PHASE TRANSITION FOR MODELS WITH CONTINUUM SET OF SPIN VALUES ON BETHE LATTICE

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Abstract. In this paper we consider models with nearest-neighbor interactions and with the set 
[0,1] of spin values, on a Bethe lattice (Cayley tree) of an arbitrary order. These models depend
on parameter $\theta$. We describe all of Gibbs measures in any right parameter $\theta$ corresponding to the
models.

Mathematics Subject Classifications (2010). 82B05, 82B20 (primary); 60K35 (second-
ary)
Key words. Cayley tree, spin value, Gibbs measures, Hammerstein’s equation, fixed point.

1. Introduction

Spin models on a graph or in a continuous spaces form a large class of systems considered
in statistical mechanics. Some of them have a real physical meaning, others have been proposed
as suitably simplified models of more complicated systems. The geometric structure of the graph
or a physical space plays an important role in such investigations. For example, in order to
study the phase transition problem on a cubic lattice $Z^d$ or in space one uses, essentially, the
Pirogov- Sinai theory; see [11] and [12]. A general methodology of phase transitions in $Z^d$ or $R^d$
was developed in [10]. On the other hand, on a Cayley tree of order $k$ one uses the theory of
Markov splitting random fields based upon the corresponding recurrent equations. In particular,
in Refs [1], [2], [13] and [16] Gibbs measures on $\Gamma_k$ have been described in terms of solutions to
the recurrent equations.

During last five years, an increasing attention was given to models with a continuum set of
spin values on a Cayley tree. Until now, one considered nearest-neighbor interactions ($J_3 = J =
\alpha = 0$, $J_1 \neq 0$) with the set of spin values $[0,1]$. The following results was achieved: splitting
Gibbs measures on a Cayley tree of order $k$ are described by solutions to a nonlinear integral
equation. For $k = 1$ (when the Cayley tree becomes a one-dimensional lattice $Z^1$) it has been
shown that the integral equation has a unique solution, implying that there is a unique Gibbs
measure. For a general $k$, a sufficient condition was found under which a periodic splitting Gibbs
measure is unique (see [6], [8], [15] and [14]).

In [7] on a Cayley tree $\Gamma_k$ of order $k \geq 2$, phase transitions were proven to exist i.e., it was
given examples of Hamiltonian of model which there exists phase transitions. Afterwards, in [9]
it was generalized the examples on $\Gamma_2$. There are some examples of models with continuum set of
spin values which there exists a phase transition on a Cayley tree of some order (see [4], [5], [7], [9]).
In [3] it was considered a model with nearest-neighbor interactions and with the set $[0,1]$ of spin
values, on a Cayley tree of order two. This model depends on two parameters $n \in \mathbb{N}$ and $\theta \in [0,1].$
Author proved that if $0 \leq \theta \leq \frac{2n+3}{2(2n+1)}$, then for the model there exists a unique translationally-invariant Gibbs measure; If $\frac{2n+3}{2(2n+1)} < \theta < 1$, then there are three translationally-invariant Gibbs
measures (i.e. phase transition occurs).
In this paper we consider models which include all of examples in [3], [7], [9] on a Cayley tree of an arbitrary order. Also we describe all of Gibbs measures corresponding to the models.

2. Preliminaries

Denote that on the bottom definitions and known results are given short. The reader can read detail in [14].

A Cayley tree $\Gamma_k = (V, L)$ of order $k \in \mathbb{N}$ is an infinite homogeneous tree, i.e., a graph without cycles, with exactly $k + 1$ edges incident to each vertices. Here $V$ is the set of vertices and $L$ that of edges (arcs). Two vertices $x$ and $y$ are called nearest neighbors if there exists an edge $l \in L$ connecting them. We will use the notation $l = (x, y)$. The distance $d(x, y), x, y \in V$, on the Cayley tree is defined by the formula

$$d(x, y) = \min\{d| x = x_0, x_1, ..., x_{d-1}, x_d = y \in V \text{ such that the pairs }$$

$$\langle x_0, x_1 \rangle, ..., \langle x_{d-1}, x_d \rangle \text{ are neighboring vertices}\}.$$

Let $x^0 \in V$ be fixed and set

$$W_n = \{x \in V \mid d(x, x^0) = n\}, \quad V_n = \{x \in V \mid d(x, x^0) \leq n\},$$

$$L_n = \{l = (x, y) \in L \mid x, y \in V_n\}.$$

The set of the direct successors of $x$ is denoted by $S(x)$, i.e.

$$S(x) = \{y \in W_{n+1} \mid d(x, y) = 1\}, \quad x \in W_n.$$

We observe that for any vertex $x \neq x^0$, $x$ has $k$ direct successors and $x^0$ has $k + 1$. Vertices $x$ and $y$ are called second neighbors, which fact is marked as $\langle x, y \rangle$, if there exist a vertex $z \in V$ such that $x, z$ and $y, z$ are nearest neighbors. We will consider only second neighbors $\langle x, y \rangle$, for which there exist $n$ such that $x, y \in W_n$. Three vertices $x$, $y$ and $z$ are called a triple of neighbors in which case we write $\langle x, y, z \rangle$, if $\langle x, y \rangle$, $\langle y, z \rangle$ are nearest neighbors and $x, z \in W_n$, $y \in W_{n-1}$, for some $n \in \mathbb{N}$.

Consider models where the spin takes values in the set $[0, 1]$, and is assigned to the vertexes of the tree. For $A \subset V$ a configuration $\sigma_A$ on $A$ is an arbitrary function $\sigma_A : A \mapsto [0, 1]$. Denote $\Omega_A = [0, 1]^A$ the set of all configurations on $A$. A configuration $\sigma$ on $V$ is then defined as a function $x \in V \mapsto \sigma(x) \in [0, 1]$; the set of all configurations is $[0, 1]^V$.

The (formal) Hamiltonian of the model is:

$$H(\sigma) = -J \sum_{\langle x, y \rangle \in L} \xi_{\sigma(x), \sigma(y)}, \quad (2.1)$$

where $J \in \mathbb{R} \setminus \{0\}$ and $\xi : (u, v) \in [0, 1]^2 \mapsto \xi_{u,v} \in \mathbb{R}$ is a given bounded, measurable function.

Let $h : x \in V \mapsto h_x = (h_{t,x}, t \in [0, 1]) \in \mathbb{R}^{[0,1]}$ be mapping of $x \in V \setminus \{x^0\}$. Given $n = 1, 2, \ldots$, consider the probability distribution $\mu^{(n)}$ on $\Omega_{V_n}$ defined by

$$\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp \left( -\beta H(\sigma_n) + \sum_{x \in W_n} h_{\sigma(x), x} \right), \quad (2.2)$$
Here, as before, \( \sigma_n : x \in V_n \mapsto \sigma(x) \) and \( Z_n \) is the corresponding partition function:

\[
Z_n = \int_{\Omega_{V_n}} \exp \left( -\beta H(\bar{\sigma}_n) + \sum_{x \in W_n} h_{\bar{\sigma}(x),x} \right) \lambda_{V_n}(d\bar{\sigma}_n). \tag{2.3}
\]

The probability distributions \( \mu^{(n)} \) are compatible if for any \( n \geq 1 \) and \( \sigma_{n-1} \in \Omega_{V_{n-1}} \):

\[
\int_{\Omega_{W_n}} \mu^{(n)}(\sigma_{n-1} \vee \omega_n) \lambda_{W_n}(d(\omega_n)) = \mu^{(n-1)}(\sigma_{n-1}). \tag{2.4}
\]

Here \( \sigma_{n-1} \vee \omega_n \in \Omega_{V_n} \) is the concatenation of \( \sigma_{n-1} \) and \( \omega_n \). In this case there exists a unique measure \( \mu \) on \( \Omega_V \) such that, for any \( n \) and \( \sigma_n \in \Omega_{V_n} \),

\[
\mu \left( \left\{ \frac{\sigma}{\sigma_{V_n}} = \sigma_n \right\} \right) = \mu^{(n)}(\sigma_n).
\]

**Definition 2.1.** The measure \( \mu \) is called splitting Gibbs measure corresponding to Hamiltonian (2.1) and function \( x \mapsto h_x, x \neq x^0 \).

The following statement describes conditions on \( h_x \) guaranteeing compatibility of the corresponding distributions \( \mu^{(n)}(\sigma_n) \).

**Proposition 2.2.** [14] The probability distributions \( \mu^{(n)}(\sigma_n), n = 1, 2, \ldots, \) in (2.2) are compatible iff for any \( x \in V \setminus \{ x^0 \} \) the following equation holds:

\[
f(t, x) = \prod_{y \in S(x)} \frac{\int_0^1 \exp(J \beta \xi_{tu}) f(u, y) du}{\int_0^1 \exp(J \beta \xi_{0u}) f(u, y) du}.
\tag{2.5}
\]

Here, and below \( f(t, x) = \exp(h_{ix} - h_{0x}), t \in [0, 1] \) and \( du = \lambda(du) \) is the Lebesgue measure.

3. Main results

Let

\[
C^+[0, 1] = \{ f \in C[0, 1] : f(x) \geq 0 \}.
\]

For every \( k \in \mathbb{N} \) we consider an integral operator \( H_k \) acting in the cone \( C^+[0, 1] \) as

\[
(H_k f)(t) = \int_0^1 K(t, u) f^k(u) du, \quad k \in \mathbb{N}.
\tag{3.1}
\]

The operator \( H_k \) is called Hammerstein’s integral operator of order \( k \). This operator is well known to generate ill-posed problems. Clearly, if \( k \geq 2 \) then \( H_k \) is a nonlinear operator.

It is known that the set of translational invariant Gibbs measures of the model (2.1) is described by the fixed points of the Hammerstein’s operator (see [3]).

For \( k \geq 2 \) in the model (2.1) and

\[
\xi_{t,u} = \xi_{t,u}(\theta, \beta) = \frac{1}{J \beta} \ln \left( 1 + \theta^{2n+1} \sqrt{4(t - \frac{1}{2})(u - \frac{1}{2})} \right), \quad t, u \in [0, 1]
\]

where \(-4^{\frac{1}{2n+1}} < \theta < 4^{\frac{1}{2n+1}}\). The for the Kernel \( K(t, u) \) of the Hammerstein’s operator \( H_k \) we have

\[
K(t, u) = 1 + \theta^{2n+1} \sqrt{4(t - \frac{1}{2})(u - \frac{1}{2})}.
\tag{3.2}
\]

Let \( t - \frac{1}{2} = x \) and \( u - \frac{1}{2} = y \), we get

\[
K(x, y) = 1 + \theta^{2n+1} \sqrt{xy}, \quad x, y \in [-0.5, 0.5].
\tag{3.3}
\]
We defined the operator $V_k : (x, y) \in R^2 \to (x', y') \in R^2$ by

$$
V_k : \begin{cases}
  x' = \frac{(2n+1)!k}{2} \sum_{i=0}^{2n+1} \frac{(-1)^i}{(2n-i)!k^{(k+i+1)!}} \frac{(x+2n+\sqrt{20y})^{k+1+i}-(-1)^{i}(x-2n+\sqrt{20y})^{k+1+i}}{2n+\sqrt{20y}}; \\
y' = \frac{(2n+1)^2(2n)!}{2^{2n+\sqrt{20y}}} \sum_{i=0}^{2n+1} \frac{(-1)^i}{(2n-i)!k^{(k+i+1)!}} \frac{(x+2n+\sqrt{20y})^{k+1+i}+(-1)^{i}(x-2n+\sqrt{20y})^{k+1+i}}{2n+\sqrt{20y}}.
\end{cases}
$$

(3.4)

**Lemma 3.1.** A function $\varphi \in C[0, 1]$ is a solution of the Hammerstein’s equation

$$
(H_k f)(t) = f(t)
$$

(3.5)

iff $\varphi(t)$ has the following form

$$
\varphi(t) = C_1 + C_2\theta^{2n+1} \sqrt{4(t - \frac{1}{2})},
$$

where $(C_1, C_2) \in R^2$ is a fixed point of the operator $V_k$ (3.4).

For $k = 2s$, $s \in N$, we denote following notations

$$
\alpha_{2i} = \frac{2n+1}{2n+2i+1} \left( \frac{1}{2} \right)^{\frac{n+i}{2n+1}}, \quad \beta_i = C_{2s}^{i}\alpha_{i+1}, \quad i, n \in N;
$$

$$
\theta_{2i+1} = \frac{(2i+1)(2n+2i+3)4^{\frac{1}{2n+1}}}{(2s-2i)(2n+2i+1)}.
$$

For $k = 2s+1$, $s \in N$, we denote following notations

$$
\alpha_{2i} = \frac{2n+1}{2n+2i+1} \left( \frac{1}{2} \right)^{\frac{n+i}{2n+1}}, \quad \beta_i = C_{2s+1}^{i+1}\alpha_{i+1}, \quad i, n \in N;
$$

$$
\theta_{2i+1} = \frac{(2i+1)(2s+2i+3)4^{\frac{1}{2n+1}}}{(2s-2i+1)(2n+2i+1)}.
$$

**Remark 3.2.** We consider the following function $\theta_x = \frac{x^2+(2n+2)x}{(x+2n+1)(x+2n+2)(x+2n+3)+2n} > 0$ we can conclude that

$$
\theta_1 < \theta_3 < \theta_5 < ... < \theta_{2s-1}.
$$

(3.6)

**Lemma 3.3.** Let $k = 2s$, $s \in N$. If the point $\gamma(x_0, y_0) \in R^+_2$ is a fixed point of (3.4), then $\gamma \in R^+_2$ and $\lambda = \frac{y}{x}$ is a root of the following equation

$$
P(\lambda) := \beta_1(\theta_1 - \theta) + \beta_3(\theta_3 - \theta)\lambda^2 + ... + \beta_{2s-1}(\theta_{2s-1} - \theta)\lambda^{2s-2} + \alpha_{2s}\lambda^{2s} = 0.
$$

(3.7)

**Proof.** Let $(x_0, y_0)$ is a fixed point of (3.4). Now, we divide the second part to first part of system (3.4) then we get following

$$
\frac{\lambda}{\theta} = \frac{C_{2s}^{1}\alpha_{2}\lambda + C_{2s}^{2}\alpha_{3}\lambda^2 + ... + C_{2s}^{2s}\alpha_{2s+1}\lambda^{2s}}{1 + C_{2s}^{1}\alpha_{1}\lambda + C_{2s}^{2}\alpha_{2}\lambda^2 + ... + C_{2s}^{2s}\alpha_{2s}\lambda^{2s}}
$$

(3.8)

where $\lambda = \frac{y}{x}$.

Let $\lambda \neq 0$. After some abbreviations we get

$$
1 - C_{2s}^{2}\alpha_{2}\theta + (C_{2s}^{2}\alpha_{2} - C_{2s}^{3}\alpha_{3}\theta)\lambda^2 + ... + (C_{2s}^{2s-2}\alpha_{2s-2} - C_{2s}^{2s-1}\alpha_{2s-1}\theta)\lambda^{2s-2} + C_{2s}^{2s}\alpha_{2s}\lambda^{2s} = 0.
$$

(3.9)
Namely,
\[ \beta_1(\theta_1 - \theta) + \beta_3(\theta_3 - \theta)\lambda^2 + \ldots + \beta_{2s-1}(\theta_{2s-1} - \theta)\lambda^{2s-2} + \alpha_{2s}\lambda^{2s} = 0, \]  
(3.10)
where \( \theta_{2i-1} = \frac{C_{2i-2}^2 \lambda - \alpha_{2i}}{C_{2i-1}^2 \lambda - \alpha_{2i}} \).

It is easy to see if \( \lambda = 0 \) then this solution corresponding to solution (1, 0) of (3.4).
\[ \square \]

Analogously, we get the following Lemma

**Lemma 3.4.** Let \( k = 2s + 1, s \in \mathbb{N} \). If the point \( \gamma(x_0, y_0) \in R^+_2 \) is a fixed point of (3.4), then \( \gamma \in \mathbb{R}^+ \) and \( \lambda = \frac{y}{x} \) is a root of the following equation
\[ Q(\lambda) := \beta_1(\theta_1 - \theta) + \beta_3(\theta_3 - \theta)\lambda^2 + \ldots + \beta_{2s-1}(\theta_{2s-1} - \theta)\lambda^{2s-2} + \beta_{2s+1}(\theta_{2s+1} - \theta)\lambda^{2s} = 0. \]  
(3.11)

**Proposition 3.5.** Let \( k = 2s, s \in \mathbb{N} \).

a) If \( \theta \leq \theta_1 \), then there is no non-trivial solution of (3.7);

b) If \( \theta > \theta_1 \), then there is exactly two (non-trivial) solutions of (3.7). These solutions are opposing.

**Proof.** Case a) of the Proposition is clearly.

b) Number of sign changes of coefficients of \( P(\lambda) \) is equal to 1. Then \( P(\lambda) \) has at most one positive solution. The second hand side we have \( P(0) < 0 \) and \( \lim_{\lambda \to \infty} P(\lambda) = +\infty \). Then by Roll’s theorem \( P(\lambda) \) has at least one positive solution. Thus, there exist \( \lambda^* > 0 \) such that \( P(\lambda^*) = 0 \). Since \( P(\lambda) \) is an even function there is only one negative solution, i.e., \( -\lambda^* \).
\[ \square \]

**Proposition 3.6.** Let \( k = 2s + 1, s \in \mathbb{N} \).

a) If \( \theta \leq \theta_1 \), then there is no non-trivial solution of (3.11);

b) If \( \theta > \theta_1 \), then there is exactly two (non-trivial) solutions of (3.11). These solutions are opposing.

**Proof.** Proof of Proposition 3.6 is similar to proof of Proposition 3.5
\[ \square \]

**Proposition 3.7.** Let \( k = 2s, s \in \mathbb{N} \).

a) Let \( -4^{\frac{1}{2n+1}} < \theta \leq \theta_1 \). Then (3.1) has only one positive fixed point: \( f(t) = 1 \).

b) Let
\[ \sum_{i=1}^{s} \frac{2^{2i-2} \beta_{2i-1} \theta_{2i-1} + \alpha_{2i}2^{2i+1}}{2^{2i+1} \beta_{2i-1}} \leq \theta \leq 4^{\frac{1}{2n+1}}. \]

Then (3.1) has exactly two positive fixed points: \( f_1(t) = 1, f_2(t) = \bar{C}(1 + \lambda^* t^{\frac{1}{2n+1}}) \), where \( \lambda^* \) is a positive solution (3.7).

c) Let
\[ \theta_1 < \theta < \sum_{i=1}^{s} \frac{2^{2i-2} \beta_{2i-1} \theta_{2i-1} + \alpha_{2i}2^{2i+1}}{2^{2i+1} \beta_{2i-1}}. \]

Then (3.1) has exactly three positive fixed points: \( f_1(t) = 1, f_2(t) = \bar{C}(1 + \lambda^* t^{\frac{1}{2n+1}}), f_3(t) = \bar{C}(1 - \lambda^* t^{\frac{1}{2n+1}}) \), where \( \lambda^* \) is a positive solution (3.7).

**Proof.** We’ll prove that case b (case a and c are similarly). From
\[ \theta \geq \sum_{i=1}^{s} \frac{2^{2i-2} \beta_{2i-1} \theta_{2i-1} + \alpha_{2i}2^{2i+1}}{2^{2i+1} \beta_{2i-1}} > \theta_1 \]
has exactly 3 solutions. They are \( \lambda_1 = 0, \lambda_2 = \lambda^* \) and \( \lambda_3 = -\lambda^* \). By definition of \( f(t) \) we get following solutions: \( f_1(t) = 1, f_2(t) = C_1(1 + \lambda^* t^{1/3n+1}) \) and \( f_3(t) = C_1(1 - \lambda^* t^{1/3n+1}) \). But it is interesting for us to find positive solutions, that’s why we need positive solutions. It’s easy to check that \( f_1(t), f_2(t) \) are positive solutions. We must check the third solution. The third solution \( f_3(t) \) be a negative if and only if \( \lambda^* \geq 2^{1/3n+1} \). Namely, it’s sufficient to check that \( P(2^{1/3n+1}) < 0 \).

Thus we have proved the following

**Theorem 3.8.** Let \( k = 2s, s \in \mathbb{N} \).

(a) If \(-4^{1/3n+1} < \theta \leq \theta_1\), then for model (2.1) on the Cayley tree of order \( k \) there exists the unique translation-invariant Gibbs measure;

(b) If

\[
\frac{\sum_{i=1}^{s+1} 2^{i-1} \beta_{2i-1} \theta_{2i-1} + \alpha_2 s 2^{2s+1}}{\sum_{i=1}^{s+1} 2^{i-1} \beta_{2i-1}} \leq \theta \leq 4^{1/3n+1},
\]

then for model (2.1) on the Cayley tree of order \( k \) there are exactly two translation-invariant Gibbs measures;

(c) If

\[ \theta_1 < \theta < \frac{\sum_{i=1}^{s+1} 2^{i-1} \beta_{2i-1} \theta_{2i-1} + \alpha_2 s 2^{2s+1}}{\sum_{i=1}^{s+1} 2^{i-1} \beta_{2i-1}}, \]

then for model (2.1) on the Cayley tree of order \( k \) there are exactly three translation-invariant Gibbs measures.

Similar to Proposition 3.7 we get the following

**Proposition 3.9.** Let \( k = 2s + 1, s \in \mathbb{N} \).

a) Let \(-4^{1/3n+1} < \theta \leq \theta_1, \theta_{2s+1} \leq \theta < 4^{1/3n+1} \). Then (3.1) has only one positive fixed point: \( f(t) = 1 \).

b) Let

\[
\frac{\sum_{i=1}^{s+1} 2^{i-1} \beta_{2i-1} \theta_{2i-1}}{\sum_{i=1}^{s+1} 2^{i-1} \beta_{2i-1}} \leq \theta < \theta_{2s+1},
\]

Then (3.1) has exactly two positive fixed points: \( f_1(t) = 1, f_2(t) = \tilde{C}(1 + \lambda^* t^{1/3n+1}) \), where \( \lambda^* \) is a positive solution (3.11).

c) Let

\[ \theta_1 < \theta < \frac{\sum_{i=1}^{s+1} 2^{i-1} \beta_{2i-1} \theta_{2i-1}}{\sum_{i=1}^{s+1} 2^{i-1} \beta_{2i-1}}, \]

Then (3.1) has exactly three positive fixed points: \( f_1(t) = 1, f_2(t) = \tilde{C}(1 + \lambda^* t^{1/3n+1}), f_3(t) = \tilde{C}(1 - \lambda^* t^{1/3n+1}) \), where \( \lambda^* \) is a positive solution (3.11).

Thus we obtain the following
Theorem 3.10. Let \( k = 2s + 1, \ s \in \mathbb{N} \).
(a) If \(-4^{s+1} < \theta < \theta_1, \ \theta_2s+1 < \theta < 4^{s+1}\), then for model (2.1) on the Cayley tree of order \( k \) there exists the unique translation-invariant Gibbs measure;
(b) If
\[
\frac{\sum_{i=1}^{s+1} 2^{2i+1} \beta_{2i-1} \theta_{2i-1}}{\sum_{i=1}^{s+1} 2^{2i+1} \beta_{2i-1}} \leq \theta < \theta_2s+1,
\]
then for model (2.1) on the Cayley tree of order \( k \) there are exactly two translation-invariant Gibbs measures;
(c) If
\[
\theta_1 < \theta < \frac{\sum_{i=1}^{s+1} 2^{2i+2} \beta_{2i-1} \theta_{2i-1}}{\sum_{i=1}^{s+1} 2^{2i+2} \beta_{2i-1}},
\]
then for model (2.1) on the Cayley tree of order \( k \) there are exactly three translation-invariant Gibbs measures.

Remark 3.11. a) For the case \( k = 2 \) Theorem 3.8 coincides with Theorem 4.2 in [4];
b) For the case \( k = 3 \) Theorem 3.10 coincides with Theorem 5.2 in [4].

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