UNIFORM APPROXIMATION BY POLYNOMIAL MODULES 
ON ARBITRARY COMPACT SUBSETS

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ABSTRACT. For a compact subset $K$ of the complex plane $\mathbb{C}$, let $P(K)$ denote the closure in $C(K)$ of analytic polynomials in $z$ and let $A(K) \subset C(K)$ denote the subalgebra of functions that are analytic in the interior of $K$. We prove that there exists a function $F \in A(K)$ such that $P(K) + P(K)F$ is uniformly dense in $A(K)$. Recent developments of analytic capacity and Cauchy transform provide us some useful tools in our proof.

1. Introduction

Let $P$ denote the set of polynomials in the complex variable $z$. For a compact subset $K$ of the complex plane $\mathbb{C}$, let $Rat(K)$ be the set of all rational functions with poles off $K$ and let $C(K)$ denote the Banach algebra of complex-valued continuous functions on $K$ with customary norm $\| \cdot \|_K$ ($\| \cdot \|_{C(K)}$, or $\| \cdot \|$). Let $P(K)$ and $R(K)$ denote the closures in $C(K)$ of $P$ and $Rat(K)$, respectively. Let $A(K) \subset C(K)$ be the algebra of functions that are analytic in $\text{Int}(K)$, the interior of $K$. For $\phi \in C(K)$, let $H(\phi, K)$ denote the closure of $P(K) + P(K)\phi$ in $C(K)$.

Let $\nu$ be a finite complex-valued Borel measure that is compactly supported in $\mathbb{C}$. For $\epsilon > 0$, $C_\epsilon(\nu)$ is defined by

$$C_\epsilon(\nu)(z) = \int_{|w-z|>\epsilon} \frac{1}{w-z} d\nu(w).$$

The (principal value) Cauchy transform of $\nu$ is defined by

$$C(\nu)(z) = \lim_{\epsilon \to 0} C_\epsilon(\nu)(z)$$

for all $z \in \mathbb{C}$ for which the limit exists. From Lemma 2.2 (2), we see that (1.2) is defined for all $z$ except for a set of zero analytic capacity. Throughout this paper, the Cauchy transform of a measure always means the principal value of the transform.

We define the analytic capacity of a compact subset $E$ by

$$\gamma(E) = \sup_{f \in \mathcal{F}} |f'(\infty)|,$$

where the supremum is taken over all those functions $f$ that are analytic in $\mathbb{C}_\infty \setminus E$ ($\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$), such that $|f(z)| \leq 1$ for all $z \in \mathbb{C}_\infty \setminus E$; and $f'(\infty) := \lim_{z \to \infty} z(f(z) - f(\infty))$. The analytic capacity of a general subset $F$ of $\mathbb{C}$ is given by:

$$\gamma(F) = \sup \{ \gamma(E) : E \subset F \text{ compact} \}.$$

Good sources for basic information about analytic capacity are [Du10], Chapter VIII of [G69], [Ga72], Chapter V of [C91], and [To14].
The continuous analytic capacity of a compact set $E \subset \mathbb{C}$ is defined as
\[ \alpha(E) = \sup |f'(\infty)| \] (1.4)
where the supremum is taken over all complex-valued functions which are continuous in $\mathbb{C}$, analytic on $\mathbb{C} \setminus E$, and satisfy $|f(z)| \leq 1$ for all $z \in \mathbb{C}$. For a general set $F$, we set
\[ \alpha(F) = \sup \{ \alpha(E) : E \subset F; \ E \text{ compact} \}. \]
The following is a simple relationship of the area measure (the Lebesgue measure) $L^2$, $\alpha$, and $\gamma$
\[ \frac{1}{4\pi} L^2(E) \leq \alpha(E)^2 \leq \gamma(E)^2 \] (1.5)
(see Theorem 3.2 on page 204 of [G69]).

The inner boundary of $K$, denoted by $\partial I K$, is the set of boundary points which do not belong to the boundary of any connected component of $\mathbb{C} \setminus K$.
The inner boundary conjecture (see [VM84], Conjecture 2) is: if $\alpha(\partial I K) = 0$, then $R(K) = A(K)$. X. Tolsa [To04] affirmatively answers the conjecture. For a compact subset $K$ with $\alpha(\partial I K) > 0$, it will be interesting to see a way to measure the size of $A(K)$ relative to $R(K)$. The following question is asked in Problem 3 of [Y19]:

**Question 1.1.** For a compact subset $K$ of $\mathbb{C}$, is there a function $F \in A(K)$ such that $R(K) + P(K)F$ is dense in $A(K)$?

The author [Y19] also provided an example of $K$ and a function $F \in A(K)$ such that $R(K) \neq A(K)$ and $R(K) + P(K)F$ is dense in $A(K)$. In this paper, we affirmatively answer Question 1.1. In fact, our main theorem below is a better version of Question 1.1.

**Main Theorem.** For a compact subset $K$ of $\mathbb{C}$, there exists a function $F \in A(K)$ such that
\[ A(K) = H(F, K). \]

Our proofs rely on remarkable results on (continuous) analytic capacity from [To03] and [To04] and modified Vitushkin approximation scheme by P. V. Paramonov [P95]. In section 2, we review some recent results of (continuous) analytic capacity and Cauchy transform that are needed in our analysis. In the section 3, we construct a finite positive measure $\eta$ with compact support in $(\text{Int}(K))^c$ such that $C \eta$ equals $F \in A(K)$ almost everywhere with respect to the area measure $L^2$. In section 4, using modified Vitushkin approximation scheme by P. V. Paramonov, we show if $C(f \eta)$ is continuous ($= F_0 \in C(K), \ L^2 - a.a.$), then $F_0 \in H(F, K)$. We prove our main theorem in section 5.

Before closing this section, we mention some previous related research results. Trent and Wang [TW81] shows if $K$ is a compact subset without interior, then $R(K) + R(K) \bar{z}$ is dense in $C(K)$. J. Verdera [V93] proves that each Dini-continuous function in $\text{clos}(A(K) + A(K) \bar{z})$ belongs to $\text{clos}(R(K) + R(K) \bar{z})$. Finally, the excellent paper [M04] proves that $R(K) + R(K) \bar{z}$ is dense in $A(K) + A(K) \bar{z}$ for any compact subset $K$. 

In [T93], using the methods developed in [T91], J. Thomson proves if \( R(K) \neq C(K) \), then \( R(K) + P(K)z \) is not dense in \( C(K) \). The author [Y94] and [Y95] study the generalized space \( R(K) + P(K)g \) and prove that for a smooth function \( g \) with \( \bar{\partial}g \neq 0 \), then \( R(K) + P(K)g \) is dense in \( A(K) + P(K)g \) if and only if \( A(K) = R(K) \). Moreover, [Y19] studies the space \( R(K) + \sum_{n=1}^{N} P(K)z^n \) and shows that \( R(K) + \sum_{n=1}^{N} P(K)z^n \) is dense in \( A(K) + \sum_{n=1}^{N} P(K)z^n \) if and only if \( A(K) = R(K) \).

2. Preliminaries

For a finite complex-valued Borel measure \( \nu \) with compact support on \( \mathbb{C} \), \( \mathcal{C}_c(\nu) \) is defined as in (1.1) and (principal value) Cauchy transform \( \mathcal{C} (\nu)(z) = \lim_{\epsilon \to 0} \mathcal{C}_\epsilon(\nu)(z) \), when the limit exists, is as in (1.2). It is well known that in the sense of distribution,

\[ \bar{\partial} \mathcal{C}(\nu) = -\pi \nu. \]  

(2.1)

The maximal Cauchy transform is defined by

\[ \mathcal{C}_*(\nu)(z) = \sup_{\epsilon > 0} |\mathcal{C}_\epsilon(\nu)(z)|. \]

Analytic capacity \( \gamma \) is defined as in (1.3). A related capacity, \( \gamma_+ \), is defined for subsets \( E \) of \( \mathbb{C} \) by:

\[ \gamma_+(E) = \sup \| \eta \|, \]

where the supremum is taken over positive measures \( \eta \) with compact support contained in \( E \) for which \( \| \mathcal{C}(\eta) \|_{L^\infty(\mathbb{C})} \leq 1 \). Since \( \mathcal{C} \eta \) is analytic in \( \mathbb{C}_\infty \setminus \text{spt}(\eta) \) and \( |(\mathcal{C}(\eta))'(\infty)| = \| \eta \| \), we have: \( \gamma_+(E) \leq \gamma(E) \) for all subsets \( E \) of \( \mathbb{C} \).

Continuous analytic capacity \( \alpha \) is defined as in (1.4). The capacity \( \alpha_+ \) of a bounded set \( E \subset \mathbb{C} \) is defined as \( \alpha_+(E) = \sup \eta(E) \), where the supremum is taken over all finite positive measures \( \eta \) supported on \( E \) such that \( \mathcal{C}(\eta) \) is a continuous function on \( \mathbb{C} \) (i.e. it coincides \( L^2 \) – a.a. with a continuous function on \( \mathbb{C} \)), with \( |\mathcal{C}(\eta)(z)| \leq 1 \). Notice that we clearly have \( \alpha_+(E) \leq \alpha(E) \), because \( |\mathcal{C}(\eta)'(\infty)| = \eta(E) \).

Given three pairwise different points \( x, y, z \in \mathbb{C} \), their Menger curvature is

\[ c(x, y, z) = \frac{1}{R(x, y, z)}, \]

where \( R(x, y, z) \) is the radius of the circumference passing through \( x, y, z \) (with \( R(x, y, z) = \infty \), \( c(x, y, z) = 0 \) if \( x, y, z \) lie on a same line). If two among these points coincide, we let \( c(x, y, z) = 0 \). For a finite positive measure \( \eta \), we set

\[ c_\eta^2(x) = \int \int c(x, y, z)^2 d\eta(y) d\eta(z) \]

and we define the curvature of \( \eta \) as

\[ c^2(\eta) = \int \int c_\eta^2(x) d\eta(x) = \int \int \int c(x, y, z)^2 d\eta(x) d\eta(y) d\eta(z) \]

and
\[ c^2_\epsilon(\eta) = \int \int \int_{\min(|z-y|,|x-z|,|x-y|) > \epsilon} c(x, y, z)^2 d\eta(x) d\eta(y) d\eta(z). \]

Set \( B(\lambda, \delta) = \{ z : |z - \lambda| < \delta \} \). For a finite complex-valued measure \( \nu \), define
\[ \Theta^*_\nu(\lambda) := \lim_{\delta \to 0} \frac{|\nu|(B(\lambda, \delta))}{\delta} \]
and
\[ \Theta_\nu(\lambda) := \lim_{\delta \to 0} \frac{|\nu|(B(\lambda, \delta))}{\delta} \]
if the limit exists.

A finite measure \( \nu \) supported in \( E \) is \( c_0 \)-linear growth if \( |\nu|(B(\lambda, \delta)) \leq c_0 \delta \) for \( \lambda \in \mathbb{C} \), denoted by \( \nu \in \Sigma(E) \) when \( c_0 = 1 \). In addition, if \( \Theta_\nu(\lambda) = 0 \), we say \( \nu \in \Sigma_0(E) \).

X. Tolsa has established the following astounding results.

**Theorem 2.1. (Tolsa’s Theorem)**

1. \( \gamma_+ \) and \( \gamma \) are equivalent. \( \alpha_+ \) and \( \alpha \) are equivalent. That is, there is an absolute positive constant \( A_T \) such that
\[ \gamma(E) \leq A_T \gamma_+(E) \tag{2.2} \]
and
\[ \alpha(E) \leq A_T \alpha_+(E). \tag{2.3} \]

2. Semiadditivity of analytic capacity:
\[ \gamma \left( \bigcup_{i=1}^m E_i \right) \leq A_T \sum_{i=1}^m \gamma(E_i) \tag{2.4} \]
and
\[ \alpha \left( \bigcup_{i=1}^m E_i \right) \leq A_T \sum_{i=1}^m \alpha(E_i) \tag{2.5} \]
where \( E_1, E_2, ..., E_m \subset \mathbb{C} \) and \( m \) could be \( \infty \).

3. There is an absolute positive constant \( C_T \) such that, for any \( a > 0 \), we have:
\[ \alpha(\{ C_*(\nu) \geq a \}) \leq \gamma(\{ C_*(\nu) \geq a \}) \leq \frac{C_T}{a} \| \nu \|. \]

**Proof.** (2.2) and (2.4) are from [To03] (also see Theorem 6.1 and Corollary 6.3 in [To14]). (2.3) and (2.5) are from [To04]. (3) follows from Proposition 2.1 of [To02] (also see [To14] Proposition 4.16). \( \square \)

Let \( \phi \) be a bounded non-negative function on \( \mathbb{R} \) supported on \([0, 1]\) with \( 0 \leq \phi(z) \leq 2 \) and \( \int \phi(|z|) d\mathcal{L}^2(z) = 1 \). Let \( \phi_\epsilon(z) = \frac{1}{\epsilon^2} \phi(\frac{|z|}{\epsilon}) \). Define the kernel function
$K_\varepsilon = -\frac{1}{z^2} * \phi_\varepsilon$ and for a finite complex-valued measure $\nu$ with compact support, define $\tilde{C}_\varepsilon \nu = K_\varepsilon * \nu$. It is easy to show that

$$K_\varepsilon(z) = -\frac{1}{z^2}, \quad |z| \geq \varepsilon$$

and $\|K_\varepsilon\|_\infty \leq \frac{C_1}{\varepsilon}$. Hence,

$$|\tilde{C}_\varepsilon \nu(\lambda) - C_\varepsilon \nu(\lambda)| = \left| \int_{|z-\lambda| \leq \varepsilon} K_\varepsilon(\lambda - z) d\nu(z) \right| \leq C_1 \frac{|\nu|(B(\lambda, \varepsilon))}{\varepsilon}. \quad (2.6)$$

Recall that the definition of Hausdorff measure (in $\mathbb{C}$). Given $d \geq 0$ and $0 < \varepsilon \leq \infty$, for $A \subset \mathbb{C}$,

$$\mathcal{H}^d(\nu) = \inf \left\{ \sum_i \text{diam}(A_i)^d : A \subset \bigcup_i A_i, \text{diam}(A_i) \leq \varepsilon \right\}.$$

The $d$-dimensional Hausdorff measure of $A$ is:

$$\mathcal{H}^d(A) = \sup_{\varepsilon > 0} \mathcal{H}^d_{\varepsilon}(A) = \lim_{\varepsilon \to 0} \mathcal{H}^d_{\varepsilon}(A).$$

For our purposes, we will use $\mathcal{H}^1$.

For a finite complex-valued measure $\nu$ with compact support, we say $C \nu$ is continuous if there exists a continuous function on $\mathbb{C}$, denoted by $F_\nu$, such that

$$C \nu(z) = F_\nu(z), \quad L^2 - \text{a.a.} \quad (2.7)$$

The following results are used throughout this paper, we list them here as a lemma.

**Lemma 2.2.** (1) Let $\mu$ be a finite positive measure with compact support in $\mathbb{C}$. For $\lambda_2 \geq \lambda_1 > 0$, let $E$ and $F$ be two bounded Borel sets such that

$$E \subset \{ z : \Theta^*_\mu(z) \geq \lambda_1 \}$$

and

$$F \subset \{ z : \lambda_1 \leq \Theta^*_\mu(z) \leq \lambda_2 \}.$$

There exist some absolute constants $c_1$, $c_2$, and $c_3$ such that $\mathcal{H}^1(E) \leq c_1 \frac{\mu(E)}{\lambda_1}$ and $\mu|_F = g \mathcal{H}^1|_F$, where $g$ is some Borel function such that $c_2 \lambda_1 \leq g(z) \leq c_3 \lambda_2$, $\mathcal{H}^1|_F - \text{a.a.}$.

(2) Suppose that $\nu$ is a finite, complex-valued Borel measure with compact support in $\mathbb{C}$. Then there exists $Q \subset \mathbb{C}$ with $\gamma(Q) = 0$ such that $\lim_{\varepsilon \to 0} C_\varepsilon(\nu)(z)$ exists for $z \in \mathbb{C} \setminus Q$.

(3) Let $\{ \nu_j \}$ be a sequence of finite complex-valued measures with compact supports. Then for $\varepsilon > 0$, there exists a Borel subset $F$ such that $\gamma(F^c) < \varepsilon$ and $C_\varepsilon(\nu_j)(z) \leq M_j < \infty$ for $z \in F$.

(4) Let $\nu$ be a compactly supported finite positive measure on $\mathbb{C}$ with $c_0$-linear growth. We have

$$\| C_\varepsilon \nu \|_{L^2(\nu)}^2 = \frac{1}{6} c_0^2(\nu) + O(\nu(\mathbb{C}))$$

where

$$|O(\nu(\mathbb{C}))| \leq C_2 c_0^2 \nu(\mathbb{C}).$$
(5) Let \( \eta \) be a finite positive measure supported on a compact subset \( E \). Suppose that \( \| \mathcal{C}\eta \|_{L^\infty(\mathbb{C})} \leq 1 \), then \( \eta \in \Sigma(E) \) and \( \| \mathcal{C}\eta \|_{L^\infty(\mathbb{C})} \leq C_2 \) for all \( \epsilon > 0 \). In particular, if \( \mathcal{C}\eta(z) \) is continuous, then \( \eta \in \Sigma_0(E) \), moreover,

\[
\eta(B(z, \delta)) \leq \delta \sup_{w \in B(z, \delta)} |F_\eta(w) - F_\eta(z)|.
\]

(6) Let \( \eta \) be a finite positive measure with compact support. Suppose that \( \| \mathcal{C}\eta \|_{L^\infty(\mathbb{C})} \leq 1 \), \( \mathcal{C}\eta(z) \) is continuous, and \( \eta(E) > 0 \). Then there exists a function \( f \) with \( 0 \leq f(z) \leq 1 \) supported on \( E \) such that \( \int f(z) \, d\eta \geq c_1 \eta(E) \) and \( \mathcal{C}(f\eta)(z) \) is continuous.

(7) For all \( E \subset \mathbb{C} \), we have

\[
\gamma(E) \approx \sup_{\eta \text{ positive}} \{ \eta(E) : \eta \in \Sigma(E), \ c^2(\eta) \leq \eta(E) \}.
\]

(8) Let \( \nu \) be a finite complex-valued measure with compact support. Let \( \eta \) be a finite positive measure supported on a compact subset \( E \). Suppose that \( \mathcal{C}\eta(z) \) is continuous and \( \mathcal{C}_*(\nu)(z) \leq C_3 < \infty \) for \( z \in E \), then

(a) \( \eta(B)^\frac{1}{2} \leq C_4 \|\nu\|^\frac{1}{2} \gamma(B) \) for all bounded Borel subsets \( B \), consequently, \( \eta(B) = 0 \) if \( \gamma(B) = 0 \); and

(b) \[
\int F_\eta(z) \, d\nu(z) = -\int \mathcal{C}(z) \, d\eta(z).
\]

Proof. (1) follows from Lemma 8.12 on page 307 in [To14].

For (2), see [To98] (also Theorem 8.1 in [To14], or see Corollary 3.1 in [ACY19]).

The proof of (3) is an application of Theorem 2.1 (2) and (3). In fact, let \( A_j = \{ \mathcal{C}_*(\nu_j)(z) \leq M_j \} \). By Theorem 2.1 (3), we can select \( M_j > 0 \) so that \( \gamma(A_j^c) < \frac{\epsilon}{2^j M_j} \). Set \( F = \cap_{j=1}^\infty A_j \). Then applying Theorem 2.1 (2), we get

\[
\gamma(F^c) \leq A_T \sum_{j=1}^\infty \gamma(A_j^c) < \epsilon.
\]

For (4), see Proposition 3.3 in [To14].

For (5), see the proof of Theorem 4.14 of [To14] on page 113 and the proof of Lemma 3.2 [To04].

(6) Using (4), we have \( c^2(\eta) \leq C_2 \|\eta\| \). Set \( A = \{ c^2_\eta(x) \geq \frac{2C_2\|\eta\|}{\eta(E)} \} \), then

\[
\eta(A) \leq \frac{\eta(E)}{2C_2\|\eta\|} \int_A c^2_\eta(x) \, d\eta(x) \leq \frac{\eta(E)}{2}.
\]

Set \( \eta_0 = \eta|_{E \setminus A} \), then \( \|\eta_0\| \geq \frac{1}{2} \eta(E) \). Hence,

\[
c^2(\eta_0) \leq \int_{E \setminus A} c^2_\eta(x) \, d\eta(x) \leq \frac{2C_2\|\eta\|}{\eta(E)} \|\eta_0\|.
\]

Let \( \mu = \left( \frac{\eta(E)}{\|\eta\|} \right)^\frac{1}{2} \eta_0 \), then

\[
c^2(\mu) = \left( \frac{\eta(E)}{\|\eta\|} \right)^\frac{1}{2} c^2(\eta_0) \leq 2C_2 \|\mu\|.
\]
Using the same proof as in Lemma 3.2 [To04] for $\mu$, we conclude that there exists a compact subset $E_0 \subset E \setminus A$ such that $\mu(E_0) \geq \frac{1}{2}\|\mu\|$, $\Theta_\mu(\lambda) = 0$ for $\lambda \in E_0$ and $c_\mu^2(z) \leq C_3$ for $z \in \mathbb{C}$. Using the same proof as in Lemma 3.4 of [To04], we find a compact subset $F \subset E_0$, $\mu(F) \geq \frac{1}{4}\|\mu\|$, and a function $f$ supported on $F$ with $0 \leq f(z) \leq 1$, and $\int f(x)d\mu(x) \geq c_1\mu(F)$ such that $C(f\mu)$ is continuous. (6) is proved.

(7) See Theorem 4.14 in [To14] and Theorem 2.1 (1).

(8) If $\eta(B) > 0$, then, by (4),

$$c^2(\eta|_B) \leq c^2(\eta) \leq C_2\|\eta\| \leq \frac{C_2\|\eta\|}{\eta(B)}\eta(B).$$

Let $\eta_B = \left(\frac{\eta(B)}{\|\eta\|}\right)^{\frac{1}{2}}\eta|_B$, hence, by (7), we have $\|\eta_B\| \leq C_4\gamma(B)$ which implies (a).

From (2.6) and (5), we have

$$|C_\epsilon\eta(\lambda) - F_\eta(\lambda)| \leq |\tilde{C}_\epsilon\eta(\lambda) - C_\epsilon\eta(\lambda)| + |\tilde{C}_\epsilon\eta(\lambda) - F_\eta(\lambda)| \leq C_1\frac{\eta(B, \epsilon)}{\epsilon} + \int |F_\eta(z) - F_\eta(\lambda)|\phi_\epsilon(\lambda - z)d\mathcal{L}^2(z) \rightarrow 0$$

as $\epsilon \rightarrow 0$. Using (2) and (8) (a), $C_\epsilon\nu(\lambda) \rightarrow C\nu(\lambda)$, $\eta - a.a.$ as $\epsilon \rightarrow 0$. Clearly,

$$\int C_\epsilon\eta(z)d\nu(z) = -\int C_\epsilon\nu(z)d\eta(z).$$

Now (8) (b) follows from Lebesgue dominated convergence theorem. □

The following lemma is due to Lemma 3.2 in [Y19].

**Lemma 2.3.** Let $\nu$ be a finite, complex-valued Borel measure that is compactly supported in $\mathbb{C}$ and assume that for some $\lambda_0$ in $\mathbb{C}$ we have:

(a) $\Theta_\nu(\lambda_0) = 0$ and

(b) $C(\nu)(\lambda_0) = \lim_{\epsilon \rightarrow 0} C_\epsilon(\nu)(\lambda_0)$ exists.

Then:

(1) $C(\nu)(\lambda) = \lim_{\epsilon \rightarrow 0} C_\epsilon(\nu)(\lambda)$ exists for $\lambda \in \mathbb{Q}^c$ with $\gamma(\mathbb{Q}) = 0$ and

(2) for all $a > 0$, there exists $\epsilon(\delta) > 0$ such that

$$\epsilon(\delta) \rightarrow 0, \text{ as } \delta \rightarrow 0,$$

and

$$\gamma(B(\lambda_0, \delta) \cap \{|C(\nu)(\lambda) - C(\nu)(\lambda_0)| > a\}) \leq \epsilon(\delta)\delta$$

where $\delta < \delta_a$ for some $\delta_a > 0$. 

3. Construction of $F$

Let $\mathbb{C} \setminus K = \bigcup_{m=0}^{\infty} U_m$, where $U_0$ is the unbounded connected component and $U_m$ is a bounded connected component for $m \geq 1$. For $m \geq 1$, let $l_m^1$ denote the vertical line passing $\lambda_m \in U_m$ and $L_m = l_m^1 \cup \text{clos}(B(\lambda_m, \delta_m))$, where $B(\lambda_m, \delta_m) \subset U_m$. Let $\lambda_m \in U_m$ and $B(\lambda_m, 2\delta_m) \subset U_m$ such that $B(\lambda_m, 2\delta_m) \cap B(\lambda_m, 2\delta_m) = \emptyset$. Let $K_m = B(\lambda_m, \delta_m)$ be a compact subset with no interior such that $B(\lambda_m, \delta_m) \setminus K_m$ is connected and $\alpha(K_m) > 0$. Let $\text{Int}(K) = \bigcup_{m=1}^{\infty} W_m$, where $W_m$ is a connected component. Let $l_m^2$ denote the vertical line passing $\lambda_m \in W_m$ and $L_m = l_m^2 \cup \text{clos}(B(\lambda_m^2, \delta_m))$, where $B(\lambda_m^2, \delta_m) \subset W_m$. Set

$$\{L_m\} = \{L_m^1\} \cup \{L_m^2\}$$

and

$$E = \partial I K \cup \bigcup_{m=0}^{\infty} K_m \setminus \left(\bigcup_{m=0}^{\infty} I_m\right).$$

(3.1)

Let $\{S_{ij}\}$ be a square partition of $\mathbb{C}$, where the length and center of the square $S_{ij}$ is $\frac{1}{2^n}$ and $s_{ij}$, respectively. That is, $\bigcup_{n=1}^{\infty} S_{ij} = \mathbb{C}$ and $\text{Int}(S_{ij}) \cap \text{Int}(S_{i_1j_1}) = \emptyset$ for $(i, j) \neq (i_1, j_1)$. Set $aS_{ij} = \{a(z - s_{ij}) + s_{ij}, z \in S_{ij}\}$. We construct a finite positive measure with compact support as the following.

By Theorem 2.1 (2), we have

$$\alpha(2S_{ij} \cap E) \geq \alpha((2S_{ij} \cap \partial E) \setminus \bigcup_{m}(2S_{ij} \cap \partial I K \cap (l_m^1 \cup l_m^2)))$$

$$\geq \frac{1}{A_T} \alpha(2S_{ij} \cap \partial I K) - \sum_{m} (\alpha(2S_{ij} \cap \partial I K \cap l_m^1) + \alpha(2S_{ij} \cap \partial I K \cap l_m^2))$$

(3.2)

$$= \frac{1}{A_T} \alpha(2S_{ij} \cap \partial I K).$$

From Theorem 2.1 (1), we find a finite positive measure $\eta_{ij}^n$ with

$$\text{spt}(\eta_{ij}^n) \subset 2S_{ij} \cap E, \mathcal{C}(\eta_{ij}^n) = F_{\eta_{ij}, L^2 - a.a.,}$$

(3.3)

where $F_{\eta_{ij}}$ (defined as in (2.7)) is continuous on $\mathbb{C}$, and, by (3.2),

$$\|\eta_{ij}^n\| \geq c_1 \alpha(2S_{ij} \cap E) \geq c_2 \alpha(2S_{ij} \cap \partial I K), \|\mathcal{C}(\eta_{ij}^n)\|_{L^\infty(\mathbb{C})} \leq 1.$$  

(3.4)

Let $M_n$ be the number of squares $S_{ij}$ with $\alpha(2S_{ij} \cap E) > 0$. We define

$$\eta_m = \frac{1}{M_n} \sum_{\alpha(2S_{ij} \cap E) > 0} \eta_{ij}^n.$$ 

The measure $\eta_m$ satisfies the following properties

(A1) $F_{\eta_m} \in A(K)$, where $F_{\eta_m}$ is as in (2.7);

(A2) $\|\mathcal{C}(\eta_m)\|_{L^\infty(\mathbb{C})} \leq 1$; and
(A3) \( \frac{F_{\eta_n}(z)-F_{\eta_n}(\lambda)}{z-\lambda} \in A(K) \) for \( \lambda \in L_m \) and \( m \geq 1 \), and
\[
\int \left| z-\frac{1}{\lambda} \right| d\eta_n(z) + \left\| \frac{F_{\eta_n}(z)-F_{\eta_n}(\lambda)}{z-\lambda} \right\|_{L^\infty(C)} \leq B_n < \infty, \quad \lambda \in \bigcup_{k=1}^{n} L_k.
\]

Let \( \{u_k\} \) be a dense set of \( \bigcup_{m=1}^{\infty} L_m \) such that \( \{u_k\} \cap L_m \) is dense in \( L_m \) for \( m \geq 1 \). Set \( f_k(z) = \frac{1}{z-u_k} \). Notice that if \( C\nu \) is continuous, then for a polynomial \( p 
\]
\[
C(p\nu)(z) = \int \frac{p(w) - p(z)}{w-z}d\nu(w) + p(z)C(\nu)(z)
\]
is also continuous. There is \( m_1 \) such that \( u_1 \in L_{m_1} \). Define
\[
d_1 = \inf_{x \in L_{m_1}, y \in \text{spt}(\eta_1)} \text{dist}(x, y) > 0.
\]
Let \( D_1 = \{x : \text{dist}(x, y) \leq \frac{1}{3}d_1, \ y \in \text{spt}(\eta_1)\} \). By the construction, we can find a polynomial \( p_{11} \) such that
\[
\left\| p_{11}(z) - \frac{1}{z-u_1} \right\|_{C(D_1)} \leq \frac{d_1}{2(\|\eta_1\| + 1)}.
\]
We have the following calculation (the maximum modulus principle is applied):
\[
\left\| \frac{F_{p_{11}\eta_1}(z) - F_{\eta_1}(u_1)}{z-u_1} \right\|_{C(D_1)} \leq \left\| \frac{F_{p_{11}\eta_1}(z) - F_{f_1\eta_1}(z)}{z-u_1} \right\|_{C(D_1)}
\]
\[
\leq \left\| \int \frac{p_{11}(w) - p_{11}(z)}{w-z}d\eta_1(w) \right\|_{C(D_1)}
\]
\[
\leq \left\| \int \frac{f_1(w) - f_1(z)}{w-z}d\eta_1(w) \right\|_{C(D_1)}
\]
\[
\leq \left( \frac{d_1}{4}\|\eta_1\| + 1 \right) \left\| p_{11} - f_1 \right\|_{C(D_1)}
\]
\[
\leq \frac{1}{2}.
\]
Set \( a_1 = \min\left(\frac{1}{2}, \frac{1}{2d_1}\right) \) and \( \xi_1 = a_1\eta_1 \). We find \( L_{m_2} \) with \( u_2 \in L_{m_2} \). Define
\[
d_2 = \inf_{x \in L_{m_1} \cup L_{m_2}, y \in \text{spt}(\eta_1) \cup \text{spt}(\eta_2)} \text{dist}(x, y) > 0.
\]
Let \( D_2 = \{x : \text{dist}(x, y) \leq \frac{1}{3}d_2, \ y \in \text{spt}(\eta_1) \cup \text{spt}(\eta_2)\} \). Let
\[
A_2 = \left\| \frac{F_{p_{11}\eta_2}(z) - F_{\eta_2}(u_1)}{z-u_1} \right\|_{C(D_2)}
\]
and
\[
a_2 = \min \left( \frac{1}{4}, \frac{1}{4A_2}, \frac{1}{4B_2} \right), \quad \xi_2 = \xi_1 + a_2\eta_2.
\]
Then
\[
\left\| \frac{F_{p_{11}\xi_2}(z) - F_{\xi_2}(u_1)}{z-u_1} \right\|_{C(D_2)} \leq \frac{1}{2} + \frac{1}{4}.
\]
Similarly, we can find two polynomials \( p_{12} \) and \( p_{22} \) such that
\[
\left\| F_{p_{12} \xi_2}(z) - \frac{F_{\xi_2}(z) - F_{\xi_2}(u_1)}{z - u_1} \right\|_{C(D_2)} \leq \frac{1}{4}
\]
and
\[
\left\| F_{p_{22} \xi_2}(z) - \frac{F_{\xi_2}(z) - F_{\xi_2}(u_2)}{z - u_2} \right\|_{C(D_2)} \leq \frac{1}{4}.
\]
We find \( L_{m_3} \) with \( u_3 \in L_{m_3} \). Define
\[
d_3 = \inf_{x \in L_{m_1} \cup L_{m_2} \cup L_{m_3}, y \in \text{spt}(\eta_{l_1}) \cup \text{spt}(\eta_{l_2}) \cup \text{spt}(\eta_{l_3})} \text{dist}(x, y) > 0.
\]
Let \( D_3 = \{ x : \text{dist}(x, y) \leq \frac{1}{2}d_3, y \in \text{spt}(\eta_{l_1}) \cup \text{spt}(\eta_{l_2}) \cup \text{spt}(\eta_{l_3}) \} \). Let
\[
A_3 = \left\| F_{p_{11} \eta_3}(z) - \frac{F_{\eta_3}(z) - F_{\eta_3}(u_1)}{z - u_1} \right\|_{C(D_3)}
\]
and
\[
a_3 = \min \left( \frac{1}{8}, \frac{1}{8A_3}, \frac{1}{8B_3} \right), \quad \xi_3 = \xi_2 + a_3 \eta_3.
\]
Then
\[
\left\| F_{p_{11} \xi_3}(z) - \frac{F_{\xi_3}(z) - F_{\xi_3}(u_1)}{z - u_1} \right\|_{C(D_3)} \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{8},
\]
\[
\left\| F_{p_{12} \xi_3}(z) - \frac{F_{\xi_3}(z) - F_{\xi_3}(u_1)}{z - u_1} \right\|_{C(D_3)} \leq \frac{1}{4} + \frac{1}{8},
\]
and
\[
\left\| F_{p_{22} \xi_3}(z) - \frac{F_{\xi_3}(z) - F_{\xi_3}(u_2)}{z - u_2} \right\|_{C(D_3)} \leq \frac{1}{4} + \frac{1}{8}.
\]
We find \( L_{m_k} \) with \( u_k \in L_{m_k} \). Define
\[
d_k = \inf_{x \in \bigcup_{i=1}^k L_{m_i}, y \in \bigcup_{i=1}^k \text{spt}(\eta_i)} \text{dist}(x, y) > 0.
\]
Let \( D_k = \{ x : \text{dist}(x, y) \leq \frac{1}{2}d_k, y \in \bigcup_{i=1}^k \text{spt}(\eta_i) \} \). Therefore, we can find polynomials \( \{ p_{kj} \}_{k \leq j} \) and positive measures \( \{ \xi_j \} \) such that, for \( k \leq l \leq j \),
\[
\left\| F_{p_{kl} \xi_{l+1}}(z) - \frac{F_{\xi_{l+1}}(z) - F_{\xi_{l+1}}(u_k)}{z - u_k} \right\|_{C(D_{l+1})} \leq \sum_{i=l}^{j+1} \frac{1}{2^i} \leq \frac{2}{2^l}.
\]
Using the maximum modulus principle, we have
\[
\left\| F_{p_{kl} \xi_{l+1}}(z) - \frac{F_{\xi_{l+1}}(z) - F_{\xi_{l+1}}(u_k)}{z - u_k} \right\|_{C(K)} \leq \frac{2}{2^l}.
\]
It is clear that, by the construction,
\[ \eta = \sum_{n=1}^{\infty} a_n \eta_n \]  
(3.5)
is well defined. Clearly, from (A1)-(A3) and for \( u \in L_m \), we conclude that \( \|C\eta\|_{L^\infty(C)} \leq 1 \), \( C\eta \) is continuous, \( C\eta(u) = F_{\eta}(u) \), \( C(\frac{\eta(z) - \eta(u)}{z - u}) = C(\frac{\eta}{w - u})(z) \mathcal{L}^2 - a.a. \), \( \|\mathcal{C}(\frac{\eta}{w - u})\|_{L^\infty(C)} \leq 1 \), and \( \mathcal{C}(\frac{\eta}{w - u})(z) \) is continuous. Then for \( k \leq l \),
\[ \left\| F_{p_k\eta}(z) - F_{\eta}(u_k) \right\|_{C(K)} \leq \frac{1}{2^{l-1}}. \]
Notice that
\[ \mathcal{C}(p_k\eta)(z) = \int \frac{p_k(w) - p_k(z)}{w - z} d\eta(w) + p_k(z) F_{\eta}(z)(\in H(F_{\eta}, K)), \mathcal{L}^2 - a.a. \]
Hence,
\[ \frac{F_{\eta}(z) - F_{\eta}(u_k)}{z - u_k} \in H(F_{\eta}, K). \]
By the construction, it is easy to show that for \( \lambda \in L_m \), there exists a sequence \( \{u_{n_k}\} \subset L_m \) with \( u_{n_k} \to \lambda \) and
\[ \left\| \frac{F_{\eta}(z) - F_{\eta}(u_{n_k})}{z - u_{n_k}} - \frac{F_{\eta}(z) - F_{\eta}(\lambda)}{z - \lambda} \right\|_{C(K)} \to 0. \]
Thus,
\[ \frac{F_{\eta}(z) - F_{\eta}(\lambda)}{z - \lambda} \in H(F_{\eta}, K). \]
for \( \lambda \in L_m \) and \( m \geq 1 \). From Lemma 2.2 (2), for \( \nu \perp H(F_{\eta}, K) \) (\( \int h(z)d\nu(z) = 0 \), for \( h \in H(F_{\eta}, K) \)), we get
\[ F_{\eta}(z)C(\nu)(z) = C(F_{\eta}\nu)(z), \gamma|_{L_m} - a.a. \]  
(3.6)
Here \( f(z) = g(z) \), \( \gamma|_{B} - a.a. \) (or \( \alpha|_{B} - a.a. \)) means there exists \( Q \subset B \) with \( \gamma(Q) = 0 \) (or \( \alpha(Q) = 0 \)) such that \( f(z) = g(z) \), \( z \in B \setminus Q \).

**Lemma 3.1.** Let \( \eta \) and \( F_{\eta} \) be as in (3.5) and (2.7). Suppose \( \nu \perp H(F_{\eta}, K) \), then
\[ C(\nu)(z) = C(F_{\eta}\nu)(z) = 0, \quad z \in U_m \]
and
\[ F_{\eta}(z)C(\nu)(z) = C(F_{\eta}\nu)(z), \mathcal{L}^2|_{W_m} - a.a. \]  
(3.7)

**Proof.** Since \( \text{spt}\eta \cap W_m = \emptyset \), \( C(\eta)(z) \) is analytic on \( W_m \). From (2.1), we have
\[ \partial(F_{\eta}(z)C(\nu)(z) - C(F_{\eta}\nu)(z)) = 0, \mathcal{L}^2|_{W_m} - a.a. \]
By Weyls lemma, there exists an analytic function \( a(z) \) on \( W_m \) such that
\[ F_{\eta}(z)C(\nu)(z) - C(F_{\eta}\nu)(z) = a(z) \mathcal{L}^2|_{W_m} - a.a. \]
From (3.6) and (1.5), we see that \( a(z) = 0 \mathcal{L}^2|_{B(\lambda_n, \delta_n)} - a.a. \), which implies (3.8).
Since $\text{spt} \eta \cap (U_m \setminus K_m) = \emptyset$, $\text{spt} \nu \cap U_m = \emptyset$, and from (3.6), $\mathcal{C}(\eta)(z)\mathcal{C}(\nu)(z) = \mathcal{C}(F_{\eta} \nu)(z)$, $z \in B(\lambda_m, \delta_m)$, we get
\[ \mathcal{C}(\eta)(z)\mathcal{C}(\nu)(z) = \mathcal{C}(F_{\eta} \nu)(z), \quad z \in U_m \setminus K_m. \]
Since $F_{\eta}(z)$ is continuous and $\mathcal{C}(\nu)(z)$ and $\mathcal{C}(F_{\eta} \nu)(z)$ are analytic on $U_m$. This implies
\[ \mathcal{C}(\eta)(z)\mathcal{C}(\nu)(z) = \mathcal{C}(F_{\eta} \nu)(z), \quad \mathcal{L}^2|_{U_m} - a.a. \]
because $\text{clos}(U_m \setminus K_m) = \text{clos}(U_m)$. From (2.1), taking $\bar{\partial}$ both sides, we get $\mathcal{C}(\nu)(z) = 0$, $\eta|_{K_m} - a.a.$ (3.7) follows. \hfill \Box

**Lemma 3.2.** Let $\eta$ and $F_{\eta}$ be as in (3.5) and (2.7). Suppose $\nu \perp H(F_{\eta}, K)$, then
\[ \mathcal{C}(\nu)(z) = \mathcal{C}(F_{\eta} \nu)(z) = 0, \quad \alpha|_{U_m} - a.a. \quad (3.9) \]

**Proof.** Let $G_m = \{z \in \partial U_m : \Theta_{\nu}(z) > \frac{1}{m}\}$. Then using Lemma 2.2 (1) for $|\nu|$, we see that $\nu|G_m$ is absolutely continuous with respect to $\mathcal{H}^1$. Therefore, if $\mu$ is a finite positive measure with compact support and $C\mu$ is continuous on $\mathcal{C}$, then by Lemma 2.2 (5), we have $\mu(G_m) = 0$ since $\mu \in \Sigma_0(\text{spt}(\mu))$. By Theorem 2.1 (1), we must have $\alpha(G_m) = 0$, which implies $\alpha(\cup_m G_m) = 0$ by Theorem 2.1 (2). Now for $\lambda_0 \in G = \{z \in \partial U_m : \Theta_{\nu}(z) = 0\}$ and $\lim_{\delta \to 0} \mathcal{C}(\nu)(\lambda_0) = \mathcal{C}(\nu)(\lambda_0)$ exists $\alpha - a.a.$ (see Lemma 2.2 (2)). Applying Lemma 2.3, we have
\[ \lim_{\delta \to 0} \frac{\gamma(B(\lambda_0, \delta) \cap \{ |\mathcal{C}(\nu)(\lambda) - \mathcal{C}(\nu)(\lambda_0)| > \epsilon \})}{\delta} = 0. \]
Let $P$ be a path stating at $\lambda_0$ such that $P \setminus \{\lambda_0\} \subset U_m$. Using Theorem 2.1 (2), we get
\[ \lim_{\delta \to 0} \frac{\gamma(B(\lambda, \delta) \cap P \cap \{ |\mathcal{C}(\nu)(\lambda) - \mathcal{C}(\nu)(\lambda_0)| \leq \epsilon \})}{\delta} \]
\[ \geq \frac{1}{A_T} \lim_{\delta \to 0} \frac{\gamma(B(\lambda, \delta) \cap P)}{\delta} - \lim_{\delta \to 0} \frac{\gamma(B(\lambda, \delta) \cap \{ |\mathcal{C}(\nu)(\lambda) - \mathcal{C}(\nu)(\lambda_0)| > \epsilon \})}{\delta} \]
\[ = \frac{1}{A_T} \lim_{\delta \to 0} \frac{\gamma(B(\lambda, \delta) \cap P)}{\delta} > 0. \]
There exists a sequence $\{\lambda_k\} \subset P \setminus \{\lambda_0\} \subset U_m$ with $\lambda_k \to \lambda_0$ and $\mathcal{C}(\nu)(\lambda_k) \to \mathcal{C}(\nu)(\lambda_0)$, which implies $\mathcal{C}(\nu)(\lambda_0) = 0$ by Lemma 3.1. Similarly, $\mathcal{C}(F_{\eta} \nu)(\lambda_0) = 0$. (3.9) is proved. \hfill \Box

**Lemma 3.3.** Let $\eta$ and $F_{\eta}$ be as in (3.5) and (2.7). Suppose $\nu \perp H(F_{\eta}, K)$, then
\[ F_{\eta}(z)\mathcal{C}(\nu)(z) = \mathcal{C}(F_{\eta} \nu)(z), \quad \mathcal{L}^2 - a.a. \quad (3.10) \]

**Proof.** From Lemma 3.1 and Lemma 3.2, since a set of zero analytic capacity is $\mathcal{L}^2$ zero set by (1.5), it remains to prove (3.10) for $\lambda \in \partial H K$. Now for $\lambda \in \partial K$ with $\int_{|z-\lambda|} d|\nu|(z) < \infty$. Set $B(\nu, \lambda, \epsilon) = \{|\mathcal{C}(\nu)(z) - \mathcal{C}(\nu)(\lambda)| > \epsilon\}$. Using Lemma 2.3, we get
\[ \lim_{\delta \to 0} \frac{\gamma(B(\lambda, \delta) \cap B(\nu, \lambda, \epsilon))}{\delta} = \lim_{\delta \to 0} \frac{\gamma(B(\lambda, \delta) \cap B(F_{\eta} \nu, \lambda, \epsilon))}{\delta} = 0. \]
It is clear that
\[
\lim_{\delta \to 0} \gamma (B(\lambda, \delta) \cap \bigcup_m (L_m \cup U_m)) > 0.
\]
By Theorem 2.1 (2), (3.6), and (3.7), we see that
\[
\lim_{\delta \to 0} \gamma (B(\lambda, \delta) \cap B(\nu, \lambda, \epsilon) \cap \{F_\eta(z)C(\nu)(z) = C(F_\eta \nu)(z)\})
\]
\[
\geq \frac{1}{A_T} \lim_{\delta \to 0} \gamma (B(\lambda, \delta) \cap \{F_\eta(z)C(\nu)(z) = C(F_\eta \nu)(z)\})
\]
\[
- \lim_{\delta \to 0} \gamma (B(\lambda, \delta) \cap B(\nu, \lambda, \epsilon)) - \lim_{\delta \to 0} \gamma (B(\lambda, \delta) \cap B(F_\eta \nu, \lambda, \epsilon))
\]
\[
\geq \frac{1}{A_T} \lim_{\delta \to 0} \gamma (B(\lambda, \delta) \cap \bigcup_m (L_m \cup U_m))
\]
\[
> 0.
\]
Therefore, there exists a sequence \{\lambda_k\} such that \(\lambda_k \to \lambda\), \(F_\eta(\lambda_k)C(\nu)(\lambda_k) = C(F_\eta \nu)(\lambda_k)\), \(F_\eta(\lambda_k) \to F_\eta(\lambda)\), \(C(\nu)(\lambda_k) \to C(\nu)(\lambda)\), and \(C(F_\eta \nu)(\lambda_k) \to C(F_\eta \nu)(\lambda)\). Hence, \(F_\eta(\lambda)C(\nu)(\lambda) = C(F_\eta \nu)(\lambda)\). The lemma is proved. \(\square\)

4. Continuous Cauchy transform

Let \(\varphi\) be a smooth function with compact support. The localization operator \(T_\varphi\) is defined by
\[
(T_\varphi f)(\lambda) = \frac{1}{\pi} \int \frac{f(z) - f(\lambda)}{z - \lambda} \partial \varphi(z) d\mathcal{L}^2(z),
\]
where \(f \in L^\infty(\mathbb{C})\). One can easily prove the following norm estimation for a continuous function \(f\):
\[
\|T_\varphi f\| \leq 4 \sup_{z_1, z_2 \in \text{spt}(\varphi)} |f(z_1) - f(z_2)| \text{diameter}(\text{spt}(\varphi)) \|\partial \varphi\|. \quad (4.1)
\]
For a finite complex-valued measure \(\nu\) with compact support, we see that
\[
T_\varphi(C\nu)(z) = C(\varphi \nu)(z), \quad \mathcal{L}^2 - a.a.. \quad (4.2)
\]
Let \(g\) be an analytic function outside the disc \(B(a, \delta)\) satisfying the condition \(g(\infty) = 0\). We consider the Laurent expansion of \(g\) centered at \(a\),
\[
g(z) = \sum_{m=1}^{\infty} \frac{c_m(g, a)}{(z - a)^m}.
\]
We define \(c_1(g) = c_1(g, a)\). \(c_1(g)\) does not depend on the choice of \(a\), while \(c_2(g, a)\) depends on \(a\). However, if \(c_1(g) = 0\), then \(c_2(g, a)\) does not depend on \(a\), in this case, we denote \(c_2(g) = c_2(g, a)\).
Theorem 4.1. Let \( \eta \) be a finite positive measure with compact support such that \( C \eta \) is continuous on \( \mathbb{C} \). Suppose that \( \mathcal{C}(f \eta) \) is continuous on \( \mathbb{C} \) for some \( f \in L^\infty(\eta) \) with \( \|f\|_{L^\infty(\eta)} \leq 1 \). Then there exists a sequence of smooth functions \( \{\phi_n\} \) with compact supports such that \( T_{\phi_n} F_{\eta} \) uniformly tends to \( F_{f \eta} \) on any compact subsets of \( \mathbb{C} \).

Let \( \delta > 0 \) and \( \delta_N(= \frac{1}{2^N}) = \frac{1}{2N+1} \delta \). The proof of Theorem 4.1 relies on modified Vitushkin approximation scheme by P. V. Paramonov in [P95]. We provide a short description of P. V. Paramonov’s ideas to process our proof.

Let \( \{\varphi_{ij}, S_{ij}\} \) be a smooth partition of unity, where the length of the square \( S_{ij} \) is \( \delta_N \), the support of \( \varphi_{ij} \) is in \( 2S_{ij} \), \( 0 \leq \varphi_{ij} \leq 1 \), \( s_{ij} \) is the center of \( S_{ij} \),

\[
\|\partial \varphi_{ij}\| \leq \frac{C_6}{\delta_N}, \quad \sum \varphi_{ij} = 1, \quad (4.3)
\]

and

\[
\bigcup_{i,j=1}^{\infty} S_{ij} = \mathbb{C}, \quad \text{Int}(S_{ij}) \cap \text{Int}(S_{i,j_1}) = \emptyset
\]

for \( (i,j) \neq (i_1,j_1) \). Define

\[
\omega(\delta_N) = \sup_{2S_{ij} \cap \text{spt}(\eta) \neq \emptyset} \left( |F_{f \eta}(z) - F_{f \eta}(w)| + |F_{\eta}(z) - F_{\eta}(w)| \right).
\]

Let \( f_{ij} = T_{\varphi_{ij}} F_{f \eta} \), then, by (4.2), we have

\[
\|f_{ij}\| \leq C_7 \omega(\delta_N) \quad (4.4)
\]

(see (4.1)), and

\[
F_{f \eta} = \sum_{ij} f_{ij} = \sum_{2S_{ij} \cap \text{spt}(\eta) \neq \emptyset} f_{ij}. \quad (4.5)
\]

For \( 2S_{ij} \cap \text{spt}(\eta) \neq \emptyset \),

\[
|c_1(f_{ij})| = \left| \int f(z) \varphi_{ij}(z) d\eta(z) \right| \leq \|\varphi_{ij}\| \eta.
\]

The standard Vitushkin approximation scheme requires to construct \( a_{ij} \) such that \( f_{ij} - a_{ij} \) has triple zeros at \( \infty \), which requires to estimate both \( c_1(a_{ij}) \) and \( c_2(a_{ij}, s_{ij}) \). The main idea of P. V. Paramonov is that one does not actually need to estimate each coefficient \( c_2(a_{ij}, s_{ij}) \). It suffices to do away (with appropriate estimates) with the sum of coefficients \( \sum_{j \in I_s} c_2(a_{ij}, s_{ij}) \) for a special partition \( \{I_s\} \) into non-intersecting groups \( I_{is} \).

Let \( \alpha_{ij} = \|\varphi_{ij}\| \eta \). Set \( \min_i = \min\{j : 2S_{ij} \cap \text{spt}(\eta) \neq \emptyset\} \) and \( \max_i = \max\{j : 2S_{ij} \cap \text{spt}(\eta) \neq \emptyset\} \). Let \( I_i = \{\min_i, \min_i + 1, \ldots, \max_i\} \). We call a subset \( I \) of \( I_i \) a complete group of indices if the following conditions are satisfied:

\[
I = \{j_s + 1, j_s + 2, \ldots, j_s + s_1, j_s + s_1 + 1, \ldots, j_s + s_1 + s_2, j_s + s_1 + s_2 + 1, \ldots, j_s + s_1 + s_2 + s_3\} \subseteq I, \]

where \( j_s, s_1, s_2, \) and \( s_3 \) are some integers.
where \( s_2 \) is chosen as in Lemma 2.7 (see below) of [P95],

\[
\delta_N \leq \sum_{j=j_*+1}^{j_*+s_1} \alpha_{ij} < \delta_N + k_1 \delta_N,
\]

and

\[
\delta_N \leq \sum_{j=j_*+s_1+s_2+1}^{j_*+s_1+s_2+s_3} \alpha_{ij} < \delta_N + k_1 \delta_N,
\]

where \( k_1(\geq 3) \) is a fixed integer.

We now present a detailed description of the procedure of partitioning \( I_i \) into groups. We split each \( I_i \) into (finitely many) non-intersecting groups \( I_{il} \), \( l = 1, ..., l_i \), as follows. Starting from the lowest index \( \min_i \) in \( I_i \) we include in \( I_{i1} \) (going upwards and without jumps in \( I_i \)) all indices until we have collected a minimal (with respect to the number of elements) complete group \( I_{i1} \). We then repeat this procedure for \( I_{i1} \), ..., \( I_{il_i-1} \) (this family can even be empty) there can remain the last portion \( I_{il_i} = I_i \setminus (I_{i1} \cup ... \cup I_{il_i-1}) \) of indices in \( I_i \), which includes no complete groups. We call this portion \( I_{il_i} \) an incomplete group (clearly, there is at most one incomplete group for each \( i \)).

Set

\[
g_{ij}^0 = \frac{\|\varphi_{ij} f \eta\|}{\|\varphi_{ij} \eta\|} T_{\varphi_{ij}} F_\eta = \frac{\|\varphi_{ij} f \eta\|}{\|\varphi_{ij} \eta\|} F_{\varphi_{ij} \eta'},
\]

(see (4.2)),

\[
\|g_{ij}^0\| \leq C_8 \omega(\delta_N)
\]

(see (4.1)), and

\[
c_1(g_{ij}^0) = c_1(f_{ij}), \quad g_{ij} = f_{ij} - g_{ij}^0.
\]

For \( |z - s_{ij}| > 3k_1 \delta_N \), using the maximum modulus principle, we get

\[
|g_{ij}(z)| \leq \frac{1}{|z - s_{ij}|^2} c_2(g_{ij}) \leq C_9 \frac{\delta_N^3}{|z - s_{ij}|^3}.
\]

and therefore,

\[
|g_{ij}(z)| \leq C_{10} \left( \frac{\delta_N}{|z - s_{ij}|^2} \alpha_{ij} := \|\varphi_{ij} \eta\| + \frac{\delta_N^3}{|z - s_{ij}|^3} \right).
\]

Let \( I = I_{il} \) be a group, define

\[
g_I = \sum_{j \in I} g_{ij}, \quad c_1(g_I) = \sum_{j \in I} c_1(g_{ij}), \quad c_2(g_I) = \sum_{j \in I} c_2(g_{ij}, s_{ij}),
\]

and let \( I'(z) = \{ j \in I : |z - s_{ij}| > 3k_1 \delta_N \} \)

\[
L_I'(z) = \sum_{j \in I'(z)} \left( \frac{\delta_N}{|z - s_{ij}|^2} \alpha_{ij} = \frac{\delta_N^3}{|z - s_{ij}|^3} \right).
\]

Define \( L_I(z) = L_I'(z) \) if \( I = I'(z) \), otherwise, \( L_I(z) = 1 + L_I'(z) \). Similar to (2.22) of [P95], we have

\[
|g_I(z)| \leq C_{11} L_I(z), \quad |g_I(z)| \leq C_{11}.
\]
The follow lemma is due to Lemma 2.7 in [P95].

**Lemma 4.2.** For each complete group $I_{il}$, there exist $s_2 = k_2 \geq 6k_1$ and a function $h_{I_{il}}$ that has the following form

$$h_{I_{il}} = \sum_{j \in I_{il}^1} \sum_{k \in I_{il}^2} \left( H^{ij}_{jk} := \frac{\delta_N}{|s_{ik} - s_{ij}|} (\lambda^j_{ik} h_{ik} - \lambda^k_{ij} h_{ij}) \right),$$

where $I_{il}^1 = (j_s + 1, \ldots, j_s + s_1)$ and $I_{il}^2 = (j_s + s_1 + s_2 + 1, \ldots, j_s + s_1 + s_2 + s_3)$, satisfying:

1. $(H1)$ $h_{ij}(z) = T_{\varphi_{ij}} F_\eta(z) = F_{\varphi_{ij}} \eta(z)$ is bounded analytic off $2S_{ij}$ satisfying
   $$|h_{ij}(z)| \leq C_{12} \omega(\delta_N), \quad c_1(h_{ij}) = -\alpha_{ij};$$
2. $(H2)$ $\lambda^k_{ij}, \lambda^j_{ik} \geq 0$ and
   $$\sum_{j \in I_{il}^1} \lambda^j_{ik} \leq 1, \quad \sum_{k \in I_{il}^2} \lambda^k_{ij} \leq 1;$$
3. $(H3)$
   $$\sum_{j \in I_{il}^1} \sum_{k \in I_{il}^2} \lambda^j_{ik} \alpha_{ij} = \delta_N, \quad \sum_{j \in I_{il}^1} \sum_{k \in I_{il}^2} \lambda^k_{ij} \alpha_{ik} = \delta_N;$$
4. $(H4)$ $c_1(H^{ij}_{jk}) = 0$, that is, $\lambda^j_{ik} \alpha_{ij} = \lambda^k_{ij} \alpha_{ij}$; and
5. $(H5)$ if $|z - s_{ij}| > 3k_1 \delta_N$ and $|z - s_{ik}| > 3k_1 \delta_N$, then
   $$|H^{ij}_{jk}(z)| \leq C_{13} \left( \frac{\lambda^j_{ij} \alpha_{ij} \delta_N}{|z - s_{ij}|^2} + \frac{\lambda^k_{ik} \alpha_{ik} \delta_N}{|z - s_{ik}|^2} \right)$$

and for all $z \in \mathbb{C}$,

$$|h_{I_{il}}(z)| \leq C_{14} L_{I_{il}}(z). \quad (4.8)$$

We rewrite (4.5) as the following

$$F_{f\eta} = \sum_{i} \sum_{l=1}^{l_i-1} (g_{I_{il}} - h_{I_{il}}) + \sum_{i} \sum_{j \in I_{il_i}} g_{ij} + f_{\delta_N}$$

where

$$f_{\delta_N} = \sum_{i} \sum_{j \in I_{il_i}} g^0_{ij} + \sum_{i} \sum_{l=1}^{l_i-1} h_{I_{il}} = T_{\varphi_{\delta_N}} F_{\eta}$$

and $\varphi_{\delta_N}$ is some smooth function with compact support. In [P95], the following is proved:

$$\sum_{i} \sum_{l=1}^{l_i-1} |g_{I_{il}} - h_{I_{il}}| + \sum_{i} |g_{I_{il_i}}| \leq C_{15}. \quad (4.9)$$

In the case of [P95], the constant $C_{15}$ is actually $C_{15} \omega(\delta_N)$, which tends to zero as $\delta_N \to 0$. However, in our case, $C_{15}$ in (4.9) is an absolute constant and does not tend to zero. Therefore, we need to modify the estimates as the following.
Let $L_1$ be a subset of complete groups $I_{il}$. Let $L_2$ be a subset of index groups $I \subset I_{il}$, $l = 1, 2, ..., l_i$ such that for each $i$, there are at most three $I \subset I_{il}$, denoted by $I^1_{il}, I^2_{il}, I^3_{il}$. We can view $L_2$ as a collection of incomplete groups with the number of groups $\leq 3$ for each $i$. Define $\Psi_{I_{il}}(z) = g_{I_{il}}(z) - h_{I_{il}}(z)$ for $1 \leq l \leq l_i - 1$. The following inequality is basically proved in [P95].

\[
\sum_{I_{il} \in L_1} |\Psi_{I_{il}}(z)| + \sum_{I_{il} \in L_2, 1 \leq k \leq 3} |g_{I^k_{il}}(z)| \leq C_{16} \min \left( 1, \left( \frac{\delta_N}{\text{dist}(z, \partial R)} \right)^{\frac{4}{3}} \right) \tag{4.10}
\]

where $R$ is a square with $R \cap S_{ij} = \emptyset$ for $j \in I$, $I \in L_1 \cup L_2$. We provide a short description below.

For $\text{dist}(z, \partial R) \leq k_1 \delta_N$, (4.10) follows from (4.9) by [P95]. Hence, we assume that $\text{dist}(z, \partial R) > k_1 \delta_N$ below. Let $l(z) \delta_N + k_1 \delta_N \leq \text{dist}(z, \partial R) < l(z) \delta_N + (k_1 + 1) \delta_N$ for $z \in R$ and $\delta_N$ small enough, where $l(z)$ is a positive integer. We need to modify the estimates in [P95] under the condition $|z - s_{ij}| > l(z) \delta_N$.

Without loss of generality, we assume that $z = 0$. In this case,

\[
|z - s_{ij}| = |0 - s_{ij}| > l(0) \delta_N.
\]

The modification of (2.34) is

\[
\sum_{j \in I} \left| \frac{\delta_N^3}{|z - s_{ij}|^3} \right|_{z=0} \leq C_{17} \sum_{l=\sqrt{\max(0,l(0)^2 - i^2)}}^{\infty} \frac{1}{(i^2 + l^2)^{\frac{3}{4}}} \leq C_{18} \int_{\sqrt{\max(0,(l(0)^2 - i^2)}}^{\infty} \frac{dt}{(i^2 + t^2)^{\frac{3}{4}}} \leq \frac{C_{19}}{\max(l(0)^2, i^2)}. \tag{4.11}
\]

Also

\[
\sum_{j \in I} \frac{\delta_N \alpha_{ij}}{|z - s_{ij}|^2} |_{z=0} \leq \frac{1}{\delta_N \max(l(0)^2, i^2)} \sum_{j \in I} \alpha_{ij} \leq \frac{C_{20}}{\max(l(0)^2, i^2)}.
\]

From (4.7), we get, for $I^k_{il} \in L_2$, $k = 1, 2, 3,$

\[
|g_{I^k_{il}}(0)| \leq C_{21} \min \left\{ 1, \frac{1}{\max(l(0)^2, i^2)} \right\}
\]

which implies that

\[
\sum_{I^k_{il} \in L_2, 1 \leq k \leq 3} |g_{I^k_{il}}(0)| \leq \frac{C_{22}}{l(0)}. \tag{4.12}
\]

Now fix a complete group $I = I_{il}$ for $1 \leq l \leq l_i - 1$. For all $i \leq 4k_2$, we estimate from (4.7) that

\[
\sum_{l=1}^{l_i-1} |\Psi_{I_{il}}(0)| \leq C_{23} \min \left\{ 1, \sum_k \frac{1}{(l(0) + k)^2} \right\} \leq C_{25} \min \left\{ 1, \frac{1}{l(0)} \right\}. \tag{4.13}
\]
We now fix $i$ and denote $d_{il} = \max_{j_1, j_2 \in I} |s_{ij_1} - s_{ij_2}| + 2k_1\delta_N$. Let $s_{il} = s_{ij_1}$, where $j_1 = \min\{j \in I\}$. We denote by $S' = \{l : d_{il} \leq |i|^{\frac{4}{3}}\delta_N, 1 \leq l \leq l_i - 1\}$ and $S'' = \{l \notin S' : 1 \leq l \leq l_i - 1\}$. Using maximum modulus principle for $(z - s_{il})^3\Psi_{il}(z)$ on $|z - s_{il}| \geq d_{il}$ and $l \in S'$, we have,

$$|\Psi_{il}(z)| \leq \frac{C_26(\delta_N|i|^{\frac{4}{3}})^3}{|z - s_{il}|^3}.$$  

Since $|0 - s_{il}| > |i|\delta_N > 3\delta_N|i|^{\frac{4}{3}}$, using (4.11), we derive that

$$\sum_{|i| > 4k_2} \sum_{l \in S'} |\Psi_{il}(0)| \leq C_{27} \sum_{|i| \geq 4k_2} |i|^{\frac{4}{3}} \min \left\{1, \frac{1}{\max(l(0)^2, i^2)} \right\} \leq C_{28} \sum_{l(0) \geq |i| \geq 4k_2} \frac{|i|^{\frac{4}{3}}}{l(0)^2} + C_{59} \sum_{|i| > l(0)} \frac{1}{|i|^{\frac{4}{3}}} \leq \frac{C_{29}}{l(0)^{\frac{4}{3}}} \tag{4.14}$$

Now for $|i| > 4k_2$ and $l \in S''$, let $b_{il}$ be the point in $\{s_{ij}, j \in I\}$ with the smallest distance to zero. By (4.7) and (4.8), we see that

$$\sum_{l \in S''} |\Psi_{il}(0)| \leq \sum_{l \in S''} \frac{C_{30} \delta_N^2}{|b_{il}|^2} + \sum_j \frac{C_{31} \delta_N^3}{|s_{ij}|^3} \leq \sum_{q=0}^{\infty} \frac{C_{30}}{\max(l(0)^2, i^2) + q^2} + \sum_j \frac{C_{31} \delta_N^3}{|s_{ij}|^3} \leq \frac{C_{32}}{\max(l(0), |i|)|i|^{\frac{4}{3}}} \int_0^\infty \frac{dt}{1 + t^2} + \sum_j \frac{C_{33} \delta_N^3}{|s_{ij}|^3} \tag{4.15}$$

Thus,

$$\sum_{|i| > 4k_2} \sum_{l \in S''} |\Psi_{il}(0)| \leq \sum_{l(0) \geq |i| \geq 4k_2} \frac{1}{l(0)^{\frac{4}{3}}} + \frac{C_{35}}{l(0)^{\frac{4}{3}}} + \frac{C_{36}}{l(0)} \leq \frac{C_{37}}{l(0)^{\frac{4}{3}}} \tag{4.16}$$

Hence, combining (4.12), (4.14), and (4.16), we prove (4.10).

Recall $\delta_N = \frac{1}{2^{n_1} = \frac{1}{2N^2+1}}\delta$, choose $N$ such that

$$N \to \infty, \quad (2N + 1)^2\omega(\delta_N) \to 0$$

as $\delta_N \to 0$ (i.e. $N \approx \frac{1}{\omega(2-n)^{\frac{1}{4}}}$. It remains to prove

$$|F_{ij}(z) - f_{\delta_N}(z)| \leq \frac{C_{37}}{N^{\frac{4}{3}}} + (2N + 1)^2\omega(\delta_N), \quad z \in \mathbb{C}. \tag{4.17}$$

Theorem 4.1 easily follows from (4.17).

Proof. (of (4.17)) Fix $S = S_{i_0,j_0}$. Let $J$ be the set of indices $(i, l)$ for $1 \leq l \leq l_i - 1$ such that there is a square $S_{ij}$ in the complete group $I_{il}$ satisfying $2S_{ij} \subset (2N + 1)S$. Let $J_0$ be the set of indices $i$ such that there is a square $S_{ij}$ in the incomplete
group $I_{il}$ satisfying $2S_{ij} \subset (2N + 1)S$. Let $J_1$ be the subset of index $i$ such that there exists $l$ with $(i, l) \in J$. From (4.10), for $z \in S$, we get

$$
\sum_{(i, l) \notin J} |g_{l_{il}}(z) - h_{l_{il}}(z)| + \sum_{i \notin J_0} |g_{l_{il}}(z)| \leq \frac{C_{38}}{N^4}.
$$

Therefore, for $z \in S$,

$$
|F_{f\eta}(z) - f_{\delta_N}(z)| 
\leq \frac{C_{39}}{N^4} + \sum_{i \in J_1} \sum_{(i, l) \in J} |g_{l_{il}}(z)| + \sum_{i \in J_1} \sum_{(i, l) \in J} |h_{l_{il}}(z)| + \sum_{i \in J_0} |g_{l_{il}}(z)|.
$$

Set $I_{il}^a = \{ j \in I_{il} : j > j_0, 2S_{ij} \setminus (2N + 1)S \neq \emptyset \}$ and $I_{il}^d = \{ j \in I_{il} : j < j_0, 2S_{ij} \setminus (2N + 1)S \neq \emptyset \}$. Let $|I|$ denote the number of indices in $I$. For each $i$ with $(i, l) \in J$ or $i \in J_0$, there is at most one $I_{il}^a$ with $I_{il}^a \neq \emptyset$ and one $I_{il}^d$ with $I_{il}^d \neq \emptyset$. Then

$$
|g_{l_{il}}(z)| \leq |g_{l_{il}}^a(z)| + |g_{l_{il}}^d(z)| + |g_{l_{il}}^{a \cup d}(z)|
\leq |g_{l_{il}}^a(z)| + |g_{l_{il}}^d(z)| + (C_7 + C_8) |I_{il} \setminus I_{il}^a \cup I_{il}^d| \omega(\delta_N)
$$

where (4.4) and (4.6) are used. Using (4.10) again, for $z \in S$, we have

$$
\sum_{i \in J_1} \sum_{(i, l) \in J} |g_{l_{il}}(z)| + \sum_{i \in J_0} |g_{l_{il}}(z)| \leq C_{40} \left( \frac{1}{N^4} + (2N + 1)^2 \omega(\delta_N) \right).
$$

Set

$$
II_i(z) := \sum_{(i, l) \in J} |h_{l_{il}}(z)|, \quad II(z) := \sum_{i \in J_1} II_i(z).
$$

For $z \in S$ and $i \in J_1$ (notice that $N > 3k_1$), if $j \in I_{il}^1 \cap (I_{il}^a \cup I_{il}^d)$ and $k \in I_{il}^2 \cap (I_{il}^a \cup I_{il}^d)$, then, by (H5),

$$
|H_{jk}^i(z)| \leq C_{13} \frac{\lambda_{ik}^k \alpha_{ij} + \lambda_{ik}^j \alpha_{ik}}{N^2 \delta_N};
$$

if $j \in I_{il}^1 \setminus (I_{il}^a \cup I_{il}^d)$ and $k \in I_{il}^2 \cap (I_{il}^a \cup I_{il}^d)$, then, by (H1) and (H4),

$$
|H_{jk}^i(z)| \leq C_{45} \left( \frac{\lambda_{ik}^j \| \varphi_{ik} \eta \|}{N \delta_N} + \lambda_{ik}^k \omega(\delta_N) \right) = C_{45} \lambda_{ik}^j \left( \frac{\| \varphi_{ik} \eta \|}{N \delta_N} + \omega(\delta_N) \right);
$$

if $j \in I_{il}^1 \cap (I_{il}^a \cup I_{il}^d)$ and $k \in I_{il}^3 \setminus (I_{il}^a \cup I_{il}^d)$, then, by (H1) and (H4),

$$
|H_{jk}^i(z)| \leq C_{45} \lambda_{ik}^j \left( \frac{\| \varphi_{ik} \eta \|}{N \delta_N} + \omega(\delta_N) \right);
$$

and if $j \in I_{il}^1 \setminus (I_{il}^a \cup I_{il}^d)$ and $k \in I_{il}^3 \setminus (I_{il}^a \cup I_{il}^d)$, then, by (H1),

$$
|H_{jk}^i(z)| \leq C_{46} (\lambda_{ik}^j + \lambda_{ij}^k) \omega(\delta_N).$$
Therefore,
\[ II_i(z) \leq \frac{C_{47}}{N^2} + C_{46} \sum_{(i,l) \in J} \sum_{j \in I^1_i \setminus (I^1_i \cup I^2_i)} \sum_{k \in I^3_i \setminus (I^1_i \cup I^2_i)} (\lambda_{ik}^j + \lambda_{kj}^i) \omega(\delta_N) \]
\[ + C_{45} \sum_{(i,l) \in J} \sum_{j \in I^1_i \cap (I^1_i \cup I^2_i)} \sum_{k \in I^3_i \setminus (I^1_i \cup I^2_i)} \lambda_{ik}^j \left( \frac{\| \varphi_{ikj} \|}{N \delta_N} + \omega(\delta_N) \right) \]
\[ + C_{45} \sum_{(i,l) \in J} \sum_{j \in I^1_i \setminus (I^1_i \cup I^2_i)} \sum_{k \in I^3_i \setminus (I^1_i \cup I^2_i)} \lambda_{kj}^i \left( \frac{\| \varphi_{ij} \|}{N \delta_N} + \omega(\delta_N) \right). \]

Using (H2), we get
\[ II_i(z) \leq \frac{C_{47}}{N^2} + C_{48} (2N + 1) \omega(\delta_N) + C_{48} \sum_{(i,l) \in J} \sum_{j \in I^1_i \setminus (I^1_i \cup I^2_i)} \| \varphi_{ij} \| \frac{1}{N \delta_N}. \]

Hence,
\[ II(z) \leq C_{49} \left( \frac{1}{N} + \frac{\eta(B(z, (2N + 1)\delta_N))}{N \delta_N} + (2N + 1)^2 \omega(\delta_N) \right). \]

Using Lemma 2.2 (5), we have
\[ \eta(B(z, (2N + 1)\delta_N)) \leq 2(2N + 1)^2 \delta_N \omega(\delta_N). \]

The proof is completed.

\[ \square \]

5. Proof of main theorem

In this section, we finish our proof of the following main theorem.

**Theorem 5.1.** Let \( \eta \) and \( F_{\eta} \) be defined as in (3.5) and (2.7). Then
\[ A(K) = H(F_{\eta}, K). \]

We need several lemmas.

**Lemma 5.2.** Let \( \eta \) and \( F_{\eta} \) be defined as in (3.5) and (2.7). Let \( \varphi \) be a smooth function with compact support. Then
\[ T_{\varphi} F_{\eta} \in H(F_{\eta}, K). \]

**Proof.** Let \( \nu \in H(F_{\eta}, K)^\perp \subset (C(K))^\ast \). Using Lemma 3.3, we have the following calculation
\[
\int T_{\varphi} F_{\eta}(z) d\nu(z) \\
= \frac{1}{\pi} \int \int \frac{F_{\eta}(z) - F_{\eta}(w)}{z - w} \bar{\partial} \varphi(w) d\mathcal{L}^2(w) d\nu(z) \\
= \frac{1}{\pi} \int (\mathcal{C}(F_{\nu})(w) - F_{\eta}(w)\mathcal{C}(\nu)(w)) \bar{\partial} \varphi(w) d\mathcal{L}^2(w) \\
= 0.
\]

The lemma is proved. \[ \square \]
Lemma 5.3. Let \( \eta \) and \( F_\eta \) be defined as in (3.5) and (2.7). Let \( \nu \in H(F_\eta, K) \subset C(K)^* \). Then

\[
\mathcal{C}\nu(z) = 0, \quad \eta - a.a.\]

Proof. Suppose that there exists a compact subset \( D \) with \( \eta(D) > 0 \) such that

\[
\text{Re}(\mathcal{C}\nu(z)) > 0, \quad z \in D.
\]

By Lemma 2.2 (3) and (8) (a), we may assume that \( \mathcal{C}_*\nu(z) \in L^\infty(\eta|_D) \). Using Lemma 2.2 (6), we can find a function \( w \) supported on \( D \) and \( 0 \leq w(x) \leq 1 \) such that \( c_1\eta(D) \leq \int w(x)d\eta(x) \) and \( \mathcal{C}(w\eta)(z) \) is continuous. From Lemma 5.2 and Theorem 4.1, we see that \( F_{w\eta} \in H(F_\eta, K) \). Using Lemma 2.2 (8) (b), we get

\[
\int \text{Re}(\mathcal{C}\nu(z))w(z)d\eta(z) = \text{Re} \left( \int \mathcal{C}\nu(z)w(z)d\eta(z) \right) = -\text{Re} \left( \int F_{w\eta}(z)d\nu(z) \right) = 0
\]

which implies that \( \text{Re}(\mathcal{C}\nu(z))w(z) = 0, \quad \eta - a.a. \). This is a contradiction. \( \square \)

Lemma 5.4. Let \( E, \eta, \) and \( F_\eta \) be defined as in (3.1), (3.5), and (2.7). Let \( S_{ij} \) be a square in the \( n \)th generation as in (4.3) and \( \alpha(2S_{ij} \cap E) > 0 \). Then there exists a finite positive measure \( \mu_{ij} \) with \( \text{spt}(\mu_{ij}) \subset 2S_{ij} \cap E \) satisfying

\[
\|\mu_{ij}\| \geq c_2\alpha(2S_{ij} \cap E), \quad \|\mathcal{C}\mu_{ij}\|_{L^\infty(\mathcal{C})} \leq 1, \quad (5.1)
\]

\( \mathcal{C}\mu_{ij}(z) \) is continuous, and

\[
F_{\mu_{ij}} \in H(F_\eta, K). \quad (5.2)
\]

Proof. Let \( \{\nu_n\} \subset H(F_\eta, K) \subset C(K)^* \) be a weak* dense subset. By Lemma 2.2 (3) and (8) (a), we find a compact subset \( E_{ij} \subset \text{spt}(\eta_{ij}^n) \), where \( \eta_{ij}^n \) is as in (3.3) and (3.4), such that

\[
\alpha(E_{ij}) \geq c_3\alpha(2S_{ij} \cap E), \quad C_*(\nu_n)(z) \leq M_n < \infty, \quad z \in E_{ij}
\]

and by Lemma 5.3, since \( \eta_{ij}^n << \eta \), we may assume that

\[
\mathcal{C}(\nu_n)(z) = 0, \quad z \in E_{ij}.
\]

Using Theorem 2.1 (1), we find a finite positive measure \( \mu_{ij} \) with \( \text{spt}(\mu_{ij}) \subset E_{ij} \) such that (5.1) holds and \( \mathcal{C}\mu_{ij}(z) \) is continuous. Applying Lemma 2.2 (8) (b), we have

\[
\int F_{\mu_{ij}}(z)d\nu_n(z) = -\int \mathcal{C}\nu_n(z)d\mu_{ij}(z) = 0.
\]

Hence, (5.2) holds. The lemma is proved. \( \square \)
Lemma 5.5. Let \( \eta \) and \( F_\eta \) be defined as in (3.5) and (2.7). Let \( S_{ij} \) be a square in the \( n \)th generation as in (4.3) and \( \alpha(2S_{ij} \cap \bigcup_n \partial U_n) > 0 \) (or \( \alpha(2S_{ij} \cap \bigcup_n U_n) > 0 \)). Then there exists a finite positive measure \( \mu_{ij} \) with \( \text{spt}(\mu_{ij}) \subset 2S_{ij} \cap \bigcup_n \partial U_n \) (or \( \text{spt}(\mu_{ij}) \subset 2S_{ij} \cap \bigcup_n U_n \)) satisfying
\[
||\mu_{ij}|| \geq c_3 \alpha(2S_{ij} \cap \bigcup_n \partial U_n)(\text{ or } \alpha(2S_{ij} \cap \bigcup_n U_n)), \quad ||C\mu_{ij}||_{L^\infty(C)} \leq 1, \quad (5.3)
\]
\( C\mu_{ij}(z) \) is continuous, and
\[
F_{\mu_{ij}} \in H(F_\eta, K). \quad (5.4)
\]

Proof. It follows from (3.7), (3.9), and the same proof of Lemma 5.4.

\( \square \)

Proof. (Theorem 5.1) By Theorem 2.1 (2) and (3.4), we get
\[
\alpha(2S_{ij} \setminus \text{Int}(K)) \leq A_T(\alpha(2S_{ij} \cap \partial T K) + \alpha(2S_{ij} \cap \bigcup_n \partial U_n) + \alpha(2S_{ij} \cap \bigcup_n U_n)) \leq C_{51}(\alpha(2S_{ij} \cap E) + \alpha(2S_{ij} \cap \bigcup_n \partial U_n) + \alpha(2S_{ij} \cap \bigcup_n U_n))
\]
Hence,
\[
\max(\alpha(2S_{ij} \cap E), \alpha(2S_{ij} \cap \bigcup_n \partial U_n), \alpha(2S_{ij} \cap \bigcup_n U_n)) \geq \frac{1}{3C_{51}} \alpha(2S_{ij} \setminus \text{Int}(K)).
\]
Therefore, from Lemma 5.4 and Lemma 5.5, there exists a finite positive measure \( \mu_{ij} \) with \( \text{spt}(\mu_{ij}) \subset 2S_{ij} \setminus \text{Int}(K) \) such that (5.1) and (5.2) hold (or (5.3) and (5.4) hold). Now for \( f \in A(K) \) and a partition of unity \( \{S_{ij}, \varphi_{ij}\} \) as in (4.3), we have
\[
\left| \int f(z) \partial \varphi_{ij}(z) d\mathcal{L}^2(z) \right| \leq C_{52} \sup_{w, z \in 2S_{ij}} |f(w) - f(z)| \alpha(2S_{ij} \setminus \text{Int}(K)).
\]
Now the theorem follows from the modified Vitushkin scheme of P. V. Paramonov in Section 4 (or see [P95]).

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