Classical height models with topological order

Christopher L Henley

Department of Physics, Cornell University, Ithaca, NY 14853-2501, USA

Received 24 November 2010, in final form 9 February 2011
Published 6 April 2011
Online at stacks.iop.org/JPhysCM/23/164212

Abstract
I discuss a family of statistical-mechanics models in which (some classes of) elements of a finite group \( G \) occupy the (directed) edges of a lattice; the product around any plaquette is constrained to be the group identity \( e \). Such a model may possess topological order, i.e. its equilibrium ensemble has distinct, symmetry-related thermodynamic components that cannot be distinguished by any local order parameter. In particular, if \( G \) is a non-Abelian group, the topological order may be non-Abelian. Criteria are given for the viability of particular models, in particular for Monte Carlo updates.

1. Introduction
‘Topological order’ [1, 2] in a system means that it has an emergent ground-state degeneracy (in the thermodynamic limit), but (in contrast to symmetry breaking), no local order parameter operator can distinguish the states. Topological order has attracted great interest over the last 20 years, since (i) it cannot (by definition) be captured by the Landau order parameter paradigm and is hence exotic from the viewpoint of traditional solid-state theory [2]; (ii) it is associated with ‘fractionalized’ excitations; (iii) it is proposed to implement qubits by the ground-state degeneracy, the coherence of which is robust against environmental perturbations [3, 4]; (iv) the formulation in terms of ground-state degeneracy makes it attractive for numerical exploration by exact diagonalization [5].

The best-known examples of topological order are quantum mechanical: the quantum Hall fluids and lattice models based on \( Z_2 \), the simplest group, such as Kitaev’s toric code [4]. Indeed, Wen [1, 2] once emphasized quantum mechanics as a defining property of topological order. But we can separate these notions: topological order (as defined above) is meaningful in a purely classical model (as developed in this paper) or in a quantum mechanical model at \( T > 0 \) [6] (so that, e.g., its renormalization group fixed points represent classical behaviors). Indeed, I would suggest that the subject of topological order skipped over more elementary examples, owing to historical accident. Compare with the history of spontaneous symmetry breaking: theorists understood classical criticality long before quantum criticality [7], and we approach quantum critical properties in light of their similarities or differences from the classical case.

Analogously, it is hoped that, in the case of topological order, classical models will (at the least) be a pedagogical aid, and that behaviors evidenced in classical models may provide a framework for conjectures about the quantum models. Disentangling classical notions and inherently quantum mechanical ones might lead to clearer (or at least different) thinking. Also, the framework in this paper naturally draws us to face hitherto unfamiliar groups—e.g. the group \( A_5 \) (see section 3)—and it might inspire the construction of quantum mechanical models involving these groups.

The explicit notion of ‘classical topological order’ was introduced and highlighted in [6], in particular the points that (i) it is characterized by ergodicity breaking, (ii) it can be implemented by hard constraints, and (iii) it must have a discrete classical dynamics—all of which applies to the models in this paper. However, much of [6] was framed in terms of the relation of the classical model to a quantum model, e.g. by taking the quantum model to a temperature at which quantum coherence is no longer important\(^1\), or by removing some Hamiltonian terms. In the present work, the model is formulated from the start as an ensemble of classical statistical mechanics, without concern for the existence of a quantum counterpart. That will permit consideration of a richer set of models (i.e. discrete non-Abelian groups \( G \)), for which we might not know how to concoct a good quantized version.

I will consider models based on either Abelian or non-Abelian discrete groups. It should be noted that abelianness

---

\(^1\) The topological order is spoiled at any \( T > 0 \) in principle, by thermally excited ‘vortices’, since they are deconfined. However, that is insignificant in practice if the vortex core energy is made large. The same would be true in our models if vortices were allowed; we sidestepped this problem by constraining the configuration space to disallow them, in effect setting the core energy to infinity.
in this paper has a different significance than in the quantum context. In the latter case, the group in question is the Berry phase (or its generalization to a unitary matrix) induced by evolution of the wavefunction from one state to an equivalent one. A specific case is the statistics of quasiparticles whose world lines braid around each other. Quantum mechanical non-Abelian statistics may be realized in models based on Abelian groups such as $Z_2$. Most of the fractional quantum Hall fluids are Abelian, but non-Abelian statistics is more suitable for quantum computation [4, 8]. Proposed realizations of non-Abelian topological order in this sense are formulated in lattice models as sums over loop coverings [9, 10].

The primary focus here is not on ideas that point to analytic solutions or to connections with the existing literature of topological order. Instead, this is meant as a generic blueprint for numerical studies. For example, the classification of models in section 3 (as summarized in the tables) is motivated by the need to select a good one for simulations, and the quantities defined in section 5 are all measurable in Monte Carlo simulations. However, the actual simulation results will be left to subsequent papers [11].

1.1. Height models

I shall realize topological order by generalizing the concept of ‘height models’. Their defining property [12–20] is the existence of a mapping directly from any allowed spin configuration $\{\sigma_i\}$ to a configuration of heights $h(r)$, wherein $h(r) - h(r')$, for adjacent sites $r$ and $r'$, is a function of the spin variables in the neighborhood (normally, either two spins on the sites $r$ and $r'$, or else one spin on site $i$ at the midpoint of the $r-r'$ bond: spins and heights commonly live on different lattices).

The ‘spins’ in the model could be any discrete degree of freedom—e.g. dimer coverings. For $\{h(r)\}$ to be well defined, it is necessary (and sufficient) that the sum of height differences is zero round any allowed plaquette configuration. Thus, a (local) spin constraint is assumed that excludes (at least) the configurations without well-defined heights, yet still allows a nonzero entropy of states $S$ in the thermodynamic limit. In the simplest cases, $h(r)$ is integer-valued.

Such models may have ‘rough’ phases, in which the coarse-grained $h(r)$ behaves as a Gaussian free field, i.e. the effective free energy of long-wavelength gradients is

$$F = \frac{1}{2} \int d^2 r |\nabla h|^2. \quad (1.1)$$

Via the apparatus of the Coulomb gas formalism [21, 22], this implies that the spin variables have power-law correlations (critical state); the topological defects may have unbinding transitions like the Kosterlitz–Thouless transition. Indeed, in two dimensions most critical states can be addressed as ‘height models’ [21] and this formalism provides an alternative route to computing exact critical exponents [17], besides conformal field theory.

Note that on a topologically nontrivial space (such as the torus periodic boundary conditions), there are nontrivial loops $\ell$, such that the net height difference (or ‘winding number’)$w_{\ell}$ added around such a loop is nonzero. It is easy to see that this is a topological invariant, in that $w_{\ell}$ is unchanged if the loop is shifted and deformed (so long as it stays topologically equivalent to the original one). Thus, if $\ell_1, \ell_2, \ldots$ are the fundamental loops, the configuration space divides up into sectors labeled by $(w_{\ell_1}, w_{\ell_2}, \ldots)$. Here, and also for the discrete-group height models introduced in the paper, a sector is each set of configurations which can be connected to each other by a succession of local spin changes (i.e. ‘updates’ in the terminology used later). Point defects may also be admitted and loops around them may also have a nontrivial $w_{\ell}$ (in which case they are topological defects).

1.2. Outline of the paper

In this paper, I first (section 2) generalize the height-model idea to the case where the height variable belongs to a discrete (finite) Abelian or non-Abelian group, thus defining a family of classical models which (in many cases) has a topological order. The models are defined by a lattice, a group, and the selected subset of group elements which are permitted values for the ‘spins’ of the model; the spins sit on the bonds. Non-Abelianess of the group has interesting consequences: for example, a collection of defects no longer has a unique net charge (section 2.3).

In sections 3 and 4, I survey the various combinations for the smallest non-Abelian groups, using the crude Pauling approximation as a figure of merit to identify the most attractive models for Monte Carlo simulation, using single-site updates. Furthermore, section 5 suggests what quantities are interesting to measure in such a simulation; however, no simulation results are reported in this paper. But some first analytic results are included in section 6, based on transfer matrices and hence implicitly one-dimensional: the main point is to shed light on how the size dependence or defect pair correlation depends on the group elements labeling the topological sector or the defects.

Finally, the conclusion (section 7) reflects on which topological behaviors are inherently quantum mechanical, and which are not (in that the same behavior can be found in classical models). Furthermore, applications are suggested, either to simulating systems with vacancy disorder, or to constructing quantum versions of the models in this family.

2. Definitions and topological behaviors

This paper is meant to introduce (and compare) a whole family of models. In this section, I define the general rules for this family, and then describe the most promising examples. (In section 4, I shall exhibit consequences for Monte Carlo updating of such models.)

2.1. Model definition

Let us take a ‘lattice’ (not necessarily Bravais, e.g. honeycomb) of sites $r$. The spins sit on the bonds of this lattice, and take values in the discrete group $G$; they are

$$\sigma(r, r') \in G. \quad (2.1)$$
where \((\mathbf{r}, \mathbf{r}')\) labels a bond of nearest neighbors. The bonds are directed; if we reverse the direction we view a bond, the spin on it turns into its inverse:

\[
\sigma(\mathbf{r}', \mathbf{r}) \equiv \sigma(\mathbf{r}, \mathbf{r}')^{-1}. \tag{2.2}
\]

Then each configuration of the spins induces a configuration of ‘heights’ \(h(\mathbf{r}) \in \mathcal{G}\), defined by

\[
h(\mathbf{r}) = \sigma(\mathbf{r}, \mathbf{r}') \ast h(\mathbf{r}'), \tag{2.3}
\]

where ‘\(\ast\)’ represents the group multiplication. Of course, \(h(\mathbf{r})\) is only defined modulo a global multiplication by some element \(\tau\), \(h'(\mathbf{r}) = h(\mathbf{r})\tau\); to make it well defined, we could arbitrarily require (say) \(h(0) = e\), where \(e\) is the group identity. One could then explicitly construct \(h(\mathbf{r})\) at the neighbors of site 0, and iteratively at their neighbors, etc; the result is independent of which bonds \((\mathbf{r}, \mathbf{r}')\) are used for this, if and only if a plaquette constraint is satisfied (equation (2.5), below).

It will be useful throughout to define the line or loop product of \(p\) spins

\[
\gamma(\ell) \equiv \sigma(\mathbf{r}_1, \mathbf{r}_p) \ast \sigma(\mathbf{r}_p, \mathbf{r}_{p-1}) \ast \cdots \ast \sigma(\mathbf{r}_2, \mathbf{r}_1). \tag{2.4}
\]

Here the loop \(\ell\) is a string of bonds \((\mathbf{r}_1, \mathbf{r}_{p+1})\) connecting end-to-end, for \(k = 0, \ldots, p\); it is a loop when \(\mathbf{r}_p = \mathbf{r}_0\).

2.1.1. Plaquette constraint. Two constraints are imposed on the spin configurations. The first is the plaquette constraint: we require the loop product around any elementary plaquette to be the identity

\[
\gamma(\ell_{\text{plaq}}) = e. \tag{2.5}
\]

This is necessary (and sufficient) for \(h(\mathbf{r})\) to be well defined.

One can define variants of any model by relaxing the plaquette constraint to allow a small number of defect plaquettes, around which the loop product is not the identity. Suppose that defects cost an energy \(\Delta\) (possibly depending on the kind of defect): then the basic, defect-free version of the model can be viewed as a Boltzmann ensemble in the limit \(T/\Delta \to 0\). On the other hand, if we imagine that there is a spin–spin interaction \(J\) that breaks the degeneracy of the states satisfying the plaquette constraint, then the basic version of the model is the \(T/J \to \infty\) limit.

Using condition (2.5) and induction (adding one plaquette at a time to the loop), the loop product must be \(\gamma(\ell) = e\) for any finite loop \(\ell\) that is contractible to a point in small steps.

2.1.2. Spin constraint. The second constraint is the spin constraint: choose a ‘spin subset’ \(\mathcal{S} \subset \mathcal{G}\) such that

\[
\sigma(\mathbf{r}, \mathbf{r}') \in \mathcal{S} \tag{2.6}
\]

everywhere. Hence, the choice of \(\mathcal{S}\) is a major part of a model’s definition; in section 3.3, below, I will discuss other desirable features of \(\mathcal{S}\). The spin constraint is retained even in versions of the model with defect plaquettes.

The constraints should respect the group and lattice symmetries. Implementing the group symmetry means requiring

\[
\sigma \in \mathcal{S} \Rightarrow u \ast \sigma \ast u^{-1} \in \mathcal{S} \tag{2.7}
\]

for any conjugating element \(u\). Thus, \(\mathcal{S}\) must be one of the group’s conjugacy classes—the simplest case—or a union of such classes. (Some non-Abelian groups, and all Abelian ones, have ‘outer’ automorphisms, symmetries which cannot be implemented by conjugacy within \(\mathcal{G}\): we may also wish to implement those symmetries too.)

To implement lattice symmetry, one asks

\[
\sigma \in \mathcal{S} \Rightarrow \sigma^{-1} \in \mathcal{S}, \tag{2.8}
\]

so the model respects inversion (around the bond’s midpoint)\(^2\).

Let us further require

\[
e \notin \mathcal{S}; \tag{2.9}
\]

thus a uniform height configuration is disallowed. Finally, and trivially,

\[
\mathcal{S}\text{ generates the full group } \mathcal{G}. \tag{2.10}
\]

(If not, I could have redefined \(\mathcal{G}\) as the subgroup generated by \(\mathcal{S}\).)

Without the spin constraint, the models would be identical to the lattice gauge models of Doucôt and Ioffe [23]. In such a model, only gauge-invariant (i.e. loop) quantities can have nonzero expectations; other correlation functions are zero, even at the nearest distance. In contrast, these group-height models (like the original height models) have nontrivial finite-size effects and local correlations: in particular, there are mediated interactions between topological defects.

Furthermore, the spin constraint allows the possibility of a long-range ordered phase, particularly if we assign different Boltzmann weights to different configurations, and we may find phase transitions as those parameters are varied. Partial orderings are also possible, and transitions might occur between different topological orders. It will be easier to explore the phenomenology of such transitions in the classical realm.

2.2. Topological sectors and topological order

For these models, a ‘sector’ means simply the configurations that can be accessed by a succession of local updates. (Here ‘local update’ means an operation that turns one valid configuration into another by changing spins in a small neighborhood of some site, as might be deployed for Monte Carlo simulation. A ‘nonlocal’ rearrangement, as developed in section 4, means that the cluster of updated sites can be arbitrarily large, and in particular could include a topologically nontrivial chain of sites that spans the periodic boundary conditions.) By this definition, ‘sectors’ are well defined in any finite system larger than the maximum update cluster. In other models and with other definitions, passing between sectors is not absolutely forbidden, so that sectors (more exactly ‘components’, like the up and down ordered phases in ferromagnets) are emergent in the thermodynamic limit.

\(^2\) In rare cases one drops the condition of including the group inverse (2.8). For example, the height representation of the square lattice dimer model is \(\mathbb{Z}\{-3, +1\}\).
In these models, sectors can be labeled by loop products $\gamma(\ell)$. As noted after (2.5), products around topologically trivial loops must give the identity, but others—e.g. through the periodic boundary conditions of a torus system—in general do not. Such loop products are not changed by local updates, and therefore must take the same value for all states in a sector. So we call sectors ‘topological’ when they are distinguished (and necessarily disconnected) by having different values of the loop product(s). The definition of topological order is that—in the thermodynamic limit—different topological sectors all become equivalent, in that they cannot be distinguished by any local expectations. Furthermore, just as the ground-state energies in different sectors should become equal in the case of quantum topological order, the free energies should become equal in our models. In place of a Landau order parameter, of quantum topological order, the free energies should become equal in our models. In place of a Landau order parameter, of quantum topological order, the free energies should become equal in our models. In place of a Landau order parameter, of quantum topological order, the free energies should become equal in our models. In place of a Landau order parameter, of quantum topological order, the free energies should become equal in our models.

Let $\ell_1, \ell_2, \ldots, \ell_g$ be the basic independent loops, where $g$ is the genus; then the loop products $\{\gamma_i = \gamma(\ell_i)\}$ (all taken from the same origin) label the possible topological sectors of configuration space. An interesting question is how many distinct sectors there are, given the system’s genus $g$.

### 2.2.1. Invariance

Before counting sectors, we need to explore invariance properties of the sector labels, in case some labels are equivalent to others. First, there is a sort of gauge freedom: if we had evaluated these loops starting from $r$ instead of from $0$, then

$$\gamma_0 \rightarrow \gamma_0' = \gamma_{0r} * \gamma_0 * \gamma_{0r}^{-1},$$

(2.11)

where $\gamma_{0r}$ is the line product along any path from the origin to $r$; notice that this same element conjugates all the distinct loops. However, just because our labeling fails to distinguish two sectors, it does not conclusively show that they are the same.

A better criterion for counting sectors as equivalent is that one can be turned into another by local updates. Keeping the same origin, but performing a single-site update (see equation (4.2), below) hitting on the origin vertex, one gets another conjugacy

$$\gamma_0 \rightarrow \gamma_0' = \tau * \gamma_0 * \tau^{-1},$$

(2.12)

where $\tau$ is now the updating multiplier.

When the group $G$ is Abelian the sector labels are invariant with respect to how we take the loop and unchanged by local updates. We can have an independent and invariant loop product $\gamma_L$ for every topologically independent loop, so the number of sectors is $n_G 2^g$ where $g$ is the system’s genus ($2g = 2$ for torus), and $n_G$ is the number of elements in the group.

Table 1 shows the number of classes $\mu_2$ and the sector count $\mu_2$ for some groups of interest.

### 2.2.2. Sector counting in the non-Abelian case

On the other hand, in the non-Abelian case, a loop product is invariant only up to a conjugacy, so we have fewer sectors. Furthermore, the allowed values of distinct loop products are not independent. Consider a square lattice model in a rectangular system cell $L_x \times L_y$, with periodic boundary conditions; let $(\gamma_x, \gamma_y)$ be the loop products along straight lines of bonds running from the origin site $(0,0)$, in the $x$ or $y$ directions, respectively. The loop running from $(0,0)$ to $(L_x,0)$ to $(L_x, L_y)$ to $(0,L_y)$ and back to $(0,0)$ contains no defect, so by inductive use of the plaquette constraint its loop product is $e$. Yet the four segments of this loop are just $\gamma_x$ and $\gamma_y$, forwards or backwards, so the loop product is

$$\gamma_x^{-1} * \gamma_y^{-1} * \gamma_x * \gamma_y = e;$$

(2.13)

in other words, $\gamma_x$ and $\gamma_y$ must commute.

So, in effect, we must define an equivalence relation $(\gamma_x, \gamma_y) \sim (\gamma_x', \gamma_y')$ whenever the pair satisfies (2.13), and each topological sector is one equivalence class. In the case of a larger genus, we extend in the obvious way to longer lists.

Some obvious kinds of equivalence classes are:

(i) $(e, e)$;

(ii) $(\omega, e)$ or $(e, \omega)$;

(iii) $(\omega, \omega^k)$ or $(\omega^k, \omega)$ for $k = 1, \ldots, m - 1$, where $\omega$ is an element (not the identity) of order $m$;

(iv) if the group has a nontrivial center $G_Z$, consisting of elements that commute with all the other elements, then if $(\gamma_x, \gamma_y)$ is a sector then $(z_x \gamma_x, z_y \gamma_y)$ is another sector, where $z_x, z_y \in G_Z$.

Table 1 shows the number of classes $\mu_1$ and the sector count $\mu_2$ for some groups of interest.

### 2.3. Defects and non-Abelian effects

We can allow the possibility (dilutely) of a plaquette that violates the plaquette constraint. The loop product around it will be called $\beta$. It is analogous to the Burgers vector of a dislocation, or of the topological defects in the usual height models based on $Z$. 

---

3 The ground-state degeneracy of the Moore–Read Pfaffian state is evaluated in [24].
Hence $\gamma$ defect 3 is moved between defects 1 and 2, the product around defects 2 and 3 is still $\gamma$. After defect 3 is moved between defects 1 and 2, the product around defects 2 and 3 is still $\gamma_2\gamma_3$. (c) After defect 3 has passed to the other side, the product around defect 2 has the new value $\gamma_2'$, which is unchanged from (b) and therefore equal to $\gamma_2\gamma_1$. Hence $\gamma_2' = \gamma_3\gamma_2\gamma_3^{-1}$, and the combined charge of defects 1 and 2 is now $\gamma_2' \gamma_1 = \gamma_1\gamma_2\gamma_3^{-1} \gamma_1$, which can be in a different class than $\gamma_2\gamma_1$.

In the case of a height model (in a rough phase), topological defects are like vortices in a two-dimensional Coulomb gas [21, 22]. They behave like $U(1)$ electric charges in a two-dimensional universe. Opposite charges feel an attractive logarithmic potential. In contrast, in the case of topological order, the attractive potential decays exponentially and the defects are deconfined [25].

If the topological order is non-Abelian, the non-commuting property of defect charges has some interesting consequences. They are not unique to topological order; this also is a long known property of defects of traditional ordered systems. This commutation is a ‘group commutator’. Such a single-defect state may conveniently allow numerical measurements of the creation free energy of a single defect. Of course, such a state is never possible for Abelian defects; in that case, a system with periodic boundary conditions must have either no defects, or at least two of them.

3. Possible models

In this section, I survey specific models, emphasizing the criteria which would make some of them particularly attractive for future investigations. To summarize section 2.1: models in this paper are specified by (i) the group $G$ (values of spins), (ii) the spin subset $S$ (values of spins), (iii) the lattice whose bonds the spins sit on.

Therefore, the models will be named in the form ‘$G(m)$lat’. Here ‘$G$’ is the group’s name, (m) is the order of the elements in the selected conjugacy class (usually that is unambiguous), and ‘lat’ abbreviates the lattice (‘tri’, ‘sq’, or ‘hc’ for triangular, square, and honeycomb). Thus ‘$S_3(2,3)$tri’ means that $G$ is the permutations of three objects, $S$ contains three pairs exchanges, as well as the two cyclic permutations (i.e. every group element except for $e$), and ‘tri’ means that the spins sit on the edges of the triangular lattice. An variant nomenclature is sometimes convenient, in which the ‘(m)’ in the label gets replaced by ‘$[\sigma_1, \sigma_2, \ldots]$’: the set $[\sigma_1, \sigma_2, \ldots]$ is simply the listing of the selected elements.

3.1. Groups

Table 1 lists the groups and spin subsets I shall be interested in. For future reference, I mention the automorphism group $A_G$ of a group $G$, which is simply its symmetry group. Each $a \in A_G$ is a permutation of the group elements preserving its structure, $a(g * g') = a(g) * a(g')$. When $G$ is non-Abelian, there is a subset of the automorphism group called the inner automorphisms, defined as the conjugations, $a_{z}(g) \equiv z * g * z^{-1}$. Obviously $a_{z}(a_{z}(g)) = a_{z * z}(g)$, so the inner automorphism subgroup is isomorphic to $G/G_Z$, where $G_Z$ (the center subgroup) consists of the elements that commute
3.1. Abelian groups. We start by considering discrete Abelian groups in this family of models. The smallest of them, $Z_2$, does not work since $S$ can have only one element. The next simplest cases are cyclic groups $Z_q$, i.e. the integers modulo $q$ under addition, although these often turn out to be height models (see section 3.2.2 below).

Beyond that we go to direct products of cyclic groups, indeed any Abelian group can be represented thus. If $S$ was also taken to be a direct product, of course the model would reduce to a superposition of non-interacting models, one for each factor. However, there are many attractive examples in which $S$ is not a direct product, in particular $Z_2 \times Z_2$ (see section 3.2.1).

3.1.2. Non-Abelian groups. The smallest non-Abelian group is $S_3$, the permutations on three objects (also isomorphic to the dihedral group $D_3$). Here, $S$ may be taken as the class of all pair permutations, the model $S_3(2)$, or as all permutations except the identity, that is $S_3(1, 2)$ in our notation.

Each of the next two smallest non-Abelian groups has eight elements. One of these is the eight-element quaternion group $Q$, i.e. the unit elements $\{\pm 1, \pm i, \pm j, \pm k\}$ from the quaternion ring. Here $S$ must be the class of the six elements not equal to $\pm 1$. The other eight-element non-Abelian group is $D_4$, the symmetry group of the square lattice.

The ‘alternating groups’ $A_4$ and $A_5$ are especially attractive for our purposes due to their high symmetry (so we can choose $S$ to be a single class containing a sizable fraction of all the group elements). They consist of the even permutations of four and five elements. Note that $A_4$ and $A_5$ are also the point groups of the (proper) rotations of a regular tetrahedron and a regular icosahedron, respectively. Being subgroups of $SO(3)$, these groups might in some sense serve as a discretization of it, just as clock models are a discretization of the XY model. That would be interesting as a way to make a connection to topological models (or gauge theories) defined in terms of Lie (i.e. continuous) groups.

Finally, $A_4$ is the smallest non-Abelian simple group, meaning it has no normal subgroups; as we shall see in a moment (section 3.2), normal subgroups are an annoyance since they tend to make the behavior more trivial than would be expected for the group $G$.

3.2. Example models

Next I shall survey the simplest examples. Most of them reduce, in some fashion, to previously known models; that is an advantage for computational studies, since old results can be used as checks. In several cases, the models in our family have ‘accidental’ topological order, i.e. beyond the group $G$. In particular, some of them have height representations.

The group and subgroup involved in our spin constraint are finite, and so is each plaquette; thus it can happen that the allowed configurations satisfy stronger constraints than those they were designed to fulfill. The first five subheadings below all, in one sense or another, reduce to known models.

3.2.1. $Z_2 \times Z_2$ and the three-coloring model. For a first example, let $G \cong Z_2 \times Z_2$, an Abelian group. Besides the identity, this group has three equivalent elements $a, b, c$; each has order 2, and the product of any two gives the third. If we treat these as a class (although they are not conjugate, since the group is Abelian), then we must choose the ‘spins’ to be that class, giving a model $Z_2 \times Z_2(2)$, i.e. the three-coloring model. Since $a = a^{-1}$, etc, we can depict the spins using three (undirected) ‘colors’ of the edges. On the square lattice this gives perhaps our simplest example (figure 2).

What about the triangular lattice case (model $Z_3 \times Z_2(2)$)? The plaquette constraint is simply that each triangle has one edge of each color: the ‘three-coloring model’ [28]. (It is usually represented on the edges of the dual (honeycomb), where the constraint says each vertex has three colors; in either case, the spins live on kagome lattice vertices, and the configurations are also the ground states of the three-state Potts...
antiferromagnet on that lattice.) This model is known to have a \( \mathbb{Z} \times \mathbb{Z} \) height representation \([17, 29]\), in addition to the finite-group height field \( h(\mathbf{r}) \) defined by \((2.3)\).

### 3.2.2. Six-vertex model.

For another example, take \( \mathcal{G} \) to be \( \mathbb{Z}_q \), with \( q > 4 \), and let the lattice \([\mathbf{r}]\) be the square lattice. Choose \( \mathcal{S} = \{+1, -1\} \). (The two elements are not the same class; they are related only by an outer automorphism.) Then the sum of spins around a plaquette can be zero (mod \( q \)) only if it is just zero, i.e. there are exactly two \(+1\) and two \(-1\) in the loop. If we express these spins on the dual (also square) lattice, as an arrow pointing outwards (respectively, inwards) wherever \( \sigma = +1 \) (respectively, \(-1\)) as the loop is traversed counterclockwise, we see that these are just the configurations of the six-vertex model—which also has an integer-valued height field\(^5\).

Since \( \mathbb{Z} \) or \( \mathbb{Z}_m \) are Abelian groups, \([+1, -1]\) is merely an outer automorphism class.

### 3.2.3. Groups with an even subgroup.

An even subgroup \( \mathcal{E} \) (with \( n_\mathcal{E} = n_\mathcal{G}/2 \) elements) has \( \mathcal{G}/\mathcal{E} \cong \mathbb{Z}_2 \). That is, any product of an even number of elements lies in \( \mathcal{E} \). Say that the spin subset \( \mathcal{S} \) consists of odd elements. (If it consisted of even elements, we would generate at most \( 2^n \mathcal{E} \) elements, we would generate at most \( \mathcal{E} \).) Notice that such a model cannot use the triangular lattice, since the plaquette rule cannot be satisfied (the plaquette product must be odd, while \( e \) is an even element).

Now if the simulation cell has even dimensions, the possible topological products \( \gamma(\ell_i) \) must lie in \( \mathcal{E} \). (Even if the cell has an odd dimension, the possible values of \( \gamma(\ell_i) \) still correspond one-to-one with elements of \( \mathcal{E} \).) Thus, the topological sector labels can only belong to the subgroup \( \mathcal{E} \).

For example, in permutation groups the even subgroup consists of even permutations. In the case of \( S_3 \), the even permutations are just \((123)\) and its powers, so \( \mathcal{E} \cong \mathbb{Z}_2 \). Consequently the model \( S_3(2)\text{sq} \) can have only Abelian topological behavior. In our list, \( D_4 \) is another group that contains an even subgroup (the proper rotations).

### 3.2.4. Groups with a center.

What if \( \mathcal{G} \) has a nontrivial center \( \mathcal{G}_Z \)? (The center is subgroup of elements that commute with every other element.) For example, if we adopt the group \( Q \) of unit axis quaternions (which has order 8), then \( Q_Z = \{+1, -1\} \cong \mathbb{Z}_2 \) and \( Q/Q_Z \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \). Thus, the model \( Q(2)\text{tri} \) projects onto configurations of the three-coloring model. (Just map \( \pm i \to a, \pm j \to b, \pm k \to c \). The group \( D_4 \) also has a center (two-fold rotations, i.e. inversions, commute with everything).

### 3.3. Criteria for models: estimates of entropy

To estimate at once the viability of many different models, I shall use very crude estimates of the entropy and (in section 4.2) updatability. Say the lattice has coordination \( z \) and the dual lattice has coordination \( z_d \), i.e. the number of sides of each plaquette. (These numbers are related by \( 1/z + 1/z_d = 1/2 \).) Also, say the group has \( n_\mathcal{G} \) elements, of which \( n_\mathcal{S} \) are in the selected subset \( \mathcal{S} \). These three parameters—\( n_\mathcal{G}, n_\mathcal{S} \), and \( z \) (or \( z_d \))—contain much of what we need to characterize the possible models. See table 2 for the parameters related to the lattice geometry, and table 1 for those related to the groups and the spin subsets.

I will use a Pauling estimate for the entropy. There are \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) edges and hence \( n_\mathcal{S} \mathbb{Z}_2 \times \mathbb{Z}_2 \) ways of placing spins independently chosen from \( \mathcal{S} \), in a hypothetical ensemble that does not (yet) enforce the plaquette constraint. If we knew the fraction of all these states that do obey the plaquette constraint, we would have the total count of allowed states and thus the desired entropy.

Pauling’s approximation is to pretend the event of satisfying the plaquette constraint is uncorrelated between plaquettes. So let \( f_e \) be the chance that a given plaquette has plaquette product equal to \( e \). Then in this approximation, the probability is \( f_e^{Nz/z_d} \) that the plaquette constraint is satisfied on all \( Nz/z_d \) plaquettes of the whole system. Thus the Pauling estimate of the ensemble entropy is

\[
\mathcal{E}^{NS_{\text{Pauling}}} = \left( n_\mathcal{S} z_d^{1/2} f_e^{z/z_d} \right)^N.
\]

The condition we must satisfy, in order to have an ensemble at all, is \( S_{\text{Pauling}} > 0 \), i.e.

\[
n_\mathcal{S} > 1/f_e^{2/z_d}.
\]

If \( z_d \) is not too small, we may estimate that the plaquette product is equally likely to be any group element, hence very crudely

\[
f_e \approx 1/n_\mathcal{G}.
\]

Less crudely, one can work out the actual probabilities that the product of \( z_d \) random group elements from the allowed set \( \mathcal{S} \) will give the identity, and these are the \( f_e \) values in table 3. Comparison of those \( f_e \) values with \( n_\mathcal{G} \) in table 1 shows that, usually, \((3.3)\) is not bad. When the product of two elements of \( \mathcal{S} \) is particularly likely to fall into one class, the true \( f_e \) deviates more from \((3.3)\), either on the low side (e.g. the model \( A_5(2)\text{tri} \) or the high side (e.g. \( A_5(5)\text{sq} \)). The extreme case is if the group contains even and odd elements, and \( \mathcal{S} \) consists of odd elements (the \( S_3(2) \) or \( D_4(m, m') \) examples in table 3). In

---

| Lattice        | Tag | \( z_d \) | \( p_{\text{th}} \) | \( p_{\text{os}} \) |
|---------------|-----|----------|----------------|----------------|
| Triangular    | tri | 6        | 0.347          | 0.500          |
| \( \sigma \)-phase | –   | 5        | 0.414          | 0.551          |
| Square        | sq  | 4        | 0.500          | 0.593          |
| Kagome        | –   | 4        | 0.524          | 0.653          |
| Honeycomb     | hc  | 3        | 0.653          | 0.697          |
| Square-octagon| –   | 3        | 0.677          | 0.730          |

---

\(^5\) If we take \( \mathcal{G} \) to be \( \mathbb{Z}_3 \), again with \( \mathcal{S} = \{+1, -1\} \), we get the configurations of the eight-vertex mode. These correspond one-to-one with a set of random spins in the dual lattice, so it is a trivial \( \mathbb{Z}_2 \) gauge model.
that case we should replace (3.3) by $f_e \approx 1/n_E = 2/nG$ if $z_d$ is even, but $f_e = 0$ if $z_d$ is odd.

There is just one entry in table 1 showing a negative Pauling entropy $S_{\text{Pauling}} < 0$, namely $A_5(2)\text{tri}$. It is convenient to explain this case in the language of the (proper) rotation group of an icosahedron, which is isomorphic to $A_5$. The only way that three two-fold elements can multiply to give the identity is when the two-fold axes are mutually orthogonal. Since the triangles share edges, any valid global configuration must use that same triad in every triangle; this entails a five-fold (5S) global symmetry breaking, since the fifteen two-fold axes of icosahedral symmetry break up into five disjoint orthogonal triads. Indeed the three used elements form a subgroup isomorphic to $Z_2 \times Z_2$ so we are back at the three-coloring model for which $S_{\text{Pauling}} > 0$. The Pauling approximation gave zero entropy only because it did not take account of the symmetry breaking and attempted to mix domains with incompatible symmetry breakings.

My purpose here is not to obtain quantitative estimates of the model’s entropy, although the Pauling estimate is sometimes surprisingly accurate. Rather, I want to compare these values between different models as a figure of merit, to aid us in guessing which models are the most interesting or the most tractable. To this end, the figures of merit are shown in table 3.

To satisfy equation (3.2), the three parameters get pushed in the following directions, but there are considerations limiting each of the three.

1. We want $n_G$ as small as possible; however, there are not so many small, discrete, non-Abelian groups: only three have $n_G \leq 8$, namely $S_5$ (permutations of three objects), $Q$ (quaternion group), or $D_4$ (point group of a square).

2. We want larger $n_S$, meaning the model is less constrained (and more tractable). In the limiting case $n_S = n_G$ the model is just a pure gauge theory, which is trivial apart from its global topological properties. On the other hand, a sufficiently large $n_S$ requires the inclusion of more than one conjugacy class in $S$, so that the spins can have inequivalent ‘flavors’. That is a aesthetically undesirable: a generic model (with unequal statistical weights) needs more parameters, and it is harder to imagine how such a model could be realized physically.

(3) We want large $z_d$, as in the honeycomb lattice. However, it is a aesthetically harder to implement a product constraint as the ground state condition of a physically plausible Hamiltonian. (When the product string is short, there are only a few symmetry-inequivalent cases for it, and it is easier to concoct a Hamiltonian term which does not reference the group multiplication, but which has those cases as its energy minimum.)

To satisfy (3.2) with a large group but $S$ consisting of just one conjugacy class, the group must have high symmetry; e.g., the alternating group $A_5$ (the proper icosahedral rotations) has $n_G = 60$ and contains conjugacy classes with 12, 15, or 20 elements, which using (2) would need $z_d > 3.30, 3.02, or 2.41$, respectively.

4. Monte Carlo updating

One essential criterion of a model is the possibility of Monte Carlo simulation. I limit consideration to the equal-weighted ensemble, in which every allowed configuration has the same weight. Then detailed balance is satisfied if the forwards and backwards rate constants are the same for any update move. But what is the minimum sufficient update move? For the six-vertex model it sufficed to reverse the arrows on the four edges of one plaquette, which changes the height field on one (dual-lattice) site, a purely local update. For the three-coloring model, the minimal update involves switching two ‘colors’ (e.g. a ↔ b) along a loop, a nonlocal update move. What happens generically for our family of models?

4.1. Cluster update move

The update move is most simply described in terms of the height function (defined in (2.3)). First pick at random a group element $\tau \neq e$ and a starting site $r_0$. Say $D$ is the domain being touched by the update. (It will be explained in a moment what determines $D$.) Then I prescribe that the update premultiplies the heights in this domain by $\tau$, so as to ‘shift’ them:

$$h'(r) = \begin{cases} \tau \cdot h(r) & \text{for } r \in D; \\ h(r) & \text{for } r \notin D. \end{cases} \quad (4.1)$$

This induces the following update of the spin configuration:

$$\sigma'(r', r) = \begin{cases} \tau \cdot \sigma(r', r) & \text{for } r', r \in D; \\ \tau \cdot \sigma(r', r) & \text{for } r' \in D, r \notin D; \\ \sigma(r', r) & \text{for } r' \notin D, r \in D; \\ \sigma(r', r) & \text{for } r, r' \notin D. \end{cases} \quad (4.2)$$

6 When both ends $r$ and $r'$ are in the cluster, the mapping $\sigma \to \sigma'$ in (4.2) is an ‘inner automorphism’ of the group, as defined in section 3.1. The update rule here can be generalized to allow outer automorphisms.
I call this a ‘gauge-like’ transformation [31]: it has the same form as a gauge transformation would, but it is valid only when an additional spin constraint is satisfied too.

If both endpoints of the bond are in \( D \), then \( \sigma' \) is conjugate to \( \sigma \) and must be legal (we include whole conjugacy classes in \( S \)).

On the other hand, where the \( (r, r') \) bond crosses the domain boundary \( \partial D \), the spin constraint is nontrivial to satisfy. Let us place an arrow along the edge from \( r \) to \( r' \) if and only if

\[
\sigma(r, r') \star r^{-1} \notin S. \tag{4.3}
\]

In other words, there is an arrow from \( r \) to \( r' \) whenever including \( r \) in \( D \) forces us to include \( r' \) as well. This arrow is not bidirectional. (It is in the case \( \tau^2 = e \).) Thus, we might have any of four possibilities (no arrows, arrows both way, or arrows one way) along each bond.

Then the update rule is to construct the arrowed percolation cluster consisting of site \( r_0 \), with the rule that site \( r' \) is included if site \( r \) is included and there is an arrow from \( r \) to \( r' \). This is the smallest possible updated domain containing \( r_0 \). Of course, we do not actually need to construct all the arrows; instead, we grow the cluster from the initial site, and construct arrows only from sites already in the cluster. (This recipe is reminiscent of the Wolff updating algorithm for an Ising model [32].)

Notice that (only) in the case where \( G \) is Abelian, (4.2) reduces to \( \sigma' = \sigma \) throughout the interior of \( D \). In other words, the update only changes spins along the boundary \( \partial D \) and thus is a loop update. In a non-Abelian model, however, the update is generally a cluster update.

In some models (see section 4.2) there is a strong chance to hit a system-spanning cluster, including most of the sites, which tends to be inefficient. (Updating all the sites is equivalent to no update.) To avoid this, a limiting size \( s_{\text{max}} \) for the update cluster \( D \) should be set; if this limit is reached, we cancel the tentative move and start over, choosing a new random \( r_0 \) and \( r \).

4.2. Numerical criteria for cluster updates

Notice that in growing a cluster from \( r_0 \), we never cared about the reverse arrows. Therefore, we obtain the same clusters as we would in an ordinary (not arrowed) percolation problem, if the occupied bond probability \( p_b \) is identified with the probability of an arrow in a pre-selected direction; that probability is simply

\[
p_b \approx 1 - \frac{n_S - 1}{n_G - 1}, \tag{4.4}
\]

if we chose the candidate updating factor \( \tau \) at random. These probabilities are shown in table 1. (When there is an even subgroup, typically \( \tau \) must be chosen from it.) In a case that any \( \tau \) works as a multiplier on any bond, I would write \( p_b = 0 \) in table 1, rather than use (4.4). In such a case, our model is locally trivial: exactly \( n_G^{n_G} \) configurations may be accessed, simply by applying one arbitrary group element at every site. In other words, there is a (locally) one-to-one mapping to the trivial model in which every site has an independent degree of freedom. That model is just the gauge model, which was studied previously in [23].

In the spirit of the Pauling approximation, let us now pretend that arrows on different bonds are uncorrelated: within that assumption, we must obtain the same cluster distribution as in the (thoroughly studied) problem of uncorrelated percolation on these lattices. It follows that the updating behavior tends to depend on the relation of \( p_b \) to the critical percolation fraction \( p_{cb} \). On the one hand, if \( p_b > p_{cb} \), then the cluster grows without limit, including a nonzero fraction of the whole system; in that case, the update move certainly is not efficient. On the other hand, if \( p_b / p_{cb} \) is too small, we never get a cluster at all, or else a single-site update (following section 4.3) would suffice. The ‘interesting’ case, when a cluster update is necessary and helpful, would be for \( p_b / p_{cb} \) to be close to or slightly less than unity.

4.3. Single-site updates

A single-site update is the case that the updated cluster \( D \) is just one site; thus only the \( z \) spins around it are updated. When \( p_b \) is much less than \( p_{cb} \), most clusters are small, and the probability \( P_1 \) of a single-site update is appreciable. If \( P_1 \) is large enough, it might be ergodic to use only single-site updates (i.e. to pick \( s_{\text{max}} = 1 \), in which case we can omit the cluster-growing algorithm. I shall concentrate on these cases, which are the easiest to simulate (and also the likeliest to extend to quantum models).

To estimate \( P_1 \), I pretend the \( z \) bonds around a site are independently occupied by randomly chosen elements of \( S \). Then

\[
P_1^{\text{est}} = 1 - \prod_a (1 - q_a), \tag{4.5}
\]

where \( \alpha \) indexes the group class (with \( n_a \) elements) that \( \tau \) might be in (excluding the identity), and I defined \( q_a \) to be the fraction of times \( \tau \star \sigma \in S \) given that \( \tau \) falls in class \( \alpha \).

To digest the implications of (4.5), let us make an even cruder version of the estimate, replacing \( q_a \) by its average over all \( a \), namely \( q_a \rightarrow 1 - p_b \). I get \( 1 - [1 - (1 - p_b)]^{n_G - 1} \) which is a lower bound on \( P_1 \) as given by (4.5). Evidently, to have a high single-site success rate, we want (i) \( n_a \) as large as possible, (ii) \( z \) as small as possible, and (iii) \( p_b \) as small as possible; in light of (4.4), the third criterion amounts to wanting \( n_S / n_G \) as large as possible. Those are the same three considerations given in section 3.3 as favoring a large entropy.

I include these estimates in table 3, particularly focusing on the models using group \( A_z \). We see from table 3 that \( P_1 \) is large enough in many cases that we can rely on single-site updates. However, whenever \( P_1 \) gets close to 1, our model is ‘too easy’ in some sense—it is practically a gauge model, with only mild constraints eliminating some of the configurations.

The entry \( P_1^{\text{est}} = 1 \) for \( A_z \) is delusory. This comes because \( q_a = 1 \) for a certain class of update multipliers, namely the order-2 class (double pairwise exchanges). If we limited ourselves to this class, indeed every update would be successful, but (it can be checked) the move would not be ergodic (does not access the whole ensemble). A similar
situation applies in the cases of \( S_3(2) \) or \( D_4(m, m') \); any \( \tau \) from the even subgroup is always accepted, while an odd \( \tau \) is never accepted; but single-site updates based on the even subgroup do not access the whole ensemble.

To implement an actual simulation, one would not want to choose \( \tau \) at random, but biased towards the group classes with a larger \( q_\alpha \) (the success fraction looking at just an isolated bond). In particular, one would omit group classes with \( q_\alpha = 0 \); if the group contains even/odd elements and \( S \) includes only one parity of element, then \( q_\alpha = 0 \) for every class of odd elements. The values of \( p_\alpha \) and \( P_\alpha^{(\text{hit})} \) in table 3 for \( S_3(2) \) and \( D_4(m, m') \) were computed assuming \( \tau \in E \).

Another way to implement a single-site update is, after choosing a random vertex \( r \), to examine the local environment of its \( z \) bonds, find the entire list of \( rs \) which can update it, and choose randomly from this list. Typically, configuration-dependent choices like this are avoided in Monte Carlo algorithms because they tend to violate detailed balance. In the present case, however, it can be checked that the number of possible \( rs \) is always the same in the old and new configurations, i.e. the rate is the same for the forward and backward steps, which is sufficient to ensure detailed balance (and an equal ensemble weight for every configuration).

4.4. Criteria for initial conditions

In height models, certain special states (e.g. the ‘columnar’ arrangement of dimers on the square lattice) were ‘ideal’ in having the maximum number of possible update moves. (For a model requiring loop updates, we might replace that criterion by ‘having the shortest typical loops.’) Certain other states (e.g. the ‘herringbone’ packing of dimers) were inert, in that no finite updates are possible (in the thermodynamic limit). These states in a height model correspond, respectively, to a zero coarse-grained gradient of the height variable, or the maximum gradient.

In the finite-group height models of this paper, the coarse-grained height gradient is undefined, but one can still construct ‘ideal’ and ‘anti-ideal’ states. It is recommended that simulation runs be started in both kinds of state, being in some sense opposite extremes of the configuration space. A diagnostic for equilibration is then whether the expectations from the two starts converge to the same values.

More exactly, rather than a single domain of an anti-ideal state, one should divide the system into two domains. Then, updates are initially possible along the domains’ border, using loops which extend across the system. (The Abelian case is assumed here.) Gradually, a larger fraction of the system’s area becomes updatable, and the loops get smaller. On the other hand, starting from an ideal state, the loops are initially small and get larger. In general, tracking the loop (or cluster) distribution is an obvious diagnostic to test for convergence to the same equilibrium state.

5. Possible measurements in simulations

In this section, I sketch how one might confirm the topological order numerically, or measure other interesting quantities, given a working Monte Carlo simulation.

5.1. Correlation functions

Correlation functions are an obvious starting point. Of course, a topological order state has exponentially decaying correlations, so this serves primarily as a negative test: we check that the system is not a height model in disguise (see section 3.2), which would have power-law correlations, and that it does not have long-range order (which can emerge even in equal-weighted entropic ensembles, or because the defining constraints are too restrictive). Correlations are also of interest near a critical point where long-range or quasi-long-range order emerges.

In models with vector spins \( s_i \), one was accustomed to evaluating the expectation of \( s_i \cdot s_j \), or occasionally its second moment. It may not be immediately obvious what to measure now. One can, of course, simply tabulate frequencies of different combinations, e.g. (for the ‘height difference’) how often \( y_{0 \rightarrow r} \) belongs to each conjugacy class. It is preferable, though, to reduce the measurements to a single (meaningful) number, and the appropriate generalization of the dot product is the trace of the matrices in the right group representation.

Thus we are led to use a character function \( \chi(x) \), where \( x \) is any group element; this is always the same within each conjugacy class of the group. I divide the actual character by the dimension of the representation, so that \( \chi(e) \equiv 1 \) for any representation, and \( |\chi(x)| \div \leq 1 \) for any element. Presumably, the best choice of representation is the one that has the largest positive \( \chi(\sigma) \) for spins (for \( \sigma \in S \)). This corresponds conceptually to using a distance metric, within the group \( G \), counting how many multiplications by some element of \( S \) are needed to take you from one element to the other one.

5.1.1. Height difference correlation. In the old ‘height models’ (sketched in section 1.1), a natural measure of fluctuations was \( \langle |h(0) - h(r)|^2 \rangle \). The natural generalization of this for the present models with finite (possibly non-Abelian) groups is

\[
C_h(r) \equiv \langle \chi(\gamma(\ell_{0\rightarrow r})) \rangle. \tag{5.1}
\]

Of course, the product \( \gamma(\ell_{0\rightarrow r}) \) is independent of which path is taken from 0 to \( r \)—provided the path does not wrap around the periodic boundary conditions.

As just noted, choosing \( \chi(\cdot) \) so that \( \chi(\sigma) \) is as close to one as possible provides that \( C_h(r) \) does express how fast the group element wanders from the identity under repeated compositions; that is the choice likeliest to give a monotonic decay with distance. If \( \gamma(\ell_{0\rightarrow r}) \) is equally likely to be any group element—which one expects for large \( r \)—then it follows that \( C(r) = 0 \).

5.1.2. Spin correlations. Similarly, we can compute

\[
G_{ij} \equiv \langle \chi(\sigma_i * \sigma_j^{-1}) \rangle. \tag{5.2}
\]

5.2. Defects

It is easy to augment the simulation to allow a defect plaquette where the plaquette constraint is violated. The same (single-site) update rules will work correctly next to the defect, but
they cannot change its position. To make a defect mobile, one can add additional update rules specific to the defect, by (say) arbitrarily choosing one bond of the plaquette and changing it to make the plaquette’s loop product be $e$ (of course, this means the loop product will not be $e$ for the plaquette on the other side of that bond, unless that was also a defect plaquette and this is the annihilation event). The simulation would normally be run with a constraint or bound on the number of defects.

The idea is to create a pair of defects, by hand, and then evaluate expectations depending on them. The first thing to measure is the distribution $P_d(R)$ of defect separations $R$. In the case of topological order, we expect deconfinement, meaning $P_0(R) \rightarrow \text{const}$ for $R > \xi$, where $\xi$ is a (not very large) correlation length. One can define an effective (entropic) potential $V(R)$ by

$$P_d(R) \propto \exp(V(R)),$$

physically, $V(R)$ is the difference in entropy due to placing the defects near to each other. In the case of a height model, $V(R) \propto \ln |R|$, and $P_d(R)$ decays to zero as a power law\(^7\).

In fact, since there are various flavors of defect labeled by different group elements $b$, one really needs to write the effective potential as

$$E = U(b) + U(b') + V_{b,b',c}(R),$$

where $b$ and $b'$ are the respective defect charges, and $c$ is the net charge of the combined defect. Here, $U(b)$ and $U(b')$ are ‘core energies’ of these respective defects; these, and usually the inter-defect potential, are functions only of the conjugacy classes of $b$, $b'$, and/or $c$. Implicit in the form (5.4) is that the exponential confinement length probably depends on all of $b$, $b'$, and $c$.

Measuring how the effective potential depends on class is more physical, since (i) it decides whether a defect is stable against decays into other defects, (ii) measurements of defect behavior (in simulations or in real systems, were any to be discovered) might be used to discover the universality class of the topological order, if that were not known. I conjecture that the dependence on $b$, $b'$, and/or $c$ is also described by a character function; it would be interesting to see if that can be explored analytically in some model.

Incidentally, since (with the appropriate boundary conditions) we can have a single defect in our system cell, that gives additional opportunities to evaluate, e.g., the core energy $U(b)$ without the complication of a second defect.

Finally, if the single-site updates of section 4.3 are not feasible, defects provide a less elaborate alternative update move, in place of the cluster update of section 4.1. Namely, we create a pair of defects and allow them to randomly walk until they annihilate. (However, if their paths differ by a loop around the periodic boundary conditions, they may be unable to annihilate.) Many Monte Carlo schemes [33, 34] are based on a similar process.

5.3. Topological sectors

The tests of topological order outlined up to here have been negative; none of them captures the positive property of topological order, which is the degeneracy of topological sectors. This can be measured in a classical simulation, if we use a (necessarily nonlocal) update which can change sectors, while satisfying the detailed balance condition. Either the cluster update of section 4.1 or the defect pair update just outlined in section 5.2 will suffice.

From the relative fraction of time spent in different topological sectors, we can infer a free energy $F_L(\gamma_x, \gamma_y)$, where $(\gamma_x, \gamma_y)$ are the loop products characterizing the sector, and $L$ refers to the system size. This is a finite-size effect, since (by definition of topological order) the difference between sectors vanishes in the thermodynamic limit; $F_L$ is expected to decay exponentially as a function of $L$.

In a similar fashion, if we allow transitions between states with and without a defect as part of the dynamics, we can evaluate the defect core energy $U(b)$. Of course, $F_L(\gamma_x, \gamma_y)$ is very analogous to $U(b)$, since $b$ is a loop product encircling the puncture where a defect sits, just as $\gamma_x$ is from the loop product encircling the system\(^8\).

6. Transfer matrix and analytic approaches

In a quantum mechanical model with topological order, the energy differences between different topological sectors decay with system size as $\exp(\text{const} \times L)$, and the correlations of a defect pair decay as $\exp(-R/\xi)$. Up to now, I have assumed without justification that this would carry over to the present classical models.

This section finally examines the basis of exponential behavior. I turn here to an analytic treatment using the (practically) one-dimensional framework of transfer matrices. First of all, this sheds some light on why the finite-size dependences, as well as the defect–defect interaction, are exponentially decaying with distance. More specifically, it clarifies the pattern of how sector-weight splittings or defect pair distributions relate to the group’s representations and symmetries.

6.1. One-dimensional model

Imagine the most trivial system which can have topological sectors: the one-dimensional version of the discrete-group height models. There can be no plaquette constraint. Our ensemble simply consists of chains of length $L$—with periodic boundary conditions—having a group element $\sigma$ placed on each link, the only constraint being that $\sigma \in S$. All $(n_S)^2$ sequences are equally likely.

If we let

$$\gamma(x) \equiv \sigma_x \ast \sigma_{x-1} \ast \cdots \ast \sigma_1$$

then the topological sectors are labeled by $\gamma(L)$. Define the $(n_S \times n_S)$ dimensional transfer matrix $T$ in the standard

---

\(^7\) Differing varieties of confinement behavior are possible, depending on whether the power is fast enough for $\int d^2R P_d(R)$ to converge at large $R$.

\(^8\) Indeed, a conformal mapping by the complex logarithm function converts the puncture geometry into a strip with periodic boundary conditions across it.
fashion: let \( T_{\gamma',\gamma} \) be the number of ways to get \( \gamma(x + 1) = \gamma' \) from \( \gamma(x) = \gamma \). Then \( (T^T)_{\gamma',\gamma} \) is the partition function (the total number of states) for the sector with \( \gamma(L) = \gamma \). Note that \( T \) commutes with permutations that implement the symmetry operations (automorphisms) of the group \( G \); hence, the eigenvalues/eigenvectors of \( T \) are classified by the representations of the automorphism group (mentioned in section 3.1). The transfer matrix has eigenvalues \( \{\Lambda_k\} \) and corresponding eigenvectors \( \{v_{k,m}\} \); the index \( m \) labels each family of symmetry-related eigenvectors belonging to the same (degenerate) eigenvalue \( \Lambda_k \).

The partition function restricted to topological sector \( \gamma \) is

\[
\sum_{k,m} [v_{k,m}]_\gamma [v_{k,m}]_e \Lambda_k^L. (6.2)
\]

Hence, in any sector the overall (entropic) free energy per unit length is \( \ln \Lambda_0 \), where \( \Lambda_0 \) is the largest eigenvalue, and \( L \)-dependent corrections depend on some larger eigenvalue \( \Lambda_1 \). For this trivial one-dimensional model, \( v_0 = (1, 1) / \sqrt{2} \). Thus \( \Lambda_1 \) is the next largest (necessarily nondegenerate) eigenvalue of the trivial representation, after \( \Lambda_0 \). More generally \( v_0 \) must be totally symmetric under all automorphisms of \( \mathcal{G} \), i.e. it belongs to the trivial representation. Indeed, \( v_0 \) also must belong to the trivial representation, since \([v_{k,m}]_\gamma \) in (6.2) is independent of \( m \), but \( \sum_m [v_{k,m}]_\gamma = 0 \) for any other representation.

Hence,

\[
P(\gamma) \approx 1 + c_\gamma(\Lambda_1 / \Lambda_0)^L \] \( P(e) \approx 1 + c_e(\Lambda_1 / \Lambda_0)^L \). (6.3)

Here

\[
c_\gamma \equiv \frac{[v_{k,m}]_\gamma [v_{k,m}]_e}{[v_{0,m}]_\gamma [v_{0,m}]_e} \] (6.4)

where \([v_{0,m}]_\gamma [v_{0,m}]_e = 1 / n_G \), for this one-dimensional model. It follows from (6.3) that, for large \( L \),

\[
\ln \frac{P(\gamma)}{P(e)} \approx (c_\gamma - c_e)e^{-L/\xi_1} \] (6.5)

where \( \exp(-1/\xi_1) \equiv |\Lambda_1 / \Lambda_0| \). Often \( \Lambda_1 < 0 \); in this case, we must add a factor \((-1)^L\) on the right-hand side of (6.5). Furthermore, at short \( L \), we may see subdominant terms with shorter decay lengths \( \xi_2 \), etc. deriving from other eigenvectors of \( T \).

6.1.1. Example. A useful example is any group \( \mathcal{G} \) when \( S = \mathcal{G} \setminus e \), i.e. every element but the identity is allowed. In this case \( T = (1, 1, \ldots, 1) \otimes (1, 1, \ldots, 1) = I \). Thus \( \Lambda_0 = n_G - 1 \) and \( \Lambda_1 = \Lambda_2 = \cdots = -1 \). Thus \( \exp(-1/\xi_1) = 1/(n_G - 1) \) and the deviations have alternating signs, i.e. the \((-1)^L\) factor is needed in (6.5). For the model \( Z_2 \times Z_2(2) \), the matrix is

\[
T = \begin{pmatrix}
1 & 2 & 2 & 2 \\
2 & 1 & 2 & 2 \\
2 & 2 & 1 & 2 \\
2 & 2 & 2 & 1
\end{pmatrix} \] (6.6)

6.1.2. Symmetry-class reduced matrix. We can classify eigenvectors as ‘symmetric’ or ‘asymmetric’ according to what representation of the automorphism group they transform under. ‘Symmetric’ eigenvectors are invariant under group symmetries, while ‘asymmetric’ eigenvectors represent a bias of the probability distribution favoring certain local patterns over other (symmetry-related) ones.

Since the sector probability ratio (6.3) is the same for all symmetry-related \( \gamma \), I believe not only \( v_0 \) but also \( v_1 \) must be totally symmetric. That affords a considerable simplification, for we can replace \( T \) by its projection \( \bar{T} \) onto the group element symmetry classes. (Such a class consists of elements that map to each under some automorphism, so these are at least as large as the conjugacy classes.) Whereas the dimension of \( T \) was the number of group elements \( n_G \), the dimension of \( \bar{T} \) is the number of group classes: \( \bar{T}_{ji} \) tells the number of times that \( \sigma \gamma \) belongs to class \( j \), if \( \gamma \) belongs to class \( i \) and \( \sigma \) runs over all \( n_S \) elements in \( S \).

For example, in the case of the group \( A_5 \), the matrix is reduced from \( 60 \times 60 \times 4 \times 4 \), with entries for orders 1 (identity), 2, 3, and 5. (There are two conjugacy classes with order 5, but they are equivalent by an outer automorphism.)

For the model \( A_5(3) \), we get

\[
\bar{T} = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 4 & 6 & 5 \\
20 & 8 & 7 & 5 \\
0 & 8 & 6 & 10
\end{pmatrix} \] (6.7)

This matrix is similar to a symmetric matrix \( D^{-1/2} \bar{T} D^{1/2} \); here \( D = \text{diag}(1, 15, 20, 24) \) for this group, or in general is the diagonal matrix with entries being the count of each class.

6.2. Sector probabilities in two dimensions?

A two-dimensional finite-group height model is also described by a transfer matrix \( T \). However, now the vector that \( T \) acts on represents all possible path products \( \gamma(x, y) \) taken to a point \((x, y)\), and thus is \((n_G)^W \) dimensional, where \( W \) is the width of the strip (in the \( y \) direction; iteration still runs in the \( x \) direction). We must replace \( c_\gamma \rightarrow c_\gamma(W) \) and \( \xi_1 \rightarrow \xi(W) \) in equation (6.5). Conceivably \( \xi(W) \rightarrow 0 \) as \( W \rightarrow \infty \), as is very well known in gapless systems, so the scaling of (6.5) as \( \exp(-L/\xi(W)) \), for a strip, does not prove exponential decay in \( d = 2 \).

Nevertheless, we can make a plausible guess to obtain a fitting form for comparison with numerics. Since all correlations are expected to be rapidly decaying, a strip of width \( W \) is like \( W/w_0 \) independent, one-dimensional strips of width \( w_0 \) in parallel. But all these strips are constrained to have the same, or equivalent, sector label \( \gamma \). The consequence is that

\[
P(\gamma) \approx \left[ 1 + c_\gamma(\Lambda_1 / \Lambda_0)^L \right]^{W/w_0} \] \( P(e) \approx 1 + c_e(\Lambda_1 / \Lambda_0)^L \) \( W/w_0 \) (6.8)

so

\[
\ln \frac{P(\gamma)}{P(e)} \approx (C_\gamma - C_e) W e^{-L/\xi} \] (6.9)

in place of (6.5), with \( C_\gamma \approx c_\gamma / w_0 \) and \( \xi \approx \xi_1 \) independent of \( W \).
My chief motivation for introducing the transfer matrix formalism is separate from such guesses about the $W$ scaling, and is much better founded. Namely, the eigenvectors for the corrections to $P(γ)$ are representations of the automorphism group. Furthermore, which representation goes with the longest correlations is probably the same as in the one-dimensional case. What really matters here is that our choice of a selected set $S$ defines a sort of metric on $G$: the distance from $g$ to $g'$ is the number of times you need to multiply by an element of $S$ to get from $g$ to $g'$. Then, the first nontrivial eigenvector $v_1$ is the mode that is slowest varying on $G$ according to this metric (apart from $v_0$ which is uniform).

The one-dimensional correlation length $ξ_1$ can be computed for any combination of $G$ and $S$ and can serve as another ‘figure of merit’ for a group. That is, in light of the previous paragraph’s argument, it should be roughly related to the true sector probability decay length for the two-dimensional model (and likely related to the defect–defect decay length as well).

6.3. Defect separations and $W = 2$ transfer matrix

Whereas the one-dimensional model already seems to capture the essence of how sector probabilities depend on system size and sector label, it does not admit topological defects and hence sheds no light on the parallel question of how $P_0(R)$ for a defect pair (as defined in section 5.2) decays with separation or depends on the respective defect charges.

Clearly, $P_0(R)$ must be associated somehow with the eigenvectors and eigenvalues of the two-dimensional transfer matrix, since all possible correlation information is expressed in it. But it is not self-evident just what kind of distortion of the ensemble is being propagated, or what sort of subdominant eigenvector: the symmetric kind (which governed the sector probabilities) or the asymmetric kind.

I will work out here a toy calculation, again using a transfer matrix, of the correlation decay due to asymmetric eigenvectors. I believe they are the ones that matter for the case of an Abelian group. In that case, the charge of a defect is a particular element: the loop product around the defect gives that same result, no matter how big the loop, and only another defect with the inverse of that charge can cancel it. In the non-Abelian case, however, these properties would seem to be defined only modulo conjugacy classes. Therefore, the picture presented here is only asserted to go with Abelian groups.

The simplest property that could influence or be influenced by a defect’s presence is the correlation of two adjacent spins on the same plaquette, i.e. sitting on bonds that make a 90° angle. A simple example is the model $Z_2 × Z_2(2)_{sq}$, in which $n_s = 3$ elements are allowed—all except the identity. These elements are $\{a, b, c\}$. Consider a plaquette with the spins on two edges specified and the remaining two spins to be assigned (there are $3^2$ unconstrained ways to do so). When adjacent edges on a plaquette have the same element, there are three ways to satisfy the plaquette constraint, but only two ways if the given adjacent edges are different. On the other hand, if we want to make a defect plaquette, there are six ways when the given spins are the same but seven ways when they are different.

| No. | $(σ_1, σ_0)$ | $(σ′_1, σ′_0)$ | $T^{(0)}$ | $T^{(α)}$ | $T^{(β)}$ | $T^{(γ)}$ |
|-----|-------------|-------------|--------|--------|--------|--------|
| 3   | (aa)        | (aa)        | 3      | 2      | 2      | 2      |
| 12  | (aa)        | (ab)        | 2      | 2      | 2      | 3      |
| 12  | (aa)        | (ba)        | 2      | 1      | 1      | 2      |
| 6   | (aa)        | (bb)        | 2      | 1      | 1      | 2      |
| 12  | (aa)        | (bc)        | 1      | 2      | 2      | 1      |
| 6   | (ab)        | (ab)        | 3      | 2      | 2      | 2      |
| 6   | (ab)        | (ba)        | 2      | 1      | 1      | 2      |
| 6   | (ab)        | (ac)        | 2      | 3      | 2      | 2      |
| 6   | (ba)        | (ca)        | 2      | 2      | 1      | 1      |
| 12  | (ab)        | (ca)        | 1      | 2      | 1      | 2      |

Let us set up a $W = 2$ strip, the narrowest kind that can capture defect correlations. This transfer matrix, unlike the previous one, refers to the actual spin configurations in each pair of bonds; we add up all the possible horizontal bonds. I assume that the upper row of plaquettes are constrained to have identity product around the plaquette. Plaquettes in the lower row can have any product—defects are permitted—with a weight $θ_0$ for the identity or $θ_a, θ_b, θ_c$ for the respective defect charges $a, b, c$. We imagine the limit in which $θ_a, b, c$ are small and ask for the corresponding defect correlations.

Although $T$ has $3^2 × 3^2 = 81$ elements, in fact there are only ten distinct kinds by symmetry, as given in table 4; ‘No.’ represents the number of times each kind occurs in the matrix. The factors $θ_0$ ≈ 1 and $θ_a ≪ 1$ are omitted in the table. To compute the matrix elements, note that when $σ_1 = σ′_1$ in the upper plaquette, the central horizontal bond may be any element (three possibilities) but if $σ_1 ≠ σ′_1$, the central bond may be only $σ_1$ or $σ′_1$ (two possibilities). There are always three possibilities for the lower horizontal bond, so the table’s rows add up to 9 or 6 depending on whether or not $σ_1 = σ′_1$.

The probability to find a defect of charge $β'$ at separation $R$, given there is a defect of charge $β$ at the origin, is then

$$P_0(R) = \frac{\text{Tr}([T^{(0)}]^{L−R−1}T^{(β′)}[T^{(0)}]^{−R−1}T^{(β)})}{\text{Tr}([T^{(0)}]^{L−R−1})}.$$ (6.10)

For a large power $M$, we can replace $[T^{(0)}]^M$ with $(Λ_0)^M v_0 ⊗ v_0 + (Λ_1)^M v_1 ⊗ v_1$. Here $v_0$ and $v_1$ are the eigenvectors belonging to the maximum and next largest eigenvalues of $T^{(0)}$. Assuming $L ≫ R ≫ 1$, we get

$$P_0(R) = \frac{\Lambda_0^{L−R−1} ∑_{k=0}^{R−1} ∑_{m=0}^{M−1} M(v_0, T^{β}v_k,M)(v_{k,m}M^Tv_0,M)v_0,M^{R−1}}{\Lambda_0^{L−1}(v_0, T^{(β′)}v_0)}.$$ (6.11)

(6.12)
Please remember, the eigenvector called \( v_1 \) here is asymmetric, and is thus not the same as the symmetric eigenvector called \( v'_1 \) in section 6.1. We see that asymptotically

\[
\ln P_d(R) \propto c(\beta) c(\beta') e^{-R/\xi}, \tag{6.15}
\]

Notice first that the decay length \( \xi \) is independent of the defect charges, but different defect charges have different projections onto this eigenmode. As is clear from the derivation, a more general form could be written, including subdominant contributions,

\[
\ln P_d(R) \propto \sum_k c_k(\beta) c_k(\beta') e^{-R/\xi_k}, \tag{6.16}
\]

where \( \xi_1 > \xi_2 > \cdots \). The later terms could be important corrections to include in fits at short \( R \), particularly when the smaller \( \xi_k \)s happen to be associated with larger coefficients \( c_k(\beta) \). Also, if \( c_k(\beta) \equiv 0 \) for certain defects, their asymptotic interaction gets carried by the first mode that has nonzero projections onto both defects.

The formulas basically apply to any width of strip. (If defects are allowed in more than one horizontal row of the plaquettes, then the defect distribution is no longer a function just of \( R \) but also of the two \( y \) coordinates; the only modification necessary is that \( T^{(\beta)} \rightarrow T^{(\beta, y)} \), labeled not only by the defect’s flavor but by its \( y \) coordinate.) I would speculate that the \( W \equiv 3 \) strip, with defects in the central row, may be a good approximation in practice, although of course there is no control parameter to make small. The basis for this speculation is simply the notion that, when we have rapid exponential decays, these are associated in the ensemble with strings connecting the defects; any influence carried by a less direct chain would be exponentially smaller in correspondence with the longer length.

### 6.4. Other approaches to \( P_d(R) \) in \( d = 2 \)

I conjecture that there is an alternative approach which is more congenial to \( d = 2 \). Namely, in the vicinity of a defect, the probabilities of local patterns have small deviations from the bulk values, which could be represented by operators \( O_2(\mathbf{r}) \) and small conjugate fields \( h_2(\mathbf{r}) \). That is, adding a Hamiltonian \( \sum_r h_2(\mathbf{r}) O_2(\mathbf{r}) \) (in the absence of the defect) would perturb the ensemble in the same way as the defect does. Note that the operator \( 'O_2(\mathbf{r})' \) is schematic, in the sense that such operators probably involve two spins at different \( \mathbf{r} \) (in light of the same logic laid out in the first paragraphs of this subsection). Then possibly some sort of mean-field approximation produces a difference equation for \( h_2(\mathbf{r}) \), the discrete analogue of Poisson’s equation \( \nabla^2 h_2(\mathbf{r}) = h_2(\mathbf{r})/\xi^2 \), which has solutions \( e^{-\mathbf{r}/\xi} / \xi^2 \). In this approach, we have a sort of small parameter in the influence of a perturbation decays as \( e^{-\mathbf{r}/\xi_0} \), which becomes arbitrarily small at sufficiently large \( R \). We can therefore rely on a linear response in that regime.

One can conceive additional approaches to \( P_d(R) \) which depend on a genuine small parameter; the difficulty is that the actual model families defined in this paper are far from that limit. For example, one could expand around the pure gauge theory: in place of the spin constraint, there would be no constraint but configurations would have a statistical weight \( \exp(\lambda \sum_{\mathbf{r}, \mathbf{r}' in S} u(\sigma(\mathbf{r}, \mathbf{r}'))) \), where \( u(\sigma) \) would penalize all \( \sigma \notin S \). In the limit \( \lambda \to 0 \), we have a pure gauge theory in which all correlation lengths \( \xi_k \) are zero, so hopefully \( \xi_0 \) would scale as a power of \( \lambda \). The models under consideration are, unfortunately, the case \( \lambda = \infty \). Still, since the topological phases are like the pure gauge models at large scales, they should be adiabatically connected and hence this approach should be qualitatively valid.

### 7. Discussion

I have put forward the notion of purely classical topological order, defined by an ergodicity breaking into sectors dependent on the topology, and not distinguishable by thermodynamic expectations of any local operator. A family of explicit models has been described, along with a suitable Monte Carlo technique, and criteria were suggested to pick out the most promising cases (in having a nontrivial and updatable ensemble of allowed states).

A framework was set up (section 2.1) to define models with three variable attributes: which group, which class(es) out of the group to select as ‘spin’ variables, and which lattice to place the model on. These are characterized by parameters—the sizes \( n_G \) and \( n_S \) of the group and the spin subset, the coordination number \( z \) of the lattice and \( z_d \) of its dual—which entered crude formulas that estimate the entropy of the model and its updatability under single-site Monte Carlo moves (sections 3.3 and 4.2), which are the only actual calculations in the paper. Groups with normal subgroups tend to be ‘less non-Abelian’, thus perhaps less attractive (section 3.2). Although topological order superficially would appear to be intrinsically featureless, there is sufficient richness of measurable functions when one considers the dependence of free energy on topological indices—finite size dependence on sector or finite distance dependence on defect separations (section 5).

In trying to connect the classical picture to the quantum theory of topological order, it is intriguing that a given two defect charges (see section 2.3) can combine in more than one way, in a classical non-Abelian model, reminiscent of fusion rules in a quantum model. If further investigation finds that the sector counting gives the same degeneracies in the classical as in the quantum case, one would conclude that this is one of the shared properties, not an intrinsically quantum one.

The physical manifestations of classical topological order and/or of non-abelianness are less striking, perhaps, than for the quantum case. Most prominent is the behavior of topological defects. Topological order implies deconfinement in the classical model for non-Abelian and Abelian cases alike. Non-abelianness (of the group) changes the rules for addition of defect charges, and braiding has physical consequences, even though there are no Berry phases in a classical model. (It must be noted, however, that the same behaviors are seen in non-Abelian defects of ordinary long-range order [26]—they are not inherent to topological order.)
Degeneracies of different topological sectors, the defining property of topological order, work differently in the non-Abelian than in the Abelian case; for example, there are far fewer distinct sectors in the non-Abelian case (section 2.2).

7.1. Quantum mechanics

Several central concepts of topological order are inherently quantum mechanical and thus have no counterpart in classical topological order. They are mainly related to phases in wavefunctions and braiding of world lines in $2 + 1$ dimensions, namely anyon and mutual statistics. Most real or imagined experiments relating to topological excitations have involved interference phenomena (e.g. tunneling in various geometries of quantum Hall fluids) and thus probe the quantum mechanical aspects of topological order.

Another feature missing in the classical models is the dual defect or quasiparticle (such as the vison [35]), which is a distortion of the phase factors in the many-body wavefunction.

A final attribute of topological orders is the nontrivial counting statistics of the excited states made by several quasiparticles, which is quantum mechanical in that it concerns the dimension of a Hilbert space. One cannot rule out the appearance of similar concepts in classical statistical mechanics—there, too, the partition function contains combinatorial factors for the placement of defects, after the other degrees of freedom have been integrated out. However, I am not aware of a classical situation in which such a nontrivial counting actually emerges.

7.2. Construction of quantum models?

Any of the classical height models with topological order may be converted into a similar quantum model if we can endow it with a ‘flipping’ move, just as classical dimer (and other) models get converted into quantum dimer models using the Rokhssar–Kivelson (RK) prescription [36]. A barrier to this is that the only generally guaranteed ‘flip’ move is a cluster update, as explained in section 4.1.

Fortunately, whenever the single-site update (section 4.3) suffices, we can define a quantum model with a simple ‘flipping’ term in the Hamiltonian, usually parametrized by an amplitude $t$, as well as a ‘potential’ term of strength $V = t$ that penalizes each flippable place. At the RK point $V = t$, the ground-state wavefunction is a superposition of all configurations in the same topological sector, with the same (equal) weighting as in the classical ensemble, and (mutually inaccessible) topological sectors are trivially degenerate. One is also free to set $V = 0$—obtaining a simpler model in which flippable sites are set $V = 0$ so that an ordered state is likely to be the outcome—or to vary $V/t$ with the hope of crossing a phase transition.

The above recipe is incomplete, in that there are many possible choices of update (labeled by the multiplier $\tau$ of section 4), and presumably all or many should be included in the ‘flipping’ term of the quantum Hamiltonian, which requires a prescription for the relative magnitudes of the coefficient to put for each class of $\tau$, as well as the relative phase factors. Presumably, a proper choice is to take the same phase factors for every term, i.e. the Hamiltonian transforms by the fully symmetric (trivial) representation of the automorphism group of $G$. Alternatively, in lucky cases, one might select a site-dependent pattern of $\tau$s so as to link the group symmetry to the lattice symmetry, in the spirit of Kitaev’s honeycomb model [4]. Another option is to include a second, quantum fluctuating field of $\tau$s which are used for the update. If the $\tau$s are derived from a second set of ‘spins’ also having the gauge-like structure of a finite-group height model, we might be able to realize dual (‘magnetic’) defects having mutual statistics with the $\sigma$-spin type (‘electric’) defects described in this paper.

7.3. Possible simulations

As laid out in section 5, several quantities, e.g. correlation functions, can be measured in classical non-Abelian height models as a test-bed, whereas the analogous calculation might be very challenging computationally in a quantum mechanical model. Of course, the answers need not be the same, but the questions may be much clearer once the classical results are in hand. First, one can create defect pairs and evaluate the histogram of their separations, which will reveal whether or not they are deconfined. Secondly, one can evaluate the probabilities of different topological sectors, which is the direct test of topological order.

Furthermore, if we generalize the models to include classical Hamiltonians (so as to weight configurations according to the Boltzmann distribution), phase transitions can be studied. Just as a standard height model may have a ‘smooth’ phase, in which one or more height components becomes locked, it seems conceivable that a discrete-group height model might have a phase in which the loop products can take values in a subgroup $G' \subset G$. If so, one might encounter critical points separating different topological phases, and characterize the critical exponents.

7.4. Dilution and effective interactions of local degrees of freedom?

A classical model might be a helpful tool for investigating the consequences of dilution disorder in a model with topological order. Each diluted site or plaquette is like a very small hole cut in the system, thereby increasing the genus and the number of topological sectors. If the hole were big, these sectors would be truly degenerate (by the definition of topological order), however this degeneracy is broken since the holes are small.

The values of $\gamma^*$ on each dilution site are local pseudospins, which are expected to have (exponentially decaying) interactions mediated by the fluid in between them, like the emergent spin-1/2 degrees of freedom in diluted spin-1 antiferromagnetic chains [37]. The ground state of such a system could be constructed by a ‘strong-disorder’ renormalization group that iteratively combines the most strongly coupled pair of pseudospins into a single effective pseudospin, as was originally done for the (exponentially decaying) antiferromagnetic coupling of the charge-bound electron spins in P-doped Si [38].

In the non-Abelian case, at least, we do not know for sure whether these interactions lead to an inert singlet phase (as in
the antiferromagnet) or could give a state with some kind of order among the pseudospins. Thus the system with defects would not be a topological liquid, and this would be a novel scenario of how order can emerge due to disorder [39].

Acknowledgments

I thank M Troyer for suggesting the problem; also L Ioffe, A Kitaev, D A Ivanov, Simon Trebst, C Castelnovo, C Chamgon, S Papanikolaou, and R Lamberty for comments and discussion. I also thank J Papaioannou and R Maimon for preliminary work on the simulation algorithm. This work was supported by NSF Grant No. DMR-1005466.

References

[1] Wen X-G 1991 Phys. Rev. B 44 2664–72
[2] Wen X-G 2004 Quantum Field Theory of Many-Body Systems (Oxford: Oxford University Press) chapters 8–10
[3] Ioffe L B, Feigel’man M V, Ioselevich A, Ivanov D, Troyer M and Blatter G 2002 Nature 415 503–6
[4] Kitaev A 2003 Ann. Phys. 303 2–30
[5] Misguich G and Lhuillier C 2005 Two-dimensional quantum antiferromagnets Frustrated Spin Systems ed H T Diep (Singapore: World Scientific); arXiv:cond-mat/0310405
[6] Castelnovo C and Chamon C 2007 Phys. Rev. B 76 174416
[7] Sachdev S 1999 Quantum Phase Transitions (Cambridge: Cambridge University Press)
[8] Freedman M, Kitaev A, Larsen M J and Wang Z 2003 Bull. Am. Math. Soc. 40 31
[9] Levin M A and Wen X-G 2005 Phys. Rev. B 71 045110
[10] Freedman M, Nayak C and Shengel K 2005 Phys. Rev. Lett. 94 066401
[11] Lamberty R Z, Papanikolaou S and Henley C L, unpublished
[12] van Beijeren H 1977 Phys. Rev. Lett. 38 993
[13] Blöte H W J and Hilhorst H J 1982 J. Phys. A: Math. Gen. 15 L631
[14] Nienhuis B, Hilhorst H J and Blöte H W J 1994 J. Phys. A: Math. Gen. 17 3559
[15] Zheng W and Sachdev S 1989 Phys. Rev. B 40 2704
[16] Kondo J and Henley C L 1995 Phys. Rev. B 52 6628
[17] Kondo J and Henley C L 1996 Nucl. Phys. B 466 540
[18] Zeng C and Henley C L 1997 Phys. Rev. B 55 14935
[19] Raghavan R, Henley C L and Arouth S L 1997 J. Stat. Phys. 86 517–50
[20] Burton J K and Henley C L 1997 J. Phys. A: Math. Gen. 30 8385–143
[21] Nienhuis B 1987 Phase Transitions and Critical Phenomena vol 11, ed C Domb and J L Lebowitz (London: Academic)
[22] Nelson D R 1983 Phase Transitions and Critical Phenomena vol 7, ed C Domb and J L Lebowitz (London: Academic)
[23] Doucot B and Ioffe L B 2005 New J. Phys. 7 187
[24] Oshikawa M, Kim Y B, Shengel K, Nayak C and Tewari S 2007 Ann. Phys. 322 1477
[25] Moessner R and Sondhi S L 2001 Phys. Rev. Lett. 86 1881–4
[26] Mermin N D 1979 Rev. Mod. Phys. 51 591
[27] Burrello M, Xu H, Mussardo G and Wan X 2010 Phys. Rev. Lett. 104 160502
[28] Baxter R J 1970 J. Math. Phys. 11 784
[29] Huse D A and Rutenberg A D 1992 Phys. Rev. B 45 7536
[30] Feng X, Youjin D and Blöte H W J 2008 Phys. Rev. E 78 031136
[31] Henley C L 2006 Phys. Rev. Lett. 96 047201
[32] Nienhuis B 1987 Rev. Mod. Phys. 62 361
[33] Oxborrow M and Henley C L 1993 Phys. Rev. B 48 6966–6998
[34] Krauth W and Moessner R 2003 Phys. Rev. B 67 064503
[35] Senthil T and Fisher M P A 2000 Phys. Rev. B 62 7850
[36] Senthil T and Fisher M P A 2001 Phys. Rev. Lett. 86 292
[37] Senthil T and Fisher M P A 2002 Phys. Rev. B 65 134521
[38] Rokhsar D S and Kivelson S 1988 Phys. Rev. Lett. 61 2376
[39] Hagiwara M, Katsumata K, Affleck I, Halperin B I and Renard J P 1990 Phys. Rev. Lett. 65 3181
[40] Bhatt R N and Lee P A 1982 Phys. Rev. Lett. 48 344
[41] Wessel S, Normand B, Sigrist M and Haas S 2001 Phys. Rev. Lett. 86 1086