Time fractional exact controllability

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Abstract

Our purpose is to adapt the Hilbert Uniqueness Method by J.-L. Lions in the case of fractional diffusion-wave equations. The main difficulty is to determine the right shape for the adjoint system, suitable for the procedure of HUM.

Keywords: Caputo and Riemann–Liouville fractional derivatives; fractional diffusion-wave equations; Hilbert Uniqueness Method.

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1 Introduction

In this paper we investigate the fractional differential equation

\[ {^C D_{0^+}^\alpha} u = \Delta u \]  

where the symbol \( {^C D_{0^+}^\alpha} u \) denotes the Caputo derivative

\[ {^C D_{0^+}^\alpha} u(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} u''(s) \, ds \]

defined in [1]. We assume that the order \( \alpha \) belongs to \((1, 2)\), so \( (1) \) is commonly called fractional diffusion-wave equation. For a comparison between Caputo derivatives and Riemann-Liouville derivatives see the survey [11].

In [13] the well-posedness has been established by means of a Fourier series approach that involves Mittag-Leffler functions. Also in [10] the proof of trace regularity properties for the weak solutions of \((1)\) is based on the representation like Fourier expansions. The paper [12] drew attention to the fact that fractional wave equations rule the propagation of mechanical diffusive waves in viscoelastic media.

The aim of this paper is to illustrate the Hilbert Uniqueness Method for \((1)\), introduced by J.-L. Lions in [8]. Several scientists show interest for HUM since it gives a general procedure to get exact controllability for distributed systems when the control acts on the boundary of the domain. This method is based on the idea that the observability of the adjoint problem is sufficient for the

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exact controllability of the assigned problem. The observability is also a necessary condition, for
detailed argumentations see [6].

Our task is to consider the following problem: for a domain \( \Omega \) and \( T > 0 \), we have to control
the system
\[
\begin{cases}
C D_{0+}^\alpha u = \Delta u & \text{in } (0, T) \times \Omega, \\
u(0) = u_t(0) = 0 & \text{in } \Omega,
\end{cases}
\]
by means of a function that acts on a boundary’s subset.

We point out that, due to the nonlocal nature of the problem, HUM allows one to achieve
exact reachability rather than exact controllability.

The main difficulty in carrying out HUM for (2) is to find the right form for the adjoint system.
Precisely, we have to change the Caputo derivative \( C D_{0+}^\alpha \) into the Riemann-Liouville derivative
\( D_{T-}^\alpha \) and consider as adjoint system associated with (2) the following problem
\[
\begin{cases}
D_{T-}^\alpha w = \Delta w & \text{in } (0, T) \times \Omega, \\
w = 0 & \text{on } (0, T) \times \partial \Omega,
\end{cases}
\]
with final data
\[
D_{T-}^{\alpha-2} w(T) = w_0, \quad D_{T-}^{\alpha-1} w(T) = w_1 \quad \text{in } \Omega.
\]
We remark that the question of the exact reachability is still unanswered, since inverse and direct
inequalities have to be proved.

Another adaptation of HUM to nonlocal problems with regular integral kernels can be found in [9].

In Section 2 we set up notations and terminology. Section 3 provides a detailed exposition of
the Hilbert Uniqueness Method for system (2).

2 Preliminaries

Let \( f \in L^1(0, T) \) \( (T > 0) \) be. The following definitions for fractional integrals and derivatives will
be used throughout the paper.

- **Riemann–Liouville fractional integrals** \( I_{0+}^\alpha f \) and \( I_{T-}^\alpha f \) of order \( \alpha > 0 \):
\[
I_{0+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} f(s) \, ds \quad \text{a.e. } t \in (0, T), \quad (3)
\]
\[
I_{T-}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s - t)^{\alpha-1} f(s) \, ds, \quad \text{a.e. } t \in (0, T), \quad (4)
\]
where \( \Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t} \, dt \) is the Euler gamma function.

- **Caputo fractional derivative** \( C D_{0+}^\alpha f \) of order \( \alpha \in (1, 2) \) when \( f' \) is absolutely continuous:
\[
C D_{0+}^\alpha f(t) = I_{0+}^{2-\alpha} f''(t) = \frac{1}{\Gamma(2 - \alpha)} \int_0^t (t - s)^{1-\alpha} f''(s) \, ds. \quad (5)
\]
– Riemann–Liouville fractional derivative $D^\alpha_{T-} f$ of order $\alpha \in (n-1, n)$, $n = 0, 1, 2$:

$$D^\alpha_{T-} f(t) = \left(\frac{d}{dt}\right)^n I^{n-\alpha}_{T-} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_t^T (s-t)^{n-\alpha-1} f(s) \, ds.$$  \hspace{1cm} (6)

In particular, we note that for $-1 < \alpha < 0$, that is $n = 0$, we have

$$D^\alpha_{T-} f = I^{-\alpha}_{T-} f.$$  \hspace{1cm} (7)

We will use later a well-known property (see e.g. [5]):

$$\int_0^T I^\alpha_{0+} f(t) g(t) \, dt = \int_0^T f(t) I^\alpha_{T-} g(t) \, dt.$$  \hspace{1cm} (8)

3 Time fractional exact controllability

Our aim is to adapt the Hilbert Uniqueness Method by J.-L. Lions [8] for fractional differential equations.

Let $T > 0$ and $\Omega$ be a bounded domain of class $C^2$ in $\mathbb{R}^N$, $N \geq 1$, with boundary $\partial \Omega$ given by $\partial \Omega = \partial \Omega_1 \cup \partial \Omega_2$ and $\partial \Omega_1 \cap \partial \Omega_2 = \emptyset$. We consider the fractional diffusion-wave equation with $\alpha \in (1, 2)$:

$$C D^\alpha_{0+} u(t, x) = \Delta u(t, x) \quad (t, x) \in (0, T) \times \Omega,$$  \hspace{1cm} (9)

with null initial conditions

$$u(0, x) = u_t(0, x) = 0 \quad x \in \Omega,$$  \hspace{1cm} (10)

and boundary conditions

$$u(t, x) = g(t, x) \quad (t, x) \in (0, T) \times \partial \Omega_1, \quad u(t, x) = 0 \quad (t, x) \in (0, T) \times \partial \Omega_2.$$  \hspace{1cm} (11)

We give the definition for a reachability problem: fixed $u_0 \in L^2(\Omega)$ and $u_1 \in H^{-1}(\Omega)$, we have to determine $g \in L^2(0, T; L^2(\partial \Omega_1))$ such that the weak solution $u$ of problem (9)-(11) verifies the final conditions

$$u(T, x) = u_0(x), \quad u_t(T, x) = u_1(x), \quad x \in \Omega.$$  \hspace{1cm} (12)

One can solve such type of problems by the Hilbert Uniqueness Method. To this end, we proceed as follows. Given $(w_0, w_1) \in (C^\infty_c(\Omega))^2$, we introduce the adjoint system of (9), that is

$$\left\{ \begin{array}{l}
D^\alpha_{T-} w(t, x) = \Delta w(t, x) \quad (t, x) \in (0, T) \times \Omega, \\
\quad w(t, x) = 0 \quad (t, x) \in (0, T) \times \partial \Omega,
\end{array} \right.$$  \hspace{1cm} (13)

with final data

$$D^\alpha_{T-} w(T, \cdot) = w_0, \quad D^{-\alpha}_{T-} w(T, \cdot) = w_1.$$  \hspace{1cm} (14)

The above problem is well-posed by means of a spectral representation and the Mittag-Leffler functions, see e.g. [5, 4, 7, 3].
Moreover, \( \alpha < 0 \). Since \( \int_0^T w(t) \, dt \) can introduce a linear operator defined as

\[
C D_{0+}^\alpha u(t, x) = \Delta u(t, x) \quad (t, x) \in (0, T) \times \Omega,
\]

and

\[
u(0, x) = u_t(0, x) = 0 \quad x \in \Omega,
\]

\[
u(t, x) = \partial_\nu w(t, x), \quad (t, x) \in (0, T) \times \partial \Omega_1, \quad u(t, x) = 0 \quad (t, x) \in (0, T) \times \partial \Omega_2.
\]

As in the non integral case, it can be proved that problem (15) admits a unique solution \( u \). We can introduce a linear operator defined as

\[
\Psi(w_0, w_1) = (- u_t(T, \cdot), u(T, \cdot)), \quad (w_0, w_1) \in (C^\infty_c(\Omega))^2.
\]

We will prove that

\[
\langle \Psi(w_0, w_1), (w_0, w_1) \rangle_{L^2(\Omega)} = \int_0^T \int_{\partial \Omega_1} |\partial_\nu w(t, x)|^2 \, dx \, dt.
\]

To this end, we multiply the equation in (15) by the solution \( w(t, x) \) of the adjoint system (13)-(14) and integrate on \((0, T) \times \Omega\), that is

\[
\int_0^T \int_{\Omega} C D_{0+}^\alpha u(t, x) w(t, x) \, dx \, dt = \int_0^T \int_{\Omega} \Delta u(t, x) w(t, x) \, dx \, dt.
\]

If we apply (8), then we obtain for any \( x \in \Omega \)

\[
\int_0^T C D_{0+}^\alpha u(t, x) w(t, x) \, dt = \int_0^T I_{T-}^{2-\alpha} u(t, x) w(t, x) \, dt = \int_0^T u_t(t, x) I_{T-}^{2-\alpha} w(t, x) \, dt.
\]

Integrating twice by parts and taking into account that \( u(0, \cdot) = u_t(0, \cdot) = 0 \), we have

\[
\int_0^T u_t(t, x) I_{T-}^{2-\alpha} w(t, x) \, dt = u_t(T, x) I_{T-}^{2-\alpha} w(T, x) - \int_0^T u_t(t, x) \frac{\partial}{\partial t} I_{T-}^{2-\alpha} w(t, x) \, dt
\]

\[
= u_t(T, x) I_{T-}^{2-\alpha} w(T, x) - u(T, x) \frac{\partial}{\partial t} I_{T-}^{2-\alpha} w(T, x) + \int_0^T u(t, x) D_{T-}^{1-\alpha} w(t, x) \, dt.
\]

Since \(-1 < \alpha - 2 < 0\), we can use (7) to get

\[
I_{T-}^{2-\alpha} w(T, x) = D_{T-}^{\alpha-2} w(T, x).
\]

Moreover, \( 0 < \alpha - 1 < 1\), so by (6) for \( n = 1 \) we have

\[
\frac{\partial}{\partial t} I_{T-}^{2-\alpha} w(T, x) = D_{T-}^{\alpha-1} w(T, x).
\]

Combining (19) with (20) yields

\[
\int_0^T C D_{0+}^\alpha u(t, x) w(t, x) \, dt
\]

\[
= u_t(T, x) D_{T-}^{\alpha-2} w(T, x) - u(T, x) D_{T-}^{\alpha-1} w(T, x) + \int_0^T u(t, x) D_{T-}^{\alpha-2} w(t, x) \, dt.
\]
On the other hand, integrating twice by parts with respect to the variable $x$ we have
\[ \int_\Omega \Delta u(t, x) w(t, x) \, dx = \int_\Omega u(t, x) \Delta w(t, x) \, dx - \int_{\partial \Omega} u(t, x) \partial_x w(t, x) \, dx. \]  
(22)

Putting (21) and (22) into (18) we get
\[ \int_0^T \int_\Omega u(t, x) D_{T-}^2 w(t, x) \, dxdt + \int_\Omega u_\tau(T, x) D_{T-}^2 w(T, x) \, dx - \int_\Omega u(T, x) D_{T-}^1 w(T, x) \, dx \]
\[ = \int_0^T \int_\Omega u(t, x) \Delta w(t, x) \, dxdt - \int_0^T \int_{\partial \Omega} u(t, x) \partial_x w(t, x) \, dxdt. \]

We recall that $w$ is the solution of the adjoint problem (13): keeping $D_{T-}^2 w = \Delta w$ in mind, from the above equation we have
\[ \int_\Omega u_\tau(T, x) D_{T-}^2 w(T, x) \, dx - \int_\Omega u(T, x) D_{T-}^1 w(T, x) \, dx = - \int_0^T \int_{\partial \Omega} u(t, x) \partial_x w(t, x) \, dxdt. \]

By the final data (14) of $w$ and the boundary conditions satisfied by $u$, see (15), we deduce that
\[ \langle \Psi(w_0, w_1), (w_0, w_1) \rangle_{L^2(\Omega)} = \int_\Omega u(T, x) w_1 \, dx - \int_\Omega u_\tau(T, x) w_0 \, dx = \int_0^T \int_{\partial \Omega} |\partial_x w(t, x)|^2 \, dxdt, \]

hence (17).

Define a semi-norm on the space $(C^\infty_c(\Omega))^2$: for $(w_0, w_1) \in (C^\infty_c(\Omega))^2$ we set
\[ \|(w_0, w_1)\|_F := \int_0^T \int_{\partial \Omega} |\partial_x w(t, x)|^2 \, dxdt. \]
(23)

We observe that $\| \cdot \|_F$ is a norm if and only if the following uniqueness theorem holds.

**Theorem 3.1** If $w$ is the solution of problem (13)–(14) such that
\[ \partial_x w(t, x) = 0, \quad \forall (t, x) \in [0, T] \times \partial \Omega, \]
then
\[ w(t, x) = 0 \quad \forall (t, x) \in [0, T] \times \Omega. \]

If theorem 3.1 holds true, then we can define the Hilbert space $F$ as the completion of $(C^\infty_c(\Omega))^2$ for the norm (23). Moreover, the operator $\Psi$ extends uniquely to a continuous operator, denoted again by $\Psi$, from $F$ to the dual space $F'$ in such a way that $\Psi : F \to F'$ is an isomorphism.

In conclusion, if we prove the uniqueness result given by theorem 3.1 and
\[ F = H^1_0(\Omega) \times L^2(\Omega) \]
with the equivalence of the respective norms, then we can solve the reachability problem (9)–(12) taking $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$. 

5
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