New kind of asymmetric integration projection operators constructed by entangled state representations and parity measurement

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By means of the technique of integration within an ordered product of operators and Dirac notation, we introduce a new kind of asymmetric integration projection operators in entangled state representations. These asymmetric projection operators are proved to be the Hermitian operator. Then, we rigorously demonstrate that they correspond to a parity measurement combined with a beam splitter when any two-mode quantum state passes through such device. Therefore we obtain a new relation between a Hermitian operator and the entangled state representation. As applications, we recover the previous results of the parity measurement in quantum metrology by our formalism.

Keywords: quantum mechanics, Dirac notation, IWOP, entangled state representation

I. INTRODUCTION

The transformation theory of quantum representations and Dirac’s ket-bra operators play important roles not only in quantum mechanics, but also in information optics and quantum optics. Especially, with the help of the technique of integration within an ordered product of operators (IWOP) \cite{1}, one can introduce many known or new quantum unitary operators by constructing asymmetric integration projection operators based on quantum representations. In this way, quantum optical version of some classical optical transformations such as optical Fresnel transformation, Hankel transformation, fractional Fourier transformation, Wigner transformation, wavelet transformation and Fresnel-hadnarm combinatorial transformation have been established \cite{2,3,4}. In addition, one can not only find some new quantum unitary operators corresponding to the known optical transformations, but also can reveal some new classical optical transformations. For example, the two-mode squeezing operator holds a natural expression in the entangled state representation \cite{4}. The complex fraction Fourier transformation has also been introduced based on the two mutually conjugate entangled state representations $|\eta\rangle$ and $|\xi\rangle$ \cite{4}.

Besides the usual coordinate and momentum representations, as well as the coherent state representation, entangled state representations are also important quantum representations \cite{4}. The concept of quantum entanglement is originated in the paper of Einstein, Podolsky and Rosen (EPR) \cite{7}, arguing on the incompleteness of quantum mechanics. According to the original idea of EPR, Fan and Klauder have introduced two kinds of entangled state representations \cite{4}. One kind of the entangled state representation $|\eta\rangle = \eta_1 + i\eta_2$ is the common eigenvector of two particles’ relative position $\hat{x}_1 - \hat{x}_2$ and total momentum $\hat{p}_1 + \hat{p}_2$, i.e., $\langle \hat{x}_1 - \hat{x}_2 | \eta \rangle = \sqrt{2}\eta_1 |\eta\rangle$, $\langle \hat{p}_1 + \hat{p}_2 | \eta \rangle = \sqrt{2}\eta_2 |\eta\rangle$. The explicit form of such common eigenvector is

$$|\eta\rangle = \exp \left[ -\frac{1}{2} |\eta|^2 + \eta \hat{a}^\dagger - \eta^* \hat{b}^\dagger + \hat{a}^\dagger \hat{b}^\dagger \right] |0\rangle_a |0\rangle_b,$$  \hspace{1cm} (1)

where $\hat{a}^\dagger$ and $\hat{b}^\dagger$ are two bosonic creation operators, and $|0\rangle_a |0\rangle_b$ is a two-mode vacuum state. The entangled state $|\eta\rangle$ is proved to be orthonormal and complete,

$$\langle \eta | \eta \rangle = \pi \delta (\eta - \eta) \delta (\eta^* - \eta^*), \int d^2\eta |\eta\rangle \langle \eta | / \pi = \hat{1},$$ \hspace{1cm} (2)

where $d^2\eta = d\eta_1 d\eta_2$. On the other hand, its conjugate state $|\xi\rangle = \xi_1 + i\xi_2$ is the common eigenvector of two particles’ center-of-mass coordinate $\hat{x}_1 + \hat{x}_2$ and relative momentum $\hat{p}_1 - \hat{p}_2$, i.e., $(\hat{x}_1 + \hat{x}_2) |\xi\rangle = \sqrt{2}\xi_1 |\xi\rangle$, $(\hat{p}_1 - \hat{p}_2) |\xi\rangle = \sqrt{2}\xi_2 |\xi\rangle$. The explicit form of this common eigenvector is

$$|\xi\rangle = \exp \left[ -\frac{1}{2} |\xi|^2 + \xi \hat{a}^\dagger + \xi^* \hat{b}^\dagger - \hat{a}^\dagger \hat{b}^\dagger \right] |0\rangle_a |0\rangle_b,$$ \hspace{1cm} (3)

which also holds orthonormal relation and completeness relation,

$$\langle \xi' | \xi \rangle = \pi \delta (\xi' - \xi) \delta (\xi'^* - \xi^*), \int d^2\xi |\xi\rangle \langle \xi | / \pi = \hat{1},$$ \hspace{1cm} (4)

where $d^2\xi = d\xi_1 d\xi_2$. By means of IWOP, these two mutually conjugate entangled state representations $|\eta\rangle$ and $|\xi\rangle$ are very useful in information optics and quantum optics \cite{1}. Nowadays, entangled state representations have been widely used in the squeezed state theory, the Wigner distribution function, the complex fraction Fourier transformation, the complex wavelet transformation, and so on \cite{2,3,4,10}. Among these useful applications, it is usually necessary to construct the corresponding asymmetric integration projection operators based on quantum representations. For example, in order to offer more qualified mother wavelets for complex wavelet transforms, Fan and
Lu introduce the following asymmetric integration projection operators [3]

\[
U(\mu, \kappa) = \frac{1}{\mu} \int \frac{d^2\eta}{\pi} \frac{\eta - \kappa}{\mu} \langle \eta | ,
\]

where \(0 \neq \mu \in \mathbb{R}\) and \(\kappa \in \mathbb{C}\). Based on Eq. (5), mother wavelets for complex wavelet transformation can be considered as a matrix element of the two-mode squeezed displaced operator \(U(\mu, \kappa)\). For detailed discussion about the applications of the entangled state representation for complex wavelet transformation, please see Ref. [2, 3]. In the case of \(\kappa = 0\), Eq. (5) is a natural representation of the two-mode squeezed operator \(U(\mu, 0) = \exp \left[ \lambda \left( \hat{a}^\dagger \hat{b}^\dagger - \hat{a} \hat{b} \right) \right]\) with \(\mu = e^\lambda\) being a squeezing parameter [4].

In the previous studies, those constructing asymmetric integration projection operators based on different quantum representations are usually quantum unitary operators. For example, for many optical transformations based on entangled state representations \(|\eta\rangle\) and \(|\xi\rangle\), those corresponding asymmetric integration projection operators are most unitary operators [2]. Different from the past, in this work we will introduce a new kind of the asymmetric integration projection operators based on the two \(|\eta\rangle\) and \(|\xi\rangle\), but these operators are Hermitian operators. Concretely, we construct the asymmetric integration projection operators as follows

\[
\int \frac{d\eta_1}{\pi} |\eta_1 + i\eta_2\rangle \langle \eta_2 + i\eta_1| = ? \quad (6)
\]
or

\[
\int \frac{d\xi_1}{\pi} |\xi_1 + i\xi_2\rangle \langle \xi_2 + i\xi_1| = ? \quad (7)
\]

Obviously, we are only swapping the real and imaginary parts of the variable \(\eta\) (or \(\xi\)) in the bra of the completeness relation of the \(|\eta\rangle\) (or \(|\xi\rangle\)), which leads to that Eqs. (6) and (7) are no longer a two-mode identity operator. According to orthonormal relations of the two \(|\eta\rangle\) and \(|\xi\rangle\) in Eqs. (2) and (4), one can easily prove that the above asymmetric integration projection operators are Hermitian operators. On the other hand, it is well known, in quantum mechanics, for a Hermitian operator, there may exist a corresponding physical measurement or observable. Enlighten by the previous work [11], we can analytically demonstrate that such Hermitian operators indeed represent to a physical measurement or observable, which is exactly the parity measurement combined with a beam splitter in quantum metrology.

This paper is organized as follows. In Sec. 2, we introduce a kind of Hermitian operators constructed by entangled state representations. In Sect. 3, we analytically prove that such Hermitian operators represent the parity measurement combined with a beam splitter. In Sec. 4, we show some applications of our results. Finally, our conclusions are presented in Sect. 5.

II. HERMITIAN OPERATORS EXPRESSED IN ENTANGLED STATE REPRESENTATION

In quantum mechanics, the IWOP is a useful tool about the transformation theory of quantum representations and the operations of quantum operators [1, 12]. In this section, we perform these asymmetric integration expressed in Eqs. (6) and (7) by IWOP. In order to facilitate the understanding the IWOP, here we will give a detailed derivation process. Note that Eq. (6) and the normal ordering form of the vacuum projective operator

\[
|\eta\rangle_\alpha |\eta\rangle_\beta |\eta\rangle_\alpha = : e^{-\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}} : ,
\]

the left of Eq. (6) can be written as

\[
\int \frac{d\eta_1}{\pi} \frac{d\eta_2}{\pi} |\eta_1 + i\eta_2\rangle \langle \eta_2 + i\eta_1| = \int \frac{d\eta_1}{\pi} \frac{d\eta_2}{\pi} e^{-\frac{1}{2} \left( \eta_1^2 + \eta_2^2 \right) + \frac{1}{2} \left( \eta_1 - i\eta_2 \right) \hat{a}^\dagger - \hat{a} \left( \eta_1 + i\eta_2 \right) \hat{b}^\dagger + \hat{b} \hat{a}^\dagger - \hat{a} \hat{b}^\dagger ,
\]

\[
\int \frac{d\eta_1}{\pi} \frac{d\eta_2}{\pi} : e^{-\frac{1}{2} \left( \eta_1^2 + \eta_2^2 \right) - (\eta_1 - i\eta_2) \hat{a}^\dagger - \hat{a} \left( \eta_1 + i\eta_2 \right) \hat{b}^\dagger + \hat{b} \hat{a}^\dagger - \hat{a} \hat{b}^\dagger} :.
\]

We can see that the right of operator \(e^{-\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}}\) are all annihilation operators, while on its left are all creation operators; therefore the whole integral is in normal ordering of operators. Then, we can rewrite Eq. (9) as the following form

\[
\int \frac{d\eta_1}{\pi} \frac{d\eta_2}{\pi} |\eta_1 + i\eta_2\rangle \langle \eta_2 + i\eta_1| = \int \frac{d\eta_1}{\pi} \frac{d\eta_2}{\pi} e^{-\left( \eta_1^2 + \eta_2^2 \right) + \eta_1 \left( \hat{a}^\dagger - \hat{a} \right) \hat{b}^\dagger + \hat{b} \hat{a}^\dagger - \hat{a} \hat{b}^\dagger}
\]

\[
\int \frac{d\eta_1}{\pi} \frac{d\eta_2}{\pi} e^{-\left( \eta_1^2 + \eta_2^2 \right) + \eta_1 \left( \hat{a}^\dagger - \hat{a} \right) \hat{b}^\dagger + \hat{b} \hat{a}^\dagger - \hat{a} \hat{b}^\dagger}.
\]

Note that the IWOP and the following mathematic integration formula

\[
\int_{-\infty}^{\infty} \exp \left[ -\alpha x^2 + \beta x \right] dx = \sqrt{\frac{\pi}{\alpha}} \exp \left( \frac{\beta^2}{4\alpha} \right) ,
\]

which holds for \(\text{Re}(\alpha) > 0\), we can directly calculate the integration in Eq. (10) and obtain

\[
\int \frac{d\eta_1}{\pi} \frac{d\eta_2}{\pi} |\eta_1 + i\eta_2\rangle \langle \eta_2 + i\eta_1| = : e^{-i\hat{a}^\dagger \hat{b}^\dagger + i\hat{a}\hat{b} - \hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}} : ,
\]

which is a concise expression of operators.

In order to get much more information from Eq. (12), we remove the restricted normal ordering symbol : , the operators’ non-commutative property manifestly appears. So let us start with Taylor expansion of \(e^{-i\hat{a}^\dagger \hat{b}^\dagger} \) and \(e^{i\hat{a}\hat{b}^\dagger}\) in Eq. (12), i.e.,

\[
e^{-i\hat{a}^\dagger \hat{b}^\dagger + i\hat{a}\hat{b}^\dagger - \hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}} := \sum_{m=0, n=0}^{\infty} \frac{\hat{a}^m \hat{b}^n}{m!} e^{-\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}} \hat{a}^\dagger \hat{b}^\dagger n \hat{a}^n : (13)
According to major properties of normally ordered product of operators which means all the bosonic creation operators $\hat{a}^\dagger$ ($\hat{b}^\dagger$) are standing on the left of annihilation operators $\hat{a}$ ($\hat{b}$) in a monomial of $\hat{a}$ ($\hat{b}$) and $\hat{a}^\dagger$ ($\hat{b}^\dagger$), we further rewrite Eq. (13) as

$$: e^{-i\hat{a}^\dagger \hat{b} + i\hat{a} \hat{b}^\dagger - \hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}}: = \sum_{m=0,n=0}^\infty i^{(n-m)} \frac{\hat{a}^m \hat{b}^n}{\sqrt{m! n!}} : e^{-i\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}} :$$

(14)

And then, applying Eq. (8), we finally have

$$: e^{-i\hat{a}^\dagger \hat{b} + i\hat{a} \hat{b}^\dagger - \hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}}: = \sum_{m=0,n=0}^\infty i^{(n-m)} \langle m|\hat{a}|n\rangle \langle n b|m\rangle \langle a| n \rangle.$$  

(15)

In Eq. (15), we have used the expression of the two-mode Fock state $|m\rangle_a |n\rangle_b = \frac{\hat{a}^m \hat{b}^n}{\sqrt{m! n!}} |0\rangle_a |0\rangle_b$. Comparing with the completeness relation of the two-mode Fock state, we can see that $m$ and $n$ in two bras are exchanged and an additional term $i^{(n-m)}$ is introduced. Here, we must point out that Eq. (6) is an asymmetric integration projection operator with continuous variables, while Eq. (15) is an asymmetric sum projection operator with discrete variables. However, they represent the same Hermitian operator.

Now, we turn to investigate the asymmetric integration projection operator based on entangled state $|\xi\rangle$. By the similar way, we can complete the integration of Eq. (17) and obtain

$$\int d\xi_1 d\xi_2 \frac{\langle \xi_1 + i \xi_2 | \langle \xi_2 + i \xi_1 |}{\pi} = : e^{i\hat{a}^\dagger \hat{b}^\dagger - i\hat{a} \hat{b}^\dagger - \hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}}: = \sum_{m=0,n=0}^\infty i^{(m-n)} \langle m|\hat{a}|n\rangle \langle n b|m\rangle \langle a| n \rangle.$$  

(17)

which is different from Eq. (12). But in essence they are a kind of Hermitian operators, which will be proved in the following. Of course, we can also cast such normal ordering operator $: e^{i\hat{a}^\dagger \hat{b} - i\hat{a} \hat{b}^\dagger - \hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}}:$ into the Fock representation, i.e.,

$$: e^{i\hat{a}^\dagger \hat{b} - i\hat{a} \hat{b}^\dagger - \hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b}}: = \sum_{m=0,n=0}^\infty i^{(m-n)} \langle m|\hat{a}|n\rangle \langle n b|m\rangle \langle a| n \rangle.$$  

III. PHYSICAL CORRESPONDENCE OF A KIND OF ASYMMETRIC INTEGRATION PROJECTION OPERATOR

In the quantum metrology, a Mach-Zehnder interferometer is a useful tool to estimate a slight variation of phase shift originating from different physical processes. A typical Mach-Zehnder interferometer is composed of two beam splitters and the phase shift between the two arms to be estimated as shown in Fig. 1. In order to extract the phase shift, one need to choose a special measurement, such as intensity measurement, homodyne measurement, and parity measurement [14]. Here, we consider parity measurement on one output of the interferometer.

Note that performing parity measurement onto one output of the interferometer is equivalent to compute the expectation value of the parity operator $\hat{P} = \exp i\pi \hat{b}^\dagger \hat{b}$ in the output state. From the perspective of metrology, all the devices after the phase shift are regarded as a measurement as shown in Fig. 1. Therefore, the parity measurement combined with a beam splitter is equivalent to the implementation of projective measurement to the state $|\psi\rangle_{ab}$ before the second beam splitter, i.e.,

$$\langle \hat{P} \rangle = \langle \psi | \hat{U}^\dagger_{BS} \left[I_A \otimes \exp i\pi \hat{b}^\dagger \hat{b} \right] \hat{U}_{BS} |\psi\rangle_{ab} = \langle \psi | \hat{\mu}_{ab} |\psi\rangle_{ab},$$  

(18)

where $I_A$ represents a unity operator of a mode of the two-mode quantum system, $\hat{U}_{BS}$ is the unitary operator for the beam splitter, and the projective operator $\hat{\mu}_{ab}$ is

$$\hat{\mu}_{ab} = \hat{U}_{BS} \left[I_A \otimes \exp i\pi \hat{b}^\dagger \hat{b} \right] \hat{U}^\dagger_{BS}.$$  

(19)

It should be pointed out that the use of $\hat{\mu}_{ab}$ here highlights the fact that parity measurement combined with a beam splitter provides a measurement scheme including all of the phase-carrying off-diagonal terms in the two-mode density matrix [15, 16]. In quantum optics, a general beam splitter which can be described by the following unitary operator [17, 18]

$$\hat{U}_{BS} = \exp \left[ \frac{\theta}{2} \left( \hat{a} e^{i\phi} - \hat{a}^\dagger e^{-i\phi} \right) \right],$$  

(20)

with the following transformation relations

$$\hat{U}_{BS} \hat{a}\hat{U}^\dagger_{BS} = \hat{a} \cos \frac{\theta}{2} + \hat{b} e^{i\phi} \sin \frac{\theta}{2},$$

$$\hat{U}_{BS} \hat{b}\hat{U}^\dagger_{BS} = \hat{b} \cos \frac{\theta}{2} - \hat{a} e^{-i\phi} \sin \frac{\theta}{2}.$$  

(21)
For a 50:50 beam splitter with \( \phi = 0 \), the projective operator \( \hat{\mu}_{ab} \) has been casted into the Fock state representation \( \hat{A} = \int d^2\alpha |\alpha\rangle_a \langle \alpha|/\pi \) in the coherent state representation and Eqs. \( \alpha \) and \( \beta \) in two bras are exchanged compared with the case of the two kets. Of course, the product state \(|\alpha\rangle_a |\beta\rangle_b \) is not a entangled state. In this work, the asymmetric integration projection operators are constructed by entangled quantum representations as shown in Eqs. \( \alpha \) and \( \beta \). Here, we are only swapping the real and imaginary parts of the variable \( \eta \) (or \( \xi \)) in the bras of the completeness relations of \( |\eta\rangle \) and \( |\xi\rangle \). Although, in form at least, the Eq. \( \alpha \) is somewhat different from Eqs. \( \alpha \) and \( \beta \), such Hermitian operators can be proved in the following that they indeed represent to the same physical measurement or observable.

Now, for our purpose, we will demonstrate that these asymmetric projection operators Eqs. \( \alpha \) and \( \beta \) also describe the parity measurement with a balanced beam splitter as shown in Fig. 1. Based on the coherent state representation, the parity operator can be expressed as

\[
(-1)^{\bar{b}} = \exp \left[ i\hat{b}^\dag \hat{b} \right] = \int \frac{d^2\beta}{\pi^2} |\beta\rangle_b \langle -\beta| ,
\]

(24)

Substituting \( \hat{A} = \int d^2\alpha |\alpha\rangle_a \langle \alpha|/\pi \) in the coherent state representation and Eqs. \( \alpha \) and \( \beta \) into Eq. \( \alpha \), we perform the integration and can obtain the normal ordering form of the projective operator \( \hat{\mu}_{ab} \) of the parity measurement

\[
\hat{\mu}_{ab} = e^{\hat{a}^\dag \cos \theta - \hat{b}^\dag \cos \theta + \hat{a}^\dag \hat{b}^\dag e^{i\phi} \sin \theta + \hat{a}^\dag \hat{b}^\dag e^{-i\phi} \sin \theta - \hat{a}^\dag \hat{b}^\dag} ,
\]

(25)

where we have used the integral formula \( \alpha \)

\[
\int \frac{d^2z}{\pi} \exp \left( \frac{\xi z^* + \xi z + \eta z^*}{\zeta} \right) = \frac{1}{\zeta} \exp \left( -\frac{\xi \eta}{\zeta} \right) ,
\]

(26)

whose convergent condition is \( \text{Re}(\zeta) < 0 \). Particularly, we consider the case of \( \theta = \pi/2 \), i.e., for the balanced beam splitter, and obtain

\[
\hat{\mu}_{ab}|_{\theta=\pi/2} = \exp \left[ \hat{a}^\dag \hat{b}^\dag e^{i\phi} + \hat{a}^\dag \hat{b}^\dag e^{-i\phi} - \hat{a}^\dag \hat{a} - \hat{b}^\dag \hat{b} \right] ,
\]

(27)

or

\[
\hat{\mu}_{ab}|_{\theta=\pi/2} = \sum_{m=0,n=0} \sum_{\alpha} e^{i(m-n)\phi} |m\rangle_a |n\rangle_b (m \langle a | n\rangle ,
\]

(28)

Eq. \( \alpha \) is the Fock representation of the parity measurement combined with a balanced beam splitter. Here, our result Eq. \( \alpha \) is somewhat different from Eq. \( \alpha \) appeared in Refs. \( \alpha \). From Eq. \( \alpha \), we can see that \( m \) and \( n \) in two bras are exchanged and an additional item \( e^{i(m-n)\phi} \) is introduced. Naturally, in the case of \( \phi = 0 \), Eq. \( \alpha \) just is that result in Refs. \( \alpha \).

Now, comparing Eq. \( \alpha \) with Eq. \( \alpha \), we easily obtain the following relation in the case of \( \phi = -\pi/2 \),

\[
\int \frac{d\eta_1 d\eta_2}{\pi}|\eta_1 + i\eta_2\rangle \langle \eta_2 + i\eta_1| = \hat{\mu}_{ab}|_{\theta=\pi/2,\phi=-\pi/2} .
\]

(29)

Similarly, in the case of \( \phi = \pi/2 \), comparing Eq. \( \alpha \) with Eq. \( \alpha \), we have

\[
\int \frac{d\xi_1 d\xi_2}{\pi}|\xi_1 + i\xi_2\rangle \langle \xi_2 + i\xi_1| = \hat{\mu}_{ab}|_{\theta=\pi/2,\phi=\pi/2} .
\]

(30)

Equations \( \alpha \) and \( \alpha \) show a new relation between a Hermitian operator and the entangled state representations, which is one important result in this work. So far, we have proved analytically that those asymmetric integration projection operators constructed by the two entangled states \( |\eta\rangle \) and \( |\xi\rangle \) indeed represent a physical measurement. That is to say, such asymmetric projection operators also correspond to a parity measurement combined with a beam splitter. Therefore, for the same measurement operators in quantum mechanics, one can cast it into different quantum representations according to the needs of the specific problems. These results not only can improve the understanding of the relations between the Hermitian operators and quantum representations, but also can bring us some convenience in dealing with the calculation related to quantum operators.

IV. SOME APPLICATIONS THE PROJECTIVE OPERATOR OF PARITY MEASUREMENT

In quantum optical metrology, besides usual intensity measurement and homodyne measurement, parity measurement has been shown to be a universal measurement scheme in phase estimation, and has been demonstrated to be the optimal measurement strategy for many input states. In order to improve the phase sensitivity, it has been proved that, with parity measurement, the Heisenberg limit can be saturated with a lossless Mach-Zehnder interferometer by using nonclassical states as the inputs, such as squeezed states \( \alpha \), NOON states \( \alpha \), and entangled coherent states \( \alpha \). As applications of our results shown in Eq. \( \alpha \) (or Eq. \( \alpha \)), in what follows we consider two specific states that have been frequently investigated in quantum optical metrology.
Case 1.— Let us first consider the input state of a Mach-Zehnder interferometer to be an NOON state [23]:

\[ |\Psi\rangle_{\text{NOON}} = \frac{1}{\sqrt{2}} (|N\rangle_a |0\rangle_b + |0\rangle_a |N\rangle_b). \] (31)

Upon leaving the phase channel and before the second beam splitter as shown in Fig. 1, the state arrives at

\[ |\psi\rangle_{ab} = \hat{U}(\varphi) |\Psi\rangle_{\text{NOON}} \]

\[ = \frac{1}{\sqrt{2}} \left( e^{i\varphi N/2} |N\rangle_a |0\rangle_b + e^{-i\varphi N/2} |0\rangle_a |N\rangle_b \right). \] (32)

where \( \hat{U}(\varphi) = \exp \left[ i\varphi (\hat{a}\hat{a} - \hat{b}\hat{b}) / 2 \right] \) denotes the two phase shifters, the angle \( \varphi \) is the phase shift between the two arms of a Mach-Zehnder interferometer to be estimated. According to Eqs. (18) and (12) (or (29)), the expectation value of the parity measurement scheme \( \langle \hat{\Pi} \rangle \) can be expressed by

\[ \langle \hat{\Pi} \rangle = \int \frac{d\eta_1 d\eta_2}{\pi} \langle \psi | \eta_1 + i\eta_2 \rangle \langle \eta_2 + i\eta_1 | \psi \rangle_{ab}. \] (33)

Note that the entangled state \( |\eta\rangle \) can be expanded in terms of two-mode Fock states as [1]

\[ |\eta\rangle = |\eta_1 + i\eta_2\rangle = e^{-|\eta|^2/2} \sum_{m,n=0}^{\infty} \frac{H_{m,n}(\eta, \eta^*)}{\sqrt{m!n!}} |m\rangle_a |n\rangle_b, \] (34)

where \( H_{m,n}(\eta, \eta^*) \) is the two-variable Hermite polynomial

\[ H_{m,n}(\xi, \eta) = \frac{\partial^{m+n}}{\partial^{m}\xi \partial^{n}\eta} \exp[-tt^* + t\xi + \xi^* t], \] (35)

Thus, substituting Eqs. (31) and (35) into Eq. (33) and noting that the mathematical integration formula shown in Eq. (11), after some simple calculation we have

\[ \langle \hat{\Pi} \rangle = \frac{iN}{2} \left[ \exp[iN\varphi] + (-1)^N \exp[-iN\varphi] \right], \] (36)

which is just that results in Ref. [23].

Case 2.— We then consider a coherent state combined with a squeezed vacuum state, i.e., \( |z\rangle_a \otimes |r\rangle_b \). For the convenience of the later calculation, we rewrite the squeezed vacuum state \( |r\rangle \) in the basis of the coherent state as follows

\[ |r\rangle_b = \text{sech}^{1/2} r \int \frac{d^2 \alpha}{\pi} e^{-|\alpha|^2 - \text{tanh}(c^2) \alpha^2} |\alpha\rangle_b, \] (37)

where \( |\alpha\rangle = \exp \left[ -|\alpha|^2 / 2 + \alpha^* \hat{b} \right] |0\rangle \) is a coherent state. When the product state \( |\psi\rangle_{in} = |z\rangle_a \otimes |\psi\rangle_{in} |r\rangle_b \) is injected into a Mach-Zehnder interferometer, upon leaving the phase channel, the state evolves as

\[ |\psi\rangle_{ab} = \hat{U}(\varphi) \hat{U}_{BS1} |z\rangle_a |r\rangle_b \]

\[ = \text{sech}^{1/2} r \int \frac{d^2 \alpha}{\pi} e^{-|\alpha|^2 - \text{tanh}(c^2) \alpha^2} \times \left( \frac{(z + i\alpha) e^{i\varphi / 2}}{\sqrt{2}} \right)_a \left( \frac{(iz + \alpha) e^{-i\varphi / 2}}{\sqrt{2}} \right)_b, \] (38)

where \( \hat{U}_{BS1} = \exp \left[ i\pi (\hat{a}^\dagger \hat{b} + \hat{a} \hat{b}^\dagger) / 4 \right] \) describes the first beam splitter. Similarly, applying Eq. (12) or (29) we have

\[ \langle \hat{\Pi} \rangle = \int \frac{d\eta_1 d\eta_2}{\pi} \langle \psi | \eta_1 + i\eta_2 \rangle \langle \eta_2 + i\eta_1 | \psi \rangle_{ab} \]

\[ = \langle \psi |_b : e^{-ia^\dagger b^\dagger + iab^\dagger - a^\dagger b - \eta^2} : | \psi \rangle_{ab}. \] (39)

Therefore, substituting Eq. (38) into Eq. (39), and noting that the following integral formula [21]

\[ \int d^2 \zeta |\zeta|^2 + f \xi + zg - 2 = \frac{1}{\sqrt{\xi^2 - 4fg}} e^{-\xi^2 + f^2 + zg^2}, \] (40)

whose convergent condition is \( \text{Re}(\xi \pm f \pm g) < 0 \) and \( \text{Re} \left( \frac{\xi^2 - fg}{\xi^2 - 4fg} \right) < 0 \), by performing this integral we have

\[ \langle \Pi_b(\varphi) \rangle_0 = e^{\frac{2(\text{cos} \phi - 1 - \text{sinh} r \text{sin} \phi)}{2(1 - \text{sinh} r \text{sin} \phi)}} \frac{\text{sech} r}{\sqrt{1 + \text{sinh}^{-2} r \text{sin}^2 \phi}}. \] (41)

which is the corresponding expectation value of the parity operator when a combination of coherent state and squeezed vacuum state is considered as input states of a Mach-Zehnder interferometer [24]. For other quantum states as the interferometer state, in general we can also obtain the expectation value of the parity measurement scheme by the above method. Therefore, our results offer a new method to investigate the parity-based phase estimation scheme with calculations of its signal.

V. CONCLUSION

In summary, we construct a new kind of asymmetric integration projection operators based on the entangled state representation, which are the Hermitian. Then, we analytically demonstrate what physical observable does such new asymmetric projection operators represent. Our results show explicitly that such Hermitian operator describes the effect of the parity measurement with a balanced beam splitter. Finally, as applications of our results, we regain the expected signal of the parity-based phase estimation scheme with some specific states that have been frequently investigated in quantum optical metrology. In a word, for some measurement operators in quantum mechanics, they may be casted into different quantum representations, which can improve the understanding of the relations between the Hermitian operators and quantum representations.

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