Abstract
One of the fundamental laws of classical statistical physics is the energy equipartition theorem which states that for each degree of freedom the mean kinetic energy $E_k$ equals $E_k = \frac{k_B T}{2}$, where $k_B$ is the Boltzmann constant and $T$ is the temperature of the system. Despite the fact that quantum mechanics has already been developed for more than 100 years, still there is no quantum counterpart of this theorem. We attempt to fill this far-reaching gap and consider the simplest system, i.e. the Caldeira–Leggett model for a free quantum Brownian particle in contact with a thermostat consisting of an infinite number of harmonic oscillators. We prove that the mean kinetic energy $E_k$ of the Brownian particle equals the mean kinetic energy $\langle E_k \rangle$ per one degree of freedom of the thermostat oscillators, i.e. $E_k = \langle E_k \rangle$. We show that this relation can be obtained from the fluctuation–dissipation theorem derived within the linear response theory and is universal in the sense that it holds true for any linear and non-linear systems in contact with a bosonic thermostat.

Keywords: quantum Brownian motion, energy equipartition, fluctuation–dissipation theorem

(Some figures may appear in colour only in the online journal)
applications of this theory in the form of the explanation of the Dulong–Petit law for the specific heat capacities of solids. Since that time the EET has become one of the most important and most useful relations exploited in various branches of natural science, including physics, chemistry and even biology [3]. For classical systems EET represents a universal relation in the sense that it does not depend on a number of particles in the system, a potential force which acts on them, any interaction between particles or the strength of coupling between the system and thermostat [4]. In contrast, this is no longer true for quantum systems. The reason is that in quantum physics observables represented by operators in general do not commute. In consequence, in quantum statistical physics, equilibrium states are characterized by the Kubo–Martin–Schwinger relation [5] which expresses a periodicity of Green’s functions in imaginary time. We can cite an example from the classic book by Feynman [6], i.e. a harmonic oscillator of mass \( m \) and eigenfrequency \( \omega_0 \) for which the Hamiltonian has the well-known form \( H = \frac{p^2}{2m} + m\omega_0^2 x^2/2 \). In this case the average kinetic energy \( E_k \) reads [6]

\[
E_k(\omega_0) = \frac{1}{2m} \langle p^2 \rangle = \frac{\hbar \omega_0}{4} \coth \frac{\hbar \omega_0}{2k_B T}
\]

and evidently depends on the potential form via the frequency \( \omega_0 \). The limit \( \omega_0 \to 0 \) corresponds to the Hamiltonian of a free particle for which one gets the classical results \( E_k(\omega_0 = 0) = k_B T/2 \). Equation (1) is derived for a system in a Gibbs canonical state and in consequence is valid only in the weak coupling limit between the oscillator and the thermostat. If it does not hold the problem is much more complicated. Moreover, in many cases the EET has been exploited far away from its domain of applicability and range of its validity, e.g. from this relation it follows that at zero temperature \( T = 0 \) the kinetic energy is zero. It is obviously not true because even at \( T = 0 \) there are quantum vacuum fluctuations of the thermostat.

From the time of Max Planck and birth of quantum physics a quantum counterpart of EET has not been explicitly formulated. In literature, both in original papers and text-books, there are various expressions for mean energy of some particular systems, see e.g. [7–13]. Here, we keep things maximally simple to present our genuine result in a transparent way. Therefore we study the most basic model of a quantum open system in the form of a one-dimensional quantum Brownian motion. Hereafter, we consider a particle of mass \( M \) subjected to the potential \( U(x) \) and interacting with a large number of independent oscillators which form the thermostat (environment) of temperature \( T \) being in an equilibrium Gibbs canonical state. The Hamiltonian of such a total system can be written as [9, 12–17] (a complete list of papers concerning this problem is too long and inevitably our choice is subjective.)

\[
H = \frac{p^2}{2M} + U(x) + \sum_i \left[ \frac{p_i^2}{2m_i} + \frac{m_i \omega_i^2}{2} \left( q_i - \frac{c_i X}{m_i \omega_i^2} \right)^2 \right],
\]

where the coordinate and momentum operators \( \{ x, p \} \) refer to the Brownian particle and \( \{ q_i, p_i \} \) are the coordinate and momentum operators of the \( i \)th heat bath oscillator of mass \( m_i \) and the eigenfrequency \( \omega_i \). The parameter \( c_i \) characterizes the interaction strength of the particle with the \( i \)th oscillator. All coordinate and momentum operators obey canonical equal-time commutation relations. From the Heisenberg equations of motion for all coordinate and momentum operators one can obtain an effective equation of motion for the particle coordinate operator \( x(t) \). It is called a generalized Langevin equation (GLE). Again, to make our work maximally transparent we now consider an elementary example, namely, a free Brownian particle for which \( U(x) = 0 \) and the GLE reads
where $\gamma(t)$ is a dissipation function (damping or memory kernel) and $\eta(t)$ denotes the random force,

$$\gamma(t) = \int_0^\infty d\omega J(\omega) \cos(\omega t),$$

$$\eta(t) = \sum_i c_i \left[ q_i(0) \cos(\omega_i t) + \frac{p_i(0)}{m_i \omega_i} \sin(\omega_i t) \right].$$

$J(\omega)$ is a spectral density of thermostat modes which contains all information on the system-thermostat coupling:

$$J(\omega) = \sum_i \frac{c_i^2}{m_i \omega_i^2} \delta(\omega - \omega_i).$$

In the standard approach it is assumed that the initial state $\rho(0)$ of the total system is uncorrelated, i.e. $\rho(0) = \rho_S(0) \otimes \rho_T(0)$, where $\rho_S$ is an arbitrary state of the Brownian particle and $\rho_T$ is an equilibrium Gibbs canonical state of thermostat of temperature $T$. Next, the thermodynamic limit is imposed meaning that the thermostat is infinitely extended and the singular dissipation relation (7), take $t \to \infty$ and the dissipation function $\gamma(t)$ is related to the memory kernel $\gamma(t)$ via the famous fluctuation–dissipation theorem [18, 19]

$$\hat{C}_\gamma(\omega) = \frac{\hbar \omega}{2} \coth \left( \frac{\hbar \omega}{2k_B T} \right) \hat{\gamma}_\eta(\omega) = 2 \mathcal{E}_k(\omega) \hat{\gamma}_\eta(\omega),$$

where $\hat{C}_\gamma(\omega)$ and $\hat{\gamma}_\eta(\omega)$ are the Fourier cosine transforms of the correlation function $C(t)$ and the dissipation function $\gamma(t)$, respectively ($\hat{\gamma}_\eta(\omega) = (2/\pi) \int_0^\infty dt g(t) \cos(\omega t)$ for any even function $g(t)$).

To calculate the average kinetic energy $E_k = \lim_{t \to \infty} \langle p^2(t) \rangle / 2M$ of the Brownian particle we solve the GLE (3) by the Laplace transform method and find the momentum

$$p(t) = \mathbf{R}(t) p(0) - \int_0^t du \mathbf{R}(t-u) \gamma(u)x(0) + \int_0^t du \mathbf{R}(t-u) \eta(u),$$

where the response function $\mathbf{R}(t)$ is determined by its Laplace transform

$$\tilde{\mathbf{R}}_L(z) = \frac{M}{Mz + \gamma_L(z)}, \quad \gamma_L(z) = \int_0^\infty dt \ e^{-z \gamma(t)}$$

and $\gamma_L(z)$ is the Laplace transform of the dissipation function $\gamma(t)$. By use of (8) we construct the symmetrized momentum–momentum correlation function $(1/2) \langle p(t) p(s) + p(s) p(t) \rangle$, utilize the fluctuation–dissipation relation (7), take $t = s$ and perform the limit $t \to \infty$, we finally obtain the expression for the average kinetic energy in the form

$$E_k = \langle \mathcal{E}_k \rangle = \int_0^\infty d\omega \ \mathcal{E}_k(\omega) \mathbb{P}(\omega),$$

$$M\ddot{x}(t) + \int_0^t du \ \gamma(t-u)\dot{x}(u) = -\gamma(t)x(0) + \eta(t),$$

$\mathcal{E}_k(\omega)$ is a dissipation function (damping or memory kernel) and $\mathbb{P}(\omega)$ denotes the random force.
where $\mathcal{E}_k(\omega)$ is given by equation (1) and
\[
P(\omega) = \frac{1}{\pi} \left[ \tilde{R}_L(\omega) + \tilde{R}_L(-i\omega) \right].
\] (11)

The formula (10) together with equation (11) constitutes a quantum analogue for partition of kinetic energy which we formulate in the following way: the mean kinetic energy $E_k$ of the Brownian particle is equal to mean kinetic energy per one degree of freedom of the thermostat free oscillators. The averaging is twofold: (i) over the Gibbs canonical state $\rho_T$ for the thermostat free (non-interacting with the Brownian particle) oscillators resulting in $\mathcal{E}_k(\omega)$ given by equation (1) and (ii) over frequencies $\omega$ of those thermostat oscillators which contribute to $E_k$ according to the probability distribution $P(\omega)$. By equations (11), (9) and (4), the function $P(\omega)$ depends on the spectral density $J(\omega)$ and in consequence the thermostat oscillators of various frequencies contribute in a greater or lesser degree to the kinetic energy $E_k$ of the Brownian particle. We discuss this aspect in the latter part of the paper.

**Theorem.** The function $P(\omega)$ defined by equation (11) is a probability measure on a positive half-line of real numbers meaning that
(A) $P(\omega) \geq 0$,
(B) $\int_0^\infty d\omega \ P(\omega) = 1$.

**Proof.**

(A) First we prove the non-negativity of $P(\omega)$. Using the definition (9) the function $P(\omega)$ can be rewritten in the form
\[
P(\omega) = \frac{2M}{\pi} \frac{A(\omega)}{A^2(\omega) + [B(\omega) - M\omega]^2}.
\] (12)

We applied the relation $\hat{\gamma}_L(i\omega) = A(\omega) - iB(\omega)$ with

\[
A(\omega) = \int_0^\infty dt \ \gamma(t) \cos(\omega t),
\] (13)

\[
B(\omega) = \int_0^\infty dt \ \gamma(t) \sin(\omega t).
\] (14)

The denominator in (12) is always positive and it is sufficient to show that the numerator $A(\omega) \geq 0$. From equation (4) we infer that $A(\omega) = (\pi/2)J(\omega)$. From equation (6) it follows that $J(\omega) \geq 0$ and the same holds true in the thermodynamic limit when $J(\omega)$ becomes a (piecewise) continuous function. Therefore $P(\omega) \geq 0$.

(B) Now we prove the normalization condition. The function $P(\omega)$ defined by equation (11) is an even function and from (11) one can obtain its equivalent form
\[
P(\omega) = \frac{2}{\pi} \int_0^\infty dt \ R(t) \cos(\omega t) \equiv \hat{R}_F(\omega)
\] (15)

which is a Fourier cosine transform of the response function $R(t)$! In turn, its inverse Fourier transform reads
\[
R(t) = \int_0^\infty d\omega \ \hat{R}_F(\omega) \cos(\omega t).
\] (16)
From the theory of Laplace transform it follows that \( \lim_{z \to \infty} \hat{f}(z) = 0 \) for any function \( f(t) \) for which the Laplace transform exists. In particular, it is also true for the function \( f(t) = \frac{\text{d}R(t)}{\text{d}t} \). Calculating its Laplace transform, we obtain the relation

\[
R(0) = \lim_{z \to \infty} z \hat{R}_L(z) = 1. \tag{17}
\]

On the other hand from (16) we get

\[
R(0) = \int_0^\infty \text{d} \omega \hat{R}_F(\omega) = \int_0^\infty \text{d} \omega P(\omega) = 1. \tag{18}
\]

So, we proved that there exists a random variable \( \xi \) for which \( P \) is its probability distribution. This random variable is interpreted as a frequency of the thermostat oscillators and equation (10) is an average value of the function \( E_k(\xi) \) of the random variable \( \xi \) (physicists frequently equate it with the integration variable). The probability distribution (11) seems to be surprisingly simple in its form as expressed by the Laplace transform (9) of the response function. The form (15) of \( P(\omega) \) looks even more simpler: it is a Fourier cosine transform of the response function \( R(t) \) which solves the GLE in equation (3).

In the following, we consider two examples of the memory kernel \( \gamma(t) \), namely the algebraically decaying dissipation function

\[
\gamma(t) = \frac{\gamma_0}{t + \tau_c}, \tag{19}
\]

and the exponentially decaying oscillations [21]

\[
\gamma(t) = \frac{\gamma_0}{\tau_c} e^{-t/\tau_c} \cos(\Omega t), \tag{20}
\]

where the parameter \( \gamma_0 \) is the strength of the system-thermostat coupling and \( \tau_c \) defines decay or relaxation time which characterizes memory effects. The case of \( \Omega = 0 \) in equation (20) corresponds to the so-called Drude model of quantum dissipation. In figure 1(a) we present the kinetic energy \( E_k \) of the free quantum Brownian particle versus thermostat temperature \( T \) for the algebraically decaying damping kernel (19). A deeper analysis of \( E_k \) is performed in [21] for the case (20) (although in a different context, see also figure 3(b) of [9] for a harmonic oscillator with the Drude model of dissipation). It supports the very well-known result that kinetic energy of the quantum Brownian particle is always greater than the classical one and at zero temperature \( T = 0 \) the kinetic energy \( E_k > 0 \).

What is indeed crucially new is shown in panel (b) of figure 1 and in figure 2 for \( \gamma(t) \) given by equation (20). We can note that the thermostat oscillators of some frequencies, say \( \tilde{\omega} \), contribute to \( E_k \) with much greater probability then the others. In panel (b) of figure 1, the most probable frequency \( \tilde{\omega} \) depends on the mass \( M \) of the Brownian particle, the coupling strength \( \gamma_0 \) and the memory time \( \tau_c \). However, it does not depend on these three parameters separately but only on their specific combination \( \alpha = M/(\tau_c \gamma_0) \). For small values of the dimensionless parameter \( \alpha \) (i.e. for long memory time or/and strong coupling), mainly high-frequency-oscillators contribute to \( E_k \). In turn, for large values of \( \alpha \) (i.e. for short memory time or/and weak coupling), mainly low-frequency-oscillators contribute to \( E_k \). A radically different behaviour is visible for the exponentially decaying oscillations in the dissipation function given by equation (20). In this case, there are three characteristic frequencies \( \varepsilon = 1/\tau_c, \mu_0 = \gamma_0/M \) and \( \Omega \). In panel (a) of figure 2, the case of the Drude model \( \Omega = 0 \) is considered and the role of the memory time \( \tau_c \) is depicted. We can see that for the long memory
time the probability distribution \( P(\omega) \) is very narrow with high peak at some low frequency \( \omega \). When \( \tau_c \) decreases (\( \varepsilon \) increases) the half-width of \( P(\omega) \) increases. It means that oscillators of a larger frequency interval contribute significantly to the kinetic energy \( E_k \). Finally, for small memory time the distribution is almost flat indicating that oscillators of almost all frequencies bring nearly equal contribution to \( E_k \).

The qualitatively new features emerge for the oscillatory dissipation function with \( \Omega > 0 \). For small \( \Omega \) the probability distribution \( P(\omega) \) is unimodal and has one maximum visible for higher frequencies \( \omega \). When \( \Omega \) increases a new extremum is born at zero frequency \( \omega = 0 \). This means that modes of low frequency starts to significantly contribute to the kinetic energy \( E_k \). For intermediate values of \( \Omega \) the probability distribution \( P(\omega) \) exhibits a clear bimodal character. Then both modes of low and high frequency play a crucial but equal role. Further increase of \( \Omega \) extinguishes the contribution of higher frequencies at the favour of the near zero frequency modes which are then the most pronounced ones.

Now, let us make a comment about the role of statistical moments \( \langle \xi^n \rangle = \int_0^\infty d\omega \omega^n P(\omega) \) of the random variable \( \xi \) distributed according to the probability density \( P(\omega) \). A caution is needed since not all moments may exist. At least the first two moments have to be finite because they have a clear physical interpretation. The first moment (i.e. the mean value of the distribution) \( \langle \xi \rangle \) is proportional to the kinetic energy \( E_k \) at zero temperature \( T = 0 \), namely,

\[
E_k(T = 0) = \frac{\hbar}{4} \langle \xi \rangle.
\]  

The second moment \( \langle \xi^2 \rangle \) is proportional to the first correction of \( E_k \) in the high temperature regime,

\[
E_k = \frac{1}{2} k_B T + \frac{\hbar^2}{24 k_B T} \langle \xi^2 \rangle.
\]
Discussion

In this part we want to make the following statements and remarks:

• We have derived the same expressions (10) and (11) by the same method for a quantum harmonic oscillator. In this case, one has to replace equation (9) by the response function of the harmonic oscillator, namely,

$$\hat{R}_L(z) = \frac{Mz}{Mz^2 + z\gamma_L(z) + M\omega_0^2}.$$  \hspace{1cm} (23)

• Equation (10) can be generalized for any quantum systems for which:

  – the relation $p(t) \propto \int_0^t \! du \, R(t-u) \eta(u)$ is satisfied,
  – the fluctuation–dissipation relation $C_F(\omega) = \mathcal{E}(\omega) \times \tilde{\gamma}_F(\omega)$ is satisfied with some function $\mathcal{E}(\omega)$.

• From the above it follows that (10) holds true within the linear response theory and can be obtained from the fluctuation–dissipation relation of the Callen–Welton type [12, 13, 18–20]. We recall that in this theory the quantum system is characterized by the Hamiltonian $\hat{H}$ and is in a thermal equilibrium state at temperature $T$ defined by the Gibbs canonical statistical operator $\rho \propto \exp[-\hat{H}/k_B T]$. Next, the external force $F(t)$ is applied to the system which develops in time under the perturbed time-dependent Hamiltonian $\hat{H} = \hat{H} - F(t)A$, where $A$ is a Hermitian operator. In the linear response approximation one can calculate fluctuations of the operator $B$. For a special choice of $A$ and $B$ one can get equation (10). We quote two examples:

1. In [8], the authors have obtained equation (4.14):

$$E_k = \frac{\hbar}{2\pi} \int_0^\infty \! d\omega \, \coth \left[ \frac{\hbar\omega}{2k_B T} \right] M\omega^2 \operatorname{Im}[\alpha(\omega + i0^+)].$$  \hspace{1cm} (24)

By comparing this equation with our formulas we see that $P(\omega) = (2/\pi) M\omega \operatorname{Im}[\alpha(\omega + i0^+)]$, where $\alpha(\omega)$ is called susceptibility.

Figure 2. Exponentially decaying oscillation of the dissipation function $\gamma(t) = (\gamma_0/\tau_c) e^{-t/\tau_c} \cos(\Omega t)$ with the oscillation frequency $\Omega$. Panel (a): the probability distribution $P(\omega)$ is presented for Drude model $\Omega = 0$ and different values of the memory time $\varepsilon = 1/\tau_c$. Panel (b): the impact of the oscillation frequency $\Omega$ on the probability density function $P(\omega)$ with $\varepsilon = 0.1$. 

J. Phys. A: Math. Theor. 52 (2019) 15LT01
2. The second example is equation (124.10) in the Landau–Lifshitz book [20] which for the momentum operator takes the form:

\[ \langle p^2 \rangle = \frac{\hbar}{\pi} \int_0^\infty d\omega \ \text{coth} \left( \frac{\hbar \omega}{2k_B T} \right) \chi''(\omega), \]  

(25)

where \( \chi''(\omega) \) is the imaginary part of the generalized susceptibility \( \chi(\omega) = \chi'(\omega) + i \chi''(\omega) \). If one knows the relation (10) together with equation (1) and compare equation (25) with them then one obtains the relation

\[ P(\omega) = \frac{2}{\pi M \omega} \chi''(\omega). \]  

(26)

Also other formulas cited in literature can be re-formulated to the form (10).

• Applying this fluctuation–dissipation relation we see that equation (10) is valid for arbitrary systems in contact with a bosonic thermostat. It means that equation (10) holds true for any potential \( U(x) \) in the Hamiltonian (2). For this class of systems the quantum partition of kinetic energy (10) is universal and the probability distribution is of the form (26), where the susceptibility \( \chi(\omega) \) is the Fourier transform of the response function \( \chi(t) \) and in more complicated cases can be calculated by the method of e.g. the retarded thermodynamic Green functions [22]:

\[ \chi(\omega) = \int_{-\infty}^{\infty} dt \ e^{i\omega t} \chi(t), \quad \chi(t) = \frac{i}{\hbar} \theta(t) \langle [p(t), p(0)] \rangle, \]  

(27)

where \( \theta(t) \) is the Heaviside step function, \( p(t) = \exp(i\tilde{H}t/\hbar) p(0) \exp(-i\tilde{H}t/\hbar) \) and averaging is over the Gibbs canonical statistical operator \( \rho \propto \exp[-\tilde{H}/k_B T] \). If \( \tilde{H} = H \) with total \( H \) in (2) then all regimes, from weak to strong coupling with thermostat, can be analyzed. However, if \( H = p^2/2M + U(x) \) then only the weak coupling limit can be considered.

• The above mentioned formula (26) establishes the relation between the probability distribution \( P(\omega) \) and the generalized susceptibility \( \chi''(\omega) \). It means that properties of the quantum environment and its coupling to a given quantum system which are characterized by \( P(\omega) \) may be experimentally inferred from the measurement of the linear response of the system to an applied perturbation, for instance electrical or magnetic. Consequently, the latter quantity may open a new pathway to study quantum open systems.

• Let us observe that equation (10) can be interpreted in the framework of superstatistics [23]. Indeed, the quantity \( E_k(\omega) \) is additionally averaged over the random variable \( \xi \), i.e. over randomly distributed frequencies of the thermostat oscillators according to the probability density \( P(\omega) \).

In conclusion, we disclose a new face of the old and well-known relations of quantum statistical physics. We propose the quantum analogue for partition of kinetic energy in the case of two exactly solvable systems, namely, a free quantum particle and a quantum harmonic oscillator. There are three new elements: the probabilistic form of equation (10), its interpretation as mean kinetic energy of the thermostat degree of freedom and information embodied in \( P(\omega) \). The proposed re-interpretation of the relation (10) has a transparent and intuitive meaning as in the classical case: the mean kinetic energy of a Brownian particle equals the mean kinetic energy of thermostat per one degree of freedom. The relation (11) is valid for arbitrary values of the system-thermostat coupling,
from weak coupling to strong coupling regimes. A particularly simple is the form of the probability density function $P(\omega)$ in the representation (15), which is the Fourier cosine transform of the response function $R(t)$. It is a challenge to extend our approach to other quantum systems and to other (spin, fermionic, ...) environments to show that mutatis mutandis the counterpart of (10) is universal and holds true for all quantum systems. The only non-trivial problem is to determine $P$. We hope that the present work stimulates such further theoretical analysis.

Acknowledgments

This work is supported by the Grants NCN 2015/19/B/ST2/02856 (PB and JL) and NCN 2017/26/D/ST2/00543 as well as the Foundation for Polish Science (FNP) Start fellowship (JS).

ORCID iDs

J Spiechowicz @ https://orcid.org/0000-0001-7569-4826
J Łuczka @ https://orcid.org/0000-0003-2296-4532

References

[1] Waterston J J 1846 Proc. R. Soc. Lond. 5 604
[2] Boltzmann L 1876 Wiener Ber. 74 553
[3] Eastwood M P, Stafford K A, Lippert R A, Jensen M, Maragakis P, Predescu C, Dror R O and Shaw D O 2010 J. Chem. Theory Comput. 6 2045
Monzel C and Sengupta K 2016 J. Phys. D: Appl. Phys. 49 243002
Hobbie R K and Roth B J 2007 Intermediate Physics for Medicine and Biology (New York: Springer)
[4] Huang K 1987 Statistical Mechanics (New York: Wiley)
[5] Kubo R 1957 J. Phys. Soc. Japan 12 570
Martin P C and Schwinger J 1959 Phys. Rev. 115 1342
[6] Feynman R P 1972 Statistical Mechanics (Reading, MA: Westview Press)
[7] Hakim V and Ambegaokar V 1985 Phys. Rev. A 32 423
[8] Ford G W, Lewis J T and O’Connell R F 1988 Ann. Phys. 185 270
[9] Grabert H, Schramm P and Ingold G L 1988 Phys. Rep. 168 115
[10] Boyanovsky D and Jasnow D 2017 Phys. Rev. A 96 062108
[11] Lampo A, Charalambous C, García-March M A and Lewenstein M 2018 Quantum 1 30
[12] Breuer H P and Petruccione F 2002 The Theory of Open Quantum Systems (New York: Oxford University Press)
[13] Weiss U 2008 Quantum Dissipative Systems (Singapore: World Scientific)
[14] Magalinskij V B 1959 J. Exp. Theor. Phys. 36 1942
[15] Nieuwenhuizen Th M and Allahverdyan A E 2002 Phys. Rev. E 66 036102
[16] Łuczka J 2005 Chaos 15 026107
[17] Hanggi P and Ingold G L 2005 Chaos 15 026105
[18] Callen H B and Welton T A 1951 Phys. Rev. 83 34
[19] Kubo R 1966 Rep. Prog. Phys. 29 255
[20] Landau L D and Lifshitz E M 1980 Statistical Physics, Part I (Oxford: Pergamon)
[21] Bialas P and Łuczka J 2018 Entropy 20 123
[22] Zubarev D N 1974 Nonequilibrium Statistical Thermodynamics (New York: Consultants Bureau)
[23] Beck C and Cohen E 2003 Physica A 322 267