GENERALIZED KÄHLER-RICCI FLOW AND THE CLASSIFICATION OF NONDEGENERATE GENERALIZED KÄHLER SURFACES

JEFFREY STREETS

ABSTRACT. We study the generalized Kähler-Ricci flow on complex surfaces with nondegenerate Poisson structure, proving long time existence and convergence of the flow to a weak hyperKähler structure.

1. Introduction

Generalized Kähler geometry and generalized Calabi-Yau structures first arose from investigations into supersymmetric sigma models [9]. These structures were rediscovered in the work of Hitchin [16], growing out of a search for special geometries defined by volume functionals on differential forms. The relationship between these points of view was elaborated upon in the thesis of Gualtieri [14]. These structures have recently attracted enormous interest in both the physics and mathematical communities as natural generalizations of Kähler Calabi-Yau structures, inheriting a rich physical and geometric theory. We will focus entirely on the “classical” description of generalized Kähler geometry (cf. [9]), i.e. not relying on the more intrinsic point of view developed by Gualtieri [14] using Courant algebroids. For our purposes a generalized Kähler manifold is a smooth manifold $M$ with a triple $(g, I, J)$ consisting of two complex structures $I$ and $J$ together with a metric $g$ which is Hermitian with respect to both. Moreover, the two Kähler forms $\omega_I$ and $\omega_J$ satisfy

$$d\omega_I = H = -d\omega_J, \quad dH = 0,$$

where the first equation defines $H$, and $d\omega_I = \sqrt{-1}(\bar{\partial} - \partial)$ with respect to the complex structure defined by $I$, and similarly for $J$.

A natural notion of Ricci flow adapted to the context of generalized Kähler geometry was introduced in work of the author and Tian [30]. We will call this flow generalized Kähler-Ricci flow (GKRF). This evolution equation was discovered in the course of our investigations into the more general “pluriclosed flow,” [28], and has the interesting feature that the complex structures must also evolve to preserve the generalized Kähler condition. Explicitly it takes the form

$$\frac{\partial}{\partial t} g = -2 \text{Rc}^g + \frac{1}{2} \mathcal{H}, \quad \frac{\partial}{\partial t} H = \Delta_g H,$$

$$\frac{\partial}{\partial t} I = L_{\theta_I} I, \quad \frac{\partial}{\partial t} J = L_{\theta_J} J,$$

(1.1)

where $H_{ij} = H_{i\bar{p}q} H_{\bar{p}q}^{\bar{j}}$, and $\theta_I, \theta_J$ are the Lee forms of the corresponding Hermitian structures. See [22] for a derivation of these equations. The metric and three-form component of the flow initially arose as the renormalization group flow for nonlinear sigma models coupled to a skew-symmetric $B$-field (cf. [24]). Supersymmetry considerations eventually related this sigma model to generalized Kähler geometry. Given this, one might expect the renormalization group flow to preserve generalized Kähler geometry. The surprising observation of [30] is that this is indeed so,
but only after introducing a further evolution equation for the complex structures themselves. It remains an interesting problem to derive these evolution equations from a Langrangian-theoretic standpoint.

A central feature of generalized Kähler geometry, observed first by Pontecorvo [25], Hitchin [17], is that the tensor \( g^{-1}[I, J] \) defines a holomorphic Poisson structure. Previously the author studied GKRF in one of the natural “extremes” of generalized Kähler geometry, namely when this Poisson structure vanishes. In this setting it was observed that the complex structures actually remain fixed, and that the flow reduces to a nonconvex fully nonlinear parabolic equation for a scalar potential function [27]. In this paper we focus entirely on the case when this Poisson structure is nondegenerate, in which case we will refer to the generalized Kähler structure itself as “nondegenerate.” It is trivial to note that GKRF will preserve this condition, at least for a short time, since it is an open condition. Whereas in the case \([I, J] = 0\) the flow of complex structures dropped out of the system, as we will see below, the evolving complex structures essentially determine the entire GKRF in the nondegenerate setting. Our main theorem gives a complete picture of the long time existence behavior of this flow in the case of dimension 4, together with a rough picture of the convergence.

**Theorem 1.1.** Let \((M^4, g, I, J)\) be a nondegenerate generalized Kähler four-manifold. The solution to generalized Kähler-Ricci flow with initial data \((g, I, J)\) exists for all time. Moreover, the associated almost hyperKähler structure \(\{\omega_{K,(i)}\} \) converges subsequentially in the \(I\)-fixed gauge to a triple of closed currents \(\{\omega_{\infty,K}^\infty\}\).

**Remark 1.2.**
1. See Definition 3.5 for the definition of the associated almost hyperKähler structure, and see Remark 2.4 for the meaning of the flow in the “\(I\)-fixed gauge”.
2. The triple of limiting currents can be interpreted as a weak hyperKähler structure. Conjecturally the flow should converge to a hyperKähler metric exponentially in the \(C^\infty\) topology but this is not yet attainable for technical reasons. In the case of tori the strong convergence follows from ([26] Theorem 1.1).
3. It had previously been observed (see Remark 3.2) that one could construct large classes of nondegenerate generalized Kähler structures by special deformations of hyperKähler structures. Theorem 1.1 roughly indicates that this the only way to construct such structures.
4. The solutions to GKRF in Theorem 1.1 are never solutions to Kähler-Ricci flow, unless the initial data is already hyperKähler in which case the Kähler-Ricci flow and generalized Kähler-Ricci flow are both fixed.

It seems natural to expect similar behavior for the generalized Kähler-Ricci flow in the nondegenerate setting in all dimensions \(n = 4k\). In particular, one might expect long time existence and convergence of the flow to a generalized Calabi-Yau structure. In dimensions greater than 4 it does not follow directly that such a structure is hyperKähler, and it would seem that more general examples should exist, although we do not know of any. While many aspects of our proof will certainly extend to higher dimensions, some key estimates exploit the low-dimensionality. One important breakthrough would be to achieve, if possible, a reduction of the flow to that of a potential function. Local constructions [22] indicate that one can express generalized Kähler structures in terms of a single potential function, but in the nondegenerate setting the objects are described as fully nonlinear expressions in the Hessian of the potential. Thus it remains far from clear if it is possible to reduce the GKRF to a scalar potential flow, as has been achieved in the setting of vanishing Poisson structure (cf. [27]). Our calculations below give hope for the possibility of such a scalar reduction, as we show for instance that all curvature quantities involved in the flow equations can be expressed in terms of the angle function between the complex structures.
The proof involves a number of a priori estimates derived using the maximum principle. Much of the structure between the two complex structures in a bihermitian triple \((g, I, J)\) is captured by the so-called angle function \(p = \text{tr}(IJ)\). As we will see in \(\S\) a certain function \(\mu\) of the angle satisfies the time-dependent heat equation precisely along the flow. This yields a priori control over the angle, and moreover a strong decay estimate for the gradient of \(\mu\). This quantity controls the torsion, yielding a priori decay of the torsion along the flow. Given this estimate, we switch points of view and study the flow merely as a solution to pluriclosed flow, and use the reduction of the flow to a parabolic flow of a \((1, 0)\)-form introduced in [26, 29]. In the presence of this torsion decay we can establish upper and lower bounds for the metric depending on a certain potential function associated to the flow. This potential function can be shown to grow linearly, showing time-dependent upper and lower bounds on the metric. We then apply the \(C^\alpha\) estimate on the metric established in [26] to obtain full regularity of the flow. Using the decay of the torsion we can derive the weak convergence statement in the sense of currents.

Here is an outline of the rest of the paper. In \(\S\) we establish background results and notation, and also review the generalized Kähler–Ricci flow. Next in \(\S\) we explain fundamental properties of nondegenerate generalized Kähler surfaces. Then in \(\S\) we develop a number of a priori estimates for the flow. Lastly in \(\S\) we establish Theorem 1.1.

Acknowledgements: The author would like to thank Richard Bamler, Joel Fine, Hans-Joachim Hein, Gang Tian, Alan Weinstein, and Qi Zhang for helpful discussions. Also, this work owes a significant intellectual debt to the series of works [7, 9, 18, 20, 21, 22] arising from mathematical physics. Moreover, I have benefited from many helpful conversations with Martin Rocek in understanding these papers. The author especially thanks Marco Gualtieri for many very useful conversations on generalized Kähler geometry. Lastly, we benefited quite significantly from Vestislav Apostolov, who provided much help in understanding his papers on bihermitian geometry and moreover suggested obtaining convergence results in the sense of currents.

2. Background

2.1. Notation and conventions. In this section we fix notation, conventions, and recall some fundamental constructions we will use in the sequel. First, given \((M^{2n}, g, J)\) a Hermitian manifold, let

\[ \omega(X, Y) = g(X, JY) \]

be the Kähler form. The Lee form is defined to be

\[ \theta = d^c \omega \circ J. \]

We let \(\nabla\) denote the Levi-Civita connection of \(g\). We will make use of two distinct Hermitian connections, in particular the Bismut connection \(\nabla^B\) (see [6]) and the Chern connection \(\nabla^C\). These are defined via

\[
\begin{align*}
g(\nabla^B_X Y, Z) &= g(\nabla_X Y, Z) + \frac{1}{2} d^c \omega(X, Y, Z), \\
g(\nabla^C_X Y, Z) &= g(\nabla_X Y, Z) + \frac{1}{2} d\omega(JX, Y, Z).
\end{align*}
\]

Here \(d^c = \sqrt{-1}(\partial - \bar{\partial})\), hence

\[ d^c \omega(X, Y, Z) = -d\omega(JX, JY, JZ). \]
We will denote the torsion tensors by $H$ and $T$ respectively, i.e.

$$H(X,Y,Z) = g(\nabla^B_X Y - \nabla^B_Y X - [X,Y],Z)$$

$$T(X,Y,Z) = g(\nabla^C_X Y - \nabla^C_Y X - [X,Y],Z).$$

Both the Bismut and Chern connections induce unitary connections on $K^{-1}_M$, with associated Ricci-type curvatures, which we denote via

$$\rho_{B,C}(X,Y) = R_{B,C}(X,Y,e_i,e_i).$$

We now record some formulas for Hermitian surfaces needed in the sequel.

**Lemma 2.1.** Let $(M^4,g,J)$ be a Hermitian surface. Then

$$H_{ijk} = -J_i^l \theta_k \omega_{lj} - J_k^l \theta_i \omega_{lj} - J^k_l \theta_i \omega_{lk}.$$  \hfill (2.2)

**Proof.** First of all, for a complex surface we have the formula $d\omega = \theta \wedge \omega$, which we express in coordinates as

$$(d\omega)_{ijk} = \theta_i \omega_{jk} + \theta_k \omega_{ij} + \theta_j \omega_{ki}.$$

Now using that $H = -d\omega(J,J,J)$, we have

$$H_{ijk} = -d\omega_{pqr} J_i^p J_j^q J_k^r$$

$$= - (\theta_p \omega_{qp} + \theta_r \omega_{pq} + \theta_q \omega_{rp}) J_i^p J_j^q J_k^r$$

$$= - J_i^p \theta_k \omega_{pq} - J_k^l \theta_i \omega_{lj} - J^k_l \theta_i \omega_{lk},$$

as required. \hfill \square

**Lemma 2.2.** (cf. [10]) Let $(M^4,g,J)$ be a Hermitian surface. Then

$$\Delta f = \Delta_C f - \langle \theta, \nabla f \rangle.$$ 

**Proof.** First observe that we can express in coordinates that

$$J_i^p d\omega_{pj} = J_i^p [\theta_p \omega_{pj} + \theta_k \omega_{kp} + \theta_j \omega_{jk}] = J_i^p \theta_k \omega_{pj} + \theta_k g_{ij} - \theta_j g_{ik}.$$ 

Hence we directly compute that

$$\Delta f = g^{ij} \nabla_i \nabla_j f$$

$$= g^{ij} \left( \partial_i \partial_j f - \Gamma^k_{ij} \partial_k f \right)$$

$$= g^{ij} \left( \partial_i \partial_j f - (\Gamma^C)^k_{ij} \partial_k f + (\Gamma - \Gamma^C)^k_{ij} \partial_k f \right)$$

$$= \Delta_C f + g^{ij} \left( -\frac{1}{2} J_i^p d\omega_{pj} g^{kl} \right) \nabla_k f$$

$$= \Delta_C f - \frac{1}{2} g^{ij} \nabla_l f \left( J_i^p \theta_p \omega_{jl} + \theta_l g_{ij} - \theta_j g_{il} \right)$$

$$= \Delta_C f - \frac{1}{2} g^{ij} \nabla_l f \left( J_i^p \theta_p g_{jql} J_q^q + \theta_l g_{ij} - \theta_j g_{il} \right)$$

$$= \Delta_C f - \langle \theta, \nabla f \rangle.$$ \hfill \square

**Lemma 2.3.** Let $(M^4,g,J)$ be a Hermitian surface. Then

$$\nabla_i J_j^k = \frac{1}{2} \left[ -g^{lk} \theta^j_i \omega_{ij}^k + J_j^p \theta^j_i \delta^k_p - g^{km} J_m^q \theta^l_q g_{ij} - \theta^l_j J_i^k \right].$$
2.2. Generalized Kähler Ricci flow. In this subsection we review the construction of generalized Kähler Ricci flow (GKRF) from [29]. To begin we review the pluriclosed flow [28]. Let \((M^{2n}, g, J)\) be a Hermitian manifold as above. We say that the metric is pluriclosed if

\[
\sqrt{-1} \partial \bar{\partial} \omega = dd^c \omega = 0.
\]

In [28] the author and Tian introduced the pluriclosed flow equation for such a metric,

\[
\frac{\partial}{\partial t} \omega = -(\rho_B)^{1,1} = \partial \partial^c \omega + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \log \det g
\]

where \(\rho_B\) is the curvature of the determinant line bundle induced by the Bismut connection as described above, and the \((1, 1)\) superscript indicates the projection onto the space of \((1, 1)\) forms. In [28] we showed that this flow preserves the pluriclosed condition and agrees with Kähler-Ricci flow when the initial data is Kähler. As exhibited in [29] Proposition 6.3, the induced pairs of metrics and Bismut torsions \((g_t, H_t)\) satisfy

\[
\frac{\partial}{\partial t} g = -2 \text{Rc}^g + \frac{1}{2} \mathcal{H} - \mathcal{L}_{\partial \partial^c} g,
\]

\[
\frac{\partial}{\partial t} H = \Delta_d H - \mathcal{L}_{\partial \partial^c} H,
\]

where \(\mathcal{H}_{ij} = H_{(\omega^q H_j)}\).

This crucial formula shows how to construct a flow which preserves generalized Kähler geometry. In particular consider \((M^{2n}, g, I, J)\) a generalized Kähler structure. Then as \((g, I)\) and \((g, J)\) are pluriclosed structures, we can construct two solutions to pluriclosed flow with these initial data, denoting the Kähler forms \(\omega_i^I, \omega_i^J\). Then let \(\phi_i^I, \phi_i^J\) denote the one parameter families of diffeomorphisms generated by \((\theta^I)^i_j, (\theta^J)^i_j\) respectively. It follows from (2.4) that both \(((\phi_i^I)^* g_t^I, (\phi_i^I)^* H_t^I)\) and \(((\phi_i^J)^* g_t^J, -(\phi_i^J)^* H_t^J)\) are solutions to

\[
\frac{\partial}{\partial t} g = -2 \text{Rc}^g + \frac{1}{2} \mathcal{H},
\]

\[
\frac{\partial}{\partial t} H = \Delta_d H,
\]

with the same initial conditions, since the original structure was generalized Kähler. It follows that

\[
(\phi_i^I)^* g_t^I = (\phi_i^I)^* g_t^J =: g_t
\]

\[
(\phi_i^I)^* H_t^I = -(\phi_i^J)^* H_t^J =: H_t
\]

defines a one-parameter family of generalized Kähler structures. To make it more manifest, observe that by construction certainly \(g_t\) is compatible with both of the integrable complex
structures \((\phi^t_I)^*I, (\phi^t_J)^*J\). Thus, in principle the two complex structures must evolve to preserve the generalized Kähler condition. Making this explicit we arrive at the generalized Kähler-Ricci flow system (cf. [30]):

\[
\frac{\partial}{\partial t} g = -2 \text{Rc}^g + \frac{1}{2} H, \quad \frac{\partial}{\partial t} H = \Delta_d H, \\
\frac{\partial}{\partial t} I = L_{\theta^*_I} I, \quad \frac{\partial}{\partial t} J = L_{\theta^*_J} J,
\]

as claimed in the introduction.

**Remark 2.4.** In obtaining estimates for the flow, we will exploit two different points of view, each of which makes performing certain calculations easier. Certain estimates will use the system (1.1) directly, which we will call a solution “in the B-field gauge.” Other times it is easier to work with pluriclosed flow directly, so we pull back the flow to the fixed complex manifold \((M^{2n}, I)\). In other words by pulling back the entire system by the family of diffeomorphisms \((\phi^t_I)^{-1}\) we return to pluriclosed flow on \((M^{2n}, I)\), which encodes everything about the GKRF except the other complex structure. But the construction above makes clear that the other complex structure is

\[J_t = [(\phi^t_I)^{-1} \circ \phi^t_J]^* J.\]

We will refer to this point of view on GKRF as occurring “in the I-fixed gauge.”

### 3. Nondegenerate generalized Kähler surfaces

In this section we record some basic properties of generalized Kähler surfaces with nondegenerate Poisson structure. First we derive special linear algebraic aspects of this structure related to the angle function, (see Definition 3.1), which plays a central role throughout what follows. Next we record some background on the Poisson structures associated to a generalized Kähler manifold, and its relationship to the construction of large families of nondegenerate generalized Kähler structures. Then we exhibit some general identities for the curvature and torsion of these structures which further emphasize the central role of the angle function, and which are essential to the analysis to follow.

#### 3.1. Linear algebraic structure

In this subsection we recall well-known fundamental linear algebraic properties of biHermitian four-manifolds. The low dimensionality results in some key simplifications which are central to the analysis to follow.

**Definition 3.1.** Given \((M^{2n}, g, I, J)\) a biHermitian manifold, let

\[p = \frac{1}{2n} \text{tr}(I \circ J)\]

denote the *angle* between \(I\) and \(J\). Observe that since \(I\) and \(J\) are both compatible with \(g\), by the Cauchy-Schwarz inequality we obtain

\[|p| = \frac{1}{2n} |\langle I, J \rangle_g| \leq \frac{1}{2n} |I|_g |J|_g = 1.\]

**Lemma 3.2.** Let \((M^{4}, g, I, J)\) be a biHermitian manifold where \(I\) and \(J\) induce the same orientation. Then

\[\{I, J\} = 2p1.\]
Proof. Since $I$ and $J$ induce the same orientation, $\omega_I$ and $\omega_J$ are both self-dual forms. Fix some point $x \in M$, and consider a $g$-orthonormal basis $\omega_1, \omega_2, \omega_3$ for self-dual two forms at $x$. Direct calculations show that the corresponding endomorphisms given by raising an index via $g$, call them $K_i$, all anticommute and satisfy the quaternion relations. Moreover, since we can express $F_I = a\omega_1 + b\omega_2 + c\omega_3$, with $a^2 + b^2 + c^2 = 1$, it follows that $I$ (and similarly $J$) are part of this quaternionic structure. In particular we may write

$$I = a_I K_1 + b_I K_2 + c_I K_3, \quad J = a_J K_1 + b_J K_2 + c_J K_3.$$  

Since the $K_i$ pairwise anticommute one then directly computes that

$$\{I, J\} = -2(a_I a_J + b_I b_J + c_I c_J)1.$$  

That is, $\{I, J\}$ is diagonal, and this forces the final equation by the definition of $p$. □

Lemma 3.3. Let $(M^4, g, I, J)$ be a biHermitian manifold where $I$ and $J$ induce the same orientation. Then

$$[I, J]^2 = 4(p^2 - 1)1.$$  

Proof. We directly compute

$$[I, J]^2 = (IJ - JJ)(IJ - JJ) = IJJI + JJJI - 21 = IJ(-JI + 2p1) + JJ(-IJ + 2p1) - 21 = 2p\{I, J\} - 41 = 4(p^2 - 1)1.$$  

□

Definition 3.4. Given $(M^{2n}, g, I, J)$ a generalized Kähler manifold, we say that it is nondegenerate if $\sigma$ defines a nondegenerate pairing on $TM$. It was observed by Pontecorvo [25] that when $n = 4$, this defines a holomorphic Poisson structure. This was extended to higher dimensions by Hitchin [17]. In particular, $\sigma$ is of type $(2, 0) + (0, 2)$ with respect to both complex structures, and is holomorphic with respect to both complex structures.

Definition 3.5. Let $(M^4, g, I, J)$ be a nondegenerate generalized Kähler structure. The associated almost hyperKähler structure is the triple $(K_0, K_1, K_2)$ where

$$K_0 = q^{-1}(IJ + p1), \quad K_1 = I, \quad K_2 = q^{-1}(J - pI),$$  

(3.1)
for \( q = \sqrt{1-p^2} \). Each \( K_i \) is an almost complex structure compatible with the given conformal class, and we will equivalently refer to the associated Kähler forms \( \{ \omega_{K_i} \} \) as the almost hyperKähler structure. Direct calculations show that any pair of \( \omega_{K_i} \) satisfies

\[
\omega_{K_i} \wedge \omega_{K_i} = \omega_{K_j} \wedge \omega_{K_j}, \quad \omega_{K_i} \wedge \omega_{K_j} = 0.
\]

Later we will need an explicit formula for the Kähler form associated to \( K_0 \). Direct calculations show that

\[
\omega_{K_0} = \Omega - i\omega_{K_1}
\]

As it turns out, this associated Kähler forms encode the almost complex structures, as made precise in the following lemma:

**Lemma 3.6.** (\cite{[2]} Lemma 5, cf. \cite{[11], [23]}) Let \( M \) be an oriented 4-manifold and \( \Phi_1, \Phi_2 \) a pair of nondegenerate real 2-forms on \( M \) satisfying the conditions

\[
\Phi_1 \wedge \Phi_1 = \Phi_2 \wedge \Phi_2, \quad \Phi_1 \wedge \Phi_2 = 0.
\]

Then there is a unique almost-complex structure \( J \) on \( M \) such that the 2-form \( \Omega = \Phi_1 - \sqrt{-1}\Phi_2 \) is of type \((2,0)\) with respect to \( J \). If moreover \( \Phi_1 \) and \( \Phi_2 \) are closed, then \( J \) is integrable and \( \Omega \) defines a holomorphic symplectic structure on \((M, J)\).

### 3.2. Local generality.

Given the existence of so much rigid holomorphic Poisson structure, one might think that nondegenerate generalized Kähler manifolds are perhaps are fully rigid, with only finite dimensional classes of examples. This is not the case, as was shown by (\cite{[2], [13]}). We follow the discussion of (\cite{[13]} Examples 6.31, 6.32). There it is shown that the specification of a nondegenerate generalized Kähler structure in dimension \( n = 4 \) with the same orientation is equivalent to specifying three closed 2-forms \( B, \omega_1, \omega_2 \) such that

\[
B \wedge \omega_1 = B \wedge \omega_2 = \omega_1 \wedge \omega_2 = \omega_1^2 + \omega_2^2 - 4B^2 = 0, \quad \omega_1^2 = \lambda \omega_2^2, \quad \lambda > 0.
\]

In particular, given this data, the generalized Kähler structure is determined by pure spinors \( e^{B+\sqrt{\lambda}\omega_1}, e^{-B+\sqrt{\lambda}\omega_2} \) (see \cite{[13]} for the pure spinor description of generalized Kähler structures).

One can use this interpretation to produce non-hyperHermitian nondegenerate generalized Kähler structures. In particular, start with a hyperKähler triple \((M, g, I, J, K)\), and let \( F_t \) be a one-parameter family of diffeomorphisms generated by a \( \omega_K \)-Hamiltonian vector field. For \( t \) sufficiently small, the forms

\[
B = \omega_K, \quad \omega_1 = \omega_I - F_t^* \omega_J, \quad \omega_2 = \omega_I + F_t^* \omega_J
\]

satisfy (3.4). Moreover, as shown in (\cite{[2]} Lemma 6), if \( f \) denotes the \( \omega_K \)-Hamiltonian function generating the \( \omega_K \) Hamiltonian vector field, then for the angle function associated to the generalized Kähler data one has

\[
\frac{\partial}{\partial t} \rho \bigg|_{t=0} = \frac{1}{2} \Delta f.
\]

Hence for any nonconstant \( f \) one produces structures with nonconstant angle, which are hence not hyperKähler (cf. Lemma 3.7 below). This same deformation was used in \cite{[2]} to produce non-hyperHermitian, strongly biHermitian, conformal classes on hyperHermitian Hopf surfaces. However, as exhibited in \cite{[2]} Corollary 2, the Gauduchon metrics in these conformal classes are never generalized Kähler, and so these do not play a role in the analysis of Theorem 1.1. Note that in this example the Poisson structure is \( \omega_K \), and is crucial to the construction of the deformation. This type of deformation, arising from the associated Poisson structure, was generalized by Goto \cite{[12]}.
Hence there are large families of nondegenerate generalized Kähler structures. As it turns out, it is possible to give a simple characterization of when such a structure on a surface is hyperKähler. This lemma is generally known (cf. [25]), and we include a simple proof based on our curvature calculations.

**Lemma 3.7.** Let \((M^4, g, I, J)\) be a nondegenerate generalized Kähler surface. Then \((M^4, g)\) is hyperKähler if and only if \(p\) is constant.

**Proof.** It follows from direct calculations that any two complex structures which are part of a hyperKähler sphere have constant angle. Conversely, if \(p\) is constant then it follows from Lemma 3.11 below that \(\theta_I = \theta_J = 0\), which since we are on a complex surface implies \(d\omega_I = 0\), i.e. that the metric is Kähler. It then follows from Lemma 3.13 that the metric is Calabi-Yau, hence hyperKähler. \(\square\)

### 3.3. Torsion and curvature identities

Here we record a number of useful identities for the torsion and curvature of generalized Kähler manifolds. Most of these identities have been previously observed in the literature, but we include the short derivations for completeness and to fix conventions/notation.

**Lemma 3.8.** ([3] Proposition 3) Let \((M^4, g, I, J)\) be a generalized Kähler manifold such that \(I\) and \(J\) induce the same orientation. Then \(\theta^I = -\theta^J\).

**Proof.** Note that \(\omega_I\) is self-dual. Moreover, since \(I\) induces the metric orientation the action of \(I\) on forms commutes with Hodge star. Hence

\[
\theta^I = Id^*\omega_I = I \star d \star \omega_I = I \star d\omega_I = \star Id\omega_I = \star H_I.
\]

Similarly one obtains \(\theta^J = \star H_J\). Since \(H_I = -H_J\) the result follows. \(\square\)

**Remark 3.9.** Given the result of Lemma 3.8, to simplify notation we will adopt the convention \(\theta = \theta^I\).

**Lemma 3.10.** (cf. [2] Lemma 7) Let \((M^4, g, I, J)\) be a nondegenerate generalized Kähler four-manifold. Then

\[
dp = \frac{1}{4} (\theta^I - \theta^J) [I, J] = \frac{1}{2} \theta[I, J].
\]

**Proof.** By Lemma 2.1 we have

\[
H_{ijk} = -J^l_i\theta_l\omega_{jk} - J^l_j\theta_l\omega_{ik} - J^l_k\theta_l\omega_{ij}.
\]
Thus we compute
\[
4 \nabla_a p = \nabla_a (I^r_s J^r_s) \\
= \nabla_a I^r_s J^r_s + I^r_s \nabla_a J^r_s \\
= \frac{1}{2} \left( (H^I_{ar} I^r_s - (H^I)_{at} J^r_s) I^r_s + \frac{1}{2} I^r_s [(H^I)_{as} J^r_t - (H^I)_{at} J^r_s] \right) \\
= -\frac{1}{2} \left[ g^{tv} \left( I^t_s \theta^v_w \omega^I_{rw} + I^t_s \theta^v_w \omega^I_{at} + I^t_s \theta^v_w \omega^I_{va} \right) \right] I^r_s J^r_s \\
+ \frac{1}{2} \left[ g^{sv} \left( I^t_s \theta^v_w \omega^I_{tv} + I^t_s \theta^v_w \omega^I_{at} + I^t_s \theta^v_w \omega^I_{va} \right) \right] I^r_s J^r_s \\
- \frac{1}{2} I^r_s \left[ g^{tv} \left( J^t_I \theta^v_w \omega^I_{sv} + J^t_I \theta^v_w \omega^I_{as} + J^t_I \theta^v_w \omega^I_{va} \right) \right] J^r_j \\
+ \frac{1}{2} I^r_s \left[ g^{sv} \left( J^t_I \theta^v_w \omega^I_{tv} + J^t_I \theta^v_w \omega^I_{at} + J^t_I \theta^v_w \omega^I_{va} \right) \right] J^r_j \\
= \sum_{i=1}^{12} A_i.
\]

Direct calculations show that \( A_1 = A_4 = A_7 = A_{10} = 0 \). On the other hand we have
\[
2A_2 = -g^{tv} I^r_s \theta^v_w \omega^I_{aw} I^s_r J^r_s = g^{tv} I^r_s \theta^v_w \omega^I_{aw} I^s_r J^r_s = g^{ws} \theta^t_w I^r_s g^{pr} J^r_s = -\theta^{l p} I^r_s J^r_s = -\theta^4(IJ)_a,
\]
\[
2A_3 = -g^{tv} I^r_s \theta^v_w \omega^I_{aw} I^s_r J^r_s = g^{tv} I^r_s \theta^v_w \omega^I_{aw} I^s_r J^r_s = g^{pr} I^r_s \theta^v_w \omega^I_{aw} J^r_s = \theta^4(IJ)_a,
\]
\[
2A_5 = g^{sv} I^r_s \theta^v_w \omega^I_{aw} I^s_r J^r_s = -g^{sv} I^r_s \theta^v_w \omega^I_{aw} I^s_r J^r_s = -\theta^{l s} I^r_s J^r_s = \theta^4(IJ)_a,
\]
\[
2A_6 = g^{sv} I^r_s \theta^v_w \omega^I_{aw} I^s_r J^r_s = -g^{sv} I^r_s \theta^v_w \omega^I_{aw} I^s_r J^r_s = \theta^{l s} I^r_s J^r_s = -\theta^4(IJ)_a,
\]
\[
2A_8 = -I^s_r g^{tv} J^t_I \theta^v_w \omega^I_{as} J^r_s = I^s_r g^{tv} J^t_I \theta^v_w \omega^I_{as} J^r_s = I^s_r g^{tr} \theta^t_I J^r_p = -\theta^{l p} I^r_s J^r_s = -\theta^4(IJ)_a,
\]
\[
2A_9 = -I^s_r g^{tv} J^t_I \theta^v_w \omega^I_{as} J^r_s = I^s_r g^{tv} J^t_I \theta^v_w \omega^I_{as} J^r_s = I^s_r g^{pr} J^t_I \theta^v_w \omega^I_{as} J^r_s = \theta^4(IJ)_a,
\]
\[
2A_{11} = I^s_r g^{tv} J^t_I \theta^v_w \omega^I_{as} J^r_s = -I^s_r g^{tv} J^t_I \theta^v_w \omega^I_{as} J^r_s = -I^s_r g^{tv} J^t_I \theta^v_w \omega^I_{as} J^r_s = I^s_r g^{tv} J^t_I \theta^v_w \omega^I_{as} J^r_s = \theta^4(IJ)_a,
\]
\[
2A_{12} = I^s_r g^{tv} J^t_I \theta^v_w \omega^I_{as} J^r_s = -I^s_r g^{tv} J^t_I \theta^v_w \omega^I_{as} J^r_s = I^s_r g^{tv} J^t_I \theta^v_w \omega^I_{as} J^r_s = -I^s_r g^{tv} J^t_I \theta^v_w \omega^I_{as} J^r_s = -\theta^4(IJ)_a.
\]

The first claimed formula follows, and the second follows from Lemma 3.8.

Lemma 3.11. Given \((M^4, g, I, J)\) a nondegenerate generalized Kähler structure, one has

\[
\theta = \frac{1}{2(p^2 - 1)} dp[I, J].
\]

Proof. Combining Lemmas 3.8 and 3.10 we have that

\[
dp[I, J] = \frac{1}{4}(\theta^I - \theta^J)[I, J]^2 = \frac{1}{2} \theta[I, J]^2 = 2(p^2 - 1) \theta.
\]

Lemma 3.12. Let \((M^4, g, I, J)\) be a nondegenerate generalized Kähler four-manifold. Then

1. \(\braket{dp, \theta} = 0\).
2. \(|\theta|^2 = \frac{1}{(1 - p^2)} |dp|^2\).
Proof. We directly compute using Lemma 3.10:

\[ \langle dp, \theta \rangle = g^{ij} dp_i \theta_j \]

\[ = g^{ij} \left[ \theta_k [I, J]^k_i \theta_j \right] \]

\[ = g^{ij} \left[ \theta_k \theta_j \left( I^k_i J^l_j - J^k_i I^l_j \right) \right] \]

\[ = \theta_k I^k_i J^l_j \theta^i - \theta_k \theta_j g^{kl} J^k_i I^l_j \]

\[ = \theta_k I^k_i J^l_j \theta^i - \theta_j I^j_i \theta^i \theta^l = 0. \]

Next using Lemma 3.11 we have

\[ |\theta|^2 = g^{ij} \theta_i \theta_j \]

\[ = g^{ij} \left( \frac{1}{2(p^2 - 1)} dp_k [I, J]^k_i \right) \left( \frac{1}{2(p^2 - 1)} dp_l [I, J]^l_j \right) \]

\[ = - \frac{1}{4(p^2 - 1)^2} dp_k g^{ki} [I, J]^k_i [I, J]^l_j dp_l \]

\[ = \frac{1}{(1 - p^2)} |dp|^2. \]

□

Lemma 3.13. Let \((M^{4n}, g, I, J)\) be a nondegenerate generalized Kähler structure. Then

\[ \rho^L_C = - dI d\log \sqrt{\det[I, J].} \]

In particular, when \(n = 1\) we have that

\[ \rho^L_C = - dI d\log(1 - p^2). \]

Proof. First we observe that since \(\mathcal{O} := \Omega^n\) is a holomorphic volume form, the Chern connection on the canonical bundle associated to the volume form \(\mathcal{O}) \wedge \mathcal{O}\) is flat. Hence

\[ \rho^L_C(\omega^n) = \rho^L_C(\mathcal{O} \wedge \mathcal{O}) - dI d\log \frac{\omega^n}{\mathcal{O} \wedge \mathcal{O}} = - dI d\log \frac{\omega^n}{\mathcal{O} \wedge \mathcal{O}} \]

Then we note that

\[ \omega^n = dV_g \]

\[ = \sqrt{\det g_{ij}} dx^1 \wedge \cdots \wedge dx^{4n} \]

\[ = \sqrt{\det (\Omega[I, J])} dx^1 \wedge \cdots \wedge dx^{4n} \]

\[ = \sqrt{\det[I, J]} Pf \Omega \]

\[ = \sqrt{\det[I, J]} \mathcal{O} \wedge \mathcal{O}. \]

In the case \(n = 1\), using Lemma 3.13 we see that

\[ \sqrt{\det[I, J]} = 4(1 - p^2). \]

Hence the second result follows. □
4. Nondegenerate Generalized Kahler Ricci flow

In this section we derive the main a priori estimates employed in the proof of Theorem 1.1. The a priori estimates roughly break into two parts. First we derive evolution equations for functions associated to the angle function in the $B$-field flow gauge. Very surprisingly, a certain function of the angle is a solution to the time-dependent heat equation with no reaction terms. Direct maximum principle arguments based on this simple evolution equation lead to a number of strong a priori estimates on the torsion, which play a central role in the proof. Second, we study the flow in the $I$-fixed gauge, utilizing a certain reduction of the pluriclosed flow to a flow for a $(1,0)$-form and a potential function to obtain further a priori estimates, including uniform equivalence of the evolving volume form.

Once these estimates are in place we can obtain the long time existence and convergence of the flow by a familiar path. In particular, one can exploit the potential function to obtain an a priori estimate for the metric shown in [26] to obtain the full regularity of the flow. Many of these estimates are not uniform as time goes to infinity, but we can exploit the decay of the torsion tensor to obtain the weak convergence claims.

4.1. A priori estimates using the angle function.

Lemma 4.1. Let $(M^4, g_t, I_t, J_t)$ be a solution to GKRF with nondegenerate initial condition in the $B$-field gauge. Then

$$\frac{\partial}{\partial t} p = \Delta p + \frac{2p|dp|^2}{1 - p^2}.$$ 

Proof. Recall that for a complex structure $J$ and a vector field $X$ we have

$$(L_X J)^I_k = X^q \nabla_q J^I_k - J^q_k \nabla_p X^I + \nabla_k X^p J^I_p.$$ 

Thus we can compute

$$\frac{\partial}{\partial t} 4p = \frac{\partial}{\partial t} (I^I_k J^k_j)$$

$$= (L_\theta I^I_k J^k_j) - I^I_k (L_\theta J)^I_k$$

$$= \left[ \theta^q \nabla_p I^I_k J^k_j - I^I_k \nabla_p J^I_k - \nabla q J^I_k \right] - I^I_k \left[ \theta^q \nabla_q J^I_k - J^q_k \nabla_p J^I_p + \nabla k \theta^q J^I_p \right]$$

$$= \theta^q \nabla_p I^I_k J^k_j J^I - I^I_k \theta^q \nabla_q J^I_k + 2 \text{tr} \left( \nabla \theta^q : [J, I] \right).$$

We observe using Lemmas 2.3 and 3.12 that

$$\theta^q \nabla_p I^I_k J^k_j J^I_k = \theta^q \left[ -g^{mk} \theta^I_q \omega_{pj} + \delta^I_q \theta^p - g^{km} \theta^k_q g_m p - \theta^p \theta^I_q \right] J^I_k$$

$$= \theta^q \left[ g^{mk} \theta^q r_j I^I_q J^k + I^I_q \theta^p J^p_j + g^{km} \theta^k_q g_m p - \theta^q \theta^q J^I_k \right]$$

$$= - \theta^q I^I_k J^I_k \theta^k + \theta^q I^I_q J^p_j \theta^p + \theta^q I^I_p J^m_q \theta^p - \theta^q \theta^q J^I_k \theta^p$$

$$= 2 \langle dp, \theta \rangle$$

$$= 0.$$

A very similar calculation yields

$$-I^I_j \theta^q \nabla_q J^I_k = 0.$$
Then we note using Lemmas 3.3 and 3.11 that

\[ 2 \text{tr} \left( \nabla \theta^2 \cdot [J, I] \right) = 2 \nabla_q \theta^p [J, I]^q_p \]

\[ = 2 \nabla_q \theta_r g^{pr} [J, I]^q_p \]

\[ = 2 \nabla_q \left[ \frac{1}{2(p^2 - 1)} \nabla_{sp} [I, J]^r_p \right] g^{pr} [J, I]^q_p \]

\[ = \nabla_q \left[ \frac{1}{p^2 - 1} \nabla_{sp} [I, J]^r_p g^{pr} [J, I]^q_p \right] - \frac{1}{p^2 - 1} \nabla_{sp} [I, J]^r_p g^{pr} \nabla_q [J, I]^q_p \]

\[ = \nabla_q \left[ \frac{1}{p^2 - 1} \nabla_{sp} g^{pr} [J, I]^r_p [J, I]^q_p \right] - 2 \theta^p \nabla_q [J, I]^q_p \]

\[ = 4 \Delta p + 2 \theta^p \nabla_q [I, J]^q_p. \]

Now we simplify the remaining term

\[ 2 \theta^p \nabla_q [I, J]^q_p = 2 \theta^p \nabla_q \left( I^q_p J^t_r - J^t_r I^q_p \right) \]

\[ = 2 \theta^p \left( (\nabla_q I^q_p) J^t_r + I^q_p \nabla_q J^t_r - (\nabla_q J^t_r) I^q_p - J^t_r (\nabla_q I^q_p) \right) \]

\[ = \sum_{i=1}^4 A_i. \]

Then

\[ A_1 = \theta^p J^r_p \left(-g^{tq} \theta^t_{r} \omega^J_{qr} + I^t_p \theta^r t \delta^q_q - g^{om} I^t_m \theta^t g_{qr} - \theta_r I^q_q \right) \]

\[ = 4 I^t_r J^r_p \theta^p \theta_{t} + \theta^p \theta_{r} I^t_p J^r_q g^q g_{qr} - \theta^p \theta_{t} J^t_r I^r_p g^{om} g_{qr} \]

\[ = 4 I^t_r J^r_p \theta^p \theta_{t} - \theta^p \theta_{t} J^t_r I^r_p - \theta^p \theta_{t} J^r_p I^r_t \]

\[ = 2 I^t_r J^r_p \theta^p \theta_{t} \]

\[ = (IJ + JI)^t_p \theta^p \theta_{t} \]

\[ = 2p |\theta|^2. \]

Next

\[ A_2 = \theta^p I^r_r \left(-g^{tq} \theta^t_{r} \omega^J_{qr} + J^t_r \theta^r t \delta^q_q - g^{om} J^t_m \theta^t g_{qr} - \theta_r J^q_q \right) \]

\[ = (\text{tr} IJ) |\theta|^2 - \theta^p I^r_r \theta^r_{t} J^t_q g_{qr} + \theta^p I^r_r g^{om} J^t_m \theta^t g_{qr} \]

\[ = (\text{tr} IJ) |\theta|^2 - 2 \theta^p \theta_{t} I^r_r J^r_q \]

\[ = 2p |\theta|^2. \]

Also

\[ A_3 = \theta^p I^r_r \left(-g^{tq} \theta^t_{r} \omega^J_{qr} + J^t_r \theta^r t \delta^q_q - g^{om} J^t_m \theta^t g_{qr} - \theta_r J^q_q \right) \]

\[ = 4 \theta^p I^r_r J^r_p \theta_{t} + \theta^p I^r_p g^{tq} \theta_{r} J^t_q g_{qr} - \theta^p I^r_p J^r_q \theta_{t} \]

\[ = 4 \theta^p I^r_r J^r_p \theta_{t} - \theta^p I^r_p J^r_q g^{om} g_{qr} - \theta^p I^r_p J^r_q \theta_{t} \]

\[ = 2p \theta_{r} I^r_r J^r_p \]

\[ = 2p |\theta|^2. \]
Lastly
\[ A_4 = -\theta p J_q^i \left( -g^{tr} \theta_t^i \omega_{qp} + \theta_{p}^i \theta_{q}^i \delta_r^t - g^{rm} \theta_{m}^i \theta_{q}^i g_{qp} - \theta_{p}^i \theta_{q}^i \right) \]
\[ = \text{tr}(IJ) |\theta|^2 - \theta p J_q^i \theta_{t}^i \omega_{qp} + \theta_{p}^i J_q^i \theta_{m}^i \theta_{r}^i \delta_r^t \]
\[ = \text{tr}(IJ) |\theta|^2 - \theta p J_q^i \theta_{t}^i \omega_{qp} - \theta_{p}^i J_q^i \theta_{m}^i \theta_{r}^i \delta_r^t \]
\[ = 2p |\theta|^2. \]

It follows that \( 2 \theta p \nabla_q[I, J]^q_p = 8p |\theta|^2 \). Hence, also applying Lemma 3.12 we obtain
\[ \frac{\partial}{\partial t} 4p = \Delta 4p + 8p |\theta|^2 = \Delta 4p + \frac{8p |dp|^2}{1 - p^2}. \]

**Lemma 4.2.** Let \((M^4, g_t, I_t, J_t)\) be a solution to GKRF with nondegenerate initial condition in the \(B\)-field gauge. Then
\[ \frac{\partial}{\partial t} \log \frac{1 + p}{1 - p} = \Delta \log \frac{1 + p}{1 - p}. \]

**Proof.** We directly compute using Lemma 4.1
\[ \frac{\partial}{\partial t} \log \frac{1 + p}{1 - p} = \frac{1 - p}{1 + p} \frac{\partial}{\partial t} \frac{1 + p}{1 - p} \]
\[ = \frac{1 - p}{1 + p} \left[ (1 - p) \frac{\partial}{\partial t} \frac{1 + p}{1 - p} - (1 + p)(- \frac{\partial}{\partial t} \frac{1 + p}{1 - p}) \right] \]
\[ = \frac{2}{(1 - p^2)} \left[ \Delta \frac{1 + p}{1 - p} + \frac{2p}{(1 - p^2)} |dp|^2 \right] \]

Similarly we have
\[ \Delta \log \frac{1 + p}{1 - p} = \nabla_i \left[ \frac{1 - p}{1 + p} \nabla_i \frac{1 + p}{1 - p} \right] \]
\[ = \nabla_i \left\{ \frac{1 - p}{1 + p} \left[ (1 - p) \nabla_i p - (1 + p)(- \nabla_i p) \right] \right\} \]
\[ = \nabla_i \left\{ \frac{2}{(1 - p^2)} \nabla_i p \right\} \]
\[ = \frac{2}{(1 - p^2)} \left[ \Delta p + \frac{2p}{(1 - p^2)} |dp|^2 \right]. \]

The result follows.

This very simple evolution equation leads to a number of crucial a priori estimates for the flow, and the evolution equations themselves are very useful in constructing test functions. As is well-known, for a solution to the heat equation against a Ricci flow background, the gradient function satisfies a particularly clean evolution equation, with the evolution of the metric exactly canceling the Ricci curvature term arising from the Bochner formula. For a solution to the \(B\)-field flow, the contribution from the positive definite tensor \(H\) makes the corresponding evolution equation even more useful.

**Lemma 4.3.** Let \((M^n, g_t, H_t)\) be a solution to (2.5), and let \(\phi_t\) be a solution to
\[ \frac{\partial}{\partial t} \phi = \Delta_{g_t} \phi. \]
Then
\[ \frac{\partial}{\partial t} |\nabla \phi|^2 = \Delta |\nabla \phi|^2 - 2 |\nabla^2 \phi|^2 - \frac{1}{2} \langle \mathcal{H}, \nabla \phi \otimes \nabla \phi \rangle. \]

**Proof.** Using the given evolution equations and the Bochner formula we have
\[ \frac{\partial}{\partial t} |\nabla \phi|^2 = \left\langle 2 \text{Rc} - \frac{1}{2} \mathcal{H}, \nabla \phi \otimes \nabla \phi \right\rangle + 2 \langle \nabla \Delta \phi, \nabla \phi \rangle \]
\[ = 2 \langle \Delta \phi, \phi \rangle - \frac{1}{2} \langle \mathcal{H}, \nabla \phi \otimes \nabla \phi \rangle \]
\[ = \Delta |\nabla \phi|^2 - 2 |\nabla^2 \phi|^2 - \frac{1}{2} \langle \mathcal{H}, \nabla \phi \otimes \nabla \phi \rangle, \]
as required. \(\square\)

**Lemma 4.4.** Let \((M^4, g)\) be a Riemannian manifold, and \(H \in \Lambda^3\). Then
\[ \mathcal{H} = |\star H|^2 g - (\ast H) \otimes (\ast H). \]

**Proof.** We express \(H = \star \alpha\), and then choose coordinates where \(g\) is the identity. It follows that
\[ \mathcal{H}_{ij} = H_{spq} H^p_j = \alpha^r (dV_g)_{ripq} \alpha^s (dV_g)_{sjpq}. \]
It is clear that for any unit vector \(v\) orthogonal to \(\alpha^\sharp\), one has \(\mathcal{H}(v, v) = |\alpha|^2\). On the other hand certainly \(\mathcal{H}(\alpha^2, \alpha^2) = 0\), and so the result follows. \(\square\)

**Lemma 4.5.** Let \((M^4, g_t, I_t, J_t)\) be a solution to GKRF with nondegenerate initial condition in the \(B\)-field gauge. Furthermore let \(\mu = \log \frac{1 + p}{1 - p}\). Then
\[ \frac{\partial}{\partial t} |\nabla \mu|^2 = \Delta |\nabla \mu|^2 - 2 |\nabla^2 \mu|^2 - \frac{1 - p^2}{8} |\nabla \mu|^4 = \Delta |\nabla \mu|^2 - 2 |\nabla^2 \mu|^2 - \frac{2}{1 - p^2} |\theta|^4. \]

**Proof.** Combining Proposition 4.2 with Lemma 4.3 yields
\[ \frac{\partial}{\partial t} |\nabla \mu|^2 = \Delta |\nabla \mu|^2 - 2 |\nabla^2 \mu|^2 - \frac{1}{2} \langle \mathcal{H}, \nabla \mu \otimes \nabla \mu \rangle. \]
We observe that in four dimensions, \(\theta = \ast H\). Moreover, \(\nabla \mu\) is a multiple of \(\nabla p\), which is orthogonal to \(\theta\) via Lemma 3.12. It follows from Lemma 4.4 that
\[ \mathcal{H}(\nabla \mu, \nabla \mu) = |\theta|^2 |\nabla \mu|^2. \]
On the other hand using the definition of \(\mu\) (cf. 4.1) and Lemma 3.12 we have
\[ |\nabla \mu|^2 = \frac{4 |dp|^2}{(1 - p^2)^2} = \frac{4}{1 - p^2} |\theta|^2, \]
and the result follows. \(\square\)

Now we derive two key a priori estimates from these evolution equations via the maximum principle.

**Proposition 4.6.** Let \((M^4, g_t, I_t, J_t)\) be a solution to GKRF with nondegenerate initial condition in the \(B\)-field gauge. Then there is a constant \(\delta = \delta(I_0, J_0)\) such that
\[ -1 < \inf p_0 \leq p_t \leq \sup p_0 < 1, \quad \sup_{M \times \{t\}} |\nabla \mu|^2 \leq \left[ (\sup |\nabla \mu_0|)^{-2} + \delta t \right]^{-1}. \]
Proof. The first inequalities follow by applying the maximum principle to the evolution equation of Lemma 4.1. For the second we first observe that \( \frac{1}{8} \inf (1 - p_t^2) \geq \frac{1}{8} \inf (1 - p_0^2) = \delta > 0 \). Then we apply the maximum principle to the result of Lemma 4.5 to show that \( \sup |\nabla \mu_t|^2 \) is bounded above by the solution to the ODE

\[
\frac{dF}{dt} = -\delta F^2, \quad F(0) = \sup |\nabla \mu_0|^2.
\]

The proposition follows. \( \square \)

4.2. Estimates from the decomposed pluriclosed flow. In this section we derive further a priori estimates for the generalized Kähler-Ricci flow, purely from the point of view of pluriclosed flow. In [29] the author and Tian observed that the pluriclosed flow reduces naturally to a degenerate parabolic flow of a \((1, 0)\)-form. In [26] we exhibited a further decomposition into a scalar flow coupled to a parabolic flow for a \((1, 0)\)-form, which naturally reduces to the parabolic complex Monge-Ampère equation when the \((1, 0)\)-form vanishes. We review this construction in our special setting below.

First, as in the reduction of Kähler-Ricci flow to the parabolic complex Monge-Ampère equation (cf. [31]), one must choose an appropriate family of background pluriclosed metrics whose Aeppli cohomology classes agree with those of the flowing metric. However, in our setting we already know that \((M^4, I)\) admits a holomorphic volume form. It follows that \( c_1 = 0 \), and so we may choose a Hermitian background metric \( h \) such that \( \rho_C(h) = 0 \). Now suppose \( \omega_t \) is a solution to pluriclosed flow on \((M^4, I)\). One can directly check using (2.3) (cf. [26] Lemma 3, with \( \mu = 0 \)) that if \( \alpha_t \) solves

\[
\frac{\partial \alpha}{\partial t} = \overline{\partial} g_t \omega_t - \frac{\sqrt{-1}}{2} \partial \log \frac{\det g_t}{\det h},
\]

then the one-parameter family of pluriclosed metrics \( \omega_\alpha = \omega_0 + \overline{\partial} \alpha + \partial \alpha \) is the given solution to pluriclosed flow.

For technical reasons in the proof of convergence, we will actually choose a different initial value of \( \alpha \), which corresponds to a different background metric, which is Kähler. First of all we claim that \((M^4, I)\) is indeed a Kähler manifold, an observation originally appearing in ([3] Proposition 2). By the Enriques-Kodaira classification of surfaces, the canonical bundle being trivial implies that \((M^4, I)\) is either a torus, a K3 surface, or a (non-Kähler) primary Kodaira surface (see [4]). However, one can rule out the existence of any kind of biHermitian structure (let alone generalized Kähler structure) on primary Kodaira surfaces by observing that it would imply the existence of 3 distinct harmonic self-dual forms, contradicting that \( b_2^+ (M) = 2 \) for such a surface (see [2] pg. 426 for more details). Since we have now shown that \((M^4, I)\) is Kähler, ([5] Theorem 12) asserts that given any pluriclosed metric \( \omega_0 \) on \( M \), we can find \( \alpha_0 \in \Lambda^{1,0} \) such that \( \omega := \omega_0 - \partial \alpha_0 - \overline{\partial} \alpha_0 \) is a Kähler metric. We then express \( \omega_\alpha = \omega + \overline{\partial} [\alpha + \alpha_0] + \partial [\alpha + \overline{\alpha_0}] \). We will always make such a choice of initial condition for \( \alpha \) without further comment.

The natural local decomposition of a pluriclosed metric as \( \omega = \partial \overline{\alpha} + \overline{\partial} \alpha \) is not canonical, as one may observe that \( \alpha + \partial f \) describes the same metric, where \( f \in C^\infty (M, \mathbb{R}) \). Due to this "gauge-invariance," the equation (4.2) is not parabolic, and admits large families of equivalent solutions. In [26] we resolved this ambiguity by giving a different description of (4.2), which is parabolic. In particular, as exhibited in ([26] Proposition 3.9, in the case the background metric
is fixed and Kähler), if one has a family of functions $f_t$ and $(1,0)$-forms $\beta_t$ which satisfy
\begin{align}
\frac{\partial}{\partial t} \beta &= \Delta_{g_t} \beta - T_{g_t} \circ \overline{\partial} \beta, \\
\frac{\partial}{\partial t} f &= \Delta_{g_t} f + \text{tr}_{g_t} g + \log \frac{\det g_t}{\det h}, \\
\alpha_0 &= \beta_0 - \sqrt{-1} \partial f_0,
\end{align}
(4.3)
then $\alpha_t := \beta_t - \sqrt{-1} \partial f_t$ is a solution to (4.2). The term $T \circ \overline{\partial} \beta$ is defined by
\[(T \circ \overline{\partial} \beta)_i = g^{lk} g_{qp} T_{ik} q \nabla_l \alpha_p.\]

We will use this decomposition to obtain two estimates crucial to Theorem 1.1. First we record a prior result:

**Lemma 4.7.** ([26] Proposition 4.4) Given a solution to (4.3) as above, one has
\[
\frac{\partial}{\partial t} |\beta|^2 = \Delta |\beta|^2 - |\nabla \beta|^2 - |\overline{\nabla} \beta|^2 - \langle Q, \beta \otimes \overline{\beta} \rangle + 2 \Re \langle \beta, T^\alpha \circ \overline{\partial} \beta \rangle,
\]
(4.4)
where
\[Q_{ij} = g^{lk} g_{qp} T_{ik} q T_{jp} p.\]
In fact the lemma above applies in any dimension, but the next corollary is special to $n = 2$.

**Corollary 4.8.** ([26] Corollary 4.5) Given a solution to (4.3) as above, one has
\[
\frac{\partial}{\partial t} |\beta|^2 \leq \Delta |\beta|^2 - |\nabla \beta|^2.
\]
In particular, one has
\[
\sup_M |\beta_t|^2 \leq \sup_M |\beta_0|^2. 
\]
(4.5)
Proof. The estimate (4.5) follows directly from ([26] Corollary 4.5) using that the background metric is Kähler. The estimate (4.6) follows directly from the maximum principle. □

The estimate (4.6) holds for any pluriclosed flow on a Kähler surface. The next two propositions require that we are studying a pluriclosed flow associated to a generalized Kähler-Ricci flow with nondegenerate initial data. In particular we will assume the evolution equations and a priori estimates of §4.1.

**Proposition 4.9.** Let $(M^4, g_t, I, J_t)$ be a solution to GKRF with nondegenerate initial data in the $I$-fixed gauge. Then there exists a constant $C = C(g_0, I_0, J_0)$ such that $C^{-1} \leq \frac{\det g}{\det g_0} \leq C$.

Proof. Using Lemmas 2.2, 3.12 and 4.2 it follows that $\mu$ satisfies, in the $I$-fixed gauge, $\mu$ satisfies
\[
\frac{\partial}{\partial t} \mu = \Delta \mu - L_{\theta^g} \mu = \Delta \mu = \Delta C \mu.
\]
A simple calculation then yields
\[
\frac{\partial}{\partial t} \mu^2 = \Delta C \mu^2 - 2 |\nabla \mu|^2 = \Delta C \mu^2 - \frac{8}{1 - p^2} |\theta|^2 \leq \Delta C \mu^2 - \delta |\theta|^2,
\]
(4.7)
for some universal constant $\delta$. On the other hand, as discussed above one has $c_1(M, I) = 0$, and hence there exists a background Hermitian metric $h$ such that $\rho_C(h) = 0$. Since we are in the $I$-fixed gauge, the metric is evolving by pluriclosed flow, and hence from [26] Lemma 6.1 we conclude that
\[
\frac{\partial}{\partial t} \log \frac{\det g}{\det h} = \Delta C \log \frac{\det g}{\det h} + |\theta|^2.
\]
Applying the maximum principle directly to this yields an a priori lower bound for the volume form of \( g \). On the other hand, setting \( \Phi = \log \frac{\det g}{\det h} + \delta^{-1} \mu^2 \) we obtain
\[
\frac{\partial}{\partial t} \Phi \leq \Delta_C \Phi.
\]
The maximum principle implies a uniform upper bound for \( \Phi \), which implies a uniform upper bound for the volume form of \( g \) since \( \mu \) is bounded above via Proposition 4.6. □

**Proposition 4.10.** Let \( (M^4, g_t, I, J_t) \) be a solution to GKRF with nondegenerate initial data in the \( I \)-fixed gauge. Given a solution to (4.3) as above, there exists a constant \( C \) depending only on the initial data so that for any existence time \( t \) one has
\[
\sup_{M \times \{t \}} \left| \frac{\partial f}{\partial t} \right| \leq C.
\]

**Proof.** We construct a test function
\[
\Phi = \frac{\partial}{\partial t} f + A_1 |\beta|^2 + A_2 |\nabla \mu|^2 + A_3 \mu^2,
\]
where the choices of constants \( A_i \) will be made explicit below. We first compute
\[
\frac{\partial}{\partial t} \frac{\partial}{\partial t} f = \frac{\partial}{\partial t} \left[ n - \text{tr}_{g_t} (\overline{\beta} + \partial \beta) + \log \frac{\det g_t}{\det h} \right]
\]
\[
= \left( \frac{\partial g_t}{\partial t} , \overline{\beta} + \partial \beta \right) - \text{tr}_{g_t} \left[ \frac{\partial}{\partial t} (\overline{\beta} + \partial \beta) \right] + \text{tr}_{g_t} \frac{\partial g_t}{\partial t}
\]
\[
= \left( \frac{\partial g_t}{\partial t} , \overline{\beta} + \partial \beta \right) + \text{tr}_{g_t} \partial \frac{\partial f_t}{\partial t}
\]
\[
= \Delta_C f_t + \left( \frac{\partial g_t}{\partial t} , \overline{\beta} + \partial \beta \right).
\]
Combining this with Lemmas 4.5, Lemma 4.7 and (4.7) yields that there is a small constant \( \delta > 0 \) such that
\[
\left( \frac{\partial}{\partial t} - \Delta_C \right) \Phi = \left( \frac{\partial g_t}{\partial t} , \overline{\beta} + \partial \beta \right) + A_1 \left[ -|\nabla \beta|^2 - |\overline{\nabla} \beta|^2 - \frac{1}{2} |T|^2 |\beta|^2 + 2 \Re \left( \beta , T^\alpha \circ \overline{\beta} \right) \right]
\]
\[
+ A_2 \left[ \theta \star \nabla |\nabla \mu|^2 - 2 |\nabla^2 \mu|^2 - \delta |\theta|^4 \right] + A_3 \left[ -\delta |\theta|^2 \right].
\]
Using (4.6) we have that
\[
2 \Re \left( \beta , T^\alpha \circ \overline{\beta} \right) \leq \epsilon \left| \overline{\nabla} \beta \right|^2 + C \epsilon^{-1} |\beta|^2 |T|^2
\]
\[
\leq \epsilon \left| \nabla \beta \right|^2 + C \epsilon^{-1} |\theta|^2.
\]
Similarly, using that \( |\theta|^2 \) is bounded we can estimate
\[
\theta \star \nabla |\nabla \mu|^2 = \nabla^2 \mu \star \theta^2
\]
\[
\leq \epsilon \left| \nabla^2 \mu \right|^2 + C \epsilon^{-1} |\theta|^4
\]
\[
\leq \epsilon \left| \nabla^2 \mu \right|^2 + C \epsilon^{-1} |\theta|^2.
\]
We also note that as \( \frac{\partial g_t}{\partial t} \) is expressed in terms of one derivative of the Lee form and the Chern-Ricci curvature, it follows from Lemma 3.13 that there is a constant \( C = C((1 - p^2)^{-1}) \) such
that
\[ \left| \frac{\partial g}{\partial t} \right|^2 \leq C \left( |\nabla^2 \mu|^2 + |\theta|^4 \right). \]
Hence we can estimate
\[ \left| \left\langle \frac{\partial g}{\partial t}, \partial \beta + \partial \overline{\beta} \right\rangle \right| \leq C \left( |\nabla^2 \mu|^2 + |\theta|^4 \right) + C |\nabla \beta|^2. \]
Putting these preliminary estimates together and choosing \( \epsilon \) sufficiently small we obtain
\[ \left( \frac{\partial}{\partial t} - \Delta_C \right) \Phi \leq |\nabla \beta|^2 \left( C - \frac{A_1}{2} \right) + |\nabla^2 \mu|^2 \left( C - \frac{A_2}{2} \right) + |\theta|^2 (C + CA_1 + CA_2 - 2\delta A_3). \]
It is now clear that if we choose \( A_1 \) and \( A_2 \) large with respect to controlled constants, then choose \( A_3 \) large with respect to controlled constants \( A_1, A_2, \) and \( \delta \) we obtain
\[ \left( \frac{\partial}{\partial t} - \Delta_C \right) \Phi \leq 0. \]
Applying the maximum principle yields an upper bound for \( \frac{\partial f}{\partial t} \), and a very similar estimate can be obtained on \( -\frac{\partial f}{\partial t} \), finishing the result. \( \square \)

**Proposition 4.11.** Given the setup above, there exists a constant \( C \) depending only on the initial data such that
\[ |f| \leq C(1 + t), \quad |\text{tr}_g(\partial \beta + \partial \overline{\beta})| \leq C. \]

*Proof.* The first estimate follows directly from Proposition 4.10. For the second we observe using Propositions 4.9 and 4.10 that
\[ |\text{tr}_g(\partial \beta + \partial \overline{\beta})| = \left| n - \frac{\partial}{\partial t} f + \log \frac{\det g}{\det h} \right| \leq C. \]
\( \square \)

**Proposition 4.12.** Given the setup above, there exists a constant \( C \) depending only on the initial data such that
\[ \text{tr}_g g_0 \leq C e^{C(f - \inf f)}. \]

*Proof.* Fix some constant \( A \), and let
\[ \Phi = \log \text{tr}_g g_0 - A(f - \inf f). \]
The function \( \Phi \) is smooth on \( M \), and Lipschitz in \( t \) due to Proposition 4.10. Using (26) Lemma 6.2 and standard estimates, and combining with (4.3) yields
\[ \frac{\partial}{\partial t} \Phi \leq \Delta \Phi + (C - A) \text{tr}_g g_0 + A \frac{\partial}{\partial t} \inf f + A \log \frac{\det g}{\det h}. \]
As we noted, \( \inf f \) is only Lipschitz in time, and so inequality holds in the sense of limsup of difference quotients. Considering this at a spatial maximum for \( \Phi \), using the result of Proposition 4.9 and choosing \( A \) sufficiently large yields
\[ \frac{\partial}{\partial t} \Phi \leq -\frac{A}{2} \text{tr}_g g_0 + C \leq 0, \]
where the last line follows if the maximum value for \( \Phi \), and hence \( \text{tr}_g g_0 \), is sufficiently large. This yields an a priori upper bound for \( \Phi \), after which the result follows. \( \square \)
5. Proof of Theorem 1.1

In this section we complete the proof of Theorem 1.1. We first establish the long time existence, and then prove a series of lemmas leading to the weak convergence statement.

**Proposition 5.1.** Let \((M^4, g, I, J)\) be a nondegenerate generalized Kähler four-manifold. The solution to generalized Kähler-Ricci flow with initial data \((g, I, J)\) exists for all time.

**Proof.** Fix \((M^4, g, I, J)\) a nondegenerate generalized Kähler four-manifold. Let \((g_t, I, J_t)\) be the solution to GKRF with this initial data in the \(I\)-fixed gauge. From Proposition 4.6, we have a priori estimates for \((1 - p^2)^{-1}\) and \(|\theta|^2\) in the \(B\)-field gauge. As these are estimates on scalar quantities associated to the time-dependent data, they hold automatically in every gauge. Next we choose a solution \((\beta_t, f_t)\) to the decomposed flow as in §4.2. Proposition 4.9 provides a uniform bound for \(|\log \det g_t - \det g_0|\). Moreover, it follows from Proposition 4.11 that \(f - \inf f \leq C(1 + t)\), hence Proposition 4.12 yields a uniform upper bound for \(\text{tr} g_t g_0\) on any finite time interval. Since the volume form is already controlled, this implies \(C^{-1} g_0 \leq g_t \leq C g_0\), on \([0, T]\), for a constant \(C(T)\).

We can now apply ([26] Theorems 1.7, 1.8), there are higher order estimates for \(g\) on any finite time interval. The claim of long time existence follows from standard arguments. \(\Box\)

**Lemma 5.2.** Let \((M^4, g_t, I, J_t)\) be a solution to (1.1) in the \(I\)-fixed gauge. Given \(a \in \Lambda^1(M)\), one has

\[
\lim_{t \to \infty} \int_M |dp|_g |a|_g dV_g = 0.
\]

**Proof.** To begin we estimate

\[
\int_M |a|_g^2 dV_g = \int_M \sqrt{-1} a \wedge \bar{a} \wedge \omega\alpha \\
\leq \int_M |a|_g^2 |\omega|_\omega \wedge \omega \\
\leq C \int_M (\text{tr} \omega \omega) \wedge \omega \\
= C \int_M \omega \wedge \omega \\
= C \int_M \omega \wedge \omega \\
= C.
\]

Then, using Propositions 4.6 and 4.9 we have that

\[
\int_M |dp|_g |a|_g dV_g \leq \left( \int_M |dp|_g^2 dV_g \right)^{\frac{1}{2}} \left( \int_M |a|_g^2 dV_g \right)^{\frac{1}{2}} \\
\leq C |dp|_g \text{Vol}(g_t) \\
\leq C t^{-1}.
\]

The lemma follows. \(\Box\)

**Proposition 5.3.** Let \((M^4, g_t, I, J_t)\) be a solution to (1.1) in the \(I\)-fixed gauge. Then \(\{\omega_K(t)\}\) converge subsequentially as \(t \to \infty\) a triple of closed currents \(\{\omega_K^\infty\}\).

**Proof.** First recall that, as explained in §4.2, we know that there is a Kähler metric \(\omega\) such that

\[
\omega_I(t) = \omega + \partial\alpha_t + \bar{\partial}\alpha_t.
\]
It follows that
\[ \int_M \omega(t) \wedge \omega = \int_M \omega \wedge \omega \leq C. \]

It follows from (\cite{8} Chapter III Proposition 1.23) that the set \( \{ \omega(t) \} \) is weakly compact in the sense of positive currents. Similarly we have
\[ \left| \int_M (\omega_J)^{1,1} \wedge \omega \right| = \left| \int_M p \omega_I \wedge \omega \right| \leq \left| \int_M \omega_I \wedge \omega \right| = C. \]

We next claim that \( (\omega_J)^{2,0+0,2} \) is bounded in \( L^\infty \). First we note that it follows from Propositions \( 4.6 \) and \( 4.9 \) that the holomorphic symplectic structure \( \Omega \) is uniformly bounded. Since \( (\omega_J)^{2,0+0,2} = q^{-2} \Omega \), Proposition \( 4.6 \) implies the \( L^\infty \) estimate. Using this and (\cite{8} Chapter III Proposition 1.23) as above we see that \( \{ \omega_J(t) \} \) is weakly compact in the sense of currents. It follows directly from (\ref{3.1}), (\ref{3.3}), and Proposition \( 4.6 \) that \( \{ \omega_K_i(t) \} \) is weakly compact for \( i = 0, 1, 2 \). Hence any sequence of times admits a subsequence converging to a triple of limiting currents \( \{ \omega_K \} \) as claimed.

Next we show that these currents are all closed. We fix \( a \in \Lambda^1 \) and compute,
\[
\left| \int_M \omega_{K_1} \wedge da \right| = \lim_{t \to \infty} \left| \int_M \omega_{K_1} \wedge da \right| \\
= \lim_{t \to \infty} \left| \int_M d\omega_I \wedge a \right| \\
\leq \lim_{t \to \infty} \int_M |T_g| |a|_g dV_g \\
\leq \lim_{t \to \infty} \int_M |dp|_g |a|_g dV_g \\
= 0,
\]
where the last line follows from Lemma \( 5.2 \). Similarly,
\[
\left| \int_M \omega_{K_2} \wedge da \right| = \lim_{t \to \infty} \left| \int_M \omega_{K_2} \wedge da \right| \\
= \lim_{t \to \infty} \left| \int_M d (q^{-1} \omega_J - p \omega_I) \wedge a \right| \\
\leq \lim_{t \to \infty} C((1 - p^2)^{-1}) \int_M |T_g| |a|_g dV_g \\
\leq \lim_{t \to \infty} C((1 - p^2)^{-1}) \int_M |dp|_g |a|_g dV_g \\
= 0,
\]
where again the last line follows from Lemma \( 5.2 \). Given this, it follows directly from (\ref{3.3}) that \( d\omega_{K_0}^\infty = 0 \). The proposition follows.

References

[1] V. Apostolov, G. Dloussky, Bihermitian metrics on Hopf surfaces Math. Res. Lett. 15 (2008), no. 5, 827-839.
[2] V. Apostolov, P. Gauduchon, G. Grantcharov, Bihermitian structures on complex surfaces, Proc. London Math. Soc. (3) 79 (1999) 414-428.
[3] V. Apostolov, M. Gualtieri, Generalized Kähler manifolds, commuting complex structures, and split tangent bundles, Comm. Math. Phys. 271 (2), 561-575.
[4] Barth, W. Hulek, K. Peters, C. Van de Ven, A. Compact complex surfaces, A series of modern surveys in mathematics, Springer, 2000, Berlin.
[5] N. Buchdahl, *On compact Kähler surfaces*, Annales de l’institut Fourier, tome 49, no. 1 (1999), 287-302.

[6] J.M. Bismut, *A local index theorem for non Kähler manifolds*, Math. Ann. 284, 681-699 (1989).

[7] J. Bogaerts, A. Sevrin, S. Van der Loo, S. Van Gils, *Properties of semi-chiral superfields*, arXiv:hep-th/9905141

[8] J.P. Demailly, *Complex analytic and differential geometry*, June 2012.

[9] S. Gates, C. Hull, M. Rocek, *Twisted multiplets and new supersymmetric non-linear σ-models*, Nuclear Physics B248 (1984) 157-186.

[10] P. Gauduchon, *Le théorème de l’excentricité nulle*, C. R. Acad. Sci. Paris, 285 (1977), 387-390.

[11] H. Geiges and J. Gonzalo, *Contact geometry and complex surfaces*, Invent. Math. 121 (1995), 147-209.

[12] R. Goto, *Deformations of generalized complex generalized Kähler structures*, J. Diff. Geom., Vol. 84 No. 3 (2010), 525-560.

[13] M. Gualtieri, *Generalized complex geometry*, Oxford D.Phil, 2003.

[14] M. Gualtieri, *Generalized complex geometry*, Ann. of Math. Vol. 174 (2011), 75-123.

[15] M. Gualtieri, *Generalized Kähler geometry*, Comm. Math. Phys, 2014, 1-35.

[16] N. Hitchin, *Generalized Calabi-Yau manifolds*, Q.J. Math. 54, no. 3, (2003) 281-308.

[17] N. Hitchin, *Instantons, Poisson structures and generalized Kähler geometry*, Comm. Math. Phys. 2006, Vol. 265, Issue 1, 131-164.

[18] C. Hull, U. Lindstrom, M. Rocek, R. von Unge, M. Zabzine, *Generalized Calabi-Yau metric and generalized Monge-Ampere equation*, JHEP, August 2010, 2010:60.

[19] S. Ivanov, S., G. Papadopolous, *Vanishing theorems and string backgrounds* Class. Quantum Grav. 18 (2001) 1089-1110.

[20] U. Lindstrom M. Rocek, R. von Unge, M. Zabzine, *Generalized Kähler manifolds and off-shell supersymmetry* Comm. Math. Phys, Feb 2007, Vol. 269, Issue 3, 833-849.

[21] U. Lindstrom M. Rocek, R. von Unge, M. Zabzine, *Linearizing generalized Kähler geometry*, JHEP, April 2007, 2007:61.

[22] U. Lindstrom M. Rocek, R. von Unge, M. Zabzine, *A potential for generalized Kähler geometry*, arXiv:hep-th/0703111, IRMA Lect. Math. Theor. Phys. 16, EMS Publ. House, Zürich (2010), 263-273.

[23] V.V. Lychagin, V.N. Rubstov and I.V. Chekalov, *A classification of Monge-Ampere equations*, Ann. Sci. Ecole. Norm. Sup. (4) 26 (1993), 281-308.

[24] J. Polchinski, *String Theory, Volume 1*, Cambridge University Press. 1998.

[25] M. Pontecorvo, *Complex structures on Riemannian four-manifolds*, Math. Ann. 309 (1997), 159-177.

[26] J. Streets, *Pluriclosed flow Born-Infeld geometry, and rigidity results for generalized Kähler manifolds*, arXiv:1502.02584, to appear Comm. PDE.

[27] J. Streets, *Pluriclosed flow on generalized Kähler manifolds with split tangent bundle*, arXiv:1405.0727, to appear Crelle’s Journal.

[28] J. Streets, G. Tian, *A parabolic flow of pluriclosed metrics*, Int. Math. Res. Notices (2010), Vol. 2010, 3101-3133.

[29] J. Streets, G. Tian, *Regularity results for the pluriclosed flow*, Geom. & Top. 17 (2013) 2389-2429.

[30] J. Streets, G. Tian, *Generalized Kähler geometry and the pluriclosed flow*, Nuc. Phys. B, Vol. 858, Issue 2, (2012) 366-376.

[31] Tian, G., Zhang, Z. *On the Kähler-Ricci flow on projective manifolds of general type*. Chinese Ann. Math. Ser. B 27 (2006), no. 2, 179-192.

E-mail address: jstreets@uci.edu

Rowland Hall, University of California, Irvine, CA 92617