A NEW METHOD TO OBTAIN RISK NEUTRAL PROBABILITY, WITHOUT STOCHASTIC CALCULUS AND PRICE MODELING, CONFIRMS THE UNIVERSAL VALIDITY OF BLACK-SCHOLES-MERTON FORMULA AND VOLATILITY’S ROLE

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Summary

A new method is proposed to obtain the risk neutral probability of share prices without stochastic calculus and price modeling, via an embedding of the price return modeling problem in Le Cam’s statistical experiments framework. Strategies-probabilities $P_{t_0,n}$ and $P_{T,n}$ are thus determined and used, respectively, for the trader selling the share’s European call option at time $t_0$ and for the buyer who may exercise it in the future, at $T$; $n$ increases with the number of share’s transactions in $[t_0, T]$. When the transaction times are dense in $[t_0, T]$ it is shown, with mild conditions, that under each of these probabilities $\log \frac{S_T}{S_{t_0}}$ has infinitely divisible distribution and in particular normal distribution for “calm” share; $S_t$ is the share’s price at time $t$. The price of the share’s call is the limit of the expected values of the call’s payoff under the translated $P_{t_0,n}$. A formula for the price is obtained. It coincides for the special case of “calm” share prices with the Black-Scholes-Merton formula with variance not necessarily proportional to $(T - t_0)$, thus confirming formula’s universal validity without model assumptions. Additional results clarify volatility’s role in the transaction and the behaviors of the trader and the buyer. Traders may use the pricing formulae after estimation of the unknown parameters.

Key words and phrases: Calm stock, European option, infinitely divisible distribution, risk neutral probability, statistical experiment, stock price-density, volatility

Running Head: Obtaining risk neutral probability and applications
1 Introduction

In this work, Le Cam’s theory of statistical experiments is used to obtain in two steps the risk neutral probability $P^*$ of a share’s price and to price the share’s European call option without stochastic calculus and price modeling. Among other results, the universal validity of the Black-Scholes-Merton (B-S-M) formula (Black and Scholes (1973), and Merton (1973)) is confirmed for “calm” stock prices, a pricing formula is obtained for non-calm stock and new insight is provided for volatility’s role and for the behaviors of the trader selling the call and of the buyer.

The buyer, at time $t_0$, of a share’s European call has at time $T(> t_0)$ the option to buy the share at predetermined price $X$. The “fair” B-S-M price $C$ is obtained when the share’s prices $\{S_t, t > 0\}$, defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$, follow a Geometric Brownian motion. However, $C$ is used extensively by traders for various price models because of its simplicity, adjusting $C$ for the demand that reflects market’s expectations.

B-S-M price $C$ is obtained by “replicating the call,” i.e. by creating a portfolio that matches the call’s payoff at $T$. This procedure guarantees $C$ does not allow arbitrage, i.e. that the call option’s buyer cannot make profit with probability 1. Alternatively, $C$ is obtained by discounting at $t_0$ the expected value of the call’s payoff, $(S_T - X)I(S_T > X)$, under the risk neutral probability $P^*$ that is equivalent to physical probability $\mathcal{P}$, for which the discounted shares’prices $\{\tilde{S}_t = e^{-rt}S_t, t > 0\}$ form a martingale; $I$ denotes the indicator function, $r = \ln(1 + i), i$ is the fixed interest (see, for example, Musiela and Rutkowski, 1997). Stochastic calculus is used with both approaches to obtain $C$ via $P^*$ but in practice $\mathcal{P}$ and $S_t$’s law under $\mathcal{P}$ are not known and $P^*$ is not always easily obtained.

Cox, Ross and Rubinstein (1979) obtained for the binomial price model B-S-M price as limiting price when the model’s parameters are properly chosen. This result, the form of $C$ which indicates the existence of contiguous sequences of probabilities and its extended use all suggest that the “fair” option’s price may be obtained for various price models as limit of expected values under a sequence of probabilities. The conjecture is proved herein without stock price modeling assumptions.

In section 2 a new 2-step method is proposed to determine $P^*$ without model assumptions for the share price via an embedding in Le Cam’s statistical experiments framework. The tools, presented in section 3, include mean-adjusted prices $S_t/ES_t, t \geq t_0$, which are densities in $(\Omega, \mathcal{F}, \mathcal{P})$ but have also mean 1 independent of $ES_t$. These are used to define two strategies-probabilities $P_{t_0,n}$ and $P_{T,n}$, respectively, for the trader who sells the call at $t_0$. 
and for the buyer, through the \( k_n \) traded stock prices at times \( t_0, t_1, \ldots, t_{k_n-1} \) in the interval \([t_0, T]\), \( T = t_{k_n} \). These probabilities are used to derive the distribution of \( \log \frac{S_T}{S_0} \) when \( k_n \) increases to infinity (with \( n \)) and price the call using \( P_{t_0,n} \) properly translated.

In modeling section 4 it is shown, with mild conditions, that when the non-random transaction times are dense in \([t_0, T]\) the distributions of \( \log \frac{S_T}{S_0} \) under \( P_{t_0,n} \) and \( P_{T,n} \) are infinitely divisible (Theorem 4.2). The prices-densities \( \{S_t/ES_t, \ t = t_0^n, t_1^n, \ldots, t_k^n\} \) form a martingale under \( P_{t_0,n} \) (Theorem 4.3); \( t_0^n = t_0, t_k^n = T \) for each \( n \). For calm stock prices, with prices-densities not changing much often, both distributions of \( \log \frac{S_T}{S_0} \) are normal (Theorem 4.4) and the role of volatility in the transaction is confirmed since the statistical experiment \( \mathcal{E}_n = \{P_{t_0,n}, P_{T,n}\} \) converges to the Gaussian experiment \( \mathcal{G} = \{\tilde{P} = N(0, 1), \tilde{Q} = N(\sigma_{[t_0,T]}, 1)\} \) (Corollary 4.3); \( \sigma_{[t_0,T]} \) is the standard deviation associated with the distribution of \( \log \frac{S_T}{S_0} \). Under \( \tilde{P} \), \( \log \frac{ES_T}{ES_0} \) follows a normal distribution with mean \(-0.5\sigma_{[t_0,T]}^2\) and variance \( \sigma_{[t_0,T]}^2 \). Conditions are also provided under which \( \sigma_{[t_0,T]}^2 = \sigma^2(T - t_0), \ \sigma > 0 \) (Proposition 6.3).

In section 5, prices of the call are obtained for the trader and the buyer. These prices are limits of discounted at \( t_0 \) expected values of the call’s payoff under \( P_{t_0,n} \) and \( P_{T,n} \), respectively, as \( n \) increases to infinity. Without loss of generality and to reduce the difficulty in the presentation \( S_T \) is discounted with \( ES_T/ES_0 \) and \( X \) with \( e^{-r(T-t_0)} \) but the discounting factors coincide when the limit distribution becomes risk neutral using translation; recall that under \( P^r \),

\[ e^{-r(T-t_0)} = s_{t_0}/EP\cdot S_T \]

(Corollaries 5.1, 5.3). Using a mapping to an equivalent risk-neutral structure, obtained by translation or change of probability \( P_{t_0,n} \) conditional on the value of the compound Poisson component of \( \log \frac{S_T}{S_0} \), the additive term \( \log \frac{ES_T}{ES_0} \) that appeared due to \( S_T \)'s discounting is replaced by \( r(T - t_0) \). For calm stock this mapping leads to the \( B-S-M \) formula with \( \sigma_{[t_0,T]}^2 \) not necessarily proportional to \( (T - t_0) \) thus proving its universal validity (Corollaries 5.2, 5.4) and justifying its frequent use for various price models that has puzzled Musiela and Rutkowski (1997, p. 111. l. -7 to l. -1). For non-calm stock, the obtained price has the same constants as the \( B-S-M \) price but the probabilities (in \( B-S-M \)) are now integrands with respect to the probability of the Poissonian component in the distribution of \( \log \frac{S_T}{S_0} \). It is reminiscent of the price in Merton (1976, p. 127) obtained when the stock’s price consists of (i) the “normal” vibration, modeled by a standard geometric Brownian motion, and (ii) the “abnormal” vibration, due for example to important new information about the stock, that is modeled by a jump process “Poisson driven”. The trader can use the obtained prices by estimating the unknown
parameters.

Additional results clarify and confirm quantitatively for calm stock that: 
a) the price the buyer is expected to pay for the option includes indeed a volatility premium (Corollaries 5.2 (ii), 5.4), and 
b) the probability $S_T$ is greater than the strike price $X$ is larger for the buyer than for the trader (Theorems 5.2 (i), (ii), 5.4 (i), (ii)).

Similar results hold for non-calm stock.

The results in this work hold also for random interest rate; see Remarks 5.1, 5.2.

Fama’s weak Efficient Market Hypothesis implies either independence or slight dependence of the share price returns (Fama, 1965, p.90, 1970, pp. 386, 414). Modeling “slight” dependence with weak dependence is acceptable in Finance (Duffie, 2010, personal communication). Thus, limiting laws obtained under independence of the price returns (used in $P_{o,n}$) remain valid under weak dependence and the same holds for the obtained prices.

The results in this work and results in Yatracos (2013) relating B-S-M price with Bayes risk and applications have been presented since 2008 in various seminars and in particular in the Third International Conference on Computational and Financial Econometrics (2009, [http://www.dcs.bbk.ac.uk/cfe09/]). Janssen and Tietje (2013) used also Le Cam’s theory of statistical experiments to discuss “the connection between mathematical finance and statistical modelling” (see the Summary) for $d$-dimensional price processes. Known results from statistical experiments are used in order to revisit financial models (p. 111, lines 22, 23). Some of the differences in their work are: a) The price process is not standardized and a risk neutral probability is assumed to exist. b) Convergence of the likelihood ratios to a normal experiment is obtained under the assumption of contiguity. c) There are no results explaining the behaviors of the trader and of the buyer. d) There is no proof of the universal validity of $B$-$S$-$M$ formula without modeling assumptions on the shares’ prices.

The theory of statistical experiments used is in Le Cam (1986, Chapters 10 and 16), Le Cam and Yang (1990, Chapters 1-4, 2000, Chapters 1-5) and in Roussas (1972, Chapter 1). A concise introduction in this material can be found in Pollard (2001). Theory of option pricing can be found, among others, in Lamberton and Lapeyre (1996) and Musiela and Rutkowski (1997). Proofs are in the Appendix.
A method to obtain risk neutral probability

The price of a European option is independent from the stock prices’ drift and depends only on the risk free interest rate. This is usually attained by finding a risk neutral probability $\mathcal{P}^*$ equivalent to the physical probability $\mathcal{P}$ for which

$$E_{\mathcal{P}^*}(\frac{S_T}{S_t} | \mathcal{F}_t) = e^{r(T-t)}; \quad (1)$$

$\{\mathcal{F}_t\}$ is the natural filtration. For example, when stock prices are log-normal Girsanov’s theorem is used to obtain $\mathcal{P}^*$ and the drift disappears obtaining (1).

For several stock price models it is not easy to determine $\mathcal{P}^*$. Moreover, the stock-price model is usually unknown. How one can proceed in this situation? We can try to obtain (1) with a different approach in two steps. Observe that the adjusted prices $\{S_t/ES_t\}$ are independent of the drift since they all have mean 1; $ES_t$ is taken under $\mathcal{P}$.

**Step 1:** Decompose $\log(S_T/S_t)$ in two components,

$$\log \frac{S_T}{S_t} = \log \frac{S_T/ES_T}{S_t/ES_t} + \log \frac{ES_T}{ES_t}. \quad (2)$$

Determine a probability $Q$ under which $S_t/ES_t$ is a martingale. There is no involvement of the interest (i.e. of $r$) in this step.

**Step 2:** Use $Q$ defined in step 1 and translate $\log \frac{S_T}{S_t}$ by $r(T-t) - \log \frac{ES_T}{ES_t}$ thus obtaining $\mathcal{P}^*$ under which $\frac{S_T}{S_t}$ satisfies (1). Probabilists prefer to obtain $\mathcal{P}^*$ with a change of probability $Q$ via $\frac{d\mathcal{P}^*}{d\mathcal{P}}$.

This new approach works for log-normal prices and allows to obtain the B-S-M price without stochastic calculus.

**Example 2.1** Let $S_t$ be a geometric Brownian motion,

$$S_t = s_0 \exp\{ (\mu - \frac{\sigma^2}{2})t + \sigma B_t \} \quad (3)$$

with $B_t$ standard Brownian motion, $t > 0$ and $s_0$ the price at $t = 0$. For $t < T$,

$$\log \frac{S_T}{S_t} = (\mu - \frac{\sigma^2}{2})(T-t) + \sigma (B_T - B_t), \quad (4)$$

and since $ES_t = s_0 \exp\{ \mu t \}, S_t/ES_t$ is a martingale under $\mathcal{P}$ (i.e. $Q = \mathcal{P}$) and you can obtain (1) via (2) either simply translating $\log \frac{S_T}{S_t}$ by $r(T-t) - \log \frac{ES_T}{ES_t}$ or with a change of probability via Lemma 6.4,

$$\frac{d\mathcal{P}^*}{d\mathcal{P}}(w) = e^{Aw+C},$$
with
\[ A = \frac{r - \mu}{\sigma^2}, \quad C = -MA - \frac{\Sigma^2 A^2}{2}, \quad M = (\mu - \frac{\sigma^2}{2})(T - t), \quad \Sigma^2 = \sigma^2(T - t). \]

It follows that under \( \mathcal{P}^* \) the law of \( \log \frac{S_T}{S_t} \) is normal with mean \( M^* = M + A\Sigma^2 = (r - \frac{\sigma^2}{2})(T - t) \) and variance \( \Sigma^{*2} = \Sigma^2 = \sigma^2(T - t) \) and (1) holds.

In Example 2.1 \( \mathcal{P}^* \) is easily obtained because the distribution of \( \log \frac{S_T}{S_t} \) is normal. Can one similarly obtain \( Q \) and \( P^* \) in other situations? Note that without share price model assumption we have only the observed share prices. It is similar to a non-parametric statistical problem where the observations have all the information. The share prices should determine the “right” \( Q \) or a sequence \( Q_n \) that will give us the distribution of \( \log \frac{S_T}{S_t} \) with an asymptotic argument via a Central Limit Theorem. The word “asymptotic” is used in the sense that there are countably infinite many transactions in each sub-interval of \( (t, T) \).

When there are “many” transactions in \([t_0, T]\), an embedding in Le Cam’s statistical experiments allows to determine the sequence of probabilities \( Q_n = P_{t,n} \) under which \( \frac{S_T}{S_t} \) satisfies (1) via the translation in Step 2 for the law of \( \log \frac{S_T}{S_t} \) under \( Q_n \).

3 The embedding, the tools and prices-densities

Let \((\Omega, \mathcal{F}, \mathcal{P})\) be the underlying probability space of the stock prices \( \{0 < S_t, \ 0 \leq t \leq T\} \), and let \( ES_t \) be the expectation with respect to \( \mathcal{P} \). Consider the process of prices-densities

\[ \{p_t = \frac{S_t}{ES_t}, \ t \in [0, T]\}. \tag{5} \]

Define on \((\Omega, \mathcal{F}, \mathcal{P})\) the (forward) probability \( P_t : \) for \( A \in \mathcal{F} \)

\[ P_t(A) = \int_A \frac{S_t(\omega)}{ES_t} \mathcal{P}(d\omega); \tag{6} \]

the derivative of \( P_t \) with respect to \( \mathcal{P} \),

\[ \frac{dP_t}{d\mathcal{P}} = \frac{S_t}{ES_t} = p_t, \ t \in [0, T]. \]
Since $S_t$ is positive, $P_t$ and $\mathcal{P}$ are mutually absolutely continuous. Thus, $P_t$ and $P_s$ are mutually absolutely continuous for each $t, s$ in $\Theta = [0, T]$, and

$$\frac{dP_t}{dP_s} = \frac{p_t}{p_s} \text{ a.s. } \mathcal{P}. $$

A binary statistical experiment $\mathcal{E} = \{P, Q\}$ with $P, Q$ probabilities on $(\hat{\Omega}, \hat{\mathcal{F}})$ (Blackwell, 1951). Le Cam (see, for example, 1986) defined $\mathcal{E}$ as Gaussian experiment when $P$ and $Q$ are equivalent and the distribution of log $\frac{dQ}{dP}$ under either $P$ or $Q$ is normal, introduced a distance $\Delta$ between experiments and proved that $\Delta$-convergence of experiments $\mathcal{E}_n = \{P_n, Q_n\}, n \geq 1$, to $\mathcal{E}$ is equivalent to weak convergence of likelihood ratios $\frac{dQ}{dP_n}$ under $P_n$ (resp. $Q_n$) to the distribution of $\frac{dQ}{dP}$ under $P$ (resp. $Q$). In analogy with the frequent weak convergence of sums of random variables to a Gaussian distribution, there is frequent $\Delta$-convergence of experiments to a Gaussian experiment.

Embed the traded stock prices in the statistical experiments framework via log $\frac{dQ}{dP_n}$ by re-expressing log $\frac{S_T}{S_{t_0}}$ as in (2) using also the intermediate prices,

$$\log \frac{S_T}{S_{t_0}} = \log \frac{S_T}{ES_T} \frac{\cdots \frac{S_{t_1}}{ES_{t_1}}}{\frac{S_{t_0}}{ES_{t_0}} \frac{S_{t_1}}{ES_{t_1}} \cdots \frac{S_{t_{k_n-1}}}{ES_{t_{k_n-1}}}} + \log \frac{ES_T}{ES_{t_0}}. \tag{7}$$

The products of normalized prices-densities $\frac{S_{t_0}}{ES_{t_0}} \cdots \frac{S_{t_{k_n-1}}}{ES_{t_{k_n-1}}}$ and $\frac{S_{t_0}}{ES_{t_0}} \cdots \frac{S_{t_n}}{ES_{t_n}}$ determine, respectively, $P_n$ and $Q_n$ in $(\Omega^{k_n}, \mathcal{F}^{k_n})$. Le Cam’s theory provides the asymptotic distribution of log $\frac{dQ_n}{dP_n}$ under $P_n$. The trader uses $P_n$ denoted in the sequel by $P_{t_0,n}$. Naturally, the buyer who acts one transaction period later than the trader uses $Q_n$, denoted $P_{T,n}$. To calculate the option’s prices the laws of log $\frac{S_T}{S_{t_0}}$ under $P_{t_0,n}$ and $P_{T,n}$ are used when $k_n \to \infty$. To obtain the “fair” price the limit law under $P_{t_0,n}$ will be translated such that (11) holds.

Consider on $(\Omega^{k_n}, \mathcal{F}^{k_n})$ the statistical experiment

$$\mathcal{E}_{k_n} = \{P_{t_0,n} = \prod_{j=0}^{k_n-1} P_{t_j}, P_{T,n} = \prod_{j=1}^{k_n} P_{t_j}\}, \tag{8}$$

with $t_0^n = t_0$ and $t_{k_n} = T$ for each $n$. $P_{t_0,n}$ is determined via (11) by

$$P_{t_0,n}(B_0 x \cdots \times B_{k_n-1}) = P_{t_0}(B_0) P_{t_1}(B_1) \cdots P_{t_{k_n-1}}(B_{k_n-1}), \tag{9}$$

for $B_j \in \mathcal{F}, j = 0, \ldots, k_n - 1$, and its extension to the product $\sigma$-field $\mathcal{F}^{k_n}$.

$P_{T,n}$ is determined similarly by prices-densities at $t_1^n, \ldots, t_{k_n}$. $P_{t_0,n}$ and $P_{T,n}$ are mutually absolutely continuous since $S_t > 0$ when $t > 0$. 

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Let
\[ Y_{n,j} = \sqrt{\frac{p^n_{t_j}}{p^n_{t_{j-1}}}} - 1, \quad U_{n,j} = \sqrt{\frac{p^n_{t_j}}{p^n_{t_{j-1}}}} - 1, \quad j = 1, \ldots, k_n, \quad (10) \]
\[ a[t_0, T] = \frac{ES_T}{ES_{t_0}}, \quad (11) \]
\[ \Lambda_k = \log \frac{p_T}{p_{t_0}} = \log \frac{\prod_{j=1}^{k_n} p^n_{t_j}}{\prod_{j=1}^{k_n} p^n_{t_{j-1}}} = 2 \sum_{j=1}^{k_n} \log(1 + Y_{n,j}) = -2 \sum_{j=1}^{k_n} \log(1 + U_{n,j}). \quad (12) \]
Rewrite \( \log(S_T/S_{t_0}) \) using (12) via (7) and (11):
\[ \log(S_T/S_{t_0}) = \Lambda_k + \log a[t_0, T]. \quad (13) \]

Experiment (8) is not specified since the prices-densities and therefore both \( P_{t_0,n} \) and \( P_{T,n} \) are all unknown but, under mild conditions, it is shown using Le Cam’s theory and (13) that when \( k_n \) increases to infinity \( \log(S_T/S_{t_0}) \) has infinitely divisible distributions under \( P_{t_0,n} \) and \( P_{T,n} \).

**Proposition 3.1** Define on \((\Omega^{k_n}, \mathcal{F}^{k_n})\) random variables
\[ \tilde{Y}_{n,j}(\omega^{(k_n)}) = Y_{n,j}(\omega_j), \quad j = 1, \ldots, k_n, \]
\( \omega^{(k_n)} = (\omega_1, \ldots, \omega_{k_n}). \) The variables \( \tilde{Y}_{n,j}, \quad j = 1, \ldots, k_n, \) are independent under \( P_{t_0,n} \) and \( P_{T,n}, \) i.e. \( Y_{n,j}(\omega_j) \) is independent of \( Y_{n,k}(\omega_k), \quad j \neq k. \) The same holds for \( U_{n,j}, \quad j = 1, \ldots, k_n, \) and \( \frac{p^n_{t_j}}{p^n_{t_{j-1}}}, \quad j = 1, \ldots, k_n. \)

**Remark 3.1** Proposition 3.1 is used to derive the asymptotic distribution of \( \Lambda_k \) (see (12)) under \( P_{t_0,n} \) and under \( P_{T,n} \) via \( \{\tilde{Y}_{n,j}, \quad j = 1, \ldots, k_n\} \) when \( n \to \infty. \)

Prices-densities and their probabilities with respect to \( \mathcal{P} \) have been already used in the Finance literature to express wealth in a new numéraire; see, for example, Detemple and Rindisbacher (2008) who attribute the notion of price-density and the obtained “forward probability measure” (like \( P_t \) in (8)) to Geman (1989) and Jamshidian (1989).

The use of prices-densities and of \( P_{t_0,n} \) and \( P_{T,n} \) is now motivated from different angles.

(i) Assume that stock prices have all the information. Just prior to \( t_0^n \) the (unobserved) information and in particular the volatility in \( (S_{t_0}^n, \ldots, S_{t_k}^n) \) is better measured by their coefficient of variation that is the variance of the
corresponding price-densities \((p_{t_{0}}^{n}, p_{t_{1}}^{n}, \ldots, p_{t_{k_{n}}^{n}}^{n})\). The trader has the information from \(t_{0}^{n} = t_{0}\) until \(t_{k_{n} - 1}^{n}\), and the buyer has the information from \(t_{1}^{n}\) until \(t_{k_{n}} = T\) expressed respectively by \((p_{t_{0}}^{n}, \ldots, p_{t_{k_{n} - 1}^{n}}^{n})\) and \((p_{t_{1}}^{n}, \ldots, p_{t_{k_{n}}^{n}}^{n})\). If both have similar information the transaction will probably occur. These information vectors are embedded in \((\Omega^{k_{n}}, F^{k_{n}})\) obtaining probabilities \(P_{t_{0},n}^{n}\) and \(P_{T,n}^{n}\). They both miss one component from the whole information \((p_{t_{0}}^{n}, p_{t_{1}}^{n}, \ldots, p_{t_{k_{n}}^{n}}^{n})\) but this additional information becomes negligible when \(k_{n}\) is large, at least in some situations (Mammen, 1986).

(ii) To compare different stocks (or assets) Modigliani and Miller (1958) used the ratio of all long term debt outstanding to the book value of all common stock, and called it “leverage” (or “financial risk”) measure. To compare values of the same stock Sprenkle (1961) used as leverage ratios of successive prices \(S_{t_{j}^{n}}/S_{t_{j-1}^{n}}\) without expectation normalization and obtained the price of a call option for a warrant that is similar to the B-S-M price with arbitrary multiplicative parameters \(k = \frac{E^{\infty}}{n_{t_{0}^{n}}}\) and \(k^{*}\) preceding, respectively, \(\Phi(d_{1})\) and \(\Phi(d_{2})\) (see form of (29)); \(k^{*}\) is a discount factor that depends on the risk of the stock (Black and Scholes, 1973, p. 639). In Sprenkle’s ratios, variances are not adjusted for the price level and the obtained price is not the B-S-M price. Using instead \(Y_{n,j}^{n}\) and \(U_{n,j}^{n}\) in (10) to measure financial risk, the B-S-M price is obtained for calm stock after translation of \(\log \frac{S_{T}}{S_{t_{0}}}\).

4 Modeling the distribution of \(\log \frac{S_{T}}{S_{t_{0}}}\)

4.1 The Modeling Assumptions

Let \(S_{t}\) be the stock value at time \(t, t \in [0, T]\), defined on \((\Omega, F, P)\). Assume

(A1) \(S_{t} > 0\) and \(ES_{t} < \infty\) for every \(t \in [t_{0}, T]\),

(A2) a countable number of transaction times in any open interval of \([t_{0}, T]\),

(A3) for the prices-densities \(p_{t_{0}}^{n}, p_{t_{1}}^{n}, \ldots, p_{t_{k_{n}}^{n}}^{n}\) with mesh size \(\delta_{n} = \sup\{t_{j}^{n} - t_{j-1}^{n}; j = 1, \ldots, k_{n}\}\), \(k_{n} = k_{n}(\delta_{n})\),

\((i)\) \(\lim_{\delta_{n} \to 0} \sup\{E_{P_{t_{j}^{n}}}^{n} (\frac{p_{t_{j}^{n}}^{n}}{p_{t_{j-1}^{n}}^{n}} - 1)^{2}, j = 1, \ldots, k_{n}\} = 0,\)

\((ii)\) \(\sup_{n} \sum_{j=1}^{k_{n}} E_{P_{t_{j}^{n}}}^{n} (\frac{p_{t_{j}^{n}}^{n}}{p_{t_{j-1}^{n}}^{n}} - 1)^{2} \leq b < \infty.\)

Assumption A1 allows in our framework the passage from stock prices to prices-densities. Assumptions A1 and A2 provide via (6) strategies \(P_{t_{0,n}}^{n}\) and
$P_{T,n}$. Assumption $A3(i)$ indicates that the contribution of the ratio $\frac{p_{tn}^n}{p_{tn}^{n-1}}$ does not affect the distribution of $\frac{P_{tn}^n}{P_{tn}^{n-1}}$, $j = 1, \ldots, k_n$. Assumption $A3(ii)$ implies that the sum of the variances of $Y_{n,j}$, $j = 1, \ldots, k_n$, (see (10)) is uniformly bounded.

**Remark 4.1** For the expectations in $A3$ it holds

$$E_{P_{tn}^{n-1}} \left( \sqrt{\frac{p_{tn}^n}{p_{tn}^{n-1}}} - 1 \right)^2 = \int (\sqrt{P_{tn}^n} - \sqrt{P_{tn}^{n-1}})^2 dP \propto H^2(P_{tn}^n, P_{tn}^{n-1});$$

$H(P_{tn}^n, P_{tn}^{n-1})$ is Hellinger’s distance of $P_{tn}^n, P_{tn}^{n-1}$ defined in (47).

### 4.2 Infinitely divisible distribution of $\log \frac{S_T}{S_{t_0}}$

Assumption $A3$ implies the sequences of distributions of $\sum_{j=1}^{k_n} Y_{n,j}$ and of $\sum_{j=1}^{k_n} U_{n,j}$ are each relatively compact both under $P_{t_0,n}$ and $P_{T,n}$. Thus, we can choose a subsequence $\{k_{n'}\}$, for which both $\sum_{j=1}^{k_{n'}} Y_{n',j}$ and $\sum_{j=1}^{k_{n'}} U_{n',j}$ converge weakly, respectively, under $P_{t_0,n'}$ and $P_{T,n'}$. Without loss of generality we will use $\{n\}$ and $\{k_n\}$ instead of $\{n'\}$ and $\{k_{n'}\}$. The next result determines the distribution of $\log \frac{S_T}{S_{t_0}}$ from its moment generating function of an infinitely divisible distribution. Recall that any infinite divisible distribution is that of the sum of two independent components, one normal and one Poissonian, and these components are unique up to a shift (Meerschaert and Scheffler, p.41).

**Theorem 4.1** (Le Cam, 1986, Proposition 2, p. 462) Assume that $A3$ holds. Then, $\Lambda_{k_n} = \log \prod_{j=1}^{k_n} \frac{p_{tn}^n}{p_{tn}^{n-1}}$ converges under $P_{t_0,n}$ in distribution to $\Lambda_{t_0}$ such that for every $s \in (0, 1)$ it holds

$$\log Ee^{s\Lambda_{t_0}} = (2\mu - \sigma^2)s + 2\sigma^2 s^2 + \int_{[-1,0) \cup (0,\infty)} [(1+y)^{2s} - 1 - 2sy]L_{t_0}(dy), \quad (14)$$

where $\mu = \lim_{n \to \infty} \sum_{j=1}^{k_n} E_{P_{tn}^n} Y_{n,j}$,

$\sigma^2 = \lim_{\tau \to 0} \lim_{n \to \infty} \sum_{j=1}^{k_n} E_{P_{tn}^n} Y_{n,j}^2 I(\{|Y_{n,j}| \leq \tau\}),$

the Lévy measure

$L_{t_0}(x) = \lim_{n \to \infty} L_{k_n}(x) = \lim_{n \to \infty} \sum_{j=1}^{k_n} E_{P_{tn}^n} Y_{n,j}^2 I(Y_{n,j} \leq x)$

and $L_{t_0}^*$ is the probability determined by the Poissonian component of $\Lambda_{t_0}$.
From (14) the normal component of $\Lambda_{t_0}$ has mean $\mu_{[t_0,T]}$ and variance $\sigma^2_{[t_0,T]}$, with
\[ \mu_{[t_0,T]} = 2\mu - \sigma^2, \quad \sigma^2_{[t_0,T]} = 4\sigma^2. \]
The mean $\mu_{[t_0,T]}$ is determined by variances since by $A3$
\[ \mu_{[t_0,T]} = -\lim_{n\to\infty} \sum_{j=1}^{n} \text{Var}(Y_{n,j}) - \sigma^2. \]

**Corollary 4.1** When $A3$ holds, the asymptotic distribution of $-\Lambda_{k_n}$ under $P_{T,n}$ has the same $\mu$ and $\sigma^2$ as $\Lambda_{t_0}$ but different Lévy measure. Under $P_{T,n}$, $\Lambda_{k_n}$ converges weakly to $\Lambda_T$, with shift component $-\mu_{[t_0,T]} = -(2\mu - \sigma^2)$, $\sigma^2_{[t_0,T]} = 4\sigma^2$ and Lévy measure $L_T$. The Poissonian component of $\Lambda_T$ has probability $L^*_T$.

The distribution of $\log \frac{S_T}{ES_T} \frac{S_{t_0}}{ES_{t_0}}$ follows from next theorem. The risk neutral probability follows from (13) with a passage to an equivalent structure by translation of $\log \frac{S_T}{ES_T} \frac{S_{t_0}}{ES_{t_0}}$ or a change of the normal probability conditional on the value of the Poissonian component. The translation’s amount is determined such that (1) holds and has the value
\[ r(T - t_0) - \log \frac{ES_T}{ES_{t_0}} - \mu_{[t_0,T]} - \frac{\sigma^2_{[t_0,T]}}{2} - \log M_{L^*_T}(1), \quad (15) \]
with $M_{L^*_T}(t)$ the moment generating function of the Poissonian component evaluated at $t = 1$.

**Theorem 4.2** When $A1 - A3$ hold, from Theorem 4.1 it follows that
\[ (i) \lim_{n\to\infty} P_{t_0,n}[\log \prod_{j=1}^{k_n} p_{j_{n-1}}] \leq x = \int \Phi \left( \frac{x - y - \mu_{[t_0,T]}}{\sigma_{[t_0,T]}} \right) L_{t_0}^*(dy), \quad (16) \]
\[ (ii) \lim_{n\to\infty} P_{T,n}[\log \prod_{j=1}^{k_n} p_{j_{n-1}}] \leq x = \int \Phi \left( \frac{x - y + \mu_{[t_0,T]}}{\sigma_{[t_0,T]}} \right) L_T^*(dy). \quad (17) \]

**Corollary 4.2** Under the assumptions of Theorem 4.2, (16) and (17) both hold when one of the terms in the product $\prod_{j=1}^{k_n} p_{j_{n-1}}$ is omitted.
Remark 4.2 Without $A3$ the distribution of $\log S_{t_0}$ is infinitely divisible when the random variables $\{Y_{n,j}, j = 1, \ldots, k_n\}$ and $\{U_{n,j}, j = 1, \ldots, k_n\}$ are uniformly asymptotically negligible. However, computational difficulties arise because centering is needed at truncated expectations that are also used in the definition of the sequences of Lévy measures $L_{n_k}$ in Theorem 4.1. The limit law is more complicated than the one presented herein (Loève (1977)).

The next result confirms the martingale property of the prices-densities.

**Theorem 4.3** When the ratios of prices-densities $\{\frac{p_{n,j}}{p_{n,j-1}}, j = 1, \ldots, k_n\}$ are independent of $S_{t_0}$, the price-densities are a martingale under $P_{t_0,n}$.

### 4.3 Calm Stock

Calm stock (or calm share price) has prices-densities $p_{t+n}$ (see (11)) that do not differ much often with respect to $P$ for small $\delta$-values, thus excluding the case of unusual jumps. To provide a quantitative definition of calm share, the difference of $p_{t+n}$ and $p_t$ over the (forward) regions

$$\{|\sqrt{p_{t+n}} - \sqrt{p_t}| > \epsilon\}, \epsilon > 0,$$

is measured by

$$\int (\sqrt{p_{t+n}} - \sqrt{p_t})^2 I(|\sqrt{p_{t+n}} - \sqrt{p_t}| > \epsilon) dP;$$

$I$ is the indicator function.

**Definition 4.1** Let $t^n_1 < \ldots < t^n_{k_n-1}$ be a partition of $(t_0 = t^n_0, T = t^n_{k_n})$, with mesh size $\delta_n = \sup\{t^n_j - t^n_{j-1}, j = 1, \ldots, k_n\}$ and $\epsilon > 0$. The stock $\{S_t\}$ is $\epsilon$-calm in $[t_0,T]$ if for any partition

$$\lim_{b_n\to0} \sum_{j=1}^{k_n} \int (\sqrt{p_{t^n_j}} - \sqrt{p_{t^n_{j-1}}})^2 I(|\sqrt{p_{t^n_j}} - \sqrt{p_{t^n_{j-1}}}| > \epsilon) dP = 0. \quad (19)$$

**Definition 4.2** The stock $\{S_t\}$ is calm in $[t_0,T]$ if it is $\epsilon$-calm for every $\epsilon > 0$.

For calm stock, the random variables $Y_{n,j}, j = 1, \ldots, k_n$ (see (10)) satisfy Lindeberg’s condition since $\epsilon$ can be written

$$\lim_{k_n\to\infty} \sum_{j=1}^{k_n} E_{P_{t^n_j-1}} Y_{n,j}^2 I(|Y_{n,j}| > \epsilon) = 0. \quad (20)$$
When \( Y_{n,j}, j = 1, \ldots, k_n \), are independent and uniformly asymptotically negligible, Lindeberg’s condition is necessary and sufficient condition for \( \sum_{j=1}^{k_n} Y_{n,j} \) to have asymptotically a normal distribution and is thus satisfied for the prices-densities of geometric Brownian motion.

**Remark 4.3** From (32) and (33) it follows that (20) and all its implications (including option pricing in section 4) hold also for \( U_{n,j}, j = 1, \ldots, k_n \), and that in (19) instead of the indicators of the forward region (18) one can use indicators of backward regions \( I(|\sqrt{p_{t_j}} - 1| > \epsilon), j = 1, \ldots, k_n \), i.e. we can define forward and backward calm stock and the obtained results hold for both.

### 4.4 Calm stock-Normal distribution for \( \log \frac{S_T}{S_{t_0}} \)

The distribution of \( \log \frac{S_T}{S_{t_0}} \) is initially obtained.

**Theorem 4.4** When \( A_1, A_2, \) and \( A_3(ii) \) hold for a calm stock in \([0, T]\), there is \( \sigma_{[t_0,T]} > 0 \) such that

(i) \( \{P_{t_0,n}\} \) and \( \{P_{T,n}\} \) are contiguous, i.e. \( \lim_{n \to \infty} P_{t_0,n}(A_n) = 0 \iff \lim_{n \to \infty} P_{T,n}(A_n) = 0 \),

\[
\lim_{n \to \infty} P_{t_0,n}[\log \prod_{j=1}^{k_n} \frac{p_{t_j}}{p_{t_j-1}} \leq x] = \Phi \left( \frac{x + \frac{\sigma^2_{[t_0,T]}}{2}}{\sigma_{[t_0,T]}^2} \right),
\]

(ii) \( \lim_{n \to \infty} P_{T,n}[\log \prod_{j=1}^{k_n} \frac{p_{t_j}}{p_{t_j-1}} \leq x] = \Phi \left( \frac{x - \frac{\sigma^2_{[t_0,T]}}{2}}{\sigma_{[t_0,T]}^2} \right),
\]

(iii) \( \sigma^2_{[t_0,T]} = 4 \lim_{\tau \to 0} \lim_{n \to \infty} \sum_{j=1}^{k_n} E_{P_{t_j-1}} \left( \sqrt{\frac{p_{t_j}}{p_{t_j-1}}} - 1 \right)^2 I(|\sqrt{\frac{p_{t_j}}{p_{t_j-1}}} - 1| \leq \tau). \)

\(
\square
\)

The risk neutral probability is obtained from Theorem 4.4 using a passage to an equivalent structure with translation of \( \log \frac{S_T}{S_{t_0}} \) by \( r(T-t_0) - \log \frac{E_S}{E_{S_{t_0}}} \) such that (11) holds. Note that for calm stock the last three terms of translation (15) vanish.

The result indicating clearly volatility’s role follows.
Corollary 4.3 Under the assumptions of Theorem 4.4, from (21) and (22) it follows that the binary experiment \( E_{k_n} = \{ P_{t_0, n}, P_{T, n} \} \) converges to the Gaussian experiment \( G = \{ P_0 = N(0, 1), P_T = N(\sigma_{[t_0, T]}, 1) \} \) when \( k_n \to \infty \). From the form of \( G \) it is clear that the volatility, \( \sigma_{[t_0, T]} \), is the determining factor in the buyer’s decision.

Remark 4.4 The parameter \( \sigma^2_{[t_0, T]} \) does not necessarily have the form \( \sigma^2(T - t_0) \) obtained, for example, with the B-S-M model. Conditions are provided for \( A_3 \) to hold and for \( \sigma^2_{[t_0, T]} \) to have the form \( \sigma^2(T - t_0) \), \( \sigma > 0 \) (Proposition 6.3).

5 Pricing the share’s European call option

5.1 Discounting with expectations’ ratios and interest

We already used \( a^{-1}[t, T] \) (see (11)) to discount the stock price \( S_T \) at \( t \) and obtain the distribution of \( \log \frac{S_T}{S_t} \). We will use it also to price the stock’s portfolio. This is not restrictive since, when calculating expected values under the martingale probability obtained from the equivalent structure, \( a^{-1}[t, T] \) coincides with \( e^{-r(T-t)} \).

The price of the call option at \( t = t_0 \) is

\[
E_{\tilde{Q}_n} a^{-1}[t_0, T] S_T I(S_T > X) - X e^{-r(T-t_0)} E_{\tilde{Q}_n} I(S_T > X); \tag{24}
\]

the expectation \( E_{\tilde{Q}_n} \) is calculated with respect to probability \( \tilde{Q}_n = P_{t_0, n}, P_{T, n} \).

To calculate the expectations for the trader and the buyer in (24) via the distribution previously obtained for \( \log \frac{S_T}{S_{t_0}} \) rewrite (24) using prices-densities (5) at the deterministic transaction times to obtain

\[
E_{\tilde{Q}_n} M_{k_n} S_{t_0} I(M_{k_n} > \frac{X}{S_{t_0}} a^{-1}[t_0, T]) - X e^{-r(T-t_0)} E_{\tilde{Q}_n} I(M_{k_n} > \frac{X}{S_{t_0}} a^{-1}[t_0, T]); \tag{25}
\]

the term

\[
M_{k_n} = \frac{S_T}{S_{t_0}} a^{-1}[t_0, T] = \frac{\prod_{j=1}^{k_n} p_{t_0}^{t_j}}{\prod_{j=1}^{k_n} p_{t_0}^{t_j-1}}. \tag{26}
\]

Note that the price can be obtained using either 1) or 2):

1) a) discount the first term in (24) with \( a^{-1}[t_0, T] \),
b) take the limit of the expectations with respect to \( P_{t_0, n} \),
c) pass by translation to the risk neutral equivalent structure and “fair” price.

2) a) discount the first term in (24) with \( e^{-r(T-t_0)} \),
b) translate “properly” $P_{t_0,n}$ to obtain $P_{t_0,n}^*$ without the term involving $a[t_0, T]$,
c) take the limit of the expectations to obtain the “fair” price.

We preferred to use 1) obtaining (25) for easier exposition, because the translation 2 b) for non-calm shares is conditional on the value of the Poissonian component and is not easily described. We also preferred not to use directly the limit risk neutral probability when taking the expected value of the interest discounted call’s payoff.

Two different situations are considered for pricing the option at $t_0$:

(P1) At $t < t_0$, the trader prices the option at $t_0$ given that $S_{t_0} = s_{t_0}$. Then, the expectations in (25) are conditional on $S_{t_0} = s_{t_0}$.

(P2) To price the option at $t_0$ when the value $S_{t_0} = s_{t_0}$ is known, the trader calculates the option’s price at $t = t_1^n > t_0$ with $t_1^n$ decreasing to $t_0$ with $n$. In (25) and (26) $t_0$ is replaced by $t_1^n$.

In both (P1) and (P2) we let in (25) $n$ and/or the number of transactions $k_n$ increase to infinity to obtain at $t_0$ the trader’s option price and a lower bound on the buyer’s price, that agree under (P1) and (P2). A translation provides the risk-neutral price and lower bound.

5.2 The Pricing Assumptions

In addition to the modeling assumptions (A1) – (A3) assume

(A4) The market consists of the stock $S$ and a riskless bond that appreciates at fixed rate $r$ and there are no dividends or transaction costs. The option is European. The buyer prefers to pay less than more.

For (P1) pricing assume

(A5) The ratios $p_{t_0,j}^n/p_{t_0,j-1}^n$, $j = 1, \ldots, k_n$ are independent of $S_{t_0}$.

For (P2) pricing assume

(A5$^*$) At $t_0$, $\lim_{n \to \infty} S_{t_1^n} = s_{t_0}$ in probability, $\lim_{n \to \infty} ES_{t_1^n} = s_{t_0}$.

In the pricing sections, to obtain the B-S-M price $\log c$ denotes $\ln c$.

5.3 Pricing a European option using (A5)

Pricing under (P1) follows. The components in (25) are initially calculated.

**Theorem 5.1** When A1 – A5 hold, the limits of the terms in (25) with $Q_n = P_{t_0,n}$, $P_{T,n}$, and $M_{k_n}$ as in (26), are:

(i) $\lim_{n \to \infty} P_{t_0,n}[M_{k_n} > \frac{X}{S_{t_0}}a^{-1}[t_0, T]|S_{t_0} = s_{t_0}]$
the trader’s price for the European option is of the form below and in particular, by passing to an equivalent structure, a generalization of cumulative distributions on the real line. The option’s price is obtained in Corollary 5.1

\begin{align*}
\text{(i)} & \lim_{n \to \infty} E_{P_{t_0,n}}[S_{t_0} M_{k_n}\ln (M_{k_n} > X_{s_{t_0}} \alpha^{-1} [t_0, T]) | S_{t_0} = s_{t_0}] = s_{t_0} \lim_{n \to \infty} P_{T,n} [M_{k_n} > X_{s_{t_0}} \alpha^{-1} [t_0, T]] \\
& = s_{t_0} \Phi \left( \frac{\log (s_{t_0}/X) + \log a[t_0, T] + \mu_{[t_0, T]} + y}{\sigma_{[t_0, T]}} \right) L^*_{t_0} (dy),
\end{align*}

\begin{align*}
\text{(ii)} & \lim E_{P_{t_0,n}} [S_{t_0} M_{k_n} \ln (M_{k_n} > X_{s_{t_0}} \alpha^{-1} [t_0, T]) | S_{t_0} = s_{t_0}] = s_{t_0} \Phi \left( \frac{\log (s_{t_0}/X) + \log a[t_0, T] + \mu_{[t_0, T]} + y}{\sigma_{[t_0, T]}} \right) L^*_T (dy),
\end{align*}

\begin{align*}
\text{(iii)} & \lim_{n \to \infty} E_{P_{t_0,n}} [S_{t_0} M_{k_n} \ln (M_{k_n} > X_{s_{t_0}} \alpha^{-1} [t_0, T]) | S_{t_0} = s_{t_0}] \\
& \geq s_{t_0} e^{-\mu_{[t_0, T]} + 5 \sigma_{[t_0, T]}^2} \int e^y \Phi \left( \frac{\log (s_{t_0}/X) + \log a[t_0, T] + \mu_{[t_0, T]} + \sigma_{[t_0, T]}^2 + y}{\sigma_{[t_0, T]}} \right) y L^*_T (dy).
\end{align*}

Let

\begin{align*}
Q_{t_0,n} (x) = P_{t_0,n} [\log \Pi_{j=1}^{k_n} p_{r,t_0,j} / p_{r,t_0,j-1} \leq x], \quad Q_{T,n} (x) = P_{T,n} [\log \Pi_{j=1}^{k_n} p_{r,T,j} / p_{r,T,j-1} \leq x],
\end{align*}

be cumulative distributions on the real line. The option’s price is obtained below and in particular, by passing to an equivalent structure, a generalization of the B-S-M price similar to the price in Merton (1976).

**Corollary 5.1** Under (P1) pricing, when A1 – A5 hold:

(i) the trader’s price for the European option is

\begin{align*}
C(t_0) = s_{t_0} \Phi \left( \frac{\log (s_{t_0}/X) + \log a[t_0, T] + \mu_{[t_0, T]} + y}{\sigma_{[t_0, T]}} \right) L^*_T (dy).
\end{align*}

(ii) a lower bound on the buyer’s price is

\begin{align*}
s_{t_0} e^{-\mu_{[t_0, T]} + 5 \sigma_{[t_0, T]}^2} \int e^y \Phi \left( \frac{\log (s_{t_0}/X) + \log a[t_0, T] + \mu_{[t_0, T]} + \sigma_{[t_0, T]}^2 + y}{\sigma_{[t_0, T]}} \right) y L^*_T (dy) - X e^{-r(T-t_0)} \int \Phi \left( \frac{\log (s_{t_0}/X) + \log a[t_0, T] + \mu_{[t_0, T]} + y}{\sigma_{[t_0, T]}} \right) L^*_T (dy).
\end{align*}

(iii) By passing to an equivalent structure either by translation (L5) or by change of the normal probability conditional on the Poissonian component log a[t_0, T] is replaced in i) and ii) via (L5) by

\begin{align*}
r(T-t_0) - \mu_{[t_0, T]} - \frac{\sigma_{[t_0, T]}^2}{2} \ln M_{L^*_T} (1),
\end{align*}

such that (1) holds and the “fair” price is obtained.
The results for calm stock follow.

**Theorem 5.2** For calm stock, under assumptions $A1, A2, A3(ii), A4, A5$:

(i) \( \lim_{n \to \infty} P_{t_0,n}[M_{kn} > \frac{X}{S_{t_0}} a^{-1}[t_0, T]|S_{t_0} = s_{t_0}] = \Phi\left(\frac{\log(s_{t_0}/X) + \log a[t_0, T] - \frac{\sigma_{[t_0,T]}^2}{2}}{\sigma_{[t_0,T]}}\right)\),

(ii) \( \lim_{n \to \infty} E_{P_{t_0,n}}[S_{t_0} M_{kn} I(M_{kn} > \frac{X}{S_{t_0}} a^{-1}[t_0, T])|S_{t_0} = s_{t_0}] = s_{t_0} \lim_{n \to \infty} P_{t_0,n}[M_{kn} > \frac{X}{s_{t_0}} a^{-1}[t_0, T]] \)

\(= s_{t_0} \Phi\left(\frac{\log(s_{t_0}/X) + \log a[t_0, T] + \frac{\sigma_{[t_0,T]}^2}{2}}{\sigma_{[t_0,T]}}\right)\),

(iii) \( \liminf_{n \to \infty} E_{P_{t_0,n}}[S_{t_0} M_{kn} I(M_{kn} > \frac{X}{S_{t_0}} a^{-1}[t_0, T])|S_{t_0} = s_{t_0}] \)

\(\geq s_{t_0} e^{\sigma_{[t_0,T]}^2} \Phi\left(\frac{\log(s_{t_0}/X) + \log a[t_0, T] + 1.5\sigma_{[t_0,T]}^2}{\sigma_{[t_0,T]}}\right)\).

Comparing (i) and (ii) it follows that the probability \( S_T \) is greater than \( X \) is larger for the buyer than for the trader. The same holds for non-calm stock from Theorem 4.1 (i), (ii) since \( \mu_{[t_0,T]} \) is less than zero.

**Corollary 5.2** For calm stock, under (P1) pricing and $A1, A2, A3(ii), A4, A5$:

(i) the trader’s price for the European option is

\[ C(t_0) = s_{t_0} \Phi(d_1) - X e^{-r(T-t_0)} \Phi(d_2) \]  

\[ = s_{t_0} \lim_{n \to \infty} P_{t_0,n}(S_T > X) - X e^{-r(T-t_0)} \lim_{n \to \infty} P_{t_0,n}(S_T > X), \]

with

\[ d_1 = \frac{\log(s_{t_0}/X) + \log a[t_0, T] + \frac{\sigma_{[t_0,T]}^2}{2}}{\sigma_{[t_0,T]}}, \quad d_2 = d_1 - \sigma_{[t_0,T]}, \]

(ii) a lower bound on the buyer’s price is

\[ s_{t_0} e^{\sigma_{[t_0,T]}^2} \Phi(\tilde{d}_1) - X e^{-r(T-t_0)} \Phi(\tilde{d}_2), \]

with

\[ \tilde{d}_1 = \frac{\log(s_{t_0}/X) + \log a[t_0, T] + 1.5\sigma_{[t_0,T]}^2}{\sigma_{[t_0,T]}}, \quad \tilde{d}_2 = \tilde{d}_1 - \sigma_{[t_0,T]}, \]
(iii) By translating either $Q_{t_0,n}$, $Q_{T,n}$ in [27] or the limit normal distributions by $r(T-t_0) - \log a_{[t_0,T]}$ or with a change of probability via Lemma 5.4 in (i) (resp. (ii)) $\log a_{[t_0,T]}$ is replaced by $r(T-t_0)$ in $d_1$ (resp. $\tilde{d}_1$), thus obtaining the B-S-M price (resp. a lower bound on the associated buyer’s price).

From (ii) it follows that buyer’s price includes a volatility premium.

**Remark 5.1** For stochastic interest rate $R$, with “expected accumulation” function $a_R[t_0,T]$ similar results can be obtained as in Corollaries [5.1] and [5.2]; in (i), (ii) with $a_R^{-1}[t_0,T]$ replacing $e^{-r(T-t_0)}$, and in (iii) by considering the equivalent structure to replace $\log a_{[t_0,T]}$ by $\log a_R[t_0,T]$.

### 5.4 Pricing a European option using ($A5^*$)

We replace $t_0$ in the previous sections by $t_1^n$ and use $P_{T,n}$ for the trader’s strategy and $P_{T,n}$ (abuse of notation) for the buyer’s strategy. $(\Omega, \mathcal{F}, \mathcal{P})$ is the probability space at $t_0$, $S_{t_0} = s_{t_0}$ a.s. $\mathcal{P}$. The asymptotic distributions previously obtained remain because of the asymptotic negligibility assumption A3(i) and Corollary 4.2. $ES_{t_0}$ is replaced by $s_{t_0}$. Instead of (25) and (26) we use

$$M_{k_n} = \frac{S_T}{S_{t_1^n}}^{-1}[t_1^n, T] = \frac{\prod_{j=2}^{k_n} R_{j-1}}{\prod_{j=2}^{k_n} R_{j-1}},$$

$$E_{Q_n} M_{k_n} S_{t_1^n} I(M_{k_n} > \frac{X}{S_{t_1^n}}^{-1}[t_1^n, T]) - X e^{-r(T-t_1^n)} E_{Q} I(M_{k_n} > \frac{X}{S_{t_1^n}}^{-1}[t_1^n, T]).$$

In this section $a_{t_0, T} = ES/T/s_{t_0}$ but $a_{t, T} = ES/T/ES_{t_1}, t > t_0$.

**Theorem 5.3** When $A1 - A5^*$ hold, the limits of the terms in (30), with $Q_n = P_{t_1^n, n}$, $P_{T,n}$, are:

(i) $\lim_{n \to \infty} P_{t_1^n, n}[M_{k_n} > \frac{X}{S_{t_1^n}}^{-1}[t_1^n, T]] = \int \Phi(\frac{\log(s_{t_0}/X) + \log a_{[t_0,T]} + y}{\sigma_{[t_0,T]}}) L_T^*(dy)$,

(ii) $\lim_{n \to \infty} E_{P_{t_1^n, n}}[S_{t_1^n} M_{k_n} I(M_{k_n} > \frac{X}{S_{t_1^n}}^{-1}[t_1^n, T])] = s_{t_0} \lim_{n \to \infty} P_{T,n}[M_{k_n} > \frac{X}{S_{t_1^n}}^{-1}[t_1^n, T]]$

$$= s_{t_0} \int \Phi(\frac{\log(s_{t_0}/X) + \log a_{[t_0,T]} - \mu_{[t_0,T]} + y}{\sigma_{[t_0,T]}}) L_T^*(dy),$$

(iii) $\lim \inf E_{P_{T,n}}[S_{t_1^n} M_{k_n} I(M_{k_n} > \frac{X}{S_{t_1^n}}^{-1}[t_1^n, T])]$
$$\geq s_0 e^{-\mu_{[t_0,T]} + \frac{5\sigma^2_{[t_0,T]}}{2}} \int e^y \Phi\left( \frac{\log(s_{t_0}a[t_0,T]/X) - \mu_{[t_0,T]} + \sigma^2_{[t_0,T]} + \frac{y}{\sigma_{[t_0,T]}}}{} \right) L_T^*(dy).$$

Let

$$Q_{t_1,n}(x) = P_{t_1,n}[\log \Pi_{j=2}^{k_n} \frac{p_{t_1}^{k_n}}{p_{t_1-1}^{k_n}} \leq x], \quad Q_{T,n}(x) = P_{T,n}[\log \Pi_{j=2}^{k_n} \frac{p_T^{k_n}}{p_{t_1-1}^{k_n}} \leq x],$$

be cumulative distributions on the real line. The option’s price is obtained below and in particular, by passing to an equivalent structure as in the previous section, a generalization of the B-S-M price similar to the price in Merton (1976).

**Corollary 5.3** Under (P2) pricing, when $A1 - A5*$ hold the obtained prices in Corollary 5.1 remain valid.

**Theorem 5.4** For calm stock, under assumptions $A1, A2, A3(ii), A4, A5*$:

(i) $\lim_{n \to \infty} P_{t_1,n}[\tilde{M}_{k_n} > \frac{X}{S_{t_1}^{n}} a^{-1}[t_1^n, T]] = \Phi\left( \frac{\log(s_{t_0}/X) + a[t_0,T] - \frac{\sigma^2_{[t_0,T]}}{2}}{} \right),$

(ii) $\lim_{n \to \infty} E_{P_{t_1,n}}[S_{t_1}^{n} \tilde{M}_{k_n} I(\tilde{M}_{k_n} > \frac{X}{S_{t_1}^{n}} a^{-1}[t_1^n, T])] = s_0 \lim_{n \to \infty} P_{T,n}[\tilde{M}_{k_n} > \frac{X}{S_{t_1}^{n}} a^{-1}[t_1^n, T]]$

$$= s_0 \Phi\left( \frac{\log(s_{t_0}/X) + a[t_0,T] + \frac{\sigma^2_{[t_0,T]}}{2}}{} \right),$$

(iii) $\liminf_{n \to \infty} E_{P_{T,n}}[S_{t_1}^{n} \tilde{M}_{k_n} I(\tilde{M}_{k_n} > \frac{X}{S_{t_1}^{n}} a^{-1}[t_1^n, T])]$

$$\geq s_0 e^{\sigma^2_{[t_0,T]} \frac{\log(s_{t_0}/X) + a[t_0,T] + 1.5\sigma^2_{[t_0,T]}}{}}.$$

**Corollary 5.4** For calm stock, under (P2) pricing and $A1, A2, A3(ii), A4, A5*$ the obtained prices in Corollary 5.2 remain valid.

**Remark 5.2** The results remain valid with stochastic interest rate $R$ as described in Remark 5.1.
6 Appendix

6.1 Proofs

Proof of Proposition 3.1 Independence of $\tilde{Y}_{n,j}, \tilde{U}_{n,j}, j = 1, \ldots, k_n$ under $P_{t_0,n}, P_T,n$

Define on $(\Omega^{k_n}, \mathcal{F}^{k_n})$ random variables

$$\tilde{Y}_{n,j}(\omega^{(k_n)}) = Y_{n,j}(\omega_j), j = 1, \ldots, k_n,$$

$$\omega^{(k_n)} = (\omega_1, \ldots, \omega_{k_n}).$$

Then, $\tilde{Y}_{n,j}, j = 1, \ldots, k_n$, are independent with respect to $P_{t_0,n}$ (defined in (9)), i.e.

$$P_{t_0,n}[\bigcap_{j=1}^{k_n}\{\tilde{Y}_{n,j} \in B_j\}] = \prod_{j=1}^{k_n} P_{t_0,n}[\tilde{Y}_{n,j} \in B_j].$$  (31)

Equality (31) follows from (9) and equality

$$\bigcap_{j=1}^{k_n}\{\omega^{(k_n)} : \tilde{Y}_{n,j}(\omega^{(k_n)}) \in B_j\} = \Omega \times \ldots \times \{\omega_j : Y_{n,j}(\omega_j) \in B_j\} \times \ldots \times \Omega, j = 1, \ldots, k_n.$$

One can confirm in the same way independence of $\tilde{Y}_{n,j}, j = 1, \ldots, k_n$, with respect to $P_T,n$. The same results hold for $\tilde{U}_{n,j}, j = 1, \ldots, k_n$ defined similarly on $(\Omega^{k_n}, \mathcal{F}^{k_n})$ from $U_{n,j}, j = 1, \ldots, k_n$. Since

$$(Y_{n,j} + 1)^2 = \frac{p^n_{t,j}}{p^n_{t,j-1}}, j = 1, \ldots, k_n,$$

independence, in the same sense, of the price-densities ratios follows.  \hfill \Box

Proposition 6.1 (i) For $Y_{n,j}, U_{n,j}, j = 1, \ldots, k_n$, in (10) and for $\epsilon$ small,

$$\sum_{j=1}^{k_n} E_{P_{t,j-1}} Y_{n,j}^2 I(|Y_{n,j}| > \frac{\epsilon}{1-2\epsilon}) \leq \sum_{j=1}^{k_n} E_{P_{t,j}} Y_{n,j}^2 I(|Y_{n,j}| > \frac{\epsilon}{1-\epsilon}) \leq \sum_{j=1}^{k_n} E_{P_{t,j-1}} Y_{n,j}^2 I(|Y_{n,j}| > \epsilon).$$  (32)

(ii) If $\sum_{j=1}^{k_n} E_{P_{t,j-1}} Y_{n,j}^2 < +\infty$, then

$$\lim_{\tau \to 0} \lim_{n \to \infty} \sum_{j=1}^{k_n} E_{P_{t,j-1}} Y_{n,j}^2 I(|Y_{n,j}| \leq \tau) = \lim_{\tau \to 0} \lim_{n \to \infty} \sum_{j=1}^{k_n} E_{P_{t,j}} Y_{n,j}^2 I(|U_{n,j}| \leq \tau).$$  (33)
Remark 6.1 Proposition 6.1 shows that if \( \{Y_{n,j}, j = 1, \ldots, k_n\} \) satisfy Lindelöf's condition the same holds for \( \{U_{n,j}, j = 1, \ldots, k_n\} \), and conversely.

Proof of Proposition 6.1
(i)
Note that for densities \( f \) and \( g \) it holds
\[
I(\sqrt{\frac{f}{g}} - 1 > \epsilon) = 1 \iff \{\sqrt{\frac{f}{g}} - 1 > \epsilon\} \text{ or } \{\sqrt{\frac{f}{g}} - 1 < -\epsilon\}.
\]
and that for small \( \epsilon \) it holds
\[
\{\sqrt{\frac{f}{g}} - 1 > \epsilon\} = \{\sqrt{\frac{g}{f}} < 1 - \frac{\epsilon}{1 + \epsilon}\} \cup \{\sqrt{\frac{g}{f}} > 1 + \frac{\epsilon}{1 - \epsilon}\} \supseteq \{\sqrt{\frac{f}{g}} - 1 > \epsilon\}.
\]

(ii) It follows from (i) since
\[
\sum_{j=1}^{k_n} E_{\tilde{P}_n} t_j^{n-1} Y_{n,j}^2 = \sum_{j=1}^{k_n} E_{\tilde{P}_n} U_{n,j}^2.
\]

Proof of Corollary 4.1
Follows from (33), the equalities \( E_{\tilde{P}_n} t_j^{n-1} Y_{n,j} = E_{\tilde{P}_n} U_{n,j}, j = 1, \ldots, k_n \), and the definition of \( L_{t_0} \) (in Theorem 4.1).

Proof of Theorem 4.2
Infinitely divisible distribution of \( \log \frac{S_{t_0}}{S_0} \)
From Theorem 4.1, the distribution of \( \Lambda_{t_0} \) (resp. \( \Lambda_T \)) is that of a sum of a normal random variable \( X \) and a Poissonian random variable \( Y \) that are independent. Parts (i) and (ii) follow by conditioning the limit distribution on the value \( Y = y \) of the Poissonian component, and then integrating with \( L_{t_0} \) and \( L_T \) respectively.

Proof of Theorem 4.3
The martingale property of prices-densities
It is enough to show that
\[
E_{\tilde{P}_{t_0,n}}[p_{n,k}^{t_j} | p_{t_j}, p_{t_j-1}, \ldots, p_{t_1}, p_{t_0}] = p_{t_j}.
\]
Observe that
\[
E_{\tilde{P}_{t_0,n}} p_{t_j} p_{n,j-1} = \int p_{t_j} p_{n,j-1} dP = \int \frac{S_{t_j}}{E S_{t_j}} dP = 1, j = 1, \ldots, k_n.
\]

From the independence of prices-densities ratios and (34)
\[
E_{\tilde{P}_{t_0,n}}[p_{n,k}^{t_j} | p_{t_j}, p_{t_j-1}, \ldots, p_{t_1}, p_{t_0}] = E_{\tilde{P}_{t_0,n}}[\prod_{m=1}^{k} \frac{p_{n,m}^{t_j}}{p_{n,m-1}^{t_j}} | p_{t_j}, p_{t_j-1}, \ldots, p_{t_1}, p_{t_0}]
\]
\[
= \prod_{m=1}^{k} E_{\tilde{P}_{t_0,n}} \frac{p_{n,m}^{t_j}}{p_{n,m-1}^{t_j}} = 1, j = 1, \ldots, k_n.
\]

To prove Theorem 4.4, the asymptotic distribution of $\Lambda_{k_n}$ (see (12)) is approximated by the asymptotic distribution of $W_{k_n} = \sum_{j=1}^{k_n} Y_{n,j} = \sum_{j=1}^{k_n} Y_{n,j}$ via next lemma that appears in Le Cam and Yang (2000); see section 5.2.7, p.92-94, Proposition 8, section 5.3 p. 95, conditions (B) and (N) and p. 108, lines -10 to -7.

**Lemma 6.1** Assume that under $P_{t_0,n}$

$$\sup\{|Y_{n,j}|, j = 1, \ldots, k_n\} \xrightarrow{n \to \infty} 0 \quad \text{in probability.}$$

When $A_3$ holds, under $P_{t_0,n}$

$$\Lambda_{k_n} - 2W_{k_n} + \sum_{j=1}^{k_n} Y_{n,j}^2 \xrightarrow{n \to \infty} 0 \quad \text{in probability.}$$

**Lemma 6.2** (see, for example, Le Cam and Yang, 2000, Proposition 1, p. 40 and p. 41, lines 14-19) For the binary experiments $\mathcal{E}_n = \{P_{0,n}, P_{1,n}\}$, assume that $\{P_{1,n}\}$ is contiguous to $\{P_{0,n}\}$ and that the asymptotic distribution of $\log M_n$ under $P_{0,n}$ is normal with mean $\mu$ and variance $\sigma^2$; $M_n = \frac{dP_{1,n}}{dP_{0,n}}$ is the density of the part of $P_{1,n}$ dominated by $P_{0,n}$. Then

(i) $\mu = -.5\sigma^2$, and
(ii) the asymptotic distribution of $\log M_n$ under $P_{1,n}$ is normal with mean $.5\sigma^2$ and variance $\sigma^2$.

**Lemma 6.3** Under $A_3$, let $F_n$ be the distribution of $W_{k_n} = \sum_{j=1}^{k_n} Y_{n,j}$. Assume that $F_n$ converges weakly to a probability distribution $F$ with Lévy representation $(\mu, \sigma^2, L)$; $\mu$ is the “shift”, $\sigma^2$ is the variance of the normal component and $L$ is the Lévy measure. Then,

a) the sequence $\{P_{T,n}\}$ is contiguous to the sequence $\{P_{t_0,n}\}$ if and only if the variance of $F$ is the limit of the variances of $F_n$.

b) the sequence $\{P_{t_0,n}\}$ is contiguous to the sequence $\{P_{T,n}\}$ if and only if $L(-1) = 0$.

**Proof.** It follows from the Corollary in Le Cam and Yang (1990, p. 47) and the Remark, p.48, lines 3-12, since $P_{t_{j-1}^0}$, $P_{t_j^0}$, $j = 1, \ldots, k_n$, do not have singular parts. \(\square\)

**Proof of Theorem 4.4** Normal distribution for $\log \frac{S_T}{S_{t_0}}$

\footnote{Abuse of notation since $Y_{n,j} = \tilde{Y}_{n,j}$, $j = 1, \ldots, k_n$.}
Under $P_{t_0,n}$ (defined in (9)),
\[ E_{P_{t_0,n}} \tilde{Y}_{n,j} = E_{P_{t_{j-1}^n}} Y_{n,j} = \int (\sqrt{p_{t_{j-1}^n}} - 1) \, dP = -h^2(P_{t_{j-1}^n}, P_{t_{j-1}^n}) = -h^2_{n,j}. \]
(36)

The equivalence of $P$, $P_{t_j^n}$, $j = 0, \ldots, k_n$, implies that
\[ E_{P_{t_0,n}} \tilde{Y}_{n,j}^2 = E_{P_{t_{j-1}^n}} Y_{n,j}^2 = \int \left( \sqrt{P_{t_{j-1}^n}} - 1 \right) \, dP = 2h^2_{n,j}, \]
(37)

\[ Var_{P_{t_0,n}} (\tilde{Y}_{n,j}) = Var_{P_{t_{j-1}^n}} (Y_{n,j}) = 2h^2_{n,j}, \]
(38)

For calm stock, from (20) it follows that
\[ P_{t_0,n} \left[ \bigcup_{j=1}^{k_n} \{ |\tilde{Y}_{n,j}| > \epsilon \} \right] \leq \sum_{j=1}^{k_n} P_{t_{j-1}^n} \left[ |Y_{n,j}| > \epsilon \right] \]
\[ \leq \frac{1}{\epsilon^2} \sum_{j=1}^{k_n} E_{P_{t_{j-1}^n}} Y_{n,j}^2 I(|Y_{n,j}| > \epsilon) \rightarrow 0, \quad \text{and} \]
\[ \sup_{1 \leq j \leq k_n} \{ |\tilde{Y}_{n,j}|, j = 1, \ldots, k_n \}_n \rightarrow 0 \]
(39)

in $P_{t_0,n}$-probability. Thus, Lemma 6.1 can be used to approximate the asymptotic distribution of $\Lambda_{k_n}$ (see (12)) with that of $W_{k_n} = \sum_{j=1}^{k_n} Y_{n,j} = \sum_{j=1}^{k_n} \tilde{Y}_{n,j}$.

Assumption A3(i) also holds for calm stocks since from (37)
\[ 2h^2_{n,j} = E_{P_{t_{j-1}^n}} Y_{n,j}^2 \leq \epsilon^2 + E_{P_{t_{j-1}^n}} Y_{n,j}^2 I(|Y_{n,j}| > \epsilon) \leq \epsilon^2 + \sum_{j=1}^{k_n} E_{P_{t_{j-1}^n}} Y_{n,j}^2 I(|Y_{n,j}| > \epsilon). \]

It follows from A3 that
\[ \sup_{1 \leq j \leq k_n} E_{P_{t_{j-1}^n}} (Y_{n,j}^2) = \sup_{1 \leq j \leq k_n} h^2_{n,j} \rightarrow 0, \]
and
\[ \sum_{j=1}^{k_n} [E_{P_{t_{j-1}^n}} Y_{n,j}]^2 = \sum_{j=1}^{k_n} h^4_{n,j} \leq \sup_{1 \leq j \leq k_n} h^2_{n,j} \sum_{j=1}^{k_n} h^2_{n,j} \rightarrow 0. \]
(40)

Since from (40) and A3(i) the truncated expectation of $Y_{n,j}$ converges to 0 as $n \rightarrow \infty$, $j = 1, \ldots, k_n$, (see, for example, Le Cam and Yang, 1990, Lemma 1, p. 34), and from A3(ii) the sum of the variances is bounded, for a calm stock
the $Y_{n,j}$’s satisfy Lindeberg’s condition and $W_n = \sum_{j=1}^{k_n} Y_{n,j}$ has distribution $F_n$ that converges weakly to a normal distribution $F$ with mean $\mu$ and variance $\sigma^2$ (see Le Cam and Yang, 1990, p.44)

$$
\mu = - \lim_{n \to \infty} \sum_{j=1}^{k_n} h_{n,j}^2, \quad \sigma^2 = \lim_{\tau \to 0} \lim_{n \to \infty} \sum_{j=1}^{k_n} E_{P_{t_{j-1}}} Y_{n,j}^2 I(|Y_{n,j}| \leq \tau).
$$

From Lemma 6.1 since

$$
\sum_{j=1}^{k_n} Y_{n,j}^2 - \sum_{j=1}^{k_n} E_{P_{t_{j-1}}} Y_{n,j}^2 \to 0
$$
in probability, it follows that the asymptotic distribution of $\Lambda_n$ is normal $N(\mu_{[t_0,T]}, \sigma^2_{[t_0,T]})$, with

$$
\mu_{[t_0,T]} = 2\mu - \sigma^2, \quad \sigma^2_{[t_0,T]} = 4\sigma^2.
$$

Asymptotic normality of $W_n$ for calm stocks implies that the limit of the variance of $F_n$ is equal to the variance of $F$ (see Le Cam and Yang, 1990, p. 49, lines -20 to -6). It follows from Lemma 6.3 a) that $\{P_{T,n}\}$ is contiguous to $\{P_{t_{0,n}}\}$, and from Lemma 6.2 (i) for the mean $\mu_{[t_0,T]}$ and the variance $\sigma^2_{[t_0,T]}$ of the asymptotic distribution of $\Lambda_n$ it holds $\mu_{[t_0,T]} = -\frac{\sigma^2_{[t_0,T]}}{2}$. Thus, from Lemma 6.3 b), $\{P_{t_{0,n}}\}$ and $\{P_{T,n}\}$ are contiguous. Since

$$
P_{T,n}[\log \frac{\Pi_{j=1}^{k_n} P_{t_{j-1}}^{n}}{\Pi_{j=1}^{k_n} P_{t_{j-1}}^{n}} \leq x] = P_{T,n}[- \log \frac{\Pi_{j=1}^{k_n} P_{t_{j-1}}^{n}}{\Pi_{j=1}^{k_n} P_{t_{j-1}}^{n}} \leq x],
$$

(22) follows from (ii) and Proposition 6.1, or simply from Lemma 6.2 (ii).

**Lemma 6.4** If a random variable $W$ has under $P$ normal distribution with mean $M$ and variance $\Sigma^2$ and

$$
\frac{dP^*}{dP} = e^{AW+C}, \quad -C = MA + \frac{\Sigma^2 A^2}{2},
$$

the distribution of $W$ under $P^*$ is normal with mean $M^*$ and variance $\Sigma^2$,

$$
M^* = M + A\Sigma^2, \quad \Sigma^2 = \Sigma^2.
$$
Proof. Follows from the moment generating function $E_{P^*} e^{\rho W}$. \hfill \Box

Proof of Theorem 5.1. Part (i) follows by taking logarithms in both sides of the inequality inside the probability, and using Theorem 4.2 (ii) and A5. For (ii) and (iii), calculation of expectations is replaced by calculations of probabilities, after change of the underlying probabilities in the expectations. (ii) From A5, mutual absolute continuity of $P_{t_0,n}$ and $P_{T,n}$ and changing probabilities we obtain

$$E_{P_{t_0,n}}[S_{t_0} M_{k_n} I(M_{k_n} > \frac{X}{S_{t_0}} a^{-1}[t_0, T]) | S_{t_0} = s_{t_0}]$$

$$= s_{t_0} E_{P_{t_0,n}}[\prod_{j=1}^{k_n} \frac{p_{t_0}^{j_n}}{p_{t_0}^{j-1}} I(\prod_{j=1}^{k_n} \frac{p_{t_0}^{j_n}}{p_{t_0}^{j-1}} > \frac{X a^{-1}[t_0, T]}{s_{t_0}}) | S_{t_0} = s_{t_0}]$$

$$= s_{t_0} E_{P_{t_0,n}}[\prod_{j=1}^{k_n} \frac{p_{t_0}^{j_n}}{p_{t_0}^{j-1}} I(\prod_{j=1}^{k_n} \frac{p_{t_0}^{j_n}}{p_{t_0}^{j-1}} > \frac{X a^{-1}[t_0, T]}{s_{t_0}})]$$

The result follows from Theorem 4.2 (ii) by conditioning the limit distribution on the value $Y = y$ of the Poissonian component.

(iii) From A5,

$$E_{P_{T,n}}[S_{t_0} M_{k_n} I(M_{k_n} > \frac{X}{S_{t_0}} a^{-1}[t_0, T]) | S_{t_0} = s_{t_0}]$$

$$= s_{t_0} E_{P_{T,n}}[\prod_{j=1}^{k_n} \frac{p_{t_0}^{j_n}}{p_{t_0}^{j-1}} I(\prod_{j=1}^{k_n} \frac{p_{t_0}^{j_n}}{p_{t_0}^{j-1}} > \frac{X a^{-1}[t_0, T]}{s_{t_0}}) | S_{t_0} = s_{t_0}]$$

$$= s_{t_0} E_{P_{T,n}}[\prod_{j=1}^{k_n} \frac{p_{t_0}^{j_n}}{p_{t_0}^{j-1}} I(\prod_{j=1}^{k_n} \frac{p_{t_0}^{j_n}}{p_{t_0}^{j-1}} > \frac{X a^{-1}[t_0, T]}{s_{t_0}})]$$

Using Skorohod’s theorem, Fatou’s Lemma and Theorem 4.2 (ii),

$$\liminf E_{P_{T,n}}[\prod_{j=1}^{k_n} \frac{p_{t_0}^{j_n}}{p_{t_0}^{j-1}} I(\prod_{j=1}^{k_n} \frac{p_{t_0}^{j_n}}{p_{t_0}^{j-1}} > \frac{X a^{-1}[t_0, T]}{s_{t_0}})]$$

$$\geq \int \int e^{x+y} I(x > \log \frac{X a^{-1}[t_0, T]}{s_{t_0}} - y) d\Phi(\frac{x + \mu[t_0, T]}{\sigma[t_0, T]}) L^r(dy).$$

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Let \( k = \log \frac{X_{a^{-1}[0,T]}}{s_{t_0}} - y \). For the integral with respect to \( x \) use (43) and (44) with 
\[
M = -\mu_{[t_0,T]}, \quad \Sigma^2 = \sigma_{[t_0,T]}^2, \quad \Sigma^* = \sigma_{[t_0,T]}^2;
\]

\[
\int_{k}^{\infty} e^x d\phi \left( \frac{x + \mu_{[t_0,T]}}{\sigma_{[t_0,T]}} \right) = e^{-\mu_{[t_0,T]} + \sigma_{[t_0,T]}^2} \int_{k}^{\infty} d\phi \left( \frac{x + \mu_{[t_0,T]} - \sigma_{[t_0,T]}^2}{\sigma_{[t_0,T]}} \right) \]

\[
= e^{-\mu_{[t_0,T]} + \sigma_{[t_0,T]}^2} \phi \left( \frac{-k - \mu_{[t_0,T]} + \sigma_{[t_0,T]}^2}{\sigma_{[t_0,T]}} \right). \]

Proof of Corollary 5.1 Follows from Theorem 5.1 and (25) for \( Q_n = P_{t_0,n}, \; P_{T,n} \). In particular, a translation conditional on the value of the compound Poisson component in the distribution of \( \log S_T \) allows to obtain (iii). \( \square \)

Proof of Theorem 5.2 Follows the line of proof of Theorem 5.1 using Theorem 4.4 instead of Theorem 4.2. \( \square \)

Proof of Corollary 5.2 Follows from Theorem 5.2 and (25) for \( Q_n = P_{t_0,n}, \; P_{T,n} \). \( \square \)

Proofs of Theorems and Corollaries in section 5.4: Follow from those for section 5.3. \( \square \)

6.2 Conditions for \( A_3 \) and \( \sigma_{[t_0,T]}^2 = \sigma^2(T - t_0) \) to hold

Differentiability conditions are provided below for \( A_3 \) to hold. These conditions hold often in parametric statistical models. Let \((\Omega, \mathcal{F}, P)\) be a probability space and let \( \rho(t), t \in [0, T] \), be a process indexed by \( t \).

Definition 6.1 The process \( \rho \) is differentiable at \( \theta \) in \( P \)-quadratic mean if there is \( U_\theta \), its derivative at \( \theta \), such that

\[
\frac{1}{\delta^2} \int [\rho(\theta + \delta) - \rho(\theta) - \delta U_\theta]^2 dP \to 0. \quad (45)
\]

When \( \theta = 0 \) (resp. \( T \)) the limit in (45) is taken for \( \delta \) positive (resp. negative).

For the prices-densities \( \{p_t, \; t \in [0, T]\} \), let

\[
\xi(t) = \sqrt{p_t}, \; t \in [0, T]. \quad (46)
\]

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Then, for the square Hellinger distance $H^2(P_t, P_\theta)$ of $P_t$ and $P_\theta$ it holds
\[
H^2(P_t, P_\theta) = .5 \int (\sqrt{P_t} - \sqrt{P_\theta})^2 dP = .5 \int \left( \frac{\xi(t)}{\xi(\theta)} - 1 \right)^2 dP_\theta.
\] (47)

Conditions for $\xi(t)$ to be quadratic mean differentiable and examples of quadratic mean differentiable densities can be found in Le Cam (1970) and Roussas (1972, Chapter 2).

**Proposition 6.2** Assume that $\xi(t)$ is $P$-quadratic mean differentiable in $[t_0, T]$ with derivative $U_\xi$, and that $\sup_{t \in [t_0, T]} E_P U_\xi^2 < \infty$. Then, $A3$ holds for the densities $p_t$, $t \in [0, T]$.

**Proof** For any $\theta$ in $[t_0, T]$ and $\delta$ small,
\[
2h^2(P_{\theta + \delta}, P_\theta) = \int \left( \frac{\xi(\theta + \delta)}{\xi(\theta)} - 1 \right)^2 dP_\theta = \int \left( \frac{\xi(\theta + \delta)}{\xi(\theta)} - 1 - \frac{\delta U_\theta}{2\xi(\theta)} \right)^2 dP_\theta
\]
\[+ \delta^2 \int \left( \frac{U_\theta}{\xi(\theta)} \right)^2 dP_\theta + \delta^2 o(1) = \delta^2 [E_P(V_\theta(\xi(\theta)))^2 + o(1)] = \delta^2 [E_P U_\theta^2 + o(1)].
\]
Thus, uniform boundedness of $E_P U_\theta^2$ implies $A3(i)$ holds.

For transaction times with small mesh size in $[t_0, T]$,
\[
2 \sum_{j=1}^{k_n} h^2(P_{t^n_j}, P_{t^n_{j-1}}) = \sum_{j=1}^{k_n} (t^n_j - t^n_{j-1})^2 E_P U_{t^n_{j-1}}^2 + o(1) \sum_{j=1}^{k_n} (t^n_j - t^n_{j-1})^2
\]
\[\leq (T - t_0)^2 \sup_{t \in [t_0, T]} E_P U_t^2 + o(1) < \infty.
\]
\[\blacksquare\]

**Proposition 6.3** Under the assumptions of Proposition 6.2, $\sigma^2_{[t_0, T]}$ depends on the spacings of the transaction times and the quadratic mean derivatives of the process $\xi$:

a) \[
\sigma^2_{[t_0, T]} = 4 \lim_{n \to \infty} \lim_{\tau \to 0} \sum_{j=1}^{k_n} (t^n_j - t^n_{j-1})^2 E_P U_{t^n_{j-1}}^2 I(|Y_{n,j}| \leq \tau), \quad \text{and}
\]
b) If $t^n_j - t^n_{j-1} = \delta t^n_{j-1}$, $\delta > 0$, and $E_P U_{t^n_{j-1}}^2 I(|Y_{n,j}| \leq \tau) = \frac{c_{\tau}}{t^n_{j-1}}$, $c_{\tau} > 0$,
\[
\sigma^2_{[t_0, T]} = 4 \delta (T - t_0) \lim_{\tau \to 0} c_{\tau}.
\]

**Proof** From the proof of Proposition 6.2
\[
E_{P_{t^n_{j-1}}} Y_{n,j}^2 I(|Y_{n,j}| \leq \tau) = (t^n_j - t^n_{j-1})^2 [E_P U_{t^n_{j-1}}^2 I(|Y_{n,j}| \leq \tau) + o(1)].
\]
a) and b) follows from 23.\[\blacksquare\]
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