Nonlinear projective filtering in a data stream

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Abstract

We introduce a modified algorithm to perform nonlinear filtering of a time series by locally linear phase space projections. Unlike previous implementations, the algorithm can be used not only for a posteriori processing but includes the possibility to perform real time filtering in a data stream. The data base that represents the phase space structure generated by the data is updated dynamically. This also allows filtering of non-stationary signals and dynamic parameter adjustment. We discuss exemplary applications, including the real time extraction of the fetal electrocardiogram from abdominal recordings.

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Nonlinear projective filtering of time series is one of the most practical outcomes of the theory of nonlinear dynamical systems as applied to real word observations. Signals with nonlinear dynamical correlations are often not handled properly by spectral processing methods. Nonlinear dynamical systems exhibiting deterministic chaos have been proposed as an alternative paradigm for the study of such complex sequences. However, for the derivation of time series methods within this framework, heavy use had to be made of the theoretical properties of deterministically chaotic systems. This seems to severely restrict the range of systems they may be applied to – after all, very few real world phenomena are adequately described by a purely deterministic time evolution. Also nonlinear filtering procedures were designed to exploit the specific structure generated by deterministic dynamics in phase space. It turns out, however, that careful use of these algorithms can give excellent results also in situations where pure determinism is violated. The reason is that when represented in a low-dimensional phase space, also non-deterministic systems may exhibit structures suitable for filtering purposes. One example is the human electro-cardiogram (ECG), a signal that shows features of nonlinear determinism but also essential stochastic components.

An introduction to deterministic chaos and dynamical systems as a framework for the analysis of time series recordings can be found in the monographs Refs. [1, 2]. Review articles on nonlinear filtering algorithms containing further references to the original literature are Refs. [3, 4]. How and why the ECG can be processed with nonlinear phase space filters is discussed in [5].

Apart from these fundamental considerations, there are also some practical issues that have so far hampered widespread use of nonlinear filters. All of the methods that have been proposed in the literature are formulated as a posteriori filters. The whole signal has to be available before a cleaned version can be computed. The actual computation is invariably quite computer time intensive. One class of methods uses a global nonlinear function to represent an approximation to the dynamics. This function has to be determined by a delicate fitting procedure and the actual filtering scheme (for example Ref. [6]) consists of an
iterative minimisation procedure in a high-dimensional space. The other class of algorithms approximates the dynamics or geometry in phase space by locally linear mappings. Here, covering phase space with small neighbourhoods is the most time consuming step, along with the need to solve a least squares or eigenvalue problem in each of these neighbourhoods. With fairly low-dimensional signals and small noise levels, fast neighbour search algorithms (see [7] for an overview) are very helpful in this regard. In any case, the posterior nature and the computational effort has been one of the major reasons why nonlinear phase space filters haven’t seen more widespread use. The purpose of this article is to introduce modifications to a locally projective noise reduction scheme that make its use in real time signal processing feasible.

In brief, we implement three main modifications to the locally projective noise reduction scheme that has been introduced in Ref. [8]. Exactly the same strategy can be applied to modify the algorithms by Sauer [9] or that by Cawley and Hsu [10]. (1) The data base of local neighbours that is needed to approximate the dynamics is restricted to points in the past, thereby rendering the filter causal. Of course, at the beginning no data base is available and noise reduction gradually becomes more effective as new points are collected. As a side effect, the curvature correction proposed in [9] and discussed in [2] can be carried out during the first sweep through the data. This will be explained below. (2) The number of neighbours required to carry out the correction for each point is limited to a number that is just sufficient for statistical stability of the local linear fits. (3) The last and most severe modification uses the fact that the dynamics is supposed to vary smoothly in phase space. Therefore, instead of determining the locally linearised dynamics at each point, the linear approximation is stored only for a collection of representative points which densely cover the observed set of points. Consequently, the local linear problem has to be solved only for points which are about to become a representative.

The algorithm we present in this paper is based on the noise reduction scheme introduced in Ref. [8]. An alternative derivation can be found in Ref. [11]. We refer the reader to these references as well as the monograph Ref. [2] for the motivation of the approach by the theory of deterministic dynamical systems. In a more general context, a scalar time series \{s_n\}, \(n = 1, \ldots, N\) can be unfolded in a multi-dimensional effective phase space using time delay coordinates \(s_n = (s_{n-(m-1)\tau}, \ldots, s_n)\) (\(\tau\) is a delay time). If \{s_n\} is a scalar observation of a deterministic dynamical system, it can be shown under certain genericity conditions [12, 13] that the reconstructed point set is a one-to-one image of the original attractor of the dynamical system. We will not explicitly assume here that there is such an underlying deterministic system. Nevertheless, general serial dependencies among the \{s_n\} will cause the delay vectors \{s_n\} to fill the available \(m\)-dimensional space in an inhomogeneous way. Linearly correlated Gaussian random variates will for example be distributed according to an anisotropic multivariate Gaussian distribution. Linear geometric filtering in phase space seeks to identify the principal directions of this distribution and project onto them. This concept is implemented by the singular systems approach, see Refs. [14, 15, 16] and many others. The present algorithm can be seen as a nonlinear generalisation of this approach which takes into account that nonlinear signals will form curved structures in delay space. In particular, noisy deterministic signals form smeared-out lower-dimensional manifolds. Nonlinear phase space filtering seeks to identify such structures and project unto them in order to reduce noise.

Thus, conceptually, the noise reduction algorithm consists of three main steps. (1) Find a low-dimensional approximation to the “attractor” described by the trajectory \{s_n\}. (2) Project each point \(s_n\) on the trajectory orthogonally onto the approximation to the attractor to produce a cleaned vector \(\hat{s}_n\). (3) Convert the sequence of cleaned vectors \(\hat{s}_n\) back into the scalar time domain to produce a cleaned time series \(\hat{s}_n\).

The low-dimensional approximation to the point set can be constructed locally in delay coordinate space using a procedure that is very similar to a local version of principal component analysis. In order to define a neighbourhood in phase space around a point \(s_n\), find
Figure 1: Local linear approximation to a one-dimensional curve. Left: approximations are not tangents but secants and all the centres-of-mass (crosses) of different neighbourhoods are shifted inward with respect to the curvature. Right: a tangent approximation is obtained by shifting the centre-of-mass outward with respect to the curvature. The open square denotes the average of the centres-of-mass of adjacent neighbourhoods, the filled square is the corrected centre-of-mass.

all of the points that are within a distance \( \epsilon \) of \( s_n \). A set of nearby points can be defined as

\[
U^{(n)}_{\Delta n} = \{ s'_n : \quad n - \Delta n \leq n', \quad ||s'_n - s_n|| < \epsilon \} 
\]

where \( ||s'_n - s_n|| \) is the phase space distance between \( s'_n \) and \( s_n \). (We use the max norm to measure distances.) In this definition we already included two modifications with respect to [8]. First, only points in the past (\( n' \leq n \)) are considered and second, neighbours are only considered that have been observed no longer than \( \Delta n \) time steps ago. The neighbourhood \( U^{(n)} \) which is actually used for noise reduction is determined by finding the largest \( \Delta n < n \) for which the number \( |U^{(n)}| \) of vectors in \( U^{(n)} \), does not exceed a specified \( U_{\text{max}} \).

The local structure of the point set is approximated by a linear subspace formed essentially by local principal components. The most straightforward implementation is to compute the local centre-of-mass

\[
\langle s \rangle^{(n)} = |U^{(n)}|^{-1} \sum_{s_{n'} \in U^{(n)}} s_{n'}
\]

where \( \sum_{s_{n'} \in U^{(n)}} \) means summation over all \( |U^{(n)}| \) vectors in \( U^{(n)} \). Then the principal components can be calculated around this point. In that case, however, the linear subspace is not tangent to the curved manifold but rather intersects with it, as illustrated in Fig. 1. Therefore, it is preferable to use a corrected centre-of-mass \( \overline{s}^{(n)} \) given by

\[
\overline{s}^{(n)} = 2 \langle s \rangle^{(n)} - |U^{(n)}|^{-1} \sum_{s_{n'} \in U^{(n)}} \langle s \rangle^{(n')}
\]

In the original implementation [8], neighbourhoods could contain future points. Thus, Eq. (3) could only be evaluated after one complete sweep through the data in which all the local centres-of-mass \( \langle s \rangle^{(n)} \) were determined. In order to compute the tangent points \( \overline{s}^{(n)} \), a second sweep through the data set was necessary. In this modified implementation, the centre-of-mass vectors \( \langle s \rangle^{(n)} \) are stored when the point with index \( n \) is processed. Thus all these vectors are available for points with index \( n' \leq n \). Thus, \( \overline{s}^{(n)} \) can be formed immediately.

Now, the local weighted covariance matrix

\[
C_{ij}^{(n)} = \sum_{s_{n'} \in U^{(n)}} |R(s_{n'} - \overline{s}^{(n)})|_i |R(s_{n'} - \overline{s}^{(n)})|_j
\]

is computed, where \( [\cdot]_i \) denotes the \( i \)-th component of the vector in brackets. As discussed in Ref. [8], the weight matrix \( R \) is chosen to be diagonal with \( R_{11} \) and \( R_{mm} \) large and all other diagonal entries \( R_{ii} = 1 \). Now determine the orthonormal eigenvectors \( c_q \) and eigenvalues of \( C^{(n)}_{ij} \) using standard matrix techniques. A \( Q \)-dimensional manifold is then
locally approximated by those $Q$ eigenvectors with the largest eigenvalues. The projected vector $\hat{s}_n$ is then given by:

$$\hat{s}_n = \tilde{S}^{(n)} + \mathbf{R}^{-1} \sum_{q=1}^Q c^q [\mathbf{c}^q \cdot \mathbf{R}(\mathbf{s}_n - \tilde{S}^{(n)})].$$

In order to translate $\{\mathbf{s}_n\}$ back into a scalar signal, we note that each scalar measurement $s_n$ appears as a component in $m$ embedding vectors, $\mathbf{s}_n, \ldots, \mathbf{s}_{n+m-1}$. The corrected scalar time series values $\hat{s}_n$ are thus obtained by averaging the corresponding components of $\hat{s}_n, \ldots, \hat{s}_{n+m-1}$.

For real time application, this means that the corrected value $\hat{s}_n$ cannot be available before $s_{n+m-1}$ has been measured and processed. Usually, however, this delay window is a very short time, at least compared to the duration of the recording, and the procedure can be regarded as effectively on-line as long as the computations necessary to obtain $\hat{s}_n$ can be carried out fast enough.

So far, we have turned the procedure into a causal filter by restricting neighbour search to points defined by measurements made in the past. By further limiting the number of neighbours searched for, we have sped up the formation of the local covariance matrices considerably. Still, as the algorithm stands, we have to solve an $(m \times m)$ eigenvalue problem for each point that is to be processed. A large fraction of this work can be avoided on the base of the assumption that the local linear structure changes smoothly over phase space. By making local linear approximations we have already assumed smoothness of the underlying manifold in the $C_1$ sense. In most physical systems, the additional assumption of $C_2$ smoothness is not less justified. We cannot, however expect that the vectors which span the principal directions vary slowly from point to point. The reason is that often some eigenvalues are nearly degenerate and change indices from point to point when they are ordered by their magnitude. We thus refrain from interpolating principal components between phase space points. Instead, we choose a length scale $h$ in phase space which is small enough such that the linear subspaces spanned by the local principal components can be regarded as effectively the same. Now we successively build up a data base of representative points for which the local points of tangency $\tilde{S}^{(n')}$, and the local principal directions $\mathbf{c}^q$, $q = 1, \ldots, Q$ have been determined already. For each new point $\mathbf{s}_n$ that is to be processed, we go through this collection of points to determine whether a representative is available closer than $h$. In that case, we use the stored tangent point and principal directions of the representative in order to perform the projections. If not, a neighbourhood is formed around $\mathbf{s}_n$ in which the eigenvalue problem is solved. The point $\mathbf{s}_n$ is then included in the list of representatives. By this procedure, all parts of the available space that is visited by observations are covered by representatives with a maximal distance $h$. If the dynamics and thus the geometry in phase space undergoes some change during the recording, new areas are visited which are automatically covered by new representatives. It is also possible to delete representatives which are older than a given time span. A realisation of the algorithm that implements all the modifications discussed in this paper is publicly available as part of the TISEAN software project \[\text{TISEAN}\].

As a first example, we show in Fig. 2 the result of applying the described procedure to a data set from an NMR laser experiment \[\text{NMR}\]. The same data has also been used in Ref. \[\text{Ref.}\]. The laser is periodically driven and once per driving cycle the envelope of the laser output is recorded. The resulting sampling rate is 91 Hz. At this rate, the modified nonlinear noise reduction scheme can be easily carried out in real time on a Pentium II processor at 200 MHz. Since further iterations can be carried out after the time corresponding to one embedding window without interfering with the previous steps, we could perform up to three iterations on a dual Pentium II workstation at 300 MHz in real time. The figure shows the result after two iterations. Projections from $m = 7$ down to $Q = 2$ dimensions were used, at least 100 neighbours were requested at $\epsilon = 200$ A/D units. The history was limited to 20000 samples, or 220 s. This fairly large data base is needed since the initial noise level is already small.
Figure 2: Result of nonlinear filtering of an NMR laser time series. An enlargement of about one quarter of the total linear extent of the attractor is shown.

Figure 3: Result of nonlinear filtering of an MCG time series. Note that the noise is not white. It contains for example contributions from non-cardiac muscle activity and fluctuations in the magnetic background field.

(less than 2% [19]) and small neighbourhoods are required to avoid curvature artefacts. The maximal distance of representative points was chosen to be $h = 120$ A/D units.

The actual acceleration resulting from the above modifications strongly depends on the situation and it is difficult to give general rules and benchmarks. Let us however study a realistic example in some detail to illustrate the main points. In electrophysiological research, it is quite attractive to augment the measurements of electric potentials with recordings of the magnetic field strength. The latter penetrate intervening tissue much more efficiently. Of particular interest are magneto-encephalographic (MEG) recordings which allow to access regions of the brain noninvasively which cannot be monitored electrically using surface electrodes. In cardiology, the magneto-cardiogram (MCG) provides additional information to the traditional electrocardiogram (ECG). A particular application is the noninvasive monitoring of the fetal heart which is otherwise complicated by shielding of the electric field by intervening tissue. A common problem with magnetic recordings, however, is that the fields are rather feeble and the measurements have to be carried out in a shielded room. Even then, noise remains a major challenge for this experimental technique. We will demonstrate in the
The result of nonlinear filtering of an MCG time series is shown in Fig. 4. Same data as Fig. 3 but in time delay representation (enlargement).

Table 1: Computation time for one iteration of the nonlinear noise reduction scheme, applied to 10 s of an MCG recording sampled at 1000 Hz. See text for details.

| method | CPU time |
|--------|----------|
| a) all neighbours | 165 s |
| b) box assisted neighbour search | 31 s |
| c) all neighbours in past | 82 s |
| d) $n - n' < 5$ s | 64 s |
| e) (d) and $U_{\text{max}} < 200$ | 15 s |
| f) (e) and reuse of representatives | 2 s |

We use an MCG recording of a normal human subject at rest. The data was kindly provided by Carsten Sternickel at the University of Bonn. The sampling rate was 1000 Hz, which is quite high. For the signal processing task, any sampling rate above about 200 Hz would be sufficient. (Below 200 Hz, the spike representing the depolarisation of the ventricle might not be resolved properly). In order to cover a significant fraction of one cardiac cycle by an embedding window, we chose an embedding with delay $\tau = 10$ ms in $m = 10$ dimensions. Neighbourhoods were formed with a radius of $\epsilon = 0.1$ (in the uncalibrated A/D units of the recording), about three times the noise level estimated by visual inspection. In Figs. 3 and 4, the result of two iterations of the noise reduction scheme is shown. We quote computation times for the processing of 10 s of MCG on a Pentium II processor at 300 MHz, determined on a PC running the Linux operating system. The timing results are summarised in Table 1. The data base of representatives for (f) was formed by assuming that the local linear subspaces are equivalent on length scales of the order of 0.06 A/D units.

Like in the case of the MCG time series demonstrated above, nonlinear phase space filtering can be successfully applied to ECG recordings [20]. The modifications of the algorithm described in the present paper allow to perform the filtering as a real time application on common personal computers without a noticeable impairment in signal quality compared to previous implementations. Since this application is similar to the MCG filtering described above, we do not need to state further details. Instead, let us discuss a related signal separation problem of practical relevance, the extraction of the fetal ECG from abdominal recordings during pregnancy [21, 22]. In order to obtain the strongest possible signal of the...
fetal cardiac activity, electrodes are usually placed on the maternal abdomen. The upper trace of Fig. 5 contains such an abdominal recording. (In this particular case one abdominal and one cervical electrode has been used; the data were kindly provided by J. F. Hofmeister, Denver [23].) Due to the large size of the maternal heart in comparison to the fetal one, the maternal ECG represents the dominant signal component. Despite a strong noise component, the spike-like ventricular complexes of the fetal signal are present. The noise floor is due to action potentials of the muscular tissue surrounding the electrodes and other measurement noise. (Note the small overall amplitude of the signal.) The nonlinear filter described in this paper can be used to extract the fetal signal in a two-step procedure: in the first step the maternal signal is cleaned up by considering the noise level to be of the magnitude of the fetal spikes. This identifies the noise and the fetal component together as a contamination of the maternal waveform. The difference between the abdominal recording and the clean maternal component then yields a noisy fetal signal. The fetal ECG can be separated from the noise in a second nonlinear noise reduction step. The result of this procedure is shown in Fig. 5. The abdominal recording (upper trace) is processed in two steps to yield a fairly clean fetal ECG (lower trace). The algorithm described in this paper allows the extraction of the fetal ECG as a real time application on a Laptop PC (Pentium processor at 133 MHz, Linux operating system) for a sampling rate of 250 Hz.

In conclusion, we have demonstrated that by certain modifications to the algorithms described in the literature, nonlinear projective noise reduction can be turned into a signal processing tool that can in many situations run in real time in a data stream. Necessary ingredients are (1) the formulation of the algorithm as a causal filter, relying only on information that is available at recording time, (2) a general speed-up of the procedure by restricting neighbour search to the immediate past (at the expense of peak noise reduction performance), and (3) further speed-up by using a data base of previously processed phase space points. By putting a time restriction on the data base for the formation of local subspaces, processing of non-stationary signals with slowly drifting parameters becomes also possible.

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