PERMUTATIONS AVOIDING SETS OF PATTERNS WITH LONG MONOTONE SUBSEQUENCES

MIKLÓS BÓNÁ AND JAY PANTONE

Abstract. We enumerate permutations that avoid all but one of the $k$ patterns of length $k$ starting with a monotone increasing subsequence of length $k - 1$. We compare the size of such permutation classes to the size of the class of permutations avoiding the monotone increasing subsequence of length $k - 1$. In most cases, we determine the exponential growth rate of these permutation classes, while in the remaining cases, we present strong numerical evidence leading to a conjectured growth rate. We also present numerical evidence that suggests a conjecture for the growth rates of these permutation classes at subexponential precision. Some of these conjectures claim that the relevant permutation classes have non-algebraic, and in one case, even non-D-finite, generating functions.

1. Introduction

We say that a permutation $p$ contains the pattern (or subsequence) $q = q_1 q_2 \cdots q_k$ if there is a $k$-element set of indices $i_1 < i_2 < \cdots < i_k$ such that $p_{i_r} < p_{i_s}$ if and only if $q_r < q_s$. If $p$ does not contain $q$, then we say that $p$ avoids $q$. For example, $p = 3752416$ contains $q = 2413$, as the first, second, fourth, and seventh entries of $p$ form the subsequence $3726$, which is order-isomorphic to $q = 2413$. A recent survey on permutation patterns by Vatter can be found in [15]. Let $Av_n(q)$ be the number of permutations of length $n$ that avoid the pattern $q$, where the length of a permutation is the number of entries in it. In general, it is very difficult to compute the numbers $Av_n(q)$, or to describe their sequence as $n$ goes to infinity.

However, the special case when $q$ is the monotone increasing pattern of length $k$ is much better understood. This is partly because the Robinson–Schensted correspondence maps $(12 \cdots k)$-avoiding permutations of length $n$ into pairs of standard Young tableaux of the same shape, on $n$ boxes, and having at most $k - 1$ columns. The number of such standard Young tableaux, and the number of their pairs, was computed by Regev [12] at great precision. He proved that for all $k \geq 2$, the asymptotic equality

$$Av_n(1234 \cdots k) \simeq \lambda_k \frac{(k - 1)^{2n}}{n^{(k^2 - 2k)/2}}$$

holds, where $\lambda_k$ is a constant given by a multiple integral.

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In particular, it follows from Regev’s results that
\[(2) \quad L(12 \cdots (k-1)) := \lim_{n \to \infty} (\Av_n(12 \cdots (k-1)))^{1/n} = (k-2)^2,\]
a fact that we will also refer to by saying that the exponential growth rate of
the sequence $\Av_n(12 \cdots (k-1))$ is $(k-2)^2$. Note that (2) is much easier to
prove than (1). See Theorem 4.10 in [6] for an easy proof of the inequality
$\Av_n(12 \cdots (k-1)) \leq (k-2)^{2n}$, which implies $L(12 \cdots (k-1)) \leq (k-2)^2$,
and see Lemma 5.3 of [5] or Theorem 1.3 of [3] for different proofs of the
inequality $L(12 \cdots (k-1)) \geq (k-2)^2$.

As monotone patterns are so well understood compared to other
patterns, it is worth taking study of permutations that do not avoid the pattern
$12 \cdots (k-1)$, but satisfy pattern avoidance conditions that significantly
restrict the ways in which a permutation can contain $12 \cdots (k-1)$. If $S$ is
a set of patterns, and the permutation $p$ avoids all patterns in $S$, then we
will say that $p$ avoids $S$, and we will write $\Av_n(S)$ for the number of such
permutations of length $n$, $\Av(S)$ for the set of such permutations of all
lengths (such a set is called a permutation class) and $\Av_n(S)$ for those such
permutations of length $n$.

Let $A_k$ be the set of $k$ patterns of length $k$ that start with an increasing
subsequence of length $k-1$. For instance,
$$A_5 = \{12345, 12354, 12453, 13452, 23451\}.$$ 
Note that a permutation $p = p_1p_2 \cdots p_n$ avoids $A_k$ if and only if the subsequence $p_1p_2 \cdots p_{n-1}$ avoids $12 \cdots (k-1)$. Therefore,
\[(3) \quad \Av_n(A_k) = n \Av_{n-1}(12 \cdots (k-1)).\]

If we remove one element of $A_k$, we find more interesting enumeration
problems. Let $A_{k,i} = A_k \setminus \{12 \cdots (i-1)(i+1) \cdots ki\}$, that is, the set $A_k$
with its element ending in $i$ removed. If $p$ avoids $A_{k,i}$, that means that
if a pattern of length $k$ that is contained in $p$ starts with an increasing
subsequence of length $k-1$, then the last entry of that pattern has to be
its $i$th largest entry. It is clear that for each $i \leq k$, the chain of inequalities
$(k-2)^2 \leq L(A_{k,i}) \leq (k-1)^2$ holds, since if a permutation avoids the
increasing pattern of length $k-1$, then it avoids $A_k$, and for all $i$, the set $A_{k,i}$
contains either the monotone pattern $12 \cdots k$, or the pattern $12 \cdots k(k-1)$,
each of which are avoided by fewer than $(k-1)^{2n}$ permutations of length $n$. See Theorem 4.10 and Exercise 4.1 in [6] for simple proofs of these upper
bounds. The interesting question is where in the interval $[(k-2)^2, (k-1)^2]$ are
the growth rates $L(A_{k,i})$ located.

Our goal in this paper is to determine the exponential growth rate $L(A_{k,i})$
of the sequence $\Av_n(A_{k,i})$, for each $i \leq k$. These growth rates fall into three
categories, depending on what $i$ is. For $2 \leq i \leq k-1$, we prove that
$L(A_{k,i}) = (k-2)^2$, so avoiding $A_{k,i}$ is just as hard (in the exponential sense)
as avoiding $12 \cdots (k-1)$. For $i = k$, we prove that $L(A_{k,k}) = (k-2)^2 + 1$. The
case of $i = 1$ is the most difficult. Note that in the case of $k = 3$, the set of
patterns to avoid is just $A_{3,1} = \{123, 132\}$, and it is well known (\cite{7}, Exercise 14.1) that $A_{5,1}(123, 132) = 2^{n-1}$. So in this case, $L(A_{5,1}) = (k - 2)^2 + 1$. On the other hand, if $k = 4$, then the set of patterns to avoid is $A_{4,1} = \{1234, 1243, 1342\}$, and the generating function of permutations avoiding that set of patterns is given in \cite{8} and could alternatively be computed using the $C$-machine framework in \cite{2}. It follows from that generating function that $L(A_{4,1}) = 2 + \sqrt{5} \approx 4.236$. We are unable to rigorously compute $L(A_{5,1})$, but we give extremely strong experimental evidence that $L(A_{5,1}) = 9$, corresponding in this $k = 5$ case to $(k - 2)^2$. 

2. When $2 \leq i \leq k - 1$

If a permutation $p$ avoids $A_{k,i}$, but contains an increasing subsequence of length $k - 1$, then the set of entries of $p$ that follow the last entry of that increasing subsequence is very restricted. This leads to the following theorem.

**Theorem 2.1.** For all $k \geq 3$, and all $2 \leq i \leq k - 1$, the equality

$$L(A_{k,i}) = (k - 2)^2$$

holds.

**Proof.** Let $p = p_1p_2 \cdots p_n \in AV_n(A_{k,i})$. For any entry $p_h$ of $p$, let the rank of $p_h$ be the length of the longest increasing subsequence of $p$ that ends in $p_h$. We define two words over the alphabet $\{1, 2, \cdots, k - 1\}$. Let $w(p)$ be the word whose $h$th letter is the rank of $p_h$, and let $z(p)$ be the word whose $h$th letter is the rank of $h$ as an entry in $p$.

Note that $i < k$, so it follows that each $p \in AV_n(A_{k,i})$ avoids the increasing pattern of length $k$, so all entries of $p$ have rank less than $k$. (This is not true when $i = k$, and that is why that case will have to be treated separately in Section 3.) Therefore, $w(p)$ and $z(p)$ will indeed be words over the mentioned alphabet. Furthermore, the map $p \rightarrow (w(p), z(p))$ is injective, since entries of a fixed rank $j$ form a decreasing sequence, hence $p$ can be recovered from its image $(w(p), z(p))$.

Let $p \in AV_n(A_{k,i})$, and again write $p = p_1p_2 \cdots p_n$. Let us take a closer look at $w(p)$. Let $j$ be the smallest index such that $w(p)_j = k - 1$. (If there is no such $j$, then $p$ avoids the increasing pattern $12 \cdots (k - 1)$, and so the number of possibilities for $p$ is less than $(k - 2)^{2n}$ as we explained in Section 1 following equation (2).) That means that there is an increasing subsequence of length $k - 1$ of $p$ ending in $p_j$. Let $a_1 < a_2 < \cdots < a_{k-1} = p_j$ be such a subsequence. If there are several such subsequences, then choose the one such that $a_{k-2}$ is maximal, then $a_{k-3}$ is maximal, and so on. Then for all entries $x$ on the right of $p_j$, the inequalities $a_{i-1} < x < a_i$ must hold, otherwise $a_1a_2 \cdots a_{k-1}x$ is a forbidden pattern. That means that all such entries of $p$ are of rank $i$ or higher, so the last $n - j$ letters of $w(p)$ are $i$ or larger. Therefore, the number of possible words $w(p)$ is at most
\[\sum_{j=1}^{n} (k-2)^{j-1}(k-i)^{n-j} \leq n(k-2)^{n-1}.\] Note that in the last estimate, we used the fact that \(i > 1.\)

Now consider \(z(p).\) As \(p_j\) is the leftmost entry of \(p\) that is of rank \(k - 1,\) and all subsequent entries of \(p\) are between \(a_{i-1}\) and \(a_i\) in value, it follows that all entries of \(p\) that are of rank \(k - 1\) except for \(p_j\) must be between \(a_{i-1}\) and \(a_i\) in value. Therefore, if \(t \notin (a_{i-1}, a_i)\), then the \(t\)th letter of \(z(p)\) cannot be \(k - 1\) (except once, the \(p_j\)th letter), while if \(t \in (a_{i-1}, a_i)\), then the \(t\)th letter of \(z(p)\) cannot be \(i - 1.\) Indeed, let us assume the entry \(t\) of \(p\) is of rank \(i - 1,\) and that \(a_{i-1} < t < a_i\) holds. Then \(t\) must be located on the left of \(a_{i-1}\) (since entries of the same rank form a decreasing subsequence), and hence, on the left of \(a_j.\) So there is an increasing subsequence in \(p\) that is of length \(i\) and whose last two entries are \(t\) and \(a_i,\) contradicting the maximality requirement of the preceding paragraph. Therefore, setting \(m = a_i - a_{i-1} - 1,\) we have fewer than \(n^3(k-2)^2m(k-2)^{n-m-3} < n^3(k-2)^{n-3}\) possibilities for \(z(p).\) Indeed, once we know the locations of \(p_j, a_{i-1}\) and \(a_i,\) we know that in those positions, \(z(p)\) has entries \(k - 1, i - 1\) and \(i,\) respectively. For each of the remaining \(n - 3\) letters of \(z(p),\) we have \(k - 2\) possibilities, because for some of them, \(k - 1\) is not a possibility, and for the rest of them, \(i - 1\) is not a possibility.

This implies that the total number of possibilities for the pair \((w(p), z(p))\) is less than \(n^4(k-2)^{2n}\), which proves our claim as we have already shown in the introduction that \(L(A_{k,i}) \geq (k-2)^2. \)

3. When \(i = k\)

The case of \(i = k\) leads to a different result.

**Theorem 3.1.** For \(k \geq 3,\) the equality

\[L(A_{k,k}) = (k - 2)^2 + 1\]

holds.

In this section, we will assume that the reader is familiar with the Robinson–Schensted correspondence. Readers who wish to learn about this correspondence can consult Chapter 3 of [13] for a thorough introduction, or Section 7.1 of [9] for a survey of some relevant facts.

**Proof.** Let \(p = p_1p_2\ldots p_n \in \mathcal{AV}_n(A_{k,k}),\) and let \(P(p)\) and \(Q(p)\) be the \(P\)-tableau and \(Q\)-tableau of \(p,\) obtained by the Robinson–Schensted correspondence. If \(p_{i+1}\) is the leftmost entry of \(p\) that is of rank \(k - 1,\) then \(p_{i+1}p_{i+2}\ldots p_n\) must be an increasing subsequence. This means that the last \(n - i\) positions that are filled in both tableaux are the \((k - 1)\)th, \(k\)th, \(\ldots,\) last positions of the first row. In particular, this implies that in \(Q(p),\) these positions are filled with the entries \(i + 1, i + 2, \ldots, n.\)

Note that this means that \(P(p)\) and \(Q(p)\) consist of two parts. One part consists of the first \(k - 2\) columns, which we will call the front, and the remaining columns, which are all of height one. We will call this second
part the tail. As we said above, in $Q(p)$, the content of this second part is known. Therefore, to specify $p$, it suffices to select the content of the tail of $P(p)$ in $\binom{n}{n-i}$ ways, then select the front of $P(p)$ and $Q(p)$ in at most $(k-2)^{2i}$ ways. This leads to the upper bound

$$Av_n(A_{k,k}) \leq \sum_{i=k-2}^{n-1} \binom{n}{n-i} (k-2)^{2i} \leq \sum_{i=0}^{n} \binom{n}{n-i} (k-2)^{2i} \leq ((k-2)^2 + 1)^n.$$ 

We still have to show that the exponential order of the sequence $Av_n(A_{k,k})$ is at least $(k-2)^2 + 1$. In order to do so, we construct $A_{k,k}$-avoiding permutations of length $n$ as follows. We choose an integer $\ell$ such that $0 \leq \ell \leq n$. Then we select an $\ell$-element subset $S$ of $[n]$. Next, we select a permutation $\pi$ on $[n]-S$ that avoids $12\cdots(k-1)$, and we postpend $\pi$ with the entries of $S$, written in increasing order, to get the permutation $p$. Note that $p$ avoids $A_{k,k}$. Indeed, patterns in $A_{k,k}$ increase until they reach an entry of rank $k-1$, then decrease. This is not possible in $p$, since the only entries of rank $k-1$ or higher are in the last $\ell$ positions, and $p$ is increasing in all those positions.

For a given $\ell$, the number of ways in which we can carry out the above steps is $\binom{n}{\ell} Av_{n-\ell}(12\cdots(k-1))$. For a given choice of $\ell$, each permutation $p$ will be obtained at most once, so we will obtain at least

$$\frac{1}{n+1} \sum_{\ell=0}^{n} \binom{n}{\ell} Av_{n-\ell}(12\cdots(k-1))$$

different permutations of length $n$ that avoid $A_{k,k}$. The division by $n+1$ is necessary because different choices of $\ell$ can lead to the same $p$.

Finally note that it follows from (1), substituting $k-1$ in the place of $k$, that there exists a constant $K_{k-1}$ such that for all positive integers $n$, the inequality

$$Av_n(12\cdots(k-1)) \geq K_{k-1} \frac{(k-2)^{2n}}{n(k^2-4k+3)/2}$$

holds. Comparing the last two displayed expressions, we see that we have constructed at least

$$\frac{K_{k-1} n^{(k^2-4k+3)/2}}{(n+1) (k-2)^{2(n-\ell)}} \leq \frac{K_{k-1}}{n^{(k^2-4k+3)/2}} ((k-2)^2 + 1)^n$$

elements of $Av_n(A_{k,k})$, proving that the exponential order of $Av_n(A_{k,k})$ is at least $(k-2)^2 + 1$. \qed
4. An injection

It follows from Theorems 2.1 and 3.1 that if \( n \) is large enough, then \( \text{Av}_n(A_{k,k-1}) < \text{Av}_n(A_{k,k}) \). In this section, we prove that \( \text{Av}_n(A_{k,k-1}) \leq \text{Av}_n(A_{k,k}) \) for every \( n \). The interest of this result lies in its proof, which is by a very simple injective map. It is rare that nontrivial inequalities between permutation class sizes can be proved injectively.

**Theorem 4.1.** For all positive integers \( n \), and all \( k \geq 3 \), the inequality
\[
\text{Av}_n(A_{k,k-1}) \leq \text{Av}_n(A_{k,k})
\]
holds.

**Proof.** Let \( p \in \text{AV}_n(A_{k,k-1}) \). Let \( p_i \) be the leftmost entry of \( p \) that is of rank \( k-1 \) if such an entry \( p_i \) exists. Then the entries \( p_{i+1}, p_{i+2}, \ldots, p_n \) must all be of rank \( k-1 \), and therefore, the subsequence \( p_ip_{i+1}\cdots p_n \) is decreasing. Now we define a map \( f : \text{AV}_n(A_{k,k-1}) \to \text{AV}_n(A_{k,k}) \) by setting \( f(p) = p \) if \( p \) does not have an entry of rank \( k-1 \), and \( f(p) = p_1p_2\cdots p_{i-1}p_ip_{i+1}\cdots p_n \) otherwise. In other words, \( f(p) \) is obtained by reversing the decreasing subsequence \( p_ip_{i+1}\cdots p_n \) of \( p \), that consists of entries of rank \( k-1 \) in \( p \).

It is then clear that \( f(p) \) avoids \( A_{k,k} \), since the only entries of rank \( k-1 \) or higher in \( f(p) \) are those in the last \( n-i+1 \) positions, and those entries are all in increasing order. Furthermore, \( f \) is injective, since given \( r \in \text{AV}_n(A_{k,k}) \), we can look at the maximal (non-extendible) increasing subsequence at the end of \( r \). That subsequence contains exactly one entry \( x \) of rank \( k-1 \). The only preimage of \( r \) under \( f \) can be obtained by reversing the subsequence of \( r \) that starts in \( x \) and goes all the way to the end of \( r \). (Note that \( f \) is not a bijection, because reversing that subsequence of \( r \) will not always result in a permutation in \( \text{AV}_n(A_{k,k-1}) \).) \( \square \)

5. When \( i = 1 \)

While we are not able to rigorously compute \( L(A_{5,1}) \), in this section we describe how to instead rigorously compute the first 642 terms of the counting sequence of \( A_{5,1} \)-avoiding permutations, from which we derive very strong numerical evidence that \( L(A_{5,1}) = 9 \).

**Conjecture 5.1.** The equality \( L(A_{5,1}) = 9 \) holds.

The Combinatorial Exploration paradigm developed by Albert, Bean, Claesson, Nadeau, Pantone, and Ulfarsson [1] is a computational framework for enumerating combinatorial objects [1]. Combinatorial Exploration is experimental in the sense that you do not know ahead of time whether it will succeed. However, when it does succeed, the output is a fully rigorous structural description of the class in the form of a combinatorial specification. From this combinatorial specification, the framework automatically produces a polynomial-time counting algorithm for the class, a system of

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1All of the relevant code is open-source and can be found on GitHub [4].
equations that the generating function for the class must satisfy, as well as other products that are not relevant here.

When applied to the permutation class $A_{5,1}$, Combinatorial Exploration finds a combinatorial specification in a few minutes, and a more favorable combinatorial specification in a few hours. The system of equations involves a main variable $x$ and two additional “catalytic” variables $y$ and $z$, and we do not know of any methods to solve it exactly, nor to extract from it any information about the asymptotic behavior of its solution. We used the resulting polynomial-time algorithm to compute the number of permutations of length $n$ in $A_{5,1}$ for $n \leq 641$ in about 20 hours. These initial terms of the counting sequence can be experimentally analyzed in several ways.

Firstly, we can use them to attempt to make a conjecture about the generating function of the counting sequence $A_{5,1}$. There are many software packages that do this kind of computation (e.g., Gfun [14] in Maple), all with various strengths and weaknesses. We used a package called GuessFunc [11] written in Maple by the second author. This package tries to fit the given initial terms to a rational, algebraic, D-finite, or differentially algebraic generating function; briefly, a generating function $f(x)$ is D-finite if it satisfies a non-trivial linear differential equation

$$p_k(x)f^{(k)}(x) + p_{k-1}(x)f^{(k-1)}(x) + \cdots + p_0(x)f(x) + q(x) = 0$$

where the coefficients $p_i(x)$ and $q(x)$ are polynomials, and $f(x)$ is differentially algebraic if there is a polynomial $P$ such that

$$P(x, f(x), f'(x), \ldots, f^{(k)}(x)) = 0.$$  

GuessFunc works, roughly, by assuming that the generating function has a particular form (e.g., D-finite of differential order 3 with polynomial coefficients of degree 12), and using the known initial terms of the counting sequence to set up a corresponding linear system of equations. If that system has a solution, that solution leads to a conjectured generating function.

Using the 642 initial terms, we were unable to make a conjecture that the generating function of $A_{5,1}$ has any of these forms. While this is not dispositive, it implies that if $A_{5,1}$ were D-finite, for example, either the differential order or the maximum degree of one of the polynomial coefficients would need to be quite large.

**Conjecture 5.2.** The generating function for $A_{5,1}$ is not D-finite.

Secondly, and more relevant to the pursuits of this work, we can apply the method of differential approximation [9, 10] to empirically estimate the asymptotic growth of the counting sequence. The method of differential approximation constructs a collection of D-finite generating functions whose initial power series coefficients match the known initial terms of the
given counting sequence (later terms are not expected to match). Then, the asymptotic behaviors of the D-finite generating functions are studied in aggregate in order to make predictions about the asymptotic behavior of the counting sequence in question. When tested on sequences whose asymptotic growth is independently already known, differential approximation shows a remarkable ability to provide very precise estimates.

Using the first 200 terms of the counting sequence of $A_{5,1}$, differential approximation predicts that the dominant singularity of its generating function (that is, the one closest to the origin) is located at $x_c \approx 0.1111111112$

indicating an exponential growth rate of $1/x_c \approx 9$ with very high precision. Further, it approximates the value of the corresponding critical exponent (a property of a given singularity) to be $\alpha \approx 1.999999990$

indicating a polynomial term of $n^{-1-\alpha} \approx n^{-3}$. As a result, we have strong experimental evidence for the following.

**Conjecture 5.3.** There exists a constant $C$ such that

$$\text{Av}_n(A_{5,1}) \sim C \cdot 9^n n^{-3}.$$  

The value of $C$ appears to be roughly $0.47$.

Such asymptotic growth, if verified, would not rule out the possibility that the generating function could be D-finite.

6. Further Experimental Results

In Section 5 we presented experimental evidence that the asymptotic growth of $A_{5,1}$ has the form $C \cdot 9^n n^{-3}$ and that the generating function of $A_{5,1}$ is non-D-finite. Combinatorial Exploration successfully produces combinatorial specifications for the other four classes of interest for $k = 5$, allowing us to compute many initial terms of the counting sequences. In this section we quickly summarize the experimental results we find for $A_{5,2}$, $A_{5,3}$, $A_{5,4}$, and $A_{5,5}$, each of which now has a known exponential growth rate due to the previous sections, as well as for $A_{6,1}$.

6.1. $A_{5,2}$. Combinatorial Exploration produces a combinatorial specification for the permutation class $A_{5,2}$ in about 5 hours. The resulting polynomial-time enumeration algorithm is slower than the one we found for $A_{5,1}$. We are only able to compute 91 terms in the counting sequence in about 5 hours using 300gb of memory. **GuessFunc** provides no conjecture for the generating function of this sequence. However, based on differential approximation, we make the following conjecture.

**Conjecture 6.1.** There exists a constant $C$ such that

$$\text{Av}_n(A_{5,2}) \sim C \cdot 9^n n^{-3}.$$
Differential approximation also shows a subdominant singularity (i.e., a singularity that is not a singularity closest to the origin) in the area of \( x \approx 0.18750 = 3/16 \). In future subsections, we will only mention subdominant singularities in cases where they are detected.

6.2. \( A_{5,3} \). For \( A_{5,3} \), we are able to calculate the first 130 terms of the counting sequence. \texttt{GuessFunc} is unable to produce a conjecture for the generating function of \( A_{5,3} \). Based again on differential approximation, we make the following conjecture.

**Conjecture 6.2.** There exists a constant \( C \) such that

\[
\text{Av}_n(A_{5,3}) \sim C \cdot 9^n n^{-3}.
\]

This time, differential approximation suggests a subdominant singularity in the area of \( x \approx 0.2 \).

6.3. \( A_{5,4} \). In this case, we can compute the first 444 terms of the counting sequence in 13 hours, using 182gb of memory. Unlike the previous cases, \texttt{GuessFunc} predicts using the first 160 terms that the generating function of \( A_{5,4} \) is D-finite with differential order 6 and maximum polynomial degree 17. (We should note here that a D-finite generating function often satisfies many different linear differential equations, and that there is normally a tradeoff in which lowering the differential order results in a higher polynomial degree, and vice versa.) We will not reproduce the differential equation here due to its size. We should note that the output of \texttt{GuessFunc} is merely a conjectured generating function, although we have a high degree of confidence in it because it was found using only the first 160 terms, and then matched nearly 300 additional terms. Conjectured generating functions can sometimes be rigorously confirmed using a “guess-and-check” approach if other information is already rigorously known, but that is not the case here.

Using differential approximation once again, we predict an asymptotic growth of the following form.

**Conjecture 6.3.** There exists a constant \( C \) such that

\[
\text{Av}(A_{5,4}) \sim C \cdot 9^n n^{-3}.
\]

6.4. \( A_{5,5} \). For \( A_{5,5} \) we have found the first 425 terms of the counting sequence using about 6.5 hours and 107gb of memory. Like the previous case, \texttt{GuessFunc} conjectures that the generating function is D-finite, this time with differential order 3 and maximum polynomial degree 8 and using only the first 55 terms. This one is small enough to print: the generating function
F(x) appears to satisfy the equation
\[
x^3(x - 1)(5x - 2)(10x - 1)(2x - 1)^2F''''(x) \\
+ x^2(2x - 1)(650x^4 - 1375x^3 + 909x^2 - 227x + 16)F''(x) \\
+ x(2x - 1)(800x^4 - 1850x^3 + 1277x^2 - 339x + 28)F'(x) \\
+ (200x^5 - 700x^4 + 716x^3 - 329x^2 + 76x - 8)F(x) \\
+ 2(5x - 2)^2 = 0
\]

Differential approximation predicts an asymptotic growth of the following form.

**Conjecture 6.4.** There exists a constant C such that
\[
\text{Av}_n(A_{5,5}) \sim C \cdot 10^n n^{-4}.
\]

The presence of the 10x - 1 factor in the coefficient of F''''(x) in the conjectured differential equation indicates the possibility (but not the certainty) of an exponential growth rate of 10 for the counting sequence, although we know already from Theorem 3.1 that the growth rate is indeed 10.

6.5. A_{6,1}. Lastly, for A_{6,1} we were only able to compute 71 terms of the counting sequence before running out of memory. We were unable to conjecture a generating function, but differential approximation suggests that the growth rate is 16. Although our confidence is not high, we announce the following conjecture.

**Conjecture 6.5.** There exists a constant C such that
\[
\text{Av}_n(A_{6,1}) \sim C \cdot 16^n n^{-13/2}.
\]

7. Further directions

The strong computational evidence obtained in this paper about the subexponential factor of the asymptotic growth of our sequences raises several intriguing questions. Answering them could shed some light on analogous problems for longer patterns as well.

First, we saw in Sections 5 and 6 that if 1 \leq i \leq 4, then there is strong numerical evidence to suggest that Av_n(A_{5,i}) \sim C_i \cdot 9^n n^{-3}, where C_i is some positive constant. This would mean that Av_n(A_{5,i}) is just a linear factor larger than Av(12\cdots(k-1)), and therefore, by formula (5), it only differs from Av_n(A_5) in a constant factor. This result would be surprising on its own, and it would also imply that the generating function of the sequence Av_n(A_{5,i}) is not algebraic if 1 \leq i \leq 4. Note that the behavior of Av_n(A_{6,1}) seems to be very similar. If its growth rate is indeed C \cdot 16^n n^{-13/2} as we suggested at the end of Section 6, then the growth rate of that sequence is one linear factor larger than that of the sequence Av_n(12345).

Second, in Section 6, we saw data suggesting that Av_n(A_{5,5}) \sim C_5 \cdot 10^n n^{-4}. We know from [12] that Av_n(1234) \sim C \cdot 9^n n^{-4}. Perhaps our proof of
Theorem 3.1 could be refined to prove this more precise asymptotic formula for \( \text{Av}_n(A_{5,5}) \). Such a result would imply that the generating function of the sequence is not algebraic.

The third, and perhaps most interesting, question is the exponential growth rate of the sequence \( \text{Av}_n(A_{k,1}) \) for general \( k \). We saw in the introduction that \( L(A_{3,1}) = (3 - 2)^2 + 1 = 2 \), and \( L(A_{4,1}) = 2 + \sqrt{5} \), which is between 4 and 5, that is, the values of \((k - 2)^2\) and \((k - 2)^2 + 1\). Numerical evidence obtained in this paper suggests that \( L(A_{5,1}) = 9 = (5 - 2)^2 \) and \( L(A_{6,1}) = 16 = (6 - 2)^2 \). This raises the following intriguing question.

**Question 7.1.** Is it true that if \( k \geq 5 \), then \( L(A_{k,1}) = (k - 2)^2 \) ?

### References

[1] M. H. Albert, C. Bean, A. Claesson, É. Nadeau, J. Pantone, and H. Ulfarsson. Combinatorial Exploration: An algorithmic framework for enumeration. [https://arxiv.org/abs/2202.07715](https://arxiv.org/abs/2202.07715), 2022.

[2] M. H. Albert, C. Homberger, J. Pantone, N. Shar, and V. Vatter, Generating permutations with restricted containers. *J. Combin. Theory Ser. A* 157 (2018), 205–232.

[3] M. H. Albert, J. Pantone, and V. Vatter, On the growth of merges and staircases of permutation classes. *Rocky Mountain J. Math.* 49 (2019), no. 2, 355–367.

[4] C. Bean, J. S. Eliasson, T. K. Magnusson, É. Nadeau, J. Pantone, H. Ulfarsson, Tilings: Combinatorial Exploration for permutation patterns. [https://github.com/PermutaTriangle/Tilings](https://github.com/PermutaTriangle/Tilings), June 2021. DOI: [https://doi.org/10.5281/zenodo.5810636](https://doi.org/10.5281/zenodo.5810636).

[5] M. Bóna, The limit of a Stanley-Wilf sequence is not always rational, and layered patterns beat monotone patterns. *J. Combin. Theory Ser. A* 110 (2005), no. 2, 223–235.

[6] M. Bóna, Combinatorics of Permutations, 2nd edition, CRC Press, 2012.

[7] M. Bóna, A Walk Through Combinatorics, 4th edition, World Scientific, 2016.

[8] D. Callan, T. Mansour, Five subsets of permutations enumerated as weak sorting permutations, *Southeast Asian Bull. Math.* 42 (2018), no. 3, 327–340.

[9] A. J. Guttmann, Asymptotic analysis of power-series expansions. In C. Domb and J. L. Lebowitz, editors, *Phase Transitions and Critical Phenomena, Vol. 13*, pages 1–234. Academic Press, London, England, 1989.

[10] J. Pantone, DiffApprox: A Maple library to predict the asymptotic behavior of counting sequences given some initial terms. [https://github.com/jaypantone/DiffApprox](https://github.com/jaypantone/DiffApprox), December 2021. DOI: [https://doi.org/10.5281/zenodo.5810652](https://doi.org/10.5281/zenodo.5810652).

[11] J. Pantone, GuessFunc: A Maple library to guess the generating function of a counting sequence given some initial terms. [https://github.com/jaypantone/guessfunc](https://github.com/jaypantone/guessfunc), December 2021. DOI: [https://doi.org/10.5281/zenodo.5810636](https://doi.org/10.5281/zenodo.5810636).

[12] A. Regev, Asymptotic values for degrees associated with strips of Young diagrams, *Advances in Mathematics*, 41 (1981), 115–136.

[13] B. Sagan, The Symmetric Group: Representations, Combinatorial Algorithms, and Symmetric Functions, (Graduate Texts in Mathematics, Vol. 203) 2nd Edition, Springer, 2001.

[14] B. Salvy and P. Zimmermann, GFUN: A Maple package for the manipulation of generating and holonomic functions in one variable. *ACM Trans. Math. Softw.*, 20(2):163–177, June 1994.

[15] V. Vatter, Permutation classes. In: Handbook of Enumerative Combinatorics, Miklós Bóna, editor, CRC Press, 2015.
