Generalized Stability Condition for Generalized and Doubly-Generalized LDPC Codes

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Abstract—In this paper, the stability condition for low-density parity-check (LDPC) codes on the binary erasure channel (BEC) is extended to generalized LDPC (GLDPC) codes and doubly-generalized LDPC (D-GLDPC) codes. It is proved that, in both cases, the stability condition only involves the component codes with minimum distance 2. The stability condition for GLDPC codes is always expressed as an upper bound to the decoding threshold. This is not possible for D-GLDPC codes, unless all the generalized variable nodes have minimum distance at least 3. Furthermore, a condition called derivative matching is defined in the paper. This condition is sufficient for a GLDPC or D-GLDPC code to achieve the stability condition with equality. If this condition is satisfied, the threshold of D-GLDPC codes (whose generalized variable nodes have all minimum distance at least 3) and GLDPC codes can be expressed in closed form.

I. INTRODUCTION

A traditional LDPC code [1] of length $N$ and dimension $K$ is graphically represented through a bipartite graph with $N$ variable nodes and $M \geq N - K$ check nodes [2]. A degree-$n$ check node of an LDPC code can be interpreted as a length-$n$ single parity-check (SPC) code, i.e., as a $(n, n-1)$ linear block code, while a degree-$n$ variable node can be interpreted as a length-$n$ repetition code, i.e., as a $(n, 1)$ linear block code.

Doubly-generalized LDPC (D-GLDPC) codes, recently introduced in [3] [4], extend the concept of LDPC codes, by allowing some variable and check nodes to be generic linear block codes instead of repetition and SPC codes, respectively. If all the variable nodes are repetition codes, then the code is a generalized LDPC (GLDPC) code [2], [5]–[8]. The codes used as variable and check nodes are called component codes of the D-GLDPC code; they will be supposed to have minimum distance $d_{\min} \geq 2$ [4]. The variable and check nodes which are not, respectively, repetition or SPC codes, are referred to as generalized nodes. The corresponding code structure is represented in Fig. 1. If $N_V$ is the number of variable nodes, then the codeword length is $N = \sum_{i=1}^{N_V} k_i$ (with $k_i$ dimension of the $i$-th variable node). An $(n, k)$ generalized variable node is connected to $n$ check nodes; $k$ of the $N$ D-GLDPC encoded bits are received by the generalized variable node, and interpreted as its $k$ information bits. An $(n, k)$ generalized check node is connected to $n$ variable nodes. If $N_C$ is the number of check nodes, then the number of parity check equations is $M = \sum_{i=1}^{N_C} (n_i - k_i)$ (with $k_i$ and $n_i$, respectively, dimension and length of the $i$-th check node). The ensembles of variable and check nodes are called variable node decoder (VND) and check node decoder (CND), respectively. For a description of the D-GLDPC codes iterative decoder on the AWGN channel and the BEC, we refer to [3], [4].

For standard LDPC ensembles, an important role is played by an inequality known as stability condition [9], [10]. For transmission on a BEC with erasure probability $q$, the stability condition establishes the following upper bound to the asymptotic threshold $q^*$ for the LDPC ensemble:

$$q^* \leq [\lambda'(0) \rho'(1)]^{-1}. \quad (1)$$

In (1), $\lambda'(0) = \lambda_2$ is the fraction of edges towards the length-2 repetition variable nodes, while $\rho'(1)$ is the derivative (computed in $x = 1$) of the function $\rho(x) = \sum_{j \geq 2} \rho_j x^{j-1}$, where $\rho_j$ is the fraction of edges connected to SPC check nodes of length $j$. The bound (1) was first developed from density evolution. However, it is possible to interpret it in a simple graphical way, by exploiting EXIT charts [11]. More specifically, the stability condition is equivalent to the following statement: For $q = q^*$, the derivative of the EXIT function $I_{E,V}(I_A, q)$ for the VND, with respect to $I_A$ and evaluated in $I_A = 1$, must be smaller than the derivative of the inverse EXIT function $I_{E,C}^{-1}(I_A)$ for the CND, evaluated in $I_A = 1$, i.e. (1) is equivalent to

$$\partial I_{E,V}(I_A, q^*)/\partial I_A|_{I_A=1} \leq d I_{E,C}^{-1}/d I_A|_{I_A=1}. \quad (2)$$

$1 I_A$ denotes the average a priori mutual information in input to the VND or to the CND.
The LDPC stability condition (1) is tight in that there exist LDPC distributions whose threshold achieves it with equality, assuming the closed form \( q^* = [X(0)\rho(1)]^{-1} \). For achieving (1) with equality, it is sufficient that the first occurrence of a tangency point between the EXIT function \( I_{E,V}(I_A, q) \) of the VND and the inverse EXIT function \( I_{E,C}^{-1}(I_A) \) of the CND appears in \( I_A = 1 \), i.e.,

\[
\begin{align*}
&\left\{ \begin{array}{l}
I_{E,V}(1, q^*) = I_{E,C}^{-1}(1) = 1 \\
\partial I_{E,V}(I_A, q^*)/\partial I_A|_{I_A=1} = dI_{E,C}^{-1}/dI_A|_{I_A=1}.
\end{array} \right.
\end{align*}
\] (3)

For LDPC codes, the first equality is always true. As proved in [4], it is always satisfied also for GLDPC and D-GLDPC codes, if all the variable and check component codes have \( d_{min} \geq 2 \), which is assumed true in this paper. Then, only the second equality will be considered in the sequel, and referred to as derivative matching condition.

In this paper, the stability condition (2), and the derivative matching condition (3) are extended to GLDPC and D-GLDPC codes. Two main results are obtained. The first is that only the component codes with \( d_{min} = 2 \), including length-2 repetition codes and SPC codes, appear in the stability condition. The second is that, for GLDPC codes satisfying the derivative matching condition, it is always possible to develop a closed-form expression of the threshold; the same expression holds also for D-GLDPC codes satisfying the derivative matching condition, if all the generalized variable nodes have \( d_{min} \geq 3 \).

II. Definitions and Basic Notation

The transmission channel is a BEC with erasure probability \( q \). Assuming a bipartite graph with random connections, the extrinsic channel, over which the messages are exchanged between the variable and check nodes, during the iterative decoding process, is modelled as a second BEC with erasure probability \( p \) [11] (depending on the decoding iteration). It is readily proved that \( I_A = 1 - p \): Since the EXIT functions will be expressed as functions of \( p \) (and \( q \) for the VND), the derivatives of the VND EXIT function and of the CND inverse EXIT function will be evaluated at \( p = 0 \) (\( I_A = 1 \)).

Under the hypothesis of random bipartite graph, the VND and CND EXIT functions can be expressed, respectively, as

\[
I_{E,V}(p, q) = \sum_{i=1}^{ \bar{T}_V } \lambda_i \cdot I_{E,V}^{(i)}(p, q)
\] (4)

\[
I_{E,C}(p) = \sum_{i=1}^{ \bar{T}_C } \rho_i \cdot I_{E,C}^{(i)}(p),
\] (5)

where \( \bar{T}_V \) and \( \bar{T}_C \) are the number of variable and check node types, \( I_{E,V}^{(i)}(p, q) \) and \( I_{E,C}^{(i)}(p) \) are the EXIT function for the \( i \)th variable node type and \( i \)th check node type, \( \lambda_i \) and \( \rho_i \) are the fractions of edges towards the variable nodes of type \( i \) and the check nodes of type \( i \).

For the scope of this work it is useful to isolate, in (4), the contribution of the repetition codes and, in (5), the contribution of the SPC codes:

\[
I_{E,V}(p, q) = \sum_{j \geq 2} (rep) \lambda_j^{(rep)} \cdot (1 - q \cdot p^{j-1}) + \sum_{i} \lambda_i \cdot I_{E,V}^{(i)}(p, q)
\]

\[
= \sum_{j \geq 2} (rep) \lambda_j^{(rep)} - q \cdot \lambda_1(p) + \sum_{i} \lambda_i \cdot I_{E,V}^{(i)}(p, q)
\]

\[
I_{E,C}(p) = \sum_{j = 1}^{ (SPC) } (rep) \rho_j^{(SPC)} \cdot (1 - p)^{j-1} + \sum_{i} \rho_i \cdot I_{E,C}^{(i)}(p)
\]

\[
= \rho_{SPC}(1 - p) + \sum_{i} \rho_i \cdot I_{E,C}^{(i)}(p).
\]

In (6), \( j \) is the length of the generic repetition variable node, \( \lambda_j^{(rep)} \) is the fraction of edges towards the repetition nodes of length \( j \), \( \lambda_1(p) \) is defined as \( \sum_{j \geq 2} \lambda_j^{(rep)} \cdot p^{j-1} \). It uses the well known EXIT function expression on the BEC, \( I_{E}(p, q) = 1 - q \cdot p^{j-1} \), for a \((j, 1)\) repetition variable node. The summation in \( i \) is over all the generalized variable node types. Analogously, in (7), \( j \) is the length of the generic SPC node, \( \rho_j^{(SPC)} \) is the fraction of edges towards the SPC nodes of length \( j \), \( \rho_{SPC}(x) \) is defined as \( \sum_{j \geq 2} \rho_j^{(SPC)} \cdot x^{j-1} \), and the expression \( I_{E,C}(p) = (1 - p)^{j-1} \), valid for a \((j, j-1)\) SPC check node, is used.

The EXIT function for an \((n, k)\) generalized variable node of a D-GLDPC code on the BEC can be expressed as

\[
I_{E}(p, q) = 1 - \frac{1}{n} \sum_{t=0}^{n-1} \sum_{z=0}^{k-1} a_{t,z} \cdot p^{t} \cdot (1 - p)^{n-t-1} \cdot q^{z} \cdot (1 - q)^{k-z},
\]

where \( a_{t,z} = [(n - t)\tilde{e}_{n-t,k-z} - (t + 1)\tilde{e}_{n-t-1,k-z}] \). The parameter \( \tilde{e}_{g,h} \) with \( g = 0, \ldots , n \) and \( h = 0, \ldots , k \) is the \((g, h)\)-th un-normalized split information function, defined as explained next. Considering a representation of the generator matrix \( G \) for the \((n, k)\) variable node, and appending to it the \((k \times k)\) identity matrix \( I_k \), \( \tilde{e}_{g,h} \) is equal to the summation of the ranks over all the possible submatrices obtained selecting \( g \) columns in \( G \) and \( h \) columns in \( I_k \). We remark that the split information functions for a generalized variable node, and then its EXIT function, heavily depend on the code representation, i.e. on the chosen generator matrix [4]. Then, the performance of the overall D-GLDPC code depends on the code representation used at the generalized variable nodes.

The EXIT function for a generalized \((n, k)\) check node of a GLDPC or D-GLDPC code on the BEC can be obtained by letting \( q \rightarrow 1 \) in (3) (no communication channel is present). The obtained expression, equivalent to [11, eq. 40], is

\[
I_{E}(p) = 1 - \frac{1}{n} \sum_{t=0}^{n-1} a_{t} p^{t} \cdot (1 - p)^{n-t-1},
\]

where \( a_{t} = (n - t)\tilde{e}_{n-t} - (t + 1)\tilde{e}_{n-t-1} \) and where for \( g = 0, \ldots , n \), \( \tilde{e}_{g} \) is the \( g\)-th un-normalized information function of the \((n, k)\) code, a concept first introduced in
It is defined as the summation of the ranks over all the possible submatrices obtained selecting \( y \) columns from the generator matrix \( G \). As opposed to the split information functions, the information functions are independent of the code representation. Thus, different code representations lead to the same EXIT function for the generalized check node. The performance of the overall D-GLDPC code is then independent of the specific representation of the generalized check nodes.

Equations (8) and (9) assume that MAP erasure correction is performed at the variable and check node.

\[ \text{III. INDEPENDENT SETS AND MINIMUM DISTANCE} \]

The development of a generalized stability condition for GLDPC and D-GLDPC codes is mostly based on a theorem proposed in this section. This theorem establishes a sufficient condition for a \((k \times (n-t))\) binary matrix, obtained by selecting \( n-t \) columns in the generator matrix \( G \) (any representation) of a \((n, k)\) linear block code, to have rank equal to \( k \). The results presented in this section can be also developed from [12]. Here, they are independently proved and formulated for the scope of the paper.

Definition 1 (independent set): Given a \((k \times n)\) binary matrix of rank \( r \), a set of \( t \) columns is said an independent set when removing these \( t \) columns from the matrix leads to a \((k \times (n-t))\) matrix with rank \( r - \Delta r < r \), for some \( 0 < \Delta r \leq t \). The number \( t \) is the size of the independent set.

If \( j \) columns are an independent set for a certain representation of the generator matrix, they are an independent set for any other representation. Moreover, removing them from any representation of the generator matrix leads to the same rank reduction \( \Delta r \). This is because all the possible representations of the generator matrix can be obtained by row summations starting from any other representation.

Lemma 1: If any representation of the generator matrix of a \((n, k)\) linear block code has an independent set of size \( j \), then the code minimum distance satisfies \( d_{\text{min}} \leq j \).

Proof: Let us suppose that \( j \) columns of a generator matrix \( G \) are an independent set of size \( j \). Then, it must be possible to perform row summations on \( G \) in order to obtain a new generator matrix representation \( G' \), in which a certain number \( \alpha \) of rows have all their 1’s in correspondence of only the columns of the independent set (see for example Fig. 2 where the first \( j \) columns are supposed an independent set, and where \( A, B \) and \( C \) are non-null matrices). Any of these \( \alpha \) rows is a valid codeword. Then, \( d_{\text{min}} \leq j \). \( \square \)

Theorem 1: Let \( G \) be any representation of the generator matrix of an \((n, k)\) linear block code. Then, the following statements are equivalent:

a) the code has minimum distance \( t \);

b) the minimum size of the independent sets of \( G \) is \( t \).

Proof: [a \Rightarrow b] If \( d_{\text{min}} = t \), then it is possible to construct a representation of \( G \) where there is at least one row with exactly \( t \) 1’s. The columns of \( G \) corresponding to these \( t \) 1’s are an independent set (of size \( t \)), because removing them from \( G \) leads to a reduction of the rank. This independent set must be of minimum size. In fact, if it existed an independent set of size \( j < t \), then from Lemma 1 it would follow \( d_{\text{min}} < t \), thus violating the hypothesis \( d_{\text{min}} = t \).

[b \Rightarrow a] Let us suppose that the minimum size of the independent sets of \( G \) is \( t \), and let us consider an independent set of size \( t \). From Lemma 1 it follows that \( d_{\text{min}} \leq t \). The proof is completed by showing that it is not possible to have \( d_{\text{min}} < t \). In fact, if \( d_{\text{min}} = j < t \) then, by reasoning in the same way as for the \([a \Rightarrow b] \) proof, it would follow that the minimum size of the independent sets of \( G \) is \( j < t \), which violates the hypothesis. \( \square \)

The present section is concluded by the following lemma (not necessary to prove Theorem 1).

Lemma 2: Let \( k < n \), and \( t \) be the minimum size of the independent sets of a \((k \times n)\) binary matrix with rank \( r \). Then, removing any independent set of size \( t \), leads to a \((k \times (n-t))\) matrix with rank \( r-1 \).

Proof: Since \( t \) is the minimum size of the independent sets of the matrix, then removing any set of \( j < t \) columns does not affect the rank. For an independent set of size \( t \), one can remove any subset of \( t-1 \) columns without reducing the rank; when removing the \( t \)-th column, the rank can only decrease by 1. \( \square \)

\[ \text{IV. STABILITY CONDITION AND DERIVATIVE MATCHING} \]

\[ \text{FOR GLDPC CODES} \]

In GLDPC codes, all the variable nodes are repetition codes. Recalling (6), and observing that in this case \( \sum_{j \geq 2}^{(\text{rep})} \lambda_j^{(r)} = 1 \), the EXIT function on the BEC for the VND is given by \( I_{E,V}(p, q) = 1 - q \lambda^{(r)}(x) \). Hence:

\[ \frac{\partial I_{E,V}(p, q)}{\partial p} \bigg|_{p=0} = -q \lambda^{(r)}_2. \]  \( (10) \)

Recalling (7), the derivative of \( I_{E,C}(p) \) at \( p = 0 \) is

\[ \frac{dI_{E,C}(p)}{dp} \bigg|_{p=0} = -\delta_{\text{SPC}}(1) + \sum_i^{(\text{gen})} \rho_i \frac{dI_{E,C}^{(i)}(p)}{dp} \bigg|_{p=0}. \]  \( (11) \)

In order to develop the previous expression, it is necessary to express the derivative of the EXIT function for the generalized
check nodes. This task can be performed by exploiting the theorem proved in Section III as explained next.

Consider an \((n, k)\) generalized check node, with EXIT function \(I_E(p)\) in the form (9). It is readily shown that the derivative of \(I_E(p)\), computed at \(p = 0\), is \(\frac{dI_E(p)}{dp}|_{p=0} = \frac{(n-1)q - a_1}{n}\). According to Theorem I, \(a_0 = 0\) if and only if the generalized check node has minimum distance \(d_{min} \geq 2\).

In fact, the generator matrix of the check node is full rank (rank \(k\)) by definition, so \(\hat{e}_n = k\). Furthermore, from Theorem II removing any single column from the generator matrix does not reduce the rank if and only if \(d_{min} \geq 2\), thus leading to \(\hat{e}_{n-k} = n k\). Then, \(a_0 = n \hat{e}_n - \hat{e}_{n-1} = n k - n k = 0\).

As recalled in Section III the hypothesis \(d_{min} \geq 2\) is always assumed in this paper. Then, it will be always assumed \(a_0 = 0\).

If \(d_{min} \geq 2\) for the check node, then \(\frac{dI_E(p)}{dp}|_{p=0} = -\frac{a_1}{n}\), where \(a_1 = (n-1)\hat{e}_{n-1} - 2\hat{e}_{n-2} = k n (n-1) - 2\hat{e}_{n-2}\). By applying again Theorem II we obtain

\[
a_1 = \begin{cases} 0 & \text{if } d_{min} \geq 3 \\ \neq 0 & \text{if } d_{min} = 2. \end{cases}
\] (12)

In fact, if \(d_{min} \geq 3\), removing any pair of columns from the generator matrix does not affect the rank. In this case \(2\hat{e}_{n-2} = 2 k (\hat{e}_n = k n (n-1)\), hence \(a_1 = 0\).

According to these results, the only generalized check nodes that give some contribution to the summation in the second member of (11) are those characterized by \(d_{min} = 2\). By recalling that all the SPC codes have minimum distance 2, we conclude that \(\frac{dI_E,C(p)}{dp}|_{p=0}\) only depends on the check nodes with \(d_{min} = 2\).

The derivative at \(p = 0\) of the EXIT function for the CND can be then expressed as

\[
\frac{dI_E,C(p)}{dp}|_{p=0} = -\rho_{SPC}(1) - \frac{d_{min} - 2}{n} \sum_{i} \frac{k_i n_i (n_i - 1) - 2 \hat{e}_{n-2}}{n_i} - \rho_{SPC}(1) - \frac{d_{min} - 2}{n} \sum_{i} \frac{2 \rho_i}{n_i} \Delta_{i n-2}^{(i)},
\] (13)

where the summation is over the generalized check node types with minimum distance 2. In the last equality, \(\Delta_{n-2}^{(i)} > 0\) is defined as follows. Let \(S_{n-2}\) be the generic \((k_i \times (n_i - 2))\) matrix obtained by selecting \(n_i - 2\) columns in the generator matrix. Then, \(\Delta_{n-2}^{(i)} = \sum S_{n-2}(k_i - \text{rank}(S_{n-2}))\), where the summation is over all the possible matrices \(S_{n-2}\). The parameter \(\Delta_{n-2}^{(i)}\) does not depend on the chosen representation for the \(i\)-th generalized check node.

The derivative at \(p = 0\) for the inverse EXIT function \(I_{E^{-1}}^{(1)}(p)\) of the CND is simply given by the inverse of (13). Then, the stability condition \(\frac{dI_{E,V}(p,q^*)}{dp}|_{p=0} \leq \frac{dI_{E,C}(p)}{dp}|_{p=0}\) for GLDPC codes leads to

\[
q^* \leq \left[\lambda_2^{(r)} \rho_{SPC}(1) + \sum_{i} \frac{d_{min} - 2}{n} \frac{2 \rho_i}{n_i} \Delta_{n-2}^{(i)}\right]^{-1}, \tag{14}
\]

an upper bound on the threshold \(q^*\) which represents a necessary condition for successful GLDPC (asymptotic) decoding.

For GLDPC codes satisfying the derivative matching condition \(\mathcal{E}\) (the first occurrence of a tangency point between \(I_{E,V}(p,q)\) and \(I_{E,C}(p)\) appears at \(p = 0\)), the threshold assumes the following simple closed form:

\[
q^* = \left[\lambda_2^{(r)} \rho_{SPC}(1) + \sum_{i} \frac{2 \rho_i}{n_i} \Delta_{n-2}^{(i)}\right]^{-1}. \tag{15}
\]

If only generalized check nodes with \(d_{min} \geq 3\) are used, the stability condition \(\mathcal{E}\) and the threshold expression (15) become, respectively,

\[
q^* \leq \left[\lambda_2^{(r)} \rho_{SPC}(1)\right]^{-1}, \tag{16}
\]

\[
q^* = \left[\lambda_2^{(r)} \rho_{SPC}(1)\right]^{-1}. \tag{17}
\]

V. STABILITY CONDITION AND DERIVATIVE MATCHING FOR D-GLDPC CODES

The derivative at \(p = 0\) of the EXIT function for the CND of D-GLDPC codes is the same as for GLDPC codes, and is expressed by (13). The derivative with respect to \(p\), at \(p = 0\), of the EXIT function for the VND is developed next.

It follows from (6):

\[
\frac{\partial I_{E,V}(p,q)}{\partial p}|_{p=0} = -q \lambda_2^{(r)} + \sum_{i} \lambda_i \frac{\partial I_{E,V}(p,q)}{\partial p}|_{p=0}.
\] (18)

In order to develop the summation on the generalized variable node types in the second term, the derivative of the EXIT function for each generalized variable node type, with respect to \(p\) and at \(p = 0\), can be computed based on (8). By defining \(f(p) = \sum_{t=0}^{n-1} a_{t,z} p^t (1 - p)^{n-1-t}\), it results

\[
\frac{\partial I_{E,C}(p,q)}{\partial p}|_{p=0} = -\frac{1}{n} \sum_{z=0}^{k} q^z (1 - q)^{k-z} \frac{df(p)}{dp}|_{p=0} = -\frac{1}{n} \sum_{z=0}^{k} q^z (1 - q)^{k-z} (a_{0,z} - a_{1,z}),
\] (19)

where the fact \(\frac{df(p)}{dp}|_{p=0} = -(n-1)a_{0,z} + a_{1,z}\) has been exploited. The previous relationship can be further developed by exploiting Theorem II. Since, by hypothesis, any variable node has minimum distance at least 2, removing any single column from the generator matrix \(G\) of the code associated to the variable node does not affect the rank of \(G\). It follows \(a_{0,z} = n \hat{e}_{n,k-z} - \hat{e}_{n-1,k-z} = k n (k^z - n k) = 0\), thus leading to

\[
\frac{\partial I_{E,C}(p,q)}{\partial p}|_{p=0} = -\frac{1}{n} \sum_{z=0}^{k} q^z (1 - q)^{k-z} a_{0,z}. \tag{20}
\]

Theorem II can be invoked again in order to show that

\[
a_{1,z} = \begin{cases} 0 & \forall z \text{ if } d_{min} \geq 3 \\ \neq 0 & \forall z \text{ if } d_{min} = 2. \end{cases}
\]

2If the derivatives are computed with respect to \(p = 1 - I_A\), the stability condition is \(\frac{dI_{E,V}(p,q^*)}{dp}|_{p=0} \geq \frac{dI_{E,C}(p)}{dp}|_{p=0}\).

3Some results about the stability condition on the BEC for GLDPC codes can be also found in [13, Appendix 7.A].
where \(d_{\text{min}}\) is the minimum distance of the \((n, k)\) code associated to the variable node under analysis. In fact, under the hypothesis \(d_{\text{min}} \geq 3\), no independent sets of size 1 and 2 are present in (any representation of) the generator matrix \(G\). Then \(a_{1, z} = \frac{(n - 1)\bar{e}_{n-1,k-1} - 2\bar{e}_{n-2,k-2}}{k(n - 1)\left(k_{-1}^{n-1}\right) - 2k\left(n_{-2}^{n-2}\right)\left(k_{-1}^{n-2}\right) = 0}.

It follows that the contribution of the generalized variable nodes to \(\frac{\partial I_{E,V}(p, q)}{\partial p}\big|_{p=0}\) is 0 if they all have minimum distance greater than or equal to 3. The only non-null contribution comes from the generalized variable nodes with minimum distance \(d_{\text{min}} = 2\), which is coherent with the fact that, among the repetition codes, only those with \(d_{\text{min}} = 2\) (i.e., the length-2 repetition codes) give a non-null contribution.

Then, (18) can be developed as

\[
\frac{\partial I_{E,V}(p, q)}{\partial p}\big|_{p=0} = -q\lambda_2^{(i)} - \sum_{i=0}^{d_{\text{min}}=2} \sum_{z=0}^{k_i} q^z (1-q)^{k_i-z} \cdot \frac{k_i n_i (n_i - 1)}{n_i} \frac{\Delta_{n-2,k-z}^{(i)}}{n_i}
\]

\[
= -q\lambda_2^{(i)} - \sum_{i=0}^{d_{\text{min}}=2} \sum_{z=0}^{k_i} q^z (1-q)^{k_i-z} \frac{2\lambda_i \Delta_{n-2,k-z}^{(i)}}{n_i} \sum_{i} \Delta_{n-2,k-z}^{(i)}. \tag{21}
\]

In the previous expression, \(\Delta_{n-2,k-z}^{(i)}\) is defined in a similar way as \(\Delta_{n-2,k-z}^{(i)}\) for GLDPC codes. Let \(S_{n-2,k-z}\) be the generic \((k_i \times (n_i - 2 + k_i - z))\) matrix obtained by selecting \(n_i - 2\) columns in the generator matrix, and \(k_i - z\) columns in the \((k_i \times k_i)\) identity matrix. Then, \(\Delta_{n-2,k-z}^{(i)} = \sum_{i} S_{n-2,k-z}^{(i)}(k_i - \text{rank}(S_{n-2,k-z}))\), where the summation is over all the possible matrices \(S_{n-2,k-z}\). Differently from \(\Delta_{n-2}^{(i)}\), the parameter \(\Delta_{n-2,k-z}^{(i)}\) depends on the code representation.

By combining (13) and (21) the stability condition \(\partial I_{E,V}(p, q^*)/\partial p\big|_{p=0} \geq dI_{E,C}(p)/dp\big|_{p=0}\) assumes the form:

\[
q^* \lambda_2^{(i)} + \sum_{i=0}^{d_{\text{min}}=2} \sum_{z=0}^{k_i} (q^*)^z (1-q^*)^{k_i-z} \frac{2\lambda_i \Delta_{n-2,k-z}^{(i)}}{n_i} \leq \left[p_{\text{SPC}}(1) + \sum_{i} \frac{2\rho_i}{n_i} \Delta_{n-2}^{(i)}\right]^{-1} - 1. \tag{22}
\]

This inequality is a necessary condition for (asymptotic) successful D-GLDPC decoding on the BEC. Differently from the GLDPC case, it is not possible to express this inequality as an upper bound on \(q^*\), because of the impossibility to factor \(q^*\) from the summation in \(z\). For D-GLDPC codes satisfying the derivative matching condition, (22) holds with equality. For the same reason, it does not lead to an explicit closed form expression of the threshold. However, if \(d_{\text{min}} \geq 3\) for all the generalized variable nodes, then any term of the summation over the generalized variable nodes is null. In this case, the same upper bound to \(q^*\) as in (15) holds:

\[
q^* \leq \left[\lambda_2^{(i)} \left(p_{\text{SPC}}(1) + \sum_{i} \frac{2\rho_i}{n_i} \Delta_{n-2}^{(i)}\right)\right]^{-1},
\]

and this inequality is achieved with equality by D-GLDPC codes satisfying the derivative matching condition. Moreover, if \(d_{\text{min}} \geq 3\) also for all the generalized check nodes, then the upper bound on \(q^*\) assumes the simple form as in (16):

\[
q^* \leq \left[\lambda_2^{(i)} \left(p_{\text{SPC}}(1) + \sum_{i} \frac{2\rho_i}{n_i} \Delta_{n-2}^{(i)}\right)\right]^{-1}.
\]

This inequality is achieved with equality by D-GLDPC codes satisfying the derivative matching condition.

VI. CONCLUSION

In this paper, a stability condition on the BEC has been derived for GLDPC and D-GLDPC codes, generalizing the inequality \(q^* \leq \left[\lambda(0)\rho(1)\right]^{-1}\), valid for standard LDPC codes. A derivative matching condition has been also defined, sufficient to achieve the stability condition with equality.

As for LDPC codes, only the variable and check nodes with minimum distance 2 are involved in the stability condition. The stability condition for GLDPC codes can be always explicitly expressed as an upper bound on the decoding threshold. D-GLDPC codes do not share this property in general; however, if all the generalized variable nodes have minimum distance at least 3, the stability condition becomes the same as for GLDPC codes. As a consequence, for GLDPC codes and for D-GLDPC codes with variable component codes of minimum distance at least 3 the decoding threshold assumes a simple closed form.

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