Solutions for a Nonhomogeneous Nonlinear Schrödinger Equation with Double Power Nonlinearity

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Abstract

We consider the problem $-\Delta u + V(x)u = f'(u) + g(x)$ in $\mathbb{R}^N$, under the assumption $\lim_{x \to \infty} V(x) = 0$, and with the nonlinear term $f$ with a double power behavior. We prove the existence two solutions when $g$ is sufficiently small and $V < 0$.

Keywords: Nonlinear Equations, Variational Methods, Orlicz Spaces

1 Perturbation of NSE

We consider the existence of solutions of the following nonhomogeneous problem

\[
\begin{align*}
-\Delta u + V(x)u &= f'(u) + g(x), \quad x \in \mathbb{R}^N; \\
E_g^V(u) &< \infty.
\end{align*}
\]

where the energy functional is defined by

\[
E_g(u) = E_g^V(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + V(x)u^2(x)dx - \int_{\mathbb{R}^N} f(u)dx - \int_{\mathbb{R}^N} g(x)u(x)dx.
\]

The nonlinearity is given by a function $f$ of double power type that is an even function $f \in C^3(\mathbb{R}, \mathbb{R})$ with $f(0) = f'(0) = f''(0) = 0$ satisfying the following requirements:

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\end{itemize}

\normalsize
1. there exist positive numbers $c_0, c_2, p, q$ with $2 < p < 2^* < q$ such that

\[
\begin{cases}
    c_0|s|^p \leq f(s) & \text{for } |s| \geq 1; \\
    c_0|s|^q < f(s) & \text{for } |s| \leq 1;
\end{cases}
\]  

\[ (f_0) \]

\[
\begin{cases}
    |f''(s)| \leq c_2|s|^{p-2} & \text{for } |s| \geq 1; \\
    |f''(s)| \leq c_2|s|^{q-2} & \text{for } |s| \leq 1;
\end{cases}
\]  

\[ (f_2) \]

2. there exists $\mu_1 > 2$ and $\mu_2 > 1$ such that, for all $s \neq 0$

\[
0 < \mu_1 f(s) \leq f'(s)s, \quad \mu_2 f'(s)s < f''(s)s^2, \quad f'''(s)s^3 > 0; \]  

\[ (f_\mu) \]

3. for any $u \in D^{1,2}$ we have

\[
f'''(u)u^3 \in L^1. \]  

\[ (f_3) \]

For example the required assumptions are satisfied by $f(s) = \frac{|s|^q}{1+|s|^{q-p}}$ with $q-p$ small enough, as shown in the appendix.

We assume $V \in L^{N/2}(\mathbb{R}^N) \cap L^t$, for some $t > N/2$ and

\[
||V||_{L^{N/2}} < S := \inf_{u \in D^{1,2}} \left( \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\left( \int_{\mathbb{R}^N} |u|^{2^*/2} \right)^{2^*/2}} \right). \]  

\[ (1) \]

Moreover, we want $V \leq 0$ and $V < 0$ on a set of positive measure.

In [18] the existence of two positive solutions $u_1, u_2 \in H^1(\mathbb{R}^N)$ of the equation $-\Delta u + u = |u|^{p-2}u + g$ is proved when $g \in L^2$ satisfies $0 \leq g \leq C \exp(-(1+\varepsilon)|x|)$, $g \neq 0$.

Recently, in [17], a similar problem for the $p$-laplacian is studied. Namely, the author proves, with variational techniques, that the problem $-\Delta u + c|u|^{p-2}u = |u|^{p-2}u + f(x, u) + h(x)$ in $\mathbb{R}^N$, where $2 \leq p < N$, $c > 0$, $h \in W^{-1,p'}(\mathbb{R}^N)$ and $f$ a is continuous superlinear function such that $f(x, 0) = 0$ and $f(x, u) = o(|u|^{p-1})$ as $|u| \to \infty$, admits two positive solutions $u_1, u_2 \in H^1(\mathbb{R}^N)$.

The existence of a positive solution of the problem $-\Delta u + u = |u|^{p-1}u + g$ on $\mathbb{R}^N$, $u(x) \to 0$ for $|x| \to \infty$, was proven in [10] when $p > \frac{N}{N-2}$ and $g \in C^{0,\alpha}(\mathbb{R}^N)$, $g \geq 0$, $g \neq 0$ and $g(x) \leq \frac{C}{(1+|x|^2)^{p-1}}$ for some $C > 0$. In [3] there is a result of multiplicity for this problem.

In [16] the author shows that the Dirichlet problem on a bounded domain $\Omega \subset \mathbb{R}^N$ in the critical case $-\Delta u = |u|^{2^*-2}u + g$ has two solutions $u_0, u_1 \in$
$H^1_0(\Omega)$, for $g$ satisfying a suitable condition, and if $g \geq 0$ then $u_0 \geq 0$ and $u_1 \geq 0$.

We are interested in studying the problem with double power nonlinearity.

In pioneering work Berestycki and Lions [7, 8] showed the existence of a positive solution in the case $V \equiv 0$ when $f''(0) = 0$, $f$ has a supercritical growth near the origin and subcritical at infinity.

More recently in the papers [2, 5, 6, 14] the double-power growth condition has been used to obtain the existence of positive solutions for different problems of the type ($P$). In particular, in [5], the authors proved that in the same hypothesis on $V$ the homogeneous problem

$$- \Delta u + Vu = f'(u)$$

has a ground state solution (i.e. least energy nontrivial solution). Other results on similar problems with the double power nonlinearity can be found in [1, 2, 12].

In this paper we prove the following theorem

**Theorem 1.** If $g \in L^{\frac{2N}{N+2}} \cap L^s$, for some $s > \frac{2N}{N+2}$, and if $||g||_{\frac{2N}{N+2}}$ is sufficiently small there exist two solutions of problem ($P$) in $D^{1,2}$. The first solution is close to 0; if also $||g||_{L^{p'} \cap L^{q'}}$ is small enough, the critical value of the second solution is close to the least energy level $m_V$ of the homogeneous problem ($P$).

Furthermore, if $g \geq 0$ the two solutions are non negative.

**Remark 2.** Indeed the hypothesis on the sign of $V$ is used only to find the second solution, but we prefer a more compact claim for the theorem. Anyway, in the proofs we focus out when we use any hypothesis.

To get the solutions of ($P$) we look for critical points of the functional $E^V_g$ constrained on the Nehari manifold

$$\mathcal{N}^V_g = \mathcal{N}_g = \{ u \in D^{1,2} : \langle \nabla E_g(u), u \rangle = 0, u \neq 0 \} = \left\{ u \in D^{1,2} \setminus 0 : \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} Vu^2 - \int_{\mathbb{R}^N} f'(u)u - \int_{\mathbb{R}^N} gu = 0 \right\}.$$ 

The study of the structure of the Nehari manifold will be a fundamental part of this paper.

This paper is organized as follows: in section 2, we recall some technical results concerning the appropriate function space required by the growth properties of the nonlinearity $f$. Moreover, we study the geometry and the properties of the Nehari manifold. In section 3, we prove a Splitting Lemma necessary to overcome the lack of compactness. This lemma is a variant for a well known result of [15]. In section 4 we prove the existence of two distinct critical points of the functional on the Nehari manifold.
2 Notations and preliminary result

We will use the following notations

- $D^{1,2} = D^{1,2}(\mathbb{R}^N) = \text{completion of } C_c^\infty(\mathbb{R}^N)$ with respect to the norm
  $$||u|| = \left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^{1/2};$$

- $||u||_V^2 = \int_{\mathbb{R}^N} |\nabla u|^2 + \int_{\mathbb{R}^N} Vu^2$; notice that, by (1), we have that $||u||_V$ is a norm in $D^{1,2}$ equivalent to the usual one;

- $2^* = \frac{2N}{N-2}$;

- $m_g = \inf_{u \in \mathcal{N}_g} E^V_g(u)$;

- $m_{1,g} = \inf_{u \in \mathcal{N}_g} E^V_g(u)$;

- $m_0 = \inf_{u \in \mathcal{N}_0^0} E^V_0(u)$; we call $\omega$ the minimizer of $E^V_0$ on $\mathcal{N}_0^0$ radially symmetric;

- $m_V = \inf_{u \in \mathcal{N}_V^0} E^V_0(u)$; we call $\bar{u}$ the minimizer of $E^V_0$ on $\mathcal{N}_V^0$;

- $\Gamma_u = \{x \in \mathbb{R}^N : |u(x)| > 1\}$;

- $|A|$ = the Lebesgue measure of the subset $A \subset \mathbb{R}^N$;

- $B_R = \{x \in \mathbb{R}^N : |x| \leq R\}$;

- $B_R^C = \mathbb{R}^N \setminus B_R$;

- $u_y(x) = u(x+y)$.

In order to study the properties of the functional $E^V_g$ and its Nehari manifold, we consider some suitable Orlicz space $L^p + L^q$, where $2 < p < 2^* < q$, related to the double power growth behavior of the function $f$. We recall some properties of these spaces to get the smoothness of the functional $E^V_g$.

Given $p \neq q$, we consider the space $L^p + L^q$ made up of the functions $v : \mathbb{R}^N \to \mathbb{R}$ such that

$$v = v_1 + v_2 \text{ with } v_1 \in L^p, v_2 \in L^q. \tag{3}$$
The space $L^p + L^q$ is a Banach space equipped with the norm:

$$||v||_{L^p + L^q} = \inf \{ ||v_1||_{L^p} + ||v_2||_{L^q} : v_1 \in L^p, v_2 \in L^q, v_1 + v_2 = v \}. \quad (4)$$

It is well known (see, for example [9]) that $L^p + L^q$ coincides with the dual of $L^{p'} \cap L^{q'}$. Then:

$$L^p + L^q = (L^{p'} \cap L^{q'})' \quad \text{with} \quad p' = \frac{p}{p-1}, \quad q' = \frac{q}{q-1}, \quad (5)$$

and we can introduce the following norm equivalent to the previous one

$$||v||_{L^p + L^q} = \inf_{\varphi \neq 0} \frac{\int v \varphi}{||\varphi||_{L^{p'}} + ||\varphi||_{L^{q'}}}. \quad (6)$$

Hereafter we recall some results useful for this paper contained in [4, 6].

**Lemma 3.** We have

1. if $v \in L^p + L^q$, the following inequalities hold:

$$\max \left( ||v||_{L^{p}(\mathbb{R}^N \setminus \Gamma_v)} - 1, \frac{1}{1 + |\Gamma_v|^{\frac{1}{\tau}}} ||v||_{L^p(\Gamma_v)} \right) \leq ||v||_{L^p + L^q} \leq \max(||v||_{L^p(\mathbb{R}^N \setminus \Gamma_v)}, ||v||_{L^p(\Gamma_v)})$$

when $\tau = \frac{p-1}{q-p}$;

2. let $\{v_n\} \subset L^p + L^q$. Then $\{v_n\}$ is bounded in $L^p + L^q$ if and only if the sequences $\{|\Gamma_{v_n}|\}$ and $\{|||v||_{L^p(\mathbb{R}^N \setminus \Gamma_{v_n})} + ||v||_{L^p(\Gamma_{v_n})}\}$ are bounded.

3. $f'$ is a bounded map from $L^p + L^q$ into $L^{p-1} \cap L^{q-1}$.

**Remark 4.** By the previous lemma we have $L^{2^*} \subset L^p + L^q$ when $2 < p < 2^* < q$. Then, by Sobolev inequality, we get the continuous embedding

$$D^{1,2}(\mathbb{R}^N) \subset L^p + L^q.$$ 

In order to prove the $C^2$ regularity of the functional $E^V_{\theta}$, we need the following lemmas proved in [9]

**Lemma 5.** If $f$ satisfies the hypothesis $[f_0]$ and $[f_2]$, we have that

1. if $\theta, u$ are bounded in $L^p + L^q$, then $f''(\theta)u$ is bounded in $L^{p'} \cap L^{q'}$;
2. \( f'' \) is a bounded map from \( L^p + L^q \) into \( L^{p/p - 2} \cap L^{q/q - 2} \);

3. \( f'' \) is a continuous map from \( L^p + L^q \) into \( L^{p/p - 2} \cap L^{q/q - 2} \);

4. the map \( (u, v) \mapsto uv \) from \( (L^p + L^q)^2 \) in \( L^{p/2} + L^{q/2} \) is bounded.

**Lemma 6.** The functional \( E^V_g \) is of class \( C^2 \) and it holds

\[
E'_g(u)[v] = \langle \nabla E^V_g(u), v \rangle = \int_{\mathbb{R}^N} \nabla u \nabla v + V uv - f'(u)v - gv; \quad (7)
\]

\[
E''_g(u)[v, w] = \int_{\mathbb{R}^N} \nabla v \nabla w + V vw - f''(u)vw. \quad (8)
\]

Moreover the Nehari manifold defined as

\[
\mathcal{N}^V_g = \left\{ u \in D^{1,2} : \int_{\mathbb{R}^N} |\nabla u|^2 + Vu^2 - f'(u)u dx - gu = 0 \right\} \quad (9)
\]

is of class \( C^1 \) and its tangent space at the point \( u \) is

\[
T_u \mathcal{N}^V_g = \left\{ v \in D^{1,2} : \int_{\mathbb{R}^N} 2 \nabla u \nabla v + 2V uv - f'(u)v dx - f''(u)uv - gv = 0 \right\}.
\]

At last, we introduce the functions

\[
\phi^0(t) = \phi_0(t) := E^V_0(tu) = \int_{\mathbb{R}^N} \frac{t^2}{2} (|\nabla u|^2 + Vu^2) - f(tu); \quad (10)
\]

\[
\phi^u(t) = \phi_g(t) := E^V_g(tu) = \phi_0(t) - t \int_{\mathbb{R}^N} gu. \quad (11)
\]

We have that

\[
\phi'_g(t) = t ||u||^2_V - \int_{\mathbb{R}^N} f'(tu)u - \int_{\mathbb{R}^N} gu; \quad (12)
\]

\[
\phi''_g(t) = ||u||^2_V - \int_{\mathbb{R}^N} f''(tu)u^2; \quad (13)
\]

\[
\phi'''_g(t) = - \int_{\mathbb{R}^N} f'''(tu)u^3. \quad (14)
\]

Notice that the conditions on \( f \) assure that also \( \phi'''_g(t) \) exists. Furthermore, if \( \frac{d}{dt} \phi_g(t) = 0 \), then \( \langle \nabla E(tu), u \rangle = 0 \), so \( tu \in \mathcal{N}^V_g \), and vice versa, so we want to find the critical points of \( \phi_g(t) \).

To study the manifold \( \mathcal{N}^V_g \) it is useful to consider the following manifold:

\[
\mathcal{V} = \left\{ w \neq 0 : G(w) := ||w||^2_V - \int_{\mathbb{R}^N} f''(w)w^2 = 0 \right\}. \quad (15)
\]
Lemma 7. We have that for all \( u \in \mathcal{D}^{1,2} \) there exists an unique \( T_u > 0 \) such that \( T_u u \in \mathcal{V} \).

Proof. We have that, using \((f_t)\) and \((f_0)\),

\[
\varphi'_0(t) = t||u||_V^2 - \int_{\mathbb{R}^N} f'(tu)u \leq t||u||_V^2 - \frac{\mu_1}{t} \int_{\mathbb{R}^N} f(tu) \leq t||u||_V^2 - t^{q-1}c_0\mu_1 \int_{|u|<1} |u|^q - t^{p-1}c_0\mu_1 \int_{|u|\geq1} |u|^p \leq (16)
\]

because \( p > 2 \). Furthermore we have that \( \varphi'_0(t) \) is strictly concave when \( t \neq 0 \), and that \( \varphi''_0(0) > 0 \), so, for every \( u \in \mathcal{D}^{1,2} \) there exist an unique maximum point \( T_u > 0 \) for the function \( \varphi'_0(t) \). Thus

\[
0 = T_u^2 \varphi''_0(T_u) = ||T_u u||_V^2 - \int_{\mathbb{R}^N} f''(T_u u)(T_u u)^2.
\]

\[
\square
\]

Proposition 8. We have that \( \inf_{w \in \mathcal{V}} ||w||_V^2 > 0 \).

Proof. By contradiction, we suppose that there exists a sequence \( \{w_n\}_n \subset \mathcal{V} \) such that \( ||w_n||_V^2 \) converges to 0. We set \( t_n = ||w_n||_V \), hence we can write \( w_n = t_n v_n \) where \( ||v_n||_V = 1 \). Remark \[4\] the sequence is bounded in \( L^p + L^q \). Since \( w_n \in \mathcal{V} \) and \( t_n \) converges to 0, we have

\[
1 = ||v_n||_V^2 = \frac{||w_n||_V^2}{t_n^2} = \frac{1}{t_n^2} \int_{\mathbb{R}^N} f''(t_n v_n)v_n^2 \leq c_2 t_n^{q-2} \int_{\mathbb{R}^N \setminus \Gamma_{v_n}} |v_n|^q + c_2 t_n^{p-2} \int_{\Gamma_{v_n}} |v_n|^p \leq c_2 t_n^{q-2} \int_{\mathbb{R}^N \setminus \Gamma_{v_n}} |v_n|^q + c_2 t_n^{p-2} \int_{\Gamma_{v_n}} |v_n|^p \leq c_2 t_n^{q-2} \int_{\mathbb{R}^N \setminus \Gamma_{v_n}} |v_n|^q + c_2 t_n^{p-2} \int_{\Gamma_{v_n}} |v_n|^p \leq c_2 t_n^{q-2} \int_{\mathbb{R}^N \setminus \Gamma_{v_n}} |v_n|^q + c_2 t_n^{p-2} \int_{\Gamma_{v_n}} \frac{|v_n|^p}{t_n^{p-1}} + c_2 t_n^{p-2} \int_{\Gamma_{v_n}} |v_n|^p \leq c_2 t_n^{q-2} \int_{\mathbb{R}^N \setminus \Gamma_{v_n}} |v_n|^q + 2c_2 t_n^{p-2} \int_{\Gamma_{v_n}} |v_n|^p.
\]
Hence we get
\[ 1 \leq c_2 t_n^{q-2} \int_{\mathbb{R}^N \setminus \Gamma_{v_n}} |v_n|^q + 2c_2 t_n^{p-2} \int_{\Gamma_{v_n}} |v_n|^p \]
and by claim 2 of Remark 3 we get the contradiction. \( \square \)

**Lemma 9.** Let \( u \in D^{1,2} \) and let \( T_u \) the unique positive number such that \( T_u u \in \mathcal{V} \). Then
\[ L = \inf_{||u||_V = 1} T_u - \int_{\mathbb{R}^N} f'(T_u u) u > 0. \]  

**Proof.** By contradiction, suppose that there exists a minimizing sequence \( u_n \), with \( ||u_n||_V = 1 \) such that \( T_{u_n} - \int f'(T_{u_n} u_n) u_n := \sigma_n \to 0 \). Let \( w_n = T_{u_n} u_n \).

We have that
\[ T_{u_n}^2 = ||w_n||_V^2 = \int f''(w_n) w_n^2, \]
because \( w_n \in \mathcal{V} \). Furthermore, by hypothesis, we have
\[ ||w_n||_V = \int f'(w_n) \frac{w_n}{||w_n||_V} + \sigma_n. \]

Thus, by (16)
\[ \mu_2 ||w_n||_V^2 = \mu_2 \int f'(w_n) w_n + \mu_2 \sigma_n ||w_n||_V < \]
\[ < \int f''(w_n) w_n^2 + \mu_2 \sigma_n ||w_n||_V = \]
\[ = ||w_n||_V^2 + \mu_2 \sigma_n ||w_n||_V. \]

So, because \( \mu_2 > 1 \) we have that
\[ 0 < (\mu_2 - 1)||w_n||_V < \mu_2 \sigma_n \to 0, \]  
that is a contradiction. \( \square \)

**Remark 10.** Obviously, by Lemma 8 we have also
\[ B := \inf_{||u||_V = 1} T_u > 0, \]
and \( B \) does not depend on \( g \).

At last we can give the following characterization of the Nehari manifold.

**Proposition 11.** Let \( ||g||_{L^{\frac{2N}{N-2}}} \), sufficiently small, and let \( u \in D^{1,2} \) with \( ||u||_V = 1 \). Then
1. If \( \int gu < 0 \), then there exists an unique \( t_1^u \) such that \( t_1^u u \in \mathcal{N}_g^V \) and \( t_0^u < t_1^u \), where \( t_0^u \) is the unique value for which \( t_0^u \in \mathcal{N}_0^V \).

2. If \( \int gu = 0 \), then there exists an unique \( t_1^u \) such that \( t_1^u u \in \mathcal{N}_g^V \) and \( t_0^u = t_1^u \).

3. If \( \int gu > 0 \), then there exist two positive numbers \( t_1^u \) and \( t_2^u \) such that \( t_1^u u \in \mathcal{N}_g^V \) and \( t_2^u < T_u < t_1^u < t_0^u \), where \( T_u \) is the unique value for which \( T_u u \in \mathcal{N}_V \).

4. \( t_1^u \) and \( t_2^u \) depend \( C^1 \) on \( g \in L^{\frac{2N}{N+2}} \) and on \( u \in D^{1,2} \setminus \{0\} \). Furthermore, fixed \( u \), we have \( t_1^u \to t_0^u \), when \( ||g||_{L^{\frac{2N}{N+2}}} \to 0 \).

Proof. 1. If \( \varphi_g'(\bar{t}) = 0 \), with \( \bar{t} \neq 0 \), by \( [f_g] \), we have that

\[
\bar{t}^2 \varphi_g''(\bar{t}) = \bar{t} \int gu + \int [\bar{t} u f'(\bar{t} u) - \bar{t}^2 u^2 f''(\bar{t} u)] < 0, \tag{19}
\]

so \( \bar{t} \) is a maximum point for \( \varphi_g \). Furthermore, we have that \( \varphi_g(0) = 0 \), \( \varphi'_g(0) > 0 \) and \( \varphi''_g(0) > 0 \).

Using \( [F_g] \) and \( [\mathcal{F}_g] \), we have

\[
\varphi_g(t) = \frac{t^2}{2}||u||^2_V - \int_{\mathbb{R}^N} f(t u) - t \int_{\mathbb{R}^N} gu \leq \frac{t^2}{2}||u||^2_V - t \int_{\mathbb{R}^N} gu - c_0 t^q \int_{|u|<1} |u|^q - c_0 t^p \int_{|u|\geq1} |u|^p \leq 0 \tag{20}
\]

because \( p > 2 \). This proves that there is exactly one \( t_1^u \) such that \( t_1^u u \in \mathcal{N}_g \); it is easy to see that \( t_0^u < t_1^u \).

2. In this case, we can prove, as in (20) that \( \varphi_g(t) \to -\infty \) when \( t \to \infty \) and that if \( \bar{t} \neq 0 \) is a critical point of \( \varphi_g \) then (19) holds. At last, consider 0 = \( \varphi_g(0) = \varphi'_g(0) < \varphi''_g(0) \), and so 0 is a local minimum for \( \varphi_g \), and we can conclude.

3. We have just proved that, for any \( u \in D^{1,2} \), we have an unique maximum point \( T_u \) of \( \varphi_g(t) \). So, if we prove that \( \int gu < \varphi_g(T_u) \) we have that there exist two numbers \( t_1^u \) and \( t_2^u \) such that \( \varphi'_g(t_1^u) = 0 \). Set \( L \) as in Lemma 9 and consider that

\[
\int gu \leq ||g||_{L^{\frac{2N}{N+2}}} ||u||_{L^{2^*}} \leq C_1 ||g||_{L^{\frac{2N}{N+2}}} ||u||_{D^{1,2}} \leq C_2 ||g||_{L^{\frac{2N}{N+2}}} ||u||_V \tag{21}
\]
Recalling that \(|u|_V = 1\), if \(|g|_{L^{2\infty}} < L\), we have exactly two positive numbers \(t_1^u\) and \(t_2^u\) such that \(\varphi_g'(t_j^u) = 0\), and \(t_1^u\) and \(t_2^u\) are respectively the maximum and the minimum point of \(\varphi_g\).

4. For Simplicity we only prove that \(t_1^u(g)\) is a \(C^1\) function. The other case is straightforward. Let us define a function \(G : \mathbb{R}^+ \times D^{1,2} \setminus \{0\} \times L^{2\infty} \rightarrow \mathbb{R}\),

\[
G : (t, u, g) \mapsto \frac{d}{dt} \varphi_g^u(t) = t||u||_V^2 - \int f'(tu)u - \int gu.
\]

We have that \(G\) is a \(C^1\) function. Let \(\bar{t}, \bar{u}, \bar{g}\) be such that \(G(\bar{t}, \bar{u}, \bar{g}) = 0\). We know that \(\frac{\partial}{\partial t} G(\bar{t}, \bar{u}, \bar{g}) = \frac{d^2}{dt^2} \varphi_g^u(\bar{t}) < 0\), thus, by the implicit function theorem there is a \(C^1\) function \(t(u, g) = t_1^u(g)\) such that \(G(t(u, g), u, g) = 0\). We have then the claimed result. 

The Nehari manifold so can be described as:

\[
\mathcal{N}_g^V = \mathcal{N}_g^{+} \cup \mathcal{N}_g^{-},
\]

where

\[
\mathcal{N}_g^{+} = \mathcal{N}_g^{+;V} := \{u \in \mathcal{N}_g^V : E_g'(u)u = 0, E_g''(u)u^2 > 0\};
\]

\[
\mathcal{N}_g^{-} = \mathcal{N}_g^{-;V} := \{u \in \mathcal{N}_g^V : E_g'(u)u = 0, E_g''(u)u^2 < 0\}.
\]

We have also that \(E_g^V > 0\) on \(\mathcal{N}_g^{-}\) and \(E_g^V < 0\) on \(\mathcal{N}_g^{+}\). Furthermore, because \(\mathcal{N}_0^V\) and \(\mathcal{V}\) are bounded away from 0, we have also that \(\inf_{u \in \mathcal{N}_0^V} ||u|| > 0\). The geometry of \(\mathcal{N}_g^V\) is represented in the following picture.
Remark 12. There exists $M > 0$ such that
\[ ||u||_V \leq M||g||_{\frac{2N}{N+2}} \text{ for any } u \in \mathcal{N}_g^+, \] (23)
indeed, by (fµ) we have
\[ \frac{1}{2}||u||_V^2 < \int f(u) + \int gu \leq \frac{1}{\mu_1} \int f'(u)u + \int gu = \frac{1}{\mu_1}||u||_V^2 + \left(1 - \frac{1}{\mu_1}\right) \int gu, \]
so
\[ \left(\frac{1}{2} - \frac{1}{\mu_1}\right)||u||_V^2 < \left(1 - \frac{1}{\mu_1}\right) \int gu. \]

3 The Splitting Lemma

We recall that a sequence \( \{u_n\}_n \in \mathcal{D}^{1,2} \) such that \( E^V_g(u) \to c \), and \( \nabla E^V_g(u) \to 0 \) is a Palais-Smale sequence at level \( c \) for \( E^V_g \).

In the same way we say that \( \{u_n\}_n \in \mathcal{N}_g^V \) such that \( E^V_g(u) \to c \), and there exists a sequence \( \varepsilon_n \to 0 \) s.t. \( |\langle \nabla E^V_g(u_n), \varphi \rangle| \leq \varepsilon_n||\varphi|| \), for all \( \varphi \in T_{u_n} \mathcal{N}_g^V \cap \mathcal{D}^{1,2} \) is a Palais-Smale sequence at level \( c \) for \( E^V_g \) restricted to \( \mathcal{N}_g^V \).

A functional \( f \) satisfies the \((PS)_c\) condition if all the Palais-Smale sequences at level \( c \) converge.

Unfortunately the functional \( E^V_g \) on \( \mathcal{N}_g^V \) does not satisfy the PS condition in all the energy range. In this section by the splitting lemma we get a description of the PS sequences for the functional \( E^V_g \).

Lemma 13. Let \( u_n \in \mathcal{N}_g \) and let \( E^V_g(u_n) \to c \). Then \( ||u_n||_V \) is bounded.

Proof. We have that
\[ ||u_n||_V^2 = \int f'(u_n)u_n + \int gu_n \] (24)
because \( u_n \in \mathcal{N}_g^V \). Furthermore, for \((f\mu)\) we have
\begin{align*}
E^V_g(u_n) &= \frac{1}{2}||u_n||_V^2 - \int f(u_n) - \int gu_n \\
&\geq \frac{1}{2}||u_n||_V^2 - \frac{1}{\mu_1} \int f'(u_n)u_n - \int gu_n = \\
&= \frac{1}{2}||u_n||_V^2 - \frac{1}{\mu_1}||u_n||_V^2 + \frac{1}{\mu_1} \int gu_n - \int gu_n = \\
&= \left(\frac{1}{2} - \frac{1}{\mu_1}\right)||u_n||_V^2 - \left(1 - \frac{1}{\mu_1}\right) \int gu_n = \\
&= ||u_n||_V^2 \left[\left(\frac{1}{2} - \frac{1}{\mu_1}\right) - \left(1 - \frac{1}{\mu_1}\right) \int g\frac{u_n}{||u_n||_V^2}\right].
\end{align*}
If $\|u_n\|_V \to \infty$ we have that
\[
\left| \int g \frac{u_n}{\|u_n\|_V^2} \right| \leq \|g\|_{L^{\frac{2N}{N+2}}} \frac{\|u\|_{L^2}}{2N} \frac{1}{\|u_n\|_V} \to 0. \tag{26}
\]
So we will have
\[
C_1 > E_g^V(u_n) \geq C_2 \|u_n\|_V^2 \to \infty \tag{27}
\]
that is a contradiction.

Lemma 14. Let $\{u_n\} \subset N_g$, and let $E_g^V(u_n) \to c$. Then, up to subsequence $u_n \to u_0$ in $D^{1,2}$. Furthermore, setting $\psi_n = u_n - u_0$ we have

1. $\|\psi_n\|_V^2 = \|u_n\|_V^2 - \|u_0\|_V^2 + o(1)$;
2. $E_g^V(\psi_n) = E_g^V(u_n) - E_g^V(u_0) + o(1)$.

Proof. By the previous lemma we have that $\|u_n\|_{D^{1,2}}$ is bounded. Then $u_n \to u_0$ and we have that
\[
\|\psi_n\|_V^2 = \|u_n\|_V^2 - \|u_0\|_V^2 + o(1).
\]
Furthermore, we have that
\[
\int f(\psi_n) = \int f(u_n) - \int f(u_0) + o(1). \tag{28}
\]
Indeed, we have the following equation, where $\tau, \theta, \sigma \in (0, 1)$
\[
\int f(u_n) - \int f(u_0) - \int f(\psi_n) = \\
= \int_{B_R} f(u_0 + \psi_n) - f(u_0) - \int_{B_R} f(\psi_n) + \\
+ \int_{B_R} f(u_0 + \psi_n) - f(\psi_n) - \int_{B_R} f(\psi_n) = \\
= \int_{B_R} f'(u_0 + \tau \psi_n) \psi_n - \int_{B_R} f(u_0) + \int_{B_R} f'((\theta u_0 + \psi_n) u_0 - \int_{B_R} f'(\sigma \psi_n) \psi_n.
\]
Using Lemma 3 we have that the terms in $B_R^C$ are arbitrarily small when $R$ is sufficiently large. Furthermore, since $\psi_n \to 0$ in $L^p(\Omega)$ for all $\Omega \subset \mathbb{R}^N$ bounded and for all $p < 2^*$, we get that
\[
\int f(u_n) - \int f(u_0) - \int f(\psi_n) \to 0.
\]
The proof follows easily. \qed
Lemma 15. Suppose that $\psi_n \to 0$ in $\mathcal{D}^{1,2}$. Then we have

$$\int V \psi_n^2 \to 0 \quad (29)$$

$$\int g \psi_n \to 0 \quad (30)$$

Proof. Again we use that $\psi_n \to 0$ in $L^p(\Omega)$ for all $\Omega \subset \mathbb{R}^N$ bounded and for all $p < 2^*$. We have that

$$\int V \psi_n^2 = \int_{B_R} V \psi_n^2 + \int_{\mathbb{R}^N \setminus B_R} V \psi_n^2 \leq ||V||_{L^r(B_R)} ||\psi_n||_{L^{2r'}(B_R)}^2 + ||V||_{L^{N/2}(\mathbb{R}^N \setminus B_R)} ||\psi_n||_{L^{2^*}(\mathbb{R}^N \setminus B_R)}^2 \to 0,$$

and that

$$\int g \psi_n = \int_{B_R} g \psi_n + \int_{\mathbb{R}^N \setminus B_R} g \psi_n \leq ||g||_{L^r(B_R)} ||\psi_n||_{L^{r'}(B_R)} + ||g||_{L^{N/2}(\mathbb{R}^N \setminus B_R)} ||\psi_n||_{L^{2^*}(\mathbb{R}^N \setminus B_R)} \to 0.$$

Lemma 16. Let $\{u_n\}_n$ a PS sequence at level $c$ for the functional $E^V_g$ restricted to the manifold $\mathcal{N}^V_g$. Then, up to a subsequence, there exist $k$ sequences of points $\{y_n^j\}_n$, $j = 1,\ldots,k$, with $|y_n^j| \to \infty$, a solution $u^0$ of the problem $-\Delta u + Vu = f'(u) + g$, and $k$ solutions $u^j$, $j = 1,\ldots,k$, of the problem $-\Delta u = f'(u)$ such that

$$u_n(x) = u^0(x) + \sum_{j=1}^k u^j(x - y_n^j) + o(1); \quad (31)$$

$$E^V_g(u_n) = E^V_g(u^0) + \sum_{j=1}^k E^V_0(u^j) + o(1). \quad (32)$$

Proof. Since $u_n$ is a PS sequence for the functional $E^V_g$ restricted to the manifold $\mathcal{N}^V_g$, then $u_n$ is a PS sequence for the functional $E^V_g$. By the Lemma 14 we have that $u_n$ converges to $u^0$ weakly in $\mathcal{D}^{1,2}$ (up to subsequence), so, given $\varphi \in C_0^{\infty}(\mathbb{R}^N)$,

$$\lim_{n \to \infty} \int \nabla u_n \cdot \nabla \varphi + Vu_n \varphi - f'(u_n) \varphi - g \varphi = 0. \quad (33)$$
It is easy to see that

\[ \int \nabla u_n \nabla \varphi + Vu_n \varphi \rightarrow \int \nabla u^0 \nabla \varphi + V u^0 \varphi. \]

Arguing as in Step 1 of [5, Lemma 3.3] we get also that, for some \(0 < \theta < 1\),

\[ \int [f'(u_n) - f'(u^0)] \varphi = \int_{\text{supp} \varphi} f''(\theta u_n + (1 - \theta) u^0)(u_n - u^0) \varphi \rightarrow 0, \quad (34) \]

as \(n \rightarrow 0\), because \(u_n - u^0 \rightarrow 0\) in \(L^p(\Omega)\), with \(\Omega\) bounded and \(p < 2^*\). So we have proved that \(u^0\) solves \(-\Delta u + Vu = f'(u) + g\).

Now we set

\[ \psi_n(x) = u_n(x) - u^0(x). \]

Then \(\psi_n \rightharpoonup 0\) weakly in \(D^{1,2}\). If \(\psi_n \nrightarrow 0\) strongly in \(D^{1,2}\), for Step 3 of [5, Lemma 3.3] we have that there exists a sequence \(\{y_n\} \subset \mathbb{R}^N\) with \(|y_n| \rightarrow \infty\) such that \(\psi_n(x + y_n) \rightarrow u^1\) in \(D^{1,2}\), and \(u^1 \neq 0\).

Because \(u^0\) is a weak solution of \((P)\) and \(u_n\) is a PS sequence for \(E^V_g\) we have that, for any \(\varphi \in C^\infty_c(\mathbb{R}^N)\),

\[ \int \nabla u_n \nabla \varphi + Vu_n \varphi - f'(u_n) \varphi - g \varphi \rightarrow 0; \]

\[ \int \nabla u^0 \nabla \varphi + V u^0 \varphi - f'(u^0) \varphi - g \varphi = 0. \]

So

\[ \int \nabla \psi_n \nabla \varphi + V \psi_n \varphi - f'(u_n) - f'(u^0) \varphi \rightarrow 0. \quad (35) \]

Using [34] we have that \(\psi_n\) is a PS sequence for the functional \(E^V_0\). Thus, for any \(\varphi \in C^\infty_c(\mathbb{R}^N)\) we have

\[ \int \nabla \psi_n(x + y_n) \nabla \varphi(x) - f'(\psi_n(x + y_n)) \varphi(x) dx = \]

\[ \int \nabla \psi_n(x) \nabla \varphi(x - y_n) - f'(\psi_n(x)) \varphi(x - y_n) dx = \]

\[ \int [f'(u_n) - f'(u^0) - f'(\psi_n)] \varphi(x - y_n) - \int V(x) \psi_n(x) \varphi(x - y_n) + o(1). \]

Using the same argument of Lemma [15] we can prove that

\[ \int V(x) \psi_n(x) v(x - y_n) \leq C \varepsilon_n ||\varphi||_{D^{1,2}}, \text{ with } \varepsilon_n \rightarrow 0; \]
We set

\[ m_g = \inf_{u \in \mathcal{N}_g} E_g^V(u) \text{ and } m_{1,g} = \inf_{u \in \mathcal{N}_g} E_g^V(u). \]
We show that there exist a solution with critical value $m_g$ and another solution with critical value $m_{1,g}$.

We set also

$$m_0 = \inf_{u \in \mathcal{N}_0^g} E_0^g(u)$$  \hspace{1cm} (37)

and we recall that there exists a positive radially symmetric function $\omega \in \mathcal{N}_0^g$ such that

$$E_0^g(\omega) = m_0 > 0.$$  \hspace{1cm} (38)

Finally, we set

$$m_V = \inf_{u \in \mathcal{N}_0^V} E_v^g(u)$$  \hspace{1cm} (39)

We know, by [5], that for any $V \leq 0$ and $V < 0$ on a set of positive measure there exists a function $\bar{u} \in \mathcal{N}_0^V$ such that

$$E_v^g(\bar{u}) = m_V$$  \hspace{1cm} (40)

and

$$0 < m_V < m_0.$$  \hspace{1cm} (41)

We prove the following results.

**Theorem 17.** There exist a $u_g \in \mathcal{N}_g^+$ such that $E_v^g(u_g) = m_g$. Furthermore, when $||g||_{L^{2N\infty}}$ is small, $u_g$ is unique.

**Proof.** By definition of $\mathcal{N}_g^+$ we have that $m_g = \inf_{u \in \mathcal{N}_g^+} E_v^g(u)$, and that $m_g < 0$.

At first we prove that $m_g > -\infty$. By contradiction, suppose that there exist a sequence $t_n > 0$ and a sequence $\{v_n\}_n \subset D^{1,2}$ with $||v_n||_V = 1$ and $t_nv_n \in \mathcal{N}_g^+$ such that

$$E_v^g(t_nv_n) = \frac{t_n^2}{2} - \int f(t_nv_n) - t_n \int gv_n \rightarrow -\infty.$$  \hspace{1cm} (42)

We have also that $t_n^2 - \int f'(t_nv_n)t_nv_n - t_n \int gv_n = 0$. So, if $t_n$ is bounded, we have

$$E_v^g(t_nv_n) = \frac{t_n^2}{2} + \int f'(t_nv_n)t_nv_n - t_n \int f(t_nv_n) \geq$$

$$\geq \frac{t_n^2}{2} + \left(1 - \frac{1}{\mu_1}\right) \int f'(t_nv_n)t_nv_n$$

that is bounded by Lemma 3. Thus we have that, up to subsequence, $t_n \rightarrow +\infty$. Finally, arguing as in (25) we have that

$$E_v^g(t_nv_n) \geq \left(1 - \frac{1}{\mu_1}\right) \frac{t_n^2}{2} - \left(1 - \frac{1}{\mu_1}\right) t_n \int gv_n \rightarrow +\infty,$$  \hspace{1cm} (43)
that is a contradiction.

Now, let \( u_n \) a minimizing sequence. For the Ekeland variational principle, we can suppose \( u_n \) be a PS sequence. For the splitting lemma there exists a \( u_g \in \mathcal{N}_g^- \) and \( k \) functions \( w^j \), \( 1 \leq j \leq k \) such that

\[
E_g^V(u_n) - E_g^V(u_g) + \sum_{j=1}^{k} E_0^0(w^j) = m_g < 0. \tag{44}
\]

We know that \( E_0^0(w^j) \geq m_0 > 0 \) for all \( j \). So, if \( k > 0 \) we will have \( E_g^V(u_n) \rightarrow m_g + \delta \) for some \( \delta > 0 \) and this is a contradiction.

So, we have

\[
u_n \rightarrow u_g \text{ in } D^{1,2}. \tag{45}
\]

Furthermore, we have \( E_g^V(u_g) = m_g < 0 \), so \( u_g \in \mathcal{N}_g^- \), and this concludes the proof of the existence.

To prove uniqueness, we argue by contradiction. If \( u_1, u_2 \) are minimizers of \( E_g^V \) on \( \mathcal{N}_g^- \), both \( u_1 \) and \( u_2 \) solve \((P)\), so we have

\[
||u_1 - u_2||_V^2 = \int (f'(u_1) - f'(u_2))(u_1 - u_2) = \int f''(\theta u_1 + (1-\theta)u_2)(u_1 - u_2)^2
\]

with \( 0 < \theta < 1 \). So

\[
||u_1 - u_2||_{L^{2^*}}^2 \leq C ||u_1 - u_2||_V^2 \leq C ||u_1 - u_2||_{L^{2^*}}^2 ||f''(\theta u_1 + (1-\theta)u_2)||_{L^{\frac{2^*}{2^*-2}}}. \tag{46}
\]

By Remark 12 we have that, if \( g \rightarrow 0 \) in \( L^{\frac{2N}{N+2}} \), then both \( u_1 \) and \( u_2 \) are small in \( L^p + L^q \), so we have that \( f''(\theta u_1 + (1-\theta)u_2) \rightarrow 0 \) in \( L^{p/p-2} \cap L^{q/q-2} \) by Lemma 5 and, by interpolation,

\[
||f''(\theta u_1 + (1-\theta)u_2)||_{L^{\frac{2^*}{2^*-2}}} \rightarrow 0,
\]

that is a contradiction. \( \square \)

**Proposition 18.** Suppose that \( g \geq 0 \). Then there exists an \( u_g \geq 0 \) in \( \mathcal{N}_g^- \) such that \( E_g^V(u_g) = m_g \).

**Proof.** Take \( u_g \) as in Theorem 17. Because \( u_g \in \mathcal{N}_g^- \) we have that \( \int gu_g > 0 \). If \( u_g \) changes sign, or \( u_g \) negative, we have that

\[
0 < \int gu_g \leq \int g|u_g| \tag{47}
\]
So, reminding that $f$ is even we have

$$E_V^g(|u_g|) = \frac{1}{2}||u_g||_V^2 - \int f(|u_g|) - \int g|u_g| \leq$$

$$\leq \frac{1}{2}||u_g||_V^2 - \int f(u_g) - \int g u_g = E_V^g(u_g).$$

We know that there exists a $\tau$ such that $\tau|u_g| \in \mathcal{N}_g^+$. Furthermore we know, by the study of $\varphi_{g|u_g}$ that $\tau$ is a local minimizer of $\varphi_{g|u_g}$, in fact, $\varphi_{g|u_g}(\tau) \leq \varphi_{g|u_g}(t)$ for all $t \in [0, \tau]$. We have

$$\frac{d}{dt}\varphi_{g|u_g}(1) = \frac{d}{dt}E_V^g(t|u_g|)_{t=1} = ||u_g||_V^2 - \int f'(|u_g|)|u_g| - \int g|u_g| \leq$$

$$\leq ||u_g||_V^2 - \int f'(u_g)u_g - \int g u_g = \frac{d}{dt}E_V^g(t u_g)_{t=1} = 0,$$

and

$$\frac{d^2}{dt^2}\varphi_{g|u_g}(1) = \frac{d^2}{dt^2}E_V^g(t|u_g|)_{t=1} = ||u_g||_V^2 - \int f''(|u_g|)|u_g|^2 =$$

$$= ||u_g||_V^2 - \int f''(u_g)u_g^2 = \frac{d^2}{dt^2}E_V^g(t u_g)_{t=1} > 0.$$

Thus $\tau \geq 1$ and

$$E_V^g(\tau|u_g|) \leq E_V^g(|u_g|) \leq E_V^g(u_g) = m_g,$$

that concludes the proof.

We want to prove that, under suitable hypothesis on $g, f$ and $V$, there exists another solution of $\mathcal{P}$ by minimizing the functional $E_V^g$ on $\mathcal{N}_g^-$. In order to prove that a minimizing sequence converges we will show that, for $g$ small,

$$m_{1,g} := \inf_{u \in \mathcal{N}_g^-} E_V^g(u) < m_g + m_0;$$

$$\text{(49)}$$

**Lemma 19.** Suppose that $V \leq 0$ and $V < 0$ on a set of positive measure. If $||g||_{L^{\frac{N}{N-2}}} \text{ sufficiently small, then there exist a } \delta > 0 \text{ such that}$

$$m_{1,g} := \inf_{u \in \mathcal{N}_g^-} E_V^g(u) < m_0 - \delta.$$

$$\text{(50)}$$

Moreover,

$$\limsup_{||g||_{L^{\frac{N}{N-2}}} \to 0} m_{1,g} \leq m_V$$

$$\text{(51)}$$
Proof. By [5] Lemma 4.4(a) and [5] Theorem 1.1 we know that there exists a $\bar{u} \in \mathcal{N}_{0}^{V}$ such that

$$E_{0}^{V}(\bar{u}) = \inf_{u \in \mathcal{N}_{0}^{V}} E_{0}^{V}(u) = m_{V} < m_{0}.$$  

We set $v = \frac{\bar{u}}{||\bar{u}||_{V}}$, so $\bar{u} = t_{0}^{*}v$. We know that there exists $t_{1}^{*} = t_{1}^{*}(g)$ such that $t_{1}^{*}v \in \mathcal{N}_{g}^{-}$ by Proposition \[11\]. Furthermore, by Proposition \[11\] we have that $t_{1}^{*} \rightarrow t_{0}^{*}$ when $||g||_{L_{\frac{2N}{\alpha+2}}} \rightarrow 0$, and so

$$m_{1,g} \leq E_{g}^{V}(t_{1}^{*}v) \rightarrow E_{0}^{V}(\bar{u}) = m_{V} < m_{0} \text{ for } ||g||_{L_{\frac{2N}{\alpha+2}}} \rightarrow 0,$$  

that concludes the proof. \qed

Theorem 20. For $||g||_{L_{\frac{2N}{\alpha+2}}} \rightarrow 0$ there exist $u_{1,g} \in \mathcal{N}_{g}^{-}$ a solution of $(\mathcal{P})$. Furthermore, if $g \geq 0$ the solution $u_{1,g}$ can be chosen positive.

Proof. By the splitting lemma, to obtain the result it is enough to show that $m_{1,g} < m_{g} + m_{0}$. In the previous lemma, we have proved that there exists a $\delta > 0$ such that $m_{1,g} < m_{0} - \delta$ for $||g||_{L_{\frac{2N}{\alpha+2}}}$ sufficiently small. By Remark \[11\] we have also that $m_{g} \rightarrow 0$ when $g \rightarrow 0$ in $L_{\frac{2N}{\alpha+2}}$. So there exists $u_{1,g} \in \mathcal{N}_{g}^{V}$ a solution of $(\mathcal{P})$. Moreover $E_{g}^{V}(u_{1,g})$ is positive, so $u_{1,g} \in \mathcal{N}_{g}^{-}$.

To prove the last claim, consider that $E_{g}^{V}(|u_{1,g}|) \leq E_{g}^{V}(u_{1,g})$. Also, there exists a $\tilde{t}$ such that $\tilde{t}|u_{1,g}| \in \mathcal{N}_{g}^{-}$. Then we have

$$m_{1,g} = E_{g}^{V}(u_{1,g}) = \max_{t} E_{g}^{V}(tu_{1,g}) \geq E_{g}^{V}(\tilde{t}u_{1,g}) \geq E_{g}^{V}(\tilde{t}|u_{1,g}|).$$  

So if $u_{1,g}$ is a solution, also $\tilde{t}|u_{1,g}| \in \mathcal{N}_{g}^{-}$ is a solution of $(\mathcal{P})$. \qed

Proposition 21. If $||g||_{L^{p} \cap L^{q}} \rightarrow 0$, then $m_{1,g} \rightarrow m_{V}$.

Proof. We take a sequence of $g_{n} \rightarrow 0$ in $L^{p'} \cap L^{q'}$. We know that for any $g_n$ there exists $u_{1,g_n}$ such that $E_{g_{n}}^{V}(u_{1,g_{n}}) = m_{1,g_n}$. For simplicity we call $u_n = u_{1,g_n}$. Also, we set $v_n = \frac{u_n}{||u_n||_{L^{p} \cap L^{q}}}$, and $u_n = t_n v_n$. We have

$$E_{g_{n}}^{V}(u_{n}) = t_n \left[ \frac{1}{2} \int f'(t_n v_n) v_n - \int \frac{f(t_n v_n)}{t_n} - \frac{1}{2} \int g_n v_n, \right]$$  

and we have that there exist a $\delta > 0$ such that $0 \leq E_{g_{n}}^{V}(u_{n}) \leq m_{V} + \delta$. Now, suppose, by contradiction, that $t_n \rightarrow \infty$. Then,

$$\frac{1}{2} \int f'(t_n v_n) v_n - \int \frac{f(t_n v_n)}{t_n} - \frac{1}{2} \int g_n v_n \rightarrow 0,$$  

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and so
\[ \frac{1}{2} \int f'(t_n v_n) v_n - \int \frac{f(t_n v_n)}{t_n} \to 0. \tag{56} \]

By \((f)_\mu\), we have that
\[ \int f'(t_n v_n) v_n - 2 \int \frac{f(t_n v_n)}{t_n} = \int f'(t_n v_n) v_n - \mu_1 \int \frac{f(t_n v_n)}{t_n} + (\mu_1 - 2) \int \frac{f(t_n v_n)}{t_n} \geq (\mu_1 - 2) \int \frac{f(t_n v_n)}{t_n}. \]

So \( \int \frac{f(t_n v_n)}{t_n} \to 0 \). Now the hypothesis on \( f \)
\[ 0 \leq c_0 t_n^{-p-1} \left[ \int_{|v_n| > 1} |v_n|^p + \int_{|v_n| < 1} |v_n|^q \right] \leq \int \frac{f(t_n v_n)}{t_n} \to 0, \tag{57} \]
so we have that both \( \int_{|v_n| > 1} |v_n|^p \) and \( \int_{|v_n| < 1} |v_n|^q \) vanish when \( n \to \infty \), and so
\[ 1 = ||v_n||_{L^p + L^q} \leq \max \left\{ \int_{|v_n| > 1} |v_n|^p, \int_{|v_n| < 1} |v_n|^q \right\} \to 0 \quad \tag{58} \]
that is a contradiction. Furthermore, by Proposition 11, we have \( t_n \) bounded away from 0. So, we have that there exists two positive constants \( c_1 \) and \( c_2 \) such that
\[ 0 < c_1 \leq t_n = ||u_n||_{L^p + L^q} \leq c_2 < \infty. \tag{59} \]

Now, let \( \tau_n \) such that \( \tau_n u_n \in \mathcal{N}_0^V \). We can show that \( \tau_n \to 1 \) when \( n \to \infty \). The main idea is that
\[ \frac{d}{dt} \phi_{g_n}(\tau_n) - \frac{d}{dt} \phi_{0}(\tau_n) = \int g_n u_n \to 0. \tag{60} \]
because \( ||u_n||_{L^p + L^q} \) is bounded and \( g_n \to 0 \) in \( L^p \cap L^q \). The details are omitted for the sake of simplicity.

Now we have that
\[ E_0^V(\tau_n u_n) - E_0^V(u_n) \to 0. \tag{61} \]
We have that $E_{g_n}^V(u_n)$ is bounded, so, up to subsequences, there exists a $d$ such that $E_{g_n}^V(u_n) \to d$ when $n \to \infty$, and, because $u_n$ is bounded in $L^p + L^q$, also $E_{d_n}^V(u_n) \to d$, and, by (61), $E_{\tau_n u_n}^V \to d$.

So, $d \geq m_V$. By Lemma 19 we know also that $d \leq m_V$ so we get the claim.

Proof of Theorem 1. By theorems 17 and 20, we have that there exists a $u_g \in N^+_g$ and $u_{1,g} \in N^-_g$ that solve (P). Furthermore, by Theorem 20 and Proposition 18 the solution can be chosen nonnegative. At least, by Remark 12 we have that $u_g \to 0$ in $D^{1,2}$ and by Proposition 21 that $m_{1,g} \to m_V$ when $g \to 0$.

A The Hypothesis on $f$

We want to prove that there exists a function that satisfies all the conditions required in the introduction.

We take the function

$$f(s) = \frac{|s|^q}{1 + |s|^{q-p}}.$$ (62)

This function is even, and it satisfies $(f_0)$. We have that, for $s > 0$

$$f'(s) = \frac{qs^{q-1} + ps^{2q-p-1}}{(1 + s^{q-p})^2},$$

$$f''(s) = \frac{s^{q-2}}{(1 + s^{q-p})^2} \left\{ q(q-1) + p(2q-p-1)s^{q-p} - \frac{2(q-p)(q + ps^{q-p})s^{q-p}}{1 + s^{q-p}} \right\}.$$ 

It’s easy to see that $f$ satisfies $(f_2)$ and the first part of $(f_0)$. We set $\mu_2 = 1 + \varepsilon > 1$; then the inequality $(1 + \varepsilon)f'(s)s < f''(s)s^2$ becomes

$$(q^2 - 2q - \varepsilon q) + p(2q - p - 2 - \varepsilon)\gamma - \frac{2(q-p)(q + p\gamma)\gamma}{1 + \gamma} > 0,$$

where $\gamma = s^{q-p}$. So, we have to prove that

$$q(q - 2 - \varepsilon) + [p(2q - p - 2 - \varepsilon) + q(2p - q - 2 - \varepsilon)]\gamma + p(p - 2 + \varepsilon)^2 > 0.$$ 

Obviously we can choose $\varepsilon$ such that $q(q - 2 - \varepsilon) > 0$ and $p(p - 2 + \varepsilon) > 0$. Furthermore, we choose $q - p$ sufficiently small such that also $2q - p - 2 - \varepsilon$ and $2p - q - 2 - \varepsilon$ are positive, so the second part of $(f_0)$ is proved.
At last we prove \([f_d]\) and that \(f''(s)s^3 > 0\). We have that, for \(s > 0\),

\[
f''(s) = \frac{6(p - q)^3s^{4q-3p-3}}{(1 + s^{p-q})^4} - \frac{6(1 + p - 2q)(p - q)^2s^{3q-2p-3}}{(1 + s^{q-p})^3} + \frac{(2p + 3p^2 + p^3 - 2q - 12pq - 6p^2q + 9q^2 + 12pq^2 - 7q^3)s^{2q-p-3}}{(1 + s^{q-p})^2} + \frac{q(2 - 3q + q^2)s^{q-3}}{1 + s^{q-p}}.
\]

We obtain that

\[
f''(s)s^3 = \frac{As^q}{1 + s^{q-p}} + \frac{Bs^{2q-p}}{(1 + s^{p-q})^2} + \frac{Cs^{3q-2p}}{(1 + s^{q-p})^3} + \frac{Ds^{4q-3p}}{(1 + s^{q-p})^4},
\]

were

\[A = q(q - 2)(q - 1); \quad B = (p - q)(2 + 3p + p^2 - 9q - 5pq + 7q^2); \quad C = 6(p - q)^2(2q - p - 1); \quad D = 6(p - q)^3.\]

We can choose \(q - p\) sufficiently small, in order to have \(B, C, D << A\). Now, set as above \(\gamma = s^{q-p}\), we have

\[
f''(s)s^3 = \frac{s^q[A + (3A + B)\gamma + (3A + 2B + C)\gamma^2 + (A + B + C + D)\gamma^3]}{(1 + s^{q-p})^4},
\]

that is positive for all \(s > 0\). So \([f_d]\) is completely proved.

Furthermore, we have that

\[
\lim_{s \to 0^+} \frac{f''(s)}{s^{q-3}} = A = q(q - 1)(q - 2) > 0, \quad (63)
\]

and

\[
\lim_{s \to +\infty} \frac{f''(s)}{s^{p-3}} = A + B + C + D = p(p - 1)(p - 2) > 0. \quad (64)
\]

So, there exists a \(c_3 > 0\) such that

\[
\begin{cases}
|f''(s)| \leq c_3|s|^{p-3} & \text{for } |s| \geq 1; \\
|f''(s)| \leq c_3|s|^{q-3} & \text{for } |s| \leq 1.
\end{cases} \quad (65)
\]

Now, let \(\Gamma = \{x \in \mathbb{R}^N : |u(x)| > 1\}\) and \(\Delta = \mathbb{R}^N \setminus \Gamma\). We have that

\[
\int_{\Gamma} f''(u)u^3 \leq \int_{\Gamma} f''(u)u^3 + \int_{\Delta} f''(u)u^3 \leq c_3 \int_{\Gamma} |u|^p + c_3 \int_{\Delta} |u|^q \leq C_1 + C_2 |u|_{L^p + L^q} \leq C_3 + C_4 |u|_{D^{1,2}} < \infty,
\]

and this proves \([f_d]\).
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