The Relation between Maxwell, Dirac and the Seiberg-Witten Equations

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Abstract

In this paper we discuss some unusual and unsuspected relations between Maxwell, Dirac and the Seiberg-Witten equations. First we investigate what is now known as the Maxwell-Dirac equivalence (MDE) of the first kind. Crucial to that proposed equivalence is the possibility of solving for $\psi$ (a representative on a given spinorial frame of a Dirac-Hestenes spinor field) the equation $F = \psi \gamma^{21} \tilde{\psi}$, where $F$ is a given electromagnetic field. Such non trivial task is presented in this paper and it permits to clarify some possible objections to the MDE which claims that no MDE may exist, because $F$ has six (real) degrees of freedom and $\psi$ has eight (real) degrees of freedom. Also, we review the generalized Maxwell equation describing charges and monopoles. The enterprise is worth even if there is no evidence until now for magnetic monopoles, because there are at least two faithful field equations that have the form of the generalized Maxwell equations. One is the generalized Hertz potential field equation (which we discuss in detail) associated with Maxwell theory and the other is a (non linear) equation (of the generalized Maxwell type) satisfied by the 2-form field part of a Dirac-Hestenes spinor field that solves the Dirac-Hestenes equation for a free electron. This is a new and surprising result, which can also be called MDE of the second kind. It strongly suggests that the electron is a composed system with more elementary “charges” of the electric and magnetic types. This finding may eventually account for the recent claims that the electron has been splitted into two electrons. Finally, we use the MDE of the first kind together with a reasonable hypothesis to give a derivation of the famous Seiberg-Witten equations

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on Minkowski spacetime. A suggestive physical interpretation for those equations is also given.

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1 Introduction

In ([1]-[5]) using standard covariant spinor fields Campolattaro proposed that Maxwell equations are equivalent to a non-linear Dirac like equation. The subject has been further developed in ([6],[8]) using the Clifford bundle formalism, which is discussed together with some of their applications in a series of papers, e.g., ([6],[10]). The crucial point in proving the mentioned equivalence (abbreviated as $MDE$ in what follows, when no confusion arises), starts once we observe that to any given representative of a Dirac-Hestenes spinor field (see more information see section 2 and for details see ([12],[14],[16],[17]))
\[ \psi \in \text{sec}[\Lambda^0(M) + \Lambda^2(M) + \Lambda^4(M)] \subset \text{sec} \mathcal{A}(M, g) \] there is associated an electromagnetic field \[ F \in \text{sec} \Lambda^2(M) \subset \text{sec} \mathcal{A}(M, g), \quad (F^2 \neq 0) \] through the Rainich-Misner theorem \([20],[6] - [8]\) by

\[ F = \psi \gamma_{21} \tilde{\psi} \tag{1} \]

Before proceeding we recall that for null fields, i.e., \(F^2 = 0\), the spinor associated with \(F\) through Eq.\((1)\) must be a Majorana spinor field \([6],[14],[15]\), but we do not need such concept in this paper. Now, since an electromagnetic field \(F\) satisfying Maxwell equation has six degrees of freedom and a Dirac-Hestenes spinor field has eight (real) degrees of freedom some authors felt uncomfortable with the approach used in \([7],[8]\) where some gauge conditions have been imposed on a nonlinear equation (equivalent to Maxwell equation), thereby transforming it into an usual linear Dirac equation (called the Dirac-Hestenes equation in the Clifford bundle formalism). The claim, e.g., in \([21]\) is that the MDE found in \((7), [8]\) cannot be general. The argument is that the imposition of gauge conditions implies that a \(\psi\) satisfying Eq.\((1)\) can have only six (real) degrees of freedom, and this implies that the Dirac-Hestenes equation corresponding to Maxwell equation can be only satisfied by a restricted class of Dirac-Hestenes spinor fields, namely the ones that have six degrees of freedom.

Incidentally, in \([21]\) it is also claimed that the generalized Maxwell equation

\[ \partial F = J_e + \gamma_5 J_m \tag{2} \]

(where \(J_e, J_m \in \text{sec} \Lambda^1(M)\)) describing the electromagnetic field generated by charges and monopoles \([9]\) cannot hold in the Clifford bundle formalism, because according to that author the formalism implies that \(J_m = 0\).

In what follows we analyze these claims of \([21]\) and prove that they are wrong (section 3). The reasons for our enterprise is that as will become clear in what follows, understanding of Eqs.\((1)\) and \((2)\) together with some reasonable hypothesis permit a derivation and even a possible physical interpretation of the famous Seiberg-Witten monopole equations \([22],[23],[26]\). So, our plan is the following: first we introduce in section 2 the mathematical formalism used in the paper, showing how to write Maxwell and Dirac equations using Clifford fields. We also introduce Weyl spinor fields and parity operators in the Clifford bundle formalism. In section 3 we prove that given \(F\) in Eq.\((1)\) we can solve that equation for \(\psi\), and we find that \(\psi\) has eight degrees of freedom, two of them being undetermined, the indetermination being related to the elements of the stability group of the spin plane \(\gamma_{21}\). This is a non trivial and beautiful result which can called inversion formula. In section 4 we introduce a generalized Maxwell equation and in section 5 we introduce the generalized Hertz equation. In section 6 we prove a mathematical Dirac-Maxwell equivalence of the first kind \([1],[8]\), thereby deriving a Dirac-Hestenes equation from the free Maxwell equations. In section 7 we introduce a new form of a mathematical Maxwell-Dirac equivalence (called MDE of the second kind) different from the one studied in section 6. This new MDE of the second kind suggests that the electron is a ‘composite’ system. To prove the Maxwell-Dirac equivalence of the second
kind we decompose a Dirac-Hestenes spinor field satisfying a Dirac-Hestenes equation in such a way that it results in a nonlinear generalized Maxwell (like) equation (Eq. (111)) satisfied by a certain Hertz potential field, mathematically represented by an object of the same mathematical nature as an electromagnetic field, i.e., $\Pi \in \sec \Lambda^2(M) \subset \sec \mathcal{O}(M, g)$. This new equivalence is very suggestive in view of the fact that there are recent (wild) speculations that the electron can be split into two components [27] (see also [28]). If this fantastic claim announced by Maris [27] is true, it is necessary to understand what is going on. The new Maxwell-Dirac equivalence presented in section 6 may eventually be useful to understand the mechanism behind the “electron splitting” into electroninos. We are not going to discuss these ideas here. Instead, we concentrate our attention in showing in section 8 that (the analogous on Minkowski spacetime) of the famous Seiberg-Witten monopole equations arises naturally from the MDE of the first kind once a reasonable hypothesis is imposed. We also present a possible coherent interpretation of that equations. Indeed, we prove that when the Dirac-Hestenes spinor field satisfying the first of Seiberg-Witten equations is an eigenvector of the parity operator, then that equation describe a pair of massless ‘monopoles’ of opposite ‘magnetic’ like charges, coupled together by its interaction electromagnetic field. Finally, in section 9 we present our conclusions.

2 Clifford and Spin-Clifford Bundles

Let $\mathcal{M} = (M, g, D)$ be Minkowski spacetime. $(M, g)$ is a four dimensional time oriented and space oriented Lorentzian manifold, with $M \simeq \mathbb{R}^4$ and $g \in \sec T^0\mathbb{R}^4$ being a Lorentzian metric of signature $(1,3)$. $T^*M$ is the cotangent bundle. $T^*M = \cup_{x \in M} T^*_x M$, $TM = \cup_{x \in M} T_x M$, and $T_x M \simeq T^*_x M \simeq \mathbb{R}^{1,3}$, where $\mathbb{R}^{1,3}$ is the Minkowski vector space. $D$ is the Levi-Civita connection of $g$, i.e., $Dg = 0$, $R(D) = 0$. Also $\mathcal{T}(D) = 0$, $R$ and $T$ being respectively the torsion and curvature tensors. Now, the Clifford bundle of differential forms $\mathcal{C}(M, g)$ is the bundle of algebras, i.e., $\mathcal{C}(M, g) = \cup_{x \in M} \mathcal{C}(T^*_x M)$, where $\forall x \in M, \mathcal{C}(T^*_x M) = \mathcal{C}_{1,3}$, the so called spacetime algebra. Recall also that $\mathcal{C}(M, g)$ is a vector bundle associated to the orthonormal frame bundle, i.e., $\mathcal{C}(M, g) = P_{SO(1,3)} \times_{ad} \mathcal{C}_{1,3}$ ([16], [17]). For any $x \in M$, $\mathcal{C}(T^*_x M)$ as a linear space over the real field $\mathbb{R}$. Moreover, $\mathcal{C}(T^*_x M)$ is isomorphic to the Cartan algebra $\Lambda(T^*_x M)$ of the cotangent space and $\Lambda(T^*_x M) = \sum_{k=0}^{4} \Lambda^k(T^*_x M)$, where $\Lambda^k(T^*_x M)$ is the $(k)$-dimensional space of $k$-forms. Then, sections of $\mathcal{C}(M, g)$ can be represented as a sum of non homogeneous differential forms. Let $\langle x^\mu \rangle$ be Lorentz coordinate functions for $M$ and let $\{e_\mu\} \in \sec FM$ (the frame bundle) be an orthonormal basis for $TM$, i.e., $g(e_\mu, e_\nu) = \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. Let $\gamma^\nu = dx^\nu \in \sec \Lambda^1(M) \subset \sec \mathcal{C}(M, g)$ ($\nu = 0, 1, 2, 3$) such that the set $\{\gamma^\nu\}$ is the dual basis of $\{e_\mu\}$. Moreover, we denote by $\hat{g}$ the metric in the cotangent bundle.
2.1 Clifford Product

The fundamental Clifford product (in what follows to be denoted by juxtaposition of symbols) is generated by $\gamma^b\gamma^b + \gamma^b\gamma^b = 2\eta^{bb}$ and if $\mathcal{C} \in \text{sec} \mathcal{O}(M,g)$ we have

$$\mathcal{C} = s + v_\mu\gamma^\mu + \frac{1}{2!}b_{\mu\nu}\gamma^\mu\gamma^\nu + \frac{1}{3!}a_{\mu\nu\rho}\gamma^\mu\gamma^\nu\gamma^\rho + p\gamma^5,$$  \hspace{1cm} (3)

where $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = dx^0dx^1dx^2dx^3$ is the volume element and $s, v_\mu, b_{\mu\nu}, a_{\mu\nu\rho}, p \in \text{sec} \bigwedge^0(M) \subset \text{sec} \mathcal{O}(M,g)$.

Let $A_r, B_s \in \text{sec} \bigwedge^r(M), B_s \in \text{sec} \bigwedge^s(M)$. For $r = s = 1$, we define the scalar product as follows:

For $a, b \in \text{sec} \bigwedge^1(M) \subset \text{sec} \mathcal{O}(M,g)$,

$$a \cdot b = \frac{1}{2}(ab + ba) = \hat{g}(a,b).$$  \hspace{1cm} (4)

We define also the exterior product $(\forall r, s = 0, 1, 2, 3)$ by

$$A_r \wedge B_s = (A_r B_s)_{r+s},$$
$$A_r \wedge B_s = (-1)^{rs}B_s \wedge A_r,$$  \hspace{1cm} (5)

where $(\cdot)_k$ is the component in $\bigwedge^k(M)$ of the Clifford field. The exterior product is extended by linearity to all sections of $\mathcal{O}(M,g)$.

For $A_r = a_1 \wedge ... \wedge a_r, B_r = b_1 \wedge ... \wedge b_r$, the scalar product is defined here as follows,

$$A_r \cdot B_r = (a_1 \wedge ... \wedge a_r) \cdot (b_1 \wedge ... \wedge b_r)$$
$$= \left| \begin{array}{ccc}
  a_1 \cdot b_1 & ... & a_1 \cdot b_r \\
  .... & ... & .... \\
  a_r \cdot b_1 & ... & a_r \cdot b_r
\end{array} \right|$$  \hspace{1cm} (6)

We agree that if $r = s = 0$, the scalar product is simple the ordinary product in the real field.

Also, if $r \neq s$, then $A_r \cdot B_s = 0$. Finally, the scalar product is extended by linearity for all sections of $\mathcal{O}(M,g)$.

For $r \leq s$, $A_r = a_1 \wedge ... \wedge a_r, B_s = b_1 \wedge ... \wedge b_s$ we define the left contraction by

$$\iota: (A_r, B_s) \mapsto A_r \iota B_s = \sum_{i_1 < ... < i_r} e^{i_1 ... i_r} (a_1 \wedge ... \wedge a_r) \cdot (b_{i_1} \wedge ... \wedge b_{i_r}) \sim b_{i_1+1} \wedge ... \wedge b_s,$$  \hspace{1cm} (7)

where $\sim$ is the reverse mapping (reversion) defined by

$$\sim: \text{sec} \bigwedge^p(M) \ni a_1 \wedge ... \wedge a_p \mapsto a_p \wedge ... \wedge a_1$$  \hspace{1cm} (8)

and extended by linearity to all sections of $\mathcal{O}(M,g)$. We agree that for $\alpha, \beta \in \text{sec} \bigwedge^0(M)$ the contraction is the ordinary (pointwise) product in the real field.
and that if $\alpha \in \sec \Lambda^0(M)$, $A_r, \in \sec \Lambda^r(M), B_s \in \sec \Lambda^s(M)$ then $(\alpha A_r)_r B_s = A_r, (\alpha B_s)$. Left contraction is extended by linearity to all pairs of elements of sections of $\mathcal{C}(M, g)$, i.e., for $A, B \in \sec \mathcal{C}(M, g)$

$$A \lrcorner B = \sum_{r,s} \langle A \rangle_r \lrcorner \langle B \rangle_s, r \leq s$$

(9)

It is also necessary to introduce the operator of right contraction denoted by $\lrcorner$. The definition is obtained from the one presenting the left contraction with the imposition that $r \geq s$ and taking into account that now if $A_r, \in \sec \Lambda^r(M), B_s \in \sec \Lambda^s(M)$ then $A_r, (\alpha B_s) = (\alpha A_r,)_r B_s$.

The main formulas used in the Clifford calculus can be obtained from the following ones (where $a \in \sec \Lambda^1(M) \subset \sec \mathcal{C}(M, g)$):

$$aB_s = a \lrcorner B_s + a \land B_s, a = B_s \land a + B_s \land a,$$

$$a \lrcorner B_s = \frac{1}{2}(aB_s - (-)^s B_s a),$$

$$A_r \lrcorner B_s = (-)^{r(s-1)} B_s \land A_r,$$

$$a \land B_s = \frac{1}{2}(aB_s + (-)^s B_s a),$$

$$A_r B_s = \langle A_r B_s \rangle_{|r-s|} + \langle A_r \lrcorner B_s \rangle_{|r-s-2|} + \ldots + \langle A_r B_s \rangle_{|r+s|}$$

$$\sum_{k=0}^{m} \langle A_r B_s \rangle_{|r-s|+2k}$$

$$A_r \cdot B_r = B_r \cdot A_r = \tilde{A}_r \lrcorner B_r = A_r \land \tilde{B}_r = \langle \tilde{A}_r \lrcorner B_r \rangle_0 = \langle A_r \cdot B_r \rangle_0$$

(10)

### 2.1.1 Hodge Star Operator

Let $\star$ be the Hodge star operator, i.e., the mapping

$$\star: \Lambda^k(M) \to \Lambda^{4-k}(M), \ A_k \mapsto \star A_k$$

where for $A_k \in \sec \Lambda^k(M) \subset \sec \mathcal{C}(M, g)$

$$[B_k \cdot A_k]_g = B_k \land \star A_k, \forall B_k \in \sec \Lambda^k(M) \subset \sec \mathcal{C}(M, g).$$

(11)

$\tau_g \in \Lambda^4(M)$ is a standard volume element. Then we can verify that

$$\star A_k = \tilde{A}_k \gamma^5.$$

(12)

### 2.1.2 Dirac Operator

Let $d$ and $\delta$ be respectively the differential and Hodge codifferential operators acting on sections of $\Lambda(M)$. If $A_p \in \sec \Lambda^p(M) \subset \sec \mathcal{C}(M, g)$, then $\delta A_p = (-)^p \star^{-1} d \star A_p$, with $\star^{-1} \star =$ identity.
The Dirac operator acting on sections of \( \mathcal{C}(M, g) \) is the invariant first order differential operator
\[
\partial = \gamma^a D_e a,
\]
where \( \{e_a\} \) is an arbitrary orthonormal basis for \( TU \subset TM \) and \( \{\gamma_b\} \) is a basis for \( T^*U \subset T^*M \) dual to the basis \( \{e_a\} \), i.e., \( \gamma^b(e_a) = \delta^b_a \), \( a, b = 0, 1, 2, 3 \). The reciprocal basis of \( \{\gamma^b\} \) is denoted \( \{\gamma_a\} \) and we have 
\[
\gamma_a \cdot \gamma_b = \eta_{ab} \quad (\eta_{ab} = \text{diag}(1, -1, -1, -1)).
\]
Also,
\[
D_e a \gamma^b = -\omega^{bc}_a \gamma_c
\]
Defining
\[
\omega_a = \frac{1}{2} \omega^{bc}_a \gamma_b \wedge \gamma_c,
\]
we have that for any \( A_p \in \text{sec} \wedge^p(M), \ p = 0, 1, 2, 3, 4 \)
\[
D_e a A = e_a + \frac{1}{2} [\omega_a, A].
\]
Using Eq.(16) we can show the very important result:
\[
\partial A_p = \partial \wedge A_p + \partial A_p = dA_p - \delta A_p,
\]
\[
\partial \wedge A_p = dA_p, \quad \partial A_p = -\delta A_p,
\]
2.2 Dirac-Hestenes Spinor Fields
Now, as is well known, an electromagnetic field is represented by \( F \in \text{sec} \wedge^2(M) \subset \text{sec} \mathcal{C}(M, g) \). How to represent the Dirac spinor fields in this formalism? We can show that Dirac-Hestenes spinor fields, do the job. We give here a short introduction to these objects (when living on Minkowski spacetime) which serves mainly the purpose of fixing notations. For a rigorous theory of these objects (using vector bundles) on a general Riemann-Cartan manifold see (17). Recall that there is a 2 : 1 mapping \( \Theta' : B \to B \) between \( B \), the set of all orthonormal ordered vector frames and \( \Theta' \), the set of all spin frames of \( T^*M \). As discussed at length in (16, 17) a spin coframe can be thought as a basis of \( T^*M \), such that two ordered basis even if consisting of the same vectors, but, with the spatial vectors differing by a \( 2\pi \) rotation are considered distinct and two ordered basis even if consisting of the same vectors, but with the spatial vector s differing by a \( 4\pi \) rotation are identified. For short, in this paper we call the spin coframes, simply spin frames. Also, vector coframes are simply called vector frames in what follows.
Consider the set \( S \) of mappings
\[
M \ni x \mapsto u(x) \in \text{Spin}_+(1, 3) \simeq Sl(2, \mathbb{C})
\]
Choose a constant spin frame \( \{\gamma_a\} \in B, \ a = 0, 1, 2, 3 \) and choose \( \Xi_0 \in \Theta' \) corresponding to a constant mapping \( u_0 \in S \). By constant we mean that the equation \( \gamma_\mu(x) = \gamma_\mu(y) \) ((\( \mu = 0, 1, 2, 3 \)) and \( u_0(x) = u_0(y), \ \forall \ x, y \in M \) has meaning.

due to the usual affine structure that can be given to Minkowski spacetime. \( \Xi_0, \Xi_u \in \Theta' \) are relate as follows
\[
u_0 s'(\Xi_0)u_0^{-1} = us'(\Xi_u)u_s^{-1} \quad (19)
\]

From now on in order to simplify the notation we take \( u_0 = 1 \). The frame \( s'(\Xi_0) = \{ \gamma_u \} \) is called the fiducial vector frame and \( \Xi_0 \) the fiducial spin frame. We note that Eq. (19) is satisfied by two such \( u \)'s differing by a signal, and of course, \( s'(\Xi_u) = s'(\Xi_{-u}) \).

Let,
\[
\mathcal{T} = \{ (\Xi_u, \Psi_{\Xi_u}) \mid u \in S, \Xi_u \in \Theta', \Psi_{\Xi_u} \in \sec \Lambda \oplus M \subset \sec \mathcal{Cl}^+(M, g) \}, \quad (20)
\]
where \( \Lambda \oplus M = \Lambda^0 M + \Lambda^2 M + \Lambda^4 M \)

We define an equivalence relation on \( \mathcal{T} \) by setting
\[
(\Xi_u, \Psi_{\Xi_u}) \sim (\Xi_{u'}, \Psi_{\Xi_{u'}}) \quad (21)
\]
if and only if
\[
us'(\Xi_u)u_s^{-1} = u's'(\Xi_{u'})u_s', \quad \Psi_{\Xi_{u'}} = \Psi_{\Xi_u} uu'^{-1}. \quad (22)
\]

**Definition:** Any equivalence class \( [(\Xi_u, \Psi_{\Xi_u})] \) will be called a Dirac-Hestenes spinor field.

Before proceeding we recall that a more rigorous definition of a DHSF as a section of a spin-Clifford bundle is given in [17]. We will not need such a sophistication in what follows.

We observe that the pairs \((\Xi_u, \Psi_{\Xi_u})\) and \((\Xi_{-u}, \Psi_{\Xi_{-u}})\) are equivalent, but the pairs \((\Xi_u, \Psi_{\Xi_u})\) and \((\Xi_{-u}, \Psi_{\Xi_{-u}})\) are not. This distinction is essential in order to give a structure of linear space (over the real numbers) to the set \( \mathcal{T} \). Indeed, such a linear structure on \( \mathcal{T} \) is defined as follows
\[
 a[(\Xi_{u_1}, \Psi_{\Xi_{u_1}})] + b[(\Xi_{u_2}, \Psi_{\Xi_{u_2}})] = [(\Xi_{u_1}, a\Psi_{\Xi_{u_1}})] + [(\Xi_{u_2}, b\Psi_{\Xi_{u_2}})], \\
 (a + b)[(\Xi_{u_1}, \Psi_{\Xi_{u_1}})] = [(\Xi_{u_1}, a\Psi_{\Xi_{u_1}})] + [(\Xi_{u_1}, b\Psi_{\Xi_{u_1}})], \\
 a, b \in \mathbb{R}. \quad (23)
\]

We can simplify the notation by recalling that every \( u \in S \) determines, of course, a unique spin frame \( \Xi_u \). Taking this into account we consider the set of all pairs \((u, \Psi_{\Xi_u}) \in S \times \sec \mathcal{Cl}^+(M, g) \)

We define an equivalence relation \( \mathcal{R} \) in \( S \times \sec \mathcal{Cl}^+(M, g) \) as follows. Two pairs \((u, \Psi_{\Xi_u}), (u', \Psi_{\Xi_{u'}}) \in \sec S \times \sec \mathcal{Cl}^+(M, g) \) are equivalent if and only if
\[
\Psi_{\Xi_{u'}}u' = \Psi_{\Xi_{u}}u \quad (24)
\]

Of course, \( s'(\Xi_{u'}) = vs'(\Xi_{u})v^{-1} \) with \( v = (u')^{-1}u \in S \). Note that the pairs \((u, \Psi_{\Xi_u})\) and \((-u, -\Psi_{\Xi_u})\) are equivalent but the pairs \((u, \Psi_{\Xi_u})\) and \((-u, -\Psi_{\Xi_u})\) are not.
Denote by $S \times \sec \mathcal{Cl}^+(M,g) / \mathcal{R}$ the quotient set of the equivalence classes generated by $\mathcal{R}$. Their elements are called Dirac-Hestenes spinors. Of course, this is the same definition as above.

From now on we simplify even more our notation. In that way, if we take two orthonormal spin frames $s^i(\Xi) = \{\gamma^\mu\}$ and $s^j(\bar{\Xi}) = \{\bar{\gamma}^\mu = R_\gamma^\mu \bar{R} = \Lambda_\mu^\nu \gamma^\nu\}$ with $\Lambda_\mu^\nu(x) \in SO^+(1,3)$ and $R(x) \in Spin^+(1,3) \forall x \in M$, $R\bar{R} = \bar{R}R = 1$, then we simply write the relation (Eq.(24)) between representatives of a Dirac-Hestenes spinor field in the two spin frames as the sections $\psi_\Xi$ and $\psi_\bar{\Xi}$ of $\mathcal{Cl}^+(M,g)$ related by

$$
\psi_\Xi = \psi_\Xi R.
$$

Recall that since $\psi_\Xi \in \sec \bigwedge^+ M \subset \sec \mathcal{Cl}^+(M,g)$, we have

$$
\psi_\Xi = s + \frac{1}{2!} b_{\mu \nu} \gamma^\mu \gamma^\nu + p\gamma^5.
$$

Note that $\psi_\Xi$ has the correct number of degrees of freedom in order to represent a Dirac spinor field and recall that the specification of $\psi_\Xi$ depends on the spin frame $\Xi$. To simplify even more our notation, when it is clear which is the spin frame $\Xi$, and no possibility of confusion arises we write simply $\psi$ instead of $\psi_\Xi$.

When $\psi \psi \neq 0$, where $\sim$ is the reversion operator, we can show that $\psi$ has the following canonical decomposition:

$$
\psi = \sqrt{\rho} e^{\beta \gamma_5/2} R,
$$

where $\rho, \beta \in \sec \bigwedge^0 (M) \subset \sec \mathcal{Cl}(M,g)$ and $R(x) \in Spin^+(1,3) \subset \mathcal{O}_{1,3}^+$, $\forall x \in M$. $\beta$ is called the Takabayasi angle. If we want to work in terms of the usual Dirac spinor field formalism, we can translate our results by choosing, for example, the standard matrix representation of the one forms $\{\gamma^\mu\}$ in $\mathbb{C}(4)$ (the algebra of the complex $4 \times 4$ matrices), and for $\psi_\Sigma$ given by Eq.(15) we have the following (standard) matrix representation $[12],[16])$:

$$
\Psi = \left( \begin{array}{cccc}
\psi_1 & -\psi_2^* & \psi_3 & -\psi_4^* \\
\psi_2 & \psi_1^* & \psi_4 & -\psi_3^* \\
\psi_3 & \psi_4^* & \psi_1 & -\psi_2^* \\
\psi_4 & -\psi_3^* & \psi_2 & \psi_1^* 
\end{array} \right).
$$

where $\psi_k(x) \in \mathbb{C}$, $k = 1, 2, 3, 4$ and for all $x \in M$.

We recall that a standard Dirac spinor field is a section $\Psi_D$ of the vector bundle $P_{Spin^+(1,3)} \times_\lambda \mathbb{C}(4)$, where $\lambda$ is the $D(\frac{3}{2},0) \oplus D(0,\frac{1}{2})$ representation of $Sl(2,\mathbb{C}) \sim Spin^+(1,3)$. For details see, e.g.,$[16],[17])$. The relation between $\Psi_D$ and $\psi$ is given by

$$
\Psi_D = \left( \begin{array}{c}
\psi_1 \\
\psi_2 \\
\psi_3 \\
\psi_4 
\end{array} \right) = \left( \begin{array}{c}
s - ib_{12} \\
-b_{13} - ib_{23} \\
-b_{03} + ip \\
-b_{01} - ib_{02} 
\end{array} \right).
$$
where \( s, b_{12}, \ldots \) are the real functions in Eq. (26) and \( * \) denotes the complex conjugation.

We recall that the even subbundle \( \mathcal{A}^{+}(M, g) \) of \( \mathcal{C}(M, g) \) is such that its typical fiber is the Pauli algebra \( \mathcal{A}_{3,0} \equiv \mathcal{A}_{3,0}^{+} \) (which is isomorphic to \( \mathbb{C}(2) \), the algebra of \( 2 \times 2 \) complex matrices). Elements of \( \mathcal{A}_{1,3}^{+} \) are called biquaternions in the old literature. The isomorphism \( \mathcal{A}_{3,0} \equiv \mathcal{A}_{1,3}^{+} \) is exhibited by putting \( \tilde{\sigma}_i = \gamma_i \gamma_0 \), whence \( \tilde{\sigma}_i \tilde{\sigma}_j + \tilde{\sigma}_j \tilde{\sigma}_i = 2 \delta_{ij} \). We recall also that the Dirac algebra is \( \mathcal{A}_{4,1} \equiv \mathbb{C}(4) \) and \( \mathcal{A}_{4,1} \equiv \mathbb{C} \otimes \mathcal{A}_{1,3} \).

Consider the complexification \( \mathcal{C}(M, g) \) of \( \mathcal{C}(g) \) called the complex Clifford bundle. Then \( \mathcal{C}(M, g) = \mathbb{C} \otimes \mathcal{C}(M, g) \) and we can verify that the typical fiber of \( \mathcal{C}(M, g) \) is \( \mathcal{C}_{4,1} = \mathbb{C} \otimes \mathcal{A}_{1,3} \), the Dirac algebra. Now let \( \{ \Delta_0, \Delta_1, \Delta_2, \Delta_3, \Delta_4 \} \subset \sec \mathcal{C}(M, g) \) be for all \( x \in M \) an orthonormal basis of \( \mathcal{C}_{4,1} \). We have,

\[
\Delta_a \Delta_b + \Delta_b \Delta_a = 2g_{ab},
\]

\( g_{ab} = \text{diag}(+1, +1, +1, +1, -1) \).

Let us identify \( \gamma_\mu = \Delta_\mu \Delta_4 \) and call \( I = \Delta_0 \Delta_1 \Delta_2 \Delta_3 \Delta_4 \). Since \( I^2 = -1 \) and \( I \) commutes with all elements of \( \mathcal{C}_{4,1} \) we identify \( I \) with \( i = \sqrt{-1} \) and \( \gamma_\mu \) with a fundamental set generating the local Clifford algebra of \( \mathcal{C}(M, g) \). Then if \( A \in \sec \mathcal{C}(M, g) \) we have

\[
A = \Phi_\mu + A^\mu_\gamma \gamma_\mu + \frac{1}{2} B^{\mu\nu}_C \gamma_\mu \gamma_\nu + \frac{1}{3!} C^{\mu\nu\rho}_C \gamma_\mu \gamma_\nu \gamma_\rho + \Phi_p \gamma_p,
\]

where \( \Phi_\mu, \Phi_p, A^\mu_\gamma, B^{\mu\nu}_C, C^{\mu\nu\rho}_C \in \sec \mathbb{C} \otimes \mathcal{A}^0(M) \subset \sec \mathcal{C}(M, g) \), i.e., \( \forall x \in M, \Phi_\mu(x), \Phi_p(x), A^\mu_\gamma(x), B^{\mu\nu}_C(x), C^{\mu\nu\rho}_C(x) \) are complex numbers.

Now, it can be verified that

\[
f = \frac{1}{2} (1 + \gamma_0) \frac{1}{2} (1 + i \gamma_1 \gamma_2); \quad f^2 = f,
\]

is a primitive idempotent field of \( \mathcal{C}(M, g) \). We can also verify without difficulty that \( i f = \gamma_2 \gamma_1 f \).

Appropriate equivalence classes (see (16, 17)) of \( \mathcal{C}(M, g) f \) are representatives of the standard Dirac spinor fields in \( \mathcal{C}(M, g) \). We can easily show that the representation of \( \Psi_D \) in \( \mathcal{C}(M, g) \) is given by

\[
\Psi_D = \psi f
\]

where \( \psi \) is the Dirac-Hestenes spinor field given by Eq. (30).

### 2.3 Weyl Spinors and Parity Operator

By definition, \( \psi \in \sec \mathcal{C}^+(M, g) \) is a representative of a Weyl spinor field (14, 15) if besides being a representative of a Dirac-Hestenes spinor field it satisfies \( \gamma_5 \psi = \pm \psi \gamma \), where \( \gamma_2 \gamma_1 = \gamma_2 \gamma_1 

The positive (negative) “eingestates” of $\gamma_5$ will be denoted $\psi_+$ ($\psi_-$). For a general $\psi \in \sec C^\ell(M, g)$ we can write

$$\psi_{\pm} = \frac{1}{2} [\psi \mp \gamma_5 \gamma_2 \psi] \quad (35)$$

Then,

$$\psi = \psi_+ + \psi_- \quad (36)$$

The parity operator $P$ in our formalism is represented in such a way that for $\psi \in \sec C^\ell(M, g)$,

$$P\psi = -\gamma_0 \psi \gamma_0 \quad (37)$$

The following Dirac-Hestenes spinor fields are eingestates of the parity operator with eingenvalues $\pm 1$:

$$P\psi^\uparrow = +\psi^\uparrow, \quad \psi^\uparrow = \gamma_0 \psi_- \gamma_0 - \psi_-$$

$$P\psi^\downarrow = -\psi^\downarrow, \quad \psi^\downarrow = \gamma_0 \psi_+ \gamma_0 + \psi_+ \quad (38)$$

### 2.4 The spin-Dirac Operator

Associated with the covariant derivative operator $D_{e_a} \ (\text{see Eq.}(14))$ acting on sections of the Clifford bundle there is a spin-covariant derivative operator $D_{e_a}^s$ acting on sections of a right spin-Clifford bundle, such that its sections are Dirac-Hestenes spinor fields. Hopefully it will be not necessary to present the details concerning this concept here (see [17]). Enough is to say that $D_{e_a}^s$ has a representative on the Clifford bundle, called $D_{e_a}^{(s)}$, such that if $\psi_\Xi$ is a representative of a Dirac-Hestenes spinor field we have

$$D_{e_a}^{(s)} \psi_\Xi = e_a(\psi_\Xi) + \frac{1}{2} \omega_a \psi_\Xi, \quad (39)$$

where $\omega_a$ has been defined by Eq. (15). The representative of the spin-Dirac operator acting on representatives of Dirac-Hestenes spinor fields is the invariant first order operator given by,

$$\partial^{(s)} = \gamma^a D_{e_a}^{(s)} \quad (40)$$

As from the definition of spin-Dirac operator we see that if we restrict our considerations to orthonormal coordinate bases $\{\gamma^\alpha = dx^\mu\}$ where $\{x^\mu\}$ are global Lorentz coordinates then $\omega_\mu = 0$ and the action of $\partial^{(s)}$ on Dirac-Hestenes spinor fields is the same as the action of $\partial$ on these fields.

### 2.5 Maxwell and Dirac-Hestenes Equations

With the mathematical tools presented above we have the following Maxwell equation,

$$\partial F = J_e \quad (41)$$
satisfied by an electromagnetic field $F \in \sec \Lambda^2(M) \subset \sec \mathcal{O}(M, g)$, and generated by a current $J_e \in \sec \Lambda^1(M) \subset \sec \mathcal{O}(M, g)$.

The Dirac-Hestenes equation in a spin frame $\Xi$ satisfied by a Dirac-Hestenes spinor field $\psi \in \sec(\Lambda^0(M) + \Lambda^2(M) + \Lambda^4(M)) \subset \sec \mathcal{O}(M, g)$ is

$$\partial \psi \gamma^2 \gamma^1 - m \psi \gamma^0 + \frac{1}{2} \gamma^a \psi \omega_a \gamma^2 \gamma^1 = 0.$$  \hspace{1cm} (42)

For what follows we restrict our considerations only for the case of orthonormal coordinate basis, in which case the Dirac-Hestenes equation reads

$$\partial \psi \gamma^2 \gamma^1 - m \psi \gamma^0 = 0.$$  \hspace{1cm} (43)

### 3 Solution of $\psi \gamma^2 \gamma^1 \bar{\psi} = F$

We now want solve Eq.(1) for $\psi$. Before proceeding we observe that on Euclidian spacetime this equation has been solved using Clifford algebra methods in [29]. Also, on Minkowski spacetime a particular solution of an equivalent equation (written in terms of biquaternions) appear in [30]. We are going to show that contrary to the claims of [21] a general solution for $\psi$ has indeed eight degrees of freedom, although two of them are arbitrary, i.e., not fixed by $F$ alone.

Once we give a solution of Eq.(1) for $\psi$, the reason for the indetermination of two of the degrees of freedom will become clear. This involves the Fierz identities, boomerangs ([12], [14], [31]) and the general theorem permitting the reconstruction of spinors from their bilinear covariants.

We start by observing that from Eq.(1) and Eq.(27) we can write

$$F = \rho e^{\beta \gamma} R_{\gamma 21} \tilde{R}.$$ \hspace{1cm} (44)

Then, defining $f = F/\rho e^{\beta \gamma}$ it follows that

$$f = R_{\gamma 21} \tilde{R}$$ \hspace{1cm} (45)

$$f^2 = -1.$$ \hspace{1cm} (46)

Now, since all objects in Eq.(44) and Eq.(45) are even we can take advantage of the isomorphism $\mathcal{O}_{3,0} \cong \mathcal{O}_{1,3}$ and making the calculations when convenient in the Pauli algebra. To this end we first write:

$$F = \frac{1}{2} F^{\mu \nu} \gamma_\mu \gamma_\nu, \quad F^{\mu \nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix},$$ \hspace{1cm} (47)

where $(E^1, E^2, E^3)$ and $(B^1, B^2, B^3)$ are respectively the Cartesian components of the electric and magnetic fields.

We now write $F$ in $\mathcal{C}\ell^+(M, g)$, the even sub-algebra of $\mathcal{C}\ell(M, g)$. The typical fiber of $\mathcal{C}\ell^+(M, g)$ (which is also a vector bundle) is isomorphic to the Pauli algebra. We put

$$\vec{\sigma}_i = \gamma_i \gamma_0, \quad i = \vec{\sigma}_1 \vec{\sigma}_2 \vec{\sigma}_3 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \gamma_5.$$ \hspace{1cm} (48)
Recall that \( i \) commutes with bivectors and since \( i^2 = -1 \) it acts like the imaginary unit \( i = \sqrt{-1} \) in \( Cl^+(M,g) \). From Eq. (47) and Eq. (48) (taking into account our previous discussion) we can write

\[
F = \vec{E} + i\vec{B},
\]

with \( \vec{E} = E^i \sigma_i, \vec{B} = B^j \sigma_j, i,j = 1,2,3 \). We can write an analogous equation for \( f \),

\[
f = \vec{e} + i\vec{b}
\]

Now, since \( F^2 \neq 0 \) and

\[
F^2 = F \cdot F + F \wedge F = (\vec{E}^2 - \vec{B}^2) + 2i(\vec{E} \cdot \vec{B})
\]

the above equations give (in the more general case where both \( I_1 = (\vec{E}^2 - \vec{B}^2) \neq 0 \) and \( I_2 = (\vec{E} \cdot \vec{B}) \neq 0 \)):

\[
\rho = \sqrt{\frac{\vec{E}^2 - \vec{B}^2}{\cos[\arctg g2\beta]}}, \quad \beta = \frac{1}{2} \arctan \left( \frac{2(\vec{E} \cdot \vec{B})}{\vec{E}^2 - \vec{B}^2} \right)
\]

Also,

\[
\vec{e} = \frac{1}{\rho}[(\vec{E} \cos \beta + \vec{B} \sin \beta)], \quad \vec{b} = \frac{1}{\rho}[(\vec{B} \cos \beta - \vec{E} \sin \beta)]
\]

### 3.1 A Particular Solution

Now, we can verify that

\[
L = \frac{e^{f_1} + f_3 \gamma_{21} + f}{\sqrt{2(1 - \gamma_5 \gamma_3)}} = \frac{\sigma_3 - i\vec{f}}{i\sqrt{2(1 - i(\vec{f} \cdot \sigma_3))}}.
\]

\[
J = f^{03} - \gamma_5 f^{12} = \vec{f} \cdot \sigma_3
\]

is a Lorentz transformation, i.e., \( LL = L \bar{L} = 1 \). Moreover, \( L \) is a particular solution of Eq. (45). Indeed,

\[
\frac{e^{f_1} + f}{\sqrt{2(1 - \gamma_5 \gamma_3)}} \frac{\gamma_{21} - f}{\sqrt{2(1 - \gamma_5 \gamma_3)}} = \frac{f[2(1 - \gamma_5 \gamma_3)]}{2(1 - \gamma_5 \gamma_3)} = f
\]

Of course, since \( f^2 = -1, \vec{e}^2 = \vec{b}^2 - 1 \) and \( \vec{e} \cdot \vec{b} = 0 \), there are only four real degrees of freedom in the Lorentz transformation \( L \). From this result in [21] it is concluded that the solution of the Eq. (1) is the Dirac-Hestenes spinor field

\[
\phi = \sqrt{\pi} e^{\gamma_5 \beta} L,
\]
which has only six degrees of freedom and thus is not equivalent to a general Dirac-Hestenes spinor field (the spinor field that must appears in the Dirac-Hestenes equation), which has eight degrees of freedom. In this way it is stated in [21] that a the MDE of first kind proposed in ([6], [8]) cannot hold. Well, although it is true that Eq. (57) is a solution of Eq. (1) it is not a general solution, but only a particular solution.

Before leaving this section we mention that there are many other Dirac-like forms of the Maxwell equations published in the literature. All are trivially related in a very simple way and in principle have nothing to do with the two kinds of MDE discussed in the present paper. See [31].

3.2 The General Solution

The general solution $R$ of Eq. (1) is trivially found. It is

$$R = LS,$$

where $L$ is the particular solution just found and $S$ is any member of the stability group of $\gamma_{21}$, i.e.,

$$S\gamma_{21}S = \gamma_{21}, \quad SS = SS = 1.$$  \hspace{-1cm}

It is trivial to find that we can parametrize the elements of the stability group as

$$S = \exp(\gamma_{03}\nu)\exp(\gamma_{21}\varphi),$$

with $0 \leq \nu < \infty$ and $0 \leq \varphi < \infty$. This shows that the most general Dirac-Hestenes spinor field that solves Eq. (1) has indeed eight degrees of freedom (as it must be the case, if the claims of ([6],[8]) are to make sense), although two degrees of freedom are arbitrary, i.e., they are like hidden variables!

Now, the reason for the indetermination of two degrees of freedom has to do with a fundamental mathematical result: the fact that a spinor can only be reconstruct through the knowledge of its bilinear covariants and the Fierz identities. Explicitly, to reconstruct a Dirac-Hestenes spinor field $\psi$, it is necessary to know also, besides the bilinear covariant given by Eq. (1), the following bilinear covariants,

$$J = \psi\gamma_{0}\tilde{\psi} \quad \text{and} \quad K = \psi\gamma_{3}\tilde{\psi}.$$  \hspace{-1cm}

Now, $J, K$ and $F$ are related trough the so called Fierz identities,

$$J^2 = \sigma^2 + \omega^2 = -K^2,$$

$$J \cdot K = 0, \quad J \wedge K = -(\omega + \gamma_5\sigma)F,$$

$$\sigma = \rho \cos \beta, \quad \omega = \rho \sin \beta.$$  \hspace{-1cm}

In the most general case when both $\sigma, \omega$ are not 0 we also have the notable identities first found by Crawford [31] (and which can be derived almost trivially

14
using the Clifford bundle formalism),

\[
\begin{align*}
F \cdot J &= \omega K \\
(\gamma_5 F) \cdot J &= \sigma K \\
F \cdot F &= \langle F \bar{F} \rangle_0 = \sigma^2 - \omega^2 \\
(\gamma_5 F) \cdot F &= 2 \sigma \omega
\end{align*}
\]  

(63)

\[
JF = (\omega + \gamma_5 \sigma)K, \quad KF = (\omega + \gamma_5 \sigma)J
\]

\[
F^2 = \omega^2 - \sigma^2 - 2\gamma_5\sigma\omega, \quad F^{-1} = KFK/(\omega^2 + \sigma^2)^2
\]  

(64)

Once we know \(\omega, \sigma, J, K\) and \(F\) we can recover the Dirac-Hestenes spinor field as follows. First, introduce a boomerang ([12], [14], [15]) \(\mathfrak{B} \in \mathcal{C}(M,g)\) given by

\[
\mathfrak{B} = \sigma + J + iF - i\gamma_5 K + \gamma_5 \omega
\]  

(65)

Then, we can construct \(\Psi = \mathfrak{B} f \in \mathcal{C}(M,g)\) (with \(f\) as in Eq.(62)) which has the following matrix representation (once the standard representation of the Dirac gamma matrices are used)

\[
\hat{\Psi} = \begin{pmatrix}
\psi_1 & 0 & 0 & 0 \\
\psi_2 & 0 & 0 & 0 \\
\psi_3 & 0 & 0 & 0 \\
\psi_4 & 0 & 0 & 0
\end{pmatrix}
\]  

(66)

Now, it can be easily verified that \(\Psi = \mathfrak{B} f\) determines the same bilinear covariants as the ones determined by \(\psi\). Note however that this spinor is not unique. In fact, \(\mathfrak{B}\) determines a class of elements \(\mathfrak{B} \eta\) where \(\eta\) is an arbitrary element of \(\mathcal{C}(M,g)\) which differs one from the other by a complex phase factor ([12], [14], [15]).

Recalling that (a representative) of a Dirac-Hestenes spinor field determines a unique element of \(\Phi \in \mathcal{C}(M)\) by \(\Phi = \psi f\), then it follows (from Eq.(63) and Eq.(65) that gives the matrix representation of \(\psi\)) that we can trivially reconstruct a \(\psi\) that solves our problem.

4 The Generalized Maxwell Equation

To comment on the basic error in [21] concerning the Clifford bundle formulation of the generalized Maxwell equation we recall the following. The generalized Maxwell equation ([9], [31]) which describes the electromagnetic field generated by charges and monopoles, can be written in the Cartan bundle as

\[
dF = K_m, \quad dG = K_e
\]  

(67)

where \(F, G \in \bigwedge^2(M)\) and \(K_m, K_e \in \bigwedge^3(M)\).
These equations are independent of any metric structure defined on the world manifold. When a metric is given and the Hodge dual operator \( \star \) is introduced it is supposed that in vacuum we have \( G = \star F \). In this case putting \( K_e = -\star J_e \) and \( K_m = \star J_m \), with \( J_e, J_m \in \sec \bigwedge^1(M) \), we can write the following equivalent set of equations

\[
\begin{align*}
  dF &= -\star J_m, \quad d \star F = -\star J_e, \quad (68) \\
  \delta(\star F) &= J_m, \quad \delta F = -J_e \quad (69) \\
  \delta(\star F) &= J_m, \quad \delta F = -J_e \\
  dF &= -\star J_m, \quad \delta F = -J_e. \quad (70)
\end{align*}
\]

Now, supposing that any \( \sec \bigwedge^j(M) \subset \sec \mathcal{C}^\ell(M, g) \) \((j = 0, 1, 2, 3, 4)\) and taking into account Eqs. (13-17) we get Eq. (2) by summing the two equations in (71), i.e.,

\[
(d - \delta)F = J_e + K_m \quad \text{or} \quad (d - \delta)\star F = -J_m + \star J_e, \quad (72)
\]

or equivalently

\[
\begin{align*}
  \partial F &= J_e + \gamma_5 J_m \quad \text{or} \quad \partial(-\gamma_5 F) = -J_m + \gamma_5 J_e. \quad (73)
\end{align*}
\]

Now, writing with the conventions of section 2, \( F = \frac{1}{2} F^\mu\nu \gamma_\mu \gamma_\nu, \quad \star F = \frac{1}{2}(\star F^\mu\nu)\gamma_\mu \gamma_\nu, \)

then generalized Maxwell equations in the form given by Eq. (69) can be written in components (in a Lorentz coordinate chart) as

\[
\partial_\mu F^{\mu\nu} = J_e^\mu, \quad \partial_\mu(\star F^{\mu\nu}) = -J^\mu_m. \quad (74)
\]

Now, assuming as in Eq. (1) that \( F = \psi \gamma_{21} \tilde{\psi} \) and taking into account the relation between \( \psi \) and the representation of the standard Dirac spinor \( \Psi_D \) given by Eq. (29), we can write Eq. (75) as

\[
\begin{align*}
  \partial_\mu \bar{\Psi}_D \left[ \gamma_\mu, \gamma_\nu \right] \Psi_D &= 2J_e^\mu, \quad \partial_\mu \bar{\Psi}_D \gamma_5 \left[ \gamma_\mu, \gamma_\nu \right] \Psi_D = -2J^\mu_m, \\
  F^{\mu\nu} &= \frac{1}{2} \bar{\Psi}_D \left[ \gamma_\mu, \gamma_\nu \right] \Psi, \quad (\star F^{\mu\nu}) = \frac{1}{2} \bar{\Psi}_D \gamma_5 \left[ \gamma_\mu, \gamma_\nu \right] \Psi. \quad (76)
\end{align*}
\]

The reverse of the first of Eqs. (73) equation reads

\[
(\partial F) = J_e - K_m. \quad (77)
\]

First summing, and then subtracting Eq. (2) with Eq. (67) we get the following equations for \( F = \psi \gamma_{21} \tilde{\psi} \),

\[
\begin{align*}
  \partial \psi \gamma_{21} \tilde{\psi} + (\partial \psi \gamma_{21} \tilde{\psi}) &= 2J_e, \quad \partial \psi \gamma_{21} \tilde{\psi} - (\partial \psi \gamma_{21} \tilde{\psi}) = 2K_m \quad (78)
\end{align*}
\]
which is equivalent to Eq.(13) in [21] (where $G$ is used for the three form of monopolar current). There, it is observed that $J_e$ is even under reversion and $K_m$ is odd. Then, it is claimed that “since reversion is a purely algebraic operation without any particular physical meaning, the monopolar current $K_m$ is necessarily zero if the Clifford formalism is assumed to provide a representation of Maxwell’s equation where the source currents $J_e$ and $K_m$ correspond to fundamental physical fields.” It is also stated that Eq.(76) and Eq.(78) imposes different constrains on the monopolar currents $J_e$ and $K_m$.

It is clear that these arguments are fallacious. Indeed, it is obvious that if any comparison is to be made, it must be done between $J_e$ and $J_m$ or between $K_e$ and $K_m$. In this case, it is obvious that both pairs of currents have the same behavior under reversion. This kind of confusion is widespread in the literature, mainly by people that works with the generalized Maxwell equation(s) in component form (Eqs.(75)).

It seems that experimentally $J_m = 0$ and the following question suggests itself: is there any real physical field governed by a equation of the type of the generalized Maxwell equation (Eq.(2)). The answer is yes.

### 5 The Generalized Hertz Potential Equation

In what follows we accept that $J_m = 0$ and take Maxwell equations for the electromagnetic field $F \in \text{sec} \bigwedge^2(M) \subset \text{sec} \mathcal{O}(M, g)$ and a current $J_e \in \text{sec} \bigwedge^1(M) \subset \text{sec} \mathcal{O}(M, g)$ as

$$\partial F = J_e. \tag{79}$$

Let $\Pi = \frac{1}{4} \Pi^{\mu\nu} \gamma_\mu \gamma_\nu = \Pi_e + i \Pi_m \in \text{sec} \bigwedge^2(M) \subset \text{sec} \mathcal{O}(M, g)$ be the so called Hertz potential ([33],[34]). We write

$$[\Pi^{\mu\nu}] = \begin{bmatrix}
0 & -\Pi_e^1 & -\Pi_e^2 & -\Pi_e^3 \\
\Pi_e^1 & 0 & -\Pi_m^3 & \Pi_m^2 \\
-\Pi_e^2 & \Pi_m^3 & 0 & -\Pi_m^1 \\
-\Pi_e^3 & -\Pi_m^2 & \Pi_m^1 & 0
\end{bmatrix}, \tag{80}$$

and define the electromagnetic potential by

$$A = -\delta \Pi \in \text{sec} \Lambda^1(T^*M) \subset \text{sec} \mathcal{O}(M, g), \tag{81}$$

Since $\delta^2 = 0$ it is clear that $A$ satisfies the Lorenz gauge condition, i.e.,

$$\delta A = 0. \tag{82}$$

Also, let

$$\gamma^5 S = d \Pi \in \text{sec} \bigwedge^3(M) \subset \text{sec} \mathcal{O}(M, g), \tag{83}$$

and call $S$, the Stratton potential. It follows also that

$$d (\gamma^5 S) = d^2 \Pi = 0. \tag{84}$$
But $d(\gamma^5 S) = \gamma^5 \delta S$ from which we get, taking into account Eq.\(76\),

\[
\delta S = 0 \tag{85}
\]

We can put Eq.\(81\) and Eq.\(83\) in the form of a single generalized Maxwell like equation, i.e.,

\[
\partial \Pi = (d - \delta) \Pi = A + \gamma^5 S = A. \tag{86}
\]

Eq.\(86\) is the equation we were looking for. It is a legitimate physical equation.

We also have,

\[
\Box \Pi = (d - \delta)^2 \Pi = dA + \gamma_5 dS. \tag{87}
\]

Next, we define the electromagnetic field by

\[
F = \partial A = \Box \Pi = dA + \gamma_5 dS = F_e + \gamma_5 F_m. \tag{88}
\]

We observe that,

\[
\Box \Pi = 0 \Rightarrow F_e = -\gamma_5 F_m. \tag{89}
\]

Now, let us calculate $\partial F$. We have,

\[
\partial F = (d - \delta) F = d^2 A + d(\gamma^5 dS) - \delta (dA) - \delta (\gamma^5 dS). \tag{90}
\]

The first and last terms in the second line of Eq.\(87\) are obviously null. Writing,

\[
J_e = -\delta dA, \text{ and } \gamma^5 J_m = -d(\gamma^5 dS), \tag{91}
\]

we get Maxwell equation

\[
\partial F = (d - \delta) F = J_e, \tag{92}
\]

if and only if the magnetic current $\gamma^5 J_m = 0$, i.e.,

\[
\delta dS = 0. \tag{93}
\]

a condition that we suppose to be satisfied in what follows. Then,

\[
\Box A = J_e = -\delta dA,
\]

\[
\Box S = 0. \tag{94}
\]

Now, we define,

\[
F_e = dA = \vec{E}_e + i\vec{B}_e, \tag{95}
\]

\[
F_m = dS = \vec{B}_m + i\vec{E}_m. \tag{96}
\]

and also

\[
F = F_e + \gamma_5 F_m = \vec{E} + i\vec{B} = (\vec{E}_e - \vec{E}_m) + i(\vec{B}_e + \vec{B}_m). \tag{97}
\]
Then, we get
\[ \Box \vec{\Pi}_e = \vec{E}, \quad \Box \vec{\Pi}_m = \vec{B}. \] (98)

It is important to keep in mind that:
\[ \Box \vec{\Pi} = 0 \Rightarrow \vec{E} = 0, \quad \vec{B} = 0. \] (99)

Nevertheless, despite this result we have,

**Hertz Theorem**
\[ \Box \vec{\Pi} = 0 \quad \Rightarrow \quad \partial \vec{F}_e = 0 \] (100)

**Proof.** We have immediately from the above equations that
\[ \partial \vec{F}_e = -\partial(\gamma_5 F_m) = -d(\gamma_5 dS) + \delta(\gamma_5 dS) = \gamma_5 d^2 S - \gamma_5 \delta dS = 0. \]  (101)

We remark that Eq.(100) has been called the Hertz theorem in [33,35] and it has been used there and also in [36-42] in order to find nontrivial *superluminal* solutions of the free Maxwell equation.

### 6 Maxwell Dirac Equivalence of First Kind

Let us consider a *generalized* Maxwell equation
\[ \partial \vec{F} = \vec{J}, \] (102)
where \( \partial = \gamma^\mu \partial_\mu \) is the Dirac operator and \( \vec{J} \) is the electromagnetic current (an electric current \( J_e \) plus a magnetic monopole current \(-\gamma_5 J_m \), where \( J_e, J_m \in \sec \bigwedge^1 M \subset \mathcal{C}(M,g) \)). We proved in section 2 that if \( F^2 \neq 0 \), then we can write
\[ \vec{F} = \psi \gamma_{21} \tilde{\psi}, \] (103)
where \( \psi \in \sec \bigwedge^1 M \subset \mathcal{C}(M,g) \) is a representative of a Dirac-Hestenes field. If we use Eq.103 in Eq.102 we get
\[ \partial (\psi \gamma_{21} \tilde{\psi}) = \gamma^\mu \partial_\mu (\psi \gamma_{21} \tilde{\psi}) = \gamma^\mu (\partial_\mu \psi \gamma_{21} \tilde{\psi} + \psi \gamma_{21} \partial_\mu \tilde{\psi}) = \vec{J}. \] (104)
from where it follows that
\[ 2 \gamma^\mu (\partial_\mu \psi \gamma_{21} \tilde{\psi})_2 = \vec{J}. \] (105)

Consider the identity
\[ \gamma^\mu (\partial_\mu \psi \gamma_{21} \tilde{\psi})_2 = \partial \psi \gamma_{21} \tilde{\psi} - \gamma^\mu (\partial_\mu \psi \gamma_{21} \tilde{\psi})_0 - \gamma^\mu (\partial_\mu \psi \gamma_{21} \tilde{\psi})_4, \] (106)
and define moreover the vectors
\[ j = \gamma^\mu (\partial_\mu \psi \gamma_{21} \tilde{\psi})_0. \] (107)
Taking into account Eqs. (104)-(108), we can rewrite Eq. (104) as

\[ \partial \psi \gamma_{21} \tilde{\psi} = \left[ \frac{1}{2} J + (j + \gamma_5 g) \right] \psi. \]  

(109)

Eq. (109) is a spinorial representation of Maxwell equation. In the case where \( \psi \) is non-singular (which corresponds to non-null electromagnetic fields) we have

\[ \partial \psi \gamma_{21} = e^{\gamma_5 \beta} \rho \left[ \frac{1}{2} J + (j + \gamma_5 g) \right] \psi. \]  

(110)

The Eq. (110) representing Maxwell equation, written in that form, does not appear to have any relationship with the Dirac-Hestenes equation (Eq. (43)). However, we shall make some algebraic modifications on it in such a way as to put it in a form that suggest a very interesting and intriguing relationship between them, and consequently a possible (?) connection between electromagnetism and quantum mechanics.

Since \( \psi \) is supposed to be non-singular (\( F \neq 0 \)) we can use the canonical decomposition of \( \psi \) and write \( \psi = \rho e^{\gamma_5 \beta} \frac{1}{2} R \), with \( \rho, \beta \in \text{sec}^{0} \mathcal{C}(M, g) \) and \( R \in \text{Spin}^{+}(1,3), \forall x \in M \). Then

\[ \partial_{\mu} \psi = \frac{1}{2} (\partial_{\mu} \ln \rho + \gamma_5 \partial_{\mu} \beta + \Omega_{\mu}) \psi, \]  

(111)

where we define the 2-form

\[ \Omega_{\mu} = 2(\partial_{\mu} R) \tilde{R}. \]  

(112)

Using this expression for \( \partial_{\mu} \psi \) into the definitions of the vectors \( j \) and \( g \) (Eqs. (107, 108)) we obtain that

\[ j = \gamma^\mu (\Omega_{\mu} \cdot S) \rho \cos \beta + \gamma_\mu (\Omega_{\mu} \cdot (\gamma_5 S)) \rho \sin \beta, \]  

(113)

\[ g = [\Omega_{\mu} \cdot (\gamma_5 S)] \rho \cos \beta - \gamma_{\mu} (\Omega_{\mu} \cdot S) \rho \sin \beta, \]  

(114)

where we define the spin 2-form \( S \) by

\[ S = \frac{1}{2} \psi \gamma_{21} \psi^{-1} = \frac{1}{2} R \gamma_{21} \tilde{R}. \]  

(115)

We now define

\[ J = \psi \gamma_0 \tilde{\psi} = \rho v = \rho R \gamma_0 R^{-1}, \]  

(116)

where \( v \) is the velocity field of the system. To continue, we define the 2-form \( \Omega = v^\mu \Omega_{\mu} \) and the scalars \( \Lambda \) and \( K \) by

\[ \Lambda = \Omega \cdot S, \]  

(117)

\[ K = \Omega \cdot (\gamma_5 S). \]  

(118)
Using these definition we have that
\[ \Omega_{\mu} \cdot S = \Lambda v_{\mu}, \]  
(119)
\[ \Omega_{\mu} \cdot (\gamma_5 S) = K v_{\mu}, \]  
(120)
and for the vectors \( j \) and \( g \) can be written as
\[ j = \Lambda \nu \cos \beta + K \nu \sin \beta = \lambda \nu, \]  
(121)
\[ g = K \nu \cos \beta - \Lambda \nu \sin \beta = \kappa \nu, \]  
(122)
where we defined
\[ \lambda = \Lambda \cos \beta + K \sin \beta, \]  
(123)
\[ \kappa = K \cos \beta - \Lambda \sin \beta. \]  
(124)

The spinorial representation of Maxwell equation is written now as
\[ \partial \psi_{\gamma 21} = e^{\gamma_5 \beta} J \psi + \lambda \psi_{\gamma 0} + \gamma_5 \kappa \psi_{\gamma 0}. \]  
(125)

Observe that there are (33–41) infinite families of non trivial solutions of Maxwell equations such that \( F^2 \neq 0 \) (which correspond to subluminal and superluminal solutions of Maxwell equation). Then, it is licit to consider the case \( J = 0 \). We have,
\[ \partial \psi_{\gamma 21} = \lambda \psi_{\gamma 0} + \gamma_5 \kappa \psi_{\gamma 0}, \]  
(126)
which is very similar to the Dirac-Hestenes equation.

In order to go a step further into the relationship between those equations, we remember that the electromagnetic field has six degrees of freedom, while a Dirac-Hestenes spinor field has eight degrees of freedom and that we proved in section 2 that two of these degrees of freedom are hidden variables. We are free therefore to impose two constraints on \( \psi \) if it is to represent an electromagnetic field. We choose these two constraints as
\[ \partial \cdot j = 0 \text{ and } \partial \cdot g = 0. \]  
(127)

Using Eqs. (121,122) these two constraints become
\[ \partial \cdot j = \rho \dot{\lambda} + \lambda \partial \cdot J = 0, \]  
(128)
\[ \partial \cdot g = \rho \dot{\kappa} + k \partial \cdot J = 0, \]  
(129)
where \( J = \rho v \) and \( \dot{\lambda} = (v \cdot \partial) \lambda, \dot{\kappa} = (v \cdot \partial) k \). These conditions imply that
\[ \kappa \lambda = \lambda \kappa \]  
(130)
which gives \( \lambda \neq 0 \):
\[ \frac{\kappa}{\lambda} = \text{const.} = -\tan \beta_0, \]  
(131)
or from Eqs. (123,124):

\[
\frac{K}{\Lambda} = \tan(\beta - \beta_0). \tag{132}
\]

Now we observe that \( \beta \) is the angle of the duality rotation from \( F \) to \( F' = e^{\gamma_5 \beta} F \). If we perform another duality rotation by \( \beta_0 \), we have \( F \mapsto e^{\gamma_5 (\beta + \beta_0)} F \), and for the Takabayasi angle \( \beta \mapsto \beta + \beta_0 \). If we work therefore with an electromagnetic field duality rotated by an additional angle \( \beta_0 \), the above relationship becomes

\[
\frac{K}{\Lambda} = \tan \beta. \tag{133}
\]

This is, of course, just a way to say that we can choose the constant \( \beta_0 \) in Eq. (131) to be zero. Now, this expression gives

\[
\lambda = \Lambda \cos \beta + \Lambda \tan \beta \sin \beta = \frac{\Lambda}{\cos \beta}, \tag{134}
\]

\[
\kappa = \Lambda \tan \beta \cos \beta - \Lambda \sin \beta = 0, \tag{135}
\]

and the spinorial representation of the Maxwell equation (Eq. (126)) becomes

\[
\partial \psi_{\gamma 21} - \lambda \psi_{\gamma 0} = 0 \tag{136}
\]

Note that \( \lambda \) is such that

\[
\rho \dot{\lambda} = -\lambda \partial \cdot J. \tag{137}
\]

The current \( J = \psi_{\gamma 0} \tilde{\psi} \) is not conserved unless \( \lambda \) is constant. If we suppose also that

\[
\partial \cdot J = 0 \tag{138}
\]

we must have

\[
\lambda = \text{const.}
\]

Now, throughout these calculations we have assumed \( \hbar = c = 1 \). We observe that in Eq. (136) \( \lambda \) has the units of \( (\text{length})^{-1} \), and if we introduce the constants \( \hbar \) and \( c \) we have to introduce another constant with unit of mass. If we denote this constant by \( m \) such that

\[
\lambda = \frac{mc}{\hbar}, \tag{139}
\]

then Eq. (136) assumes a form which is identical to Dirac-Hestenes equation:

\[
\partial \psi_{\gamma 21} - \frac{mc}{\hbar} \psi_{\gamma 0} = 0. \tag{140}
\]

It is true that we didn’t prove that Eq. (140) is really Dirac equation since the constant \( m \) has to be identified in this case with the electron’s mass, and we do not have any good physical argument to make that identification, until now. In resume, Eq. (140) has been obtained from Maxwell equation by imposing some gauge conditions allowed by the hidden parameters in the solution of Eq. (1) for \( \psi \) in terms of \( F \). In view of that, it seems more appropriate instead of using the term mathematical Maxwell-Dirac equivalence of first kind to talk about a
correspondence between that equations under which the two extra degrees of freedom of the Dirac-Hestenes spinor field are treated as hidden variables.

To end this section we observe that it is to earlier to know if the above results are of some physical value or only a mathematical curiosity. Let us wait...

7 Maxwell-Dirac Equivalence of Second Kind

We now look for a Hertz potential field $\Pi \in \text{sec} \wedge^2(M)$ satisfying the following (non linear) equation

$$\partial \Pi = (\partial \mathfrak{S} + m \mathfrak{P} \gamma_3 + m \langle \Pi \gamma_{012} \rangle_1 + \gamma_5 (\partial \mathfrak{P} + m \mathfrak{S} \gamma_3 - \gamma_5 \langle m \Pi \gamma_{012} \rangle_3) \quad (141)$$

where $\mathfrak{S}, \mathfrak{P} \in \text{sec} \wedge^0(M)$, and $m$ is a constant. According to section 5 the electromagnetic and Stratton potentials are

$$A = \partial \mathfrak{S} + m \mathfrak{P} \gamma_3 + m \langle \Pi \gamma_{012} \rangle_1, \quad (142)$$

$$\gamma_5 S = \gamma_5 (\partial \mathfrak{P} + m \mathfrak{S} \gamma_3 - \gamma_5 \langle m \Pi \gamma_{012} \rangle_3), \quad (143)$$

and must satisfy the following subsidiary conditions,

$$\Box (\partial \mathfrak{S} + m \mathfrak{P} \gamma_3 + m \langle \Pi \gamma_{012} \rangle_1) = J_e \quad (144)$$

$$\Box (\gamma_5 (\partial \mathfrak{P} + m \mathfrak{S} \gamma_3 - \gamma_5 \langle m \Pi \gamma_{012} \rangle_3)) = 0, \quad (145)$$

$$\Box \mathfrak{S} + m \partial \cdot \langle \Pi \gamma_{012} \rangle_1 = 0, \quad (146)$$

$$\Box \mathfrak{P} - m \partial \cdot (\gamma_5 \langle \Pi \gamma_{012} \rangle_3) = 0. \quad (147)$$

Now, in the Clifford bundle formalism, as we already explained above, the following sum is a legitimate operation

$$\psi = -\mathfrak{S} + \Pi + \gamma_5 \mathfrak{P} \quad (148)$$

and according to the results of section 2 defines $\psi$ as a (representative) of some Dirac-Hestenes spinor field. Now, we can verify that $\psi$ satisfies the equation

$$\partial \psi \gamma_{21} - m \psi \gamma_0 = 0 \quad (149)$$

which is as we already know a representative of the standard Dirac equation (for a free electron) in the Clifford bundle, which is a Dirac-Hestenes equation (Eq.(43)), written in an orthonormal coordinate spin frame.

The above developments suggest (consistently with the spirit of the generalized Hertz potential theory developed in section 5) the following interpretation.

The Hertz potential field $\Pi$ generates the real electromagnetic field of the electron (The question of the physical dimensions of the Dirac-Hestenes and Maxwell fields is discussed in [8].) Moreover, the above developments suggest that the electron is “composed” of two “fundamental” currents, one of electric type and the other of magnetic type circulating at the ultra microscopic level, which generate the observed electric charge and magnetic moment of the electron. Then,
it may be the case, as speculated by Maris [27], that the electromagnetic field of the electron can be spliced into two parts, each corresponding to a new kind of subelectron type particle, the electrino. Of course, the above developments leaves open the possibility to generate electrinos of fractional charges. We still study more properties of the above system in another paper.

8 Seiberg-Witten Equations

As it is well known, the original Seiberg-Witten (monopole) equations have been written in euclidean “spacetime” and for the self dual part of the field $F$. However, on Minkowski spacetime, of course, there are no self dual electromagnetic fields. Indeed, Eq.(12) implies that the unique solution (on Minkowski spacetime) of the equation $\ast F = F$ is $F = 0$. This is the main reason for the difficulties in interpreting that equations in this case, and indeed in [28] it was attempted an interpretation of that equations only for the case of euclidean manifolds. Here we want to derive and to give a possible interpretation to that equations based on a reasonable assumption.

Now, the analogous of Seiberg-Witten monopole equations read in the Clifford bundle formalism and on Minkowski spacetime as

\[
\begin{align*}
\partial \psi & \gamma_{21} - A \psi = 0 \\
F & = \frac{1}{2} \psi \gamma_{21} \psi \\
F & = dA \tag{150}
\end{align*}
\]

where $\psi \in \text{sec} \mathcal{C}+ (M, g)$ is a Dirac-Hestenes spinor field, $A \in \text{sec} \Lambda^1 (M) \subset \text{sec} \mathcal{C} (M, g)$ is an electromagnetic vector potential and $F \in \text{sec} \Lambda^2 (M) \subset \text{sec} \mathcal{C} (M, g)$ is an electromagnetic field.

Our intention in this section is:

(a) To use the Maxwell Dirac-Equivalence of the first kind (proved in section 7) and an additional hypothesis to be discussed below to derive the Seiberg-Witten equations on Minkowski spacetime.

(b) to give a (possible) physical interpretation for that equations.

8.1 Derivation of Seiberg-Witten Equations

Step 1. We assume that the electromagnetic field $F$ appearing in the second of the Seiberg-Witten equations satisfy the free Maxwell equation, i.e., $\partial F = 0$.

Step 2. We use the Maxwell-Dirac equivalence of the first kind proved in section 6 to obtain Eq.(138).

\[
\partial \psi \gamma_{21} - \lambda \psi \gamma_0 = 0 \tag{151}
\]

Step 3. We introduce the ansatz

\[
A = \lambda \psi \gamma_0 \psi^{-1}. \tag{152}
\]
This means that the electromagnetic potential (in our geometrical units) is identified with a multiply of the velocity field defined through Eq. (116). Under this condition Eq. (151) becomes

\[ \partial \psi_{\gamma 21} - A \psi = 0, \quad (153) \]

which is the first Seiberg-Witten equation!

### 8.2 A Possible Interpretation of the Seiberg-Witten Equations

Well, it is time to find an interpretation for Eq. (153). In order to do that we recall from section 2.5 that if \( \psi_{\pm} \) are Weyl spinor fields (as defined through Eq. (34), then \( \psi_{\pm} \) satisfy a Weyl equation, i.e.,

\[ \partial \psi_{\pm} = 0. \quad (154) \]

Consider now, the equation for \( \psi_{+} \) coupled with an electromagnetic field \( A = gB \in \sec \Lambda^1(M) \subset \sec \mathcal{C}(M,g) \), i.e.,

\[ \partial \psi_{+ \gamma 21} + gB \psi_{+} = 0. \quad (155) \]

This equation is invariant under the gauge transformations

\[ \psi_{+} \mapsto \psi_{+} e^{\gamma 5 \theta}; B \mapsto B + \partial \theta. \quad (156) \]

Also, the equation for \( \psi_{-} \) coupled with an electromagnetic field \( gB \in \sec \Lambda^1(M) \) is

\[ \partial \psi_{\gamma 21} + gB \psi_{-} = 0. \quad (157) \]

which is invariant under the gauge transformations

\[ \psi_{-} \mapsto \psi_{-} e^{\gamma 5 \theta}; B \mapsto B - \partial \theta. \quad (158) \]

showing clearly that the fields \( \psi_{+} \) and \( \psi_{-} \) carry opposite ‘charges’. Consider now the Dirac-Hestenes spinor fields \( \psi^\uparrow, \psi^\downarrow \) given by Eq. (38) which are eigenvectors of the parity operator and look for solutions of Eq. (153) such that \( \psi = \psi^\uparrow \). We have,

\[ \partial \psi_{\uparrow \gamma 21} + gB \psi_{\uparrow} = 0 \quad (159) \]

which separates in two equations,

\[ \partial \psi_{\uparrow} + g\gamma_5 B \psi_{\uparrow} = 0; \quad \partial \psi_{\downarrow} - g\gamma_5 B \psi_{\downarrow} = 0. \quad (160) \]

These results show that when a Dirac-Hestenes spinor field associated with the first of the Seiberg-Witten equations is in an eigenstate of the parity operator, that spinor field describes a pair of particles with opposite ‘charges’. We interpret these particles (following Lochack [42], that suggested that an equation equivalent to Eq. (160) describe massless monopoles of opposite ‘charges’).
as being *massless* ‘monopoles’ in *auto-interaction*. Observe that our proposed interaction is also consistent with the third of Seiberg-Witten equations, for \( F = dA \) implies a *null* magnetic current.

It is now well known that Seiberg-Witten equations have non trivial solutions on Minkowski manifolds (see [25]). From the above results, in particular, taking into account the inversion formula (Eq. (56)) it seems to be possible to find whole family of solutions for the Seiberg-Witten equations, which has been here derived from a Maxwell-Dirac equivalence of first kind (proved in section 6) with the additional hypothesis that electromagnetic potential \( A \) is parallel to the velocity field \( v \) (Eq. (152)) of the system described by Eq. (116). We conclude that a consistent set of Seiberg-Witten equations on Minkowski spacetime must be

\[
\begin{aligned}
\partial \psi & \gamma_{21} - A \psi = 0 \\
F & = \frac{1}{2} \psi \gamma_{21} \tilde{\psi} \\
F & = dA \\
A & = \lambda \psi \gamma_{0} \psi^{-1}
\end{aligned}
\]  

(161)

9 Conclusions

In this paper we exhibit two different kinds of possible Maxwell-Dirac equivalences (*MDE*). Although many will find the ideas presented above speculative from the physical point of view, we hope that they may become important, at least from a mathematical point of view. Indeed, not to long ago, researching solutions of the free Maxwell equation \( \partial F = 0 \) satisfying the constraint \( F^2 \neq 0 \) (a necessary condition for derivation of a *MDE* of the first kind) conducd to the discovery of families of *superluminal* solutions of Maxwell equations and also of all the main linear relativistic equations of theoretical Physics ([34],[42]). The study of the *MDE* of the second kind reveal an unsuspected interpretation of the Dirac equation, namely that the electron seems to be a composed system build up from the self interaction of two currents of ‘electrical’ and ‘magnetic’ types. Of course, it is to earlier to say if this discovery has any physical significance. We showed also, that by using the *MDE* of the first kind together with a reasonable hypothesis we can shed light on the meaning of Seiberg-Witten monopole equations on Minkowski spacetime. We hope that the results just found may be an indication that Seiberg-Witten equations (which are a fundamental key in the study of the topology of four manifolds equipped with an *euclidean* metric tensor), may play an important role in Physics, whose arena where phenomena occur is a *Lorentzian* manifold.

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References

[1] A. A. Campolattaro, New spinor representation of Maxwell equations 1. Generalities, *Int. J. Theor. Phys.* **19**, 99-126 (1980).

[2] A. A. Campolattaro, New spinor representation of Maxwell equations 2. Generalities, *Int. J. Theor. Phys.* **19**, 127-138 (1980).

[3] A. A. Campolattaro, Generalized Maxwell equations and quantum mechanics 1. Dirac equation for the free electron, *Int. J. Theor. Phys.* **19**, 141-155 (1980).

[4] A. A. Campolattaro, Generalized Maxwell equations and quantum mechanics 2. Generalized Dirac equation, *Int. J. Theor. Phys.* **19**, 477-482 (1980).

[5] A. A. Campolattaro, From classical electrodynamics to relativistic quantum mechanics, in J. Keller and Z. Owczew (eds.), *The theory of the electron*, *Adv. Appl. Clifford Algebras* **7**(S), 167-173 (1997).

[6] J. Vaz, Jr. and W. A. Rodrigues, Jr., Equivalence of Dirac and Maxwell equations and quantum mechanics, *Int. J. Theor. Phys.* **32**, 945-949 (1993).

[7] J. Vaz, Jr. and W. A. Rodrigues, Jr., Maxwell and Dirac theory as an already unified theory, in J. Keller and Z. Owczew (eds.), *The theory of the electron*, *Adv. Appl. Clifford Algebras* **7**(S), 369-385 (1997).

[8] W. A. Rodrigues, Jr., and J. Vaz, Jr, From electromagnetism to relativistic quantum mechanics, *Found. Phys.* **28**, 789-814 (1998).

[9] A. Maia, Jr., E. Recami, W. A. Rodrigues, Jr., and M. A. F. Rosa, Magnetic Monopoles without strings in the Kähler-Clifford algebra bundle: a geometrical interpretation, *J. Math. Phys.* **31**, 502-505 (1990).

[10] Q. A. G. Souza and W. A. Rodrigues, Jr., in P. Letelier and W. A. Rodrigues, Jr., (eds.), *The Dirac operator and the structure of Riemann-Cartan-Weyl spaces*, *Gravitation. The Spacetime Structure*, pp. 179-212, World Sci. Publ. Co., Singapore, 1994.

[11] W. A. Rodrigues, Jr., Q. A. G. Souza, The Clifford bundle and the nature of the gravitational field, *Found. Phys.* **1465-1490** (1993).

[12] W. A. Rodrigues, Jr., Q. A. G. Souza, J. Vaz, Jr. and P. Lounesto, Dirac-Hestenes spinor fields on Riemann-Cartan manifolds, *Int. J. Theor. Phys.* **35**, 1849-1990 (1996).

[13] W. A. Rodrigues, Jr., Q. A. G. Souza and J. Vaz, Jr., Spinor fields and superfields as equivalence classes of exterior algebra fields, in R. Ablamowicz and P. Lounesto (eds.), *Clifford Algebras and Spinor Structure*, pp. 177-198, Kluwer Acad. Publ., Dordrecht (1995).
[14] P. Lounesto, Clifford algebras and Hestenes spinors, Found. Phys. 23, 1203-1237 (1993).

[15] P. Lounesto, Clifford algebras, relativity and quantum mechanics, in P. Letelier and W. A. Rodrigues, Jr. (eds.), Gravitation. The Spacetime Structure, pp. 50-81, World Sci. Publ. Co. Singapore, 1994.

[16] W. A. Rodrigues, Jr., Algebraic and Dirac-Hestenes spinors and spinor fields, RP 56/02IMECC-UNICAMP, (2002), math-phys/0212030

[17] R. A. Mosna and W. A. Rodrigues, Jr., The bundles of algebraic and Dirac-Hestenes spinor fields, RP 57/02IMECC-UNICAMP, (2002) math.ph/0212033

[18] V. L. Figueiredo, W. A. Rodrigues Jr. and E. C. de Oliveira, Clifford algebras and the hidden geometrical nature of spinors, Algebras, Groups and Geometries 7, 153-198 (1990).

[19] V. L. Figueiredo, W. A. Rodrigues Jr. and E. C. de Oliveira, Covariant, algebraic and operator spinors, Int. J. Theor. Phys. 29, 371-395 (1990).

[20] G. Rainich, Electrodynamics and general relativity theory, Am. Math. Soc. Trans. 27, 106-136 (1925).

[21] A. Gsponer, On the “equivalence” of Maxwell and Dirac equations, in publ. Int. J. Theor. Phys. 41, 689-694 (2002).

[22] N. Seiberg and E. Witten, Monopoles, duality and chiral symmetry breaking in N = 2 QCD, Nucl. Phys. B 431, 581-640 (1994).

[23] C. Nash, Topology and Physics-a historical essay, preprint arXiv:hep-th/9709135v4 (1997).

[24] L. I. Nicolaescu, Notes on Seiberg-Witten theory, Graduate Studies in Mathematics 28, Am. Math. Soc., Providence, Rodhe Island, 2000.

[25] G. L. Naber, Topology, geometry and gauge fields interactions, Applied Math. Sci. 141, Springer-Velarg, New York, 2000.

[26] H. Maris, On the fission of elementary particles and evidence for fractional electrons in liquid helium, J. Low Temp. Phys. 120, 173-204 (2000).

[27] M. Chown, Double or quit, New Scientist no. 2000 (October 14), 24-27 (2000).

[28] J. Vaz, Jr., Clifford Algebras and Witten’s monopole equations, in B. N. Apanasov, S. B. Bradlow, W. A. Rodrigues, Jr. and K. K. Uhlenbeck (eds.), Geometry, Topology and Physics, W. de Gruyter, Berlin, 1997.

[29] F. Gursey, Contribution to the quaternion formalism in special relativity, Rev. Fac. Sci. Instambul A 20, 149-171 (1956).
[30] J. Crawford, On the algebra of Dirac bispinors densities: factorization and inversion theorems, *J. Math. Phys.* **26**, 1439-1441 (1985).

[31] W. A. Rodrigues, Jr. and E. C. de Oliveira, Maxwell and Dirac equations in the Clifford and Spin-Clifford bundles, *Int. J. Theor. Phys.* **29**, 397-412 (1990).

[32] J. Stratton, *Electrodynamics theory*, McGraw-Hill, New York, 1941.

[33] A. L. Trovon de Carvalho and W. A. Rodrigues, Jr., The non sequitur mathematics and physics of the “new electrodynamics” proposed by the AIAS group, *Random Oper. and Stoch. Equ.* **9**, 161-202 (2001).

[34] W. A. Rodrigues, Jr. and J. Vaz, Jr., Subluminal and superluminal solutions in vacuum of the Maxwell equations and the massless Dirac equation, in H. Keller and Z. Olszewcz (eds.), The theory of the electron (Proc. of the Conference: The theory of the electron, Mexico, 1995) *Adv. Appl. Clifford Algebras* **7** (S), 457-466 (1997).

[35] W. A. Rodrigues, Jr. and J. E. Majorino, A unified theory for construction of arbitrary speed \(0 \leq v < \infty\) solutions of the relativistic wave equations, *Random Opr. and Stch. Equ.* **4**, 355-400 (1996).

[36] W. A. Rodrigues, Jr. and J. Y. Lu, On the existence of undistorted progressive waves (UPWs) of arbitrary speeds \(0 \leq v < \infty\) in nature, *Found. Phys.* **27**, 435-508 (1997).

[37] E. C. de Oliveira and W. A. Rodrigues, Jr. Superluminal electromagnetic waves in free space, *Ann. der Physik* **7**, 654-659 (1998).

[38] J. E. Majorino and W. A. Rodrigues, Jr., What is Superluminal Wave Motion? *Sci. and Tech. Mag.* **2**(4), (1999), [http://www.ime.unicamp.br/research/1999/rp59-99.html](http://www.ime.unicamp.br/research/1999/rp59-99.html)

[39] W. A. Rodrigues, Jr., D. S. Thober and A. L. Xavier, Jr., Causal explanation of observed superluminal behavior of microwave propagation in free space, *Phys. Lett. A* **284**, 217-224 (2001).

[40] E. C. de Oliveira and W. A. Rodrigues, Jr., D. S. Thober and A. L. Xavier, Jr., Thoughtful comments on ‘Bessel beams’ and signal propagation, *Phys. Lett. A* **284**, 269-303 (2001).

[41] E. C. de Oliveira and W. A. Rodrigues, Jr., Finite energy superluminal solutions of Maxwell equations, *Phys. Lett. A* **291**, 367-370 (2001).

[42] G. Lochak, Wave equation for a magnetic monopole, *Int. J. Theor. Phys.* **24**, 1019-1050 (1985).