Four-qubit pure states as fermionic states

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(Dated: May 11, 2014)

The embedding of the n-qubit space into the n-fermion space with 2n modes is a widely used method in studying various aspects of these systems. This simple mapping raises a crucial question: does the embedding preserve the entanglement structure? It is known that the answer is affirmative for n = 2 and n = 3. That is, under either local unitary (LU) operations or with respect to stochastic local operations and classical communication (SLOCC), there is a one-to-one correspondence between the 2- (or 3-)qubit orbits and the 2- (or 3-)fermion orbits with 4 (or 6) modes. However these results do not generalize as the mapping from the n-qubit orbits to the n-fermion orbits with 2n modes is no longer surjective for n > 3. Here we consider the case of n = 4. We show that surprisingly, the orbit mapping from qubits to fermions remains injective under SLOCC, and a similar result holds under LU for generic orbits. As a byproduct, we obtain a complete answer to the problem of SLOCC equivalence of pure 4-qubit states.

PACS numbers: 03.65.Ud, 03.67.Mn

It is well-known that any n-qubit pure state $|\psi\rangle$ can be ‘viewed’ as an n-fermion state with 2n modes. The underlying reason is that one can ‘pair’ those 2n modes to obtain n pairs, and then allow only one mode of each pair to be ‘occupied’ by a fermion (i.e. ‘single occupancy’). This simple mapping has been used widely as a technique to study various aspects of qubits and fermionic systems, such as the general relationship between fermionic systems and spin systems [1], the QMA-completeness of the $N$-representability problem [2], the black hole/qubit correspondence [3], and ground state properties of fermionic systems with local Hamiltonians [4].

Despite the success of this simple embedding from the n-qubit space into an n-fermion space with 2n modes, the crucial question whether the mapping preserves the entanglement structure remains unclear. At first glance, this seems quite implausible as after the embedding the qubit local group is only a ‘small’ subgroup of the fermionic local group. However, it was shown in [5, 6] that the LU orbits of the 2-fermion system with 4 modes are in a one-to-one correspondence with the 2-qubit LU orbits, based on the fermionic version of Schmidt decomposition, and the result also holds when considering SLOCC orbits. The n = 3 SLOCC case was discussed in [7], and surprisingly, the one-to-one correspondence of orbits stays intact. In fact, the mathematical problem has been studied in multilinear algebra and matrix analysis for many years [8, 9]. For the n = 3 LU case, related studies in the $N$-representability community for the 3-fermion system gave some hints [10–12], and recently it is shown that the mysterious one-to-one relationship between orbits remains [13].

A simple dimension counting shows that the one-to-one relationship between orbits does not generalize for n > 3 [14]. There are indeed more fermionic orbits than qubit orbits. However, it is natural to ask whether the fermionic local groups can mix any two locally inequivalent qubit states. Yet this seems to be tough question, even for the n = 4 case, for several reasons: the 4-qubit orbits under SLOCC received various controversial treatments, manifesting that this by itself is not a simple problem [15–23]; under SLOCC there are infinitely many orbits for n = 4, while for n = 3 there are only finitely many orbits; unlike the LU case, knowing only the invariants does not solve the SLOCC classification problem; unlike the n = 3 case, only little is known for LU invariants in the n = 4 case.

In this work, we handle the n = 4 case. We show that surprisingly, two inequivalent 4-qubit states under the qubit SLOCC group $SL_2^4 \times S_4$ (including qubit permutations) remain inequivalent under the fermionic SLOCC group $SL_8$. We ignore the constant factor introduced from replacing the ‘true’ SLOCC group by matrices with determinant one, as the corresponding success probability of the SLOCC protocol is not important for our discussion. Our proof relies on the celebrated theorem of Kostant and Rallis that for an (infinitesimal) symmetric space, any vector admits a unique Jordan decomposition into a semisimple part and a nilpotent part which commute [24]. We examine separately the semisimple, nilpotent, and mixed orbits. As a byproduct, we complete the SLOCC classification for 4-qubit states given in [15, 18]. Furthermore, we show that generically, two inequivalent 4-qubit states under the LU group $SU_2^4 \times S_4$ remain inequivalent under the fermionic LU group $SU_8$.

The setting—We denote by V an 8-dimensional complex Hilbert space, for which we fix an orthonormal basis $\{|i\rangle: i = 1, \ldots, 8\}$ and the orthogonal decomposition $V = V_1 \oplus V_2 \oplus V_3 \oplus V_4$ into four 2-dimensional subspaces $V_i := \text{span}\{2i-1, 2i\}$. The exterior power $\Lambda^4(V)$ is the Hilbert space of a fermionic system consisting of 4 fermions with 8 modes. The 4-vectors $e_{ijkl} := |i \land j \land k \land l\rangle, 1 \leq i < j < k < l \leq 8$, form an orthonormal basis of $\Lambda^4(V)$. We shall view the tensor product $H := V_1 \otimes V_2 \otimes V_3 \otimes V_4$ as the Hilbert space of
We identify this tensor product with the subspace 
\[ W := V_1 \wedge V_2 \wedge V_3 \wedge V_4 \] of \( \wedge^4(V) \) via the isometric embedding
\[
|ijkl \rangle \mapsto e_{1+i,3+j,5+k,7+l}, \quad i, j, k, l \in \{0, 1\}.
\] (1)

Physically, under this embedding, any 4-qubit state can be viewed as a fermionic single occupancy vector (SOV). One can imagine each of the subspaces \( V_i \) as a localized site (or an atomic orbit) with two electron spin states, as illustrated in Fig. 1. Then for an SOV, each \( V_i \) can only be 'occupied' by a single fermion. The SOV space is then identified with

\[
\text{SOV space} = \{ \psi \mid \psi \text{ is a fermionic SOV} \}.
\]

Based on this identification, we can extend these families to fermionic states to fermionic SOVs, it is found that there is a one-to-one correspondence between the qubit orbits and fermionic orbits under both SLOCC and LU (including qubit permutations) [5, 7, 13]. However, these results will not directly generalize to more than 3 qubits, as there exist fermionic orbits which do not meet the SOV space [14].

Our study is motivated by the following intriguing question: if two pure 4-qubit states are inequivalent under the qubit local group enlarged by permutations, \( SU \times S_4 \) (or \( SL \times S_4 \)), do they remain inequivalent under the fermionic group \( SU_8 \) (or \( SL_8 \)) after the embedding into the fermionic system?

Notice that a similar question was asked in [25], without considering qubit permutations. In that case simple counterexamples can be found, where qubit states which are not equivalent under \( SL_4 \times S^2 \) become equivalent under the fermionic \( SL_{2^n} \). However, the question becomes much harder when qubit permutations are considered.

Surprisingly, we shall show that the answer to our question is affirmative for the SLOCC case, and almost always affirmative for the LU case. We shall first discuss the relatively simpler case of SLOCC.

The SLOCC case—For convenience, let us introduce the following notation. For any \( (SL \times S_4) \)-orbit \( \mathcal{O} \subset \mathcal{H} \) we shall denote by \( \mathcal{O} \) the unique \( SL_8 \)-orbit in \( \wedge^4(V) \) which contains \( \mathcal{O} \). Our main result is then given by the following theorem.

**Theorem 1.** If \( \mathcal{O}_1 \neq \mathcal{O}_2 \) are two \( (SL \times S_4) \)-orbits in \( \mathcal{H} \), then \( \mathcal{O}_1 \neq \mathcal{O}_2 \).

Note that the \( SL_8 \)-orbits in \( \wedge^4(V) \) were classified in [26], and the \( SL \times S_4 \)-orbits were first classified in [15], giving the well-known results of nine families. There are subsequent treatments in [16–23], and we shall use the nine families obtained in [18] which provides some corrections to [15].

**The case of closed orbits**—To prove Theorem 1, we first consider the closed orbits. This is natural since almost all orbits are closed. We shall show that in this case the answer to our question is affirmative, as given by the following

**Observation 1.** Let \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) be two different closed \( SL \times S_4 \)-orbits. Then, after the embedding \( \mathcal{H} \to \wedge^4(V) \) into the fermionic system, the enlarged orbits \( \mathcal{O}_1 \) and \( \mathcal{O}_2 \) are also closed and different.

In order to show this, we shall use some facts from the theory of invariants (see e.g. [27]).

Let \( A \) be the algebra of complex polynomial functions on \( \wedge^4(V) \) which are invariant under the action of \( SL_8 \). Let \( \wedge^4(V)/SL_8 \) denote the affine variety attached to the algebra \( A \). It is known that \( A \) is isomorphic to the polynomial algebra over \( \mathbb{C} \) in seven variables (see e.g. [28, Sec. I.3]). Consequently, as an affine variety, \( \wedge^4(V)/SL_8 \) is isomorphic to the affine space \( \mathbb{C}^7 \). The importance of this variety is that it parametrizes the closed \( SL_8 \)-orbits in \( \wedge^4(V) \). More precisely, each closed orbit is represented by a point in the variety, each point in the variety corresponds to some closed orbit, and different points represent different closed orbits.

Similarly, let \( B \) be the algebra of complex polynomial functions on \( \mathcal{H} \) which are invariant under the action of \( SL \times S_4 \). It is also known [18] that \( B \) is isomorphic to the polynomial algebra over \( \mathbb{C} \) in four variables. Consequently, the affine variety \( \mathcal{H}/(SL \times S_4) \) attached to \( B \) is isomorphic to the affine space \( \mathbb{C}^4 \). This variety parametrizes the closed \( SL \times S_4 \)-orbits in \( \mathcal{H} \).

Note that, due to our embedding (1), the group \( SL \times S_4 \) is a subgroup of \( SL_8 \). For any polynomial function \( f : \wedge^4(V) \to \mathbb{C} \) we shall denote by \( f' \) its restriction to the subspace \( \mathcal{H} \). Note that if \( f \in A \) then \( f' \in B \). This restriction map \( A \to B \) is a homomorphism of algebras. By working with explicit generators of the algebras \( A \) and \( B \), for details see Appendix A, we shall prove that this homomorphism is onto and consequently the corresponding morphism of varieties

\[
\Phi : \mathbb{C}^4 \cong \otimes V_i/(SL \times S_4) \to \wedge^4(V)/SL_8 \cong \mathbb{C}^7
\] (2)

is injective.

We now claim that if \( \psi \in \mathcal{O} \), where \( \mathcal{O} \) is a closed \( (SL \times S_4) \)-orbit in \( \mathcal{H} \), then the orbit \( \mathcal{O} = SL_8 \cdot \psi \) is also closed. To show
Among the nilpotent orbits, only the trivial orbit \( O \subseteq H \) is contained in \( \Lambda^4(V) \). This means that the linear map \( \theta \) in [26], on the exceptional complex simple Lie algebra \( \mathfrak{e}_7 \), having \( \mathfrak{sl}_8 \) and \( \mathfrak{p} \) as its \(+1\) and \(-1\) eigenspaces, respectively, is an involutory automorphism of \( \mathfrak{e}_7 \). Moreover, the subspace \( \mathfrak{p} \) can be identified with \( \Lambda^4(V) \) as an \( \mathfrak{sl}_8 \)-module. The closed \( \mathfrak{sl}_8 \)-orbits in \( \Lambda^4(V) \) are precisely those that meet the Cartan subspace \( \mathfrak{c} \subseteq \mathfrak{p} \) (\( \mathfrak{c} \) has dimension seven and is unique up to the action of \( \mathfrak{sl}_8 \)). In [26, Sect. 3.1], the following basis elements of \( \mathfrak{c} \) are given:

\[
\begin{align*}
p_1 & : = e_{1234} + e_{5678}, \\
p_2 & : = e_{1357} + e_{6824}, \\
p_3 & : = e_{1562} + e_{8437}, \\
p_4 & : = e_{1683} + e_{4752}, \\
p_5 & : = e_{1845} + e_{7263}, \\
p_6 & : = e_{1476} + e_{2385}, \\
p_7 & : = e_{1728} + e_{3546}.
\end{align*}
\] (4)

The closed \( \mathfrak{sl}_8 \times \mathfrak{sl}_8 \)-orbits are those that belong to the family 1 in [18, Table 7]. These are precisely the orbits that meet the 4-dimensional subspace \( \mathfrak{a} \) spanned by the vectors

\[
\begin{align*}
|0000\rangle + |1111\rangle, & \quad |0011\rangle + |1100\rangle, \\
|0101\rangle + |1010\rangle, & \quad |0110\rangle + |1001\rangle.
\end{align*}
\] (5)

After the embedding into the fermionic space, these vectors become \( p_2, p_4, p_5, \) and \( -p_6 \), respectively. Since this subspace is contained in \( \mathfrak{c} \), our claim follows.

We summarize this in the following diagram,

\[
\begin{array}{c}
|0000\rangle + |1111\rangle, \\
|0011\rangle + |1100\rangle, \\
|0101\rangle + |1010\rangle, \\
|0110\rangle + |1001\rangle.
\end{array}
\] (6)

where \( \mathfrak{e}_7, \mathfrak{sl}_8, \mathfrak{sl}_2^{\times 4} \) are the corresponding Lie algebras, and the arrows \( \uparrow \) indicate the corresponding embeddings. In Appendix B we provide more details about these embeddings by using the Dynkin diagrams.

We can now show Observation 1. We have just shown that the orbits \( O_i \) are closed and that \( \Phi(O_i) = O_i \) for \( i = 1,2 \). Since \( \Phi \) is one-to-one and \( O_1 \neq O_2 \), we have \( \Phi(O_1) \neq \Phi(O_2) \), i.e., \( O_1 \neq O_2 \).

The case of nilpotent orbits—An \( \mathfrak{sl}_8 \)-orbit \( O \subseteq H \) is called nilpotent if its closure contains the zero vector. One defines similarly the nilpotent \( \mathfrak{sl}_8 \)-orbits in \( \Lambda^4(V) \). In the previous section we have shown that if \( O \) is closed then \( \hat{O} \) is closed as well. On the other hand, it is immediate from the definition, that if \( O \) is nilpotent then \( \hat{O} \) is nilpotent, too. Among the nilpotent orbits, only the trivial orbit \( \{0\} \) is closed.

There are exactly 9 nilpotent \( \mathfrak{sl}_8 \times \mathfrak{sl}_8 \)-orbits (including the trivial orbit \( \{0\} \)). The representatives of these 9 orbits can be obtained from [18, Table 7] by setting the parameters \( a, b, c, d \) (if any) to 0. That table classifies the \( \mathfrak{sl}_8 \times \mathfrak{sl}_8 \)-orbits in \( \mathfrak{H} \) into 9 families depending on the complex parameters \( a, b, c, d \). They are numbered by integers \( 1, 2, 3, 6, 9, 10, 12, 14, \) and \( 16 \). The last three families consist of a single orbit, which is nilpotent. Two states belonging to different families do not belong to the same \( \mathfrak{sl}_8 \times \mathfrak{sl}_8 \)-orbit [18, Theorem 3.6]. In particular, the 9 nilpotent orbits are pairwise distinct.

The main result of this section is the following observation.

**Observation 2.** If \( O_1 \) and \( O_2 \) are nilpotent \( \mathfrak{sl}_8 \times \mathfrak{sl}_8 \)-orbits and \( O_1 \neq O_2 \), then \( \hat{O}_1 \neq \hat{O}_2 \).

To show this observation, clearly we may assume that both \( O_1 \) and \( O_2 \) are non-zero orbits. We shall say that an \( \mathfrak{sl}_8 \)-orbit in \( \Lambda^4(V) \) is an SOV orbit if it meets \( \mathfrak{H} \). Trivially, if \( \mathfrak{O} \) is an \( \mathfrak{sl}_8 \times \mathfrak{sl}_8 \)-orbit, then \( \hat{\mathfrak{O}} \) is an SOV orbit. However, there exist \( \mathfrak{sl}_8 \)-orbits in \( \Lambda^4(V) \) which are not SOV [14]. To show the observation, it suffices to show that there are at least 8 non-zero nilpotent \( \mathfrak{sl}_8 \)-orbits in \( \Lambda^4(V) \) which are SOV.

There are exactly 94 non-zero nilpotent \( \mathfrak{sl}_8 \)-orbits in \( \Lambda^4(V) \), see [26, Table 2] (as well as also [29, Table XI] for an independent derivation carried out in a different context). Both enumerations make use of the decomposition (3), where \( \mathfrak{p} = \Lambda^4(V) \). The nilpotent \( \mathfrak{sl}_8 \)-orbits in \( \Lambda^4(V) \) are then classified by constructing representatives of the so-called normal \( \mathfrak{sl}_8 \)-triples. These are non-zero triples \((H, E, F)\) with \( H \in \mathfrak{sl}_8 \) and \( E, F \in \Lambda^4(V) \) such that

\[ [H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H. \] (7)

In [26, Table 2], the elements \( H \in \mathfrak{sl}_8 \) are given as diagonal matrices, but the elements \( E \) and \( F \) were not computed. They can be computed by using Eqs. (7).

Our computations show that the orbits 1, 2, 5, 6, 9, 20, 44, and 50 in [26, Table 2] are SOV orbits. The results are summarized in Table I, where the first column gives the label of the orbit in [26, Table 2], and the second column gives the label of the family from [18, Table 7] whose nilpotent orbit (obtained by setting \( a = b = c = d = 0 \)) is contained in the orbit given in the first column. The third and fourth columns list the corresponding elements \( H \) and \( E \).

Note that the choice of the elements \( E \) and \( F \) is not unique. In addition to (7), we have imposed the condition that

\[ F = \sigma E, \] (8)

where \( \sigma \) is another involutory automorphism of the Lie algebra \( \mathfrak{e}_7 \) which commutes with \( \theta \). The action of \( \sigma \) on \( \mathfrak{sl}_8 \) is given by \( \sigma(X) = -X^T \), where \( T \) denotes the transposition, and on \( \Lambda^4(V) \) it is specified by the images of the basis elements

\[ \sigma: e_{ijkl} \mapsto e_{9-i,9-k,9-j,9-l}. \] (9)

Note that \( \sigma H = -H \) because each \( H \) is a diagonal matrix.

The general case—To finish the proof of Theorem 1, we shall use some results from Kostant and Rallis [24]. Any \( \psi \in \Lambda^4(V) \) admits a unique Jordan decomposition \( \psi = \psi_s + \psi_n \), i.e., such that \( \psi_s \) is semisimple (as an element of the Lie algebra \( \mathfrak{e}_7 \)), \( \psi_n \) is nilpotent, and they commute (that is, their Lie bracket vanishes, \( [\psi_s, \psi_n] = 0 \)). An element \( \psi \in \Lambda^4(V) \) is semisimple if and only if the orbit \( \mathfrak{sl}_8 \cdot \psi \) is closed, which
As a byproduct, Theorem 2 gives a complete answer to the problem of SLOCC equivalence of pure 4-qubit states, providing a much simpler criterion for checking the SLOCC equivalence of pure 4-qubit states than the one proposed in [18, Section 4]. To prove this theorem, we need to examine all 9 nine families. The case of closed orbits has already been studied in [18], and the assertion for the families 12, 14, and 16 holds trivially. The detailed analysis of the families 2, 3, 6, 9, and 10 is given in Appendix B.

The LU case—Having fully solved the SLOCC case, we now move to the LU case. This is much harder, however we managed to deal with the generic orbits, which in fact cover almost all orbits. Some facts from our previous discussion of the SLOCC case will be used in the proof.

Let $f(a, b, c, d) = (a^2 - b^2)(a^2 - c^2)(a^2 - d^2)(b^2 - c^2)(b^2 - d^2)(c^2 - d^2)$, a polynomial in four complex variables $a, b, c, d$. Furthermore, let

$$\Lambda = \{ g \cdot (ap_2 + bp_4 + cp_5 - dp_6) : g \in \mathbb{S}L, f(a, b, c, d) \neq 0 \} \subseteq W. $$

We observe that $\Lambda$ contains an open dense subset of $W$ which is also $\mathbb{S}L \times S_4$ invariant. To prove it, we shall view $f$ as a polynomial function $f: \mathbb{C}^4 \rightarrow \mathbb{C}$ by considering $a, b, c, d$ as coordinates in $\mathbb{C}$ with respect to the basis $(p_2, p_4, p_5, -p_6)$. Then the polynomial $f^2$ extends (uniquely) to an $g \in \mathbb{C}^4$. Indeed, on $a$ we have $2f^2 = 2(3\Sigma^3 - 2\Pi^2)$, where $\Sigma$ and $\Pi$ are the generators of $B$ of degree 8 and 12 from [18]. Recall that the 4-qubit hyperdeterminant, Det, is a homogeneous polynomial $W \rightarrow \mathbb{C}$ of degree 24 which is $\mathbb{S}L \times S_4$ invariant, i.e., Det $\in B$. The set $\Omega = \{ \psi \in W : \text{Det}(\psi) \neq 0 \}$ is open, dense, and $\mathbb{S}L \times S_4$ invariant subset of $W$. It is known [31, Section III] that each $\mathbb{S}L$-orbit, which is contained in $\Omega$, meets $\Lambda$. It follows that the set of all $\psi \in \Omega$ such that $\Sigma(\psi)^3 \neq 2\Pi(\psi)^2$ is contained in $\Lambda$, and clearly it is open and dense in $W$.

**Theorem 3.** Let $\phi \in \Lambda$ and let $U \in \text{SU}(8)$ be such that $\psi := U \cdot \phi \in W$. Then there exists $U' \in \text{SU} \times S_4$ such that $\psi = U' \cdot \phi$.

The proof of this theorem will be given in Appendix C.

**Summary**—We have shown that the embedding of the space of 4-qubit pure states into the 4-fermion space of 8 modes preserves the entanglement structure under the natural fermionic SLOCC group $\text{SU}_8$, which is also the case for generic orbits under the fermionic LU group $\text{SU}_8$. This surprising property of the 4-qubit states, following already known facts for 2- and 3-qubit systems, reveals interesting connection between qubit and fermionic systems, providing new perspectives on the entanglement structures of both systems. One can naturally ask what happens for other LU orbits, and in the more general case of $n$ qubits. We believe that the discussion of these difficult, but intriguing question shall shed light on insights of new physics in these many-body systems.

**Acknowledgements**—LC was mainly supported by MITACS and NSERC. The CQT is funded by the Singapore MoE and the NRF as part of the Research Centres of Excellence.
programme. DD was supported in part by an NSERC Discovery Grant. BZ is supported by NSERC and CIFAR.

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Appendix A: The algebras $\mathcal{A}$ and $\mathcal{B}$

This section discusses explicit generators of the algebras $\mathcal{A}$ and $\mathcal{B}$ of polynomial invariants for the action of $SL_4$ on $\Lambda^4(V)$ and the action of $SL \rtimes S_4$ on $\otimes^4_{i=1} V_i$, respectively, and their relationship.

The seven generators of $\mathcal{A}$ have degrees 2, 6, 8, 10, 12, 14, and 18. They were computed by A. A. Katanova in [28]. Explicitly, they are given by the formulae

$$ f_{2n}(\psi) = \text{tr} A(\psi)^{2n}, \quad (n = 1, 3, 4, 5, 6, 7, 9) \quad (10) $$

where $A(\psi)$ is a $28 \times 28$ matrix whose entries are quadratic forms in the components of $\psi$. Once we have chosen these generators, we obtain an explicit identification of the variety $\Lambda^4(V)/SL_4$ with the affine space $\mathbb{C}^7$: given a closed $SL_4$-orbit $O \subseteq \Lambda^4(V)$ we chose a point $\psi \in O$ and assign to $O$ the point in $\mathbb{C}^7$ with coordinates $f_{2n}(\psi), n = 1, 3, 4, 5, 6, 7, 9, 10, 12, 14, 18$. The four generators of the algebra $\mathcal{B}$ have degrees 2, 6, 8, and 12. They were computed first in [18] and recently another set of generators was computed in [31]. Explicit computation with these generators show that the restriction map $\mathcal{A} \to \mathcal{B}$ induces an isomorphism of the subalgebra $\mathbb{C}[f_2, f_6, f_8, f_{12}]$ of $\mathcal{A}$ onto $\mathcal{B}$, i.e., we have $\mathcal{B} = \mathbb{C}[f_2, f_6, f_8, f_{12}]$, where $f_i$ denotes the restriction of $i \in \mathcal{A}$ to $\mathcal{B}$. Consequently, the restrictions $f_{10}^i, f_{14}^i, f_{18}^i$ can be expressed as polynomials in $f_2^i, f_6^i, f_8^i, f_{12}^i$. Explicitly we have obtained the formulae

$$ 2^9 \cdot 3^6 \cdot 5^1 \cdot 7^1 \cdot 9^1 = f_2^i (7 f_2^i - 2^5 \cdot 7 \cdot 9 f_2^i f_6^i + 2^6 \cdot 3^5 f_8^i), \quad (11) $$

$$ 2^{14} \cdot 3^7 \cdot 5^1 \cdot 7^1 \cdot 11^1 \cdot 317 f_2^i f_6^i - 11 \cdot 251 f_7^i - 2^{10} \cdot 3^2 \cdot 7 \cdot 11 \cdot 13 f_2^i f_6^i + 2^{11} \cdot 3^4 \cdot 7 \cdot 11 f_2 f_6^i + 2^{11} \cdot 3^3 \cdot 7 \cdot 11 f_6 f_8^i - 2^6 \cdot 3^2 \cdot 7 \cdot 11 \cdot 103 f_3^i f_8^i, \quad (12) $$

$$ 2^{19} \cdot 3^8 \cdot 5^2 \cdot 7^1 \cdot 9^1 = -5^2 \cdot 13903 f_2^i + 2^7 \cdot 5 \cdot 89 \cdot 1609 f_2^i f_6^i - 2^4 \cdot 3^2 \cdot 5 \cdot 8989 f_2^i f_6^i + 2^{12} \cdot 3^2 \cdot 37 \cdot 109 f_2^i f_6^i + 2^{10} \cdot 5^2 \cdot 7^2 \cdot 13513 f_3^i f_6^i - 2^{15} \cdot 3^6 \cdot 349 f_2^i f_6^i + 2^{12} \cdot 3^3 \cdot 331 f_3^i f_8^i + 2^{21} \cdot 3^3 \cdot 5 f_6^i + 2^{12} \cdot 5 \cdot 71 \cdot 127 \cdot 1409 f_6^i f_{12}^i. \quad (13) $$

The image of the morphism $\Phi$ can be described as the graph of the morphism $\mathbb{C}^4 \to \mathbb{C}^3$ given by the above three equations. More precisely, we have to substitute $f_2^i, f_6^i, f_8^i, f_{12}^i$ with complex coordinates $z_1, z_2, z_3, z_4$ and $f_{10}^i, f_{14}^i, f_{18}^i$ with $z_5, z_6, z_7$, respectively, to obtain the formulae expressing $z_5, z_6, z_7$ as polynomial functions in $z_1, z_2, z_3, z_4$. 
Appendix B: Dynkin diagrams for \((\mathfrak{so}_8, \mathfrak{sl}_2^4) \subseteq (e_7, \mathfrak{sl}_6)\)

In this section we describe the embedding of the symmetric space \((\mathfrak{so}_8, \mathfrak{sl}_2^4)\) into the larger symmetric space \((e_7, \mathfrak{sl}_6)\) by using the root system and the root space decomposition of \(e_7\).

The simple roots of \(e_7\) are \(\alpha_1, \alpha_2, \ldots, \alpha_7\). The simple roots of the subalgebra \(\mathfrak{sl}_6\) are \(-\alpha_0, \alpha_1, \alpha_3, \alpha_4, \ldots, \alpha_7\), where \(\alpha_0 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 5\alpha_5 + 2\alpha_6 + \alpha_7\) is the highest root of \(e_7\). These seven roots are labeled in white to indicate that the corresponding root vectors belong to \(\mathfrak{sl}_6\), the \(+1\) eigenspace of the involution \(\theta\). The roots \(\alpha_2\) and \(\alpha_7\) are printed in black because the corresponding root vectors belong to the subspace \(p = \wedge^4(V)\), the \(-1\) eigenspace of \(\theta\).

\[
\beta = \alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6
\]

FIG. 2: Embedding of \(\mathfrak{so}_8\) into \(e_7\)

The roots \(\alpha_3, \beta, \alpha_5, \alpha_7\) form the Dynkin diagram of the subalgebra \(\mathfrak{so}_8\). It is interesting that \(\alpha_0\) is also the highest root of this \(\mathfrak{so}_8\). The intersection \(\mathfrak{so}_8 \cap \mathfrak{sl}_6\) is the Lie algebra \(\mathfrak{sl}_2^4\) of SL. The simple roots of this subalgebra are \(-\alpha_0, \alpha_3, \alpha_5\) and \(\alpha_7\).

Moreover, \(\alpha_2\) is the lowest weight of the SL\(_8\) module \(\wedge^4(V)\), and \(\beta = (-\alpha_0 + \alpha_3 + \alpha_5 + \alpha_7)/2\) is the lowest weight of the SL module \(\otimes^4_{i=1} V_i\).

We remark that the point-wise stabilizer of \(e\) in SL\(_8\), i.e., the group

\[
A = \{a \in \text{SL}_8 : a \cdot p_i = p_i \ \forall i = 1, \ldots, 7\}
\]

is the three-qubit Pauli group of order 256. The centre \(Z(A)\) of \(A\) is generated by \(iI_8\), where \(i^2 = -1\) and \(I_8\) is the identity of \(\text{SL}_8\). Hence the action of \(A\) on \(\wedge^4 V\) is the abstract group \(A/Z(A)\) which is an elementary Abelian group of order \(2^9\). This agrees with [27, Summary Table p. 261, No. 18].

Appendix C: Proof of Theorem 2

This section proves Theorem 2. We explicitly list the orbits from [18, Table 7] with non-trivial Jordan decomposition as in Table II.

As already discussed, it suffices to show that two states \(|\psi(a, b, c)\rangle\) and \(|\psi(a', b', c')\rangle\) from the same family in Table II are in the same \((\text{SL} \times S_4)\)-orbit if and only if all invariants \(f' \in B\) agree.

If the invariants do not agree, then the states are obviously in different orbits. In order to show sufficiency, assume that the invariants agree, i.e., \(g_j(a, b, c) = g_j(a', b', c')\) for \(j = 2, 6, 8, 12\), where \(g_j(a, b, c) = f_j(\psi(a, b, c))\) and the polynomials \(f'_j\) are the generators of the algebra \(B\). For each family, we obtain a system of polynomial equations. The corresponding radical ideal is generated by the polynomials listed in Table III. Computing the primary decomposition of the ideals, we find that there are linear relations between the triples of variables \((a, b, c)\) and \((a', b', c')\) given by finite groups (see Table IV).

| no. | representative \(|\psi\rangle\) |
|-----|------------------|
| 2   | \[
\frac{a+c-1}{2}((|0000\rangle + |1111\rangle) + \frac{a+c+1}{2}(|0011\rangle + |1100\rangle) + \frac{b-c+1}{2}(|0101\rangle + |1010\rangle) + \frac{b-c-1}{2}(|1110\rangle + |1001\rangle) + \frac{b}{2}(|0111\rangle + |1011\rangle + |0101\rangle + |1000\rangle + |1110\rangle) + |0010\rangle - |0100\rangle - |1011\rangle - |1101\rangle)\]
| 3   | \[
\frac{b+c+1}{2}(|0000\rangle + |1111\rangle + |0101\rangle + |1010\rangle) + \frac{b-c+1}{2}(|0101\rangle + |1010\rangle) + \frac{b-c-1}{2}(|0011\rangle + |1100\rangle) + \frac{b}{2}(|0111\rangle + |1011\rangle + |0101\rangle + |1000\rangle + |1110\rangle) + |0010\rangle - |0100\rangle - |1011\rangle - |1101\rangle)\]
| 4   | \[
\frac{a}{2}(|0000\rangle + |1010\rangle + |0101\rangle + |1000\rangle + |1101\rangle) + \frac{b}{2}(|0011\rangle + |1101\rangle + |0010\rangle + |1110\rangle) + \frac{c}{2}(|0001\rangle + |1111\rangle + |0100\rangle + |1011\rangle) + \frac{d}{2}(|0010\rangle + |1100\rangle + |0101\rangle + |1010\rangle) + |0000\rangle - |0010\rangle - |0100\rangle - |0110\rangle - |0001\rangle - |1000\rangle - |1100\rangle - |1101\rangle)\]

TABLE II: Orbits from [18, Table 7] with non-trivial Jordan decomposition.

| no. | generators of the radical ideal |
|-----|----------------------------------|
| 2   | \[
((c + c') (c + c') (c - a' / 2 - b / 2) (c - a / 2 + b / 2) \
\times (c + a' / 2 - b / 2) (c + a / 2 + b / 2) + b^2 c^2 - c^2 - b^2 a^2 + c a^2 + c^2 b^2 - 2 b^2 c^2 + c^2 b^2 + c^2 a^2 + c^2 c^2 - b^2 c^2 - c^2 + a^2 + b^2 + 2 c^2 - a^2 - b^2 + 2 c^2\]
| 3   | \[
(b - a') (b + a') (b - a') (b + a') + a^2 + b^2 - a^2 - b^2\]
| 4   | \[
(b - b') (b + b') (b - a') (b + a') + a^2 + b^2 - a^2 - b^2\]
| 6   | \[
(b - b') (b + b') (b - a' / 2 - b / 2) (b - a' / 2 + b / 2) \times (b + a' / 2 - b / 2) (b + a' / 2 + b / 2) + a^2 + b^2 - a^2 - b^2\]
| 9   | \[
(a - a') (a + a') + a^2\]
| 10  | \[
(a - a') (a + a') + a^2\]

TABLE III: Generators of the radical of the ideal generated by \(g_j(a', b', c') - g_j(a, b, c)\).

Assume that for the states \(|\psi^{(\mu)}(a, b, c)\rangle\) and \(|\psi^{(\mu)}(a', b', c')\rangle\) from the same family \(\mu = 2, 3, 6, 9, 10\) all polynomial invariants agree. In the following, we show...
TABLE IV: Symmetries of the varieties corresponding to identical invariants.

that the linear transformations on the variables $a, b, c$ corresponding to the generators of the groups in Table IV can be realized by operations from $\text{SL} \rtimes S_4$ on the states.

Family 2 Direct computation shows that (i) applying the transformation

$$
\frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & -i \\ -i & 1 \end{array} \right) \otimes \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & -i \\ -i & 1 \end{array} \right)
\otimes \frac{1}{\sqrt{2}} \left( \begin{array}{cc} i & 1 \\ -1 & -i \end{array} \right) \otimes \frac{1}{\sqrt{2}} \left( \begin{array}{cc} -i & -1 \\ 1 & i \end{array} \right)
$$

followed by a permutation of the last two qubits maps $|\psi^{(2)}(a, b, c)\rangle$ to $|\psi^{(2)}(b, a, c)\rangle$; (ii) swapping the two middle qubits maps the state $|\psi^{(2)}(a, b, c)\rangle$ to the state $|\psi^{(2)}((a + b)/2 + c, (a + b)/2 - c, (a - b)/2)\rangle$.

Family 3 Direct computations shows that (i) the states $|\psi^{(3)}(a, b)\rangle$ and $|\psi^{(3)}(a, -b)\rangle$ are related by the transformation $I_2 \otimes I_2 \otimes (i\sigma_y) \otimes (i\sigma_y)$; (ii) the states $|\psi^{(3)}(a, b)\rangle$ and $|\psi^{(3)}(b, a)\rangle$ are related by the transformation $M^{-1} \otimes M^{-1} \otimes M \otimes M$, where

$$M = \frac{1}{\sqrt{2}} \left( \begin{array}{cc} 1 & i \\ i & 1 \end{array} \right).$$

Family 6 Direct computation shows that swapping the two middle qubits maps the state $|\psi^{(6)}(a, b)\rangle$ to the state $|\psi^{(6)}((a + 3b)/2, (a - b)/2)\rangle$. Furthermore, the following calculation shows that applying the transformation $T_1 = I_2 \otimes I_2 \otimes (i\sigma_y) \otimes (i\sigma_y)$, followed by swapping the first two qubits maps the state $|\psi^{(6)}(a, b)\rangle$ to the state $|\psi^{(6)}(a, -b)\rangle$:

$$|\psi^{(6)}(a, b)\rangle = \frac{a + b}{2} (|0000\rangle + |1111\rangle + b(|0101\rangle + |1010\rangle) + i(|1001\rangle - |0101\rangle) + \frac{a - b}{2} (|0011\rangle + |1100\rangle) + \frac{1}{2} (|0010\rangle + |0100\rangle + |1011\rangle + |1101\rangle) - |0011\rangle - |1101\rangle)$$

$$\tau = (12) \frac{a + b}{2} (|0011\rangle + |1100\rangle) - b(|0110\rangle + |1001\rangle) - i(|0100\rangle - |0110\rangle) + \frac{a - b}{2} (|0000\rangle + |1111\rangle) + \frac{1}{2} (-|0001\rangle + |0111\rangle + |1000\rangle - |1110\rangle) + \frac{1}{2} (-|0011\rangle + |1001\rangle + |1010\rangle - |1100\rangle) + |0010\rangle - |0100\rangle + |1011\rangle + |1101\rangle)$$

Family 9 Direct computation shows that the states $|\psi^{(9)}(a)\rangle$ and $|\psi^{(9)}(a)\rangle$ are related by the transformation $I_2 \otimes (i\sigma_z) \otimes I_2 \otimes (i\sigma_z)$.

Family 10 Direct computation shows that the states $|\psi^{(10)}(a)\rangle$ and $|\psi^{(10)}(-a)\rangle$ are related by the transformation $M^{-4}$ where $M$ is given in (16). In summary, we have shown that two states which belong to the same family and for which all polynomial invariants agree are in the same $(\text{SL} \rtimes S_4)$-orbit. The finite groups in Table IV define relations on the space of parameters $(a, b, c)$.

Appendix D: Proof of Theorem 3

In this section we prove Theorem 3.

By the hypothesis we have $\phi = g \cdot a$ for some $g \in \text{SL}$ and some $\alpha = ap_2 + bp_4 + cp_5 - dp_6$, where $a, b, c, d \in \mathbb{C}$ and $a^2, b^2, c^2, d^2$ are pairwise distinct. After setting $v_i = g \cdot |i\rangle$, $i = 1, \ldots, 8$, we have

$$\phi = a(v_1 \wedge v_2 \wedge v_5 \wedge v_7 + v_2 \wedge v_4 \wedge v_6 \wedge v_8) + b(v_1 \wedge v_3 \wedge v_6 \wedge v_8 + v_2 \wedge v_4 \wedge v_5 \wedge v_7) + c(v_1 \wedge v_3 \wedge v_5 \wedge v_8 + v_2 \wedge v_3 \wedge v_6 \wedge v_7) + d(v_1 \wedge v_4 \wedge v_6 \wedge v_7 + v_2 \wedge v_3 \wedge v_5 \wedge v_8).$$

We set $u_i = U^{|i\rangle}$ for $i = 1, \ldots, 8$, and so $\{u_i\}$ is an orthonormal basis of $V$. We have

$$v_j = \sum_{i=1}^{8} x_{ij} u_i, \quad j = 1, \ldots, 8$$
where \( x_{ij} = \langle u_i | v_j \rangle = \langle i | U g | j \rangle \). Thus \( X := (x_{ij}) \) equals the matrix \( U g \), and we partition it into 16 blocks \( X_{kl} \) of size \( 2 \times 2 \).

For convenience, set \( e_{ij} = |i \wedge j\rangle \). Since \( \psi = U \cdot \phi \in W \), the partial inner product \( \langle e_{2k-1,2k} | \psi \rangle \) vanishes for \( k = 1,2,3,4 \). Equivalently, we have

\[
\langle u_{2k-1} \wedge u_{2k} | \phi \rangle = 0, \quad k = 1,2,3,4. \tag{18}
\]

Let us consider this equation for \( k = 1 \). After expanding the partial inner product by using the formula given in [14, Eq. (2)], we obtain a linear combination of the bivectors \( v_i \wedge v_j \) with \( 1 \leq i < j \leq 8 \). The bivectors \( v_{2k-1} \wedge v_{2k} \) do not occur in this expansion. The coefficients of the other 24 bivectors \( v_i \wedge v_j \) must be 0, and so we obtain 24 equations. Each pair of the parameters \( a,b,c,d \) occurs in exactly four of these equations. For instance, the four equations in which only \( a \) and \( b \) occur are the following:

\[
\begin{align*}
\langle v_1 \wedge v_3 \rangle & : \quad aD_{57} + bD_{68} = 0, \\
\langle v_2 \wedge v_4 \rangle & : \quad bD_{57} + aD_{68} = 0, \\
\langle v_5 \wedge v_7 \rangle & : \quad aD_{13} + bD_{24} = 0, \\
\langle v_6 \wedge v_8 \rangle & : \quad bD_{13} + aD_{24} = 0,
\end{align*}
\]

where \( D_{ij} = x_{1i}x_{2j} - x_{1j}x_{2i} \). Since \( a^2 \neq b^2 \), the first two equations imply that \( D_{57} = D_{68} = 0 \) and the last two imply that \( D_{13} = D_{24} = 0 \). One obtains similar results by using the other five pairs of the parameters \( a,b,c,d \). The final result is that all \( D_{ij}, i < j \), vanish except possibly \( D_{12}, D_{34}, D_{56}, \) and \( D_{78} \). As \( X \) is invertible, at least one of these four minors does not vanish. It is now easy to see that exactly one of the blocks \( X_{kl} \) is invertible, and all others vanish.

By applying the same arguments to the other three equations in (18), we deduce that in each row and each column of blocks in \( X \) exactly one block is invertible and all others vanish. Since \( X = U g \) and \( g \in \text{SL} \) is block-diagonal, it follows that the unitary matrix \( U = X g^{-1} \) has a permuted block structure, i.e., \( U \in U(2)^4 \rtimes S_4 \). For suitable \( S = \bigoplus_{k=1}^4 \lambda_k I_2 \), with \( \prod \lambda_k = 1 \), we have \( U S \in SU \rtimes S_4 \). By using the facts that \( S g = g S \) and \( S \cdot w = w \) for all \( w \in W \), we obtain that \( U S \cdot \phi = U S g \cdot \alpha = U g \cdot \alpha = \psi \). Thus we can take \( U' = U S \) to complete the proof.