Exact Solitonic Solutions of the One-Dimensional Gross-Pitaevskii Equation with a Time-Dependent Harmonic Potential and Interatomic Interaction

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We derive exact solitonic solutions of the one-dimensional time-dependent Gross-Pitaevskii equation with time-dependent strengths of the harmonic external potential and the interatomic interaction. The time-dependence of the external potential and interatomic interaction are given in terms of a general function of time. For an oscillating strength of the external potential, the solutions correspond to breathing single and multiple solitons. The amplitude and frequency of the oscillating potential can be used to control the dynamics of the center of mass of the solitons. For certain values of these parameters, the solitons can be trapped at the center of the harmonic potential.

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I. INTRODUCTION

Since the experimental realization of solitons in Bose-Einstein condensates \[1, 2, 3, 4, 5, 6\], intense interest in their properties has emerged \[7, 8, 9, 10, 11, 12, 13\]. Recently, exact solitonic solutions of the time-dependent Gross-Pitaevskii equation that describes the behavior of the condensate, have been obtained \[13, 14, 15, 16, 17, 18\]. We have shown in \[19\], that exact solutions of the Gross-Pitaevskii equation can not be obtained for general external potential and interatomic interaction. For the Gross-pitaevskii equation to be solved exactly, the external potential and the interatomic interaction must be related. As an example on such a relation, the external potential in the Gross-Pitaevskii equation solved in Ref. \[16\] is an expulsive harmonic potential and the interatomic interaction has an exponential prefactor that grows with time with a rate that is related to the strength of the harmonic potential. Our investigation in \[19\] has provided a mathematical proof for the existence of such a correlation between the external potential and the interatomic interaction.

From an experimental point of view, it would be more interesting to find exact solutions for the external potentials and interatomic interactions used in the experiments. For instance, it would be more interesting to have exact solutions for the realistic case of an upright harmonic external potential and constant interatomic interaction rather than for an inverted harmonic potential and exponentially growing interatomic interaction strength as in \[16\]. It turns out, however, that for the Gross-Pitaevskii equation to be solved exactly with an upright harmonic potential, the interatomic interaction will have to be oscillatory and divergent at regular times \[19\].

In this paper, we use the Darboux transformation method \[20\] to investigate the possibility of obtaining exact solutions of a time-dependent Gross-Pitaevskii equation with strengths of the harmonic external potential and interatomic interaction that are general functions of time. It turns out that some special cases of such a general form of harmonic potential and interatomic interaction can be easily realized experimentally by controlling the strength of the magnetic field that provides the harmonic trap and using Feshbach management of interatomic interaction \[21\]. One interesting special case corresponds to an oscillating strength of the harmonic potential and practically constant strength of the interatomic interaction. The amplitude and frequency of the oscillating external potential can be used to delay the escape of the soliton from the harmonic potential or even to trap it at the center.

The rest of the paper is organized as follows. In the next section, we present the general form of the Gross-Pitaevskii equation to be solved. In section \[\text{III}\] we use the Darboux transformation method to derive the exact solutions. In section \[\text{IV}\] we present and discuss the properties of the solutions in the special case of an oscillating external potential. We end in section \[\text{V}\] with a summary of our main conclusions.

II. THE GROSS-PITAEVKII EQUATION

When the confinement of the Bose-Einstein condensate is much larger in, say the \(y\) and \(z\) (transverse) directions, compared to the confinement in the \(x\) direction, the system can be considered effectively one-dimensional along the \(x\) direction. The three-dimensional Gross-Pitaevskii equation can then be integrated over the \(y\) and \(z\) directions to
result in the following one-dimensional Gross-Pitaevskii equation \[23\]

\[
i \frac{\partial \psi(x, t)}{\partial t} = -\frac{\partial^2 \psi(x, t)}{\partial x^2} - \frac{1}{4} \lambda \rho(t) x^2 \psi(x, t) - 2a q(t) |\psi(x, t)|^2 \psi(x, t).
\]

(1)

Here \(a\) is the \(s\)-wave scattering length, and \(\lambda = 2\omega_x / \omega_\perp\), where \(\omega_x\) and \(\omega_\perp\) are the characteristic frequencies of the harmonic trapping potential in the \(x\) and transverse directions, respectively. In the last equation, length is scaled to \(a_\perp\), and \(\psi(x, t)\) to \(1/\sqrt{2a_\perp}\), where \(a_\perp = \sqrt{\hbar/m \omega_\perp}\) is the characteristic length of the harmonic trap in the transverse direction. The dimensionless general functions \(\rho(t)\) and \(q(t)\) are introduced to account for the time-dependencies of the strengths of the trapping potential and the interatomic interaction.

For Eq. (1) to be solved exactly using the Darboux transformation method, the functions \(\rho(t)\) and \(q(t)\) must be parametrically related to each other through a general function \(g(t)\) \[19\]. In this case, the Gross-Pitaevskii equation, that we derive exact solutions for, takes the form

\[
i \frac{\partial \psi(x, t)}{\partial t} = -\frac{\partial^2 \psi(x, t)}{\partial x^2} - \frac{1}{4} \lambda \left[\lambda g(t)^2 - \dot{g}(t)\right] x^2 \psi(x, t) - 2a e^{2c_1 + \lambda} \int g(t) dt |\psi(x, t)|^2 \psi(x, t),
\]

where \(c_1\) is an arbitrary constant.

### III. DARBOUX TRANSFORMATION AND THE EXACT SOLUTIONS

In the Darboux transformation method, we start by finding a linear system of equations for an auxiliary field \(\Psi(x, t)\) such that Eq. (2) is its consistency condition \[19\]. \[20\]. We find that Eq. (2) corresponds to the consistency condition of the following linear system \[23\]:

\[
\Psi_x = \zeta J \Psi \Lambda + P \Psi,
\]

(3)

\[
\Psi_t = 2i \xi^2 J \Psi \Lambda^2 + 2i \xi P \Psi \Lambda + \lambda x g \xi J \Psi \Lambda + W \Psi,
\]

(4)

where,

\[
\Psi(x, t) = \begin{pmatrix} \psi_1(x, t) & \psi_2(x, t) \\ \phi_1(x, t) & \phi_2(x, t) \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]

\[
\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad P = \begin{pmatrix} -\sqrt{\pi Q(x, t)} & 0 \\ 0 & \sqrt{\pi Q(x, t)} \end{pmatrix},
\]

\[
W = \begin{pmatrix} ia |Q(x, t)|^2 & \sqrt{\pi \lambda x g(t)} Q(x, t) + i \sqrt{\pi} Q_x(x, t) \\ -i a |Q(x, t)|^2 & -\sqrt{\pi \lambda x g(t)} Q^*(x, t) + i \sqrt{\pi} Q^*_x(x, t) \end{pmatrix},
\]

and \(\zeta(t) = \exp(\int \lambda g(t) dt)\).

The constants \(\lambda_1\) and \(\lambda_2\) are arbitrary constants. The subscripts \(x\) and \(t\) denote partial derivatives with respect to \(x\) and \(t\), respectively. The function \(Q(x, t)\) is related to the wave function through \(Q(x, t) = \psi(x, t) / \sqrt{\xi e^{c_1 + \lambda x g^2/4}}\).

Equation (2) is obtained from the consistency condition \(\Psi_{xt} = \Psi_{tx}\).

The linear system of 8 equations, Eqs. (3) and (4), reduces to an equivalent system of 4 equations with nontrivial solutions by making the following substitutions: \(\lambda_1 = -\lambda_2, \psi_2 = \phi_1^*, \) and \(\phi_2 = -\psi_1^*\). To be able to solve this reduced linear system we need to know an exact (seed) solution of Eq. (2). Following our previous approach of finding seed solutions \[19\], we find the following seed solution to Eq. (2):

\[
\psi_0(x, t) = \exp \left[ c_2 + \lambda g/2 + i \left( e^{\lambda g} k_0 x - \frac{2A^2 a - k_0^2}{2\lambda} - \frac{1}{4} \lambda g x^2 + 2A^2 a \int \left( e^{2(c_1+c_2+\lambda g)} + k_0^2 e^{2\lambda g} \right) dt \right) \right],
\]

(5)

where \(c_2, A\) and \(k_0\) are arbitrary constants and \(\dot{g} = dg/dt\).

The Darboux transformation can now be applied to the linear system to generate a new solution of Eq. (2) as follows \[20\]:

\[
\psi(x, t) = \psi_0(x, t) + 2(\lambda_1 + \lambda_1^*) e^{-c_1-i\lambda x g^2/4+(\lambda/2)\int \frac{\phi_1 \psi_1^*}{(|\phi_1|^2 + |\psi_1|^2)}}
\]

(6)

Solving the linear system (3) and (4) using the seed solution Eq. (5) and then substituting for \(\psi_0(x, t), \psi_1(x, t),\) and \(\phi_1(x, t)\) in the last equation, we obtain the following new exact solution to Eq. (2).
\[ \psi(x, t) = \eta \frac{1}{\sqrt{4A}} e^{-i\eta^2 x^2 / 4A} \left\{ A e^{i \theta_1} + 4\lambda_1 e^{i\theta_1 + i\theta_2} (2i c_3 A \sqrt{ae} e^{2\alpha \eta} + c_4 q e^{-2i\theta_1 + \Delta_r \sqrt{\eta}})(c_3 q e^{2\alpha \eta} - 2i A \sqrt{ac} e^{-i\theta_1 + \Delta_r \sqrt{\eta}}) / \left[ c_3^2 e^{2\alpha \eta - \Delta_r \sqrt{\eta} x} + c_4^2 e^{-2\alpha \eta + \Delta_r \sqrt{\eta} x} - 4 A c_1 c_2 \sqrt{a}(2\lambda_1 - \Delta_r)(\cos 2\theta_1 + \cos 2\theta_2) + (\Delta_1 - 2\lambda_1)(\sin 2\theta_1 + \sin 2\theta_2)} \right\} \],

where

\[ q_1 = i(2A^2 a(2\lambda_1 - 1) + k_0(k_0(1 - 2\lambda_\eta) + 2\lambda \sqrt{\eta}))/2\lambda, \]

\[ q_2 = -2\alpha \eta - \Delta_r \sqrt{\eta} x, \]

\[ q_3 = \Delta_1 + k_0 + i(\Delta_1 + 2\lambda_1 - 2\lambda_1 r), \]

\[ \theta_1 = \Delta_1 + k_0 + 2\lambda_1 - 2\lambda_1 r, \]

\[ \theta_2 = (k_0^2 (1 - 2\lambda_1))/4\lambda + 2A^2 a(1 - 2\lambda_\eta) + 2\Delta_1 \lambda(2\lambda_1 - k_0 - \sqrt{\eta})/2\lambda, \]

\[ \theta_3 = \sqrt{\eta} \Delta_1 x, \]

\[ \theta_4 = \sqrt{\eta} \Delta_1 x, \]

\[ \theta_5 = k_0^2 (\Delta_1 - 2\lambda_1 i)^2 + (\Delta_1 - 2\lambda_1 r)^2 + 4\Delta_1 r + k_0(2\Delta_1 - 4\lambda_1 + k_0), \]

\[ \alpha = \Delta_1 + k_0 + i(\Delta_1 + 2\lambda_1 - 2\lambda_1 r), \]

\[ \eta = \int e^{2\lambda_\eta dt}. \]

the subscripts \( r \) and \( i \) denote real and imaginary parts, respectively, and \( c_3 \) and \( c_4 \) are arbitrary constants. Further reduction and simplification of this new general solution as performed in [13] will not be considered here. Furthermore, we do not attempt to obtain all classes of solitonic solutions. Instead, the focus in this paper will be on the effect of the parameters of the oscillating trapping potential on the dynamics of single and multiple solitons which are described by Eq. (7).

**IV. SPECIAL CASE: SOLITONS IN AN OSCILLATING HARMONIC TRAP**

In this section we derive from the general solution found in the previous section solitonic solutions for an oscillating harmonic trapping potential. To this end, we take

\[ \eta(t) = \int e^{\alpha_1 + \alpha_2 \sin(\omega t + \delta)} dt, \]

where \( \alpha_1, \alpha_2, \omega, \) and \( \delta \) are constants. Using the last equation and \( \eta = \int e^{2\lambda_\eta dt} \) to substitute for \( g \) in Eq. (2), the Gross-Pitaevskii equation takes the form

\[ \frac{i \partial \psi(x, t)}{\partial t} = \left\{ -\frac{\partial^2}{\partial x^2} - \frac{1}{8} \alpha_2 \omega^2 \left( \sin(\omega t + \delta) + \frac{1}{2} \alpha_2 \cos(\omega t + \delta)^2 \right) x^2 - 2ae^{2\lambda_1 + (\alpha_2 \sin(\omega t + \delta) + \delta)} |\psi(x, t)|^2 \right\} \psi(x, t). \]

It should be noted that the specific choice of \( \eta(t) = \exp(2\lambda t) \) leads to the Gross-Pitaevskii equation of Ref. [16] with an expulsive harmonic potential. Simulating the time dependencies of the harmonic potential and the interatomic interaction of Eq. (9) may be difficult experimentally, but if we choose the parameters such that \( c_1 \gg \alpha_1 \gg \alpha_2 \), the interatomic interaction can be considered practically as constant, and the strength of the harmonic potential will be oscillating as \( \sin(\omega t + \delta) \). It should be noted, however, that even in this limit, the contribution of the positive \( \cos(\omega t + \delta)^2 \) term will result in that the trapping potential will spend more time being an inverted parabola than being an upright parabola. Over a large enough time interval, this will lead to expelling the solitons away from the center of the trap. This is indeed the picture that we get when we plot the density of a single-soliton solution as shown in Fig. 1. It is clear from this figure that the soliton is being expelled out from the center of the condensate at \( x = 0 \). The oscillation in the trajectory of the soliton is due to the oscillating trapping potential. The discontinuous appearing of the soliton’s peak density is due to the interaction with the background. This trajectory can be also extracted from the general solution Eq. (7) by considering the term \((\Gamma_3 e^{2\alpha \eta - \sqrt{\eta} \Delta_r x} + \Gamma_4 e^{-2\alpha \eta + \sqrt{\eta} \Delta_r x}) \) in the denominator. At the peaks of the oscillation in Fig. 1, this is the dominant term that determines the position of the peak of the soliton.
Specifically, the peaks are given by the condition $2\alpha \eta - \sqrt{\eta \Delta_r} x = 0$. Using this condition to plot $x$ vs. $t$ in Fig. 2, we obtain a curve that is identical to the soliton trajectory in Fig. 1. The mean slope of this curve is proportional to $\alpha/2\Delta_r$. Hence, the rate at which the soliton leaves the center of the trapping potential can be delayed by choosing the parameters and the arbitrary constants such that $\alpha/2\Delta_r$ is small.

For the special case of $\alpha = 0$ the center of mass of the soliton will be located at $x = 0$ at all times. The oscillating trapping potential will result only in oscillations in the width and peak density of the solitons. The solitons in this case remain trapped at the center of the trapping potential. We show this case in Figs. 3, 4 where we see a multi-soliton solution with central soliton being pinned at $x = 0$ and off-central ones oscillating around their initial positions. In Fig. 4 we use the same parameters as in Fig. 3 but with a doubled frequency $\omega$. This shows clearly that the oscillations of the off-centered solitons are due to the oscillation in the trapping potential. In Fig. 5 we show that for some values of the parameters the dynamics of the peak soliton density can be drastic such that the soliton disappears in the background and reappears at regular discrete times. Figure 6 is similar to Fig. 5 but with a larger number of solitons.

V. CONCLUSIONS

We have found exact single and multi solitonic solutions of a time-dependent Gross-Pitaevskii equation with time-dependent amplitudes of harmonic trapping potential and interatomic interaction. We considered the interesting special case of sinusoidally oscillating strength of the harmonic trapping potential. We focused on the effect of the frequency and amplitude of the oscillating trapping potential on the dynamics of the solitons. The parameters can be chosen such that the soliton is trapped at the center of the condensate. For a typical $^{87}$Rb condensate with $10^5$ atoms and trapping frequency of order 100 Hz, the Thomas-Fermi size of the condensate is $R_{TF} = (Na/\delta)^{1/5} \approx 10\delta$ [24]. This means that the width of the solitons of Figs. 3, 4 are of the order of the size of a nonsolitonic condensate while the width of the solitons in Figs 5-6 is smaller than the size of the nonsolitonic condensate.

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FIG. 1: (Left) Spatiotemporal contour plot of solitons density profile. (Right) Soliton density profile at $t = 0$. The values of the parameters used in this plot are: $-c_2 = c_1 = 10$, $c_3 = c_4 = 1$, $\lambda_{1i} = 2$, $\lambda_{1r} = 1$, $A = 0.1$, $a = 0.2$, $k_0 = 0$, $\lambda = 1$, $\omega = 0.1$, $\delta = 0$, $\alpha_1 = -4$, $\alpha_2 = 0.3$.

FIG. 2: Trajectory of the soliton peak density. The values of the parameters used here are the same as those of Fig.1.
FIG. 3: (Left) Spatiotemporal contour plot of solitons density profile. (Right) Soliton density profile at \( t = 0 \). The values of the parameters used in this plot are: \(-c_2 = c_1 = 10, c_3 = c_4 = 1, \lambda_{1i} = \lambda_{1r} = 1, A = 2, \alpha = 0.9, k_0 \sim 5.03, \lambda = \omega = 1, \delta = 0, \alpha_1 = -6, \alpha_2 = 0.3\). The value of \( k_0 \) is the solution of \( \alpha = 0 \) with respect to \( k_0 \).
FIG. 4: Spatiotemporal contour plot of density solitons density profile. The values of the parameters used in this plot are the same as those of Fig. 3 but with $\omega = 2$. 
FIG. 5: (Left) Spatiotemporal contour plot of solitons density profile. (Right) Soliton density profile at $t = 0$. The values of the parameters used in this plot are: $-c_2 = c_1 = 10$, $c_3 = c_4 = 1$, $\lambda_{1t} = 0$, $\lambda_{tr} = 1$, $A = 2$, $a = 0.9$, $k_0 = 0$, $\lambda = 1$, $\omega = 0.01$, $\delta = 0$, $\alpha_1 = -2$, $\alpha_2 = 0.3$. 
FIG. 6: (Left) Spatiotemporal contour plot of solitons density profile. (Right) Soliton density profile at $t = 0$. The values of the parameters used in this plot are the same as those of Fig.3 but with $\alpha_1 = -1$. 