Inverse Co-even Domination of Graphs

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Abstract

The purpose of this paper is to introduce a new inverse domination parameter in the graphs it is called inverse co-even domination number. Some properties of the theory to this definition were only touched. Also, many properties and limitations on this definition are determined. Additionally, some properties of inverse co-even domination number for some certain graphs and its complement are founded, such as regular, complete, path, cycle, wheel, complete bipartite, and star.

Mathematical subject classification: 05C69

Keywords: Domination number, Co-even domination number, Inverse co-even domination number.

1. Introduction

In this work, all graphs are undirected, simple, and finite. In a graph $G = (V(G), E(G))$ with vertex set $V(G)$ and edge set $E(G)$. The number of edges that incident to the vertex $v$ is constituted the degree of $v$, and denoted by $deg(v)$. There are special kinds of vertices depend on its degree as an example an isolated vertex if $d(v) = 0$ and a pendent vertex if $d(v) = 1$. The minimum and maximum degree of $G$ denoted by $\delta(G)$ and $\Delta(G)$, respectively. In case $\Delta(G) = \delta(G)$, $G$ is called a regular graph [1], [2]. The concept of domination in graphs has led to solving many problems in the great a huge range of various fields in graph theory as topological graph [3] and labeled graph [4,5] and others.

Now, if there is $D \subseteq V$ such that $N[D] \equiv V$, then this set is called a dominating. Furthermore, if $D$ has no proper subset say $F$ such that $F$ is a dominating, then $D$ is called minimal. $MDS(G)$ refers to the set of all minimal dominating of a graph $G$. The minimum cardinality all member in the set $MDS(G)$ is called the domination number of $G$ and denoted by $\gamma(G)$ [6]. After this, there are many different definitions of parameter domination of graphs are introduced as in [7-9].
Furthermore, the notion of the inverse domination number is discussed. Kulli and Sgarkanti [10] introduced this concept and finding it by this way if \( D \) belongs to \( MDS(G) \) and there is another dominating set disjoint from the set \( D \) say \( H \) then \( H \) is called the inverse dominating respect to \( D \). Again, the minimum cardinality of all these is called inverse domination number.

In recent years many researchers worked of this notion, readers may be referred to [11-14]

Now, a new domination parameter in the graphs it is called co-even domination number denoted by \( \gamma_{coe}(G) \) and it is inverse (ICED) denoted by \( \gamma_{coe}^{-1}(G) \) are defined below. Some properties for some certain graphs of inverse co-even domination number are been discussed. Also, some bounds for \( \gamma_{coe}^{-1}(G) \) are obtained and characterized the graphs obtaining those bounds.

**Definition 2.1**[9]. Let \( G = (V, E) \) be a graph and \( D \subseteq V \) such that the set \( D \) is dominating. The set \( D \) is called co-even dominating (CEDS) if each vertex \( v \in V - D \) has an even degree. The co-even domination number is the cardinality of minimum CEDSD and denoted by \( \gamma_{coe}(G) \) and the set \( D \) is called \( \gamma_{coe}^{-1}(G) \) set.

**Definition 2.2.** Consider \( D \) be \( \gamma_{coe}^{-1}(G) \) set of a graph \( G \). If \( V - D \) contains a set \( F \) belong to CEDS(G), then \( F \) is called inverse co-even dominating set (ICEDS) of \( G \) with respect to the set \( D \) and is denoted \( D^{-1} \). The minimum cardinality of these sets is called inverse co-even domination number and is denoted by \( \gamma_{coe}^{-1}(G) \).

**Proposition 2.3.**[9]. Assume that \( G = (n, m) \) be a graph and \( D \) is a CEDS, then

1. All vertices of odd or zero degrees belong to every co-even dominating set.
2. \( \deg(v) \geq 2 \), for all \( v \in V - D \)
3. If \( G \) is \( r \)-regular graph then \( \gamma_{coe}(G) = \left\{ \begin{array}{ll} n, & \text{if } r \text{ is odd} \\ \gamma(G), & \text{if } r \text{ is even} \end{array} \right. \)
4. \( \gamma(G) \leq \gamma_{coe}(G) \).

**Theorem 2.4.** If \( G = (n, m) \) is a graph has \( \gamma_{coe}(G) \) and \( \gamma_{coe}^{-1}(G) \), then

\[
2n - 2\gamma_{coe}(G) - \gamma_{coe}^{-1}(G) \leq m \leq \frac{n(n-1)}{2}.
\]

**proof.** Firstly, to prove \( m \geq 2n - 2\gamma_{coe}(G) - \gamma_{coe}^{-1}(G) \). Since \( G \) has co-even domination number \( \gamma_{coe}(G) \), then there is a \( D \) is \( \gamma_{coe}^{-1}(G) \) set. Thus, each vertex in the set \( V - D \) is adjacent to at least one vertex in \( D \). Therefore, there are at least \( n - \gamma_{coe}(G) \) edges. Now, there are two subcases that depend on the set \( V - D - D^{-1} \) as follows.

**Subcase 1.** If \( V - D - D^{-1} = \emptyset \), then there are at least \( (n - \gamma_{coe}(G))/2 \) edge joining every two vertices pairwise in \( V - D \) to keep that every vertex in \( V - D \) has even degree. Similarly, there
are at least \((n - \gamma_{\text{coe}}^{-1}(G))/2\). Therefore, the minimum edges in this case is \(n - \gamma_{\text{coe}}(G) + (n - \gamma_{\text{coe}}^{-1}(G))/2 + (n - \gamma_{\text{coe}}^{-1}(G))/2 = 2n - \gamma_{\text{coe}}(G) - (\gamma_{\text{coe}}(G) + \gamma_{\text{coe}}^{-1}(G))/2\).

**Subcase 2.** If \(V - D - D^{-1} \neq \emptyset\), then there is a set \(T = V - D - D^{-1}\). Since the set \(D^{-1}\) is co-even dominating, there is at least one edge join each vertex in \(T\) with some vertices in \(D^{-1}\). Therefore, the minimum edges in this case is \(2n - 2\gamma_{\text{coe}}(G) - \gamma_{\text{coe}}^{-1}(G) < 2n - \gamma_{\text{coe}}(G) - (\gamma_{\text{coe}}(G) + \gamma_{\text{coe}}^{-1}(G))/2\). Thus,

\[
m \geq 2n - 2\gamma_{\text{coe}}(G) - \gamma_{\text{coe}}^{-1}(G).
\]

Secondly, it is easy to see that \(m \leq \frac{n(n-1)}{2}\) when the graph \(G\) is complete. Therefore, the proof is done. □

**Proposition 2.5.** Assume that \(G = (n, m)\) be a graph and \(D\) is a CEDS, then

1) If \(G\) has a vertex of degree zero or odd, then \(G\) has no inverse co-even dominating set.

2) If \(G\) is a null, path, star, or wheel then \(G\) has no inverse ICEDS if \(r\) is odd. Otherwise, \(G\) has an ICEDS.

3) If \(G\) has an inverse co-even dominating set \(D^{-1}\), then \(\deg(v) \geq 2\) for all \(v \in V\).

4) \(\gamma_{\text{coe}}(G) \leq \gamma_{\text{coe}}^{-1}(G)\).

5) \(1 \leq \gamma_{\text{coe}}^{-1}(G) \leq n - \gamma_{\text{coe}}(G)\).

**Proof.** 1) From Proposition 2.3.(1), all vertices of odd or zero degrees belong to every co-even dominating set. Thus, there is no ICEDS, since if there is a dominating set \(F\) in \(V - D\), then all vertices of odd or zero degrees lie outside of \(F\). Therefore, \(F\) is not co-even dominating set, that means \(F\) is not inverse co-even dominating set.

2) If \(G\) is \(r - \text{regular}\) and \(r\) is odd, then there is a unique co-even dominating set contains all vertices in \(G\). Thus, there is no inverse co-dominating set. Otherwise, \(G\) has an inverse co-even dominating set.

3) It is clear that \(\deg(v) \geq 2\) for all \(v \in V - D\), Proposition 2.3.(2). Now, since \(G\) has an inverse co-even dominating set \(D^{-1}\), then \(\deg(v) \geq 2\), for all \(v \in V - D^{-1}\). Therefore, the result is obtained.

4) and 5) The proof is straightforward. □

**Proposition 2.6.** Assume that \(G = (n, m)\) be a graph and \(D\) is a CEDS, then

1) If \(G\) be a null, path, star, or wheel then \(G\) has no inverse co-dominating set.

2) \(\gamma_{\text{coe}}^{-1}(C_n) = \left\lfloor \frac{n}{3} \right\rfloor\), where \(C_n\) is a cycle of order \(n\).
3) \( \gamma_{\text{coe}}^{-1}(K_n) = 1 \), if \( n \equiv 1 \pmod{2} \). Otherwise \( K_n \) has no ICEDS where \( K_n \) is complete of order \( n \); \( n \geq 3 \).

4) \( \gamma_{\text{coe}}^{-1}(K_{m,n}) = 2 \), if \( m \) and \( n \) are even. Otherwise \( K_{m,n} \) has no ICEDS where \( K_{m,n} \) is complete bipartite.

**Proof.** 1) If \( G \) is null, path, or star, then \( G \) has a vertex of degree one or zero, then \( G \) has no ICEDS according to Proposition 2.3(1). Now, if \( G \) is a star, then \( D \) contains all vertices in \( G \) or all vertices except the center vertex. In both cases \( G \) has no inverse co-even dominating set.

2) Since \( C_n \) is 2-regular graph, then \( \gamma_{\text{coe}}^{-1}(C_n) = \gamma_{\text{coe}}^{-1}(C_n) = \left\lceil \frac{n}{2} \right\rceil \).

3) If \( n \equiv 1 \pmod{2} \), then all vertices of even degree. Thus, it is obvious that \( \gamma_{\text{coe}}^{-1}(K_n) = 1 \).

4) According to Proposition 2.3.(2), \( \deg(C) \geq 2 \), for all \( C \in G \). This case occurs when \( m \) and \( n \) are even. Thus, we can take one vertex from each sets \( V_1 \) and \( V_2 \) different from the tow vertices in \( D \) to dominate all vertices where \( V_1 \) and \( V_2 \) are the partite set in \( G \). Therefore, the result is obtained.

\( \square \)

**Proposition 2.7.** Assume that \( G \) be a graph has \( \gamma_{\text{coe}}^{-1}(G) \) and let \( D \) be a \( \gamma_{\text{coe}}^{-1}\)-set. If for each vertex \( u \in D \), such that \( \langle N[u] \rangle \) is a complete subgraph induced by \( N[u] \) of order at least three, then \( \gamma_{\text{coe}}^{-1}(G) = \gamma_{\text{coe}}^{-1}(G) = \left\lceil \frac{n}{3} \right\rceil \).

**Proof.** Let \( u_1 \in D \), then by the hypothesis \( \langle N[u_1] \rangle \) is a complete of order at least three. Now, to prove that each vertex says \( u_1 \) adjacent to \( u_1 \) must belong to \( V - D \). Suppose that \( u_i \) belongs to the set \( D \), then there are two different cases.

**Case 1.** If \( u_i \) adjacent to only vertices in \( N[u_1] \), this leads to a contradiction with the minimum of \( D \).

**Case 2.** If \( u_i \) adjacent to a vertex or more in \( V - D \), then the subgraph induced by set \( N[u_1] \) union with the set of these vertices must be complete subgraph, since \( \langle N[u_1] \rangle \) is complete. Again, this leads to a contradiction with the minimum of \( D \).

Thus, from two cases above, \( \langle D \rangle \) is independent, and all vertices which dominated by a vertex in \( D \) must belong to \( V - D \). Thus, the induced subgraph of these vertices with a vertex in \( D \) constitutes a complete subgraph. Therefore, each vertex in \( V - D \) has an even degree, so it is clear that \( \gamma_{\text{coe}}^{-1}(G) = \gamma_{\text{coe}}(G) \). \( \square \)

**Proposition 2.8.** Assume that \( D \) be any \( \gamma_{\text{coe}}^{-1}\)-set and \( V - D \) is independent with condition every vertex of graph \( G \) has even degree, then \( \gamma_{\text{coe}}^{-1}(G) + \gamma_{\text{co}}(G) = n \).

**Proof.** Let \( D \) be any \( \gamma_{\text{coe}}^{-1}\)-set and \( V - D \) is an independent set, then \( V - D \) is MCEDS, since it has no proper CEDS and for each vertex belong to \( V - (V - D) \equiv D \) has even degree. Thus, the proof is done. \( \square \)

**Proposition 2.9.** Assume that \( G \) be a \((n, m)\)-graph with \( \gamma_{\text{coe}}^{-1}(G) = \gamma_{\text{co}}(G) \), then \( m \geq 2n - 3\gamma_{\text{coe}}(G) \).
Proof. Assume that $D$ and $D^{-1}$ be minimum CEDS and ICEDS of $G$ respectively. Then each vertex in the set $(V - D - D^{-1})$ has even degree by definition of minimum co-even dominating set. Thus, the minimum number of vertices that adjacent to a vertex in the set $(V - D - D^{-1})$ is two, one of them is adjacent to a vertex in $D$ and another is adjacent to a vertex in $D^{-1}$. Furthermore, the minimum number of edges between the two sets occurs where each vertex of them adjacent to only one vertex of the other. Thus, $m \geq |V - D - D^{-1}| + |D| = 2(n - 2y_{coe}(G)) + y_{coe}(G) = 2n - 3y_{coe}(G)$. □

3. Inverse Co-even dominating set in the complement of a graph

Proposition 3.1. Assume that $G$ be a graph, then

1) $y_{coe}(C_n) = y_{coe}(\overline{C_n}) = 2$, if $n$ is odd; $n \neq 3$, Otherwise has no inverse co-even dominating set

2) $y_{coe}(K_{m,n}) = y_{coe}(\overline{K_{m,n}}) = 2$, if $m$ and $n$ are odd, Otherwise has no inverse co-even dominating set.

Proof. 1) If $n$ is odd; $n \neq 3$, then the degree of all vertices in $\overline{C_n}$ are even. Also, any adjacent vertices in $C_n$ have dominated all vertices in $\overline{C_n}$. Therefore, $y_{coe}(\overline{C_n}) = y_{coe}(\overline{C_n}) = 2$. Otherwise, each vertex in $\overline{C_n}$ has odd degree, then according to proposition 2.5(1), $G$ has no inverse co-even dominating set.

2) It is obvious that $\overline{K_{m,n}} = K_m \cup K_n$, so if $m$ and $n$ are odd in $G$, then each vertex of $K_m$ or $K_n$, has even degree. Therefore, we can take two vertices one of them from $K_m$ and the other from $K_n$ such that these vertices different from the two vertices which chooses in the minimum dominating set $D$ that taken $y_{coe}(\overline{K_{m,n}})$. Thus, in this case $y_{coe}(\overline{K_{m,n}}) = 2$. Otherwise the graph $\overline{K_{m,n}}$ has no inverse co-even dominating according to proposition 2.5(1). □

Observation 3.2. A graph $G$ has no ICEDS, if $G \cong \overline{P_n}$, $\overline{S_n}$, $\overline{W_n}$, $\overline{K_n}$.

Proof. It is obvious from proposition 2.5(1). □

4. Conclusion

Throughout, this a new parameter of domination is called the inverse co-even domination number is introduced. Some of the results of this number are obtained for certain graphs as a path, cycle, star, wheel, regular, complete, and complete bipartite and it is complement of each of them.

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