On some geometric properties of sets related to the $d$-Hausdorff contents.*

A. I. Tyulenev †

Abstract

Let $S \subset \mathbb{R}^n$ be a closed $d$-thick set for some $d \in (0, n)$, i.e. there is $\lambda \in (0, 1]$ such that the $d$-Hausdorff content $\mathcal{H}^d_{\infty}(S \cap Q) \geq \lambda l(Q)^d$ for all cubes $Q \subset \mathbb{R}^n$ centered in $x \in S$ with side lengths $l(Q) \in (0, 1]$. For such sets we establish a characterization of the well-known porosity condition in terms of the $d$-Hausdorff contents. In the case of an arbitrary set $S \subset \mathbb{R}^n$ given a cube $Q$, we show that the "smallness" of the $d$-Hausdorff content $\mathcal{H}^d_{\infty}(Q \cap S)$ implies existence of a Borel set $U \subset Q \setminus S$ with $\mathcal{H}^n(U) \approx \mathcal{H}^n(Q)$. The sharpness of the results is illustrated by several examples.

Mathematical Subject Classification 28A12, 28A78

1 Introduction

The Hausdorff measures and the Hausdorff contents play an important role in many areas of analysis. However, the later is more flexible in some sense. Indeed, in comparison with the Hausdorff measures the Hausdorff contents allow to work with sets composed of pieces of different dimensions. In what follows, given a number $d \in [0, n]$ and a set $E \subset \mathbb{R}^n$, by the symbols $\mathcal{H}^d(E)$ and $\mathcal{H}^d_{\infty}(E)$ we will denote the $d$-Hausdorff measure and the $d$-Hausdorff content of $E$ respectively (see the next section for the precise definitions).

Fix a nonempty set $E \subset \mathbb{R}^n$ and parameters $d \in [0, n]$ and $\lambda \in (0, 1]$. Here and in the sequel by a cube $Q$ we always mean a closed cube with sides parallel to the coordinate axes, more precisely $Q(x, l) := \prod_{i=1}^n [x_i - l/2, x_i + l/2]$. We say that a cube $Q = Q(x, l)$ is $(d, \lambda)$-thick with respect to the set $E$ if

$$\mathcal{H}^d_{\infty}(Q(x, l) \cap E) \geq \lambda l^d;$$

we say that $Q$ is $(d, \lambda)$-thin with respect to the set $S$ if

$$\mathcal{H}^d_{\infty}(Q(x, l) \cap E) < \lambda l^d.$$

Finally, we say that a set $E$ is $(d, \lambda)$-thick provided that for every $x \in E$ and every $l \in (0, 1]$ the cube $Q(x, l)$ is $(d, \lambda)$-thick with respect to the set $E$. As far as we know $d$-thick sets were firstly introduced by V. Rychkov [5] in 2000. In fact, the concept of $d$-thick sets is a natural and far reaching generalization of the concept of the so-called Ahlfors-David $d$-regular sets. Recall that

---

*Keywords: Porous sets, Hausdorff content, $d$-thick sets, Frostman measures
†Steklov Mathematical Institute of Russian Academy of Sciences (Moscow). E-mails: tyulenev-math@yandex.ru, tyulenev@mi.ras.ru
given \( d \in (0, n] \), a closed set \( S \subset \mathbb{R}^n \) is said to be Ahlfors-David \( d \)-regular if there are constants \( C_S^1 > 0 \) and \( C_S^2 > 0 \) such that

\[
C_S^1 l^{d} \leq \mathcal{H}^d(x, l) \cap S \leq C_S^2 l^{d} \quad \text{for all } \ x \in S \text{ and all } \ l \in (0, 1]. \tag{1.1}
\]

In the sequel, condition (1.1) will be usually called the Ahlfors-David \( d \)-regularity condition. It is easy to show that given a number \( d \in (0, n] \), every Ahlfors-David \( d \)-regular set is \( d \)-thick (see Lemma 2.2 in [9] for the proof). The converse is false [8], [9]. For example, any path-connected set \( \Omega \subset \mathbb{R}^n \) containing at least two distinct points is \( 1 \)-thick [9] (see Example 2.1 therein). On the other hand, it is easy to build planar rectifiable curves that fail to satisfy Ahlfors-David \( 1 \)-regularity condition [8] and hence, fail to satisfy Ahlfors-David \( d \)-regularity condition for any \( d \in (0, 2] \). Furthermore, one can easily construct a domain \( \Omega \subset \mathbb{R}^n \) with a cusp (see Example 6.3 in [9]) that fails to satisfy the Ahlfors-David \( n \)-regularity condition (and hence fail to satisfy the Ahlfors-David \( d \)-regularity condition for any \( d \in (0, n] \)). Recently, it was discovered [5], [9], [10], [8] that \( d \)-thick sets can be effectively used in the theory of traces of function spaces. For example, in [9] for any given closed \( d \)-thick set \( S \subset \mathbb{R}^n \) with \( d \in (0, n] \) and any \( p \in (\max\{1, n - d\}, \infty) \) an exact description of traces of functions \( F \in W^{1,p}_p(\mathbb{R}^n) \) to the set \( S \) was obtained.

On the other hand, in the theory of traces of function spaces on different subsets of \( \mathbb{R}^n \), the so-called porous sets are also play an important role [10], [11], [12], [13], [14], [15], [16], [17], [18], [19]. As far as we know, the term "porous set" has been firstly used by E. P. Dolzenko [14]. Porous sets have a lot of interesting applications in many other areas of analysis. The corresponding literature is so huge that we just mention only groundbreaking papers [12], [13], the beautiful survey [21] and the monograph [19] (see chapter 11 therein) where the reader can find some historical references, classical results and applications of porous sets. We slightly modify a modern definition of porous sets that is commonly used in the literature [21]. First of all, given a set \( S \subset \mathbb{R}^n \) and a parameter \( \tau \in (0, 1] \) we say that a cube \( Q \) is \( \tau \)-porous with respect to \( S \) if there is a cube \( Q' \subset Q \setminus S \) with \( \text{diam } Q' \geq \tau \text{diam } Q \). Given a parameter \( \tau \in (0, 1/2] \), a set \( S \subset \mathbb{R}^n \) is said to be \( \tau \)-porous if for every \( x \in S \) and every \( l \in (0, 1] \), the cube \( Q = Q(x, l) \) is \( \tau \)-porous with respect to the set \( S \). It was observed by A. Jonsson [16] (see also Proposition 9.18 in [11]) that if a closed set \( S \subset \mathbb{R}^n \) satisfies the Ahlfors-David \( d \)-regularity condition with \( d \in (0, n) \), then it is \( \tau \)-porous for some \( \tau \in (0, 1) \) depending only on \( d \) and \( n \). Recall also that a set \( S \subset \mathbb{R}^n \) is porous if, and only if, its Assouad dimension is strictly less than \( n \) [17].

The facts mentioned above give a motivation for a more deep study of geometric properties of \( d \)-thick sets. In this paper we are going to extend observations of A. Jonsson made for the Ahlfors-David \( d \)-regular sets to the case of \( d \)-thick sets. Note that in proving porosity properties of Ahlfors-David \( d \)-regular sets both lower and upper bounds in (1.1) play an important roles. On the other hand, the \( d \)-thick condition appeals only to a lower bound because the upper bounds for the Hausdorff contents hold true automatically. Hence, without a nontrivial conditions controlling upper bounds of the corresponding contents it is no chance to obtain porosity properties of \( d \)-thick sets. This motivates to introduce the following concept. Given a set \( S \subset \mathbb{R}^n \) and parameters \( \tau, \lambda \in (0, 1] \), \( d \in (0, n] \), we say that a cube \( Q \) is \( (\tau, d, \lambda) \)-sparse with respect to \( S \) if there is a \( (d, \lambda) \)-thin with respect to \( S \) cube \( Q' \subset Q \) with \( l(Q') = \tau l(Q) \). We also would like to underline an importance of the parameter \( \tau \) which is responsible for a some sort of flexibility in choosing \( (d, \lambda) \)-thin cubes \( Q' \subset Q \) of smaller than \( Q \) size. Indeed, in the case \( \tau = 1 \) examples of \( d \)-planes in \( \mathbb{R}^n \) would make the corresponding definition unreasonable.

One of the main results of this paper reads as follows.

**Theorem 1.1.** Let \( S \subset \mathbb{R}^n \) be a \( d \)-thick set for some \( d \in (0, n] \) and \( \lambda \in (0, 1] \). Then, there
exists a constant \( c = c(n,d,\lambda) \in (0,1) \) such that for each \( \tau \in (0,1) \) the following properties hold:

1. every \( \tau \)-porous with respect to the set \( S \) cube \( Q = Q(x,l) \), \( l \in (0,1) \) is \( (\tau,d,1) \)-sparse with respect to \( S \);
2. every \( (\tau,d,2^{-n-2d}) \)-sparse with respect to the set \( S \) cube \( Q = Q(x,l) \), \( l \in (0,1) \) is \( c \tau \)-porous with respect to \( S \).

Note that Theorem 1.1.1 is sharp. Namely, as we will see in Example 6.1 the \( d \)-thick condition cannot be dropped down. In fact for arbitrary sets \( S \subset \mathbb{R}^n \) we can establish a more rough and much more simple result. Given a number \( \gamma \in (0,1) \), we say that a cube \( Q = Q(x,l) \) is \( \gamma \)-hollow with respect to the set \( S \) if there is a Borel set \( U \subset \mathbb{R}^n \setminus S \) with \( \mathcal{H}^n(U) \geq \gamma l^n \).

**Theorem 1.2.** Let \( S \subset \mathbb{R}^n \) be a closed set and let \( d \in (0,n] \), \( \lambda \in (0,1) \). Then, for each \( \tau \in (0,1) \) every \( (\tau,d,\lambda) \)-sparse with respect to the set \( S \) cube \( Q = Q(x,l) \), \( l \in (0,1) \) is \( (1 - \lambda^2)\tau \)-hollow with respect to \( S \).

Using much more delicate tools introduced in section 4 of this paper we can establish the following result. It is interesting by itself but we keep in mind its future applications in the theory of traces of function spaces. Given a set \( S \subset \mathbb{R}^n \) and parameters \( d \in (0,n] \), \( \lambda \in (0,1] \), \( \rho \geq 1 \), we define

\[
\tilde{S}(d,\lambda,\rho) := \bigcup \rho Q,
\]

where the union is taken over all \((d,\lambda)\)-thick with respect to \( S \) cubes \( Q \) with \( 0 < l(Q) \leq \delta \).

**Theorem 1.3.** Let \( d \in (0,n] \), \( \lambda \in (0,1) \) and \( \rho \geq 1 \). Let \( S \subset \mathbb{R}^n \) be an arbitrary set. Then, there exist constants \( c = c(n,d) \in (0,1] \) and \( \delta = \delta(n,d,\lambda,\rho) \in (0,1) \) such that for each \( \tau \in (0,1] \) for every \( (\tau,d,\frac{1}{2^{d+s+n}}) \)-sparse with respect to \( S \) cube \( Q = Q(x,l) \), \( l \in (0,1] \) and any \( \delta \in (0,\delta] \) the set

\[
W_\delta := (Q \setminus S) \setminus \tilde{S}(d,\lambda,\rho,\delta l)
\]

is a \( c \tau \)-cavity of \( Q \) with respect to the set \( S \).

As far as we know Theorems 1.1.1 and 1.3 are new. In some sense they can be considered as complementary results to the well known classical results relating the porosity properties and dimensions [22], [13], [23]. The papers above contain results claiming that a "sufficiently nice porosity" of a give set \( E \subset \mathbb{R}^n \) implies that its Hausdorff dimension has to have an upper bound strictly less that \( n \). Roughly speaking, results of this paper go in the opposite direction.

The paper organised as follows. Section 2 contains an elementary background. Section 3 is devoted to some properties of \( d \)-thick sets. Section 4 is a technical core of the paper. In section 5 we prove Theorems 1.1.1.3 Finally, section 6 contains simple examples demonstrating the sharpness of the main theorems.

## 2 Preliminaries

Throughout the paper \( C, C_1, C_2, \ldots \) will be generic positive constants. These constants can change even in a single string of estimates. The dependence of a constant on certain parameters is expressed, for example, by the notation \( C = C(n,p,k) \). We write \( A \approx B \) if there is a constant \( C \geq 1 \) such that \( A/C \leq B \leq CA \). Given a number \( c \in \mathbb{R} \) we denote by \( \lfloor c \rfloor \) the integer part of \( c \).

If no otherwise stated we let \( \mathbb{R}^n \) denote the linear space of all strings \( x = (x_1,\ldots,x_n) \) of real numbers equipped with the uniform norm \( \| \cdot \|_\infty \), i.e. \( \| x \|_\infty := \max \{ |x_1|,\ldots,|x_n| \} \). Given a set \( E \subset \mathbb{R}^n \) we denote by \( \overline{E} \) and \( E^c \) the closure and the complement (in \( \mathbb{R}^n \)) of \( E \) respectively. Given a set \( E \subset \mathbb{R}^n \) we will always denote by \( \chi_E \) the characteristic function of \( E \).

If no otherwise stated all cubes are assumed to be closed with sides parallel to the coordinate axes. More precisely, for any \( x \in \mathbb{R}^n \) and \( l \geq 0 \) we set \( Q(x,l) := \prod_{i=1}^n [x_i - l/2, x_i + l/2] \). Given
a cube $Q$, we will denote by $l(Q)$ the diameter of $Q$ computed in $\| \cdot \|_\infty$-norm, i.e. its side length. By a dyadic cube we mean an arbitrary closed cube $Q_{k,m} := \prod_{i=1}^n \left[ \frac{m_i}{2^k}, \frac{m_i+1}{2^k} \right]$ with $k \in \mathbb{Z}$ and $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$. Given $k \in \mathbb{Z}$, by $D_k$ we denote the family of all closed dyadic cubes with side lengths equal $2^{-k}$. For any $k \in \mathbb{Z}$ and $m \in \mathbb{Z}^n$ we set $b_k(m) := \{ m' \in \mathbb{Z}^n : Q_{k,m} \subset 3Q_{k,m'} \}$. We say that two dyadic cubes $Q_{k,m}$ and $Q_{k,m'}$ are neighboring provided that $m' \in b_k(m)$ and $m \neq m'$.

In the sequel by a measure we mean only nonnegative Borel measure on $\mathbb{R}^n$. By $\mathcal{L}^n$ we denote the classical $n$-dimensional Lebesgue measure in $\mathbb{R}^n$. We say that a set $E \subset \mathbb{R}^n$ is measurable if it is measurable with respect to $\mathcal{L}^n$.

In what follows we will use the following abbreviations. Given a family of cubes $\{ Q_\alpha \}_{\alpha \in \mathcal{I}}$ in $\mathbb{R}^n$, we set

$$l_\alpha := \text{diam } Q_\alpha = l(Q_\alpha), \quad \alpha \in \mathcal{I}.$$ 

More generally, given a sequence of families of cubes $\{ \{ Q_{\alpha}^s \}_{\alpha \in \mathcal{I}} \}_{s \in \mathbb{N}_0}$, we set

$$l_\alpha^s := l(Q_{\alpha}^s), \quad s \in \mathbb{N}_0 \quad \text{and} \quad \alpha \in \mathcal{I}^s.$$ 

Given a family $\{ Q_\alpha \}_{\alpha \in \mathcal{I}}$ of cubes in $\mathbb{R}^n$ and a cube $Q \subset \mathbb{R}^n$, we define the restriction of the index set $\mathcal{I}$ to the cube $Q$ as

$$\mathcal{I}|_Q := \{ \alpha \in \mathcal{I} : Q_\alpha \subset Q \}. $$

In the sequel we will commonly use the following partial order on the set of all families of non-overlapping (i.e. different dyadic cubes have disjoint interiors) dyadic cubes. Given two non-overlapping families $\{ Q_\alpha \}_{\alpha \in \mathcal{I}}, \mathcal{I} \subset \mathbb{N}_0 \times \mathbb{Z}^n$ and $\{ Q'_\alpha \}_{\alpha \in \mathcal{I}'}, \mathcal{I}' \subset \mathbb{N}_0 \times \mathbb{Z}^n$ of dyadic cubes, we write $\{ Q_\alpha \}_{\alpha \in \mathcal{I}} \preceq \{ Q'_\alpha \}_{\alpha \in \mathcal{I}'}$ provided that for every $\alpha' \in \mathcal{I}'$ there exists a unique $\alpha \in \mathcal{I}$ such that $Q_\alpha \supset Q'_\alpha$. If in addition $l_\alpha > l_{\alpha'}$ for all such $\alpha$ and $\alpha'$ we write $\{ Q_\alpha \}_{\alpha \in \mathcal{I}} < \{ Q'_\alpha \}_{\alpha \in \mathcal{I}'}$. We say that two families of dyadic non-overlapping cubes $\{ Q_\alpha \}_{\alpha \in \mathcal{I}}$ and $\{ Q'_{\alpha'} \}_{\alpha' \in \mathcal{I}'}$ are comparable provided that

$$\text{ either } \{ Q_\alpha \}_{\alpha \in \mathcal{I}} \preceq \{ Q'_{\alpha'} \}_{\alpha' \in \mathcal{I}'} \quad \text{or} \quad \{ Q'_{\alpha'} \}_{\alpha' \in \mathcal{I}'} \preceq \{ Q_\alpha \}_{\alpha \in \mathcal{I}}.$$ 

Otherwise we call the corresponding families incomparable.

Given a set $E \subset \mathbb{R}^n$, by a covering of the set $E$ we mean a family $\{ U_\beta \}_{\beta \in \mathcal{J}}$ of subsets of $\mathbb{R}^n$ such that $E \subset \bigcup_{\beta \in \mathcal{J}} U_\beta$. Given a set $E \subset \mathbb{R}^n$, by a dyadic non-overlapping covering of the set $E$ we mean a family $\{ Q_\alpha \}_{\alpha \in \mathcal{I}}, \mathcal{I} \subset \mathbb{N}_0 \times \mathbb{Z}^n$ of closed dyadic cubes such that $\{ Q_\alpha \}_{\alpha \in \mathcal{I}}$ is a covering of $E$ and different cubes have disjoint interiors.

In this paper we will work not only with the classical Hausdorff measures and contents but also with its corresponding dyadic analogs.

**Definition 2.1.** Let $E \subset \mathbb{R}^n$ be a nonempty set and $d \in [0, n]$. For any $\delta \in (0, \infty]$ we set

$$\mathcal{H}_\delta^d(E) := \inf_{\beta \in \mathcal{J}} \sum (\text{diam } U_\beta)^d, \quad \mathcal{H}_\delta^d(E) := \inf_{\alpha \in \mathcal{I}} (l_\alpha)^d, \quad (2.1)$$

where in the definition of $\mathcal{H}_\delta^d(E)$ the infimum is taken over all at most countable coverings $\{ U_\beta \}_{\beta \in \mathcal{J}}$ of the set $E$ by sets $U_\beta$ with $\text{diam } U_\beta < \delta$ for all $\beta \in \mathcal{J}$ and in the definition of $\mathcal{H}_\delta^d(E)$ the infimum is taken over all dyadic non-overlapping coverings $\{ Q_\alpha \}_{\alpha \in \mathcal{I}}$ of the set $E$ by dyadic cubes with $l_\alpha < \delta$ for all $\alpha \in \mathcal{I}$. The value $\mathcal{H}_\delta^d(E)$ is called the $d$-Hausdorff content of the set $E$. The value $\mathcal{H}_\infty^d(E)$ is called the dyadic $d$-Hausdorff content of the set $E$. We define the $d$-Hausdorff measure and the dyadic $d$-Hausdorff measure of the set $E$ as $\mathcal{H}^d(E) := \lim_{\delta \to 0} \mathcal{H}_\delta^d(E)$ and $\mathcal{H}^d(E) := \lim_{\delta \to 0} \mathcal{H}_\delta^d(E)$ respectively.

**Remark 2.1.** Given a set $E \subset \mathbb{R}^n$ and a parameter $\delta > 0$, it is easy to see that up to some universal constants $\mathcal{H}_\delta^d(E)$ and $\mathcal{H}_\delta^d(E)$ give the same values of the corresponding contents. More precisely, we have

$$\mathcal{H}_\delta^d(E) \leq \mathcal{H}_\delta^d(E) \leq 2^n \mathcal{H}_\delta^d(E). \quad (2.2)$$
We say that a dyadic non-overlapping family is optimal provided that there exists a set $E$ with diam $U < \delta$, $\beta \in \mathcal{I}$ we obtain (we set $k^\beta := k(U_\beta)$ for all $\beta \in \mathcal{J}$)

$$
\sum_{\beta \in \mathcal{J}} \sum_{\mathbb{Z}^n} 2^{-d k^\beta} \leq 2^n \sum_{\beta \in \mathcal{J}} (\text{diam } U_\beta)^d.
$$

(2.3)

Taking an infimum over all coverings $\{U_\beta\}_{\beta \in \mathcal{J}}$ of the set $E$ with the corresponding properties in both sides of (2.3) and taking into account (2.1) we get the second inequality in (2.2).

**Definition 2.2.** Let $d \in [0, n]$ and $E \subset \mathbb{R}^n$ be an arbitrary set with $\mathcal{H}_\infty^d(E) > 0$. We say that a family $\{U_\beta\}_{\beta \in \mathcal{J}}$ is a $d$-almost covering of the set $E$ if there exists a set $E' \subset E$ with $\mathcal{H}_\infty^d(E') = 0$ such that the family $\{U_\beta\}_{\beta \in \mathcal{J}}$ is a covering of the set $E \setminus E'$.

**Definition 2.3.** Let $d \in (0, n]$ and $E \subset \mathbb{R}^n$ be a set with $\mathcal{H}_\infty^d(E) \in (0, +\infty)$. Given $\varepsilon > 0$, we say that a $d$-almost covering $\{U_\beta\}_{\beta \in \mathcal{J}}$ of the set $E$ is $\varepsilon$-optimal provided that

$$
\sum_{\beta \in \mathcal{J}} (\text{diam } U_\beta)^d \leq (1 + \varepsilon) \mathcal{H}_\infty^d(E).
$$

We say that a dyadic non-overlapping $d$-almost covering $\{Q_\alpha\}_{\alpha \in \mathcal{I}}$ of the set $E$ is (dyadically) $\varepsilon$-optimal provided that

$$
\sum_{\alpha \in \mathcal{I}} (l_\alpha)^d \leq (1 + \varepsilon) \tilde{\mathcal{H}}_\infty^d(E).
$$

We say that a dyadic $\varepsilon$-optimal (dyadically) non-overlapping $d$-almost covering $\{Q_\alpha\}_{\alpha \in \mathcal{I}}$ of the set $E$ is maximal (in the sense of side lengths of the corresponding cubes) if $\{Q_\alpha\}_{\alpha \in \mathcal{I}} \supseteq \{Q_\alpha\}_{\alpha \in \mathcal{I}}$ for any $\varepsilon$-optimal dyadic non-overlapping $d$-almost covering $\{Q_\alpha\}_{\alpha \in \mathcal{I}}$ of $E$ comparable with $\{Q_\alpha\}_{\alpha \in \mathcal{I}}$.

**Remark 2.2.** Let $d \in (0, n]$ and $E \subset \mathbb{R}^n$ be a set with $\mathcal{H}_\infty^d(E) \in (0, +\infty)$. It is easy to show that a maximal $\varepsilon$-optimal dyadic non-overlapping $d$-almost covering of the set $E$ exists. Note however that it is not unique in general. The point is that there can exist several incomparable maximal $\varepsilon$-optimal dyadic non-overlapping $d$-almost coverings of the set $E$.

**Proposition 2.1.** Let $E \subset \mathbb{R}^n$ be a set. Let $d \in (0, n]$, $\lambda \in (0, 1]$ and let $Q = Q(x, l)$ be a $(d, \lambda)$-dyadically thin with respect to $E$ cube. Then, there is $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and any $\varepsilon$-optimal dyadic non-overlapping $d$-almost covering $\{Q_\alpha\}_{\alpha \in \mathcal{I}}$ of the set $Q \cap S$ it holds

$$
k_\alpha := -\log_2 l_\alpha \geq [-\log_2 (\lambda^{1 \over d})] + 1 \quad \text{for every } \alpha \in \mathcal{I}.
$$

(2.4)

**Proof.** We set $\gamma := \tilde{\mathcal{H}}_\infty^d(Q \cap E)$ for brevity. By the assumptions we have $\gamma < \lambda^d$. Take $\varepsilon_0 > 0$ so small that $(1 + \varepsilon_0) \gamma < \lambda^d$. Hence, taking $\varepsilon \in (0, \varepsilon_0)$ and an arbitrary $\varepsilon$-optimal dyadic non-overlapping $d$-almost covering $\{Q_\alpha\}_{\alpha \in \mathcal{I}}$ of the set $Q \cap E$ we clearly get

$$
(l_\alpha)^d \leq \sum_{\alpha' \in \mathcal{I}} (l_{\alpha'})^d < \lambda^d \quad \text{for every } \alpha \in \mathcal{I}.
$$

Since the cubes $Q_\alpha$, $\alpha \in \mathcal{I}$ are dyadic we obtain (2.4)
3 Thick sets and Frostman-type measures

In this small section we recall again the corresponding definitions of $d$-thick sets from the introduction and establish their characteristic properties described in terms of special regular sequences of measures.

**Definition 3.1.** Let $d \in (0, n]$ and $\lambda \in (0, 1]$. Let $E \subset \mathbb{R}^n$ be a set with $H_\infty^d (E) > 0$. We say that a cube $Q = Q(x, l)$ with $l \in (0, 1]$ is $(d, \lambda)$-thick with respect to the set $E$ if

$$H_\infty^d (Q \cap E) \geq \lambda l^d. \hspace{1cm} (3.1)$$

We say that a cube $Q = Q(x, l)$ with $l \in (0, 1]$ is $(d, \lambda)$-thin with respect to the set $E$ if

$$H_\infty^d (Q \cap E) < \lambda l^d. \hspace{1cm} (3.2)$$

We say that a cube $Q$ is $(d, \lambda)$-dyadically thick with respect to the set $E$ if (3.1) holds true with $H_\infty^d (Q \cap E)$ replaced by $H_\infty^{d, \lambda} (Q \cap E)$. Similarly, we say that a cube $Q$ is $(d, \lambda)$-dyadically thin with respect to the set $E$ if (3.2) holds true with $H_\infty^d (Q \cap E)$ replaced by $H_\infty^{d, \lambda} (Q \cap E)$.

We recall also the concept of sparse cubes.

**Definition 3.2.** Let $d \in (0, n]$ and $\tau, \lambda \in (0, 1]$. Let $E \subset \mathbb{R}^n$ be a set with $H_\infty^d (E) > 0$. We say that a cube $Q$ with $l(Q) \in (0, 1]$ is $(\tau, d, \lambda)$-sparse with respect to the set $E$ if there exists a $(d, \lambda)$-thin with respect to $E$ cube $Q' \subset Q$ with $l(Q') = \tau l(Q)$.

The following elementary property will be quite useful in the sequel. It exhibits relations between $(d, \lambda)$-thick (thin) cubes and $(d, \lambda)$-dyadically thick (thin) cubes respectively.

**Proposition 3.1.** Let $d \in (0, n)$, $\lambda \in (0, 1)$ and let $E \subset \mathbb{R}^n$ be a Borel set with $H_\infty^d (E) > 0$. Let $Q = Q(x, l)$ be a cube with $l \in (0, 1]$ and let $k := [- \log_2 l(Q)] \in \mathbb{N}_0$. Then, the following holds:

(i) if $Q$ is $(d, \lambda)$-(dyadically) thick with respect to $E$, then there is a $(d, \frac{\lambda}{2^{d(j+1)}})$-(dyadically) thick with respect to $E$ dyadic cube $Q_{k,m} \subset Q$ such that $Q_{k,m} \cap Q \neq \emptyset$;

(ii) if $Q$ is $(d, \frac{\lambda}{2^{d(j+1)}})$-(dyadically) thin with respect to $E$, then every dyadic cube $Q_{k+j,m} \subset Q$ is $(d, \lambda)$-(dyadically) thin with respect to $E$.

Proof. For the number $k$, by its very definition, we have $l(Q) \in [2^{-k}, 2^{-k+1})$. Since $Q$ is assumed to be closed there are at most $3^n$ dyadic cubes $Q_{k,m}$ such that $Q_{k,m} \cap Q \neq \emptyset$.

To prove the first claim assume the contrary. Then using the subadditivity property of the $d$-Hausdorff content $H_\infty^d$ we get

$$H_\infty^d (Q \cap E) \leq \sum_{Q_{k,m} \cap Q \neq \emptyset} H_\infty^d (Q_{k,m} \cap E) < 3^n \frac{\lambda}{3^n} 2^{-kd} < \lambda l(Q)^d.$$

This in contradiction with the assumption that $Q$ is $(d, \lambda)$-thick with respect to $E$.

To prove the second claim assume the contrary, i.e. that there is a $(d, \lambda)$-thick with respect to $E$ dyadic cube $Q_{k+j,m} \subset Q$. Due to monotonicity of the $d$-Hausdorff content $H_\infty^d$ we get by definition of the number $k$

$$H_\infty^d (Q \cap E) \geq H_\infty^d (Q_{k+j,m} \cap E) \geq \lambda 2^{-(k+j)d} \geq \frac{\lambda l^d}{2^{d(j+1)}}.$$

This is in contradiction with the assumption that $Q$ is $(d, \frac{\lambda}{2^{d(j+1)}})$-thin with respect to the set $E$.

The corresponding dyadic analogs of the claims can be proved similarly. \qed
As was already mentioned in the introduction, to the best of our knowledge the following concept was firstly introduced in [5].

**Definition 3.3.** Let $d \in [0, n]$ and $\lambda \in (0, 1)$. A set $E \subset \mathbb{R}^n$ is called $(d, \lambda)$-thick if there exists a locally finite Borel measure $\mu$ with the characteristic property of the Ahlfors-David $(1 \leq \lambda)$-regular sets is well known. It follows immediately from Theorem 1 in section 1.2, Chapter 1 of the monograph [1].

**Proposition 3.2.** Given $d \in (0, n]$, a closed set $S \subset \mathbb{R}^n$ is Ahlfors-David $d$-regular if there exists a locally finite Borel measure $\mu$ and a constant $C_\mu > 0$ depending only on $d$, $C^1_S$ and $C^2_S$ such that

$$
\frac{r^d}{C_\mu} \leq \mu(Q(x, r) \cap S) \leq C_\mu dr^d \quad \text{for all } x \in S \quad \text{and all } r \in (0, 1].
$$

Unfortunately, in comparison with Ahlfors-David $d$-regular sets $d$-thick sets can not be characterized with the help of the unique "nice" measure. Instead of this we recall the following concept which was firstly introduced in [5].

**Definition 3.4.** Let $d \in (0, n]$ and $S \subset \mathbb{R}^n$ be a closed set with $\mathcal{H}^d(S) > 0$. We say that a sequence of Borel measures $\{\mu_k\} = \{\mu_k\}_{k \in \mathbb{N}_0}$ is $d$-regular on $S$ provided that the following conditions hold:

1. for every $k \in \mathbb{N}_0$

$$
\text{supp} \mu_k = S;
$$

2. there exists a constant $C^1 > 0$ such that for each $k \in \mathbb{N}_0$

$$
\mu_k(Q(x, l)) \leq C^1 l^d \quad \text{for every } x \in \mathbb{R}^n \quad \text{and every } l \in (0, 2^{-k}];
$$

3. there exists a constant $0 < C^2 \leq C^1$ such that for each $k \in \mathbb{N}_0$

$$
\mu_k(Q(x, l) \cap S) \geq C^2 l^d \quad \text{for every } x \in S \quad \text{and every } l \in [2^{-k}, 1];
$$

4. for every $k \in \mathbb{N}_0$ it holds $\mu_k = w_k \mu_0$ with $w_k \in L^\infty(S, \mu_0)$ and

$$
2^{d-n} w_{k+1}(x) \leq w_k(x) \leq w_{k+1}(x) \quad \text{for } \mu_0 - \text{a.e. } x \in S.
$$

**Remark 3.2.** It is easy to see that there exists the minimal among all constants $C^1 > 0$ for which (3.5) holds, we denote it by $C^1_{(\mu_k)}$. Similarly, there exists the maximum among all constants $C^2 > 0$ for which (3.6) holds, we denote it by $C^2_{(\mu_k)}$.

We believe that the following result is interesting by itself. It gives the characterization of $d$-thick sets $S$ in terms of $d$-regular on $S$ sequences of measures and looks like the corresponding analog of Proposition 3.2.

As far as we know, this is the first result of such type in the literature.

**Theorem 3.1.** Let $d \in (0, n]$. A closed set $S \subset \mathbb{R}^n$ is $(d, \lambda)$-thick for some $\lambda \in (0, 1)$ if and only if there exists a $d$-regular on $S$ sequence of measures $\{\mu_k\} = \{\mu_k\}_{k \in \mathbb{N}_0}$.

**Proof.** The fact that for any closed $d$-thick set $S$ there exists a $d$-regular on $S$ sequence of measures was established in [5] (see Corollary 3.1 therein).

To prove the converse fix a cube $Q(x, l)$ with $x \in S$ and $l \in (0, 1]$. Fix a sufficiently small number $\varepsilon > 0$ and take an $\varepsilon$-optimal covering $\{U_\beta\}_{\beta \in J}$ of the set $Q(x, l) \cap S$. Clearly, $\text{diam } U_\beta \leq l$
for all $\beta \in \mathcal{J}$. Furthermore, for every $\beta \in \mathcal{J}$ there is a cube $Q_\beta \supset U_\beta$ with $l_\beta = \text{diam} \ U_\beta$. Hence, if $k \in \mathbb{N}_0$ is the maximal integer $\geq l$ we can combine observations above with (3.4)–(3.7) and get

$$C^2_{\{\mu_k\}} \leq \mu_{k+1}(Q(x,l)) \leq \sum_{\beta \in \mathcal{J}} \mu_{k+1}(U_\beta) \leq 2^{n-d} \sum_{\beta \in \mathcal{J}} \mu_k(Q_\beta)$$

$$\leq 2^{n-d} C^1_{\{\mu_k\}} \sum_{\beta \in \mathcal{J}} (l_\beta)^d \leq 2^{n-d}(1 + \varepsilon) C^1_{\{\mu_k\}} \mathcal{H}_\infty^d(Q(x,l) \cap S).$$

Since $\varepsilon > 0$ can be chosen arbitrarily, this proves that $Q(x,l)$ is $(d,\lambda)$-thick with respect to $S$ for every

$$0 < \lambda < \frac{C^2_{\{\mu_k\}}}{2^{n-d} C^1_{\{\mu_k\}}}.$$

\[ \square \]

4 Special coverings and hollow cubes

The following concept will be extremely useful.

**Definition 4.1.** Let $d \in (0, n]$ and $\lambda \in (0, 1]$. Let $S \subset \mathbb{R}^n$ be a set with $\mathcal{H}_\infty^d(S) > 0$. We say that a family $\mathcal{F} := \{Q_\alpha\}_{\alpha \in \mathcal{I}}$, $\mathcal{I} \subset \mathbb{N}_0 \times \mathbb{Z}^n$ is a dyadic non-overlapping $(d,\lambda)$-thick $d$-almost covering of the set $S$ if $\mathcal{F}$ is a dyadic non-overlapping $d$-almost covering of $S$ and for every $\alpha \in \mathcal{I}$ the cube $Q_\alpha$ is $(d,\lambda)$-dyadically thick w.r.t. $S$.

Recall that given a sequence of families of dyadic cubes $\{(Q_\alpha^s)_{\alpha \in \mathcal{I}}\}_{s \in \mathbb{N}_0}$ we set $l^+_s := l(Q_\alpha^s)$ for brevity. As far as we know the following concept has never been explicitly formulated in the literature.

**Definition 4.2.** Let $d \in (0, n]$ and $\lambda \in (0, 1]$. Let $S \subset \mathbb{R}^n$ be a set with $\mathcal{H}_\infty^d(S) > 0$. We say that a sequence $\tilde{\mathcal{A}}^s = \{\tilde{A}^s\}_{s \in \mathbb{N}_0}$ is $(d,\lambda)$-nice for $S$ provided that the following conditions hold:

1. $\tilde{A}_0 := \{\alpha \in \{0\} \times \mathbb{Z}^n : \mathcal{H}_\infty^d(S \cap Q_\alpha) > 0\}$ and $\tilde{A}^s \subset \mathbb{N} \times \mathbb{Z}^n$ for every $s \in \mathbb{N}$;
2. for every $s \in \mathbb{N}$ the family $\{Q_\alpha^s\}_{\alpha \in \tilde{A}^s}$ is a dyadic non-overlapping $(d,\lambda)$-thick $d$-almost covering of the set $S$;
3. $\{Q_\alpha^s\}_{\alpha \in \tilde{A}^s} \prec \{Q_{\alpha'}^{s+1}\}_{\alpha' \in \tilde{A}^{s+1}}$ for each $s \in \mathbb{N}$;
4. for each $s \in \mathbb{N}_0$, for every $\alpha \in \tilde{A}^s$ and every dyadic cube $Q \subset Q_\alpha^s$

$$\sum_{\alpha' \in \tilde{A}^{s+1}} (l^{s+1}_{\alpha'})^d \leq \begin{cases} 2^{n-d}(l(Q))^d, & \text{if } Q = Q_\alpha^s \text{ and } Q \text{ is } (d,1) - \text{dyadically thick w.r.t. } S; \\ (l(Q))^d & \text{in other cases.} \end{cases}$$

(4.1)

The following result is in fact Lemma 2.1 of Netrusov’s paper [18] adapted for our framework. We present the full proof to make our paper self-contained (it repeats almost verbatim the corresponding proof from [18]). Furthermore, we hope that this proof will clarify the driving ideas of this paper for the reader.

**Lemma 4.1.** Let $d \in (0, n]$ and $\lambda \in (0, 1]$. Let $S \subset \mathbb{R}^n$ be an arbitrary set with $\mathcal{H}_\infty^d(S) > 0$. Let $Q = Q_{\alpha_0}$, $\alpha_0 \in \mathbb{N}_0 \times \mathbb{Z}^n$ be a dyadic cube with

$$0 < \mathcal{H}_\infty^d(Q_{\alpha_0} \cap S) < 1.$$  

(4.2)
Then, there exists a dyadic non-overlapping \((d, \lambda)-\text{thick} d\)-almost covering \(\{Q_\alpha\}_{\alpha \in \hat{A}}\) of the set \(S \cap Q_{\alpha_0}\) such that:

1. for every \(\alpha \in \hat{A}\) it holds
   \[ l_\alpha \leq \frac{l_{\alpha_0}}{2}; \]  
   \[ (4.3) \]

2. for every dyadic cube \(Q \subset Q_{\alpha_0}\)
   \[ \sum_{\alpha \in \hat{A}} (l_\alpha)^d \leq (l(Q))^d. \]  
   \[ (4.4) \]

**Proof.** We split the proof into two steps.

**Step 1.** Recall Definition 2.3 and fix an \(\varepsilon > 0\) such that
\[ 0 < \tau := \frac{\varepsilon}{1 - \lambda} < 1 \]  
and consider an index set \(\mathcal{I}_{\alpha_0} \subset \mathbb{N}_0 \times \mathbb{Z}^n\) such that \(\{Q_\alpha\}_{\alpha \in \mathcal{I}_{\alpha_0}}\) is a maximal (dyadically) \(\varepsilon\)-optimal dyadic non-overlapping \(d\)-almost covering of \(S \cap Q_{\alpha_0}\). By Proposition 2.1 (decreasing \(\varepsilon > 0\) if necessary) we have
\[ l_\alpha \leq l_{\alpha_0}/2 \quad \text{for every} \quad \alpha \in \mathcal{I}_{\alpha_0}. \]  
\[ (4.6) \]

By the symbol \(\mathcal{I}^1_{\alpha_0} \subset \mathcal{I}_{\alpha_0}\) we denote the set of all indices corresponding to \((d, \lambda)\)-dyadically thick with respect to \(S\) dyadic cubes and set \(\tilde{\mathcal{I}}^1_{\alpha_0} := \mathcal{I}_{\alpha_0} \setminus \mathcal{I}^1_{\alpha_0}\). Using Definition 2.3 and subadditivity of the dyadic \(d\)-Hausdorff content we clearly have
\[ \sum_{\alpha \in \mathcal{I}^1_{\alpha_0}} (l_\alpha)^d / (1 + \varepsilon) + \sum_{\alpha \in \tilde{\mathcal{I}}^1_{\alpha_0}} (l_\alpha)^d / (1 + \varepsilon) = \sum_{\alpha \in \mathcal{I}_{\alpha_0}} (l_\alpha)^d / (1 + \varepsilon) \leq \tilde{H}^d_{\infty}(Q_{\alpha_0} \cap S) \]  
\[ \leq \tilde{H}^d_{\infty} \left( \bigcup_{\alpha \in \mathcal{I}^1_{\alpha_0}} Q_\alpha \cap S \right) + \sum_{\alpha \in \tilde{\mathcal{I}}^1_{\alpha_0}} \tilde{H}^d_{\infty}(Q_\alpha \cap S) \]  
\[ \leq \tilde{H}^d_{\infty} \left( \bigcup_{\alpha \in \mathcal{I}^1_{\alpha_0}} Q_\alpha \cap S \right) + \lambda \sum_{\alpha \in \mathcal{I}^1_{\alpha_0}} (l_\alpha)^d \leq \sum_{\alpha \in \mathcal{I}^1_{\alpha_0}} (l_\alpha)^d + \lambda \sum_{\alpha \in \tilde{\mathcal{I}}^1_{\alpha_0}} (l_\alpha)^d. \]

Hence, we obtain
\[ \varepsilon \sum_{\alpha \in \mathcal{I}^1_{\alpha_0}} (l_\alpha)^d \geq \left(1 - \lambda(1 + \varepsilon)\right) \sum_{\alpha \in \mathcal{I}^1_{\alpha_0}} (l_\alpha)^d. \]  
\[ (4.7) \]

As a result, using subadditivity of \(\tilde{H}^d_{\infty}\) and Definition 2.3 we deduce from (4.7)
\[ \tilde{H}^d_{\infty} \left( S \setminus \bigcup_{\alpha \in \mathcal{I}^1_{\alpha_0}} Q_\alpha \right) \leq \sum_{\alpha \in \tilde{\mathcal{I}}^1_{\alpha_0}} (l_\alpha)^d \]  
\[ \leq \frac{\varepsilon}{(1 - \lambda)(1 + \varepsilon)} \sum_{\alpha \in \mathcal{I}_{\alpha_0}} (l_\alpha)^d \leq \tau \tilde{H}^d_{\infty}(Q_{\alpha_0} \cap S). \]  
\[ (4.8) \]

Note that according to the construction of the index set \(\mathcal{I}_{\alpha_0}\) and Definition 2.3 we obtain for every dyadic cube \(Q \subset Q_{\alpha_0}\)
\[ \sum_{\alpha \in \mathcal{I}_{\alpha_0}, Q_\alpha \subset Q} (l_\alpha)^d \leq (l(Q))^d. \]  
\[ (4.9) \]
Indeed, otherwise if we change our dyadic non-overlapping \(d\)-almost covering and take \(Q\) instead of all cubes \(Q_\alpha \subset Q\) we obtain an \(\varepsilon\)-optimal dyadic non-overlapping \(d\)-almost covering of \(S \cap Q_\alpha\) which contradicts the maximality of the family \(\{Q_\alpha\}_{\alpha \in I_{\alpha}}\).

Step 2. Suppose that given a \((d, 1)\)-dyadically thin with respect to \(S\) dyadic cube \(Q_\alpha\), we have already built for some \(k_0 \in \mathbb{N}\) and for every \(j \in \{1, \ldots, k_0\}\) the index sets \(I_{\alpha j}^k\) and \(\tilde{I}_{\alpha j}^k\) such that:

(i) \(I_{\alpha 0}^1 \subset \cdots \subset I_{\alpha 0}^{k_0}\);
(ii) for every \(\alpha \in I_{\alpha 0}^{k_0}\) the cube \(Q_\alpha\) is \((d, \lambda)\)-dyadically thick with respect to \(S\);
(iii) for every \(\alpha' \in I_{\alpha 0}^{k_0}\) the cube \(Q_{\alpha'}\) is \((d, \lambda)\)-dyadically thin with respect to \(S\);
(iv) for every dyadic cube \(Q \subset Q_\alpha\) we have

\[
\sum_{\alpha \in \tilde{I}_{\alpha 0}^{k_0} \cup \tilde{I}_{\alpha 0}^{k_0}} (l_\alpha)^d \leq (l(Q))^d. \tag{4.10}
\]

For every \(\alpha \in \tilde{I}_{\alpha 0}^{k_0}\) we repeat the same arguments as those given at the first step above to obtain the index sets \(I_{\alpha 1}^1, \tilde{I}_{\alpha 1}^1\) such that:

\[
Q_\alpha \cap S \subset \bigcup_{\alpha' \in \tilde{I}_{\alpha 1}^1} Q_{\alpha'} \quad \text{and} \quad \sum_{\alpha' \in \tilde{I}_{\alpha 1}^1} (l_\alpha')^d \leq \tau(l_\alpha)^d. \tag{4.11}
\]

Furthermore, for every dyadic cube \(Q \subset Q_\alpha\)

\[
\sum_{\alpha' \in \tilde{I}_{\alpha 1}^1 \cup \tilde{I}_{\alpha 0}^{k_0}} (l_\alpha')^d \leq (l(Q))^d. \tag{4.12}
\]

We define now

\[
I_{\alpha 0}^{k_0+1} := \left( \bigcup_{\alpha \in \tilde{I}_{\alpha 0}^{k_0}} I_{\alpha 1}^1 \right) \cup I_{\alpha 0}^{k_0} \quad \text{and} \quad \tilde{I}_{\alpha 0}^{k_0+1} := \bigcup_{\alpha \in \tilde{I}_{\alpha 0}^{k_0}} \tilde{I}_{\alpha 1}^1.
\]

It is clear that conditions (i)–(iii) are satisfied with \(k_0\) replaced by \(k_0 + 1\). It remains to verify that (4.10) holds true with \(k_0 + 1\) instead of \(k_0\). Indeed, combining (4.12) with (4.10) we obtain for any dyadic cube \(Q \subset Q_\alpha\)

\[
\sum_{\alpha \in \tilde{I}_{\alpha 0}^{k_0+1} \cup \tilde{I}_{\alpha 0}^{k_0+1}} (l_\alpha)^d = \sum_{\alpha \in \tilde{I}_{\alpha 0}^{k_0} \cup \tilde{I}_{\alpha 0}^{k_0}} (l_\alpha)^d + \sum_{\alpha \in \tilde{I}_{\alpha 0}^{k_0}} (l_\alpha)^d \leq (l(Q))^d. \tag{4.13}
\]

Hence, we establish that (4.10) holds true with \(k_0 + 1\) instead of \(k_0\).

As a result, by induction we built the sequences \(\{I_{\alpha 0}^k\}_{k \in \mathbb{N}}\) and \(\{\tilde{I}_{\alpha 0}^k\}_{k \in \mathbb{N}}\) such that for each \(k \in \mathbb{N}\) conditions (i)–(iv) are satisfied with \(k_0\) replaced by \(k\). We set

\[
\tilde{A}_{\alpha 0} := \bigcup_{k \in \mathbb{N}} I_{\alpha 0}^k. \tag{4.14}
\]

Note also that according to our construction estimate (4.13) implies for every \(k \in \mathbb{N}\)

\[
\bigcup_{\alpha \in \tilde{I}_{\alpha 0}^{k+1}} Q_\alpha \subset \bigcup_{\alpha \in \tilde{I}_{\alpha 0}^k} Q_\alpha \quad \text{and} \quad \sum_{\alpha \in \tilde{I}_{\alpha 0}^k} (l_\alpha)^d < \tau^k \tilde{H}_\infty^d (Q_\alpha_0 \cap S). \tag{4.14}
\]
Since $\tau \in (0,1)$, this leads to
\[
\mathcal{H}_\infty^d \left( (Q_{00} \cap S) \setminus \bigcup_{\alpha \in \hat{A}_{00}} Q_\alpha \right) \leq \lim_{k \to \infty} \sum_{\alpha \in \hat{J}^k} (l_\alpha)^d = 0
\]
and hence $\{Q_\alpha\}_{\alpha \in \hat{A}_{00}}$ is a dyadic non-overlapping $(d, \lambda)$-thick $d$-almost covering of the set $S \cap Q_{00}$.

Furthermore, combining (4.14) and (4.15) with inequality (4.10) in which $k_0$ is replaced by any $k \in \mathbb{N}$ we get
\[
\sum_{\alpha \in \hat{A}_{00}} (l_\alpha)^d = \lim_{k \to \infty} \sum_{\alpha \in \hat{J}^k} (l_\alpha)^d \leq (l(Q))^d
\]
for every dyadic cube $Q \subset Q_{00}$.

Based on Lemma 4.1 we can establish on of the main results of this section.

**Theorem 4.1.** Let $d \in (0, n]$ and $\lambda \in (0, 1)$. Let $S \subset \mathbb{R}^n$ be an arbitrary set with $\mathcal{H}_\infty^d(S) > 0$.
Then, there exists a $(d, \lambda)$-nice for $S$ sequence $\{\hat{A}^s\} \in \mathbb{N}_0 := \{\hat{A}^s(S, d, \lambda)\}_{s \in \mathbb{N}_0}$ of index sets.

**Proof.** We built the desirable sequence by induction. Clearly, the family $\{Q^0_{0}\}_{\alpha \in \hat{A}^0}$ is comprised by all dyadic cubes $Q_{0,m}$, $m \in \mathbb{Z}^n$ for each of which $\mathcal{H}_\infty^d(Q_{0,m} \cap S) > 0$. Fix an arbitrary index $\alpha_0 \in \hat{A}^0$. There are two cases to be considered.

In the first case
\[
\lambda(l_{\alpha_0})^d \leq \mathcal{H}_\infty^d(Q_{00} \cap S) < (l_{\alpha_0})^d.
\]
We apply Lemma 4.1 to the cube $Q_{\alpha_0}$ and obtain an index set $\hat{A}_{\alpha_0}$ satisfying (4.13) and (4.14).

In the second case
\[
\mathcal{H}_\infty^d(Q_{00} \cap S) = (l_{\alpha_0})^d,
\]
i.e. the cube $Q_{\alpha_0}$ is $(d, 1)$-dyadically thick with respect to $S$. Divide $Q_{\alpha_0}$ into $2^n$ cubes and select those of them whose intersection with $S$ is nonempty. Let $J^g_{\alpha_0}$ and $J^b_{\alpha_0}$ be the index sets corresponding to $(d, \lambda)$-dyadically thick with respect to $S$ and $(d, \lambda)$-dyadically thin with respect to $S$ dyadic cubes respectively from those just selected. For each $\tilde{\alpha} \in J^b_{\alpha_0}$ we apply Lemma 4.1. This gives index sets $\hat{A}_{\tilde{\alpha}}$, $\tilde{\alpha} \in J^b_{\alpha_0}$ satisfying items (1) and (2) of Lemma 4.1 in which $\hat{A}_{\alpha_0}$ is replaced by $\hat{A}_{\tilde{\alpha}}$. We set
\[
\hat{A}_{\alpha_0} := \left( \bigcup_{\tilde{\alpha} \in J^b_{\alpha_0}} \hat{A}_{\tilde{\alpha}} \right) \cup J^g_{\alpha_0}.
\]
Hence, using (4.14) with $\tilde{\alpha}$ instead of $\alpha_0$ and taking into account that card $J^g_{\alpha_0} \cup J^b_{\alpha_0} \leq 2^n$ we obtain
\[
\sum_{Q_{0'} \subset Q_{\alpha_0}} (l_{\alpha'})^d \leq \sum_{\tilde{\alpha} \in J^b_{\alpha_0}} \sum_{\alpha' \in \hat{A}_{\tilde{\alpha}}} (l_{\alpha'})^d + \sum_{\tilde{\alpha} \in J^g_{\alpha_0}} (l_{\tilde{\alpha}})^d + \sum_{\tilde{\alpha} \in J^b_{\alpha_0}} (l_{\tilde{\alpha}})^d \leq \sum_{\tilde{\alpha} \in J^b_{\alpha_0}} (l_{\tilde{\alpha}})^d \leq \text{card}(J^g_{\alpha_0} \cup J^b_{\alpha_0}) \left( \frac{l_{\alpha_0}}{2} \right)^d \leq 2^{n-d}(l_{\alpha_0})^d.
\]
We define
\[
\hat{A}^1 := \bigcup_{\alpha \in \hat{A}^0} \hat{A}_\alpha.
\]
From the construction it follows that item (1) of Definition 4.2 holds true, items (3) and (4) of Definition 4.2 are verified for $s = 0$ and, finally, that item (2) is verified for $s = 1$. 

11
Suppose that we have already built for some \( \alpha_0 \in \mathbb{N}_0 \) index sets \( \tilde{A}^0, ..., \tilde{A}^{j_0 + 1} \) such that items (3) and (4) of Definition 4.2 are satisfied for any \( s \in \{0, ..., j_0\} \) and item (2) is satisfied for each \( s \in \{1, ..., j_0 + 1\} \). To make an inductive step it is sufficient to repeat for each \( \alpha \in \tilde{A}^{j_0 + 1} \) line by line all arguments above that were used for the index \( \alpha_0 \in \tilde{A}^0 \). Then we define

\[
\tilde{A}^{j_0 + 2} := \bigcup_{\alpha \in \tilde{A}^{j_0 + 1}} \tilde{A}_\alpha.
\]

This clearly implies that items (3) and (4) of Definition 4.2 are satisfied for each \( s \in \{0, ..., j_0 + 1\} \) and item (2) is satisfied for each \( s \in \{1, ..., j_0 + 2\} \). This completes the inductive step and hence finishes the proof.

**Definition 4.3.** Let \( d \in (0, n] \) and \( \lambda \in (0, 1) \). Let \( S \subset \mathbb{R}^n \) be an arbitrary set with \( \tilde{H}^d_\infty(S) > 0 \). We say that the index set \( A = A(S, d, \lambda) \subset \mathbb{N}_0 \times \mathbb{Z}^n \) is \((d, \lambda)\)-fundamental for \( S \) if \( \{Q_\alpha\}_{\alpha \in A} \) is the family of all \((d, \lambda)\)-dyadically thick with respect to \( S \) dyadic cubes.

As far as we know in the present form the following result has never been formulated in the literature. It gives a nice stratification of \((d, \lambda)\)-fundamental for \( S \) index sets.

**Theorem 4.2.** Let \( d \in (0, n] \) and \( \lambda \in (0, 1) \). Let \( S \subset \mathbb{R}^n \) be an arbitrary set with \( \tilde{H}^d_\infty(S) > 0 \). Let \( A = A(S, d, \lambda) \subset \mathbb{N}_0 \times \mathbb{Z}^n \) be the \((d, \lambda)\)-fundamental for \( S \) index set. Then, there exists a unique sequence of index sets \( \{A^s\}_{s \in \mathbb{N}} = \{A^s(S, d, \lambda)\}_{s \in \mathbb{N}} \) satisfying the following conditions:

1. \( A = \bigcup_{s \in \mathbb{N}} A^s \);
2. for every \( s \in \mathbb{N} \) the family \( C^s := \{Q_\alpha\}_{\alpha \in A^s} \) is a dyadic non-overlapping \((d, \lambda)\)-thick \( d \)-almost covering of the set \( S \);
3. \( \{Q_\alpha\}_{\alpha \in A^s} \prec \{Q_\alpha\}_{\alpha \in A^{s+1}} \) for every \( s \in \mathbb{N} \);
4. Assume that for some \( \alpha \in A^s \), \( \alpha' \in A^{s+1} \) there is a dyadic cube \( Q_{k,m} \) such that

\[
Q_{\alpha'} \subset Q_{k,m} \subset Q\alpha \quad \text{and} \quad 2^{-k} \in \left( l_{\alpha'}^{-1}, l_{\alpha}^{-1}\right),
\]

then the cube \( Q_{k,m} \) is \((d, \lambda)\)-dyadically thin with respect to \( S \).

**Proof.** First of all given an index \( \alpha \in \mathbb{N}_0 \times \mathbb{Z}^n \) with \( \tilde{H}^d_\infty(Q_\alpha \cap S) > 0 \), we denote by \( A_\alpha \) the index set corresponding to all maximal \((d, \lambda)\)-dyadically thick we respect to \( S \) dyadic cubes whose side lengths are strictly less than \( l_\alpha \). Then, the following properties clearly hold true for every \( \alpha \in \mathbb{N}_0 \times \mathbb{Z}^n \):

- (A) \( A_\alpha \subset A \);
- (B) \( \{Q_\alpha\} \prec \{Q_\alpha\}_{\alpha' \in A_\alpha} \);
- (C) the family \( \{Q_\alpha\}_{\alpha' \in A_\alpha} \) is a dyadic non-overlapping \((d, \lambda)\)-thick \( d \)-almost covering of \( Q_\alpha \cap S \).

Properties (A) and (B) are clear by the construction. To establish (C) take a \((d, \lambda)\)-nice for \( S \) sequence \( \{\tilde{A}^s\}_{s \in \mathbb{N}_0} := \{\tilde{A}^s(S, d, \lambda)\}_{s \in \mathbb{N}_0} \) of index sets. Let \( j_0 \in \mathbb{N}_0 \) be the first number for which \( \{Q_\alpha\} \prec \{Q_\alpha\}_{\alpha' \in \tilde{A}^{j_0} \cap Q_\alpha} \). Note that by the construction \( \{Q_\alpha\}_{\alpha' \in A_\alpha} \prec \{Q_\alpha\}_{\alpha' \in \tilde{A}^{j_0} \cap Q_\alpha} \). Since \( \{Q_\alpha\}_{\alpha' \in \tilde{A}^{j_0} \cap Q_\alpha} \) is a dyadic non-overlapping \( d \)-almost covering of the set \( S \cap Q_\alpha \) we can complete the proof of Property (C).

We build the desirable sequence of index sets \( \{A^s\}_{s \in \mathbb{N}_0} \) by induction.

The base of induction. We set

\[
T^0_s := \{\alpha \in \{0\} \times \mathbb{Z}^n : Q_\alpha \text{ is } (d, \lambda)\text{-dyadically thick with respect to } S\},
\]

\[
T^0_b := \{\alpha \in \{0\} \times \mathbb{Z}^n : 0 < \tilde{H}^d_\infty(Q_\alpha \cap S) < \lambda\}.
\]

Now we set

\[
A^1 := T^0_s \bigcup \left( \bigcup_{\alpha \in T^0_b} A_\alpha \right).
\]
It follows immediately from the construction that the family \( \{Q_\alpha\}_{\alpha \in A^1} \) is a dyadic non-overlapping \((d, \lambda)\)-thick \(d\)-almost covering of the set \(S\).

The induction step. Suppose that for some \(j_0 \in \mathbb{N}\) we have already built the index sets \(\{A^s\}_{s=1}^{j_0}\). We set

\[
A^{j_0 + 1} := \bigcup_{\alpha \in A^{j_0}} A_\alpha.
\]

As a result by induction we obtain all index sets \(A^s, s \in \mathbb{N}\).

It follows immediately from the construction that \(A^s \subset A\) and \(\{Q_\alpha\}_{\alpha \in A^s} \prec \{Q_\alpha\}_{\alpha \in A^{j_0 + 1}}\) for all \(s \in \mathbb{N}\). Furthermore, for each \(s \in \mathbb{N}\) the family \(\{Q_\alpha\}_{\alpha \in A^s}\) is a dyadic non-overlapping \((d, \lambda)\)-thick \(d\)-almost covering of \(S\). Suppose now that there exist \(j \in \mathbb{N}, \alpha \in A^j\), \(\alpha' \in A^{j+1}\) and a cube \(Q_{k,m}\) such that

\[
Q_{\alpha'} \subset Q_{k,m} \subset Q_\alpha \quad \text{and} \quad 2^{-k} \in (l_{\alpha'}, l_\alpha).
\]

Note that the cube \(Q_{k,m}\) has to be \((d, \lambda)\)-dyadically thin with respect to \(S\) because otherwise it contradicts the construction of the index set \(A_\alpha\). To complete the proof it is sufficient to note that item (4) just verified in combination with the construction of the index sets \(A^s, s \in \mathbb{N}\) gives (1), i.e.

\[
A = \bigcup_{s \in \mathbb{N}} A^s.
\]

**Definition 4.4.** Let \(d \in (0, n]\) and \(\lambda \in (0, 1)\). Let \(S \subset \mathbb{R}^n\) be a set with \(\widetilde{H}_d^\lambda(S) > 0\). Let \(A = A(S, d, \lambda)\) be the \((d, \lambda)\)-fundamental for \(S\) index set. Let \(\{A^s\}_{s \in \mathbb{N}} = \{A^s(S, d, \lambda)\}_{s \in \mathbb{N}}\) be the sequence of index sets constructed in Theorem 4.2. We call \(\{A^s\}_{s \in \mathbb{N}}\) the canonical decomposition of the index set \(A\).

The following result will be important in proving Theorem 5.2 however we believe that it can be interesting by itself. It reflects combinatorial properties of dyadic \((d, \lambda)\)-dyadically thick with respect to a given set \(S \subset \mathbb{R}^n\) cubes.

**Theorem 4.3.** Let \(d \in (0, n]\) and \(\lambda_1, \lambda_2 \in (0, 1)\). Let \(S \subset \mathbb{R}^n\) be a set with \(\mathcal{H}_d^\lambda(S) > 0\). Let \(A(\lambda_1) := A(S, d, \lambda_1)\) and \(A(\lambda_2) := A(S, d, \lambda_2)\) be the \((d, \lambda_1)\)-fundamental for \(S\) and \((d, \lambda_2)\)-fundamental for \(S\) index sets respectively. Let \(\{A^s(\lambda_1)\}_{s \in \mathbb{N}}\) be the canonical decomposition of the index set \(A(\lambda_1)\). Let \(\{\hat{A}^s\} := \{\hat{A}^s(S, d, \lambda_1)\}_{s \in \mathbb{N}}\) be a \((d, \lambda)\)-nice for \(S\) sequence of index sets. Let \(Q_{a_0}\) be a dyadic cube and \(j_0 \in \mathbb{N}_0\) be the first number for which \(\{Q_\alpha\} \supseteq \{Q_\alpha\}_{\hat{A}^0|_{Q_{a_0}}}\). Assume that \(I \subset A(\lambda_2)\) is an index set satisfying the following conditions:

1. \(\text{int} Q_\alpha \cap \text{int} Q_{\alpha'} = \emptyset\) for any \(\alpha, \alpha' \in I\) with \(\alpha \neq \alpha'\);
2. \(\{Q_\alpha\}_{\alpha \in I} \supseteq \{Q_\alpha\}_{\hat{A}^0|_{Q_{a_0}}}\).

Then,

\[
\sum_{\alpha \in I} (l_\alpha)^d \leq \begin{cases} 
2^{n-d(d_{a_0})^d} & Q_{a_0} \text{ is } (d, 1) \text{- dyadically thick with respect to } S; \\
\frac{(l_{a_0})^d}{\lambda_2^d} & Q_{a_0} \text{ is } (d, 1) \text{- dyadically thin with respect to } S.
\end{cases}
\] (4.18)

**Proof.** In the case \(\{\alpha_0\} = \hat{A}^0_{Q_{a_0}}\) we clearly get \(I = \{\alpha_0\}\) and hence (4.18) trivially holds true.

Suppose now that \(\{Q_{a_0}\} \prec \{Q_\alpha\}_{\hat{A}^0|_{Q_{a_0}}}\). Note that for every \(\alpha \in I\) the family \(\{Q_\alpha\}_{\hat{A}^0|_{Q_{a_0}}}\) is a dyadic non-overlapping \(d\)-almost covering of the set \(S \cap Q_\alpha\). Hence, taking into account \(I \subset A(\lambda_2)\)
and using (4.1) we obtain
\[
\sum_{\alpha \in \mathcal{I}} (l_\alpha)^d \leq \lambda_2^{-1} \sum_{\alpha \in \mathcal{I}} \tilde{H}_\infty^d(Q_\alpha \cap S) \\
\leq \sum_{\alpha \in \mathcal{I}} \lambda_2^{-1} \sum_{\tilde{\alpha} \in \mathcal{A}_0|Q_\alpha} (l_{\tilde{\alpha}})^d \leq \lambda_2^{-1} \sum_{\tilde{\alpha} \in \mathcal{A}_0|Q_{\alpha_0}} (l_{\tilde{\alpha}})^d \leq \frac{c}{\lambda_2^2} (l_{\alpha_0})^d,
\]
(4.19)
where \(c = 1\) in the case when \(Q_{\alpha_0}\) is \((d, 1)\)-dyadically thin with respect to \(S\) or \(c = 2^{n-d}\) in the case when \(Q_{\alpha_0}\) is \((d, 1)\)-dyadically thick with respect to \(S\).

\[ 5 \text{ Main results} \]

In this section we firstly prove Theorems 1.1 and 1.2. After that we prove a result which can be interesting by itself. Furthermore, it will be a keystone in proving Theorem 1.3.

The next proposition follows immediately from the very definition of dyadic cubes. Nevertheless, we present the details.

**Proposition 5.1.** Let \(Q_{k,m}\) be a dyadic cube and let \(j \in \mathbb{N}, j \geq 3\). Then there are \((2^j - 4)^n\) dyadic cubes \(Q_{k+j,m'}\) such that \(5Q_{k+j,m'} \subset Q_{k,m}\).

**Proof.** Let \(Q_{k,m} := \prod_{i=1}^n [m_i/2^k, m_i + 1/2^k]\) for some \(k \in \mathbb{N}_0\) and \(m \in \mathbb{Z}^n\). Note that \(5Q_{k+j,m'} \subset Q_{k,m}\) if and only if \(m'_i \in \{2^j(m_i + 2), ..., 2^j(m_i + 1) - 2\}\) for each \(i \in \{1, ..., n\}\). Hence, the amount off dyadic cubes \(Q_{k+j,m'}\) satisfying the corresponding conditions is exactly \((2^j - 4)^n\).

The following assertion is elementary. Nevertheless, it will be the keystone in proving Theorem 1.1 below.

**Lemma 5.1.** Let \(S \subset \mathbb{R}^n\) be a set. Let \(d \in (0, n)\) and \(\lambda \in (0, 1]\). Let \(Q = Q_{k,m}\) be a \((d, \lambda)\)-dyadically thin with respect to \(S\) dyadic cube. Then, there exists a \((d, 2^{d-n}\lambda)\)-dyadically thin with respect to \(S\) dyadic cube \(Q_{k+1,m'} \subset Q_{k,m}\).

**Proof.** By the assumption of the lemma we have \(\tilde{H}_\infty^d(Q \cap S) < \lambda 2^{-kd}\). Hence, there exists an index set \(\mathcal{I} := \mathcal{I}(Q) \subset \mathbb{N}_0 \times \mathbb{Z}^n\) such that for the family of dyadic cubes \(\{Q_\alpha\}_{\alpha \in \mathcal{I}}\) it holds
\[
\sum_{\alpha \in \mathcal{I}} (l_\alpha)^d < \lambda 2^{-kd}.
\]
(5.1)
Suppose that all dyadic cubes \(Q_{k+1,m'} \subset Q\) are \((d, 2^{d-n}\lambda)\)-dyadically thick with respect to \(S\). Hence, by the definition of the dyadic \(d\)-Hausdorff content and by Proposition 2.1 we have \(l_\alpha \leq 2^{-1-k}\), \(\alpha \in \mathcal{I}\) and furthermore
\[
\sum_{\alpha \in \mathcal{I}} (l_\alpha)^d \geq \tilde{H}_\infty^d(Q_{k+1,m'} \cap S) \geq \lambda 2^{d-n}2^{-(k+1)d}\ 
\text{for every } Q_{k+1,m'} \subset Q.
\]
Using this, we take into account that \( \text{card}\{m' \in \mathbb{Z}^n : Q_{k+1,m'} \subset Q_{k,m}\} = 2^n \) and get a contradiction with (5.1)

\[
\sum_{\alpha \in \mathcal{I}} (l_\alpha)^d = \sum_{Q_{k+1,m'} \subset Q} \sum_{\alpha \in \mathcal{I}} (l_\alpha)^d \geq \sum_{Q_{k,m} \subset Q} \lambda^{2d-n-2(k+1)d} = \lambda^{-kd}.
\]

\[\square\]

**Theorem 5.1.** Let \( S \subset \mathbb{R}^n \) be a closed \((d, \lambda)\)-thick set for some \( d \in (0,n) \) and \( \lambda \in (0,1) \). Then, there exists a number \( \sigma = \sigma(n, d, \lambda) \in (0,1) \) such that every \((d,1)\)-dyadically thin with respect to \( S \) dyadic cube \( Q = Q_{k,m} \) is \( \sigma \)-porous with respect to \( S \).

**Proof.** We fix a dyadic cube \( Q = Q_{k,m} \) and set \( l := l(Q) = 2^{-k} \) for brevity. We split the proof into several steps.

**Step 1.** Using Lemma 5.1 several times we obtain a sequence of nested dyadic cubes

\[
Q_{k,m^0} \supset \ldots \supset Q_{k+s,m^s} \supset \ldots
\]

such that \( Q_{k+s,m^s} \) is \((d, 2^{(d-n)s})\)-dyadically thin with respect to \( S \) for every \( s \in \mathbb{N}_0 \). We set

\[
j := j(n, \lambda) := \left\lfloor \log_2 \left( \frac{15}{\lambda n} + 4 \right) \right\rfloor + 1, \quad s := s(d, n, \lambda) := \left\lfloor \frac{j d}{n - d} \right\rfloor + 1.
\]

We clearly get

\[
\frac{(2^j - 4)n^2}{15} \geq 1, \quad \bar{\lambda} := 2^{s(d-n)} < \frac{1}{2^jd}.
\]

(5.2)

**Step 2.** Let \( \mathcal{E}_j \) be the set of all indices \( m' \in \mathbb{Z}^n \) for each of which \( 5Q_{k+s+j,m'} \subset Q_{k+s,m^s} \) and \( Q_{k+s+j,m'} \cap S \neq \emptyset \). We set \( M_j := \text{card} \mathcal{E}_j \). We also define the set

\[
\mathcal{F}_j := \{ m \in \mathcal{B}^{k+j+s}(d, \lambda/3^n, S) : Q_{k+j+s,m} \subset Q_{k+s,m^s} \}.
\]

We claim that

\[
\text{card} \mathcal{F}_j \geq \frac{M_j}{5^n}.
\]

(5.3)

Indeed, if \( m' \in \mathcal{E}_j \) then, there is a point \( x' \in S \cap Q_{k+s+j,m'} \). Since \( S \) is \((d, \lambda)\)-thick the cube \( Q(x', 2^{-k-s-j}) \) is \((d, \lambda)\)-thick with respect to \( S \). By Proposition 3.1 there is a dyadic cube \( Q_{k+s+j,m''} \) such that \( Q_{k+s+j,m''} \cap Q(x', 2^{-k-s-j}) \neq \emptyset \) and \( Q_{k+s+j,m''} \) is \((d, \lambda/3^n)\)-dyadically thick with respect to \( S \). It is easy to see that \( Q_{k+s+j,m''} \subset 5Q_{k+s+j,m'} \). Since \( m' \in \mathcal{E}_j \) this implies that \( m'' \in \mathcal{F}_j \). Hence, we obtain a map \( \mathcal{G} : \mathcal{E}_j \rightarrow \mathcal{F}_j \). On the other hand, \( Q_{k+s+j,m''} \subset 5Q_{k+s+j,m'} \). This gives

\[
\text{card} \mathcal{G}^{-1}(m'') \leq 5^n \quad \text{for every } \quad m'' \in \mathcal{F}_j.
\]

This inequality immediately gives (5.3).

**Step 3.** Consider two cases. In the first case \( M_j < (2^j - 4)^n \). Hence, by Proposition 5.1 there exists a cube \( Q_{k+s+j,m'} \) whose intersection with \( S \) is empty. Hence, taking \( \sigma = 2^{-j-s} \) we complete this case.

In the second case \( M_j = (2^j - 4)^n \). On the one hand, \( Q_{k+s,m^s} \) is \((d, \bar{\lambda})\)-dyadically thin with respect to \( S \) by the construction. By Proposition 2.1 we have \( l_\alpha \leq 2^{-k-s-j-1} \). Hence, using (5.2) and (5.3) we obtain

\[
\sum_{\alpha \in \mathcal{I}} (l_\alpha)^d \geq \sum_{m' \in \mathcal{F}_j} \sum_{\alpha \in \mathcal{I}} (l_\alpha)^d \geq \frac{M_j}{5^n} \lambda \left( \frac{l}{2^{j+s}} \right)^d > \bar{\lambda} \left( \frac{l}{2^s} \right)^d.
\]

(5.4)
This contradiction completes the consideration of the second case with \( \sigma = 2^{-j-s} \).

As a result, the theorem is proved with \( \sigma(n, d, \lambda) = 2^{-j(n, \lambda) - s(n, d, \lambda)} \).

**Proposition 5.2.** Let \( S \subseteq \mathbb{R}^n \) be a set. Let \( d \in (0, n], \lambda, \tau \in (0, 1] \) and let \( Q = Q(x, l) \) be a \((\tau, d, \lambda, \frac{\lambda}{2^{d+1}n})\)-sparse with respect to \( S \) cube. Then, there is a \((d, \lambda)\)-dyadically thin with respect to \( S \) dyadic cube \( Q_{k,m} \) with \( 2^{-k} \geq \frac{\tau l}{\lambda} \).

**Proof.** By definition there is a cube \( Q' \subseteq Q \) with \( l(Q') = \tau l(Q) \) which is \((d, \frac{\lambda}{2^{d+1}n})\)-thin with respect to \( S \). By Remark 2.1 the cube \( Q \) is \((d, \lambda^2 - 2d)\)-dyadically thin with respect to \( S \). We set \( k := k(l, \tau) := \lceil \log_2(\tau l) \rceil + 1 \). Hence, application of Proposition 4.1 with \( j = 1 \) gives existence of a \((d, \lambda)\)-dyadically thin with respect to \( S \) dyadic cube \( Q_{k,m} \subseteq Q' \).

Now we are ready to prove one of the main results.

**Proof of Theorem 1.1.** The first claim follows immediately from the corresponding definitions.

To prove the second claim we fix an arbitrary \((\tau, d, 2^{-n-2d})\)-sparse with respect to \( S \) cube \( Q \) with \( l(Q) \in (0, 1] \). Using Proposition 5.2 we find a \((d, 1)\)-dyadically thin with respect to \( S \) dyadic cube \( Q_{k,m} \subseteq Q \) with \( 2^{-k} \geq \frac{\tau l}{\lambda} \). Application of Theorem 5.1 gives the claim with \( c = \frac{\sigma(n, d, \lambda)}{4} \).

Recall the corresponding definition from the introduction.

**Definition 5.1.** Given a set \( S \subseteq \mathbb{R}^n \), a cube \( Q \) and a parameter \( \gamma \in (0, 1] \), we say that a set \( U \subseteq Q \) is a \( \gamma \)-cavity of the cube \( Q \) with respect to the set \( S \) if

\[
U \subseteq Q \setminus S \quad \text{and} \quad \mathcal{H}^n(U \setminus S) \geq \gamma(l(Q))^n.
\]

We say that \( Q \) is \( \gamma \)-hollow we the respect to the set \( S \) if there exists a \( \gamma \)-cavity \( U \subseteq Q \) with respect to \( S \).

The second main result is extremely simple. However as we will see in section 6 we can not hope to obtain more than this.

**Proof of Theorem 1.2.** We fix an arbitrary \((\tau, d, \lambda)\)-sparse with respect to \( S \) cube \( Q = Q(x, l) \) with \( l \in (0, 1] \). By Definition 5.2 there is a cube \( Q' \subseteq Q \) with \( l(Q') = \tau l \) and \( \mathcal{H}^n_\infty < \lambda^n \). By Definition 2.1 there is an at most countable covering \( \{U_\beta\}_{\beta \in \mathcal{J}} \) of the set \( Q' \cap S \) such that

\[
\sum_{\beta \in \mathcal{J}} (\text{diam} U_\beta)^d < \lambda l^d.
\]

Clearly \( l_\beta < \lambda^{\frac{1}{n}} \tau l \) for all \( \beta \in \mathcal{J} \). Hence, we get

\[
\sum_{\beta \in \mathcal{J}} (l_\beta)^n < \lambda^{\frac{n}{d} - 1} \tau^{n-d} \sum_{\alpha \in \mathcal{I}} (l_\alpha)^d \leq \lambda^{\frac{n}{d}} (\tau l)^n.
\]

Since \( \lambda < 1 \) this proves that the cube \( Q \) is \((1 - \lambda^{\frac{n}{d}})\tau\)-hollow with respect to the set \( S \).

The theorem is proved. \( \square \)

To prove Theorem 1.3 we need much more delicate tools. The following result will be the keystone for that. It is interesting by itself. Furthermore, we believe that it can be useful in the future investigations.

**Theorem 5.2.** Let \( d \in (0, n], \lambda \in (0, 1) \) and \( \rho \geq 1 \). Let \( S \subseteq \mathbb{R}^n \) be a Borel set with \( \mathcal{H}^n_\infty(S) > 0 \) and let \( \mathcal{A} := \mathcal{A}(S, d, \lambda) \) be the corresponding \((d, \lambda)\)-fundamental for \( S \) index set. Assume that a dyadic cube \( Q = Q_{k,m} \) is \((d, 1)\)-dyadically thin with respect to \( S \). For each \( \zeta \in \mathbb{N}_0 \) we define

\[
\tilde{U}_\zeta := Q \setminus \bigcup_{\alpha \in \mathcal{A}} \rho Q_{\alpha^\zeta}.
\]

16
Then, for every $\gamma \in (0, 1 - 2^{d-n})$ there exist $\kappa = \kappa(\gamma, n, d, \rho) \in \mathbb{N}$ such that for any $\kappa > \kappa$ the set

$$U_\kappa := (Q \setminus S) \cap \tilde{U}_\kappa$$

is a $\gamma$-cavity of the cube $Q$ with respect to $S$.

Proof. During the proof we use the shorthand $B^j := B^j(S, d, \lambda)$ for each $j \in \mathbb{N}_0$. Let $\{A^s\}_{s \in \mathbb{N}} := \{A^s(S, d, \lambda)\}_{s \in \mathbb{N}}$ be the canonical decomposition of the index set $A$. We also fix a $(d, \lambda)$-nice for $S$ sequence of index sets $\{\hat{A}^s\}_{s \in \mathbb{N}_0} = \{\hat{A}^s(S, d, \lambda)\}_{s \in \mathbb{N}_0}$. From item (2) of Theorem 4.2 it follows that

$$H^d_\infty(S \setminus \tilde{U}_\kappa) = 0$$

for any $\kappa \in \mathbb{N}_0$.

Hence, given a parameter $\gamma \in (0, 1 - 2^{d-n})$ (which we assume to be fixed during the proof) it is sufficient to establish existence of $\kappa = \kappa(\gamma, n, d, \lambda, \rho) \in \mathbb{N}$ such that

$$H^n(\tilde{U}_\kappa) \geq \tau H^n(Q)$$

for any $\kappa > \kappa$. (5.6)

We split the proof into several steps.

Step 1. Taking into account Proposition 2.1 we can take the first number $s_0 \in \mathbb{N}_0$ for which

$$\{Q\} \prec \{Q^s\}_{s = s_0+1}$$

(5.7)

Fix an arbitrary $\theta \in (0, 1)$ so small that

$$(1 + \theta)^n 2^{d-n} < 1.$$  

(5.8)

Note that if dyadic cubes $Q_{j_1,m_1}$ and $Q_{j_2,m_2}$ with $j_1, j_2 \in \mathbb{N}_0$ and $m_1, m_2 \in \mathbb{Z}^n$ are such that:

(i) $Q_{j_1,m_1} \subset Q_{j_2,m_2}$,
(ii) $j_1 - j_2 > \log_2(\frac{\rho}{1 + \theta})$,
then we have the inclusion

$$\rho Q_{j_1,m_1} \subset (1 + \theta)Q_{j_2,m_2}.$$  

(5.9)

We set

$$\kappa^* := \kappa^*(\rho, \theta) := \left\lfloor \log_2(\frac{\rho}{1 + \theta}) \right\rfloor + 2,$$

fix $\kappa > 2\kappa^*$ big enough (to be specified later) and $\kappa > \kappa$. We split the family

$$\mathcal{I} := \{\alpha \in A : \rho Q_\alpha \cap Q \neq \emptyset \text{ and } l_\alpha \leq 2^{-\kappa l(Q)}\}$$

into three subfamilies, more precisely we set

$$\mathcal{I}_1 := \{\alpha \in \mathcal{I} : \text{int } Q_\alpha \subset \mathbb{R}^n \setminus Q\},$$
$$\mathcal{I}_2 := \{\alpha \in \mathcal{I} \setminus \mathcal{I}_1 : \text{there exists } \alpha' \in \hat{A}^{\kappa^*+1} \text{ s.t. } Q_{\alpha'} \subset Q_\alpha\},$$
$$\mathcal{I}_3 := \{\alpha \in \mathcal{I} \setminus \mathcal{I}_1 : \text{there exists } \alpha' \in \hat{A}^{\kappa^*+1} \text{ s.t. } Q_\alpha \subset Q_{\alpha'} \text{ and } l_\alpha < l_{\alpha'}\}.$$  

It follows directly from the construction that

$$\mathcal{I} \subset \mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_3.$$  

(5.10)

Step 2. By the construction $2\kappa > 2\rho$. Hence, it is clear that

$$\bigcup_{\alpha \in \mathcal{I}_1} (\rho Q_\alpha \cap Q) \subset Q \setminus \left(1 - \frac{2\rho}{2\kappa}\right)Q.$$
Hence,
\[ H^n \left( \bigcup_{\alpha \in J_2} (\rho Q_\alpha \cap Q) \right) \leq (l(Q))^n \left( 1 - \left( \frac{2\rho}{2^\varepsilon} \right)^n \right). \]  \hfill (5.11)

**Step 3.** By definition of the index set \( I_2 \) we have that if \( \text{int} Q_\alpha \cap \text{int} Q_{\alpha'} \neq \emptyset \) for some \( \alpha \in I_2 \) and \( \alpha' \in \hat{A}^{s_0+1} \) then we necessary have \( Q_{\alpha'} \subset Q_\alpha \). Let \( J_2 \subset I_2 \) be the index set labeling maximal dyadic cubes from the family \( \{Q_\alpha\}_{\alpha \in I_2} \). We obviously have
\[ \bigcup_{\alpha \in I_2} Q_\alpha \subset \bigcup_{\alpha \in J_2} Q_\alpha. \]

Since \( Q \) is \((d,1)\)-dyadically thin with respect to \( S \), application of Theorem 4.3 gives
\[ H^n \left( \bigcup_{\alpha \in I_2} \rho Q_\alpha \right) \leq H^n \left( \bigcup_{\alpha \in J_2} cQ_\alpha \right) \leq \sum_{\alpha \in J_2} \rho^n (l(Q))^{n-d} \sum_{\alpha \in J_2} l_\alpha^d \leq \frac{\rho^n}{2^{\varepsilon(n-d)\lambda}} (l(Q))^n. \]  \hfill (5.12)

**Step 4.** Without loss of generality we may assume that \( \varepsilon \) is even. By the construction there is an index set \( J_3 \subset \hat{A}^{s_0+1} \) such that for every \( \alpha \in J_3 \) there is a unique \( \alpha' \in J_3 \) such that \( Q_\alpha \subset Q_{\alpha'} \subset Q \). We split the index set \( J_3 \) into two parts, more precisely we set
\[ J_3^1 := \{ \alpha' \in J_3 : l_{\alpha'} \geq 2^{-\frac{\varepsilon}{2}} l(Q) \}, \quad J_3^2 := \{ \alpha' \in J_3 : l_{\alpha'} < 2^{-\frac{\varepsilon}{2}} l(Q) \}. \]

The key observation is that since \( \varepsilon > 2\varepsilon^* \) we have by (5.9) that
\[ Q_\alpha \subset Q_{\alpha'} \quad \text{implies} \quad \rho Q_\alpha \subset (1 + \theta)Q_{\alpha'} \quad \text{for every} \quad \alpha \in J_3 \quad \text{and} \quad \alpha' \in J_3^1. \]

Hence, we get
\[ \bigcup_{\alpha \in I_3} \rho Q_\alpha \subset \left( \bigcup_{\alpha' \in J_3^1} (1 + \theta)Q_{\alpha'} \right) \bigcup \left( \bigcup_{\alpha' \in J_3^2} \rho Q_{\alpha'} \right). \]  \hfill (5.13)

Recall again that the cube \( Q \) is \((d,1)\)-dyadically thin with respect to \( S \). Hence, arguing as in (5.12) and using (5.11) we obtain
\[ H^n \left( \bigcup_{\alpha' \in J_3^2} \rho Q_{\alpha'} \right) \leq \sum_{\alpha' \in J_3^2} \rho^n (l_{\alpha'})^{n-d} (l_{\alpha'})^d \leq \rho^n \frac{(l(Q))^{n-d}}{2^{(n-d)\lambda}} (l(Q))^n. \]  \hfill (5.14)

On the other hand, by (5.7) we have \( l_{\alpha'} \leq l(Q)/2 \) for every \( \alpha' \in J_3 \). Hence, taking into account that \( Q \) is \((d,1)\)-dyadically thin with respect to \( S \) we have by the second inequality in (5.11)
\[ H^n \left( \bigcup_{\alpha' \in J_3^1} (1 + \theta)Q_{\alpha'} \right) \leq (1 + \theta)^n \sum_{\alpha' \in J_3^1} (l_{\alpha'})^n \leq (1 + \theta)^n 2^{d-n} (l(Q))^n. \]  \hfill (5.15)
Step 5. Collecting (5.10)–(5.15) we deduce that
\[ \mathcal{H}^n(Q \setminus \tilde{U}_\kappa) \leq (1 - \gamma)(l(Q))^n, \]
\[ \gamma(\theta, \kappa) := \left(1 - \frac{2\rho}{2^n}\right)^n - (1 + \theta)^n 2^{d-n} - \frac{\rho^n}{2^{(n-d)\kappa}} - \frac{\rho^n}{2^{(n-d)\kappa}}. \quad (5.16) \]

In order to finish the proof we recall (5.8) and note that
\[ \lim_{\theta \to 0} \lim_{\kappa \to \infty} \gamma(\theta, \kappa) = 1 - 2^{d-n}. \]

Hence, taking $\theta > 0$ small enough and then taking $\kappa$ big enough we get $\gamma(\theta, \kappa) > \gamma$ for all $\kappa > \kappa$.

\[ \square \]

Proof of theorem 1.3. First of all we note that if $Q'$ is a $(d, \lambda)$-thick with respect to $S$ cube, then, by Proposition 3.1 there is a $(d, \lambda/3^n)$-dyadically thick with respect to $S$ dyadic cube $Q_{k,m} \cap Q' \neq \emptyset$ with $k = \lfloor \log_2 l(Q') \rfloor$. Clearly, $\rho Q' \subset 3\rho Q_{k,m}$. By Proposition 5.2 there is a $(d, 1)$-dyadically thin with respect to $S$ dyadic cube $Q_{j,m} \subset Q$ with $2^{-j} \geq \frac{\kappa}{4} l(Q)$. Hence, we apply Theorem 5.2 to the cube $Q_{j,m}$ with $3\rho$ instead of $\rho$ and $\lambda/3^n$ instead of $\lambda$. This gives the desirable claim with the parameters
\[ c(n, d) = 4^{-n} \frac{(1 - 2^{d-n})}{2}, \quad \delta(n, d, \lambda, \rho) := \exp\left(-\kappa\left(1 - \frac{2^{d-n}}{2}, n, d, \frac{\lambda}{3^n}, 3\rho\right) \ln 2\right). \]

\[ \square \]

Remark 5.1. We would like to draw the reader attention again to the fact that since for any $\lambda \in (0, 1)$ the family of all $(d, \lambda)$-(dyadically) thick with respect to a given set $S \subset \mathbb{R}^n$ cubes covers the set $S$ up to a negligible set $S'$ of $\mathcal{H}^d$-measure zero, Theorem 5.2 and Theorem 1.3 are more deep than Theorem 1.2. Furthermore, we believe that the proof of Theorem 5.2 can be useful in the further investigations.

\[ \square \]

6 Examples

We present elementary examples demonstrating the sharpness of Theorems 1.1, 1.3 and 5.2.

The following example shows that the $d$-thick condition in the statement of Theorem 1.1 can not be dropped.

Example 6.1. Let $K \subset [0, 1]$ be the standard middle-third Cantor set. For each $s \in \mathbb{N}$ we define $K_s := \{2^{-s}x : x \in K\}$. We set
\[ F_s := \bigcup_{i=0}^{2^s-1} \left(\frac{i}{2^s} + K_{2^s}\right). \]

Obviously, given $s \in \mathbb{N}$, the maximal size of closed 1-dimensional cubes $Q \subset [0, 1] \setminus F_s$ is less than $2^{-s}$. If $d = \log 2 / \log 3$ we clearly have
\[ \mathcal{H}^d(F_s) \leq \frac{2^s}{2^{ds}} \rightarrow 0, \quad s \rightarrow \infty. \quad (6.1) \]

Finally, we define
\[ F := \bigcup_{s=0}^{\infty} \left((1 - \frac{1}{2^s}) + \frac{1}{2^{s+1}} F_{s+1}\right). \]
It follows from (6.1) that the set $F$ is not $(d, \lambda)$-thick for any $\lambda \in (0, 1)$. On the other hand, any dyadic interval $[1 - \frac{2}{2^s}, 1 - \frac{1}{2^s}]$, $s \in \mathbb{N}$ can be $\tau$-porous only with $\tau < 2^{-s}$. As a result, using (6.1) again it is easy to see that for any $c, \tau \in (0, 1)$ there is a one-dimensional cube $Q' \subset [0, 1]$ such that $Q'$ is $(\tau, \frac{\ln 2}{\ln 3}, 1/8)$-sparse with respect to $F$ but $Q$ is not $\tau c$-porous with respect to $F$. □

The following example exhibits the sharpness of Theorem 5.2.

**Example 6.2.** If $d = n$ the first claim of Theorem 1.2 is no longer true. Indeed, consider a dyadic cube $Q_{k,m} = Q(x, 2^{-k})$. Consider a cube $Q(x, l) := S$ with $l \in (0, 2^{-k})$. Clearly, the cube $Q_{k,m}$ is $(n, 1)$-dyadically thin with respect to $S$ independently on the choice of $l$. At the same time $H^n(Q \setminus Q(x, l)) \to 0$, $l \to 2^{-k}$. □

Now we show that the restriction $d < n$ is essential in Theorem 1.3.

**Example 6.3.** For every $s \in \mathbb{N}$ we define

$$G_s := \bigcup_{i=0}^{2^s-1} \left[ \frac{i}{2^s}, \frac{i+1}{2^s} + \frac{1}{10} \frac{1}{2^s} \right].$$

Finally, we define

$$G := \bigcup_{s=0}^{\infty} \left( (1 - \frac{1}{2^s}) + \frac{1}{2s+1} G_{s+1} \right).$$

It is easy to see that for any $\kappa \in \mathbb{N}$ the cube $[1 - 2^{-\kappa+1}, 1 - 2^{-\kappa}]$ is $1/8$-sparse with respect to the set $G$. But

$$[1 - 2^{-\kappa+1}, 1 - 2^{-\kappa}] \setminus \bigcup_{\substack{l(Q') \leq 2^{-2\kappa} \\text{for} \ Q' \in G}} Q' = \emptyset.$$  (6.3)

□

**References**

[1] A. Jonsson, H. Wallin, Function Spaces on Subsets of $\mathbb{R}^n$, Harwood Acad. Publ., London, 1984, Mathematical Reports, Volume 2, Part 1.

[2] K. J. Falconer. *The geometry of fractal sets*, volume 85 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 1986.

[3] H. Triebel, *The Structure of Functions*, Birkhauser, Basel, 2001.

[4] L. Ilnatsyeva and A. V. Vähäkangas, Characterization of traces of smooth functions on Ahlfors regular sets, J. Funct. Anal. 265 (2013), No 9, pp 1870–1915.

[5] V. S. Rychkov, Linear extension operators for restrictions of function spaces to irregular open sets, Studia. Math. 140, 141–162 (2000).

[6] P. Shvartsman, Sobolev $W^1_p$-spaces on closed subsets of $\mathbb{R}^n$, Advances in Math. 220, No. 6 (2009), 1842–1922.

[7] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, NJ, 1970.

[8] A. Tyulenev Restrictions of Sobolev $W^1_p(\mathbb{R}^2)$-spaces to planar rectifiable curves, arXiv preprint arXiv:2010.05286.
[9] A. Tyulenev and S. Vodop’yanov, Sobolev $W^1_p$-spaces on $d$-thick closed subsets of $\mathbb{R}^n$, Sb. Math. 211:6 (2020), 786–837.

[10] A. Tyulenev and S. Vodop’yanov, On the Whitney problem for weighted Sobolev spaces, Doklady Mathematics 95 (1), 79–83.

[11] H. Triebel, The Structure of Functions, Birkhauser, Basel, 2001.

[12] J. Väisälä, Porous sets and quasisymmetric maps, Trans. Amer. Math. Soc. 299 (1987), 525–533.

[13] P. Koskela and S. Rohde, Hausdorff Dimension and mean porosity, Math. Ann. 309 (1997), 593–609.

[14] E. P. Dolzenko, Boundary properties of arbitrary functions, Izv. Akad. Nauk. SSSR Ser. Mat. 31 (1967), 3–14. (Russian)

[15] S. Bechtel and M. Egert. Interpolation theory for Sobolev functions with partially vanishing trace on irregular open sets. J. Fourier Anal. Appl. 25 (2019), no. 5, 2733–2781.

[16] A. Jonsson, Atomic decomposition of Besov spaces on closed sets, in: Function Spaces, Differential Operators and Non-Linear Analysis, in: Teubner-Texte Math., vol. 133, Teubner, Leipzig, 1993, pp. 285–289.

[17] J. Luukkainen, Assouad dimension: antifractal metrization, porous sets, and homogeneous measures, J. Korean Math. Soc. 35 (1998) 23–76.

[18] Yu. V. Netrusov, Spectral synthesis in the Sobolev space associated with integral metric, Investigations on linear operators and function theory. Part 22, Zap. Nauchn. Sem. POMI, 217, POMI, St. Petersburg, 1994, 92–111; J. Math. Sci. (New York), 85:2 (1997), 1814–1826.

[19] P. Mattila, Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability, vol. 44. Cambridge University Press, Cambridge (1999)

[20] D. Lučić, T. Rajala, J. Takanen, Dimension estimates for the boundary of planar Sobolev extension domains, preprint (2020), [arXiv:2006.14213].

[21] P. Shmerkin, Porosity, dimension, and local entropies: a survey, preprint (2011), [arXiv:1110.5682].

[22] A. Salli, On the Minkowski dimension of strongly porous fractal sets in $\mathbb{R}^n$. Proc. London Math. Soc. (3), 62(2):353–372, 1991.

[23] T. Nieminen, Generalized mean porosity and dimension. Ann. Acad. Sci. Fenn. Math., 31(1):143–172, 2006.