A REMARK ON THE FREENESS CONDITION OF SUZUKI’S CORRESPONDENCE THEOREM FOR INTERMEDIATE C*-ALGEBRAS

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Abstract. Let $\Gamma$ be a discrete group satisfying the approximation property (AP). Let $X$, $Y$ be $\Gamma$-spaces and $\pi: Y \to X$ be a proper factor map which is injective on the non-free part. We prove the one-to-one correspondence between intermediate $\Gamma$-$C^*$-algebras of $C_0(X) \times_\pi \Gamma \subset C_0(Y) \rtimes \Gamma$ and intermediate $\Gamma$-$C^*$-algebras of $C_0(X) \subset C_0(Y)$. This is a generalization of Suzuki’s theorem that proves the statement for free actions.

1. Introduction

Let $\Gamma$ be a discrete group and $X$, $Y$ be locally compact spaces on which $\Gamma$ acts. Let $\pi: Y \to X$ be a proper factor map. We study the relation between intermediate $\Gamma$-$C^*$-algebras of $C_0(X) \subset C_0(Y)$ and intermediate $C^*$-algebras of $C_0(X) \times_\pi \Gamma \subset C_0(Y) \rtimes \Gamma$.

Inclusions of operator algebras play an important role in many subjects including operator theory and knot theory. Structures of subalgebras of $C^*$-algebras have been studied by many hands ($\text{ILP98}$, $\text{Izu02}$, $\text{GK96}$, $\text{Zac01}$, $\text{Zsi00}$, etc.).

A Galois correspondence theorem in operator algebras refers to a type of structure results for subalgebras of crossed products and fixed point subalgebras of operator algebras. This is proved in many cases. More precisely, a Galois correspondence is that for an operator algebra $M$ on which a compact group $G$ (or a discrete group $\Gamma$) acts, there exists a one-to-one correspondence between intermediate operator algebras of $M^G \subset M$ and closed subgroups of $G$ (or a one-to-one correspondence between intermediate operator algebras of $M \subset \rtimes \Gamma$ and subgroups of $\Gamma$). Izumi, Longo and Popa $\text{ILP98}$ prove the Galois correspondence for a factor $M$ on which a compact group $G$ acts minimally (or a discrete group $\Gamma$ acts outerly). In $\text{Izu02}$, Izumi proves the Galois correspondence for a simple $\sigma$-unital $C^*$-algebra on which a finite group acts outerly.

Ge and Kadison $\text{GK96}$ prove the tensor splitting theorem that for every factor $M$ and every von Neumann subalgebra $N$, every von Neumann subalgebra of $M \bar{\otimes} N$ which contains $M$ is of the form $M \bar{\otimes} N_0$ for some von Neumann subalgebra $N_0$ of $N$. In the case of simple $C^*$-algebras, the tensor splitting theorem is established under some conditions (see $\text{Zac01}$, $\text{Zsi00}$).

Suzuki proves the following theorem among others in $\text{Suz18}$.

Theorem 1 (Suzuki, $\text{Suz18}$, Main Theorem ($C^*$-case)). Let $\Gamma$ be a discrete group satisfying the AP. Let $X$, $Y$ be $\Gamma$-spaces on which $\Gamma$ acts freely and $\pi$ be a proper factor map from $Y$ to $X$. Then the map

$$C_0(Z) \to C_0(Z) \rtimes_\Gamma \Gamma$$

gives a lattice isomorphism between the lattice of intermediate extensions of $\pi$ and that of intermediate $\Gamma$-$C^*$-algebras of $C_0(X) \rtimes_\Gamma \Gamma \subset C_0(Y) \rtimes_\Gamma \Gamma$.

The freeness condition cannot be removed in general (see $\text{Suz18}$ Proposition 2.6). The following theorem generalizes the above result of Suzuki by relaxing the freeness condition.

Theorem 2 (Theorem 9). Let $\Gamma$ be a discrete group satisfying the AP. Let $X$, $Y$ be $\Gamma$-spaces and $\pi$ be a proper factor map from $Y$ to $X$ with the following condition: for every element $x$ in $X$ fixed by some non-neutral element of $\Gamma$, one has $|\pi^{-1}(x)| = 1$. Then the map

$$C_0(Z) \to C_0(Z) \rtimes_\Gamma \Gamma$$

gives a lattice isomorphism between the lattice of intermediate extensions of $\pi$ and that of intermediate $\Gamma$-$C^*$-algebras of $C_0(X) \rtimes_\Gamma \Gamma \subset C_0(Y) \rtimes_\Gamma \Gamma$. 

Lemma 7. Let \( K \) be a compact Hausdorff space and \( \pi \) be a continuous map from \( Y \) to \( X \). Let \( K \) be a compact subset of \( Y \) satisfying \( K = \pi^{-1}(\pi(K)) \) and \( U \) be an open neighborhood of \( K \). Then there exists an open neighborhood \( V \) of \( \pi(K) \) satisfying \( \pi^{-1}(V) \subset U \).

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2. Preliminaries

We say \( X \) is a \( \Gamma \)-space if \( X \) is a locally compact space equipped with a \( \Gamma \)-action by homeomorphisms.

Definition 3. For a \( \Gamma \)-space \( X \) and an element \( g \) of \( \Gamma \), we denote by \( \text{Fix}_X(g) \) the set of fixed points of \( g \), i.e. \( \{ x \in X \mid g.x = x \} \). Also, we denote by \( S_X \) the set of all elements in \( X \) which have non-trivial stabilizers.

Definition 4. Let \( X, Y \) be \( \Gamma \)-spaces. A map \( \pi \) from \( Y \) to \( X \) is said to be a factor map if it is a \( \Gamma \)-equivariant quotient map from \( Y \) to \( X \). We also refer to \( \pi \) as an extension. A factor map is proper if the preimage of every compact set is compact.

The approximation property (AP) has been introduced for locally compact groups by Haagerup–Kraus [HK92]. In the discrete case, the AP is weaker than weak amenability and stronger than exactness. See [HK92] and Section 12 of [BO08] for details.

Let \( X \) be a \( \Gamma \)-space. There exists a canonical conditional expectation \( E \) from \( C_0(X) \rtimes_r \Gamma \) onto \( C_0(X) \) defined by

\[
E(fu_g) = \begin{cases} f & \text{if } g = e \\ 0 & \text{if } g \neq e, \end{cases}
\]

for \( f \in C_0(X) \) and \( g \in \Gamma \). Note that \( E \) is faithful (see [BO08], Chapter 4.1).

The following proposition plays an important role in the proof of the main theorem.

Proposition 5 ([Suz17], Proposition 3.4). Let \( \Gamma \) be a group satisfying the AP. Let \( A \) be a \( \Gamma \)-\( C^* \)-algebra and let \( X \) be a closed subspace of \( A \). Let \( x \) be an element of \( A \rtimes_r \Gamma \) satisfying \( E(xu_g) \in X \) for all \( g \in \Gamma \). Then \( x \) is contained in the closed subspace

\[
X \rtimes_r \Gamma := \overline{\text{span}} \{ xu_g \mid x \in X, g \in \Gamma \}.
\]

3. Main theorem

In this section, we prove the Main Theorem (Theorem 3).

Proposition 6. Let \( X, Y \) be \( \Gamma \)-spaces and \( \pi \) be a \( \Gamma \)-equivariant quotient map from \( Y \) to \( X \). Assume for every element \( x \in S_X \), one has \( |\pi^{-1}(x)| = 1 \). Then for every subset \( C \) of \( S_Y \), one has \( \pi^{-1} \circ \pi(C) = C \).

Proof. Let \( \varphi : S_Y \to S_X \) be the restriction of \( \pi \) to \( S_Y \). It suffices to show that \( \varphi \) is bijective. For every \( x \in S_X \), there exists \( e \neq g \in \Gamma \), \( x = g.x \). Since \( \pi \) is surjective, there exists \( y \in Y \) satisfying \( \pi(y) = x \). If \( y \neq y \), then one has \( |\pi^{-1}(x)| \geq 2 \), which contradicts the assumption. Hence, \( g.y = y \), that is \( y \in S_Y \). Thus, \( \varphi \) is surjective. Injectivity follows from the assumption. 

Lemma 7. Let \( X, Y \) be compact Hausdorff spaces and \( \pi \) be a continuous map from \( Y \) to \( X \). Let \( K \) be a compact subset of \( Y \) satisfying \( K = \pi^{-1}(\pi(K)) \) and \( U \) be an open neighborhood of \( K \). Then there exists an open neighborhood \( V \) of \( \pi(K) \) satisfying \( \pi^{-1}(V) \subset U \).
Proof. Suppose there exists no open neighborhood. Since $X$ is Hausdorff, there exists a decreasing net $\{V_i\}_{i \in I}$ of open sets such that $\bigcap_i V_i = \pi(K)$. By the assumption, for every $i \in I$, one has $\pi^{-1}(V_i) \not\subset U$. Hence, there exist $x_i \in V_i$ and $y_i \in Y$ such that $\pi(y_i) = x_i$ and $y_i \not\in U$. Since $Y$ is compact, there exists a subnet $\{y_j\}_{j \in J}$ and $y \in Y$ such that $y_j \to y$. Since $\pi$ is continuous, one has $\pi(y_j) \to \pi(y)$. Then one has $\pi(y) \in \pi(K)$. Hence, one has $y \in K$. Since $U$ is an open neighborhood of $K$, there exists $j_0 \in J$ such that $y_{j_0} \in U$. This contradicts $y_j \not\in U$ for every $j \in J$.

We say that $A \subset B$ is a non-degenerate inclusion of $\mathcal{C}^*$-algebras if every (or, equivalently, some) approximate unit of $A$ is an approximate unit of $B$. If $A \subset B$ is non-degenerate, the inclusion extends to an inclusion $M(A) \subset M(B)$ (see [Lan95, Proposition 2.1]).

We use the following lemma to show the non-compact case of Theorem 9.

**Lemma 8.** Let $A \subset B$ be a non-degenerate inclusion of $\mathcal{C}^*$-algebras. Let $C$ be a $\mathcal{C}^*$-subalgebra of the multiplier algebra $M(A)$. The map

$$D \mapsto D + C =: D^\wedge$$

defines a one-to-one correspondence between intermediate $\mathcal{C}^*$-algebras $A \subset D \subset B$ and intermediate $\mathcal{C}^*$-algebras $A + C \subset D \subset B + C$, with its inverse map given by

$$D \mapsto D \cap B =: D^\vee.$$

**Proof.** Let $D$ be an intermediate $\mathcal{C}^*$-algebra of $A \subset B$. We first note that $C \subset M(D)$ and $D^\wedge = D + C = Q_D^{-1} \cap Q_D(C) \subset M(D)$ is a $\mathcal{C}^*$-subalgebra, where $Q_D : M(D) \to M(D)/D$ is the quotient map. Let $\{u_i\}$ be an approximate unit of $A$. By the non-degeneracy of the inclusion, $\{u_i\}$ is an approximate unit of $B$.

Since $D \subset B$, one has $D \subset (D^\wedge)^\vee$. Let $a$ be an element of $(D^\wedge)^\vee$. Since $D^\wedge = D + C$, there exist $d \in D$ and $c \in C$ such that $a = d + c$. Since $u_i c \in A$, one has $u_i(d + c) \in D + A = D$. Since $\{u_i\}$ is an approximate unit for $B$, one has $a \in D$. Hence, one has $(D^\wedge)^\vee = D$.

Let $D$ be an intermediate $\mathcal{C}^*$-algebra of $A + C \subset B + C$. Since $C \subset D$, one has $(D^\vee)^\wedge \subset D$. Let $a$ be an element of $D$. Since $a \in B + C$, there exist $b \in B$ and $c \in C$ such that $a = b + c$. Since $u_i b \in B$ and $u_i c \in A$, one has $u_i(b + c) \in B \cap D$. Since $c - u_i c \in C + A$, one has $u_i b + c = u_i(b + c) + (c - u_i c) \in (D \cap B) + C = (D^\wedge)^\vee$. Hence, one has $(D^\vee)^\wedge$.

**Theorem 9.** Let $\Gamma$ be a discrete group satisfying the AP. Let $X$, $Y$ be $\Gamma$-spaces and $\pi$ be a proper factor map from $Y$ to $X$ such that $|\pi^{-1}(x)| = 1$ for every element $x$ of $\mathcal{S}_X$. We regard $C_0(X)$ as a $\Gamma$-$\mathcal{C}^*$-subalgebra of $C_0(Y)$ via $\pi$. Then the map

$$C_0(Z) \mapsto C_0(Z) \rtimes_r \Gamma$$

gives a lattice isomorphism between the lattice of intermediate extensions of $\pi$ and that of intermediate $\mathcal{C}^*$-algebras of $C_0(X)$, $\Gamma \subset C_0(Y) \rtimes_r \Gamma$.

**Proof.** We first show this theorem when $X$ and $Y$ are compact. Let $E$ be the canonical conditional expectation from $C(Y) \rtimes_r \Gamma$ to $C(Y)$. Let $a \in C(Y) \rtimes_r \Gamma$. Let $\varepsilon > 0$ be given. There exist $n \in \mathbb{N} \cup \{0\}$, $t_k \in \Gamma$, and $f_k \in C(Y)$ ($k \in \{0, \ldots, n\}$) such that $\|a - \sum_{k=0}^n f_k u_k t_k\| < \varepsilon$, $t_0 = e$, and $t_k \neq e$ ($k \in \{1, \ldots, n\}$).

Since $\text{Fix}_X(t)$ is closed for every $t \in \Gamma$, for each $k \in \{1, 2, \ldots, n\}$, there exists $\tilde{f}_k \in C(X)$ such that $f_k = \pi_* (\tilde{f}_k)$ on $\text{Fix}_Y(t_k)$ by the Tietze extension theorem and Lemma 6.

For each $k \in \{1, \ldots, n\}$, since $\text{Fix}_Y(t_k)$ is compact, there exists an open set $W_k$ of $Y$ such that $\text{Fix}_Y(t_k) \subset W_k$ and $|f_k - \pi_* (\tilde{f}_k)| < \varepsilon/n$ on $W_k$. By Proposition 6 and Lemma 7, there exists an open neighborhood $U_k^0$ of $\pi(\text{Fix}_X(t_k))$ in $X$ such that $\pi^{-1}(U_k^0) \subset W_k$.

For each $k \in \{1, 2, \ldots, n\}$, $t_k$ acts on $X \setminus U_k^0$ freely. Hence, there exist a finite subset $J'_k \subset \mathbb{N}$ and a finite open covering $\{U_{k,j}^0\}_{j \in J'_k}$ of $X \setminus U_k^0$ such that for every $j \in J'_k$, $t_k U_{k,j}^0 \cap U_{k,j}^0 = \emptyset$. Let $J_k = J'_k \cup \{0\}$ and $I = J_1 \times J_2 \times \cdots \times J_n$. For every $i = (j_1, j_2, \ldots, j_n) \in I$, we define $V_i$ to be $U_1^{j_1} \cap U_2^{j_2} \cap \cdots \cap U_n^{j_n}$. We remark that for every $k \in \{1, 2, \ldots, n\}$ and every $i = (j_1, j_2, \ldots, j_n) \in I$ with $j_k \neq 0$, one has $t_k U_i \cap U_i = \emptyset$. We also have $\bigcup_{i=(j_1,j_2,\ldots,j_n)} U_i = U_k^0$. For the open covering
It follows from the Whitehead lemma \[Whi48\] that for each \(k \in \{1, 2, \ldots, n\}\) and every \(i = (j_1, j_2, \ldots, j_n) \in I\) with \(j_k \neq 0\), one has \(h_i(t_kh_i) = 0\). We define a c.c.p map \(\Phi : C(Y) \times_r \Gamma \to C(X) \times_r \Gamma\) by the map \(a \mapsto \sum_{i \in I} h_i^a\).

For each \(k \in \{1, 2, \ldots, n\}\), we define \(g_k := \Phi(u_{t_k}^*)u_k = \sum_{i \in I} h_i^a(t_kh_i^a) \in C(X)\). For \(k \neq 0\), since \(\text{supp}(g_k) \subset U_k^0\) and \(|g_k| \leq 1\), one has \(f_kg_k \approx_{\varepsilon/n} \tilde{f}_kg_k\). Therefore,

\[
\Phi(a) \approx_{\varepsilon} \Phi\left(\sum_{k=0}^{n} f_kg_k u_k\right) = \sum_{k=0}^{n} \tilde{f}_kg_k u_k = f_0 + \sum_{k=1}^{n} \tilde{f}_kg_k u_k \approx_{(\varepsilon/n)\cdot n} f_0 + \sum_{k=1}^{n} \tilde{f}_kg_k u_k
\]

Since \(\Phi(a) \in C^*(a, C(X))\) and \(\sum_{k=1}^{n} \tilde{f}_kg_k u_k \in C(X) \times_r \Gamma\), one has

\[
E(a) \approx_{\varepsilon} f_0 \approx_{2\varepsilon} \Phi(a) - \sum_{k=1}^{n} \tilde{f}_kg_k u_k \in C^*(a, C(X) \times_r \Gamma).
\]

Since \(\varepsilon > 0\) was arbitrary, one has \(E(a) \in C^*(a, C(X) \times_r \Gamma)\).

Let \(D\) be an intermediate \(C^*\)-algebra of \(C(X) \times_r \Gamma \subset C(Y) \times_r \Gamma\). Then, by the result shown in the previous paragraph, one has \(E(D) \subset D\). By Proposition 5 for every intermediate \(C^*\)-algebra \(D\) of \(C(X) \times_r \Gamma \subset C(Y) \times_r \Gamma\), one has \(D = E(D) \times_r \Gamma\).

Next, we show this theorem in the case where \(X\) and \(Y\) are noncompact. Let \(\tilde{X} = X \sqcup \{x_\infty\}\), \(\tilde{Y} = Y \sqcup \{y_\infty\}\) be the one-point compactifications of \(X\), \(Y\) respectively. Let \(\tilde{\pi} : \tilde{Y} \to \tilde{X}\) denote the continuous extension of \(\pi\). Since \(\tilde{\pi}^{-1}(\{x_\infty\}) = \{y_\infty\}\), \(\tilde{\pi}\) satisfies the assumption of this theorem.

We will use a one-to-one correspondence between intermediate \(C^*\)-algebras of \(C_0(X) \times_r \Gamma \subset C_0(Y) \times_r \Gamma\) and that of \(C(\tilde{X}) \times_r \Gamma \subset C(\tilde{Y}) \times_r \Gamma\). Let \(\tilde{E}\) be the canonical conditional expectation from \(C(\tilde{Y}) \times_r \Gamma\) to \(C(\tilde{Y})\). Let \(D\) be an intermediate \(C^*\)-algebra of \(C_0(X) \times_r \Gamma \subset C_0(Y) \times_r \Gamma\). Since \(D^\land := D + C_0^*\Gamma\) is an intermediate \(C^*\)-algebra of \(C(\tilde{X}) \times_r \Gamma \subset C(\tilde{Y}) \times_r \Gamma\), one has \(D^\land = \tilde{E}(D^\land) \times_r \Gamma\). Let \((D^\land)^\lor = D^\land \cap (C_0(Y) \times_r \Gamma)\). By Lemma 8 and Proposition 5 one has

\[
D = (D^\land)^\lor = (\tilde{E}(D^\land) \times_r \Gamma) \cap (C_0(Y) \times_r \Gamma) = (\tilde{E}(D^\land) \cap C_0(Y)) \times_r \Gamma.
\]

\[\square\]

4. Examples

**Example 10** (branched covering). Let \(X\) be a complex plane \(\mathbb{C}\). Let \(k\) be an integer greater than or equal to 2 and \(Y = \{(z, w) \in \mathbb{C} \times \mathbb{C} \mid w^k = z\}\) be a Riemann surface of the \(k\)-th square root. Let \(\pi\) be a projection from \(Y\) to \(X\), i.e., \(Y \ni (z, w) \mapsto z \in X\). Let \(\alpha \in \mathbb{R} \setminus \mathbb{Q}\). We define \(\mathbb{Z}\)-actions on \(X\), \(Y\) by the following: for each \(n \in \mathbb{Z}\),

\[
X \ni z \mapsto ze^{2\pi i \alpha n} \in X
\]

\[
Y \ni (z, w) \mapsto (ze^{2\pi i \alpha n}, we^{2\pi i \alpha n}) \in Y.
\]

Then \(\pi\) is a \(\mathbb{Z}\)-equivariant proper quotient map and for every \(n \in \mathbb{Z}\), one has \(\text{Fix}_Y(n) = \{(0, 0)\}\) and \(\text{Fix}_X(n) = \{0\}\).

We show that the assumption of \(\pi\) in Theorem 9 is closed under taking the direct product. It follows from the Whitehead lemma \[WhiIS\] that \(\pi_1 \times \pi_2\) in Lemma 12 is quotient. For the reader’s convenience, we include the proof.
Lemma 11. Let $Y$ be a topological space and $X$ be a locally compact space. Let $\pi$ be a surjective proper continuous map and $U$ be a subset of $X$. If $\pi^{-1}(U)$ is open, $U$ is open.

Proof. Take an element $x$ in $U$. Let $V$ be a relative compact open neighborhood of $x$. It suffices to show $U \cap V$ is open. Since $\pi^{-1}(U \cap V) = \pi^{-1}(U) \cap \pi^{-1}(V)$, $\pi^{-1}(U \cap V)$ is open. By continuity and properness, $\overline{V} \cap (U \cap V)^c = \pi(\pi^{-1}(\overline{V}) \cap \pi^{-1}(U \cap V)^c)$ is compact. We remark that $X = (U \cap V) \cup (\overline{V} \cap (U \cap V)^c) \cup V^c$.

Since $\overline{V} \cap (U \cap V)^c$ and $V^c$ is closed, $((\overline{V} \cap (U \cap V)^c) \cup V^c)$ is closed. Then $U \cap V$ is open. □

Lemma 12. For each $i \in \{1, 2\}$, let $X_i$, $Y_i$ be $\Gamma$-spaces and $\pi_i : Y_i \to X_i$ be a proper factor maps. We see $X_1 \times X_2$ and $Y_1 \times Y_2$ are $\Gamma$-spaces with diagonal actions. Then $\pi := \pi_1 \times \pi_2 : Y_1 \times Y_2 \to X_1 \times X_2$ is a proper factor map.

Proof. It suffices to check properness and quotientness. We will show properness. Let $C \subset Y_1 \times X_2$ be a compact subset. For each $i \in \{1, 2\}$, let $p_i : Y_1 \times Y_2 \to Y_i$ be $i$-th projections. By the $\pi_i$’s continuity, for each $i \in \{1, 2\}$, $p_i(C)$ is compact. Since for each $i$, $\pi_i^{-1}(p_i(C))$ is compact, $\pi^{-1}(C) \subset \pi_1^{-1}(p_1(C)) \times \pi_2^{-1}(p_2(C))$ is compact.

We will show quotientness. Let $U \subset X$. Since $\pi$ is continuous, if $U$ is open, then $\pi^{-1}(U)$ is open. By Lemma 11 if $\pi^{-1}(U)$ is open, then $U$ is open. □

Example 13. For each $i \in \{1, 2\}$, let $X_i$, $Y_i$ be $\Gamma$-spaces and $\pi_i : Y_i \to X_i$ be a $\Gamma$-equivariant proper factor map. By the above lemma, if for every $i \in \{1, 2\}$, $\pi_i$ is a proper factor map such that $|\pi_i^{-1}(x)| = 1$ for every element $x$ in $S_{X_i}$, then so is $\pi_1 \times \pi_2 : Y_1 \times Y_2 \to X_1 \times X_2$.

We show that some compactifications satisfy the assumption in the main theorem. For a locally compact space $X$, we denote by $\beta X$ the Stone-Čech compactification of $X$. The following proposition is easily seen from the universal property of the Stone-Čech compactification and Lemma 11.

Proposition 14. Let $\varphi : Y \to X$ be a proper quotient map between locally compact spaces $X$ and $Y$. Let $\tilde{X} = X \cup \partial X$ be a compactification of $X$. Then there exists a quotient map $\beta \varphi : \beta Y \to \tilde{X}$ extending $\varphi$.

Remark 15. By the above proposition, one has

$$(\beta \varphi)(\beta Y \setminus Y) = \partial X \text{ and } \beta \varphi(Y) = X.$$  

It follows that $(\beta \varphi)_*(C(\tilde{X})) \cap C_0(Y) = (\beta \varphi)_*(C_0(X))$.

Proposition 16. With the same assumptions as the previous proposition, we define $\tilde{Y}$ to be the character space of the $C^\ast$-subalgebra of $C(\beta Y)$ generated by $(\beta \varphi)_*(C(\tilde{X}))$ and $C_0(Y)$. Then one has

$$C(\tilde{Y})/C_0(Y) \cong \beta \varphi_*(C(\partial X)).$$

Proof.

$$C(\tilde{Y})/C_0(Y) = ((\beta \varphi)_*(C(\tilde{X})) + C_0(Y))/C_0(Y)$$

$$\cong (\beta \varphi)_*(C(\tilde{X}))/((\beta \varphi)_*(C(\tilde{X})) \cap C_0(Y))$$

$$= (\beta \varphi)_*(C(\tilde{X}))/\varphi_*(C_0(X))$$

$$\cong \beta \varphi_*(C(\partial X)).$$

Remark 17. Let $\partial Y$ to be $\tilde{Y} \setminus Y$. Then one has

$$C(\partial Y) \cong \beta \varphi_*(C(\partial X)) \text{ and } \beta \varphi(\partial Y) \cong \partial X.$$  

For a compact space $X$ and an open subset $U$ of $X$, we consider the $C^\ast$-subalgebra of $\ell_\infty(X)$ generated by $C(X)$ and the characteristic function $\chi_{\overline{U}}$. 

5
Proposition 18. Let $X$ be a compact space and $U$ be an open subset of $X$. Let $p = \chi_U \in \ell_\infty(X)$, then one has $\mathbb{C}^*(\mathbb{C}(X), p) \cong \mathbb{C}(\mathbb{C}) \oplus \mathbb{C}(X \setminus U)$.

Proof. Let $V$ be a subset of $X$. Let $q = \chi_V$ be the characteristic function in $\ell_\infty(X)$. We will show $\mathbb{C}(X)q \cong \mathbb{C}(\overline{V})$. We define $\varphi : \mathbb{C}(X)q \to \mathbb{C}(\overline{V})$ by $f q \mapsto f |_{\overline{V}}$ for $f \in \mathbb{C}(X)$. The well-definedness and the injectivity of $\varphi$ follows from continuity. The surjectivity of $\varphi$ follows from the Tietze’s extension theorem. Hence, $\varphi$ is an isomorphism.

Remark 19. Under the above assumption, let $\tilde{X}$ be the character space of $\mathbb{C}^*(\mathbb{C}(X), p)$ and $\iota : \mathbb{C}(X) \to C(\tilde{X})$ be the inclusion of $\mathbb{C}(X)$ to $C(\tilde{X})$. By the above proposition, one has $\tilde{X} \cong \overline{U} \cup (X \setminus U)$. Hence, $\pi := \iota_* : \tilde{X} \to X$ is a proper quotient map, and for every $x \in X \setminus \partial U$, one has $|\pi^{-1}(x)| = 1$. Furthermore, for every $x \in X$, $x$ belong to $\partial U$ if and only if one has $|\pi^{-1}(x)| = 2$.

Example 20. Let $X$ be a compact $\Gamma$-space, $U$ be an open subset of $X$ and $p = \chi_U \in \ell_\infty(X)$. Let $X$ be a $\Gamma$-space $\tilde{X}$ to be the character space of $\mathbb{C}^*(\mathbb{C}(X), \Gamma \cdot p)$, $\iota : \mathbb{C}(X) \to C(\tilde{X})$ be the inclusion of $\mathbb{C}(X)$ to $C(\tilde{X})$ and $\pi := \iota_* : \tilde{X} \to X$. Then $\pi$ is a proper factor map. If $S_X \cap \Gamma \cdot \partial U = \emptyset$, then for every $x \in S_X$, one has $|\pi^{-1}(x)| = 1$.

Proof. We prove $|\pi^{-1}(x)| = 1$ for every $x \in S_X$. It suffices to show that for every $x \in S_X$, one has $x \in \Gamma \cdot \partial U$ if and only if $|\pi^{-1}(x)| \geq 2$.

Let $x \in \Gamma \cdot \partial U$. There exists $g \in \Gamma$ s.t. $x \in g \cdot \partial U$. Let $X_1$ be the character space of $\mathbb{C}^*(\mathbb{C}(X), g \cdot p)$. Let $\iota_1$ be the inclusion of $\mathbb{C}(X)$ to $\mathbb{C}^*(\mathbb{C}(X), g \cdot p)$ and $\pi_2$ be the inclusion of $\mathbb{C}^*(\mathbb{C}(X), g \cdot p)$ to $C(X)$. Let $\pi_1 = (\iota_1)_* : X_1 \to X$ and $\pi_2 = (\iota_2)_* : \tilde{X} \to X_1$. Since $\pi_1^{-1}(x) = \pi_2^{-1}(\pi_1^{-1}(x))$, one has $|\pi_1^{-1}(x)| = |\pi_2^{-1}(\pi_1^{-1}(x))| \geq |\pi_1^{-1}(x)| = 2$ by the above remark.

Let $x$ be an element of $X$ such that $|\pi^{-1}(x)| \geq 2$. There exist distinct elements $y_1$ and $y_2$ in $X$ such that $\pi(y_1) = \pi(y_2) = x$. Let $\Gamma = \{g_n\}_{n \in \mathbb{N}}$. Let $A_n$ be a $\Gamma$-subalgebra of $\mathbb{C}(X)$ generated by $\mathbb{C}(X)$ and $\{g_n \cdot p\}_{k=1}^n$. Then $\cup_{n} A_n$ is dense in $\mathbb{C}(\tilde{X})$. So, there exist $n \in \mathbb{N}$ and $a \in A_n$ such that $y_1(a) \neq y_2(a)$. We regard $y_1, y_2$ as characters on $A_n$. By the above remark, there exists $g \in \Gamma$ such that $x \in g \cdot \partial U$.

The assumption in the above remark is satisfied in many cases including the following case.

Example 21. Let $X$ be a compact metric $\Gamma$-space. Assume $S_X$ is separable and the Lebesgue covering dimension of $S_X$ is zero. Let $A, B$ be disjoint closed subsets in $X$. Then, by the second separation theorem (see [Eng78, 1.5.13]), there exists an open set $V$ in $X$ s.t. $A \subset U$, $B \subset U^c$ and $\partial U \cap S_X = \emptyset$. Since $S_X$ is $\Gamma$-invariant, one has $S_X \cap \Gamma \cdot \partial U = \emptyset$.

We construct another example. We are grateful to Yuhei Suzuki for letting us know the following example. We say an element $g$ in a free group is indivisible if it is not a proper power of some element in the free group. For every element $x$ in a free group, we denote by $x^\infty$ the limit of $x^n$ in the Gromov compactification of the free group.

Remark 22. We consider the free group $\mathbb{F}_d$ of rank $d$, where $1 < d < \infty$, and its action on its Gromov boundary $\partial \mathbb{F}_d$. Let $x$ be an element of $\mathbb{F}_d$. The stabilizer group of $x^\infty$ is a cyclic group and we denote by $y$ its generator. Then $y$ is indivisible. Also, one has either $x^\infty = y^\infty$ or $x^\infty = y^{-\infty}$.

Example 23. Let $\mathbb{F}_d := \langle a_1, \ldots, a_d \rangle$ be the free group of rank $d$, where $1 < d < \infty$. Let $\partial \mathbb{F}_d$ denote the Gromov boundary. Let $\Gamma$ be a non-trivial normal subgroup of $\mathbb{F}_d$. Then the $\Gamma$-action on $\partial \mathbb{F}_d$ is topologically free and minimal. Let $T$ be a subset of $\mathbb{F}_d$ which consists of indivisible
elements such that \( |\{ C_{x,t} \} | t \in T \) \( < \infty \). Assume that for every non-zero \( n \in \mathbb{Z} \) and for every \( t \in T, t^n \notin \Gamma \). Let

\[
R_{T, \Gamma} := \{(x, x) \mid x \in \partial F_d \} \cup \{(g.t^\infty, g.t^{-\infty}) \mid g \in \Gamma, t \in T \cup T^{-1}\}.
\]

Then \( R_{T, \Gamma} \) is a \( \Gamma \)-invariant equivalence relation on \( \partial F_d \). Also, \( \partial F_d / R_{T, \Gamma} \) is Hausdorff.

Furthermore, the quotient map \( \pi: \partial F_d \rightarrow \partial F_d / R_{T, \Gamma} \) is a proper quotient map satisfying for every \( x \in \partial F_d / R_{T, \Gamma} \) with a non-trivial stabilizer, \( |\pi^{-1}(x)| = 1 \).

**Proof.** Since the \( \Gamma \)-action on \( \partial F_d \) is topologically free, the \( \Gamma \)-action on \( \partial F_d \) is topologically free. We will show the minimality of the \( \Gamma \)-action. Let \( S := \{a_1, \ldots, a_d\} \), which is a finite generating set of \( \partial F_d \). We regard \( \partial F_d \) as the set of infinite reduced words of \( F_d \) (see [BOOS 5.1]). Let \( x = x_1x_2 \cdots \) and \( y = y_1y_2 \cdots \) be elements in \( \partial F_d \), where \( x_i, y_i \in S \cup S^{-1} \). Let \( \gamma \in \Gamma \) be a non-trivial element. For each \( n \in \mathbb{N} \), there exists \( z_n \in S \cup S^{-1} \) such that \( |y_nz_n| > |y_n| \) and \( |z_n\gamma| > |\gamma| \), where \( |\cdot| \) is the length function on \( \Gamma \) determined by \( S \). Let \( w_n = (y_1 \cdots y_n)z_n\gamma z_n^{-1}(y_1 \cdots y_n)^{-1} \in \partial F_d \). Then one has \( w_n \rightarrow y \). Also, by the normality of \( \Gamma \), \( w_n \) belongs to \( \Gamma \). Hence, the \( \Gamma \)-action on \( \partial F_d \) is minimal.

Since \( |\{ C_{x,t} \} | t \in T \) \( < \infty \), there exists a finite subset \( T' \) of \( T \) such that \( \{ C_{x,t} \} \mid t \in T \} = \{ C_{x,t} \} \mid t \in T' \}. So, one has \( R_{T, \Gamma} \subset R_{T', \Gamma} \). Hence, in the similar way as [SIZ17 Lemma 4.4], we can show that \( R_{T, \Gamma} \) is a \( \Gamma \)-invariant equivalence relation on \( \partial F_d \) and \( \partial F_d / R_{T, \Gamma} \) is Hausdorff.

We will show that \( \pi \) satisfies that for every \( x \in \partial F_d / R_{T, \Gamma} \) with a non-trivial stabilizer, one has \( |\pi^{-1}(x)| = 1 \). Let \( x \in \partial F_d / R_{T, \Gamma} \) satisfying \( \gamma.x = x \) for some non-neutral element \( \gamma \in \Gamma \). We remark each equivalence class of \( \Gamma \)-action contains at most two elements. Suppose there exist \( g \in \Gamma \) and \( t \in T \cup T^{-1} \) such that \( |\pi^{-1}(x)| = \{g.t^\infty, g.t^{-\infty}\} \). Then one has \( \gamma.g.t^\infty \in \{g.t^\infty, g.t^{-\infty}\} \). Since there exists no element \( h \in F_d \) with \( h.t^\infty = \gamma.t^\infty \), we may assume \( \gamma.g.t^\infty = g.t^\infty \). Since \( g^{-1}g.t^\infty \) fixes \( t^\infty \), there exist a non-zero integer \( n \) such that \( g^{-1}g.t^n = t^n \). This contradicts the assumption. Hence, one has \( |\pi^{-1}(x)| = 1 \).

By the following proposition and Theorem 19, one has a one-to-one correspondence between intermediate \( C^* \)-algebras of \( C(\partial F_d / R_{T, \Gamma}) \times_{\Gamma} \Gamma \subset C(\partial F_d) \times_{\Gamma} \) and subsets of \( T \).

**Proposition 24.** In addition to the above condition, assume for distinct elements \( s, t \in T \), one has \( C_{\Gamma}(s) \cap \{t, t^{-1}\} = \emptyset \). Then the map

\[
(\rho: \partial F_d \rightarrow Z) \mapsto S_{\rho}.
\]

where \( S_{\rho} := \{t \in T \mid |\rho^{-1} \circ \rho(t)| = 2\} \), gives a one-to-one correspondence between intermediate extensions of \( \pi \) and subsets of \( T \) with its inverse map given by

\[
S \mapsto (\rho_S: \partial F_d \rightarrow \partial F_d / R_{S, \Gamma}),
\]

where \( \rho_S: \partial F_d \rightarrow \partial F_d / R_{S, \Gamma} \) is the quotient map.

**Proof.** Let \( (\rho, \rho') \) be an intermediate extension of \( \pi \), where \( \rho: \partial F_d \rightarrow Z \) and \( \rho': Z \rightarrow \partial F_d / R_{T, \Gamma} \) are factor maps such that \( \rho' \circ \rho = \pi \). Since \( \rho_{S_{\rho}} \) is surjective, we define the \( \Gamma \)-equivariant map \( f: \partial F_d / R_{S_{\rho}, \Gamma} \rightarrow Z \) by \( x \mapsto \rho(y) \) for some \( y \in \partial F_d \) with \( \rho_{S_{\rho}}(y) = x \). We first check this map is well-defined. Let \( x \in \partial F_d / R_{S_{\rho}, \Gamma} \). Let \( y_1 \) and \( y_2 \) be elements in \( \partial F_d \) such that \( \rho_{S_{\rho}}(y_1) = \rho_{S_{\rho}}(y_2) = x \). We may assume \( y_1 \neq y_2 \). By the definition of \( R_{S_{\rho}, \Gamma} \), we may assume \( y_1 = g.s^\infty \) and \( y_2 = g.s^{-\infty} \) for some \( g \in \Gamma \) and some \( s \in S_{\rho} \). By the definition of \( R_{S_{\rho}, \Gamma} \), one has \( (g.s^\infty, g.s^{-\infty}) \in R_{S_{\rho}, \Gamma} \). Hence, one has \( \rho(y_1) = \rho(y_2) \). Hence, \( f \) is well-defined. Similarly, we can check \( f \) is \( \Gamma \)-equivariant.

We will show \( f \) is a continuous bijective map, that is an extension of \( f \), one has \( \rho = f \circ \rho_{S_{\rho}} \). Hence, \( f \) is surjective. Also, since \( \rho \) and \( \rho_{S_{\rho}} \) are quotient maps, \( f \) is continuous.

We will show the injectivity of \( f \). Let \( x_1, x_2 \) be elements in \( \partial F_d / R_{S_{\rho}, \Gamma} \) with \( f(x_1) = f(x_2) \). There exist \( y_1 \) and \( y_2 \) in \( \partial F_d \) such that \( \rho_{S_{\rho}}(y_i) = x_i \) for each \( i \in \{1, 2\} \). Then one has \( \rho(y_1) = \rho(y_2) \). We may assume \( y_1 \neq y_2 \). By the definition of \( S_{\rho} \), one has \( (y_1, y_2) \in R_{S_{\rho}, \Gamma} \). So, one has \( x_1 \neq x_2 \). Hence, \( f \) is injective. Hence, \( \partial F_d / R_{S_{\rho}, \Gamma} \cong Z \) as an intermediate extension of \( \pi \).
Let $S$ be a subset of $T$. We will show that $S = S_{PS}$. Let $s ∈ S$. Since $(s^∞, s^−∞) ∈ R_{S,Γ}$, one has $s ∈ S_{PS}$. Let $s ∈ S_{PS}$. Then one has $(s^∞, s^−∞) ∈ R_{S,Γ}$. So, there exist $g ∈ Γ$ and $s' ∈ S$ such that $s^∞ = g.s^∞$ or $s^−∞ = g.s^−∞$. Suppose $s ≠ s'$. Since $gsg^{-1}$ fixes $g.s^∞$ and $g.s^−∞$, and $s'$ is indivisible, then there exist $n ∈ Z \setminus \{0\}$ such that $s^n = gsg^{-1}$ by Remark. Since $s$ is indivisible, one has $n = ±1$. This contradicts the assumption, that is $C_T(s) ∩ \{s', s^{-1}\} = ∅$, where $[s]$ is the $Γ$-conjugacy class of $s$. Hence, one has $s = s' ∈ S$.

**Corollary 25.** In the above conditions, the map

$$S ↦ C(∂F_d/R_{S,Γ}) ∼_Γ \Gamma$$

provides a lattice isomorphism between the lattice of subsets of $T$ and that of intermediate $C^*$-algebras of $C(∂F_d/R_{T,Γ}) ∼_Γ C(∂F_d) ∼_Γ Γ$.

In many cases, we can construct an infinite subset $T$ of $F_d$ satisfying the above conditions as follows.

**Proposition 26.** Let $F_d$ be the free group of rank $d$ ($d > 1$). Let $Γ$ be a normal subgroup of $F_d$ such that the quotient group $F_d/Γ$ is not virtually cyclic. Then there exists an infinite subset $T$ satisfying the following properties:

1. Every element of $T$ is indivisible.
2. For every $0 ≠ n ∈ Z$ and $t ∈ T$, one has $t^n ∉ Γ$.
3. For every distinct elements $s, t ∈ T$, one has $C_T(s) ∩ \{t, t^{-1}\} = ∅$.
4. $|C_{F_Γ}(t)| = 1$.
5. $|C_T(t)| = 1$.

**Proof.** Since $F_d/Γ$ is not virtually cyclic, there exists an indivisible element $t$ of $F_d$ such that for every $0 ≠ n ∈ Z$, $t^n ∉ Γ$. We denote by $(t)$ the subgroup of $F_d$ generated by $t$. Let $S$ be a subset of $F_d$ such that $\bigcup_{s ∈ S} C_T(s^{t^{-1}}) = C_{F_d}(t)$. We show $|S| = |[F_d/Γ : (t)Γ]|$, where $|F_d/Γ : (t)Γ|$ is the index of $(t)Γ$ in $F_d/Γ$. Let $s, s' ∈ F_d$. Let $C_T(s^{t^{-1}}) = C_T(s's'^{-1})$. Then there exists $g ∈ Γ$ such that $gst^{-1}g^{-1} = s's'^{-1}$. Hence, one has $s'^{-1}gst = ts^{-1}gs$. Since the centralizer of $t$ is $(t)$, one has $s'^{-1}gst ∉ (t)$. Hence, one has $sΓ = s'Γ$. Hence, one has $|S| = |F_d/Γ : (t)Γ|$. Since $F_d/Γ$ is not virtually cyclic, one has $[F_d/Γ : (t)Γ] = ∞$. Hence, one has $|S| = ∞$.

Let $T = \{sts^{-1} | s ∈ S\}$. By construction, $T$ satisfies the properties (4) and (5). Since $Γ$ is normal and $t$ is indivisible, $T$ satisfies the properties (1) and (2). We show $T$ satisfies the property (3). Let distinct elements $s, t ∈ S$. By construction, one has $t ∉ C_T(s)$. Suppose there exists $g ∈ Γ$ such that $t^{-1} = gst^{-1}gs$. Then one has $t^{-∞} = gs.t^{-∞}$, which is a contradiction, since there exists no element $h ∈ F_d$ such that $h.t^{-∞} = t^{-∞}$.

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