What is entanglement?

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Abstract

I conjecture that only those states of light whose Wigner function is positive are real states, and give arguments suggesting that this is not a serious restriction. Hence it follows that the Wigner formalism in quantum optics is capable of interpretation as a classical wave field with the addition of a zeropoint contribution. Thus entanglement between pairs of photons with a common origin occurs because the two light signals have amplitudes and phases, both below and above the zeropoint intensity level, which are correlated with each other.
I. What states of light are real?

I shall begin with the following

**Conjecture 1** Real states of light have a positive Wigner function.

Probably the reader does not believe that this conjecture may be true. Therefore I shall rewrite it in a more acceptable form:

**Conjecture 2** In all experiments where the conventional quantum interpretation involves states of light with negative Wigner functions, the interpretation would require only positive Wigner functions if all sources of noise, and other nonidealities, are taken into account.

A. The quantum states of light and their Wigner representation

The ”states” which form the basis for the quantum optical description of the light field are the **Fock states**, which are generated by applying the **creation operators** $\hat{a}_{k,\lambda}^\dagger$ to the hilbert-space vector

$$|0\rangle = \prod_{k,\lambda} |0_{k,\lambda}\rangle ,$$

(1)

which represents the vacuum. The whole Fock space is then spanned by the set of vectors

$$|\{n_{k,\lambda}\}\rangle = \prod_{k,\lambda} |n_{k,\lambda}\rangle = \prod_{k,\lambda} \frac{1}{\sqrt{n_{k,\lambda}!}} (\hat{a}_{k,\lambda}^\dagger)^{n_{k,\lambda}} |0_{k,\lambda}\rangle ,$$

(2)

which represents a state having $n_{k,\lambda}$ photons of wave number $k$ and polarization $\lambda$. The latter index takes either the value 1 or 2. The most general **pure state** is a superposition of these, that is

$$|\Phi\rangle = \sum \phi(\{n_{k,\lambda}\}) |\{n_{k,\lambda}\}\rangle , \quad \sum |\phi|^2 = 1 .$$

(3)

The full set of quantum states is obtained by extending the set $|\Phi\rangle$ to **mixtures** of the form

$$\hat{\rho} = \sum |\Phi\rangle P_\phi \langle \Phi| , \quad 0 \leq P_\phi \leq 1 , \quad \sum P_\phi = 1 .$$
The Wigner function of this state is defined as

$$W_\rho(\{\alpha_{k,\lambda}\}) = Tr \left[ \hat{\rho} \hat{W}(\{\alpha_{k,\lambda}\}) \right],$$

(4)

where \(\{\alpha_{k,\lambda}\}\) are a set of complex variables, representing the amplitudes of the radiation modes, and

$$\hat{W}(\{\alpha_{k,\lambda}\}) = \prod_{k,\lambda} \frac{1}{\pi^2} \int \exp \left[ \xi_{k,\lambda}(\hat{a}_{k,\lambda}^\dagger - \alpha_{k,\lambda}^*) - \xi_{k,\lambda}^*(\hat{a}_{k,\lambda} - \alpha_{k,\lambda}) \right] d^2\xi_{k,\lambda}.$$  

(The integration should be performed with respect to the real and imaginary parts of every complex variable.) For instance, the Wigner function of the vacuum is

$$W_0(\{\alpha_{k,\lambda}\}) = \prod_{k,\lambda} (2/\pi) \exp \left( -2|\alpha_{k,\lambda}|^2 \right).$$

(5)

In the Hilbert-space representation the electric and magnetic fields of the radiation are usually expanded in normal modes, the coefficients of the expansion being the creation, \(\hat{a}_{k,\lambda}^\dagger\), and annihilation, \(\hat{a}_{k,\lambda}\), operators of photons. In the Wigner representation these operators become the amplitudes of the modes, which are random variables with a distribution given by the Wigner function (assumed positive). For instance the expansion of the electric field in free space is

$$E(r, t) = \sum_{k,\lambda} \sqrt{\frac{2\hbar\omega}{L^3}} \Re [\alpha_{k,\lambda} e_{k,\lambda} \exp (i\hat{k}.r - i\omega t)], \omega \equiv c|\vec{k}|,$$

(6)

where \(\hbar\) is Planck’s constant, \(e_{k,\lambda}\) the polarization vector and \(L^3\) the normalization volume.

The Wigner function in nonrelativistic quantum mechanics plays the role of a pseudoprobability distribution; its marginals with respect to position and momentum separately give the quantum probabilities for each of these variables, but the function itself is not positive definite. There are great difficulties in interpreting the Wigner function as a true probability distribution in quantum mechanics. Nevertheless I propose that for the light field the Wigner function may be assumed positive for all physically realizable states. This assumption will not put too big difficulties for the interpretation of actual experiments, as is argued in the next subsection.
B. When is the Wigner function positive?

The Fock states (3) form a basis appropriate for the solution of the Maxwell equations fulfilled by the electromagnetic field. They play the same role as, for instance, the functions \{\sin(nx)\} in the solution of the diffusion equation in one dimension,

\[
\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2},
\]

\(f(x, t)\) being the density of the diffusing matter. With boundary conditions \(f(0, t) = f(\pi, t) = 0\), the solution is

\[
f(x, t) = \sum c_n \sin(nx) \exp(-n^2t),
\]

where the coefficients \(c_n\) may be obtained from the initial condition \(f(x, 0)\). We need the whole set of functions \{\sin(nx)\} with integer \(n\) in order to be able to express the solution of the problem as a Fourier series (this requirement is called completeness of the basis), but it is obvious that not all series of the form (7) may represent real physical states. In particular the functions \{\sin(nx)\} themselves are not positive definite (except for \(n = 1\)) and cannot represent densities. Similarly we may assume that the ”Fock states” do not correspond to physically realizable states (except (1)) although all of them are necessary in order to represent every possible physical state of the radiation field.

Our assumption contrasts with the frequent interpretation of experiments in terms of one-photon states. The one-photon state \(\ket{1}\) has the Wigner function

\[
W_1(\alpha) = (4|\alpha|^2 - 1)W_0(\alpha),
\]

where \(W_0(\alpha)\) is the vacuum Wigner function for a single mode (5). Taking into account the complete set of modes, including the “unoccupied” ones, the Wigner function may be written as

\[
W_1(\{\alpha_{k,\lambda}\}) = (4|\alpha_{k,\lambda}|^2 - 1) W_0(\{\alpha_{k,\lambda}\}).
\]

Of course this is not always positive.

Actually the experimental situations in which something like a one-photon state has been reported, must necessarily involve the observations of wave packets rather than single-mode signals. Indeed, the latter, which fill
the whole of space and time, are not at all physical objects. A wave packet has the Hilbert-space representation

$$|\zeta(x)\rangle = \sum_{k,\lambda} \zeta_{k,\lambda} e^{ikx} a_{k,\lambda}^\dagger |0\rangle ,$$

where \{\zeta_{k,\lambda}\} are a set of random, but not independent variables satisfying

$$\sum_{k,\lambda} |\zeta_{k,\lambda}|^2 = 1 ,$$

and which, furthermore, are nonzero only for a set of vectors \(\mathbf{k}\) falling within a small ellipsoidal region centred at \(\mathbf{k}'\).

The Wigner function of (8) is not positive. However, it is necessary to bear in mind that there is no way of controlling the moment at which such a packet is emitted, in the atomic-cascade situation used for such experiments, and this may be taken into account by forming an appropriate mixture of such wave-packet states. We have been able to show that the one-photon state becomes a mixture having a positive Wigner function. The proof will not be reproduced here, but a hint may be got by realizing that the mixed state with density operator

$$\hat{\rho} = \frac{1}{2} |1\rangle\langle 1| + \frac{1}{2} |0\rangle\langle 0|$$

has a positive Wigner function, namely

$$\mathcal{W}_1(\{a_{k',\lambda'}\}) = 2 |a_{k,\lambda}|^2 \mathcal{W}_0(\{a_{k',\lambda'}\}) .$$

Incidentally, with our approach it is possible to explain, in terms of pure waves, one of the most dramatic instances of "corpuscular behaviour" of light, the anticorrelation after a beam splitter. But the wave-particle duality of light will not be treated further in this article.

In summary I challenge the current, although rarely explicit, assumption that all density operators represent physically realizable states. In contrast I propose that only those states having a positive Wigner function may be realizable. On the other hand one may ask whether there are real states such that their (positive) probability distribution, \(W\), is not a Wigner function, that is no density operator \(\hat{\rho}\) exists leading to \(W\) via eq. (4). Such states might violate the Heisenberg uncertainty inequalities which, in our approach, derive
from the existence of a minimal noise which may be controlled in experiments. Thus I think that the answer is in the negative, in the sense that the amount of noise in the preparation procedure cannot be reduced below the Heisenberg inequalities limit. Indeed, it will be usually well above that limit. In any case the question will not be studied here further.

C. Quantum pure states which are real

Pure states, of the form \( \langle \alpha \rangle \), rarely have a positive Wigner function. Indeed it is known that if the Wigner function is positive, it is gaussian. The proof was given in [2] for a single mode and generalized to many modes in [3]. In contrast, no general rule is known in the case of quantum mixed states.

In the rest of this article we shall treat only the restricted, but important, kind of real states having a gaussian Wigner function. For simplicity we shall consider the ideal (i.e., unphysical) situation where only a single mode of the field contains photons, so that the set \( \{ n_{k,\lambda} \} \) contains a single member, and the number states are designated simply \(|0\rangle, |1\rangle, |2\rangle, \ldots \). The full Wigner function will be given by the product of the single-mode function with \( W_0(\alpha_{k,\lambda}) \) for each of the “unoccupied” modes. The generalization to many modes should not be difficult.

The most general single-mode gaussian Wigner function may be written in terms of two real parameters, A and B, and a complex one, a, as follows

\[
W(\alpha) = \frac{\sqrt{AB}}{\pi} \exp \left\{ -A(\text{Re}\alpha - \text{Re}a)^2 - B(\text{Im}\alpha - \text{Im}a)^2 \right\}.
\]

(9)

If \( AB = 4 \), \( W(\alpha) \) corresponds to a quantum pure state and if \( AB > 4 \) to a mixed state. Values such that \( AB < 4 \) do not provide a Wigner function, that is no density operator exists whence \( W(\alpha) \) may obtained via (4).

Among pure states the case \( A = B \), that is

\[
W_a(\alpha) = (2/\pi) \exp \left\{ -2|\alpha - a|^2 \right\},
\]

(10)

is called coherent state, which is an idealized form of the continuous-wave laser if \( a = 0 \) (and represents the vacuum \( |\rangle \) if \( a \neq 0 \)). The case \( A \neq B \), that is

\[
W_{a,s}(\alpha) = (2/\pi) \exp[-2e^{2s}(\text{Re}\alpha - \text{Re}a)^2 - 2e^{-2s}(\text{Im}\alpha - \text{Im}a)^2],
\]

6
with $s$ real is called squeezed state. The hilbert-space representations of these two states are, respectively,

$$|a⟩ = \exp \left( a^{\dagger}a^* - a^*a \right) |0⟩ = \sum_n \frac{a^n}{\sqrt{n!}} \exp \left( -|a|^2/2 \right) |n⟩,$$

and

$$|a, s⟩ = \exp \left( s(a^{\dagger^2} - \hat{a}^2) \right) |a⟩. \quad (12)$$

States (11) and (12) are the only single-mode quantum pure states which are real according to our criterion.

D. Stochastic optics

If all real states of light have a positive Wigner function, we may interpret that function as an actual probability distribution of the amplitudes of the radiation modes. Thus quantum optics becomes a disguised stochastic theory, where the states of light are probability distributions defined on the set of possible realizations of the electromagnetic field. We propose the name stochastic optics for the stochastic interpretation of quantum optics derived from the Wigner function. From another point of view, the stochastic interpretation provides an explicit hidden variables theory where the amplitudes of the electromagnetic field are the “hidden” variables!

The most dramatic consequence of stochastic optics is that the vacuum is no longer empty, but filled with a random electromagnetic radiation having an energy $\frac{1}{2}\hbar\omega$ per radiation mode, on the average, as is shown in eq.(5). That radiation corresponds precisely to the additional term introduced by Max Planck in his second radiation law (see e.g.10). The picture that emerges is that space contains a random background of electromagnetic waves providing what we shall call a zeropoint field (ZPF).

The question whether that theory should be named classical is a matter of taste. It might be said classical in the sense that it is a pure wave theory, essentially the same developed during the XIX Century. Photons are just wavepackets, usually localized in the form of needles of radiation (see eq.(8)), superimposed to the ZPF. Nevertheless the theory departs from classical optics in the assumption that there exists a fundamental noise, the ZPF, which cannot be eliminated even at zero Kelvin. I prefer to remain closer to the current nomenclature and say that stochastic optics is not a classical theory.
II. What states of light are classical?

A. Zeropoint field and detection theory

The ZPF is usually not directly observable, although indirectly it may produce observable effects, as explained below. Observable signals consist of additional radiation on top of the "sea" of ZPF. Crucial for the stochastic interpretation is the assumption that the ZPF has precisely the same nature as signals. Therefore explaining why the ZPF is not directly observable, e.g. by firing photon detectors, is a non-trivial problem which will not be studied here (see, e.g. [1]). In the following we state the problem more precisely.

The hilbert-space formalism of detection is based on normal ordering, that is putting creation operators to the left and annihilation operators to the right. Let us begin with the ideal case where the radiation field is represented by a single mode with amplitude $\alpha$. Then the detection probability per unit time is

$$P^q \propto Tr [\hat{\rho} \hat{a}^{\dagger} \hat{a}^{\dagger}] = \frac{1}{2} Tr [\hat{\rho} \{\hat{a}^{\dagger} \hat{a} + \hat{a} \hat{a}^{\dagger} - 1\}]$$

$$= \int W(\{\alpha\}) \left[|\alpha|^2 - \frac{1}{2}\right] d^2\alpha \equiv \langle |\alpha|^2 - \frac{1}{2} \rangle_w = n \quad (13)$$

where $\hat{\rho}$ is the state of the radiation, $W(\alpha)$ the corresponding Wigner function, and $n$ is the "mean number of photons" in the state. The first equality derives from the use of the commutation relations and the second is the passage to the Wigner representation. The symbol $\langle \rangle_w$ means the average of the quantity inside weighted with the Wigner function of the state.

In the general case of many modes and working in the Heisenberg picture, it is straightforward to get, for a point-like detector [2]

$$P^q(\mathbf{r}, t) \propto \int W(\{\alpha_k\}) [I(\mathbf{r}, t; \{\alpha_k\}) - I_0] d^2N \alpha_k = \langle I - I_0 \rangle, \quad (14)$$

where

$$I(\mathbf{r}, t; \{\alpha_k\}) = c c_0 |E(\mathbf{r}, t; \{\alpha_k\})|^2 \quad (15)$$

is the intensity for a realization of the field at the position and time $(\mathbf{r}, t)$. $W(\{\alpha_k\})$ is the Wigner function of the initial state, $N$ is the number of modes (we should take the limit $N \to \infty$ at some appropriate moment) and $I_0$ is the mean intensity of the ZPF. (We use the "calligraphic" $I$ for the (random variable) intensity in order to distinguish it from the (nonrandom) average intensity.
Writing the electric field, $E$, in terms of the initial amplitudes of the normal modes, $\{\alpha_k\}$, is usually straightforward although lengthy. It should be made for every particular experiment (almost all performed experiments with light produced by parametric down conversion have been studied with the Wigner function formalism in the series of articles\textsuperscript{12-15}).

Eq.\textsuperscript{(14)} may be interpreted as stating that the detector has a threshold so that it only detects the part of the field which is above the average ZPF, that is the detector removes the ZPF. The quantum rule \textsuperscript{(14)} is just to subtract the mean, a formal procedure which cannot be physical because it gives rise to “negative probabilities”. The problem is not the huge value of the zeropoint energy (the ZPF intensity is about $10^5 \text{w/cm}^2$ in the visible range), because the threshold intensity $I_0$ cancels precisely that intensity. The problem lies in the fluctuations of the intensity. For the weak light signals of typical quantum-optical experiments the fluctuations of $I$ may be such that $I < I_0$. This problem may be solved as discussed elsewhere,\textsuperscript{11} but it will not be considered here further.

B. Classical and nonclassical states of light

The fact that nonclassicality derives from the existence of the ZPF provides a criterion to classify the states of the radiation field. We shall call classical (nonclassical) those states where the ZPF is irrelevant (relevant). More specifically we will consider a state as classical if the total field $E(r,t)$ can be decomposed into two independent parts, $E_0(r,t)$ and $E_1(r,t)$ representing ZPF and signal respectively.\textsuperscript{6} When this is the case, all optical phenomena are associated with the signal alone, and the ZPF may be ignored altogether, as is the situation in classical optics. In particular detectors remove precisely the ZPF, see \textsuperscript{(14)}. The independence of the fields implies that the corresponding amplitudes, $\{\alpha_{0k,\lambda}\}$ and $\{\alpha_{1k,\lambda}\}$, are independent random variables. If the probability densities of these are $W_0(\{\alpha_{0k,\lambda}\})$ and $W_1(\{\alpha_{1k,\lambda}\})$, then the density of the total field will be their convolution, that is

$$W(\{\alpha_{k,\lambda}\}) = \int W_1(\{\beta_{k,\lambda}\})W_0(\{\alpha_{k,\lambda} - \beta_{k,\lambda}\})d^{2N}\beta.$$ 

It is well known that the Wigner function of a state of light may be obtained by means of the convolution of the (Glauber) P-function and the
Wigner function of the vacuum state when the former exists, that is

$$W(\{\alpha_{k,\lambda}\}) = \int P(\{\beta_{k,\lambda}\}) W_0(\{\alpha_{k,\lambda} - \beta_{k,\lambda}\}) d^{2N} \beta.$$  \hspace{1cm} (16)

If we identify $W_1(\{\alpha_{k,\lambda}\})$ with $P(\{\alpha_{k,\lambda}\})$ it is clear that the decomposition of the stochastic field $E(r,t)$ into two independent parts requires that a P-function exists which is positive definite. So, to summarize, the existence of a positive P-function is a necessary and sufficient condition for being able to decompose the field into independent signal and ZPF parts. This is our criterion for a state of light to be classical and it agrees precisely with the standard quantum-optical definition of classical state.

There is only one classical "pure" state, namely the coherent state (the vacuum state being a particular case). In the ideal situation of a single mode its P-function is

$$P(\alpha) = \delta^2 (\alpha - a),$$

$\delta^2$ being the two-dimensional Dirac's delta. This leads to the interpretation that the coherent state represents a deterministic (non-random) signal superimposed to the ZPF. The Wigner function of the state is given in eq. (10).

An important kind of classical "mixed" state is chaotic light, whose P-function is

$$P(\{\alpha_{k,\lambda}\}) = \prod_{k,\lambda} \frac{1}{\pi n_{k,\lambda}} \exp \left( -\frac{|\alpha_{k,\lambda}|^2}{n_{k,\lambda}} \right),$$  \hspace{1cm} (17)

and the Wigner function

$$P(\{\alpha_{k,\lambda}\}) = \prod_{k,\lambda} \frac{2}{\pi (2n_{k,\lambda} + 1)} \exp \left( -\frac{2|\alpha_{k,\lambda}|^2}{2n_{k,\lambda} + 1} \right).$$  \hspace{1cm} (18)

The quantity $n_{k,\lambda}$ is, in quantum language, the mean number of photons in the mode $k,\lambda$, as may be easily proved from the final equality in (13). A particular case of chaotic is thermal light, where the dependence of $n$ on the frequency $\omega = |kc|$ is given by Planck's law.

### III. Entanglement is correlation between quantum fluctuations

We are in a position to justify the essential conclusion of the present paper, stated in the title of this section. But I shall not attempt here a general
A. Gaussian Wigner functions involving two radiation modes

We are interested in studying a real state of light consisting of radiation in two separated regions of space (see section 1 for the meaning of real). Light in every region will contain many modes but, for the sake of clarity, we shall study an example involving only two modes with amplitudes labelled $\alpha_{k,\lambda}$ and $\beta_{k',\lambda'}$, assuming that in all other modes we have just ZPF. A more physical state would correspond to having two wave packets, the first (second) containing many modes with wavevectors close to $k$ ($k'$). For simplicity we will remove the subindices and label the amplitudes $\alpha$ and $\beta$ in the following.

We shall consider a classical two-modes state whose marginal for every mode corresponds to chaotic light (17). Its P-function is gaussian

$$P_{12}(\alpha, \beta) = \frac{ab - c^2}{\pi^2} \exp \left\{ -a |\alpha|^2 - b |\beta|^2 + c (\alpha \beta^* + \beta \alpha^*) \right\},$$

(19)

$a, b$ and $c$ being three real numbers fulfilling $a, b > 0, c^2 < ab$. The latter condition guarantees that $P$ goes to zero at infinite, a necessary condition for normalizability. In order to simplify the argument I will consider a slightly less general state by assuming $a = b$, although the generalization to $a \neq b$ is trivial. The intensity of the radiation, above the ZPF, is measured by the integral (see (13))

$$2n = \int P_{12}(\alpha, \beta) \left( |\alpha|^2 + |\beta|^2 \right) d^2 \alpha d^2 \beta = \frac{2a}{a^2 - c^2},$$

(20)

where $n$ represents the mean number of photons per mode.

The Wigner function of this state is given by the convolution with the vacuum Wigner function (see (14).) We shall write it in terms of $n$ and a correlation parameter, $x$, defined by

$$x = \frac{2c}{2a + a^2 - c^2}.$$

(21)
We get

\[ W_{12}(\alpha, \beta) = \frac{1-x^2}{\pi^2A^2} \exp\left\{-A \left[|\alpha|^2 + |\beta|^2 - x(\alpha\beta^* + \beta\alpha^*)\right]\right\}, \quad A \equiv \frac{2}{(2n+1)(1-x^2)} \]

This Wigner function represents a classical state if \(|c| < a\), which implies a restriction on the range of values of \(x\). In fact, from (20) and (21) we obtain

\[ a = \frac{4n}{4n^2 - x^2 (2n+1)^2}. \]

The condition \(a > 0\) leads to

\[ \text{classical : } n \geq 0, |x| < \frac{2n}{2n+1}, \quad (23) \]

A transparent interpretation of these results emerges in our approach. Remembering eq. (16), we see that (22) is the probability distribution of the sum of two random variables representing the signal and the ZPF, respectively. In the signal, whose distribution is (19), there is correlation between the two modes. In contrast in the ZPF, whose distribution is (5), all modes are uncorrelated (the distribution is a product of single-mode terms). In summary, we have a correlation involving the signal but not the ZPF. This should be called a classical correlation.

The constraints for the state (22) to be real are weaker than (23), namely

\[ \text{real : } n \geq 0, |x| < 1. \]

(Another condition for the state to be real is that \(W\) may be obtained from a density operator via (4), which is true in this case but will not be proved here). On the other hand the marginal of (22),

\[ W_1(\alpha) = \frac{2}{\pi (2n+1)} \exp\left\{-\frac{2|\alpha|^2}{(2n+1)}\right\}, \]

represents a classical state. I shall call entangled any state of two modes which is not classical but has classical marginals, that is

\[ \text{entangled : } n \geq 0, \frac{2n}{2n+1} < |x| < 1. \quad (24) \]
(I propose this as a sufficient condition, not excluding the possibility of entangled states with nonclassical marginals). We see that such states involve a correlation between the two modes which is larger than the correlation of any classical state (given by (23)). This is because not only the signal but also the ZPF or ”quantum vacuum fluctuations” are correlated in the entangled state.

"Two-photon entanglement" similar to this one (although involving many modes) occurs in the process of parametric down conversion. In fact, light produced in that way consists of two separated beams whose Wigner functions are gaussian but the full state is not classical. That is, every beam alone consists of chaotic light of the type (18), but the two beams are entangled.

The reader may realize that entanglement becomes less relevant when the intensity \( n \) (the mean number of photons) is large because the interval of \( x \) in (24) becomes narrow. The opposite is true if \( n \) is small. This is to be expected because large \( n \) corresponds to the classical limit.

**B. Entanglement and Bell’s inequalities**

There is no agreement about the definition of entanglement for mixed quantum states. A fashionable criterion is the violation of a Bell inequality. In actual experiments the inequalities tested have never been genuine Bell inequalities, derived using only general properties of local hidden variables, but inequalities involving auxiliary assumptions. The violation of one of such inequalities does not imply the refutation of local realism, a fact qualified as ”existence of loopholes”. Nevertheless we may consider the experiments as valid tests of entanglement. In practice any test involves measuring a coincidence detection rate as a function of some controllable angular parameter, \( \phi \). The inequalities are violated if the measured coincidence rate, \( R_{12} \), is of the form

\[
R_{12} = \text{const.} \times (1 + V \cos \phi),
\]

with the visibility or contrast, \( V \), greater than some limit, usually 0.71.

The most frequent tests involve polarization correlation. In order to study polarization we need to take \( \alpha \) and \( \beta \) as (complex) two-dimensional vectors or, what is equivalent, to double the number of modes. Introducing the polarization vectors, \( e_{k,\lambda} \), see (6), the vector amplitude \( \alpha \) might be written

\[
\alpha = \alpha_x e_{k,1} + \alpha_y e_{k,2}, \quad |\alpha|^2 = |\alpha_x|^2 + |\alpha_y|^2,
\]
and similar for $\beta$. The Wigner function will be, instead of (22),
\[ W_{12}(\alpha, \beta) = \frac{(1 - x^2)^2}{\pi^4} A^4 \exp\left\{ -A \left[ |\alpha|^2 + |\beta|^2 - x (\alpha \cdot \beta^* + \beta \cdot \alpha^*) \right] \right\}, \]
\[ A \equiv \frac{2}{(2n + 1) (1 - x^2)}. \] 

We shall consider that the amplitudes $\alpha$ and $\beta$ correspond to two polarized light beams arriving at two polarization analyzers at angles $\phi_1$ and $\phi_2$, respectively. In what follows we may ignore the ZPF in all modes except those included in (26). Consequently we assume that the amplitudes emerging from the polarizers are given by Malus law, that is
\[ \lambda = (\alpha \cdot u_1), \mu = (\beta \cdot u_2), \]
the vector $u_1$ having components $(\cos \phi_1, \sin \phi_1)$ and similar for $u_2$. Note that the scalar amplitudes $\lambda$ and $\mu$ correspond to modes with polarization in the directions of $u_1$ and $u_2$, respectively. As said above we ignore the modes with polarization perpendicular to $u_1$ or $u_2$, which contain just ZPF.

The coincidence detection rate in two detectors placed after the polarizers may be calculated by a straightforward generalization of (13), namely
\[ R_{12} \propto \int W_{12}(\alpha, \beta) \left( |\lambda|^2 - \frac{1}{2} \right) \left( |\mu|^2 - \frac{1}{2} \right) d^2x d^2y d^2\alpha d^2\beta. \]

The integration is trivial using eq. (26) and we get
\[ R_{12} \propto n^2 + \frac{1}{2} \left( n + \frac{1}{2} \right)^2 x^2 \left[ 1 + \cos (2\phi_1 - 2\phi_2) \right]. \]

The visibility is
\[ V = \frac{\frac{1}{2} \left( n + \frac{1}{2} \right)^2 x^2}{n^2 + \frac{1}{2} \left( n + \frac{1}{2} \right)^2 x^2}, \]
which is greater than $\frac{1}{3}$ for entangled states, fulfilling condition (24), but it is smaller than $\frac{1}{3}$ for classical states, where condition (23) holds true. In particular, the limit $V = 0.71$ may be surpassed if $n < 0.42$ (but this is specific for the gaussian Wigner functions studied in this paper.)

We see that high visibility is only possible with weak signals. I conjecture that the signal weakness, combined with the necessity of removing efficiently
the ZPF, gives rise to the difficulties for performing "loophole-free" tests of Bell’s inequalities. This problem will be studied elsewhere.

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