Recovery of the Center-of-Mass Galilean invariance of a NR-QCD motivated Hamiltonian

D. A. Amor-Quiroz\textsuperscript{1}, P. O. Hess\textsuperscript{1}, O. Civitarese\textsuperscript{2} and T. Yepez-Martinez\textsuperscript{1}

\textsuperscript{1}Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México Circuito Exterior, C.U., Apartado postal 70-543, 04510 Mexico City, Mexico.
\textsuperscript{2}Departamento de Física, Universidad Nacional de La Plata, C.C.67 (1900), La Plata, Argentina.

E-mail: arturo.amor@nucleares.unam.mx, hess@nucleares.unam.mx, osvaldo.civitarese@fisica.unlp.edu.ar, tochtli.yepez@nucleares.unam.mx

Abstract. We evaluate the use of 3D-harmonic oscillator eigenfunctions in the expansion of the fermionic fields of an effective QCD Hamiltonian with the $1/r$ plus a linearly confining potential. Since both the potential and the functions involved in such expansion are non-relativistic, a modified ter-Vehn method is applied to recover the galilean invariance of the center of mass of the mesonic states built with the use of the Random Phase Approximation. Finally, a preliminary calculation for the mesons below 1 GeV is presented.

1. Introduction

The Random Phase Approximation (RPA) was introduced in 1953 by Bohm and Pines to describe oscillations in a degenerate electron gas [1]. It consists of a harmonic mapping (bosonization) of fermionic pairs that allows for a correlated ground state, which turns out to be composed of an even number of pairs. The RPA is well-known for preserving (in principle) the symmetries of the total Hamiltonian [2] such as translational and Galilean invariance, though these symmetries can be broken due to some of the following reasons:

a) Use of empirical (non-self-consistent) single-particle energies.
b) Use of interactions that are not translational invariant.
c) Truncation of the configuration space.

In the present document we discuss how to restore the center of mass (CM) symmetries of the RPA calculations applied to the model presented in [3, 4], in which the fermionic fields are expanded in 3D-harmonic oscillator (3D-h.o.) eigenfunctions for solving an effective QCD Hamiltonian with a static potential. Such basis presents many analytical results that simplify intermediate computations [5, 6], though the involved wavefunctions are non-relativistic and thus they prevent the RPA calculations from maintaining the appropriate symmetries.

In [5, 6] the use of Talmi-Moshinsky (TM) transformations was proposed as a tool to identify and remove the spurious excitations originated from frame-dependant dynamics [7, 8]. This could be achieved by restricting the RPA calculations to a subspace of two-particle functions where the bosonized pairs do not present CM excitations. Such procedure has the advantage of
reducing the size of the involved matrices and therefore requires less computing power. However, this approach may come to be too restrictive as in RPA calculations the CM excitations have to be removed not only from pairs but for any number of fermions. If, for example, there are two pairs in the final state, each pair should be able to move with its own CM but only the CM from the two pairs together has to be substracted, that is, from a 4-particle system.

Consequently, in the present document we prefer to evaluate the use of a modified Meyer-ter-Vehn method [9] which consists of treating the RPA equations in such a way that it is possible to separate the spurious states.

2. Brief description of the model

The model presented in [3] consists of an effective QCD Hamiltonian inspired in the Coulomb gauge formalism, where the quarks interact through a static potential, which simulates the effect of the gluons.

In the same way as in QED, implementing the Coulomb gauge gives rise to a local and instantaneous potential. Lattice calculations have shown that this potential is approximately of the form $-\frac{\alpha}{r} + \beta r$ [10], where $r = |x - y|$ is the relative distance between two points of the color charge densities. Such interaction is known in the literature as Cornell potential and its linear component reproduces the phenomenon of confinement, since it makes energetically less favorable to increase the distance between color charges. Because of this, the Cornell potential simulates the effect of gluons, which makes possible the treatment of a simplified theory where the only fields involved are fermionic.

This approach to the problem is equivalent to replacing the Faddeev-Popov operator by the static potential [11]

\[
\langle x | \nabla \cdot \mathcal{D}^{-1} \frac{1}{\nabla \cdot \mathcal{D}} | y \rangle \rightarrow V(|x - y|) = -\frac{\alpha}{|x - y|} + \beta |x - y| .
\]

Thus, the effective Hamiltonian to be solved in the present model is given by

\[
H = H_{Kq} + H_{mq} + H^{q-coulomb}_{Coulomb} \\
H_{Kq} = \int \psi^\dagger(x)\left[-i\alpha \cdot \nabla\psi(x)\right]dx \\
H_{mq} = \int \psi^\dagger(x)\left[\beta m_0\psi(x)\right]dx \\
H^{q-coulomb}_{Coulomb} = -\frac{1}{2} \int \rho_c^{(q)}(x)V(|x - y|)\rho_c^{(q)c}(y)dxdy ,
\]

where $\alpha_i$ and $\beta$ are related to the Dirac matrices $\gamma^\mu$ via $\alpha_i = \gamma^0\gamma^i$ and $\beta = \gamma^0$, while $m_0$ is the bare mass of the quark. For the purpose of this work, the bare mass for the $u$ and $d$ quarks is assumed to be the same though lower than the mass of the quark $s$, i.e.,

\[
(m_0)^{YT} = m_{u,d}\delta_{Y,\frac{1}{2}}\delta_{T,\frac{1}{2}} + m_s\delta_{Y,-\frac{3}{2}}\delta_{T,0} .
\]

On the other hand,

\[
\rho^{(q)c} = \sum_{c'c} \psi^\dagger_{c'}T_{c'c}\psi_{c'}f ,
\]

is the color charge density restricted to the quark sector $(q)$ only.

The main characteristic of the model is that the spatial part of the fermionic fields is expanded in the non-relativistic 3D-h.o. basis in the coordinate space [12], i.e., the angular functions are
the spherical harmonics while the radial part is given by

\[ R_{nl}(r) = N_{nl} r^l e^{-\frac{1}{2} \gamma (\lambda^2 - 1)} L_n^l(\lambda r^2) \]

\[ N_{nl} = \left( \frac{2(n!)}{\Gamma(n+l+\frac{3}{2})} \right)^{\frac{1}{2}} \lambda^{l+\frac{3}{2}} \]

\[ n = \frac{N-l}{2}, \quad (5) \]

where \( L_n^l(x) \) are the associated Laguerre polynomials and \( \gamma = \frac{1}{\sqrt{\lambda}} \) is the harmonic oscillator length. The latter is not assumed as a free parameter, as it is chosen via a variational principle in which the ground state energy of the RPA calculations is minimized.

Afterwards, the TM transformations are introduced to refer the couplings of two-points wavefunctions to the Jacobi coordinates \( R = \frac{1}{\sqrt{2}} (x + y) \) and \( r = \frac{1}{\sqrt{2}} (x - y) \)

\[ [\Psi_{n_a l_a a}(x) \otimes \Psi_{n_b l_b b}(y)]^L_M = \sum_{n_r, n_R L} \langle n_r l_r r; n_R l_R R; L|n_a l_a a; L \rangle \langle n_r l_r r| \otimes \Psi_{n_R l_R R}(R) \rangle^L_M. \]

Such transformations allow for analytical integration, as the only integrals involved in the matrix elements of the potential are of the form \[ \int_0^{\infty} x^2 dr R_{n_r l_r r}(r)V(\sqrt{2}r)R_{n_r l_r r}(r) \]

while the dependence on the \( R \) coordinate sums out. It is to be noted that the definition of the Jacobi coordinates do not coincide with the CM and relative coordinates, though their definitions are closely related for pairs composed of particles with equal masses.

Once the Hamiltonian and the basis are set, the RPA method is applied to diagonalize the effective Hamiltonian while removing the spurious CM excitations as explained in Section 3. Later, an a posteriori flavor mixing procedure [13] is set for the \( \eta \) and \( \eta' \) pseudoscalar mesons. Finally, the resulting calculations are compared with those published in [3].

3. The ter-Vehn method

Here we present a generalization of the Meyer-ter-Vehn method which consists in allowing the operators to couple in pairs to a certain total representation.

We start by assuming a particle-hole formulation given by the fermionic operators

\[ b_{a\mu}^\dagger = q_{a\mu}^\dagger, \quad d_{a\mu} = q_{-a\mu}^\dagger, \]

where the operators belong to a given representation \( a \) with possible magnetic projections \( \mu \); whereas the index \( \pm \frac{1}{2} \) refers to a particle (hole) operator above (below) the Fermi level in such manner that, by the Pauli exclusion principle, any attempt to create a particle below this level will vanish

\[ q_{-\frac{1}{2} a\mu}^\dagger |0\rangle = 0. \]

The creation and annihilation operators of a particle-hole pair are defined as couplings to a certain total representation \( (\Gamma \mu) \) as follows

\[ \gamma_{a b; \Gamma \mu}^\dagger = \sum_{\mu a \mu b} \langle a \mu_b | \bar{b} \mu_b; \Gamma \mu \rangle b_{a \mu_a}^\dagger d_{b \mu_b}^\dagger, \]

\[ \gamma_{a b; \Gamma \mu} = (\gamma_{a b; \Gamma \mu}^\dagger)^\dagger, \quad (10) \]
as the shorthand notation for the Clebsch-Gordan (CG) coefficients refers to the product of an arbitrary number of different yet independent algebras. For example, in the present model the pair operators represent non-colored meson-like states described by the $SU_f(2) \times SU_f(2) \times SU_c(3)$, thus giving the following structure for the CG:

$$\langle a\mu_a, b\mu_b | \Gamma \mu \rangle = \langle j_a m_a, j_b - m_b | JM_J \rangle \langle T_a T_{za}, T_b - T_{zb} | TT_z \rangle \langle (10)c_a, (01)c_b | (00)0 \rangle_1 .$$

By doing this, one has to be careful in defining the transformation rules for lowering or rising the indices.

$$\gamma^\dagger a\bar{b}\mu \Gamma \mu = \sum_{\mu_a, \mu_b}(\bar{a}\mu_a, b\mu_b | \Gamma \mu \rangle b^\dagger \bar{a}\mu_a d^b\mu_b = (-1)^\bar{\phi}_\mu \gamma^\dagger a\bar{b};\Gamma \mu \rangle \Gamma \mu \mu .$$

The phase $(-1)^\bar{\phi}_\mu$ will be fully determined by the CG involved in the couplings, though they will generically depend on the total magnetic projection $\mu$ and not on its components $\mu_a$ or $\mu_b$ [14].

With the above definitions, the standard RPA equations are given by

$$\sum_{ab} \left( A^{(\Gamma \mu)}_{(ab)(a' b')} B^{(\Gamma \mu)}_{(ab)(a' b')} \right) \left( \begin{array}{cc} X_a^{\alpha, \Gamma \mu} & Y_a^{\alpha, \Gamma \mu} \\ -Y_a^{\alpha, \Gamma \mu} & X_a^{\alpha, \Gamma \mu} \end{array} \right) \delta_{\Gamma \Gamma'} \delta_{\mu' \mu} = \hbar \Omega_{\alpha}^{\Gamma} \left( X_a^{\alpha, \Gamma \mu} Y_a^{\alpha, \Gamma \mu} - Y_a^{\alpha, \Gamma \mu} X_a^{\alpha, \Gamma \mu} \right) \delta_{\Gamma \Gamma'} \delta_{\mu' \mu} .$$

where $A$ and $B$ are the forward and backward matrices respectively. The ter-Vehn method allows us to remove the couplings between intrinsic and spurious states from the Hamiltonian. This is equivalent to solving the same system in equation (13) but modifying the forward and backward matrices as shown below

$$A^{(\Gamma \mu)}_{(ab)(a' b')} = A^{(\Gamma \mu)}_{(ab)(a' b')} - \frac{1}{2} \left( Z^{(R)}_{(ab)(a' b')} + Z^{(P)}_{(ab)(a' b')} \right)$$

$$B^{(\Gamma \mu)}_{(ab)(a' b')} = B^{(\Gamma \mu)}_{(ab)(a' b')} - \frac{1}{2} \left( Z^{(R)}_{(ab)(a' b')} - Z^{(P)}_{(ab)(a' b')} \right) .$$

The structure of the $Z$ coefficients is given in [17].

Such method does not remove the spurious state corresponding to pure oscillations of the CM coordinate. However, this mode is approximately decoupled and appears at a finite energy. In the particular case of the present model, the energy of the pure CM state turns to be of the order of 3 GeV and thus it lies far above the mesonic states evaluated by this method (i.e., below 1 GeV) [17].

4. Results

An exploratory spectrum of the mesons below 1 GeV was already shown in [3], though the CM contribution was not eliminated there and thus compromises the self-consistency of the found solutions. However, such document can be regarded as a feasibility test for further calculations.

Here we present some preliminary calculations that were obtained by implementing the ter-Vehn method to the same fit used in [3], whose values are given in Table 1. This has the purpose of evaluating the CM contributions to the previous results.

The fit in [3] was chosen to reproduce the mass of the pion-like states, while the flavor mixing interaction $H_{FM}$ was selected to reproduce as best as possible the $\eta'$ meson.
Table 1. Set of parameters for performing the RPA calculations.

| Parameter | Value |
|-----------|-------|
| $m_{u,d}[GeV]$ | 0.05 |
| $m_s[GeV]$ | 0.31 |
| $\alpha$ | 0.16 |
| $\beta[GeV^2]$ | 0.40 |
| $H_{FM}[GeV]$ | 0.170 |

Table 2. Experimental and RPA masses [MeV] for pseudoscalar and vector mesons for the Set of parameters from Table 1 after having removed the CM via the ter-Vehn method and implementing an \textit{a posteriori} flavor mixing procedure for the $\eta$ and $\eta'$ mesons.

| Meson-like state | Exp. value | RPA (ref. [3]) | RPA (CM-removed) |
|------------------|------------|----------------|------------------|
| $\pi$            | 139        | 136            | 144              |
| $K$              | 495        | 465            | 476              |
| $\eta$           | 547        | 356            | 363              |
| $\eta'$          | 957        | 958            | 963              |
| $\rho$           | 770        | 676            | 679              |
| $\omega$         | 782        | 676            | 694              |
| $K^*$            | 892        | 942            | 941              |
| $\phi$           | 1020       | 1145           | 1200             |

It can be observed from Table 2 that the CM motion may be neglected for states such as the $K^*$, though its contribution can reach up to 55 MeV for the $\phi$ meson. Nevertheless, the most noticeable feature of the CM removal procedure is that it breaks the degeneracy from the $\rho$ and $\omega$ states by rising the latter as it should. This is due to a small flavor mixing allowed by the structure of the vacuum in the ter-Vehn method, even if the Hamiltonian preserves the flavor symmetry.

5. Conclusions

The results shown in Table 2 exhibit that even without removing the spurious states coming from the CM motion, a fairly reasonable reproduction of the mesons below 1 GeV can be achieved. Nonetheless, the involved physics are improved by the ter-Vehn method and thus, a refitting of the whole parameters is strongly encouraged.

The physics of the model can also be improved, for example, by introducing a dynamical chiral symmetry breaking or by adding spin-spin interactions. Subsequently, the model can be tested over multiquark systems [15] and by computing decaying widths [16].

The advantages obtained up to the current state of the model are summarized below:

- It is a simple yet semi-realistic model. Even though the spectra can be improved, its current state is relatively good.
- The harmonic oscillator basis allows analytic integration, while the integrals involved in the interaction matrix elements are well behaved.
- The ter-Vehn method restores the Galilean and traslational symmetries broken due to the selection of the basis.
- Decoupling the spurious states gives rise to a small flavor mixing evident in the $\omega$-$\rho$ splitting.

Because of the above, the conclusion is that the model is feasible but requires further research to be able to compete with the current approaches for non-perturbative QCD.

Details of the present calculations will be given in [17].
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