AUTOMORPHISMS OF INFINITE JOHNSON GRAPH

MARK PANKOV

Abstract. We consider the infinite Johnson graph $J_\infty$ whose vertex set consists of all subsets $X \subset \mathbb{N}$ satisfying $|X| = |\mathbb{N} \setminus X| = \infty$ and whose edges are pairs of such subsets $X, Y$ satisfying $|X \setminus Y| = |Y \setminus X| = 1$. An automorphism of $J_\infty$ is said to be regular if it is induced by a permutation on $\mathbb{N}$ or it is the composition of the automorphism induced by a permutation on $\mathbb{N}$ and the automorphism $X \mapsto \mathbb{N} \setminus X$. The graph $J_\infty$ admits non-regular automorphisms.

Our first result states that the restriction of every automorphism of $J_\infty$ to any connected component ($J_\infty$ is not connected) coincides with the restriction of a regular automorphism. The second result is a characterization of regular automorphisms of $J_\infty$ as order preserving and order reversing bijective transformations of the vertex set of $J_\infty$ (the vertex set is partially ordered by the inclusion relation). As an application, we describe automorphisms of the associated infinite Kneser graph.

1. Introduction

1.1. Classical Grassmann and Johnson graphs. Let $V$ be an $n$-dimensional vector space (over a division ring) and $n < \infty$. The Grassmann graph $\Gamma_k(V)$ is the graph whose vertex set is the Grassmannian $G_k(V)$ formed by all $k$-dimensional subspaces of $V$ and whose edges are pairs of $k$-dimensional subspaces with $(k-1)$-dimensional intersections (in what follows, two vertices of a graph joined by an edge will be called adjacent). The graph $\Gamma_k(V)$ is connected. By duality, $\Gamma_k(V)$ is isomorphic to $\Gamma_{n-k}(V^*)$ ($V^*$ is the vector space dual to $V$). Classical Chow’s theorem \cite{5} states that every automorphism of $\Gamma_k(V)$, $1 < k < n - 1$, is induced by a semilinear automorphism of $V$ or a semilinear isomorphism of $V$ to $V^*$; the second possibility can be realized only in the case when $n = 2k$. If $k = 1, n - 1$ then any two distinct vertices of $\Gamma_k(V)$ are adjacent and any bijective transformation of the vertex set is an automorphism of $\Gamma_k(V)$. We refer \cite{11} for more information concerning Grassmann graphs.

The Johnson graph $J(n, k)$ is formed by all $k$-element subsets of $\{1, \ldots, n\}$, two such subsets are adjacent if their intersection consists of $k - 1$ elements. This graph admits a natural isometric embedding in $\Gamma_k(V)$. Consider a base $B$ of the vector space $V$ and the subset of $\mathcal{G}_k(V)$ formed by all $k$-dimensional subspaces spanned by subsets of $B$. Subsets of such type are called apartments of $\mathcal{G}_k(V)$ (see \cite{11} for motivations of this term). Every apartment of $\mathcal{G}_k(V)$ is the image of an isometric embedding of $J(n, k)$ in $\Gamma_k(V)$. However, the image of every isometric embedding of $J(n, k)$ in $\Gamma_k(V)$ is an apartment of $\mathcal{G}_k(V)$ only in the case when $n = 2k$. This follows from the classification of isometric embeddings of Johnson graphs $J(l, m)$ in $\Gamma_l(V)$ \cite{12}. The graphs $J(n, k)$ and $J(n, n - k)$ are isomorphic: the mapping $*$ transferring every subset $X \subset \{1, \ldots, n\}$ to the complement $\{1, \ldots, n\} \setminus X$ defines

2000 Mathematics Subject Classification. 05C63, 05C50.
an isomorphism between these graphs (in the case when \( n = 2k \), this is an automorphism of \( J(n, k) \)). It is not difficult to prove that every automorphism of \( J(n, k) \) is induced by a permutation on \( \{1, \ldots, n\} \) or \( n = 2k \) and it is the composition of the automorphism \( \ast \) and the automorphism induced by a permutation on \( \{1, \ldots, n\} \) (an analog of Chow’s theorem).

So, we can say that \( J(n, k) \) is a "thin prototype" of \( \Gamma_k(V) \). Different characterizations of Grassmann and Johnson graphs can be found in \([6, 9, 10]\), see also Sections 9.1 and 9.3 in \[3\].

1.2. Grassmann graphs of infinite-dimensional vector spaces. Now, suppose that \( V \) is a vector space of infinite dimension \( \aleph_0 \). Grassmannians of \( V \) can be defined as the orbits of the action of the linear group \( \text{GL}(V) \) on the set of all proper subspaces of \( V \). By \([11]\), there are the following three types of Grassmannians:

- \( G_k(V) \) formed by all subspaces of dimension \( k \in \mathbb{N} \),
- \( G^k(V) \) formed by all subspaces of codimension \( k \in \mathbb{N} \),
- \( G_\infty(V) \) formed by all subspaces of infinite dimension and codimension.

Let \( G \) be one of these Grassmannians. We say that \( S, U \in G \) are adjacent if

\[
\dim(S/(S \cap U)) = \dim(U/(S \cap U)) = 1.
\]

The associated Grassmann graph, it will be denoted by \( \Gamma_k(V), \Gamma^k(V) \) or \( \Gamma_\infty(V) \) (respectively), is the graph whose vertex set is \( G \) and whose edges are pairs of adjacent elements.

The graph \( \Gamma_k(V) \) is connected and every automorphism of \( \Gamma_k(V) \) is induced by a semilinear automorphism of \( V \) \([11]\). By duality (see, for example, \([1, 11]\)), \( \Gamma^k(V) \) is canonically isomorphic to \( \Gamma_k(V^*) \). Thus \( \Gamma^k(V) \) is connected and every automorphism of \( \Gamma^k(V) \) is induced by a semilinear automorphism of \( V^* \). The graph \( \Gamma_\infty(V) \) is not connected. It admits automorphisms whose restrictions to distinct connected components are induced by distinct semilinear isomorphisms \([2]\). There is the following open problem \([8]\).

**Problem.** Describe the restrictions of automorphisms of \( \Gamma_\infty(V) \) to connected components.

The idea used to prove Chow’s theorem can not be exploited by many reasons, for example, by the fact that the vector spaces \( V \) and \( V^* \) are non-isomorphic (\( \dim V < \dim V^* \)) \([1]\).

1.3. In this paper, a weak version of this problem will be solved. We consider the infinite Johnson graph \( J_\infty \) — a thin prototype of \( \Gamma_\infty(V) \). The vertex set of \( J_\infty \) is formed by all subsets \( X \subseteq \mathbb{N} \) satisfying \( |X| = |\mathbb{N} \setminus X| = \infty \), two such subsets \( X, Y \) are adjacent if

\[
|X \setminus Y| = |Y \setminus X| = 1.
\]

There is a natural isometric embedding of \( J_\infty \) in \( \Gamma_\infty(V) \): for any infinite linearly independent subset \( B \subseteq V \) consider the restriction of the graph \( \Gamma_\infty(V) \) to the set formed by all elements of \( G_\infty(V) \) spanned by subsets of \( B \).

An automorphism of \( J_\infty \) will be called regular if it is induced by a permutation on \( \mathbb{N} \) or it is the composition of the automorphism induced by a permutation on \( \mathbb{N} \) and the automorphism \( X \rightarrow \mathbb{N} \setminus X \). The graph \( J_\infty \) is not connected and admits non-regular automorphisms (a simple modification of the example from \([2]\)). Our first result (Theorem 1) states that the restriction of every automorphism of \( J_\infty \) to
any connected component of \( J_\infty \) coincides with the restriction of a regular automorphism. The vertex set of \( J_\infty \) is partially ordered by the inclusion relation. The second result (Theorem 2 and Corollary 1) is a characterization of regular automorphisms of \( J_\infty \) as order preserving and order reversing bijective transformations of the vertex set. As an application of Theorem 2, we show that every automorphism of the associated infinite Kneser graph \( K_\infty \) is induced by a permutation on \( \mathbb{N} \).

Some general information concerning automorphisms of graphs can be found in [4].

\section{Infinite Johnson graphs}

\subsection{Definition}

Our definition of infinite Johnson graphs is similar to the definition of Grassmann graphs of infinite-dimensional vector spaces given in the previous section.

Denote by \( S_\infty \) the group of all permutations on \( \mathbb{N} \) and consider the action of this group on the set of all proper subsets of \( \mathbb{N} \). The associated orbits are of the following three types:

1. the set consisting of all \( X \subset \mathbb{N} \) such that \( |X| = k \) (\( k \) is a given natural number),
2. the set consisting of all \( X \subset \mathbb{N} \) such that \( |\mathbb{N} \setminus X| = k \) (as in the previous case, \( k \) is a given natural number),
3. the set consisting of all \( X \subset \mathbb{N} \) such that \( |X| = |\mathbb{N} \setminus X| = \infty \).

Let \( J \) be one of these orbits. We say that \( X,Y \in J \) are adjacent if

\[ |X \setminus Y| = |Y \setminus X| = 1 \]

(in the case (1), this condition is equivalent to the equality \( |X \cap Y| = k - 1 \)). The associated Johnson graph, we will denote it by \( J_k, J^k \) or \( J_\infty \) (respectively), is the graph whose vertex set is \( J \) and whose edges are pairs of adjacent elements.

\subsection{Some remarks on \( J_k \) and \( J^k \)}

The mapping * transferring every subset \( X \subset \mathbb{N} \) to the complement \( \mathbb{N} \setminus X \) defines an isomorphism between \( J^k \) and \( J_k \). The structure of \( J_k \) is rather similar to the structure of finite Johnson graphs. This graph is connected. The distance between \( X,Y \in J_k \) is equal to \( |X \setminus Y| = |Y \setminus X| \) and the diameter of \( J_k \) is \( k \). Maximal cliques of \( J_k \) are the following two types:

- the star \( St(A), A \in J_{k-1} \), consisting of all vertices of \( J_k \) containing \( A \),
- the top \( T(B), B \in J_{k+1} \), consisting of all vertices of \( J_k \) contained in \( B \).

Every automorphism \( f \) of \( J_k \) preserves the class of maximal cliques (stars and tops). Every top consists of precisely \( k + 1 \) vertices and every star contains an infinite number of vertices; this means that stars go to stars and tops go to tops. In particular, \( f \) induces a bijective transformation of the vertex set of \( J_{k-1} \). This transformation is an automorphism of \( J_{k-1} \), since two stars in \( J_k \) have a non-zero intersection (consisting of precisely one vertex) if and only if the associated vertices of \( J_{k-1} \) are adjacent. So, \( f \) induces an automorphism of \( J_{k-1} \). Step by step, we come to a permutation on \( \mathbb{N} \) (an automorphism of \( J_1 \)). This permutation induces \( f \). Now, suppose that \( g \) is an automorphism of \( J^k \). Then \( h = * g * \) is an automorphism of \( J_k \). Hence \( h \) is induced by a permutation \( s \in S_\infty \) and an easy verification shows that \( g = * h * \) also is induced by \( s \). Therefore, \textit{all automorphisms of the Johnson graphs} \( J_k \) and \( J^k \) are induced by permutations on \( \mathbb{N} \).
2.3. Basic properties of \( J_\infty \). The graph \( J_\infty \) is not connected. The connected component containing \( X \in J_\infty \) will be denoted by \( J(X) \); it consists of all \( Y \in J_\infty \) satisfying

\[
|X \setminus Y| = |Y \setminus X| < \infty.
\]

Any two connected components of \( J_\infty \) are isomorphic (every permutation on \( \mathbb{N} \) induces an automorphism of \( J_\infty \), we consider a permutation transferring \( X \in J_\infty \) to \( Y \in J_\infty \), the associated automorphism of \( J_\infty \) sends \( J(X) \) to \( J(Y) \)).

The graph \( J_\infty \) contains an infinite number of connected components. If \( X \in J_\infty \) and \( A \) is a finite subset of \( X \) then \( X \setminus A \) is a vertex of \( J_\infty \) which does not belong to \( J(X) \). So, \( X, Y \in J_\infty \) belong to distinct connected components if they are incident subsets of \( \mathbb{N} (X \subset Y \text{ or } Y \subset X) \).

Let \( X \in J_\infty \). The star \( St(X) \) consists of all \( Y \in J_\infty \) containing \( X \) and satisfying \( |Y \setminus X| = 1 \). Similarly, the top \( T(X) \) is formed by all \( Y \in J_\infty \) contained in \( X \) and such that \( |X \setminus Y| = 1 \). Clearly, \( St(X) \) and \( T(X) \) both are maximal cliques of \( J_\infty \) and it is easy to see that every maximal clique of \( J_\infty \) is a star or a top.

The automorphisms of \( J_\infty \) induced by permutations on \( \mathbb{N} \) map stars to stars and tops to tops. The automorphism \( * \) (sending every \( X \in J_\infty \) to \( \mathbb{N} \setminus X \)) transfers stars to tops and tops to stars.

3. Automorphisms of \( J_\infty \)

3.1. Main results. Recall that an automorphism of \( J_\infty \) is regular if it is induced by a permutation on \( \mathbb{N} \) or it is the composition of the automorphism \( * \) and the automorphism induced by a permutation on \( \mathbb{N} \). Note that \( *f = f* \) for every automorphism \( f \) of \( J_\infty \) induced by a permutation on \( \mathbb{N} \).

The vertex set of \( J_\infty \) is partially ordered by the inclusion relation. We say that a bijective transformation \( f \) of the vertex set is order preserving or order reversing if it satisfies the condition

\[
X \subset Y \iff f(X) \subset f(Y) \quad \forall X, Y \in J_\infty
\]

or the condition

\[
X \subset Y \iff f(Y) \subset f(X) \quad \forall X, Y \in J_\infty,
\]

respectively. Every automorphism of \( J_\infty \) induced by a permutation on \( \mathbb{N} \) is order preserving. The automorphism \( * \) is order reversing. Therefore, every regular automorphism of \( J_\infty \) is order preserving or order reversing; in particular, all regular automorphisms of \( J_\infty \) preserve the incidence relation.

Now we modify the example from [2] mentioned above and establish the existence of non-regular automorphisms of \( J_\infty \).

Example 1. Let \( A \in J_\infty \) and \( B \) be a vertex of the connected component \( J(A) \). We take any permutation \( s \in S_\infty \) sending \( A \) to \( B \). This permutation preserves \( J(A) \) and we define

\[
f(X) := \begin{cases} 
  s(X) & X \in J(A) \\
  X & X \in J_\infty \setminus J(A).
\end{cases}
\]

This is an automorphism of \( J_\infty \). We choose \( Y \in J_\infty \) which is a proper subset of \( A \) non-incident with \( B \). It is clear that \( Y \notin J(A) \), thus \( f(Y) = Y \). This means that \( f \) does not preserve the incidence relation (\( A \) and \( Y \) are incident, but \( f(A) = B \) and \( f(Y) = Y \) are non-incident). Therefore, the automorphism \( f \) is non-regular.
Our main result is the following.

**Theorem 1.** The restriction of every automorphism of $J_\infty$ to any connected component of $J_\infty$ coincides with the restriction of a regular automorphism to this connected component.

The second result is a characterization of regular automorphisms.

**Theorem 2.** Every order preserving bijective transformation of the vertex set of $J_\infty$ is the automorphism of $J_\infty$ induced by a permutation on $\mathbb{N}$.

Observe that for every order reversing bijective transformation $f$ of the vertex set of $J_\infty$ the mapping $^*f$ is order preserving. Thus, as a direct consequence of Theorem 2, we get the following characterization of regular automorphisms of $J_\infty$.

**Corollary 1.** The group of all regular automorphisms of $J_\infty$ coincides with the group formed by all order preserving and order reversing bijective transformations of the vertex set of $J_\infty$.

### 3.2. Application: automorphisms of the infinite Kneser graph

Recall that the Kneser graph $K(n, k)$ and the Johnson graph $J(n, k)$ have the same vertex set; two vertices of $K(n, k)$ are adjacent if they are disjoint subsets of $\{1, \ldots, n\}$ (here we assume that $k < n - k$). Every automorphism of $K(n, k)$ is induced by a permutation on $\{1, \ldots, n\}$. This follows from the Erdős–Ko–Rado theorem; see Section 7.8 in [7].

Consider the infinite Kneser graph $K_\infty$ corresponding to the Johnson graph $J_\infty$. The vertex set of this graph coincides with the vertex set of $J_\infty$ and two vertices of $K_\infty$ are adjacent if they are disjoint subsets of $\mathbb{N}$. This graph is a thin prototype of so-called distant graph defined for a vector space of dimension $\aleph_0$ [2]. It is not difficult to prove that $K_\infty$ is a connected graph of diameter 3.

**Corollary 2.** Every automorphism of $K_\infty$ is induced by a permutation on $\mathbb{N}$.

**Proof.** For every $X \in K_\infty$ denote by $X^o$ the set of all vertices of $K_\infty$ adjacent with $X$. If $X, Y \in K_\infty$ then

$$X \subset Y \iff Y^o \subset X^o.$$  

This implies that every automorphism of $K_\infty$ is an order preserving transformation of the vertex set of $K_\infty$. Since $K_\infty$ and $J_\infty$ have the same vertex set, Theorem 2 gives the claim. \(\square\)

### 4. Proof of Theorem 1

Let $A \in J_\infty$ and $f$ be the restriction of an automorphism of $J_\infty$ to the connected component $J(A)$. Then $f(J(A))$ is a connected component of $J_\infty$. It is clear that $f$ transfers maximal cliques of $J_\infty$ (stars and tops) contained in $J(A)$ to maximal cliques contained in $f(J(A))$.

**Lemma 1.** One of the following possibilities is realized:

(A) $f$ transfers stars to stars and tops to tops,

(B) $f$ transfers stars to tops and tops to stars.

**Proof.** We will use the following facts:
• The intersection of two distinct stars \( St(X) \) and \( St(Y) \) is empty or contains precisely one vertex; the second possibility is realized only in the case when \( X, Y \) are adjacent vertices of \( J_\infty \). The same holds for the intersection of two distinct tops.

• The intersection of a star \( St(X) \) and a top \( T(Y) \) is empty or consists of precisely two vertices; the second possibility is realized only in the case when \( X \subset Y \) and \( |Y \setminus X| = 2 \).

The proof is a direct verification.

Suppose that \( J(A) \) contains a star \( St(X), X \in J_\infty \) such that \( f(St(X)) \) is a star. Consider any \( Y \in J_\infty \) adjacent with \( X \). We choose \( Z \in J_\infty \) satisfying \( X \cup Y \subset Z \) and \( |Z \setminus (X \cup Y)| = 1 \).

Then \( |St(X) \cap T(Z)| = |St(Y) \cap T(Z)| = 2 \) and \( |f(St(X)) \cap f(T(Z))| = |f(St(Y)) \cap f(T(Z))| = 2 \).

Since \( St(X) \) goes to a star, the latter equality guarantees that \( f(T(Z)) \) is a top and \( f(St(Y)) \) is a star.

So, \( f(St(Y)) \) is a star for every \( Y \in J_\infty \) adjacent with \( X \). Now consider an arbitrary \( Y \in J_\infty \) such that the star \( St(Y) \) is contained in \( J(A) \). We take any \( C_0 \in St(X), C \in St(Y) \) and consider a path \( C_0, C_1, \ldots, C_i = C \) in \( J(A) \) (a path joining \( C_0 \) and \( C \) exists, since \( J(A) \) is a connected component). Then \( X, C_0 \cap C_1, C_1 \cap C_2, \ldots, C_{i-1} \cap C_i, Y \)

is a path in \( J_\infty \) (possible \( X \) coincides with \( C_0 \cap C_1 \) or \( C_{i-1} \cap C_i \) coincides with \( Y \)). It was established above that \( St(C_0 \cap C_1) \) goes to a star. Then, by the same arguments, the image of \( St(C_1 \cap C_2) \) is a star. Step by step, we get that \( f(St(Y)) \) is a star. Similarly, we establish that tops go to tops.

If \( f \) transfers every star to a top then the same arguments show that tops go to stars. \( \square \)

**Proposition 1.** In the case (A), \( f \) is induced by a permutation on \( \mathbb{N} \), i.e. there exists \( s \in S_\infty \) such that

\[
 f(U) = s(U) \quad \forall U \in J(A).
\]

Proposition 1 will be proved in two steps — Lemmas 2 and 3. In each of these lemmas, we assume that \( f \) satisfies (A).

For every \( X \in J_\infty \) we denote by \( X^\sim \) the set consisting of \( X \) and all vertices of \( J_\infty \) adjacent with \( X \).

**Lemma 2.** For every \( X \in J(A) \) the restriction of \( f \) to \( X^\sim \) is induced by a permutation on \( \mathbb{N} \).

**Proof.** We can suppose that \( f(X) = X \) (otherwise, we consider \( tf \), where \( t \in S_\infty \) transfers \( f(X) \) to \( X \)). In this case, the restriction of \( f \) to \( X^\sim \) is a bijective transformation of \( X^\sim \).
A star $St(U)$ is contained in $X^\sim$ if and only if
\[(1) \quad U \subset X \quad \text{and} \quad |X \setminus U| = 1.\]
Thus $f$ defines a permutation on the set of all $U$ satisfying $\text{(1)}$. By Subsection 2.2, this permutation is induced by a certain permutation $s$ on $X$.

Now we extend $s : X \to X$ to a permutation on $\mathbb{N}$. Let $n \in \mathbb{N} \setminus X$. We choose $Y \in X^\sim$ containing $n$. Since $n \not\in X$, we have $X \neq Y$ and $X,Y$ are adjacent. This means that $n$ is unique element of $Y \setminus X$. Since $f(Y)$ and $f(X) \neq X$ are adjacent, $f(Y) \setminus X$ consists of precisely one element. We denote this number by $s(n)$.

Show that our definition of $s(n)$ does not depend on $Y$. Let us take any $Z \in X^\sim \setminus \{Y\}$ containing $n$. Since $Y$ and $Z$ both are adjacent to $X$, we have
\[|X \setminus (X \cap Y)| = |X \setminus (X \cap Z)| = 1.\]
If $X \cap Y$ coincides with $X \cap Z$ then $Y = Z$ (recall that $n$ belongs to both $Y, Z$ and $n \not\in X$). Therefore, $X \cap Y$ and $X \cap Z$ are adjacent vertices of $J_{\infty}$. The latter guarantees that $Y$ and $Z$ are adjacent. Thus
\[(2) \quad f(Y) = \{s(n)\} \cup (X \cap f(Y)) \quad \text{and} \quad f(Z) = \{n'\} \cup (X \cap f(Z))\]
are adjacent (here $n'$ is unique element of $f(Z) \setminus X$). Note that
\[(3) \quad X \cap f(Y) \neq X \cap f(Z).\]
Indeed, the equality
\[X \cap f(Y) = X \cap f(Z)\]
implies the existence of a star containing
\[f(X) = X, f(Y), f(Z);\]
however, there is no star containing $X, Y, Z$ (these vertices are contained in a top). Since $f(Y)$ and $f(Z)$ are adjacent, $\text{(2)}$ and $\text{(3)}$ show that $s(n) = n'$.

It is clear that $s : \mathbb{N} \to \mathbb{N}$ is a permutation on $\mathbb{N}$ and $f(U) = s(U)$ for every $U \in X^\sim$. \hfill \square

So, for every $X \in J(A)$ there is a permutation $s_X \in S_{\infty}$ such that
\[f(U) = s_X(U) \quad \forall U \in X^\sim.\]

**Lemma 3.** If $X, Y \in J(A)$ are adjacent then $s_X = s_Y$.

**Proof.** Suppose that
\[X = \{n\} \cup (X \cap Y) \quad \text{and} \quad Y = \{m\} \cup (X \cap Y).\]
We can assume that $f(X) = X$ and $f(Y) = Y$. Indeed, in the general case we have
\[f(X) = \{n'\} \cup (f(X) \cap f(Y)), \quad f(Y) = \{m'\} \cup (f(X) \cap f(Y))\]
and consider $tf$, where $t \in S_{\infty}$ transfers $n', m'$ and $f(X) \cap f(Y)$ to $n, m$ and $X \cap Y$, respectively.

It is easy to see that
\[X^\sim \cap Y^\sim = St(X \cap Y) \cup T(X \cup Y).\]
We have
\[s_X(X \cap Y) = s_X(X) \cap s_X(Y) = f(X) \cap f(Y) = X \cap Y.\]
Similarly, we get
\[s_Y(X \cap Y) = X \cap Y.\]
Then \( s_X(n) = s_Y(n) = n \) and \( s_X(m) = s_Y(m) = m \).

Let \( k \in \mathbb{N} \setminus (X \cap Y) \). Then
\[
U := \{k\} \cup (X \cap Y) \in \text{St}(X \cap Y) \subset X^\sim \cap Y^\sim
\]
and
\[
s_X(U) = \{s_X(k)\} \cup (X \cap Y), \quad s_Y(U) = \{s_Y(k)\} \cup (X \cap Y).
\]
The equality
\[
s_X(U) = f(U) = s_Y(U)
\]
shows that \( s_X(k) = s_Y(k) \).

Let \( k \in X \cap Y \). Then
\[
W := \{n, m\} \cup [(X \cap Y) \setminus \{k\}] \in T(X \cup Y) \subset X^\sim \cap Y^\sim
\]
and
\[
s_X(W) = \{n, m\} \cup [(X \cap Y) \setminus \{s_X(k)\}], \quad s_Y(W) = \{n, m\} \cup [(X \cap Y) \setminus \{s_Y(k)\}].
\]
The equality
\[
s_X(W) = f(W) = s_Y(W)
\]
implies that \( s_X(k) = s_Y(k) \). \( \square \)

By connectedness, Lemma 3 guarantees that \( s_X = s_Y \) for all \( X, Y \in J(A) \).

Proposition 1 is proved.

In the case (B), the mapping \( *f \) transfers stars to stars and tops to tops; hence it is induced by a permutation on \( \mathbb{N} \). Thus \( f \) is the composition of \( * \) and the mapping induced by a permutation on \( \mathbb{N} \).

5. Proof of Theorem 2

Let \( f \) be an order preserving bijective transformation of the vertex set of \( J_\infty \).

**Lemma 4.** Let \( \{X_i\}_{i \in I} \) be a family of vertices of \( J_\infty \) (possible infinite) such that
\[
X := \bigcap_{i \in I} X_i \in J_\infty
\]
Then
\[
f(X) = \bigcap_{i \in I} f(X_i).
\]

**Proof.** Since \( f \) is order preserving, \( f(X) \) is contained in every \( f(X_i) \) and we have (4)
\[
f(X) \subset \bigcap_{i \in I} f(X_i).
\]
The inclusion
\[
f(X) \subset \bigcap_{i \in I} f(X_i) \subset f(X_i)
\]
and the fact that \( f(X), f(X_i) \) are vertices of \( J_\infty \) guarantee that
\[
X' := \bigcap_{i \in I} f(X_i) \in J_\infty.
\]
The inverse mapping \( f^{-1} \) is order preserving and \( f^{-1}(X') \) is contained in every \( X_i \).

Thus
\[
f^{-1}(X') \subset X \quad \text{and} \quad X' \subset f(X).
\]
By (4), \( f(X) \subset X' \). Therefore, \( f(X) = X' \). \( \square \)
Lemma 5. If $X, Y \in J_\infty$, $Y \subset X$ and $|X \setminus Y| = 1$ then

$$|f(X) \setminus f(Y)| = 1.$$ 

Proof. It is clear that $f(Y)$ is a proper subset of $f(X)$. If $|f(X) \setminus f(Y)| > 1$ then there exists $Z \in J_\infty \setminus \{f(X), f(Y)\}$ such that

$$f(Y) \subset Z \subset f(X).$$

Then

$$Y \subset f^{-1}(Z) \subset X.$$ 

Since $|X \setminus Y| = 1$, the latter inclusions mean that $f^{-1}(Z)$ coincides with $X$ or $Y$, a contradiction. \qed

Lemma 6. For every $X \in J_\infty$ there exists a bijective mapping $s : X \to f(X)$ such that $f(Y) = s(Y)$ for every $Y \in J_\infty$ contained in $X$.

Proof. We restrict ourself to the case then $f(X) = X$ (in the general case, we consider the mapping $tf$ with $t \in S_\infty$ transferring $f(X)$ to $X$). Denote by $\mathcal{X}$ the set of all $Y \subset X$ satisfying $|X \setminus Y| = 1$. All elements of $\mathcal{X}$ are vertices of $J_\infty$ and, by Lemma 5, $f$ defines a permutation on $\mathcal{X}$. By Subsection 2.2, this permutation is induced by a certain permutation $s$ on $X$, i.e.

$$f(Y) = s(Y) \quad \forall Y \in \mathcal{X}.$$ 

Every $Y \in J_\infty$ contained in $X$ can be presented as the intersection of a family $\{Y_i\}_{i \in I}$ of elements from $\mathcal{X}$ (possible infinite). Then

$$f(Y) = \bigcap_{i \in I} f(Y_i) = \bigcap_{i \in I} s(Y_i) = s(Y)$$

(the first equality follows from Lemma 4). \qed

For every $n \in \mathbb{N}$ denote by $[n]$ the set of all vertices of $J_\infty$ containing $n$.

Lemma 7. For every $n \in \mathbb{N}$ there exists $s(n) \in \mathbb{N}$ such that

$$f([n]) = [s(n)].$$

Proof. We take $Y_1, Y_2 \in J_\infty$ satisfying

$$Y_1 \cup Y_2 \in J_\infty \quad \text{and} \quad Y_1 \cap Y_2 = \{n\}.$$ 

Let $X := Y_1 \cup Y_2$. Lemma 5 implies the existence of a bijection $s : X \to f(X)$ such that $f(Y) = s(Y)$ for every $Y \in J_\infty$ contained in $X$. Then

$$f(Y_1) \cap f(Y_2) = s(Y_1) \cap s(Y_2) = s(Y_1 \cap Y_2) = \{s(n)\}.$$ 

We show that the number $s(n)$ is as required.

Let $Z \in [n]$. If $Z$ has an infinite intersection with $X$ then $Z \cap X$ is an element of $[n]$ contained in $X$ and

$$f(Z \cap X) = s(Z \cap X)$$

contains $s(n)$. The inclusion

$$f(Z \cap X) \subset f(Z)$$

guarantees that $f(Z) \in [s(n)]$.

Suppose that $Z \cap X$ is finite. In this case, we decompose $Z \setminus X$ in the disjoint union of two infinite subsets $A, B$ and define

$$T := (Z \cap X) \cup A.$$
Then $T \in J_\infty$; moreover,

$$n \in T \subset Z \text{ and } X' := X \cup T \in J_\infty.$$ 

By Lemma, there exists a bijection $s' : X' \to f(X')$ such that $f(Y) = s'(Y)$ for every $Y \in J_\infty$ contained in $X'$. Since

$$Y_1 \cap Y_2 \cap T \neq \emptyset$$

(this intersection contains $n$), we have

$$f(Y_1) \cap f(Y_2) \cap f(T) = s'(Y_1) \cap s'(Y_2) \cap s'(T) = s'(Y_1 \cap Y_2 \cap T) \neq \emptyset.$$ 

On the other hand,

$$f(Y_1) \cap f(Y_2) = \{s(n)\}.$$ 

Hence $f(T)$ contains $s(n)$ and the inclusion $f(T) \subset f(Z)$ implies that $f(Z)$ belongs to $[s(n)]$.

So, we obtain that $f([n]) \subset [s(n)]$. Applying the same arguments to the transformation $f^{-1}$, we get the inverse inclusion. □

The mapping $n \to s(n)$ is a permutation on $\mathbb{N}$ and $f$ is the automorphism of $J_\infty$ induced by this permutation.

REFERENCES

[1] Baer R., Projective Geometry and Linear Algebra, Academic Press, New York 1952.
[2] Blunck A., Havlicek H., On bijections that preserve complementarity of subspaces, Discrete Math. 301(2005), 46-56.
[3] Brouwer A.E., Cohen A.M., Neumaier A., Distance-Regular Graphs, Springer-Verlag, New York, 1989.
[4] Cameron P.J., Automorphisms of graphs, Topics in Algebraic Graph Theory, Encyclopedia of Mathematics and Its Applications 102, Cambridge University Press, 2005.
[5] Chow W.L., On the geometry of algebraic homogeneous spaces, Ann. of Math. 50(1949), 32-67.
[6] Fu T.-S., Huang T., A unified approach to a characterization of Grassmann graphs and bilinear form graphs, European J. Combin. 15(1994), 363-373.
[7] Godsil C., Royle G., Algebraic graph theory, Graduate Texts in Math. 207, Springer-Verlag, New York, 2001.
[8] Havlicek H., Private communication, 2003.
[9] Metsch K., A characterization of Grassmann graphs, European J. Combin. 16(1995), 639-644.
[10] Numata M., A characterization of Grassmann and Johnson graphs, J. Combin. Theory, Ser. B 48(1990), 178-190.
[11] Pankov M., Grassmannians of classical buildings, Algebra and Discrete Math. Series 2, World Scientific, 2010.
[12] Pankov M., Isometric embeddings of Johnson graphs in Grassmann graphs, J. Algebraic Combin., accepted (see online first articles).

Department of Mathematics and Informatics, University of Warmia and Mazury, Żolnierka 14A, 10-561 Olsztyn, Poland

E-mail address: markpankov@gmail.com, pankov@matman.uwm.edu.pl