Random Average Sampling and Reconstruction in Shift-Invariant Subspaces of Mixed Lebesgue Spaces

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Abstract. In this paper, the problem of reconstruction of signals in mixed Lebesgue spaces from their random average samples has been studied. Probabilistic sampling inequalities for certain subsets of shift-invariant spaces have been derived. It is shown that the probabilities increase to one when the sample size increases. Further, explicit reconstruction formulae for signals in these subsets have been obtained for which the numerical simulations have also been performed.

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1. Introduction

In information theory, it is highly desirable to gain maximum information about a function (or signal) using the least available data. Image processing, data analysis, computer tomography, bio-engineering and artificial intelligence are a few fields which often deal with sampling and reconstruction problems. In 1949, the celebrated Shannon sampling theorem was proved which turned out to be a milestone in this field of study and set the foundation for information theory. Over these years, the theory of sampling has been intensively studied. For a detailed survey on the theory, we refer to Butzer and Stens [7]. Further, the theory has been developed for shift-invariant spaces with a single generator as well as with multiple generators in various contexts such as regular, irregular, average and multi-channel sampling by several authors.
The average sampling is a useful and a practical model of sampling when the physical devices fail to measure the exact value of a signal at a given time. A suitable mathematical model for the measurement process of the sample \( f(x) \) at the location \( x \) is given by considering the local average of the form 
\[
(f \ast \psi)(x) = \int_{K} f(x - y) \psi(y) dy,
\]
where \( \psi \) reflects the properties of the aperture device used for sampling. In such models, it is enough to consider the values near the sampling locations for the average samples. Hence, the averaging function \( \psi \) is mostly assumed to be compactly supported. Sun and Zhou gave the reconstruction formulae from local convoluted samples for band-limited functions in [19, 20], for spline subspaces in [21, 22] and for shift-invariant subspaces with symmetric averaging functions in [23]. The average sampling and reconstruction algorithms for shift-invariant subspaces were also studied in [1, 8, 13]. In [18], the average sampling has been analyzed for a reproducing kernel subspace of \( L^p \)-spaces. Li et al. gave the reconstruction formula for the functions in local shift-invariant subspaces of \( L^2(\mathbb{R}^d) \) from average random samples in [16].

Random sampling is another type of sampling which has been used in practical applications such as learning theory, image filtering and compressed sensing. In [2], Bass and Gröchenig discussed random sampling for multivariate trigonometric polynomials. They obtained the probabilistic sampling inequality for band-limited functions on \( \mathbb{R}^d \) in [3, 4]. Random sampling in shift-invariant spaces was studied in [9, 25, 27]. Yang and Tao in [26] studied random sampling and gave an approximation model for signals having bounded derivatives. Random sampling for reproducing kernel subspaces was analyzed in [11, 15, 17]. The random sampling theory was also extended to mixed Lebesgue spaces which are generalized versions of the Lebesgue spaces. In this context, the random sampling for reproducing kernel subspaces was studied in [10] and that for multiply generated shift-invariant subspaces was analyzed in [12]. For a detailed study of the mixed Lebesgue spaces, we refer to [5].

This paper deals with the study of random average sampling for functions in mixed Lebesgue spaces using a probabilistic approach. For \( 1 \leq p, q < \infty \), let \( L^{p,q}(\mathbb{R} \times \mathbb{R}^d) \) denote the mixed Lebesgue space, which consists of complex valued measurable functions on \( \mathbb{R} \times \mathbb{R}^d \) such that
\[
\left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}^d} |f(x,y)|^q dy \right)^{p/q} dx \right)^{1/p} < \infty
\]
and let \( L^{\infty,\infty}(\mathbb{R} \times \mathbb{R}^d) \) denote the set of all complex valued measurable functions on \( \mathbb{R}^{d+1} \) such that \( \|f\|_{L^{\infty,\infty}(\mathbb{R} \times \mathbb{R}^d)} := \text{ess sup} |f| < \infty \).

Similarly, for \( 1 \leq p, q < \infty \), \( l^{p,q}(\mathbb{Z} \times \mathbb{Z}^d) \) denotes the space of all complex sequences \( c = \{c(k_1, k_2)\}_{(k_1, k_2) \in \mathbb{Z} \times \mathbb{Z}^d} \) such that
\[ \|c\|_{l^p,q} := \left( \sum_{k_1 \in \mathbb{Z}} \left( \sum_{k_2 \in \mathbb{Z}^d} |c(k_1, k_2)|^q \right)^{p/q} \right)^{1/p} < \infty \]

and \( l^\infty,\infty(\mathbb{Z} \times \mathbb{Z}^d) \) denotes the space of all complex sequences on \( \mathbb{Z}^{d+1} \) such that \( \|c\|_{l^\infty,\infty} := \sup_{k \in \mathbb{Z}^{d+1}} |c(k)| < \infty \). We observe that \( L^{p,p}(\mathbb{R} \times \mathbb{R}^d) = L^p(\mathbb{R}^{d+1}) \) and \( l^{p,p}(\mathbb{Z} \times \mathbb{Z}^d) = l^p(\mathbb{Z}^{d+1}) \) for \( 1 < p < \infty \).

We consider a multiply generated shift-invariant subspace of the mixed Lebesgue space \( L^{p,q}(\mathbb{R} \times \mathbb{R}^d), 1 < p, q < \infty \), of the form

\[ V^{p,q}(\Phi) = \left\{ \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}^d} c^T(k_1, k_2) \Phi(\cdot - k_1, \cdot - k_2) : c \in (l^{p,q}(\mathbb{Z} \times \mathbb{Z}^d))^r \right\}, \]

where \( \Phi = (\phi_1, \phi_2, \ldots, \phi_r)^T \) with \( \phi_i \in L^{p,q}(\mathbb{R} \times \mathbb{R}^d) \) and \( c = (c_1, c_2, \ldots, c_r)^T \).

We prove sampling inequalities for certain subsets of \( V^{p,q}(\Phi) \) and estimate the probabilities with which they hold. Our results show that the probability tends to one as the number of samples increases. Further, using these sampling inequalities, we derive explicit reconstruction formulae. We also simulate the reconstruction numerically for certain examples and obtain the error estimates.

### 2. Preliminaries

The subsets of \( V^{p,q}(\Phi) \) that are needed for our analysis are defined as follows: Let \( C_K \) denote the compact set \([-K_1, K_1] \times [-K_2, K_2]^d \subseteq \mathbb{R} \times \mathbb{R}^d \) for some positive constants \( K_1 \) and \( K_2 \). For a positive integer \( N \), \( V^{p,q}_N(\Phi) \) denotes the finite dimensional subspace of \( V^{p,q}(\Phi) \) given by

\[ V^{p,q}_N(\Phi) := \left\{ \sum_{i=1}^r \sum_{|k_1| \leq N} \sum_{|k_2| \leq N} c_i(k_1, k_2) \phi_i(\cdot - k_1, \cdot - k_2) : c_i \in L^{p,q}([-N, N] \times [-N, N]^d) \right\}, \]

where for a multi-index \( k = (k_1, k_2, \ldots, k_d) \in \mathbb{Z}^d, |k| := \max_{1 \leq i \leq d} |k_i|. \) The unit ball of the above space is

\[ V^{p,q,*}_N(\Phi) := \left\{ g \in V^{p,q}_N(\Phi) : \|g\|_{L^{p,q}(\mathbb{R} \times \mathbb{R}^d)} = 1 \right\}. \]

For \( \omega > 0 \) and \( \psi \in L^1(\mathbb{R}^{d+1}) \), the subset \( V^{p,q}_{N,\omega,\psi}(\Phi) \) is defined as

\[ V^{p,q}_{N,\omega,\psi}(\Phi) := \left\{ f \in V^{p,q}_N(\Phi) : \|f \ast \psi\|_{L^{p,q}(C_K)} \geq \omega \right\}. \]
and its unit ball is denoted by $V_{N,\omega,\psi}^{p,q}(\Phi)$. Further, for $0 < \mu \leq 1$, we define
\[
V_{N,\psi}^{p,q}(\Phi, \mu, C_K) := \left\{ f \in V_{N}^{p,q}(\Phi) : \mu \|\psi\|_{L^1(\mathbb{R}^{d+1})} \|f\|_{L^p,q(\mathbb{R} \times \mathbb{R}^d)} \leq \int_{C_K} \|(f * \psi)(x,y)\|dxdy \right\}. \tag{2.1}
\]

Also, for $0 < \delta < 1$, we define the sets $V_{\psi}^{p,q}(\Phi, \delta, C_K)$ and $V_{\psi}^{p,q}(\Phi, \delta, C_K)$ as
\[
V_{\psi}^{p,q}(\Phi, \delta, C_K) := \left\{ f \in V_{\psi}^{p,q}(\Phi) : \|f\|_{L^p,q(C_K)} \geq (1 - \delta) \|f\|_{L^p,q(\mathbb{R} \times \mathbb{R}^d)} \right\}
\]
and
\[
V_{\psi}^{p,q}(\Phi, \delta, C_K) := \left\{ f \in V_{\psi}^{p,q}(\Phi, \delta, C_K) : \|f \ast \psi\|_{L^{p,q}(C_K)} \geq (1 - \delta) \|\psi\|_{L^{1,1}(C_K)} \|f\|_{L^{p,q}(\mathbb{R} \times \mathbb{R}^d)} \right\}
\]
respectively and $V_{\psi}^{p,q,*}(\Phi, \delta, C_K)$ as the unit ball of $V_{\psi}^{p,q}(\Phi, \delta, C_K)$. In fact, $V_{\psi}^{p,q}(\Phi, \delta, C_K)$ consists of the signals in $V_{\psi}^{p,q}(\Phi)$ whose energy is concentrated on the set $C_K$.

We make the following assumptions for our study:

(A1) The generators $\phi_1, \phi_2, \ldots, \phi_r$ have stable shifts, i.e., there exist positive constants $\alpha_1$ and $\alpha_2$ such that
\[
\alpha_1 \|c \|_{L^{p,q}} \leq \left\| \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}^d} c^T(k_1, k_2) \Phi(-k_1, -k_2) \right\|_{L^{p,q}(\mathbb{R} \times \mathbb{R}^d)} \leq \alpha_2 \|c\|_{L^{p,q}}, \tag{2.2}
\]
for $c \in L^{p,q}(\mathbb{Z} \times \mathbb{Z}^d)^r$ and $\|c\|_{L^{p,q}} = \sum_{i=1}^r \|c_i\|_{L^{p,q}} < \infty$.

Also $\phi_i$’s are continuous with polynomial decay, i.e.,
\[
|\phi_i(x, y)| \leq \frac{\tilde{c}}{(1 + |x|)^{s_1}(1 + |y|)^{s_2}}, \quad (x, y) \in (\mathbb{R} \times \mathbb{R}^d), \tag{2.3}
\]
where $\tilde{c}$ is a positive constant and $s_1, s_2$ are positive constants satisfying $s_1, s_2 > d + 1 - \frac{1}{p} - \frac{d}{q}$.

(A2) The averaging function $\psi \in L^{1}(\mathbb{R}^{d+1})$ with supp$(\psi) \subseteq C_K$.

(A3) There exists a probability density function $\rho$ defined over $C_K$ such that
\[
C_{\rho,1} \leq \rho(x, y) \leq C_{\rho,2} \tag{2.4}
\]
for all $(x, y) \in C_K$, where $C_{\rho,1}$ and $C_{\rho,2}$ are positive constants.

We provide some examples of generators satisfying the assumption (A1) in Sect. 5.

### 3. Random Average Sampling Inequalities

In this section, we shall show that the sampling inequalities hold with certain probabilities for functions in $V_{N,\omega,\psi}^{p,q}(\Phi), V_{N,\psi}^{p,q}(\Phi, \mu, C_K)$ and $V_{\psi}^{p,q}(\Phi, \delta, C_K)$. The result for $V_{N,\omega,\psi}^{p,q}(\Phi)$ states as follows:
Theorem 3.1. Let $\Phi, \psi$ and $\rho$ satisfy assumptions (A1), (A2) and (A3) respectively. Further, let $\{(x_j, y_k)\}_{j,k \in \mathbb{N}}$ be a sequence of i.i.d. random variables that are drawn from a general probability distribution over $C_K = [-K_1, K_1] \times [-K_2, K_2]^d$ and whose density function is $\rho$. Then for any $\gamma \in (0, 1)$, $0 < \omega \leq \|\psi\|_{L^{1,1}(C_K)}$ and

$$nm > \left( \frac{54r\sqrt{2}(\ln 2)(2N+1)^{(d+1)}\|\psi\|_{L^{1,1}(C_K)}}{(2K_1)(q-1)(2K_2)^{d(p-1)}} \right)^2 \times 2 \left( \frac{\gamma \mathcal{C}_{\rho,1} (c^* \|\psi\|_{L^{1,1}(C_K)})^{(1-pq)} \omega^{pq}}{(2K_1)(q-1)(2K_2)^{d(p-1)}} \right) + 81\|\psi\|_{L^{1,1}(C_K)},$$

the sampling inequality

$$\mathcal{A}_{\gamma,\omega} \|f\|_{L^{p,q}(\mathbb{R} \times \mathbb{R}^d)} \leq \left\| (f \ast \psi)(x_j, y_k) \right\|_{L^{p,q}} \leq \mathcal{B}_{\gamma,\omega} \|f\|_{L^{p,q}(\mathbb{R} \times \mathbb{R}^d)}$$

holds with probability at least $1 - \mathcal{A}_1 e^{-(n \omega \beta_1)} - \mathcal{A}_2 e^{-(n \omega \beta_2)}$ for every $f$ in $V_{N,\omega,\psi}(\Phi)$, where

$$\mathcal{A}_1 = 2 \exp \left( r(2N+1)^{(d+1)} \ln \left( 4c^* + 1 \right) \right),$$

$$\beta_1 = \frac{(2K_1)^{(1-q)}(2K_2)^{d(1-p)}}{6(2K_1)(q-1)(2K_2)^{d(p-1)} + \gamma \mathcal{C}_{\rho,1} \left( \frac{\omega}{c^* \|\psi\|_{L^{1,1}(C_K)}} \right)^{pq}},$$

$$\mathcal{A}_2 = \frac{4 \left( (2c^* + \frac{1}{4})(c^* + \frac{1}{4}) \right)^r (2N+1)^{(d+1)}}{3r(\ln 2)^2(2N+1)^{(d+1)}},$$

$$\beta_2 = \frac{(2K_1)^{(1-q)}(2K_2)^{d(1-p)}}{18\sqrt{2} \left( 81(2K_1)(q-1)(2K_2)^{d(p-1)} + 2\gamma \mathcal{C}_{\rho,1} \left( \frac{\omega}{c^* \|\psi\|_{L^{1,1}(C_K)}} \right)^{pq} \right)} + \gamma \mathcal{C}_{\rho,1} \left( \frac{\omega}{c^* \|\psi\|_{L^{1,1}(C_K)}} \right)^{pq} c^*,$$

$$c^* = \frac{4\tilde{c}}{2^{\frac{(p+q)}{pq}}} \alpha_1 \left( \sum_{k_1 \in \mathbb{Z}} \frac{1}{(1+|k_1|)} \left( \frac{\alpha_1}{p-\gamma} \right)^{\frac{p-1}{p}} \left( \sum_{k_2 \in \mathbb{Z}^d} \frac{1}{(1+|k_2|)} \left( \frac{\alpha_1}{q-\gamma} \right)^{\frac{q-1}{q}} \right) \right).$$
In order to prove Theorem 3.1, we consider for \( f \in V^{p,q}(\Phi) \) and a sequence of i.i.d. random variables \( \{(x_j, y_k)\}_{j,k \in \mathbb{N}} \) that are drawn from a general probability distribution over \( C_K \) whose density function is \( \rho \), the random variables \( Y_{j,k}(f), j,k \in \mathbb{N} \), defined by

\[
Y_{j,k}(f) = |(f * \psi)(x_j, y_k)| - \int_{C_K} \rho(x,y)|(f * \psi)(x, y)|\,dxdy
\]  

(3.3)

and a few of its properties, namely its expectation, variance and certain norm estimates. The proofs of these properties and the subsequent analysis require the following variants of Young’s inequality.

**Lemma 3.2.** Let \( 1 \leq p, q, r \leq \infty \) with \( \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1 \). Suppose \( f \in L^p(\mathbb{R}^d), g \in L^q(\mathbb{R}^d) \) with \( \text{supp}(g) \subset C_K \), where \( C_K \) denotes the cube \([-K, K]^d\). Then,

\[
\|f * g\|_{L^r(C_K)} \leq \|f\|_{L^p(C_{2K})} \|g\|_{L^q(C_K)}.
\]  

(3.4)

**Proof.** Let \( 1 < p, q < \infty \) and \( p', q' \) be their conjugate exponents respectively. Consider for \( x \in C_K \),

\[
|(f * g)(x)|
\]

\[
\leq \int_{\mathbb{R}^d} |f(x - y)||g(y)|\,dy
\]

\[= \int_{C_K} |f(x - y)||g(y)|\,dy
\]

\[= \int_{C_K} |f(x - y)|^{\frac{p}{q}}|f(x - y)|^{(1 - \frac{p}{q})}|g(y)|^{\frac{q}{p}}|g(y)|^{(1 - \frac{q}{p})}\,dy
\]

\[= \int_{C_K} (|f(x - y)|^{p}|g(y)|^{q})^{\frac{1}{p}}|f(x - y)|^{(1 - \frac{p}{q})}|g(y)|^{(1 - \frac{q}{p})}\,dy.
\]

As \( \frac{1}{p'} + \frac{1}{q'} + \frac{1}{r} = 1 \), we get

\[
|(f * g)(x)|
\]

\[
\leq \left( \int_{C_K} (|f(x - y)|^{p}|g(y)|^{q})^{\frac{1}{p'}}\,dy \right)^{\frac{r}{p'}} \left( \int_{C_K} (|f(x - y)|^{(1 - \frac{p}{q})})^{q'}\,dy \right)^{\frac{1}{q'}}
\]

\[\times \left( \int_{C_K} (|g(y)|^{(1 - \frac{q}{p})})^{p'}\,dy \right)^{\frac{1}{p'}}
\]

by Holder’s inequality. Now using the relations \( q = p'(1 - \frac{q}{p}) \) and \( p = q'(1 - \frac{p}{q}) \), we obtain

\[
|(f * g)(x)|
\]

\[\leq \left( \int_{C_K} |f(x - y)|^{p'}|g(y)|^{q'}\,dy \right)^{\frac{1}{p'}} \left( \int_{C_K} |f(x - y)|^{p}\,dy \right)^{\frac{1}{q'}} \left( \int_{C_K} |g(y)|^{q}\,dy \right)^{\frac{1}{p'}}
\]
Let \( \text{Lemma 3.3.} \)

thereby proving the desired inequality (3.4). The proof of (3.4) for the other case is obvious.

The version of Young’s inequality for mixed Lebesgue spaces is given in [14]. The analogous result for the compact subset \( C_K = [-K_1, K_1] \times [-K_2, K_2]^d \) is given below.

**Lemma 3.3.** Let \( 1 < p, q < \infty, f \in L^{p,q}(\mathbb{R} \times \mathbb{R}^d) \) and \( g \in L^1(\mathbb{R}^{d+1}) \) with \( \text{supp}(g) \subset C_K \). Then, we have

\[
\|f * g\|_{L^{p,q}(C_K)} \leq \|f\|_{L^{p,q}(C_{2K})} \|g\|_{L^1(C_K)}. \tag{3.5}
\]

Also

\[
\|f * g\|_{L^{\infty, \infty}(C_K)} \leq \|f\|_{L^{\infty, \infty}(C_{2K})} \|g\|_{L^1(C_K)}. \tag{3.6}
\]

**Proof.** For a fixed \( x \in [-K_1, K_1] \), consider the functions \( f_x, g_x \) on \([-K_2, K_2]^d\), given by \( f_x(y) = f(x, y) \) and \( g_x(y) = g(x, y), y \in [-K_2, K_2]^d \).

Then,

\[
\|f * g\|_{L^{p,q}(C_K)}^p = \int_{[-K_1, K_1]} \left( \int_{[-K_2, K_2]^d} \left| \int_{\mathbb{R}^d} f(x - x', y - y') g(x', y') dy' \right|^q dx \right)^{\frac{p}{q}} dx
\]

\[
= \int_{[-K_1, K_1]} \left( \int_{[-K_2, K_2]^d} \left| \int_{\mathbb{R}^d} (f_{x-x'} * g_{x'})(y) dy \right|^q dy \right)^{\frac{p}{q}} dx
\]

\[
= \int_{[-K_1, K_1]} \left( \int_{[-K_2, K_2]^d} \left( \int_{[-K_1, K_1]} (f_{x-x'} * g_{x'})(y) dy \right)^q \right)^{\frac{p}{q}} dx
\]
Similarly, from (3.7), it then follows that

\[ \left\| \int_{[-K_1, K_1]} \left( f_{x-x'} * g_{x'} \right)(\cdot) dx' \right\|^p_{L^q([-K_2, K_2]^d)} \ dx. \]

Applying the Minkowski’s integral inequality and Lemma 3.2, we have

\[
\left\| \int_{[-K_1, K_1]} \left( f_{x-x'} * g_{x'} \right)(\cdot) dx' \right\|_{L^q([-K_2, K_2]^d)} \leq \int_{[-K_1, K_1]} \left\| \left( f_{x-x'} * g_{x'} \right)(\cdot) \right\|_{L^q([-K_2, K_2]^d)} dx' \leq \int_{[-K_1, K_1]} \| f_{x-x'} \|_{L^q([-2K_2, 2K_2]^d)} \| g_{x'} \|_{L^1([-K_2, K_2]^d)} dx'.
\]

For \( x \in [-K_1, K_1] \), let \( \tilde{f}(x) = \| f_x \|_{L^q([-2K_2, 2K_2]^d)} \) and \( \tilde{g}(x) = \| g_x \|_{L^1([-K_2, K_2]^d)} \). Then \( \text{supp}(\tilde{g}) \subset [-K_1, K_1] \) and we have

\[
\| f * g \|_{L^{p,q}(C_K)}^p \leq \int_{[-K_1, K_1]} \left\| \int_{[-K_1, K_1]} \left( f_{x-x'} * g_{x'} \right)(\cdot) dx' \right\|_{L^q([-K_2, K_2]^d)}^p dx \leq \int_{[-K_1, K_1]} \left( \int_{[-K_1, K_1]} \right. \left( f(x-x') \tilde{g}(x') \right) dx' \right) dx
\]

\[
= \int_{[-K_1, K_1]} |(\tilde{f} * \tilde{g})(x)|^p dx = \| \tilde{f} * \tilde{g} \|_{L^p([-K_1, K_1])}^p.
\]
Appealing to Lemma 3.2 once again, we get

\[
\| f * g \|_{L^{p,q}(C_K)}^p \leq \| \tilde{f} \|_{L^p([-2K_1, 2K_1])}^p \| \tilde{g} \|_{L^1([-K_1, K_1])}^p.
\]

Further we observe that

\[
\| \tilde{f} \|_{L^p([-2K_1, 2K_1])}^p = \int_{[-2K_1, 2K_1]} \| f_x \|_{L^q([-2K_2, 2K_2]^d)} dx = \int_{[-2K_1, 2K_1]} \left( \int_{[-2K_2, 2K_2]^d} |f(x, y)|^q dy \right)^{\frac{p}{q}} dx = \| f \|_{L^{p,q}(C_{2K})}^p.
\]

Similarly,

\[
\| \tilde{g} \|_{L^1([-K_1, K_1])}^p = \left( \int_{[-K_1, K_1]} \| g_x \|_{L^1([-K_2, K_2]^d)} dx \right)^p = \left( \int_{[-K_1, K_1]} \int_{[-K_2, 2K_2]^d} |g(x, y)| dy dx \right)^p = \| g \|_{L^{1,1}(C_K)}^p.
\]

From (3.7), it then follows that

\[
\| f * g \|_{L^{p,q}(C_K)} \leq \| f \|_{L^{p,q}(C_{2K})} \| g \|_{L^{1,1}(C_K)}.
\]
The proof of (3.6) is obvious.

Now we shall look into the properties satisfied by the random variables $Y_{j,k}(f)$ defined in (3.3).

(1) The expectation $\mathbb{E}[Y_{j,k}(f)] = 0$. \hfill (3.8)

(2) $\|Y_{j,k}(f)\|_{L^\infty,\infty} \leq \|f\|_{L^\infty,\infty(C_{2K})}\|\psi\|_{L^{1,1}(C_K)}$. \hfill (3.9)

Proof. As defined in (3.3), we have

\[
\sup_{j,k} \left| \mathbb{E}[Y_{j,k}(f)] \right| = \sup_{j,k} \left| \int_{C_K} \rho(x,y) \left| (f \ast \psi)(x,y) \right| dx \right| 
\]

by (3.6).

Now we have

\[
\mathbb{E}[Y_{j,k}(f)] = 0.
\]

Proof. Consider,

\[
\mathbb{E}[Y_{j,k}(f) - Y_{j,k}(g)] = 0.
\]

Proof. It follows from (3.8) that $\mathbb{E}[Y_{j,k}(f)] = \mathbb{E}[Y_{j,k}(f)^2]$. Now

\[
\mathbb{E}[Y_{j,k}(f)^2] = \mathbb{E}[\left| (f \ast \psi)(x,y) \right|^2] + \mathbb{E}\left[ \left( \int_{C_K} \rho(x,y) \left| (f \ast \psi)(x,y) \right| dx \right)^2 \right] 
\]

\[
-2 \mathbb{E}\left[ \left| (f \ast \psi)(x,y) \right| \left( \int_{C_K} \rho(x,y) \left| (f \ast \psi)(x,y) \right| dx \right) \right] 
\]

\[
= \mathbb{E}[\left| (f \ast \psi)(x,y) \right|^2] - \left( \int_{C_K} \rho(x,y) \left| (f \ast \psi)(x,y) \right| dx \right)^2.
\]
which leads to the inequality
\[
\text{Var}(Y_{j,k}(f)) \leq \mathbb{E}[|(f \ast \psi)(x_j, y_k)|^2] \\
\leq \|f \ast \psi\|_{L^{\infty,\infty}(C_K)}^2 \\
\leq \|f\|_{L^{\infty,\infty}(C_{2K})}^2 \|\psi\|_{L^{1,1}(C_K)}^2,
\]
by (3.6).

(5) \[\text{Var}(Y_{j,k}(f) - Y_{j,k}(g)) \leq 4\|f - g\|_{L^{\infty,\infty}(C_{2K})}^2 \|\psi\|_{L^{1,1}(C_K)}^2.\]

Proof.
\[
\text{Var}(Y_{j,k}(f) - Y_{j,k}(g)) \\
= \mathbb{E}[(Y_{j,k}(f) - Y_{j,k}(g))^2] \\
\leq \mathbb{E}\left[\left((f - g) \ast \psi)(x_j, y_k)\right)^2 + \int_{C_K} \rho(x, y) \left|((f - g) \ast \psi)(x, y)\right| dxdy\right]^2 \\
= \mathbb{E}\left[\left((f - g) \ast \psi)(x_j, y_k)\right]^2 + \left(\int_{C_K} \rho(x, y) \left|((f - g) \ast \psi)(x, y)\right| dxdy\right)^2 \\
2 \left(\int_{C_K} \rho(x, y) \left|((f - g) \ast \psi)(x, y)\right| dxdy\right) \mathbb{E}\left[\left((f - g) \ast \psi)(x_j, y_k)\right] \\
\leq 4\|f - g\|_{L^{\infty,\infty}(C_K)}^2 \|\psi\|_{L^{1,1}(C_K)}^2. \\
\]

To proceed further, we shall consider the following lemmas. The relation between the \(L^{\infty,\infty}\) and \(L^{p,q}\) norms of functions in \(V_{N}^{p,q}(\Phi)\) is given below.

Lemma 3.4 [12]. Suppose that \(\Phi\) satisfies (2.2) and (2.3). Then for every function \(f \in V_{N}^{p,q}(\Phi)\), we have
\[
\|f\|_{L^{\infty,\infty}(\mathbb{R} \times \mathbb{R}^d)} \leq c'\|f\|_{L^{p,q}(\mathbb{R} \times \mathbb{R}^d)},
\]
where
\[
c' = \frac{2(\frac{1}{p'} + \frac{1}{q'})}{\alpha_1} \left(\sum_{k_1 \in \mathbb{Z}} \frac{1}{(1 + |k_1|)^{\alpha_1 p'}}\right)^{\frac{1}{p'}} \left(\sum_{k_2 \in \mathbb{Z}^d} \frac{1}{(1 + |k_2|)^{\alpha_2 q'}}\right)^{\frac{1}{q'}}
\]
and \(p', q'\) denote the conjugate exponents of \(p\) and \(q\) respectively.

As the sampling inequalities are given in a probabilistic sense, it is vital to consider the notion of covering numbers which help in the estimation of the probability.

For a compact set \(C\) in a metric space, its covering number \(\mathcal{N}(C, \epsilon), \epsilon > 0\) is the least number of balls of radius \(\epsilon\) needed to cover \(C\). The following lemma gives an upper bound for the covering number of \(V_{N}^{p,q}(\Phi)\).
Lemma 3.5 [12]. Suppose that $\Phi$ satisfies (2.2) and (2.3). Then the covering number of $V_N^{p,q,*}(\Phi)$ with respect to $\| \cdot \|_{L_\infty,\infty(\mathbb{R} \times \mathbb{R}^d)}$ is bounded by

$$\mathcal{N}(V_N^{p,q,*}(\Phi), \epsilon) \leq \exp \left( r(2N + 1)^{(d+1)} \ln \left( \frac{2c'}{\epsilon} + 1 \right) \right),$$

where $c'$ as in Lemma 3.4.

Remark 3.6. In the above lemma, $r(2N + 1)^{(d+1)}$ is the dimension of the space $V_N^{p,q,*}(\Phi)$ and hence if complex valued functions are considered, then this value is to be replaced by $2r(2N + 1)^{(d+1)}$.

The analogue of the Bernstein’s inequality for multivariate random variables is as follows (see [6,12]).

Lemma 3.7. Let $Z_{j,k}$ ($j = 1, 2, \ldots, n; k = 1, 2, \ldots, m$) be independent random variables with expected values $\mathbb{E}[Z_{j,k}] = 0$ for all $j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, m$. Assume that $\text{Var}(Z_{j,k}) \leq \sigma^2$ and $|Z_{j,k}| \leq M$ almost surely for all $j = 1, 2, \ldots, n$ and $k = 1, 2, \ldots, m$. Then for any $\lambda \geq 0$,

$$\text{Prob} \left( \left| \sum_{j=1}^{n} \sum_{k=1}^{m} Z_{j,k} \right| \geq \lambda \right) \leq 2 \exp \left( -\frac{\lambda^2}{2nm\sigma^2 + \frac{2}{3}M\lambda} \right).$$

We shall now state and prove a crucial lemma for our analysis.

Lemma 3.8. Let $\Phi, \psi$ and $\rho$ be as in the hypothesis of Theorem 3.1 and $Y_{j,k}$ be as defined in (3.3). Then for any $m, n, N \in \mathbb{N}$ and $\lambda > 54r\sqrt{2(\ln 2)(2N + 1)}^{(d+1)} \left( 1 + \left( 1 + \frac{3nm}{2r\sqrt{2(\ln 2)(2N + 1)}^{(d+1)}} \right)^{\frac{3}{2}} \right) \|\psi\|_{L_1,1(C_K)}$, the following inequality holds:

$$\text{Prob} \left( \sup_{f \in V_N^{p,q,*}(\Phi)} \left| \sum_{j=1}^{n} \sum_{k=1}^{m} Y_{j,k}(f) \right| \geq \lambda \right) \leq A_1 \exp \left( -\frac{\lambda^2}{4c^*\|\psi\|_{L_1,1(C_K)}(2nm\sigma^*\|\psi\|_{L_1,1(C_K)} + \frac{\lambda}{2})} \right) + A_2 \exp \left( -\frac{\lambda^2}{18\sqrt{2}\|\psi\|_{L_1,1(C_K)}(81nm\|\psi\|_{L_1,1(C_K)} + 2\lambda)} \right),$$

(3.10)

where

$$A_1 = 2 \exp \left( r(2N + 1)^{(d+1)} \ln (4c^* + 1) \right),$$

$$A_2 = \frac{4 \left( (2c^* + \frac{1}{2})(c^* + \frac{1}{2}) \right)^{r(2N + 1)^{(d+1)}}}{3r(\ln 2)^2(2N + 1)^{(d+1)}}$$

and $c^*$ is as in Theorem 3.1.
Proof. For \( l \in \mathbb{N} \), let \( \mathcal{E}_l \) be a set of all centers of \( \mathcal{N}(V_N^{p,q,*}(\Phi),2^{-l}) \) balls of radius \( 2^{-l} \) covering \( V_N^{p,q,*}(\Phi) \) with respect to \( \| \cdot \|_{L^\infty,\infty(\mathbb{R} \times \mathbb{R}^d)} \). Then for any \( f \in V_N^{p,q,*}(\Phi) \), there exists \( f_l \in \mathcal{E}_l \) such that \( \| f - f_l \|_{L^\infty,\infty(\mathbb{R} \times \mathbb{R}^d)} < 2^{-l} \).

Using inequality (3.9), it can be easily seen that

\[
Y_{j,k}(f) = Y_{j,k}(f_1) + (Y_{j,k}(f_2) - Y_{j,k}(f_1)) + (Y_{j,k}(f_3) - Y_{j,k}(f_2)) + \cdots, \tag{3.11}
\]

for every \( j,k \in \mathbb{N} \).

Now, we define a sequence of events \( \{\omega_l\}_{l \in \mathbb{N}} \) as follows:

\[
\omega_1 = \left\{ \text{there exists } f_1 \in \mathcal{E}_1 \text{ such that } \left| \sum_{j=1}^n \sum_{k=1}^m Y_{j,k}(f_1) \right| \geq \frac{\lambda}{2} \right\}
\]

and for \( l \geq 2 \),

\[
\omega_l = \left\{ \text{there exist } f_1 \in \mathcal{E}_1 \text{ and } f_{l-1} \in \mathcal{E}_{l-1} \text{ with } \| f_l - f_{l-1} \|_{L^\infty,\infty(\mathbb{R} \times \mathbb{R}^d)} \leq 3 \cdot 2^{-l} \text{ such that } \left| \sum_{j=1}^n \sum_{k=1}^m (Y_{j,k}(f_l) - Y_{j,k}(f_{l-1})) \right| \geq \frac{\lambda}{2l^2} \right\}.
\]

If \( \sup_{f \in V_N^{p,q,*}(\Phi)} \left| \sum_{j=1}^n \sum_{k=1}^m Y_{j,k}(f) \right| \geq \lambda \), then the event \( \omega_l \) holds for some \( l \in \mathbb{N} \). Suppose not, using (3.11), we have

\[
\left| \sum_{j=1}^n \sum_{k=1}^m Y_{j,k}(f) \right| \leq \left| \sum_{j=1}^n \sum_{k=1}^m Y_{j,k}(f_1) \right| + \sum_{l=2}^\infty \left| \sum_{j=1}^n \sum_{k=1}^m (Y_{j,k}(f_l) - Y_{j,k}(f_{l-1})) \right|
\]

\[
\leq \frac{\lambda}{2} + \sum_{l=2}^\infty \frac{\lambda}{2l^2} = \frac{\pi^2 \lambda}{12} < \lambda,
\]

as our choice of \( f_l \) for a given \( f \in V_N^{p,q,*}(\Phi) \) satisfies \( \| f_l - f_{l-1} \|_{L^\infty,\infty(\mathbb{R} \times \mathbb{R}^d)} \leq 3 \cdot 2^{-l} \). This gives a contradiction.

Now, in order to estimate the required probability, we shall estimate the probability of each of the events \( \omega_l, l \in \mathbb{N} \). As \( |\mathcal{E}_l| = \mathcal{N}(V_N^{p,q,*}(\Phi),2^{-l}), l \in \mathbb{N} \), we have

\[
\text{Prob}(\omega_1) \leq \mathcal{N} \left( V_N^{p,q,*}(\Phi), \frac{1}{2} \right) \cdot \text{Prob} \left( \left| \sum_{j=1}^n \sum_{k=1}^m Y_{j,k}(f) \right| \geq \frac{\lambda}{2} : f \in \mathcal{E}_1 \right)
\]

and for \( l \geq 2 \),

\[
\text{Prob}(\omega_l)
\leq \mathcal{N} \left( V_N^{p,q,*}(\Phi), 2^{-l} \right) \cdot \mathcal{N} \left( V_N^{p,q,*}(\Phi), 2^{-l+1} \right)
\times \text{Prob} \left( \left| \sum_{j=1}^n \sum_{k=1}^m Y_{j,k}(f_l) - Y_{j,k}(f_{l-1}) \right| \geq \frac{\lambda}{2l^2} : f_l \in \mathcal{E}_l, f_{l-1} \in \mathcal{E}_{l-1}, \| f_l - f_{l-1} \|_{L^\infty,\infty(\mathbb{R} \times \mathbb{R}^d)} \leq 3 \cdot 2^{-l} \right).
\]
Consider,

\[
\text{Prob} \left( \left| \sum_{j=1}^{n} \sum_{k=1}^{m} Y_{j,k}(f) \right| \geq \frac{\lambda}{2} : f \in \mathcal{E}_1 \right) \leq 2 \exp \left( -\frac{(\frac{\lambda}{2})^2}{2nm \| \text{Var}(Y_{j,k}(f)) \|_{l^{\infty, \infty}} + \frac{3}{2} \| Y_{j,k}(f) \|_{l^{\infty, \infty}} \frac{\lambda}{2}} \right).
\]

\[
\leq 2 \exp \left( -\frac{\lambda^2}{8nm \| f \|^2_{L^{\infty, \infty}(C_{2K})} \| \psi \|^2_{L^{1,1}(C_K)} + \frac{4\lambda c^*}{3} \| f \|_{L^{\infty, \infty}(C_{2K})} \| \psi \|_{L^{1,1}(C_K)} } \right),
\]

using Lemma 3.10 and properties (1), (2) and (4) of $Y_{j,k}(f)$.

Further, using Lemma 3.4,

\[
\text{Prob} \left( \left| \sum_{j=1}^{n} \sum_{k=1}^{m} Y_{j,k}(f) \right| \geq \frac{\lambda}{2} : f \in \mathcal{E}_1 \right) \leq 2 \exp \left( -\frac{\lambda^2}{8nm (c^*)^2 \| \psi \|^2_{L^{1,1}(C_K)} + \frac{4\lambda c^*}{3} \| \psi \|_{L^{1,1}(C_K)} } \right).
\]

where $c^*$ is as in Theorem 3.1. By Lemma 3.5,

\[
\mathcal{N} \left( V^{p,q,*}_N(\Phi), \frac{1}{2} \right) \leq \exp \left( r(2N + 1)^{(d+1)} \ln \left( 4c^* + 1 \right) \right).
\]

Hence,

\[
\text{Prob}(\omega_1) \leq 2 \exp \left( -\frac{\lambda^2}{4c^* \| \psi \|_{L^{1,1}(C_K)} \left( 2nmc^* \| \psi \|_{L^{1,1}(C_K)} + \frac{\lambda}{3} \right) } \right).
\]

(3.12)

For $l \geq 2$, let $f_l \in \mathcal{E}_l, f_{l-1} \in \mathcal{E}_{l-1}$ and $\| f_l - f_{l-1} \|_{L^{\infty, \infty}(\mathbb{R} \times \mathbb{R}^d)} \leq 3 \cdot 2^{-l}$. Then

\[
\| Y_{j,k}(f_l) - Y_{j,k}(f_{l-1}) \|_{l^{\infty, \infty}(C_K)} \leq 6 \cdot 2^{-l} \| \psi \|_{L^{1,1}(C_K)}
\]

and

\[
\| \text{Var}(Y_{j,k}(f_l) - Y_{j,k}(f_{l-1})) \|_{l^{\infty, \infty}(C_K)} \leq 36 \cdot 2^{-2l} \| \psi \|^2_{L^{1,1}(C_K)},
\]

using properties (3) and (5) of $Y_{j,k}(f)$. It then follows from property (1) of $Y_{j,k}(f)$ and Lemma 3.10 that

\[
\text{Prob} \left( \left| \sum_{j=1}^{n} \sum_{k=1}^{m} \left( Y_{j,k}(f_l) - Y_{j,k}(f_{l-1}) \right) \right| \geq \frac{\lambda}{2l^2} \right)
\]
\[ \leq 2 \exp \left( - \frac{\lambda^2 2^l}{4l^2 \left( 72nm(2^{-l/2})\|\psi\|_{L^{1,1}(C_K)}^2 + 2\lambda\|\psi\|_{L^{1,1}(C_K)} \right) \right) \]

\[ \leq 2 \exp \left( - \frac{\nu 2^l}{l^2} \right), \]

where \( \nu = \frac{\lambda^2}{4 \left( 81nm\|\psi\|_{L^{1,1}(C_K)}^2 + 2\lambda\|\psi\|_{L^{1,1}(C_K)} \right)} \).

In the above calculation we used the fact that \( l^2 2^{-l} \leq \frac{9}{8} \) for \( l \in \mathbb{N} \).

Now by Lemma 3.5 and further simplifications we obtain,

\[ N \left( V^{p,q,*}_N (\Phi), 2^{-l} \right) N \left( V^{p,q,*}_N (\Phi), 2^{(-l+1)} \right) \]

\[ \leq \exp \left[ (\ln \left( (2c^* + 2^{-l}) (2c^* + 2^{(-l+1)}) \right) + 2 \ln 2^l - \ln 2) r(2N + 1)^{(d+1)} \right] \]

\[ \leq \exp \left[ (\ln \left( (2c^* + \frac{1}{4}) (2c^* + \frac{1}{4}) \right) + 2 \ln 2^l - \ln 2) r(2N + 1)^{(d+1)} \right] \]

\[ = \exp \left[ (\ln \left( (2c^* + \frac{1}{4}) (c^* + \frac{1}{4}) \right) + 2 \ln 2^l) r(2N + 1)^{(d+1)} \right]. \]

Then the estimate for the probability that the event \( \omega_l \) holds for some \( l \geq 2 \) is given by

\[ \text{Prob} \left( \bigcup_{l=2}^{\infty} \omega_l \right) \]

\[ \leq \sum_{l=2}^{\infty} \exp \left[ (\ln \left( (2c^* + \frac{1}{4}) (c^* + \frac{1}{4}) \right) + 2 \ln 2^l) r(2N + 1)^{(d+1)} \right] 2 \exp \left( - \frac{\nu 2^l}{l^2} \right) \]

\[ \leq 2 \left( (2c^* + \frac{1}{4}) (c^* + \frac{1}{4}) \right)^{r(2N+1)^{(d+1)}} \sum_{l=2}^{\infty} \exp \left[ - \nu 2^l \left( 2^l - \frac{l\alpha}{\nu 2^l} \right) \right], \quad (3.13) \]

where \( \alpha = 2r(\ln 2)(2N + 1)^{(d+1)} \).

It is easy to see that \( \min_{l \geq 2} \frac{2^l}{\nu 2^l} = \frac{2}{9} \) and \( \max_{l \geq 2} \frac{l\alpha}{\nu 2^l} = \frac{3}{8} \).

Hence,

\[ \frac{2^l}{\nu 2^l} - \frac{l\alpha}{\nu 2^l} \geq \frac{2}{9} - \frac{6r\sqrt{2}(\ln 2)(2N + 1)^{(d+1)}\|\psi\|_{L^{1,1}(C_K)} \left( 81nm\|\psi\|_{L^{1,1}(C_K)} + 2\lambda \right)}{\lambda^2}. \]

Now, using the assumption on \( \lambda \), it can be verified that

\[ \lambda^2 - 2 \left( 54r\sqrt{2}(\ln 2)(2N + 1)^{(d+1)}\|\psi\|_{L^{1,1}(C_K)} \right) \lambda \]

\[ - 54 \times 81r\sqrt{2}(\ln 2)(2N + 1)^{(d+1)}nm\|\psi\|_{L^{1,1}(C_K)}^2 > 0. \]

Further, this implies that
\[
\frac{2}{9} - 6r \sqrt{2} (\ln 2)(2N + 1)^{(d+1)} \left( \frac{81nm\|\psi\|^2_{L^{1,1}(C_K)} + 2\lambda\|\psi\|_{L^{1,1}(C_K)}}{\lambda^2} \right) > \frac{1}{9}.
\]

(3.14)

So,

\[
\frac{2\frac{s}{l^2}}{l^2} - \frac{\lambda_{\alpha}^2}{\nu^2} < \frac{1}{9}.
\]

Using the above inequality in (3.13), we obtain

\[
\text{Prob} \left( \bigcup_{l=2}^{\infty} \omega_l \right) \leq 2 \left( (2c^* + \frac{1}{4}) (c^* + \frac{1}{4}) \right)^{r(2N+1)(d+1)} \sum_{l=2}^{\infty} e^{-\frac{\nu l^2}{9}}.
\]

(3.15)

Applying integral test,

\[
\sum_{l=2}^{\infty} e^{-\frac{\nu l^2}{9}} \leq \int_{1}^{\infty} e^{-\frac{\nu l^2}{9}} \, dx \leq \frac{1}{\sqrt{2} \ln \sqrt{2}} \int_{\sqrt{2}}^{\infty} e^{-\frac{u^2}{2}} \, du = \frac{9e^{-\sqrt{2}}}{{2\nu} \ln \sqrt{2}}.
\]

From (3.14), it also follows that

\[
\nu = \frac{\lambda^2}{4 \left( 81nm\|\psi\|^2_{L^{1,1}(C_K)} + 2\lambda\|\psi\|_{L^{1,1}(C_K)} \right)} > 27r \sqrt{2}(\ln \sqrt{2})(2N + 1)^{(d+1)},
\]

and so

\[
\sum_{l=2}^{\infty} e^{-\frac{\nu l^2}{9}} \leq \frac{2}{3r(\ln 2)^2(2N + 1)^{(d+1)}} \exp \left( -\frac{\sqrt{2}\nu}{9} \right)
\]

\[
= \frac{2}{3r(\ln 2)^2(2N + 1)^{(d+1)}} \exp \left( -\frac{\lambda^2}{18\sqrt{2}\|\psi\|_{L^{1,1}(C_K)} \left( 81nm\|\psi\|_{L^{1,1}(C_K)} + 2\lambda \right)} \right).
\]

Therefore, from (3.15) we get

\[
\text{Prob} \left( \bigcup_{l=2}^{\infty} \omega_l \right) \leq \frac{4 \left( (2c^* + \frac{1}{4}) (c^* + \frac{1}{4}) \right)^{r(2N+1)(d+1)}}{3r(\ln 2)^2(2N + 1)^{(d+1)}}
\]

\[
\times \exp \left( -\frac{\lambda^2}{18\sqrt{2}\|\psi\|_{L^{1,1}(C_K)} \left( 81nm\|\psi\|_{L^{1,1}(C_K)} + 2\lambda \right)} \right).
\]

(3.16)

The required inequality (3.10) now follows from (3.12) and (3.16).

Now, we are in a position to prove Theorem 3.1.
Proof of Theorem 3.1. As $f \in V_{N,\omega,\psi}^{p,q}(\Phi)$ satisfies (3.2) if and only if $f \in V_{N,\omega,\psi}^{p,q,*}(\Phi)$. For $\lambda > 0$, and the random variable

$$\sup_{f \in V_{N,\omega,\psi}^{p,q,*}(\Phi)} \left| \sum_{j=1}^{n} \sum_{k=1}^{m} Y_{j,k}(f) \right|,$$

consider the event

$$E = \left\{ \sup_{f \in V_{N,\omega,\psi}^{p,q,*}(\Phi)} \left| \sum_{j=1}^{n} \sum_{k=1}^{m} Y_{j,k}(f) \right| \geq \lambda \right\}$$

and its complement

$$E^c = \left\{ -\lambda \leq \sum_{j=1}^{n} \sum_{k=1}^{m} Y_{j,k}(f) \leq \lambda, \ \forall \ f \in V_{N,\omega,\psi}^{p,q,*}(\Phi) \right\}.$$ 

The above event $E^c$ can also be written as

$$E^c = \left\{ nm \int_{C_{K}} \rho(x,y) |(f * \psi)(x,y)| \, dx \, dy - \lambda \leq \sum_{j=1}^{n} \sum_{k=1}^{m} |(f * \psi)(x_j,y_k)| \leq \sum_{j=1}^{n} \sum_{k=1}^{m} |(f * \psi)(x_j,y_k)| + \lambda, \ \forall \ f \in V_{N,\omega,\psi}^{p,q,*}(\Phi) \right\},$$

by (3.3).

First, we shall show that

$$\left\| \{(f * \psi)(x_j,y_k)\}_{j=1,2,\ldots;n; \ k=1,2,\ldots,m} \right\|_{p,q} \leq \sum_{j=1}^{n} \sum_{k=1}^{m} |(f * \psi)(x_j,y_k)| \leq nm \left( \frac{2-1}{q} \right) n \left( \frac{p-1}{p} \right) \left\| \{(f * \psi)(x_j,y_k)\}_{j=1,2,\ldots;n; \ k=1,2,\ldots,m} \right\|_{p,q}.$$ ...

As $p, q > 1$, we have

$$\sum_{k=1}^{m} |(f * \psi)(x_j,y_k)|^q \leq \left( \sum_{k=1}^{m} |(f * \psi)(x_j,y_k)| \right)^q$$

which in turn gives

$$\sum_{j=1}^{n} \left( \sum_{k=1}^{m} |(f * \psi)(x_j,y_k)|^q \right)^{\frac{p}{q}} \leq \sum_{j=1}^{n} \left( \sum_{k=1}^{m} |(f * \psi)(x_j,y_k)| \right)^p.$$
\[
\leq \left( \sum_{j=1}^{n} \sum_{k=1}^{m} |(f \ast \psi)(x_j, y_k)| \right)^p.
\]

So,
\[
\| \{(f \ast \psi)(x_j, y_k)\}_{j=1,2,\ldots,n; k=1,2,\ldots,m} \|_{L^p,q} \leq \sum_{j=1}^{n} \sum_{k=1}^{m} |(f \ast \psi)(x_j, y_k)|.
\]

For the other inequality, we apply the Holder’s inequality twice.

Suppose the event \( E^c \) holds true. Then, by (3.17) and (2.4), for \( f \in V_{\alpha,\omega,\psi}(\Phi) \),
\[
\| \{(f \ast \psi)(x_j, y_k)\}_{j=1,2,\ldots,n; k=1,2,\ldots,m} \|_{L^p,q} \leq \sum_{j=1}^{n} \sum_{k=1}^{m} |(f \ast \psi)(x_j, y_k)| \leq \int_{C_K} \rho(x,y) \| (f \ast \psi)(x,y) \| dx dy + \lambda.
\]

Now, by repeatedly applying Holder’s inequality, we get
\[
\| f \ast \psi \|_{L^{1,1}(C_K)} \leq (2K_2)^d \left( \frac{q-1}{q} \right) \int_{[-K_2,K_2]^d} \| (f \ast \psi)(x,y) \|^{q-1} dy dx \leq (2K_2)^d \left( \frac{q-1}{q} \right) \| f \ast \psi \|_{L^p,q(C_K)}.
\]

Further by (3.5),
\[
\| f \ast \psi \|_{L^{1,1}(C_K)} \leq (2K_1)^d \left( \frac{p-1}{p} \right) (2K_2)^d \left( \frac{q-1}{q} \right) \| \psi \|_{L^{1,1}(C_K)}.
\]

Then, from (3.18)
\[
\| \{(f \ast \psi)(x_j, y_k)\}_{j=1,2,\ldots,n; k=1,2,\ldots,m} \|_{L^p,q} \leq \sum_{j=1}^{n} \sum_{k=1}^{m} |(f \ast \psi)(x_j, y_k)| \leq \int_{C_K} \rho(x,y) \| (f \ast \psi)(x,y) \| dx dy + \lambda.
\]

Similarly,
\[
\| \{(f \ast \psi)(x_j, y_k)\}_{j=1,2,\ldots,n; k=1,2,\ldots,m} \|_{L^p,q} \geq \sum_{j=1}^{n} \sum_{k=1}^{m} |(f \ast \psi)(x_j, y_k)| \geq \left( \frac{1-p}{p} \right) m \left( \frac{1-q}{q} \right) \| f \ast \psi \|_{L^{1,1}(C_K)} - \lambda.
\]
In order to obtain a lower bound for \(\|f * \psi\|_{L^{1,1}(C_K)}\), consider
\[
\|f * \psi\|_{L^{p,q}(C_K)} \leq (2K_1)^{\left(\frac{q-1}{pq}\right)} \left( \int_{[-K_1,K_1]} \left( \int_{[-K_2,K_2]^d} |(f * \psi)(x,y)|^q \, dy \right)^p \, dx \right)^{\frac{1}{pq}} \\
\leq (2K_1)^{\left(\frac{q-1}{pq}\right)} \|f * \psi\|_{L^{\infty,\infty}(C_K)} \times \left( \int_{[-K_2,K_2]^d} \left( \int_{[-K_1,K_1]} |(f * \psi)(x,y)|^p \, dx \right)^{\frac{1}{p}} \, dy \right)^{\frac{1}{q}} \\
\leq (2K_1)^{\left(\frac{q-1}{pq}\right)} \|f * \psi\|_{L^{\infty,\infty}(C_K)} \times \left( \int_{[-K_2,K_2]^d} \left( \int_{[-K_1,K_1]} |(f * \psi)(x,y)| \, dx \right)^{\frac{1}{p}} \, dy \right)^{\frac{1}{q}}.
\]
by Holder’s inequality. Furthermore, by Minkowski’s integral inequality,
\[
\|f * \psi\|_{L^{p,q}(C_K)} \leq (2K_1)^{\left(\frac{q-1}{pq}\right)} (2K_2)^d \left( \frac{p-1}{pq} \right) \|f * \psi\|_{L^{\infty,\infty}(C_K)} \left(\int_{C_K} |(f * \psi)(x,y)| \, dx \, dy \right)^{\frac{1}{pq}}.
\]
Once again applying Holder’s inequality, we have
\[
\|f * \psi\|_{L^{p,q}(C_K)} \leq (2K_1)^{\left(\frac{q-1}{pq}\right)} (2K_2)^d \left( \frac{p-1}{pq} \right) \left( c^* \|\psi\|_{L^{1,1}(C_K)} \left(\int_{C_K} |(f * \psi)(x,y)| \, dx \, dy \right)^{\frac{1}{pq}} \right).
\]
Now, by (3.6) and Lemma 3.4,
\[
\|f * \psi\|_{L^{p,q}(C_K)} \leq (2K_1)^{\left(\frac{q-1}{pq}\right)} (2K_2)^d \left( \frac{p-1}{pq} \right) \left( c^* \|\psi\|_{L^{1,1}(C_K)} \left(\int_{C_K} |(f * \psi)(x,y)| \, dx \, dy \right)^{\frac{1}{pq}} \right).
\]
Moreover, as \(f \in V_{N,\omega,\psi}^p(\Phi)\), we obtain
\[
\|f * \psi\|_{L^{1,1}(C_K)} \geq (2K_1)^{(1-q)} (2K_2)^d (1-p) \left( c^* \|\psi\|_{L^{1,1}(C_K)} \left(\int_{C_K} |(f * \psi)(x,y)| \, dx \, dy \right)^{\frac{1}{pq}} \right) \omega^{pq}.
\]
Hence,
\[
\left\{ \{f * \psi)(x_j,y_k)\}_{j=1,2,...,n; k=1,2,...,m} \right\}_{L^{p,q}} \geq n^{\left(\frac{1-p}{p}\right)} m^{\left(\frac{1-q}{q}\right)} \left( \frac{c^* \|\psi\|_{L^{1,1}(C_K)} \left(\int_{C_K} |(f * \psi)(x,y)| \, dx \, dy \right)^{\frac{1}{pq}} \right) \omega^{pq}.
\]
\[
\geq n^{\left(\frac{1-p}{p}\right)} m^{\left(\frac{1-q}{q}\right)} \left( \frac{c^* \|\psi\|_{L^{1,1}(C_K)} \left(\int_{C_K} |(f * \psi)(x,y)| \, dx \, dy \right)^{\frac{1}{pq}} \right) \omega^{pq}.
\]
In view of the above inequality and (3.20), the event $E^c$ is contained in the event

$$
\tilde{E} = \left\{ r \left( \frac{1-c}{p} \right) m \left( \frac{1-p}{q} \right) \left( \frac{nm (c^* \| \psi \|_{L^{1,1}(C_K)})^{(1-p)q \omega^p q}}{(2K_1)(q-1)(2K_2)(d(p-1)) C_{p,q} \rho, \lambda - \lambda} \right) \right.
$$

$$
\leq \left\| \left\{ (f \ast \psi)(x_j, y_k) \right\} \right\|_{l^p, q}
$$

$$
\leq nm \frac{C_{p,q} \| \psi \|_{L^{1,1}(C_K)}}{(2K_1)^{\left( \frac{1-p}{p} \right)}(2K_2)^{d(q-p)}} + \lambda,
$$

$$\forall f \in V_{N, \omega, \psi}^p (\Phi)
$$

and so $\text{Prob}(\tilde{E}) \geq 1 - \text{Prob}(E)$.

For $\lambda = \frac{\gamma C_{p,q} (c^* \| \psi \|_{L^{1,1}(C_K)})^{(1-p)q \omega^p q}}{(2K_1)(q-1)(2K_2)^{d(p-1)}}$, we note that the inequality in the event $\tilde{E}$ is the required sampling inequality (3.2). Also, for $n, m$ satisfying (3.1), the above $\lambda$ satisfies the hypothesis of Lemma 3.8. In fact, taking

$$M_1 = \frac{\gamma C_{p,q} (c^* \| \psi \|_{L^{1,1}(C_K)})^{(1-p)q \omega^p q}}{(2K_1)(q-1)(2K_2)^{d(p-1)}}$$

and

$$M_2 = 2r \sqrt{2}(\ln 2)(2N + 1)^{(d+1)}$$

in (3.1) for the sake of simplicity, we have

$$nm > \frac{27M_2 \| \psi \|_{L^{1,1}(C_K)}}{M_1^2} \left( 2M_1 + 81 \| \psi \|_{L^{1,1}(C_K)} \right)$$

iff $M_1^2(nm)^2 > \left( 54M_1 M_2 \| \psi \|_{L^{1,1}(C_K)} + \frac{3}{M_2} \left( 27M_2 \| \psi \|_{L^{1,1}(C_K)} \right)^2 \right) nm$

iff $\left( M_1 nm - 27M_2 \| \psi \|_{L^{1,1}(C_K)} \right)^2 > \left( 27M_2 \| \psi \|_{L^{1,1}(C_K)} \right)^2 \left( 1 + \frac{3nm}{M_2} \right)$

iff $M_1 nm > 27M_2 \| \psi \|_{L^{1,1}(C_K)} \left( 1 + \left( 1 + \frac{3nm}{M_2} \right)^{\frac{1}{2}} \right)$

iff $\lambda > \frac{54r \sqrt{2}(\ln 2)(2N + 1)^{(d+1)} \left( 1 + \left( 1 + \frac{3nm}{2r \sqrt{2}(\ln 2)(2N + 1)^{(d+1)}} \right)^{\frac{1}{2}} \right)}{\| \psi \|_{L^{1,1}(C_K)}}$.\)
Therefore, for \( f \in V_{N,\omega,\psi}^{p,q}(\Phi) \), the sampling inequality holds with probability at least:

\[
1 - A_1 \exp \left( -nm \frac{(2K_1)^{(1-q)}(2K_2)^{d(1-p)}}{6(2K_1)^{(q-1)}(2K_2)^{d(p-1)} + \gamma C_{p,1}} \left( \frac{\sqrt{3}}{2} \frac{\gamma C_{p,1}}{\omega} \left( \frac{\psi \| L_{1,1}(C_K) \|_{p,q}}{c^*} \right)^2 \right) \right)
\]

\[
- A_2 \exp \left( -nm \frac{(2K_1)^{(1-q)}(2K_2)^{d(1-p)}}{18\sqrt{2}} \left( 81(2K_1)^{(q-1)}(2K_2)^{d(p-1)} + 2\gamma C_{p,1} \left( \frac{\omega \| L_{1,1}(C_K) \|_{p,q}}{c^*} \right)^2 \right) \right)
\]

**Remark 3.9.** If \( f \in V_{N,\omega,\psi}^{p,q}(\Phi) \) but \( \frac{f}{\| f \|_{L^{p,q}(\mathbb{R} \times \mathbb{R}^d)}} \) does not belong to the same, then the above proof may be carried out analogously for the set \( V_{N,\omega,\psi}^{p,q}(\Phi, \theta) \), \( \theta > 0 \), defined by

\[
V_{N,\omega,\psi}^{p,q}(\Phi, \theta) := \{ f \in V_{N}^{p,q}(\Phi) : \| f \|_{L^{p,q}(C_K)} \geq \omega \text{ and } \| f \|_{L^{p,q}(\mathbb{R} \times \mathbb{R}^d)} \leq \theta \}.
\]

For such functions, \( \frac{f}{\| f \|_{L^{p,q}(\mathbb{R} \times \mathbb{R}^d)}} \in V_{N,\omega,\psi}^{p,q}(\Phi) \).

The following theorem provides a sampling inequality for another subset \( V_{N,\omega,\psi}^{p,q}(\Phi, \mu, C_K) \) of \( L^{p,q}(\mathbb{R} \times \mathbb{R}^d) \), defined as in (2.1).

**Theorem 3.10.** Let \( \Phi, \psi, \rho \) and the i.i.d random variables \( \{(x_j, y_k)\}_{j, k \in \mathbb{N}} \) over \( C_K \) be as in the hypothesis of Theorem 3.1. For \( N \in \mathbb{N}, 0 < \mu \leq 1 \) and \( 0 < \eta < \mu C_{p,1} \), let \( m, n \in \mathbb{N} \) be such that

\[
mn > \frac{54r \sqrt{2}(\ln 2)(2N + 1)^{(d+1)}}{\eta} \left( 2 + \frac{81}{\eta} \right).
\]

Then, the sampling inequality for the functions \( f \in V_{N,\psi}^{p,q}(\Phi, \mu, C_K) \), namely,

\[
(nm\| \psi \|_{L^{1,1}(C_K)}(\mu C_{p,1} - \eta) \| f \|_{L^{p,q}(\mathbb{R} \times \mathbb{R}^d)}) \leq \sum_{j=1}^{n} \sum_{k=1}^{m} |(f \ast \psi)(x_j, y_k)|
\]

\[
\leq \left( nm\| \psi \|_{L^{1,1}(C_K)} \left( C_{p,2}(2K_1)^{(p-1)}(2K_2)^{\frac{d(p-1)}{q}} + \eta \right) \| f \|_{L^{p,q}(\mathbb{R} \times \mathbb{R}^d)} \right)
\]

holds with probability at least

\[
1 - A_1 \exp \left( -nm \frac{3\eta^2}{4c^* (6c^* + \eta)} \right) - A_2 \exp \left( -nm \frac{\eta^2}{18\sqrt{2}(81 + 2\eta)} \right),
\]

where \( A_1, A_2 \) and \( c^* \) are as in Lemma 3.8.
Proof. Let \( f \in V_{N,\psi}^{p,q}(\Phi, \mu, C_K) \) and \( \tilde{f} = \frac{f}{\|f\|_{L^p,q(R \times \mathbb{R}d)}} \). Suppose we assume that
\[
\left| \sum_{j=1}^{n} \sum_{k=1}^{m} Y_{j,k}(\tilde{f}) \right| \leq nm\eta \|\psi\|_{L^{1,1}(C_K)},
\]
where \( Y_{j,k} \) is as in (3.3). In other words,
\[
\left| \sum_{j=1}^{n} \sum_{k=1}^{m} (\tilde{f} \ast \psi)(x_j, y_k) \right| \leq nm\eta \|\psi\|_{L^{1,1}(C_K)}.
\]
By assumption \((A_3)\), this reduces to
\[
\left| \sum_{j=1}^{n} \sum_{k=1}^{m} (\tilde{f} \ast \psi)(x_j, y_k) \right| \leq nm\eta \|\psi\|_{L^{1,1}(C_K)}.
\]
Further, making use of the definition of \( V_{N,\psi}^{p,q}(\Phi, \mu, C_K) \) and the assumption \((A_2)\) for the lower inequality and (3.19) for the upper inequality, we obtain
\[
\left| \sum_{j=1}^{n} \sum_{k=1}^{m} (\tilde{f} \ast \psi)(x_j, y_k) \right| \leq nm\eta \|\psi\|_{L^{1,1}(C_K)}.
\]
Rewriting in terms of \( f \), this turns out to be the required sampling inequality (3.22), which, by our initial assumption, holds with probability at least
\[
Prob \left( \sup_{\tilde{f} \in V_{N,\psi}^{p,q}(\Phi, \mu, C_K)} \left| \sum_{j=1}^{n} \sum_{k=1}^{m} Y_{j,k}(\tilde{f}) \right| \leq nm\eta \|\psi\|_{L^{1,1}(C_K)} \right) \geq 1 - Prob \left( \sup_{\tilde{f} \in V_{N,\psi}^{p,q}(\Phi)} \left| \sum_{j=1}^{n} \sum_{k=1}^{m} Y_{j,k}(\tilde{f}) \right| \geq nm\eta \|\psi\|_{L^{1,1}(C_K)} \right).
\]
Now, by the assumption on \( n, m \) in the hypothesis, we have
\[
\frac{nm}{r\sqrt{2}(\ln 2)(2N + 1)^{(d+1)}} > \frac{54}{\eta} \left( 2 + \frac{81}{\eta} \right)
\]
and so
\[
\left( \frac{nm}{r\sqrt{2}(\ln 2)(2N + 1)^{(d+1)}} \right)^2 \frac{\eta^2}{2916} > \frac{\eta}{27} + \frac{3}{2}.
\]
After a straightforward calculation, we get
\[
\frac{nmm\eta}{54r\sqrt{2}(\ln 2)(2N + 1)^{(d+1)} - 1}^2 > 1 + \frac{3nm}{2r\sqrt{2}(\ln 2)(2N + 1)^{(d+1)}}.
\]
which then implies that
\[
\frac{nmm\eta\|\psi\|_{L^{1,1}(C_K)}}{54r\sqrt{2}(\ln 2)(2N + 1)^{(d+1)}} \times \left( 1 + \left( 1 + \frac{3nm}{2r\sqrt{2}(\ln 2)(2N + 1)^{(d+1)}} \right)^{\frac{1}{2}} \right) \|\psi\|_{L^{1,1}(C_K)}.
\]

We may now appeal to Lemma 3.8 in order to estimate the probability. By taking \( \lambda = nmm\eta\|\psi\|_{L^{1,1}(C_K)} \) in the lemma and simplifying further, the result is proved.

We shall now consider the sampling inequality for the set \( V_{p,q}^{\psi,q}(\Phi, \delta, C_K) \). This requires the following finite dimensional approximation.

**Lemma 3.11** [12]. Let \( 1 < p, q < \infty, \frac{1}{p} + \frac{1}{p'} = 1 \) and \( \frac{1}{q} + \frac{1}{q'} = 1 \). Let \( f \in V_{p,q}^{\psi,q}(\Phi) \) and \( \| f \|_{L^{p,q}(\mathbb{R}^d)} = 1 \), wherein \( \Phi \) satisfies the assumption (A1). Also let \( C_K = [-K_1, K_1] \times [-K_2, K_2]^d \) and \( s = \min\{s_1, s_2\} + \frac{1}{p} + \frac{d}{q} - (d + 1) \). Then for any \( \epsilon_1, \epsilon_2 > 0 \), there exists \( f_N \in V_{p,q}^{\psi,q}(\Phi) \) such that
\[
\| f - f_N \|_{L^{p,q}(C_K)} \leq \epsilon_1 \text{ if } N \geq N_1(K_1, K_2, \epsilon_1) = \max\{K_1, K_2\}
\]
\[
+ \left( \tilde{c}(K_1)^{\frac{1}{p}}(K_2)^{\frac{d}{q}p'}(1 + K_2)^{\left( \frac{1}{q'} + \frac{1}{p'} \right)2(d+1)} \right) \frac{\alpha_1 \epsilon_1 (s_2q' - d)^{\frac{1}{p'}}}{\alpha_1 \epsilon_1 (s_1p' - 1)^{\frac{1}{p'}}}
\]
\[
+ \left( \tilde{c}(K_1)^{\frac{1}{p}}(K_2)^{\frac{d}{q}p'}(1 + K_2)^{\left( \frac{1}{q'} + \frac{1}{p'} \right)2(d+1)} \right) \frac{\epsilon_1 (s_2q' - d)^{\frac{1}{p'}}}{\alpha_1 \epsilon_1 (s_1p' - 1)^{\frac{1}{p'}}}
\]
and
\[
\| f - f_N \|_{L^{\infty,\infty}(C_K)} \leq \epsilon_2 \text{ if } N \geq N_2(K_1, K_2, \epsilon_2) = \max\{K_1, K_2\}
\]
\[
+ \left( \tilde{c}d^{\frac{1}{p'}}(1 + K_2)^{\left( \frac{1}{q'} + \frac{1}{p'} \right)2\left( \frac{1}{p'} + \frac{d}{q'} \right)} \right) \frac{\alpha_1 \epsilon_2 (s_2q' - d)^{\frac{1}{p'}}}{\alpha_1 \epsilon_2 (s_1p' - 1)^{\frac{1}{p'}}}.
\]
Let $\Phi, \psi, \{(x_j, y_k)\}_{j, k \in \mathbb{N}}$, cuboid $C_K = [-K_1, K_1] \times [-K_2, K_2]^d$ and the probability density function $\rho$ be as in the hypothesis of Theorem (3.1). Then for $0 < \delta < 1$, $0 < \epsilon < 1 - \delta$ and $0 < \gamma < 1 - \frac{\epsilon}{(1 - \delta - \epsilon)^{1 + pq}}$, the sampling inequality
\[
A \|f\|_{L^{p,q}(\mathbb{R} \times \mathbb{R}^d)} \leq \left\{ \left\| (f * \psi)(x_j, y_k) \right\|_{\mathbb{R}^d} \right\|_{p,q} \leq B \|f\|_{L^{p,q}(\mathbb{R} \times \mathbb{R}^d)}
\] (3.23)
holds uniformly for all $f \in V_{\psi}^{p,q}(\Phi, \delta, C_K)$ with probability at least
\[
1 - A_1 e^{(-nm\beta_1)} - A_2 e^{(-nm\beta_2)}
\]
provided $n, m$ satisfies (3.1), where
\[
A = \frac{C_{\rho,1}(c^*)^{(1-pq)} \|\psi\|_{L^{1,1}(C_K)}}{(2K_1)(q-1)(2K_2)^{d(p-1)}} \cdot \frac{1}{n^\frac{1}{p} m^\frac{1}{q}}
\]
\[
B = \frac{\alpha_2 \|\psi\|_{L^{1,1}(C_K)}}{\alpha_1} \left( \frac{C_{\rho,2}}{(2K_1)(1-q)(2K_2)^{d(p-1)}} \right) + \frac{\gamma C_{\rho,1}(c^*)^{(1-pq)} \|\psi\|_{L^{1,1}(C_K)}}{(2K_1)(q-1)(2K_2)^{d(p-1)}} \cdot \frac{1}{n^\frac{1}{p} m^\frac{1}{q}}
\]
the constants $c^*, A_1, A_2, \beta_1$ and $\beta_2$ are as in Theorem 3.1 with $\omega = (1 - \delta - \epsilon)\|\psi\|_{L^{1,1}(C_K)}$. The constant $N$ appearing in $A_1$ and $A_2$ is given by
\[
N = \max \left\{ N_1(2K_1, 2K_2, \epsilon), N_2(2K_1, 2K_2, \frac{c C_{\rho,1}(c^*)^{(1-pq)}}{(2K_1)(q-1)(2K_2)^{d(p-1)}}) \right\}
\]
where $N_1$ and $N_2$ are as in Lemma 3.11.

Proof. Let $f \in V_{\psi}^{p,q}(\Phi, \delta, C_K)$. Without loss of generality, we may assume that $\|f\|_{L^{p,q}(\mathbb{R} \times \mathbb{R}^d)} = 1$. We may write $f$ as
\[
f = \sum_{i=1}^{r} \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}^d} c_i(k_1, k_2) \phi_i(\cdot - k_1, \cdot - k_2).
\]
Let $0 < \delta < 1$ and $0 < \epsilon < 1 - \delta$. By the proof of Lemma 3.11, the element
\[
f_N = \sum_{i=1}^{r} \sum_{|k_1| \leq N} \sum_{|k_2| \leq N} c_i(k_1, k_2) \phi_i(\cdot - k_1, \cdot - k_2) \in V_{\psi}^{p,q}(\Phi)
\]
satisfies
\[
\|f - f_N\|_{L^{p,q}(C_{2K})} \leq \epsilon
\] (3.24)
and
\[
\|f - f_N\|_{L^{\infty,\infty}(C_{2K})} \leq \frac{c C_{\rho,1}(c^*)^{(1-pq)}}{(2K_1)(q-1)(2K_2)^{d(p-1)}},
\] (3.25)
where $N = \max \left\{ N_1(2K_1, 2K_2, \epsilon), N_2 \left( 2K_1, 2K_2, \frac{\epsilon C_{\rho,1}(c^*)^{(1-pq)}}{(2K_1)^{(q-1)(2K_2)^{d(p-1)}}} \right) \right\}$.

Furthermore, in view of Lemma 3.3, for the known averaging function $\psi$, we have the following

$$
\|(f - f_N) * \psi\|_{L^{p,q}(C_K)} \leq \|f - f_N\|_{L^{p,q}(C_{2K})}\|\psi\|_{L^{1,1}(C_K)}.
$$

Using (3.24) and (3.25), these in turn reduce to

$$
\|(f - f_N) * \psi\|_{L^{p,q}(C_K)} \leq \epsilon \|\psi\|_{L^{1,1}(C_K)} \quad \text{(3.26)}
$$

and

$$
\|(f - f_N) * \psi\|_{L^{\infty,\infty}(C_K)} \leq \frac{\epsilon C_{\rho,1}(c^*)^{(1-pq)}\|\psi\|_{L^{1,1}(C_K)}}{(2K_1)^{(q-1)(2K_2)^{d(p-1)}}} \quad \text{(3.27)}
$$

From the above inequality (3.27), we also have

$$
\left\| \left\{ (f - f_N) * \psi \right\}(x_j, y_k) \right\|_{p,q} \leq n \frac{1}{m} \frac{\|f - f_N\|_{L^{p,q}(C_{2K})}\|\psi\|_{L^{1,1}(C_K)}}{(2K_1)^{(q-1)(2K_2)^{d(p-1)}}},
$$

which leads to the inequality

$$
\left\| \left\{ (f_N * \psi)(x_j, y_k) \right\} \right\|_{p,q} \leq \left\| \left\{ (f * \psi)(x_j, y_k) \right\} \right\|_{p,q} - n \frac{1}{m} \frac{\|f - f_N\|_{L^{p,q}(C_{2K})}\|\psi\|_{L^{1,1}(C_K)}}{(2K_1)^{(q-1)(2K_2)^{d(p-1)}}} + n \frac{1}{m} \frac{\|f_N * \psi\|_{L^{p,q}(C_{2K})}\|\psi\|_{L^{1,1}(C_K)}}{(2K_1)^{(q-1)(2K_2)^{d(p-1)}}}.
$$

(3.28)

Also, (3.26) together with the definition of $V^{p,q}_{\psi}(\Phi, \delta, C_K)$ implies

$$
\|f_N * \psi\|_{L^{p,q}(C_K)} \geq \|f * \psi\|_{L^{p,q}(C_K)} - \epsilon \|\psi\|_{L^{1,1}(C_K)} \geq (1 - \delta - \epsilon)\|\psi\|_{L^{1,1}(C_K)},
$$

which shows that $f_N \in V^{p,q}_{\psi}(\Phi, \omega, \psi)$ with $\omega = (1 - \delta - \epsilon)\|\psi\|_{L^{1,1}(C_K)}$.

In the presence of Theorem 3.1, the sampling inequality for $f_N$, namely

$$
\mathcal{A}_{\gamma, \omega}\|f_N\|_{L^{p,q}(\mathbb{R} \times \mathbb{R}^d)} \leq \left\| \left\{ (f_N * \psi)(x_j, y_k) \right\} \right\|_{p,q} \leq \mathcal{B}_{\gamma, \omega}\|f_N\|_{L^{p,q}(\mathbb{R} \times \mathbb{R}^d)} \quad \text{(3.29)}
$$

holds with probability at least $1 - \mathcal{A}_1 e^{-(nm\beta_1)} - \mathcal{A}_2 e^{-(nm\beta_2)}$.

Then using this sampling inequality in the lower inequality of (3.28), we obtain

$$
\left\| \left\{ (f * \psi)(x_j, y_k) \right\} \right\|_{p,q}
$$
So, \( f \in V^p,q(\Phi, \delta, C_K) \), it follows that
\[
\|f\|_{L^p,q(C_K)} \leq \epsilon \text{ and together with the fact that } f \in V^p,q(\Phi, \delta, C_K), \text{ we obtain }
\]
\[
\|f\|_{L^p,q(C_K)} \geq \|f\|_{L^p,q(C_K)} - \epsilon \geq 1 - \delta - \epsilon.
\]

Now, from (3.24), clearly \( \|f - f_N\|_{L^p,q(C_K)} \leq \epsilon \) and together with the fact that \( f \in V^p,q(\Phi, \delta, C_K) \), we obtain
\[
\|f\|_{L^p,q(C_K)} \geq \|f\|_{L^p,q(C_K)} - \epsilon \geq 1 - \delta - \epsilon.
\]

Therefore, the sampling inequality (3.23) holds with probability at least \( 1 - A_1e^{-(nm\beta_1)} - A_2e^{-(nm\beta_2)} \), where \( \beta_1 \) and \( \beta_2 \) are as in Theorem 3.1 with \( \omega = (1 - \delta - \epsilon)\|\psi\|_{L^{1,1}(C_K)} \).

Remark 3.13. We observe that the probabilities with which the sampling inequalities, proved in Theorems 3.1, 3.10 and 3.12, hold approach one when the sample size tends to infinity.

4. Reconstruction Using Random Average Samples

In this section, we give reconstruction formulae for functions in the signal classes \( V^p,q_{\omega, \psi}(\Phi) \) and \( V^p,q_{\psi}(\Phi, \mu, C_K) \).
Theorem 4.1. Let $\Phi$ and $\psi$ satisfy the assumptions (A1) and (A2) respectively. Suppose $\{(x_j, y_k)\}_{j,k\in \mathbb{N}}$ is a sequence of i.i.d. random variables that are drawn from a general probability distribution over the cuboid $C_K = [-K_1, K_1] \times [-K_2, K_2]^d$ with the density function $\rho$ satisfying the assumption (A3). If

$$\left\| \sum_{|k_1| \leq N} \sum_{|k_2| \leq N} c^T(k_1, k_2)(\Phi \ast \psi)\cdot (-k_1, \cdot -k_2) \right\|_{L^p,q(C_K)} \geq \beta \|c\|_{L^p,q} \tag{4.1}$$

holds for all $c \in (L^p,q([-N, N] \times [-N, N]^d))^r$, $N \in \mathbb{N}$ and for some positive constant $\beta$, then for any $\gamma \in (0, 1)$, there exists a finite sequence of functions $\{G_{j,k} : (j,k) \in \{1,2,\ldots,n\} \times \{1,2,\ldots,m\}\}$ such that

$$f(x,y) = \sum_{j=1}^{n} \sum_{k=1}^{m} (f \ast \psi)(x_j, y_k)G_{j,k}(x,y)$$

holds for all $f \in V_{N}^{p,q}(\Phi)$ with probability at least

$$1 - A_1 \exp \left( -nm \frac{(2K_1)^{(1-q)}(2K_2)^d(1-p)}{6(2K_1)(q-1)(2K_2)^d(p-1) + \gamma C_{p,1} \left( \frac{\beta(\alpha_2c^*)^{-1}}{\|\psi\|_{L^1,1(C_K)}} \right)^{pq} c^*} \right) - A_2 \exp \left( -nm \frac{(2K_1)^{(1-q)}(2K_2)^d(1-p)}{18\sqrt{2} (81(2K_1)(q-1)(2K_2)^d(p-1) + 2\gamma C_{p,1} \left( \frac{\beta(\alpha_2c^*)^{-1}}{\|\psi\|_{L^1,1(C_K)}} \right)^{pq} c^*)} \right), \tag{4.2}$$

where $c^*, A_1$ and $A_2$ are positive constants as in Theorem 3.1.

Proof. We observe that it is enough to prove the reconstruction formula for $f \in V_{N}^{p,q,*}(\Phi)$. Let $f \in V_{N}^{p,q,*}(\Phi)$ be given by

$$f = \sum_{i=1}^{r} \sum_{|k_1| \leq N} \sum_{|k_2| \leq N} c_i(k_1, k_2)\phi_i(\cdot - k_1, \cdot - k_2). \tag{4.3}$$

Then for $1 \leq j \leq n, 1 \leq k \leq m$ with $n, m \in \mathbb{N}$ satisfying (3.1),

$$(f \ast \psi)(x_j,y_k) = \sum_{i=1}^{r} \sum_{|k_1| \leq N} \sum_{|k_2| \leq N} c_i(k_1, k_2)(\phi_i \ast \psi)(x_j - k_1, y_k - k_2), \tag{4.4}$$

which can be represented as a matrix equation as follows:

For $1 \leq j \leq n, 1 \leq k \leq m$, let $F_{jk} := (f \ast \psi)(x_j, y_k)$ and further for $k_1 \in Z, k_2 \in Z^d$, let $M_{jk}(k_1, k_2)$ denote the $r$-tuple

$$(m_{1,k_1,k_2}(x_j, y_k), m_{2,k_1,k_2}(x_j, y_k), \ldots, m_{r,k_1,k_2}(x_j, y_k)), \tag{4.5}$$

where $m_{i,k_1,k_2}(x_j, y_k) := (\phi_i \ast \psi)(x_j - k_1, y_k - k_2), 1 \leq i \leq r.$
For fixed $j, k$ and $k_1$, let
\[ M_{jk}(k_1) := \{ M_{jk}(k_1, k_2) : k_2 \in \mathbb{Z}^d; |k_2| \leq N \} \]
and
\[ M_{jk} := \{ M_{jk}(k_1) : k_1 \in \mathbb{Z}; |k_1| \leq N \}, \]
for a given $j, k$. Similarly, we consider
\[
c(k_1, k_2) := \left( c_1(k_1, k_2), c_2(k_1, k_2), \ldots, c_r(k_1, k_2) \right),
\]
\[
c(k_1) := \{ c(k_1, k_2) : k_2 \in \mathbb{Z}^d, |k_2| \leq N \},
\]
\[
c(k_1) := \{ c(k_1) : k_1 \in \mathbb{Z}, |k_1| \leq N \},
\]
where $c_i(k_1, k_2), i = 1, 2, \ldots, r$ are the coefficients in the expression (4.3). Then (4.4) can be rewritten as
\[
F_{jk} = \sum_{i=1}^{r} \sum_{|k_1| \leq N} \sum_{|k_2| \leq N} c_i(k_1, k_2) m_{i,k_1,k_2}(x_j, y_k)
\]
\[
= \sum_{|k_1| \leq N} \sum_{|k_2| \leq N} M_{jk}(k_1, k_2) c(k_1, k_2)
\]
\[
= \sum_{|k_1| \leq N} M_{jk}(k_1) c(k_1)
\]
\[
= M_{jk} c
\]
\[
= (M c)_{jk},
\]
where $M$ denotes the $nm \times (2N + 1)$ matrix whose rows are given by $M_{jk}$. Taking $F$ to be the $nm \times 1$ column vector with the entries $F_{jk}$, the matrix equation representing the system of equations in (4.4) is
\[
M c = F.
\]
In order to solve for the unknown vector $c$, we make use of the sampling inequality (3.2). Now by (4.1) and (2.2), we have
\[
\| f * \psi \|_{L^p,q(C_K)} \geq \beta \| c \|_{L^p,q} \geq \frac{\beta}{\alpha_2} \| f \|_{L^p,q(\mathbb{R} \times \mathbb{R}^d)} = \frac{\beta}{\alpha_2},
\]
which shows that $f \in V_{N,\omega,\psi}(\Phi)$ with $0 < \omega = \frac{\beta}{\alpha_2}$. Further, $\frac{\beta}{\alpha_2} \leq \| f * \psi \|_{L^p,q(C_K)} \leq \| \psi \|_{L^{1,1}(C_K)},$ by (3.5).
Therefore, an application of Theorem 3.1 gives
\[
A_{\gamma, \frac{\beta}{\alpha_2}} \| f \|_{L^p,q(\mathbb{R} \times \mathbb{R}^d)} \leq \left\| \{ f * \psi \}_k \right\|_{L^p,q(\mathbb{R} \times \mathbb{R}^d)} \forall f \in V_{N,\omega,\psi} \Phi
\]
which holds with probability atleast (4.2). Using (4.6) and (2.2) in (4.7), gives
\[
\| M c \|_{L^p,q} \geq A_{\gamma, \frac{\beta}{\alpha_2}} \alpha_1 \| c \|_{L^p,q} \forall c \in \left( L^p,q([-N, N] \times [-N,N]^d) \right)^r, N \in \mathbb{N}.
\]
The above inequality implies that $M$ is an injective operator on $\left( L^p,q([-N, N] \times [-N,N]^d) \right)^r$. Denoting the conjugate transpose of $M$ by $M^*$,
we also have $M^*M$ is injective and hence invertible. From (4.6), it then follows that $c = (M^*M)^{-1}M^*F = (\tilde{M})^TF$, where $\tilde{M} = M(M^TM)^{-1}$.

From the definition of $\tilde{M}$, we observe that the matrix $\tilde{M}$ has the same structure as that of $M$. We shall use the notations $\tilde{M}_{jk}, \tilde{M}_{jk}(k_1, k_2)$ for the entities analogous to those of $M$ and we denote the $i^{th}$ coordinate of the $r$-tuple $\tilde{M}_{jk}(k_1, k_2)$ by $\left(\tilde{M}_{jk}(k_1, k_2)\right)_i$.

We may then explicitly write the unknown coefficients as follows. For $k_1 \in \mathbb{Z}, |k_1| \leq N$, we have

$$c(k_1) = \sum_{j,k} \tilde{M}_{jk}(k_1) F_{jk}$$

and for $k_2 \in \mathbb{Z}^d, |k_2| \leq N$, $c(k_1, k_2) = \sum_{j,k} \tilde{M}_{jk}(k_1, k_2) F_{jk}$ whose $i^{th}$ coordinate is

$$c_i(k_1, k_2) = \sum_{j,k} (\tilde{M}_{jk}(k_1, k_2))_i F_{jk}, \quad i \in \{1, 2, \ldots, r\}.$$ 

Substituting for $c_i(k_1, k_2)$ in (4.3), we obtain

$$f = \sum_{j,k} F_{jk} \sum_{|k_1| \leq N} \sum_{|k_2| \leq N} \sum_{i=1}^{r} \left(\tilde{M}_{jk}(k_1, k_2)\right)_i \phi_i(\cdot - k_1, \cdot - k_2).$$

Now, let $\tilde{\Phi}(k_1, k_2)$ denote the $r$-tuple of functions

$$\left(\phi_1(\cdot - k_1, \cdot - k_2), \phi_2(\cdot - k_1, \cdot - k_2), \ldots, \phi_r(\cdot - k_1, \cdot - k_2)\right).$$

Further, let $\tilde{\Phi}(k_1)$ and $\tilde{\Phi}$ be defined just as $c(k_1)$ and $c$ are defined in (4.5). Then

$$f = \sum_{j,k} F_{jk} (\tilde{M}\tilde{\Phi})_{jk}.$$ 

In other words,

$$f(x, y) = \sum_{j,k} (f \ast \psi)(x_j, y_k) G_{j,k}(x, y), \quad (x, y) \in \mathbb{R} \times \mathbb{R}^d,$$

where $G_{j,k} := (\tilde{M}\tilde{\Phi})_{jk}$, which holds with probability at least (4.2).

Using the sampling inequality provided in Theorem 3.10, we shall give a reconstruction formula when $V_{N,p,q}(\Phi) = V_{N,\psi}(\Phi, \mu, C_K)$. 

**Theorem 4.2.** Let $\Phi$, $\psi$ and $\rho$ satisfy the assumptions $(A_1)$, $(A_2)$ and $(A_3)$ respectively. Also, let $\{(x_j, y_k)\}_{j,k \in \mathbb{N}}$ denote a sequence of i.i.d random variables over a cuboid $C_K \subset \mathbb{R} \times \mathbb{R}^d$ drawn from a probability distribution with probability density function $\rho$. For $N \in \mathbb{N}, 0 < \mu \leq 1$ and $0 < \eta < \mu \mathcal{C}_{p,1}$,
let \( m, n \in \mathbb{N} \) be such that (3.21) is satisfied. Then, there exist functions \( \{G_{j,k}\}_{j=1,\ldots,n; k=1,\ldots,m} \) such that every \( f \in V_{N,\psi}^{p,q}(\Phi, \mu, C_K) \) can be reconstructed by

\[
f(x, y) = \sum_{j=1}^{n} \sum_{k=1}^{m} (f * \psi)(x_j, y_k) G_{j,k}(x, y), \quad (x, y) \in \mathbb{R} \times \mathbb{R}^d
\]

with probability at least

\[
1 - A_1 \exp \left( -nm \frac{3\eta^2}{4c^* (6c^* + \eta)} \right) - A_2 \exp \left( -nm \frac{\eta^2}{18\sqrt{2}(81 + 2\eta)} \right),
\]

where \( A_1, A_2 \) and \( c^* \) are as in Lemma 3.8.

**Proof.** Let \( f \in V_{N,\psi}^{p,q}(\Phi, \mu, C_K) \). Then \( f \) may be expressed as a finite linear combination of the translates of \( \phi_i \)'s and by (2.2), there exist positive constants \( \alpha_1 \) and \( \alpha_2 \) such that

\[
\alpha_1 \|c\|_{l^{p,q}} \leq \|f\|_{L^{p,q}(\mathbb{R} \times \mathbb{R}^d)} \leq \alpha_2 \|c\|_{l^{p,q}},
\]

where \( c \in (l^{p,q}(\mathbb{Z} \times \mathbb{Z}^d))^r \) denotes the finite coefficient vector. By the sampling inequality (3.22) in Theorem 3.10 as well as the above inequality, we may say that

\[
a_1 \|c\|_{l^{p,q}} \leq \sum_{j=1}^{n} \sum_{k=1}^{m} |(f * \psi)(x_j, y_k)| \leq a_2 \|c\|_{l^{p,q}}, \quad a_1, a_2 > 0
\]

holds with a certain probability, dependent on the number of samples. As in the proof of Theorem 4.1, we then have

\[
a_1 \|c\|_{l^{p,q}} \leq \|Mc\|_{l^{1,1}} \leq a_2 \|c\|_{l^{p,q}}, \quad \forall c,
\]

where \( M \) is the \( nm \times (2N + 1) \) matrix with entries involving the values of the translates of the convoluted generators at \((x_j, y_k), 1 \leq j \leq n, 1 \leq k \leq m \). Therefore, \( M^*M \) is invertible, where \( M^* \) denotes the conjugate transpose of \( M \). The rest of the proof follows as in Theorem 4.1.

5. Examples and Numerical Simulation

In this section, we give some examples of generators \( \phi_i \) which satisfy the assumption \((A_1)\). We also validate the results obtained in the previous section numerically using some of these examples.

For \( n \in \mathbb{N} \), we consider the cardinal B-spline of degree \( n \) defined on \( \mathbb{R} \) by

\[
B_n(x) = (B_0 * B_0 * \cdots * B_0)(x), \quad (n \text{ convolutions}),
\]

where \( B_0(x) = \chi_{[-\frac{1}{2}, \frac{1}{2}]}(x) \). Then its Fourier transform \( \hat{B}_n \) is given by

\[
\hat{B}_n(\omega) = \left( \frac{\sin(\pi \omega)}{\pi \omega} \right)^{n+1}, \quad \text{where the Fourier transform of a function } f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)
\]
is defined as \( \hat{f}(\omega) := \int_{\mathbb{R}^d} f(x)e^{-2\pi i \langle \omega, x \rangle} dx \) and it is extended to \( L^2(\mathbb{R}^d) \) as a unitary operator using the density argument.

**Example 5.1.** We define \( \phi : \mathbb{R}^2 \rightarrow \mathbb{R} \) by \( \phi(x, y) := B_n(x)B_n(y), n \in \mathbb{N} \).

Clearly \( \phi \) satisfies the decay condition in the assumption (A1). We shall show that the above \( \phi \) satisfies the stable shifts condition in (A1) for \( r = 1, p = q = 2 \) and \( d = 1 \). In order to do that we consider,

\[
\left\| \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} c(k_1, k_2) \phi(\cdot - k_1, \cdot - k_2) \right\|^2_{L^2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} \left| \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} c(k_1, k_2)e^{-2\pi i \langle (k_1, k_2), \omega \rangle} \hat{\phi}(\omega) \right|^2 d\omega = \int_{[0,1]^2} \left| \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} c(k_1, k_2)e^{-2\pi i \langle (k_1, k_2), \omega \rangle} \right|^2 \left| \sum_{j \in \mathbb{Z}^2} \hat{\phi}(\omega + j) \right|^2 d\omega. \tag{5.1}
\]

Let \( P_\phi(\omega) = \sum_{j \in \mathbb{Z}^2} \left| \hat{\phi}(\omega + j) \right|^2 \). Now,

\[
|\hat{\phi}(\omega_1, \omega_2)| = \left| \hat{B}_0(\omega_1) \right|^{n+1} \left| \hat{B}_0(\omega_2) \right|^{n+1} \leq \left| \hat{B}_0(\omega_1) \right| \left| \hat{B}_0(\omega_2) \right| \quad \text{and}
\]

\[
P_\phi(\omega) \leq \left( \sum_{j_1 \in \mathbb{Z}} \left| \hat{B}_0(\omega_1 + j_1) \right|^2 \right) \left( \sum_{j_2 \in \mathbb{Z}} \left| \hat{B}_0(\omega_2 + j_2) \right|^2 \right) = 1.
\]

Also, by the periodicity of \( P_\phi \),

\[
P_\phi(\omega) \geq \inf_{|\omega_1| \leq \frac{1}{2}, |\omega_2| \leq \frac{1}{2}} |P_\phi(\omega)| \geq \inf_{|\omega_1| \leq \frac{1}{2}, |\omega_2| \leq \frac{1}{2}} |\hat{\phi}(\omega)|^2 \geq \inf_{|\omega_1| \leq \frac{1}{2}} \left| \frac{\sin(\pi \omega_1)}{\pi \omega_1} \right|^{2(n+1)} \inf_{|\omega_2| \leq \frac{1}{2}} \left| \frac{\sin(\pi \omega_2)}{\pi \omega_2} \right|^{2(n+1)} = \left( \frac{2}{\pi} \right)^{4(n+1)}.
\]

Further,

\[
\int_{[0,1]^2} \left| \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} c(k_1, k_2)e^{-2\pi i \langle (k_1, k_2), \omega \rangle} \right|^2 d\omega = \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} |c(k_1, k_2)|^2 = \|c\|_{l^2}^2.
\]
Using the above relation and the bounds for \( P_\phi \) in (5.1), we obtain
\[
\left( \frac{2}{\pi} \right)^{2(n+1)} \|c\|_2^2 \leq \left\| \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} c(k_1, k_2) \phi(\cdot - k_1, \cdot - k_2) \right\|_{L^2(\mathbb{R}^2)}^2 \leq \|c\|_2^2.
\]

**Example 5.2.** For \( r, N \in \mathbb{N} \), let \( u_i, v_i \in \mathbb{Z} \) be such that \(|u_i - u_j| > 2N\) and \(|v_i - v_j| > 2N\) for \( 1 \leq i, j \leq r, i \neq j \). Let \( \phi_t(x, y) = B_n(x - u_i)B_n(y - v_i), n \in \mathbb{N} \) for \( i = 1, 2, \ldots, r \).

Then we have
\[
\left\| \sum_{|k_1| \leq N} \sum_{|k_2| \leq N} \sum_{i=1}^{r} c_i(k_1, k_2) \phi_i(\cdot - k_1, \cdot - k_2) \right\|_{L^2(\mathbb{R}^2)}^2 = \int_{\mathbb{R}^2} \left\| \sum_{i=1}^{r} \sum_{|k_1| \leq N} \sum_{|k_2| \leq N} c_i(k_1, k_2) e^{-2\pi i(k_1\omega_1 + k_2\omega_2 + u_i\omega_1 + v_i\omega_2)} \hat{B}_n(\omega_1)\hat{B}_n(\omega_2) \right\|^2 \ d\omega_1 d\omega_2 = \int_{[0,1]^2} \left\| \sum_{i=1}^{r} \sum_{|k_1| \leq N} \sum_{|k_2| \leq N} c_i(k_1, k_2) e^{-2\pi i((k_1 + u_i)\omega_1 + (k_2 + v_i)\omega_2)} \right\|^2 P_B(\omega_1, \omega_2) \ d\omega_1 d\omega_2,
\]
where \( P_B(\omega_1, \omega_2) = \sum_{(j_1, j_2) \in \mathbb{Z}^2} |\hat{B}_n(\omega_1 + j_1)|^2 |\hat{B}_n(\omega_2 + j_2)|^2 \). As in the previous example, we get \( \left( \frac{2}{\pi} \right)^{4(n+1)} \leq P_B(\omega_1, \omega_2) \leq 1 \).

As \( \{ (k_1 + u_i, k_2 + v_i) : |k_1| \leq N, |k_2| \leq N, 1 \leq i \leq r \} \) is a set of distinct elements, the Fourier coefficients of
\[
\sum_{i=1}^{r} \sum_{|k_1| \leq N} \sum_{|k_2| \leq N} c_i(k_1, k_2) e^{-2\pi i((k_1 + u_i)\omega_1 + (k_2 + v_i)\omega_2)}
\]
are \( c_i(k_1, k_2) \) and hence
\[
\int_{[0,1]^2} \left\| \sum_{i=1}^{r} \sum_{|k_1| \leq N} \sum_{|k_2| \leq N} c_i(k_1, k_2) e^{-2\pi i((k_1 + u_i)\omega_1 + (k_2 + v_i)\omega_2)} \right\|^2 d\omega = \sum_{i=1}^{r} \left\| c_i \right\|_{L^2}^2.
\]

It follows that
\[
\frac{1}{\sqrt{r}} \|c\|_{L^2, 2} \leq \left( \sum_{i=1}^{r} \left\| c_i \right\|_{L^2}^2 \right)^{\frac{1}{2}} \leq \sum_{i=1}^{r} \left\| c_i \right\|_{L^2} = \|c\|_{L^2, 2},
\]
by applying Cauchy-Schwarz inequality for the lower inequality.
Hence, we get
\[
\left(\frac{2}{\pi}\right)^{2(n+1)} \frac{1}{\sqrt{r}} \|c\|_{l^2, 2}^2 \leq \left\| \sum_{|k_1| \leq N} \sum_{|k_2| \leq N} \sum_{i} c_i(k_1, k_2) \phi_i(\cdot - k_1, \cdot - k_2) \right\|_{L^2(\mathbb{R}^2)} \\
\leq \|c\|_{l^2, 2} \]
and therefore the generators \(\{\phi_i : 1 \leq i \leq r\}\) satisfy the stable shifts condition in assumption (A1) which is being applied in the reconstruction of functions in \(V^{p,q}_{N,\omega,\psi}(\Phi)\) and \(V^{p,q}_{N,\psi}(\Phi, \mu, C_K)\) for \(p = q = 2\). The decay condition in assumption (A1) can be easily verified for \(s_1, s_2 > 1\).

**Example 5.3.** Let \(\phi_1\) and \(\phi_2\) be two compactly supported continuous functions such that \(\phi_1(\cdot - k_1, \cdot - k_2)\) is orthogonal to \(\phi_2(\cdot - k_1', \cdot - k_2')\) for every \(k_1, k_1', k_2, k_2' \in \mathbb{Z}\) and there exist positive constants \(A_1, B_1, A_2\) and \(B_2\) such that \(A_1 \leq P_{\phi_1}(\omega) \leq B_1\) a.e. and \(A_2 \leq P_{\phi_2}(\omega) \leq B_2\) a.e.

Using the orthogonality of the translates of \(\phi_1\) and \(\phi_2\) and then proceeding as in Example 5.1, we have
\[
\left\| \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} \sum_{i=1}^{2} c_i(k_1, k_2) \phi_i(\cdot - k_1, \cdot - k_2) \right\|_{L^2(\mathbb{R}^2)}^2 = \left\| \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} c_1(k_1, k_2) \phi_1(\cdot - k_1, \cdot - k_2) \right\|_{L^2(\mathbb{R}^2)}^2 + \left\| \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} c_2(k_1, k_2) \phi_2(\cdot - k_1, \cdot - k_2) \right\|_{L^2(\mathbb{R}^2)}^2 \\
\leq B_1 \|c_1\|_{l^2, 2}^2 + B_2 \|c_2\|_{l^2, 2}^2 \\
\leq B \left( \|c_1\|_{l^2}^2 + \|c_1\|_{l^2}^2 \right)^2 \leq B \|c\|_{l^2, 2}^2,
\]
where \(B = \max\{B_1, B_2\}\).

For the lower inequality, we have
\[
\left\| \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} \sum_{i=1}^{2} c_i(k_1, k_2) \phi_i(\cdot - k_1, \cdot - k_2) \right\|_{L^2(\mathbb{R}^2)}^2 = \left\| \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} c_1(k_1, k_2) \phi_1(\cdot - k_1, \cdot - k_2) \right\|_{L^2(\mathbb{R}^2)}^2 + \left\| \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} c_2(k_1, k_2) \phi_2(\cdot - k_1, \cdot - k_2) \right\|_{L^2(\mathbb{R}^2)}^2.
\]
\[ \geq A_1 \|c_1\|_2^2 + A_2 \|c_2\|_2^2 \]

\[ \geq A (\|c_1\|_2^2 + \|c_2\|_2^2) , \]

where \( A = \min\{A_1, A_2\} \).

Further using Cauchy-Schwarz inequality on the numbers \( \|c_1\|_2 \) and \( \|c_2\|_2 \), we obtain

\[ \left\| \sum_{k_1 \in \mathbb{Z}} \sum_{k_1 \in \mathbb{Z}} \sum_{i=1}^2 c_i(k_1, k_2) \phi_i(\cdot - k_1, \cdot - k_2) \right\|_{L^2(\mathbb{R}^2)} \geq \left( \frac{A}{2} \right)^\frac{1}{2} \|c\|_{l^2}. \]

Thus the generators \( \{\phi_1, \phi_2\} \) have stable shifts. The decay condition is obviously true for \( s_1, s_2 > 1 \).

For numerical implementations, we consider the cuboid \( C_K = [-2.5, 2.5] \times [-2.5, 2.5] \). The function \( f(x, y) \) and the averaging function \( \psi(x, y) \) are as defined below:

\[ f(x, y) = 3B_2(x)B_2(y - 1) - 5B_2(x + 1)B_2(y) \]

\[ \psi(x, y) = \chi_{[-\frac{1}{8}, \frac{1}{8}] \times [-\frac{1}{8}, \frac{1}{8}]}(x, y). \]

One can easily verify that the functions \( f \) and \( \psi \) satisfy the conditions of Theorem 4.1. Using the reconstruction formula provided in Theorem 4.1, the simulation is performed for various values of the sample size \( nm \). The graphical representations of the function \( f \) and its reconstructed version \( \tilde{f} \) corresponding to 25 (\( m = 5 \) and \( n = 5 \)) random samples are shown in Figs. 1 and 2 respectively.

The reconstruction error \( \|f - \tilde{f}\| \) is also computed with respect to \( L^\infty, L^1 \) and \( L^2 \) norms for various sample sizes. The numerical results are presented in Table 1.

\[ \text{Figure 1. The 3D plot of the function } f(x, y) \]
Figure 2. The 3D plot of $\tilde{f}$ corresponding to 25 samples $(n = 5, m = 5)$

Table 1. The reconstruction error $\|f - \tilde{f}\|

| Sample size | Reconstruction error |
|-------------|----------------------|
| $n$ | $m$ | $\|f - \tilde{f}\|_{L^\infty(C_K)}$ | $\|f - \tilde{f}\|_{L^1(C_K)}$ | $\|f - \tilde{f}\|_{L^2(C_K)}$ |
| 5 | 5 | $2.3315 e^{-15}$ | $7.6548 e^{-15}$ | $7.7786 e^{-30}$ |
| 7 | 7 | $2.2204 e^{-15}$ | $3.8640 e^{-15}$ | $2.5115 e^{-30}$ |
| 10 | 10 | $1.3323 e^{-15}$ | $3.1535 e^{-15}$ | $1.5283 e^{-30}$ |

Further, to test Theorem 4.2 numerically, we consider $C_K = [-3, 3] \times [-3, 3]$, $f(x, y) = B_1(x)B_1(y) + 3B_1(x - 1)B_1(y - 1)$ and $\psi(x, y) = \chi_{[\frac{1}{2}, \frac{3}{2}] \times [\frac{1}{2}, \frac{3}{2}]}$. It can be easily shown that $f \in V^{2,2}_{1,\psi}(\Phi, \mu, C_K)$, where $0 < \mu \leq 1$. The reconstruction formula in Theorem 4.2 has been used for reconstructing the function $f$ for different sample sizes. Figures 3 and 4 show the function $f$ and its reconstructed version $\tilde{f}$ corresponding to 25 random samples $(m = 5, n = 5)$. The errors in $L^\infty$, $L^1$ and $L^2$ norms are calculated and the numerical values are given in Table 2.

Note: After completing our work, we noticed that a similar setting has been considered in [24], wherein the sampling inequalities for the sets $V^{p,q}_{N,\omega,\psi}(\Phi)$ and $V^{p,q}_{\psi}(\Phi, \delta, C_K)$ have been obtained. We have derived sampling inequalities for $V^{p,q}_{N,\omega,\psi}(\Phi)$, $V^{p,q}_{N,\psi}(\Phi, \mu, C_K)$ and $V^{p,q}_{\psi}(\Phi, \delta, C_K)$ and our estimates are significantly different and more accurate. Moreover, we have also given the minimum
Figure 3. The 3D plot of the function $f(x, y)$

Figure 4. The 3D plot of $\tilde{f}$ for $n=5, m=5$

Table 2. The reconstruction error $\|f - \tilde{f}\|$

| Sample size | Reconstruction error | $\|f - \tilde{f}\|_{L^{\infty}(C_K)}$ | $\|f - \tilde{f}\|_{L^{1}(C_K)}$ | $\|f - \tilde{f}\|_{L^{2}(C_K)}$ |
|-------------|----------------------|---------------------------------|---------------------------------|---------------------------------|
| $n=5$       | $m=5$                | $7.8930 \ e^{-14}$              | $1.6052 \ e^{-13}$             | $4.7277 \ e^{-27}$             |
| $n=7$       | $m=7$                | $5.5227 \ e^{-14}$              | $5.5811 \ e^{-14}$             | $1.8715 \ e^{-27}$             |
| $n=10$      | $m=10$               | $8.8818 \ e^{-16}$              | $2.3891 \ e^{-15}$             | $7.3617 \ e^{-31}$             |
number of samples required for the sampling inequalities to hold with positive probabilities. Further, we have provided the reconstruction formulae for functions in $V_{N,\omega,\psi}^{p,q}(\Phi)$ and $V_{N,\psi}^{p,q}(\Phi,\mu,C_K)$. These results have also been tested for some examples using numerical simulations.

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Declarations

Conflict of interest The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] Aldroubi, A., Sun, Q., Tang, W.S.: Convolution, average sampling, and a Calderon resolution of the identity for shift-invariant spaces. J. Fourier Anal. Appl. 11(2), 215–244 (2005)
[2] Bass, R.F., Gröchenig, K.: Random sampling of multivariate trigonometric polynomials. SIAM J. Math. Anal. 36(3), 773–795 (2004)
[3] Bass, R.F., Gröchenig, K.: Random sampling of bandlimited functions. Israel J. Math. 177(1), 1–28 (2010)
[4] Bass, R.F., Gröchenig, K.: Relevant sampling of band-limited functions. Ill. J. Math. 57(1), 43–58 (2013)
[5] Benedek, A., Panzone, R.: The space $L_p$, with mixed norm. Duke Math. J. 28(3), 301–324 (1961)
[6] Bennett, G.: Probability inequalities for the sum of independent random variables. J. Am. Stat. Assoc. 57(297), 33–45 (1962)
[7] Butzer, P.L., Stens, R.L.: Sampling theory for not necessarily band-limited functions: A historical overview. SIAM Rev. 34(1), 40–53 (1992)
[8] Devaraj, P., Yugesh, S.: A local weighted average sampling and reconstruction theorem over shift invariant subspaces. RM 71, 319–332 (2017)
[9] Führ, H., Xian, J.: Relevant sampling in finitely generated shift-invariant spaces. J. Approx. Theory 240, 1–15 (2019)
[10] Goyal, P. Patel, D. and Sivananthan, S.: Random sampling in reproducing kernel subspace of mixed Lebesgue spaces, arXiv:2102.08632v1

[11] Jiang, Y. and Li, W.: Random sampling in weighted reproducing kernel subspaces of $L^p(\mathbb{R}^d)$, arXiv:2003.02993

[12] Jiang, Y., Li, W.: Random sampling in multiply generated shift-invariant subspaces of mixed Lebesgue spaces $L^{p,q}(\mathbb{R} \times \mathbb{R}^d)$, J. Comput. Appl. Math. 386, 113237 (2021)

[13] Kang, S., Kwon, K.H.: Generalized average sampling in shift invariant spaces. J. Math. Anal. Appl. 377, 70–78 (2011)

[14] Li, R., Liu, B., Liu, R., Zhang, Q.Y.: The $L^{p,q}$-stability of the shifts of finitely many functions in mixed Lebesgue spaces $L^{p,q}(\mathbb{R}^{d+1})$. Acta Mathematica Sinica, English Series. 34(6), 1001–1014 (2018)

[15] Li, Y., Sun, Q., Xian, J.: Random sampling and reconstruction of concentrated signals in a reproducing kernel space. Appl. Comput. Harmon. Anal. 54, 273–302 (2021)

[16] Li, Y., Wen, J., Xian, J.: Reconstruction from convolution random sampling in local shift invariant spaces. Inverse Prob. 35, 125008 (2019)

[17] Patel, D., Sampath, S.: Random sampling in reproducing kernel subspaces of $L^p(\mathbb{R}^n)$. J. Math. Anal. Appl. 491, 124270 (2020)

[18] Nashed, M.Z., Sun, Q., Xian, J.: Convolution sampling and reconstruction of signals in a reproducing kernel subspace. Proc. Am. Math. Soc. 141(6), 1995–2007 (2013)

[19] Sun, W., Zhou, X.: Reconstruction of band-limited functions from local averages. Constr. Approx. 18, 205–222 (2002)

[20] Sun, W., Zhou, X.: Reconstruction of band-limited signals from local averages. IEEE Trans. Inf. Theory 48, 2955–2963 (2002)

[21] Sun, W., Zhou, X.: Average sampling in spline subspaces. Appl. Math. Lett. 15, 233–237 (2002)

[22] Sun, W., Zhou, X.: Reconstruction of functions in spline subspaces from local averages. Proc. Am. Math. Soc. 131(8), 2561–2571 (2003)

[23] Sun, W., Zhou, X.: Average sampling in shift invariant subspaces with symmetric averaging functions. J. Math. Anal. Appl. 287(1), 279–295 (2003)

[24] Wang, S.: The random convolution sampling stability in multiply generated shift invariant subspaces of weighted mixed Lebesgue spaces. AIMS Math. 7(2), 1707–1725 (2021)

[25] Yang, J.: Random sampling and reconstruction in multiply generated shift-invariant spaces. Anal. Appl. 17(2), 323–347 (2019)

[26] Yang, J., Tao, X.: Random sampling and approximation of signals with bounded derivatives. J Inequal. Appl. 107, 1–14 (2019)

[27] Yang, J., Wei, W.: Random sampling in shift invariant spaces. J. Math. Anal. Appl. 398(1), 26–34 (2013)
