Second-order topological insulator in a coinless discrete-time quantum walk

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Higher-order topological insulators not only exhibit exotic bulk-boundary correspondence principle, but also have an important application in quantum computing. However, they have never been achieved in quantum walk. In this paper, we construct a two-dimensional coinless discrete-time quantum walk to simulate second-order topological insulator with zero-dimensional corner states. We show that both of the corner and edge states can be observed through the probability distribution of the walker after multi-step discrete-time quantum walks. Furthermore, we demonstrate the robustness of the topological corner states by introducing the static disorder. Finally, we propose a possible experimental implementation to realize this discrete-time quantum walk in a three-dimensional integrated photonic circuits. Our work offers a new route to explore exotic higher-order topological matters using discrete-time quantum walks.

I. INTRODUCTION

Quantum walk, which describes the propagation of quantum particles on a lattice [1–3], is a quantum version of classical random walk. Due to its simplicity and high controllability, quantum walk has become a powerful tool for universal quantum computing [4, 5] and quantum simulation [6–9]. Inspired by an original theoretical paper of Kitagawa [10], discrete-time quantum walk (DTQW) has become an outstanding platform for simulating various topological phenomena [11–19]. In particular, the topological edge states and winding numbers have been detected by both unitary [20–26] and non-unitary [27–31] one-dimensional DTQWs. For the two-dimensional case, the one-dimensional edge states have been observed without [32, 33] and with [34] the synthetic gauge field. Very recently, the Chern number has been successfully probed by an anomalous displacement [35].

Notice that the current research on topological features of DTQWs only focus on the simulation of the first-order topological insulator, which supports topological protected states in the (d−1)-dimensional boundary for a d-dimension system. Recently, higher-order topological insulators, which have lower-dimensional gapless boundary states, have attracted much attention in both theory [36–40] and experiment [41–46]. Physically, these higher-order topological insulators exhibit an exotic bulk-boundary correspondence principle that an nth-order topological insulator supports gapless (d−n) dimensional boundary states. Moreover, these gapless boundary states can support nontrivial fractional quasi-particles (such as parafermion or Ising anyon etc.), providing a new architecture for quantum information processing and quantum computing [47, 48]. Nevertheless, these interesting higher-order topological phases have never been achieved in quantum walks.

In this paper, we introduce a two-dimensional coinless DTQW to simulate a second-order topological insulator, which hosts the zero-dimensional corner states and one-dimensional edge states. We show that both of the corner and edge states can be observed through the probability distribution of the walker after multi-step DTQWs. Furthermore, we demonstrate the robustness of the topological corner states by introducing the static disorder. Finally, we propose a possible experimental implementation to realize this discrete-time quantum walk in a three-dimensional integrated photonic circuits. Since the coupling and phase between each two lattice sites at each step of DTQWs can be adjusted independently, our proposal can be extended directly to realize other exotic higher-order topological insulators, such as the non-Hermitian [49–51] and Floquet higher-order topological insulators [52–56], which have not been observed in experiments. Our work offers a new route to explore exotic topological matters using DTQWs.

This paper is organized as follows. In Sec. II, we introduce a two-dimensional coinless DTQW. In Sec. III, we demonstrate the existence of the zero-dimensional corner states and the one-dimensional edge states through calculating the spectra and the topological invariant. In Sec. IV, we show the probability distributions of the walker after multiple-step DTQWs. We also verify the robustness of the corner states by introducing the static disorder. In Sec. V, we propose a possible experimental implementation in a three-dimensional integrated photonic circuits. Finally, we give the summarization in Sec. VI.
II. A TWO-DIMENSIONAL COINLESS DTQW

We begin to introduce a two-dimensional SSH model with $\pi$-flux per plaquette, which is governed by the Hamiltonian

$$H = \sum_{x,y} (t_x a_{x+1,y}^\dagger a_{x,y} + t_y e^{i\pi} a_{x,y+1}^\dagger a_{x,y}) + \text{H.c.} \quad (1)$$

where $a_{x,y}^\dagger$ ($a_{x,y}$) is the creation (annihilation) operator of a spinless particle at the site $(x, y)$, $t_x(y) = t + (-1)^x(y)\delta t$ are the two types of hopping amplitudes in the $x$ ($y$) direction, respectively, and can be defined as $J_1 = t - \delta t$, $J_2 = t + \delta t$, and H.c. is the Hermitian conjugate. This Hamiltonian can host the topological-protected corner states [36, 37]. In the following, we will construct a two-dimensional coinless DTQW to simulate the second-order topological insulator, based on the Hamiltonian (1).

We firstly divide it into four parts

$$H = H_{2y} + H_{1y} + H_{2x} + H_{1x}, \quad (2)$$

where $H_{1x}$ ($H_{2x}$) and $H_{1y}$ ($H_{2y}$) are the intracellular (intercellular) hoppings along the $x$ and $y$ directions, respectively. Then, we construct a one-step operator of a DTQW as

$$U_{\text{step}} = \sum_{x=0}^{M/2-1} V_{2x+1} \otimes I_y, \quad \text{Eq.} \, (3)$$

For simplicity, we use the units $\Delta T = h = 1$ hereafter. For the Hamiltonian (1), these four substep operators are chosen as

$$U_1 = \sum_{x=0}^{M/2-1} V_{2x+1} \otimes I_y, \quad \text{Eq.} \, (4)$$

$$U_2 = (|1\rangle_x \langle 1| + |M\rangle_x (M)) \otimes I_y + \sum_{x=1}^{M/2-1} V_{2x} \otimes I_y, \quad \text{Eq.} \, (5)$$

$$U_3 = \sum_{x=1}^{M} \sum_{y=0}^{M/2-1} |x\rangle_x \langle x| \otimes V_{2y+1}, \quad \text{Eq.} \, (6)$$

$$U_4 = I_x \otimes (|1\rangle_y \langle 1| + |M\rangle_y (M)) + \sum_{x=1}^{M} \sum_{y=1}^{M/2-1} |x\rangle_x \langle x| \otimes V_{2y}, \quad \text{Eq.} \, (7)$$

where the translation operators in the $x$ and $y$ directions are defined as

$$V_x = \cos \left(\frac{\pi r}{4}\right) |x\rangle_x \langle x| + |x+1\rangle_x \langle x+1| - \quad \text{Eq.} \, (8)$$

$$i \sin \left(\frac{\pi r}{4}\right) |x+1\rangle_x \langle x| + |x\rangle_x \langle x+1|,$$

$$V_y = \cos \left(\frac{\pi r}{4}\right) |y\rangle_y \langle y| + |y+1\rangle_y \langle y+1| - \quad \text{Eq.} \, (9)$$

$$i \sin \left(\frac{\pi r}{4}\right) [e^{i\pi} |y+1\rangle_y \langle y| + e^{i\pi} |y\rangle_y \langle y+1|].$$

This DTQW is implemented in the Hilbert space $|x\rangle \otimes |y\rangle$, with $x \in \{1, M\}$ and $y \in \{1, M\}$ $(M$ is even). The operator $I_{x(y)}$ denotes a $M \times M$ identity matrix in the subHilbert space $|x\rangle \langle y|$. It should be emphasized that in order to generate the $\pi$-flux per plaquette, here we have added key phase factors of the translation operator $V_y$. By applying the one-step operator in Eq. (3) many times, a multi-step DTQW can be realized, as shown schematically in Fig. 1, and the topologically protected corner states can be explored, as will be shown.

III. SPECTRA AND TOPOLOGICAL INVARIANT

In order to illustrate the topological features of this DTQW, here we discuss the spectra and the topological invariant. In Fig. 2(a), we plot the quasienergy spectra of the effective Hamiltonian, $H_{\text{eff}} = \ln U_{\text{step}}$, under the open boundary condition. This figure shows clearly that the gapless zero-energy and gapped nonzero-energy states can occur. Moreover, the gapless zero-energy states are four-degenerate and separated from the bulk states by a large energy gap, while these four-degenerate gapped nonzero-energy states are separated from the bulk states only with a tiny gap, as shown in Fig. 2(b). When we increase the parameter $J_1$, this tiny bandgap disappears quickly. In Fig. 2(c), we plot the collective distributions of these four-degenerate zero- and nonzero-energy states, which are indeed localized at the
The quasienergy spectra of the effective Hamiltonian, $H_{\text{eff}} = i \ln U_{\text{step}}$, as a function of the parameter $J_1$. (b) The quasienergy spectrum with $J_1 = 0.1$. There are four-degenerate zero- and nonzero-energy states, denoted respectively by the red and blue points. (c) Left: the collective distribution of the four-degenerate zero-energy corner-localized states. Right: the collective distribution of the four-degenerate nonzero-energy edge-localized states with the state numbers $\{203, 204, 205, 206\}$. Here the lattice size is chosen as $20 \times 20$ and the parameter $J_2 = 1$. The appearance of the zero-dimensional corner states can be attributed to the second-order bulk topology, which is described by introducing the Wannier bands and the nested Wilson loops [36, 37]. Generally speaking, the complete characterization of the second-order topology for a Floquet system gives a pair of $Z_2$ invariant, which can predict the appearance of zero- and $\pi$-corner states [52–56]. For our model, only one $Z_2$ invariant is enough with the absence of the $\pi$-corner states.

To construct the topological invariant, we consider the eigenstates of the one-step operator in momentum representation

$$U_{\text{step}}(k)|E_k\rangle = e^{-iE_k}|E_k\rangle,$$

where two gapped bands with the quasienergy $\pm E_k$ are doubly degenerate, respectively, and the eigenstates can be denoted as $|+ E_k^0\rangle$ and $|+ E_k^1\rangle$ ($|- E_k^0\rangle$ and $|- E_k^1\rangle$) for upper (lower) bands. When the lower two bands are filled, the Wilson loop operator in the $x$ direction is defined as

$$W_{x,k} = F_{x,k+(N_x-1)\Delta k_x e_x} \cdots F_{x,k+\Delta k_x e_x} F_{x,k},$$

where $F_{x,k}$ is a $2 \times 2$ matrix with element $[F_{x,k}]_{mn} = \langle - E_k^m | - E_k^n \rangle / (m, n = 1, 2)$, $e_x$ is the unit vector in the $x$ direction, and $\Delta k_x = 2\pi/N_x$. The 2D Brillouin zone is discretized by using the interval $(2\pi/N_x, 2\pi/N_y)$, such that there are $(N_x + 1)(N_y + 1)$ k-points in total. With the periodic boundary condition, $| - E_k^0 \rangle = | - E_k^0 + 2\pi \rangle$, we diagonalize Eq. (11) as

$$W_{x,k} |v^j_{x,k}\rangle = e^{i2\pi v^j_x(k_y)} |v^j_{x,k}\rangle,$$

where $j = \pm$ denotes two Wannier bands. These Wannier bands carry their own topological invariants, which can be evaluated by calculating the nested Wilson loops.

We firstly construct the Wannier states

$$|w^\pm_{x,k}\rangle = \sum_{n=1,2} [v^\pm_{x,k}]^n |v^\pm_{x,k}\rangle,$$

where $[v^\pm_{x,k}]^n$ denotes the $n$-th element of the 2-component spinor $|v^\pm_{x,k}\rangle$. Then, with the periodic boundary condition, $|w^\pm_{x,k}\rangle = |w^\pm_{x,k+2\pi e_y}\rangle$, the nested Wilson loops along $k_y$ in the Wannier bands $v^\pm_{x,k}$ are

$$W_{y,k} = F_{y,k+(N_y-1)\Delta k_y e_y} \cdots F_{y,k+\Delta k_y e_y} F_{y,k}^\pm,$$

where $F_{y,k} = \langle w^\pm_{x,k+\Delta k_y e_y} | w^\pm_{x,k}\rangle e_y$ is the unit vector in the $y$ direction, and $\Delta k_y = 2\pi/N_y$. Through Eq. (14), we obtain the nested polarization as

$$p^\pm_y = -\frac{i}{2\pi N_x} \sum_{k_x} Log|W_{y,k}^\pm|.$$

Similarly, we can also obtain the nested polarization $p^\pm_y$ from the nest Wilson loops in the $y$ direction. Finally,
the topological quadrupole phase is characterized by a $\mathbb{Z}_2$ invariant \[ \nu = 4p_y^x p_y^x. \] (16)

By choosing $N_x = N_y = 50$, we numerically calculate the topological invariant $\nu$ by using the above procedure. As shown in Fig. 3, we find that the topological invariant $\nu$ is quantized to be 0 or 1, which corresponds to the trivial or topological phases, respectively.

IV. OBSERVATION OF CORNER AND EDGE STATES

In this section, we mainly show that the corner and edge states can be observed experimentally through the probability distribution of the walker after a multi-step DTQW. It is well known that the probability distributions of multi-step DTQWs exhibit the ballistic behaviors \[ [1], \] which are entirely different from the diffusive behaviors of the classic version. Utilizing this feature, we can demonstrate the existence of the corner and edge states through the local behavior of the probability distribution of the walker.

In the first case, we tune the parameter $J_1 = 1.5$, which corresponds to a trivial phase. We initialize the walker at one corner of the lattice $(x,y) = (1,1)$ or one edge of the lattice with $(x,y) = (1,2)$. Since the system does not support any localized states, the probability of the walker spreads ballistically into the bulk as increasing the number of the DTQW steps; see Figs. 4(a) and 4(b). Then we tune the parameter to $J_1 = 0.1$, which corresponds to a quadrupole topological phase supporting the localized corner and edge states; see Fig. 2(c). In such case, when this initial state is prepared at one corner of the lattice $(x,y) = (1,1)$, since it has a large overlap with the corner state, the most part of the walker’s wave packet remains localized near $(x,y) = (1,1)$ as increasing the step of the DTQW; see Fig. 4(c). In Fig. 4(d), we initialize the walker at one edge of the lattice with $(x,y) = (1,2)$. Sim-
Similarly, this initial state has a large overlap with the edge states, and we can also observe a large nonvanishing localization at one edge of the lattice. Since the edge states have a vanishing distribution at the corners of the lattice [see Fig. 2(c)], the walker only localizes at one edge of the lattice.

The observable properties of the corner states are robust against small fluctuations with the second-order topological protection. Conversely, the observable properties of the edge states are not robust due to the tiny gap from the bulk states. To support this claim, we add the static disorder into the evolution processing. The one-step operator for the static disorder is introduced as

$$U_{\text{total}} = U_{\text{step}} \times U_{\text{dis}},$$

(17)

with

$$U_{\text{dis}} = \sum_{x,y} e^{i \delta_{x,y}} |x,y\rangle \langle x,y|.$$  (18)

Here $\delta_{x,y}$ is chosen randomly from the interval $[-W/2, W/2]$, where $W$ is the disorder strength.

Figure 5 shows the probabilities $P_c(N)$ and $P_e(N)$ of the walker remained respectively at the corner state or the edge state without and with the static disorder. When the walker is initialized at one corner of the lattice $(x,y) = (1,1)$, it has a stable large probability at this corner as increasing the step of the DTQW, and this corner probability $P_c(N)$ is robust against the static disorder; see Fig. 5(a). When the walker is initialized at one edge of the lattice $(x,y) = (1,2)$, it also has a large probability at this edge. In Fig. 5(b), we also show that the edge probability $P_e(N)$ decreases when the static disorder is added, demonstrating that the edge states are not robust. This decrease is more evident as we increase the parameter $J_1$. The results for other three corners or edges of the lattice are similar and thus not shown here.

V. POSSIBLE EXPERIMENTAL IMPLEMENTATION

Finally, we propose a possible scheme to realize this DTQW of Eq. (3) in a three-dimensional integrated photonic circuit [57], where a single photon acts as a walker and a single waveguide can indicate a two-dimensional lattice site in the $x$ and $y$ directions. The waveguides are extended in the $z$ direction, corresponding to the time dimension of the DTQW. The key to realize this DTQW in experiments is how to achieve the specific translation operators $V_x$ and $V_y$. (a) A single-layer waveguide structure for realizing a beam splitter (BS). (b) A double-layer waveguide structure for realizing a phase-shifted beam splitter (PBS).

FIG. 5: (a) The probability $P_c(N)$ at the corner $(x,y) = (1,1)$ when the walker is initialized at the same corner. (b) The probability $P_e(N)$ at the right edge of the lattice when the walker is initialized at $(x,y) = (1,2)$. The parameter $J_1 = 0.1$. The solid lines represent the case without the static disorder and the dashed lines represent the case $W = 2.5$ averaged over the 100 disordered realizations.

FIG. 6: Scheme of waveguide structure for realizing the translation operators $V_x$ and $V_y$. (a) A single-layer waveguide structure for realizing a beam splitter (BS). (b) A double-layer waveguide structure for realizing a phase-shifted beam splitter (PBS).
splitter matrix, respectively. Fortunately, we can realize these translation operators in an integrated photonic circuit with the directional coupler geometry [58], where two waveguides are brought close together for a certain interaction length and coupled by an evanescent field. In the following, we will show how to implement the translation operators $V_x$ and $V_y$ with the single- and double-layer waveguide structures, respectively.

The translation operator $V_x$ can be realized by a directional coupler; see Fig. 6(a). The standard coupled mode theory [59] gives a transfer matrix as

$$T_1(z) = \begin{pmatrix} \cos(Kz) & -i\sin(Kz) \\ -i\sin(Kz) & \cos(Kz) \end{pmatrix}, \quad (19)$$

which can be used to realize Eq. (8). According to Eq. (19), the parameter in $V_x$ can be adjusted through altering the coupling coefficient $K$ and the interaction length $z$.

Due to the current technology of full phase-shift controllability between two waveguides [60, 61], we can introduce an arbitrary phase in the first (or second) row of $T_1(z)$. However, the phases required in $V_y$ are at the off-diagonal elements of the matrix, which indicates that we can not realize the translation operator $V_y$ directly by a single directional coupler. Thus, we design a double-layer waveguide structure to overcome this limitation. As shown in Fig. 6(b), if a single photon pulse is input from the port labeled by $|y\rangle^n$ (or $|y+1\rangle^n$), it will go through the upper (or lower) layer waveguide structure and obtain a phase $\Phi$ (or $-\Phi$). According to the coupled mode theory, the total transfer matrix governed by this double-layer waveguide structure is

$$T_2(z) = \begin{pmatrix} \cos(Kz) & -i\sin(Kz)e^{i\Phi} \\ -i\sin(Kz)e^{-i\Phi} & \cos(Kz) \end{pmatrix}, \quad (20)$$

which is exactly the phased-shifted beam-splitter matrix in Eq. (9). Thus, the experimental implementation of $U_{\text{step}}$ is possible with the current technology of the three-dimensional waveguide architecture [62–65]. The phase $\Phi$ can be chosen arbitrarily and is here taken as $\Phi = m\pi$, where $m$ is an integer. When $m$ is even, the transfer matrix $T_2(z)$ reduces to $T_1(z)$. That is, a single-layer waveguide structure is enough for this case.

\section{VI. CONCLUSIONS}

In summary, we have constructed a two-dimensional coinless DTQW to simulate the second-order topological insulator. We have shown that both of the corner and edge states can be observed through the probability distribution independently. Furthermore, we have demonstrated the robustness of the topological corner states by introducing the static disorder. Finally, we have proposed a possible experimental implementation in a three-dimensional integrated photonic circuits. Since the coupling and phase between each two lattice sites at each step of DTQWs can be adjusted independently, our scheme can be generalized directly to realize the non-Hermitian [49–51] and Floquet higher-order topological insulators [52–56]. Our work offers a new route to explore exotic higher-order topological matters using DTQWs.

\section{VII. ACKNOWLEDGMENTS}

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