Distributed Projected Subgradient Method for Weakly Convex Optimization

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Abstract—The stochastic subgradient method is a widely-used algorithm for solving large-scale optimization problems arising in machine learning. Often these problems are neither smooth nor convex. Recently, Davis et al. [1], [2] characterized the convergence of the stochastic subgradient method for the weakly convex case, which encompasses many important applications (e.g., robust phase retrieval, blind deconvolution, biconvex compressive sensing, and dictionary learning). In practice, distributed implementations of the projected stochastic subgradient method (stoDPSM) are used to speed-up risk minimization. In this paper, we propose a distributed implementation of the stochastic subgradient method with a theoretical guarantee. Specifically, we show the global convergence of stoDPSM using the Moreau envelope stationarity measure. Furthermore, under a so-called sharpness condition, we show that deterministic DPSM (with a proper initialization) converges linearly to the sharp minima, using geometrically diminishing step-size. We provide numerical experiments to support our theoretical analysis.

I. INTRODUCTION

Optimization in multi-agent networks has received a great deal of attention in the past few years in control, signal processing, and machine learning. A wide range of networked problems such as distributed detection [3], estimation [4], and localization [5] can be formulated via distributed optimization with applications in wireless sensor networks [6], robotic networks [7], power networks [8], and social networks [9]. In such decentralized frameworks, a number of agents in a network need to accomplish a global task, which is formulated as an optimization. Each individual agent, however, has limited information about the objective function. Therefore, agents locally interact with each other to solve the global problem. Decentralized techniques have gained popularity over time due to robustness to individual failures, imposing low computational burden on individual agents, and promoting privacy.

In a (constrained) multi-agent optimization, we deal with a problem of the form

$$\min_x f(x) = \frac{1}{N} \sum_{i=1}^{N} f_i(x) \quad \text{s.t.} \quad x \in \mathcal{X},$$  \hspace{1cm} (I.1)

where $\mathcal{X} \subset \mathbb{R}^n$ is a closed convex set known to all agents, and $f_i(x)$ is only available to agent $i$. Given this partial knowledge, agents must communicate with each other to minimize $f(x)$. There exists a large body of works on distributed convex optimization, where each $f_i(x)$ is convex (see e.g., the seminal work of [10] and its following papers). The literature has witnessed various algorithms for solving (I.1), which come with theoretical guarantees. In this paper, we depart from the classical convex setting and focus on weakly convex and nonsmooth problems. In particular, we assume that every $f_i(x)$ is $\rho$–weakly convex $^1$ and the subgradient $\|\partial f_i(x)\|$ is uniformly bounded.

Weakly convex problems play a key role in important machine learning applications such as robust phase retrieval [11]–[13], blind deconvolution [14], [15], biconvex compressive sensing [16], and dictionary learning [1]. Recently, Davis et al. [1], [2] characterized the convergence of the stochastic subgradient method for the weakly convex case. However, training in a single device the machine learning models aforementioned can take a significant amount of time. In practice, distributed implementations of the stochastic subgradient method (stoDPSM) are used to speed-up training time (see e.g. [17], [18]).

In this paper we focus on developing a theoretical convergence result for the distributed projected subgradient method (DPSM)

$$x_{i,k+1} = \text{Proj}_{\mathcal{X}} \left( \sum_{j=1}^{N} a_{i,j}(k)x_{j,k} - \alpha_k g_{i,k} \right),$$  \hspace{1cm} (I.2)

where $\text{Proj}_{\mathcal{X}}$ denotes the orthogonal projection onto $\mathcal{X}$, $a_{i,j}(k)$ is the weight agents $i$ associates to information received from agents $j$ at time $k$, $\alpha_k$ is the stepsize, and $g_{i,k}$ is any subgradient of $f_i$ evaluated at $\sum_{j=1}^{N} a_{i,j}(k)x_{j,k}$. This algorithmic scheme was first proposed for unconstrained distributed convex optimization [10] and it was then extended to the constrained scenario [19], [20]. Under standard assumptions on the weights $a_{i,j}(k)$ and $\alpha_k$, it can be shown that (I.2) converges to the minimizer of $f(x)$ in (I.1) for convex problems.

$^1$I.e., there exists a convex function $h_i(x)$ such that $f_i(x) = h_i(x) - \frac{\rho}{2}\|x\|^2$.
Nevertheless, the weakly convex problem (which is essentially non-convex) has not been addressed in the literature. We (i) provide global convergence results for DPSM using the notion of Moreau envelope and show that the infimum of its gradient approaches zero with a rate \( O(k^{-1/4}) \); (ii) show a linear convergence rate of DPSM under a so-called sharpness condition; (iii) extend the global convergence results to the stochastic setting (stoDPSM). This paper is the first work providing convergence analysis of distributed subgradient for weakly convex, non-smooth problems, extending classical convex analysis to non-convex setting.

A. Related work

We now briefly review the existing work on distributed (sub)gradient method. When \( f_i \) is convex and non-smooth, the distributed subgradient method converges in terms of function value [10] in unconstrained scenario and the distributed stochastic subgradient projection algorithms [21] can deal with a common constraint. In the case that each agent only knows its own constraint information, convergence guarantee was proved in [19]. In all above results, diminishing stepsize (square-summable but nonsummable) is required and the convergence for constrained problem is measured by the distance between the sequence and optimal set, i.e., the limiting of \( \text{dist}(x_{i,k}, X^*) \) is zero for any \( i \), where \( X^* \) is the optimal point set. The square-summable condition was relaxed in [20], provided the optimum set is bounded. Moreover, a convergence rate of \( O(1/\sqrt{k}) \) is shown in [20] if the stepsize is set to \( \alpha_k = O(1/k) \) for strongly convex \( f_i \). For smooth convex and unconstrained case, the convergence is established in [22]. We refer to the survey [23] for a complete review of decentralized optimization of convex problems.

Whilst \( f_i \) is nonconvex and its gradient \( \nabla f_i \) is Lipschitz continuous, \( f_i(x) \) is automatically weakly convex. Plenary algorithms have been proposed for the constrained setting. For example, the convergence of distributed projected stochastic gradient was founded in [24]; an ergodic convergence rate was established in [25] for proximal gradient method. The nonsmooth objective function in [25] is in the composite form, i.e., \( f_i(x) = g_i(x) + h_i(x) \), where \( g_i \) is smooth but \( h_i \) is nonsmooth with easy proximal mapping. In contrast, the nonsmooth objective in (1.1) generally doesn’t follow an easy proximal mapping. While the computation of subgradient of \( f_i \) is inexpensive, algorithm (1.2) is a better choice. Recently, when \( f_i \) is weakly convex, the centralized proximal-type subgradient methods have been shown to converge in finite time in terms of a stationarity measure using Moreau envelope (see [1], [12], [26], [27]). Under the presence of sharpness property, the centralized subgradient converges linearly in a local region [2]. In this paper, we focus on establishing the convergence of the distributed version of the projected subgradient algorithm. Specifically, we show convergence for the average point \( \bar{x}_k = 1/N \sum_{i=1}^N x_{i,k} \) using the Moreau envelope. With the sharpness regularity condition, the local linear convergence rate for \( \sum_{i=1}^N \|x_{i,k} - x^*\|^2 \) and \( \|\bar{x}_k - x^*\| \) is established, where \( x^* \) is the global sharp minimizer. We summarize the convergence results for distributed projected subgradient method in table I.

II. Preliminaries

Notation: We use \( \langle x, y \rangle = x^\top y \) to denote the Euclidean inner product and \( \|x\| \) to denote the Euclidean norm of \( x \). We denote by \( \partial h(x) \) the subgradient set of a convex function \( h(x) \). Abusing notation, we use \( \langle \partial h(x), y \rangle \) to denote the inner product of any elements of \( \partial h(x) \) and a vector \( y \).

A. Network Model

We consider a time-varying network of agents that can exchange information locally. To model the network, we use a time-varying graph \( (\mathcal{V}, E_k) \), where \( \mathcal{V} = \{1, \ldots, N\} \) denotes the set of nodes and \( E_k \subseteq N \times N \) is the set of links connecting nodes at time \( k \). Let \( A(k) = [a_{i,j}(k)] \) denote the matrix of weights associated with links in the graph at time \( k > 0 \). For node \( i \), \( N_k(i) \) denotes the neighborhood of \( i \) in which \( a_{i,j}(k) \) > 0. Define \( \Phi(k, s) = A(s)A(s+1) \cdots A(k-1)A(k) \) for \( k \geq s \), \( \Phi(k,k) = A(k) \) and \( \Phi(k,s) = I \) for \( k < s \).

B. Weak Convexity and Optimality Measure

We assume the local objective \( f_i(x) \) in (1.1) is \( \rho \)-weakly convex for some \( \rho \geq 0 \); i.e., there exists a convex function \( h_i(x) \) such that \( f_i(x) = h_i(x) - \frac{\rho}{2}\|x\|^2 \). Although \( f_i(x) \) is not convex, we may define its subdifferential by

\[
\partial f_i(x) = \partial h_i(x) - \rho x, \quad \forall x \in \mathcal{X}; \quad \text{(II.1)}
\]

(see [28]). Here, \( \partial h_i(x) \) is the subdifferential in the convex sense. The following lemma states an equivalent definition of weakly convex functions and strongly convex functions.

**Lemma II.1.** If \( f(x) \) is \( \rho \)-weakly convex and \( g(x) \) is \( \tau \)-strongly convex in \( \mathbb{R}^n \), then for \( x_1, \ldots, x_m \in \mathbb{R}^n \), it follows that

\[
f(\sum_{i=1}^m a_i x_i) \leq \sum_{i=1}^m a_i f(x_i) + \frac{\rho}{2} \sum_{i=1}^{m-1} \sum_{j=i+1}^m a_i a_j \|x_i - x_j\|^2 \quad \text{(II.2)}
\]

and

\[
g(\sum_{i=1}^m a_i x_i) \leq \sum_{i=1}^m a_i g(x_i) - \frac{\tau}{2} \sum_{i=1}^{m-1} \sum_{j=i+1}^m a_i a_j \|x_i - x_j\|^2, \quad \text{(II.3)}
\]
where \( \sum_{i=1}^{m} a_i = 1 \) and \( a_i \geq 0 \) for all \( i \).

To analyze DPSM, we follow the framework of [1], where authors propose a novel convergence analysis for centralized subgradient method. We extend this analysis to the distributed case for weakly convex problems. Since there exist different stationary points in nonconvex problems, neither the suboptimal objective value nor the distance to the optimum set tend to be good measures for analysis. On the other hand, the subgradient of the objective is not continuous, which renders it difficult to analyze the convergence of the subgradient norm. A surrogate stationary measure for problem (I.1) was thus defined using the Moreau envelope in [1]. We briefly review it in the sequel.

Recall that if \( f_i(x) \) is \( \rho \)-weakly convex, iff we have the following inequality [1, Lemma 2.1]

\[
f_i(y) \geq f_i(x) + \partial f_i(x)^\top (y - x) - \frac{\rho}{2} \|y - x\|^2. \tag{II.4}
\]

This inequality is also known as prox-regular inequality introduced in [29]. Therefore, \( f(x) = 1/N \sum_{i=1}^{N} f_i(x) \) is also \( \rho \)-weakly convex. Let \( \varphi(x) = f(x) + \mathbb{1}_X(x) \), where \( \mathbb{1}_X \) is the indicator function of \( X \). Define the Moreau envelope [30] as

\[
\varphi_\lambda(x) := \min_{y \in \mathbb{R}^n} \varphi(y) + \frac{1}{2\lambda} \|y - x\|^2, \quad t < 1/\rho.
\]

The Moreau envelope is a \( C^1 \) smooth approximation to the nonsmooth function \( f(x) \) over \( X \). Since \( f_i \) is \( \rho \)-weakly convex, the minimization problem above is strictly convex and the minimizer is unique. We denote it as

\[
\hat{x} = \arg\min_{y \in \mathbb{R}^n} \varphi(y) + \frac{1}{2t} \|y - x\|^2,
\]

and the mapping \( \text{prox}_{\lambda f}(x) := \hat{x} \) is called the proximal mapping. We here omit \( X \) in the notation \( \text{prox}_{\lambda f} \) but remember the proximal mapping is related to \( X \). The proximal mapping is only used in our analysis but not computed in the algorithm. From the optimality condition of \( \hat{x} \), one has

\[
0 \in \partial f(\hat{x}) + \partial \mathbb{1}_X(\hat{x}) + \frac{1}{t}(\hat{x} - x).
\]

It follows that

\[
\text{dist}(0, \partial f(\hat{x}) + \partial \mathbb{1}_X(\hat{x})) \leq \frac{1}{t} \|\hat{x} - x\|.
\]

\( \mathbb{1}_X(x) = 0 \) when \( x \in X \), and \( \mathbb{1}_X(x) = \infty \) otherwise.

Therefore, if \( \frac{1}{t} \|\hat{x} - x\| \leq \varepsilon \), then \( \hat{x} \) is \( \varepsilon \)-stationary and \( x \) is close to the \( \varepsilon \)-stationary point \( \hat{x} \). We also have [30]

\[
\|\nabla \varphi_t(x)\| = \frac{1}{t} \|\hat{x} - x\|, \tag{II.5}
\]

which can be used as near-stationarity measure of \( x \).

### C. Assumptions

In this part, we introduce the assumptions used for our analysis. To begin with, we define the average vector

\[
\bar{x}_k := \frac{1}{N} \sum_{i=1}^{N} x_{i,k},
\]

and

\[
v_{i,k} := \sum_{j \in N_i(k)} a_{i,j}(k) x_{j,k}.
\]

Then, the iteration in DPSM (I.2) can be rewritten as

\[
x_{i,k+1} = \text{Proj}_{X} (v_{i,k} - \alpha_k g_{i,k}), \tag{II.6}
\]

where \( g_{i,k} \in \partial f_i(v_{i,k}) \) is any element of the subdifferential set. Unlike the centralized algorithm, the distributed update does not rely on a fusion center and computing \( v_{i,k}, g_{i,k} \) can be done in a decentralized manner. The following assumptions on the network are commonly adopted in the literature [10], [19], [31].

**Assumption 1 (Weights rule).** There exists a scalar \( \eta \in (0,1) \) such that for all \( i, j \in \{1, \ldots, N\} \),

- \( a_{i,i}(k) \geq \eta \) for all \( k \geq 0 \).
- If \( a_{i,j}(k) > 0 \) then \( a_{i,j}(k) \geq \eta \).

**Assumption 2 (Doubly stochasticity).** The weight matrix \( A(k) \) is doubly stochastic (i.e., \( \sum_j a_{i,j}(k) = \sum_j a_{j,i}(k) = 1, \forall i, k \)).

**Assumption 3 (Connectivity).** The graph \( (V, E_{\infty}) \) is strongly connected, where \( E_{\infty} \) is the set of edges \( (j, i) \) representing agent pairs communicating directly infinitely many times, i.e., \( E_{\infty} = \{(j, i) : (j, i) \in E_k \text{ for infinitely many indices } k\} \).

**Assumption 4 (Bounded Intercommunication Interval).** There exists an integer \( B \geq 1 \) such that for every \( (j, i) \in E_{\infty} \), agent \( j \) sends its information to the neighboring agent \( i \) at least once every \( B \) consecutive time slots, i.e., at time \( t_k \) or at time \( t_k + 1 \) or . . . or (at latest) at time \( t_k + B - 1 \) for any \( k \geq 0 \).
We also need some assumptions on the function $f(x)$.

**Assumption 5.** $f(x)$ is lower bounded. Every $f_i$ is $\rho-$weakly convex and we have $||\partial f_i(x)|| \leq L$ for $x \in \mathcal{X}$.

The bounded subgradient holds if $f_i(x)$ is globally $L-$Lipschitz continuous on $\mathbb{R}^n$ or $\mathcal{X}$ [32]. One common class of weakly convex function is $f(x) = h(c(x))$, where $h$ is convex and Lipschitz and $c$ is smooth with Lipschitzian Jacobian $\mathcal{L}$. However, such $f(x) = h(c(x))$ is usually locally Lipschitz continuous. A common assumption to resolve this issue is that $\mathcal{X}$ is compact or the sequence $\{x_{i,k}\}$ is bounded. Then, the boundedness of subgradient is equivalent to the $L-$Lipschitz of $f_i(x)$. Such assumptions are usually needed in centralized algorithms; see [1], [27].

Last, the stepsize for the global convergence of DPSM should be non-summable and diminishing.

**Assumption 6.** The stepsize $\alpha_k > 0$ satisfies

$$\sum_{k=0}^{\infty} \alpha_k = \infty \text{ and } \lim_{k \to \infty} \alpha_k = 0.$$  

A commonly used stepsize sequence satisfying Assumption 6 can be as follows

$$\alpha_k = \frac{1}{k^q}, \text{ where } q \in (0, 1].$$

**D. Technical lemmas**

In this part, we introduce some necessary lemmas. All proofs can be found in the appendix.

**Lemma II.2.** Under Assumptions 1 to 4, there exist constants $c > 0$ and $\lambda \in (0, 1)$ such that

$$\|\Phi(k, s) - \frac{1}{N}1_1^T\|_{op} \leq c\lambda^{k-s},$$

where $\| \cdot \|_{op}$ is the matrix operator norm.

**Lemma II.3.** Let $\lambda \in (0, 1)$ and $\{\gamma_k\}$ be a positive sequence. Suppose $\sum_{k=0}^{\infty} \gamma_k = \infty$ and $\lim_{k \to \infty} \gamma_k = 0$. Considering the convolution sequence $\sum_{k=0}^{T-1} \lambda^k \gamma_{T-k-1}$, we have

$$\sum_{k=0}^{T-1} \lambda^k \gamma_{T-k-1} = O(\gamma_{T-1}).$$ (II.7)

For convergence analysis, we should show that the deviation of individual errors from the mean $\|x_{i,k} - \bar{x}_k\|$ goes to zero. We define a vector $\Delta_k$ where $\Delta_{k,i} := x_{i,k} - \bar{x}_k$. That is, $\Delta_k \in \mathbb{R}^N$ is the vector formed by stacking all individual deviations from the mean. The following inequality (II.8) for $\Delta_k$ was established in [19], [20], [23] for convex problems. We show that it still holds for the weakly convex case.

**Lemma II.4.** Under Assumptions 1 to 6, for the distributed projected subgradient algorithm (II.6), we have the following consensus result

$$\lim_{k \to \infty} ||\Delta_{k,i}|| = 0, \forall i.$$ (II.8)

Furthermore, similar to the result of [20, Proposition 8], the convergence rate can be characterized as follows.

**Lemma II.5.** Under Assumptions 1 to 6, for the distributed projected subgradient algorithm (II.6), we have the following error rate

$$||\Delta_k||^2 = O(\alpha_k^2).$$ (II.9)

We also have the following well-known property of the projection onto convex sets.

**Lemma II.6.** [19] For convex closed set $\mathcal{X}$, it follows that $\forall y \in \mathcal{X}$

$$||\text{Proj}_{\mathcal{X}}(y) - y||^2 \leq ||y - \text{Proj}_{\mathcal{X}}(y)||^2.$$

We further have the following property of the proximal mapping. Although the proximal mapping is not nonexpansive when $f(x)$ is weakly convex, it is still Lipschitz continuous.

**Lemma II.7.** If $f(x)$ is $\rho-$weakly convex, then the proximal mapping with $t < 1/\rho$ satisfies

$$||\text{prox}_f(x_1) - \text{prox}_f(x_2)|| \leq \frac{1}{1-t\rho}||x_1 - x_2||,$$

$\forall x_1, x_2 \in \mathcal{X}.$

**III. MAIN RESULTS AND CONVERGENCE ANALYSIS**

In this section, we state the main convergence results of DPSM. First, if the Assumptions 1 to 6 hold, the Moreau envelope sequence $\{\phi_i(\bar{x}_k)\}$ converges and the infimum of its gradient converges to 0. Second, under the sharpness condition, DPSM converges linearly with geometrically diminishing stepsize in a neighborhood of the sharp minimizer. Finally, we also provide the convergence result of distributed projected stochastic subgradient method.

**A. Global Convergence**

We now establish the first convergence result. The following lemma states the improvement after one iteration of the algorithm (II.6).

**Lemma III.1** (One-step improvement). Let

$$s_k := \arg\min_{y \in \mathcal{X}} f(y) + \frac{1}{2t}||y - \bar{x}_k||^2,$$

$$v_{i,k} = \sum_{j \in N_i(k)} a_{i,j}(k)x_{j,k},$$

$$\hat{v}_{i,k} := \arg\min_{y \in \mathcal{X}} f(y) + \frac{1}{2t}||y - v_{i,k}||^2.$$
Under Assumption 5, we have
\[
\sum_{i=1}^{N} \|x_{i,k+1} - \hat{v}_{i,k}\|^2 \leq \sum_{i=1}^{N} \|v_{i,k} - \hat{v}_{i,k}\|^2
+ 2\alpha_k \left( N\left(\frac{-1}{2t} + \rho\right)\|\bar{x}_k - s_k\|^2 + \frac{L(2-t\rho)}{1-t\rho} \sum_{i=1}^{N} \|x_{i,k} - \bar{x}_k\| \right.
+ 2\rho(1 + \frac{1}{(1-t\rho)^2}) \sum_{i=1}^{N} \|v_{i,k} - \bar{x}_k\| \right) + N\lambda^2\alpha_k^2.
\] (III.1)

By invoking Lemma II.4, the distance \(\|x_{i,k} - \bar{x}_k\|\) goes to zero for all \(i\) when \(\alpha_k\) goes to zero. Note that for \(t < 1/(2\rho)\), the term \((-\frac{1}{2t} + \rho)\|\bar{x}_k - s_k\|^2\) in (III.1) is strictly negative if \(\|\bar{x}_k - s_k\|^2\) is not zero. Then, we may have
\[
\sum_{k=1}^{\infty} \|x_{i,k+1} - \hat{v}_{i,k}\|^2 < \sum_{k=1}^{\infty} \|v_{i,k} - \hat{v}_{i,k}\|^2,
\]
which means that compared to \(v_{i,k}\), the new point \(x_{i,k+1}\) is closer to \(\hat{v}_{i,k}\). Hence, the algorithm continues to make progress. Now, we present our first main result, which shows the decay of the gradient of the Moreau envelope (optimality measure).

**Theorem III.2.** Let \(t < \frac{1}{2\rho}\) and \(\{x_{i,k}\}\) be the sequence of projected subgradient method for solving problem (I.1). Under Assumptions 1 to 6,
1. If \(\sum_{k=0}^{\infty} \alpha_k^2 < \infty\), there exists \(\bar{\varphi}_t\) such that
\[
\lim_{k \to \infty} \varphi_t(x_{i,k}) = \lim_{k \to \infty} \varphi_t(\bar{x}_k) = \bar{\varphi}_t;
\]
2. There exists \(b_k = O(\alpha_k^2)\) such that
\[
\inf_k \|\nabla \varphi_t(\bar{x}_k)\| \leq \frac{2}{1 - 2t\rho} \sum_{k=0}^{\infty} \alpha_k.
\]

Statement (1) of Theorem III.2 suggests that the Moreau envelope function value converges at the mean \(\bar{x}_k\) if \(\sum_{k=0}^{\infty} \alpha_k^2 < \infty\). Statement (2) is the decentralized counterpart of the centralization algorithm established in [1]. Due to Assumption 6, it implies that
\[
\lim_{k \to \infty} \inf_k \|\nabla \varphi_t(\bar{x}_k)\| = 0.
\]
It also provides the convergence rate of \(\inf_k \|\nabla \varphi_t(\bar{x}_k)\|^2\). For example, if \(\alpha_k = O(1/\sqrt{k})\), we have
\[
\inf_{k=1,2,\ldots,T} \|\nabla \varphi_t(\bar{x}_k)\|^2 \leq O\left(\frac{\log T}{\sqrt{T}}\right),
\]
for sufficiently large \(T\). Compared with the centralized algorithm [1], we have an extra term \(\sum_{k} b_k\) in the upper bound, which is the cost of decentralization as it has constants involving network parameters.

**B. Local Convergence Rate with Sharpness Property**

In this section, we discuss the convergence rate of the Algorithm I.2 under the presence of sharpness property. The definition of sharpness is given as follows.

**Definition III.3 (Sharpness).** A function \(f : \mathcal{X} \to \mathbb{R}\) possesses the local sharpness property, if there exist constants \(\beta > 0\) and \(B > 0\) such that the following inequality holds for the minimizer \(x^*\) of \(f(x)\)
\[
f(x) - \min f \geq \beta \|x - x^*\|, \quad \forall x \in \mathcal{B},
\]
where \(\mathcal{B} = \{x \in \mathcal{X} : \|x - x^*\| \leq B\}\). Furthermore, \(x^*\) is called the local sharp minimizer of \(f\).

It has been shown the centralized subgradient method converges linearly in the neighborhood of the sharp minimizer \([2, 11, 27]\), if the Polyak stepsize \([34]\) or geometrically diminishing stepsize \([35]\) is adopted. The Polyak stepsize \([34]\) and geometrically diminishing stepsize were firstly proposed for convex problems and they also work for weakly convex problems. Since the Polyak stepsize needs the knowledge of the optimal function value, we will only consider the geometrically diminishing stepsize, i.e, \(\alpha_k = \mu_0 \gamma^k\), where \(\mu_0 > 0\) and \(\gamma \in (0, 1)\) are constants decided by the problem parameters. Under some conditions, we can show the linear rate of DPSM in Theorem III.5. The proof idea is as follows. As the sharpness is a property of the whole function \(f(x)\), we can only use the sharpness inequality at \(\bar{x}_k\). We firstly need to estimate the deviation from mean \(\|\Delta_k\|\) when using the geometrically diminishing stepsize.

**Lemma III.4.** Let the stepsize \(\alpha_k\) in Algorithm I.2 be
\[
\alpha_k = \mu_0 \gamma^k, \quad k \geq 0, \quad \text{where} \quad \mu_0 > 0, \quad \gamma \geq \lambda^\delta, \quad \delta \in (0, 1)
\]
and \(\lambda\) is the parameter given in Lemma I.2. Then, \(\|\Delta_k\| = O(\alpha_k)\).

**Proof.** From inequality (A.7) in the proof of Lemma I.4 and the fact \(\gamma \geq \lambda^\delta\), we have
\[
\|\Delta_{k+1}\| \leq c \lambda^k \|\Delta_0\| + c \sqrt{N} L \sum_{l=0}^{k-1} \lambda^{k-1-l} \alpha_l + \sqrt{N} L \alpha_k
\leq \left( \frac{c}{\lambda} \|\Delta_0\| + \sqrt{N} L \left( \frac{c \gamma^{1/2} - 1 + \lambda \mu_0}{1 - \gamma^{1/2}} \right) \right) \gamma^{k+1}.
\]
(III.3)

**Assumption 7.** Let \(x_{1,0}, \ldots, x_{N,0}\) be the initial points in Algorithm I.6. Given any constants \(\lambda \in (\lambda, 1)\) and \(\eta \geq \sqrt{2}\), define
\[
e_0 := \min \left\{ \max \left\{ \frac{\beta}{\mu_0}, \frac{1}{N} \sum_{i=1}^{N} \|x_{i,0} - x^*\|^2 \right\}, \frac{B}{\eta} \right\},
a := \frac{2(L + \beta) L}{\lambda^2},
q := \frac{2B}{\eta} e_0 - q e_0 - \frac{2(L + \beta) c}{\sqrt{N} \lambda} \|\Delta_0\|,
\]
where \(c, \lambda\) are constants given in Lemma II.2, \(\beta\) and \(B\) are defined in (II.2), \(\rho\) is the weak-convexity parameter, and \(L\) is the bound on subgradients. Let the stepsize in Algorithm II.6 be given by \(\alpha_k = \mu_0 \gamma^k\), where \(0 < \mu_0 \leq \min\left(\frac{2\beta^2}{\rho^2 c^2}, 10\sqrt{N(\lambda + L^2 + \frac{1}{\eta^2})}\right)\) and \(\gamma \in (0, 1)\).

We use the stepsize assumption above to prove the following theorem.

**Theorem III.5.** Let \(N \geq 2\) and \(x^*\) be a local sharp minimizer of problem (I.1). Suppose the initial points \(x_{1,0}, \ldots, x_{N,0}\) in Algorithm II.6 satisfy for all \(i \in \{1, \ldots, N\}\) the three constraints

\[
\sum_{i=1}^{N} \|x_{i,0} - x^*\|^2 \leq \frac{N}{\eta^2} \min\left\{\left(\frac{2\beta}{\rho}\right)^2, B^2\right\},
\]

\[
\|x_{i,0} - x^*\|^2 \leq \frac{\rho^2}{N} \sum_{i=1}^{N} \|x_{i,0} - x^*\|^2,
\]

\[
\|\Delta_0\| < \frac{2\beta \epsilon_0 - \rho \epsilon_0}{2(L + \beta) c},
\]

where \(c, \lambda\) are constants given in Lemma II.2. Under Assumptions 1 to 5 and \(\delta > 0\) such that for \(\gamma = \lambda^\delta\), we have

\[
\sum_{i=1}^{N} \|x_{i,k} - x^*\|^2 \leq N \gamma^{2k} \epsilon_0^2 \tag{III.4}
\]

and

\[
\|x_{i,k} - x^*\|^2 \leq \eta^2 \gamma^{2k} \epsilon_0^2 \tag{III.5}
\]

for any sequence \(\{x_{i,k}\}\) generated by Algorithm II.6.

The following comments about the theorem are in order:

1. The convergence rate \(\gamma = \lambda^\delta\) is the same as the decaying rate of stepsize. But it cannot be smaller than \(\lambda\), which is the convergence rate of the consensus.

2. For the centralized subgradient method \([2], [27]\), the local linear rate is established in the tube

\[
\mathcal{T} = \{x : \text{dist}(x, \mathcal{X}_i) \leq \frac{2\beta}{\rho}\},
\]

where \(\mathcal{X}_i\) is the set of the global sharp minimizers. In Theorem III.5, the initialization constraints ensure that the individual initial points are close enough to each other as well as a sharp minimizer (local convergence). Moreover, since we use the local sharpness property, the local region should be included in \(\mathcal{B}\).

3. An immediate corollary of Theorem III.5 is that \(\|\bar{x}_k - x^*\|^2 \leq 1/N \sum_{i=1}^{N} \|x_{i,k} - x^*\|^2 \leq \gamma^{2k} \epsilon_0^2\) under the same conditions.

4. If \(f_i(x)\) is convex, i.e., \(\rho = 0\), then the condition \(\epsilon_0 \leq \frac{\beta}{\rho}\) can be removed. The weak convexity parameter \(\rho\) restricts the initialization region, which is also clearly stated for centralized subgradient method \([2]\).

### C. Distributed Projected Stochastic Subgradient Method

In some settings, the function \(f_i(x)\) at local agent is given by \(f_i(x) = \frac{1}{m_i} \sum_{j=1}^{m_i} f_{i,j}\), where \(m_i\) is an extremely large number and \(f_{i,j}\) is \(\rho\)-weakly convex. Therefore, it is forbidden to compute the subgradient of \(f_i(x)\) in each iteration. In contrast to the algorithm (I.2), the distributed stochastic projected subgradient method iterates as follows

\[
x_{i,k+1} = \text{Proj}_\chi (v_{i,k} - \alpha_k \xi_{i,k}),
\]

where \(\alpha_k > 0\) is the stepsize,

\[
v_{i,k} = \sum_{j \in \mathcal{N}(k)} a_{i,j}(k)x_{j,k},
\]

and \(\xi_{i,k}\) satisfies \(\mathbb{E}\xi_{i,k} \in \partial f_i(v_{i,k})\). In practice, for each \(i\), we uniformly randomly select index \(i_t \in \{1, 2, \ldots, m_i\}\) and set \(\xi_{i,t} \in \partial f_{i,t}(v_{i,k})\).

We assume that \(\mathbb{E}\|\xi_{i,k}\|^2 \leq L^2\) for all \(i, k\), which is standard as in [1]. We have the following convergence result for distributed projected stochastic subgradient method (III.6).

**Theorem III.6.** Let \(t < \frac{1}{\beta}\) and \(\{x_{i,k}\}\) be the sequence of algorithm (III.6). Under Assumptions 1 to 6, there exists a subsequence of \(\{\|\nabla \varphi_t(\bar{x}_k)\|\}\) converging to zero, i.e.,

\[
\liminf_{k \to \infty} \mathbb{E}\|\nabla \varphi_t(\bar{x}_k)\| = 0.
\]

### IV. Numerical Experiment

We conduct simulations on robust phase retrieval problem

\[
\min_{x \in \mathbb{R}^n} f(x) = \frac{1}{N} \sum_{i=1}^{N} \left(\frac{1}{m} \sum_{j=1}^{m} |\langle w_{i,j}, x \rangle|^2 - y_{i,j}\right). \tag{IV.1}
\]

The problem is to recover the random signal \(\tilde{x} \in \mathbb{R}^n\) using Gaussian measurements \(w_{i,j}\). We generate the measurements \(w_{i,j}\) and the observations \(y_{i,j}\) following the work [12]. For simplicity, we only consider the noiseless case. More specifically, the ground truth \(\tilde{x}\) is drawn from \(\mathcal{N}(0, I_n)\) and \(y_{i,j} = \langle w_{i,j}, \tilde{x} \rangle^2\), where \(w_{i,j}\) are i.i.d standard Gaussian random variables. As suggested by [12], the recovery rate is 100\% when \(N \times m \geq 2.7n\) for the proximal linear algorithm. Therefore, we use \(N \times m \geq 3n\) for subgradient method in all tests. All initialization follows from the procedure proposed in [12, Section 4.2] and we set \(x_{1,0} = x_{2,0} = \ldots = x_{N,0}\). We generate an Erdős-Rényi model \(G(N, 0.3)\) and \(A(k) = A\) is time-invariant Metropolis Hasting matrix associated
with the graph. Therefore, we have $\lambda$ is the second largest eigenvalue of $A$ in Lemma II.2.

The robust phase retrieval formulation (IV.1) was shown to be weakly convex [12] and have sharpness property w.h.p under mild probabilistic assumptions in [12], [13]. Different from the definition III.3, the sharpness condition is given by

$$f(x) - \min f \geq \kappa \| x - \tilde{x} \| \| x + \tilde{x} \|,$$

where $\kappa > 0$ is some numerical number. Hence, $\pm \tilde{x}$ are also the global minimizers.

We denote $\mathcal{X}^*$ as the global minimizers set and $\text{dist}(x, \mathcal{X}^*)$ denotes the distance between $x$ and $\mathcal{X}^*$. The global minimizers set is $\{ \tilde{x}, -\tilde{x} \}$. According to [11, Lemma 3.1], there is no other critical points in the tube $\{ x : \text{dist}(x, \mathcal{X}^*) \leq \frac{2\kappa}{\rho} \}$. Since 0 is also a critical point to the population function $f_P(x) = E_a[ \langle a, x \rangle^2 - \langle a, \tilde{x} \rangle^2 ]$ [11, Theorem 5.1]. We have $\{ x : \| x - \tilde{x} \| \leq \frac{2\kappa}{\rho} \} \cap \{ x : \| x + \tilde{x} \| \leq \frac{2\kappa}{\rho} \} = \emptyset$. To satisfy definition III.3, we let $\beta = \kappa \| \tilde{x} \|$ and choose $B = \{ x : \| x - x^* \| \leq \frac{2\beta}{\rho} \}$, where $x^*$ is $\tilde{x}$ or $-\tilde{x}$ and the sign is decided by the initialization.

**Synthetic data** Firstly, we solve the robust phase retrieval problem (IV.1) by stochastic DPSM using diminishing stepsize. In each epoch $K$, the stepsize is set to $\alpha_K = O(1/K)$ or $\alpha_K = O(1/\sqrt{K})$. We plot the log distance v.s. epoch $K$ in Figure 1. We see that stochastic DPSM converges to the global minimizer.

Secondly, we demonstrate the linear rate of DPSM. Like the centralized subgradient method [2], $\mu_0$ and $\gamma$ should be tuned for performance. In Figure 2, 'CSub' represents the centralized subgradient method [2]. We see that $\gamma = 0.8$ works for CSub but not for DPSM, since the smallest $\gamma$ is 0.99 for $\mu_0 = 4/N$. For $\mu_0 = 5/N$, $\gamma = 0.99$, DPSM does not converge, so the largest $\mu_0$ may be $4/N$. This indicates that convergence rate of DPSM is slower than CSub. And $\gamma = 0.99 > \lambda = 0.8818$ also demonstrates that the convergence rate cannot be faster than consensus. Although, the DPSM is not faster than CSub in the iteration number, DPSM has the advantages of parallel computation. And if the data number $m \times N$ is extremely large, the computation of the whole subgradient is not affordable. We also test the case $m = 1$, i.e., there is only single data at each node. In Figure 3, we observe similar performance as in Figure 2.

**Real-world image** We use digit images from the MNIST data set [36]. The gray image dimension is $n = 28 \times 28 = 784$ and we set $m = 84$, $N = 28$ so that the number of Gaussian measurements is $m \times N = 3 \times n$. Other settings are the same as previous synthetic data. In Figure 4, we show the original, initial guess and recovered image. We see that the recovery is identical to the true image. The convergence plot is shown in Figure 5.

**V. Conclusion**

We analysed the (stochastic) distributed subgradient method for solving constrained weakly convex optimization. Under standard assumptions on the connectivity of the network and the weights, we presented the global convergence of the average point using Moreau...
envelope. Moreover, we proved a linear convergence rate under the sharpness property. Numerical results on robust phase retrieval illustrate our theory.

A natural extension of this work is to consider the directed network. For example, the convergence of directed distributed subgradient method for convex problems was analyzed in [37]. It will also be interesting to see whether it is possible to deal with different constraints at each local node; see the convex constraints in [19]. Finally, it will be worth considering non-convex constraints (e.g., sphere constraint [38]).

APPENDIX

Proof of Lemma II.1. We prove it by induction. For $m = 2$, let $y = a_1 x_1 + a_2 x_2$, where $a_1 + a_2 = 1$, $a_1 \geq 0$ and $a_2 \geq 0$. From the subgradient inequality (II.4), we have

$$f(x_1) \geq f(y) + \langle \partial f(y), x_1 - y \rangle - \frac{\rho}{2} \| x_1 - y \|^2$$

and

$$f(x_2) \geq f(y) + \langle \partial f(y), x_2 - y \rangle - \frac{\rho}{2} \| x_2 - y \|^2. $$

Multiplying the two above inequalities by $a_1$ and $a_2$, respectively, and summing them, yield

$$a_1 f(x_1) + a_2 f(x_2) \geq f(y) - \frac{\rho}{2} a_1 a_2 \| x_1 - x_2 \|^2.$$

Similarly, we also have

$$a_1 g(x_1) + a_2 g(x_2) \geq g(y) + \frac{\tau}{2} a_1 a_2 \| x_1 - x_2 \|^2.$$

Therefore, inequality (II.2) holds for $m = 2$. Suppose they hold for $m = k$. For $m = k + 1$, let $z = \sum_{i=1}^{k+1} a_i x_i$ and $b = \sum_{i=1}^{k} a_i$. We have

$$f(z) = f(b \sum_{i=1}^{k} \frac{a_i}{b} x_i + a_{k+1} x_{k+1})$$

$$\leq b f(\sum_{i=1}^{k} \frac{a_i}{b} x_i) + a_{k+1} f(x_{k+1}) +$$

$$\frac{\rho}{2} a_{k+1} b \| \sum_{i=1}^{k} \frac{a_i}{b} (x_i - x_{k+1}) \|^2$$

$$\leq b \left( \sum_{i=1}^{k} \frac{a_i}{b} f(x_i) + \frac{\rho}{2} \sum_{i=1}^{k} \sum_{j=i+1}^{k} \frac{a_i a_j}{b^2} \| x_i - x_j \|^2 \right)$$

$$+ a_{k+1} f(x_{k+1}) + \frac{\rho}{2} a_{k+1} b \| \sum_{i=1}^{k} \frac{a_i}{b} (x_i - x_{k+1}) \|^2,$$

(A.1)

where the first inequality follows from $b + a_{k+1} = 1$ and the second from assumption step. Notice that since $\| . \|^2$ is $2$–strongly convex, it follows from the assumption for strongly convex function that

$$\| \sum_{i=1}^{k} \frac{a_i}{b} (x_i - x_{k+1}) \|^2$$

$$\leq \sum_{i=1}^{k} \frac{a_i}{b} \| x_i - x_{k+1} \|^2 - \sum_{i=1}^{k} \sum_{j=i+1}^{k} \frac{a_i a_j}{b^2} \| x_i - x_j \|^2.$$

Substituting it into (A.1) yields

$$f(z) \leq \sum_{i=1}^{k} a_i f(x_i)$$

$$+ \frac{\rho}{2} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \left( \frac{a_i a_j}{b} - \frac{a_{k+1} a_i a_j}{b} \right) \| x_i - x_j \|^2$$

$$+ a_{k+1} f(x_{k+1}) + \frac{\rho}{2} \sum_{i=1}^{k} \sum_{j=i+1}^{k+1} a_i a_j \| x_i - x_{k+1} \|^2$$

$$= \sum_{i=1}^{k+1} a_i f(x_i) + \frac{\rho}{2} \sum_{i=1}^{k} \sum_{j=i+1}^{k+1} a_i a_j \| x_i - x_j \|^2,$$

where we use $a_{k+1} = 1 - b$ in the equality. Therefore, inequality (II.2) holds for $m = k + 1$. Using the same argument and noticing that $-\| . \|^2$ is $2$–weakly convex, we have that (II.3) also holds for $m = k + 1$. Hence, we obtain the desired results.

Proof of Lemma II.2. It is shown in [10, Proposition 1] that there exist $\eta$ such that

$$\left| \frac{\Phi(k, s)}{N} \right|^j \leq \frac{1}{N} \left| 1 + \frac{\eta - B_0}{1 - \eta B_0} (1 - \eta B_0)^{(k-s)/B_0} \right|.$$
for all $s$ and $k$ with $k \geq s$, where $[\Phi(k, s)]_{i,j}$ denotes the $i$-th row and $j$-th column element of $\Phi(k, s)$, $B_0 = (N - 1)B$ and $B$ is the intercommunication interval bound of Assumption 3. By using the matrix norm inequality

$$
\|A\|_F \leq \|A\|_\infty \leq N\|A\|_\infty
$$

for any symmetric real matrix $A \in \mathbb{R}^{N \times N}$, where $\|A\|_F$ is the Frobenius norm and $\|A\|_\infty = \max_{i,j} |[A]_{i,j}|$, we have the desired result, where $c = 2N^{1+\eta B_0}$ and $\lambda = (1 - \eta B_0)B_0^{-1}$.

Proof of Lemma II.3. Since $\lim_{T \to \infty} \gamma_T = 0$, there exists $M > 0$ such that $\gamma_k$ is uniformly bounded, i.e., $\gamma_k \leq M, \forall k \geq 0$. For each $T$, we have $\lambda^k \leq \gamma_T - 1$, for any $k \geq K_0(T) := \left\lceil \log \frac{\gamma_T - 1}{\log x} \right\rceil$. It follows that

$$
\begin{align*}
T-1 \sum_{k=0}^{K_0(T)-1} \lambda^k \gamma_T - k - 1 & = \sum_{k=0}^{K_0(T)-1} \lambda^k \gamma_T - k - 1 + \sum_{k=K_0(T)}^{T-1} \lambda^k \gamma_T - k - 1 \\
& \leq \frac{1}{1 - \lambda} \frac{\max_{0 \leq k \leq K_0(T)-1} \gamma_T - k - 1}{1 - \lambda} \cdot M
\end{align*}
$$

Recall $\sum_{T=0}^{\infty} \gamma_T = 0$ and $\sum_{T} \gamma_T = \infty$. It is clear that $K_0(T) = \left\lceil \log \frac{\gamma_T - 1}{\log x} \right\rceil = o(T)$, otherwise $\gamma_T - 1$ may be decreasing at least as a geometric sequence, which contradicts $\sum_{T=0}^{\infty} \gamma_T = 0$. Then, we have

$$
\lim_{T \to \infty} S_T = \frac{\gamma_T - 1}{\log x} = 1.
$$

Therefore, $\sum_{k=0}^{T-1} \lambda^k \gamma_T - k - 1 = O(\gamma_T - 1)$ holds for sufficiently large $T$ and thus we have the desired result.

Proof of Lemma II.4 and Lemma II.5. The inequality (II.8) is the same as [19, Lemma 8]. We provide the proof for completeness. Without loss of generality, we assume $n = 1$. Define

$$
\begin{align*}
x_k & = [x_{1,k}, x_{2,k}, \ldots, x_{N,k}], \\
v_k & = [v_{1,k}, v_{2,k}, \ldots, v_{N,k}], \\
e_k & = [e_{1,k}, e_{2,k}, \ldots, e_{N,k}],
\end{align*}
$$

where $e_{i,k} = \text{Proj}_{\mathcal{X}}(v_{i,k} - \alpha_k g_{i,k}) - \hat{v}_{i,k}$. The iteration (II.6) can be rewritten as

$$
x_{k+1} = v_k + e_k = A(k)x_k + e_k.
$$

That is, the iteration is split into a linear term $A(k)x_k$ and a nonlinear term $e_k$. Using Lemma II.6 and Assumption 5, it follows that

$$
\|e_i\| \leq \|v_i - \alpha_k g_i\| - \|v_i\| \leq \alpha_k^2 L^2.
$$

Therefore, we have

$$
\|e_k\| \leq \sqrt{NL\alpha_k}.
$$

Let $J = \frac{1}{l}11^T$, where $1 \in \mathbb{R}^N$ is a column vector with all elements 1. Then, $\Delta_k = x_k - Jx_k$. We have

$$
\begin{align*}
\Delta_{k+1} & = (I - J)x_{k+1} - (I - J)x_k \\
& = (I - J)A(k)x_k + (I - J)e_k \\
& = A(k)x_k - A(k)Jx_k + (I - J)e_k \\
& = A(k)\Delta_k + (I - J)e_k,
\end{align*}
$$

where the third equality is due to $JA(k) = J = A(k)J$. Therefore, the following recursion holds for $k \geq s \geq 0$

$$
\Delta_{k+1} = \Phi(k, s)\Delta_s + \sum_{l=s}^{k-1} \Phi(k, l+1)(I - J)e_l + (I - J)e_k.
$$

Since $1^T 1 = (I - J)e_1 = 0$, $\forall l$, we have

$$
\Delta_{k+1} = (\Phi(k, s) - J)\Delta_s + \sum_{l=s}^{k-1} (\Phi(k, l+1) - J)(I - J)e_l + (I - J)e_k.
$$

It follows from Lemma II.2 that there exist $c > 0$ and $\lambda \in (0, 1)$, where $\lambda$ is independent of $k$, such that

$$
\|\Delta_{k+1}\| \leq c\lambda^k\|\Delta_0\| + c\sqrt{NL}\sum_{l=0}^{k-1} \lambda^{k-l-1} \alpha_l + \sqrt{NL}\alpha_k.
$$

With the Lemma II.3, we have (II.9) as desired.

Proof of Lemma II.7. Let $1_{\mathcal{X}}(\cdot)$ be the indicator function of $\mathcal{X}$. Denote $y_1 = \text{prox}_f(x_1)$ and $y_2 = \text{prox}_f(x_2)$. From optimality condition, there exists $v_1 \in \partial f(y_1)$ and $v_2 \in \partial f(y_2)$ such that

$$
\begin{align*}
\frac{1}{t}(y_1 - x_1) + \partial f(y_1) + v_1 &= 0 \\
\frac{1}{t}(y_2 - x_2) + \partial f(y_2) + v_2 &= 0
\end{align*}
$$

It follows that

$$
\frac{1}{t}((y_1 - x_1) - (y_2 - x_2), y_1 - y_2)
$$

and

$$
\rho\|y_1 - y_2\|^2,
$$

where the last inequality is due to [1, Lemma 2.1] and the convexity of $1_{\mathcal{X}}(\cdot)$. Therefore, the claimed inequality follows from Cauchy-Schwarz inequality.

Proof of Lemma III.1. The following inequality holds because of the non-expansiveness of the projector

$$
\|x_{i,k+1} - \hat{v}_{i,k}\|^2 = \|\text{Proj}_{\mathcal{X}}(v_{i,k} - \alpha_k g_{i,k}) - \hat{v}_{i,k}\|^2
$$

$$
\leq \|v_{i,k} - \alpha_k g_{i,k} - \hat{v}_{i,k}\|^2
$$

$$
= \|v_{i,k} - \hat{v}_{i,k}\|^2 - 2\alpha_k\langle v_{i,k} - \hat{v}_{i,k}, g_{i,k}\rangle
$$

$$
+ \alpha_k^2\|g_{i,k}\|^2.
$$
Recall the weak convexity of $f_i$ and the boundedness of $g_{i,k}$. It follows that
\[
\|x_{i,k+1} - \hat{v}_{i,k}\|^2 \\
\leq \|v_{i,k} - \hat{v}_{i,k}\|^2 + 2\alpha_k(f_i(\hat{v}_{i,k}) - f_i(v_{i,k})) \\
+ \frac{\rho}{2}\|v_{i,k} - \hat{v}_{i,k}\|^2 + L^2\alpha_k^2.
\] (A.8)

Using the Lipschitz continuity of $f_i$ and Lemma II.7, we have
\[
f_i(\hat{v}_{i,k}) - f_i(v_{i,k}) \\
f_i(\hat{v}_{i,k}) - f_i(s_k) + f_i(s_k) - f_i(\bar{x}_k) + f_i(\bar{x}_k) - f_i(v_{i,k}) \\
\leq \frac{1}{t}\|v_{i,k} - s_k\| + f_i(s_k) - f_i(\bar{x}_k) + f_i(\bar{x}_k) - f_i(v_{i,k}) \\
\leq \frac{1}{t-t\rho} + 1)\|v_{i,k} - s_k\| + f_i(s_k) - f_i(\bar{x}_k) \\
\leq \frac{L(2-t\rho)}{1-t\rho} \sum_{j=1}^{N} a_{i,j}(k)\|x_{j,k} - \bar{x}_k\| + f_i(s_k) - f_i(\bar{x}_k) \\
\leq \frac{\rho}{2}\|v_{i,k} - \hat{v}_{i,k}\|^2 \\
+ \frac{\rho}{2}\|v_{i,k} - \hat{v}_{i,k}\|^2 + \alpha_k f_i(v_{i,k}) \\
\leq \rho\|\bar{x}_k - s_k\|^2 + \rho\|\bar{x}_k - \bar{x}_k + s_k - \hat{v}_{i,k}\|^2 \\
\leq \rho\|\bar{x}_k - s_k\|^2 + 2\rho(1 + \frac{1}{1-t\rho})\|v_{i,k} - \bar{x}_k\|^2 \\
\leq \rho\|\bar{x}_k - s_k\|^2 + 2\rho(1 + \frac{1}{1-t\rho}) \sum_{j=1}^{N} a_{i,j}(k)\|x_{j,k} - \bar{x}_k\|^2.
\] (A.9)

Summing inequalities (A.9) and (A.10) for $i = 1, \ldots, N$, yields
\[
\sum_{i=1}^{N} f_i(\hat{v}_{i,k}) - f_i(v_{i,k}) + \frac{\rho}{2}\|v_{i,k} - \hat{v}_{i,k}\|^2 \\
\leq \frac{L(2-t\rho)}{1-t\rho} \sum_{i=1}^{N} \|x_{i,k} - \bar{x}_k\| + N(f(s_k) - f(\bar{x}_k)) \\
+ \frac{N\rho}{2}\|\bar{x}_k - s_k\|^2 + 2\rho(1 + \frac{1}{1-t\rho}) \sum_{i=1}^{N} \|x_{i,k} - \bar{x}_k\|^2.
\] (A.11)

From the definition of $s_k$, if $t < \frac{1}{2\rho}$, one has
\[
f(s_k) - f(\bar{x}_k) + \rho\|\bar{x}_k - s_k\|^2 \\
= f(s_k) - f(\bar{x}_k) + \frac{(1/2t \cdot 1/2t + \rho)}{2t}\|\bar{x}_k - s_k\|^2 \\
\leq (\frac{1}{2t} + \rho)\|\bar{x}_k - s_k\|^2.
\] (A.12)

Therefore, we have (III.1) by combining (A.8), (A.11) and (A.12).
where the first inequality is because of the definition of \( \tilde{v}_{i,k} \) and \( \sum_{j=1}^{N} a_{i,j}(k) \tilde{x}_{j,k} \in X \), the second inequality follows from inequality (II.2) in Lemma II.1 and the convexity of \( \| \cdot \|^2 \) and the last inequality holds due to Lemma II.7. Letting \( \tilde{\varphi}_{t,k+1} := \frac{1}{N} \sum_{i=1}^{N} \varphi_t(x_{i,k+1}) \) together with (A.15) gives

\[
\begin{align*}
\tilde{\varphi}_{t,k+1} \leq & \tilde{\varphi}_{t,k} + \frac{\rho}{2N(1 - t \rho)^2} \sum_{i=1}^{N} \sum_{j=1}^{N-1} \sum_{l=j+1}^{N} a_{i,j}(k) a_{i,l}(k) \| x_{j,k} - x_{l,k} \|^2 \\
&+ \frac{\alpha_k}{t} \left( \rho - \frac{1}{2t} \right) \| \tilde{x}_k - s_k \|^2 \\
&+ \frac{L(2 - t \rho)}{N(1 - t \rho)} \sum_{i=1}^{N} \| x_{i,k} - \tilde{x}_k \| \\
&+ \frac{2\rho}{N} \left( 1 + \frac{1}{1 - t \rho} \right) \sum_{i=1}^{N} \| x_{i,k} - \tilde{x}_k \| \right) + \frac{L^2 \alpha_k^2}{2t}
\leq \tilde{\varphi}_{t,k} + b_k + \frac{L^2 \alpha_k^2}{2t},
\end{align*}
\]

where

\[
b_k := \frac{\rho}{2N(1 - t \rho)^2} \sum_{i=1}^{N} \sum_{j=1}^{N-1} \sum_{l=j+1}^{N} a_{i,j}(k) a_{i,l}(k) \| x_{j,k} - x_{l,k} \|^2 \\
+ \frac{\alpha_k}{t} \left( \rho - \frac{1}{2t} \right) \| \tilde{x}_k - s_k \|^2 \\
+ \frac{L(2 - t \rho)}{N(1 - t \rho)} \sum_{i=1}^{N} \| x_{i,k} - \tilde{x}_k \| \\
+ \frac{2\rho}{N} \left( 1 + \frac{1}{1 - t \rho} \right) \sum_{i=1}^{N} \| x_{i,k} - \tilde{x}_k \| \right).
\]

and the last inequality in (A.16) follows from \(-\frac{1}{2t} + \rho < 0\). By invoking Lemma II.5, we have

\[
\begin{align*}
\alpha_k & \sum_{i=1}^{N} \| x_{i,k} - \tilde{x}_k \| = O(\alpha_k^2), \\
\alpha_k & \sum_{i=1}^{N} \| x_{i,k} - \tilde{x}_k \| = O(\alpha_k^2)
\end{align*}
\]

(A.18)

and thus \( b_k = O(\alpha_k^2) \). Because \( f(x) \) is lower bounded on \( X \), we have \( \varphi_t(x) \) is also lower bounded on \( X \). From (A.16) it follows that

\[
\tilde{\varphi}_{t,k+1} - \inf_{X} \varphi_t(x) \leq \tilde{\varphi}_{t,k} - \inf_{X} \varphi_t(x) + O(\alpha_k^2).
\]

Since \( \sum_{k=0}^{\infty} \alpha_k^2 < \infty \), using Lemma 2\(^3\) in [39, Chapter 2.2] we have \( \{ \tilde{\varphi}_{t,k} \} \) converges to some value \( \tilde{\varphi}_t \).

Recall that \( \varphi_t(x) \) is continuous differentiable. Since \( \| x_{i,k} - \tilde{x}_k \| \to 0 \), it follows that

\[
|\tilde{\varphi}_{t,k} - \varphi_t(x_{i,k})|^2 \to 0
\]

and

\[
|\tilde{\varphi}_{t,k} - \varphi_t(x_{i,k})|^2 = \left| \frac{1}{N} \sum_{i=1}^{N} \varphi_t(x_{i,k}) - \varphi_t(\tilde{x}_k) \right|^2
\leq \frac{2}{N^2} \sum_{i=1}^{N} \left| \varphi_t(x_{i,k}) - \varphi_t(\tilde{x}_k) \right|^2
\to 0.
\]

Thus, \( \varphi_t(\tilde{x}_k) \to \tilde{\varphi}_t \).

(2). The inequality (A.16) can be re-written as

\[
\frac{\alpha_k}{t} \left( \frac{1}{2} - \rho \right) \| \tilde{x}_k - s_k \|^2 \leq \tilde{\varphi}_{t,k} - \tilde{\varphi}_{t,k+1} + b_k + \frac{L^2 \alpha_k^2}{2t},
\]

(A.20)

Using (A.20), we have

\[
\sum_{k=0}^{\infty} \alpha_k \frac{1}{t} - \rho \| \tilde{x}_k - s_k \|^2 \\leq \tilde{\varphi}_{t,0} - \tilde{\varphi}_t + \sum_{k=0}^{\infty} b_k + \sum_{k=0}^{\infty} \frac{L^2 \alpha_k^2}{2t}.
\]

Dividing both sides by \( \sum_{k=0}^{\infty} \alpha_k \) yields

\[
\inf_{k=1,2,...,\infty} \| \nabla \varphi_t(\tilde{x}_k) \|^2 \leq \frac{2}{1 - 2t \rho} \sum_{k=0}^{\infty} b_k + \sum_{k=0}^{\infty} \frac{L^2 \alpha_k^2}{2t}.
\]

If \( \alpha_k = O(1/\sqrt{k}) \), for sufficiently large \( T \) we have

\[
\inf_{k=1,2,...,T} \| \nabla \varphi_t(\tilde{x}_k) \|^2 \leq O\left( \frac{\log T}{\sqrt{T}} \right).
\]

Before proving Theorem III.5, we need the following technical lemma.

**Lemma A.1.** Given \( a > 0, 0 < 2b \leq a \) and \( c \geq 1 \), the lower bound of the minimum value in (\( P_N \)) is given by

\[
\min_{x_1, \ldots, x_N} \left\{ -\frac{1}{2} \sum_{i=1}^{N} (x_i^2 - 2bx_i) \right\}
\]

s.t. \( \sum_{i=1}^{N} x_i^2 \leq Na^2 \), \( 0 \leq x_i \leq ca \), \( \forall i \).

\(^3\)The lemma is stated as follows. Let \( u_{k+1} \geq 0 \) and let \( u_{k+1} \leq (1 + \alpha_k)u_k + \beta_k \), \( \sum_{k=0}^{\infty} \alpha_k < \infty \), \( \sum_{k=0}^{\infty} \beta_k < \infty \). Then \( u_k \to u \geq 0 \).
Proof of Lemma A.1. The dual function is given by
\[
g(\lambda) := \min_{0 \leq x \leq ca} -\frac{1}{2} \sum_{i=1}^{N} (a_i^2 - 2b_i x_i) + \lambda (\sum_{i=1}^{N} a_i^2 - Na^2),
\]
where \(\lambda \geq 0\). We have
\[
g(\lambda) = N \cdot \min_{0 \leq x \leq ca} \{(\lambda - \frac{1}{2}) x^2 + bx\} - \lambda Na^2
\]
\[
= \begin{cases} 
N \left[(\lambda - \frac{1}{2}) c^2 a^2 + cba \right] - \lambda Na^2 
& \text{if } \lambda \geq \frac{1}{2} - \frac{b}{ca}, \\
0 
& \text{otherwise}.
\end{cases}
\]
Note that \(\frac{1}{2} - \frac{b}{ca} \geq 0\). Therefore, we have
\[
\max_{\lambda \geq 0} g(\lambda) = g \left( \frac{1}{2} - \frac{b}{ca} \right) = \frac{1}{2} Na^2 + \frac{Nba}{c}.
\]
The weak duality implies the desired result.

**Proof of Theorem III.5.** We prove it by induction. By the definition of \(e_0\) and the assumptions on \(k = 0\), we have \(\sum_{i=1}^{N} \|x_{i,0} - x^*\|^2 \leq N\gamma e_0^2\) and \(\|x_{i,0} - x^*\|^2 \leq \eta e_0, \forall i \in [N] := \{1, \ldots, N\}\). Assume that (III.4) and (III.5) hold for \(k \geq 0\). For \(k + 1\), we have
\[
\sum_{i=1}^{N} \|x_{i,k+1} - x^*\|^2 
\leq \sum_{i=1}^{N} \|v_{i,k} - \alpha_k g_i x_{i,k} - x^*\|^2 
\leq \sum_{i=1}^{N} (\|v_{i,k} - x^*\|^2 - 2\alpha_k \langle v_{i,k} - x^*, g_i x_{i,k} \rangle) + NL_2^2 \alpha_k^2 
\leq \sum_{i=1}^{N} (\|v_{i,k} - x^*\|^2 - 2\alpha_k (f_i(x_{i,k}) - f_i(x^*)) + \alpha_k \rho \|v_{i,k} - x^*\|^2) + NL_2^2 \alpha_k^2 
= \sum_{i=1}^{N} (\|v_{i,k} - x^*\|^2 - 2\alpha_k (f_i(x_{i,k}) - f_i(x^*)) + f_i(x_{i,k}) - f_i(x^*)) + \alpha_k \rho \|v_{i,k} - x^*\|^2) + NL_2^2 \alpha_k^2 
\leq \sum_{i=1}^{N} (\|v_{i,k} - x^*\|^2 - 2\alpha_k \|v_{i,k} - x^*\|^2 + 2L \alpha_k \|v_{i,k} - \overline{x}_k\|) 
- 2N \beta \alpha_k \|\overline{x}_k - x^*\| + NL_2^2 \alpha_k^2,
\]
where the third inequality follows from the weak convexity and the last one is due to the sharpness property and Lipschitz continuity of \(f_i\). Using the convexity of \(\|\cdot\|^2\) and \(\|\cdot\|\) and the stochasticity of columns of \(A(k)\), we have
\[
\sum_{i=1}^{N} \|x_{i,k+1} - x^*\|^2 
\leq \sum_{i=1}^{N} ((1 + \rho \alpha_k) \|x_{i,k} - x^*\|^2 + 2L \alpha_k \|x_{i,k} - \overline{x}_k\|) 
- 2N \beta \alpha_k \|\overline{x}_k - x^*\| + NL_2^2 \alpha_k^2 
\leq \sum_{i=1}^{N} ((1 + \rho \alpha_k) \|x_{i,k} - x^*\|^2 + 2L \alpha_k \|x_{i,k} - x^*\|) 
+ 2L + \beta \alpha_k \sum_{i=1}^{N} \|\overline{x}_k - x_{i,k}\| + NL_2^2 \alpha_k^2
\]
\[
\leq \sum_{i=1}^{N} ((1 + \rho \alpha_k) \|x_{i,k} - x^*\|^2 + 2L \alpha_k \|x_{i,k} - x^*\|) 
+ 2\sqrt{N} (L + \beta) \alpha_k \sum_{i=1}^{N} \|\overline{x}_k - x_{i,k}\| + NL_2^2 \alpha_k^2
\]
\[+ \frac{2 \sqrt{N} (L + \beta) L}{\lambda^2} \left( \frac{\gamma^2 \beta - 1}{1 - \gamma^2 \beta - 1} + \lambda \right) \alpha_k^2 + NL_2^2 \alpha_k^2,
\]
(A.21)

where we use \(\|\overline{x}_k - x^*\| \geq \|x_{i,k} - x^*\| - \|\overline{x}_k - x_{i,k}\|\) and \(\|\cdot\|_1 \leq \sqrt{N} \|\cdot\|_{\infty}\) in the second inequality. The last inequality is due to (III.3). Recall the induction assumption that \(\sum_{i=1}^{N} \|x_{i,k} - x^*\|^2 \leq Ne_0^2 \gamma^{2k}\) and \(\|x_{i,k} - x^*\| \leq \eta e_0 \gamma^k\). Since
\[
\mu_0 \leq \frac{e_0}{2\beta - \rho \mu_0},
\]
we have \(2\beta \alpha_k \mu_0 = \frac{2\beta \alpha_k \mu_0}{1 + \rho \mu_0} \leq \eta e_0 \gamma^k\). By invoking Lemma A.1 (letting \(a = e_0 \gamma^k\), \(b = \frac{3 \beta \alpha_k}{1 + \rho \mu_0}\) and \(c = \eta\) in the lemma), we deduce that
\[
(1 + \rho \mu_0) \sum_{i=1}^{N} \left(\|x_{i,k} - x^*\|^2 - \frac{2\beta \alpha_k}{1 + \rho \mu_0} \|x_{i,k} - x^*\| \right) 
\leq (1 + \rho \mu_0) Ne_0^2 \gamma^{2k} - 2N \beta \alpha_k e_0 \gamma^{2k}\]
This, together with (A.21) yields
\[
\sum_{i=1}^{N} \|x_{i,k+1} - x^*\|^2 
\leq (1 + \rho \mu_0) Ne_0^2 \gamma^{2k} - 2N \beta \mu_0 e_0 \gamma^{2k}
\]
\[+ \frac{2 \sqrt{N} (L + \beta) \lambda}{\lambda^2} \left( \frac{\gamma^2 \beta - 1}{1 - \gamma^2 \beta - 1} + \lambda \right) \mu_0 (e_0 \gamma^{2k})^2 + NL_2^2 \mu_0 \gamma^{2k}\]
\[= N \gamma e_0^2 \left(1 + \left(\rho - \frac{2 \beta}{\eta e_0} + \frac{2 (L + \beta) \lambda}{\sqrt{N} \lambda e_0^2} \right) \mu_0 \right) + \frac{2 (L + \beta) L}{\lambda^2} \left( \frac{(\gamma^2 \beta - 1)}{1 - \gamma^2 \beta - 1} + \lambda \right) L^2 \mu_0 \gamma^{2k}\]
\[= N \gamma e_0^2 \left(1 - \frac{q}{e_0} \mu_0 + \frac{a \gamma^2 \beta - 1}{1 - \gamma^2 \beta - 1} + a \lambda + L^2 \mu_0 \right),\]
\[ a = \frac{2(L + \beta)c}{\lambda^2}, \quad q = \frac{2\beta}{\lambda} c_0 - \rho c_0^2 - \frac{2(L + \beta)c}{\gamma N} \| A_0 \| . \]

Since \( \gamma \in (0, 1) \), if we have the following two conditions:

1) \( q > 0 \)

2) \( 1 > \gamma^2 \geq 1 - \frac{q \mu_0}{c_0^2} + \frac{ac_0^{1/4 - 1}}{1 - \gamma^{1/\delta - 1}} + a \lambda + \frac{L^2}{e_0^2} \mu_0^2, \quad (A.23) \)

the result follows

\[ \sum_{i=1}^{N} \| x_{i,k+1} - x^* \|^2 \leq N \gamma^{2(k+1)} e_0^2. \]

Proof of Condition 1) Since \( e_0 \leq \frac{2\beta}{\rho \gamma} \) and

\[ \| \Delta_0 \| < \frac{2\beta e_0 - \rho c_0^2}{2(L + \beta)c} \lambda, \quad (A.24) \]

we have \( q > 0 \).

Proof of Condition 2) To ensure (A.23), it is sufficient to show

\[ 1 > \gamma^2 \geq 1 - \frac{q \mu_0}{10c_0^2 \sqrt{N}} + \frac{ac_0^{1/4 - 1}}{1 - \gamma^{1/\delta - 1}} + a \lambda + \frac{L^2}{e_0^2} \mu_0^2, \quad (A.25) \]

for some \( \gamma \in (0, 1) \), which is equivalent to

\[ -\gamma^{1/\delta + 1} + \gamma^2 + (1 - \frac{q \mu_0}{10c_0^2 \sqrt{N}} + \frac{ac_0^{1/4 - 1}}{1 - \gamma^{1/\delta - 1}} + a \lambda + \frac{L^2}{e_0^2} \mu_0^2) \geq 0, \]

if we multiply by \( (1 - \gamma^{1/\delta - 1}) \) and re-arrange the terms.

Consider the function

\[ \phi(\gamma) = -\gamma^{1/\delta + 1} + (1 - \frac{q \mu_0}{10c_0^2 \sqrt{N}} + \frac{ac_0^{1/4 - 1}}{1 - \gamma^{1/\delta - 1}} + a \lambda + \frac{L^2}{e_0^2} \mu_0^2) \gamma^{1/\delta - 1} + \gamma^2 - (1 - \frac{q \mu_0}{10c_0^2 \sqrt{N}} + \frac{ac_0^{1/4 - 1}}{1 - \gamma^{1/\delta - 1}} + a \lambda + \frac{L^2}{e_0^2} \mu_0^2). \]

Our goal is to find \( 1 > \delta > 0 \) such that \( \phi(\lambda^\delta) \geq 0 \) when \( \mu_0 > 0 \).

Letting \( \epsilon := -\frac{q \mu_0}{10c_0^2 \sqrt{N}} + \frac{ac_0^{1/4 - 1}}{1 - \gamma^{1/\delta - 1}} + a \lambda + \frac{L^2}{e_0^2} \mu_0^2 \), we have \( \epsilon < 0 \) since \( 0 < \mu_0 < \frac{1}{10(\alpha \lambda + L^2)^2} \sqrt{N} \) due to Assumption 7. By the same token we have

\[ -\frac{ac_0^{1/4 - 1}}{e_0^2} \mu_0^2 \geq \left( \frac{1}{\Lambda} - 1 \right) \epsilon, \quad \text{as} \quad \mu_0 \leq \frac{q}{10 \sqrt{N}(\alpha \lambda + L^2 + \frac{ac_0}{\gamma N})}. \quad (A.26) \]

Therefore, if \( 0 < \mu_0 \leq \frac{q}{10 \sqrt{N}(\alpha \lambda + L^2 + \frac{ac_0}{\gamma N})} \), we have

\[ (1 - \lambda^{2\delta} + \epsilon - \frac{ac_0 \mu_0^2}{e_0^2}) \lambda^{1-\delta} + \lambda^{2\delta} - (1 + \epsilon) \]

\[ \geq (1 - \lambda^{2\delta} + \frac{1}{\Lambda} \lambda^{1-\delta} + \lambda^{2\delta} - (1 + \epsilon) \]

\[ = (1 - \lambda^{2\delta})(\lambda^{1-\delta} - 1) + \frac{1}{\Lambda} \lambda^{1-\delta} \epsilon - \epsilon. \]

It is clear for every \( \lambda \in (0, 1) \) that

\[ (1 - \lambda^{2\delta})(\lambda^{1-\delta} - 1) + \frac{1}{\Lambda} \lambda^{1-\delta} \epsilon \to \frac{1}{\Lambda} \lambda \epsilon \quad \text{as} \quad \delta \to 0. \]

Therefore, there exists sufficiently small \( \delta > 0 \) such that \( \phi(\lambda^\delta) \geq \frac{1}{\Lambda} \lambda \epsilon - \epsilon > 0 \), since \( \Lambda > \lambda \) and \( \epsilon < 0 \).

Combining (A.22), (A.26) and (A.24), we have

\[ \sum_{i=1}^{N} \| x_{i,k+1} - x^* \|^2 \leq Ne_0^2 \gamma^{2k+2}, \]

if \( 0 < \mu_0 \leq \min\{ \frac{e_0^2}{\rho c_0^2}, \frac{q}{10 \sqrt{N}(\alpha \lambda + L^2 + \frac{ac_0}{\gamma N})} \} \), \( \gamma = \lambda^\delta \) and \( \| \Delta_0 \| < \frac{\beta e_0 - \rho c_0^2}{2(L + \beta)c} \lambda \).

Lastly, we need to verify (III.5) for \( k + 1 \). Since \( \| x_{i,k+1} - x^* \|^2 \leq \frac{\gamma}{10} \sum_{i=1}^{N} \| x_{i,k+1} - x^* \|^2 \leq e_0^2 \gamma^{2k+2} \), it follows from (III.3) that

\[ \| x_{i,k+1} - x^* \|^2 \leq \| x_{i,k+1} - \bar{x}_{k+1} \| + \| \bar{x}_{k+1} - x^* \| \]

\[ \leq \| \Delta_{k+1} \| + e_0 \gamma^{k+1} \]

\[ \leq \left( \frac{e_0 \| \Delta_0 \|}{\lambda^2} + \frac{\sqrt{N}L}{\lambda^2} \left( c \frac{\gamma^{\frac{\lambda}{\gamma^2}}}{1 - \gamma} + \lambda \right) \mu_0 \right)^{\gamma^{k+1} + e_0 \gamma^{k+1}} \]

Using (A.25), one has

\[ (c \frac{\gamma^{\frac{\lambda}{\gamma^2}}}{1 - \gamma} + \lambda) \mu_0 < \frac{q}{10 \sqrt{N} a} \leq \frac{\beta e_0}{5 \sqrt{N} a} \]

Therefore, we have

\[ \| x_{i,k+1} - x^* \|^2 \leq \left( \frac{\beta e_0 - \rho c_0^2}{2(L + \beta)c} \lambda \right)^{\gamma^{k+1} + e_0 \gamma^{k+1}}. \]

Since \( \| \Delta_0 \| < \left( \frac{\beta e_0 - \rho c_0^2}{2(L + \beta)c} \right)^2 \lambda \), it follows that

\[ \| x_{i,k+1} - x^* \|^2 \leq \left( \frac{\beta e_0 - \rho c_0^2}{2(L + \beta)c} + \frac{\beta}{10(L + \beta)c_0} \right)^{\gamma^{k+1} + e_0 \gamma^{k+1}} \]

\[ \leq \left( \frac{1}{2} + \frac{21}{20} \right) e_0 \gamma^{k+1}, \]

where the second inequality follows from \( \beta \leq L \) and the last inequality holds since \( \eta \geq \sqrt{\gamma} \).

Proof of Theorem III.6. The proof is quite similar to that of algorithm (I.2). We explain the main steps below.

First, we have the same consensus lemma as Lemma II.4.
Substituting $z = \hat{v}_{i,k}$ into (A.13) and taking expectation conditioned on $k$, we obtain
\[
\mathbb{E} \varphi_1(x_{i,k+1}) \\
\leq f(\hat{v}_{i,k}) + \frac{1}{2} \mathbb{E} \left[ \|x_{i,k+1} - \hat{v}_{i,k}\|^2 \right] \\
\leq f(\hat{v}_{i,k}) + \frac{1}{2} \mathbb{E} \left[ \|\hat{v}_{i,k} - \hat{v}_{i,k} + \hat{v}_{i,k} - v_{i,k} - \xi_{i,k}\|^2 \right] + \frac{\alpha_k}{k} \mathbb{E} \left[ \|\xi_{i,k}\|^2 \right] \\
\leq \varphi_1(v_{i,k}) - \frac{\alpha_k}{k} \langle \hat{v}_{i,k} - \hat{v}_{i,k}, g_{i,k} \rangle + \frac{\alpha_k}{k} L^2. \\
\tag{A.27}
\]
Then, the remaining parts of the proof are the same as that of Theorem III.2. 

REFERENCES

[1] D. Davis and D. Drusvyatskiy, “Stochastic model-based minimization of weakly convex functions,” SIAM Journal on Optimization, vol. 29, no. 1, pp. 207–239, 2019.
[2] D. Davis, D. Drusvyatskiy, K. J. MacPhee, and C. Paquette, “Subgradient methods for sharp weakly convex functions,” Journal of Optimization Theory and Applications, vol. 179, no. 3, pp. 962–982, 2018.
[3] S. Shahrampour, A. Rakhlin, and A. Jadbabaie, “Distributed detection: Finite-time analysis and impact of network topology,” IEEE Transactions on Automatic Control, vol. 61, no. 11, pp. 3256–3268, 2016.
[4] S. Kar, J. M. Moura, and K. Ramanan, “Distributed parameter estimation in sensor networks: Nonlinear observation models and imperfect communication,” IEEE Transactions on Information Theory, vol. 58, no. 6, pp. 3575–3605, 2012.
[5] N. Atanasov, R. Tron, V. M. Preciado, and G. J. Pappas, “Joint estimation and localization in sensor networks,” in 53rd IEEE Conference on Decision and Control, pp. 6875–6882, IEEE, 2014.
[6] G. Mateos and G. B. Giannakis, “Distributed recursive least-squares: Stability and performance analysis,” IEEE Transactions on Signal Processing, vol. 60, no. 7, pp. 3740–3754, 2012.
[7] F. Bullo, J. Cortes, and S. Martinez, Distributed control of robotic networks: a mathematical approach to motion coordination algorithms. Princeton University Press, 2009.
[8] S. Bolognani, R. Carli, G. Cavaro, and S. Zampieri, “Distributed reactive power feedback control for voltage regulation and loss minimization,” IEEE Transactions on Automatic Control, vol. 60, no. 4, pp. 966–981, 2014.
[9] D. Acemoglu, A. Nedic, and A. Ozdaglar, “Convergence of rule-of-thumb learning rules in social networks,” in 2008 47th IEEE Conference on Decision and Control, pp. 1714–1720, IEEE, 2008.
[10] A. Nedic and A. Ozdaglar, “Distributed subgradient methods for multi-agent optimization,” IEEE Transactions on Automatic Control, vol. 54, no. 1, pp. 48–61, 2009.
[11] D. Davis, D. Drusvyatskiy, and C. Paquette, “The nonsmooth landscape of phase retrieval,” IMA Journal of Numerical Analysis, 2020.
[12] J. C. Duchi and F. Ruan, “Solving (most) of a set of quadratic inequalities: Composite optimization for robust phase retrieval,” Information and Inference: A Journal of the IMA, vol. 8, no. 3, pp. 471–529, 2019.
[13] Y. C. Eldar and S. Mendelson, “Phase retrieval: Stability and recovery guarantees,” Applied and Computational Harmonic Analysis, vol. 36, no. 3, pp. 473–494, 2014.
[14] T. F. Chan and C.-K. Wong, “Total variation blind deconvolution,” IEEE transactions on Image Processing, vol. 7, no. 3, pp. 370–375, 1998.
[15] A. Levin, Y. Weiss, F. Durban, and W. T. Freeman, “Understanding blind deconvolution algorithms,” IEEE transactions on pattern analysis and machine intelligence, vol. 33, no. 12, pp. 2354–2367, 2011.
[16] S. Ling and T. Strohmer, “Self-calibration and biconvex compressive sensing,” Inverse Problems, vol. 31, no. 11, p. 115002, 2015.
[17] M. Abadi, P. Barham, J. Chen, Z. Chen, and A. e. a. Davis, “Tensorflow: A system for large-scale machine learning,” in 12th USENIX Symposium on Operating Systems Design and Implementation, 2016.
[18] H. Zhang, Z. Zheng, S. Xu, W. Dai, Q. Ho, X. Liang, Z. Hu, J. Wei, P. Xie, and E. Xing, “Poseidon: An efficient communication architecture for distributed deep learning on gpu clusters,” in USENIX Annual Technical Conference, 2017.
[19] A. Nedic, A. Ozdaglar, and P. A. Parrilo, “Constrained consensus and optimization in multi-agent networks,” IEEE Transactions on Automatic Control, vol. 55, no. 4, pp. 922–938, 2010.
[20] S. Liu, Z. Qiu, and L. Xie, “Convergence rate analysis of distributed optimization with projected subgradient algorithm,” Automatica, vol. 83, pp. 162–169, 2017.
[21] S. S. Ram, A. Nedić, and V. V. Veeravalli, “Distributed stochastic subgradient projection algorithms for convex optimization,” Journal of optimization theory and applications, vol. 147, no. 3, pp. 516–545, 2010.
[22] K. Yuan, Q. Ling, and W. Yin, “On the convergence of decentralized gradient descent,” SIAM Journal on Optimization, vol. 26, no. 3, pp. 1835–1854, 2016.
[23] A. Nedić, A. Olshovsky, and M. G. Rabbat, “Network topology and communication-computation tradeoffs in decentralized optimization,” Proceedings of the IEEE, vol. 106, no. 5, pp. 953–976, 2018.
[24] P. Bianchi and J. Jakubowicz, “Convergence of a multi-agent projected stochastic gradient algorithm for non-convex optimization,” IEEE transactions on automatic control, vol. 58, no. 2, pp. 391–405, 2012.
[25] J. Zeng and W. Yin, “On nonconvex decentralized gradient descent,” IEEE Transactions on signal processing, vol. 66, no. 11, pp. 2834–2848, 2018.
[26] D. Davis and B. Grimmer, “Proximally guided stochastic subgradient methods for nonsmooth, nonconvex problems,” SIAM Journal on Optimization, vol. 29, no. 3, pp. 1908–1930, 2019.
[27] X. Li, Z. Zhu, A. M.-C. So, and J. D. Lee, “Incremental methods for weakly convex optimization,” arXiv preprint arXiv:1907.11687, 2019.
[28] J.-P. Vial, “Strong and weak convexity of sets and functions,” Mathematics of Operations Research, vol. 8, no. 2, pp. 231–259, 1983.
[29] R. Poliquin and R. Rockafellar, “Prox-regular functions in variational analysis,” Transactions of the American Mathematical Society, vol. 348, no. 5, pp. 1805–1838, 1996.
[30] R. Rockafellar, Convex Analysis. Princeton: Princeton University Press, 1970.
[31] J. N. Tsitsiklis, “Problems in decentralized decision making and computation,” tech. rep., Massachusetts Inst of Tech Cambridge Lab for Information and Decision Systems, 1984.
[32] R. Rockafellar and R. J.-B. Wets, Variational analysis, vol. 317. Springer Science & Business Media, 2009.
[33] D. Drusvyatskiy and C. Paquette, “Efficiency of minimizing compositions of convex functions and smooth maps,” Mathematical Programming, vol. 178, no. 1-2, pp. 503–558, 2019.
[34] B. T. Polyak, “Minimization of unsmooth functionals,” USSR Computational Mathematics and Mathematical Physics, vol. 9, no. 3, pp. 14–29, 1969.
[35] J.-L. Goffin, “On convergence rates of subgradient optimization methods,” Mathematical programming, vol. 13, no. 1, pp. 329–347, 1977.
[36] Y. LeCun, L. Bottou, Y. Bengio, and P. Haffner, “Gradient-based learning applied to document recognition,” Proceedings of the IEEE, vol. 86, no. 11, pp. 2278–2324, 1998.
[37] C. Xi and U. A. Khan, “Distributed subgradient projection algorithm over directed graphs,” *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 3986–3992, 2016.

[38] X. Li, S. Chen, Z. Deng, Q. Qu, Z. Zhu, and A. M. C. So, “Weakly convex optimization over stiefel manifold using riemannian subgradient-type methods,” *arXiv preprint arXiv:1911.05047*, 2019.

[39] B. T. Polyak, “Introduction to optimization. optimization software,” *Inc., Publications Division, New York*, vol. 1, 1987.