BAXTER OPERATORS AND ASYMPTOTIC REPRESENTATIONS

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Abstract. We introduce a category \( \mathcal{O} \) of representations of the elliptic quantum group associated with \( \mathfrak{sl}_2 \) with well-behaved \( q \)-character theory. We derive separation of variables relations for asymptotic representations in the Grothendieck ring of this category. Baxter \( Q \)-operators are obtained as transfer matrices for asymptotic representations and obey \( TQ \)-relations as a consequence of the relations in \( K_0(\mathcal{O}) \).

1. Introduction

In Baxter’s formulation \([1]\) the problem of solving exactly solvable models of statistical mechanics reduces to finding the common spectrum of a family \( T(z) \) of commuting endomorphisms of a vector space depending on a complex spectral parameter \( z \). For example in the six- or eight-vertex model the vector space is a tensor product of copies of \( \mathbb{C}^2 \). The Bethe ansatz is a celebrated method to compute eigenvectors and eigenvalues of the transfer matrices \( T(z) \). In this method one seeks eigenvectors among a specific family of vectors depending on parameters, called Bethe roots. The condition for the parameters to yield eigenvectors is a system of algebraic equations—the Bethe ansatz equations—for the Bethe roots, and one gets an implicit description of the eigenvectors and eigenvalues. This was done by Lieb \([30]\) for the ice model, a special case of the six-vertex model, adapting the method of Bethe \([6]\), who had considered the Heisenberg spin chain. Baxter \([1]\) devised another method to obtain the Bethe ansatz equation by directly computing the eigenvalues. The method also works for models where the Bethe ansatz fails, such as the eight-vertex model, but does not give direct information on the eigenvectors. One uses a new set of commuting endomorphisms \( Q(z) \)—the \( Q \)-operator—that commute with the transfer matrices and obey a functional relation, the Baxter \( TQ \)-relation:

\[
T(z)Q(z) = \phi(z)Q(z+h) + \phi(z+h)Q(z-h),
\]

for some specific function \( \phi \) (\( h \) is a parameter of the model). Moreover \( T(z) \) and \( Q(z) \) are entire functions of \( z \) with known functional behaviour: for example in the eight-vertex model \( Q(z) \) has theta function-like double periodicity properties. The common eigenvalues obey the same equation and the functional relation determines their form: for example in the eight-vertex model the double periodicity property implies that eigenvalues of \( Q(z) \) have the form \( \prod_{i=1}^n \theta(z-z_i) \) where \( \theta \) is the odd Jacobi theta function for some elliptic curve depending on the parameters of the model and \( z_i \) are unknowns to be determined. Inserting \( z = z_i \) in Baxter’s \( TQ \)-relation we obtain the Bethe ansatz equations

\[
\prod_{j,j \neq i} \frac{\theta(z_i - z_j + h)}{\theta(z_i - z_j - h)} = \frac{\phi(z_i + h)}{\phi(z_i)}, \quad i = 1, \ldots, n.
\]

Each solution of this system of \( n \) equations for \( n \) unknowns gives a candidate for an eigenvalue of \( Q \), from which the corresponding eigenvalue of \( T \) can be computed from the \( TQ \)-relation.
The approach to exactly solvable models based on transfer matrices and Bethe ansatz was extended to a variety of models of statistical mechanics, quantum integrable models and quantum field theory. It was developed into a full-fledged theory by the Leningrad school under the names Quantum Inverse Scattering Method, and Algebraic Bethe Ansatz, leading to the theory of quantum groups, see the lecture notes [14] for a review. In modern terminology, if we have a (suitable) pair \( V, W \), of representations of a quantum group, for example an affine quantum enveloping algebra, we obtain an \( R \)-matrix \( R_{V,W} \), an endomorphism of \( V \otimes W \), and transfer matrices are traces over one of the tensor factors, say \( W \), called auxiliary space: 
\[
t_W = \text{tr}_W R_{V,W}.
\]

The basic fact, that follows from the Yang–Baxter equation, is that varying \( W \) we get commuting endomorphisms of \( V \), and the transfer matrix \( T(z) \) of the six-vertex model is obtained for \( W \) a two-dimensional representation of the quantum affine algebra of \( sl_2 \) with evaluation parameter \( z \) and \( V \) a tensor product of two-dimensional representations.

The representation theory meaning of the Baxter \( Q \)-operator was understood much later: Bazhanov, Lukyanov and Zamolodchikov [4, 5] gave a construction of a \( Q \)-operator in a quantum field theory context as a transfer matrix for a certain infinite dimensional auxiliary space. Frenkel and Hernandez constructed \( Q \)-operators for arbitrary untwisted affine quantum enveloping algebras as transfer matrices defined as traces over certain representations of a Borel subalgebra of the quantum loop algebra. These pre-fundamental representations belong to a category that had previously been introduced and studied by Hernandez and Jimbo [23]. The Baxter \( TQ \)-relations follow then from relations in the Grothendieck ring of this category. The point is that the \( R \)-matrix is given by the action of the tensor product of two opposite Borel subalgebras, so that \( R_{V,W} \) makes sense if the auxiliary space \( W \) is a representation of a Borel subalgebra.

The goal of this paper is to extend these results to the elliptic quantum group associated to \( sl_2 \). One new feature is the appearance of the dynamical parameter, so that transfer matrices are now difference operators acting on functions of the dynamical parameters and it requires some care to extend the constructions to this case. A more serious new difficulty is that the \( R \)-matrix does not have the same triangular structure as in the affine case, so it does not make sense to consider representations of Borel subalgebras. We show however that, at least in the \( sl_2 \)-case considered here, one can construct \( Q \)-operators as transfer matrices for certain infinite dimensional representations belonging to a suitable abelian category \( \mathcal{O} \). The objects of this category have well-defined \( q \)-characters [27], [20], and this allows us to derive relations in the Grothendieck ring, from which \( TQ \)-relations are obtained.

The paper is structured as follows. In Section 2 we review the theory of the elliptic quantum group associated to \( sl_2 \), discuss the notion of eigenvalues for difference operators, and define asymptotic representations. A category \( \mathcal{O} \) of representations containing all asymptotic representations is constructed in Section 3. We define the \( q \)-character map and show that it is an injective ring homomorphism from the Grothendieck ring \( K_0(\mathcal{O}) \) to a suitable commutative ring \( \mathcal{M}_q \) and deduce relations among asymptotic representations in \( K_0(\mathcal{O}) \). We identify highest weights of simple objects in \( \mathcal{O} \) in Section 4 and describe the classes of finite dimensional modules in \( K_0(\mathcal{O}) \) in terms of asymptotic representations. Transfer matrices associated with representations in \( \mathcal{O} \) are constructed in Section 5. In particular we construct the \( Q \)-operator and prove its \( TQ \)-relations using the relations found in Section 3. Future directions are indicated in Section 6. The same construction also gives \( Q \)-operators for the Yangian of \( sl_2 \) as transfer matrices associated with asymptotic Yangian representations. We indicate this in the Appendix.
2. The elliptic quantum group and its representations

Let \( \mathbb{M} \) be the field of meromorphic functions \( g(x) \) on \( x \in \mathbb{C} \). It contains the subfield \( \mathbb{C} \) of constant functions. We work mostly with \( \mathbb{M} \)-vector spaces. An \( \mathbb{M} \)-linear (or more generally \( \mathbb{C} \)-linear) map \( \Phi \) of two \( \mathbb{M} \)-vector spaces will also be denoted by \( \Phi(x) \) if the dependence on \( x \) needs to be precised. Fix \( \hbar \in \mathbb{C}^\times \).

2.1. Meromorphic eigenvalues. To work with the difference operators in representations of the elliptic quantum group, let us introduce a category \( \mathcal{F} \). An object in \( \mathcal{F} \) is a triplet \( (V, K_+(z), K_-(z)) \) with the following conditions:

(F1) \( V \) is a finite-dimensional \( \mathbb{M} \)-vector space, and \( K_\pm(z) : V \rightarrow V \) are \( \mathbb{C} \)-linear maps with parameter \( z \in \mathbb{C} \) (maybe not well-defined on all of \( \mathbb{C} \));

(F2) if \( (v_j)_{1 \leq j \leq n} \) is an \( \mathbb{M} \)-basis of \( V \), then there exist meromorphic functions \( a_\pm(z;x) \) on \( (z,x) \in \mathbb{C}^2 \) for \( 1 \leq i,j \leq n \) such that

\[
K_\pm(z) \left( \sum_{j=1}^n a_j(x)v_j \right) = \sum_{i,j=1}^n a_\pm(z;x)g_{ij}(x \pm \hbar) v_i \quad \forall g_{ij}(x) \in \mathbb{M}.
\]

A morphism \( (V, K_+(z), K_-(z)) \rightarrow (V', K'_+(z), K'_-(z)) \) in \( \mathcal{F} \) is an \( \mathbb{M} \)-linear map \( \Phi : V \rightarrow V' \) such that \( \Phi K_\pm(z) = K'_\pm(z)\Phi : V \rightarrow V' \). When there is no confusion, for simplicity we also denote \((V, K_+(z), K_-(z))\) by \( V \).

\( \text{Hom}_\mathcal{F}(V, V') \) is a sub-\( \mathbb{C} \)-vector space of \( \text{Hom}_\mathbb{M}(V, V') \), making \( \mathcal{F} \) into an abelian category. The (co)kernel of a morphism \( \Phi : V \rightarrow V' \) in \( \mathcal{F} \) is induced from that of the \( \mathbb{M} \)-linear map \( \Phi : V \rightarrow V' \). By induction on \( \dim \mathbb{M}(V) \), every object \( V \) in \( \mathcal{F} \) admits a Jordan-Hölder series: a sequence of subobjects \( V = V_n \supset V_{n-1} \cdots \supset V_1 \supset V_0 = 0 \) such that each quotient object \( V_i/V_{i+1} \) is simple.

Let \( a^\pm(x) \in \mathbb{M}^\times \). Define \( \mathbb{M}[a^+(z), a^-(z)] := (\mathbb{M}, K_+(z), K_-(z)) \in \mathcal{F} \) by

\[
K_\pm(z)(g(x)) = g(x \pm \hbar)a^\pm(z) \quad \text{for } g(x) \in \mathbb{M}.
\]

**Definition 2.1.** \( \mathcal{F}_{\text{mer}} \) is the full subcategory of \( \mathcal{F} \) consisting of objects \( V \) whose quotient objects in Jordan-Hölder series are isomorphic to the \( \mathbb{M}[a^+(z), a^-(z)] \) for \( a^\pm(x) \in \mathbb{M}^\times \).

From the definition it is not difficult to show

**Lemma 2.2.** \( \mathcal{F}_{\text{mer}} \) is an abelian subcategory of \( \mathcal{F} \). An object \( V \) of \( \mathcal{F} \) is in \( \mathcal{F}_{\text{mer}} \) if and only if there exists an \( \mathbb{M} \)-basis \( (v_i)_{1 \leq i \leq n} \) of \( V \) with respect to which the matrices \( (a_\pm(z;x)i,j)_{1 \leq i,j \leq n} \) in Equation 2.1 are upper triangular whose diagonals are independent of \( x \) and non-zero.

Notice that: \( v_i \in V \) is an eigenvector of the \( \mathbb{C} \)-linear maps \( K_\pm(z) \) of eigenvalue \( a^\pm_i(z;x) = a^\pm_i(z) \) respectively; \( (V,K_-(z))^{-1}, K_+(z))^{-1} \in \mathcal{F}_{\text{mer}} \).

**Lemma 2.3.** \( \mathbb{M}[a^+_1(z), a^-_1(z)] \cong M[a^+_2(z), a^-_2(z)] \) in \( \mathcal{F} \) if and only if there exists \( c \in \mathbb{C}^\times \) such that \( a^+_2(z) = c^{\pm 1}a^+_1(z) \).

2.2. Algebraic notions. We briefly recall the notion of \( \hbar \)-algebras\(^1\) from \( \textbf{[11]} \), with \( \hbar \) being the one-dimensional complex Lie algebra and \( \gamma = -\hbar \).

Let \( \mathcal{X} \) denote the category whose objects are \( \mathbb{C} \)-graded \( \mathbb{M} \)-vector spaces \( X = \oplus_{n \in \mathbb{Z}} X[n] \), and morphisms are \( \mathbb{M} \)-linear maps which preserve the \( \mathbb{C} \)-gradings. If \( X[\alpha] \neq 0 \), then \( \alpha \) is called a weight of \( X \), non-zero vectors in \( X[\alpha] \) are of weight \( \alpha \), and \( X[\alpha] \) is the weight space of weight \( \alpha \). Let \( wt(X) \) be the set of weights of \( X \).

\(^1\)Functions in \( \mathbb{M} \) have variable \( x \). Be aware of the change of variables \( x \mapsto z \) here.

\(^2\)We use \( \hbar \) to avoid confusion of \( \mathbb{C} \)-algebras in the usual sense.
For $X,Y \in \mathcal{V}$ define their dynamical tensor product $X \hat{\otimes} Y \in \mathcal{V}$ as follows. For $\alpha, \beta \in \mathbb{C}$, let $X[\alpha] \hat{\otimes} Y[\beta]$ be the usual tensor product of $\mathbb{C}$-vector spaces $X[\alpha] \otimes \mathbb{C} Y[\beta]$ modulo the relation \[ (2.2) \quad g(x)v \otimes_C w = v \otimes_C g(x + \beta h)w \quad \text{for} \quad v \in X[\alpha], \quad w \in Y[\beta], \quad g(x) \in \mathbb{M}. \]

Let $\otimes$ denote the image of $\otimes_C$ under the quotient. $X[\alpha] \hat{\otimes} Y[\beta]$ becomes an $\mathbb{M}$-vector space by setting $g(x)(v \otimes w) = v \otimes g(x)w$. For $\gamma \in \mathbb{C}$, set $(X \otimes Y)[\gamma]$ to be the direct sum of the $X[\alpha] \hat{\otimes} Y[\beta]$ over all such $\alpha, \beta \in \mathbb{C}$ that $\alpha + \beta = \gamma$.

Following [11, §4.1], an $\mathfrak{h}$-algebra is a unital associative algebra $A$ over $\mathbb{C}$, endowed with $\mathbb{C}$-bigrading $A = \oplus_{\alpha, \beta \in \mathbb{C}} A_{\alpha, \beta}$ which respects the algebra structure, and two $\mathbb{C}$-algebra embeddings $\mu_\alpha, \mu_\beta : \mathbb{M} \rightarrow A_{0,0}$, called the left and right moment maps, such that for $a \in A_{\alpha, \beta}$ and $g(x) \in \mathbb{M}$:

$$\mu_\alpha(g(x))a = a\mu_\alpha(g(x - ah)), \quad \mu_\beta(g(x))a = a\mu_\beta(g(x - bh)),$$

A morphism of $\mathfrak{h}$-algebras is a $\mathbb{C}$-algebra homomorphism preserving the moment maps. From two $\mathfrak{h}$-algebras $A, B$ we construct their tensor product $A \hat{\otimes} B$ as follows. For $\alpha, \beta, \gamma \in \mathbb{C}$, let $A_{\alpha, \beta} \hat{\otimes} B_{\beta, \gamma}$ be $A_{\alpha, \beta} \otimes \mathbb{C} B_{\beta, \gamma}$ modulo the relation

$$\mu_\alpha^A(g(x))a \otimes_C b = a \otimes_C B_{\beta, \gamma}^B(g(x))b \quad \text{for} \quad a \in A_{\alpha, \beta}, \quad b \in B_{\beta, \gamma}, \quad g(x) \in \mathbb{M}.$$

$(A \hat{\otimes} B)_{\alpha, \gamma}$ is the direct sum of the $A_{\alpha, \beta} \hat{\otimes} B_{\beta, \gamma}$ over $\beta \in \mathbb{C}$. Multiplication in $A \hat{\otimes} B$ is induced by $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$. The moment maps are given by ($\hat{\otimes}$ denotes the image of $\otimes_C$ under the quotient $\otimes_C \rightarrow \hat{\otimes}$)

$$\mu^A_{\alpha} : g(x) \mapsto \mu^A_{\alpha}(g(x)) \hat{\otimes} 1, \quad \mu^B_{\beta} : g(x) \mapsto 1 \hat{\otimes} \mu^B_{\beta}(g(x)) \quad \text{for} \quad g(x) \in \mathbb{M}.$$

To $X \in \mathcal{V}$ is attached an $\mathfrak{h}$-algebra $D_X$, a $\mathbb{C}$-subalgebra of $\text{End}_C(X)$ as in [11, §4.2]. For $\alpha, \beta \in \mathbb{C}$, the subspace $(D_X)_{\alpha, \beta}$ consists of such $\Phi$ in $\text{End}_C(X)$ that:

$$\Phi(X[\gamma]) \subseteq X[\gamma + \beta - \alpha], \quad \Phi(g(x)v) = g(x + \beta h)\Phi(v)$$

for $\gamma \in \text{wt}(X)$, $v \in X$ and $g(x) \in \mathbb{M}$. The moment maps $\mu_\alpha, \mu_\beta$ are defined by:

$$\mu_\beta(g(x))v = g(x)v, \quad \mu_\alpha(g(x))v = g(x + ho)v \quad \text{for} \quad v \in X[\alpha], \quad g(x) \in \mathbb{M}.$$

Let $X, Y \in \mathcal{V}$. For $\Phi \in (D_X)_{\alpha, \beta}$ and $\Psi \in (D_Y)_{\beta, \gamma}$, the $\mathbb{C}$-linear map

$$\Pi : X \otimes_C Y \rightarrow X \hat{\otimes} Y, \quad v \otimes_C w \mapsto \Phi(v) \hat{\otimes} \Psi(w)$$

respects Relation (2.2). Indeed, for $w \in Y[\delta]$ and $g(x) \in \mathbb{M}$:

$$\Pi(g(x)v \otimes_C w) = \Phi(g(x)v) \hat{\otimes} \Psi(w) = g(x + \beta h)\Phi(v) \hat{\otimes} \Psi(w) = \Phi(v) \otimes_C g(x + \beta h + (\delta + \gamma - \beta)h)\Psi(w) = \Pi(v \otimes_C g(x + \delta h)w).$$

$\Pi$ induces the $\mathbb{C}$-linear map $\Phi \hat{\otimes} \Psi : X \otimes Y \rightarrow X \hat{\otimes} Y$ which is easily shown to be in $(D_X \hat{\otimes} Y)_{\alpha, \gamma}$. As in [11, Lemma 4.3]:

**Lemma 2.4.** $\Phi \hat{\otimes} \Psi \mapsto \Phi \hat{\otimes} \Psi$ extends uniquely to a morphism of $\mathfrak{h}$-algebras $\theta_{XY} : D_X \hat{\otimes} D_Y \rightarrow D_X \hat{\otimes} Y$.

### 2.3. Elliptic quantum group

Fix a complex number $\tau \in \mathbb{C}$ with $\text{Im}(\tau) > 0$. Define the Jacobi theta function

$$\theta(z) := -\sum_{j=-\infty}^{\infty} \exp \left( i\pi (j + \frac{1}{2})^2 \tau + 2i \pi (j + \frac{1}{2})(z + \frac{1}{2}) \right), \quad i = \sqrt{-1}.$$

It is an entire function on $z \in \mathbb{C}$ with zeros lying on the lattice $\mathbb{Z} + \tau \mathbb{Z}$ and

$$\theta(z + 1) = -\theta(z), \quad \theta(z + \tau) = -e^{-\pi \tau - 2\pi \tau z} \theta(z), \quad \theta(-z) = -\theta(z).$$

---

3 This dynamical tensor product and $\Phi \hat{\otimes} \Psi$ in Lemma 2.4 are slightly different from [11, §3.1].
Assume from now on that $\mathbb{Z} + \mathbb{Z}r$ and $h\mathbb{Z}$ only intersect at 0. Unless otherwise stated (in the appendix), $\otimes$ denotes the ordinary tensor product $\otimes_M$ of $M$-vector spaces and $M$-linear maps.

Let $V \in V$ be such that $V = \mathbb{M}v_+ \oplus \mathbb{M}v_-$ with $v_\pm$ being of weight $\pm 1$ respectively. Define the $\text{End}_M(V^{\otimes 3})$-valued meromorphic function on $z \in C$ by its matrix with respect to the $M$-basis $(v_+ \otimes v_+, v_+ \otimes v_-, v_- \otimes v_-)$:

$$R(z; x) = \begin{pmatrix}
1 & 0 & 0
0 & \frac{\theta(z)\theta(z+h)\theta(z-h)}{\theta(z)\theta(z)\theta(h)} & 0
0 & 0 & \frac{\theta(z+h)\theta(z)}{\theta(z)\theta(h)}
0 & 0 & 1
\end{pmatrix}.$$  

(2.3)

It is the matrix $R^+(z, x)$ in [10] (93)]. Applying a suitable gauge transformation in [10] Lemma 8.1, one obtains $R^+(x, z, -\frac{h}{2}, \tau)$ in [13] §1.

$R(z; x)$ satisfies the quantum dynamical Yang–Baxter equation in [11]:

$$R^{12}(z - w; x + hh^{(3)})R^{13}(z; x)R^{23}(w; x + hh^{(1)}) = R^{23}(w; x)R^{13}(z; x + hh^{(2)})R^{12}(z - w; x) \in \text{End}_M(V^{\otimes 3}).$$  

(2.4)

Here $R^{12}(z; x + hh^{(3)})$ means that if $w_1, w_2 \in V$ and $w_3 \in V[\alpha]$, then

$$R^{12}(z; x + hh^{(3)})(w_1 \otimes w_2 \otimes w_3) = R(z; x + h\alpha)(w_1 \otimes w_2) \otimes w_3.$$  

The other symbols have a similar meaning.

The elliptic quantum group $E = E_{r, \lambda}(\mathfrak{g}_2)$ is the operator algebra in [13] §3, or equivalently the dynamical quantum group associated to $R(z; x)$ in [11] §4.4. It is an $h$-algebra generated by the $L_{ij}(z) \in E_{i1, j1}$ with $i, j \in \{\pm\}$ subject to the relation

$$\mu_1(R^{12}(z - w; x))L^{13}(z)L^{23}(w) = L^{23}(w)L^{13}(z)\mu_2(R^{12}(z - w; x)).$$  

(2.5)

By [13] §1 and [11] Proposition 4.2, there is an $h$-algebra morphism

$$\Delta : E \rightarrow E \otimes E, \quad L_{ij}(z) \mapsto \sum_{k = \pm} L_{ik}(z) \otimes L_{kj}(z)$$  

(2.6)

which is co-associative $(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta$. For $u \in C$,

$$\Psi_u : E \rightarrow E, \quad L_{ij}(z) \mapsto L_{ij}(z + uh)$$  

(2.7)

extends uniquely to an automorphism of $h$-algebra.

A representation of $E$ on $X \in V$ is an $h$-algebra morphism $\rho : E \rightarrow D_X$ which depends meromorphically on $z \in C$. More precisely, it consists of four operators $L^\alpha_{ij}(z) \in (D_X)_{i1, j1}$ for $i, j \in \{\pm\}$ with parameter $z \in C$ such that: with respect to an $M$-basis of $X$, for any $v \in X$, the coefficients of the $L^\alpha_{ij}(z)v$ are meromorphic functions on $z, x$: Equation (2.6) holds in $D_X$ with $\mu_1, \mu_2$ being moment maps in $D_X$. We also call $X$ an $E$-module.

If $(\rho, X)$, $(\sigma, Y)$ are representations of $E$, then Equation (2.6) together with Lemma (2.4) endows $X \otimes Y$ with a representation $\theta_{XY} : (\rho \otimes \sigma) \circ \Delta$.

Let $(\rho, X)$ be a representation. Fix a weight basis $(w_\alpha)$ of $X$. For $i, j \in \{\pm\}$, define $L^\alpha_{ij}(z; x) \in \text{End}_M(X)$ by $L^\alpha_{ij}(z; x)w_\alpha = L^\alpha_{ij}(z)w_\alpha$ for all $\alpha$. Set

$$L^\alpha_{ij}(z; x) := \sum_{i, j \in \{\pm\}} E_{ij} \otimes L^\alpha_{ij}(z; x) = \begin{pmatrix} L^\alpha_{+j}(z; x) & L^\alpha_{-j}(z; x) \\ L^\alpha_{-j}(z; x) & L^\alpha_{+j}(z; x) \end{pmatrix} \in \text{End}_M(V \otimes X)$$

(2.8)

4Strictly speaking $E$ is not well-defined as an $h$-algebra. Nevertheless, we are only concerned with representations and Equations (2.6)–(2.7) make sense. Equations (2.8), (2.9) and [11] (4.4.3) are coherent as $R^\alpha_{ij}(z; x) = R^\alpha_{ij}(z; x + ph + qh)$ for $i, j, p, q \in \{\pm\}$. 


where $E_{ij} \in \text{End}_\mathbb{H}(V)$ is $v_k \mapsto \delta_{jk} v_i$. That $\rho$ is an $\hbar$-algebra morphism is equivalent to the following RLL relation [11 Proposition 4.5]:

\[
\begin{align*}
R^{12}(z - w; x + hh(3))L^{X,13}(z; x)L^{X,23}(w; x + hh(1)) &= L^{X,23}(w; x)L^{X,13}(z; x + hh(2))R^{12}(z - w; x) \in \text{End}_{\mathbb{H}}(V \otimes V \otimes X). \\

\text{For } i, j, m, n \in \{\pm\}, \text{ let } R^{ij}_{mn}(z; x) \text{ be the coefficient of } v_m \otimes v_n \text{ in } R(z; x)(v_i \otimes v_j). \text{ Then Equation (2.8) means the following identities for all } i, j, m, n \in \{\pm\},
\end{align*}
\]

\[
\sum_{p,q} R^{pq}_{mn}(z - w; x + hh)L^{X}_{pi}(z; x)L^{X}_{qj}(w; x + ih) = \sum_{p,q} L^{X}_{pq}(w; x)L^{X}_{mp}(z; x + qh)R^{ij}_{pq}(z - w; x) \in \text{End}_{\mathbb{H}}(X).
\]

By Equation (2.4), $L^X(z; x) := R(z; x)$ affords a representation of $E$ on $V$.

### 2.4. Asymptotic representations.

Let $\ell \in \mathbb{C}$ and $W^\ell = \oplus_{j=0}^\infty \mathbb{M}v_j$ with $v_j$ being of weight $\ell - 2j$ so that $W^\ell \in V$. Let $L^\ell(z; x) \in \text{End}_\mathbb{H}(V \otimes W^\ell)$ be the matrix

\[
\left( \sum_{j=0}^\infty E_{j}\theta(z(\ell - j + 1)h)\theta(z(\ell - j + 1)h) \right) = \left( \sum_{j=0}^\infty E_{j}\theta(z(\ell + j + 1)h)\theta(z(\ell - j + 1)h) \right) = \left( \sum_{j=0}^\infty E_{j}\theta(z(\ell + j + 1)h)\theta(z(\ell - j + 1)h) \right).
\]

Here $E_{ij} \in \text{End}_\mathbb{H}(W^\ell)$ is $w_k \mapsto \delta_{jk} v_i$.

**Proposition 2.5.** $L^\ell(z; x)$ and the basis $(v_j)$ define a representation of $E$ on $W^\ell$.

**Proof.** Assume $n \in \mathbb{Z}_{\geq 0}$. Let $V^n = \oplus_{j=0}^n \mathbb{M}v_j$ with $v_j$ being of weight $n - 2j$. Under the correspondence $[u] \mapsto \theta(hu), r \mapsto h^{-1}, u \mapsto h^{-1}z, P = s \mapsto h^{-1}x$, the matrix $R^\ell(u, s)$ in [23 (2.18)] is identified with $R(z; x)$. From [28] (2.19)–(2.20) & Theorem 4.14 (setting $\varphi_n(u - v) = -1$ in $\pi_{n,q,-n}$) one obtains a representation $\rho^n$ of $E$ on $V^n$ whose matrix with respect to the basis $(v_j)_{0 \leq j \leq n}$ is

\[
\begin{pmatrix}
\theta(z(\ell + j + 1)h)\theta(z(\ell + j + 1)h) & \cdots & \theta(z(\ell + 1)h)\theta(z(\ell + 1)h) \\
\cdots & \cdots & \cdots \\
\theta(z(\ell + n)h)\theta(z(\ell + n)h) & \cdots & \theta(\ell)\theta(\ell)
\end{pmatrix} \in \text{End}_\mathbb{H}(V \otimes V^n).
\]

Here $h(v_j) = (n - 2j)v_j, S^h(v_j) = v_{j+1}$ and $v_{-1} = v_{n+1} = 0$. Set

\[
v'_0 := v_0, \quad v'_j := \frac{\theta(h(\ell + 2j)h)\cdots \theta(\ell)\theta(\ell)}{\theta(nh(\ell - 1)h)\cdots \theta((n - j + 1)h)\theta((n - j + 1)h)} v_j \text{ for } 1 \leq j \leq n.
\]

Then $(v'_j)$ forms another basis of $V^n$. Let $L'^n(z; x)$ be the matrix of $\rho^n$ with respect to $(v'_j)$. Let us view $V^n$ as a subspace of $W^n$ by $v'_j \mapsto v_j$ for $0 \leq j \leq n$.

(I) $L'^{pq}_{mn}(z; x)v_n = L^{pq}_{mn}(z; x)v_n$ for $p, q \in \{\pm\}$ with $(p, q) \neq (+, -)$.

(II) $L'^{pq}_{mn}(z; x)v_j = L^{pq}_{mn}(z; x)v_j$ for $0 \leq j < n$.

Our goal is prove Equation (2.8) with $L^X(z; x) = L^\ell(z; x)$ and $X = W^\ell$. Fix $i, j, m, n \in \{\pm\}$ and $k \in \mathbb{Z}_{\geq 0}$. Let $L$ (resp. $R$) be the left-hand side (resp. right-hand side) of Equation (2.8) applied to $w_k$. We are reduced to prove $L = R$. Notice that $L, R$ are linear combinations of $w_k, w_{k+1}, w_{k+2} \in V^{k+2} \subset W^\ell$ whose coefficients are meromorphic functions on $z, w, x, \ell \in \mathbb{C}$. (Set $w_{-1} = w_{-2} = 0$.) We shall fix $z, w, x \in \mathbb{C}$ to be generic and view $L = L(\ell)$ and $R = R(\ell)$ as vector-valued meromorphic functions on $\ell$. By (I)–(II), $L(n) = R(n)$ for $n < k + 2$.

For two vector-valued meromorphic functions $f(\ell), g(\ell)$ on $\ell$, let us write $f(\ell) \sim g(\ell)$ if there exist $N \in \mathbb{Z}_{\geq 0}$ and $a \in \mathbb{C}$ such that for $H = f, g$:

\[
H(\ell + h^{-1}) = (\ell)^N H(\ell), \quad H(\ell + h^{-1}\tau) = (\ell)^N e^{-\pi i(N\tau + 2Nh + a)} H(\ell).
\]
Assume $L(\ell) \sim R(\ell)$. The coefficients of $L(\ell) - R(\ell)$ are products of the $\theta(\ell h + b)^{\pm 1}$ for $b \in \mathbb{C}$ with constant functions. Since $\theta(\ell h + b)$ cannot have zeroes at large enough integers, $L(\ell) \sim R(\ell)$. We are led to prove $L(\ell) \sim R(\ell)$. \\
If $m = n = -$, then $R, L$ are independent of $\ell$ as is so the second row of the square matrix $L'(z; x)$. If $m = n = +$, then $L(\ell), R(\ell) \sim \theta(\ell h + z + h)\theta(\ell h + w + h)$, because the first row of $L'(z; x) \sim \theta(\ell h + z + h)$.

For $(n, n) = (+, -)$, we have $L(\ell) = R^{+}_{-+}L_{++}L_{-j} + R^{+}_{-+}L_{--}L_{++}$ and
\[
R^{+}_{-+}L_{++}L_{-j} \sim \frac{\theta(\ell h + x + (1 - 2i - 2j)h)\theta(\ell h + x - (1 + 2i + 2j)h)}{\theta(\ell h + x - (2i + 2j)h)^2}\theta(\ell h + z + h),
\]
\[
R^{+}_{-+}L_{--}L_{++} \sim \frac{\theta(\ell h + z - w + x - (2i + 2j)h)}{\theta(\ell h + w + h)}\theta(\ell h + w + h) \sim \theta(\ell h + z + h).
\]

Here for simplicity $i \in \{\pm\}$ is taken as $i1 \in \{\pm 1\}$ and the factors irrelevant to $\ell$ have been omitted. $R(\ell)$ is a sum of the $L_{-q}L_{-p}R_{pq}$. Since $L_{-q}, R_{pq} \sim 1$ and $L_{++} \sim \theta(\ell h + z + h)$, we have $R(\ell) \sim \theta(\ell h + z + h) \sim L(\ell)$. For $(n, n) = (-, +)$, similarly $L(\ell) \sim \theta(\ell h + w + h) \sim R(\ell)$. □

One could have deduced the $W^\ell$ from the evaluation modules in [15, §4] by suitable gauge transformations. We took an indirect approach by analyzing the $\ell$-dependence of matrix coefficients. This is to be compared with [23, Proposition 4.5] and [35, Lemma 5.1] where matrix coefficients are Laurent polynomials in $\ell := q^k$ in the case of quantum affine (super)algebras.

Remark 2.6. If $\ell \notin \mathbb{Z}_{2^\infty} + h^{-1}(\mathbb{Z} + \mathbb{Z}r)$, then $W^\ell$ is simple. If $\ell \in l + h^{-1}(\mathbb{Z} + \mathbb{Z}r)$ with $l \in \mathbb{Z}_{2^\infty}$, then $V^\ell := \oplus_{j = 0}^l \mathbb{M}v_j$ is a sub-$\mathcal{E}$-module of $W^\ell$ and it is contained in any non-zero sub-$\mathcal{E}$-module of $W^\ell$. In other words, $V^\ell$ is the simple socle of $W^\ell$.

Definition 2.7. Let $\ell, u \in \mathbb{C}$. The asymptotic representation $W^\ell,u$ is the pullback of the representation $W^\ell$ in Proposition 2.5 by $\Psi_0$ in Equation (2.7). $\ell$ is called the spin parameter and $u$ the spectral parameter.

3. Category $\mathcal{O}$ and $q$-characters

We introduce a tensor category $\mathcal{O}$ of representations of $\mathcal{E}$ containing all the $W^\ell,u$, and study its Grothendieck ring, which turns out to be commutative.

Let $X$ be an $\mathcal{E}$-module. For $\alpha \in \text{wt}(X)$ choose an $\mathbb{M}$-basis $(v^\alpha_j)_{1 \leq j \leq s}$ of $X[\alpha]$. For $1 \leq i, j \leq r$ let $a^\alpha_{ij}(z; x)$ be the coefficient of $v^\alpha_i$ in $L^\alpha_{-j}(z)v^\alpha_j$; these are meromorphic functions on $z, x \in \mathbb{C}$ by Section 2.3. We say that $L^\alpha_{-j}(z)$ is invertible for generic $z \in \mathbb{C}$ if $\text{det}(a^\alpha_{ij}(z; x))_{1 \leq i, j \leq r}$ is a non-zero meromorphic function for all $\alpha \in \text{wt}(X)$.

As a two-by-two matrix over $D_X$, $L^\alpha(z)$ admits a Gauss decomposition:
\[
L^\alpha(z) := \begin{pmatrix} L^\alpha_{++}(z) & L^\alpha_{+-}(z) \\ L^\alpha_{-+}(z) & L^\alpha_{--}(z) \end{pmatrix} = \begin{pmatrix} 1 & F^\alpha(z) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} K^\alpha_{++}(z) & 0 \\ 0 & K^\alpha_{--}(z) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ E^\alpha(z) & 1 \end{pmatrix}.
\]

By definition, we have $K^\alpha(z) = L^\alpha_{--}(z) \in (D_X)_{-1,-1}$ and
\[
K^\alpha_{++}(z) \in (D_X)_{1,1}, \quad E^\alpha(z) \in (D_X)_{0,2}, \quad F^\alpha(z) \in (D_X)_{2,0}.
\]

It follows that $(X, K^\alpha_{++}(z), K^\alpha_{--}(z))_{X[\alpha]} \in \mathcal{F}$ for $\alpha \in \text{wt}(X)$.

Definition 3.1. A representation $X$ of $\mathcal{E}$ is said to be in category $\mathcal{O}$ if:

1. the weight spaces $X[\alpha]$ are finite-dimensional over $\mathbb{M}$;
2. there exist $\alpha_1, \alpha_2, \cdots, \alpha_r \in \mathbb{C}$ such that $\text{wt}(X) \subseteq \bigcup_{\ell = 1}^r (\alpha_j + 2\mathbb{Z}_{\leq 0})$;
3. $L^\alpha_{--}(z)$ is invertible for generic $z \in \mathbb{C}$;
4. for $\alpha \in \text{wt}(X)$ we have $(X, K^\alpha_{++}(z), K^\alpha_{--}(z))_{X[\alpha]} \in \mathcal{F}_{\text{mer}}$. 


A morphism of two representations \( X, Y \) in \( \mathcal{O} \) is an \( M \)-linear map \( \Phi : X \rightarrow Y \) such that \( \Phi L^X_{ij}(z) = L^Y_{ij}(z) \Phi \) for all \( i, j \in \{ \pm \} \).

When there is no confusion, we drop the superscript \( X \) from \( K^X_E, F^X, L^X_{ij} \).

**Lemma 3.2.** \( \mathcal{O} \) is an abelian category.

**Proof.** This follows from Lemma 2.2. \( \square \)

**Proposition 3.3.** The representation \( \mathcal{W}^t \) in Proposition 2.5 is in category \( \mathcal{O} \).

**Proof.** Since \( L_{-\ldots}(z)w_j = \theta(z + (j + 1)h)w_j \) for all \( j \in \mathbb{Z}_{\geq 0} \), \( L_{-\ldots}(z) \) is invertible. Conditions (1)-(3) in Definition 3.1 are true. \( E, F, K_- \) are easy to compute:

\[
K_-(z)w_j = \theta(z + (j + 1)h)w_j, \quad E(z)w_j = -\frac{\theta(z - x + (j - 1)h)\theta(jh)}{\theta(z + jh)\theta(x + h)}w_{j-1},
\]

\[
F(z)w_j = \frac{\theta(z + x + (\ell - j)h)\theta((\ell - j)h)}{\theta(z + (j + 1)h)\theta(x + (\ell - 2j - 1)h)}w_{j+1}.
\]

For \( K_+ \), let us be in the situation of the proof of Proposition 2.5. From [28, Theorem 4.13] one observes that: for \( t \in \mathbb{Z}_{\geq 0} \),

\[
K_+^{V_1}(z)K_-^{V_1}(z - h)v'_j = \theta(z + (l + 1)h)\theta(z)v'_j \quad \text{for } 0 \leq j \leq l.
\]

It is therefore enough to prove the above identity for \( \mathcal{W}^t \). In other words,

\[
(L_{++}(z) - L_{-\ldots}(z)L_{-\ldots}(z)^{-1}L_{++}(z))L_{-\ldots}(z - h)w_j = \theta(z + (l + 1)h)\theta(z)w_j.
\]

Let \( z, x \in \mathbb{C} \) be generic. The left hand side is \( g_j(\ell)w_j \) where \( g_j(\ell) \) is an entire function on \( \ell \in \mathbb{C} \) and \( g_j(\ell) \sim \theta(\ell h + z + h) \). From the embedding \( V^n \subset W^n \) we deduce that \( g_j(n) = \theta(nh + z + h)\theta(z) \) for all \( n \in \mathbb{Z}_{\geq 3} \). This forces \( g_j(\ell) = \theta(z + (\ell + 1)h)\theta(z) \) for all \( \ell \in \mathbb{C} \). \( \square \)

**Remark 3.4.** Let \( X \in \mathcal{O} \). Then \( K^X_E(z)K^X_Y(z - h) \in (D_X)_{0, 0} \) commutes with the \( L^X_Y(w) \in D_X \); see [15, Theorem 13], [10, Remark 10] and [29, Corollary E.24].

Next we adapt the \( q \)-character theory of Knight [27] and Frenkel–Reshetikhin [20] to the category \( \mathcal{O} \). Motivated by Lemma 2.4, let \( \mathcal{M} \) be the quotient of the set of pairs \((a^+(z), a^-(z))\) of non-zero meromorphic functions on \( z \in \mathbb{C} \) by the relation

\[(a^+(z), a^-(z)) \equiv (ca^+(z), c^{-1}a^-(z)) \quad \text{for } c \in \mathbb{C}^\times.
\]

The isomorphism class of \((a^+(z), a^-(z))\) is denoted by \([a^+(z), a^-(z)] \in \mathcal{M}\). Make \( \mathcal{M} \) into a group by component-wise multiplication. Introduce formal symbols \( t^a \) for \( a \in \mathcal{C} \). Define the set \( \mathcal{M}_t \), whose elements are formal sums (possibly infinite)

\[
\sum_{a \in \mathcal{C}} \sum_{m \in \mathcal{M}} c_{m, a} mt^a \quad \text{with coefficients } c_{m, a} \in \mathbb{Z} \text{ such that:}
\]

(M1) there exist \( a_1, a_2, \ldots, a_r \in \mathcal{C} \) such that the coefficient of \( mt^a \) is non-zero only if \( a \in \bigcup_{j=1}^r (a_j + 2\mathbb{Z}_\leq 0) \);

(M2) for all \( a \in \mathcal{C} \), the number of terms \( mt^a \) with non-zero coefficients is finite.

Make \( \mathcal{M}_t \) into a ring: addition is the usual one of formal sums; multiplication is induced by \((mt^a)(mt'^b) = (mn^b)t^{a+b} \) for \( m, n, m', n' \in \mathcal{M} \) and \( a, b, a', b' \in \mathcal{C} \).

**Definition 3.5.** Let \( X \) be in category \( \mathcal{O} \). For \( a \in \text{wt}(X) \) choose an \( M \)-basis \((v^\alpha_i)_{1 \leq i \leq r_a} \) of \( X[a] \) such that: the coefficients \( a^{\pm}_{ij}(z; x) \) of \( v^\alpha_i \) in \( K^X_{ij}(z)v^\alpha_j \) form upper triangular matrices \((a^{\pm}_{ij}(z; x))_{1 \leq i \leq r_a} \) with diagonals \( a^{\pm}_{ii}(z; x) = a^{\pm}_{ii}(z) \neq 0 \) being independent of \( x \). The \( q \)-character of \( X \) is defined to be

\[
\chi_q(X) := \sum_{a \in \text{wt}(X)} \sum_{i=1}^{r_a} [a^{+}_{ii}(z), a^{-}_{ii}(z)]t^a \in \mathcal{M}_t.
\]

**Lemma 3.6.** \( \chi_q(X) \) is independent of the choice of basis \((v^\alpha_i) \) of \( X \).
Proof. By Lemma 2.2 and Condition (4) in Definition 3.1, such a basis \((v_i^a)\) exists. Conditions (1)–(2) in Definition 3.1 imply (M1)–(M2) for \(\chi_q(X)\), which is an element in \(\mathcal{M}_0\). From the upper triangular property we deduce that: in the Grothendieck group \(K_0(\mathcal{F}_{\text{mer}})\) of the abelian category \(\mathcal{F}_{\text{mer}}\), the isomorphism class of \((X, K^+_i(z), K^-_i(z))\) is the sum of those of irreducible objects \(\mathcal{M}[a_{ii}^+ (z), a_{ii}^- (z)]\), which corresponds to the second summation in \(\chi_q(X)\) and is therefore independent of the choice of basis \((v_i^a)\) by Lemma 2.3. 

Example 3.7. Let \(\ell \in \mathbb{C}\). From the proof of Proposition 3.3 we see that

\[
\chi_q(\mathcal{W}^\ell) = \sum_{j=0}^{\infty} \frac{\theta(z + (\ell + 1)h)\theta(z)}{\theta(z + jh)} \ell^{-2j} = [\theta(z + (\ell + 1)h), \theta(z + h)]t^\ell \times \chi_q(\mathcal{W}^0) + \ldots.
\]

In particular, \(\chi_q(\mathcal{W}^\ell) = [\theta(z + (\ell + 1)h), \theta(z + h)]t^\ell \times \chi_q(\mathcal{W}^0)\).

Let \(X\) be in category \(\mathcal{O}\). A non-zero weight vector \(v\) is called a highest weight vector if \(L_{-\lambda}(z)v = 0\) and \(K_{\pm}(z)v = a_{\mp}^{\pm}(z)v\) where \(a_{\pm}^{\pm}(z)\) are meromorphic functions on \(z\) (independent of \(x\)). Call \(V\) a highest weight module if it is \(M\)-linearly spanned by \(L_{\pm}(-\zeta_1)L_{\pm}(-\zeta_2)\cdots L_{\pm}(-\zeta_n)\) where \(n \in \mathbb{Z}_{\geq 0}, \zeta_1, \zeta_2, \ldots, \zeta_n \in \mathbb{C}\) are generic, and \(v \in X[a]\) is a highest weight vector; the monomial \([a_+^{\pm}(z), a_-^{\pm}(z)]t^\alpha M \in \mathcal{M}_0\) is called the highest weight of \(X\) (and of \(v\)).

For example, \(w_0 \in \mathcal{W}^\ell\) is a highest weight vector. \(\mathcal{W}^\ell\) is of highest weight if and only if \(\ell \in \mathbb{Z}_{\geq 0} + h(-1)(\mathbb{Z} + \mathbb{Z}r)\). See Remark 2.3.

Lemma 3.8. Simple objects in category \(\mathcal{O}\) are of highest weight. Two such objects are isomorphic if and only if their highest weights are the same, if and only if their \(q\)-characters are the same.

Proof. Let \(S\) be a simple object in category \(\mathcal{O}\). By Condition (2) of Definition 3.1, there exists \(\alpha \in \text{wt}(S)\) such that \(S[\alpha + n] = 0\) for all \(n \in \mathbb{Z}_{\geq 0}\). This implies \(L_{-\lambda}(z)S[\alpha] = 0\) and \(K_{\pm}(z)S[\alpha] = L_{\pm}(z[S[\alpha]]\). Since \((S[\alpha], K_+, K_-) \in \mathcal{F}_{\text{mer}}\), there exists \(0 \neq v \in S[\alpha]\) such that \(K_+(z)v = a_+^{\pm}(z)v\) with \(a_+^{\pm}(z) \in \mathcal{M}\). So \(v\) is a highest weight vector. Since \(S\) is simple, it is spanned by the \(L_{i_{1,j_1}}(z_1)L_{i_{2,j_2}}(z_2)\cdots L_{i_{m,j_m}}(z_m)v\). By Equation 2.9 and Lemma 2.4, one may assume \((i_s, j_s) = (\pm)\) for all \(s \leq n\). This proves that \(S\) is of highest weight. \(\chi_q(S)\) \([a_+^{\pm}(z), a_-^{\pm}(z)]t^\alpha M \in \mathcal{M}_0\) is of the form \([a_+^{\pm}(z), a_-^{\pm}(z)]t^\alpha M \in \mathcal{M}_0\) as the leading term.

Assume that \(mt^\alpha\) is the highest weight of two simple objects \(S_1, S_2\) in category \(\mathcal{O}\). Consider the \(E\)-module \(V = S_1 \oplus S_2\) with natural projections \(\pi_i : V \rightarrow S_i\). Let \(v_1 \in S_1\) be highest weight vectors. Then \((v_1, v_2) \in V\) is also a highest weight vector, which generates a highest weight submodule \(W\) of \(V\). Since \(\pi_1(v_1, v_2) = v_1\), \(\pi_1|W : W \rightarrow S_1\) are non-zero and hence surjective as the \(S_1\) are simple. This implies that the \(S_i\) are simple quotients of \(W\). Being a highest weight module, \(W\) has a unique simple quotient. This implies \(S_1 \cong S_2\). 

For \(mt^\alpha \in \mathcal{M}_0\), a highest weight of an object in category \(\mathcal{O}\), fix a simple object \(S(mt^\alpha) \in \mathcal{O}\) of highest weight \(mt^\alpha\). Let \(S\) be the set of all such \(S(mt^\alpha)\).

Define the completed Grothendieck group \(K_0(\mathcal{O})\): elements are formal sums (possibly infinite) \(\sum_{S \in \mathcal{O}} n_S[S]\) with coefficients \(n_S \in \mathbb{Z}\) such that \(\oplus_{S \in \mathcal{S}} S^{[n_S]} \in \mathcal{O}\); addition is the usual one of formal sums. For \(X \in \mathcal{O}\) and \(S \in \mathcal{S}\), the multiplicity \(m_{S,X} \in \mathbb{Z}_{\geq 0}\) of \(S\) in \(X\) is a well-defined due to Definition 3.1 (1)–(2), as in the case.

\footnote{This is stronger than the notion of highest weight in [16] §3.}
of Kac–Moody algebras [20] §9.6. Furthermore \([X] := \sum_{S \in \mathcal{S}} m_{S,X} [S] \in K_0(\mathcal{O}).\) By Definition 3.8 \([X] \mapsto \chi_q(X)\) extends uniquely to a morphism of additive groups \(\chi_q : K_0(\mathcal{O}) \rightarrow \mathcal{M}_t.\)

**Proposition 3.9.** If \(X, Y \in \mathcal{O}\), then the tensor product representation \(X \otimes Y\) is in category \(\mathcal{O}\) and \(\chi_q(X \otimes Y) = \chi_q(X)\chi_q(Y).\)

**Proof.** Conditions (1)–(2) in Definition 3.1 are clear for the representation \(X \otimes Y\). The proof of Condition (3) for \(X \otimes Y\) is similar to [20] §2.4. For \(\alpha, \beta \in \mathbb{C}\), choose ordered bases \((v^\alpha_i)_{1 \leq i \leq r_\alpha}\) and \((w^\beta_j)_{1 \leq j \leq s_\beta}\) for \(X[\alpha]\) and \(Y[\beta]\) respectively as in Definition 3.5. Order the basis \((v^\alpha_i \otimes w^\beta_j)_{\alpha, \beta, i, j}\) of \(X \otimes Y\) such that:

(a) \(v^\alpha_{i_1} \otimes w^\beta_{j_1} \leq v^\alpha_{i_2} \otimes w^\beta_{j_2}\) if \(i_1 \leq i_2 \leq \alpha_\) and \(j_1 \leq j_2 \leq s_\beta;\)

(b) \(v^\alpha_{i} \otimes w^\beta_{j} \prec v^\alpha_{i+\ell} \otimes w^\beta_{j+\ell-2}\) if \(i \leq r_\alpha+\ell\), \(j \leq s_\beta-\ell, k \leq r_\alpha\) and \(l \leq s_\beta.\)

By Equation (2.6) and Lemma 2.4 in \(D_{X \otimes Y}\) we have

\[K^X \otimes Y(z) = K^X(z) \otimes K^Y(z) + L^X_{-\ell}(z) \otimes L^Y_{\ell}(z).\]

Since \(L^X_{-\ell}(z) \in (D_X)_{-1, 1}\) and \(L^Y_{\ell}(z) \in (D_Y)_{1, -1}\), the ordered basis \((v^\alpha_i \otimes w^\beta_j)\) induces an upper triangular matrix for \(K^X \otimes Y(z)\), whose diagonal element associated to \(v^\alpha_i \otimes w^\beta_j\) is the product of those associated to \(v^\alpha_i, w^\beta_j\). So \(K^X \otimes Y(z)\) is invertible.

Conditions (3)–(4) for \(X\) indicate that the \(2 \times 2\) matrix \(L^X(z)\) is invertible. Set

\[L^X(z)^{-1} = \begin{pmatrix} L^X_{++}(z) & L^X_{+-}(z) \\ L^X_{-+}(z) & L^X_{--}(z) \end{pmatrix}.\]

Then \(L^X_{ij}(z) \in (D_X)_{i, j}\) for \(i, j \in \{\pm\}\) and \(L^X_{++}(z) = K^X(z)^{-1}.\) By Equation (2.6), \(L^X \otimes Y(z)\) is also invertible, and we have in \(D_{X \otimes Y}:\)

\[K^X \otimes Y(z)^{-1} = K^X(z)^{-1} \otimes K^Y(z)^{-1} + L^X_{++}(z) \otimes L^Y_{++}(z).\]

So similar arguments for \(K^X \otimes Y(z)\) work for \(K^X \otimes Y(z)^{-1}.\) In particular Condition (4) is true for \(X \otimes Y\) and the \(q\)-character formulas match. \(\square\)

\(K_0(\mathcal{O})\) is a ring with multiplication induced from \([X][Y] := [X \otimes Y]\) for \(X, Y \in \mathcal{O}.\)

**Corollary 3.10.** \(\chi_q : K_0(\mathcal{O}) \rightarrow \mathcal{M}_t\) is an injective homomorphism of rings. In particular, \(K_0(\mathcal{O})\) is a commutative ring.

The proof of injectivity is standard as in [20] Theorem 3 (1)], making use of Lemma 3.8. We are able to prove the first main result of this paper.

**Theorem 3.11.** \([W^{f,0} \otimes W^{u,0}] = [W^{f-u,0} \otimes W^{u,0}] \in K_0(\mathcal{O})\) for \(f, u \in \mathbb{C}.\)

**Proof.** Let us make explicit the variable \(z\) in elements of \(\mathcal{M}_t.\) Set \(m(z) := \chi_q(W^{0,0}).\) From Example 3.7 and Equation (2.7) we obtain

\[\chi_q(W^{f,u}) = m(z + uh) [\theta(z + (f + u + 1)h), \theta(z + (u + 1)h)]^f\]

and \(\chi_q(W^{f,0} \otimes W^{u,0}) = \chi_q(W^{f-u,0} \otimes W^{u,0}).\) Conclude by Corollary 3.10. \(\square\)

The spin/spectral parameters in asymptotic representations are interchangeable. This observation will lead to functional relations in Section 4.

## 4. Factorization of simple representations

We prove more general facts on simple objects in category \(\mathcal{O}\).

By Lemma 3.8 simple objects in category \(\mathcal{O}\) are parametrized by their highest weights. The following result, stated without proof in [15] Theorem 9, describes all such highest weights as elements in \(\mathcal{M}_t.\)
Applying Equation (2.9) with \( w \) By setting one-dimensional representations of highest weight \( g \)

\[ \text{Theorem 4.1.} \]

\[ O \]

Such a non-zero meromorphic function must be of the form in the theorem.

\[ \alpha, \beta \]

\[ \text{The following result and its proof are adapted from half of [31, Proposition 3.6].} \]

\[ \alpha, \beta \]

\[ \theta(z-w+x+\alpha h)\theta(h) \]

\[ \theta(z-w+x+\alpha h) \]

\[ \theta(z-w+x) \]

\[ \theta(h) \]

\[ \sum_{k=1}^{n} A_k(z; x + h)B_k(w; x). \]

Set \( h(z) \) \( = \frac{a^+(z)}{a^-(z)} \). Multiply both sides of the identity by \( \frac{\theta(z-w+h)}{\theta(z-w)\theta(x+\alpha h)}; \)

\[ \frac{\theta(z-w+x+\alpha h)}{\theta(x+\alpha h)}h(w) - \frac{\theta(z-w+x)}{\theta(x)}h(z) = \frac{\theta(z-w)}{\theta(h)} \sum_{k=1}^{n} A_k(z; x + h)B_k(w; x). \]

By setting \( w = z + 1 \) and \( w = z + \tau \) respectively, we obtain

\[ h(z+1) = h(z), \quad h(z+\tau) = e^{-2\pi i \alpha k} h(z). \]

Such a non-zero meromorphic function must be of the form in the theorem.

The proof of the “if” part is standard, by taking tensor products of the \( W^{t, u} \) and one-dimensional representations of highest weight \( [g(z), g(z)]^t h \) for \( g(x) \in \mathbb{M}^x. \) \( \Box \)

For \( \alpha, \beta \in \mathbb{C} \), let \( L(\alpha, \beta) := S(\theta(z+\alpha h), \theta(z+\beta h))^{t, \beta} \); by Remark 4.1, it is the submodule of \( W^{t, \beta, \beta-1} \) generated by \( w_0 \). Define

\[ \Sigma(\alpha, \beta) := \left\{ \begin{array}{ll} \{\beta + p \mid 0 \leq p \leq l - 1, p \in \mathbb{Z}\} & \text{if } \alpha - \beta \in l + h^{-1}(\mathbb{Z} + \mathbb{Z}) \tau, l \in \mathbb{Z}_{\geq 0}, \\ \{\beta + p \mid p \in \mathbb{Z}_{\geq 0}\} & \text{otherwise.} \end{array} \right. \]

If \( p > 0 \), then \( w_p \in L(\alpha, \beta) \) implies \( \beta + p - 1 \in \Sigma(\alpha, \beta) \).

The following result and its proof are adapted from half of [31] Proposition 3.6].

**Proposition 4.2.** Let \( n \in \mathbb{Z}_{>0} \) and \( \alpha_1, \alpha_2, \ldots, \alpha_n, \beta_1, \beta_2, \ldots, \beta_n \in \mathbb{C} \) be such that: \( \alpha_j - \Sigma(\alpha_i, \beta_i) \) and \( h^{-1}(\mathbb{Z} + \mathbb{Z}) \tau \) do not intersect for all \( 1 \leq i < j \leq n \). Then the tensor product \( L(\alpha_1, \beta_1) \otimes L(\alpha_2, \beta_2) \otimes \cdots \otimes L(\alpha_n, \beta_n) \) contains a unique non-zero vector (up to scalar product by \( \mathbb{M} \)) annihilated by \( L_{-+}(z) \).

**Proof.** By induction on \( n \): for \( n = 1 \) this is trivial as \( L(\alpha_1, \beta_1) \) is simple. Let \( n > 1 \). Suppose \( 0 \neq v \in \bigotimes_{j=1}^{n} L(\alpha_j, \beta_j) \) is annihilated by \( L_{-+}(z) \). Set \( \mathcal{L} := \bigotimes_{j=1}^{n} L(\alpha_j, \beta_j) \) and \( v = \sum_{r=0}^{p} v_r \otimes v_r \in W^{t, \beta_1, \beta_1-1} \otimes \mathcal{L} \) with \( v_r \in \mathcal{L} \) for \( 0 \leq r \leq p \) and \( v_p \neq 0 \). Computing the term \( w_p \otimes \mathcal{L} \) in \( L_{-+}(z)v = 0 \) gives \( L_{-+}(z)v_p = 0 \). The induction hypothesis applied to \( \mathcal{L} \), one may assume \( v_p = w_0^{n-1} \). If \( p = 0 \) then \( v = w_0^n \) and we are done. If \( p > 0 \), then \( \beta_1 + p - 1 \in \Sigma(\alpha_1, \beta_1) \). Consider the
component \( w_{p-1} \otimes L \) in \( L_{-}(z)v = 0 \):

\[
\prod_{j=2}^{n} \theta(z + \alpha_j \hbar) \left( \frac{\theta(z + (\beta_1 - 1 + p)h - x)\theta(ph)w_{p-1} \otimes v_p}{\theta(x)} \right) = \theta(z + (p - 1 + \beta_1)h) (w_{p-1} \otimes L_{-}(z)v_{p-1}) .
\]

By Proposition 2.2 \( L_{-}(z)v_{p-1} \in L \) is entire on \( z \). Set \( z = -(p - 1 + \beta_1)h \). Then \( \prod_{j=2}^{n} \theta((\alpha_j - (\beta_1 + p - 1))h) = 0 \), and \( \alpha_j - (\beta_1 + p - 1) \in h^{-1}(\mathbb{Z} + \mathbb{Z}r) \) for some \( 2 \leq j \leq n \), in contradiction with the assumption \( \beta_1 + p - 1 \in \Sigma(\alpha_1, \beta_1) \). \( \square \)

**Proposition 4.3.** Let \( n, \alpha_i, \beta_i \) be as in Proposition 4.2. Assume: \( \beta_1 + \Sigma(-\beta_i, -\alpha_i) \) and \( h^{-1}(\mathbb{Z} + \mathbb{Z}r) \) do not intersect for all \( 1 \leq i < j \leq n \). Then the tensor product \( L(\alpha_1, \beta_1) \otimes L(\alpha_2, \beta_2) \otimes \cdots \otimes L(\alpha_n, \beta_n) \) is a highest weight module.

**Proof.** Use induction on \( n \). For \( n = 1 \) this is obvious. Let \( n > 1 \). Assume that \( L := \otimes_{j=2}^{n} L(\alpha_j, \beta_j) \) is of highest weight, with highest weight vector \( w_0 \otimes \cdot \cdot \cdot \otimes L(\alpha_n, \beta_n) \) being a highest weight module. Let us prove by induction on \( p \in \mathbb{Z}_{\geq 0} \) with \( w_p \in L(\alpha_1, \beta_1) \) that \( w_p \otimes w_0 \otimes \cdot \cdot \cdot \otimes \cdot \cdot \cdot \in S \). For \( p = 0 \) this is trivial. Let \( p > 0 \) and \( w_p \in L(\alpha_1, \beta_1) \). Then \( \beta_1 + p - 1 \in \Sigma(\alpha_1, \beta_1) \) and \( \theta(\alpha_1 - \beta_1 - p + 1)h) \neq 0 \). By induction hypothesis \( w_{p-1} \otimes w_0 \otimes \cdot \cdot \cdot \in S \). Applying \( L_{-}(z) \) to this vector gives

\[
\prod_{j=2}^{n} \theta(z + (\alpha_1 - p + 1)h)\theta(x + (1 - p)h)\theta(x + (\alpha_1 - \beta_1 - 2p + 3)) w_{p-1} \otimes L_{-}(z)w_0 \otimes \cdot \cdot \cdot
\]

\[
+ \prod_{j=2}^{n} \theta(z + \beta_j h) \left( \frac{\theta(z + (\alpha_1 - p)h)\theta((\alpha_1 - \beta_1 - p + 1)h)}{\theta(x + (\alpha_1 - \beta_1 - 2p + 1)h)} w_p \otimes w_0 \otimes \cdot \cdot \cdot \right) .
\]

\( L_{-}(z)w_0 \otimes \cdot \cdot \cdot \in L \) being entire on \( z \), set \( z = -(\alpha_1 - p + 1)h \) so that the first term vanishes. By assumption, the second term is non-zero. So \( w_p \otimes w_0 \otimes \cdot \cdot \cdot \in S \). This proves \( L(\alpha_1, \beta_1) \otimes w_0 \otimes \cdot \cdot \cdot \subseteq S \). The vectors \( v \in L \) such that \( L(\alpha_1, \beta_1) \otimes v \subseteq S \) form a submodule \( L' \) of \( L \). Since \( L \) is generated by \( w_0 \otimes \cdot \cdot \cdot \), we have that \( L' = L \) and \( \otimes_{j=1}^{n} L(\alpha_j, \beta_j) \) is a highest weight module. \( \square \)

Our proof is similar to those of [9, Lemma 4.10] and [32, Theorem 4.6] for quantum affine \( sl_2 \). The Vandermonde determinant arguments in loc. cit. can be simplified and strengthened by Weyl modules, as indicated in the proof of [8, Theorem 2.6(iii)]; see a closer situation in [34, Proposition 5.2].

**Remark 4.4.** One might define the twisted dual \( X^\vee \) of \( X \in \mathcal{O} \) as in [35, §3] and [34, §7]; the pullback of the usual Hopf dual by a Cartan involution which permutes highest weights and lowest weights. Presumably \( L(\alpha, \beta) \cong L(-\beta, -\alpha) \) up to tensor product by one-dimensional modules, and Propositions 4.2, 4.3 are dual to each other. The Hopf dual of \( \mathcal{E} \)-modules was discussed in [13, §11].

The following result is parallel to [31, Proposition 3.6] and [32, Corollary 4.7].

**Corollary 4.5.** In Theorem 4.1, there exist a rearrangement of the \( \alpha_k, \beta_k \) and a one-dimensional \( \mathcal{E} \)-module \( D \) of weight zero such that the simple module of highest weight \( m \alpha \) is isomorphic to \( D \otimes L(\alpha_1, \beta_1) \otimes L(\alpha_2, \beta_2) \otimes \cdots \otimes L(\alpha_n, \beta_n) \).

**Proof.** We follow the arguments right after [31, (3.19)]. Notice that the integer part \( [z] \) of an element \( z \in \mathbb{Z} + h^{-1}(\mathbb{Z} + \mathbb{Z}r) \) is well-defined by the projection \( \mathbb{Z} + h^{-1}(\mathbb{Z} + \mathbb{Z}r) \rightarrow \mathbb{Z} \), \( n + h^{-1}x \rightarrow n \) for \( n \in \mathbb{Z} \), \( x \in \mathbb{Z} + \mathbb{Z}r \). One can rearrange the \( \alpha_k, \beta_k \) such that: for every \( 1 \leq k \leq n \), if

\[
X_k := \{ \alpha_p - \beta_q \mid k \leq p, q \leq n \} \cap (\mathbb{Z} + h^{-1}(\mathbb{Z} + \mathbb{Z}r)) \neq \emptyset,
\]
then \( \alpha_k - \beta_k \in X_k \) and \( |\alpha_k - \beta_k| \leq |z| \) for all \( z \in X_k \). The conditions in Propositions \( 4.2 \) and \( 4.3 \) are fulfilled. \( \otimes_{k=1}^n L(\alpha_k, \beta_k) \) is of highest weight and contains a unique highest weight vector; it must be simple by Lemma \( 5.3 \).

Set \( g(z) = a^{-1}(z) \prod_{k=1}^n (z + \alpha_k)^{-1} \) and let \( D \) be the simple module of highest weight \( [g(z), g(z)] \). Then \( D \) is one-dimensional, and \( D \otimes (\otimes_{k=1}^n L(\alpha_k, \beta_k)) \) is a simple module of highest weight \( ml^n \). This completes the proof.

As a consequence, we obtain a highest weight classification of finite-dimensional simple modules in \( \mathcal{O} \); its proof is identical to that of \( [31, \text{Proposition 3.7}] \).

**Corollary 4.6.** The simple module in Theorem \( 4.2 \) is finite-dimensional if and only if: after a rearrangement of the \( \alpha_k, \beta_k \) we have \( \alpha_k - \beta_k \in \mathbb{Z}_{\geq 0} + h^{-1}(\mathbb{Z} + \mathbb{Z} \tau) \) for all \( 1 \leq k \leq n \).

In \( [28, \text{Theorem 4.11}] \) there is a similar classification in the formal setting, based on the link between quantum affine algebras and elliptic quantum groups via difference equations. See also \( [7, \text{Theorem 5.1}] \) for a different approach.

**Corollary 4.7.** Let \( X \in \mathcal{O} \) be finite-dimensional. In the fractional ring of \( K_0(\mathcal{O}) \), \( [X] \) is a Laurent polynomial in the \( \left( \frac{\chi_q(W^j)}{\chi_q(W^{-j})} \right)_{j \in \mathbb{C}} \) whose coefficients are \( \mathbb{Z} \)-linear combinations of the isomorphism classes of one-dimensional objects in \( \mathcal{O} \).

**Proof.** Since \( [X] \in K_0(\mathcal{O}) \) is a sum of isomorphism classes of finite-dimensional simple objects, in view of Corollaries \( 4.2 \) \( 4.3 \) we may assume \( X = V \) where \( l \in \mathbb{Z}_{\geq 0} \) and \( V \) is the simple socle of \( W^l \); see Remark \( 4.17 \). By Example \( 5.7 \)

\[
\chi_q(V^l) = \sum_{j=0}^l \left[ \frac{\theta(z + (j + 1)h)\theta(z)}{\theta(z + (j + 1)h)} \right]^{l-2j} \sum_{j=0}^l \chi_q(D_j) \frac{\chi_q(W^j)}{\chi_q(W^{-j})} \frac{\chi_q(W^{-1})}{\chi_q(W^{-1})}.
\]

\( D_j \) is the one-dimensional \( \mathcal{E} \)-module of highest weight \( [\theta(z + (j + 1)h), \theta(z + (j + 1)h)] \) for \( 0 \leq j \leq l \). Replacing \( \chi_q(\cdot) \) by \( [\cdot] \) we obtain the desired result.

Corollary \( 4.7 \) corresponds to \( [18, \text{Theorem 4.8}] \), and can be viewed as generalized Baxter relations in the category \( \mathcal{O} \).

5. Transfer matrices and functional relations

Fix an even positive integer \( L \in 2\mathbb{Z}_{>0} \) and \( a_1, a_2, \ldots, a_L \in \mathbb{C} \setminus (\mathbb{Z} + \mathbb{Z}\tau) \). Let \( V_L \) be the \( \mathbb{M} \)-linear subspace of \( V^\otimes L \) spanned by the \( v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_L} := v_{i_L} \) such that \( \sum_{i=1}^L i_j = 0 \); i.e., there are as many positive \( i \)'s as negative \( i \)'s. Following \( [16] \) we construct an action of \( K_0(\mathcal{O}) \) on \( V_L \) by difference operators.

Introduce formal variables \( p^a \) for \( a \in \mathbb{C} \) such that \( p^a p^b = p^{a+b} \). For \( a \in \mathbb{C} \), define \( T_a = (D_{V_L})_{-a,-a} \) by \( T_a(v_{i_L}) = v_{i_L} \); see also \( [31, \text{\S 4.2}] \).

Let \( X \in \mathcal{O} \). Associate to two basis vectors \( v_{i_L}, v_{j_L} \in V_L \) the operator

\[
L^X_{ij}(z) := L_{i_1,j_1}(z + a_1 - h)L_{i_2,j_2}(z + a_2 - h) \cdots L_{i_L,j_L}(z + a_L - h) \in (DX)_{0,0}.
\]

Since \( (DX)_{0,0} \subseteq \text{End}_\mathbb{M}(X) \), one can take trace of \( L^X_{ij}(z) \) over weight spaces.

**Definition 5.1.** The transfer matrix associated to \( X \in \mathcal{O} \) is the formal sum

\[
t_X(z; p) := \sum_{a \in \text{wt}(X)} p^a T_a \sum_{i_L} E_{i_L} \text{Tr}_{X[a]} \left( L^X_{ij}(z) |_{X[a]} \right).
\]

Here the \( E_{i_L} \in \text{End}_\mathbb{M}(V_L) \) are elementary matrices associated to the \( v_{i_L} \).
The coefficients of the $p^n T_n$ are $\text{End}_g(V_L)$-valued meromorphic functions on $z \in \mathbb{C}$. When $X$ is finite-dimensional, one may take $p = e^w$ for a certain complex number $w$ and $p^n = e^{nw}$. Then $t_X(z; p) \in D_{V_L} \subseteq \text{End}_g(V_L)$.

**Remark 5.2.** Consider the $\mathcal{E}$-module $V$ constructed after Equation (2.7). $t_V(z; 1)$ can be identified with the transfer matrix $T(z)$ in [16, Theorem 1] with $W$ being the tensor product $\Psi_{a_1 h^{-1}}(V) \otimes \Psi_{a_2 h^{-1}}(V) \otimes \cdots \otimes \Psi_{a_k h^{-1}}(V)$.

**Example 5.3.** Let $D \in \mathcal{O}$ be one-dimensional of highest weight $mt^\alpha$. By Theorem 4.4 and Corollary 4.5, there exists $g(x) \in \mathbb{M}^+$ such that $mt^\alpha = |g(z), g(z)|^{0}$. It follows that $t_D(z; p) = \prod_{i=1}^L g(z + a_i - \hbar)$.

**Proposition 5.4.** Let $X, Y \in \mathcal{O}$ and $t, u \in \mathbb{C}$. The following equations hold.

(i) $t_{\Psi_X^{-1}}(z; p) = t_X(z + uh; p)$.

(ii) $t_X(z; p) t_Y(z; p) = t_{X \otimes Y}(z; p)$.

(iii) $t_W(z; p) t_{W'}(z + uh; p) = t_{W \circ W'}(z + uh; p) t_{W''}(z; p)$.

(iv) $t_X(z; p) t_Y(w; p) = t_Y(w; p) t_X(z; p)$.

**Proof.** (i) is clear from Equation (2.7). For $v_\perp$ and $v_\parallel$ two basis vectors in $V_L$, $L^X \otimes Y(z) = \sum_{k_1, k_2, \ldots, k_L \in \{\pm 1\}} L^X_{k_1}(z + a_k - \hbar) \otimes L^Y_{k_2}(z + a_k - \hbar)$. The terms with $\sum_{i} k_i \neq 0$ disappear in $t_{X \otimes Y}(z; p)$, as they have zero trace over the weight spaces. (ii) is based on the following identity: let $\alpha \in \text{wt}(X), \beta \in \text{wt}(Y)$, $T_{t \alpha + \beta} \text{Tr}_{X[\alpha], Y[\beta]}(f \otimes g) = T_{\alpha} \text{Tr}_{X[\alpha]}(f) T_{\beta} \text{Tr}_{Y[\beta]}(g)$ for $f \in (D_X)_{0, t}, g \in (D_Y)_{0, 0}$. (i), (ii) and Theorem 1.1 imply (iii), (iv) is proved in the same way as [18, Theorem 5.3], using the commutativity of $K_0(\mathcal{O})$ in Corollary 5.11.

Notice that for $\ell \in \mathbb{C}$, the $L^W_{\ell}$ are entire on $z$. The coefficients of $p^n T_\ell$ in $t_W(z; p)$ are $\text{End}_g(V_L)$-valued entire functions on $z$.

**Definition 5.5.** The Baxter $Q$-operator is $Q(z; p) := t_{W_{\ell=1}}(0; p)$.

**Theorem 5.6.** (i) $t_{W_{\ell=1}}(z; p) = Q(z; p)$ for $\ell \in \mathbb{C}$.

(ii) Assume $a_1 = a_2 = \cdots = a_L = a$ and $L = 2n$. Then

$$
\tilde{Q}(z + 1; p) = (-1)^n \tilde{Q}(z; p), \quad \tilde{Q}(z + \tau; p) = (-1)^n e^{-n\pi(x + 2z + 2a)} \tilde{Q}(z; p),
$$

where $\tilde{Q}(z; p) := p^{-zh^{-1}} T_{-zh^{-1}} Q(z; p)$.

**Proof.** The formal power series $t_W(z; p)$ and $Q(z; p)$ are both invertible. Replacing $(\ell, u, z)$ in Proposition 5.3 (iii) by $(zh^{-1} + \ell, zh^{-1}, 0)$ we obtain

$$
Q(z + \ell h; p) t_W(z; p) = t_W(z; p) Q(z; p),
$$

which leads to (i) by the commutativity of transfer matrices.

Consider the representation $\mathcal{W}$ of $\mathcal{E}$ on the same underlying $\mathbb{M}$-vector space $W := \bigoplus_{k=0}^L \mathbb{M} u_k$ in Proposition 2.5. The second row of $L^L(z; x)$ is independent of $\ell$, while the first row are $\text{End}_g(W)$-valued entire functions $g(\ell)$ on $\ell$ satisfying...
$g(l) \sim \theta(\ell h + z + h)$. Under the assumption $a_l = a$ for $1 \leq l \leq L$, the $L^{W_{m-1}}_L(0)$, as $\text{End}_{\mathbb{C}}(W)$-valued entire functions $h(\ell)$ on $\ell$, satisfy $h(\ell) \sim \theta(\ell + a)^n$ because the number of $1 \leq \ell \leq 2n$ with $i_\ell = \pm n$. From

$$\tilde{Q}(z;p) = \sum_{k=0}^{\infty} p^{-2k} T_{-2k} \sum_{i=1}^{\infty} E_{i,L} T_{ibw_1} \left( L^{W_{m-1}}_L(0) \right)$$

we obtain the desired double periodicity in (ii).

Remark 5.7. In Theorem 5.7 (i) is to be compared with [13 Corollary 4.6], and (ii) with [13 Theorem 5.9]. Combining the proof of Corollary 4.7 with Theorem 5.9 (i), we conclude as in [13 Theorem 4.8] that for $n \in \mathbb{Z}_{>0}$:

$$(5.10) \quad t_{V^*}(z;p) = \sum_{j=0}^{n} \frac{Q(z + nh;p)Q(z - nh;p)}{Q(z + jh;p)Q(z + (j - 1)h;p)} \prod_{l=1}^{L} \theta(z + ai + jh).$$

Remark 5.8. It is important to understand the convergence of $\tilde{Q}(z;p)$ with respect to $p$, as in [13 Remark 5.12(ii)]. For example, let $L = 2$. With respect to the basis $(v_+ \otimes v_-; v_- \otimes v_+)$ of $V_2$ is the two-by-two matrix $\tilde{Q}(z;p) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$:

$$A = \sum_{j=0}^{\infty} p^{-2j} T_{-2j} \frac{\theta(z + a_1 - jh)\theta(z + x + (-j + 1)h)\theta(x - jh)}{\theta(x)\theta(z + x + (-2j + 1)h)},$$

$$B = -\sum_{j=1}^{\infty} p^{-2j} T_{-2j} \frac{\theta(z + a_1 + x - jh)\theta(z - (j - 1)h)\theta(a_2 - x + jh)\theta(jh)}{\theta(x - h)},$$

$$C = -\sum_{j=1}^{\infty} p^{-2j} T_{-2j} \frac{\theta(a_1 - x + jh)\theta((j + 1)h)\theta(z + a_2 + x - jh)\theta(z - jh)}{\theta(z + x - 2jh)},$$

$$D = \sum_{j=0}^{\infty} p^{-2j} T_{-2j} \frac{\theta(a_1 + jh)\theta(z + a_2 - jh)\theta(z + x + jh)\theta(x - (j + 1)h)}{\theta(x - h)\theta(z + x + 2jh)}.$$

The convergence of $A, B, C, D$ in $p$ is still unclear to us.

Remark 5.9. Assume $a_1 = a_2 = \cdots = a_L = a$ and $L = 2n$. Suppose that $\tilde{Q}(z;p)$ converges for certain $p = e^w$. If $Q(z;p)$ has an eigenvalue $q(z)$ which is independent of $x$, then by Theorem 5.6 (ii), $q(z) = cp^{2n-1} \prod_{i=1}^{n} \theta(z - z_i)$ where $c \in \mathbb{C}^*$ and $z_i \in \mathbb{C}$. Since $t_{V^*}(z;p)$ is entire on $z$, by Equation (5.10) we have:

$$(5.11) \quad p^2 \left( \frac{\theta(z_k + a)}{\theta(z_k + a + h)} \right)^{2n} \prod_{1 \leq j \leq n, j \neq k} \frac{\theta(z_k - z_j - h)}{\theta(z_k - z_j + h)} \quad \text{for} \quad 1 \leq k \leq n,$$

$$(5.12) \quad z_1 + z_2 + \cdots + z_n - na \in \mathbb{Z} + \mathbb{Z} r.$$

The first equation was deduced in [13] from Bethe ansatz, and the second is Baxter’s sum rule for homogeneous models $(a_1 = a_2 = \cdots = a_L)$, to be compared with [33] for the eight-vertex model based on representations of the Sklyanin’s elliptic algebra (the elliptic quantum group associated with $\mathfrak{sl}_2$ of vertex type [25]).

6. Further discussions

In this paper, for the elliptic quantum group associated to $\mathfrak{sl}_2$, we have introduced a category $\mathcal{O}$ of representations and studied in details the asymptotic representations $W^{W_{m-1}}$. The $W^{W_{m-1}}$ satisfy a “separation of variables” identity (Theorem 5.11) and are used to construct the elliptic Baxter $Q$-operator (Definition 5.5).
For an affine quantum group (of an arbitrary complex finite-dimensional simple Lie algebra), Baxter Q-operators have been constructed in [18, §4] as transfer matrices over positive pre-fundamental representations, generalizing previous works of Bazhanov–Lukyanov–Zamolodchikov on \( sl_2 \) [4, 5]. The pre-fundamental representations can only be defined over a Borel subalgebra instead of the full quantum group [23]. Let us comment that in the elliptic case the Borel subalgebras and their pre-fundamental representations are still unavailable.

The ideas of the paper should work for affine quantum groups and Yangians, leading to a second construction of Q-operators without reference to Borel subalgebras [23, 18] or double Yangians [3]. As an illustrating example, in the appendix we discuss the two definitions of Q-operators for the Yangian of \( sl_2 \). In general, asymptotic representations should be in corresponding categories \( O \) [22, §3] (non-integrable); see [35, §5] in the case of quantum affine \( gl(m|n) \).

We plan to work on elliptic quantum groups for Lie (super)algebras of higher ranks [7, 21, 29]. The asymptotic representations might be deduced from those of affine quantum groups by twistors [25] or by difference equations [22, 29]. They might also be realized directly as “asymptotic limits” of finite-dimensional representations [23, 35]; see a particular example for \( E_\tau, \hbar (gl_N) \) in [7, §3.4].

An important property of Q-operators is that their roots satisfy the Bethe ansatz equations. For elliptic quantum \( sl_2 \), this is derived from the two-dimensional irreducible representations; see Remark 5.9. In higher ranks, such two-dimensional representations do not exist, and we should consider infinite-dimensional representations, as indicated in the recent works on affine quantum groups [24, 19, 13] and quantum toroidal \( gl_1 \) [12].

At last let us mention another work [17] on quantum integrable systems with boundary conditions. In loc. cit., the Q-operator is the Sklyanin transfer matrix of the pre-fundamental representation of the Yangian of \( sl_2 \) (see Example A.3), but the proof of TQ relations is less representation-theoretic. It remains open what a tensor category \( O \) of representations would be in this context.

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Appendix A. Baxter operators for the Yangian

In this appendix we study the asymptotic representations of the Yangian algebra \( Y(gl_2) \) and use their spin parameter to define the Baxter Q-operator. Then we compare our Q-operator with that in [2]; see Remark A.7. The results (category \( O \), \( q \)-characters, functional relations of transfer matrices, etc.) are almost identical to the elliptic case. Most of their proofs will be omitted.

Throughout this section, vector spaces, linear maps and tensor products are over \( \mathbb{C} \). Let us first define the Yangian. Fix a basis \((v_1, v_2)\) of the vector space \( \mathbb{C}^2 \). With respect to the basis \((v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2)\) of \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) set

\[
R(z) := \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \frac{z+1}{z} & 0 & 0 \\
0 & 0 & \frac{z+1}{z} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \in \text{End}(\mathbb{C}^2)^{\otimes 2}.
\]

It is well-known that \( R \) verifies the quantum Yang–Baxter equation in \( \text{End}(\mathbb{C}^2)^{\otimes 3} \):

\[
R^{12}(z-w)R^{13}(z)R^{23}(w) = R^{23}(w)R^{13}(z)R^{12}(z-w).
\]
Let \( r \in \mathbb{Z}_{\geq 0} \). The Yangian \( Y^r \) is an algebra generated by \( t_{ij}^{(n)} \) with \( 1 \leq i, j \leq 2 \) and \( n \in \mathbb{Z}_{\leq r} \). Introduce generating functions \( T(z) \in \text{End}(\mathbb{C}^2) \otimes Y^r((z^{-1})) \) where the \( E_{ij} \in \text{End}(\mathbb{C}^2) \) are elementary matrices for \( 1 \leq i, j \leq 2 \). The defining relation of \( Y^r \) is the following identity in \( \text{End}(\mathbb{C}^2) \otimes Y^r((z^{-1})) \):

\[
R^{12}(z-w)T^{13}(z)T^{23}(w) = T^{23}(w)T^{13}(z)R^{12}(z-w).
\]

(A.13)

One has natural projections of algebras \( Y^{r+1} \to Y^r : t_{ij}^{(n)} \to t_{ij}^{(n)} \). To \( r, s \in \mathbb{Z}_{\geq 0} \) is attached an algebra homomorphism

\[
(A.14) \quad \Delta^{rs} : Y^{r+s} \to Y^r \otimes Y^s, \quad T(z) \mapsto T_{12}(z)T_{13}(z).
\]

The \( \Delta^{rs} \) are co-associative: \((1 \otimes \Delta^{rs})\Delta^{r,s+q} = (\Delta^{rs} \otimes 1)\Delta^{r+s,q}\). There is a one-parameter family \( \{ \Psi_u : Y^r \to Y^r \}_{u \in \mathbb{C}} \) of algebra automorphisms defined by

\[
\Psi_u : \quad t_{ij}(z) \mapsto t_{ij}(z + u) \in Y^r((z^{-1})).
\]

The algebra \( Y^r \) is \( Z \)-graded with: \( |t_{ij}^{(n)}|_Z = j - i \). A graded representation of \( Y^r \) is a \( Z \)-graded vector space \( V = \oplus_{n \in \mathbb{Z}} V_n \) (weight space decomposition) endowed with a morphism of graded algebras \( \rho : Y^r \to \text{End}(V) \). Call \( V \) a graded \( Y^r \)-module, and let \( \text{wt}(V) \) denote the set of weights of \( V \) as in Section 2.2. Write \( t_{ij}^{(n)}(z) = \rho(t_{ij}(z)) \in \text{End}(V)((z^{-1})) \) and \( T^\nu(z) = (1 \otimes \rho)(T(z)) \).

If \( (\rho, V) \) and \( (\sigma, W) \) are graded representations of \( Y^r \) and of \( Y^s \) respectively, then \( (\rho \circ \sigma) \circ \Delta^{rs} \) makes \( V \otimes W \) into a graded representation of \( Y^{r+s} \).

**Example A.1.** Let \( m \in \mathbb{Z}_{\geq 0} \). Let \( V^m \) be the vector space with basis \( (w_i)_{0 \leq i \leq m} \). Set \( w_i \) to be of weight \( i \). There is a graded \( Y^1 \)-module structure on \( V^m \):

\[
t_{11}(z)w_i = (z + m - i)w_i, \quad t_{12}(z)w_i = (m - i)w_{i+1},
\]

\[
t_{21}(z)w_i = iw_{i-1}, \quad t_{22}(z)w_i = (z + i)w_i.
\]

Here \( w_{-1} = w_{m+1} = 0 \) by convention. This comes from the evaluation map

\[
ev : Y^1 \to U(\mathfrak{gl}_2), \quad t_{ij}(z) \mapsto \delta_{ij}z - E_{ij},
\]

where \( U(\mathfrak{gl}_2) \) denotes the universal enveloping algebra of \( \mathfrak{gl}_2 \); see e.g. [31] (2.5).

**Example A.2.** Let \( V^\infty \) be the vector space with basis \( (w_i : i \in \mathbb{Z}_{\geq 0}) \) and with \( w_i \) being of weight \( i \). Fix \( \ell \in \mathbb{C} \). In the formulas of the \( t_{ij}(z)w_k \) in Example A.1 one can replace \( m \) by \( \ell \) everywhere, leading to a graded \( Y^1 \)-module structure on \( V^\infty \), denoted by \( W^\ell \). If \( \ell \in \mathbb{Z}_{\geq 0} \), then \( V^\ell \) is the simple socle of \( W^\ell \) under the natural embedding \( V^\ell \subset V^\infty, w_i \mapsto w_i \).

For \( \ell, \mu \in \mathbb{C} \), define the asymptotic representation \( W^{\ell,\mu} := \Psi^\ell_\mu(W^\ell) \).

**Example A.3.** There is another graded \( Y^1 \)-module structure on \( V^\infty \):

\[
t_{11}(z)w_i = (z - i)w_i, \quad t_{12}(z)w_i = -w_{i+1}, \quad t_{21}(z)w_i = iw_{i-1}, \quad t_{22}(z)w_i = w_i.
\]

This module, denoted by \( W \), was constructed in [23] (3.37) as a Fock space representation of the harmonic oscillator algebra.\footnote{The ordinary Yangian \( Y(\mathfrak{gl}_2) \) is the quotient of \( Y^0 \) by \( t_{ij}^{(0)} = \delta_{ij} \).}

A graded \( Y^r \)-module \( V \) is said to be in category \( \mathcal{O}^r \) if:

(i) \( \text{wt}(V) \) is bounded below and all the weight spaces are finite-dimensional;

(ii) \( T^\nu(z) \in \text{End}(\mathbb{C}^2 \otimes V)((z^{-1})) \) and \( t_{ij}^{(n)}(z) \in \text{End}(V)((z^{-1})) \) are invertible as matrix-valued Laurent series.

\footnote{\text{Tensoring} \( W^{\ell,\mu} \) with the one-dimensional module \( t_{ij}(z) = \delta_{ij}z^{-1} \) in \( \mathcal{O}^0 \) results in a module over \( Y(\mathfrak{gl}_2) \). It is unknown how to transform \( W \) into a \( Y(\mathfrak{gl}_2) \)-module.}
The projections $Y^{r+1} \rightarrow Y^r$ imply $O^0 \subseteq O^1 \subseteq O^2 \subseteq \cdots$.

By condition (ii), the two-by-two matrix $T^Y(z)$ admits a Gauss decomposition:

$$T^Y(z) = \begin{pmatrix} 1 & f^Y(z) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} K^Y(z) & 0 \\ 0 & K^Y_2(z) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^Y(z) & 1 \end{pmatrix}.$$ 

The $K^Y(z)$ are invertible and $K^Y(z)K^Y(w) = K^Y(w)K^Y(z)$.

For $\alpha \in \text{wt}(V)$, the commuting family of linear operators $K^Y(z)|_{V_\alpha}$ decomposes the finite-dimensional vector space $V_\alpha$ into a direct sum of the generalized eigenspaces $V_{(g_1, g_2, \alpha)}$ where $g_1(z), g_2(z) \in \mathbb{C}((z^{-1}))^\times$ and

$$V_{(g_1, g_2, \alpha)} := \{v \in V_\alpha \mid (K^Y(z) - g_1(z))^{N+1}v = 0 \text{ for } N > \dim V_\alpha, \ i = 1, 2\}.$$ 

Let $R$ be the group ring over $\mathbb{Z}$ of the multiplicative group $\mathbb{C}((z^{-1}))^\times \times \mathbb{C}((z^{-1}))^\times$; elements in $R$ are $\mathbb{Z}$-linear combinations of the $[g_1, g_2]$ with $g_1 \in \mathbb{C}((z^{-1}))^\times$. Introduce a formal variable $p$. The $q$-character of $V$ is a Laurent series in $p$ with coefficients in $R$; see \cite{27} §2 or \cite{22} §7.4 with $p = 1$,

$$\chi_q^Y(V) := \sum_{\alpha \in \text{wt}(V)} \sum_{g_1, g_2 \in \mathbb{C}((z^{-1}))^\times} [g_1, g_2] \dim V_{(g_1, g_2, \alpha)} \in R((p)).$$

Notice that $\chi_q^Y(V)$ is independent of the choice of $r$ such that $V \in O^r$.

Let $K_0(O^r)$ denote the completed Grothendieck group of the abelian category $O^r$. The $q$-character map induces an injective morphism of additive groups $\chi_q^Y : K_0(O^r) \rightarrow R((p))$. Furthermore, for $V \in O^r$ and $W \in O^s$ we have

$$V \otimes W \in O^{r+s} \quad \text{and} \quad \chi_q^Y(V \otimes W) = \chi_q^Y(V)\chi_q^Y(W).$$

**Remark A.4.** The $Y^1$-modules $V^m$, $W^l$, and $W$ in Examples A.1 and A.3 are in $O^1$:

$$\chi_q^Y(V^m) = [z + m, z] \sum_{i=0}^m p^i \left[ \frac{z-1}{z+i-1}, \frac{z+i}{z} \right],$$

$$\chi_q^Y(W^l) = [z + \ell, z] \sum_{i=0}^\infty p^i \left[ \frac{z-1}{z+i-1}, \frac{z+i}{z} \right] = [z + \ell, z] \chi_q^Y(W^0),$$

$$\chi_q^Y(W) = \sum_{i=0}^\infty p^i |z, 1| = \frac{[z, 1]}{1 - p}.$$ 

As a consequence of the injectivity of $q$-characters, the same identity as in Theorem 5.11 holds: $[W^{u, 0} \otimes W^{u, u}] = [W^{u-u, 0} \otimes W^{u, u}]$ in $K_0(O^2)$.

Fix a positive integer $L \in \mathbb{Z}_{>0}$ and $a_1, a_2, \ldots, a_L \in \mathbb{C}^\times$. Define the quantum space $V_L := (\mathbb{C}^2)^{\otimes L}$. We identify 12-strings $i_1i_2 \cdots i_L$ of length $L$ with the vector $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_L} \in V_L$. For $0 \leq s \leq L$, let $V_L^s$ be the subspace of $V_L$ spanned by the $i_1i_2 \cdots i_L$ where $1$ appears exactly $s$ times.

Let $W \in O^r$. Define the **monodromy matrix** in $\text{End}(V_L \otimes W)((z^{-1}))$

$$T^{W, L}(z) := T^W(z + a_1)_{1, L+1}T^W(z + a_2)_{2, L+1} \cdots T^W(z + a_L)_{L, L+1}.$$ 

The transfer matrix is obtained by taking twisted trace over $W$: $t_W(z; p) := \sum_{\alpha \in \text{wt}(W)} p^\alpha (1_{V_L} \otimes \text{Tr}_{W_\alpha})(T^{W, L}(z)) \in \text{End}(V_L)((z^{-1}, p))$.

Proposition 5.3 holds by setting $h = 1$. In particular, the transfer matrices are commuting operators in $V_L$. From the weight grading we observe that $V_L^s$ is stable by $t_W(z; p)$ for $0 \leq s \leq L$. Notice that $t_W(z; p) \in \text{End}(V^\infty)[z, \ell]$ in Examples A.1 and A.2. The following definition makes sense.

**Definition A.5.** The Baxter Q-operator is defined to be $Q(z; p) := t_W(0; p)$.
Theorem A.6. For $0 \leq s \leq L$, the $\text{End}(V^s_L)[[p]]$-valued polynomial $Q(z;p)|_{V^s_L}$ in $z$ is of degree $s$.

Proof. In the two-by-two matrix $T^{W^t}(z)$, the second row is independent of $\ell$, and the first row is a polynomial in $\ell$ of degree $s$. So in

$$T^{W^t,\ell}(0) = \sum_{i,j} E_{i,j1} \otimes E_{i,j2} \cdots \otimes E_{i,j,L} \otimes W^t_{i,j1}(a_1) W^t_{i,j2}(a_2) \cdots W^t_{i,jL}(a_L)$$

only the terms $W^t_{i,j}(a_i)$ with $i_1 = 1$ raise possibly the power of $z$ by $1$. In $V^s_L$, the number of such $i_1$ is $s$. So $Q(z;p)|_{V^s_L}$ is a polynomial in $z$ of degree $s$. We follow the idea of [13, §5.3] to prove that $Q(z;0)|_{V^s_L}$ is of degree $s$.

Let us order the basis $(i_1i_2\cdots i_L)$ of $V^s_L$ as follows: $i_1i_2\cdots i_L < j_1j_2\cdots j_L$ if there exists $1 \leq t < L$ such that $i_t = j_t$ and $i_{t+1} = j_{t+1}$ for all $t > t$. This is a total ordering. Furthermore, notice that for $i < i$, the last tensor factor in the above summation annihilates $w_0$. This means that $Q(z;0)$ is upper triangular with respect to this ordering. Its diagonal term associated to $i_1i_2\cdots i_L \in V^s_L$ is $\prod_{i=1}^L (a_i + \delta_{i,1} z)$, which is a polynomial in $z$ of degree $s$ as $a_i \neq 0$ for $1 \leq i \leq L$. □

Theorem 5.3 (i) still holds by taking $h = 1$. From Remark A.4 we deduce functional relations between the $t_{V^m}(z;p)$ and $Q(z;p)$:

$$\chi^\mathcal{Y}_q(V^m) = \sum_{i=0}^m p^i [z + i, z + i + 1] \chi^\mathcal{Y}_q(W^m) \chi^\mathcal{Y}_q(W^{-1})$$

for $m \in \mathbb{Z}_{\geq 0}$.

In terms of transfer matrices the case $m = 1$ leads to the Baxter TQ relation

$$t_{V^1}(z;p) = \frac{Q(z + 1;p)}{Q(z;p)} \prod_{i=1}^L (z + a_i) + \frac{Q(z - 1;p)}{Q(z;p)} \prod_{i=1}^L (z + a_i + 1).$$

The convergence of $Q(z;p)$ with respect to $p$ is much simpler than in the elliptic case. Indeed, the matrix coefficients of $Q(z;p)$ with respect to the basis $(i)$ are linear combinations of the power series $\sum_{i=0}^\infty p^i s^i$ with $0 \leq s \leq L$; these are rational functions in $p$ whose only possible pole is at $p = 1$.

Remark A.7. Let us compare $Q(z;p)$ with the $Q$-operator $Q_+(z)$ in [2] (3.51). Indeed, $Q_+(z)$ can be identified with the transfer matrix $t_W(z;p)|_{p=0, r=0}$ associated to $W$ in Example A.3. By Remark A.4 $\frac{W^t}{W^s}$ is an $\text{End}(V^s_L)[[p]]$-valued polynomial in $z$ of degree $s$, the coefficient of $z$ being $\frac{1}{1 - p}$. Let $A_+^s(p) \in \text{End}(V^s_L)[[p]]$ be the coefficient of $z^s$ in $Q(z;p)|_{V^s_L}$. The above equality implies

$$Q(z;p)|_{V^s_L} = (1 - p)A_+^s(p) \times t_W(z;p)|_{V^s_L}$$

for $0 \leq s \leq L$.

Remark A.8. From the proof of Theorem A.6 we see that: $A_+^s(0)$ is upper triangular whose diagonal associated to $i$ is $\prod_{i,j=2} a_{i,j}$. Under genericity conditions on the $a_{i,j}$, one can assume that $A_+^s(p)$ is diagonalizable and its eigenvalues are all of multiplicity one. Then $Q(z;p)|_{V^s_L}$ and the $t_W(z;p)|_{V^s_L}$ are all diagonalizable. For example, take $L = 2$ and $s = 1$. With respect to the basis $(21, 12)$ of $V^1_L$,

$$A_+^1(p) = \frac{1}{1 - p} \left( a_1 + \frac{p}{1 - p} a_2 + \frac{1}{1 - p} a_1 \right).$$
It has two distinct eigenvalues if and only if \((a_1 - a_2)^2 + \frac{4p}{1-p^2} \neq 0\), if and only if the following Bethe ansatz equation in \(z\) has two distinct solutions:

\[
p(z + a_1 + 1) (z + a_2 + 1) = (z + a_1) (z + a_2).
\]

Let \(z = z_1 \in \mathbb{C}\) be one of its solution. Then

\[
v(z_1) := (z_1 + a_1 + 1)v_2 \otimes v_1 + (z_1 + a_2)v_1 \otimes v_2
\]

is a common eigenvector of \(A^2_2(p), Q(z; p)\) of eigenvalues respectively

\[
\lambda(z_1) := \frac{a_1}{1-p} + \frac{p}{1-p^2} + \frac{1}{(1-p)^2} \frac{z_1 + a_2}{z_1 + a_1 + 1},
\]

\[
\lambda(z_1) \times (z - z_1).
\]

View \(V_2\) as the graded \(Y^2\)-module \(\Psi^\ast_{\Omega}(V^1) \otimes \Psi^\ast_{\Omega}(V^1)\) by \(w_i \otimes w_j = v_{i+1} \otimes v_{j+1}\). Then \(v(z_1) = t_{21}(z_1) v_2 \otimes \Psi^\ast_{\Omega}(V^1)\) comes from Algebraic Bethe Ansatz \(\cite{14} \S 4\).

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