THE LAPLACE TRANSFORM OF THE SECOND MOMENT IN THE GAUSS CIRCLE PROBLEM

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Abstract. The Gauss circle problem concerns the difference $P_2(n)$ between the area of a circle of radius $\sqrt{n}$ and the number of lattice points it contains. In this paper, we study the Dirichlet series with coefficients $P_2(n)^2$, and prove that this series has meromorphic continuation to $\mathbb{C}$. Using this series, we prove that the Laplace transform of $P_2(n)^2$ satisfies

$$\int_0^\infty P_2(t)^2 e^{-t/X} \, dt = CX^{3/2} - X + O(X^{1/2 + \epsilon}),$$

which gives a power-savings improvement to a previous result of Ivić [Ivi96].

Similarly, we study the meromorphic continuation of the Dirichlet series associated to the correlations $r_2(n + h)r_2(n)$, where $h$ is fixed and $r_2(n)$ denotes the number of representations of $n$ as a sum of two squares. We use this Dirichlet series to prove asymptotics for $\sum_{n \geq 1} r_2(n + h)r_2(n)e^{-n/X}$, and to provide an additional evaluation of the leading coefficient in the asymptotic for $\sum_{n \leq X} r_2(n + h)r_2(n)$.

1. Introduction

A classic result of Gauss states that the number $S_2(R)$ of integer lattice points contained in a circle of radius $\sqrt{R}$ is well-approximated by the circle’s area. To quantify the accuracy of this estimate, one defines the lattice point discrepancy

$$P_2(R) := S_2(R) - \pi R = \sum_{n \leq R} r_2(n) - \pi R,$$

in which $r_2(n)$ denotes the number of representations of $n$ as a sum of two integer squares.

The famous Gauss circle problem is the pursuit of the minimal $\alpha$ for which $P_2(R) \ll R^{\alpha + \epsilon}$ for all $\epsilon > 0$. Pointwise, the greatest improvement to the trivial bound $P_2(R) \ll \sqrt{R}$ of Gauss is due to Huxley [Hux03], who proved

$$P_2(R) \ll R^{131/416} (\log R)^{18637/8320} \quad \left(\frac{131}{416} = 0.31490\ldots\right)$$

using his variant of the “discrete Hardy-Littlewood circle method.”

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Lower bounds in the form of $\Omega_\pm$-results suggest the well-known conjecture $P_2(R) \ll R^{1/4+\epsilon}$. This conjecture is also supported by various on-average results, including mean square estimates, which are estimates of the form

$$\int_0^R P_2(t)^2 dt = \left( \frac{1}{3\pi} \sum_{n \geq 1} \frac{r_2^2(n)}{n^{3/2}} \right) R^2 + Q(R),$$

where $Q(R)$ is an error term. The current best bound for $Q(R)$ is due to Nowak [Now04], who showed that $Q(R) \ll R(\log R)^{3/2} \log \log R$.

In [Ivi96], Ivić considered the Laplace transform of $P_2(R)^2$ (as well as the second moment of the error in the Dirichlet divisor problem) and proved

$$\int_0^\infty P_2(t)^2 e^{-t/R} dt = \frac{1}{4} \left( \frac{R}{\pi} \right)^{3/2} \sum_{n \geq 1} \frac{r_2^2(n)}{n^{3/2}} - R + O_\epsilon(R^{1/2+\epsilon}),$$

where $\alpha$ is chosen such that the convolution estimate

$$\sum_{n \leq R} r_2(n)r_2(n+h) = C_\lambda R + O(R^{1/2+\epsilon})$$

holds uniformly for $h \leq \sqrt{X}$. In this way, improved asymptotics for the convolution sum (1.3) lead to sharper asymptotics for the Laplace transform of $P_2(n)^2$. In [Ivi01], Ivić proved that $\alpha \leq \frac{2}{3}$ using recent results of Chamizo [Cha99] to adapt an argument of Motohashi concerning convolution sums in the divisor problem [Mot94]. Correspondingly, the current best error term in the Laplace transform for $P_2(R)^2$ in (1.2) is $O(R^{2/3+\epsilon})$.

The primary result in this article is the following theorem, which establishes an improved error term in the above mean square Laplace transform.

**Theorem 1.1.** For any $\epsilon > 0$,

$$\int_0^\infty P_2(t)^2 e^{-t/R} dt = \frac{1}{4} \left( \frac{R}{\pi} \right)^{3/2} \sum_{n \geq 1} \frac{r_2^2(n)}{n^{3/2}} - R + O_\epsilon(R^{1/2+\epsilon}).$$

Due to a line of spectral poles appearing in our analysis, we conjecture moreover that the exponent $\frac{1}{2}$ in this new error term is optimal.

We approach this problem by investigating the Dirichlet series associated to $S_2(R)^2$ and $P_2(R)^2$, defined by

$$D(s, S_2 \times S_2) := \sum_{n \geq 1} \frac{S_2(n)^2}{n^{s+2}}, \quad D(s, P_2 \times P_2) := \sum_{n \geq 1} \frac{P_2(n)^2}{n^s}. $$

These Dirichlet series have been partially analyzed before. For example, a recent paper of Furuya and Tanigawa [FT14] builds upon the earlier work of Ivić to give a partial meromorphic continuation of the Dirichlet series $D(s, P_2 \times P_2)$. In this paper, techniques developed in [HKLDW17a], [HKLDW17c], and [HKLDW17b] are applied to derive the full meromorphic continuation of $D(s, P_2 \times P_2)$.
Let $B_k(\sqrt{R})$ denote the $k$-dimensional ball of radius $\sqrt{R}$, let $r_k(n)$ denote the number of representations of $n$ as a sum of $k$ squares, and define

$$S_k(R) := \sum_{n \leq R} r_k(n), \quad P_k(R) := \sum_{n \leq R} r_k(n) - \text{Vol } B_k(\sqrt{R}).$$

Estimating $P_k(R)$ represents the $k$-dimensional analogue of the Gauss circle problem, described in detail in the survey article [IKKN06]. In [HKLDW17b], the authors showed that for $k \geq 3$, the Dirichlet series $D(s, S_k \times S_k)$ and $D(s, P_k \times P_k)$ have meromorphic continuation to the complex plane. These continuations were used to prove $k$-dimensional analogues of (1.1) and (1.2) in the case $k \geq 3$.

The techniques used in [HKLDW17b] to understand $D(s, P_k \times P_k)$ for $k \geq 3$ break down in the case $k = 2$. In this article, we show how to modify previous methods to address the dimension 2 case. This culminates in Theorem 5.1, which describes the meromorphic continuation of $D(s, P_2 \times P_2)$ to the entire complex plane.

The techniques of this paper can also be used to give explicit meromorphic continuation the shifted convolution Dirichlet series,

$$D_2(s; h) := \sum_{n \geq 0} \frac{r_2(n)r_2(n + h)}{(n + h)^s}.$$

These shifted convolutions give a new way to understand Chamizo’s asymptotic (1.3) and give a new derivation of the constant $C_h$. With the aid of exponential smoothing, particularly strong versions of (1.3) are attainable.

**Theorem 1.2.** For any $\epsilon > 0$,

$$\sum_{n \geq 1} r_2(n)r_2(n + h)e^{-n/X} = C_hX + O_{h}(X^{\frac{3}{2} + \epsilon}e^{h/X}h^{\Theta}).$$

Here, $\Theta$ refers to the best progress towards the non-archimedean Ramanujan conjecture. A full statement of this result, including a non-trivial estimate for the corresponding sharp sum (1.3) and a new evaluation of $C_h$, is given in Theorem 7.1. While Chamizo also used spectral techniques, including trace-type formulas, to evaluate the leading coefficient, our methods are very different.

Although $D_2(s; h)$ can be used to provide bounds for the shifted convolution sum (1.3), the authors have not been able to improve known bounds on the shifted convolution sum itself. However, by summing over both $n$ and $h$, we gain deep understanding of

$$Z_2(s, w) := \sum_{h \geq 1} \sum_{n \geq 0} \frac{r_2(n)r_2(n + h)}{(n + h)^2h^w},$$

which can be used to recognize significant cancellation within $D(s, P_k \times P_k)$ for application in many problems.
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2. Decomposition of $D(s, S_2 \times S_2)$ and $D(s, P_2 \times P_2)$

In this section, we show that the meromorphic properties of $D(s, P_2 \times P_2)$ can be recovered from the meromorphic properties of $D(s, S_2 \times S_2)$. We then decompose $D(s, S_2 \times S_2)$ into a sum of simpler functions that we analyze in later sections. The methodology of this section is extremely similar to section §2 of [HKLDW17b], so we sketch the proofs and focus on the differences.

**Proposition 2.1.** The Dirichlet series $D(s, P_2 \times P_2)$ and $D(s, S_2 \times S_2)$ are related through the equality

$$D(s, P_2 \times P_2) = D(s - 2, S_2 \times S_2) + \pi^2 \zeta(s - 2) - 2\pi \zeta(s - 1) - 2\pi L(s - 1, \theta^2)$$

$$- \frac{2\pi}{2\pi i} \int_{(s)} L(s - 1 - z, \theta^2) \zeta(z) \frac{\Gamma(z) \Gamma(s - 1 - z)}{\Gamma(s - 1)} \, dz,$$

when $\sigma > 1$ and $\Re s > \sigma$, where $L(s, \theta^2)$ is the normalized $L$-function

$$L(s, \theta^2) := \sum_{n \geq 1} \frac{r_2(n)}{n^s} = 4 \zeta[4i](s) = 4 \zeta(s) L(s, \chi), \quad (2.1)$$

and $\chi = (\frac{-1}{\cdot})$ is the non-trivial Dirichlet character of modulus 4.

In Proposition 2.1 and throughout the paper, we use the common notation

$$\frac{1}{2\pi i} \int_{(s)} f(z) \, dz := \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\sigma + it) \, dt.$$

**Proof.** Note that $P_2(n)^2$ and $S_2(n)^2$ are related by the formula

$$P_2(n)^2 = S_2(n)^2 - 2\pi n S_2(n) + \pi^2 n^2.$$

Divide by $n^s$, sum over $n \geq 1$, and simplify. For the middle term, note that

$$\sum_{n \geq 1} \frac{S_2(n)}{n^{s-1}} = \sum_{n \geq 1} \frac{1 + r_2(n)}{n^{s-1}} + \sum_{n \geq 1} \sum_{m=1}^{n-1} \frac{r_2(m)}{n^{s-1}}$$

$$= \zeta(s - 1) + L(s - 1, \theta^2) + \sum_{m,h \geq 1} \frac{r_2(m)}{(m + h)^{s-1}}.$$

We decouple $m$ and $n$ in the final sum with the Mellin-Barnes identity

$$\frac{1}{(m + h)^s} = \frac{1}{2\pi i} \int_{(s)} \frac{1}{m^{s-z} h^z} \frac{\Gamma(z) \Gamma(s - z)}{\Gamma(s)} \, dz, \quad (\sigma > 0, \Re s > \sigma),$$

given in [GR15, 6.422(3)]. The rest of the simplification follows as in the proof of [HKLDW17b, Proposition 2.1].
To prove (2.1), we first recognize \( L(s, \theta^2) \) in terms of the Dedekind zeta function \( \zeta_{\mathbb{Z}[i]}(s) \) by grouping elements of the principal ideal domain \( \mathbb{Z}[i] \) by norm. The final equality \( \zeta_{\mathbb{Z}[i]}(s) = \zeta(s)L(s, \chi) \) follows from the theory of split primes in \( \mathbb{Z}[i] \).

\[ \square \]

Remark 2.2. Simple factorizations for \( L(s, \theta^k) \) are only known for \( k = 2, 4, 6, 8 \). Simplification along the lines of (2.1) was therefore not available in the previous work [HKLDW17b] for general \( k \geq 3 \).

As in [HKLDW17b, Proposition 2.2] or [HKLDW17a, Proposition 3.1], we can decompose \( D(s, S_2 \times S_2) \) into the sum of a function \( W_2(s) \) and an associated Mellin-Barnes integral.

**Proposition 2.3.** The Dirichlet series associated to \( S_k(n)^2 \) decomposes into

\[
D(s, S_2 \times S_2) = \zeta(s + 2) + W_2(s) + \frac{1}{2\pi i} \int_{(\sigma)} W_2(s - z) \zeta(z) \frac{\Gamma(z)\Gamma(s + 2 - z)}{\Gamma(s + 2)} \, dz
\]

for \( \text{Re} \, s > 2 \) and \( 1 < \sigma < \text{Re}(s - 1) \), in which

\[
W_2(s) = \frac{16\zeta(s + 2)^2 L(s + 2, \chi)^2}{(1 + 2^{-s-2})\zeta(2s + 4)} + 2Z_2(s + 2, 0),
\]

\[
Z_2(s, w) = \sum_{h \geq 1} \sum_{n \geq 0} \frac{r_2(n + h)r_2(n)}{(n + h)^s h^w}.
\]

Here \( Z_2(s, w) \) converges locally normally for \( \text{Re} \, s > 2 \) and \( \text{Re} \, w \geq 0 \).

Proof. We first note a beautiful result of Borwein and Choi [BC03], that

\[
\sum_{n \geq 1} \frac{r_2(n)^2}{n^s} = \frac{16\zeta(s)^2 L(s, \chi)^2}{(1 + 2^{-s})\zeta(2s)}.
\]  

(2.2)

Given this, the proof of [HKLDW17b, Proposition 2.2] applies verbatim. \( \square \)

In \( W_2(s) \), the first term \( \sum r_2(n)^2 n^{-s-2} \) has a double pole at \( s = -1 \), coming from the factor \( \zeta^2(s) \) in the numerator. This behavior is unique to the dimension 2 case, as the rightmost pole of the analogous function, \( \sum r_k(n)^2 n^{-s-k} \), is simple for all \( k \geq 3 \).

3. **Meromorphic Continuation of \( D_2(s; h) \) and \( Z_2(s, w) \)**

In this section, we explain how to obtain the meromorphic continuations of the singly-summed shifted convolutions

\[
D_2(s; h) := \sum_{n \geq 0} \frac{r_2(n + h)r_2(n)}{(n + h)^s},
\]

as well as the doubly-summed shifted convolution

\[
Z_2(s, w) := \sum_{h \geq 1} \sum_{n \geq 0} \frac{r_2(n + h)r_2(n)}{(n + h)^s h^w} = \sum_{h \geq 1} \frac{D_2(s; h)}{h^w}.
\]
These constructions follow analogous work in [HH16] and [HKLDW17b]. However, a major distinction between the traditional Gauss circle problem and the generalized Gauss circle problems in dimension \( k \geq 3 \) becomes apparent in this section.

Let \( P_h(z, s) \) denote the Poincaré series
\[
P_h(z, s) = \sum_{\Gamma \setminus \Gamma_0(4)} \text{Im}(\gamma z)s e^{2\pi i h \gamma z},
\]
and let \( \theta(z) \) denote the standard theta function
\[
\theta(z) = \sum_{n \in \mathbb{Z}} e^{2\pi i n^2 z},
\]
which is a modular form of weight \( \frac{1}{2} \) on \( \Gamma_0(4) \). A classic unfolding argument shows that for \( \text{Re } s \) sufficiently large,
\[
\langle |\theta^2| y \cdot P_h(z, s) \rangle = \int_{\Gamma_0(4) \setminus \mathcal{H}} |\theta^2(z)|^2 y P_h(z, s) d\mu(z) = \frac{\Gamma(s)D_2(s; h)}{(4\pi)^s}, \tag{3.1}
\]
where \( \langle \cdot, \cdot \rangle \) denotes the Petersson inner product on \( \Gamma_0(4) \) and \( d\mu(z) = \frac{dx dy}{y^2} \) is the corresponding Haar measure. Dividing by \( h^w \) and summing over \( h \) recovers \( Z_2(s, w) \).

To understand the meromorphic properties of \( D_2(s; h) \) and \( Z_2(s, w) \), we perform a spectral expansion on \( P_h(z, s) \). However, it is not possible to immediately replace \( P_h \) by its spectral expansion in the inner product because \( |\theta^2(z)|^2 y \notin L^2(\Gamma_0(4) \setminus \mathcal{H}) \). It is necessary to modify \( |\theta^2|^2 y \) to be square integrable. In [HKLDW17b], this was accomplished by subtracting appropriate Eisenstein series evaluated at specific parameters. But in dimension 2, the naïve choices of Eisenstein series would be evaluated at poles, so it is necessary to present a new approach.

3.1. Modifying \( |\theta^2|^2 y \) to be Square Integrable. Let \( E_a(z, s) \) denote the Eisenstein series associated to the cusp \( a \) of \( \Gamma_0(4) \setminus \mathcal{H} \), given by
\[
E_a(z, s) = \sum_{\gamma \in \Gamma_a \setminus \Gamma_0(4)} \text{Im}(\sigma_a^{-1} \gamma z)^s,
\]
where \( \Gamma_a \subset \Gamma_0(4) \) is the stabilizer of the cusp \( a \), and \( \sigma_a \in \text{PSL}_2(\mathbb{R}) \) satisfies \( \sigma_a \infty = a \) and induces an isomorphism \( \Gamma_a \cong \Gamma_\infty \) through conjugation. The quotient \( \Gamma_0(4) \setminus \mathcal{H} \) has three cusps, which can be represented as \( \infty, \frac{1}{2}, \frac{1}{2} \).

Lemma 3.1. Define \( V(z) \) by
\[
V(z) = |\theta^2(z)|^2 \text{Im}(z) - \text{const}_{u=1} E_\infty(z, u) - \text{const}_{u=1} E_0(z, u), \tag{3.2}
\]
where \( \text{const}_{u=c} f(u) \) refers to the constant term in the Laurent expansion of \( f(u) \) expanded at \( u = c \). Then \( V(z) \in L^2(\Gamma_0(4) \setminus \mathcal{H}) \).

The use of constant terms in Laurent expansions of Eisenstein series to modify the growth of functions at cusps is not new, and has been used for example in [HKKrL16, §6] and [LD17, §5] in a similar manner.
Proof. In [HKLDW17b, §3], it is shown that
\[ |\theta^2(z)|^2 \text{Im}(z) = |\theta^2(\sigma_0 z)|^2 \text{Im}(\sigma_0 z) = y(1 + O(e^{-2\pi y})) \]
and that \( \theta(z) \) has exponential decay at the cusp \( \frac{1}{2} \).

The Eisenstein series \( E_a(z, s) \) have Fourier expansions of the form
\[
E_a(\sigma_b z, s) = \delta_{[a-b]} y^s + \pi \frac{1}{2} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \varphi_{a0}(s) y^{1-s} + \frac{2\pi^2 y^{\frac{3}{2}}}{\Gamma(s)} \sum_{n \neq 0} \varphi_{a0}(s)|n|^{s - \frac{1}{2}} K_{s - \frac{1}{2}}(2\pi |n| y)e(nx),
\] (3.3)
in which the coefficients \( \varphi_{a0}(s) \) are described in [DI83], for example. Here and throughout, we use \( \delta_{[\text{condition}] \text{ true}} \) as a Kronecker delta, which is 1 if the condition is true and is otherwise 0.

When \( b = \infty \), we will write these coefficients as \( \varphi_{an}(s) \). As described in [HKLDW17b, §3.3], these coefficients are given by
\[
\varphi_{0h}(t) = \frac{\sigma^{(2)}_{1-2t}(h)}{4\zeta^{(2)}(2t)}, \quad \varphi_{\frac{h}{2}}(t) = \frac{(-1)^h \sigma^{(2)}_{1-2t}(h)}{4\zeta^{(2)}(2t)}, \quad \varphi_{\infty h}(t) = \frac{2^{-4t} \sigma^{(2)}_{1-2t}(\frac{h}{2}) - 2^{-4t} \sigma^{(2)}_{1-2t}(\frac{h}{2})}{\zeta^{(2)}(2t)},
\] (3.4)
where \( \zeta^{(2)}(t) \) is the Riemann zeta function with its 2-factor removed, \( \sigma_v(h) \) is the sum of divisors function, and \( \sigma_v^{(2)}(h) \) is the sum of odd-divisors function.

From the expansion (3.3) and asymptotics of the \( K \)-Bessel function, we see that
\[
E_a(\sigma_b z, u) = \delta_{[a-b]} y^u + \pi \frac{1}{2} \frac{\Gamma(u - \frac{1}{2})}{\Gamma(u)} \varphi_{a0}(u) y^{1-u} + O(e^{-2\pi y}).
\] (3.5)

It is therefore natural to attempt to mollify the growth of \( |\theta^2(z)|^2 y \) at the 0 and \( \infty \) cusps by subtracting \( E_{\infty}(z, 1) \) and \( E_0(z, 1) \), but both \( E_{\infty}(z, u) \) and \( E_0(z, u) \) have poles at \( u = 1 \). In particular, \( \varphi_{a0}(u) \) has a simple pole at \( u = 1 \) in both cases. Referring to (3.3) and (3.5), it is clear that \( \text{const}_{u=1} E_{\infty}(z, u) \) has leading term \( y \), and secondary terms that are logarithmic and constant in \( y \), and is otherwise of rapid decay (and similarly for \( E_0 \) with respect to the 0 cusp). As the constant terms of these Laurent expansion are modular, we conclude that
\[ \mathcal{V}(z) := |\theta^2(z)|^2 y - \text{const}_{u=1} E_{\infty}(z, u) - \text{const}_{u=1} E_0(z, u) \in L^2(\mathcal{T}_0(4) \backslash \mathcal{H}), \]
which proves the lemma. \( \square \)

### 3.2. Modified Inner Product Representation

We will use the modified function \( \mathcal{V}(z) \) instead of \( |\theta^2(z)|^2 y \) to study the meromorphic properties of \( Z_2(s, w) \). Replacing (3.1) with \( \mathcal{V} \) shows that
\[
\langle \mathcal{V}, P_h(\cdot, \bar{s}) \rangle = \frac{\Gamma(s)}{(4\pi)^s} D_2(s; h) - \langle \text{const}_{u=1} (E_{\infty}(\cdot, u) + E_0(\cdot, u)), P_h(\cdot, \bar{s}) \rangle.
\]
The inner product of the Eisenstein series against the Poincaré series can be directly computed (by unfolding and applying [GR15, 6.621(3)]) to be

\[
\langle E_a(\cdot, u), P_h(\cdot, s) \rangle = 2\pi u^{\frac{s+u}{2}} h^{s+u-1} \frac{\Gamma(s+u-1)\Gamma(s-u)}{\Gamma(s)\Gamma(u)} \varphi_{ah}(u) \Gamma(s+u-1)\Gamma(s-u),
\]

provided that \( \text{Re} s + u - 1 > 0 \) and that \( \text{Re} u \) is sufficiently large. The equality (3.6) may be subsequently extended by meromorphic continuation.

After some simplification, we have

\[
\langle \text{const } u = 1 (E\infty(\cdot, u) + E_0(\cdot, u)), P_h(\cdot, s) \rangle = \text{const } u = 1 \langle E\infty(\cdot, u) + E_0(\cdot, u), P_h(\cdot, s) \rangle = \frac{\pi}{(4\pi h)^{s-1}} \Gamma(s-1) \left( \varphi_{\infty h}(1) + \varphi_{0h}(1) \right).
\]

Here we have used that the coefficients \( \varphi_{ah}(u) \) are holomorphic at \( u = 1 \) as long as \( h \geq 1 \), as can be seen from (3.4).

This shows that

\[
D_2(s; h) = \frac{(4\pi)^2}{\Gamma(s)} \langle V, P_h(\cdot, s) \rangle + 4\pi^2 s^{-1} \frac{\varphi_{\infty h}(1) + \varphi_{0h}(1)}{h^{s-1}}.
\]

Dividing by \( h^w \) and summing over \( h \geq 1 \) gives that

\[
Z_2(s, w) = \frac{(4\pi)^2}{\Gamma(s)} \sum_{h \geq 1} \frac{\langle V, P_h(\cdot, s) \rangle}{h^w} + 4\pi^2 s^{-1} \sum_{h \geq 1} \frac{\varphi_{\infty h}(1) + \varphi_{0h}(1)}{h^{s+w-1}}.
\]

**Remark 3.2.** The difference between the expansion (3.8) and its higher-dimensional analogue from equation (3.7) in [HKLDW17b] is purely technical, and these expressions should be directly compared. Indeed, the remainder of the description of the meromorphic properties of \( Z_2 \) is essentially the same as the description of \( Z_k \) for \( k \geq 3 \), except at times when restricting to even dimension allows for greater simplification.

### 3.3. Spectral Expansion

We now provide a spectral expansion of the Poincaré series \( P_h(z, s) \) and insert this expansion into (3.7) and (3.8). Regarding \( V \) as a generic modular, square-integrable function, this is identical to the spectral expansion that appears in [HKLDW17b]. We introduce the necessary notation to describe and state the spectral expansion, but we defer to [HKLDW17b, §3.2] for the proof.

The Poincaré series \( P_h(z, s) \) has a spectral expansion (as given in [IK04, Theorem 15.5]) of the form

\[
P_h(z, s) = \sum_j \langle P_h(\cdot, s), \mu_j \rangle \mu_j(z) + \sum_a \frac{V}{4\pi} \int_{-\infty}^{\infty} \langle P_h(\cdot, s), E_a(\cdot, \frac{1}{2} + it) \rangle E_a(z, \frac{1}{2} + it) \, dt,
\]

in which \( V \) is the volume of the fundamental domain for \( \Gamma_0(4)\backslash \mathcal{H} \), \( a \) ranges over the three cusps of \( \Gamma_0(4)\backslash \mathcal{H} \), and \( \{\mu_j\} \) denotes an orthonormal basis of
Hecke-Maass forms for $L^2(\Gamma_0(4)\backslash \mathcal{H})$ with associated types $\frac{1}{2} + it_j$. These Maass forms admit Fourier expansions

$$\mu_j(z) = \sum_{n \neq 0} \rho_j(n)y^\frac{1}{2}K_{it_j}(2\pi|n|y)e(nx),$$

where $e(x) = e^{2\pi ix}$, and have associated eigenvalues $\lambda_j(n)$ and $L$-functions

$$L(s, \mu_j) = \sum_{n \geq 1} \frac{\rho_j(n)}{n^s}.$$ 

Inserting the spectral expansion (3.9) into the expression for $D_2(s; h)$ in (3.7) and the expression for $Z_2(s, w)$ in (3.8) proves the following theorem.

**Theorem 3.3.** For $\text{Re } s$ sufficiently large, the singly-summed shifted convolution $D_2(s; h)$ can be written as

$$D_2(s; h) = 4\pi^2 \frac{(\varphi_{\infty h}(1) + \varphi_{0 h}(1))}{s - 1}$$
$$+ \frac{2\pi}{h^s} \sum_j \rho_j(h)G(s, it_j)\langle \mathcal{V}, \mu_j \rangle$$
$$+ \sum_a \frac{2\pi V}{2\pi i} \int (0) \frac{\varphi_{ah} \left( \frac{1}{2} - z \right) G(s, z)}{h^s + \frac{1}{2} - z \Gamma \left( \frac{1}{2} + z \right)} \langle \mathcal{V}, E_a(\cdot, \frac{1}{2} - z) \rangle dz,$$

and for $\text{Re } w$ also sufficiently large, the doubly-summed shifted convolution $Z_2(s, w)$ can be written as

$$Z_2(s, w) = 4\pi^2 \frac{(\varphi_{\infty h}(1) + \varphi_{0 h}(1))}{s - 1}$$
$$+ \sum_j \left[ L(s + w - \frac{1}{2}, \mu_j)G(s, it_j)\langle \mathcal{V}, \mu_j \rangle \right]$$
$$+ \sum_a \frac{2\pi V}{2\pi i} \int (0) \frac{G(s, z) \pi^{\frac{1}{2} + z}}{\Gamma \left( \frac{1}{2} + z \right)} \sum_{h \geq 1} \frac{\varphi_{ah} \left( \frac{1}{2} - z \right)}{h^{s+w-z+\frac{1}{2}}} \langle \mathcal{V}, E_a(\cdot, \frac{1}{2} - z) \rangle dz.$$ 

In both expressions, $G(s, z)$ denotes the collected gamma factors

$$G(s, z) := \frac{\Gamma(s - \frac{1}{2} + z)\Gamma(s - \frac{1}{2} - z)}{\Gamma(s)^2}.$$ 

We refer to first lines of (3.10) and (3.11) as the “non-spectral part,” to the second lines as the “discrete part,” and to the third lines as the “continuous part” of the spectrum of $D_2$ or $Z_2$, respectively.

### 3.4. Meromorphic Continuation of $D_2(s; h)$ and $Z_2(s, w)$

The description of the meromorphic continuation of $Z_2(s, w)$ can be obtained from the meromorphic continuation of $Z_k(s, w)$ as given in [HKLDW17b, §3.3] by specializing to $k = 2$, using the modified $\mathcal{V}$ as defined in (3.2), and tracking
changes in the non-spectral part. The single shifted convolution \( D_2(s; h) \) is described only implicitly there, so we consider it explicitly here.

**Lemma 3.4.** The single-sum shifted convolution \( D_2(s; h) \) has meromorphic continuation to \( \mathbb{C} \). The rightmost pole occurs at \( s = 1 \), coming from the non-spectral part. The function \( D_s(s; h) \) is otherwise analytic in \( \Re s > \frac{1}{2} \), though on the line \( \Re s = \frac{1}{2} \) there is a line of poles appearing in the discrete part of the spectrum of \( D_2 \).

We consider the non-spectral, discrete, and continuous parts separately. As there is only a single sum, the analysis is significantly simpler than the analysis of \( Z_2(s, w) \).

**Proof.** The meromorphic continuation of the non-spectral part of \( D_2(s; h) \) is trivial, and we see a unique simple pole at \( s = 1 \) with residue

\[
\text{Res}_{s=1} = 4\pi^2(\varphi_{\infty h}(1) + \varphi_{0h}(1)).
\] (3.12)

In the discrete part of the spectrum, there are poles at \( s = \frac{1}{2} \pm it_j \) coming from the gamma factors in \( G(s, it_j) \). As Selberg’s Eigenvalue Conjecture is known for \( \Gamma_0(4) \) [Hux85], it is known that \( \sup_j \{ \Re it_j \} = 0 \). Note that for any fixed \( s \), the gamma factors \( G(s, it_j) \) have exponential decay in \( t_j \), so the sum converges absolutely.

The integrand of the continuous part of the spectrum has poles at \( s = \frac{1}{2} \pm z \) due to the gamma factors in \( G(s, z) \). Note that for any fixed \( s \), the gamma factors \( G(s, z) \) have exponential decay in \( z \), so that the integral converges absolutely. Proving the meromorphic continuation of the continuous part of the spectrum is subtle, but the methodology of [HKLDW17a, §4.4.2] or [HKLDW17b, §3.3.3] of iteratively shifting lines of integration and picking up residual terms applies here. However, since there is a line of poles from the discrete spectrum on the line \( \Re s = \frac{1}{2} \), and since we are integrating along the line \( \Re z = 0 \), it is not necessary to pursue a detailed understanding of the meromorphic properties for \( \Re s \leq \frac{1}{2} \) to complete the proof of the lemma. \( \Box \)

**Remark 3.5.** It is interesting to note that each individual \( D_2(s; h) \) has poles at \( s = \frac{1}{2} \pm it_j \) from the discrete spectrum, while the complete sum \( Z_2(s, 0) \) does not. That is, by averaging over \( h \), the leading line of poles vanishes. This phenomenon was observed by Chamizo [Cha99, §4] and featured in the proof of the current best bound for \( \sum_{n \leq X} r(n) r(n+h) \) of Ivić (as described in (1.3) in the introduction).

We now translate [HKLDW17b, Lemma 3.3], a summary of the meromorphic behavior of \( Z_k(s, w) \), into the dimension \( k = 2 \) case. The function \( Z_2(s, 0) \) will be further analyzed in sections §4.3-4.4.

**Lemma 3.6.** The doubly-summed shifted convolution \( Z_2(s, w) \) has meromorphic continuation to \( \mathbb{C}^2 \). In particular, the specialized shifted convolution \( Z_2(s, 0) \) has meromorphic continuation to the plane. For \( \Re s > -\frac{1}{2} \),
all poles of \(Z_2(s, 0)\) come from the non-spectral part (which has a simple pole at \(s = 2\) and a double pole at \(s = 1\)) and the continuous part of the spectrum (whose poles appear within the residual terms \(R_j\), as defined in (3.16)).

The non-spectral part. The non-spectral part can be described explicitly by computing the Dirichlet series associated to the coefficients \(\varphi_{ah}(t)\). These computations were performed in [HKLDW17b, (3.12)], and we have

\[
\sum_{h \geq 1} \varphi_{0h}(t) = \frac{\zeta(w)\zeta(2)(w-1+2t)}{4t^2},
\]
\[
\sum_{h \geq 1} \varphi_{1h}(t) = \frac{(2^{1-w}-1)\zeta(w)\zeta(2)(w-1+2t)}{4t^2},
\]
\[
\sum_{h \geq 1} \varphi_{\infty h}(t) = \frac{\zeta(w)\zeta(w-1+2t)}{2^{4w}\zeta(2)(2t)} \left( \frac{1}{4^{w-1}} - \frac{1}{2^{w-1}} \right).
\]

Thus the non-spectral part, as it appears in (3.11), can be written as

\[
8\zeta(s+w-1)\zeta(s+w) \frac{(w-1+2t)}{(s-1)(1-2^{1-s-w}+4^{1-s-w})}.
\]

This has clear meromorphic continuation to \(\mathbb{C}^2\). Specializing to \(w = 0\), we note a simple pole at \(s = 2\) and a double pole at \(s = 1\).

The discrete spectrum. The discrete part of the spectrum in (3.11) has clear meromorphic continuation to the plane, coming from the meromorphic continuations of the \(L\)-functions \(L(s, \mu_j)\) and the gamma functions. Note that for any fixed \(s\) away from poles, the gamma factor \(G(s, it_j)\) has exponential decay in \(t_j\) and the sum over \(t_j\) converges absolutely. Specializing to \(w = 0\), we now analyze the poles. The first line of apparent poles at \(s = \frac{1}{2} \pm it_j\) do not actually occur. For odd Maass forms \(\mu_j\), the inner products \(\langle \mathcal{V}, \mu_j \rangle\) vanish (as noted in [HKLDW17a, §4.2]). For even Maass forms \(\mu_j\), the apparent poles are cancelled by trivial zeros of \(L(s, \mu_j)\), as \(L(-2m \pm it_j, \mu_j) = 0\) for any \(m \in \mathbb{Z}_{\geq 0}\). Thus the discrete part of the spectrum is analytic for \(\text{Re} \ s > -\frac{1}{2}\) and has poles at \(s = \frac{1}{2} \pm it_j = -m\) for \(m\) odd, \(m \in \mathbb{Z}_{>0}\).

Remark 3.7. Poor understanding of the growth of the discrete inner products \(\langle \mathcal{V}, \mu \rangle\) represents the main obstacle in improving sharp second moments for \(P_k(n)^2\). It is therefore of note that the inner product \(\langle \mathcal{V}, \mu_j \rangle\) factors as

\[
\langle \mathcal{V}, \mu_j \rangle = \frac{8}{\pi} L(\frac{1}{2}, \mu_j) \tilde{L}(\frac{1}{2}, \mu_j \times \chi) \Gamma(\frac{1}{2} + it_j) \Gamma(\frac{1}{2} - it_j)
\]

in dimension \(k = 2\), where \(\chi\) is the nontrivial character mod 4, as before, and

\[
\tilde{L}(s, \mu_j \times \chi) = \sum_{n \geq 1} \frac{\chi(n)\lambda_j(n)}{n^s}.
\]
This identity follows from the observation that $y^{1/2} \theta^2(z) = E^1_{\infty}(z, \frac{1}{2})$, where $E^1_{\infty}(z, s)$ is the weight-one Eisenstein series for the cusp at infinity, and so the inner product becomes a special value of a Rankin-Selberg convolution which we then simplify. A similar construction can be obtained in the $k = 4$ case.

Also note that (3.15) gives an alternative proof that $\langle V, \mu_j \rangle = 0$ when $\mu_j$ is odd.

The continuous spectrum. The continuous part of the spectrum is the most nuanced. For convenience, we rewrite the continuous component as

$$\frac{2\pi V}{2\pi i} \sum_a \int_{(0)} \frac{G(s, z)\pi^{\frac{1}{2} + z}}{\Gamma(\frac{1}{2} + z)} \zeta_a(s + w, z) \langle V, E_a(\cdot, 1 - \frac{z}{2}) \rangle \, dz,$$

in which $\zeta_a(s, z)$ is defined by

$$\zeta_a(s, z) = \sum_{h \geq 1} \frac{\zeta_{2h}(s) - z}{h^{s+w-\frac{1}{2} - z}}.$$

We describe these Dirichlet series explicitly via (3.13) as

$$\zeta_0(s + w, z) = \frac{\zeta(s + w - \frac{1}{2} - z)\zeta^{(2)}(s + w - \frac{1}{2} + z)}{2^{1+2z}\zeta^{(2)}(1 + 2z)},$$

$$\zeta_{\frac{1}{2}}(s + w, z) = \frac{\zeta(s + w - \frac{1}{2} - z)\zeta^{(2)}(s + w - \frac{1}{2} + z)}{2^{1+2z}\zeta^{(2)}(1 + 2z)} \left( \frac{2^z}{2^{s+w-\frac{1}{2}} - 1} \right),$$

$$\zeta_{\infty}(s + w, z) = \frac{\zeta(s + w - \frac{1}{2} - z)\zeta(s + w - \frac{1}{2} + z)}{2^{2+4z}\zeta^{(2)}(1 + 2z)} \left( \frac{4^z}{4^{s+w-\frac{1}{2}} - 2^z} - \frac{2^z}{2^{s+w-\frac{1}{2}}} \right).$$

The integrand within the continuous component has apparent poles when $s + w - \frac{1}{2} \pm z = 1$ and when $s = \frac{1}{2} \pm z - j$ for $j \in \mathbb{Z}_{\geq 0}$. In [HKLDW17b, §3.3.3], it is proved that it is possible to meromorphically continue the continuous component past these apparent poles. These apparent poles do not contribute poles at the expected locations, but instead introduce additional residual terms in the meromorphic continuation. Overall, in the cases when $\text{Re}(s + w) \neq \frac{3}{2}$ and $\text{Re}(s) \neq \frac{1}{2} - j$, the meromorphic continuation of the continuous component is given by

$$\langle V, E_a(\cdot, 1 - \frac{z}{2}) \rangle + \sum_{j=0}^{\lfloor \frac{1}{2} - \text{Re}(s) \rfloor} (\mathcal{R}^+_{s \rightarrow j}(s, w) - \mathcal{R}^-_{s \rightarrow j}(s, w)).$$
The terms $R^+_1$ and $R^-_1$ denote residual terms coming from apparent poles in the zeta functions in the continuous component. These are given by

$$R^+_1(s, w) = 2\pi V \Res_{z=\frac{3}{2}-s-w} \frac{G(s, z)\pi^{\frac{1}{2}+z}}{\Gamma(\frac{1}{2}+z)} \sum_a \zeta_a(s+w, z) \langle V, E_a(\cdot, \frac{1}{2} \mp z) \rangle$$

and, as described in (3.16), only appear when $\Re(s+w) < \frac{3}{2}$. The terms $R^+_j$ and $R^-_j$ denote residual terms coming from apparent poles from the gamma functions in the continuous component and are given by

$$R^-_j(s, w) = 2\pi V \sum_a \Res_{z=\frac{1}{2}+j-s} \frac{G(s, z)\pi^{\frac{1}{2}+z}}{\Gamma(\frac{1}{2}+z)} \zeta_a(s+w, z) \langle V, E_a(\cdot, \frac{1}{2} \mp z) \rangle.$$ 

In the cases when $\Re(s+w) = \frac{3}{2}$ and $\Re(s) = \frac{1}{2} - j$, (3.16) is slightly altered, mainly that the line of integration for the integral term is bent slightly to the right into the zero-free region of $\zeta(1-2z)$, and we only have the corresponding $R^-$ residue for that line.

Note that for any fixed $s$ and $w$, only finitely many residual terms $R^\pm_j$ appear in the meromorphic continuation (3.16). As each residual term has meromorphic continuation to $\mathbb{C}^2$ (coming from the meromorphic continuations of the zeta function, gamma function, and Eisenstein series), we conclude that the continuous part of the spectrum admits meromorphic continuation to $\mathbb{C}^2$.

4. Analytic behavior of $W_2(s)$

Recall from Proposition 2.3 that $W_2(s)$ is defined by

$$W_2(s) = \frac{16\zeta(s+2)^2 L(s+2, \chi)^2}{(1+2^{-s-2})\zeta(2s+4)} + 2Z_2(s+2, 0).$$

In this section, we will study the meromorphic properties of $W_2(s)$. As described in §3.4, the discrete spectrum of $Z_2(s, 0)$ has infinitely many poles on the line $\Re s = -\frac{1}{2}$. Thus we restrict our analysis of $W_2(s)$ to the half-plane $\Re s > -\frac{3}{2}$.

Our analysis follows the decomposition of $W_2(s)$ into diagonal, non-spectral, discrete, and continuous parts. When these observations are combined, we conclude the following theorem.

**Theorem 4.1.** The function $W_2(s)$ is meromorphic in $\mathbb{C}$ and analytic in the right half-plane $\Re s > 0$. The rightmost pole of $W_2(s)$ occurs at $s = 0$, with residue $2\pi^2$, coming from the non-spectral part. The function $W_2(s)$ is otherwise analytic in $\Re s > -\frac{3}{2}$, with the exception of a pole at $s = -\frac{3}{2}$ with residue

$$\Res_{s=-\frac{3}{2}} W_2(s) = \frac{8(4-\sqrt{2})\zeta(\frac{3}{2})^2 L(\frac{3}{2}, \chi)^2}{7\pi^2 \zeta(3)} \approx 1.27046 77438,$$

coming from the continuous part of the spectrum.
4.1. The Diagonal Part. We first consider the first term in $W_2(s)$,

$$
\frac{16\zeta(s+2)L(s+2,\chi)^2}{(1+2^{-s-2})\zeta(2s+4)},
$$

which we call the “diagonal part.” Using well-known properties of $\zeta(s)$ and $L(s,\chi)$, we see that the diagonal part is analytic in the right half-plane $\text{Re} \, s > -1$. There is a double pole at $s = -1$ coming from $\zeta(s+2)^2$ with principal part

$$
\frac{4}{(s+1)^2} + \frac{8\gamma + \frac{4}{3}\log 2 + \frac{32}{\pi}L'(1,\chi) - \frac{48}{\pi^2}\zeta'(2)}{s+1},
$$

in which we’ve used the evaluation $L(1,\chi) = \pi/4$ to simplify.

The diagonal part has infinitely many simple poles on the line $\text{Re} \, s = -2$, coming from $1 + 2^{-s-2} = 0$ as well as infinitely many poles at the zeros of $\zeta(2s+4)$. Note that $\zeta(2s+4)^{-1}$ is analytic for $\text{Re} \, s > -\frac{3}{2}$.

Remark 4.2. As in [HKLDW17a, HKLDW17b], the diagonal part perfectly cancels with a pair of residual terms from the continuous spectrum once $\text{Re} \, s < -\frac{3}{2}$. Thus the poles coming from zeros of $(1 + 2^{-s-2})\zeta(2s+4)$ will not affect our analysis of $W_2(s)$.

4.2. The Non-Spectral Part. As shown in (3.14), the non-spectral part of $W_2(s)$ is given by

$$
\mathcal{E}_2(s) := \frac{16\zeta(s+1)\zeta(s+2)}{s+1}(1-2^{-s-1}+4^{-s-1}).
$$

The meromorphic behavior of $\mathcal{E}_2(s)$ is determined by the behavior of $\zeta(s)$. This term has a simple pole at $s = 0$ with residue $2\pi^2$, a double pole at $s = -1$ with principal part

$$
-\frac{8}{(s+1)^2} - \frac{8(\gamma + \log \pi)}{s+1},
$$

and is otherwise analytic.

4.3. The Discrete Spectral Part. The discrete part of $W_2(s)$ is analytic for $\text{Re} \, s > -\frac{5}{2}$, where we focus our analysis. On the line $\text{Re} \, s = -\frac{5}{2}$, the discrete part has a line of poles at $s = -\frac{5}{2} \pm it_j$, where $\frac{1}{4} + t_j^2$ denotes a discrete eigenvalue of the Laplace-Beltrami operator on $\Gamma_0(4) \backslash \mathcal{H}$.

4.4. The Continuous Spectral Part. As shown in section §3.4, infinitely many residual terms $\mathcal{R}_{\pm,j}(s,w)$ appear in the meromorphic continuation of the continuous part of $Z_2(s,w)$. However, the only residual terms that appear in the half-plane $\text{Re} \, s > -\frac{5}{2}$ are $\mathcal{R}_1^\pm$ and $\mathcal{R}_0^\pm$.

In [HKLDW17b, Lemma 4.3], it is shown that

$$
\mathcal{R}_1^+(s,0) = -\mathcal{R}_1^-(s,0).
$$
The proof applies in the case \( k = 2 \) as well, and (4.2) shows that the total contribution of \( R^+_{1} - R^−_{1} \) within \( 2Z_2(s + 2, 0) \) is given by

\[
4R^+_{1}(s + 2, 0) = \frac{4V \Gamma(s + \frac{3}{2}) \pi^{s+\frac{3}{2}} \langle V, E_0(\cdot, -\overline{\tau}) \rangle}{\Gamma(s + 2)^2}.
\]

We relate the inner product \( \langle V, E_0(\cdot, -\overline{\tau}) \rangle \) to the diagonal part through Gupta’s generalization [DG00] of Zagier’s regularized Rankin–Selberg construction [Zag81]. As in [HKLDW17b, §4.1], Gupta and Zagier give the equality

\[
\langle V, E_0(\cdot, \tau) \rangle = \frac{\Gamma(s)}{(4\pi)^s V} \sum_{m \geq 1} \frac{r_2(m)^2}{m^s} = \frac{16\Gamma(s)\zeta(s)^2 L(s, \chi)^2}{(4\pi)^s V (1 + 2^{-s}) \zeta(2s)^2},
\]

valid initially for \( 0 < \text{Re} \ s < 1 \) and extended through analytic continuation. Note that we have used the identity (2.2) of Borwein and Choi for the second equality. It follows that

\[
4R^+_{1}(s + 2, 0) = \frac{4^{s+3} \pi^{2s+3} \zeta(s+1) \pi^{s+2} \zeta(2s) \zeta(2s)}{(s+1)^2(1+2^s) \zeta(2s)}. \]

Recall from section §3.4 that the residual terms \( R^\pm_{1}(s + 2, 0) \) only contribute when \( \text{Re} \ s < -\frac{1}{2} \). It therefore suffices to study \( R^+_{1}(s + 2, 0) \) in the strip \( \text{Re} \ s \in (-\frac{5}{2}, -\frac{1}{2}) \). In this region, the only poles come from \( \Gamma(s + \frac{3}{2}) \) and \( \zeta(-s) \). There is a double pole at \( s = -1 \) with principal part

\[
\frac{4}{(s+1)^2} + \frac{24 \log \pi - 4 \log 2 + 144 \zeta'(2)/\pi^2}{3(s+1)} - \frac{32L'(1, \chi)}{\pi(s+1)},
\]

as well as a simple pole at \( s = -\frac{3}{2} \), coming from \( \Gamma(s + \frac{3}{2}) \), with residue

\[
\frac{8(4 - \sqrt{2}) \zeta(\frac{3}{2})^2 L(\frac{3}{2}, \chi)^2}{7\pi^2 \zeta(3)}.
\]

The next pair of residual terms also satisfy \( R^+_{0}(s, 0) = -R^−_{0}(s, 0) \). As with \( R^+_{1} \), we see that the total contribution of \( R^+_{0} - R^−_{0} \) within \( 2Z_2(s + 2, 0) \) is given by

\[
4R^+_{0}(s + 2, 0) = \frac{(4\pi)^s V \langle V, E_{\infty}(\cdot, \overline{\tau} + 2) \rangle}{\Gamma(s + 2)} - \frac{16\zeta^2(s + 2) L(s + 2, \chi)^2}{(1 + 2^{-s-2}) \zeta(2s + 4)}.
\]

Thus the contribution from \( R^+_{0} \) exactly cancels with the diagonal part (4.1) in the left half-plane \( \text{Re} \ s < -\frac{3}{2} \), as stated in Remark 4.2.

5. Analysis of \( D(s, P_2 \times P_2) \)

In this section we begin our study of \( D(s, P_2 \times P_2) \), with an emphasis on the behavior of its leading poles. By analogy with \( D(s, P_k \times P_k) \) in dimensions \( k \geq 3 \), one should expect a large amount of cancellation in the rightmost poles and residues of the components of \( D(s, P_2 \times P_2) \).
Table 1. Summary of Polar Data in the Half-Plane \( \text{Re } s > \frac{1}{2} \)

| Pole Location | Line | Contributing Term | Residue |
|---------------|------|-------------------|---------|
| \( s = 3 \)   |      | \( \pi^2 \zeta(s - 2) \)                  | \( \pi^2 \)      |
| \( s = 3 \)   | (5.1)| \( \frac{\varphi_2(s-3)}{s-1} \), from \( W_2(s-3) \) | \( \pi^2 \)      |
| \( s = 3 \)   | (5.4)| \( -2\pi \frac{L(s-2, \theta^2)}{s-2} \) | \( -2\pi^2 \)  |
| \( s = 2 \)   | (5.1)| \( \mathcal{E}_2(s-2) \), from \( W_2(s-2) \) | \( 2\pi^2 \)   |
| \( s = 2 \)   | (5.2)| \( -2\pi \zeta(s - 1) \)                   | \( -2\pi \)     |
| \( s = 2 \)   | (5.3)| \( -2\pi L(s - 1, \theta^2) \)             | \( -2\pi^2 \)   |
| \( s = 2 \)   | (5.4)| \( 2\pi \frac{L(s-1, \theta^2)}{s-2} \)   | \( \pi^2 \)     |
| \( s = 2 \)   | (5.4)| \( -2\pi \frac{L(s-2, \theta^2)}{s-2} \) | \( -2\pi L(0, \theta^2) \) |
| \( s = \frac{3}{2} \)| (5.3)| \( \frac{4\Re(z -1, 0)}{s-1} \), from \( \frac{W_2(s-3)}{s-1} \) \( \frac{16(4-\sqrt{2})\zeta(3)^2 \log(\frac{s}{\theta})^2}{7\pi^2 \zeta(3)} \) | \( \frac{\pi^2}{8} \)  |
| \( s = 1 \)   | (5.1)| \( \zeta(s) \)                              | 1              |
| \( s = 1 \)   | (5.3)| \( \frac{W_2(s-3)}{s-1} \)                 | \( W_2(-2) \)   |
| \( s = 1 \)   | (5.3)| \( \frac{s\varphi_2(s-1)}{12} \)          | \( \frac{s^2}{6} \) |

Combining Proposition 2.1, which relates \( D(s, P_2 \times P_2) \) and \( D(s, S_2 \times S_2) \), with Proposition 2.3, which relates \( D(s, S_2 \times S_2) \) to \( \zeta(s) \) and \( W_2(s) \), yields the following unified expression for \( D(s, P_2 \times P_2) \):

\[
D(s, P_2 \times P_2) = \zeta(s) + W_2(s - 2) + \pi^2 \zeta(s - 2) - 2\pi \zeta(s - 1) - 2\pi L(s - 1, \theta^2)
\]

\[
- \frac{1}{2\pi i} \int_{\sigma} W_2(s - 2 - z) \zeta(z) \frac{\Gamma(z) \Gamma(s - z)}{\Gamma(s)} \, dz
\]

\[
- \frac{2\pi}{2\pi i} \int_{\sigma} L(s - 1 - z, \theta^2) \zeta(z) \frac{\Gamma(z) \Gamma(s - 1 - z)}{\Gamma(s - 1)} \, dz,
\]

initially valid for \( \text{Re } s \gg 1 \) and \( \sigma \in (1, \text{Re } s) \). We restrict our analysis of \( D(s, P_2 \times P_2) \) to the half-plane \( \text{Re } s > \frac{1}{2} \) so as to avoid a line of poles appearing in the discrete part of the spectrum of \( (5.3) \). For each line \( (5.1) \)–\( (5.4) \), we study the locations and residues of poles for \( \text{Re } s > \frac{1}{2} \). This information is collected in Table 1 for easy reference.

**Poles from terms in (5.1) and (5.2).** These two lines contain simple \( L \)-functions and \( W_2(s - 2) \), so our polar data is either classically known or given by Theorem 4.1.
Poles from terms in (5.3). To understand the meromorphic properties of the integral, shift the line of integration (σ) left to (−3 + ε) for some small ε > 0. There are poles at z = 1 from ζ(z) as well as poles at z = 0 and z = −1 from Γ(z). By Cauchy’s residue theorem, line (5.3) can be written as

\[
\frac{1}{2\pi i} \int_{(−3+ε)} W_2(s - 2 - z)\zeta(z) \frac{Γ(z)Γ(s - z)}{Γ(s)} dz
\]

\[
+ \frac{W_2(s - 3)}{s - 1} - \frac{W_2(s - 2)}{2} + \frac{sW_2(s - 1)}{12}.
\]

The shifted integral is analytic in the right half-plane Re s > −1 + ε, and the poles of the extracted residue terms can be understood from the poles of W_2(s) as described in Theorem 4.1.

Poles from terms in (5.4). As above, shift the line of integration (σ) to (−3 + ε) to show that the integral in (5.4) is given by

\[
-\frac{2\pi}{2\pi i} \int_{(−3+ε)} L(s - 1 - z, \theta^2)\zeta(z) \frac{Γ(z)Γ(s - 1 - z)}{Γ(s - 1)} dz
\]

\[
-2\pi \left( \frac{L(s - 2, \theta^2)}{s - 2} - \frac{L(s - 1, \theta^2)}{2} + \frac{L(s, \theta^2)(s - 1)}{12} \right).
\]

The shifted integral is analytic for Re s > −1 + ε, and the poles of the z-residues can be understood using the identity \( L(s, \theta^2) = 4\zeta(s)L(s, \chi) \) noted in (2.1).

5.1. Examination of Poles and Their Cancellation. With reference to Table 1, we see that the residues of the poles at s = 3 cancel, so that \( D(s, P_2 \times P_s) \) is analytic for Re s > 2. To examine the potential pole at s = 2, we compute

\[
-2\pi L(0, 0) = -8\pi \zeta(0)L(0, \chi) = 2\pi,
\]

in which we’ve used that \( \zeta(0) = -1/2 \) and that \( L(0, \chi) = 1/2 \). Referring to Table 1, we see that the residues of the poles at z = 2 cancel as well.

The pole at \( s = \frac{3}{2} \) clearly does not cancel, and represents the leading pole of \( D(s, P_2 \times P_2) \).

To understand the residue at s = 1, we must compute \( W_2(-2) \). This calculation is simplified by the observation that \( R_0^\pm \) perfectly cancels with the diagonal part in this region, so both can be ignored. The contribution from the non-spectral term is \( E_2(-2) = -2 \). The contribution from \( R_1^\pm \) is \( 4R_1^\pm(0, 0) \), which vanishes since \( R_1^\pm \) has \( \Gamma(s + 2)^2 \) in its denominator. Similarly, the discrete spectrum, which appears here as

\[
2 \cdot 2\pi \sum_j L(s + 2 - \frac{1}{2}, \mu_j)G(s + 2, it_j) \langle \nu, \mu_j \rangle,
\]

vanishes since \( G(0, it_j) = 0 \). Thus \( W_2(-2) = -2 \), and it follows that the pole at \( D(s, P_2 \times P_2) \) at s = 1 has residue \( \frac{\pi^2}{6} - 1 \).
From these observations we derive the following theorem.

**Theorem 5.1.** The Dirichlet series $D(s, P_2 \times P_2)$, originally defined in the right half-plane $\Re s > 3$ by the series

$$\sum_{m=1}^{\infty} \frac{P_2(m)^2}{m^s},$$

has meromorphic continuation to $\mathbb{C}$ given by (5.1)–(5.4). It is analytic in the right half-plane $\Re s > \frac{3}{2}$ and has a pole at $s = \frac{3}{2}$ with residue

$$C_2^2 := \frac{16(4 - \sqrt{2})\zeta(\frac{3}{2})^2L(\frac{3}{2}, \chi)^2}{7\pi^2\zeta(3)}.$$  

The function $D(s, P_2 \times P_2)$ has a second simple pole at $s = 1$ with residue $\frac{\pi^2}{6} - 1$ and is otherwise analytic in the right half-plane $\Re s > \frac{1}{2}$.

**Corollary 5.2.** The Dirichlet series $D(s, S_2 \times S_2)$ has meromorphic continuation to the plane, attainable from Theorem 5.1 and Proposition 2.1.

**Remark 5.3.** Much of the analysis of $D(s, P_k \times P_k)$ in [HKLDW17b] carries over to $D(s, P_2 \times P_2)$, which makes it possible to identify key differences in the meromorphic behavior of $D(s, P_k \times P_k)$ between the cases $k = 2$ and $k \geq 3$. Notably, the leading pole at $s = \frac{3}{2}$ in the dimension 2 case corresponds to a “traveling pole” at $s = \frac{5-k}{2}$ in dimension $k$ which contributes to the rightmost pole at $s = 1$ in dimension 3 and is otherwise non-dominant.

Movement of this pole relative to a fixed pole at $s = 1$ accounts for the apparent phase change in the generalized Gauss circle problem between dimensions $k = 2$ and $k \geq 3$.

### 6. Second Moment Analysis

In this section, we produce estimates for the discrete Laplace transform

$$\sum_{n \geq 1} P_2(n)^2 e^{-n/X}$$

and the continuous Laplace transform $\int_0^\infty P_2(t)^2 e^{-t/X} dt$.

We do this by estimating the integral

$$\frac{1}{2\pi i} \int_{(4)} D(s, P_2 \times P_2)X^s \Gamma(s) ds = \sum_{n \geq 1} P_2(n)^2 e^{-n/X}$$

using the meromorphic information from Theorem 5.1.

**Theorem 6.1.** We have

$$\sum_{n \geq 1} P_2(n)^2 e^{-n/X} = C_2^2 \Gamma(\frac{3}{2})X^{\frac{3}{2}} + \left(\frac{\pi^2}{6} - 1\right)X + O(X^{\frac{3}{2} + \epsilon})$$

for any $\epsilon > 0$, in which $C_2^2$ is the constant defined in Theorem 5.1.

**Proof.** The proof of [HKLDW17b, Theorem 6.3] for dimensions $k \geq 3$ applies, mutatis mutandis, in the dimension $k = 2$ case. Briefly, after making the necessary modification to $V$ as in (3.2), it is possible to show that $W_2(s)$ is of moderate growth in vertical strips. Then [HKLDW17b, Lemma 6.2]
shows that the Mellin-Barnes integral appearing in the decomposition of 
$D(s, S_2 \times S_2)$ from Proposition 2.3 is also of moderate growth. It then suf-
fices to shift the line of integration to $Re s = \frac{1}{2} + \epsilon$ and account for the 
residues stated in Theorem 5.1. □

As in [HKLDW17b, §8], it is possible to use Theorem 5.1 and Theorem 6.1 to 
produce an asymptotic for the continuous Laplace transform.

**Theorem 6.2.** The Laplace transform of the second moment of the lattice 
point discrepancy satisfies

$$
\int_0^\infty P_2(t)^2 e^{-t/X} \, dt = C_3 \Gamma \left(\frac{3}{2}\right) X^{3/2} - X + O(1),
$$

for any $\epsilon > 0$, where $C_3$ is defined as in Theorem 5.1.

To prove this, we use the identity

$$
P_2(t)^2 = P_2([t])^2 + \pi^2([t] - t)^2 + 2\pi P_2([t])(|t| - t)
$$

and consider the Laplace transform of each term in (6.1) in turn.

**Lemma 6.3** (First term in the Laplace transform of (6.1)). We have

$$
\int_0^\infty P_2([t])^2 e^{-t/X} \, dt = C_3 \Gamma \left(\frac{3}{2}\right) X^{3/2} + \left(\frac{\pi^2}{6} - 1\right) X + O(1).
$$

**Proof.** This follows from the direct computation

$$
\int_0^\infty P_2([t])^2 e^{-t/X} \, dt = X (1 - e^{-1/X}) \sum_{n \geq 0} P_2(n)^2 e^{-n/X}
$$

and Theorem 6.1. □

An estimate for the Laplace transform of the second term in (6.1) follows 
by specializing [HKLDW17b, Lemma 8.4] to the case $k = 2$. However, a far 
simpler proof is available when the dimension is even.

**Lemma 6.4** (Second term in the Laplace transform of (6.1)). We have

$$
\pi^2 \int_0^\infty ([t] - t)^2 e^{-t/X} \, dt = \frac{\pi^2}{3} X + O(1).
$$

**Proof.** We compute

$$
\pi^2 \int_0^\infty ([t] - t)^2 e^{-t/X} \, dt = \pi^2 \sum_{n \geq 0} \int_0^{n+1} (n^2 - 2nt + t^2) e^{-t/X} \, dt
$$

$$
= \pi^2 e^{-1/X} \left(2e^{1/X} X^2 - 2X^2 - 2X - 1\right) \sum_{n \geq 0} e^{-n/X}.
$$

Summing the geometric series, computing a series expansion at infinity, and 
simplifying completes the proof. □
The analysis of the third term relies on the meromorphic properties of the Dirichlet series with coefficients \( P_2(n) \). Again, proofs in general dimension \( k \geq 3 \) greatly simplify in dimension 2.

**Lemma 6.5** (Third term in the Laplace transform of (6.1)). We have

\[
2\pi \int_0^\infty P_2([t])([t] - t) e^{-t/X} \, dt = -\frac{\pi^2}{2} X + O(X^\epsilon).
\]

**Proof.** Splitting the bounds of integration at integers and summing gives

\[
I_3 := 2\pi \int_0^\infty P_2([t])([t] - t) e^{-t/X} \, dt = 2\pi \sum_{n \geq 0} P_2(n) \int_n^{n+1} (n - t) e^{-t/X} \, dt
\]

\[
= -2\pi X \left( X - X e^{-1/X} - e^{-1/X} \right) \sum_{n \geq 0} P_2(n) e^{-n/X},
\]

By Mellin inversion, we rewrite this as

\[
I_3 = -2\pi X \left( X - X e^{-1/X} - e^{-1/X} \right) \frac{1}{2\pi i} \int_\sigma D(s, P_2) X^s \Gamma(s) \, ds,
\]

in which \( D(s, P_2) := \sum_{n \geq 1} P_2(n) n^{-s} \) denotes the Dirichlet series associated to \( P_2(n) \).

Modifying the proof of Proposition 2.1, we see that the Dirichlet series \( D(s, P_2) \) can be written

\[
D(s, P_2) = \zeta(s) + \frac{1}{2} L(s, \theta^2) - \pi \zeta(s - 1) + \frac{L(s - 1, \theta^2)}{s - 1}
\]

\[
+ \frac{1}{2\pi i} \int_{-1+\epsilon/2} L(s - z, \theta^2) \zeta(z) \frac{\Gamma(z) \Gamma(s - z)}{\Gamma(s)} \, dz.
\]

The integral is holomorphic for \( \text{Re} \, s > \epsilon/2 \). The apparent pole in \( D(s, P_2) \) at \( s = 2 \) cancels (as \( \text{Res}_{s = 1} L(s, \theta^2) = \pi \)), while the pole at \( s = 1 \) has residue \( -\frac{\pi^2}{2} \) (using the evaluation \( L(0, \theta^2) = -1 \) as in (5.5)). By shifting the line of integration \( \sigma \) to \( \epsilon \), we conclude that

\[
I_3 = -2\pi X \left( X - X e^{-1/X} - e^{-1/X} \right) \left( \frac{\pi^2}{2} X + O(X^\epsilon) \right).
\]

Series expansion and simplification completes the proof. \( \Box \)

Combining Lemmas 6.3, 6.4, and 6.5 gives a proof of Theorem 6.2.

7. **Estimates for Correlation Sums**

Recall the definition

\[
D_2(s; h) := \sum_{n \geq 0} \frac{r_2(n + h)}{(n + h)^s}.
\]

In section §3.4, we saw that \( D_2(s; h) \) has a meromorphic continuation to \( \mathbb{C} \) given by (3.10). Further, \( D_2 \) has a pole at \( s = 1 \) with residue given by (3.12), and is otherwise analytic for \( \text{Re} \, s > \frac{1}{2} \).
In this section, we use this information to produce smooth and sharp estimates for the shifted convolution sums on $r_2(n)r_2(n + h)$ and prove the following theorem.

**Theorem 7.1.** Write $h = 2^a h'$ where $2 \nmid h'$. For any $\varepsilon > 0$, we have

$$
\sum_{n \geq 1} r_2(n)r_2(n + h)e^{-n/X} = C_hX + O_t(\frac{X^{\frac{1}{2}+\varepsilon}e^{h/X\theta h}}{h}) \tag{7.1}
$$

where $\Theta \leq \frac{2}{\lambda}$ denotes the best progress towards the (non-archimedean) Ramanujan conjecture, the implicit constant in the error term is independent of $h$, and where

$$
C_h = 4\pi^2(\varphi_{\infty h}(1) + \varphi_0 h(1)) = 2^{a+1} - 3\frac{8\sigma_1(h')}{h} = \frac{8(-1)^h}{h}\sum_{d|h}(-1)^d d. \tag{7.2}
$$

The error term in (7.1) is $O(X^{\frac{1}{2}+\varepsilon})$, uniformly for $h \ll X$.

Correspondingly, we have the weak sharp estimate

$$
\sum_{n \leq X} r_2(n + h)r_2(n) = C_hX + O_\lambda((X + h)^{1-\lambda} + h) \tag{7.3}
$$

for some $\lambda > 0$, for the same constant $C_h$ as above.

An argument similar to that used in the proofs of Theorems 6.1 and 6.2 shows that $D_2(s; h)$ is of moderate growth in vertical strips. It is therefore straightforward to estimate the integral

$$
\frac{1}{2\pi i} \int_{(4)} D_2(s; h)X^s\Gamma(s)ds = \sum_{n \geq 1} r_2(n + h)r_2(n)e^{-(n+h)/X} \tag{7.4}
$$

by shifting contours. Our analysis follows the decomposition of $D_2(s; h)$ given in (3.10). The non-spectral part of $D_2(s; h)$ can be evaluated explicitly through the Mellin inversion identity

$$
\frac{1}{2\pi i} \int_{(4)} \frac{4\pi^2}{s-1} \frac{\varphi_{\infty h}(1) + \varphi_0 h(1)}{h^{s-1}}X^s\Gamma(s)ds = 4\pi^2(\varphi_{\infty h}(1) + \varphi_0 h(1))Xe^{-h/X}.
$$

To bound the contribution from the discrete and continuous components, we shift the line of $s$-integration to $(\frac{1}{2} + \varepsilon)$ and bound the integrals. The $h$-dependence within the discrete spectrum is determined by $\rho_j(h)/h^{\frac{s}{2} - \frac{1}{2}}$. In [HH16, §4], it is shown that $\rho_j(h)e^{-\frac{1}{2}\|t_j\|} \ll h^{\Theta + \varepsilon/2}$ on average over $t_j$, where $\Theta \leq \frac{2}{\lambda}$ denotes the best progress towards the (non-archimedean) Ramanujan conjecture, and hence $\rho_j(h)/h^{s - \frac{1}{2}} \ll_j h^{\Theta - \varepsilon/2}$. It follows that the discrete contribution is $O_t(X^{\frac{1}{2}+\varepsilon}h^{\Theta})$.

Similarly, the $h$-dependence within the continuous spectrum is determined by $\varphi_{ah}(\frac{1}{2} - z)/h^{s - \frac{1}{2} - z}$. From the estimate $\zeta(1 + it)^{-1} \ll \log(1 + |t|)$, we obtain $\varphi_{ah}(\frac{1}{2} + it) \ll d(h)\log(1 + |t|)$. Thus the continuous contribution is

$$
O_t\left(\frac{d(h)}{h}X^{\frac{1}{2}+\varepsilon}\right) = O_t(X^{\frac{1}{2}+\varepsilon}).
$$
We conclude that
\[\sum_{n \geq 1} r_2(n + h)r_2(n)e^{-(n+h)/X} = C_hXe^{-h/X} + O(e^{X^{1/2+\epsilon}h^0}),\]
in which \(C_h := 4\pi^2(\varphi_{\infty h}(1) + \varphi_{0h}(1))\). Multiplying by \(e^{h/X}\) proves (7.1).

Techniques in [LD17, §5] or [HKLDW17b, §7] show how to transform the smoothed estimate (7.1) into (7.3) (and also short-interval type estimates) using the fact that \(D_2(s; h)\) is of moderate growth in vertical strips.

It remains to prove that \(C_h\) may be written using the alternate expressions given in (7.2). We first show that \(C_h\) is of the form presented in [Cha99, Theorem 3.2].

**Lemma 7.2.** Suppose \(h = 2^\alpha h'\) where \(2 \nmid h'\). Then
\[C_h = 4\pi^2(\varphi_{\infty h}(1) + \varphi_{0h}(1)) = \frac{8\sigma_1(h')}{h} |2^{\alpha+1} - 3|. \tag{7.5}\]

**Proof.** By (3.4), \(C_h\) can be written as
\[4\pi^2 \left( \frac{\sigma_2(h)}{4\zeta(2)} + \frac{\frac{1}{2}\sigma_1(h') - \frac{1}{2}\sigma_1(\frac{h}{2})}{\zeta(2)} \right) = 8\left(\sigma_2(h) + \sigma_1(h') - \frac{1}{2}\sigma_1(\frac{h}{2})\right),\]
where \(\sigma_1(x) = 0\) if \(x\) is not a positive integer. We note that
\[\sigma_1(h) = \sum_{d|h} \frac{1}{d} = \sum_{d|h} \frac{d}{h} = \frac{1}{h} \sum_{d|h} d = \frac{\sigma_1(h)}{h}\]
and that \(\sigma_2(h) = \sigma_2(h')\). If \(h\) is odd, then (7.5) is immediate. Similarly, if \(h = 2h'\), then (7.5) can be easily checked.

Suppose \(h = 2^\alpha h'\) with \(\alpha \geq 2\). Then
\[8\left(\sigma_2(h) + \sigma_1(h') - \frac{1}{2}\sigma_1(\frac{h}{2})\right) = 8\left(\frac{2^\alpha \sigma_1(h')}{h} + \frac{4\sigma_1(h')\sigma_1(2^{\alpha-2})}{h} - \frac{\sigma_1(h')\sigma_1(2^{\alpha-1})}{h}\right).\]
Using \(\sigma_1(2^k) = 2^{k+1} - 1\) and simplifying completes the proof. \(\square\)

A second lemma completes the proof of (7.2) by demonstrating that the leading coefficient \(C_h\) is the same as presented by Ivić [Ivi96, Ivi01].

**Lemma 7.3.** Suppose \(h = 2^\alpha h'\) where \(2 \nmid h'\). Then
\[8 \frac{\sigma_1(h')}{h} |2^{\alpha+1} - 3| = 8 \frac{(-1)^h}{h} \sum_{d|h} (-1)^{d/d}.\]

**Proof.** If \(h\) is odd, then the lemma is immediate. Suppose that \(h = 2^\alpha h'\) where \(\alpha \geq 1\). For any divisor \(d\) of \(h\), we can write \(d' = 2^j d'\) where \(d'\) is
odd. Splitting up the divisors \( d \) into classes \( 2^j d' \) in the sum, rearranging, and swapping the order of summation shows that

\[
8 \frac{(-1)^h}{h} \sum_{d|h} (-1)^d d = \frac{8}{h} \sum_{j=0}^{\alpha} \sum_{d'|h'} (-1)^{(2^j d')2^j d'}
\]

\[
= \frac{8}{h} \sum_{d'|h'} d' \left( -2 + \sum_{j=1}^{\alpha} 2^j \right) = \frac{8 \sigma_1(h')}{h} (2^{\alpha+1} - 3),
\]

which completes the proof. \( \square \)

**Remark 7.4.** Theorem 7.1 recovers the leading term evaluation of the correlation sums investigated by Chamizo [Cha99] and Ivić [Ivi96], as well as some power savings. Naïve estimates for the amount of power savings following from the techniques in this paper are weaker than the estimates achieved by Chamizo and Ivić, but it may be possible to use the approach outlined here to improve this bound.

**References**

[BC03] J. M. Borwein and S. K. K. Choi. On Dirichlet Series for Sums of Squares. *The Ramanujan Journal*, 7(1):95–127, 2003.

[Cha99] Fernando Chamizo. Correlated sums of \( r(n) \). *J. Math. Soc. Japan*, 51(1):237–252, 1999.

[DG00] Shamita Dutta Gupta. The Rankin-Selberg method on congruence subgroups. *Illinois J. Math.*, 44(1):95–103, 2000.

[DI83] J.-M. Deshouillers and H. Iwaniec. Kloosterman sums and Fourier coefficients of cusp forms. *Invent. Math.*, 70(2):219–288, 1982/83.

[FT14] Jun Furuya and Yoshio Tanigawa. On integrals and Dirichlet series obtained from the error term in the circle problem. *Funct. Approx. Comment. Math.*, 51(2):303–333, 2014.

[GR15] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. Elsevier/Academic Press, Amsterdam, eighth edition, 2015. Translated from the Russian, Translation edited and with a preface by Daniel Zwillinger and Victor Moll, Revised from the seventh edition [MR2360010].

[HH16] Jeff Hoffstein and Thomas A. Hulse. Multiple Dirichlet series and shifted convolutions. *J. Number Theory*, 161:457–533, 2016. With an appendix by Andre Reznikov.

[HKKrL16] Thomas A. Hulse, Chan Ieong Kuan, Eren Mehmet Kıral, and Li-Mei Lim. Counting square discriminants. *J. Number Theory*, 162:255–274, 2016.

[HKLDW17a] Thomas A. Hulse, Chan Ieong Kuan, David Lowry-Duda, and Alexander Walker. The second moment of sums of coefficients of cusp forms. *J. Number Theory*, 173:304–331, 2017.
[HKLDW17b] Thomas A. Hulse, Chan Ieong Kuan, David Lowry-Duda, and Alexander Walker. Second moments in the generalized Gauss circle problem, 2017.

[HKLDW17c] Thomas A. Hulse, Chan Ieong Kuan, David Lowry-Duda, and Alexander Walker. Short-interval averages of sums of Fourier coefficients of cusp forms. J. Number Theory, 173:394–415, 2017.

[Hux85] M. N. Huxley. Introduction to Kloostermania. In Elementary and analytic theory of numbers (Warsaw, 1982), volume 17 of Banach Center Publ., pages 217–306. PWN, Warsaw, 1985.

[Hux03] M. N. Huxley. Exponential sums and lattice points. III. Proc. London Math. Soc. (3), 87(3):591–609, 2003.

[IK04] Henryk Iwaniec and Emmanuel Kowalski. Analytic number theory, volume 53 of American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 2004.

[IKKN06] A. Ivić, E. Krätzel, M. Kühleitner, and W. G. Nowak. Lattice points in large regions and related arithmetic functions: recent developments in a very classic topic. In Elementare und analytische Zahlentheorie, volume 20 of Schr. Wiss. Ges. Johann Wolfgang Goethe Univ. Frankfurt am Main, pages 89–128. Franz Steiner Verlag Stuttgart, Stuttgart, 2006.

[Ivi96] A. Ivić. The Laplace transform of the square in the circle and divisor problems. Studia Sci. Math. Hungar., 32(1-2):181–205, 1996.

[Ivi01] A. Ivić. A note on the Laplace transform of the square in the circle problem. Studia Sci. Math. Hungar., 37(3-4):391–399, 2001.

[LD17] David Lowry-Duda. On Some Variants of the Gauss Circle Problem. PhD thesis, Brown University, 5 2017. https://arxiv.org/abs/1704.02376.

[Mot94] Yōichi Motohashi. The binary additive divisor problem. Ann. Sci. École Norm. Sup. (4), 27(5):529–572, 1994.

[Now04] Werner Georg Nowak. Lattice points in a circle: an improved mean-square asymptotics. Acta Arith., 113(3):259–272, 2004.

[Zag81] Don Zagier. The Rankin-Selberg method for automorphic functions which are not of rapid decay. J. Fac. Sci. Univ. Tokyo Sect. IA Math., 28(3):415–437 (1982), 1981.