Adaptive Data Fusion for Multi-task Non-smooth Optimization

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October 2022

Abstract
We study the problem of multi-task non-smooth optimization that arises ubiquitously in statistical learning, decision-making and risk management. We develop a data fusion approach that adaptively leverages commonalities among a large number of objectives to improve sample efficiency while tackling their unknown heterogeneities. We provide sharp statistical guarantees for our approach. Numerical experiments on both synthetic and real data demonstrate significant advantages of our approach over benchmarks.

1 INTRODUCTION
In most machine-learning contexts, algorithm developers and theorists are concerned with solving a single task or optimizing a single metric at a time. Nonetheless, even in the big data era, the datasets are expensive and oftentimes collected for a large number of tasks, and models based on a single task likely hit the performance ceiling due to the limited sample size without fully exploiting the dataset featuring multiple tasks. For instance, in inventory management, the hype cycle of technology is getting shortened. It is increasingly critical for retailers to recognize the consumption patterns of customers as early as possible, so as to minimize the cost caused by backordering and holding. Since the selling data is limited at the early stage of the operations, decision making can generally be challenging. Nevertheless, a retailer usually sells multiple products in a store or manages multiple stores that sell the same product. They naturally define a group of related tasks. The retailer may effectively pool the datasets together to obtain a better estimation and decision. By sharing representations between related tasks, multi-task learning conceptually helps the model generalize better on individual tasks (Caruana, 1997).

That being said, the relatedness of the tasks is implicit and hard to quantify. An oversighted or misspecified relationship among tasks could adversely hurt the performance of data pooling across multiple tasks. When the tasks are highly distinct, naive multi-task learning procedures could underperform single-task learning (STL) ones which tackle each task separately.

To utilize the commonalities of datasets related to multiple tasks, statistical models can help us develop a family of reliable approaches with theoretical quantification of multi-task performance while adapting to the unknown task relatedness. To this end, we propose a data fusion approach to tackle unknown heterogeneities among the tasks and improve the data utilization for each individual task. Particularly, consider $m$ tasks of empirical risk minimization where the $j$-th task is to minimize
the empirical risk \( f_j(\theta) = \frac{1}{n_j} \sum_{i=1}^{n_j} \ell(\theta, \xi_{ji}) \) over the parameter of interest \( \theta \) from data \( \{\xi_{ji}\}_{i=1}^{n_j} \sim \mathcal{P}_j \). We propose to minimize an augmented objective

\[
F_{w,\lambda}(\Theta, \beta) = \sum_{j=1}^{m} w_j \left[ f_j(\theta_j) + \lambda_j \|\theta_j - \beta\|_2 \right]
\]

jointly over task-specific estimators \( \Theta = (\theta_1, \cdots, \theta_m) \in \mathbb{R}^{d \times m} \) and a multi-task center \( \beta \in \mathbb{R}^d \) to be learned. Weighting hyperparameters \( w_j \) are specified to reflect the importance of the information regarding each individual task (e.g. \( w_j = n_j \)). The regularization term \( \lambda_j \|\theta_j - \beta\|_2 \) drives the estimator of each individual task \( \theta_j \) towards a common center \( \beta \), with strength parameterized by \( (\lambda_1, \cdots, \lambda_m) \). It is straightforward to see that our method interpolates between minimizing the risk of the individual task as \( \lambda_j \) approaches zero, and a robust pooling of the individual minimizers as \( \lambda_j \) increases.

Our key contribution is the analysis of the aforementioned procedure in a wide range of problems where the losses \( \{f_j\}_{j=1}^{m} \) are convex but allowed to be nonsmooth. We prove that the proposed estimator automatically adapts to the unknown similarity among the tasks. In our motivating example, the cost function is naturally a piecewise linear nonsmooth convex function which is closely related to quantile regression. Other examples include linear max-margin classifiers as well as threshold regression models. Among these models, since the objective function is not differentiable at many places, technical challenges arise in the uniform concentration results and convergence rates as the subgradient is now a set-valued mapping and not continuous. Nonetheless, with statistical modeling, a theoretical analysis under such scenarios becomes possible.

In addition to the theoretical guarantees, we experiment with the numerical procedures on both synthetic data and a real-world dataset of the newsvendor problem in Section 5. The experiment reveals a steady and reliable benefit of the performance of the proposed method over benchmark ones, with significant improvement over STL where the data are scarce, and over blindly pooling the data together. This proposed method offers a reliable procedure for practitioners to leverage the possible relatedness between tasks in inventory decision-making, financial risk management, and many other applications.

1.1 Related Work

Multi-task learning based on parameter augmentation, such as the introduction of the common center \( \beta \) in our method, has achieved great empirical success (Evgeniou and Pontil, 2004; Jalali et al., 2013; Chen et al., 2011). Our estimator originates from the framework of Adaptive and Robust Multi-task Learning (ARMUL) proposed by Duan and Wang (2022), while we relaxed the smoothness and strong convexity condition on the empirical risks \( f_j \), such that we can extend the analysis to many real-world applications from statistical learning to inventory decision-making, to financial risk management. The motivating inventory management example, often known as the data-driven newsvendor problem, can be expressed as a quantile regression problem with the quantile level determined by a ratio of per unit holding cost versus the backordering one (Levi et al., 2007, 2015; Ban and Rudin, 2019). The objective function, also known as the “check function”, is convex but not differentiable. These applications coincide with the classical quantile regression in statistics and econometrics literature, dated back to Koenker and Bassett Jr (1978), which estimates the conditional quantile of the response variable across values of predictive covariates. Besides the aforementioned newsvendor problems in inventory management, quantile regression finds a wide range of applications in survival data analysis (Koenker and Geling, 2001; Wang and Wang, 2014), financial risk management (Engle and Manganelli, 1999; Rockafellar et al., 2000) and many other applications.
fields. We refer the reader to Koenker (2005); Koenker et al. (2017) for an extensive overview of quantile regression.

Immense applications rising in various fields call for the need of generalization to the nonsmooth objectives in multi-task learning. Despite its practical importance, MTL for nonsmooth objectives remains largely unexplored in statistical learning, with several exceptions including Fan et al. (2016); Chao et al. (2021); Kan et al. (2022). In contrast to our framework, they studied quantile regression with multivariate response variables in linear models and neural networks, respectively, and treated each variable as a different task sharing the same observed covariates. Since their models are based on a shared covariate for multiple tasks, they typically imposed factor structure and augmented the objective with a rank-based regularization on the $\Theta$ matrix. On the contrary, our framework features different covariates in each task. As such, we regularize the objective function with a penalty driving towards a robust central of all tasks and utilize the information to jointly optimize over the individual estimators and the intrinsic central. Our analysis also complements and generalizes limited existing literature on nonsmooth quantile regression in large-scale or distributed datasets (Volgushev et al., 2019; Chen et al., 2021) which considered only the quantile regression under homogeneous tasks. Several other existing works considered a similar framework as ours in an empirical Bayesian argument. Gupta and Kallus (2022) developed a data pooling procedure for data-driven newsvendor problems that shrinks the empirical distribution of each individual task towards a weighted global empirical distribution according to an anchor distribution. The data distributions there have finite supports. Mukherjee et al. (2015) focused on the Gaussian setting, studied the predictive risk instead of estimation error, and proposed a shrinkage estimator towards a data-driven location simultaneously optimized.

1.2 Notation

The constants $c_1, c_2, C, C_1, C_2, \cdots$ may differ from line to line. We use $[n]$ as a shorthand for $\{1, 2, \cdots, n\}$ and $|\cdot|$ to denote the absolute value of a real number or cardinality of a set. $||A|| = \sup_{||x||_2=1}||Ax||_2$ denotes the spectral norm. Let $\mathbb{Z}_+$ be the set of positive integers and $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$. Define $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$ for $x \in \mathbb{R}$. Define $B(x, r) = \{ y \in \mathbb{R}^d : ||y - x||_2 \leq r \}$ for $x \in \mathbb{R}^d$ and $r \geq 0$.

2 PROBLEM FORMULATION

Let $m \in \mathbb{Z}_+$ be the number of tasks and $\mathcal{X}$ be the sample space, and let $\ell: \mathbb{R}^d \times \mathcal{X} \rightarrow \mathbb{R}$ be a (non-smooth) loss function. For every $j \in [m]$, let $\mathcal{P}_j$ be a probability distribution over $\mathcal{X}$ and $\mathcal{D}_j = \{\xi_{ji}\}_{i=1}^{n_j}$ be $n_j$ independent samples drawn from $\mathcal{P}_j$. The $j$-th task is to estimate the population loss minimizer

$$\theta^*_j \in \arg\min_{\theta \in \mathbb{R}^d} \mathbb{E}_{\xi \sim \mathcal{P}_j} \ell(\theta, \xi)$$

from the data. Denote by $\Theta^* = (\theta^*_1, \cdots, \theta^*_m) \in \mathbb{R}^{d \times m}$ the parameter matrix.

Define the empirical loss function of the $j$-th task as

$$f_j(\theta) = \frac{1}{n_j} \sum_{i=1}^{n_j} \ell(\theta, \xi_{ji}).$$

Two straightforward strategies are single-task learning (STL) and data pooling (DP). The former corresponds to solving the individual tasks separately, i.e.,

$$\hat{\theta}_j = \arg\min_{\theta \in \mathbb{R}^d} f_j(\theta), \quad \forall j \in [m].$$
The latter corresponds to merging all datasets to train a single model, i.e.,

\[ \hat{\theta}_1 = \cdots = \hat{\theta}_m = \arg\min_{\theta \in \mathbb{R}^d} \sum_{j=1}^m n_j f_j(\theta). \]

These two strategies have intrinsic shortcomings: STL does not take full advantage of the data available, while DP has a high risk of model misspecification. To resolve this issue, define

\[ F_{w,\lambda}(\Theta, \beta) = \sum_{j=1}^m w_j [f_j(\theta_j) + \lambda_j ||\theta_j - \beta||_2], \quad (2.1) \]

where \( \Theta = (\theta_1, \cdots, \theta_m) \in \mathbb{R}^{d \times m}, \beta \in \mathbb{R}^d, \ w = (w_1, \cdots, w_m) \in \mathbb{R}^m_+ \) are weight parameters (e.g. \( w_j = n_j \)), and \( \lambda = (\lambda_1, \cdots, \lambda_m) \in \mathbb{R}^m_+ \) are penalty parameters. We propose to solve an augmented program

\[ (\hat{\Theta}, \hat{\beta}) \in \arg\min_{\Theta \in \mathbb{R}^{d \times m}, \beta \in \mathbb{R}^d} F_{w,\lambda}(\Theta, \beta) \quad (2.2) \]

where each task receives its own estimate \( \hat{\theta}_j \) while the penalty terms shrink \( \hat{\theta}_j \)'s toward \( \hat{\beta} \) to promote similarity among the estimated models. This is a convex program so long as \{\( f_j \)\}_{j=1}^m \) are all convex.

If we choose \( \lambda = 0 \), we return to the STL setting; if we choose sufficiently large \( \lambda \), the cusp of the \( \ell_2 \) penalty at zero enforces strict equality \( \hat{\theta}_j = \hat{\beta} \) for all \( j \in [m] \), effectively pooling all the data. Therefore, it is desirable to choose a suitable \( \lambda \) such that we can automatically adapt to whichever situation proves more suitable.

Note that (2.2) belongs to the framework of Adaptive and Robust Multi-task Learning (ARMUL) proposed by Duan and Wang (2022). However, theoretical guarantees of ARMUL require \{\( f_j \)\}_{j=1}^m \) to be smooth and locally strongly convex near the minimizers. As such, scenarios where ARMUL is powerful includes, for example, multi-task linear regression and multi-task logistic regression. In contrast, our theoretical results relax the smoothness condition and extend to scenarios where \{\( f_j \)\}_{j=1}^m \) are non-smooth, which are ubiquitous in statistical learning and operations research.

3 EXAMPLES

Here we introduce three motivating examples in statistical learning and operations management.

3.1 Newsvendor Problem

Suppose a retailer sells a perishable good that needs to be prepared/stocked/ordered in advance. Let \( D \) be a random variable representing the market demand. The retailer needs to decide a quantity \( q \) of goods to prepare (e.g. raw materials to buy, food to defrost) ahead of time in order to minimize the expected cost a combination of the backorder/underage and holding/overage costs as follows,

\[ \mathbb{E}_D \left[ b(D - q)^+ + h(D - q)^- \right], \]

where \( b \) and \( h \) are backorder and holding costs per unit, respectively.

In practice, the distribution of the demand \( D \) is not known beforehand. Instead, the information available is a set of independent random samples \{\( D_i \)\}_{i=1}^n \) drawn from that. We can estimate \( q(\tau) \) by \( \hat{q} \in \arg\min_{q \in \mathbb{R}} f(\theta) \), where the objective function

\[ f(q) = \frac{1}{n} \sum_{i=1}^n [b(D_i - q)^+ + h(D_i - q)^-] \]
is non-smooth. If we define the check loss \( \rho_\tau(z) = (1 - \tau)z^- + \tau z^+ \), then \( f(q) \) is proportional to \( \frac{1}{n} \sum_{i=1}^{n} \rho_\tau(D_i - q) \) with \( \tau = b/(b + h) \). The solution \( \hat{q} \) is the \( \tau \)-th sample quantile of the data \( \{D_i\}^n_{i=1} \).

The above classical newsvendor problems assumed that the holding cost and the backordering cost grow linearly with regard to quantity surplus and deficit, respectively. We can relax this assumption to cases where the two costs are replaced with \( B((D - q)^+) \) and \( H((D - q)^-) \) with general functions \( B, H : \mathbb{R}_+ \to \mathbb{R}_+ \) that are convex, non-decreasing, and satisfy \( B(0) = H(0) \). Given the data \( \{D_i\}^n_{i=1} \), it is natural to estimate the best linear decision rule by minimizing the loss function

\[
 f(q) = \frac{1}{n} \sum_{i=1}^{n} \left[ B((D_i - q)^+) + H((D_i - q)^-) \right].
\]

In modern newsvendor problems, the data for a specific product at one store can be quite limited. Fortunately, multiple products in the same store or multiple stores in a nearby region have similar sales patterns. A joint analysis of the datasets by multi-task learning facilitates decision making.

### 3.2 Quantile Regression

Denote by \( F_{Y|X} \) the conditional CDF of a response \( Y \in \mathbb{R} \) given covariates \( X \in \mathbb{R}^d \). Define the \( \tau \)-th conditional quantile of \( Y \) given \( X \in \mathbb{R}^d \) as \( Q_{Y|X}(\tau) = \inf \{ y : F_{Y|X}(y) \geq \tau \} \). Assume that \( Q_{Y|X}(\tau) = X^\top \theta^* \) holds for some \( \theta^* \in \mathbb{R}^d \). Given \( n \) i.i.d. samples \( \{(x_i, y_i)\}^n_{i=1} \) from some joint distribution \( P \), we can estimate \( \theta^* \) by \( \hat{\theta} \in \arg\min_{\theta \in \mathbb{R}^d} f(\theta) \), where the objective

\[
 f(\theta) = \frac{1}{n} \sum_{i=1}^{n} \rho_\tau(y_i - x_i^\top \theta)
\]

is non-smooth. This is the quantile regression in statistics (Koenker and Bassett Jr, 1978) which targets the conditional quantile of the response. In contrast, least squares regression aims to estimate the conditional mean. When we collect data from multiple populations (e.g. different geographical locations), multi-task learning helps utilize their commonality while tackling the heterogeneity.

### 3.3 Support Vector Machine

Consider a binary classification problem where one wants to predict the label \( Y \in \{\pm 1\} \) from covariates \( X \in \mathbb{R}^d \). A popular method for training linear classifiers of the form \( X \mapsto \text{sgn}(X^\top \theta) \) is the support vector machine (SVM) (Cortes and Vapnik, 1995). Given the data \( \{(x_i, y_i)\}^n_{i=1} \), the (soft-margin) SVM amounts to minimizing the empirical loss function below:

\[
 f(\theta) = \frac{1}{n} \sum_{i=1}^{n} (1 - y_i x_i^\top \theta)^+ + \mu \|\theta\|_2^2,
\]

where \( \mu \geq 0 \) is a penalty parameter. Here \( f \) is non-smooth. SVM has demonstrated superior performance in binary classification problems. In multi-class and multi-label settings, multi-task SVM is a popular approach where each task is to distinguish a pair of classes.

### 4 THEORETICAL ANALYSIS

In this section, we provide a non-asymptotic analysis of (2.2). Of particular interest to us is to generalize the results of Duan and Wang (2022) to non-smooth empirical loss functions. While the
emirical loss functions \( \{ f_j \}_{j=1}^m \) could be non-smooth, in many cases their population versions (expectations), \( \{ F_j \}_{j=1}^m \), have desirable properties such as first-order smoothness and strong convexity. Intuitively, \( f_j \) and \( F_j \) are “close”, and we can leverage this closeness to bound estimation errors. In general, this \( F_j \) can be any function that is close to \( f_j \) and enjoys the aforementioned properties.

The study under statistical settings is built upon the deterministic results in Appendix A, which could be of independent interest. See Appendix C for this section’s proofs.

To analyze the estimation error under statistical settings, we assume \( n_j = n, w_j = 1, \) and \( \lambda_j = \lambda \) for all \( j \in [m] \). In addition, assume that \( \ell : \mathbb{R}^d \times \mathcal{X} \to \mathbb{R} \) is convex in its first argument and let \( l : \mathbb{R}^d \times \mathcal{X} \to \mathbb{R}^d \) be a vector-valued function such that for every \( \theta \in \mathbb{R}^d \) and \( \xi \in \mathcal{X} \), \( l(\theta, \xi) \) belongs to the subdifferential \( \partial \ell(\theta, \xi) \) of \( \ell \). We make the following assumptions.

**Assumption 4.1 (Concentration).** There exists an absolute constant \( c \) such that \( ||l(\theta, \xi)||_2 \leq \sigma \leq c \) for all \( \theta \in \mathbb{R}^d, \xi \sim \mathcal{P}_j, \) and \( j \in [m] \).

**Assumption 4.2 (Regularity).** Let \( F_j(\theta) = \mathbb{E}_{\xi \sim \mathcal{P}_j} \ell(\theta, \xi) \). Suppose that \( F_j \) is twice differentiable on \( \theta \), and denote \( \Sigma_j(\theta) = \nabla^2 F_j(\theta) \). There exists a constant \( C_1 > 0 \) such that
\[
||\Sigma_j(\theta_1) - \Sigma_j(\theta_2)|| \leq C_1||\theta_1 - \theta_2||_2, \quad \forall \theta_1, \theta_2 \in \mathbb{R}^d, \quad j \in [m].
\]
Furthermore, there exist \( \theta^* \in \mathbb{R}^d \) and constants \( C_1 \in (0, 1), M > 0 \) such that for any \( \theta \in B(\theta^*, M) \) and \( j \in [m] \), all eigenvalues of \( \Sigma_j(\theta) \) belong to \((c_1, 1/c_1)\).

**Assumption 4.3 (Variability of \( l \)).** Assume one of the followings hold.

1. There is a function \( U : \mathcal{X} \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) such that
\[
||v^\top(l(\theta_1, \xi) - l(\theta_2, \xi))|| \leq U(\xi, v, \theta_1, \theta_2)||\theta_1 - \theta_2||_2
\]
holds for all \( v \in \mathbb{R}^d \). \( U(\xi, v, \theta_1, \theta_2) \) satisfies that
\[
\sup_{||v||_2=1} \sup_{\theta_1, \theta_2} \mathbb{E}_{\xi \sim \mathcal{P}_j} \exp(t_0 U(\xi, v, \theta_1, \theta_2)) \leq C,
\]
\[
\sup_{||v||_2=1} \mathbb{E}_{\xi \sim \mathcal{P}_j} \sup_{\theta_1, \theta_2} U(\xi, v, \theta_1, \theta_2) \leq d^{c_2},
\]
for some \( c_2, t_0, C > 0 \). Furthermore, assume that \( \sup_{||v||_2=1} \mathbb{E}_{\xi \sim \mathcal{P}_j} \exp(t_0 ||v^\top l(\theta^*, \xi)||) \leq C \) holds for some constants \( t_0, C > 0 \).

2. For some constants \( c_2, c_3, r > 0 \),
\[
\sup_{\theta_1 \in B(\theta^*, r)} \sup_{\theta_2 \in B(\theta^*, r)} \mathbb{E}_{\xi \sim \mathcal{P}_j} \frac{||l(\theta_1, \xi) - l(\theta_2, \xi)||_2^4}{n^{c_3}Z} \leq \frac{d^{c_2}}{n^{c_3}Z}
\]
for any large \( Z > 0 \). Also,
\[
\sup_{||v||_2=1} \mathbb{E}_{\xi \sim \mathcal{P}_j} \left\{ v^\top |l(\theta_1, \xi) - l(\theta_2, \xi)|^2 \right\} \leq C ||\theta_1 - \theta_2||_2
\]
and \( \sup_{||v||_2=1} \mathbb{E}_{\xi \sim \mathcal{P}_j} \sup_{\theta} \exp(t_0 ||v^\top l(\theta, \xi)||) \leq C \) for some \( t_0, C > 0 \).
Assumption 4.1 assumes that there exists a subgradient of the loss function that is subgaussian. Assumption 4.2 requires that the Hessian of the population risk satisfies certain continuity condition and is bounded below and above near its minimizer. Assumption 4.3 concerns the variability of the subgradient function $l$ which is standard in the literature of nonsmooth statistical learning (Chen et al., 2021). It is easy to verify that the examples satisfy the above assumptions under general regularity conditions on the covariates.

Note that, with regard to the empirical loss function, the assumptions above only concern first-order conditions, which are weaker than the second-order condition required in Duan and Wang (2022), and thus they can apply to more general settings. In our case, this allows us to extend our analyses to non-smooth empirical loss functions. Further, our assumptions only target one subgradient of the empirical loss function. While a gradient may not always exist for the empirical loss function, a subgradient always exists. As it turns out, conclusion about the closeness between one subgradient and its expectation is sufficient for conclusion on a uniform closeness between all subgradients and their expectation. See Appendix A for more details.

Define $\tilde{\Theta} = (\tilde{\theta}_1, \cdots, \tilde{\theta}_m)$ as the estimators from STL, i.e., $\tilde{\theta}_j = \arg\min_{\theta \in \mathbb{R}^d} f_j(\theta)$. We have the following result on the closeness between $\hat{\theta}_j$ and $\tilde{\theta}_j$, ensuring the former’s fidelity to its associated dataset $D_j$.

**Theorem 4.1** (Personalization). Let Assumptions 4.1, 4.2, and 4.3 hold. There exist constants $C_1$ and $C_2$ such that, when $\lambda < \rho M/4$, the following holds with probability at least $1 - C_1 n^{-d}$:

$$
||\tilde{\theta}_j - \theta^*_j||_2 \leq C_2 \sigma \sqrt{\frac{d \log n + \log m}{n}},
$$

$$
||\hat{\theta}_j - \tilde{\theta}_j||_2 \leq C_2 \left( \lambda + \sigma \sqrt{\frac{d \log n + \log m}{n}} \right)
$$

for all $j \in [m]$.

Note that the output of $\hat{\theta}_j$ of (2.2) always satisfies

$$
\hat{\theta}_j \in \arg\min_{\theta \in \mathbb{R}^d} \left\{ f_j(\theta) + \lambda||\theta - \hat{\beta}||_2 \right\}, \quad \forall j \in [m]. \tag{4.1}
$$

Therefore, $\hat{\theta}_j$ and $\tilde{\theta}_j$ minimize similar functions. The penalty term $\lambda||\theta - \hat{\beta}||_2$ in (4.1) can be viewed as a perturbation added to the objective function $f_j$, and Theorem 4.1 tells us that it can only perturb the minimizer by a limited amount decided by the penalty level $\lambda$. Intuitively, when the empirical loss function $f_j$ is “close” to a strongly convex function in a neighborhood of its minimizer $\tilde{\theta}_j$, the Lipschitz penalty function does not make much difference. Theorem 4.1 guarantees the fidelity of our approach (2.2) to individual datasets for general $M$-estimation.

By Assumption 4.1, we have $\sigma \lesssim 1$. Theorem 4.1 implies that when $\lambda \lesssim \sigma \sqrt{\frac{d \log n + \log m}{n}}$, the bound $||\hat{\theta}_j - \theta^*_j||_2 \lesssim \sigma \sqrt{\frac{d \log n + \log m}{n}}$ simultaneously holds for all $j \in [m]$ with high probability. In that case, our approach (2.2) achieves the same estimation error rate of STL up to logarithmic factors.

In the definition and theorem to follow, we consider all the tasks and study the adaptivity and robustness of (2.2).

**Assumption 4.4** (Task Relatedness). There exist $\varepsilon, \delta \geq 0$ and subset $S \subseteq [m]$ such that

$$
|S^c| \leq \varepsilon m \quad \text{and} \quad \min_{\theta \in \mathbb{R}^d} \max_{j \in S} ||\theta^*_j - \theta||_2 \leq \delta.
$$
It is worth pointing out that any $m$ tasks are $(0, \max_j \|\theta_j^*\|_2)$-related. Smaller $\varepsilon$ and $\delta$ imply stronger similarity among the tasks. When all but a small proportion of $\{\theta_j^*\}_{j=1}^m$ are close to each other, the following theorem shows that a single choice of $\lambda$ can automatically enforce an appropriate degree of relatedness among the learned models, while tolerating a reasonable fraction of exceptional tasks that are dissimilar to others.

**Theorem 4.2 (Adaptivity).** Let Assumptions 4.1, 4.2, 4.3 and 4.4 hold. When $n^{d(m-1)m^{m-d}} \geq 1$, there exist positive constants $\{C_i\}_{i=0}^4$ such that, if

$$C_1\sigma \sqrt{d \log n + \log m} < \lambda < C_2 \sigma$$

and $0 \leq \varepsilon < C_3$, the following bounds hold with probability at least $1 - C_4(m^{-d} + 1)n^{-d}$:

$$\max_{j \in S} \|\hat{\theta}_j - \theta_j^*\|_2 \leq C_0 \left( \sigma \sqrt{d \log mn} + \min \{\lambda, \delta\} + \varepsilon \lambda \right),$$

$$\max_{j \in S^c} \|\hat{\theta}_j - \theta_j^*\|_2 \leq C_0 \lambda.$$

Moreover, there exists a constant $C_5$ such that under the conditions $\varepsilon = 0$ and $C_5 \sigma \sqrt{n} < \sigma \sqrt{d \log n + \log m}$, we have $\hat{\theta}_1 = \cdots = \hat{\theta}_m = \arg\min_{\theta \in \mathbb{R}^d} \{\sum_{j=1}^m f_j(\theta)\}$.

It is worth pointing out that the same error bound on $\max_{j \in S} \|\hat{\theta}_j - \theta_j^*\|_2$ holds even if the data of the tasks in $S^c$ have been arbitrarily contaminated.

Theorem 4.2 simultaneously controls the estimation errors for all individual tasks in $S$ and suggests choosing $\lambda = C\sigma \sqrt{d \log n + \log m}$ for some constant $C$. In practice, this $C$ can be selected by cross-validation. This allows us to choose a single $\lambda$ to achieve minimax optimality (up to a logarithmic factor), matching the minimax lower bound in Duan and Wang (2022). Indeed, that lower bound is proved for a class of smooth losses that are included by our general function classes.

For any $\varepsilon$ and $\delta$, a simple bound $\|\hat{\theta}_j - \theta_j^*\|_2 \lesssim \lambda$ always holds for all $j \in [m]$, which echoes Theorem 4.1. In comparison, Theorem 4.2 implies more refined results. When $\delta = \varepsilon = 0$, all target parameters are the same and $S = [m]$. Data pooling becomes a natural approach, whose error rate is $O(\sigma \sqrt{d/mn})$. Our approach (2.2) has the same rate (up to a logarithmic factor). When $\varepsilon = 0$ and $\delta$ grows from 0 to $+\infty$, the error bounds smoothly transit from those for data pooling to those for STL. See Section 5 for illustration.

The second term $\min \{\lambda, \delta\}$ is non-decreasing in the discrepancy $\delta$ among $\{\theta_j^*\}_{j \in S}$. It increases first and then flattens out, never exceeding the error rate of STL. Combined with the first term, when $\varepsilon = 0$, our approach (2.2) achieves the smaller error rate between data pooling and STL. When $\varepsilon > 0$, the third term $\varepsilon \lambda$ is the price we pay for not knowing the index set $S^c$ for outlier tasks.

In summary, the theoretical investigation yields a principled approach for choosing a single regularization parameter $\lambda$ for all tasks. The resulting estimators automatically adapt to unknown task relatedness and are robust against a certain fraction of outlier tasks.

**5 NUMERICAL EXPERIMENTS**

We conduct experiments on synthetic and real data to test our approach in various scenarios. Below we present descriptions and key findings. The curves and error bands show the means and their 95% confidence intervals computed from 100 independent runs, respectively.
5.1 Synthetic Data

We first generate synthetic data for multi-task quantile regression. The number of tasks is $m = 50$. For every $j \in [m]$, the dataset $D_j$ consists of $n = 200$ samples $\{(x_{ji}, y_{ji})\}_{i=1}^n$. The covariate vectors $\{x_{ji}\}_{(i,j) \in [n] \times [m]}$ are i.i.d. from the 20-dimensional standard normal distribution, given which we sample each response $y_{ji} = x_{ji}^T \gamma_j^* + \epsilon_{ji}$ from a linear model with noise term $\epsilon_{ji} \sim N(0,0.25)$ being independent of the covariates. The coefficient vectors $\{\gamma_j^*\}_{j=1}^m \subseteq \mathbb{R}^{20}$ are generated according to the prescribed level of task relatedness defined in Assumption 4.4. For every $\epsilon \in \{0,0.1\}$ and $\delta \in \{0,0.2,0.4,\ldots,2\}$ we use the procedure below to get $m$ tasks that are $(\epsilon,\delta)$-related and share the same signal strength.

- Select $\epsilon m$ tasks uniformly at random and let $S$ be the index set of unselected tasks;
- Draw $m$ i.i.d. random vectors $\{\eta_j\}_{j=1}^m$ uniformly from the unit sphere, and set $\gamma_j^* = 2\eta_j$ for all $j \notin S$;
- For each $j \in S$, set $\gamma_j^* = (2 \cos \alpha)e_1 + (2 \sin \alpha)\eta_j$, where $e_1 = (1,0,\cdots,0)$ and $\alpha = 2 \arcsin(\delta/4)$.

We have $\|\gamma_j^*-2e_1\|_2 = \delta, \forall j \in S$ and $\|\gamma_j^*\|_2 = 2, \forall j \in [m]$.

Our target quantile level is $\tau = 0.9$. Given the covariates $x_{ji}$, the $\tau$-th quantile of the response $y_{ji}$ is $x_{ji}^T \gamma_j^* + 0.5\Phi^{-1}(\tau)$, where $\Phi$ is the cumulative distribution function of $N(0,1)$. In quantile regression, we add an all-one covariate and enlarge the input dimension to $d = 21$. For the $j$-th task, the true coefficients in the quantile function are $\theta_j^* = (0.5\Phi^{-1}(\tau), \gamma_j^{*\top}) \in \mathbb{R}^{21}$. For any algorithm that produces estimates $\{\hat{\theta}_j\}_{j=1}^m$ of $\{\theta_j^*\}_{j=1}^m$, we compute the maximum estimation error $\max_{j \in [m]} \|\hat{\theta}_j - \theta_j^*\|_2$ and its restricted version $\max_{j \in S} \|\hat{\theta}_j - \theta_j^*\|_2$ on the subset $S$ containing similar tasks (if $\epsilon > 0$).

Following our theories, we set the regularization parameter $\lambda$ for our approach to be $C\sqrt{d/n}$ and select $C$ from $\{0.1,0.2,\cdots,1\}$ by 5-fold cross-validation. We compare the approach with single-task learning (STL) and data pooling (DP). Figures 1 and 2 demonstrate how the estimation errors grow with the heterogeneity parameter $\delta$.

![Figure 1: Impact of task relatedness when $\epsilon = 0$.](image)

The simulations confirm the theoretical guarantees Theorem 4.2 for our proposed method. When $\epsilon = 0$ and $\delta$ is small, it behaves similarly as DP. As $\delta$ increases, the new method tackles the
Figure 2: Impact of task relatedness when $\varepsilon = 0.2$. $x$-axis: $\delta$. $y$-axis: $\max_{j \in S} ||\hat{\theta}_j - \theta^*_j||_2$ (left) or $\max_{j \in [m]} ||\hat{\theta}_j - \theta^*_j||_2$ (right). Red solid lines: new approach. Blue triangles: DP. Black dashed lines: STL.

heterogeneity and never underperform STL, while DP’s estimation error grows rapidly. When $\varepsilon = 0.2$, the new method works well on the set $S$ of related tasks, and DP makes huge errors due to the $\varepsilon$-fraction of exceptional tasks. The two panels of Figure 2 imply that the new method behaves similarly as STL on $S^c$. This agrees with Theorem 4.1 that our estimates for individual tasks are never too far from the corresponding empirical loss minimizers. Data pooling performs poorly on $S^c$.

5.2 Real Data

We apply the proposed method to a data-driven newsvendor problem. We use a real-world dataset made publicly available by Buttler et al. (2022) that contains sales data at 35 different stores in a local bakery chain over a period of 1215 days, from January 2016 to April 2019. According to the authors, every evening each store orders products to be delivered the next morning from a central factory. Unsold goods will be disposed of at the end of the day. The authors use the sales data as the demand because all products are everyday items with typically high stock levels, which makes censored demand unlikely. The dataset also contains information about the weather, promotions, holidays, calendric (e.g. year, month, weekday) and lag features (e.g. mean demand over the past 7 days). To utilize the features we generalize the classical newsvendor problem in Example 3.1 into a covariate-assisted data-driven newsvendor problem.

In particular, recall that $\{D_i\}_{i=1}^n \subseteq \mathbb{R}$ are the realized daily market demands and $\{x_i\}_{i=1}^n \subseteq \mathbb{R}^d$ are the covariates for the corresponding days. Suppose we want to decide the ordering quantity $q_i$ using a linear combination of the $d$-dimension features, as $q_i = x_i^\top \theta$ with a coefficient parameter $\theta \in \mathbb{R}^d$ to be determined. Assume that any leftover at the end of the day leads to a holding cost of $\$h$ per unit. Meanwhile, any demand that cannot be satisfied results in a backorder cost of $\$b$ per unit. The cost on the $i$-th day is $h(D_i - q_i)^+ + b(D_i - q_i)^-$ dollars, which is proportional to the check loss $\rho_\tau(D_i - q_i)$ with $\tau = b/(b + h)$. We can estimate the best $\theta$ with the minimum expected
cost through minimizing the following nonsmooth objective function,

\[ f(\theta) = \frac{1}{n} \sum_{i=1}^{n} \left[ b(D_i - x_i^\top \theta)^+ + h(D_i - x_i^\top \theta)^- \right]. \]

It can also be viewed as a quantile regression problem with \( \tau = b/(b + h) \).

We study the first product in the dataset, which is sold at \( m = 32 \) stores. Each store needs a model that decides its order quantity every day to minimize the cost. Throughout our experiments, we fix \( \tau = 0.9 \). For every \( j \in [m] \), the \( j \)-th store has historical data \( \{(x_{ji}, D_{ji})\}_{i=1}^{n} \), where \( x_{ji} \) consists of real-valued covariates available before the \( i \)-th day, and \( D_{ji} \) is the demand on that day. We use 19 covariates and add an all-one covariate. Therefore, \( x_{ji} \) has dimension \( d = 20 \). We focus on linear decision rules of the form \( x_{ji} \mapsto x_{ji}^\top \theta_j \), where \( \theta_j \in \mathbb{R}^d \). The problem is formulated as multi-task quantile regression. The loss function of task \( j \) is \( f_j(\theta) = \frac{1}{n} \sum_{i=1}^{n} \rho_\tau(D_{ji} - x_{ji}^\top \theta) \). Same as our experiments on synthetic data, here we also compare the new method with single-task learning (STL) and data pooling (DP).

Our testing set consists of all the data in 2019 (four months). For each \( k \in [12] \), we implement all methods on the data over the \( k \) months before 2019. The penalty parameter for our new method is \( \lambda = C \sqrt{d/n} \) with \( C \in \{0.1, 0.2, \cdots, 1\} \). We run the method on the first 80% of the training data for each \( C \) and evaluate them on the rest 20%. Then, we choose the one with the lowest validation error, refit the models on the whole training set. We measure the performance of three methods by their average testing losses over all the \( m \) tasks, which are proportional to the average daily costs of those decision rules.

Figure 3 reveals how the testing losses decrease as more training data become available. In particular, the new method is always the best. When there are only one or two month’s data for training, both the new method and DP outperform STL. Then the curve of DP flattens out, as its model misspecification error dominates the statistical error. The new method and STL benefit from increased sample size. The former is significantly better by a large margin when there are at most 8 months’ data for training. The two approaches have little difference when the training set is sufficiently large. Therefore, our approach is always a good choice, especially when the data are scarce.
6 DISCUSSIONS

We have studied a simple approach for multi-task optimization problems with possibly nonsmooth loss, theoretically proved its adaptivity to the unknown task relatedness, and demonstrated its power on real data. There are several directions we plan to pursue in future research. We will develop efficient algorithms for solving the multi-task non-smooth optimization problems we studied. The algorithms should fit for distributed computing architectures and preserve the privacy of individual dataset owners. We will also develop statistical tools for uncertainty quantification in the multi-task setting.
A DETERMINISTIC RESULTS

In this section, we present deterministic results on (2.2); see Appendix B for the proofs.

Definition A.1 (Regularity). Let \( \theta^* \in \mathbb{R}^d \), \( 0 < M \leq \infty \), \( 0 < \rho \leq L < \infty \), \( F : \mathbb{R}^d \to \mathbb{R} \), and \( 0 \leq \zeta < \infty \). A convex function \( f : \mathbb{R}^d \to \mathbb{R} \) is said to be \((\theta^*, M, \rho, L, F, \zeta)\)-regular if
- \( F \) is convex and twice differentiable;
- \( \rho I \preceq \nabla^2 F(\theta) \preceq L I \) holds for all \( \theta \in B(\theta^*, M) \);
- \( \sup_{\theta \in B(\theta^*, M)} ||f - \nabla F(\theta)||_2 \leq \zeta \leq \rho M/2 \) where \( f : \mathbb{R}^d \to \mathbb{R} \) is such that for every \( \theta \in \mathbb{R}^d \), \( f(\theta) \in \partial f(\theta) \);
- \( ||\nabla F(\theta^*)||_2 \leq \rho M/2 - \zeta \).

Theorem A.1 (Personalization). If \( f_j \) is \((\theta^*_j, M, \rho, L, F_j, \zeta_j)\)-regular and \( 0 \leq \lambda < \rho M/2 - \zeta_j \), then

\[
||\hat{\theta}_j - \tilde{\theta}_j||_2 \leq \frac{\lambda}{\rho} + \frac{\zeta_j}{\rho},
\]

\[
||\tilde{\theta}_j - \theta^*_j||_2 \leq \frac{||\nabla F(\theta^*_j)||_2}{\rho} + \frac{\zeta_j}{\rho}.
\]

In most cases, \( ||\nabla F(\theta^*_j)||_2 \) is 0 since the \( F_j \) that we relate \( f_j \) with would be the population loss function and \( \theta^*_j \) is the population loss minimizer. As such, the optimality gap of STL is due to \( \zeta_j \), a uniform first-order upper-bound.

Definition A.2 (Task relatedness). Let \( \varepsilon, \delta, \zeta_S \geq 0 \), \( \{\theta^*_j\}_{j=1}^m \subseteq \mathbb{R}^d \), \( 0 < M \leq \infty \), \( 0 < \rho \leq L < +\infty \), and \( S \subseteq [m] \). \( \{f_j\}_{j=1}^m \) are said to be \((\varepsilon, \delta, \zeta_S)\)-related with regularity parameters \((\{\theta^*_j\}_{j=1}^m, M, \rho, L, S, \{F_j\}_{j \in S}, \{\zeta_j\}_{j \in S}) \) if
- for any \( j \in S \), \( f_j \) is \((\theta^*_j, M, \rho, L, F_j, \zeta_j)\)-related (Definition A.1);
- \( \min_{\theta \in \mathbb{R}^d} \max_{j \in S} ||\theta^*_j - \theta||_2 \leq \delta \);
- \( |S^c| / |S| \leq \varepsilon \);
- \( \sup_{\theta \in B(\theta^*, M)} ||g_S(\theta) - \nabla G_S(\theta)||_2 \leq \zeta_S \leq \sum_{j \in S} \zeta_j \), where \( G_S = \sum_{j \in S} F_j \) and \( g_S : \mathbb{R}^d \to \mathbb{R}^d \) is such that for every \( \theta \in \mathbb{R}^d \), \( g_S(\theta) \in \partial \sum_{j \in S} f_j(\theta) \).

Theorem A.2 (Adaptivity and Robustness). Let \( \{f_j\}_{j=1}^m \) be \((\varepsilon, \delta, \zeta_S)\)-related with regularity parameters \((\{\theta^*_j\}_{j=1}^m, M, \rho, L, S, \{F_j\}_{j \in S}, \{\zeta_j\}_{j \in S}) \). Define \( \kappa = L/\rho \). Suppose \( \kappa \varepsilon < 1 \) and

\[
\frac{5\kappa}{1 - \kappa \varepsilon} \max_{j \in S} \{(||\nabla F_j(\theta^*_j)||_2 + \zeta_j)\} < \lambda < \frac{\rho M}{2}. \tag{A.1}
\]

Then, the estimators \( \{\hat{\theta}_j\}_{j \in S} \) in (2.2) satisfy

\[
||\hat{\theta}_j - \theta^*_j||_2 \leq \frac{||\sum_{k \in S} \nabla F_k(\theta^*_j)||_2}{\rho |S|} + \frac{7}{(1 - \kappa \varepsilon)} \min \left\{ \frac{3\kappa^2 \delta, 2 \lambda}{5 \rho} \right\} + \frac{\varepsilon \lambda}{\rho} + \frac{2 \zeta_S}{\rho |S|}.
\]

Moreover, there exists a constant \( C \) such that under the conditions \( \varepsilon = 0 \) and \( C \kappa L \delta < \lambda \), we have \( \hat{\theta}_1 = \cdots = \hat{\theta}_m = \arg\min_{\theta \in \mathbb{R}^d} \{\sum_{j=1}^m f_j(\theta)\} \).
B PROOF OF DETERMINISTIC RESULTS

B.1 Proof of Theorem A.1

By Lemma D.1 and $||\nabla F_j(\theta_*^j)||_2 \leq \rho M/2 - \zeta_j$, any minimizer $\tilde{\theta}_j$ of $f_j$ satisfies $||\tilde{\theta}_j - \theta_*^j||_2 \leq ||\nabla F_j(\theta_*^j)||_2/\rho + \zeta_j/\rho < M/2$. Hence $\nabla^2 F_j(\theta) \succeq \rho I$ holds for all $\theta \in B(\tilde{\theta}_j, M/2)$. By Lemma D.2 and $\lambda < \rho M/2 - \zeta_j$, $||\tilde{\theta}_j - \theta_j||_2 \leq \lambda/\rho + \zeta_j/\rho$.

B.2 Proof of Theorem A.2

First, note that the task-relatedness, combined with Lemma D.5, yields

\[ B \text{ PROOF OF DETERMINISTIC RESULTS} \]

\[ \sup_{\theta \in B(\theta^*, M)} \sup_{g_S \in \partial g_S(\theta)} ||g_S - \nabla G_S(\theta)||_2 \leq \zeta_S, \]

\[ \sup_{\theta \in B(\theta^*, M)} \sup_{f_j \in \partial f_j(\theta)} ||f_j - \nabla F_j(\theta)||_2 \leq \zeta_j, \quad \forall j \in S, \]

where $g_S = \sum_{j \in S} f_j$ and $G_S = \sum_{j \in S} F_j$.

Define $\gamma = \max_{j \in S} \{ ||\nabla F_j(\theta_j^*)||_2^2 + \zeta_j \}$. We first assume

\[ 3L\delta < \gamma + \frac{1 - \kappa \epsilon}{5\kappa} \lambda. \] (B.1)

Define $\theta^* = \arg\min_{\theta \in \mathbb{R}^d} \max_{j \in S} ||\theta_j^* - \theta||_2$, and recall that $\min_{\theta \in \mathbb{R}} \max_{j \in S} \{ ||\theta_j^* - \theta||_2 \} \leq \delta$. From (A.1) we have that $\gamma < \rho M/10$ and $\lambda < \rho M/2$. When

\[ 3L\delta < \gamma + \frac{1 - \kappa \epsilon}{5\kappa} \lambda < \gamma + \lambda < \frac{3}{5} LM, \]

we have $\delta < M/5$; thus $\max_{j \in S} ||\theta_j^* - \theta^*||_2 < M/5$. Recall that, for any $j \in S$, we have the sub-regularity condition that $\nabla^2 F(\theta) \succeq L I$, $\forall \theta \in B(\theta_j^*, M)$; thus we have $\nabla^2 F(\theta) \succeq L I$, $\forall \theta \in B(\theta^*, 4M/5)$. This leads to $||\nabla F_j(\theta_j^*) - \nabla F_j(\theta^*)||_2 \leq \delta$, $\forall j \in S$. Define $\eta = \max_{j \in S} \{ ||\nabla F_j(\theta^*)||_2 + \zeta_j \}$.

By triangle inequality,

\[ \eta \leq \gamma + \max_{j \in S} \{ ||\nabla F_j(\theta_j^*) - \nabla F_j(\theta^*)||_2 \} \leq \gamma + \lambda \delta + \frac{4}{3} \gamma + \frac{1 - \kappa \epsilon}{15\kappa} \lambda, \]

where the last inequality results from (B.1). Consequently, we have

\[ \frac{3\kappa \eta}{1 - \kappa \epsilon} < \frac{4\kappa \gamma}{1 - \kappa \epsilon} + \frac{\lambda}{5} < \frac{\rho M}{2} < LM. \]

By Lemma B.3, $\tilde{\theta}_j = \tilde{\beta}$ for all $j \in S$ and

\[ ||\tilde{\beta} - \theta^*||_2 \leq \frac{\epsilon \lambda}{\rho} + \frac{\sum_{j \in S} \nabla F_j(\theta^*)}{\rho |S|} + \frac{2\zeta_S}{\rho |S|}, \]

For any $j \in S$, $||\tilde{\theta}_j - \theta_j^*||_2 \leq ||\tilde{\beta} - \theta^*||_2 + ||\theta^* - \theta_j^*|| \leq ||\tilde{\beta} - \theta^*||_2 + \delta$. Also,

\[ \frac{||\sum_{j \in S} \nabla F_j(\theta^*)||_2}{\rho |S|} \leq \frac{||\sum_{j \in S} \nabla F_j(\theta_j^*)||_2}{\rho |S|} + \frac{L \delta}{\rho} \leq \frac{||\sum_{j \in S} \nabla F_j(\theta_j^*)||_2}{\rho |S|} + \kappa \delta. \]
Based on the above estimates and noting that $\kappa \geq 1$, we have

$$||\hat{\theta}_j - \theta_j^*||_2 \leq \left\| \sum_{k \in S} \nabla F_k(\theta_k^*) \right\|_2 \rho |S| + 2\kappa \delta + \frac{\varepsilon \lambda}{\rho} + \frac{2\zeta S}{\rho |S|}, \quad \forall j \in S.$$ 

Now, note that condition (A.1) forces $\lambda > 5\kappa \gamma$. The above result implies that when $3L\delta < \gamma + \frac{1 - \kappa \varepsilon}{\rho \kappa} \lambda$, for any $j \in S$,

$$||\hat{\theta}_j - \theta_j^*||_2 \leq \frac{2\gamma + \lambda}{\rho} \leq \frac{2}{5\kappa} + 1 \leq \frac{7\lambda}{5\rho},$$

where the second inequality is due to condition (A.1). Denote

$$U = \frac{\left\| \sum_{k \in S} \nabla F_k(\theta_k^*) \right\|_2}{\rho |S|} + \frac{\varepsilon \lambda}{\rho} + \frac{2\zeta S}{\rho |S|}.$$ 

To summarize, for any $j \in S$ we have

$$||\hat{\theta}_j - \theta_j^*||_2 \leq \frac{\left\| \sum_{k \in S} \nabla F_k(\theta_k^*) \right\|_2}{\rho |S|} + \frac{\varepsilon \lambda}{\rho} + \frac{2\zeta S}{\rho |S|} + \frac{7\kappa}{5\kappa} \cdot \frac{1}{1 - \kappa \varepsilon} \min \left\{ 3L\delta, \gamma + \frac{1 - \kappa \varepsilon}{5\kappa} \lambda \right\}.$$ 

The relation between $\hat{\theta}_j$ and $\arg\min_{\theta \in \mathbb{R}^d} \{ \sum_{j=1}^m f_j(\theta) \}$ can be derived from Lemma B.3.

### B.3 Supporting Lemmas for Deterministic Results

**Lemma B.1.** Let $\{f_j\}_{j=1}^m$ and $\{F_j\}_{j=1}^m$ be convex. Suppose $F_j$ is twice differentiable for all $j \in [m]$, and there exist $\theta^* \in \mathbb{R}^d$ and $0 < M < \infty$ such that for all $j \in [m]$ 

$$\rho_j I \preceq \nabla^2 F_j(\theta) \preceq L_j I, \quad \forall \theta \in B(\theta^*, M)$$

with some $0 < \rho_j \leq L_j < +\infty$. Define

$$f_0 = \sum_{j=1}^m f_j \square (\lambda_j \| \cdot \|_2), \quad g_0 = \sum_{j=1}^m f_j, \quad G_0 = \sum_{j=1}^m F_j,$$

and denote $\rho_0 = \sum_{j=1}^m \rho_j$. If, for some $0 \leq \zeta_0, \zeta_1, \ldots, \zeta_m < +\infty$,

$$\sup_{\theta \in B(\theta^*, M)} \sup_{g_0 \in \partial g_0(\theta)} \|g_0 - \nabla G_0(\theta)\|_2 \leq \zeta_0,$$

$$\sup_{\theta \in B(\theta^*, M)} \sup_{f_j \in \partial f_j(\theta)} \|f_j - \nabla F_j(\theta)\|_2 \leq \zeta_j, \quad \forall j \in [m],$$

15
and

\[
\|\nabla F_j(\theta^*)\|_2 + \zeta_j + \frac{2L_j}{\rho_0} \left(\left\|\sum_{k=1}^{m} \nabla F_k(\theta^*)\right\|_2 + \zeta_0\right) < \lambda_j < \|\nabla F_j(\theta^*)\|_2 + \zeta_j + L_j M
\]

for all \( j \in [m] \), then \( \tilde{\theta}_1 = \cdots = \tilde{\theta}_m = \tilde{\beta} = \tilde{\theta} \),

\[
f_0(\theta) = g_0(\theta), \quad \forall \theta \in B(\theta^*, R),
\]

\[
\|\tilde{\beta} - \theta^*\|_2 \leq \frac{\|\sum_{k=1}^{m} \nabla F_k(\theta^*)\|_2}{\rho_0} + \frac{\zeta_0}{\rho_0}
\]

where \( R = \min_{j \in [m]} \{ (\lambda_j - \|\nabla F_j(\theta^*)\|_2 - \zeta_j) / L_j \} \).

**Proof of Lemma B.1.** By assumption, \((\lambda_j - \|\nabla F_j(\theta^*)\|_2 - \zeta_j) / L_j < M \) for any \( j \in [m] \). Since \( \lambda_j > \|\nabla F_j(\theta^*)\|_2 + \zeta_j \), by Lemma D.3, we have \( f_j = f_j(\lambda_j \|\cdot\|_2) \) in \( B(\theta^*, \lambda_j - \|\nabla F_j(\theta^*)\|_2 - \zeta_j) / L_j \). Then, \( f_0 = g_0 \in B(\theta^*, R) \).

Since \( \nabla^2 F_j(\theta) \succeq \rho_j I \) for any \( \theta \in B(\theta^*, M) \), we have

\[
F_j(\theta) - F_j(\theta^*) \geq \frac{\rho_j}{2} \|\theta - \theta^*\|^2 + \langle \nabla F_j(\theta^*), \theta - \theta^* \rangle,
\]

and thus

\[
G_0(\theta) - G_0(\theta^*) = \sum_{j=1}^{m} F_j(\theta) - F_j(\theta^*)
\]

\[
\geq \frac{1}{2} \left( \sum_{j=1}^{m} \rho_j \right) \|\theta - \theta^*\|^2 + \left\langle \sum_{j=1}^{m} \nabla F_j(\theta^*), \theta - \theta^* \right\rangle
\]

\[
= \frac{\rho_0}{2} \|\theta - \theta^*\|^2 + \langle \nabla G_0(\theta^*), \theta - \theta^* \rangle, \quad \forall \theta \in B(\theta^*, M),
\]

from which we have \( \nabla^2 G_0(\theta^*) \succeq \rho_0 I \) for all \( \theta \in B(\theta^*, M) \). By assumption, we have

\[
\frac{2}{\rho_0} \left( \left\|\sum_{k=1}^{m} \nabla F_k(\theta^*)\right\|_2 + \zeta_0\right) \leq \frac{\lambda_j - \|\nabla F_j(\theta^*)\|_2 - \zeta_j}{L_j}, \quad \forall j \in [m].
\]

Taking the minimum over \( j \) on both sides, we have

\[
\left\|\sum_{j=1}^{m} \nabla F_j(\theta^*)\right\|_2 \leq \frac{1}{2} R \rho_0 - \zeta_0.
\]

By Lemma D.1,

\[
\tilde{\beta} = \arg\min_{\beta \in \mathbb{R}^d} f_0(\beta) = \arg\min_{\theta \in \mathbb{R}^d} g_0(\theta) = \tilde{\theta} \subseteq B\left(\theta^*, \frac{\left\|\sum_{j=1}^{m} \nabla F_j(\theta^*)\right\|_2}{\rho_0} + \frac{\zeta_0}{\rho_0}\right).
\]

Finally, \( \tilde{\theta}_j = \tilde{\beta} \) follows from \( \tilde{\theta}_j \in \arg\min_{\theta \in \mathbb{R}^d} \left\{ f_j(\theta) + \lambda_j \|\theta - \tilde{\beta}\|_2 \right\} \), \( \tilde{\theta}_j \in B(\theta^*, R) \), and Lemma D.3. We have completed the proof. \( \square \)
Lemma B.2 (Robustness). Let \( \{f_j\}_{j=1}^m \) be convex. Suppose there exists \( S \subseteq [m], \theta^{*} \in \mathbb{R}^d \) and \( 0 < M < \infty \) such that for all \( j \in S \), there exists a twice differentiable convex function \( F_j \) such that
\[
\rho_j I \preceq \nabla^2 F_j(\theta) \preceq L_j I, \quad \forall \theta \in B(\theta^{*}, M)
\]
with some \( 0 < \rho_j \leq L_j < +\infty \). Define
\[
f_S = \sum_{j \in S} f_j(\theta) \quad \text{and} \quad G_S = \sum_{j \in S} F_j,
\]
and denote
\[
\rho_S = \sum_{j \in S} \rho_j, \quad \tilde{\theta}_S = \arg\min_{\theta \in B(\theta^{*}, M)} f_S(\theta), \quad \lambda_S = \sum_{j \in S^c} \lambda_j.
\]
If, for some \( 0 \leq \zeta_j < +\infty, j \in S \), and some \( 0 \leq \zeta_S < +\infty \),
\[
\sup_{\theta \in B(\theta^{*}, M)} \sup_{f_j(\theta) \in \partial f_j(\theta)} \|f_j - \nabla f_j(\theta)\|_2 \leq \zeta_j, \quad \forall j \in S,
\]
and
\[
\|\nabla F_j(\theta^*)\|_2 + \zeta_j + \frac{2L_j}{\rho_S} \left( \|\sum_{k \in S} \nabla F_k(\theta^*)\|_2 + \zeta_S \right) + \frac{L_j \lambda_S}{\rho_S} < \lambda_j < \|\nabla F_j(\theta^*)\|_2 + \zeta_j + L_j M
\]
for all \( j \in S \), then
\[
\|\tilde{\theta}_S - \theta^*\|_2 \leq \frac{\|\sum_{k \in S} \nabla F_k(\theta^*)\|_2}{\rho_S} + \frac{\zeta_S}{\rho_S},
\]
\( \tilde{\theta}_j = \tilde{\beta} \) for \( j \in S \), and
\[
\|\tilde{\beta} - \tilde{\theta}_S\|_2 \leq \frac{\lambda_S}{\rho_S} + \frac{\zeta_S}{\rho_S}.
\]

Proof of Lemma B.2. Define \( R = \min_{j \in S} \{ (\lambda_j - \|\nabla F_j(\theta^*)\|_2 - \zeta_j)/L_j \} \). By Lemma B.1 and its proof, we have
\[
f_S(\theta) = g_S(\theta), \quad \forall \theta \in B(\theta^*, R),
\]
\[
\tilde{\theta}_S = \arg\min_{\theta \in \mathbb{R}^d} g_S(\theta) \subseteq B\left( \theta^*, \|\sum_{k \in S} \nabla F_k(\theta^*)\|_2 + \frac{\zeta_S}{\rho_S} \right),
\]
and \( \nabla^2 G_S(\theta) \succeq \rho_S I \) for all \( \theta \in B(\theta^*, R) \). Define \( f_0 = \sum_{j=1}^m f_j \). By Lemma D.4 we have that
\[
f_0 - f_S = \sum_{j \in S^c} f_j(\theta) \quad \text{is convex and } \lambda_{S^c}-\text{Lipschitz.}
\]
Note that
\[
R > \frac{1}{\rho_S} \left( 2 \|\sum_{k \in S} \nabla F_k(\theta^*)\|_2 + 2\zeta_S + \lambda_S \right),
\]
and thus
\[
\frac{\lambda_{S^c}}{\rho_S} < R - \frac{2\|\sum_{k \in S} \nabla F_k(\theta^*)\|_2}{\rho_S} - \frac{\zeta_S}{\rho_S}.
\]
Denote the right-hand side above as $R_{S_c}$. Since $\lambda_{S_c} < \rho S_{R_{S_c}} - \zeta_S$ and $G_S$ is strongly convex in $B(\theta^*, R_{S_c}) \subseteq B(\theta^*, R)$, we can control the effect of $f_0 - f_S$ by Lemma D.2:

$$\|\hat{\theta} - \hat{\theta}_S\|_2 \leq \frac{\lambda_{S_c}}{\rho S} + \frac{\zeta_S}{\rho S}.$$ 

Finally, note that $\hat{\theta}_j \in \text{argmin}_{\theta \in \mathbb{R}^d} \left\{ f_j(\theta) + \lambda_j \| \theta - \hat{\theta} \|_2 \right\}$ for all $j \in S$. Since, for all $j \in S$, $\lambda_j > \| \nabla F_j(\theta^*) \|_2 + \zeta_j$ and

$$\| \hat{\theta} - \theta^* \|_2 \leq \| \hat{\theta}_S - \theta^* \|_2 + \| \hat{\theta} - \hat{\theta}_S \|_2$$

$$\leq \frac{\| \sum_{k \in S} \nabla F_k(\theta^*) \|_2}{\rho S} + \frac{2 \zeta_S}{\rho S} + \frac{\lambda_{S_c}}{\rho S}$$

$$\leq \frac{\lambda_j - \| \nabla F_j(\theta^*) \|_2 - \zeta_j}{L_j} < M,$$

$\hat{\theta}_j = \hat{\theta}$ for all $j \in S$ by Lemma D.3. \hfill \Box

**Lemma B.3.** Let $\{f_j\}_{j=1}^m$ be convex. Suppose there exists $S \subseteq [m]$, $\theta^* \in \mathbb{R}^d$, $0 < M < \infty$ and $0 < \rho \leq L < \infty$ such that for all $j \in S$, there exists a twice differentiable convex function $F_j$ such that

$$\rho I \preceq \nabla^2 F_j(\theta) \preceq L I, \quad \forall \theta \in B(\theta^*, M).$$

Define $g_S = \sum_{j \in S} f_j$, $G_S = \sum_{j \in S} F_j$, and denote

$$\tilde{\theta}_S \in \text{argmin}_{\theta \in \mathbb{R}^d} g_S(\theta), \quad \eta = \max_{j \in S} \{ \| \nabla F_j(\theta^*) \|_2 + \zeta_j \}, \quad \kappa = \frac{L}{\rho}, \quad \varepsilon = \frac{|S_c|}{|S|}.$$

Further suppose that, for some $0 \leq \zeta_j < +\infty$, $j \in S$, and some $0 \leq \zeta_S \leq \sum_{j \in S} \zeta_j$,

$$\sup_{\theta \in B(\theta^*, M)} \sup_{g_S \in \partial g_S(\theta)} \| g_S - \nabla G_S(\theta) \|_2 \leq \zeta_S,$$

$$\sup_{\theta \in B(\theta^*, M)} \sup_{f_j \in \partial f_j(\theta)} \| f_j - \nabla F_j(\theta) \|_2 \leq \zeta_j, \quad \forall j \in S.$$

Take $\lambda_j = \lambda$ for all $j \in [m]$ and some $\lambda > 0$. If $\kappa \varepsilon < 1$ and

$$\frac{3 \kappa \eta}{1 - \kappa \varepsilon} < \lambda < LM,$$

then $\tilde{\theta}_j = \hat{\theta}$ for $j \in S$, and

$$\| \hat{\theta} - \tilde{\theta}_S \|_2 \leq \frac{\sum_{j \in S_c} \lambda_j + \zeta_S}{\rho |S|} \leq \frac{\varepsilon \lambda}{\rho |S|} + \frac{\zeta_S}{\rho |S|},$$

$$\| \tilde{\theta}_S - \theta^* \|_2 \leq \frac{\| \sum_{k \in S} \nabla F_k(\theta^*) \|_2}{\rho |S|} + \frac{\zeta_S}{\rho |S|}.$$

**Proof of Lemma B.3.** From the assumption $\lambda > \frac{3 \kappa \eta}{1 - \kappa \varepsilon}$ we get $\lambda > 3 \kappa \eta + \kappa \varepsilon \lambda$ and for all $j \in S$,

$$\| \nabla F_j(\theta^*) \|_2 + \zeta_j + \frac{2L}{\rho |S|} \left( \frac{\| \sum_{k \in S} \nabla F_k(\theta^*) \|_2 + \zeta_S}{\rho |S|} + \frac{L \lambda |S_c|}{\rho |S|} \right)$$

$$\leq \eta + 2 \kappa \eta + \kappa \varepsilon \lambda \leq 3 \kappa \eta + \kappa \varepsilon \lambda < \lambda = \lambda_j.$$

Note that $\lambda_j = \lambda \leq LM$ for all $j \in S$. The proof is finished by Lemma B.2 and its proof. Note that $\text{argmin}_{\theta \in \mathbb{R}^d} \sum_{j \in S} f_j(\theta) = \text{argmin}_{\theta \in \mathbb{R}^d} \sum_{j \in S} f_j(\theta)$. \hfill \Box
C PROOF OF SECTION 4

C.1 Proof of Theorem 4.1

The results follow immediately from Theorem A.1 and Corollary C.1.

C.2 Proof of Theorem 4.2

Take populations risks as \( \{F_j\}_{j=1}^m \). By assumptions, \( \rho, L, M > 1 \) and \( \frac{\varepsilon}{1 - \varepsilon} \lesssim 1 \). Then, by Theorem A.2, there exist positive constants \( \{C_j\}_{j=0}^2 \) such that when

\[
C_1 \max_j \zeta_j < \lambda < C_2,
\]

we have for all \( j \in S \),

\[
||\hat{\theta}_j - \theta_j^*||_2 \leq C_0 \left( \frac{\zeta_j}{m} + \min \{\delta, \lambda\} + \varepsilon \lambda \right).
\]

Thus, we are to determine the order of \( \max_{j \in S} \{\zeta_j\} \) and \( \zeta_S \). Let \( l : \mathbb{R}^d \times \mathcal{X} \to \mathbb{R}^d \) be such that for every \( \theta \in \mathbb{R}^d \), \( l(\theta, \xi) \in \partial \ell(\theta, \xi) \). For any \( \{D_j\}_{j=1}^m \), denote \( f_j(\theta) = \frac{1}{n} \sum_{i=1}^n l(\theta, \xi_{ji}) \) for all \( j \in [m] \), and \( F(\theta) = \sum_{j=1}^m f_j(\theta) \). By Corollary C.1, for some positive constants \( c_1 \) and \( c_2 \), the following holds with probability at least \( 1 - c_1 n^{-d} \):

\[
\sup_{\theta \in B(\theta^*, r)} ||f_j(\theta) - \mathbb{E} f_j(\theta)||_2 \leq c_2 \sigma \sqrt{\frac{\log n + \log m}{n}}, \quad \forall j \in S.
\]

By Corollary C.2, for some positive constants \( c_3 \) and \( c_4 \), the following holds with probability at least \( 1 - c_4 (mn)^{-d} \):

\[
\sup_{\theta \in B(\theta^*, r)} ||F(\theta) - \mathbb{E} F(\theta)||_2 \leq c_4 \sigma \sqrt{\frac{mn \log mn}{n}}.
\]

When \( 1 \leq (n^d)^{m-1} m^{m-d} \), we have \( \zeta_S \leq m \cdot \max_{j \in S} \zeta_j \). Theorem 4.1 applied to the tasks in \( S^c \) yields

\[
||\hat{\theta}_j - \theta_j^*||_2 \leq C_0 \lambda, \quad \forall j \in S^c.
\]

The relation between \( \hat{\theta}_j \) and \( \arg\min_{\theta \in \mathbb{R}^d} \left( \sum_{j=1}^m f_j(\theta) \right) \) can be derived from Lemma A.2. We finish the proof by taking union bounds and redefining the constants.

C.3 Supporting Lemmas for Section 4

Lemma C.1 (Uniform First-Order Condition). Define \( R(\theta) = \sum_{k=1}^K [l(\theta, \xi_k) - \mathbb{E} l(\theta, \xi_k)] \), where \( \{\xi_k\}_{k=1}^K \) are independent and \( l \) satisfies Assumptions 4.1, 4.2 and 4.3. Choose some constant \( 0 < r < \infty \). Then, there exist constants \( c_1, c_2 > 0 \) such that with probability at least \( 1 - c_1 K^{-d} \),

\[
\sup_{\theta \in B(\theta_0, r)} ||R(\theta)||_2 \leq c_2 \sigma \sqrt{d K \log K}, \quad \forall \theta_0 \in \mathbb{R}^d.
\]

Proof of Lemma C.1. By assumption, \( l(\theta, \xi_k) \) is subgaussian for all \( \theta \) and \( k \) and \( ||l(\theta, \xi_k)||_{\psi_2} \leq \sigma \). Thus, \( R(\theta) \) is the sum of \( K \) independent centered subgaussian random vectors and \( ||R(\theta)||_{\psi_2} \lesssim \sigma \sqrt{K} \). By Theorem 2.1 of Hsu et al. (2012), for some \( c > 0 \),

\[
\mathbb{P} \left[ ||R(\theta)||_2^2 > c^2 K \sigma^2 \left( d + 2 \sqrt{d} l + 2 l \right) \right] \leq e^{-t}, \quad \forall t \geq 0.
\]
Since $2\sqrt{dt} \leq d + t$, 
\[
\mathbb{P}\left[\|R(\theta)\|_2 > c\sigma \sqrt{K(d + t)}\right] \leq e^{-t}, \quad \forall t \geq 0.
\]

Similar to the proof of Lemma 5.2 in Vershynin (2012), $\forall \varepsilon > 0$, an $\varepsilon$-net $N_\varepsilon$ over $B(\theta_0, r)$ satisfies 
\[
|N_\varepsilon| \leq \left(1 + \frac{2r}{\varepsilon}\right)^d.
\]

By union bounds, 
\[
\mathbb{P}\left[ \max_{\theta \in N_\varepsilon} \|R(\theta)\|_2 \leq c\sigma \sqrt{K(d + t)} \right] \geq 1 - \left(1 + \frac{2r}{\varepsilon}\right)^d e^{-t}, \quad \forall t \geq 0.
\]

Let $t = d \log K$, $\varepsilon = r \sqrt{K \log K}$. We have 
\[
\left(1 + \frac{2r}{\varepsilon}\right)^d e^{-t} = \left(1 + \frac{2}{\sqrt{K \log K}}\right)^d K^{-d} \leq cK^{-d}.
\]

Thus, 
\[
\mathbb{P}\left[ \max_{\theta \in N_\varepsilon} \|R(\theta)\|_2 \leq c_2 \sigma \sqrt{dK \log K} \right] \geq 1 - c_1 K^{-d}.
\]

By the proof of Proposition 3.4 in Chen et al. (2021), with probability at least $1 - c_1 K^{-d}$, 
\[
\|R(\theta_1) - R(\theta_2)\|_2 \leq c_3 \sqrt{dK \log K},
\]

for any $\theta_1 \in B(\theta_0, r)$, $\theta_2 \in N_\varepsilon$ such that $\|\theta_1 - \theta_2\|_2 \leq \varepsilon$. Taking union bounds over the two events, we have, with probability at least $1 - c_1 K^{-d}$, 
\[
\sup_{\theta \in B(\theta_0, r)} \|R(\theta)\|_2 \leq c_2 \sigma \sqrt{dK \log K}
\]

for some $c_1, c_2 > 0$. \hfill \qed

**Corollary C.1** (Maximum of $\{\zeta_j\}_{j \in S}$). Choose some constant $0 < r < \infty$. There exist positive constants $c_1$ and $c_2$ such that, with probability at least $1 - c_1 n^{-d}$, 
\[
\sup_{\theta \in B(\theta^*, r)} \|f_j(\theta) - \mathbb{E}f_j(\theta)\|_2 \leq c_2 \sigma \sqrt{d \log n + \log m \over n}, \quad \forall j \in S.
\]

*Proof of Corollary C.1.* The proof is almost identical to the proof for Lemma C.1. We can set $K = n$, $t = d \log n + \log |S|$ for all tasks in $S$. Taking union bounds and dividing by $n$ on both sides yield the result. \hfill \qed

**Corollary C.2** (Order of $\zeta_S$). Choose some constant $0 < r < \infty$. There exist positive constants $c_1$ and $c_2$ such that, with probability at least $1 - c_1 (mn)^{-d}$, 
\[
\sup_{\theta \in B(\theta^*, r)} \|F(\theta) - \mathbb{E}F(\theta)\|_2 \leq c_2 \sigma \sqrt{dm \log mn \over n}.
\]

*Proof of Corollary C.2.* The proof is almost identical to the proof for Lemma C.1. Setting $K = |S| n$ and dividing by $n$ on both sides yield the result. \hfill \qed
D TECHNICAL LEMMAS

Lemma D.1. Let $F, f : \mathbb{R}^d \to \mathbb{R}$ be convex. Denote $x^* \in \text{argmin}_{x \in \mathbb{R}^d} F(x)$ and $\bar{x} \in \text{argmin}_{x \in \mathbb{R}^d} f(x)$. Suppose there exist $x_0 \in \mathbb{R}^d$, $G_0 \in \partial F(x_0)$, $0 < r, \rho < \infty$ and $\zeta \geq 0$ such that $\|G_0\|_2 \leq r \rho / 2 - \zeta$,

$$F(x) - F(x_0) \geq \langle G_0, x - x_0 \rangle + \frac{\rho}{2} \|x - x_0\|_2^2, \quad \forall x \in B(x_0, r),$$

and

$$\sup_{x \in B(x_0, r)} \left\{ \sup_{g \in \partial f(x), G \in \partial F(x)} \|g - G\|_2 \right\} \leq \zeta.$$

Then,

$$\|x_0 - x^*\|_2 \leq \frac{2\|G_0\|_2}{\rho} \quad \text{and} \quad \|x_0 - \bar{x}\|_2 \leq \frac{2\|G_0\|_2}{\rho} + \frac{2\zeta}{\rho}.$$

Furthermore, if $\nabla^2 F(x) \succeq \rho I$ for all $x \in B(x_0, r)$, then $x^*$ is unique, and we have

$$\|x_0 - x^*\|_2 \leq \frac{\|\nabla F(x_0)\|_2}{\rho} \quad \text{and} \quad \|x_0 - \bar{x}\|_2 \leq \frac{\|\nabla F(x_0)\|_2}{\rho} + \frac{\zeta}{\rho}.$$

Proof of Lemma D.1. Let $G(x), g(x)$ be subgradients of $F(x), f(x)$, respectively, and define $x_t = (1 - t)x_0 + tx$. We have

$$f(x) - f(x_0) = \langle x - x_0, \int_0^1 g(x_t)dt \rangle$$

$$F(x) - F(x_0) = \langle x - x_0, \int_0^1 G(x_t)dt \rangle$$

This yields

$$f(x) - f(x_0) \geq F(x) - F(x_0) - \|x - x_0\|_2 \left\| \int_0^1 [g(x_t) - G(x_t)]dt \right\|_2$$

$$\geq \frac{\rho}{2} \|x - x_0\|_2 - \|G_0\|_2 \|x - x_0\|_2 - \|x - x_0\|_2 \int_0^1 \|g(x_t) - G(x_t)\|_2 dt.$$  

When $\|x - x_0\|_2 < r$,

$$\int_0^1 \|g(x_t) - G(x_t)\|_2 dt \leq \sup_{x \in B(x_0, r)} \left\{ \sup_{g \in \partial f(x), G \in \partial F(x)} \|g - G\|_2 \right\} \leq \zeta,$$

we have

$$f(x) - f(x_0) \geq \frac{\rho}{2} \|x - x_0\|_2^2 - (\|G_0\|_2 + \zeta) \|x - x_0\|_2$$

$$\geq \frac{\rho}{2} \|x - x_0\|_2 \left( \|x - x_0\|_2 - \frac{2\|G_0\|_2}{\rho} - \frac{2\zeta}{\rho} \right), \quad \forall x \in B(x_0, r).$$

Hence $f(x) - f(x_0) > 0$ when $2\|G_0\|_2 / \rho + 2\zeta / \rho < \|x - x_0\|_2 \leq r$. When $\|x - x_0\|_2 > r$, there exists $z = (1 - t)x_0 + tx$ for some $t \in (0, 1)$ such that $\|z - x_0\|_2 = r$. By $f(z) > f(x_0)$ and the convexity of $f$, we have

$$f(x_0) < f(z) \leq (1 - t)f(x_0) + tf(x)$$

and thus $f(x) > f(x_0)$. Therefore, $\text{argmin}_{x \in \mathbb{R}^d} f(x) \subseteq B(x_0, 2\|G_0\|_2 / \rho + 2\zeta / \rho)$. By a similar argument, $\text{argmin}_{x \in \mathbb{R}^d} F(x) \subseteq B(x_0, 2\|G_0\|_2 / \rho)$.  

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Now, suppose that $\nabla^2 F(x) \succeq \rho I$ for all $x \in B(x_0, r)$. From $\arg\min_{x \in \mathbb{R}^d} F(x) \subseteq B(x_0, 2\|G_0\|_2/\rho) \subseteq B(x_0, r)$ and the strong convexity of $F$ therein we get the uniqueness of $x^*$. Then, $\nabla F(x^*) = 0$, $\|x^* - x_0\|_2 \leq r$, and from
\[
\|\nabla F(x_0)\|_2 = \|\nabla F(x_0) - \nabla F(x^*)\|_2 \\
= \left\| \left( \int_0^1 \nabla^2 F((1-t)x^* + tx_0) dt \right) (x_0 - x^*) \right\|_2 \geq \rho \|x_0 - x^*\|_2
\]
we have $\|x_0 - x^*\|_2 \leq \|\nabla F(x_0)\|_2/\rho$. Finally, by the definition of $\tilde{x}$, $0 \in \partial f(\tilde{x})$. We have
\[
\|\nabla F(x_0)\|_2 \geq \|\nabla F(x_0) - \nabla F(\tilde{x})\|_2 - \|\nabla F(\tilde{x}) - 0\|_2 \\
\geq \left\| \left( \int_0^1 \nabla^2 F((1-t)\tilde{x} + tx_0) dt \right) (x_0 - \tilde{x}) \right\|_2 - \sup_{g \in \partial f(\tilde{x})} \|g - \nabla F(\tilde{x})\|_2 \\
\geq \rho \|x_0 - \tilde{x}\|_2 - \zeta,
\]
from which we have $\|x_0 - \tilde{x}\|_2 \leq \|\nabla F(x_0)\|_2/\rho + \zeta/\rho$. We have completed the proof. \hfill \qed

**Lemma D.2.** Let $F, f : \mathbb{R}^d \to \mathbb{R}$ be convex functions and $x^* = \arg\min_{x \in \mathbb{R}^d} F(x)$. Suppose $F$ is differentiable and
\[
\rho I \preceq \nabla^2 F(x), \quad \forall x \in B(x^*, r)
\]
holds for some $0 < \rho < \infty$ and $0 < r < \infty$. If, for some $\zeta \geq 0$,
\[
\sup_{x \in B(x^*, r)} \sup_{f \in \partial f(x)} \|f - \nabla F(x)\|_2 \leq \zeta,
\]
then
\[
\|f\|_2 \geq \rho \min \{\|x - x^*\|_2, r\} - \zeta, \quad \forall f \in \partial f(x). \tag{D.1}
\]
If $g : \mathbb{R}^d \to \mathbb{R}$ is convex and $\lambda$-Lipschitz for some $0 \leq \lambda < \rho r - \zeta$, then all minimizers of $f(x) + g(x)$ belong to $B(x^*, \lambda/\rho + \zeta/\rho)$.

**Proof of Lemma D.2.** The optimality of $x^*$ and the strong convexity of $F$ near $x^*$ implies $\nabla F(x^*) = 0$. Choose any $f \in \partial f(x)$. If $0 < \|x - x^*\|_2 \leq r$, then
\[
\|f\|_2 \|x - x^*\|_2 \geq \langle f, x - x^* \rangle \\
= \langle f - \nabla F(x), x - x^* \rangle + \langle \nabla F(x) - \nabla F(x^*), x - x^* \rangle \\
\geq - \sup_{f \in \partial f(x)} \|f - \nabla F(x)\|_2 \|x - x^*\|_2 + \left\langle \left( \int_0^1 \nabla^2 F((1-t)x^* + tx) dt \right) (x - x^*), x - x^* \right\rangle \\
\geq \rho \|x - x^*\|_2^2 - \zeta \|x - x^*\|_2
\]
and $\|f\|_2 \geq \rho \|x - x^*\|_2 - \zeta$. If $\|x - x^*\|_2 > r$, there exists $z = (1-t)x^* + tx$ for some $t \in (0, 1)$ such that $\|z - x^*\|_2 = r$. By the convexity of $f$, $\langle f - f', x - z \rangle \geq 0$ and hence $\langle f - f', x - x^* \rangle \geq 0$. Then,
\[
\|f\|_2 \|x - x^*\|_2 \geq \langle f, x - x^* \rangle \\
= \langle f - f', x - x^* \rangle + \langle f' - \nabla F(z), x - x^* \rangle + \langle \nabla F(z) - \nabla F(x^*), x - x^* \rangle \\
\geq - \sup_{f' \in \partial f(z)} \|f' - \nabla F(z)\|_2 \|x - x^*\|_2 + \left\langle \left( \int_0^1 \nabla^2 F((1-t)x^* + tz) dt \right) (z - x^*), x - x^* \right\rangle
\]
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\[ \geq pr \|x - x^*\|_2 - \zeta \|x - x^*\|_2 \]
and \[ \|f\|_2 \geq pr - \zeta. \] We have verified (D.1).

Choose any \( \hat{x} \in \text{argmin}_{x \in \mathbb{R}^d} \{ f(x) + g(x) \} \). There exist \( f \in \partial f(\hat{x}) \) and \( g \in \partial g(\hat{x}) \) such that \( f + g = 0 \). The Lipschitz property of \( g \) yields \[ \|f\|_2 = \|g\|_2 \leq \lambda. \] Since \( \lambda < pr - \zeta \), we obtain from (D.1) that
\[ pr - \zeta > \lambda \geq \|f\|_2 \geq \rho \min \{\|\hat{x} - x^*\|_2, r\} - \zeta, \]
which leads to \[ \|\hat{x} - x^*\|_2 \leq \lambda/\rho + \zeta/\rho. \] We have completed the proof. \( \square \)

**Lemma D.3.** Let \( F, f : \mathbb{R}^d \rightarrow \mathbb{R} \) be convex functions. Suppose \( F \) is twice differentiable and
\[ \nabla^2 F(x) \preceq LI, \quad \forall x \in B(x^*, M) \]
holds for some \( x^* \in \mathbb{R} \), \( 0 < L < \infty \) and \( 0 < M < \infty \). If, for some \( \zeta \geq 0 \) and \( \lambda > 0 \),
\[ \sup_{x \in B(x^*, M)} \sup_{f \in \partial f(x)} \|f - \nabla F(x)\|_2 \leq \zeta, \]
and \( \lambda > \|\nabla F(x^*)\|_2 + \zeta \), then
\[ f(x) = f(\hat{x})(\lambda \cdot \|\cdot\|_2)(x) \quad \text{and} \quad \text{argmin}_{x \in \mathbb{R}^d} \{ f(x') + \lambda \|x - x'\|_2 \} = x \]
hold for all \( x \in B(x^*, \min \{(\lambda - \|\nabla F(x^*)\|_2 - \zeta)/L, M\}) \).

**Proof of Lemma D.3.** For any \( x \) such that \[ \|x - x^*\|_2 \leq \min \{(\lambda - \|\nabla F(x^*)\|_2 - \zeta)/L, M\}, \]
we have
\[ \sup_{f \in \partial f(x)} \|f\|_2 \leq \sup_{f \in \partial f(x)} \|f - \nabla F(x)\|_2 + \|\nabla F(x) - \nabla F(x^*)\|_2 \]
\[ \leq \zeta + L \|x - x^*\|_2 + \|\nabla F(x^*)\|_2 \leq \lambda. \]
Recall \( f(\hat{x})(\lambda \cdot \|\cdot\|_2)(x) \) is convex, \( \inf_{x \in \mathbb{R}^d} f(x') + \lambda \|x - x'\|_2 \). Define \( h(x') = f(x') + \lambda \|x - x'\|_2 \). Since \[ \|f\|_2 \leq \lambda \] for any \( f \in \partial f(x) \), it follows from \( \partial \|x-x'\|_2 \) that \( 0 \in \partial h(x') \). Thus, \[ \text{argmin}_{x \in \mathbb{R}^d} \{ f(x') + \lambda \|x - x'\|_2 \} = x, \]
and \( f(x) = f(\hat{x})(\lambda \cdot \|\cdot\|_2)(x). \) \( \square \)

**Lemma D.4.** If \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) is convex, \( \inf_{x \in \mathbb{R}^d} f(x) > -\infty, g : \mathbb{R}^d \rightarrow \mathbb{R} \) is convex and \( L \)-Lipschitz with respect to a norm \( \|\cdot\| \) for some \( L \geq 0 \), then \( f \circ g \) is convex and \( L \)-Lipschitz with respect to \( \|\cdot\| \).

**Proof of Lemma D.4.** The lemma is directly taken from Lemma E.4 of Duan and Wang (2022). \( \square \)

**Lemma D.5.** Let \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) be a convex function, \( g : \mathbb{R}^d \rightarrow \mathbb{R}^d \) be a continuous vector field, and \( \Omega \subseteq \mathbb{R}^d \) be an open set. Choose any \( f : \mathbb{R}^d \rightarrow \mathbb{R}^d \) such that for every \( x \in \mathbb{R}^d \), \( f(x) \in \partial f(x) \). Then,
\[ \sup_{x \in \Omega} \sup_{h \in \partial f(x)} \|h - g(x)\|_2 = \sup_{x \in \Omega} \|f(x) - g(x)\|_2. \]

**Proof of Lemma D.5.** Define \( M = \sup_{x \in \Omega} \|f(x) - g(x)\|_2 \). The claim is trivially true when \( M = \infty \). Below we assume that \( M < \infty \). Choose an arbitrary \( x \in \Omega \) and any \( h \in \partial f(x) \). It suffices to prove that \[ \|h - g(x)\|_2 \leq M. \] Let \( v = (h - g(x))/(h - g(x))\|_2 \) if \( h \neq g(x) \); otherwise, let \( v \) be any unit-norm vector. By construction, \[ \|h - g(x)\|_2 = (h - g(x), v). \] Define a univariate function \( F(t) = f(x + tv), \ t \in \mathbb{R} \). It is convex and satisfies
\[ \partial F(t) = \{u : u \in \partial f(x + tv)\}. \]
In particular, we have \( \langle h, v \rangle \in \partial F(0) \) and \( \langle f(x + n^{-1}v), v \rangle \in \partial F(n^{-1}), \forall n \). By the convexity of \( F \), \( \{\langle f(x + n^{-1}v), v \rangle\}_{n=1}^{\infty} \) is non-increasing and \( \langle h, v \rangle \leq \langle f(x + n^{-1}v), v \rangle, \forall n \). Therefore,

\[
\|h - g(x)\|_2 = \langle h - g(x), v \rangle \leq \lim_{n \to \infty} \langle f(x + n^{-1}v), v \rangle - \langle g(x), v \rangle.
\]

The continuity of \( g \) yields \( g(x) = \lim_{n \to \infty} g(x + n^{-1}v) \) and

\[
\|h - g(x)\|_2 \leq \lim_{n \to \infty} \langle f(x + n^{-1}v) - g(x + n^{-1}v), v \rangle. \tag{D.2}
\]

Define \( B(r) = \{y \in \mathbb{R}^d : \|y - x\|_2 \leq r \} \) for any \( r \geq 0 \). Since \( x \in \Omega \) and \( \Omega \) is open, there exists \( \delta > 0 \) such that \( B(\delta) \subseteq \Omega \). For any \( n > 1/\delta \), we have \( x + n^{-1}v \in B(\delta) \subseteq \Omega \) and thus

\[
\langle f(x + n^{-1}v) - g(x + n^{-1}v), v \rangle \leq \|f(x + n^{-1}v) - g(x + n^{-1}v)\|_2 \\
\leq \sup_{y \in \Omega} \|f(y) - g(y)\|_2 = M. \tag{D.3}
\]

The inequalities (D.2) and (D.3) imply that \( \|h - g(x)\|_2 \leq M. \)

\[\square\]

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