Approximation-Free Prescribed Performance Control With Prescribed Input Constraints

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Abstract—This letter considers the tracking control problem for an unknown nonlinear system with time-varying bounded disturbance subjected to a prescribed performance and input constraints. When performance and input constraints are specified simultaneously for such a problem, a trade-off is inevitable. Consequently, a feasibility condition for prescribing performance and input constraints is devised to address such difficulties of arbitrary prescription. In addition, an approximation-free controller with low complexity is proposed, which ensures that the constraints are never violated, provided that the feasibility condition holds. Finally, simulation results corroborate the effectiveness of the proposed controller.

Index Terms—Constraint, input constraint, prescribed performance, low-complexity, approximation free.

I. INTRODUCTION

DIFFERENT methods have been developed by academic and professional researchers for designing controllers for nonlinear systems. Despite these efforts, designing controllers for systems subjected to constraints and unknown time-varying disturbances remains challenging. Various constraints arise in most practical systems, including performance constraints, saturation, physical stoppages, and safety requirements. Therefore, constraints cannot be avoided while designing controllers for practical systems. For controller design, constraints can typically be prescribed in two forms: prescribed performance constraints (PPC) on some variable (such as tracking error) and prescribed input constraints (PIC).

A wide variety of methods have been developed to address PPC, including: reference governors [1], model predictive control [2], funnel control [3], barrier Lyapunov functions (BLF) [4], prescribed performance control [5], control barrier functions [6], and extremum seeking control [7]. As far as the literature is concerned, BLF has been extensively used in dealing with constraints. That’s because its design methodology allows it to incorporate many Lyapunov-based nonlinear control techniques. However, for uncertain, unknown systems, the design lacks simplicity. The approach described in [5] is well-suited to a diverse set of situations [8] since it is low-complexity, approximation-free, and robust. Much research and development have gone into the controller design for nonlinear systems that undergo input saturation. One can refer to the results in [9], [10], [11]. It follows that controller design for nonlinear systems subjected to either PPC or PIC is a well-established field of study. Improving performance with limited resources is always difficult. Same with PPC and PIC [12], PPC aims for lower steady-state error, safe transient response, and fast convergence of tracking error. In contrast, PIC focuses on actuator safety or control effort minimization. Thus, there is a trade-off in prescribing performance and input constraints simultaneously. To get the best out of this trade-off, one must come up with a feasibility condition, and indeed, such a condition will depend on system dynamics and bounds on various control system parameters such as disturbance, reference trajectory, and initial operating range. For example, a large external disturbance or a desired trajectory with a large upper bound will inevitably necessitate the same level of opposing control command, which may extend beyond the PIC [13]. Thus, before prescribing input constraints, one must look for the feasible condition for PIC. Further, many practical systems always operate in some specified regions where they are controllable under PIC [12]. In the presence of PIC, one cannot globally stabilize the unstable system. There is always a feasible set of initial conditions for PIC. Also, global results are not attained in many PPC studies [14], [15], [16] on tracking error. The prescribed performance function choice depends on the constrained variable’s initial state. In [17], global results were achieved by transiently relaxing the PPC. However, as discussed, it makes no sense to pursue global results when there is a PIC. In addition, arbitrary PPC makes no sense because there may be an initial condition of error variable within the initial bounds of PPC that does not belong to the set of initial conditions that are feasible for PIC. Therefore, we must seek a viable PPC for a PIC. Notably, few results are available addressing PPC and PIC [18], [19], [20], [21]. In [18], works are done for linear systems, nonlinear systems in [19], [21]. Also, in [18], [19], [20] authors relax the PPC whenever the input saturation is active, and in [21] assumptions are made on the existence of a feasible set of control input for a given initial conditions and actuator saturation limit.

Motivated by the above discussions and aforementioned works, a controller has been developed in this letter with the following listed contributions:
II. PRELIMINARIES AND PROBLEM FORMULATION

Notations: We denote the set of real, positive real, non-negative real, and non-negative real numbers by \( \mathbb{R}, \mathbb{R}^+, \mathbb{R}_{0}^+, \) and \( \mathbb{N} \), respectively. \( \mathbb{N}_n = \{1, \ldots, n\} \) and \( n \) is positive integer. \( L^\infty \) represents the set of all essentially bounded measurable functions. For \( x(t) \in \mathbb{R}, x \uparrow a : x \) approaches a real value \( a \) from the left side, \( x \downarrow a : x \) approaches a real value \( a \) from the right side, and \( x^{(n)} \) represents \( n \)th time derivative of signal \( x \).

Consider a class of strict-feedback nonlinear system

\[
\begin{align*}
\dot{\xi}_i &= \xi_{i+1}, \quad \forall i \in \mathbb{N}_{n-1}, \\
\dot{\xi}_n &= f(\xi) + g(\xi)v + d, \\
y &= \xi_1,
\end{align*}
\]

where \( \xi(t) = [\xi_1(t), \ldots, \xi_n(t)]^T \in \mathbb{R}^n \) is the state vector, \( f : \mathbb{R}^n \to \mathbb{R} \) is the unknown smooth nonlinear function, \( g : \mathbb{R}^n \to \mathbb{R} \) is the unknown control coefficient, \( d(t) \in \mathbb{R}^n \) is the unknown piecewise continuous bounded disturbance, \( v(t) \in \mathbb{U} \subseteq \mathbb{R} \) and \( y(t) \in \mathbb{R} \) are the input and output of the system, respectively.

Before moving to the control goal, it is necessary to define the class of PPC and PIC dealt with in this letter. Let \( \bar{u} \in \mathbb{R}^+ \) be a prescribed constraint on input, i.e., PIC, and \( \psi : \mathbb{R}_{0}^+ \to \mathbb{R}^+ \) is PPC defined as \( \psi(t) := \psi_0 e^{-\mu t} + \psi_\infty \) where \( \psi_0 \) is a positive constant, and \( \psi_\infty \) and \( \mu \) are positive and nonnegative constants, represent the bounds on the steady-state error and the decay rate of the tracking error, respectively.

Control goal: The control problem is to design a control law \( v \) such that (i) the output \( \xi_1(t) \) track the desired output \( \bar{\xi}(t) \in \mathbb{R} \), \( \forall t \in \mathbb{R}^+ \), (ii) input follow its PIC, i.e., \( |v| < \bar{u} \) and output tracking error defined as \( \bar{\varepsilon} := \xi_1 - \bar{\xi}_d \), follow its PPC, i.e., \( |\bar{\varepsilon}| < \bar{\psi} \), \( \forall t \in \mathbb{R}^+ \), and (iii) all the closed-loop signals are bounded.

In addition, one of our problems will be to seek the feasibility condition for the PIC and PPC. Obtaining such a feasibility condition will necessitate specific knowledge of the system’s dynamics, disturbances, and tracking performance parameters in terms of their upper bounds on the signals. A few assumptions are required for this are listed below.

Assumption 1 [22], [23], [24], [25]: The unknown map \( f \) satisfies the Lipschitz continuity condition, that is, for all \( x, x' \in \mathbb{R}^n \), there exists a constant \( k_f \in \mathbb{R}^+ \) such that the following holds

\[
|f(x) - f(x')| \leq k_f||x - x'||_p^p,
\]

where \( k_f \) is a known Lipschitz constant and \( || \cdot ||_p^p \) known as the \( p \)th norm in the \( \mathbb{R}^n \).

Note that one can use the Lipschitz constant inference approaches proposed in [26], [27], [28] to estimate the Lipschitz constant of unknown dynamics from a finite number of data collected from the system.

Assumption 2: There exist a known constant \( g > 0 \) and a constant \( \bar{g} \geq g \), such that \( \bar{g} \leq g(x) \leq \bar{g} \) for all \( x \in \mathbb{R}^n \).

Assumption 3: There exists known constant \( d \geq 0 \) such that disturbances \( |d(t)| \leq d \) for all \( t \in \mathbb{R}^+_0 \).

Assumption 4: For a given desired trajectory \( \bar{x}_d \), there exists a constant \( \bar{\xi}_d > 0 \), such that \( ||\bar{\xi}_d(t)||_\infty < \bar{\xi}_d \), for all \( t \in \mathbb{R}^+_0 \) for \( \bar{\xi}_d = [\bar{\xi}_d, \bar{\xi}_d^{(1)}, \ldots, \bar{\xi}_d^{(n-1)}]^T \).

III. CONTROLLER DESIGN

This section proposes a robust approximation-free controller for (1). To begin the controller design, we define a filtered tracking error,

\[
r := \lambda_1 \bar{\varepsilon} + \lambda_2 \bar{\varepsilon}^2 + \cdots + \lambda_{n-1} \bar{\varepsilon}^{(n-2)} + \bar{\varepsilon}^{(n-1)},
\]

where \( \lambda_i, \forall i \in \mathbb{N}_{n-1} \) is a strictly positive constant and following the definition of output tracking error mentioned in the problem statement,

\[
\bar{\varepsilon}^{(i-1)} = \bar{\xi} - \bar{\xi}_d^{(i-1)}, \quad \forall i \in \mathbb{N}_n.
\]

Taking the time derivative of (2) and using (3), one has

\[
\dot{r} = \lambda_1 \bar{\varepsilon}^{(1)} + \lambda_2 \bar{\varepsilon}^{(2)} + \cdots + \lambda_{n-1} \bar{\varepsilon}^{(n-1)} + \bar{\xi}_d - \bar{\xi}_d^{(n)}.
\]

Using (4) and (5), closed-loop dynamics can be written as

\[
\dot{r} = \psi + g(\xi)v + d - \bar{\xi}_d^{(n)},
\]

where \( \phi = \sum_{i=1}^{n-1} \lambda_i \bar{\varepsilon}^{(i)} \).

Consider a non-increasing smooth function \( \psi_r : \mathbb{R}_0^+ \to \mathbb{R}_0^+ \) as a virtual performance constraint (VPC) over \( r \), defined as

\[
\psi_r(t) := \psi_0 e^{-\mu t} + \psi_\infty, \quad \forall t \in \mathbb{R}_0^+,
\]

where \( \psi_\infty, \psi_0 \) and \( \mu \) have similar attributes as of \( \psi_\infty, \psi_0 \) and \( \mu \) for PPC. Note that, in (6), \( \psi_r \) and \( \psi_r \) are bounded for all \( t \in \mathbb{R}_0^+ \) and the bounds are given as

\[
\psi_\infty \leq \psi_r \leq \psi_0 + \psi_\infty, \quad \text{and}
\]

\[
-\mu r \psi_0 \leq \psi_r \leq 0.
\]

The control input is designed as

\[
v = -\frac{2\bar{u}}{\pi} \arctan\left(\frac{\pi r}{2 \tan\left(\frac{\pi \psi_r}{2}\right)}\right),
\]

where \( \bar{u} \in \mathbb{R}^+ \) is PIC, \( r \) and \( \psi_r \) are as mentioned in (2) and (6), respectively. In (9), \( r \) is designed using (2) and (3), with

\[
\lambda_i = \frac{(n-1)!}{a^{n-1} d}, \quad a > \mu, \quad \forall i \in \mathbb{N}_{n-1}.
\]

and \( \psi_r \), i.e., VPC is chosen based on the PPC defined in the problem statement, as follows

\[
\mu_\psi = \mu,
\]

where \( \mu_\psi \) denotes the Laplace transform, and \( s \) is the Laplace variable.
ψ_0 = (a - μ_r)^{n-1} ψ_0, \quad (12)
ψ_r^∞ = d^{n-1} ψ_∞. \quad (13)

Remark 1: In (9), one can readily observe that the designed controller is simple and easy to implement. Also, it is a model-free and approximation-free controller.

IV. PRELIMINARIES FOR THE STABILITY ANALYSIS

In this section, first, a few results will be established, which will motivate the idea behind the selection of parameters of VPC (ψ_r) in (11)-(13) based on PPC (ψ_0). Further, a few lemmas will be presented, which will be later used in stability analysis. The lemmas are as follows.

Lemma 1: Consider the signals X(t) ∈ ℝ and Z(t) ∈ ℝ, such that |X(t)| < X_0 e^{-μ_r t} + X_∞, where have similar attributes as of ψ_∞, ψ_0 and μ for PPC. If z = \frac{d}{1+ω_p} x, where z = L(Z(t)), x = L(X(t)), and a > μ_r, p \in \mathbb{Z}_+, then |Z(t)| < Z_0 e^{-μ_r t} + Z_∞, with Z_0 = X_0 (\frac{2a-μ_r}{a-μ_r})^p and Z_∞ = \frac{2a}{\tau} ψ_∞.

Lemma 2: If |r| < ψ_r and \lambda_i = (n-i)^a_i, a > μ_r is a positive design constant, then for all t ∈ ℝ_0^+ and ∀t ∈ [0, 1, . . . , n - 1],

\begin{align*}
|\xi^0(t)| < \frac{(2a - μ_r)^i ψ_0}{(a - μ_r)^{n-i}} e^{-μ_r t} + \frac{2i ψ_∞}{a^{n-i}}. \quad (14)
\end{align*}

Remark 2: In (14), substituting the parameters given in (11)-(13) for i = 0, yields |\xi| < ψ_0 e^{-μ_r t} + ψ_∞, or |\xi| < ψ_r, one of our control goals. Hence, if we can make filtered tracking error r to follow its VPC ψ_r, or the hypothesis of the above lemma, i.e., |r| < ψ_r, then the goal will be achieved. To achieve the same, control input is designed in (9), based on filtered tracking error, VPC and PIC.

Corollary 1: If |r| < ψ_r and \lambda_i = (n-i)^a_i, a > μ_r is a positive design constant, then

\begin{align*}
\dot{r} < ψ_0 c_1 + ψ_r c_2 + \tilde{\xi}_c d_3 + \tilde{\xi}_d + g u, \quad (15)
\end{align*}

where c_1 = \frac{\zeta k m^{t/\nu_0}(2a - μ_r)^{n-1}}{(a - μ_r)^n}, c_2 = \frac{\zeta k m^{t/\nu_0}(a - μ_r)^{n-1}}{a^{n-1}}, and c_3 = k n^{1/\nu_0} + 1 are positive constants, with \zeta = \frac{2a - μ_r}{2a - μ_r}, (3a - μ_r)^{n-1} - (2a - μ_r)^{n-1} and \zeta = 2a(3a - μ_r)^{n-1} - (2a - μ_r)^{n-1}.

Lemma 3: If the filtered tracking error r given in (2) is transgressing its upper bound ψ_r mentioned in (6), then (r-ψ_r) will approach 0 from the left side and

\begin{align*}
\lim_{(r-ψ_r)^0} \dot{r} \geq -μ_r ψ_r. \quad (16)
\end{align*}

Lemma 4: If the filtered tracking error r given in (2) is transgressing its upper bound −ψ_r mentioned in (6), then (r+ψ_r) will approach 0 from the right side, and

\begin{align*}
\lim_{(r+ψ_r)^0} \dot{r} \leq μ_r ψ_r. \quad (17)
\end{align*}

The proofs of above lemmas are given in the Appendix.

V. STABILITY ANALYSIS

In this section, stability analysis will be shown based on the results of the lemmas presented in the previous section.

Theorem 1: Consider the system (1) satisfying Assumptions 1-4. The control input designed in (9) can ensure that the closed-loop system has following properties: i) output will follow its desired trajectory, ii) tracking error and input will never transgress its PPC and PIC, respectively, and iii) all the closed-loop signals will remain bounded, provided the following feasibility conditions for PIC and PPC are true.

PIC: \( \ddot{u} > \frac{1}{\xi}(ψ_r c_2 + \tilde{\xi}_c d_3 + \tilde{d}), \) \quad (18)

PPC: \( |r(0)|(a - μ_r)^{1-n} < ψ_0 < \frac{g u - ψ_r c_2 - \tilde{\xi}_c d_3 - \tilde{d}}{(1 + μ_r)(a - μ_r)^{n-1}}. \) \quad (19)

Proof: Stability analysis is done using proof-by-contradiction. To begin with the proof, we will first establish propositional statement P1 as follows.

P1: If the (18) and (19) is true, input is designed as (9), and initially r is within its designed constraints ψ_r, then there exists at least a time instant at which r violates its constraints, or,

\( \exists t_j \quad \text{such that} \quad |r(t_j)| > ψ_r(t_j), \forall t_j \in (t_i, t_i + t_i, \ldots, t_i). \)

where \( t_i < t_{i+1} \) represent ith instant of violation of performance constraint, \( i \in \mathbb{N} \), and \( n \in \mathbb{N} \). We are now prepared for the proof.

Remark 3: To start with proof by contradiction, we will assume P1 is true. We will find later on that it will contradict the result of Lemma 3 and 4.

Suppose that P1 is true, then we have the following.

\( |r(t)| < ψ_r(t), \forall t \in [0, t_1]. \) \quad (20)

Suppose that at the instant of time \( t_1 \), the tracking error is transgressing its performance constraints (i.e., upper or lower bounds). With the following analysis, we will see that error never transgresses its performance constraints.

Noting (20), and using (15) of Corollary 1, for all \( t \in [0, t_1] \), we have

\( \dot{r} < ψ_0 c_1 + ψ_r c_2 + \tilde{\xi}_c d_3 + g u + \tilde{d}, \) \quad (21)

\( \dot{r} > -ψ_0 c_1 - ψ_r c_2 - \tilde{\xi}_c d_3 + g u - \tilde{d}. \) \quad (22)

Following (9), we infer that

\begin{align*}
\lim_{(r-ψ_r)^0} \nu &= -\nu, \quad (23)
\lim_{(r+ψ_r)^0} \nu &= \nu. \quad (24)
\end{align*}

Consequently, following Assumption 2, we have

\begin{align*}
-\tilde{g} u &\leq \lim_{(r-ψ_r)^0} g u \leq -\tilde{g} u, \quad (25)
\tilde{g} u &\leq \lim_{(r+ψ_r)^0} g u \leq \tilde{g} u. \quad (26)
\end{align*}

Now using (21) we have \( \lim_{(r-ψ_r)^0} \dot{r} < ψ_0 c_1 + ψ_r c_2 + \tilde{\xi}_c d_3 - \lim_{(r-ψ_r)^0} g u + \tilde{d} \). Further Using (25), we can infer that for all \( t \in [0, t_1] \),

\begin{align*}
\lim_{(r-ψ_r)^0} \dot{r} < ψ_0 c_1 + ψ_r c_2 + \tilde{\xi}_c d_3 - \tilde{g} u + \tilde{d}. \quad (27)
\end{align*}

Similarly, using (26) and (22), \( \forall t \in [0, t_1] \), we obtain

\begin{align*}
\lim_{(r+ψ_r)^0} \dot{r} > -ψ_0 c_1 - ψ_r c_2 - \tilde{\xi}_c d_3 + \tilde{g} u - \tilde{d}. \quad (28)
\end{align*}

Now, recalling (11), (12), and (19), it follows

\( ψ_0 c_1 + μ_r \leq \tilde{g} u - ψ_r c_2 - \tilde{\xi}_c d_3 - \tilde{d}. \) \quad (29)
Further, (29) can be written as,
\[ \dot{\psi}_0 c_1 + \psi_\infty c_2 + \xi_d c_3 - g \tilde{v} + \tilde{d} < -\mu_c \psi_0, \text{ or} \]
\[ -\psi_0 c_1 - \psi_\infty c_2 - \xi_d c_3 + g \tilde{v} - \tilde{d} > \mu_r \psi_0. \]
Now incorporating (30) in (27), and (31) in (28), it can be inferred that over \([0, t_1]\)
\[ \lim \inf_{(r-\dot{r})^+} \dot{r} < -\mu_c \psi_0, \quad \lim \sup_{(r+\dot{r})^+} \dot{r} > \mu_r \psi_0. \]
Recalling lemmas 3 and 4, it can be inferred that (32) contradicts (4), and (33) contradicts (17). Hence, over \([0, t_1]\), tracking error will never approach its performance constraints. Consequently, it can be concluded that there is no \(t_1\) in which the \(r\) violates its designed constraint \(\psi_r\). Since there does not exist the first instance of violation of the designed constraint, there does not exist any time at which \(r\) will violate its constraints \(\psi_r\). Therefore, it can be concluded that \(P1\) is false. Now following (11), (12) and (19), it follows \(\psi_0 > \{|0|\}\), and following (6), \(\psi_r(0) > |0|\). Thus initially, \(r\) is within its designed VPC (\(\psi_r\)), and further noting that Proposition P1 is false, we have the following:
\[ |r(t)| < \psi_r(t), \quad \forall t \geq 0. \]  
Now, following Lemma 2 and using (10)-(11), we have
\[ |\tilde{\xi}(i)| < (2a - \mu_r)^i |\psi_0 e^{-\mu t}| + (2a)^i |\psi_\infty|, \quad i \in [0, 1, \ldots, n-1]. \]
Using (35), it can be concluded that \(\tilde{\xi}^{(i)}\) will converge asymptotically to a set, \(\Gamma_i := \{\tilde{\xi}^{(i)} \in \mathbb{R} : |\tilde{\xi}^{(i)}| < (2a)^i |\psi_\infty|\}\) and output tracking error \(\tilde{\xi}\) will follow its PPC, i.e., \(\psi_r(t), \forall t \in \mathbb{R}_+^0\). Now, we will seek the boundedness of all the closed-loop signals.

Following (34) and (35) we have \(r \in L^\infty\) and \(\tilde{\xi}^{(i)} \in L^\infty\). Consequently, following assumption 4 and recalling from (3) that \(\xi_l \in \xi^{(L-1)} + \xi^{(L-2)}\), we have \(\tilde{\xi}_l \in L^\infty, \forall l \in \mathbb{N}_l\). Knowing the fact that \(f(\xi)\) in (1) is a smooth nonlinear function. As a result, we have \(f(\xi) \in L^\infty\). Also, it is straightforward to follow from (9) that if \(|r| < \psi_r, then |v| < \tilde{v}\), thus we have \(v \in L^\infty\). Following (1) and (5), and with the help of established boundedness of the signal, and Assumptions 2 and 3, that \(g(\xi)\) and disturbance are bounded, we have \(\tilde{\xi}_l \in L^\infty, \forall l \in \mathbb{N}_l\) and \(\dot{r} \in L^\infty\), respectively. Thus, all closed-loop signals are bounded. This completes the proof.

VI. SIMULATION RESULTS AND DISCUSSION

In this section, a simulation study is presented to show the effectiveness of the proposed approach. Consider a control-affine nonlinear system
\[ \dot{\xi}_1 = \xi_2, \]
\[ \dot{\xi}_2 = -0.5(\sin \xi_1 + \xi_2) + (3 + \cos \xi_2)v + d, \]
\[ y = \xi_1, \]
where \(\xi(t) \in \mathbb{R}, v(t) \in U \in \mathbb{R}\) and \(y\) are the state, the input, and the output of the system (36), respectively, and \(d(t) = 0.5 \sin 2t\) is a disturbance. The desired output is \(\xi_d(t) = 0.5 \sin t\). For (36), correspondingly, following (1), we can note that \(f(\xi) = -0.5(\sin \xi_1 + \xi_2)\) and \(g(\xi) = 3 + \cos \xi_2\), and are assumed to be unknown. For (36), one can readily obtain \(k_1 = 0.5, g = 2, d = 0.5\), and for the given desired output, we have \(\xi_d = 0.5\). The design parameter \(a\) is chosen as \(a = 2\), accordingly following corollary 1, \(c_1 = 9, c_2 = 6, \) and \(c_3 = 2\). Now, following the feasibility conditions (18), we have PIC: \(\tilde{v} > 0.78\). The goal is to design control law \(\nu\) such that the output tracks the desired trajectory without transgressing PIC: \(\tilde{v} = 6, \) and PPC: \(\psi_r = \psi_0 e^{-\mu t} + \psi_\infty\), (with \(\psi_0 = 1, \psi_\infty = 0.01\) and \(\mu = 1\)), on tracking error. It can be easily verified using (19) that PPC satisfies its feasibility condition for a given PIC, i.e., \(\psi_r < 1.1\). The controller is designed using (9), \(\nu = -\frac{2d}{\pi} \arctan\left(\frac{\xi_1 - \xi_1}{\xi_2 - \xi_2}\right)\), where, as mentioned in (2) \(r = \lambda_1 \xi_1 + \xi_1, \) with \(\xi_1 = \xi_1 - \xi_d\) and \(\xi_2 = \xi_2 - \xi_d\) as mentioned in (2), and \(\psi_r = \psi_0 e^{-\mu t} + \psi_\infty\). The parameter \(\mu, \psi_0, \psi_\infty, \) and \(\lambda_1\) are given by (11)-(10), the aforementioned parameters, and the simulation study is done for two sets of initial conditions, i.e., \(\xi(0) = [0.4, 0.29]^T\) and \(\xi(0) = [0.6, 0.29]^T\). For both sets of initial conditions, it can be observed from Fig. 1 that the output tracks the desired trajectory along with its tracking error following the PPC. Also, from Fig. 2, it can be seen that that input follows its PIC. Further, it can be observed from Fig. 2 the filtered tracking errors follow its VPC. It is to note that, since \(\xi_d(0) = 0\) and \(\xi_0(0) = 0.5\), so with change in the initial condition \(\xi_0(0) = [0.4, 0.29]^T\) to \([0.4, 0.29]^T\), \(\xi(0) = [0.4, 0.29]^T\) changes from \([0.4, -0.21]^T\) to \([0.6, -0.21]^T\), respectively. Consequently, \(r(0)\) changes from 0.59 to 0.99, and also \(|r(0)| (a - \mu)^{1-n}\) changes from 0.59 to 0.99. It can be calculated that a further increase in \(\xi_1(0)\) from 0.6 will violate the feasibility condition \(|r(0)| (a - \mu)^{1-n} < \psi_0\), also it can be observed from Fig. 2 that initially control input is near to its PIC, thus motivating
the feasibility condition. The observation made from Fig. 1 and 2 was as expected and stated in Theorem 1.

VII. CONCLUSION

A controller has been proposed for the tracking problem of control affine nonlinear system subjected to PPC and PIC. The structure of the controller is simple as it does not require any adaptive laws, calculation of any derivatives, system knowledge or approximation. Hence, the controller is easy to implement and an approximation-free controller. Also, the derived feasibility condition for the prescription of constraint restricts arbitrary prescription. The simulation results confirm these facts. In future, the work will be extended for multiagent systems.

APPENDIX A

PROOF OF LEMMAS AND BINOMIAL IDENTITY

A. Proof for \( \sum_{i=1}^{n-1} (-i)^m \binom{n}{m} h^m = h((a + h)^{n-1} - h^n) \)

Using the binomial identity \( \binom{n}{m} = \binom{m}{m} \), we have \( \sum_{i=1}^{n-1} (-i)^m \binom{n}{m} h^m = \sum_{i=1}^{n-1} (-i)^m a^m h^m \). Further using the binomial identity \( \binom{n}{m} = \binom{m}{m} + \binom{m}{m+1} \), it can be written as \( \sum_{i=1}^{n-1} (-i)^m a^m h^m = \sum_{i=1}^{n-1} (-i)^m (a^n - h^n a^m) \). Substituting \( \sum_{i=1}^{n-1} (-i)^m a^m h^m = a(a + h)^{n-1} - a^n \), it can be written as \( \sum_{i=1}^{n-1} (-i)^m a^m h^m = (a + h)(a + h) - h^n (a + h)^{n-1} - a^n \). Further simplifying, we have \( \sum_{i=1}^{n-1} (-i)^m a^m h^m = h((a + h)^{n-1} - h^n) \).

B. Proof of Lemma 1

The proof is in three steps as follows:

Step 1: Let \( z = \frac{1}{(s + a)^p} \), then it can be represented as a signal passing through a series of low pass filters as shown in the figure below:

\[
\begin{array}{cccccc}
 & & x & & & \\
& & & \frac{1}{s + a} & & \\
& & & & \frac{1}{s + a} & \\
& & & & \frac{1}{s + a} & \\
& & & & \frac{1}{s + a} & \\
P' \text{ blocks} & & & & & z \\
\end{array}
\]

Let \( z_1 \) be the output of the first filter, then \( Z_1(t) = L^{-1}(z_1) \), can be written as \( Z_1(t) = \int_0^t e^{-a(t-\tau)} X(\tau)d\tau \). Since \( |X(t)| < X_0 e^{-\mu t} + X_\infty \), thus we have \( |Z_1(t)| < \int_0^t e^{-a(t-\tau)} (X_0 e^{-\mu \tau} + X_\infty) d\tau \). Simplifying it, we have \( |Z_1(t)| < \frac{X_0}{(a - \mu)} (e^{-\mu t} - e^{-at}) + X_\infty \). Further it can be written as \( |Z_1(t)| < \frac{X_0}{(a - \mu)} e^{-\mu t} + X_\infty \).

Recursively following the above steps \( p' \) times, it can be easily found that \( |Z(t)| < \frac{X_0}{(a - \mu)^p} e^{-\mu t} + X_\infty \).

Step 2: Similar to step 1, if \( z = \frac{1}{(s + a)^q} \), it can be represented as a signal passing through a series of filters, as shown in the figure below.

\[
\begin{array}{cccccc}
 & & x & & & \\
& & & \frac{1}{s + a} & & \\
& & & & \frac{1}{s + a} & \\
& & & & \frac{1}{s + a} & \\
& & & & \frac{1}{s + a} & \\
Q \text{ blocks} & & & & & z \\
\end{array}
\]

Let \( z_1 \) be the output of the first filter, then we can write \( z_1 = x(1 - \frac{a}{s + a}) \). Further, we have

\[
|Z_1(t)| = |L^{-1}(z_1)| < |X(t)| + a \int_0^t e^{-a(t-\tau)} X(\tau) d\tau. \tag{37}
\]

For the second term of (37), performing a similar analysis as done in step 1, we have \( |Z_1(t)| < X_0 (\frac{2a}{(a - \mu)^q} e^{-\mu t} + 2X_\infty \).

Recursively following the above steps \( q' \) times, we have \( |Z(t)| < X_0 (\frac{2a}{(a - \mu)^q} e^{-\mu t} + 2q' X_\infty \).

Step 3: Following the figure below and step 1 with \( p' = p - q \), we can easily obtain the bounds of \( Z_1(t) \).

\[
\begin{array}{cccccc}
 & & & & & \\
& & & & \frac{1}{s + a} & \\
& & & & \frac{1}{s + a} & \\
& & & & \frac{1}{s + a} & \\
& & & & \frac{1}{s + a} & \\
P - q \text{ blocks} & & & & & z \\
\end{array}
\]

Further, following step 2 with \( q' = q \), it is straightforward to prove the given result.

C. Proof of Lemma 2

Using the Laplace transformation, (2) can be written as

\[
L(\xi(t)) = \frac{L(\nu)}{(s + a)^p} + \int \frac{\nu}{(s + a)^p} e^{-a(t-\tau)} dx d\tau. \tag{38}
\]

Following the hypothesis and Lemma 1, one can deduce from (38) that for \( \forall t \in \mathbb{R}_+, \) and in \( \{0, 1, \ldots, n - 1\} \), \( \|\xi^k(t)\| < \frac{(2a - \mu)\|\nu\| e^{-\mu t}}{(a - \mu)^{k+1}} + 2\|\nu\| e^{-at} \).

D. Proof of Corollary 1

Following \( \phi \) in (5) and using Lemma 2, we have

\[
|\phi| < \frac{|\|\nu\| e^{-at} + \sum_{k=1}^{n-1} \frac{n-k}{(n-1)!} a^{n-k} (2a - \mu)\|\nu\|}{(a - \mu)^{k+1}} + \frac{|\|\nu\| e^{-at} + \sum_{k=1}^{n-1} \frac{n-k}{(n-1)!} a^{n-k} (2a - \mu)\|\nu\|}{(a - \mu)^{k+1}}. \tag{39}
\]

Further, in (5), \( \xi(t) \in \mathbb{R}^n \), i.e., finite-dimensional vector space, so all norms are equivalent or one can find constant \( c_1 \) such that \( ||\xi(t)||_{p^*} \leq c_1 ||\xi(t)||_{\infty} \), for all \( t \in \mathbb{R}_+ \). Further, using Holder inequality, one can find \( c_1 = n^{1/p^*} \), holds the equivalence relation. Now using the Assumption 1, we have

\[
|f(\xi(t))| \leq k n^{1/p^*} ||\xi(t)||_{\infty}. \tag{40}
\]

Let \( \tilde{\xi} = [\tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_n] \), then following (3), we have \( \tilde{\xi} = \xi - \xi_d \). Substituting \( \xi = \tilde{\xi} + \xi_d \) in (40), and applying triangular inequality, we have \( |f(\xi(t))| \leq k \).
Further, it is straightforward to write
\[
\phi + f(\xi) + d - \xi_d^{(n)} \leq |\phi| + |f(\xi)| + |d| + |\xi_d^{(n)}|.
\]  
(42)

Using (39) and (41), and following Assumptions 3 and 4, one can have the following inequality
\[
|\phi + f(\xi) + d - \xi_d^{(n)}| < |\psi_r(\xi_1) + \psi_t(\xi_2) + \xi_d^{(3)} + \tilde{d}|
\]  
(43)

Further, using (43) in (5), one gets (15).

### E. Proof of Lemma 3 and Lemma 4

It is straightforward to assume that before transgressing any bounds, the tracking error must be within its prescribed performance bounds (i.e., $-\psi_r < r < \psi_r$). This implies that $-2\psi_r < r - \psi_r < 0$. Thus, we can analyze that if $r$ is transgressing its upper bound, i.e., $\psi_r$, then $(r - \psi_r)$ will approach 0 from the left side. Consequently, it is easy to know that when $(r - \psi_r)$ approaches 0 from the left side, the time derivative of $(r - \psi_r)$ will be greater than equal to 0. As a result, we have
\[
\lim_{(r - \psi_r) \to 0} \dot{r} \geq \psi_r.
\]  
(44)

Noting (8), we can infer from (44) that $\lim_{(r - \psi_r) \to 0} \dot{r} \geq -\mu\psi_0$. Hence Lemma 3 is proven. Similarly, we can prove Lemma 4.

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