Functional flows in QED and the modified Ward–Takahashi identity

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Abstract

In the functional renormalisation group approach to gauge theory, the Ward–Takahashi identity is modified due to the presence of an infrared cutoff term. It take the most accessible form for the Wilsonian effective action. In the present work we solve these identities, partially, for the Wilsonian effective action of QED. In particular, we compute the longitudinal part of the photon two point vertex function as a momentum-dependent function in the presence of the cutoff $k$. The resultant Wilsonian effective action carries form factors that originate from the modified Ward–Takahashi identity. We show how this result carries over to the one-particle-irreducible effective action.

Keywords: functional renormalisation group, Ward–Takahashi identity, QED, Wilsonian effective action, 1PI effective action

1. Introduction

The functional renormalisation group (FRG) approach,\textsuperscript{[1–5]} has been successfully applied to various field theoretical problems as a robust non-perturbative method. Applications range from quantum gravity, high energy physics and QCD to problems in condensed matter physics, and non-equilibrium physics.

The FRG approach is based on a flow equation for a generating functional of the theory at hand, and hence can be formulated in terms of a coupled set of integro-differential equations for correlation functions. The full, field-dependent propagator takes a pivotal rôle in these formulations. In gauge theories, such a set-up requires a gauge fixing, and gauge invariance is carried by the Slavnov–Taylor (STI) or Ward–Takahashi identity (WTI), henceforth both summarised as WTI. In the presence of an infrared momentum cutoff $k$ the WTI
survives as modified Ward–Takahashi identity (mWTI), [6–18]. The standard WTI is recovered in the limit of vanishing infrared cutoff.

Still, even in the presence of the regulator the mWTI can be formulated as standard WTI with a modified generator of gauge transformations, [13, 16–18]. For more details on the properties of such a modified generator see [19]. This concise form of the mWTI as a symmetry identity is very useful for the construction of closed solutions of the mWTI. In the present work we use it for the Wilsonian effective action $S_k$, the generating functional for amputated connected correlation functions.

In summary, the FRG approach to gauge theories is based on master equations for the scale dependence of the generating functionals, the functional flow equations, as well as master equations for the symmetries, the mWTI. In terms of the Wilsonian effective action, $S_k$, both master equations are finite sums of linear and bilinear forms in $S_k^{(n)}$, reading

$$D[\phi] S_k^{(n)}[\phi] = \sum c_n S_k^{(n)}[\phi] + \sum c_{nn} S_k^{(n)}[\phi] S_k^{(m)}[\phi],$$

with operator coefficients $c_n$ and $c_{nn}$. For the flow equation of $S_k$, [4, 20, 21], $D[\phi]$ stands for the scale derivative $D[\phi] = k \partial_k$. In the case of the mWTI, $D[\phi]$ stands for the linear generator of symmetry transformations.

The form of (1) entails that an expansion of the master equations in terms of fields leads to relatively simple hierarchies of multi-linear equations for the expansion coefficients of the Wilsonian effective action $S_k^{(n)}$. Equation (1) is even amiable to closed partial solutions in terms of general field dependencies. This holds in particular for the formulation of the mWTI as an unbroken symmetry as put forward in [13, 16–18].

In turn, the related master equation [5] for the 1PI effective action $\Gamma[\Phi]$, the Wetterich equation, has the form

$$D[\phi] \Gamma_k[\Phi] = \sum c_n \Gamma_k^{(n)}[\Phi] G[\Phi],$$

with

$$G[\Phi] = \frac{1}{\Gamma_k^{(2)}[\Phi] + R_k},$$

see also [22, 23]. Note that the propagator $G$ in (2) and (3) is $k$-dependent. We drop any reference to this for the sake of a better readability of the equations. The 1PI master equation always involves the propagator $G[\Phi]$. The propagator relates to $S_k^{(2)} - (S_k^{(1)})^2$ which underlines the similarity of both sets of master equations. For more details on progress in gauge theories including gravity we refer the reader to the reviews [15, 18, 24–32].

It is the inverse in (2) which makes solutions to symmetry identities in a closed form less easily accessible as for the Wilsonian effective action. Moreover, already the derivation of symmetry identities and algebraic manipulations are structurally simpler with (1).

On the other hand, the numerical stability of solutions of the flow equations of the type (2) is qualitatively better. Again it is the inverse in (2) that triggers this difference. For example, for large momenta the propagator $G$ decays with the dispersion of the theory at hand. This is trivially achieved by the form $1/(\Gamma_k^{(2)} + R_k)$ for the 1PI formulation, while it requires non-trivial cancellations between the $S_k^{(2)}$ and $(S_k^{(1)})^2$ in the Wilsonian effective action framework.

The above observations on the properties of (1) and (2) suggest a combined use of both equations within approximations to the effective actions $S_k$ and $\Gamma_k$:

(i) One partially solves the algebraically tractable mWTI based on (1).

(ii) The results are translated from the Wilsonian effective action to the 1PI effective action via the Legendre transformation connecting both actions.
The flows for the correlation functions are solved in terms of the flow equation for the 1PI effective action.

On the level of the correlation functions or vertices the above strategy entails that we deduce algebraic relations between \( G_F^{(n)}(k_0 = 0) \) and \( f = S^{(n)}(k_0 = 0) \) via the Legendre transformation. This allows us to translate the symmetry constraints for \( S^{(n)}(k_0 = 0) \) to similar ones for \( G_F^{(n)}(k_0 = 0) \). Finally, we use these relations in the flow equation (2). In summary this leads to symmetry-consistent approximations to the flows of the interacting parts of the 1PI correlation functions \( G_F^{(n)}(k_0 = 0) \).

The main purpose of the present paper is to make progress on the above programme in terms of (i) and (ii) in QED: first we provide a solution for the mWTI for QED with a massless fermion for the Wilsonian effective action. This result is then translated to the 1PI effective action. Part (iii) of the programme, the solution of the symmetry-enhanced flow equations and its analysis will be reported in a separate paper.

This paper is organised as follows. In section 2, the relation between the Wilson and 1PI effective actions as well as their flow equations are reviewed. We also present a brief derivation of the mWTI for QED. In section 3 we describe the truncation scheme used in the current work. In section 4 we solve the mWTI for the truncated Wilson action and map the result onto the 1PI effective action. A summary and discussion are given in section 5. Several appendices contain the technical details.

### 2. Functional flows and the mWTI

In this section we give a brief derivation of the flows for the Wilson and one-particle irreducible (1PI) effective actions, \( S_I \) and \( I_F \) respectively. These are the generating functionals of amputated connected and 1PI correlation functions. If applied to gauge theories, the regularisation procedure leads to modified symmetry identities, that are also introduced here.

The derivations in the present section are kept general. In the present work, however, we are predominantly interested in QED, so we shall use it as an explicit example for the general relations derived below. Applications of the FRG to Abelian gauge theories range from QED to the Abelian Higgs models as effective theories for high \( T_c \) superconductors, see e.g. \([9, 16, 33–41]\). Its classical gauge-fixed action is given by

\[
S_{cl}[\varphi] = \frac{1}{4} \int \frac{d^4 \varphi}{(2 \pi)^4} \left[ F_{\mu \nu}^2(a) + \int \bar{\psi}(i \gamma^\mu \mathcal{B} (a) + i m_0) \psi + \frac{1}{2 \kappa_0} \int (\partial_\mu a_\mu)^2 \right] \tag{4}
\]

with

\[
\mathcal{B}_\mu(a) = \gamma_\mu D_\mu(a), \quad D_\mu(a) = \partial_\mu + i e_0 a_\mu, \tag{5}
\]

and \( \left\{ \gamma_\mu, \gamma_\nu \right\} = 2 \delta_{\mu \nu} \).

The second line in (4) constitutes the gauge fixing sector with a general covariant gauge fixing and the trivial ghost term in QED. The classical action (4) depends on the bare couplings (or parameters) \( e_0, m_0, \kappa_0 \). The field \( \varphi \) comprises all fields

\[
\varphi = (a_\mu, \psi, \overline{\psi}, c, \bar{c}), \tag{6}
\]

including the decoupled ghost fields. The latter only plays a rôle for the BRST transformation introduced later.
2.1. Wilson and 1PI effective actions and their flows

We consider a theory in four-dimensional Euclidean space, which has a gauge symmetry written as a BRST symmetry. It is described by a gauge-fixed action \( \mathcal{S}[\varphi] \), a functional of fields \( \varphi^A \) which collectively represent gauge and matter fields as well as ghosts and anti-ghosts. The index \( A \) denotes the Lorentz indices of gauge fields, the spinor indices of the fermions, and other indices distinguishing different types of generic fields. The Grassmann parity for \( \varphi^A \) is expressed as \( \epsilon(\varphi^A) = \epsilon^A \epsilon_A = 0 \) (1) if the field \( \varphi^A \) is Grassmann even (odd).

In FRG approach, we introduce an IR cutoff \( k \) through positive functions that behave as
\[
K^A(p) \rightarrow \begin{cases} 
1 & (p^2 < k^2), \\
0 & (p^2 > k^2). 
\end{cases}
\]
The functions go to zero sufficiently rapidly as \( p^2 \to \infty \). For simplicity, we write the functions as \( K^A(p) \) in the rest of the paper. The action \( \mathcal{S} \) defined at some UV scale is given as a sum the kinetic and interaction terms
\[
\mathcal{S}[\varphi] = \frac{1}{2} \varphi \cdot D \cdot \varphi + \mathcal{S}_I[\varphi],
\]
where we have used a condensed matrix notation in momentum space. For example, the kinetic term in (8) has the explicit form
\[
\varphi \cdot D \cdot \varphi = \int \frac{d^4p}{(2\pi)^4} \varphi^A(-p)D_{AB}(p)\varphi^B(p).
\]

Due to the presence of the regulator functions (7) the Wilsonian effective action \( S_k \) is the scale-dependent generating functional of amputated connected correlation functions. The scale-dependent generating functional for 1PI correlation functions, \( G_k \) is obtained via a Legendre transform from \( S_k \). The scale-dependence of \( S_k \), \( G_k \) is encoded in flow equations, i.e., the Polchinski equation and Wetterich equation respectively.

In the main text of the present work we resort to normalised fields and couplings in the Wilsonian effective action \( S_k \),
\[
\hat{\phi} = Z_{\phi}^{1/2} \phi, \\
g = Z_g g_0,
\]
where \( g_0 \) are \( k \)-independent couplings defined at some UV scale. For QED with the classical action (4) we have the parameters \( \bar{g}_0 = (\mu_0, m_0, \xi_0) \) and the normalised couplings read
\[
e = Z_e e_0, \\
m = Z_m m_0, \\
\xi = Z_\xi \xi_0.
\]
The rescaling (10) facilitates the access to scaling properties, as the scale-dependent effective actions \( S_k \) and \( G_k \) have the same renormalisation group equation as the full effective actions at \( k = 0 \), see [15], with
\[
\frac{d}{d\mu} \mathcal{G} = 0, \quad \text{where} \quad \mathcal{G} = S_k, G_k, S_0, G_0.
\]
Moreover, the related expansion coefficients in powers of \( \hat{\phi} \), the amputated connected correlation functions or their 1PI parts, are renormalisation group invariant.

For the sake of accessibility of the current work and due to the minor modifications of the derivation in the presence of the rescaled fields we recall the derivation of the flow of the Wilsonian effective action in appendix A. Since the Wilson action is a functional of
(Z^A)^{1/2} \phi^A$, we may extract contributions of the anomalous dimensions from $S_k(Z^{1/2}\phi)$. The final equation for $S_k$ in terms of the rescaled fields $\phi$ reads, see (A10)

$$\partial_t S_k|_{\phi} = - \frac{\partial}{\partial \phi^A} \left[ \left( \partial_t \log K \right)^A - \frac{1}{2} \eta_A \right] \frac{\partial^2 S_k}{\partial \phi^A} + (-)^{\frac{1}{2}} \frac{1}{2} \left[ \partial_t K - \eta K (1 - K) \right] D^{-1} [AB] \times \frac{\partial^2 S_k}{\partial \phi^B \partial \phi^A} - \frac{\partial^2 S_k}{\partial \phi^B \partial \phi^A},$$

with the anomalous dimensions

$$\eta_A = - \partial_t \log Z^A,$$

for $\phi^A$. For the flow equation of the interaction part $S_{I,k}$, see (A11) in the appendix A.

The Wilson action $S_k$ is the generating functional of the connected (amputated) cutoff correlation functions. Already its 1PI part, the scale-dependent effective action, $\Gamma_k$, carries all the information about the correlation functions of the theory at hand. It is a part of the full Legendre effective action, $\tilde{\Gamma}_k$, which is obtained via the Legendre transformation of the Wilson effective action:

$$\tilde{\Gamma}_k[\phi] = \Gamma_k[\phi] + \frac{1}{2} \phi \cdot R_k \cdot \phi,$$

It is the standard scale-dependent effective action, $\Gamma_k$, that is used in most applications. The regulator $R_k$ relates to $K$ with

$$R_k = \frac{K}{1 - K} D,$$

A particularly concise iterative relation exists between the interaction part of $\tilde{\Gamma}_k$,

$$\tilde{\Gamma}_{I,k}[\phi] = \Gamma_{I,k}[\phi] - \frac{1}{2} \phi \cdot D \cdot \phi,$$

and the interaction part of $S_k$,

$$\tilde{\Gamma}_{I,k}[\phi] = S_{I,k}[\phi] - \frac{1}{2} (\phi - \phi) \cdot (G^{(0)})^{-1} \cdot (\phi - \phi),$$

where $(G^{(0)})^{AB} = ((1 - K)D^{-1})^{AB}$ are the high momentum propagators. This relation is most conveniently written in terms of the two-point functions of the Wilsonian effective action and the 1PI effective action. They comprise the propagators of the theory at hand. These two-point functions are the basic building blocks of the functional flow equations and read

$$(S^{(2)}_{I,k} \phi)]_{AB} = \frac{\partial^2 S_{I,k}}{\partial \phi^A \partial \phi^B}, \quad (\Gamma^{(2)}_{I,k} \phi)_{AB} = \frac{\partial^2 \Gamma_{I,k}}{\partial \phi^A \partial \phi^B},$$

where $\partial^l$ and $\partial^r$ stand for left- and right-derivatives respectively. Then, (18) leads us to the relation

$$S^{(2)}_{I,k}[\tilde{\phi}] = (G^{(0)})^{1-1} - (G^{(0)})^{-1} \cdot G[\tilde{\phi}] \cdot (G^{(0)})^{-1},$$

with the full propagator

$$G[\tilde{\phi}] = (\Gamma^{(2)}_{I,k}[\tilde{\phi}] + R_k)^{-1},$$

see also (3). Note that we recover $G^{(0)}$ if we drop the contributions from the interaction part in (21). Using (20) and the relation between $\tilde{\phi}$ and $\phi^A$,.
the flow equation for $S_k$ is easily transformed into one for $\Gamma_k$.

\[
\left( \partial_t - \frac{1}{2} \gamma_4, \frac{\delta}{\delta \Phi_4} \right) \Gamma_k \left[ \Phi \right] = \frac{1}{2} \operatorname{Tr} G \left[ \Phi \right] \cdot \left( \partial_t - \eta \right) \cdot R_k.
\]

Equation (23) shows the relation between derivatives of $\Gamma_k$ and the inverse of the second field derivative of $\Gamma_k$ already discussed in the introduction. This structure is also present in the mWTI, and suggests a solution of the mWTI in terms of $S_k$, and its insertion into $G_k$.

Finally we remark that the Wilson action $S_{Ik}$ can be iteratively expanded in terms of the Legendre action $G_{Ik}$, their field derivatives and the cutoff (high energy) propagators $G_0$:

\[
S_{Ik} [\hat{\phi}] = \Gamma_{Ik} [\hat{\phi}] - G^{(0)} \cdot S^{(1)}_{Ik} + \frac{1}{2} S^{(1)}_{Ik} \cdot G^{(0)} \cdot S^{(1)}_{Ik} + \frac{1}{2} \Gamma^{(1)}_{Ik} \cdot G^{(0)} \cdot \Gamma^{(1)}_{Ik} + \cdots.
\]

The tree expansion in (24) will be used to construct $S_{Ik}$ from $\Gamma_{Ik}$ in QED.

2.2. Derivation of the mWTI

From now on we concentrate on QED. In the present section we briefly recapitulate the derivation of the mWTI. The BRST transformations $\delta \varphi^A$ take the form

\[
\delta \varphi^A = (R_1)^A_B \varphi^B + (R_2)^A_B \varphi^B c,
\]

where $(R_1)^A_B$ and $(R_2)^A_B$ are field independent functions and $c$ is the ghost field. The classical BRST transformations of the photon and the fermion fields are described by the first and the second terms in (25), respectively. Note that the transformations in (25) are linear in the field except for the presence of a free ghost field denoted as $c$.

Even for linear gauge symmetries, the BRST transformations for the IR fields $\delta \Phi^A$ become nonlinear: the BRST transformations for the IR fields get modified due to interactions generated by the integration over the higher momentum modes. The free ghost and the anti-ghost field do not contribute to the modifications. The full BRST transformation of the IR field $\hat{\phi}$ can be rewritten similarly to the classical transformation (25) in terms of the mean field $\Phi$ in (22):

\[
\delta \hat{\phi} = K \left( R_1^A_B \hat{\Phi}^B \right) + K \left( R_2^A_B \hat{\Phi}^B c \right).
\]

This has been detailed in [18] in terms of a composite field language; see appendix B for a few details, including the changes of the coefficients from (25) to (26) owing to the wave functions renormalisation. Using (B1) and (26), we obtain the mWTI

\[
\Sigma_k [\hat{\phi}] = S^{(1)}_{Ik} \cdot \delta \hat{\phi} + \operatorname{Tr} K \cdot R_1^A_B \cdot G^{(0)}_{Ik} \cdot S^{(2)}_{Ik} \cdot c = 0.
\]

2.3. The mWTI for the Wilson action of QED

So far we have briefly recalled the mWTI and its derivation. Now we turn to their application to QED in a given approximation. To this end, we first fix the coefficient functions $R_1^A_B$ and
\( R^{A}_{BC} \) from the classical BRST transformations for the UV fields \( \delta \varphi^A = R^{B}_{A} \varphi^B + R^{B}_{BC} \varphi^B \varphi^C \):

\[
\begin{align*}
\delta_{\xi_{C}} a_{\mu}(p) &= - i p_{\mu} c(p), \\
\delta_{\xi_{C}} \psi(p) &= - i e_0 \int_q \psi(q) c(p - q), \\
\delta_{\xi_{C}} \tilde{\psi}(-p) &= i e_0 \int_q \tilde{\psi}(-q) c(q - p), \\
\delta_{\xi_{C}} \tilde{c}(p) &= \xi_0^{-1} p_{\mu} a_{\mu}(p),
\end{align*}
\]

(28)

where the bare gauge coupling \( e_0 \) and gauge parameter \( \xi_0 \) are \( k \) independent constants. Then as shown in appendix B.2, the quantum BRST transformations derived from (26) are given for the renormalised fields

\[
\tilde{\psi} = (a, \psi, \tilde{\psi}, c, \tilde{c}).
\]
(29)

The ghost and anti-ghost are free fields, and have no genuine wave function renormalisation. This entails the natural choice \( Z_{c} = Z_{\tilde{c}} = 1 \). For convenience we choose

\[
Z_{c} = 1/Z_{c}, \quad Z_{\tilde{c}} = Z_{\tilde{c}}, \quad \text{with} \quad e = Z_{c} e_0,
\]
(30)

With this rescaling we are lead to

\[
\begin{align*}
\delta a_{\mu}(p) &= - K(p) Z_{c} Z_{\tilde{c}}^{1/2} i p_{\mu} C(p), \\
\delta \psi(p) &= - K(p) i e \int_q \psi(q) C(p - q), \\
\delta \tilde{\psi}(-p) &= K(p) i e \int_q \tilde{\psi}(-q) C(q - p), \\
\delta \tilde{c}(p) &= K(p) Z_{c} Z_{\tilde{c}}^{1/2} \xi^{-1} p_{\mu} A_{\mu}(p),
\end{align*}
\]
(31)

in terms of the composite mean fields

\[
\tilde{\Phi} = (A_{\mu}, \Psi, \tilde{\Psi}, C, \tilde{C}),
\]
(32)

defined in (22). In (31) the wave function renormalisations are absorbed in the fields. The only remnant is the product \( Z_{c} Z_{\tilde{c}}^{1/2} \) of the coupling renormalisation function and the wave function renormalisation of the photon. If this product is set to unity we are left with the classical transformation except for the occurrence of the composite mean fields on the right-hand side. Hence, the rescaling (30) make already apparent the standard relations. The composite mean fields in QED follow from their general definition in (22) as

\[
\begin{align*}
A_{\mu}(p) &= K^{-1}(p) a_{\mu}(p) - (G_G^{(0)})_{\mu\nu} \frac{\partial S_{G}}{\partial a_{\nu}(-p)}, \\
\Psi(p) &= K^{-1}(p) \psi(p) - (G_{F}^{(0)})_{\mu\nu} \frac{\partial S_{F}}{\partial \psi(-p)}, \\
\Psi(-p) &= K^{-1}(p) \tilde{\psi}(-p) - \frac{\partial S_{F}}{\partial \tilde{\psi}(p)} G_{F}^{(0)}.
\end{align*}
\]
(33)

Note that \( C(p) = c(p), \tilde{C}(p) = \tilde{c}(p) \) with the same wave function renormalisation (30), since \( c \) and \( \tilde{c} \) are free fields. From (27), we obtain the WT operator for QED
\[
\Sigma_k[\phi] = Z_x Z_3^{1/2} \int_\mathcal{P} \left[ \frac{\partial^2 S_k}{\partial \phi_k(p)^2} - \frac{\partial S_k}{\partial \phi_k(p)} \right]
- \frac{1}{\xi} \int_\mathcal{P} \left[ \frac{\partial S_k}{\partial \psi_\lambda(q)} \right] \left[ \frac{\partial S_k}{\partial \psi_\lambda(-q)} \right] \left[ \frac{\partial S_k}{\partial \psi_\lambda(-p)} \right] \left[ \frac{\partial S_k}{\partial \psi_\lambda(-r)} \right] \left( q-p \right), \tag{34}
\]

where
\[
U(-q, p) = \left[ K(q) G^{(0)}_F(p) - K(p) G^{(0)}_F(q) \right]. \tag{35}
\]

The bare gauge parameter \( \xi_0 = \xi / Z_3 \) and the gauge coupling \( e_0 = e / Z_e \) are \( k \)-independent and one can show that \( \Sigma_k[\phi] \) in (34) itself is a composite operator as defined in [15, 18], and satisfies the flow equation for composite operators discussed in appendix B.1.

### 3. The truncated Wilson action

In this section, we construct the interaction part of the Wilson action, \( S_{I,k} \), within a suitable truncation. Since \( S_{I,k} \) is related to the generating functional of connected correlation function obtained by integrating over high energy modes of the original theory, it can be constructed in terms of its 1PI part, \( \Gamma_{I,k} \). The latter is quite useful for the study of the RG flow. Therefore, we first provide a truncated form of \( \Gamma_{I,k} \) and then construct \( S_{I,k} \) via the Legendre transformation or equivalently the tree expansion (24).

In addition to corrections to the primetricallydivergent correlation functions, the two-point functions and the gauge coupling, we also introduce the four-Fermi couplings. These couplings are the lowest order of higher dimensional matter interactions, that are generated from the primetrically divergent correlation functions within one RG-step. Here we need them in order to close our approximation scheme. In terms of the interaction part \( \Gamma_{I,k} \) of the 1PI effective action this truncation corresponds to corrections to the electron and photon two-point functions, the self-energy, \( \hbar^{(\psi)} \), and the vacuum polarisation, \( \hbar^{(aa)} \), respectively, the electron-photon vertex and its quantum corrections, \( \hbar^{(\psi\phi)} \), and the four-electron scattering vertex \( \hbar^{(\psi\psi\psi\psi)} \). Schematically this leads to

\[
\Gamma_{I,k}[\Phi] = \frac{1}{2} \left[ \hbar^{(aa)} . A^2 + \hbar^{(\psi\psi)} . \bar{\psi} \psi \right]
+ \hbar^{(\psi\phi)} . \bar{\psi} \gamma^\mu \psi + \hbar^{(\psi\psi\psi\psi)} . \bar{\psi} \psi \bar{\psi} \psi, \tag{36}
\]

where the powers in the fields in (36) stand for the tensor products. The first line comprises the corrections to the kinetic terms of photon and electrons, and the second term comprises the interaction terms. The full expression including all momentum dependencies and Lorentz indices is given in appendix C. Correspondingly, the schematic expression for the interaction part of the Wilsonian effective action reads
\[ S_{[k]} \approx \frac{1}{2} (\bar{G}_G^{-1} \cdot \bar{h}^{(aa)}) \cdot (\bar{G}_G a \bar{G}_G a) \]
\[ + (\bar{G}_F^{-1} \cdot \bar{h}^{(\psi\psi)}) \cdot (\bar{G}_F \psi \bar{G}_F \psi) \]
\[ + \bar{h}^{(\psi\psi)} \cdot (\bar{G}_G a \bar{G}_F \psi \bar{G}_F \psi) \]
\[ + \bar{h}^{(\psi\psi)} \cdot (\bar{G}_F \psi \bar{G}_F \psi \bar{G}_F \psi), \]
(37)

and the explicit expression with momentum dependence and Lorentz indices can be found in appendix C, (C3). The form (37) makes apparent that the Wilsonian effective action generates amputated connected correlation functions. The explicit \( \bar{G} \)-factors are arranged such that the 1PI-parts of the vertex coefficients \( \bar{h} \) agree with the 1PI coefficient functions \( h \) in (36). Hence, for connected correlation functions that only contain 1PI parts, \( h \) and \( \bar{h} \) agree. This holds true for two and three-point functions, see also the relations (40). In turn, for higher correlation functions, \( \bar{h} \) can be easily expanded in \( h \) according to the diagrammatic expressions of connected correlation functions in terms of 1PI correlations. Due to the definition of the fields the propagators in (37) are normalised with the inverse bare propagators, leading to

\[ \bar{G}(p) \equiv G(p)(G^{(0)}(p))^{-1}, \]

where the \( k \)-dependence of \( \bar{G} \) is implicit. For photons and electrons this reads more explicitly

\[ (\bar{G}_G)_{\mu\nu}(p) = \left[ \frac{1}{1 + h^{(aa)}G^{(0)}_{\mu\nu}} \right](p), \]

\[ (\bar{G}_F)_{\alpha\beta}(p) = \left[ \frac{1}{1 + G^{(0)}_{\alpha\beta}h^{(\psi\psi)}} \right](p). \]

(39)

As discussed above we have for the two point functions

\[ \bar{h}^{(aa)}_{\mu\nu}(p) = h^{(aa)}_{\mu\nu}(p), \quad \bar{h}^{(\psi\psi)}_{\alpha\beta}(p) = h^{(\psi\psi)}_{\alpha\beta}(p), \]

(40a)

and for the three point function

\[ \bar{h}^{(\psi\psi)}_{\alpha\beta\mu}(p_1, p_2) = h^{(\psi\psi)}_{\alpha\beta\mu}(p_1, p_2) = h_{\epsilon}(p_1, p_2)^{\gamma_{\mu,\alpha\beta}}, \]

(40b)

For the four-Fermi interaction we have a genuine connected part which is not 1PI:

\[ \bar{h}^{(\psi\psi\psi\psi)}_{\alpha\beta\gamma\delta}(p_1, p_2, p_3) = h^{(\psi\psi\psi\psi)}_{\alpha\beta\gamma\delta}(p_1, p_2, p_3) \]
\[ - \frac{1}{2} h^{(\psi\psi)}_{\beta\gamma\delta}(p_1, p_2) h^{(\psi\psi)}_{\alpha\beta\mu}(p_3)(G_G)_{\mu\nu}(p_1 + p_2). \]

(40c)

The four-Fermi term \( \bar{h}^{(\psi\psi\psi\psi)} \) contains a one-photon exchange contribution with two three-point vertices as given in the last term in (40), in addition to the one proportional to four-point function \( h^{(\psi\psi\psi\psi)} \).

In this paper we concentrate on massless fermions, and hence we have chiral symmetry. Furthermore, we only take into account the classical tensor structure of the photon–electron vertex, \( \gamma_{\mu} \), i.e.

\[ h^{(\psi\psi)}_{\alpha\beta}(p) = \sigma(p) \rho, \]
\[ h^{(\psi\psi)}_{\alpha\beta\gamma}(p_1, p_2) = - e h_{\epsilon}(p_1, p_2)^{\gamma_{\mu,\alpha\beta}}, \]

(41)
where σ(p) and h_v(p_1, p_2) are form factors. In (40c), we have not specified the form of the four-Fermi interaction. In the present work we take the standard chiral form which already follows from one RG-step. It can be written as a combination of two terms. The first one reads

\[
\frac{1}{2k^2} \int_{p_1, p_2} \{ h_5(s, t, u) \{ (\bar{\Psi}(p_1)\Psi(p_2))((\bar{\Psi}(p_3)\Psi(p_4))
- (\bar{\Psi}(p_1)\gamma_5\Psi(p_2))((\bar{\Psi}(p_3)\gamma_5\Psi(p_4))
+ h_V(s, t, u) \{ (\bar{\Psi}(p_1)\gamma_{\mu}\Psi(p_2))((\bar{\Psi}(p_3)\gamma_{\mu}\Psi(p_4))
+ (\bar{\Psi}(p_1)\gamma_5\gamma_{\mu}\Psi(p_2))((\bar{\Psi}(p_3)\gamma_5\gamma_{\mu}\Psi(p_4)) \} \}. \tag{42}
\]

while the second one follows as

\[
\frac{1}{2k^4} \int_{p_1, p_2} h_V(s, t, u)(p_1 + p_3)(p_2 + p_3)\nu \times \{ (\bar{\Psi}(p_1)\gamma_{\mu}\Psi(p_2))((\bar{\Psi}(p_3)\gamma_{\mu}\Psi(p_4))
+ (\bar{\Psi}(p_1)\gamma_5\gamma_{\mu}\Psi(p_2))((\bar{\Psi}(p_3)\gamma_5\gamma_{\mu}\Psi(p_4)) \}. \tag{43}
\]

Here we have included the chiral invariant four-Fermi interactions with form factors which are functions of \( s = (p_1 + p_2)^2, t = (p_1 + p_3)^2, \) and \( u = (p_1 + p_3)^2. \) As four-Fermi terms, we introduced a derivative vector coupling other than commonly used scalar and vector couplings. We will see shortly how these higher dimensional operators with the form factors affect relations among lower dimensional operators via non-trivial loop contributions in the mWTI. From (43) with the use of the Legendre transformation, we obtain the interaction part of the truncated Wilson action which is given in appendix C.

4. Constraints from the mWTI

Having constructed the Wilson action (C3), we now derive the relations for the couplings, resulting from the mWTI, \( \Sigma_k = 0. \) When the WT operator \( \Sigma_k \) is expanded as polynomials of the fields, \( \Sigma_k = 0 \) leads to a number of coupled relations for the couplings and form factors in the Wilson action. We substitute \( \Sigma_k \) given by (C3) into (34) and find coefficients of operators \( a_{\mu}c \) and \( \bar{\psi}\psi/c \) in \( \Sigma_k = 0. \) In this manner, we obtain two WT relations (D1) and (D3). Furthermore, we assume locality of the fermionic bilinear term and the gauge interaction

\[
\sigma(p) = 0, \quad h_v(p, q) = 1. \tag{44}
\]

The first WT relation out of \( a_{\mu}c \) terms is given by

\[
p_\mu h_{\mu\nu}^{(a)}(p) = p_\nu \mathcal{L}(p) = \frac{e^2}{Z_\mu Z_\nu^{1/2}} \int_q \text{Tr} [U(p - q, q)_{\nu}], \tag{45}
\]

where \( \mathcal{L} \) is the longitudinal part of \( h_{\mu\nu}^{(a)} \)

\[
h_{\mu\nu}^{(a)}(p) = P_T^\mu T(p) + P_L^\mu \mathcal{L}(p), \tag{46}
\]

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We next consider the $\psi\psi/c$ terms and obtain the second WT relation as shown in appendix C

\begin{equation}
(e - e Z_e Z_3^{1/2}) (p - q) - 2 e \int \left[ \frac{1}{k^2} \left\{ \left(\hbar S - 2h\nu\right) (l^2, (p + q + l)^2, (p - q)^2) \right\} 
- 2h\nu ((p - q)^2, (p + q + l)^2, l^2) \} + e^2 (1 - K(l)) T (l^2) + \frac{1}{k^2} \{2(p - q)^2
\times h\nu (l^2, (p + q + l)^2, k^2) + l^2 h\nu ((p - q)^2, (p + q + l)^2, l^2) \} \right] 
\times U(-q - l, p + l) 
- e \int \left[ \frac{2}{k^2} h\nu ((p - q)^2, (p + q + l)^2, l^2) + e^2 \frac{(1 - K(l))}{l^2} \{ T (l^2) - \xi L (l^2) \} \right] \right] 
\times IU (-q - l, p + l) f = 0. \tag{47}
\end{equation}

In writing (47), we used the expressions for the photon full propagator as well as $(\hat{G}_G)_{\mu\nu}$ defined in (38)

\begin{equation}
(\hat{G}_G)_{\mu\nu} (p) = (1 - K(p)) (T(p) P_{\mu\nu}^T + \xi L(p) P_{\mu\nu}^L), \tag{48}
\end{equation}

\begin{equation}
(\hat{G}_G)_{\mu\nu} (p) = p^2 (T(p) P_{\mu\nu}^T + L(p) P_{\mu\nu}^L). \tag{49}
\end{equation}

Note that (47) consists of terms with and without the integration over the momentum $l$. The first term without the integration is the tree term and the rest is one-loop terms. It is important to realise that we call a term as the tree or loop term in reference to the Wilson action $S_k$ that is obtained after integrating out all the modes with their momenta above the cutoff. Even a tree term contains the contributions of the higher momentum modes.

The first term in (47) is proportional to $(p - q)$ with a momentum independent factor. In contrast to this tree term, the rest consist of loop integrals which lead to some functions of momenta $p$ and $q$. It is reasonable therefore to require that these two different kinds of contributions vanish separately.

Vanishing of the first term gives

\begin{equation}
Z_3 Z_3^{1/2} = 1, \tag{50}
\end{equation}

where the constant $Z_3$ for finite renormalisation of the gauge coupling may be defined as $e = e(k) = Z_3 e(\mu) = Z_3 e_0$. This corresponds exactly to the well-known identity

\begin{equation}
Z_1 \equiv Z_3 Z_3^{1/2} Z_e = Z_2. \tag{51}
\end{equation}

Therefore, the standard relation $Z_1 = Z_2$ which ensures the charge universality remains unchanged in our realisation of gauge symmetry in QED.

On the other hand, there are two independent integrals containing $U$ and $fUf$. If we demand the integrands to vanish we arrive at two non-trivial constraints:

\begin{equation}
\frac{1}{k^2} \left\{ \left(\hbar S - 2h\nu\right) (l^2, (p + q + l)^2, (p - q)^2) \right\} 
- 2h\nu ((p - q)^2, (p + q + l)^2, l^2) \} - e^2 (1 - K(p - q)) \{ T ((p - q)^2) - \xi L((p - q)^2) \} - e^2 \frac{(1 - K(l))}{l^2} \{ T (l^2) - \xi L (l^2) \}, \tag{52}
\end{equation}
where (50), a result from the 2nd WT relation, is used. These imply that the form factors $h_S$ and $h_V$ are functions of two variables, $h_S(s, t, u) = h_S(s, u)$ and $h_V(s, t, u) = h_V(s, u)$, while $h_V$ is a function of a single variable, $h_V(s, t, u) = h_V(u)$. It is now clear that the momentum dependent form factors in the four-Fermi interactions are needed to cancel the photon exchange contributions in the mWTI for the present assumptions $\sigma = 0$ and $h_v = 1$ in (44).

Let us turn to the first WT relation (45). For a specific choice of the cutoff function

$$K(p) = \exp(-p^2/k^2),$$

the rhs of (45) can be calculated analytically as shown in appendix D. We obtain the analytic expression of the longitudinal component of the photon two point function

$$\mathcal{L}(p) = -e^2\frac{k^2}{2\pi^2p^4} \left[ 1 - \exp(-p^2/2) \right] \left[ \frac{1}{2} \exp(-\bar{p}^2/2) \right],$$

where $\bar{p}^2 = p^2/k^2$. In the limits of $\bar{p}^2 \rightarrow 0$ and $\bar{p}^2 \rightarrow \infty$, $\mathcal{L}(p)$ behaves as

$$\mathcal{L}(p) \sim \frac{e^2k^2}{2\pi^2} \times \begin{cases} 3/8 - \bar{p}^2/12 & \bar{p}^2 \rightarrow 0, \\ 1/\bar{p}^2 & \bar{p}^2 \rightarrow \infty. \end{cases}$$

The derivative expansion would give us $\mathcal{L}(p) \sim 3e^2k^2/16\pi^2$, the first term on the first line. We also note that $\mathcal{L}(p) \sim e^2k^2/2\pi^2p^4$ for any non-zero $p^2$ in the limit of $k^2 \rightarrow 0$.

5. Conclusions

In the FRG approach to gauge theories, gauge symmetry is encoded in the mWTI. A truncation to the effective action has to satisfy this mWTI. For the Wilsonian effective action these identities can be cast into a simple form of an unbroken symmetry identity, see [13, 16–18]. For the 1PI effective action the mWTI has a less convenient form due to the presence of the full field-dependent propagator. In turn, the flow equation for the 1PI effective action has a remarkable numerical stability that originates in the dependence on the full field-dependent propagator. As discussed in the introduction, this suggests a combined use of the mWTI for the Wilsonian effective action and the flow equation of the 1PI effective action.

In the present work we have put forward this approach for QED with a massless electron. We have partially solved the mWTI for the Wilsonian effective action. This solution inevitably leads to momentum dependence couplings, or form factors. For the sake of simplicity, we introduced form factors only to the photon two point function and four fermi interactions, while we ignored those in the fermion two point function and the gauge interaction. Even if we included the latter, however, the generic structure observed in this paper would be the same: higher order correlation functions would have been still determined from lower order ones. Our solutions to the relations are different from the one obtained by an approximation in the spirit of derivative expansion. Finally, the related truncation for the 1PI effective action has been calculated via the Legendre transformation, also featuring momentum-dependent vertices.

Note also that the present approach is easily extended to non-Abelian gauge theories. There, in Landau gauge a BRST-consistent solution of the flow equation is at the root of the
dynamical generation of the gluonic mass gap, see [43, 44]. We hope to report on such an extension in near future.

In summary, in the present work we have used the mWTIs to relate couplings in the truncated Wilsonian and 1PI effective action, working out part (i) and (ii) of the programme put forward in the introduction, see page two. In particular, we found that a particular linear combination of the four fermi couplings can be written in terms of the photon two point function as in (53). It is left to solve the flow equation for the remaining vertex functions. This also allows to resolve the question, whether the above WT relations are compatible with the solution of the flow equation in the present truncation. Such a compatibility is at the root of the overall consistency of the present approximation. This discussion, and the solution of the gauge symmetry-consistent flows, is deferred to a forthcoming paper.

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Appendix A. Flow equation

Here and in the following appendices, the notation before the renormalisation (11) is used so that we observe clearly how various quantities are renormalised from the original definition of the Wilson action to be given in (A2) and (A3).

In order to define the Wilson action, we introduce IR fields $\phi^A$ in addition to the original fields $\varphi^A$ and rewrite the generating functional as
\[
Z[J] = \int D\varphi \exp \left( -S[\varphi] + J \cdot \varphi \right) = \int D\varphi D\phi \exp \left[ -S[\varphi] + J \cdot \varphi - \frac{1}{2} (\phi - K \varphi - J (1 - K) D^{-1}) \right. \\
\left. \cdot \frac{D}{K (1 - K)} \cdot (\phi - K \varphi - (-)^{(1)} D^{-1}(1 - K) J) \right] = N[J] \int D\phi \exp \left[ -\frac{1}{2} \phi \cdot K^{-1} D \cdot \phi - S_{I,k}[\phi] + J \cdot K^{-1} \phi \right].
\]

where interactions of $\phi$ are generated as
\[
\exp[-S_{I,k}[\phi]] = \int D\chi \exp \left[ -\frac{1}{2} \chi \cdot (1 - K)^{-1} D \cdot \chi - S_I[\phi + \chi] \right].
\]

On the second line of (A1), a Gaussian integral over $\phi$ is inserted into the partition function. Changing the order of integrals over the fields $\chi = \varphi - \phi$ and $\phi$, we define the Wilson action
\[
S_I[\phi] = \frac{1}{2} \phi \cdot K^{-1} D \cdot \phi - S_{I,k}[\phi].
\]

The original fields $\varphi^A$ are decomposed into the IR fields $\phi^A$ with propagator $KD^{-1}$ and the UV fields $\chi^A$ with propagator $(1 - K)D^{-1}$.
At this stage, we introduce $Z$ factors for IR fields and their source terms by rescaling $\phi \rightarrow Z_{\phi}^{-1/2} \phi$, $J_{\phi} \rightarrow Z_{\phi}^{-1/2}J_{\phi}$. The Wilson action takes the form

$$S_k[Z_{\phi}^{-1/2}] = \frac{1}{2} \phi^A A^A (K^A)^{-1} D_{AB} \phi^B + S_{A,k}[Z_{\phi}^{-1/2}].$$

(A4)

The partition function for the Wilson action

$$Z_{\phi}[J] = \int D\phi \exp \left[ -S_k[Z_{\phi}^{-1/2}] + J \cdot K^{-1} \phi \right],$$

(A5)

is related to that for the original one as

$$Z_{\phi}[J] = N[Z^{-1/2}] Z_{\phi}[J],$$

(A6)

where the normalisation factor is given by

$$N[Z^{-1/2}] = \exp \left[ -(-\gamma' \langle \mu \rangle) \frac{1}{2} I_{A} \left( \frac{1}{Z} \right)^A (D^{-1})^A_{AB} J_{B} \right].$$

(A7)

The Polchinski flow equation is obtained from the requirement that the partition function $Z_{\phi}[J]$ does not depend on the cutoff $k = e^\gamma$: $k \partial_k Z_{\phi}[J] = \partial_\gamma Z_{\phi}[J] = \partial_\gamma (N[Z^{-1/2}] Z_{\phi}[J]) = 0$. It is straightforward to obtain

$$\partial_\gamma S_k[Z_{\phi}^{-1/2}] = -\phi^A (\partial_\gamma \log K)^A \frac{\partial S_k}{\partial \phi^A}
+ (-\gamma') \frac{1}{2} \left[ Z^{-1} \{ \partial_\gamma K - \eta K (1 - K) \} D^{-1} \right]^{AB}
\times \left[ \frac{\partial^\gamma S_k}{\partial \phi^B} \frac{\partial S_k}{\partial \phi^A} - \frac{\partial^\gamma S_k}{\partial \phi^B} \frac{\partial S_k}{\partial \phi^A} \right],$$

(A8)

where

$$\eta_A = -\partial_\gamma \log Z^A,$$

(A9)

are the anomalous dimensions for $\phi^A$. Since the Wilson action is a functional of $(Z^A)^{1/2} \phi^A$, we may extract contributions of the anomalous dimensions from $S_k(Z_{\phi}^{-1/2})$:

$$\partial_\gamma S_k[\phi] = -\tilde{\phi}^A \left[ (\partial_\gamma \log K)^A - \frac{1}{2} \eta_A \right] \frac{\partial S_k}{\partial \tilde{\phi}^A}$$

$$+ (-\gamma') \frac{1}{2} \left[ \{ \partial_\gamma K - \eta K (1 - K) \} D^{-1} \right]^{AB}
\times \left[ \frac{\partial \tilde{\phi}^A}{\partial \phi^B} \frac{\partial S_k}{\partial \phi^A} - \frac{\partial \tilde{\phi}^A}{\partial \phi^B} \frac{\partial S_k}{\partial \phi^A} \right],$$

(A10)
For the interaction part $S_{I,k}$, the flow equation reads [18, 45]
\[
\partial_i S_{I,k}[\phi] = \frac{1}{2} \partial^A \eta^A D_{AB} \partial^B + \phi^A \left( \frac{1}{2} \eta - \eta (1 - K) \right) \partial^A S_{I,k} + (-) \frac{1}{2} \left( \left[ \partial_i K - \eta K (1 - K) \right] D^{-1} \right)^A B \\
\times \left[ \frac{\partial^A S_{I,k}}{\partial \phi^B} \frac{\partial^B S_{I,k}}{\partial \phi^A} - \frac{\partial^A \partial^B S_{I,k}}{\partial \phi^A} \right]
\]  
\text{(A11)}

**Appendix B. Derivation of the WT identity**

We consider the realisation of gauge (BRST) symmetry for the Wilson action. It is the WT operator that signals the presence of symmetry. We will show that it takes the form
\[
\Sigma_k [\phi] = \partial^A S \delta \phi^A - (-) \frac{\partial^A \delta \phi^A}{\partial \phi^A}, \quad \text{(B1)}
\]
where $\delta \phi^A$ denote the BRST transformations for $\phi^A$. The first and the second terms in (B1) are the changes of the Wilson action and the path integral measure under the transformation, respectively. When the condition
\[
\Sigma_k [\phi] = 0, \quad \text{(B2)}
\]
holds, we have the symmetry at the quantum level. The path integration over the high momentum modes produces the corrections to the gauge or BRST transformation at the scale $k$.

We now derive the WT operator (B1) and construct the BRST transformations for the IR fields, $\delta \phi^A$: starting from the symmetry of the UV action $S[\varphi]$, we will find its modification due to the presence of the cutoff.

Consider a change of variables, $\varphi^A \to \varphi^A = \varphi^A + \delta \varphi^A \lambda$, under the BRST transformation
\[
\delta \varphi^A = R^A [\varphi]. \quad \text{(B3)}
\]
It induces a change of the UV action
\[
S[\varphi] \to S[\varphi] + \frac{\partial^A S}{\partial \varphi^A} \delta \varphi^A \lambda, \quad \text{(B4)}
\]
as well as a change of the functional measure
\[
D\varphi \to D\varphi \left( 1 + (-) \frac{\partial^A \delta \varphi^A}{\partial \varphi^A} \lambda \right), \quad \text{(B5)}
\]
The invariance of the functional integral
\[
Z_\varphi[J] = \int D\varphi' \exp (-S[\varphi'] + J \cdot \varphi') = Z_\varphi[J] \\
+ \int D\varphi \left[ - \frac{\partial^A S}{\partial \varphi^A} \delta \varphi^A + (-) A^A \frac{\partial^A}{\partial \varphi^A} \delta \varphi^A + J \cdot \delta \varphi \right] \lambda \\
\times \exp (-S[\varphi] + J \cdot \varphi), \quad \text{(B6)}
\]
leads to a relation
\[
\int \mathcal{D}\varphi \, \Sigma[\varphi] \exp(-S[\varphi] + J \cdot \varphi) = \int \mathcal{D}\varphi \, J^A R^A[\varphi] \exp(-S[\varphi] + J \cdot \varphi) = J_A R^A[\partial J/\partial J] Z[J].
\] (B7)

Here
\[
\Sigma[\varphi] = \frac{\partial^2 S}{\partial^2 \varphi^A} \delta \varphi^A - (-\gamma A \frac{\partial^2 \varphi^A}{\partial^2 \varphi^A}).
\] (B8)

denotes the WT operator for the UV action $\mathcal{S}$. Using the relation between partition functions $Z[J] = N [Z^{-1/2}J] Z[J]$, we obtain the WT operator for the Wilson action from the following calculation.
\[
J_A R^A[\partial J/\partial J] Z[J] = J_A R^A[\partial J/\partial J] N [Z^{-1/2}J] Z[J] = N [Z^{-1/2}J] \int \mathcal{D}\phi \, \Sigma[\phi] \exp(-S[\phi] + J \cdot K^{-1} \phi).
\] (B9)

In the second expression in (B9), derivatives $R^A[\partial J/\partial J]$ act on $N [Z^{-1/2}J]$ as well as the partition function. This generates the modified BRST transformation $\delta \phi^A$. It is easy to confirm that $\Sigma[\phi]$ in (B9) agrees with the expression in (B1).

### B.1. Composite operators

The WT operator $\Sigma[\phi]$ is characterised by composite operators. We summarise some results on them [18]. An operator $O_k[\phi]$ is called a composite operator if it fulfills a RG flow equation
\[
\partial_t O_k[\phi] = - \phi^A (\partial_t \log K)^A \frac{\partial^2 O_k}{\partial \phi^A} + \frac{\partial^2 S}{\partial \phi^A} [Z^{-1} \{ \partial_t K - \eta K (1 - K) \} D^{-1}]^A \frac{\partial^2 O_k}{\partial \phi^B} - \frac{1}{2} [Z^{-1} \{ \partial_t K - \eta K (1 - K) \} D^{-1}]^{AC} \frac{\partial^2 \partial_t O_k}{\partial \phi^C \partial \phi^A}.
\] (B10)

This flow equation takes the same form as a variation of the Polchinski equation (A8) for an infinitesimal deformation of the Wilson action, $\Delta S_k$. The composite fields $[\phi^A]_k$ defined as
\[
[\phi^A]_k \equiv (K^{-1} \phi)^A - (Z^{-1} G^{(0)})^{AB} \frac{\partial S_k}{\partial \phi^B}
\]
\[
= \phi^A - (Z^{-1} G^{(0)})^{AB} \frac{\partial S_k}{\partial \phi^B},
\] (B11)
play an important role in constructing $\Sigma_k$. Note that $[\phi^A]_k$ equal the full mean fields $\Phi$ in the 1PI language defined in (22). Then, (26) is obtained from (25) simply by multiplying the function $K$ and replacing classical fields with the corresponding composite operators [18]. In addition to $[\phi^A]_k$, $K^A \partial S_k/\partial \phi^A$ and the WT operator $\Sigma_k$ itself are composite operators. As a result, once the identity $\Sigma_k[\phi] = 0$ is shown at some scale $\hat{k}$, $\Sigma_k[\phi]$ vanishes at any scale $k$. Therefore, the WT identity can be used to define a gauge invariant subspace in the theory space.

### B.2. The WT identity for the Wilson action for QED

We will describe the construction of the quantum BRST transformation and the WT operator for QED with fields $\phi^A = (\phi_i, \overline{\psi}_i, \psi_i, c, \overline{c})$. Though we use the same notations for the
component fields as for $\phi$, here they all represent the unrenormalised fields. The kinetic part of the Wilson at the scale $k$ is given by

$$S_{0,k} = \int_p K^{-1}(p) \left[ \frac{Z_2}{2} a_\nu(-p) p^2 \left\{ \delta_{\mu\nu} - \left( 1 - \xi^{-1} \right) \frac{p_\mu p_\nu}{p^2} \right\} a_\nu(p) + \bar{c}(-p) i p^2 c(p) \right]$$

$$+ \int_p K^{-1}(p) Z_2 \bar{\psi}(-p) \slash{p} \psi(p)$$

$$= \frac{1}{2} (K^A)^{-1} Z_A \phi^A D_{AB} \phi^B,$$

where $Z_2$ and $Z_3$ are the renormalisation constants of the fermion and photon fields respectively. For simplicity, the fermion is chosen to be massless and we have the chiral symmetry. The matrix $D_{AB}$

$$D_{AB}(p) \equiv \begin{pmatrix} (D_G)_{\mu\nu}(p) & 0 & 0 \\ 0 & (D_F)_{\lambda\beta}(p) & 0 \\ 0 & 0 & (D_F^{T})_{\alpha\delta}(p) \end{pmatrix},$$

has the components

$$(D_G)_{\mu\nu}(p) = p^2 (P_{\mu\nu}^T + \xi^{-1} P_{\mu\nu}^L),$$

with

$$P_{\mu\nu}^T = \delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}, \quad P_{\mu\nu}^L = \frac{p_\mu p_\nu}{p^2},$$

and

$$(D_F)_{\lambda\beta}(p) = (\slash{p})_{\lambda\beta}, \quad (D_F^{T})_{\alpha\delta}(p) = (\slash{p}^{T})_{\alpha\delta}.$$}

The high energy propagators are expressed as

$$(G^{(0)})^{AB}(p) = (1 - K)(D^{-1})^{AB}(p)$$

$$= \begin{pmatrix} (G^{(0)}_G)_{\mu\nu}(p) & 0 & 0 \\ 0 & (G^{(0)}_F)_{\lambda\beta}(p) & 0 \\ 0 & 0 & (G^{(0)}_F^{T})_{\alpha\delta}(p) \end{pmatrix},$$

where

$$(G^{(0)}_G)_{\mu\nu}(p) = \frac{1 - K}{p^2} (P_{\mu\nu}^T + \xi P_{\mu\nu}^L),$$

$$(G^{(0)}_F)_{\lambda\beta}(p) = (1 - K) \left( \frac{1}{p} \right)_{\lambda\beta},$$

$$(G^{(0)}_F^{T})_{\alpha\delta}(p) = (1 - K) \left( \frac{1}{p^{T}} \right)_{\alpha\delta}. (B18)$$
By starting from (28), the quantum BRST transformation for QED is obtained by multiplying $K(p)$ and replacing the fields by their composite operators on the rhs. This procedure is based on the observation that $K^{-1}\delta\phi^A$ are composite operators [18]

$$
K^{-1}\delta a_\mu(p) = -i p_\mu c(p),
K^{-1}\delta\psi(p) = -i e_0 \int_q [\bar{\psi}(q)]_k c(p - q),
K^{-1}\delta\bar{\psi}(-p) = i e_0 \int_q [\bar{\psi}(-q)]_k c(q - p),
K^{-1}\delta\bar{c}(-p) = \xi^{-1}_0 p_\mu [a_\mu(p)]_k.
$$

(B19)

Here $[\phi^A]^k$ are composite fields given in (B11). Written in terms of $\bar{\phi}$ and $\bar{\Phi}$, we obtain (31). By substituting (B19) into (B1), we obtain the WT operator for QED

$$
\Sigma_k[\phi] = \int_p \left[ K(p) \frac{\partial S_k}{\partial a_\mu(p)} (-i p_\mu c(p)) + K(p) \frac{\partial S_k}{\partial c(p)} \right]
$$

$$
\times \left\{ Z_3 \xi^{-1}_0 p_\mu K^{-1}(p) a_\mu(p) - \frac{(1 - K(p))}{p^2} p_\mu \frac{\partial S_k}{\partial a_\mu(-p)} \right\}
$$

$$
- i e_0 \int_{p,q} \left[ \frac{\partial^2 S_k}{\partial \bar{\psi}_\alpha(p)} \frac{K(q)}{K(p)} \bar{\psi}_\alpha(p) - K(q) \frac{\partial^2 S_k}{\partial \bar{\psi}_\alpha(-p)} \right] c(q - p)
$$

$$
- i e_0 \int_{p,q} U_{\bar{\beta}\alpha}(-q, p) \left[ \frac{\partial^2 S_k}{\partial \bar{\psi}_\alpha(-p)} \frac{\partial S_k}{\partial \bar{\psi}_\beta(q)} - \frac{\partial^2 S_k}{\partial \bar{\psi}_\alpha(-p)} \frac{\partial S_k}{\partial \bar{\psi}_\beta(q)} \right] c(q - p)
$$

$$
= \int_p \left[ \frac{\partial S_k}{\partial a_\mu(p)} (-i p_\mu c(p)) + Z_3 \frac{\partial^2 S_k}{\partial c(p)} \xi^{-1}_0 p_\mu a_\mu(p) \right]
$$

$$
- i e_0 \int_{p,q} \left[ \frac{\partial^2 S_k}{\partial \bar{\psi}_\alpha(p)} \frac{K(q)}{K(p)} \bar{\psi}_\alpha(p) - K(q) \frac{\partial^2 S_k}{\partial \bar{\psi}_\alpha(-p)} \right] c(q - p)
$$

$$
- i e_0 \int_{p,q} U_{\bar{\beta}\alpha}(-q, p) \left[ \frac{\partial^2 S_k}{\partial \bar{\psi}_\alpha(-p)} \frac{\partial S_k}{\partial \bar{\psi}_\beta(q)} - \frac{\partial^2 S_k}{\partial \bar{\psi}_\alpha(-p)} \frac{\partial S_k}{\partial \bar{\psi}_\beta(q)} \right] c(q - p),
$$

$$
= \int_p \frac{\partial S_k}{\partial a_\mu(p)} (-i p_\mu c(p)) + i e_0 Z_2 \int_{p,q} [\bar{\psi}(-q)(\nabla - q)\psi(p)] c(q - p)
$$

$$
- i e_0 \int_{p,q} \left[ \frac{\partial^2 S_k}{\partial \bar{\psi}_\alpha(p)} \frac{K(q)}{K(p)} \bar{\psi}_\alpha(p) - K(q) \frac{\partial^2 S_k}{\partial \bar{\psi}_\alpha(-p)} \right] c(q - p)
$$

$$
+ i e_0 Z_2 \int_{p,q} \left[ \frac{\partial^2 S_k}{\partial \bar{\psi}_\alpha(q)} U_{\bar{\beta}\alpha}(-q, p) K^{-1}(p) (\nabla \psi(p))_k
$$

$$
+ K^{-1}(q)[\bar{\psi}(-q)g]_k U_{\bar{\beta}\alpha}(-q, p) \frac{\partial^2 S_k}{\partial \bar{\psi}_\alpha(-p)} \right] c(q - p)
$$

$$
- i e_0 \int_{p,q} U_{\bar{\beta}\alpha}(-q, p) \left[ \frac{\partial^2 S_k}{\partial \bar{\psi}_\alpha(-p)} \frac{\partial^2 S_k}{\partial \bar{\psi}_\beta(q)} - \frac{\partial^2 S_k}{\partial \bar{\psi}_\alpha(-p)} \frac{\partial^2 S_k}{\partial \bar{\psi}_\beta(q)} \right] c(q - p),
$$

(B20)
where
\[ U(-q, p)_{\beta\alpha} = Z_{\xi}^{-1} [K(q)G_F^{(0)}(p) - K(p)G_F^{(0)}(q)]_{\beta\alpha}. \] (B21)

In the last expression of (B20), the WT operator \( \Sigma_k[\phi] \) is expressed in terms of the interaction part of the Wilson action, \( S_{ik} \). Since the gauge parameter \( Z_3\xi^{-1} = \xi_0^{-1} \) and gauge coupling \( e_0 \) are \( k \)-independent, \( \Sigma_k[\phi] \) remains a composite operator.

Now, in order to write (B19) and (B20) with the renormalised fields \( \bar{\phi} \), we make a rescaling \( a_\mu \rightarrow Z_3^{-1/2} a_\mu, \ \bar{\psi} \rightarrow Z_2^{-1/2} \bar{\psi}, \ \psi \rightarrow Z_2^{-1/2} \psi, \ c \rightarrow Z_3^1 c, \ \bar{c} \rightarrow Z_3^{-1} \bar{c}, \) where \( Z_c = e/e_0 \). Then, we obtain the BRST transformations (31) and the WT identity (34).

Appendix C. Wilsonian effective action for QED

In section 3 we have discussed the approximation to the 1PI effective action and the Wilsonian effective action used in the present work. With all momentum dependence and Lorentz indices the schematic expression (36) is given by

\[
\Gamma_{1,k}[\bar{\phi}] = Z_3^{-1/2} \int_p A_\mu(-p)A_\nu(p)h^{(\mu\nu)}(p) + Z_2 \int_p \bar{\Psi}\delta(-p)\Psi(p)h^{(\psi\psi)}(p)
+ Z_2 Z_3^{-1/2} \int_{p_1, p_2} \bar{\Psi}_\alpha(p_1)A_\mu(-(p_1 + p_2))\Psi_\alpha(p_2)h^{(\mu\nu)}(p_1, p_2)
+ Z_2^2 \int_{p_1, p_2, p_3} \bar{\Psi}_\alpha(p_1)\Psi_\beta(p_2)\bar{\Psi}_\beta(p_3)\Psi_\beta(-(p_1 + p_2 + p_3))h^{(\psi\psi\psi)}(p_1, p_2, p_3),
\] (C1)

Correspondingly, \( S_{1,k} \) is given by

\[
S_{1,k}[\phi] = Z_3^{-1/2} \int_p a_\mu(-p)a_\nu(p)[\bar{G}_G(p)h^{(\mu\nu)}(p)]_{\alpha\beta}
+ Z_2 Z_3^{-1/2} \int_{p_1, p_2} \bar{\psi}_\alpha(p_1)a_\mu(-(p_1 + p_2))\psi_\beta(p_2)
\times \left[ \prod_{i=1}^4 \{ \bar{G}_F(p_i) \bar{G}_G(-(p_1 + p_2))h^{(\mu\nu)}(p_1, p_2) \} \right]_{\alpha\beta}
+ Z_2^2 \int_{p_1, p_2, p_3} \bar{\psi}_\alpha(p_1)\psi_\beta(p_2)\bar{\psi}_\beta(p_3)\psi_\beta(-(p_1 + p_2 + p_3))
\times \left[ \prod_{i=1}^4 \{ \bar{G}_F(p_i) \} h^{(\psi\psi\psi)}(p_1, p_2, p_3) \right]_{\alpha\beta}. \] (C2)

In this work we further reduce the general tensor structure of the interaction coefficients \( h^{(\psi\psi)}, \ h^{(\psi\psi\psi)} \), see (40a)–(40c) and (41)–(43). This leads to
\[ S_{l,k} = \int_p \left[ \frac{Z_3}{2} \langle \psi | (1 + \sigma(p) - g(1 + \sigma(q))) | \psi \rangle - eZ_2 Z_3^{1/2} \int_{q,p} h_e(-p, q) \tilde{\sigma}(p)(1 + \sigma(q)) \psi(p) \right] \]
\[- eZ_2 Z_3^{1/2} \int_{q,p} h_e(-p, q) \tilde{\sigma}(p)(1 + \sigma(q)) \psi(p) \]
\[+ \frac{Z_3^2}{2} \int_{p_1,...,p_3} \left[ \prod_{i=1}^4 \tilde{\sigma}(p_i) \right] \left( \tilde{\psi}(p_1) \tilde{\psi}(p_2) \tilde{\psi}(p_3) \tilde{\psi}(p_4) \right) \]
\[- (\tilde{\psi}(p_1) \gamma_5 \tilde{\psi}(p_2)) (\tilde{\psi}(p_3) \gamma_5 \tilde{\psi}(p_4)) \]
\[+ \frac{h_v}{k^2} \left[ (\tilde{\psi}(p_1) \gamma_5 \tilde{\psi}(p_2)) (\tilde{\psi}(p_3) \gamma_5 \tilde{\psi}(p_4)) \right] \]
\[- e^2 h_v(p_1, p_2) (\tilde{\psi}(p_3) \gamma_5 \psi(p_2)) (\tilde{\psi}(p_3) \gamma_5 \psi(p_4)) (G_G)_{\mu \nu}(p_1 + p_2) \]
\[+ \frac{Z_3^2}{2k^2} \int_{p_1,...,p_3} \left[ \prod_{i=1}^4 \tilde{\sigma}(p_i) \right] h_v(s, t, u)(p_1 + p_3) \left[ (\tilde{\psi}(p_1) \gamma_5 \tilde{\psi}(p_2)) (\tilde{\psi}(p_3) \gamma_5 \tilde{\psi}(p_4)) \right] \]
\[\times \left[ (\tilde{\psi}(p_1) \gamma_5 \tilde{\psi}(p_2)) (\tilde{\psi}(p_3) \gamma_5 \tilde{\psi}(p_4)) \right].\]  
\[\text{(C3)}\]

with \( p_4 = -(p_1 + p_2 + p_3) \) and
\[\tilde{\sigma}(p) = \frac{1}{1 + (1 - K(p)) \sigma(p)}.\]  
\[\text{(C4)}\]

**Appendix D. Derivation of (47) and (55)**

Before using our ansatz (44), the second WT relation takes the following form:
\[ e_0 Z_3 [ \mu(1 + \sigma(p)) - g(1 + \sigma(q))] - eZ_2 Z_3^{1/2} h_e(-q, p) \gamma_5 (p - q), (G_G)_{\mu \nu}(p - q) \]
\[- e_0 Z_2^2 \int_l \left[ 2h_v \mu^2, (p + q + l)^2, (p - q)^2 \right] U(-q - l, p + l) \]
\[+ 2h_v \mu^2, (p + q + l)^2, (p - q)^2 \gamma_5 U(-q - l, p + l) \gamma_5 \mu \]
\[- h_v \mu^2, (p - q)^2, (p + q + l)^2, l^2 \Tr[U(-q - l, p + l) \gamma_5 \mu] \tilde{\sigma}(p + l) \tilde{\sigma}(q + l) \]
\[+ e_0 Z_2^2 \int_l \left[ 2h_v \mu^2, (p + q + l)^2, (p - q)^2 \right] (\mu - q) U(-q - l, p + l) (\mu - q) \]
\[- h_v \mu^2, (p - q)^2, (p + q + l)^2, l^2 \left[ \Tr[U(-q - l, p + l)] \tilde{\sigma}(p + l) \tilde{\sigma}(q + l) \right. \]
\[+ e_0 e^2 Z_2 Z_3 \int_l \gamma_5 \mu \Tr[U(p - q - l, \mu) \gamma_5] (G_G)_{\mu \nu}(p - q) \]
\[- h_e(-l, p) h_e(-q, -p + q + l) \gamma_5 U(p - q - l, l) \gamma_5 (G_G)_{\mu \nu}(p - l) \] \[= 0.\]  
\[\text{(D1)}\]
In (D1), one-particle reducible contributions are summed up to give
\[
eq Z \frac{1}{2} h_{\varepsilon}(q, p) \gamma_{\mu}(p - q) (\bar{G}_G)_{\mu \nu}(p - q) + e_0 e \frac{1}{2} Z \frac{1}{2} h_{\varepsilon}(q, p) (\bar{G}_G)_{\mu \nu}(p - q) \\
\times \int_l Tr[U(p - q - l, l) \gamma_{\mu}] h_{\varepsilon}(\bar{l} - q - p + q + l) \sigma(l) \bar{\sigma}(p - q - l) \\
= Z \frac{1}{2} h_{\varepsilon}(q, p) \gamma_{\mu} \left[ (\bar{G}_G)_{\mu \nu}(p - q) + \frac{\xi \{(1 - K) \xi\} (p - q)}{(p - q)^2 + \eta \{(1 - K) \xi\} (p - q)} \right] \\
= Z \frac{1}{2} h_{\varepsilon}(q, p) (p - q) \gamma_{\mu}.
\]

Here we have used the extended form of the first WT relation with form factors $h_{\varepsilon}(p, q)$ and $\sigma(p)$:
\[
p_{\mu} \mathcal{L}(p) = e_0 e \frac{1}{2} Z \frac{1}{2} h_{\varepsilon}(p - q, q) \gamma_{\mu} \\
\times \int_l Tr[U(p - q - l, l) \gamma_{\mu}] h_{\varepsilon}(\bar{l} - q - p + q + l) \sigma(l) \bar{\sigma}(p - q - l).
\]

Note that the propagators $G_F$ and $G_G$ contain the inverse of $Z$ factors, $U \propto Z^{-1}$, $G_G \propto Z^{-1}$. Using the locality assumption $\sigma(p) = 0$ ($\bar{\sigma} = 1$), $h_{\varepsilon}(p, q) = 1$, the relation $e = Z e_0$, and some trace relations such as $\int_l Tr[U(-q, p)] = 2l^2 U(-q, p)$, we obtain (47).

Let us consider the first WT relation (D3), which reduces to
\[
p_{\mu} \mathcal{L}(p) = e^2 Z \frac{1}{2} \int_q Tr[U(p - q, q) \gamma_{\mu}] \\
\times [K(q + p) - K(q - p)]_{\mu} \frac{1 - K(q)}{q^2}.
\]

with our present assumption (44). Using the Gaussian function for the cutoff function (54) and the integral representation of the modified Bessel function
\[
\int_0^\pi d\theta \exp(2\rho \bar{q} \cos \theta) \sin^2 \theta = \frac{\pi}{2\rho \bar{q}} I_1(2\rho \bar{q}),
\]
we have
\[
\int_q \exp(-\bar{q}^2 + 2\bar{p} \cdot q) \left(1 - \exp(-\bar{q}^2)\right) \\
= \frac{1}{8\pi^2 \rho} \int_0^\infty dq \left[ \exp(-q^2) - \exp(-2q^2) \right] I_1(2\rho q) \\
= \frac{1}{16\pi^2 \rho^2} \left[ \exp(\bar{p}^2) - \exp(\bar{p}^2/2) \right].
\]

where $\bar{p}^2 = p^2/k^2$ and $q^2 = q^2/k^2$. From (D4) with (D6), we obtain (55).

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