \circ \lambda = m \lambda, \quad \gamma' \circ \lambda = \nu \lambda,
\]

\[
q_{\gamma,m\lambda}(t) = \lambda^{-1} \left[ \cos m \lambda t + \sin \nu \lambda t \right]; \\
\circ = <: \gamma \circ \lambda = \nu \lambda, \quad \gamma' \circ \lambda = m \lambda
\]

for high frequency resolving scaling (as \( \lambda \to 0 \)), any of

\[
q_{\gamma,m\lambda}(t) = \lambda^{-1} \left[ \cos \nu \lambda t + \sin m \lambda t \right]; \\
\circ = >: \gamma \circ \lambda = \nu \lambda, \quad \gamma' \circ \lambda = \nu \lambda,
\]

\[
q_{\gamma,m\lambda}(t) = \lambda^{-1} \left[ \cos m \lambda t + \sin \nu \lambda t \right]; \\
\circ = <: \gamma \circ \lambda = \nu \lambda, \quad \gamma' \circ \lambda = \nu \lambda
\]

for low frequency resolving scaling, while any of

\[
q_{\gamma,m\lambda}(t) = \lambda^{-1} \left[ \cos \nu \lambda t + \sin m \lambda t \right]; \\
\circ = >: \gamma \circ \lambda = m \lambda, \quad \gamma' \circ \lambda = \nu \lambda,
\]

\[
q_{\gamma,m\lambda}(t) = \lambda^{-1} \left[ \cos m \lambda t + \sin \nu \lambda t \right]; \\
\circ = <: \gamma \circ \lambda = \nu \lambda, \quad \gamma' \circ \lambda = m \lambda
\]

for mixed (high and low) frequency resolving scaling.

Next consider the sequence \( (q_{\gamma\lambda}) \) that supports the bases (4) when \( R_\circ = (R \setminus \{ 0 \}) \), and let \( \circ \) be any one of the six symbols \( >, <, \top, \bot, \triangleright, \triangleleft \), to invoke the following double parameterized Riemann-Lebesgue lemma (RLL).

**Lemma 1** (see [7]). For \( \Gamma_{\gamma\lambda}[\cdot] = \int_R q_{\gamma\lambda}(t) dt \), when \( m \in Z \),

\[
\lim_{|m| \to \infty} \Gamma_{\gamma\lambda}[f] = 0, \tag{8}
\]

for all \( f \in \mathcal{F}(R) \), for any scaling factor \( \lambda \in R_\circ \), and for all six \( \circ \) symbols.

This lemma justifies the existence and guarantees the convergence of the scale-frequency transform

\[
F_{\gamma\lambda}(\lambda, m) := \int_R f(t) q_{\gamma\lambda}(t) dt = \langle f, q_{\gamma\lambda}(t) \rangle, \tag{9}
\]

for all \( \lambda \in R_\circ \), of \( f \in \mathcal{F}(R) \) with respect to \( q_{\gamma\lambda} \). Make then use of the notation

\[
\{ F_\gamma(\mu) \} = \int_R f(t) \left\{ \cos \mu t \over \sin \mu t \right\} dt \tag{10}
\]

in

\[
\tilde{f}(\mu) = \int_R f(t) e^{-i\mu t} dt = \mathcal{F} [f(t)] \tag{11}
\]

to rewrite it as

\[
\tilde{f}(\mu) = F_\gamma(\mu) - i F_\delta(\mu). \tag{12}
\]

That is,

\[
F_\gamma(\mu) = \text{Re} \left[ \tilde{f}(\mu) \right], \quad F_\delta(\mu) = - \text{Im} \left[ \tilde{f}(\mu) \right]. \tag{13}
\]

Unlike the Fourier transformation \( \tilde{f}(\mu) \), which is a complex function of \( \mu \), \( F_\gamma(\lambda, m) \) is a real function of the scaled frequencies

\[
\rho_\gamma = \gamma \circ \lambda, \quad \rho_\delta = \gamma' \circ \lambda. \tag{14}
\]

**Theorem 2.** \( F_\gamma^{-1}(\lambda, m) = f(t) \in \mathcal{F}(R) \) satisfies

\[
f(t) = \frac{1}{2\pi} \int_R F_\gamma(\mu_\gamma) \cos \mu_\gamma t d\mu_\gamma + \frac{1}{2\pi} \int_R F_\delta(\rho_\gamma) \sin \rho_\gamma t d\rho_\gamma. \tag{15}
\]

**Proof.** The above result follows directly from the structural linearity of

\[
F_\gamma(\lambda, m) = \frac{1}{\lambda} F_\gamma(\mu_\gamma) + \frac{1}{\lambda} F_\delta(\rho_\gamma) \tag{16}
\]
and the linearity of its inverse transformation. □
utilization of anyone of the components of $\hat{f}(\mu)$ in (13); namely,

$$F_{\Box} (\lambda, m) \iff \begin{cases} F_{\Box c}^{-1} (\lambda, m) \\ F_{\Box s}^{-1} (\lambda, m) \end{cases} = F_{\Box}^{-1} (\lambda, m) = f(t) ,$$

when $\left\{ F_{\Box c} (\mu) \right\}$ is utilized.

Interestingly, this results in 12 distinct scale-frequency forms for the inverse Fourier transform $f(t)$.

Obviously, the symbols $\downarrow, \uparrow$ correspond to scaling sines and cosines in the same way, while $\triangleright, \triangleleft$ or $\nearrow, \searrow$ correspond to scaling cosines and sines in different ways. It should be noted here that although a Fourier transform is virtually unaffected by a shift of $f$ in time (which mixes the roles of the cosines and sines), the corresponding scale-frequency transform could change drastically. This raises the question on the rather unclear merits of different-way scalings, to the favour of same-way ($\downarrow, \uparrow$) scalings. It is believed, however, that different-way scalings can particularly reveal their merits when applied to discontinuous $f \in \mathcal{F}(\mathbb{R})$.

**Theorem 3** (high frequency resolving scaling). When $\Box = \triangleright$ in $F_{\Box} (\lambda, m) \iff f(t)$, the inverse is either

$$f(t) = F_{\triangleright c}^{-1} (\lambda, m)$$

$$= \frac{1}{2\pi M} \sum_{m=0}^{M-1} \int_{\mathbb{R}} m F_{c} (m\lambda) \cos (\lambda + 1) m t d\lambda$$

$$- \frac{1}{2\pi M} \sum_{m=0}^{M-1} \int_{\mathbb{R}} m \left[ F_{c} (m\lambda) + F_{s} ([m + \varsigma] \lambda) - F_{s} (m\lambda) \right] \times \sin (\lambda + 1) m t d\lambda ,$$

(18)

or the dual representation

$$f(t) = F_{\triangleright s}^{-1} (\lambda, m)$$

$$= \frac{1}{2\pi M}$$

$$\times \sum_{m=0}^{M-1} \int_{\mathbb{R}} (m + \varsigma) \left[ F_{c} (m\lambda) + F_{s} ([m + \varsigma] \lambda) - F_{s} (m\lambda) \right] \times \cos (\lambda m t d\lambda - \frac{1}{2\pi M}$$

$$\times \sum_{m=0}^{M-1} \int_{\mathbb{R}} (m + \varsigma) F_{s} ([m + \varsigma] \lambda) \sin (\lambda m t d\lambda .$$

(19)

Proof. By incorporation of

$$F_{c} (\lambda, m) = \frac{1}{\lambda} F_{c} (\mu_{c}) + \frac{1}{\lambda} F_{s} (\rho_{s})$$

(20)
in the details of the proofs of Theorem 1 and Corollary 1 of [7].

A similar theorem holds also for the case when $\Box = <$. **Theorem 4** (low frequency resolving scaling). For $\Box = \nearrow$ in $F_{\Box} (\lambda, m) \iff f(t)$, the inverse is either

$$f(t) = F_{\nearrow c}^{-1} (\lambda, m)$$

$$= -\frac{1}{2\pi M} \sum_{m=0}^{M-1} \int_{\mathbb{R}} m F_{c} (m\lambda) \cos (\frac{1}{\lambda} + 1) m t (\ln \lambda)$$

$$+ \frac{1}{2\pi M} \sum_{m=0}^{M-1} \int_{\mathbb{R}} m \left[ \frac{1}{\lambda} F_{c} (m\lambda) + \frac{1}{\lambda} F_{s} (\frac{m + \varsigma}{\lambda}) - \frac{1}{\lambda} F_{s} (\frac{m}{\lambda}) \right] \times \sin (\frac{1}{\lambda} + 1) m t (\ln \lambda) ,$$

(21)

or the dual representation

$$f(t) = F_{\nearrow s}^{-1} (\lambda, m)$$

$$= \frac{1}{2\pi M} \sum_{m=0}^{M-1} \int_{\mathbb{R}} (m + \varsigma) \frac{1}{\lambda} F_{c} (m\lambda) + \frac{1}{\lambda} F_{s} (\frac{m + \varsigma}{\lambda}) - \frac{1}{\lambda} F_{s} (\frac{m}{\lambda}) \times \cos (\frac{m + \varsigma}{\lambda} + m) t (\ln \lambda)$$

$$+ \frac{1}{2\pi M} \sum_{m=0}^{M-1} \int_{\mathbb{R}} (m + \varsigma) \frac{1}{\lambda} F_{s} (\frac{m + \varsigma}{\lambda}) \times \sin (\frac{m + \varsigma}{\lambda} + m) t (\ln \lambda)$$

(22)

when the logarithmic integrals are conceived in the Principal Value sense.

Proof. By incorporation of $F_{c} (\lambda, m) = (1/\lambda) F_{c} (\mu_{c}) + (1/\lambda) F_{s} (\rho_{s})$ in the details of the proofs of Theorem 2 and Corollary 2 of [7].

A similar theorem holds also for the case when $\Box = \searrow$.

3. Discretized Scale-Frequency Series Inverse

Here we discretize the $\lambda$ axis uniformly over $\mathbb{R}_{\o}$ according to $\lambda_{k} = k\delta \in [-\Lambda, \Lambda] \subset \mathbb{R}$, where $\Lambda$ is the scale factor band.
width, which is not apriori known. $k \in [-N, N] \setminus \{0\} = W_0 \subset \mathbb{Z}$, with $\delta = \Lambda/N$ as the scale sample size, which can be arbitrarily small or large. In the special case when $\Lambda = N$, $\delta = 1$.

3.1. Discretized High Frequency Resolving Scaling. The above mentioned discretization of the scale variable, with $k$ as a scaling factor number, when $\square = \triangleright$, leads to

$$F_c (k\delta, m) = \frac{1}{k\delta} F_c (m\delta) + \frac{1}{k\delta} F_c ([m + \zeta] k\delta),$$

(23)

the pertaining phase

$$\varphi_{k,m\delta} = m\delta + m,$$

(24)

and to the directly related discretized notation,

$$A_{k,m} = \frac{\delta}{2\pi M} m \text{Re} \left[ \tilde{f} (m\delta) \right],$$

(25)

$$B_{k,m} = \frac{\delta}{2\pi M} m \left[ \text{Im} \left[ \tilde{f} (m\delta) \right] + \text{Re} \left[ \tilde{f} ([m + \zeta] k\delta) \right] \right],$$

(26)

Alternatively, for the dual representation (19), we employ

$$\varphi_{k,m\delta}^+ = (m + \zeta) k\delta + m,$$

(27)

$$A_{k,m}^+ = \frac{\delta}{2\pi M} (m + \zeta) \left[ \text{Re} \left[ \tilde{f} (m\delta) \right] - \text{Im} \left[ \tilde{f} ([m + \zeta] k\delta) \right] \right] + \text{Im} \left[ \tilde{f} (m\delta) \right],$$

(28)

$$B_{k,m}^+ = \frac{\delta}{2\pi M} (m + \zeta) \text{Im} \left[ \tilde{f} ([m + \zeta] k\delta) \right],$$

(29)

to state one of the main results of this work.

**Theorem 5** (high frequency resolving discrete scaling). When $\square = \triangleright$ in $F_c (\lambda, m) \leftrightarrow f (t)$, the DSF series inverse is either

$$f (t) \approx \mathfrak{D} F^{-1}_{c\triangleright} (\lambda, m)$$

$$= \sum_{k \in W_0} \sum_{m=0}^{M-1} A_{k,m} \cos \varphi_{k,m\delta} t + B_{k,m} \sin \varphi_{k,m\delta} t,$$

(30)

or the dual representation

$$f (t) \approx \mathfrak{D} F^{-1}_{c\downarrow} (\lambda, m)$$

$$= \sum_{k \in W_0} \sum_{m=0}^{M-1} A_{k,m}^+ \cos \varphi_{k,m\delta}^+ t + B_{k,m}^+ \sin \varphi_{k,m\delta}^+ t.$$  

(31)

**Proof.** By considering (13) when following the same arguments of the proofs of Theorem 3 and Corollary 3 of [7].

A similar result holds also for the case when $\square = \downarrow$.

3.2. Discretized Low Frequency Resolving Scaling. Here $\square = \triangleright$ is associated, by discretizing the scale factor, with

$$\psi_{k,m,\delta}^{\downarrow} = \frac{m}{k\delta} + m,$$

(32)

$$\mathfrak{A}_{k,m} = - \frac{\delta}{2\pi M} m \left( \frac{1}{k\delta} \right)^2 \text{Re} \left[ \tilde{f} \left( \frac{m}{k\delta} \right) \right],$$

(33)

$$\mathfrak{B}_{k,m} = \frac{\delta}{2\pi M} m \left( \frac{1}{k\delta} \right)^2 \times \left\{ \text{Re} \left[ \tilde{f} \left( \frac{m}{k\delta} \right) \right] - \text{Im} \left[ \tilde{f} \left( \frac{[m + \zeta]}{k\delta} \right) \right] \right\}.$$  

(34)

Alternatively, for the dual representation (22), we employ

$$\psi_{k,m,\delta}^{\uparrow} = \frac{(m + \zeta)}{k\delta} + m,$$

(35)

$$\mathfrak{A}_{k,m}^+ = \frac{\delta}{2\pi M} (m + \zeta) \left( \frac{1}{k\delta} \right)^2 \times \left\{ \text{Re} \left[ \tilde{f} \left( \frac{m}{k\delta} \right) \right] - \text{Im} \left[ \tilde{f} \left( \frac{[m + \zeta]}{k\delta} \right) \right] + \text{Im} \left[ \tilde{f} \left( \frac{m}{k\delta} \right) \right] \right\},$$

(36)

$$\mathfrak{B}_{k,m}^+ = - \frac{\delta}{2\pi M} (m + \zeta) \left( \frac{1}{k\delta} \right)^2 \text{Im} \left[ \tilde{f} \left( \frac{[m + \zeta]}{k\delta} \right) \right] \text{Im} \left[ \tilde{f} \left( \frac{m}{k\delta} \right) \right] + \text{Im} \left[ \tilde{f} \left( \frac{m}{k\delta} \right) \right] \text{Im} \left[ \tilde{f} \left( \frac{[m + \zeta]}{k\delta} \right) \right].$$

(37)

to state the following main result of this work.

**Theorem 6** (low frequency resolving discrete scaling). When $\square = \triangleright$ in $F_c (\lambda, m) \leftrightarrow f (t)$, the DSF series inverse is either

$$f (t) \approx \mathfrak{D} F^{-1}_{c\triangleright} (\lambda, m)$$

$$= \sum_{k \in W_0} \sum_{m=0}^{M-1} \mathfrak{A}_{k,m} \cos \psi_{k,m\delta} t + \mathfrak{B}_{k,m} \sin \psi_{k,m\delta} t,$$

(38)

or the dual representation

$$f (t) \approx \mathfrak{D} F^{-1}_{c\downarrow} (\lambda, m)$$

$$= \sum_{k \in W_0} \sum_{m=0}^{M-1} \mathfrak{A}_{k,m}^+ \cos \psi_{k,m\delta}^+ t + \mathfrak{B}_{k,m}^+ \sin \psi_{k,m\delta}^+ t.$$  

(39)

**Proof.** By considering (13) when following the same arguments of the proofs of Theorem 4 and Corollary 4 of [7].

A similar result holds also for the case when $\square = \downarrow$.

4. Demonstrative Application

To illustrate the applicability of the previously reported results, we give here three illustrative examples of the high
frequency resolving scaling in recovering three well-known simple Fourier preimages.

Example 7. The even real signal \( f(t) = e^{-|t|} \) has a real image, \( \hat{f}(\mu) = 2/(1 + \mu^2) \); that is, \( \text{Im}[\hat{f}(\mu)] = 0 \). According to (30) it admits the following family of nonharmonic series expansions:

\[
e^{-|t|} = \mathcal{D}F^{-1}_{\infty} (\lambda, m)
\]

\[
= \delta \sum_{k+N}^{M-1} \sum_{m=0}^{m} \frac{m}{1 + m^2 k^2 \delta^2} \cos (k\delta + 1) mt
\]

\[
+ \left\{ \frac{m}{1 + m^2 k^2 \delta^2} - \frac{m}{1 + m^2 \delta} \right\} \sin (k\delta + 1) mt,
\]

(40)

for various values of \( \delta, \mu, M, \) and \( N \). It should be noted here that \( \delta N \) defines the unknown scale factor band width \( \Lambda \).

Obviously when \( \mu = 0 \), the expansions become purely cosinoidal, and the dual DSF expansions \( \mathcal{D}F^{-1}_{\infty} (\lambda, m) \) can similarly be constructed according to (31).

Example 8. For the odd signal \( g(t) = t e^{-|t|} \) we have a purely imaginary image, \( \hat{g}(\mu) = (-4\mu/(1 + \mu^2)^2) \); that is, \( \text{Re}[\hat{g}(\mu)] = 0 \). Consequently, according to (30) it admits the following family of nonharmonic sinusoidal series expansions:

\[
t e^{-|t|} = \mathcal{D}G^{-1}_{\infty} (\lambda, m)
\]

\[
= -\frac{2\delta}{\pi M} \sum_{k+N}^{M-1} \sum_{m=0}^{m} \frac{m [m + \mu] k\delta}{1 + [m + \mu] k^2 \delta^2} \sin (k\delta + 1) mt,
\]

(41)

in addition to the dual expansions \( \mathcal{D}G^{-1}_{\infty} (\lambda, m) \).

Example 9. The nonsymmetric real signal \( h(t) = (t + 1)e^{-|t|} \) has the complex image \( \hat{h}(\mu) = 2/(1 + \mu^2) - (4\mu/(1 + \mu^2)^2) i \); then it admits, by (30), the family of nonharmonic series expansions:

\[
(t + 1) e^{-|t|} = \mathcal{D}H^{-1}_{\infty} (\lambda, m)
\]

\[
= \frac{\delta}{\pi M} \sum_{k+N}^{M-1} \sum_{m=0}^{m} \frac{m}{1 + m^2 k^2 \delta^2} \cos (k\delta + 1) mt
\]

\[
- \left\{ \frac{m}{1 + m^2 k^2 \delta^2} + \frac{2m [m + \mu] k\delta}{1 + [m + \mu] k^2 \delta^2} \right\} \sin (k\delta + 1) mt,
\]

(42)

in addition to the dual DSF expansions \( \mathcal{D}H^{-1}_{\infty} (\lambda, m) \).

5. Rate-Optimization of the DSF Series Inverse

In DSF series inversion of \( \hat{f}(\mu) \), the number \( M \) of frequency points is decided by the need for satisfactory followup of the variation of \( \hat{f}(\mu) \) over its effective compact support. In contrast, the number of scaling points \( 2N = 2\Lambda/\delta \in \mathbb{Z} \) remains rather arbitrary.

Besides, these series can be made rate-optimal in the sense that the derivative of a certain norm of them will vanish at a high rate as \( M \) and \( N \) tend to infinity. Clearly when \( N \to \infty, \delta \to 0 \) for a given \( \Lambda \). The two free parameters that can be optimized in this sense are (i) the scale factor step, \( \delta \), for a given \( N \), and (ii) the frequency gap size, \( \mu \), for a given \( M \).

Here we will study this optimization problem for the case of the DSF series inverse \( \mathcal{D}F^{-1}_{\infty} (\lambda, m) \), (30), without loss of generality.

Definition 10. The DSF norm \( \|f\|_2 \) of \( f(t) \in \mathcal{F}(R) \) is the sum of the amplitudes of all scaled-harmonics appearing in the \( \mathcal{D}F^{-1}_{\infty} (\lambda, m) \) expansion; that is,

\[
\|f\|_2 = \sum_{k,N}^{M-1} \sum_{m=0}^{m} \left\| A_{k,m} \right\|^2 + \left\| B_{k,m} \right\|^2 \]

(43)

which, for certain \( \delta \) and \( \mu \), may even not be closed in the space \( L^2(R) \), and despite the fact that the coefficients \( A_{k,m} \) and \( B_{k,m} \) do satisfy the scale-frequency Riemann-Lebesgue lemma \[7\]. Nevertheless, \( \|f\|_2 \) can easily be proved to satisfy the following defining properties of a norm:

\[
\begin{align*}
(i) \|c f\|_2 &= |c| \|f\|_2, \ c \in \mathbb{R} \text{ (absolute homogeneity)}, \\
(ii) \|f + g\|_2 &\leq \|f\|_2 + \|g\|_2 \text{ (subadditivity)}, \\
(iii) \|f\|_2 &= 0, \text{ if and only if } f(t) = 0 \text{ (separating points)}. \\
\end{align*}
\]

Moreover, as is well known, for piecewise continuous functions, the usual Fourier series (not necessarily (30)) may, in general, not converge absolutely and then the norm \( \|f\|_2 \) becomes unbounded. Besides, for sufficiently smooth functions, the pertaining Fourier coefficients satisfy the Riemann-Lebesgue lemma and this causes \( \|f\|_2 \) to approach the uniform norm \( \|f\| = \sup \|f(t)\| \) as an upper bound, namely,

\[
\|f\| \leq \|f\|_2.
\]

(45)

Relation (45) is called Lyaponov’s norm inequality.
Conjecture 11. Like \( \| f \|_* \), the DSF norm \( \| f \|_c \) also satisfies
\[
\| f \| \leq \| f \|_c. 
\]
(46)

Proposition 12. Stationarization of \( \| f \|_c \) with respect to \( \delta \) and \( \varsigma \) leads to stabilizing the DSF series inverse.

Proof. Consider, without loss of generality, \( \| f \|_c \) for \( \mathcal{D} \mathcal{F}^{-1}_c(\lambda, m) \):
\[
\| f \|_c = \sum_{k=-N \atop k \neq 0}^{N-1} \sum_{m=0}^{M-1} \sqrt{A_{km}^2 + B_{km}^2},
\]
(47)
Substitution of (25) and (26) for \( A_{km}^2 + B_{km}^2 \) in (47), when adopting the simplifying notation
\[
\tilde{f}_\delta = \tilde{f} \left( \frac{m}{k \delta} \right), \quad \tilde{f}_{\delta, \varsigma} = \tilde{f} \left( [m + \varsigma] k \delta \right),
\]
leads to
\[
A_{km}^2 + B_{km}^2 = m^2 \frac{\delta^2}{4\pi^2 M^2} \times \left\{ \text{Re} \left[ \tilde{f}_{\delta, \varsigma} \right] + \text{Im} \left[ \tilde{f}_{\delta, \varsigma} \right] \right\} 
\times \left\{ \text{Re} \left[ \tilde{f}_{\delta} \right] + \text{Im} \left[ \tilde{f}_{\delta} \right] - 2 \text{Re} \left[ \tilde{f}_{\delta} \right] + 2 \text{Re}^2 \left[ \tilde{f}_{\delta} \right] \right\},
\]
(49)
then to
\[
\| f \|_c = \frac{\delta}{2\pi M} \times \sum_{k=-N \atop k \neq 0}^{N-1} \sum_{m=0}^{M-1} m \times \left\{ \text{Re} \left[ \tilde{f}_{\delta, \varsigma} \right] + \text{Im} \left[ \tilde{f}_{\delta, \varsigma} \right] \right\} 
\times \left\{ \text{Re} \left[ \tilde{f}_{\delta} \right] + \text{Im} \left[ \tilde{f}_{\delta} \right] - 2 \text{Re} \left[ \tilde{f}_{\delta} \right] \right\} 
+ 2 \text{Re}^2 \left[ \tilde{f}_{\delta} \right] \right\}^{1/2}.
\]
(50)
Stationarization of \( \| f \|_c \) with respect to \( \delta \) and \( \varsigma \) leads, respectively, to
\[
\frac{\partial}{\partial \delta} \| f \|_c = \frac{\delta^2}{4\pi^2 M^2},
\]
Obviously, increasing \( M \) and \( N \) (or decreasing \( \delta \)), which increases the smoothness of \( \mathcal{D} \mathcal{F}^{-1}_c(\lambda, m) \), accelerates the convergence of the series in (51) and (52) to zero. This is accompanied by accelerating the stationarization of \( (\partial/\partial \delta) \| f \|_c \) and \( (\partial/\partial \varsigma) \| f \|_c \), with a rate \( M^{-2} N^{-2} \), and to stabilizing \( \mathcal{D} \mathcal{F}^{-1}_c(\lambda, m) \) with respect to variations in \( \delta \) and \( \varsigma \). This is a fact that illustrates the consistency of this proposition and its underlying Conjecture II. The same arguments are also applicable to the rate optimization of every other \( \mathcal{D} \mathcal{F}^{-1}_c(\lambda, m) \).

Lemma 13. A rate-optimal \((\delta, \varsigma)\) pair for the DSF series inverse \( \mathcal{D} \mathcal{F}^{-1}_c(\lambda, m) \) exists when the system (51)-(52) is satisfied.

Example 14. To illustrate the previous lemma, let us revisit Example 7 with \( \tilde{f}(\mu) = 2/(1 + \mu^2) \), to determine the rate-optimal values of \( \delta \) and \( \varsigma \) for it. In the notation of Proposition 12, we have
\[
\text{Re} \left[ \tilde{f}_{\delta} \right] = \frac{2}{1 + m^2 k^2 \delta^2}, \quad \text{Im} \left[ \tilde{f}_{\delta} \right] = 0.
\]
(53)
First, since
\[
\frac{\partial}{\partial \delta} \text{Re} \left[ \tilde{f}_{\delta} \right] = -4 m^2 k^2 \delta \left( 1 + m^2 k^2 \delta^2 \right)^2, \quad \frac{\partial}{\partial \delta} \text{Im} \left[ \tilde{f}_{\delta} \right] = -4 m^2 k^2 \delta \left( 1 + m^2 k^2 \delta^2 \right)^2.
\]
(54)
then
\[
\langle \Re\{\hat{f}_{\delta,c}\} - \Re\{\hat{f}_{\delta}\}\rangle = -2c_\delta [m + c_\delta] (1 + [m + c_\delta]^2 k^2 \delta^2)^{1/2},
\]
\[
\frac{\partial}{\partial \delta} \langle \Re\{\hat{f}_{\delta,c}\} - \Re\{\hat{f}_{\delta}\}\rangle = -4 \left\langle \frac{[m + c_\delta]^2}{(1 + [m + c_\delta]^2 k^2 \delta^2)^{1/2}} - \frac{m^2}{(1 + m^2 k^2 \delta^2)^{1/2}} \right\rangle k^2 \delta.
\]
(55)

Consider further these relations in (51) to rewrite it as
\[
\frac{\partial}{\partial \delta} \|f\|_\infty = \frac{2\delta^3}{\pi^2 M^2 \sum_{k=-N}^{N} \sum_{m=0}^{M-1} A_{k,m}^2 + B_{k,m}^2 (1 + m^2 k^2 \delta^2)}
\]
\[
\times \left\{ \frac{c_\delta [2m + c_\delta]}{(1 + [m + c_\delta]^2 k^2 \delta^2)^{1/2}} \right\}
\]
\[
\times \left\langle \frac{[m + c_\delta]^2}{(1 + [m + c_\delta]^2 k^2 \delta^2)^{1/2}} - \frac{m^2}{(1 + m^2 k^2 \delta^2)^{1/2}} \right\rangle
\]
(56)

for the rate-optimal $c$. 

Despite its complexity, the previous relation appears as a nonlinearly weighted negative mean of $m$, which may always be approximated by some fraction of $M$, like $-M/2$, say. Finally, the rate-optimal $(\delta, c)$ pair for the $DF^{-1}(\lambda, m)$ inverse emerges as the solution of the simultaneous system of nonlinear equations (56)-(59), a solution that may not necessarily be unique.

### 6. Recursive Maximum-Entropy Reconstruction of the DSF Series Inverse

The possible nonuniqueness of the rate-optimal $(\delta, c)$ pair of Lemma 13 calls for imposing an additional weak (inequality) constraint to select a robust value for this pair, which may account for all approximations that are involved in the development of $DF^{-1}(\lambda, m)$. The constraint must be the satisfaction of the Heisenberg uncertainty principle (HUP); see, for example, [9], when rate-optimizing $\delta$ and $c$. The HUP essentially states that $f(t) = DF^{-1}(\lambda, m)$ and $\tilde{f}(\mu)$ cannot be both arbitrarily concentrated. In information theory, the definitive measure of concentration of a probability density function, $\phi(t)$, is the Boltzmann entropy
\[
H(\phi) = - \int_{R} \phi(t) \ln \phi(t) dt.
\]
(60)

Hirschman Jr. [10, 11] observed in 1957 that Weyl's formulation of the HUP is a consequence of a stronger version of the inequality of this principle, namely, the entropy version
\[
H(\phi) \geq \frac{\ln e}{2},
\]
(61)
when $\|\phi\|_2 = 1$ and $\tilde{\phi}(\mu) \geq 0$. Hirschman also conjectured that the minimizers of the above sharp inequality were Gaussians, as in the case of the HUP.
To state the main result of this section, we adopt the notation

\[ Q = \int_R \mathcal{D}F^{-1}_\infty (\lambda, m) \, dt, \]

(62)

\[ \|f\|_2^2 = \int_R \|\hat{f}\|^2 \, dt, \]

the square of the \( p = 2 \) norm, which satisfies Parseval’s identity: \( \|f\|_2 = \|\hat{f}\|_2 \).

**Definition 15.** The DSF Taxi-cab norm \( \|f\|_2 \) of \( f(t) \in \mathcal{F}(R) \) is the sum of the absolute values of all scaled-harmonics appearing in the \( \mathcal{D}F^{-1}_\infty (\lambda, m) \) expansion; that is,

\[ \|f\|_2 = \sum_{k=-N}^N \sum_{m=0}^{M-1} |A_{k,m\ell}| + |B_{k,m\ell}|, \]

(63)

where \( A_{k,m\ell} \) and \( B_{k,m\ell} \) are, respectively, defined by (25)-(26) or (27)-(28) and (33)-(34) or (36)-(37).

**Lemma 16.** A rate-optimal pair \((\delta, \zeta)\) for \( \mathcal{D}F^{-1}_\infty (\lambda, m) \) with a known \( Q \) should satisfy

\[
1 \| \mathcal{D}F^{-1}_\infty (\lambda, m) \|_\infty^2 \left[ \ln 1 \| \mathcal{D}F^{-1}_\infty (\lambda, m) \|_\infty^2 - \ln Q^2 \right] > \frac{Q^4 \ln (e/2)}{H \left( Q^{-2} |\hat{f}|^2 \right) - \|\hat{f}\|^2 \ln Q^2}. 
\]

(64)

Proof. Conceive \( f(t)/Q \) as a probability density function \( \phi(t) \). Moreover, if \( f(t)/Q \in \mathcal{F}(R) \), then \( f(t)/Q \approx \hat{f}(\mu)/Q \). Hence in the notation of our DSF series \( \mathcal{D}F^{-1}_\infty (\lambda, m) \), even when it is not necessarily \( > 0 \), the previous entropy version for the uncertainty principle writes as

\[
H \left( Q^{-2} |\mathcal{D}F^{-1}_\infty (\lambda, m)|^2 \right) \geq \frac{\ln (e/2)}{H \left( Q^{-2} |\hat{f}|^2 \right)},
\]

(65)

where

\[
H \left( Q^{-2} |\mathcal{D}F^{-1}_\infty (\lambda, m)|^2 \right) = 2 \int_R \left[ Q^{-2} \sum_{k \in W_n}^M \sum_{m=0}^{M-1} A_{k,m} \cos \phi_{k,m,\ell}t + B_{k,m} \sin \phi_{k,m,\ell}t \right]^2 \]

\[ \times \ln \left[ Q^{-2} \sum_{k \in W_n}^M \sum_{m=0}^{M-1} A_{k,m} \cos \phi_{k,m,\ell}t + B_{k,m} \sin \phi_{k,m,\ell}t \right] \, dt,
\]

(66)

\[
H \left( Q^{-2} |\hat{f}|^2 \right) = Q^{-2} \left[ H \left( |\hat{f}|^2 \right) - \|\hat{f}\|^2 \ln Q^2 \right].
\]

(67)

Clearly

\[
H \left( Q^{-2} |\mathcal{D}F^{-1}_\infty (\lambda, m)|^2 \right) < \left[ Q^{-2} \sum_{k \in W_n}^M \sum_{m=0}^{M-1} (A_{k,m} + B_{k,m})^2 \right]^2 \]

\[ \times \ln \left[ Q^{-2} \sum_{k \in W_n}^M \sum_{m=0}^{M-1} (A_{k,m} + B_{k,m})^2 \right] T(k, m; \delta),
\]

where

\[
T(k, m; \delta) = \begin{cases} \frac{2\pi}{m(k\delta + 1)}, & m \neq 0 \\ 0, & m = 0. \end{cases}
\]

(69)

Then replacement of \( T(k, m; \delta) \) by \( 1 \) in the previous inequality reduces it to

\[
H \left( Q^{-2} |\mathcal{D}F^{-1}_\infty (\lambda, m)|^2 \right) < \left[ Q^{-2} \sum_{k \in W_n}^M \sum_{m=0}^{M-1} (A_{k,m} + B_{k,m})^2 \right]^2 \]

\[ \times \ln \left[ Q^{-2} \sum_{k \in W_n}^M \sum_{m=0}^{M-1} (A_{k,m} + B_{k,m})^2 \right].
\]

(70)

Now since

\[
\left[ Q^{-2} \sum_{k \in W_n}^M \sum_{m=0}^{M-1} (A_{k,m} + B_{k,m})^2 \right] \leq \left[ \sum_{k=-N}^N \sum_{m=0}^{M-1} \frac{A_{k,m}}{Q} + \frac{B_{k,m}}{Q} \right]^2,
\]

(71)

then

\[
H \left( Q^{-2} |\mathcal{D}F^{-1}_\infty (\lambda, m)|^2 \right) < \left[ \sum_{k=-N}^N \sum_{m=0}^{M-1} \frac{A_{k,m}}{Q} + \frac{B_{k,m}}{Q} \right]^2 \]

\[ \times \ln \left[ \sum_{k=-N}^N \sum_{m=0}^{M-1} \frac{A_{k,m}}{Q} + \frac{B_{k,m}}{Q} \right]^2.
\]

(72)

and relation (65) becomes

\[
\left[ \sum_{k=-N}^N \sum_{m=0}^{M-1} \frac{A_{k,m}}{Q} + \frac{B_{k,m}}{Q} \right]^2 \leq \left[ \sum_{k=-N}^N \sum_{m=0}^{M-1} \frac{A_{k,m}}{Q} + \frac{B_{k,m}}{Q} \right]^2,
\]

(73)

Furthermore, since

\[
\| Q^{-1} \mathcal{D}F^{-1}_\infty (\lambda, m) \|_\infty^2 = Q^{-2} \| \mathcal{D}F^{-1}_\infty (\lambda, m) \|_\infty^2,
\]

(74)

then the left-hand side of (72) becomes

\[
Q^{-2} \| \mathcal{D}F^{-1}_\infty (\lambda, m) \|_\infty^2 \ln 1 \| \mathcal{D}F^{-1}_\infty (\lambda, m) \|_\infty^2 - \ln Q^2.
\]

(75)

Finally, consideration of this relation together with (67) in (72) leads to the required result.

□
Since the entropy $H(Q^{-2}|\mathfrak{D}F_{r,\infty}^{-1}(\lambda, m)|^2)$ is a function of the $(\delta, \varsigma)$ free parameters, then variation of these parameters can always be forced towards maximization of this entropy. As is well known, see, for example, [12], a maximum-entropy $\mathfrak{D}F_{r,\infty}^{-1}(\lambda, m)$ should be a most certain inverse of $\hat{f} (\mu)$ and with best achievable smoothness.

Theorem 17. A robust rate-optimal pair $(\delta, \varsigma)$, for $\mathfrak{D}F_{r,\infty}^{-1}(\lambda, m)$ with a known $Q$, is a solution to the nonlinear programming problem

Minimize $\|Q^{-1}\mathfrak{D}F_{r,\infty}^{-1}(\lambda, m)\|^2_b \ln \|Q^{-1}\mathfrak{D}F_{r,\infty}^{-1}(\lambda, m)\|^2_b$  
Subject to: (64), (51)-(52).

(76)

Proof. By consideration of relation (72) of Lemma 16 when maximizing the entropy $H(Q^{-2}|\mathfrak{D}F_{r,\infty}^{-1}(\lambda, m)|^2)$, subject to the inequality constraint (64) and the equality constraints (51) and (52).

$Q$, however, is not apriori known. Hence, we may assume that $\delta_r, \varsigma_r$ correspond to an $r$th iteration, $Q_r$, for $Q$ and $\mathfrak{D}F_{r,\infty}^{-1}(\lambda; m; \delta_r, \varsigma_r)$ is $\mathfrak{D}F_{r,\infty}^{-1}(\lambda, m)$ that corresponds to $Q_r$. These concepts invoke a recursive determination process, indexed by $r = 0, 1, 2, 3, \ldots, L$, with $L$ as the termination number, defined, namely, as

$$Q_{r+1} - Q_r \leq \epsilon, \quad (77)$$

where $0 \leq \epsilon \ll 1$ is an acceptable tolerance.

Proposition 18. A robust rate-optimal pair $(\delta, \varsigma) = \lim_{r \to -1} (\delta_r, \varsigma_r)$ for $\mathfrak{D}F_{r,\infty}^{-1}(\lambda, m)$ exists when $(\delta_r, \varsigma_r)$ is a solution to the recursive nonlinear programming problem

Minimize $\|Q^{-1}\mathfrak{D}F_{r,\infty}^{-1}(\lambda, m; \delta_r, \varsigma_r)\|^2_b \ln \|Q^{-1}\mathfrak{D}F_{r,\infty}^{-1}(\lambda, m; \delta_r, \varsigma_r)\|^2_b$

Subject to: (64), (51)-(52),

with $\mathfrak{D}F_{r,\infty}^{-1}(\lambda; m; \delta_r, \varsigma_r)$ replacing $\mathfrak{D}F_{r,\infty}^{-1}(\lambda, m)$,

$$Q_{r+1} = \int_0^L \mathfrak{D}F_{r,\infty}^{-1}(\lambda, m; \delta_r, \varsigma_r) dt, \quad (78)$$

and arbitrary $Q_0 > 0$.

Due to convexity of the inequality constraint (64), the nonlinear programming problem of Theorem 17 should be solvable by the method of Kuhn-Tucker multipliers [13]. The rather large computational complexity of the above proposition is comparable to that of the alternative wavelet analysis. It should, however, not represent a difficulty in our present times, when powerful computing resources are available or accessible almost everywhere.

7. Conclusion

In this paper, we have demonstrated how evaluation of the inverse Fourier transform can be variably scaled. The reported DSF representation, for, say, high frequency resolving scaling, bypasses the need for quadratures in computing (1). Instead, a repeated evaluation of the real and imaginary parts of $\hat{f}$, when evaluating the $A_{k,m}$ and $B_{k,m}$ (or the $\hat{A}_{k,m}$ and $\hat{B}_{k,m}$) coefficients, of (24)–(26), is needed at a large enough number of scale-frequency points $(m, k)$, as illustrated in Sections 4 and 5. Low frequency resolving scaling also requires a similar computational effort with the real and imaginary parts of $\hat{f}$ when evaluating the $\hat{A}_{k,m}$ and $\hat{B}_{k,m}$ (or the $\hat{A}_{k,m}^+$ and $\hat{B}_{k,m}^+$) coefficients of (32)–(34).

Determination of the rate-optimal scaling factor step $\delta$ and frequency gap number $\varsigma$, in DSF series has been demonstrated to invoke solving a recursive nonlinear programming problem that maximizes entropy in the framework of the uncertainty principle. It should be emphasized that the scale-frequency approach to inversion of the Fourier transform is a new concept that is not necessarily fast. In that sense it is not expected to pose a challenge to the FFT algorithm. Its error analysis is, however, of crucial importance and will be a subject for a future publication.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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