Continuity of Discounted Values for Setup-Cost Inventory Control

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Abstract

This paper proves continuity of value functions in discounted periodic-review single-commodity total-cost inventory control problems with fixed ordering costs, possibly bounded inventory storage capacity, and possibly bounded order sizes for finite and infinite horizons. In each of these constrained models, the finite and infinite-horizon value functions are continuous, there exist deterministic Markov optimal finite-horizon policies, and there exist stationary deterministic Markov optimal infinite-horizon policies. For models with bounded inventory storage and unbounded order sizes, this paper also characterizes the conditions under which (s,

1 Introduction

Periodic review inventory control studies have a rich history in operations research beginning with Arrow et al. [1] and Dvoretzky et al. [8]. The classical results, including on multistage problems, are summarized in the texts Bensoussan [3], Porteus [24], Simchi-Levi et al. [27], Zipkin [31]. One of the principal methods of studying inventory control problems is the analysis of (s, S) policies. Under an (s, S) policy, the controller only orders when the inventory level drops below s, and the amount that is ordered returns the inventory level to S. Since the seminal paper of Scarf [26], (s, S) policies have been shown to be optimal in a variety of problem formulations; see, e.g., Veinott and Wagner [28] for the case of discrete demand, Zabel [30] and Iglehart [20] for the case of continuous demand. Recent developments in the theory of Markov decision processes (MDPs) have improved the understanding of the periodic-review single-commodity inventory control problem with setup costs, unbounded storage capacity, unbounded order sizes, and backordering; see Feinberg [9], Feinberg and Lewis [11]. A complete description of the optimality of (s, S) policies for convex holding

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costs in this setting is given in Feinberg and Liang [12], where additional references can be found. For the optimality of $(s,S)$ policies under the average-cost criteria, see Feinberg and Lewis [11].

This paper studies continuity of values under two additional constraints to the inventory problem with fixed order costs: possibly bounded storage capacity and possibly bounded order sizes. Bounded storage models were first considered in Hartley [17], but remain an active field of research; see Jiang et al. [21] and the references therein. Studies of models with bounded orders include Xie [29], Bensoussan et al. [4], Bartoszewicz and Latosiński [2]. We also consider the version of the model with lost sales, in which unrealized demand is lost. Feinberg and Liang [12] proved continuity of the finite and discounted infinite horizon value functions for the setup cost inventory models with unbounded storage, unbounded order sizes, and backorders by studying optimal policies and in particular, optimal $(s,S)$ policies. One difficulty that models with possibly bounded storage and possibly bounded order sizes and models with lost sales present is that, for these models, $(s,S)$ policies may not be optimal or even feasible, and continuity of value functions must be established via other means.

Continuity of values is an important property for practical applications in two major ways. For approximate methods based on discretization of continuous state and action spaces, continuity of values is often required for convergence of solutions to the approximate problems to the solutions of the true problem as the discretization is refined. In addition, for problems where the controller only has access to a noisy or otherwise partially observable system transition model, continuity of values is necessary for empirical consistency of the solutions to the estimated models. For general overviews of these and related issues, see Kara and Yüksel [22], Kara et al. [23].

The main results of this paper concern continuity of values in the infinite horizon versions of these problems. Theorem 2.4, whose main results were obtained in [16, Theorem 14] as a corollary of a generalized version of Berge’s maximum theorem, establishes continuity of values. Theorem 3.1 establishes continuity of values in the infinite horizon for unbounded order constraints. While the case of backorders with unbounded storage capacity was studied in [12], the results for models with backorders with limited storage capacity and models with lost sales are new. Theorem 3.4 characterizes the optimality of $(s_t, S_t)$ and $(s, S)$ policies for problems with backorders and bounded storage capacity. Theorem 4.1 establishes continuity of values for bounded orders.

This paper is organized as follows. Section 2 defines the stochastic periodic-review single-commodity inventory control problem with possibly bounded storage capacity and possibly bounded order sizes, and it provides results on continuity of values in the finite horizon setting. Section 3 considers inventory models with unbounded order sizes, and it establishes the continuity of values in the infinite-horizon and characterizes the conditions under which $(s_t, S_t)$ and $(s, S)$ policies are optimal in the finite and infinite-horizon, respectively. Section 4 studies models with bounded orders and establishes continuity of values in the infinite-horizon.
2 Model Description

In this work, \( \mathbb{R} \) denotes the real numbers, \( \mathbb{R}_+ := [0, +\infty) \) the nonnegative real numbers, \( \mathbb{Z} \) the integers, and \( \mathbb{N} := \{0, 1, 2, \ldots\} \) the natural numbers. The *stochastic periodic-review setup-cost inventory model* is defined as follows. At times \( t \in \mathbb{N} \), a controller views the current inventory of a single commodity and makes an ordering decision. Lead times are zero, so the orders are filled instantaneously prior to the realization of demand. The cost of ordering is paid at the time of delivery of the order. For problems with backorders, any demand is satisfied and unmet demand is backlogged. For problems with lost sales, the demand is satisfied up to the available inventory level, and if the demand is greater than this level, the unmet demand is lost. After the demand is satisfied, the controller views the remaining inventory and pays holding (i.e., excess inventory) or backorder (i.e., negative inventory) costs, and the process continues. The demand and order quantity are assumed to be nonnegative.

Let \( a \wedge b := \min\{a, b\} \) and \( a \vee b := \max\{a, b\} \). The inventory model is characterized by the following parameters:

1. \( K > 0 \) is a fixed ordering cost, paid whenever the order size is nonzero;
2. \( \overline{c} > 0 \) is the per-unit ordering cost;
3. \( \{D_t : t = 1, 2, \ldots\} \) is a sequence of i.i.d. nonnegative finite random variables representing the demand at periods 0, 1, \ldots, where for such a random variable \( D \), we assume that \( \mathbb{E}D < +\infty \) and \( P(D > 0) > 0 \);
4. \( T : \mathbb{R} \to \mathbb{R} \) is the backorder rule given by \( T(x) := 0 \vee x \) for the inventory model with lost sales, and \( T(x) := x \) for the inventory model with backorders;
5. \( h : \mathbb{R} \to \mathbb{R}_+ \) is convex and continuous holding/backorder cost function with \( h(x) \to +\infty \) as \( |x| \to +\infty \), where there is no loss in generality to assume that \( \inf h = 0 \);
6. \( \alpha \in [0, 1) \) is the discount factor;
7. \( \overline{a} > 0 \) is the maximum order size, where \( \overline{a} = +\infty \) means any finite order size may be placed;
8. \( \overline{x} > 0 \) is the maximum storage size, where \( \overline{x} = +\infty \) means any finite amount of inventory may be stored.

The model with \( \overline{a} = +\infty \) (\( \overline{a} < +\infty \)) is called an unbounded (bounded) orders model. The model with \( \overline{x} = +\infty \) (\( \overline{x} < +\infty \)) is called an unbounded (bounded) storage model.

The inventory model can be formulated as an MDP in the following way. Let the state space \( \mathbb{X} \) represent the amount of available inventory and the action space \( \mathbb{A} := [0, +\infty) \) represent order sizes. The multifunction of feasible actions \( A : \mathbb{X} \to 2^\mathbb{A} \setminus \{\emptyset\} \), where \( A(x) \subset \mathbb{A} \) represents the possible inventory order sizes at the state \( x \in \mathbb{X} \) and captures the constraints implied by the values of \( \overline{a} \) and \( \overline{x} \). With each combination of the pair \( (\overline{a}, \overline{x}) \), there are four models characterized by the state space \( \mathbb{X} \) and feasible actions \( A(x) \), listed below.
1. (U) Unbounded orders, unbounded storage: \( \mathbb{X} := \mathbb{R} \) for problems with backorders and \( \mathbb{X} := [0, +\infty) \) for problems with lost sales, and \( A(x) := [0, +\infty) \) for each \( x \in \mathbb{X} \).

2. (BO) Bounded orders, unbounded storage: \( \mathbb{X} := \mathbb{R} \) for problems with backorders and \( \mathbb{X} := [0, +\infty) \) for problems with lost sales, and \( A(x) = [0, a] \) for each \( x \in \mathbb{X} \).

3. (BS) Unbounded orders, bounded storage: \( \mathbb{X} := (-\infty, \bar{x}] \) for problems with backorders and \( \mathbb{X} := [0, \bar{x}] \) for problems with lost sales, and \( A(x) = [0, 0 \vee (\bar{x} - x)] \) for each \( x \in \mathbb{X} \).

4. (BOS) Bounded orders, unbounded storage: \( \mathbb{X} := (-\infty, \bar{x}] \) for problems with backorders and \( \mathbb{X} := [0, \bar{x}] \) for problems with lost sales, and \( A(x) = [0, a] \cap [0, 0 \vee (\bar{x} - x)] \) for each \( x \in \mathbb{X} \).

In the unbounded orders, unbounded storage model (U), \( A \) is not compact valued. In the bounded storage, unbounded orders model (BO), the image \( A(X) = \bigcup_{x \in \mathbb{X}} A(x) = (-\infty, \bar{x}] \) is not a compact set in the model with backorders, and \( A(\mathbb{X}) = [0, \bar{x}] \) in the model with lost sales. In the bounded orders, unbounded storage model (BS) and the bounded orders, bounded storage model (BOS), \( A \) is compact valued and has a compact image. In each model, \( A \) is upper semicontinuous and lower semicontinuous as a multifunction, and \( \text{Gr}(A) \) is closed. The graphs of the feasible actions \( A \) in each of the models (U, BO, BS, BOS) for models with backorders are given in Figure 1.

The dynamics of the system are defined by the equation

\[
x_{t+1} = T(x_t + a_t - D_{t+1}), \quad t \in \mathbb{N}
\]

where \( x_t \) and \( a_t \) denote the current inventory level and the amount ordered at period \( t \). The transition probability \( q(dx_{t+1}|x_t, a_t) \) for the MDP defined by the dynamics above is

\[
q(B|x_t, a_t) = P(T(x_t + a_t - D_{t+1}) \in B)
\]

for each measurable \( B \subset \mathbb{R} \). The one-step expected cost is

\[
c(x,a) := K1_{\{a>0\}} + \bar{c}a + \mathbb{E}h(T(x + a - D)), \quad x \in \mathbb{X}, a \in A
\]

where \( 1_B \) is the indicator of the set \( B \). We observe that \( c \) is inf-compact.
An example of a function $h$ satisfying the above conditions is $h(x) = |x|$, or with slight generalization $h(x) = h_1x^+ + h_2x^-$, where $x^+ := 0 \lor x$ and $x^- := - (0 \land x)$, where $h_1, h_2 > 0$.

Let $H_t = (X \times A)^t \times X$ be the space of histories of the system up to period $t = 0, 1, \ldots$. A (possibly randomized) control at $t = 0, 1, \ldots$ is a transition probability $\phi_t : H_t \to \mathcal{P}(A)$. A policy $\phi$ is a sequence $(\phi_0, \phi_1, \ldots)$ of controls. The policy $\phi$ is Markov if all controls $\phi_t$ depend only on the current state and time. A Markov policy is stationary if all controls only depend on the state (and not time). A policy $\phi$ is nonrandomized if each control $\phi_t$ is concentrated at a single point in $A$. In this latter case, there is no loss in generality to write $\phi_t : X \to A$.

For an horizon $N \in \mathbb{N} \cup \{+\infty\}$, the expected total discounted costs is defined as

$$v_{N, \alpha}^\pi(x) := \mathbb{E}_x \left[ \sum_{t=0}^{N-1} \alpha^t c(x_t, a_t) \right], \quad x \in X,$$

(4)

where $v_{0, \alpha}^\pi(x) := 0 \ (x \in X)$. When $N = +\infty$, (4) defines the infinite-horizon expected total discounted cost, which is denoted by $v_{\alpha}^\pi(x)$. Let $v_{N, \alpha} := \inf_{\pi} v_{N, \alpha}^\pi(x)$ and $v_{\alpha} := \inf_{\pi} v_{\alpha}^\pi(x), \ x \in X$. A policy $\pi$ is called $N$-stage optimal if $v_{N, \alpha}^\pi = v_{N, \alpha}$ and discount-optimal if $v_{\alpha}^\pi = v_{\alpha}$.

For a function $f : X \to \mathbb{R}$ and a subset $C \subset X$, we denote the epigraph of $f$ restricted to $C$ by

$$\text{epi}_C f = \{(x, \lambda) : x \in C, f(x) \leq \lambda\}$$

(5)

where $\text{epi} f := \text{epi}_X f$ and, for each $\lambda \in \mathbb{R}$, the sublevel set of $f$ restricted to $C$ by

$$\mathcal{D}_f(\lambda; C) = \{x \in C : f(x) \leq \lambda\}$$

(6)

where $\mathcal{D}_f : \mathbb{R} \times \mathbb{R} \to \mathcal{P}(X)$. For a multifunction $\Gamma : X \to Y$ and subset $K \subset X$, the graph of $\Gamma$ on $K$ is denoted by $\text{Gr}_K(\Gamma) := \{(x, y) : x \in K, y \in \Gamma(x)\}$, and $\text{Gr}(\Gamma) := \text{Gr}_X(\Gamma)$.

We recall the following standard definitions.

**Definition 2.1.** A function $f : \mathbb{R} \to \mathbb{R}$ is convex on a set $X \subset \mathbb{R}$, if for all $x, y \in X$ and $\theta \in [0, 1]$,

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y).$$

(7)

It is $K$-convex on $X$ if $x \leq y$ implies

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y) + \theta K.$$  

(8)

Every convex function is 0-convex. A function is convex on $X$ if and only if $\text{epi}_X f$ is convex. The next definition was introduced in Feinberg et al. [13] in the conditions of Assumption W*, but was formulated and studied as a separate property in the later works Feinberg et al. [14, 15], Feinberg and Kasyanov [10], Feinberg et al. [16] with applications relevant to inventory control in Feinberg [9].

**Definition 2.2** (Feinberg [9, Definition 2.1]). For metric spaces $X$ and $Y$, a multifunction $\Gamma : X \to Y$ is...
2^Y \setminus \{\emptyset\}$, and function $f : X \times Y \to \mathbb{R} \cup \{+\infty\}$, the function $f$ is $\mathbb{K}$-inf-compact on $\text{Gr}(\Gamma)$ if, for each nonempty compact set $K \subset X$ and $\lambda \in \mathbb{R}$, the set $D_f(\lambda; K \times Y)$ is compact.

Finally, following the presentation of Rockafellar and Wets [25, Chapter 4], denote the collection of all infinite subsets of $\mathbb{N}$ by $\mathcal{N}_\infty^\# = \{N \subset \mathbb{N} : N \text{ is infinite}\}$, and let $X$ be a subset of a Euclidean space. For a sequence of sets $C_n \subset X$, the outer limit of $\{C_n\}_{n \in \mathbb{N}}$ is defined as

$$
\limsup_{n \to \infty} C_n = \{x \in X : \exists N \in \mathcal{N}_\infty^\#, \exists x_n \in C_n (n \in N), x_n \to x\},
$$

which is composed of all limit points of sequences $\{x_n\}_{n \in \mathbb{N}}$ with $x_n \in C_n$ for each $n \in \mathbb{N}$. We observe that the outer limit is always a closed subset of $X$.

Theorem 2.3 shows that Assumption $W^*$ holds for each of the models $(U, BO, BS, BOS)$ with lost sales and with backorders. Therefore, the functions $v_{t,\alpha}$ and $v_{\alpha}$ satisfy the optimality equations according to Feinberg et al. [13, Theorem 2]. Similarly to the model $(U)$ with backorders [19, 5, 27, 9], the optimality equations can be written as:

$$
v_{t+1,\alpha}(x) = \min\{G_{t,\alpha}(x), K + \min_{a \in A(x)} G_{t,\alpha}(x + a)\} - \bar{c}x \quad x \in \mathbb{X}, \quad t = 0, 1, \ldots
$$

$$
v_{\alpha}(x) = \min\{G_{\alpha}(x), K + \min_{a \in A(x)} G_{\alpha}(x + a)\} - \bar{c}x \quad x \in \mathbb{X},
$$

Assumption $(W^*)$. The following hold.

1. The function $c$ is $\mathbb{K}$-inf-compact on $\text{Gr}(A)$ and bounded below;

2. the transition probability $q(\cdot| x, a)$ is weakly continuous in $(x, a) \in \text{Gr}(A)$.

Theorem 2.3. Assumption $(W^*)$ holds for each inventory model $(U, BO, BS, BOS)$ with lost sales and with backorders.

Proof. In each inventory model $(U, BO, BS, BOS)$, with lost sales and with backorders, the action sets are continuous and $\text{Gr}(A)$ is closed. Since $c$ is nonnegative, it is bounded below by 0. To show that $c$ is $\mathbb{K}$-inf-compact, let $K \subset \mathbb{X}$ be compact, and let $\lambda \in \mathbb{R}$. Then since $\text{Gr}(A)$ is closed,

$$
D_f(\lambda; \text{Gr}_K(A)) = \{(x, a) : x \in K, a \in A(x), c(x, a) \leq \lambda\} \subset K \times [0, \lambda \bar{c}^{-1}]
$$

is a closed subset of a compact set; hence compact, so $c$ is $\mathbb{K}$-inf-compact. To show that $(x, a) \mapsto q(\cdot| x, a)$ is continuous, it suffices to observe that the model dynamics is described by $x_{t+1} = f(x_t, a_t, D_{t+1})$, where $f(x, a, D) = T(x + a - D)$ is a continuous function. According to Hernández-Lerma [18, p. 92], $q$ is weakly continuous. 

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$$

$$
v_{\alpha}(x) = \min\{G_{\alpha}(x), K + \min_{a \in A(x)} G_{\alpha}(x + a)\} - \bar{c}x \quad x \in \mathbb{X},
$$

6
where
\[ G_{t,\alpha}(x) := cx + \mathbb{E}h(T(x - D)) + \mathbb{E}v_{t,\alpha}(T(x - D)), \quad x \in \mathbb{X}, \quad t = 0, 1, \ldots \]  
(13)

\[ G_{\alpha}(x) := cx + \mathbb{E}h(T(x - D)) + \mathbb{E}v_{\alpha}(T(x - D)), \quad x \in \mathbb{X}, \]  
(14)

In addition, since \( v_{t,\alpha} \uparrow v_{\alpha} \) and \( G_{t,\alpha} \uparrow G_{\alpha} \), and since \( v_{t,\alpha}(x) \geq \mathbb{E}h(x - D) \) for each \( x \in \mathbb{X} \) and \( t \geq 1 \), it follows that
\[ \lim_{x \to +\infty} G_{t,\alpha}(x) = \lim_{x \to +\infty} v_{t,\alpha}(x) = \lim_{x \to +\infty} G_{\alpha}(x) = \lim_{x \to +\infty} v_{\alpha}(x) = +\infty \]  
(15)

Assumption W* implies that the finite horizon value functions are lower semicontinuous, which follows from the general MDP theory Feinberg et al. [13, Theorem 2]. In fact, both \( v_{N,\alpha} \) and \( G_{N,\alpha} \) are continuous for each \( N = 0, 1, \ldots \), as stated in the following theorem. This continuity is a corollary of a generalized form of Berge’s maximum theorem, which holds for discontinuous one-step cost functions. Feinberg et al. [16, Theorem 14] states the continuity of the functions \( v_{N,\alpha} \) for models (U, BO, BS, BOS), and its proof established continuity of \( x \mapsto \mathbb{E}v_{N,\alpha}(x - D) \); see additional references there. The following theorem restates Feinberg et al. [16, Theorem 14] with the additional claim that \( G_{N,\alpha} \) are also continuous.

**Theorem 2.4** (cf. Feinberg et al. [16, Theorem 14]). For each inventory model (U, BO, BS, BOS) with lost sales and with backorders, and for each \( N = 0, 1, \ldots \), the functions \( v_{N,\alpha} \), \( x \mapsto \mathbb{E}v_{N,\alpha}(T(x - D)) \), and \( G_{N,\alpha} \) are continuous, and there is an optimal \( N \)-stage deterministic Markov policy \( \phi^N = (\phi_0, \ldots, \phi_{N-1}) \).

**Proof.** Let \( N \in \mathbb{N} \) be arbitrary. The continuity of \( v_{N,\alpha} \) and existence of \( \phi^N \) follow directly from the statement of Theorem 14 in Feinberg et al. [16]. The proof of that theorem also establishes that \( x \mapsto \mathbb{E}v_{N,\alpha}(T(x - D)) \) is continuous. Then the function \( G_{N,\alpha} \) defined in (13) is the sum of continuous functions; hence, continuous.

The following lemma is a useful result that was established in the proof of Feinberg and Liang [12, Theorem 5.3] for the inventory model (U) with backorders. The proof there also holds for the inventory models (BO, BS, BOS) with lost sales and with backorders, so we reproduce it here in the form of the lemma.

**Lemma 2.5** (cf. Feinberg and Liang [12, Theorem 5.3]). For each inventory model (U, BO, BS, BOS) with lost sales and with backorders, suppose the function \( v_{\alpha} \) is continuous. Then the function \( G_{\alpha} \) is also continuous.

**Proof.** For \( x_0 \in \mathbb{X} \), we define the function \( g_{\alpha}(x) = v_{\alpha}(x \lor (x_0 + 1)) + \mathbb{E}(x \lor (x_0 + 1)) \), which is continuous and bounded. Therefore, \( x \mapsto \mathbb{E}g_{\alpha}(T(x - D)) \) is continuous. Furthermore, \( G_{\alpha}(x) = (1 - \alpha)cx + \mathbb{E}h(T(x - D)) + \alpha\mathbb{E}g_{\alpha}(T(x - D)) + \alpha\mathbb{E}[D] \) is a sum of continuous functions for each
$x \in (-\infty, x_0 + 1]$, so $G_\alpha$ is continuous on $(-\infty, x_0 + 1]$, and in particular $G_\alpha$ is continuous at $x_0$. Since $x_0 \in X$ was arbitrary, the result follows.

We recall the definition of an $(s, S)$ policy.

**Definition 2.6.** For each $t = 0, 1, \ldots$, let $s_t \leq S_t$ be finite numbers. An $(s_t, S_t)$ control at step $t$ is the function $\phi_t(x_t) = (S_t - x_t)1_{(x_t < s_t)}$. A Markov policy $\phi$ is an $(s_t, S_t)$ policy if $\phi = (\phi_0, \phi_1, \ldots)$ are all $(s_t, S_t)$ controls for each $t$. A policy is called an $(s, S)$ policy if it is a stationary Markov $(s_t, S_t)$ policy.

The following proposition is a classic fact about $K$-convex functions that connects them to $(s, S)$ policies; see, e.g., Bertsekas [5, Lemma 4.2.1(d)] or Simchi-Levi et al. [27, Lemma 9.3.2(d)]

**Proposition 2.7** (Bertsekas [5, Lemma 4.2.1(d)]). Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, inf-compact $K$-convex function. Let

$$S_f \in \text{arg min}\{f(x) : x \in X\},$$

$$s_f = \text{min}\{x \in X : f(x) \leq K + f(S_f)\},$$

and observe that $-\infty < s_f \leq S_f < +\infty$. Then

1. $g(S_f) + K < g(x)$ for all $x < s_f$,
2. $g(x)$ is decreasing on $(-\infty, s_f]$, so $g(s_f) < g(x)$ for all $x < s_f$,
3. $g(x) \leq g(z) + K$ for all $s_f \leq x \leq z$.

Proposition 2.7 is the mechanism by which $(s, S)$ polices can be shown to be optimal for $K$-convex functions $f$. In particular, Statements 1 and 2 imply that for each $x < s_f$, it is preferable to order $S_f - x$ than to order nothing, and Statement 3 implies that for each $x \geq s_f$, it is always optimal not to order.

### 3 Models with Unbounded Order Sizes

In this section we consider the unbounded orders models $(U, BS)$; that is, order sizes are unbounded, and the capacity can be either finite or unbounded. We consider both cases of backorders and lost sales. The following theorem shows that the discounted infinite horizon value function $v_\alpha$ is continuous. We remark that the proof does not require precise knowledge of the structure of optimal $N$-stage policies, nor discount-optimal (infinite horizon) policies.

**Theorem 3.1.** For the models $(U, BS)$ with lost sales and with backorders, the functions $v_\alpha$ and $G_\alpha$ are finite and continuous for each $\alpha \in [0, 1)$.
Proof. The continuity of \( v_\alpha \) and \( G_\alpha \) for the model (U) with backorders is the result Feinberg and Liang [12, Theorem 5.3], but the following argument will apply to both models (U, BS) with lost sales and with backorders. This is because the policy \( \psi(x) := 0 \vee -x \), which only refills backorders, is feasible in each model. (With lost sales, \( \psi(x) = 0 \) for all \( x \in \mathbb{R} \).)

We will show that \( v_{N,\alpha} \to v_\alpha \) uniformly on the sets \([-M, M]\) for each \( M > 0 \). Then continuity of \( v_\alpha \) will follow from continuity of \( v_{N,\alpha} \) and from the uniform limit theorem; see, e.g., Rockafellar and Wets [25, Proposition 7.15(b)]. For each \( N = 0, 1, \ldots \) consider the policy \( \phi \) which plays an \( N \)-stage optimal policy for stages \( t = 0, 1, \ldots N - 1 \), and then plays the policy \( \psi \) for \( t = N, N + 1, \ldots \). Then

\[
v_\phi(x) = \mathbb{E}_x^\phi \sum_{t=0}^{N-1} \alpha^t c(x_t, a_t) + \mathbb{E}_x^\phi \sum_{t=N}^{\infty} \alpha^t c(x_t, a_t) = v_N(x) + \mathbb{E}_x^\phi \sum_{t=N}^{\infty} \alpha^t c(x_t, a_t). \tag{18}
\]

We observe that \( h(T(z)) \leq h(z) \) for all \( z \in \mathbb{R} \). Therefore, establishing the following inequalities for the inventory model with backorders will by comparison establish them for the model with lost sales. For \( t \geq N \), the inequality

\[
\mathbb{E}_x^\phi c(x_t, a_t) \leq K + \mathbb{E}_x^\phi \{ |x_N| + \mathbb{E}[ (-D) + h(|x| - D)] \}
\]

holds. Furthermore,

\[
|x| - (t - N + 1)\mathbb{E}D \leq \mathbb{E}_x^\phi |x_N| \leq |x|,
\]

so we obtain the bound

\[
v_\phi(x) \leq v_N(x) + \sum_{t=N}^{\infty} \alpha^t \mathbb{E}_x^\phi \{ K + \mathbb{E}[ (2|t| + 2(t - N + 1)\mathbb{E}D) + h((-D) + h(|t| - D)) \}
\]

\[
= v_N(x) + \frac{\alpha^N}{1 - \alpha} \left( K + c(|x|) + \mathbb{E}[h(-D) + h(|x| - D)] \right) + \frac{2\mathbb{E}D}{1 - \alpha}
\]

\[
\leq v_N(x) + \gamma \alpha^N (1 + 2|x|)
\]

for \( \gamma = (1 - \alpha)^{-1}(K + \mathbb{E}[h(-D) + h(|x| - D)] + 2\mathbb{E}D(1 - \alpha)^{-1}) \). Thus

\[
\sup_{|x| \leq M} (v_\phi(x) - v_{N,\alpha}(x)) \leq \sup_{|x| \leq M} \left( v_\phi(x) - v_{N,\alpha}(x) \right) \leq \gamma (1 + 2M) \alpha^N,
\tag{20}
\]

which converges to 0 as \( N \to +\infty \). Hence, the uniform limit theorem implies \( v_\alpha \) is continuous, and Lemma 2.5 implies \( G_\alpha \) is also continuous. \( \square \)

A classic fact among the properties of \( K \)-convex functions is that the function

\[
g(x) := \min_{a \geq 0} \{ K 1_{\{a > 0\}} + f(x + a) \}
\]

is \( K \) convex, if \( f \) is \( K \)-convex; see, e.g., Simchi-Levi et al. [27, Proposition 9.3.3]. The immediate application of this fact is to the unbounded orders, unbounded storage inventory model (U), but (21) also holds when the minimization is taken over all \( a \in [0, \mathbb{E} \vee (x - x)] \), which is shown in the following lemma.
Lemma 3.2. Let \( f \) be a \( K \)-convex function. Then the function
\[
g(x) := \min_{0 \leq a \leq x} \left\{ K 1_{\{a > 0\}} + f(x+a) \right\}
\]  
(22)
is also \( K \)-convex.

Remark 1. In the formulation of Proposition 9.3.3 in Simchi-Levi et al. [27], the coefficient in front of the indicator \( 1_{\{a > 0\}} \) is an arbitrary \( Q > 0 \) instead of \( K \). The similar generalization holds for (22), but it is not used in the current paper.

Proof of Lemma 3.2. The proof of Proposition 9.3.3 in Simchi-Levi et al. [27] actually suffices for the modified equation (22), but we reproduce it in order to deal with the added constraint explicitly. The property that the proof in Simchi-Levi et al. [27, Proposition 9.3.3] requires throughout the argument is that, if \( x_0 \leq x_1 \) and \( a \geq 0 \), then \( a + (x_1 - x_0) \geq 0 \). The related property holds for (22), since if \( 0 \leq a \leq \overline{x} - x_1 \), then \( 0 \leq a + (x_1 - x_0) \leq \overline{x} - x_0 \). This leaves the basic argument unchanged.

Let \( E = \{ x : g(x) = f(x) \} \) and \( O = \{ x : g(x) < f(x) \} \). Let \( x_0 \leq x_1 \), let \( \theta \in [0, 1] \), and denote \( x_\theta := (1 - \theta)x_0 + \theta x_1 \). We consider four cases.

Case 1: \( x_0, x_1 \in E \). Then since \( a = 0 \) is feasible for all \( x \),
\[
g(x_\theta) \leq f(x_\theta) \leq (1 - \theta)f(x_0) + \theta f(x_1) + \theta K = (1 - \theta)g(x_0) + \theta g(x_1) + \theta K.
\]

Case 2: \( x_0, x_1 \in O \). Fix \( 0 \leq a_0 \leq \overline{x} - x_1 \) and \( 0 \leq a_1 \overline{x} - x_1 \) such that \( g(x_0) = K + f(x_0 + a_0) \) and \( g(x_1) = K + f(x_1 + a_1) \). Now, the inequality \( x_0 + a_0 \leq x_1 + a_1 \) holds. Indeed, if \( x_0 + a_0 < x_1 \), this is immediate. Otherwise, if \( x_0 + a_0 \geq x_1 \), it follows that \( a_0 + (x_0 - x_1) \leq \overline{x} - x_1 \). Therefore, \( a_1 \geq x_0 + a_0 - x_1 \), and the inequality holds. Furthermore, \( x_\theta \leq x_\theta + a_\theta \). Thus
\[
g(x_\theta) \leq K + f(x_\theta + a_\theta)
\]
\[
\leq (1 - \theta)(K + f(x_0 + a_0)) + \theta(K + f(x_1 + a_1)) + \theta K
\]
\[
= (1 - \theta)g(x_0) + \theta g(x_1) + \theta K.
\]

Case 3: \( x_0 \in E, x_1 \in O \). Fix \( 0 \leq a_1 \leq \overline{x} - x_1 \) such that \( g(x_1) = K + f(x_1 + a_1) \), and fix \( \nu \) such that \( x_\theta = (1 - \nu)x_0 + \nu(x_1 + a_1) \). We observe that \( \nu \leq \theta \), and
\[
g(x_\theta) \leq f(x_\theta)
\]
\[
\leq (1 - \nu)f(x_0) + \nu f(x_1 + a_1) + \nu K
\]
\[
= (1 - \theta)g(x_0) + \theta g(x_1) + (\theta - \nu)(f(x_0) - f(x_1 + a_1) - K)
\]
\[
\leq (1 - \theta)g(x_0) + \theta g(x_1)
\]
\[
\leq (1 - \theta)g(x_0) + \theta g(x_1) + \theta K,
\]
where the third inequality follows from the following observation: since \( f(x_0) = g(x_0) \leq K + f(x_0 + a) \) for all \( 0 \leq a \leq \overline{x} - x_0 \), and since \( 0 \leq a_1 + (x_1 - x_0) \leq \overline{x} - x_0 \), then \( f(x_0) \leq K + f(x_1 + a_1) \).
Case 4: $x_0 \in O$, $x_1 \in E$. Fix $0 \leq a_0 \leq \bar{x} - x_0$ such that $g(x_0) = K + f(x_0 + a_0)$. We observe that $0 \leq x_1 - x_0 \leq \bar{x} - x_0$. Therefore, $g(x_0) \leq K + f(x_1)$, which implies $f(x_0 + a_0) \leq f(x_1)$. Now, if $x_\theta \leq x_0 + a_0$, then $0 \leq a_0 + x_0 - x_\theta \leq \bar{x} - x_\theta$, which implies

$$
g(x_\theta) \leq K + f(x_0 + a_0)$$

$$= (1 - \theta)(K + f(x_0 + a_0)) + \theta f(x_1) + \theta(K + f(x_0) - f(x_1))$$

$$\leq (1 - \theta)g(x_0) + \theta g(x_1) + \theta K,$$

where the final inequality follows from $f(x_0 + a_0) \leq f(x_1)$. On the other hand, if $x_\theta \geq y_0$, then fix $\nu \in [0, 1]$ such that $x_\theta = (1 - \nu)y_0 + \nu x_1$, and we observe that $\nu \leq \theta$. Then it follows that

$$g(x_\theta) \leq f(x_\theta)$$

$$\leq (1 - \nu)f(y_0) + \nu f(x_1) + \nu K$$

$$= (1 - \theta)g(x_0) + \theta g(x_1) + \nu K + (\theta - \nu)(f(y_0) - f(x_1)) - (1 - \theta)K$$

$$\leq (1 - \theta)g(x_0) + \theta g(x_1) + \theta K,$$

where the final inequality follows again from $f(x_0 + a_0) \leq f(x_1)$.

The above four cases exhaust all combinations of $x_0$ and $x_1$, so $g$ is indeed $K$-convex.

The $K$-convexity of $g$ defined in (21) has direct applications to the structure of optimal policies for the unbounded orders, unbounded storage inventory model (U) with backorders. Lemma 3.2 now extends these applications to the unbounded orders, bounded storage model (BS) with backorders.

Lemma 3.3. For the models (U, BS) with backorders, the functions $G_{t,\alpha}^t$, $t = 0, 1, \ldots$, and $G_{\alpha}$, $\alpha \in [0, 1]$, are $K$-convex, and there exists $\alpha^* \in [0, 1)$ and $t_0 = 1, 2, \ldots$ such that

1. for each $\alpha \in [0, \alpha^*)$, $G_{t,\alpha}$ and $G_{\alpha}$ are convex and nondecreasing functions for $t = 0, 1, \ldots$;

2. for each $\alpha \in [\alpha^*, 1)$, $G_{t,\alpha}$ is convex and nondecreasing for $t = 0, 1, \ldots, t_0 - 1$, and $G_{t,\alpha}$ and $G_{\alpha}$ inf-compact for $t = t_0, t_0 + 1, \ldots$

Proof. Denote by $v_{t,\alpha}^U, v_{\alpha}^U, G_{t,\alpha}^U, G_{\alpha}^U$ the functions defined in (11)-(14) for the model (U), and similarly $v_{t,\alpha}^{BS}, v_{\alpha}^{BS}, G_{t,\alpha}^{BS}, G_{\alpha}^{BS}$ for the model (BS). Then the inequalities

$$v_{t,\alpha}^U \geq v_{t,\alpha}^{BS}, \quad v_{\alpha}^U \geq v_{\alpha}^{BS}, \quad G_{t,\alpha}^U \geq G_{t,\alpha}^{BS}, \quad G_{\alpha}^U \geq G_{\alpha}^{BS}$$

(23)

hold for each $t = 0, 1, \ldots$ since $S \subset S'$ implies $\inf S \geq \inf S'$. To verify that $G_{t,\alpha}^{BS}, G_{\alpha}^{BS}$ are inf-compact, it suffices in view of (15) and Theorem 3.1 to verify that $\lim_{x \to -\infty} G_{t,\alpha}^{BS}(x) = \lim_{x \to -\infty} G_{\alpha}^{BS}(x) = +\infty$. On the other hand, if $G_{t,\alpha}^U$ is convex and nondecreasing for some $\alpha$ and $t$, then according to Feinberg and Liang [12, Lemma 4.6], then the optimality equation (11) is achieved with $\alpha = 0$, and so $G_{t,\alpha}^{BS} = G_{t,\alpha}^U$ and $G_{\alpha}^{BS} = G_{\alpha}^U$, and hence each of these functions is convex (and $K$-convex).
Suppose there exists $\alpha^*$ and $t_0$ such that $G_{t_0,\alpha}^U$ is $K$-convex and $\lim_{x \to -\infty} G_{t_0,\alpha}^U(x) = +\infty$ for each $\alpha \geq \alpha^*$. Then $G_{t_0,\alpha}^{BS}$ is also $K$-convex, and $G_{t_0,\alpha}^{BS}(x) \geq G_{t_0,\alpha}^U(x) \to +\infty$ as $x \to -\infty$. It follows that $G_{t,\alpha}^{BS}$ is $K$-convex for each $t = t_0, t_0 + 1, \ldots$, and hence $G_{t,\alpha}^{BS}$ is also $K$-convex. Let 
\[ \alpha = 1 + c^{-1} \lim_{x \to -\infty} h(x)/x. \] If $\alpha < 0$, then according to Feinberg and Lewis [11, Lemma 6.11], the function $G_{t,\alpha}^U$ is $K$-convex and inf-compact for each $\alpha \geq \alpha^* := 0$. If $\alpha \geq 0$, then according to Feinberg and Liang [12, Theorem 4.2(ii)] there exists $t_0 < +\infty$ such that $G_{t_0,\alpha}^U$ is convex, hence, $K$-convex, and nondecreasing for $t = 0, 1, \ldots, t_0 - 1$, and $G_{t,\alpha}^U$ and $G_{t,\alpha}^U$ are convex, hence $K$-convex, and nondecreasing for $t = t_0, t_0 + 1, \ldots$. Hence, by the argument of the previous paragraph, Statement 2 holds.

The next theorem describes optimal policies for models with unbounded order sizes (U, BS) with backorders. The discount-optimality of stationary $(s, S)$ policies holds for sufficiently large discount factors $\alpha < 1$. Feinberg and Liang [12, Theorem 4.3] describe optimal policies for the model (U) with backorders, and the following theorem extends this result to the model (BS) with backorders.

**Theorem 3.4.** For the inventory models $(U, BS)$ with backorders, there exists $\alpha^* \in [0, 1)$ and $t_0 = 0, 1, \ldots$ such that, for each $\alpha \geq \alpha^*$ the following statements hold:

1. For each $N = 0, 1, \ldots$,
   
   (a) if $N < t_0$, then the policy which never orders is $N$-stage optimal;
   
   (b) if $N \geq t_0$, then the policy which never orders at steps $t = N - t_0, \ldots, N - 1$, and is an $(s_t, S_t)$ policy for the pair $s_t := s_{G_{N-t-1,\alpha}}$ and $S_t := S_{G_{N-t-1,\alpha}}$ defined in (17)-(16) at steps $t = 0, 1, \ldots, N - t_0 - 1$ is $N$-stage optimal.

2. For the pair $(s_\alpha, S_\alpha)$ defined in (17)-(16) for the function $f(x) := G_\alpha(x)$, the $(s_\alpha, S_\alpha)$ policy is discount optimal. In addition, the sequence $\{(s_N, S_N)\}_{N=t_0,t_0+1,\ldots}$ defined in Statement 1 is bounded, and each of its limit points $(s'_\alpha, S'_\alpha)$ is discount optimal.

**Proof.** For the model (U), the theorem follows from Feinberg and Liang [12, Theorem 4.3], so we establish the result for the model (BS). To prove Statement 1, we apply Lemma 3.3 to obtain $\alpha^*$ and $t_0$ such that $G_{N,\alpha}$ is $K$-convex and inf-compact for each $\alpha \in [\alpha^*, 1)$ and $N = t_0, t_0 + 1, \ldots$. Then by Proposition 2.7, for each $t = 0, 1, \ldots, N - t_0 - 1$, we set $s_t := s_{G_{N-t-1,\alpha}}$ and $S_t := S_{G_{N-t-1,\alpha}}$, since each $G_{t_0,\alpha}, \ldots, G_{N-1,\alpha}$ is inf-compact and $K$-convex. For $t = N - t_0, \ldots, N - 1$, Lemma 3.3 implies that the function $G_{t,\alpha}$ is convex and nondecreasing, so the policy which never orders is optimal.

For $\alpha^*$ defined in the above paragraph, Lemma 3.3 implies that $G_\alpha$ is $K$-convex and inf-compact. Therefore, by Proposition 2.7, we set $s_\alpha := s_{G_\alpha}$ and $S_\alpha := S_{G_\alpha}$, and the stationary policy $(s_\alpha, S_\alpha)$ is optimal. Furthermore, since each $G_{t,\alpha}$ is continuous with $G_{t,\alpha} \uparrow G_\alpha$, according to Rockafellar and Wets [25, Proposition 7.4(d), Theorem 7.33], the inclusion

\[
\{(s_t, S_t)\}_{t=t_0}^{\infty} \subset \limsup_{t \to \infty} \{x \in X : G_{t,\alpha}(x) \leq K + \inf G_{t,\alpha}\} \subset \{x \in X : G_\alpha(x) \leq K + \inf G_\alpha\}
\]  

(24)
holds. The set on the right-hand side of (24) is compact, so the sequence on the left-hand side of (24) is contained in a compact set; hence, every limit point of \( \{(s_t, S_t)\}_{t=t_0}^\infty \) is discount optimal, and Statement 2 is proved.

\[ \square \]

## 4 Models with Bounded Order Sizes

In this section we consider the bounded orders, unbounded storage (BO) or bounded orders, bounded storage (BOS) inventory models with lost sales and with backorders. We shall consider the additional assumptions that \( h(x) = O(|x|^p) \) and \( D \in L^p \) for some \( p \geq 1 \). This assumption is natural to applications and common in the inventory control literature; see, e.g., [6, pp. 43-44] and Chen and Simchi-Levi [7, Assumptions 4-5]. The following theorem is the main result of the section, which shows that the value function is continuous. For both of these models. Unlike Theorem 3.1, the value function in this setting may take on infinite values.

**Theorem 4.1.** For the model (BO, BOS) the functions \( v_\alpha \) and \( G_\alpha \) are finite and continuous for each \( \alpha \in [0, 1) \) in the following cases:

1. models with lost sales;
2. models with backorders, when \( h(x) = O(|x|^p) \), and \( \mathbb{E}[D^p] < +\infty \) for some \( p \geq 1 \).

**Proof.** As with Theorem 3.1, we will show that \( v_{N,\alpha} \rightarrow v_\alpha \) uniformly on the sets \([-M, M]\) for each \( M > 0 \), and continuity will follow from the uniform limit theorem; see, e.g., Rockafellar and Wets [25, Proposition 7.15(b)]. For each \( N = 0, 1, \ldots \) consider the policy \( \phi \) which plays an \( N \)-stage optimal policy for stages \( t = 0, 1, \ldots N-1 \), and then plays the constant policy 0 for \( t = N, N+1, \ldots \). Then

\[
\phi^\rho_\alpha(x) = \mathbb{E}_x^{\phi} \sum_{t=0}^{N-1} \alpha_t c(x_t, a_t) + \mathbb{E}_x^{\phi} \sum_{t=N}^\infty \alpha_t c(x_t, a_t) = v_N(x) + \mathbb{E}_x^{\phi} \sum_{t=N}^\infty \alpha_t c(x_t, a_t). \tag{25}
\]

To prove that \( v_\alpha \) is finite and continuous in Case 1, we first show that there exists \( \tilde{x} \geq 0 \) such that it is optimal not to order at \( x \geq \tilde{x} \) at each time \( t = 0, 1, \ldots \). Now, \( v_\alpha(0) = 0 \), since \( h(0) = 0 \) and the policy that never orders is optimal. Similarly, \( \mathbb{E} v_\alpha(T(-D)) = v_\alpha(0) = 0 \). Let \( x^0 := \max \arg \min \{v_\alpha(x) : x \geq 0\} \) and \( x^1 := \max \arg \min \{\mathbb{E} v_\alpha(T(x-D)) : x \geq 0\} \). Since \( \lim_{x \to +\infty} v_\alpha(x) = \lim_{x \to +\infty} \mathbb{E} v_\alpha(T(x-D)) = +\infty \), we conclude that \( x^0, x^1 < +\infty \). Furthermore, for \( x \geq \max\{x^0, x^1\} \), it is optimal not to order; hence, set \( \tilde{x} = \max\{x^0, x^1\} \). Since \( 0 \leq \mathbb{E}_x^{\phi} x_N < \tilde{x} \), and since \( h(x_t) \leq h(\tilde{x}) \) for each \( t = 0, 1, \ldots \), it follows that \( \mathbb{E}_x^{\phi} \alpha_t^t c(x_t, a_t) \leq \alpha^t h(\tilde{x}) \) for each \( t \geq N \), and hence

\[
\mathbb{E}_x^{\phi} \sum_{t=N}^\infty \alpha_t^t c(x_t, a_t) \leq \frac{\alpha^N}{1-\alpha} h(\tilde{x}). \tag{26}
\]

Thus \( v_\alpha^\rho(x) \leq v_N(x) + \frac{\alpha^N}{1-\alpha} h(\tilde{x}) \), and therefore

\[
\sup_{|x| \leq M} (v_\alpha(x) - v_{N,\alpha}(x)) \leq \sup_{|x| \leq M} (v_\alpha^\rho(x) - v_{N,\alpha}(x)) \leq \frac{\alpha^N}{1-\alpha} h(\tilde{x}). \tag{27}
\]
which converges to 0 as \( N \to +\infty \).

We now consider Case 2. We observe that \( h(T(z)) \leq h(z) \) for all \( z \in \mathbb{X} \). Furthermore, \( h(x) = 1 + \rho |x|^p \) for some \( \rho > 0 \). Therefore, establishing the following inequalities for the inventory model with backorders will by comparison establish them for the model with lost sales. Because \( 0 \leq \phi_t(x) \leq \bar{a} \) for each \( t = 0, 1, \ldots, N - 1 \), it follows that

\[
|x| - N\mathbb{E}D \leq \mathbb{E}^\phi_N |x_N| \leq |x| + N\bar{a}
\]  

so \( \mathbb{E}^\phi_N |x_N| \leq |x| + N(\bar{a} + \mathbb{E}D) \). Then

\[
\mathbb{E}h(x_N - \sum_{i=N+1}^{t+1} D_i) \leq \mathbb{E}\left( 1 + \rho |x_N| + \sum_{i=N+1}^{t+1} |D_i|^p \right) \leq \mathbb{E}\left( 1 + \rho(1 + t - N + 1)^p |x_N|^p + \sum_{i=N+1}^{t+1} |D_i|^p \right) = 1 + \rho(1 + t - N + 1)^p (|x + N(\bar{a} + \mathbb{E}D)|^p + (t - N + 1)\mathbb{E}[D^p]) \]

for large \( \eta > 0 \). Therefore,

\[
\mathbb{E}_t^\phi \alpha^t c(x_t, a_t) \leq \alpha^t (K + |x + N\bar{a} + 1 + \eta(t - N)^p N^p)
\]

for and each \( t = N, N+1, \ldots \), which implies that

\[
\mathbb{E}_T^\phi \sum_{t=N}^{\infty} \alpha^t c(x_t, a_t) \leq \sum_{t=N}^{\infty} \alpha^t (K + |x + N\bar{a} + 1 + \eta(t - N)^p N^p) \leq \frac{\zeta \alpha^N}{1 - \alpha} (|x| + 1)
\]

for \( \zeta > 0 \) which does not depend on \( x \) or \( N \). Thus

\[
v_\phi^\alpha(x) \leq v_N(x) + \frac{\zeta \alpha^N}{1 - \alpha} (|x| + 1)
\]

and therefore

\[
\sup_{|x| \leq M} (v_\alpha(x) - v_{N,\alpha}(x)) \leq \sup_{|x| \leq M} \left( \frac{\zeta \alpha^N}{1 - \alpha} (M + 1) \right),
\]

which converges to 0 as \( N \to +\infty \). Therefore, according to the uniform limit theorem, \( v_\alpha \) is continuous, and Lemma 2.5 implies \( G_\alpha \) is also continuous.

The following example demonstrates that, if the assumptions of case (ii) in Theorem (4.1) does not hold, then it is possible that \( v_\alpha(x) = G_\alpha(x) = +\infty \) for some \( x \in \mathbb{X} \).

**Example 4.2.** Consider the model with \( D \) is almost surely constant \( d > \bar{a} \). For \( h(x) = \alpha^{-x^2} \), and
for any feasible policy $\phi$,

$$v^\phi_\alpha(x) = \sum_{t=0}^{\infty} \alpha^t c(x_t, a_t) \geq \sum_{t=0}^{\infty} \alpha^t h(x_{t+1}) = \sum_{t=0}^{\infty} \alpha^t - x_{t+1}^2. \quad (36)$$

For $t = 0, 1, \ldots$, it follows that

$$x - dt \leq x_t - d \leq x_{t+1} \leq x_t - (d - \bar{a}) \leq x - (d - \bar{a})t,$$

so in the limit

$$\lim_{t \to \infty} t - x_{t+1}^2 \leq \lim_{t \to \infty} t - [x - (d - \bar{a})t]^2 = -\infty, \quad (37)$$

which implies the on the right hand side of (36) diverges for each $x$.

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