On Higher Regularized Traces of a Differential Operator with Bounded Operator Coefficient Given in a Finite Interval

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Abstract. In this work, we find a higher regularized trace formula for a regular Sturm–Liouville differential operator with operator coefficient.

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1. Introduction and Preliminaries

The history of regularized trace formulas for ordinary differential operators goes back to the year 1953. In that year, Gelfand and Levitan [13] studied the boundary value problem

\[-y'' + q(x)y = \lambda y, \quad y'(0) = y'() = 0,\]

with \(q(x) \in C[0, \pi]\). They found the formula

\[\sum_{n=0}^{\infty} (\lambda_n - \mu_n) = \frac{1}{4}(q(0) + q()),\]

under the assumption \(\int_0^\pi q(x)dx = 0\), where the \(\mu_n\) are the eigenvalues of this problem. \(\lambda_n = n^2\) are the eigenvalues of the same problem with \(q(x) = 0\). After the pinioneeing work by Gelfand–Levitan, there was a growing interest and many scientists used the same method to obtain the regularized traces of ordinary differential operators. Later, Dikii [6] gave another proof of Gelfand–Levitan’s formula from a different point of view. Afterward, Dikii [7] and Gelfand [12] made significant progress in literature by computing regularized sums of powers of eigenvalues. Later on, Levitan [16] calculated the regularized traces of Sturm–Liouville Problem with a new method. This research led to Faddeev [9], who connected the trace theory with singular
differential operators. Gasimov [11] made the first study combining singular operators with discrete spectrum. In the sequel, in many other papers such as [8,10,17,19] regularized traces of various scalar differential operators were investigated. In the references [2,3,14,18] the regularized trace formulas of Sturm–Liouville operators with operator coefficient were found.

In this paper, we compute the higher-order regularized trace of Sturm–Liouville operator. Considering this study as a continuation of the chaining historical development studies that we gave above, the calculation of regularized trace of higher order will take its place in the literature as a generalization. We open the discussion by recalling some facts:

Let $H$ be an infinite-dimensional separable Hilbert space. We will denote the inner product and norm by $(.,.)_H$ and $\|\cdot\|_H$ in $H$. Let $H_1 = L^2(0,\pi; H)$ be the set of all strongly measurable functions $f$ defined on $[0,\pi]$ and taking values in the space $H$. The following conditions hold for all functions $f$ in $H_1$:

1. The scalar function $(f(x),g)_H$ is Lebesgue measurable on the interval $[0,\pi]$, for every $g \in H$,
2. $\int_0^\pi \|f(x)\|_H^2 \, dx < \infty$.

$H_1$ is a normed linear space. We will denote the inner product and norm by $(.,.)$ and $\|\cdot\|$ in $H_1$. If the inner product is defined by

$$(f_1,f_2) = \int_0^\pi (f_1(x),f_2(x))_H \, dx,$$

for two arbitrary elements $f_1$ and $f_2$ of $H_1$, then the space $H_1$ becomes a separable Hilbert space, [15]. Consider Sturm–Liouville problems formed by

$L_0 y = -y''$, \hspace{1cm} y'(0) = y' (\pi) = 0 \hspace{1cm} (1.1)

$L y = -y'' + Q(x)y$, \hspace{1cm} y'(0) = y' (\pi) = 0 \hspace{1cm} (0 \leq x \leq \pi), \hspace{1cm} (1.2)$

in $H_1$. Assume that the operator function $Q(x)$ in (1.2) satisfies the conditions:

(Q1) $Q(x)$ has $(2k-2)^{th}$ order weak derivatives $(k \geq 2)$ and $Q^{(i)}(x) : H \to H$ $(i = 0,1,2,\ldots,2k-2)$ are self-adjoint kernel operators for every $x \in [0,\pi]$,

(Q2) $\|Q\| < \frac{1}{2}$,

(Q3) There is an orthonormal basis $\{\varphi_n\}_{n=1}^\infty$ of $H$ such that $\sum_{n=1}^\infty \|Q(x)\varphi_n\| < \infty$,

(Q4) The functions $\|Q^{(i)}(x)\|_{\sigma_1(H)}$ $(i = 0,1,2,\ldots,2k-2)$ are both bounded and measurable in $[0,\pi]$.

Here, $\sigma_1(H)$ denotes the Banach space consisting of kernel operators from $H$ to $H$. The spectrum of the operator $L_0$, denoted by $\sigma(L_0)$ is the set $\{m^2\}_{m=0}^\infty$. Every point of this set is an eigenvalue of $L_0$ with infinite multiplicity [14]. The eigenvectors corresponding to $m^2$ have the form:

$$\psi_{mn} = K_m \varphi_n \cos mx,$$
where \( m = 0, 1, 2, \ldots \); \( n = 1, 2, \ldots \) and
\[
K_m = \begin{cases} 
\frac{1}{\sqrt{\pi}}, & m = 0 \\
\sqrt{\frac{2}{\pi}}, & m = 1, 2, \ldots 
\end{cases}
\]
Let \( R_0^0 \) and \( R_\lambda \) be resolvents of the operator \( L_0 \) and \( L \), respectively. Since the operator \( Q(x) \) satisfies the condition (Q3), \( QR_\lambda^0 : H_1 \rightarrow H_1 \) is a kernel operator for every \( \lambda \notin \{ m^2 \}_{m=0}^\infty \) [5]. One can prove that the spectrum of the operator \( L \) is a subset of the union of pairs of disjoint intervals \([m^2 - \frac{1}{2}, m^2 + \frac{1}{2}]\) \((m = 0, 1, 2, \ldots)\). \( m^2 \) can be an eigenvalue of \( L \) with finite or infinite multiplicity and \( \lim_{n \to \infty} \lambda_{mn} = m^2 \), [14]. Here, \( \{ \lambda_{mn} \}_{n=1}^\infty \) are eigenvalues of \( L \) belonging to the interval \([m^2 - \frac{1}{2}, m^2 + \frac{1}{2}]\). Each point different from \( m^2 \) of the spectrum of \( L \) is an isolated eigenvalue with finite multiplicity, [4]. Since \( QR_\lambda^0 \) is a kernel operator and
\[
R_\lambda - R_0^0 = -R_\lambda QR_\lambda^0, \tag{1.4}
\]
we have \( R_\lambda - R_0^0 \in \sigma_1(H_1) \) for each \( \lambda \) which belongs to the resolvent set of \( L \). In this case, from [5] we obtain the formula
\[
\text{tr} \left( R_\lambda - R_0^0 \right) = \sum_{m=0}^\infty \sum_{n=1}^\infty \left( \frac{1}{\lambda_{mn} - \lambda} - \frac{1}{m^2 - \lambda} \right). \tag{1.5}
\]

2. Main Results

The main purpose of this note is to obtain the higher-order trace formula for Sturm–Liouville problem (1.2).

**Theorem 2.1.** If the operator function \( Q(x) \) satisfies the conditions (Q1)–(Q4), then the \( k \)th regularized trace formula of \( L \) is:
\[
\sum_{m=0}^\infty \left\{ \sum_{n=1}^\infty \left( \lambda_{mn}^k - m^{2k} \right) - k \sum_{j=2}^{2k+2} (-1)^j j^{-1} \text{Res}_{\lambda=m^2} \left[ \lambda^{k-1} \text{tr}(QR_\lambda^0)^j \right] \right. \right.
\]
\[
- k \pi^{-1} m^{2k-2} \int_0^\pi \text{tr} Q(x) \, dx - 4k \pi^{-1} m^{2k} \sum_{i=2}^k m^{-2i} a_i \bigg\} = (-1)^{k-1} \frac{k}{4\pi} \left( \text{tr} Q^{(2k-2)}(\pi) + \text{tr} Q^{(2k-2)}(0) \right) + \frac{2}{\pi} k a_k. \tag{2.1}
\]
Here, \( a_i = (-1)^i 2^{-2i} \left( \text{tr} Q^{(2i-3)}(\pi) - \text{tr} Q^{(2i-3)}(0) \right) \) \((i = 2, 3, \ldots, k)\).

**Theorem 2.2.** Let \( Q(x) \) satisfy conditions
\[
\begin{align*}
(Q5) \quad & Q^{(2i-3)}(0) = Q^{(2i-3)}(\pi) = 0 \quad (2 \leq i \leq k), \\
(Q6) \quad & \int_0^\pi \text{tr} Q(x) \, dx = 0.
\end{align*}
\]
If \( Q(x) \) satisfies the conditions (Q1)–(Q6), then the formula (2.1) becomes:
\[
\sum_{m=0}^\infty \left\{ \sum_{n=1}^\infty \left( \lambda_{mn}^k - m^{2k} \right) - k \sum_{j=2}^{2k+2} (-1)^j j^{-1} \text{Res}_{\lambda=m^2} \left[ \lambda^{k-1} \text{tr}(QR_\lambda^0)^j \right] \right\} \right. \right.
\]
\[
= (-1)^{k-1} k 2^{-2k} \left[ \text{tr} Q^{(2k-2)}(0) + \text{tr} Q^{(2k-2)}(\pi) \right]. \tag{2.2}
\]
3. Proofs

If we multiply the equality (1.5) by $\frac{\lambda_k}{2\pi i}$, with $k \geq 2$, $k \in \mathbb{Z}$ and integrate on the circle $|\lambda| = b_p = p^2 + p$ (p $\geq 1$), then we have

$$
\frac{1}{2\pi i} \int_{|\lambda|=b_p} \lambda^k \text{tr}(R_\lambda - R_\lambda^0) \, d\lambda = \sum_{m=0}^{p} \sum_{n=1}^{\infty} (m^{2k} - \lambda_{mn}^k).
$$

(3.1)

Using the formula (1.4), we obtain

$$
R_\lambda - R_\lambda^0 = \sum_{j=1}^{N} (-1)^j R_\lambda^0 (QR_\lambda^0)^j + (-1)^{N+1} R_\lambda (QR_\lambda^0)^{N+1},
$$

where $N \geq 1$ is any natural number. Using the last equality, we rewrite (3.1) as follows:

$$
\sum_{m=0}^{p} \sum_{n=1}^{\infty} (\lambda_{mn}^k - m^{2k}) = \sum_{j=1}^{N} M_{pj} + M_{p}^{(N)},
$$

(3.2)

here,

$$
M_{pj} = \frac{(-1)^j}{2\pi i} \int_{|\lambda|=b_p} \lambda^k \text{tr} \left[ R_\lambda^0 (QR_\lambda^0)^j \right] d\lambda,
$$

(3.3)

$$
M_{p}^{(N)} = \frac{(-1)^N}{2\pi i} \int_{|\lambda|=b_p} \lambda^k \text{tr} \left[ R_\lambda (QR_\lambda^0)^{N+1} \right] d\lambda.
$$

(3.4)

**Lemma 3.1.** If the operator function $Q(x)$ satisfies the condition $(Q3)$, then

$$
M_{pj} = \frac{(-1)^j k}{2\pi i j} \int_{|\lambda|=b_p} \lambda^{k-1} \text{tr} (QR_\lambda^0)^j d\lambda.
$$

(3.5)

**Proof.** Using $QR_\lambda^0$ as a kernel operator for every $\lambda \neq m^2$ in the space $\sigma_1(H_1)$, one can prove that $QR_\lambda^0$ is analytic with respect to the norm in the space $\sigma_1(H_1)$ in the domain $C \setminus \{m^2\}$ and

$$
\text{tr} \left\{ [(QR_\lambda^0)^j]' \right\} = j \text{tr} \left[ (QR_\lambda^0)' (QR_\lambda^0)^{j-1} \right].
$$

(3.6)

If we consider the equation $(QR_\lambda^0)' = (QR_\lambda^0)^2$, then we state the formula (3.6):

$$
\text{tr} \left\{ [(QR_\lambda^0)^j]' \right\} = j \text{tr} \left[ R_\lambda^0 (QR_\lambda^0)^j \right].
$$

(3.7)

If we substitute the formula (3.7) in (3.3), then we have

$$
M_{pj} = \frac{(-1)^j}{2\pi i j} \int_{|\lambda|=b_p} \lambda^k \text{tr} \left\{ [(QR_\lambda^0)^j]' \right\} d\lambda.
$$
From the last equality, we have

\[ M_{pj} = \frac{(-1)^{j+1}}{2\pi i j} \int_{|\lambda|=b_p} \text{tr} \left\{ \lambda^{k}(QR^{0}_{\lambda})^{j} \right\} \, d\lambda \]

\[ = \frac{(-1)^{j}}{2\pi i j} \int_{|\lambda|=b_p} k\lambda^{k-1}(QR^{0}_{\lambda})^{j} \, d\lambda \]

\[ + \frac{(-1)^{j+1}}{2\pi i j} \int_{|\lambda|=b_p} \text{tr} \left\{ \lambda^{k}(QR^{0}_{\lambda})^{j} \right\} \, d\lambda. \quad (3.8) \]

Moreover, we can show that

\[ \text{tr} \left\{ \lambda^{k}(QR^{0}_{\lambda})^{j} \right\} = \left\{ \text{tr} \lambda^{k}(QR^{0}_{\lambda})^{j} \right\}'. \]

Therefore, we have

\[ \int_{|\lambda|=b_p} \left\{ \text{tr} \left\{ \lambda^{k}(QR^{0}_{\lambda})^{j} \right\} \right\}' \, d\lambda = \int_{|\lambda|=b_p} \left\{ \text{tr} \lambda^{k}(QR^{0}_{\lambda})^{j} \right\}' \, d\lambda. \quad (3.9) \]

We write the integral on the right-hand side of the equality (3.9) as follows:

\[ \int_{|\lambda|=b_p} \left\{ \text{tr} \lambda^{k}(QR^{0}_{\lambda})^{j} \right\} ' \, d\lambda = \int_{|\lambda|=b_p, \text{Im}\lambda \geq 0} \left\{ \text{tr} \lambda^{k}(QR^{0}_{\lambda})^{j} \right\} ' \, d\lambda \]

\[ + \int_{|\lambda|=b_p, \text{Im}\lambda \leq 0} \left\{ \text{tr} \lambda^{k}(QR^{0}_{\lambda})^{j} \right\} ' \, d\lambda. \quad (3.10) \]

Let \( \varepsilon_{0} \) be a constant which satisfies the inequality \( 0 < \varepsilon_{0} < b_{p} - (p^{2} + p) \).

Knowing that the function \( \text{tr} \lambda^{k}(QR^{0}_{\lambda})^{j} \) is analytic on simply connected domains \( G_{1} \) and \( G_{2} \) defined by

\[ G_{1} = \{ \lambda \in C : b_{p} - \varepsilon_{0} < |\lambda| < b_{p} + \varepsilon_{0}, \text{Im}\lambda > -\varepsilon_{0} \} \]

\[ G_{2} = \{ \lambda \in C : b_{p} - \varepsilon_{0} < |\lambda| < b_{p} + \varepsilon_{0}, \text{Im}\lambda < -\varepsilon_{0} \} . \]

The sets \( \{ \lambda \in C : |\lambda| = b_{p}, \text{Im}\lambda \geq 0 \} \) and \( \{ \lambda \in C : |\lambda| = b_{p}, \text{Im}\lambda \leq 0 \} \) are subdomains of \( G_{1} \) and \( G_{2} \), respectively. After calculating the integrals on the right side in (3.10) on these subdomains, we rewrite equality (3.10):

\[ \int_{|\lambda|=b_p} \left\{ \text{tr} \lambda^{k}(QR^{0}_{\lambda})^{j} \right\} ' \, d\lambda = \text{tr} \left[ -b_{p}(QR^{0}_{b_{p}})^{j} \right] - \text{tr} \left[ b_{p}(QR^{0}_{b_{p}})^{j} \right] \]

\[ + \text{tr} \left[ b_{p}(QR^{0}_{b_{p}})^{j} \right] - \text{tr} \left[ -b_{p}(QR^{0}_{b_{p}})^{j} \right] = 0. \quad (3.11) \]

From (3.8), (3.9) and (3.11), we find

\[ M_{pj} = \frac{(-1)^{j}k}{2\pi i j} \int_{|\lambda|=b_p} \lambda^{k-1}(QR^{0}_{\lambda})^{j} \, d\lambda. \]
Now, we are at the position to prove the main results:

Proof of Theorem 2.1: According to Lemma 3.1

\[
M_{p1} = \frac{-k}{2\pi i} \int_{|\lambda|=b_p} \lambda^{k-1} \text{tr}(QR_0^0) d\lambda. \tag{3.12}
\]

Since \( \{\psi_{mn}\}_{m=0, n=1}^\infty \) is an orthonormal basis of the space \( H_1 \), we obtain

\[
M_{p1} = \frac{-k}{2\pi i} \int_{|\lambda|=b_p} \lambda^{k-1} \sum_{m=0}^\infty \sum_{n=1}^\infty (QR_0^0 \psi_{mn}, \psi_{mn}) d\lambda
\]

\[
= k \sum_{m=0}^\infty \sum_{n=1}^\infty m^{2k-2} (QR_0^0 \psi_{mn}, \psi_{mn})
\]

\[
= k \sum_{m=1}^p \sum_{n=1}^\infty m^{2k-2} \frac{\pi}{2} \int_0^\pi (Q(x)\varphi_n, \varphi_n)_{H} \cos^2 mx \, dx
\]

\[
= k \sum_{m=1}^p \sum_{n=1}^\infty m^{2k-2} \left[\frac{\pi}{2} \int_0^\pi (Q(x)\varphi_n, \varphi_n)_{H} \left(1 + \cos 2mx\right) dx \right]
\]

\[
= \frac{k}{\pi} \sum_{m=1}^p m^{2k-2} \int_0^\pi \text{tr}Q(x) dx + \frac{k}{\pi} \sum_{m=1}^p m^{2k-2} \int_0^\pi \text{tr}Q(x) \cos 2mx dx
\]

(3.13)

for \( k \geq 2 \). Now, we evaluate the second integral in (3.13):

\[
\int_0^\pi \text{tr}Q(x) \cos 2mx dx = \left[ \frac{1}{2m} \sin 2mx \text{tr}Q(x) \right]_0^\pi - \frac{1}{2m} \int_0^\pi \text{tr}Q'(x) \sin 2mx dx
\]

\[
= -\frac{1}{2m} \int_0^\pi \text{tr}Q'(x) \sin 2mx dx
\]

\[
= \frac{1}{4m^2} \left( \text{tr}Q'(\pi) - \text{tr}Q'(0) \right) - \frac{1}{4m^2} \int_0^\pi \text{tr}Q''(x) \cos 2mx dx.
\]

If we continue the process in a similar way, then we obtain the result of the integration:

\[
\int_0^\pi \text{tr}Q(x) \cos 2mx dx = \sum_{i=2}^k \frac{(-1)^i}{(2m)^{2i-2}} \left( \text{tr}Q^{(2i-3)}(\pi) - \text{tr}Q^{(2i-3)}(0) \right)
\]

\[
+ \frac{(-1)^{k-1}}{(2m)^{2k-2}} \int_0^\pi \text{tr}Q^{(2k-2)}(x) \cos 2mx \, dx. \tag{3.14}
\]
Putting (3.14) into (3.13):

\[ M_{p1} = \frac{k}{\pi} \sum_{m=1}^{p} m^{2k-2} \int_{0}^{\pi} \text{tr} Q(x) \, dx \]

\[ + \frac{k}{\pi} \sum_{m=1}^{p} \sum_{i=2}^{k} (-1)^{i} 2^{2-2i} m^{2k-2i} \left( \text{tr} Q^{(2i-3)}(\pi) - \text{tr} Q^{(2i-3)}(0) \right) \]

\[ - \frac{k}{\pi} \sum_{m=1}^{p} (-1)^{k} 2^{2-2k} \int_{0}^{\pi} \text{tr} Q^{(2k-2)}(x) \cos 2mx \, dx. \] (3.15)

From (3.2) and (3.15), we have

\[ \sum_{m=0}^{p} \left\{ \sum_{n=1}^{\infty} \left( \lambda_{mn}^{k} - m^{2k} \right) - \sum_{j=2}^{N} M_{pj} - k\pi^{-1} m^{2k-2} \int_{0}^{\pi} \text{tr} Q(x) \, dx \right. \]

\[ \left. - k\pi^{-1} \sum_{i=2}^{k} (-1)^{i} 2^{2-2i} m^{2k-2i} \left( \text{tr} Q^{(2i-3)}(\pi) - \text{tr} Q^{(2i-3)}(0) \right) \right\} \]

\[ = -k\pi^{-1} \sum_{m=1}^{p} (-1)^{k} 2^{2-2k} \int_{0}^{\pi} \text{tr} Q^{(2k-2)}(x) \cos 2mx \, dx + M_{p}^{(N)}. \] (3.16)

Similar to [4], one can show that

\[ | M_{p}^{N} | \leq \text{const} \, p^{1+2k-N}. \] (3.17)

From (3.17), we obtain

\[ \lim_{p \to \infty} M_{p}^{(2k+2)} = 0. \] (3.18)

On the other hand, from [1] we have

\[ \frac{1}{\pi} \sum_{m=1}^{\infty} \int_{0}^{\pi} \text{tr} Q^{(2k-2)}(x) \cos 2mx \, dx \]

\[ = \frac{1}{4} \left[ \text{tr} Q^{(2k-2)}(0) + \text{tr} Q^{(2k-2)}(\pi) \right] - \frac{1}{2\pi} \left[ \text{tr} Q^{(2k-3)}(\pi) - \text{tr} Q^{(2k-3)}(0) \right]. \] (3.19)

Furthermore, since the function \( \lambda^{k-1} \text{tr} (QR_{0})^{j} \) on right side of the equality (3.5) is analytic on the domain \( \lambda \neq m^{2} \quad (m = 0, 1, 2, \ldots) \), Residue Theorem is satisfied:

\[ \int_{|\lambda|=b_{p}} \lambda^{k-1} \text{tr} (QR_{0})^{j} d\lambda = 2\pi i \sum_{m=0}^{p} \text{Res}_{\lambda=m^{2}} [\lambda^{k-1} \text{tr} (QR_{0})^{j}]. \]

Therefore, we get the formula (3.5) as follows:

\[ M_{pj} = (-1)^{j} \frac{k}{j} \sum_{m=0}^{p} \text{Res}_{\lambda=m^{2}} [\lambda^{k-1} \text{tr} (QR_{0})^{j}]. \] (3.20)

If we consider the formulas (3.18), (3.19), (3.20) and take limit in (3.16) as \( p \to \infty \), then we obtain kth order regularized trace formula of the operator
\[ L: \sum_{m=0}^{\infty} \left\{ \sum_{n=1}^{\infty} (\lambda_{mn}^k - m^{2k}) - k \sum_{j=2}^{2k+2} (-1)^j j^{-1} \text{Res}_{\lambda=m^2} [\lambda^{k-1} \text{tr}(QR_\lambda^0)^j] \right. \]
\[ - k \pi^{-1} m^{2k-2} \int_0^\pi \text{tr}Q(x) \, dx - 4k \pi^{-1} m^{2k} \sum_{i=2}^{k} m^{-2i} a_i \right\} \]
\[ = (-1)^{k-1} k \frac{\pi}{k} \left( \text{tr}Q(2k-2)(0) + \text{tr}Q(2k-2)(\pi) \right) + \frac{2}{\pi} k a_k, \quad (3.21) \]

here,
\[ a_i = (-1)^i 2^{-2i} (\text{tr}Q(2i-3)(\pi) - \text{tr}Q(2i-3)(0)) \quad (i = 2, 3, \ldots, k). \quad (3.22) \]

This completes the proof. □

**Proof of Theorem 2.2.** If we consider the condition (Q5):
\[ Q^{(2i-3)}(0) = Q^{(2i-3)}(\pi) = 0 \quad (2 \leq i \leq k), \]
on the equality (3.22), we get \[ a_i = 0 \quad (i = 2, 3, \ldots, k). \]
If we use the condition (Q6):
\[ \int_0^\pi \text{tr}Q(x) \, dx = 0 \]
for the integral on the left side of the formula (3.21), then we have
\[ \sum_{m=0}^{\infty} \left\{ \sum_{n=1}^{\infty} (\lambda_{mn}^k - m^{2k}) - k \sum_{j=2}^{2k+2} (-1)^j j^{-1} \text{Res}_{\lambda=m^2} [\lambda^{k-1} \text{tr}(QR_\lambda^0)^j] \right\} \]
\[ = (-1)^{k-1} k 2^{-2k} \left[ \text{tr}Q^{(2k-2)}(0) + \text{tr}Q^{(2k-2)}(\pi) \right]. \]

This concludes the proof. □

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**Compliance with Ethical Standards**

**Conflict of interest** The authors declare that they have no conflict of interest.

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